ON CHAPTER XII IN
CARTAN’S ”LEÇONS SUR LA
GÉOMÉTRIE DES ESPACES DE
RIEMANN”.

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Abstract

One shows that Cartan’s method of adapted frames in Chapter XII of his famous treatise of Riemannian geometry, leads to a classification theorem of homogeneous Riemannian manifolds. Examples of classification in 3D dimensions obtained by Cartan are given using this powerful method.

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1 Introduction.

The first edition of Cartan’s Riemannian geometry textbook (some think, the most important text on Riemannian geometry to this day), was based on Cartan’s lectures in 1925. An augmented edition appeared in 1946 [second edition, Gauthier-Villars, Paris, 1946]. In view of the great influence which this work has exerted on the subsequent development of Differential Geometry. The book was translated in English and in Russian, and carefully reviewed in Math Reviews in 1985. That review, which was presenting Cartan’s work to a worldwide audience, had an important omission: Cartan’s contribution in Chapter XII: Groups of Isometries of a Riemannian Manifold. To these days the elegance and the richness in ideas adapted frames that were at the center of that book chapter, are still little known to the general community of mathematicians, and surprisingly enough, are rarely quoted by Riemannian geometry experts. In this paper we will go to over that Chapter with having also on mind more recent developments of Cartan’s ideas.

2 Simply transitive groups of isometries of Riemannian manifolds

The objective of Chapter XII in [C46] was to (i) determine those Lie groups that act as groups of isometries of a Riemannian manifold, and (ii) for such a Lie group, find the Riemannian manifolds admitting that group as a group of isometries.
Assume $K$ is a Lie group, subgroup of the group $I_gM$ of isometries of the Riemannian manifold $(M, g)$. For each point $x \in M$, the orbit of $x$ (called by Cartan trajectory) is $K(x) = \{k(x), k \in K\}$. Cartan introduces the method of adapted frames. For each $u \in OM_x$, Cartan defines the adapted frame field to be frame field defined on the orbit $K(x)$ as follows:

$$U(kx) = (d_xk)(u), \forall k \in K.$$  \hspace{1cm} (1)

The group $K$ is transitive if $K(x) = M$, which is same as saying that $(M, g)$ is a Riemannian homogeneous space. Cartan considers first the particular case when the group $K$ is simply transitive. In this case in the isotropy group $G = K_x$, subgroup of elements in $K$ that keep the point $x$ fixed, is trivial, and the manifold $M$ and $K$ are in a one to one correspondence, thus $M$ has a Lie group structure. Moreover the adapted frame field $U$ defined in (1) determines the Riemannian metric on $M$, since if we define $\Theta_u$ to be the dual form of the orthoframe field $U$, then the Riemannian metric $g$ is given by

$$g = ||\Theta_u||^2.$$  \hspace{1cm} (2)

Assume $M$ has dimension $n$ and let

$$\Theta_u(k(x) = \theta^1(k(x))U_1(k(x)) + \cdots + \theta^n(k(x))U_n(k(x)).$$  \hspace{1cm} (3)

Then since the form $\Theta_u$ is invariant ($\forall k \in K, L^*_k\Theta_u = \Theta_u$), it follows that $d\Theta_u$ is also invariant, therefore

$$d\theta^i = \frac{1}{2}C^i_{jk}\theta^j \wedge \theta^k,$$  \hspace{1cm} (4)
where $C_{jk}^i$ are constants with $C_{jk}^i + C_{kj}^i = 0$.

On the other hand the connection forms associated with the frame field $\Theta_u$ are defined by

$$d\theta^i + \theta^i_j \wedge \theta^j = 0, \theta^i_j + \theta^j_i = 0, \forall i, j = 1, \ldots, n.$$  \hspace{1cm} (5)

From (4), (5) Cartan obtains

$$\theta^i_j = \frac{1}{2} C_{jk}^i \theta^k.$$  \hspace{1cm} (6)

On the other hand the curvature forms associated with the frame field $\Theta_u$ are defined by

$$\Omega^i_j = d\theta^i_j + \frac{1}{2} \theta^i_k \wedge \theta^k_j = 0, \Omega^i_j + \Omega^j_i = 0, \forall i, j = 1, \ldots, n,$$  \hspace{1cm} (7)

and the Riemann-Christoffel curvature coefficients $R_{jkr}^i$ with respect to this frame are given by

$$\Omega^i_j = \frac{1}{2} R_{jkr}^i \theta^k \wedge \theta^r, R_{jkr}^i + R_{jrk}^i = 0, \forall i, j, k, r = 1, \ldots, n,$$  \hspace{1cm} (8)

and since we work in an orthogonal frame field, $R_{jkr}^i = R_{ijkr}$ From equations (5)-(8) Cartan obtains

$$R_{ijkr} = -\frac{1}{4} C_{ij}^m C_{krm}.$$  \hspace{1cm} (9)

Example 2.1 In Chapter XII Cartan considers the particular example of the unit sphere $S^3 = \{x = (x^1, x^2, x^3, x^4) \in \mathbb{R}^4, \|x\|^2 = 1$, and considers the Pfaff forms

$$\sigma_1 = -x^2 dx^1 + x^1 dx^2 - x^4 dx^3 + x^3 dx^4$$
$$\sigma_2 = -x^3 dx^1 + x^4 dx^2 + x^1 dx^3 - x^2 dx^4$$
$$\sigma_3 = -x^4 dx^1 - x^3 dx^2 + x^2 dx^3 + x^1 dx^4$$  \hspace{1cm} (10)
that correspond to an adamle frame field corresponding to a simply transitive group of isometries. Milnor ([M76]) showed that any left invariant metric on $S^3$ is of the form $g_\lambda$, where

$$g_\lambda = 4\left((\lambda_2\lambda_3)^{-1}\sigma_1^2 + (\lambda_1\lambda_3)^{-1}\sigma_2^2 + (\lambda_1\lambda_2)^{-1}\sigma_3^2\right).$$  

(11)

3 Riemannian manifolds that admit a multiply transitive group of isometries.

The second part of Chapter XII in [C46] is dedicated to transitive groups of isometries $K$ of a Riemannian manifold $(M, g)$ for which the dimension of the isotropy group $H = K_x$ is positive. As a starting point, Cartan the isotropic representation $g = \lambda_u(h)$ ([1]) of the isotropy algebra $\mathfrak{h}$ as a Lie subalgebra of $\mathfrak{o}(n)$. Note that the Lie group $K$ is embedded in the orthoframe bundle $O_gM$, via

$$\phi_u(k) = (d_xk)(u), \forall k \in K.$$  

(12)

Let $\theta \in \Omega^1(O_gM, \mathbb{R}^n), \omega \in \Omega^1(O_gM, \mathfrak{o}(n))$ be the dual, respectively the Levi-Civita connection form of $(M, g)$. The vector valued differential forms $\theta_u = \phi_u^*\theta, \omega_u = \phi_u^*\omega$ are left-invariant differential forms on $K$, with $\theta_u \in \Omega^1(K, \mathbb{R}^n), \omega_u \in \Omega^1(K, \mathfrak{o}(n))$, and if $g^\perp$ is the orthocomplement of $g$ in $\mathfrak{o}(n)$, then the left invariant forms $\theta_u, \omega_{u,g}$ on $K$ are linearly independent, and the Levi-Civita connection form, then the $\phi_u$ pull-back of the structure equations,

$$d\theta + \omega \wedge \theta = 0, \Omega = d\omega + \frac{1}{2}\omega \wedge \omega,$$  

(13)
yield the Maurer-Cartan equations of $K$.

$$d\theta_u + \omega_u \wedge \theta_u = 0, \Omega_u = d\omega_u + \frac{1}{2}\omega_u \wedge \omega_u,$$  \hspace{1cm} (14)

If $\omega_u = \omega_{u, g} \oplus \omega_{u, g^\perp}$ is the decomposition of $\omega_u$ with respect to the splitting $\mathfrak{o}(n) = \mathfrak{g} \oplus \mathfrak{g}^\perp$, the $\mathfrak{g}^\perp$-component $\omega_{u, g^\perp}$ depends on $\theta_u$ only as shown by Cartan, therefore there is a $\mathfrak{g}$-invariant linear map $\Gamma : \mathbb{R}^n \to \mathfrak{g}^\perp$, such that $\omega_{u, g^\perp} = \Gamma(\theta_u)$. Similarly, if $\Omega_u = \Omega_{u, g} \oplus \Omega_{u, g^\perp}$, there is a $\mathfrak{g}$-invariant bilinear map $\bar{\Omega} : \mathbb{R}^n \times \mathbb{R}^n :\to \mathfrak{g}$, such that $\Omega_{u, g} = \bar{\Omega}(\theta_u, \theta_u)$.

The Lie algebra equations, dual to (14) are:

$$T(X, Y) = \Gamma(Y)X - \Gamma(X)Y$$

$$\bar{\Omega}(X, Y) = \bar{\Omega} - [\Gamma(X), \Gamma(Y)]_{\mathfrak{g}}$$

$$[\xi, \eta] = [\xi, \eta]$$

$$[\xi, X] = \xi(X)$$

$$[X, Y] = -T(X, Y) - \bar{\Omega}(X, Y),$$  \hspace{1cm} (15)

leading us to the following definition

**Definition 3.1** An $n$-dimensional Cartan triple in a triple $(\mathfrak{g}, \Gamma, \bar{\Omega})$ where $\Gamma : \mathbb{R}^n \to \mathfrak{g}^\perp$ is a $\mathfrak{g}$-invariant linear map and

$$\bar{\Omega} : \mathbb{R}^n \times \mathbb{R}^n \to \mathfrak{g}$$

is a $\mathfrak{g}$-invariant bilinear map such that if we define $T, \bar{\Omega}$ and the “taller” bracket $[\cdot, \cdot]$ by (15), then $[\cdot, \cdot]$ yields a Lie algebra structure on $\mathfrak{t}(\mathfrak{g}, \Gamma, \bar{\Omega}) = \mathfrak{g} \oplus \mathbb{R}^n$, provided some identities in $\bar{\Omega}$ and $\Gamma$, resulting from the Jacobi identities for that Lie algebra hold true.

Here $[\cdot, \cdot]$ is the commutator of two matrices and $\xi_{\mathfrak{g}}$ is the $\mathfrak{g}$-component of $\xi$ with respect to the decomposition $\mathfrak{o}(n) = \mathfrak{g} \oplus \mathfrak{g}^\perp$. 

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3.1 Maximal closed Cartan triples and full groups of isometries of Riemannian homogeneous spaces

Cartan triples were introduced by Patrangenaru [P94], who used them to classify metrically Riemannian homogeneous spaces. Note that for each Cartan triple \((g, \Gamma, \Omega)\), \(g\) is a Lie subalgebra of \(\mathfrak{t} = \mathfrak{t}(g, \Gamma, \Omega)\). Let \(K\) be the simply connected Lie group of Lie algebra \(\mathfrak{t}\) and let \(G\) be the connected Lie subgroup of \(K\), whose Lie algebra is \(g\). The Cartan triple \((g, \Gamma, \Omega)\) is said to be closed if \(G\) is a closed subset of \(K\).

Assume \(\mathcal{R}_1, \mathcal{R}_2\) are two Cartan triples, such that for \(j = 1, 2, \mathcal{R}_j = (g_j, \Gamma_j, \Omega_j)\). We say that \(\mathcal{R}_1 \leq \mathcal{R}_2\) if there is a vector subspace \(a\) of \(g_2\) such that

\[
g_1 \subseteq g_2 = g_1 \oplus a, g_1^\perp = g_2^\perp \oplus a
\]

(16)

and with respect to the decompositions in (3) we have

\[
\Gamma_1 = \Gamma_2 \oplus \Gamma_a, \quad (17)
\]

\[
\Omega_2 = \Omega_1 \oplus \Omega_a, \quad (18)
\]

where \(\Gamma_a\) is the \(a\)-component of \(\Gamma_1\) and \(\Omega_a\) is the \(a\)-component of \(\Omega_2\) with respect to the corresponding decomposition in (3).

Let \(\mathcal{C}_n\) be the set of all \(n\)-dimensional Cartan triples and let \(\mathcal{M}_n\) be the set of maximal Cartan triples in \((\mathcal{C}_n, \leq)\). The orthogonal group \(O(n)\) acts on the right on \(\mathcal{C}_n\), leaving \(\mathcal{M}_n\) invariant. This action \(A_n\)
is given by $A_n((g, \Gamma, \Omega), a) = (g', \Gamma', \Omega')$, where

$$g' = Ad(a^{-1})g,$$

$$\Gamma'(\cdot) = Ad(a^{-1})\Gamma(a(\cdot)),$$

$$\Omega'(\cdot, \cdot) = Ad(a^{-1})\Omega(a(\cdot), a(\cdot)).$$

The following result is an immediate consequence of the main result in [P94].

**Theorem 3.1** ([P94]). There is a one to one correspondence between isometry classes of simply connected $n$-dimensional Riemannian homogeneous spaces and $A_n$-orbits of closed Cartan triples in $\mathcal{M}_n$.

In this correspondence, if $(M, g)$ is a Riemannian homogeneous space, if $u \in O_gM_x$ is a fixed orthoframe at a given point $x \in M$, if $\mathfrak{t}(M)_x$ is the Lie algebra of Killing vector fields on $M$ vanishing at $x$, and if $\lambda_u$ is the linear isotropic representation, then one may take $g = \lambda_u(\mathfrak{t}(M)_x)$, and $T$ and $\Omega$ in equation (2) are the **torsion and the $g$-component of the curvature** of the Ambrose-Singer connection. Conversely a simply connected Riemannian homogeneous space corresponding to a closed maximal Cartan triple is usually called a **geometric realization** of that Cartan triple. As a manifold is the quotient space $K/G$, where $K$ is the simply connected Lie group of Lie algebra $\mathfrak{t}$ and $G$ is the Lie subgroup corresponding to $g$. The subgroup $G$ is simply connected by the exact homotopy sequence of the fibration $G \subseteq K \rightarrow K/G$. The metric is defined by the Euclidean norm in $\mathbb{R}^n$, and, by the definition of the Cartan triple is $ad(g)$-invariant, hence $K/G$ is a reductive homogeneous space [KN69].
Remark 3.1 Theorem 3.1 in the line of Cartan’s algorithmic program of classification of Riemannian homogeneous spaces in dimension $n$. That program was based on the following three steps:
(i) Determine all the closed Lie subgroups of $O(n)$.
(ii) For a given Lie subgroup $G \subseteq O(n)$, determine all the Lie groups $K$ acting as transitive groups of isometries of a Riemannian manifold with isotropy group $G$, and
(iii) Given a Lie group $K$ as in (ii), determine all the homogeneous spaces $(M, g)$ admitting $K$ as a group of isometries.

4 The 3D Riemannian homogeneous spaces with a multiply transitive group of isometries.

Let $(e^i_j)_{i,j=1,...,n}$ be the natural basis of $\mathfrak{gl}(n, \mathbb{R})$; the matrices $e^i_j$ act on the natural basis of $\mathbb{R}^n$ as linear endomorphisms by:

$$(e^i_j)(e_k) = \delta^i_k e_j, \forall i, j, k = 1, n. \quad (22)$$

An orthogonal basis of $\mathfrak{o}(n)$ with respect to the Killing form is given by

$$f^i_j = e^i_j - e^j_i, \forall (i, j), 1 \leq j < i \leq n. \quad (23)$$

Cartan considered in a separate section the case $n = 3$. As a first step in the Cartan triple method, note that any nontrivial, proper Lie subalgebra of $\mathfrak{o}(3)$ is conjugated to $\mathfrak{g}_1 = \mathbb{R} f^2_1$. Therefore, in the multiply
transitive case, by Theorem 3.1 it suffices to list only $A_3$-orbits of the following “special” Cartan triples:

- $(\mathfrak{o}(3), 0, \Omega)$, and
- $(\mathfrak{g}_1, \Gamma, \Omega)$.

An $(\mathfrak{o}(3), 0, \Omega)$ Cartan triple corresponds to a Riemannian manifold whose group of isometries is six dimensional. In this case a geometric realization of such a Cartan triple has constant curvature ([K72], Theorem 3.1). A three dimensional simply connected Riemannian manifold $(M, g)$ with $\dim(I(M, g)) = 6$ and positive sectional curvature is isometric to a round sphere $S^3_R$ of radius $R > 0$, the $\mathfrak{o}(3)$-curvature of which has the form $\Omega = R^{-2}\Omega_1$. Here

$$\Omega_1(x, y) = (x^1 y^2 - x^2 y^1) f_1^2 + (x^1 y^3 - x^3 y^1) f_3^2 + (x^2 y^3 - x^3 y^2) f_3^2. \quad (24)$$

Élie Cartan’s application of his method of adapted frames in 3D, in our terminology amounts to listing the $(\mathfrak{g}_1, \Gamma, \Omega)$ - Cartan triples. Cartan’s results are as follows:

$$\Gamma(e_1) = af_3^3 + bf_2^3,$$

$$\Gamma(e_2) = -bf_3^3 + af_2^3,$$

$$\Gamma(e_3) = 0, \quad (25)$$

$$\Omega(e_1, e_2) = kf_1^2,$$

$$\Omega(e_1, e_3) = \Omega(e_2, e_3) = 0,$$

where

$$a(k + a^2 + b^2) = ab = 0. \quad (26)$$

If $a \neq 0$, from equations (25)-(26) with $a = \text{Span}(f_3^3, f_2^3)$, one can show that the Cartan triple is smaller than $(\mathfrak{o}(3), 0, -a^2\Omega_1)$, thus a
geometric realization has constant negative sectional curvature. Note that in the case of a symmetric space $M$ the Ambrose-Singer connection and the Riemannian connection coincide as connections on $M$, therefore one can read both the isometry group and the curvature operator from the Cartan triple (see equations (2)).

If $a = 0$, a straightforward computation shows that a geometric realization has the Ricci quadratic form given by

$$Ric(x) = (k + b^2)((x^1)^2 + (x^2)^2) + 2b^2(x^3)^2.$$  \hfill (27)

Since the sectional curvature is positive if and only the Ricci quadratic form is positively defined, it follows that a geometric realization of such a Cartan triple has positive sectional curvature only if $b \neq 0$ and $k > -b^2$. Moreover, Cartan \cite{C46} showed that in this case a geometric realization is topologically a three dimensional sphere. Also note that such a Cartan triple is maximal iff $b \neq 0, b^2 \neq k > -b^2$. From equation (25) it follows that the isometry group of a geometric realization of such a Cartan triple is a 4 dimensional extension of the the group $S^3$ of unit quaternions by the group of unit complex numbers $S^1$. Indeed from (2) and (12), with $a = 0$, it follows that

$$\begin{align*}
\left[ e_1, e_2 \right] &= -2be_3 - (k + b^2)f_1^2, \\
\left[ e_1, e_3 \right] &= be_2, \\
\left[ e_2, e_3 \right] &= -be_1, \\
\left[ f_1^2, e_1 \right] &= -e_2, \\
\left[ f_1^2, e_2 \right] &= e_1, \\
\left[ f_1^2, e_3 \right] &= 0,
\end{align*}$$  \hfill (28)

showing that if $K$ is the simply connected Lie group of Lie algebra $\mathfrak{k} = \{ \mathfrak{g}_1, \Gamma, \Omega \}$, then $\mathfrak{s} = \text{Span}(e_1, e_2, 2be_3 + (k + b^2)f_1^2)$ is a Lie subalgebra of $\mathfrak{k}$ that is transverse to $\mathfrak{g}_1$. From the general Cartan triple
approach \[P94\] it follows that \( \mathfrak{s} \) is isomorphic to a transitive Killing algebra of the geometric realization \( M \) of \((\mathfrak{g}_1, \Gamma, \Omega)\), that is \( \mathfrak{s} \) can be identified with a Lie algebra of Killing vector fields on \( M \), such that for any point \( x \in M \), the evaluation map \( ev_x : \mathfrak{s} \to T_x M \) \( ev_x(\xi) = \xi(x) \) is onto. Since \( \text{dim}\mathfrak{s} = \text{dim}(M) \), it follows that \( ev_x \) is an isomorphism.

Thus the connected Lie subgroup \( S \) of \( K \) of Lie subalgebra \( \mathfrak{s} \) acts transitively on \( M \), with a discrete isotropy group (fiber \( F \) of the projection \( \pi : S \to M, \pi(g) = g(x) \)). Note that from the structural equations of this Lie subalgebra, it turns out that \( S \) is isomorphic to \( S^3 \), the group of unit quaternions. From the exact homotopy sequence of this fibration, since \( S \) is connected and simply connected we get \( \pi_0(F) = 0 \), that is \( F \) is the fiber is the trivial subgroup of \( S \), showing that \( \pi \) is a diffeomorphism. Note that and since \( K \) is compact, the isotropy group (in this case the Lie subgroup of \( K \) whose Lie algebra is \( \mathbb{R}f_1^2 \)) is a closed one dimensional subgroup of a compact Lie group and therefore it is isomorphic to \( S^1 \).

Élie Cartan (C46, p.305) integrated the structural equations (15) and calculated the Riemannian structure of the corresponding homogeneous spaces.

**Definition 4.1** A geometric realization of \((\mathfrak{g}_1, \Gamma, \Omega)\), with \( a = 0, b \neq 0 \) and \( b^2 \neq k > -b^2 \) is said to be a Cartan sphere.

Using (15) one can construct an isomorphism of Lie algebras \( \phi : \mathfrak{k}(\mathfrak{g}_1, \Gamma, \Omega) \to \mathfrak{u}(2) \), with \( \phi(\mathfrak{s}) = \mathfrak{su}(2) \). Shankar [S2001] mentions that \( S^3 \) (regarded here as \( SU(2) \) ) admits Riemannian homogeneous structures ( not normal ) with \( U(2) \) as the connected component of
the full group of isometries. These are the only possible 4 dimensional full groups of isometries of simply connected three dimensional Riemannian homogeneous manifolds with positive sectional curvature. Therefore we have following result.

**Proposition 4.1** The Cartan spheres correspond to the three dimensional Riemannian homogeneous manifolds Shankar’s list, whose isometry groups are locally isomorphic to $U(2)$.

At the end of this section Cartan obtained the following result.

**Theorem 4.1** (Cartan) Any geometric realization of a Cartan triple in 3D that admits a four dimensional transitive group of isometry is homeomorphic to the Euclidean space, a product of a round sphere and an Euclidean line or a 3D sphere. Those geometric realizations of positive scalar curvature are spheres, but there are 3D spheres, geometric realizations of 3D Cartan triples that do not have a positive scalar curvature.

**Remark 4.1** Theorem 4.1 was later extended by in [P96] who showed that geometric realizations of 3D Cartan triples, including triples with $g = 0$, can be deformed to one of the eight 3D geometries of Thurston, that are the building blocks of 3D manifolds according to Thurston and Perelman.
5 General intransitive groups of isometries. Groups whose orbits are curves or surfaces

In the last two sections of Chapter XII, Cartan considers the case of intransitive groups of isometries. If $K$ is an intransitive group of isometries of an $n + k$ dimensional Riemannian manifold $(M, g)$, and $K(x)$ be an orbit of maximum dimension $n$ of $K$, then $K$ is transitive on $K(x)$, therefore Cartan shows that one can select coordinates $x^1, \ldots, x^n, y^1, \ldots, y^r$, locally around $K(x)$ such that $y^a, a = 1, \ldots, r$ are invariants of the group $K$ and for fixed $y^1, \ldots, y^r$, the group acts transitively on the submanifold described by $x^1, \ldots, x^n$. This is a foliation of $M$ by homogeneous spaces. Finally in Chapter XII of [C46], Cartan describes the form of the metric tensor, in case the maximal orbits are curves or surfaces.

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