SYMPLECTIC NON-SQUEEZING FOR THE CUBIC NONLINEAR
KLIEG-GORDON EQUATION ON $\mathbb{T}^3$

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Abstract. We consider the periodic defocusing cubic nonlinear Klein-Gordon equation in three dimensions in the symplectic phase space $H^{1/2} (\mathbb{T}^3) \times H^{-1/2} (\mathbb{T}^3)$. This space is at the critical regularity for this equation and we note that there is no uniform control on the local time of existence for arbitrary initial data. We prove a local in time non-squeezing result and a global in time non-squeezing result for certain open subsets of the phase space, with no smallness condition on the size of the initial data. Analogously to the work of Burq and Tzvetkov [10], we first define a set of full measure with respect to a suitable randomization of the initial data on which the flow of this equation is globally defined. The proof then relies on an approximation result for the flow, which uses probabilistic estimates for the nonlinear component of the flow map as well as deterministic stability theory, and Gromov's non-squeezing theorem. For the global in time result, we also use the infinite dimensional symplectic capacity defined by Kuksin [21].

1. Introduction

We consider the behaviour of solutions to the initial-value problem for the periodic defocusing cubic nonlinear Klein-Gordon equation

$$\begin{align*}
\begin{cases}
u_{tt} - \Delta u + u + u^3 &= 0, \quad u : \mathbb{R} \times \mathbb{T}^3 \rightarrow \mathbb{R} \\
(u, \partial_t u)|_{t=0} &= (u_0, u_1) \in H^{1/2} (\mathbb{T}^3) \times H^{-1/2} (\mathbb{T}^3) =: \mathcal{H}^{1/2} (\mathbb{T}^3),
\end{cases}
\end{align*}$$

(1.1)

where $H^{1/2} (\mathbb{T}^3)$ is the usual inhomogeneous Sobolev space. Although there is no scaling symmetry for the nonlinear Klein-Gordon equation, the nonlinear wave equation

$$u_{tt} - \Delta u + u^p = 0$$

(1.2)

has a scaling symmetry

$$u(t, x) \mapsto u^\lambda (t, x) := \lambda^{2/(p-1)} u(\lambda t, \lambda x)$$

and the scale invariant critical space corresponds to the homogeneous Sobolev space at regularity $s_c := \frac{d}{2} - \frac{2}{p-1}$. Since ill-posedness and blowup are normally associated with high frequencies or short time scales, the nonlinear term dominates the mass term and one can still regard $s_c$ as the critical regularity for (1.1). Note that when $d = 3$, and $p = 3$, we have $s_c = \frac{1}{2}$, hence we are interested in the behaviour of solutions of (1.1) in the critical space.

Local strong solutions to (1.1) can be constructed by adapting the works of [21] to the compact setting, using the Strichartz estimates for the nonlinear Klein-Gordon equation [36]. However, due to the critical nature of this problem, the local time of existence for solutions depends not only on the norm of the initial data but also on the profile, and moreover, the critical regularity for this equation does not correspond to a conserved quantity. Resultingly, the global wellposedness of this equation is still an open problem. The best known results on Euclidean space are for

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subcritical regularities $s \geq 3/4$. In \[26\], global wellposedness is proved for the nonlinear Klein-Gordon equation by working in Besov spaces and adapting Bourgain’s high-low argument \[5\] and arguments from \[20\]. Once again, these results can be modified for the compact setting.

In this work, we will study the qualitative behaviour of solutions to (1.1) by investigating symplectic non-squeezing for the flow of this equation. In the finite dimensional setting, Gromov’s non-squeezing theorem states that there is no symplectic embedding of a ball into a cylinder unless the radius of the ball is less than or equal to that of the cylinder. There is a natural symplectic structure on Sobolev spaces, and for infinite dimensional Hamiltonian equations, there are specific regularities where this structure is compatible with the flow of the equation. Consequently, infinite dimensional Hamiltonian equations can be realized, at least formally, as a symplectic flow on these phase spaces. For Hamiltonian equations, symplectic non-squeezing is connected to the problem of weak turbulence, which examines, for example, whether the energy of a given solution concentrates at high frequencies over time. Specifically, symplectic non-squeezing for Hamiltonian equations addresses the question of whether or not the Fourier coefficients of solutions can become arbitrarily small after long time and whether solutions exhibit asymptotic completeness, see Section 2 for details.

In finite dimensions, Gromov discovered the first example of a symplectic capacity, namely the Darboux width. Gromov proved his non-squeezing theorem by showing that the Darboux width is invariant under the flow of a symplectomorphism. Subsequently, there were other (comparable but not necessarily equivalent) definitions of symplectic capacities, developed by Ekeland-Hofer, Viterbo, and Hofer-Zehnder among others. See \[18\] and references therein for a more thorough account of these developments.

The study of infinite dimensional symplectic capacities and non-squeezing for nonlinear Hamiltonian PDEs was initiated by Kuksin \[21\]. He extended the definition of the Hofer-Zehnder capacity to infinite dimensional Hilbert spaces and proved the invariance of this capacity under the flow of certain Hamiltonian equations provided the flow maps for the equation are of the form

$$\Phi(t) = \text{linear operator} + \text{compact smooth operator}.$$ 

This infinite dimensional symplectic capacity, $c$, on the Hilbert space inherits the finite dimensional normalization

$$c(B_r(u_*)) = c(C_r(z; k_0)) = \pi r^2,$$

where $B_r(u_*)$ is the infinite dimensional ball centered at $u_*$ in the Hilbert space, and $C_r(z; k_0)$, the infinite dimensional cylinder (see (1.13) for the precise definition of the cylinder in our context). The proof of this normalization in infinite dimensions is an adaptation of the original proof by Hofer and Zehnder which can be found in \[18\], see \[21\] for details of the infinite dimensional argument. Consequently, if a flow map $\Phi$ preserves capacities, one can conclude that squeezing is impossible, namely

$$\Phi(t)(B_R(u_*)) \subset C_r(z; k_0) \Rightarrow R \leq r.$$

Several examples of nonlinear Klein-Gordon equations with weak nonlinearities can readily be shown to satisfy (1.3), see \[21\]. Symplectic non-squeezing was later proved for certain subcritical nonlinear Klein-Gordon equations in \[2\] using Kuksin’s framework, see also \[34\]. Bourgain later extended these results to the cubic NLS in dimension one in \[7\], where the flow is not a compact perturbation of the linear flow. There, the argument follows from approximating the full equation by a finite dimensional flow and applying Gromov’s finite dimensional non-squeezing result to this approximate flow. Symplectic non-squeezing was also proven for the KdV \[16\]. In this situation,
there is a lack of smoothing estimates in the symplectic space which would allow the infinite dimensional KdV flow to be easily approximated by a finite-dimensional Hamiltonian flow. To resolve this issue, the authors of [16] invert the Miura transform to work on the level of the modified KdV equation, for which stronger estimates can be established.

As we will see in Section 2, the symplectic phase space for any nonlinear Klein-Gordon equation is \( H^{1/2}(\mathbb{T}^d) \) for any dimension \( d \geq 1 \). In particular, for the cubic nonlinear Klein-Gordon equation in dimension three, the symplectic space is at the critical regularity, and this presents some serious obstructions to using simple modifications of the existing arguments. Kuksin’s approach requires some additional regularity in the compactness estimates. In light of ill-posedness results below the critical space, for instance the results of Christ-Colliander-Tao [13] can be adapted to (1.1) and also [22] and [19], there is no way to gain the additional regularity needed. Bourgain’s argument in [7] uses an iteration scheme in which one needs uniform control over time-steps of the iteration, and, once again, this seems to be a genuine obstruction to applying this argument at the critical regularity. Finally, the arguments of [16] depend heavily on the structure of the KdV equation.

In order to circumvent these difficulties, we will use a probabilistic approach. Ultimately, however, we obtain the following deterministic non-squeezing result:

**Theorem 1.1.** Let \( \Phi \) denote the flow of the cubic nonlinear Klein-Gordon equation (1.1). Fix \( 0 < R, k_0 \in \mathbb{Z}^3, z \in \mathbb{C}, \) and \( u_* \in H^{1/2}(\mathbb{T}^3) \). For all \( 0 < \eta < R \), there exists \( N \equiv N(\eta, \|u_*\|, R, k_0) \) and \( \sigma \equiv \sigma(\eta, \|u_*\|, R, k_0) > 0 \) such that for any \( r < R - \eta \), and any \( 0 \leq t \leq \sigma \),

\[
\Phi(t)(\Pi_{2N}B_R(u_*)) \not\subseteq C_r(z; k_0).
\]

**Remark 1.1.** Even though we use probabilistic techniques, we are able to exploit the available stability theory in order to obtain the deterministic statement in Theorem 1.1. More precisely, once we prove probabilistically that there exists a set of full measure on which the flow is globally defined, we are able to define the flow locally in time on all of \( \Pi_{2N}B_R(u_*) \). Once this deterministic flow has been defined, we can once again use the critical stability theory in order to obtain the necessary estimates for the flow on \( \Pi_{2N}B_R(u_*) \), which yield Theorem 1.1.

**Remark 1.2.** For the cubic nonlinear Klein-Gordon equation in the critical space, not only is there no established global flow, but the local time of existence depends on the profile of the initial data. Hence, a priori we cannot define \( \Phi(t) \) for any positive times \( t > 0 \) on any infinite set as the infimum over the local times of existence for initial data in that set might be zero. We note that although Theorem 1.1 is local, the length of the time interval does not depend on the profile of the initial data as is typically the case in the critical setting. We fix the projection in (1.5) at frequency \( 2N \) so that we have enough control to define the flow map \( \Phi(t) \) for \( t \in [0, \sigma] \). If a global flow can be defined on this space, then this theorem yields local non-squeezing for the full ball \( B_R(u_*) \).

**Remark 1.3.** The parameter \( \eta \) in Theorem 1.1 corresponds to the control we can obtain over the radius of the cylinder. If we demand better control over the radius, this theorem only holds for shorter time scales.

In recent years, probabilistic tools have been extremely helpful in closing the gap between the scaling prediction and existing deterministic results in various situations, even in certain supercritical regimes. In particular, much progress has been made in studying invariant Gibbs measures for Hamiltonian PDEs. Such measures have been studied in the work of Zhidkov [42, 43, 44], Lebowitz, Rose and Speer [23] and subsequently by many others. In [13], working...
with the Gibbs measure introduced in \cite{23}. Bourgain proved the existence of a Hamiltonian flow acting in the support of this measure for the nonlinear Schrödinger equation in one and two spatial dimensions. Bourgain then used this invariance to prove almost sure global wellposedness for these equations in the support of the measure. In \cite{11,12}, Burq and Tzvetkov consider the cubic nonlinear wave equation on a three-dimensional compact manifold. They constructed large sets of initial data of super-critical regularity that lead to local solutions, using a randomization procedure which relies on expansion of the initial data with respect to an orthonormal basis of eigenfunctions of the Laplacian. Together with invariant measure considerations they also prove almost sure global existence for the cubic nonlinear wave equation on the three-dimensional unit ball. Many further results in this direction have been obtained in recent years, see \cite{8}, \cite{4}, \cite{12}, \cite{39}, \cite{40}, \cite{38}, \cite{31}, \cite{27}, \cite{29}, \cite{33}, \cite{6}, \cite{41} and references therein.

While it is desirable to establish the existence of an invariant Gibbs measure, in many situations there are serious technical difficulties associated with defining such a measure. Most notably, in dimension $d \geq 3$ it is only possible to define a Gibbs measure for initial data in very rough Sobolev spaces. In such spaces, the multilinear analysis necessary for wellposedness arguments is not available.

In the absence of an invariant measure, other approaches have been developed to show almost sure global existence for periodic super-critical equations via a suitable randomization of the initial data. Energy estimates are used by Nahmod, Pavlović and Staffilani \cite{28} in the context of the periodic Navier-Stokes equation in two and three dimensions and by Burq and Tzvetkov \cite{10} for the three-dimensional periodic defocusing cubic nonlinear wave equation. Colliander and Oh \cite{15} adapt Bourgain’s high-low frequency decomposition \cite{5} to prove almost sure global existence of solutions to the one-dimensional periodic defocusing cubic nonlinear Schrödinger equation below $L^2(T)$. See also \cite{25} and \cite{32} for almost sure global wellposedness of the nonlinear wave equation of power type on Euclidean space.

To prove Theorem 1.1, we rely on an adaptation of the almost sure global wellposedness result from \cite{10} in order to work on a set of full measure with respect to a suitable randomization of the initial data, $\Sigma \subset H^{1/2}(\mathbb{T}^3)$, on which the nonlinear Klein-Gordon equation is globally wellposed. We will show that for a certain nested sequence of subsets $\Sigma_\lambda \subset \Sigma$, the flow of this equation effectively has the form (1.3). Before stating our main results, we will describe the randomization procedure for the initial data.

Let $\{(h_k,l_k)\}_{k \in \mathbb{Z}^3}$ be a sequence of independent, zero-mean, real-valued Gaussian random variables on a probability space $(\Omega,\mathcal{A},\mathbb{P})$. Fix $(\phi_0,\phi_1) \in H^s(\mathbb{T}^3)$, and define a randomization map $\Omega \times H^s \to H^s$ by

$$
(\omega,(\phi_0,\phi_1)) \mapsto (\phi_0^\omega,\phi_1^\omega) := \left( \sum_{k \in \mathbb{Z}^3} h_k(\omega)\hat{\phi}_0(k)e^{ik \cdot x}, \sum_{k \in \mathbb{Z}^3} l_k(\omega)\hat{\phi}_1(k)e^{ik \cdot x} \right).
$$

We could similarly take non-Gaussian random variables which satisfy suitable boundedness conditions on their distributions. The quantity above is understood as a limit in $L^2(\Omega;H^s(\mathbb{T}^3))$. We note that for any $(\phi_0,\phi_1)$, the map (1.6) induces a probability measure on $H^s$, given by

$$
\mu_{(\phi_0,\phi_1)}(A) = \mathbb{P}(\omega \in \Omega : (\phi_0^\omega,\phi_1^\omega) \in A).
$$

We denote by $\mathcal{M}^s$ the set of such measures:

$$
\mathcal{M}^s := \{ \mu_{(\phi_0,\phi_1)} : (\phi_0,\phi_1) \in H^s \}.
$$
We note that the support of any $\mu \in M^s$ is contained in $H^s$. Furthermore, if for some $s_1 > s$ we have that $(\phi_0, \phi_1) \notin H^{s_1}$ then the induced measure satisfies $\mu(H^{s_1}) = 0$. In other words, this randomization procedure does not regularize at the level of Sobolev spaces. Moreover, if $\phi_0$ and $\phi_1$ have all their Fourier coefficients nonzero, then the support of the corresponding measure $\mu_{(\phi_0,\phi_1)}$ is $H^s$, that is, $\mu_{(\phi_0,\phi_1)}$ charges every open set in $H^s$ with positive measure. As a consequence, for such a measure, sets of full $\mu$ measure are dense. See [11] for details.

The arguments of Theorem 2 in [10], which treats the cubic nonlinear wave equation, apply to the cubic nonlinear Klein-Gordon equation with the slight modification that one must consider the inhomogeneous energy functional
\begin{equation}
E(w) = \frac{1}{2} \int |\nabla w|^2 + w^2 + (w_t)^2 + \frac{1}{2} w^4.
\end{equation}
We denote by $S(t)$ the free evolution for (1.1), given by
\begin{equation}
S(t)(u_0, u_1) = \cos(t\langle \nabla \rangle)u_0 + \frac{\sin(t\langle \nabla \rangle)}{\langle \nabla \rangle} u_1,
\end{equation}
and we state Theorem 2 from [10] adapted to our situation.

**Theorem 1.2.** Let $M = T^3$ with the flat metric and fix $\mu \in M^s$, $0 < s < 1$. Then there exists a full $\mu$ measure set $\Sigma \subset H^s(T^3)$ such that for every $(u_0, u_1) \in \Sigma$, there exists a unique global solution $u$ of the nonlinear Klein-Gordon equation
\begin{equation}
\begin{aligned}
&u_{tt} - \Delta u + u + u^3 = 0, \quad u : \mathbb{R} \times T^3 \to \mathbb{R} \\
&(u, \partial_t u) \big|_{t=0} = (u_0, u_1)
\end{aligned}
\end{equation}
satisfying
\[(u(t), u_t(t)) \in (S(t)(u_0, u_1), \partial_t S(t)(u_0, u_1)) + C(\mathbb{R}; H^1_x(T^3)).\]

Furthermore, if we denote by
\[
\Phi(t)(u_0, u_1) \equiv (u(t), \partial_t u(t))
\]
the flow thus defined, the set $\Sigma$ is invariant under the flow $\Phi(t)$:
\[
\Phi(t)(\Sigma) = \Sigma \quad \forall t \in \mathbb{R}.
\]

Finally, for any $\varepsilon > 0$, there exist $C, c, \theta > 0$ such that for every $(u_0, u_1) \in \Sigma$ there exists $M = M(u_0, u_1) > 0$ such that the global solution of (1.8) given by
\[
u(t) = S(t)(u_0, u_1) + w(t)
\]
satisfies
\begin{equation}
\|(w(t), \partial_t w(t))\|_{H^1} \leq C(M + |t|)^{1-s+\varepsilon}
\end{equation}
and furthermore, for each $K > 0$,
\[
\mu((u_0, u_1) \in \Sigma : M > K) \leq Ce^{-cK^\theta}.
\]

More specifically, the set $\Sigma$ of full measure we will work with is defined by
\begin{equation}
(1.10) \quad \Sigma = \{(u_0, u_1) \in H^{1/2} : \|S(t)(u_0, u_1)\|_{L^6_x(T^3)}^6 \in L^1_{\text{loc}}(\mathbb{R}_t), \|S(t)(u_0, u_1)\|_{L^\infty_x(T^3)} \in L^1_{\text{loc}}(\mathbb{R}_t)\}.
\end{equation}
The set specified in Theorem 1.2 actually imposes an $L^3_{\text{loc}} L^6_x$ condition on the free evolution, however, it will be more convenient to work with the above definition. It is clear that the condition in (1.10) is stronger but does not change any of the arguments in [10]. This set has full measure
with respect to any \( \mu \in \mathcal{M}^\ast \) which implies, in particular, that \( \Sigma \) is not comprised of initial data which are smoother at the level of Sobolev spaces. We will work on the nested subsets \( \Sigma_\lambda \subset \Sigma \), which we will define in a following section. The union of these subsets is all of \( \Sigma \) and there exists \( C, c, \theta > 0 \) so that for any \( \lambda > 0 \), they satisfy
\[
\mu(\Sigma_\lambda) \geq 1 - C e^{-c\lambda^\theta}.
\]

Let \( \tilde{\Phi} \) denote the nonlinear component of the flow map for the cubic nonlinear Klein-Gordon equation. On these subsets, we are able to prove a probabilistic adaptation of Kuksin’s argument from [21], and prove bounds of the form
\[
\|\tilde{\Phi}(t)(u_0, u_1)\|_{L^\infty_t \mathcal{H}^2_x(S(0, T) \times \mathbb{T}^3)} \lesssim \|(u_0, u_1)\|_{\mathcal{H}_x^{s_1}}, \quad (u_0, u_1) \in \Sigma_\lambda
\]
for some \( s_1 < \frac{1}{2} < s_2 \).

Once we have established the probabilistic bounds on the nonlinear component of the flow map, our argument has two key components. The first is an approximation estimate for the flow map \( \Phi(\cdot) \) restricted to the finite dimensional subspaces defined by projecting onto the first \( 2N \) frequencies. We will show in Section 4 that this flow is a symplectomorphism when restricted to the finite dimensional subspaces defined by projecting onto the first \( 2N \) frequencies. Consequently, we are able to show that this equation preserves infinite dimensional capacities. Moreover, we will show that this equation provides a good approximation to the full nonlinear Klein-Gordon equation. More precisely, we will prove the following proposition.

**Proposition 1.3.** Let \( \Phi \) denote the flow map for the cubic nonlinear Klein-Gordon equation (1.11) and \( \Phi_N \) the flow map for (1.11). Then for any \( T > 0 \) and for every \( (u_0, u_1) \in \Sigma_\lambda \cap B_R \),
\[
\sup_{t \in [0, T]} \|\Phi(t)(u_0, u_1) - \Phi_N(t)(u_0, u_1)\|_{\mathcal{H}_x^{s_2}(\mathbb{T}^3)} \leq C(\lambda, T, R) \varepsilon_1(N)
\]
with \( \varepsilon_1(N) \to 0 \) as \( N \to \infty \).

We note that this is a global approximation result and there is no restriction on the size of the initial data.

The second component in our argument is showing that locally, if one restricts to a low frequency window, this approximation still holds with uniform bounds on an open set containing \( \Sigma_\lambda \). We construct these open sets using the deterministic critical stability theory. We will prove the following theorem.

**Theorem 1.4.** Let \( \Phi \) denote the flow of the cubic nonlinear Klein-Gordon equation (1.11) and \( \Phi_N \) the flow of (1.11). Fix \( R > 0 \) and \( N \in \mathbb{N} \). Then for all sufficiently small \( \varepsilon_0 > 0 \), there exists \( \sigma \equiv \sigma(R, \varepsilon_0, N) \) such that for any initial data \( (u_0, u_1) \in B_R \)
\[
\sup_{t \in [0, \sigma]} \|\Phi(t)(u_0, u_1) - \Phi_N(t)(u_0, u_1)\|_{\mathcal{H}_x^{s_2}(\mathbb{T}^3)} \leq C(R) [\varepsilon_1(N) + \varepsilon_0]
\]
with \( \lim_{N \to \infty} \varepsilon_1(N) = 0 \).
Remark 1.4. Once again, the projection operators in (1.12) allow us to gain the necessary control via the stability theory in order to obtain convergence on all of \( \Pi_2 \mathcal{B}_R \). It is the combination of Proposition 1.3 with the stability theory which enables us to get a deterministic non-squeezing theorem, regardless of the fact that Proposition 1.3 is a probabilistic result.

Remark 1.5. While we can prove global convergence for the flow maps for initial data in \( \Sigma_\lambda \), these sets are not open and have a rather complicated structure. This poses serious problems for the proof of the non-squeezing theorem for several reasons, among which is the fact that it is not at all clear how to compute the capacities \( c(\Sigma_\lambda \cap \mathcal{B}_R) \), or even ensure that they are positive since the subsets \( \Sigma_\lambda \) have empty interior. In order to upgrade the approximation to open subsets, one must localize in time. We have no prediction at the moment for an estimate of this capacity using Kukšin’s definition. Ultimately, it seems likely that a probabilistic definition of the capacity might be a more suitable approach. This is work in progress.

Remark 1.6. As an easy consequence of the arguments used to prove Theorem 1.4 we obtain wellposedness for long times on open subsets, as well as topological genericity for the set of initial data for which this equation is globally wellposed. Moreover, these arguments can be modified and combined with Gaussian measure considerations to obtain some qualitative information about the initial data which fail to lie in these open subsets.

As in [7] and [16], we will prove non-squeezing for the full equation by using this approximation theorem and Gromov’s finite dimensional non-squeezing theorem, which we quote here for the symplectic space \( (\mathbb{R}^{2N}, \omega_0) \), where \( \omega_0 \) is the standard symplectic form on \( \mathbb{R}^{2N} \), see [18] for details.

**Theorem 1.5** (Gromov’s non-squeezing theorem, [17]). Let \( R \) and \( r > 0 \), \( z \in \mathbb{C} \), \( 0 < k_0 \leq N \), and \( x_* \in \mathbb{R}^{2N} \). If there is a symplectomorphism \( \phi \) defined on \( \mathcal{B}_R(x_*) \subset \mathbb{R}^{2N} \) so that
\[
\phi(\mathcal{B}_R(x_*)) \subset C_r(z; k_0)
\]
then \( r \geq R \).

We would like to take \( N \to \infty \) in Theorem 1.5. To do so, we use Theorem 1.4 to obtain the necessary uniform local control. We define the infinite dimensional ball
\[
\mathcal{B}_r(u_*):= \left\{ u \in \mathcal{H}^{1/2}(\mathbb{T}^3) : \| u - u_* \|_{\mathcal{H}^{1/2}} \leq r \right\},
\]
and for \( z = (z_1, z_2) \in \mathbb{C} \) and \( k_0 \in \mathbb{Z}^3 \), the infinite dimensional cylinder
\[
C_r(z; k_0):= \left\{ (u_1, u_2) \in \mathcal{H}^{1/2}(\mathbb{T}^3) : \langle k_0 \rangle \hat{u}_1(k_0) - z_1^2 + \langle k_0 \rangle^{-1} \hat{u}_2(k_0) - z_2^2 \leq r^2 \right\}.
\]

The proof of Theorem 1.5 follows easily once we have proven Theorem 1.3. Indeed, fix parameters \( R > 0 \), \( k_0 \in \mathbb{Z}^3 \), \( z \in \mathbb{C} \), \( u_* \in \mathcal{H}^{1/2} \) and \( 0 < \eta < R \). For \( R_1 > 0 \) large we have the inclusion \( \mathcal{B}_R(u_*) \subset \mathcal{B}_{R_1} \), and we fix \( N > |k_0| \) sufficiently large and \( \varepsilon_0 > 0 \) sufficiently small so that for \( C(R_1) \) and \( \varepsilon_1(N) \) as in (1.6) we have
\[
C(R_1)\varepsilon_1(N) < \frac{\eta}{4} \quad \text{and} \quad C(R_1)\varepsilon_0 < \frac{\eta}{4}.
\]

Let \( \sigma \) be such that the conclusions of Theorem 1.4 hold. Then for all \( (u_0, u_1) \in \mathcal{B}_{R_1} \), we have
\[
sup_{t \in [0, \sigma]} \| \Phi(t)\Pi_{2N}(u_0, u_1) - \Phi(t)\Pi_{2N}(u_0, u_1) \|_{\mathcal{H}_z^{1/2}} < \frac{\eta}{2}
\]
Let \( r < R - \eta \) and define
\[
\varepsilon := \frac{R - r}{2} > \frac{\eta}{2}.
\]
By Theorem 1.5 there exists some \((u_0, u_1) \in \Pi_{2N} B_R(u_*)\) such that
\[
||k_0\|_H^N(u_0, u_1)(k_0) - z_k||^2 + ||k_0\|_H^{-1}(\partial_t F_N(u_0, u_1)(k_0) - z_2||^2 \right)^{1/2} > r + \varepsilon.
\]
Hence by triangle inequality, (1.14) and (1.15) we obtain
\[
\left(\langle|k_0|\Phi_N(\sigma)(u_0, u_1)(k_0) - z_k||^2 + \langle|k_0|\Phi_N(\sigma)(u_0, u_1)(k_0) - z_2||^2 \right)^{1/2} > r + \varepsilon - \frac{\eta}{2} > r,
\]
which concludes the proof.

Finally, we present one last version of the non-squeezing theorem. In contrast to Theorem 1.1, we do not need to consider the finite dimensional projection of the ball and the result we obtain is for large times. In exchange for this, however, we must restrict ourselves to initial data sufficiently close to elements of \(\Sigma_\lambda\) and the control we obtain over the diameter of the cylinder is not as good.

**Theorem 1.6.** Fix \(\mu \in M^{1/2}\) and let \(\Phi\) denote the flow of the cubic nonlinear Klein-Gordon equation (1.1). Let \(R, T > 0, k_0 \in \mathbb{Z}^3, z \in \mathbb{C},\) and \(u_* \in H^{1/2}(\mathbb{T}^3)\). Then there exists \(\theta > 0\) such that for every \(\varepsilon > 0\) there exists an open set \(U_\varepsilon\) with
\[
\mu(U_\varepsilon) \geq 1 - e^{-1/\varepsilon^\theta}
\]
and such that
\[
\Phi(T)(U_\varepsilon \cap B_R(u_*)) \subseteq C_r(z; k_0),
\]
for all \(r > 0\) with \(\pi r^2 < c(U_\varepsilon \cap B_R(u_*)).\)

**Remark 1.7.** The capacity \(c(U_\varepsilon \cap B_R(u_*))\) is positive since it is the capacity of an open set. In practice, the sets \(U_\varepsilon\) will be constructed by taking the \(\rho\)-fattening of the subsets \(\Sigma_\lambda\), see Section 4 for details. At the moment, we do not have a better bound for this capacity than the trivial bound for open sets. The proof of Theorem 1.6 is the only place where we use the infinite dimensional symplectic capacity. We note that similarly to Theorem 1.1, this yields a weak global non-squeezing result in the case that the flow can be defined on the interval \([0, T]\) for all initial data in \(B_R(u_*)\).

**Organization of Paper.** In Section 2 we provide some background on Symplectic Hilbert spaces and the relation of non-squeezing to the energy transition problem and we introduce the capacity we will work with. In Section 3 we collect some deterministic and probabilistic facts. In Section 4 we prove local and global properties of solutions to the full equation (1.1) and similarly for the equation with truncated nonlinearity (1.11). In Section 5 we prove the boundedness assumptions on the flow maps of these equations. In Section 6 we prove Proposition 1.3. Finally in Section 7 we prove Theorem 1.4 and Theorem 1.6.

**Notation.** We write \(X \lesssim Y\) to denote \(X \leq CY\) for some \(C > 0\) which depends only on fixed parameters. Occasionally we will also use the notation \(H^s := H^s \times H^{s-1}\), where this product space is endowed with the obvious norm. We let \(B_R(u_*)\) denote the ball of radius \(R\) centered at \(u_*\) and occasionally we will use the shorthand \(B_R := B_R(0)\). We let \(C_r(z; k)\) denote the cylinder centered at \(z \in \mathbb{C}\) of radius \(r\) in the \(k\)-th frequency. We define the Fourier projection \(\Pi_K\) on \(H^s\) as the projection onto frequencies \(|k| \leq K\) and we define \(\Pi_{> K} = I - \Pi_K\).

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2. Symplectic Hilbert spaces

We begin with some background on symplectic Hilbert spaces. We follow the exposition in [21, 16]. Consider a Hilbert space $H$ with scalar product $\langle \cdot, \cdot \rangle$ and a symplectic form $\omega_0$ on $H$. Let $J$ be an almost complex structure on $H$ which is compatible with the Hilbert space inner product, that is, a bounded self-adjoint operator with $J^2 = -1$ such that for all $z, v \in H$,

$$\omega_0(z, v) = \langle z, Jv \rangle.$$

We say the pair $(H, \omega_0)$ is the symplectic phase space for a PDE with Hamiltonian $H[z(t)]$ if the PDE can be written as

$$\dot{z}(t) = J \nabla H[z(t)].$$

(2.1)

Here, $\nabla$ is the usual gradient with respect to the Hilbert space inner product, defined by

$$\langle v, \nabla H[z] \rangle \equiv \frac{d}{d\varepsilon} H[z + \varepsilon v] \bigg|_{\varepsilon = 0}.$$

The expression (2.1) is equivalent to the condition

$$\omega_0(v, \dot{z}(t)) = \omega_0(v, J \nabla H[z(t)]) = \langle v, \nabla H[z] \rangle = \frac{d}{d\varepsilon} H[z + \varepsilon v] \bigg|_{\varepsilon = 0}.$$

(2.2)

Let $B := (1 - \Delta)^{1/2}$ and consider the Hilbert space $H^{1/2}(\mathbb{T}^3)$ with the usual scalar product

$$\langle (z_1, z_2), (v_1, v_2) \rangle \frac{1}{2} := \int_{\mathbb{T}^3} Bz_1 \cdot v_2 + \int_{\mathbb{T}^3} B^{-1}z_2 \cdot v_1$$

For $(u, u_t) = (z_1, z_2)$ we can rewrite (1.1) as the system of first order equations

$$\begin{cases}
(z_1)_t = z_2 \\
(z_2)_t = (1 - \Delta)z_1 - (z_1)^3.
\end{cases}$$

(2.3)

Define the skew symmetric linear operator

$$J : H^{1/2}(\mathbb{T}^3) \to H^{1/2}(\mathbb{T}^3), \quad J = \begin{pmatrix} 0 & B^{-1} \\ -B & 0 \end{pmatrix}$$

then $J$ is an almost complex structure on $H^{1/2}(\mathbb{T}^3)$ compatible with the symplectic form

$$\omega_{1/2}(z, v) := \int_{\mathbb{T}^3} z_1 \cdot v_2 - \int_{\mathbb{T}^3} z_2 \cdot v_1,$$

that is, setting $z := (z_1, z_2)$ and $v = (v_1, v_2)$, we have $\omega_{1/2}(z, v) = \langle z, Jv \rangle_{1/2}$. We set

$$\mathcal{N}[z] := \frac{1}{4} \int |z_1|^4 \quad \text{and} \quad A := B^2 \oplus I$$

the cubic nonlinear Klein-Gordon equation (2.3) can be written in Hamiltonian form as

$$\dot{z} = J(Az + \nabla \mathcal{N}[z]).$$

(2.5)

Equivalently we can write $\dot{z} = J \nabla H(z)$ for the Hamiltonian

$$H(z) = \frac{1}{2} \int |\nabla z_1|^2 + \frac{1}{2} \int |z_1|^2 + \frac{1}{2} \int |z_2|^2 + \frac{1}{4} \int |z_1|^4.$$

We will often abuse notation by setting $u = (u, u_t)$, and we will write

$$\dot{u} = J(Au + \nabla \mathcal{N}[u]).$$

We will switch between formulations (2.5) and (1.1) of the cubic nonlinear Klein-Gordon equation.
2.1. An infinite dimensional symplectic capacity. Kuksin’s construction of a symplectic capacity for an open set $\mathcal{O}$ is based on finite dimensional approximations of this set. It is an infinite dimensional analogue of the Hofer-Zehnder capacity [18]. Before defining this capacity, we first recall the definition of a symplectic capacity on a symplectic Hilbert space $(\mathcal{H}, \omega)$.

**Definition 2.1.** A symplectic capacity, $c$, on $(\mathcal{H}, \omega)$ is a function on open sets $O \subset \mathcal{H}$ which takes values in $[0, \infty)$ and has the following properties:

1) **Translational invariant:** $c(O) = c(O + \xi)$ for $\xi \in \mathcal{H}$.
2) **Monotonicity:** $c(O_1) \geq c(O_2)$ if $O_1 \supseteq O_2$.
3) **2-homogeneity:** $c(\tau O) = \tau^2 c(O)$.
4) **Nontriviality:** $0 < c(O) < \infty$ if $O \neq \emptyset$ is bounded.

For a given Darboux basis of $\mathcal{H}$, let $\mathcal{H}_N$ denote the span of the first $N$ basis vectors. Similarly, we use the notation $O_N$ for any subset $O$ projected onto these basis vectors. We collect a few definitions.

**Definition 2.2 (Admissible function).** Consider a smooth function $f \in C^\infty(O)$ and let $m > 0$. The function $f$ is called $m$-admissible if

i) $0 \leq f \leq m$ everywhere.
ii) $f \equiv 0$ in a nonempty subdomain of $O$.
iii) $f|_{\partial O} \equiv m$ and the set $\{f < m\}$ is bounded and the distance from this set to $\partial O$ is $d(f) > 0$.

**Definition 2.3 (Fast function).** Let $f_N := f|_{O_N}$ and consider the corresponding Hamiltonian vector field $V_{f_N}$, that is, for $z, v \in \mathcal{H}_N$, we have

$$\omega(V_{f_N}(z), v) = \nabla f_N(z)v.$$ 

A periodic trajectory of $V_{f_N}$ is called fast if it is not a stationary point and its period $T$ satisfies $T \leq 1$. An admissible function $f$ is called fast if there exists $N_0(f)$ such that for all $N \geq N_0$, the vector field $V_{f_N}$ has a fast solution.

**Remark 2.1.** In light of the fact that $J^2 = -I$, we also have the representation

$$V_{f_N}(z) = J \nabla f_N(z)$$

With these definitions, we are now ready to state the definition of a capacity $c$ on $\mathcal{H}$.

**Definition 2.4.** For an open, nonempty domain $O \subset \mathcal{H}$, its capacity $c(O)$ equals

$$c(O) = \inf\{m_* \mid \text{each } m\text{-admissible function with } m > m_* \text{ is fast}\}$$

In [21], it is shown that this definition satisfies the axioms of a capacity, that is the criteria of Definition 2.1 and while the construction of this capacity depends on the choice of Darboux basis, if one chooses another basis which is quadratically close to the first, then the capacity does not change.

---

1 a Darboux basis is a basis of $\mathcal{H}$, $(u_1, \ldots, v_1, \ldots)$ such that $\omega(u_i, v_i) = 1$ and all other pairings are zero.
2 that is some other Darboux basis $\{\psi^\pm_j\}$ such that $\sum_i \|\psi^\pm_j - \psi^\pm_j\|^2 < \infty$
2.2. non-squeezing and transfer of energy to higher Fourier modes. We will explain briefly the heuristics of how non-squeezing relates to the transfer of energy to higher Fourier modes for Hamiltonian equations. The energy transition problem investigates the following question: for a solution $u$ to a Hamiltonian equation with Fourier series
\[ u(x,t) = \sum_{k \in \mathbb{Z}^3} \hat{u}_k(t) e^{ikx}, \quad \hat{u}_k(t) \in \mathbb{R}, \]
is it true that typically $|\hat{u}_k(t)|$ decay with $t$ for small $k$? Equivalently, let $\mathcal{H}$ denote the phase space for the given equation, then this becomes a question about whether for some ball $B \subset \mathcal{H}$, for $k \in \mathbb{Z}^3$ and for $\theta \ll 1$ one has
\[ |\hat{u}_k(t)| \leq \theta, \quad t = t_k \gg 1 \]
for solutions with $u(0) \in B$. If $\mathcal{H}$ is a Sobolev space and the equation we consider has a conserved energy at the level of $\mathcal{H}^1$, as is the case for the nonlinear Klein-Gordon equation, we can interpret (2.6) as a statement that the higher Sobolev norms of a solution grow in time. This is due to the fact that if there is decay of certain Fourier modes, but, roughly speaking, the $\mathcal{H}^1$ norm is conserved, then some other higher Sobolev norm must compensate by growing below. Now let $\Phi$ denote the flow map of the Hamiltonian equation and rewrite (2.6) as
\[ \Phi(t)(B) \subset \{ u \in \mathcal{H} : |u_k(t)| \leq \theta \} \]
for fixed $k$. If the radius of the ball $B$ is $R > 0$, then by non-squeezing, this would be impossible unless $R \leq \theta$. Thus, non-squeezing implies a negative answer to the question of the decay of Fourier modes.

3. Preliminaries

3.1. Deterministic preliminaries.

**Definition 3.1.** A pair of real numbers $(q,r)$, is called admissible provided $2 \leq q \leq +\infty$, $2 \leq r < +\infty$,
\[ \frac{1}{q} + \frac{3}{r} = 1 \quad \text{and} \quad \frac{1}{q} + \frac{1}{r} \leq \frac{1}{2}. \]
We call a pair $(\tilde{q}',\tilde{r}')$ a conjugate admissible pair if
\[ \frac{1}{\tilde{q}} + \frac{1}{\tilde{q}'} = 1 = \frac{1}{\tilde{r}} + \frac{1}{\tilde{r}'} \]
for some admissible pair $(\tilde{q},\tilde{r})$.

**Remark 3.1.** In fact, a larger range of exponents are admissible for the nonlinear Klein-Gordon equation than the range stated above, which only coincides with the admissible exponents for the nonlinear wave equation. Since the above estimates are all we will need in our arguments, we refrain from stating the full range of Strichartz exponents for simplicity. For a full formulation, see for instance [30].

**Proposition 3.2 (Strichartz estimates).** Let $u$ be a solution to the inhomogeneous equation
\[ u_{tt} - \Delta u + u = F(u), \quad (u,\partial_t u)|_{t=0} = (u_0, u_1) \]
on $[0,T]$ for $T \leq 1$. Let $(q,r)$ be an admissible pair, then
\[ \|u(t,\partial_t u)\|_{L_t^q L_x^r([0,T] \times \mathbb{T}^3)} + \|u\|_{L_t^q L_x^r([0,T] \times \mathbb{T}^3)} \lesssim \|(u_0, u_1)\|_{H^{1/2}(\mathbb{T}^3)} + \|F\|_{L_t^p L_x^q([0,T] \times \mathbb{T}^3)} \]
where \((\tilde{p}', \tilde{q}')\) is a conjugate admissible pair.

We recall the definition of \(X^{s,b}(\mathbb{R} \times \mathbb{T}^3)\) spaces, with norm
\[
\|u\|_{X^{s,b}(\mathbb{R} \times \mathbb{T}^3)} = \|\langle n \rangle^s |\tau|^{-\frac{d}{2}} \langle n,\tau \rangle^b \hat{u}(n,\tau)\|_{L^2_x L^4_t}.
\]
For \(b > \frac{1}{2}\), \(X^{s,b}\) embeds into \(C_t H^s_x\). We will also work with the local spaces \(X^{s,b,\delta}\), which are defined by the norm
\[
\|u\|_{X^{s,b,\delta}} = \inf\{\|\tilde{u}\|_{X^{s,b}(\mathbb{R} \times \mathbb{T}^3)} : \tilde{u}|_{[-\delta,\delta]} = u\}.
\]
We recall the fact that free solutions to (1.1) lie in \(X^{s,b}\).

**Lemma 3.3.** Let \(f \in H^s\) for \(s \in \mathbb{R}\) and let \(S(t)\) denote the free evolutions for the Klein-Gordon equation. Then for any Schwartz time cutoff \(\eta \in S_x(\mathbb{R})\),
\[
\|\eta(t)S(t)f\|_{X^{s,b}(\mathbb{R} \times \mathbb{T}^d)} \leq C(\eta,b)\|f\|_{H^s(\mathbb{T}^d)}.
\]

We state the following interpolation estimate for the Strichartz estimates for the nonlinear Klein-Gordon equation which we will use to estimate frequency localized functions in \(X^{s,b}\) spaces, see [2] for this formulation of the estimates.

**Proposition 3.4 (Strichartz).** Let \(I \subset \mathbb{R}\). There exists a constant \(C \equiv C(I) > 0\) such that
\[
\left\| \sum_{|n| \sim N} a(n)e^{i(x \cdot n + t(n))} \right\|_{L^4(I \times \mathbb{T}^3)} \leq C(I)N^{1/2} \left( \sum |a(n)|^2 \right)^{1/2}.
\]
By Hölder’s inequality we have
\[
\left\| \sum_{|n| \sim N} \int d\lambda \frac{c(n,\tau)}{|\tau| - \langle n \rangle^{\frac{d}{2}+\theta}} e^{i(x \cdot n + t)\lambda} \right\|_{L^4(I \times \mathbb{T}^3)} \leq C(I)N^{1/2} \left( \int \sum |c(n,\tau)|^2 d\tau \right)^{1/2}
\]
and by interpolating with Parseval’s identity we obtain that for \(2 \leq r \leq 4\)
\[
(3.4) \left\| \sum_{|n| \sim N} \int d\lambda \frac{c(n,\tau)}{|\tau| - \langle n \rangle^{\frac{d}{2}+\frac{\theta}{r}}} e^{i(x \cdot n + t)\lambda} \right\|_{L^r(\mathbb{T} \times \mathbb{T}^3)} \leq C(I)N^{\theta/2} \left( \int \sum |c(n,\tau)|^2 d\tau \right)^{1/2}
\]
where \(\theta = 2 - \frac{4}{r}\).

We will also record some facts about the projection operator we use to define the truncated equation. Let \(\psi\) be a smooth bump function with \(\text{supp } \psi \subset (-2,2)\) and \(\psi \equiv 1\) on \((-1,1)\). We introduce the smooth spectral projector
\[
(3.5) \quad S_N(u)(x) \equiv \psi(-N^{-2} \Delta)(u)(x) = \hat{u}(0) + \sum_{n \in \mathbb{Z}^d_1} \psi\left(\frac{|n|^2}{N^2}\right) \hat{u}(n)e^{i(n \cdot x)}
\]
In [9], these operators were used to define an approximate equation for the cubic nonlinear wave equation. We will similarly use these operators to truncate the nonlinearity of the cubic nonlinear Klein-Gordon equation. We use this smoothed projection instead of the standard truncation because we will need to exploit the fact that this family of operators has uniform \(L^p\) bounds.
Lemma 3.5. Let $M$ be a compact Riemannian manifold and let $\Delta$ be the Laplace Beltrami operator on $M$. Let $1 \leq p \leq \infty$. Then $S_N \equiv \psi(-N^{-2} \Delta) : L^p(M) \to L^p(M)$ is continuous and there exists $C > 0$ such that for every $N \geq 1$,
\[ \|S_N\|_{L^p \to L^p} \leq C. \]
Moreover, for all $f \in L^p(M)$, $S_N f \to f$ in $L^p$ as $n \to \infty$.

Proof. See [35, Theorem 2.1]. \qed

Finally, we will need the following identity to prove the symplectic properties for the truncated nonlinear Klein-Gordon equation.

Lemma 3.6. Let $K$ be large enough so that $\Pi K S_N = S_N$, then
\[ \int_{T^3} S_N(S_N u)^3 \Pi K \partial_t v = \int_{T^3} (S_N u)^3 S_N \partial_t v \]

3.2. Probabilistic preliminaries. We will record some of the basic probabilistic results about the randomization procedure. Most of these estimates are consequences of the classical estimates of Paley-Zygmund for random Fourier series on the torus. These estimates were used heavily in the works of Burq-Tzvetkov, see especially [9] for proofs.

Proposition 3.7 (Large deviation estimate; Lemma 3.1 in [11]). Let $\{l_n\}_{n=1}^{\infty}$ be a sequence of real-valued independent random variables with associated distributions $\{\mu_n\}_{n=1}^{\infty}$ on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Assume that the distributions satisfy the property that there exists $c > 0$ such that
\[ \int_{-\infty}^{+\infty} e^{\gamma x} d\mu_n(x) \leq e^{c\gamma^2} \text{ for all } \gamma \in \mathbb{R} \text{ and for all } n \in \mathbb{N}. \]
Then there exists $c > 0$ such that for every $\lambda > 0$ and every sequence $\{c_n\}_{n=1}^{\infty} \in \ell^2(\mathbb{N})$ of complex numbers,
\[ \mathbb{P}\left( \omega : \left| \sum_{n=1}^{\infty} c_n l_n(\omega) \right| > \lambda \right) \leq 2e^{-\frac{c\lambda^2}{\sum_n |c_n|^2}}. \]
As a consequence there exists $C > 0$ such that for every $p \geq 2$ and every $\{c_n\}_{n=1}^{\infty} \in \ell^2(\mathbb{N})$,
\[ \left\| \sum_{n=1}^{\infty} c_n l_n(\omega) \right\|_{L^p(\Omega)} \leq C\sqrt{p} \left( \sum_{n=1}^{\infty} |c_n|^2 \right)^{1/2}. \]

Remark 3.2. Gaussian iid or Bernoulli random variables satisfy the exponential moment assumption in the statement of Proposition 3.7.

This large deviation estimate is the key component in the proof of the following corollary, which states that the free evolution of randomized initial data satisfies almost surely better integrability properties.

Corollary 3.8 (Corollary A.5, [11]). Fix $\mu \in \mathcal{M}^s$ and suppose $\mu$ is induced via the map (1.6) for $(\phi_0, \phi_1) \in \mathcal{H}^s$. Let $2 \leq p_1 < \infty$, $2 \leq p_2 \leq \infty$ and $\delta > 1 + \frac{1}{p_1}$ and $0 < \sigma \leq s$ Then there exist constants $C, c > 0$ such that for every $\lambda > 0$,
\[ \mu\left( \{(v_0, v_1) \in \mathcal{H}^s(\mathbb{T}^3) : \|t\|^{-\delta} S(t)(v_0, v_1)\|_{L^{p_1}_t L^{p_2}_{x,x}(\mathbb{T}^3)} > \lambda \} \right) \leq C e^{-c\lambda^2/\|\langle \phi_0, \phi_1 \rangle \|_{\mathcal{H}^s}^2}. \]

The following result is a simple consequence of Sobolev embedding.
Lemma 3.9. Let $T > 0$ fixed and $2 \leq p < \infty$ large. Let $2 \leq r < \infty$ be chosen sufficiently large so that $s > \max(3/r, p)$ and let $\delta > 1 + \frac{2}{r}$. There exists $C \equiv C(T) > 0$ such that for every $\lambda > 0$, and every $(v_0, v_1)$ which satisfies

\begin{equation}
\| \langle t \rangle^{-\delta} (1 - \Delta)^{s/2} S(t)(v_0, v_1) \|_{L^p_t\big(\mathbb{R} \times T^3\big)} \leq K
\end{equation}

we have

\[ \| S(t)(v_0, v_1) \|_{L^p_{t,x}([0, T] \times T^3)} \leq CK. \]

Proof. Let $T < 1$ for notational simplicity. By Sobolev embedding,

\[ \int_0^T \| S(t)(v_0, v_1) \|_{L^p_{t,x}}^p \leq \int_0^1 \| S(t)(v_0, v_1) \|_{W^{1,r}}^p \leq \left( \int_0^1 \langle t \rangle^{-\delta \frac{2p}{r-p}} \right)^{\frac{r-p}{r}} \left( \int_0^1 \langle t \rangle^{-\delta} \| S(t)(v_0, v_1) \|_{W^{1,r}}^{r} \right)^{\frac{1}{r}} \leq CK. \]

From Corollary 3.8 and Lemma 3.9 we conclude that the set of initial data which satisfies good local $L^p$ bounds has full $\mu$ measure.

Corollary 3.10. Fix $\mu \in M^s$ and let $2 \leq p < \infty$. Then for a set of full $\mu$ measure,

\[ \| S(t)(v_0, v_1)(t) \|_{L^p_{t,x}(T^3)} \in L^1_{\text{loc}}(\mathbb{R}_t) \]

Proof. Suppose $\mu$ is induced via the map \( (u_0, u_1) \rightarrow \mathcal{H}^s \). By \( 3.8 \), we can bound

\begin{align*}
\mu \left( \left\{ \| \langle t \rangle^{-\delta} S(t)(v_0, v_1) \|_{L^p_tL^s_{x}(\mathbb{R} \times T^3)} = \infty \right\} \right) &\leq \mu \left( \bigcap_{\lambda > 0} \left\{ \| \langle t \rangle^{-\delta} S(t)(v_0, v_1) \|_{L^p_tL^s_{x}(\mathbb{R} \times T^3)} > \lambda \right\} \right) \\
&\leq \lim_{\lambda \rightarrow \infty} C e^{-c\lambda^2/\| (u_0, u_1) \|^2_{\mathcal{H}^s}} = 0.
\end{align*}

Hence, for almost all $(v_0, v_1) \in \mathcal{H}^s$, there exists some $\lambda > 0$ such that

\[ (v_0, v_1) \in \left\{ \| \langle t \rangle^{-\delta} S(t)(v_0, v_1) \|_{L^p_tL^s_{x}(\mathbb{R} \times T^3)} \leq \lambda \right\}, \]

and the result follows by Corollary 3.9.

4. Wellposedness theory

We record some global bounds on the solution to the cubic nonlinear Klein-Gordon \( (1.1) \). The only new component in this statement is the bounds on the $L^4$ norm of the solution, which follows by a similar argument to the proof of Proposition 3.1 in \( [10] \). We include the proofs of these statements in Appendix 13.

Proposition 4.1. Let $0 < s < 1$ and let $\mu \in M^s$. Then for any $\varepsilon > 0$, there exist $C, c, \theta > 0$ such that for every $(u_0, u_1) \in \Sigma$, there exists $M = M(u_0, u_1) > 0$ such that the global solution $u$ to the cubic nonlinear Klein-Gordon equation \( (1.1) \) satisfies

\[ u(t) = S(t)(u_0, u_1) + w(t) \]

\[ \| (w(t), \partial_t w(t)) \|_{\mathcal{H}^s} \leq C \left( M + |t| \right)^{1+\varepsilon} \]

\[ \| u \|_{L^4(T^3)} \leq C \left( M^{1/2} + |t| \right)^{\frac{1}{2}+\varepsilon} \]

and furthermore $\mu((u_0, u_1) \in \Sigma : M > \lambda) \leq Ce^{-c\lambda^\theta}$. 
We now turn to studying the global wellposedness and symplectic properties of the approximating equation (4.1). Let $S_N$ be the smooth projection operator defined in (3.5) and we can write the equation with truncated nonlinearity in Hamiltonian form as
\[ \dot{u}_N = J(Au_N + \nabla N^N[u_N]), \]
where
\[ N^N[u] := \frac{1}{4} \int |S_Nu|^4 \quad \text{and} \quad A := (I - \Delta) \oplus I. \]

For $K$ sufficiently large so that $\Pi_K S_N = S_N$, this equation is equivalent to the uncoupled system
\[ \begin{aligned}
\partial_t \Pi_K u_N &= J(A \Pi_K u_N + S_N \nabla N[S_N u_N]), \quad (t, x) \in \mathbb{R} \times T^3 \\
(\Pi_K u_N, \partial_t \Pi_K u_N)|_{t=0} &= (\Pi_K u_0, \Pi_K u_1) \\
\Pi_{\geq K}(u_N) &= S(t)(\Pi_{\geq K} u_0, \Pi_{\geq K} u_1)
\end{aligned} \tag{4.2} \]

**Remark 4.1.** As remarked in the introduction, (4.2) is a nonlinear flow for low frequencies and a decoupled linear evolution for high frequencies. We note that the solution $u_N$ is supported on all frequencies. We will nonetheless call this the truncated flow for simplicity even though this defines a flow on the whole space.

Global wellposedness for (4.2) follows from local wellposedness by observing that the linear evolution is globally defined and the energy functional
\[ H_N(\Pi_K(u_N, (u_N)_t)) = \frac{1}{2} \int_{T^3} |\nabla_x \Pi_K u_1|^2 + (\Pi_K u_N)^2 + (\Pi_K (u_N)_t)^2 + \frac{1}{4} \int_{T^3} (S_N(u_N))^4 \]
is well defined and conserved under the flow of (4.2) for bounded frequency components. Note that the bounds on the solution depend on the energy of the initial data and hence they are not uniform in the truncation parameter. Nonetheless, Burq-Tzvetkov [9] proved that if one restricts to initial data $(u_0, u_1) \in \Sigma$, then the nonlinear components of the solutions to the cubic nonlinear wave equation satisfy uniform bounds. As was the case in Theorem (1.2), the proof of these uniform bounds follows for the nonlinear Klein-Gordon equation from the arguments in [9] with only minor modifications.

**Proposition 4.2** (Proposition 3.1, [10]). Let $0 < s < 1$ and let $\mu \in M^s$. Then for any $\varepsilon > 0$, there exist $C, c, \theta > 0$ such that for every $(u_0, u_1) \in \Sigma$, there exists $M = M(u_0, u_1) > 0$ such that the family of global solutions $(u_N)_{N \in \mathbb{N}}$ to (4.2) satisfies
\[
\begin{aligned}
&u_N(t) = S(t)(u_0, u_1) + w_N(t) \\
&\|(w_N(t), \partial_t w_N(t))\|_{H^1} \leq C(M + |t|)^{1+\varepsilon} \\
&\|S_N u_N\|_{L^4(T^3)} \leq C(M^{1/2} + |t|)^{1+\varepsilon}
\end{aligned}
\]

and furthermore $\mu((u_0, u_1) \in \Sigma : M > \lambda) \leq C e^{-c \lambda^\theta}$.

The truncated flow maps also preserve symplectic capacities. This is an easy consequence of the fact that when restricted to bounded frequencies, these maps are finite dimensional symplectomorphisms.

**Proposition 4.3.** The flow maps $\Phi_N(t)$ preserve symplectic capacities $c(\mathcal{O})$ for any domain $\mathcal{O} \subset \mathcal{H}^{1/2}(T^3)$. 

Proof. Let \( K \) be large enough so that \( \Pi_K S_N = S_N \) and consider the Hamiltonian \((4.3)\). For \((v_1, v_2) \in \Pi_K H^{1/2}(\mathbb{T}^3)\) we use Lemma 3.6 to compute
\[
\frac{d}{d\varepsilon} H_N(\Pi_K(z_1, z_2) + \varepsilon(v_1, v_2)) \bigg|_{\varepsilon=0} = \frac{d}{d\varepsilon} \left[ \int_{\mathbb{T}^3} (\Pi_K z_1 + \varepsilon v_1)^2 + |\nabla_x (\Pi_K z_1 + \varepsilon v_1)|^2 + (\Pi_K z_2 + \varepsilon v_2)^2 + \frac{1}{4} \int_{\mathbb{T}^3} (S_N(z_1 + \varepsilon v_1))^4 \right]_{\varepsilon=0}
\]
\[
= \int_{\mathbb{T}^3} (\Pi_K z_1)v_1 + \nabla_x (\Pi_K z_1) \nabla_x v_1 + (\Pi_K z_2)v_2 + (S_N z_1)^3 S_N(v_1)
\]
\[
= \int_{\mathbb{T}^3} (\Pi_K z_1)v_1 + \nabla_x (\Pi_K z_1) \nabla_x v_1 + (\Pi_K z_2)v_2 + S_N(S_N z_1)^3 \Pi_K v_1
\]
\[
= \int_{\mathbb{T}^3} \partial_t (\Pi_K z_2)v_1 + \partial_t (\Pi_K z_1)v_2
\]
\[
= \omega_{\frac{1}{2}}((v_1, v_2), (\partial_t \Pi_K z_1, \partial_t \Pi_K z_2)),
\]
Thus the maps \( \Phi_N \) are symplectomorphisms on \( \Pi_K H^{1/2} \), denote this restricted map by \( \tilde{\Phi}_N \). Since the flow decouples low and high frequencies, we can write
\[
\Phi_N(u) = \tilde{\Phi}_N(\Pi_K u) + e^{tJ\Lambda} \Pi_{\geq K} u = e^{tJ\Lambda} (e^{-tJ\Lambda} \tilde{\Phi}_N(\Pi_K u) + \Pi_{\geq K} u)
\]
The invariance of the symplectic capacity under the flow of \( \Phi_N \) follows from [21] Lemma 5], noting that \( e^{-tJ\Lambda} \tilde{\Phi}_N \) is also a symplectomorphism and that \( e^{tJ\Lambda} \) is an isometry, hence it preserves admissible and fast functions. \( \square \)

**Definition of \( \Sigma_\lambda \).** For \((u_0, u_1) \in \Sigma\), we denote the global solution to \((1.1)\) by
\[
u(t) = S(t)(u_0, u_1) + w(t).
\]
Fix \( \varepsilon > 0 \) and let \( C > 0 \) be as in Proposition 4.1 and Proposition 4.2 and define
\[
E_\lambda := \{(u_0, u_1) \in \Sigma : ||(w(t), \partial_t w(t))||_{H^1} \leq C(\lambda + |t|)^{1+\varepsilon}\}
\]
\[
H_\lambda := \{(u_0, u_1) \in \Sigma : ||(w_N(t), \partial_t w_N(t))||_{H^1} \leq C(\lambda + |t|)^{1+\varepsilon}\}
\]
\[
J_\lambda := \{(u_0, u_1) \in \Sigma : ||u||_{L^4_t} \leq C(\lambda + |t|)^{1+\varepsilon}\}
\]
\[
K_\lambda := \{(u_0, u_1) \in \Sigma : ||u_N||_{L^4_t} \leq C(\lambda + |t|)^{1+\varepsilon}\}.
\]
We set
\[
(4.4) \quad \Sigma_\lambda := E_\lambda \cap H_\lambda \cap J_\lambda \cap K_\lambda
\]

**Proposition 4.4.** Fix \( \mu \in M^{1/2} \) and let \( \Sigma_\lambda \) be as defined in \((4.4)\). Then there exists \( C, \varepsilon, \theta > 0 \) such that
\[
(4.5) \quad \mu(\Sigma_\lambda) \geq 1 - Ce^{-c\lambda^\theta}
\]
and for any \( T > 0 \), there exists \( C_T > 0 \) such that for every \((u_0, u_1) \in \Sigma_\lambda\), we have
\[
(4.6) \quad ||S(t)(u_0, u_1)||_{H^\mu_{\varepsilon, t}(\mathbb{T}^3)} \leq C_T \lambda.
\]
Proof. Since
\[
\Sigma_\lambda := E_\lambda \cap H_\lambda \cap J_\lambda \cap K_\lambda
\]
we will estimate the probabilities of each of these sets individually. Suppose that \( \mu \) is induced by the randomization of \( (u_0, u_1) \in H^{1/2} \). By Proposition 4.2 and Proposition 4.1 there exists \( C, c > 0 \) such that
\[
\mu(E_\lambda \cap H_\lambda \cap J_\lambda \cap K_\lambda) \geq 1 - Ce^{-c\lambda^6},
\]
and by Corollary 3.8
\[
\mu(H_\lambda) \geq 1 - Ce^{-c\lambda^2/\|\phi_0, \phi_1\|_{H^s}^2},
\]
which yields (4.5). Finally, (4.6) follows immediately from the definition of \( \Sigma \) in (1.10).

5. Probabilistic bounds for the nonlinear component of the flow

In this section, we will show boundedness for the nonlinear component of the cubic nonlinear Klein-Gordon equation on the subsets \( \Sigma_\lambda \subset \Sigma \). We will begin with the proof of a local boundedness property for the nonlinearity \( \nabla N[u] \). The argument is based on Strichartz estimates together with the improved averaging effects for the free evolution of initial data \((u_0, u_1) \in \Sigma_\lambda\), where \( \Sigma_\lambda \) was defined in (4.4), as well as the uniform bounds on the nonlinear component of those global solutions from Proposition 4.1 and Proposition 4.2. In [9], Bourgain treats subcritical nonlinear Klein-Gordon equations with estimates in local \( X^{s,b} \) spaces. The reason Bourgain’s estimates fail at the critical regularity is because Strichartz estimates are not available to work at regularities \( s_1 < \frac{3}{2} \), which is necessary criterion to use Kuksin’s framework. We will get around this constraint by using the improved probabilistic bounds. Recall that we are considering the equation
\[
\dot{u} = J(Au + \nabla N[u]),
\]
where we use the abuse of notation \( u = (u_0, u_1) \) and
\[
N[u] := \frac{1}{4} \int |u|^4 \quad \text{and} \quad A := (I - \Delta) \oplus I.
\]
We wish to obtain bounds on \( \|\nabla N[u(t)]\|_{X^{s_2-\frac{1}{2}+\delta}} \) for solutions to the cubic nonlinear Klein-Gordon equation (1.1) with initial data \((u_0, u_1) \in \Sigma_\lambda\). We will perform the estimates with \( b = \frac{1}{2} + \) and hence, we need to estimate the expression
\[
\left( \sum_n \int d\tau \frac{|\tilde{f}(n, \tau)|^2}{\langle n \rangle^{2(1-s_2)} \langle |\tau| - \langle n \rangle \rangle^{2(1-b)}} \right)^{\frac{1}{2}}
\]
for \( f = u^3 \) and \( b > 1/2 \). Define
\[
c(n, \tau) = \langle n \rangle^{s_1} \langle |\tau| - \langle n \rangle \rangle^b |\tilde{u}(n, \tau)|,
\]
then \( \|c\|_{L^2} = \|u\|_{X^{s_1,b}} \). By duality, (5.2) can be estimated by
\[
\sum_{n=n_1+n_2+n_3} \int_{\tau = \tau_1 + \tau_2 + \tau_3} \prod_{i=1}^3 \frac{c(n_i, \tau_i)}{\langle n_i \rangle^{s_1} \langle |\tau_i| - \langle n_i \rangle \rangle^b \langle n \rangle^{1-s_2} \langle |\tau| - \langle n \rangle \rangle^{1-b}} \langle d(n, \tau) \rangle \int d\tau
\]
where \( \|d\|_{L^2(\mathbb{T}^2)} \leq 1 \) and the function \( u \) in the expression for \( c \) will be taken to be either
(I) the free evolution \( S(t)(u_0, u_1) \) of initial data \((u_0, u_1) \in \Sigma_\lambda \subset H^{1/2} \), or
(II) the nonlinear component, \( w(t) \), of a global solution to (1.1) with initial data \((u_0, u_1) \in \Sigma_\lambda\).
We will refer to these as type (I) or type (II) functions.

We restrict the $n_i$ and $n$ to dyadic regions $|n_i| \sim N_i$ and $|n| \sim N$. We will implicitly insert a time cut-off with each function but we will omit the notation. The ordering of the size of the frequencies will not play a role in this argument. Letting

$$\hat{F}_i(n_i, \tau_i) = \frac{c(n_i, \tau_i)}{|\tau_i - \langle n_i \rangle|^b} \chi_{|n_i| \sim N_i}, \quad \hat{G}(n, \tau) = \frac{d(n, \tau)}{|\tau - \langle n \rangle|^{1-b}} \chi_{|n| \sim N}$$

we will need to estimate expressions of the form

$$\int_{R_1} \int_{T^3} \prod_{i=1}^3 F_i \cdot G \, dx \, dt.$$  \hfill (5.5)

We will use expressions (5.5) and (5.3) as starting points in proofs of the subsequent propositions. Recall that $B_R$ is the ball of radius $R$ centered at zero.

5.1. Boundedness of the flow map.

**Proposition 5.1** (Local boundedness). Consider the cubic nonlinear Klein-Gordon equation (1.1). Then there exists $s_1 < \frac{1}{2} < s_2$ with $s_1, s_2$ sufficiently close to 1/2 such that for any $\lambda, R, T > 0$, for every $(u_0, u_1) \in \Sigma \cap B_R$ and for any interval $J \subset [0, T]$ with $|J| = \delta$, the nonlinearity satisfies the bound

$$\|\nabla N[u]\|_{X^{s_2 - 1, \frac{1}{2} + s, \delta}(J \times T^3)} \leq C(\lambda, R, T) \delta^{\epsilon} \left( 1 + \|u\|_{X^{s_1, \frac{1}{2} + s, \delta}(J \times T^3)}^{9/4} + \|u\|_{X^{s_1, \frac{1}{2} + s, \delta}(J \times T^3)} \right).$$

where $(u, \partial_t u)$ is the global solution to the cubic nonlinear Klein-Gordon equation (1.1) with initial data $(u_0, u_1)$.

**Proof.** For any solution

$$u(t) = S(t)(u_0, u_1) + w(t),$$

our computations will yield (5.10) with the nonlinear component of the solutions, $w$, on the right-hand side instead of $u$. Now, for any $s, b \in \mathbb{R}$ and any interval $J \subset [0, T]$ with $|J| = \delta$ and $\inf J = t_0$, and $\eta(t)$ a Schwartz time-cutoff adapted to that interval we have

$$\|\eta(t)w\|_{X^{s, b}(J \times T^3)} = \|\eta(t)(u - S(t)(u_0, u_1))\|_{X^{s, b}(J \times T^3)}$$

$$\lesssim \|\eta(t)u\|_{X^{s, b}(J \times T^3)} + \|\eta(t)S(t)(u_0, u_1)\|_{X^{s, b}(J \times T^3)}$$

$$\lesssim \|\eta(t)u\|_{X^{s, b}(J \times T^3)} + \|(u(t_0), \partial_t u(t_0))\|_{H^s}$$

Now, we can bound

$$\|(u(t_0), \partial_t u(t_0))\|_{H^s} \lesssim \|(u_0, u_1)\|_{H^s} + \|(w, \partial w)\|_{L^\infty([0, T] \times T^3)}$$

and since the terms on the right-hand side of (5.10) are computed with $s = s_1 < 1/2$, then by the choice of $\Sigma$ from (4.4), we have that for any such $J \subset [0, T]$,

$$\|\eta(t)w\|_{X^{s, b}(J \times T^3)} \lesssim \|\eta(t)u\|_{X^{s, b}(J \times T^3)} + C(\lambda, R, T)$$

which will yield (5.10). The key point here is that on any interval $[0, T]$, the choice of $\Sigma$ yields uniform control on bounds of the Sobolev norm of solutions. We will not repeat these considerations in subsequent propositions.

We will analyze the different combinations of $u_i$ systematically. The argument only depends on the number of $u_i$ which are of type (I) or type (II), so we will only present one combination from each case.
• Case (A): All $u_i$ of type (II). Since
\[ \| F_i^\alpha F_i^{1-\alpha} \|_{L^{4+\varepsilon_1}} \lesssim \| F_i \|_{L^{p_0}} \| F_i \|_{L^{(1-\alpha)}} \]
for $p = \frac{(4+\varepsilon_1)^5}{4+\varepsilon_1}$, we use Hölder’s inequality, Strichartz estimates and Sobolev embedding to estimate by
\[ (N_1 N_2 N_3)^{-s_1} N^{-(1-s_2)} \int_{T_3} \prod_{i=1}^3 F_i \cdot G \, dx \, dt \]
\[ \lesssim (N_1 N_2 N_3)^{-s_1} N^{-(1-s_2)} \prod_{i=1}^3 \| F_i \|_{L^{p_0}} \| F_i \|_{L^{(1-\alpha)}} \| G \|_{L^{4-3\varepsilon_1}} . \]

Taking $\alpha = 1/4$, we have $(1 - \alpha)5 < 4$ and for $\varepsilon_1$ sufficiently small, $p\alpha < 6$. Provided
\[ (5.7) \]
we can bound
\[ \frac{1}{\langle |n| - |\tau| \rangle^{1-b}} = \frac{1}{\langle |n| - |\tau| \rangle^{1-b - \frac{a_2}{2} + \frac{a_2}{2}} + \langle |n| - |\tau| \rangle^{\frac{a_2}{2}}} \lesssim \frac{1}{\langle |n| - |\tau| \rangle^{\frac{a_2}{2}}} \]
and we can apply Strichartz estimates (3.4) with $r = \frac{15}{4}$ and $\theta_1 = 14/15$ for the $F_i$ and by Strichartz estimates with $r = 4 - 3\varepsilon_1$ and $\theta_2 = \frac{4-6\varepsilon_1}{4-3\varepsilon_1}$ for $G$. Hence, using Sobolev embedding as well, we obtain
\[ \lesssim (N_1 N_2 N_3)^{\frac{21}{30} - s_1} N^{-(1 - \frac{12}{15} - \frac{4-6\varepsilon_1}{4-3\varepsilon_1} - s_2)} \prod_{i=1}^3 \| u_i \|_{H^1(T^3)}^\frac{1}{14} \prod_{i=1}^3 \| c_i \|_{L^2}^{3/4} . \]

The expression on the right-hand side is summable for dyadic values of $N_i$ and $N$. Thus, since $(u_0, u_1) \in \Sigma_\lambda$, we obtain
\[ (5.8) \]
\[ \| F_i \|_{L^{p_0}} \| F_i \|_{L^{(1-\alpha)}} \lesssim (\lambda + T)^{\frac{3}{4} + \frac{3}{2} \beta} \prod_{i=1}^3 \| u_i \|_{X^{\ast_1, \beta}}^{\frac{3}{2}} . \]

• Case (B): $u_1$ of type (I) and $u_2, u_3$ of type (II) In this case, we use Hölder’s inequality and Strichartz estimates (3.4) with $r = \frac{15}{4}$ and $\theta = \frac{8}{9}$, and provided
\[ (5.7) \]

which holds if (5.7) holds, we can bound
\[ (5.18) \lesssim (N_1 N_2 N_3)^{-s_1} N^{-(1-s_2)} \int F_1 F_2 F_3 \cdot G \, dx \, dt \]
\[ \lesssim (N_1 N_2 N_3)^{-s_1} N^{-(1-s_2)} \| F_1 \|_{L^6} \| F_2 \|_{L^{18} / 15} \| F_3 \|_{L^{18} / 15} \| G \|_{L^{18} / 15} \]
\[ \lesssim (N_1)^{-s_1} (N_2 N_3)^{\frac{9}{2} - s_1} N^{-(\frac{9}{2} - s_2)} \| F_1 \|_{L^6} \| u_2 \|_{X^{\ast_1, \beta}} \| u_3 \|_{X^{\ast_1, \beta}} \]
which is again summable for dyadic $N$ and $N_i$, yielding
\[ (5.9) \]
\[ (5.18) \lesssim \lambda \| u_2 \|_{X^{\ast_1, \beta}} \| u_3 \|_{X^{\ast_1, \beta}} \]
• Case (C): \( u_1, u_2 \) of type (I) and \( u_3 \) of type (II) In this case, we use Hölder’s inequality and Strichartz estimates \((5.21)\) with \( r = 3 \) and \( \theta = \frac{2}{3} \) and

\[ 1 - b > \frac{1}{3}, \]

which holds if \((5.7)\) holds and estimate

\[ \int F_1 F_2 F_3 \cdot G dx \]

\[ \lesssim (N_1 N_2 N_3)^{-s_1} N^{-(1-s_2)} \int F_1 F_2 F_3 \cdot G dx \]

\[ \lesssim (N_1 N_2 N_3)^{-s_1} N^{-(1-s_2)} \|F_1\|_{L^6} \|F_2\|_{L^6} \|F_3\|_{L^6} \|G\|_{L^3} \]

\[ \lesssim (N_1 N_2)^{-s_1} (N_3)^{\frac{1}{2}-s_1} N^{-(\frac{s}{2}-s_2)} \|F_1\|_{L^6} \|F_2\|_{L^6} \|u_3\|_{X^{s_1,b}} \]

which is again summable for dyadic \( N \) and \( N_i \), yielding

\[ \int F_1 F_2 F_3 \cdot G dx \lesssim \lambda^2 \|u_3\|_{X^{s_1,b}} \]

• Case (D): All \( u_i \) of type (I) In this case we estimate

\[ \int F_1 F_2 F_3 \cdot G dx \]

\[ \lesssim (N_1 N_2 N_3)^{-s_1} N^{-(1-s_2)} \int F_1 F_2 F_3 \cdot G dx \]

\[ \lesssim (N_1 N_2 N_3)^{-s_1} N^{-(1-s_2)} \|F_1\|_{L^6} \|F_2\|_{L^6} \|F_3\|_{L^6} \|G\|_{L^2} \]

\[ \lesssim (N_1 N_2)^{-s_1} (N_3)^{-s_1} N^{-(1-s_2)} \|F_1\|_{L^6} \|F_2\|_{L^6} \|F_3\|_{L^6} \]

which is again summable for dyadic \( N \) and \( N_i \), yielding

\[ \int F_1 F_2 F_3 \cdot G dx \lesssim \lambda^2. \]

Finally, we note that for \( -\frac{1}{2} < - (1-b) \), we have

\[ \|\nabla u\|_{X^{s_2-1, -(1-b)}, \delta} \lesssim \delta^c \|\nabla u\|_{X^{s_2-1, -(1-b)}, \delta} \]

where the implicit constant depends only on \( \varepsilon \). Hence, to obtain the time factor in the estimates, we can perform the previous estimates with \( b + \varepsilon \) with \( b = \frac{1}{2} + \) sufficiently close to \( \frac{1}{2} \) so as to ensure that \((5.7)\) continues to hold. Combining \((5.8)\), \((5.9)\), \((5.10)\), \((5.11)\) and \((5.12)\) yields

\[ \|\nabla u\|_{X^{s_2-1, -(1-b)}, \delta} \lesssim C(\lambda, R) \delta^c \left( 1 + \|u\|_{X^{s_1, \frac{1}{2} + \delta}}^2 + \|u\|_{X^{s_1, \frac{1}{2} + \delta}}^{9/4} \right). \]

\[ \|\nabla u_N\|_{X^{s_2-1, \frac{1}{2} + \delta}} \lesssim C(\lambda, R, T) \delta^c \left( 1 + \|u_N\|_{X^{s_1, \frac{1}{2} + \delta}}^2 + \|u_N\|_{X^{s_1, \frac{1}{2} + \delta}}^{9/4} \right). \]

Indeed, by the choice of \( \Sigma \), the nonlinear component of the solution \( w_N \) will satisfy the same bounds as the nonlinear component, \( w \), of the solution to the full equation. Since the linear components of \( u \) and \( u_N \) are the same, certainly we have the same bounds on \( S(t)(u_0, u_1) \) and we can repeat the arguments in the previous proof to obtain \((5.13)\).

For the truncated equation, we recall that

\[ \dot{u}_N = J \left( Au_N + \nabla u_N \right). \]
where again we use the abuse of notation $u_N = (u_N, \partial_t u_N)$, and
\begin{equation}
(5.15) \quad \mathcal{N}^N[u_N] := \frac{1}{4} \int |S_N u_N|^4 \quad \text{and} \quad A := (I - \Delta) \oplus I.
\end{equation}

A direct consequence of Proposition 4.2, Proposition 5.1 and our choice of $\Sigma_\lambda$ is the following local boundedness result for the truncated Klein-Gordon equation. It is important to note that the bounds we obtain are uniform in the truncation parameter.

**Proposition 5.2** (Local boundedness for the truncated equation). Consider the cubic nonlinear Klein-Gordon equation with truncated nonlinearity $\text{(1.11)}$. Then there exists $s_1 < \frac{1}{2} < s_2$ with $s_1, s_2$ sufficiently close to $1/2$ such that for any $\lambda, R, T > 0$, for every $(u_0, u_1) \in \Sigma_\lambda \cap B_R$ and for any interval $J \subset [0, T]$ with $|J| = \delta$, the truncated nonlinearity satisfies the bound
\begin{equation}
(5.16) \quad \|
abla \mathcal{N}^N[u_N]\|_{X^{s_2 - 1, -\frac{1}{2} +, \delta}(J \times \mathbb{T}^3)} \leq C(\lambda, R, T) \delta^c \left(1 + \|u_N\|_{X^{s_1 + \frac{1}{2} +, \delta}(J \times \mathbb{T}^3)}^2 + \|u_N\|_{X^{s_1 + \frac{1}{2} +, \delta}(J \times \mathbb{T}^3)}^{9/4}\right),
\end{equation}
where $(u_N, \partial_t u_N)$ is the global solution to $\text{(1.11)}$ with initial data $(u_0, u_1)$.

### 5.2. Continuity estimates for the flow map.

We need the following continuity-type estimate when we compare the full nonlinearity on solutions of the full flow to solutions of the truncated equation.

**Proposition 5.3.** Consider the cubic nonlinear Klein-Gordon equation $\text{(1.11)}$ and the cubic nonlinear Klein-Gordon equation with truncated nonlinearity $\text{(1.11)}$. Then there exists $s_1 < \frac{1}{2} < s_2$ with $s_1, s_2$ sufficiently close to $1/2$ such that for any $\lambda, R, T > 0$, for every $(u_0, u_1) \in \Sigma_\lambda \cap B_R$ and for any interval $J \subset [0, T]$ with $|J| = \delta$, the nonlinearity satisfies the bound
\begin{equation}
(5.17) \quad \|
abla \mathcal{N}[u] - \nabla \mathcal{N}[u_N]\|_{X^{-\frac{1}{4} - \frac{1}{2} +, \delta}(J \times \mathbb{T}^3)} \leq C(\lambda, R, T) \delta^c \left(1 + \|u - u_N\|_{X^{\frac{1}{2} +, \frac{1}{2} +, \delta}(J \times \mathbb{T}^3)} + \|u\|_{X^{s_1 + \frac{1}{2} +, \delta}(J \times \mathbb{T}^3)}^6 + \|u_N\|_{X^{s_1 + \frac{1}{2} +, \delta}(J \times \mathbb{T}^3)}^6\right),
\end{equation}
where $(u, \partial_t u)$ and $(u_N, \partial_t u_N)$ to the full and truncated equations, respectively with initial data $(u_0, u_1)$.

**Proof.** As in the proof of the Proposition 5.1 in light of (5.12) we will take $b = \frac{1}{2} +$ for $b$ sufficiently close to $\frac{1}{2}$ so that the time localization yields the $\delta^c$ factor in (5.17). We first note that
\[|(u)^3 - (u_N)^3| \lesssim |u - u_N|^2 + |u_N|^2\]
hence these estimates are similar to those in Proposition 5.1 but we will take $u_1 = |u - u_N|$ which is always of type (II) since the solutions $u$ and $u_N$ have the same linear component. More precisely, once again we estimate the expression
\begin{equation}
(5.18) \quad \sum_{n=n_1+n_2+n_3} \int_{\tau=\tau_1+\tau_2+\tau_3} \frac{c_1(n_1, \tau_1)}{(n_1)^{\frac{3}{2}}} \langle \tau_1 - \langle n_1 \rangle \rangle^{\frac{1}{2} +} \prod_{i=2}^3 \frac{c_i(n_i, \tau_i)}{(n_i)^{\frac{3}{2}}} \langle \tau_i - \langle n_i \rangle \rangle^{\frac{1}{2} +} \langle \tau \rangle^{\frac{3}{2}} \langle \tau - \langle n \rangle \rangle^{\frac{3}{2}} d(n, \tau).
\end{equation}
where \( \|d\|_{L^2(T^2)} \leq 1 \) and as before the function \( u \) in the expression for \( c \) is either of type (I) or type (II), with \( u_1 \) always of type (II).

As above, we restrict the \( n_i \) and \( n \) to dyadic regions \( |n_i| \sim N_i \) and \( |n| \sim N \). We will implicitly insert a time cut-off with each function but we will omit the notation. The ordering of the size of the frequencies will not play a role in this argument. Similarly, we take \( F_i \) and \( G_i \) to be

\[
\tilde{F}_i(n_i, \tau_i) = \frac{c(n_i, \tau_i)}{\langle |\tau_i| - |n_i| \rangle^{\frac{1}{2}+\varepsilon}} \chi_{\{n_i | \sim N_i \}}, \quad \tilde{G}(n, \tau) = \frac{d(n, \tau)}{\langle |\tau| - |n| \rangle^{\frac{1}{2}+\varepsilon}} \chi_{\{|n| \sim N \}}
\]

The key difference between this proof and the proof of Proposition 5.1 is that in each case, we will estimate \( u_1 \) in \( X^{\frac{2}{3}, \frac{1}{3}, \delta} \) instead of using \( X^{s_1, \frac{2}{3}, \delta} \) which we use for the other functions.

- **Case (A): All \( u_i \) of type (I)** Once again, we recall that since
  \[
  \| F_i^\alpha F_i^{1-\alpha} \|_{L^{4_\varepsilon_1}} \lesssim \| F_i \|^\alpha_{L^\infty} \| F_i \|^{1-\alpha}_{L^{(1-\alpha)\varepsilon_1}}
  \]
  for \( p = \frac{(4+4\varepsilon_1)}{1-\varepsilon_1} \), we use H"older’s inequality, Strichartz estimates and Sobolev embedding to estimate (5.18) by

\[
N_1^{-\frac{1}{2}}(N_2 N_3)^{-s_1} N^{-\frac{1}{2}} \int_{T_3^2} \prod_{i=1}^{3} F_i \cdot G \, dx \, dt
\]

\[
\lesssim N_1^{-\frac{1}{2}}(N_1 N_2 N_3)^{-s_1} N^{-\frac{1}{2}} \| F_i \|^3_{L^{4_\varepsilon_1}} \| F_i \|^{1/2}_{L^6} \| G \|_{L^{4_\varepsilon_1}}.
\]

Taking \( \alpha = 1/4 \), we have \( (1 - \alpha)\varepsilon_1 < 4 \) and for \( \varepsilon_1 \) sufficiently small, \( p\alpha < 6 \). By Sobolev embedding and Strichartz estimates (3.4) with \( r = \frac{15}{4} \) and \( \theta_1 = 14/15 \) for the \( F_2, F_3 \) and by Strichartz estimates with \( r = 4 - \varepsilon_1 \) and \( \theta_2 = \frac{4-2\varepsilon_1}{4-\varepsilon_1} \) for \( F_1 \) and \( G \) we obtain

\[
\lesssim (N_1 N)^{-1} \frac{1}{2} (N_2 N_3)^{\frac{21}{20} - s_1} \| F_i \|^3_{L^{4_\varepsilon_1}} \| u_i \|^{1/4}_{H^1(T^3)} \| c_i \|^{3/4}_{L^2}
\]

The expression on the right-hand side of the inequality is summable for dyadic values of \( N_i \) and \( N \) provided \( \varepsilon_1 > \theta \). Thus we obtain

(5.19)

\[
(5.18) \lesssim (\lambda + T)^{\frac{1}{2}+\theta} \| u_1 \|_{X^{\frac{1}{2}, \frac{1}{2}, \delta}} \prod_{i=2}^{3} \| u_i \|_{X^{\frac{1}{2}, \frac{1}{2}, \delta}}.
\]

- **Case (B): \( u_2 \) of type (I) and \( u_1, u_3 \) of type (II)** In this case, we use H"older’s inequality and Strichartz estimates (3.4) with \( r = \frac{15}{8} \) and \( \theta = \frac{8}{9} \) and estimate

(5.18) \( \lesssim N_1^{-\frac{1}{2}} (N_2 N_3)^{-s_1} N^{-\frac{1}{2}} \int F_1 F_2 F_3 \cdot G \, dx \, dt \)

\[
\lesssim N_1^{-\frac{1}{2}} (N_2 N_3)^{-s_1} N^{-\frac{1}{2}} \| F_1 \|_{L^{18}} \| F_2 \|_{L^{18}} \| F_3 \|_{L^{18}} \| G \|_{L^{18}}
\]

\[
\lesssim N_1^{-\frac{1}{2}} (N_2 N_3)^{-s_1} N^{-\frac{1}{2}} \| F_2 \|_{L^{6}} \| u_1 \|_{X^{\frac{1}{2}, \frac{1}{2}, \delta}} \| u_3 \|_{X^{s_1, \frac{1}{2}, \delta}}
\]

which is again summable for dyadic \( N \) and \( N_i \), yielding

(5.20)

\[
(5.18) \lesssim \lambda \| u_1 \|_{X^{\frac{1}{2}, \frac{1}{2}, \delta}} \| u_3 \|_{X^{s_1, \frac{1}{2}, \delta}}
\]
• **Case (C):** \(u_2, u_3\) of type (I) and \(u_1\) of type (II) In this case, we use Hölder’s inequality and Strichartz estimates [11] with \(r = 3\) and \(\theta = \frac{\alpha}{3}\) in [11] and estimate

\[
\|\nabla N[u] - \nabla N[S_N u]\|_{X^{s+1.5+\delta_{(J\times T^3)}}} \lesssim C(\lambda, R, T) \delta^c N^{-\theta} \left( 1 + \|u\|_{X^{s+1.5+\delta_{(J\times T^3)}}}^2 + \|u\|_{X^{s+1.5+\delta_{(J\times T^3)}}}^6 \right),
\]

where \((u, \partial_t u)\) is the global solution to the cubic nonlinear Klein-Gordon equation (1.1) with initial data \((u_0, u_1)\).

**Proof.** Once again, we use the inequality.

\[
|\langle N[u] - N[S_N u]\rangle| \lesssim |(I - S_N)u|(|u|^2 + |S_N u|^2),
\]

We perform the same arguments as in the Proposition 5.1 but we will always put the high-frequency term, \((I - S_N)u\), in \(X^{s+1.5+\delta}\), even for the linear evolution. For instance, consider the case where all the \(u_i\) are of type (I). We once again estimate the expression

\[
\sum_{n=n_1+n_2+n_3} \int_{\tau_1+\tau_2+\tau_3} \frac{c_1(n_1, \tau_1)}{\langle \tau_1 \rangle^{2+}} \prod_{i=2}^3 \frac{c_i(n_i, \tau_i)}{\langle \tau_i \rangle^{2+}} \frac{d(n, \tau)}{\langle \tau \rangle^{2+}}
\]

and we obtain

\[
\|\nabla N[u] - \nabla N[S_N u]\|_{X^{s+1.5+\delta_{(J\times T^3)}}} \lesssim C(\lambda, R, T) \delta^c N^{-\theta} \left( 1 + \|u\|_{X^{s+1.5+\delta_{(J\times T^3)}}}^2 + \|u\|_{X^{s+1.5+\delta_{(J\times T^3)}}}^6 \right),
\]

which is summable for dyadic \(N\) and \(N_i\), By Strichartz estimates, recalling that we set

\[
u_1 = (1 - S_N)S(t)(u_0, u_1),
\]
we obtain
\[ \|u_1\|_{X^{s_1, \frac{1}{3}+\delta}} = \|(1 - S_N)S(t)(u_0, u_1)\|_{X^{s_1, \frac{1}{3}+\delta}} \lesssim N^{-\theta}\|S(t)(u_0, u_1)\|_{X^{\frac{1}{2}, \frac{1}{3}+\delta}} \lesssim N^{-\theta}\|(u_0, u_1)\|_{H^{\frac{1}{2}}(\mathbb{T}^3)}. \]
which yields the desired estimate. The other combinations of functions follow analogously. \qed

6. Probabilistic approximation of the flow of the NLKG

This section is devoted to the proof of the approximation of the flow map for the cubic nonlinear Klein-Gordon equation by the flow of the nonlinear Klein-Gordon equation with truncated nonlinearity. We will use here the probabilistic boundedness estimates from Section 5.

**Proposition 6.1.** Let \( \Phi \) denote the flow of the cubic nonlinear Klein-Gordon equation
\[ \dot{u} = J(Au + \nabla N[u]) \]
and \( \Phi_N \) the flow of the cubic nonlinear Klein-Gordon equation with truncated nonlinearity
\[ \dot{u}_N = J(Au_N + \nabla N[u_N]). \]

Fix \( R, T, \lambda > 0 \). Then for every \((u_0, u_1) \in \Sigma_\lambda \cap B_R\),
\[ \sup_{t \in [0, T]} \|\Phi(t)(u_0, u_1) - \Phi_N(t)(u_0, u_1)\|_{H^{\frac{1}{2}}(\mathbb{T}^3)} \leq C(\lambda, T, R) \varepsilon_1(N) \]
with \( \varepsilon_1(N) \to 0 \) as \( N \to \infty \).

**Proof.** Fix \( R, T, \lambda > 0 \) and let \( \Sigma_\lambda \) be as defined in (4.4). We need to estimate the difference \( \Phi - \Phi_N \) for initial data \((u_0, u_1) \in \Sigma_\lambda \cap B_R\). Fix such a \((u_0, u_1)\) and let \( u(t) \) and \( u_N(t) \) denote the corresponding solutions to the full and truncated equations, respectively. By the choice of \( \Sigma_\lambda \),
\begin{align*}
\|S(t)(u_0, u_1)\|_{L^6([0, T]; L^6(\mathbb{T}^3))} &< C_T \lambda \\
\|\dot{w}(\partial_t w)\|_{L^\infty([0, T]; H^1(\mathbb{T}^3))} &< C(\lambda + T)^{1+}, \\
\|\partial_t w_N\|_{L^\infty([0, T]; H^1(\mathbb{T}^3))} &< C(\lambda + T)^{1+},
\end{align*}
where, as usual, \( \dot{w}(t) \) and \( w_N(t) \) are the nonlinear components of the global solutions \( u(t) \) and \( u_N(t) \), respectively. Note in particular, for any subinterval \( J \subset [0, T] \), these bounds hold uniformly. Furthermore, for \( |J| = \delta \), Proposition 5.1 and the inhomogeneous estimate for the Klein-Gordon equation yield
\[ \|w\|_{X^{s_2, \frac{1}{3}+\delta}} \leq \|\nabla N[u]\|_{X^{s_2-1, \frac{1}{3}+\delta}} \leq C(\lambda, R, T) \delta^c \left(1 + \|u\|_{X^{s_1, \frac{1}{3}+\delta}}^2 + \|u\|_{X^{s_1, \frac{1}{3}+\delta}}^\frac{9}{4}\right), \]
hence if \( \inf J = t_0 \), by Lemma 3.3 we obtain
\[ \|u\|_{X^{s_1, \frac{1}{3}+\delta}} \leq \|S(t)(u_0, u_1)\|_{X^{s_1, \frac{1}{3}+\delta}} + \|w\|_{X^{s_2, \frac{1}{3}+\delta}} \leq C\|(u(t_0), \partial_t u(t_0))\|_{H^{s_1}} + \|\nabla N[u]\|_{X^{s_2-1, \frac{1}{3}+\delta}}. \]
Using the uniform bounds from (6.1) and the homogeneous estimate, Lemma 3.3 and similar considerations to those at the start of the proof of Proposition 5.1 we obtain
\[ \|(u(t_0), \partial_t u(t_0))\|_{H^{s_1}} \lesssim C(\lambda, R, T) \]
(6.2)

hence
\[ \|u\|_{X^{s_1, \frac{1}{3}+\delta}} \leq C(\lambda, R, T) + C(\lambda, R, T) \delta^c \left(1 + \|u\|_{X^{s_1, \frac{1}{3}+\delta}}^2 + \|u\|_{X^{s_1, \frac{1}{3}+\delta}}^\frac{9}{4}\right), \]
(6.3)
Finally, by (5.13) we have that the right-hand side is finite, we have hence the inhomogeneous estimate yields and by Proposition 5.2, this argument yields the same result for $\Phi$ (6.4) Thus by taking $\delta \equiv \delta(\lambda, R, T) > 0$ sufficiently small, independent of $J \subset [0, T]$, we obtain that (6.4) \[ \|\Phi(t)(u_0, u_1)\|_{X^{s_1, \frac{1}{2}+\delta}((J \times \mathbb{T}^3)} + \|\partial_t \Phi(t)(u_0, u_1)\|_{X^{s_1-1, \frac{1}{2}+\delta}((J \times \mathbb{T}^3)} \leq C(\lambda, T, R). \] and by Proposition 5.2 this argument yields the same result for $\Phi(t)$ with the same $\delta > 0$. Now, define \[ (\phi, \phi_t) := (u - u_N, \partial_t u - \partial_t u_N) \] then we have \[ \frac{d}{dt} (\phi, \phi_t) - JA(\phi, \phi_t) = J \left( \nabla N[u] - \nabla N[u_N] \right) \] Set \[ \Delta_1 := \nabla N[u] - \nabla N[u_N], \] \[ \Delta_2 := \nabla N[u_N] - \nabla N[S_N u_N], \] \[ \Delta_3 := (I - S_N) \nabla N[S_N u_N], \] and fix $J \subset [0, T]$ with $|J| = \delta$. We will estimate \[ \|\Delta_1\|_{X^{-\frac{1}{2}, -\frac{1}{2}+\delta}(J \times \mathbb{T}^3)} + \|\Delta_2\|_{X^{-\frac{1}{2}, -\frac{1}{2}+\delta}(J \times \mathbb{T}^3)} + \|\Delta_3\|_{X^{-\frac{1}{2}, -\frac{1}{2}+\delta}(J \times \mathbb{T}^3)} \] By Proposition 5.3 we can bound $\Delta_1$ by \[ \|\Delta_1\|_{X^{-\frac{1}{2}, -\frac{1}{2}+\delta}} \leq C(\lambda, R, T) \delta^c \|u - u_N\|_{X^{s_1, \frac{1}{2}+\delta}} \left[ \left( 1 + \|u\|_{X^{s_1, \frac{1}{2}+\delta}} \right) + \left( 1 + \|u\|_{X^{s_1+1, \frac{1}{2}+\delta}} \right) \right]. \] For the second term, Proposition 5.4 and Remark 5.1 yields \[ \|\Delta_2\|_{X^{s_2-1, -\frac{1}{2}+\delta}} \leq C(\lambda, R, T) \delta^c N^{-\theta} \left( 1 + \|u_N\|_{X^{s_1, 1+\delta}} \right). \] Finally, by (5.13) we have \[ \|\Delta_3\|_{X^{-\frac{1}{2}, -\frac{1}{2}+\delta}} \leq N^{-\theta} \|\nabla N[S_N u_N]\|_{X^{s_2-1, -\frac{1}{2}+\delta}} \leq C(\lambda, R, T) \delta^c N^{-\theta} \left( 1 + \|u_N\|_{X^{s_1, 1+\delta}} \right). \] In the second and third terms, we used the observation that for any $N \in \mathbb{N}$, $s \in \mathbb{R}$ and any $v$ such that the right-hand side is finite, we have \[ \|S_N v\|_{X^{s, \frac{1}{2}+}} \leq \|v\|_{X^{s, \frac{1}{2}+}}. \] Let $\Delta := \Delta_1 + \Delta_2 + \Delta_3$, then by Duhamel’s formula, \[ (\phi(t), \partial_t \phi(t)) = \int_0^t e^{-\tau JA(\Delta(\tau), \partial_t \Delta(\tau))} d\tau, \] hence the inhomogeneous estimate yields \begin{equation} (6.5) \|\phi\|_{X^{s, \frac{1}{2}+\delta}} \leq \|\Delta_1\|_{X^{-\frac{1}{2}, -\frac{1}{2}+\delta}} + \|\Delta_2\|_{X^{-\frac{1}{2}, -\frac{1}{2}+\delta}} + \|\Delta_3\|_{X^{-\frac{1}{2}, -\frac{1}{2}+\delta}}, \end{equation}
and together with (6.4) and the similar result for $\Phi_N$ we can bound (6.5) by
\[
\|\phi\|_{L^2_{\delta}([0,T] \times \mathbb{T}^3)} \leq C(\lambda, T, R) \delta^c \|\phi\|_{L^2_{\delta}([0,T] \times \mathbb{T}^3)} + C(\lambda, T, R) \delta^c N^{-\theta},
\]
and similarly for the time derivative component. Thus for $\delta > 0$ sufficiently small (and independent of $N$) we obtain
\[
\|(\phi, \partial_t \phi)\|_{L^\infty_{x}H^{1/2}_{\delta}([0,T] \times \mathbb{T}^3)} \leq C_1(\lambda, R, T, \varepsilon_1(N)),
\]
with $\lim_{N \to \infty} \varepsilon_1(N) = 0$. We conclude the proof by noting that (6.7) depends only on the length of the subinterval, hence we can split the interval $[0, T]$ into sufficiently small subintervals to obtain the bound (6.7) on each piece. \hfill $\square$

As a corollary of Proposition 6.1, we obtain the following decomposition for the flow map of the cubic nonlinear Klein-Gordon equation on the subsets $\Sigma \subset \Sigma$. This should be compared to the result 21 Lemma 3]. Although we will not use this corollary in this work, we believe that such a decomposition will be necessary in order to prove the invariance of the capacity so we include it as an easy consequence.

**Corollary 6.2.** Let the assumptions of Proposition 7.1 hold. Fix $\varepsilon > 0$, then for any $\lambda, R, T > 0$, $N \in \mathbb{N}$ and $(u_0, u_1) \in \Sigma_{\lambda} \cap B_R$ we have
\[
\Phi(t)(u_0, u_1) = e^{tJA}(I + \widetilde{\Phi}_\varepsilon(t))(I + \widetilde{\Phi}_N(t))(u_0, u_1), \quad \text{for } t \in [0, T]
\]
where $(I + \widetilde{\Phi}_N)$ is a smooth symplectomorphisms on $H^{1/2}(\mathbb{T}^3)$. Moreover, for any $N \geq N_0(\lambda, R, T)$ sufficiently large, $\widetilde{\Phi}_\varepsilon$ satisfies
\[
\|\widetilde{\Phi}_\varepsilon(g)\|_{H^{1/2}_{\delta}} \leq \varepsilon \quad \text{for } g \in (I + \widetilde{\Phi}_N)(\Sigma_{\lambda} \cap B_R).
\]

**Proof.** By the previous proposition, for every $\lambda, R, T, \varepsilon > 0$ we can choose $N \geq N_1(\lambda, R, T, \varepsilon)$ so that the nonlinear components, $\widetilde{\Phi}$ and $\widetilde{\Phi}_N$, of the flow maps satisfy
\[
\|\widetilde{\Phi}(t) - \widetilde{\Phi}_N(t)\|(u_0, u_1)\|_{L^\infty_{x}H^{1/2}_{\delta}([0,T] \times \mathbb{T}^3)} < \varepsilon.
\]
Writing
\[
\Phi = e^{tJA}(I + \widetilde{\Phi}_N + (\widetilde{\Phi} - \widetilde{\Phi}_N))(I + \widetilde{\Phi}_N)^{-1}(I + \widetilde{\Phi}_N)
= e^{tJA}(I + (\widetilde{\Phi} - \widetilde{\Phi}_N) \circ (I + \widetilde{\Phi}_N)^{-1})(I + \widetilde{\Phi}_N)
\]
and by setting $\widetilde{\Phi}_\varepsilon := (\widetilde{\Phi} - \widetilde{\Phi}_N) \circ (I + \widetilde{\Phi}_N)^{-1}$ we obtain the bound (6.9) from (6.10). \hfill $\square$

7. Approximation of the flow on open sets

The goal of this section is to prove Theorem 1.4. We define the $\rho$ - fattening of a set $V$ by
\[
V_\rho := \bigcup_{u \in V} B_{\rho}(u).
\]
A first step will be to use the critical local theory from Appendix A to conclude that there exists some $\rho \equiv \rho(\lambda, T)$ such that we obtain the same result with uniform bounds on the open set $\Sigma_{\lambda, \rho}$, which is $\rho$ - fattening of $\Sigma_{\lambda}$.

A key component of our argument is the critical long-time stability theory which allows us to upgrade the sets of large measure where the approximation holds to open sets. Stability arguments first appeared in the context of the three-dimensional energy critical nonlinear Schrödinger equation in 14, see also 37. For the nonlinear Klein-Gordon equation in periodic settings, some
care is required as the Strichartz estimates need to be localized in time. Nonetheless, they follow in a similar manner from the Strichartz estimates of Proposition 3.2, see Appendix A. One modification we will present are the stability arguments adapted to the nonlinear Klein-Gordon equation with truncated nonlinearity. Importantly, we will be able to choose the small parameters in these arguments uniformly in the truncation parameter. In the sequel, we will use the notation

$$\|u\|_{X(I \times T^3)} := \|u\|_{L^4_t(I \times T^3)}.$$  

**Proposition 7.1.** Let $\Phi$ denote the flow of the cubic nonlinear Klein-Gordon equation (1.1) and $\Phi_N$ the flow of the cubic nonlinear Klein-Gordon equation with truncated nonlinearity (1.11). Fix $T, R > 0$, $K, N \in \mathbb{N}$ with $K < N$. Let $\lambda > 0$ be such that $\Sigma_\lambda \cap B_R \neq \emptyset$, then there exists $\gamma_1, \gamma_2 > 0$ and $\rho \equiv \rho(\lambda, T) > 0$ sufficiently small such that for every $(u_0, u_1) \in \Sigma_{\lambda, \rho} \cap B_R$ and every $I \subseteq [0, T]$,

$$\sup_{t \in I} \|S_K(\Phi(t)(u_0, u_1) - \Phi_N(t)(u_0, u_1))\|_{H^{1/2}(\mathbb{T}^3)} \leq C(\lambda, R, T) [\varepsilon_1(N) + |I|^{\gamma_1} K^{\gamma_2} \rho]$$

with $\lim_{N \to \infty} \varepsilon_1(N) = 0$. Moreover, we can choose $\rho \equiv \rho(\lambda) > 0$ uniformly for all $T \leq 1$.

**Proof.** Let $\lambda > 0$ such that $\Sigma_\lambda \cap B_R \neq \emptyset$. Fix $(v_0, v_1) \in \Sigma_\lambda \cap B_R$ and we will prove that there exists $\rho \equiv \rho(\lambda, T) > 0$ such that for every $(u_0, u_1) \in B_\rho(v_0, v_1)$

$$\sup_{t \in [0, T]} \|S_K[\Phi(t)(u_0, u_1) - \Phi_N(t)(u_0, u_1)]\|_{H^{1/2}(\mathbb{T}^3)} \leq C(\lambda, R, T) [\varepsilon_1(N) + |I|^{\gamma_1} K^{\gamma_2} \rho].$$

For such $(v_0, v_1)$ then we have unique global solutions $v$ and $v_N$ with initial data $(v_0, v_1)$ to the equations (1.1) and (1.11), respectively. By the choice of $\Sigma_\lambda$, there exists some $M > 0$ depending only on $\lambda, T$ such that

$$\|v_N\|_{X([0, T] \times T^3)} \leq M$$

$$\|v\|_{X([0, T] \times T^3)} \leq M.$$  

Let $\rho_1 \equiv \rho_1(M, T) \geq 0$ be small enough so that the conclusions of Lemma A.2 and Lemma A.4 hold. Let $(u_0, u_1) \in B_\rho(v_0, v_1)$, then by Lemma A.2 and Lemma A.4 there exists solutions

$$(7.3) \quad u := \Phi(t)(u_0, u_1)$$

$$(7.4) \quad u_N := \Phi_N(t)(u_0, u_1)$$

to equations (1.1) and (1.11), respectively, on $[0, T]$. Using Proposition 7.1 and letting $\bar{\Phi}$ and $\bar{\Phi}_N$ denote the nonlinear components of the flow maps for the full equation (1.1) and the equation with truncated nonlinearity (1.11) we estimate

$$\sup_{t \in I} \|S_K(\Phi(t)(u_0, u_1) - \Phi_N(t)(u_0, u_1))\|_{H^{1/2}(\mathbb{T}^3)} = \sup_{t \in I} \|S_K(\Phi(t)(u_0, u_1) - \bar{\Phi}(t)(u_0, u_1))\|_{H^{1/2}(\mathbb{T}^3)}$$

$$\leq \sup_{t \in I} \|S_K(\Phi(t)(v_0, v_1) - \bar{\Phi}(t)(v_0, v_1))\|_{H^{1/2}(\mathbb{T}^3)}$$

$$+ \sup_{t \in I} \|S_K(\bar{\Phi}(t)(v_0, v_1) - \bar{\Phi}(t)(u_0, u_1))\|_{H^{1/2}(\mathbb{T}^3)}$$

$$+ \sup_{t \in I} \|S_K(\bar{\Phi}_N(t)(v_0, v_1) - \bar{\Phi}_N(t)(u_0, u_1))\|_{H^{1/2}(\mathbb{T}^3)}$$

$$\leq C(\lambda, R, T) \varepsilon_1(N) + A + B.$$
We will start by estimating term $A$. We choose a conjugate admissible pair $(q', r')$, specified by
\[ 3 = \frac{1}{q'} + \frac{3}{r'} \]
so that $q' = 4/3 - \alpha$ for some $0 < \alpha < 4/3$. Then the admissibility relation yields
\[ r' = \frac{4 - 3\alpha}{3 - 3\alpha} > \frac{4}{3} \]
for all $\alpha > 0$. We write
\[ r' = \frac{4}{3} + \beta, \quad \beta = \frac{\alpha}{3 - 3\alpha}. \]

Then, with the conjugate admissible pair $(q', r')$, we use Strichartz estimates to bound
\[
\sup_{t \in I} \| S_K \left( \Phi(t)(v_0, v_1) - \Phi(t)(u_0, u_1) \right) \|_{\dot{H}^{1/2}(\mathbb{T}^3)} \lesssim \| S_K [u^3 - v^3] \|_{L_t^{4/3} L_x^{3/2 + \beta}(I \times \mathbb{T}^3)}
\]
We use Hölder’s inequality in time to obtain a small power in the length of the interval
\[
\| S_K [u^3 - v^3] \|_{L_t^{4/3} L_x^{3/2 + \beta}(I \times \mathbb{T}^3)} \lesssim |I|^{\gamma_1} \| S_K [u^3 - v^3] \|_{L_t^{4/3} L_x^{3/2 + \beta}([0,T] \times \mathbb{T}^3)}
\]
and by Sobolev embedding in space with
\[
\frac{3}{4 + 3\beta} + \frac{\gamma_2}{3} = \frac{3}{4}
\]
and Bernstein’s inequality, we obtain the bound
\[
\| S_K [u^3 - v^3] \|_{L_t^{4/3} L_x^{3/2 + \beta}(I \times \mathbb{T}^3)} \lesssim |I|^{\gamma_1} \| (1 - \Delta)^{\gamma_2/2} S_K [u^3 - v^3] \|_{L_t^{4/3} L_x^{3/2 + \beta}([0,T] \times \mathbb{T}^3)}
\]
\[
\lesssim |I|^{\gamma_1} K^{\gamma_2} \| S_K [u^3 - v^3] \|_{L_t^{4/3} L_x^{3/2 + \beta}([0,T] \times \mathbb{T}^3)}
\]
Using the boundedness of the smooth projection operators from Lemma 35 and the bound 3.8 from Lemma A.2 we obtain
\[ A \lesssim |I|^{\gamma_1} K^{\gamma_2} C(M, T) \rho \]
A similar computation yields the same for bound for $B$. Thus, for all $(u_0, u_1) \in B_\rho(v_0, v_1)$, we have 
\[
\sup_{t \in I} \| S_K [\Phi(t)(u_0, u_1) - \Phi_N(t)(u_0, u_1)] \|_{\dot{H}^{1/2}(\mathbb{T}^3)} \lesssim C(\lambda, R, T) \varepsilon_1(N) + |I|^{\gamma_1} K^{\gamma_2} \rho,
\]
as required. Finally, the dependence of the constants from the stability lemmas on $T$ only results from the constant in Strichartz estimates, so we can choose $\rho$ uniformly for $T \leq 1$.

Remark 7.1. A crucial component of the previous proof is the fact that we only need to compare the nonlinear components of the solutions as the free evolutions are the same for both $\Phi$ and $\Phi_N$. This allows us to perform the same estimates on any subinterval of $[0,T]$ without having to iterate the bounds for the initial data. As a result, we obtain the following corollary which yields convergence on the whole interval.

**Corollary 7.2.** Let $\Phi$ denote the flow of the cubic nonlinear Klein-Gordon equation (1.11) and $\Phi_N$ the flow of the cubic nonlinear Klein-Gordon equation with truncated nonlinearity (1.11). Fix $T, R > 0$, and $N, K \in \mathbb{N}$ with $K < N$. Let $\lambda > 0$ be such that $\Sigma_{\lambda} \cap B_R \neq \emptyset$, then there exists $\gamma_1, \gamma_2 > 0$ and $\rho \equiv \rho(\lambda, T) > 0$ sufficiently small such that for every $(u_0, u_1) \in \Sigma_{\lambda, \rho} \cap B_R$,
\[
\sup_{t \in [0, T]} \| S_K (\Phi(t)(u_0, u_1) - \Phi_N(t)(u_0, u_1)) \|_{H^{1/2}} \leq C(\lambda, R, T)\varepsilon_1(N)
\]
with \( \lim_{N \to \infty} \varepsilon_1(N) = 0 \).

**Proof.** Let \( \lambda, R, T > 0 \) and \( \rho(\lambda, T) > 0 \) as in Proposition 7.1. Then for \((u_0, u_1) \in \Sigma_{\lambda, \rho} \cap B_R \), any \( \delta > 0 \) we partition \([0, T]\) into disjoint intervals \( I_j \) of length \( \delta > 0 \). Then for any \( K \in \mathbb{N} \), Proposition 7.1 yields
\[
\sup_{t \in [0, T]} \| S_K (\Phi(t)(u_0, u_1) - \Phi_N(t)(u_0, u_1)) \|_{\mathcal{H}^{1/2}} \\
= \max_j \sup_{t \in I_j} \| S_K (\Phi(t)(u_0, u_1) - \Phi_N(t)(u_0, u_1)) \|_{\mathcal{H}^{1/2}} \\
\leq C(\lambda, R, T) [\varepsilon_1(N) + \delta^{\gamma_1} K^{\gamma_2} \rho],
\]
and taking \( \delta \to 0 \) we obtain the result. \( \square \)

7.1. **Proof of Theorem 1.4.** We recall the statement of this theorem.

**Theorem 7.3.** Let \( \Phi \) denote the flow of the cubic nonlinear Klein-Gordon equation \((1.1)\) and \( \Phi_N \) the flow of \((1.11)\). Fix \( R > 0 \) and \( \lambda, \rho \) to equations \((1.1)\) and \((1.11)\) with initial data \((u_0, u_1) \in B_R \).

**Proof.** Fix \( R > 0 \), \( I = [0, \sigma] \) for some \( \sigma > 0 \) to be fixed, and let \( \lambda > 0 \) be sufficiently large so that we can find \((v_0, v_1) \in \Sigma_{\lambda} \cap B_R \). Noting that for all \( N \in \mathbb{N} \), we have \( S_{2N}(v_0, v_1) \in \Sigma_{\lambda} \cap B_R \), there exists some constant \( M > 0 \), depending only on \( \lambda \), such that the corresponding global solutions \( v \) and \( v_N \) to equations \((1.1)\) and \((1.11)\) with initial data \( S_{2N}(v_0, v_1) \) satisfy
\[
\| v \|_{X(I \times \mathbb{T}^3)} \leq M, \\
\| v_N \|_{X(I \times \mathbb{T}^3)} \leq M.
\]

Let \((u_0, u_1) \in B_R \), then
\[
\| S(t) \Pi_{2N}(v_0 - u_0, v_1 - u_1) \|_{L^4_{t,x}(I \times \mathbb{T}^3)} \leq |I|^{1/4} \sup_{t \in I} \| S(t) \Pi_{2N}(v_0 - u_0, v_1 - u_1) \|_{L^4_x(\mathbb{T}^3)} \\
\leq |I|^{1/4} \sup_{t \in I} \| S(t) \Pi_{2N}(v_0 - u_0, v_1 - u_1) \|_{H^3_x(\mathbb{T}^3)} \\
\lesssim |I|^{1/4} N^{1/2} \| (v_0 - u_0, v_1 - u_1) \|_{H^{3/2}_x(\mathbb{T}^3)} \\
\lesssim |I|^{1/4} N^{1/2} R.
\]

Let \( \rho_1(M, R, I) \) be as in the stability lemma and recall that the dependence of \( \rho_1 \) on \( I \) arises only because of the constant in Strichartz estimates. In particular, for \( \sigma \leq 1 \) we can choose \( \rho_1(M, R) \) uniformly. Let \( 0 < \varepsilon_0 < \rho_1 \), then setting
\[
\sigma \approx N^{-2} R^{-4} \varepsilon_0,
\]
the smallness condition \((A.5)\) of Lemma \( A.2 \) is met and we conclude that for \( t \in [0, \sigma] \), solutions
\[
u(t) := \Phi(t) \Pi_{2N}(u_0, u_1) \\
u(t) := \Phi_N(t) \Pi_{2N}(u_0, u_1)
\]
exist to equations \((1.1)\) and \((1.11)\), respectively. Moreover, we conclude from \((A.8)\) that
\[
\| u^3 - v^3 \|_{L^{3/2}(I \times \mathbb{T}^3)} \leq C(M) \varepsilon_0.
\]
Hence, by Strichartz estimates, the nonlinear components of the solutions satisfy

\begin{equation}
\sup_{t \in I} \| \tilde{\Phi}(t)\Pi_{2N}(v_0, v_1) - \tilde{\Phi}(t)\Pi_{2N}(u_0, u_1) \|_{H^{1/2}(\mathbb{T}^3)} \leq \| u^3 - v^3 \|_{L^{4/3}_t(L^3_x)} \leq C(M) \varepsilon_0.
\end{equation}

We can estimate (7.6) using the triangle inequality as in (7.5), and we can similarly conclude the analogue of (7.7) for the term with \( \Phi_N \). Thus, for all \((u_0, u_1) \in B_R\), we obtain

\[ \sup_{t \in [0, \sigma]} \| \Phi(t)\Pi_{2N}(u_0, u_1) - \Phi_N(t)\Pi_{2N}(u_0, u_1) \|_{H^{1/2}(\mathbb{T}^3)} < C(R) [\varepsilon_1(N) + \varepsilon_0]. \]

\[ \square \]

7.2. Proof of Theorem 1.6. The proof of this theorem follows from Proposition 7.4 and the argument used to prove Theorem 1.1. We will prove the following statement, from which we obtain Theorem 1.6 readily.

**Theorem 7.4.** Let \( \Phi \) denote the flow of the cubic nonlinear Klein-Gordon equation (1.1). Fix \( T, R > 0, k_0 \in \mathbb{Z}^3, z \in \mathbb{C}, \) and \( u_* \in H^{1/2}(\mathbb{T}^3) \) and let \( \lambda > 0 > 0 \) be such that \( \Sigma_{\lambda, \rho} \cap B_R(u_*) \neq \emptyset \). Then for all \( 0 < \rho \equiv \rho(\lambda, T) \) sufficiently small,

\begin{equation}
\Phi(T)(\Sigma_{\lambda, \rho} \cap B_R(u_*)) \not\subset C_r(z; k_0).
\end{equation}

for all \( r > 0 \) with \( \pi r^2 < \varepsilon = c(\Sigma_{\lambda, \rho} \cap B_R(u_*)) \).

**Proof of Theorem 1.6** Fix \( T, R > 0, k_0 \in \mathbb{Z}^3, z \in \mathbb{C}, \) and \( u_* \in H^{1/2}(\mathbb{T}^3) \). Let \( \| u_* \| =: R_1 \), then Proposition 7.4 yields \( \gamma_1, \gamma_2 > 0 \) and \( \rho \equiv \rho(\lambda, T) > 0 \) such that for any \((u_0, u_1) \in \Sigma_{\lambda, \rho} \cap B_{R_1}\), and any \( K, N \in \mathbb{N} \) with \( |k_0| < K < N \),

\begin{equation}
\sup_{t \in [0, T]} \| S_K(\Phi(t)u_0, u_1) - \Phi_N(t)(u_0, u_1) \|_{H^{1/2}} \leq \varepsilon_1(N) \leq C(\lambda, R_1, T) \varepsilon_1(N)
\end{equation}

with \( \lim_{N \to \infty} \varepsilon_1(N) = 0 \). Let \( \pi r^2 < c(\Sigma_{\lambda, \rho} \cap B_R(u_*)) \) and let

\[ \varepsilon = \frac{\pi^{-1}c(\Sigma_{\lambda, \rho} \cap B_R(u_*)) - r}{2} \]

Fix \( N \in \mathbb{N} \) sufficiently large so that for any \((u_0, u_1) \in \Sigma_{\lambda, \rho} \cap B_{R_1}\), we obtain from (7.9) that

\begin{equation}
\sup_{t \in [0, T]} \| S_K(\Phi(t)u_0, u_1) - \Phi_N(t)(u_0, u_1) \|_{H^{1/2}} \leq \varepsilon < \frac{r}{2}.
\end{equation}

Since \( \Phi_N \) preserves capacities by Proposition 1.3, we have the equality

\[ c(\Phi_N(t)(\Sigma_{\lambda, \rho} \cap B_R(u_*))) = c(\Sigma_{\lambda, \rho} \cap B_R(u_*)) \]

for all \( t \in \mathbb{R} \). Thus for \( T \) as above, we can find some \((u_0, u_1) \in \Sigma_{\lambda, \rho} \cap B_R(u_*) \) such that

\[ \left( |k_0| \Phi_N(T)(u_0, u_1)(k_0) - z|^2 + |k_0|^{-1} |\partial_t \Phi_N(T)(u_0, u_1)(k_0) - z|^2 \right)^{1/2} > r + \varepsilon. \]

and since \((u_0, u_1) \in \Sigma_{\lambda, \rho} \cap B_{R_1}\), we conclude by the triangle inequality and (7.10) that

\[ \left( |k_0| \Phi(T)(u_0, u_1)(k_0) - z|^2 + |k_0|^{-1} |\partial_t \Phi(T)(u_0, u_1)(k_0) - z|^2 \right)^{1/2} > r. \]

which completes the proof. \[ \square \]
Appendix A. Stability Arguments

This appendix is devoted the proofs of the critical stability lemmata for the cubic nonlinear Klein-Gordon equation. As usual, these statements are proved first for solutions which have sufficiently small Strichartz bounds, we will call these short-time stability arguments. In order to conclude the statement for arbitrarily large bounds, one needs to divide a given time interval into subintervals such that the norm of the solution is sufficiently small on each subinterval, then the statement follows from an iteration argument. We include these proofs as we would like to make explicit the dependence of the constants on the various parameters involved.

Lemma A.1 (Short-time stability). Let $I \subset \mathbb{R}$ a compact time interval and $t_0 \in I$. Let $v$ be a solution defined on $I \times \mathbb{R}^d$ of the Cauchy problem

$$\begin{cases} v_t - \Delta v + v + v^3 = e \\ (v, \partial_t v)|_{t=t_0} = (v_0, v_1) \in \mathcal{H}^{1/2}(\mathbb{T}^3). \end{cases}$$

Let $(u(t_0), \partial_t u(t_0)) = (u_0, u_1) \in \mathcal{H}^{1/2}(\mathbb{T}^3)$ be such that

$$||(v_0 - u_0, v_1 - u_1)||_{\mathcal{H}^{1/2}(\mathbb{T}^3)} \leq K_1$$

for some $K_1 > 0$. Suppose also that we have the smallness conditions

$$\|v\|_{X(I \times \mathbb{T}^3)} \leq \rho_0$$

$$\|S(t-t_0)(v_0 - u_0, v_1 - u_1)\|_{X(I \times \mathbb{T}^3)} \leq \rho$$

$$\|e\|_{L^q_t L^r_x(I \times \mathbb{T}^3)} \leq \rho,$$

for some $0 < \rho < \rho_0(K_1)$ a small constant and $(\tilde{q}', \tilde{r}')$ a conjugate admissible pair. Then there exists a unique solution $(u(t), \partial_t u(t))$ to the cubic nonlinear Klein-Gordon equation on $I \times \mathbb{T}^3$ with initial data $(u_0, u_1)$ at time $t_0$ and $C \equiv C(I) \geq 1$ which satisfies

(A.1) $\|v - u\|_{X(I \times \mathbb{T}^3)} \leq C\rho$

(A.2) $\|v^3 - u^3\|_{L^{4/3}_t H^{1/2}_x(I \times \mathbb{T}^3)} \leq C\rho$

(A.3) $\|(v - u, \partial_t u - \partial_t v)\|_{L^\infty_t H^{1/2}_x(I \times \mathbb{T}^3)} \leq C\rho$

for all admissible pairs $(q, r)$. Furthermore, the dependence of the constant $C$ on time arises only from the constant in the localized Strichartz estimates.

Proof. Without loss of generality, let $t_0 = \inf I$ and let $w := v - u$, then $w$ satisfies the Cauchy problem

(A.4) $$\begin{cases} w_{tt} - \Delta w + w + w^3 - (w + v)^3 = e \\ (w, \partial_t w)|_{t=t_0} = (v_0 - u_0, v_1 - u_1) \end{cases}$$

By Strichartz estimates, Hölder’s inequality and the assumptions above,

$$\|w\|_{X(I \times \mathbb{T}^3)} \leq C \left( \|S(t)(v_0 - u_0, v_1 - u_1)\|_{X(I \times \mathbb{T}^3)} + \|(w + v)^3 - v^3\|_{L^{4/3}_{t,x}(I \times \mathbb{T}^3)} + \|e\|_{L^\tilde{q}_t L^\tilde{r}_x(I \times \mathbb{T}^3)} \right)$$

$$\leq C \left( 2\rho + (\rho_0)^2 \|w\|_{X(I \times \mathbb{T}^3)} + \|w\|_{X(I \times \mathbb{T}^3)}^3 \right)$$

hence a continuity argument yields (A.1) provided $\rho_0$ is sufficiently small. In particular

$$\|(w + v)^3 - v^3\|_{L^{4/3}_{t,x}(I \times \mathbb{T}^3)} \leq C\rho$$
for such \( \rho_0 \). From (A.1) we have
\[
\| (w, \partial_t w) \|_{L_t^\infty H_x^{1/2} (I \times \mathbb{T}^3)} + \| w \|_{L_t^q L_x^r (I \times \mathbb{T}^3)} + \| w \|_{X (I \times \mathbb{T}^3)} \\
\leq C \left( K_1 + \| (w + v)^3 \|_{L_t^{4/3} L_x^3 (I \times \mathbb{T}^3)} + \| e \|_{L_t^q L_x^r (I \times \mathbb{T}^3)} \right) \\
\leq C (K_1 + \rho + C \rho),
\]
hence we obtain (A.3) provided \( \rho_0 \equiv \rho_0 (K_1) \) is chosen sufficiently small. \( \square \)

**Lemma A.2** (Long-time stability). Let \( I \subset \mathbb{R} \) a compact time interval and \( t_0 \in I \). Let \( v \) be a solution defined on \( I \times \mathbb{T}^3 \) of the Cauchy problem
\[
\left\{ \begin{array}{l}
\begin{align*}
&v_t - \Delta v + v + v^3 = e \\
&\left( v, \partial_t v \right) \mid_{t = t_0} = (v_0, v_1) \in H^{1/2} (\mathbb{T}^3).
\end{align*}
\end{array} \right.
\]
Suppose that
\[
\| v \|_{X (I \times \mathbb{T}^3)} \leq L
\]
for some constant \( L > 0 \). Let \( t_0 \in I \) and let \( (u(t_0), \partial_t u (t_0)) = (u_0, u_1) \in H^{1/2} (\mathbb{T}^3) \) be such that
\[
\| (v_0 - u_0, v_1 - u_1) \|_{H^{1/2} (\mathbb{T}^3)} \leq K_1
\]
for some \( K_1 > 0 \). Suppose also that we have the smallness conditions
\[
\begin{align*}
&\| S(t - t_0) (v_0 - u_0, v_1 - u_1) \|_{X (I \times \mathbb{T}^3)} \leq \rho \\
&\| e \|_{L_t^q L_x^r (I \times \mathbb{T}^3)} \leq \rho,
\end{align*}
\]
for some \( 0 < \rho < \rho_1 (K_1, I, L) \) a small constant and \( (q', r') \) a conjugate admissible pair. Then there exists a unique solution \( (u(t), \partial_t u (t)) \) to the cubic nonlinear Klein-Gordon equation on \( I \times \mathbb{T}^3 \) with initial data \( (u_0, u_1) \) at time \( t_0 \) and \( C \equiv C (I, L) \geq 1 \) which satisfies
\[
\begin{align*}
&\| v - u \|_{X (I \times \mathbb{T}^3)} \leq C \rho \\
&\| v^3 - u^3 \|_{L_t^{4/3} L_x^3 (I \times \mathbb{T}^3)} \leq C \rho \\
&\| (v - u, \partial_t u - \partial_t v) \|_{L_t^\infty L_x^{1/2} (I \times \mathbb{T}^3)} + \| v - u \|_{L_t^q L_x^r (I \times \mathbb{T}^3)} \leq C K_1
\end{align*}
\]
admissible pairs \( (q, r) \). Moreover, the dependence of the constants \( C \) and \( \rho_1 \) on time arise solely from the constant in the time localized Strichartz inequality.

**Proof.** Fix \( I \subset \mathbb{R} \) and let \( \rho_0 \equiv \rho_0 (2K_1) > 0 \) be as in Lemma A.1. This will allow for some growth in the argument. We divide the time interval \( I \) into \( J \sim \left( 1 + \frac{L}{\rho_0} \right)^4 \) subintervals \( I_j = [t_j, t_{j+1}] \) such that
\[
\| v \|_{X (I_j)} \leq \rho_0,
\]
letting \( w := u - v \), we can apply the previous lemma on the first interval to obtain
\[
\begin{align*}
&\| v - u \|_{X (I_0 \times \mathbb{T}^3)} \leq C \rho \\
&\| v^3 - u^3 \|_{L_t^{4/3} L_x^3 (I \times \mathbb{T}^3)} \leq C \rho \\
&\| (v - u, \partial_t v - \partial_t u) \|_{L_t^\infty H_x^{1/2} (I_0 \times \mathbb{T}^3)} + \| v - u \|_{L_t^q L_x^r (I_0 \times \mathbb{T}^3)} \leq C K_1.
\end{align*}
\]
We would like to apply this argument iteratively to claim that
\[ \|v - u\|_{X(I_j \times T^3)} \leq C(j)\rho \]
\[ \|v^3 - u^3\|_{L^{4/3}(I \times T^3)} \leq C(j)\rho \]
\[ \|(v - u, \partial_t v - \partial_t u)\|_{L^\infty H^{1/2}_{x,L^1}(I_j \times T^3)} + \|v - u\|_{L^{q'}_{t,x}(I_j \times T^3)} \leq C(j)K_1. \]
In order to do this, we need to ensure that for each \( t_j \) we have
\[ \|(v(t_j) - u(t_j), \partial_t v(t_j) - \partial_t u(t_j))\|_{H^{1/2}(T^3)} \leq 2K_1 \]
and
\[ \|S(t - t_j)(u(t_j) - v(t_j))\|_{X(I_j \times T^3)} < C(I, L)\rho \]
We prove these statements by induction. For (A.11) we use Strichartz estimates and we bound
\[ \|(w(t_j), \partial_t w(t_j))\|_{H^{1/2}_{x,T^3}} \leq \|(w(t_0), \partial_t w(t_0))\|_{H^{1/2}(T^3)} + \|(w + v)^3 - v^3\|_{L^{4/3}_{t,x}([0,t_j-1] \times T^3)} + \|e\|_{L^{q'}_t L^{q''}_x} \]
\[ \leq K_1 + \sum_{k=0}^{j-1} C(k)\rho + \rho \]
and similarly
\[ \|S(t - t_j)(u(t_j) - v(t_j))\|_{X(I_j \times T^3)} \leq 2\rho + \sum_{k=0}^{j-1} C(k)\rho \]
so the conclusion follows by choosing \( \rho_1(K_1, I, L) \) sufficiently small. \qed

**Lemma A.3** (Short-time stability for the truncated equation). Let \( I \subset \mathbb{R} \) a compact time interval and \( t_0 \in I \). Let \( v_N \) be a solution defined on \( I \times T^3 \) of the Cauchy problem
\[
\begin{cases}
(v_N)_t - \Delta v_N + v_N + S_N(S_N v_N)^3 = e \\
(v_N, \partial_t v_N)|_{t=t_0} = (v_0, v_1) \in H^{1/2}(T^3).
\end{cases}
\]
Let \( t_0 \in I \) and let \((u_N(t_0), \partial_t u_N(t_0)) = (u_0, u_1) \in H^{1/2}(T^3)\) be such that
\[ \|(v_0 - u_0, v_1 - u_1)\|_{H^{1/2}(T^3)} \leq K_1 \]
for some \( K_1 > 0 \). Suppose also that we have the smallness conditions
\[ \|S_N v_N\|_{X(I \times T^3)} \leq \rho_0 \]
\[ \|S(t - t_0)(v_0 - u_0, v_1 - u_1)\|_{X(I \times T^3)} \leq \rho \]
\[ \|e\|_{L^{q'}_t L^{q''}_x(I \times T^3)} \leq \rho, \]
for some \( 0 < \rho < \rho_0(K_1) \) a small constant and \((\tilde{q}', \tilde{r}')\) a conjugate admissible pair. Then there exists a unique solution \((u(t), \partial_t u(t))\) to the truncated cubic nonlinear Klein-Gordon equation on \( I \times T^3 \) with initial data \((u_0, u_1)\) at time \( t_0 \) and \( C \equiv C(I) \geq 1 \) which satisfies
\[ \|v_N - u_N\|_{X(I \times T^3)} \leq C\rho \]
\[ \|(v_N)^3 - (u_N)^3\|_{L^{4/3}(I \times T^3)} \leq C\rho \]
\[ \|(v_N - u_N, \partial_t u_N - \partial_t u_N)\|_{L^\infty H^{1/2}_{x,L^1}(I \times T^3)} + \|v_N - u_N\|_{L^{q'}_{t,x}(I \times T^3)} \leq C K_1 \]
for any admissible pair \((q, r)\). Furthermore, the dependence of \( C \) on time arises only from the constant in the localized Strichartz estimates.
Proof. Without loss of generality, let \( t_0 = \inf I \). Let \( w_N = v_N - u_N \), then \( w_N \) satisfies the Cauchy problem
\[
\begin{aligned}
(w_N)_t - \Delta w_N + w_N + S_N(S_Nv_N)^3 - S_N[w_N + v_N]^3 &= e \\
(w_N, \partial_t w_N)|_{t=0} &= (v_0 - u_0, v_1 - u_1)
\end{aligned}
\]
(A.16)

By Strichartz estimates, Hölder’s inequality, the boundedness of \( S_N \) and the assumptions above,
\[
\|w_N\|_{X(I \times \mathbb{T}^3)}
\leq C \left( \|S(t)(v_0 - u_0, v_1 - u_1)\|_{X(I \times \mathbb{T}^3)} + \|(S_Nv_N)^3 - [S_N(w_N + v_N)]^3\|_{L^{4/3}_t(L^{6'}_x(I \times \mathbb{T}^3))} + \|e\|_{L^q_t(L^{r'}_x(I \times \mathbb{T}^3))} \right)
\leq C \left( 2\rho + \|S_Nv_N\|_{X(I \times \mathbb{T}^3)}^3 + \|w_N\|_{X(I \times \mathbb{T}^3)}^3 \right)
\leq C \left( \rho_0 + \|w_N\|_{X(I \times \mathbb{T}^3)}^3 + \|w_N\|_{X(I \times \mathbb{T}^3)}^3 \right)
\]

hence a continuity argument yields (A.13) provided \( \rho_0 \) is sufficiently small. Similarly
\[
\|(w_N, \partial_t w_N)\|_{L^q_t(H^{1/2}_x(I \times \mathbb{T}^3))} + \|w_N\|_{L^q_t(L^{r'}_x(I \times \mathbb{T}^3))} + \|w_N\|_{X(I \times \mathbb{T}^3)}
\leq C \left( \|(v_0 - u_0, v_1 - u_1)\|_{H^{1/2}(\mathbb{T}^3)} + \|(S_Nv_N)^3 - [S_N(w_N + v_N)]^3\|_{L^{4/3}_t(L^{6'}_x(I \times \mathbb{T}^3))} + \|e\|_{L^q_t(L^{r'}_x(I \times \mathbb{T}^3))} \right)
\leq C \left( K_1 + \rho_0 \|w_N\|_{X(I \times \mathbb{T}^3)} + \|w_N\|_{X(I \times \mathbb{T}^3)}^3 \right)
\]

We conclude (A.15) by a continuity argument for \( \rho_0 \equiv \rho_0(K_1) > 0 \) sufficiently small. \( \square \)

From this, the same proof as in Lemma yields the long-time stability argument for the truncated equation with bounds uniform in the truncation parameter.

**Lemma A.4 (Long-time stability).** Let \( I \subset \mathbb{R} \) a compact time interval and \( t_0 \in I \). Let \( v_N \) be a solution defined on \( I \times \mathbb{T}^3 \) of the Cauchy problem
\[
\begin{aligned}
(v_N)_t - \Delta v_N + v_N + S_N(S_Nv_N)^3 &= e \\
(v_N, \partial_t v_N)|_{t=t_0} &= (v_0, v_1) \in H^{1/2}(\mathbb{T}^3).
\end{aligned}
\]
Suppose that
\[
\|S_Nv_N\|_{X(I \times \mathbb{T}^3)} \leq L
\]
for some constant \( L > 0 \). Let \( (u_N(t_0), \partial_t u_N(t_0)) = (u_0, u_1) \in H^{1/2}(\mathbb{T}^3) \) be such that
\[
\|(v_0 - u_0, v_1 - u_1)\|_{H^{1/2}(\mathbb{T}^3)} \leq K_1
\]
for some \( K_1 > 0 \). Suppose also that we have the smallness conditions
\[
\|S(t - t_0)(v_0 - u_0, v_1 - u_1)\|_{X(I \times \mathbb{T}^3)} \leq \rho \]
\[
\|e\|_{L^q_t(L^{r'}_x(I \times \mathbb{T}^3))} \leq \rho,
\]
for some \( 0 < \rho < \rho_1(K_1, I, L) \) a small constant and any \((q', r')\) a conjugate admissible pair. Then there exists a unique solution \( (u(t), \partial_t u(t)) \) to the truncated cubic nonlinear Klein-Gordon equation...
on $I \times \mathbb{T}^3$ with initial data $(u_0, u_1)$ at time $t_0$ and $C \equiv C(I) \geq 1$ which satisfies

$$\|v_N - u_N\|_{X(I \times \mathbb{T}^3)} \leq C \rho$$

(A.17)

$$\|(v_N)^3 - (u_N)^3\|_{L^4/3}(I \times \mathbb{T}^3) \leq C \rho$$

(A.18)

$$\|(v_N - u_N, \partial_t u_N - \partial_t v_N)\|_{L^\infty_t H^{1/2}_x(I \times \mathbb{T}^3)} + \|v_N - u_N\|_{L^\infty_t L^4_x(I \times \mathbb{T}^3)} \leq C K_1$$

(A.19)

Furthermore, the dependence of $C$ on time arises only from the constant in the localized Strichartz estimates.

APPENDIX B. ALMOST SURE BOUNDS ON GLOBAL SOLUTIONS OF THE CUBIC NLKG

This appendix is devoted to the proof of Proposition 14. I recall the statement.

**Proposition B.1.** Let $0 < s < 1$ and let $\mu \in M^s$. Then for any $\varepsilon > 0$, there exist $C, c, \theta > 0$ such that for every $(u_0, u_1) \in \Sigma$, there exists $M > 0$ such that the family of global solution $u$ to cubic nonlinear Klein-Gordon equation (1.1) satisfies

$$u(t) = S(t)(u_0, u_1) + w(t)$$

(B.1)

$$\|(w(t), \partial_t w(t))\|_{H^s} \leq C(M + |t|)^{1+\varepsilon}$$

(B.2)

and furthermore $\mu((u_0, u_1) \in \Sigma : M > \lambda) \leq C e^{-c\lambda^\theta}$.

**Proof.** Fix $\mu \in M^s$. Following the proofs of Proposition 4.1 in [10] and Lemma 2.2 from [9], fix $\varepsilon > 0$, $\rho > \frac{1}{2}$ and $\tilde{\rho} > 1/3$ and define

$$F_N := \{(v_0, v_1) \in \Sigma : \|\Pi^N(v_0, v_1)\|_{H^s} \leq N^{1-s+\varepsilon}\}$$

$$G_N := \{(v_0, v_1) \in \Sigma : \|\Pi^N v_0\|_{L^4} \leq N^{\varepsilon}\}$$

$$H_N := \{(v_0, v_1) \in \Sigma : \|(t)^{-\rho} \Pi^N(v_0, v_1)\|_{L^2_t L^\infty_x(\mathbb{R} \times \mathbb{T}^3)} \leq N^{\varepsilon-s}\}$$

$$K_N := \{(v_0, v_1) \in \Sigma : \|(t)^{-\tilde{\rho}} \Pi^N(v_0, v_1)\|_{L^2_t L^6_x(\mathbb{R} \times \mathbb{T}^3)} \leq N^{\varepsilon-s}\}$$

$$R_M := \{(u_0, u_1) \in \Sigma : \|(t)^{-\rho} \Pi^N S(t)(u_0, u_1)\|_{L^\infty_t L^4_x(\mathbb{R} \times \mathbb{T}^3)} \leq N^{\varepsilon-s}\}.$$

We let

$$E_N = F_N \cap G_N \cap H_N \cap K_N \cap R_N.$$

The bounds on the measure of $E_N$ follow from Proposition 4.1 in [10] and Lemma 2.2 in [9]. We consider the inhomogeneous energy functional (1.7)

$$\mathcal{E}(w(t)) = \frac{1}{2} \int |\nabla w|^2 + \frac{1}{2} \int |w|^2 + \frac{1}{2} \int |w_t|^2 + \frac{1}{4} \int (w)^4.$$

Fix $(v_0, v_1) \in E_N$ and let $v_N$ denote the solution to

$$(v_N)_t + \Delta v_N + v_N + (v_N + S(t)\Pi^N(v_0, v_1))^3 = 0, \quad (v_N, \partial_t v_N)|_{t=0} = \Pi^N(v_0, v_1)$$

then

$$\mathcal{E}(v_N(t))^{1/2} \leq CN^{1-s+\varepsilon}.$$
Since $E$ controls the $H^1$ norm, we no longer need to project away from constants to obtain (B.1). To prove (B.2) note that by the definition of $R_N$, we have
\[
\|u_N(t)\|_{L^4(T^3)} \leq \|v_N(t)\|_{L^4(T^3)} + \|S(t)\Pi^N(v_0,v_1)\|_{L^4(T^3)} \\
\leq E(v_N(t))^{1/4} + CN^{-s+\varepsilon} \\
\leq CN^{1-\frac{1}{4s}+\varepsilon}.
\]
The conclusion then follows as in Proposition 4.1 in [10].
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