On the gravitational instability of a dissipative medium.

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This paper shows that the ordinary Jeans wave number can be obtained as a limiting case of a more general approach that includes dissipative effects. Corrections to the Jeans critical mass associated to viscosity are established. Some possible implications of the results are finally discussed.

I. INTRODUCTION

Structure formation in fluid systems is closely related to the existence of growing modes while considering density fluctuations in the hydrodynamical regime. Relevant growing modes at very small wave numbers have long ago been identified in gravitational systems, leading to key concepts in present astrophysics such as the Jeans wave number and the Jeans mass \(^{[1]}\). These parameters correspond to the conditions that a density fluctuation must satisfy in order to grow up to the point of forming a structure.

Entropy production plays an important role in realistic descriptions of thermodynamical processes, and some discussion of its physical sources may yield interesting results in the analysis of the time evolution of statistical fluctuations in astrophysical systems. This work is mainly devoted to the study of the dispersion relation that arises from the fluctuation analysis of the thermodynamical variables of a simple dissipative fluid under the influence of a gravitational field. The analysis is reviewed in the case of non-vanishing viscosity and different features of the fluid are established. In particular, this work shows the possibility of viscosity-affected growing modes for some astrophysical fluids, which may be of interest for both astrophysicists and cosmologists.

This paper is divided as follows: section two is dedicated to the hydrodynamical model analysis and the establishment of the dispersion relation in which the main results are based. Section three is devoted to the mathematical treatment needed for its solution. The Jeans instability is recovered in the presence of viscosity for typical situations present in protogalaxies. Finally, a discussion of the results obtained and an outline of future work are included in section four of the paper.

II. HYDRODYNAMIC MODEL

In order to establish the basic equations that lie at the core of this work, we remind the reader that the so-called Navier-Stokes-Fourier equations of hydrodynamics for a simple fluid arise from the structure of what is now called Linear Irreversible Thermodynamics (LIT) by supplementing the two conservation equations for mass and momentum, respectively and the balance equation for the internal energy, with additional information. Indeed, these equations read \(^{[3]}\):

\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x^i}(\rho u^i) = 0 \tag{1}
\]

\[
\rho \frac{D u^i}{Dt} + \frac{\partial \Xi^{ij}}{\partial x^j} = F^i \tag{2}
\]

\[
\rho \frac{D \varepsilon}{Dt} + \frac{\partial Q^j}{\partial x^j} = -\Xi^{ij} \frac{\partial u_i}{\partial x^j} \tag{3}
\]
where \( \frac{\partial}{\partial t} = \frac{\partial}{\partial t} + u_i \frac{\partial}{\partial x_i} \).

Here, \( \rho(x^j, t), \) \( u^i(x^j, t) \) and \( \varepsilon(x^j, t) \) are the local density, velocity and internal energy, respectively, \( \Xi^{i j} \) the momentum current (or stress tensor) and \( Q^j \) the heat flux. All indices run from 1 to 3. In general, for isotropic fluids, \( \Xi^{i j} = p \delta^{i j} + \tau^{i j} \), where \( p \) is the local hydrostatic pressure and \( \tau^{i j} \) the viscous tensor \((\delta^{ij} \) is the unitary dyadic).

Notice that Eqs. \([4]-[7]\) contain fifteen unknowns (including the pressure) and there are only five equations, so the system is not well determined. If we arbitrarily choose to describe the states of the fluid through the set of variables \( \rho(x^j, t), u^i(x^j, t), T(x^j, t), \) where \( T \) is the local temperature, we need nine dynamic equations of state (or constitutive equations) relating \( \Xi^{i j} \) and \( Q^j \) to the state variables plus two local equations of state \( p = p(\rho, T) \) and \( \varepsilon = \varepsilon(\rho, T) \). If according to the tenets of LIT we choose the so-called linear constitutive laws, namely

\[
\tau^{i j} = -2\eta \sigma^{i j} - \zeta \delta^{i j}
\]

(4)

\[
Q^j = -\kappa \delta^{i j} \frac{\partial T}{\partial x^i}
\]

(5)

with \( \eta \) and \( \zeta \), the shear and bulk viscosities, respectively, \( \sigma^{i j} \) the symmetrical traceless part of the velocity gradient, \( \theta = \frac{\partial u_0}{\partial t} \), and \( \kappa \) being the thermal conductivity.

Eqs. \([4]-[7]\) are the well known constitutive equations of Navier-Stokes and Fourier, respectively. Substitution of these equations into Eqs. \([4]-[7]\) yields a set of second order in space, first order in time non-linear coupled set of partial differential equations for the chosen variables \( \rho, u^i \) and \( T \). This set, which the reader may seek in the literature \([4]-[7]\), is the so called Navier-Stokes-Fourier system of hydrodynamic equations. The non-linearities appearing in such equations have two sources, the inertial terms \( u^i \frac{\partial}{\partial x^i} \) arising from the hydrodynamic time derivatives, plus quadratic terms in the gradients of velocity arising from Eqs. \([4]-[7]\). Moreover, it should be mentioned that this set of equations is consistent with the second law of thermodynamics, the Clausius uncompensated heat, or entropy production, is strictly positive definite.

Nevertheless, for the purpose of this paper, this set of equations is too complicated. In order to deal with fluctuations around the equilibrium state, one assures that for any of two state variables, call them \( X(x^j, t) \) one can write that,

\[
X(x^j, t) = X_o + \delta X(x^j, t)
\]

(6)

where \( X_o \) is the equilibrium value of \( X \) and \( \delta X \) the corresponding fluctuation. Neglecting all terms of order \((\delta X(x^j, t))^2\) and higher in the NSF non-linear set one finds the linearized NSF equations of hydrodynamics \([4]-[7]\),

\[
\frac{\partial}{\partial t} (\delta \rho) + \rho \theta = 0
\]

(7)

\[
\rho \frac{\partial u_k}{\partial t} = - \frac{1}{\rho_o \kappa_T} \frac{\partial}{\partial x^k} (\delta \rho) - \frac{\beta}{\kappa_T} \frac{\partial}{\partial x^k} (\delta T) + 2\eta \delta^{ij} \frac{\partial u_k}{\partial x^j} - (\frac{2}{3} \eta - \zeta) \frac{\partial}{\partial x^k}(\theta) + F_k
\]

(8)

\[
\frac{\partial}{\partial t} (\delta T) = D_T \delta^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} \delta T - \frac{\beta T_o}{\rho_o C_p \kappa_T} \theta
\]

(9)

since \( u_{k o} = 0, u_k = \delta u_k \) and \( \delta = (\frac{\partial}{\partial x_o}) \rho_o, \kappa_T = \rho_o (\frac{\partial p_o}{\partial \rho_o}) T_o \) and \( D_T = \frac{k}{\rho_o C_v} \) is the thermal diffusivity. \( C_p \) and \( C_v \) are the specific heats at constant pressure and constant volume, respectively.

Note now that from Eq. \([8]\), \( u_k \) uncouples from the hydrodynamic modes, whence, one formally arrives at the desired set, namely:

\[
\frac{\partial}{\partial t} (\delta \rho) + \rho \theta = 0
\]

(10)

\[
\rho \frac{\partial \theta}{\partial t} = - \frac{1}{\rho_o \kappa_T} \delta^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} (\delta \rho) - \frac{\beta}{\kappa_T} \delta^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} (\delta T) + D_o \delta^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} (\theta) - \delta^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} (\zeta) - \frac{\beta T_o}{\rho_o C_p \kappa_T} \theta
\]

(11)

\[
\frac{\partial}{\partial t} (\delta T) = D_T \delta^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} (\delta T) - \frac{\beta T_o}{\rho_o C_p \kappa_T} \theta
\]

(12)
assuming that $F_k = -\frac{\partial \delta \phi}{\partial s}$. Also, $D_v = (\frac{\delta \rho + \gamma}{\rho})$. Now, we notice that $\frac{\partial}{\partial t} (\delta \rho) = -\rho_o \theta$ so we can reduce this set to only two equations, Eq. (11) and Eq. (12). We now perform a few minor transformations introducing the speed of sound of the fluid $C_s^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$ through the relationship

$$k_T = \frac{\gamma k_s}{C_v k_s}$$

where $k_s$ is the adiabatic compressibility $\left(\frac{\partial p}{\partial \rho}\right)_s$ whence, $k_T = \frac{c_s^2}{k_s}$. This leaves us finally with the set:

$$-\frac{\partial^2 (\delta \rho)}{\partial t^2} + \frac{C_s^2}{\gamma} \delta^{ij} \frac{\partial (\delta \rho)}{\partial x^i \partial x^j} + \frac{C_o^2 \beta}{\gamma} \delta^{ij} \frac{\partial (\delta T)}{\partial x^i \partial x^j} + D_v \delta^{ij} \frac{\partial}{\partial x^i} \left(\frac{\partial (\delta \rho)}{\partial x^j}\right) + \rho_o \frac{\partial (\varphi)}{\partial x^i \partial x^j} = 0$$

(14)

$$\frac{\partial}{\partial t} (\delta T) = D_T \delta^{ij} \frac{\partial (\delta T)}{\partial x^i \partial x^j} - \frac{\gamma - 1}{\beta} \frac{\partial (\delta \rho)}{\partial t} = 0$$

(15)

where we have used that $C_p - C_v = \frac{\rho_s T_0}{\delta x^2}$.

Eqs. (14,15) form a set of coupled equations for the density and temperature fluctuations in the fluid under the action of a conservative force whose nature need not to be specified for the time being. They are the basis for studying the properties of the time correlation functions of thermodynamic fluctuations. Those of the density will be of particular interest here.

### III. SOLUTION TO THE HYDRODYNAMIC EQUATIONS

The results derived in the previous section are far from being new. Aside from the term $\nabla^2 \delta \phi$ which arises from the presence of an external conservative force, they are identical to the ones that have been widely discussed in the literature. The question here is if the gravitational potential introduces any substantial modification in the correlation functions for the thermodynamic fluctuations. To examine this possibility we recall that, if we are considering only the fluctuations, the gravitational potential satisfies the Poisson equation:

$$\delta^{ij} \frac{\partial (\varphi)}{\partial x^i \partial x^j} = -4\pi G \delta \rho$$

(16)

Now, equation (15) reads

$$-\frac{\partial^2 (\delta \rho)}{\partial t^2} + \frac{C_s^2}{\gamma} \delta^{ij} \frac{\partial (\delta \rho)}{\partial x^i \partial x^j} + \frac{C_o^2 \beta}{\gamma} \delta^{ij} \frac{\partial (\delta T)}{\partial x^i \partial x^j} + D_v \delta^{ij} \frac{\partial}{\partial x^i} \left(\frac{\partial (\delta \rho)}{\partial x^j}\right) - 4\pi G \rho_o \delta \rho = 0$$

(17)

The solution to Eqs. (15) and (17) proceeds in the standard fashion. We reduce them to a set of algebraic equations by taking their Laplace-Fourier transform, choose to set the static temperature fluctuations equal to zero, eliminate the temperature leading to an equation for $\delta \rho (\vec{k}, s)$, which is the ratio of two polynomials in $s$ . To compute $\delta \rho (\vec{k}, t)$ one must take the inverse Laplace transform of the former quantity, which demands the knowledge of the roots of the denominator, which is a cubic equation in $s^3$ (dispersion equation). Next, we proceed with intuition, we assume that the Rayleigh peak is not substantially modified by the external potential, so that according to the standard theory, one of the roots of the cubic equation is $s = -D_T k^2$, where $D_T$ is the thermal diffusivity. This allows us to simplify the denominator to the form,

$$s + D_T k^2 (s^2 + D_v k^2 s + C_o^2 k^2 - 4\pi G \rho_o) = 0$$

(18)

the two roots of the quadratic equation are:

$$s_{1,2} = -\frac{D_v k^2}{2} \pm i \left[\frac{(C_o^2 k^2 - 4\pi G \rho_o) - D^2 v k^4}{4}\right]^{1/2}$$

(19)

If $4\pi G \rho_o = 0$ and viscosity dominates over the term in $k^2$, this result reduces to the one giving rise to the standard Brillouin peaks which correspond to density fluctuations of the type $s \pm 2 k$. 


\[ \delta \rho (\vec{k}, t) = \delta \rho (\vec{k}, 0) \frac{1}{\gamma} e^{-D_v k^2 t} \cos \left( C_o k t \right) \]  

(20)

which are the acoustic modes damped by the Stokes-Kirchhoff factor \( D_v \).

On the other hand, if \( 4\pi G \rho_o \neq 0 \), the threshold value for \( k \) distinguishing between damped oscillations and growing modes is given by

\[ (C_o^2 k^2 - 4\pi G \rho_o) - \frac{D_v^2 k^4}{4} = 0 \]  

(21)

or,

\[ k^2 = \frac{2C_o^2}{D_v^2} \left( 1 \pm \sqrt{1 - \frac{4\pi G \rho_o D_v^2}{C_o^4}} \right) \]  

(22)

Eq. (22) is a generalization of Jeans wave number when dissipative effects due to viscosity are non-negligible, and is the main result of the paper. We can note that, if \( \frac{4\pi G \rho_o D_v^2}{C_o^4} \ll 1 \), and taking the – sign for the square root, we have:

\[ k^2 \approx \frac{2C_o^2}{D_v^2} \left[ 1 - 1 + \frac{1}{2} \left( \frac{4\pi G \rho_o D_v^2}{C_o^4} \right) \right] \]  

(23)

or

\[ k^2 \approx \frac{4\pi G \rho_o}{C_o^2} = K_J^2 \]  

(24)

This is the square value of the Jeans wave number \[8\]. It was derived by Jeans in 1902 and rederived in many other waves by several authors \[8\]. Here we simply show that it is almost a trivial consequence of LIT.

Let us expand the square root in Eq. (22) up to second order in order to have a better grasp of the effects of viscosity on the Jeans wave number. Now we have that,

\[ k^2 \approx \frac{2C_o^2}{D_v^2} \left[ 1 - \frac{2\pi G \rho_o D_v^2}{C_o^4} \right] - \frac{1}{8} \left( \frac{4\pi G \rho_o D_v^2}{C_o^4} \right)^2 \]  

(25)

or, using Eq. (24):

\[ k^2 \approx K_J^2 + \frac{4D_v^2}{C_o^4} (\pi G \rho_o)^2 = K_J^2 (1 + 2\frac{D_v^2}{C_o^4} \pi G \rho_o) \]  

(26)

so that the dimensionless number

\[ \varepsilon = \frac{2D_v^2}{C_o^4} \pi G \rho_o = \frac{2 (\frac{4}{3} \eta + \varsigma)^2 \pi G}{\rho_o C_o^4} \]  

(27)

is a good measure of the second order corrections to the Jeans wave number due to viscous effects.

For the corrected Jeans mass we have \[8\]:

\[ M_c = \frac{4}{3} \pi \left( \frac{\pi}{k} \right)^3 = \frac{4}{3} \pi \left( \frac{\pi}{K_J (1 + \frac{2D_v^2}{C_o^4} \pi G \rho_o)} \right)^3 \rho_o = M_J (1 + \frac{2D_v^2}{C_o^4} \pi G \rho_o)^{-\frac{3}{2}} \]  

(28)

or

\[ M_c = M_J (1 + \varepsilon)^{-\frac{3}{2}} \]  

(29)

where the standard Jeans mass \( M_J \) is given by \[8\]:

4
\[ M_J = \frac{4}{3} \pi \left( \frac{\pi}{K_J} \right)^3 \rho_o = \frac{4}{3} \pi \left( \frac{\pi}{4 \pi G \rho_o} C_o \right)^3 \rho_o = \frac{(\pi)^{\frac{1}{2}}}{6 \left((\rho_o)^{\frac{1}{2}} (G)^{\frac{1}{2}} \right)} C_o^3 \]  

(30)

One should compare these results with Weinberg \[2\] and Peebles p. 108 \[3\]. The question here is if in the primeval Universe, the plasma had a substantial viscosity.

Eq. (30) deserves more attention. As shown by Weinberg \[2\] a similar result follows from a more intuitive calculation addressing the behavior of the dispersion relation in sound waves. Using numerical values for the density of barions (protons) and the velocity of sound in a background of photons and protons, he is able to estimate the temperature range for which the protogalactic mass can grow to \(10^{11}\) solar masses. Similar reasoning may be applied to Eq. (29) and, of course, predict a new \(M_J\). The question here is if the viscous corrections have any real meaning at all or they just turn out to be an academic curiosity. For instance, in accretion disks \[9\] \(\rho_o \approx 4 \times 10^{-4} \text{ kg m}^{-3}\), \(\eta\) can be estimated as high as \(10^{10} \text{ Pa s}\) and \(C_o \approx 5 \times 10^{14} \text{ m s}^{-1}\). This would yield a value of \(D_o \approx 10^{14} \text{ m}^2 \text{ s}^{-1}\), so that

\[ \varepsilon = \frac{2D_o^2}{C_o^4} \pi G \rho_o \approx 10^{-1} \]

the estimated correction is at the order of 10%.

### IV. FINAL REMARKS

Dissipation is always present in real processes. Whenever the thermodynamical functions of a self-gravitating viscous system fluctuate around an equilibrium value, dissipation will affect the general equilibrium condition for the stability of those fluctuations. We have been able to establish a simple expression, Eq. (27) that measures how important this correction will be, assuming that the Rayleigh peak remains essentially unaffected by the gravitational field. If this were not the case, then a full treatment of the dispersion relation would be needed, involving interesting effects of the gravitational field in the Rayleigh-Brillouin spectrum, for the case of damped systems. This modifications may have some importance in astrophysics, since natural astrophysical masers are already well-known \[10\] and its scattering in fairly dense systems will provide information about the transport properties of the scatterer.

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