Stein approximation for multidimensional Poisson random measures by third cumulant expansions

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Abstract

We obtain Stein approximation bounds for stochastic integrals with respect to a Poisson random measure over $\mathbb{R}^d$, $d \geq 2$. This approach relies on third cumulant Edgeworth-type expansions based on derivation operators defined by the Malliavin calculus for Poisson random measures. The use of third cumulants can exhibit faster convergence rates than the standard Berry-Esseen rate for some sequences of Poisson stochastic integrals.

Key words: Stein approximation; multidimensional Poisson random measures; Poisson stochastic integrals; cumulants; Malliavin calculus; Edgeworth expansions.

Mathematics Subject Classification: 62E17; 60H07; 60H05.

1 Introduction

Stein approximation bounds for stochastic integrals with respect to a Poisson random measure have been obtained in [12] using finite difference operators on the Poisson space. In this paper we derive related bounds for compensated Poisson stochastic integrals $\delta(u) := \int_{\mathbb{R}^d} u_x(\gamma(dx) - \lambda(dx))$ of compactly supported processes $(u_x)_{x \in \mathbb{R}^d}$ with

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respect to a Poisson random measure $\gamma(dx)$ with intensity the Lebesgue measure $\lambda(dx)$ on $\mathbb{R}^d$, $d \geq 2$. In contrast with [12], our approach is based on derivation operators and Edgeworth type expansions that involve the third cumulant of Poisson stochastic integrals, and can result into faster convergence rates, see e.g. (1.4) below.

Edgeworth type expansions have been obtained on the Wiener space in [9], [5], by a construction of cumulant operators based on the inverse $L^{-1}$ of the Ornstein-Uhlenbeck operator, extending the results of [10] on Stein approximation and Berry-Esseen bounds.

In Proposition 4.1 we derive Edgeworth type expansions of the form

$$
E[\delta(u)g(\delta(u))] = E[\|u\|_{L^2(\mathbb{R}^d)}^2 g'(\delta(u))] + \sum_{k=2}^{n} E\left[g^{(k)}(\delta(u))\Gamma^u_{k+1}1\right] + E\left[\left|g^{(n+1)}(\delta(u))R^u_n\right|\right] 
$$

(1.1)

when the random field $(u_x)_{x \in \mathbb{R}^d}$ is predictable with respect to a given total order on $\mathbb{R}^d$, where $\Gamma^u_k$ is a cumulant type operator and $R^u_n$ is a remainder term, defined using the derivation operators of the Malliavin calculus on the Poisson space.

Based on (1.1), in Corollary 5.2 we deduce Stein approximation bounds of the form

$$
d(\delta(u), \mathcal{N}) \leq |1 - \text{Var}[\delta(u)]| + \sqrt{\text{Var}[\|u\|_{L^2(\mathbb{R}^d)}^2]} \]
+ E\left[\left|\int_{\mathbb{R}^d} u_x^2 \lambda(dx) + \left\langle u, D \int_{\mathbb{R}^d} u_x^2 \lambda(dx) \right\rangle_{L^2(\mathbb{R}^d)} \right| \right] + E\left[|R^u_n|\right],
$$

where $D$ is a gradient operator acting on Poisson functionals, and $\mathcal{N} \simeq \mathcal{N}(0,1)$ is a standard Gaussian random variable, see also Proposition 5.1. Here,

$$
d(F, G) := \sup_{h \in \mathcal{L}} |E[h(F)] - E[h(G)]|
$$

is the Wasserstein distance between the laws of two random variables $F$ and $G$, where $\mathcal{L}$ denotes the class of 1-Lipschitz functions on $\mathbb{R}$.

In particular, when $f$ is a differentiable deterministic function with support in the
closed centered ball $B(R) := B(0; R)$ with radius $R > 0$ we obtain bounds of the form

$$d \left( \int_{\mathbb{R}^d} f(x)(\gamma(dx) - \lambda(dx)), \mathcal{N} \right) \leq \left| 1 - \|f\|_{L^2(\mathbb{R}^d)}^2 \right| + \left| \int_{\mathbb{R}^d} f^3(x) \lambda(dx) \right|$$

$$+ 8(K_d v_d R)^2 \|f\|_{L^2(\mathbb{R}^d)} \|\nabla^{\mathbb{R}^d} f\|_{L^\infty(\mathbb{R}^d, \mathbb{R}^d)},$$

where $v_d$ denotes the volume of the unit ball in $\mathbb{R}^d$ and $K_d > 0$ is a constant depending only on $d \geq 2$. The bound (1.2) can be compared to the classical Stein bound

$$d \left( \int_{\mathbb{R}^d} f(x)(\gamma(dx) - \lambda(dx)), \mathcal{N} \right) \leq \left| 1 - \|f\|_{L^2(\mathbb{R}^d)}^2 \right| + \int_{\mathbb{R}^d} |f^3(x)| \lambda(dx),$$

for compensated Poisson stochastic integrals, see Corollary 3.4 of [12], which involves the $L^3(\mathbb{R}^d)$ norm of $f$ instead of third cumulant $\kappa_f^3 = \int_{\mathbb{R}^d} f^3(x) \lambda(dx)$ of $\int_{\mathbb{R}^d} f(x)(\gamma(dx) - \lambda(dx))$, and relies on the use of finite difference operators, see Theorem 3.1 of [12] and § 4.2 of [4].

For example when $f_k, k \geq 1$, is a radial function given on $B(Rk^{1/d})$ by

$$f_k(x) := \frac{1}{C_k \sqrt{k}} g \left( \frac{|x|_{\mathbb{R}^d}}{k^{1/d}} \right),$$

where $g \in C^1_0([0, R])$ is continuously differentiable on $[0, R]$, and

$$C^2 := \int_0^R g^2(r)r^{d-1}dr < \infty,$$

so that $\|f_k\|_{L^2(B(Rk^{1/d}))} = 1$, the bound (1.3) yields the standard Berry-Esseen convergence rate

$$d \left( \int_{B(Rk^{1/d})} f_k(x)(\gamma(dx) - \lambda(dx)), \mathcal{N} \right) \leq \frac{v_d}{C^3 \sqrt{k}} \int_0^R |g(r)|^2 r^{d-1}dr, \quad k \geq 1.$$

While (1.2) does not improve on (1.3) when the function $f$ has constant sign, if $g$ satisfies the condition

$$\int_0^R g^3(r)r^{d-1}dr = 0,$$

then the third cumulant bound (1.2) yields the $O(1/k)$ convergence rate

$$d \left( \int_{B(Rk^{1/d})} f_k(x)(\gamma(dx) - \lambda(dx)), \mathcal{N} \right) \leq \frac{2(2K_d v_d R)^2}{kC^2} \|f'\|_\infty^2,$$

(1.4)
which improves on the standard Berry-Esseen rate, see Section 5.

In Sections 2 and 3 we recall some background material on the Malliavin calculus and differential geometry on the Poisson space, by revisiting the approach of [13], [14] using the recent constructions of [1] and references therein on the solution of the divergence problem. In Section 4 we derive Edgeworth type expansions for the compensated Poisson stochastic integral \( \delta(u) \), based on a family of cumulant operators that are associated to the random field \((u_x)_{x \in \mathbb{R}^d}\). In Section 5 we derive Stein type approximation bounds for stochastic integrals, with deterministic examples.

While this paper is dealing with Poisson random measures on \( \mathbb{R}^d \) with \( d \geq 2 \), the special case \( d = 1 \) requires a different treatment for the standard Poisson process on the real half line, see [15], and the \( d \)-dimensional setting of the present paper shows significant differences with the one-dimensional case.

**Preliminaries**

Let \( d \geq 2 \) and \( 0 < R < R' := 2R \). We recall the existence of a \( C^\infty \) kernel function \( G_\eta : B(R') \times B(R') \rightarrow \mathbb{R}^d \) defined as

\[
G_\eta(x, y) := \int_0^1 \frac{(x - y)}{s} \eta \left( y + \frac{x - y}{s} \right) ds, \quad x, y \in B(R'),
\]

where \( \eta \in C^\infty_0(B(R')) \) is such that \( \int_{B(R)} \eta(x) dx = 1 \), see [1], and satisfying the following properties:

i) The kernel \( G_\eta(x, y) \) satisfies the bound

\[
|G_\eta(x, y)|_{\mathbb{R}^d} \leq \frac{K_d}{|x - y|_\mathbb{R}^{d-1}}, \quad x, y \in B(R'),
\]

for a constant \( K_d > 0 \) depending only on \( d \), see Lemma 2.1 of [1], by choosing \( K_d \) and the function \( \eta \in C^\infty_c(B(R')) \) therein so that \( \|\eta\|_\infty \leq (d - 1)K_d(R')^{-d} \).

ii) For any \( p > 1 \) and \( g \in L^p(B(R')) \) the function

\[
f(x) := \int_{B(R')} G_\eta(x, y)g(y) \lambda(dy), \quad x \in B(R'),
\]
satisfies the bound
\[ \|f\|_{L^p(B(R');\mathbb{R}^d)} \leq K_d v_d R' \|g\|_{L^p(B(R'))}, \quad p > 1, \quad (1.6) \]
which follows from Young’s inequality and (1.5), cf. Theorem 2.4 in [1].

iii) For any \( h \in C^\infty_0(B(R')) \) we have the relation
\[ h(y) - \int_{B(R') \setminus B(R)} h(x) \eta(x) \lambda(dx) = \int_{B(R')} \langle G_\eta(x, y), \nabla^d_x h(x) \rangle \mathbb{R}^d \lambda(dx), \quad y \in B(R'), \quad (1.7) \]
\( \text{cf. Lemma 2.2 in [1], by taking } \eta \in C^\infty_c(B(R') \setminus B(R)). \) In particular, when \( h \in C^\infty_0(B(R)) \) we have
\[ h(y) = \int_{B(R')} \langle G_\eta(x, y), \nabla^d_x h(x) \rangle \mathbb{R}^d \lambda(dx), \quad y \in B(R'). \quad (1.8) \]

An extension of the framework of this paper by replacing \( B(R) \) with a compact \( d \)-dimensional Riemannian manifold \( M \) and \( \lambda(dx) \) with the volume element of \( M \) requires the Laplacian \( \mathcal{L} = \text{div}^M \nabla^M \) to be invertible on \( C^\infty_c(M) \), with
\[ \mathcal{L}^{-1} u(x) = \int_M g(x, y) u(y) \lambda(dy), \quad x \in M, \quad u \in C^\infty_c(M), \]
where \( g(x, y) \) is the heat kernel on \( M \). In this case we can define \( G_\eta(x, y) \in \mathbb{R}^d \) as
\[ G_\eta(x, y) = \nabla^M_x g(x, y), \quad \lambda \otimes \lambda(dx, dy) - \text{a.e.} \]
with the relation
\[ \nabla^M_x \mathcal{L}^{-1} u(x) = \int_M u(y) G_\eta(x, y) \lambda(dy) \in T_x M, \quad x \in M, \quad u \in C^\infty_c(M), \]
from which the divergence inversion relation (1.8) holds by duality.

2 Gradient, divergence and covariance derivative

There exists different notions of gradient and divergence operators for functionals of Poisson random measures. The operators of [2], [16], [7], and their associated integration by parts formula rely on an \( \mathbb{R}^d \)-valued gradient for random functionals and
a divergence operator which is associated to the non-compensated Poisson stochastic integral of the divergence of $\mathbb{R}^d$-valued random fields. This particularity, together with a lack of a suitable commutation relation between gradient and divergence operators on Poisson functionals, makes this framework difficult to use for a direct analysis of Poisson stochastic integrals, while it has found applications to statistical estimation and sensitivity analysis [7], [16].

In this paper we use the construction of [13], [14] which relies on real-valued tangent processes and of a divergence operator that directly extends the compensated Poisson stochastic integral. This framework also allows for simple commutation relations between gradient and divergence operators using the deterministic inner product in $L^2(\mathbb{R}^d, \lambda)$, see Proposition 2.6, and it naturally involves the Poisson cumulants, see Definition 3.2 and Relation (3.6).

**Gradient operator**

In the sequel we consider a Poisson random measure $\gamma(dx)$ on a probability space $(\Omega, \mathcal{F}, P)$ and we let $\{X_1, \ldots, X_n\}$ denote the configuration points of $\gamma(dx)$ when $B(R)$ contains $n$ points in the configuration $\gamma$, i.e. when $\gamma(B(R)) = n$.

**Definition 2.1** Given $A$ a closed subset of $B(R')$, we let $\mathcal{S}_A$ denote the set of random functionals $F_A$ of the form

\[
F_A = \sum_{n=0}^{\infty} \mathbf{1}_{\{\gamma(B(R))=n\}} f_n (X_1, \ldots, X_n),
\]

where $f_0 \in \mathbb{R}$ and $(f_n)_{n \geq 1}$ is a sequence of functions satisfying the following conditions:

- for all $n \geq 1$, $f_n \in C^\infty_c (A^n)$ is a symmetric function in $n$ variables,

- for all $n \geq 1$ and $i = 1, \ldots, n$ we have the continuity condition

\[
f_n (x_1, \ldots, x_n) = f_{n-1} (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n),
\]

for all $x_1, \ldots, x_n \in B(R')$ such that $|x_i|_{\mathbb{R}^d} \geq R$.

We also let $\mathcal{S}$ denote the union of the sets $\mathcal{S}_A$ over the closed subsets $A$ of $B(R')$. 
The gradient operator $D$ is defined on random functionals $F \in \mathcal{S}$ of the form (2.1) as
\[D_y F := \sum_{n=1}^{\infty} \mathbf{1}_{\{\gamma(B(R))=n\}} \sum_{i=1}^{n} \langle G_{\eta}(X_i, y), \nabla_{x_i} f(X_1, \ldots, X_n) \rangle_{\mathbb{R}^d}, \quad (2.3)\]
y $\in B(R)$. For any $F \in \mathcal{S}$, by (1.5) we have $DF \in L^1(\Omega \times B(R))$ from the bound
\[
E \left[ \int_{B(R)} |D_x F| \lambda(dx) \right] \leq \|\nabla_{\mathbb{R}^d} f\|_{\mathbb{R}^d} \infty E \left[ \int_{B(R)} \int_{B(R)} |G_{\eta}(x, y)| \gamma(dx) \lambda(dy) \right] = \|\nabla_{\mathbb{R}^d} f\|_{\mathbb{R}^d} \infty \int_{B(R)} \int_{B(R)} \frac{1}{|x - y|^{d-1}} \lambda(dx) \lambda(dy) \\
= K_d v_d^2 R^d \|\nabla_{\mathbb{R}^d} f\|_{\mathbb{R}^d} \infty < \infty.
\]

**Poisson-Skorohod integral**

We let $\mathcal{U}_0$ denote the space of simple random fields of the form
\[u = \sum_{i=1}^{n} g_i G_i, \quad n \geq 1, \quad (2.4)\]
with $G_i \in \mathcal{S}_{A_i}$ and $g_i \in C^\infty_0(B(R))$, $i = 1, \ldots, n$.

**Definition 2.2** We define the Poisson-Skorohod integral $\delta(u)$ of $u \in \mathcal{U}_0$ of the form (2.4) as
\[
\delta(u) := \sum_{i=1}^{n} \left( G_i \int_{B(R)} g_i(x)(\gamma(dx) - \lambda(dx)) - \langle g_i, DG_i \rangle_{L^2(B(R))} \right). \quad (2.5)
\]

In particular, for $h \in C^\infty_0(B(R))$ we have
\[
\delta(h) = \int_{B(R)} h(x)(\gamma(dx) - \lambda(dx)).
\]

The proof of the next proposition, cf. Proposition 8.5.1 in [13] and Proposition 5.1 in [14], is given in the appendix.

**Proposition 2.3** The operators $D$ and $\delta$ satisfy the duality relation
\[
E[\langle u, DF \rangle_{L^2(B(R))}] = E[F \delta(u)], \quad F \in \mathcal{S}, \quad u \in \mathcal{U}_0. \quad (2.6)
\]
As a consequence of Proposition 2.3 and the denseness of $S$ in $L^1(\Omega)$ and that of $U_0$ in $L^1(\Omega \times B(R))$, the gradient operator $D$ is closable in the sense that if $(F_n)_{n \in \mathbb{N}} \subset S$ tends to zero in $L^2(\Omega)$ and $(DF_n)_{n \in \mathbb{N}}$ converges to $U$ in $L^1(\Omega \times B(R))$, then $U = 0$ a.e.. Similarly, the divergence operator $\delta$ is closable in the sense that if $(u_n)_{n \in \mathbb{N}} \subset U_0$ tends to zero in $L^2(\Omega \times B(R))$ and $(\delta(u_n))_{n \in \mathbb{N}}$ converges to $G$ in $L^1(\Omega)$, then $G = 0$ a.s..

The gradient operator $D$ defines the Sobolev space $\mathbb{D}^{1,1}$ with the Sobolev norm

$$
\|F\|_{\mathbb{D}^{1,1}} := \|F\|_{L^2(\Omega)} + \|DF\|_{L^1(\Omega \times B(R))}, \quad F \in S.
$$

In the sequel we fix a total order $\preceq$ on $B(R)$ and consider the space $\mathcal{P}_0 \subset U_0$ of simple predictable random field of the form

$$
u := \sum_{i=1}^{n} g_i F_i,
$$

such that the supports of $g_1, \ldots, g_n$ satisfy

$$
\text{Supp } (g_i) \preceq \cdots \preceq \text{Supp } (g_n) \quad \text{and} \quad F_i \in S_{A_i},
$$

where $\text{Supp } (g_1) \cup \cdots \cup \text{Supp } (g_{i-1}) \subset A_i \subset B(R')$ and $A_i \preceq \text{Supp } (g_i)$, $i = 1, \ldots, n$.

Such random fields are predictable in the sense of e.g. § 5 of [8] and references therein.

We will also assume that the order $\preceq$ is compatible with the kernel $G_\eta$ in the sense that

$$
G_\eta(x, y) = 0 \quad \text{for all} \quad x, y \in B(R) \quad \text{such that} \quad x \preceq y.
$$

Under the compatibility condition (2.8) we have in particular

$$
D_y F = 0, \quad y \in B(R), \quad A \preceq y, \quad F \in S_A.
$$

Moreover, if $u \in \mathcal{P}_0$ is a predictable random field of the form (2.7) we note that by (2.3) and the compatibility condition (2.8) we have

$$
D_y F_i = 0, \quad A_i \preceq y, \quad i = 1, \ldots, n,
$$
hence
\[ D_y u_x = 0, \quad x \preceq y, \quad x, y \in B(R). \] (2.9)

**Example.** The order \( \preceq \) defined by
\[ x = (x^{(1)}, \ldots, x^{(d)}) \preceq y = (y^{(1)}, \ldots, y^{(d)}) \iff x^{(1)} \leq y^{(1)} \] (2.10)
is compatible with the kernel \( G_\eta \) provided that the support of \( \eta \) is contained in
\[ \{ x = (x^{(1)}, \ldots, x^{(d)}) \in B(R') \setminus B(R) : x^{(1)} > R \}. \]

The proof of the next Proposition 2.4 is given in the appendix.

**Proposition 2.4** The Poisson-Skorohod integral of \( u = (u_x)_{x \in B(R)} \) in the space \( \mathcal{P}_0 \) of simple predictable random fields satisfies the relation
\[ \delta(u) = \int_{B(R)} u_x (\gamma(dx) - \lambda(dx)), \] (2.11)
which extends to the closure of \( \mathcal{P}_0 \) in \( L^2(\Omega \times B(R)) \) by density and the isometry relation
\[ E[\delta(u)^2] = E \left[ \int_{B(R)} u_x^2 \lambda(dx) \right], \quad u \in \mathcal{P}_0. \] (2.12)

**Covariant derivative**

In addition to the gradient operator \( D \), we will also need the following notion of covariant derivative operator \( \tilde{\nabla} \) defined on stochastic processes that are viewed as tangent processes on the Poisson space \( \Omega \), see [14].

**Definition 2.5** Let the operator \( \tilde{\nabla} \) be defined on \( u \in \mathcal{P}_0 \) as
\[ \tilde{\nabla}_y u_x := D_y u_x + \langle G_\eta(x, y), \nabla_x^R u_x \rangle_{\mathbb{R}^d}, \quad x, y \in B(R). \]

We note that from the compatibility condition (2.8) and Relation (2.9) we also have
\[ \tilde{\nabla}_y u_x = 0, \quad x \preceq y, \quad x, y \in B(R). \] (2.13)

From the bound
\[ E \left[ \int_{B(R) \times B(R)} \left| \tilde{\nabla}_x u_y \right| \lambda(dx) \lambda(dy) \right] \]
\[ \leq \|Du\|_{L^1(\Omega \times B(R) \times B(R))} + E \left[ \int_{B(R) \times B(R)} |\langle G_0(x, y), \nabla^d_x u_x \rangle_{\mathbb{R}^d}| \lambda(dx) \lambda(dy) \right] \]

\[ \leq \|Du\|_{L^1(\Omega \times B(R) \times B(R))} + K_d E \left[ \int_{B(R) \times B(R)} \frac{1}{|x-y|^{d-1}} |\nabla^d_x u_x|_{\mathbb{R}^d} \lambda(dx) \lambda(dy) \right] \]

\[ \leq \|Du\|_{L^1(\Omega \times B(R) \times B(R))} + K_d v_d R^d E \left[ \int_{B(R)} |\nabla^d_x u_x|_{\mathbb{R}^d} \lambda(dx) \right] \]

\[ = \|Du\|_{L^1(\Omega \times B(R) \times B(R))} + K_d v_d R^d \|\nabla^d u\|_{L^1(\Omega \times B(R) \times \mathbb{R}^d)}, \]

we check that \( \tilde{\nabla} \) extends to the Sobolev space \( \tilde{\mathbb{D}}^{1,1}_0 \) of predictable random fields defined as the completion of \( \mathcal{P}_0 \) under the Sobolev norm

\[ \|u\|_{\tilde{\mathbb{D}}^{1,1}_0} := \|u\|_{L^2(\Omega, W^{1,1}_0(B(R)))} + \|Du\|_{L^1(\Omega \times B(R) \times B(R))}, \quad u \in \mathcal{P}_0, \]

where \( W^{1,p}_0(B(R)) \) is the first order Sobolev space completion of \( \mathcal{C}_0^\infty(B(R)) \) under the norm

\[ \|f\|_{W^{1,p}(B(R))} := \|f\|_{L^p(B(R))} + \|\nabla^d f\|_{L^p(B(R) \times \mathbb{R}^d)}, \quad p \geq 1. \]

**Commutation relation**

In the sequel, we denote by \( \tilde{\mathbb{D}}^{1,\infty}_0 \) the set of predictable random fields \( u \) in \( \tilde{\mathbb{D}}^{1,1}_0 \) that are bounded together with their covariant derivative \( \nabla u \).

**Proposition 2.6** For \( u \in \tilde{\mathbb{D}}^{1,\infty}_0 \) a predictable random field, we have the commutation relation

\[ D_y \delta(u) = u(y) + \delta(\tilde{\nabla}_y u), \quad y \in B(R). \quad (2.14) \]

**Proof.** Taking \( h \in \mathcal{C}_0^\infty(B(R)) \), we have \( \delta(h) \in \mathcal{S} \) and

\[ D_y \delta(h) = D_y \int_{B(R)} h(y)(\gamma(dx) - \lambda(dx)) \]

\[ = \int_{B(R)} \langle G_\eta(x, y), \nabla^d_x h(x) \rangle_{\mathbb{R}^d} \gamma(dx) \]

\[ = \int_{B(R)} \langle G_\eta(x, y), \nabla^d_x h(x) \rangle_{\mathbb{R}^d} \lambda(dx) + \delta(\tilde{\nabla}_y h) \]

\[ = h(y) + \delta(\tilde{\nabla}_y h). \]

where we applied (1.8). Next, taking \( u = hF \in \mathcal{P}_0 \) a simple predictable random field, we check that \( \delta(u) \in \mathcal{S} \), and by (2.5) or (6.3) we have

\[ D_y \delta(Fh) = D_y (F \delta(h) - \langle h, DF \rangle_{L^2(B(R))}) \]

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\[ D_y (F \delta(h)) = \delta(h)D_y F + FD_y \delta(h) \]
\[ = \delta(h)D_y F + F(h(y) + \delta(\tilde{\nabla}_y h)) \]
\[ = Fh(y) + \delta(hD_y F + F\tilde{\nabla}_y h) \]
\[ = Fh(y) + \delta(\tilde{\nabla}_y (Fh)) \]
\[ = u_y + \delta(\tilde{\nabla}_y u), \quad y \in B(R). \]

We conclude by the denseness of \( P_0 \) in \( \tilde{\mathcal{D}}_0^{1,1} \) and by the closability of \( \tilde{\nabla}, D \) and \( \delta \).

\[ \square \]

3 Cumulant operators

In the sequel, given \( h \) in the standard Sobolev space \( W^{1,p}(B(R)) \) on \( B(R) \) and \( f \in L^q(B(R)) \) with \( 1 = p^{-1} + q^{-1}, p, q \in [1, \infty] \), we define

\[ (\tilde{\nabla} h)_{f_x} := \int_{B(R)} f(y) \tilde{\nabla}_y h(x) \lambda(dy) = \int_{B(R)} f(y) \langle G_\eta(x,y), \nabla^{\mathbb{R}^d} h(x) \rangle_{\mathbb{R}^d} \lambda(dy), \] \quad (3.1)

\( x \in B(R) \). More generally, given \( k \geq 1 \) and \( u \in \tilde{\mathcal{D}}_0^{1,1} \) a predictable random field, we let the operator \( (\tilde{\nabla} u)^k \) be defined in the sense of matrix powers with continuous indices, as

\[ (\tilde{\nabla} u)^k f_y = \int_{B(R)} \cdots \int_{B(R)} (\tilde{\nabla}_{x_k} u_{x_y} \tilde{\nabla}_{x_{k-1}} u_{x_k} \cdots \tilde{\nabla}_{x_1} u_{x_2}) f_{x_1} \lambda(dx_1) \cdots \lambda(dx_k), \]

\( y \in B(R), f \in L^2(B(R)) \).

**Proposition 3.1** For any \( n \in \mathbb{N}, p > 1, r \in [0, 1], h \in W^{1,p/(1-r)^{n-1/r}}(B(R)) \) and \( f \in L^{p/(1-r)^n}(B(R)) \) we have the bound

\[ \| (\tilde{\nabla} h)^n f \|_{L^p(B(R))} \leq (K_d v_d R')^n \| f \|_{L^{p/(1-r)^n}(B(R))} \prod_{j=1}^n \| \nabla^{\mathbb{R}^d} h \|_{L^{p/(1-r)^{j-1/r}}(B(R);\mathbb{R}^d)}. \] \quad (3.2)

**Proof.** For \( n = 1 \) we have

\[ \| (\tilde{\nabla} h)^1 f \|_{L^p(B(R))} = \left( \int_{B(R)} \int_{B(R)} f(y) \tilde{\nabla}_y h(x) \lambda(dy) \right)^{1/p} \lambda(dx) \]
\[
\begin{align*}
&= \int_{B(R)} \int_{B(R)} f(y) \langle G_\eta(x, y), \nabla_x^d h(x) \rangle \mathbb{R}^d \lambda(dy) \right|^p \lambda(dx) \\
&= \int_{B(R)} \left( \int_{B(R)} f(y) G_\eta(x, y) \lambda(dy) \right) \right|^p \nabla_x^d h(x) \mathbb{R}^d \lambda(dx) \\
&\leq \int_{B(R)} \left( \int_{B(R)} f(y) G_\eta(x, y) \lambda(dy) \right) \left( \int_{B(R)} \nabla_x^d h(x) \mathbb{R}^d \lambda(dx) \right)^{1-r} \\
&\leq (K_d v_d R')^p \| f \|_{L^p(B(R))} \| \nabla_x^d h \|_{L^{p/(1-r)}(B(R))}^p \\
\end{align*}
\]

where we used the bound (1.6). Next, assuming that (3.2) holds at the rank \( n \geq 1 \) and using (3.3), we have

\[
\| (\tilde{\nabla} h)^n f \|_{L^p(B(R))} = \| (\tilde{\nabla} h)^{n-1} (\tilde{\nabla} h) f \|_{L^p(B(R))} \\
\leq (K_d v_d R')^n \| (\tilde{\nabla} h) f \|_{L^p(B(R))} \prod_{j=1}^n \| \nabla_x^d h \|_{L^{p/(1-r)}(B(R))} \\
\leq (K_d v_d R')^{n+1} \| f \|_{L^p(B(R))} \prod_{j=1}^{n+1} \| \nabla_x^d h \|_{L^{p/(1-r)}(B(R))},
\]

and we conclude to (3.2) by induction. \( \square \)

In particular, for \( r = 0, f \in L^p(B(R)), p > 1, \) and \( h \in W^{1,1}(B(R)) \) the argument of Proposition 3.1 shows that

\[
\| (\tilde{\nabla} h)^n f \|_{L^p(B(R))} \leq (K_d v_d R')^n \| f \|_{L^p(B(R))} \| \nabla_x^d h \|_{L^\infty(B(R))}^n, \quad n \in \mathbb{N}.
\]

We note that for \( u \in \mathbb{D}_0^{1,\infty} \) a predictable random field, the random field \((\tilde{\nabla} u)u \in \mathbb{D}_0^{1,\infty}\) is also predictable from (2.13) and (3.1).

In the next definition we construct a family of cumulant operators which differs from the one introduced in [11] on the Wiener space.

**Definition 3.2** Given \( k \geq 2 \) and \( u \in \mathbb{D}_0^{1,\infty} \) a predictable random field we define the operators \( \Gamma_k^u : \mathbb{D}_{1,1} \rightarrow L^1(\Omega) \) by

\[
\Gamma_k^u F := F \langle (\tilde{\nabla} u)^{k-1} u, u \rangle_{L^2(B(R))} + \langle (\tilde{\nabla} u)^{k-1} u, DF \rangle_{L^2(B(R))}, \quad F \in \mathbb{D}_{1,1}.
\]

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We note that for \( h \) in the space \( W^{1,\infty}(B(R)) \) of bounded functions in \( W^{1,1}(B(R)) \), and \( f \in L^p(B(R)), \ p > 1, \ m \geq 1 \), we have

\[
\langle h^m, (\tilde{\nabla} h) f \rangle_{L^2(B(R))} = \int_{B(R)} h^m(x) \int_{B(R)} f(y) \langle G_\eta(x, y), \nabla_x^d h(x) \rangle_{\mathbb{R}^d} \lambda(dy) \lambda(dx)
\]

\[
= \frac{1}{m + 1} \int_{B(R)} f(y) \langle \eta_{\eta}(x, y), \nabla_x^d h^{m+1}(x) \rangle_{\mathbb{R}^d} \lambda(dy) \lambda(dx)
\]

where we applied (1.7), hence

\[
\langle h^m, (\tilde{\nabla} h)^{n+1} f \rangle_{L^2(B(R))} = \frac{1}{m + 1} \int_{B(R)} h^{m+1}(x)(\tilde{\nabla} h)^n f(x) \lambda(dx),
\]

which implies by induction

\[
\langle (\tilde{\nabla} h)^n f, h^m \rangle_{L^2(B(R))} = \frac{m!}{(m + n)!} \int_{B(R)} h^{m+n}(x)f(x) \lambda(dx).
\]

In Lemma 3.3 we generalize this identity to \( h \) a random field.

**Lemma 3.3** For \( n \in \mathbb{N}, \ m \geq 1, \ u \in \tilde{B}_{0}^{1,\infty} \) a predictable random field and \( f \in L^p(B(R)), \ p > 1, \) we have

\[
\langle (\tilde{\nabla} u)^n f, u^m \rangle_{L^2(B(R))} = \frac{m!}{(m + n)!} \int_{B(R)} u^{m+n} f(x) \lambda(dx) \tag{3.4}
\]

\[
+ \sum_{k=1}^{n} \frac{m!}{(m + k)!} \left\langle (\tilde{\nabla} u)^{n-k} f, D \int_{B(R)} u^{m+k} \lambda(dx) \right\rangle_{L^2(B(R))}.
\]

**Proof.** Using the adjoint \( \tilde{\nabla}^* u \) of \( \tilde{\nabla} u \) on \( L^2(B(R)) \) given by

\[
(\tilde{\nabla}^* u)v_y := \int_{B(R)} (\tilde{\nabla}_y u_x)v_x \lambda(dx), \ y \in B(R), \ v \in L^2(B(R)),
\]

with the duality relation

\[
\langle v, (\tilde{\nabla}^* u) h \rangle_{L^2(B(R))} = \langle ((\tilde{\nabla} u)v, h \rangle_{L^2(B(R))}, \ h, v \in L^2(B(R)),
\]

we will show by induction on \( k = 0, 1, \ldots, n \) that

\[
(\tilde{\nabla}^* u)^n u^m_{x_0} = \int_{B(R)} \cdots \int_{B(R)} u^m_{x_0} \tilde{\nabla}_{x_0} u_{x_1} \tilde{\nabla}_{x_1} u_{x_2} \cdots \tilde{\nabla}_{x_{n-1}} u_{x_n} \lambda(dx_1) \cdots \lambda(dx_n)
\]

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\[
= \sum_{i=1}^{k} \frac{m!}{(m+i)!} \int_{B(R)} \cdots \int_{B(R)} \tilde{v}_{x_0} u_{x_1} \cdots \tilde{v}_{x_{n-i}} u_{x_{n-i}} D_{x_{n-i}} u_{x_{n-i+1}}^m \lambda(dx_1) \cdots \lambda(dx_{n-i}) \\
+ \frac{m!}{(m+k)!} \int_{B(R)} \cdots \int_{B(R)} u_{x_{n-k}}^m \tilde{v}_{x_0} u_{x_1} \cdots \tilde{v}_{x_{n-k-1}} u_{x_{n-k}} \lambda(dx_1) \cdots \lambda(dx_{n-k}) \tag{3.5}
\]

By (3.1), this relation holds for \( k = 0 \). Next, assuming that the identity (3.5) holds for some \( k \in \{0, 1, \ldots, n-1\} \), and using the relation

\[
\tilde{v}_{x_{n-k-1}} u_{x_{n-k}} = D_{x_{n-k-1}} u_{x_{n-k}} + \langle G_\eta(x_{n-k}, x_{n-k}), \tilde{v}_{x_{n-k}} u_{x_{n-k}} \rangle_{\mathbb{R}^d}, \quad x_{n-k-1}, x_{n-k} \in B(R),
\]

we have

\[
(\tilde{v}^* u)^n u_{x_0}
\]

\[
= \sum_{i=1}^{k} \frac{m!}{(m+i)!} \int_{B(R)} \cdots \int_{B(R)} \tilde{v}_{x_0} u_{x_1} \cdots \tilde{v}_{x_{n-i}} u_{x_{n-i}} D_{x_{n-i}} u_{x_{n-i+1}}^m \lambda(dx_1) \cdots \lambda(dx_{n-i}) \\
+ \frac{m!}{(m+k)!} \int_{B(R)} \cdots \int_{B(R)} u_{x_{n-k}}^m \tilde{v}_{x_0} u_{x_1} \cdots \tilde{v}_{x_{n-k-1}} u_{x_{n-k}} \lambda(dx_1) \cdots \lambda(dx_{n-k}) \\
= \sum_{i=1}^{k} \frac{m!}{(m+i)!} \int_{B(R)} \cdots \int_{B(R)} \tilde{v}_{x_0} u_{x_1} \cdots \tilde{v}_{x_{n-i}} u_{x_{n-i}} D_{x_{n-i}} u_{x_{n-i+1}}^m \lambda(dx_1) \cdots \lambda(dx_{n-i}) \\
+ \frac{m!}{(m+k)!} \int_{B(R)} \cdots \int_{B(R)} u_{x_{n-k}}^m \tilde{v}_{x_0} u_{x_1} \cdots \tilde{v}_{x_{n-k-1}} u_{x_{n-k}} \lambda(dx_1) \cdots \lambda(dx_{n-k}) \\
+ \frac{m!}{(m+k)!} \int_{B(R)} \cdots \int_{B(R)} \langle G_\eta(x_{n-k}, x_{n-k}), \tilde{v}_{x_{n-k}} u_{x_{n-k}} \rangle_{\mathbb{R}^d} \times u_{x_{n-k}}^{m+k-2} \tilde{v}_{x_0} u_{x_1} \cdots \tilde{v}_{x_{n-2-k}} u_{x_{n-1-k}} \lambda(dx_1) \cdots \lambda(dx_{n-k}) \\
= \sum_{i=1}^{k} \frac{m!}{(m+i)!} \int_{B(R)} \cdots \int_{B(R)} \tilde{v}_{x_0} u_{x_1} \cdots \tilde{v}_{x_{n-i}} u_{x_{n-i}} D_{x_{n-i}} u_{x_{n-i+1}}^m \lambda(dx_1) \cdots \lambda(dx_{n-i}) \\
+ \frac{m!}{(m+k+1)!} \int_{B(R)} \cdots \int_{B(R)} \tilde{v}_{x_0} u_{x_1} \cdots \tilde{v}_{x_{n-k}} u_{x_{n-k-1}} D_{x_{n-k-1}} u_{x_{n-k}}^{m+k} \lambda(dx_1) \cdots \lambda(dx_{n-k}) \\
+ \frac{m!}{(m+k+1)!} \int_{B(R)} \cdots \int_{B(R)} \tilde{v}_{x_0} u_{x_1} \cdots \tilde{v}_{x_{n-k-2}} u_{x_{n-k-1}} \times \int_{B(R)} \langle G_\eta(x, x_{n-k-1}), \nabla_{x_{n-k-1}}^{m+k} u_{x} \rangle_{\mathbb{R}^d} \lambda(dx) \lambda(dx_1) \cdots \lambda(dx_{n-k-1}) \\
= \sum_{i=1}^{k+1} \frac{m!}{(m+i)!} \int_{B(R)} \cdots \int_{B(R)} \tilde{v}_{x_0} u_{x_1} \cdots \tilde{v}_{x_{n-i}} u_{x_{n-i}} D_{x_{n-i}} u_{x_{n-i+1}}^m \lambda(dx_1) \cdots \lambda(dx_{n-i}) \\
+ \frac{m!}{(m+k+1)!} \int_{B(R)} \cdots \int_{B(R)} u_{x_{n-k}}^{m+k+1} \tilde{v}_{x_0} u_{x_1} \cdots \tilde{v}_{x_{n-k-2}} u_{x_{n-k-1}} \lambda(dx_1) \cdots \lambda(dx_{n-k-1})
\[ \sum_{i=1}^{k+1} \frac{m!}{(m+i)!} (\tilde{\nabla}^* u)^{n-i} D_{x_0} \int_{B(R)} u_s^{m+i} \lambda(ds) + \frac{m!}{(m+k+1)!} (\tilde{\nabla}^* u)^{n-k-1} u_{x_0}^{m+k+1}, \]

which shows by induction that (3.5) holds at the rank \( k = n \), in particular we have

\[ (\tilde{\nabla}^* u)^m u_x^m = \frac{m!}{(m+k)!} u_x^{m+k} + \sum_{i=2}^{n+1} \frac{m!}{(m+i-1)!} (\tilde{\nabla}^* u)^{n+1-i} D_x \int_{B(R)} u_i^{n+i-1} \lambda(dy), \]

\( x \in B(R) \), which yields (3.4) by integration with respect to \( x \in B(R) \) and duality.

\[ \square \]

As a consequence of Lemma 3.3 we have

\[ \Gamma^h_k 1 = \int_{B(R)} \frac{u_x^k}{(k-1)!} \lambda(dx) + \sum_{i=2}^{k-1} \frac{1}{i!} \left( \tilde{\nabla} u \right)^{k-1-i} u D \int_{B(R)} u_x^i \lambda(dx) \right), \]

\( k \geq 2 \). Hence when \( h \in W^{1,p}(B(R)) \), \( p > 1 \), is a deterministic function such that \( \|\nabla^d h\|_\infty < \infty \), we find the relation

\[ \Gamma^h_k 1 = \frac{1}{(k-1)!} \int_{B(R)} h^k(x) \lambda(dx) = \frac{1}{(k-1)!} \kappa^h_k, \quad k \geq 2, \tag{3.6} \]

which shows that \( \Gamma^h_k 1 \) coincides with the cumulant \( \kappa^h_k = \int_{B(R)} h^k(x) \lambda(dx) \) of order \( k \geq 2 \) of the Poisson stochastic integral \( \int_{B(R)} h(x)(\gamma(dx) - \lambda(dx)) \).

## 4 Edgeworth type expansions

Classical Edgeworth series provide expansion of the cumulative distribution function \( P(F \leq x) \) of a centered random variable \( F \) with \( E[F^2] = 1 \) around the Gaussian cumulative distribution function \( \Phi(x) \), using the cumulants \( (\kappa_n)_{n \geq 1} \) of a random variable \( F \) and Hermite polynomials. Edgeworth type expansions of the form

\[ E[F^g(F)] = \sum_{l=1}^n \frac{\kappa_{l+1}}{l!} E[g^{(l)}(F)] + E[g^{(n+1)}(F) \Gamma_{n+1} F], \quad n \geq 1, \]

for \( F \) a centered random variable, have been obtained by the Malliavin calculus in [9], where \( \Gamma_{n+1} \) is a cumulant type operator on the Wiener space such that \( n! E[\Gamma_n F] \) coincides with the cumulant \( \kappa_{n+1} \) of order \( n + 1 \) of \( F \), \( n \in \mathbb{N} \), cf. [11], extending the
results of [3] to the Wiener space.

In this section we establish an Edgeworth type expansion of any finite order with an explicit remainder term for the compensated Poisson stochastic integral $\delta(u)$ of a predictable random field $(u_x)_{x \in B(R)}$. In the sequel we let $\langle \cdot, \cdot \rangle$ denote $\langle \cdot, \cdot \rangle_{L^2(B(R))}$, except if stated otherwise.

Before proceeding to the statement of general expansions in Proposition 4.1, we illustrate the method with the derivation of an expansion of order one for a deterministic integrand $f$. By the duality relation (2.6) between $D$ and $\delta$, the chain rule of derivation for $D$ and the commutation relation (2.14) we get, for $g \in C^2_b(\mathbb{R})$ and $f \in W^{1,1}_0(B(R))$ such that $\|\nabla^{\mathbb{R}^d} f\|_\infty < \infty$,

$$E[\delta(f)g(\delta(f))] = E[(f, D\delta(f))g'(\delta(f))]$$
$$= E[(f, f)g'(\delta(f))] + E[(f, \delta(\widetilde{\nabla} f))g'(\delta(f))]$$
$$= E[(f, f)g'(\delta(f))] + E[\langle \widetilde{\nabla} f, D(g'(\delta(f)))f \rangle]$$
$$= E[(f, f)g'(\delta(f))] + E[\langle \widetilde{\nabla} f, D\delta(f) \rangle g''(\delta(f))]$$
$$= E[(f, f)g'(\delta(f))] + \frac{1}{2} \int_{B(R)} f^3(x) \lambda(dx) E[g''(\delta(f))] + E[\langle (\widetilde{\nabla} f, \delta(\widetilde{\nabla} f))g''(\delta(f)) \rangle]$$
$$= \kappa_f^3 E[g'(\delta(f))] + \frac{1}{2} \kappa_3^f E[g''(\delta(f))] + E[g''(\delta(f))\delta((\widetilde{\nabla} f)^2 f)],$$

since by Lemma 3.3 we have

$$\langle (\widetilde{\nabla} f, f) \rangle = \frac{1}{2} \int_{B(R)} f^3(x) \lambda(dx) = \frac{1}{2} \kappa_3^f.$$

In the next proposition we derive general Edgeworth type expansions for predictable integrand processes $(u_x)_{x \in \mathbb{R}^d}$.

**Proposition 4.1** Let $u \in \widetilde{D}_0^{1,\infty}$ and $n \geq 0$. For all $g \in C_b^{n+1}(\mathbb{R})$ and bounded $G \in D_{1,1}$ we have

$$E[G\delta(u)g(\delta(u))] = E[(u, DG)g(\delta(u))] + \sum_{k=1}^n E[g^{(k)}(\delta(u))\Gamma_k^{u} G]$$

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+ E \left[ Gg^{(n+1)}(\delta(u)) \left( \int_{B(R)} \frac{u_x^{n+2}}{(n+1)!} \lambda(dx) + \sum_{k=2}^{n+1} \left( (\nabla u)^{n+1-k}u, D \int_{B(R)} \frac{u_x^k}{k!} \lambda(dx) \right) \right) \right] \\
+ E \left[ Gg^{(n+1)}(\delta(u)) (\nabla u)^n u, \delta(\nabla^* u) \right].

**Proof.** By the duality relation (2.6) between $D$ and $\delta$, the chain rule of derivation for $D$ and the commutation relation (2.14), we get

\[
E[G((\nabla u)^k u, D\delta(u))g(\delta(u))] - E[G((\nabla u)^{k+1} u, D\delta(u))g'(\delta(u))]
\]

\[
= E[G((\nabla u)^k u, u)g(\delta(u))] + E[G((\nabla u)^k u, \delta(\nabla^* u))g(\delta(u))] - E[G((\nabla u)^{k+1} u, D\delta(u))g'(\delta(u))]
\]

\[
= E[G((\nabla u)^k u, u)g(\delta(u))] + E[(\nabla^* u, D Gg(\delta(u))(\nabla u)^k u)] - E[G((\nabla u)^{k+1} u, D\delta(u))g'(\delta(u))]
\]

\[
= E[G((\nabla u)^k u, u)g(\delta(u))] + E[\langle (\nabla u)^{k+1} u, DG \rangle g(\delta(u))] + E[G(\nabla^* u, D((\nabla u)^k u))g(\delta(u))]
\]

\[
= E[g(\delta(u)) \Gamma_{k+2}^u G],
\]

where we used (2.9) and (2.13). Therefore, we have

\[
E[G\delta(u)g(\delta(u))] = E[\langle u, D(Gg(\delta(u))) \rangle]
\]

\[
= E[G\langle u, D\delta(u) \rangle g'(\delta(u))] + E[\langle u, D G \rangle g(\delta(u))]
\]

\[
= E[\langle u, D G \rangle g(\delta(u))] + E[Gg^{(n+1)}(\delta(u)) \langle (\nabla u)^n u, D\delta(u) \rangle]
\]

\[
+ \sum_{k=0}^{n-1} \left( E[Gg^{(k+1)}(\delta(u)) \langle (\nabla u)^k u, D\delta(u) \rangle] - E[Gg^{(k+2)}(\delta(u)) \langle (\nabla u)^{k+1} u, D\delta(u) \rangle] \right)
\]

\[
= E[\langle u, D G \rangle g(\delta(u))] + \sum_{k=1}^{n} E[g^{(k)}(\delta(u)) \Gamma_{k+1}^u G] + E[Gg^{(n+1)}(\delta(u)) \langle (\nabla u)^n u, D\delta(u) \rangle]
\]

\[
= E[\langle u, D G \rangle g(\delta(u))] + \sum_{k=1}^{n} E[g^{(k)}(\delta(u)) \Gamma_{k+1}^u G]
\]

\[
+ E[Gg^{(n+1)}(\delta(u)) \langle (\nabla u)^n u, u \rangle] + E[Gg^{(n+1)}(\delta(u)) \langle (\nabla u)^n u, \delta(\nabla^* u) \rangle],
\]

and we conclude by Lemma 3.3. \qed

When $f \in W^{1,1}_0(B(R))$ is a deterministic function such that $\|\nabla^d f\|_{\infty} < \infty$, and $g \in C^\infty_b(\mathbb{R})$, Proposition 4.1 shows that

\[
E[\delta(f)g(\delta(f))]
\]

\[
= \sum_{k=1}^{n+1} \frac{1}{k!} \int_{B(R)} f^{k+1}(x) \lambda(dx) E[g^{(k)}(\delta(f))] + E[g^{(n+1)}(\delta(f)) \langle (\nabla f)^n f, \delta(\nabla^* f) \rangle]
\]

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\[ \sum_{k=1}^{n+1} \frac{1}{k!} k f_k E[g^{(k)}(\delta(f))] + E[g^{(n+1)}(\delta(f)) \delta((\nabla f)^{n+1} f)], \quad n \geq 0, \]

with, by Proposition 3.1 applied with \( p = 2 \) and \( r = 0 \),

\[
E[\delta((\nabla f)^{n+1} f)] \leq \sqrt{E[\nabla^2((\nabla f)^{n+1} f)^2]} = \|((\nabla f)^{n+1} f)\|_{L^2(B(R))} \leq (K_d v d R')^{n+1} \|f\|_{L^2(B(R))} \|\nabla f\|_{L^\infty(B(R);\mathbb{R}^d)}.
\]

In addition, as \( n \) tends to \(+\infty\) we have

\[
E[\delta(f)g(\delta(f))] = \sum_{k=1}^{\infty} \frac{1}{k!} \int_{B(R)} f^{k+1}(x) \lambda(dx) E[g^{(k)}(\delta(f))] \\
= \sum_{k=1}^{\infty} \frac{1}{k!} \int_{B(R)} f^{k+1}(x) \lambda(dx) E[g^{(k)}(\delta(f))] \\
= E\left[ \int_{B(R)} f(x)(g(\delta(f) + f(x)) - g(\delta(f)))\lambda(dx) \right]
\]

provided that the derivatives of \( g \) decay fast enough, which is a particular instance of the standard integration by parts identity for finite difference operators on the Poisson space, see e.g. Lemma 2.9 in [12] or Lemma 5 in [4].

### 5 Stein approximation

Applying Proposition 4.1 with \( n = 0 \) and \( G = 1 \) to the solution \( g_x \) of the Stein equation

\[
1_{(-\infty,x]}(z) - \Phi(z) = g'_x(z) - zg_x(z), \quad z \in \mathbb{R},
\]

and \( u \in \tilde{D}_0^{1,1} \) a predictable random field this gives the expansion

\[
P(\delta(u) \leq x) - \Phi(x) = E[g'_x(\delta(u))\langle u, u \rangle - \delta(u)g_x(\delta(u))] \\
= E[(1 - \langle u, u \rangle)g'_x(\delta(u))] + E[\langle u, \delta(\nabla u) \rangle g'_x(\delta(u))],
\]

around the Gaussian cumulative distribution function \( \Phi(x) \), with \( \|g_x\|_\infty \leq \sqrt{2\pi}/4 \) and \( \|g'_x\|_\infty \leq 1 \), \( x \in \mathbb{R} \), by Lemma 2.2-(v) of [6]. The next result applies Proposition 4.1 with \( n = 1 \) and \( G = 1 \).
Proposition 5.1 For any random field $u \in \mathbb{W}_0^{1,\infty}$ we have

\[
d(\delta(u), \mathcal{N}) \\
\leq E[|1 - \langle u, u \rangle - \langle \nabla^* u, Du \rangle|] + E \left[ \left| \int_{B(R)} u_x^2 \lambda(dx) + \left( u, D \int_{B(R)} u_x^2 \lambda(dx) \right) \right| \right] \\
+ 2E \left[ |\langle (\nabla u) u, \delta(\nabla^* u) \rangle| \right].
\]  

(5.1)

Proof. For $n = 1$ and $G = 1$, Proposition 4.1 shows that

\[
E[\delta(u)g(\delta(u))] = E[g'(\delta(u))((u, u) + \langle \nabla^* u, Du \rangle)] \\
+ \frac{1}{2} E \left[ g''(\delta(u)) \left( \int_{B(R)} u_x^2 \lambda(dx) + \left( u, D \int_{B(R)} u_x^2 \lambda(dx) \right) \right) \right] \\
+ E[g''(\delta(u))\langle (\nabla u) u, \delta(\nabla^* u) \rangle].
\]

Let $h : \mathbb{R} \to [0, 1]$ be a continuous function with bounded derivative. Using the solution $g_h \in C^1_b(\mathbb{R})$ of the Stein equation

\[
h(z) - E[h(\mathcal{N})] = g'(z) - zg(z), \quad z \in \mathbb{R},
\]

with the bounds $\|g'_h\|_\infty \leq h'_\infty$ and $\|g''_h\|_\infty \leq 2\|h'_\infty\|, x \in \mathbb{R}$, cf. Lemma 1.2-(v) of [10] and references therein, we have

\[
E[h(\delta(u))] - E[h(\mathcal{N})] = E[\delta(u)g_h(\delta(u)) - g'_h(\delta(u))] \\
= E[g'_h(\delta(u))((u, u) + \langle \nabla^* u, Du \rangle) - 1] \\
+ \frac{1}{2} E \left[ g''(\delta(u)) \left( \int_{B(R)} u_x^2 \lambda(dx) + \left( u, D \int_{B(R)} u_x^2 \lambda(dx) \right) \right) \right] \\
+ 2E[g''_h(\delta(u))\langle (\nabla u) u, \delta(\nabla^* u) \rangle],
\]

hence

\[
|E[\delta(u)h(\delta(u))] - E[h(\mathcal{N})]| \leq \|h'_\infty\| \quad E[|1 - \langle u, u \rangle - \langle \nabla^* u, Du \rangle|] \\
+ \|h'_\infty\| E \left[ \left| \int_{B(R)} u_x^2 \lambda(dx) + \left( u, D \int_{B(R)} u_x^2 \lambda(dx) \right) \right| \right] \\
+ 2\|h'_\infty\| E \left[ |\langle (\nabla u) u, \delta(\nabla^* u) \rangle| \right],
\]

which yields (5.1). \qed

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As a consequence of Proposition 5.1 and the Itô isometry (2.12) we have the following corollary.

**Corollary 5.2** For \( u \in \overline{D}_0^{1, \infty} \) we have

\[
d(\delta(u), \mathcal{N}) \leq |1 - \text{Var}[\delta(u)]| + \sqrt{\text{Var}[||u||^2_{L^2(B(R))}]}
\]

\[
+ E \left[ \int_{B(R)} u^3_x \lambda(dx) + \left< u, D \int_{B(R)} u^2_x \lambda(dx) \right> \right]
\]

\[
+ E[|\langle \nabla u, D\delta \rangle| + 2E[|\langle (\nabla u, \delta) \rangle|].
\]

**Proof.** By the Itô isometry (2.12) we have

\[
\text{Var}[\delta(u)] = E \left[ \left( \int_{B(R)} u_x (\gamma(dx) - \lambda(dx)) \right)^2 \right] = E[\langle u, u \rangle],
\]

hence

\[
E[|1 - \langle u, u \rangle - \langle \nabla^* u, D\delta \rangle|]
\]

\[
\leq E[|1 - E[\langle u, u \rangle]| + E[|\langle u, u \rangle - E[\langle u, u \rangle]|] + E[|\langle \nabla^* u, D\delta \rangle|]
\]

\[
= |1 - \text{Var}[\delta(u)]| + \sqrt{\text{Var}[|\langle u, u \rangle|^2] + E[|\langle \nabla^* u, D\delta \rangle|]
\]

\[
= |1 - \text{Var}[\delta(u)]| + \sqrt{\text{Var}[||u||^2_{L^2(B(R))}] + E[|\langle \nabla^* u, D\delta \rangle|].
\]

\[
\Box
\]

In particular, when \( \text{Var}[\delta(u)] = 1 \), Corollary 5.2 shows that

\[
d(\delta(u), \mathcal{N}) \leq \sqrt{\text{Var}[||u||^2_{L^2(B(R))}] + E \left[ \int_{B(R)} u^3_x \lambda(dx) + \left< u, D \int_{B(R)} u^2_x \lambda(dx) \right> \right]
\]

\[
+ E[|\langle \nabla^* u, D\delta \rangle| + 2E[|\langle (\nabla u, \delta) \rangle|].
\]

When \( f \in W_0^{1, \infty}(B(R)) \) is a deterministic function we have

\[
\text{Var}[\delta(f)] = E \left[ \left( \int_{B(R)} f(x)(\gamma(dx) - \lambda(dx)) \right)^2 \right] = \int_{B(R)} f^2(x) \lambda(dx),
\]

and Corollary 5.1 shows that

\[
d(\delta(f), \mathcal{N}) \leq \left| 1 - \int_{B(R)} f^2(x) \lambda(dx) \right| + \left| \int_{B(R)} f^3(x) \lambda(dx) \right| + 2E[|\delta((\nabla f)^2)|].
\]
Given the bound
\[ E[|\delta((\tilde{\nabla} f)^2 f)|] \leq \sqrt{E[|\delta((\nabla f)^2 f)|^2]} \]
\[ = \|((\nabla f)^2 f\|_{L^2(B(R))} \]
\[ \leq (K_d v_d R')^2 \|f\|_{L^2(B(R))} \|\nabla^{R^d} f\|^2_{L^\infty(B(R)\times R^d)} \]

obtained from Proposition 3.1 with \( p = 2 \) and \( r = 0 \), \( f \in W_0^{1,\infty}(B(R)) \), we have the following corollary.

**Corollary 5.3** For \( f \in W_0^{1,\infty}(B(R)) \) we have
\[
d \left( \int_{B(R)} f(x)(\gamma(dx) - \lambda(dx)), N \right) \leq \left| 1 - \|f\|^2_{L^2(B(R))} \right| + \left| \int_{B(R)} f^3(x) \lambda(dx) \right| + 2(K_d v_d R')^2 \|f\|_{L^2(B(R))} \|\nabla^{R^d} f\|^2_{L^\infty(B(R)\times R^d)}.\]

In particular, if \( \|f\|_{L^2(B(R))} = 1 \) we find
\[
d \left( \int_{B(R)} f(x)(\gamma(dx) - \lambda(dx)), N \right) \leq \left| \int_{B(R)} f^3(x) \lambda(dx) \right| + 2(K_d v_d R')^2 \|\nabla^{R^d} f\|^2_{L^\infty(B(R)\times R^d)}.\]

As an example, consider \( f_k \) given on \( B(Rk^{1/d}) \) by
\[
f_k(x) := \frac{1}{C \sqrt{k}} g \left( \frac{|x|_{\mathbb{R}^d}}{k^{1/d}} \right),\]
where \( g \in C^1_0([0, R]) \) and
\[ C^2 := v_d \int_0^R g^2(r) r^{d-1} dr, \]
so that \( f_k \in L^2(B(Rk^{1/d})) \) with
\[
\|f\|^2_{L^2(B(Rk^{1/d})} = \frac{v_d}{C^2 k^{1/d}} \int_0^{Rk^{1/d}} g^2 \left( \frac{r}{k^{1/d}} \right) r^{d-1} dr = \frac{v_d}{C^2} \int_0^R g^2(r) r^{d-1} dr = 1, \]
and
\[
\int_{B(Rk^{1/d})} f_k^3(x) dx = \frac{1}{C^3 k^{3/2}} \int_0^{Rk^{1/d}} g^3(r k^{-1/d}) r^{d-1} dr = \frac{1}{C^3 \sqrt{k}} \int_0^R g^3(r) r^{d-1} dr, \]
k \geq 1. We have
\[
\|\nabla^{R^d} f_k\|^2_{L^\infty(B(R)\times R^d)} \leq \frac{\|g'\|^2_{L^\infty} d}{C^2 k^{1+2/d}}, \]
hence
\[
d \left( \int_{B(R)} f_k(x)(\gamma(dx) - \lambda(dx)), N \right) \leq \left| \int_{B(R)} f_k^3(x) \lambda(dx) \right| + \frac{2(K_d v_d R^d)^2 d}{k^{1+2/d} C^2} \left\| g' \right\|_\infty^2
\leq \frac{v_d}{C^3 \sqrt{k}} \int_0^R g^3(r)r^{d-1}dr + \frac{2(K_d v_d R^d)^2 d}{k C^2} \left\| g' \right\|_\infty^2.
\]

In particular, if \( g \) satisfies the condition
\[
\int_0^R g^3(r)r^{d-1}dr = 0,
\]
then we find the \( O(1/k) \) convergence rate
\[
d \left( \int_{B(R)} f_k(x)(\gamma(dx) - \lambda(dx)), N \right) \leq \frac{2(K_d v_d R^d)^2 d}{k C^2}, \quad k \geq 1.
\]

6 Appendix

Proof of Proposition 2.3.

As a consequence of (1.7) and (2.2) we have

\[
f_n (x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) - f_{n-1} (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)
\]

\[
= f_n (x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) - f_{n-1} (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \int_{B(R') \setminus B(R)} \eta(x) \lambda(dx)
\]

\[
= f_n (x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) - \int_{B(R') \setminus B(R)} \eta(x)f_n (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \lambda(dx)
\]

\[
= \int_{B(R')} \langle G(x_i, y), \nabla_{x_i} f_n (x_1, \ldots, x_n) \rangle_{\mathbb{R}^d} \lambda(dx_i)
\]

\[
= \int_{B(R')} \langle G(x_i, y), \nabla_{x_i} f_n (x_1, \ldots, x_n) \rangle_{\mathbb{R}^d} \lambda(dx_i), \quad \text{(6.1)}
\]

\( x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n \in B(R') \). Recall that for all \( F \in \mathcal{S} \) of the form (2.1) we have

\[
E[F] = e^{-B(R)} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{B(R)} \cdots \int_{B(R)} f_n (x_1, \ldots, x_n) \lambda(dx_1) \cdots \lambda(dx_n).
\]

Hence, using (6.1), for \( g \in \mathcal{C}_0^1(B(R)) \) and \( F \) of the form (2.1) we have

\[
E \left[ \int_{B(R)} g(y) D_y F \lambda(dy) \right]
\]

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\begin{align*}
&= E \left[ \sum_{n=1}^{\infty} \mathbb{1}_{\{\gamma(B(R))=n\}} \sum_{i=1}^{n} \int_{B(R)} g(y) \langle \mathcal{G}_n(X_i, y), \nabla X f (X_1, \ldots, X_n) \rangle \lambda(dy) \right] \\
&= e^{-\lambda(B(R))} \sum_{n=1}^{\infty} \frac{1}{n!} \int_{B(R)} \cdots \int_{B(R)} \sum_{i=1}^{n} \int_{B(R)} g(y) \langle \mathcal{G}_n(x_i, y), \nabla X f (x_1, \ldots, x_n) \rangle \lambda(dy) \lambda(dx_1) \cdots \lambda(dx_n) \\
&= e^{-\lambda(B(R))} \sum_{n=1}^{\infty} \frac{1}{n!} \int_{B(R)} \cdots \int_{B(R)} \left( \sum_{i=1}^{n} g(x_i) - \int_{B(R)} g(y) \lambda(dy) \right) f_n(x_1, \ldots, x_n) \lambda(dx_1) \cdots \lambda(dx_n) \\
&= E \left[ F \left( \int_{B(R)} g(x)(\gamma(dx) - \lambda(dx)) \right) \right].
\end{align*}

Next, for $u$ of the form (2.4), we check by a standard argument that

\begin{align*}
E[\langle u, DF \rangle_{L^2(B(R))}] &= \sum_{i=1}^{n} E[G_i \langle g_i, DF \rangle_{L^2(B(R))}] \\
&= \sum_{i=1}^{n} \left( E[\langle g_i, D(FG_i) \rangle_{L^2(B(R))}] - F \langle g_i, DG_i \rangle_{L^2(B(R))} \right) \\
&= E \left[ F \sum_{i=1}^{n} \left( G_i \int_{B(R)} g_i(x)(\gamma(dx) - \lambda(dx)) - \langle g_i, DG_i \rangle_{L^2(B(R))} \right) \right] \\
&= E[F \delta(u)].
\end{align*}

\textbf{Proof of Proposition 2.4.} Taking $u \in \mathcal{P}_0$ a predictable random field of the form (2.7) we note that by (2.3) and the compatibility condition (2.10) we have

$$g_i(y)D_y F_i = 0, \quad y \in B(R), \quad i = 1, \ldots, n,$$

hence by (2.5) we have

$$\delta(u) = \delta \left( \sum_{i=1}^{n} F_i g_i \right) = \sum_{i=1}^{n} F_i \delta (g_i) \quad (6.3)$$
\[ \begin{align*}
&= \sum_{i=1}^{n} F_i \int_{B(R)} g_i(x) (\gamma(dx) - \lambda(dx)) \\
&= \int_{B(R)} u_x(\gamma(dx) - \lambda(dx)),
\end{align*} \]

showing that \( \delta(u) \) coincides with the Poisson stochastic integral of \((u_x)_{x \in B(R)}\). Regarding the isometry relation (2.12), we have

\[
E[\delta(u)^2] = E \left[ \left( \sum_{i=1}^{n} F_i \int_{B(R)} g_i(x) (\gamma(dx) - \lambda(dx)) \right)^2 \right] \\
= E \left[ \sum_{i,j=1}^{n} F_i F_j \int_{B(R)} g_i(x) (\gamma(dx) - \lambda(dx)) \int_{B(R)} g_j(x) (\gamma(dx) - \lambda(dx)) \right] \\
= 2E \left[ \sum_{1 \leq i < j \leq n} F_i \int_{B(R)} g_i(x) (\gamma(dx) - \lambda(dx)) F_j \int_{B(R)} g_j(x) (\gamma(dx) - \lambda(dx)) \right] \\
+ E \left[ \sum_{i=1}^{n} F_i^2 \left( \int_{B(R)} g_i(x) (\gamma(dx) - \lambda(dx)) \right)^2 \right] \\
= E \left[ \sum_{i=1}^{n} F_i^2 \int_{B(R)} g_i^2(x) \lambda(dx) \right] \\
= E \left[ \int_{B(R)} u^2(x) \lambda(dx) \right],
\]

which shows that (2.11) extends to the closure of \( \mathcal{P}_0 \) in \( L^2(\Omega \times B(R)) \) by density and a Cauchy sequence argument. \( \square \)

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