POWER-LAW LOCALIZATION FOR ALMOST-PERIODIC OPERATORS: A NASH-MOSER ITERATION APPROACH

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Abstract. In this paper we develop a Nash-Moser type iteration approach to prove (inverse) power-law localization for some $d$-dimensional discrete almost-periodic operators with polynomial long-range hopping. We also provide a quantitative lower bound on the regularity of the hopping. As an application, some results of [Sar82, Pös83, Cra83, BLS83] are generalized to the polynomial hopping case.

1. Introduction

The localization-delocalization transition theory for almost-periodic operators has been one of the central themes in mathematical physics over the years (see [Sim82, Bou05, MJ17, Dam17] and references therein). While considerable progress has been made towards almost-periodic models with Laplacian hopping or even exponential long-range hopping, much less is known for the polynomial hopping ones. We call an operator exhibits the Anderson localization if it has pure point spectrum together with a complete set of exponentially localized eigenfunctions. Apparently, Anderson localization may occur only for operators with exponential long-range hopping (including the Laplacian case). When considering polynomial hopping ones, we would like to expect instead the power-law localization, which means such operators have pure point spectrum together with a complete set of polynomially localized eigenfunctions. The aim of present paper is to develop a Nash-Moser type iteration approach to prove (inverse) power-law localization for some $d$-dimensional almost-periodic operators.

More precisely, let us start with the discrete operator

$$H = \varepsilon T_\phi + d_\delta^{i'}, \quad i, i' \in \mathbb{Z}^d, \quad \varepsilon \geq 0,$$

(1.1)

where $(d_i)_{i \in \mathbb{Z}^d}$ is a $d$-dimensional sequence and $T_\phi$ is the long-range hopping operator defined by

$$(T_\phi u)_i = \sum_{j \in \mathbb{Z}^d} \phi_{i-j} u_j, \quad u = (u_i)_{i \in \mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d)$$

with a symbol $\phi = (\phi_i)_{i \in \mathbb{Z}^d} \in \mathbb{R}^{\mathbb{Z}^d}$. If we let $\phi_i = \delta_{i e}$ with $|e|_1 = \sum_{i=1}^d |e_i| = 1$, then (1.1) becomes the standard discrete Schrödinger operator and we write $T_\phi = \Delta$. In the present we focus on the almost-periodic sequences $(d_i)_{i \in \mathbb{Z}^d}$, and some special examples include $d_i = \tan(\pi i \cdot \vec{\omega})$, $d_i = \exp(2\pi \sqrt{-1} i \cdot \vec{\omega})$, $d_i = (i \cdot \vec{\omega} \mod 1)$ with
\[ \mathbf{i} \cdot \vec{\omega} = \sum_{i=1}^{d} i \cdot \omega_i, \quad \vec{\omega} \in \mathbb{R}^d. \] If \( \varepsilon = 0 \), we know that the spectrum of \( H \) is pure point and \( \{ \delta_i \}_{i \in \mathbb{Z}^d} \) forms a complete set of eigenfunctions. A natural question is whether such localization preserves or not for \( \varepsilon \neq 0 \). It turns out this is quite a delicate problem and the localization depends sensitively on certain parameters associated to the operators ([Bou05, MJ17, Dam17]). However, it is an intuition that \( H \) could be “diagonalizable” and thus shows localization if the coupling \( \varepsilon \) is small enough and \((d_i)_{i \in \mathbb{Z}^d}\) is reasonably “separated”. In this context the celebrated Kolmogorov-Arnold-Moser (KAM) ([Kol54, Arn63, Mos62]) method becomes a good candidate to handle such issues.

Indeed, ever since Dinaburg-Sinai [DS75] first introduced the KAM method in the field of almost-periodic operators, it has become a very powerful tool to achieve both delocalization and localization (see e.g. [Cra83, BLS83, Pöss83, MP84, Sin87, Eli92, CD93, Eli97, AFK11] just for a few). Particularly, in the 1-dimensional Laplacian hopping case, the KAM method can be modified to give reducibility of corresponding transfer matrix, and thus to obtain some delocalization results in the “small” potentials case. Since the remarkable Aubry-André duality [AA80], the delocalization at the small potential may imply localization of its duality at the large potential [BLT83, JK16, AYZ17].

An alternative method is to “diagonalize” the infinite matrix \( H \) directly [Cra83, BLS83, Pöss83, CD93, Eli97]. This idea was first introduced by Craig [Cra83]. In [Cra83], Craig performed an inverse spectral procedure relied on a modified KAM method, and obtained the existence of almost-periodic Schrödinger operators satisfying the Anderson localization. Let \( D = \text{diag}_{i \in \mathbb{Z}^d}(d_i) \). The core of the proof in [Cra83] is to find a unitary transformation and a diagonal operator \( D' \) so that

\[ Q^{-1}(\varepsilon \Delta + D')Q = D, \]

where \( Q \) is derived from the limit of a sequence of invertible operators in the KAM iteration steps. To get such transformations, Craig imposed some non-resonant condition on \((d_i)_{i \in \mathbb{Z}^d} \), i.e.,

\[ |d_i - d_j| \geq \Omega(|i - j|) \text{ for } \forall \ i \neq j, \tag{1.2} \]

where \( |i| = \sup_{1 \leq v \leq d} |i_v| \) and \( \Omega : \mathbb{R}_+ \to \mathbb{R}_+ \) is some weight function which decays slower than the exponential one ([Rüs80]). The special case that \( \Omega(t) = \gamma t^{-\tau} \) with some \( \tau > d, \gamma > 0 \) corresponds to the standard Diophantine condition. At every iteration step, some new diagonal operator will emerge, which would not satisfy the condition (1.2) in general. Then Craig placed those new diagonal terms in \( D' \), and as a result, only inverse spectral type results were obtained. Later, Bellissard-Lima-Scoppola [BLS83] dealt with the direct problem with

\[ |\varphi_i| \leq Ce^{-\rho|i|} \text{ for some } C > 0, \ \rho > 0 \text{ and } \forall \ i \in \mathbb{Z}^d. \tag{1.3} \]

They observed that for some special almost-periodic potentials, the condition (1.2) is stable under small perturbations. Then by using again the KAM method, they showed for such potentials \((d_i)_{i \in \mathbb{Z}^d}\), there exists a unitary operator \( Q \) and some diagonal operator \( D' \) so that for \( |\varepsilon| \ll 1 \),

\[ Q^{-1}(\varepsilon T_\phi + D)Q = D'. \tag{1.4} \]

This implies in particular that \( H \) has pure point spectrum. Since in this KAM procedure \( Q \approx I \) (with \( I \) being the identity operator) in the analyticity norm,
the eigenfunctions of $D'$ under the transformation $Q$ form a complete set of exponentially localized eigenfunctions of $H$. While the potentials of [BLS83] seem restricted, they contain in fact the Maryland potential [GFP82] and Sarnak’s potential [Sar82] as special cases. Subsequently, Pöschel presented a general KAM approach (in the setting of translation invariant Banach algebras) and provided new examples of limit-periodic potentials having Anderson localization. Pöschel’s proof also requires the operator to satisfy both (1.2) and (1.3).

All results as mentioned above concern operators with exponential long-range hopping. In this paper we try to generalize some results of [Cra83, BLS83, Pöss83] to the polynomial long-range hopping case, i.e.,

$$|\phi_i| \leq |i|^{-s} \text{ for some } s > 0 \text{ and } \forall \ i \in \mathbb{Z}^d \setminus \{0\}.$$  

We assume additionally $(d_i)_{i \in \mathbb{Z}^d}$ satisfies the Diophantine condition (this is reasonable since we have a slower decay of the hopping). We again want to find invertible transformation $Q$ and some diagonal operator $D'$ so that (1.4) holds true. The transformation $Q$ should be the limit of some sequence of invertible operators $Q_k$ along the iteration steps in some operator norm. In [BLS83, Pöss83], they introduced an exponential norm for the Toeplitz type operator $A$:

$$\|A\|_a = \sum_{k \in \mathbb{Z}^d} |A_k| e^{a|k|}, \ a > 0,$$

where $A_k$ denotes the k-diagonal of $A$ (see (2.2) in the following for details). Thus at the $k$-th KAM step, it needs to find $Q_k$ and $D_{k-1}$ so that

$$Q_k^{-1} \left( \varepsilon T_\phi + D + \sum_{i=1}^k D_{i-1} \right) Q_k = D + R_k, \ \|R_k\|_{\rho_k} = O(\varepsilon^d),$$  

(1.5)

where $\inf_{k \geq 0} \rho_k \geq \rho/2$. The key point is to determine $Q_k, D_{k-1}$ which turn out to be the solutions of so called homological equations. Since the small divisors difficulty, one may lose some regularity at each iteration step. Fortunately, the condition (1.2) and the analytic norm permit a loss of order $\delta$ for arbitrary $\delta > 0$. This combined with the sup-exponential smallness of the Newton error will lead to the convergence of $Q_k$ and $\sum_{i=1}^k D_{i-1}$ in $\|\cdot\|_{\rho/2}$ eventually, and then the exponential decay preserves! However, when we deal with the polynomial one, we must use a Sobolev type norm

$$\|A\|_s^2 = \sum_{k \in \mathbb{Z}^d} |A_k|^2 (k)^{2s}, \ \langle k \rangle = \max\{1, |k|\}, \ s > 0.$$  

If we want to perform a similar scheme as in [BLS83, Pöss83], there comes a serious difficulty: Since we are in the polynomial case, the loss of regularity when solving the homological equations is of order $\tau > d$ which is fixed at each step! We can imagine such iterations must fail in finite steps (i.e., a loss of all regularities). This motivates us to employ instead the Nash-Moser iteration scheme. Hence at the $k$-th step we would like to get

$$Q_k^{-1} \left( \varepsilon \sum_{i=1}^k T_{i-1} + D + \sum_{i=1}^k D_{i-1} \right) Q_k = D + R'_k + R''_k,$$

where $T_i = (S_{\theta_i} - S_{\theta_{i-1}}) T_\phi$ and $S_{\theta}$ denotes the smoothing operator (see Definition 3.2 for details). As compared with (1.5), our reminder consists of two parts.
Namely, the \( R_k' \) is the usual “square error”, while \( R_k'' \) represents a new “substitution error” which comes from a further smoothing procedure when we try to solve the homological equations. It is actually the \( R_k'' \) that dominates the convergence of the scheme. The total reminder in our scheme is now an exponentially small one, rather than a sup-exponential one as in (1.5). In the iteration scheme, another key ingredient called the tame type property (of the Sobolev norm) plays an essential role. Roughly speaking, the tame property means in the present that the \( s \)-norm of the product \( \prod_{i=1}^{n} A_i \) is bounded above by a linear function of \( \|A_i\|_s (i = 1, \cdots, n) \).

We would like to remark that our proof is a true Nash-Moser type iteration ([Nas56, Mos61]), and in particular within the spirit of the scheme introduced by Hörmander [Hör76, Hör77] (see also [AG07] for an excellent exposition). The above argument is a very schematic, only formal, overview of the proof. It is our main purpose here to actually develop the whole iteration scheme from scratch. We also want to point out that we can prove a quantitative (probably sharp) lower bound on the regularity required by \( T \), which is given by

\[
\|T\|_s < \infty \text{ with } s > \tau + d.
\]

Similar regularity bound has been previously encountered in localization theory of disordered models (cf. [AM93]). Finally, while we can’t handle the general analytic quasi-periodic potentials, we believe the method may be improved to resolve at least some special cases, such as the almost Mathieu one (i.e., \( d_i = \cos 2\pi (i \cdot \omega) \)).

Once the above reducibility is obtained via the Nash-Moser type iteration, we can apply it to some interesting examples of [Cra83, Sar82, Pös83, BLS83] and establish the (inverse) power-law localization.

1.1. Structure of paper. The structure of paper is then as follows. The main results are introduced in §2 and some preliminaries (including the tame property and smoothing operator) are presented in §3. The main part of the paper is §4, where a Nash-Moser type iteration scheme is developed. A complete Nash-Moser iteration type theorem in our setting is stated in §5. The proofs of our main theorems are contained in §6. Some useful estimates are given in the appendix.

2. Main results

We present our results in the language of translation invariant Banach algebra as in Pöschel [Pös83].

2.1. Translation invariant Banach algebra. Our start point is a Banach algebra \( \mathcal{B}, \| \cdot \|_\mathcal{B} \) of complex \( d \)-dimensional sequences \( (a_i)_{i \in \mathbb{Z}^d} \). The operations are pointwise addition and multiplication of sequences. We assume the sequence \( (1)_{i \in \mathbb{Z}^d} \) belongs to \( \mathcal{B} \) and has norm 1. Denote by \( \sigma_j, j \in \mathbb{Z}^d \) the translations on \( \mathcal{B} \), which are defined by \( \sigma_j a = a_{i-j} \) for each \( a \in \mathcal{B} \). We assume further that \( \mathcal{B} \) is translation invariant, i.e., \( \|\sigma_j a\|_\mathcal{B} = \|a\|_\mathcal{B} \) for all \( j \in \mathbb{Z}^d \) and \( a \in \mathcal{B} \).

It was proven in [Pös83] that the translation invariance implies for every \( a \in \mathcal{B}, \)

\[
\|a\|_\infty = \sup_{i \in \mathbb{Z}^d} |a_i| \leq \|a\|_\mathcal{B}.
\]
2.2. The \((\tau, \gamma)\)-distal sequence. Let \(p = (p_i)_{i \in \mathbb{Z}^d}\) be an arbitrary complex sequence. Fix \(\tau > 0, \gamma > 0\). We call \(p\) a \((\tau, \gamma)\)-distal sequence for \(\mathcal{B}\) if for all \(k \in \mathbb{Z}^d \setminus \{0\}\),

\[
(p - \sigma_k p)^{-1} \in \mathcal{B}, \quad \|(p - \sigma_k p)^{-1}\|_{\mathcal{B}} \leq \gamma^{-1}|k|^{\tau}
\]

with \((p - \sigma_k p)^{-1} = \frac{1}{p_i - p_{i-k}}\) and \(|k| = \sup_{1 \leq \nu \leq d} |k_{\nu}|\). We remark that \(p\) itself is not necessarily in \(\mathcal{B}\).

Denote by \(DC_{\mathcal{B}}(\tau, \gamma)\) the set of all \((\tau, \gamma)\)-distal sequence for \(\mathcal{B}\).

2.3. The Sobolev Toeplitz operator. Now we define the Toeplitz operator with polynomial off-diagonal decay. Let \(\mathcal{M}\) be the set of all infinite matrices \(A = (a_{i,j})_{i,j \in \mathbb{Z}^d}\) satisfying for every \(k \in \mathbb{Z}^d\),

\[
A_k = (a_{i,k-i})_{i \in \mathbb{Z}^d} \in \mathcal{B},
\]

where \(A_k\) is called a \(k\)-diagonal of \(A\). We can then define the Toeplitz operator of Sobolev type. For any \(s \in [0, +\infty)\) and \(A \in \mathcal{M}\), define

\[
\|A\|_s^2 = \sum_{k \in \mathbb{Z}^d} \|A_k\|_{\mathcal{B}}^2 (\langle k \rangle)^{2s}, \quad \langle k \rangle = \max\{1, |k|\}. \tag{2.2}
\]

Then let

\[
\mathcal{M}^s = \{A \in \mathcal{M} : \|A\|_s < \infty\}, \quad \mathcal{M}^\infty = \bigcap_{s \geq 0} \mathcal{M}^s.
\]

For any \(A = (a_{i,j})_{i,j \in \mathbb{Z}^d}\), let \(\overline{A} = \text{diag}_{i \in \mathbb{Z}^d}(a_{i,i})\) be the main diagonal part of \(A\). Denote by \(\mathcal{M}_0^\infty = \{A \in \mathcal{M}^\infty : A = \overline{A}\}\) the subspace of all diagonal operators.

2.4. The main results. Let \(D = \text{diag}_{i \in \mathbb{Z}^d}(d_i)\). Denote \(D \in DC_{\mathcal{B}}(\tau, \gamma)\) if \((d_i)_{i \in \mathbb{Z}^d} \in DC_{\mathcal{B}}(\tau, \gamma)\).

Fix \(I \in \mathcal{M}\) to be the identity operator. The first main result is

**Theorem 2.1.** Fix \(\delta > 0, \alpha_0 > d/2, \tau > 0, \gamma > 0\). Let

\[
D \in DC_{\mathcal{B}}(\tau, \gamma), \quad T \in \mathcal{M}^{\alpha+\delta}, \quad \alpha > 0.
\]

Then for

\[
\alpha > \alpha_0 + \tau + 7\delta,
\]

there exists some \(\epsilon_0 = \epsilon_0(\delta, \tau, \gamma, \alpha_0, \alpha) > 0\) such that the following holds true. If \(\|T\|_{\alpha+\delta} \leq \epsilon_0\), then there exist invertible \(Q_+ \in \mathcal{M}^{\alpha-\tau-7\delta}\) and some \(D_+ \in \mathcal{M}^0\) such that

\[
Q_+^{-1}(T + D + D_+)Q_+ = D
\]

with

\[
\|Q_+ - I\|_{\alpha-\tau-7\delta} \leq C\|T\|_{\alpha+\delta}^\delta, \quad \|Q_+^{-1} - I\|_{\alpha-\tau-7\delta} \leq C\|T\|_{\alpha+\delta}^\delta, \quad \|D_+\|_0 \leq C\|T\|_{\alpha+\delta}^\delta.
\]
where $C = C(\delta, \tau, \alpha_0, \alpha) > 0$. Moreover, if both $T$ and $D$ are real symmetric, then
\[ U^{-1}(T + D + D_+)U = D, \]
\[ \|U - I\|_{\alpha - \tau - 7\delta} \leq C\|T\|_{\alpha + 4\delta}, \]
where $U = Q_+(Q_+^tQ_+)^{-\frac{1}{2}}$ is a unitary operator with $Q_+^t$ denoting the transpose of $Q_+$.

**Remark 2.1.** We would like to remark that we have obtained a quantitative (probably sharp) lower bound on the regularity of $T$. Namely,
\[ T \in \mathcal{M}^s \text{ with } s > d/2 + \tau. \] (2.3)
In fact, we have that $\delta > 0$ can be arbitrary. If $s > d/2 + \tau$, we let $\zeta = s - d/2 - \tau > 0$, $\alpha = d/2 + \tau + \zeta/10$, $\delta = \zeta/1000$, $\alpha_0 = d/2 + \zeta/10$. Then it is easy to check that all the conditions of the theorem are satisfied in this case. Indeed, the lower bound $d/2 + \tau$ can be explained as follows: $d/2$ gives the tame information, and $\tau$ the loss of “derivatives”.

The second main result is

**Theorem 2.2.** Fix $\delta > 0, \alpha_0 > d/2, \tau > 0, \gamma > 0$. Let
\[ T \in \mathcal{M}^{\alpha + 4\delta}, \alpha > 0 \]
Assume $D + D' \in DC_{\mathcal{B}}(\tau, \gamma)$ for any $D' \in \mathcal{M}_0^{\infty}$ with $\|D'\|_0 \leq \eta$ ($\eta > 0$). Then for
\[ \alpha > \alpha_0 + \tau + 7\delta, \]
there exists some $\epsilon_0 = \epsilon_0(\eta, \delta, \tau, \gamma, \alpha_0, \alpha) > 0$ such that the following holds true. If $\|T\|_{\alpha + 4\delta} \leq \epsilon_0$, then there exist invertible $Q_+ \in \mathcal{M}^{\alpha - \tau - 7\delta}$ and some $D_+ \in \mathcal{M}_0^{\infty}$ such that
\[ Q_+^{-1}(T + D)Q_+ = D + D_+ \]
with
\[ \|Q_+ - I\|_{\alpha - \tau - 7\delta} \leq C\|T\|_{\alpha + 4\delta}, \]
\[ \|Q_+^{-1} - I\|_{\alpha - \tau - 7\delta} \leq C\|T\|_{\alpha + 4\delta}, \]
\[ \|D_+\|_0 \leq C\|T\|_{\alpha + 4\delta}, \]
where $C = C(\delta, \tau, \alpha_0, \alpha) > 0$. Moreover, if both $T$ and $D$ are real symmetric, then
\[ U^{-1}(T + D)U = D + D_+, \]
\[ \|U - I\|_{\alpha - \tau - 7\delta} \leq C\|T\|_{\alpha + 4\delta}, \]
where $U = Q_+(Q_+^tQ_+)^{-\frac{1}{2}}$ is a unitary operator.

**Remark 2.2.** This theorem shows that if the non-resonant condition is stable under small perturbations, then we can treat the direct problem and obtain the reducibility of the original operator $T + D$. 
2.5. **Power-law localization.** Now we can apply the above theorems to obtain the (inverse) power-law localization.

Consider the polynomial long-range hopping operator

\[(T_{\phi}u)_i = \sum_{j \in \mathbb{Z}^d} \phi_{i-j} u_j, \quad \phi = (\phi_i)_{i \in \mathbb{Z}^d} \in \mathbb{R}^{\mathbb{Z}^d},\]

with \(|\phi_i| \leq |i|^{-s} \text{ for } i \in \mathbb{Z}^d \setminus \{0\}, \phi_0 = 0.\]

Obviously, \(T_{\phi} \in \mathcal{M}^{s'}\) for \(0 \leq s' < s - d/2.\)

The first application is

**Corollary 2.3.** Fix \(\alpha_0 > d/2, \tau > 0, \gamma > 0\) and \(\delta > 0.\) Assume that \(T_{\phi}\) is given by (2.4) and (2.5) with

\[s > \alpha_0 + \tau + d/2 + 12\delta.\]

Let \(D = \text{diag}_{i \in \mathbb{Z}^d}(d_i) \in D C_{\mathbb{R}}(\tau, \gamma).\) Then there is \(\varepsilon_0 = \varepsilon_0(s, d, \delta, \tau, \gamma, \alpha_0) > 0\) so that for \(\|T_{\phi}\|_{s-d/2-\delta} \leq \varepsilon_0,\) the following holds true. There exists some \(D' = \text{diag}_{i \in \mathbb{Z}^d}(d'_i)\) with

\[D' - D \in \mathcal{M}_0^\infty, \|D - D'\|_0 \leq C\|T_{\phi}\|_{s-d/2-\delta}^{-1} \text{ satisfying}\]

\[H' = T_{\phi} + D'\]

has pure point spectrum with a complete set of polynomially localized eigenfunctions \(\{c_k\}_{k \in \mathbb{Z}^d}\) satisfying

\[|(c_k)_i| \leq 2(|i - k|)^{-s+\tau+d/2+12\delta} \text{ for } i, k \in \mathbb{Z}^d.\]

Moreover, if \((d_i)_{i \in \mathbb{Z}^d}\) is a real-valued sequence, then the spectrum of \(H'\) is equal to the closure of the sequence \((d_i)_{i \in \mathbb{Z}^d}\) in \(\mathbb{R}.\)

**Remark 2.3.** Recalling (2.3) and (2.5), we also have a quantitative lower bound on the regularity of \(\phi,\) i.e.,

\[|\phi_i| \leq |i|^{-s} \text{ with } s > d + \tau.\]

This is interesting if one is familiar with the random power-law localization. Consider now \(H_\omega = \varepsilon T_{\phi} + d_i(\omega) \delta_i\), where \((d_i(\omega))_{i \in \mathbb{Z}^d}\) is a sequence of i.i.d. random variables having uniform distribution on \([0, 1].\) Then by developing the remarkable fractional moment method, Aizenman-Molchanov [AM93] showed that for

\[H_\omega\]

exhibits power-law localization for small \(\varepsilon\) and a.e. \(\omega.\)

**Proof.** We apply Theorem 2.1 with

\[\alpha = s - d/2 - 5\delta.\]

Since (2.6) and (2.5), we have

\[T_{\phi} \in \mathcal{M}^{s-d/2-\delta} = \mathcal{M}^{\alpha+4\delta},\]

\[\alpha = s - d/2 - 5\delta > \alpha_0 + \tau + 7\delta.\]

Hence using Theorem 2.1 implies that if \(\|T_{\phi}\|_{s-d/2-\delta} \ll 1,\) there are \(Q_+ \in \mathcal{M}, D_+ \in \mathcal{M}_0^\infty\) so that

\[Q_+^{-1}(T_{\phi} + D + D_+)Q_+ = D\]
and
\[ \|D_+\|_0 \leq C\|T_\phi\|_{s-d/2-\delta}^{\frac{1}{1-\alpha_0}}, \quad (2.7) \]
\[ \|Q_+ - I\|_{s-d/2-\tau-12\delta} \leq C\|T_\phi\|_{s-d/2-\delta}^{\frac{1}{1-\alpha_0}}. \quad (2.8) \]

We let \( D' = D + D_+ \). Then the estimate on \( D' - D \) follows from (2.7).

Next, note that \( D \) is a diagonal operator. Then the standard basis \( \{\delta_k\}_{k \in \mathbb{Z}^d} \) of \( \ell^2(\mathbb{Z}^d) \) is a complete set of eigenfunctions of \( D \) with eigenvalues \( \{d_k\}_{k \in \mathbb{Z}^d} \). Then
\[ H'Q_+\delta_k = Q_+D\delta_k = d_kQ_+\delta_k, \]
which implies \( \{c_k\}_{k \in \mathbb{Z}^d} \) (\( c_k = Q_+\delta_k \)) may be a complete set of eigenfunctions of \( H' \). Indeed, we obtain since (2.8)
\[ \|Q_+\|_{s-d/2-\tau-12\delta} \leq 2 \text{ if } \|T_\phi\|_{s-d/2-\delta} \ll 1, \]
which together with (2.1) implies
\[ |(Q_+\delta_k)_1| = |(Q_+)_{i,k}| = |(Q_+)_{i-k}(i)| \leq \|((Q_+)_{i-k})\|_{\infty} \leq \|Q_+\|_{s-d/2-\tau-12\delta}(i-k)^{-s+d/2+\tau+12\delta} \leq 2(i-k)^{-s+d/2+\tau+12\delta}, \]
where \( (Q_+)_{i-k} \) denotes the \( i-k \) diagonal of \( Q_+ \). Since
\[ s - d/2 - \tau - 12\delta > \alpha_0 > d/2, \]
we have \( c_k \in \ell^2(\mathbb{Z}^d) \). This shows that \( \{c_k\}_{k \in \mathbb{Z}^d} \) are eigenfunctions of \( H' \). We then show the completeness. Denote by \( (\cdot, \cdot) \) the standard inner product on \( \ell^2(\mathbb{Z}^d) \). Let \( \varphi \in \ell^2(\mathbb{Z}^d) \) and suppose for all \( k \in \mathbb{Z}^d \), \( (\varphi, c_k) = 0 \). It suffices to show \( \varphi = 0 \). Assume \( \varphi \neq 0 \). Choose a \( k \in \mathbb{Z}^d \) so that \( |\varphi_{k'}| \leq |\varphi_k| \) for all \( k' \in \mathbb{Z}^d \). Then \( \varphi_k \neq 0 \) However, we have
\[ 0 = (\varphi, c_k) = \varphi_k + (\varphi, (Q_+ - I)\delta_k), \]
which together with (2.8) implies
\[ |\varphi_k| = |(\varphi, (Q_+ - I)\delta_k)| \leq |\varphi_k| \sum_{j \in \mathbb{Z}^d} |(Q_+ - I)_{j,k}| \leq |\varphi_k| \sum_{j \in \mathbb{Z}^d} \|Q_+ - I\|_{j-k} \leq \|Q_+ - I\|_{\alpha_0} |\varphi_k| \sum_{j \in \mathbb{Z}^d} (j-k)^{-\alpha_0} \leq C\|T_\phi\|_{s-d/2-\delta}^{\frac{1}{1-\alpha_0}} |\varphi_k| = o(1)|\varphi_k|. \]
This contradicts \( \varphi_k \neq 0 \). We have proven the completeness.

Finally, if \( (d_k)_{k \in \mathbb{Z}^d} \in \mathbb{R}^{\mathbb{Z}^d} \), then \( Q_+ \) can be replaced with a unitary operator. Consequently, the spectrum of \( H' \) is equal to that of \( D \).

This proves Corollary 2.3. \( \square \)

**Corollary 2.4.** Fix \( \alpha_0 > d/2, \tau > 0, \gamma > 0 \) and \( \delta > 0 \). Assume that \( T_\phi \) is given by (2.4) and (2.5) with
\[ s > \alpha_0 + \tau + d/2 + 12\delta. \]
Let \( D = \text{diag}_{i \in \mathbb{Z}^d}(d_i) \in DC_\mathbb{N}(\tau, \gamma) \). Assume further \((p_i + d_i)_{i \in \mathbb{Z}^d} \in DC_\mathbb{N}(\tau, \gamma)\) for each \( P = \text{diag}_{i \in \mathbb{Z}^d}(p_i) \in \mathcal{M}_0^\infty\) satisfying \( \|P\|_0 \leq \eta, \eta > 0 \). Then there is \( \epsilon_0 = \epsilon_0(\eta, s, d, \delta, \tau, \gamma, \alpha_0) > 0 \) so that for \( \|T_\phi\|_{s-d/2-\delta} \leq \epsilon_0 \), the following holds true. There exists some \( D' = \text{diag}_{i \in \mathbb{Z}^d}(d'_i) \) with

\[
D' - D \in \mathcal{M}_0^\infty, \|D - D'\|_0 \leq C\|T_\phi\|_{s-d/2-\delta}
\]

such that

\[
H = T_\phi + D
\]

has pure point spectrum with a complete set of polynomially localized eigenfunctions \( \{e_k\}_{k \in \mathbb{Z}^d} \) satisfying

\[
|\langle e_k, i \rangle| \leq 2|i - k|^{-s + \tau + d/2 + 12\delta} \text{ for } i, k \in \mathbb{Z}^d.
\]

Moreover, if \((d_i)_{i \in \mathbb{Z}^d}\) is a real-valued sequence, then the spectrum of \( H \) is equal to the closure of the sequence \((d'_i)_{i \in \mathbb{Z}^d}\) in \( \mathbb{R} \).

**Proof.** Recalling Theorem 2.2, the proof is similar to that of Corollary 2.3. We omit the details here. \( \square \)

2.6. **Application to almost-periodic operators.** In this section we apply our results to some concrete examples of almost-periodic operators and prove corresponding (inverse) power-law localization.

**Example 2.5.** We revisit Pöschel’s limit-periodic potentials. Let \( \mathcal{P} \) denote the set of all \((a_i)_{i \in \mathbb{Z}^d}\) with period \( 2^n, n \geq 0 \), i.e., \( a_i = a_j \) if \( i - j \in 2^n \mathbb{Z}^d \). Define \( \mathfrak{L} \) to be the closure of \( \mathcal{P} \) in the \( \|\cdot\|_{\infty} \) norm. Then \( \mathfrak{L} \) is a translation invariant Banach algebra. In the following we generalize Pöschel’s results to the polynomial long-range hopping case: There exists some polynomial long-range hopping limit-periodic operator of which the spectrum is pure point and can be either \([0, 1] \), or the standard Cantor set. Precisely, we have

**Corollary 2.6.** Fix \( \alpha_0 > d/2 \) and \( \delta > 0 \). Assume that \( T_\phi \) is given by (2.4) and (2.5) with

\[
s > \alpha_0 + (\log_2 3 + 1/2)d + 12\delta.
\]

Then there is \( \epsilon_0 = \epsilon_0(s, d, \delta, \alpha_0) > 0 \) so that for \( \|T_\phi\|_{s-d/2-\delta} \leq \epsilon_0 \), the following holds true. There exists some \( d' \in \mathfrak{L} \) such that

\[
H' = T_\phi + d'_i \delta_{iW}
\]

has pure point spectrum with a complete set of polynomially localized eigenfunctions \( \{e_k\}_{k \in \mathbb{Z}^d} \) satisfying

\[
|\langle e_k, i \rangle| \leq 2|i - k|^{-s + (\log_2 3 + 1/2)d + 12\delta} \text{ for } i, k \in \mathbb{Z}^d.
\]

Moreover, the spectrum of \( H' \) can be either \([0, 1] \), or be a Cantor set of zero Lebesgue measure.

**Remark 2.4.** We refer to [Avi09, DG10, DG11, DG13] for more results on Schrödinger operators with limit-periodic potentials.
Proof. It needs to construct limit-periodic sequences and then apply Corollary 2.3.

Such sequences were introduced by Pöschel [Pös83]. First, for any $v > 0$, define characteristic function $\chi_{A_v} : \mathbb{Z} \to \{0, 1\}$, where $A_v = \bigcup_{n \in \mathbb{Z}} [n2^v, n2^v + 2^{v-1})$ for even $v$, and $A_v = \bigcup_{n \in \mathbb{Z}} [n2^v + 2^{v-1}, n2^v + 2^v)$ for odd $v$. Then the desired sequences $d'' = (d''_i)_{i \in \mathbb{Z}^d}$ and $d''' = (d'''_i)_{i \in \mathbb{Z}^d}$ are defined by

$$d''_i = \sum_{v=1}^{d} \sum_{u=1}^{d} \chi_{A_v}(i_u)2^{-(v-1)d-u}, \enspace i = (i_1, \ldots, i_d) \in \mathbb{Z}^d,$$

$$d'''_i = 2 \sum_{v=1}^{d} \sum_{u=1}^{d} \chi_{A_v}(i_u)3^{-(v-1)d-u}, \enspace i = (i_1, \ldots, i_d) \in \mathbb{Z}^d.$$ 

Pöschel (see Example 1 and 2 in [Pös83]) has proven that $d'' \in \mathfrak{C}$ is a $(d, 16^{-d})$-distal sequence for $\mathfrak{C}$, and $d'''$ is a $(d \log_2 3, 3^{-d})$-distal one. Moreover, $\{d''_i\}$ is dense in $[0, 1]$, and $\{d'''_i\}$ is dense in the standard Cantor set of real numbers in $[0, 1]$.

Finally, it suffices to apply Corollary 2.3 with $\tau = d \log_2 3, \gamma = 16^{-d}$ and $D = \text{diag}_{i \in \mathbb{Z}^d}(d''_i)$ or $D = \text{diag}_{i \in \mathbb{Z}^d}(d'''_i)$. \hfill \Box

In the following we apply Corollary 2.3 and 2.4 to the quasi-periodic operators case. For $x \in \mathbb{R}, \vec{\omega} = (\omega_1, \cdots, \omega_d) \in \mathbb{R}^d, i \in \mathbb{Z}^d$, define

$$\|x\|_{\mathbb{R}/\mathbb{Z}} = \inf_{k \in \mathbb{Z}} |x - k|, \enspace i \cdot \vec{\omega} = \sum_{v=1}^{d} i_v \omega_v.$$ 

We have

**Corollary 2.7.** Fix $\alpha_0 > d/2, \tau > 0, \gamma > 0$ and $\delta > 0$. Assume that $T_\phi$ is given by (2.4) and (2.5) with

$$s > \alpha_0 + \tau + d/2 + 12\delta.$$ 

Let $(\mathfrak{C}, \| \cdot \|_{\mathfrak{C}})$ be a translation invariant Banach algebra of functions defined on $\mathbb{C}$, and let $\vec{\omega} \in \mathbb{R}^d$. If $f$ is an arbitrary function satisfying

$$(f - \sigma_{1, \vec{\omega}} f)^{-1} \in \mathfrak{C} \text{ and } \| (f - \sigma_{1, \vec{\omega}} f)^{-1} \|_{\mathfrak{C}} \leq \gamma^{-1} |i|^\tau \text{ for all } i \in \mathbb{Z}^d \setminus \{0\} \quad (2.9)$$

with $(\sigma_{1, \vec{\omega}} f)(x) = f(x - i \cdot \vec{\omega})$, then there is some $\epsilon_0 = \epsilon_0(s, d, \delta, \tau, \gamma, \alpha_0) > 0$ so that for $\|T_\phi\|_{s-d/2-\delta} \leq \epsilon_0$, the following holds true. There exists a function $f'$ with $\|f' - f\|_{\mathfrak{C}} \leq C\|T_\phi\|_{s-d/2-\delta}$ such that

$$H' = T_\phi + f'(i \cdot \vec{\omega})\delta_{i \vec{\omega}}$$

has pure point spectrum with a complete set of polynomially localized eigenfunctions $\{\epsilon_k\}_{k \in \mathbb{Z}^d}$ satisfying

$$|\epsilon_k|_i \leq 2|k - j|^{-s + \tau + d/2 + 12\delta} \text{ for } i, j \in \mathbb{Z}^d.$$ 

Moreover, if $f$ is a real-valued function, then spectrum of $H'$ is equal to the closure of the sequence $(f(i \cdot \vec{\omega}))_{i \in \mathbb{Z}^d}$ in $\mathbb{R}$.

If the condition (2.9) holds for all $f'$ with $\|f' - f\|_{\mathfrak{C}} \leq \eta$, then the above statement also holds true with the role of $f, f'$ interchanged.

**Proof.** As done in Pöschel [Pös83], we first define sequence Banach algebra $(\mathfrak{C}, \| \cdot \|_{\mathfrak{C}})$ via the following: Let $\mathfrak{B}$ be the set of all sequence $(a_i)_{i \in \mathbb{Z}^d}$ satisfying $a_i = f(i \cdot \vec{\omega})$ for some $f \in \mathfrak{C}$. Define $\|a\|_{\mathfrak{B}} = \inf_{f \in \mathfrak{C}} f(i \cdot \vec{\omega}) = a_i \|f\|_{\mathfrak{C}}$. Pöschel has proven that
\((\mathfrak{B}; \cdot; \|\cdot\|_\mathfrak{B})\) is a translation invariant Banach algebra. Thus to prove the corollary, it suffices to use Corollary 2.3 and 2.4 in this setting. \(\square\)

In the following we assume \(\tilde{\omega}\) satisfies the Diophantine condition
\[
\|i \cdot \tilde{\omega}\|_{\mathbb{R}/\mathbb{Z}} \geq \frac{\gamma}{|i|^\sigma} \quad \text{for } \forall \ i \in \mathbb{Z}^d \setminus \{0\}. \tag{2.10}
\]

**Example 2.8** (Meromorphic potential, [BLS83, Sar82]). This type of potentials was first introduced by [BLS83], which includes the Maryland potential [GFP82] and Sarnak’s potential [Sar82] as special cases. More precisely, if \(r > 0\), \(\mathcal{H}_r\) denotes the set of period 1 holomorphic functions on \(\mathcal{D}_r = \{z \in \mathbb{C} : |\Im z| < r\}\) satisfying
\[
\|f\|_{\mathcal{H}_r} : = \sup_{z \in \mathcal{D}_r} \left( |f(z)| + \frac{|df(z)|}{dz} \right) < \infty.
\]
Then \((\mathcal{H}_r, \|\cdot\|_{\mathcal{H}_r})\) becomes a translation invariant Banach algebra (see [Pös83]).

Let \(\mathfrak{Y}_r\) denote the set of period 1 meromorphic functions on \(\mathcal{D}_r\) satisfying
\[
f \in \mathfrak{Y}_r \iff \exists \ c = c(r) > 0 \ s.t., \ \inf_{x \in \mathbb{R}, \ z \in \mathcal{D}_r} |f(z) - f(z + x)| \geq c|x|_{\mathbb{R}/\mathbb{Z}}.
\]
We have

**Lemma 2.9** ([BLS83]). Fix \(r > 0\) and let \(\tilde{\omega}\) satisfy (2.10). If \(f \in \mathfrak{Y}_r, \ 0 < r_1 < r\), then there are \(C > 0, \eta > 0\) so that for every \(g \in \mathcal{H}_r\) with \(\|g\|_{\mathcal{H}_r} \leq \eta\), one has
\[
f_1 = f + g \in \mathfrak{Y}_{r-r_1}\text{ and } \quad \|f_1 - \sigma_{i\cdot\tilde{\omega}} f_1\|_{\mathcal{H}_{r-r_1}} \leq C\eta^{-1}|i|^\sigma \quad \text{for } \forall \ i \neq 0.
\]

Moreover, the functions \(\tan(\pi z), \ \exp(2\pi \sqrt{-1} z) \in \mathfrak{Y}_r\) for any \(r > 0\).

In this example we deal with
\[
H = T_\phi + f(i \cdot \tilde{\omega})\delta_W, \quad f \in \mathfrak{Y}_r \quad (r > 0). \tag{2.11}
\]

Combining Corollary 2.7 and Lemma 2.9 yields

**Corollary 2.10.** Fix \(\alpha_0 > d/2, \tau > 0, \gamma > 0\) and \(\delta > 0\). Assume that \(T_\phi\) is given by (2.4) and (2.5) with
\[
s > \alpha_0 + \tau + d/2 + 12\delta.
\]
Let \(\tilde{\omega}\) satisfy (2.10), and let \(H\) be defined by (2.11). Then there is some \(\epsilon_0 = \epsilon_0(f, r, s, d, \delta, \tau, \gamma, \alpha_0) > 0\) so that for \(\|T_\phi\|_{s-d/2-\delta} \leq \epsilon_0\), \(H\) has pure point spectrum with a complete set of polynomially localized eigenfunctions \(\{e_k\}_{k \in \mathbb{Z}^d}\) satisfying
\[
|\langle e_k | i \rangle| \leq 2(i - k)^{-s+\tau+d/2+12\delta} \quad \text{for } i, k \in \mathbb{Z}^d.
\]
In particular, the above statements hold true for
\[
H = T_\phi + \tan(\pi i \cdot \tilde{\omega})\delta_W
\]
and
\[
H = T_\phi + \exp(2\pi \sqrt{-1} i \cdot \tilde{\omega})\delta_W.
\]

**Remark 2.5.** (1). The Maryland type potential \(\tan(i \cdot \tilde{\omega})\) was first introduced in [GFP82]. It is well-known that the Maryland type models (even with an exponential long-range hopping) are solvable ones [GFP82, FP84, Sim85]. We refer to [JL17] for an almost complete description of the spectral types for the standard Maryland model (i.e., the 1D Schrödinger operator with the Maryland potential).
Very recently, a Rayleigh-Schrödinger perturbation type method was developed by [KPS20] to prove Anderson localization for some quasi-periodic operators with Maryland type potentials.

(2). The Schrödinger operator with the potential \(\exp(2\pi\sqrt{-1} \cdot \vec{\omega})\) was first introduced in [Sar82], and is a typical non-normal operator. While the spectral theorem is not available for the non-normal operator, we can still define pure point spectrum for it: We say an arbitrary operator on \(L^2(\mathbb{Z}^d)\) is of pure point spectrum if it has a complete set of eigenfunctions.

Example 2.11 (Discontinuous potential, [Cra83]). Let \(\mathcal{I}\) denote the space of all functions of period 1 and bounded variation with the norm

\[ \|f\|_3 = \sup_{x \in \mathbb{R}} |f(x)| + \|f\|_{BV}, \]

where \(\| \cdot \|_{BV}\) denotes the standard total variation. Then \((\mathcal{I}, \| \cdot \|_3)\) is a translation invariant Banach algebra.

Obviously, a typical example of such a potential is \(f(x) = x \mod 1\) which has been previously studied by Craig [Cra83].

In this example we study

\[ T_\phi + (i \cdot \vec{\omega}) \mod 1)\delta_{i'}. \]

Lemma 2.12 ([Pös83]). Let \(\vec{\omega}\) satisfy (2.10), and let \(f(x) = x \mod 1\). Then

\[ (f - \sigma_i \cdot \vec{\omega})^{-1} \in \mathcal{I}, \quad \|(f - \sigma_i \cdot \vec{\omega})^{-1}\|_3 \leq \gamma^{-1}|i|^r \text{ for } i \neq 0. \]

Combining Corollary 2.7 and Lemma 2.12 yields

Corollary 2.13. Fix \(\alpha_0 > d/2, \tau > 0, \gamma > 0\) and \(\delta > 0\). Assume that \(T_\phi\) is given by (2.4) and (2.5) with

\[ s > \alpha_0 + \tau + d/2 + 12\delta. \]

Let \(\vec{\omega}\) satisfy (2.10). Then there is \(\epsilon_0 = \epsilon_0(s, d, \delta, \tau, \gamma, \alpha_0) > 0\) so that for \(\|T_\phi\|_{s-d/2-\delta} \leq \epsilon_0\), the following holds true. There exists some \(d' \in \mathcal{I}\) with

\[ \|(x \mod 1) - d'\|_3 \leq C\|T_\phi\|_{s-d/2-\delta} \]

such that

\[ H' = T_\phi + d'(i \cdot \vec{\omega})\delta_{i'} \]

has pure point spectrum with a complete set of polynomially localized eigenfunctions \(\{e_k\}_{k \in \mathbb{Z}^d}\) satisfying

\[ |(e_k)|_i \leq 2(i - k)^{-s+\tau+d/2+12\delta} \text{ for } i, k \in \mathbb{Z}^d. \]

Moreover, the point spectrum of \(H'\) is just the set \(\{(i \cdot \vec{\omega}) \mod 1 : i \in \mathbb{Z}^d\}\).

Remark 2.6. A natural generation of the potential \(x \mod 1\) is the so called Lipschitz monotone potential, which was introduced recently by [JK19]. In [JK19], the authors proved all couplings localization for the 1D quasi-periodic Schrödinger operator with a Lipschitz monotone potential.
3. Preliminaries

3.1. The notation.
- Denote by $I$ the identity operator on $\mathcal{M}$.
- For two operators $X, Y$, we define $[X, Y] = XY - YX$.
- By $C = C(f) > 0$, we mean the constant $C$ depends only on $f$.
- Typically, we write $x \in \mathbb{R}^d$ for a vector, and $x \in \mathbb{R}$ for a scalar. If $k \in \mathbb{Z}^d$, then let
  \[ |k| = \max_{1 \leq v \leq d} |k_v|, \quad (k) = \max\{1, |k|\}. \]
- By $\delta_{ij}$ we denote the Kronecker delta. Namely, $\delta_{ii} = 1$, and $\delta_{ij} = 0$ if $i \neq j$.
- By $\text{diag}_{i \in \mathbb{Z}^d}(d_i)$ we denote the diagonal matrix whose diagonal elements are $(d_i)_{i \in \mathbb{Z}^d}$.
- For any $A = (a_{ij})_{i,j \in \mathbb{Z}^d}$, let $\overline{A} = \text{diag}_{i \in \mathbb{Z}^d}(a_{ii})$ be the main diagonal part of $A$.
- By $0 \leq x \ll y$ we mean there is a small $c > 0$ so that $x \leq cy$.
- Throughout the paper $\alpha_0 > d/2, \gamma > 0$ are fixed.

3.2. Tame property. The Sobolev norm defined by (2.2) has the following important tame property (which is also called the interpolation property). Such tame property is not shared by the exponential one (see [Pös83]).

Lemma 3.1. Fix $\alpha_0 > d/2$. Then for any $s \geq \alpha_0$ and $X, Y \in \mathcal{M}^s$, we have
\[
\|XY\|_s \leq K_0 \|X\|_{\alpha_0} \|Y\|_s + K_1 \|X\|_s \|Y\|_{\alpha_0},
\]
where
\[
K_0 = \sqrt{20 \sum_{k \in \mathbb{Z}^d} (k)^{-2\alpha_0}}, \quad (3.2)
\]
\[
K_1(s) = (1 - 10^{-\frac{s}{2}})^{-s} \sqrt{2 \sum_{k \in \mathbb{Z}^d} (k)^{-2\alpha_0}}. \quad (3.3)
\]

Remark 3.1. Obviously, $K_1(\alpha_0, s)$ is bounded for $s$ is a bounded interval. This Sobolev type norm was previously introduced by Berti-Bolle [BB13] to prove the existence of finitely smoothing quasi-periodic solutions for some nonlinear Schrödinger equations. Recently, such norm was also used by Shi [Shi21] to give a multi-scale proof of power-law localization for some random operators.

Proof. For a detailed proof, we refer to the appendix.

3.3. Smoothing operator. The smoothing operator plays an essential role in the Nash-Moser iteration scheme. In the present context we have

Definition 3.2. Fix $\theta > 0$. Define the smoothing operator $S_\theta : \mathcal{M} \to \mathcal{M}^\infty$ by
\[
(S_\theta X)_{i,j} = X_{i,j} \text{ for } |i - j| \leq \theta,
\]
\[
(S_\theta X)_{i,j} = 0 \text{ for } |i - j| > \theta.
\]
Lemma 3.3. Fix $\theta > 0$. Then for $X \in M^{s'}$, we have
\begin{align*}
\|S_\theta X\|_s &\leq \theta^{s-s'} \|X\|_{s'} \quad \text{for } s \geq s' \geq 0, \\
\|(I - S_\theta)X\|_s &\leq \theta^{s-s'} \|X\|_{s'} \quad \text{for } 0 \leq s \leq s'.
\end{align*}

Proof. Recalling the Definition 3.2 and (2.2), the proof is trivial. \hfill \Box

4. The Nash-Moser scheme

The main scheme is to find a sequence of invertible operators $Q_0, \ldots, Q_k, \ldots$ such that
\[ Q_k^{-1} \left( \sum_{l=1}^{k} T_{l-1} + D + \sum_{l=1}^{k} D_{l-1} \right) Q_k = D + R_k, \]
where
\[ T_l = (S_{\theta_l} - S_{\theta_{l-1}}) T, \quad \theta_l = \theta_0 \Theta^l, \]
where $\theta_0, \Theta > 1$ will be specified later. We will show $\sum_{l=1}^{k} T_{l-1} \to T, Q_k \to Q$, $\sum_{l=1}^{k} D_{l-1} \to D$ as $k \to \infty$ in the $s$-norm for some $s > 0$.

The key difficulty in the above scheme is the so called loss of derivatives when we determine $Q_k$. Since we are in the Sobolev case, we have a $\tau$-order loss of derivative at each iteration step. Such a loss is not presented in the analytic case (the loss in the analytic norm is of order $\delta$ for arbitrary $\delta > 0$, cf. [Pös83, BLS83]).

4.1. The iteration step. We first deal with the iteration step. We will work with $s$-Sobolev norms with
\[ s \in [\alpha_0, \alpha_1], \]
where $\alpha_0 > d/2$ is fixed and $\alpha_1$ will be specified later.

We also assume\(^1\)
\[ \theta_l = \theta_0 \Theta^l, \quad T_l = (S_{\theta_l} - S_{\theta_{l-1}}) T \quad (l \geq 1). \tag{4.1} \]
where $\theta_0 > 1$, $\Theta > 1$ will be specified later. We have the following iteration proposition.

Proposition 4.1. Fix $\alpha_0 > d/2, \tau > 0, \gamma > 0, \delta > 0$. Assume that
\[ \alpha > \tau + \alpha_0 + 7\delta, \quad \alpha_1 \geq 2\alpha + \delta, \]
Assume further that
\[ \|T\|_{\alpha+4\delta} \leq 1, \quad \Theta \geq \max\{8^{2/\delta}(K_0 + K_1(\alpha_0))^{4/\delta}, 10^{1/\delta}, 10^{1/\alpha_0}\}. \]
If for $1 \leq l \leq k$, there exist $W_l, R_l, Q_l \in M$ and $D_{l-1} \in M_0^\infty$ so that
\[ Q_l^{-1} H_l Q_l = D + R_l, \quad H_l = \sum_{i=1}^{l} (T_{i-1} + D_{i-1}) + D \]
\[^1\text{The exponential scale of smoothing operator was first introduced by Klainerman [Kla80].}\]
with

\[ Q_0 = I, D_0 = 0, T_0 = S_{\theta_0}T, \]
\[ Q_l = Q_{l-1}V_l, V_l = I + W_l, \]
\[ \|W_l\| \leq \theta_{l-1}^{\tau_0 - \Theta + 4\delta} \text{ for } s \in [\alpha_0, \alpha_1], \]
\[ \|V_l^{-1} - I\| \leq 2K_1(s)\theta_{l-1}^{\tau_0 - \Theta + 4\delta} \text{ for } s \in [\alpha_0, \alpha_1], \]
\[ \|R_l\| \leq \theta_{l-1}^{\alpha_0 - \alpha} \text{ for } s \in [\alpha_0, \alpha_1], \]
\[ \|D_{l-1}\| \leq 3\theta_{l-1}^{\alpha_0 - \alpha}, \]

where \( K_1(s), T_l \) are defined by (3.3) and (4.1) respectively. Then there is some \( C = C(\delta, \Theta, \tau, \gamma, \alpha, \alpha_0, \alpha_1) > 0 \) such that, for \( \theta_0 \geq C \), there exist \( W_{k+1}, R_{k+1}, Q_{k+1} \in \mathcal{M} \) and \( D_k \in \mathcal{M}_0^\infty \) so that

\[ Q_{k+1}^{-1}H_{k+1}Q_{k+1} = D + R_{k+1}, \]
\[ H_{k+1} = H_k + T_k + D_k \]

(4.2)

We first do some formal computations. Note that for \( 1 \leq l \leq k \), all the \( W_l, R_l, Q_l, D_{l-1} \) have been constructed so that

\[ Q_k^{-1}H_kQ_k = D + R_k. \]

We try to determine \( W_{k+1}, R_{k+1}, D_k \) so that (4.2) holds. For this purpose, we set

\[ V_{k+1}^{-1}Q_k^{-1}(H_k + T_k + D_k)Q_kV_{k+1} \]
\[ = V_{k+1}^{-1}Q_k^{-1}H_kQ_kV_{k+1} + V_{k+1}^{-1}Q_k^{-1}(T_k + D_k)Q_kV_{k+1} \]
\[ = V_{k+1}^{-1}(D + R_k)V_{k+1} + V_{k+1}^{-1}Q_k^{-1}(T_k + D_k)Q_kV_{k+1} + V_{k+1}^{-1}Q_k^{-1}(D + R_k)V_{k+1}. \]

We hope that \( V_{k+1} = I + W_{k+1} \) and \( W_{k+1} \) would be of order \( O(\|R_k\|_s) \). Then

\[ V_{k+1}^{-1}(D + R_k)V_{k+1} = D + [D, W_{k+1}] + R_k + R_{(1)}, \]

where

\[ R_{(1)} = V_{k+1}^{-1}(D + R_k)V_{k+1} - D - [D, W_{k+1}] - R_k \]
\[ = V_{k+1}^{-1}DV_{k+1} - D - [D, W_{k+1}] + V_{k+1}^{-1}R_kV_{k+1} - R_k \]
\[ = (V_{k+1}^{-1} - I)[D, W_{k+1}] + (V_{k+1}^{-1} - I)R_kW_{k+1} \]
\[ + (V_{k+1}^{-1} - I)R_k + R_kW_{k+1}, \]

(4.3)

Then using the decomposition of \( T \) (into \( T_l \)) may yield \( \|T_k\|_s = O(\|R_k\|_s) \). As a result, we have

\[ V_{k+1}^{-1}Q_k^{-1}(T_k + D_k)Q_kV_{k+1} = Q_k^{-1}(T_k + D_k)Q_k + R_{(2)}, \]

(4.4)
Proof. We first show that for all $n \geq 1$, $s \geq \alpha_0$, we have

$$
\|X^n\|_{\alpha_0} \leq C_0^{n-1}\|X\|_{\alpha_0}^n,
$$

(4.12)

and

$$
\|X^n\|_s \leq n^2C_0^nK_1(s)\|X\|_{\alpha_0}^{n-1}\|X\|_s.
$$

(4.13)

Proof. We first show that for all $n \geq 1$, (4.10) holds true. We prove it by induction. Using Lemma 3.1 with $s = \alpha_0$ yields for $n = 2$,

$$
\|X_1X_2\|_{\alpha_0} \leq (K_0 + K_1(\alpha_0))\|X_1\|_{\alpha_0}\|X_2\|_{\alpha_0}
= C_0\|X_1\|_{\alpha_0}\|X_2\|_{\alpha_0}.
$$
Assume that
\[
\| \prod_{i=1}^{n} X_i \|_{\alpha_0} \leq C_0^{n-1} \prod_{i=1}^{n} \| X_i \|_{\alpha_0},
\]
(4.14)

Then by using Lemma 3.1 again, we obtain since (4.14)
\[
\| \prod_{i=1}^{n+1} X_i \|_{\alpha_0} \leq K_0 \| X_{n+1} \|_{\alpha_0} \| \prod_{i=1}^{n} X_i \|_{\alpha_0} + K_1(\alpha_0) \| \prod_{i=1}^{n} X_i \|_{\alpha_0} \| X_{n+1} \|_{\alpha_0}
\]
\[
\leq C_0 \| \prod_{i=1}^{n} X_i \|_{\alpha_0} \| X_{n+1} \|_{\alpha_0}
\]
\[
\leq C_0^n \prod_{i=1}^{n+1} \| X_i \|_{\alpha_0}.
\]

This proves (4.10) and then (4.12).

Next we try to prove (4.11). We also prove it by induction. For \( n = 2 \), we have by Lemma 3.1, \( K_0 > 1, K_1 > 1 \) that
\[
\| X_1 X_2 \|_s \leq C_0 K_1(s) (\| X_1 \|_{\alpha_0} \| X_2 \|_s + \| X_1 \|_s \| X_2 \|_{\alpha_0}).
\]

Assume that
\[
\| \prod_{i=1}^{n} X_i \|_s \leq nC_0^m K_1(s) \sum_{i=1}^{n} \left( \prod_{j \neq i} \| X_j \|_{\alpha_0} \right) \| X_i \|_s.
\]
(4.15)

Then by Lemma 3.1, (4.10) and (4.15), we get
\[
\| \prod_{i=1}^{n+1} X_i \|_s \leq K_0 \| X_1 \|_{\alpha_0} \| \prod_{i=2}^{n+1} X_i \|_s + K_1(\alpha_0) \| X_1 \|_s \| \prod_{i=2}^{n+1} X_i \|_{\alpha_0}
\]
\[
\leq nK_0 C_0^n K_1(s) \sum_{i=2}^{n+1} \left( \prod_{2 \leq j \neq i \leq n+1} \| X_j \|_{\alpha_0} \right) \| X_i \|_s \| X_1 \|_{\alpha_0}
\]
\[
+ C_0^{n-1} K_1(s) \prod_{i=2}^{n+1} \| X_i \|_{\alpha_0} \| X_1 \|_s
\]
\[
\leq (nC_0^n K_1(s) + C_0^{n-1} K_1(s)) \sum_{i=1}^{n+1} \left( \prod_{1 \leq j \neq i \leq n+1} \| X_j \|_{\alpha_0} \right) \| X_i \|_s
\]
\[
\leq (n + 1)C_0^{n+1} K_1(s) \sum_{i=1}^{n+1} \left( \prod_{1 \leq j \neq i \leq n+1} \| X_j \|_{\alpha_0} \right) \| X_i \|_s.
\]

Obviously, (4.13) is a direct corollary of (4.11).

The proof is finished. \( \square \)

Now we are ready to estimate \( Q_l^{-1}, Q_l \) (1 \( \leq l \leq k \))

**Lemma 4.3.** Assume that
\[
\Theta^{\delta/2} \geq 8C_0^2,
\]
(4.16)
\[
\alpha > \alpha_0 + \tau + 5\delta,
\]
(4.17)
where $C_0$ is given by (4.9). Then there exists
\[ C = C(\delta, \tau, \alpha_0, \alpha_1) > 0 \]
such that the following holds: If $\theta_0 \geq C$, then for all $1 \leq l \leq k$, we have
\[ \|Q_l^{-1}\|_s \leq \theta_{l-1}^{(s-\alpha+\tau+4\delta)+\delta} \text{ for } s \in [\alpha_0, \alpha_1], \]  
(4.18)
\[ \|Q_l\|_s \leq \theta_{l-1}^{(s-\alpha+\tau+4\delta)+\delta} \text{ for } s \in [\alpha_0, \alpha_1], \]  
(4.19)
\[ \|Q_l^{-1} - Q_{l-1}^{-1}\|_s \leq \theta_{l-1}^{s-\alpha+\tau+6\delta} \text{ for } s \in [\alpha_0, \alpha_1], \]  
(4.20)
\[ \|Q_l - Q_{l-1}\|_s \leq \theta_{l-1}^{s-\alpha+\tau+6\delta} \text{ for } s \in [\alpha_0, \alpha_1], \]  
(4.21)
where $x_+ = x$ if $x \geq 0$ and $x_+ = 0$ if $x < 0$.

Proof. By induction assumptions, we have for $1 \leq l \leq k$,
\[ \|V_l^{-1}\|_s \leq 1 + 2K_1(s)\theta_{l-1}^{s-\alpha+\tau+4\delta}, \quad \|V_l^{-1} - I\|_s \leq 2K_1(s)\theta_{l-1}^{s-\alpha+\tau+4\delta}, \]  
(4.22)
which together with (4.11) yields
\[ \|Q_l^{-1}\|_s \leq \prod_{i=1}^l \|V_i^{-1}\|_s \leq lC_0^lK_1(s) \sum_{i=1}^l \left( \prod_{j \neq i} \|V_j^{-1}\|_s \right) \|V_i^{-1}\|_s \]
\[ \leq lC_0^l\theta_{l-1}^{s-\alpha+\tau+4\delta}, \]  
(4.23)
Let
\[ 1 < C_0 \leq \theta_0^{\delta/2}. \]
Since (4.16), we have for any $l \geq 2$,
\[ l^2C_0^l2^{l-1} = (l^2\theta_0^{\delta/2})l^{-1} \leq (8C_0^2)^{l-1} \]
\[ \leq (\theta_0^{\delta/2}2^{\delta/2})l^{-1} = \theta_l^{\delta/2}. \]  
(4.24)
Since (4.23) and (4.24), we can obtain for
\[ \alpha_0 \leq s < \alpha - \tau - 4\delta, \]
that
\[ \|Q_l^{-1}\|_s \leq l^2C_0^l2^{l-1}K_1(s)(1 + 2K_1(s)\theta_0^{s-\alpha+\tau+4\delta}) \]
\[ \leq l^2C_0^l2^{l-1}K_1(s)(1 + 2K_1(s)) \]
\[ \leq K_1(s)(1 + 2K_1(s))\theta_{l-1}^{\delta/2} \]
\[ \leq K_1(s)(1 + 2K_1(s))\theta_{l-1}^{\delta/2}\theta_{l-1}^{\delta/2} \]
\[ \leq \theta_{l-1}^{\delta/2} \text{ (if } \theta_0 \geq C(\delta, \tau, \alpha_0, \alpha_1) > 0). \]  
(4.25)
Similarly, for
\[ \alpha - \tau - 4\delta \leq s \leq \alpha_1, \]
we obtain
\[
\|Q_l^{-1}\|_s \leq l^2 C_0^l 2^{l-1} K_1(s)(1 + 2 K_1(s) \theta_l^{-\alpha + \tau + 4\delta})
\leq 3 K_1^2(s) \theta_l^{-\alpha + \tau + 4\delta}
\leq 3 K_1^2(s) \theta_0^{-\alpha + \tau + 5\delta}
\leq \theta_l^{-\alpha + \tau + 5\delta} \text{ (if } \theta_0 \geq C(\delta, \tau, \alpha_0, \alpha_1) > 0). \tag{4.26}
\]
Combining (4.25) and (4.26) implies (4.18). The estimate of \(Q_l\) is similar and easier.

We now turn to the differences. Note that
\[
\text{Estimate of } D
\]
This finishes the proof.

Finally, if \(\theta_0 \geq C(\delta, \tau, \alpha_0, \alpha_1) > 0\), then
\[
\|Q_l^{-1} - Q_{l-1}^{-1}\|_s = \|(I + W_l)^{-1} - I\|_s
\leq 2 K_1(s) \theta_0^{s - \alpha + \tau + 4\delta}
\leq \theta_0^{s - \alpha + \tau + 6\delta},
\]
which implies (4.20). The estimate of \(Q_l - Q_{l-1}\) (i.e., (4.21)) is similar and easier.

This finishes the proof. \(\square\)

**Estimate of \(D_k\)**

Note that if \(X \in \mathcal{M}_0^\infty\), then \(\|X\|_s = \|X\|_0\) for any \(s \geq 0\). Obviously, \(\mathcal{M}_0^\infty\) is a Banach space in the \(\|\cdot\|_{\alpha_0}\)-norm.

**Lemma 4.4.** Let \(Q, Q^{-1} \in \mathcal{M}^\alpha_0\) satisfy
\[
C_0\|Q - I\|_{\alpha_0} \leq \frac{1}{10}, \quad C_0\|Q^{-1} - I\|_{\alpha_0} \leq \frac{1}{10}, \tag{4.27}
\]
where \(C_0\) is given by (4.9). Then for any \(P, P' \in \mathcal{M}^\alpha_0\), the equation
\[
Q^{-1}(X + P)Q + P' = 0 \tag{4.28}
\]
has a unique solution \(X \in \mathcal{M}_0^\infty\) with
\[
\|X\|_{\alpha_0} \leq 2(\|Q^{-1}PQ\|_{\alpha_0} + \|P'\|_{\alpha_0}). \tag{4.29}
\]
Proof. The proof is based on the Banach fixed point theorem. It is easy to see (4.28) is equivalent to
\[ X - (Q^{-1}(X + P)Q + P') = X. \] (4.30)

Define a map \( f : \mathcal{M}_0^\infty \to \mathcal{M}_0^\infty \) given by
\[ f(X) = X - (Q^{-1}(X + P)Q + P'). \]

To prove the existence and uniqueness of the solution of (4.30), it suffices to show \( f \) is a contractive map. Obviously, we have
\[ Y - Q^{-1}YQ = -(Q^{-1} - I)Y(Q - I) - (Q^{-1} - I)Y - Y(Q - I). \] (4.31)

Let \( X', X'' \in \mathcal{M}_0^\infty \) be arbitrary and \( Y = X' - X'' \). This combining with (4.31) and Lemma 4.5 implies
\[
\|f(X') - f(X'')\|_{\alpha_0} = \|\left[(Q^{-1} - I)Y(Q - I) + (Q^{-1} - I)Y + Y(Q - I)\right]\|_{\alpha_0}
\leq \|\left[(Q^{-1} - I)Y(Q - I) + (Q^{-1} - I)Y + Y(Q - I)\right]\|_{\alpha_0}
\leq C_0\|Q - I\|_{\alpha_0}\|Q^{-1} - I\|_{\alpha_0}\|Y\|_{\alpha_0}
+ C_0\|Q - I\|_{\alpha_0}\|Y\|_{\alpha_0} + C_0\|Q^{-1} - I\|_{\alpha_0}\|Y\|_{\alpha_0}
\leq \frac{1}{2}\|Y\|_{\alpha_0} = \frac{1}{2}\|X' - X''\|_{\alpha_0} \text{ (since (4.27))}. \]

We have shown that \( f \) is a contractive map, and thus the existence and uniqueness of \( X \).

Now we estimate the solution \( X \). From (4.31) and (4.27) and Lemma 3.1, we obtain
\[
\|X\|_{\alpha_0} = \|X - Q^{-1}XQ - Q^{-1}PQ - P'\|_{\alpha_0}
\leq \|(Q^{-1} - I)X(Q - I) + (Q^{-1} - I)X + X(Q - I)\|_{\alpha_0}
\leq C_0\|Q - I\|_{\alpha_0}\|Q^{-1} - I\|_{\alpha_0}\|X\|_{\alpha_0}
+ C_0\|Q^{-1}PQ\|_{\alpha_0} + C_0\|P'\|_{\alpha_0},
\]
which implies (4.29).

We finish the proof. \( \square \)

Next, we estimate the decomposition \( T_k = (S_{\theta_k} - S_{\theta_{k-1}})T \).

Lemma 4.5. We have that
\[ \|T_k\|_s \leq \theta_k^{s-s'}\|T\|_{s'} \text{ for } s \geq s' \geq 0, \] (4.32)
\[ \|T_k\|_s \leq \theta_k^{s-s'}\|T\|_{s'} \text{ for } \alpha_0 \leq s \leq s'. \] (4.33)

In particular, if
\[ \|T\|_{s'+\delta} \leq 1, \theta_0 \geq \Theta^{(s'+\delta-\alpha_0)/\delta}, \] (4.34)
then
\[ \|T_k\|_s \leq \theta_k^{s-s'-\delta} \text{ for } s \geq s' + \delta, \] (4.35)
\[ \|T_k\|_s \leq \theta_k^{s-s'} \text{ for } s \in [\alpha_0, s' + \delta). \] (4.36)
Proof. The proof of (4.32) and (4.33) is trivial and relies on properties of smoothing operators (i.e., (3.4) and (3.5) of Lemma 3.3). Now we turn to the proof of (4.35) and (4.36). Note that \(|T| \leq 1 + \epsilon\), which implies (4.35). While for \(\alpha_0 \leq s < s' + \delta\), we obtain
\[
\|T_k\|_s \leq \theta_{k-1}^{s-s'-\delta} = \Theta^{s'-s} \theta_{k-1}^{s'-s} \\
\leq \theta_{k-1}^{s-s'} + \alpha_0 \theta_{k-1}^{s'-s'} \\
\leq \theta_{k-1}^{s-s'},
\]
where in the last inequality we use (4.34).

Now we estimate \(D_k\).

Lemma 4.6. Let
\[
\kappa = -\alpha + \alpha_0 + \tau + 6\delta < 0, \quad (4.37) \\
\|T\|_{\alpha + 4\delta} \leq 1, \quad (4.38) \\
\theta_0^\delta \geq C^{\alpha + 4\delta - \alpha_0}. \quad (4.39)
\]
Then there is a \(C = C(\kappa, \delta, \alpha_0) > 0\) such that if \(\theta_0 \geq C\), the homological equation (4.6) admits a unique solution \(D_k \in M_{\infty}\) satisfying
\[
\|D_k\|_{\alpha_0} \leq 3\theta_k^{\alpha_0 - \alpha}.
\]

Proof. The proof is based on Lemma 4.3 and 4.4. We set \(P = T_k, P' = R_k\) and \(Q = Q_k\) in Lemma 4.4. It suffices to check the conditions (4.27). From Lemma 4.3, we have since (4.20) and (4.16)
\[
\|Q_k^{-1} - I\|_{\alpha_0} \leq \sum_{i=1}^{k} \theta_{i-1}^{\kappa} = \theta_0^{\kappa} \sum_{i=1}^{k} \Theta^{(i-1)\kappa} \\
\leq \theta_0^{\kappa} \sum_{i=1}^{\infty} (8C_0^{2})^{2(i-1)} \\
\leq \theta_0^{\kappa} C(\kappa, \delta, \alpha_0) \leq \frac{1}{10C_0},
\]
where in the last inequality we use \(\theta_0 \geq C(\delta, \kappa, \alpha_0) > 0\). Similarly, we obtain \(|Q_k - I|_{\alpha_0} \leq \frac{1}{10C_0}\). Thus the conditions of Lemma 4.4 are satisfied.

Now let \(s' = \alpha + 3\delta\) in (4.36) of Lemma 4.5. We have since (4.38) and Lemma 4.5
\[
\|P\|_{s} = \|T_k\|_{s} \leq \theta_k^{s-\alpha-3\delta} \text{ for } s \in [\alpha_0, \alpha + 4\delta],
\]
which together with Lemma 4.4 implies
\[
\|D_k\|_{\alpha_0} \leq 2(\|Q_k^{-1}PQ_k\|_{\alpha_0} + \|P'\|_{\alpha_0}) \\
\leq 2C_0^{2}\|Q_k\|_{\alpha_0} \|Q_k^{-1}\|_{\alpha_0} \|P\|_{\alpha_0} + 2\theta_k^{\alpha_0 - \alpha} \\
\leq 2C_0^{2}\theta_k^{\delta} \|P\|_{\alpha_0} + 2\theta_k^{\alpha_0 - \alpha} \\
\leq 4C_0^{2}\theta_k^{\delta} \theta_k^{\alpha_0 - \alpha - 3\delta} + 2\theta_k^{\alpha_0 - \alpha} \\
= 4C_0^{2}\theta_0^{\delta} \theta_k^{\alpha_0 - \alpha} + 2\theta_k^{\alpha_0 - \alpha} \leq 3\theta_k^{\alpha_0 - \alpha},
\]
where in the last inequality we use \(\theta_0 \geq C(\delta, \alpha_0) > 0\).
Estimate of $W_{k+1}$

Now we try to solve the homological equation (4.4). The loss of derivatives appears in this step. We first establish some useful estimates.

**Lemma 4.7.** Under the assumptions of Lemma 4.6 and assuming further
\[
\kappa_1 = \alpha_0 - \alpha + \tau + 7\delta < 0,
\] (4.40)
then we have
\[
\|Q_k^{-1}T_kQ_k\|_s \leq \theta_k^{s-\alpha} \text{ for } s \in [\alpha_0, \alpha_1],
\] (4.41)
\[
\|Q_k^{-1}D_kQ_k\|_s \leq \theta_k^{\alpha_0-\alpha+3\delta} \text{ for } s \in [\alpha_0, \alpha - \tau - 4\delta],
\] (4.42)
\[
\|Q_k^{-1}D_kQ_k\|_s \leq \theta_k^{s-\alpha} \text{ for } s \in [\alpha - \tau - 4\delta, \alpha_1].
\] (4.43)

**Proof.** From (4.11), (4.35) and (4.36), we obtain for $s \in [\alpha_0, \alpha_1],$
\[
\|Q_k^{-1}T_kQ_k\|_s \leq 3C_0^3\|Q_k^{-1}\|_{\alpha_0}\|Q_k^{-1}\|_{\alpha_0}\|T_k\|_s
+ 3C_0^3\|Q_k\|_{\alpha_0}\|T_k\|_s\|Q_k^{-1}\|_s
+ 3C_0^3\|Q_k^{-1}\|_{\alpha_0}\|T_k\|_s\|Q_k\|_s
\leq 3C_0^3\theta_k^{s-\alpha-3\delta}\tau_k\theta_k^{-1} + 6C_0^3\theta_k^{s-\alpha-3\delta}(s-\alpha + \tau + 4\delta)_+ + 2\delta
\leq 3C_0^3\theta_k^{s-\alpha-\delta} + 6C_0^3\theta_k^{s-\alpha-\delta} + 6C_0^3\theta_k^{s-\alpha-\delta} (\text{since } \kappa_1 < 0)
\leq 9C_0^3\theta_k^{-\delta}\theta_k^{s-\alpha} \leq \theta_k^{s-\alpha} \text{ (if } \theta_0 \geq C(\delta, \alpha_0) > 0),
\]
which implies (4.41). Similarly, we obtain
\[
\|Q_k^{-1}D_kQ_k\|_s \leq 27C_0^3\theta_k^{-\delta}\theta_k^{\alpha_0-\alpha+3\delta} \leq \theta_k^{\alpha_0-\alpha+3\delta} \text{ if } \theta_0 \geq C(\delta, \alpha_0) > 0.
\]
Then if $\alpha_0 \leq s < \alpha - \tau - 4\delta,$ we obtain since (4.44)
\[
\|Q_k^{-1}D_kQ_k\|_s \leq 27C_0^3\theta_k^{-\delta}\theta_k^{\alpha_0-\alpha+3\delta} \leq \theta_k^{\alpha_0-\alpha+3\delta} \text{ if } \theta_0 \geq C(\delta, \alpha_0) > 0.
\]
If $\alpha - \tau - 4\delta \leq s \leq \alpha_1,$ then
\[
\alpha_0 - \alpha + 2\delta = s - \alpha + (-s + \alpha_0 + 2\delta)
\leq s - \alpha + (\alpha_0 - \alpha + \tau + 6\delta)
\leq s - \alpha - \delta,
\] (4.45)
\[
(s - \alpha + \tau + 4\delta)_+ + \alpha_0 - \alpha + 2\delta = s - \alpha + (\alpha_0 - \alpha + \tau + 6\delta)
\leq s - \alpha - \delta,
\] (4.46)
where in (4.45) and (4.46) we use the fact $\kappa_1 < 0.$ Hence if $\alpha - \tau - 4\delta \leq s \leq \alpha_1,$ we have
\[
\|Q_k^{-1}D_kQ_k\|_s \leq 27C_0^3\theta_k^{-\delta}\theta_k^{\alpha_0-\alpha} \leq \theta_k^{\alpha_0-\alpha} \text{ if } \theta_0 \geq C(\delta, \alpha_0) > 0.
\]
This proves (4.42) and (4.43).
We are ready to solve the homological equation \((4.7)\). Write
\[
G = Q_k^{-1}(T_k + D_k)Q_k + R_k. 
\] (4.47)
Since Lemma 4.6, we have \(\overline{G} = 0\). By Lemma 4.7, we obtain
\[
\|G\|_s \leq 3\theta_k^{-\alpha+3\delta} \text{ for } s \in [\alpha_0, \alpha_1]. 
\] (4.48)

**Lemma 4.8.** The homological equation \((4.7)\) admits a unique solution \(W = W_{k+1}\) which is given by
\[
W_{i,j} = \frac{(S_{\theta_k+1}G)_{i,j}}{d_j - d_i} \text{ for } i \neq j, \quad W_{k+1} = 0. 
\] (4.49)

Under the assumptions of Lemma 4.7 and assuming further
\[
\theta_k^6 \geq 3\gamma^{-1}\Theta^\tau, 
\] (4.50)
then
\[
\|W_{k+1}\|_s \leq \theta_k^{\alpha+\tau+4\delta} \text{ for } s \in [\alpha_0, \alpha_1]. 
\] (4.51)

**Proof.** Note that \([D,W]_{i,j} = (d_i - d_j)W_{i,j}, d_i - d_j \neq 0 \text{ for } i \neq j, \text{ and } \overline{G} = 0\). Then \((4.49)\) follows.

We then turn to the estimate. Recall that \(h = (d_i)_{i \in \mathbb{Z}^d}\) is a \((\tau, \gamma)\)-distal sequence, i.e.,
\[
\|(h - \sigma_p h)^{-1}\|_{\mathcal{B}} \leq \gamma^{-1}\langle p \rangle^\tau \text{ for } p \neq 0. 
\] (4.52)
Let \(G_p\) be the \(p\)-diagonal of \(G\) (i.e., \(G_p(i) = G_{i, i-p}\)). Then we obtain since \((4.52)\) and \((2.2)\)
\[
\|W\|_s^2 \leq \sum_{0 < |p| \leq \theta_k+1} \|G_p\|^2_{\mathcal{B}}\|(h - \sigma_p h)^{-1}\|^2_{\mathcal{B}}\langle p \rangle^{2s} 
\leq \gamma^{-2} \sum_{0 < |p| \leq \theta_k+1} \|G_p\|^2_{\mathcal{B}}\langle p \rangle^{2s+2\tau} 
= \gamma^{-2}\|S_{\theta_k+1}G\|^2_{s+\tau},
\]
which implies
\[
\|W_{k+1}\|_s \leq \gamma^{-1}\|S_{\theta_k+1}G\|_{s+\tau}. 
\]

Thus recalling \((4.48)\), we have for \(\alpha_0 \leq s \leq \alpha_1 - \tau\) (i.e., \(s + \tau \leq \alpha_1\),
\[
\|W_{k+1}\|_s \leq \|G\|_{s+\tau} 
\leq 3\gamma^{-1}\theta_k^{\alpha+\tau+3\delta} 
\leq 3\gamma^{-1}\theta_0^{-\delta}\theta_k^{\alpha+\tau+4\delta} 
\leq \theta_k^{\alpha+\tau+4\delta} \text{ (since } (4.50)\)).
\]
If \(\alpha_1 - \tau \leq s \leq \alpha_1\) (i.e., \(s + \tau \in [\alpha_1, \alpha_1 + \tau]\)), then we have
\[
\|W_{k+1}\|_s \leq \gamma^{-1}\|S_{\theta_k+1}G\|_{s+\tau} 
\leq \gamma^{-1}\theta_k^{\alpha+\tau+3\delta} 
\leq 3\gamma^{-1}\theta_k^{\alpha+\tau+3\delta} 
= 3\gamma^{-1}\Theta\theta_0^{-\delta}\theta_k^{\alpha+\tau+4\delta} 
\leq \theta_k^{\alpha+\tau+4\delta} \text{ (since } (4.50)\)).
\]
This proves (4.51). □

**Estimate of $V_{k+1}^{-1}$**

We first introduce a perturbation argument.

**Lemma 4.9.** Let $C_0$ be given by (4.9) and $W \in \mathcal{M}^*$, $s \geq \alpha_0$. If

$$4C_0^2\|W\|_{\alpha_0} \leq 1/2,$$

(4.53)

then we have that $V = I + W$ is invertible in $\mathcal{M}^*$ and

$$\|V^{-1}\|_s \leq 1 + 2K_1(s)\|W\|_s \quad (s > \alpha_0),$$

$$\|V^{-1}\|_{\alpha_0} \leq 2,$$

$$\|V^{-1} - I\|_s \leq 2K_1(s)\|W\|_s,$$

where $K_1(s)$ is given by (3.3).

**Proof.** The proof is based on the Neumann series argument and Lemma 4.2. From (4.11), we have for $n \geq 2$,

$$\|W^n\|_s \leq n^2C_0^2K_1(s)\|W\|_{\alpha_0}^{n-1}\|W\|_s$$

$$\leq (4C_0^2\|W\|_{\alpha_0})^{n-1}K_1(s)\|W\|_s,$$

(4.54)

where we use the fact $n^2 \leq 4^{n-1}$ and $n \geq 2$. Then by (4.53) and (4.54),

$$\|W^n\|_s \leq 2^{-(n-1)}K_1(s)\|W\|_s$$

for $n \geq 2$,

which implies

$$\sum_{n=0}^{\infty} \|W^n\|_s \leq 1 + \|W\|_s + \sum_{n=2}^{\infty} 2^{-(n-1)}K_1(s)\|W\|_s$$

$$\leq 1 + 2K_1(s)\|W\|_s < \infty,$$

$$\sum_{n=1}^{\infty} \|W^n\|_s \leq \|W\|_s + \sum_{n=2}^{\infty} 2^{-(n-1)}K_1(s)\|W\|_s$$

$$\leq 2K_1(s)\|W\|_s.$$

Since $\mathcal{M}^*$ is a Banach space, applying the standard Neumann series argument shows

$$(I + W)^{-1} = \sum_{n=0}^{\infty} (-W)^n \in \mathcal{M}^*, \quad \|(I + W)^{-1}\|_s \leq 1 + 2K_1(s)\|W\|_s,$$

$$\|(I + W)^{-1} - I\|_s \leq 2K_1(s)\|W\|_s, \quad \|(I + W)^{-1}\|_{\alpha_0} \leq 2.$$

□

Then we estimate $V_{k+1}^{-1}$.

**Lemma 4.10.** Under the assumptions of Lemma 4.8, we have

$$\|V_{k+1}^{-1}\|_s \leq 1 + 2K_1(s)\theta_k^{s-\alpha_0+\tau+4\delta} \text{ for } s \in [\alpha_0, \alpha_1],$$

(4.55)

$$\|V_{k+1}^{-1} - I\|_s \leq 2K_1(s)\theta_k^{s-\alpha_0+\tau+4\delta} \text{ for } s \in [\alpha_0, \alpha_1].$$

(4.56)

**Proof.** This is a direct consequence of Lemma 4.9 and 4.8. □
Now we estimate the reminders. Recalling (4.8), we have

\[
R_{k+1} = R'_{k+1} + R''_{k+1},
\]

\[
R'_{k+1} = (I - S_{\theta_{k+1}}) \left( Q_k^{-1} (T_k + D_k) Q_k + R_k \right),
\]

\[
R''_{k+1} = R_{(1)} + R_{(2)},
\]

where \(R_{(1)}, R_{(2)}\) are given by (4.3) and (4.5), respectively. Note that \(R'_{k+1}\) comes from the smoothing procedure, while \(R''_{k+1}\) is the standard Newton error.

We first estimate \(R'_{k+1}\).

**Lemma 4.11.** Under the assumptions of Lemma 4.8 and assuming further

\[
\alpha_1 \geq 2\alpha + \delta, \quad (4.57)
\]

\[
\max (\Theta^{-\alpha_0}, \Theta^{-\delta}) \leq 1/10. \quad (4.58)
\]

Then for \(\theta_0 \geq C(\delta, \tau, \alpha_0) > 0\), we have

\[
\left\| R'_{k+1} \right\|_s \leq \frac{1}{2} \theta_{s+1}^{\alpha - \alpha_0} \text{ for } s \in [\alpha_0, \alpha_1]. \quad (4.59)
\]

**Proof.** First, we estimate \(\left\| (I - S_{\theta_{k+1}}) R_k \right\|_s\). We have two cases.

**Case 1.** \(s \in [\alpha + \delta, \alpha_1]\). In this case we have

\[
\left\| (I - S_{\theta_{k+1}}) R_k \right\|_s \leq \left\| R_k \right\|_s \leq \theta_k^{s - \alpha}
\]

\[
= \Theta^{s - \alpha} \theta_{k+1}^{\alpha - \alpha_0}
\]

\[
\leq \Theta^{s - \delta} \theta_{k+1}^{\alpha - \alpha_0}
\]

\[
\leq \frac{1}{10} \theta_{s+1}^{\alpha - \alpha_0} \text{ (since (4.58)).} \quad (4.60)
\]

**Case 2.** \(s \in [\alpha_0, \alpha + \delta]\). In this case we have since (4.57)

\[
\left\| (I - S_{\theta_{k+1}}) R_k \right\|_s \leq \theta_k^{s - \alpha} \left\| R_k \right\|_{s + \alpha}
\]

\[
\leq \theta_k^{s - \alpha} \theta_s^{\alpha}
\]

\[
= \Theta^{s - \alpha} \theta_{k+1}^{\alpha - \alpha_0}
\]

\[
\leq \Theta^{s - \alpha} \theta_{k+1}^{\alpha - \alpha_0} \leq \frac{1}{10} \theta_{s+1}^{\alpha - \alpha_0} \text{ (since (4.58)).} \quad (4.61)
\]

Next, recalling (4.41), we can prove similarly

\[
\left\| (I - S_{\theta_{k+1}}) Q_k^{-1} T_k Q_k \right\|_s \leq \frac{1}{10} \theta_{s+1}^{\alpha - \alpha_0} \text{ for } s \in [\alpha_0, \alpha_1].
\]

Finally, for \(\left\| (I - S_{\theta_{k+1}}) Q_k^{-1} D_k Q_k \right\|_s\), we have

**Case 1.** \(s \in [\alpha + \delta, \alpha_1]\). Similar to the proof of (4.60), we have in this case

\[
\left\| (I - S_{\theta_{k+1}}) Q_k^{-1} T_k Q_k \right\|_s \leq \frac{1}{10} \theta_{s+1}^{\alpha - \alpha_0}.
\]

**Case 2.** \(s \in [\alpha - \tau - 4\delta, \alpha + \delta]\). Similar to the proof of (4.61), we have in this case

\[
\left\| (I - S_{\theta_{k+1}}) Q_k^{-1} D_k Q_k \right\|_s \leq \frac{1}{10} \theta_{s+1}^{\alpha - \alpha_0}.
\]
Case 3. \( s \in [\alpha_0, \alpha - \tau - 4\delta] \). In this case we have since \( s + \alpha \geq \alpha_0 + \alpha > \alpha - \tau - 4\delta \)

\[
\| (I - S_{\theta_{k+1}}) Q_0^{-1} D_k Q_k \|_s \leq \theta_{k+1}^{-\alpha} \| Q_0^{-1} D_k Q_k \|_{s+\alpha}
\]

\[
\leq \theta_{k+1}^{-\alpha} \theta_{k+1}^s \quad \text{(since (4.43))}
\]

\[
\leq \Theta^{-\alpha} \theta_s \leq \frac{1}{10} \theta_{k+1}^{s-\alpha} \quad \text{(if } \Theta^{-\alpha} \leq 1/10 \text{).}
\]

This finishes the proof of (4.59). \( \square \)

Now we estimate \( R_{k+1}'' \). We have

Lemma 4.12. Under the assumptions of Lemma 4.8 and assuming further

\[
\theta_0^{-\kappa_1} \geq C(\alpha_0, \alpha_1) \Theta^{\alpha-\alpha_0-\kappa_1}, \quad (4.62)
\]

where \( \kappa_1 < 0 \) is given by (4.40). Then we have

\[
\| R_{k+1}'' \|_s \leq \frac{1}{2} \theta_{k+1}^{s-\alpha} \quad \text{for } s \in [\alpha_0, \alpha_1].
\]

Proof. From (4.48) and Lemma 4.8, we obtain

\[
\| [D, W_{k+1}] \|_s \leq \| S_{\theta_{k+1}} G \|_s
\]

\[
\leq \| G \|_s \leq 3 \theta_k^{s-\alpha+3\delta} \quad (s \in [\alpha_0, \alpha_1]),
\]

which together with Lemma 3.1 and Lemma 4.10 implies for \( s \in [\alpha_0, \alpha_1] \),

\[
\| R(1) \|_s = \| (V_0^{-1} - I)[D, W_{k+1}] \|_s
\]

\[
+ \| (V_0^{-1} - I) R_k W_{k+1} \|_s
\]

\[
+ \| (V_0^{-1} - I) R_k W_{k+1} \|_s + \| R_k W_{k+1} \|_s
\]

\[
\leq C(s, \alpha_0) \theta_{k-2\alpha+\alpha_0+\gamma+7\delta}
\]

\[
+ C(s, \alpha_0) \theta_{k-3\alpha+2\alpha+2\gamma+8\delta}
\]

\[
+ C(s, \alpha_0) \theta_{k-2\alpha+\alpha_0+\gamma+4\delta}
\]

\[
\leq C(s, \alpha_0) \theta_k^{-\alpha+\kappa_1} \quad \text{(since } \kappa_1 < 0 \text{)}
\]

\[
\leq C(s, \alpha_0) \Theta^{-(s-\alpha+\kappa_1)} \theta_k^{s-\alpha+\kappa_1}
\]

\[
\leq C(\alpha_1, \alpha_0) \Theta^{\alpha-\alpha_0} \theta_0^{s-\alpha} \theta_{k-1}^{s-\alpha}
\]

\[
\leq \frac{1}{4} \theta_{k+1}^{s-\alpha} \quad \text{(since (4.62)).}
\]

Similarly, we have by recalling

\[
\| Q_k^{-1} (T_k + D_k) Q_k \|_s \leq 2 \theta_k^{s-\alpha+3\delta} \quad \text{for } s \in [\alpha_0, \alpha_1]
\]
that
\[
\|R_{(2)}\|_s = \|(V_{k+1}^- - I)Q_k^{-1}(T_k + D_k)Q_kW_{k+1}\|_s \\
+ \|(V_{k+1}^- - I)Q_k^{-1}(T_k + D_k)Q\|_s \\
+ \|Q_k^{-1}(T_k + D_k)Q_kW_{k+1}\|_s
\]
\[
\leq C(s, \alpha_0)\theta_k^{s-3\alpha+2\alpha_0+2\tau+11\delta} \\
+ C(s, \alpha_0)\theta_k^{-2\alpha+\alpha+\tau+7\delta} \\
\leq C(s, \alpha_0)\theta_k^{s-\alpha+\kappa_1} \text{ (since } \kappa_1 < 0) \\
\leq \frac{1}{4}\theta_{k+1}^{s-\alpha}.
\]

This finishes the proof. \(\square\)

Combining Lemma 4.8, 4.10, 4.11 and 4.12 implies the proposition. \(\square\)

4.2. The initial step. Now we turn to the initial step.

**Proposition 4.13.** Assume that
\[
-\alpha + \alpha_0 + \tau + 3\delta < 0, \quad (4.63)
\]
\[
\|T\|_{\alpha+3\delta} \leq \theta_0^{\alpha_0-\alpha} \leq 1, \quad (4.64)
\]
\[
\theta_0^\delta \geq C(\alpha_0, \alpha_1)\Theta^{\alpha_0-\alpha+\delta}. \quad (4.65)
\]

Then there is some \(C = C(\delta, \tau, \gamma, \alpha_0, \alpha_1) > 0\) such that for \(\theta_0 \geq C\), there exist \(W_1, R_1 \in \mathcal{M}\) with \(V_1 = I + W_1\) such that
\[
V_1^{-1}(T_0 + D)V_1 = D + R_1, \quad (4.66)
\]

where
\[
\|W_1\|_s \leq \theta_0^{s-\alpha+\tau+\delta} \text{ for } s \in [\alpha_0, \alpha_1], \quad (4.67)
\]
\[
\|V_1^{-1} - I\|_s \leq \theta_0^{s-\alpha+\tau} \text{ for } s \in [\alpha_0, \alpha_1], \quad (4.68)
\]
\[
\|V_1^{-1} - I\|_s \leq \theta_0^{s-\alpha+\tau+2\delta} \text{ for } s \in [\alpha_0, \alpha_1], \quad (4.69)
\]
\[
\|R_1\|_s \leq \theta_1^{s-\alpha} \text{ for } s \in [\alpha_0, \alpha_1]. \quad (4.70)
\]

**Proof.** Note that \(\overline{T} = 0\). Then
\[
[D, W_1] + T_0 = 0
\]

has a unique solution \(W_1\). We also have
\[
V_1^{-1}(T_0 + D)V_1 - D = V_1^{-1}([D, W_1] + T_0V_1) \\
= V_1^{-1}T_0W_1 = R_1, \quad (4.71)
\]

which yields (4.66).

Next, we estimate \(W_1, V_1, R_1\). Because of \(T_0 = S_{\theta_0}T\), we obtain for \(s \geq \alpha + 3\delta\),
\[
\|T_0\|_s \leq \theta_0^{s-\alpha-3\delta}\|T\|_{\alpha+3\delta} \leq \theta_0^{s-\alpha-3\delta} \quad \text{(since } (4.64))\). \quad (4.72)
\]

If \(s < \alpha + 3\delta\), we also have by (4.64)
\[
\|T_0\|_s \leq \|T\|_{\alpha+3\delta} \leq \theta_0^{s-\alpha} \leq \theta_0^{s-\alpha} \text{ (since } s \geq \alpha_0),
\]

where
which together with (4.72) implies
\[ ||T_0||_s \leq \theta_0^{s-\alpha} \text{ for } s \in [\alpha_0, \alpha_1]. \] (4.73)

Similar to the proof of Lemma 4.8, we obtain
\[ ||W_1||_s \leq \gamma^{-1} \varepsilon ||T_0||_{s+\tau}, \]
which implies for \( s + \tau \in [\alpha_0 + \tau, \alpha_1] \) and \( \theta_0 \geq C(\gamma, \delta) > 0, \)
\[ ||W_1||_s \leq \gamma^{-1} \theta_0^{-\delta} \theta_0^{-\alpha+\tau+\delta} \]
\[ \leq \theta_0^{s-\alpha+\tau+\delta}. \]

If \( s + \tau \in [\alpha_1, \alpha_1 + \tau] \), we have since (4.64)
\[ ||W_1||_s \leq \gamma^{-1} \theta_0^{-\delta} ||T_0||_s \]
\[ \leq \gamma^{-1} \theta_0^{-\delta + s - \alpha} \]
\[ \leq \theta_0^{s-\alpha+\tau+\delta}. \]

This proves (4.67). Now, since Lemma 4.9, we have for \( \theta_0 \geq C(\delta, \tau, \alpha_0, \alpha_1) > 0, \)
\[ ||V_1^{-1}||_s \leq 1 + 2K_1(s)\theta_0^{s-\alpha+\tau+\delta} \leq \theta_0^{(s-\alpha+\tau+\delta)+2\delta}, \]
\[ ||V_1^{-1} - I||_s \leq 2K_1(s)\theta_0^{s-\alpha+\tau+\delta} \leq \theta_0^{s-\alpha+\tau+2\delta}, \]
which yields (4.68) and (4.69). Finally, we estimate \( R_1 \). From (4.71), (4.73), (4.68) and Lemma 3.1, we have
\[ ||R_1||_s = ||V_1^{-1}T_0W_1||_s \]
\[ \leq C(\alpha_0, s)\theta_0^{(s-\alpha+\tau+\delta)+2\alpha+2\alpha_0+\tau+2\delta} \]
\[ + C(\alpha_0, s)\theta_0^{-2\alpha+\alpha_0+\tau+2\delta}. \] (4.74)

If \( \alpha_0 \leq s < \alpha - \tau - \delta \), then we obtain since (4.63)
\[ (s-\alpha+\tau+\delta)_+ - 2\alpha + 2\alpha_0 + \tau + 2\delta = -2\alpha + 2\alpha_0 + \tau + 2\delta \]
\[ = s - \alpha + (-\alpha + 2\alpha_0 + \tau + 2\delta - s) \]
\[ \leq s - \alpha + (-\alpha + \alpha_0 + \tau + 2\delta) \text{ (since } -s \leq -\alpha_0) \]
\[ \leq s - \alpha - \delta. \]

If \( \alpha - \tau - \delta \leq s \leq \alpha_1 \), we also have by (4.63)
\[ (s-\alpha+\tau+\delta)_+ - 2\alpha + 2\alpha_0 + \tau + 2\delta = s - 3\alpha + 2\alpha_0 + 2\tau + 3\delta \]
\[ = s - \alpha + (-2\alpha + 2\alpha_0 + 2\tau + 3\delta) \]
\[ \leq s - \alpha - 3\delta. \]

Thus recalling (4.74), we have
\[ ||R_1||_s \leq C(\alpha_0, s)\theta_0^{s-\alpha-\delta} \]
\[ = C(\alpha_0, s)\theta_0^{-\alpha}\theta_0^{s-\alpha-\delta} \]
\[ \leq C(\alpha_0, \alpha_1)\Theta^{\alpha+\delta-\alpha_0}\theta_0^{-\delta}\theta_1^{s-\alpha} \]
\[ \leq \theta_1^{s-\alpha} \text{ (since (4.65))}. \]

This proves (4.70).
5. The iteration theorem

In this section we combine Proposition 4.1 and 4.13 to establish the iteration theorem.

Fix any \( \delta > 0, \alpha_0 > d/2, \tau > 0, \gamma > 0 \). We collect the conditions imposed on parameters \( \Theta, \alpha, \alpha_1, \theta_0 \). Then

- We let (recalling (4.16) and (4.58))
  \[
  \Theta = \Theta(\delta, \alpha_0) = \max\{8^{2/\delta}C_0^{4/\delta}, 10^{1/\delta}, 10^{1/\alpha_0}\},
  \]
  where \( C_0 = C_0(\alpha_0) \) is given by (4.9).
- We let (recalling (4.40))
  \[
  \alpha > \alpha_0 + \tau + 7\delta.
  \]
- We let (recalling (4.57))
  \[
  \alpha_1 \geq 2\alpha + \delta.
  \]
- We let (recalling (4.38) and (4.64))
  \[
  \|T\|_{\alpha+4\delta} \leq \theta_0^{\alpha_0-\alpha} \leq 1.
  \]
- We also assume \( \theta_0 \) is large enough, i.e.,
  \[
  \theta_0 \geq C(\delta, \tau, \gamma, \alpha, \alpha_0, \alpha_1) > 0.
  \]

**Theorem 5.1.** Fix \( \delta > 0, \alpha_0 > d/2, \tau > 0, \gamma > 0 \). Assume that

\[
\begin{align*}
\alpha &> \alpha_0 + \tau + 7\delta, \\
\alpha_1 &\geq 2\alpha + \delta, \\
\Theta & = \Theta(\delta, \alpha_0) = \max\{8^{2/\delta}C_0^{4/\delta}, 10^{1/\delta}, 10^{1/\alpha_0}\}, \\
\|T\|_{\alpha+4\delta} &\leq \theta_0^{\alpha_0-\alpha} \leq 1.
\end{align*}
\]

Then there is some \( C = C(\delta, \tau, \gamma, \alpha, \alpha_0, \alpha_1) > 0 \) such that for \( \theta_0 \geq C \), the following holds true: For any \( k \geq 1 \) and \( 1 \leq l \leq k \), there exist \( W_l, R_l, Q_l \in \mathcal{M} \) and \( D_{l-1} \in \mathcal{M}_0^\infty \) so that

\[
Q_l^{-1}H_lQ_l = D + R_l, \quad H_l = \sum_{i=1}^{l}(T_{i-1} + D_{i-1}) + D
\]

with

\[
\begin{align*}
Q_0 &= I, D_0 = 0, T_0 = S_{\theta_0}T, \\
Q_l &= Q_{l-1}V_l, V_l = I + W_l, \\
\|W_l\|_s &\leq \theta_{l-1}^{s-\alpha+\tau+4\delta} \text{ for } s \in [\alpha_0, \alpha_1], \\
\|V_l^{-1} - I\|_s &\leq 2K_1(s)\theta_{l-1}^{s-\alpha+\tau+4\delta} \text{ for } s \in [\alpha_0, \alpha_1], \\
\|R_l\|_s &\leq \theta_{l}^{s-\alpha} \text{ for } s \in [\alpha_0, \alpha_1], \\
\|D_{l-1}\|_0 &\leq 3\theta_{l-1}^{\alpha_0-\alpha},
\end{align*}
\]

where \( K_1(s), T_l \) are defined by (3.3) and (4.1) respectively.

**Proof.** The proof is based on a combination of Proposition 4.1 and 4.13. \( \square \)
6. Proofs of Theorem 2.1 and 2.2

Proof of Theorem 2.1. It suffices to prove the convergence of

\[ Q_k, Q_k^{-1}, \sum_{i=1}^{k} D_{l-1}, \sum_{i=1}^{k} T_{l-1} \text{ (as } k \to \infty) \]

appeared in Theorem 5.1. We set

\[ \|T\|_{\alpha+\delta} = \theta_0^{\alpha_0 - \alpha}, \]

i.e.,

\[ \theta_0^{-\delta} = \|T\|_{\alpha+\delta}. \]

We first prove the convergence and estimate of \( Q_k, Q_k^{-1} \). From the proof of Lemma 4.3, we have

\[ \|Q_k^{-1} - Q_l^{-1}\| \leq \theta_k^{-\delta} \leq \sum_{l \geq 1} \theta_l^{-\delta} \leq \Theta_0^{-\delta} < \infty, \]

which implies

\[ Q' = I + \sum_{l \geq 1} (Q_l^{-1} - Q_l^{-1}) \in M^{\alpha-\tau-\delta}, \]

\[ \|Q' - I\|_{\alpha-\tau-\delta} \leq C(\delta, \alpha_0) \theta_0^{-\delta} \leq C\|T\|_{\alpha+\delta}^{-\delta}, \]

\[ \|Q_k^{-1} - Q'\|_{\alpha-\tau-\delta} \leq \sum_{l \geq k+1} \theta_l^{-\delta} \to 0 \text{ (as } k \to \infty). \]

Similarly, we also have there is some \( Q_+ \in M^{\alpha-\tau-\delta} \) so that

\[ \|Q_+ - I\|_{\alpha-\tau-\delta} \leq C(\delta, \alpha_0) \theta_0^{-\delta} \leq C\|T\|_{\alpha+\delta}^{-\delta}, \]

\[ \|Q_k - Q_+\|_{\alpha-\tau-\delta} \to 0 \text{ (as } k \to \infty). \]

It is easy to see \( Q' = Q_+^{-1} \).

We then show the convergence of \( \sum_{l=1}^{k} D_{l-1} \). Recalling (5.1), we obtain

\[ \sum_{l \geq 1} \|D_{l-1}\|_0 \leq \sum_{l \geq 1} 3\theta_0^{\alpha_0 - \alpha} \leq 3\theta_0^{\alpha_0 - \alpha} \sum_{l \geq 0} \Theta_0^{-(\alpha - \alpha_0)l} \]

\[ \leq C(\delta, \alpha, \alpha_0) \theta_0^{\alpha_0 - \alpha}. \]

Then there is some \( D_+ \in M_0^{\infty} \) so that

\[ \|D_+\|_0 \leq C(\delta, \alpha, \alpha_0) \theta_0^{\alpha_0 - \alpha} \leq C\|T\|_{\alpha+\delta}^{-\delta}, \]

\[ \|\sum_{l=1}^{k} D_{l-1} - D_+\|_0 \leq \sum_{l \geq k+1} 3\theta_0^{\alpha_0 - \alpha} \to 0 \text{ (as } k \to \infty). \]
Considering $\sum_{l=1}^{k} T_{l-1}$, we have

$$\sum_{l=1}^{k} T_{l-1} = S_{\theta_{k-1}} T,$$

which implies

$$\| T - \sum_{l=1}^{k} T_{l-1} \|_{\alpha-\tau-\delta} = \| (I - S_{\theta_{k-1}}) T \|_{\alpha-\tau-\delta}$$

$$\leq \theta_{k-1}^{-\tau-\delta} \| T \|_{\alpha} \rightarrow 0 \text{ (as } k \rightarrow \infty).$$

Obviously, we obtain

$$\| R_k \|_{\alpha-\delta} \leq \theta_{k}^{-\tau-\delta} \rightarrow 0 \text{ (as } k \rightarrow \infty).$$

Next, we will show

$$Q_{-1}^{-1}(T + D + D_+)Q_{+} = D. \quad (6.3)$$

As mentioned above, $D$ is not necessary in $\mathcal{M}$. However, $(6.3)$ is equivalent to

$$T + D_+ = Q_{+} DQ_{-1}^{-1} - D = -[D, Q_{+}]Q_{+}^{-1}.$$ 

We will show $[D, Q_{+}] \in \mathcal{M}^{\alpha-\tau-\delta}$. Note that

$$\sum_{l=0}^{k} T_l + \sum_{l=0}^{k} D_l = -[D, Q_{k+1}]Q_{k+1}^{-1} + Q_{k+1} R_{k+1} Q_{k+1}^{-1},$$

where

$$[D, Q_{l+1}] = [D, Q_l (I + W_{l+1})] = [D, Q_l] V_{l+1} + Q_l [D, W_{l+1}], \quad [D, Q_0] = 0. \quad (6.4)$$

Obviously, $[D, Q_{+}] \in \mathcal{M}$. It suffices to prove

$$\lim_{k \rightarrow \infty} \| Q_{k+1} R_{k+1} Q_{k+1}^{-1} \|_{\alpha-\tau-\delta} = 0,$$

$$\lim_{k \rightarrow \infty} \| [D, Q_{+} - Q_{k+1}] \|_{\alpha-\tau-\delta} = 0.$$

It is easy to see

$$\| Q_{k+1} R_{k+1} Q_{k+1}^{-1} \|_{\alpha-\tau-\delta} \leq C(\delta, \tau, \alpha) \theta_{k}^{-\tau-5\delta} \rightarrow 0 \text{ (as } k \rightarrow \infty).$$

From $(6.4)$ and Theorem 5.1, we have

$$\| [D, Q_l - Q_{l-1}] \|_{\alpha-\tau-\delta} \leq C \| [D, Q_{l-1}] \|_{\alpha-\tau-7\delta} \theta_{l-1}^{(\alpha-\tau-\delta-\alpha)+\tau+4\delta}$$

$$+ C \theta_{l-1}^{(\alpha-\tau-\delta)-\alpha+\delta}$$

$$\leq C \| [D, Q_{l-1}] \|_{\alpha-\tau-7\delta} \theta_{l-1}^{-3\delta} + C \theta_{l-1}^{-3\delta}$$

$$\leq \| [D, Q_{l-1}] \|_{\alpha-\tau-7\delta} \theta_{l-1}^{-2\delta} + \theta_{l-1}^{-2\delta} \text{ (if } \theta_0 > C).$$

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Let \( a_l = \|[D, Q_l]\|_{\alpha - \tau - 7\delta} \). Then \( 0 \leq a_1 \leq \theta_0^{-7\delta} \) and

\[
a_l \leq (1 + \theta_{l-1}^{-2\delta})a_{l-1} + \theta_{l-1}^{-2\delta} \\
\leq 2a_{l-1} + \theta_{l-1}^{-2\delta} \quad \text{(since } \theta_0 > C) \\
\leq 2^2a_{l-2} + 2\theta_{l-1}^{-2\delta} + \theta_{l-2}^{-2\delta} \\
\leq \cdots \\
\leq 2^{l-1}a_1 + 2^{l-2}\theta_1^{-2\delta} + 2^{l-3}\theta_2^{-2\delta} + \cdots + \theta_{l-1}^{-2\delta} \\
\leq \theta_{l-1}^{-\delta}.
\]

As a result,

\[
\|[D, Q_l] - [D, Q_{l-1}]\|_{\alpha - \tau - 7\delta} \leq \theta_{l-1}^{-\delta} + \theta_{l-1}^{-2\delta} \leq 2\theta_{l-1}^{-\delta},
\]

which implies

\[
\|[D, Q_+ - Q_{k+1}]\|_{\alpha - \tau - 7\delta} \leq \sum_{l \geq k+2} \|[D, Q_l] - [D, Q_{l-1}]\|_{\alpha - \tau - 7\delta} \\
\leq 2 \sum_{l \geq k+2} \theta_{l-1}^{-\delta} \\
\leq C\theta_{k+1}^{-\delta} \to 0 \quad \text{(as } k \to \infty).\]

Thus \([D, Q_+] = [D, Q_+] + [D, Q_+ - Q_{k+1}] \in \mathcal{M}^\alpha_{\alpha - \tau - 7\delta}\) and (6.3) follows.

The remaining is to show if both \( T \) and \( D \) are real symmetric, then \( Q_+ \) can be improved to become a unitary operator. Suppose now that

\[
(d_i)_{i \in \mathbb{Z}^d} \in \mathbb{R}^{\mathbb{Z}^d}, \quad T^i = T.
\]

From (4.6), we know that \( D_+ \in \mathbb{R}^{\mathbb{Z}^d} \). Thus by taking transpose on both sides of (6.3), we obtain

\[
Q_+DQ_+^{-1} = D + D_+ + T = (Q_+^t)^{-1}DQ_+^t,
\]

which implies

\[
D(Q_+^tQ_+) = (Q_+^tQ_+)D.
\]

As a result, we have

\[
(Q_+^tQ_+)_{i,j}(d_i - d_j) = 0. \quad (6.5)
\]

However, since \((d_i)_{i \in \mathbb{Z}^d}\) is a \((\tau, \gamma)\)-distal sequence, we obtain in particular

\[
(d_i - d_j) \neq 0 \quad \text{for } i \neq j,
\]

which together with (6.5) yields

\[
(Q_+^tQ_+)_{i,j} = 0 \quad \text{for } i \neq j.
\]

Thus we have shown \(Q_+^tQ_+ \in \mathcal{M}_0^\infty\). This means that we can set

\[
U = Q_+(Q_+^tQ_+)^{-\frac{1}{2}}.
\]
It is easy to check that $U$ is a unitary operator and
\[
U^{-1}(T + D + D_+)U = (Q_+^i Q_+)\frac{1}{2}(Q_+^{-1}(T + D + D_+)Q_+)\frac{1}{2} = (Q_+^i Q_+)\frac{1}{2} D (Q_+^i Q_+)\frac{1}{2} = D,
\]
where in the last inequality we use the fact that $Q_+^i Q_+$ is a diagonal operator.

Finally, we estimate $U^t, U$. First, we observe that
\[
Q_+^t + Q_+ = I + (Q_+ - I)(Q_+ - I) + (Q_+ - I) + (Q_+ - I) := I + P.
\]
From (6.1) and (6.2), we have
\[
\|P\|_0 \leq C\theta_0^{-\delta}.
\]
(6.6)
Since $Q_+^t + Q_+$ is diagonal, $P$ is diagonal as well. Then we can let $P = \text{diag}_{i \in \mathbb{Z}^d}(p_i)$.

It also follows from (6.6) that
\[
\sup_{i \in \mathbb{Z}^d} |p_i| \leq C\theta_0^{-\delta}.
\]
Consequently, we have for $\theta_0 \geq C$,
\[
(Q_+^t Q_+)^{-\frac{1}{2}} = \text{diag}_{i \in \mathbb{Z}^d}(\sqrt{1 + p_i})
= \text{diag}_{i \in \mathbb{Z}^d}\left(1 + \frac{1}{2}p_i + O(p_i^2)\right)
= I + P',
\]
where $P' \in M_\infty$ satisfies $\|P'\|_0 \leq C\theta_0^{-\delta}$. Hence we have since Lemma 3.1
\[
\|U^t - I\|_{\alpha - \tau - 7\delta} = \|U - I\|_{\alpha - \tau - 7\delta}
= \|Q_+(Q_+^t Q_+)\frac{1}{2} - I\|_{\alpha - \tau - 7\delta}
= \|Q_+ - I - Q_+ P'\|_{\alpha - \tau - 7\delta}
\leq \|Q_+ - I\|_{\alpha - \tau - 7\delta} + \|Q_+ P'\|_{\alpha - \tau - 7\delta}
\leq C\theta_0^{-\delta} + C(\tau, \alpha, \alpha_0)\|Q_+\|_{\alpha - \tau - 7\delta}\|P'\|_{\alpha - \tau - 7\delta}
\leq C\theta_0^{-\delta} + C\|Q_+\|_{\alpha - \tau - 7\delta}\theta_0^{-\delta}
\leq C\theta_0^{-\delta} \leq C\|T\|_{\alpha + 4\delta}^{-\delta}.
\]
This proves Theorem 2.1. \hfill \square

Proof of Theorem 2.2. The proof is similar to that of Theorem 2.1. Assume that at the $k$-th iteration step we have
\[
Q_k^{-1}\left(\sum_{i=1}^{k} T_{i-1} + D\right) Q_k = D + \sum_{i=1}^{k} D_{i-1} + R_k,
\]
where $D_{l-1} \in \mathcal{M}_0^\infty$ ($1 \leq l \leq k$). We want to find $Q_{k+1} = Q_k(I + W_{k+1}) \in \mathcal{M}$, $D_k \in \mathcal{M}_0^\infty$ so that

$$Q_{k+1}^{-1} \left( \sum_{l=1}^{k+1} T_{l-1} + D \right) Q_{k+1} = D + \sum_{l=1}^{k+1} D_{l-1} + R_{k+1}.$$

Note that

$$Q_{k+1}^{-1} \left( \sum_{l=1}^{k} T_{l-1} + T_k + D \right) Q_{k+1}$$

$$= (I + W_{k+1})^{-1} \left( D + \sum_{l=1}^{k} D_{l-1} + R_k \right) (I + W_{k+1})$$

$$+ Q_{k+1}^{-1}(T_k)Q_{k+1}$$

$$= D + \sum_{l=1}^{k} D_{l-1} + [D + \sum_{l=1}^{k} D_{l-1}, W_{k+1}] + R_k' + R_{k+1},$$

where $S_{\theta_k}, R_k' = R_k$ and $\|R_k'\|_s = O(\|R_k\|_s)$. Our aim is to eliminate terms of order $O(\|R_k\|_s)$. Then it needs to solve the new homological equation

$$\left[ D + \sum_{l=1}^{k} D_{l-1}, W_{k+1} \right] + R_k' - \overline{R_k} = 0,$$

(6.7)

which then implies

$$D_k = \overline{R_k'}.$$

As compared with (4.7), the main part of (6.7) becomes $D + \sum_{l=1}^{k} D_{l-1}$, rather than $D$! Fortunately, we have a much stronger assumption, i.e., $D + D' \in DC_B(\tau, \gamma)$ for any $D' \in \mathcal{M}_0^\infty$ with $\|D'\|_0 \leq \eta$ ($\eta > 0$). It is easy to see

$$D + \sum_{l=1}^{k} D_{l-1} \in DC_B(\tau, \gamma)$$

if $\theta_0 \geq C(\eta) > 0$. As a result, the equation (6.7) can be solved almost the same as that of (4.7).

Once the equation (6.7) is solved and estimated, the remaining issue is just to perform a similar iteration as that in proving Theorem 2.1. Thus we omit the details here.

\[\square\]

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Appendix A.

In this appendix we prove Lemma 3.1.

Proof of Lemma 3.1. The proof is standard and is based on the Hölder inequality. For any \( X \in \mathcal{M}^s \), recall that

\[
X_k = (X_k(i))_{i \in Z^d}, \quad X_k(i) = X_{i,i-k}.
\]

We first show if \( Z = XY \), then

\[
Z_k = \sum_{j \in Z^d} X_j(\sigma_j Y_{k-j}). \tag{A.1}
\]

In fact, we have

\[
Z_k(i) = Z_{i,i-k} = \sum_{j \in Z^d} X_{i,i-j}Y_{i-j,i-k} = \sum_{j \in Z^d} X_j(i)(\sigma_j Y_{k-j})(i),
\]

which implies (A.1).

Next, from (2.2), we obtain since (A.1)

\[
\|Z\|^2 = \sum_{k \in Z^d} \|Z_k\|^2_B(k)^{2s} = \sum_{k \in Z^d} \|X_j(\sigma_j Y_{k-j})\|^2_B(k)^{2s} \leq \sum_{k \in Z^d} \left( \sum_{j \in Z^d} \|X_j\|_B \|\sigma_j Y_{k-j}\|_B \right)^2(k)^{2s} = \sum_{k \in Z^d} \left( \sum_{j \in Z^d} \|X_j\|_B \|Y_{k-j}\|_B \right)^2(k)^{2s},
\]

where in the last equality we use the translation invariance of \( \mathcal{B} \). Let \( a_k = \|X_k\|_B, b_k = \|Y_k\|_B \). It suffices to study the sum

\[
\left( \sum_{j \in Z^d} a_j b_{k-j} \right)^2(k)^{2s}.
\]

We have the following two cases.

Case 1. \( j \in I_k := \{ j \in Z^d : (k)^{2s}(k-j)^{-2s} \leq 10 \} \). In this case we have

\[
(k)^{2s}(k-j)^{-2s} (j)^{-2\alpha_0} \leq 10(j)^{-2\alpha_0}. \tag{A.2}
\]
Hence we have by using the Hölder inequality
\[
\sum_{\mathbf{k} \in \mathbb{Z}^d} \left( \sum_{j \notin \mathcal{I}_k} a_j b_{\mathbf{k} - j} \right)^2 \langle \mathbf{k} \rangle^{2s} \leq \sum_{\mathbf{k} \in \mathbb{Z}^d} \left( \sum_{j \in \mathcal{I}_k} a_j^2 \langle \mathbf{j} \rangle^{2\alpha_0} \right) \left( \sum_{j \in \mathcal{I}_k} b_{\mathbf{k} - j}^2 \langle \mathbf{j} \rangle^{-2\alpha_0} \right) \langle \mathbf{k} \rangle^{2s}
\]
\[
\leq \|X\|_{\alpha_0}^2 \sum_{\mathbf{k} \in \mathbb{Z}^d} \left( \sum_{j \in \mathcal{I}_k} b_{\mathbf{k} - j}^2 \langle \mathbf{k} - \mathbf{j} \rangle^{2s} \langle \mathbf{k} \rangle^{-2\alpha_0} \langle \mathbf{k} \rangle^{2s} \langle \mathbf{k} - \mathbf{j} \rangle^{-2s} \right)
\]
\[
\leq 10\|X\|_{\alpha_0}^2 \sum_{\mathbf{k} \in \mathbb{Z}^d} \left( \sum_{j \in \mathcal{I}_k} b_{\mathbf{k} - j}^2 \langle \mathbf{k} - \mathbf{j} \rangle^{2s} \langle \mathbf{j} \rangle^{-2\alpha_0} \right) \quad (\text{since } (A.2))
\]
\[
\leq 10\|X\|_{\alpha_0}^2 \sum_{\mathbf{k} \in \mathbb{Z}^d} \langle \mathbf{j} \rangle^{-2\alpha_0} \left( \sum_{j \in \mathcal{I}_k} b_{\mathbf{k} - j}^2 \langle \mathbf{k} \rangle^{2s} \right)
\]
\[
\leq M_0^2 \|X\|_{\alpha_0}^2 \|Y\|_{\alpha_0}^2,
\]
where \(M_0 = \sqrt{\frac{10}{\sum_{\mathbf{k} \in \mathbb{Z}^d} \langle \mathbf{k} \rangle^{-2\alpha_0}} < \infty \) since \(\alpha_0 > d/2\).

**Case 2.** \(j \notin \mathcal{I}_k\). In this case we must have \(\mathbf{k} \neq \mathbf{0}\). Then
\[
\langle \mathbf{k} \rangle > 10^{\frac{1}{d}} (\mathbf{k} - \mathbf{j}) > 10^{\frac{1}{d}} |\mathbf{k} - \mathbf{j}| \geq 10^{\frac{1}{d}} (|\mathbf{k} - \mathbf{j}|) \geq 10^{\frac{1}{d}} ((\mathbf{k} - \mathbf{j}))
\]
which yields
\[
\langle \mathbf{j} \rangle^{-2s} \leq (1 - 10^{-\frac{1}{d}})^{-2s} \langle \mathbf{k} \rangle^{-2s}. \quad (A.3)
\]
Thus using again the Hölder inequality implies
\[
\sum_{\mathbf{k} \in \mathbb{Z}^d} \left( \sum_{j \notin \mathcal{I}_k} a_j b_{\mathbf{k} - j} \right)^2 \langle \mathbf{k} \rangle^{2s} \leq \sum_{\mathbf{k} \in \mathbb{Z}^d} \left( \sum_{j \neg \in \mathcal{I}_k} a_j^2 \langle \mathbf{j} \rangle^{2\alpha_0} \right) \left( \sum_{j \notin \mathcal{I}_k} b_{\mathbf{k} - j}^2 \langle \mathbf{k} - \mathbf{j} \rangle^{-2\alpha_0} \langle \mathbf{j} \rangle^{-2s} \right) \langle \mathbf{k} \rangle^{2s}
\]
\[
\leq \|Y\|_{\alpha_0}^2 \sum_{\mathbf{k} \in \mathbb{Z}^d} \left( \sum_{j \notin \mathcal{I}_k} a_j^2 \langle \mathbf{j} \rangle^{2s} \langle \mathbf{k} - \mathbf{j} \rangle^{-2\alpha_0} \langle \mathbf{j} \rangle^{-2s} \right) \langle \mathbf{k} \rangle^{2s}
\]
\[
\leq (1 - 10^{-\frac{1}{d}})^{-2s} \|Y\|_{\alpha_0}^2 \sum_{\mathbf{k} \in \mathbb{Z}^d} \left( \sum_{j \notin \mathcal{I}_k} a_j^2 \langle \mathbf{j} \rangle^{2s} \langle \mathbf{k} - \mathbf{j} \rangle^{-2\alpha_0} \right) \quad (\text{since } (A.3))
\]
\[
\leq (1 - 10^{-\frac{1}{d}})^{-2s} \|Y\|_{\alpha_0}^2 \sum_{\mathbf{k} \in \mathbb{Z}^d} a_j^2 \langle \mathbf{j} \rangle^{2s} \left( \sum_{\mathbf{k} \in \mathbb{Z}^d} \langle \mathbf{k} - \mathbf{j} \rangle^{-2\alpha_0} \right)
\]
\[
\leq M_1(s) \|Y\|_{\alpha_0}^2 \|X\|_{\alpha_0}^2,
\]
where \(M_1(s) = (1 - 10^{-\frac{1}{d}})^{-s} \sqrt{\sum_{\mathbf{k} \in \mathbb{Z}^d} \langle \mathbf{k} \rangle^{-2\alpha_0}} < \infty \) since \(\alpha_0 > d/2\).

Combining **Case 1** and **Case 2** implies
\[
\|XY\|_s \leq \sqrt{2M_0^2(\alpha_0) \|X\|_{\alpha_0}^2 \|Y\|_s^2 + 2M_1^2(s) \|Y\|_{\alpha_0}^2 \|X\|_s^2}
\]
\[
\leq K_0(\alpha_0) \|X\|_{\alpha_0} \|Y\|_s + K_1(s) \|X\|_s \|Y\|_{\alpha_0}.
\]
which proves Lemma 3.1, where \(K_0 = \sqrt{2}M_0\), \(K_1(s) = \sqrt{2}M_1(s)\).
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