A two-dimensional Gauss-Kuzmin theorem for $N$-continued fraction expansions

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Abstract

A two-dimensional Gauss-Kuzmin theorem for $N$-continued fraction expansions is shown. More exactly, we obtain a Gauss-Kuzmin theorem related to the natural extension of the measure-dynamical system corresponding to these expansions. Then, using characteristic properties of the transition operator associated with the random system with complete connections underlying $N$-continued fractions on the Banach space of complex-valued functions of bounded variation we derive explicit lower and upper bounds for the convergence rate of the distribution function to its limit.

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1 Introduction

The purpose of this paper is to show a two-dimensional Gauss-Kuzmin theorem for $N$-continued fraction expansions introduced by Burger et al. [1], and studied by Dajani et al. [4].

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In the last twenty-five years some versions of the so-called two-dimensional theorem related to the Gauss-Kuzmin problem are obtained for different types of continued fraction expansions. For example, the case of the regular continued fraction (RCF) expansion was extensively studied in [2], [5] and [7]. For Hurwitz’ singular continued fractions there are known the results obtained in [3] and [11]. Also, in [12] Sebe proved the first two-dimensional Gauss-Kuzmin theorem which leads to an estimate of the approximation error by the expansion algorithm in the grotesque continued fraction.

Our study concludes the series of papers [8] and [9] where the second author discussed some metric properties of these expansions. Moreover he investigated the associated Perron-Frobenius operator and proved a Gauss-Kuzmin theorem for $N$-continued fraction applying the method of random systems with complete connections (RSCC) by Iosifescu [6].

The outline of this paper is as follows. In Section 2 we present notions and preliminary results to be used in the sequel. We mention that the infinite-order-chain representation of the sequence of the partial quotients of the $N$-continued fraction expansion allows a concise formulation of the results obtained. Section 3 presents a Gauss-Kuzmin theorem related to the natural extension [10] of the measure-dynamical system corresponding to $N$-continued fraction expansions. Section 4 is devoted to derive explicit lower and upper bounds of the error which provide a more refined estimate of the convergence rate. The key role in this section is played by the transition operator associated with the RSCC underlying $N$-continued fraction on the Banach space of function of bounded variation. The last section collects some concluding remarks and give interesting numerical calculations.

2 Preliminaries

Fix an integer $N \geq 1$. The measure-theoretical dynamical system $(I, B_I, T_N)$ is defined as follows: $I := [0, 1]$, $B_I$ denotes the $\sigma$-algebra of all Borel subsets of $I$, and $T_N$ is the transformation

\[
T_N : I \to I; \quad T_N(x) := \begin{cases} \frac{N}{x} - \left\lfloor \frac{N}{x} \right\rfloor & \text{if } x \in I, \\ 0 & \text{if } x = 0. \end{cases}
\]
In addition, we write \((I, \mathcal{B}_I, G_N, T_N)\) as \((I, \mathcal{B}_I, T_N)\) with the following probability measure \(G_N\) on \((I, \mathcal{B}_I)\):

\[
G_N(A) := \frac{1}{\log \frac{N+1}{N}} \int_A \frac{dx}{x + N}, \quad A \in \mathcal{B}_I. \tag{2.2}
\]

The measure \(G_N\) is \(T_N\)-invariant, i.e., \(G_N(T_N^{-1}(A)) = G_N(A)\) for any \(A \in \mathcal{B}_I\) and the dynamical system \((I, \mathcal{B}_I, G_N, T_N)\) is ergodic. For any \(0 < x < 1\) put \(a_n(x) = a_1(T_N^{n-1}(x)), \ n \in \mathbb{N}_+ := \{1, 2, \ldots\}\), with \(T_N^0(x) = x\). Then every irrational \(0 < x < 1\) can be written in the form

\[
x = \frac{N}{a_1 + \frac{N}{a_2 + \frac{N}{a_3 + \cdots}}} := [a_1, a_2, \ldots]_N \tag{2.3}
\]

where \(a_n\)'s are non-negative integers, \(a_n \geq N, \ n \in \mathbb{N}_+\). We will call (2.3) the \(N\)-continued fraction expansion of \(x\) and \(p_n(x)/q_n(x) := [a_1, a_2, a_3, \ldots, a_n]_N\) the \(n\)-th order convergent of \(x \in I\). Then \(p_n(x)/q_n(x) \to x, \ n \to \infty\). Here \(p_n\)'s and \(q_n\)'s satisfy for \(n \in \mathbb{N}_+\) the following:

\[
p_n(x) := a_np_{n-1} + Np_{n-2}, \ n \geq 2 \tag{2.4}
\]

\[
q_n(x) := a_nq_{n-1} + Nq_{n-2}, \ n \geq 1 \tag{2.5}
\]

with \(p_0(x) := 0, q_0(x) := 1, p_1(x) := 1, q_1(x) := 0, p_1(x) := N, q_1(x) := a_1\). One easily shows that

\[
\left| x - \frac{p_n(x)}{q_n(x)} \right| < \frac{N^n}{q_n^2(x)} < \frac{1}{N^n}, \ n \geq 1. \tag{2.6}
\]

For any \(n \in \mathbb{N}_+\) and \(i^{(n)} = (i_1, \ldots, i_n) \in \Lambda^n, \Lambda := \{N, N+1, \ldots\}\), we will say that

\[
I_N(i^{(n)}) = \{x \in I \setminus \mathbb{Q} : a_k(x) = i_k \text{ for } k = 1, \ldots, n\} \tag{2.7}
\]

is the fundamental interval of rank \(n\) and make the convention that \(I_N(i^{(0)}) = I \setminus \mathbb{Q}\). For example, for any \(i \in \Lambda,\) we have

\[
I_N(i) = \{x \in I \setminus \mathbb{Q} : a_1 = i\} = (I \setminus \mathbb{Q}) \cap \left(\mathbb{N} \cap \left(\frac{N}{i + 1}, \frac{N}{i}\right)\right). \tag{2.8}
\]

For any \(n \in \mathbb{N}_+\) we have

\[
\lambda(T_N^n < x|a_1, \ldots, a_n) = \frac{(s_n + N)x}{s_n x + N}, \quad x \in I \tag{2.9}
\]
where \( s_n := Nq^{n-1} = [a_n, a_{n-1}, \ldots, a_2, a_1]_N \), \( n \geq 1 \), \( s_0 := 0 \) and \( \lambda \) is the Lebesgue measure on \( I \). Equation (2.9) is the Brodén-Borel-Lévy formula for this type of expansion. It allows us to determine the probability distribution of \((a_n)_{n \in \mathbb{N}_+}\) under \( \lambda \). For any \( i \in \Lambda \), we have

\[
\lambda(a_1 = i) = \frac{N}{i(i + 1)}, \quad \lambda(a_{n+1} = i|a_1, \ldots, a_n) = V_{N,i}(s_n) \quad (2.10)
\]

where

\[
V_{N,i}(x) := \frac{x + N}{(x + i)(x + i + 1)}. \quad (2.11)
\]

Hence the invariant probability measure of the transformation \( T_N \) is \( G_N \), the sequence \((a_n)_{n \in \mathbb{N}_+}\) is strictly stationary on \((I, \mathcal{B}_I, G_N)\). As such, a doubly infinite version of it, say \((a_l)_{l \in \mathbb{Z}}\), should exist on a richer probability space. Indeed, such a version can be effectively constructed on the natural extension \((I^2, \mathcal{B}_I^2, G_N, T_N)\) of \((I, \mathcal{B}_I, G_N, T_N)\) where the transformation \( T_N \) is defined on the square space \((I^2, \mathcal{B}_I^2) := (I, \mathcal{B}_I) \times (I, \mathcal{B}_I)\) by

\[
T_N(x, y) := \left( T_N(x), \frac{N}{a_1(x) + y} \right) \quad (2.12)
\]

and the extended measure \( \overline{G}_N \) is defined by

\[
\overline{G}_N(B) := \frac{1}{\log \left( \frac{N+1}{N} \right)} \int_B \frac{Ndx\,dy}{(xy + N)^2}, \quad B \in \mathcal{B}_I^2. \quad (2.13)
\]

The transformation \( \overline{T}_N \) is bijective on \( I^2 \) with the inverse

\[
(\overline{T}_N)^{-1}(x, y) = \left( \frac{N}{a_1(y) + x}, T_N(y) \right), \quad (x, y) \in I^2. \quad (2.14)
\]

Iterations of (2.12) and (2.14) are given as follows for each \( n \geq 2 \):

\[
(\overline{T}_N)^n(x, y) = (T^N_n(x), [a_n(x), a_{n-1}(x), \ldots, a_2(x), a_1(x) + y]_N) \quad (n \geq 1)
\]

\[
(\overline{T}_N)^{-n}(x, y) = ([a_n(y), a_{n-1}(y), \ldots, a_2(y), a_1(y) + x]_N, T^N_n(y)).
\]

We define extended incomplete quotients \( \overline{a}_l(x, y), l \in \mathbb{Z} \) at \((x, y) \in I^2\) by

\[
\overline{a}_l(x, y) := \overline{a}_1 \left( (\overline{T}_N)^{l-1}(x, y) \right), \quad l \in \mathbb{Z}, \quad (2.15)
\]

with \( \overline{a}_1(x, y) = a_1(x) \). Clearly, for \( n \in \mathbb{N}_+ \) and any \((x, y) \in I^2\), \( \overline{a}_n(x, y) = a_n(x) \), \( \overline{a}_0(x, y) = a_1(y) \), \( \overline{a}_{-n}(x, y) = a_{n+1}(y) \). Since the extended measure
\( G_N \) is preserved by \( T_N \), that is, \( G_N = G_N(T_N)^{-1} \), the doubly infinite sequence \((a_{\ell}(x, y))_{\ell \in \mathbb{Z}}\) is strictly stationary under \( G_N \). The dependence structure of \((a_{\ell})_{\ell \in \mathbb{Z}}\) follows from the fact that

\[
G_N([0, x] \times I | \sigma_0, \sigma_{-1}, \ldots) = \frac{(N + a)x}{ax + N} \quad G_N\text{-a.s.} \tag{2.16}
\]

for any \( x \in I \), where \( a := [\sigma_0, \sigma_{-1}, \ldots]_N \). Hence

\[
G_N(\sigma_1 = i | \sigma_0, \sigma_{-1}, \ldots) = V_{N,i}(a) \quad G_N\text{-a.s.} \tag{2.17}
\]

where \( V_{N,i} \) as in (2.11). We thus see that \((a_{\ell})_{\ell \in \mathbb{Z}}\) is an infinite-order-chain in the theory of dependence with complete connections \([6]\).

Put \( \overline{\sigma}_\ell = [\sigma_{\ell}, \sigma_{\ell-1}, \ldots]_N, \ell \in \mathbb{Z} \). Note that \( \overline{\sigma}_\ell = N/(\overline{\sigma}_{\ell-1} + \sigma_\ell), \ell \in \mathbb{Z} \). It follows from (2.17) that \((\overline{\sigma}_\ell)_{\ell \in \mathbb{Z}}\) is a strictly stationary \(I\)-valued Markov process on \((I^2, B_I^2, G_N)\) with the following transition mechanism: from state \( \overline{\sigma} \in I \) the only possible transitions are those to states \( N/(\overline{\sigma} + i), i \in \Lambda \), the transition probability being \( V_{N,i}(\overline{\sigma}) \). Clearly, whatever \( \ell \in \mathbb{Z} \) we have

\[
G_N(\overline{\sigma}_\ell < x) = G_N(\overline{\sigma}_0 < x) = G_N(I \times [0, x]) = G_N([0, x]), x \in I.
\]

Motivated by (2.16) we shall consider the family of (conditional) probability measures \((G_{N,a})_a\) on \( B_I \) defined by their distribution functions

\[
G_{N,a}([0, x]) = \frac{(N + a)x}{ax + N}, \quad x \in I, a \in I. \tag{2.18}
\]

In particular, \( G_{N,0} \) is the Lebesgue measure on \( I \). For any \( a \in I \) put \( s_{0,a} = a \) and \( s_{n,a} = N/(s_{n-1,a} + a_n), n \in \mathbb{N}_+ \). These facts lead us to the random system with compete connections \( \{(I, B_I), (\Lambda, \mathcal{P}(\Lambda)), u, V\} \), where \( \mathcal{P}(\Lambda) \) is the power set of \( \Lambda, u : I \times \Lambda \to I \) is defined as

\[
u(x, i) := u(x) = u_{N,i}(x) = \frac{N}{x + i} \tag{2.19}
\]

and \( V : I \times \Lambda \to I \) is defined as

\[
V(x, i) := V_{N,i}(x) = v_i(x) \tag{2.20}
\]

with \( V_{N,i} \) as in (2.11), for all \( x \in I \) and \( i \in \Lambda \).

Then \((s_{n,a})_{n \in \mathbb{N}_+}\) is an \( I \)-valued Markov chain on \((I, B_I, G_{N,a})\) which starts from \( s_{0,a} = a, a \in I \), and has the following transition mechanism: from state \( s \in I \) the only possible transitions are those to states \( N/(s + i) \) with the corresponding transition probability \( V_{N,i}(s), i \in \Lambda \). Let \( B(I) \) denote the Banach space of all bounded \( I \)-measurable complex-valued functions defined
on $I$ which is a Banach space under the supremum norm. The transition operator of $(s_n,a)_{n \in \mathbb{N}_+}$ takes $f \in B(I)$ into the function defined by

$$E_a(f(s_{n+1},a)|s_n,a = s) = \sum_{i \in \Lambda} V_{N,i}(s)f\left(\frac{N}{s+i}\right) = \{Uf\}(s) \text{ for any } s \in I,$$

where $E_a$ stands for the mean-value operator with respect to the probability measure $G_{N,a}$, whatever $a \in I$, and $U$ is the Perron-Frobenius operator of $(I, B_I, G_N, T_N)$ [9].

In connection with the operator $U$, if we define $U^\infty f = \int_I f(x)G_N(dx)$, $f \in B(I)$, then we have

$$U^\infty U^nf = U^\infty f, \text{ for any } f \in B(I) \text{ and } n \in \mathbb{N}_+.$$ (2.21)

Note that $G_{N,a}(A|a_1, \ldots, a_m) = G_{N,s_{m,a}}(T_N^m(A))$, for all $a \in I$, $A \in \sigma(a_{m+1}, \ldots)$ and $m \in \mathbb{N}_+$. In particular, it follows that the Brodén-Borel-Lévy formula holds under $G_{N,a}$ for any $a \in I$, that is,

$$G_{N,a}(T_N^m < x|a_1, \ldots, a_m) = \frac{(s_{m,a} + N)x}{s_{m,a}x + N}, \quad x \in I, m \in \mathbb{N}_+.$$ (2.22)

Note also that $(s_{n,a})_{n \in \mathbb{N}}$ under $G_{N,a}$ is a version of $(\overline{s}_{n})_{n \in \mathbb{N}}$ under $G_N(\cdot|\overline{s}_0 = a)$ for any $a \in I$.

### 3 A two-dimensional Gauss-Kuzmin theorem

The problem of finding the asymptotic behaviour of $T_N^{-n}(A)$ as $n \rightarrow \infty$, $A \in B_I$, represents the Gauss-Kuzmin-type problem for $N$-continued fraction expansions. We define the functions $(F_n)_{n \in \mathbb{N}}$ on $I$ by

$$F_0(x) := x, \quad F_n(x) := \lambda(T_N^n < x), n \geq 1.$$ (3.1)

The essential argument of the proof is the Gauss-Kuzmin-type equation which in this case is

$$F_{n+1}(x) = \sum_{i \in \Lambda} \left( F_n \left( \frac{N}{i} \right) - F_n \left( \frac{N}{i+1} \right) \right)$$ (3.2)

for $x \in I$ and $n \in \mathbb{N}$. The measure $G_N$ defined in (2.22) is an eigenfunction of (3.2), namely, if we put $F_n(x) = \log \left( \frac{x+N}{x} \right)$, $x \in I$, we obtain $F_{n+1}(x) = \log \left( \frac{x+N}{x} \right)$. Let us recall now a slightly modified version of Gauss-Kuzmin theorem for $T_N$ proved in [8].
Theorem 3.1. Let \((I, B_I, G_N, T_N)\) and \(F_n\) be as above. Then for any \(x \in I\),

\[
F_n(x) = G_N([0, x]) + \mathcal{O}(q^n) \tag{3.3}
\]

with \(0 < q < 1\).

In this section we will derive a Gauss-Kuzmin theorem related to the natural extension \((I^2, B^2_I, G_N, T_N)\) defined in Section 2.

For any \(n \in \mathbb{N}^+\) and \(x, y \in I\), let us define \(\Delta_{x,y} = [0, x] \times [0, y]\) and the functions \(F_n(x, y)\) by

\[
F_n(x, y) := \lambda((T_N)^n \in \Delta_{x,y}) , \tag{3.4}
\]

where \(\lambda\) is the Lebesgue measure on \(I^2\). Then the following holds.

Theorem 3.2. (A Gauss-Kuzmin theorem for \(T_N\)) For every \(n \geq 2\) and \((x, y) \in I^2\) one has

\[
F_n(x, y) = \frac{1}{\log \left(\frac{N+1}{N}\right)} \log \left(\frac{xy + N}{N}\right) + \mathcal{O}(q^n) \tag{3.5}
\]

with \(0 < q < 1\).

For any \(0 < y \leq 1\), put \(\ell_1 := \left\lfloor \frac{N}{y} \right\rfloor\). Then \((T_N)^{n+1}(x, y) \in \Delta_{x,y}\) is equivalent to

\[
(T_N)^n \in \left( \bigcup_{i \geq \ell_1 + N} \left[ \frac{N}{x+i}, \frac{N}{i} \right] \times [0, 1] \right) \cup \left( \left[ \frac{N}{x+\ell_1}, \frac{N}{\ell_1} \right] \times \left[ \frac{N}{y} - \ell_1, 1 \right] \right) .
\]

Now, from (3.4) we have the following recursion formula:

\[
F_{n+1}(x, y) = \sum_{i \geq \ell_1} \left( F_n \left( \frac{N}{i}, 1 \right) - F_n \left( \frac{N}{x+i}, 1 \right) \right)
- \left( F_n \left( \frac{N}{\ell_1}, \frac{N}{y} - \ell_1 \right) - F_n \left( \frac{N}{x+\ell_1}, \frac{N}{y} - \ell_1 \right) \right) . \tag{3.6}
\]

The measure \(G_N\) defined in (2.13) is an eigenfunction of (3.6), namely, if we put \(F_n(x, y) = \log \left( \frac{xy+N}{N} \right)\), \(x, y \in I\), we obtain \(F_{n+1}(x, y) = \log \left( \frac{xy+N}{N} \right)\).
Lemma 3.3. Let \( n \in \mathbb{N}, n \geq 2 \) and let \( y \in I \cap \mathbb{Q} \) with \( y = [\ell_1, \ldots, \ell_d]_N, \ell_1, \ldots, \ell_d \in \Lambda, \ell_d \geq N + 1 \), where \( d \leq \lfloor n/(N+1) \rfloor \). Then for every \( x, x^* \in [0, 1) \) with \( x^* < x \),

\[
\left| F_n(x, y) - F_n(x^*, y) - \frac{1}{\log \left( \frac{N+1}{N} \right)} \log \left( \frac{xy + N}{x^*y + N} \right) \right| < C \lambda(\Delta_{x,y} \setminus \Delta_{x^*,y}) q^{n-d},
\]

where \( C \) is an universal constant.

Proof. Let \( y_0 = y, y_i := [\ell_{i+1}, \ldots, \ell_1]_N, i = 1, \ldots, d - 1, \) and \( y_d = 0 \). Applying (3.6) one gets

\[
F_n(x, y) - F_n(x^*, y) = \sum_{i \geq \ell_1} \left( F_{n-1} \left( \frac{N}{x^* + i}, 1 \right) - F_{n-1} \left( \frac{N}{x + i}, 1 \right) \right) + \left( F_{n-1} \left( \frac{N}{x + \ell_1}, y_1 \right) - F_{n-1} \left( \frac{N}{x^* + \ell_1}, y_1 \right) \right) \quad \text{(3.7)}
\]

Now for each \( B \in \mathcal{B}_1^2 \) one has

\[
\frac{N}{(N+1)^2 \log \left( \frac{N+1}{N} \right)} \lambda(B) \leq G_N(B) \leq \frac{1}{N \log \left( \frac{N+1}{N} \right)} \lambda(B). \quad \text{(3.8)}
\]

Now from (2.8), (3.8) and the fact that \( T_N \) is \( G_N \)-invariant, it follows that:

\[
\sum_{i \geq \ell_1} \left( \frac{N}{x^* + i} - \frac{N}{x + i} \right) = \sum_{i \geq \ell_1} \lambda \left( \left[ 0, \frac{N}{x + i} \right] \times \left[ 0, \frac{N}{x^* + i} \right] \right) \leq (N+1)^2 \log \left( \frac{N+1}{N} \right) \sum_{i \geq \ell_1} \lambda(I_N(i)) \leq \frac{(N+1)^2}{N} (x - x^*) \sum_{i \geq \ell_1} \lambda(I_N(i)) \leq \frac{(N+1)^2}{N} (x - x^*) N \ell_1 \leq \frac{(N+1)^3}{N} (x - x^*) y \leq \frac{(N+1)^3}{N} \lambda(\Delta_{x,y} \setminus \Delta_{x^*,y}). \quad \text{(3.9)}
\]

For every \( 2 \leq k \leq d \), a similar analysis leads to

\[
\sum_{i \geq \ell_k} |[i, \ell_{k-1}, \ldots, x^* + \ell_1]_N - [i, \ell_{k-1}, \ldots, x + \ell_1]_N| \leq \ldots
\]
\[(N+1)^2 \log \left( \frac{N + 1}{N} \right) \sum_{i \geq \ell_k} G_N \left( (I_N)^k [i, \ell_{k-1}, \ldots, x^* + \ell_1]_N, [i, \ell_{k-1}, \ldots, x + \ell_1]_N \times [0, 1] \right) \]
\leq \frac{(N+1)^2}{N} \sum_{i \geq \ell_k} \lambda((x^*, x) \times I_N (\ell_1, \ldots, \ell_{k-1}, i)) \\
\leq \frac{(N+1)^2}{N} (x - x^*) \lambda(I_N (\ell_1, \ldots, \ell_{k-1})) \\
\leq \frac{(N+1)^3}{N} (x - x^*) y = \frac{(N+1)^3}{N} \lambda(\Delta_{x,y} \setminus \Delta_{x^*,y}). \tag{3.10}

Since \(F_n(x, 1) = F_n(x)\), from Theorem 3.1 it follows that

\[
\sum_{i \geq \ell_1} \left( F_{n-1} \left( \frac{N}{x^* + i}, 1 \right) - F_{n-1} \left( \frac{N}{x + i}, 1 \right) \right) = \\
\sum_{i \geq \ell_1} \left( G_N \left( \left[ \frac{N}{x + i}, \frac{N}{x^* + i} \right] \right) + \left( \frac{N}{x^* + i} - \frac{N}{x + i} \right) O(q^{n-1}) \right) = \\
\frac{1}{\log \left( \frac{N+1}{N} \right)} \sum_{i \geq \ell_1} \log \left( \frac{x^* + i + 1}{x + i + 1} \cdot \frac{x + i}{x^* + i} \right) + \\
\frac{(N+1)^3}{N} \lambda(\Delta_{x,y} \setminus \Delta_{x^*,y}) O(q^{n-1}) \leq \\
\frac{1}{\log \left( \frac{N+1}{N} \right)} \cdot \log \left( \frac{x + \ell_1}{x^* + \ell_1} \right) + \frac{(N+1)^3}{N} \lambda(\Delta_{x,y} \setminus \Delta_{x^*,y}) O(q^{n-1}). \tag{3.11}
\]

Now from (3.7), (3.11), we have:

\[
F_n(x, y) - F_n(x^*, y) = \frac{1}{\log \left( \frac{N+1}{N} \right)} \log \left( \frac{x + \ell_1}{x^* + \ell_1} \right) + \\
\frac{(N+1)^3}{N} \lambda(\Delta_{x,y} \setminus \Delta_{x^*,y}) O(q^{n-1}) + \\
\frac{1}{\log \left( \frac{N+1}{N} \right)} \log \left( \frac{\ell_2 + \frac{N}{x^* + \ell_1}}{\ell_2 + \frac{N}{x + \ell_1}} \right) + \frac{(N+1)^3}{N} \lambda(\Delta_{x,y} \setminus \Delta_{x^*,y}) O(q^{n-2}) + \\
F_{n-2} \left( \frac{N}{\ell_2 + \frac{N}{x^* + \ell_1}}, y_2 \right) - F_{n-2} \left( \frac{N}{\ell_2 + \frac{N}{x + \ell_1}}, y_2 \right).
\]
Applying (3.7) \( d \)-times and tacking into account that \( y_d = 0 \), we get

\[
\frac{1}{\log \left( \frac{x+y}{N} \right)} \log \left( \frac{x+\ell_1}{x^*+\ell_1} \right) \frac{F_n(x,y) - F_n(x^*,y)}{x^*+\ell_1} = \frac{F_n(x,y) - F_n(x^*,y)}{x^*+\ell_1} = \frac{x+y}{x^*+\ell_1} \log \left( \frac{x+y}{x^*+\ell_1} \right) + \frac{(N+1)^4}{N} \log \left( \frac{x+y}{x^*+\ell_1} \right) \frac{\lambda(\Delta_{x,y} \setminus \Delta_{x^*,y})}{\lambda(\Delta_{x,y} \setminus \Delta_{x^*,y})} \left( \mathcal{O}(q^{n-1}) + \cdots + \mathcal{O}(q^{n-d}) \right).
\]

If \( p_d \) and \( q_d \) are as in (2.4) and (2.5) with \( a_1 = x+\ell_1 \) and \( a_d = \ell_d, \ d \geq 2 \), then \( [\ell_i, \ldots, \ell_2, x+\ell_1]_N = N^{q_i - 1}/q_i, \ i = 1, \ldots, d. \) Since \( q_0 = 1 \), thus we have

\[
(x + \ell_1)([x + \ell_1]_N + \ell_2) \cdots ([\ell_d-1, \ldots, \ell_2, x + \ell_1]_N + \ell_d) = q_d.
\]

Let \( p_d^* \) and \( q_d^* \) are as in (2.4) and (2.5), with \( a_1 = x^*+\ell_1 \) and \( a_d = \ell_d, \ d \geq 2 \). Note that \( p_d = p_d^* \). Thus we find that

\[
\frac{(x + \ell_1)([x + \ell_1]_N + \ell_2) \cdots ([\ell_d-1, \ldots, \ell_2, x + \ell_1]_N + \ell_d)}{(x^* + \ell_1)([x^* + \ell_1]_N + \ell_2) \cdots ([\ell_d-1, \ldots, \ell_2, x^* + \ell_1]_N + \ell_d)} = \frac{q_d}{q_d^*} = \frac{p_d^* q_d}{q_d^* p_d} = \frac{x + \ell_1 + [\ell_2, \ldots, \ell_d]_N}{x^* + \ell_1 + [\ell_2, \ldots, \ell_d]_N} \frac{x + \frac{N}{q_d^*}}{x^* + \frac{N}{q_d^*}} = \frac{x y + N}{x^* y + N}.
\]

Therefore,

\[
F_n(x,y) - F_n(x^*,y) = \frac{1}{\log \left( \frac{x+y}{N} \right)} \log \left( \frac{x+y}{x^*+\ell_1} \right) + \frac{(N+1)^3}{N} \lambda(\Delta_{x,y} \setminus \Delta_{x^*,y}) \mathcal{O}(q^{n-1}) + \cdots + \mathcal{O}(q^{n-d})
\]

which completes the proof. \( \square \)

**Proof of Theorem 3.2** Since \( \Delta_{x,pd/qd} \subset \Delta_{x,y} \) and \( F_n(x,y) = \lambda \left( (T_N)^{-n} (\Delta_{x,y}) \right) \), from (2.6), (3.8) and the fact that \( T_N \) is \( G_N \)-invariant, we find that

\[
F_n(x,y) - F_n(x,y) = \lambda \left( (T_N)^{-n} (\Delta_{x,y}) \setminus (T_N)^{-n} (\Delta_{x,pd/qd}) \right) \\
\leq \frac{(N+1)^2}{N} \log \left( \frac{N+1}{N} \right) G_N \left( (T_N)^{-n} (\Delta_{x,y}) \setminus (T_N)^{-n} (\Delta_{x,pd/qd}) \right) \\
\leq \left( \frac{N+1}{N} \right)^2 \lambda \left( [0,x] \times \left[ \frac{p_d}{q_d}, y \right] \right) \\
\leq \left( \frac{N+1}{N} \right)^2 x \left| y - \frac{p_d}{q_d} \right| \leq \left( \frac{N+1}{N} \right)^2 \cdot \frac{x}{N^n}.
\]

(3.12)
Since for every fixed \( x \in [0, 1] \) the function \( y \mapsto \log \left( \frac{xy + N}{N} \right) \) is a differentiable on \([0, 1]\), by the \( Mean \ Value \ Theorem \) we have

\[
\left| \log \left( \frac{xy + N}{N} \right) - \log \left( \frac{x \frac{p_d}{q_d} + N}{N} \right) \right| = \left| y - \frac{p_d}{q_d} \frac{x}{\xi + N} \right| \leq x \left| y - \frac{p_d}{q_d} \right| \leq \frac{x}{N^{n^*}}. \tag{3.13}
\]

where \( p_d/q_d \leq \xi \leq y \). From Lemma 3.3, (3.12) and (3.13), we have

\[
\left| F_n(x, y) - \frac{1}{\log \left( \frac{N+1}{N} \right)} \log \left( \frac{xy + N}{N} \right) \right| \leq \left| F_n(x, y) - F_n \left( x, \frac{p_d}{q_d} \right) \right|
\]
\[
+ \left| F_n \left( x, \frac{p_d}{q_d} \right) - F_n(0, \frac{p_d}{q_d}) \right| - \frac{1}{\log \left( \frac{N+1}{N} \right)} \log \left( \frac{x \frac{p_d}{q_d} + N}{N} \right)
\]
\[
+ \frac{1}{\log \left( \frac{N+1}{N} \right)} \left| \log \left( \frac{xy + N}{N} \right) - \log \left( \frac{x \frac{p_d}{q_d} + N}{N} \right) \right|
\]
\[
\leq \left( \frac{N+1}{N} \right)^2 \cdot \frac{x}{N^n} + Cq^{n-d} + \frac{x}{N^n}
\]

which completes the proof. \( \Box \)

4 \hspace{1em} \textbf{Improving result}

In this section we shall estimate the error term

\[
e_{n,a}(x, y) = G_{n,a} \left( T_n^n \in [0, x], s_n,a \in [0, y] \right) - \frac{1}{\log \left( \frac{N+1}{N} \right)} \log \left( \frac{xy + N}{N} \right)
\]

for any \( a, x, y \in \mathcal{I} \) and \( n \in \mathbb{N} \).

In the main result of this section, Theorem 4.7, we shall derive lower and upper bounds (not depending on \( a \in \mathcal{I} \)) of the supremum

\[
\sup_{x \in \mathcal{I}, y \in \mathcal{I}} \left| e_{n,a}(x, y) \right|, \quad a \in \mathcal{I}, \tag{4.1}
\]

which provide a more refined estimate of the convergence rate involved. First, we obtain a lower bound for the error, which suggests the exact convergence rate of \( G_{n,a} \left( s_n,a \in [0, y] \right) \) to \( G_N ([0, y]) \) as \( n \to \infty \) for all \( a \in \mathcal{I} \).
Theorem 4.1. Whatever $a \in I$ we have

$$\frac{1}{2}v_N(a)(1) \leq \sup_{y \in I} |G_{N,a}(s_n,a) - G_N([0,y])|$$

(4.2)

for all $n \in \mathbb{N}_+$.

**Proof.** First, using the continuity of $G_N([0,y])$ with respect to $y$ it is easy to see that

$$\sup_{y \in I} |G_{N,a}(s_n,a) - G_N([0,y])| = \sup_{y \in I} |G_{N,a}(s_n,a) - G_N([0,y])|$$

for all $a \in I$ and $n \in \mathbb{N}$. Second, whatever $s \in I$ we have

$$G_{N,a}(s_n,a) = G_{N,a}(s_n,a) - G_N([0,s]) - (G_{N,a}(s_n,a) - G_N([0,s])) \leq \sup_{y \in I} |G_{N,a}(s_n,a) - G_N([0,y])| + \sup_{y \in I} |G_{N,a}(s_n,a) - G_N([0,y])|$$

$$= 2 \sup_{y \in I} |G_{N,a}(s_n,a) - G_N([0,y])|.$$}

Hence

$$\sup_{y \in I} |G_{N,a}(s_n,a) - G_N([0,y]) - G_N([0,y])| \geq \frac{1}{2} \sup_{s \in I} G_{N,a}(s_n,a = s),$$

for all $a \in I$ and $n \in \mathbb{N}$. For any $n \in \mathbb{N}_+$ we have

$$U^nf(y) = \sum_{i_1, \ldots, i_n} v_{i_1, \ldots, i_n}(x) f(u_{i_1, \ldots, i_n}(x))$$

(4.3)

where $u_{i_1, \ldots, i_n} = u_{i_n} \circ \ldots \circ u_{i_1}$, $v_{i_1, \ldots, i_n}(x) = v_{i_1}(x)v_{i_2}(u_{i_1}(x)) \ldots v_{i_n}(u_{i_{n-1} \ldots i_1}(x))$, $n \geq 2$, and the functions $u_i$ and $v_i$, $i \in \Lambda$, are defined in (2.19) and (2.20). Now, we obtain

$$G_{N,a}(s_n,a = [i_1, \ldots, i_2, i_1 + a]_N) = G_{N,a}(I_N(i(a))) = v_{i_1, \ldots, i_n}(a),$$

$n \geq 2,$

$$G_{N,a}(s_1,a = \frac{N}{i_1 + a}) = G_{N,a}(I_N(i_1)) = v_{i_1}(a)$$

for all $a \in I$ and $i_1, \ldots, i_n \in \Lambda$. 

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Putting \( p_n(i_1, \ldots, i_n) / q_n(i_1, \ldots, i_n) = [i_1, \ldots, i_n]_N \)
we get

\[
v_{i_1 \ldots i_n}(x) = \frac{(x + N)^{n-1}}{q_{n-1}(i_2, \ldots, i_n)(x + i_1) + p_{n-1}(i_2, \ldots, i_n)} \times \frac{1}{q_n(i_2, \ldots, i_n, N)(x + i_1) + p_n(i_2, \ldots, i_n, N)}
\]

(4.4)

for all \( i_n \in \Lambda, n \geq 2, \) and \( a \in I. \)

By (4.4) we have

\[
\sup_{s \in I} G_{N,a}(s_{n,a} = s) = v_{N(n)}(a) \quad (4.5)
\]

where we write \( N(n) \) for \((i_1, \ldots, i_n)\) with \( i_1 = \ldots = i_n = N, n \in \mathbb{N}_+. \)

Also by (4.4) we have

\[
v_{N(n)}(a) = \frac{(a + N)^{n-1}}{q_{n-1}(N, \ldots, N)(a + N) + p_{n-1}(N, \ldots, N) \quad (n-1 \text{ times})} \times \frac{1}{q_n(N, \ldots, N, N)(a + N) + p_n(N, \ldots, N, N) \quad n \text{ times}}.
\]

It is easy to see that \( v_{N(n)}(\cdot) \) is a decreasing function. Therefore

\[
\sup_{s \in I} G_{N,a}(s_{n,a} = s) \geq v_{N(n)}(1)
\]

for all \( a \in I. \)

\[\square\]

**Theorem 4.2.** (The lower bound) Whatever \( a \in I \) we have

\[
\frac{1}{2} v_{N(n)}(1) \leq \sup_{x \in I, y \in I} \left| G_{N,a}(T^n_N \in [0, x], s_{n,a} \in [0, y]) - \frac{1}{\log \left( \frac{N+1}{N} \right)} \log \left( \frac{xy + N}{N} \right) \right|
\]

for all \( n \in \mathbb{N}_+. \)
Proof. Whatever $a \in I$ and $n \in \mathbb{N}_+$, by Theorem 4.1 we have

$$\sup_{x \in I, y \in I} \left| G_{N,a} (T_N^n \in [0, x], s_{n,a} \in [0, y]) - \frac{1}{\log \left( \frac{N+1}{N} \right)} \log \left( \frac{xy + N}{N} \right) \right| \geq \sup_{y \in I} \left| G_{N,a} (T_N^n \in I, s_{n,a} \in [0, y]) - \frac{1}{\log \left( \frac{N+1}{N} \right)} \log \left( \frac{y + N}{N} \right) \right| = \sup_{y \in I} \left| G_{N,a} (s_{n,a} \in [0, y]) - G_N ([0, y]) \right| \geq \frac{1}{2} v_{N(n)}(1).$$

□

Remark 4.3. Here

$$v_{N(n)}(1) = \frac{(1 + N)N^{n+1}}{q_{n+1}q_{n+2}}, \quad n \in \mathbb{N}_+,$$

where $q_n = N(q_{n-1} + q_{n-2})$, $n \in \mathbb{N}_+$, with $q_{-1} = 0$ and $q_0 = 1$. It is easy to see that

$$q_n = \frac{1}{\sqrt{N^2 + 4N}} \left[ \left( \frac{N + \sqrt{N^2 + 4N}}{2} \right)^{n+1} - \left( \frac{N - \sqrt{N^2 + 4N}}{2} \right)^{n+1} \right].$$

It should be noted that Theorem 4.2 in connection with the limit

$$\lim_{n \to \infty} \left( \frac{1}{2} v_{N(n)}(1) \right)^{1/n} = \frac{2}{N + \sqrt{N^2 + 4N} + 2}$$

leads to an estimate of the order of magnitude of the error $e_{n,a}(x, y)$.

It is known that for the RCF-expansion [3] the exact order of magnitude of the supremum there is $O(g^{2n})$ with $g = (\sqrt{5} - 1)/2$, $g^2 = (3 - \sqrt{5})/2 = 0.38196\ldots$. Note that for $N = 1$, $\lim_{n \to \infty} \left( \frac{1}{2} v_{N(n)}(1) \right)^{1/n} = g^2$.

In what follows we study the transition operator associated with the RSCC underlying $N$-continued fraction on the Banach space of complex-valued functions of bounded variation. The characteristic properties of this operator are used to derive an explicit upper bound for $\sup_{x \in I, y \in I} |e_{n,a}(x, y)|$, $a \in I$.

Let $BV(I)$ the Banach space of complex-valued functions $f$ of bounded variation on $I$ under the norm

$$\|f\|_v := \text{var} f + |f|.$$
Remember that the variation $\text{var}_A f$ over $A \subset I$ of $f \in B(I)$ is defined as

$$
\sup \sum_{i=1}^{k} |f(t_i) - f(t_{i+1})|
$$

the supremum being taken over all $t_1 < \cdots < t_k \in A$, $k \geq 2$. We write simply $\text{var} f$ for $\text{var}_I f$ and if $\text{var} f < \infty$, then $f$ is called a function of bounded variation.

We start by proving the following elementary result.

**Proposition 4.4.** For any $f \in BV(I)$ we have

$$
\text{var} U f \leq \frac{1}{N+1} \cdot \text{var} f.
$$

**Proof.** Put $v_i(x) = (x + N) \left( \frac{1}{x+i} - \frac{1}{x+i+1} \right), i \in \Lambda$. We have

$$
v'_i(x) = \frac{1}{(x+i)(x+i+1)} \left[ 1 - \frac{(x+N)(2x+2i+1)}{(x+i)(x+i+1)} \right], i \in \Lambda.
$$

It follows that $v'_i(x) < 0, x \in I, i \in \Lambda$. Hence

$$
\text{var} v_i = v_i(0) - v_i(1), i \in \Lambda
$$

$$
|v_i| = \sup_{x \in I} v_i(x) = v_i(0) = \frac{N}{i(i+1)}, i \in \Lambda.
$$

Thus

$$
\sup_{i \in \Lambda} |v_i| = v_N(0) = \frac{1}{N+1}.
$$

Also,

$$
\sum_{i \in \Lambda} \text{var} v_i = \sum_{i \in \Lambda} \left( \frac{N}{i(i+1)} - \frac{N+1}{(i+1)(i+2)} \right) = 0.
$$

We have

$$
\text{var} U f = \text{var} \sum_{i \in \Lambda} v_i \cdot (f \circ u_i) \leq \sum_{i \in \Lambda} \text{var} (v_i \cdot (f \circ u_i))
$$

$$
\leq \sum_{i \in \Lambda} |v_i| \text{var}(f \circ u_i) + \sum_{i \in \Lambda} |f \circ u_i| \text{var} v_i
$$

$$
\leq \left( \sup_{i \in \Lambda} |v_i| \right) \sum_{i \in \Lambda} \text{var}(f \circ u_i) + |f| \sum_{i \in \Lambda} \text{var} v_i
$$

$$
\leq \frac{1}{N+1} \cdot \text{var} f
$$

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because
\[
\sum_{i \in \Lambda} \var(f \circ u_i) = \sum_{i \in \Lambda} \var[\frac{N}{N+1}] f = \var f.
\]

\[\square\]

**Corollary 4.5.** For any \( f \in BV(I) \) and for all \( n \in \mathbb{N} \) we have
\[
\var U^n f \leq \frac{1}{(N + 1)^n} \cdot \var f,
\]
(4.6)
\[
|U^n f - U^\infty f| \leq \frac{1}{(N + 1)^n} \cdot \var f.
\]
(4.7)

**Proof.** Note that for any \( f \in BV(I) \) and \( u \in I \) we have
\[
|f(u)| - \left| \int_I f(x)G_N(dx) \right| \leq \left| f(u) - \int_I f(x)G_N(dx) \right| = \left| \int_I (f(u) - f(x))G_N(dx) \right| \leq \var f,
\]
whence
\[
|f| \leq \left| \int_I f(x)G_N(dx) \right| + \var f, \quad f \in BV(I).
\]
(4.8)

Finally, (2.21) and (4.8) imply that
\[
|U^n f - U^\infty f| \leq \var (U^n f - U^\infty f) = \var U^n f.
\]
for all \( n \in \mathbb{N} \) and \( f \in BV(I) \), which leads to (4.7). \[\square\]

**Theorem 4.6.** (The upper bound) Whatever \( a \in I \) we have
\[
\sup_{x \in I, y \in I} \left| G_{N,a}(T^a_N \in [0, x], s_{n,a} \in [0, y]) - \frac{1}{\log \left( \frac{N+1}{N} \right)} \log \left( \frac{xy + N}{N} \right) \right| \leq \frac{1}{(N + 1)^n}
\]
for all \( n \in \mathbb{N} \).

**Proof.** Let \( F_{n,a}(y) = G_{N,a}(s_{n,a} \leq y) \) and \( H_{n,a}(y) = F_{n,a}(y) - G_N([0, y]) \), \( a, y \in I, n \in \mathbb{N} \). As we have noted \( U \) is the transition operator of the Markov chain \((s_{n,a})_{n \in \mathbb{N}}\). For any \( y \in I \) consider the function \( f_y \) defined on \( I \) as
\[
f_y(a) := \begin{cases} 
1 & \text{if } 0 \leq a \leq y, \\
0 & \text{if } y < a \leq 1.
\end{cases}
\]
Hence
\[
U^n f_y(a) = E_a (f_y(s_{n,a})|s_{0,a} = a) = G_{N,a}(s_{n,a} \leq y)
\]
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for all \( a, y \in I, \ n \in \mathbb{N} \). As

\[
U^\infty f_y = \int_I f_y(a)G_N(da) = G_N([0, y]), \quad y \in I.
\]

It follows from Corollary 4.5 that

\[
|H_{n,a}(y)| = |G_{N,a}(s_{n,a} \leq y) - G_N([0, y])| = |U^n f_y(a) - U^\infty f_y| \leq \frac{1}{(N + 1)^n} \text{var } f_y = \frac{1}{(N + 1)^n} \tag{4.9}
\]

for all \( a, y \in I, \ n \in \mathbb{N} \). By (2.22), for all \( a \in I, \ x, y \in I \) and \( n \in \mathbb{N} \) we have

\[
G_{N,a}(T^a_N \in [0, x], s_{n,a} \in [0, y]) = \int_0^y G_{N,a}(T^a_N \in [0, x]| s_{n,a} = z) \, dF_{n,a}(z)
\]

\[
= \int_0^y \frac{(z + N)x}{zx + N} \, dF_{n,a}(z) = \int_0^y \frac{(z + N)x}{zx + N} \, dG_N(z) + \int_0^y \frac{(z + N)x}{zx + N} \, dH_{n,a}(z)
\]

\[
= \frac{1}{\log \left( \frac{N+1}{N} \right)} \log \left( \frac{xy + N}{N} \right) + \frac{(z + N)x}{zx + N} H_{n,a}(z) - \int_0^y \frac{Nx(1-x)}{(zx + N)^2} H_{n,a}(z) \, dz.
\]

Hence, by (4.9)

\[
\left| G_{N,a}(T^a_N \in [0, x], s_{n,a} \in [0, y]) - \frac{1}{\log \left( \frac{N+1}{N} \right)} \log \left( \frac{xy + N}{N} \right) \right| \leq \frac{1}{(N + 1)^n} \log \left( \frac{N+1}{N} \right) \log \left( \frac{xy + N}{N} \right)
\]

for all \( a, x, y \in I \) and \( n \in \mathbb{N} \). \( \Box \)

Combining Theorem 4.2 with Theorem 4.6 we obtain Theorem 4.7.

**Theorem 4.7.** Whatever \( a \in I \) we have

\[
\frac{1}{2^{\nu_N(a)}}(1) \leq
\sup_{x \in I, y \in I} \left| G_{N,a}(T^a_N \in [0, x], s_{n,a} \in [0, y]) - \frac{1}{\log \left( \frac{N+1}{N} \right)} \log \left( \frac{xy + N}{N} \right) \right| \leq \frac{1}{(N + 1)^n}
\]

for all \( n \in \mathbb{N}_+ \).

**Remark 4.8.** Theorem 4.7 implies that the convergence rate is \( O(\alpha^n) \), with

\[
\frac{2}{N + \sqrt{N^2 + 4N} + 2} \leq \alpha \leq \frac{1}{N + 1}.
\]
For example, we have

| $N$ | $g^2 = 0.381966 \leq \alpha \leq 0.5$ |
|-----|--------------------------------------|
| $N = 2$ | $0.267949192 \leq \alpha \leq 0.333333\ldots$ |
| $N = 5$ | $0.145898033 \leq \alpha \leq 0.166666\ldots$ |
| $N = 10$ | $0.083920216 \leq \alpha \leq 0.090909\ldots$ |
| $N = 100$ | $0.009804864 \leq \alpha \leq 0.00990099$ |
| $N = 1000$ | $0.000998004 \leq \alpha \leq 0.000999$ |
| $N = 10000$ | $0.00009998 \leq \alpha \leq 0.00009999$ |

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