ON THE FINITENESS OF TWISTS OF IRREDUCIBLE SYMPLECTIC VARIETIES

TEPPEI TAKAMATSU

Abstract. Irreducible symplectic varieties are higher-dimensional analogues of K3 surfaces. In this paper, we prove the finiteness of twists of irreducible symplectic varieties via a fixed finite field extension of characteristic 0. The main ingredient of the proof is the cone conjecture for irreducible symplectic varieties, which was proved by Markman and Amerik–Verbitsky. As byproducts, we also discuss the cone conjecture over non-closed fields by Bright–Logan–van Luijk’s method. We also give an application to the finiteness of derived equivalent twists. Moreover, we discuss the case of K3 surfaces or Enriques surfaces over fields of positive characteristic.

1. Introduction

The classification of varieties is one of the main purposes of algebraic geometry. Over an algebraically closed field, there are many such classification theorems. However, over a general base field, such classification problems become difficult since there are many twisted forms of varieties. In this paper, we study the finiteness of twists of varieties, especially for irreducible symplectic varieties. Let $k$ be a field, and $k'$ a finite field extension of $k$. Let $X$ be a variety over $k$. We denote the set of $k$-isomorphism classes of varieties over $k$ which are isomorphic to $X_{k'} := X \times_k k'$ after the base change to $k'$ by $\text{Tw}_{k'/k}(X)$. In general, the set $\text{Tw}_{k'/k}(X)$ is not necessarily finite even when $X$ is smooth and projective over $k$. For example, if $X$ is a projective space $\mathbb{P}^n_k$ over a number field $k$ and $k'/k$ is a non-trivial extension, then one can prove that $\text{Tw}_{k'/k}(X)$ is infinite by the Morita equivalence and the Chebotarev’s density theorem. On the other hand, if $k'/k$ is a Galois extension and $X_{k'}$ has an automorphism group of finite order, then the Galois cohomological argument asserts that $\text{Tw}_{k'/k}(X)$ is finite. In particular, $\text{Tw}_{k'/k}(X)$ is finite if $X$ is general type varieties of characteristic 0. Therefore, this finiteness reflects certain finiteness of the automorphism group of a variety, and it seems to be most interesting when the Kodaira dimension of $X$ is 0.

The main theorem of this paper is the following.

Theorem 1.0.1 (Theorem 3.1.2, Theorem 4.3.6). Let $k'/k$ be a finite extension of fields, $X$ a variety over $k$. Then $\text{Tw}_{k'/k}(X)$ is a finite set in the following cases.

1. $\text{char } k \neq 2$, and $X$ is a K3 surface over $k$.
2. $\text{char } k = 2$, and $X$ is a non-supersingular K3 surface over $k$.
3. $\text{char } k = 0$, and $X$ is an irreducible symplectic variety over $k$.

Moreover, in the above cases, we also prove that $\# \text{Tw}_{k'/k}(X)$ is bounded by a constant which depends only on $[k': k]$, the isometry class of geometric Néron–Severi lattice, and the deformation class of $X$ (see Theorem 5.0.1 for precise statements).

Theorem 1.0.1 is a generalization of Cattaneo–Fu’s work ([CF19]) on the finiteness of real forms of irreducible symplectic varieties. By the example of projective spaces, we
can say that the finiteness of $\text{Tw}_{k'/k}$ is more interesting when $k$ is a global field rather than a local field. Indeed, when $k$ is a global field, as in [Tak20a, Corollary 6.2.1], such finiteness is one of the evidence of the finiteness of varieties with cohomologies of bounded ramification (cohomological formulation of the Shafarevich conjecture).

We also give such finiteness for Enriques surfaces of arbitrary characteristic, by using Liedtke’s lifting result for Enriques surfaces [Lie15]. This finiteness can be seen as one of the evidence of several finiteness properties of the automorphism group of Enriques surfaces of characteristic 2.

**Theorem 1.0.2** (Theorem 3.2.5). Let $k'/k$ be a finite extension of perfect fields of characteristic 2, $X$ an Enriques surface over $k$. Then $\text{Tw}_{k'/k}(X)$ is a finite set.

Moreover, as an application, we give the following finiteness of derived equivalent twists. We recall that varieties are derived equivalent if there is an equivalence between the bounded derived categories of coherent sheaves. We denote the set of $k$-isomorphism classes of varieties over $k$ which are derived equivalent to $X$ and isomorphic to $X_{\overline{k}}$ after the base change to an algebraic closure $\overline{k}$ by $\text{Tw}^D(X)$.

**Corollary 1.0.3** (Theorem 6.0.5). In the case (1) or (2) in Theorem 1.0.1, the set $\text{Tw}^D(X)$ is a finite set.

We also prove the same finiteness for some irreducible symplectic varieties, including $K3^{[n]}$-type, generalized Kummer type, and $OG_{10}$-type (see Theorem 6.0.5 and Corollary 6.0.6 for precise statements). Combining Corollary 1.0.3 with [BM01, Corollary 1.2] and [LO15, Theorem 1.1], we have the following finiteness over non-closed fields.

**Corollary 1.0.4.** Let $X$ be a $K3$ surface over a field $k$ of characteristic $\neq 2$. Then there exist only finitely many $k$-isomorphism classes of smooth projective varieties $Y$ over $k$ which are derived equivalent to $X$.

Now we give some comments on proofs of Theorem 1.0.1 for irreducible symplectic varieties. In Cattaneo–Fu’s work [CF19], first, they study the Klein automorphism group of irreducible symplectic varieties by using the cone conjecture for irreducible symplectic varieties, and then they see the Galois cohomology directly. In our situation, there is no direct analogue of the Klein automorphism group. Thus to avoid the difficulty of the group structure and the Galois cohomological arguments, we will take a different approach, though we use the cone conjecture for irreducible symplectic varieties too. The main idea is reducing the problem to the finiteness of twists of quasi-polarized irreducible symplectic variety. In this reduction, we need two steps. First, we should bound the polarization degree of irreducible symplectic variety. Secondary, we need the finiteness of the set of polarizations of bounded degree modulo automorphisms for irreducible symplectic variety. Both steps are related to the cone conjecture for irreducible symplectic varieties, proved by Markman [Mar11] and Amerik and Verbitsky [AV17], [AV16] when the base field is algebraically closed (see also Markman and Yoshioka’s work [MY15]). As byproducts, we also prove the following cone conjecture for irreducible symplectic varieties over non-closed fields of characteristic 0.

**Theorem 1.0.5** (Theorem 4.1.4, Theorem 4.2.7). Let $k$ be a field of characteristic 0. Let $X$ be an irreducible symplectic variety over $k$.

1. The action of $\text{Bir}(X)$ on $\mathcal{MV}^+_X$ admits a rational polyhedral fundamental domain.
(2) The action of Aut(X) on Nef^+_X admits a rational polyhedral fundamental domain.

To prove Theorem 1.0.5 we follow the method given by Bright–Logan–van Luijk [BLvL19], where they proved the cone conjecture for K3 surfaces over non-closed fields of characteristic away from 2.

The outline of this paper is as follows. In Section 2 we introduce the notation of several cones of varieties, and we also recall the definition of almost abelian groups. Moreover, we prove the finiteness of the Galois cohomology of an almost abelian group, which will be used for irreducible symplectic varieties of the geometric Picard rank 2. In Section 3 we prove the finiteness of twists for K3 surfaces and Enriques surfaces. In Section 4 we will prove the birational and automorphism cone conjectures for irreducible symplectic variety following the method by Bright–Logan–van Luijk, and prove the main theorem. In Section 5 we argue the uniform boundedness of Tw_{k'/k}(X). In Section 6, we recall definitions and generalities of derived equivalent varieties, and we prove the finiteness of Tw^D(X).

Acknowledgements. The author is deeply grateful to his advisor Naoki Imai for deep encouragement and helpful advice. The author also would like to thank Tetsushi Ito for helpful comments on Lemma 4.3.1. Moreover, the author would like to thank Yohsuke Matsuzawa, Shou Yoshikawa, Kenta Hashizume, Alexei N. Skorobogatov, Yuki Yamamoto for helpful suggestions. The author was supported by JSPS KAKENHI Grant number JP19J22795.

2. Preliminaries

2.1. Definitions and notations. First, we recall the definition of several cones of varieties, and (quasi-)polarizations.

Definition 2.1.1. Let k be a field. Let X be a smooth projective variety over k. Let Λ_X be a torsion free part of Néron–Severi lattice (Pic_X/k / Pic_0^0_X/k)(k). Here, we denote the Picard functor of X over k and its identity component by Pic_X/k and Pic_0^0_X/k.

1. The positive cone C_X ⊂ Λ_X is the connected component of the cone of all elements λ ∈ Λ_X with (λ, λ) > 0 satisfying that C_X contains the ample divisor classes. We denote the closure of C_X in Λ_X by C_X. We denote the convex hull of C_X ∩ Λ_X, Q in Λ_X, R by C^+_X.

2. The ample cone Amp_X ⊂ C_X is the cone generated by all ample divisor classes. We denote the closure of Amp_X in Λ_X by Nef_X. We denote the convex hull of Nef_X ∩ Λ_X, Q in Λ_X, R by Nef^+_X.

3. The movable cone MV_X ⊂ C_X is the cone generated by movable divisor classes. Here, we say a divisor D of X is movable if the base locus of the linear system of D has codimension ≥ 2. We denote the closure of MV_X in Λ_X by MV_X. We denote the convex hull of MV_X ∩ Λ_X, Q in Λ_X, R by MV^+_X.

4. A polarization (resp. quasi-polarization) on X is an element L ∈ Pic_X/k(k) which is ample (resp. nef and big).

Lemma 2.1.2. Let k be a field, and k_s the separable closure of k. Let X be a smooth projective variety over k. Then the following hold.

\footnote{Note that Pic_X/k(k) is not necessarily equal to Pic(X), but Pic_X/k(k) ⊗_Z Q = Pic(X) ⊗_Z Q holds in general.}
(1) $\text{Amp}_X = \text{Amp}_{X_{ks}} \cap \Lambda_{X,R}$.
(2) $\text{Nef}_X = \text{Nef}_{X_{ks}} \cap \Lambda_{X,R}$.
(3) $\mathcal{MV}_X = \mathcal{MV}_{X_{ks}} \cap \Lambda_{X,R}$.
(4) $\mathcal{MV}_X = \mathcal{MV}_{X_{ks}} \cap \Lambda_{X,R}$.

Proof. First, we prove the assertions (1) and (3). Take an element $x = \sum a_i x_i$ of the right-hand side of (1) (resp. (3)), where $a_i$ is a positive real number and $x_i \in \Lambda_{X_{ks}}$ is an ample divisor class (resp. movable divisor class). Take a finite Galois extension $k'/k$ such that $x_i \in \Lambda_{X_{k'}}$ for any $i$. We put $x_i' = \sum_{\sigma \in \text{Gal}(k'/k)} \sigma(x_i) / \#(\text{Gal}(k'/k))$, which is also an ample divisor class (resp. movable divisor class). Then we have $x = \sum a_i x_i'$, which is an element of the left-hand side. The assertions (2) and (4) follow from [BLvL19, Lemma 3.8]. □

Finally, we introduce a notation of a set of twists.

Definition 2.1.3. Let $k$ be a field, and $X$ a variety over $k$. Let $k'/k$ be an extension of fields. We put

$$\text{Tw}_{k'/k}(X) := \{Y : \text{variety over } k \mid Y_{k'} \simeq_{k'} X_{k'} \}/k-\text{isom}.$$ 

2.2. almost abelian groups. In this section, we prove the finiteness of the group cohomology of an almost abelian group, which will be used to prove Theorem 4.3.6 for an irreducible symplectic variety of geometric Picard rank 2. Moreover, combining with Oguiso’s result, we also propose the finiteness of twists for certain Calabi–Yau threefolds.

First, we recall the definition of almost abelian groups, which was defined by Oguiso.

Definition 2.2.1 (cf. [Ogu08, Section 8]). A group $G$ is an almost abelian group of finite rank $r$ if there exists a normal subgroup $G_0$ of $G$ of finite index, such that there exists a finite subgroup $K$ of $G_0$ with the exact sequence

$$(1) \quad 1 \rightarrow K \rightarrow G_0 \rightarrow \mathbb{Z}^r \rightarrow 1.$$ 

Lemma 2.2.2. Let $\Gamma$ be a finite group. Let $G$ be a group admitting a $\Gamma$-action $\sigma$. Then the Galois cohomology $H^1(\Gamma, G)$ is finite in the following cases.

(1) $G$ is a finite group
(2) $G$ is a free abelian group of finite rank.
(3) There exists an exact sequence

$$1 \rightarrow K \rightarrow G \rightarrow \mathbb{Z}^r \rightarrow 1$$

with a finite normal subgroup $K$ of $G$ and a non-negative integer $r$.

Proof. (1) is trivial.

(2) See [Fu15, Corollary 4.1.7] for example.

(3) For any $c \in H^1(\Gamma, G)$ and its representative cocycle $c$, we have the group action $\sigma_c: \Gamma \rightarrow \text{Aut}(G)$ given by $\sigma_c(\gamma)(g) = c(\gamma) \sigma(\gamma)(g) c(\gamma)^{-1}$. We denote the $\Gamma$-group $G$ with $\sigma_c$ by $\text{c}G$. Since $K$ is a normal subgroup and $\sigma$-stable, it is also $\sigma_c$-stable. Moreover, we have a natural $\Gamma$-action on $\mathbb{Z}^r$ which is also given by conjugation by $c$ after the original $\Gamma$-action, i.e. it is actually given by the original $\Gamma$-action. We also denote these actions on $K$ and $\mathbb{Z}^r$ by $\sigma_c$. Then we have a bijection

$$H^1(\Gamma, G) \rightarrow H^1(\Gamma, \text{c}G)$$
which sends a class \( d: \Gamma \to G \) to \( d \cdot c^{-1} \), and we also have a similar bijection
\[
H^1(\Gamma, \mathbb{Z}^r) \to H^1(\Gamma, c^r \mathbb{Z}^r)
\]
which is compatible with the first bijection. Now we have a commutative diagram of a set
\[
\begin{array}{ccc}
H^1(\Gamma, K) & \mapsto & H^1(\Gamma, G) \\
\downarrow \cong & & \downarrow \cong \\
H^1(\Gamma, c^r K) & \mapsto & H^1(\Gamma, c^r G)
\end{array}
\]
where the horizontal sequences are exact sequences of pointed sets. By (1) and (2), four terms sit on the right or the left sides are finite sets. Therefore, the image of \( p \) is a finite set. On the other hand, each fiber \( p^{-1}(p(c)) \) is bijectively send to \( p^{-1}(1) \), which is a finite set since the bottom left side is a finite set. Therefore, we have \( H^1(\Gamma, G) \) is a finite set. □

**Proposition 2.2.3.** Let \( \Gamma \) be a finite group. Let \( G \) be an almost abelian group of finite rank \( r \) admitting \( \Gamma \)-action \( \sigma: \Gamma \to \text{Aut}(G) \). Then the group cohomology \( H^1(\Gamma, G) \) is finite.

**Proof.** Let \( G^0 \) be as in Definition 2.2.1. We put
\[
G^1 := \bigcap_{\gamma \in \Gamma} \gamma G^0.
\]
We note that \( G^1 \) is a normal subgroup of \( G \) of finite index, since
\[
[G^0: G^0 \cap \gamma(G^0)] \leq [G: \gamma(G^0)].
\]
Then the exact sequence (I) induces the exact sequence
\[
1 \to K \cap G^1 \to G^1 \to \mathbb{Z}^r.
\]
Since \( G^1 \) is of finite index in \( G^0 \), the image of the right arrow is also of finite index, and so it is isomorphic to \( \mathbb{Z}^r \). Therefore, replacing \( G^0 \) with \( G^1 \), we may assume that \( G^0 \) is stable under the \( \Gamma \)-action.

For any \( c \in H^1(\Gamma, G) \), we have a corresponding action \( \sigma_c \) on \( G \). Then \( G^0 \) is \( \sigma_c \)-stable since it is normal and \( \sigma \)-stable. Now we have a commutative diagram of a set
\[
\begin{array}{ccc}
H^1(\Gamma, G^0) & \mapsto & H^1(\Gamma, G) \\
\downarrow \cong & & \downarrow \cong \\
H^1(\Gamma, c^r G^0) & \mapsto & H^1(\Gamma, c^r G)
\end{array}
\]
where the horizontal sequences are exact sequences of pointed sets. Therefore, by Lemma 2.2.2 the image of \( p' \) is finite. Moreover, each fiber \( p'^{-1}(p'(c)) \) is finite since the bottom left side is finite by Lemma 2.2.2. Now we have the desired finiteness. □

**Remark 2.2.4.** By the proof of Proposition 2.2.3, the number \( \#H^1(\Gamma, G) \) is bounded above by the constant which depends only on the order \( \#\Gamma \), the index of \( G^0 \) in \( G \), the order \( \#K \), and \( r \) in Definition 2.2.1. Here, we note that the number of all the isomorphism classes of \( \Gamma \)-module \( \mathbb{Z}^r \) is bounded above by \( r \) and \( \#\Gamma \) by [Bor63, Section 5, (a)], and thus so is \( \#H^1(\Gamma, \mathbb{Z}^r) \).
Definition 2.2.5. Let $k$ be a field. A Calabi–Yau variety over $k$ is a smooth projective variety satisfying that $\omega_{X/k} \cong \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$.

Corollary 2.2.6. Let $k'/k$ be a finite extension of fields of characteristic 0, $X$ a variety over $k$. Then the set $\text{Tw}_{k'/k}(X)$ is finite in the following cases.

1. $X$ is a Calabi–Yau threefolds satisfying that $\text{rk Pic}(X_{k'}) \leq 3$.
2. $X$ is a Calabi–Yau varieties of odd dimension satisfying that $\text{rk Pic}(X_{k'}) \leq 2$.

Proof. First, take a finitely generated subfield $k_0 \subset k$ and a variety $X_0$ over $k_0$ such that $X_{0,k} \cong X$. Fix an embedding $ι: k_0 \hookrightarrow \mathbb{C}$. Since a Calabi–Yau variety has no infinitesimal automorphism, the automorphism scheme $\text{Aut}(X_{k'})$ is unramified over $k_0$. Thus we have $$\text{Aut}(X_{k'}) \cong \text{Aut}(X_{0,k_0}) \cong \text{Aut}(X_{0,\mathbb{C}}).$$ Therefore, $\text{Aut}(X_{k'})$ is finite or almost abelian of rank 1 by [Ogu14, Theorem 1.2] and [LOP18, Theorem 1.1]. Since $\text{Aut}(X_{k'}) \subset \text{Aut}(X_{\mathbb{C}})$, the group $\text{Aut}(X_{k'})$ is also finite or almost abelian of rank 1. By Proposition 2.2.3, the set $\text{Tw}_{k'/k}(X)$ is a finite set.

3. The case of surfaces

3.1. K3 surfaces. In this subsection, we discuss the finiteness of $\text{Tw}_{k'/k}$ in the case of K3 surfaces. First of all, we recall the definition of K3 surfaces.

Definition 3.1.1. Let $k$ be a field. A K3 surface over $k$ is a smooth projective surface $X$ over $k$ satisfying that $\omega_{X/k} \cong \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$. A K3 surface $X$ over $k$ is called supersingular if the Picard rank of $X_{k'}$ is 22.

The main theorem of this section is the following.

Theorem 3.1.2. Let $k$ be a field, and $X$ a K3 surface over $k$. Let $k'/k$ be a finite extension of fields. Suppose that the characteristic of $k$ is away from 2 or $X$ is not supersingular. Then, the set $\text{Tw}_{k'/k}(X)$ is a finite set.

Remark 3.1.3. If $k'/k$ is a solvable extension, then Theorem 3.1.2 can be proved by the Galois cohomological argument as in the proof in [LMS14, Proposition 2.4.1].

To prove Theorem 3.1.2 we recall the cone conjecture for K3 surfaces.

Theorem 3.1.4 (Cone conjecture). Let $k$ be a field, and $X$ a K3 surface over $k$. Suppose that the characteristic of $k$ is not equal to 2 or $X$ is not supersingular. Then, the action of $\text{Aut}(X)$ on $\text{Nef}_X^+$ admits a rational polyhedral fundamental domain.

Proof. See [BLvL19, Corollary 3.15] for the case of $\text{char } k \neq 2$. The same proof works if $\text{char } k = 2$ and $X$ is not supersingular (see also remarks after [LM18, Theorem 6.1]).

The cone conjecture implies the finiteness of polarizations, by the following easy lemma.

Lemma 3.1.5. Let $Λ$ be a $\mathbb{Z}$-lattice (i.e. $\mathbb{Z}$-module with a symmetric bilinear pairing valued in $\mathbb{Z}$) of index $(1, p-1)$, $C_Λ \subset Λ_\mathbb{R}$ a positive cone (i.e. one connected component of the cone of all elements $λ \in Λ_\mathbb{R}$ with $(λ, λ) > 0$), $C_Λ^*$ the closure of $C_Λ$, and $Π \subset C_Λ^*$ a rational polyhedron. We fix a positive integer $d \in \mathbb{Z}$. Then there exist only finitely many integral elements of degree $d$ in $Π$. In particular, in the setting of Theorem 3.1.4 the set of quasi-polarizations on $X$ of degree $d$ modulo $\text{Aut}(X)$ is finite.
Proof. This is a well-known argument (see [Huy16, Chapter 8, Corollary 4.10] for the ample case), but for the sake of completeness, we include it. The second statement follows by applying \( \Lambda = \Lambda_X \). Therefore, we will prove the first statement. First, we note that for any \( y,z \in \Omega_X \), we have \((y,z) \geq 0\). Let \( x_1, \ldots, x_n \in \Pi \cap \Lambda \) be a nonzero generator of a rational polyhedron. We put \( b_{i,j} := (x_i, x_j) \), and \( M := \max b_{i,j} \). Let \( x = \sum_{i=1}^n a_i x_i \in \Pi \cap \Lambda \) be a degree \( d \) element. We have

\[
(x, x_i) = \sum_{1 \leq j \leq n} a_j b_{i,j} > 0
\]

for any \( 1 \leq i \leq n \). Indeed, if \( (x, x_i) = 0 \), then we have \((x_i, x_i) < 0 \) by the assumption on index, and it contradicts \((x_i, x_i) \geq 0\). Since the left-hand side is a positive integer, there exists \( 1 \leq j_i \leq n \) such that \( a_{j_i} \geq n^{-1}M^{-1} \) and \( b_{j_i,j_i} > 0 \) (so \( b_{j_i,j_i} \geq 1 \)). On the other hand, since

\[
(x, x) = \sum_{i,j=1,\ldots,n} a_i a_j b_{i,j} = d,
\]

we have \( a_i a_j b_{i,j} \leq d \). Therefore, we have \( a_i n^{-1}M^{-1} \leq d \). Therefore, such \( x \) should be located on a bounded domain that is independent of \( x \), so it finishes the proof. \( \square \)

Lemma 3.1.6. Let \( k \) be a field, and \( k' \) a separable algebraic extension of \( k \). Let \( X \) be a K3 surface over \( k \). Then there exists a positive number \( d \) such that for any \( Y \in \text{Tw}_{k'/k}(X) \), the variety \( Y \) admits a degree \( d \) polarization.

Proof. We may assume that \( k' \) is the separable closure of \( k \). Note that \( \text{Pic}_{Y/k} \) is isomorphic to \( \text{Pic}_{X/k} \) after taking the base changes to \( k' \). Let \( L \) be a \( \mathbb{Z} \)-lattice \( \text{Pic}_{X_{\overline{k}}}(\overline{k}) = \text{Pic}_{X'/\overline{k}}(k') \) (see [Ito18, Lemma 3.1]). Then the set of conjugacy class of actions \( \text{Gal}(k'/k) \to \text{O}(L) \) is finite by [Bor63, Section 5, (a)]. Take \( Y_1, Y_2 \in \text{Tw}_{k'/k}(X) \) such that there exists a lattice isometry \( \phi : \text{Pic}_{Y_1/k}(k) \simeq \text{Pic}_{Y_2/k}(k) \) which is induced by an isometry \( \overline{\phi} : \text{Pic}_{Y_1/k}(k') \simeq \text{Pic}_{Y_2/k}(k') \). It is enough to show that \( Y_1 \) and \( Y_2 \) admit a polarization of the same degree. Without loss of generality, we may assume \( \phi \) preserves a positive cone. Take an ample class \( y \) in \( \text{Pic}_{Y_1/k}(k) \). By [BLvL19, Proposition 3.7], there exists \( r \in R_{\overline{\gamma}} \) such that \( r \circ \phi(y) \in \text{Nef}_{\overline{\gamma}} \). Here, the group \( R_{\overline{\gamma}} \) is the Galois fixed part of the Weyl group of \( Y_{2,k'} \) (see [BLvL19, Definition 3.3] for precise definition). We note that \( y \) has nonzero intersection numbers with any \((-2)\)-classes in \( \text{Pic}_{Y_1/k}(k') \) (i.e. \( x \in \text{Pic}_{Y_1/k}(k') \) such that \((x,x) = -2\)). Therefore, the nef class \( r \circ \phi(y) \in \text{Pic}_{Y_2/k}(k) \) has a non-zero intersection number with any \((-2)\)-classes in \( \text{Pic}_{Y_2/k}(k') \) and thus it is also an ample class. \( \square \)

Proof of Theorem 3.1.2. First, we note that we may assume \( k'/k \) is a finite Galois extension. Indeed, since an infinitesimal automorphism of a K3 surface is trivial, the automorphism scheme \( \text{Aut}_{X/k} \) is an unramified scheme over \( k \). Therefore, for any K3 surface \( X, Y \) over \( k \), the isomorphism scheme \( \text{Isom}_{X,Y/k} \) is an unramified scheme over \( k \), since \( \text{Isom}_{X_{\overline{k}}, Y_{\overline{k}}/\overline{k}} \) is empty or non-canonically isomorphic to \( \text{Aut}_{X_{\overline{k}}/\overline{k}} \). Therefore, for the separable closure \( k'' \) of \( k \) in \( k' \), we have \( \text{Tw}_{k'/k}(X) = \text{Tw}_{k''/k}(X) \).

In the following, we suppose that \( k'/k \) is a finite Galois extension. Let \( d \) be an integer as in Lemma 3.1.6. For any \( Y \in \text{Tw}_{k'/k}(X) \), take a degree \( d \) polarization \( L_Y \). By Lemma 3.1.5, we can take \( M_1, \ldots, M_m \in \text{Pic}_{X/k}(k') \) be a complete system of representatives of the set of polarizations of degree \( d \) on \( X_{k'} \) modulo \( \text{Aut}(X_{k'}) \). We
put
\[ T_i := \left\{ (Y, L) \bigg| \begin{array}{l} Y: \text{K3 surface over } k, \\ L: \text{polarization on } Y, \\ (Y, L)_{k'} \simeq (X_{k'}, M_i) \end{array} \right\} /k\text{-isom.} \]

Then \( Y \mapsto (Y, L_Y) \) gives an inclusion of sets \( \text{Tw}_{k'/k}(X) \hookrightarrow \bigcup_{1 \leq i \leq n} T_i \). Therefore, it suffices to show the finiteness of \( T_i \). This follows from the finiteness of automorphism of polarized K3 surfaces (e.g. see [Huy16, Chapter 5, Proposition 3.3]).

### 3.2. Enriques surfaces

First, we recall the definition of an Enriques surface.

**Definition 3.2.1.** Let \( k \) be a field. An *Enriques surface over \( k \)* is a smooth projective surface over \( k \) satisfying \( \omega_{X_{k'/k}} \equiv \mathcal{O}_{X_{k'}} \) and \( b_2(X_{k'}) = 10 \), where \( \equiv \) denotes the numerical equivalence.

In this section, we prove the finiteness of twists via a finite extension for Enriques surfaces. If the base field has characteristic \( \neq 2 \), the problem is reduced to the case of K3 surfaces since an automorphism group of a K3 surface has at most finitely many conjugacy classes of Enriques involution (see [Tak20b, Lemma 3.6]). But if the base field has characteristic \( 2 \), K3 double cover may not exist in general. To avoid this difficulty, we use the following lifting result given by Liedtke.

**Theorem 3.2.2.** Let \( k \) be a field of characteristic 2. Let \( X \) be an Enriques surface over \( k \). There exists a finite separable extension \( k'/k \) and a complete discrete valuation ring \( R \) of mixed characteristic \((0, 2)\) with residue field \( k' \) such that there exists a smooth projective scheme \( \mathcal{X} \) over \( k \) such that \( \mathcal{X}_{k'} \simeq X_{k'} \).

**Proof.** If \( k = \overline{k} \), this theorem follows from [Lie15, Theorem 4.10]. By [Lie15, Proposition 3.4], replacing \( k \) with a finite separable extension if necessary, we may assume that \( X \) admits a line bundle \( L \) such that \( L_{\overline{k}} \in \text{Pic}(X) \) gives a Cossec–Verra polarization in the sense of [Lie15, Definition 3.2]. As in the argument in [Lie15, Theorem 3.1], \( L \) defines a birational morphism \( \nu: X \rightarrow X' \), where \( X'_{\overline{k}} \) is an Enriques surface with worst Du Val singularities, and \( X' \) admits an ample Cossec–Verra polarization \( \nu_*(L) \).

By Artin’s simultaneous resolution, it is enough to construct a lifting of \( X' \). Take a complete discrete valuation ring \( R \) of mixed characteristic \((0, 2)\) with residue field \( k \) satisfying that \( \sqrt{2} \in R \) is a uniformizer. We note that \( \text{Pic}^r_{X'/k} \) is written as \( G_{b,a} \) with \( a, b \in k \) with \( ab = 2 \) by the Oort–Tate classification. As in the proof of [Lie15, Theorem 4.9], we can find lifts \( a', b' \in R \) with \( a'b' = 2 \). Therefore, we have a lift of \( \text{Pic}^r_{X'/k} \) to \( R \). By [Lie15, Theorem 4.7], the variety \( X' \) admits a formal lift to \( R \), which is algebraizable by [Tak20b, Lemma 2.4] (see also [Lie15, Proposition 4.4]).

**Remark 3.2.3.** In Theorem 3.2.2, the generic fiber \( X_\eta := X_{\text{Frac}(R)} \) is automatically an Enriques surface. Indeed, the second Betti number of \( X_\eta \) is 10 by the proper and smooth base change theorems. Moreover, for any curve \( C \subset X_\eta \), we have \( K_{X_{k'/k}} \cdot C = K_{X_\eta} \cdot C_s = 0 \), where \( C_s \) is the special fiber of the closure of \( C \subset X \).

**Lemma 3.2.4.** Let \( k \) be a field of characteristic 2. Let \( X \) be an Enriques surface over \( k \). Then there exists a finite extension \( k'' \) of \( k \) satisfying that for any extension \( k'' \) of \( k' \) and any positive integer \( d \), the set of polarizations of degree \( d \) on \( X_{k''} \) modulo \( \text{Aut}(X_{k''}/k'') \) is finite.
Replacing \( \varphi \) argument as in [BLvL19, Proposition 3.7] (see also the proof of Proposition 4.0.6). Here, \( X \) characteristic 0 lift \( R \). In the following, we replace \( k' \) the subgroup generated by reflections with respect to \( R \) fundamental domain with respect to the action of \( \text{Aut}(X_{\ell'}) \). Let \( R'' \) with residue field \( R' \) in \( F' \). Then for any extension \( k''/k' \), there exists a complete discrete valuation ring \( R'' \) with residue field \( k'' \) satisfying that \( R' \subset R'' \). Let \( F'' \) be the fraction field of \( R'' \). Now by Lemma 3.1.3 and the cone conjecture for Enriques surfaces of characteristic 0 ([Kaw97, Theorem 2.1]), the set \( O \) of \( \text{Aut}(X_{F''}) \)-orbit in degree 4d ample line bundles on \( X_{F''} \) is a finite set. We define the subset \( O' \subset O \) to be the set of orbits in \( O \) which contains an ample line bundle \( L \) on \( X_{F''} \) satisfying \( \text{sp}(L) \) is ample. Here, \( \text{sp}: \text{Pic}(X_{F''}) \to \text{Pic}(X_{k''}) \)

is the natural specialization map. Take a complete system of representatives \( M_1, \ldots, M_n \) of \( O' \) such that \( \text{sp}(M_i) \) is ample for \( 1 \leq i \leq n \). Note that the set of elements in \( \text{Pic}_{X_{F''/k''}}(k'') \) two times of which appear in specializations of \( M_1, \ldots, M_n \) is a finite set. We put this set as \( \{L_1, \ldots, L_m\} \). We will prove that this set represents all the elements of polarizations of degree \( d \) on \( X_{k''} \) modulo \( \text{Aut}(X_{k''}) \). Take a polarization \( L \) of degree \( d \) on \( X_{k''} \). Take a lift \( M \in \text{Pic}(X_{F''}) = \text{Pic}(X_{F''}) \) of \( L'' \). Then \( M \) is a polarization of degree 4d on \( X_{F''} \). Therefore, there exists an automorphism \( \gamma \in \text{Aut}(X_{F''}) \) and \( 1 \leq i \leq n \) such that \( (\gamma(M) = M_i \). Taking the specialization, we have \( \gamma \text{sp}(M_i) \). Here, \( \gamma \in \text{Aut}(X_{k''}) \) is a specialization of \( \gamma \) which exists by [MM64, Corollary 1]. We note that the specialization of automorphism is compatible with sp by the argument in the proof of [LM18, Theorem 2.1]. Therefore, \( \gamma \text{sp}(L) \) is one of an element of \( \{L_1, \ldots, L_m\} \).

Theorem 3.2.5. Let \( k \) be a perfect field of characteristic 2, and \( X \) an Enriques surface over \( k \). Let \( k'/k \) be a finite extension of fields. Then the set \( \text{Tw}_{k'/k}(X) \) is finite.

Proof. We may assume that \( k'/k \) is a Galois extension. As in the argument in Lemma 3.1.6 the set of lattice isometry classes of \( \Lambda_Y \) for \( Y \in \text{Tw}_{k'/k}(X) \) is a finite set. We denote this finite set by \( S \). We fix \( Y \in \text{Tw}_{k'/k}(X) \). First, by [Huy16, Chapter 8, Remark 2.2], there exists a polarization \( L_Y \) satisfying the following: For any \((-2)\)-class \( x \) (i.e. \( x \in \text{Pic}(Y^\vee) \) with \( (x, x) = -2 \)) satisfying \( x^\perp \not\in \Lambda_Y \), we have \( (L, x) \neq 0 \). Take a \( Y' \in \text{Tw}_{k'/k}(k) \) with an isometry \( \phi: \Lambda_Y \simeq \Lambda_{Y'} \). Let \( R_{Y'} \subset O(\Lambda_{Y'}) \) be the subgroup generated by reflections with respect to \((-2)\)-curves. Then \( \text{Nef}_{Y'} \) is a fundamental domain with respect to the action of \( R_{Y'} \) on the positive cone \( C_Y \) by [CDL+21, Proposition 2.2.1]. We put \( R_Y' := R_{Y'}^{\text{Gal}(\overline{k}/k)} \subset O(\Lambda_{Y'}) \). Now we can show that \( \text{Nef}_{Y'} \) is a fundamental domain with respect to the action of \( R_{Y'} \) by the same argument as in [BLvL19, Proposition 3.7] (see also the proof of Proposition 4.0.6). Replacing \( \phi \), we may assume that \( \phi(L_Y) \subset C_{Y'} \). Therefore, there exists \( r \in R_{Y'} \) such that \( r \circ \phi(L_Y) \) is a quasi-polarization of \( Y' \). Moreover, by the choice of \( L \), the element \( r \circ \phi(L_Y) \) has positive intersection with any \((-2)\)-curve on \( Y^\vee \) (since there exists a polarization on \( Y' \)). Thus, by [CDL+21, Proposition 2.1.5], \( r \circ \phi(L_Y) \) is a polarization on \( Y' \). Therefore, there exists an integer \( d \) such that \( Y \in \text{Tw}_{k'/k}(X) \) admits a polarization \( M_Y \) of degree \( d \) (we can take \( d \) as the least common multiple of \( (L_Y, L_Y) \) for \( [Y] \in S \)).
By Lemma 3.2.4, we may assume that the set of polarizations of degree $d$ on $X_{k'}$ modulo $\text{Aut}(X_{k'})$ is a finite set. We take a complete system of representatives $L_1, \ldots, L_m \in \Lambda_{X_{k'}}$ of this set. Now we put

$$T_i := \left\{ (Y, L) \mid \begin{array}{l}
Y: \text{Enriques surface over } k, \\
L: \text{polarization on } Y, \\
(Y, L)_{k'} \simeq (X_{k'}, L_i)
\end{array} \right\} /k\text{-isom.}$$

Then $Y \mapsto (Y, M_Y)$ gives an inclusion of sets $T_{k'/k}(X) \to \bigsqcup_{1 \leq i \leq m} T_i$. On the other hand, each $T_i$ is a finite set by [Bri18, Proposition 2.26] and [Dol16, Theorem 3]. Therefore, it finishes the proof. \hfill $\square$

4. Cone conjecture for irreducible symplectic varieties

In this section, we prove certain rational variants of the cone conjecture for irreducible symplectic varieties.

**Definition 4.0.1.**

(1) Let $k$ be a field of characteristic 0. Let $X$ be a smooth projective variety over $k$. We say $X$ is an irreducible symplectic variety if $X_k$ is simply connected of even dimension $2r$ and there exists $\omega \in H^0(X, \Omega^2_{X/k})$ uniquely up to constant, such that $\omega^{\otimes n}$ vanishes nowhere.

(2) Let $k$ be a subfield of $\mathbb{C}$. Let $X$ be an irreducible symplectic variety over $k$. It is known that there exists a unique natural integral non-degenerate symmetric primitive bilinear pairing $q$ on $H^2(X, \mathbb{Z})$, of signature $(3, b_2(X\mathbb{C}) - 3)$ satisfying that there exists a positive rational number $\beta_X$ such that

$$q(\alpha, \alpha)^n = \beta_X \int_X \alpha^{2n}$$

for any $\alpha \in H^2(X, \mathbb{Z})$, and that $q(\alpha, \alpha)$ is positive for an ample class $\alpha \in H^2(X, \mathbb{Z})$ (the second condition is automatic if $b_2(X\mathbb{C}) \neq 6$). We refer to $q$ as the Beauville–Bogomolov form, and we denote this pairing $q$ by $(\ast, \ast)$ for short. We note that $q$ is $\text{Aut}(\mathbb{C}/k)$-invariant by the argument in [Yan19, Proposition 2.1.5].

(3) Let $k$ be a field of characteristic 0. Let $X$ be an irreducible symplectic variety over $k$. Since $X$ is defined over a finitely generated subfield $k' \subset k$ (i.e. there exists an irreducible symplectic variety $Y$ over $k'$ such that $Y_k \simeq X$), by fixing an embedding $k' \subset \mathbb{C}$, we have an isomorphism $H^2_{k'}(X_{\overline{k}}, \mathbb{Z}) \simeq H^2(Y, \mathbb{Z}) \otimes \mathbb{Z}$. Thorough this isomorphism, we can associate an integral non-degenerate symmetric bilinear pairing $q$ on $H^2_{k'}(X_{\overline{k}}, \mathbb{Z})$. By the argument in [Yan19, Proposition 2.1.5] (see also [Yan19, Lemma 2.1.1]), $q$ is independent of the choice of $k'$ and an embedding $k' \subset \mathbb{C}$, and $q$ is also $\text{Gal}(\overline{k}/k)$-invariant. We denote this pairing by $(\ast, \ast)$. We also denote the $\mathbb{Z}$-valued symmetric bilinear pairing on $\Lambda_X$ which is given by the composition of the Chern character and $q$ by $(\ast, \ast)$.

**Remark 4.0.2.** Let $X_1, X_2$ be an irreducible symplectic variety over a field $k$ of characteristic 0. Let $f: X_1 \dashrightarrow X_2$ be a birational map. Since $K_{X_i}$ are trivial and $X_i$ are terminal, the map $f$ is small (i.e. isomorphic in codimension 1). Indeed, if we take a smooth projective variety $Y$ over $k$ with birational morphisms $\pi_1: Y \to X_1$ and $\pi_2: Y \to X_2$, then the set of $\pi_i$-exceptional prime divisors are no other than the set of prime divisors $E \subset Y$ such that $\text{ord}_E(K_Y - \pi_i^*(K_{X_i})) > 0$, which do not depend on $i$. Therefore, one can pull back a line bundle on $X_2$ and an element of $\Lambda_{X_2}$ via $f$. 
Definition 4.0.3. Let $k$ be an algebraically closed field of characteristic $0$. Let $X$ be an irreducible symplectic variety over $k$.

(1) A prime divisor $E \subset X$ is exceptional if $(E, E) < 0$. We denote the set of prime exceptional divisors by $\mathcal{P}_{\text{ex}}(X)$.

(2) By the same argument as in [Mar11, Proposition 6.2], a prime exceptional divisor $E$ defines the reflection $r_E \in O(\Lambda_X)$. We denote the subgroup of $O(\Lambda_X)$ generated by the reflections with respect to prime exceptional divisors by $W_{\text{Exc}}$. Note that if $k = \mathbb{C}$, our definition is the same as [Mar11, Definition 6.8] by [Mar11, Theorem 6.18 (3) (5), Lemma 6.23 (2)].

Proposition 4.0.4. Let $k$ be an algebraically closed field of characteristic $0$. Let $X$ be an irreducible symplectic variety over $k$. Then $\overline{\mathcal{M}V}_X \cap C_X$ is a fundamental domain for the action of $W_{\text{Exc}}$ on $C_X$, cut out by closed half-space associated to elements of $\mathcal{P}_{\text{ex}}(X)$.

Proof. See [Mar11, Lemma 6.22].

Definition 4.0.5. Let $k$ be a field of characteristic $0$. Let $X$ be an irreducible symplectic variety over $k$. As in Definition 4.0.3, we have the subgroup $W_{\text{Exc}} \subset O(\Lambda_X^k)$. Define $R_X$ to be the fixed part $W_{\text{Exc}}^k$, where $G_k$ denotes the absolute Galois group of $k$. We note that $R_X$ acts on $\Lambda_X$.

Proposition 4.0.6. Let $k$ be a field of characteristic $0$. Let $X$ be an irreducible symplectic variety over $k$. Then $\overline{\mathcal{M}V}_X \cap C_X$ is a fundamental domain for the action of $R_X$ on $C_X$.

Proof. This follows from the same argument as in [BLvL19, Proposition 3.7], but we include it. First, we prove that any class $x$ of $C_X$ are $R_X$-equivalent to an element of $\overline{\mathcal{M}V}_X \cap C_X$. If we suppose that $x$ has trivial stabilizer in $R_X$, then there exists a unique $g \in W_{\text{Exc}}$ such that $gx \in \overline{\mathcal{M}V}_X \cap C_X$. For any $\sigma \in G_k$, we have

$$(\sigma g)(x) = \sigma g(\sigma^{-1}(x)) = \sigma(g(x)) \in \overline{\mathcal{M}V}_X \cap C_X,$$

since the Galois action preserves $\mathcal{M}V_X$. By the assumption, we have $\sigma g = g$, and thus $g \in R_X$. Next, we consider the case where $x \in C_X$ has a non-trivial stabilizer. By Proposition 4.0.3, $x$ lies in the following walls

$$\bigcup_{E \in \mathcal{P}_{\text{ex}}(X)} (E^\perp \cap C_X) \subset C_X.$$

By the argument in [Huy16, Chapter 8, Remark 2.3], this union is locally finite. Therefore, one can take a sequence $(x_n)$ of elements of $C_X$ such that $x_n$ converge to $x$ and all $x_n$ lying in the interior of the same chamber of $C_X$ with respect to the above walls. Then there exists a unique $g \in R_X$ with $g(x_n) \in \overline{\mathcal{M}V}_X \cap C_X$, so is $g(x)$.

Now it suffices to prove that $\overline{\mathcal{M}V}_X \cap C_X$ intersects the translate by any non-trivial element of $R_X$ only in its boundary. This follows from Proposition 4.0.4 and [BLvL19, Lemma 3.8].

4.1. Birational cone conjecture.

Proposition 4.1.1. Let $k$ be a field of characteristic $0$. Let $X$ be an irreducible symplectic variety over $k$. Let

$$\rho: \text{Bir}(X) \times R_X \to O(\Lambda_X)$$
be the natural action. Then the image of $\rho$ is a finite index subgroup, and the kernel of $\rho$ is a finite subgroup of $\text{Aut}(X) \subset \text{Bir}(X)$.

**Proof.** First, we note that a birational automorphism that sends a very ample class to a very ample class is an automorphism and an automorphism group of a quasi-polarized irreducible symplectic variety is a finite group ([Bri18, Proposition 2.26]). Therefore, the case of $k = \mathbb{C}$ follows from [Mar11, Proposition 6.18, Lemma 6.23]. Since an irreducible symplectic variety have no infinitesimal birational automorphisms ([Bla17, Theorem 3.8]) or infinitesimal line bundles, the case of $k = \mathbb{C}$ follows. Indeed, one can take a finitely generated subfield $k'$ of $k$ such that $X$ comes from $X'$ over $k'$. Take an embedding $k' \hookrightarrow \mathbb{C}$. Then $\text{Bir}(X) = \text{Bir}(X_{k'}) = \text{Bir}(X_{\mathbb{C}})$ by the above fact. On the other hand, we have $\Lambda_X = \Lambda_{X_{k'}} = \Lambda_{\mathbb{C}}$ by the above fact. Now we have $R_X = R_{X_{k'}} = R_{X_{\mathbb{C}}}$ and we can reduce to the case of $k = \mathbb{C}$.

In the following, we treat a general case. We know that

$$\rho' : \text{Bir}(X_{\mathbb{C}}) \times R_{X_{\mathbb{C}}} \rightarrow O(\Lambda_{X_{\mathbb{C}}})$$

has the finite kernel and the image of finite index. We note that a birational automorphism group satisfies a Galois descent property $\text{Bir}(X_{\mathbb{C}})^{G_k} = \text{Bir}(X)$. Since $\text{Bir}(X_{\mathbb{C}})$ is finitely generated group ([BST12 Theorem 2]), the action of $G_k$ on $\text{Bir}(X_{\mathbb{C}})$ factors through a finite quotient. Similarly, since $\Lambda_{X_{\mathbb{C}}}$ is finitely generated group, the actions of $G_k$ on $R_{X_{\mathbb{C}}}$ and $O(\Lambda_{X_{\mathbb{C}}})$ factors through a finite quotient. Therefore, by [BLvL19, Lemma 3.12], the restriction

$$\text{Bir}(X) \times R_X \rightarrow O(\Lambda_X)^{G_k}$$

has the finite kernel and the image of finite index. By [BLvL19, Proposition 2.2 (2)], $O(\Lambda_X)^{G_k}$ is of finite index in $O(\Lambda_X)$. Here, the group $O(\Lambda_X)$ is the subgroup of $O(\Lambda_{X_{\mathbb{C}}})$ stabilizing $\Lambda_X$ as a subset. Moreover, by [BLvL19, Proposition 2.2 (1), (2)], the map $O(\Lambda_{X_{\mathbb{C}}}) \rightarrow O(\Lambda_X)$ has the finite kernel and the image of finite index. Therefore the morphism

$$\text{Bir}(X) \times R_X \rightarrow O(\Lambda_X)$$

has the finite kernel and the image of finite index. \qed

We will study about the structure of $R_X$ by following the method of [BLvL19].

**Definition 4.1.2.**  
(1) Let $W$ be a group, and $T \subset W$ a set of elements of order 2 which generates $W$ as a group. For $t_i, t_j \in T$, we denote the order of $t_i t_j$ by $n_{i,j}$ (if the order is infinite, we set $n_{i,j} = 0$). We say $(W, T)$ is a Coxeter system if $W$ has the following representation.

$$\langle t \in T \mid t^2 = 1, (t_i t_j)^{n_{i,j}} = 1 \text{ if } n_{i,j} \neq 0 \rangle.$$

(2) Let $(W, T)$ be a Coxeter system. Let $G$ be a graph with vertices $T$ and such that $t_i, t_j$ are adjacent in $G$ if and only if $t_i$ does not commute with $t_j$. We label an integer to each edge between $t_i$ and $t_j$ by $n_{i,j} - 2$ for $n_{i,j} > 0$ and 0 otherwise. We say this graph $G$ is the Coxeter–Dynkin diagram of the Coxeter system $(W, T)$.

(3) Let $(W, T)$ be a Coxeter system. The length $\ell(w)$ of $w \in W$ is the length of a shortest word consists of elements of $T$ that represents $w$. If $W$ is finite, there exists a unique $w_0 \in W$ such that $\ell(w_0)$ is greater than $\ell(w)$ for any $w \neq w_0$. We say $w_0$ is the longest element of $(W, T)$. 

\[ \]
The following is an analogue of [BLvL19, Proposition 3.6, Remark 3.9].

**Proposition 4.1.3.** Let $k$ be a field of characteristic 0. Let $X$ be an irreducible symplectic variety over $k$. Let $I$ be a Galois orbit of prime exceptional divisors on $X^\text{ex}_k$.

We denote the subgroup of $W_{\text{Exc}}$ generated by the reflections with respect to $E \in I$ by $W_I$. Then the following holds.

1. If $W_I$ is finite, then one of the following holds.
   a. For any $E_1, E_2 \in I$ with $E_1 \neq E_2$, we have $(E_1, E_2) = 0$.
   b. For any $E_1 \in I$, there exists a unique $E_2 \in I$ such that $E_1 \neq E_2$ and $(E_1, E_2) \neq 0$. Moreover, in this case, we have $(E_1, E_1) = -2(E_1, E_2)$.

   We denote the set of Galois orbits $I$ of prime exceptional divisors such that $W_I$ is finite by $F$.

2. For $I \in F$, we denote the longest element of the Coxeter system $(W_I, I)$ by $r_I$. Then $(R_X, \{r_I | I \in F\})$ is a Coxeter system. Moreover, $r_I$ is given by the reflection with respect to the sum $C_I$ of all classes in $I$.

3. A class $\lambda \in C_X$ lies in $\overline{\mathcal{M}} V_X$ if and only if $(\lambda, E) \geq 0$ for any $E \in I$ and $I \in F$.

**Proof.** First, we note that $(W_{\text{Exc}}, \mathcal{P}_{\text{Exc}})$ and $(W, I)$ are Coxeter systems (see [Hec18, Section 5.4] and [BB05, Proposition 2.4.1 (i)]).

1. Let $I$ be a Galois orbit of prime exceptional divisors such that $W_I$ is finite. Take $E_1, E_2 \in I$ as $E_1 \neq E_2$. Then by [Mar11, Theorem 5.8], their classes in $\Lambda_X$ are different, so their classes are linearly independent. We put $\alpha := (E_1, E_2)$ and $\beta := (E_1, E_1) = (E_2, E_2)$. Note that $r_{E_1}$ and $r_{E_2}$ stabilize the $\mathbb{R}$-vector space $W := \mathbb{R}E_1 \oplus \mathbb{R}E_2$, and their representation matrix is the following.

\[
\begin{pmatrix}
-1 & -\frac{2}{\alpha} \\
0 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 0 \\
-\frac{2}{\alpha} & -1
\end{pmatrix}
\]

Therefore, we have

\[
\begin{pmatrix}
-1 & -\frac{4}{\beta} \\
-\frac{2}{\beta} & -1
\end{pmatrix}
\]

Since $W_I$ is finite and the above matrix has $\mathbb{Q}$-coefficient, $r_{E_1}r_{E_2}|_W$ has the order 2, 3, or 4. By seeing a trace, we can show that the order cannot be 4. Moreover, we have the order is 2 if and only if $\alpha = 0$, and the order is 3 if and only if $\beta = -2\alpha$. On the other hand, since the Coxeter–Dynkin diagram of $(W_I, I)$ is a finite union of finite trees ([BB05, Exercise 1.4]), there exists a vertex of degree $\leq 1$. Since $G_k$ acts transitively on the diagram, we have either every vertex have degree 0 or every vertex have degree 1. Therefore, if $W_I$ is finite, then (a) or (b) holds. Therefore, it finishes the proof of part (1).

2. The first part follows from [GI14, Theorem 1]. We will show the second part. First, we consider the case (a). We write $I$ as $\{E_1, \ldots, E_r\}$. Then $W_I$ is isomorphic to the Coxeter group $A_r^\prime$. Therefore, their longest element is given by $r_{E_1}r_{E_2}\cdots r_{E_r} = r_{C_I}$. Next, we consider the case (b). We write $I$ as $\{E_1, E'_1, \ldots, E_r, E'_r\}$, where $(E_i, E'_i) \neq 0$. Then $W_I$ is isomorphic to the Coxeter group $A_2^\prime$, and their longest element is given by
(r_{E_1}r_{E_2}r_{E_1}) \cdots (r_{E_1}r_{E_2}r_{E_1}) = r_{E_1} + r_{E_2} + \cdots r_{E_1} + r_{C_1}. Therefore, it finishes the proof of part (2).

(3) Let $F' \subset \Lambda_{X_0}$ be a set of all the classes $x \in \Lambda_{X_0}$ such that $(x, x) < 0$. By the argument in [Huy16] Remark 8.2.3, the following union is locally finite.

$$\bigcup_{x \in F'} (x^\perp \cap C_{X_0}) \subset C_{X_0}.$$ 

Let $I$ be a Galois orbit of prime exceptional divisors such that $W_I$ is infinite. Suppose that there exists a $\lambda \in C_X$ such that $(\lambda, E) = 0$ for some $E \in I$. Then $\lambda$ is orthogonal to every element in $I$. Therefore, $\lambda$ is fixed by the action of $W_I$, and therefore $\lambda$ is orthogonal to $w(E)$ for any $E \in I, w \in W_I$. Since $wr_Ew^{-1} = r_w(E)$, by [Spe09] Theorem 1, the set $W_I E$ is infinite for some $E \in I$. Therefore, $\lambda$ is orthogonal to infinitely many elements in $F'$, which is a contradiction. Thus, for any $E \in I$, the wall $E^\perp$ does not meet with $C_X$. Combining with Proposition 4.0.4, it finishes the proof.

The main theorem in this subsection is the following.

**Theorem 4.1.4.** Let $k$ be a field of characteristic 0. Let $X$ be an irreducible symplectic variety over $k$. Then there exists a rational polyhedral cone $\Pi$ in $\mathcal{MV}^+_X$ which is a fundamental domain for the action of $\Gamma_{\text{Bir}(X)}$ on $\mathcal{MV}^+_X$. Here, $\Gamma_{\text{Bir}(X)}$ means the image of the birational automorphism group via the natural action $\text{Bir}(X) \to O(\Lambda_{X, k})$.

**Proof.** Let $\Gamma$ be the image of $\rho : \text{Bir}(X) \cong R_X \to O(\Lambda_X)$, which is an arithmetic subgroup of $O(\Lambda_X)$. We choose an ample class $y \in C_X \cap \Lambda$. The set

$$\Pi := \{x \in C^+_X \mid (\gamma x, y) \geq (x, y) \text{ for all } \gamma \in \Gamma\}$$

is rational polyhedral as in the proof of [BLyL19] Corollary 3.15. Let $I$ be a Galois orbit of prime exceptional divisors such that $W_I$ is finite. Let $C_I$ be as in Proposition 4.1.3. Taking $\gamma = r_{C_I}$, one can show that $x \in \Pi$ satisfies $(x, C_I) \geq 0$. By Proposition 4.1.3 (3), $\Pi$ is contained in $\mathcal{MV}^+_X$. Now we will show that $\Pi$ satisfies desired properties. For any $x \in \mathcal{MV}^+_X$, one can find $\phi \in \text{Bir}(X)$ and $r \in R_X$ such that $r\phi(x)$ lies in $\Pi \subset \mathcal{MV}^+_X$. On the other hand, $\phi(x) \in \mathcal{MV}^+_X$. Therefore, by Proposition 4.0.4, we have $r\phi(x) = \phi(x) \in \Pi$. It finishes the proof. \qed

### 4.2. Automorphism cone conjecture.

In this subsection, let $k$ be a field of characteristic 0, $X$ an irreducible symplectic variety over $k$. Let $\Sigma \subset \Lambda_{X_0}$ be the set

$$\Sigma := \left\{f_*(e) \mid e \in \text{Nef}_X \text{ is integral, primitive, and extremal, and } f : Y \dashrightarrow X_0 \text{ is a birational map of irreducible symplectic varieties over } k\right\}.$$ 

Here, the cone $\text{Nef}_X \subset \Lambda_{X, k}$ is a dual cone of $\text{Nef}_Y$. We note that any element in $\Sigma$ is a primitive MBM class in the sense of [AV17] Definition 2.14.

**Definition 4.2.1.** Let $\mathcal{MV}^9_{X_0}$ be the interior of the movable cone. We define the birational ample cone of $X_0$ by

$$\mathcal{BA}_{X_0} := \mathcal{MV}^9_{X_0} \setminus (\cup_{\lambda \in \Sigma} \lambda^\perp \cap \mathcal{MV}_{X_0}^9).$$

We denote the intersection $\mathcal{BA}_{X_0} \cap \Lambda_{X, k}$ by $\mathcal{BA}_X$.

**Proposition 4.2.2.** Let $f : Y \dashrightarrow X$ be a birational map of irreducible symplectic varieties over $k$. Then the image $f_*(\text{Amp}_Y)$ is a connected component of $\mathcal{BA}_X$. Moreover, every connected component of $\mathcal{BA}_X$ is of this form.
Proof. The case of $k = \overline{k}$ follows from [MY15 Proposition 2.1]. Therefore, we will treat a general case. We note that

$$f_{\overline{k},*}(\text{Amp}_{\overline{k}}) \cap \Lambda = f_*(\text{Amp}_Y).$$

Indeed, an element $f_{\overline{k},*}(x)$ of the left-hand side is Galois invariant, and since $f$ is defined over $k$, $x \in \text{Amp}_{\overline{k}}$ is also Galois invariant. Therefore, Lemma 2.1.2 gives desired equality. Therefore, the first part of Proposition 4.2.2 follows from the case of $k = \overline{k}$, since $f_*(\text{Amp}_Y)$ is connected.

We will show the remaining part of Proposition 4.2.2. Let $A$ be a connected component of $\mathcal{B}A_X$. From the case of $k = \overline{k}$, there exists a birational map $f : Z \rightarrow X_{\overline{k}}$ of irreducible symplectic varieties over $\overline{k}$, such that $A$ is contained in $f_*(\text{Amp}_Z) \subset \mathcal{B}A_{X_{\overline{k}}}$. For any $\sigma \in G_k$, by the $k$-structure of $X$, we have an isomorphism

$$\phi_{\sigma} : X_{\overline{k}} \rightarrow \sigma(X_{\overline{k}}).$$

On the other hand, we have a birational map $\sigma f : \sigma(Z) \rightarrow \sigma(X_{\overline{k}})$. We want to show that $\psi_{\sigma} = \sigma f^{-1} \circ \phi_{\sigma} \circ f$ is an actual isomorphism, so that $\psi_{\sigma}$ gives $k$-structure on $Z$. Take an element $L \in A$, and let $M \in \text{Amp}_Z$ such that $f_*(M) = L$. Note that there exist classes $L' = \sigma(L)$ on $\sigma(X_{\overline{k}})$, $M' = \sigma(M)$ on $\sigma(Z)$. Then we have

$$\psi_{\sigma,*}(M) = \sigma f^{-1} \circ \phi_{\sigma,*}(L) = \sigma f^{-1}(L') = M'.$$

Here, the second equality follows from $L \in \Lambda_{X_{\overline{k}}}$, therefore, $\psi_{\sigma}$ pulls back an ample class to an ample class, thus $\psi_{\sigma}$ is an isomorphism (indeed, in this case, $\psi_{\sigma}$ pulls back a very ample class to a very ample class). Therefore, $Z$ descends to a variety $Y$ over $k$, and by the definition of the $k$-structure on $Y$, a birational map $f$ is defined over $k$. Combining with the equality (2), it finishes the proof.

Theorem 4.2.3 ([AV17 Theorem 5.3], [AV16 Corollary 1.4]). Suppose that $b_2(X_{\overline{k}}) \geq 5$. Then $\Sigma$ have bounded Beauville–Bogomolov square.

In the following, we suppose that $b_2(X_{\overline{k}}) \geq 5$. Let $\Pi \subset \mathcal{M}Y^+_X$ be a rational polyhedral fundamental domain as in Theorem 4.1.4. By Theorem 4.2.3 and [MY15 Proposition 3.4], the set

$$\{ \lambda \in \Sigma \mid \lambda \cap \Pi \cap C_X \neq \emptyset \}$$

is finite. This set divides $\Pi$ into a finite union of closed rational polyhedral subcones. We denote these subcones by

$$\Pi_j, \quad j \in J.$$

Lemma 4.2.4. Let $f : Y \rightarrow X$ be a birational map of irreducible symplectic varieties over $k$. Let $g$ be an element of $\text{Bir}(X)$, such that $g_*(\Pi_j)$ intersects $f_*(\text{Amp}_Y)$. Then $g(\Pi_j) = f_*(\text{Nef}_Y) \cap g(\Pi)$.

Proof. The proof is the same as in [MY15 Lemma 2.3], but we include the proof. By replacing $f$ by $g^{-1} \circ f$, we may assume that $g = 1$. We denote the interior of $\Pi_j$ (resp. $\Pi$) by $\Pi^0_j$ (resp. $\Pi^0$). It suffices to prove the equality

$$\Pi^0_j = f_*(\text{Amp}_Y) \cap \Pi^0.$$

The inclusion $\subset$ follows from Proposition 4.2.2. Another inclusion also follows since the right-hand side is convex, so connected and disjoint from the hyperplane $\lambda^\perp$ for every $\lambda \in \Sigma$ by Proposition 4.2.2. \qed
Let \( f: Y \rightarrow X \) be a birational map of irreducible symplectic varieties over \( k \). Let \( J_f \subseteq J \) be the subset consisting of indices \( j \) such that their exists \( g_j \in \text{Bir}(X) \) such that \( g_j(\Pi_j) \) is contained in \( f_*(\text{Nef}_Y) \).

**Lemma 4.2.5.** Let \( f_i: Y_i \rightarrow X \) \((i = 1, 2)\) be a birational map of irreducible symplectic varieties over \( k \). Then the following hold.

1. \( Y_i \) is isomorphic to \( Y_2 \) over \( k \) if and only if \( J_{f_1} = J_{f_2} \).
2. If \( J_{f_1} \cap J_{f_2} \) is nonempty, then we have \( J_{f_1} = J_{f_2} \).

**Proof.** The proof is the same as in [MY15, Lemma 2.4], but we include the proof. For (1), first suppose that there exists an isomorphism \( \phi: Y_1 \rightarrow Y_2 \). Then \( \psi := f_2 f_1^{-1} \) sends \( f_1(\text{Nef}_{Y_1}) \) to \( f_2(\text{Nef}_{Y_2}) \). Thus for any \( g \in \text{Bir}(X) \) and \( j \in J \), \( g(\Pi_j) \subseteq f_1(\text{Nef}_{Y_1}) \) if and only if \( \psi g(\Pi_j) \subseteq f_2(\text{Nef}_{Y_2}) \). Thus we have \( J_{f_1} = J_{f_2} \). For the remaining part of (1) and (2), suppose that there exists an element \( j \in J_{f_1} \cap J_{f_2} \). Then there exist \( h_i \in \text{Bir}(X) \) such that \( h_i(\Pi_j) \subseteq f_1(\text{Nef}_{Y_1}) \). Thus \( f_2^{-1} h_2 h_1^{-1} f_1 \) maps some ample class to an ample class. Therefore, \( f_2^{-1} h_2 h_1^{-1} f_1 \) is an isomorphism, and it finishes the proof. \( \square \)

For a birational map \( f: Y \rightarrow X \) of irreducible symplectic varieties, we put \( J_Y := J_f \) which depends only on an isomorphism class of \( Y \) by Lemma 4.2.5.

**Lemma 4.2.6.** For any \( j \in J_X \), the set

\[ \{ g \in \text{Bir}(X) \mid g(\Pi_j) \subseteq \text{Nef}_X \} \]

is an \( \text{Aut}(X) \)-coset, i.e. it is equal to \( \text{Aut}(X) g_j \) for some \( g_j \in \text{Bir}(X) \). Moreover, \( \text{Nef}_X^j \) is the union of \( \text{Aut}(X) \)-translates of finitely many rational polyhedral subcones \( g_j(\Pi_j) \) \((j \in J_X)\).

**Proof.** The proof is the same as in [MY15, Lemma 2.6, Corollary 2.7], but we include the proof. Suppose that \( g, h \in \text{Bir}(X) \) satisfy \( g(\Pi_j), h(\Pi_j) \subseteq \text{Nef}_X \). Take a class \( \alpha \) in the interior of \( \Pi_j \). Then \( g(\alpha) \) and \( h(\alpha) \) is an ample class, so \( gh^{-1} \in \text{Aut}(X) \). Thus we have the first assertion. By Lemma 4.2.4, the cone \( \text{Nef}_X^j \) is the union of \( \text{Bir}(X) \)-translates of the \( \Pi_j \) intersecting its interior. By the first assertion, this is a union of \( \text{Aut}(X) \)-translate of \( g_j(\Pi_j) \). Therefore, we have the second assertion. \( \square \)

Now we have the following main theorem in this subsection.

**Theorem 4.2.7.** Let \( k \) be a field of characteristic 0. Let \( X \) be an irreducible symplectic variety over \( k \). Suppose that \( b_2(X_{\mathbb{Q}}) \geq 5 \). Then the following hold.

1. The set \( \mathfrak{B}_X \) of k-isomorphism classes of irreducible symplectic varieties in the k-birational class of \( X \) is finite.
2. There exists a rational polyhedral cone \( D \) in \( \text{Nef}_X^+ \) which is a fundamental domain for the action of \( \Gamma_{\text{Aut}(X)} \). Here, \( \Gamma_{\text{Aut}(X)} \) means the image of the automorphism group of \( X \) via the natural action \( \text{Aut}(X) \rightarrow O(A_X, \mathbb{R}) \).

**Proof.** (1) follows from Lemma 4.2.5 since the map \( \mathfrak{B}_X \rightarrow 2^J; Y \mapsto J_Y \) induces an injection.

(2) We choose an ample class \( y \in C_X \cap \Lambda \). We put

\[ D := \{ x \in \text{Nef}_X \mid (\gamma x, y) \geq (x, y) \text{ for all } \gamma \in \Gamma_{\text{Aut}(X)} \} \]

Then \( D \) is a rational polyhedral by [Tot10, Lemma 2.2] and Lemma 4.2.6. \( \square \)
Remark 4.2.8. Theorem [AV17, 5.7] (2) also holds for the case where $b_2(X_F) \leq 4$. Indeed, if $\Lambda_X$ is of rank 1, the cone conjecture is trivial. On the other hand, if $\Lambda_X$ is of rank 2, Then $\Lambda_X = \Lambda_X$, thus by the argument below [AV17, Theorem 5.6], the cone conjecture also holds.

4.3. Proof of the main theorem. First, we will prove an analogue of Lemma 3.1.6

Lemma 4.3.1. Let $k$ be a field of characteristic 0, and $X$ an irreducible symplectic variety over $k$ with $b_2(X_F) \geq 5$. Then there exists a positive integer $d$ such that there exists a polarization $L_Y$ on $Y$ with $(L_Y, L_Y) = d$ for any irreducible symplectic variety $Y$ satisfying the following conditions.

1. There exists a finitely generated subfield $K \subset k$ field embeddings $\iota_1, \iota_2: K \hookrightarrow \mathbb{C}$, and irreducible symplectic variety $X', Y'$ over $K$ with $X'_k \simeq X$ and $Y'_k \simeq Y$, such that the complex manifolds $X_{i_1, k}$ and $Y_{i_2, k}$ are homeomorphic.

2. There exits an isometry $\Phi: \Lambda_{X_F} \simeq \Lambda_{Y_F}$ which induces $\Lambda_X \simeq \Lambda_Y$.

Proof. By [AV17, Theorem 5.3] and [AV16, Corollary 1.4], there exists a positive number $N$ such that any primitive MBM class on $X_k^+$ has the self Beauville square greater than $-N$, for any irreducible symplectic variety $Y$ over $k$ satisfying the condition (1). We put

$$O^+(\Lambda_{X_F}, \Lambda_X) := \{ g \in O^+(\Lambda_{X_F}) \mid g\Lambda_X = \Lambda_X \}.$$ 

We also put

$$S := C_X^+ \setminus \left\{ v^+ \mid v \in \Lambda_{X_F}, (v, v) > -N, v^+ \not\in \Lambda_X \right\} \cup \bigcup_{g \in O^+(\Lambda_{X_F}, \Lambda_X)} \text{Fix}(g) \quad \text{s.t. } 1 \neq \rho \in O^+(\Lambda_X)$$

Then there is a natural action $O^+(\Lambda_{X_F}, \Lambda_X)$ on $\pi_0(S)$. We will see that there exist only finitely many orbits of this action. Since the image of $O^+(\Lambda_{X_F}, \Lambda_X)$ in $O^+(\Lambda_X)$ is of finite index by [BLvL19, Proposition 2.2], there exists a rational polyhedral fundamental domain $P \subset C_X^+$ of the action of $O^+(\Lambda_{X_F}, \Lambda_X)$ on $\Lambda_X$. By [MY13, Proposition 3.4] and [Huy16, Chapter 8, Remark 2.2],

$$P \setminus \left\{ v^+ \mid v \in \Lambda_{X_F}, (v, v) > -N, v^+ \not\in \Lambda_X \right\} \cup \bigcup_{g \in O^+(\Lambda_{X_F}, \Lambda_X)} \text{Fix}(g) \quad \text{s.t. } 1 \neq \rho \in O^+(\Lambda_X)$$

has only finitely many connected components $W_1, \ldots, W_m$. There exists a connected component $V_1, \ldots, V_m$ of $S$ such that $W_i \subset V_i$ for $1 \leq i \leq m$. Then we can show that $V_1, \ldots, V_m$ represent all the $O^+(\Lambda_{X_F}, \Lambda_X)$-orbits of $\pi_0(S)$. Indeed, for any connected component $V$ of $S$, there exists an element $g \in O^+(\Lambda_{X_F})$ such that $gV \cap P^o \neq \phi$. We have $gV \cap W_i \neq \phi$ for some $i$, and then we have $V_j = gV$.

Next, take an element $L_i \in V_i \cap \Lambda_X \cap C_X$. We put $d_i := (L_i, L_i)$, and $d := \max_{1 \leq i \leq m} d_i$. Take an irreducible symplectic variety $Y$ over $k$, with an isometry $\Phi: \Lambda_{X_F} \simeq \Lambda_{Y_F}$ which induces $\Lambda_X \simeq \Lambda_Y$. We may assume that $\Phi(C_X) = C_Y$. By Proposition 4.5.6 Definition.
and Proposition 4.2.2, the boundary $\partial(\Phi^{-1}(\text{Amp}_Y)) \cap C_X$ is contained in

$$\left\{ v^+ \mid v \in \Lambda_X^+, (v, v) > -N, v^+ \not\in \Lambda_X \right\} \cup \bigcup_{g \in O^+(\Lambda_X^+, \Lambda_X)} \text{Fix}(g).$$

Therefore, there exists a connected component $V$ of $S$ such that $V \subset \Phi^{-1}(\text{Amp}_Y)$. There exists an element $g \in O^+(\Lambda_X^+, \Lambda_X)$ and $1 \leq i \leq m$ such that $gV = V_i$. Then $L_{i,Y} := \Phi(g^{-1}L_i)$ gives an ample line bundle on $Y$ with $(L_{i,Y}, L_{i,Y}) = d_i \leq d$. Replacing $d$ by $d!$, it finishes the proof.

**Remark 4.3.2.** Let $X$ be an irreducible symplectic variety over $k$ with a polarization of Beauville-Bogomolov square $d$, and $Y$ an irreducible symplectic variety over $k$ such that there exists an isometry $\Lambda_X^+ \cong \Lambda_Y^+$ which induces $\Lambda_X \cong \Lambda_Y$. Then $Y$ does not necessarily admit a polarization of Beauville-Bogomolov square $d$. However, by using the action of $R_Y$ (see Proposition 4.0.6), there exists an element $L \in \text{MV}_Y \cap C_Y$. Therefore, by applying [MZ13, Section 4], there exists an irreducible symplectic variety $Z$ over $k$ which is birational to $Y$, and $Z$ admits a quasi-polarization of Beauville-Bogomolov square $d$. (Note that tough [MZ13] only treats the case of $k = \mathbb{C}$, the same result over general $k$ holds since we can run a similar MMP as in [MZ13] in our case. Indeed, the cone theorem and the contraction theorem hold by the same proof as in [KM98, Theorem 3.7], and the existence of flips can be reduced to $\mathbb{C}$-case. Moreover, the termination of flips also can be proved by the same argument as in [MZ13 Section 4], and the irreducible symplecteness of output variety can be checked after the base change to $\mathbb{C}$, by the same argument as in [MZ13 Section 4].)

**Lemma 4.3.3.** Let $k$ be a field of characteristic 0, $k'$ an algebraic field extension of $k$, and $X$ an irreducible symplectic variety over $k$ with $b_2(X) \geq 5$.. Then there exists a positive number $d$ satisfying the following. For any irreducible symplectic variety $Y$ over $k$ such that $Y_{k'}$ is birational to $X_{k'}$, there exists a polarization of Beauville-Bogomolov square $d$ on $Y$.

**Proof.** We may assume that $k'$ is an algebraic closure of $k$. By the assumption, we have a $\mathbb{Z}$-lattice isometry between $\Lambda_X$ and $\Lambda_{Y_{k'}}$. As in the argument in Lemma 3.1.6 by [Bor63, Section 5, (a)], the candidates of isomorphism classes of Gal($k'/k$)-$\mathbb{Z}$-lattice $\Lambda_{Y_{k'}}$ is only finitely many. Therefore, the desired statement follows from Lemma 4.3.1 and [Huy99, Theorem 4.6].

Next, we will prove the following analogue of Lemma 3.1.5 as a Corollary of cone conjectures.

**Lemma 4.3.4.** Let $k$ be a field of characteristic 0. Let $X$ be an irreducible symplectic variety over $k$. Fix a positive integer $d$. Then the set of quasi-polarizations on $X$ of Beauville–Bogomolov square $d$ modulo Aut$(X)$ is a finite set.

**Proof.** This follows from the automorphism cone conjecture and the same argument as in Lemma 3.1.5. The following lemma is used for treating $b_2(X) \leq 4$. 


Lemma 4.3.5. Let $k$ be a field of characteristic $0$. Let $X$ be an irreducible symplectic variety over $k$. Suppose that the rank of $\Lambda_{X_k}$ is 2. Then the automorphism group $\text{Aut}(X)$ is either a finite group or an almost finite group of rank 1 (see Definition 2.2.1).

Proof. Since $\text{Aut}(X) \subset \text{Aut}(X_k)$, we may assume that $k$ is algebraically closed. We note that the natural morphism $\rho: \text{Aut}(X) \to O(\Lambda_X)$ has the finite kernel. Then applying [Ogu08, Proposition 8.3 (2)] to an exact sequence

$$1 \to \text{ker} \rho \to \text{Aut}(X) \to \text{Im} \rho \to 1,$$

we can reduce the problem to $\text{Im} \rho$. Moreover, applying [Ogu08, Proposition 8.3 (1)] to an exact sequence

$$1 \to \text{Im} \rho \cap SO(\Lambda_X) \to \text{Im} \rho \to \text{Im} \text{det} \to 1,$$

we can reduce the problem to $\text{Im} \rho \cap SO(\Lambda_X)$. On the other hand, $SO(\Lambda_X)$ is a finite group or an almost finite group of rank 1 by [Dic57, Theorem 87]. Thus it finishes the proof. □

In this section, we prove the following theorem.

Theorem 4.3.6. Let $k$ be a field of characteristic $0$. Let $k'/k$ be a finite extension of fields. Let $X$ be an irreducible symplectic variety over $k$. Then the set $\text{Tw}_{k'/k}(X)$ is finite.

Proof. Taking the Galois closure, we may assume that $k'/k$ is a finite Galois extension. First, we consider the case where $b_2(X_{k'}) \geq 5$. Let $d$ be a positive integer as in Lemma 4.3.3. Then any $Y \in \text{Tw}_{k'/k}(X)$ admits a polarization $M_Y$ of Beauville–Bogomolov square $d$. Moreover, by Lemma 4.3.4 we can take a complete system of representatives $M_1, \ldots, M_m \in \Lambda_{X_{k'}}$ of polarizations on $X_{k'}$ of Beauville–Bogomolov square $d$ modulo $\text{Aut}(X_{k'})$.

For $1 \leq i \leq m$, we put

$$T_i := \left\{ (Y, M) \mid \begin{array}{l} Y: \text{irreducible symplectic variety over } k \\ M: \text{polarization on } Y \\ (Y, M)_{k'} \simeq (X_{k'}, M_i) \end{array} \right\}/k\text{-isom.}$$

Then we have an injective morphism of sets

$$\text{Tw}_{k'/k}(X) \to \bigsqcup_i T_i; Y \mapsto (Y, M_Y),$$

where $M_Y$ is taken as the above argument. The finiteness of $T_i$ follows from the finiteness of the automorphism group of polarized irreducible symplectic variety ([Bri18 Proposition 2.26]). Therefore, $\text{Tw}_{k'/k}(X)$ is a finite set.

Next, we treat the case of $b_2(X_{k'}) \leq 4$. In this case, the Picard rank of $X_{k'}$ is 1 or 2. If the Picard rank of $X_{k'}$ is 1, then the finiteness is reduced to the polarized case. If the Picard rank of $X_{k'}$ is 2, by Lemma 4.3.5 and Proposition 2.2.3 $\text{Tw}_{k'/k}(X)$ is also finite. It finishes the proof of Theorem 4.3.6. □

5. Uniform bounds

In this section, we prove the following uniformness result for $\text{Tw}_{k'/k}(X)$.

Theorem 5.0.1. Let $k'/k$ be a finite extension of fields of characteristic $p$. 
(1) In Theorem 5.1.3, the cardinality of the set Tw\textsubscript{k’/k}(X) is bounded above by a constant which depends only on \( p \), \([k’: k]\) and \(\text{disc}(\Lambda_{X_{\mathbb{C}}})\).

(2) Suppose that \( k \) is a subfield of \( \mathbb{C} \). Let \( X \) be an irreducible symplectic variety over \( k \). Then the order of the set Tw\textsubscript{k’/k}(X) is bounded above by a constant which depends only on \([k’: k]\), \(\text{disc}(\Lambda_{X_{\mathbb{C}}},q)\) and the deformation class of \( X_{\mathbb{C}} \).

Proof. First, by the effective base point free theorem [Ko93, Theorem 1.1], \( 2(n+2)!nL \) is base point free. We put \( m:=2(n+2)!n \). Since \( \text{Bir}(X, L) \subset \text{Bir}(X, mL) \), we will establish a bound of \( \text{Bir}(X, mL) \). We denote the natural action on the linear system \( mL \) by \( \rho \): \( \text{Bir}(X, mL) \to \text{PGL}(H^0(X, mL)) \). First, we will bound the order of cyclic subgroups of \( \text{Bir}(X, mL) \). Let \( \langle g \rangle = G \subset \text{Bir}(X, mL) \) be a cyclic subgroup. Let \( \bar{g} \in \text{GL}(H^0(X, mL)) \) be a lift of \( \rho(g) \). Let \( v \in H^0(X, mL) \) be an eigenvector of \( \bar{g} \). Then the corresponding effective divisor \( D_v \in mL \) satisfies \( g_v(D_v) = D_v \), i.e. \( G \subset \text{Bir}(X, D_v) \). Here, we denote the set of automorphisms of \( X \) preserving the Cartier divisor \( D \) by \( \text{Aut}(X, D) \). Therefore, by [Kov01, Theorem 1.2.3], we have

\[
\#G \leq (m^nL^n)^{n+1}.
\]

Note that in [Kov01], it is supposed that \((X, D_v)\) is log canonically polarized, i.e. \( D_v \) is ample. But the same proof works in our situation. Now we note that the contraction \( \phi_{mL} : X \to Y \) induces the inclusion \( \text{Aut}(X, mL) \to \text{Aut}(Y, O(1)) \), in particular \( \text{Aut}(X, mL) \) acts on a degree \( m^nL^n \) projective subvariety \( Y \) faithfully. Therefore, by [Sza96, main bound], we have a bound

\[
\#\text{Aut}(X, mL) \leq (m^nL^n)^{16n^3}.
\]

Therefore, it finishes the proof. \( \square \)

5.1. Bounds of automorphism groups. In this subsection, we establish a uniform bound of canonical quasi-polarized variety, based on the work of Kovács.

Proposition 5.1.1. Let \( k \) be an algebraically closed field of characteristic 0. Let \( X \) be a smooth projective \( n \)-dimensional variety over \( k \) with the trivial canonical bundle. Let \( L \) be a nef big line bundle on \( X \). Then the order of the group \( \text{Bir}(X, L) \) is finite and bounded above by a constant which depends only on \( n \) and \( L^n \).

Proof. By [Ogu79, Corollary 2.5] and [Keu16, Theorem 1.4], the order of any finite subgroup of \( \text{Aut}(X) \) is bounded by \( 42^{22} \). Here, we use that \( 1+nM_{22}(\mathbb{Z}_e) \subset \text{GL}_{22}(\mathbb{Z}_e) \) is torsion free if \( n = \ell^2 = 4 \) or \( n = \ell \geq 3 \). Throughout the proof, we note that we may fix the lattice isometry class of \( \Lambda_{X_{\mathbb{C}}} \) since the set of isometry classes of \( \mathbb{Z} \)-lattices with bounded rank and discriminant is finite ([Cas82, ch. 9, Theorem 1.1]). The proof is accomplished by seeing constants appearing in the proof of Theorem 1.3.6 carefully.

Corollary 5.1.2. Let \( k, X, n, L \) be as in Proposition 5.1.1. Let \( k’/k \) be a finite extension of fields of characteristic 0. Then \#Tw\textsubscript{k’/k}(X, L) is bounded above by a constant which depends only on \( n \), \( L^n \), and \([k’: k]\).

Remark 5.1.3. Let \( X \) be a K3 surface over an algebraically closed field \( k \). Since \( \text{Aut}(X) \to \text{O}(H^2_{\text{et}}(X, \mathbb{Q}_l)) \) is injective for any \( \ell \neq p \) by [Ogu79, Corollary 2.5] and [Keu16, Theorem 1.4], the order of any finite subgroup of \( \text{Aut}(X) \) is bounded by \( 42^{22} \). Here, we use that \( 1+nM_{22}(\mathbb{Z}_e) \subset \text{GL}_{22}(\mathbb{Z}_e) \) is torsion free if \( n = \ell^2 = 4 \) or \( n = \ell \geq 3 \).
5.2. Proof of Theorem 5.0.1 (1). In this subsection, we follow the notation in Theorem 5.0.1 (1). For simplicity, if some real number $c$ admits an upper bound which depends only on $p$, $[k': k]$, and $\text{disc}(\Lambda_{X_{k'}})$, then we say that $c$ is uniformly bounded.

To begin with, as in the beginning of the proof of Theorem 5.1.2 we may assume that $k'$ is a finite separable extension. Moreover, we may assume that $\Lambda_{X_{k'}} = \Lambda_{X_{k_s}} = \Lambda_{X_{k'}}$. Indeed, an order of a torsion subgroup of $\text{GL}(\Lambda_{X_{k_s}})$ is bounded by $\#\text{GL}_p(\mathbb{F}_3)$, since $1 + 3M_p(\mathbb{Z}_3) \subset \text{GL}_p(\mathbb{Z}_3)$ is a torsion free subgroup.

Next, we will show that we may assume $\text{Aut}(X_{k'}) = \text{Aut}(X_{k_s})(= \text{Aut}(X_{k'}))$. We need the following lemma. Let $\rho: \text{Aut}(X_{k'}) \ltimes R_{X_{k'}} \to O(\Lambda_{X_{k'}})$ be the natural morphism. Here, we write $R_{X_{k'}}$ for the subgroup of $O(\Lambda_{X_{k'}})$ generated by reflections by $-2$ classes as in [BLvL19]. We note that $R_{X_{k'}}$ is equal to $R_{X_{k_s}}$, which is defined similarly, by [BLvL19] Corollary 3.2. Therefore, the absolute Galois group $G_k$ acts on $\text{Aut}(X_{k'})$, $R_{X_{k'}}$, or $O(\Lambda_{X_{k'}})$.

**Lemma 5.2.1.** The index of $\text{Im} \rho$ in $O(\Lambda_{X_{k_s}})$ is bounded above by a constant which depends only on the lattice isometry class of $\Lambda_{X_{k_s}} = \Lambda_{X_{k'}}$.

*Proof.* In the characteristic 0 case, this follows from the proof of [Huy16] Chapter 8, Theorem 4.2]. Indeed, any element in $\ker(O(\Lambda_{X_{k'}}) \to O(\Lambda_{X_{k_s}}/\Lambda_{X_{k'}}))$ comes from the image of $\rho$. Therefore, we will treat the positive characteristic case. If $X$ is supersingular, then this finiteness of the index essentially follows from the crystalline Torelli theorem [BL18]. In this case, by the proof of [LM18] Proposition 5.2], the index $[O(\Lambda_{X_{k'}}): \rho(\text{Aut}(X_{k'}) \ltimes R_{X_{k'}})]$ is absolutely bounded. Indeed, the index

$$[O^+(\Lambda_{X_{k'}})/R_{X_{k'}}: \text{Im}(\text{Aut}(X_{k'}) \to O^+(\Lambda_{X_{k'}})/R_{X_{k'}})]$$

is at most $\#\text{GL}(\Lambda_{X_{k'}} \otimes_{\mathbb{Z}} \mathbb{F}_p)$.

On the other hand, if $X$ has finite height, there exists a projective K3 families $\mathcal{X} \to W(k)$ with $X_{k'} \simeq X_{k'}$ such that the restriction map $\text{Pic}(\mathcal{X}) \to \text{Pic}(X_{k'})$ is an isomorphism. In this case, we have $\Lambda_{X_{k'}} \simeq \text{Pic}(\mathcal{X}) \simeq \Lambda_{X_{k'}}$. Here, we put $F := \text{Frac}(W(k'))$. Therefore, by the case of char $k = 0$, the index

$$[\Lambda_{X_{k'}}: \text{Im}(\text{Aut}(\mathcal{X}_{k'}) \ltimes R_{X_{k'}} \to O(\Lambda_{X_{k'}}))]$$

is uniformly bounded. Since the map

$$\text{Aut}(\mathcal{X}_{k'}) \ltimes R_{\Lambda_{k'}} \to O(\Lambda_{X_{k'}})$$

factors through the specialization map $\text{Aut}(\mathcal{X}_{k'}) \ltimes R_{\Lambda_{k'}} \to \text{Aut}(X_{k'}) \ltimes R_{X_{k'}}$ by [LM18] Theorem 2.1], it finishes the proof. \hfill $\Box$

**Lemma 5.2.2.** The number of generators of the image of $\rho$ is uniformly bounded. In particular, the number of generators of $\text{Aut}(X_{k'}) \ltimes R_{X_{k'}}$ and $\text{Aut}(X_{k'})$ are uniformly bounded.

*Proof.* The first statement follows from Lemma 5.2.1 and the Schreier index formula. The second statement follows from the first statement since the order of $\ker \rho$ is uniformly bounded by Remark 5.1.3 \hfill $\Box$

Since the order of $\ker \rho$ is uniformly bounded by Remark 5.1.3, we may assume that $\ker \rho$ is $G_{k'}$-trivial. Let $\gamma_1, \ldots, \gamma_N$ be generators of $\text{Aut}(X_{k_s})$. For any $\sigma \in G_{k'}$, we have $\sigma(\gamma_i) = \gamma_i\delta_{i, \sigma}$ for $\delta_{i, \sigma} \in \ker \rho$. Then the map $\sigma \mapsto \delta_{i, \sigma}$ gives a group morphism $\psi_i: G_{k'} \to \ker \rho$. We note that the index of $\cap_i \ker \psi_i$ in $G_{k'}$ is uniformly bounded, since
so are $N$ and the order of $\ker \rho$. Replacing $k'$ by a finite field extension corresponding to $\cap_i \ker \psi_i$, we may assume that $\Aut(X_{k'}) = \Aut(X_{k'_i})$.

We need the following lemma to bound the polarization degree uniformly.

**Lemma 5.2.3.** In the statement of Lemma 5.1.4 we can take a positive integer $d$ which depends only on the lattice isometry class of $\Lambda_{X_T}$ (does not depend on $[k': k]$).

**Proof.** The number of candidates of isomorphism classes of $\mathbb{Z}$-lattices $\Lambda_{Y}$ in Lemma 3.1.6 depend on the lattice isometry class of $\Lambda_{X_T}$. On the other hand, as in the proof of Lemma 3.1.6, the set of degree of polarizations on $Y$ depends only on the lattice isometry class of $\Lambda_{Y}$. Therefore, it finishes the proof.

In the proof of Theorem 3.1.2, first, we took a complete system of representatives $M_1, \ldots, M_m \in \Lambda_{X_{k'}}$ of polarizations on $X_{k'}$ of degree $d$. Here, $d$ is taken as in Lemma 5.2.3. Next, we defined the injective map $T_{w_{k'}/k}(X) \rightarrow \bigcup_{1 \leq i \leq m} T_i; Y \mapsto (Y, L_Y)$. Here, $T_i$ is a set of isomorphism classes of $k$-forms of $(X_{k'}, M_i)$ (see the proof of Theorem 3.1.2). Therefore, it is enough to show that the integer $m$ and the cardinalities of $T_i$ are uniformly bounded. Note that the cardinalities of $T_i$ are bounded by a constant which depends only on $[k': k]$ by Remark 5.1.3. Therefore, we will consider the bound of the integer $m$. Since $\Nef(X_T)$ is a fundamental domain with respect to the action of $R_{X_T}$, the problem can be reduced to bound the cardinality of the following set

$$\{ \lambda \in \Lambda_{X_T} \mid (\lambda, \lambda) = d \}/ \Aut(X_T) \ltimes R_{X_T}.$$

We may fix the lattice isometry class of $\Lambda_{X_T}$. For any lattice isometry class $\Lambda$, one can fix a rational polyhedral fundamental domain $D_\Lambda \subset C_\Lambda^+$ with respect to the action of $O^+(\Lambda)$. Here, $C_\Lambda^+$ is the convex hull of $\overline{C_\Lambda} \cap \Lambda_{Q}$, and $C_\Lambda$ is the positive cone of $\Lambda$. Moreover, we put $O^+(\Lambda_{X_T}) = \bigcup_{l \in L} \Gamma g_l$, where $\Gamma := \Im \rho$. We note that the cardinality of the index set $L$ is uniformly bounded by Lemma 5.2.1. Then $\bigcup_{l \in L} g_l D_{\Lambda_{X_T}}$ is a fundamental domain with respect to the action of $\Aut(X_T) \ltimes R_{X_T}$. Therefore, we have the desired uniform boundedness.

### 5.3 Proof of Theorem 5.0.1 (2)

In this subsection, we follow the notation in Theorem 5.0.1 (2). For simplicity, if some real number admits upper bound which depends only on $[k': k]$, $\text{disc}(\Lambda_{X_T}, q)$, and the deformation class of $X_T$, then we say that $c$ is uniformly bounded.

To begin with, we may assume $\Lambda_{X_{k'}} = \Lambda_{X_T}$. Indeed, an order of a torsion subgroup of $\text{GL}(\Lambda_{X_T})$ is bounded by $\# \text{GL}(\Lambda_{X_{k'}}, \mathbb{F}_2)$, since $1 + 3 \text{End}(\Lambda_{X_{k'}, \mathbb{Z}_3}) \subset \text{GL}(\Lambda_{X_{k'}})$ is a torsion free subgroup. We note that the Picard number of $X_{k'}$ is bounded above by the second Betti number of $X_T$, which depend only on a deformation class of $X$. Moreover, we will show that we may assume $\text{Bir}(X_{k'}) = \text{Bir}(X_T)$. To prove this, we propose the following lemma.

**Lemma 5.3.1.** In the statement of Lemma 4.3.3 we can take a positive integer $d$ which depends only on the lattice isometry class of $\Lambda_{X_T}$ and the deformation class of $X_C$ (not on $[k': k]$).

**Proof.** The number of candidates of isomorphism classes of $\text{Gal}(k'/k)\mathbb{Z}$-lattices $\Lambda_{Y_T}$ in Lemma 4.3.3 depends only on the lattice isometry class of $\text{Pic}(X_T)$. Moreover, the integer $N$ appearing in the proof of Lemma 4.3.1 depends only on the deformation class of $X_C$. Therefore, the desired statement follows from the proof of Lemma 4.3.1 and Lemma 4.3.3.
Take a positive integer $d$ in Lemma 5.3.1. Then $X$ admits a polarization $L$ of Beauville–Bogomolov square $d$. As before, let $\rho: \text{Bir}(X_\mathbb{F}) \ltimes R_{X_\mathbb{F}} \rightarrow O(\Lambda_{X_\mathbb{F}})$ be the natural action. The action of $\text{Gal}(\overline{k}/k')$ on the right hand side is trivial, and $\ker \rho \subset \text{Bir}(X_\mathbb{F})$ is contained in $\text{Bir}(Y, L)$, whose order is uniformly bounded by Lemma 5.1.1. Therefore, we may assume that $\ker \rho$ is $\text{Gal}(\overline{k}/k')$-trivial.

**Lemma 5.3.2.** The index of $\text{Im} \rho$ in $O(\Lambda_{X_\mathbb{F}})$ depends only on the lattice isometry class of $\Lambda_{X_\mathbb{F}}$ and the deformation class of $X_\mathbb{C}$.

**Proof.** We may assume $\overline{k} = \mathbb{C}$. As in the proof of [Mar11, Lemma 6.23], to prove Lemma 5.3.2, it suffices to bound the following.

1. The index of $\text{Im}(O_\text{Hdg}^+(H^2(X_\mathbb{C}, \mathbb{Z})) \rightarrow O^+(\Lambda_{X_\mathbb{C}}))$ in $O^+(\Lambda_{X_\mathbb{C}})$.

2. The index of $\text{Mon}^2(X_\mathbb{C})$ in $O^+(H^2(X_\mathbb{C}, \mathbb{Z}))$.

Here, $O_\text{Hdg}^+(H^2(X_\mathbb{C}, \mathbb{Z})) \subset O^+(H^2(X_\mathbb{C}, \mathbb{Z}))$ is the subgroups of elements which preserve the Hodge structure, and $\text{Mon}^2(X_\mathbb{C})$ is the monodromy group restricted on the second cohomology (see [Mar11, Definition 1.1]). (1) depends on the lattice isometry class of $\text{disc} \Lambda_{X_\mathbb{F}}$ (see [Huy16, Chapter 14, Proposition 2.6]). (2) clearly depends on the deformation class of $X_\mathbb{C}$. Therefore, it finishes the proof.

**Lemma 5.3.3.** The number of generators of the image of $\rho$ is uniformly bounded. In particular, the number of generators of $\text{Bir}(X_\mathbb{F}) \ltimes R_{X_\mathbb{F}}$ and $\text{Bir}(X_\mathbb{F})$ are uniformly bounded.

**Proof.** The first statement follows from Lemma 5.3.2 and the Schreier index formula. The second statement follows from the first statement since the order of $\ker \rho$ is uniformly bounded.

Let $\gamma_1, \ldots, \gamma_N$ be generators of $\text{Bir}(X_\mathbb{F})$. For any $\sigma \in \text{Gal}(\overline{k}/k')$, we have $\sigma(\gamma_i) = \gamma_i \delta_{i, \sigma}$ for $\delta_{i, \sigma} \in \ker \rho$. Then the map $\sigma ightarrow \delta_{i, \sigma}$ gives a group morphism $\psi_i: \text{Gal}(\overline{k}/k') \rightarrow \ker \rho$. We note that the index of $\bigcap_i \ker \psi_i$ in $\text{Gal}(\overline{k}/k')$ is uniformly bounded, since so are $N$ and the order of $\ker \rho$. Replacing $k'$ by a finite field extension corresponding to $\bigcap_i \ker \psi_i$, we may assume that $\text{Bir}(X_\mathbb{F}) = \text{Bir}(X_{k'})$.

Next, we will recall the argument in Theorem 4.3.1. First, we will consider the case of $b_2(X_\mathbb{F}) \geq 5$. We take a complete system of representatives $M_1, \ldots, M_m \in \Lambda_{X_\mathbb{F}}$ of polarizations on $X_{k'}$ of Beauville–Bogomolov square $d$. Here, $d$ is taken as in Lemma 5.3.1. We have a injective morphism of sets $\Psi: \text{Tw}_{k'/k}(X) \rightarrow \bigsqcup T_i$, where $T_i$ is the set of twists of $(X_{k'}, M_i)$ via $k'/k$. Therefore, we should bound the following constants.

1. The cardinality of each set $T_i$.

2. The positive integers $m$.

**Bound of (1).** This follows from Proposition 5.1.1.

**Bound of (2).** In the following, we use the same notation $D_\lambda$ as in the proof of Theorem 5.0.1 (1). Since a birational automorphism which sends a polarization to a polarization is an automorphism, we need to bound the set of degree $d$ elements $\lambda \in \mathcal{MV}(X_\mathbb{F})^* \cap \Lambda_{X_\mathbb{F}}$ modulo birational automorphisms. Since $\mathcal{MV}(X_\mathbb{F})$ is a fundamental domain with respect to the action of $R_{X_\mathbb{F}}$, the problem can be reduced to bound the cardinality of the following set.

$$\{ \lambda \in \Lambda_{X_\mathbb{F}} \mid (\lambda, \lambda) = d \} / \text{Bir}(X_\mathbb{F}) \ltimes R_{X_\mathbb{F}}$$
Let $\rho$: Bir$(X_\mathbb{T}) \ltimes R_{X_\mathbb{T}} \rightarrow O(\Lambda_{X_\mathbb{T}})$ be the natural map and $\Gamma$ the image of $\rho$. Moreover, we put $O^+(\Lambda_{X_\mathbb{T}}) = \bigcup_{l \in L} \Gamma g_l$. We note that the cardinality of the index set $L$ is uniformly bounded by Lemma 5.3.2. Then $\bigcup_{l \in L} g_l D_{\Lambda_{X_\mathbb{T}}}$ is a fundamental domain with respect to the action of Bir$(X_\mathbb{T}) \ltimes R_{X_\mathbb{T}}$, and we have the desired uniform boundedness.

Finally, we consider the case where $\beta_2(X_\mathbb{T}) \leq 4$. If the Picard rank of $X_\mathbb{T}$ is 1, then we have desired finiteness by Proposition 5.1.1 since we may fix the lattice isometry class of $\Lambda_{X_\mathbb{T}}$. If the Picard rank of $X_\mathbb{T}$ is 2, by Remark 2.2.4 and the proof of [Ogu08, Proposition 8.3 (1), (2)], it is enough to bound $\# \ker \rho$ and the order of torsion part of SO$(\Lambda_{X_\mathbb{T}})$. The former is uniformly bounded as in the argument in the beginning of this subsection, and the latter is also uniformly bounded since we may fix the lattice isometry class of $\Lambda_{X_\mathbb{T}}$. Therefore, it finishes the proof.

6. ON THE DERIVED EQUIVALENT TWISTS

In this section, we prove the finiteness of derived equivalent twists, as an application of the finiteness of twists via finite extensions.

**Proposition 6.0.1.** Let $k$ be a field. Let $X, Y$ be smooth projective varieties over $k$. We denote their bounded derived categories of coherent sheaves by $D_b(X), D_b(Y)$. Let $p$ (resp., $q$) be the first projection $X \times_k Y \rightarrow X$ (resp., the second projection $X \times_k Y \rightarrow Y$). Suppose that $X$ and $Y$ are derived equivalent, i.e., there exists an equivalence $F$: $D_b(X) \cong D_b(Y)$. Then there exists a perfect complex $P \in D_b(X \times_k Y)$ unique up to isomorphism, which is called the Fourier–Mukai kernel, such that $F$ is written as

$$\Phi_P: D_b(X) \rightarrow D_b(Y); \varepsilon \mapsto Rq_*(Lp^*\varepsilon \otimes P).$$

**Proof.** See [Huy06, Corollary 5.17].

**Proposition 6.0.2.** Let $k, X, Y, p, q, P$ be as in Proposition 6.0.1. We put

$$v(P) := \text{ch}(P) \cdot \sqrt{\text{td}(X \times_k Y)},$$

where $\text{ch}(P)$ is the Chern character of $P$ and $\text{td}(X \times_k Y)$ is the Todd character of the tangent bundle $T_{X \times_k Y}/k$. We put the cohomological Fourier–Mukai transform as

$$\Phi^\text{et}_{P,\ell}: H^*(X_\mathbb{F}, \mathbb{Q}_\ell) \rightarrow H^*(X_\mathbb{F}, \mathbb{Q}_\ell) \alpha \mapsto Rp_*(v(P)Lq^*(\alpha)).$$

Then $\Phi^\text{et}_{P,\ell}$ gives a $\text{Gal}(k_s/k)$-equivariant isomorphism of $\mathbb{Q}_\ell$-vector space. Moreover, there exists an integer $N$ which depends on the dimension of $X$, such that for any $\ell > N$, the isomorphism $\Phi^\text{et}_{P,\ell}$ induces an isomorphism $H^*_\text{et}(X, \mathbb{Z}_\ell) \rightarrow H^*_\text{et}(Y, \mathbb{Z}_\ell)$.

**Proof.** For the first statement, see [Huy06, Remark 5.30], [LO15, Section 2]. The second statement follows from the definition of the Chern character and the Todd character.

**Remark 6.0.3.** As in [Huy06, Lemma 10.6], in the case of K3 surfaces, the class $v(P)$ has an integral coefficient.

**Definition 6.0.4.** Let $k$ be a field of characteristic 0, $X$ an irreducible symplectic variety over $k$, $\ell$ a prime number, and $n$ a positive number. We denote the torsion-free part of $\bigoplus H^i_\text{et}(X_\mathbb{T}, \mathbb{Z}_\ell)$ by $H^i_\ell(X_\mathbb{T})$. Let $\mathcal{L}$ be the abstract $\mathbb{Z}_\ell$-lattice which is
isomorphic to $H^*_+(X_{\mathbb{T}})$. We define a \textit{level $\ell^n$ structure of the full cohomology of $X$} as a $G_k$-invariant $\text{GL}(\mathcal{L}, \ell^n)$-orbit $\alpha$ of an isomorphism over $\mathbb{Z}_\ell$

$$\mathcal{L} \simeq H^*_+(X_{\mathbb{T}}).$$

Here, we put $\text{GL}(\mathcal{L}, \ell^n)$ as a principal level $\ell^n$-subgroup

$$\ker(\text{GL}(\mathcal{L}) \to \text{GL}(\mathcal{L} \otimes \mathbb{Z}_\ell/\ell^n \mathbb{Z}_\ell)).$$

We note that for any quasi-polarization $M$ of $X$, any automorphism of $(X, M, \alpha)$ acts trivially on $H^*_+(X_{\mathbb{T}})$ if $\ell^n \geq 3$, since $\text{GL}(\mathcal{L}, \ell^n)$ have no non-trivial torsion elements.

**Theorem 6.0.5.** Let $k$ be a field. Let $X$ be a smooth projective variety over $k$. We put

$$\text{Tw}^D(X) := \left\{ Y : \text{variety over } k \bigg| X_{\mathbb{T}} \simeq Y_{\mathbb{T}}, \right.$$ \left. X \text{ is derived equivalent to } Y \right\} /k\text{-isom.}$$

Then $\text{Tw}^D(X)$ is finite in the following cases.

1. $X$ is a K3 surface over $k$, and the characteristic of $k$ is not equal to 2.
2. $X$ is a K3 surface over $k$, and $X$ is not supersingular.
3. $k$ is of characteristic 0, and $X$ is irreducible symplectic variety over $k$ with the second Betti number $b_2(X_{\mathbb{T}}) \geq 5$ such that

$$\text{Aut}(X_{\mathbb{T}}) \to \text{Aut}(H^*_+(X_{\mathbb{T}}, \mathbb{Q}_\ell))$$

is injective for any prime number $\ell$.

In particular, in the case of (1), the isomorphism classes of K3 surfaces over $k$ which are derived equivalent to $X$ are finitely many.

**Proof.** The last statement for the case (1) follows from the finiteness of $\text{Tw}^D$ and the corresponding statement over algebraically closed field given by Bridgeland–Maciocia and Lieblich–Olsson ([BM01, Corollary 1.2] and [LO15, Theorem 1.1]).

In the following, we fix a prime number $\ell > \max\{N, 2\}$, where $N$ is given by applying \ref{6.0.2} to $X$.

First, we will prove the assertions (1) and (2). Since the isomorphism functor is unramified (see the proof of Theorem \ref{3.1.2}), we have

$$\text{Tw}^D(X) = \left\{ Y : \text{variety over } k \bigg| X_{k_s} \simeq Y_{k_s}, \right.$$ \left. X \text{ is derived equivalent to } Y \right\}$$

for any K3 surfaces $X$ over $k$. Here, we denote the separable closure of $k$ by $k_s$. By Lemma \ref{5.1.6} there exists a positive integer $d$ such that every $Y \in \text{Tw}^D(X)$ admit a polarization $M_Y$ of degree $d$. Moreover, for any $Y_1, Y_2 \in \text{Tw}^D(X)$, we have a Galois equivariant isomorphism

$$\Phi_{\ell, \ell'}: H^*_+(Y_{1, \mathbb{T}}, \mathbb{Z}_\ell) \simeq H^*_+(Y_{2, \mathbb{T}}, \mathbb{Z}_\ell)$$

by the choice of $\ell$. Therefore, we can take a finite Galois extension $k'/k$ such that $Y_{k_s}$ admits a level $\ell$-structure $\alpha_{Y_{k_s}}$ on the full cohomology for any $Y \in \text{Tw}^D(X)$. As in the proof of Theorem \ref{3.1.2} by Lemma \ref{5.1.5} we can take a complete system of representatives $M_1, \ldots, M_m \in \text{Pic}_{\mathcal{X}/k}(k_s)$ of polarizations of $X_{k_s}$ of degree $d$ modulo
Aut}_{X/k}(k_s). Moreover, we denote possible level structures on the full cohomology of \((X_{k_s})\) by \((\alpha_1, \ldots, \alpha_M)\). We put

\[
T_{i,j} := \left\{ (Y, M, \alpha) \middle| \begin{array}{l}
Y: \text{K3 surface over } k', \\
M: \text{polarization of } X, \\
\alpha: \text{level } \ell\text{-structure on the full cohomology of } Y, \\
(Y, M_{k_s}, \alpha_{k_s}) \simeq_\ell (X_{k_s}, M_i, \alpha_i)
\end{array} \right\}.
\]

Now we have a map of sets

\[
\text{Tw}^D(X) \to \bigsqcup_{i,j} T_{i,j}; Y \mapsto (Y_{k'}, M_{Y,k'}, \alpha_{Y,k'}).
\]

This map has finite fiber by Theorem 5.1.2 in the cases (1) and (2). On the other hand, the automorphism group of a polarized K3 surface with a level structure on the full cohomology is trivial since the map \(\text{Aut}_{X/k}(k_s) \to H^2_{et}(X_{k_s}, \mathbb{Z}_\ell)\) is injective as in Remark 5.1.3 (see also Definition 6.0.4). Therefore, each set \(T_{i,j}\) is a singleton. Now we have the desired finiteness.

Next, we will prove the assertion (3). By Lemma 4.3.3 there exists a positive integer \(d\) such that for any \(Y \in \text{Tw}^D(X)\), \(Y\) admits a polarization \(M_Y\) of Beauville–Bogomolov square \(d\). As before, we can take a finite Galois extension \(k'/k\) such that \(Y_{k'}\) admits a level \(\ell\)-structure \(\alpha_{Y,k'}\) on the full cohomology for any \(Y \in \text{Tw}^D(X)\).

As in the proof of Theorem 4.3.6 by Lemma 4.3.4 we can take a complete system of representatives \(M_1, \ldots, M_m \in \Lambda_{X/k}\) of polarizations on \(X_{k}\) of Beauville–Bogomolov square \(d\). Moreover, we denote possible level \(\ell\)-structures on the full cohomology of \(Y_{k'}\) by \(\alpha_1, \ldots, \alpha_M\). We put

\[
T_{i,j} := \left\{ (Y, M, \alpha) \middle| \begin{array}{l}
Y: \text{irreducible symplectic variety over } k', \\
M: \text{polarization on } Y, \\
\alpha: \text{level } \ell\text{-structure on the full cohomology of } (Y, M), \\
(Y, M_{k'}, \alpha_{k'}) \simeq_\ell (X_{k'}, M_i, \alpha_i)
\end{array} \right\}.
\]

Now we have a map of sets

\[
\text{Tw}^D(X) \to \bigsqcup_{i,j} T_{i,j}; Y \mapsto (Y_{k'}, M_{Y,k'}, \alpha_{Y,k'}).
\]

This map has finite fiber by Theorem 4.3.6. On the other hand, by the assumption, the automorphism group of a polarized irreducible symplectic variety with a level structure on the full cohomology is trivial (see the remark in Definition 6.0.4). Therefore, each set \(T_{i,j}\) is a singleton, and we have the desired finiteness.

\[\square\]

**Corollary 6.0.6.** Let \(k\) be a field of characteristic 0, and \(X\) is irreducible symplectic variety over \(k\). Suppose that there exist a subfield \(K \subset k\), an embedding \(K \hookrightarrow \mathbb{C}\), and an irreducible symplectic variety \(X'\) over \(K\) with \(X'_{\mathbb{C}} \simeq X\) such that \(X_{\mathbb{C}}\) is deformation equivalent to one of the following.

1. The \(n\)-points Hilbert scheme \(S^{[n]}\) of a K3 surface \(S\) over \(\mathbb{C}\) (in this case, we say that \(X\) is of \(K3^{[n]}\)-type).
2. The fiber of the summation morphism \(A^{[n+1]}\to A\) over \(0 \in A\), where \(A^{[n+1]}\) is a \(n+1\)-points Hilbert scheme of an abelian surface over \(\mathbb{C}\) (in this case, we say that \(X\) is if generalized Kummer-type).
3. O Grady’s 10-dimensional varieties over \(\mathbb{C}\) (see [O’G99]) (in this case, we say that \(X\) is \(OG_{10}\)-type).
Then $Tw^D(X)$ is a finite set. In particular, in the case of (1), the isomorphism classes of $K3^{[n]}$-type varieties over $k$ which are derived equivalent to $X$ are finitely many.

Proof. This follows from Theorem 6.0.5, the argument in the proof of [Ogu20, Theorem 1.3], [Ogu20, Theorem 5.1], and [MW17, Theorem 3.1]. The last statement follows from the Lefschetz principle and [Bec21, Proposition 9.4].

□

Remark 6.0.7. For a K3 surface $X$ over $k$, the map $\text{Aut}(X) \to \text{GL}(H^2_\text{et}(X, \mathbb{Z}_\ell))$ is injective. However, for general irreducible symplectic varieties over characteristic 0 fields, this map is no longer injective (see [Ogu20, Theorem 1.2], [MW17, Theorem 5.2]). On the other hand, the author does not know any counterexample to the condition in Theorem 6.0.5.(3) (see also [MW17, Remark 6.9]).

References

[AV16] E. Amerik and M. Verbitsky, Collections of parabolic orbits in homogeneous spaces, homogeneous dynamics and hyperkähler geometry, preprint (2016), arXiv:1604.03927.

[AV17] E. Amerik and M. Verbitsky, Morrison-Kawamata cone conjecture for hyperkähler manifolds, Ann. Sci. Éc. Norm. Supér. (4) 50 (2017), no. 4, 973–993.

[BB05] A. Björner and F. Brenti, Combinatorics of Coxeter groups, Graduate Texts in Mathematics, vol. 231, Springer, New York, 2005.

[Bec21] T. Beckmann, Derived categories of hyper-Kähler manifolds: extended Mukai vector and integral structure, preprint (2021), arXiv:2103.13382.

[BL18] D. Bragg and M. Lieblich, Twistor spaces for supersingular K3 surfaces, preprint (2018), arXiv:1804.07282.

[Bla17] J. Blanc, Algebraic structures of groups of birational transformations, Algebraic groups: structure and actions, Proc. Sympos. Pure Math., vol. 94, Amer. Math. Soc., Providence, RI, 2017, pp. 17–30.

[BLvL19] M. Bright, A. Logan, and R. v. Luijk, Finiteness results for K3 surfaces over arbitrary fields, European Journal of Mathematics (2019).

[BM01] T. Bridgeland and A. M. Maciocia, Complex surfaces with equivalent derived categories, Math. Z. 236 (2001), no. 4, 677–697.

[Bor63] A. Borel, Arithmetic properties of linear algebraic groups, Proc. Internat. Congr. Mathematicians (Stockholm, 1962), Inst. Mittag-Leffler, Djursholm, 1963, pp. 10–22.

[Bri18] M. Brion, Notes on automorphism groups of projective varieties, 2018.

[BS12] S. Boissière and A. Sarti, A note on automorphisms and birational transformations of holomorphic symplectic manifolds, Proc. Amer. Math. Soc. 140 (2012), no. 12, 4053–4062.

[Car82] J. W. S. Cassels, Rational quadratic forms, Proceedings of the International Mathematical Conference, Singapore 1981 (Singapore, 1981), North-Holland Math. Stud., vol. 74, North-Holland, Amsterdam-New York, 1982, pp. 9–26.

[CDL⁺21] F. Cossec, I. Dolgachev, C. Liedtke, et al., Enriques surfaces 1, preprint (2021).

[CF19] A. Cattaneo and L. Fu, Finiteness of Klein actions and real structures on compact hyperkähler manifolds, Math. Ann. 375 (2019), no. 3-4, 1783–1822.

[Dic57] L. Dickson, Introduction to the theory of numbers, Dover books on advanced mathematics, Dover Publications, 1957.

[Dol84] I. Dolgachev, On automorphisms of Enriques surfaces, Invent. Math. 76 (1984), no. 1, 163–177.

[Dol16] I. V. Dolgachev, A brief introduction to Enriques surfaces. Development of moduli theory—Kyoto 2013, Adv. Stud. Pure Math., vol. 69, Math. Soc. Japan, [Tokyo], 2016, pp. 1–32.

[Fu15] L. Fu, Étale cohomology theory, revised ed., Nankai Tracts in Mathematics, vol. 14, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2015.

[GI14] M. Geck and L. Iancu, Coxeter groups and automorphisms, preprint (2014), arXiv:1412.5428.

[Hec18] G. Heckman, Coxeter groups.
[Huy99] D. Huybrechts, *Compact hyperkähler manifolds: basic results*, Inventiones mathematicae 135 (1999), no. 1, 63–113.

[Huy06] D. Huybrechts, *Fourier-Mukai transforms in algebraic geometry*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, Oxford, 2006.

[Huy16] D. Huybrechts, *Lectures on K3 surfaces*, Cambridge Studies in Advanced Mathematics, vol. 158, Cambridge University Press, Cambridge, 2016.

[Ito18] K. Ito, *Finiteness of Brauer groups of K3 surfaces in characteristic 2*, Int. J. Number Theory 14 (2018), no. 6, 1813–1825.

[Kaw97] Y. Kawamata, *On the cone of divisors of Calabi-Yau fiber spaces*, Internat. J. Math. 8 (1997), no. 5, 665–687.

[Keu16] J. Keum, *Orders of automorphisms of K3 surfaces*, Adv. Math. 303 (2016), 39–87.

[KM98] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.

[Kol93] J. Kollár, *Effective base point freeness*, Math. Ann. 296 (1993), no. 4, 595–605.

[Kov01] S. J. Kovács, *Number of automorphisms of principally polarized abelian varieties*, Advances in algebraic geometry motivated by physics (Lowell, MA, 2000), Contemp. Math., vol. 276, Amer. Math. Soc., Providence, RI, 2001, pp. 3–7.

[Lie15] C. Liedtke, *Arithmetic moduli and lifting of Enriques surfaces*, J. Reine Angew. Math. 706 (2015), 35–65.

[LM18] M. Lieblich and D. Maulik, *A note on the cone conjecture for K3 surfaces in positive characteristic*, Math. Res. Lett. 25 (2018), no. 6, 1879–1891.

[LMS14] M. Lieblich, D. Maulik, and A. Snowden, *Finiteness of K3 surfaces and the Tate conjecture*, Ann. Sci. Éc. Norm. Supér. (4) 47 (2014), no. 2, 285–308.

[LO15] M. Lieblich and M. Olsson, *Fourier-Mukai partners of K3 surfaces in positive characteristic*, Ann. Sci. Éc. Norm. Supér. (4) 48 (2015), no. 5, 1001–1033.

[LOP18] V. Lazić, K. Oguiso, and T. Peternell, *The Morrison-Kawamata cone conjecture and abundance on Ricci flat manifolds*, Uniformization, Riemann-Hilbert correspondence, Calabi-Yau manifolds & Picard-Fuchs equations, Adv. Lect. Math. (ALM), vol. 42, Int. Press, Somerville, MA, 2018, pp. 157–185.

[Mar11] E. Markman, *A survey of Torelli and monodromy results for holomorphic-symplectic varieties*, Complex and differential geometry, Springer Proc. Math., vol. 8, Springer, Heidelberg, 2011, pp. 257–322.

[MM64] T. Matsusaka and D. Mumford, *Two fundamental theorems on deformations of polarized varieties*, Amer. J. Math. 86 (1964), 668–684.

[MW17] G. Mongardi and M. Wandel, *Automorphisms of O’Grady’s manifolds acting trivially on cohomology*, Algebr. Geom. 4 (2017), no. 1, 104–119.

[MY15] E. Markman and K. Yoshioka, *A proof of the Kawamata-Morrison cone conjecture for holomorphic symplectic varieties of K3[n] or generalized Kummer deformation type*, Int. Math. Res. Not. IMRN (2015), no. 24, 13563–13574.

[MZ13] D. Matsushita and D.-Q. Zhang, *Zariski F-decomposition and Lagrangian fibration on hyperkähler manifolds*, Math. Res. Lett. 20 (2013), no. 5, 951–959.

[O’G99] K. G. O’Grady, *Desingularized moduli spaces of sheaves on a K3*, J. Reine Angew. Math. 512 (1999), 49–117.

[Ogu79] A. Ogus, *Supersingular K3 crystals*, Journées de Géométrie Algébrique de Rennes (Rennes, 1978), Vol. II, Astérisque, vol. 64, Soc. Math. France, Paris, 1979, pp. 3–86.

[Ogu08] K. Oguiso, *Bimeromorphic automorphism groups of non-projective hyperkähler manifolds—a note inspired by C. T. McMullen*, J. Differential Geom. 78 (2008), no. 1, 163–191.

[Ogu14] ———, *Automorphism groups of Calabi-Yau manifolds of Picard number 2*, J. Algebraic Geom. 23 (2014), no. 4, 775–795.

[Ogu20] ———, *No cohomologically trivial nontrivial automorphism of generalized Kummer manifolds*, Nagoya Math. J. 239 (2020), 110–122.

[Spe09] D. E. Speyer, *Powers of Coxeter elements in infinite groups are reduced*, Proc. Amer. Math. Soc. 137 (2009), no. 4, 1295–1302.
[Sza96] E. Szabó, *Bounding automorphism groups*, Math. Ann. **304** (1996), no. 4, 801–811.

[Tak20a] T. Takamatsu, *On a cohomological generalization of the Shafarevich conjecture for K3 surfaces*, Algebra Number Theory **14** (2020), no. 9, 2505–2531.

[Tak20b] ———, *On the Shafarevich conjecture for Enriques surfaces*, Math. Z. (2020).

[Tot10] B. Totaro, *The cone conjecture for Calabi-Yau pairs in dimension 2*, Duke Math. J. **154** (2010), no. 2, 241–263.

[Yan19] Z. Yang, *On Irreducible Symplectic Varieties of K3[^n]-type in Positive Characteristic*, preprint (2019), arXiv:1911.05053

Graduate School of Mathematical Sciences, The University of Tokyo, Komaba, Tokyo, 153-8914, Japan

Email address: teppei@ms.u-tokyo.ac.jp