The Terwilliger algebra of the incidence graphs of Johnson geometry

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Submitted: Jan 4, 2012; Accepted: Oct 8, 2013; Published: Oct 21, 2013
Mathematics Subject Classifications: 05E30

Abstract
In 2007, Levstein and Maldonado computed the Terwilliger algebra of the Johnson graph $J(n, m)$ when $3m \leq n$. It is well known that the halved graphs of the incidence graph $J(n, m, m + 1)$ of Johnson geometry are Johnson graphs. In this paper, we determine the Terwilliger algebra of $J(n, m, m + 1)$ when $3m \leq n$, give two bases of this algebra, and calculate its dimension.

Keywords: Terwilliger algebra; Johnson graph; incidence graph; Johnson geometry

1 Introduction
Let $\Gamma = (X, R)$ denote a simple connected graph with the vertex set $X$ and the edge set $R$. Suppose $\text{Mat}_X(\mathbb{C})$ denotes the algebra over the complex number field $\mathbb{C}$ consisting of

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all matrices whose rows and columns are indexed by elements of $X$. For vertices $x$ and $y$, $\partial(x, y)$ denotes the distance between $x$ and $y$, i.e., the length of a shortest path connecting $x$ and $y$. Fix a vertex $x \in X$. Let $D(x) = \max\{\partial(x, y) \mid y \in X\}$ denote the diameter with respect to $x$. For each $i \in \{0, 1, \ldots, D(x)\}$, let $\Gamma_i(x) = \{y \in X \mid \partial(x, y) = i\}$ and define $E^*_i = E^*_i(x)$ to be the diagonal matrix in $Mat_X(\mathbb{C})$ with yy-entry
\[
(E^*_i)_{yy} = \begin{cases} 
1, & \text{if } y \in \Gamma_i(x), \\
0, & \text{otherwise}.
\end{cases}
\]

The subalgebra $\mathcal{T} = \mathcal{T}(x)$ of $Mat_X(\mathbb{C})$ generated by the adjacency matrix $A$ of $\Gamma$ and $E^*_{0}, E^*_{1}, \ldots, E^*_{D(x)}$ is called the Terwilliger algebra of $\Gamma$ with respect to $x$. Let $\mathbb{C}^X$ denote the vector space over $\mathbb{C}$ consisting of all column vectors whose coordinates are indexed by $X$. A $\mathcal{T}$-module is any subspace $W$ of $\mathbb{C}^X$ such that $\mathcal{T}W \subseteq W$. We call a nonzero $\mathcal{T}$-module irreducible if it does not properly contain a nonzero $\mathcal{T}$-module. An irreducible $\mathcal{T}$-module $W$ is thin if $\dim E^*_iW \leq 1$ for every $i$, and the graph $\Gamma$ is said to be thin with respect to $x$ if every irreducible $\mathcal{T}(x)$-module is thin.

Terwilliger [13, 14, 15] initiated the study of the Terwilliger algebra of an association scheme, which has been an important tool in studying structures of an association scheme. For more information, see [4, 5, 6]. The Terwilliger algebras of group schemes were discussed in [11, 2]. The Terwilliger algebras of some distance-regular graphs have been determined; see [17] for strongly regular graphs, [8] for Hypercubes, [11] for Hamming graphs, [12] for Johnson graphs, [10] for odd graphs.

Let $\Omega$ be a set of cardinality $n$ and let $\binom{\Omega}{i}$ denote the collection of all $i$-subsets of $\Omega$. Suppose $m$ is a nonnegative integer with $m + 1 \leq n$. The incidence graph $J(n, m, m + 1)$ of Johnson geometry is a bipartite graph with a bipartition $\binom{\Omega}{m} \cup \binom{\Omega}{m+1}$, where $y \in \binom{\Omega}{m}$ and $z \in \binom{\Omega}{m+1}$ are adjacent if $y \subseteq z$. The graph $J(n, m, m + 1)$ is distance-biregular (see [3]). It is well known that the halved graphs of $J(n, m, m + 1)$ are Johnson graphs.

Levstein and Maldonado [12] determined the Terwilliger algebra of the Johnson graph $J(n, m)$ when $3m \leq n$. In this paper we shall determine the Terwilliger algebra of $J(n, m, m + 1)$ with respect to $x \in \binom{\Omega}{m}$ when $n \geq 3m$. In Section 2, we introduce some useful identities for intersection matrices. In Section 3, the Terwilliger algebra of $J(n, m, m + 1)$ is described. In Section 4, we give two bases of this algebra and compute its dimension.

## 2 Intersecion matrices

In this section we shall introduce intersection matrices and some related identities.

Let $V$ be a set of cardinality $v$. The inclusion matrix $W_{i,j}(v)$ is a binary matrix whose rows and columns are indexed by elements of $\binom{V}{i}$ and $\binom{V}{j}$, respectively, with the $yz$-entry defined by
\[
(W_{i,j}(v))_{yz} = \begin{cases} 
1, & \text{if } y \subseteq z, \\
0, & \text{otherwise}.
\end{cases}
\]
Observe that
\[ W_{i,j}(v)W_{j,k}(v) = \begin{pmatrix} k - i \\ j - i \end{pmatrix} W_{i,k}(v). \] (1)

Let \( H^I_{i,j}(v) \) be a binary matrix whose rows and columns are indexed by elements of \( \binom{\Gamma}{i} \) and \( \binom{\Gamma}{j} \), respectively, and the \( yz \)-entry is defined by
\[ (H^I_{i,j}(v))_{yz} = \begin{cases} 1, & \text{if } |y \cap z| = l, \\ 0, & \text{otherwise}. \end{cases} \]

Define
\[ C^I_{i,j}(v) = \sum_{g=l}^{\min(i,j)} \begin{pmatrix} g \\ l \end{pmatrix} H^g_{i,j}(v). \] (2)

In order to simplify the notation, we write \( W_{i,j} \) for \( W_{i,j}(v) \) when \( v \) is clear from context, and do the same for \( H^I_{i,j}(v) \) and \( C^I_{i,j}(v) \). The matrices \( W_{i,j}, H^I_{i,j} \) and \( C^I_{i,j} \) are intersection matrices introduced in [7].

Observe \( C^I_{i,j} \) is the all-one matrix and
\[ C^{\min(i,j)}_{i,j} = \begin{cases} W^T_{j,i}, & \text{if } i > j, \\ W_{i,j}, & \text{otherwise}. \end{cases} \]

**Lemma 2.1** [7] Let \( V \) be a set of cardinality \( v \). Write \( W_{i,j} = W_{i,j}(v) \) and \( C^I_{i,j} = C^I_{i,j}(v) \). Then
\[ C^I_{i,j}C^*_j = \sum_{h=\max(0,l+s-j)}^{\min(l,s)} \begin{pmatrix} v - l - s \\ j - l - s + h \end{pmatrix} \begin{pmatrix} i - h \\ l - h \end{pmatrix} \begin{pmatrix} k - h \\ s - h \end{pmatrix} C^h_{i,k}. \]

In particular, the following hold:
(i) \( W^T_{i,j}W_{i,k} = C^I_{j,k} \);
(ii) \( C^I_{i,j}W_{j,k} = \begin{pmatrix} k - l \end{pmatrix} C^I_{i,k} \);
(iii) \( W_{i,k}W^T_{j,k} = \sum_{l=\max(0,i+j-k)}^{\min(l,i)} \begin{pmatrix} v - i - j \\ k - i + j + l \end{pmatrix} C^I_{i,j} \);
(iv) \( W_{i,j}C^I_{j,k} = \sum_{h=\max(0,l+s-j)}^{\min(l,i)} \begin{pmatrix} v - l - i \\ j - l - s + h \end{pmatrix} \begin{pmatrix} k - h \\ l - k \end{pmatrix} C^h_{i,k} \).

Fix \( x \in \binom{\Gamma}{m} \). We then consider the adjacency matrix \( A \) of \( J(n, m, m + 1) \) as a block matrix with respect to the partition \( \{x\} \cup \Gamma_1(x) \cup \cdots \cup \Gamma_{D(x)}(x) \). Let \( A_{i,j} \) be the submatrix of \( A \) with rows indexed by vertices of \( \Gamma_i(x) \) and columns indexed by vertices of \( \Gamma_j(x) \).

**Lemma 2.2** Given two vertices \( x, y \) of \( J(n, m, m + 1) \). If \( x \in \binom{\Gamma}{m} \), then
\[ \partial(x, y) = \begin{cases} 2i, & \text{if } |y| = m \text{ and } |x \cap y| = m - i, \\ 2i + 1, & \text{if } |y| = m + 1 \text{ and } |x \cap y| = m - i. \end{cases} \]

In particular, \( D(x) = \min(2m + 1, 2n - 2m) \).
Proof. Immediate from [9, Lemma 2.2]. □

Lemma 2.3 Let $I_{(x)}$ be the identity matrix of size $\binom{n}{x}$. Then

$$A_{i,j} = 0, \quad \text{if } 0 \leq i \leq j \leq D(x) \text{ and } i \neq j - 1;$$

$$A_{2i,2i+1} = I_{\binom{m}{x}} \otimes W_{i,i+1}(n-m), \quad \text{if } 0 \leq i \leq \left\lfloor \frac{D(x) - 1}{2} \right\rfloor;$$

$$A_{2i+1,2i+2} = W_{m-i-1,m-i}(m) \otimes I_{\binom{n-m}{i+1}}, \quad \text{if } 0 \leq i \leq \left\lfloor \frac{D(x)}{2} \right\rfloor - 1,$$

where “$\otimes$” denotes the Kronecker product of matrices.

Proof. (3) is directed.

Pick $y \in \Gamma_{2i}(x)$, $z \in \Gamma_{2i+1}(x)$. By Lemma 2.2 we have $|y| = m$, $|z| = m + 1$, $|x \cap y| = |x \cap z| = m - i$. Suppose $y = \alpha_{m-i} \cup \beta_i$, $z = \alpha'_{m-i} \cup \beta'_{i+1}$, where $\alpha_{m-i}$ and $\alpha'_{m-i} \in \binom{x}{m}$, while $\beta_i \in \binom{n-x}{i}$ and $\beta'_{i+1} \in \binom{n-x}{i+1}$. Then

$$(A_{2i,2i+1})_{yz} = 1 \iff \alpha_{m-i} = \alpha'_{m-i} \text{ and } \beta_i \subseteq \beta'_{i+1} \iff (I_{\binom{m}{x}} \otimes W_{i,i+1}(n-m))_{yz} = 1,$$

which leads to (4).

Similarly, (5) holds. □

3 The Terwilliger algebra

Let $n \geq 3m$ and $X$ denote the vertex set of $J(n,m,m+1)$. Fix $x \in \binom{0}{m}$. In this section we shall determine the Terwilliger algebra $\mathcal{T} = \mathcal{T}(x)$ of $J(n,m,m+1)$. Hereafter the ground set of all matrices $C_{p,q}(m)$ is $x$ and that of $C_{p,q}^l(n-m)$ is $\Omega \setminus x$.

For $i,j \in \{0,1,\ldots,2m+1\}$, let $\mathcal{M}_{i,j}$ be the vector space spanned by

$C_{\left\lfloor \frac{i}{2} \right\rfloor,\left\lfloor \frac{j}{2} \right\rfloor}^l(m) \otimes C_{\left\lceil \frac{i}{2} \right\rceil,\left\lceil \frac{j}{2} \right\rceil}^s(n-m),$

where

$$0 \leq l \leq \min(m - \left\lfloor \frac{i}{2} \right\rfloor, m - \left\lfloor \frac{j}{2} \right\rfloor), \quad 0 \leq s \leq \min(\left\lceil \frac{i}{2} \right\rceil, \left\lceil \frac{j}{2} \right\rceil).$$

Write

$$\mathcal{M} = \bigoplus_{i,j=0}^{2m+1} L(\mathcal{M}_{i,j}),$$

where $L(\mathcal{M}_{i,j}) = \{ L(M) \in \text{Mat}_X(\mathbb{C}) \mid M \in \mathcal{M}_{i,j} \}$, and

$$L(M)_{\Gamma_k(x) \times \Gamma_l(x)} = \begin{cases} M, & \text{if } k = i \text{ and } l = j, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $\mathcal{M}$ is a vector space. By Lemma 2.1 $\mathcal{M}$ is an algebra. In the remaining of this section we shall prove $\mathcal{T} = \mathcal{M}$. 

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Lemma 3.1 The Terwilliger algebra $T$ is a subalgebra of $\mathcal{M}$.

Proof. By Lemma 2.3 we have $A \in \mathcal{M}$. For $0 \leq i \leq 2m + 1$, since

$$E_i^* = E_i^*(x) = L(C_{m-\lfloor \frac{i}{2} \rfloor, m-\lfloor \frac{i}{2} \rfloor}(n - m)) \in \mathcal{M},$$

we get $T \subseteq \mathcal{M}$.

For $i, j \in \{0, 1, \ldots, 2m + 1\}$, let $T_{i,j} = \{M_{i,j} \mid M \in T\}$, where $M_{i,j}$ is the submatrix of $M$ with rows indexed by vertices of $\Gamma_i(x)$ and columns indexed by vertices of $\Gamma_j(x)$. Since $T$ is an algebra, each $T_{i,j}$ is a vector space. Since $TE_i^* T \subseteq T$, $(TE_i^* T)_{i,k} \subseteq T_{i,k}$, which gives

$$T_{i,j} T_{j,k} \subseteq T_{i,k}. \tag{7}$$

Since $A, E_i^* \in T$, we have $E_i^* A E_i^* A E_i^* \cdots A E_i^* A E_i^* \in E_i^* T E_i^*$, from which it follows that

$$A_{i_1, i_2} A_{i_2, i_3} \cdots A_{i_{p-2}, i_{p-1}} A_{i_{p-1}, i_p} \in T_{i_1, i_p}, \tag{8}$$

where $0 \leq i_s \leq 2m + 1$ for any $s \in \{1, \ldots, p\}$.

Note that

$$W^T_{m-\lfloor \frac{i}{2} \rfloor, m-\lfloor \frac{i}{2} \rfloor}(m) \otimes W_{\lfloor \frac{i}{2} \rfloor, \lfloor \frac{i}{2} \rfloor}(n - m) = I_{(m-\lfloor \frac{i}{2} \rfloor)} \otimes I_{(n-m)}.$$  

By Lemma 2.3 and (1), for $h + 1 \leq k$, one gets

$$A_{h, h+1} \cdots A_{k-1, k} = ([k/2] - [h/2])![\lceil k/2 \rceil - \lceil h/2 \rceil]! W^T_{m-\lfloor \frac{i}{2} \rfloor, m-\lfloor \frac{i}{2} \rfloor}(m) \otimes W_{\lfloor \frac{i}{2} \rfloor, \lfloor \frac{i}{2} \rfloor}(n - m).$$

Hence, by (8), for $h \leq k$, we have

$$W^T_{m-\lfloor \frac{i}{2} \rfloor, m-\lfloor \frac{i}{2} \rfloor}(m) \otimes W_{\lfloor \frac{i}{2} \rfloor, \lfloor \frac{i}{2} \rfloor}(n - m) \in T_{h,k}. \tag{9}$$

Lemma 3.2 For $2i + 2 \leq j \leq 2m + 1$ and $0 \leq s \leq i + 1$, we have

$$C_{m-\lfloor \frac{i}{2} \rfloor, m-\lfloor \frac{i}{2} \rfloor}^s(n - m) \in T_{2i+2,j}. \tag{10}$$

Proof. We use induction on $s$ ($s$ decreasing from $i + 1$ to 0). Since

$$C_{m-\lfloor \frac{i}{2} \rfloor, m-\lfloor \frac{i}{2} \rfloor}^s(n - m) \otimes C_{i+1, \lfloor \frac{i}{2} \rfloor}^s(n - m) = W^T_{m-\lfloor \frac{i}{2} \rfloor, m-\lfloor \frac{i}{2} \rfloor}(m) \otimes W_{i+1, \lfloor \frac{i}{2} \rfloor}(n - m),$$

by (9), (10) holds for $2i + 2 \leq j \leq 2m + 1$ and $s = i + 1$.

Assume that $C_{m-\lfloor \frac{i}{2} \rfloor, m-\lfloor \frac{i}{2} \rfloor}^s(n - m) \otimes C_{i+1, \lfloor \frac{i}{2} \rfloor}^s(n - m) \in T_{2i+2,j}$. By (7) and (8) we obtain

$$(C_{m-\lfloor \frac{i}{2} \rfloor, m-\lfloor \frac{i}{2} \rfloor}^s(n - m) \otimes C_{i+1, \lfloor \frac{i}{2} \rfloor}^s(n - m)) (A_{j,j+1} A_{j+1,j}) \in T_{2i+2,j} T_{j, j} \subseteq T_{2i+2,j}, \tag{11}$$

$$(C_{m-\lfloor \frac{i}{2} \rfloor, m-\lfloor \frac{i}{2} \rfloor}^s(n - m) \otimes C_{i+1, \lfloor \frac{i}{2} \rfloor}^s(n - m)) (A_{j,j-1} A_{j-1,j}) \in T_{2i+2,j} T_{j, j} \subseteq T_{2i+2,j}. \tag{12}$$
When $j$ is even, by Lemma 2.3 Lemma 2.1 (9) leads to

$$aC_{m-i-1,m-\frac{i}{2}}^\frac{1}{2} (m) \otimes C_{i+1,\frac{i}{2}}^s (n-m) + bC_{m-i-1,m-\frac{i}{2}}^\frac{1}{2} (m) \otimes C_{i+1,\frac{i}{2}}^{s-1} (n-m) \in T_{2i+2,j},$$

where $a = (n-m-s-\frac{i}{2})(\frac{i}{2} - s + 1)$ and $b = (i-s+2)(\frac{i}{2} - s + 1)$. Similarly when $j$ is odd, (12) yields that

$$a'C_{m-i-1,m-\frac{i}{2}}^\frac{1}{2} (m) \otimes C_{i+1,\frac{i}{2}}^s (n-m) + b'C_{m-i-1,m-\frac{i}{2}}^\frac{1}{2} (m) \otimes C_{i+1,\frac{i}{2}}^{s-1} (n-m)$$

belongs to $T_{2i+2,j}$, where $a' = (n-m-s-\lfloor \frac{i}{2} \rfloor + 1)(\lfloor \frac{i}{2} \rfloor - s)$ and $b' = (i-s+2)(\lfloor \frac{i}{2} \rfloor - s + 1)$. Since $s \leq i+1 \leq \lfloor \frac{i}{2} \rfloor$, $b \neq 0$ and $b' \neq 0$. Thus we have $C_{m-i-1,m-\frac{i}{2}}^\frac{1}{2} (m) \otimes C_{i+1,\frac{i}{2}}^{s-1} (n-m) \in T_{2i+2,j}$.

Hence the desired result follows.  

\[ \square \]

**Lemma 3.3** The algebra $M$ is a subalgebra of $T$.

**Proof.** During this proof we will omit the symbol $(m)$ from matrices in front of “$\otimes$”, and omit $(n-m)$ from matrices behind “$\otimes$”.

In order to get the desired conclusion, we only need to show that $M_{i,j} \subseteq T_{i,j}$ for $i, j \in \{0, 1, \ldots , 2m+1\}$. Write $M_{i,j} = \{ M^T \mid M \in M_{i,j} \}$ and $T_{i,j} = \{ M^T \mid M \in T_{i,j} \}$. Since $M_{i,j} = M_{i,j}^T$ and $T_{i,j} = T_{i,j}^T$, it suffices to prove $M_{i,j} \subseteq T_{i,j}$ for $i \leq j$. We use induction on $i$.

**Step 1.** We show that $M_{0,j} \subseteq T_{0,j}$ for $0 \leq j \leq 2m+1$.

According to (6), the subspace $M_{0,j}$ is spanned by $C_{m,m-\frac{j}{2}}^l \otimes C_{0,\frac{j}{2}}^0$, where $0 \leq l \leq m - \frac{j}{2}$. Since

$$C_{m,m-\frac{j}{2}}^l \otimes C_{0,\frac{j}{2}}^0 = \begin{pmatrix} m - \frac{j}{2} \\ l \end{pmatrix} W_{W_{m-\frac{j}{2},m}} W_{0,\frac{j}{2}}$$

for any $l \in \{0, 1, \ldots , m - \frac{j}{2}\}$, we get $M_{0,j} \subseteq T_{0,j}$ from (9).

**Step 2.** Assume that $M_{p,j} \subseteq T_{p,j}$ for $p \leq 2i$. We will show that $M_{2i+1,j} \subseteq T_{2i+1,j}$ and $M_{2i+2,j} \subseteq T_{2i+2,j}$.

**Step 2.1.** We show that $M_{2i+1,j} \subseteq T_{2i+1,j}$ for $2i+1 \leq j \leq 2m+1$.

It suffices to prove

$$C_{m-i,m-\frac{i}{2}}^l \otimes C_{i+1,\frac{i}{2}}^s \in T_{2i+1,j},$$

where $0 \leq l \leq m - \frac{i}{2}$, $0 \leq s \leq i+1$.

By induction hypothesis,

$$C_{m-i,m-\frac{i}{2}}^l \otimes C_{i,\frac{i}{2}}^s \in M_{2i,j} \subseteq T_{2i,j},$$

where $a = (n-m-s-\frac{i}{2})(\frac{i}{2} - s + 1)$ and $b = (i-s+2)(\frac{i}{2} - s + 1)$. Similarly when $j$ is odd, (12) yields that

$$a'C_{m-i-1,m-\frac{i}{2}}^\frac{1}{2} (m) \otimes C_{i+1,\frac{i}{2}}^s (n-m) + b'C_{m-i-1,m-\frac{i}{2}}^\frac{1}{2} (m) \otimes C_{i+1,\frac{i}{2}}^{s-1} (n-m)$$

belongs to $T_{2i+2,j}$, where $a' = (n-m-s-\lfloor \frac{i}{2} \rfloor + 1)(\lfloor \frac{i}{2} \rfloor - s)$ and $b' = (i-s+2)(\lfloor \frac{i}{2} \rfloor - s + 1)$. Since $s \leq i+1 \leq \lfloor \frac{i}{2} \rfloor$, $b \neq 0$ and $b' \neq 0$. Thus we have $C_{m-i-1,m-\frac{i}{2}}^\frac{1}{2} (m) \otimes C_{i+1,\frac{i}{2}}^{s-1} (n-m) \in T_{2i+2,j}$.

Hence the desired result follows.  

\[ \square \]
for \(0 \leq l \leq m - \lfloor \frac{j}{2} \rfloor\), \(0 \leq s \leq i\). Since
\[
A_{2i,2i+1}^T = I_{(m-i)}^T \otimes W_{i,i+1}^T \in \mathcal{M}_{2i,2i+1}^T \subseteq \mathcal{T}_{2i,2i+1}^T,
\]
we have
\[
(I_{(m-i)}^T \otimes W_{i,i+1}^T)(C_{m-i,m-\lfloor \frac{j}{2} \rfloor}^d \otimes C_{i,\lfloor \frac{j}{2} \rfloor}^s) \in \mathcal{T}_{2i,2i+1}^T \subseteq \mathcal{T}_{2i+1,j}.
\]
By Lemma 2.1, (13) holds for \(0 \leq l \leq m - \lfloor \frac{j}{2} \rfloor\) and \(0 \leq s \leq i\).

Next we shall show that (13) holds for \(0 \leq l \leq m - \lfloor \frac{j}{2} \rfloor\) and \(s = i + 1\).

By (9), for \(j \leq k \leq 2m + 1\),
\[
(W_{m-\lfloor \frac{j}{2} \rfloor,m-i}^T \otimes W_{i+1,\lfloor \frac{j}{2} \rfloor}^T)(W_{m-\lfloor \frac{j}{2} \rfloor,m-\lfloor \frac{j}{2} \rfloor}^T \otimes W_{\lfloor \frac{j}{2} \rfloor,\lfloor \frac{j}{2} \rfloor}^T)(W_{m-\lfloor \frac{j}{2} \rfloor,m-\lfloor \frac{j}{2} \rfloor}^T \otimes W_{\lfloor \frac{j}{2} \rfloor,\lfloor \frac{j}{2} \rfloor}^T)
\]
belongs to \(\mathcal{T}_{2i+1,j}\). By Lemma 2.1,
\[
aC_{m-i,m-\lfloor \frac{j}{2} \rfloor}^{m-\lfloor \frac{j}{2} \rfloor} \otimes \left( \sum_{k=\max(0,i+1+\lfloor \frac{j}{2} \rfloor)}^{i+1} \left( \frac{n-m-i-1-\lfloor \frac{j}{2} \rfloor}{\lfloor \frac{k}{2} \rfloor - \lfloor \frac{j}{2} \rfloor} \right) C_{i+1,\lfloor \frac{j}{2} \rfloor} \right)
\]
belongs to \(\mathcal{T}_{2i+1,j}\), where \(a = \left( \frac{\lfloor \frac{j}{2} \rfloor - i}{\lfloor \frac{j}{2} \rfloor - i-1} \right) \neq 0\). Since (13) holds for \(0 \leq l \leq m - \lfloor \frac{j}{2} \rfloor\) and \(0 \leq s \leq i\), one has
\[
\left( \frac{n-m-i-1-\lfloor \frac{j}{2} \rfloor}{\lfloor \frac{k}{2} \rfloor - \lfloor \frac{j}{2} \rfloor} \right) C_{m-i,m-\lfloor \frac{j}{2} \rfloor}^{m-\lfloor \frac{j}{2} \rfloor} \otimes C_{i+1,\lfloor \frac{j}{2} \rfloor} \subseteq \mathcal{T}_{2i+1,j}.
\]
Since \(0 \leq 2i+1 \leq j \leq k-1 \leq 2m-1\) and \(n \geq 3m\), we get
\[
n-m-i-1-\lfloor \frac{j}{2} \rfloor \geq n-m-m-\lfloor \frac{j}{2} \rfloor \geq m-\lfloor \frac{j}{2} \rfloor \geq \lfloor \frac{k}{2} \rfloor - \lfloor \frac{j}{2} \rfloor \geq 0,
\]
and so \(\left( \frac{n-m-i-1-\lfloor \frac{j}{2} \rfloor}{\lfloor \frac{k}{2} \rfloor - \lfloor \frac{j}{2} \rfloor} \right) \neq 0\). Hence (13) holds for \(0 \leq l \leq m - \lfloor \frac{j}{2} \rfloor\) and \(s = i + 1\).

**Step 2.2.** We show that \(\mathcal{M}_{2i+2,j} \subseteq \mathcal{T}_{2i+2,j}\) for \(2i+2 \leq j \leq 2m+1\).

It suffices to prove
\[
C_{m-i,m-\lfloor \frac{j}{2} \rfloor}^l \otimes C_{i+1,\lfloor \frac{j}{2} \rfloor}^s \subseteq \mathcal{T}_{2i+2,j}, \quad 0 \leq l \leq m - \lfloor \frac{j}{2} \rfloor, \quad 0 \leq s \leq i + 1.
\]
By the inductive assumption, for \(0 \leq l \leq m - \lfloor \frac{j}{2} \rfloor\) and \(0 \leq s \leq i + 1\),
\[
C_{m-i,m-\lfloor \frac{j}{2} \rfloor}^l \otimes C_{i+1,\lfloor \frac{j}{2} \rfloor}^s \in \mathcal{M}_{2i+1,j} \subseteq \mathcal{T}_{2i+1,j}.
\]
Since
\[
A_{2i+2,2i+2}^T = W_{m-i-1,m-i} \otimes I_{(n-m-i)}^T \in \mathcal{T}_{2i+2}^T,
\]
by (7) we have
\[ (W_{m-i-1,m-i} \otimes I_{i+1}) (C^d_{m-i,m-\lfloor \frac{i}{2} \rfloor} \otimes C^s_{i+1,\lceil \frac{i}{2} \rceil}) \in \mathcal{T}_{2i+1,2i+2} \subseteq \mathcal{T}_{2i+2,j}. \] (16)

By Lemma 2.1,
\[ W_{m-i-1,m-i} C^d_{m-i,m-\lfloor \frac{i}{2} \rfloor} = (i + 1 - l)C^d_{m-i-1,m-\lfloor \frac{i}{2} \rfloor} + (m - \lfloor \frac{j}{2} \rfloor - l + 1)C^{d-1}_{m-i-1,m-\lfloor \frac{i}{2} \rfloor}. \]

Thus (16) implies that
\[ ((i + 1 - l)C^d_{m-i-1,m-\lfloor \frac{i}{2} \rfloor} + (m - \lfloor \frac{j}{2} \rfloor - l + 1)C^{d-1}_{m-i-1,m-\lfloor \frac{i}{2} \rfloor}) \otimes C^s_{i+1,\lceil \frac{i}{2} \rceil} \] (17)
belongs to \( \mathcal{T}_{2i+2,j} \), where \( 0 \leq l \leq m - \lfloor \frac{i}{2} \rfloor, 0 \leq s \leq i + 1 \). Since the coefficient of \( C^{d-1}_{m-i-1,m-\lfloor \frac{i}{2} \rfloor} \otimes C^s_{i+1,\lceil \frac{i}{2} \rceil} \) in (17) is \( m - \lfloor \frac{j}{2} \rfloor - l + 1 \neq 0 \), by Lemma 3.2 we get (15).

Hence the desired result follows.

**Theorem 3.4** Fix \( x \in \binom{\Omega}{m} \). Let \( \mathcal{T} \) be the Terwilliger algebra of \( J(n,m,m+1) \) with respect to \( x \) and \( \mathcal{M} \) be the algebra defined in (6). If \( n \geq 3m \), then \( \mathcal{T} = \mathcal{M} \).

**Proof.** Combining Lemmas 3.1 and 3.3, the desired result follows. \( \square \)

The condition \( n \geq 3m \) guarantees the coefficient of \( C^{m-\lfloor \frac{i}{2} \rfloor}_{m-i,m-\lfloor \frac{i}{2} \rfloor} \otimes C^s_{i+1,\lceil \frac{i}{2} \rceil} \) in (14) is non-zero. It seems to be interesting to determine the Terwilliger algebra of \( J(n,m,m+1) \) without this assumption.

**Theorem 3.5** ([16, Theorem 13]) Let \( \Gamma = (X,R) \) be a graph and \( \mathcal{T} \) be the Terwilliger algebra of \( \Gamma \) with respect to a vertex \( x \). If \( E^*_i \mathcal{T} E^*_i \) is symmetric for any \( i \in \{0, 1, \ldots, D(x)\} \), then \( \Gamma \) is thin with respect to \( x \).

**Corollary 3.6** With reference to Theorem 3.4, \( J(n,m,m+1) \) is thin with respect to \( x \).

**Proof.** By Theorem 3.4, for any \( i \in \{0, 1, \ldots, D(x)\} \), the subspace \( E^*_i \mathcal{T} E^*_i \) is spanned by
\[ L(C^d_{m-\lfloor \frac{i}{2} \rfloor,m-\lfloor \frac{i}{2} \rfloor}(m) \otimes C^s_{\lfloor \frac{i}{2} \rfloor,\lceil \frac{i}{2} \rceil}(n-m)), \]
where \( 0 \leq l \leq m - \lfloor \frac{i}{2} \rfloor, 0 \leq s \leq \lceil \frac{i}{2} \rceil \). Since each element of \( E^*_i \mathcal{T} E^*_i \) is symmetric, we get the conclusion from Theorem 3.5. \( \square \)
4 Two bases of the Terwilliger algebra

In this section we shall determine two bases of the Terwilliger algebra $T$ in Theorem 3.4. Set

$$G_{i,j} = \{ g \mid H^g_{m-\lfloor \frac{i}{2} \rfloor, m-\lfloor \frac{j}{2} \rfloor} (m) \neq 0 \}, \quad R_{i,j} = \{ r \mid H^r_{\lceil \frac{i}{2} \rceil, \lceil \frac{j}{2} \rceil} (n-m) \neq 0 \}.$$ 

Theorem 4.1 Let $T$ be as in Theorem 3.4. Then

$$\{L(H^g_{m-\lfloor \frac{i}{2} \rfloor, m-\lfloor \frac{j}{2} \rfloor} (m) \otimes H^r_{\lceil \frac{i}{2} \rceil, \lceil \frac{j}{2} \rceil} (n-m)), \ g \in G_{i,j}, \ r \in R_{i,j} \}_{i,j=0}^{2m+1}$$

as well as

$$\{L(C^l_{m-\lfloor \frac{i}{2} \rfloor, m-\lfloor \frac{j}{2} \rfloor} (m) \otimes C^s_{\lceil \frac{i}{2} \rceil, \lceil \frac{j}{2} \rceil} (n-m)), \ l \in G_{i,j}, \ s \in R_{i,j} \}_{i,j=0}^{2m+1}$$

are two bases of $T$.

Proof. Without loss of generality, suppose $i \leq j$. We have $H^I_{i,j}(v) \neq 0$ if and only if $\max(0, i+j-v) \leq l \leq \min(i, j)$, so $\left[ \frac{i}{2} \right] - |R_{i,j}| + 1 \leq r \leq \left[ \frac{i}{2} \right]$, when $r \in R_{i,j}$. By (2) we obtain

$$C^r_{\lceil \frac{i}{2} \rceil, \lceil \frac{j}{2} \rceil} (n-m) = \sum_{h=r}^{\left\lfloor \frac{i}{2} \right\rfloor} \left( \begin{array}{c} h \\ r \end{array} \right) H^h_{\lceil \frac{i}{2} \rceil, \lceil \frac{j}{2} \rceil} (n-m),$$

which implies that $H^r_{\lceil \frac{i}{2} \rceil, \lceil \frac{j}{2} \rceil} (n-m)$ is a linear combination of $\{C^s_{\lceil \frac{i}{2} \rceil, \lceil \frac{j}{2} \rceil} (n-m)\}_{s \in R_{i,j}}$ for any $r \in R_{i,j}$. Similarly, $H^g_{m-\lfloor \frac{i}{2} \rfloor, m-\lfloor \frac{j}{2} \rfloor} (m)$ can be expressed as a linear combination of $\{C^l_{m-\lfloor \frac{i}{2} \rfloor, m-\lfloor \frac{j}{2} \rfloor} (m)\}_{l \in G_{i,j}}$. Hence every element of

$$\{H^g_{m-\lfloor \frac{i}{2} \rfloor, m-\lfloor \frac{j}{2} \rfloor} (m) \otimes H^r_{\lceil \frac{i}{2} \rceil, \lceil \frac{j}{2} \rceil} (n-m) \}_{g \in G_{i,j}, r \in R_{i,j}}$$

belongs to $M_{i,j}$. Again by (2), for $0 \leq l \leq m - \lfloor \frac{i}{2} \rfloor$ and $0 \leq s \leq \lfloor \frac{j}{2} \rfloor$,

$$C^l_{m-\lfloor \frac{i}{2} \rfloor, m-\lfloor \frac{j}{2} \rfloor} (m) \otimes C^s_{\lceil \frac{i}{2} \rceil, \lceil \frac{j}{2} \rceil} (n-m)$$

$$= \left( \sum_{g=l}^{m-\lfloor \frac{i}{2} \rfloor} \left( \begin{array}{c} g \\ l \end{array} \right) H^g_{m-\lfloor \frac{i}{2} \rfloor, m-\lfloor \frac{j}{2} \rfloor} (m) \right) \otimes \left( \sum_{s=s}^{\left\lceil \frac{j}{2} \right\rceil} \left( \begin{array}{c} \left\lceil \frac{j}{2} \right\rceil \\ s \end{array} \right) H^r_{\lceil \frac{i}{2} \rceil, \lceil \frac{j}{2} \rceil} (n-m) \right).$$

Observe that (21) are linearly independent, so (21) is a basis of $M_{i,j}$. Therefore (18) is a basis of $T$.

Furthermore, by (20) we get $\{C^l_{m-\lfloor \frac{i}{2} \rfloor, m-\lfloor \frac{j}{2} \rfloor} (m) \otimes C^s_{\lceil \frac{i}{2} \rceil, \lceil \frac{j}{2} \rceil} (n-m)\}_{l \in G_{i,j}, s \in R_{i,j}}$ is also a basis of $M_{i,j}$, from which it follows that (19) is a basis of $T$. \qed
Corollary 4.2 With reference to Theorem 3.4 we get the dimension of $T$ is

$$
\dim T = \begin{cases} 
\frac{1}{12} (m+1)(m+2)(m+3)(3m+10) - 4, & \text{if } n = 3m, \\
\frac{1}{12} (m+1)(m+2)(m+3)(3m+10) - 1, & \text{if } n = 3m + 1, \\
\frac{1}{12} (m+1)(m+2)(m+3)(3m+10), & \text{if } n \geq 3m + 2.
\end{cases}
$$

Proof. By Theorem 4.1

$$
\dim T = \sum_{i,j=0}^{2m+1} |G_{i,j}| |R_{i,j}|
= \sum_{i,j=0}^{2m+1} (\min(m - \lfloor \frac{i}{2} \rfloor, m - \lfloor \frac{j}{2} \rfloor) - \max(0, m - \lfloor \frac{i}{2} \rfloor - \lfloor \frac{j}{2} \rfloor) + 1)
\times (\min(\lceil \frac{i}{2} \rceil, \lceil \frac{j}{2} \rceil) - \max(0, \lfloor \frac{i}{2} \rfloor + \lceil \frac{j}{2} \rceil - n + m) + 1).
$$

By zigzag calculation, we get the desired result. \qed

5 Concluding Remark

We conclude this paper with the following remarks:

(i) Let $\Omega$ be a set of cardinality $n$ and let $J(n, m)$ be the Johnson graph based on $\Omega$ with $n \geq 3m$. Fix an $m$-subset $x$ of $\Omega$. Let $T' = T'(x)$ and $T = T(x)$ be the Terwilliger algebra of $J(n, m)$ and $J(n, m, m+1)$ with respect to $x$, respectively. Since $\bigoplus_{i,j=0}^{m} E_{2i}(x) T E_{2j}(x)$ is an algebra, $\{L(H_{m-i,m-j}^g) \otimes H_{i,j}^r(n - m), \ g \in G_{2i,2j}, \ r \in R_{2i,2j}\}_{i,j=0}^{m}$ is a basis of $\bigoplus_{i,j=0}^{m} E_{2i}(x) T E_{2j}(x)$ by Theorem 4.1. By [12, Definition 4.2, Lemma 4.4, Theorem 5.9] this basis coincides with that of $T'$, which implies that $T' \simeq \bigoplus_{i,j=0}^{m} E_{2i}(x) T E_{2j}(x)$.

(ii) Using the same method, the Terwilliger algebra of $J(n, m, m+1)$ with respect to an $(m+1)$-subset may be determined.

Acknowledgement

We are indebted to the anonymous reviewer for his detailed reports. We would like to thank Professor Hiroshi Suzuki for proposing this problem. This research is supported by NSFC (11301270, 11271047).

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