THE MINKOWSKI’S INEQUALITY BY MEANS OF A GENERALIZED FRACTIONAL INTEGRAL

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ABSTRACT. We use the definition of a fractional integral, recently proposed by Katugampola, to establish a generalization of the reverse Minkowski’s inequality. We show two new theorems associated with this inequality, as well as state and show other inequalities related to this fractional operator.

Keywords: Minkowski’s inequality, Generalized fractional integral.
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1. Introduction

Studies involving integral inequalities are important in several areas of science: mathematics, physics, engineering, among others, in particular we mention: initial value problem, linear transformation stability, integral-differential equations, and impulse equations [1, 2].

The space of \( p \)-integrable functions \( L^p(a, b) \) play a relevant role in the study of inequalities involving integrals and sums. Further, it is possible to extend this space of \( p \)-integrable functions, to the space of the measurable Lebesgue functions, denoted by \( X^p(a, b) \), in which the space \( L^p(a, b) \) is contained [3]. Thus, new results involving integral inequalities have been possible and consequently, some applications have been made [1, 2]. We mention few of them, the inequalities of: Minkowski, Hölder, Hardy, Hermite-Hadamard, Jensen, among others [4, 5, 6, 7, 8, 9, 10].

On the other hand, non-integer order calculus, usually referred to as fractional calculus, is used to generalize of integrals and derivatives, in particular integrals involving inequalities. There are many definitions of fractional integrals, for example: Riemann-Liouville, Hadamard, Liouville, Weyl, Erdryi-Kober and Katugampola [3, 11, 12, 13]. Recently, Khalil et al. [14] and Adeljawad [15], introduced the local conformable fractional integrals and derivatives. From such fractional integrals, one obtains generalizations of the inequalities: Hadamard, Hermite-Hadamard, Opial, Gruss, Ostrowski, among others [16, 17, 18, 19, 20, 21, 22].

Recently, Katugampola [23] proposed a fractional integral unifying other well known ones: Riemann-Liouville, Hadamard, Weyl, Liouville and Erdlyi-Kober. Motivated by this formulation, we present a generalization of the reverse Minkowski’s inequality [24, 25, 26], using the
fractional integral introduced by Katugampola. We point out that studies in this direction, involving fractional integrals, are growing in several branches of mathematics [18, 27, 28].

The work is organized as follows: In section 2, we present the definition of the fractional integral, as well as its particular cases. We present the main theorems involving the reverse Minkowski’s inequality, as well as the suitable spaces for such definitions. In section 3, our main result, we propose the reverse Minkowski’s inequality using the fractional integral. In section 4, we discuss other inequalities involving this fractional integral. Concluding remarks close the article.

2. Preliminaries

In this section, we present the reverse Minkowski’s inequality theorem associated with the classical Riemann integral and its respective generalization via Riemann-Liouville and Hadamard fractional integrals. In addition, we present the fractional integral introduced by Katugampola, and we conclude with a theorem in order to recover particular cases.

Erhan et al. [5] address the inequalities of Hermite-Hadamard and reverse Minkowski for two functions \( f \) and \( g \) by means of the classical Riemann integral. On the other hand, Lazhar [7] also proposed a work related to the inequality involving integrals, that is, Hardy’s inequality and the reverse Minkowski’s inequality. Two theorems below were motivation for the works performed so far, via the Riemann-Liouville and Hadamard integrals, involving the reverse Minkowski’s inequality.

**Definition 1.** The space \( X^p_c(a, b) \) \((c \in \mathbb{R}, 1 \leq p \leq \infty)\) consists of those complex-valued Lebesgue measurable functions \( f \) on \((a, b)\), for which \( \|f\|_{X^p_c} < \infty \) with

\[
\|f\|_{X^p_c} = \left( \int_a^b |x^c f(x)|^p \frac{dx}{x} \right)^{1/p} \quad (1 \leq p < \infty)
\]

and

\[
\|f\|_{X^\infty_c} = \sup_{x \in (a, b)} \left[ x^c |f(x)| \right].
\]

In particular, when \( c = 1/p \) the space \( X^p_c(a, b) \) coincides with the space \( L^p(a, b) \) [3].

**Theorem 1.** Let \( f, g \in L^p(a, b) \) be two positive functions, with \( 1 \leq p \leq \infty \), \( 0 < \int_a^b f^p(t) \, dt < \infty \) and \( 0 < \int_a^b g^p(t) \, dt < \infty \). If \( 0 < m \leq \frac{f(t)}{g(t)} \leq M \), for \( m, M \in \mathbb{R}_+^* \) and \( \forall t \in [a, b] \), then

\[
(2.1) \quad \left( \int_a^b f^p(t) \, dt \right)^{\frac{1}{p}} + \left( \int_a^b g^p(t) \, dt \right)^{\frac{1}{p}} \leq c_1 \left( \int_a^b (f^p + g^p)(t) \, dt \right)^{\frac{1}{p}},
\]

with \( c_1 = \frac{M (m + 1) + (M + 1)}{(m + 1) (M + 1)} \) [3].
Theorem 2. Let \( f, g \in L^p(a, b) \) be two positive functions, with \( 1 \leq p \leq \infty \), \( 0 < \int_a^b f^p (t) \, dt < \infty \) and \( 0 < \int_a^b g^p (t) \, dt < \infty \). If \( 0 < m \leq \frac{f(t)}{g(t)} \leq M \), for \( m, M \in \mathbb{R}_+ \) and \( \forall t \in [a, b] \), then

\[
\left( \int_a^b f^p (t) \, dt \right) \frac{2}{p} + \left( \int_a^b g^p (t) \, dx \right) \frac{2}{p} \geq c_2 \left( \int_a^b f^p (t) \, dx \right) \frac{1}{p} \left( \int_a^b g^p (t) \, dx \right) \frac{1}{p},
\]

with \( c_2 = \frac{(M + 1)(m + 1)}{M} - 2 \). \[5\]

We present the definitions of the fractional integrals that will be useful in the development of the article: Riemann-Liouville fractional integral, Hadamard integral, Erdlyi-Kober integral, Katugampola integral, Weyl integral and Liouville integral.

Definition 2. Let \([a, b] (-\infty < a < b < \infty)\) be a finite interval on the real-axis \( \mathbb{R} \). The Riemann-Liouville fractional integrals (left-sided and right-sided) of order \( \alpha \in \mathbb{C} \), \( \text{Re}(\alpha) > 0 \), are defined by

\[
J^{\alpha}_{a+} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x f(t) \left( \frac{x}{t} \right)^{1-\alpha} \, dt, \quad x > a
\]

and

\[
J^{\alpha}_{b-} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b f(t) \left( \frac{t}{x} \right)^{1-\alpha} \, dt, \quad x < b,
\]

respectively \[3,12\].

Definition 3. Let \((a, b) (0 \leq a < b < \infty)\) be a finite or infinite interval on the half-axis \( \mathbb{R}^+ \). The Hadamard fractional integrals (left-sided and right-sided) of order \( \alpha \in \mathbb{C} \), \( \text{Re}(\alpha) > 0 \) of a real function \( f \in L^p(a, b) \) are defined by

\[
H^{\alpha}_{a+} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \left( \log \frac{x}{t} \right)^{1-\alpha} \frac{f(t)}{t} \, dt, \quad a < x < b
\]

and

\[
H^{\alpha}_{b-} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \left( \log \frac{t}{x} \right)^{1-\alpha} \frac{f(t)}{t} \, dt, \quad a < x < b,
\]

respectively \[3,12\].

Definition 4. Let \((a, b) (-\infty \leq a < b \leq \infty)\) be a finite or infinite interval or half-axis \( \mathbb{R}^+ \). Also let \( \text{Re}(\alpha) > 0 \), \( \sigma > 0 \) and \( \eta \in \mathbb{C} \). The Erdlyi-Kober fractional integrals (left-sided and right-sided) of order \( \alpha \in \mathbb{C} \) of a real function \( f \in L^p(a, b) \) are defined by

\[
I^{\alpha}_{a+},_{\sigma,\eta} f(x) := \frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \int_a^x \frac{t^{\sigma(\eta+1)-1}}{(x^\sigma - t^\sigma)^{1-\alpha}} f(t) \, dt, \quad 0 \leq a < x < b \leq \infty
\]
and

\[(2.8) \quad I_{\alpha, \eta}^\alpha f(x) := \frac{\sigma x^{\eta}}{\Gamma(\alpha)} \int_x^b \frac{t^{\eta(1-\alpha)-1}}{(t^\sigma - x^\sigma)^{1-\alpha}} f(t) \, dt, \quad 0 < a < x < b \leq \infty,\]

respectively [3, 12].

**Definition 5.** Let \([a, b] \subset \mathbb{R}\) be a finite interval. Then the Katugampola fractional integrals (left-sided and right-sided) of order \(\alpha \in \mathbb{C}, \rho > 0, \text{Re}(\alpha) > 0\) of a real function \(f \in X^p_c(a, b)\) are defined by

\[(2.9) \quad \rho I_{\alpha}^\alpha f(x) := \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{t^{\rho-1}}{(x^\rho - t^\rho)^{1-\alpha}} f(t) \, dt, \quad x > a\]

and

\[(2.10) \quad \rho I_{\alpha}^\alpha f(x) := \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{t^{\rho-1}}{(t^\rho - x^\rho)^{1-\alpha}} f(t) \, dt, \quad x < b,\]

respectively [13].

**Definition 6.** The Weyl fractional integrals of order \(\alpha \in \mathbb{C}, \text{Re}(\alpha) > 0\) of a real function \(f\) locally integrated into \((-\infty, \infty)\) being \(-\infty \leq x \leq \infty\) are defined by

\[(2.11) \quad x W_\infty^\alpha = x I_\infty^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x \frac{f(t)}{(x - t)^{1-\alpha}} \, dt\]

and

\[(2.12) \quad -\infty W_x^\alpha = -\infty I_x^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} \frac{f(t)}{(t - x)^{1-\alpha}} \, dt,\]

respectively [29].

**Definition 7.** Let a continuous function by parts in \(\mathbb{R} = (-\infty, \infty)\). The Liouville fractional integrals (left-sided and right-sided) of order \(\alpha \in \mathbb{C}, \text{Re}(\alpha) > 0\) of a real function \(f\), are defined by

\[(2.13) \quad I_+^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x \frac{f(t)}{(x - t)^{1-\alpha}} \, dt\]

and

\[(2.14) \quad I_-^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_x^{\infty} \frac{f(t)}{(t - x)^{1-\alpha}} \, dt,\]

respectively [3, 12].

Zoubir [25] established the reverse Minkowski’s inequality and another result that refers to the inequality via Riemann-Liouville fractional integral according to the following two theorems.
In 2014, Chinchane et al. [31] and recently Chinchane [32], established the reverse Minkowski’s inequality via fractional integral of Saigo and the integrals above mentioned. Finally, we introduce this integral and with a theorem we study their respective particular cases.

\[ \left( J^\alpha f^p(x) \right)^{\frac{1}{p}} + \left( J^\alpha g^p(x) \right)^{\frac{1}{p}} \leq c_1 \left( J^\alpha (f + g)^p(x) \right)^{\frac{1}{p}}, \]

where \( c_1 = \frac{M (m + 1) + (M + 1)}{(m + 1) (M + 1)} \)

**Theorem 3.** Let \( \alpha > 0, p \geq 1 \) and \( f, g \) two positive functions in \([0, \infty)\), such that \( \forall x > 0, J^\alpha f^p(x) < \infty \) and \( J^\alpha g^p(x) < \infty \). If \( 0 < m \leq \frac{f(t)}{g(t)} \leq M \), for \( m, M \in \mathbb{R}^*_+ \) and \( \forall t \in [0, x] \), then

\[ \left( J^\alpha f^p(x) \right)^{\frac{1}{p}} + \left( J^\alpha g^p(x) \right)^{\frac{1}{p}} \leq c_1 \left( J^\alpha (f + g)^p(x) \right)^{\frac{1}{p}}, \]

where \( c_1 = \frac{M (m + 1) + (M + 1)}{(m + 1) (M + 1)} \)

**Theorem 4.** Let \( \alpha > 0, p \geq 1 \) and \( f, g \) two positive functions in \([0, \infty)\), such that \( \forall x > 0, J^\alpha f^p(x) < \infty \) and \( J^\alpha g^p(x) < \infty \). If \( 0 < m \leq \frac{f(t)}{g(t)} \leq M \), for \( m, M \in \mathbb{R}^*_+ \) and \( \forall t \in [0, x] \), then

\[ \left( J^\alpha f^p(x) \right)^{\frac{1}{p}} + \left( J^\alpha g^p(x) \right)^{\frac{1}{p}} \geq c_2 \left( J^\alpha f^p(x) \right)^{\frac{1}{p}} \left( J^\alpha g^p(x) \right)^{\frac{1}{p}}, \]

where \( c_2 = \frac{(M + 1) (m + 1)}{M} - 2 \)

In 2014, Chinchane et al. [26] and Sabrina et al. [30] also established the reverse Minkowski’s inequality via Hadamard fractional integral as in two theorems below.

**Theorem 5.** Let \( \alpha > 0, p \geq 1 \) and \( f, g \) two positive functions in \([0, \infty)\), such that \( \forall x > 0, H^\alpha f^p(x) < \infty \) and \( H^\alpha g^p(x) < \infty \). If \( 0 < m \leq \frac{f(t)}{g(t)} \leq M \), for \( m, M \in \mathbb{R}^*_+ \) and \( \forall t \in [0, x] \), then

\[ \left( H^\alpha f^p(x) \right)^{\frac{1}{p}} + \left( H^\alpha g^p(x) \right)^{\frac{1}{p}} \leq c_1 \left( H^\alpha (f + g)^p(x) \right)^{\frac{1}{p}}, \]

where \( c_1 = \frac{M (m + 1) + (M + 1)}{(m + 1) (M + 1)} \)

**Theorem 6.** Let \( \alpha > 0, p \geq 1 \) and \( f, g \) two positive functions in \([0, \infty)\), such that \( \forall x > 0, H^\alpha f^p(x) < \infty \) and \( H^\alpha g^p(x) < \infty \). If \( 0 < m \leq \frac{f(t)}{g(t)} \leq M \), for \( m, M \in \mathbb{R}^*_+ \) and \( \forall t \in [0, x] \), then

\[ \left( H^\alpha f^p(x) \right)^{\frac{1}{p}} + \left( H^\alpha g^p(x) \right)^{\frac{1}{p}} \geq c_2 \left( H^\alpha f^p(x) \right)^{\frac{1}{p}} \left( H^\alpha g^p(x) \right)^{\frac{1}{p}}, \]

where \( c_2 = \frac{(M + 1) (m + 1)}{M} - 2 \)

In 2014 Chinchane et al. [31] and recently Chinchane [32], established the reverse Minkowski’s inequality via fractional integral of Saigo and the \( k \)-fractional integral, respectively.

In 2017, Katugampola [23] introduced a fractional integral that unifies the six fractional integrals above mentioned. Finally, we introduce this integral and with a theorem we study their respective particular cases.
Definition 8. Let $\varphi \in X^p_c(a,b)$, $\alpha > 0$ and $\beta, \rho, \eta, \kappa \in \mathbb{R}$. Then, the fractional integrals of a function $f$, left and right, are given by

$$
\rho I_\alpha^{a+,\eta,\kappa} \varphi(x) := \frac{\rho^{1-\beta} x^\kappa}{\Gamma(\alpha)} \int_a^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho - \tau^\rho)^{1-\alpha}} \varphi(\tau) d\tau, \quad 0 \leq a < x < b \leq \infty
$$

and

$$
\rho I_{b-,\eta,\kappa} \varphi(x) := \frac{\rho^{1-\beta} x^{\rho\eta}}{\Gamma(\alpha)} \int_x^b \frac{\tau^{\kappa+\rho-1}}{(\tau^\rho - x^\rho)^{1-\alpha}} \varphi(\tau) d\tau, \quad 0 \leq a < x < b \leq \infty
$$

respectively, if integrals exist \[23\].

From now on, let’s work only with the integral on the left, Eq.\(2.19\), because with the right integral we have a similar treatment.

Theorem 7. Let $\alpha > 0$ and $\beta, \rho, \eta, \kappa \in \mathbb{R}$. Then for $f \in X^p_c(a,b)$, with $a < x < b$, we have \[23\]:

1. For $\kappa = 0, \eta = 0$ and the limit $\rho \to 1$, at Eq.\(2.19\), we get the Riemann-Liouville fractional integral, i.e; Eq.\(2.3\).
2. With $\beta = \alpha, \kappa = 0, \eta = 0$, we take the limit $\rho \to 0^+$ and using the $\ell$'Hospital role, at Eq.\(2.19\), we get the Hadamard fractional integral, i.e; Eq.\(2.5\).
3. In the case $\beta = 0$ and $\kappa = -\rho(\alpha+\eta)$, at Eq.\(2.19\), we get the Erdlyi-Kober fractional integral, i.e; Eq.\(2.7\).
4. For $\beta = \alpha, \kappa = 0$ and $\eta = 0$, at Eq.\(2.19\), we get Katugampola fractional integral, i.e; Eq.\(2.9\).
5. With $\kappa = 0, \eta = 0, a = -\infty$ and take the limit $\rho \to 1$, at Eq.\(2.19\), we get Weyl fractional integral, i.e; Eq.\(2.11\).
6. With $\kappa = 0, \eta = 0, a = 0$ and take the limit $\rho \to 1$, at Eq.\(2.19\), we get Liouville fractional integral, i.e; Eq.\(2.13\).

3. Reverse Minkowski fractional integral inequality

In this section, our main contribution, we establish and prove the reverse Minkowski’s inequality via generalized fractional integral Eq.\(2.19\) and a theorem that refers to the reverse Minkowski’s inequality.

Theorem 8. Let $\alpha > 0$, $\rho, \eta, \kappa, \beta \in \mathbb{R}$ and $p \geq 1$. Let $f, g \in X^p_c(a,x)$ two positive functions in $[0,\infty)$, such that $\forall x > a$, $\rho I_\alpha^{a+,\eta,\kappa} f^p(x) < \infty$ and $\rho I_\alpha^{a+,\eta,\kappa} g^p(x) < \infty$. If $0 < m \leq \frac{f(t)}{g(t)} \leq M$, for $m, M \in \mathbb{R}^*_+$ and $\forall t \in [a,x]$, then

$$
\left(\rho I_\alpha^{a+,\eta,\kappa} f^p(x)\right)^{\frac{1}{p}} + \left(\rho I_\alpha^{a+,\eta,\kappa} g^p(x)\right)^{\frac{1}{p}} \leq c_1 \left(\rho I_\alpha^{a+,\eta,\kappa} (f + g)^p(x)\right)^{\frac{1}{p}},
$$

where

$$
\rho I_\alpha^{a+,\eta,\kappa} \varphi(x) := \frac{\rho^{1-\beta} x^\kappa}{\Gamma(\alpha)} \int_a^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho - \tau^\rho)^{1-\alpha}} \varphi(\tau) d\tau, \quad 0 \leq a < x < b \leq \infty
$$

and

$$
\rho I_{b-,\eta,\kappa} \varphi(x) := \frac{\rho^{1-\beta} x^{\rho\eta}}{\Gamma(\alpha)} \int_x^b \frac{\tau^{\kappa+\rho-1}}{(\tau^\rho - x^\rho)^{1-\alpha}} \varphi(\tau) d\tau, \quad 0 \leq a < x < b \leq \infty
$$
with \( c_1 = \frac{M (m + 1) + (M + 1)}{(m + 1)(M + 1)} \).

**Proof.** Using the condition \( \frac{f(t)}{g(t)} \leq M, \ t \in [a, x] \), we can write
\[
f(t) \leq M(f(t) + g(t)) - Mf(t),
\]
which implies,
\[
(M + 1)^p f^p(t) \leq M^p (f(t) + g(t))^p.
\]

Multiplying by \( \frac{\rho^{1-\beta} x^\kappa t^{\rho(\eta+1)-1}}{\Gamma(\alpha) (x^\rho - t^\rho)^{1-\alpha}} \) both sides of Eq. (3.2) and integrating with respect to the variable \( t \), we have
\[
(M + 1)^p \frac{\rho^{1-\beta} x^\kappa}{\Gamma(\alpha)} \int_a^x \frac{t^{\rho(\eta+1)-1}}{(x^\rho - t^\rho)^{1-\alpha}} f^p(t) \, dt \leq \frac{M^p \rho^{1-\beta} x^\kappa}{\Gamma(\alpha)} \int_a^x \frac{t^{\rho(\eta+1)-1}}{(x^\rho - t^\rho)^{1-\alpha}} (f + g)^p(t) \, dt.
\]

Consequently, we can write
\[
\left( \rho T_{a+\alpha,\eta,\kappa}^p (x) \right)^{\frac{1}{p}} \leq \frac{M}{M + 1} \left( \rho T_{a+\alpha,\eta,\kappa}^p (f + g)(x) \right)^{\frac{1}{p}}.
\]

On the other hand, as \( mg(t) \leq f(t) \), follow
\[
(1 + \frac{1}{m})^p g^p(t) \leq \left( \frac{1}{m} \right)^p (f(t) + g(t))^p.
\]

Further, multiplying by \( \frac{\rho^{1-\beta} x^\kappa t^{\rho(\eta+1)-1}}{\Gamma(\alpha) (x^\rho - t^\rho)^{1-\alpha}} \) both sides of Eq. (3.5) and integrating with respect to the variable \( t \), we have
\[
\left( \rho T_{a+\alpha,\eta,\kappa}^p (t) \right)^{\frac{1}{p}} \leq \frac{1}{m + 1} \left( \rho T_{a+\alpha,\eta,\kappa}^p (f + g)(t) \right)^{\frac{1}{p}}.
\]

From Eq. (3.4) and Eq. (3.6), the result follows.

Eq. (3.1) is the so-called reverse Minkowski’s inequality associated with the Katugampola fractional integral.

**Theorem 9.** Let \( \alpha > 0, \ \rho, \eta, \kappa, \beta \in \mathbb{R} \) and \( p \geq 1 \). Let \( f, g \in X^p(a, x) \) be two positive functions in \([0, \infty)\), such that \( \forall x > a, \ \rho T_{a+\alpha,\eta,\kappa}^p (x) < \infty \) and \( \rho T_{a+\alpha,\eta,\kappa}^p (x) < \infty \). If
0 < m \leq \frac{f(t)}{g(t)} \leq M, \text{ for } m, M \in \mathbb{R}_+^* \text{ and } \forall t \in [a, x], \text{ then}

(3.7) \quad \left( \rho \mathcal{T}_{a+}^{\alpha,\beta,\eta,\kappa} f^p(x) \right)^{\frac{1}{p}} + \left( \rho \mathcal{T}_{a+}^{\alpha,\beta,\eta,\kappa} g^p(x) \right)^{\frac{1}{p}} \geq c_2 \left( \rho \mathcal{T}_{a+}^{\alpha,\beta,\eta,\kappa} f^p(x) \right)^{\frac{1}{p}} \left( \rho \mathcal{T}_{a+}^{\alpha,\beta,\eta,\kappa} g^p(x) \right)^{\frac{1}{p}}

with \( c_2 = \frac{(M + 1)(m + 1)}{M} - 2 \).

Proof. Carrying out the product between Eq. (3.4) and Eq. (3.6), we have

(3.8) \quad \frac{(M + 1)(m + 1)}{M} \left( \rho \mathcal{T}_{a+}^{\alpha,\beta,\eta,\kappa} f^p(x) \right)^{\frac{1}{p}} \left( \rho \mathcal{T}_{a+}^{\alpha,\beta,\eta,\kappa} g^p(x) \right)^{\frac{1}{p}} \leq \left( \rho \mathcal{T}_{a+}^{\alpha,\beta,\eta,\kappa} (f + g)^p(x) \right)^{\frac{2}{p}}.

Using the Minkowski’s inequality, on the right side of Eq. (3.8), we have

(3.9) \quad \frac{(M + 1)(m + 1)}{M} \left( \rho \mathcal{T}_{a+}^{\alpha,\beta,\eta,\kappa} f^p(x) \right)^{\frac{1}{p}} \left( \rho \mathcal{T}_{a+}^{\alpha,\beta,\eta,\kappa} g^p(x) \right)^{\frac{1}{p}} \leq \left( \left( \rho \mathcal{T}_{a+}^{\alpha,\beta,\eta,\kappa} f^p(x) \right)^{\frac{1}{p}} + \left( \rho \mathcal{T}_{a+}^{\alpha,\beta,\eta,\kappa} g^p(x) \right)^{\frac{1}{p}} \right)^{2}.

So, from Eq. (3.9), we conclude that

\[ \frac{(M + 1)(m + 1)}{M} - 2 \left( \rho \mathcal{T}_{a+}^{\alpha,\beta,\eta,\kappa} f^p(x) \right)^{\frac{1}{p}} \left( \rho \mathcal{T}_{a+}^{\alpha,\beta,\eta,\kappa} g^p(x) \right)^{\frac{1}{p}} \leq \left( \rho \mathcal{T}_{a+}^{\alpha,\beta,\eta,\kappa} f^p(x) \right)^{\frac{2}{p}} + \left( \rho \mathcal{T}_{a+}^{\alpha,\beta,\eta,\kappa} g^p(x) \right)^{\frac{2}{p}}. \]

Note that, if \( \beta = \alpha, \kappa = 0, \eta = 0 \) and the limit \( \rho \to 1 \), in Eq. (2.19), we recover Riemann-Liouville fractional integral, Eq. (2.3). In this sense, choosing \( a+ = 0 \), and substituting in Theorem 8 and Theorem 9, we obtain, as particular cases, the respective Theorem 3 and Theorem 4, which correspond to the inequality via Riemann-Liouville fractional integral. On the other hand, if \( \beta = \alpha, \kappa = 0, \eta = 0 \), and the limit \( \rho \to 0+ \) and using the \( \ell \)Hospital rule, in Eq. (2.19), we obtain the Hadamard fractional integral, Eq. (2.5). Similarly, choosing \( \alpha = 1 \) and substituting in Theorem 8 and Theorem 9, we obtain, as particular cases, the Theorem 5 and Theorem 6, respectively.

4. Other fractional integral inequalities

In this section we generalize the results discussed by Chinchane [32], Sulaiman [33] and Sroysang [34] on the reverse Minkowski’s inequality via Riemann integral, using the fractional integral proposed by Katugampola [23].
Theorem 10. Let $\alpha > 0$, $\rho, \eta, \kappa, \beta \in \mathbb{R}$, $p \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $f, g \in X^p(a, x)$ be two positive functions in $[0, \infty)$, such that $\forall x > a$, $\rho T_{a+}^{\alpha, \beta} f(x) < \infty$ and $\rho T_{a+}^{\alpha, \beta} g(x) < \infty$. If $0 < m \leq \frac{f(t)}{g(t)} \leq M$, for $m, M \in \mathbb{R}^*_+$ and $\forall t \in [a, x]$, then

\[
\left( \rho T_{a+}^{\alpha, \beta} f(x) \right)^{\frac{1}{p}} \left( \rho T_{a+}^{\alpha, \beta} g(x) \right)^{\frac{1}{q}} \leq \left( \frac{M}{m} \right)^{\frac{1}{pq}} \left( \rho T_{a+}^{\alpha, \beta} f^\frac{1}{\rho^q} (x) g^\frac{1}{\rho^p} (x) \right).
\]

Proof. Using the condition $\frac{f(t)}{g(t)} \leq M$, $t \in [a, x]$ with $x > a$, we have

\[
f(t) \leq Mg(t) \Rightarrow g^\frac{1}{\rho^p} (t) \geq M^{-\frac{1}{\rho}} f^\frac{1}{\rho^q} (t).
\]

Multiplying by $f^\frac{1}{\rho^q} (t)$ both sides of Eq.\(\text{(4.2)}\), we can rewrite it as follows

\[
f^\frac{1}{\rho^q} (t) g^\frac{1}{\rho^p} (t) \geq M^{-\frac{1}{\rho}} f (t).
\]

Now, multiplying by $\frac{\rho^{1-\beta} x^\kappa \mu (\eta+1)-1}{\Gamma (\alpha) (x^\rho - t^\rho)^{1-\alpha}}$ both sides of Eq.\(\text{(4.3)}\) and integrating with respect to the variable $t$, we have

\[
\int_a^x \frac{\rho^{1-\beta} x^\kappa \mu (\eta+1)-1}{\Gamma (\alpha) (x^\rho - t^\rho)^{1-\alpha}} M^{-\frac{1}{\rho}} f (t) dt \leq \int_a^x \frac{\rho^{1-\beta} x^\kappa \mu (\eta+1)-1}{\Gamma (\alpha) (x^\rho - t^\rho)^{1-\alpha}} f^\frac{1}{\rho^q} (t) g^\frac{1}{\rho^p} (t) dt.
\]

So, the inequality follows

\[
M^{-\frac{1}{pq}} \left( \rho T_{a+}^{\alpha, \beta} f(x) \right)^{\frac{1}{p}} \leq \left( \rho T_{a+}^{\alpha, \beta} f^\frac{1}{\rho^q} (x) g^\frac{1}{\rho^p} (x) \right)^{\frac{1}{p}}.
\]

On the order hand, we have

\[
m^\frac{1}{\rho^p} g^\frac{1}{\rho^p} (t) \leq f^\frac{1}{\rho^q} (t), \quad x > a.
\]

Multiplying by $g^\frac{1}{\rho^p} (t)$ both sides of Eq.\(\text{(4.6)}\) and using the relation $\frac{1}{p} + \frac{1}{q} = 1$, we have

\[
m^\frac{1}{\rho^p} f^\frac{1}{\rho^q} (t) \leq g^\frac{1}{\rho^p} (t).
\]

Multiplying by $\frac{\rho^{1-\beta} x^\kappa \mu (\eta+1)-1}{\Gamma (\alpha) (x^\rho - t^\rho)^{1-\alpha}}$ both sides of Eq.\(\text{(4.7)}\) and integrating with respect to the variable $t$, we have

\[
m^\frac{1}{\rho^p} \left( \rho T_{a+}^{\alpha, \beta} g(x) \right)^{\frac{1}{q}} \leq \left( \rho T_{a+}^{\alpha, \beta} f^\frac{1}{\rho^q} (x) g^\frac{1}{\rho^p} (x) \right)^{\frac{1}{q}}.
\]
Evaluating the product between Eq. (4.5) and Eq. (4.8) and using the relation \( \frac{1}{p} + \frac{1}{q} = 1 \), we conclude that
\[
\left( \int_a x \rho \tau_{a+\eta,\kappa} f(x) \right)^{\frac{1}{p}} \left( \int_a x \rho \tau_{a+\eta,\kappa} g(x) \right)^{\frac{1}{q}} \leq \left( \frac{M}{m} \right)^{\frac{1}{p}} \left( \int_a x \rho \tau_{a+\eta,\kappa} f^{\frac{1}{p}}(x) g^{\frac{1}{q}}(x) \right).
\]

\[\square\]

**Theorem 11.** Let \( \alpha > 0 \), \( \rho, \eta, \kappa, \beta \in \mathbb{R} \), \( p \geq 1 \) and \( \frac{1}{p} + \frac{1}{q} = 1 \). Let \( f, g \in X^p(a, x) \) be two positive functions in \([0, \infty)\), such that \( \forall x > a \), \( \tau_{a+\eta,\kappa} f^{\rho(x+)}(x) < \infty \), \( \tau_{a+\eta,\kappa} f^{\rho(x+)}(x) < \infty \), \( \tau_{a+\eta,\kappa} g^p(x) < \infty \) and \( \tau_{a+\eta,\kappa} g^q(x) < \infty \). If \( 0 < m \leq \frac{f(t)}{g(t)} \leq M \), for \( m, M \in \mathbb{R}^* \) and \( \forall t \in [a, x] \), then
\[
\tau_{a+\eta,\kappa} f^p(x) g^q(x) \leq c_3 \left( \tau_{a+\eta,\kappa} (f^p + g^q)(x) \right) + c_4 \left( \tau_{a+\eta,\kappa} (f^q + g^q)(x) \right),
\]
with \( c_3 = \frac{2^{p-1}M^p}{p(M+1)^p} \) and \( c_4 = \frac{2^{q-1}}{q(m+1)^q} \).

**Proof.** Using the hypothesis, we have the following identity
\[
(M+1)^p f^p(t) \leq M^p (f+g)^p(t).
\]

Multiplying by \( \frac{\rho^{1-\beta} x^{\alpha+\beta(\eta+1)-1}}{\Gamma(\alpha)(x^\beta - t^\beta)^{1-\alpha}} \) both sides of Eq. (4.10) and integrating with respect to the variable \( t \), we get
\[
\int_a^x \frac{\rho^{1-\beta} x^{\alpha+\beta(\eta+1)-1}}{\Gamma(\alpha)(x^\beta - t^\beta)^{1-\alpha}} (M+1)^p f^p(t) dt \leq \int_a^x \frac{\rho^{1-\beta} x^{\alpha+\beta(\eta+1)-1}}{\Gamma(\alpha)(x^\beta - t^\beta)^{1-\alpha}} M^p (f+g)^p(t) dt.
\]

In this way, we have
\[
\tau_{a+\eta,\kappa} f^p(x) \leq \frac{M^p}{(M+1)^p} \tau_{a+\eta,\kappa} (f+g)^p(x).
\]

On the other hand, as \( 0 < m < \frac{f(t)}{g(t)} \), \( t \in (a, x) \), we have
\[
(m+1)^q g^q(t) \leq (f+g)^q(t).
\]

Again, multiplying by \( \frac{\rho^{1-\beta} x^{\alpha+\beta(\eta+1)-1}}{\Gamma(\alpha)(x^\beta - t^\beta)^{1-\alpha}} \) both sides of Eq. (4.12) and integrating with respect to the variable \( t \), we get
\[
\tau_{a+\eta,\kappa} g^q(x) \leq \frac{1}{(m+1)^q} \tau_{a+\eta,\kappa} (f+g)^q(x).
\]
Considering Young’s inequality, \(35\)
\[
(4.14) \quad f(t)g(t) \leq \frac{f^p(t)}{p} + \frac{g^q(t)}{q},
\]
multiplying by \(\frac{\rho^{1-\beta}x^\kappa t^{\rho(\eta+1)-1}}{\Gamma(\alpha)(x^\rho - t^\rho)^{1-\alpha}}\) both sides of Eq. (4.14) and integrating with respect to the variable \(t\), we have
\[
(4.15) \quad \rho^{\alpha,\beta}I_{a+\eta,\kappa}(fg)(x) \leq \frac{1}{p} \left(\rho^{\alpha,\beta}I_{a+\eta,\kappa}f^p(x)\right) + \frac{1}{q} \left(\rho^{\alpha,\beta}I_{a+\eta,\kappa}g^q(x)\right).
\]
Thus, using Eq. (4.11), Eq. (4.13) and Eq. (4.15), we get
\[
(4.16) \quad \rho^{\alpha,\beta}I_{a+\eta,\kappa}(fg)(x) \leq \frac{1}{p} \left(\rho^{\alpha,\beta}I_{a+\eta,\kappa}f^p(x)\right) + \frac{1}{q} \left(\rho^{\alpha,\beta}I_{a+\eta,\kappa}g^q(x)\right) 
\leq \frac{M^p}{p(M+1)^p} \left(\rho^{\alpha,\beta}I_{a+\eta,\kappa}f^p(x)\right) + \frac{1}{q(m+1)^q} \left(\rho^{\alpha,\beta}I_{a+\eta,\kappa}g^q(x)\right).
\]
Using the following inequality, \((a+b)^r \leq 2^{p-1}(a^r + b^r), r > 1, a, b \geq 0\), we get
\[
(4.17) \quad \rho^{\alpha,\beta}I_{a+\eta,\kappa}(f+g)^p(x) \leq 2^{p-1}\rho^{\alpha,\beta}I_{a+\eta,\kappa}(f^p + g^p)(x)
\]
and
\[
(4.18) \quad \rho^{\alpha,\beta}I_{a+\eta,\kappa}(f+g)^q(x) \leq 2^{q-1}\rho^{\alpha,\beta}I_{a+\eta,\kappa}(f^q + g^q)(x).
\]
Thus, replacing Eq. (4.17) and Eq. (4.18) at Eq. (4.16), we conclude that
\[
(4.19) \quad \rho^{\alpha,\beta}I_{a+\eta,\kappa}(fg)(x) \leq \frac{2^{p-1}M^p}{p(M+1)^p} \left(\rho^{\alpha,\beta}I_{a+\eta,\kappa}(f^p + g^p)(x)\right) + \frac{2^{q-1}}{q(m+1)^q} \left(\rho^{\alpha,\beta}I_{a+\eta,\kappa}(f^q + g^q)(x)\right).
\]

**Theorem 12.** Let \(\alpha > 0, \rho, \eta, \kappa, \beta \in \mathbb{R}\) and \(p \geq 1\). Let \(f, g \in X^p(a, x)\) be two positive functions in \([0, \infty)\), such that \(\forall x > a, \rho^{\alpha,\beta}I_{a+\eta,\kappa}f^p(x) < \infty\) and \(\rho^{\alpha,\beta}I_{a+\eta,\kappa}g^p(x) < \infty\). If \(0 < m \leq \frac{f(t)}{g(t)} \leq M, \) for \(m, M \in \mathbb{R}^*_+\) and \(\forall t \in [a, x]\), then
\[
\frac{M+1}{M-c} \left(\rho^{\alpha,\beta}I_{a+\eta,\kappa}(f(x) - cg(x))\right)^\frac{1}{p} \leq \left(\rho^{\alpha,\beta}I_{a+\eta,\kappa}f^p(x)\right)^\frac{1}{p} + \left(\rho^{\alpha,\beta}I_{a+\eta,\kappa}g^p(x)\right)^\frac{1}{p} \leq \frac{m+1}{m-c} \left(\rho^{\alpha,\beta}I_{a+\eta,\kappa}(f(x) - cg(x))\right)^\frac{1}{p}.
\]
Proof. By hypothesis $0 < c < m \leq M$, so

$$mc \leq Mc \Rightarrow mc + m \leq mc + M \leq Mc + M \Rightarrow (M + 1)(m - c) \leq (m + 1)(M - c).$$

Thus, we conclude that

$$\frac{M + 1}{M - c} \leq \frac{m + 1}{m - c}.$$

Also, we have

$$m - c \leq \frac{f(t) - cg(t)}{g(t)} \leq M - c$$

which implies,

$$(4.20) \quad \frac{(f(t) - cg(t))^p}{(M - c)^p} \leq g^p(t) \leq \frac{(f(t) - cg(t))^p}{(m - c)^p}.$$ 

Again, we have

$$\frac{1}{M} \leq \frac{g(t)}{f(t)} \leq \frac{m - c}{cm} \Rightarrow \frac{m - c}{cf(t)} \leq \frac{M - c}{cM},$$

which implies,

$$(4.21) \quad \left(\frac{M}{M - c}\right)^p (f(t) - cg(t))^p \leq f^p(t) \leq \left(\frac{m}{m - c}\right)^p (f(t) - cg(t))^p.$$ 

Multiplying by $\frac{\rho^{1-\beta}x^{k+\rho(\eta+1)-1}}{\Gamma(\alpha)(x^\rho - t^\rho)^{1-\alpha}}$ both sides of Eq. (4.20) and integrating with respect to the variable $t$, we have

$$\int_a^x \frac{\rho^{1-\beta}x^{k+\rho(\eta+1)-1}}{\Gamma(\alpha)(x^\rho - t^\rho)^{1-\alpha}} \frac{(f(t) - cg(t))^p}{(M - c)^p} dt \leq \int_a^x \frac{\rho^{1-\beta}x^{k+\rho(\eta+1)-1}}{\Gamma(\alpha)(x^\rho - t^\rho)^{1-\alpha}} g^p(t) dt \leq \int_a^x \frac{\rho^{1-\beta}x^{k+\rho(\eta+1)-1}}{\Gamma(\alpha)(x^\rho - t^\rho)^{1-\alpha}} \frac{(f(t) - cg(t))^p}{(m - c)^p} dt.$$

In this way, we obtain

$$(4.22) \quad \frac{1}{M - c} \left(\rho^{\alpha+\beta}T^{\alpha,\eta,\kappa}(f(x) - cg(x))^p\right)^{\frac{1}{p}} \leq \left(\rho^{\alpha+\beta}T^{\alpha,\eta,\kappa}g^p(x)\right)^{\frac{1}{p}} \leq \frac{1}{m - c} \left(\rho^{\alpha+\beta}T^{\alpha,\eta,\kappa}(f(x) - cg(x))^p\right)^{\frac{1}{p}}.$$
Realizing the same procedure as in Eq. (4.21), we have

\begin{equation}
\frac{M}{M-c} \left( \rho I_{\alpha+\eta,\kappa} \left( f(x) - cg(x) \right)^p \right)^{\frac{1}{p}} \leq \left( \rho I_{\alpha+\eta,\kappa}^{\alpha,\beta} f^p(x) \right)^{\frac{1}{p}} \leq \frac{m}{m-c} \left( \rho I_{\alpha+\eta,\kappa} \left( f(x) - cg(x) \right)^p \right)^{\frac{1}{p}}.
\end{equation} (4.23)

Adding Eq. (4.22) and Eq. (4.23), we conclude that

\begin{equation}
\frac{M + 1}{M - c} \left( \rho I_{\alpha+\eta,\kappa} \left( f(x) - cg(x) \right)^p \right)^{\frac{1}{p}} \leq \left( \rho I_{\alpha+\eta,\kappa}^{\alpha,\beta} f^p(x) \right)^{\frac{1}{p}} + \left( \rho I_{\alpha+\eta,\kappa}^{\alpha,\beta} g^p(x) \right)^{\frac{1}{p}} \leq \frac{m + 1}{m - c} \left( \rho I_{\alpha+\eta,\kappa} \left( f(x) - cg(x) \right)^p \right)^{\frac{1}{p}}.
\end{equation}

Theorem 13. Let \( \alpha > 0, \rho, \eta, \kappa, \beta \in \mathbb{R} \) and \( p \geq 1 \). Let \( f, g \in X^p(a, x) \) be two positive functions in \([0, \infty)\), such that \( \forall x > a, \rho I_{\alpha+\eta,\kappa} \left( f^p(x) < \infty \right) \) and \( \rho I_{\alpha+\eta,\kappa} \left( g^p(x) < \infty \right) \). If \( 0 \leq a \leq f(t) \leq A \) and \( 0 \leq b \leq g(t) \leq B \), \( \forall t \in [a, x] \), then

\begin{equation}
\rho I_{\alpha+\eta,\kappa}^{\alpha,\beta} f^p(x) + \rho I_{\alpha+\eta,\kappa}^{\alpha,\beta} g^p(x) \leq c_5 \left( \rho I_{\alpha+\eta,\kappa} \left( f(x) + g(x) \right)^p(x) \right)^{\frac{1}{p}},
\end{equation} (4.24)

with \( c_5 = \frac{A(a + B) + B(A + b)}{(A + b)(a + B)} \).

Proof. By hypothesis, it follows that

\begin{equation}
\frac{1}{B} \leq \frac{1}{g(t)} \leq \frac{1}{b}.
\end{equation} (4.25)

Realizing the product between Eq. (4.24) and \( 0 < a \leq f(t) \leq A \), we have

\begin{equation}
\frac{a}{B} \leq \frac{f(t)}{g(t)} \leq \frac{A}{b}.
\end{equation} (4.26)

From Eq. (4.26), we get

\begin{equation}
g^p(t) \leq \left( \frac{B}{a + B} \right)^p (f(t) + g(t))^p
\end{equation} (4.27)

and

\begin{equation}
f^p(t) \leq \left( \frac{A}{b + A} \right)^p (f(t) + g(t))^p.
\end{equation} (4.28)
Multiplying by $\frac{\rho^{1-\beta} x^{\rho(\eta+1)-1}}{\Gamma(\alpha)(x^\rho - t^\rho)^{1-\alpha}}$ both sides of Eq. (4.27) and integrating with respect to the variable $t$, we have

$$\int_a^x \frac{\rho^{1-\beta} x^{\rho(\eta+1)-1}}{\Gamma(\alpha)(x^\rho - t^\rho)^{1-\alpha}} g^p(t) \, dt \leq \int_a^x \frac{\rho^{1-\beta} x^{\rho(\eta+1)-1}}{\Gamma(\alpha)(x^\rho - t^\rho)^{1-\alpha}} \left( \frac{B}{a+B} \right)^p (f(t) + g(t))^p \, dt.$$  

Thus, it follows that

$$\left( \frac{\rho T_{a+\eta,\kappa}^\alpha}{\rho T_{a+\eta,\kappa}^\alpha} f(x) \right)^{\frac{1}{p}} \leq \frac{B}{a+B} \left( \frac{\rho T_{a+\eta,\kappa}^\alpha}{\rho T_{a+\eta,\kappa}^\alpha} (f + g)^p(x) \right)^{\frac{1}{p}}.$$  

Similarly, we perform the calculations for Eq. (4.28), we get

$$\left( \frac{\rho T_{a+\eta,\kappa}^\alpha}{\rho T_{a+\eta,\kappa}^\alpha} g^p(x) \right)^{\frac{1}{p}} \leq \frac{A}{b+A} \left( \frac{\rho T_{a+\eta,\kappa}^\alpha}{\rho T_{a+\eta,\kappa}^\alpha} (f + g)^p(x) \right)^{\frac{1}{p}}.$$  

Adding Eq. (4.29) and Eq. (4.30), we conclude that

$$\left( \frac{\rho T_{a+\eta,\kappa}^\alpha}{\rho T_{a+\eta,\kappa}^\alpha} f(x) \right)^{\frac{1}{p}} + \left( \frac{\rho T_{a+\eta,\kappa}^\alpha}{\rho T_{a+\eta,\kappa}^\alpha} g^p(x) \right)^{\frac{1}{p}} \leq \frac{A(a + B) + B(b + A)}{(a + B)(b + A)} \left( \frac{\rho T_{a+\eta,\kappa}^\alpha}{\rho T_{a+\eta,\kappa}^\alpha} (f + g)^p(x) \right)^{\frac{1}{p}}.$$  

**Theorem 14.** Let $\alpha > 0$ and $\rho, \eta, \kappa, \beta \in \mathbb{R}$. Let $f, g \in X_p^\alpha(a, x)$ be two positive functions in $[0, \infty)$, such that $\forall x > a$, $\frac{\rho T_{a+\eta,\kappa}^\alpha}{\rho T_{a+\eta,\kappa}^\alpha} f(x) < \infty$ and $\frac{\rho T_{a+\eta,\kappa}^\alpha}{\rho T_{a+\eta,\kappa}^\alpha} g(x) < \infty$. If $0 < m \leq \frac{f(t)}{g(t)} \leq M$, for $m, M \in \mathbb{R}_+^*$ and $\forall t \in [a, x]$, then

$$\frac{1}{M} \left( \frac{\rho T_{a+\eta,\kappa}^\alpha}{\rho T_{a+\eta,\kappa}^\alpha} f(x) \right) \leq \frac{1}{(m+1)(M+1)} \left( \frac{\rho T_{a+\eta,\kappa}^\alpha}{\rho T_{a+\eta,\kappa}^\alpha} (f + g)^2(x) \right) \leq \frac{1}{m} \left( \frac{\rho T_{a+\eta,\kappa}^\alpha}{\rho T_{a+\eta,\kappa}^\alpha} f(x) \right) g(x).$$

**Proof.** Being $0 < m \leq \frac{f(t)}{g(t)} \leq M$, $\forall t \in [a, x]$, we have

$$g(t) (m+1) \leq g(t) + f(t) \leq g(t) (M+1).$$

Also, it follows that $\frac{1}{M} \leq \frac{g(t)}{f(t)} \leq \frac{1}{m}$, which implies,

$$g(t) \left( \frac{M+1}{M} \right) \leq g(t) + f(t) \leq g(t) \left( \frac{m+1}{m} \right).$$

Evaluating the product between Eq. (4.32) and Eq. (4.33), we have

$$\frac{f(t) g(t)}{M} \leq \frac{(g(t) + f(t))^2}{(m+1)(M+1)} \leq \frac{f(t) g(t)}{m}.$$
Multiplying by \( \frac{1}{M} \frac{1}{\Gamma (\alpha)} \frac{1}{(x^\rho - t^\rho)^{1-\alpha}} \) both sides of Eq. (4.34) and integrating with respect to the variable \( t \), we have
\[
\frac{1}{M} \frac{1}{\Gamma (\alpha)} \int_a^x \frac{1}{(x^\rho - t^\rho)^{1-\alpha}} f(t) g(t) \, dt
\leq \frac{1}{(m+1)(M+1)} \left( \rho \mathcal{T}^{\alpha, \beta}_{a, \eta, \kappa} (f(x) + g(x))^2 \right)
\leq \frac{1}{m} \left( \rho \mathcal{T}^{\alpha, \beta}_{a, \eta, \kappa} f(x) g(x) \right).
\]

**Theorem 15.** Let \( \alpha > 0, \rho, \eta, \kappa, \beta \in \mathbb{R} \) and \( p \geq 1 \). Let \( f, g \in X^p(a, x) \) be two positive functions in \([0, \infty)\), such that \( \forall x > a, \rho \mathcal{T}^{\alpha, \beta}_{a, \eta, \kappa} f^p(x) < \infty \) and \( \rho \mathcal{T}^{\alpha, \beta}_{a, \eta, \kappa} g^p(x) < \infty \). If
\[
0 < m \leq \frac{f(t)}{g(t)} \leq M, \text{ for } m, M \in \mathbb{R}^*_+ \text{ and } \forall t \in [a, x],
\]
then
\[
\left( \rho \mathcal{T}^{\alpha, \beta}_{a, \eta, \kappa} f^p(x) \right)^{\frac{1}{p}} + \left( \rho \mathcal{T}^{\alpha, \beta}_{a, \eta, \kappa} g^p(x) \right)^{\frac{1}{p}} \leq 2 \left( \rho \mathcal{T}^{\alpha, \beta}_{a, \eta, \kappa} h^p(f(x), g(x)) \right)^{\frac{1}{p}},
\]
where \( h(f(x), g(x)) = \max \left\{ M \left[ \frac{M}{m} + 1 \frac{f(x) - M g(x)}{g(x)} \right], \frac{(m+M) g(x) - f(x)}{m} \right\} \).

**Proof.** From the hypothesis, \( 0 < m \leq \frac{f(t)}{g(t)} \leq M, \forall t \in [a, x] \), we have
\[
0 < m \leq M + m - \frac{f(t)}{g(t)}
\]
and
\[
M + m - \frac{f(t)}{g(t)} \leq M.
\]
Thus, using Eq. (4.35) and Eq. (4.36), we get
\[
g(t) \leq \frac{(M + m) g(t) - f(t)}{m} \leq h(f(t), g(t)),
\]
where \( h(f(t), g(t)) = \max \left\{ M \left[ \frac{M}{m} + 1 \frac{f(t) - M g(t)}{g(t)} \right], \frac{(M+m) g(t) - f(t)}{m} \right\} \).
Using the hypothesis, it follows that $0 < \frac{1}{M} \leq \frac{g(t)}{f(t)} \leq \frac{1}{m}$. In this way, we obtain

\[\frac{1}{M} \leq \frac{1}{M} + \frac{1}{m} - \frac{g(t)}{f(t)}\]

and

\[\frac{1}{M} + \frac{1}{m} - \frac{g(t)}{f(t)} \leq \frac{1}{m}.\]

Then, from Eq.(4.38) and Eq.(4.39), we have

\[\frac{1}{M} \leq \frac{\left(\frac{1}{m} + \frac{1}{M}\right)f(t) - g(t)}{f(t)} \leq \frac{1}{m},\]

which can be rewrite as

\[f(t) \leq M\left(\frac{1}{m} + \frac{1}{M}\right)f(t) - Mg(t)\]

\[= \frac{M(M + m)f(t) - M^2mg(t)}{Mm}\]

\[= \frac{\left(M + 1\right)f(t) - Mg(t)}{\left(M + 1\right)f(t) - Mg(t)}\]

\[\leq M \left[\frac{\left(M + 1\right)f(t) - Mg(t)}{\left(M + 1\right)f(t) - Mg(t)}\right]\]

\[\leq h(f(t), g(t)).\]

Thus, using Eq.(4.37) and Eq.(4.40), we can write

\[f^p(t) \leq h^p(f(t), g(t))\]

and

\[g^p(t) \leq h^p(f(t), g(t)).\]

Multiplying by $\frac{\rho^{1-\alpha}x^\kappa t^{\rho(\eta+1)-1}}{\Gamma(\alpha)(x^\rho - t^\rho)^{1-\alpha}}$ both sides of Eq.(4.41) and integrating with respect to the variable $t$, we have

\[\int_a^x \frac{\rho^{1-\alpha}x^\kappa t^{\rho(\eta+1)-1}}{\Gamma(\alpha)(x^\rho - t^\rho)^{1-\alpha}}f^p(t)\,dt \leq \int_a^x \frac{\rho^{1-\alpha}x^\kappa t^{\rho(\eta+1)-1}}{\Gamma(\alpha)(x^\rho - t^\rho)^{1-\alpha}}h^p(f(t), g(t))\,dt.\]

In this way, we obtain

\[\left(\rho^{\alpha,\beta} T_{\alpha,\kappa} f^p(x)\right)^\frac{1}{p} \leq \left(\rho^{\alpha,\beta} T_{\alpha,\kappa} h^p(f(x), g(x))\right)^\frac{1}{p}.\]
Using the same procedure as above, for Eq. \((4.42)\), we have

\[
\left( \rho T_{a^+,\eta,\kappa}^{\alpha,\beta} f^p(x) \right)^{\frac{1}{p}} \leq \left( \rho T_{a^+,\eta,\kappa}^{\alpha,\beta} h^p(f(x), g(x)) \right)^{\frac{1}{p}}.
\]

Thus, using Eq. \((4.43)\) and Eq. \((4.44)\), we conclude that

\[
\left( \rho T_{a^+,\eta,\kappa}^{\alpha,\beta} f^p(x) \right)^{\frac{1}{p}} + \left( \rho T_{a^+,\eta,\kappa}^{\alpha,\beta} g^p(x) \right)^{\frac{1}{p}} \leq 2 \left( \rho T_{a^+,\eta,\kappa}^{\alpha,\beta} h^p(f(x), g(x)) \right)^{\frac{1}{p}}.
\]

Using Eq. \((2.19)\) and Theorem 7 with the convenient conditions for each respective fractional integral, we have the previous theorems, that is, Theorem 10 to Theorem 15 introduced and demonstrated above, contain as particular cases, each result involving the following fractional integrals: Riemann-Liouville, Hadamard, Liouville, Weyl, Edrélyi-Kober, and Katugampola.

5. Concluding remarks

After a brief introduction to the fractional integral, proposed by Katugampola and fractional integrals in the sense of Riemann-Liouville and Hadamard, we generalize the reverse Minkowski’s inequality obtaining, as a particular case, the inequality involving the fractional integral in the Riemann-Liouville sense and Hadamard sense \([23]\). We also show other inequalities using the Katugampola fractional integral. The application of this fractional integral can be used to generalize several inequalities, among them, we mention the Gruss-type inequality, recently introduced and proved \([30]\). A continuation of this work, with this formulation of fractional integral, consists in generalize the inequalities of Hermite-Hadamard and Hermite-Hadamard-Fjer. Moreover, we will discuss inequalities via \(M\)-fractional integral according to \([37]\).

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