Close-to-convexity and Starlikeness of Analytic Functions

See Keong Lee, V. Ravichandran, and Shamani Supramaniam

Abstract. For functions \( f(z) = z^p + a_{n+1}z^{p+1} + \cdots \) defined on the open unit disk, the condition \( \Re\left( f'(z)/z^{p-1} \right) > 0 \) is sufficient for close-to-convexity of \( f \). By making use of this result, several sufficient conditions for close-to-convexity are investigated and relevant connections with previously known results are indicated.

1. Introduction

Let \( D := \{ z \in \mathbb{C} : |z| < 1 \} \) be the open unit disk and \( \mathcal{S}_{p,n} \) be the class of all analytic functions \( f : D \to \mathbb{C} \) of the form

\[
f(z) = z^p + a_{n+1}z^{p+1} + \cdots
\]

with \( \mathcal{S} := \mathcal{S}_{1,1} \). For studies related to multivalent functions, see [5,7,10]. Singh and Singh [15] obtained several interesting conditions for functions \( f \in \mathcal{S} \) satisfying inequalities involving \( f'(z) \) and \( zf''(z) \) to be univalent or starlike in \( D \). Owa et al. [11] generalized the results of Singh and Singh [15] and also obtained several sufficient conditions for close-to-convexity, starlikeness and convexity of functions \( f \in \mathcal{S} \). In fact, they have proved the following theorems.

Theorem 1.1. [11] Theorems 1-3 Let \( 0 \leq \alpha < 1 \) and \( \beta, \gamma \geq 0 \). If \( f \in \mathcal{S} \), then

\[
\Re\left( 1 + \frac{zf''(z)}{f'(z)} \right) > \frac{1 + 3\alpha}{2(1 + \alpha)} \implies \Re\left( f'(z) \right) > \frac{1 + \alpha}{2},
\]

\[
\Re\left( 1 + \frac{zf''(z)}{f'(z)} \right) < \frac{3 + 2\alpha}{2 + \alpha} \implies |f'(z) - 1| < 1 + \alpha,
\]

\[
|f'(z) - 1|^\beta |zf''(z)|^\gamma < \frac{(1 - \alpha)^{\beta + \gamma}}{2^{\beta + 2\gamma}} \implies \Re\left( f'(z) \right) > \frac{1 + \alpha}{2}.
\]

Theorem 1.2. [11] Theorem 4 Let \( 1 < \lambda < 3 \). If \( f \in \mathcal{S} \), then

\[
\Re\left( 1 + \frac{zf''(z)}{f'(z)} \right) < \begin{cases} 
\frac{5\lambda - 1}{2(\lambda + 1)}, & 1 < \lambda \leq 2; \\
\frac{\lambda + 1}{2(\lambda - 1)}, & 2 < \lambda < 3,
\end{cases} \implies zf'(z)/f(z) < \frac{\lambda(1 - z)}{\lambda - z}.
\]
In this present paper, the above results are extended for functions \( f \in \mathcal{A}_{p,n} \) and in particular for functions in \( \mathcal{A}_{1,n} \).

2. Close-to-convexity and Starlikeness

For \( f \in \mathcal{A} \), the condition \( \text{Re} f'(z) > 0 \) implies close-to-convexity and univalence of \( f \). Similarly, for \( f \in \mathcal{A}_{p,1} \), the inequality \( \text{Re}(f'(z)/z^{p-1}) > 0 \) implies \( p \)-valency of \( f \). See [17, 18]. From this result, the functions satisfying the hypothesis of Theorems 2.1–2.3 are \( p \)-valent in \( D \). A function \( f \in \mathcal{A}_{p,1} \) is close-to-convex if there is a \( p \)-valent convex function \( \phi \) such that \( \text{Re}(f'(z)/\phi(z)) > 0 \). Also they are all close-to-convex with respect to \( \phi(z) = z^p \).

**Theorem 2.1.** If the function \( f \in \mathcal{A}_{p,n} \) satisfies the inequality

\[
\text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \frac{(2p-n) + \alpha(2p+n)}{2(\alpha+1)},
\]

then

\[
\text{Re}\left(\frac{f'(z)}{pz^{p-1}}\right) > \frac{1 + \alpha}{2}.
\]

For the proof of our main results, we need the following lemma.

**Lemma 2.1.** [6, Lemma 2.2a] Let \( z_0 \in \mathbb{D} \) and \( r_0 = |z_0| \). Let \( f(z) = a_n z^n + a_{n+1} z^{n+1} + \cdots \) be continuous on \( \overline{B}_{r_0} \) and analytic on \( \mathbb{D}_{r_0} \cup \{z_0\} \) with \( f(z) \neq 0 \) and \( n \geq 1 \). If

\[
|f(z_0)| = \max\{|f(z)| : z \in \mathbb{D}_{r_0} \},
\]

then there exists an \( m \geq n \) such that

1. \( \frac{z_0 f'(z_0)}{f(z_0)} = m \), and
2. \( \text{Re}\left(\frac{z_0 f''(z_0)}{f'(z_0)} + 1 \right) \geq m \).

**Proof of Theorem 2.1.** Let the function \( w \) be defined by

\[
f'(z) = \frac{p}{pz^{p-1}} + \frac{\alpha w(z)}{1 + \alpha w(z)} - \frac{zw'(z)}{1 + w(z)}.
\]

Then clearly \( w \) is analytic in \( \mathbb{D} \) with \( w(0) = 0 \). From (2.2), some computation yields

\[
1 + \frac{zf''(z)}{f'(z)} = p + \frac{\alpha zw'(z)}{1 + \alpha w(z)} - \frac{zw'(z)}{1 + w(z)}.
\]

Suppose there exists a point \( z_0 \in \mathbb{D} \) such that

\[
|w(z_0)| = 1 \text{ and } |w(z)| < 1 \text{ when } |z| < |z_0|.
\]

Then by applying Lemma 2.1 there exists \( m \geq n \) such that

\[
z_0 w'(z_0) = mw(z_0), \quad (w(z_0) = e^{i\theta}; \theta \in \mathbb{R}).
\]
Thus, by using (2.3) and (2.4), it follows that

\[
\Re\left(\frac{1 + \frac{z_0 f''(z_0)}{f''(z_0)}}{p + \Re\left(\frac{\alpha m w(z_0)}{1 + \alpha w(z_0)}\right) - \Re\left(\frac{m w(z_0)}{1 + w(z_0)}\right)}\right)
\]

\[
= p + \Re\left(\frac{\alpha m w(z_0)}{1 + \alpha w(z_0)}\right) - \Re\left(\frac{m e^{i\theta}}{1 + e^{i\theta}}\right)
\]

\[
= p + \alpha m (\alpha + \cos \theta) - \frac{m}{2}
\]

\[
\leq \frac{(2p - n) + \alpha(2p + n)}{2(\alpha + 1)}
\]

which contradicts the hypothesis (2.1). It follows that \( |w(z)| < 1 \), that is

\[
\left|\frac{f'(z)}{p z^{p-1}} - 1\right| < 1.
\]

This evidently completes the proof of Theorem 2.1.

Owa [13] shows that a function \( f \in A_{p,1} \) satisfying \( \Re(1 + z f''(z)/f'(z)) < p + 1/2 \) implies \( f \) is \( p \)-valently starlike. Our next theorem investigates the close-to-convexity of this type of functions. For related results, see [4, 14, 19].

**Theorem 2.2.** If the function \( f \in A_{p,n} \) satisfies the inequality

\[
\Re\left(1 + \frac{z f''(z)}{f'(z)}\right) < \frac{(p + n) \alpha + (2p + n)}{(\alpha + 2)},
\]

then

\[
\left|\frac{f'(z)}{p z^{p-1}} - 1\right| < 1 + \alpha.
\]

**Proof.** Consider the function \( w \) defined by

\[
\frac{f'(z)}{p z^{p-1}} = (1 + \alpha) w(z) + 1.
\]

Then clearly \( w \) is analytic in \( \mathbb{D} \) with \( w(0) = 0 \). From (2.6), some computation yields

\[
1 + \frac{z f''(z)}{f'(z)} = p + \frac{(1 + \alpha) z w'(z)}{(1 + \alpha) w(z) + 1}.
\]

Suppose there exists a point \( z_0 \in \mathbb{D} \) such that

\[
|w(z_0)| = 1 \quad \text{and} \quad |w(z)| < 1 \quad \text{when} \quad |z| < |z_0|.
\]

Then by applying Lemma 2.1 there exists \( m \geq n \) such that

\[
z_0 w'(z_0) = mw(z_0), \quad (w(z_0) = e^{i\theta}; \theta \in \mathbb{R}).
\]
Thus, by using (2.7) and (2.8), it follows that
\[
\text{Re} \left( 1 + \frac{z_0 f''(z_0)}{f'(z_0)} \right) = p + \text{Re} \left( \frac{(1 + \alpha)z_0 w'(z_0)}{(1 + \alpha)w(z_0) + 1} \right)
\]
\[
= p + \text{Re} \left( \frac{(1 + \alpha)me^{i\theta}}{(1 + \alpha)e^{i\theta} + 1} \right)
\]
\[
= p + \frac{m(1 + \alpha)(1 + \alpha + \cos \theta)}{1 + (1 + \alpha)^2 + 2(1 + \alpha)\cos \theta}
\]
\[
\geq \frac{(p + n)\alpha + (2p + n)}{(\alpha + 2)},
\]
which contradicts the hypothesis (2.5). It follows that \(|w(z)| < 1\), that is,
\[
\left| f'(z) \right|_{pz^p - 1} < 1 + \alpha.
\]

This evidently completes the proof of Theorem 2.2. □

Owa [12] has also showed that a function \( f \in A \) satisfying
\[
\left| \frac{f'(z)}{g'(z)} - 1 \right|^{\beta} |zf''(z)/g'(z) - zf'(z)g''(z)/(g'(z))^2|^{\gamma} < (1 + \alpha)^{\beta + \gamma}
\]
for \( 0 \leq \alpha < 1, \beta \geq 0, \gamma \geq 0 \) and \( g \) a convex function, is close-to-convex. Also, see [3]. Our next theorem investigates the close-to-convexity of similar class of functions.

**Theorem 2.3.** If \( f \in A_{p,n} \), then

\[
\left| \frac{f'(z)}{pz^p - 1} - 1 \right|^{\beta} \left| \frac{f''(z)}{z^p - 2} - (p - 1) \frac{f'(z)}{z^p - 1} \right|^{\gamma} < \left( \frac{pn}{(1 - \alpha)^{\beta + \gamma}} \right)
\]

implies
\[
\text{Re} \left( \frac{f'(z)}{pz^p - 1} \right) > \frac{1 + \alpha}{2},
\]
and

\[
\left| \frac{f'(z)}{pz^p - 1} - 1 \right|^{\beta} \left| \frac{f''(z)}{z^p - 2} - (p - 1) \frac{f'(z)}{z^p - 1} \right|^{\gamma} < (pn)^{\gamma}|1 - \alpha|^{\beta + \gamma}
\]

implies
\[
\left| \frac{f'(z)}{pz^p - 1} - 1 \right| < 1 - \alpha.
\]

**Proof.** For the function \( w \) defined by

\[
\frac{f'(z)}{pz^p - 1} = \frac{1 + \alpha w(z)}{1 + w(z)},
\]
we can rewrite (2.11) to yield
\[
\frac{f'(z)}{pz^p - 1} - 1 = \frac{(\alpha - 1)w(z)}{1 + w(z)}.
\]
For the second implication in the proof, consider the function

\[(2.12) \quad \left| \frac{f'(z)}{p z^{p-1}} - 1 \right|^{\beta} = \frac{|w(z)|^\beta |1 - \alpha|^\beta}{|1 + w(z)|^\beta}. \]

By some computation, it is evident that

\[ \frac{f''(z)}{z^{p-2}} - (p-1) \frac{f'(z)}{z^{p-1}} = \frac{p(\alpha - 1)zw'(z)}{(1 + w(z))^2} \]

or

\[(2.13) \quad \left| \frac{f''(z)}{z^{p-2}} - (p-1) \frac{f'(z)}{z^{p-1}} \right|^{\gamma} = \frac{p^\gamma |zw'(z)|^\gamma |1 - \alpha|^\gamma}{|1 + w(z)|^{2\gamma}}. \]

From (2.12) and (2.13), it follows that

\[ \left| \frac{f'(z)}{p z^{p-1}} - 1 \right|^{\beta} \left| \frac{f''(z)}{z^{p-2}} - (p-1) \frac{f'(z)}{z^{p-1}} \right|^{\gamma} = \frac{p^\gamma |w(z)|^\beta |1 - \alpha|^\beta + \gamma |zw'(z)|^\gamma}{|1 + w(z)|^{\beta + 2\gamma}}. \]

Suppose there exists a point \( z_0 \in \mathbb{D} \) such that

\[ |w(z_0)| = 1 \text{ and } |w(z)| < 1 \text{ when } |z| < |z_0|. \]

Then (2.14) and Lemma 2.1 yield

\[ \left| \frac{f'(z_0)}{p z_0^{p-1}} - 1 \right|^{\beta} \left| \frac{f''(z_0)}{z_0^{p-2}} - (p-1) \frac{f'(z_0)}{z_0^{p-1}} \right|^{\gamma} = \frac{p^\gamma (1 - \alpha)^{\beta + \gamma} |w(z_0)|^\beta |mw(z_0)|^\gamma}{|1 + e^{i\theta}|^{\beta + 2\gamma}} \]

\[ \geq \frac{p^\gamma n^\gamma (1 - \alpha)^{\beta + \gamma}}{2^{\beta + 2\gamma}}, \]

which contradicts the hypothesis (2.9). Hence \( |w(z)| < 1 \), which implies

\[ \left| \frac{1 - f'(z)}{p z^{p-1}} - \alpha \right| < 1, \]

or equivalently

\[ \Re \left( \frac{f'(z)}{p z^{p-1}} \right) > \frac{1 + \alpha}{2}. \]

For the second implication in the proof, consider the function \( w \) defined by

\[(2.14) \quad \frac{f'(z)}{p z^{p-1}} = 1 + (1 - \alpha)w(z). \]

Then

\[(2.15) \quad \left| \frac{f'(z)}{p z^{p-1}} - 1 \right|^{\beta} = |1 - \alpha|^\beta |w(z)|^\beta \]

and

\[(2.16) \quad \left| \frac{f''(z)}{z^{p-2}} - (p-1) \frac{f'(z)}{z^{p-1}} \right|^{\gamma} = p^\gamma |zw'(z)|^\gamma |1 - \alpha|^\gamma. \]
From (2.15) and (2.16), it is clear that
\[
\left| \frac{f''(z)}{p^{\beta}z^{p-1}} - 1 \right| = \left| \frac{f''(z)}{p^{\beta}z^{p-1}} - (p - 1) \frac{f'(z)}{p^{\beta}z^{p-1}} \right| = p^\gamma |w(z)| |1 - \alpha|^\gamma |zw'(z)|^\gamma.
\]
Suppose there exists a point \(z_0 \in \mathbb{D}\) such that
\[
|w(z_0)| = 1 \quad \text{and} \quad |w(z)| < 1 \quad \text{when} \quad |z| < |z_0|.
\]
Then by applying Lemma 2.1 and using (2.4), it follows that
\[
\left| \frac{f'(z_0)}{pz_0^{\beta}z_0^{p-1}} - 1 \right| = \left| \frac{f''(z_0)}{p^{\beta+\gamma}z_0^{p-1}} \right| = p^\gamma |w(z_0)| |1 - \alpha|^\gamma |zw'(z_0)|^\gamma
\]
\[
= p^\gamma m^\gamma |1 - \alpha|^\gamma \geq (pm)^\gamma |1 - \alpha|^\gamma,
\]
which contradicts the hypothesis (2.10). Hence \(|w(z)| < 1\) and this implies
\[
\left| \frac{f'(z)}{pz^{\beta}z^{p-1}} - 1 \right| < 1 - \alpha.
\]
Thus the proof is complete. \(\square\)

In next theorem, we need the concept of subordination. Let \(f\) and \(g\) be analytic functions defined on \(\mathbb{D}\). Then \(f\) is subordinate to \(g\), written \(f \prec g\), provided there is an analytic function \(w : \mathbb{D} \to \mathbb{D}\) with \(w(0) = 0\) such that \(f = g \circ w\).

**Theorem 2.4.** Let \(\lambda_1\) and \(\lambda_2\) be given by
\[
\lambda_1 = \frac{2n + 4(2p - 1)}{4 + n - 2p + \sqrt{16n + n^2 + 32p - 12np - 28p^2}},
\]
\[
\lambda_2 = \frac{2n + 4(2p - 1)}{-n + 2p + \sqrt{16 - 8n + n^2 - 48p + 4np + 36p^2}},
\]
and \(\lambda_1 < \lambda < \lambda_2\). If the function \(f \in \mathcal{A}_{p,n}\) satisfies the inequality
\[
\text{Re} \left(1 + \frac{zf''(z)}{f'(z)}\right) < \begin{cases} 
\frac{2(1-p)\lambda^2 + (2-2p-n)}{2(\lambda + 1)}, & \lambda_1 < \lambda \leq \frac{p+n}{p}; \\
\frac{2(1-p)\lambda^2 + (2+2p+n)}{2(\lambda - 1)}, & \frac{p+n}{p} < \lambda < \lambda_2;
\end{cases}
\]
then
\[
\frac{zf'(z)}{pf(z)} < \frac{\lambda(1-z)}{\lambda - z}.
\]

**Proof.** Let us define \(w\) by
\[
\frac{zf'(z)}{pf(z)} = \frac{\lambda(1-w(z))}{\lambda - w(z)}.
\]
By doing the logarithmic differentiation on (2.19), we get
\[ 1 + \frac{zf''(z)}{f'(z)} = \frac{p\lambda(1-w(z))}{\lambda - z} - \frac{zw'(z)}{1-w(z)} + \lambda - w(z). \]

Assume that there exists a point \( z_0 \in \mathbb{D} \) such that \( |w(z_0)| = 1 \) and \( |w(z)| < 1 \) when \( |z| < |z_0| \). By applying Lemma 2.1 as in Theorem 2.1, it follows that
\[
\text{Re} \left( 1 + \frac{z_0f''(z_0)}{f'(z_0)} \right) = \text{Re} \left( \frac{p\lambda(1-e^{i\theta})}{\lambda - e^{i\theta}} \right) - \text{Re} \left( \frac{me^{i\theta}}{1-e^{i\theta}} \right) + \text{Re} \left( \frac{me^{i\theta}}{\lambda - e^{i\theta}} \right)
\]
\[
= \frac{p\lambda(\lambda + 1)(1 - \cos \theta)}{\lambda^2 + 1 - 2\lambda \cos \theta} + \frac{m(\lambda \cos \theta - 1)}{\lambda^2 + 1 - 2\lambda \cos \theta}
\]
\[
= \frac{\lambda + 1}{2}(2 - p) + \frac{(\lambda^2 - 1)(p + m) - p\lambda}{2(\lambda^2 + 1 - 2\lambda \cos \theta)},
\]
which yields the inequality
\[
\text{Re} \left( 1 + \frac{z_0f''(z_0)}{f'(z_0)} \right) \geq \begin{cases} 
\frac{2(1-p)\lambda^2 + (4+n)\lambda + (2-2p-n)}{2(\lambda+1)}, & \lambda_1 < \lambda \leq \frac{p+n}{p}; \\
\frac{2(1-p)\lambda^2 + n\lambda + (2-2p+n)}{2(\lambda-1)}, & \frac{p+n}{p} < \lambda < \lambda_2.
\end{cases}
\]

Since (2.20) obviously contradicts hypothesis (2.17), it follows that \( |w(z)| < 1 \). This proves the subordination (2.18).

**Remark 2.1.** The subordination (2.18) can be written in equivalent form as
\[
\left| \frac{\lambda(f'(z)/f(z) - 1)}{zf''(z)/f'(z) - \lambda} \right| < 1,
\]
or by further computation, as
\[
\left| \frac{1}{p} \frac{zf''(z)}{f'(z)} - \frac{\lambda}{\lambda + 1} \right| < \frac{\lambda}{\lambda + 1}.
\]
The last inequality shows that \( f \) is starlike in \( \mathbb{D} \).

**Remark 2.2.** When \( p = 1 \) and \( n = 1 \), Theorems 2.1, 2.4 reduce to Theorems 1.1 and 1.2.

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E-mail address: sklee@cs.usm.my

E-mail address: vravi@maths.du.ac.in

E-mail address: sham105@hotmail.com