LOCAL RIGIDITY OF MANIFOLDS WITH HYPERBOLIC CUSPS
I. LINEAR THEORY AND MICROLOCAL TOOLS

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Abstract. In this paper, we implement linear PDE tools on manifolds with hyperbolic cusps and globally negative sectional curvature in order to study length rigidity problems. We obtain the spectral rigidity of such manifolds. We also prove several results which are classical in the case of compact manifolds, and are usual tools in the study of the marked length spectrum.

1. Introduction

1.1. Spectral rigidity. In this article, we are interested in the linear version of the marked length spectrum rigidity problem, namely the question of infinitesimal spectral rigidity as it was originally studied in [GK80]. We recall that a manifold \((M, g)\) is said to be spectrally rigid if any smooth isospectral deformation \((g_\varepsilon)_{\varepsilon \in (-1, 1)}\) of the metric \(g\) is trivial, namely there exists an isotopy \((\phi_\varepsilon)_{\varepsilon \in (-1, 1)}\) such that \(\phi_\varepsilon^* g_\varepsilon = g\).

In the case of a closed manifold, this usually boils down to proving that the X-ray transform \(I_2^g\) — that is, the integration of symmetric 2-tensors along closed geodesics in \((M, g)\) — on symmetric solenoidal or divergence-free 2-tensors is injective. This will be called solenoidal injectivity in the rest of the paper.

The solenoidal injectivity of this operator \(I_2^g\) was first obtained for negatively-curved closed surfaces by Guillemin-Kazhdan in their celebrated paper [GK80]. More generally, their proof works for tensors of any order \(m \in \mathbb{N}\). This result was then extended by Croke-Sharafutdinov [CS98] to negatively-curved closed manifolds of arbitrary dimension. More recently, [PSU14] obtained the solenoidal injectivity of \(I_2^g\) for any Anosov Riemannian surfaces \((M, g)\), namely surfaces for which the geodesic flow is Anosov or uniformly hyperbolic on the unit tangent bundle \(SM\). Guillarmou [Gui17] then extended the result on Anosov surfaces to tensors of arbitrary order \(m \in \mathbb{N}\). More generally, it is conjectured that the X-ray transform \(I_m^g\) is solenoidal injective on closed Anosov Riemannian manifolds but the question remains open in dimension \(\geq 3\).

In this article, we are interested in the solenoidal injectivity of \(I_2^g\) on noncompact complete manifolds of negative curvature whose ends are real hyperbolic cusps. This does not seem to have been considered before in the literature and we hope to use to study the local marked length spectrum rigidity of such manifolds in a second article.
More precisely, the case we will consider will be that of a complete negatively-curved Riemannian manifold \((M,g)\) with a finite numbers of ends of the form \(Z_{a,\Lambda} = [a, +\infty] \times (\mathbb{R}^d / \Lambda)_\theta\), where \(a > 0\), and \(\Lambda\) is a crystallographic group with covolume 1. On this end, we have the metric \(g = y^{-2}(dy^2 + d\theta^2)\). The sectional curvature of \(g\) is constant equal to \(-1\), and the volume of \(Z_{a,\Lambda}\) is finite. All ends with finite volume and curvature \(-1\) take this form. In dimension two, all cusps are the same (we must have \(\Lambda = \mathbb{Z}\)). However, in higher dimensions, if \(\Lambda\) and \(\Lambda'\) are not in the same orbit of \(SO(d, \mathbb{Z})\), \(Z_{a,\Lambda}\) and \(Z_{a',\Lambda'}\) are never isometric. In the following, we will sometimes call cusp manifolds such manifolds. Up to taking a finite cover, we can always assume that each \(\Lambda\) is a lattice in \(\mathbb{R}^d\).

![Figure 1. A surface with three cusps.](image)

In our case, we denote by \(C\) the set of hyperbolic free homotopy classes on \(M\), which is in one-to-one correspondance with the set of hyperbolic conjugacy classes of \(\pi_1(M, \cdot)\). From elementary Riemannian geometry, since the flow is Anosov, we know that for each such class \(c \in C\) of \(C^1\) curves on \(M\), there is a unique representant \(\gamma_g(c)\) which is a geodesic for \(g\). If \(h\) is a symmetric 2-tensor, we define its X-ray transform by

\[
I^2_g h(c) = \frac{1}{\ell(\gamma_g(c))} \int_0^{\ell(\gamma_g(c))} h_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t)) dt,
\]

where \(\gamma\) is a parametrization by arc-length. We will prove the

**Theorem 1.** Let \((M^{d+1}, g)\) be a negatively-curved complete manifold whose ends are real hyperbolic cusps. Let \(-\kappa_0 < 0\) be the maximum of the sectional curvature. Then, for all \(\alpha \in (0, 1)\) and \(\beta \in (0, \sqrt{-\kappa_0} \alpha)\), the X-ray transform \(I^2_g\) is injective on

\[
y^\beta C^\alpha(M, S^2 T^* M) \cap H^1(M, S^2 T^* M) \cap \ker D^*
\]

Here, \(D^*\) denotes the divergence on 2-tensors: as usual, a tensor \(f\) is declared to be solenoidal if and only if \(D^* f = 0\). It is defined as the formal transpose (for the
$L^2$-scalar product) of the operator $D := \sigma \circ \nabla$ acting on 1-forms, where $\nabla$ is the Levi-Civita connection and $\sigma$ is the operator of symmetrization of 2-tensors.

In turn, the previous Theorem implies the spectral rigidity for smooth compactly supported isospectral deformations.

**Corollary 1.1.** Let $(M^{d+1}, g)$ be a negatively curved complete manifold whose ends are real hyperbolic cusps. Let $(g_\varepsilon)_{\varepsilon \in (-1,1)}$ be a smooth isospectral deformation of $g = g_0$ with compact support in $M$. Then, there exists an isotopy $(\phi_\varepsilon)_{\varepsilon \in (-1,1)}$ such that $\phi_\varepsilon^* g_\varepsilon = g$.

Theorem 1 is the first step towards proving the local rigidity of the marked length spectrum on such manifolds, as the X-ray transform on symmetric 2-tensors turns out to be the differential of the marked length spectrum.

In order to prove Theorem 1, we will need — together with a Livsic-type theorem which does not really differ from the compact case — to study the decomposition of symmetric 2-tensors into a potential part and a solenoidal part (or divergence-free part). Namely, we will need to prove that any symmetric 2-tensor $f$ can be written as $f = Dp + h$, where $p$ is a 1-form and $h$ is solenoidal. The existence of such a decomposition relies on the analytic properties of the elliptic differential operator $D$ and in particular on the existence of a parametrix with compact remainder. Since the manifold $M$ is not compact, this theory is made harder (smoothing operators are no longer compact) and one has to resort to a careful analysis of the behaviour of the operator on the infinite ends of the manifold. A large part of this article is devoted to this study as the next paragraph explains.

1.2. **Pseudodifferential calculus on manifolds with hyperbolic cusps.** A careful study of the operators on the infinite ends of the models will thus be needed. The relevant techniques are that of Melrose's b-calculus which we will adapt to our setting. We insist on the fact that we hope to use the framework developed here in a second article in order to study the nonlinear problem. While the operators $D$ and $D^* D$ studied in this first article are very likely to belong to the “fibered cusp calculus” introduced by Mazzeo-Melrose [MM98], we rather chose to expand the microlocal calculus developed in [Bon16] and [GW17] and this for two main reasons.

First of all, for the nonlinear problem, we intend to use the resolvent of the generator $X$ of the geodesic flow on the unit tangent bundle $SM$ as it was studied in [GW17]. Since $X$ is not elliptic, it is not straightforward that the techniques of Melrose [Mel93] can be applied to study its analytic properties and to prove, in particular, the meromorphic extension of $(X \pm \tau)^{-1}$ to the whole complex plane. It was the purpose of [GW17] to expand the relevant calculus introduced in [Bon16] in order to deal with such an operator.
Secondly, we will mostly be interested in the analytic behaviour of the operator $D^* D$ on weighted Hölder-Zygmund spaces. On the one hand, this does not seem to have been considered so far by the microlocal school working on noncompact manifolds with cusps and for which we refer to [M83, MM98, Vai01]. In particular, we prove boundedness results of pseudodifferential operators on such manifolds and show how to construct a parametrix on these spaces modulo a compact remainder. On the other hand, boundedness of pseudodifferential operators on manifolds with bounded geometry seems to have been considered by various authors (see [Skr98, Tay97] for instance). Roughly speaking, this assumption asserts that the manifold is uniformly comparable to $\mathbb{R}^{d+1}$ and that the usual results known on $\mathbb{R}^d$ can be transferred to such manifolds. However, in our case, since the radius of injectivity collapses to 0 in the cusps, we are not dealing with a bounded geometry and we cannot use such results. We refer to Section §2.5 for a more extensive discussion.

In the core of this article, we will be working with admissible fibered cusps (see Definition 2.1) and our theory of inversion will be phrased in this general context. Moreover, we will choose Sobolev spaces with careful weights, depending on the zero and non-zero Fourier modes in the $\theta$-variable. As for the introduction, we state a simpler version of our main theorem of inversion (see Theorem 3) in the case where the fiber over the cusp is trivial (it is a point) and the operator acts on distributions on $M$.

**Theorem 2.** Let $(M^{d+1}, g)$ be a negatively-curved complete manifold whose ends are real hyperbolic cusps. Let $P$ be a pseudodifferential operator on $M$. Assume that it is $(\rho_-, \rho_+)-L^2$ (resp. $-L^\infty$)-admissible in the sense of Definition 3.2 (resp. Definition 4.3). Also assume that it is uniformly elliptic in the sense of Definition 2.3. Then then there is a discrete set $S \subset (\rho_-, \rho_+) \setminus S$, there is an operator $Q_I$ such that

$$PQ_I - 1 \quad \text{and} \quad Q_I P - 1$$

are compact operators on $y^{e^{-d/2}}H^s(M)$ (resp. $y^\rho C^*_s(M)$) for $\rho \in I$, $s \in \mathbb{R}$. In particular, $P$ is Fredholm on these spaces.

The spaces $H^s(M)$ are the usual Sobolev spaces built from the metric. The spaces $C^*_s(M)$ are the Hölder-Zygmund spaces, introduced in Section §4. They coincide with the usual Hölder spaces $C^s(M)$ built from the distance (induced by the metric) for $s \in (0, 1)$. In a sense that will be made precise, to be admissible means that $P$ commutes with local isometries of the cusps, modulo compact operators.

By this rather comprehensive treatment, we hope to provide a complete theory of inversion of elliptic pseudodifferential operators on manifolds with hyperbolic cusps when acting on Hölder-Zygmund and Sobolev spaces.
1.3. Outline of the paper. In Section §2, we introduce the basic functional spaces and the class of pseudodifferential operators we will be working with. Section §3 is dedicated to the notion of *indicial operator* and to the inversion of an elliptic pseudodifferential operators on weighted Sobolev spaces. In Section §4, we prove boundedness properties of our class of pseudodifferential operators on Hölder-Zygmund spaces. We also show how to invert elliptic operators in the calculus on weighted Hölder-Zygmund spaces.

In the last Section §5, we show how the previous theory can be applied to the operators $\nabla_S$ (the gradient of the Sasaki metric on the unit tangent bundle $SM$), $D$ and $D^*D$. This will provide the decomposition of tensors into a potential and a solenoidal part. We also obtain a Livsic Theorem (see Theorem 4) which is rather similar to the compact case. In the end, gathering all these different pieces together, we will deduce Theorem 1.

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2. Pseudo-differential operators

2.1. Main result. Since manifolds with cusps are non-compact, one has to introduce new techniques (compared with the compact case) to solve PDE problems. However here, the lack of compactness is in some sense only one dimensional, so that many problems can be solved with a one dimensional scattering approach.

An important remark is that we will rely on constructions from [GW17], itself based on [Bon16]. In the former paper, the techniques from Melrose [Mel93] had to be adapted to deal with operators that are not elliptic. In Section §2.5, we will compare our setup to that of Mazzeo and Melrose’s *fibred cusp calculus*.

Since we want to state results in some generality, we will consider in this whole section the following setup: we are given a non-compact manifold $N$ with a finite number of ends $N_\ell$, which take the form

\begin{equation}
Z_{\ell,a} \times F_\ell.
\end{equation}

Here, $Z_{\ell,a} = \{ z \in Z_\ell \mid y(z) > a \}$, and

\[ Z_\ell = [0, +\infty) \times \left( \mathbb{R}^d / \Lambda_\ell \right)_\theta. \]

In all generality, $\Lambda_\ell \subset O(d) \ltimes \mathbb{R}^d$ is a crystallographic group. However, according to Bieberbach’s Theorem, up to taking a finite cover, we can assume that $\Lambda \subset \mathbb{R}^d$ is a
lattice of translations. We will work with that case, and check that the results are stable by taking quotients under free actions of finite groups of isometries.

The slice \((F, g)\) is a compact Riemannian manifold. We will use the variables \(z = (x, \zeta) \in Z \times F\) and \(x = (y, \theta) \in [a, +\infty) \times \mathbb{R}^d / \Lambda\). We assume that \(N\) is endowed with a metric \(g\), equal over the cusps to

\[
\frac{dy^2 + d\theta^2}{y^2} + g_{F, i}.
\]

We will also have a vector bundle \(L \rightarrow N\), and will assume that for each \(\ell\), there is a vector bundle \(L_\ell \rightarrow F\), so that

\[
L|_{N_\ell} \simeq Z_\ell \times L_\ell.
\]

Whenever \(L\) is a hermitian vector bundle with metric \(g_L\), a compatible connection \(\nabla^L\) is one that satisfies

\[
X g_L(Y, Z) = g_L(\nabla^L_X Y, Z) + g_L(Y, \nabla^L_X Z).
\]

Taking advantage of the product structure, we impose that when \(X\) is tangent to \(Z\),

\[
\nabla^L_X Y(x, \zeta) = d_x Y(X) + A_x(X) \cdot Y,
\]

where the connection form \(A_x(X)\) is an anti-symmetric endomorphism depending linearly on \(X\), and \(A(y \partial_y), A(y \partial_\theta)\) do not depend on \(y, \theta\). In particular, we get that the curvature of \(\nabla^L\) is bounded, as are all its derivatives.

**Definition 2.1.** Such data \((L \rightarrow N, g, g_L, \nabla^L)\) will be called an admissible bundle.

Given a cusp manifold \((M, g)\), the bundle of differential forms over \(M\) is an admissible bundle. Since the tangent bundle of a cusp is trivial, any linearly constructed bundle over \(M\) is admissible. For example, the bundle of forms over the Grassmann bundle of \(M\), or over the unit cosphere bundle \(S^* M\).

Throughout, the paper, we will mainly be using Sobolev spaces or Hölder-Zygmund spaces. As usual, when dealing with non-compact manifolds, weighted spaces will play an important role. The Sobolev spaces \(H^{s, \rho_0, \rho_\perp}(L)\) defined for \(s, \rho_0, \rho_\perp \in \mathbb{R}\) are \(H^s\)-based Sobolev spaces (see §2.2 for the definition of Sobolev norms) with weight \(y^{\rho_0}\) on the zero Fourier mode (in the \(\theta\) variable) and \(y^{\rho_\perp}\) for the non-zero Fourier modes. We refer to Definition 3.1 for an exact definition.

We are going to prove the following result:

**Theorem 3.** Let \(L\) be an admissible bundle in the sense of Definition 2.1. Assume that \(L\) is endowed with a pseudo-differential operator \(P\). Assume that it is \((\rho_-, \rho_+) - L^2\) (resp. \(-L^\infty\))-admissible in the sense of Definitions 3.2 (resp. Definition 4.3). Also assume that it is uniformly elliptic in the sense of Definition 2.3. Then there is
a discrete set $S \subset (\rho_-, \rho_+) \text{ such that for each connected component } I := (\rho^-_I, \rho^+_I) \subset (\rho_-, \rho_+) \setminus S, \text{ there is an operator } Q_I \text{ that is } I\text{-admissible, such that}

$$
PQ_I - 1 \text{ and } Q_I P - 1
$$

are bounded as operators

$$
H^{-N, \rho^+} - \epsilon - d/2, \rho_\perp (L) \to H^{N, \rho^+} + \epsilon - d/2, \rho_\perp (L),
$$

(resp. $y^{\rho^+} - \epsilon C^{N} - y^{\rho^+} + C^{N}$) for all $N > 0$ and $\epsilon > 0$ small enough. In particular, $P$ is Fredholm with same index on each space $H^{s, \rho_0 - d/2, \rho_\perp}$ (resp. $y^{\rho_0} C^s$) for $s \in \mathbb{R}, \rho_0 \in I, \rho_\perp \in \mathbb{R}$.

There is no particular reason for an elliptic pseudo-differential operator to be Fredholm on a non-compact manifolds, even if the ellipticity is uniform at infinity. One has to introduce some kind of ellipticity or boundary condition at infinity, which depends on the geometry. In our case, this will take the following form. We will require that our operators commute with the generators of local isometries of the cusp, that is $y \partial_y + y \partial_\theta$ and $\partial_\theta$. We will be able to allow this to hold modulo compact operators.

Under this assumption, the general strategy goes as follows: first, one inverts $P$ modulo a smoothing remainder that is not compact; by compact injection of $H^s \hookrightarrow H^{s'}$ for $s > s'$ on the orthogonal of the $\theta$-zeroth Fourier mode (see Lemma 3.1), it is sufficient to explicitly invert the operator acting on sections not depending on $\theta$. As in b-calculus, this is done by introducing an indicial operator $I_Z(P)$ (see §3.3) which is a convolution operator in the $r = \log y$ variable, defined on “the model at infinity” and acting on sections that are independent of $\theta$. The set $S$ can be computed by hand, as will be explained in Corollary 3.1: it consists of the real parts of the indicial roots of the indicial family $I_Z(P, \lambda)$.

2.2. Functional spaces. Let $f$ be a function on $N$. We define for an integer $k \geq 0$:

$$
\|f\|_{C^k}(z) = \sup_{0 \leq j \leq k} \|\nabla^j f(z)\|,
$$

and $C^k(N)$ is the space of functions such that this is uniformly bounded in $z \in N$. We write $f \in C^\infty(N)$ if all the derivatives of $f$ are bounded. If $f$ is infinitely many times differentiable, but its derivatives are not bounded, we simply say that $f$ is smooth.

The Christoffel coefficients of the metric in the cusp in the frame

$$
X_y := y \partial_y, \ X_\theta := y \partial_\theta, \ X_\zeta := \partial_\zeta
$$
are independent of \((y, \theta)\). As a consequence, in the cusp, there are uniform constants such that
\[
\|f\|_{C^k(z)} \asymp \sup_{|\alpha|} |X_\alpha f(z)|,
\]
(here, \(\alpha\) is a multiindex valued in \(\{y, \theta, \zeta\}\).) Let \(0 < \alpha < 1\). We will write \(f \in C^\alpha(N)\) if:
\[
\|f\|_{C^\alpha} := \sup_{x \in N} |f(z)| + \sup_{z, z' \in N, z \neq z'} \frac{|f(z) - f(z')|}{d(z, z')^\alpha} = \|f\|_{\infty} + \|f\|_\alpha < \infty
\]
In particular, a function \(f\) may be \(\alpha\)-Hölder continuous, with a uniform Hölder constant of continuity (i.e. \(\|f\|_\alpha < \infty\)), but may not be in \(C^\alpha(N)\) if \(\|f\|_{\infty} = \infty\) for instance. It also makes sense to define \(C^\alpha\) for \(\alpha \in \mathbb{R}_+ \setminus \mathbb{N}\) by asking that \(f \in C^{[\alpha]}(N)\) and that the \([\alpha]\)-th derivatives of \(f\) are \(\alpha - [\alpha]\) Hölder-continuous.

The Lebesgue spaces \(L^p(N)\), for \(p \geq 1\), are the usual spaces defined with respect to the measure \(d\mu = y^{-d-1}dyd\theta d\text{vol}(\zeta)\) induced by the metric. For \(s \in \mathbb{R}\), we define (via the spectral theorem):
\[
\|f\|_{H^s(N)} := \|(-\Delta + 1)^s f\|_{L^2(N)},
\]
and \(H^s(N)\) is the completion of \(C^\infty(N)\) with respect to this norm. We will abuse notations, and denote by \(y\) also a smooth extension to \(N\) of the coordinates defined in the cusps; we will assume this extension is positive. For the reader to get familiar with these spaces, let us mention the following embedding lemmas.

**Lemma 2.1.** Let \(0 \leq s < s' < 1\) and \(\rho - d/2 < \rho'\). Then \(y^\rho C^{s'}(N) \hookrightarrow y^{\rho'} H^s(N)\) is a continuous embedding.

**Lemma 2.2.** Let \(k \in \mathbb{N}, s > \frac{d+1}{2} + k\). Then \(y^{-d/2} H^s(N) \hookrightarrow C^k(N)\) is a continuous embedding.

The shift by \(y^{d/2}\) will often appear throughout the article and is due to the fact that Sobolev spaces are built from the \(L^2\) space induced by the hyperbolic measure \(dyd\theta d\text{vol}(\zeta)/y^{d+1}\). We will prove (and even refine) these embedding lemmas in Section §4.3.

2.3. **Pseudo-differential operators on cusps.** Before we can start the proof of the Theorem, we have to introduce some spaces and some algebras of operators. We want to consider the action of operators on sections of \(L \to N\) or more generally from sections of \(L_1 \to N\) to sections of \(L_2 \to N\) where \(L_{1,2}\) are admissible bundles. In the paper [GW17], an algebra of semi-classical operators was described using results from [Bon16]; it consisted of families of operators depending on a small parameter \(h > 0\). In this paper, most of the time, we will be using classical operators, which is equivalent to fixing the value of \(h\) to 1.
To describe the class we will be using, it will suffice to say which types of smoothing remainders we will allow, and which quantization we will manipulate.

Our class of smoothing operators will be the class \( \Psi^{-\infty}_{\text{small}}(L_1, L_2)(= \Psi^{-\infty, L_2}_\text{small}(L_1, L_2)) \) of operators \( R \) that are bounded as

\[
R: y^\rho H^{-N}(N, L_1) \to y^\rho H^N(N, L_2),
\]

for any \( \rho \in \mathbb{R}, N \geq 0 \). These are called \( L^2\)-small smoothing operators.

In the compact part, we will use usual pseudo-differential operators with symbols \( \sigma \) in the Kohn-Nirenberg class, satisfying usual estimates of the form

\[
|\partial_x^\alpha \partial_\xi^\beta \sigma| \leq C_{\alpha, \beta} (\xi)^{m-|\beta|}.
\]

It suffices now to explain what we will be calling a pseudo-differential operator in the \textit{ends}. For this, we consider one end, and we drop the \( \ell \)'s. Instead of quantizing \( Z_\ell \), we work with the full cusp \( Z \).

Let us denote by \( \text{Op}^w \) the usual Weyl quantization on \( \mathbb{R}^{d+1} \times \mathbb{R}^k \). Given \( \chi \in C_c^\infty \) equal to 1 around 0, and \( a \in S'(\mathbb{R}^{2d+2k+2}) \), we denote by \( \text{Op}^w(a)_\chi \) the operator whose kernel is

\[
K(y, \theta, x; y', \theta', x') = \chi \left[ \frac{y'}{y} - 1 \right] K_{\text{Op}^w(a)}(y, \theta, x; y', \theta', x').
\]

Next, we can associate \( a \in C^\infty(T^*(Z \times \mathbb{R}^k), \mathcal{L}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2})) \) with its periodic lift

\[
\tilde{a} \in C^\infty(T^*(\mathbb{R}^d \times \mathbb{R}_y^d \times \mathbb{R}_\xi^k), \mathcal{L}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2})).
\]

(supported for \( y > 0 \)). Linear changes of variable have an explicit action on the Weyl quantization on \( \mathbb{R}^{d+1+k} \). We deduce that if \( f \in C^\infty(Z \times \mathbb{R}^k, \mathbb{R}^{n_1}) \), denoting by \( \tilde{f} \) the periodic lift to \( \mathbb{R}^{d+1} \times \mathbb{R}^k \), \( \text{Op}^w(\tilde{a})_\chi \tilde{f} \) is again periodic. In particular, \( \text{Op}^w(\tilde{a})_\chi \) defines an operator from compactly supported smooth sections of \( \mathbb{R}^{n_1} \to Z \times \mathbb{R}^k \) to distributional sections of \( \mathbb{R}^{n_1} \to Z \times \mathbb{R}^k \).

As a consequence, it makes sense to set

\[
\text{Op}_{\mathbb{R}^d}(a) f = y^{(d+1)/2} \text{Op}^w(a)_\chi [y^{-(d+1)/2} f].
\]

Using a partition of unity on \( F_\ell \), we can globalize this to a Weyl quantization \( \text{Op}_{\mathbb{R}^{n_1}, L_1 \to L_2}^w \), and then on the whole manifold \( \text{Op}_{\mathbb{R}^{n_1}, L_1 \to L_2}^w \) — the arguments in [Zwo12, Section 14.2.3] apply. We will write Op this Weyl quantization on the whole manifold. Since \( F \) is compact, one check that the resulting operators are uniformly properly supported above each cusp.

Now, we need to say more about the symbol estimates that we will require. By \( \langle \xi \rangle \), we refer to the Japanese bracket of \( \xi \) with respect to the natural metric \( g^* \) on \( T^* N \), which is equivalent to \( g_{z_1}^* + g_{F_\ell}^* \). We denote by \( Y, J, \eta \) the dual variables to \( y, \theta, \zeta \). In the case \( F_\ell \) is a point, \( \langle \xi \rangle = \sqrt{1 + y^2 |\zeta|^2} \).
Definition 2.2. A symbol of order $m$ is a smooth section $a$ of $\mathcal{L}(L_1, L_2) \to T^*N$, that satisfies the usual estimates over $N_0$, and above each $N_\ell$, and in local charts in $F_\ell$, for each $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$, there is a constant $C > 0$:

$$\left| (y \partial_y)^\alpha (y \partial_\theta)^\beta (\partial_\xi)^\gamma (y^{-1} \partial_j)^{\alpha'} (y^{-1} \partial_\eta)^{\beta'} (\partial_\eta)^{\gamma'} a \right|_{\mathcal{L}(L_1, L_2)} \leq C \langle \xi \rangle^{m-\alpha'-|\beta'|-|\gamma'|}.$$

This does not actually depend on the order in which the derivatives were taken. We write $a \in S^m(T^*N, \mathcal{L}(L_1, L_2))$.

We denote by $\Psi^m_{\text{small}}(N, L_1 \to L_2) (= \Psi^{m, L^2}_{\text{small}}(N, L_1 \to L_2))$ the class of operators of the form

$$\text{Op}(a) + R,$$

with $R \in \Psi^{-\infty}_{\text{small}}$ and $a \in S^m$.

2.4. Microlocal calculus. The following basic results hold

Proposition 2.1. Consider $a \in S^m(T^*N, \mathcal{L}(L_1, L_2))$, and $b \in S^{m'}(T^*N, \mathcal{L}(L_2, L_3))$. Then

1. $\text{Op}(a)$ is continuous from $y^\rho H^s(N, L_1)$ to $y^\rho H^{s-m}(N, L_2)$ for all $s, \rho \in \mathbb{R}$.
2. $\text{Op}(a) \text{Op}(b) \in \Psi^{m+m'}_{\text{small}}$, and

$$\text{Op}(a) \text{Op}(b) = \text{Op}(ab) + O_{\Psi^{m+m'-1}_{\text{small}}}(1).$$

Proof. So far, we can only do the proof of (1) in the case that $s, m$ are integers because we do not know the nature of the operator $(-\Delta + 1)^s$. Using classical results in the compact part, we can restrict our attention to the cusps, and further to the case of $\text{Op}_{\mathbb{R}^k}$. The case when $k = 0$ was dealt with in [Bon16]. As was explained in Appendix A of [GW17], the proofs therein adapt readily to the case $k \geq 1$. We will come back to the case that $s, m \notin \mathbb{Z}$ at the end of this subsection. □

Definition 2.3. Let $a \in S^m(T^*N, \mathcal{L}(L_1, L_2))$. We will say that $a$ is uniformly elliptic in there is a constant $c > 0$ such that for every $(x, \xi) \in T^*N$ with $\|\xi\| > 1/c$ and $u \in L^1_{1,x}$,

$$\|a(x, \xi)u\|_{L^2_{x,2}} \geq c\langle \xi \rangle^m_x \|u\|_{L^1_{1,x}}.$$

Proposition 2.2. Let $a \in S^m(T^*N, \mathcal{L}(L_1, L_2))$ be uniformly elliptic. Then we can find $Q \in \Psi^{-m}(N, \mathcal{L}(L_2, L_1))$ such that

$$Q \text{Op}(a) = 1 + R,$$

with $R \in \Psi^{-\infty}_{\text{small}}(N, \mathcal{L}(L_1, L_1))$.

Before going on with the proof, observe that the remainder here is not a compact operator, contrary to the case of a compact manifold.
Proof. Here, we can apply the usual parametrix construction. First one can choose a \( q_0 \in S^{-m}(T^*N, \mathcal{L}(L_2, L_1)) \) such that for \( \|\xi\| > 2/c, \)
\[
q_0 a = 1_{L_1}
\]
Then
\[
\text{Op}(q_0) \text{Op}(a) = 1 + \text{Op}(r_1) + R_1.
\]
Here, \( r_1 \in S^{-1}(T^*N, \mathcal{L}(L_1)) \), and \( R_1 \) is small smoothing. Then
\[
\text{Op}((1 - r_1)q_0) \text{Op}(a) = 1 + \text{Op}(r_2) + R_2,
\]
where \( r_2 \in S^{-2}(T^*N, \mathcal{L}(L_1)) \) and \( R_2 \) is again small smoothing. Now, we can iterate this construction, and find a formal solution \( \text{Op}(\tilde{q}) \) with
\[
\tilde{q} = q_0 - \sum_{i \geq 0} r_i q_0.
\]
(the sum is formal, it does not converge). Then, by means of a Borel summation, one can find an actual symbol \( q \in S^{-m}(T^*N, \mathcal{L}(L_2, L_1)) \) such that as \( |\xi| \to \infty \), uniformly in \( x \),
\[
q \sim q_0 - \sum_{i \geq 0} r_i q_0.
\]
As a consequence one gets
\[
\text{Op}(q) \text{Op}(a) = 1 + R,
\]
with \( R \) small smoothing.

We can now prove Proposition 2.1.

Proof of Proposition 2.1. The Laplacian defined by the Friedrichs extension of the quadratic form
\[
\int_N g^L(-\Delta^L f, f) d\text{vol}_g = \int_N \|\nabla \cdot f\|^2
\]
is uniformly elliptic. Given \( N > 0 \), by adding a large constant \( h_N^{-2} \), we can obtain a symbol \( \sigma \) such that
\[
\text{Op}(\sigma)(-\Delta^L + h_N^{-2}) = h_N^{-2} 1_{L} + O_{\Psi_{\text{small}}}(1).
\]
Following arguments as in the proof of [Zwo12, Theorem 14.8, p.358], one deduces that for each \( s \in \mathbb{R} \), there is a uniformly elliptic symbol \( \sigma_s \) of order \( s \) such that
\[
H^s(N, L) = \text{Op}(\sigma_s)L^2(N, L),
\]
with equivalent norms. Together with the product stability of pseudo-differential operators, this finishes the proof of Proposition 2.1.

In the following, we will write \( \Lambda_{-s} = \text{Op}(\sigma_s) \).
2.5. **Fibred cusp calculus.** To study Fredholm properties of differential operators on ends of the type (1), the so-called *fibred-cusp calculus* was introduced by Mazzeo and Melrose in [MM98]. We will explain here why it does not suit our needs entirely, reason for developing our arguments from scratch.

The algebra of pseudo-differential operators we have just introduced is an extension of an algebra of *differential* operators. The latter is itself the algebra generated by \( \mathcal{V}_0 \), the Lie algebra of vector fields of the form

\[ ay\partial_y + by\partial_\theta + X(\partial_\zeta), \]

where the coefficients \( a, b, X \) are \( C^\infty \)-bounded on \( Z \times F \). A crucial observation is that the Laplacian associated to the metric of \( Z \times F \) is in this algebra.

Let us recall on the other hand the setup of the *Fibred-Cusp Calculus* developed by Mazzeo-Melrose [MM98]. We have a manifold \( N' \) whose boundary has a finite number of components. Those have a neighbourhood of the form

\[ [0,\epsilon]_u \times X, \]

with a bundle map \( p : X \to F_\zeta \). The generic coordinate in \( p^{-1}(\zeta) \) is denoted \( \theta \). The fibred cusp algebra \( \Psi_{fc}^{\text{diff}} \) is the algebra of differential operators generated by the algebra \( \mathcal{V}_{fc} \) of vector fields of the form

\[ au^2\partial_u + bu\partial_\zeta + c\partial_\theta, \]

where \( a, b, c \) are \( C^\infty \) functions of \( u, \zeta, \theta \) (including at \( u = 0 \)). In our case, with \( u = 1/y \), we can see that if \( V \in \mathcal{V}_0 \), \( uV \in \mathcal{V}_{fc} \). However, if \( V \in \mathcal{V}_{fc} \), \( (1/u)V \) is not necessarily in \( \mathcal{V}_0 \).

The purpose of [MM98] was to analyze whether operators in \( \Psi_{fc}^{\text{diff}} \) have parametrices modulo compact remainders when acting on \( L^2(N') \). This involves the inversion of an *indicial operator*, which is a family of operators \( \hat{P}(\zeta, \eta) \), parametrized by \( (\zeta, \eta) \in T^*F \), acting on the fiber \( p^{-1}(\zeta) \) (here \( \mathbb{R}^d/\Lambda \)).

If \( P \) is a differential operator of order \( m \) in our class, \( u^mP \in \Psi_{fc}^{\text{diff}} \), so one could apply the results in [MM98]. However, here follows two reasons why this is not satisfying for our purposes.

- In the case that \( P \) is not differential, but pseudo-differential of varying order, it is not quite obvious what would replace the correspondence \( P \mapsto u^mP \). This is crucial when dealing with anisotropic spaces as in [GW17]. This will intervene in our second paper.
- We are able to deal with Hölder-Zygmund spaces (instead of \( L^2(N') \)). As far as we know, this has not been done before with fibred-cusp calculus.

Since we are dealing with a much smaller class than the whole fibred cusp calculus, the criterion for being Fredholm is also simpler. Indeed, we only need to invert a
family of operators $I(P, \lambda)$, with $\lambda \in i\mathbb{R}$, each such operator acting on $F$ (the base instead of the fiber).

In the general case of the fibred cusp calculus, one does not require that the fibers $p^{-1}(\zeta)$ are flat manifolds. Let us explain why this is crucial in our case. The central point is to have a space of vector fields that is stable under Lie brackets (a Lie algebra). If $yX_1$ and $yX_2$ are two vector fields tangent to the fibers, so that $X_1$ and $X_2$ a smooth up to the boundary, we compute

$$[yX_1, yX_2] = y^2 [X_1, X_2].$$

In particular, we can only allow vector fields $X_1, X_2$ such that their Lie bracket are $O(1/y)$ as $y \to +\infty$. If we also require that they do not all vanish themselves as $y \to +\infty$, this is a very strong condition on the fibers. It probably implies that the curvature of the fibers goes to 0 as $y \to +\infty$.

This was the reason for Mazzeo and Melrose to study the algebra $V_{fc}$. It also suggests that our techniques could be extended to the fibred cusp case, with the assumption that there are family of vector fields in the fibers $p^{-1}(\zeta)$ which are asymptotically parallel. This would be verified if these fibers are almost flat manifolds. For example the case of complex-hyperbolic cusps. We leave this to future investigations, and refer to [Gro78] and [BBC12].

To close this section, let us explain why it should not be surprising that the fibred cusp calculus does not behave very well with propagators. Indeed, consider some propagator $e^{itP}$. In its microlocal properties, the hamilton flow of the principal symbol of $P$ will appear. It is then important that the class of symbols considered is stable under the action of this flow. In the compact case, to prove such a statement, one relies on the usual statement that if $\varphi$ is a smooth flow, there is some $\lambda > 0$ such that for $t \in \mathbb{R}$.

$$\|f \circ \varphi_t\|_{C^m} \leq C_n e^{\lambda |t|} \|f\|_{C^m}.$$  

However, the proof of this statement on a manifold uses crucially the fact that the metric has bounded curvature, and bounded covariant derivatives of its curvature tensor. The crux of the problem is then that the curvature of a metric in the form

$$\frac{dy^2 + g_{y,\theta,\zeta}(d\theta)}{y^2} + g_{y,\zeta}(d\zeta)$$

does not even have bounded curvature in general.

In particular, there is no reason that propagators of general fibered-cusp operator propagate singularities in a nice fashion at infinity. The examples built in [DPPS15] show even that in the case that the curvature, or its derivatives, are not bounded, new dynamical phenomenon appear.
3. Parametrices modulo compact operators: Sobolev case.

3.1. Black-box formalism. Here again, we follow arguments exposed in [GW17].

Associated to each cusp $Z$, we have extension and restriction operators defined in the following way. Start by letting

$$\Pi_Z f := \int f \, d\theta.$$  

Given $f \in \mathcal{D}'(]a, +\infty[ \times F_Z, L_Z)$, we obtain a distribution $E_Z f \in \mathcal{D}'(N, L)$ by setting

$$E_Z f(\phi) = f(\Pi_Z \phi).$$

Conversely, given $f \in \mathcal{D}'(N, L)$, we obtain $P_Z f \in \mathcal{D}'(]a, +\infty[ \times F_Z, L_Z)$ by setting

$$P_Z f(\phi) = f(E_Z \phi).$$

Given $\chi \in C^\infty([a, +\infty[)$ which is locally constant around $a$, we define

$$Z(\chi) f := \sum_Z \chi(a)(1 - E_Z P_Z) f + E_Z(\chi P_Z f).$$

The operators $E_Z$, $P_Z$ and $Z(\chi)$ together form a black box formalism, as it was introduced by Sjöstrand and Zworski in [SZ91].

**Definition 3.1.** We pick a function $\tilde{y} \in C^\infty([a, +\infty[)$ such that $\tilde{y}(y) = y$ for $y > 3a$, and $\tilde{y}(y < 2a) = 1$. Then we define for $s, \rho_0, \rho_\perp \in \mathbb{R}$,

$$H^{s, \rho_0, \rho_\perp}(N, L) = Z(\tilde{y}^{\rho_0 - \rho_\perp})(y^{\rho_\perp} H^s).$$

These are weighted Sobolev spaces, with weight $y^{\rho_0}$ on the zero Fourier mode and weight $y^{\rho_\perp}$ on the non-zero Fourier modes.

Note that we take the same weight on each cusps, this will suffice for our purposes. To obtain compact remainders in parametrices, the following observation going back to [LP76] is essential: for any $\rho_\perp \in \mathbb{R}, s > s'$, the restriction of the injection $y^{\rho_\perp} H^s(N, L) \hookrightarrow y^{\rho_\perp} H^{s'}(N, L)$ to non-constant Fourier modes is compact.

**Lemma 3.1.** If $\chi \in C^\infty([a, +\infty[)$ is a smooth cutoff function such that $\chi \equiv 1$ for $y > 2a$ and vanishing around $y = a$, then for all $s > s'$:

$$1 - E_Z \chi P_Z : H^{s, \rho_0, \rho_\perp}(N, L) \to H^{s', -\infty, \rho_\perp}(N, L)$$

is compact.

By this, we mean that for any $N > 0$, the operator

$$1 - E_Z \chi P_Z : H^{s, \rho_0, \rho_\perp}(N, L) \to H^{s', -N, \rho_\perp}(N, L)$$

is compact.
Proof. The value of $\rho_0$ is inessential here, so we take $\rho_0 = \rho_\perp = \rho$. Since $[1 - \mathcal{E}_Z\chi\mathcal{P}_Z, y^\rho] = 0$ sufficiently high in the cusp, the lemma boils down to the case $\rho = 0$. For the sake of simplicity, we assume that there is a single cusp and that $L \to N$ is the trivial bundle $N \times \mathbb{R} \to N$, the general case is handled in a similar fashion. Let $\psi_n \in C_c^\infty(N)$ be a smooth cutoff function such that $\psi_n \equiv 1$ on $y < n$ and $\psi_n \equiv 0$ on $y > 2n$. The operators of injection

$$T_n := \psi_n(1 - \mathcal{E}_Z\chi\mathcal{P}_Z) \in \mathcal{L}(H^s(N), H^{s'}(N))$$

are compact, so it is sufficient to prove that the injection

$$T := 1 - \mathcal{E}_Z\chi\mathcal{P}_Z \in \mathcal{L}(H^s(N), H^{s'}(N))$$

is the norm-limit of the operators $T_n$. In other words, if we can prove that for all $n \in \mathbb{N}$, there exits a constant $C_n > 0$ such that: for all $f \in H^s(N)$ such that $\chi\mathcal{P}_Zf \equiv 0$ (we denote by $H^s_\delta(N)$ the space of such functions endowed with the norm $\| \cdot \|_{H^s}$), we have

$$\|(1 - \psi_n)f\|_{H^{s'}} \leq C_n\|f\|_{H^s},$$

and that $C_n \to_{n \to +\infty} 0$, then we are done. Using Wirtinger’s inequality, one can obtain like in [GW17, Lemma 4.9] that

$$\|1 - \psi_n\|_{\mathcal{L}(H^s_\delta, H^{s'}_\delta)} \leq C/n$$

for some uniform constant $C > 0$ (depending on the lattice $\Lambda$). Since we trivially have $\|1 - \psi_n\|_{\mathcal{L}(H^s_\delta, H^{s'}_\delta)} \leq 1$, we obtain by interpolation that $\|1 - \psi_n\|_{\mathcal{L}(H^s_\delta, H^{s'}_\delta)} \leq (C/n)^{1-s}$ for all $s \in [0, 1]$. Since $\|1 - \psi_n\|_{\mathcal{L}(H^s_\delta, H^{s'}_\delta)} \leq 1$ for all $k \in \mathbb{Z}$, we can interpolate once again to conclude.

Lemma 3.2. Consider $\rho_\perp \in \mathbb{R}$, $\rho_0 < \rho_\perp$, and $s > s'$. Then $H^{s, \rho_0, \rho_\perp}(N, L) \hookrightarrow H^{s', \rho_0, \rho_\perp}(N, L)$ is a compact injection.

Proof. One can write $f = (1 - \mathcal{E}_Z\chi\mathcal{P}_Z)f + \mathcal{E}_Z\chi\mathcal{P}_Zf$. The first term is dealt by applying the previous lemma. As to $\mathcal{E}_Z\chi\mathcal{P}_Zf$, this is a classical lemma on $\mathbb{R}$. □

Eventually, we will need this last lemma:

Lemma 3.3. Consider $\rho_\perp, \rho'_\perp \in \mathbb{R}$, $\rho_0 \in \mathbb{R}, s, s' \in \mathbb{R}$ such that $s > s', \rho_\perp > \rho'_\perp$. Then $H^{s, \rho_0, \rho_\perp} \hookrightarrow H^{s', \rho_0, \rho'_\perp}$ is a continuous embedding.

Proof. Once again, decomposing in zero and non-zero Fourier modes and using interpolation estimates, it is sufficient to prove that $yH^1 \hookrightarrow L^2$ is a continuous embedding on functions with zero Fourier mode. But:

$$\|f\|_{H^1}^2 = \|y^{-1}f\|_{H^1}^2 \leq \|y^{-1}f\|_{L^2}^2 + \|y\partial_y(y^{-1}f)\|_{L^2}^2 + \|y\partial_\theta(y^{-1}f)\|_{L^2}^2 + \|\partial_\xi(y^{-1}f)\|_{L^2}^2$$
Using Wirtinger's inequality for functions with zero integral, we can control the term \( \|y\partial_y(y^{-1}f)\|_{L^2}^2 = \|\partial_y f\|_{L^2}^2 \geq \|f\|_{L^2}^2 \) and this provides the sought estimate. \( \square \)

It will be more convenient for zeroth Fourier modes to use the variable \( r = \log y \). The following lemma is crucial:

**Lemma 3.4.** Consider \( \chi \in C^\infty([a, +\infty[) \), constant for \( y > 2a \), and vanishing around \( y = a \). Then the following maps are bounded

\[
H^{s,\rho_0,\rho_1}(N, L) \ni f \mapsto \chi P_Z f \in e^{(\rho_0+d/2)r} H^s(\mathbb{R} \times F_Z, L_Z);
\]

\[
e^{\rho r} H^s(\mathbb{R} \times F_Z, L_Z) \ni f \mapsto E_Z(\chi f) \in H^{s,\rho_0-d/2,-\infty}(N, L),
\]

where \( r = \log y \), and \( H^s(\mathbb{R} \times F_Z, L) \) is the usual Sobolev space, built from the \( L^2 \) space induced by the measure \( drd\text{vol}_{F_Z}(\zeta) \).

We insist on the fact that there is a shift of \( -d/2 \) due to the fact that we are considering the usual euclidean measure when working in the \( r \)-variable. We will prove this below after Proposition 3.1.

### 3.2. Admissible operators

We can now introduce the class of admissible operators.

**Definition 3.2.** Consider \( A \in \Psi^m_{\text{small}}(N, L(L_1, L_2)) \) and \( I^m_Z(A) \in \Psi^m(\mathbb{R} \times F_Z, L_Z) \) a convolution operator in the \( r \)-variable. We will say that \( A \) is a \( \mathbb{R}\text{-}L^2\)-admissible operator with indicial operator \( I^m_Z(A) \) if the following holds. There exists a cutoff function \( \chi \in C^\infty([a, +\infty[) \) (depending on \( A \)), such that \( \chi \) is supported for \( y > 2a \), equal to 1 for \( y > C \) for some \( C > 2a \),

\[
\chi[A, \partial_y] \chi \text{ and } E_Z \chi [P_Z A E_Z - I^m_Z(A)] \chi P_Z,
\]

are operators bounded from \( y^N H^{-N} \) to \( y^{-N} H^N \), for all \( N \in \mathbb{N} \). The operator \( I^m_Z(A) \) is independent of \( \chi \).

When \( \rho > \rho' \), the unique convolution operator that is bounded from \( e^{\rho r} L^2(dr) \) to \( e^{\rho' r} L^2(dr) \) is the null operator. It follows that the indicial operator associated to a \( L^2 \) admissible operator is necessarily unique. Modulo compact remainders, the first condition in (5) mean that the operator \( A \) preserves the \( \theta \)-Fourier modes; the second condition implies that sufficiently high in the cusp, \( A \) is a convolution operator in the \( r = \log y \) variable when acting on the zeroth Fourier mode. In particular, if \( B \) is a compactly supported pseudodifferential operator, \( B \) is admissible, and \( I^m_Z(B) = 0 \).

Observe that in general, if \( P \in \Psi^m \), then in the cusp, \( \chi[P, \partial_y] \chi \) is in \( y^{-\infty} \Psi^m \). Indeed, its symbol can be expressed with derivatives of the symbol of \( P \), that include at least one derivative \( \partial_y \). However, if \( \sigma \in S^m \), \( \partial_y \sigma \in y^{-\infty} S^m \). What we gain with our assumption is that the order becomes \( -\infty \).
An important consequence of the definition is that if $A$ is admissible, then
\begin{equation}
\chi P_Z A [1 - E_Z \chi P_Z], \text{ and } \chi [1 - E_Z P_Z] A P_Z \chi
\end{equation}
both are continuous from $y^N H^{-N}$ to $y^{-N} H^N$. For the first one, let $K$ be the inverse of $\partial_\theta$ in $\{f \in L^2(\mathbb{R}^d/\Lambda), \int f = 0\}$. Abusing notation a little, we consider its action on the cusps; it is then bounded on every $H^{s, \rho, \rho \perp}$. Then
\[0 = \partial_\theta \chi P_Z A [1 - E_Z P] \chi K = \chi P_Z A [1 - E_Z P] \chi + \chi P_Z [\partial_\theta, A] [1 - E_Z P] \chi K,
\]
which proves the first assertion in (6) by using the assumption (5) on $[\partial_\theta, A]$. However the conditions in (6) are not necessarily stable under products, nor under taking parametrices.

**Proposition 3.1.** Consider $A = \text{Op}(\sigma)$. Then the first operator in equation (5) satisfies the required conditions if $\partial_\theta \sigma = 0$. Additionally, the second one also does in each cusp if,
\[\tilde{\sigma} : (r, z; \lambda, \eta) \mapsto \int \sigma|_Z (e^r, \theta, \zeta; e^{-r} \lambda, J = 0, \eta) d\theta,
\]
does not depend on $r$. In that case, the operator $I_Z(A)$ is pseudo-differential, properly supported, and its principal symbol is $\tilde{\sigma}$. Both these conditions are satisfied when $\sigma$ is invariant by local isometries of the cusp.

Finally, an operator $A$ is $L^2$ admissible if and only if it is of the form $\text{Op}(\sigma) + B + R$, where $\sigma$ satisfies the conditions above, $R$ is $L^2$ admissible smoothing, and $B$ is a compactly supported pseudo-differential operator. We deduce that the set of $L^2$ admissible operators is stable by composition.

From the decomposition (2), we deduce that $\nabla^L$ is a geometric operator. More generally, all the differential operators that can be defined completely locally using only the metric structure are bound to be properly supported geometric operators. For example, the Laplacian or the Levi-Civita connection. In the following, the operators $D$ and $D^* D$ will be local differential operators, so they will be properly supported geometric operators in the sense of the previous definition.

**Proof of Proposition 3.1.** Again, it suffices to work directly with $\text{Op}_U$ on $Z \times U$. First, we observe that when $\partial_\theta \sigma = 0$, $\text{Op}_U(\sigma)$ commutes with $\partial_\theta$. Reciprocally, if $[\partial_\theta, \text{Op}(\sigma)]$ is bounded from $y^N H^{-N}$ to $y^{-N} H^N$, it implies that $\partial_\theta \sigma \in y^{-\infty} S^{-\infty}$. In particular, we can replace $\sigma$ by $\int \sigma d\theta$, and this only adds a negligible correction. For the second condition, one has to do a change of variables. For details, we refer to [GW17, Section 4.1].

Now, we can prove Lemma 3.4.
Proof of Lemma 3.4. Recall that \( H_s = \Lambda_s L^2 \) with \( \Lambda_s = \text{Op}(\sigma_s) \). Since the symbols \( \sigma_s \) were built in a parametrix construction for the Laplacian, they are invariant under local isometries. In particular, \( \Lambda_s \) is \( \mathbb{R} \)-admissible. Additionally, its restriction to the zeroth Fourier mode acts as an pseudo-differential operator of order \( s \) (as will be seen in detail in Lemma 4.11). Since it is uniformly properly supported, conjugation with \( y^\rho \) does not change these properties. It follows that one can restrict to the case that \( s = \rho_0 = \rho_\perp = 0 \).

Now, it boils down to the observation that the volume measure on the cusp is
\[
y^{-d-1}d\theta dy = e^{-rd}d\theta dr \quad \text{with} \quad r = \log y.
\]

\[\Box\]

Lemma 3.5. Let \( A \) be an admissible pseudodifferential operator of order \( m \in \mathbb{R} \). Then \( A \) is bounded as an operator between \( H^{s+m,\rho_0,\rho_\perp} \) and \( H^{s,\rho_0,\rho_\perp} \), for all \( s, \rho_0, \rho_\perp \in \mathbb{R} \).

Proof. We decompose the operator in four terms:
\[
A = (1 - E_Z \chi_P Z) A (1 - E_Z \chi_P Z) f + E_Z \chi_P Z A (1 - E_Z \chi_P Z) f + (1 - E_Z \chi_P Z) A E_Z \chi_P Z f + E_Z \chi_P Z A E_Z \chi_P Z f
\]
The first term is bounded as a map \( H^{s+m,\rho_0,\rho_\perp} \rightarrow H^{s,\rho_0,\rho_\perp} \), where we have used the boundedness of \( A \) obtained in Proposition 2.1. By (6), the second and third terms are immediately bounded. As to the last term, it is dealt exactly like the first term. \[\Box\]

3.3. Indicial resolvent. Let us consider a \( \mathbb{R} \)-L^2 admissible operator \( A \) of order \( m \), and introduce
\[
I_Z(A, \lambda) f(\zeta) = e^{-\lambda r} I_Z(A) \left[ e^{\lambda r'} f(\zeta') \right]
\]
Since \( A \) is small, this defines a holomorphic family of operators on \( F_Z \); it is called the Indicial family associated to \( A \).

Lemma 3.6. The Indicial family is a homomorphism in the sense that for all \( \mathbb{R} \)-L^2 admissible operators \( P \) and \( Q \), and for all \( \lambda \in \mathbb{C} \),
\[
I_Z(PQ, \lambda) = I_Z(P, \lambda) I_Z(Q, \lambda) \quad I_Z(P + Q, \lambda) = I_Z(P, \lambda) + I_Z(Q, \lambda)
\]
Proof. The only non-trivial part of this statement is that if \( P, Q \) are admissible, 
\[ I_Z(PQ) = I_Z(P)I_Z(Q). \]
To this end, we write (abusing notations for an instant)
\[
P_ZPQ\varepsilon_Z = P_ZP(\varepsilon_ZP_Z + 1 - \varepsilon_ZP_Z)Q\varepsilon_Z \\
= P_ZP\varepsilon_ZP_Z\varepsilon_Z \quad \text{mod (compact)} \\
= I_Z(P)I_Z(Q) \quad \text{mod (compact)}.
\]
\[ \square \]

Lemma 3.7. Assume \( A \) is an elliptic \( \mathbb{R} \)-\( L^2 \) admissible operator of order \( m \). Then 
for each \( \lambda \in \mathbb{C} \), \( I_Z(A, \lambda) \) is an elliptic pseudo-differential operator of order \( m \), and 
\( I_Z(A, \lambda)^{-1} \) is a meromorphic family of pseudo-differential operators of order \( -m \). Its 
poles are called indicial roots of \( A \) (at \( Z \)). The set 
\[ \{ \Re s \mid s \text{ is an indicial root} \} \]
is discrete in \( \mathbb{R} \).

Proof. The fact that \( I_Z(A, \lambda) \) is a pseudo-differential operator follows from a direct 
computation. One can actually compute the principal symbol of \( I_Z(A, \lambda) \). It does 
not depend on \( \lambda \):
\[
z, \eta \mapsto \sigma(A)(e^r, \theta, \xi, 0, 0, \eta).
\]
In particular, if \( A \) was elliptic, so is \( I_Z(A, \lambda) \). However, we will need some uniformity 
in the ellipticity. We can assume that \( A \) decomposes as \( \text{Op}(\sigma) + R \) (the compactly 
supported pseudo-differential operator does not contribute to the indicial family).
Let us deal with both parts separately. Let us write
\[
I_Z(R)f(r, \zeta) = \int_{\mathbb{R} \times F_Z} K(r - r', \zeta, \zeta') f(r', \zeta') dr' d\zeta',
\]
so that the kernel of \( I_Z(R, \lambda) \) is
\[ \hat{K}(-i\lambda, \zeta, \zeta'), \]
the Fourier transform being taken in the first variable. Since \( R \) is smoothing and 
\( \mathbb{R} \)-\( L^2 \) admissible, for any \( N, k > 0, \rho \in \mathbb{R} \) and \( T > 0 \), we let \( u(r, \zeta) = e^{-\rho r} (-1)^k \delta(r - T)\delta^{(k)}(\zeta, \zeta'') \). Then,
\[
e^{\rho r}P_ZR\varepsilon_Zu = e^{\rho r}I_Z(R)u + O_{H^{N,-\infty}}(1)
\]
The left hand side is valued in all \( H^{N,-d/2} \), \( N > 0 \), with bounds uniform in \( \zeta'' \).
According to Lemma 4.8, it is thus contained in \( C^k \), \( k \geq 0 \). However, the first term 
in the RHS is \( e^{\rho(r-T)} \delta^{(k)}(r - T, \zeta, \zeta'') \). With \( r = r_0 + T, r_0 \) fixed, and \( T \to +\infty \),
we deduce that for all \( \rho \in \mathbb{R} \) \( e^{\rho r}K(r, \zeta, \zeta') \) is \( C^k \) (in the Banach sense).

Estimating thus the Fourier transform, we deduce that \( I_Z(R, \lambda) \) is a \( O((1 + |\Re \lambda|)^{-\infty}) \) Sobolev-smoothing operator on \( L_Z \to F_Z \), locally uniformly in \( \Re \lambda \), in
the sense that for all $N \in \mathbb{N}$, for all $s, s' \in \mathbb{R}$, for all $a < b$ there exists a constant $C_{N,s,s',a,b} > 0$ such that $\|I_Z(R, \lambda)\|_{L(H^s, H^{s'})} \leq \frac{C_{N,s,s',a,b}}{(1 + |\Re \lambda|)^N}$ for all $a < \Re \lambda < b$.

We now consider a general $L^2$ admissible operator $A$, like in Proposition 3.1. Let $Q$ be a parametrix for $I$ so that it is $L^2$-admissible. Then, by Lemma 3.6, $I_Z(A, \lambda)I_Z(Q, \lambda) = 1 + I_Z(R, \lambda)$. From the discussion before, $1 + I_Z(R, \lambda)$ is an analytic Fredholm family, which is eventually invertible when $|\Im \lambda|$ becomes large. It satisfies the assumptions of the Fredholm Analytic theorem. As a consequence, $I_Z(Q, \lambda)(1 + I_Z(R, \lambda))^{-1}$ is a meromorphic family of bounded operators, and where it is bounded, it is equal to $I_Z(A, \lambda)^{-1}$. □

Now, we want to invert $I_Z(A)$ from the knowledge of $I_Z(A, \lambda)^{-1}$. Pick a $\rho \in \mathbb{R}$ such that $I_Z(A, \lambda)^{-1}$ has no pole on $\{\Re \lambda = \rho\}$, and consider the operator $S_\rho$ whose kernel is

$$
\int_{\Re \lambda = \rho} e^{\lambda(r-r')}I_Z(A, \lambda)^{-1}d\lambda.
$$

Then one finds that $S_\rho$ is bounded from $e^{\rho r}H^s(\mathbb{R} \times F_Z)$ to $e^{\rho r}H^{s+m}(\mathbb{R} \times F_Z)$, and

$$
I_Z(A)S_\rho = 1.
$$

Since $I_Z(A, \lambda)^{-1}$ is holomorphic, one also get by contour deformation that $S_\rho$ does not change when $\rho$ varies continuously without crossing the real part of an indicial root, so that given a connected component $I$ of $\mathbb{R} \setminus \{\Re \lambda \mid I_Z(A, \lambda)$ is not invertible$, \}$, we denote $S_I$ the inverse.

### 3.4. General Sobolev admissible operators.

When dealing with differential operators, whose kernel is supported exactly on the diagonal, the assumption that one can work with spaces $H^{s,\rho_0,\rho_\perp}$ for any $\rho_0, \rho_\perp \in \mathbb{R}$ is not very important. However, we will be dealing with pseudo-differential operators that are not properly supported. We will also be dealing parametrices, which cannot be $\mathbb{R}$-admissible since some poles appear.

**Definition 3.3.** Let $\rho_+ > \rho_-$. We say that an operator $A$ is $(\rho_-, \rho_+) - L^2$-admissible of order $m$ if it can be decomposed as $A = A_{\text{comp}} + A_{\text{cusp}} + R$, where $A_{\text{comp}}$ is a compactly supported pseudo-differential operator of order $m$, $A_{\text{cusp}} = \text{Op}(\sigma)$ with $\sigma$ satisfying the conditions of Proposition 3.1. Finally, $R$ is $(\rho_-, \rho_+) - L^2$-smoothing admissible:

1. For all $\rho_0 \in ]\rho_-, \rho_+[; \rho_\perp \in \mathbb{R}$ and $N > 0$, $R$ is bounded from $H^{-N,\rho_0-d/2,\rho_\perp}$ to $H^{N,\rho_0-d/2,\rho_\perp}$,
2. For all $\rho_\perp \in \mathbb{R}$ and $N, \epsilon > 0$, $[\partial_\theta, R]$ is bounded from $H^{-N,\rho_+ - d/2 - \epsilon,\rho_\perp}$ to $H^{N,\rho_+ - d/2 + \epsilon,\rho_\perp}$.
(3) There is a convolution operator $I_Z(R)$ and $C > a$ such that

$$\chi_C p Z R \chi_C \mathcal{E}_Z - \chi_C I_Z(R) \chi_C$$

is an operator bounded from $e^{\rho - \varepsilon} H^{-N}(\mathbb{R} \times F_Z)$ to $e^{\rho + \varepsilon} H^{N}(\mathbb{R} \times F_Z)$ for all $N, \varepsilon > 0$.

The difference between being $\mathbb{R}$-admissible and $(\rho_-, \rho_+)$-admissible lies only in the behaviour on the zeroth Fourier mode in the cusps, where certain asymptotic behaviour is allowed. In the other Fourier modes in $\theta$, all exponential behaviours are allowed.

Each $(\rho_-, \rho_+)$-L2 admissible operator $A$ is associated with a convolution operator $I_Z(A)$ in each cusp. We can also define the indicial family $I_Z(A, \lambda)$, which is holomorphic in the strip

$$\mathbb{C}_{\rho_-, \rho_+} := \{ \lambda \in \mathbb{C}, \Re \lambda \in (\rho_-, \rho_+) \}.$$

**Proposition 3.2.** The set of $(\rho_-, \rho_+)$-L2 admissible operators is an algebra of operators, and the indicial family is also an algebra homomorphism.

The proof is the same as that of Proposition 3.1 and Lemma 3.6. The proof of Lemma 3.7 still applies, albeit in $\mathbb{C}_{\rho_-, \rho_+}$ instead of $\mathbb{C}$, so we can still define the set of indicial roots, and the indicial inverses $S_I$.

### 3.5. Improving Sobolev parametrices.

In this section, we will prove Theorem 3 in the case that the operator is $(\rho_-, \rho_+)$-L2-admissible (except the part about the Fredholm index that we will deal with in the next section). Recall that we for Proposition 2.2, we built a symbol $q$ such that $A \text{Op}(q) - 1$ and $\text{Op}(q)A - 1$ are smoothing operators. From Lemma 3.2, we deduce that it would suffice to improve $\text{Op}(q)$ only with respect to the action on the zeroth Fourier coefficient in the cusps. Since the symbol $q$ was built using symbolic calculus, we deduce directly that $\text{Op}(q)$ is $\mathbb{R}$-L2-admissible. Consider an open interval $I$ which is a connected component of $\mathbb{R} \setminus \{ \Re \lambda \mid \lambda \text{ is an indicial root} \}$,

and the corresponding inverse $S_I$ of $I_Z(A)$. Then set

$$Q_I := \text{Op}(q) + \sum_Z \mathcal{E}_Z \chi_C [S_I - I_Z(\text{Op}(q))] \chi_C p Z,$$

which is now a $I - L^2$-admissible pseudodifferential operator (indeed, we have corrected the indicial part of $\text{Op}(q)$, by an $I - L^2$ admissible smoothing operator). We now write $Q_I A = 1 + R'_I$ and we aim to prove that $R'_I$ is compact on $H^{s, \rho_0 - d/2, \rho_\perp}$ for all $s \in \mathbb{R}, \rho_0 \in I, \rho_\perp \in \mathbb{R}$. By stability by composition of admissible pseudodifferential operators (see Proposition 3.2), we know that $R'_I$ is a smoothing admissible operators. Moreover, the operator $Q_I$ was chosen so that $I_Z(R'_I) = 0$ (this can
be checked using the calculation rules of Lemma 3.6). As a consequence, thanks to Lemma 3.2, the proof of Theorem 3 (except the Fredholm properties) now boils down to the following Lemma:

**Lemma 3.8.** Let \( A \in \Psi^{-m}(N, \mathcal{L}(L)) \) be a \((\rho_-, \rho_+)-L^2\) admissible pseudodifferential operator such that \( I_Z(A) = 0 \). Then \( A \) is bounded from \( H^{s_0, \rho_0, -d/2, \rho_+} \) to \( H^{s_0, \rho_0, -d/2, \rho_+} \) for \( \rho_0, \rho_0' \in (\rho_-, \rho_+) \), \( \rho_+ \in \mathbb{R} \).

**Proof.** Let \( \chi \) be a smooth cutoff function in the cusp. Then:

\[
Af = (1 - E_Z \chi P_Z)A(1 - E_Z \chi P_Z)f + E_Z \chi P_Z A(1 - E_Z \chi P_Z)f \equiv (1 - E_Z \chi P_Z)A(1 - E_Z \chi P_Z)f
\]

By definition of being admissible, the first three terms directly satisfy the announced bounds. The last one also does since we have assumed that \( I_Z(A) = 0 \). \( \square \)

**Corollary 3.1.** The set of \( \rho_0 \in (\rho_-, \rho_+) \) for which one cannot build such a parametrix is given by the real part of the set

\[
\{ \lambda \mid \Re \lambda \in (\rho_-, \rho_+) \text{, } I_Z(A, \lambda) \text{ is not invertible} \}.
\]

### 3.6. Fredholm index of elliptic operators.

**Lemma 3.9.** For all \( s, \rho_0, \rho_+ \in \mathbb{R} \), one can identity via the \( L^2 \) scalar product the spaces \( (H^{s, \rho_0, \rho_+})' \simeq H^{-s, -\rho_0, -\rho_+} \).

**Proof.** We have to prove that the bilinear map

\[
C_c^\infty(N, L) \times C_c^\infty(N, L) \ni (u, v) \mapsto \langle u, v \rangle = \int_N g_N(u, v) d \text{vol}_N(z)
\]

extends boundedly as a map \( H^{s, \rho_0, \rho_+} \times H^{-s, -\rho_0, -\rho_+} \to \mathbb{C} \). Up to a smoothing order modification of \( \Lambda_s \) which we denote by \( \Lambda'_s \), we can assume that \( \Lambda_{-s}' \Lambda'_s = 1 \). Then, for \( u, v \in C_c^\infty(N, L) \), one has \( \langle u, v \rangle = \langle \Lambda_{-s}' \Lambda'_s u, v \rangle = \langle \Lambda_s u, \Lambda'_s v \rangle \). By Lemma 3.5, since \( \Lambda_{\pm s} \) is admissible, \( \Lambda_{\pm s} : H^{s, \rho_0, \rho_+} \to H^{0, \rho_0, \rho_+} \) is bounded. The boundedness of (7) on \( H^{0, \rho_0, \rho_+} \times H^{0, -\rho_0, -\rho_+} \to \mathbb{C} \) is immediate (these are \( L^2 \) spaces with weight \( y^\rho_0 \) on the zeroth Fourier mode and \( y^{\rho_1} \) on the non-zero modes) and thus:

\[
\| \Lambda_s u, \Lambda'_s v \| \lesssim \| \Lambda_s u \|_{H^{0, \rho_0, \rho_+}} \| \Lambda'_s v \|_{H^{0, -\rho_0, -\rho_+}} \lesssim \| u \|_{H^{s, \rho_0, \rho_+}} \| v \|_{H^{-s, -\rho_0, -\rho_+}}
\]

We then conclude by density of \( C_c^\infty(N, L) \). \( \square \)

In the following, we will denote by \( P^* \) the formal adjoint of a pseudodifferential operator \( P \). An immediate computation shows that

\[
I_Z(P^*, \lambda) = I_Z(P, d - \lambda^*).
\]

As a consequence, \( \lambda \) is an indicial root of \( P \) if and only if \( d - \lambda \) is an indicial root of \( P^* \).
Proposition 3.3. Let \( P \) be a \((\rho_-,\rho_+)\)-L^2 admissible elliptic pseudodifferential operator of order \( m \in \mathbb{R} \). Let \( I \) be a connected component in \((\rho_-,\rho_+)\) not containing the real part of any indicial root. Then \( P \) is Fredholm as a bounded operator \( H^{s+m,\rho_0-d/2,\rho_+} \to H^{s,\rho_0-d/2,\rho_\perp} \) with \( s \in \mathbb{R}, \rho_0 \in I, \rho_\perp \in \mathbb{R} \). The index does not depend on \( s,\rho_0,\rho_\perp \) in that range.

Proof. We write \( I = (\rho^I_-,\rho^I_+) \). First, from the parametrix construction, and the compactness of the relevant spaces, we deduce that the kernel of \( P \) is finite dimensional on each of those spaces (and is actually always the same). Indeed, we have

\[
QP = 1 + K,
\]

with \( K \) mapping \( H^{-N,\rho^I_--\epsilon-d/2,\rho_\perp} \to H^{N,\rho^I_+-\epsilon-d/2,\rho_\perp} \) for any \( N > 0, \epsilon > 0 \) small enough and any \( \rho_\perp \in \mathbb{R} \). In particular, by the compact embeddings of Lemma 3.2, we know that \( K \) is compact on \( H^{s,\rho_0-d/2,\rho_+} \) for any \( s \in \mathbb{R}, \rho_0 \in I, \rho_\perp \in \mathbb{R} \). We deduce that the kernel of \( 1 + K \) is finite dimensional. Moreover, by Lemma 3.3 given \( N > 0 \) and \( \rho_\perp \in \mathbb{R} \), we have for \( N' > N \) large enough \( \rho^I_+ > \rho_\perp \) large enough that

\[
H^{N',\rho_0-d/2,\rho_\perp} \hookrightarrow H^{N,\rho_0-d/2,\rho^I_+},
\]

and this implies that the kernel of \( P \) is contained in the intersection of all the spaces \( H^{s,\rho_0-d/2,\rho_+}, s \in \mathbb{R}, \rho_0 \in I, \rho_\perp \in \mathbb{R} \). In particular, the kernel of \( P \), which is contained in the kernel of \( 1 + K \) satisfies the same result, and its dimension does not depend on the space. Eventually, using Lemma 3.9, we can consider the same argument for the adjoint \( P^* \) (to obtain the codimension of the image of \( P \)), and this closes the proof. \( \square \)

4. Pseudo-differential operators for Hölder-Zygmund spaces on cusps

In this section, we are going to prove that the class of pseudodifferential operators defined in the previous section is bounded on the Hölder-Zygmund spaces \( C^s \) (see below for a definition). On a compact manifold, this is a well-known fact and we refer to the arguments before [Tay97, Equation (8.22)] for more details. In our case, there are subtleties coming from the non-compactness of the manifold. First, just as for the scale of Sobolev spaces \( H^s \) (built from the Laplacian induced by the metric), we need to correctly define the Hölder-Zygmund spaces so that they take into account the geometry at infinity of the manifold, namely the hyperbolic cusps. This is done via a Littlewood-Paley decomposition that encapsulates the hyperbolic behaviour. At this stage, we insist on the fact that the euclidean Littlewood-Paley decomposition is rather remarkable insofar as it only involves Fourier multipliers (and not “real” pseudodifferential operators), which truly simplify all the computations.
This is not the case in the hyperbolic world and some rather tedious integrals have to be estimated.

Then, we will be able to prove that the previously defined pseudodifferential operators of order $m \in \mathbb{R}$ map continuously $C^m_s$ to $C^s$, just as in the compact setting. Since we can always split the operator in different parts that are properly supported in cusps or in a fixed compact subset of the manifold (modulo a smoothing operator), we can directly restrict ourselves to operators supported in a cusp as long as we know that smoothing operators enjoy the boundedness property.

Finally, we will prove Theorem 3 in the $L^\infty$ case.

4.1. Definitions and properties. In the paper [Bon16], only Sobolev spaces were considered. So we will have to prove several basic results of boundedness of the calculus, acting now on Hölder-Zygmund spaces. We will give the proofs in the case of cusps, and leave the details of extending to products of cusps with compact manifolds to the reader.

We consider a smooth cutoff function $\psi \in C_0^\infty(\mathbb{R})$ such that $\psi(s) = 1$ for $|s| \leq 1$ and $\psi(s) = 0$ for $|s| \geq 2$. We define for $j \in \mathbb{N}^*$,

$$\varphi_j(x, \xi) = \psi(2^{-j}(\xi)) - \psi(2^{-j+1}(\xi)),$$

where $\langle \xi \rangle := \sqrt{1 + y^2|\xi|^2}$ and here $|\xi|$ is the euclidean norm of the vector $\xi \in \mathbb{R}^{d+1}$.

Observe that

$$\text{supp } \varphi_j \subset \{(x, \xi) \in \mathbb{H}^{d+1} \times \mathbb{R}^{d+1} \mid 2^{j-1} \leq \langle \xi \rangle \leq 2^{j+1}\}.$$  

Then, with $\varphi_0 = \psi((\xi))$, $\sum_{j=0}^{+\infty} \varphi_j(x, \xi) = 1$. We introduce the

**Definition 4.1.** We define the Hölder-Zygmund space of order $s$ as:

$$C^s_*(Z) := \{u \in \Delta^N L^\infty(Z) + L^\infty(Z) \mid \|u\|_{C^s_*} < \infty\},$$

where:

$$\|u\|_{C^s_*} := \sup_{j \in \mathbb{N}} 2^{js}\|\text{Op}(\varphi_j)u\|_{L^\infty(Z)}$$

and $N = 0$ for $s > 0$ and $N > (|s| + d + 1)/2$ when $s \leq 0$.

One can check that the definition of these spaces do not depend on the choice of the initial function $\psi$ (as long as it satisfies the aforementioned properties). This mainly follows from Lemma 4.3. Note that, although a cutoff function $\chi$ around the “diagonal” $y = y'$ has been introduced in (4) in the quantization $\text{Op}$, we still have $1 = \sum_{j \in \mathbb{N}} \text{Op}(\varphi_j)$. Thus, given $u \in C^s_*$ with $s > 0$, one has $u = \sum_{j \in \mathbb{N}} \text{Op}(\varphi_j)u$, with normal convergence in $L^\infty$ and

$$\|u\|_{L^\infty} \leq \sum_{j \in \mathbb{N}} \|\text{Op}(\varphi_j)u\|_{L^\infty} \leq \sum_{j \in \mathbb{N}} 2^{-js} 2^{js}\|\text{Op}(\varphi_j)u\|_{L^\infty} \leq \|u\|_{C^s_*}$$
It can be checked that this definition locally coincides with the usual definition of Hölder-Zygmund spaces on a compact manifold, that is for \( s \notin \mathbb{N} \), \( C^s_* \) contains the functions that have \([s]\) derivatives which are locally \( L^\infty \) and such that the \([s]\)-th derivatives are \( s-[s] \) Hölder continuous. Indeed, if we choose a function \( f \) that is localized in a strip \( y \in [a, b] \), then the size of the annulus in the Paley-Littlewood decomposition is uniform in \( y \) and can be estimated in terms of \( a \) and \( b \), so the definition of the Hölder-Zygmund spaces boils down to that of \( \mathbb{R}^{d+1} \). This will be made precise in Proposition 4.2.

**Definition 4.2.** We will say that an operator \( R \) is small Zygmund-smoothing, and write \( R \in \Psi^{-\infty, L^\infty}_{\text{small}}(N, L) \) if

\[
R : y^\rho C^s_*(N, L) \to y^\rho C^{s'}_*(N, L)
\]

is bounded for any \( \rho \in \mathbb{R}, s, s' \in \mathbb{R} \). We will denote by \( \Psi^m_{\text{small}}(N, L) \) the operators that decompose as \( \text{Op}(\sigma) + R \), with \( \sigma \in S^m \) and \( R \in \Psi^{-\infty, L^\infty}_{\text{small}}(N, L) \).

We have the equivalent of Proposition 2.1:

**Proposition 4.1.** Let \( P = \text{Op}(\sigma) \) be a pseudodifferential operator in the class \( \Psi^m(N, L_1 \to L_2) \). Then:

\[
P : y^\rho C^{s+m}_*(N, L_1) \to y^\rho C^s_*(N, L_2),
\]

is bounded for \( s \in \mathbb{R} \). If \( \sigma' \in S^{m'} \) is another symbol,

\[
\text{Op}(\sigma) \text{Op}(\sigma') = \text{Op}(\sigma\sigma') + O_{\Psi^m_{\text{small}} L^\infty}(1).
\]

As usual, since we added a cutoff function on the kernel of the operator around the diagonal \( y = y' \), the statement boils down to \( \rho = 0 \), which we are going to prove in the next paragraph.

### 4.2. Basic boundedness.

The first step here is to derive a bound on \( L^\infty \) spaces. We follow the notations in [Bon16]−, denoting the lifting of functions on \( Z \) to periodic functions in \( \mathbb{H}^{d+1} \). If \( f \) is a function on the full cusp \( Z \), then for \( P = \text{Op}(\sigma) \), one has:

\[
P f(x) = \int_{\mathbb{H}^{d+1}} \chi(y'/y - 1) \left( \frac{y}{y'} \right)^{\frac{d+1}{2}} \ K^w_{\sigma}(y, \theta, y', \theta') f(y', \theta') dy' d\theta',
\]

where the kernel \( K^w_{\sigma} \) can be written:

\[
K^w_{\sigma}(x, x') = \int_{\mathbb{R}^{d+1}} e^{i(x-x', \xi)} \sigma \left( \frac{x + x'}{2}, \xi \right) d\xi
\]

For \( s \in \mathbb{N} \), this does not exactly coincide with the set of functions that have exactly \([s]\) derivatives in \( L^\infty \).
If \( P : L^\infty(Z) \to L^\infty(Z) \) is bounded, then:

\[
\|P\|_{L^\infty(L^\infty)} \leq \sup_{(y, \theta) \in \mathbb{R}^{d+1}} \int_{\mathbb{R}^{d+1}} \chi(y'/y - 1) \left( \frac{y}{y'} \right)^{\frac{d+1}{2}} |K^w_{\sigma}(y, \theta, y', \theta')| dy' d\theta'
\]

(10)

\[
\lesssim \sup_{(y, \theta) \in \mathbb{R}^{d+1}} \int_{y' = cy}^{y' = y/c} \int_{\theta' \in \mathbb{R}^d} |K^w_{\sigma}(y, \theta, y', \theta')| dy' d\theta'.
\]

Thus, we will look for bounds on \( |K^w_{\sigma}(y, \theta, y', \theta')| \). A rather immediate computation shows that:

\[
x_i - x'_i \frac{K^w_{\sigma}}{i(y + y')/2} = K^w_{\sigma} X_i \sigma,
\]

where \( x = (x_0, x_1, \ldots, x_d) = (y, \theta) \) and \( X_0 = y^{-1} \partial_y, X_i = y^{-1} \partial_{J_i} \) for \( i = 1, \ldots, d \) and we will iterate many times this equality, denoting \( X^\alpha = X^\alpha_0 \ldots X^\alpha_d \) for each multiindices \( \alpha \). Since

\[
|K^w_{\sigma}(y, \theta, y', \theta')| \lesssim \int_{\mathbb{R}^{d+1}} |\sigma((x + x')/2, \xi)| d\xi,
\]

we also get

\[
|K^w_{\sigma}(y, \theta, y', \theta')| \lesssim \left| \frac{x - x'}{y + y'} \right|^{-\alpha} \int_{\mathbb{R}^{d+1}} |X^\alpha \sigma| d\xi.
\]

**Lemma 4.1.** Let \( \sigma \in S^{-m} \) with \( m > d + 1 \). Then \( \text{Op}(\sigma) \) is bounded on \( L^\infty \).

**Proof.** Under the assumptions, \( \sigma \) is integrable in \( \xi \), and so are its derivatives. In particular, we get for all multiindices \( \alpha \),

\[
|K^w_{\sigma}(y, \theta, y', \theta')| \lesssim C_{\alpha} \left( \frac{y + y'}{y - x'} \right)^{\alpha}.
\]

From this we deduce

\[
|K^w_{\sigma}(y, \theta, y', \theta')| \lesssim \frac{1}{(y + y')^{d+1}} \frac{1}{1 + \left| \frac{\theta - \theta'}{y + y'} \right|^{d+1}}
\]

and

\[
\| \text{Op}(\sigma) \|_{L^\infty \to L^\infty} \lesssim \sup_y \int_{y/C}^{yC} dy' \int_{\mathbb{R}^d} d\theta \frac{1}{(y + y')^{d+1}} \frac{1}{1 + \left| \frac{\theta}{y + y'} \right|^{d+1}}
\]

\[
\lesssim \sup_y \int_{y/C}^{yC} dy' \frac{1}{y + y'} < \infty.
\]

□
We now use the previous dyadic partition of unity. Given a symbol $\sigma \in S^m$, we define $\sigma_j := \sigma \varphi_j \in S^{-\infty}$. Observe that

$$P = \text{Op}(\sigma) = \sum_{j=0}^{+\infty} \text{Op}(\sigma \varphi_j) = \sum_{j=0}^{+\infty} P_j,$$

where $P_j := \text{Op}(\sigma_j)$. We will need the following refined version of the previous lemma:

**Lemma 4.2.** Assume that $\sigma \in S^m$. Then, $\|P_j\|_{L^\infty(L^\infty)} \lesssim 2^{jm}$

In particular, if $u \in L^\infty$, we find that $u \in C^0_*$ (but the converse is not true!).

*Proof.* The proof is similar to the proof of Lemma 4.1, but we have to be careful to obtain the right bound in terms of power of $2^j$. Since $\varphi_j$ has support in $\{2^{j-1} \leq \langle \xi \rangle \leq 2^{j+1}\}$, the kernel $K^w_{\sigma_j}$ of $P_j$ satisfies:

$$|K^w_{\sigma_j}(x, x')| \lesssim \int_{\{2^{j-1} \leq \langle \xi \rangle \leq 2^{j+1}\}} \langle \xi \rangle^m d\xi \lesssim \frac{2^{j(m+d+1)}}{(y + y')^{d+1}}$$

Differentiating in $\xi$, we get for all multiindices $\alpha$,

$$|K^w_{\sigma_j}| \lesssim \left| \frac{y + y'}{x - x'} \right|^\alpha \frac{2^{j(m-|\alpha|+d+1)}}{(y + y')^{d+1}}$$

Combining with (11) (we iterate the equality $k'$ times in $y$ and $k$ times in $\theta$ that is in each $\theta_i$ coordinate), we obtain:

$$|K^w_{\sigma_j}(x, x')| \lesssim \frac{2^{j(m+d+1)}}{(y + y')^{d+1}} \left( 1 + 2^{jk'} \left| \frac{y - y'}{y + y'} \right|^{k'} + 2^{jk} \left| \frac{\theta - \theta'}{y + y'} \right|^k \right)$$
Then, integrating in (10), we obtain:
\[
\|P_j\|_{L(L^\infty,L^\infty)} \lesssim \sup_{(y,\theta) \in \mathbb{H}^{d+1}} \int_{y' = y/C} y' \int_{\theta' \in \mathbb{R}^d} |K^w_\sigma(y,\theta,y',\theta')| dy' d\theta'
\]
\[
\lesssim 2^{j(m+d+1)} \sup_{(y,\theta) \in \mathbb{H}^{d+1}} \int_{y' = y/C} y' \int_{\theta' \in \mathbb{R}^d} \frac{dy' d\theta'}{(y + y')^{d+1}} \left(1 + 2^{jk'} \left| \frac{y - y'}{y + y'} \right|^{k'} + 2^{jk} \left| \frac{\theta - \theta'}{y + y'} \right|^k \right)
\]
\[
\lesssim 2^{j(m+d+1)} \sup_{(y,\theta) \in \mathbb{H}^{d+1}} 2^{-jd} \int_{y' = y/C} y' \int_{\theta' \in \mathbb{R}^d} \frac{dy'}{(y + y') \left(1 + 2^{jk'} \left| \frac{y - y'}{y + y'} \right|^{k'} \right)^{1-d/k}}
\]
\[
\lesssim 2^{j(m+1)} \int_{1/C}^C \frac{1}{(1 + u) \left(1 + 2^{jk'} \left| \frac{u-1}{u+1} \right|^{k'} \right)^{1-d/k}} du,
\]
where we have done the change of variable \(u = y'/y\). We let \(v = 2^{j} \frac{1-u}{1+u}\), so that \(u = (1 - 2^{-j}v)/(1 + 2^{-j}v)\),
\[
\frac{1}{(1 + u)} = (1 + 2^{-j}v)/2, \quad du = -\frac{2^{1-j}}{(1 + 2^{-j}v)^2} dv.
\]
and we get the bound
\[
\int_{1/C}^C \frac{1}{(1 + u) \left(1 + 2^{jk'} \left| \frac{u-1}{u+1} \right|^{k'} \right)^{1-d/k}} du \lesssim 2^{-j} \int_{-2^{j(C-1)/(C+1)}}^{2^{j(C-1)/(C+1)}} \frac{1}{(1 + |v|^{k'})^{1-d/k}} dv \lesssim 2^{-j}.
\]
Let now \(k = d+1\) and \(k' = d+2\). We can bound the term \(1/(1 + 2^{-j}v)\) by \((C+1)/2\), and we get
\[
\|P_j\|_{L(L^\infty,L^\infty)} \lesssim 2^{jm} \int_R \frac{dv}{(1 + |v|^{d+2})^{1/(d+1)}} \lesssim 2^{jm}.
\]
Here, it was crucial that the kernel is uniformly properly supported. □

**Lemma 4.3.** Let \(\sigma \in S^m\). For all \(N \in \mathbb{N}\), there exists a constant \(C_N > 0\) such that for all integers \(j,k \in \mathbb{N}\) such that \(|j-k| \geq 3\),
\[
\|P_j \text{Op}(\varphi_k)\|_{L(L^\infty,L^\infty)}, \|\text{Op}(\varphi_k)P_j\|_{L(L^\infty,L^\infty)} \leq C_N 2^{-N \max(j,k)},
\]
where \(P_j = \text{Op}(\sigma \varphi_j)\).
Proof. This is a rather tedious computation and we only give the key ingredients. It is actually harmless to assume that \( \sigma = 1 \), which we will assume to hold for the sake of simplicity. We use [Bon16, Proposition 1.19]. We know that

\[
\text{Op}(\varphi_j) \text{Op}(\varphi_k) f(x) = \int_{x' \in \mathbb{H}^{d+1}} \left( \frac{y}{y'} \right)^{\frac{d+1}{2}} K^w_{\varphi_j \varphi_k}(x, x') f(x') dx'
\]

where, by definition,

\[
K^w_{\varphi_j \varphi_k}(x, x') = \int e^{i(x-x',\xi)} \varphi_j \varphi_k \left( \frac{x + x'}{2}, \xi \right) d\xi
\]

and

\[
\varphi_j \varphi_k(x, \xi) = 2^{-2d-2} \int e^{2i(-(x-x_1,\xi-\xi_1)+(x-x_2,\xi-\xi_2))} \varphi_j(x_2, \xi_1) \varphi_k(x_1, \xi_2) \chi(y, y_1, y_2) dx_1 dx_2 d\xi_1 d\xi_2,
\]

where, for fixed \( y, \chi(y, \cdot, \cdot) \) is supported in the rectangle \( \{ y/C \leq y_1, y_2 \leq yC \} \) (\( C \) not depending on \( y \)). To prove the claimed boundedness estimate, it is thus sufficient to prove that

\[
\sup_{x \in \mathbb{H}^{d+1}} \int_{x' \in \mathbb{H}^{d+1}} \left( \frac{y}{y'} \right)^{\frac{d+1}{2}} |K^w_{\varphi_j \varphi_k}(x, x')| dx' \lesssim C_\alpha 2^{-N \max(j,k)},
\]

and we certainly need bounds on the kernel \( K^w_{\varphi_j \varphi_k} \). First observe that it is supported in some region \( \{ y/C' \leq y' \leq yC' \} \) so, as before, the term \( (y/y')^{\frac{d+1}{2}} \) is harmless in the integral. Then, we follow the same strategy as in the proof of Lemma 4.2. We deduce that it suffices to obtain bounds of the form

\[
|K^w_{X^\alpha(\varphi_j \varphi_k)}|, |K^w_{\varphi_j \varphi_k}| \lesssim C_\alpha 2^{-N \max(j,k)}(y + y')^{d+1},
\]

for \( |\alpha| \leq d + 2 \).

For the sake of simplicity, we only deal with the bound on \( |K^w_{\varphi_j \varphi_k}| \), the others being similar. To obtain a bound on this kernel, it is sufficient to prove that \( |\varphi_j \varphi_k(x, \xi)| \lesssim C_\alpha 2^{-N \max(j,k)}(\xi)^{-N} \) (where \( N \) has to be chosen large enough). Indeed, one then obtains:

\[
|K^w_{\varphi_j \varphi_k}(x, x')| \lesssim C_\alpha 2^{-N \max(j,k)} \int_{\mathbb{H}^{d+1}} \frac{d\xi}{\left( 1 + \left( \frac{y + y'}{2} \right)^2 |\xi|^2 \right)^{N/2}} \lesssim C_\alpha 2^{-N \max(j,k)} (y + y')^{d+1}.
\]

We denote by \( y_1 D_{x_1, i} := \frac{y_1}{2i} \partial_{x_1, i} \) the operator of derivation and we use in (16) the identity

\[
(1 + y_1^2|\xi - \xi_1|^2)^{-N} (1 + y_1^2 D_{x_1}^2)^N (e^{2i(x-x_1,\xi-\xi_1)}) = e^{2i(x-x_1,\xi-\xi_1)}
\]
where $D_{x_1}^2 = \sum_i D_{x_{1,i}}^2$. In terms of Japanese bracket, this can be rewritten shortly

$\langle \xi - \xi_1 \rangle^{-2N} \langle D_{x_1} \rangle^{2N} (e^{2i(x-x_1,\xi-\xi_1)}) = e^{2i(x-x_1,\xi-\xi_1)}$. We thus obtain:

$$\varphi_j \varphi_k(x, \xi) = 2^{-2d-2} \int e^{2i((x-x_1,\xi-\xi_1)+(x-x_2,\xi-\xi_2))} \langle \xi - \xi_1 \rangle^{-2N} \langle \xi - \xi_2 \rangle^{-2N}$$

$$\langle D_{x_1} \rangle^{2N} \langle D_{x_2} \rangle^{2N} (\varphi_j(x_2, \xi_1)\varphi_k(x_1, \xi_2)\chi(y, y_1, y_2)) dx_1 dx_2 d\xi_1 d\xi_2,$$

We also need to use this trick in the $x$ variable (more precisely on the $\theta$ variable) to ensure absolute convergence of this integral. This yields the formula:

$$\varphi_j \varphi_k(x, \xi) = 2^{-2d-2} \int e^{2i((x-x_1,\xi-\xi_1)+(x-x_2,\xi-\xi_2))}$$

$$\langle \theta - \theta_1 \rangle^{-2M} \langle \theta - \theta_2 \rangle^{-2M} \langle D_{J_1} \rangle^{2M} \langle D_{J_2} \rangle^{2M}$$

$$\left[ \langle \xi - \xi_1 \rangle^{-2N} \langle \xi - \xi_2 \rangle^{-2N} \langle D_{x_1} \rangle^{2N} \langle D_{x_2} \rangle^{2N} (\varphi_j(x_2, \xi_1)\varphi_k(x_1, \xi_2)\chi(y, y_1, y_2)) \right]$$

dx_1 dx_2 d\xi_1 d\xi_2,$$

where $M$ is chosen large enough. We here need to clarify a few things. First of all, the notation is a bit hazardous insofar as $\langle \theta - \theta_1 \rangle^2 := 1 + \frac{\theta_1 - \theta \cdot}{\theta_1}$ this time. This comes from the fact that the natural operation of differentiation (which preserves the symbol class) is $\langle D_{J_1} \rangle^2 := 1 + \sum_{i=1}^d(y_i^{-1} \partial_{J_{i,i}})^2$. If ones formally develops the previous formula, one obtains a large number of terms involving derivatives — coming from the brackets

$$\langle D_{J_1} \rangle^{2M} \langle D_{J_2} \rangle^{2M} \langle D_{x_1} \rangle^{2N} \langle D_{x_2} \rangle^{2N}$$

— of $\varphi_j$ and $\varphi_k$. These derivatives obviously do not change the supports of these functions and can only better the estimate (there is a $2^{-j}$ that pops up out of the formula each time one differentiates, stemming from the very definition of $\varphi_j$). As a consequence, it is actually sufficient to bound the integral if one forget about these brackets of differentiation. We are thus left to bound

$$\int e^{2i((x-x_1,\xi-\xi_1)+(x-x_2,\xi-\xi_2))} \langle \theta - \theta_1 \rangle^{-2M} \langle \theta - \theta_2 \rangle^{-2M} \langle \xi - \xi_1 \rangle^{-2N} \langle \xi - \xi_2 \rangle^{-2N} \varphi_j(x_2, \xi_1)\varphi_k(x_1, \xi_2)\chi(y, y_1, y_2) dx_1 dx_2 d\xi_1 d\xi_2.$$

We can now assume without loss of generality that $k \geq j + 3$. Then, $\varphi_j$ and $\varphi_k$ are supported in two distinct annulus whose interdistance is bounded below by $2^{k-1} - 2^{j+1} \geq 2^{k-2}$. Using this fact, one can bound the integrand by

$$\langle \xi - \xi_1 \rangle^{-2N} \langle \xi - \xi_2 \rangle^{-2N} \langle \theta - \theta_1 \rangle^{-2M} \langle \theta - \theta_2 \rangle^{-2M} \chi(y, y_1, y_2)$$

$$\lesssim C N 2^{-Nk} \langle \xi \rangle^{-4N} \langle \theta - \theta_1 \rangle^{-2M} \langle \theta - \theta_2 \rangle^{-2M} \chi(y, y_1, y_2),$$

where the last bracket is $\langle \xi \rangle := \sqrt{1 + y^2 |\xi|^2}$. (The estimates actually come out with a Japanese bracket in terms of $y_{1,2}$ but these are uniformly comparable to the Japanese
bracket in terms of \( y \) because \( \chi \) is supported in the region \( \{ y/C \leq y_1, 2 \leq y C \} \). We thus obtain:

\[
\left| \int e^{2i(\langle x-x_1, \xi-\xi_1 \rangle + \langle x-x_2, \xi-\xi_2 \rangle)} (\theta - \theta_1)^{-2M} (\theta - \theta_2)^{-2M} \langle \xi - \xi_1 \rangle^{-2N} \langle \xi - \xi_2 \rangle^{-2N} \varphi_j(x_2, \xi_1) \varphi_k(x_1, \xi_2) \chi(y, y_1, y_2) dx_1 dx_2 d\xi_1 d\xi_2 \right|
\]

\[
\lesssim C_N 2^{-Nk} \langle \xi \rangle^{-4N} \int_{\substack{x_1 \in \mathbb{R}^{d+1} \\ 
\quad x_2 \in \mathbb{R}^{d+1} \\ 
\quad 2^{j-1} \leq \langle \xi_1 \rangle \leq 2^{j+1} \\ 
\quad 2^{k-1} \leq \langle \xi_2 \rangle \leq 2^{k+1} \}} (\theta - \theta_1)^{-2M} (\theta - \theta_2)^{-2M} \chi(y, y_1, y_2) d\xi_1 d\xi_2 dx_1 dx_2
\]

We simply use a volume bound of the annulus (the ball in which it is contained actually) for the \( \xi_1, \xi_2 \) integrals which provides:

\[
\int_{2^{j-1} \leq \langle \xi_1 \rangle \leq 2^{j+1}} d\xi_1 \lesssim 2^{(d+1)/y_1^{d+1}}
\]

As a consequence, the bound in the previous integral becomes:

\[
C_N 2^{-Nk} \langle \xi \rangle^{-4N} \int_{\substack{x_1 \in \mathbb{R}^{d+1} \\ 
\quad x_2 \in \mathbb{R}^{d+1} \\ 
\quad 2^{j-1} \leq \langle \xi_1 \rangle \leq 2^{j+1} \\ 
\quad 2^{k-1} \leq \langle \xi_2 \rangle \leq 2^{k+1} \}} (\theta - \theta_1)^{-2M} (\theta - \theta_2)^{-2M} \chi(y, y_1, y_2) \frac{dx_1 dx_2}{y_1^{d+1} y_2^{d+1}}
\]

Now, the last integral can be bounded by

\[
\int_{y_1 = y/C}^{Cy} \int_{\theta_1 \in \mathbb{R}^d} \int_{y_2 = y/C}^{Cy} \int_{\theta_2 \in \mathbb{R}^d} (\theta - \theta_1)^{-2M} (\theta - \theta_2)^{-2M} \frac{dx_1 dx_2}{y_1^{d+1} y_2^{d+1}} \lesssim 1,
\]

where \( M \) is large enough, which eventually yields the estimate

\[
|\varphi_j \nabla \varphi_k(x, \xi)| \lesssim C_N 2^{-Nk} 2^{(j+k)(d+1)} \langle \xi \rangle^{-4N}.
\]

Since \( N \) was chosen arbitrary, we can always take it large enough so that it swallows the term \( 2^{(j+k)(d+1)} \). In the end, concluding by symmetry of \( j \) and \( k \), we obtain the sought estimate

\[
|\varphi_j \nabla \varphi_k(x, \xi)| \lesssim C_N 2^{-N \max(j, k)} \langle \xi \rangle^{-N}.
\]

This implies the estimate on the kernel \( K^w_{\varphi_j \nabla \varphi_k} \) and concludes the proof. \( \square \)
**Remark 4.1.** Following the same scheme of proof, one can also obtain the independence of the definition of the Hölder-Zygmund spaces with respect to the cutoff function $\psi$ chosen at the beginning. If $\tilde{\psi} \in C_0^\infty(\mathbb{R})$ is another cutoff function such that $\tilde{\psi} \equiv 1$ on $[-a,a]$ and $\tilde{\psi} \equiv 0$ on $\mathbb{R} \setminus [-b,b]$ (and $0 < a < b$), we denote by $\text{Op}(\tilde{\psi}_j)$ the operators built from $\tilde{\psi}$ like in (9). Then, in order to show the equivalence of the $C^*_s$- and $\tilde{C}^*_s$-norms respectively built from $\psi$ or $\tilde{\psi}$, one has to compute quantities like $\| \text{Op}(\psi_j) \text{Op}(\tilde{\psi}_k) \|_{L(L^\infty,L^\infty)}$. If $k \in \mathbb{N}$ is fixed, then the terms $\text{Op}(\psi_j) \text{Op}(\tilde{\psi}_k)$ “interact” (in the sense that one will not be able to obtain a fast decay estimate like (18)) for $j \in [k-1+\lfloor \log_2(a) \rfloor, k+1+\lceil \log_2(b) \rceil]$. We can improperly call these terms “diagonal terms”. Note that the number of such terms is independent of both $j$ and $k$. The content of Lemma 4.3 can be interpreted by saying that when taking the same cutoff function (that is $\psi = \tilde{\psi}$), the diagonal terms are $\{j,k \in \mathbb{N} | |j-k| \leq 2\}$. In the following, we will use the definition of Hölder-Zygmund spaces with the rescaled cutoff functions $\tilde{\psi}_h := \psi(h \cdot)$. The diagonal terms are then shifted by $\log_2(h^{-1})$.

A consequence of the previous Lemma is the following estimate. Note that it is not needed for the proof of Proposition 4.1 but will appear shortly after when comparing the Hölder-Zygmund spaces $C^*_s$ with the usual spaces $C^s$.

**Lemma 4.4.** Let $P = \text{Op}(\sigma)$ for some $\sigma \in S^m$, $m \in \mathbb{R}$ and let $0 < s < m$. Then, there exists a constant $C > 0$ such that for all $j \in \mathbb{N}$:

$$\| P \text{Op}(\varphi_j) \|_{L(C^*_s,L^\infty)} \leq C 2^{-j(s-m)}$$

**Proof.** This is a rather straightforward computation, using Lemma 4.3:

$$\| P \text{Op}(\varphi_j) f \|_{L^\infty} \lesssim \sum_{k \in \mathbb{N}} \| P_k \text{Op}(\varphi_j) f \|_{L^\infty}$$

$$\lesssim \sum_{|k-j| \geq 3} \| P_k \text{Op}(\varphi_j) f \|_{L^\infty} + \sum_{|k-j| \leq 2} \| P_k \text{Op}(\varphi_j) f \|_{L^\infty}$$

$$\lesssim \sum_{|k-j| \geq 3} C_N 2^{-N \max(j,k)} \| f \|_{L^\infty} + 2^{jm} \| \text{Op}(\varphi_j) f \|_{L^\infty}$$

$$\lesssim \| f \|_{L^\infty} + 2^{-j(s-m)} 2^{js} \| \text{Op}(\varphi_j) f \|_{L^\infty}$$

$$\lesssim 2^{-j(s-m)} \| f \|_{C^*_s},$$

where $N \geq 1$ is arbitrary. \hfill \Box

We can now start the proof of Proposition 4.1.
Proof of Proposition 4.1, case $s + m > 0$, $s > 0$. We look at:

$$\| \text{Op}(\varphi_j) Pu \|_{L^\infty} \lesssim \sum_{|j - k| \geq 3} \| \text{Op}(\varphi_j) P_k u \|_{L^\infty} + \| \text{Op}(\varphi_j) \sum_{|j - k| \leq 2} P_k u \|_{L^\infty}.$$ 

The first term can be bounded using Lemma 4.3 and for $N \geq [s] + 1$:

$$\sup_{j \in \mathbb{N}} 2^j \sum_{|j - k| \geq 3} \| \text{Op}(\varphi_j) P_k u \|_{L^\infty} \lesssim \| u \|_{L^\infty} \lesssim \| u \|_{C^{s+m}}.$$ 

Concerning the second term, we use the same trick, writing $u_k := \text{Op}(\varphi_k) u$.

$$\| \text{Op}(\varphi_j) \sum_{|j - k| \leq 2} P_k u \|_{L^\infty} \lesssim \sum_{|j - k| \leq 2} \sum_{|j - l| \geq 5} \| P_k u_l \|_{L^\infty} + \sum_{|j - k| \leq 2} \sum_{|j - l| \leq 4} \| P_k u_l \|_{L^\infty}.$$ 

The first term can be bounded just like before, using Lemma 4.3. As to the second term, we use Lemma 4.2, which gives that

$$\sup_{j \in \mathbb{N}} 2^j \sum_{|j - k| \leq 2} \sum_{|j - l| \leq 4} \| P_k u_l \|_{L^\infty} \lesssim \| u \|_{C^{s+m}}.$$ 

Combining the previous inequalities, we obtain the desired result. Observe that the proof above also gives that for $P \in \Psi_m$, $m \in \mathbb{R}$,

$$\| Pu \|_{C^{s-m}} \lesssim \| u \|_{L^\infty}. \quad \square$$

Next, we want to deal with the case of negative $s$. To this end, we need to have some rough space on which our operators are bounded. Consider the space of distributions (for some constant $h > 0$ small enough).

$$C^{-2n} := (-h^2 \Delta + 1)^n L^\infty.$$ 

equipped with the norm

$$\| u \| := \inf \{ \| v \|_{L^\infty} \mid (-h^2 \Delta + 1)^n v = u \}.$$ 

Lemma 4.5. For $n \geq 1$ and $h$ small enough, $s > 0$, and $\sigma \in S^{-2n+1-s}$, $\text{Op}(\sigma)$ is bounded on $C^{-2n}$. Also, for $n > n'$, $C^{-2n'} \subset C^{-2n}$.

Proof. First of all, we prove that $L^\infty \subset C^{-2n}$. To this effect, we consider parametrices

$$(-h^2 \Delta + 1)^n \text{Op}(q_n) = 1 + h^N \text{Op}'(r_n),$$
with \( q_n \) of order \(-2n \), and \( r_n \) of order \(-N \). Taking \( N \) larger than \( d + 1 \), by Lemma 4.1, \( \text{Op}(r_n) \) is bounded on \( L^\infty \) and \( \text{Op}(q_n) \) is bounded from \( L^\infty \) to \( C^s_{2n} \subset L^\infty \) by the previous Lemma. We get that for \( v \in L^\infty \),
\[
(-h^2 \Delta + 1)^n \text{Op}(q_n)(1 + h^N \text{Op}(r_n))^{-1} v = v,
\]
the inverse being defined by Neumann series for \( h \) small enough and \( P_n \) is of order \(-2n \) so \( P_n v \in C^s_{2n} \subset L^\infty \). The inclusion \( C^{2n'} \subset C^{2n} \) follows decomposing \((-h^2 \Delta + 1)^n = (-h^2 \Delta + 1)^{n'}(-h^2 \Delta + 1)^{n-n'}\).

For \( f = (-h^2 \Delta + 1)^n \tilde{f} \in C^{2n} \) (with \( \tilde{f} \in L^\infty \)), observe that
\[
\text{Op} \sigma f = \text{Op} \sigma (-h^2 \Delta + 1)^n \tilde{f} = \text{Op} \sigma' (-h^2 \Delta + 1)^n \text{Op} \sigma \tilde{f},
\]
with \( \sigma \in S^{2n+1-s} \) — here \( \text{Op} \sigma' \) is a quantization with cutoffs around the diagonal with a larger support and \( \sigma' \in S^{-s} \). By the last remark in the proof of the previous lemma, this is in \( C^s + (-h^2 \Delta + 1)^n C^s_{2n+s-1} \subset L^\infty + (-h^2 \Delta + 1)^n L^\infty \subset C^{2n} \).

Proof of Proposition 4.1, general case. Given \( p \in S^m \) and \( n \), we can build parametrices
\[
(-h^2 \Delta + 1)^k \text{Op}(q_k) \text{Op}(p) = \text{Op}(p) + \text{Op}(r_k),
\]
with \( q_n \in S^{-2k}, r_n \in S^{-2n-d-1} \). With \( k \geq n + (m+d+1)/2 \), we get that for \( u \in C^{2n} \),
\[
\text{Op}(p)u = (-h^2 \Delta + 1)^k \text{Op}(q_k) \text{Op}(p)u - \text{Op}(r_k)u \in C^{2n+k} + C^{2n} = C^{2n+k}.
\]
In particular, \( \text{Op}(p) \) is continuous from \( C^{2n} \) to \( C^{-4n-2[(m-d-1)/2]} \). Next, inspecting the proof of Lemma 4.3, we find that it also applies to the spaces \( C^{2n} \). In particular, we obtain that for all \( n \geq 0 \), and every \( s \in \mathbb{R} \),
\[
\|\text{Op}(p)u\|_{C^s} \leq C\|u\|_{C^s+m} + C\|u\|_{C^{-2n}}.
\]
So far, we have proved that for \( n \geq 0, s \in \mathbb{R}, m \in \mathbb{R} \), \( \text{Op}(p) \) is continuous as a map
\[
\{u \in C^{2n} \mid \|u\|_{C^s+m} < \infty \} \to \{u \in C^{-4n-2[(m-d-1)/2]} \mid \|u\|_{C^s} < \infty \}.
\]
We would like to replace \(-4n - 2[(m - d - 1)/2] \) by a number that only depends on \( s \). To this end, we pick \( u \in C^{-4n-2[(m-d-1)/2]} \) such that \( \|u\|_{C^s} < \infty \). First off, if \( s > 0 \), then \( u \in L^\infty \). So we assume that \( s \leq 0 \). Then for all \( \epsilon > 0 \), using the estimate (19),
\[
\|\text{Op}(\xi^{s-\epsilon})u\|_{C^s} < \infty.
\]
Using parametrices again, we can find \( r_N \in S^{-s-\epsilon} \) and \( q \in S^{s+\epsilon} \) so that
\[
u = \text{Op}(q_{s+\epsilon}) \text{Op}(\xi^{s-\epsilon})u + \text{Op}(r_N)u.
\]
Since \( \text{Op}(r_N)u, \text{Op}(\xi^{s-\epsilon})u \in L^\infty \), we can apply the first part of the proof and obtain \( u \in C^{-2[(s+\epsilon+d+1)/2]} \).
4.3. Correspondance between Hölder-Zygmund spaces and usual Hölder spaces. We prove that the Hölder-Zygmund spaces $C^s(Z)$ coincide with the usual spaces $C^s(Z)$ when $s \in \mathbb{R}_+ \setminus \mathbb{N}$.

Proposition 4.2. For all $s \in \mathbb{R}_+ \setminus \mathbb{N}$, $C^s(Z) = C^s(Z)$ and more precisely
\[
\|f\|_{C^s(Z)} \approx \|f\|_{C^s(Z)}.
\]

For the sake of simplicity, we prove the previous proposition in the case $s \in (0, 1)$, the general case being handled in a similar fashion. This will require a preliminary

Lemma 4.6. There exists a constant $C > 0$ such that for all $j \in \mathbb{N}$:
\[
\|\text{Op}(\varphi_j)\|_{L^\infty} \leq C2^{-j}.
\]

Proof. Let us start by giving an explicit expression:
\[
\text{Op}(\varphi_j) = \int_{\mathbb{R}^{d+1}} \int_{\mathbb{R}^{d+1}} \chi(y'/y - 1)(y/y') \frac{d+1}{2} e^{i(x-x',\xi)} \sigma_j \left( \frac{y + y'}{2}, \xi \right) d\xi dy' d\theta.
\]

Since there is no dependence in $\theta$, we can remove $\theta$ and get
\[
\int_{y'=0}^{+\infty} \int_{Y=-\infty}^{+\infty} \chi(y'/y - 1)(y/y') \frac{d+1}{2} e^{i(y-y',Y)} \sigma_j \left( \frac{y + y'}{2}, Y, J = 0 \right) dY dy'.
\]

That is,
\[
\int_{y'=0}^{+\infty} \int_{Y=-\infty}^{+\infty} \chi(y'/y - 1)(y/y') \frac{d+1}{2} e^{i(y-y',Y)} \sigma_j \left( \frac{y + y'}{2}, Y, J = 0 \right)
\]
\[
\psi \left( 2^{-j} \sqrt{1 + \left( \frac{y + y'}{2} \right)^2} \right) - \psi \left( 2^{-j+1} \sqrt{1 + \left( \frac{y + y'}{2} \right)^2} \right) dY dy'.
\]

Making the change of variables $u = \frac{y + y'}{2}$, we get the following expression:
\[
2 \times \int_{u=y/2}^{+\infty} \int_{Y=-\infty}^{+\infty} \chi(2u/y - 2) \left( \frac{y}{2u - y} \right) \frac{d+1}{2} e^{2i(y-u, Y)}
\]
\[
\left[ \psi \left( 2^{-j} \sqrt{1 + (uY)^2} \right) - \psi \left( 2^{-j+1} \sqrt{1 + (uY)^2} \right) \right] dY du.
\]

(20)

It is sufficient to prove that each term in this difference is bounded by $C2^{-j}$. Let us deal with the first one for instance. For the sake of simplicity, we also forget about the factors
\[
\chi(2u/y - 2) \left( \frac{y}{2u - y} \right) \frac{d+1}{2}
\]
since, in the end, this will amount to integrating in the $y'$ variable for $y' \in [y/C', yC']$, for some uniform constant $C' > 0$. Using the identity
\[
\left( \frac{i \partial_Y}{2u} \right) (e^{2i(y-u,Y)}) = e^{2i(y-u,Y)},
\]
we can thus estimate the first term in (20)

\[
\int_{y=C'}^{y=C'} \int_{Y=-\infty}^{+\infty} e^{2i(y-u,Y)} \psi \left( 2^{-j} \sqrt{1 + (uY)^2} \right) dY du
\]

\[
= \int_{u=y/C'}^{u=y/C'} \int_{Y=-\infty}^{+\infty} e^{2i(y-u,Y)} \left( \frac{i \partial_Y}{2u} \right)^2 (e^{2i(y-u,Y)}) \psi \left( 2^{-j} \sqrt{1 + (uY)^2} \right) dY du
\]

\[
= \int_{u=y/C'}^{u=y/C'} \int_{Y=-\infty}^{+\infty} e^{2i(y-u,Y/u)} \left( \frac{i \partial_Y}{2} \right)^2 \left[ \psi \left( 2^{-j} \sqrt{1 + (uY/u)^2} \right) \right] dY du/u
\]

\[
= -\frac{2-j}{4} \int_{u=y/C'}^{u=y/C'} \int_{Y=-\infty}^{+\infty} e^{2i(y-u,Y/u)} \left[ \frac{1}{(1 + Y^2)^{3/2}} \psi' \left( 2^{-j} \sqrt{1 + Y^2} \right) + 2^{-j} \frac{Y}{\sqrt{1 + Y^2}} \psi'' \left( 2^{-j} \sqrt{1 + Y^2} \right) \right] dY du/u
\]

Once again, we only estimate the first term in the previous sum, the second one being handled in the same fashion. By definition, $\psi$ is supported in the ball of radius 2, thus:

\[
\left| 2^{-j} \int_{u=y/C'}^{u=y/C'} \int_{Y=-\infty}^{+\infty} e^{2i(y-u,Y/u)} \frac{1}{(1 + Y^2)^{3/2}} \psi'(2^{-j} \sqrt{1 + Y^2}) dY du/u \right|
\]

\[
\lesssim 2^{-j} \int_{u=y/C'}^{u=y/C'} \int_{Y=-\infty}^{+\infty} \frac{1}{(1 + Y^2)^{3/2}} \psi'(2^{-j} \sqrt{1 + Y^2}) dY du/u
\]

\[
\lesssim 2^{-j} \int_{u=y/C'}^{u=y/C'} \int_{|Y| \leq 2} \frac{dY}{(1 + Y^2)^{3/2}} du/u \lesssim 2^{-j} \int_{u=y/C'}^{u=y/C'} du/u \lesssim 2^{-j}
\]

This concludes the proof of the Lemma. \qed

We can now prove Proposition 4.2:
Proof. We first prove that there exists $C > 0$ such that for all functions $f \in C^s_*$, \[ \|f\|_{C^s} \leq C\|f\|_{C^s}. \] For $x, x' \in Z$ such that $d(x, x') \leq 1$, we write:

\[ |f(x) - f(x')| = \left| \sum_{j \in \mathbb{N}} (\text{Op}(\varphi_j)f)(x) - (\text{Op}(\varphi_j)f)(x') \right| \]
\[ \leq \sum_{j \in \mathbb{N}} |(\text{Op}(\varphi_j)f)(x) - (\text{Op}(\varphi_j)f)(x')| \]

Let $N \in \mathbb{N} \setminus \{0\}$ be the unique integer such that $2^{-N} \leq d(x, x') \leq 2^{-N+1}$. We split the previous sum between $j \geq N$ and $j < N$. First:

\[ \sum_{j \geq N} |(\text{Op}(\varphi_j)f)(x) - (\text{Op}(\varphi_j)f)(x')| \lesssim \sum_{j \in \mathbb{N}} \|\text{Op}(\varphi_j)f\|_{L^\infty} \]
\[ \lesssim \sum_{j \geq N} 2^{-js}\|f\|_{C^s} \]
\[ \lesssim 2^{-sN}\|f\|_{C^s} \lesssim \|f\|_{C^s}d(x, x')^s \]

Now, using Lemma 4.4 with $P = \nabla$ (note that $0 < s < m = 1$), one has:

\[ \sum_{j < N} |(\text{Op}(\varphi_j)f)(x) - (\text{Op}(\varphi_j)f)(x')| \lesssim \|\nabla\text{Op}(\varphi_j)f\|_{L^\infty}d(x, x') \]
\[ \lesssim 2^{-j(s-1)}\|f\|_{C^s}d(x, x') \lesssim \|f\|_{C^s}d(x, x')^s \]

Eventually, using the obvious estimate $\|f\|_{L^\infty} \lesssim \|f\|_{C^s}$, one obtains $\|f\|_{C^s} \lesssim \|f\|_{C^s}$. Let us now prove the other estimate. We start with:

\[ \text{Op}(\varphi_j)f(x) = \int_{\mathbb{H}^{d+1}} \chi(y'/y - 1)(y/y')^{\frac{d+1}{2}} K_{\varphi_j}^w(x, x')f(x')dx' \]
\[ = \int_{\mathbb{H}^{d+1}} \chi(y'/y - 1)(y/y')^{\frac{d+1}{2}} K_{\varphi_j}^w(x, x')(f(x') - f(x))dx' \]
\[ + f(x) \text{Op}(\varphi_j)1 \]

According to Lemma 4.6, the last term is bounded by $\lesssim \|f\|_{L^\infty} 2^{-j} \lesssim \|f\|_{C^s} 2^{-j}$. As to the first term, using the Hölder property of $f$:

\[ \left| \int_{\mathbb{H}^{d+1}} \chi(y'/y - 1)(y/y')^{\frac{d+1}{2}} K_{\varphi_j}^w(x, x')(f(x') - f(x))dx' \right| \]
\[ \lesssim \int_{\mathbb{H}^{d+1}} \chi(y'/y - 1)(y/y')^{\frac{d+1}{2}} \left| K_{\varphi_j}^w(x, x') \right| d(x, x')^s dx' \|f\|_{C^s} \]

Now, following the exact same arguments as the ones developed in Lemma 4.2 and using the crucial fact that on the support of the kernel of the pseudodifferential operator
(namely for \( y' \in [y/C, yC] \)) one can bound the distance \( d(x, x') \lesssim \log(y/y') + \frac{|\vartheta - \vartheta'|}{y} \), one can prove the estimate

\[
\sup_{x \in \mathbb{H}^{d+1}} \int_{\mathbb{H}^{d+1}} \chi(y'/y - 1)(y/y')^{d+1} \left| K^w_{\varphi_j}(x, x') \right| d(x, x')^s dx' \lesssim 2^{-js}
\]

The sought estimate \( \| f \|^2_{C^s} \lesssim \| f \|^2_{C^s} \) then follows immediately.

\[\square\]

4.4. Embedding estimates. Using the Paley-Littlewood decompositions in the cusps, we are going to prove the embedding estimates. We can actually strengthen them to the following two Lemmas:

**Lemma 4.7.** For all \( s, s' \in \mathbb{R} \) such that \( s' > s \), \( \rho, \rho' \in \mathbb{R} \) such that \( \rho' > \rho - d/2 \),

\[ y^\rho C^s(N, L) \hookrightarrow y^{\rho'} H^s(N, L) \]

is a continuous embedding.

In our notations, \( y^{\rho'} H^s = H^{s, \rho', \rho'} \).

**Lemma 4.8.** For all \( s, \rho \in \mathbb{R} \),

\[ y^\rho H^s(N, L) \hookrightarrow y^{\rho + d/2} C^{s-(d+1)/2}(N, L) \]

is a continuous embeddings.

Observe that the two lemmas are locally true so that it is sufficient to prove them when the function is supported on a single fibered cusp. The key lemma here is the following

**Lemma 4.9.** For all \( s \in \mathbb{R} \),

\[ \| u \|^2_{H^s(N)} \asymp \sum_{j \in \mathbb{N}} \| \text{Op}(\varphi_j)u \|^2_{L^2(Z)} 4^js \]

**Proof.** The proof is done using semiclassical estimates and then concluding by equivalence of norms when \( h \) is bounded away from 0. For \( h > 0 \), we start from

\[ \| u \|^2_{H^s_h(N)} \asymp \| \text{Op}_h(\langle \xi \rangle^s)u \|^2_{L^2} \]

\[ = \sum_{j,k} \langle \text{Op}_h(\langle \xi \rangle^s \varphi_j)u, \text{Op}_h(\langle \xi \rangle^s \varphi_k)u \rangle \]

\[ = \sum_{|j-k| \leq 2} \langle \text{Op}_h(\langle \xi \rangle^s \varphi_j)u, \text{Op}_h(\langle \xi \rangle^s \varphi_k)u \rangle + \sum_{|j-k| \geq 3} \langle \text{Op}_h(\langle \xi \rangle^s \varphi_k) \text{Op}_h(\langle \xi \rangle^s \varphi_j)u, u \rangle . \]

The first term is obviously bounded by \( \lesssim \sum_j \| \text{Op}_h(\langle \xi \rangle^s \varphi_j)u \|^2_{L^2(Z)} \). To bound the last term we can first use the estimate (18) in the proof of Lemma 4.3 which yields

\[ \langle \text{Op}_h(\langle \xi \rangle^s \varphi_k) \text{Op}_h(\langle \xi \rangle^s \varphi_j)u, u \rangle \leq C_N 2^{-N \max(j,k)} \| u \|^2_{H^s_h(N)} , \]
where $N > |s|$ is taken arbitrary large and thus $\sum_{|j-k|\geq 3} \langle \text{Op}_h(\langle \xi \rangle^s \varphi_k) \text{Op}_h(\langle \xi \rangle^s \varphi_j) u, u \rangle \lesssim \|u\|^2_{H^{-N}_h(Z)}$. Now, we also have that
\[
\|u\|^2_{H^{-N}_h} = \| \sum_j \text{Op}_h(\langle \xi \rangle^{-N} \varphi_j) u \|^2_{L^2} \\
\lesssim \sum_j \langle \text{Op}_h(\langle \xi \rangle^{-N} \varphi_j) u \rangle^2 \\
\lesssim \sum_j \| \text{Op}_h(\langle \xi \rangle^s \varphi_j) u \|^2_{L^2} + h \|u\|^2_{H^{-N}_h(Z)} \\
\lesssim \sum_j \| \text{Op}_h(\langle \xi \rangle^s \varphi_j) u \|^2_{L^2} + h \|u\|^2_{H^{-N}_h(Z)},
\]
where the penultimate inequality follows from Gårding’s inequality [GW17, Lemma A.15] for symbols of order $-(2N - 1)$ since $2^j \langle \xi \rangle^{-N-s} \langle \xi \rangle^s \varphi_j \in S^{-(2N-1)}$ is controlled by $\lesssim \langle \xi \rangle^s \varphi_j$. For $h$ small enough, we can swallow the term $h \|u\|^2_{H^{-N}_h(Z)}$ in the left-hand side and we eventually obtain that $\|u\|^2_{H^{-N}_h} \lesssim \sum_j \| \text{Op}_h(\langle \xi \rangle^s \varphi_j) u \|^2_{L^2}$, where the constant hidden in the $\lesssim$ notation is independent of $h$. Actually, since $\langle \xi \rangle^s \varphi_j \lesssim 2^j \varphi_j$, the same arguments involving Gårding’s inequality also yield
\[
\|u\|^2_{H^{-N}_h} \lesssim \sum_j \|2^j \varphi_j \text{Op}_h(\varphi_j) u \|^2_{L^2},
\]
On the other hand,
\[
\sum_j \| \text{Op}_h(\langle \xi \rangle^s \varphi_j) u \|^2_{L^2(Z)} = \langle \sum_j \text{Op}_h(\langle \xi \rangle^s \varphi_j)^2 u, u \rangle.
\]
Using expansions for products, we find that this is $\lesssim \langle \text{Op}_h(\langle \xi \rangle^{2s} \sum \varphi_j^2) u, u \rangle$. This itself is controlled by the $H^s_h$ norm. Eventually, we conclude by equivalence of norms when $h$ is bounded away from 0 (see Remark 4.1).

We will also need the following observation: $\text{Op}(\varphi_j)(y^s u) = y^s \text{Op}(\varphi_j)(u)$ for some other quantization $\text{Op}'$ (the cutoff function $\chi(y'/y - 1)$ in the quantization $\text{Op}$ is changed to $(y'/y)^s \chi(y'/y - 1)$). In the following proof, we will denote by $\text{Op}'$ and $\text{Op}''$ other quantizations than $\text{Op}$ which are produced by multiplying the cutoff function $\chi$ by some power of $y'/y$. Eventually, one last remark is that Proposition 4.2 imply in particular that the spaces $C^s_*(N)$ defined for $s \in \mathbb{R}_+ \setminus \mathbb{N}$ do not depend on the choice of the cutoff function $\chi$ in the quantization (insofar as they can be identified to the usual Hölder spaces $C^s(N)$).
Proof of Lemma 4.7. We fix $\rho < \rho' + d/2$ and $\varepsilon > 0$ small enough so that $\rho < \rho' + d/2 - \varepsilon$. Then:

$$\|u\|_{y_{\rho'} H^s}^2 = \sum_{j \in \mathbb{N}} \|\text{Op}(\varphi_j)(y^{\rho'} u)\|_{L^2}^2 4^j s'$$

$$\lesssim \sum_{j \in \mathbb{N}} \|y^{\rho'} \text{Op}(\varphi_j) u\|_{L^2}^2 4^j s'$$

$$\lesssim \sum_{j \in \mathbb{N}} \|y^{\rho'-d/2+\varepsilon} \text{Op}(\varphi_j) u\|_{L^\infty}^2 4^j s'$$

$$\lesssim \sum_{j \in \mathbb{N}} 4^{j(s-s')} \|\text{Op}''(\varphi_j)(y^{\rho'-d/2+\varepsilon} u)\|_{L^\infty}^2 4^j s' \lesssim \|u\|_{y^{\rho'+d/2-\varepsilon} C^{s'_*}}^2 \lesssim \|u\|_{y^{\rho'} C^{s'_*}}^2,$$

since $s < s'$.

Proof of Lemma 4.8. Let us sketch the proof for the embedding $y^{-d/2} H^{(d+1+\varepsilon)/2} \hookrightarrow C^0$, the general case being handled in the same fashion with a little bit more work. We start by computing a $L^1 \to L^\infty$ norm for $\text{Op}(\sigma)$ when $\sigma \in S^{-(d+1+\varepsilon)}$. We find

$$\|\text{Op}(\sigma)\|_{y^\rho L^1 \to L^\infty}^2 \leq \sup_{x,x'} y^{d+1} y'^{2\rho} \left| \sum_{\gamma \in \Lambda} K_w^\sigma(x, y', \theta' + \gamma) \right|.$$

Going through the arguments of proof for equation (14), we deduce that

$$|K_w^\sigma(x, y', \theta')| \leq \left[(y + y')^{d+1} \left(1 + \left|\frac{y - y'}{y + y'}\right|^{k'} + \left|\frac{\theta - \theta'}{y + y'}\right|^{k}\right)\right]^{-1}.$$

As a consequence, we have to estimate:

$$\sum_{\gamma \in \Lambda} |K_w^\sigma(x, y', \theta' + \gamma)| \leq \sum_{\gamma \in \Lambda} \left[(y + y')^{d+1} \left(1 + \left|\frac{y - y'}{y + y'}\right|^{k'} + \left|\frac{\theta - \theta' + \gamma}{y + y'}\right|^{k}\right)\right]^{-1}$$

$$\leq \left[(y + y')^{d+1} \left(1 + \left|\frac{y - y'}{y + y'}\right|^{k'}\right)\right]^{-1} \sum_{\gamma \in \Lambda} \left[1 + \left|\frac{\theta - \theta' + \gamma}{y + y'}\right|^{k}\right]^{-1}.$$
Since \( y + y' > a \) the function in the sum has bounded variation, so we can apply a series-integral comparison, and replace it by the integral.

\[
\leq \frac{C(y + y')^d}{(y + y')^{d+1}} \left( 1 + \frac{|y - y'|}{|y + y'|} \right) \int_{\gamma \in \mathbb{R}^d} \left[ 1 + \frac{|y - y'|}{|y + y'|}^{k'} \right]^{-1} \left[ 1 + \frac{|y - y'|}{|y + y'|}^{k''} \right]^{-1} dy
\]

We deduce that

\[
\| \text{Op}(\varphi_j) \|_{y^d L^1 \rightarrow L^\infty}^2 \leq \sup_{x,x'} y^{d+1} y'^p \left( y + y' \right) \left( 1 + \frac{|y - y'|}{|y + y'|} \right)^{1-d/k} \left[ 1 + \frac{|y - y'|}{|y + y'|}^{k''} \right]^{-1} .
\]

This is bounded for \( \rho = -d \). We conclude that \( \text{Op}(\sigma) \) is bounded from \( y^{-d} L^1 \) to \( L^\infty \).

Now, we recall that for \( h > 0 \) small enough, \( (-\Delta + h^{-2})^{-\rho(d+1+\epsilon)/2} = \text{Op}(\sigma_{d+1+\epsilon}) + R \), with \( R \) smoothing, and \( \sigma_{d+1+\epsilon} \in S^{-d-1-\epsilon} \). For \( f \in y^{-d} W^{d+1+\epsilon,1} \), writing

\[
f = (-\Delta + h^{-2})^{-\rho(d+1+\epsilon)/2} \left(-\Delta + h^{-2}\right)^{+(d+1+\epsilon)/2} \Big|_{y^{-d} L^1} f,
\]

we deduce that \( y^{-d} W^{d+1+\epsilon,1} \hookrightarrow C^0 \) for \( \epsilon > 0 \). By interpolation, we then deduce that \( y^{-d/2} W^{(d+1+\epsilon)/2,2} = y^{-d/2} H^{(d+1+\epsilon)/2} \hookrightarrow C^0 \).

4.5. Improving parametrices II. In this section, we will explain how one can prove Theorem 3 in the case of operators acting on Hölder-Zygmund spaces on cusps.

Let us gather the conditions for an operator to be \( \mathbb{R}L^\infty \)-admissible.

**Definition 4.3.** Let \( A \in \Psi^m_{\text{small}}(N, L) \), and for each cusp \( Z, I_Z(A) \in \Psi^m(\mathbb{R} \times F_Z, L_Z) \) a pseudo-differential convolution operator. We will say that \( A \) is \( \mathbb{R}L^\infty \)-admissible with indicial operator \( I_Z(A) \) if the following holds. There exist some cutoff function \( \chi \in C^\infty([a, +\infty]) \), such that \( \chi \equiv 1 \) on \( y > C \) for some \( C > 2a \),

\[
\chi[\partial_\theta, A] \chi \quad \text{and} \quad E_Z \chi [\mathcal{P}_Z A \mathcal{E}_Z - I_Z(A)] \chi \mathcal{P}_Z,
\]

are operators bounded from \( y^N C^*_s - N \) to \( y^{-N} C^*_s \), for all \( N \in \mathbb{N} \). The operator \( I_Z(A) \) is independent of \( \chi \).

In the proof of Theorem 3 in the case of \( L^2 \)-admissible operators, the main ingredients were the existence of the inverse of the indicial operator and the compactness of some injections. Translating the proof to the case of Hölder-Zygmund spaces, the compactness of the corresponding injections is still assured.
Lemma 4.10. For any $\rho \in \mathbb{R}$, $s > s'$, the restriction of the injection $y^\rho C^s(N, L) \hookrightarrow y^\rho C^{s'}(N, L)$ to non-constant Fourier modes is compact. In other words, if $\chi \in C^\infty([a, +\infty[)$ is a smooth cutoff function such that $\chi \equiv 1$ for $y > 2a$ and vanishing around $y = a$, then

$$1 - E_Z \chi \mathcal{P}_Z : y^\rho C^s(N, L) \to y^\rho C^{s'}(N, L)$$

is compact.

Proof. We follow the proof of Lemma 3.1. As in that proof, it is sufficient to prove that $\|(1 - \psi_n)f\|_{C^Q} \leq C/n\|f\|_{C^s_{\rho_0}}$ for some $s_0 > 0, C > 0$ and then to conclude by interpolation. Since $L^\infty \hookrightarrow C^0$ and $C^{1+\epsilon} \hookrightarrow C^1$ (for any $\epsilon > 0$), it is therefore sufficient to prove that $\|(1 - \psi_n)f\|_{L^\infty} \leq C/n\|f\|_{C^1}$. By Poincaré-Wirtinger’s inequality, there exists a constant $C > 0$ (only depending on the lattice $\Lambda$) such that for any $f$ such that $\int f d\theta = 0$, $\|f(y)\|_{L^\infty(\mathbb{T}^d)} \leq C\|\hat{\partial}_y f(y)\|_{L^\infty(\mathbb{T}^d)}$, for all $y > a$. Thus, $\|(1 - \psi_n)f(y)\|_{L^\infty(\mathbb{T}^d)} \leq C/n\|\hat{\partial}_y f(y)\|_{L^\infty(\mathbb{T}^d)}$ and passing to the supremum in $y$, we obtain the sought result. □

The fact that the indicial operator has a bounded inverse is however a bit more subtle. For simplicity, assume there are no indicial roots in $\{ \Re \lambda \in I \} \supset i\mathbb{R}$, and consider the action of

$$S_I = \int_{i\mathbb{R}} e^{\lambda(r-r')}(I_Z(A, \lambda))^{-1} d\lambda,$$

on $C^s(\mathbb{R} \times F_Z)$. While the action of convolution operators on $L^2$ spaces is very convenient to analyze, it is not so easy for Hölder-Zygmund spaces. First, from the computations in the proof of Lemma 3.4, we deduce that the $C^s$ spaces of $L \to N$, correspond with the usual $C^s$ spaces of $L_Z \to \mathbb{R} \times F_Z$.

Next, we prove the following lemma

Lemma 4.11. Assume that $\text{Op}(\sigma)$ is admissible. Then $I_Z(\text{Op}(\sigma), \lambda)$ is $\langle \Im \lambda \rangle^{-1}$-semiclassically elliptic, i.e it can be written as $\text{Op}_h(\tilde{\sigma}) + \mathcal{O}(h^\infty)$, where $h = \langle \Im \lambda \rangle^{-1}$, the remainder is a smoothing operator, and both $\tilde{\sigma}$ and $1/\tilde{\sigma}$ are symbols.

Proof. Let us express the kernel of $I_Z(\text{Op}(\sigma))$ (in local charts in $F_Z$) as

$$\int e^{i\Phi(r, r', z, \eta)}\chi(r - r')\tilde{\sigma} \left( \frac{z + z'}{2}, \lambda, \eta \right) \frac{2e^{(r+r')/2}}{e^r + e^{r'}} \frac{d\eta d\lambda}{(2\pi)^{1+d}},$$

with

$$\Phi = \langle z - z', \eta \rangle + 2\lambda \tanh \frac{r - r'}{2}.$$

As a consequence, $I_Z(\text{Op}(\sigma), \lambda) = \text{Op}(\sigma_\lambda)$ with

$$\sigma_\lambda = \frac{1}{2\pi} \int e^{-\lambda u + 2\mu \tanh \frac{u}{2}} \chi(u) \frac{\sigma(z, \mu, \eta) d\mu}{\cosh \frac{u}{2}}.$$
This integral is stationary at $\mu = i\Re \lambda$, $u = 0$, with compact support in $u$, and symbolic estimates in $\mu$. So we get $\sigma_\lambda \in S^m$, with the refined estimates

$$\tag{23} |\partial^\alpha_x \partial^\beta_t \sigma_\lambda| \leq C_{\alpha, \beta}(1 + |\Re \lambda|^2 + |\eta|^2)^{(m-\beta)/2},$$

with constants $C_{\alpha, \beta}$ locally uniform in $\Re \lambda$. We deduce from this that $I_Z(\text{Op}(\sigma), \lambda)$ is semi-classical with parameter $h = (\Re \lambda)^{-1}$. Since $\sigma$ was elliptic, we also get for $|\lambda|^2 + |\eta|^2 > 1/c^2$:

$$\sigma_\lambda = \sigma(z, \lambda, \eta) \left( 1 + \mathcal{O}\left( \frac{|\Re \lambda|}{(|\Re \lambda|^2 + |\eta|^2)^{1/2}} \right) \right).$$

As a consequence, $I_Z(\text{Op}(\sigma), \lambda)$ is elliptic for all $\lambda$, and is semi-classical elliptic as $h \to 0$, so it is invertible for $h$ small enough. \hfill \Box

From this, we deduce that $I_Z(A, \lambda)^{-1}$ is also pseudo-differential, and $(\Re \lambda)^{-1}$-semi-classically elliptic and that $S_I$ is pseudo-differential. More precisely, we recall from the proof of Lemma 3.7 that if $QA = 1 + R$ is a first parametrix for $A$, then we can write for $|\Re \lambda| \gg 0$ large enough

$$I_Z(A, \lambda)^{-1} = I_Z(Q, \lambda)(1 + I_Z(R, \lambda))^{-1} = I_Z(Q, \lambda) + I_Z(Q, \lambda)I_Z(R, \lambda)(1 + I_Z(R, \lambda))^{-1} = \text{Op}(\sigma_\lambda) + R_\lambda,$$

where $\sigma_\lambda \in S^{-m}$ satisfies the symbolic estimates (23) with $m$ replaced by $-m$ and $R_\lambda$ is a $\mathcal{O}((\Re \lambda)^{-\infty})$ smoothing operator. Note that, in (23), $\sigma_\lambda$ also satisfies the symbolic estimate when differentiating with respect to $\lambda$. Writing $\tilde{\sigma}(\lambda, z, \eta) := \sigma_\lambda(z, \eta)$, we have that $\tilde{\sigma} \in S^{-m}(\mathbb{R} \times F_Z)$ (and is independent of $r$).

We write $S_I = S_I^{(1)} + S_I^{(2)}$, the operators respectively obtained from the contributions of $\text{Op}(\sigma_\lambda)$ and $R_\lambda$ in the formula (22). Choosing local patches in $F_Z$, we can write

$$S_I^{(1)} f(r, \zeta) = \int_{\mathbb{R} \times \mathbb{R}^n} e^{i\lambda(r-r')} e^{i(\zeta-z') \cdot \eta} \tilde{\sigma}(\lambda, z, \xi) f(r', \zeta') dr' d\zeta' d\lambda d\eta,$$

and this is a classical pseudodifferential operator of order $-m$ on $R \times F_Z$ which is bounded as a map $C^*_s(\mathbb{R} \times F_Z) \to C^*_s(\mathbb{R} \times F_Z)$.

It remains to study $S_I^{(2)}$. For the sake of simplicity, we will confuse in our notations the operator and its kernel. We pick $z, z' \in F_Z$ and $r > 1$. When $|\rho| < \epsilon$, $S_I^{(2)}(r, z, z')$ is a classical pseudodifferential operator of order $-m$ on $R \times F_Z$ which is bounded as a map $C^*_s(\mathbb{R} \times F_Z) \to C^*_s(\mathbb{R} \times F_Z)$.

$$S_I^{(2)}(r, z, z') = \int_{\mathbb{R}} e^{itr} R_{it}(z, z') dt = e^{itr} \int_{\mathbb{R}} e^{itr} R_{it+\rho}(z, z') dt,$$

where $R_{it+\rho}$ is $\mathcal{O}((t^{-\infty})$ in $C^\infty(F_Z \times F_Z)$, for $|\rho| < \epsilon$. We deduce that $S_I^{(2)}(r, z, z')$ is $\mathcal{O}(e^{-\epsilon |r|})$ in $C^\infty(\mathbb{R} \times F_Z \times F_Z)$. In particular, $S_I^{(2)}$ acts boundedly as a map $C^*_s(\mathbb{R} \times F_Z) \to C^*_s(\mathbb{R} \times F_Z)$. Now that we have checked that $S_I$ is bounded on
the appropriate spaces, the proof of Section §3.5 applies. This finishes the proof of Theorem 3.

4.6. **Fredholm index of elliptic operators II.** We now state a result concerning the Fredholm index of elliptic operators acting on Hölder-Zygmund spaces. It is similar to Proposition 3.3.

**Proposition 4.3.** Let $P$ be a $(\rho_-, \rho_+)$-$L^\infty$ and $-L^2$ admissible elliptic pseudodifferential operator of order $m \in \mathbb{R}$. Let $I$ be a connected component in $(\rho_-, \rho_+)$ not containing any indicial root. Then, the Fredholm index of the bounded operator $P : y^\rho C_s^{n+m} \to y^\rho C_s^n$ is independent of $s \in \mathbb{R}, \rho \in I$. Moreover, the Fredholm index coincides with that of Proposition 3.3, that is of $P$ acting on Sobolev spaces $H^{s+m, \rho-d/2, \rho_+} \to H^{s, \rho-d/2, \rho_+}$, for $s, \rho_+ \in \mathbb{R}$.

**Proof.** This is a rather straightforward consequence of Proposition 3.3 combined with the embedding estimates of Lemma 4.7 and Lemma 4.8. \qed

This concludes the proof of Theorem 3.

5. **X-ray transform and symmetric tensors**

In this Section, we apply the previous theory of inversion of elliptic pseudodifferential operators to the three operators $\nabla_S, D$ and $D^*D$ and prove that the X-ray transform is solenoidal injective on 2-tensors.

5.1. **Gradient of the Sasaki metric.** A first step towards the Livsic Theorem 4 is the analytic study of the gradient $\nabla_S$ induced by the Sasaki metric $g_S$ (itself induced by $g$) on the unit tangent bundle $SM$ of $(M, g)$.

We recall that the tangent bundle to $SM$ can be decomposed according to:

$$T(SM) = \mathbb{V} \oplus^\bot \mathbb{H} \oplus^\bot \mathbb{R}X,$$

where $\mathbb{H}$ is the horizontal bundle, $\mathbb{V}$ is the vertical bundle and $SM$ is endowed with the Sasaki metric $g_S$. If $\pi : TM \to M$ denotes the projection on the base, then $d\pi : \mathbb{H} \oplus^\bot \mathbb{R}X \to TM$ is an isomorphism, and there also exists an isomorphism $\mathcal{K} : \mathbb{V} \to TM$ called the connection map. We denote by $\nabla_S$ the Levi-Civita connection induced by the Sasaki metric $g_S$ on $SM$. Given $u \in C^\infty(SM)$, one can decompose its gradient according to:

$$\nabla_S u = \nabla^v u + \nabla^h u + Xu \cdot X,$$

where $\nabla^v, \nabla^h$ are the respective vertical and horizontal gradients (the orthogonal projection of the gradient on the vertical and horizontal bundles), i.e. $\nabla^v u \in \mathbb{V}, \nabla^h u \in \mathbb{H}$. 
Lemma 5.1. The gradient $\nabla_S : C^\infty(SM) \to C^\infty(SM, T(SM))$ is an elliptic $\mathbb{R}$-$L^2$ and $\mathbb{R}$-$L^\infty$ admissible differential operator of order 1. Its only indicial root is 0. Moreover, there exists two $[0, +\infty]$-$L^2$ and $L^\infty$ admissible pseudodifferential operators $Q, R$ of order $-1, -\infty$ such that:

$$Q \nabla_S = 1 + R$$

with $R$ bounded from $H^{-N,-d/2+\rho,\rho_\perp}$ to $H^{N,-d/2+\epsilon,\rho_\perp}$ and from $y^\rho C^\ast$ to $y^\rho C^\ast$ for all $d/2 > \epsilon > 0, N \in \mathbb{N}, \rho > 0, \rho_\perp \in \mathbb{R}$.

Proof. The fact that $\nabla_S$ is an elliptic admissible differential operator of order 1 is immediate. We compute its indicial operator. Let $TZ \simeq [a, +\infty) \times \mathbb{T}^d \times \mathbb{R} \times \mathbb{R}^d$ be a global trivialization of the tangent space to the cusp with coordinates $(y, \theta, v_y, v_\theta)$. Let $f \in C^\infty(\mathbb{R}^{d+1})$ be a smooth 0-homogeneous function. Then:

$$y^{-\lambda} \nabla_S (fy^\lambda) = \nabla^v f + \lambda f d\pi^{-1}(y \partial_y) + \sum_{i=1}^{d} (R_i f) d\pi^{-1}(y \partial_{v_i})$$

where $R_i := -v_{\theta_i} \partial_{v_y} + v_y \partial_{v_{\theta_i}}$ and $\nabla_S$ actually denotes the gradient on the whole tangent bundle $TM$. We set $I(Q, \lambda)(Z) := \lambda^{-1} g_S(Z, d\pi^{-1}(y \partial_y))$. Then:

$$I(Q, \lambda) I(\nabla_S, \lambda) f = f$$

The only indicial root of $\nabla_S$ is thus $\lambda = 0$. □

5.2. Exact Livsic theorem. We recall that $\mathcal{C}$ is the set of hyperbolic free homotopy classes on $M$ and that for each such class $c \in \mathcal{C}$ of $C^1$ curves on $M$, there is a unique representant $\gamma_g(c)$ which is a geodesic for $g$.

In this section, we prove an exact Livsic theorem asserting that a function whose integrals over closed geodesic vanish is a coboundary, namely a derivative in the flow direction. For $f \in C^0(SM)$, we can define

$$I^g f(c) = \frac{1}{\ell(\gamma_g(c))} \int_0^{\ell(\gamma_g(c))} f(\gamma(t), \dot{\gamma}(t)) dt,$$

for $c \in \mathcal{C}$.

Theorem 4 (Livsic theorem). Let $(M^{d+1}, g)$ be a negatively-curved complete manifold whose ends are real hyperbolic cusps. Denote by $-\kappa_0$ the maximum of the sectional curvature. Let $0 < \alpha < 1$ and $0 < \beta < \sqrt{-\kappa_0} \alpha$. Let $f \in y^\beta C^\alpha(SM) \cap H^1(SM)$ such that $I^g f = 0$. Then there exists $u \in y^\beta C^\alpha(SM) \cap H^1(SM)$ such that $f = Xu$. Moreover, $\nabla^v Xu, \nabla_X \nabla^v u \in L^2(SM)$ and $u$ thus satisfies the Pestov identity (Lemma 5.2).
We will denote by $N_\perp$ the subbundle of $TM \to SM$ whose fiber at $(x,v) \in SM$ is given by $N_\perp(x,v) := \{v\}^\perp$. Using the maps $d\pi$ and $\mathcal{K}$, the vectors $\nabla^v_hu$ can be identified with elements of $N_\perp$, i.e. $\mathcal{K}(\nabla^v u), d\pi(\nabla^h u) \in N_\perp$. For the sake of simplicity, we will drop the notation of these projection maps in the following and consider $\nabla^v_hu$ as elements of $N_\perp$. Before starting with the proof of the Livsic Theorem 4, we recall the celebrated Pestov identity:

**Lemma 5.2 (Pestov identity).** Let $(M^{d+1},g)$ be a cusp manifold. Let $u \in H^2(SM)$. Then

$$\|\nabla^v Xu\|^2 = \|\nabla_X \nabla^v u\|^2 - \int_{SM} \kappa(v, \nabla^v u)\|\nabla^v u\|^2 d\mu(x,v) + d\|Xu\|^2,$$

where $\kappa$ is the sectional curvature.

In the compact case, the proof is based on the integration of local commutator formulas and clever integration by parts (see [PSU15, Proposition 2.2]). Since the manifold has finite volume and no boundary, the proof is identical and we do not reproduce it here. By a density argument and using the fact that the sectional curvature is pinched negative, assuming only $\nabla^v Xu \in L^2(SM)$, we deduce that $\nabla_X \nabla^v u, \nabla^v u \in L^2(SM)$ and

$$\|\nabla_X \nabla^v u\|, \|\nabla^v u\| \lesssim \|\nabla^v Xu\|.$$

**Proof of Theorem 4.** In this proof, we will first build $u$, and then determine its exact regularity. For the construction, we follow the usual tactics, but we give the details since we want to let the Hölder constant grow at infinity. For the sake of simplicity, we will denote by $y : M \to \mathbb{R}_+$ a smooth extension of the height function (initially defined in the cusps) to the whole unit tangent of the manifold, such that $0 < c < y$ is uniformly bounded from below and $y \leq a$ on $M \setminus \bigcup_t Z_t$. The case of uniformly Hölder functions was dealt with in [PPS15, Remark 3.1]. Since the flow is transitive, we pick a point with dense orbit $x_0$, and define

$$u(\varphi_t(x_0)) = \int_0^t f(\varphi_s(x_0))ds.$$

Obviously, we have $Xu = f$, so it remains to prove that it is locally uniformly Hölder to consider the extension of $u$ to $SM$. Pick $x_1 = \varphi_t(x_0)$ and $x_2 = \varphi_{t'}(x_0)$, with $t' > t$. Pick $\epsilon > 0$, and assume that $d(x_1, x_2) = \epsilon$. By the Shadowing Lemma, there is a periodic point $x'$ with $d(x_1, x') < \epsilon$ and period $T < |t' - t| + C\epsilon$, for some uniform constant $C > 0$ depending on the dynamics, which shadows the segment $(\varphi_s(x_0))_{s \in [t,t']}$. Moreover, there exists a time $\tau \leq C\epsilon$ such that we have the following estimate:

$$d(\varphi_s(\varphi_{\tau}(x_1)), \varphi_s(x')) \leq C\epsilon e^{-\sqrt{\kappa}_0 \min\{|s|,|t' - t|\}}$$

(25)
This is a classical bound in hyperbolic dynamics (see [HF, Proposition 6.2.4] for instance). The constant $\sqrt{K_0}$ follows from the fact the maximum of the curvature is related to the lowest expansion rate of the flow (see [Kli95, Theorem 3.9.1] for instance).

Then, using the assumption that $\int_0^T f(\varphi_s(x'))ds = 0$, we write:

$$u(x_2) - u(x_1) = \int_0^{t' - t} f(\varphi_s(x_1))ds$$
$$= \int_0^{t' - t} f(\varphi_s(\varphi_t(x_1))) - f(\varphi_s(x'))ds - \int_{t' - t}^T f(\varphi_s(x'))ds + \int_0^T f(\varphi_s(x_1))ds$$

The last two terms are immediately bounded by $\lesssim \epsilon y(x_1)^\beta$. As to the first one, it is controlled by $\lesssim \int_0^{t' - t} y(\varphi_s(x'))^\beta d(\varphi_s(x_1), \varphi_s(x'))^\alpha$ using the assumption on $f$. Let us find an upper bound on $y(\varphi_s(x'))$. Of course, when a segment of the trajectory $(\varphi_s(x'))_{s \in [0,T]}$ is included in a compact part of the manifold (say of height $y \leq \alpha$), $y(\varphi_s(x'))$ is uniformly bounded by $\alpha$, so the only interesting part is when the trajectory is contained in the cusps. In time $|t' - t|$, the segment $(\varphi_s(x'))_{s \in [0,T]}$ has started and returned at height $y(x_1)$. Thus, it can only go up to a height

$$y(\varphi_s(x')) \leq e^{\min(s,|t' - t| - s)} y(x_1). \tag{26}$$

Combining (25) and (26), this leads to:

$$\int_0^{t' - t} y(\varphi_s(x'))^\beta d(\varphi_s(x_1), \varphi_s(x'))^\alpha$$
$$\lesssim \int_0^{t' - t} y(x_1)^\beta e^{\beta \min(s,|t' - t| - s)} d(x_1, x_2)^\alpha e^{-\alpha \sqrt{K_0} \min(s,(t' - t) - s)} ds$$
$$\lesssim y(x_1)^\beta d(x_1, x_2)^\alpha \int_0^{t' - t} e^{(\beta - \alpha \sqrt{K_0}) \min(s,|t' - t| - s)} ds$$

As long as $\sqrt{K_0} \alpha > \beta$, this is uniformly bounded as $|t' - t| \to +\infty$. In particular, we conclude that $u$ is $y^\beta C^\alpha$, and we can thus extend it to a global $y^\beta C^\alpha$ function on $SM$.

We now have to prove that $u \in H^1(SM)$ and to this end, we will use a kind of bootstrap argument. Since $f \in H^1(SM)$ and $f = Xu$, we obtain that $\nabla^v Xu \in L^2(SM)$. Moreover, as discussed after the Pestov identity, we obtain directly that $\nabla_x \nabla^v u, \nabla^v u \in L^2(SM)$.

By using the commutator identity $[X, \nabla^v] = -\nabla^h$ (see [PSU15, Lemma 2.1]), we deduce $\nabla^h u \in L^2(SM)$. Thus, $\nabla_s u \in L^2$. By Lemma 5.1, we deduce that $u \in H^1(SM)$ \hfill $\Box$
5.3. **X-ray transform and symmetric tensors.** Although we will mostly use 1- and 2-tensors, it is convenient to introduce notations for general symmetric tensors. We will be using the injection

$$\pi_m : v \in C^\infty(M, SM) \to v \otimes \cdots \otimes v \in C^\infty(M, SM^\otimes m).$$

Given a symmetric $m$-tensor $h \in C^\infty(M, S^m(T^*M))$, we can define a function on $SM$ by pulling it back via $\pi_m$:

$$\pi_m^* h : (x, v) \mapsto h_x(v \otimes \cdots \otimes v).$$

**Definition 5.1.** The X-ray transform on symmetric $m$-tensors is defined in the same way as for $C^0$ functions on $SM$: if $h$ is a symmetric $m$-tensor,

$$I_m^0 h(c) = \frac{1}{\ell(\gamma(c))} \int_0^{\ell(\gamma(c))} \pi_m^* h(\gamma(t), \dot{\gamma}(t)) dt,$$

where $t \mapsto \gamma(t)$ is a parametrization by arclength, $c \in C$.

Given a symmetric $m$-tensor $h$, we can consider its covariant derivative $\nabla h$, which is a section of $T^* M \otimes S^m(T^* M) \to M$.

If $\sigma$ denotes the symmetrization operator from $\otimes^{m+1} T^* M$ to $S^{m+1}(T^* M)$, we define the symmetric derivative as

$$Dh = \sigma(\nabla h) \in C^\infty(M, S^{m+1}(T^* M)).$$

Given $x \in M$, the pointwise scalar product for tensors in $\otimes^m T_x^* M$ is defined by

$$\langle v_1^* \otimes \cdots \otimes v_m^*, w_1^* \otimes \cdots \otimes w_m^* \rangle_x = \prod_{j=1}^m g(v_j, w_j),$$

where $v_j, w_j \in T_x M$ and $v_j^*, w_j^*$ denotes the dual vector given by the musical isomorphism. We can then endow the spaces $C^\infty(M, S^m(T^* M))$ with the scalar product

$$(27) \quad \langle h_1, h_2 \rangle = \int_M \langle h_1(x), h_2(x) \rangle_x d\text{vol}(x)$$

We obtain a global scalar product on $C^\infty(M, S^m(T^* M))$ by declaring that whenever $m \neq m'$, $C^\infty(M, S^m(T^* M))$ is orthogonal to $C^\infty(M, S^{m'}(T^* M))$. Following conventions we denote by $-D^*$ the adjoint of $D$ with respect to this scalar product. One can compute that for a tensor $T$, for any orthogonal frame $e_1, \ldots, e_{d+1}$,

$$D^* T(\cdot) = \text{Tr}(\nabla T)(\cdot) = \sum_i \nabla_{e_i} T(e_i, \cdot).$$

The operator $D^*$ is called the *divergence*, and one can check that it maps symmetric tensors to symmetric tensors.
Definition 5.2. Let $f$ be a tensor so that $D^* f = 0$. Then we say that $f$ is solenoidal.

We can also define $\pi_{m*}$, which is the formal adjoint of $\pi^*_m$ --- with respect to the usual scalar product on $L^2(SM)$. Moreover, one can check that

$$\pi_{m+1}^* D = X \pi_m^*.$$  

Through $\pi_m^*$ we obtain another scalar product on symmetric tensors:

$$[u, v] = \int_{SM} \pi_m^* u \pi_m^* v.$$  

Representing $[u, v] = \langle Au, v \rangle$, one can check that there are universal constants $C_m > 0$ such that

$$\|A\| \leq C_m,$$

$$\|A^{-1}\| \leq C_m$$

when restricted to $m$-tensors.

In the following, we will restrict our study to 1- and 2-tensors but it is very likely that most of the results still hold for tensors of general order $m \in \mathbb{N}$. As in the compact case, we obtain:

Lemma 5.3. The symmetric derivative $D$ is $\mathbb{R}$-$L^2$ and $\mathbb{R}$-$L^\infty$ admissible. Its only indicial root is $-1$. Additionally, it is injective on $\gamma^p H^s$ and $\gamma^p C^s$ for all $p, s \in \mathbb{R}$. In particular, there is a $1 - 1, +\infty[-L^2$ (resp. $1 - 1, +\infty[-L^\infty$) admissible pseudo-differential operators $Q, R$ of order $-1, -\infty$ such that

$$QD = 1 + R.$$

In particular, the image of $D$ is closed.

Proof. Since $D$ is a differential operator, it makes no difference to work with Sobolev or Hölder-Zygmund spaces. The first step is to prove that $D$ is uniformly elliptic. We deal with the general case $m \geq 0$. By taking local coordinates around a point $(x, \xi) \in T^* M \setminus \{0\}$ for instance, one can compute the principal symbol of the operator $D$ which is $\sigma(D)(x, \xi) : u \mapsto \sigma(\xi \otimes u)$, where $u \in S^m(T^*_x M)$ (see [Sha94, Theorem 3.3.2]). Then, using the fact that the antisymmetric part of $\xi \otimes u$ vanishes in the integral:

$$\|\sigma(D)u\|^2 \geq C_m^{-1} \int_{S^d} \langle \xi, v \rangle^2 \pi^*_m u^2(v)dv = C_m^{-1} |\xi|^2 \int_{S^d} \langle \xi/|\xi|, v \rangle^2 \pi^*_m u^2(v)dv > 0,$$

unless $u \equiv 0$. Since $S^m(T^*_x M)$ is finite dimensional, the map

$$(u, \xi/|\xi|) \mapsto \|\sigma(D)(x, \xi/|\xi|)u\|,$$

defined on the compact set $\{u \in S^m, |u|^2 = 1\} \times S^d$ is bounded and attains its lower bound $C^2 > 0$ (which is independent of $x$). Thus $\|\sigma(x, \xi)u\| \geq C|\xi\|\|u\|$, so the operator is uniformly elliptic.

Next, let us give a word on the injectivity of $D$. Consider a 1-form $f$ such that $Df = 0$, and $f$ is either in some $\gamma^p H^s$ or some $\gamma^p C^s$. Then $f$ is smooth by the elliptic regularity Theorem. As a consequence $\pi_1^* f$ is a smooth function on $SM$. Recall that
\[ X\pi^*_1 f = \pi^*_2 D f = 0. \] Additionally, the geodesic flow admits a dense orbit; we deduce that \( \pi^*_1 f \) is a constant. However, since \( f \) is a 1-form, \( \pi^*_1 f(x, -v) = -\pi^*_1 f(x, v) \) for all \( (x, v) \in SM \), thus \( f = 0 \).

Now, we recall the results from Section §3. Since \( D \) is a differential operator that is invariant under local isometries, it is a \( \mathbb{R} \) admissible elliptic operator. In particular, it suffices to determine whether its associated indicial operator \( I_Z(D, \lambda) \) has a left inverse. In the present case, since \( D \) is an operator on sections of a bundle over \( M \), the indicial operator is just a matrix. We consider a 1-form \( \alpha \) in the cusp in the form

\[
y^\lambda \left[ a \frac{dy}{y} + \sum b_i \frac{d\theta_i}{y} \right]
\]

Then we find that

\[
D\alpha = y^\lambda \left[ a \left( \frac{dy^2}{y^2} - \sum \frac{d\theta_i^2}{y^2} \right) + \sum b_i (\lambda + 1) \frac{d\theta_i dy + dy d\theta_i}{y^2} \right].
\]

The matrix \( I_Z(D, \lambda) \) is thus the transpose of

\[
\begin{pmatrix}
\lambda & -1 & -1 & \ldots & -1 & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & 0 & 0 & 2(\lambda + 1) & 0 & \ldots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2(\lambda + 1) & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 2(\lambda + 1)
\end{pmatrix}
\]

In particular, with

\[
J(\lambda) = \begin{pmatrix}
(\lambda + 1)^{-1} & - (\lambda + 1)^{-1} & 0 & \ldots & 0 & \ldots & 0 \\
0 & 0 & 0 & 0 & (2(\lambda + 1))^{-1} & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & \ldots & (2(\lambda + 1))^{-1}
\end{pmatrix}
\]

we get

\[
J(\lambda)I_Z(D, \lambda) = I; \quad \|J^{-1}\| = \mathcal{O}(|\lambda|^{-1}) \text{ as } \Im \lambda \to \pm \infty.
\]

We deduce that \( D \) has \(-1\) for sole indicial root. As a consequence, we can apply Lemma 3:

\[
(28) \quad QD = 1 + R,
\]

with \( R \) bounded from \( H^{s, \rho} \) to \( H^{N, -d/2 - 1 + \epsilon} \) and from \( C^{s, \rho}_* \) to \( C^{s, -1 + \epsilon}_* \), for all \( d/2 > \epsilon > 0, s \in \mathbb{R}, \rho > -d/2 + 1 \).

Let us now prove that the image of \( D \) is closed (for the Sobolev spaces, the case of Hölder-Zygmund spaces is similar). This is rather classical argument once one has an inverse for the operator modulo a compact remainder, but we reproduce it here for the reader's convenience. For a sequence \( (u_n) \) of elements of \( H^{1+s} \) such that \( Du_n \to f \in H^s \), \( QD u_n = u_n + Ku_n \) also converges since \( Q \) is continuous. By
extraction, since $K$ is compact we can assume that $R(u_n/\|u_n\|)$ converges also, to some $v$. Then, we have

$$u_n + \|u_n\|(v + o(1)) = Qf + o(1).$$

Assume that $\|u_n\|$ is bounded. Then we obtain that $u_n$ itself converges in $H^{1+s}$, to some $u$, and $Du = f$. Otherwise, we can decompose $u_n = \lambda_n v + w_n$, with $w_n \perp v$, $w_n$ bounded and $\lambda_n \to \infty$. We deduce that $Rv = -v$, and $QDu_n = QDw_n$, so that we can extract $w_n$ to make it converge to some $w$, and $Dw = f$. □

Since the image of $D$ is closed, it is the orthogonal of the kernel of $D^*$, and each $f \in H^s(M, S^2(T^*M))$ can be written as

$$f = f^s + Du,$$

with $D^*f^s = 0$, and $f^s \in H^s(M, S^2(T^*M))$, $u \in H^{1+s}(M, S^1(T^*M))$. The tensor $f^s$ is called the divergence-free part or the solenoidal part of $f$, and $Du$ the exact part or the potential part of $f$. This can be naturally generalized to tensors of any order and Hölder-Zygmund spaces, following the same scheme of proof.

To close this section, remark that the X-ray transform satisfies $IX = 0$ and thus $0 = IX\pi^*_m = I\pi^*_mD = ImD$. Thus in general it is impossible to recover the exact part $Dp$ of a tensor $f$ from the knowledge of $Imf$. We will say that the X-ray is solenoidal injective on smooth symmetric $m$-tensors if it is injective when restricted to ker $D^*$.

5.4. Projection on solenoidal tensors. In this section, we will study the symmetric Laplacian on 1-forms, that is the operator $\Delta := D^*D$ acting on sections of $S^1(T^*M) \to M$. We will denote by $\lambda^\pm_d = d/2 \pm \sqrt{d^2 + 4}$. 

**Lemma 5.4.** For all $s \in \mathbb{R}, \rho \in ]\lambda^+_{d}, \lambda^-_{d}[, \rho_\perp \in \mathbb{R}$, the operator $\Delta$ is invertible on the spaces $H^{s,\rho-d/2,\rho_\perp}(M, S^1(T^*M))$ and on $y^\rho C^s_s(M, S^1(T^*M))$. Its inverse $\Delta^{-1}$ is a pseudodifferential operator of order $-2$.

**Proof.** The operator $\Delta = D^*D$ is elliptic since $D$ is elliptic, and it is also invariant under local isometries, and differential. In particular, it is $\mathbb{R}$-$L^2$ and $\mathbb{R}$-$L^\infty$ admissible, so we can apply Theorem 3. Let us compute its indicial operator: we find

$$I(\Delta, \lambda) \left( a dy \right) = (\lambda^2 - \lambda d - d)a dy$$

$$I(\Delta, \lambda) \left( b_i d\theta^i_\perp \right) = \frac{1}{2}(\lambda + 1)(\lambda - (d + 1))b_i \frac{d\theta^i_\perp}{dy}$$
\( I(\Delta, \lambda) \) is a diagonal matrix which is invertible for

\[
\lambda \notin \left\{ -1, d + 1, \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + d} \right\}
\]

The interval \( \lambda_d^-, \lambda_d^+ \) does not contain other any roots, so we can apply directly Theorem 3, and get a pseudo-differential operator of order \(-2\), \( Q \), bounded on the relevant Sobolev and Hölder-Zygmund spaces such that

\[
(29) \quad Q\Delta = 1 + K,
\]

with \( K \) bounded from \( y^\rho H^{-N} \) to \( y^{-\rho} H^N \), \( y^{\rho+d/2} C_{-}^{-N} \) to \( y^{d/2-\rho} C_{+}^{N} \) for all \( \rho \in [0, \lambda_d^+ - d/2] \). We can also do this on the other side:

\[
(30) \quad \Delta Q = 1 + K',
\]

\( K' \) satisfying the same bounds. We deduce that \( \Delta \) is Fredholm. Additionally, from the parametrix equation, we find that any element of its kernel (on any Sobolev or Hölder-Zygmund space we are considering) has to lie in \( L^2(SM) \). However, on \( L^2 \), \( \Delta u = 0 \) implies \( Du = 0 \), and \( u = 0 \). Additionally, on \( L^2 \), \( \Delta \) is self-adjoint, so it is invertible and its Fredholm index is 0. We then conclude using Propositions 3.3 and 4.3.

\[\square\]

As a consequence, we obtain the

**Lemma 5.5.** \( \pi_{\ker D^*} = 1 - D\Delta^{-1}D^* \) is the orthogonal projection on solenoidal tensors. It is a \( \lambda_d^-, \lambda_d^+ \)-admissible operator operator of order 0.

### 5.5. Solenoidal injectivity of the X-ray transform.

We now prove Theorem 1. As usual, the proof relies on the Pestov identity combined with the Livsic theorem. It follows exactly that of [CS98]; nevertheless, we thought it was wiser to include it insofar as we only work in \( H^1 \) regularity on a noncompact manifold (where as [CS98] is written in smooth regularity on a compact manifold).

We recall that there exists a canonical splitting

\[
T_{(x,v)}(TM) = V_{(x,v)} \oplus H_{(x,v)},
\]

where \( (x, v) \in TM \) which is orthogonal for the Sasaki metric. We insist on the fact that we now work on the whole tangent bundle \( TM \) and no longer on the unit tangent bundle \( SM \). As a consequence, the horizontal space \( H \) is the same but the vertical space \( V \) sees its dimension increased by 1. These two spaces are identified to the tangent vector space \( T_xM \) via the maps \( dx \) and \( K \).

Given \( u \in C^\infty(TM) \), we can write \( \nabla_S u = \nabla^v u + \nabla^h u \), where \( \nabla^v u \in V, \nabla^h u \in H \). We denote by \( \text{div}^{v,h} \) the formal adjoints of the operators \( \nabla_S^{v,h} \).
Proof. We first start with an elementary inequality. Let \( u \in C^\infty(SM) \). We extend \( u \) to \( TM \setminus \{0\} \) by 1-homogeneity. The local Pestov identity \([CS98, Equation (2.14)]\) at \((x,v) \in TM\) reads:
\[
2 \langle \nabla^h u, \nabla^v(Xu) \rangle = |\nabla^h u|^2 + \text{div}^h Y + \text{div}^v Z - \langle R(v,\nabla^v u)v, \nabla^v u \rangle
\]
where
\[
Y := \langle \nabla^h u, \nabla^v u \rangle v - \langle v, \nabla^h u \rangle \nabla^v u \\
Z := \langle v, \nabla^h u \rangle \nabla^h u
\]
Moreover, \( \langle v, Z \rangle = |Xu|^2 \). Integrating over \( SM \) and using the Green-Ostrogradskii formula \([Sha94, Theorem 3.6.3]\) together with the assumption that the curvature is nonpositive, we obtain:
\[
(\text{31}) \quad \int_{SM} \|\nabla^h u\|^2 d\mu \leq 2 \int_{SM} \langle \nabla^h u, \nabla^v(Xu) \rangle d\mu - (3 + d) \int_{SM} \langle v, Z \rangle d\mu
\]
Note that by a density argument, the previous formula extends to functions \( u \in H^1(SM) \) such that \( \nabla^v(Xu) \in L^2(SM) \).

We now consider the case where \( \pi_m^* f = Xu \) with \( f \in H^1 \) (and thus \( u \in H^1 \) and \( \nabla^v(Xu) \in L^2 \) by the arguments given in the proof of Livsic theorem). Following \([CS98, Equation (2.18)]\), one obtains the following equality almost-everywhere in \( TM \):
\[
2 \langle \nabla^h, \nabla^v(Xu) \rangle = \text{div}^h W - 4 \times u\pi_m^*(D^*f),
\]
with \( W(x,v) = 4u(x,v)(f_x(\cdot, v, \ldots, v))^\sharp \) (where \( \sharp : T^*M \to TM \) is the musical isomorphism). In (31), this yields
\[
(\text{32}) \quad \int_{SM} \left( |\nabla^h u|^2 + (3 + d)|Xu|^2 \right) d\mu \leq -4 \int_{SM} u\pi_m^*(D^*f)d\mu
\]

We now assume that \( f \) is a symmetric \( m \)-tensor in
\[
y^\beta C^\alpha(M, S^m(T^*M)) \cap H^1(M, S^m(T^*M)),
\]
such that \( D^*f = 0 \) and \( I_m(f) = 0 \). By the Livsic Theorem 4, there exists \( u \in y^\beta C^\alpha(SM) \cap H^1(SM) \) such that \( \pi_m^* f = Xu \). By (32), we obtain \( Xu = 0 \), thus \( \tilde{f} = 0 \). \( \square \)

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YANNICK GUEDES BONTHONNEAU AND THIBAULT LEFEUVRE

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