Tests of Universality in AdS/QCD

Joshua Erlich

Department of Physics, College of William and Mary, Williamsburg, Virginia 23187-8795

Christopher Westenberger

Department of Physics, The College of New Jersey, Ewing, New Jersey 08618-1104

Estimates of the light hadron masses, decay constants and couplings in AdS/QCD models are generally more accurate than should have been expected. Certain predictions based on the AdS/CFT correspondence, such as the ratio of the equilibrium viscosity to entropy density, are universal and therefore provide firm experimental tests of these models. Other observables, while not completely universal, may be relatively insensitive to model details. We calculate the dependence of a number of low-energy hadronic observables on details of the hard-wall AdS/QCD model. In particular, we vary the infrared boundary conditions, the 5D gauge coupling, and the mass of the field responsible for chiral symmetry breaking, while holding fixed a small number of observables. We also find a generalized Gell-Mann–Oakes–Renner relation which helps to justify the identification of model parameters with the product of physical quark mass and chiral condensate as per the AdS/CFT correspondence.

Introduction—AdS/QCD is a class of extra-dimensional models in which the light mesons of quantum chromodynamics (QCD) are interpreted as Kaluza-Klein modes. In recent years, AdS/QCD models have proved surprisingly successful at reproducing low-energy hadronic observables such as masses and decay constants. Most low-energy observables that have been calculated in AdS/QCD models agree with experiment at the 10-20% level.

AdS/QCD was originally motivated by the AdS/CFT correspondence in the context of models that confine with chiral symmetry breaking, although QCD with three colors certainly does not have a weakly-coupled AdS/CFT dual via this correspondence. While it is possible to reproduce a Regge-like spectrum at high energies, AdS/QCD models generally make poor predictions when extrapolated to high energies (for a recent discussion, see Refs. [7, 8]). These models typically fail above 2 GeV, where the QCD coupling is small and the AdS/CFT correspondence suggests that a classical extra-dimensional description should not provide a good approximation to QCD.

Top-down AdS/QCD models arise from string theory, with the D4-D8 system describing a gauge theory which is confining with non-Abelian chiral symmetry breaking in QCD. So far all models rooted in string theory contain states not included in the QCD spectrum. Furthermore, the classical limit of the AdS/CFT correspondence requires the number of colors, N, and the 't Hooft coupling, \( g_s^2 N \), to be large. Nonetheless, predictions of top-down AdS/QCD models, naively extrapolated to three colors, fare relatively well when compared to experimental data. Bottom-up models are also motivated by the AdS/CFT correspondence, but are more phenomenological and allow more freedom to match QCD data. The AdS/CFT correspondence has also motivated extra-dimensional models of electroweak symmetry breaking and condensed matter systems. With a bit of hindsight, we address the question of why AdS/QCD models have been so successful for modeling static properties of the light hadrons.

Chiral symmetry breaking and QCD sum rules are the underlying principles of this approach, as in Refs. [12, 13]. Both top-down models such as the D4-D8 system and bottom-up models such as the hard-wall and soft-wall models generally make similar predictions for low-energy observables, despite the vastly different effective (4+1)-dimensional spacetime geometries. Certain predictions are known to be exactly universal in models based on the classical limit of the AdS/CFT correspondence, such as the shear viscosity-to-entropy density ratio in thermal equilibrium \( \eta/s = 1/(4\pi) \) (which turns out to be in agreement up to a factor of a few with experimental estimates for the quark-gluon plasma from RHIC data). Model-independent predictions serve as crucial tests of the AdS/QCD approach. Conversely, it may be that the success of AdS/QCD models below 2 GeV is the result of universality of predictions in that regime.

Focusing on hard-wall models, we will demonstrate that a number of low-energy observables continue to provide an unreasonably good fit to data as the squared mass \( m_X^2 \) of the field responsible for chiral symmetry breaking in the model is varied over a reasonable range, hinting at universality of AdS/QCD predictions for those observables. However, we find some sensitivity to the 5D gauge coupling \( g_5 \) and the boundary conditions arising from localized kinetic terms for the 5D gauge fields. Our comparison to experimental data suggests that in the space of these parameters, there is a saddle point in the root-mean-squared error near the values determined by naive application of the AdS/CFT correspondence while matching to the ultraviolet (UV). The fact that we can successfully extrapolate certain results from low energies where QCD is strongly coupled, to high energies where QCD is weakly coupled, remains mysterious. We will also generalize a derivation of the Gell-Mann–Oakes–Renner relation for the pion mass and decay constant in terms of AdS/QCD model parameters.
AdS/QCD—In this section we review the hard-wall AdS/QCD model. We would like to identify the towers of radial excitations of mesons in QCD with Kaluza-Klein modes of fields in an extra dimension. The spins, parities, and charges of the QCD states correspond to the spins, parities, and charges of the Kaluza-Klein modes. The charges for now will refer to SU(2) isospin representations.

To reflect the approximate SU(2) x SU(2) chiral symmetry of the light quarks, we begin with a SU(2) x SU(2) gauge theory in some (4+1)-dimensional (5D) spacetime background. The Kaluza-Klein decomposition of the gauge fields should include a massive tower of modes, but no zero mode. The global chiral symmetry of the (3+1)-dimensional (4D) theory is lifted to a higher-dimensional gauge invariance in the spirit of hidden local symmetry [17], as in the deconstructed extra-dimensional model of Son and Stephanov [18]. The bulk 5D metric and the background profile serve 4D Lorentz invariance in the effective 4D theory. The metric takes the form

\[ ds^2 = e^{-A(z)} (dx_\mu dx^\mu - dz^2), \]

where the Greek spacetime indices run from 0 to 3, and the function \( A(z) \) describes the warping of the spacetime. The geometry is manifestly isometric under 4D Lorentz transformations acting on \( x^\mu \) but not \( z \).

A 5D scalar field charged under the SU(2) x SU(2) gauge group is introduced in order to dynamically break the diagonal SU(2) isospin subgroup. Alternatively, or in addition, the boundary conditions can be chosen such as to break the chiral symmetry [19, 20], but we will not consider this possibility here. The 5D action is,

\[
S_{5D} = -\frac{1}{8\kappa^2_5} \int d^5x \sqrt{-g} \left( L_{MN} a^{\alpha} + R_{MN} R^{\alpha} \right) g^{MP} g^{NQ} + \text{Tr} \left( |D_M X|^2 - m_X^2 |X|^2 \right),
\]

where the capital Latin spacetime indices run from 0 to 4, the gauge index \( \alpha \) runs from 1 to 3, and \( m_X^2 \) is the squared mass of the scalar field \( X \). \( L_{MN} \) and \( R_{MN} \) are the field strength tensors of the two SU(2) gauge group factors. We normalize the gauge fields as in Ref. [12].

4D parity corresponds to the transformation \( x^i \rightarrow -x^i, \ i = 1, 2, 3 \), together with an exchange \( L_{M}^a \leftrightarrow R_{M}^a \). The spectrum contains vector mesons, axial vector mesons, pseudoscalars, and scalars. If we allow gravity to fluctuate, then there are also spin-two modes that can be identified with a tower of spin-two glueballs [13]. With a given geometry (specified by \( A(z) \)), the free parameters in the model are \( g_s, z_m, m_X \), and the background profile \( v(z) \), which should solve the equation of motion for the field \( X \) in the background spacetime. Since the equation of motion for \( X \) is a second order differential equation, there are two parameters in its solution. The field \( X \) carries the quantum numbers of the quark bilinear \( \bar{q}_L q_R \). The AdS/CFT correspondence identifies the non-normalizable solution (i.e. the solution with infinite action) with the source for an operator charged under the chiral symmetry like the field \( X \) [21, 22], i.e. the quark mass \( m_q \) up to an overall normalization; the normalizable solution (i.e. the solution with finite action) is related to the expectation value of the corresponding operator [23, 24], in this case the chiral condensate \( \sigma = \langle \bar{q}_L q_R \rangle = 2 \text{Re} \langle \bar{q}_L q_R \rangle \), where \( q \) here is either the up or down quark. These identifications follow from the AdS/CFT correspondence, but we need not a priori make these identifications in order to define the AdS/QCD model. We also do not include the gravitational backreaction of the background scalar field [21, 22].

There are a number of ways one could choose the function \( A(z) \) in the metric. We could choose the geometry so as to obtain the physical spectrum of vector mesons, or to at least obtain the linear confining spectrum as in the soft-wall model [6]. Alternatively, although we do not expect the model to accurately describe QCD at high energies, we may choose the geometry so as to best match the high-energy behavior of certain correlation functions of QCD. The choice of the AdS geometry in the UV region (where the warp factor \( \exp[-A(z)] = R^2/z^2 \) is largest) is equivalent to an assumption of conformal behavior of QCD at high energies compared to the mass of the rho meson. In the hard-wall model the geometry is assumed for simplicity to be a slice of AdS between a UV cutoff length \( z = \epsilon \rightarrow 0 \) and an IR length scale \( z = z_m \). There is still freedom in specifying boundary conditions for 5D fields in the IR region of the geometry. The slice of Anti-de Sitter space with boundary conditions of the form \( L_{zz}(x, z_m) = R_{zz}(x, z_m) = 0 \) for the gauge fields leads to a spectrum and decay constants which agree with a Padé approximation to the two-point function of isospin currents about a point in the deep Euclidean regime [27]. In the spirit of deconstructed extra dimensions, we can observe the emergence of the radial direction of Anti-de Sitter space as the order of the Padé approximation becomes large [28]. In order to reproduce power corrections to correlation functions in the UV, including the dependence on the running coupling, we could add higher dimension operators to the action and modify the geometry away from Anti-de Sitter space in the UV region [29]. However, for simplicity we will focus on the simple hard-wall model in our investigation, with the geometry described above: a slice of Anti-de Sitter space described by the metric in Eq. (1) with \( A(z) = \log(z^2/R^2) \), between \( z = \epsilon \) and \( z = z_m \). The AdS curvature \( R \) can be absorbed into the definitions of \( g_s \) and \( m_X \), so we will generally rescale those parameters and set \( R = 1 \).

If we naively follow the AdS/CFT correspondence we can fix the squared mass \( m_X^2 \) by its relation to the conformal dimension of the quark bilinear in the ultraviolet as in Refs. [12, 13]. According to the AdS/CFT correspon-
dence, the mass of the field $X$ is related to the scaling dimension $\Delta$ of the corresponding operator, $T q$, via

$$m_X^2 = \Delta(\Delta - 4).$$

(3)

In the ultraviolet $\Delta = 3$, and we would infer $m_X^2 = -3$. However, the AdS/CFT correspondence does not directly apply, and this identification of the $X$ mass ignores renormalization effects which tend to reduce the scaling dimension at lower energies. We will refer to $\Delta$ as the effective scaling dimension, and we will allow $\Delta$ to vary in our analysis. The solution to the equations of motion for the background $X$-field takes the form $X(z) = v(z)1$, with

$$v(z) = m_q z^4 - \Delta + \frac{\sigma}{4(\Delta - 2)} z^\Delta.$$

(4)

We assume in this analysis that $\Delta > 2$. To the extent that the model parameters are related to the physical quark mass and chiral condensate, the factor of $1/(4(\Delta - 2))$ multiplying the condensate $\sigma$ follows from the AdS/CFT correspondence, and we assume here that the parameters $m_q$ and $\sigma$ are real (35).

**Correlation Functions and Observables** — We calculate seven observables: $m_\pi$, $f_\pi$, $m_\rho$, $f_\rho$, $m_\sigma$ and $g_\sigma \pi \pi$, where $m_\pi$ are meson masses, $F$'s are the corresponding decay constants, and $g_\sigma \pi \pi$ describes the coupling of the $\rho$ to pions. For this analysis we choose to fix the parameters $z_m$, $m_q$ and $\sigma$ by matching to $m_\rho = 775.8$ MeV, $f_\pi = 92.4$ MeV, and $m_\sigma = 139.6$ MeV.

Kaluza-Klein modes are the nonvanishing solutions to the classical equations of motion subject to the appropriate boundary conditions. We work in the axial gauge $L_\mu^z = R_\mu^z = 0$ and consider transverse fluctuations with $\delta H L_\mu^z = \delta \rho R_\mu^z = 0$. The transverse part of the diagonal SU(2) gauge fields $V_\mu^a = (L_\mu^a + R_\mu^a)_{\perp}/2$ satisfy the linearized equation of motion,

$$\left[\partial_z \left(\frac{1}{z} \partial_z V_\mu^a / z, \partial_z V_\mu^a / z\right)\right]_{\perp} = 0.$$

(5)

The Kaluza-Klein modes, Fourier transformed in 3+1 dimensions, are solutions of the form

$$V_\mu^a(q_n, z) = \epsilon^a(q_n) \psi_n(z),$$

(6)

where $q_n^2 = m_n^2$ is the squared mass of the $n^{th}$ Kaluza-Klein mode, and $\psi_n(z)$ satisfies the boundary conditions $\psi_n(0) = 0$. The wavefunction $\psi_n(z)$ is normalized so that the kinetic term for the Kaluza-Klein mode is canonically normalized in the effective 4D theory. There is some arbitrariness in the choice of boundary conditions at $z = z_m$. In order that the boundary conditions not break the chiral symmetry, we typically choose gauge-invariant conditions of the form $L_{\mu\nu}^a(x, z_m) = R_{\mu\nu}^a(x, z_m) = 0$ as in Refs. (12 [13]). However, this choice is somewhat ad hoc, so in the following section we will consider a more general class of gauge-invariant boundary conditions, obtained by the addition of localized gauge kinetic terms.

To calculate decay constants for the vector Kaluza-Klein modes, we couple to a background zero-mode for the diagonal isospin gauge field, which is normalized if $\epsilon \neq 0$ (36). The effective 4D action is obtained by expanding the 5D action in terms of the Kaluza-Klein modes and integrating over the radial coordinate $z$. The rho mesons are identified with the lightest Kaluza-Klein mode of $V_\mu^a$. The coefficient of the term bilinear in the Kaluza-Klein mode and the zero mode corresponds to the vacuum-to-one-particle matrix element of the isospin current, and defines the decay constant for that mode (4). It follows that,

$$F_\rho^2 = \frac{1}{g_5^2} \left(\psi_1(\epsilon)/\epsilon\right)^2.$$

(7)

The correlator of isospin currents can be written in terms of the masses and decay constants. In the approximation of narrow QCD resonances, the correlation function of the product of two vector currents takes the form

$$i \int d^4x \left(\mathcal{J}_\mu^a(x) \mathcal{J}_\nu^b(0)\right) e^{iq x} = \delta^{ab} \sum_n \frac{F_n^2}{q^2 - m_n^2} \left(\epsilon_{\mu\nu} - \frac{g_\rho \rho}{m_\rho^2}\right).$$

(8)

The bulk-to-boundary propagator is the solution to the Fourier-transformed classical equation of motion of the form $V_\mu^a(q, z) = V_\mu^a(q) V(q, z)$, subject to the boundary condition $V(q, \epsilon) = 1$. In terms $V(q, z)$, the sum over modes in Eq. (8) can be written (10, (12, 13),

$$\sum_n \frac{F_n^2}{q^2 - m_n^2} = \frac{1}{g_5^2} \frac{\partial_z V(q, z)}{z} \bigg|_{z=\epsilon}.$$

(9)

For large $-q^2$ the correlator is logarithmic as in QCD, and matching to the perturbative result would determine $g_5 = 2\pi$ (12). However, since we don’t expect the model to be valid in the ultraviolet regime, it is reasonable to allow $g_5$ to vary.

Decay constants for the axial-vector modes are determined similarly. The axial-vector field is the combination

$$A_\mu^a = (L_\mu^a - R_\mu^a)/2.$$ 

The transverse part of the axial-vector field $A^a_{\perp \mu}$ satisfies the equation of motion, Fourier transformed in 3+1 dimensions,

$$\left[\partial_z \left(\frac{1}{z} \partial_z A_\mu^a / z\right) + \frac{q^2}{z^3} A_\mu^a - \frac{4g_5^2 v(z)^2}{z^3} A_\mu^a\right]_{\perp} = 0.$$

(10)

The $a_1$ meson corresponds to the solution satisfying $A_{\perp \mu}^a(\epsilon) = 0$ and whatever we choose for the IR boundary conditions $A_{\perp \mu}(z = z_m) = 0$ as in Refs. (12 [13]). However, this choice is somewhat ad hoc, so in the following section we will consider a more general class of gauge-invariant boundary conditions, obtained by the addition of localized gauge kinetic terms.

To calculate decay constants for the vector Kaluza-Klein modes, we couple to a background zero-mode for the diagonal isospin gauge field, which is normalized if $\epsilon \neq 0$ (36). The effective 4D action is obtained by expanding the 5D action in terms of the Kaluza-Klein modes and integrating over the radial coordinate $z$. The rho mesons are identified with the lightest Kaluza-Klein mode of $V_\mu^a$. The coefficient of the term bilinear in the Kaluza-Klein mode and the zero mode corresponds to the vacuum-to-one-particle matrix element of the isospin current, and defines the decay constant for that mode (4). It follows that,

$$F_\rho^2 = \frac{1}{g_5^2} \left(\psi_1(\epsilon)/\epsilon\right)^2.$$

(7)

The correlator of isospin currents can be written in terms of the masses and decay constants. In the approximation of narrow QCD resonances, the correlation function of the product of two vector currents takes the form

$$i \int d^4x \left(\mathcal{J}_\mu^a(x) \mathcal{J}_\nu^b(0)\right) e^{iq x} = \delta^{ab} \sum_n \frac{F_n^2}{q^2 - m_n^2} \left(\epsilon_{\mu\nu} - \frac{g_\rho \rho}{m_\rho^2}\right).$$

(8)

The bulk-to-boundary propagator is the solution to the Fourier-transformed classical equation of motion of the form $V_\mu^a(q, z) = V_\mu^a(q) V(q, z)$, subject to the boundary condition $V(q, \epsilon) = 1$. In terms $V(q, z)$, the sum over modes in Eq. (8) can be written (10, (12, 13),

$$\sum_n \frac{F_n^2}{q^2 - m_n^2} = \frac{1}{g_5^2} \frac{\partial_z V(q, z)}{z} \bigg|_{z=\epsilon}.$$

(9)

For large $-q^2$ the correlator is logarithmic as in QCD, and matching to the perturbative result would determine $g_5 = 2\pi$ (12). However, since we don’t expect the model to be valid in the ultraviolet regime, it is reasonable to allow $g_5$ to vary.

Decay constants for the axial-vector modes are determined similarly. The axial-vector field is the combination

$$A_\mu^a = (L_\mu^a - R_\mu^a)/2.$$ 

The transverse part of the axial-vector field $A^a_{\perp \mu}$ satisfies the equation of motion, Fourier transformed in 3+1 dimensions,

$$\left[\partial_z \left(\frac{1}{z} \partial_z A_\mu^a / z\right) + \frac{q^2}{z^3} A_\mu^a - \frac{4g_5^2 v(z)^2}{z^3} A_\mu^a\right]_{\perp} = 0.$$

(10)

The $a_1$ meson corresponds to the solution satisfying $A_{\perp \mu}^a(\epsilon) = 0$ and whatever we choose for the IR boundary conditions $A_{\perp \mu}(z = z_m) = 0$ as in Refs. (12 [13]). However, this choice is somewhat ad hoc, so in the following section we will consider a more general class of gauge-invariant boundary conditions, obtained by the addition of localized gauge kinetic terms.
\[\nu(z)\delta^{ab} \exp(2i\pi^a T^a), \text{ where } T^a \text{ generate the broken part of the chiral symmetry group. In the } A_1 = 0 \text{ gauge the equations of motion, Fourier transformed in } 3+1 \text{ dimensions, are,} \]
\[\partial_{z} \left( \frac{1}{z} \partial_{z} \phi^a \right) + \frac{4g_s^2 \nu(z)^2}{z^3} (\pi^a - \phi^a) = 0, \quad (12)\]
\[-q^2 \partial_{z} \phi^a + \frac{4g_s^2 \nu(z)^2}{z^2} \partial_{z} \pi^a = 0. \quad (13)\]

The pion corresponds to the solution satisfying the boundary conditions \(\varphi(\epsilon) = \pi(\epsilon) = 0\) and \(\varphi(z_m) = 0\). Because the longitudinal component of the gauge fields do not appear in the \((3+1)\)-dimensional components of the field strengths \(L_{\mu\nu}\) and \(R_{\mu\nu}\) at the quadratic level, the boundary conditions for the longitudinal components will not be modified when we generalize the boundary conditions in a gauge-invariant manner. Couplings between mesons are determined from the terms in the effective 4D action which couple the corresponding Kaluza-Klein modes. We will include the coupling of the rho to two pions, \(g_{\rho\pi\pi}\), in our fits. In terms of the fields \(\pi\) and \(\varphi\) the coupling is given by \(\text{Eq. 13}\),

\[g_{\rho\pi\pi} = g_{5} \int dz \psi_{\rho}(z) \left( \frac{\varphi'(z)^2}{g_{5}^2 z^2} + \frac{4\nu(z)^2 (\pi - \varphi)^2}{z^3} \right), \quad (14)\]

where \(\pi\) and \(\varphi\) are normalized so that the pion has a canonical kinetic term in the effective 4D theory.

**Varying Boundary Conditions**— In order to determine how predictions of the seven aforementioned observables vary as a function of boundary conditions in the IR, we first consider the \(\rho\) wave functions subject to \(\psi_{\rho}(\epsilon) = 0\) and \(a m_{\rho}^2 \psi_{\rho}(z_m) - b \partial_z \psi_{\rho}(z_m) = 0\), where \(a, b \in \mathbb{R}^+\). The latter boundary condition is consistent with the general form,

\[a \partial^2 \psi_{\rho}(x, z_m) - b V_{\psi_{\rho}}(x, z_m) = 0, \quad (15)\]

where \(V_{\psi_{\rho}}(x, z_m)\) is the field strength tensor for the diagonal gauge field \(V_{\psi_{\rho}}^M\). (Our analysis here is linearized in the fields, so there is no ambiguity from the quadratic parts of the field strength.)

The boundary conditions arise from the addition of localized gauge kinetic terms on the IR boundary:

\[S_{3D} \to S_{3D} - \frac{1}{8g_{5}^2 z_m} \int d^4z \frac{a}{b} \left( L_{\mu\nu} L^{\mu\nu} + R_{\mu\nu} R^{\mu\nu} \right). \quad (16)\]

In addition to modifying the boundary conditions as in Eq. (15), the localized kinetic term has the consequence of rescaling the normalization of the kinetic terms for the Kaluza-Klein modes in the effective 4D theory, so to restore its canonical normalization the \(\rho\) wavefunction satisfies,

\[\int_{z_m}^{z} \frac{dz}{z} \psi_{\rho}(z)^2 + \frac{a}{b z_m} \psi_{\rho}(z_m)^2 = 1, \quad (17)\]

and similarly for the \(a_1\) wavefunction.

For the time being we fix \(g_5 = 2\pi\) and effective scaling dimension \(\Delta = 3\). For any given value of \(b\), we fix \(m_{\rho} = 775.8\) MeV, and use the IR boundary condition to fix \(z_m\). By fixing \(f_\pi = 92.4\) MeV, and \(m_\pi = 139.6\) MeV, we determine the parameters \(m_q\) and \(\sigma\). We then calculate \(F_\rho, m_{a_1}, F_{a_1}\) and \(g_{\rho\pi\pi}\) for the chosen value of \(b\), which are plotted in Fig. [1]. The Neumann boundary condition \(L_{\mu\nu} = R_{\mu\nu} = 0\) corresponds to the limit \(b/a \to \infty\).

We define the RMS error as \(\text{Eq. 12}\),

\[\varepsilon_{\text{RMS}} = \sqrt{\sum_{\text{observables}} \left( \frac{\text{theory} - \text{expt}}{\text{expt}} \right)^2}, \quad (18)\]

where \(n_{\text{DOF}} = \#\) observables \(-\#\) parameters. For the experimental values we took the central values listed in Ref. [12]. In our calculations we considered \(b/\pi \in [0.56, 20.0\, \text{GeV}]\), and observed that the RMS error of the model (calculated using \(F_{a_1}^{1/2}\) for \(m_{a_1}, F_{a_1}^{1/2}\) but not \(g_{\rho\pi\pi}\)) ranged from values as low as 3.2 percent when \(b/a \approx 2.5\) GeV to 30 percent for smaller values of \(b/a\). The RMS error is higher if we include \(g_{\rho\pi\pi}\), as seen in Fig. [2]. This is consistent with the intuition that \(g_{\rho\pi\pi}\) is more sensative to the details of the model, and will also depend on higher-dimension operators that have not been included in the action.

**Varying \(g_5\)**— By fixing Neumann boundary conditions in the IR, we now determine how predictions of the observables vary as a function of the 5D gauge coupling \(g_5\). Demanding that \(m_\rho = 775.8\) MeV fixes \(z_m = 1/(323\, \text{MeV})\). Next, we determine \(m_q\) and \(\sigma\) by fixing \(f_\pi = 92.4\) MeV, \(m_\pi = 139.6\) MeV. The observables \(F_\rho, m_{a_1}, F_{a_1}\) and \(g_{\rho\pi\pi}\) are plotted in Fig. [3]. In our calculations we considered \(g_5 \in [4.8, 9.9]\), and observed that the RMS error of model ranged from values as low as 14.85 percent to 60.7 percent and is visualized in Fig. [4]. Curiously, the value of \(g_5\) obtained by matching to the ultraviolet, \(g_5 = 2\pi\), is near a minimum of the RMS error. This may be used as evidence in favor of the naive matching.

**Varying \(X\) mass**— The mass of the field \(X\) is related to the effective scaling dimension \(\Delta\) by Eq. [10]. Allowing \(\Delta\) to vary while fixing the AdS/QCD predictions for \(m_\pi\) and \(f_\pi\) introduces a dependence of the model parameters \(m_q\) and \(\sigma\) on \(\Delta\). The parameter \(z_m\) is determined by the rho mass, which is not affected by \(\Delta\). Fig. [5] shows the dependence of the observables on \(\Delta\). Fig. [6] shows how the RMS error varies with \(\Delta\).

**A Comment on Chiral Symmetry**— In Fig. [7] we plot \(10m_q\) and \(\sigma\), fit by \(m_q\) and \(f_\pi\), as functions of \(\Delta\). (We rescale \(m_q\) by a factor of ten for visibility in the plot.) Note that as the effective scaling dimension \(\Delta\) decreases from its classical value of three, the parameter \(m_q\) quickly increases. The chiral symmetry breaking scale in the model is set by \(\sigma\) and \(z_m\). The chiral limit is the regime in which the quark mass is much smaller than the chiral symmetry breaking scale. To the extent that \(m_q\) represents the light quark masses, this would seem to imply a
deviation from the chiral limit as $\Delta$ decreases. It is curious that the fit to data disfavors the region of parameter space $m_X^2 \approx -3$, as demonstrated in Fig. 6, although even $m_X^2 = -3$ provides a good fit.

The Gell-Mann–Oakes–Renner (GOR) relation [31] reflects the pattern of chiral symmetry breaking. In terms of the quark mass $m_q$ and the magnitude of the chiral condensate $\sigma$, the GOR relation is,

$$m^2_X f^2_\pi = 2m_q \sigma.$$  \hspace{1cm} (19)

In Ref. [12], a derivation of the GOR relation was given in which $m_q$ and $\sigma$ were the parameters of the hard-wall model, with effective $\mathcal{O}g$ scaling dimension $\Delta = 3$. The derivation, which we generalize below, is independent of the 5D gauge coupling $g_5$ and the IR boundary conditions. However, the normalization of the GOR relation depends on the $\Delta$-dependent scaling of $\sigma$ as in Eq. (4). To see this, we recall the derivation of the GOR relation from Ref. [12].

The transverse part of the axial vector field, $A_{\perp \mu}$, satisfies the equation of motion Eq. (10). If $q^2 = 0$ then $\pi(z) = \text{constant}$ is a solution to Eq. (13), while Eq. (10) and Eq. (12) become identical. Hence, to leading order in $m_\pi^2$ we can write the solution for the pion in terms of the bulk-to-boundary propagator $A(q, z)$, as

$$\varphi(z) \approx A(0, z) - 1.$$  \hspace{1cm} (20)

With $v(z) = (m_q z^{4-\Delta} + \sigma/(4\Delta - 8) z^\Delta)$, the function $u^3/v(u)^2$ is negligible except when $u \sim (m_q/\sigma)^{\frac{1}{2\Delta - 4}}$. For such small $u$ we can replace $u \rightarrow \epsilon$ in the remainder of the integrand, which we recognize as $-f^2_\pi$ from Eq. (11). For $z \gg (m_q/\sigma)^{\frac{1}{2\Delta - 4}}$, the integral becomes constant, with $\pi(z) \rightarrow -1$ and $\varphi(z) = A(0, z) - 1$ solving the equations of motion. With these approximations, the integral in Eq. (20) becomes,

$$\pi(z_m) \approx -\frac{m^2_q f^2_\pi}{2m_q \sigma} \approx -1,$$  \hspace{1cm} (21)

which finally yields the GOR relation, Eq. (19). We test the GOR relation in Fig. 8. In addition to the dependence of the model parameters on the effective scaling dimension $\Delta$ in the analysis above, it was recently stressed in Ref. [32] that the operator dual to the field $X$ via the AdS/CFT correspondence is rescaled from $g_L g_R$ by an $N_c$-dependent factor. However, that factor would rescale $m_q$ and $\sigma$ oppositely such that the product $m_q \sigma$ is invariant, and the GOR relation as expressed above continues to hold.

The classical value of $\Delta$ using the AdS/CFT correspondence, $\Delta = 3$, is near a local maximum of the RMS error.
The error is dominated by $g_{\rho\pi\pi}$, which improves if we allow $\Delta$ to increase significantly from its classical value, contrary to the intuition from asymptotic freedom. On the other hand, as mentioned earlier the coupling $g_{\rho\pi\pi}$ is sensitive to higher-dimension operators in the 5D action which have not been included in this analysis, whereas the quadratic observables (i.e. masses and decay constants) are not.

Conclusions— We have studied the dependence of low-energy hadronic observables on parameters of the hard-wall AdS/QCD model. The 5D gauge coupling and X-field mass obtained by matching to the ultraviolet, $g_5 = 2\pi$ and $m_X^2 = -3$, seem to lie near a saddle point of the RMS error of the model. The value $g_5 = 2\pi$ is preferred by data, but the model is in even better agreement with data if $m_X^2$ is away from its UV-matched value. We have reproduced the Gell-Mann–Oakes–Renner (GOR) relation in terms of the model parameters $m_q$ and $\sigma$, defined with an $m_X$-dependent scaling consistent with the AdS/CFT correspondence. We found some dependence of the observables on the choice of boundary conditions for the 5D gauge fields, and the model can be made to better fit experimental data by an appropriate choice of boundary conditions. Our results help to justify the matching of AdS/QCD to current correlators in the ultraviolet, and support some aspects of AdS/QCD universality.

Acknowledgments— We thank Aleksey Cherman, Tom Cohen, Leandro Da Rold, Hovhannes Grigoryan, Ami Katz, and Matthew Reece for useful discussions. The work of JE was supported in part by NSF grants PHY-0504442 and PHY-0757481. Part of this work was completed during the Research Experience for Undergraduates program at the College of William and Mary. Some of this work was reported at the 8th conference on Quark Confinement and the Hadron Spectrum [33].

[1] J. M. Maldacena, Adv. Theor. Math. Phys. 2, 231 (1998).
[2] E. Witten, Adv. Theor. Math. Phys. 2, 505 (1998) [arXiv:hep-th/9803131].
[3] J. Polchinski and M. J. Strassler, [arXiv:hep-th/0003136].
[4] I. R. Klebanov and M. J. Strassler, JHEP 0008, 052 (2000) [arXiv:hep-th/0007191].
FIG. 5: Hard-wall model predictions as function of $\Delta$.

FIG. 6: RMS Error as function of $\Delta$. The error is dominated by $g_{\rho\pi\pi}$.

FIG. 7: The model parameters $10m_q$ and $\sigma^{1/3}$ as a function of the effective scaling dimension $\Delta$.

FIG. 8: Test of the Gell-Mann–Oakes–Renner (GOR) relation. The curve is the model prediction, with $m_q$ and $\sigma$ fit to the experimental values of $m_\pi$ and $f_\pi$. The curve is consistently below, but close to, the approximate value 1 dictated by the GOR relation.

[5] M. Kruczenski, D. Mateos, R. C. Myers and D. J. Winters, JHEP 0405, 041 (2004) [arXiv:hep-th/0311270].
[6] A. Karch, E. Katz, D. T. Son and M. A. Stephanov, Phys. Rev. D 74, 015005 (2006) [arXiv:hep-ph/0602229].
[7] M. J. Strassler, [arXiv:0801.0629] [hep-ph].
[8] C. Csaki, M. Reece and J. Terning, [arXiv:0811.3001] [hep-ph].
[9] T. Sakai and S. Sugimoto, Prog. Theor. Phys. 113, 843 (2005) [arXiv:hep-th/0412141]; Prog. Theor. Phys. 114, 1083 (2005) [arXiv:hep-th/0507073].
[10] S. Hong, S. Yoon and M. J. Strassler, JHEP 0604, 003 (2006) [arXiv:hep-th/0602018].
[11] G. F. de Teramond and S. J. Brodsky, Phys. Rev. Lett. 94, 201601 (2005) [arXiv:hep-th/0501022].
[12] J. Erlich, E. Katz, D. T. Son and M. A. Stephanov, Phys. Rev. Lett. 95, 261602 (2005) [arXiv:hep-ph/0501128].
[13] L. Da Rold and A. Pomarol, Nucl. Phys. B 721, 79 (2005) [arXiv:hep-ph/0501218].
[14] J. Hirn and V. Sanz, Phys. Rev. Lett. 97, 121803 (2006) [arXiv:hep-ph/0606086]; C. D. Carone, J. Erlich and J. A. Tan, Phys. Rev. D 75, 075005 (2007) [arXiv:hep-ph/0612242]; C. D. Carone, J. Erlich and M. Sher, Phys. Rev. D 76, 015015 (2007) [arXiv:0704.3084 [hep-th]]; C. D. Carone, J. Erlich and M. Sher, Phys. Rev. D 78, 015001 (2008) [arXiv:0802.3702 [hep-ph]]; D. K. Hong and H. U. Yee, Phys. Rev. D 74, 015011 (2006) [arXiv:hep-ph/0602177].

[15] D. T. Son, Phys. Rev. D 78, 046003 (2008) [arXiv:0804.3972 [hep-th]]; A. Adams, K. Balasubramanian and J. McGreevy, JHEP 0811, 059 (2008) [arXiv:0807.1111 [hep-th]]; S. S. Gubser and S. S. Pufu, JHEP 0811, 033 (2008) [arXiv:0805.2960 [hep-th]]; M. M. Roberts and S. A. Hartnoll, JHEP 0808, 035 (2008) [arXiv:0805.3898 [hep-th]].

[16] P. Kovtun, D. T. Son and A. O. Starinets, Phys. Rev. Lett. 94, 111601 (2005) [arXiv:hep-th/0405231].

[17] M. Bando, T. Kugo, S. Uehara, K. Yamawaki and T. Yanagida, Phys. Rev. Lett. 54, 1215 (1985).

[18] D. T. Son and M. A. Stephanov, Phys. Rev. D 69, 065020 (2004).

[19] C. Csaki, C. Grojean, H. Murayama, L. Pilo and J. Terning, Phys. Rev. D 69, 055006 (2004) [arXiv:hep-th/0306152].

[20] J. Erlich and V. Sanz, Phys. Rev. Lett. 97, 121803 (2006) [arXiv:hep-ph/0606086].

[21] E. Witten, Adv. Theor. Math. Phys. 2, 253 (1998) [arXiv:hep-th/9802150].

[22] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Phys. Lett. B 428, 105 (1998) [arXiv:hep-th/9802109].

[23] V. Balasubramanian, P. Kraus, A. E. Lawrence and S. P. Trivedi, Phys. Rev. D 59, 104021 (1999) [arXiv:hep-th/9805208].

[24] I. R. Klebanov and E. Witten, Nucl. Phys. B 556, 89 (1999) [arXiv:hep-th/9905104].

[25] C. Csaki and M. Reece, JHEP 0705, 062 (2007) [arXiv:hep-th/0608266].

[26] K. Ghoroku, N. Maru, M. Tachibana and M. Yahiro, Phys. Lett. B 633, 602 (2006) [arXiv:hep-ph/0510334].

[27] M. Shifman, arXiv:hep-ph/0507246.

[28] J. Erlich, G. D. Kribs and I. Low, Phys. Rev. D 73, 096001 (2006) [arXiv:hep-th/0602110].

[29] J. Hirn, N. Rius and V. Sanz, Phys. Rev. D 73, 085005 (2006) [arXiv:hep-ph/0512240].

[30] D. Z. Freedman, S. D. Mathur, A. Matusis and L. Rastelli, Nucl. Phys. B 546, 96 (1999) [arXiv:hep-th/9804058].

[31] M. Gell-Mann, R. J. Oakes and B. Renner, Phys. Rev. 175, 2195 (1968).

[32] A. Cherman, T. D. Cohen and E. S. Werbos, arXiv:0804.1096 [hep-ph].

[33] J. Erlich, arXiv:0812.3976 [hep-ph].

[34] However, in top-down models the glueballs propagate in the full (9+1)-dimensional spacetime, while the mesons are confined to flavor branes.

[35] A factor of $1/(2\Delta - 4)$ appears in the AdS/CFT relation between the coefficient of the normalizable solution $z^\Delta$ and the expectation value of the corresponding operator. (We thank Tom Cohen and Aleksey Cherman for pointing out that the correct normalization appears in Ref. [24]. This normalization differs from that in Ref. [23] by a subtle rescaling first noticed by requiring consistency of the AdS/CFT correspondence with Ward identities [30].) The additional factor of $1/2$ in Eq. (4) appears because the chiral condensate is $\langle \bar{q}q \rangle = \langle \bar{q}_L q_R + \bar{q}_R q_L \rangle$, while the operator that transforms under the chiral symmetry like the field $X$ is $\bar{q}_L q_R$ without the addition of its hermitian conjugate. If we assume that $m_q$ is normalized to be the quark mass and $\sigma$ has vanishing imaginary part, then according to the AdS/CFT correspondence $\sigma$ is the properly normalized chiral condensate.

[36] More precisely, the zero mode is in the spectrum if Neumann boundary conditions are imposed at $z = \epsilon$, but the Neumann and Dirichlet conditions become degenerate while the zero mode decouples from the spectrum as $\epsilon \to 0$. Also note that the zero mode is consistent with our generalized boundary conditions at $z = z_m$. 