Understanding d’Alembert’s principle: System of Pendulums

Subhankar Ray

Department of Physics, Jadavpur University, Calcutta 700 032, India

J. Shamanna

Department of Physics, University of Calcutta, Calcutta 700 009, India

(Dated: May 26, 2006)

Lagrangian mechanics uses d’Alembert’s principle of zero virtual work as an important starting point. The orthogonality of the force of constraint and virtual displacement is emphasized in literature, without a clear warning that this is true usually for a single particle system. For a system of particles connected by constraints, it is shown, that the virtual work of the entire system is zero, even though the virtual displacements of the particles are not perpendicular to the respective constraint forces. It is also demonstrated why d’Alembert’s principle involves virtual work rather than the work done by constraint forces on allowed displacements.

PACS numbers: 45.45.20.Jj, 01.40.Fk

Keywords: d’Alembert’s principle, Lagrangian mechanics, Virtual work, Holonomic, Non-holonomic, Scleronomous, Rheonomous constraints

I. INTRODUCTION

The principle of zero work by constraint forces on virtual displacement, also known as d’Alembert’s principle, is an important step in formulating and solving a mechanical problem with constraints. In the simple systems widely used in literature, e.g., a single particle rolling down a frictionless incline, or a simple pendulum with inextensible ideal string, the force of constraint is perpendicular to the virtual displacement. This results in zero virtual work by constraint forces. It is often tempting to assume that the constraint forces are always orthogonal to respective virtual displacements, even for a system of particles. d’Alembert’s principle then seems to be a consequence of this orthogonality.

In this article we study two simple systems: a double pendulum and an N-pendulum. In these systems it is observed that, the virtual displacements are not perpendicular to the respective constraint forces acting on individual particles (pendulum bobs). However, d’Alembert’s principle of zero virtual work still holds for the systems as a whole. In these problems, the principle of zero virtual work is a consequence of, (i) the relation between the virtual displacements of coupled components (neighbouring bobs), and (ii) the appearance of (Newtonian) action-reaction pairs in forces of constraint between neighbouring particles. Thus d’Alembert’s principle is more subtle and involved than it is often thought to be. Greenwood has rightly said, “... workless constraints do no work on the system as a whole in an arbitrary virtual displacement. Quite possibly, however, the workless constraint forces will do work on individual particles of the system.”

II. DOUBLE PENDULUM

Let us first consider a double pendulum, with inextensible ideal strings of lengths $L_1$ and $L_2$. Let $r_1$ and $r_2$ denote the instantaneous positions of the pendulum bobs $P_1$ and $P_2$, with respect to the point of suspension (see figure 1). When the system is suspended from a stationary support; the holonomic, scleronomous constraint equations are,

$$|r_1| = L_1, \quad |r_2 - r_1| = L_2. \tag{1}$$

The equations for allowed velocities are obtained by differentiating (1) with respect to $t$,

$$r_1 \cdot v_1 = 0, \quad (r_2 - r_1) \cdot (v_2 - v_1) = 0. \tag{2}$$

Thus, the allowed velocity $v_1$ of $P_1$ is orthogonal to its position vector $r_1$. The relative velocity of the second bob $(v_2 - v_1)$ with respect to the first, is orthogonal to the relative position vector $(r_2 - r_1)$. Let $\mathbf{n}_1$ and $\mathbf{n}_2$ be unit vectors orthogonal to $(r_1)$ and $(r_2 - r_1)$ respectively,

$$\mathbf{n}_1 \cdot r_1 = 0 \quad \text{and} \quad \mathbf{n}_2 \cdot (r_2 - r_1) = 0. \tag{3}$$

From (2) and (3) we obtain the allowed velocities as,

$$v_1 = b_1 \mathbf{n}_1, \quad v_2 - v_1 = b_2 \mathbf{n}_2 \quad \Rightarrow \quad v_2 = v_1 + b_2 \mathbf{n}_2 \tag{4}$$
where $b_1$ and $b_2$ are arbitrary real constants, denoting the magnitude of the relevant vectors.

\[ T \]

The virtual velocities are defined as a difference between two allowed velocities, $\ddot{v}_k = v_k - v'_k \hat{n}$. The allowed displacements $dr_k$, and the virtual displacements $\delta r_k$ are then given by,

\[
dr_1 = v_1 dt = dq_1 \hat{n}_1, \\
\delta r_1 = dr_1 - dr'_1 = (dq_1 - dq'_1) \hat{n}_1 = \delta q_1 \hat{n}_1.
\]

\[ (5) \]

\[
dr_2 = v_2 dt = v_1 dt + b_2 \hat{n}_2 dt = dr_1 + dq_2 \hat{n}_2, \\
\delta r_2 = (dr_1 + dq_2 \hat{n}_2) - (dr'_1 + dq'_2 \hat{n}_2) = \delta r_1 + \delta q_2 \hat{n}_2.
\]

\[ (6) \]

It may be noted that, under the given holonomic, scleronomous constraints (velocity and time independent), the sets of allowed and virtual displacements are equivalent.

A set of allowed displacements $\{dr_1, dr_2\}$ is obtained by a specific choice of the numbers $\{b_1, b_2\}$ or $\{dq_1, dq_2\}$. By making different choices of the set $\{dq_1, dq_2\}$ we get a whole family of allowed displacements, $\{dr_k\}$, where,

$\{dr_k\} \Rightarrow \{dr_1, dr_2\}_{\{b_1, b_2\}} = \{dr_1, dr_2\}_{\{dq_1, dq_2\}}$.

Or more precisely,

$\{dr_k\} \Rightarrow \{\{dr_1, dr_2\}_{\{b_1, b_2\}}\}_{b_1, b_2 \in \Re}$

where $\Re$ is the set of real numbers. Similarly, by choosing different set of quantities $\{\delta q_1, \delta q_2\}$ we get the family of virtual displacements, $\{\delta r_k\}$, where,

$\{\delta r_k\} \Rightarrow \{\delta r_1, \delta r_2\}_{\{\delta q_1, \delta q_2\}}$.

Thus, it is easy to see that for holonomic, scleronomous constraints, the set of all possible allowed displacements is the same as the set of all possible virtual displacements. This is in agreement with the fact that for holonomic, scleronomous systems, the sets $\{dr_k\}$ and $\{\delta r_k\}$ satisfy the same equations, namely,

\[
r_1 \cdot dr_1 = 0, \\
(r_2 - r_1) \cdot (dr_2 - dr_1) = 0 \\
r_1 \cdot \delta r_1 = 0, \\
(r_2 - r_1) \cdot (\delta r_2 - \delta r_1) = 0
\]

As the pendulums are suspended by inextensible ideal strings, one may assume that the tensions in the strings act along their lengths. This essentially implies that there is no shear in the string to transmit transverse force. Thus the tension $T_1$ is along $-r_1$ and tension $T_2$ is along $r_1 - r_2$. Hence, the virtual displacement $\delta r_1$ for the first pendulum is perpendicular to the tension $T_1$, but the virtual displacement $\delta r_2$ of the second pendulum is not perpendicular to $T_2$.

\[
T_1 \cdot \delta r_1 = 0 \\
T_2 \cdot \delta r_2 = T_2 \cdot (\delta r_1 + \delta q_2 \hat{n}_2) = T_2 \cdot \delta r_1 + (T_2 \cdot \hat{n}_2) \delta q_2 = T_2 \cdot \delta r_1
\]

\[ (7) \]

FIG. 1: Double pendulum: (a) position vectors, (b) orthogonality of tensions to part of virtual displacements, (c) cancellation of part of virtual work related to action-reaction pair (tension)
Let us introduce unit vectors $\hat{v}_1$ from (15) and (16), the allowed velocities are given by $n_P r_T$. The equations for allowed velocities, obtained by differentiating the above equations, are shown in figure 2. The constraint equations for this system are,

\[
\delta W_1 = R_1 \cdot \delta r_1 = (T_1 - T_2) \cdot \delta r_1 = -T_2 \cdot \delta r_1 \neq 0
\]
\[
\delta W_2 = R_2 \cdot \delta r_2 = T_2 \cdot (\delta r_1 + \delta q_2 \hat{n}_2) = T_2 \cdot \delta r_1 \neq 0.
\]

(8)

Although neither $\delta W_1$ nor $\delta W_2$ is zero, their sum adds up to zero. Thus the “equal and opposite” Newtonian reaction comes to our rescue, and we have a cancellation in the total virtual work.

\[
\delta W_1 + \delta W_2 = R_1 \cdot \delta r_1 + R_2 \cdot \delta r_2 = -T_2 \cdot \delta r_1 + T_2 \cdot \delta r_1 = 0
\]

(9)

This shows that in the case of a double pendulum with stationary support, d’Alembert’s principle utilizes the equal and opposite nature of the tensions between neighbouring bobs (Newtonian reaction-action pair).

Let us now consider the double pendulum with a moving point of suspension. This gives us a system with a rheonomous constraint. Let the velocity of the point of suspension be $v_0$. The constraint equations in this case are,

\[
|r_1 - v_0 t| = L_1,
|r_2 - v_0 t| = L_2
\]

(10)

The only non-trivial modification is that, the virtual displacements are no longer equivalent to the allowed displacements.

For the first bob $P_1$,

\[
\delta r_1 = \frac{v_1 dt}{v_0 dt + dq_1 \hat{n}_1},
\]

\[
\delta r_1 = \delta r_1 - \delta r_1' = (\delta q_0 dt + dq_1 \hat{n}_1) - (v_0 dt + dq_1' \hat{n}_1) = (dq_1 - dq_1' \hat{n}_1) = \delta q_1 \hat{n}_1.
\]

(11)

Therefore virtual displacement $\delta r_1$ is a vector along $\hat{n}_1$, whereas allowed displacement $\delta r_1$ is sum of a vector along $\hat{n}_1$ and a vector along $v_0$. For the second bob $P_2$,

\[
\delta r_2 = \frac{v_2 dt}{v_0 dt + dq_1 \hat{n}_1 + dq_2 \hat{n}_2},
\]

\[
\delta r_2 = (\delta q_0 dt + dq_1 \hat{n}_1 + dq_2 \hat{n}_2) - (v_0 dt + dq_1' \hat{n}_1 + dq_2' \hat{n}_2) = \delta q_1 \hat{n}_1 + \delta q_2 \hat{n}_2.
\]

(12)

Thus $\delta r_1$ and $\delta r_2$ are not equivalent to $\delta r_1$ and $\delta r_2$. However the relation between $\delta r_1$ and $\delta r_2$ remains the same as in the case of a double pendulum with stationary support.

\[
\delta r_2 = \delta r_1 + \delta q_2 \hat{n}_2.
\]

(13)

Hence the above inferences, in particular, \[\text{7}, \text{8}, \text{9}\] are true even in this case.

### III. \textit{N}-PENDULUM

It is instructive to repeat the above exercise for a system of $N$-pendulum joined end to end by inextensible, ideal strings. Let $r_1$, $r_2$, \ldots, $r_N$ denote the instantaneous position vectors of pendulum bobs $P_1$, $P_2$, \ldots, $P_N$ respectively, as shown in figure 2. The constraint equations for this system are,

\[
|r_1| = L_1,
|r_k - r_{k-1}| = L_k, \quad k = 2, 3, \ldots, N.
\]

(14)

The equations for allowed velocities, obtained by differentiating the above equations, are

\[
r_1 \cdot v_1 = 0,
(r_k - r_{k-1}) \cdot (v_k - v_{k-1}) = 0, \quad k = 2, 3, \ldots, N.
\]

(15)

Let us introduce unit vectors $\hat{n}_1, \hat{n}_2, \ldots, \hat{n}_N$, where $\hat{n}_k$ is normal to the relative position of the bob $P_k$ with respect to $P_{k-1}$, i.e., $(r_k - r_{k-1})$.

\[
\hat{n}_1 \cdot r_1 = 0,
\hat{n}_k \cdot (r_k - r_{k-1}) = 0, \quad k = 2, 3, \ldots, N.
\]

(16)

From \[\text{15} \text{and} \text{16}\], the allowed velocities are given by

\[
v_1 = b_1 \hat{n}_1, \quad v_k - v_{k-1} = b_k \hat{n}_k \quad \Rightarrow \quad v_k = v_{k-1} + b_k \hat{n}_k, \quad k = 2, 3, \ldots, N.
\]

(17)
where \( \{b_k, k = 1, \ldots, N\} \) are a set of real constants denoting the magnitude of the relevant vectors. The allowed displacements \( dr_k \) and the virtual displacements \( \delta r_k \) are,

\[
\begin{align*}
&dr_1 = b_1 \hat{n}_1 dt = dq_1 \hat{n}_1, \\
&dr_2 = dr_1 + b_2 \hat{n}_2 dt = dr_1 + dq_2 \hat{n}_2, \\
&\vdots \\
&dr_N = dr_{N-1} + b_N \hat{n}_N dt = dr_{N-1} + dq_N \hat{n}_N,
\end{align*}
\]

\[
\begin{align*}
&\delta r_1 = \delta q_1 \hat{n}_1, \\
&\delta r_2 = \delta r_1 + \delta q_2 \hat{n}_2, \\
&\vdots \\
&\delta r_N = \delta r_{N-1} + \delta q_N \hat{n}_N
\end{align*}
\]

As is noted in the previous section for case of double pendulum, due to the holonomic, scleronomous nature of the constraints, the sets of allowed and virtual displacements are equivalent. From the above equations one can see that the virtual displacement of each pendulum (\( \delta r_k \)) is a vector sum of the virtual displacement of the previous pendulum (\( \delta r_{k-1} \)) and a component along the unit normal \( \hat{n}_k \). Hence the virtual displacements (with the exception of \( \delta r_1 \)) are not orthogonal to the corresponding relative position vectors.

Let us now consider the constraint forces on each individual pendulum bob \( \mathcal{P}_k \). The bob \( \mathcal{P}_k \) is pulled towards its point of suspension (the previous bob \( \mathcal{P}_{k-1} \)) by a tension \( T_k \) along \( (r_k - r_{k-1}) \). The next pendulum bob, \( \mathcal{P}_{k+1} \), is pulled towards \( \mathcal{P}_k \) by a tension \( T_{k+1} \) along \( (r_k - r_{k+1}) \). In response to this, a reaction force \( (-T_{k+1}) \) acts on the bob \( \mathcal{P}_k \) along \( (r_{k+1} - r_k) \). Thus between any two neighbouring pendulum bobs, there exists a pair of equal and opposite action-reaction forces. The total force on \( \mathcal{P}_k \) is \( (T_k - T_{k+1}) \) for \( k = 1, 2, \ldots, N-1 \). However, for the last pendulum \( \mathcal{P}_N \), the net constraint force is \( T_N \).

The virtual work done by the constraint forces at different system points (particle positions) are,

\[
\begin{align*}
&\delta W_1 = R_1 \cdot \delta r_1 = (T_1 - T_2) \cdot \delta r_1 = T_1 \cdot \delta r_1 - T_2 \cdot \delta r_1 \\
&\delta W_k = R_k \cdot \delta r_k = (T_k - T_{k+1}) \cdot \delta r_k = T_k \cdot \delta r_k - T_{k+1} \cdot \delta r_k \quad k = 2, \ldots, N-1 \\
&\delta W_N = R_N \cdot \delta r_N = T_N \cdot (\delta r_{N-1} + \delta q_N \hat{n}_N)
\end{align*}
\]

As the strings of the pendulums are ideal, the tension \( T_k \) acts along the length of the string, i.e., \( (r_{k-1} - r_k) \). Thus
the tension $T_k$ is normal to the unit vector $\hat{n}_k$. The above virtual work elements become,

$$\delta W_1 = -T_2 \cdot \delta r_1, \quad \delta W_k = T_k \cdot \delta r_{k-1} - T_{k+1} \cdot \delta r_k, \quad \delta W_N = T_N \cdot \delta r_{N-1}. \quad (19)$$

It is clear that the virtual work at each system point $\mathcal{P}_k$ is non-zero. However if we sum the virtual work at all these system points, we observe a mutual cancellation and the total virtual work vanishes.

$$\sum_{k=1}^N \delta W_k = -T_2 \cdot \delta r_1 + (T_2 \cdot \delta r_1 - T_3 \cdot \delta r_2) + \ldots + (T_{j-1} \cdot \delta r_{j-2} - T_j \cdot \delta r_{j-1}) + (T_{j+1} \cdot \delta r_j - T_{j+2} \cdot \delta r_{j+1}) + \ldots + (T_{N-1} \cdot \delta r_{N-2} - T_N \cdot \delta r_{N-1}) + T_N \cdot \delta r_N = 0 \quad (20)$$

This vanishing of total virtual work is a consequence of (i) definition of virtual displacement, (ii) appearance of action-reaction pairs in the forces of constraint. The virtual work connected to each bob $\mathcal{P}_k$, is composed of three parts, (i) virtual work by the tension $T_k$ on the component of virtual displacement $\delta q_k \hat{n}_k$ orthogonal to the relative position vector, (ii) virtual work by the tension $T_k$ on part of the virtual displacement $\delta r_{k-1}$ related to that of the previous bob, and (iii) virtual work by the reaction tension $(-T_{k+1})$ (acting towards the next bob $\mathcal{P}_{k+1}$) on the virtual displacement $\delta r_k = \delta r_{k-1} + \delta q_k \hat{n}_k$. The first component for each $\mathcal{P}_k$ vanishes because of orthogonality of the related force and virtual displacement. This is shown schematically in figure 2(b). Due to the “equal and opposite” nature of action reaction pairs, and existence of a common term in the virtual displacement of neighbouring bobs, the other terms for each bob cancel with the related terms of its neighbours. Shaded areas in figure 2(c) illustrate this cancellation.

| System Pendulums with fixed string length holonomic constraints | Stationary support scleronomous $(\delta r_0 \sim \delta q_0)^\dagger$ | Moving support rheonomous $(\delta r_k \neq \delta q_k)^*$ |
|---|---|---|
| Simple pendulum | $\mathbf{R} = \mathbf{T}$ | $\mathbf{T} \cdot \delta \mathbf{r} = 0$ | $\mathbf{R} \cdot \delta \mathbf{r} = 0$ |
| $\mathbf{T} \cdot \delta \mathbf{r} = 0$ | $\mathbf{R} \cdot \delta \mathbf{r} = 0$ | $\mathbf{R} \cdot \delta \mathbf{r} = 0$ |

$\dagger$: $\delta r_k$ and $\delta q_k$ are equivalent if the constraints are both holonomic and scleronomous.

$\ast$: $\delta r_k$ and $\delta q_k$ are not equivalent if the constraints are non-holonomic and/or rheonomous.

$\mathbf{R}_k = (\mathbf{T}_k - \mathbf{T}_{k+1}), \ k = 1, \ldots, N - 1$

$\mathbf{R}_N = \mathbf{T}_N$

Thus $\delta \mathbf{r}_k$ is a vector along $\hat{n}_k$, whereas $\delta \mathbf{r}_1$ is sum of a vector along $\hat{n}_1$ and a vector along $\mathbf{v}_0$. For subsequent bobs

Let us now study the $N$-pendulum when its point of suspension is moving with velocity $\mathbf{v}_0$. The system now has a rheonomous constraint as well,

$$|\mathbf{r}_1 - \mathbf{v}_0 t| = L_1, \quad |\mathbf{r}_k - \mathbf{r}_{k-1}| = L_k, \quad k = 2, 3, \ldots, N. \quad (21)$$

In accordance with the previous section, for $N$-pendulum with moving support, the only non-trivial modification is that, the virtual displacements are no longer equivalent with the allowed displacements. For the first bob $\mathcal{P}_1$,

$$\delta \mathbf{r}_1 = \mathbf{v}_1 dt + \delta q_1 \hat{n}_1, \quad \delta \mathbf{r}_1 = \mathbf{v}_0 dt + \delta q_1 \hat{n}_1 - (\mathbf{v}_0 dt + \delta q_1' \hat{n}_1) = (\delta q_1 - \delta q_1') \hat{n}_1 = \delta q_1 \hat{n}_1. \quad (22)$$

Thus $\delta \mathbf{r}_1$ is a vector along $\hat{n}_1$, whereas $\delta \mathbf{r}_1$ is sum of a vector along $\hat{n}_1$ and a vector along $\mathbf{v}_0$. For subsequent bobs
\[ \mathcal{P}_k, \]
\[
\begin{align*}
\dot{r}_k &= v_k dt = v_{k-1} dt + b_k \dot{n}_k dt = v_0 dt + dq_1 \dot{n}_1 + dq_2 \dot{n}_2 + \cdots + b_k \dot{n}_k dt \\
\delta r_k &= (q_0 dt + dq_1 \dot{n}_1 + dq_2 \dot{n}_2 + \cdots + dq_k \dot{n}_k) - (q_0 dt + dq_1' \dot{n}_1 + dq_2' \dot{n}_2 + \cdots + dq_k' \dot{n}_k) \\
&= \delta q_1 \dot{n}_1 + \delta q_2 \dot{n}_2 + \cdots + \delta q_k \dot{n}_k.
\end{align*}
\]

Thus \( \delta r_k \) and \( d r_k \) are not necessarily equivalent. However the relation between \( \delta r_{k-1} \) and \( \delta r_k \) remains the same as in the previous case of \( N \) pendulum with stationary support.

\[
\delta r_k = \delta r_{k-1} + \delta q_k \dot{n}_k.
\]

Hence the above inferences, in particular, (18), (19), (20) are true even for an \( N \)-pendulum with moving support.

The adjacent table summarizes the results presented in sections II and III.

### IV. CONCLUSION

The zero virtual work principle of d’Alembert identifies a special class of constraints, which is available in nature, and is solvable. Two noteworthy features of d’Alembert’s principle are, (i) it involves the virtual work \(( R_k \cdot \delta r_k )\), i.e., work done by constraint forces on virtual displacement \( \delta r_k \) and not on allowed displacement \( d r_k \), and (ii) the total virtual work for the entire system vanishes, i.e., \( \sum ( R_k \cdot \delta r_k ) = 0 \), though virtual work on individual particles of the system need not be zero \(( R_k \cdot \delta r_k ) \neq 0 \). For holonomic (velocity independent) and scleronomous (time independent) constraints, e.g., pendulum with stationary support, the allowed and virtual displacements are collinear and hence a distinction between work done on allowed displacement and that on virtual displacement is not possible. For understanding the nature of this distinction one needs to study a system which is either non-holonomic or rheonomous. A pendulum with moving support, or a particle sliding down a moving frictionless inclined plane are examples of simple rheonomous systems. In constrained systems involving a single particle, the second feature mentioned above, in reference to d’Alembert’s principle, becomes irrelevant. As there is only one particle, there is no summation in virtual work, and \( ( R \cdot \delta r ) = 0 \). This implies that the force of constraint \( R \) is normal to the virtual displacement \( \delta r \). In order to really appreciate the importance of summation in the total virtual work, one needs to study system of particles involving several constraints. The double pendulum and \( N \)-pendulum, particularly with moving support, present two of the simplest systems illustrating the subtlety of d’Alembert’s principle.

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* Electronic address: sray@rediffmail.com, sray@phys.jdvu.ac.in (S. Ray)
† Electronic address: jshamanna@rediffmail.com (J. Shamanna)
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