Isolated Horizon structures in quasiequilibrium black hole initial data

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I. GENERAL PROBLEM AND SPECIFIC GOAL

Here we discuss the use of the whole structure of an isolated horizon in order to set inner boundary conditions for constructing quasiequilibrium (binary) black hole initial data, when solving the full set of Einstein equations under a quasiequilibrium approximation. This extends existing proposals in the literature that exploit partially nonexpanding horizon notion, the first level in the hierarchy considered in instantaneous equilibrium. Here we propose the use of the full isolated horizon structure when solving the elliptic system resulting from the complete set of conformal 3+1 Einstein equations under a quasiequilibrium ansatz prescription. We argue that a set of geometric inner boundary conditions for this extended elliptic system then follows, determining the shape of the excision surface.

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A. 3+1 decompositions.

We consider 3+1 foliations \{\Sigma_t\} of spacetimes \((M, g_{ab})\), with Levi-Civita connection \(\nabla_a\). We denote by \(n^a\) the timelike unit normal to the spacelike \(\Sigma_t\) and \(\beta^a\) the shift vector. We denote by \(\gamma_{ab}\) the induced metric on \(\Sigma_t\). The evolution vector is \(\ell^a = N n^a + \beta^a\), with \(N\) the lapse function and \(\beta^a\) the shift vector. We denote by \(K_{ab}\) the induced metric on \(\Sigma_t\), i.e. \(\gamma_{ab} = g_{ab} + n_a n_b\), and by \(D_a\) its associated Levi-Civita connection. The sign convention for the extrinsic curvature of \((\Sigma_t, \gamma_{ab})\) inside \((M, g_{ab})\) is \(K_{ab} := -\frac{1}{2}\mathcal{L}_n \gamma_{ab} = -\gamma_a^c \nabla_c n_b\).

B. Closed 2-surfaces.

Given a 2-surface \(S_t \subset \Sigma_t\), \(s^a\) is the spacelike unit normal vector pointing "outwards" (towards infinity in the asymptotically flat case) and tangent to \(\Sigma_t\). The normal plane \(T_p S_t\) at \(p\) in \(S_t\) is spanned by \(n^a\) and \(k^a\). Alternatively, one can span \(T_p S_t\) in terms of the "outgoing" and "ingoing" null normals, respectively denoted as \(\ell^a\) and \(k^a\), satisfying \(k^a \ell_a = -1\). Directions defined by \(\ell^a\) and \(k^a\) are uniquely determined, but a boost-normalization freedom remains. We can then write \(\ell^a = f \cdot (n^a + s^a)\) and \(k^a = \frac{1}{2f}(n^a - s^a)\), up to a factor \(f\). The induced metric on \(S_t\) is given by: \(g_{ab} = g_{ab} + k_a \ell_b + \ell_a k_b = g_{ab} + n_a n_b - s_a s_b = \gamma_{ab} - s_a s_b\). The Levi-Civita connection associated with \(g_{ab}\) is \(\mathcal{D}_a\). We define the second fundamental tensor of \((S_t, g_{ab})\) as \(K_{ab} := q^d a q^e b \nabla_d q^f c\), so that

\[
K_{ab} = n^c \Theta_{ab}^{(n)} + s^c \Theta_{ab}^{(s)} = k^c \Theta_{ab}^{(k)} + \ell^c \Theta_{ab}^{(\ell)},
\]
to \( \Theta_{ab}^{(s)}; H_{ab} \equiv q^c_a q^d_b \mathcal{D}_c s_d = \Theta_{ab}^{(s)} \). Extrinsic curvature information of \((S, q_{ab}) \) in \((M, g_{ab})\) is completed by the normal fundamental forms associated with normal vectors \( v^a \). We shall employ the 1-form \( \Omega_a := k^c q^d_\alpha \nabla_d \ell_c \).

### II. ISOLATED HORIZONS

Isolated horizons [8] provide a setting for black hole horizons in equilibrium inside a dynamical spacetime. The minimal notion of equilibrium is given by a NEH. A NEH is a \( S^2 \times \mathbb{R} \) null hypersurface \( \mathcal{H} \), on which the Einstein equation holds under a certain energy condition, and that is sliced by marginally (outer) trapped surfaces. That is, the expansion associated with \( \ell^a \) vanishes on \( \mathcal{H} \): \( \theta^a = \epsilon^{ab} \Theta_{ab} = 0 \). The geometry of a NEH is characterized by the pair \((\mathcal{H}, \nabla_a)\), where \( q_{ab} \) is the induced null metric on \( \mathcal{H} \) and \( \nabla_a \) is the unique connection (not a Levi-Civita one) induced from the ambient spacetime connection. The connection \( \nabla_a \) characterizes the extrinsic geometry of the NEH tube. Expression \( \nabla_a \ell^b = \omega^a_b \ell^b \) defines a 1-form \( \omega^a_b \) intrinsic to \( \mathcal{H} \), for each \( \ell^a \). A surface gravity \( \kappa(\ell) \) is then defined from the acceleration of \( \ell^a \), i.e. \( \ell^a \nabla_a \ell^a = \kappa(\ell) \ell^a \). In terms of the normal fundamental form introduced above, we can write \( \omega^a_b = \Omega_a - \kappa(\ell) q_a \) (cf. Eq. (5.35) in [4]).

A hierarchy on \( \mathcal{H} \) results from the progressive demand of geometry invariance under the \( \ell^a \) (evolution) flow:

(i) A NEH is characterized by the time invariance of the intrinsic geometry \( q_{ab} \); \( \mathcal{L}_\epsilon q_{ab} = 2 \Theta_{ab} = 0 \).

(ii) A weakly isolated horizon (WHI) is a NEH, together with an equivalence class of null normals \( \ell^a \), for which the 1-form \( \omega^a_b \) is time-invariant: \( \mathcal{L}_\epsilon \omega^a_b = 0 \). This is equivalent to the surface gravity constancy: \( \nabla_a \kappa(\ell) = 0 \).

(iii) An isolated horizon (IH) is a WHI on which the whole extrinsic geometry is time-invariant: \( \mathcal{L}_\ell^b \nabla_a = 0 \).

NEH and IH equilibrium levels represent genuine restrictions on the geometry of \( \mathcal{H} \) as a hypersurface in \((M, g_{ab}) \) [this is not the case for a WIH, that can always be constructed on a NEH by an appropriate choice of \( \ell^a \)]. Writing \( \Theta_{ab} = \sigma_{ab} + \frac{1}{2} \theta(\ell) q_{ab} \), NEH conditions translate into the vanishing of the shear \( \sigma_{ab} \) and expansion \( \theta(\ell) \):

\[
\Theta_{ab} = 0 \iff \theta(\ell) = 0 \quad , \quad \sigma_{ab} = 0 . \quad (2)
\]

These NEH conditions fix part of the extrinsic curvature of \( \mathcal{S}_L \) (cf. Equation (1) and have been considered in [5] \& [6]). In the next equilibrium level, (strongly) IH conditions can be expressed as (cf. Eq. (5.2) in [3] or Eq. (9.4) in [7])

\[
\kappa(\ell) \Theta_{ab}^{(k)} = \frac{1}{2} \left( 2 D_a \Omega_b^{(k)} + 2 D_b \Omega_a^{(k)} + \Omega_a^{(k)} \Omega_b^{(k)} \right) - \frac{1}{2} R_{ab} + 4 \pi \left( q^c_a q^d_b T_{cd} - \frac{T}{2} g_{ab} \right) , \quad (3)
\]

where \( 2 R_{ab} \) is the \( q_{ab} \)-Ricci tensor, \( T_{ab} \) is the stress-energy tensor, and \( T = g^c d T_{cd} \). Condition (3) represents a geometric constraint on the IH data (cf. discussion in [3]). From Eq. (1), NEH conditions (2) together with the IH constraint (3) fix completely the second fundamental form \( K_{ab} \) of \( \mathcal{S}_L \). Surface gravity \( \kappa(\ell) \) in (3) must be set to a constant value, \( \kappa(\ell) = \kappa_s \). This entails no loss of generality, due to the gauge freedom in the WIH structure \( \mathcal{A} [3, 4] \). Inserting in (3) the 3+1 expressions of \( \Omega_a \) and \( \Theta_{ab}^{(k)} \)

\[
\Omega_a = - q^c_a s_c K_{cd} + 2 D_a \ln f ,
\]

\[
\Theta_{ab}^{(k)} = - \frac{1}{2f} (H_{ab} + q^c_a q^d_b K_{cd}) , \quad (4)
\]

a constraint on the 3+1 fields evaluated on \( \mathcal{S}_L \) follows.

From the trace of (4), expansion \( \theta(k) = \Theta_{cd}^{(k)} q^{cd} \) satisfies

\[
\theta(k) = \frac{1}{\kappa_s} \left( 2 D^a \Omega^b_c + \Omega^b_c \Omega^a_c - \frac{1}{2} R + 4 \pi \left( q^c d T_{cd} - T \right) \right) . \quad (5)
\]

Adapting \( f = f (s^a + s^a) \) to the 3+1 slicing, i.e. \( f = N \), Eq. (5) becomes the condition (11.24) in [2] for the lapse \( N \). The traceless part of (5) completely fixes the ingoing shear \( \sigma_{ab}^{(k)} \). From the equality between the two independent expressions from (3) and (4) for

\[
\sigma_{ab}^{(k)} = \Theta_{ab}^{(k)} - \frac{1}{2} \theta(k) q_{ab} , \quad (6)
\]

it follows a geometric condition on 2 degrees of freedom associated with the traceless part of the extrinsic curvature of the excised surface \( S \) as embedded in \( \Sigma \).

### III. QUASI-EQUILIBRIUM APPROXIMATIONS TO EINSTEIN EQUATIONS

Initial data for Cauchy evolutions of Einstein equations are constructed by solving the constraint equations. Under a conformal ansatz, constraints are cast as a scalar and a vector elliptic equations. In the XCTS approach [1], an additional elliptic equation for the (conformal) lapse follows from maximal slicing. Part of the initial data must be chosen freely. An approach to this consists in solving the whole set of Einstein equations under a quasiequilibrium approximation (e.g. [1, 2, 3]). We briefly review the approach in [2]. A fiducial time-independent flat metric \( f_{ab} \), \( \partial_t f_{ab} = 0 \), with connection \( \mathcal{D}_a \) is introduced. Then, a conformal decomposition of the data \( (\gamma_{ab}, K_{ab}) \) is performed by choosing a conformal representative \( \tilde{\gamma}_{ab} \) through the unimodular condition \( \partial_t \tilde{\gamma}_{ab} = \partial_t f_{ab} \). We write \( \gamma_{ab} = \Psi^{-1} \tilde{\gamma}_{ab} \), and \( K_{ab} = \Psi^{-4} \tilde{A}_{ab} + \frac{1}{2} K \tilde{\gamma}_{ab} \), with \( K = \gamma_{cd} K_{cd} \) and \( \tilde{A}_{ab} \)

\[
\tilde{A}_{ab} = \frac{1}{2N} \left( D_a D_b \psi + \tilde{D}^a b_{ab} - \frac{2}{3} \tilde{D}^a c \tilde{D}^b D_c \psi \right) + \partial_t \tilde{\gamma}_{ab} \) . \quad (7)
\]
where $\tilde{D}_a$ is the Levi-Civita connection associated with $\gamma_{ab}$. The constrained evolution scheme in \cite{2} gives rise to a mixed elliptic-hyperbolic system whose elliptic subsystem is given by the following equations. The Hamiltonian constraint becomes an equation for $\Psi$ whereas the momentum constraint translates into an equation for $\beta^a$. Maximal slicing and a Dirac-like gauge condition, namely preservation in time of $K = 0$ and $D_c\gamma^{ca} = 0$, are considered in \cite{2}. From $K = 0$ an elliptic equation for the lapse $N$ follows. The resulting XCTS-like elliptic subsystem on $\Psi, \beta^a$ and $N$ is formally expressed as

$$L \Psi = S_\Psi, \quad L_{\beta^a} = S_\beta^a, \quad L_N N = S_N,$$  \hspace{1cm} (8)

with the elliptic operators $L_\Psi$, $L_\beta^a$ and $L_N$ and the sources $S_\Psi$, $S_\beta^a$ and $S_N$ (cf. \cite{2}). The hyperbolic part consists of a wavelike equation on $\tilde{\gamma}_{ab}$

$$\frac{\partial^2 \tilde{\gamma}_{ab}}{\partial t^2} - \frac{N^2}{\Psi^4} \tilde{\gamma}_{ab} - 2L_\beta \frac{\partial \tilde{\gamma}_{ab}}{\partial t} + L_\beta L_\gamma \tilde{\gamma}_{ab} = S_\gamma, \hspace{1cm} (9)$$

with $S_\gamma$ not depending on second derivatives of $\tilde{\gamma}_{ab}$. The quasiequilibrium scheme follows by setting in (9) the values of $\partial \tilde{\gamma}_{ab}$ and $\frac{\partial S_\gamma}{\partial t}$ to appropriate a priori prescribed quantities. Then, Eqs. (9) and (8) define an extended elliptic system. In this brief paper we discuss neither outer boundary conditions nor bulk quasiequilibrium prescriptions \cite{1,2,3}, and focus on inner boundary conditions derived from IH structures when using excision.

A. NEH inner boundary conditions

NEH conditions \cite{2} and the gauge adaptation of the coordinate system to the excision tube can be used to fix four inner conditions in the elliptic subsystem \cite{8}. We conformally rescale the relevant objects on $S$: $\tilde{q}_{ab} = \Psi^{-1}q_{ab}$, with connection $2\tilde{D}_a$, $s^a = \Psi^{-2}s^a$ and the 2+1 decomposition of the shift $\beta^a = \beta^a_{\perp s^a} + \beta^o_{s^a}$, with $\beta^o_{\perp s^a} = 0$ and $\beta^a_{\perp s^a} = \beta^a_{s^a}$. Condition $\theta^{(a)} = 0$ in \cite{2} translates into

$$4\tilde{s}^c \tilde{D}_c \psi + \tilde{D}_c \tilde{s}^c \psi = - \Psi^3 \tilde{A}_{ab} \tilde{s}^c \tilde{s}^d + \frac{2}{3} \Psi K.$$ \hspace{1cm} (10)

Coordinate adaptation to the horizon, namely $t^a = t^a + \beta^o_{s^a}$, and the vanishing of $\sigma^{(a)}_{ab}$ in \cite{2} [here we use $\partial \tilde{q}_{ab} = 0$; cf. \cite{3} for general expressions] become conditions \cite{7}

$$\beta^a_{\perp} = N, \quad 2\tilde{D}_a \beta^o_{\perp} + 2\tilde{D}_b \beta^o_{\perp} - (\tilde{D}_c \beta^o_{\perp}) \tilde{q}_{ab} = 0.$$ \hspace{1cm} (11)

In the XCTS context, a fifth boundary condition, generally interpreted as a condition on the lapse $N$, is chosen arbitrarily to complete the elliptic system \cite{8}.

B. IH inner boundary conditions

IH boundary conditions \cite{3} represent three geometric conditions, to be set together with the NEH ones. As discussed above, under the $\ell^a$-normalization choice $f = N$, IH equation (5) on $\theta^{(k)}$ becomes a condition on $N$. Therefore, all inner boundary conditions for the system \cite{8} are determined. The only remaining freedom, necessary to avoid degeneracies, is in the choice of the constant $\kappa_o$.

Regarding the elliptic system \cite{9} on $\tilde{\gamma}_{ab}$, we need to assess five boundary conditions (see \cite{2} for the first discussion of this issue). The spatial gauge determines three of the degrees of freedom associated with $\tilde{\gamma}_{ab}$ on $S$ (cf. \cite{4,10} for two alternative perspectives on this, both based on the use of Dirac-like gauges $D_a\gamma^{ca} = 0$).

IH condition (9) can then be used to set inner conditions for the remaining 2 degrees of freedom of $\tilde{\gamma}_{ab}$. Condition (9) can be seen as a (nonlinear) Robin condition on two of the degrees of freedom of $\tilde{\gamma}_{ab}$ (or $\tilde{q}_{ab}$). This follows from: i) the complete determination of $H_{ab}$ from joined NEH and IH conditions, ii) the writing $H_{ab} = \Psi^2(H_{ab} + 2s^c\tilde{D}_c \ln \Psi q_{ab})$ [where $H_{ab}$ is the extrinsic curvature of $(S, \tilde{q}_{ab})$ into $(\Sigma, \tilde{\gamma}_{ab})$, and iii) the expression of $\tilde{H}_{ab}$ as the radial derivative $\tilde{H}_{ab} = \frac{1}{2}\Psi q_{ab} L \tilde{q}_{cd}$. The explicit form for the IH geometric boundary conditions is cumbersome, but it follows straightforwardly from the expression of \cite{11} in terms of conformal quantities (we assume here $\partial \tilde{q}_{ab} = 0$, but cf. Eqs. (10.105) and (10.106) in \cite{7} for general expressions)

$$\Theta^{(k)}_{ab} = - \frac{1}{2N^2} \left\{ (N + \beta^o)\Psi^{-2}H_{ab}^o + \frac{1}{2} \left[ \tilde{D}^a \beta^o_b + \tilde{D}^b \beta^o_a \right] \right\}$$

$$+ \left[ 2\Psi^{-2} s^c \tilde{D}_c \ln \Psi + \frac{1}{3} \left( NK - \beta^o \Psi^{-2} H - s^c \tilde{D}_c (\beta^o \Psi^{-2}) \right. \right.$$}

$$\left. \left. - 2\tilde{D}_c \beta^c_{\perp} + \beta^c_{\perp s^a} \tilde{D}_d \tilde{s}_c \right) \tilde{q}_{ab} \right\},$$ \hspace{1cm} (12)

$$\Omega_{a} = 2\tilde{D}_a \ln N + \frac{\Psi^2}{\sqrt{2N}} \left\{ (\beta^o \Psi^{-2})^2 s^c \tilde{D}_c \tilde{s}_a - 2\tilde{D}_a (\beta^o \Psi^{-2}) \right.$$}

$$+ \tilde{H}_{ac} \tilde{\beta^o_{\perp}} - \tilde{q}_{ac} \tilde{s}^d \tilde{D}_d \tilde{\beta}^o_{\perp} \right\}.$$ \hspace{1cm} (13)

IV. DISCUSSION

NEH conditions on an excised surface $S$ determine three geometric conditions. (Strongly) IH conditions determine three additional geometric conditions, up to a freely chosen constant (the surface gravity $\kappa_o$). In geometric terms, they fully determine the second fundamental form $K_{ab}$ of $S$ in a spacetime $(M, g_{ab})$. In a 3+1 description, this translates into the determination of both the extrinsic curvature $H_{ab}$ of $S$ in $\Sigma$ and the projection onto $S$ of the extrinsic curvature $K_{ab}$ of $\Sigma$ in $M$.

These conditions can be expressed in terms of initial data $(\gamma_{ab}, K_{ab})$ on a 3-slice $\Sigma$. This nontrivial feature permits their use as inner boundary conditions in the construction of initial data. We have illustrated this in the particular case of an elliptic system resulting from a conformal decomposition of Einstein equations under a quasiequilibrium ansatz. Inner boundary conditions for a XCTS-like elliptic system (five equations) are determined from NEH conditions (three conditions), together
with two additional ones: (i) the gauge adaptation of the coordinate system to the excised tube, and (ii) the IH condition for the elliptic system on the unimodular conformal metric $\tilde{\gamma}_{ab}$ follow from the spatial gauge (three conditions) and the traceless part of the IH conditions, i.e. the IH expression for $\sigma^{(k)}_{ab}$ (two conditions). Our main conclusion is the following: the full IH structure determines geometric conditions for a black hole in instantaneous equilibrium that fix (the physical) part of the inner boundary conditions of the conformal metric $\tilde{\gamma}_{ab}$. In particular, they fully determine the shape of the excision surface $\mathcal{S}$ in $\Sigma$.

IH conditions may have some further physical interest. Under maximal slicing and Dirac-like gauges, the only remaining freedom in the discussed system is the choice of the constant $\kappa_o$. This determines a one-parameter family of horizon foliations that fixes the inherent boost ambiguity in the IH description. On physical grounds, the fixing of the horizon-boost is expected to be related to a (quasilocal) linear momentum of the black hole. The matching of the latter with the prescription following from a post-Newtonian expansion could eventually be used to fix a preferred value of the parameter $\kappa_o$.

There is a number of important caveats in the present approach. First, we have only addressed issues of geometric nature, with no reference whatsoever to the analytic well-posedness of the considered elliptic boundary conditions. Second, the physical convenience of using (strongly) IH in the initial data construction can be called into question. They may represent too stringent conditions in certain realistic astrophysical situations where the use of NEH boundary conditions could prove to be enough. Third, the implementation of (strongly) IH conditions can be challenging from a numerical point of view. In the context of the last two caveats, the free (effective) inner boundary conditions for $\tilde{\gamma}_{ab}$ proposed in [3] represent an interesting alternative, at least in generic cases. In contrast with IH conditions, prescribing the shape of $\mathcal{S}$, conditions in [3] fix the (conformal) intrinsic geometry of $\mathcal{S}$. Technically, they are considerably simpler than IH conditions. All these issues must be assessed numerically.

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