Existence of spanning $\mathcal{F}$-free subgraphs with large minimum degree

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Abstract

Let $\mathcal{F}$ be a family of fixed graphs and let $d$ be large enough. For every $d$-regular graph $G$, we study the existence of a spanning $\mathcal{F}$-free subgraph of $G$ with large minimum degree. This problem is well-understood if $\mathcal{F}$ does not contain bipartite graphs. Here we provide asymptotically tight results for many families of bipartite graphs such as cycles or complete bipartite graphs.

Let $G = (V, E)$ be a $d$-regular graph on $n$ vertices. It is easy to see that every such graph has a spanning triangle-free (in particular, bipartite) subgraph $H$, with minimum degree at least $d/2$. Let $\mathcal{F}$ denote a family of fixed graphs. We say that $G$ is $\mathcal{F}$-free if for every $F \in \mathcal{F}$, $G$ does not contain any subgraph isomorphic to $F$. Then, a natural question is which is the largest minimum degree of a spanning $\mathcal{F}$-free subgraph $H$ of $G$. Let $\text{ex}(n, \mathcal{F})$ be the maximum number of edges in an $\mathcal{F}$-free graph on $n$ vertices. Since the complete graph on $n = d + 1$ vertices, denoted by $K_n$, is $d$-regular, the largest minimum degree a spanning $\mathcal{F}$-free subgraph $H$ of $K_n$ is at most $2\text{ex}(d+1, \mathcal{F})/(d+1)$. In [8], Foucaud, Krivelevich and Perarnau conjectured the following.

Conjecture 1 ([8]). Let $\mathcal{F}$ be a family of fixed graphs and let $d$ be large enough. Then for every $d$-regular graph $G$, there exists a spanning $\mathcal{F}$-free subgraph $H \subseteq G$ with minimum degree $\delta(H) = \Omega(\text{ex}(d, \mathcal{F})/d)$.

If the chromatic number of $\mathcal{F}$ is at least 3 (no bipartite graphs in $\mathcal{F}$), it is easy to verify the conjecture. Kun [11] showed that if $\mathcal{F} = \{C_3, \ldots, C_{g-1}\}$ then every $d$-regular graph $G$ admits a spanning subgraph with minimum degree $\Omega(d^{1/g})$. This gives a lower bound on the minimum degree which it is still far from the conjectured one. A better non explicit result is obtained in [8]. In particular, if $\mathcal{F} = \{C_3, C_4, C_5\}$, it is showed that Conjecture 1 holds up to a logarithmic factor. A similar problem has been studied when $\mathcal{F}$ is composed by complete bipartite graphs in [3].

In this paper we prove that the conjecture is true for a large number of families $\mathcal{F}$ with $\chi(\mathcal{F}) = 2$.

For any two graphs $F$ and $G$, we say that $\varphi : V(F) \to V(G)$ is an homomorphism if $uv \in E(F)$ implies $\varphi(u)\varphi(v) \in E(G)$. We say that $\varphi$ is locally injective if, for every $v \in F$, the restriction of $\varphi$ onto the

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neighbors of \( v \) in \( F \) is injective. Let \( \text{hom}^*(F, \mathcal{G}) \) be the number of locally injective homomorphisms from \( F \) to \( \mathcal{G} \). Observe that the condition \( \text{hom}^*(F, \mathcal{G}) = 0 \), for every \( F \in \mathcal{F} \), is stronger than \( \mathcal{F} \)-freeness. Any copy of \( F \) in \( \mathcal{G} \) induces an injective homomorphism from \( F \) to \( \mathcal{G} \) which, in particular, is also locally injective. We call a family of fixed graphs \( \mathcal{F} \) closed if for every graph \( \mathcal{G} \) we have: \( \text{hom}^*(F, \mathcal{G}) = 0 \) for every \( F \in \mathcal{F} \) if and only if \( \mathcal{G} \) is \( \mathcal{F} \)-free.

Here we prove that Conjecture 1 holds for every closed family \( \mathcal{F} \) of graphs.

**Theorem 2.** Let \( \mathcal{F} \) be a closed family of graphs and let \( d \) be large enough. Then for every \( d \)-regular graph \( G \), there exists a spanning \( \mathcal{F} \)-free subgraph \( H \subseteq G \) with minimum degree \( \delta(H) = \Omega(\text{ex}(d, \mathcal{F})/d) \).

The proof goes as follows. First, we select a large bipartite subgraph of \( G \) with stable sets \( A \) and \( B \). Then we consider a graph \( \mathcal{G} \) of order \( \alpha d \) for some large constant \( \alpha \), such that \( \text{hom}^*(F, \mathcal{G}) = 0 \), for every \( F \in \mathcal{F} \), and \( \delta(\mathcal{G}) = \Omega(\text{ex}(d, \mathcal{F})/d) \). We color \( A \) in a two steps procedure. First, we randomly assign a color to each vertex \( a \in A \) and we either remove the color, or keep it and delete some dangerous edges incident to \( a \) from \( G \). We complete the coloring by deterministically assigning colors to the uncolored vertices in \( A \). After that, we color \( B \) and remove edges from \( G \) iteratively in a similar fashion. During all the procedure, we make sure that all the degrees stay large enough and that no color appears more than once in each neighborhood, that is neighborhoods are rainbowly colored. Finally, we use this coloring to embed the resulting subgraph to \( \mathcal{G} \). According to this embedding, from all the edges still remaining we only keep the ones whose embedding agrees with the edges in \( \mathcal{G} \). This final subgraph satisfies the desired properties.

As a corollary, we get explicit results for some important families \( \mathcal{F} \) with \( \chi(\mathcal{F}) = 2 \). The problem of determining \( \text{ex}(d, \mathcal{F}) \) when \( \mathcal{F} \) contains bipartite graphs has attracted a lot of attention (see [9] for a complete survey on the topic). Here we use some well-known constructions in extremal graph theory, to provide explicit corollaries of Theorem 2.

- Let \( \mathcal{F} = \{C_3, \ldots, C_{2r+1}\} \). Observe that \( \mathcal{F} \) is a closed family. However, the asymptotic order of \( \text{ex}(d, \mathcal{F}) \) is not known in general. Using the Erdős-Rényi random graph \( G(n, p) \) for some good probability \( p = p(n, r) \), it is easy to show the existence of \( \mathcal{F} \)-free graphs of order \( d \) and many edges. Thus, by Theorem 2 we have that every \( d \)-regular graph \( G \) admits a spanning subgraph with girth \( g \geq 2r + 2 \) and minimum degree \( \Omega(d^{1/(2r-1)}) = \Omega(d^{1/(g-3)}) \). This improves the result of Kun [11]. Nevertheless, for any family of graphs \( \mathcal{F} \), lower bounds on \( \text{ex}(d, \mathcal{F}) \) obtained from mimicking random graphs do not seem to be tight.

- Let \( \mathcal{F} = \{C_3, C_4, C_5\} \). From the constructions of \( C_4 \)-free graphs provided by Erdős, Rényi and Sós [6] and Brown [2] one can obtain an \( \mathcal{F} \)-free graph of order \( d \) and \( \Omega(d^{3/2}) \) edges. Again, by Theorem 2 for every \( d \)-regular graph \( G \) we can show the existence of a spanning subgraph with girth at least 6 and minimum degree \( \Omega(\sqrt{d}) \). Extremal constructions for graphs with girth at least 8 and 12 are also known [13, 12]. From them we can obtain tight explicit lower bounds for \( \mathcal{F} = \{C_3, \ldots, C_7\} \) and \( \mathcal{F} = \{C_3, \ldots, C_{11}\} \).

- Let \( \mathcal{F} = \{C_{2q}\} \) where \( q > 2 \) is a prime. While \( \mathcal{F} \) is not close, \( \mathcal{F}' = \{C_p, C_{2p}\} \) is. Moreover, since \( q \) is odd, \( \text{ex}(d, \mathcal{F}) = \Theta(\text{ex}(d, \mathcal{F}')) \). Thus, Conjecture 1 is true for these families.
• Let $F = \{K_{a,b}\}$ with $a \leq b$. Since $K_{a,b}$ has diameter two, each locally injective homomorphism of $K_{a,b}$ onto a graph $G$, is also injective. Thus, if $\text{hom}^*(K_{a,b}, G) = 0$, then $G$ is $K_{a,b}$-free, and $F$ is closed. In particular, any family composed by complete bipartite graphs is closed. It is conjectured that $\text{ex}(d, K_{a,b}) = \Theta(n^{2-1/a})$. While the upper bound has been proved for all values of $a$ and $b$ (Kovári, Sós and Turán [10]), the lower bound is still widely open. However, it is known to be true in the following cases: $a = 2$ and $b \geq 2$, $a = 3$ and $b \geq 3$ (Brown [2]), and $b > (a - 1)!$ (Alon, Rónyai and Szabó [1]). Using these results we can get tight explicit lower bounds on the existence of subgraphs with large minimum degree and inducing no complete bipartite subgraph of a given size.

• Let $F = \{Q_3\}$, where $Q_s$ is the $s$-dimensional hypercube. The family $F$ is closed: again, every locally injective homomorphisms is also injective. This is not true when we consider $s \geq 4$. It is conjectured in [7] that $\text{ex}(d, Q_3) = \Theta(d^{8/5})$, but no better bound that $\text{ex}(d, Q_3) \geq \text{ex}(d, C_4) = \Omega(d^{3/2})$ is known.

Theorem 2 solves in the affirmative Conjecture 1 for families of graphs satisfying a “local” condition. In order to solve the conjecture for every family of graph $F$, one needs to extend the idea of local injectivity to injectivity. In terms of colorings, one needs to prove the existence of a spanning subgraph $H$ with large minimum degree and a coloring $\chi$ such that all the copies of $F$ in $H$ are rainbow. This is stronger than the rainbow condition we impose here in the neighborhoods of vertices in $H$.

Finally, Theorem 2 studies the case when $G$ is $d$-regular. Similar results in terms of the maximum and minimum degree have been given in [8]. We believe that the same techniques used here could be extended to the non regular case.

1 Probabilistic tools

Here we introduce some standard tools from the probabilistic method that can be found in [14].

Lemma 3 (Chernoff’s inequality). For any $0 \leq t \leq np$:

$$\Pr(|\text{Bin}(n, p) - np| > t) < 2e^{-t^2/3np}.$$  

Lemma 4 (Talagrand’s inequality). Let $X$ be a nonnegative random variable not identically 0, which is determined by $n$ independent trails $T_1, \ldots, T_n$ and satisfying the following for some $c_1, c_2 > 0$:

• changing the outcome of any one trial can affect $X$ by at most $c_1$, and

• for any $s$, if $X \geq s$ then there is a set of at most $c_2s$ trials whose outcomes certify that $X \geq s$,

then for any $0 \leq t \leq E(X)$,

$$\Pr(|X - E(X)| > t + 60c_1 \sqrt{c_2E(X)}) \leq 4e^{-t^2/(8c_2^2c_2E(X))}.$$  

Lemma 5 (McDiarmid’s inequality). Let $X$ be a nonnegative random variable not identically 0, which is determined by $n$ independent trails $T_1, \ldots, T_n$ and $m$ independent permutations $\pi_1, \ldots, \pi_m$, and satisfying the following for some $c_1, c_2 > 0$:

- changing the outcome of any one trial can affect $X$ by at most $c_1$,
- interchanging two elements in any one permutation can affect $X$ by at most $c_1$, and
- for any $s$, if $X \geq s$ then there is a set of at most $c_2s$ trials whose outcomes certify that $X \geq s$,

then for any $0 \leq t \leq \mathbb{E}(X)$,

$$\Pr(|X - \mathbb{E}(X)| > t + 60c_1 \sqrt{c_2 \mathbb{E}(X)}) \leq 4e^{-t^2/(8c_1^2c_2\mathbb{E}(X))}.$$ 

Lemma 6 (Lovász Local Lemma). Consider a set $\mathcal{E}$ of events such that for each $E \in \mathcal{E}$

- $\Pr(E) \leq p < 1$, and
- $E$ is mutually independent from the set of all but at most $D$ of other events.

If $4pD \leq 1$, then with positive probability, none of the events in $\mathcal{E}$ occur.

2 Proof of Theorem 2

For every $v \in V(G)$ we denote by $N_G(v)$ the set of vertices adjacent to $v$ in $G$, by $d_G(v) = |N_G(v)|$ the degree of $v$ in $G$ and by $N_G^2(v)$ the set of vertices at distance two from $v$ in $G$. If the graph $G$ is clear from the context, we use $N(v)$, $d(v)$ and $N^2(v)$. We denote by $\Delta(G)$ and by $\delta(G)$ the maximum and the minimum degree of $G$. We denote by $\chi$ a vertex (partial) coloring of a graph $G$. We use $\chi(G)$ to denote the chromatic number of $G$, and for every family of graphs $\mathcal{F}$, we also use $\chi(\mathcal{F}) = \min_{F \in \mathcal{F}} \chi(F)$. For every vertex $v \in V$, we denote by $\chi(v)$ its color and for every set $S \subseteq V(G)$, by $\chi(S)$ the set of colors appearing in $S$. We call $S \subseteq V$ rainbow if $\chi(u) \neq \chi(v)$ for every $u, v \in S$, $u \neq v$, that were assigned a color by $\chi$.

First, we may assume that $\chi(\mathcal{F}) = 2$. Otherwise, Theorem 2 can be easily proven (see Proposition 1 in [3] for a stronger version of it).

We use the technique of almost-regularization to obtain an $\mathcal{F}$-free graph $G$ where all the degrees are similar. For every $\beta \geq 1$, we say that $G$ is $\beta$-almost regular if $\Delta(G) \leq \beta \delta(G)$. Erdős and Simonovits [7] showed the following. For every $\gamma \in (0, 1)$, there exists some $\beta = \beta(\gamma)$ such that if the maximum number of edges of an $\mathcal{F}$-free $\beta$-almost regular graph of order $m$ is $O(m^{1+\gamma})$, then $\text{ex}(m, \mathcal{F}) = O(m^{1+\gamma})$.

Otherwise stated, for every family $\mathcal{F}$ with $\chi(\mathcal{F}) = 2$, there exists an $\mathcal{F}$-free graph $G$ with $\Delta(G) \leq \beta \delta(G)$, for some constant $\beta = \beta(\gamma)$. In particular, this implies that $\delta(G) = \Omega(\text{ex}(m, \mathcal{F})/m)$

Observe that we may also assume that $\mathcal{F}$ does not contain any forest. Suppose that $\mathcal{F}$ contains a forest $T$. Since $\text{ex}(d, \mathcal{F}) \leq \text{ex}(d, T) = O(d)$, then Theorem 2 is trivially true. By the result of Erdős
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in [3, 4], if $\mathcal{F}$ does not contain any forest, there exists a constant $\varepsilon_0 = \varepsilon_0(\mathcal{F}) > 0$ such that for every large enough $m$, $\text{ex}(m, \mathcal{F}) > m^{1+\varepsilon_0}$. Thus, the graph $\mathcal{G}$ satisfies $\delta(\mathcal{G}) = \Omega(\text{ex}(m, \mathcal{F})/m) > m^{\varepsilon}$, for some $\varepsilon < \varepsilon_0$.

If $\mathcal{F}$ is closed, such $\mathcal{G}$ also satisfies $\text{hom}^*(F, \mathcal{G}) = 0$, for every $F \in \mathcal{F}$.

Finally, observe that for every constant $\alpha > 0$,

$$\frac{\text{ex}(\alpha d, \mathcal{F})}{\alpha d} = O\left(\frac{\text{ex}(d, \mathcal{F})}{d}\right).$$

Then, the following proposition implies Theorem 2.

**Proposition 7.** Let $\mathcal{F}$ be a family of graphs with $\chi(\mathcal{F}) = 2$ that does not contain any forest. Let $G$ be a $d$-regular graph. Suppose that there exists $\varepsilon > 0$, $\beta > 0$, $\alpha \geq 32\beta$ and a graph $\mathcal{G}$ on $\alpha d$ vertices such that,

- $G$ is $\beta$-almost regular,
- $G$ has minimum degree $\delta(G) > d^\varepsilon$, and
- $\text{hom}^*(F, \mathcal{G}) = 0$, for every $F \in \mathcal{F}$.

Then there exists a spanning $\mathcal{F}$-free subgraph $H \subseteq G$ with $\delta(H) = \Omega(\delta(G))$.

### 2.1 Outline of the proof of Proposition [7]

First of all, observe that we can consider $G$ to be bipartite. If this is not the case, we can always find a bipartite spanning subgraph where every vertex has at least half of its degree in $G$. Thus we obtain a new graph which is bipartite and where every degree is asymptotically the same as in $G$. Let $A$ and $B$ be the two stable sets of $G$.

We construct a spanning subgraph $H$ and an $\alpha d$ coloring $\chi$ of $H$ in two phases. By the construction of $H$ and $\chi$ we will deduce that $H$ is a spanning subgraph of $G$ satisfying the desired properties, thus proving Proposition [7].

On Phase I we will color the vertices in $A$. First, we will assign a random color to every vertex $a \in A$. If there are too many vertices $b \in N(a)$ such that the color of $a$ appears more that once in $N(b)$, we will uncolor $a$. Otherwise it retains its color and we delete the dangerous edges incident to $a$. After this random partial coloring of $A$ we will deterministically assign a color to all the uncolored vertices in $A$ and delete the corresponding dangerous edges. At the end of Phase I, with positive probability, we will obtain a subgraph satisfying the properties in the Lemma stated below.

On Phase II we will extend the partial coloring to $B$. In the first part, we will iteratively color the vertices in $B$ until no vertex in $A$ has too many uncolored neighbors. At each iteration a partial coloring is obtained using the same idea used in Phase I: we randomly color all the uncolored vertices,
uncolor a vertex $b \in B$ that was assigned a color in this iteration if the previous condition holds and otherwise retain the color in $b$ and delete all the dangerous edges. In this case we also delete an edge $ab$ if $\chi(a)\chi(b) \notin E(\mathcal{G})$. We will prove that after each iteration, a large proportion of the uncolored vertices in $B$ retain the color they were assigned. Once there are not many uncolored vertices in each neighborhood of $a \in A$, we finish this phase by deterministically assigning every uncolored vertex a color that minimizes the number of conflict edges.

After Phase II, the subgraph $H$ and the coloring $\chi$ obtained satisfy with positive probability,

- for every $v \in V(H)$, $d_H(v) = \Omega(\delta(\mathcal{G}))$,
- for every $v \in V(H)$, $N_H(v)$ is rainbow, and
- for every $uv \in E(H)$, $\chi(u)\chi(v) \in E(\mathcal{G})$.

From these properties, it is easy to deduce that the subgraph $H$ satisfies the statement of Proposition 7.

### 2.2 Preliminary lemma

For every graph $G$, partial coloring $\chi$ of $G$ and $v \in V$, we define

$$\text{Bad}(v, \chi, G) = |\{u \in N_G(v) : \exists v' \in N_G(u), v \neq v' \text{ and } \chi(v') = \chi(v)\}|$$

that is, the number of vertices $u \in N_G(v)$ such that $\chi(v)$ appears more than once in $N_G(u)$. This quantity will be crucial all along the paper.

For a random map $\chi$ from $V(G)$ to $V(\mathcal{G})$, let $p$ be the probability that an edge $uv \in E(G)$ gets mapped to an edge in $E(\mathcal{G})$, i.e. $\chi(u)\chi(v) \in E(\mathcal{G})$. Then we have,

$$\frac{\delta(G)}{ad} = p^- \leq p = \frac{2|E(\mathcal{G})|}{(ad)^2} \leq p^+ = \frac{\Delta(G)}{ad}.$$

Observe that if $\mathcal{G}$ is $\beta$-almost regular, $p^+ \leq \beta p^-$. We will use the following lemma to show that the minimum degree in both $A$ and $B$ is large after the first iteration of each phase of the coloring procedure.

**Lemma 8.** Let $G$ be a bipartite graph with stable sets $X$ and $Y$ and maximum degree $d$. Let $\chi$ be a random coloring of $Y$ where each vertex is assigned a color from $V(\mathcal{G})$ independently and uniformly at random. For every $x \in X$ and $c \in V(\mathcal{G})$ let $W_{x,c}$ be the number of $y \in N(x)$ such that

1. $y$ is the only vertex with color $\chi(y)$ in $N(x)$,

2. $\text{Bad}(y, \chi, G) \leq \frac{d}{\sqrt{a}}$, and

3. $c\chi(y) \in E(\mathcal{G})$. 

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Suppose that \( d \leq 2d_G(x) \). Then,

\[
\Pr \left( W_{x,c} \leq \frac{p^{-d_G(x)}}{2} \right) = e^{-\Omega(d_G)}.
\]

**Proof.** Let \( W'_{x,c} \) be the number of neighbors \( y \in N(x) \) such that at least one of this two conditions holds

\[
\begin{align*}
&\bullet \; (E1) \exists y' \in N(x), y' \neq y, \chi(y) = \chi(y'). \\
&\bullet \; (E2) \left| \{ x' \in N_G(y) \setminus \{ x \} : \exists y' \in N_G(x'), y \neq y' \text{ and } \chi(y') = \chi(y) \right| \geq \frac{d}{\sqrt{\alpha}} - 1.
\end{align*}
\]

Observe that if condition \((E2)\) is not satisfied, then \( \text{Bad}(y, \chi, G) \leq \frac{d}{\sqrt{\alpha}} \).

Let us first show that \( W'_{x,c} \) is concentrated around its expected value. First of all, we will fix the coloring of \( Y \setminus N(x) \) and prove that for any such coloring \( W'_{x,c} \) is concentrated.

The probability that a vertex \( y \) is alone in its color class within \( N(x) \) is \( (1 - \frac{1}{\alpha d})^{d_G(x)} \).

We call a color class \( c' \) dangerous for \( y \in N(x) \) if \((E2)\) is satisfied when \( \chi(y) = c' \). The fact that \( c' \) is dangerous for \( y \) does only depend on the coloring of \( Y \setminus N(x) \) which is already fixed. In particular, since there are at most \( d^2 \) vertices in \( N^2(y) \), the number of dangerous color classes is at most \( \frac{d^2}{(d/\sqrt{\alpha}) - 1} \leq 2\sqrt{\alpha d} \). Thus, the probability that a vertex \( y \in N(x) \) is in one of these bad color classes is at most \( 2/\sqrt{\alpha} \).

Therefore,

\[
E(W'_{x,c}) \leq \left( \left( 1 - \left(1 - \frac{1}{\alpha d} \right)^{d_G(x)} \right) + \frac{2}{\sqrt{\alpha}} \right) d_G(x) \leq \frac{3d_G(x)}{\sqrt{\alpha}}.
\]

If we condition on the coloring given to \( Y \setminus N(x) \), \( W'_{x,c} \) only depends on the colors assigned to \( N(x) \).

Changing the color of \( y \in N(x) \) from \( c' \) to \( c'' \) can change by at most \( 2 \) the value of \( W'_{x,c} \). This change can create or destroy at most two vertices with the property \((E1)\). It can also be the case that \( c'' \) is a dangerous color class for \( y \) and that \( c' \) is not (or viceversa); in this case the change is at most \( 1 \). On the other hand, if we have \( W'_{x,c} \geq s \), given that the coloring in \( Y \setminus N(x) \) is fixed, there exists a set of \( s \) choices of colors that certify \( W'_{x,c} \geq s \).

By Talagrand’s inequality with \( c_1 = 2 \) and \( c_2 = 1 \), we have

\[
\Pr \left( W'_{x,c} \geq \frac{4d_G(x)}{\sqrt{\alpha}} \right) \leq \Pr \left( \left| W_{x,c} - E(W_{x,c}) \right| \geq \frac{d_G(x)}{\sqrt{\alpha}} \right) = e^{-\Omega(d_G)}.
\]

Observe that coloring each vertex in \( Y \) with a color \( c \in V(\mathcal{G}) \) chosen independently and uniformly at random, is equivalent to color \( Y \) in the same way and then permute the colors in the color classes according to a permutation \( \pi \) of length \( |V(\mathcal{G})| \) chosen uniformly at random. These two steps can be done independently and thus we can analyse them also separately. In the first step the color classes are set, while, in the second one each color class is assigned a color according to \( \pi \). Observe that the
condition \( c\chi(y) \in \mathbb{E}(G) \) only depends on the second step, while the other two conditions just depend on the color classes.

Let \( M \) be the set of vertices \( y \in N(x) \) such that conditions (E1) and (E2) are not satisfied. Notice that the set \( M \) is fully determined by the first step of the coloring. Since \( \pi \) is a uniformly chosen permutation, the set of \( |M| \) colors assigned to \( M \) is chosen uniformly from all the sets of size \( |M| \subseteq [\alpha d] \).

Then, for every \( J \subset Y \cap N(x) \) with \( |J| = j \),

\[
\mathbb{E}(W_{x,c}|M = J) = \sum_{y \in J} \Pr(c\chi(y) \in \mathbb{E}(G)|M = J) = \sum_{y \in J} \Pr(c\chi(y) \in \mathbb{E}(G)) \geq p^{-j}.
\]

Recall that, by the independence of the two step coloring, to show concentration of \( W_{x,c} \) we only need to focus on the second step. Once the color classes have been set, a swap between two positions of \( \pi \) can change \( W_{x,c} \) by at most 2 and if \( W_{x,c} \geq s \) we can certify it by only giving the permutation \( \pi \). We can use McDiarmid’s inequality to prove that for all \( 0 < \beta < 1 \)

\[
\Pr \left( W_{x,c} \leq (1 - \beta)p^{-j}|M = J \right) \leq \Pr \left( |W_{x,c} - \mathbb{E}(W_{x,c})| \geq \beta p^{-j}|M = J \right) = e^{-\Omega(p^{-j})}. \tag{2}
\]

Now, using (1) and (2)

\[
\Pr \left( W_{x,c} \leq \frac{p^{-d_G(x)}}{2} \right) = \sum_{J \subseteq Y \cap N(x)} \Pr \left( W_{x,c} \leq \frac{p^{-d_G(x)}}{2}\big|M = J \right) \Pr(M = J) \\
\leq \sum_{J \subseteq Y \cap N(x)} \Pr \left( W_{x,c} \leq \frac{p^{-d_G(x)}}{2}\big|M = J \right) \Pr(M = J) + e^{-\Omega(d_G(x))} \\
\leq \sum_{J \subseteq Y \cap N(x)} \Pr \left( W_{x,c} \leq \frac{p^{-j}}{2(1 - \frac{1}{\sqrt{\alpha}})}\big|M = J \right) \Pr(M = J) + e^{-\Omega(d_G(x))} \\
\leq e^{-\Omega(\delta(G))} \sum_{J \subseteq Y \cap N(x)} \Pr(M = J) + e^{-\Omega(d_G(x))} = e^{-\Omega(\delta(G))}.
\]

\[\square\]

2.3 Phase I: Coloring A

We will color the set \( A \) as follows:

1. For every \( a \in A \), let \( \chi_0(a) = c \), where \( c \in V(G) \) is chosen independently and uniformly at random. Uncolor \( a \in A \) if \( \text{Bad}(a, \chi_0, G) \geq \frac{d}{\sqrt{\alpha}} \). Let \( A_0 \) be the set of colored vertices in \( A \).
2. Delete all the edges between \( a \in A_0 \) and \( N_G(a) \) that may cause conflicts, i.e. delete \( ab \) if \( \exists a' \in A_0 \) such that \( a'b \in E(G) \), \( a' \neq a \) and \( \chi'_0(a) = \chi'_0(a') \). Let \( H'_0 \) be the subgraph obtained by removing these edges from \( G \).

3. Consider an arbitrary order of the vertices in \( A \setminus A_0 = (a_1, \ldots, a_s) \), where \( s = |A \setminus A_0| \).

4. For every \( i \) from 1 to \( s \),
   
   (a) Assign to \( a_i \) the color \( c \in V(G) \) that minimizes \( \text{Bad}(a_i, \chi'_{i-1}, H'_{i-1}) \). Let \( \chi'_i \) be the partial coloring of \( A \) obtained from \( \chi'_{i-1} \) and the colored vertex \( a_i \).
   
   (b) Delete all the edges between \( a_i \) and \( N_{H'_i}(a_i) \) that may cause conflicts, i.e. delete \( a_ib \) if \( \exists a' \in A \) such that \( a'b \in E(H'_{i-1}) \), \( a' \neq a_i \) and \( \chi'_i(a_i) = \chi'_i(a') \). Let \( H'_i \) be the subgraph obtained by removing these edges from \( H'_{i-1} \).

**Lemma 9.** With positive probability, after Phase I we have a coloring \( \chi'_s \) of \( A \) and a spanning subgraph \( H'_s \subseteq G \) with the following properties,

- \((P1)\) for every \( a \in A \), \( d_{H'_i}(a) \geq d/2 \),
- \((P2)\) for every \( b \in B \), \( N_{H'_i}(b) \) is rainbow, and
- \((P3)\) for every \( b \in B \) and \( c \in V(G) \), there are at least \( \delta(G)/2\alpha \) vertices \( a \in N_{H'_i}(b) \) such that \( \chi'_s(a)c \in E(G) \).

**Proof.** Let us first show that \((P1)\) is satisfied. If \( a \in A_0 \), we have \( d_{H'_i}(a) = d_{H'_0}(a) \geq (1 - 1/\sqrt{\alpha})d \), since by the choice of \( \chi'_0(a) \) we deleted at most \( d/\sqrt{\alpha} \) edges incident to \( a \). On the other hand, for every \( a_i \in A \setminus A_0 \), since there are at most \( d^2 \) edges incident to \( N_{H'_{i-1}}(a) \), there exists a color \( c \in V(G) \) (recall that \( |V(G)| = \alpha d \)) such that if \( \chi_i(a_i) = c \)

\[
\text{Bad}(a_i, \chi'_{i}, H'_{i-1}) = |\{b \in N_{H'_{i-1}}(a_i) : \exists a' \in N_{H'_{i-1}}(b), a \neq a' \text{ and } \chi'_i(a') = \chi'_i(a)\}| \leq \frac{d}{\alpha}
\]

Thus, \( d_{H'_i}(a_i) = d_{H'_i}(a_i) \geq (1 - 1/\alpha)d \) for all \( a_i \in A \setminus A_0 \). If \( \alpha \) is large enough, for all \( a \in A \), \( d_{H'_i}(a) \geq d/2 \).

Property \((P2)\) holds since when a color is retained on a vertex \( a \in A \), we delete the edges that make this color appear more than once within the neighbors of \( b \in N_{H'_i}(a) \).

Let us show that \((P3)\) holds with positive probability. In order to show it, we will look only on the first partial coloring \( \chi'_0 \). For every \( b \in B \) and \( c \in V(G) \), let \( E_{b,c} \) be the event that there are at most \( \delta(G)/2\alpha = p^{-d_G(b)/2} \) vertices \( a \in N_{H'_i}(b) \) such that \( a \) is the only vertex with color \( \chi'_0(a) \) in \( N_{H'_0}(b) \), \( a \) does not get uncolored (i.e. \( \text{Bad}(a, \chi'_0, G) \geq \frac{d}{\sqrt{\alpha}} \)) and \( \chi'_0(a)c \in E(G) \).

By applying Lemma \( \Xi \) to \( G \) with \( X = B, Y = A, \chi = \chi'_0 \) and \( x = b \), \( \Pr(E_{b,c}) = e^{-\Omega(\delta(G))} = e^{-\Omega(d^4)} \). Notice that \( E_{b,c} \) is mutually independent from the other events \( E_{b',c'} \) where both \( b' \) and \( c' \) are at distance larger than 6 from \( b \) and \( c \) in the respective graphs \( G \) and \( \mathcal{G} \). Then, each event is mutually
We will color the set $B$ by coloring the vertices of $H_{i-1}$, none of the events in $E_{b,c}$ holds with positive probability, provided that $d$ is large enough. By coloring the vertices $a_i \in A \setminus A_0$ and deleting edges incident to $a_i$, one can neither decrease the degree of $b$ in $A_0$ nor change the fact that a given $a \in A_0$ is the only neighbor of $b$ with color $\chi_0(a)$. Thus, with positive probability (P3) is satisfied.

2.4 Phase II: Coloring $B$

We will color the set $B$ as follows:

1. Let $B_0 = B$, $H_0 = H'_s$ and $\chi_0 = \chi'_s$. For every $a \in A$, let $B_0(a) = B \cap N_{H_0}(a)$.

2. For all $i$ from 1 to $\tau$ such that $|B_\tau(a)| \leq d^\eps/2$ for every $a \in A$,

   (a) Construct $\chi_i$ as follows. For every $v \in A \cup (B \setminus B_{i-1})$, let $\chi_i(v) = \chi_{i-1}(v)$ and for every $b \in B_{i-1}$, let $\chi_i(b) = c$, where $c \in V(G)$ is chosen independently and uniformly at random.

   (b) Uncolor $b \in B_{i-1}$ if

   $$\text{Bad}(b, \chi_{i}, H_{i-1}) \geq \frac{1}{2}|N_{H_{i-1}}(b) \cap \chi_{i}^{-1}(N_G(\chi_i(b)))|.$$  

   Let $B_i$ be the set of uncolored vertices.

   (c) Construct $H_i$ from $H_{i-1}$ by deleting the following edges: for every $b \in B_{i-1} \setminus B_i$, delete all the edges between $b$ and $N_{H_{i-1}}(b)$ that either may cause conflicts (i.e. delete $ab$ if $\exists b' \in B$ such that $ab' \in E(H_{i-1})$, $b' \neq b$ and $\chi_i(b) = \chi_i(b')$) or such that $\chi_i(a)\chi_i(b) \not\in E(G)$.

   Let $B_i(a) = B_i \cap N_{H_i}(a)$.

3. Consider an arbitrary order of the vertices in $B \setminus B_\tau = (b_{\tau+1}, \ldots, b_t)$, where $t = |B \setminus B_\tau| + \tau$.

4. For every $i$ from $\tau + 1$ to $t$,

   (a) Assign to $b_i$ the color $c \in V(G)$ that minimizes $\text{Bad}(b_i, \chi_{i-1}, H_{i-1})$. Let $\chi_i$ be the partial coloring of $V(G)$ obtained from $\chi_{i-1}$ and the colored vertex $b_i$.

   (b) Construct $H_i$ from $H_{i-1}$ by deleting all the edges between $b_i$ and $N_{H_{i-1}}(b_i)$ that either may cause conflicts (i.e. delete $ab_i$ if $\exists b' \in B$ such that $ab' \in E(H_{i-1})$, $b' \neq b_i$ and $\chi_i(b_i) = \chi_i(b')$); or such that $\chi_i(a)\chi_i(b_i) \not\in E(G)$.

5. Let $\chi = \chi_t$ and $H = H_t$.

Lemma 10. With positive probability, Phase II of the coloring procedure ends and provides a spanning subgraph $H$ of $G$ and a coloring $\chi$ of $H$ with the following properties,

- (Q1) for every $v \in V(H)$, $d_H(v) \geq \frac{\delta(G)}{4\alpha}$.

- (Q2) for every $v \in V(H)$, $N_H(v)$ is rainbow, and
• (Q3) for every \( uv \in E(H) \), \( \chi(u)\chi(v) \in E(G) \).

Let \( \text{GoodNeighs}_i(a) \) be the number of colored vertices \( b \in B_{i-1}(a) \setminus B_i(a) \) such that \( \chi_i(a)\chi_i(b) \in E(G) \). Recall that \( p^+ = \Delta(G)/\alpha d \).

We will make sure that, for every \( 1 \leq i < \tau \), the two following conditions are satisfied:

• (C1) for every \( a \in A \), \( |B_i(a)| \leq \frac{d}{\alpha^{i/2}} \), and

• (C2) for every \( a \in A \), \( \text{GoodNeighs}_i(a) \leq \max\{2p^+|B_{i-1}(a)|, d^{i/2}\} \).

In particular, we will also make sure that after the first iteration, the following condition is satisfied:

• (C3) for every \( a \in A \), its degree in \( B \setminus B_1 \) is at least \( \frac{\delta(G)}{\alpha} \).

Let \( D_a = \{|B_i(a)| \geq \frac{d}{\alpha^{i/2}}\} \).

**Claim 11.** For every \( 1 \leq i < \tau \) and for every \( a \in A \), if (C1) holds up to iteration \( i-1 \) then

\[
\Pr(D_a) = e^{-\Omega(d^{i/2})}.
\]

**Proof.** For every vertex \( b \in B_{i-1}(a) \), every color \( c \in V(G) \) and every coloring of \( B \setminus B_{i-1}(a) \), let \( x = |N_{H_{i-1}}(b) \cap \chi_i^{-1}(N_G(c))| \). In the same fashion as in the proof of Lemma \( \text{S} \) we call a color class \( c \) dangerous for \( b \in B_{i-1}(a) \) if

\[
|\{a' \in N_{H_{i-1}}(b) \setminus \{a\} : \exists b' \in N_{H_{i-1}}(a'), b \neq b' \text{ and } \chi_i(b') = c\}| \geq x/2 - 1.
\]

Let \( X \) be the number of vertices \( b \in B_{i-1}(a) \) that get a dangerous color. Observe that \( |B_i(a)| \leq X \): if \( b \in B_{i-1}(a) \) is colored with a non dangerous color, then \( \text{Bad}(b, \chi_i, H_{i-1}) \leq x/2 \) independently from the coloring on \( B_{i-1}(a) \) and \( b \) retains its color. Thus, it suffices to show that \( X \) is concentrated around its expected value. Again, we will fix the coloring on \( B \setminus B_{i-1}(a) \) and show that for any such coloring \( X \) is concentrated.

There are at most \( xd \) vertices in \( N^2_{H_{i-1}}(b) \) and hence, the number of dangerous color classes is at most \( \frac{xd}{x/2-1} \leq 3d \). In particular, the probability that a vertex \( b \in B_{i-1}(a) \) is colored with a dangerous color is at most \( 3/\alpha \). Since (C1) holds up to iteration \( i-1 \), there are at most \( |B_{i-1}(a)| \leq d/\alpha^{(i-1)/2} \) candidates for \( X \). Thus, \( E(X) \leq \frac{d}{\alpha}|B_{i-1}(a)| \leq \frac{3d}{\alpha^{(i+1)/2}} \).

Since the coloring of \( B \setminus B_{i-1}(a) \) is given, \( X \) only depends on the colors assigned to \( B_{i-1}(a) \). Recall that the set of dangerous colors for a given \( b \in B_{i-1}(a) \) is fixed by the choices in \( B \setminus B_{i-1}(a) \). If we change the color of \( b \in B_{i-1}(a) \), \( X \) can change by at most 1 (just if the new color is dangerous for \( b \) but the previous one was not, or vice versa). Moreover, if \( X \geq s \) there exists a set of \( s \) colored vertices (the ones that receive a dangerous color) that certify \( X \geq s \). Applying Talagrand’s inequality to \( X \) with \( c_1 = 1 \) and \( c_2 = 1 \),

\[
\Pr(D_a) = \Pr\left(X \geq \frac{d}{\alpha^{i/2}}\right) \leq \Pr\left(|X - E(X)| > \left(1 - \frac{3}{\sqrt{\alpha}}\right)\frac{d}{\alpha^{i/2}}\right) = e^{-\Omega(d/\alpha^{i/2})} = e^{-\Omega(d^{i/2})},
\]
if $\alpha$ is large enough.

Let $E_a = \{\text{GoodNeighs}_a(a) \geq \max\{2p^{+}|B_{i-1}(a)|, d^{\delta/2}\}\}$.  

**Claim 12.** For every $a \in A$, 

$$\Pr(E_a) = e^{-\Omega(d^{\delta/4})}.$$  

**Proof.** Let $Z_a$ be the number of vertices $b \in B_{i-1}(a)$ such that $\chi_i(a) \chi_i(b) \in E(G)$, independently from the fact that $b$ retains or not its color. Observe that when $b$ is assigned a random color $c$, it satisfies $\chi_i(a) c \in E(G)$ with probability at most $p^+$. Since the choice of the colors for $b \in B_{i-1}(a)$ is done independently, $Z_a$ follows a binomial distribution with $|B_{i-1}(a)|$ trials and probability at most $p^+$. In particular, $\mathbb{E}(Z_a) \leq p^+|B_{i-1}(a)|$.

Suppose first that $|B_{i-1}(a)| \geq \frac{d^{\delta/4}}{p^+} = \alpha d^{1+\epsilon/4}/\Delta(G)$. By Chernoff’s inequality,

$$\Pr\left(Z_a \geq 2p^+|B_{i-1}(a)|\right) \leq \Pr\left(|Z_a - \mathbb{E}(Z_a)| \geq p^+|B_{i-1}(a)|\right) = e^{-\Omega(p^+|B_{i-1}(a)|)} = e^{-\Omega(d^{\delta/4})}.$$  

Suppose now that $|B_{i-1}(a)| < \frac{d^{\delta/4}}{p^+}$. Then, $\mathbb{E}(Z_a) < d^{\delta/4}$ and also by Chernoff’s inequality,

$$\Pr\left(Z_a \geq d^{\delta/2}\right) \leq \Pr\left(|Z_a - \mathbb{E}(Z_a)| \geq (1-o(1))d^{\delta/2}\right) \leq \Pr\left(|Z_a - \mathbb{E}(Z_a)| \geq d^{\delta/4}\right) = e^{-\Omega(d^{\delta/4})}.$$  

Since some of the elements counted in $Z_a$ can get uncolored, $\text{GoodNeighs}_a(a) \leq Z_a$ and the claim follows.

Let $F_a = \{d_{H_1}(a) \leq \frac{\delta(G)}{4\alpha}\}$.

**Claim 13.** For every $a \in A$, 

$$\Pr(F_a) = e^{-\Omega(d)}.$$  

**Proof.** Recall that, by (P1), for every $a \in A$, $d_{H_0}(a) \geq d/2$. By Lemma 8 in $G = H_0$ with $X = A$, $Y = B$, $\chi = \chi_1$, $x = a$ and $c = \chi_1(a)$ we have that the degree of each vertex $a \in A$ to $B \setminus B_1$ in $H_1$ is at least $\frac{p^{-}(d/2)}{2} = \frac{\delta(G)}{4\alpha}$, with probability $e^{-\Omega(d)}$.

**Claim 14.** Conditions (C1), (C2) and (C3) hold after the first iteration with positive probability.

**Proof.** By Claim 11, 12 and 13, we have $\Pr(D_a) = e^{-\Omega(d^{\delta/2})}$, $\Pr(E_a) = e^{-\Omega(d^{\delta/4})}$ and $\Pr(F_a) = e^{-\Omega(d)}$ respectively. On the other hand, $D_a$, $E_a$, $F_a$ are mutually independent from every other $D_{a'}$, $E_{a'}$ and $F_{a'}$ such that $a'$ is at distance larger than 6 from $a'$ in $H_0$. Thus, each event is mutually independent from all but at most $3d^6$ other events. Since $e^{-\Omega(d^{\delta/4})} = o(d^6)$, the local lemma allows us to show that after the first iteration (C1), (C2) and (C3) hold with positive probability, provided that $d$ is large enough.

An exact argument suffices to show the following.
Claim 15. Let \(2 \leq i < \tau\). If (C1) and (C2) hold for every \(1 \leq j < i\), then (C1) and (C2) hold after the \(i\)-th iteration with positive probability.

Now we proceed to prove Lemma 10.

Proof of Lemma 10. Observe that (Q2) and (Q3) are satisfied deterministically, by Phases I and II of the coloring procedure. By Claim 13 Phase II ends with positive probability. It just remains to show that (Q1) is satisfied with positive probability at the end of Phase II.

By Claim 14 condition (C3) is satisfied after the first iteration. Hence,

\[
d_H(a) \geq |N_H(a) \cap (B \setminus B_1)| = |N_{H_1}(a) \cap (B \setminus B_1)| \geq \frac{\delta(G)}{4\alpha}.
\]

Let us show that the degrees of \(b \in B\) in \(H\) are also large. First notice that the degree of a vertex \(b \in B\) only decreases in the iteration when \(b\) retains its color.

If \(b \in B \setminus B_\tau\), by (P3), for every \(c \in V(G)\), \(|N_{H_0}(b) \cap \chi_0^{-1}(N_G(c))| \geq p^{-d/2} = \delta(G)/2\alpha\). By construction of the coloring procedure, in the iteration when \(b\) retains its color, at most half of the edges incident to \(b\) are deleted. Hence, for every \(b \in B \setminus B_\tau\)

\[
d_H(b) \geq \frac{\delta(G)}{4\alpha}.
\]

Now consider \(b_i \in B_\tau\). Phase II assigns to \(b_i\) the color that minimizes \(\text{Bad}(b_i, \chi_{i-1}, H_{i-1})\).

First we show that the degree of every \(a \in A\) in \(H\) is not too large. By Claim 14 and 15 (C1) holds at each iteration, which implies that the number of uncolored neighbors decreases exponentially fast. In particular, \(\tau = O(\log d)\). Using again that (C1) and (C2) are satisfied until iteration \(\tau\), we have,

\[
|N_H(a) \cap (B \setminus B_\tau)| \leq \sum_{i=1}^{\tau} \text{GoodNeighs}_i(a) \leq 2p^+|B_1(a)| + O(d^{\varepsilon/2} \log d) \leq \left(\sum_{i=1}^{\infty} \frac{1}{\alpha^{i/2}}\right) \frac{2\Delta(G)}{\alpha d} |B_0(a)| + O(d^{\varepsilon/2} \log d) \leq \frac{4\Delta(G)}{\alpha},
\]

provided that \(d\) and \(\alpha\) are large enough. Here we also used that \(|B_0(a)| \leq d\).

We will use the former upper bound to lower bound the degree of \(b_i\) in \(H\). By (3) and since \(|B_{i-1}(a)| \leq |B_\tau(a)| \leq d^{\varepsilon/2}\) for any \(i > \tau\), there are at most \((4\Delta(G)/\alpha + d^{\varepsilon/2})d \leq 8\Delta(G)d/\alpha\) vertices in \(N_{H_{i-1}}(b_i)\). Thus, there is a color \(c \in V(G)\) such that

\[
\text{Bad}(b_i, \chi_{i-1}, H_{i-1}) = |\{a \in N_{H_{i-1}}(b_i) : \exists b' \in N_{H_{i-1}}(b_i), b_i \neq b' \text{ and } \chi_{i-1}(b') = c\}| \leq \frac{8\Delta(G)}{\alpha^2}.
\]
We set $\chi_i(b_i) = c$ and delete all the edges $ab_i$ for some $a \in A$ that may cause conflicts (at most $8\Delta(G)/\alpha^2$). Now observe that by the almost regularity hypothesis $\Delta(G) \leq \beta\delta(G) \leq \frac{\alpha}{32}\delta(G)$. Since property $(P3)$ holds for every $b \in B$ and every color $c \in V(G)$, the degree of $b_i$ in $H$ is

$$d_H(b_i) \geq \frac{\delta(G)}{2\alpha} - \frac{8\Delta(G)}{\alpha^2} \geq \frac{\delta(G)}{2\alpha} - \frac{8\beta\delta(G)}{\alpha^2} \geq \frac{\delta(G)}{4\alpha}.$$ 

**Proof of Proposition** Let $\alpha > 0$ be a large positive constant. Let $H$ and $\chi$ be the spanning subgraph and the partial coloring of it obtained by applying the coloring procedure. By Lemma[10] with positive probability $N_H(v)$ is rainbow for every $v \in V$. Suppose that $H$ contains a copy of $F \in F$. Consider the coloring of $F$ induced by the colors given by $\chi$ to the copy of $F$ in $H$. By the properties of $\chi$ it induces an homomorphism from $F$ to $G$. Moreover, since for every vertex $v \in V(F)$, $N_F(v)$ is rainbow, it induces a locally injective homomorphism. However, by the hypothesis of the theorem, $\text{hom}^*(F, G) = 0$, getting a contradiction. Thus, $H$ is $F$-free. Finally, by Lemma[10] $\delta(H) = \Omega(\delta(G))$. 

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