Hamiltonian vector fields on Weil bundles

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Abstract
Let $M$ be a paracompact smooth manifold, $A$ a Weil algebra and $M^A$ the associated Weil bundle. In this paper, we give a characterization of hamiltonian field on $M^A$ in the case of Poisson manifold and of Symplectic manifold.

Keywords: Weil algebra, Weil bundle, Poisson manifold, Hamiltonian vector fields

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1 Introduction

In what follows, we denote $M$, a paracompact differentiable manifold of dimension $n$, $C^\infty(M)$ the algebra of smooth functions on $M$ and $A$ a local algebra in the sense of André Weil i.e a real commutative algebra of finite dimension, with unit, and with an unique maximal ideal $m$ of codimension 1 over $\mathbb{R}$[14]. In this case, there exists an integer $h$ such that $m^{h+1} = (0)$ and $m^h \neq (0)$. The integer $h$ is the height of $A$. Also we have $A = \mathbb{R} \oplus m$.

We recall that a near point of $x \in M$ of kind $A$ is a morphism of algebras
$$\xi : C^\infty(M) \rightarrow A$$

such that
$$\xi(f) - f(x) \in m$$

for any $f \in C^\infty(M)$. We denote $M^A_x$ the set of near points of $x \in M$ of kind $A$ and $M^A = \bigcup_{x \in M} M^A_x$ the manifold of infinitely near points on $M$ of kind $A$ and
$$\pi_M : M^A \rightarrow M$$

the projection which assigns every infinitely near point to $x \in M$ to its origin $x$. The triplet $(M^A, \pi_M, M)$ defines a bundle called bundle of infinitely near points or simply Weil bundle[4].

When $M$ and $N$ are smooth manifolds and when $h : M \rightarrow N$ is a differentiable map of class $C^\infty$, then the map
$$h^A : M^A \rightarrow N^A, \xi \mapsto h^A(\xi)$$

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such that for all \( g \) in \( C^\infty(N) \),
\[
[h^A(\xi)](g) = \xi(g \circ h)
\]
is differentiable. Thus, for \( f \in C^\infty(M) \), the map
\[
f^A : M^A \rightarrow \mathbb{R}^A = A, \xi \mapsto [f^A(\xi)](id_{\mathbb{R}}) = \xi(id_{\mathbb{R}} \circ f) = \xi(f)
\]
is differentiable of class \( C^\infty \). The set, \( C^\infty(M^A, A) \) of smooth functions on \( M^A \) with values on \( A \), is a commutative algebra over \( A \) with unit and the map
\[
C^\infty(M) \rightarrow C^\infty(M^A, A), f \mapsto f^A
\]
is an injective morphism of algebras. Then, we have [2]:
\[
(f + g)^A = f^A + g^A; (\lambda \cdot f)^A = \lambda \cdot f^A; (f \cdot g)^A = f^A \cdot g^A.
\]
In [2] and [9], we showed that the following assertions are equivalent:

1. A vector field on \( M^A \) is a differentiable section of the tangent bundle \((TM^A, \pi_{M^A}, M^A)\).
2. A vector field on \( M^A \) is a derivation of \( C^\infty(M^A) \).
3. A vector field on \( M^A \) is a derivation of \( C^\infty(M^A, A) \) which is \( A \)-linear.
4. A vector field on \( M^A \) is a linear map \( X : C^\infty(M) \rightarrow C^\infty(M^A, A) \) such that
\[
X(f \cdot g) = X(f) \cdot g^A + f^A \cdot X(g), \text{ for any } f, g \in C^\infty(M).
\]

In all what follows, we denote \( \mathcal{X}(M^A) \) the set of vector fields on \( M^A \) and \( \text{Der}_A[C^\infty(M^A, A)] \) the set of \( A \)-linear maps
\[
X : C^\infty(M^A, A) \rightarrow C^\infty(M^A, A)
\]
such that
\[
X(\varphi \cdot \psi) = X(\varphi) \cdot \psi + \varphi \cdot X(\psi), \text{ for any } \varphi, \psi \in C^\infty(M^A, A).
\]
Then [9],
\[
\mathcal{X}(M^A) = \text{Der}_A[C^\infty(M^A, A)].
\]
The map
\[
\mathcal{X}(M^A) \times \mathcal{X}(M^A) \rightarrow \mathcal{X}(M^A), (X, Y) \mapsto [X, Y] = X \circ Y - Y \circ X
\]
is skew-symmetric \( A \)-bilinear and defines a structure of an \( A \)-Lie algebra over \( \mathcal{X}(M^A) \). If
\[
\theta : C^\infty(M) \rightarrow C^\infty(M),
\]
is a vector field on \( M \), then there exists one and only one \( A \)-linear derivation,
\[
\theta^A : C^\infty(M^A, A) \rightarrow C^\infty(M^A, A)
\]
called prolongation of the vector field \( \theta \), such that
\[
\theta^A(f^A) = [\theta(f)]^A, \text{ for any } f \in C^\infty(M).
\]
If \( \theta, \theta_1 \) and \( \theta_2 \) are vector fields on \( M \) and if \( f \in C^\infty(M) \), then we have:
1. \((\theta_1 + \theta_2)^A = \theta_1^A + \theta_2^A\),
2. \((f \cdot \theta)^A = f^A \cdot \theta^A\),
3. \([\theta_1, \theta_2]^A = [\theta_1^A, \theta_2^A]\).

The map

\[ X(M) \to \text{Der}_A[C^\infty(M^A, A)], \theta \mapsto \theta^A \]

is an injective morphism of \(\mathbb{R}\)-Lie algebras.

## 2 Hamiltonian vector fields on weil bundles

### 2.1 Structure of \(A\)-Poisson manifold on \(M^A\) when \(M\) is a Poisson manifold

We recall that a Poisson structure on a smooth manifold \(M\) is due to the existence of a bracket \(\{,\}\) on \(C^\infty(M)\) such that the pair \((C^\infty(M), \{,\})\) is a real Lie algebra such that, for any \(f \in C^\infty(M)\) the map

\[ \text{ad}(f) : C^\infty(M) \to C^\infty(M), g \mapsto \{f, g\} \]

is a derivation of commutative algebra i.e

\[ \{f, g \cdot h\} = \{f, g\} \cdot h + g \cdot \{f, h\} \]

for \(f, g, h \in C^\infty(M)\). In this case we say that \(M\) is a Poisson manifold and \(C^\infty(M)\) is a Poisson algebra [12], [13].

We denote

\[ C^\infty(M) \to \text{Der}_\mathbb{R}[C^\infty(M)], f \mapsto \text{ad}(f), \]

the adjoint representation and \(d_{\text{ad}}\) the operator of cohomology associated to this representation. For any \(p \in \mathbb{N}\),

\[ \Lambda^p_{\text{Pois}}(M) = L^p_{\text{sk}}(C^\infty(M), C^\infty(M)) \]

denotes the \(C^\infty(M)\)-module of skew-symmetric multilinear forms of degree \(p\) from \(C^\infty(M)\) into \(C^\infty(M)\). We have

\[ \Lambda^p_{\text{Pois}}(M) = C^\infty(M). \]

When \(M\) is a smooth manifold, \(A\) a weil algebra and \(M^A\) the associated Weil bundle, the \(A\)-algebra \(C^\infty(M^A, A)\) is a Poisson algebra over \(A\) if there exists a bracket \(\{,\}\) on \(C^\infty(M^A, A)\) such that the pair \((C^\infty(M^A, A), \{,\})\) is a Lie algebra over \(A\) satisfying

\[ \{\varphi_1 \cdot \varphi_2, \varphi_3\} = \{\varphi_1, \varphi_3\} \cdot \varphi_2 + \varphi_1 \cdot \{\varphi_2, \varphi_3\} \]

for any \(\varphi_1, \varphi_2, \varphi_3 \in C^\infty(M^A, A)\) [11].

When \(M\) is a Poisson manifold with bracket \(\{,\}\), for any \(f \in C^\infty(M)\), let

\[ [\text{ad}(f)]^A : C^\infty(M) \to C^\infty(M^A, A), g \mapsto \{f, g\}^A, \]

be the prolongation of the vector field \(\text{ad}(f)\) and let

\[ [\widehat{\text{ad}(f)}]^A : C^\infty(M^A, A) \to C^\infty(M^A, A) \]

be the prolongation of the vector field \(\widehat{\text{ad}(f)}\).
be the unique $A$-linear derivation such that
\[
[\text{ad}(f)]^A(g^A) = [\text{ad}(f)]^A(g) = [f, g]^A
\]
for any $g \in C^\infty(M)$.

For $\varphi \in C^\infty(M^A, A)$, the application
\[
\tau_\varphi : C^\infty(M) \longrightarrow C^\infty(M^A, A), \quad f \longmapsto -[\text{ad}(f)]^A(\varphi)
\]
is a vector field on $M^A$ considered as derivation of $C^\infty(M)$ into $C^\infty(M^A, A)$ and
\[
\tilde{\tau}_\varphi : C^\infty(M^A, A) \longrightarrow C^\infty(M^A, A)
\]
the unique $A$-linear derivation (vector field) such that
\[
\tilde{\tau}_\varphi(f^A) = \tau_\varphi(f) = -[\text{ad}(f)]^A(\varphi)
\]
for any $f \in C^\infty(M)$. We have for $f \in C^\infty(M)$,
\[
\tilde{\tau}_f^A = [\text{ad}(f)]^A,
\]
and for $\varphi, \psi \in C^\infty(M^A, A)$ and for $a \in A$,
\[
\tilde{\tau}_{\varphi + \psi} = \tilde{\tau}_\varphi + \tilde{\tau}_\psi; \tilde{\tau}_{a \varphi} = a \cdot \tilde{\tau}_\varphi; \tilde{\tau}_\varphi \psi = \varphi \cdot \tilde{\tau}_\psi + \psi \cdot \tilde{\tau}_\varphi.
\]
For any $\varphi, \psi \in C^\infty(M^A, A)$, we let
\[
\{\varphi, \psi\}_A = \tilde{\tau}_\varphi(\psi).
\]
In [1], we have show that this bracket defines a structure of $A$-Poisson algebra on $C^\infty(M^A, A)$.

Thus, when $M$ is a Poisson manifold with bracket $\{,\}$, then $\{,\}_A$ is the prolongation on $M^A$ of the structure of Poisson on $M$ defined by $\{,\}$.

The map
\[
C^\infty(M^A, A) \longrightarrow \text{Der}_A[C^\infty(M^A, A)], \varphi \longmapsto \tilde{\tau}_\varphi,
\]
is a representation from $C^\infty(M^A, A)$ into $C^\infty(M^A, A)$. We denote $\tilde{\text{d}}_A$ the cohomology operator associated to this adjoint representation [9].

For any $p \in \mathbb{N}$, $\Lambda^p_{\text{Pois}}(M^A, \sim_A) = \Lambda^p_{\text{sk}}[C^\infty(M^A, A), C^\infty(M^A, A)]$ denotes the $C^\infty(M^A, A)$-module of skew-symmetric multilinear forms of degree $p$ on $C^\infty(M^A, A)$ into $C^\infty(M^A, A)$. We have
\[
\Lambda^0_{\text{Pois}}(M^A, \sim_A) = C^\infty(M^A, A).
\]
We denote
\[
\Lambda_{\text{Pois}}(M^A, \sim_A) = \bigoplus_{p=0}^\infty \Lambda^p_{\text{Pois}}(M^A, \sim_A).
\]
For $\Omega \in \Lambda^p_{\text{Pois}}(M^A, \sim_A)$ and $\varphi_1, \varphi_2, \ldots, \varphi_{p+1} \in C^\infty(M^A, A)$, we have
\[
\tilde{\text{d}}_A \Omega(\varphi_1, \ldots, \varphi_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i-1} \tilde{\tau}_{\varphi_i}[\Omega(\varphi_1, \ldots, \widehat{\varphi_i}, \ldots, \varphi_{p+1})]
\]
\[
+ \sum_{1 \leq i < j \leq p+1} (-1)^{i+j} \Omega([\varphi_i, \varphi_j], \varphi_1, \ldots, \widehat{\varphi_i}, \ldots, \widehat{\varphi_j}, \ldots, \varphi_{p+1})
\]
where $\widehat{\varphi_i}$ means that the term $\varphi_i$ is omitted.

**Proposition 1.** For any $\eta \in \Lambda^p_{\text{Pois}}(M)$, we have
\[
\tilde{\text{d}}_A(\eta^A) = (d\eta)^A.
\]
3 Hamiltonian vector fields on weil bundles

When \( M \) is a Poisson manifold with bracket \( \{,\} \), a vector field
\[
\theta : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)
\]

1. is locally hamiltonian if \( \theta \) is closed for the cohomology associated with the adjoint representation
\[
ad : \mathcal{C}^\infty(M) \to \text{Der} [\mathcal{C}^\infty(M)]
\]
i.e. \( d_{\text{ad}}\theta = 0 \).

2. is globally hamiltonian if \( \theta \) is exact for the cohomology associated with the adjoint representation
\[
ad : \mathcal{C}^\infty(M) \to \text{Der} [\mathcal{C}^\infty(M)]
\]
i.e. there exists \( f \in \mathcal{C}^\infty(M) \) such that \( \theta = d_{\text{ad}}(f) \).

Thus, a vector field
\[
X : \mathcal{C}^\infty(M^A, A) \to \mathcal{C}^\infty(M^A, A)
\]

1. is locally hamiltonian if \( X \) is closed for the cohomology associated with the adjoint representation
\[
\bar{\tau}_\varphi : \mathcal{C}^\infty(M^A, A) \to \text{Der}_A [\mathcal{C}^\infty(M^A, A)]
\]
i.e. \( \bar{d}_A X = 0 \).

2. is globally hamiltonian if \( X \) is exact for the cohomology associated with the adjoint representation
\[
\bar{\tau}_\varphi : \mathcal{C}^\infty(M^A, A) \to \text{Der}_A [\mathcal{C}^\infty(M^A, A)]
\]
i.e. there exists \( \varphi \in \mathcal{C}^\infty(M^A, A) \) such that \( X = \bar{d}_A(\varphi) \).

**Proposition 2.** When \( M \) is a Poisson manifold with bracket \( \{,\} \), then a vector field
\[
\theta : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)
\]
is locally hamiltonian if and only if the vector field
\[
\theta^A : \mathcal{C}^\infty(M^A, A) \to \mathcal{C}^\infty(M^A, A).
\]
is locally hamiltonian.

**Proof.** Indeed, for any \( \eta \in \Lambda^p_{\text{Pois}}(M) \), we have
\[
\bar{d}_A(\eta^A) = (d_{\text{ad}}\eta)^A.
\]
In particular, for \( p = 1 \), we have
\[
\bar{d}_A(\theta^A) = (d_{\text{ad}}\theta)^A.
\]
Thus, \( d_{\text{ad}}\theta = 0 \) if and only if \( \bar{d}_A(\theta^A) = 0 \). \( \square \)
**Proposition 3.** When $M^A$ is a $A$-Poisson manifold with bracket $\{,\}_A$, then a vector field

$$X : C^\infty(M^A, A) \rightarrow C^\infty(M^A, A)$$

locally hamiltonian is a derivation of the Poisson $A$-algebra $C^\infty(M^A, A)$.

**Proof.** We have

$$\tilde{d}_A X : C^\infty(M^A, A) \times C^\infty(M^A, A) \rightarrow C^\infty(M^A, A)$$

$(\varphi, \psi) \mapsto (\tilde{d}_A X)(\varphi, \psi)$

and if $\tilde{d}_A X = 0$, then for any $\varphi, \psi \in C^\infty(M^A, A)$,

$$0 = (\tilde{d}_A X)(\varphi, \psi) = \tilde{\tau}_\varphi [X(\psi)] - \tilde{\tau}_\psi [X(\varphi)] - X(\{\varphi, \psi\}_A)$$

$$= \{\varphi, X(\psi)\}_A - \{\psi, X(\varphi)\}_A - X(\{\varphi, \psi\}_A)$$

i.e.

$$X(\{\varphi, \psi\}_A) = \{X(\varphi), \psi\}_A + \{\varphi, X(\psi)\}_A.$$

That ends the proof. \(\square\)

**Proposition 4.** Let $M$ be a Poisson manifold with bracket $\{,\}$. If a vector field

$$\theta : C^\infty(M) \rightarrow C^\infty(M)$$

is globally hamiltonian then the vector field

$$\theta^A : C^\infty(M^A, A) \rightarrow C^\infty(M^A, A)$$

is globally hamiltonian.

**Proof.** Based on the assumptions, there exists $f \in C^\infty(M)$ such that $\theta = d_{ad}(f)$. Thus,

$$\theta^A = [ad(f)]^A$$

$$= \tilde{d}_A (f^A).$$

Thus, $\theta = d_{ad}(f)$ then $\theta^A = \tilde{d}_A (f^A)$ is globally hamiltonian. \(\square\)

**Proposition 5.** When $M^A$ is a $A$-Poisson manifold with bracket $\{,\}_A$, then a vector field

$$X : C^\infty(M^A, A) \rightarrow C^\infty(M^A, A)$$

globally hamiltonian is the derivation interior of the Poisson $A$-algebra $C^\infty(M^A, A)$.

**Proof.** If the vector field

$$X : C^\infty(M^A, A) \rightarrow C^\infty(M^A, A)$$

is globally hamiltonian, there exists $\varphi \in C^\infty(M^A, A)$ such that $X = \tilde{d}_A \varphi$. For any $\psi \in C^\infty(M^A, A)$, we have

$$X(\psi) = (\tilde{d}_A \varphi)(\psi)$$

$$= \tilde{\tau}_\varphi (\psi)$$

$$= \{\varphi, \psi\}_A$$

i.e. $X = ad(\varphi)$. where

$$ad(\varphi) : C^\infty(M^A, A) \rightarrow C^\infty(M^A, A), \psi \mapsto \{\varphi, \psi\}_A$$

Thus, $X$ is globally hamiltonian if there exists $\varphi \in C^\infty(M^A, A)$ such that $X = \tilde{\tau}_\varphi = ad(\varphi)$ i.e. $X$ is the interior derivation of the Poisson $A$-algebra $C^\infty(M^A, A)$. \(\square\)
3.1 Hamiltonian vector fields on $M^A$ when $M$ is a symplectic manifold

When $(M, \Omega)$ is a symplectic manifold, then $(M^A, \Omega^A)$ is a symplectic $A$-manifold \[1\].

For any $f \in C^\infty(M)$, we denote $X_f$ the unique vector field on $M$ such that

\[ i_{X_f} \Omega = df \]

where

\[ d : \Lambda(M) \rightarrow \Lambda(M) \]

is the operator of de Rham cohomology. We denote

\[ d^A : \Lambda(M^A, A) \rightarrow \Lambda(M^A, A) \]

the operator of cohomology associated with the representation

\[ \chi(M^A) \rightarrow \text{Der} \left[ C^\infty(M^A, A) \right], X \mapsto X. \]

For $\varphi \in C^\infty(M^A, A)$, we denote $X_\varphi$ the unique vector field on $M^A$, considered as a derivation of $C^\infty(M^A, A)$ into $C^\infty(M^A, A)$, such that

\[ i_{X_\varphi} \Omega^A = d^A(\varphi). \]

The bracket

\[ \{\varphi, \psi\}_{\Omega^A} = -\Omega^A(X_\varphi, X_\psi) = X_\varphi(\psi) \]

defines a structure of $A$-Poisson manifold on $M^A$ and for any $f \in C^\infty(M)$,

\[ X_f^A = (X_f)^A. \]

\[ i_{(X_f)^A} \Omega^A = i_{X_f} \Omega^A. \]

We deduce that \[1\]:

**Theorem 6.** If $(M, \Omega)$ is a symplectic manifold, the structure of $A$-Poisson manifold on $M^A$ defined by $\Omega^A$ coincide with the prolongation on $M^A$ of the Poisson structure on $M$ defined by the symplectic form $\Omega$ i.e for any $\varphi \in C^\infty(M^A, A)$, $\tilde{\tau}_\varphi = X_\varphi$.

Therefore, for any $\varphi, \psi \in C^\infty(M^A, A)$, we have

\[ \{\varphi, \psi\}_{\Omega^A} = \{\varphi, \psi\}_{\lambda}. \]

**Proposition 7.** If $\omega$ is a differential form on $M$ and if $\theta$ is a vector field on $M$, then

\[ (i_\theta \omega)^A = i_{\theta^A}(\omega^A). \]

**Proof.** If the degree of $\omega$ is $p$, then $(i_\theta \omega)^A$ is the unique differential $A$-form of degree $p - 1$ such that

\[ (i_\theta \omega)^A(\theta^1, ..., \theta^A_{p-1}) = \left[ (i_\theta \omega)(\theta_1, ..., \theta_{p-1}) \right]^A = \left[ \omega(\theta, \theta_1, ..., \theta_{p-1}) \right]^A. \]
for any $\theta_1, \theta_2, \ldots, \theta_{p-1} \in \mathfrak{X}(M)$. As $i_{\theta'}(\omega^A)$ is of degree $p - 1$ and is such that

$$i_{\theta'}(\omega^A)[\theta_1, \ldots, \theta_{p-1}] = \omega^A(\theta^A, \theta_1^A, \ldots, \theta_{p-1}^A)$$

$$= \left[\omega(\theta, \theta_1, \ldots, \theta_{p-1})\right]^A$$

for any $\theta_1, \theta_2, \ldots, \theta_{p-1} \in \mathfrak{X}(M)$, we conclude that $(i_{\theta'}\omega)^A = i_{\theta'}(\omega^A)$. \hfill \square

When $(M, \Omega)$ is a symplectic manifold,

1. a vector field $\theta$ on $M$ is locally hamiltonian if the form $i_\theta \Omega$ is closed for the de Rham cohomology and $\theta$ is globally hamiltonian if there exists $f \in C^\infty(M)$ such that $i_\theta \Omega = -d(f)$, i.e. the form $i_\theta \Omega$ is $\mathfrak{d}$-exact.

2. a vector field $X$ on $M^A$ is locally hamiltonian if the form $i_X \Omega^A$ is $d^A$-closed and $X$ is globally hamiltonian if there exists $\varphi \in C^\infty(M^A, A)$ such that $i_X \Omega^A = -d^A(\varphi)$, i.e. the form $i_X \Omega^A$ is $d^A$-exact.

**Proposition 8.** A vector field $\theta : C^\infty(M) \to C^\infty(M)$ on a symplectic manifold $M$ is locally hamiltonian, if and only if $\theta^A : C^\infty(M^A, A) \to C^\infty(M^A, A)$ is a locally hamiltonian vector field.

**Proof.** For any $\theta \in \mathfrak{X}(M)$, we have

$$d^A(i_{\theta'}\Omega^A) = d^A[(i_{\theta'})^A]$$

$$= [d(i_{\theta'})]^A.$$ 

Thus, $\theta$ is locally hamiltonian, i.e. $d(i_{\theta'}\Omega) = 0$ if and only if, $d^A(i_{\theta'}\Omega^A) = 0$ i.e. $\theta^A : C^\infty(M^A, A) \to C^\infty(M^A, A)$ is a locally hamiltonian vector field. \hfill \square

**Theorem 9.** A vector field $X : C^\infty(M^A, A) \to C^\infty(M^A, A)$ on $M^A$ locally hamiltonian is a derivation of the $A$-Lie algebra induced by the $A$-structure of Poisson defined by the symplectic $A$-manifold $(M^A, \Omega^A)$.

**Proof.** Let $(M^A, \Omega^A)$ be a symplectic manifold. For any $\varphi, \psi \in C^\infty(M^A, A)$,

$$\{\varphi, \psi\}_\Omega = -\Omega^A(X_\varphi, X_\psi)$$

$$= X_\varphi(\psi)$$

If $X$ is locally hamiltonian vector field, we have $d^A(i_X\Omega^A) = 0$ i.e. for any $X$ and $Y \in \mathfrak{X}(M^A),

$$d^A(i_X\Omega^A)(X, Y) = 0.$$ 

In particular, for any $X_\varphi$ and $X_\psi$, we have

$$0 = (d^A(i_X\Omega^A))(X_\varphi, X_\psi)$$

$$= X_\varphi[i_X\Omega^A(X_\psi)] - X_\psi[i_X\Omega^A(X_\varphi)] - i_X\Omega^A([X_\varphi, X_\psi])$$

Therefore

$$i_X\Omega^A([X_\varphi, X_\psi]) = X_\varphi[i_X\Omega^A(X_\psi)] - X_\psi[i_X\Omega^A(X_\varphi)]$$

i.e

$$\Omega^A(X_\varphi, X_\psi) = X_\varphi[\Omega^A(X, X_\psi)] - X_\psi[\Omega^A(X, X_\varphi)]$$

Hence

$$X(\{\varphi, \psi\}_\Omega) = [X(\varphi), \psi]_\Omega + \{\varphi, X(\psi)\}_\Omega.$$

That ends the proof. \hfill \square
Proposition 10. Let \((M, \Omega)\) be a symplectic manifold. If a vector field
\[
\theta : C^\infty(M) \rightarrow C^\infty(M)
\]
is globally hamiltonian then the vector field
\[
\theta^A : C^\infty(M^A, A) \rightarrow C^\infty(M^A, A)
\]
is globally hamiltonian.

Proof. If \(\theta\) is globally hamiltonian, then there exists \(f \in C^\infty(M)\) such that \(i_\theta \Omega = -d(f)\). Then,
\[
(i_\theta \Omega)^A = [-d(f)]^A
\]
\[
= d^A(f^A)
\]
Thus
\[
i_\theta^A \Omega^A = d^A(f^A).
\]
i.e \(\theta^A\) is globally hamiltonian. \(\square\)

References

[1] Bossoto, B.G.R. Okassa, E. (2012). A-Poisson structures on Weil bundles, Int. J. Contemp. Math. Sciences, vol.7, n°16, 785-803.

[2] Bossoto, B.G.R. Okassa, E. (2008). Champs de vecteurs et formes différentielles sur une variété des points proches, Arch. math. (BRNO), Tomus44, 159-171.

[3] Helgason, S. (1962). Differential Geometry and symmetric spaces, New York; Academic Press, 1962

[4] Kolár, I, Michor, P. W. and Slovak, J. (1993) Natural Operations in Differential Geometry, Springer-Verlag, Berlin.

[5] Koszul, J.L., Ramanan, S. (1960). Lectures On Fibre Bundles and Differential Geometry, Tata Institute of Fundamental Research, Bombay.

[6] Lichnerowicz, A. (1977). Les variétés de Poisson et leurs algèbres de Lie associées, J. Diff. Geom., 12, 253–300.

[7] Morimoto, A. (1976). Prolongation of connections to bundles of infinitely near points, J. Diff. Geom, 11, 479-498.

[8] Nkou, V.B. Bossoito, B.G.R., Okassa, E. (2015). New characterization of vector field on Weil bundles, Theoretical Mathematics & Applications, vol.5, no.2, 1-17.

[9] Nkou, V.B. Bossoito, B.G.R. (2014). Cohomology associated to a Poisson structure on Weil bundles, Int.Math. Forum, vol. 9, no. 7, 305- 316.

[10] Okassa, E. (1986-1987). Prolongement des champs de vecteurs à des variétés des points proches, Ann. Fac. Sci. Toulouse Math. VIII (3), 346-366.

[11] Pham-Mau-Quan, F. (1969). Introduction à la géométrie des variétés différentiables, Dunod Paris.
[12] Vaisman, I. (1995). *Second order Hamiltonian vector fields on tangent bundles*, Differential Geom., 5, 153–170.

[13] Vaisman, I. (1994). *Lectures on the Geometry of Poisson Manifolds*, in: Progress in Math., 118, Basel Birkhäuser.

[14] Weil A. (1953). *Théorie des points proches sur les variétés différentiables*, Colloq. Géom. Diff. Strasbourg, 111-117.