INVARIANT HYPERSURFACES OF ENDOMORPHISMS OF THE
PROJECTIVE 3-SPACE

DE-QI ZHANG

Abstract. We consider surjective endomorphisms $f$ of degree $> 1$ on the projective
$n$-space $\mathbb{P}^n$ with $n = 3$, and $f^{-1}$-stable hypersurfaces $V$. We show that $V$ is a hyperplane
(i.e., $\deg(V) = 1$) but with four possible exceptions; it is conjectured that $\deg(V) = 1$
for any $n \geq 2$; cf. [7], [3].

Dedicated to Prof. Miyanishi on the occasion of his 70th birthday

1. Introduction

We work over the field $\mathbb{C}$ of complex numbers. In this paper, we study properties of
$f^{-1}$-stable prime divisors of $X$ for endomorphisms $f : \mathbb{P}^3 \to \mathbb{P}^3$. Below is our main result.

Theorem 1.1. Let $f : \mathbb{P}^3 \to \mathbb{P}^3$ be an endomorphism of degree $> 1$ and $V$ an irreducible
hypersurface such that $f^{-1}(V) = V$. Then either $\deg(V) = 1$, i.e., $V$ is a hyperplane, or
$V$ equals one of the four cubic hypersurfaces $V_i = \{S_i = 0\}$, where $S_i$’s are as follows,
with suitable projective coordinates:

1. $S_1 = X_3^3 + X_0X_1X_2$;
2. $S_2 = X_3^3X_3 + X_0X_1^2 + X_2^3$;
3. $S_3 = X_3^2X_2 + X_1^2X_3$;
4. $S_4 = X_0X_1X_2 + X_0^3X_3 + X_1^3$.

We are unable to rule out the four cases in Theorem 1.1 and do not know whether there
is any endomorphism $f_{V_i} : V_i \to V_i$ of $\deg(f_{V_i}) > 1$ for $i = 2, 3$ or 4, but see Examples 1.5
(for $V_1$) and 1.2 below.

Example 1.2. There are many endomorphisms $f_{V_i} : V_i \to V_i$ of $\deg(f_{V_i}) > 1$ for the
normalization $V'$ of $V = V_i (i = 3, 4)$, where $V' \simeq \mathbb{F}_1$ in either case (cf. Remark 1.3 below).
Conjecture 1.4 below asserts that $f_{V_i}$ is not lifted from any endomorphism $f : \mathbb{P}^3 \to \mathbb{P}^3$
restricted to the non-normal cubic surface $V$. Indeed, consider the endomorphism
$f_{V_2} : \mathbb{P}^2 \to \mathbb{P}^2 ([X_0, X_1, X_2] \to [X_0^q, X_1^q, X_2^q])$ with $q \geq 2$. It lifts to an endomorphism

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$f_{F_1} : F_1 \rightarrow F_1$ of $\deg(f_{F_1}) = q^2$, where $F_1 \rightarrow \mathbb{P}^2$ is the blowup of the point $[0, 0, 1]$ fixed by $f_{F_1}^{-1}$.

**Remark 1.3.** Below are some remarks about Theorem 1.1.

1. The non-normal locus of $V_i$ $(i = 3, 4)$ is a single line $C$ and stabilized by $f^{-1}$. Let $\sigma : V'_i \rightarrow V_i$ $(i = 3, 4)$ be the normalization. Then $V'_i$ is the (smooth) Hirzebruch surface $F_1$ (i.e., the one-point blowup of $\mathbb{P}^2$; see [1, Theorem 1.5], [16]) with the conductor $\sigma^{-1}(C) \subset V'_i$ a smooth section at infinity (for $V_3$), and the union of the negative section and a fibre (for $V_4$), respectively. $f|V_i$ lifts to a (polarized) endomorphism $f_{V'_i} : V'_i \rightarrow V'_i$.

2. $V_1$ (resp. $V_2$) is unique as a normal cubic (or degree three del Pezzo) surface of Picard number one and with the singular locus $\text{Sing} V_1 = 3A_2$ (resp. $\text{Sing} V_2 = E_6$); see [17, Theorem 1.2], and [19, Theorem 4.4] for the anti-canonical embedding of $V_i$ in $\mathbb{P}^3$. $V_1$ contains exactly three lines of triangle-shaped whose three vertices form the singular locus of $V_1$. And $V_2$ contains a single line on which lies its unique singular point. $f^{-3}$ fixes the singular point(s) of $V_i$ $(i = 1, 2)$.

3. $f^{-1}$ (or its positive power) does not stabilize the only line $L$ on $V_2$ by using [14, Theorem 4.3.1] since the pair $(V_2, L)$ is not log canonical at the singular point of $V_2$. For $V_1$, we do not know whether $f^{-1}$ (or its power) stabilizes the three lines.

1.4. **A motivating conjecture.** Here are some motivations for our paper. It is conjectured that *every hypersurface $V \subset \mathbb{P}^n$ stabilized by the inverse $f^{-1}$ of an endomorphism $f : \mathbb{P}^n \rightarrow \mathbb{P}^n$ of $\deg(f) > 1$, is linear*. This conjecture is still open when $n \geq 3$ and $V$ is singular, since the proof of [3] is incomplete as we were informed by an author. The smooth hypersurface case was settled in the affirmative in any dimension by Cerveau - Lins Neto [4] and independently by Beauville [2]. See also [15, Theorem 1.5 in arXiv version], [18], and [19] for related results.

By Theorem 1.1, this conjecture is true when $n = 3$ but with four exceptional cubic surfaces $V_i$ which we could not rule out.

From the dynamics point of view, as seen in Dinh-Sibony [5, Theorem 1.3, Corollary 1.4], $f : \mathbb{P}^n \rightarrow \mathbb{P}^n$ behaves nicely exactly outside those $f^{-1}$-stabilized subvarieties. We refer to Fornaess-Sibony [7], and [5] for further references.

A smooth hypersurface $X$ in $\mathbb{P}^{n+1}$ with $\deg(X) \geq 3$ and $n \geq 2$, has no endomorphism $f_X : X \rightarrow X$ of degree $> 1$ (cf. [4, 2, Theorem]). However, singular $X$ may have plenty of endomorphisms $f_X$ of arbitrary degrees as shown in Example 1.5 below. Conjecture 1.4 asserts that such $f_X$ can not be extended to an endomorphism of $\mathbb{P}^{n+1}$. 
Example 1.5. We now construct many polarized endomorphisms for some degree \( n + 1 \) hypersurface \( X \subset \mathbb{P}^{n+1} \), with \( X \) isomorphic to the \( V_1 \) in Theorem 1.1 when \( n = 2 \). Let

\[
f = (F_0, \ldots, F_n) : \mathbb{P}^n \to \mathbb{P}^n
\]

\((n \geq 2)\), with \( F_i = F_i(X_0, \ldots, X_n) \) homogeneous, be any endomorphism of degree \( q^n > 1 \), such that \( f^{-1}(S) = S \) for a reduced degree \( n + 1 \) hypersurface \( S = \{S(X_0, \ldots, X_n) = 0\} \). So \( S \) must be normal crossing and linear: \( S = \sum_{i=0}^{n} S_i \) (cf. [15, Theorem 1.5 in arXiv version]). Thus we may assume that \( f = (X_q^0, \ldots, X_q^n) \) and \( S_i = \{X_i = 0\} \). The relation \( S \sim (n + 1)H \) with \( H \subset \mathbb{P}^n \) a hyperplane, defines

\[
\pi : X = Spec \oplus_{i=0}^n O(-iH) \to \mathbb{P}^n
\]

which is a Galois \( \mathbb{Z}/(n + 1) \)-cover branched over \( S \) so that \( \pi^*S_i = (n + 1)T_i \) with the restriction \( \pi|T_i : T_i \to S_i \) an isomorphism.

This \( X \) is identifiable with the degree \( n + 1 \) hypersurface

\[
\{Z^{n+1} = S(X_0, \ldots, X_n)\} \subset \mathbb{P}^{n+1}
\]

and has singularity of type \( z^{n+1} = xy \) over the intersection points of \( S \) locally defined as \( xy = 0 \). Thus, when \( n = 2 \), we have \( \text{Sing} X = 3A_2 \) and \( X \) is isomorphic to the \( V_1 \) in Theorem 1.1 (cf. Remark 1.3). We may assume that

\[
f^*S(X_0, \ldots, X_n) = S(X_0, \ldots, X_n)^q
\]

after replacing \( S(X_0, \ldots, X_n) \) by a scalar multiple, so \( f \) lifts to an endomorphism

\[
g = (Z^q, F_0, \ldots, F_n)
\]

of \( \mathbb{P}^{n+1} \) (with homogeneous coordinates \([Z, X_0, \ldots, X_n]\)), stabilizing \( X \), so that

\[
g_X := g|X : X \to X
\]

is a polarized endomorphism of \( \text{deg}(g_X) = q^n \) (cf. [15, Lemma 2.1]). Note that \( g^{-1}(X) \) is the union of \( q \) distinct hypersurfaces

\[
\{Z = \zeta^iS(X_0, \ldots, X_n)\} \subset \mathbb{P}^{n+1}
\]

(all isomorphic to \( X \)), where \( \zeta := \exp(2\pi \sqrt{-1}/q) \).

This \( X \) has only Kawamata log terminal singularities and \( \text{Pic} X = (\text{Pic} \mathbb{P}^{n+1}) | X \) \((n \geq 2)\) is of rank one, using Lefschetz type theorem [12, Example 3.1.25] when \( n \geq 3 \). We have \( f^{-1}(S_i) = S_i \) and \( g_X^{-1}(T_i) = T_i \), where \( 0 \leq i \leq n \).

When \( n = 2 \), the relation \((n + 1)(T_i - T_0) \sim 0 \) gives rise to an étale-in-codimension-one \( \mathbb{Z}/(n + 1) \)-cover

\[
\tau : \mathbb{P}^n \simeq \tilde{X} \to X
\]
so that $\sum_{i=0}^n \tau^* T_i$ is a union of $n+1$ normal crossing hyperplanes; indeed, $\tau$ restricted over $X \setminus \text{Sing} X$, is its universal cover (cf. [13] Lemma 6)), so that $g_X$ lifts up to $\tilde{X}$. A similar result seems to be true for $n \geq 3$, by considering the ‘composite’ of the $\mathbb{Z}/(n+1)$-covers given by $(n+1)(T_i - T_0) \sim 0$ ($1 \leq i < n$).

2. Proofs of Theorem 1.1 and Remark 1.3

We use the standard notation in Hartshorne’s book and [11].

2.1. We now prove Theorem 1.1 and Remark 1.3. By [15, Theorem 1.5 in arXiv version], we may assume that $V \subset \mathbb{P}^3$ is an irreducible rational singular cubic hypersurface.

We first consider the case where $V$ is non-normal. Such $V$ is classified in [6, Theorem 9.2.1] to the effect that either $V = V_i$ ($i = 3, 4$) or $V$ is a cone over a nodal or cuspidal rational planar cubic curve $B$. The description in Remark 1.3 on $V_3, V_4$ and their normalizations, is given in [16, Theorem 1.1], [1, Theorem 1.5, Case (C), (E1)]; the $f^{-1}$-invariance of the non-normal locus $C$ is proved in [15, Proposition 5.4 in arXiv version].

We consider and will rule out the case where $V$ is a cone over $B$. Since $V$ is normal crossing in codimension 1 (cf. [15, Theorem 1.5 or Proposition 5.4 in arXiv version]), the base $B$ of the cone $V$, and $L \subset V$ the generating line lying over the node of $B$. Then $f_V := f|V$ satisfies the assertion that $f_V^{-1}(P) = P$. Indeed, the normalization $V'$ of $V$ is a cone over a smooth rational (twisted) cubic curve (in $\mathbb{P}^3$), i.e., the contraction of the $(-3)$-curve on the Hirzebruch surface $\mathbb{F}_3$ of degree 3; $f_V$ lifts to an endomorphism $f_{V'}$ of $V'$ so that the conductor $C' \subset V'$ is preserved by $f_{V'}^{-1}$ (cf. [15, Proposition 5.4 in arXiv version]) and consists of two distinct generating lines $L_i$ (lying over $L$). Thus $f_{V'}^{-1}$ fixes the vertex $L_1 \cap L_2$ (lying over $P$). Hence $f_V^{-1}(P) = P$ as asserted.

By [15, Lemma 5.9 in arXiv version], $f : \mathbb{P}^3 \to \mathbb{P}^3$ (with $\text{deg}(f) = q^3 > 1$ say) descends, via the projection $\mathbb{P}^3 \to \mathbb{P}^2$ from the point $P$, to an endomorphism $h : \mathbb{P}^2 \to \mathbb{P}^2$ with $\text{deg}(h) = q^2 > 1$ so that $h^{-1}(B) = B$. This and $\text{deg}(B) = 3 > 1$ contradict the linearity property of $h^{-1}$-stable curves in $\mathbb{P}^2$ (see e.g. Theorem 1.5 and the references in [15 arXiv version]).

2.2. Next we consider the case where $V \subset \mathbb{P}^3$ is a normal rational singular cubic hypersurface. By the adjunction formula,

$$-K_V = -(K_{\mathbb{P}^3} + V)|V \sim H|V$$
which is ample, where \( H \subset \mathbb{P}^3 \) is a hyperplane. Since \( K_V \) is a Cartier divisor, \( V \) has only Du Val (or rational double, or ADE) singularities. Let \( \sigma : V' \to V \) be the minimal resolution. Then

\[
K_{V'} = \sigma^* K_V \sim \sigma^*(-H|V).
\]

For \( f : \mathbb{P}^3 \to \mathbb{P}^3 \), we can apply the result below to \( f_V := f|_V \).

**Lemma 2.3.** Let \( V \subset \mathbb{P}^3 \) be a normal cubic surface, and \( f_V : V \to V \) an endomorphism such that \( f_V^*(H|V) \sim qH|V \) for some \( q > 1 \) and the hyperplane \( H \subset \mathbb{P}^3 \). Let

\[
S(V) = \{ G \subset V \mid G : \text{irreducible}, G^2 < 0 \}
\]

be the set of negative curves on \( V \), and set

\[
E_V := \sum_{E \in S(V)} E.
\]

Replacing \( f_V \) by its positive power, we have:

1. If \( f_V^* G \equiv aG \) for some Weil divisor \( G \neq 0 \), then \( a = q \). We have \( f_V^*(L|V) \sim q(L|V) \) for every divisor \( L \) on \( \mathbb{P}^3 \). Especially, \( \deg(f_V) = q^2 \); \( K_V \sim -H|V \) satisfies \( f_V^* K_V \sim qK_V \).
2. \( S(V) \) is a finite set. \( f_V^* E = qE \) for every \( E \in S(V) \). So \( f_V^* E_V = qE_V \).
3. A curve \( E \subset V \) is a line in \( \mathbb{P}^3 \) if and only if \( E \) is equal to \( \sigma(E') \) for some \((-1)\)-curve \( E' \subset V' \).
4. Every curve \( E \in S(V) \) is a line in \( \mathbb{P}^3 \).
5. We have

\[
K_V + E_V = f_V^*(K_V + E_V) + \Delta
\]

for some effective divisor \( \Delta \) containing no line in \( S(V) \), so that the ramification divisor

\[
R_{f_V} = (q - 1)E_V + \Delta.
\]

In particular, the cardinality \( \#S(V) \leq 3 \), and the equality holds exactly when \( K_V + E_V \sim_{\mathbb{Q}} 0 \); in this case, \( f_V \) is étale outside the three lines of \( S(V) \) and \( f_V^{-1}(\text{Sing } V) \).
Proof. For (1) and (2), we refer to [15, Lemma 2.1] and [14, Proposition 3.6.8] and note that $L \sim bH$ for some integer $b$.

(3) We may assume that $E' \simeq \mathbb{P}^1$, where $E' := \sigma'(E)$ is the proper transform of $E$. (3) is true because $E$ is a line if and only if

$$1 = E.H|_V (= E'.\sigma^*(H|_V) = E'(-K_V)),$$

and by the genus formula $-2 = 2g(E') - 2 = (E')^2 + E'.K_V$.

(4) $E' := \sigma'(E)$ satisfies $E'.K_V = E.K_V < 0$ and $(E')^2 \leq E'.\sigma^*E = E^2 < 0$. Hence $E'$ is a $(-1)$-curve by the genus formula. Thus (4) follows from (3).

(5) The first part is true because, by (2), the ramification divisor $R_{f|V} = (q - 1)E_V +$ (other effective divisors). Also, by (1) and (2), $\Delta \sim (1 - q)(K_V + E_V)$. Since $K_V.E = -1$ for every $E \in S(V)$ (by (4)), we have

$$0 \leq -K_V.\Delta = -K_V.(1 - q)(K_V + E_V) = (q - 1)(3 - \#S(V)).$$

Now the second part of (5) follows from this and the fact that $\Delta = 0$ if and only if $-K_V.\Delta = 0$ since $-K_V$ is ample. The last part of (5) follows from the purity of branch loci and the description of $R_{f|V}$ in (5). \hfill \square

2.4. We now prove Theorem 1.1 and Remark 1.3 for the normal cubic surface $V$. We use the notation in Lemma 2.3. Suppose that the Picard number

$$\rho := \rho(V) \geq 3.$$

Since $K_V$ is not nef and by the minimal model program for klt surfaces, there is a composite

$$V = V_\rho \xrightarrow{\tau_{\rho}} V_{\rho - 1} \cdots \xrightarrow{\tau_2} V_2$$

of birational extremal contractions such that

$$\rho(V_i) = i.$$

Let

$$E_i \subset V_i$$

be the exceptional (irreducible) divisor of $\tau_i : V_i \to V_{i - 1}$. Since $V$ is Du Val, either $E_i$ is contained in the smooth locus $V_i \setminus \text{Sing}(V_i)$ and is a $(-1)$-curve, or $E_i$ contains exactly one singular point

$$P_i \in \text{Sing} V_i$$

of type $A_{n_i}$ so that $\tau_i(E_i) \in V_{i - 1}$ is a smooth point. In particular, every $V_i$ is still Du Val. Let

$$V'_i \to V_i$$
be the minimal resolution. Since $-K_{V_i}$ is the pushforward of the ample divisor $-K_{V_V}$, it is ample. So $V_i$ is still a Gorenstein del Pezzo surface. Noting that $K_{V'_i}$ is the pullback of $K_{V_i}$, we have

$$(K^2_{V_{i-1}}) = K^2_{V_{i-1}} = K^2_{V_i} + (n_i + 1) \geq 3 + (0 + 1) = 4$$

for all $3 \leq i \leq \rho$.

Note that the proper transform

$$E_i(V) \subset V$$

of $E_i \subset V_i$ is a negative curve. Since $E_i(V)$ stabilizes every negative curve in $S(V)$ and especially $E_i(V)$ (when $f$ is replaced by its positive power, as seen in Lemma 2.3), $f_V$ descends to

$$f_i : V_i \rightarrow V_i.$$ 

The $V'_i$ and $S(V'_2)$, the set of negative curves on $V'_2$, are classified in [17, Figure 6]. Since $K^2_{V_2} \geq 4$, $(V'_2, S(V'_2))$ is as described in one of the last 10 cases in [ibid.]. For example, we write

$$V_2 = V_2(2A_2 + A_1)$$

if $\text{Sing} V_2$ consist of two points of type $A_2$ and one point of type $A_1$.

Except the four cases

$$V_2(D_4), V_2(4A_4), V_2(A_3), V_2(2A_1)$$

in [ibid.], exactly two $(-1)$-curves in $S(V'_2)$ map to intersecting negative curves

$$M_i \in S(V_2)$$

so that

$$S(V) = \{E_\rho, M_1(V), M_2(V)\}$$

with

$$M_i(V) \subset V$$

the proper transform of $M_i$, so

$$K_V + E_\rho + M_1(V) + M_2(V) \sim_\mathbb{Q} 0$$

(cf. Lemma 2.3) and hence $K_{V_2} + M_1 + M_2 \sim_\mathbb{Q} 0$, which is impossible by a simple calculation and blowing down $V'_2$ to its relative minimal model.

For each of the above four exceptional cases, we may assume that $f_2^{-1}$ stabilizes both extremal rays $\mathbb{R}_+[M_i]$ of the closed cone $\overline{\text{NE}}(V_2)$ of effective 1-cycles, with

$$M_i \subset V_2$$
the image of some \((-1\)-curve on \(V')\), where both extremal rays are of fibre type in the cases \(V_2(D_4)\) and \(V_2(4A_1)\), where the first (resp. second) is of fibre type (resp. birational type) in the cases \(V_2(A_3)\) and \(V_2(2A_1)\). Let
\[ F_i(\sim 2M_i) \]
with \(i = 1, 2\), or with \(i = 1\) only, be the fibre of the extremal fibration
\[ \varphi_i = \Phi_{|2M_i|} : V \to B_i \simeq \mathbb{P}^1 \]
passing through the point \((\tau_3 \circ \cdots \circ \tau_\rho)(E_\rho)\). Then the proper transform
\[ F_i(V) \subset V \]
of \(F_i\) is a negative curve so that
\[ E_V = F_1(V) + F_2(V) + E_\rho \]
in the cases \(V_2(D_4)\) and \(V_2(4A_1)\), and
\[ E_V = F_1(V) + M_2(V) + E_\rho \]
in the cases \(V_2(A_3)\) and \(V_2(2A_1)\) (cf. Lemma 2.3). Then \(K_V + E_V \sim_Q 0\), and hence \(K_{V_2} + F_1 + F_2 \sim_Q 0\) or \(K_{V_2} + F_1 + M_2 \sim_Q 0\) where the latter is impossible by a simple calculation as in the early paragraph. Thus
\[ K_{V_2} + F_1 + F_2 \sim_Q 0. \]
By making use of Lemma 2.3 (1) or (2), \(f_2^* F_i = q F_i\), and \(f_2\) descends to an endomorphism
\[ f_{B_1} : B_1 \to B_1 \]
of degree \(q\). Thus the ramification divisor of \(f_{B_1}\) is of degree \(2(q - 1)\) by the Hurwitz formula, and is hence equal to
\[ (q - 1)P + \sum (b_i - 1)P_i \]
with
\[ \sum (b_i - 1) = q - 1 \]
where \(P \in B_1\) so that \(F_1\) lies over \(P\). But then
\[ R_{f_2} \geq (q - 1)(F_1 + F_2) + \sum (b_i - 1)F'_i \]
where \(F'_i\) are fibres of \(\varphi_1\) lying over \(P_i\), so that
\[ K_{V_2} + F_1 + F_2 \geq f_2^* (K_{V_2} + F_1 + F_2) + \sum (b_i - 1)F'_i \]
which is impossible since \(K_{V_2} + F_1 + F_2 \sim_Q 0\).
2.5. Consider the case $\rho(V) = 2$. Then the minimal resolution

$$V' \to V$$

and its negative curves are described in one of the first five cases in [17, Figure 6].

For the case $V = V(A_3)$, two $(-1)$-curves on $V'$ map to two negative curves

$$M_1, M_2$$
on $V$. Note that $f_V^*(M_i) = q^*M_i$ (see Lemma 2.3). There is a contraction

$$V \to \mathbb{P}^2$$
of $M_1$ so that the image of $M_2$ is a plane conic preserved by $f_p^{-1}$ where

$$f_p : \mathbb{P}^2 \to \mathbb{P}^2$$
is the descent of $f_V$ (of degree $q^2 > 1$), contradicting [15, Theorem 1.5(4) in arXiv version].

For the case $V(2A_2 + A_1)$, there are exactly five $(-1)$-curves

$$M'_i \subset V'$$
with $M_i \subset V$ their images. Moreover, $M'_1M'_2 = 1$ and both $M_i$ $(i = 1, 2)$ are negative curves on $V$; each $M'_j$ ($j = 3, 4$) meets the isolated $(-2)$-curve; $M_1$ and $M_3$ (reap. $M_2$ and $M_4$) meet the same component of one (resp. another) $(-2)$-chain of type $A_2$. We have

$$M_1 + M_2 \sim 2L$$for some integral Weil divisor $L$, by considering a relative minimal model of $V'$. In fact, $M_1 + 3M_2 \sim 4M_3$. Since $f_V^*(M_1 + M_2) = q(M_1 + M_2)$ (see Lemma 2.3), $f_V$ lifts to some

$$g : U \to V.$$

Here the double cover (given by the relation $M_1 + M_2 \sim 2L$)

$$\pi : U = Spec \bigoplus_{i=0}^1 \mathcal{O}(-iL) \to V$$
is branched along $M_1 + M_2$. Indeed, when $2 \nmid q$, the normalization

$$\hat{U}$$
of the fibre product of $\pi : U \to V$ and $f_V : V \to V$ is isomorphic to $U$ and we take $g$ to be the first projection $\hat{U} \to U$; when $2 \mid q$, we have $\hat{U} = V \bigsqcup V$ and let $g$ be the composite of $\pi : U \to V$, the inclusion $V \cup \emptyset \to V \bigsqcup V$ and the first projection $\hat{U} \to U$. Now $\text{Sing } U$ consists of a type $A_1$ singularity lying over $M_1 \cap M_2$ and four points in $\pi^{-1}(\text{Sing } V)$ of type $A_1, A_1, \frac{1}{2}(1, 1)$ and $\frac{1}{3}(1, 1)$, and every $M_j$ $(j = 3, 4)$ splits into two negative curves on $U$ which are hence preserved by $g^{-1}$ (as in Lemma 2.3 after $f_V$ is replaced by its positive power). Thus $f_{V^{-1}}(M_i) = M_i$ $(1 \leq i \leq 4)$. As in the proof of Lemma 2.3,
$E'_V := M_1 + M_2 + M_3$ satisfies $K_V + E'_V \sim_{\mathbb{Q}} 0$ and $f_V$ is étale over $V \setminus (E'_V \cup \text{Sing } V)$. The latter contradicts the fact that $f'_V^* M_4 = qM_4$ and hence $f_V$ has ramification index $q$ along $M_4$.

For $V = V(A_4 + A_1)$, there are exactly four $(-1)$-curves

$$M'_i \subset V'$$

with $M_i \subset V$ their images. We may so label that the five $(-2)$-curves and $M'_j$ $(j = 1, 2, 3)$ form a simple loop:

$$M'_1 - (-2) - (-2) - (-2) - (-2) - M'_2 - M'_3 - (-2) - M'_1;$$

both $M_j$ $(j = 2, 3)$ are negative curves on $V$. We can verify that $M_4 + 2M_2 \sim 3(M_5 - M_2)$, and $M_1 + M_2 \sim M_5$, where $M_5$ is the image of a curve $M'_5 \simeq \mathbb{P}^1$ and $M'_2$ is the smooth fibre (passing through the intersection point of $M'_3$ and the isolated $(-2)$-curve) of the $\mathbb{P}^1$-fibration on $V'$ with a singular fibre consisting of $M'_1$, $M'_2$ and the $(-2)$-chain of type $A_4$ sitting in between them. As in the case $V(2A_2 + A_1)$, $f_V$ lifts to

$$g : U \to U$$

on the triple cover $U$ defined by the relation $M_3 + 2M_2 \sim 3(M_5 - M_2)$, so that each $M_i$ $(i = 4, 5)$ splits into three negative curves on $U$ preserved by $g^{-1}$. Hence $f^{-1}_V(M_i) = M_i$ $(i = 2, \ldots, 5)$. But then for $E'_V := M_2 + M_3 + M_4$ we have $K_V + E'_V \sim_{\mathbb{Q}} 0$ as in the proof of Lemma 2.3 so that $f_V$ is étale over $V \setminus (E'_V \cup \text{Sing } V)$, contradicting the fact that $f_V$ has ramification index $q$ along $M_5$.

For $V = V(D_5)$, the lonely $(-1)$-curve $M'_1$ and the intersecting $(-1)$-curves $M'_2 \cup M'_3$ on $V'$ satisfy $2M_1 \sim M_2 + M_3$ where

$$M_i \subset V$$

denotes the image of $M'_i$ (indeed, the three $M'_i$ together with the five $(-2)$-curves form the support of two singular fibres and a section in a $\mathbb{P}^1$-fibration). As in the case $V(2A_2 + A_1)$, $f_V$ lifts to

$$g : U \to U$$

on the double cover $U$ defined by the relation $2M_1 \sim M_2 + M_3$, so that $M_4$ splits into two negative curves on $U$ preserved by $g^{-1}$. Thus $f_V^{-1}(M_1) = M_1$. Hence $(V, M_1)$ is log canonical (cf. [14 Theorem 4.3.1]). But $(V, M_1)$ is not log canonical because $M'_1$ meets the $(-2)$-tree of type $D_5$ in a manner different from the classification of [10 Theorem 9.6]. We reach a contradiction.

For $V = V(A_3 + 2A_1)$, let $M'_1 \subset V'$ be the $(-1)$-curve meeting the middle component of the $(-2)$-chain of type $A_3$, let $M'_2$ be the $(-1)$-curve meeting two isolated $(-2)$-curves,
and let $M'_2$ be the $(-1)$-curve meeting both $M'_1$ and $M'_3$. Then the images
\[ M_i \subset V \]
of $M'_i$ satisfy $2M_1 \sim 2M_3$ (indeed, $M'_1$, $M'_3$ and the five $(-2)$-curves form the support of two singular fibres in some $\mathbb{P}^1$-fibration). The relation $2(M_1 - M_3) \sim 0$ defines a double cover
\[ \pi : U = \text{Spec} \oplus_{i=0}^1 \mathcal{O}(-i(M_1 - M_3)) \rightarrow V \]
étable over $V \setminus \text{Sing} V$. In fact, $\pi$ restricted over $V \setminus \text{Sing} V$, is the universal cover over it, so $U$ is again a Gorenstein del Pezzo surface and hence the irregularity $q(U) = 0$. Thus $f_V$ lifts to
\[ g : U \rightarrow U. \]

Now $\pi^{-1}(\text{Sing} V)$ consists of two smooth points and the unique singular point of $U$ (of type $A_1$), and each $\pi^*M_i$ ($i = 1, 2$) splits into two negative curves $M_i(1)$, $M_i(2)$ on $U$ preserved by $g^{-1}$; thus $f_V^*M_i = qM_i$ and $g^*M_i(j) = qM_i(j)$ (as in Lemma \[2.3\] (1)).

We assert that $f_V^{-1}$ permutes members of the pencil
\[ \Lambda := |M_1 + M_2|. \]
It suffices to show that $g^{-1}$ permutes members of the irreducible pencil
\[ \Lambda_U \]
(parametrized by $\mathbb{P}^1$ for $q(U) = 0$) which is the pullback of $\Lambda$. Now $\pi^*(M_1 + M_2)$ splits into two members
\[ M_1(j) + M_2(j) = \text{div}(\xi_j) \]
(in local equation; $j = 1, 2$) which are preserved by $g^{-1}$ and span $\Lambda_U$. We may assume that $g^*\xi_j = \xi_j^q$ after replacing the equation by a scalar multiple. Then the $g^*$-pullback of every member $\text{div}(a\xi_1 + b\xi_2)$ in $\Lambda_U$ is equal to $\text{div}(a\xi_1^q + b\xi_2^q)$ and hence is the union of members in $\Lambda_U$ because we can factorize $a\xi_1^q + b\xi_2^q$ as a product of linear forms in $\xi_1$, $\xi_2$. This proves the assertion.

By the assertion and since $f_V^*(M_1 + M_2) = q(M_1 + M_2)$, $f_V$ descends to an endomorphism
\[ f_B : B \rightarrow B \]
of degree $q$ on the curve $B \simeq \mathbb{P}^1$ parametrizing the pencil $\Lambda$. We have $f_B^*P_0 = qP_0$ for the point parametrizing the member $M_1 + M_2$ of $\Lambda$. Write $K_B = f_B^*K_B + R_{f_B}$, where the ramification divisor
\[ R_{f_B} = (q - 1)P_0 + \Delta_B \]
with $\Delta_B = \sum (b_i - 1)Q_j$ of degree $q - 1$ for some $b_i \geq 2$. Thus the ramification divisor
\[ R_{f_V} = (q - 1)(M_1 + M_2) + \Delta_V \]
with $\Delta_V = \sum (b_i - 1)F_i$ (other effective divisor), where

$$F_i \in \Lambda$$

is parametrized by $Q_i$. On the other hand, one can verify that $-K_V \sim 2M_1 + M_2$, by blowing down to a relative minimal model of $V'$; indeed, $M_2$ is a double section of the $\mathbb{P}^1$-fibration

$$\varphi := \Phi_{2M_1} : V \to \mathbb{P}^1.$$ 

So $-M_1 \sim K_V + M_1 + M_2 = f_V^*(K_V + M_1 + M_2) + \Delta_V$ and hence, by Lemma 2.3 (1),

$$(b_1 - 1)F_1 \leq \Delta_V \sim (1 - q)(K_V + M_1 + M_2) \sim (q - 1)M_1.$$ 

This is impossible because $F_1$ is horizontal to the half fibre $M_1$ of $\varphi$. Indeed, $F_1.M_1 = (M_1 + M_2).M_1 = M_2.M_1 = 1$.

2.6. Consider the last case $\rho(V) = 1$. Since $K_V^2 = 3$, we have

$$V = V(3A_2), V(E_6), \text{ or } V(A_1 + A_5),$$

and the minimal resolution

$$V' \to V$$

and the negative curves on $V'$ are described in [17, Figure 5]. For the first two cases, $V$ is isomorphic to $V_i$ ($i = 1$, or 2) in Theorem [13] by the uniqueness result in [17, Theorem 1.2] and by [9, Theorem 4.4].

For $V = V(A_1 + A_5)$, the images

$$M_i \subset V$$

of the two $(-1)$-curves $M'_i \subset V'$ satisfy $2M_1 \sim 2M_2$; indeed, $M'_i$ together with the six $(-2)$-curves form the support of two singular fibres and a section in some $\mathbb{P}^1$-fibration. Let

$$\pi : U \to V$$

be the double cover given by the relation $2(M_1 - M_2) \sim 0$. In fact, $\pi$ restricted over $V \setminus \text{Sing } V$, is the universal cover over it. So $f_V$ lifts to

$$g : U \to U.$$ 

As in the case $V(A_3 + 2A_1)$, if we let $M'_1$ be the one meeting the second component of the $(-2)$-chain of type $A_5$, then $\pi^*M_1$ splits into two negative curves on $U$ preserved by $g^{-1}$. Thus $f_V^{-1}(M_1) = M_1$, and, as in the case $V(D_5)$ above, contradicts [14, Theorem 4.3.1] and [10, Theorem 9.6].

This completes the proof of Theorem [13] for normal cubic surfaces and hence the whole of Theorem [13]. To determine the equations of $V_i$ ($i = 1, 2$), we can check that the equations in Theorem [13] possess the right combination of singularities and then use
the very ampleness of \(-K_{V_i}\) to embed \(V_i\) in \(\mathbb{P}^3\) as in [9] and the uniqueness of \(V_i\) up to isomorphism, and hence up to projective transformation by [9] (cf. [17, Theorem 1.2]).

2.7. Now we prove Remark 1.3. From Lemma 2.3 till now, we did not assume the hypothesis (*) that \(f_V\) is the restriction of some \(f: \mathbb{P}^3 \to \mathbb{P}^3\) whose inverse stabilizes \(V\). From now on till the end of the paper, we assume this hypothesis (*).

For \(V = V(E_6)\) or \(V(3A_2)\), the relation \(V \sim 3H\) defines a triple cover

\[
\pi: X = \text{Spec} \bigoplus_{i=0}^{2} \mathcal{O}(-iH) \to V
\]

branched along \(V\). Then

\[
X = \{Z^3 = V(X_0, \ldots, X_3)\} \subset \mathbb{P}^4
\]

is a cubic hypersurface, where we let \(V(X_0, \ldots, V_3)\) be the cubic form defining \(V \subset \mathbb{P}^3\). Our \(\pi^{-1}\) restricts to a bijection \(\pi^{-1}: \text{Sing} V \to \text{Sing} X\). As in Example 1.5, \(f\) lifts to \((Z^q, f): \mathbb{P}^4 \to \mathbb{P}^4\) stabilizing \(X\), so that the restriction

\[
g = (Z^q, f)|X: X \to X
\]

is also a lifting of \(f\). By the Lefschetz type theorem [12 Example 3.1.25], \(\text{Pic}(X) = (\text{Pic}(\mathbb{P}^4))|X\).

For \(V = V(E_6)\), \(V^\prime\) contains only one \((-1)\)-curve \(M^\prime\) and hence \(V\) contains only one line \(M\) (the image of \(M^\prime\)) by Lemma 2.3. Note that

\[
\{Q\} := \text{Sing} V \subset M.
\]

Let

\[
\Pi_{MM} \subset \mathbb{P}^3
\]

(say given by \(X_3 = 0\)) be the unique plane such that

\[
\Pi_{MM}|V = 3M.
\]

Indeed, \(3M\) belongs to the complete linear system \(|H|V|\), and the exact sequence

\[
0 \to \mathcal{O}(-2H) \to \mathcal{O}(H) \to \mathcal{O}_V(H) \to 0
\]

and the vanishing of \(H^1(\mathbb{P}^3, -2H)\) (e.g. by the Kodaira vanishing) imply

\[
H^0(\mathbb{P}^3, \mathcal{O}(H)) \simeq H^0(V, \mathcal{O}_V(H)).
\]

Our \(\pi^*\Pi_{MM}\) is a union of three 2-planes

\[
L_i \subset \mathbb{P}^4
\]

because the restriction of \(\pi\) over \(\Pi_{MM}\) is given by the equation

\[
Z^3 = V(X_0, \ldots, X_3)|\Pi_{MM} = M(X_0, \ldots, X_2)^3
\]
where $M(X_0, \ldots, X_2)$ is a linear equation of $M \subset \Pi_{ MMM}$. This and the fact that $\pi^* \Pi_{ MMM}$ is a generator of $\text{Pic}(X) = (\text{Pic}(\mathbb{P}^3)) | X$, imply that the Weil divisor $L_1$ is not a Cartier divisor on $X$. Since $\text{Sing} \ X$ consists of a single point $P$ lying over $\{Q\} = \text{Sing} \ V$, $L_1$ is not Cartier at $P$ and hence $X$ is not factorial at $P$. Thus $X$ is not $\mathbb{Q}$-factorial at $P$ because the local $\pi_1$ of $P$ is trivial by a result of Milnor (cf. the proof of [10, Lemma 5.1]). Hence $g^{-1}(P)$ contains no smooth point (cf. [11, Lemma 5.16]) and must be equal to $\text{Sing} \ X = \{P\}$. Thus $f^{-1}(Q) = Q$ because $\pi^{-1}(Q) = P$.

2.8. Before we treat the case $V(3A_2)$, we make some remarks. Up to isomorphism, there is only one $V(3A_2)$ (cf. [17, Theorem 1.2]). Set $V := V(3A_2)$. There is a Gorenstein del Pezzo surface $W$ such that $\rho(W) = 1$, $\text{Sing} \ W$ consists of four points $\beta_i$ of Du Val type $A_2$,

$$\pi_i(W \setminus \text{Sing} \ W) = (\mathbb{Z}/(3))^\oplus 2$$

and there is a Galois triple cover $V \to W$ étale over $W \setminus \{\beta_1, \beta_2, \beta_3\}$ so that a generator $h \in \text{Gal}(V/W)$ permutes the three singular points of $V$ lying over $\beta_4$ (cf. [13, Figure 1, Lemma 6]). Since the embedding $V \subset \mathbb{P}^3$ is given by the complete linear system $| -K_V|$ (cf. [9]) and $h^*(-K_V) \sim -K_V$, our $h$ extends to a projective transformation of $\mathbb{P}^3$, also denoted as $h$. Since $h(V) = V$,

$$h^*V(X_0, \ldots, X_3) = cV(X_0, \ldots, X_3)$$

for some nonzero constant $c$. This $h$ lifts to a projective transformation of $\mathbb{P}^4$, also denoted as $h$, stabilizing the above triple cover $X \subset \mathbb{P}^3$ of $\mathbb{P}^3$ by defining $h^*Z = \sqrt{c}Z$. Then this $h$ permutes the three singular points of $X$ lying over $\text{Sing} \ V$.

2.9. For $V = V(3A_2)$, $V'$ has exactly three $(-1)$-curves $M'_i$ and their images $M_i$ are therefore the only lines on $V$ (cf. Lemma 2.3). The graph $\sum M_i$ is triangle-shaped whose vertices (the intersection $M_i \cap M_j$) are the three points in $\text{Sing} \ V$. The sum of the three $(-1)$-curves $M'_i$ and three $(-2)$-chains of type $A_2$ is linearly equivalent to $-K_{V'}$ and hence $\sum M_i \sim -K_V$; also $2M_a \sim M_b + M_c$ so long $\{a, b, c\} = \{1, 2, 3\}$; indeed, the three $M'_i$ and the six $(-2)$-curves form the support of two singular fibres and two cross-sections of some $\mathbb{P}^1$-fibration. Thus $3M_i \sim -K_V \sim H|V$. As argued in the case $V(E_6)$, there is a unique hyperplane $\Pi_i$ such that

$$\Pi_i|V = 3M_i$$

our $\pi^* \Pi_i$ is a union of three 2-planes $L_{iij}$ in $\mathbb{P}^4$ (sharing a line lying over $M_i \subset \mathbb{P}^3$), $L_{i1}$ is not a Cartier divisor on $X$, the $X$ is not $\mathbb{Q}$-factorial at least at one of the two points (and hence at both points, since the above $h$ permutes $\text{Sing} \ X$) in $L_{i1} \cap \text{Sing} \ X$ (lying
over $M_i \cap \text{Sing } V$), and $g^{-1}(\text{Sing } X) = \text{Sing } X$. Thus $f^{-1}(\text{Sing } V) = \text{Sing } V$. Hence $f^{-3}$ fixes each point in $\text{Sing } V$.

This completes the proof of Remark 1.3 for normal cubic surfaces and hence the whole of Remark 1.3.

**Remark 2.10.** The proof of Theorem 1.1 actually shows: if $f_V : V \rightarrow V$ is an endomorphism (not necessarily the restriction of some $f : \mathbb{P}^3 \rightarrow \mathbb{P}^3$) of $\deg(f_V) > 1$ of a Gorenstein normal del Pezzo surface with $K_V^2 = 3$ (i.e., a normal cubic surface), then $V$ is equal to $V_1$ or $V_2$ in Theorem 1.1 in suitable projective coordinates.

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Department of Mathematics, National University of Singapore
10 Lower Kent Ridge Road, Singapore 119076
E-mail address: matzdq@nus.edu.sg