A New Look at Levi-Civita connection in noncommutative geometry

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Abstract

We give a new definition of Levi-Civita connection for a noncommutative pseudo-Riemannian metric on a noncommutative manifold given by a spectral triple and prove the existence-uniqueness result for a class of modules of one forms over a large class of noncommutative manifolds, including Connes-Landi deformations of spectral triples on the Connes-Dubois Violette-Rieffel-deformation of a compact manifold equipped with a toral action satisfying some mild assumptions on the action. The assumptions on the action are general enough to accommodate any free as well as any ergodic action. The existence an uniqueness of Levi Civita connection is also seen to hold for a spectral triple on quantum Heisenberg manifolds. As an application, we compute the Ricci and scalar curvature for a general conformal perturbation of the canonical metric on the noncommutative 2-torus as well as for a natural metric on the quantum Heisenberg manifold.

1 Introduction

The concepts of connection and curvature occupy a central place in any form of geometry, classical or noncommutative. In noncommutative geometry (NCG for short) a la Connes, there have been several attempts over the last few years to formulate and study analogues of curvature. There seem to be mainly two different approaches to this problem so far:

(a) formulating an analogue of Levi-Civita connection and computing the corresponding curvature operator, in particular scalar and Ricci curvatures (see, e.g. [21], [9]);

or

(b) defining Ricci and scalar curvature directly through an asymptotic expansion of the noncommutative Laplacian (see [12], [14], [15], [17], [18], [19], [11], [25], [26], [13], [20] etc.).

In the classical case, at least for a compact Riemannian manifold, these two approaches turn out to be equivalent. In the first approach, one gets the Levi-Civita connection and the full curvature operator, which are important in their own right. In NCG, the definition of Levi-Civita connection given in [21] seemed to face an obstacle because it was not possible to get a unique Levi-Civita (i.e. both torsion-less and metric-compatible in a suitable sense) connection in some standard examples such as the quantum Heisenberg manifolds, as observed by the authors of [6] (see also the appendix B of [23]). In a recent article, J. Rosenberg ([31]) proposed an alternative definition giving an existence and uniqueness theorem for some noncommutative manifolds including the noncommutative tori. This has been followed by several computations of scalar curvature by a number of authors (e.g. [28], [1]). In [1], the definition of Rosenberg has been extended to the case of (noncommutative) pseudo-Riemannian metrics as well. However, one drawback of this approach is that it uses spaces of vector fields instead of one-forms. Somehow, vector fields in NCG are not as well-behaved as in the case of classical geometry. In fact, they do not form a module over the underlying (noncommutative) smooth algebra. The simple description of vector fields, i.e. derivations of the smooth algebra of the noncommutative tori, which played a crucial role in the success of Rosenberg’s approach for such noncommutative manifolds, hold only for spectral triples equivariant w.r.t. a toral action which is also ergodic on the underlying C∗ algebra. On the other hand, the space of one-forms is quite well-behaved in NCG and it does have a natural module structure over the noncommutative algebra of smooth functions. In algebra and algebraic geometry, including noncommutative algebraic geometry, the notion of connection on a module is quite familiar and standard. In fact, the definition of [21] was given in terms of modules of one-forms but the main problem there (in our opinion) was the formulation of a Riemannian metric as an inner product and consequently, the definition of metric-compatibility of a connection.
The aim of the present article is to give a new definition of Levi-Civita connection for a noncommutative pseudo-Riemannian metric and prove existence of a unique Levi-Civita connection for an arbitrary pseudo-metric (to be explained later) on a very large class of noncommutative manifolds, which do include all the Rieffel-deformations of classical compact Riemannian manifolds obtained by toral actions satisfying some mild conditions. Moreover, our existence-uniqueness result covers the case of quantum Heisenberg manifold for which the approach of [21] did not succeed. In some sense, we have combined the approaches of [21] and [31] as we work in the setting of one-forms instead of vector fields but define a pseudo-metric to be a symmetric, bilinear non-degenerate form instead of a sesquilinear inner product. Such an approach was also taken in [23], though in the set-up of bicovariant differential calculi (in the sense of Woronowicz) on quantum groups, which made the authors restrict their attention to invariant metrics and equivariant connections (w.r.t. the given quantum group co-action) only.

Using our approach, we also compute the Ricci and scalar curvatures for the conformal perturbation of the canonical metric on the noncommutative 2-torus. It is interesting to note that most of the examples considered in [26] on Rieffel-deformation of classical manifolds also belong to the class for which our existence-uniqueness result applies, hence one can compute the scalar curvature of the Levi-Civita connection obtained from our approach. This leads to an important open question, which we plan to investigate in future: do the scalar curvatures computed from our Levi-Civita connection agree with the one obtained in [26] through an asymptotic expansion? The difference between the class of examples covered in [31], [26] (and the references therein), [23] and ours is that the former works deal with Levi Civita connections for a particular metric while our work deals with a particular class of modules (see Proposition 4.13) and arbitrary pseudo-Riemannian metric on that module.

It should also be mentioned that there is a very different, mostly algebraic, approach by S. Majid and his co-authors ([21], [3], [4], [2] etc. and references therein). We think our approach needs to be modified, by replacing the symmetric tensor product used here with something like a more general braided tensor product, to include some of the examples considered by them. We hope to look into this in future.

Let us discuss the plan of the article. We begin with some generalities on vector spaces, bimodules and linear maps (Sections 2 and 3). In particular, we focus on a certain class of bimodules called ‘centered bimodules’ and the flip map on the tensor product of two copies of such bimodules. In Section 4, we formulate the notion of Levi-Civita (torsion-free and metric compatible) connections in the framework of noncommutative geometry under certain natural assumptions (Assumption 1 - IV) on the module of one forms. Then we state and prove the main result giving the existence and uniqueness of the Levi-Civita connection in Section 5. Section 6 is devoted to the verification of our assumptions for the one forms of the Connes-Landi spectral triples on a large class of noncommutative manifolds obtained by Rieffel deformation of classical Riemannian manifolds. In Section 7 we apply our results to an interesting class of spectral triples on the quantum Heisenberg manifold proving the existence of a unique Levi-Civita connection. In the last section of the paper, we define and compute the Ricci and scalar curvatures for some examples including an arbitrary ‘conformal deformation’ of the canonical metric on the noncommutative 2-torus.

We fix some notations which we will follow. Throughout the article, \( \mathcal{A} \) will denote a complex unital *-subalgebra of a \( C^* \) algebra. \( \mathcal{Z}(\mathcal{A}) \) will denote the center of \( \mathcal{A} \). For a subset \( S \) of a right \( \mathcal{A} \)-module \( \mathcal{E} \), \( SA \) will denote \( \text{span\{sa : s \in S, a \in \mathcal{A} \}} \). For right \( \mathcal{A} \) modules \( \mathcal{E} \) and \( \mathcal{F} \), \( \text{Hom}_\mathcal{A}(\mathcal{E}, \mathcal{F}) \) will denote the set of all right \( \mathcal{A} \)-linear maps from \( \mathcal{E} \) to \( \mathcal{F} \). We will use the notations \( \text{End}_\mathcal{A}(\mathcal{E}) \) and \( \mathcal{E}_\mathcal{A} \) for \( \text{Hom}_\mathcal{A}(\mathcal{E}, \mathcal{E}) \) and \( \text{Hom}_\mathcal{A}(\mathcal{E}, \mathcal{A}) \) respectively. The tensor product over the complex numbers \( \mathbb{C} \) is denoted by \( \otimes_{\mathbb{C}} \) while the notation \( \otimes_{\mathcal{A}} \) will denote the tensor product over the algebra \( \mathcal{A} \).

If \( \mathcal{E} \) and \( \mathcal{F} \) are bimodules, \( \text{Hom}_\mathcal{A}(\mathcal{E}, \mathcal{F}) \) has a left \( \mathcal{A} \)-module structures given by left multiplication by elements of \( \mathcal{A} \), i.e., for elements \( a \) in \( \mathcal{A} \), \( e \) in \( \mathcal{E} \) and \( T \) in \( \text{Hom}_\mathcal{A}(\mathcal{E}, \mathcal{F}) \), \( (aT)(e) := aT(e) \in \mathcal{F} \). The right \( \mathcal{A} \)-module structure on \( \text{Hom}_\mathcal{A}(\mathcal{E}, \mathcal{F}) \) is given by \( Ta(e) = T(ae) \). Lastly, for a linear map \( T \) between suitable modules, \( \text{Ran}(T) \) will denote the Range of \( T \).
2 Generality on the flip map on vector spaces

Let $V$ be a complex vector space and $\sigma^C$ denotes the map from $V \otimes_C V \to V \otimes_C V$ defined on simple tensors by the formula $\sigma^C(v \otimes w) = w \otimes_C v$. We will use the maps $\sigma^C_{|2} := \sigma^C \otimes_C \text{id}_V$, $\sigma^C_{|23} := \text{id}_V \otimes_C \sigma^C$ and $\sigma^C_{|13} := \sigma^C_{|12} \sigma^C_{|23}$.

Then the map $P^C := \frac{\sigma^C + 1}{2}$ is an idempotent. We will denote $\text{Ran}(P^C)$ by $V \otimes_C \text{sym} V$. We will need the maps $P^C_{|12} := P^C \otimes_C \text{id}_V$ and $P^C_{|23} := \text{id}_V \otimes_C P^C$. Thus, for elements $v_1, v_2, v_3$ in $V$, $P^C_{|12}(v_1 \otimes v_2 \otimes v_3) = \frac{1}{2} (v_1 \otimes v_2 + v_2 \otimes v_1) \otimes v_3$ and $P^C_{|23}(v_1 \otimes v_2 \otimes v_3) = v_1 \otimes \frac{1}{2} (v_2 \otimes v_3 + v_3 \otimes v_2)$.

The proof of the following proposition is basically a reformulation of the proof of the existence and uniqueness of Levi Civita connections for pseudo-Riemannian manifolds.

**Proposition 2.1** If $V$ is a vector space, then each of the following the maps is an isomorphism of vector spaces.

$P^C_{|12}|_{\text{Ran}(P^C_{|23})} : \text{Ran}(P^C_{|23}) = V \otimes_C (V \otimes_C \text{sym} V) \to \text{Ran}(P^C_{|12}) = (V \otimes_C \text{sym} V) \otimes_C V$

$P^C_{|23}|_{\text{Ran}(P^C_{|12})} : \text{Ran}(P^C_{|12}) = (V \otimes_C \text{sym} V) \otimes_C V \to \text{Ran}(P^C_{|23}) = V \otimes_C (V \otimes_C \text{sym} V)$

**Proof:** We prove the statement about the first of the two maps since the proof for the other map is similar. Let us begin by proving that the first map is one-one. Let $X \in \text{Ran}(P^C_{|23})$ such that $P^C_{|12}(X) = 0$. That is, $\sigma^C_{|23}(X) = X$ and $\sigma^C_{|12}(X) = -X$. Now, it is easy to verify the following braid relations:

$$\sigma^C_{|12}\sigma^C_{|23}\sigma^C_{|12} = \sigma^C_{|23}\sigma^C_{|12}\sigma^C_{|23}. \quad (1)$$

But we have $\sigma^C_{|12}\sigma^C_{|12}\sigma^C_{|12}(X) = -\sigma^C_{|12}\sigma^C_{|23}(X) = -\sigma^C_{|23}(X) = X$. On the other hand, $\sigma^C_{|23}\sigma^C_{|12}\sigma^C_{|23}(X) = \sigma^C_{|23}\sigma^C_{|12}(X) = -\sigma^C_{|23}(X) = -X$. This implies, $X = -X$, i.e. $X = 0$. Thus, the map $P^C_{|12}|_{\text{Ran}(P^C_{|23})}$ is injective.

Now we come to surjectivity. If $V$ is finite dimensional, surjectivity follows since $\text{Ran}(P^C_{|23})$ and $\text{Ran}(P^C_{|12})$ are of the same dimension. In the general case, given any $\xi \in (V \otimes_C \text{sym} V) \otimes_C V$ such that $\sigma^C_{|23}(\xi) = \xi$, there exists a natural number $n$ and linearly independent elements $e_1, e_2, \ldots, e_n$ of $V$ such that $\xi$ belongs to $(K \otimes_C \text{sym} K) \otimes_C K$, where $K := \text{span}\{e_1, e_2, \ldots, e_n\}$. If $P^C_{|K_{12}}$ denotes the map $P^C_{|12}|_{K \otimes_C K \otimes_C K}$, then by the surjectivity of $P^C_{|K_{12}}|_{\text{Ran}(P^C_{|23})}$ for finite dimensional vector spaces, there exists $\eta \in K \otimes_C (K \otimes_C \text{sym} K)$ such that $P^C_{|K_{12}}(\eta) = \xi$. Since $\xi$ is arbitrary, the proof of surjectivity is complete. \(\square\)

3 Generality on bimodules and maps on them

3.1 Some basic definitions and facts

We will say that a subset $S$ of a right $A$-module $E$ is right $A$-total in $E$ if the right $A$-linear span of $S$ equals $E$.

We record the following well-known facts without proof which we will use throughout the article without mentioning.

**Proposition 3.1** We have the following:

1. Let $S$ be a right $A$-total subset of a right $A$-module $E$. If $T_1$ and $T_2$ are two right $A$-linear maps from $E$ to another right $A$-module $F$ such that they agree on $S$, then they agree everywhere on $E$.

2. Let $E$ and $F$ be $A - A$-bimodules which are finitely generated and projective as both left and right $A$-modules. Then for elements $e_i \in E$, $f \in F$ and $\phi_i \in F \phi^A$, the map $\xi_{E,F} : E \otimes_A F^* \to \text{Hom}_A(F,E)$ defined by $\xi_{E,F}(\sum_i e_i \otimes_A \phi_i)(f) = \sum_i e_i \phi_i(f)$ defines an isomorphism of $A$-bimodules.
3.2 Centered modules

We recall the concept of centered bimodules and the canonical flip map from Section 6 of [32].

**Definition 3.2** The center of an $A-A$-bimodule $E$ is defined to be the set $Z(E) = \{ e \in E : ea = ae \forall a \in A \}$. $E$ is called centered if $Z(E)$ is right $A$-total in $E$, i.e., the right $A$-linear span of $Z(E)$ equals $E$.

It is easy to see that $Z(E)$ is a $Z(A)$-bimodule.

For a related notion of central bimodules, we refer to the paper [16] of Dubois-Violette and Michor. It is easy to see that a centered bimodule in the sense of Skeide ([32]) is a central bimodule in the sense of Dubios-Violette and Michor ([16]), i.e., if $E$ is a centered module in the sense of Definition 3.2, then $e.a = a.e$ for all $e$ in $E$ and for all $a$ in $Z(A)$.

**Example 3.3** If $A = C^\infty(M)$ for some compact manifold $M$, and $\Gamma(E)$ the $A-A$-bimodule of sections of some smooth vector bundle $E$ on $M$, then $\Gamma(E)$ is centered. In particular, the $A-A$-bimodule $\Omega^k(M)$ of $k$-forms on $M$ is centered.

If $E$ is a free bimodule of the form $A \otimes \mathbb{C}^n$ and $\{ e_i : i = 1, 2, \cdots, n \}$ is the canonical basis of $\mathbb{C}^n$, then $E$ is centered with $Z(E) = \{ a \otimes e_i : i = 1, 2, \cdots, n, a \in Z(A) \}$.

As an immediate corollary to the definition of a centered module, we observe that

**Lemma 3.4** Let $E$ be a centered $A-A$ bimodule. If $a' \in Z(A)$, then $a'e = ea'$ for all $e$ in $E$.

**Proof:** Since $E$ is centered, we can write $e$ as a finite sum $e = \sum e_i a_i$ where $e_i \in Z(E)$ and $a_i \in A$. Thus,

$$a'e = \sum a'e_i a_i = \sum e_i a' a_i = \sum e_i a_i a' = e.a',$$

where we have used the fact that $e_i \in Z(E)$ and $a' \in Z(A)$.

3.3 The canonical flip and symmetrization maps

Let $E$ be a centered $A-A$ bimodule. It follows from Theorem 6.10 of [32] that there exists a unique $A-A$ bimodule isomorphism $\sigma^{can}$ from $E \otimes_A E$ to itself such that $\sigma^{can}(\omega \otimes_A \eta) = \eta \otimes_A \omega$ for all $\omega, \eta \in Z(E)$. It follows from the construction of $\sigma^{can}$ that $(\sigma^{can})^2 = \text{id}$.

Since $\sigma^{can}$ is a bimodule map, the maps $\sigma_{12}^{can}, \sigma_{23}^{can}, \sigma_{13}^{can} : E \otimes_A E \otimes_A E \to E \otimes_A E \otimes_A E$ defined as $\sigma_{12}^{can} := \sigma_{12} \circ \sigma^{can} \circ \sigma^{can}$, $\sigma_{23}^{can} := \sigma_{23} \circ \sigma^{can} \circ \sigma^{can}$ respectively are well defined bimodule morphisms.

Moreover, let $P_{sym}^{can} \in \text{End}(E \otimes_A E)$ be defined by $P_{sym}^{can} := \sigma_{12}^{can} + \sigma_{23}^{can} - 2 \sigma_{13}^{can}$. Then $P_{sym}^{can}$ is an $A-A$-bilinear idempotent.

We make the following simple observation.

**Lemma 3.5** We have $\sigma^{can}(\omega \otimes_A e) = e \otimes_A \omega$ and $\sigma^{can}(e \otimes_A \omega) = \omega \otimes_A e$ for all $\omega \in Z(E)$ and $e \in E$.

**Proof:** Since $E$ is central and $\sigma$ is right linear, it is enough to prove the lemma for elements $e$ of the form $\eta b$ where $\eta \in Z(E)$ and $b \in A$.

We compute $\sigma^{can}(\omega \otimes_A \eta b) = \sigma^{can}(\omega \otimes_A \eta)b = (\eta \otimes_A \omega)b = \eta \otimes_A \omega b = \eta b \otimes_A \omega = e \otimes_A \omega$.

The other equality follows similarly.

We conclude this section with the following:

**Lemma 3.6** Let $P_{ij}^{can} := \frac{1}{2}(1 + \sigma_{ij}^{can})$, $(i,j) = (12),(13),(23)$. Then the following maps are bimodule isomorphisms:

$$P_{12}^{can} \mid_{\text{Ran}(P_{23}^{can})} : \text{Ran}(P_{23}^{can}) \to \text{Ran}(P_{12}^{can}) \text{, } P_{23}^{can} \mid_{\text{Ran}(P_{12}^{can})} : \text{Ran}(P_{12}^{can}) \to \text{Ran}(P_{23}^{can}) \text{.}$$
4 Formulation of the Levi-Civita problem

4.1 Space of forms, pseudo-Riemannian metrics and connections

Let \((\mathcal{A}, \mathcal{H}, D)\) be a spectral triple of compact type. Let \(d_D\) denote the canonical derivation on \(\mathcal{A}\) defined by

\[ d_D(\cdot) = \sqrt{-1}[D, \cdot]. \]

We fix the spectral triple and henceforth denote \(d_D\) simply by \(d\). We refer to [7], [21], [24] for definition and detailed discussion on the bimodule of noncommutative differential forms. However, we only need to consider the spaces of one and two forms, to be denoted by \(\Omega^1(\mathcal{A})\) and \(\Omega^2(\mathcal{A})\) respectively. They are defined as follows. The space \(\Omega^1(\mathcal{A})\) is the linear span of elements of the form \(a[D, b], a, b \in \mathcal{A}\), in \(\mathcal{B}(\mathcal{H})\). It is clearly a right \(\mathcal{A}\)-module. Any element \(\omega\) is an element of \(\mathcal{B}(\mathcal{H})\), we have a natural multiplication map \(m_0 : \Omega^1(\mathcal{A}) \otimes \mathcal{A} \rightarrow (\omega \otimes \eta) \mapsto \omega \eta \in \mathcal{B}(\mathcal{H})\). Let \(J\) be the module of junk forms, defined to be the right \(\mathcal{A}\)-submodule of the \(\text{Im}(m_0)\) spanned by elements of the form \(\sum [D, a][D, b]\) (finite sum) such that \(\sum a_i[D, b_i] = 0\) \((a_i, b_i \in \mathcal{A})\). We define \(\Omega^1(\mathcal{A}) = \text{Im}(m_0)/J\) and let \(m : \Omega^1(\mathcal{A}) \otimes \mathcal{A} \rightarrow \Omega^2(\mathcal{A})\) be the composition of \(m_0\) and the quotient map from \(\text{Im}(m_0)\) to \(\Omega^2(\mathcal{A})\).

Let us denote \(\Omega^1(\mathcal{A})\) by the symbol \(\mathcal{E}\) from now on. We will need to make some assumptions (Assumption I, II, III, IV) for defining the metric compatibility of a connection. We introduce them one by one.

**Assumption I:** \(\mathcal{E}\) is finitely generated and projective both as a left and right \(\mathcal{A}\)-module. Moreover, the \(\mathcal{A} - \mathcal{A}\) bimodule \(\mathcal{E} \otimes_\mathcal{A} \mathcal{E}\) admits a splitting as a direct sum of right \(\mathcal{A}\) modules: \(\text{Ker}(m) \oplus \mathcal{F}\), where \(\mathcal{F} \cong \text{Im}(m)\).

This assumption is motivated by the decomposition in the classical case that we have:

**Proposition 4.1** If \(\mathcal{A}\) is a subalgebra of \(\mathcal{C}^\infty(M)\), then \(\Omega^1(\mathcal{A}) \otimes \mathcal{A} \Omega^1(\mathcal{A})\) has a decomposition as above. In fact, if \(\sigma_{\text{can}}\) denotes the canonical flip on \(\Omega^1(\mathcal{A}) \otimes \mathcal{A} \Omega^1(\mathcal{A})\) discussed before, then \(\text{Ker}(m) = \text{Ker}(1 - \sigma_{\text{can}})\) and \(\mathcal{F} = \text{Ker}(1 + \sigma_{\text{can}})\).

**Definition 4.2** We will denote by \(P_{\text{sym}} \in \text{Hom}(\mathcal{E} \otimes_\mathcal{A} \mathcal{E}, \mathcal{E} \otimes_\mathcal{A} \mathcal{E})\) the projection onto \(\mathcal{E} \otimes_\mathcal{A} \mathcal{E} := \text{Ker}(m)\). Moreover, \(\sigma\) will denote the map \(2P_{\text{sym}} - 1\).

**Lemma 4.3** We have the following:

1. \(m\) is an \(\mathcal{A} - \mathcal{A}\) bilinear map.
2. \(\mathcal{E} \otimes_\mathcal{A} \mathcal{E} := \text{Ker}(m)\) and \(\text{Im}(m)\) are \(\mathcal{A} - \mathcal{A}\) bimodules.
3. \(P_{\text{sym}}\) and hence \(\sigma\) are \(\mathcal{A} - \mathcal{A}\) bimodule maps.
4. \(P_{\text{sym}}^2 = P_{\text{sym}}\) and \(\sigma^2 = 1\).
Proof: It suffices to prove the first statement only, i.e. $m$ is bilinear, as the other statements are straightforward consequences of it. Clearly, the map $m_0$ is a bilinear map, so we have to prove that the quotient map from $\text{Im}(m_0)$ to $\Omega^2_D(A)$ is bilinear. Again, for this it is enough to prove that $J$ is closed under left $A$-module multiplication since by definition, $J$ is closed under right $A$ multiplication. To this end, let $\sum_i a_i[D,b_i] = 0$ be a finite sum, where $a_i,b_i \in A$. For $c \in A$, $c \sum_i [D,a_i][D,b_i] = \sum_i [D,c][D,a_i][D,b_i] - [D,c] \sum_i a_i[D,b_i] = \sum_i [D,ca_i][D,b_i]$, as $\sum_i a_i[D,b_i] = 0$. But since $\sum_i ca_i[D,b_i] = 0$, we have $\sum_i [D,ca_i][D,b_i] \in J$ which implies that $c \sum_i a_i[D,b_i] \in J$. □

Thus, we have the following result which we will use throughout the sequel.

**Corollary 4.4** Under the above assumptions, since $\sigma$ is a bimodule map, the maps $\sigma_{12}, \sigma_{23}, \sigma_{13} : E \otimes_A E \otimes_A E \otimes_A E$ defined by $\sigma_{12} := \sigma \otimes_A \text{id}$, $\sigma_{23} := \text{id} \otimes_A \sigma$ and $\sigma_{13} := \sigma_{12} \sigma_{23} \sigma_{12}$ respectively are well defined bimodule morphisms.

**Proposition 4.5** When $A = C^\infty(M)$, $E \otimes^\text{sym}_A E$ coincides with the usual symmetric tensor product of one forms on $M$ and for $\omega, \eta \in \Omega^1(M)$, $P\text{sym}(\omega \otimes_A \eta) = \frac{1}{2}(\omega \otimes_A \eta + \eta \otimes_A \omega)$.

Proof: This follows from the well-known fact that in the classical case, the multiplication map is nothing but the exterior or wedge product of forms, which satisfies $m(\omega \otimes_A \eta) = -m(\eta \otimes_A \omega)$ for $\omega, \eta \in \Omega^1(C^\infty(M))$. □

We now recall the definition of a connection in the framework of noncommutative geometry (21, 7).

**Definition 4.6** A (right) connection on the noncommutative manifold given by the above spectral triple is a $C^\ast$-linear map $\nabla : E \to E \otimes_A E$ satisfying

$$\nabla(\omega a) = \nabla(\omega)a + \omega \otimes_A da.$$ 

It is called torsionless if $m \circ \nabla = d$.

**Remark 4.7** All our definitions and results can be easily modified for the set up of left connections.

Throughout this article, a connection will mean a right connection on $E$ as above.

Next, we want to introduce a noncommutative analogue of pseudo-Riemannian metric. In classical differential geometry, a Riemannian metric on a manifold $M$ is a smooth, positive definite, symmetric bilinear form on the tangent (or, equivalently, co-tangent) bundle. One can extend it to the complexification of the tangent/co-tangent spaces in two ways: either as a sesquilinear pairing (inner product) on the module of one-forms, which is conjugate-linear in one variable and linear in the other, or, as a complex bilinear form, i.e. a $C^\infty(M)$-linear map on $\Omega^1(M) \otimes_{C^\infty(M)} \Omega^1(M)$. Somehow, the first of these two alternatives seemed to be more popular in the literature so far to formulate a noncommutative analogue of metric, with some exceptions like the formulation in [23] in the framework of bicovariant differential calculi on quantum groups. One advantage of defining a (Riemannian) metric for noncommutative manifold as a non-degenerate sesquilinear pairing (i.e. inner product) taking value in the underlying $C^\ast$ algebra is that one can use the rich and popular theory of Hilbert modules. However, when one wants to deal with pseudo-Riemannian metrics, there is no assumption of positive definiteness and the relative advantage of sesquilinear extension over the bilinear extension no longer exists. Moreover, the existence and uniqueness of classical Levi-Civita connection for a classical manifold do not need any positive definiteness and hold for an arbitrary pseudo-Riemannian metric. For this reason, it makes sense to consider bilinear non-degenerate forms as pseudo-Riemannian metrics in the noncommutative set-up. However, in classical case there is no difference between right module maps or bimodule maps, as the left and right $C^\infty(M)$-actions on the module of forms coincide. This is no longer true in the noncommutative framework. In fact, as we will see, requiring a pseudo-metric to be a bimodule map restricts the choice of metrics. It is reasonable to require one-sided (right/left) $A$-linearity only. For this reason, we give the following definition:
Definition 4.8 A pseudo-Riemannian metric \( g \) on \( E := \Omega^1_D(A) \) given by a spectral triple is an element of \( \text{Hom}_A(E \otimes_A E, A) \) such that

(i) \( g \) is symmetric, i.e., \( g \sigma = g \).

(ii) \( g \) is non-degenerate, i.e., the right \( A \)-linear map \( V_g : E \to E^* \) defined by \( V_{g}(\omega)(\eta) = g(\omega \otimes_A \eta) \) is an isomorphism of right \( A \)-modules.

We will say that a pseudo-Riemannian metric \( g \) is a pseudo-Riemannian bi-metric if in addition, \( g \) is an \( A-A \) bimodule map.

Remark 4.9 Our definition of nondegeneracy of \( g \) is stronger than the definition given by most of the authors who required only the injectivity of \( V_g \). However, in the classical situation, i.e., when \( A = C^\infty(M) \), these two definitions are equivalent as \( V_g \) is a bundle map from \( T^*M \) to \( (T^*M)^* \cong TM \) in that case and the fibers are finite dimensional.

To compare our definition of a pseudo-Riemannian metric with that of \([21, 31] \) and \([1] \), let us consider the case when \( E \) is a free bimodule of rank \( n \), i.e., \( A \otimes_C \mathbb{C}^n \). Let \( e_i, i = 1, \ldots, n \) be the standard basis of \( \mathbb{C}^n \). A pseudo-metric in our sense is determined by an invertible element \( A := ((g_{ij})) \) of \( M_n(A) \), where \( g_{ij} = g((1 \otimes e_i) \otimes_A (1 \otimes e_j)) \) and \( g((a \otimes e_i) \otimes_A (b \otimes e_j)) = g_{ij}ab \) for all \( a, b \in A \). On the other hand, a pseudo-metric in the sense of \([1] \) corresponding to \( A \) will be given by the sesquilinear pairing \( << a \otimes e_i, b \otimes e_j >> = a^*g_{ij}b \). Thus, there is a one-to-one correspondence between these two notions of pseudo-metric at least for the case when \( E \) is a free bimodule. In fact, they do agree in a sense on the basis elements. But their extensions are quite different as maps.

As we have already remarked, the authors of \([23] \) also chose to define a noncommutative metric as a symmetric, nondegenerate bilinear form. They formulated a notion of the Levi-Civita connections and computed such connections for a number of important examples. However, they worked in the framework of bicovariant differential calculus a la Woronowicz and hence restricted themselves to invariant metrics and bi-equivariant connections only. Thus, in their set-up, it is not possible to perturb the given metric by an arbitrary invertible element of the ‘smooth algebra’, which seems to be of great interest in the study of ‘modular curvature’ taken up recently by many authors.

Let us make few more assumptions for the rest of the article.

Assumption II: The map \( \sigma \) coincides with the canonical flip \( \sigma^{\text{can}} \) discussed before on \( Z(E) \times Z(E) \), i.e., if \( \omega, \eta \) are in \( Z(E) \), then \( \sigma(\omega \otimes_A \eta) = \eta \otimes_A \omega \).

Assumption III: \( Z(E) \) is finitely generated and projective over \( Z(A) \) and moreover, the map \( u^E : Z(E) \otimes_Z Z(A) \to E \) defined by

\[
u^E \left( \sum_i e'_i \otimes_Z Z(A) \right)(a_i) = \sum_i e'_ia_i\]

is an isomorphism of vector spaces.

Assumption IV: There exists a nondegenerate pseudo-Riemannian bi-metric \( g_0 \) on \( E \).

We will prove the existence and uniqueness of Levi Civita connections for a spectral triple such that the corresponding space of one forms \( E := \Omega^1_D(A) \) satisfies Assumption I - Assumption IV. In particular, we have the following set of examples.

Example 4.10 (i) Assumptions I - IV are satisfied for classical manifolds: here, \( Z(E) = \Omega^1(M) \), \( Z(A) = A = C^\infty(M) \).

(ii) We will prove in Subsection \([23] \) that all the above assumptions are valid for Rieffel-deformations of
classical manifolds under mild assumptions on the toral actions used for deformation, giving abundant supply of noncommutative and interesting examples.

(iii) In Section 4, we will see that the space of one forms on the quantum Heisenberg manifolds satisfy Assumption I - IV.

(iv) We say that $E$ is free as a bimodule if $E \cong \mathbb{C}^n \otimes_A A$ as a right $A$ module and the left $A$-module structure on $E$ is given by $a(e_i \otimes b) = e_i \otimes ab$ ( $\{e_i : i = 1, 2, \cdots n\}$ is a basis of $\mathbb{C}^n$ and $a, b \in A$. ) In this case, Assumption III is valid ( see Corollary 4.18 ). Moreover, Assumption IV holds by choosing any $n \times n$ invertible scalar matrix to define a pseudo-Riemannian bi-metric.

Now we discuss the corollaries to the above assumptions which will be crucially used later on.

Lemma 4.11 For a noncommutative manifold satisfying Assumption I, there is a torsion-less connection, say $\nabla_0$.

Proof: By Assumption I, there is a sub-bimodule $F$ of $E \otimes_A E$ and a bimodule isomorphism, say $Q$, from $F = \text{Im}(1 - P_{\text{sym}})$ to $\text{Im}(m) = \overline{\Omega^2_f}(A)$, satisfying $Q((1 - P_{\text{sym}})(\beta)) = m(\beta)$ for $\beta \in E \otimes_A E$. Moreover, as $E$ is finitely generated and projective, we can find a free rank $n$ ( right ) module $A \otimes \mathbb{C} \mathbb{C}^n$ containing $E$ as a complemented right submodule. Let $p$ be an idempotent in $M_n(A) \cong \text{Hom}_A(A \otimes \mathbb{C} \mathbb{C}^n, A \otimes \mathbb{C} \mathbb{C}^n)$ such that $E = p(A \otimes \mathbb{C} \mathbb{C}^n)$. Let $e_i, i = 1, \ldots, n$, be the standard basis of $\mathbb{C}^n$ and define $\nabla_0 : A \otimes \mathbb{C} \mathbb{C}^n \to E \otimes_A E$ by

$$\nabla_0(e_i) := Q^{-1}(d(pe_i))a + pe_i \otimes da, \quad i = 1, \ldots, n, a \in A.$$ 

Then $\nabla_0 = \tilde{\nabla}_0|_E$, (2) is clearly a torsion-less connection on $E$. $\Box$

As a trivial consequence of Assumption II, Lemma 3.5 and the relation $g\sigma = g$, it follows that $g(\omega \otimes_A \eta) = g(\eta \otimes_A \omega)$ if either $\omega$ or $\eta$ is in $Z(E)$. (3)

Next, we come to Assumption III. It is clear from the definitions that the map $vE$ is left $Z(A)$, right $A$-linear. Moreover, define a left $A$, right $Z(A)$-linear map $vE : A \otimes_Z(A) Z(E) \to E$ by

$$vE \left( \sum_i a_i \otimes_Z(A) e_i \right) = \sum_i a_i e_i.$$ 

Lemma 4.12 If $E$ satisfies Assumption III, $vE$ is also an isomorphism.

Proof: Consider the map $p : Z(E) \times A \to A \otimes_Z(A) Z(E)$ given by $(e, a) \mapsto (a \otimes_Z(A) e)$. It is clear that $p(ea', a) = p(e, a')$, so that we get a well-defined map $\pi : Z(E) \otimes_Z(A) A \to A \otimes_Z(A) Z(E)$ given by $(e \otimes_Z(A) a) \mapsto (a \otimes_Z(A) e)$. It is in fact an isomorphism, with the inverse map, say $q$, given by $(a \otimes_Z(A) e) \mapsto (e \otimes_Z(A)a)$. Observe that $vE = vE \circ q$, hence $vE$ is an isomorphism as well. $\Box$

We are grateful to Ulrich Krahmer for kindly pointed out the following implication of Assumption III:

Proposition 4.13 Let us endow $Z(E) \otimes_Z(A) A$ with an $A - A$ bimodule structure defined by $b(e \otimes_Z(A) a)'c = e \otimes_Z(A) ba'c$, where $e \in E, a' \in Z(A), b, c \in A$. If $E$ satisfies Assumption III, then we have the following isomorphism of $A - A$-bimodules:

$$E \cong A \otimes_Z(A) Z(E) \cong Z(E) \otimes_Z(A) A.$$ 

8
Proposition 4.16  

It is easy to see that \( \varepsilon^F \) defines an \( \mathcal{A} - \mathcal{A} \) bimodule isomorphism. The other isomorphism follows by using the map \( v^F \). ⊓⊔

**Corollary 4.14**  

The right \( \mathcal{A} \)-linear span of \( \mathcal{L}(\mathcal{E}) \) is total in \( \mathcal{E} \), i.e, \( \mathcal{E} \) is centered.

Moreover, we have:

**Lemma 4.15**  

Let \( \mathcal{E} \) and \( \mathcal{F} \) be \( \mathcal{A} - \mathcal{A} \)-bimodules satisfying Assumption III.

1. Let \( \mathcal{F} \) be any \( \mathcal{A} - \mathcal{A} \)-bimodule. Then the map \( T_{\mathcal{E},\mathcal{F}}^B := ((\varepsilon^F)^{-1} \otimes_\mathcal{A} \text{id}_\mathcal{F}) \) defines a right \( \mathcal{A} \)-module isomorphism from \( \mathcal{E} \otimes_\mathcal{A} \mathcal{F} \) to \( \mathcal{Z}(\mathcal{E}) \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{F} \), which is also left \( \mathcal{Z}(\mathcal{A}) \)-linear. Similarly, the map \( T_{\mathcal{E},\mathcal{F}}^R := (\text{id}_\mathcal{E} \otimes_\mathcal{A} (\varepsilon^F)^{-1}) \) defines a left \( \mathcal{A} \)-linear isomorphism from \( \mathcal{F} \otimes_\mathcal{A} \mathcal{E} \) to \( \mathcal{F} \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{Z}(\mathcal{E}) \). (2) If \( \mathcal{F}_0 \) is a left \( \mathcal{Z}(\mathcal{A}) \)-submodule of \( \mathcal{F} \) which is also right \( \mathcal{A} \)-total, then \( (T_{\mathcal{E},\mathcal{F}_0}^R)^{-1}(\mathcal{Z}(\mathcal{E}) \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{F}_0) \) is right \( \mathcal{A} \)-total in \( \mathcal{E} \otimes_\mathcal{A} \mathcal{F} \). Similarly, \( (T_{\mathcal{E},\mathcal{F}_0}^R)^{-1}(\mathcal{F}_1 \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{Z}(\mathcal{E})) \) is left \( \mathcal{A} \)-total in \( \mathcal{F} \otimes_\mathcal{A} \mathcal{E} \), for a left \( \mathcal{A} \)-total right \( \mathcal{Z}(\mathcal{A}) \) submodule \( \mathcal{F}_1 \) of \( \mathcal{F} \).

**Proof:** To start with, we note that both the maps \( (\varepsilon^F)^{-1} \otimes_\mathcal{A} \text{id}_\mathcal{F} \) and \( (\text{id}_\mathcal{E} \otimes_\mathcal{A} (\varepsilon^F)^{-1}) \) are well defined since \( \varepsilon^F \) is right \( \mathcal{A} \)-linear and \( \varepsilon^F \) is left \( \mathcal{A} \)-linear. We have \( \mathcal{F} \cong \mathcal{F} \otimes_\mathcal{A} \mathcal{A} \cong \mathcal{A} \otimes_\mathcal{A} \mathcal{F} \), so \( \mathcal{E} \otimes_\mathcal{A} \mathcal{F} \cong (\mathcal{Z}(\mathcal{E}) \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{A}) \otimes_\mathcal{A} \mathcal{F} \cong \mathcal{Z}(\mathcal{E}) \otimes_{\mathcal{Z}(\mathcal{A})} (\mathcal{A} \otimes_\mathcal{A} \mathcal{F}) \cong \mathcal{Z}(\mathcal{E}) \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{F} \) is the required isomorphism. The other part of (1) follows similarly. For (2), we observe that \( \mathcal{Z}(\mathcal{E}) \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{F}_0 \mathcal{A} = \mathcal{Z}(\mathcal{E}) \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{F} \) and \( \mathcal{A} \mathcal{F}_1 \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{Z}(\mathcal{E}) = \mathcal{F} \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{Z}(\mathcal{E}) \). Thus, (2) follows by applying (1).

⊔

We end the discussion on Assumption III by showing that one can replace \( \mathcal{Z}(\mathcal{E}) \) and \( \mathcal{Z}(\mathcal{A}) \) in Assumption III by a possibly smaller submodule \( \mathcal{E}' \subseteq \mathcal{Z}(\mathcal{E}) \) and a subalgebra \( \mathcal{A}' \subseteq \mathcal{Z}(\mathcal{A}) \).

Consider the following:

**Assumption III’:** There exists a unital \( * \)-subalgebra \( \mathcal{A}' \) of \( \mathcal{Z}(\mathcal{A}) \) and an \( \mathcal{A}' \)-submodule \( \mathcal{E}' \) of \( \mathcal{Z}(\mathcal{E}) \) such that \( \mathcal{E}' \) is projective and finitely generated over \( \mathcal{A}' \) and the map

\[
\varepsilon^\mathcal{E}_{\mathcal{E}'} : \mathcal{E}' \otimes_{\mathcal{A}'} \mathcal{A}' \to \mathcal{E},
\]

defined by

\[
\varepsilon^\mathcal{E}_{\mathcal{E}'} \left( \sum_i e_i' \otimes_{\mathcal{A}'} a_i \right) = \sum_i e_i' a_i
\]

is an isomorphism of vector spaces.

If we take \( \mathcal{A}' \) and \( \mathcal{E}' \) to be \( \mathcal{Z}(\mathcal{A}) \) and \( \mathcal{Z}(\mathcal{E}) \) respectively, Assumption III’ is merely Assumption III. Let us show that the converse holds as well, i.e, Assumption III and Assumption III’ are equivalent.

**Proposition 4.16**  

If Assumption III’ holds, we have \( \mathcal{Z}(\mathcal{E}) \cong \mathcal{E}' \otimes_{\mathcal{A}'} \mathcal{Z}(\mathcal{A}) \).

**Proof:**  

We begin by observing that an analogue of Proposition 4.13 shows that \( \mathcal{E} \cong \mathcal{E}' \otimes_{\mathcal{A}'} \mathcal{A} \) as bimodules where the bimodule structure of \( \mathcal{E}' \otimes_{\mathcal{A}'} \mathcal{A} \) is defined by \( z_1(e' \otimes_{\mathcal{A}'} a_1)z_2 = e' \otimes_{\mathcal{A}'} z_1 a_1 z_2 \). Thus, \( \{ \sum_i e_i' \otimes_{\mathcal{A}'} a_i : e_i' \in \mathcal{E}', a_i \in \mathcal{Z}(\mathcal{A}) \} \subseteq \mathcal{Z}(\mathcal{E}' \otimes_{\mathcal{A}'} \mathcal{A}) \).

For the reverse inclusion, let us suppose that there exists a free \( \mathcal{A}' \) module \( \mathcal{F} \) and an idempotent \( P \) on \( \mathcal{F} \) such that \( P(\mathcal{F}) = \mathcal{E}' \). Let \( m_1, m_2, \ldots, m_n \) be a basis of \( \mathcal{F} \). Therefore,

\[
\mathcal{E} \cong \mathcal{E}' \otimes_{\mathcal{A}'} \mathcal{A} = P(\mathcal{F}) \otimes_{\mathcal{A}'} \mathcal{A} = (P \otimes_{\mathcal{A}'} \text{id}_{\mathcal{A}})(\mathcal{F} \otimes_{\mathcal{A}'} \mathcal{A}).
\]

Clearly, \( P \otimes_{\mathcal{A}'} \text{id}_{\mathcal{A}} \) is an idempotent on \( \mathcal{F} \otimes_{\mathcal{A}'} \mathcal{A} \) and thus for all \( y \in \mathcal{E}' \otimes_{\mathcal{A}'} \mathcal{A} \subseteq \mathcal{F} \otimes_{\mathcal{A}'} \mathcal{A} \), we have

\[
(P \otimes_{\mathcal{A}'} \text{id}_{\mathcal{A}})(y) = y.
\]

(4)
On the other hand, \( \mathcal{Z}(\mathcal{E}' \otimes_{A'} \mathcal{A}) \) is also a submodule of \( \mathcal{F} \otimes_{A} \mathcal{A} \) and if \( x \) is an element of \( \mathcal{Z}(\mathcal{E}' \otimes_{A'} \mathcal{A}) \), there exists unique elements \( a_i \in \mathcal{A} \) such that \( x = \sum_i m_i \otimes_{A'} a_i \). Since \( xb = bx \) for all \( b \in \mathcal{A} \), we see that \( a_i \in \mathcal{Z}(\mathcal{A}) \) for all \( i \). Hence,

\[
(P \otimes_{A'} \text{id}_\mathcal{A})(x) = \sum_i (P \otimes_{A'} \text{id}_\mathcal{A})(m_i \otimes_{A'} a_i) = \sum_i P(m_i) \otimes_{A'} a_i \in \mathcal{E}' \otimes_{A'} \mathcal{Z}(\mathcal{A}).
\]

But by (4), \( (P \otimes_{A'} \text{id}_\mathcal{A})(x) = x \) so that \( x \in \mathcal{E}' \otimes_{A'} \mathcal{Z}(\mathcal{A}). \) Since \( x \) is an arbitrary element of \( \mathcal{Z}(\mathcal{E}' \otimes_{A'} \mathcal{A}) \cong \mathcal{Z}(\mathcal{E}) \), this completes the proof. \( \square \)

This gives the following:

**Proposition 4.17 Assumption III and Assumption III’ are equivalent.**

**Proof:** By Proposition 4.16 we have \( \mathcal{Z}(\mathcal{E}) \cong \mathcal{E}' \otimes_{A'} \mathcal{Z}(\mathcal{A}) \) via the multiplication map. Therefore,

\[
\mathcal{Z}(\mathcal{E}) \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{A} \cong \mathcal{E}' \otimes_{A'} \mathcal{Z}(\mathcal{A}) \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{A} = \mathcal{E}' \otimes_{A'} \mathcal{A} \cong \mathcal{E}.
\]

\( \square \)

**Corollary 4.18** If \( \mathcal{E} \) is free as a right \( A \)-module of finite rank so that \( \mathcal{E} = \mathbb{C}^n \otimes_{\mathbb{C}} \mathcal{A} \) and the left \( A \) module structure on \( \mathcal{E} \) is defined as \( a(e_i \otimes \mathbb{C} \mathcal{A}) = e_i \otimes \mathbb{C} \mathcal{A} ab \), where \( e_1, e_2, \cdots, e_n \) is a basis of \( \mathbb{C}^n \) and \( a, b \in \mathcal{A} \), then Assumption III holds.

**Proof:** We note that the elements \( e_1, e_2, \cdots, e_n \) belong to \( \mathcal{Z}(\mathcal{E}) \). Take \( \mathcal{E}' \) to be the complex linear span of \( e_1, e_2, \cdots, e_n \) and \( A' = \mathbb{C} \). Since \( \mathcal{E}' \) is a free module over \( A' \), we can reach the desired conclusion by using Proposition 4.17. \( \square \)

We note the following results for future use.

**Lemma 4.19** 1. If \( g_0 \) is a pseudo-Riemannian bi-metric, then \( g_0(\omega \otimes_{A} \eta) \in \mathcal{Z}(\mathcal{A}) \) if \( \omega, \eta \) belong to \( \mathcal{Z}(\mathcal{E}) \).

2. If \( g \) and \( g_0 \) are pseudo-Riemannian metrics, then \( V^{-1}_g V_{g_0}(\mathcal{Z}(\mathcal{E})) \) is right \( A \)-total in \( \mathcal{E} \).

3. Let \( S \) be a subset of \( \mathcal{Z}(\mathcal{E}) \) which is right \( A \)-total in \( \mathcal{E} \). If \( g \) is a pseudo-Riemannian metric then \( g(\beta \otimes_{A} \eta) = 0 \) for all \( \eta \in S \) if and only if \( \beta = 0 \). The same conclusion holds if \( g(\eta \otimes_{A} \beta) = 0 \) for all \( \eta \in S \).

4. \( V_g \) is left \( \mathcal{Z}(\mathcal{A}) \)-linear.

**Proof:** The first claim is a trivial consequence of the assumption that \( g_0 \) is an \( A \)-\( A \) bimodule map. Indeed, since \( \omega, \eta \) are in \( \mathcal{Z}(\mathcal{E}) \),

\[
g_0(\omega \otimes_{A} \eta)a = g_0(\omega \otimes_{A} \eta a) = g_0(\omega \otimes_{A} a \eta) = g_0(\omega a \otimes_{A} \eta) = a g_0(\omega \otimes_{A} \eta).
\]

For part 2, we see that

\[
V^{-1}_g V_{g_0}(\mathcal{Z}(\mathcal{E})) = V^{-1}_g (V_{g_0}(\mathcal{Z}(\mathcal{E}))) = V^{-1}_g (V_{g_0}(\mathcal{E})) = V^{-1}_g (\mathcal{E}) = \mathcal{E}.
\]

Now we prove the third assertion. The equation \( g(\beta \otimes_{A} \eta) = 0 \) implies \( V_g(\beta)(\eta) = 0 \). Since this holds for all \( \eta \) in \( S \) which is right \( A \)-total, we have \( V_g(\beta)(\mathcal{E}) = 0 \) by right \( A \)-linearity of \( V_g \). Thus \( V_g(\beta) = 0 \) and thus \( \beta = 0 \) by non-degeneracy of \( g \). The second claim follows by observing that \( g(\eta \otimes_{A} \beta) = g(\beta \otimes_{A} \eta) \) since \( \eta \in S \subseteq \mathcal{Z}(\mathcal{E}) \), \( g \sigma = g \) and then applying (4). \( \square \)

For proving 4., let \( a' \in \mathcal{Z}(\mathcal{A}) \), \( \xi, \eta \in \mathcal{Z}(\mathcal{E}) \).

\[
\begin{align*}
\text{We have } V_g(a' \xi)(\eta) &= g(a' \xi \otimes_{A} \eta) = g(\xi a' \otimes_{A} \eta) = g(\xi \otimes_{A} a' \eta) \\
&= g(\xi \otimes_{A} \eta a') = g(\xi \otimes_{A} \eta)a' = a' g(\xi \otimes_{A} \eta) \\
&= a' V_g(\xi)(\eta),
\end{align*}
\]

where we have used the fact that \( a' \in \mathcal{Z}(\mathcal{A}) \). As \( \mathcal{Z}(\mathcal{E}) \) is right \( A \)-total, 4. follows. \( \square \)
4.2 Classical Levi-Civita connection revisited

We first recast the classical Levi-Civita connection for a pseudo-Riemannian metric in a language suitable for us. Let $(M, g)$ be a pseudo-Riemannian manifold and $\mathcal{A} = C^{\infty}(M)$. Fix a torsionless connection $\nabla_0$. In the classical case, one usually works on the real field only, but one can as well complexity spaces and modules involved, which we do here. The pseudo-metric can be identified as a symmetric bilinear form on the complexified space of one-forms $\Omega^1(M)$, i.e., as a map $g : \Omega^1(M) \circledcirc \mathcal{A} \circledcirc \Omega^1(M) \to \mathcal{A}$, satisfying $g_\sigma = g$ where $\sigma$ denotes the map $\sigma^{can}$ as before. The Levi-Civita connection $\nabla$ for $(M, g)$ is the unique torsionless connection, which also satisfies the following

$$g_{13}(\nabla(\omega) \circledcirc \mathcal{A} \eta) + g_{13}(\nabla(\eta) \circledcirc \mathcal{A} \omega) = dg(\omega \circledcirc \mathcal{A} \eta),$$

for all $\omega, \eta$ in $\Omega^1(M)$, where $g_{13} = (g \circledcirc \mathcal{id}_{\mathcal{A}})_{13}$. To put this in a form suitable for us, we give the following definition:

**Definition 4.20** Let $q : \Omega^1(M) \circledcirc \mathcal{A} \Omega^1(M) \circledcirc \mathcal{A} \Omega^1(M) \to \Omega^1(M) \circledcirc \mathcal{A} \Omega^1(M) \circledcirc \mathcal{A} \Omega^1(M)$ be the natural quotient map. Then for a $\mathbb{C}$-linear map $X$ from $\Omega^1(M)$ to $\Omega^1(M) \circledcirc \mathcal{A} \Omega^1(M)$, we define an element $\Pi_q(X) \in \text{Hom}_{\mathbb{C}}(\Omega^1(M) \circledcirc \mathcal{A} \Omega^1(M), \Omega^1(M))$, by the following:

$$\Pi_q(X)(\omega \circledcirc \mathcal{A} \eta) = g_{13}(X(\omega) \circledcirc \mathcal{A} \eta) + g_{13}(X(\eta) \circledcirc \mathcal{A} \omega).$$

Thus, $\Pi_q(X)(\omega \circledcirc \mathcal{A} \eta) \in \Omega^1(M)$, where $g_{13} = (g \circledcirc \mathcal{id}_{\mathcal{A}})_{13}$. To put this in a form suitable for us, we give the following definition:

**Definition 4.21** If $X$ is a connection on $\Omega^1(M)$, $\Pi_q(X)$ defines an element of $\text{Hom}_{\mathbb{C}}(\Omega^1(M) \circledcirc \mathcal{A} \Omega^1(M), \Omega^1(M))$, to be denoted by $\Pi_q(X)$ again.

**Proof:** The proof is omitted because we will prove a similar result in a more general setting in the next subsection. □

Then, the Levi-Civita connection $\nabla$ satisfies $\Pi_q(\nabla)(\cdot, \cdot) = dg(\cdot, \cdot)$. Indeed, since both $\nabla$ and $\nabla_0$ are torsionless, $L = \nabla - \nabla_0$ is a right $\mathcal{A}$-linear map with the range in $\Omega^1(M) \circledcirc \mathcal{A} \Omega^1(M)$ and we have

$$\Pi_q(L)(\cdot, \cdot) = dg(\cdot, \cdot) - \Pi_q(\nabla_0)(\cdot, \cdot).$$

Denote by $\Phi_q$ the restriction of $\Pi_q$ on the space $\text{Hom}_{\mathcal{A}}(\Omega^1(M), \Omega^1(M) \circledcirc \mathcal{A}^{sym} \Omega^1(M))$ and let $S_q(\cdot, \cdot) = dg(\cdot, \cdot) - \Pi_q(\nabla_0)(\cdot, \cdot)$. Then, the classical Levi-Civita theorem can be stated in the following form:

**Proposition 4.22** There is a unique solution $L \in \text{Hom}_{\mathcal{A}}(\Omega^1(M), \Omega^1(M) \circledcirc \mathcal{A}^{sym} \Omega^1(M))$ of the equation $\Phi_q(L) = S_q$, and $\nabla = L + \nabla_0$ is the Levi-Civita connection for $g$.

4.3 Defining metric-compatibility of a connection

Let $g$ be a pseudo-Riemannian metric on the noncommutative manifold satisfying the Assumption I to Assumption IV as before. Motivated by Proposition 4.22, we want to define an analogue of the map $\Pi_q(T)$ in the noncommutative set-up. To this end, let $\nabla$ be a connection and let $\Pi_q(\nabla) : \mathcal{Z}(\mathcal{E}) \circledcirc \mathcal{E}(\mathcal{E}) \to \mathcal{E}$ be the map given by

$$\Pi_q(\nabla)(\omega \circledcirc \mathcal{A} \eta) = (g \circledcirc \mathcal{id}_{\mathcal{A}})_{13}(\nabla(\omega) \circledcirc \mathcal{A} \eta + \nabla(\eta) \circledcirc \mathcal{A} \omega).$$

Then we have
Lemma 4.23 $\Pi^0_y(\nabla)(\omega a' \otimes_C \eta) = \Pi^0_y(\nabla)(\omega \otimes_C a' \eta)$ for all $a' \in Z(A)$ and $\omega, \eta \in Z(\mathcal{E})$.

Proof: Write $\nabla(\eta) = \sum_i \eta_i^{(1)} \otimes_A \eta_i^{(2)}$, $\eta_i^{(1)}, \eta_i^{(2)} \in \mathcal{E}$, where the sum has finitely many terms. We have $\sigma_{23}(\omega \otimes_A da' \otimes_A \eta) = \omega \otimes_A \eta \otimes_A da', \sigma_{23}(\nabla(\eta)a' \otimes_A \omega) = \sum_i \eta_i^{(1)} \otimes_A \omega \otimes_A \eta_i^{(2)} a'$. Using these, we get

$$\Pi^0_y(\nabla)(\omega a' \otimes_C \eta) = (g \otimes_A \text{id})\sigma_{23}(\nabla(\omega)a' \otimes_A \eta + \omega \otimes_A da' \otimes_A \eta + \nabla(\eta) \otimes_A \omega a')$$

$$= (g \otimes_A \text{id})\sigma_{23}(\nabla(\omega) \otimes_A \eta a') + g(\omega \otimes_A \eta)da' + \sum_i g(\eta_i^{(1)} \otimes_A \omega)\eta_i^{(2)}a'$$

$$= (g \otimes_A \text{id})\sigma_{23}(\nabla(\omega) \otimes_A \eta a') + g(\omega \otimes_A \eta)da'$$

$$= \Pi^0_y(\nabla)(\omega \otimes_C \eta a')$$

$$= \Pi^0_y(\nabla)(\omega \otimes_C a' \eta).$$

Hence $\Pi^0_y(\nabla)$ descends to a map on $Z(\mathcal{E}) \otimes_Z(A) Z(\mathcal{E})$, to be denoted by the same notation. From the connection property of $\nabla$, we can easily verify the following:

Lemma 4.24 $\Pi^0_y(\nabla)(\omega \otimes_{Z(A)} \eta a') = \Pi^0_y(\nabla)(\omega \otimes_{Z(A)} \eta)a' + g(\omega \otimes_A \eta)da'$ for all $\omega, \eta \in Z(\mathcal{E})$ and $a \in Z(A)$.

Thus, the map $S^0_y(\nabla) : Z(\mathcal{E}) \otimes_Z(A) Z(\mathcal{E}) \to \mathcal{E}$ given by:

$$S^0_y(\nabla)(\omega \otimes_{Z(A)} \eta) = dg(\omega \otimes_A \eta) - \Pi^0_y(\nabla)(\omega \otimes_{Z(A)} \eta)$$

is right $Z(A)$-linear. Now we define

Definition 4.25 $S_y(\nabla) := u \circ (S^0_y(\nabla) \otimes_Z(A) \text{id}_A) \circ (\text{id}_Z(\mathcal{E}) \otimes_{Z(\mathcal{E})} T^L_{\mathcal{E}A}) \circ (T^L_{\mathcal{E}A} \otimes_Z \text{id}_\mathcal{E}) : \mathcal{E} \otimes_A \mathcal{E} \to \mathcal{E}$, where $u : \mathcal{E} \otimes_Z(A) A \to \mathcal{E}$ denotes the map defined by $u(\sum_i e_i \otimes_{Z(\mathcal{E})} a_i) = \sum_i e_i a_i$.

A simple book-keeping using the definition of the maps involved gives us the following:

Proposition 4.26 $S_y(\nabla)(\omega a \otimes_A \eta b) = S^0_y(\nabla)(\omega \otimes_{Z(A)} \eta)ab$, for all $\omega, \eta \in Z(\mathcal{E})$ and $a, b \in A$. We also set $\Pi_y(\nabla)(\omega \otimes_A \eta) = dg(\omega \otimes_A \eta) - S_y(\nabla)(\omega \otimes_A \eta)$.

We can now state the definition of compatibility of $\nabla$ with the pseudo-Riemannian metric $g$.

Definition 4.27 A connection $\nabla$ is called compatible with the pseudo-Riemannian metric $g$ if for all $\omega, \eta \in \mathcal{E}$, we have $\Pi_y(\nabla)(\omega \otimes_A \eta) = dg(\omega \otimes_A \eta)$.

It is clear from the discussions in Subsection 4.2 that this definition coincides with that of the classical case.
Definition 4.28 We say that a connection $\nabla$ is Levi-Civita for $g$ if it is torsionless as well as compatible with $g$. If there is unique Levi-Civita connection for a pseudo-Riemannian metric $g$ then we say that $g$ has the Levi-Civita property. We say that the underlying noncommutative manifold is Levi-Civita, or that it satisfies the Levi-Civita property if every pseudo-Riemannian metric $g$ on it has the Levi-Civita property.

Let us now define a right $\mathcal{A}$-linear map

$$\Phi_g : \text{Hom}_A(\mathcal{E}, \mathcal{E} \otimes_A \mathcal{E}) \rightarrow \text{Hom}_A(\mathcal{E} \otimes_A \mathcal{E}, \mathcal{E})$$

by

$$\Phi_g(L) = (g \otimes_A \text{id}) \sigma(1 + \sigma).$$

Remark 4.29 (i) From Proposition 4.26, it is easy to see that $S_g(\nabla) = S_g(\nabla) \circ \sigma$ for any connection $\nabla$, as $S_g^0$ has this property on $\mathcal{Z}(\mathcal{E} \otimes_A \mathcal{E})$, which is right $\mathcal{A}$-total in $\mathcal{E} \otimes_A \mathcal{E}$. Thus, $S_g(\nabla)$ is determined by its restriction on $\mathcal{E} \otimes_A \mathcal{E}$.

(ii) Moreover, for any two torsionless connections $\nabla, \nabla'$, their difference $\nabla - \nabla' \in \text{Hom}_A(\mathcal{E}, \mathcal{E} \otimes_A \mathcal{E})$, 

$$(S_g(\nabla) - S_g(\nabla')) |_{\mathcal{E} \otimes_A \mathcal{E}} = \Phi_g(\nabla - \nabla').$$

Let us now state and prove the key theorem using which we will prove the existence and uniqueness of Levi-Civita connection in this article.

Theorem 4.30 Let $g$ be a pseudo-Riemannian metric. If the map $\Phi_g$ defined above is an isomorphism of right $\mathcal{A}$-modules then the Riemannian pseudo-metric $g$ has the Levi-Civita property.

Proof: Recall the torsionless connection $\nabla_0$. Clearly, $\nabla$ is torsionless if and only if $L := \nabla - \nabla_0$ has range in $\mathcal{E} \otimes_A \mathcal{E}$. Moreover, $\nabla$ is compatible with $g$ if and only if $S_g(\nabla) = 0$, or,

$$S_g(\nabla) - S_g(\nabla_0) = -S_g(\nabla_0).$$

By Remark 4.29, the above equality holds if and only if it holds on $\mathcal{E} \otimes_A \mathcal{E}$, which can be rewritten as

$$\Phi_g(L) = -S_g(\nabla_0),$$

where $L = \nabla - \nabla_0$. If $\Phi_g$ is an isomorphism, $\nabla = \nabla_0 - \Phi_g^{-1}(S_g(\nabla_0))$ is the unique Levi-Civita connection. This proves the theorem. □

5 Existence and uniqueness of Levi-Civita connection

In this section, we prove that Assumptions 1 - IV suffice for existence of a unique Levi-Civita connection. We continue to denote $\Omega^1_{\mathcal{F}}(\mathcal{A})$ by the symbol $\mathcal{E}$, i.e, we prove that $\Phi_g : \text{Hom}_A(\mathcal{E}, \mathcal{E} \otimes_A \mathcal{E}) \rightarrow \text{Hom}_A(\mathcal{E} \otimes_A \mathcal{E}, \mathcal{E}, \mathcal{E})$ is an isomorphism of right $\mathcal{A}$-modules.

We begin by the following definition:

Definition 5.1 Let $\mathcal{E}$ and $\mathcal{F}$ be $\mathcal{A}-\mathcal{A}$-bimodules and suppose $g$ be a pseudo-Riemannian metric on $\mathcal{F}$. Then for $e \in \mathcal{E}$ and $f \in \mathcal{F}$, we define the element $S_{e,f}$ of $\text{Hom}_A(\mathcal{F}, \mathcal{E})$ by the formula

$$S_{e,f}(f_1) = eg(f \otimes_A f_1).$$

Then, by Proposition 3.1 we see that

$$\zeta_{\mathcal{E},\mathcal{F}}(e \otimes_A V_g(f)) = S_{e,f}.$$

We will denote $V_g^{-1}V_{g_0}$ by the symbol $\pi_{g,g_0}$.

Then, we have

$$g(\pi(\omega) \otimes_A \eta) = g_0(\omega \otimes_A \eta).$$
Proposition 5.2 1. There exists a right $A$-linear map $\hat{\cdot} : \text{Hom}_A(\mathcal{E}, \mathcal{E}) \to \text{Hom}_A(\mathcal{E} \otimes_A \mathcal{E}, A)$ by $B \mapsto \hat{B}$, where $\hat{B}$ is defined by:

$$\hat{B}(\omega \otimes A \eta) = g(B(\omega) \otimes A \eta).$$

2. There exists a right $A$-linear map $\cup$ from $\text{Hom}_A(\mathcal{E} \otimes_A \mathcal{E}, A) \to \text{Hom}_A(\mathcal{E}, \mathcal{E})$ defined by $C \mapsto C^\cup$ where $C^\cup(\omega)$ is defined to be the unique element in $\mathcal{E}$ satisfying the equation

$$g(C^\cup(\omega) \otimes A \eta) = C(\omega \otimes A \eta).$$

3. The maps $\hat{\cdot}$ and $\cup$ are inverses of one another.

4. The maps $\hat{\cdot}$ and $\cup$ are left $Z(A)$ linear.

**Proof:** It is easy to see that $C^\cup(\omega)$ is nothing but $V_g^{-1}$ applied on the element $C(\omega, \cdot) \in \mathcal{E}^*$. The rest can be verified by straightforward and direct computations. □

We define the map

$$\Gamma : \mathcal{E} \otimes_A \mathcal{E}^* \to \text{Hom}_A(\mathcal{E} \otimes_A \mathcal{E}, A) \quad \text{by} \quad \Gamma = \hat{\cdot} \circ \zeta_{\mathcal{E}, \mathcal{E}}.$$ 

Then we have the following:

**Lemma 5.3** $\Gamma$ is a left $Z(A)$-linear map and a right $A$-linear isomorphism.

Moreover, the map $(\text{id}_{Z(\mathcal{E})} \otimes Z(\mathcal{A})) \Gamma : Z(\mathcal{E}) \otimes Z(\mathcal{A}) \otimes \mathcal{E}^* \to Z(\mathcal{E}) \otimes Z(\mathcal{A}) \text{Hom}_A(\mathcal{E} \otimes A, \mathcal{E}, A)$ satisfies

$$(\text{id}_{Z(\mathcal{E})} \otimes Z(\mathcal{A})) \Gamma(\omega_1 \otimes Z(\mathcal{A}) \omega_2 \otimes \mathcal{A} V_g(\pi_{g,g_0}(\omega_1))) \sigma(\omega \otimes A \eta) = \omega_1(\omega_2 g(\pi_{g,g_0}(\omega_1) \otimes A \eta) \otimes A \omega),$$

where $\omega_1, \omega_2, \omega_3, \omega, \eta \in Z(\mathcal{E})$.

**Proof:** The first part of the statement is a straightforward consequence of the definition of $\Gamma$. To see the other part of the statement, note that, as $\omega, \eta \in Z(\mathcal{E})$, $\sigma(\omega \otimes A \eta) = \eta \otimes A \omega$. Now, by definition of $\hat{\cdot}$ as well as using the observation $\zeta_{\mathcal{E}, \mathcal{E}}(e \otimes A \mathcal{V}(f)) = S_{e,f}$ made before, we have $\Gamma(\omega_2 \otimes Z(\mathcal{A}) V_g(\pi_{g,g_0}(\omega_1)))(\eta \otimes A \omega) = g(S_{\omega_2, \pi_{g,g_0}(\omega_1)}(\eta) \otimes A \omega) = g(\omega_2 g(\pi_{g,g_0}(\omega_1) \otimes A \eta) \otimes A \omega)$, from which the required expression easily follows. □

**Definition 5.4** $\Psi_g : \text{Hom}_A(\mathcal{E}, \mathcal{E} \otimes A) \to \text{Hom}_A(\mathcal{E} \otimes A, \mathcal{E})$ is defined by

$$\Psi_g := \zeta_{\mathcal{E}, \mathcal{E} \otimes A} T_{\mathcal{E}, \text{Hom}_A(\mathcal{E} \otimes A, A)}^{-1} (\text{id}_{Z(\mathcal{E})} \otimes Z(\mathcal{A})) \Gamma \zeta_{\mathcal{E}, \mathcal{E}}^{-1}.$$

Then $\Psi_g$ is an $Z(A)$-$A$ bimodule isomorphism. We also denote by $\Theta_g$ the inverse of the isomorphism $\Psi_g$.

**Lemma 5.5** It is easy to see that if $g$ is a pseudo-Riemannian metric on such a module $\mathcal{E}$, then the right $A$-linear map $V_g$ is actually $Z(A)$-bilinear and hence so is $V_g^{-1}$.

Thus, the map $\text{id}_{Z(\mathcal{E})} \otimes Z(\mathcal{A}) V_g^{-1} : Z(\mathcal{E}) \otimes Z(\mathcal{A}) \mathcal{E}^*$ makes sense. We make two definitions at this point.

**Definition 5.6** We define $\tau : \mathcal{E} \otimes A \mathcal{E}^* \to \mathcal{E} \otimes A \mathcal{E}$ and $\xi : \text{Hom}_A(\mathcal{E}, \mathcal{E} \otimes A) \to \mathcal{E} \otimes A \mathcal{E} \otimes A \mathcal{E}$ by the following formulas:

$$\tau := (T_{\mathcal{E}, \mathcal{E}})\Gamma^{-1} (\text{id}_{Z(\mathcal{E})} \otimes Z(\mathcal{A}) V_g^{-1}) T_{\mathcal{E}, \mathcal{E}}^\tau,$$

$$\xi = (T_{\mathcal{E}, \mathcal{E} \otimes A}^{-1} (\text{id}_{Z(\mathcal{E})} \otimes Z(A)) \tau) (\text{id}_{\mathcal{E} \otimes A} T_{\mathcal{E}, \mathcal{E} \otimes A}^\tau \zeta_{\mathcal{E}, \mathcal{E}}^{-1}).$$

We have the following proposition.

**Proposition 5.7** $\tau : \mathcal{E} \otimes A \mathcal{E}^* \to \mathcal{E} \otimes A \mathcal{E}$ defines an isomorphism, which is left $Z(A)$ and right $A$ linear. Moreover, $\xi$ also defines an $Z(A)$-$A$ bimodule isomorphism from $\text{Hom}_A(\mathcal{E}, \mathcal{E} \otimes A)$ to $\mathcal{E} \otimes A \mathcal{E} \otimes A \mathcal{E}$. 

14
Proof: It is clear from the observation that each of the constituent maps in the definition of \( \tau \) and \( \xi \) is \( \mathcal{Z}(A) - A \)-linear isomorphism. □

It is easy to see the following result:

**Lemma 5.8** Let \( \sigma L \in \text{Hom}_A(\mathcal{E}, \mathcal{E} \otimes_A \mathcal{E}) \) be defined by \((\sigma L)(e) = \sigma(L(e)). Then \( \xi(\sigma L) = \sigma_{12}\xi(L). In particular, if \( L = \sigma L, \) then \( \xi(L) = \sigma_{12}\xi(L). \)

Now, by part (2) of Lemma 5.8, \( \mathcal{Z}(\mathcal{E}) \otimes_{\mathcal{Z}(A)} \mathcal{Z}(\mathcal{E}) \) is both left and right \( A \)-total in \( \mathcal{E} \otimes_A \mathcal{E}. \) Also, recall \( \pi_{g,0}, \) from \( \mathcal{E} \) to \( \mathcal{E} \) and the fact that it is left \( \mathcal{Z}(A) \) right \( A \)-linear. Further, \( \pi_{g,0}(\mathcal{Z}(\mathcal{E})) \) is left \( \mathcal{Z}(A) \) submodule which is right \( A \)-total, so by (2) of Lemma 5.8, \( \mathcal{Z}(\mathcal{E}) \otimes_{\mathcal{Z}(A)} \pi_{g,0}(\mathcal{Z}(\mathcal{E})) \) is right \( A \)-total in \( \mathcal{E} \otimes_A \mathcal{E}. \)

Moreover, we have

**Lemma 5.9** Let \( L \in \text{Hom}_A(\mathcal{E}, \mathcal{E} \otimes_A \mathcal{E}), \) \( T = \Psi_g(L), T' := T \circ \sigma, L' := \Theta_g(T'). \) Then

\[
\xi(L') = \sigma_{23}\xi(L).
\]

Proof: As the maps \( \Theta_g, \Psi_g, T \mapsto T', L \mapsto L' \) are both \( A \)-linear and by Lemma 5.8, \( \mathcal{Z}(\mathcal{E}) \otimes_{\mathcal{Z}(A)} \mathcal{Z}(\mathcal{E}) \) is both left and right \( A \)-total in \( \mathcal{E} \otimes_A \mathcal{E} \), it is enough to prove the statement of the lemma for \( L \) of the form \( \omega_1 \otimes_A \omega_2 \otimes_A V_g(\pi_{g,0}(\omega_3)) \) in \( \text{Hom}_A(\mathcal{E}, \mathcal{E} \otimes_A \mathcal{E}) \cong \mathcal{E} \otimes_A \mathcal{E} \otimes_A \mathcal{E}^*, \) where \( \omega_i \in \mathcal{Z}(\mathcal{E}) \) for all \( i. \) We claim \( L' = \omega_1 \otimes_A \pi_{g,0}(\omega_3) \otimes_A V_g(\omega_2)\), which will complete the proof. This claim is equivalent to the following:

\[
\Psi_g(\omega_1 \otimes_A \omega_2 \otimes_A V_g(\pi_{g,0}(\omega_3))) \circ \sigma = \Psi_g(\omega_1 \otimes_A \pi_{g,0}(\omega_3) \otimes_A V_g(\omega_2))
\]

It suffices to check this on the right \( A \)-total subset \( \mathcal{Z}(\mathcal{E}) \otimes_{\mathcal{Z}(A)} \mathcal{Z}(\mathcal{E}) \), which we will do now. Take \( \omega, \eta \in \mathcal{Z}(\mathcal{E}) \) and observe that, by definition of \( \sigma \) and \( \pi_{g,0}, \)

\[
\Psi_g(\omega_1 \otimes_A \omega_2 \otimes_A V_g(\pi_{g,0}(\omega_3)))(\sigma(\omega \otimes_A \eta)) = \omega_1 g(\omega_2 g(\pi_{g,0}(\omega_3) \otimes_A \eta) \otimes_A \omega) = \omega_1 g(\omega_2 g_0(\omega_3, \eta) \otimes_A \omega)
\]

\[
= \omega_1 g(\omega_2 \otimes_A \omega) g_0(\omega_3 \otimes_A \eta)
\]

\[
= \omega_1 g(\pi_{g,0}(\omega_3) \otimes_A \eta) g(\omega_2 \otimes_A \omega)
\]

\[
= \omega_1 g(g(\pi_{g,0}(\omega_3) \otimes_A \eta) g(\omega_2 \otimes_A \omega) \eta)
\]

\[
= \Psi_g(\omega_1 \otimes_A \pi_{g,0}(\omega_3) \otimes_A V_g(\omega_2)) (\omega \otimes_A \eta).
\]

Here we have used the fact that \( g_0(\omega_2 \otimes_A \omega) \) is in the center of the algebra \( A \) as \( \omega_2, \omega \in \mathcal{Z}(\mathcal{E}) \) and \( g_0 \) is a pseudo-Riemannian bi-metric □

**Lemma 5.10** For \( L \in \text{Hom}_A(\mathcal{E}, \mathcal{E} \otimes_A^{X_m} \mathcal{E}) \), we have

\[
\Phi_g(L) = \Psi_g(L) \circ (1 + \sigma).
\]

Proof: Indeed, as in the proof of Lemma 5.9, it is enough to check this for \( L \) of the form \( \omega_1 \otimes_A \omega_2 \otimes_A V_g(\pi_{g,0}(\omega_3)). \) Moreover, following the lines of computations of that proof, we have

\[
\Psi_g(\sigma \circ L)(\omega \otimes_A \eta)) = \omega_2 g(\omega_1 \otimes_A \eta) g_0(\omega_3 \otimes_A \omega),
\]

for all \( \omega, \eta \in \mathcal{Z}(\mathcal{E}). \) On the other hand, we have \((g \otimes_A \text{id})\sigma_{23}(L(\omega) \otimes_A \eta) = g(\omega_1 \otimes_A \eta) g_0(\omega_3 \otimes_A \omega) = \omega_2 g(\omega_1 \otimes_A \eta) g_0(\omega_3 \otimes_A \omega), \) as \( \omega_2 \) is in the center of \( \mathcal{E}. \) By the right linearity of the maps involved, this proves

\[
\Psi_g(\sigma \circ L) = (g \otimes_A \text{id}) \sigma_{23}(L \otimes_A \text{id}).
\]

Now the statement of the lemma follows immediately. □

We are now in a position to prove the main result.
Theorem 5.11 Under the Assumptions I - IV, or alternatively, Assumption I,II, III', IV, the map \( \Phi_g \) is an isomorphism of right \( \mathcal{A} \) modules, hence a unique Levi-Civita connection exists.

Proof: Combining Lemma 5.9 and Lemma 5.10 we see that the proof of the theorem is equivalent to proving that the map \( \xi(L) \Rightarrow (1 + \sigma_{23})\xi(L) \) gives an isomorphism when \( L \) is varying over \( \text{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\text{sym}} \mathcal{E}) \), i.e. \( \xi(L) \) is varying over \( (\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}) \otimes_{\mathcal{A}} \mathcal{E} \). But we have already seen (Lemma 5.6 and Assumption II) that it is an isomorphism of \( \mathcal{A} \)-modules.

For the proof under Assumption I,II, III', IV, it is easy to see that all the arguments for the proof of this theorem go through verbatim if we replace \( \mathcal{E} \) and \( \mathcal{Z}(\mathcal{A}) \) by \( \mathcal{E}' \) and \( \mathcal{A}' \) respectively. \( \square \)

In the next two sections, we will prove that there exists a unique Levi Civita connection for the space of one forms on a large class of Rieffel deformed manifolds and also for the “sub Riemannian metric” on the quantum Heisenberg manifold studied in [6].

6 Verification of assumptions for Connes-Landi-deformed spectral triples

In this section, we prove that the spectral submodule (see Definition 6.4) of the space of one forms of the Connes-Landi spectral triples on Connes-Dubois Violette-Rieffel-deformations of a large class of compact manifolds satisfy the Levi-Civita property. The examples include the cases when the manifold is free or ergodic. This covers a lot of interesting examples, including the cases when the manifold is a compact homogeneous space associated to some compact semisimple Lie group \( G \) and the \( T^n \)-action comes from a toral subgroup of \( G \).

In the first subsection, we prove some preparatory results on the fixed point algebra of a \( \ast \)-algebra under the action of a compact abelian Lie group. Here, we verify that Assumption I,II, III, IV hold for the spectral submodule \( \mathcal{E} \) as in Definition 6.4 provided the action of the group is isometric and free. In the next subsection, we describe the Rieffel deformation of unital algebras and their modules under the action of \( T^n \). In the final subsection, we prove that under our assumptions, the deformed module \( \mathcal{E}_\theta \) on the Rieffel deformed manifold satisfies the conditions Assumptions I - IV so that by virtue of Theorem 5.11 there always exists a unique Levi Civita connection for every pseudo-Riemannian metric on \( \mathcal{E}_\theta \).

We recall that for any action of \( T^n \) on a module \( \mathcal{E} \) (or an algebra \( \mathcal{C} \)), the spectral subspace corresponding to a character \( \mu_m \equiv (m_1, \ldots, m_n) \in \mathbb{T}^n \cong \mathbb{Z}^n \), denoted by \( \mathcal{E}_{\mu_m} \) (respectively \( \mathcal{C}_{\mu_m} \)), consists of all \( \xi \) such that \( \sigma_t(\xi) = \chi_m(t)\xi \) for all \( t = (t_1, \ldots, t_n) \in \mathbb{T}^n \), where \( \chi_m(t) := t_1^{m_1} \cdots t_n^{m_n} \). It is easily seen that, \( \mathcal{C}_{\mu_m} \mathcal{C}_{\mu_n} \subseteq \mathcal{C}_{\mu_{m+n}} \) and also we have \( \mathcal{E}_{\mu_m} \mathcal{C}_{\mu_n} \subseteq \mathcal{E}_{\mu_{m+n}} \), for any equivariant \( \mathcal{C} \) bimodule \( \mathcal{E} \). The linear span of all the \( \mathcal{C}_{\mu_m} \mathcal{C}_{\mu_n} \) comprise the so-called ‘spectral subalgebra’ for the action.

Let \( G \) be a group. Let us recall that a spectral triple \((\mathcal{A}, \mathcal{H}, D)\) is called \( G \)-equivariant if there exists a unitary representation \( U \) of \( G \) on \( \mathcal{H} \) such that \( U_g D = DU_g \). Moreover, we recall the following well known fact (see [8] for the details).

Proposition 6.1 Suppose that \( M \) is a compact Riemannian manifold with an isometric action of the \( n \)-torus \( T^n \) on \( M \). Consider the spectral triple \((C^\infty(M), \mathcal{H}, d + d^*)\) where \( \mathcal{H} \) is the Hilbert space of forms and \( d \) is the de-rham differential on \( \mathcal{H} \). The \( T^n \) action on smooth forms extends to a unitary representation on \( \mathcal{H} \) and the spectral triple is equivariant w.r.t this representation of \( T^n \). In particular, if \( U \) denotes the representation of \( T^n \) on \( \mathcal{H} \), \( \beta \) the action of \( T^n \) on \( C^\infty(M) \) and \( \delta(\cdot) = [d + d^* , \cdot] \), then

\[
U_t(f\delta(g)) = \alpha_t(f)U_t(\delta(g)) = \alpha_t(f)\delta(\alpha_t(g)) \quad \forall \ t \in T^n.
\]

In this set up, it is easy to see the following result:

Lemma 6.2 If \( \mathcal{A} \) is a subalgebra of \( C^\infty(M) \) kept invariant by the action of a compact group \( G \) acting by isometries on \( M \), then the map \( m : \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \rightarrow \Omega^2(\mathcal{A}) \) is \( G \)-equivariant.
As an immediate corollary, we have

**Corollary 6.3** With the notations of Lemma 6.3, Ker(m) is invariant under the action of G. Moreover, if \( \Omega^1(A) \otimes A \Omega^1(A) = \text{Ker}(m) \oplus F \) gives the decomposition as in Proposition 4.1, then F is also kept invariant by G.

**Proof:** The G-invariance of Ker(m) follows from the G-invariance of m. Moreover, from Proposition 4.1, \( F = \text{Ker}(1 - \sigma^{\text{can}}) \). Since \( \sigma^{\text{can}} \) is G-equivariant, F is G-invariant. \( \square \)

Throughout this section, we will follow the notations introduced in the following definition.

**Definition 6.4** Let M be a compact Riemannian manifold as in Proposition 6.7. Then \( \mathcal{C} \) will denote the \( \mathbb{T}^n \)-equivariant spectral submodule of \( \text{Span}(\mathcal{E}_k : k \in \mathbb{Z}) \) where \( \mathcal{E}_k \) denotes the k-th spectral subspace of \( \Omega^1(M) \). Similarly, \( \mathcal{C} \) will denote the spectral subalgebra \( \text{Span}(\mathcal{C}_k : k \in \mathbb{Z}) \) where \( \mathcal{C}_k \) is the k-th spectral subspace of \( C^\infty(M) \). Finally, \( \mathcal{E}' \) (respectively \( \mathcal{C}' \)) will denote the fixed point submodule of (respectively, subalgebra) \( \mathcal{E}_0 \) (respectively \( \mathcal{C}_0 \)).

**Remark 6.5** Since the representation of \( \mathbb{T}^n \), say U, commutes with \( d + d^* \), it is easy to see that \( U_t(\mathcal{E}) \subseteq \mathcal{E} \) for all \( t \in \mathbb{T}^n \). Moreover, it is easy to see that the space of one forms for the spectral triple \((\mathcal{C}, \mathcal{H}, d + d^*)\) is precisely \( \mathcal{E} \).

6.1 Some results on the fixed point algebra

We have already seen that for a compact Riemannian manifold \( M \), Assumption I - IV are easily satisfied since \( A = \mathcal{Z}(A) = C^\infty(M) \) and \( \mathcal{E} = \mathcal{Z}(\mathcal{E}) = \Omega^1(M) \). However, in order to prove our desired result for Rieffel deformation of \( C(M) \) (if it makes sense), we need to prove something more. Indeed, the aim of this subsection is to prove Theorem 6.7 (for a generalized version, see Remark 6.13) which says that if the \( \mathbb{T}^n \) action on \( M \) is free, then Assumption I - IV hold for the module \( \mathcal{E} \). To achieve this, we first prove that Assumption III' as in Subsection 4.1 to hold with \( \mathcal{E}' \) and \( \mathcal{C}' \) as in Definition 6.3. Then we apply Proposition 4.1.7 to derive Assumption III for the module \( \mathcal{E} \).

Thus, we will make use of the notation \( u^E_{\mathcal{E}} : \mathcal{E}_0 \otimes \mathcal{C}_m \rightarrow \mathcal{E} \) introduced in Subsection 4.1. The key ingredients for proving Theorem 6.1.2 are Lemma 6.6 and Theorem 6.7 which we state and prove now.

**Lemma 6.6** If for each \( m \in \mathbb{Z}^n \), we can find \( a_1, \ldots, a_k \in \mathcal{C}_m \) and \( b_1, \ldots, b_k \in \mathcal{C}_{-m} \) (k depends on \( m \) ) such that \( \sum_i b_i a_i = 1 \), then the map \( u^E_{\mathcal{E}} \) is an isomorphism.

**Proof:** We need to prove that under the above assumption, the map \( u^E_{\mathcal{E}} \) has a right \( \mathcal{A} \)-linear inverse. However, since \( u^E_{\mathcal{E}} \) is right \( \mathcal{C} \)-linear to start with, it suffices to prove that \( u^E_{\mathcal{E}} \) defines an isomorphism of vector spaces. Hence, it is sufficient to prove that \( \forall \mathcal{m}_m \), the restriction \( p^E_{\mathcal{m}} \) of \( u^E_{\mathcal{E}} \) to \( \mathcal{E}_0 \otimes \mathcal{C}_m \) is a vector space isomorphism onto its image \( \mathcal{E}_m \). Then the map

\[
q^E_{\mathcal{m}} : \mathcal{E}_m \rightarrow \mathcal{E}_0 \otimes \mathcal{C}_m \text{ defined by } q^E_{\mathcal{m}}(e) := \sum_i e b_i \otimes \mathcal{C}_m a_i
\]

satisfies \( p^E_{\mathcal{m}} \circ q^E_{\mathcal{m}} = \text{id} \). On the other hand, as \( b_i a_i \in \mathcal{C}_m \) if \( a \in \mathcal{C}_m \), we have

\[
q^E_{\mathcal{m}} \circ p^E_{\mathcal{m}}(e \otimes \mathcal{C}_m a) = \sum_i e a b_i \otimes \mathcal{C}_m a_i = e \otimes \mathcal{C}_m a_i \sum b_i a_i = e \otimes \mathcal{C}_m a.
\]

This finishes the proof of the lemma. \( \square \)

**Theorem 6.7** Let \( M \) be a compact connected Riemannian manifold and let \( \alpha \) be a smooth action of a compact abelian Lie group \( G \) on \( M \). Then there exists a dense open submanifold \( M_0 \) on \( M \) on which \( \alpha \) restricts to a free action of \( G \) and moreover, the canonical map \( M_0 \rightarrow M_0/G \) is a principal \( G/H \)-bundle, where \( H \) is the maximal isotropy subgroup of \( G \). If the action \( \alpha \) on \( M \) is free to start with, we can take \( M_0 \) to be \( M \) itself.

17
**Proof:** Let $H$ be the subgroup given by Theorem 3.1 of Chapter IV of [5] such that $G/H$ is the principal orbit type. Thus, all isotropy subgroups of $G$ are contained in $H$. Take $M_0 = M(H)$ in the notation of [5] which is the set of all points with orbit types $G/H$, i.e. the isotropy subgroup at the point being conjugate to $H$. Note that as $G$ is abelian, any two subgroups which are conjugate are actually equal, so the isotropy groups at each point of $M_0$ is $H$. Moreover, as $N(H) = G$, we conclude by Theorem 3.3, Chapter IV of [5] that the canonical projection map $M_0 \to M_0/H$ makes it a fiber bundle with the fibre and structure group both being $G/H$, which means it is a $G/H$ principal bundle. It also follows from Theorem 3.1, Chapter IV of [5] that $M_0$ is open and dense in $M$. If the action $\alpha$ is free, then clearly, $H$ is the trivial group and so we can take $M_0$ to be $M$. \(\square\)

Let us use the symbol $\chi_m$ for the character $m \in \hat{K}$ for the sake of clarity. We shall identify the bimodule $E_m$ with the bimodule of sections of a certain vector bundle over $M$.

**Lemma 6.8** Let $M$ be a smooth Riemannian manifold equipped with a smooth and free action $\alpha$ of a compact abelian Lie group $K$ such that $\pi: M \to M/K$ is a principal $K$-bundle. Let $M \times \chi_m \mathbb{C} \to M/K$ denotes the associated vector bundle $\{M \to M/K\}$ corresponding to the character $\chi_m$.

Then the space of all smooth functions $f$ on $M$ satisfying $f(\alpha_t(x)) = \chi_m(t)f(x)$ is in one to one correspondence with the set of all smooth sections of the vector bundle $M \times \chi_m \mathbb{C} \to M/K$.

**Proof:** The elements of the total space of the associated vector bundle $M \times \chi_m \mathbb{C}$ is given by equivalence class $[y, \lambda]$ of $(y, \lambda) \in M \times \mathbb{C}$ such that $(y, \lambda) \sim (y, t, \pi_m(t^{-1})\lambda)$ for all $t \in K$. Now, for $f \in E_m$, we can define a section of the above vector bundle $s_f$ by

$$s_f([x]) = [x, f(x)],$$

where $[x]$ denotes the class of the point $x$ in $M/K$. We need to show that this is well defined. But for any $t \in K$, $s_f([xt]) = [xt, f(x)] = [xt, \chi_m(t)f(x)] = [xt, \chi_m(t^{-1})f(x)] = [x, f(x)]$. This proves the well definedness. Similarly, given a section $s$ of the above vector bundle we can define a function $f_s$ on $M$ by $f_s(x) = \lambda_x$ where $\lambda_x \in \mathbb{C}$ is such that $s([x]) = [x, \lambda_x]$. Clearly, $\lambda_x$ is uniquely determined, because the $K$ action is free. Moreover, $[x, \lambda_x] = [xt, \chi_m(t^{-1})\lambda_x]$ implies $\lambda_{xt} = \chi_m(t)\lambda_x$, i.e. $f_s \in E_m$.

Finally, it is easy to verify that the maps $f \mapsto s_f$ and $s \mapsto f_s$ are inverses of one another, completing the proof. \(\square\)

**Lemma 6.9** For a complex smooth Hermitian vector bundle over a compact manifold $M$ there are finitely many smooth sections $s_i$’s such that $\sum <s_i, s_i> = 1$ where $<\cdot, \cdot>$ denotes the $C^\infty(M)$- valued inner product coming from the Hermitian structure.

**Proof:** Corresponding to a finite open cover $\{U_i, i = 1, \ldots, l\}$ choose finitely many smooth sections $\gamma_i$ which are non zero on $U_i$. Then choosing a smooth partition of unity $\psi_i, i = 1, \ldots, l$, we can construct $t_i = \psi_i\gamma_i$’s so that $t = \sum <t_i, t_i>$ is nowhere zero. The sections $s_i = \frac{1}{t_i}$ satisfy the conditions of the lemma. \(\square\)

This gives us the following:

**Proposition 6.10** Suppose $M$ is a compact Riemannian manifold equipped with a free and isometric action of a compact abelian Lie group $K$. Then the map $u^*\xi$ is an isomorphism.

**Proof:** Without loss of generality, we can assume $M$ to be connected. In general, if $M$ has $k$ connected components $M_1, M_2, \ldots, M_k$, the module $\mathcal{E}$ decomposes as $\mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_k$, where $\mathcal{E}_i$ is the linear span of spectral subspaces of $\Omega^1(M_i)$, and it is suffices to prove that for all $i$, $u^*\mathcal{E}_i$ is an isomorphism from $(E_i)_{\mathbb{R}} \otimes [C_0, C]$ onto $\mathcal{E}_i$. We use Theorem 6.1. Since the action of $K$ on $M$ is free, theorem 6.1 implies that $M \to M/K$ is a principal $K$-bundle. Therefore, for the sections $s_i$ as in Lemma 6.9 we have functions $f_{s_i}$ in $C_m$ by Lemma 6.8. By the definition of $f_{s_i}$ and the relation $\sum <s_i, s_i> = 1$, it follows that

$$\sum_i f_{s_i}^2 = 1.$$
Since $f_{s_i}$ belongs to $C_{m_i}$, the function $f_{s_i}$ belongs to $C_{-m_i}$. Thus, we can apply Lemma 6.6 to deduce the conclusion of the theorem. □

**Lemma 6.11** Suppose that the $T \equiv T^n$ action on $M$ is free. Then $E'$ is a projective and finitely generated module over $C'$.

**Proof**: For a module $F$ equipped with an action of $T^n$, let us denote the $T^n$ invariant submodule of $F$ by the symbol $F^{T^n}$. We recall that by Theorem 6.7, $M/T$ is a smooth compact manifold and $M$ is a principal $T$-bundle over $M/T$. Let $\pi$ denote the projection map from $M$ onto $M/T$. Given any point in $M$, we can find a $T$-invariant open neighborhood $U$ which is $T$-equivariantly diffeomorphic with $U/T \times T$. Moreover, we can choose $U$ in such a way that $U/T$ is the domain of a local coordinate chart for the manifold $M/T$, say $U = \pi^{-1}(V)$, where $V$ is the domain of some local chart for $M/T$. This gives the following isomorphism:

$$\Omega^1(U)^T \cong \Omega^1(U/T) \otimes_C \Omega^1(T)^T \cong \Omega^1(U/T) \otimes_C \mathcal{L},$$

$\mathcal{L}$ being the complexified Lie algebra of $T$ which is nothing but $C^n$ in this case. Thus, the conclusion of Theorem 6.12 also holds in this case.

**Theorem 6.12** In the notations of Definition 6.4, the module $E$ satisfies Assumption I - IV.

**Proof**: Assumption I, II, and IV clearly hold for the module $E$. By a combination of Proposition 6.10 and Lemma 6.11 and Proposition 4.17, we deduce that Assumption III holds for $E$. □

**Remark 6.13** It is not difficult to see that Proposition 6.10 and Lemma 6.11 also holds if we assume that the action is foliated, i.e., the isotropy subgroup at any point on the manifold is equal to a subgroup $K' \subseteq T^n$. Thus, the conclusion of Theorem 6.12 also holds in this case.

### 6.2 Some generalities on Rieffel-deformation

Our main reference for Rieffel deformation of a $C^*$ algebra endowed with a strongly continuous action by $T^n$ is [30]. However, we will also need to use equivalent descriptions of this deformation given in [8, 10, 26].

Let $C^\infty(T^n)$ be the Frechet algebra corresponding to the noncommutative $n$-torus, where the deformation parameter is given by a real, $n \times n$ skew symmetric matrix $\theta = ((\theta_{kl}))$. We denote the canonical $T^n$-action on this algebra by $\tau$ and let $A(T^n_\theta)$ denote the canonical ‘polynomial subalgebra’, i.e., the dense unital *-subalgebra of $C^\infty(T^n_\theta)$) generated by polynomials in $U_i$ and their inverses, where $U_i, i = 1, \ldots, n$ are the canonical unitary ‘coordinates’, i.e., elements satisfying

$$U_k U_i = \exp(2\pi i \theta_{ki}) U_i U_k. \quad (6)$$

**Definition 6.14** For any unital algebra $C$ equipped with a $T^n$-action $\beta$, the corresponding deformation denoted by $C_\theta$ is defined to be the fixed point subalgebra $(C \otimes_C A(T^n_\theta))^{\tau \times \tau^{-1}}$ of the algebra $C \otimes_C A(T^n_\theta)$ with respect to the action $\beta \times \tau^{-1}$.

Let $U^\theta_{m, m}$ denotes the canonical generators of $(A(T^n_\theta))_{\mathbb{m}}$. Then an arbitrary element of $C_\theta$ is a finite sum of the form $\sum_{m} a_{m} \otimes_C U^\theta_{m, m}$ where $a_{m}$ belongs to $(C)_{\mathbb{m}}$. Here and henceforth, $\sum_{m}$ will always denote a finite sum.

Moreover, the group $T^n$ acts on $C_\theta$ by the following formula:

$$\beta^\theta_{m}(a_{m} \otimes_C U^\theta_{m, m}) = \beta_{i}(a_{m}) \otimes_C U^\theta_{m, m}. \quad (7)$$

In particular, if $(C_\theta)_{k}$ denotes the $k$-th spectral subspace for the action of $T^n$ on $C_\theta$, then elements of $(C_\theta)_{k}$ are spanned by elements of the form $a_k \otimes_C U^\theta_{k, k}$, where $a_k$ belongs to $C_k$.

Similarly, one can deform equivariant bimodules as follows [30]:
Definition 6.15 Suppose $\mathcal{C}, \mathcal{D}$ are unital algebras equipped with actions $\sigma^1, \sigma^2$ of $\mathbb{T}^n$. Let $\mathcal{E}$ be a $\mathcal{C} - \mathcal{D}$ $\mathbb{T}^n$ equivariant bimodule, i.e. $\mathcal{E}$ has a $\mathbb{T}^n$ action $\sigma^i$ such that for $a \in \mathcal{C}$, $b \in \mathcal{D}$, $e \in \mathcal{E}$, $\sigma^i(a.e.b) = \sigma^i(a)\sigma^i(e)\sigma^i(b)$. Then the $\theta$-deformation of $\mathcal{E}$ is a $\mathcal{C}_\theta - \mathcal{D}_\theta$ bimodule $\mathcal{E}_\theta$ defined to be the fixed point submodule of $\mathcal{E} \otimes_{\mathbb{C}} \mathcal{A}(\mathbb{T}^n_\theta)$. This has a natural bimodule structure.

Moreover, the module $\mathcal{E}_\theta$ also admits an action of $\mathbb{T}^n$ exactly as above.

We have the following easy consequence of the definitions above:

Lemma 6.16 Let $\mathcal{C}$ be an algebra equipped with $\mathbb{T}^n$-action and $\mathcal{E}, \mathcal{F}$ be equivariant $\mathcal{C}$-bimodules, in the sense discussed above. Let $L : \mathcal{E} \to \mathcal{F}$ be an $\mathbb{T}^n$-equivariant (i.e. commutes with the respective toral actions) $\mathcal{C}$ bimodule map. The $L$ induces a $\mathcal{C}_\theta$ bimodule map $L_\theta : \mathcal{E}_\theta \to \mathcal{F}_\theta$ which is the restriction of the map $(L \otimes_{\mathbb{C}} \text{id}) : \mathcal{E} \otimes_{\mathbb{C}} \mathcal{A}(\mathbb{T}^n_\theta) \to \mathcal{F} \otimes_{\mathbb{C}} \mathcal{A}(\mathbb{T}^n_\theta)$ to the fixed point submodule. If $L$ is a $\mathcal{C}$ bimodule isomorphism, then $L_\theta$ will be a $\mathcal{C}_\theta$-bimodule isomorphism. If $\mathcal{E}$ and $\mathcal{F}$ are algebras in particular, then $L_\theta$ is an algebra homomorphism.

Now suppose that $\text{Ker}(L)$ is complemented as a $\mathcal{C}$ bimodule in $\mathcal{E}$, i.e. there exists a bimodule $M \subseteq \mathcal{E}$ such that $\mathcal{E} \cong \text{Ker}(L) \oplus M$. Then

1. $\text{Ker}(L)$ is invariant under the action of $\mathbb{T}^n$.
2. $M \cong \text{Im}(L)$.
3. If $M$ is $\mathbb{T}^n$-invariant, then $\mathcal{E}_\theta = \text{Ker}(L_\theta) \oplus M_\theta$ and $M_\theta \cong \text{Im}(L_\theta)$.
4. If $\mathcal{F} = \mathcal{E}$ and $L$ is an idempotent, then $L_\theta$ is also idempotent.

The following lemma will also be of use to us.

Lemma 6.17 Let $\mathcal{C}$ be an algebra equipped with $\mathbb{T}^n$-action and $\mathcal{E}, \mathcal{F}$ be equivariant $\mathcal{C}$-bimodules, in the sense discussed above. Then $(\mathcal{E} \otimes_{\mathbb{C}} \mathcal{F})_\theta \cong \mathcal{E}_\theta \otimes_{\mathbb{C}_\theta} \mathcal{F}_\theta$ as $\mathcal{C}_\theta$ bimodules.

An alternative description of this deformation procedure (30, 26) is as follows. Suppose the algebra under consideration is equipped with the action of $\mathbb{T}^n$ so that we have the spectral subspaces as defined in the beginning of this section. For a skew symmetric $n \times n$ matrix $\theta$, let $\chi_\theta$ denote the bicharacter on $\mathbb{Z}^n \times \mathbb{Z}^n$ defined by

$$\chi_\theta(x_1, x_2) = e^{\pi i < x_1, \theta x_2 >}.$$  \hfill (8)

The deformed algebra $\mathcal{C}_\theta$ is $\mathcal{C}$ as a vector space but the multiplication rule is deformed as follows: If $a = \sum_a a_a, b = \sum_b b_b$ in $\mathcal{C}$ where $a_a \in \mathcal{C}_{\bar{a}}, b_b \in \mathcal{C}_{\bar{b}}$, then the deformed multiplication $\times_\theta$ is defined by

$$a \times_\theta b = \sum_{\bar{a}, \bar{b}} \chi_\theta(\bar{a}, \bar{b}) a_a b_b.$$  \hfill (9)

The deformation of the module $\mathcal{E}_\theta$ is defined in a similar way.

Now we recall the Connes-Landi deformation (10) of a spectral triple and its associated space of forms. We will work in the set up of Proposition 6.1. In particular, $\mathcal{C}$ and $\mathcal{E}$ will denote the spectral subalgebra and the spectral submodule respectively as in Definition 6.4. The spectral triple with which we are concerned is $(\mathcal{C}, \mathcal{H}, d + d^*)$. By Remark 6.5, the action $\beta$ keeps $\mathcal{E}$ invariant. Then we have the following:

Theorem 6.18 Viewing the algebra $\mathcal{C}_\theta$ as in (9), we define $\pi^\theta : \mathcal{E} \to \mathcal{B}(\mathcal{H})$ by

$$\pi^\theta(e)(h) = \sum_{m, n \in \mathbb{Z}^n} \chi_\theta(m, n) e_m(h_n),$$

where $e = \sum_m e_m, a = \sum_a a_a$.

Then $\pi^\theta(\mathcal{E}) \cong \mathcal{E}_\theta.$  \hfill (10)
We note that an analogous formula defines a representation of \( C_\theta \) on \( \mathcal{H} \), to be denoted by \( \pi_\theta \) again. Moreover, \((C_\theta, \mathcal{H}, d + d^*)\) defines a spectral triple. Moreover, \( \mathcal{E}_\theta \) and \( \tilde{\mathcal{O}}^2(C_\theta) \) are canonically isomorphic as \( C_\theta\)-bimodules with \((\tilde{\mathcal{O}}^1(C))_\theta \) and \((\tilde{\mathcal{O}}^2(C))_\theta \) respectively. If \( \delta : C \to \mathcal{E} \) denotes the map which sends \( a \) to \([d + d^*, a] \), then we have a deformed map \( \delta_\theta \) from \( C_\theta \) to \( \mathcal{E}_\theta \).

**Proof:** The representation of \( C_\theta \) comes from Proposition 10 by taking \( \mathcal{E} = C \). For the proof that \((C_\theta, \mathcal{H}, d + d^*)\) is a spectral triple, we refer to [5]. For the isomorphism \( \pi^\theta(\mathcal{E}) \cong \mathcal{E}_\theta \), we refer to Proposition 2.8 of [20].

Next, we observe that by virtue of Remark 6.19, \( \mathcal{E} = \tilde{\mathcal{O}}^1(C) \) and \( \tilde{\mathcal{O}}^2(C) \) can be deformed. Then the isomorphism follows by using the identification of Proposition 10. The last assertion follows by observing that the map \( \delta \) is \( \mathbb{T}^n \)-equivariant. \( \square \)

Henceforth we will make the identifications \( \mathcal{E}_\theta \cong (\tilde{\mathcal{O}}^1(C))_\theta \), \( \tilde{\mathcal{O}}^2(C_\theta) \cong (\tilde{\mathcal{O}}^2(C))_\theta \) without explicitly mentioning.

### 6.3 Levi-Civita property of the one forms on the Connes-Landi deformed spectral triple

We will continue to use the notations introduced in Definition 6.4. The goal of this subsection is to apply the results deduced in the last two subsections for proving Theorem 6.3. We first claim that \( \mathcal{E}_\theta = \tilde{\mathcal{O}}^1(C_\theta) \) always satisfies Assumptions I, II, IV. After that, we will verify Assumption III under the additional assumption that the action is free.

**Lemma 6.19** Ker(\( m_\theta \)) is complemented in \( \mathcal{E}_\theta \otimes_{C_\theta} \mathcal{E}_\theta \).

**Proof:** This follows by applying Lemma 6.16 and Corollary 6.3 applied to the \( \mathbb{T}^n \)-equivariant map \( m_\theta \). \( \square \)

Now we would like to verify Assumption IV. For that, we first recall the following definition:

**Definition 6.20** Let \( \mathcal{F} \) and \( \mathcal{F}' \) be two \( \mathcal{A} \) bimodules admitting actions by a group \( \mathbb{T}^n \) and denoted by \( \beta \) and \( \beta' \) respectively. Then \( \text{Hom}_\mathcal{A}(\mathcal{F}, \mathcal{F}') \) admits a natural \( \mathbb{T}^n \) action \( \gamma \) defined by

\[
(\gamma_z . T)(e) = \beta_z'(T(\beta_z^{-1}. e)).
\]

Here, \( z, T \) and \( e \) belong to \( \mathbb{T}^n \), \( \text{Hom}_\mathcal{A}(\mathcal{F}, \mathcal{F}) \) and \( \mathcal{E} \) respectively.

**Lemma 6.21** In the set up of Definition 6.20, assume furthermore that \( \mathcal{A} \) admits an action \( \alpha \) of \( \mathbb{T}^n \) and \( \beta, \beta' \) are both \( \alpha \)-equivariant. Then we have

\[
\gamma_z(Ta)(\omega) = (\gamma_z(T)(\alpha_z(a)))(\omega) \quad \text{and} \quad \gamma_z(aT)(\omega) = \alpha_z(a)\gamma_z(T)(\omega)
\]

for \( a \) in \( \mathcal{A} \) and \( \omega \) in \( \mathcal{E} \).

**Proof:** We compute

\[
\gamma_z(Ta)(\omega) = \beta'_z((Ta)(\beta_z^{-1}(\omega))) = \beta'_z(T(a\beta_z^{-1}(\omega))) = \beta'_z(T(\beta_z^{-1}((\alpha_z(a))\omega))) = \gamma_z(T)((\alpha_z(a))\omega) = (\gamma_z(T)(\alpha_z(a)))(\omega).
\]

The other equality follows similarly. \( \square \)

Let \( U \) denote the representation of \( \mathbb{T}^n \) on the Hilbert space of forms \( \mathcal{H} \) and \( \beta \) denote the action of \( \mathbb{T}^n \) on \( C^\infty(M) \). By the usual averaging trick, we choose and fix a \( \mathbb{T}^n \)-invariant Riemannian metric \( g \) on \( M \), i.e., for all \( \omega, \eta \) in \( \mathcal{E} = \tilde{\mathcal{O}}^1(C) \), we have

\[
g(U_z(\omega) \otimes c U_z(\eta)) = \beta_z(g(\omega \otimes c \eta)). \quad (11)
\]
By using the $\mathbb{T}^n$-invariance of $g$, it is easy to see that the map $V_g : E \to E^*$ is $\mathbb{T}^n$-equivariant. Thus, we have an $\mathcal{C}_\theta$-bimodule isomorphism $(V_g)_\theta$ from $E_\theta$ to $(E^*_\theta)$.

We view $g$ as an element of $\text{Hom}_\mathbb{C}(\mathcal{E} \otimes \mathcal{E}, \mathcal{C})$. We can equip $\mathcal{E} \otimes \mathcal{E}$ with the diagonal action of $\mathbb{T}^n$. Therefore, by Definition 6.20, we have an action of $\mathbb{T}^n$ on $\text{Hom}_\mathbb{C}(\mathcal{E} \otimes \mathcal{E}, \mathcal{C})$. Since by (11) $g$ is equivariant, we have a deformed map $g_\theta \in \text{Hom}_\mathbb{C}(\mathcal{E} \otimes \mathcal{E}, \mathcal{E}_\theta)$, where we have used the fact that an arbitrary element of $E$ is defined to be the $\mathbb{T}^n$-equivariant. Thus, we have

**Proposition 6.22** $g_\theta$ is an element of $\text{Hom}_\mathbb{C}(\mathcal{E}_\theta \otimes \mathcal{E}_\theta, \mathcal{E}_\theta)$ defined by

$$g_\theta((\sum_k e_k \otimes C U_{-k}^\theta) \otimes C_\theta (\sum_l e_l \otimes U_{-l}^\theta)) = \sum_k g(e_k \otimes A e_l) U_{-k}^\theta U_{-l}^\theta$$

**Proof:** The formula for $g_\theta$ follows from the definition of $g_\theta$ as in Lemma 6.10. □

The map $g_\theta$ is the candidate for the pseudo-Riemannian metric on $E_\theta = \mathcal{E}_\theta^\dagger(\mathcal{C}_\theta)$ required by Assumption IV. We prove this in a series of steps.

To begin with, we observe the following:

**Lemma 6.23** The map $u^{\mathcal{E}^*} : (\mathcal{E}^*)' \otimes \mathcal{C} \to \mathcal{E}^*$ is an isomorphism. Moreover, both the maps $u^\mathcal{E}$ and $u^{\mathcal{E}^*}$ are both $\mathbb{T}^n$-equivariant.

**Proof:** $(\mathcal{E}^*)'$ is the fixed point set of the action of $\mathbb{T}^n$ on $\mathcal{E}^*$. Thus, the isomorphism of $u^{\mathcal{E}^*}$ follows from Lemma 6.23. The second statement follows by simple computations. □

**Proposition 6.24** $(\mathcal{E}^*_\theta) \cong (\mathcal{E}_\theta)^*$ as $\mathcal{C}_\theta - \mathcal{C}_\theta$ bimodules.

**Proof:** Let $e^*_k$ denote an element belonging to the spectral subspace of $\mathcal{E}^*$ corresponding to $k$ in $\mathbb{T}^n$. Then an arbitrary element of $\mathcal{E}^*_\theta$ is a finite sum of the form $\sum_k e^*_k \otimes C U_{-k}^\theta$.

Then we define a map $T^\mathcal{E}_\theta : (\mathcal{E}^*_\theta) \to (\mathcal{E}^*_\theta)^*$ by

$$T^\mathcal{E}_\theta((\sum_k e^*_k \otimes C U_{-k}^\theta)) = \sum_k e^*_k U_{-k}^\theta U_{-k}^\theta,$$

where we have used the fact that an arbitrary element of $E_\theta$ is a finite sum of the form $\sum_k e_k \otimes C U_{-k}^\theta$.

Now, since $e^*_k \in (\mathcal{E}^*_\theta)$ and $e_k \in \mathcal{E}^*$ it follows that $e^*_k(e_l) \in \mathcal{A}_{k+l}$, i.e., the $(k+l)$-th spectral subspace of $\mathcal{C}$. Thus, by definition, $\sum_k e^*_k(e_l) \otimes C U_{-k}^\theta U_{-l}^\theta \in \mathcal{C}_\theta$ implying that $T^\mathcal{E}_\theta((\sum_k e^*_k \otimes C U_{-k}^\theta)) \in (\mathcal{E}_\theta)^*$.

We claim that the map $T^\mathcal{E}$ is equivariant w.r.t the $\mathbb{T}^n$-action on $(\mathcal{E}^*)_\theta$ and $(\mathcal{E}_\theta)^*$, i.e., if $\gamma$ and $\gamma'$ denote the actions on $(\mathcal{E}^*)_\theta$ and $(\mathcal{E}_\theta)^*$ respectively, then

$$T^\mathcal{E}_\theta((\gamma z)((\sum_k e^*_k \otimes C U_{-k}^\theta))) = \gamma'_z(T^\mathcal{E}_\theta((\sum_k e^*_k \otimes C U_{-k}^\theta))).$$

(13)
Indeed, if \( v \) and \( u \) denote the actions of \( \mathbb{T}^n \) on \( E^* \) and \( E \) respectively, then
\[
T^E_\theta(\sum_k e_k^* \otimes C U^\theta_{-k}) (e_l^* \otimes C U^\theta_{-l}) = T^E_\theta(\sum_k v_k(e_k^* \otimes C U^\theta_{-k}) (e_l^* \otimes C U^\theta_{-l})
\]
\[
= \sum_k v_k(e_k^*) (e_l^*) U^\theta_{-k} U^\theta_{-l}
\]
\[
= \sum_k \alpha_k(e_k^* (u_{-k} e_l^*)) \otimes C U^\theta_{-k} U^\theta_{-l}
\]
\[
= \alpha_k(e_k^* (u_{-k} e_l^*)) \otimes C U^\theta_{-k}
\]
\[
= \alpha_k(T^E_\theta(e_k^* \otimes C U^\theta_{-k})(u_{-k} e_l^* \otimes C U^\theta_{-l}))
\]
\[
= \gamma_k(T^E_\theta(\sum_k e_k^* \otimes C U^\theta_{-k}) (e_l^* \otimes C U^\theta_{-l})).
\]

This proves (13).

Thus, we have a well defined equivariant morphism
\[
T^E_\theta : (E^*_\theta)_{-\theta} \to ((E\theta)_{-\theta})^* \cong E^*,
\]
and subsequently, a morphism
\[(T^E_\theta)_\theta : (E^*_\theta)_{\theta} \cong (E\theta)^*_\theta \to E^*_{\theta}._\theta.
\]

Finally, it is easy to check that the maps \( T^E_\theta \) and \( (T^E_\theta)_\theta \) are inverses of one another. This finishes the proof.

\[\Box\]

**Proposition 6.25** For all \( e \) in \( \mathcal{Z}(E_\theta) \),
\[
(V_g)_\theta(e) = V_{g\theta}(e)
\]
and hence the map \( V_{g\theta} : E_\theta \to (E^*_\theta)^* \) is an isomorphism.

**Proof:** By the equivariance of \( g \), it easily follows that the map \( V_g \) is equivariant and hence the map \( (V_g)_\theta \) is an element of \( \text{Hom}_{A_{\mathbb{T}}} (E_\theta, (E^*)_\theta) \). By virtue of Proposition 6.24, we can view \( (V_g)_\theta \) as an element of \( \text{Hom}_{C_{\theta}} (E_\theta, (E^*_\theta)^*) \). Thus, (14) makes sense. Moreover, since \( V_g \) is an isomorphism from \( E \) to \( E^* \), Lemma 6.16 implies that \( (V_g)_\theta \) is an isomorphism from \( E_\theta \) to \( (E^*_\theta)^* \). Thus, the isomorphism of \( V_{g\theta} \) follows from (14).

Viewing \( E_\theta \) as the fixed point submodule \( (E \otimes C A(T^\theta)_{\mathbb{T}})^{\theta}_{\mathbb{T}} \) and using the isomorphism of Proposition 6.24, the action of \( (V_g)_\theta \) as an element of \( (E^*_\theta)^* \) is defined by
\[
(V_g)_\theta(\sum_k e_k^* \otimes C U^\theta_{-k}) (\sum_l e_l^* \otimes C U^\theta_{-l}) = \sum_{i,k} V_{g\theta}(e_k^*) (e_l^*) \otimes C U^\theta_{-k} U^\theta_{-l}.
\]
Here, \( e_k, e_l \) belong to the spectral subspaces corresponding to \( k, l \in \mathbb{T}^n \) respectively. Thus, we compute
\[
(V_g)_\theta(\sum_k e_k^* \otimes C U^\theta_{-k}) (\sum_l e_l^* \otimes C U^\theta_{-l}) = \sum_{k,l} g(e_k^* \otimes A e_l^*) U^\theta_{-k} U^\theta_{-l}
\]
\[
= g(\sum_k e_k^* \otimes C U^\theta_{-k}) \otimes C (\sum_l e_l^* \otimes C U^\theta_{-l}) \quad (\text{by (12)})
\]
\[
= V_{g\theta}(\sum_k e_k^* \otimes C U^\theta_{-k}) (\sum_l e_l^* \otimes C U^\theta_{-l}).
\]

This proves (14) and finishes the proof. \[\Box\]

**Corollary 6.26**

We have \( (E^*_\theta)^* \cong (E^*)^\prime \otimes \mathcal{Z}(C) C_\theta \).
Proof: By Lemma 6.23 the map \( u^\ast \) defines an isomorphism from \( (\mathcal{E}^\ast)_\theta \) to \( (\mathcal{E}^\ast)_\theta \otimes \mathcal{Z}(C)_\theta \mathcal{C}_\theta \). However, as observed in the proof of Lemma 6.23 the action of \( T^n \) on \( (\mathcal{E}^\ast)_\theta \) is trivial and so is the action of \( T^n \) on \( \mathcal{Z}(C) \). Thus, \( (\mathcal{E}^\ast)_\theta \cong (\mathcal{E}^\ast)' \) and \( \mathcal{Z}(C)_\theta \cong \mathcal{Z}(C) \). Moreover, by Proposition 6.24 \( (\mathcal{E}^\ast)_\theta \cong (\mathcal{E}^\ast)^\ast \). Combining all these facts, we have the result. □

**Proposition 6.27** \( g_\theta \) is a noncommutative pseudo-Riemannian bi-metric on \( \mathcal{E}_\theta \).

**Proof**: Clearly, \( \sigma (= 2P_{\text{sym}} - 1) \) is \( T^n \)-equivariant, and as \( g\sigma = g \), we have \( g_\theta \circ \sigma_\theta = g_\theta \) too, i.e. \( g_\theta \) is symmetric. It is also clear that \( g_\theta \) is a bimodule map. Finally, by Proposition 6.24 \( V_{g_\theta} \) is nondegenerate. □

**Theorem 6.28** The deformed noncommutative manifold satisfies Assumptions I, IV.

Now, we make use of the additional assumption on the toral action and prove:

**Theorem 6.29** If the \( T^n \) action is free, then the deformed noncommutative manifold satisfies Assumptions II, III as well.

**Proof**: From Theorem 6.10 we see that the map \( u \) of Assumption III is a vector space isomorphism. But it is also a \( T^n \)-equivariant map, hence induces an isomorphism between the corresponding deformed modules. Thus, in particular, \( \mathcal{E}_\theta \) is centered. Moreover, as \( \sigma = \sigma^{\text{can}} \) for the classical case and \( \sigma, \sigma^{\text{can}} \) are \( T^n \)-equivariant, the corresponding deformed maps too agree, verifying Assumption II. □

This gives us the following result:

**Theorem 6.30** Suppose that \( M \) is a compact Riemannian manifold such that \( T^n \) acts by isometries on \( M \). Assume that this action by \( T^n \) is free and \( K \) is the isotropy subgroup for this action. Suppose \( C \) denotes the spectral subalgebra of \( C^\infty(M) \) under the action of \( T^n \). Then the space of one forms on the deformed algebra \( A_\theta \) satisfies the Levi-Civita property, that is, any noncommutative pseudo-Riemannian metric on the deformed manifold admits a unique Levi-Civita connection.

### 7 Existence and uniqueness of the Levi-Civita connection for quantum Heisenberg manifold

In this section, we consider the example of the quantum Heisenberg manifolds introduced in [29]. In [6], a spectral triple and the corresponding space of forms were studied. However, it turned out that with a particular choice of a metric and the definition of the metric compatibility of the connection in the sense of [21], there exists no connection on the space of one forms which is both torsionless and compatible with the metric. We will see that with our definition of metric compatibility of a connection, every pseudo-Riemannian metric on this noncommutative manifold admits a unique Levi-Civita connection. Moreover, the spectral triple which we will consider is not a Connes-Landi deformation and hence this illustrates that our theory can accommodate some examples beyond the Connes-Landi deformations of classical manifolds.

The description of the Dirac operator and the space of one forms require the Pauli spin matrices denoted by \( \sigma_1, \sigma_2, \sigma_3 \) in [6]. However, in this section as well as Subsection 5.3 we will use the symbols \( e_1, e_2, e_3 \) for \( \sigma_1, \sigma_2, \sigma_3 \) respectively. In particular, we have the following relations for \( e_1, e_2, e_3 \):

\[
e_j^2 = 1, e_je_k = -e_ke_j, e_1e_2 = \sqrt{-1}e_3, e_2e_3 = \sqrt{-1}e_1, e_1e_3 = \sqrt{-1}e_2.
\]

(15)

Moreover, we will denote a generic element of \( \Omega^1_D(\mathcal{A}) \) by \( \sum_j e_j a_j \) instead of \( \sum_j a_j \otimes \sigma_j \) as done in [6]. Lastly, we are going to work with right connections instead of left connections as done in [6].

The symbol \( \mathcal{A} \) will denote the algebra of smooth functions on the quantum Heisenberg manifold. The algebra \( \mathcal{A} \) admits an action of the Heisenberg group. \( \tau \) will denote a certain state on \( \mathcal{A} \) invariant under
the action of the Heisenberg group. Let $X_1, X_2, X_3$ denote the canonical basis of the Lie algebra of the Heisenberg group so that we have associated self-adjoint operators $d_{X_j}$ on $L^2(A, \tau) \otimes \mathbb{C}^3$ in the natural way. Then the triple $(A, L^2(A, \tau) \otimes \mathbb{C}^2, D)$ defines a spectral triple on $A$ where $A$ is represented on $L^2(A, \tau) \otimes \mathbb{C}^2$ diagonally and the Dirac operator $D$ is defined as

$$D = \sum_j d_{X_j} \otimes \mathbb{C} \gamma_j,$$

where $\{\gamma_j : j = 1, 2, 3\}$ are self-adjoint $3 \times 3$ matrices satisfying $\gamma_i \gamma_j + \gamma_j \gamma_i = 2 \delta_{ij}$.

The following lemma is a direct consequence of the proof of Proposition 9 of [6].

**Lemma 7.1** For all $F$ in $A$,

$$d(F) = \sum_{j=1}^{3} e_j \partial_j(F),$$

where $\partial_1(F) = \frac{\partial f}{\partial x}$, $\partial_2(F) = -2\pi \sqrt{-1} \partial x F + \frac{\partial F}{\partial y}$, $\partial_3(F) = -2\pi \sqrt{-1} \partial y F$

for some $\alpha$ greater than 1. The derivations $\partial_1, \partial_2, \partial_3$ satisfy the following relation:

$$[\partial_1, \partial_2] = [\partial_2, \partial_3] = 0, \quad [\partial_1, \partial_2] = \partial_3. \quad (16)$$

The space of one forms and two forms on $A$ are as follows:

**Proposition 7.2** The module of one forms $E := \Omega^1_D(A)$ is a free module generated by $e_1, e_2, e_3$. Moreover, $e_1, e_2, e_3$ are central elements. The module $J$ of junk forms as introduced in Subsection 4.4 is equal to $A$ and the space of two forms $\Omega^2_D(A) = A \oplus A \oplus A$.

**Proof:** The space of one forms are described in Proposition 21 of [6]. The fact that $e_1, e_2, e_3$ are central can be easily seen from the definition of the representation of $A$ on $L^2(A, \tau) \otimes \mathbb{C}^2$. The statement about the two forms follow from Proposition 22 of the same paper. \(\square\)

**Theorem 7.3** For any pseudo-Riemannian metric on $E$, there exists a unique Levi-Civita connection on the module $E = \Omega^1_D(A)$.

**Proof:** We need to check that Assumption I - IV are satisfied for the module $E$. We will use the fact that $e_i$ are central elements throughout the proof. Moreover, $m, m_0, J, F$ and $P_{sym}$ will be as in Subsection 4.1. We recall that $\Omega^2_D(A) = (\text{Im}(m_0))/J$. By virtue of (15) and the fact that $J = A$ (Proposition 7.2), it is easy to see that $\text{Ker}(m)$ is spanned by $\{e_i \otimes_A e_j + e_j \otimes_A e_i : i, j = 1, 2, 3\}$ and $F = \{e_i \otimes_A e_j - e_j \otimes_A e_i : i < j, i = 1, 2, 3\}$. Clearly, $E \otimes_A E = \text{Ker}(m) \oplus F$ so that Assumption 1 is satisfied.

For checking Assumption II, let us recall that $P_{sym}$ is the projection onto $\text{Ker}(m)$ so that by the proof of Assumption I above, we have

$$P_{sym}(e_i \otimes_A e_j - e_j \otimes_A e_i) = 0, \quad P_{sym}(e_i \otimes_A e_j + e_j \otimes_A e_i) = e_i \otimes_A e_j + e_j \otimes_A e_i$$

and thus $2P_{sym}(e_i \otimes_A e_j) = e_i \otimes_A e_j + e_j \otimes_A e_i$.

Therefore, $\sigma(e_i \otimes_A e_j) = (2P_{sym} - 1)(e_i \otimes_A e_j) = e_j \otimes_A e_i$.

This proves Assumption II.

Assumption III is clearly satisfied by using Corollary 4.18 since $e_1, e_2, e_3 \in Z(E)$. Finally, the metric $g$ defined by

$$g(e_i \otimes_A e_j) = \delta_{ij}1 \quad (17)$$

satisfies Assumption IV. \(\square\)

We end this section with the computation of the Levi-Civita connection for the metric defined in (17). Since $E := \Omega^1_D(A)$ is a free module with generators $e_1, e_2, e_3$, any connection on $E$ is determined by its action on $e_1, e_2, e_3$. Our choice of the torsionless connection $\nabla_0$ is defined as follows:
Definition 7.4
\[ \nabla_0(e_j) = 0 \quad \text{for} \quad j = 1, 2; \nabla_0(e_3) = -e_1 \otimes_A e_2. \quad (18) \]

The proof of the following proposition is a verbatim adaptation of the proof of Proposition 31 of [6] with the only difference that we use right connections instead of left connections.

**Proposition 7.5** \( \nabla_0 \) is a torsionless connection on \( E \).

The next lemma will be needed in the proof of Theorem 7.7.

**Lemma 7.6** Let \( g \) be the metric defined on \( E \) by the formula
\[ g(e_i \otimes_A e_j) = \delta_{ij}. \]
Then we have the following:
\[ \Phi^g_{\nabla_0}(e_i \otimes_A e_j) = -\sum_m e_m T_{ij}^m, \]
where, \( T_{i3}^2 = T_{3i}^2 = \frac{1}{2} \) and \( T_{ij}^m = 0 \) otherwise.

**Proof:** By definition of \( \Phi^g_{\nabla_0} \),
\[ \Phi^g_{\nabla_0}(e_i \otimes_A e_j) = \frac{1}{2}(g \otimes_A \text{id})\sigma_{23}(\nabla_0 \otimes_A \text{id})(e_i \otimes_A e_j + e_j \otimes_A e_i). \]
Clearly, \( T_{ij}^m = T_{ji}^m \).

From (18), it is immediate that for all \( i, j \in \{1, 2\} \),
\[ \Phi^g_{\nabla_0}(e_i \otimes_A e_j) = 0. \]
Moreover,
\[ \Phi^g_{\nabla_0}(e_1 \otimes_A e_3) = \frac{1}{2}(g \otimes_A \text{id})\sigma_{23}(\nabla_0(e_1) \otimes_A e_3 + \nabla_0(e_3) \otimes_A e_1) \]
\[ = -\frac{1}{2}(g \otimes_A \text{id})\sigma_{23}(e_1 \otimes_A e_2) \]
\[ = \frac{1}{2}(g \otimes_A \text{id})(e_1 \otimes_A e_1) \]
\[ = \frac{1}{2} e_2. \]
Thus,
\[ T_{i3}^2 = T_{3i}^2 = \frac{1}{2}; \quad T_{13}^3 = T_{31}^3 = 0. \]

Next,
\[ \Phi^g_{\nabla_0}(e_3 \otimes_A e_2) = \frac{1}{2}(g \otimes_A \text{id})\sigma_{23}(\nabla_0(e_2) \otimes_A e_3 + \nabla_0(e_3) \otimes_A e_2) \]
\[ = -\frac{1}{2}(g \otimes_A \text{id})\sigma_{23}(e_1 \otimes_A e_2) \]
\[ = 0. \]
Thus, for all \( m = 1, 2, 3 \), \( T_{ij}^m = 0 \). The rest of the \( T_{ij}^m \) can be computed by using (19). \( \square \)

Now, we are ready to compute the explicit form of the Levi-Civita connection for the metric \( g \) on the module \( E \).
Theorem 7.7  Let $\mathcal{E} = \Omega^1_0(\mathcal{A})$ be the module of one forms for the quantum Heisenberg manifold $\mathcal{A}$ with generators $e_1, e_2, e_3$ as above. Let us consider the metric $g$ on $\mathcal{E}$ determined by $g(e_i \otimes_A e_j) = \delta_{ij}$. Then there exists a unique Levi-Civita connection $\nabla$ on $\mathcal{E}$ given by

$$\nabla = \nabla_0 + L,$$

where $\nabla_0$ is the torsionless connection on $\mathcal{E}$ as in [15], $L : \mathcal{E} \to \mathcal{E} \otimes_A \mathcal{E}$ is defined by $L(e_j) = \sum_{i,m} e_i \otimes_A e_m L^j_{im}$,

$$L^j_{im} = \frac{1}{2} (T^m_{ij} + T^i_{jm} - T^j_{mi}), \quad (20)$$

where the elements $\{T^m_{ij} : i, j, m = 1, 2, 3\}$ are as in Lemma 7.6

More precisely, the non zero $L^1_{im}$ are as follows:

$$L^1_{23} = L^1_{32} = 0.25, \quad L^2_{13} = L^2_{31} = -0.25, \quad L^3_{12} = L^3_{21} = 0.25.$$  

If $\nabla$ is given by

$$\nabla(e_i) = \sum_{j,k} e_j \otimes_A e_k \Gamma^i_{jk}, \quad (21)$$

then the non zero $\Gamma^i_{jk}$ are as follows:

$$\Gamma^1_{23} = \Gamma^1_{32} = 0.25, \quad \Gamma^2_{13} = \Gamma^2_{31} = -0.25, \quad \Gamma^3_{12} = -0.75, \quad \Gamma^3_{21} = 0.25.$$  

Proof:  The proof is a consequence of the defining condition of the Levi-Civita connection:

$$\Phi_g(L)(e_i \otimes_A e_j) = dg(e_i \otimes_A e_j) - \Phi_g(\nabla_0)(e_i \otimes_A e_j), \quad (22)$$

$$mL(e_i) = 0. \quad (23)$$

The equation $(23)$ holds since

$$mL(e_i) = m\nabla(e_i) - m\nabla_0(e_i) = d(e_i) - d(e_i) = 0,$$

as $\nabla$ and $\nabla_0$ are both torsionless connections. Now, as $\sigma(e_i \otimes_A e_j) = e_j \otimes_A e_i$ and $L(e_j) = \sum_{i,m} e_i \otimes_A e_m L^j_{im}$, we conclude that

$$L^j_{im} = L^j_{mi} \quad \forall i, j, m. \quad (24)$$

Next, by virtue of the relation $g(e_i \otimes_A e_j) = \delta_{ij} 1_A$, $(22)$ is equivalent to

$$\Phi_g(L)(e_i \otimes_A e_j) = -\Phi_g(\nabla_0)(e_i \otimes_A e_j) = \sum_m e_m T^m_{ij}.$$  

Since $\Phi_g(L)(e_i \otimes_A e_j) = \frac{1}{2} \sum_y e_y(L^i_{j,y} + L^j_{i,y})$,

in the notation of Lemma 7.6 we have

$$L^i_{j,m} + L^j_{i,m} = T^m_{ij}. \quad (25)$$

Interchanging $(i, j, m)$ with $(j, m, i)$ and $(m, i, j)$, we have respectively:

$$L^i_{m,j} + L^j_{m,i} = T^i_{jm}, \quad (26)$$

$$L^i_{j,m} + L^j_{i,m} = T^i_{mj}. \quad (27)$$

Now, by $(25)$ + $(20)$ - $(27)$ and $(24)$, we have

$$L^j_{i,m} = \frac{1}{2} (T^m_{ij} + T^i_{jm} - T^j_{mi}),$$

which proves $(20)$. The expressions for $L^i_{j,m}$ and $\Gamma^i_{jk}$ follow from $(20)$ and $(21)$.  \(\square\)
8 Computation of the Ricci and scalar curvature

In this section, we apply the theory developed so far for computing the curvature of the Levi Civita connection of a conformally deformed metric. In the next subsection, we will derive a general formula for the Christoffel symbols (see Definition 8.3) of the Levi Civita connection for the conformal deformation of a metric on a class of free modules. This will be used to compute the Ricci and scalar curvature for the module of one forms for the canonical spectral triple on the noncommutative torus. The last subsection will deal with the computation of the curvature for the space of one forms on the quantum Heisenberg manifold studied in [6].

Let us start by defining the notions of Ricci and scalar curvature under Assumption I–Assumption IV. For this, we need a couple of definitions.

Let us consider the following automorphism

\[ \rho : \mathcal{E} \otimes \mathcal{A} \mathcal{E}^* \to \mathcal{E}^* \otimes \mathcal{A} \mathcal{E} \]

defined by \( \rho := (T^R_{\mathcal{E}, \mathcal{E}})^{-1} \) flip \( T^L_{\mathcal{E}, \mathcal{E}} \), where flip : \( \mathcal{E}' \otimes \mathcal{A}' \mathcal{E}^* \to \mathcal{E}^* \otimes \mathcal{A}' \mathcal{E}' \) is the map given by flip \( (e' \otimes \mathcal{A}' \phi) = \phi \otimes \mathcal{A}' e' \) which is clearly well defined and an isomorphism.

We will also need the map:

\[ \mathcal{E} \otimes \mathcal{C} \mathcal{E} \to \mathcal{E} \otimes \mathcal{A} \mathcal{E} \otimes \mathcal{A} \mathcal{E} \]

defined by \( \omega \otimes \mathcal{C} \eta \mapsto \nabla \omega \otimes \mathcal{A} \eta - \omega \otimes \mathcal{A} Q^{-1}(d\eta) \), where \( Q \) is the isomorphism from Im(1 - \( P_{\text{sym}} \)) to Im(\( \tilde{\Omega}^2_D(A) \)) as in Lemma 4.11.

Using the Leibniz rules for \( \nabla \) and \( d \), one can easily verify that the above map descends to a map on \( \mathcal{E} \otimes \mathcal{A} \mathcal{E} \), to be denoted by \( H \).

Then as in [6], and references therein, we define the “curvature operator” as follows:

\[ R(\nabla) := H \circ \nabla : \mathcal{E} \to \mathcal{E} \otimes \mathcal{A} \mathcal{E} \otimes \mathcal{A} \mathcal{E} \]

and consider

\[ \Theta := (\sigma_{23} \otimes \mathcal{A}) \text{id}_{\mathcal{E}} \cdot \mathcal{E} \otimes \mathcal{A} \mathcal{E} \otimes \mathcal{A} \mathcal{E} \]

where \( \sigma_{23} = \text{id}_{\mathcal{E}} \otimes \mathcal{A} \sigma : \mathcal{E} \otimes \mathcal{A} \mathcal{E} \otimes \mathcal{A} \mathcal{E} \to \mathcal{E} \otimes \mathcal{A} \mathcal{E} \to \mathcal{E} \).

**Definition 8.1** The Ricci curvature \( \text{Ric} \) is defined as the element in \( \mathcal{E} \otimes \mathcal{A} \mathcal{E} \) given by

\[ \text{Ric} := (\text{id}_{\mathcal{E}} \otimes \mathcal{A} \text{id})_{\mathcal{E}}(\Theta) \]

where \( \text{ev} : \mathcal{E}^* \otimes \mathcal{A} \mathcal{E} \to \mathcal{A} \) is the \( \mathcal{A} - \mathcal{A} \)-bilinear map sending \( e^* \otimes \mathcal{A} f \) to \( e^*(f) \) for all \( e^* \in \mathcal{E}^* \) and \( f \in \mathcal{E} \).

The scalar curvature \( \text{Scal} \) is defined as:

\[ \text{Scal} := \text{ev}(V_g \otimes \mathcal{A} \text{id})(\text{Ric}) \in \mathcal{A}. \]

**Remark 8.2** It is easy to see that in the classical case, i.e., when \( \mathcal{E} = \tilde{\Omega}^2_D(A) \) and \( \mathcal{A} = C^\infty(M) \), the above definitions of Ricci and Scalar curvature do coincide with the usual notions.

In case our module \( \mathcal{E} \) is free, one can make the following definition:

**Definition 8.3** Suppose that \( \mathcal{E} \) is a free module with a basis \( \{e_1, e_2, \cdots e_n\} \in Z(\mathcal{E}) \). Then we can talk about the “Christoffel symbols” \( \Gamma_{jk}^i \in \mathcal{A} \) given by

\[ \nabla(e_i) = \sum_{j,k} e_j \otimes \mathcal{A} e_k \Gamma_{jk}^i. \]

Then we have the following result which will be used in the next subsections.
Proposition 8.4 If $\mathcal{E}$ is a free module with a basis $\{e_1, e_2, \ldots, e_n\} \in \mathcal{Z}(\mathcal{E})$ satisfying Assumption I – IV such that $d(e_i) = 0$ for all $i = 1, 2, \ldots, n$, then we have the following:

1. There exist derivations $\partial_j : \mathcal{E} \to \mathcal{A}$, $j = 1, 2, \ldots, n$, such that
   \[ da = \sum_j e_j \partial_j(a). \] (31)

2. $m(e_i \otimes_A e_j) = e_i \otimes_A e_j - e_j \otimes_A e_i$. Moreover,
   \[ \Gamma^p_{kl} = \Gamma^p_{lk}. \] (32)

3. If $\nabla$ is a torsion free connection, then
   \[ R(\nabla)(e_i) = \sum_{j,k,l} e_j \otimes_A e_k \otimes_A e_p r^j_{klt} \] where $r^j_{klt} = \sum_p [(\Gamma^p_{jk}, p) - \partial_l (\Gamma^i_{jk}) + \partial_k (\Gamma^i_{jl})]$. The Ricci tensor $\text{Ric}$ is given by
   \[ \text{Ric}(e_j, e_l) = \sum_{i} \sum_p (\Gamma^p_{ij}, p) - \partial_l (\Gamma^i_{jl}) + \partial_k (\Gamma^i_{jl})]. \]

The Scalar curvature $\text{Scal} = \sum_{j,l} g(e_j \otimes_A e_l) \text{Ric}(e_j, e_l)$. 

\[ \text{Proof:} \] The first assertion follows by the using the fact that the operator $d$ is a derivation and the basis elements $e_i$ belong to $\mathcal{Z}(\mathcal{E})$. For the proof of the second assertion, we use the fact that $\sigma(e_i \otimes_A e_j) = e_j \otimes_A e_i$ by Assumption II so that $m(e_i \otimes_A e_j) = -m(e_j \otimes_A e_i)$. Since $\nabla$ is torsion free, we have
   \[ m \nabla(e_i) = d(e_i) \] and hence
   \[ \sum_{j,k} < m(e_j \otimes_A e_k), (\Gamma^i_{jk} - \Gamma^i_{kj}) > = 0. \]

Thus, the proof of part 2. will be complete once we prove that $\{m(e_j \otimes_A e_k) : j, k, j < k\}$ is a linearly independent set. Since $\ker(m) = \text{Ran}(1 + \sigma)$, $\{e_i \otimes_A e_j + e_j \otimes_A e_i : i, j\}$ is a basis of $\ker(m)$. Therefore, the set $\mathcal{F}$ as in Assumption I has as basis $\{e_i \otimes_A e_j - e_j \otimes_A e_i : i < j\}$. Since $m : \mathcal{F} \to \text{im}(m)$ is an isomorphism, we see that $\{m(e_j \otimes_A e_k) : j, k, j < k\}$ is linearly independent.

The third assertion can now be proved by simple computations using (32) and the equations (30), (31). \qed

8.1 The Christoffel symbols for the conformal deformation of a bi-metric on a class of free modules

Let $\mathcal{E}$ be an $\mathcal{A} - \mathcal{A}$ bimodule such that Assumption I – Assumption IV of Section 4 are satisfied. In particular, we have a pseudo-Riemannian bi-metric $g_0$ and by Theorem 5.11 $g_0$ admits a unique Levi-Civita connection which we will denote by $\nabla_0$. Thus, we have
   \[ dg_0 = \Phi_{g_0}(\nabla_0). \] (33)

Now let $k$ be an invertible element of $\mathcal{A}$ and $g = kg_0$ be the conformally deformed metric. We want to compute the curvature tensor of the Levi-Civita connection $\nabla$ for the metric $g$.

In the notations and definitions of Section 4, we have
   \[ \nabla = \nabla_0 + L \] where $\Phi_g(L) = dg - \Phi_g(\nabla_0)$. Then we have
   \[ \nabla = \nabla_0 + \Phi^{-1}_g(dg - \Phi_g(\nabla_0)). \] (34)

29
At this point, let us recall that for \( \mathcal{A} - \mathcal{A} \) bimodules \( \mathcal{E} \) and \( \mathcal{F} \), \( \text{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{F}) \) is an \( \mathcal{A} - \mathcal{A} \) bimodule via the following formulas:

\[
(a.T)(e) = a.T(e), \quad (T.a)(e) = T(ae) \quad \text{for all} \ a \in \mathcal{A}, \ e \in \mathcal{E}, \ T \in \text{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{F}).
\]

With this convention, consider the map

\[
L_k : \text{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}}^\text{sym} \mathcal{E}) \rightarrow \text{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}}^\text{sym} \mathcal{E})
\]

obtained by left multiplication by the element \( k \). Then we have the following:

**Proposition 8.5** \( \nabla = \nabla_0 + \Phi^{-1}_0 L_k^{-1}(dk.g_0) \).

**Proof:** To begin with, we observe that \( g = k g_0 \) implies that

\[
\Phi_g = L_k \Phi_{g_0}. \quad \text{Thus} \quad \Phi^{-1}_g = \Phi^{-1}_{g_0} L_k^{-1} = \Phi^{-1}_{g_0} L_{k^{-1}}.
\]

Then from (33), we have

\[
\nabla = \nabla_0 + \Phi^{-1}_{g_0} L_k^{-1}(dk.g_0 + k.dg_0 - L_k \Phi_{g_0}(\nabla_0)),
\]

where we have used Leibniz rule. However, (33) implies that

\[
\nabla = \nabla_0 + \Phi^{-1}_{g_0} L_{k^{-1}}(dk.g_0).
\]

\( \square \)

In order to compute the curvature tensor, we need a better understanding of the map \( \Phi^{-1}_{g_0} \). This will be achieved in Lemma 8.11. Before that, we will need the following definition and the subsequent results (Lemma 8.10 in particular) for the proof of Lemma 8.11

**Definition 8.6** Let \( g_0 \) be a nondegenerate Riemannian bi-metric on \( \mathcal{E} \). Then we define a map

\[
g^{(2)}_0 : (\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}) \otimes_{\mathcal{A}} (\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}) \rightarrow \mathcal{A} \text{ by the formula } g^{(2)}_0((\omega \otimes_{\mathcal{A}} \eta) \otimes_{\mathcal{A}} (\omega' \otimes_{\mathcal{A}} \eta')) = g_0(\omega \otimes_{\mathcal{A}} \eta \otimes_{\mathcal{A}} \omega' \otimes_{\mathcal{A}} \eta').
\]

The following observation is going to be useful for us.

**Lemma 8.7** 1. \( g^{(2)}_0 \) is an \( \mathcal{A} \)-bilinear map.
2. \( g^{(2)}_0(\theta \otimes_{\mathcal{A}} \theta') = 0 \quad \forall \theta' \) implies that \( \theta = 0 \).
3. For all \( \omega, \eta, \omega', \eta' \in \mathcal{E} \), we have \( g^{(2)}_0(\sigma(\omega \otimes_{\mathcal{A}} \eta) \otimes_{\mathcal{A}} (\omega' \otimes_{\mathcal{A}} \eta')) = g^{(2)}_0((\omega \otimes_{\mathcal{A}} \eta) \otimes_{\mathcal{A}} \sigma(\omega' \otimes_{\mathcal{A}} \eta')).
\]

**Proof:** The first assertion is trivial to see and hence we omit its proof. For the second assertion, let us suppose that for \( i = 1, 2, \cdots n \), there exist \( \omega_i, \eta_i \in \mathcal{E} \) such that for all \( \omega', \eta' \in \mathcal{E} \),

\[
g^{(2)}_0((\sum_i \omega_i \otimes_{\mathcal{A}} \eta_i) \otimes_{\mathcal{A}} (\omega' \otimes_{\mathcal{A}} \eta')) = 0.
\]

Then by the definition of \( g^{(2)}_0 \), we see that

\[
V'_{g_0}(\sum_i \omega_i g_0(\eta_i \otimes_{\mathcal{A}} \omega')) = 0.
\]

By nondegeneracy of \( g_0 \), we conclude that

\[
\sum_i \omega_i g_0(\eta_i \otimes_{\mathcal{A}} \omega') = 0.
\]

Thus, \( \zeta_{\mathcal{E}, \mathcal{E}}(\sum_i \omega_i \otimes_{\mathcal{A}} \eta_i)(\omega') = 0 \) for all \( \omega' \in \mathcal{E} \) implying that \( \sum_i \omega_i \otimes_{\mathcal{A}} \eta_i = 0 \).
Now we come to the proof of 3. We begin by observing that the set \( \mathcal{Z}(\mathcal{E}) \otimes_\mathcal{A} \mathcal{Z}(\mathcal{E}) \) is both left and right \( \mathcal{A} \)-total in \( \mathcal{E} \otimes_\mathcal{A} \mathcal{E} \) and the map \( g_0^{(2)} \) is both left and right \( \mathcal{A} \)-linear, hence it is enough to prove the equality for \( \omega, \eta, \omega', \eta' \in \mathcal{Z}(\mathcal{E}) \). But this follows from a simple computation since

\[
g_0^{(2)}(\sigma(\omega \otimes_\mathcal{A} \eta) \otimes_\mathcal{A} (\omega' \otimes_\mathcal{A} \eta')) = g_0(\eta g_0(\omega \otimes_\mathcal{A} \omega') \otimes_\mathcal{A} \eta') = g_0(\omega \otimes_\mathcal{A} \omega') g_0(\eta \otimes_\mathcal{A} \eta') = g_0(\eta g_0(\eta \otimes_\mathcal{A} \eta') \otimes_\mathcal{A} \omega') = g_0^{(2)}((\omega \otimes_\mathcal{A} \eta) \otimes_\mathcal{A} \sigma(\omega' \otimes_\mathcal{A} \eta')).
\]

\[\square\]

**Definition 8.8** For a pseudo-Riemannian bi-metric \( g_0 \), we define an element \( \Omega_{g_0} \in \mathcal{E} \otimes_\mathcal{A} \mathcal{E} \) by

\[
\Omega_{g_0} = (\text{id}_\mathcal{E} \otimes_\mathcal{A} V_g^{-1})\zeta_{\mathcal{E},\mathcal{E}}^{-1}(\text{id}_\mathcal{E}).
\]

Note that the map \((\text{id}_\mathcal{E} \otimes_\mathcal{A} V_g^{-1})\) used in the above Definition is well defined since \( V_g \) is bilinear.

**Lemma 8.9** \( \Omega_{g_0} \) is an element of \( \mathcal{E} \otimes_\mathcal{A}^{\text{sym}} \mathcal{E} \).

**Proof:** We need to prove that \( \sigma \Omega_{g_0} = \Omega_{g_0} \). To this end, we observe that by virtue of part 2. of Lemma \[8.7\] and the right \( \mathcal{A} \)-totality of the set \( \mathcal{Z}(\mathcal{E}) \otimes_\mathcal{A} \mathcal{Z}(\mathcal{E}) \) in \( \mathcal{E} \otimes_\mathcal{A} \mathcal{E} \), it is enough to prove that for all \( \omega', \eta' \in \mathcal{Z}(\mathcal{E}) \),

\[
g_0^{(2)}(\sigma \Omega_{g_0} \otimes_\mathcal{A} (\omega' \otimes_\mathcal{A} \eta')) = g_0^{(2)}(\Omega_{g_0} \otimes_\mathcal{A} (\omega' \otimes_\mathcal{A} \eta')).
\]

Now, the definition of \( \Omega_{g_0} \) implies that for all \( \omega \in \mathcal{E} \),

\[
\zeta_{\mathcal{E},\mathcal{E}}(\text{id}_\mathcal{E} \otimes_\mathcal{A} V_g)\Omega_{g_0}(\omega) = \omega.
\]

Let us write \( \Omega_{g_0} = \sum_i v_i \otimes_\mathcal{A} w_i a_i \) for some \( v_i \in \mathcal{E}, w_i \in \mathcal{Z}(\mathcal{E}) \) and \( e_i \in \mathcal{E} \). Then the above equation translates to

\[
\sum_i v_i g_0(w_i a_i \otimes_\mathcal{A} \omega) = \omega.
\]

(35)

If \( \omega', \eta' \in \mathcal{Z}(\mathcal{E}) \), part 3. of Lemma \[8.7\] implies that

\[
g_0^{(2)}(\sigma \Omega_{g_0} \otimes_\mathcal{A} (\omega' \otimes_\mathcal{A} \eta')) = g_0^{(2)}(\Omega_{g_0} \otimes_\mathcal{A} (\eta' \otimes_\mathcal{A} \omega'))
\]

\[
= g_0^{(2)}(\sum_i (v_i \otimes_\mathcal{A} w_i a_i) \otimes_\mathcal{A} (\eta' \otimes_\mathcal{A} \omega'))
\]

\[
= g_0(\sum_i v_i g_0(w_i a_i \otimes_\mathcal{A} \eta') \otimes_\mathcal{A} \omega')
\]

\[
= g_0(\eta' \otimes_\mathcal{A} \omega')
\]

\[
= g_0(\omega' \otimes_\mathcal{A} \eta')
\]

\[
= g_0(\sum_i v_i g_0(w_i a_i \otimes_\mathcal{A} \omega') \otimes_\mathcal{A} \eta')
\]

\[
= g_0^{(2)}(\Omega_{g_0} \otimes_\mathcal{A} (\omega' \otimes_\mathcal{A} \eta')).
\]

where we have used (35) twice. This finishes the proof. \[\square\]

**Lemma 8.10**

We have \((g_0 \otimes_\mathcal{A} \text{id})\sigma_{23}(\Omega_{g_0} \otimes_\mathcal{A} \eta) = \eta\).
Proof: Let us continue writing \( \Omega_{g_0} \) as \( \sum_i v_i \otimes_A w_i a_i \) (finitely many terms) for some \( v_i \in \mathcal{E}, w_i \in \mathcal{Z}(\mathcal{E}) \) and \( e_i \in \mathcal{E} \) so that the relation \( \sigma \Omega_{g_0} = \Omega_{g_0} \) (as obtained from Lemma 8.13) implies that

\[
\sum_i v_i \otimes_A w_i a_i = \sum_i w_i \otimes_A v_i a_i.
\]

(36)

Now, for \( \eta \in \mathcal{Z}(\mathcal{E}) \), we see that

\[
(g_0 \otimes \text{id}) \sigma_{23}(\Omega_{g_0} \otimes_A \eta) = (g_0 \otimes \text{id})(\sum_i v_i \otimes_A \eta \otimes_A w_i a_i) = \sum_i g_0(v_i \otimes_A \eta) w_i a_i.
\]

Therefore, for all \( \eta \in \mathcal{Z}(\mathcal{E}) \) and \( \eta' \in \mathcal{E} \), we have

\[
g_0((g_0 \otimes_A \text{id}) \sigma_{23}(\Omega_{g_0} \otimes_A \eta) \otimes_A \eta') = g_0\left(\sum_i g_0(v_i \otimes_A \eta) w_i a_i \otimes_A \eta'\right)
\]

\[
= g_0\left(\sum_i \left(\sum_i w_i g_0(v_i \otimes_A \eta) \otimes_A a_i \eta'\right)\right)
\]

\[
= g_0(2) \left(\sum_i \left(\sum_i (w_i \otimes_A v_i) \otimes_A (\eta \otimes_A a_i \eta')\right)\right)
\]

\[
= g_0(2) \left(\sum_i (v_i \otimes_A w_i a_i) \otimes_A (\eta \otimes_A \eta')\right) \quad \text{(by (36))}
\]

\[
= g_0(\sum_i v_i g_0(w_i a_i \otimes_A \eta) \otimes_A \eta')
\]

\[
= g_0(\eta \otimes_A \eta') \quad \text{(by (35)).}
\]

Hence, for all \( \eta \in \mathcal{Z}(\mathcal{E}) \) and for all \( \eta' \in \mathcal{E} \), we get

\[
g_0((g_0 \otimes_A \text{id}) \sigma_{23}(\Omega_{g_0} \otimes_A \eta) \otimes_A \eta') = g_0(\eta \otimes_A \eta').
\]

By the nondegeneracy of \( g_0 \), we can conclude that for all \( \eta \in \mathcal{Z}(\mathcal{E}) \),

\[
(g_0 \otimes_A \text{id}) \sigma_{23}(\Omega_{g_0} \otimes_A \eta) = \eta.
\]

Since \( \mathcal{Z}(\mathcal{E}) \) is right \( \mathcal{A} \)-total in \( \mathcal{E} \), this finishes the proof for all \( \eta \in \mathcal{E} \). \( \square \)

Having obtained the above results, we are now in a position to prove the following lemma.

Lemma 8.11 For a pseudo-Riemannian bi-metric \( g_0 \) and and element \( \beta \) in \( \mathcal{E} \), let us define the map

\[
T_\beta : \mathcal{E} \otimes_A \mathcal{E} \rightarrow \mathcal{E} \text{ by } T_\beta(\omega \otimes_A \eta) = \beta g_0(\omega \otimes_A \eta).
\]

Then \( (\Phi_{g_0}^{-1}(T_\beta))(\omega) = P_{\text{sym}}(\beta \otimes_A \omega) - \frac{1}{2} \Omega_{g_0} g_0(\beta \otimes_A \omega) \),

where \( \Omega_{g_0} \) is as in Definition 8.8.

Proof: We first observe that since \( g_0 \) is a bi-metric, \( T_\beta \in \text{Hom}(\mathcal{E} \otimes_A \mathcal{E}, \mathcal{E}) \) and hence it makes sense to apply \( \Phi_{g_0}^{-1} \) on \( T_\beta \).

For a fixed \( \beta \in \mathcal{E} \), let us define \( L_\beta^1 \in \text{Hom}_A(\mathcal{E}, \mathcal{E} \otimes_A \mathcal{E}) \) as

\[
L_\beta^1(\omega) = P_{\text{sym}}(\beta \otimes_A \omega) - \frac{1}{2} \Omega_{g_0} g_0(\beta \otimes_A \omega).
\]

32
We want to prove $\Phi_{g_0} L^1_\beta = T_\beta$. So for $\omega, \eta \in Z(\mathcal{E})$, we compute

$$\Phi_{g_0} L^1_\beta (\omega \otimes_A \eta) = (g_0 \otimes_A id) \sigma_{23}(L^1_\beta \otimes_A id)(\omega \otimes_A \eta + \eta \otimes_A \omega)$$

$$= (g_0 \otimes_A id) \sigma_{23}(L^1_\beta(\omega \otimes_A \eta + L^1_\beta(\eta) \otimes_A \omega)$$

$$= (g_0 \otimes_A id) \sigma_{23}(P^{sym}(\beta \otimes \omega) \otimes_A \eta - \frac{1}{2} \Omega_{g_0} g_0(\beta \otimes_A \omega) \otimes_A \eta$$

$$+ P^{sym}(\beta \otimes_A \eta) \otimes \omega - \frac{1}{2} \Omega_{g_0} g_0(\beta \otimes_A \eta) \otimes_A \omega)$$

$$= \frac{1}{2}(g_0 \otimes_A id) \sigma_{23}(\beta \otimes_A \omega \otimes_A \eta + \omega \otimes_A \beta \otimes_A \eta$$

$$+ \beta \otimes_A \eta \otimes_A \omega + \eta \otimes_A \beta \otimes_A \omega - \Omega_{g_0} g_0(\beta \otimes_A \omega) \otimes_A \eta$$

$$- \Omega_{g_0} g_0(\beta \otimes_A \eta) \otimes_A \omega)$$

$$= \frac{1}{2}(g_0(\beta \otimes_A \eta) \omega + g_0(\omega \otimes_A \eta) \beta + g_0(\beta \otimes_A \omega) \eta$$

$$+ g_0(\eta \otimes_A \omega) \beta - (g_0 \otimes_A id) \sigma_{23}(\Omega_{g_0} \otimes_A \eta) g_0(\beta \otimes_A \omega)$$

$$- (g_0 \otimes_A id) \sigma_{23}(\Omega_{g_0} \otimes_A \omega) g_0(\beta \otimes_A \eta)$$

Now using Lemma 8.10 and the fact that $\omega, \eta \in Z(\mathcal{E})$, the expression reduces to $2g_0(\eta \otimes_A \omega) \beta$. However, since $g_0$ is a bimetric and $\omega, \eta \in Z(\mathcal{E})$, by part 1 of Lemma 4.19, $g_0(\eta \otimes_A \omega) \beta = \beta g_0(\eta \otimes_A \omega)$. Hence for all $\omega, \eta \in Z(\mathcal{E})$,

$$\Phi_{g_0}(L^1_\beta (\omega \otimes \eta) = T_\beta(\omega \otimes \eta).$$

Since the set $Z(\mathcal{E}) \otimes Z(\mathcal{A}) Z(\mathcal{E})$ is right $\mathcal{A}$-total in $\mathcal{E} \otimes_A \mathcal{E}$, this finishes the proof. □

As an immediate corollary, we have:

**Proposition 8.12**

$$\nabla(\omega) = \nabla_0(\omega) + \frac{1}{2} g^{-1} \partial k^{-1} \partial(\omega) + \frac{1}{2} \omega \otimes_A k^{-1} \partial k - \frac{1}{2} \Omega_{g_0} g_0(k^{-1} \partial k \otimes_A \omega).$$

**Proof:** The result follows by combining Lemma 8.11 with Proposition 8.9 □

Now we are prepared to state and prove the main result of this subsection:

**Proposition 8.13** If $\mathcal{E} = \mathcal{L}^1(\mathcal{A})$ is a free module with a basis $\{e_1, e_2, \cdots e_n\} \in Z(\mathcal{E})$ satisfying Assumption I - IV such that $d(e_i) = 0$ for all $i = 1, 2, \cdots n$, $\nabla_0(e_i) = 0$ for all $i = 1, 2, \cdots n$. Suppose that $g_0$ is a pseudo-Riemannian bi-metric on $\mathcal{E}$ such that $g_0(e_1 \otimes e_1) = \delta_{11}$. Then the Christoffel symbols of the Levi-Civita connection are given by:

$$\Gamma^i_{jl} = \frac{1}{2}(\delta_{il} k^{-1} \partial j(k) + \delta_{lj} k^{-1} \partial i(k) - \delta_{il} k^{-1} \partial j(k)).$$

**Proof:** To begin with, we claim that under our assumptions, $\Omega_{g_0} = \sum_i e_i \otimes_A e_i$. Indeed, for a fixed $k_0$, if we let $\Omega_{g_0}$ to be $\sum_{i,j} e_i \otimes_A e_j a_{ij}$ for some $a_{ij}$ in $\mathcal{A}$, then

$$(g_0 \otimes_A id) \sigma_{23}(\Omega_{g_0} \otimes_A e_{k_0}) = (g_0 \otimes_A id)(\sum_{i,j} e_i \otimes_A e_{k_0} \otimes_A e_j a_{ij}) = \sum_j e_j a_{k_0,j},$$

and thus by Lemma 8.10 we deduce that $a_{k_0 k_0} = 1$ and $a_{k_0 j} = 0$ if $j \neq k_0$. This proves the claim.

Now we apply Proposition 8.12 and the fact $\nabla_0(e_i) = 0$ to have

$$\nabla(e_i) = \frac{1}{2} \sum_j k^{-1} e_j \partial j(k) \otimes_A e_i + \sum_j e_i \otimes_A k^{-1} e_j \partial j(k) - \frac{1}{2} \sum_j e_i \otimes_A e_i g_0(k^{-1} \sum_j e_j \partial j(k) \otimes_A e_i))$$

$$= \sum_{j,l} e_j \otimes_A e_l(\frac{1}{2} \delta_{il} k^{-1} \partial j(k) + \frac{1}{2} \delta_{lj} k^{-1} \partial i(k) - \frac{1}{2} \delta_{il} k^{-1} \partial j(k)).$$

□
8.2 Computation of curvature for the conformally deformed metric on the non-commutative torus

We recall that the noncommutative 2-torus \( C(\mathbb{T}_2^2) \) is the universal \( C^* \) algebra generated by two unitaries \( U \) and \( V \) satisfying \( UV = e^{2\pi i \theta} VU \) where \( \theta \) is a number in \([0, 1]\). The spectral subalgebra for the \( T^2 \) action \( C_\theta = A(\mathbb{T}_2^2) \) is the dense \(*\)-subalgebra generated by \( U \) and \( V \). In fact, this can be realized as the Rieffel deformation of the spectral subalgebra \( C \subseteq C^\infty(T^2) \). The \( 2 \times 2 \) skew-symmetric matrix corresponding to the deformation is \( \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} \). As discussed in Theorem 6.18, the module \( E_\theta \) of one-forms for the Connes-Landi deformed spectral triple on \( C_\theta = A(\mathbb{T}_2^2) \) is a deformation of the module of one-forms on the spectral submodule \( E \) of \( C^\infty(T^2) \) for the action by the group \( T^2 \) (see Definition 6.4). Moreover, by the results of Section 6 Assumption I-IV are satisfied. One can have the following concrete description of the spectral subalgebra for the \( \mathbb{T}_2^2 \) metric on the non-commutative torus.

The next proposition will be needed in the computation of the scalar curvature of the noncommutative torus.

**Proposition 8.14** The module \( E_\theta \) is freely generated by the central elements

\[ e_1 = 1 \otimes \mathbb{C} \gamma_1, \quad e_2 = 1 \otimes \mathbb{C} \gamma_2. \]

**Proof:** The fact that \( E_\theta \) is generated by \( e_1 \) and \( e_2 \) is a well known result (see [22] for example). They are elements of \( \mathcal{Z}(E_\theta) \) indeed. \( \square \)

**Lemma 8.15**

\[ d(e_1) = d(e_2) = 0. \]

**Proof:** It is easy to see that \( dU = \sqrt{-1}e_1U \), \( dV = \sqrt{-1}e_2V \). As \( e_1, e_2 \in \mathcal{Z}(E) \) and \( \sigma = \sigma_{can} \) on \( \mathcal{Z}(E) \times \mathcal{Z}(E) \), we have \( m(e_i \otimes_A e_i) = 0 \) for \( i = 1, 2 \). By Leibniz rule, we have \( d^2U = d(e_1)U - e_1dU = 0. \) However, since \( dU = \sqrt{-1}e_1U \), we have \( 0 = d(e_1)U - \sqrt{-1}m(e_1 \otimes_A e_1)U \); so that \( d(e_1)U \) and hence \( d(e_1) \) equals 0. Similarly, \( d(e_2) = 0. \) \( \square \)

**Remark 8.16** From the equalities \( dU = \sqrt{-1}e_1U \), \( dV = \sqrt{-1}e_2V \), it follows that the derivations \( \partial_1 \) and \( \partial_2 \) as in part 1. of Proposition 8.2 are given by the following formulas:

\[ \partial_1(U) = \sqrt{-1}U, \quad \partial_1(V) = 0, \quad \partial_2(U) = 0, \quad \partial_2(V) = \sqrt{-1}V. \]

From this formulas, it can be easily checked that \( \partial_1 \) and \( \partial_2 \) commute.
The bilinear metric on $\mathcal{A}(\Omega^2)_{\theta_0}$ is defined by identifying $\mathcal{E}_\theta$ as a subset of $\mathcal{A}(\Omega^2)_{\theta_0} \otimes_{\mathbb{C}} M_2(\mathbb{C})$.

**Definition 8.17** Let $(g_0)_\theta$ be the metric on $\mathcal{E}_\theta$ defined by $(g_0)_\theta(\omega \otimes \zeta) = (\tau \otimes \text{Tr}_{M_2(\mathbb{C})})(\omega^* \zeta)$.

It is easy to see that

$$(g_0)_\theta(e_i \otimes_A e_j) = \delta_{ij}1_A. \quad (38)$$

For the choice of the connection $\nabla_0$, we will use the one constructed in (2) of Lemma 4.11 where $\tilde{\nabla}_0$ is an in (4.11). Since our module $\mathcal{E} = \Omega^1_D(\mathcal{A})$ is free, in the notation of Lemma 4.11 we see that $\nabla_0 = Q^{-1}d$. In particular,

$$\nabla_0(e_i) = 0 \text{ since } d(e_i) = 0 \text{ by Lemma 8.15} \quad (39)$$

**Theorem 8.18**

$$\text{Ric}(e_1, e_1) = \text{Ric}(e_2, e_2) = -\frac{1}{2}(k^{-1}(\partial_1^2 + \partial_2^2)(k) + \partial_1(k^{-1})\partial_1(k) + \partial_2(k^{-1})\partial_2(k)).$$

$$\text{Ric}(e_1, e_2) = -\text{Ric}(e_2, e_1) = \frac{1}{2}(\partial_1(k^{-1})\partial_2(k) - \partial_2(k^{-1})\partial_1(k)).$$

$$\text{Scal} = -(\partial_1^2 + \partial_2^2)(k) - k(\partial_2(k^{-1})\partial_2(k) + \partial_1(k^{-1})\partial_1(k)).$$

**Proof:** Due to our choice of $\sigma$ and Lemma 8.13, we can apply Proposition 5.4 and hence $R(\nabla)(e_i)$, Ricci curvature and scalar curvature are as in 3. of that Proposition. Moreover, by (39), the Christoffel symbols are as in (49). Thus, we have:

$$\Gamma_{11}^1 = \frac{1}{2}k^{-1}\partial_1(k), \quad \Gamma_{12}^1 = \frac{1}{2}k^{-1}\partial_2(k),$$

$$\Gamma_{21}^1 = -\frac{1}{2}k^{-1}\partial_2(k), \quad \Gamma_{22}^1 = \frac{1}{2}k^{-1}\partial_2(k), \quad \Gamma_{12}^2 = \Gamma_{21}^2 = \frac{1}{2}k^{-1}\partial_1(k),$$

and we have used (52).

Using these formulas for Christoffel symbols, we can compute

$$\text{Ric}(e_1, e_1) = \sum_{i, p=1}^2 (\Gamma_{11}^p \Gamma_{p1}^i - \Gamma_{11}^i \Gamma_{p1}^p) - \sum_{i=1}^2 (\partial_1(\Gamma_{11}^i) - \partial_i(\Gamma_{11}^i))$$

$$= \Gamma_{12} \Gamma_{11} - \Gamma_{11} \Gamma_{12} + \Gamma_{12} \Gamma_{21} - \Gamma_{12} \Gamma_{22} - \frac{1}{2}(\partial_1(k^{-1}\partial_1(k)) + \partial_2(k^{-1}\partial_2(k)))$$

$$= 0 - \frac{1}{2}(\partial_1(k^{-1})\partial_1(k) - \frac{1}{2}k^{-1}\partial_1^2(k) - \frac{1}{2}k^{-1}\partial_2^2(k))$$

$$= -\frac{1}{2}(\partial_1^2 + \partial_2^2)(k) + \partial_1(k^{-1})\partial_1(k) + \partial_2(k^{-1})\partial_2(k)).$$

The computations for $\text{Ric}(e_2, e_2)$, $\text{Ric}(e_1, e_2)$ and $\text{Ric}(e_2, e_1)$ are similar and hence omitted. The only extra ingredient in the computation of $\text{Ric}(e_1, e_2)$ and $\text{Ric}(e_2, e_1)$ is that the derivations $\partial_1$ and $\partial_2$ commute as was remarked in Remark 8.16.

Finally, using the formula of the scalar curvature in 3. of Proposition 5.4 and the equation (48), we get that

$$\text{Scal} = \sum_{j,l} k g_0(e_j \otimes_A e_l) \text{Ric}(e_j, e_l)$$

$$= \sum_{j} k \text{Ric}(e_j, e_j)$$

$$= -k(k^{-1}(\partial_1^2 + \partial_2^2)(k) + \partial_2(k^{-1})\partial_2(k) + \partial_1(k^{-1})\partial_1(k))$$

$$= -(\partial_1^2 + \partial_2^2)(k) - k(\partial_2(k^{-1})\partial_2(k) + \partial_1(k^{-1})\partial_1(k)).$$

$\Box$
8.3 Computation of the curvature for the example of the quantum Heisenberg manifold

In this subsection, we compute the curvature of the Levi Civita connection on the quantum Heisenberg manifold computed in Section 7. We will continue to follow the notations used in that section.

**Theorem 8.19** Let ∇ denote the Levi-Civita connection for the metric g on the module E of one forms over the quantum Heisenberg manifold A. The the Ricci and scalar curvature of ∇ are as follows:

\[ \text{Ric}(e_1, e_1) = 0.125, \quad \text{Ric}(e_2, e_2) = \text{Ric}(e_3, e_3) = -0.125, \]

\[ \text{Ric}(e_1, e_2) = \text{Ric}(e_1, e_3) = \text{Ric}(e_2, e_1) = \text{Ric}(e_2, e_3) = \text{Ric}(e_3, e_1) = \text{Ric}(e_3, e_2) = 0. \]

\[ \text{Scal} = -0.125. \]

**Proof:** The proof follows by a direct computation using Theorem 7.7 and the formulas of Ric(e_j, e_l) and Scal in Proposition 8.4.

**Remark 8.20** We note that the quantum Heisenberg manifold has a constant negative scalar curvature and moreover, the curvature is independent of the choice of the parameter \( \alpha \) used to define the Dirac operator.

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