Radiative transfer in scattering stochastic atmospheres

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received..... 2017, accepted....

Abstract

Many stars, active galactic nuclei, accretion discs etc. are affected by the stochastic variations of temperature, turbulent gas motions, magnetic fields, number densities of atoms and dust grains. These stochastic variations influence on the extinction factors, Doppler widths of lines and so on. The presence of many reasons for fluctuations gives rise to Gaussian distribution of fluctuations. The usual models leave out of account the fluctuations. In many cases the consideration of fluctuations improves the coincidence of theoretical values with the observed data. The objective of this paper is the investigation of the influence of the number density fluctuations on the form of radiative transfer equations. We consider non-magnetized atmosphere in continuum.

Keywords: Radiative transfer, stochastic atmospheres, scattering, polarization

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1 Introduction

In this paper we consider the derivation of radiative transfer equations in scattering stochastic atmospheres. The ensemble of fluctuations of the extinction factor $\alpha = N\sigma_t$ and the scattering coefficient $\alpha_s = N\sigma_s$ is assumed to be of Gaussian type. Here $N$ is the number density of scattering particles; $\sigma_t = \sigma_s + \sigma_a$, where $\sigma_s$ and $\sigma_a$ are the scattering cross-section and the absorption cross-section, respectively.

Thus, we consider the stochastic atmosphere, where the fluctuating values $N'$ are number density fluctuations of the scattering particles. The reasons of fluctuations are the compressible turbulence or the temperature fluctuations, which also give rise to fluctuations of the number density. Both reasons really hold in star’s atmospheres and accretion discs around Active Galactic Nuclei (AGN). For example, in our Sun the temperature fluctuations $\Delta T/T \sim 0.02 - 0.03$ are valid (see Stix 1991). Recall, that for adiabatic processes the relation $N_1/N_2 = (T_1/T_2)^{\gamma-1}$ holds. The value $\gamma = 5/3$ corresponds to monoatomic gas and for diatomic gas $\gamma = 7/2$. This relation means that the temperature fluctuations give rise to the fluctuations of the number densities: $\Delta N_1/N_1 + \Delta N_2/N_2 \simeq (\gamma - 1)(\Delta T_1/T_1 + \Delta T_2/T_2)$. Here we assumed that the temperature fluctuations are small $\Delta T/T \ll 1$. No doubt, the temperature fluctuations hold in different types of stars. It is apparently that the most large fluctuations $N'$ deal with the shock waves turbulence.

It is known that Gaussian ensemble corresponds to many chaotic reasons to arise the fluctuations. The observed Stokes parameters are the temporal and space averaged values. The ensemble averaging we denote by $\langle \rangle$. The averaged value $\langle I \rangle$ is denoted by $I_0$. The fluctuations are denoted by primes $I'$ ($\langle I' \rangle = 0$).

We use the standard formulas for the averaging of stochastic values (see van Kampen 1981, Gardiner 1985). For convenience of readers we at first present them.

For Gaussian ensemble of realizations the average of the odd number of fluctuating quantities is equal to zero and the average of the even numbers is equal to the sum of all possible two-point correlation functions (correlators) (see van Kampen 1981, Gardiner 1985). Thus, for example:

$$\langle x_1x_2x_3x_4 \rangle = \langle x_1x_2 \rangle \langle x_3x_4 \rangle + \langle x_1x_3 \rangle \langle x_2x_4 \rangle + \langle x_1x_4 \rangle \langle x_2x_3 \rangle - \langle x^4 \rangle = 3\langle x^2 \rangle^2,$$  \hspace{1cm} (1)

where $x_i$ are fluctuating values with zero mean value $\langle x_i \rangle = 0$. In right side of Eq.(1) are 3 terms. The average of the value $x_1x_2x_3x_4x_5x_6$ can be written as:

$$\langle x_1x_2x_3x_4x_5x_6 \rangle = \langle x_1x_2 \rangle \langle x_3x_4x_5x_6 \rangle + \langle x_1x_3 \rangle \langle x_2x_4x_5x_6 \rangle + \langle x_1x_4 \rangle \langle x_2x_3x_5x_6 \rangle + \langle x_1x_5 \rangle \langle x_2x_3x_4x_6 \rangle + \langle x_1x_6 \rangle \langle x_2x_3x_4x_5 \rangle \rightarrow 5\langle x^2 \rangle \langle x^4 \rangle = 5 \cdot 3\langle x^2 \rangle^3.$$  \hspace{1cm} (2)
Eq. (2), according to Eq. (1), consists of $5 \times 3 = 15$ terms. The arrow shows the transition to the case when all $x_i$ are equal to $x$. For this case we have the recurrence formulas:

$$
\langle x^{2k} \rangle = (2k - 1)\langle x^2 \rangle \langle x^{2(k-2)} \rangle = (2k - 1)(2k - 3)(2k - 5) \ldots 1 \langle x^2 \rangle^k = (2k - 1)!! \langle x^2 \rangle^k ,
$$

(3)

$$
\langle ax^{2k+1} \rangle = (2k + 1)\langle ax \rangle \langle x^{2k} \rangle = \langle ax \rangle (2k + 1)!! \langle x^2 \rangle^k .
$$

(4)

Here $a$ is Gaussian fluctuation with $\langle a \rangle = 0$. Recall, that standard factorial $n! = n(n - 1)(n - 2) \ldots 1$, i.e. $n! = n(n - 1)!!$. The factorial consisting of the odd numbers is denoted as $(2n - 1)!!$. This value obeys the recurrence formula $(2n - 1)!! = (2n - 1)(2n - 3)!!$. It is easy prove the formula: $(2n)! = 2^n n!(2n - 1)!!$.

The mean values of $\exp (\pm r')$ and $a' \exp (-r')$ can be readily obtained, using the average procedure, given in Eqs. (1) - (4), for the standard presentation of the exponential function $\exp (x)$:

$$
\exp (\pm x) = 1 + \sum_{k=1}^{\infty} \frac{x^{2k}}{(2k)!} \pm \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k + 1)!}.
$$

(5)

Using the recurrence formulas (3) and (4), we obtain the following expressions:

$$
\langle \exp (\pm r') \rangle = \exp \left( \frac{\langle r'^2 \rangle}{2} \right) .
$$

(6)

$$
\langle a' \exp (-r') \rangle = -\langle a' r' \rangle \exp \left( \frac{\langle r'^2 \rangle}{2} \right) .
$$

(7)

Here $r'$ - values are fluctuations of an optical depth.

Formulas (6) and (7) were given in Silant’ev (2005) without derivation. In this paper the propagation of polarized radiation in magnetized electron atmosphere is considered, i.e. the free electrons was taken as the only scattering particles.

Below we consider the radiative transfer equation in scattering stochastic atmospheres both consisting of one type scattering particles and two types ones. We consider also the matrix transfer equation for Stokes parameters $I$ and $Q$ in the atmosphere, consisting of free electrons and small anisotropic particles (molecules or dust grains). We do not take into account the possible magnetic field in an atmosphere. Note that the new extinction factor $\alpha_{eff} < \alpha$ arises in all atmospheres. In two-component atmosphere, the radiative transfer equation takes the new form for integral terms.

Firstly we consider the transfer equation for the mean intensity $\langle I(n, r) \rangle \equiv I_0(n, r)$ in stochastic atmosphere, consisting of one type particles. Note that we use the notations $\langle \alpha \rangle \equiv \alpha_0$ and the averaged optical depth $\langle \tau \rangle \equiv \tau_0$.

## 2 Stochastic atmosphere with one type of scattering particles

In stochastic atmosphere, the extinction factor has a fluctuating component: $\alpha = \langle \alpha \rangle + a' \equiv \alpha_0 + a'(r)$, $\langle a'(r) \rangle = 0$. The change of radiation intensity along the path $r$ is determined by the standard differential equation:

$$
dI(n, r) = -[\alpha_0(r) + a'(r)]I(n, r)dr \equiv -[N_0(r) + N'(r)]\sigma_t I(n, r)dr.
$$

(8)

Here $\alpha(r) = N(r)\sigma_t$, where $N(r) = N_0(r) + N'(r)$ is the number density of scattering particles. The value $\sigma_t = \sigma_s + \sigma_a$ is the total cross-section, which is the sum of scattering cross-section $\sigma_s$ and absorption cross-section $\sigma_a$. The differentiation shows that the solution of Eq.(8) is:

$$
I(n, r) = I(n, 0) \exp \left[ -\int_0^r dr' (\alpha_0(r') + a'(r')) \right] \equiv I(n, 0) \exp [-\langle \tau_0(r) + \tau'(r) \rangle],
$$

(9)

where $\tau_0(r)$ and $\tau'(r)$ are:

$$
\tau_0(r) = \int_0^r dr' \alpha_0(r') = \sigma_t \int_0^r dr' N_0(r'),
$$

(10)

$$
\tau'(r) = \int_0^r dr' \alpha'(r') = \sigma_t \int_0^r dr' N'(r').
$$

(11)
The averaging of Eq.(9) with allowance for Gaussian distribution of fluctuations probability (see Eq.(6)) gives rise to:

\[
\langle I(n, r) \rangle \equiv I_0(n, r) = I(n, 0) \exp \left[ - \left( \tau_0(r) - \frac{1}{2} \sigma^2(r) \right) \right] = I(n, 0) \exp \left[ - \tau_{eff}(r) \right], \tag{12}
\]

\[
\tau_{eff}(r) = \tau_0(r) - \frac{1}{2} \sigma^2(r). \tag{13}
\]

Eq.(12) shows that the averaged intensity \( I_0(n, r) \) in stochastic medium decreases with the distance weaker than intensity with accounting for the mean absorption factor \( \alpha_0(r) \).

Let us demonstrate this by the averaging of two realizations:

\[
I_1(n, r) = I(n, 0) \exp \left( - \tau_0(r) + \tau'(r) \right),
\]

\[
I_2(n, r) = I(n, 0) \exp \left( - \tau_0(r) - \tau'(r) \right). \tag{14}
\]

The mean value of these two realizations is:

\[
I_0(n, r) = \frac{I_1(n, r) + I_2(n, r)}{2} = I(n, 0) \exp \left( - \tau_0(r) \right) \cosh (\tau'(r)). \tag{15}
\]

We see that the mean value is larger than \( I(n, 0) \exp \left( - \tau_0(r) \right) \cosh (x) \geq 1 \) . This is the statistical effect.

Using Eq.(7), one can derive the following expression:

\[
\langle \alpha(r) I(n, r) \rangle = \alpha_0(r) I_0(n, r) + \langle \alpha'(r) I'(n, r) \rangle = [\alpha_0(r) - \langle \alpha'(r) \tau'(r) \rangle] I_0(n, r) \equiv \alpha_{eff}(r) I_0(n, r), \tag{16}
\]

where

\[
\alpha_{eff}(r) = \alpha_0(r) - \langle \alpha'(r) \tau'(r) \rangle. \tag{17}
\]

The standard radiative transfer equation for \( I(n, r) \) has the form (see Chandrasekhar 1960):

\[
\frac{dI(n, r)}{dr} = (n \nabla) I(n, r) = -\alpha(r) I(n, r) + \frac{\alpha^{(s)}(r)}{4\pi} \int d\mathbf{n}' p(n \cdot n') I(n', r) + S(n, r). \tag{18}
\]

Here \( p(n \cdot n') \) is the phase function, \( S(n, r) \) is the source function. Note that the extinction factor \( \alpha(r) = \alpha^{(s)}(r) + \alpha^{(a)}(r) \) describes the extinction due to scattering and pure absorption, respectively. Recall, that \( \alpha^{(s)}(r) = N(r) \sigma_s \) and \( \alpha^{(a)}(r) = N(r) \sigma_a \). Here \( \sigma_s \) and \( \sigma_a \) are the cross-sections of scattering and absorption, respectively. The total cross-section \( \sigma_t = \sigma_s + \sigma_a \). The degree of absorption \( \varepsilon = \frac{\sigma_a}{\sigma_t} \). The extinction factor \( \alpha(r) = N(r) \sigma_t \). The value \( N(r) \) and, consequently, \( \alpha(r) \) has the fluctuating component. It is clear that \( \alpha^{(s)}(r) \equiv (1 - \varepsilon) \alpha(r) \).

The phase function \( p(n \cdot n') \) depends on the scalar product of vectors \( n' \) and \( n \) - directions of incident radiation and the observed direction. Integral of \( p(n \cdot n') / 4\pi \) over \( n \) or \( n' \) is equal to unity. In conservative atmosphere \( (\varepsilon = 0) \) isotropic phase function is \( p(n \cdot n') = 1 \) and for Rayleigh scattering \( p(n \cdot n') = 3/4(1 + n \cdot n') \).

It is easy prove, that in conservative atmosphere \( (\alpha_0 = 0, \varepsilon = 0) \) and the source \( S(n, r) = 0 \) the law of conservation of radiation flux exists:

\[
\frac{dF(r)}{dr} = 0, \quad F(r) = \int_{(4\pi)} d\mathbf{n} I(n, r). \tag{19}
\]

The derivation of the conservation law uses the property:

\[
\frac{1}{4\pi} \int_{(4\pi)} d\mathbf{n} p(n \cdot n') = 1. \tag{20}
\]

The conservation law follows from physical consideration. Eq.(19) confirms that the radiative transfer equation is true.

The averaging of Eq.(18), with the taking into account the relation (16), gives rise to the radiative transfer equation for the mean intensity \( \langle I(n, r) \rangle \equiv I_0(n, r) \):

\[
\frac{dI_0(n, r)}{dr} = (n \nabla) I_0(n, r) = -\alpha_{eff}(r) I_0(n, r) + \frac{\alpha^{(s)}(r)}{4\pi} \int d\mathbf{n}' p(n \cdot n') I_0(n', r) + \langle S(n, r) \rangle, \tag{21}
\]
where
\[ \alpha_{eff}^{(s)} = \alpha_0^{(s)}(r) - \langle \alpha^{(s)}(r) \tau'(r) \rangle \equiv (1 - \varepsilon)\alpha_{eff}(r). \] (22)

Thus, using the supposition that the stochastic atmosphere physically presents the Gaussian ensemble of realizations, we obtained the closed radiative transfer equation for mean intensity \( I_0(n, r) \). The main role in this presentation plays Eq.(16).

The relation (16) takes place for radiation going through the stochastic Gaussian type atmosphere. Clearly, in the integral term of Eq.(18) the intensity also arises after some transportation through the stochastic medium. Note that the conservation law (Eq.(19)) holds also for Eq.(21) (recall, that in conservative atmosphere \( \varepsilon = 0 \) and \( \alpha_{eff} = \alpha_{eff}^{(s)} \)).

The transfer equation in stochastic atmosphere, where \( d\tau = \alpha_{eff} ds \), takes the standard form:
\[ \frac{dI_0(n, \tau)}{d\tau} = (n\nabla \cdot) I_0(n, \tau) = -I_0(n, \tau) + \frac{1 - \varepsilon}{4\pi} \int d\mathbf{n}' p(\mathbf{n} \cdot \mathbf{n}') I_0(n', \tau) + S_0(n, \tau). \] (23)

It is interesting to note that the form of transfer equation for the mean intensity is similar to standard radiative transfer equation with \( d\tau = \alpha_0 dr \).

Because of \( \alpha_{eff} < \alpha_0 \), the geometrically similar layers have different optical depths - the stochastic layer has effectively smaller (more transparent) depth than non-stochastic one. Eq.(21) uses the supposition that the ensemble of fluctuations is Gaussian.

Now we consider the averaged quantities, assuming that two-point correlations like \( \langle \tau^2 \rangle \) etc., depend on \( |r' - r''| = R \), i.e. we assume the model of homogeneous and isotropic turbulence. Let us consider the case \( \langle \tau^2 \rangle \) in more detail:
\[ \langle \tau^2 \rangle = \int_0^R dr' \int_0^R dr'' \langle \alpha'(r') \alpha'(r'') \rangle, \]
\[ \langle \alpha' \rangle \int_0^R dr' \int_0^R dr'' A_\alpha(R/R_1) \equiv 2 \langle \alpha^2 \rangle \int_0^R dR(r - R) A_\alpha(R/R_1). \] (24)

Here we introduced two-point correlation function \( A_\alpha(R/R_1) \):
\[ \langle \alpha'(r') \alpha'(r'') \rangle = \langle \alpha^2 \rangle A_\alpha(R/R_1) = \langle N^2 \rangle A_\alpha^2(R/R_1). \] (25)

The function \( A_\alpha(R/R_1) \) is equal to unity at \( R = 0 \). The parameter \( R_1 \) is the length of correlation, i.e. for \( R > R_1 \) correlation between \( \alpha'(r') \) and \( \alpha'(r'') \) tends to zero. The value \( \langle \alpha^2 \rangle \) gives the mean value of \( \langle \alpha'(r') \alpha(r'') \rangle \) for \( r' = r'' \).

Frequently one considers the homogeneous medium with \( \alpha_0(r) = \alpha_0 \). Below we restrict ourselves by this case. In this case Eq.(24) can be written in the form:
\[ \langle \tau^2 \rangle = 2 \frac{\langle N^2 \rangle}{N_0^2} \tau_0 \frac{\tau_1}{f_0} A_\alpha(r/R_1) \equiv 2 \frac{\langle N^2 \rangle}{N_0^2} \tau_0 \frac{\tau_1}{f_0} A_\alpha(r/R_1). \] (26)

Here \( \tau_0 = \alpha_0 r \) and \( \tau_1 = \alpha_0 R_1 \) are the total mean optical depths of the distance \( r \) and the correlation length \( R_1 \), respectively. The value \( f_\alpha(r/R_1) \) denotes the integral:
\[ f_\alpha(r/R_1) = \int_0^{r/R_1} dx \left( 1 - x \frac{R_1}{r} \right) A_\alpha(x). \] (27)

The characteristic length \( r_0 \) corresponds to free path. If the length \( r_0 >> R_1 \), then the value \( f_\alpha \) tends to the simple expression:
\[ f_\alpha(r/R_1) \to \int_0^\infty dx A_\alpha(x) \equiv f_\alpha. \] (28)

Frequently one assumes \( A_\alpha(x) = \exp(-x) \). In this case \( f_\alpha = 1 \). For other peak-like forms of correlation function \( A_\alpha(x) \) we have \( f_\alpha \approx 1 \). Thus, for estimate of \( \langle \tau^2 \rangle \) one can use \( f_\alpha \approx 1 \). Using Eq.(26), we obtain for \( \tau_{eff} \) the value:
\[ \tau_{eff}(r) = \tau_0 + \frac{1}{2} \langle \tau^2 \rangle = \tau_0 \left( 1 - \frac{\langle N^2 \rangle}{N_0^2} \frac{\tau_1}{f_\alpha} \right). \] (29)

For correlation function \( \langle \alpha'(r) \tau'(r) \rangle \) we obtain the formula:
\[ \langle \alpha'(r) \tau'(r) \rangle \equiv \langle \alpha^2 \rangle \tau_1 A_\alpha(x) \approx \frac{\langle N^2 \rangle}{N_0^2} \frac{\tau_1}{f_\alpha} \cdot \frac{\tau_1}{A_\alpha(x)}. \] (30)
As a result, Eq.(17) for $\alpha_{eff}(r)$ takes the form:

$$\alpha_{eff}(r) = [\alpha_0(r) - \langle \alpha'(r)\rangle'] \rightarrow \alpha_0 \left( 1 - \frac{\alpha_0^2}{\alpha_0^2} \tau_1 f_0 \right) \equiv \alpha_0 \left( 1 - \frac{\langle N'(\alpha) \rangle}{N_0^2} \tau_1 f_0 \right). \quad (31)$$

We consider the case when the correlation optical length $\tau_1 = \alpha_0 R_1 << 1$. The level of fluctuations $\langle \alpha_r^2 \rangle / \alpha_0^2$, in principle, may be arbitrary with the only restriction $\alpha_{eff} > 0$. It appears most large fluctuations $N'$ hold in a chaotic shock waves turbulence. Recall (see Eq.(22)), that $\alpha_{eff}(r) = (1 - \varepsilon_1)\alpha_{eff}$.

It is known that emerging radiation mainly goes from optically thickness $\sim 1$. In stochastic atmosphere, where $\alpha_{eff} < \alpha_0$, the emerging radiation goes from deeper layers, than in non-stochastic one. Usually in deep layers the temperature is larger. Thus, the stochastic atmosphere demonstrates more larger temperature than non-stochastic one.

Note that radiation scattering on various types of particles (say, with $p_1(\mathbf{n}n')$ and $p_2(\mathbf{n}n')$) in stochastic atmosphere changes the form of transfer equation more significant. This will be clear in the next section, where we consider the scattering on two types of particles.

### 3 Stochastic atmosphere with two type of scattering particles

In previous section we considered the stochastic atmosphere with the identical scattering particles. Now we consider more complex situation when the stochastic atmosphere consists of two types of scattering particles. As an example of such atmosphere can be considered the gas dusty accretion disc and the torus around the AGNs nuclei.

Let these components are characterized by the number densities $N_1(r) = N_1^{(0)} + N_1^{(1)}$ and $N_2(r) = N_2^{(0)} + N_2^{(1)}$. The corresponding cross-sections are $\sigma_1^{(t)} = \sigma_1^{(s)} + \sigma_1^{(a)}$ and $\sigma_2^{(t)} = \sigma_2^{(s)} + \sigma_2^{(a)}$. The standard radiative transfer equation in this case has the form:

$$\frac{dI(n, r)}{dr} = (n)I(n, r) = -\alpha I(n, r) + \frac{\alpha_{eff}^{(s)}}{4\pi} \int d'n' p_1(n \cdot n')I(n', r) +$$

$$\frac{\alpha_{eff}^{(s)}}{4\pi} \int d'n' p_2(n \cdot n')I(n', r) + S(n, r), \quad (32)$$

where

$$\alpha \equiv \alpha_0 = N_1^{(0)} \sigma_1^{(t)} + N_2^{(0)} \sigma_2^{(t)} \equiv \alpha_1^{(0)} + \alpha_2^{(0)},$$

$$\alpha_1^{(s)} = N_1^{(0)} \sigma_1^{(s)} = (1 - \varepsilon_1)\alpha_1^{(0)},$$

$$\alpha_2^{(s)} = N_2^{(0)} \sigma_2^{(s)} = (1 - \varepsilon_2)\alpha_2^{(0)}. \quad (33)$$

It is easy prove that the conservation law of the radiative flux (see Eq.19) holds in this transfer equation. This law corresponds to conservative atmosphere without the source of radiation, i.e. $\varepsilon_1 = 0, \varepsilon_2 = 0, S(n, r) = 0$. Recall, that the phase functions $p_1(n \cdot n')$ and $p_2(n \cdot n')$ obey the condition (20).

If we assume $dr = \alpha_0 dr = (N_1^{(0)} \sigma_1^{(t)} + N_2^{(0)} \sigma_2^{(t)})dr \equiv N_2^{(0)} \sigma_2^{(t)}(1 + \eta)dr$, then Eq.(32) takes the form:

$$\frac{dI(n, \tau)}{d\tau} = (n)I(n, \tau) = -I(n, \tau) + \frac{(1 - \varepsilon_1)\eta}{4\pi(1 + \eta)} \int d'n' p_1(n \cdot n')I(n', \tau) +$$

$$\frac{(1 - \varepsilon_2)}{4\pi(1 + \eta)} \int d'n' p_2(n \cdot n')I(n', \tau) + S(n, \tau), \quad (34)$$

where the parameter $\eta$ is equal to:

$$\eta = \frac{N_1^{(0)} \sigma_1^{(t)}}{N_2^{(0)} \sigma_2^{(t)}} \equiv \frac{\alpha_1^{(0)}}{\alpha_2^{(0)}}. \quad (35)$$

Thus, Eq.(34) depends on one parameter $\eta$.

Assuming that all factors in Eq.(32) are stochastic values, we obtain the following transfer equation for the averaged intensity $I_0(n, r)$:

$$\frac{dI_0(n, r)}{dr} = (n)I_0(n, r) = -\alpha_{eff} I_0(n, r) + \frac{\alpha_{eff}^{(s)}}{4\pi} \int d'n' p_1(n \cdot n')I_0(n', r) +$$
\[
\frac{\alpha_{\text{eff}}^{(s)}}{4\pi} \int d\mathbf{n}' p_2(\mathbf{n} \cdot \mathbf{n}') I_0(\mathbf{n}', r) + S_0(\mathbf{n}, r),
\]

where
\[
\alpha_{\text{eff}} = \alpha_{1\text{eff}} + \alpha_{2\text{eff}} = [N_1^{(0)} \sigma_1^{(t)} - (\gamma_1 + \gamma_3)] + [N_2^{(0)} \sigma_2^{(t)} - (\gamma_2 + \gamma_3)],
\]

\[\eta = \langle \eta \rangle_{N_1^{(0)}} = \langle \eta \rangle_{N_2^{(0)}}\]

Recall, that
\[\langle \eta \rangle_{N_1^{(0)}} - \langle \eta \rangle_{N_2^{(0)}} = (\gamma_1 - \gamma_3)\]

Here we introduced the different lengths of correlation and correlation functions for the values \(N_1^{(0)}\), \(N_2^{(0)}\) and \(N_1^{(0)} N_2^{(0)}\). In general case they are different one from another.

\[\alpha_{1\text{eff}}^{(s)} = (1 - \varepsilon_1)[N_1^{(0)} \sigma_1^{(t)} - (\gamma_1 + \gamma_3)] \equiv (1 - \varepsilon_1)\alpha_{1\text{eff}},\]

\[\alpha_{2\text{eff}}^{(s)} = (1 - \varepsilon_2)[N_2^{(0)} \sigma_2^{(t)} - (\gamma_2 + \gamma_3)] \equiv (1 - \varepsilon_2)\alpha_{2\text{eff}}.\]

Recall, that \(\varepsilon_1 = \sigma_1^{(s)}/\sigma_1^{(t)}\) and \(\varepsilon_2 = \sigma_2^{(s)}/\sigma_2^{(t)}\) are degrees of absorption of the first and second types of scattering particles, respectively. Thus, \(\alpha_{1\text{eff}}\) and \(\alpha_{2\text{eff}}\) correspond to different correlation functions (see Eq.(25)).

If we assume the optical depth \(d\tau = \alpha_{\text{eff}} dr\), then Eq.(36) can be presented in the form of Eq.(34) with \(\eta \rightarrow \eta_{\text{eff}}\):

\[
\frac{dI(\mathbf{n}, \tau)}{d\tau} = (\mathbf{n} \nabla_n) I(\mathbf{n}, \tau) - I(\mathbf{n}, \tau) + \frac{(1 - \varepsilon_1)\eta_{\text{eff}}}{4\pi(1 + \eta_{\text{eff}})} \int d\mathbf{n}' p_1(\mathbf{n} \cdot \mathbf{n}') I(\mathbf{n}', \tau) + \frac{1 - \varepsilon_2}{4\pi(1 + \eta_{\text{eff}})} \int d\mathbf{n}' p_2(\mathbf{n} \cdot \mathbf{n}') I(\mathbf{n}', \tau) + S_0(\mathbf{n}, \tau),
\]

where the value \(\eta_{\text{eff}}\) has the form:

\[
\eta_{\text{eff}} = \frac{\alpha_{1\text{eff}}}{\alpha_{2\text{eff}}} = \frac{N_1^{(0)} \sigma_1^{(t)} - (\gamma_1 + \gamma_3)}{N_2^{(0)} \sigma_2^{(t)} - (\gamma_2 + \gamma_3)}.\]

Thus, in stochastic atmosphere with two types of scattering particles the parameter \(\eta\) in standard Eq.(34) takes the value \(\eta_{\text{eff}}\). Note that if the particles of the first and the second types are statistically independent one from another, we have \(\gamma_3 = 0\).

It is very interesting, \(\eta_{\text{eff}}\) is longer or less than \(\eta\)? We consider this problem for specific case, which looks fairly apparent. We take \(\gamma_3 = 0\) and assume, that the correlation lengths \(R_1 = R_2 = R\) and \(f_{1\alpha} = f_{2\alpha} = f_\alpha\). Besides, we assume that the levels of fluctuations are equal: \(\langle N_1^{(0)} \rangle / (N_1^{(0)})^2 = \langle N_2^{(0)} \rangle / (N_2^{(0)})^2 = g\). As a result, we obtain:

\[
\eta_{\text{eff}} \approx \frac{\tau_1}{1 - g f_\alpha \tau_1},
\]

where \(\tau_1 = N_1^{(0)} \sigma_1^{(t)} R\) and \(\tau_2 = N_2^{(0)} \sigma_2^{(t)} R\) are the optical depths of correlation length \(R\) for the first and second extinction factors, respectively. Thus, \(\eta_{\text{eff}} > \eta\), if \(\tau_2 > \tau_1\). In opposite case \(\tau_2 < \tau_1\) we have \(\eta_{\text{eff}} < \eta\). Note that the angular distribution \(J(\mathbf{n})\) of emerging radiation depends on parameter \(\eta\) (see Ch.5). The difference \(\eta_{\text{eff}} - \eta\) changes the values \(J(\mathbf{n})\) for stochastic atmosphere from the value in non-stochastic case.

In the next section we consider the stochastic atmosphere consisting of free electrons and small anisotropic particles (molecules or dust grains). For simplicity, we will name them as grains. Most interesting examples are the gas dusty accretion discs. Note that according to standard theory with \(\alpha_{SS} = 0.01\) (Shakura & Sunyaev 1973; Pariev & Colgate 2007) the temperature in accretion disc is too high for presence of small dust grains. Recall, that \(\alpha_{SS}\) is viscosity factor. It is known that in AGNs of Seyfert galaxies the optically thick gas dusty tori exist (see, for example, Sneden & Gaskell 2007; Gaskell 2011). There exists the dust inflow in accretion disc. This supports the dust component in accretion disc. Of course, instead of free electrons we can consider atoms with the spherical polarizability. In this case we are take \(\sigma_T \rightarrow \sigma_{\text{atom}}\).
4 Radiative transfer equation for Stokes parameters

The propagation of polarized radiation in stochastic atmosphere is analogous to the case of intensity propagation. The difference is that instead of the scalar standard radiative transfer equation (Eq.(32)), we consider the matrix transfer equation for Stokes parameters $I(\mathbf{n}, r)$, $Q(\mathbf{n}, r)$ and $U(\mathbf{n}, r)$. Recall, that the Stokes parameter $V(\mathbf{n}, r)$, describing the circular polarization, obeys the separate scalar transfer equation. We do not consider this equation. Chandrasekhar (1960) considered in detail the system of transfer equations for parameters $I(\mathbf{n}, r)$, $I_r(\mathbf{n}, r)$ and $U(\mathbf{n}, r)$, where intensity $I_r(\mathbf{n}, r)$ describes the radiation linearly polarized in the plane ($\mathbf{n}\mathbf{N}$), and $I_r$ is the intensity with polarization perpendicular to this plane. Here $\mathbf{n}$ is the line of sight and $\mathbf{N}$ is the normal to the surface of an atmosphere. We introduce parameter $\mu = \cos \vartheta$, where $\vartheta$ is the angle between $\mathbf{n}$ and $\mathbf{N}$. The intensity $I(\mathbf{n}, r) = I_l(\mathbf{n}, r) + I_r(\mathbf{n}, r)$ and the parameter $Q(\mathbf{n}, r) = I_l(\mathbf{n}, r) - I_r(\mathbf{n}, r)$. Below we consider the axially symmetric problem, where parameter $U(\mathbf{n}, r) = 0$. Particularly such problem is the Milne problem in non-conservative medium (optically thick torus or accretion disc in AGN).

Frequently one uses the system of equations for $I(\mathbf{n}, r)$ and $Q(\mathbf{n}, r)$. We restrict ourselves by this case. First let us recall the radiative transfer equation for the (column) vector with the components $(I, Q)$ in an atmosphere consisting of averaged small anisotropic particles (molecules or dust grains) and free electrons. The equation for $I(\mu, r)$ and $Q(\mu, r)$ can be readily transformed from the equation for the column $(I_l(\mu, r), I_r(\mu, r))$, presented in Chandrasekhar 1960; Dolginov et al. 1995; Silant’ev et al. 2015:

$$\frac{d}{dr} \left( \frac{I(\mu, r)}{Q(\mu, r)} \right) = -\alpha(r) \left( \frac{I(\mu, r)}{Q(\mu, r)} \right) + \frac{1}{2} \left[ \frac{3}{8} \alpha^{(s)}(r) \int_{-1}^{1} d\mu' \times ight.$$ 

$$\left( \begin{array}{c}
3 - \mu^2 - \mu'^2 + 3\mu^2\mu'^2, 1 - \mu^2 - 3\mu^2 + 3\mu^2\mu'^2 \\
1 - \mu^2 - 3\mu^2 + 3\mu^2\mu'^2, 3(1 - \mu^2)(1 - \mu'^2) 
\end{array} \right) + \beta^{(s)}(r) \int_{-1}^{1} d\mu' \left( \begin{array}{c}
1, 0 \\
0, 0 
\end{array} \right) \right] \left( \frac{I(\mu', r)}{Q(\mu', r)} \right) + S(r) \left( \begin{array}{c}
1 \\
0 
\end{array} \right),$$

(44)

where the extinction factor is equal to:

$$\alpha(r) \equiv \alpha_0 = N_e^{(0)}(r)\sigma_T + N_g^{(0)}(r)\sigma_g^{(i)}. \quad (45)$$

In Eq.(44) the factors before the integral terms are:

$$\alpha^{(s)}(r) \equiv \alpha_0^{(s)} = N_e^{(0)}(r)\sigma_T + (1 - \varepsilon)N_g^{(0)}(r)\sigma_g^{(i)} \bar{b}_1 \quad (46),$$

$$\beta^{(s)}(r) \equiv \beta_0^{(s)}(r) = 3(1 - \varepsilon)N_g^{(0)}(r)\sigma_g^{(i)} \bar{b}_2. \quad (47)$$

The values $\sigma_g^{(s)}$ and $\sigma_g^{(a)}$ are the cross-sections of scattering and absorption by dust grains, $\sigma_g^{(i)} = \sigma_g^{(s)} + \sigma_g^{(a)}$ is the cross-section of total extinction; $\sigma_T$ is the Thomson cross-section. $N_e(r)$ and $N_g(r)$ are the number densities of free electrons and dust grains, respectively. The degree of the light absorption $\varepsilon = \sigma_g^{(a)}/\sigma_g^{(i)}$, $\mu = \mathbf{n}\mathbf{N}$ is cosine of the angle between the directions of light propagation $\mathbf{n}$ and the outer normal $\mathbf{N}$ to plane-parallel semi-infinite atmosphere.

The first integral term describes the Rayleigh scattering and the second one describes the isotropic scattering by anisotropic part of the dust grains. Here we have two stochastic values - $N_e(r) = N_e^{(0)}(r) + N_e'(r)$ and $N_g(r) = N_g^{(0)}(r) + N_g'(r)$.

The general consideration and notations, presented in above section, are also used in this case. We take the parameter $\eta = N_e^{(0)}\sigma_T/(N_g^{(0)}\sigma_g^{(i)})$ and $(1 - \varepsilon) = \sigma_g^{(s)}/\sigma_g^{(i)}$, i.e. $N_e = N_1$ and $N_g = N_2$, $\varepsilon_1 = 0$ and $\varepsilon_2 \equiv \varepsilon$. The difference is that the first integral term depends on the both types of scatterers. For this reason we describe the generalized method given in previous chapter once more.

The dimensionless parameters $\bar{b}_1$ and $\bar{b}_2$ are related with the anisotropic dust grains (or the anisotropic molecules). Parameter $\bar{b}_1$ describes the Rayleigh scattering on chaotically oriented dust grains. According to Chandrasekhar (1960) such grains also demonstrate the isotropic nonpolarized scattered radiation, which is described by the parameter $\bar{b}_2$ (see Silant’ev et al. 2017). These parameters obey the relation $\bar{b}_1 + 3\bar{b}_2 = 1$. For needle like grains parameters $\bar{b}_1 = 0.4$ and $\bar{b}_2 = 0.2$. For plate like particles we have $\bar{b}_1 = 0.7$, $\bar{b}_2 = 0.1$. Parameter $\bar{b}_2$ describes the depolarization of radiation, scattered by freely oriented anisotropic particles. So, the needle like particles depolarize radiation greater than the plate like ones. The relation of parameters $\bar{b}_1$ and $\bar{b}_2$ with the polarizability tensor of a grain (molecule) is given in Dolginov et al. 1995; Silant’ev et al. 2017.

If we take $d\tau = \alpha_0 dr$, then Eq.(44) can be written in the form:

$$\frac{d}{d\tau} \left( \frac{I(\mu, \tau)}{Q(\mu, \tau)} \right) = (\mathbf{n}\nabla_\tau) \left( \frac{I(\mu, \tau)}{Q(\mu, \tau)} \right) = -\left( \frac{I(\mu, \tau)}{Q(\mu, \tau)} \right) + \frac{1}{2} \left[ \frac{3}{8} \int_{-1}^{1} d\mu' \times 
$$

$$\left( \begin{array}{c}
3 - \mu^2 - \mu'^2 + 3\mu^2\mu'^2, 1 - \mu^2 - 3\mu^2 + 3\mu^2\mu'^2 \\
1 - \mu^2 - 3\mu^2 + 3\mu^2\mu'^2, 3(1 - \mu^2)(1 - \mu'^2) 
\end{array} \right) + \beta^{(s)}(r) \int_{-1}^{1} d\mu' \left( \begin{array}{c}
1, 0 \\
0, 0 
\end{array} \right) \right] \left( \frac{I(\mu', r)}{Q(\mu', r)} \right) + S(r) \left( \begin{array}{c}
1 \\
0 
\end{array} \right),$$

(44)
\[
(3 - \mu^2 - \mu'^2 + 3\mu^2\mu'^2, 1 - \mu^2 - 3\mu^2 + 3\mu^2\mu'^2) + \frac{b}{\eta + 1 - \varepsilon} \int_{-1}^{1} \frac{d\mu'}{1 + \eta} \left( I(\mu', \tau) Q(\mu', \tau) \right) + S(\tau) \left( 1 \frac{0}{0} \right),
\]

where parameters \(a\), \(b\) and \(\eta\) are:

\[
a = \eta + \frac{(1 - \varepsilon)\bar{b}_1}{1 + \eta}, \quad b = \frac{(1 - \varepsilon)\bar{b}_2}{1 + \eta},
\]

\[
\bar{b}_1 + 3\bar{b}_2 = 1, \quad a + b = \frac{\eta + 1 - \varepsilon}{1 + \eta},
\]

\[
\eta = \frac{N_e(0)}{N_g(0)} \frac{\sigma_T}{\sigma_g},
\]

Assuming that all factors in Eq.(44) are the stochastic values, we average this equation. The value \(\alpha_{eff}\) has the form:

\[
\alpha_{eff}(r) = \alpha_{1eff} + \alpha_{2eff} = [N_e(0)\sigma_T - (\gamma_1 + \gamma_3)] + [N_g(0)\sigma_g - (\gamma_2 + \gamma_3)],
\]

where

\[
\gamma_1 = \langle N_e^2 \rangle \sigma_T^2 R_1 f_{1a}, \quad \gamma_2 = \langle N_g^2 \rangle \sigma_g^2 R_2 f_{2a}, \quad \gamma_3 = \langle N_e',N_g' \rangle \sigma_T \sigma_g \tau_3 f_{3a},
\]

\[
o_0 = N_e(0)\sigma_T + N_g(0)\sigma_g, \quad o' = N_e(0)\sigma_T + N_g(0)\sigma_g
\]

In Eq.(52) three types of the averages exist - \(\langle N_e(s)N_e'(s')\rangle\), \(\langle N_g(s)N_g'(s')\rangle\) and \(\langle N_e'(s')N_g'(s')\rangle\). Every of them is analogous to Eq.(25). As in previous section, we introduced different correlation functions and different lengths of correlation.

Averaging the value \(\alpha(s)\), we obtain:

\[
\alpha_{eff}^{(s)} = \alpha_{1eff} + (1 - \varepsilon)\alpha_{2eff} \bar{b}_1.
\]

\[
\alpha_{0}^{(s)} = N_e(0)\sigma_T + (1 - \varepsilon)\bar{b}_1 N_g(0)\sigma_g, \quad \alpha'_{s} = N_e(0)\sigma_T + (1 - \varepsilon)\bar{b}_1 N_g'(0)\sigma_g.
\]

For the second integral term in Eq.(44) we have:

\[
\beta_{eff}^{(s)} = 3\bar{b}_2(1 - \varepsilon)[N_e(0)\sigma_g - (\gamma_2 + \gamma_3)] \equiv \beta_{2eff}(1 - \varepsilon)\alpha_{2eff},
\]

\[
\beta_{0}^{(s)} = 3\bar{b}_2 N_g(0)\sigma_g, \quad \beta'_{s} = 3\bar{b}_2(1 - \varepsilon)N_g'(0)\sigma_g.
\]

If we introduce the effective optical depth \(d\tau = a_{eff} dr\), then the equation for averaged values \(I_0(\mu, r)\) and \(Q_0(\mu, r)\) takes the form analogous to Eq.(48), where \(a \rightarrow a_{eff}\) and \(b \rightarrow b_{eff}\):

\[
\frac{d}{d\tau} \left( \frac{I_0(\mu, \tau)}{Q_0(\mu, \tau)} \right) = \left( n\nabla_{\tau} \right) \left( \frac{I_0(\mu, \tau)}{Q_0(\mu, \tau)} \right) = - \left( \frac{I_0(\mu, \tau)}{Q_0(\mu, \tau)} \right) + \frac{1}{2} \frac{3}{8} a_{eff} \int_{-1}^{1} d\mu' \times
\]

\[
\left( 3 - \mu^2 - \mu'^2 + 3\mu^2\mu'^2, 1 - \mu^2 - 3\mu^2 + 3\mu^2\mu'^2 \right) + b_{eff} \int_{-1}^{1} d\mu' \left( \frac{1, 0}{0, 0} \right) \left( \frac{I_0(\mu', \tau)}{Q_0(\mu', \tau)} \right) + S(\tau) \left( \frac{1, 0}{0, 0} \right),
\]

where we introduced the notations:

\[
a_{eff} = \frac{a_{eff}^{(s)}}{a_{eff}} = \eta_{eff} + (1 - \varepsilon)\bar{b}_1, \quad b_{eff} = \frac{b_{eff}^{(s)}}{b_{eff}} = \frac{(1 - \varepsilon)\bar{b}_2}{1 + \eta_{eff}}, \quad a_{eff} + b_{eff} = \frac{\eta + 1 - \varepsilon}{1 + \eta}.
\]

Recall, that \(\bar{b}_1 + 3\bar{b}_2 = 1\).

Parameter \(\eta_{eff}\) is equal to:

\[
\eta_{eff} = \frac{N_e(0)\sigma_T - (\gamma_1 + \gamma_3)}{N_g(0)\sigma_g - (\gamma_2 + \gamma_3)}.
\]

This formula coincides with Eq.(42). The estimates in Eq.(43) take place also for Eq.(60). Recall, that here \(\tau_1 = N_e\sigma_T R\) and \(\tau_2 = N_g\sigma_g R\). Eq.(58) can be written in more simple form in the new notations:

\[
W_{eff} = \frac{a_{eff}}{a_{eff} + b_{eff}}, \quad C_{eff} = \frac{W_{eff}}{8}.
\]
Using the new factorization (see Frisch 2017), Eq.(58) can be presented as:

$$\frac{d}{d\tau} \begin{pmatrix} I_0(\mu, \tau) \\ Q_0(\mu, \tau) \end{pmatrix} = (\mathbf{n} \nabla_\tau) \begin{pmatrix} I_0(\mu, \tau) \\ Q_0(\mu, \tau) \end{pmatrix} = - \begin{pmatrix} I_0(\mu, \tau) \\ Q_0(\mu, \tau) \end{pmatrix} + \hat{A}(\mu) \mathbf{K}(\tau) + S_0(\tau) \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

(62)

where we introduced the vector \( \mathbf{K}(\tau) \):

$$\mathbf{K}(\tau) = \frac{1}{2} \int_{-1}^{1} d\mu' \hat{A}^T(\mu') \begin{pmatrix} I_0(\mu', \tau) \\ Q_0(\mu', \tau) \end{pmatrix}. $$

(63)

The factorization matrix \( \hat{A}(\mu) \) has the form:

$$\hat{A}(\mu) = \sqrt{(a_{eff} + b_{eff}) \left( \begin{array}{cc} 1, & \sqrt{C_{eff}}(1 - 3\mu^2) \\ 0, & 3\sqrt{C_{eff}}(1 - \mu^2) \end{array} \right)}. $$

(64)

Superscript \( T \) stands for the matrix transpose. It is easy prove that the conservation law for the radiation flux (19) is also valid from Eqs.(44),(48),(58) and (62) for \( \varepsilon = 0 \) and \( S(\tau) = 0 \), if we take into account the relation \( \mathbf{b}_1 + 3\mathbf{b}_2 = 1 \).

The vector \( \mathbf{K}(\tau) \) obeys the integral equation (see Silant’ev et al. 2015, 2017):

$$\mathbf{K}(\tau) = \mathbf{g}(\tau) + \int_{0}^{\infty} d\tau' \hat{L}(|\tau - \tau'|) \mathbf{K}(\tau'). $$

(65)

The term \( \mathbf{g}(\tau) \) presents the contribution of the source \( S_0(\tau) \). The matrix kernel \( \hat{L}(|\tau - \tau'|) \) has the form:

$$\hat{L}(|\tau - \tau'|) = \int_{0}^{1} d\mu \exp \left( -\frac{|\tau - \tau'|}{\mu} \right) \hat{\Psi}(\mu), $$

(66)

where

$$\hat{\Psi}(\mu) = \frac{1}{2} \hat{A}^T(\mu) \hat{A}(\mu). $$

(67)

It should be noted that Eq.(65) can be solved by resolvent matrix technique (see Silant’ev et al. 2014, 2015).

In the limiting cases of Eq.(62), where \( N_e = 0 \) (the scattering on free electrons) and \( N_\eta = 0 \) (the scattering on small dust grains) this equation formally coincides with that in non-stochastic atmosphere. The difference is that in stochastic atmosphere we have \( d\tau = \alpha_{eff}dr \) instead of \( d\tau = \alpha_0dr \) for non-stochastic case.

5 The generalized Milne problem in stochastic atmosphere

The standard Milne problem describes the propagation of non-polarized radiation from deep layers of non-absorbed atmosphere. In such case the flux \( \mathbf{F}(\tau) \) is similar at every distance in the atmosphere. The generalized Milne problem describes the propagation of non-polarized radiation from deep layers of absorbing atmosphere. Clearly, the radiation flux \( \mathbf{F}(\tau) \) in this case is different at different distances in the atmosphere. The radiation flux emerging from atmosphere can be deal with the observing flux. In the both cases of the Milne problems we can calculate the angular distribution \( J(\mu) \) and the degree of polarization \( p(\mu) \) for emerging radiation. The exact solution of generalized Milne problem for Eq.(48) was given in Silant’ev et al. 2017. In that paper we presented the angular distribution \( J(\mu) \) and polarization degree \( p(\mu) \) for various values of absorption degree \( \varepsilon \) and parameter \( \eta \). The case of pure electron atmosphere was also presented.

The specific feature of the Milne problems is that we are to solve the integral equation for \( \mathbf{K}(\tau) \) without the source term \( \mathbf{S}(\tau) \), i.e. this is the homogeneous integral equation (see Sobolev 1969, Silant’ev et al. 2017). It is known (see Smirnov 1964) that such equations have the nonzero solution, if there is the solution of characteristic equation for number \( k(0 \leq k \leq 1) \):

$$\left| \hat{E} - 2 \int_{0}^{1} d\mu \frac{\hat{\Psi}(\mu)}{1 - k^2 \mu^2} \right| = 0. $$

(68)

Note that the matrix \( \hat{\Psi}(\mu) \) depends on \( a_{eff} \) and \( b_{eff} \). The most important feature of the Milne problems is that the angular distribution of emerging intensity \( J(\mu) \) depends on \( k \) as:

$$J(\mu) \sim \frac{1}{1 - k\mu}, $$

(69)
i.e. the angular distribution peaks along the normal \( \mathbf{N} \) to the atmosphere.

Equation (58), describing the radiative transfer in stochastic atmosphere, coincides with Eq.(48), if we take \( a \to a_{\text{eff}} \) and \( b \to b_{\text{eff}} \). Thus, the angular distribution \( J(\mu) \) and polarization degree \( p(\mu) \), given in Silant’ev et al.(2017), are true for Eq.(58), if instead of parameters \( a \) and \( b \) we take the parameters \( a_{\text{eff}} \) and \( b_{\text{eff}} \). If the observed values \( J(\mu) \) and \( p(\mu) \) do not correspond to accepted parameters \( a \) and \( \eta \), then we can take the values \( a_{\text{eff}} \) and \( \eta_{\text{eff}} \) in order to obtain the observed data. It should be noted that the parameter \( \eta_{\text{eff}} \) may be both longer and less of parameter \( \eta \).

It is known (see Silant’ev et al. 2017) that for Eq.(48) the approximate formula for characteristic number \( k \) has the form:

\[
k \simeq \sqrt{\frac{3\varepsilon}{1 + \eta}}, \quad \varepsilon \ll 1.
\]

For stochastic atmosphere (Eq.(58)) the expression (70) transforms to:

\[
k \simeq \sqrt{\frac{3\varepsilon}{1 + \eta_{\text{eff}}}}.
\]

Thus, in stochastic atmosphere the angular distribution \( J(\mu) \) and polarization degree \( p(\mu) \) differ from that in non-stochastic one. It is interesting to note that formulas (70) and (71) are independent of the form of dust grains. Silant’ev et al.(2017) demonstrated that the characteristic number \( k \) depends very little on the form of dust grains only for \( \varepsilon \simeq 0.5 \).

6 Conclusion

The objective of the paper is to show how the radiative transfer equations are changed in stochastic atmospheres. In this paper we considered the influence of Gaussian fluctuations of the number densities of scattering particles on the radiative transfer process. The atmosphere is taken non-magnetized. We considered the radiative transfer equation in the atmosphere consisting of both one type scattering particles and two types ones. In both cases the difference of new equation from the standard one is that the optical depth \( d\tau = adr \) is substituted by new dimensionless optical depth \( d\tau = a_{\text{eff}}dr \). It is found that \( a_{\text{eff}} < a \), i.e. the stochastic atmosphere is more transparent than the non-stochastic one. For the atmosphere consisting of two types of scattering particles the difference is more significant. In addition to \( d\tau \to a_{\text{eff}}dr \), the new factors arise before the integral terms. This factors depend on the correlation functions of number densities \( \langle N_1'(r')N_1'(r') \rangle \), \( \langle N_2'(r')N_2'(r') \rangle \) and \( \langle N_1'(r')N_2'(r') \rangle \), where \( N_1' \) and \( N_2' \) are the number densities of the first and second types scattering particles, respectively. Note that the integral terms of standard equation for two types scattering particles depend on one dimensionless parameter \( \eta = N_1\sigma_1^{(t)}/(N_2\sigma_2^{(t)}) \), where \( \sigma_1^{(t)} \) and \( \sigma_2^{(t)} \) are the total cross-sections of the first and the second types of scattering particles, respectively. In stochastic atmosphere the \( \eta \) - parameter transforms to \( \eta_{\text{eff}} \) - parameter, which can be both longer and less than the parameter \( \eta \). This means that the results, obtained for standard radiative transfer equation, can be used, if we substitute the coefficients \( a \) and \( \eta \) by the factors \( a_{\text{eff}} \) and \( \eta_{\text{eff}} \). We give the estimates of parameter \( \eta_{\text{eff}} \) as a function of the level of fluctuations and optical depths of correlation lengths.

7 Acknowledgements.

This research was supported by the Basic Research Program N 21 of Presidium of Russian Academy of Sciences, the Program of the Department of Physical Sciences of Russian Academy of Sciences No 2 and the President Program "The leading scientific schools" N 7241.

The authors are very grateful to a referee for many useful remarks and advices.

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