Excitation of Photons by Inflationary Gravitons

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ABSTRACT

We use a recent result for the graviton contribution to the one loop vacuum polarization to solve the effective field equations for dynamical photons on de Sitter background. Our results show that the electric field experiences a secular enhancement proportional to the number of inflationary e-foldings. We discuss the minimum this establishes for primordial inflation to seed cosmic magnetic fields.

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1 Introduction

Because photons have zero mass it does not take much to affect the long wavelength modes. This has lead many to suspect that the explosive expansion of spacetime during primordial inflation might help to explain cosmic magnetic fields [1]. However, the Maxwell Lagrangian is conformally invariant, which means that free photons cannot locally sense the expansion of spacetime. The search for an inflationary connection has prompted investigations of explicit conformal breaking terms which might be present in the effective action [2]. Quantum effects from the conformal anomaly have been also studied [3].

Conformal breaking from other particles can be communicated to photons. No one knows the gravitational couplings of the charged partners of the Standard Model Higgs (which become the longitudinal polarizations of the $W^{\pm}$ at low energies) but it has been suggested that the inflationary production of minimally coupled Higgs scalars could endow the photon with a mass during inflation [4, 5, 6], and that this might seed the ubiquitous cosmic magnetic fields of the current epoch [7, 8]. An explicit one loop computation of the massless charged scalar contribution to the vacuum polarization on de Sitter background [9, 10] has confirmed the photon mass conjecture [11], although more work needs to be done to connect this to cosmic magnetic fields [12]. Similar one loop results pertain as well when the scalar has a small mass [13, 14].

Because inflation produces more and more charged scalars as time progresses (provided they are light and nearly minimally coupled) the effective photon mass grows. The scalar mass remains small during this process [15, 16, 17] until a static, nonperturbative limit is eventually reached [18]. The vacuum energy drops while this occurs [19] and there are dramatic changes in the electrodynamic forces exerted by point charges and current dipoles [20].

The effects of charged, minimally coupled scalars are fascinating but dependent upon assumptions about the unknown conformal coupling of the Higgs. Gravitons also break conformal invariance so they too can communicate the violence of primordial inflation to the photon sector [21, 22, 23]. Graviton effects are weaker because they are mediated through derivative interactions, but they are universal. Hence they serve to establish the minimum level at which primordial inflation must affect electromagnetism. The purpose of this paper is to complete the derivation of these minimum effects.
Our technique is based on a recent dimensionally regulated and fully renormalized computation of the one loop graviton contribution to the vacuum polarization $i[\Pi^\mu(x; x')]$ on de Sitter background [24]. We use this to quantum correct Maxwell’s equation,

$$\partial_\nu \left[ \sqrt{-g} g^\mu\rho g^{\nu\sigma} F_{\rho\sigma}(x) \right] + \int d^4 x' \left[ \mu^{\nu\sigma} \right](x; x') A_\nu(x') = J^\nu(x), \quad (1)$$

where $g_{\mu\nu}$ is the de Sitter metric, $F_{\rho\sigma} \equiv \partial_\rho A_\sigma - \partial_\sigma A_\rho$ is the usual field strength tensor and $J^\mu(x)$ is the current density. With $J^\mu(x) \neq 0$ one can study how inflationary gravitons alter the electrodynamic response to standard sources, as has recently been done for a point charge and for a point magnetic dipole with the following results [25]:

- An observer co-moving with respect to the sources (hence at an exponentially increasing physical distance) perceives the magnitude of the point charge to increase linearly with co-moving time and logarithmically with the co-moving position;
- The co-moving observer reports only a negative logarithmic spatial variation in the one loop field of the magnetic dipole;
- An observer at fixed invariant distance from the sources perceives no secular change of the point charge; and
- The static observer reports a secular enhancement of the magnetic dipole moment.

For our study we set $J^\mu(x) = 0$ and work out the one loop corrections to dynamical photons.

This paper has four sections of which the first is this Introduction. In section 2 we use the vacuum polarization [24, 25] to derive an equation for the one loop correction to spatial plane wave photon mode functions. This equation is solved in section 3. In section 4 we discuss the minimum our result sets for inflation to seed cosmic magnetic fields.

2 Effective Mode Equation for Photons

The purpose of this section is to convert the quantum corrected Maxwell equation (1) into a simple relation for the one loop corrections to the mode
function of a plane wave photon on de Sitter background. We first specialize to plane wave photons at one loop order. Then we discuss the restrictions imposed by cosmology, by effective field theory and by our lack of knowledge about the initial state.

2.1 Perturbative Formulation

We work on spatially flat sections of de Sitter in conformal coordinates,

\[ ds^2 = a^2(\eta) \left( -d\eta^2 + d\vec{x} \cdot d\vec{x} \right), \]

where \( a(\eta) = -\frac{1}{H\eta} = e^{Ht} \) is the scale factor and \( H \) is the Hubble parameter. Hence the metric can be written as \( g_{\mu\nu} = a^2 \eta_{\mu\nu} \), where \( \eta_{\mu\nu} \) is the Minkowski metric. Because the Maxwell Lagrangian is conformally invariant, all the scale factors cancel in the rightmost term of (1) and we can express it as \( \partial_\nu F^\nu\mu(x) \), where we raise and lower indices with the Minkowski metric, \( F^\nu\mu \equiv \eta^{\nu\rho} \eta^{\mu\sigma} F_{\rho\sigma} \).

We follow [24] in employing a noncovariant representation for the vacuum polarization [26],

\[ i[\mu\Pi^\nu](x; x') = (\eta^{\mu\rho} \eta^{\nu\sigma} - \eta^{\mu\sigma} \eta^{\nu\rho}) \partial_\rho \partial'_\sigma F(x; x') \]

\[ + (\eta^{\mu\nu} \eta^{\rho\sigma} - \eta^{\mu\sigma} \eta^{\nu\rho}) \partial_\rho \partial'_\sigma G(x; x'), \]

where \( \eta^{\mu\nu} \equiv \eta^{\mu\nu} + \delta^\mu_0 \delta^\nu_0 \) is the purely spatial part of the Minkowski metric. (For the transformation to a covariant representation see [27].) Substituting (3) into (1) with \( J^\mu = 0 \) and partially integrating the primed derivatives on the right hand side gives,

\[ \partial_\nu F^\nu\mu(x) = -\partial_\nu \int d^4 x' \left\{ iF(x; x') F^\nu\mu(x') + iG(x; x') F^{\bar{\eta}\nu}(x') \right\}. \]

Here a barred index on any tensor means that its 0-component is zero, for example, \( V^\mu V_\nu = V^\mu - \delta^\mu_0 V^0 \).

Because the structure functions are only known to one loop order there is no alternative to expanding in powers of the quantum gravitational loop counting parameter \( \kappa^2 \equiv 16\pi G \),

\[ F^{\nu\mu}(x) = F^{\nu\mu}_{(0)}(x) + \kappa^2 F^{\nu\mu}_{(1)}(x) + \kappa^4 F^{\nu\mu}_{(2)}(x) + \ldots, \]

\[ F(x; x') = 0 + \kappa^2 F_{(1)}(x; x') + \kappa^4 F_{(2)}(x; x') + \ldots, \]

\[ G(x; x') = 0 + \kappa^2 G_{(1)}(x; x') + \kappa^4 G_{(2)}(x; x') + \ldots. \]
Substituting these expansions into (4) and segregating terms of the same order in $\kappa^2$ gives the tree order and one loop relations,

$$\partial_\nu F^{\nu\mu}_{(0)}(x) = 0 ,$$

$$\partial_\nu F^{\nu\mu}_{(1)}(x) = -\partial_\nu \int d^4x' \left\{ iF_{(1)}(x;x')F^{\nu\mu}_{(0)}(x') + iG_{(1)}(x;x')F^{\nu\mu}_{(0)}(x') \right\} .$$ (9)

For plane wave photons with wave vector $\vec{k}$ and transverse polarization vector $\vec{\varepsilon}(\vec{k}, \lambda)$, the tree order field strengths are,

$$F^{0i}_{(0)}(x) = -\partial_0 u(\eta, k) \times \varepsilon^i e^{i\vec{k}\cdot\vec{x}} , \quad F^{ij}_{(0)}(x) = u(\eta, k) \times i[k^i \varepsilon^j - k^j \varepsilon^i] e^{i\vec{k}\cdot\vec{x}} .$$ (10)

By conformal invariance the tree order mode function is the same in de Sitter conformal coordinates as it is in flat space, $u(\eta, k) = e^{-ik\eta}/\sqrt{2k}$. Because the structure functions depend only on the differences of the spatial coordinates $\Delta \vec{x} \equiv \vec{x} - \vec{x}'$, one can see from equation (9) that the one loop field strengths take the same form (10) as the tree order ones,

$$F^{0i}_{(1)}(x) = -\partial_0 \Delta u(\eta, k) \times \varepsilon^i e^{i\vec{k}\cdot\Delta\vec{x}} , \quad F^{ij}_{(1)}(x) = \Delta u(\eta, k) \times i[k^i \varepsilon^j - k^j \varepsilon^i] e^{i\vec{k}\cdot\Delta\vec{x}} .$$ (11)

(This same form is valid to all orders.) Substituting (11) into (9) and making a few simplifications results in an equation for the one loop correction $\Delta u(\eta, k)$ to the photon mode function,

$$(\partial_0^2 + k^2) \Delta u(\eta, k) = i\kappa \partial_0 \int d^4x' iF_{(1)}(x;x')u(\eta', k)e^{-i\vec{k}\cdot\Delta\vec{x}}$$

$$-k^2 \int d^4x' \left[ iF_{(1)}(x;x') + iG_{(1)}(x;x') \right] u(\eta', k)e^{i\vec{k}\cdot\Delta\vec{x}} .$$ (12)

### 2.2 Schwinger-Keldysh Formalism

The treatment we have given so far applies as well for traditional quantum field theory on flat space (for example, see [28]). However, it is important to understand that the usual effective field equations describe matrix elements of the field operator between states which are free in the asymptotic past and future. These in-out matrix elements provide a correct description of scattering processes in flat space, but they make little sense in cosmology because the universe began with a singularity at some finite time and no one knows how (or even if) it will end. Persisting with the in-out effective field equations for inflationary cosmology would result in two embarrassments from the nonlocal source term on the right hand side of expression (9):
• Because the in-out structure functions do not vanish for $x^\mu$ outside the past light-cone of $x^\mu$ the right hand side of (12) would be dominated by contributions from the far future when the inflated 3-volume is much larger;

• Because the in-out structure functions are complex the result would not be real, even if the tree order field strengths $F^{\mu\nu}_0(x)$ are real.

The more meaningful effective field to study for cosmology is the true expectation value of the field operator in the presence of some state which is released at a finite time. The appropriate field equations for studying expectation values are those of the Schwinger-Keldysh formalism \([29, 30, 31, 32, 33, 34, 35, 36, 37]\). The associated one loop structure functions are \([25]\),

\[
iF(1) = \frac{-1}{8\pi^2} \left\{ H^2 \left[ \ln(a) + \alpha \right] - \left[ \ln(a) \frac{1}{3a^2} - \frac{\beta}{a^2} \right] (\partial^2 + 2Ha\partial_0) + \frac{H}{3a}\partial_0 \right\} \delta^4(x-x') + \frac{a^{-1}\partial^6}{384\pi^3} \left\{ \frac{\theta(\Delta\eta - \Delta x)}{a'} \left( \ln \left[ \frac{1}{4}H^2(\Delta\eta^2 - \Delta x^2) \right] - 1 \right) \right\} - \frac{H^2}{32\pi^3} \left\{ \left[ \frac{\partial^4}{4} + \partial^2 \partial_0^2 \right] \theta(\Delta\eta - \Delta x) \right\},
\]

\[
iG(1) = \frac{H^2}{6\pi^2} \left[ \ln(a) + \frac{3}{4}\gamma \right] \delta^4(x-x') + \frac{H^2 \partial^4}{96\pi^3} \left\{ \theta(\Delta\eta - \Delta x) \left( \ln \left[ \frac{1}{4}H^2(\Delta\eta^2 - \Delta x^2) \right] - 1 \right) \right\},
\]

In these and subsequent expressions the coordinate separations are $\Delta\eta \equiv \eta - \eta'$ and $\Delta x \equiv \|\vec{x} - \vec{x}'\|$ and the flat space d’Alembertian is $\partial^2 \equiv \eta^\mu\eta_\nu \partial_\mu \partial_\nu = -\partial_0^2 + \nabla^2$. Note that expressions (13-14) are manifestly real, and that the factors of $\theta(\Delta\eta - \Delta x)$ make each term vanish whenever the point $x^\mu$ strays outside the past light-cone of $x^\mu$.\(^1\) These are important features of the Schwinger-Keldysh formalism which the in-out formalism lacks.

The constants $\alpha$, $\beta$ and $\gamma$ which appear in expressions (13-14) represent the arbitrary finite parts of the three higher derivative counterterms which were needed to renormalize the vacuum polarization \([24]\) because Einstein + Maxwell is not perturbatively renormalizable \([38, 39]\). No physical principle

\(^1\)One consequence is that spatial integration by parts produces no surface terms in the Schwinger-Keldysh formalism. Partial integration in time can and does produce surface terms at the initial time.
can fix these constants because those counterterms cannot actually be present in fundamental theory. They are the price we must pay for using Einstein + Maxwell as a low energy effective field theory. In contrast, the logarithms of the scale factor with which the three constants are paired,

\[ \ln(a) + \alpha , \quad \frac{1}{3} \ln(a) - \beta , \quad \ln(a) + \frac{3}{4} \gamma , \]  

(15)

represent unique and reliable predictions of the theory which must persist in whatever is the correct ultraviolet completion of Einstein + Maxwell. At late times these logarithms dwarf the unknown constants, which means that we can make reliable predictions in the late time regime.

Another limit to the generality of our formalism is that the structure functions (13-14) were computed without correcting the free vacuum state. For in-out matrix elements we typically do not worry about correcting the states because infinite time evolution is supposed to accomplish this in the weak operator sense. However, when the universe is released at a finite time one must include at least the perturbative corrections to the initial state. In the Schwinger-Keldysh formalism these corrections show up as new interaction vertices on the initial value surface [40]. Unlike the finite parts of the higher derivative counterterms, it is perfectly possible to work these corrections out [41]. However, there is no point to doing so because they give rise to surface terms which fall off like powers of the inflationary scale factor. We shall assume that these corrections simply serve to cancel the surface terms which would arise, at various points, from partial integrations.

3 Solving the Equation

The purpose of this section is to solve equation (12) for \( \Delta u(\eta, k) \) in the late time limit for which reliable predictions can be made. The first step is integrating the various constituents of the structure functions (13-14) against tree order plane wave photons,

\[ I_1(\eta, k) = -\frac{a^{-1}(\partial_0^2 + k^2)^3}{384 \pi^3} \int d^4x' \Theta \left[ \ln \left( \frac{H^2}{4} \left( \Delta \eta'^2 - \Delta x'^2 \right) \right) - 1 \right] u(\eta', k) e^{-ik \cdot \Delta x'} , \]  

(16)

\[ I_2(\eta, k) = -\frac{H^2(\partial_0^2 + k^2)^2}{128 \pi^3} \int d^4x' \Theta \left[ \ln \left( \frac{H^2}{4} \left( \Delta \eta'^2 - \Delta x'^2 \right) \right) - 1 \right] u(\eta', k) e^{-ik \cdot \Delta x'} , \]  

(17)

\[ I_3(\eta, k) = \frac{H^2(\partial_0^2 + k^2) \partial_0^2}{32 \pi^3} \int d^4x' \Theta \left[ \ln \left( \frac{H^2}{4} \left( \Delta \eta'^2 - \Delta x'^2 \right) \right) + 1 \right] u(\eta', k) e^{-ik \cdot \Delta x'} , \]  

(18)
\[ I_4(\eta, k) = -\frac{H^2 \ln(a)}{8\pi^2} \int d^4x' \delta^4(x-x') u(\eta', k) e^{-i\vec{k} \cdot \Delta \vec{x}'}, \]  
\[ I_5(\eta, k) = \frac{-a^{-2} \ln(a)(\partial_0^2 + k^2)}{24\pi^2} \int d^4x' \delta^4(x-x') u(\eta', k) e^{-i\vec{k} \cdot \Delta \vec{x}'}, \]  
\[ I_6(\eta, k) = \frac{H a^{-1} \ln(a) \partial_0}{12\pi^2} \int d^4x' \delta^4(x-x') u(\eta', k) e^{-i\vec{k} \cdot \Delta \vec{x}'}, \]  
\[ I_7(\eta, k) = \frac{H a^{-1} \partial_0}{24\pi^2} \int d^4x' \delta^4(x-x') u(\eta', k) e^{-i\vec{k} \cdot \Delta \vec{x}'}, \]

where, to save space, we have defined the causality-enforcing \( \Theta \)-function as \( \Theta \equiv \theta(\Delta \eta - \Delta x) \). The local contributions \( I_{4-7}(\eta, k) \) are given in section 3.1, then the more difficult nonlocal contributions \( I_{1-3}(\eta, k) \) are evaluated in section 3.2. The dominant contribution at late times turns out to be from \( I_4(\eta, k) \). In section 3.3 we use it to compute the leading behavior of \( \Delta u(\eta, k) \) for late times, which means \( \eta \to 0^- \).

### 3.1 Local Contributions

The local contributions (19-22), are trivial because of the delta function,

\[ I_4(\eta, k) = -\frac{H^2 \ln(a)}{8\pi^2} \times u(\eta, k), \]  
\[ I_5(\eta, k) = 0, \]  
\[ I_6(\eta, k) = -\frac{ikH \ln(a)}{12\pi^2 a} \times u(\eta, k), \]  
\[ I_7(\eta, k) = \frac{ikH}{24\pi^2 a} \times u(\eta, k). \]

In reaching (24-26) we have used the form of the tree order mode function,

\[ u(\eta, k) \equiv \frac{e^{-ik\eta}}{\sqrt{2k}} \Rightarrow \partial_0 u(\eta, k) = iku(\eta, k). \]  

### 3.2 Nonlocal Contributions

The first step is to perform the angular integrations using the formula,

\[ \int d\delta x' \Theta f(\Delta x)e^{-i\vec{k} \cdot \Delta \vec{x}} = 4\pi \int_0^{\Delta \eta} dr r^2 f(r) \frac{\sin(kr)}{kr}. \]  

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Employing relation (28) in (16-18) allows us to write,

\[
I_1 = -\frac{a^{-1}(\partial_0^2 + k^2)^3}{96\pi^2 k} \int_{\eta_i}^{\eta} d\eta' \frac{u(\eta', k)}{a'} \int_0^{\Delta \eta} drr \sin(kr) \left\{ \ln \left[ \frac{H^2}{4} (\Delta \eta^2 - r^2) \right] - 1 \right\},
\]

\[
I_2 = -\frac{H^2(\partial_0^2 + k^2)^2}{32\pi^2 k} \int_{\eta_i}^{\eta} d\eta' u(\eta', k) \int_0^{\Delta \eta} drr \sin(kr) \left\{ \ln \left[ \frac{H^2}{4} (\Delta \eta^2 - r^2) \right] - 1 \right\},
\]

\[
I_3 = \frac{H^2(\partial_0^2 + k^2)^2}{8\pi^2 k} \int_{\eta_i}^{\eta} d\eta' u(\eta', k) \int_0^{\Delta \eta} drr \sin(kr) \left\{ \ln \left[ \frac{H^2}{4} (\Delta \eta^2 - r^2) \right] + 1 \right\},
\]

where \( \eta_i \equiv -H^{-1} \) denotes the initial time. The next step is to perform the two independent radial integrations,

\[
J_1(\Delta \eta, k) \equiv \int_0^{\Delta \eta} drr \sin(kr) = \frac{T(k \Delta \eta)}{k^2},
\]

\[
J_2(\Delta \eta, k) \equiv \int_0^{\Delta \eta} drr \sin(kr) \ln \left[ \frac{H^2}{4} (\Delta \eta^2 - r^2) \right],
\]

\[
= 2 \ln \left( \frac{H \Delta \eta}{k} \right) T(k \Delta \eta) + \frac{1}{k^2} \int_0^{\Delta \eta} \frac{dt}{t} \left\{ T[k \Delta \eta(1 - 2t)] - T(k \Delta \eta) \right\},
\]

where \( T(x) \equiv \sin(x) - x \cos(x) = \frac{1}{6}x^3 + O(x^5) \). This allows us to express the integrals \( I_{1-3}(\eta, k) \) as,

\[
I_1(\eta, k) = -\frac{a^{-1}(\partial_0^2 + k^2)^3}{96\pi^2 k} \int_{\eta_i}^{\eta} d\eta' \frac{u(\eta', k)}{a'} \left\{ J_2(\Delta \eta, k) - J_1(\Delta \eta, k) \right\},
\]

\[
I_2(\eta, k) = -\frac{H^2(\partial_0^2 + k^2)^2}{32\pi^2 k} \int_{\eta_i}^{\eta} d\eta' u(\eta', k) \left\{ J_2(\Delta \eta, k) - J_1(\Delta \eta, k) \right\},
\]

\[
I_3(\eta, k) = \frac{H^2(\partial_0^2 + k^2)^2}{8\pi^2 k} \int_{\eta_i}^{\eta} d\eta' u(\eta', k) \left\{ J_2(\Delta \eta, k) + J_1(\Delta \eta, k) \right\}.
\]

Because the various integrands of (35-37) vanish like \( \Delta \eta^3 \ln(\Delta \eta) \) at \( \eta' = \eta \) we can pass one factor of the differential operator \( (\partial_0^2 + k^2) \) through the integration to act on \( J_1(\Delta \eta, k) \) and \( J_2(\Delta \eta, k) \),

\[
I_1(\eta, k) = -\frac{a^{-1}(\partial_0^2 + k^2)^2}{48\pi^2 k} \int_{\eta_i}^{\eta} d\eta' \frac{u(\eta', k)}{a'} \left\{ 2 \sin(k \Delta \eta) \ln(H \Delta \eta) \right. \\
+ \left. \int_0^{\Delta \eta} \frac{dt}{t} \left[ \sin[k \Delta \eta(1 - 2t)] - \sin(k \Delta \eta) \right] \right\},
\]

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We can pass one more derivative through the integration using the identity, equation in the late time limit. Absence of these corrections is one reason we can only use the effective mode not having included perturbative corrections to the initial state [41]. The at the initial time. This, and the lower initial value divergences, derive from space result [28]. The inverse factor of $\Delta S_{\text{Sitter}}$ corrections. The first integral (43) is a red-shifted version of the flat $I$ $I$ $I$ $I$ $I$ $I$

$$I_2(\eta, k) = -\frac{H^2(\partial_0^2 + k^2)}{16\pi^2 k} \int_{\eta_1}^{\eta} d\eta' u(\eta', k) \left\{ 2 \sin(k\Delta \eta) \ln(H\Delta \eta) + \int_0^{1/t} \frac{dt}{t} \left[ \sin[k\Delta \eta(1-2t)] - \sin(k\Delta \eta) \right] \right\}, \quad (39)$$

$$I_3(\eta, k) = \frac{H^2\partial_0^2}{4\pi^2 k} \int_{\eta_1}^{\eta} d\eta' u(\eta', k) \left\{ 2 \sin(k\Delta \eta) \ln(H\Delta \eta) + 2 \sin(k\Delta \eta) + \int_0^{1/t} \frac{dt}{t} \left[ \sin[k\Delta \eta(1-2t)] - \sin(k\Delta \eta) \right] \right\}. \quad (40)$$

We can pass one more derivative through the integration using the identities,

$$(\partial_0^2 + k^2) \left[ u(\eta', k) f(\Delta \eta) \right] = -(\partial_0 - ik) \times \partial_0' \left[ u(\eta', k) f(\Delta \eta) \right], \quad (41)$$

$$(\partial_0^2 + k^2) \left[ \frac{u(\eta', k)}{a'} f(\Delta \eta) \right] = -(\partial_0^2 + k^2)(\partial_0 - ik) \times \partial_0' \left[ \frac{u(\eta', k)}{a'} f(\Delta \eta) \right] + H(\partial_0 - ik)^2 \times \partial_0' \left[ u(\eta', k) f(\Delta \eta) \right]. \quad (42)$$

The final answers depend upon $\Delta \eta_i \equiv \eta - \eta_i = H^{-1}(1 - \frac{1}{a})$,

$$I_1(\eta, k) = \frac{H^2 u(\eta, k)}{48\pi^2 a} \left\{ \left[ \frac{1 + 2ik\Delta \eta_i + e^{2ik\Delta \eta_i}}{H^2\Delta \eta_i^2} \right] + \left[ \frac{1 + e^{2ik\Delta \eta_i}}{H\Delta \eta_i} \right] - \frac{4ik}{H} \ln(H\Delta \eta_i) - \frac{2ik}{H} \int_0^{1/t} \frac{dt}{t} \left[ e^{2ik\Delta \eta_i t} - 1 \right] \right\}, \quad (43)$$

$$I_2(\eta, k) = \frac{H^2 u(\eta, k)}{16\pi^2} \left\{ -2 \ln(H\Delta \eta_i) - \int_0^{1/t} \frac{dt}{t} \left[ e^{2ik\Delta \eta_i t} - 1 \right] \right\}, \quad (44)$$

$$I_3(\eta, k) = \frac{H^2 u(\eta, k)}{16\pi^2} \left\{ \left[ 6 - 4ik\Delta \eta_i + 2e^{2ik\Delta \eta_i} \right] \ln(H\Delta \eta_i) + e^{2ik\Delta \eta_i} + 7 - 2ik\Delta \eta_i + \int_0^{1/t} \frac{dt}{t} \left[ (3 - 2ik\Delta \eta_i)(e^{2ik\Delta \eta_i t} - 1) + e^{2ik\Delta \eta_i}(e^{-2ik\Delta \eta_i t} - 1) \right] \right\}. \quad (45)$$

Although we have simplified the derivations, the results (44-45) can be read off from an earlier analysis of the graviton corrections to massless corrections [42]. These, and the local contributions of section 3.1, are the intrinsically de Sitter corrections. The first integral (43) is a red-shifted version of the flat space result [28]. The inverse factor of $\Delta \eta_i^2$ it contains diverges quadratically at the initial time. This, and the lower initial value divergences, derive from not having included perturbative corrections to the initial state [41]. The absence of these corrections is one reason we can only use the effective mode equation in the late time limit.
3.3 Late Time Solution

By substituting the structure functions (13-14) into expression (12), and then comparing with our master integrals (16-22), we see that the one loop correction to the photon mode function obeys the equation,

\[(\partial_0^2 + k^2) \Delta u(\eta, k) = ik\partial_0 \left[ I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 \right] \]
\[+ k^2 \left[ -I_1 + \frac{1}{3} I_2 - I_3 + \frac{1}{3} I_4 - I_5 + I_6 + I_7 \right]. \tag{46} \]

Recall that we can only solve the equation reliably in the late time regime. Table 1 gives the late time limiting forms of the master integrals \( I_1 - I_7(\eta, k) \).

From the table we see that the dominant effect derives from the time derivative of \( I_4(\eta, k) \),

\[(\partial_0^2 + k^2) \Delta u(\eta, k) = -\frac{ikH^3 a}{8\pi^2} \times u(\eta, k) + O\left(\ln(a)\right). \tag{47} \]

Hence we find,

\[\Delta u(\eta, k) = \frac{ikH \ln(a)}{8\pi^2} \times u(\eta, k) + O\left(\frac{1}{a}\right). \tag{48} \]
It follows that the one loop field strengths are,

\[ \kappa^2 F_{(1)}^{0i}(x) = \frac{\kappa^2 H^2}{8\pi^2} \left\{ \ln(a) + O(1) \right\} \times F_{(0)}^{0i}(x), \quad (49) \]

\[ \kappa^2 F_{(1)}^{ij}(x) = \frac{\kappa^2 H^2}{8\pi^2} \left\{ \frac{ik \ln(a)}{Ha} + O\left(\frac{1}{a}\right) \right\} \times F_{(0)}^{ij}(x). \quad (50) \]

4 Discussion

We have employed a previous computation of the one loop contribution to the vacuum polarization from inflationary gravitons [24] to derive what happens to photons during primordial inflation. Our results (49-50) for the field strengths show that the electric field experiences a secular enhancement, relative to its classical value. In contrast, the one loop magnetic field is weaker than its classical counterpart. Both results are consistent with the one loop photon wave function (48) relaxing to zero less slowly (by one factor of \(\ln(a)\)) than the classical mode function approaches a constant.

The enhancement we find seems to derive from the buffeting of photons by inflationary gravitons. Even though the photon’s kinetic energy redshifts to zero, its spin does not and this permits it to continue interacting with inflationary gravitons even at late times. The same \(\ln(a)\) enhancement was found for massless fermions [43, 42, 45], and was explicitly tied to the spin interaction [44]. In contrast, massless, minimally coupled scalars neither experience any significant effect from inflationary gravitons [46, 47], nor do they induce a significant effect on inflationary gravitons [48, 49, 50]. Gravitons also have spin so it would be very interesting to see what they do to themselves, as well as to the force of gravity.

An interesting technical detail concerns the comparison of our full one loop computations with the result previously derived using the Hartree approximation [24]. Both calculations give the same time dependence, confirming the general reliability of the Hartree approximation for predicting the functional form. However, the sign of our exact computation differs from that of the Hartree result. This emphasizes the need for making exact computations, and has clear implications for gravitons [51].

One consequence of our result is that quantum gravitational perturbation theory must break down after an enormous number of e-foldings \(\ln(a) \sim 1/\kappa^2 H^2 \sim 10^{10}\). It is also interesting to work out what our result says for the
possibility of inflation seeding cosmic magnetic fields. We can use the 0-point energy to estimate the number of photons created by inflationary gravitons. Because the co-moving time $t$ is related to the conformal time $\eta$ by $dt = a d\eta$, the physical Hamiltonian (which generates evolution in $t$) is $1/a$ times the conformal one. Without inflationary gravitons the physical 0-point energy in a single plane wave photon of wave vector $\vec{k}$ can be computed using the tree order field strengths (10),

$$\frac{1}{2a} \left[ F_{(0)}^{0i} \times F_{(0)}^{0i*} + \frac{1}{2} F_{(0)}^{ij} \times F_{(0)}^{ij*} \right] = \frac{k^2 uu^*}{2a} = \frac{k}{2a} . \tag{51}$$

We can find the effect of inflationary gravitons by using the full quantum-corrected field strengths in (51) and then substituting our one loop results (49-50),

$$\frac{1}{2a} \left[ F^{0i} \times F^{0i*} + \frac{1}{2} F^{ij} \times F^{ij*} \right] \longrightarrow \left\{ 1 + \frac{\kappa^2 H^2 \ln(a)}{8\pi^2} + O(\kappa^4) \right\} \times \frac{k}{2a} . \tag{52}$$

The occupation number $N(\eta, k)$ is defined by equating the 0-point energy (52) to $(N + \frac{1}{2}) \times \frac{k}{a}$,

$$N = \frac{\kappa^2 H^2 \ln(a)}{16\pi^2} + O(\kappa^4) . \tag{53}$$

The remainder of the computation is the same as the analysis [12] for the much larger effect from inflationary scalars (if the Higgs is minimally coupled and still light at inflationary energy scales). Substituting our value (53) for the occupation number into equation (118) of that paper results in the following estimate for the magnetic strength on scale $\ell_0$,

$$B^2(\ell_0) \sim \frac{\hbar^2 G H_I^2 N_I}{8\pi^3 c^4 \ell_0^4} , \tag{54}$$

where $H_I$ is the inflationary Hubble parameter and $N_I$ is the value reached by $\ln(a)$. Plugging in the numbers with $N_I \sim 50$ and $\ell_0 \sim 10$ kpc gives,

$$B(\ell_0) \sim 10^{-61} \text{ Gauss} . \tag{55}$$

This is far too small to have seeded today’s cosmic magnetic fields, but it does serve to establish the absolute minimum effect which must be present from inflation.
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