Continuum Damage Mechanics Approach for Modeling Cumulative-Damage Model

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1.Introduction

Fatigue failure should be considered in the engineering design to ensure safety and reliability during service life [1]. Fatigue failure assessment plays an important role in the design of engineering structures since it ensures the safety of engineering structures during their service lives. Since its initiation by Wohler in 1860, despite enormous efforts, reliable and consistent fatigue models applicable to complex loading history are still under development [2]. Substantial engineering structures in the industry fail by fatigue.

Engineering structures in the industry are usually subjected to variable loading. Variable loading, including variable amplitude and loading paths, is related to the fatigue life of engineering structures. Comparing the fatigue on constant loading, the fatigue on variable loading is always additionally investigated for the interactive effect associated with two adjacent load steps on fatigue life. For constant-amplitude and path loading, several researchers have proposed various fatigue criteria and described the effect of additional hardening behavior related to nonproportional loading paths [3], mean stress [4], and stress gradients [5, 6] on high-cycle fatigue. In general, those approaches can be roughly categorized into three, namely, strain energy method [7], critical plane approach [8, 9], stress-field intensity [10]. Stress gradients, well known as a factor affecting fatigue strength of metals, can be predicted by the theory of stress-field intensity since it was initially proposed aiming to the damage domain. Qylafku [11], Taylor [12], and Zeng [13] employed the stress-field intensity concept to model several fatigue criteria. Despite the significant efforts made on stress-field intensity methods, there is no general consensus as to the suitability of various loading situations, e.g., nonproportional loading. For high-cycle fatigue under variable loading, it has been reported that loading history significantly affects fatigue life, and several damage-accumulation models have been proposed. Despite the significant efforts made on damage-accumulation models, none has
gained widespread acceptance. Among them, Miner’s damage-accumulation rule remains the most commonly adopted in practice [14]. Zhao [15] proposed a corrected Miner’s damage-accumulation rule to improve the accuracy of prediction, while life predictions’ accuracy has still been found unsatisfactory [16]. Drawbacks associated with the rules include their inability to express definite effects of the loading sequence on fatigue life [17]. Some other researchers, including Morrow et al. [18], Wang et al. [19], and Zhu et al. [20] established damage-accumulation models aiming at definite effects. Wang’s rule was validated by multiaxial variable loading tests. Notably, damage parameters and fatigue life under constant-amplitude loading were employed to model Morrow’s plastic work interaction rule and Wang’s rule. Zhu et al. focused on isodamage lines-based methods and proposed a new nonlinear fatigue and Wang’s rule. Zhu et al. based methods and proposed a new nonlinear fatigue damage-accumulation model. Nevertheless, Xia et al. [21] reported that either damage parameter or fatigue life under constant-amplitude loading cannot reveal the definite effect of the loading sequence on fatigue life.

Initiated by Kachanov in 1958, continuum damage mechanics has been employed in modeling cumulative-damage models. Shang et al. [22], Hua et al. [23], and Yuan [24] proposed cumulative-damage models based on the damage-evolution model proposed by Lemaitre, which predicts the life expectancy of engineering structures subjected to variable loading. However, many material parameters in these models have to be identified, making them inconvenient for applications.

Here, we propose a new accumulative-damage model based on continuum damage mechanics, to improve the abovementioned shortcomings. First, the equivalent completely reversed stress amplitude, which accounts for the effect of additional hardening behavior related to nonproportional loading paths, mean stress, stress gradients, and loading history on high-cycle fatigue under variable loading, is elaborated using the stress-field intensity concept. The effect of loading history on fatigue life is established by combing damage parameters and fatigue life under constant-amplitude loading. Here, the equivalent completely reversed stress amplitude under single-stage loading is called the damage parameters. By introducing equivalent completely reversed stress amplitudes into the damage-evolution model, a new cumulative-damage model is established. The obtained rule compares very well with experimental data in the literature, and it is consistent with the previously proposed models, including the Palmgren–Miner law [17], corrected Palmgren–Miner law [15], Morrow’s plastic work interaction rule [18], and Wang’s rule [19]. Moreover, only one parameter is evaluated in our model, and it is very simple to obtain the model parameter.

2. Equivalent Completely Reversed Stress Amplitude

High-cycle fatigue failure is a local phenomenon, and the locality, also called the critical domain, is taken as a research object in stress-field intensity concepts. Nevertheless, the application of stress-field intensity concepts in investigating multiaxial high-cycle fatigue, especially under nonproportional loading, has not been extensively studied. Here, we propose damage parameters, based on continuum damage mechanics and stress-field intensity concepts, to model the cumulative-damage model.

Continuum damage mechanics has provided a method for analyzing damage development by introducing a damage variable into the constitutive stress-strain relationship. As one of the outstanding examples, Lemaitre et al. [25] proposed a differential equation for damage development. Distinguishingly, the damage development equation can be expressed as follows for the components subjected to fully reversed tension:

\[
\frac{dD}{dN} = \frac{\eta \sigma_{eq}^{2p+2}}{(1 - D)^{2p+2}}
\]

To develop the damage development equation available for multiaxial high-cycle fatigue, equation (1) is modified as follows:

\[
\frac{dD}{dN} = \frac{\eta (1 - cr - cr \sin \theta) \sigma_{eq}^{2p+2}}{(1 - D)^{2p+2}},
\]

where \(\sigma_{eq}\) is the damage parameter describing the effect of additional hardening behavior related to nonproportional loading paths and mean stress on high-cycle fatigue under single-stage constant loading. The damage parameter is expressed as equation (3), as proposed by Freitas et al. [26]. The parameter \(c\) proposed by Yao et al. [27], which is employed when considering the effect of stress gradients, is also utilized here, and it is expressed as equation (4). Considering the variety of \(c\) under situations where the mechanical component is subjected to nonproportional loading, \(c\) is calculated when \(\sigma_{eq,\text{max}}\) reaches the maximum value. Certainly,

\[
\sigma_{eq} = \frac{f - 1}{\tau - 1} \left[ \frac{\sigma_y^2}{3} + \tau^2 + \left( 3 - \frac{\sqrt{3} f - 1}{\tau - 1} \right) \sigma_{H,\text{max}} \right]^{1/2}
\]

\[
c = \frac{1}{\max \left[ \sigma_{eq,\text{max}}(t) \right]} \frac{d \sigma_{eq}}{d \tau}
\]

the damage evolution equation can also be obtained from the explicit expression of damage variables as follows:

\[
D = 1 - \left[ 1 - \frac{N_{-1f}}{N_f} \right]^{1/(2p+3)}
\]

where \(N_f\) is the fatigue life of a point located at the critical domain and \(N_{-1f}\) is the predicted life equivalent to that of structural components subjected to fully reversed tensile loading. Both are, respectively, expressed as follows:

\[
N_f = \frac{M \sigma_{eq}^{(2p+2)}}{(2p + 3)(1 - cr - cr \sin \theta)} \left[ 1 - (1 - D_c)^{2p+3} \right],
\]

\[
N_{-1f} = \frac{M \sigma_{eq}^{(2p+2)}}{2p + 3} \left[ 1 - (1 - D_c)^{2p+3} \right].
\]
Introducing stress-field intensity concepts, the average superior limit of the intrinsic damage dissipation work [28] in the critical domain can be obtained as the left side of equation (8). The hypothetical condition that the same average superior limit of the intrinsic damage dissipation work in the critical domain implies the same fatigue life is captured to formulate the damage parameters in our model. To simplify, the average superior limit of the intrinsic damage dissipation work in the critical domain can be obtained as the right side of the following equation:

\[
\frac{1}{V} \int_V \left( \int_0^D Y_{\text{max}} dD \right) dV = \int_0^{D_1} \frac{\sigma_{\text{eq}}^2}{2E(1-D)} dD. \tag{8}
\]

Then, expanding equation (8), the following equation is obtained:

\[
\frac{1}{V} \int_V \left[ \frac{\sigma_{\text{eq}}^2 R_y}{2E(1-D)} - \frac{\sigma_{\text{eq}}^2 R_y}{2E} \right] dV = \frac{\sigma_{\text{eq}}^2}{2E(1-D_1)} - \frac{\sigma_{\text{eq}}^2}{2E}.
\]

\[
R_y = \frac{2(1+y)}{3} + 3(1-2y)\left( \frac{\sigma_{\text{eq}}}{\sigma_{\text{eq}}} \right)^2.
\tag{9}
\]

Substituting equations (5)–(7) into equation (9) and employing first-order approximation, the damage parameters are finally expressed as follows. Notably, several material constants are contracted to consider the self-consistency of the damage parameters:

\[
\sigma_{\text{eq}} = \left( \frac{1}{V} \int_V \left[ \frac{\sigma_{\text{eq}}^2 R_y}{2E} (1-cr - cr \sin \theta) \right] dV \right)^{1/2}.
\tag{10}
\]

Numerous fatigue experiments of variable amplitude have shown that loading history has a remarkable effect on fatigue life. For structural components subjected to uniaxial loading, different loading sequence makes Miner’s cumulative critical value fall in different regions. When structural components are subjected to a high-low loading sequence, Miner’s cumulative critical value falls below one, and conversely, for a low-high loading sequence, the value is greater than one. Here, by combing the damage parameters, the equivalent completely reversed stress amplitude considering the loading sequence is proposed as follows:

\[
\sigma_{\text{eq,seq,1}} = \sigma_{\text{eq,seq,1}} \left( \frac{n_{i-1}}{N_{f_1}} \right)^{\frac{1}{1/2p+3}} (1/2p+2)^{1/2p+3}, \quad (i \geq 2),
\]

\[
\phi_i = \left( N_{f_1} / N_{f_1} \right)^{\frac{1}{1/2p+3}} (1/2p+2)^{1/2p+3},
\tag{11}
\]

where \( \phi_i \) is the contraction towards the partial variables in equation (11).

3. Proposed Cumulative-Damage Model

For smooth structural components subjected to single-stage fully reversed tensile loading, the damage-evolution equation can be obtained from the explicit expression of the damage variable:

\[
\begin{align*}
D &= 1 - \left[ 1 - \frac{n_{i-1}}{N_{f_1}} \right]^{1/2p+3}, \\
N_{f_0} &= \frac{M\sigma_{\text{eq}}^{2p+2}}{2p+3} \left[ 1 - (1-D)^{2p+3} \right].
\end{align*}
\tag{12}
\]

Substituting equation (11) into (12), the expression of the damage-evolution associated with ith loading step can be obtained as follows:

\[
D_i = 1 - \left[ 1 - \left( \frac{n_{i-1}}{N_{f_i}} \right) \right]^{1/2p+3}, \quad (i \geq 2). \tag{13}
\]

Then, the expression of damage-evolution associated with the first loading step (equation (13)) should be rewritten as follows:

\[
D_1 = 1 - \left( \frac{n_{i-1}}{N_{f_1}} \right) \right]^{1/2p+3}. \tag{14}
\]

For structural components subjected to two-step loading, the expression of damage-evolution associated with the second loading step can be expressed as follows:

\[
D_2 = 1 - \left[ 1 - \left( \frac{n_{i-1}}{N_{f_1}} \right) \right]^{1/2p+3}. \tag{15}
\]

Based on the damage-equivalent concepts [19], if the fatigue life and applied cycles under \( i-1 \)th constant-amplitude loading are \( N_{f_i} \) and \( n_{i-1} \), then the damage caused by \( n_{i-1} \) cycle loading can be equivalent to that caused by \( n_i \) cycle loading at the ith level, i.e., \( D_i = D_{i-1} \). Then, the equivalent damage of the first loading step can be obtained through the second loading step by combining equations (14) and (15). The equivalent relation can be expressed as follows:

\[
\frac{n_{i+1}}{N_{f_2}} = \left( \frac{n_{i}}{N_{f_1}} \right)^{\phi_i}. \tag{16}
\]

Applying the superposable fatigue-life condition to a single-loading step, \( N_{f_2} = n_1 + n_2 \), and the cumulative-damage model for two steps loading can be established as follows:

\[
\left( \frac{n_1}{N_{f_1}} \right)^{\phi_i} + \frac{n_2}{N_{f_2}} = 1. \tag{17}
\]
critical value fall in different regions. The principle associated with the conclusion is expressed in equation (18). For structural components subjected to high-low two-step uniaxial loading (fully reversed tensile loading), Miner’s cumulative critical value falls below one, and conversely, the sum is greater than one:

\[
\begin{align*}
\frac{n_1}{N_{f1}} + \frac{n_2}{N_{f2}} &\geq \left( \frac{n_1}{N_{f1}} \right)^{\phi_1} + \left( \frac{n_2}{N_{f2}} \right)^{\phi_1} = 1, \quad (\sigma_{a2} \geq \sigma_{a1}, N_{f1} \geq N_{f2}), \\
\frac{n_1}{N_{f1}} + \frac{n_2}{N_{f2}} &< \left( \frac{n_1}{N_{f1}} \right)^{\phi_1} + \left( \frac{n_2}{N_{f2}} \right)^{\phi_1} = 1, \quad (\sigma_{a2} < \sigma_{a1}, N_{f1} < N_{f2}).
\end{align*}
\]

(18)

Furthermore, for structural components subjected to three-step loading, the expression of damage-evolution associated with the third loading step can be expressed as equation (19), and it is obtained from equation (13). The damage-evolution associated with the second loading step can be obtained from equation (15), and it is explicitly expressed in equation (20):

\[
D_3 = 1 - \left[ 1 - \left( \frac{n_3}{N_{f2}} \right)^{1-\phi_1} \left( \frac{n_3}{N_{f2}} \right)^{-1/2p+3} \right],
\]

(19)

\[
D_2 = 1 - \left[ 1 - \left( \frac{n_1}{N_{f1}} \right)^{1-\phi_1} \left( \frac{n_1}{N_{f1}} \right)^{-1/2p+3} \right].
\]

(20)

Then, one can obtain the cumulative-damage model for three-step loading based on the damage-equivalent concepts. The resulting cumulative-damage model for three-step loading is expressed as follows:

\[
\left( \frac{n_1}{N_{f1}} \right)^{1-\phi_1} \left\{ \left( \frac{n_2}{N_{f2}} \right)^{\phi_1} \left( \frac{n_3}{N_{f2}} \right)^{\phi_1-1} + \frac{n_3}{N_{f2}} \right\} + \frac{n_3}{N_{f2}} = 1.
\]

(21)

Similarly, the recurrence formula of the ith level loading can be expressed as follows:

\[
\begin{align*}
\frac{n_i}{N_{f_i}} + \frac{n_j}{N_{f_j}} &= 1, \quad (i \geq 3), \\
\left( \frac{n_{i-2}}{N_{f_{i-2}}} \right)^{1-\phi_1} \left\{ \left( \frac{n_{i-1}}{N_{f_{i-1}}} \right)^{\phi_1} \left( \frac{n_{i-1}}{N_{f_{i-1}}} \right)^{\phi_1-1} + \frac{n_{i-1}}{N_{f_{i-1}}} \right\} &= \frac{n_i}{N_{f_i}}, \\
\frac{n_i}{N_{f_i}} &= \left( \frac{n_{i-1}}{N_{f_{i-1}}} \right)^{\phi_1}.
\end{align*}
\]

(22)

The application of the new cumulative-damage model for predicting the fatigue life of structural components under variable loading involves identifying the material parameter \( p \). Material parameter \( p \) can be obtained by analyzing the experimental data associated with a specimen subjected to fully reversed tensile loading employing the least square method. Certainly, the resulting identification model (equation (23)) is derived by combing equation (12) and the least-square method:

\[
p = \frac{\sum_{i=1}^{f} \log N_{-1f,i} \sum_{i=1}^{j} \log \sigma_{-1a,i} - j \sum_{i=1}^{f} \log N_{-1f,i} \log \sigma_{-1a,i}}{2j \sum_{i=1}^{f} (\log \sigma_{-1a,i})^2} - 1.
\]

(23)

4. Evaluation by the Experimental Data

4.1. Uniaxial Loading Condition. The uniaxial two-level step-loading test data for C35, SAE4130, and 7050-T7451 (Table 1) were used to evaluate the proposed damage-cumulative model. For better comparison, existing popular cumulative-damage models, including Palmgren–Miner law, corrected Palmgren–Miner law, and Wang’s rule, were employed to predict the fatigue life under the loading conditions. All experimental data and material parameters required in these models are listed in Table 2.

For better comparison, the total fatigue life under uniaxial two-level step loading and that predicted using the Palmgren–Miner law, corrected Palmgren–Miner law, Wang’s rule, and the proposed model are plotted on the same coordinate plane (Figure 1). 97.8% of the predicted data fall within an error factor of 2, and the proportion for the data predicted by the other models is 78.2%, 80.4%, and 82.6% for Palmgren–Miner law, corrected Palmgren–Miner law, and Wang’s rule, respectively. To predict fatigue life for uniaxial two-level loading, the proposed model yielded better results than other tested models.

4.2. Multiaxial Loading Condition. To evaluate the proposed model for predicting structural components under multiaxial variable loading, multiaxial two-level and two-stage block-loading test data for LY12CZ (Table 3) were used. The cylindrical specimens were subjected to combined torsion and tension. First, the equivalent completely reversed stress amplitude in the specimen under the stress condition was evaluated. Then, the proposed cumulative-damage model was employed to predict the fatigue life of the cylindrical specimen under multiaxial variable loading.

Considering the cylindrical specimen subjected to combined torsion and tension, at each point located in the critical domain of the specimen, we define the coordinate system (Figure 2). In this frame, the stress state under combined torsion and tension is described by the following components and aimed at each point located in the critical domain:
\[ \begin{align*}
\sigma(r_D, \theta, t) &= \sigma_a \sin \omega t + \sigma_m = \sigma(t), \\
\tau(r_D, \theta, t) &= [\tau_a \sin(\omega t - \delta) + \tau_m] \sqrt{1 + \frac{r_D^2}{R^2} - \frac{2r_D \cos \theta}{R}} = \tau(t) \sqrt{1 + \frac{r_D^2}{R^2} - \frac{2r_D \cos \theta}{R}}.
\end{align*} \] (24)

### Table 1: Uniaxial two-level step-loading test data and predicted life using the tested rules [29, 30].

| Materials | $\sigma_a$ | $\sigma_a$ | $n_i/N_{f1}$ | Min. | C. Miner | Wan. | Pro. | Exp. |
|-----------|-----------|------------|---------------|------|----------|------|------|------|
| C35       | 353       | 275        | 0.1           | 689200 | 461200   | 645282 | 181625 | 353280 |
| C35       | 353       | 275        | 0.25          | 589800 | 355000   | 522964 | 124743 | 226560 |
| C35       | 353       | 275        | 0.5           | 406000 | 178000   | 350078 | 84092  | 108840 |
| C35       | 353       | 275        | 0.75          | 229000 | 1000     | 195578 | 63672  | 80040  |
| C35       | 353       | 275        | 0.1           | 695000 | 467000   | 665427 | 282970 | 425960 |
| C35       | 353       | 275        | 0.25          | 597500 | 369500   | 555952 | 205401 | 209140 |
| C35       | 353       | 275        | 0.5           | 435000 | 207000   | 395480 | 149872 | 203200 |
| C35       | 353       | 275        | 0.75          | 272500 | 44500    | 248607 | 123416 | 141780 |
| C35       | 294       | 353        | 0.1           | 365200 | 245200   | 347626 | 159006 | 218400 |
| C35       | 294       | 353        | 0.25          | 313000 | 193000   | 288484 | 114354 | 200600 |
| C35       | 294       | 353        | 0.5           | 226000 | 106000   | 202826 | 80373  | 129200 |
| C35       | 294       | 353        | 0.75          | 139000 | 9000     | 125034 | 62535  | 80600  |
| C35       | 294       | 353        | 0.1           | 371000 | 251000   | 360633 | 233743 | 245800 |
| C35       | 294       | 353        | 0.25          | 327500 | 207500   | 312617 | 182448 | 214300 |
| C35       | 294       | 353        | 0.5           | 255000 | 135000   | 240633 | 149837 | 173000 |
| C35       | 294       | 353        | 0.75          | 182500 | 62500    | 173737 | 121174 | 137700 |
| C35       | 353       | 294        | 0.1           | 122800 | 107200   | 125055 | 128000 | 134240 |
| C35       | 353       | 294        | 0.25          | 229000 | 213400   | 232769 | 242000 | 243560 |
| C35       | 353       | 294        | 0.5           | 178000 | 106000   | 141485 | 149837 | 156600 |
| C35       | 353       | 294        | 0.75          | 108000 | 56200    | 125034 | 112000 | 129200 |

Note: Min.: Palmgren–Miner damage cumulative law, C. Miner: corrected Palmgren–Miner damage cumulative law, Wan.: Wang YY’s damage cumulative law, Pro.: our proposal, Exp.: median fatigue experimental life, 7050-7050-T7451 aluminum alloy.
Table 2: Experimental data and material parameters required in predicted models [29, 30].

| Materials     | $\sigma_{-1}$ | $N_{-1/f_1}$ | $\sigma_{-1}$ | $N_{-1/f_2}$ | $\sigma_{-1}$ | $N_{-1/f_3}$ | $\sigma_{-1}$ | $N_{-1/f_4}$ | $\sigma_1$ | $p$ |
|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|------------|-----|
| C35           | 353           | 52000         | 334           | 110000        | 294           | 400000        | 275           | 760000        | 216        | 4.30|
| SAE4130       | 648           | 53500         | 607           | 110000        | 565           | 224000        | 552           | 282000        | 391        | 4.17|
| 7050-T7451    | 176           | 27027         | 133           | 61400         | 85            | 225800        | -             | -             | -          | 23  |

Figure 1: Assessment for the proposed model by means of three kinds of materials. (a) Fatigue life predictions using test models for C35 under two-level step loading. (b) Fatigue life predictions using test models for SAE4130 under two-level step loading. (c) Fatigue life predictions using test models for 7050-T7451 under two-level step loading.
Then, von Mises equivalent stress is expressed as follows (equation (24)):

$$
\sigma_{eq}(r_D, \theta, t) = \sqrt{\sigma^2(t) + 3r^2(t)} \left( 1 + \frac{r_D^2}{R^2} - \frac{2r_D \cos \theta}{R} \right).
$$

Considering $r_D \ll R$, the following approximate equation for equation (25) is established:

$$
\sigma_{eq}(r_D, \theta, t) \approx \sigma_{eq}(0, 0, t) = \sqrt{\sigma^2(t) + 3r^2(t)}.
$$

Substituting equation (26) into (4), the coefficient $c$ will be zero, which expresses the effect of the stress gradient on
fatigue life. Based on the same approximate principle, the following approximate equations are established:

\[ \sigma_{eq}(r_{D}, \theta) = \frac{f_{e}}{r_{e}} \sqrt{\frac{\sigma_{a}^{2} + \tau_{a}^{2}}{3}} + \left(1 - \frac{\sqrt{3} f_{e}}{3 r_{e}} \right) \left(\sigma_{a} + \sigma_{m} \right), \]

(27)

\[ \langle \sigma_{eq}^{2}(r_{D}, \theta, t) \rangle_{\text{max}} \approx \max \left[ \frac{2 (1 + \nu)}{3} \sigma_{0}^{2} (0, 0, t) + 3 (1 - 2 \nu) \sigma_{m}^{2} (t) \right]. \]

(28)

Ultimately, the equivalent completely reversed stress amplitude for the cylindrical specimen subjected to combined torsion and tension can be obtained by substituting equations (26)–(28) into equation (10):

\[ \sigma_{eq} = \left[ \langle \sigma_{eq}^{2} R_{\gamma} \rangle_{\text{max}} \right]^{1/2} \left( \frac{\sigma_{a}}{r_{e}} \right)^{2p+2} \left( f(r_{D}) \right)^{2p+2} (1 + \nu) (1 - cr - cr \sin \theta) \frac{\sqrt{\tau_{e}^{2} + \tau_{m}^{2}}}{2} \]

(29)

Note that equation (29) cannot be applied in the case of pure torsion since a large error is obtained by making \( \nu = 0 \). However, one can immediately establish the equivalent completely reversed stress amplitude under pure torsion and torsion from equation (10):

\[ \sigma = \frac{1}{\sqrt{V}} \int_{V} \left( \frac{\sigma_{a}}{r_{e}} \right)^{2p+2} \left( f(r_{D}) \right)^{2p+2} (1 + \nu) (1 - cr - cr \sin \theta) \sqrt{\tau_{e}^{2} + \tau_{m}^{2}} \frac{\sqrt{\tau_{e}^{2} + \tau_{m}^{2}}}{2} \]

(30)

Analogously, the following equation for expressing the equivalent completely reversed stress amplitude under pure torsion can be obtained:

\[ \sigma_{eq} = \frac{r_{a}^{p+1/p+2} \left( r_{a} + r_{m} \right)^{1/(p+2)}}{1 \sqrt{V} \int_{V} \left( \frac{\sigma_{a}}{r_{e}} \right)^{2p+2} \left( f(r_{D}) \right)^{2p+2} (1 + \nu) (1 - cr - cr \sin \theta) \sqrt{\tau_{e}^{2} + \tau_{m}^{2}} \frac{\sqrt{\tau_{e}^{2} + \tau_{m}^{2}}}{2} } \]

(31)

A concise form of equation (31) can be obtained by substituting equation (30) into equation (31):

\[ \sigma_{eq} = \frac{r_{a}^{p+1/p+2} \left( r_{a} + r_{m} \right)^{1/(p+2)}}{1 \sqrt{V} \int_{V} \left( \frac{\sigma_{a}}{r_{e}} \right)^{2p+2} \left( f(r_{D}) \right)^{2p+2} (1 + \nu) (1 - cr - cr \sin \theta) \sqrt{\tau_{e}^{2} + \tau_{m}^{2}} \frac{\sqrt{\tau_{e}^{2} + \tau_{m}^{2}}}{2} } \]

(32)

Therefore, the proposed cumulative-damage model for predicting the fatigue life of structural components under multiaxial two-level variable loading and combined torsion and tension can be explicitly expressed as follows:
Engineering structures are usually subjected to block program sequence loading. It is important to study certain fatigue issues, such as cumulative-damage under program sequence loading. Hence, we investigated cumulative-damage under multiaxial two-stage block loading and combined torsion and tension using the proposed model. Based on damage-equivalent concepts, the damage caused by \( n_{i-1} \) cycle loading is equivalent to that caused by \( n_i' \) cycle loading at the \( i \)th level. The variable \( Z_i \) expresses the ratio of the absorbed fatigue \( n_i' \) with the total fatigue life under single-stage loading, and its expression is derived from equations (22) and (33) as follows:

\[
Z_i = \begin{cases} 
\left( \frac{n_i}{N_{f1}} \right)^{\theta_{i-1}} + \sum_{k=1}^{i-2} n_k \left( \frac{n_i}{N_{f1}} \right)^{\phi_i} \left( \frac{n_{i-1}}{N_{fi-1}} \right)^{\phi_{i-1}}, & (i \geq 3), \\
\left( \frac{n_i}{N_{f1}} \right)^{\theta_{i-1}} \left( \frac{n_i}{N_{fi}} \right)^{\phi_i} + \sum_{k=1}^{i-2} n_k \left( \frac{n_i}{N_{fi}} \right)^{\phi_i} \left( \frac{n_{i-1}}{N_{fi-1}} \right)^{\phi_{i-1}}, & (i \geq 1) 
\end{cases}
\]

For the fatigue-life prediction under program sequence loading, we may be ignorant of the level loading causing component failure, and consequently, \( Z_i \) cannot be calculated. However, the sequence of \( Z_i \), i.e., \( Z_2, Z_3, \ldots \), can be calculated from equation (34). According to the sequence of \( Z_i \), component failure will result at the \( i - 1 \)th level due to fatigue once \( Z_i \geq 1 \). There may be two cases of cumulative-damage at \( i - 1 \)th level, even under at \( Z_i \geq 1 \). The predicted fatigue life for two cases can be, respectively, expressed as follows:

\[
N_{\text{predicted}} = \begin{cases} 
\sum_{k=1}^{i-2} n_k + n_{i-1}, & \text{if } Z_{i-1} + \frac{n_{i-1}}{N_{fi-1}} \leq 1, \\
\left( 1 - Z_{i-1} \right) N_{fi-1} + \sum_{k=1}^{i-2} n_k, & \text{if } Z_{i-1} + \frac{n_{i-1}}{N_{fi-1}} > 1.
\end{cases}
\]

The proposed model was assessed for predicting the fatigue life of structural components under multiaxial two-level and two-stage block loading and combined torsion and tension using relevant data for LY12CZ (Table 3). The proposed model for predicting fatigue life under two kinds of loading is expressed in equations (33) and (35). For better comparison with other cumulative-damage models, including the Palmgren–Miner law, Morrow’s plastic work interaction rule, and Wang’s rule, the models were synchronously used to predict the fatigue life of LY12CZ, and all text data under single-stage loadings and material parameter required in these models are listed in Tables 4 and 5. Notably, multiaxial equivalent stress parameters are employed in Morrow’s plastic work interaction rule. However, in the proposed model, the equivalent stress parameters are replaced by Matake’s critical plane stress parameters [32], and the interaction exponent is set to \(-0.45\) [33].

The predicted fatigue life of LY12CZ under multiaxial two-level step and two-stage block loading based on the test cumulative-damage models is shown in Figure 3. The percentage of the predicted data falling within the factor of 2.05 scatter band is 80%, 80%, 88%, and 96% for the Palmgren–Miner law, Morrow’s plastic work interaction rule, Wang’s rule, and our model, respectively. For predicting the fatigue life of LY12CZ under multiaxial variable loading, including two-level step loading and two-stage block loading, our model yielded results comparable to those of the test cumulative-damage models.
5. Conclusions

Here, we propose a cumulative-damage model based on continuum damage mechanics to evaluate the fatigue life of smooth structural components under axial and multiaxial variable loading. The conclusions can be obtained as follows:

(1) The model is very competitive with the existing test cumulative-damage models. It adopts a kind of equivalent completely reversed stress amplitude that expresses the effect of mean stress, stress gradients, loading history, and additional hardening behavior related to nonproportional loading paths on high-cycle fatigue under variable loading.

(2) Only one parameter is evaluated for the application of our model, and it is very simple to obtain the model parameter which is a simple function on the slope of $S$-$N$ curve of materials.

(3) The concept of equivalent completely reversed stress amplitude provides a new conception to investigate the cumulative damage for structural components under variable loading. Cumulative damage during fatigue progress and fatigue life can be precisely predicted by combining the equivalent completely reversed stress amplitude and damage development equation.

### Abbreviations

| $V$          | Volume of the critical domain |
|--------------|-------------------------------|
| $\sigma_{1\text{eq}}$, $\sigma_{-1\text{eq}}$, and $\sigma_{-1\text{ms}}$ | Stress amplitude under fully reversed loading in tension, damage parameters, and equivalent completely reversed stress amplitude, respectively |
| $D$          | Internal damage variable |
| $\sigma_{\text{eq,max}}$ | Maximum von Mises equivalent stress in the critical domain |
| $r$ and $\theta$ | Polar diameter within polar coordinates and polar angle within polar coordinates, respectively |
| $M$ and $p$  | Material parameters dependent on the slope of $S$-$N$ curve |
| $\sigma_{-1\text{eq},i}$ and $\sigma_{-1\text{ms},i}$ | Damage parameters for the $i$th level loading and equivalent completely reversed stress amplitude for the $i$th level loading, respectively |

Table 4: Test results under single-stage loadings for LY12CZ [21, 31].

| Loading paths | $\sigma_a$ | $\sigma_m$ | $\tau_a$ | $\tau_m$ | $\delta$ | $N_f$ | $\sigma_{-1\text{eq}}$ | $\sigma_{\text{Matsue}}$ |
|---------------|-------------|-------------|----------|----------|---------|-------|-----------------------|------------------------|
| A1            | 247.52      | 0           | 142.91   | 0        | 0       | 16831 | 333                   | 242                    |
| A2            | 176.81      | 0           | 102.08   | 0        | 0       | 37769 | 238                   | 173                    |
| A3            | 250         | 0           | 0        | 0        | 0       | 56316 | 250                   | 179                    |
| A4            | 0           | 0           | 144.3    | 0        | 0       | 75299 | 204                   | 146                    |
| A5            | 0           | 0           | 202.08   | 0        | 0       | 8911  | 285                   | 202                    |
| A6            | 350         | 0           | 0        | 0        | 0       | 31725 | 350                   | 250                    |
| B1            | 247.52      | 0           | 142.91   | 0        | 45      | 10229 | 326                   | 246                    |
| B2            | 176.81      | 0           | 102.08   | 0        | 45      | 348899| 234                   | 176                    |
| B3            | 158         | 0           | 125      | 0        | 45      | 57004 | 246                   | 188                    |
| C1            | 247.52      | 0           | 142.91   | 0        | 90      | 11067 | 308                   | 246                    |
| C2            | 176.81      | 0           | 102.08   | 0        | 90      | 85684 | 220                   | 176                    |
| C3            | 250         | 0           | 144.34   | 0        | 90      | 4634  | 312                   | 248                    |

Table 5: Material parameters required in the proposed model for LY12CZ [31].

| Materials | $\sigma_{a1}$ | $N_{f1}$ | $\sigma_{a2}$ | $N_{f2}$ | $\sigma_{a3}$ | $N_{f3}$ | $\sigma_{-1}$ | $p$ |
|-----------|---------------|----------|---------------|----------|---------------|----------|---------------|-----|
| LY12CZ    | 350           | 31725    | 300           | 140670   | 250           | 272123   | 169           | 2.15|

Figure 3: Fatigue life predictions for LY12CZ under multiaxial variable loading based on test damage cumulative models.
$N_{fi}$ and $n_{fi}$: Fatigue life and applied cycles, respectively, under the $i$th level constant-amplitude loading

$\sigma_{-1}$: Endurance limits in reversed tension

$n_{-1}$: Applied cycles under fully reversed loading in tension

$D_i$: Internal damage variable of the $i$th step loading

$D'_i$ and $n'_i$: Equivalent internal damage variable and applied cycles, respectively, based on damage-equivalent concepts, respectively

$\sigma_{ai}$: $i$th level stress amplitude under fully reversed loading in tension

$\sigma_{-1a}$ and $N_{-1a}$: Stress amplitude and fatigue life plotted in S-N curve

$\sigma_{\text{Matake}}$: Damage parameters proposed by Matake et al.

$E$: Elastic modulus

$\sigma_{\text{eq}}$: von Mises equivalent stress

$R_{\gamma}$: Three axis factor

$\sigma_{H\text{max}}$: Maximum hydrostatic stress in one cycle

$\langle \sigma \rangle_{\text{max}}$: Maximum value in symbol

$Y_{\text{max}}$: Maximum generalized force of damage driving in one cycle

$\nu$: Poisson ratio

$\sigma_H$: Hydrostatic stress

$f(r_D)$: Stress distribution function in the critical domain

$N_{\text{predicted}}$: Predicted life of smooth specimen

$D_{\text{cr}}$: The critical value of damage

$\sigma_{ai}$, $\sigma_m$, $r_a$, and $r_m$: Normal stress amplitude, normal mean stress, shear stress amplitude, and shear mean stress, respectively

$\delta$: Phase difference between normal loading path and shear-loading path

$r$: Radius of smooth specimen

$r_D$: Polar diameter of the critical domain.

### Data Availability

The data are from previously reported studies, and these prior studies are cited at relevant places within the text as references [21, 29–31].

### Conflicts of Interest

The authors declare that they have no conflicts of interest.

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