Cosmology of Light Moduli

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Abstract

In string/$M$-theory with a large compactification radius, some axion-like moduli can be much lighter than the gravitino. Generic moduli in gauge-mediated supersymmetry breaking models also have a mass far below the weak scale. Motivated by these, we examine the cosmological implications of light moduli for the mass range from the weak scale to an extremely small scale of $\mathcal{O}(10^{-26})$ eV, and obtain an upper bound on the initial moduli misalignment for both cases with and without a late entropy production.
I. INTRODUCTION

A very weakly-interacting light scalar field $\phi$ can be cosmologically troublesome. In the early universe with the Hubble expansion parameter $H$ much bigger than the scalar boson mass $m_\phi$, this scalar field would take an initial value $\delta \phi$ significantly different from its vacuum value at present. Due to this misalignment, $\phi$ starts to coherently oscillate at a later time and subsequently its energy density behaves as a non-relativistic matter. Depending upon its lifetime, such a coherently oscillating scalar field may spoil the big bang nucleosynthesis or produce too much gamma- or $X$-rays through its late-time decay, or may overclose the universe.

The most well-known and first studied example of such a light scalar boson is the invisible axion which has been introduced to solve the strong CP problem \cite{1, 2}. The mass and the typical misalignment of the invisible axion are determined by a single unknown parameter, the axion decay constant $f_a$ which corresponds to the scale of spontaneous $U(1)_{PQ}$-breaking, as $\delta a \approx f_a$ and $m_a \approx f_\pi m_\pi / f_a$ where $f_\pi$ and $m_\pi$ are the pion decay constant and the pion mass, respectively. If there is no entropy production after the QCD phase transition, the requirement that the relic axion mass density does not overclose the universe leads to the famous constraint $f_a \lesssim 10^{12}$ GeV. It was later realized that generic hidden sector supergravity model predicts a cosmologically troublesome light scalar field, the Polonyi field, whose typical misalignment is given by the Planck scale $M_P = 1/\sqrt{8\pi G_N} \approx 2 \times 10^{18}$ GeV \cite{3}. Again in the absence of a late time entropy production after the Polonyi field starts to oscillate, it would completely spoil the successful big-bang nucleosynthesis unless the Polonyi mass $m_\phi \gtrsim 40$ TeV.

This Polonyi problem has been recently revived in the context of string theory \cite{4}. The (approximately) degenerate string vacua are described by moduli fields whose typical misalignments are given by either the string scale or the compactification scale which is close to $M_P$. In the previous studies, the moduli masses were assumed to be of order the gravitino mass $m_{3/2}$ which would be of order of the weak scale if the supersymmetry breaking
is transmitted to the supersymmetric standard model sector by gravitational interactions. However due to approximate non-linear global $U(1)$ symmetries in string/$M$-theory, some of the axion-like moduli in string theory can be much lighter than the gravitino \[^5\]. In particular, in models with a large compactification radius, the mass of such light axion-like moduli behaves as $m_\phi \approx e^{-\pi \text{Re}(T)m_3/2}$ where $T$ is the overall modulus field whose value corresponds to the compactification radius-squared in the heterotic string length unit. As we will discuss in section II, $\text{Re}(T)$ can be as large as $\mathcal{O}(\frac{1}{\alpha_{\text{GUT}}})$ in the $M$-theory limit, implying that $m_\phi$ can be extremely smaller than $m_3/2$, for instance as small as $10^{-34}m_3/2$.

Although not as dramatic as the axion-like moduli in string/$M$-theory, generic moduli in models with a gauge-mediated supersymmetry breaking can also be much lighter than the weak scale \[^6\]. In gauge-mediated models, the supersymmetry breaking scale $\Lambda$ is related to the weak scale $M_W$ by some loop suppression factor: $M_W = (\frac{\pi}{\alpha})^l \Lambda$ where the integer $l$ counts the number of loops involved in transmitting supersymmetry breaking to the supersymmetric standard model sector. Then for a reasonable value of $l$, the moduli mass $m_\phi \approx \Lambda^2/M_P$ would be in the range far below the weak scale.

Motivated by the above observations, in this paper we wish to study some cosmological aspects of a generic light modulus $\phi$ with an arbitrary mass in the range from the weak scale to an extremely small mass scale of $\mathcal{O}(10^{-26})$ eV. It turns out that no meaningful cosmological bound is obtained for the moduli mass below $10^{-26}$ eV. In fact, moduli cosmology in gauge-mediated supersymmetry breaking models has been discussed recently \[^7\], but only for a rather narrow mass range $m_\phi = 10$ keV $- 1$ GeV. The organization of this paper goes as follows. In section II, we discuss in more detail the masses of the light axion-like moduli in string/$M$-theory to make our motivation more clear. In section III, we examine some generic features of the moduli dynamics in the early universe to see how it depends on the parameters involved and also identify the initial moduli misalignment $\delta \phi$. We then use the known cosmological and astrophysical observations to obtain an upper bound of the moduli misalignment $\delta \phi$ in section IV. In this regard, we first consider the case that there is no entropy production after the moduli oscillation begins and summarize the results in Figure
1. As is well known, the light moduli density (relative to the entropy density) can be diluted if there occurs an entropy production after the moduli oscillation begins. We thus finally discuss the dilution of the light moduli density in various possible cosmological scenarios with a late entropy production, including the case that the entropy-producing field $\varphi$ is a massive moduli with $m_\varphi \gtrsim \mathcal{O}(40)$ TeV and also the case that $\varphi$ corresponds to the flaton field triggering thermal inflation [8]. The relaxed bounds on $\delta\phi$ for the cases with a late entropy production are depicted in Figures 2 and 3.

II. LIGHT AXION-LIKE MODULI IN STRING/M-THEORY

To make our motivation more clear, let us discuss in more detail the masses of axion-like moduli in compactified string/M-theory. As is well known, the theory predicts the dilaton superfield $S$ and the Kähler moduli superfields $T_I$ ($I = 1, 2, ..., h_{1,1}$) for generic compactifications preserving the four-dimensional supersymmetry [9]. The scalar components $\text{Re}(S)$ and $\text{Re}(T_I)$ of these superfields determine the four-dimensional gauge coupling constant and the size (and also the shape) of the internal six manifold, respectively. For an isotropic six manifold, we have $\langle \text{Re}(T_I) \rangle \approx \langle \text{Re}(T) \rangle$ where $\text{Re}(T)$ denotes the overall modulus whose VEV corresponds to the radius-squared of the internal six manifold in the heterotic string length unit. The pseudoscalar components $\text{Im}(S)$ and $\text{Im}(T_I)$ are often called the model-independent axion and the model-dependent Kähler axions, respectively [10,11]. These axion-like moduli are periodic variables and we normalize them by imposing the periodicity conditions:

$$\text{Im}(S) \equiv \text{Im}(S) + 1, \quad \text{Im}(T_I) \equiv \text{Im}(T_I) + 1.$$  \hspace{1cm} (1)

Under this normalization, the scalar components are given by [12]

$$\text{Re}(S) \approx \frac{1}{\alpha_{\text{GUT}}} \approx 4\pi e^{-2D} \frac{V}{(2\alpha')^3} \approx 2(4\pi \kappa^2)^{-2/3} V,$$

$$\text{Re}(T) \approx \frac{6^{1/3} V^{1/3}}{4\pi^2 2\alpha'} \approx 6^{1/3}(4\pi \kappa^2)^{-1/3} \pi \rho V^{1/3},$$ \hspace{1cm} (2)
where $e^{2D}$ is the heterotic string dilaton, $V$ is the internal space volume, $\kappa^2$ is the eleven-dimensional gravitational coupling constant, and finally $\pi \rho$ denotes the length of the eleventh segment in $M$-theory. The above relations, together with $M_P = 2\pi \rho \kappa^{-2} V$, show that the heterotic string coupling $e^{2D}$ scales as $\alpha_{\text{GUT}} \langle \text{Re}(T) \rangle^3$ and the eleventh length $\rho$ scales as $M_P^{-1} \langle \text{Re}(T) \rangle^{3/2}$. Inserting the proper numerical coefficients, it is easy to see that for $\alpha_{\text{GUT}} \approx 1/25$ and $M_P \approx 2 \times 10^{18}$ GeV, the large radius limit with $\langle \text{Re}(T) \rangle \gg 1$ corresponds to the $M$-theory limit with a strong heterotic string coupling, which can be described by an eleven-dimensional supergravity on a manifold with boundary. Inserting the proper numerical coefficients, it is easy to see that for $\alpha_{\text{GUT}} \approx 1/25$ and $M_P \approx 2 \times 10^{18}$ GeV, the large radius limit with $\langle \text{Re}(T) \rangle \gg 1$ corresponds to the $M$-theory limit with a strong heterotic string coupling, which can be described by an eleven-dimensional supergravity on a manifold with boundary. For the superfield normalization determined by the periodicity condition (1), the holomorphic gauge kinetic functions satisfy the relation \[ 4\pi f_{E_8} - 4\pi f_{E_8}' \approx \sum_I l_I T_I \] (3) where $l_I$’s are integer coefficients, and $f_{E_8}$ and $f_{E_8}'$ denote the gauge kinetic functions for $E_8$ and $E_8'$, respectively. Here the gauge kinetic functions are normalized as $\langle \text{Re}(f_a) \rangle = 1/g_a^2$ and $\langle \text{Im}(f_a) \rangle = \theta_a/8\pi^2$, where $g_a$ and $\theta_a$ denote the gauge coupling constant and the vacuum angle for the $a$-th gauge group, respectively. Note that for integer $l_I$ the relation (3) is consistent with the periodicity condition (1) and the periodic vacuum angles $\theta_a \equiv \theta_a + 2\pi$. The relation (3) suggests that, even in the $M$-theory limit, $\text{Re}(T) \approx \text{Re}(T_I)$ can not be arbitrarily large, but is constrained not to significantly exceed $4\pi \text{Re}(f_{E_8}) \approx \frac{1}{\alpha_{\text{GUT}}}$. For compactifications on a smooth Calabi-Yau manifold with vanishing $E_8'$ field-strength, it turns out that at least one of $l_I$’s is positive and other $l_I$’s are still non-negative integers. This then leads to the upper limit:

$$\langle \text{Re}(T) \rangle \lesssim \frac{1}{\alpha_{\text{GUT}}}$$ (4)

which corresponds to the lower limit on the Newton’s constant $G_N$ discussed in [14], and also to the lower limit on the Kaluza-Klein scale $M_{\text{KK}} \gtrsim \alpha_{\text{GUT}}/\sqrt{G_N}$ discussed in [15].

The Kähler potential of the effective supergravity model depends upon $\text{Re}(S)$ and also $\text{Re}(T_I)$ with unsuppressed coefficients of order unity. As a result, once the four-dimensional
supersymmetry is broken, $\text{Re}(S)$ and $\text{Re}(\mathcal{T}_I)$ receive the masses of order $m_{3/2}$ from the supergravity scalar potential. However the masses of the axion-like moduli $\text{Im}(S)$ and $\text{Im}(\mathcal{T}_I)$ are constrained by the approximate non-linear global $U(1)$ symmetries defined as

$$U(1)_S : \text{Im}(S) \to \text{Im}(S) + \alpha_S, \quad U(1)_I : \text{Im}(\mathcal{T}_I) \to \text{Im}(\mathcal{T}_I) + \alpha_I,$$

where $\alpha_S$ and $\alpha_I$ denote arbitrary real constants. In the limit where one (combination) of these $U(1)$’s becomes an exact symmetry, the corresponding axion-like field becomes an exact Goldstone boson and thus is massless. In string/$M$-theory, these $U(1)$-symmetries are explicitly broken either by the Yang-Mills axial anomaly or by the world-sheet (membrane) instanton effects \[16\]. If the hidden sector gauge interactions provide a dynamical seed for supersymmetry breaking, the $U(1)$-breaking by the hidden sector Yang-Mills axial anomaly is so strong that the linear combination of $\text{Im}(S)$ and $\text{Im}(\mathcal{T}_I)$ which couples to the hidden sector anomaly get the masses of order $m_{3/2}$. However most of the known compactification models allow a combination of $U(1)_S$ and $U(1)_I$ which is free from the hidden sector anomaly, but still explicitly broken by the world-sheet instanton effects and/or by the observable sector anomaly (mainly the QCD anomaly) \[11,16\].

Let $\phi$, being a linear combination of $\text{Im}(S)$ and $\text{Im}(\mathcal{T}_I)$, denote the axion-like moduli for the combination of $U(1)_S$ and $U(1)_I$ which is free from the hidden sector anomaly. Then the effective potential $V_\phi$ of this axion-like moduli includes first of all the contribution from the world-sheet instanton effects which is estimated to be \[5,12\]

$$V_{WS} \approx e^{-2\pi\langle\text{Re}(\mathcal{T})\rangle}m_{3/2}^2M_p^2.$$  

(6)

If $\phi$ couples to the observable sector QCD anomaly, $V_\phi$ would include also the contribution from the QCD anomaly, $V_{QCD} \approx f_\pi^2m_\pi^2$. In fact, one could argue based on supersymmetry and the periodicity condition (4) for the axion-like moduli that there is no other type of contribution to $V_\phi$ \[12\].

For compactifications with $\langle\text{Re}(\mathcal{T})\rangle \lesssim 17$, $V_\phi$ is dominated by the world-sheet instanton contribution \[8\]. (Here and in the following, we assume $m_{3/2} \approx 1$ TeV for the simplicity of the discussion.) Then the mass of the axion-like moduli field $\phi$ is estimated to be
However if \( \langle \text{Re}(T) \rangle \gtrsim 17 \) and also \( \phi \) couples to the QCD anomaly, \( V_\phi \) is dominated by the QCD contribution, leading to \( m_\phi \approx f_\pi m_\pi / M_P \). (If \( \langle \text{Re}(T) \rangle \) is even bigger than about 20, we have \( V_{WS} \lesssim 10^{-9} V_{QCD} \) and then this \( \phi \) can be identified as the QCD axion solving the strong CP problem \[3,12\].) In models with \( h_{1,1} > 1 \), there can be a combination of \( U(1)_S \) and \( U(1)_I \) which is free from both the hidden sector Yang-Mills axial anomaly and the observable sector QCD anomaly. The axion-like moduli field for this combination does not couple to the QCD anomaly and then its mass is given by Eq. (7) even for \( \langle \text{Re}(T) \rangle \gtrsim 17 \).

The above discussion, particularly Eq. (7), implies that \( m_\phi \) is highly sensitive to the compactification radius which is measured by \( \langle \text{Re}(T) \rangle \). It can be extremely smaller than \( m_{3/2} \) if the compactification radius is large enough to have \( \langle \text{Re}(T) \rangle \gg 1 \). As was noted in the discussions above \[4\], a large value of \( \langle \text{Re}(T) \rangle \) is allowed in the \( M \)-theory limit but it is constrained not to significantly exceed \( \frac{1}{\alpha_{GUT}} \approx 25 \). Then for the range \( 0 < \langle \text{Re}(T) \rangle \lesssim 25 \), \( m_\phi \) can be anywhere between \( m_{3/2} \) and the extremely small mass \( 10^{-34} m_{3/2} \). Note that even when \( \langle \text{Re}(T) \rangle \approx 1 \) for which the weakly-coupled heterotic string theory provides a sensible description for the dynamics above \( M_P \), the axion-like moduli mass \( m_\phi \) can be smaller than \( m_{3/2} \) by one or two orders of magnitudes.

### III. MODULI DYNAMICS IN THE EARLY UNIVERSE

In the previous section, we have noted that, in string/\( M \)-theory with a large compactification radius, some of the axion-like moduli can be much lighter than the gravitino mass. Motivated by this observation, in this and next sections, we study cosmological aspects of a generic light modulus \( \phi \) with an arbitrary mass below the weak scale.

Moduli dynamics in the early universe would be governed by the free energy density \( V_{\text{eff}} \) which depends not only on \( \phi \) but also on other scalar fields \( \Phi \) and the radiation temperature \( T \). Expanding \( V_{\text{eff}} \) around the present moduli VEV which is set to zero, one generically has
\[ V_{\text{eff}}(\phi, \Phi, T) = \Omega_0(\Phi, T) + \Omega_1(\Phi, T)\phi + \frac{1}{2}\Omega_2(\Phi, T)\phi^2 + ..., \]  

where the moduli-tadpole \( \Omega_1 \) arises since the expansion is made around the present moduli VEV, not around the minimum of \( V_{\text{eff}} \) in the early universe. Obviously at present, \( \Omega_n \) are given by

\[ \Omega_0(\Phi_0, T_0) = \Omega_1(\Phi_0, T_0) = 0, \quad \Omega_2(\Phi_0, T_0) = m_\phi^2, \]

where \( \Phi_0, T_0, \) and \( m_\phi \) denote the present VEV of \( \Phi \), the present temperature, and the present mass of \( \phi \), respectively, and the vanishing of \( \Omega_0(\Phi_0, T_0) \) comes from the vanishing (or negligibly small) cosmological constant at present.

In the early universe, \( \Phi \) and \( T \) can take values far away from the present ones. As a result, \( \Omega_n \)'s in the early universe can significantly differ from their present values. For instance, when \( H \gg m_\phi \), the effective moduli mass \( \sqrt{\Omega_2} \) can be of order \( H \) and thus much bigger than \( m_\phi \).

To proceed, let us parameterize the free energy density (8) as follows [17]:

\[ V_{\text{eff}}(\phi, T) = \frac{1}{2}m_\phi^2\phi^2 + \frac{1}{2}c^2H^2(\phi - \phi_1)^2 + ..., \]

where \( m_\phi \) denotes the moduli mass at present and the ellipsis denote the irrelevant terms. This parameterization is useful since in most cases of interest, \( c \) and \( \phi_1 \) are approximately time-independent constants, which would greatly simplify the analysis. Obviously the coefficients \( c \) and \( \phi_1 \) describes the non-derivative interactions of \( \phi \) with the background energy density in the early universe, e.g. the radiation or the inflaton energy density. Generically \( \phi_1 \) is expected to be of order \( M_P \). The value of \( c \) measures how strongly \( \phi \) couples to supersymmetry breaking environment and thus

\[ c \approx m_\phi/m_{3/2}. \]  

It was noted in Ref. [18] that in the case of \( c \gg 1 \), the initial moduli misalignment can be rapidly damped away. In this paper, we consider only the case \( c \lesssim O(1) \) which appears to be
more natural. Note that for axion-like moduli discussed in the previous section, $c \approx e^{-\pi \text{Re}(T)}$ can be very small.

Treating $c$ and $\phi_1$ as constants, the equation of motion for $\phi$ is given by

$$\ddot{\phi} + 3H \dot{\phi} + (m_\phi^2 + c^2 H^2) \phi = c^2 H^2 \phi_1 .$$

(12)

In the following, we wish to examine the evolution of $\phi$ by solving this equation with the initial value taken at some initial time $t_i$:

$$\phi(t_i) = \phi_i , \quad \dot{\phi}(t_i) = 0 .$$

(13)

Let us first consider the evolution of $\phi$ during inflationary era for which the Hubble parameter $H$ is roughly a constant. The solution of Eqs. (12) and (13) is easily found to be as follows: For $\left( \frac{3}{2} H \right)^2 > m_\phi^2 + c^2 H^2$,

$$\phi(t) = \phi_{\text{min}} + (\phi_i - \phi_{\text{min}}) \left[ \frac{1 + \beta}{2 \beta} e^{-\frac{3(1-\beta)}{2} H(t-t_0)} - \frac{1 - \beta}{2 \beta} e^{-\frac{3(1+\beta)}{2} H(t-t_0)} \right] ,$$

(14)

while for $\left( \frac{3}{2} H \right)^2 < m_\phi^2 + c^2 H^2$,

$$\phi(t) = \phi_{\text{min}} + (\phi_i - \phi_{\text{min}}) e^{-\frac{3H}{2 m_\phi} (t-t_0)} \left[ \cos[\beta' \phi_\phi(t-t_0)] - \frac{3H}{2 \beta' m_\phi} \sin[\beta' \phi_\phi(t-t_0)] \right] ,$$

(15)

where $\beta = \sqrt{1 - \frac{3}{4} (c^2 + \frac{m_\phi^2}{H^2})}$, $\beta' = \sqrt{1 - \left( \frac{3}{4} - c^2 \right) \frac{H^2}{m_\phi^2}}$, and $\phi_{\text{min}}$ is the temporal minimum of the effective potential which is given by

$$\phi_{\text{min}} = \frac{c^2 H^2}{m_\phi^2 + c^2 H^2} \phi_1 .$$

(16)

Let us consider some interesting limits of the above solution. If $m_\phi \ll H$ and $c \ll 1$, we have $\beta \approx 1 - \frac{3}{9} (c^2 + \frac{m_\phi^2}{H^2})$ and then the modulus value after the inflation ($\phi_f$) is given by

$$\phi_f \approx \phi_i - \frac{N_e}{3} (\phi_i - \phi_{\text{min}}) \left( \frac{H^2}{m_\phi^2 + c^2 H^2} \right) ,$$

(17)

where $N_e$ is the number of e-folding. In the above, we assumed that $c$ and $m_\phi/H$ are small enough so that $N_e \ll \frac{H^2}{c H^2 + m_\phi^2}$. Thus simply speaking, in the case of $m_\phi \ll H$ and $c \ll 1$, $\phi$ is frozen at its initial value $\phi_i$. This is what we thought to happen during inflation before
realizing the role of $H^2$ term in Eq. (10). But for $m_\phi \ll H$ and $c \sim 1$, we have the final modulus value

$$
\phi_f \approx \phi_{\min} + (\phi_i - \phi_{\min}) \times \mathcal{O}(e^{-3N_e/2}),
$$

(18)

showing that $\phi$ rapidly approaches to the temporal minimum $\phi_{\min} \simeq \phi_1$ independently of the value $\phi_i$ before the inflation. Finally for $m_\phi \gg H$, $\phi$ exponentially approaches to $\phi_{\min} \simeq c^2 H^2 m_\phi^2 \phi_1$, but with an exponentially decreasing oscillatory tail. This is what happens in so-called thermal inflation [8] which will be discussed in more detail in the next section.

Let us now consider the evolution of $\phi$ during the radiation-dominated (RD) or matter-dominated (MD) era for which the Hubble parameter is given by $H = p/t$ where $p = \frac{1}{2}$ (RD), $\frac{2}{3}$ (MD). In RD or MD era, the solution of Eq. (12) is given by

$$
\phi(z) = p^2 c^2 \phi_1 \frac{S_{\alpha-1,\nu}(z)}{z^\alpha} + C_1 \frac{J_\nu(z)}{z^\alpha} + C_2 \frac{Y_\nu(z)}{z^\alpha},
$$

(19)

where $\nu^2 = \alpha^2 - p^2 c^2 \geq 0$ for $\alpha = \frac{1}{2}(3p - 1)$ and $z = m_\phi t$. Here $J_\nu(z)$ and $Y_\nu(z)$ are Bessel functions, and $S_{\mu,\nu}(z)$ is Lommel function which is defined by

$$
S_{\mu,\nu}(z) = \frac{\pi}{2} \left[ \nu Y_\nu(z) \int_0^z y^\mu J_\nu(y) \, dy - J_\nu(z) \int_0^z y^\mu Y_\nu(y) \, dy \right] \\
+ 2^{\nu-1} \Gamma\left(\frac{\mu - \nu + 1}{2}\right) \Gamma\left(\frac{\mu + \nu + 1}{2}\right) \left\{ \sin\left(\frac{\mu - \nu}{2} \pi\right) J_\nu(z) - \cos\left(\frac{\mu - \nu}{2} \pi\right) Y_\nu(z) \right\}.
$$

(20)

The coefficients $C_1$ and $C_2$ are fixed by the initial condition (13):

$$
C_1 = A_1 \phi_i + B_1 c^2 \phi_1,
$$

$$
C_2 = A_2 \phi_i + B_2 c^2 \phi_1
$$

(21)

where

$$
A_1 = \frac{\pi}{2} z_i^\alpha \left\{ z_i Y'_\nu(z_i) - \alpha Y_\nu(z_i) \right\},
$$

$$
A_2 = -\frac{\pi}{2} z_i^\alpha \left\{ z_i J'_\nu(z_i) - \alpha J_\nu(z_i) \right\},
$$

$$
B_1 = -\frac{\pi}{2} z_i \left\{ Y'_\nu(z_i) S_{\alpha-1,\nu}(z_i) - Y_\nu(z_i) S'_{\alpha-1,\nu}(z_i) \right\} p^2,
$$

$$
B_2 = \frac{\pi}{2} z_i \left\{ J'_\nu(z_i) S_{\alpha-1,\nu}(z_i) - J_\nu(z_i) S'_{\alpha-1,\nu}(z_i) \right\} p^2.
$$

(22)
Here \( z_i = m_\phi t_i \) and the prime denotes the differentiation with respect to \( z \).

From the solution (19), we can calculate the moduli abundance coming from the coherent oscillation. For \( z \gg 1 \), \( \phi(z) \) is dominated by the oscillating tail

\[
\phi(z) \approx \left( \frac{2}{\pi} \right)^{1/2} z^{-\frac{3}{2}} \{ C_1 \cos[z - (\nu + \frac{1}{2}) \frac{\pi}{2}] + C_2 \sin[z - (\nu + \frac{1}{2}) \frac{\pi}{2}] \}.
\]  

The energy density of this oscillating modulus is given by

\[
\rho_\phi = \frac{1}{2} m_\phi^2 (\phi'^2 + \phi^2) \approx \left( \frac{2}{\pi} \right) (C_1^2 + C_2^2) m_\phi^2 z^{-3p}
\]  

and normalizing it by the entropy density \( s = \frac{2\pi}{45} g_* T^3 \), we find

\[
m_\phi Y_\phi \equiv \frac{\rho_\phi}{s} = \frac{45}{2\pi^2 g_*} (C_1^2 + C_2^2) \frac{m_\phi^2}{z^{3p} T^3},
\]

where \( g_* \) denotes the effective number of the relativistic degrees of freedom at \( T \).

Most of the cosmological implications of \( \phi \) is in fact determined by the oscillation amplitude:

\[
\delta \phi \equiv \left( \frac{C_1^2 + C_2^2}{\pi} \right)^{1/2}
\]

which we call the initial moduli misalignment throughout this paper. The oscillation coefficients \( C_1 \) and \( C_2 \) are determined by the modulus value \( \phi_i = \phi(t_i) \) at an initial time \( t_i \) with \( \phi'(t_i) = 0 \), and also by the two dynamical parameters \( c \) and \( \phi_1 \) in the free energy density (10) which governs the moduli dynamics at later time \( t > t_i \). Depending upon the periods under consideration, \( z_i = m_\phi t_i \) may be chosen to be either very small, or of order unity, or very large. It turns out that the coefficients \( A_{1,2} \) and \( B_{1,2} \) in Eq. (22) are essentially of order one for \( z_i \sim 1 \) and an arbitrary value of \( c \lesssim 1 \). Thus roughly speaking, \( C_1 \) and \( C_2 \) are linear combinations of \( \phi_i \) (at \( t_i \approx m_\phi^{-1} \)) and \( c^2 \phi_1 \) with coefficients of order one. At any rate, using Eqs. (25) and (24), any cosmological bound on \( \delta \phi \) can be translated into a constraint on the parameter set \( (\phi_i, c^2 \phi_1) \).

If the universe were radiation-dominated when \( \phi \)-oscillation begins at \( t \sim m_\phi^{-1} \) and there is no entropy production since then, we have \( z = \left( \frac{45}{2\pi^2 g_*} \right)^{1/2} \frac{m_\phi M_P}{T^2} \), and thus
\[ m_\phi Y_\phi = \left( \frac{45}{2\pi^2 g_*} \right)^{1/4} \left( \frac{\delta \phi}{M_P} \right)^2 (m_\phi M_P)^{1/2} . \] 

(27)

In other case that the universe was matter-dominated, for instance by the inflaton oscillation or by the heavy moduli oscillation, at the moment when \( \phi \)-oscillation begins, one has to take into account the subsequent entropy production due to the out-of-equilibrium decays of the inflaton or heavy moduli. Assuming that the whole matter energy is converted into the radiation with the reheat temperature \( T_R \), the modulus energy density normalized by the entropy density is given by Eq. (25) evaluated at the reheat time \( t_R = \left( \frac{40}{\pi^2 g_*} \right)^{1/2} \frac{M_P}{T_R} \).

\[ m_\phi Y_\phi = 3 \left( \frac{\delta \phi}{M_P} \right)^2 T_R . \] 

(28)

IV. THE CONSTRAINTS ON THE INITIAL MISALIGNMENT

A. The cosmological moduli problem

In the previous section, we obtained the moduli energy density normalized by the entropy density when the coherent oscillation begins during the RD era:

\[ m_\phi Y_\phi \approx g_*^{-1/4} \left( \frac{\delta \phi}{M_P} \right)^2 (m_\phi M_P)^{1/2} \approx 6 \times 10^8 \left( \frac{m_\phi}{\text{GeV}} \right)^{1/2} \left( \frac{\delta \phi}{M_P} \right)^2 \text{GeV} . \] 

(29)

A coherently oscillating modulus can dominate the energy density of the universe unless \( m_\phi Y_\phi \) is less than the temperature \( T_{\text{EQ}} \approx 3 \text{ eV} \) of matter-radiation equality. This implies that one has to worry about the over-produced moduli which would contradict with the cosmological observations as long as the moduli mass is in the range

\[ m_\phi \gtrsim 10^{-26} \text{ eV} . \] 

(30)

On the other hand, sufficiently heavy moduli decaying before about one second do not affect the standard prediction of the big-bang nucleosynthesis, and thus would not contradict with the currently known cosmological observations. Having interactions suppressed by \( M_P \), the moduli lifetime is estimated to be
\[ \tau_\phi \approx \xi \times 10^{14} \left( \frac{m_\phi}{\text{GeV}} \right)^{-3} \text{sec}, \]  

where \( \xi \), being roughly of order one, is a coefficient which accounts for the ambiguity in our estimate of the lifetime. For moduli lighter than \( \sim 40 \text{ TeV} \), their decay products may change the abundance of the light elements after or during the nucleosynthesis, the spectrum of cosmic background radiation, or the observed \( \gamma \) and X-ray backgrounds \cite{19}. Moduli whose lifetime is longer than the age of the universe would overclose the universe unless \( m_\phi Y_\phi \lesssim 3 \text{eV} \). This means for the moduli mass \( m_\phi \lesssim 0.1 \text{ GeV} \), the initial misalignment is constrained as

\[ \frac{\delta \phi}{M_P} \lesssim 4 \times 10^{-9} \left( \frac{m_\phi}{0.1 \text{ GeV}} \right)^{-1/4}. \]  

unless there is a late entropy production after \( \phi \) starts to oscillate.

Based on the analysis of \cite{19}, the recently reported X-ray background \cite{20} and Eq. (32), we obtain the constraints on \( \delta \phi \) arising from these considerations again under the assumption that there is no entropy production after \( \phi \)-oscillation begins at \( t \sim m_\phi^{-1} \). The results are summarized in Figure 1 showing the cosmological upper limit on the initial moduli misalignment \( \delta \phi \) for the moduli mass \( m_\phi \) below \( \sim 40 \text{ TeV} \). The line (0) comes from Eq. (32) which is required for the moduli not to overclose the universe, and the line (1) from the recently reported X-ray background. The lines (a)-(h) are obtained from the observed \( \gamma \)-ray background, the spectrum of cosmic microwave background radiation, and also the light element abundances.

The results of Figure 1 show that, in the absence of a late time entropy production, the initial moduli misalignment is required to be very small compared to its natural value \( \sim M_P \), for instance \( \delta \phi \lesssim 10^{-6} M_P \) for \( m_\phi \approx 1 \text{ eV} \), \( \delta \phi \lesssim 10^{-10} M_P \) for \( m_\phi \approx 1 \text{ MeV} \), \( \delta \phi \lesssim 10^{-11} M_P \) for \( m_\phi \approx 1 \text{ GeV} \). Let us recall that \( \delta \phi \) is determined by the modulus value \( \phi_i \) at an initial time \( t_i \) with \( \phi'(t_i) = 0 \), and also the dynamical parameters \( (c, \phi_1) \) in the free energy density which would govern the moduli dynamics at time \( t \gtrsim t_i \). (See Eqs. (21)–(26).) Its order of magnitude is roughly given by the bigger one among \( \phi_i \) and \( c^2 \phi_1 \) for the initial time.
\[ t_i \approx m_\phi^{-1}. \] Its natural value would be of order \( M_P \) which then falls at far above the upper limits in Figure 1 for most range of the moduli mass.

**B. Dilution by heavy moduli decays**

If there is an entropy production during the period after \( \phi \) starts to oscillate but well before \( \phi \) decays, the moduli energy density is diluted as \( Y_\phi \rightarrow Y_\phi/\Delta \) where \( \Delta = S_{\text{after}}/S_{\text{before}} \) denotes the entropy production factor. Since \( Y_\phi \propto \delta \phi^2 \), this obviously leads to the relaxation of the constraints on the initial moduli misalignment which will be discussed below.

Usually the entropy production is due to out-of-equilibrium decays of non-relativistic particles which appear in the form of another coherently oscillating scalar field \( \varphi \). In order to be compatible with the big-bang nucleosynthesis, this entropy-producing scalar field \( \varphi \) is required to decay before the nucleosynthesis with the reheat temperature \( T_R \gtrsim 6\,\text{MeV} \) [21]. Also note that if the light moduli \( \phi \) is so light that its oscillation begins after the entropy production by \( \varphi \) is over, i.e. \( m_\phi \lesssim g_*^{1/2}T_R^2/M_P \), \( Y_\phi \) is not affected by the entropy production by \( \varphi \). Thus the energy density of oscillating \( \phi \) can be diluted only for the moduli mass

\[
m_\phi \gtrsim 5 \times 10^{-14} \left( \frac{T_R}{6\,\text{MeV}} \right)^2 \text{eV}.
\]

(33)

To be more specific, let us consider the interesting possibility that the entropy-producing field \( \varphi \) is a massive moduli \( \varphi \) with \( \delta \varphi \sim M_P \) and \( m_\varphi \sim 40\,\text{TeV} \) which would give a maximal entropy production with \( T_R \approx 6\,\text{MeV} \). This heavy modulus dominates the energy density of the universe as soon as its coherent oscillation begins at \( T \approx 10^{11} \,\text{GeV} \). The light moduli which are lighter than \( \sim 40\,\text{TeV} \) but heavier than the bound in Eq. (33) start to coherently oscillate during the matter dominated era by the heavy modulus oscillation. Then we can apply Eq. (28) and get the abundance

\[
m_\phi Y_\phi \approx T_R \left( \frac{\delta \phi}{M_P} \right)^2,
\]

(34)

which is diluted compared to Eq. (27) by a factor \( S_{\text{after}}/S_{\text{before}} = (m_\phi M_P)^{1/2}/g_*^{1/4}T_R \). It is rather straightforward to derive the cosmological limits on the light moduli misalignment
δφ for the diluted moduli density (34) as we did for the case without any late entropy production. The results for the case of $T_R = 6\, \text{MeV}$, i.e. maximal dilution, are shown in Figure 2. It shows that the entropy production by heavy moduli decays ameliorates but does not completely solve the cosmological moduli problem.

C. Dilution by thermal inflation

The most efficient way to dilute the dangerous light moduli $\phi$ is to have a late inflation since inflation dilutes the moduli density both by the spatial expansion and by the large amount entropy production. The most natural framework for a late inflation would be the so-called thermal inflation models [8].

In thermal inflation models, the entropy-producing field $\phi$ corresponds to a flaton field parameterizing a flat direction in supersymmetric models. This flat direction is lifted by the soft breaking mass and also by the Planck scale suppressed non-renormalizable terms, leading to the following (renormalization group improved) effective potential at zero temperature:

$$V_\phi = V_0 - m_\phi^2 |\phi|^2 + \frac{|\phi|^{2n+4}}{M_P^{2n}},$$

(35)

where $n$ is a model-dependent integer and the negative mass-squared can arise as a consequence of radiative corrections associated with the strong Yukawa coupling of $\phi$. The true vacuum expectation value is $\langle \phi \rangle \approx (m_\phi M_P)^{1/(n+1)}$ and $V_0$ is adjusted to $V_0 \approx m_\phi^2 \langle \phi \rangle^2 \approx m_\phi^{2n+4} M_P^{2n}$ in order for the true vacuum energy density to vanish. At high temperature $T \gg m_\phi$, the effective flaton mass-squared including the thermal contribution of $\mathcal{O}(T^2)$ is positive and thus $\langle \phi \rangle_{T \gg m_\phi} = 0$. For the period of $m_\phi \lesssim T \lesssim V_0^{1/4}$, the universe is vacuum-dominated, yielding an exponential expansion with the number of e-foldings ($N_e$) which is determined by $e^{N_e} \approx 0.42(100/g_*)^{1/4}(M_P/m_\phi)^{n/2n+2}$. At $T \approx m_\phi$, the flaton starts to roll down to its true vacuum value $\langle \phi \rangle$, and then the vacuum energy density $V_0$ is converted into the energy density of coherently oscillating flaton field. The oscillating flatons eventually decay and are converted into the radiation with the reheat temperature
\[ T_R \approx 1.7 g_R^{-1/4} \sqrt{M_P \Gamma_\varphi} \approx 0.1 \gamma^{1/2} g_R^{-1/4} m_\varphi \left( \frac{m_\varphi}{M_P} \right)^{\frac{5n-1}{2(n+2)}}, \] (36)

where \( \gamma \) is introduced to parameterize the \( \varphi \)-decay width \( \Gamma_\varphi = \gamma m_\varphi^3 / 64\pi \langle \varphi \rangle^2 \) and \( g_R \) denotes the effective number of the relativistic degrees of freedom at \( T_R \). As was discussed in \[22\], \( \varphi \) can couple to ordinary matter through its mixing with Higgses, and then the most efficient decay channel is the decay into stop and anti-stop pair. Assuming \( m_\varphi > 2m_\tilde{t} \) so that \( \varphi \) can decay into stop pairs \( \tilde{t} \) and \( \tilde{t}^* \), we have roughly \( \gamma \approx (2m_\tilde{t}/m_\varphi)^4 \). Although it can be a quite small number (particularly when \( m_\varphi \gg m_\tilde{t} \)), we assume here \( \gamma \approx 1 \) as a conservative choice.

The entropy production factor of thermal inflation is given by

\[ \frac{S_{\text{after}}}{S_{\text{before}}} \approx \frac{V_0}{3T_R m_\varphi^3} \approx 0.1 \gamma^{-\frac{1}{2}} \left( \frac{M_P}{m_\varphi} \right)^{\frac{5n-1}{2(n+2)}}. \] (37)

From Eq. (36) we see that to achieve \( T_R \gtrsim 6 \text{ MeV} \) it is required that \( \gamma^{1/2} m_\varphi \gtrsim 100 \text{ MeV} \), \( \gamma^{3/7} m_\varphi \gtrsim 60 \text{ GeV} \), and \( \gamma^{2/5} m_\varphi \gtrsim 700 \text{ GeV} \) for \( n = 1, 2, 3 \) respectively. Under this restriction, the maximal entropy production factor is \( 2 \times 10^{18}, 6 \times 10^{23} \gamma^{1/7}, 8 \times 10^{27} \gamma^{1/5} \) respectively.

The light moduli \( \phi \) can start to oscillate either before or after the thermal inflation depending upon their masses. Obviously the moduli oscillation should start before thermal inflation occurs in order for thermal inflation to sufficiently dilute the moduli density. This gives a lower bound on the moduli mass for which thermal inflation leads to the sufficient dilution of moduli density: \( m_\phi \gtrsim H_{\text{TI}} \approx (V_0 / M_P^2)^{1/2} \approx 100 \text{ eV}, 100 \text{ keV}, 1 \text{ MeV} \) for \( n = 1, 2, 3 \) respectively, where \( H_{\text{TI}} \) is the Hubble expansion parameter during thermal inflation. The moduli lighter than this bound but start to oscillate before the flaton decay are also diluted somewhat by the entropy production due to the flaton decay. For this mass range the analysis of the previous section can be applied. For \( m_\phi \approx H_{\text{TI}} \), we need a detailed analysis, but we will not concern such details. In Figure 3(a), we showed the relaxed constraints on the initial misalignment \( \delta \phi \) when the moduli density is maximally diluted by thermal inflation.

Although thermal inflation provides a huge entropy and thus dilute the moduli energy density due to the initial misalignment, it can cause an additional misalignment induced
by a shifted minimum of the free energy density during the inflation. As was noted in the discussions below Eq. (18), during the thermal inflation period, the shifted minimum is given by $\phi_{\text{min}} \approx \frac{c H_{T1}^2}{m_\phi^2} \phi_1$. For $H_{T1} \lesssim m_\phi$, this shifted minimum leads to an energy density $[8,7]$.

$$(m_\phi Y_\phi)_{T1} \approx \frac{c^4 T_R V_0}{m_\phi^2 M_P^2} \left( \frac{\phi_1}{M_P} \right)^2,$$

where $c$ and $\phi_1$ are those in the free energy density $[10]$ during thermal inflation. We showed the resulting constraints on $\phi_1$ in Figure 3(b) for the case that $c \sim 1$ over the entire moduli mass range, and in Figure 3(c) for the more plausible case that $c \approx m_\phi / m_{3/2}$ with $m_{3/2} = 100 \text{ GeV}$.

V. CONCLUSION

In this paper, we have discussed the possibility of light moduli having a mass far below the weak scale, and examined the cosmological bounds on the initial moduli misalignment for the mass range $40 \text{ TeV} \sim 10^{-26} \text{ eV}$. A very light moduli can arise as an axion-like moduli in string/$M$-theory with a large compactification radius, $m_\phi \approx e^{-\pi \text{Re}(T)} m_{3/2}$ with $\text{Re}(T) = 1 \sim \frac{1}{\alpha_{\text{GUT}}}$. Also generic moduli in gauge-mediated supersymmetry breaking models, can have a mass in the range $10 \text{ eV} \sim 1 \text{ GeV}$. We then studied the cosmological evolution of a generic light modulus $\phi$ to quantify its relic energy density which is determined by the initial misalignment $\delta \phi$. The initial misalignment $\delta \phi$ is set by the modulus value $\phi_i$ at an initial time $t_i$ with $\phi'(t_i) = 0$ and also by the dynamical parameter $c^2 \phi_1$ in the moduli free energy density $[10]$ at $t \gtrsim t_i$.

For the case that there is no entropy production after $\phi$-oscillation begins at $t \sim m_\phi^{-1}$ (but of course before $\phi$ decays), the bounds on the misalignment $\delta \phi$ coming from various astronomical and cosmological observations are shown in Figure 1. We then discussed how much such bounds can be relaxed by a late entropy dumping which can be driven typically by e.g. out-of-equilibrium decay of another heavy moduli [Figure 2] or by a late vacuum domination like thermal inflation [Figure 3]. The bound on the initial misalignment turned
out to be severe so that $\delta \phi \approx M_P$ is not allowed in most cases. Thermal inflation appears to be efficiently relax the bound, however still $\delta \phi \approx M_P$ is allowed only for a limited moduli mass range as shown in Figure 3.

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REFERENCES

[1] For a comprehensive review on axion, see J. E. Kim, Phys. Rep. 150, 1 (1987).

[2] J. Preskill, M. B. Wise and F. Wilczek, Phys. Lett. B120, 127 (1983); L. P. Abbot and P. Sikivie, Phys. Lett. B120, 133 (1983); M. Dine and W. Fischler, Phys. Lett. B120, 137 (1983).

[3] G. D. Coughlan, W. Fischler, E. W. Kolb, S. Raby and G. G. Ross, Phys. Lett. B131, 59 (1983).

[4] J. Ellis, D. V. Nanopoulos and M. Quiros, Phys. Lett. B174, 176 (1986); B. de Carlos, J. A. Casas, F. Quevedo and E. Roulet, Phys. Lett. B318, 447 (1993); T. Banks, D. B. Kaplan and A. E. Nelson, Phys. Rev. D49, 779 (1994).

[5] T. Banks and M. Dine, Nucl. Phys. B479, 173 (1996).

[6] A. de Gouvea, T. Moroi and H. Murayama, Phys. Rev. D56, 1281 (1997)

[7] J. Hashiba, M. Kawasaki and T. Yanagida, hep-ph/9708220; T. Asaka, J. Hashiba, M. Kawasaki and T. Yanagida, hep-ph/9711501.

[8] D. H. Lyth and E. D. Stewart, Phys. Rev. Lett. 75, 201 (1995).

[9] M. Green, J. Schwarz and E. Witten, Superstring Theory (Cambridge University Press, Cambridge, England, 1987).

[10] E. Witten, Phys. Lett. B149, 351 (1984).

[11] K. Choi and J. E. Kim, Phys. Lett. B154, 393 (1985); Phys. Lett. B165, 71 (1985).

[12] K. Choi, “Axions and the Strong CP Problem in M-theory”, hep-th/9706171.

[13] P. Horava and E. Witten, Nucl. Phys. B460, 506 (1996); Nucl. Phys. B475, 94 (1996).

[14] E. Witten, Nucl. Phys. B471, 135 (1996).
[15] E. Caceres, V. S. Kaplunovsky and I. M. Mandelberg, “Large-Volume String Compactifications, Revisited”, hep-th/9606036.

[16] M. Dine, N. Seiberg, X.-G. Wen, and E. Witten, Nucl. Phys. B289, 319 (1987); Nucl. Phys. B278, 769 (1986).

[17] D. H. Lyth and E. D. Stewart, Phys. Rev. D53, 1784 (1996).

[18] A. Linde, Phys. Rev. D53, 4129 (1996).

[19] J. Ellis, G. B. Gelmini, J. L. Lopez, D. V. Nanopoulos and S. Sarkar, Nucl. Phys. B373, 399 (1992).

[20] K.C. Gendreau, et al., Pub. Astro. Soc. Japan 47, L5 (1995); D.E. Gruber, in The X-ray Background, Cambridge Univ. Press (1992), edited by X. Barcons and A.C. Fabian; S.C. Kappadath, et al., in Proc. 24th Int. Cosmic Ray Conf. Vol.2 (1995) p.230; A.A. Zdziarski, Mon. Not. R. Astron. Soc. 281, L9 (1996).

[21] G. Lazarides, R. K. Schaefer, D. Seckel and Q. Shafi, Nucl. Phys. B346, 193 (1990); M. Kawasaki, T. Moroi and T. Yanagida, Phys. Lett. B383, 313 (1996).

[22] K. Choi, E. J. Chun and J. E. Kim, Phys. Lett. B430, 209 (1997).
FIG. 1. Constraints on the initial misalignment of moduli. We take $\Omega_0 = 1$, $h = 0.7$ and the lifetime $\tau = 10^{14} \left(\frac{m_\phi}{1\text{GeV}}\right)^{-3}\text{sec}$. The line (a) comes from the lower bound on the age of the universe, $\Omega_\phi h^2 \leq 1$, which is unavoidable even for very light moduli. The lines from (b) to (j) are for the case that moduli decay to photons or charged particles: (b) Recently reported X-ray background, (c) Observed $\gamma$-ray background, (d) Distorsion of CMBR, $\mu > 8 \times 10^{-3}$, (e) Photoproduction, $(D+^3\text{He})/H > 10^{-4}$, (f) Photodestruction, $D/H < 10^{-5}$, (g) Hadronic shower, $(D+^3\text{He})/H > 10^{-4}$, (h) Hadronic shower, $Y_p(^4\text{He}) > 0.25$, (i) Entropy production, $(D+^3\text{He})/H > 10^{-4}$, (j) Entropy production, $Y_p(^4\text{He}) > 0.25$. The line (a) can be straightforwardly extended to the region $m_\phi < 1\text{eV}$. 

$\tau$/sec

$\frac{\delta \phi}{M_P}$

$m_\phi(\text{GeV})$
FIG. 2. Relaxed bounds on the initial misalignment of light moduli when there is an entropy production due to the heavy modulus decay which yields the reheating temperature $T_{RH} = 6$ MeV. Again the line at $m_\phi \sim 1$ eV can be straightforwardly extended to the region $m_\phi < 1$ eV.

\[ \frac{\delta \phi}{M_P} \]

\[ \tau (\sec) \]

\[ m_\phi (\text{GeV}) \]

FIG. 3. Constraint on the misalignment of moduli when there exists thermal inflation with $T_R = 10$ MeV: (a) Constraint on the initial misalignment of moduli, (b) Constraint on the misalignment ($\delta \phi = \phi_1$) induced by the shifted minimum during thermal inflation when $c = 1$ over the entire mass range, (c) Constraint on the misalignment ($\delta \phi = \phi_1$) induced by thermal inflation for $c = m_\phi/m_{3/2}$ with $m_{3/2} = 100$ GeV.
