A Test for the Zero Mean Hypothesis in Cosmology

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One working hypothesis on which analyses of cosmological data are based is the zero ensemble mean hypothesis, which is related to the statistical homogeneity of cosmological perturbations. This hypothesis, however, should be tested by observational data in the current era of precision cosmology. Herein, we test the hypothesis by analyzing recent, foreground-reduced cosmic microwave background (CMB) maps, combining the spherical harmonic coefficients of the masked CMB temperature anisotropies in such a way that the combined variables can be treated as statistically independent samples. We find evidence against the zero mean hypothesis in two particular ranges of multipoles, with significance levels of $2.5\sigma$ and $3.1\sigma$ in the multipole ranges of $\ell \approx 61$-86 and 213-256, respectively, for both the Planck and Wilkinson Microwave Anisotropy Probe maps. The latter signal is consistent with our previous result found by using brute-force Monte-Carlo simulations. However, within the method employed in this paper we conclude that the zero mean hypothesis is consistent with the current CMB data on the basis of Stouffer’s weighted $Z$ statistics, which takes multiple testing into account.

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I. INTRODUCTION

Recent precise measurements of anisotropies of the cosmic microwave background (CMB) as well as a number of probes of the large scale structure (LSS) of the universe have led us to the standard, concordant model of cosmology. In the standard cosmological model, the universe contains small density fluctuations on top of otherwise flat, homogeneous, and isotropic space-time. The density fluctuations are thought to be generated through quantum fluctuations in the accelerating expansion phase in the early universe, i.e., inflation. An important diagnostic characteristic of inflation models is that they predict statistically homogeneous and isotropic Gaussian fluctuations with a near scale-invariant power spectrum (for a review, see [1]).

Among these features, the Gaussianity and approximate scale invariance have been intensively tested by a number of observations and verified with high significance [3–5], while the statistical homogeneity and isotropy have been less tested and often assumed implicitly in cosmological analyses [6]. Recently, the test of statistical isotropy has attracted much attention [8], after the authors of [8] found hints for the breaking of statistical isotropy in the Wilkinson Microwave Anisotropy Probe (WMAP) CMB anisotropy data (see also, [12, 13]). The existence of the statistical anisotropy has been confirmed by Planck data [14], and more recently Adrami et al. have found the statistical anisotropy at the $3.3\sigma$ level by measuring the local variance, using the 1000 available Planck Full Focal Plane simulations [15]. On the other hand, no evidence has been found in a sample of luminous red galaxies observed by the Sloan Digital Sky Survey [16].

In this paper, we test statistical homogeneity using recent full-sky CMB temperature maps provided by the WMAP and Planck satellites. Specifically, we test the null hypothesis that the means of cosmological perturbations are zero in spherical harmonic space, which corresponds to the usual Fourier space in three-dimensional space. It is understood that the zero mean hypothesis is related to statistical homogeneity as follows [17]. We usually assume that because of the cosmological principle, perturbation variables, such as the CMB temperature, can be decomposed into a space-independent background value and perturbations, as $T = T_0(t) + \delta T(t, \vec{x})$ with

$$\langle \delta T(t, \vec{x}) \rangle = 0.$$  \hspace{1cm} (1)

Note that this decomposition can be done only if the expectation value of $T$ is constant, i.e., $\langle T(t, \vec{x}) \rangle = \text{const.}$ with fixed time. It is always possible to satisfy Eq. (1) even if the expectation is not constant, but in that case

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the background temperature cannot be space-independent. The condition that this expectation value is constant is equivalent to the statistically homogeneous (stationary) condition of the mean,

$$\langle \delta T(t, \vec{x}) \rangle = \langle \delta T(t, \vec{x} + \vec{X}) \rangle \quad \forall \vec{X} \in \mathbb{R}^3 .$$

(2)

Hence the zero mean hypothesis is equivalent to the statistical homogeneity of the mean. The temperature anisotropy can be written, in the linearized theory, as

$$\delta T(\vec{x}, \hat{n}, t) = \int \frac{d^3 k}{(2\pi)^3} T(\vec{k}, \hat{n}) \phi(\vec{k}) e^{i \vec{k} \cdot \vec{x}} ,$$

(3)

where \( \phi(\vec{k}) \) are the Fourier modes of the initial density perturbations and \( T(\vec{k}, \hat{n}) \) is the linear transfer function that relates the initial density perturbations to the currently observed temperature anisotropies. Because the temperature fluctuations are observed on the sphere, it is common practice to express them with real spherical harmonic coefficients

$$a_{\ell m} = \int d^2 \hat{n} \delta T(\vec{x} = 0, \hat{n}, t) R_{\ell m}(\hat{n}) ,$$

(4)

where \( R_{\ell m} \) is the real set of spherical harmonics. Because the transfer function in Eq. (3) is completely determined by the cosmological perturbation theory given a cosmological model, the zero mean condition \( \langle \delta T(t, \vec{x}) \rangle = 0 \) is equivalent to the conditions \( \langle \phi(\vec{k}) \rangle = \langle a_{\ell m} \rangle = 0 \). In the following, we test whether the condition \( \langle a_{\ell m} \rangle = 0 \) is satisfied using the latest CMB data.

Studies on the zero mean hypothesis using CMB data can be found in Refs. [17] and [18]. One complication when testing the zero mean hypothesis with CMB data is the existence of the mask, which suppresses foreground contamination but generates correlations between the samples. Picon analyzed CMB data by constructing \( \nu \)-vectors that disentangle the correlations [17], while Kashino et al. utilized Monte-Carlo simulations to take into account the correlations in the samples [18]. In this paper we build on the work of [17], and extend the analysis toward higher multipoles using the latest CMB data from Planck.

II. METHOD

Herein we summarize the method developed by [17]. Observed temperature fluctuations, \( \delta T(\hat{n})_{\text{obs}} \), involve a convolution with the detector beam window \( B \) and the pixel smoothing kernel \( K \) and can be expressed as

$$\delta T_{\text{obs}}(\hat{n}) = M(\hat{n}) \left( K * [B * \delta T_{\text{CMB}}(\hat{n}) + N(\hat{n})] \right) ,$$

(5)

where \( \delta T_{\text{CMB}}(\hat{n}) \) is the CMB signal we want to estimate, \( N(\hat{n}) \) is instrument noise and \( M(\hat{n}) \) is the mask. Here we have omitted the foreground assuming that \( M(\hat{n}) \) can successfully mask the foreground contaminated regions. In spherical harmonic space, the above equation is expressed as

$$a_{\ell m}^{\text{obs}} = \sum_{\ell' m'} M_{\ell m; \ell' m'} a_{\ell' m'}^{\text{full sky}} ,$$

(6)

where \( a_{\ell m}^{\text{full sky}} = K_{\ell} B_{\ell} a_{\ell m}^{\text{CMB}} + K_{\ell} n_{\ell m} \) consists of the spherical harmonic coefficients of the sky, including signal and noise, and \( M_{\ell m; \ell' m'} \) is the mask-coupling matrix. Because instrumental noise is expected to be well described by a Gaussian distribution as shown in the WMAP [19] and Planck papers, ensemble averages of \( a_{\ell m}^{\text{obs}} \) are zero if those of \( a_{\ell m}^{\text{CMB}} \) are zero. Considering an axisymmetric mask that satisfies \( M_{\ell m; \ell' m'} = M_{\ell m; \ell m} \delta_{m m'} \) and using matrix notation for a fixed \( m \), the above equation can be expressed as

$$a_{m}^{\text{obs}} = M \cdot a_{m}^{\text{full sky}} ,$$

(7)

where the dot represents inner product over multipoles \( \ell \). To remove the effect of the mask from the observed spherical harmonic coefficients and disentangle the coupling, consider \( m \)-independent \( \nu \)-vectors that satisfy the relation

$$\check{\nu}^t = \check{v}^t M ,$$

(8)

where

$$\nu_{m} = \left\{ \begin{array}{ll} v_{\ell} & \text{for } |m| \leq \ell_{\text{min}} \text{ and } \ell_{\text{min}} \leq \ell \leq \ell_{\text{max}} \\ 0 & \text{(otherwise)} \end{array} \right. .$$

(9)
Let us construct a variable \(d_m\) as the dot product of \(\vec{v}\) and \(\vec{a}_{\text{obs}}\)

\[
d_m \equiv \vec{v}^\dagger \cdot \vec{a}_{\text{obs}} = \vec{v}^\dagger \mathbf{M} \vec{a}_{\text{full sky}} = \vec{v}^\dagger \cdot \vec{a}_{\text{full sky}} ,
\]

for \(|m| < m_{\text{max}}\). The new stochastic variable \(d_m\) has the following properties:

1. Foreground insensitive, because we work on \(a_{\ell m}^{\text{obs}}\), the spherical harmonic coefficients of the masked sky
2. Statistically independent, because they are constructed as a linear combination of statistically independent variables \(a_{\ell m}^{\text{CMB}}\) and \(n_{\ell m}\)
3. Gaussian with zero mean, if \(a_{\ell m}^{\text{CMB}}\) and \(n_{\ell m}\) are as well
4. Having \(m\)-independent variance, where \(\sigma^2 = \sum_{\ell_{\text{max}}}^{\ell_{\text{min}}} K_\ell (B_\ell^2 C_\ell + N_\ell) v_\ell^2\).

Owing to these properties, we can formulate a simple statistical test of the zero mean hypothesis.

To obtain the \(v\)-vectors, it is convenient to work in pixel space. In pixel space, Eq. (8) is written as

\[
(1 - M(\hat{n})) \vec{v}(\hat{n}) = 0 .
\]

Substituting \(v(\hat{n}) = \sum v_{\ell m} Y_{\ell m}(\hat{n})\) and defining the matrix

\[
D_{\ell i} = (1 - M(\hat{n}_i)) \sum_{|m| \leq \ell_{\text{min}}} Y_{\ell m}(\hat{n}_i) ,
\]

where \(i\) runs over all pixels and \(\ell_{\text{min}} \leq \ell \leq \ell_{\text{max}}\), we can rewrite the system of equations in Eq. (8) as

\[
D \vec{v} = 0 ,
\]

or, in component notation, \(\sum_{\ell} D_{\ell i} v_\ell = 0\). The dimension of the matrix \(D\) is \((\ell_{\text{max}} - \ell_{\text{min}} + 1, N_{\text{pix}})\), where we have used the Healpix pixelization scheme with \(N_{\text{side}} = 256\), and therefore \(N_{\text{pix}} = 786432\). We find an approximate solution of this system of equations using singular value decomposition (SVD).

### III. RESULT

We tested the zero mean hypothesis with the stochastic variable \(d_m\), which is constructed as a linear combination of \(a_{\ell m}\) given by Eq. (10). To perform the inner product in Eq. (10), we divide the multipoles into bins [17], and the ranges of these bins are shown in Fig. 1. In the figure, we show histograms of the variable \(d_m\), constructed from the WMAP (red) and Planck (black) maps. The variable \(d_m\) is normalized by the sample variance \(\sigma\). It is evident from the figure that the examined multipole range the Planck and WMAP maps give consistent results. The distributions are consistent with a Gaussian distribution, and the means of the distributions are zero, except for possible deviations for the multipole ranges of \(\ell \approx 61 - 86\) and \(\ell \approx 213 - 256\).

In Fig. 2, we depict the result of the test showing how many sigmas the observed data deviate from the zero mean. Here we perform a simple test assuming Gaussian statistics as follows. We estimate the mean from the sample by computing \(\bar{d} = \sum_m d_m / (2\ell_{\text{min}} + 1)\) and then obtain the error in the estimate of the mean from the formula

\[
\sigma(\bar{d}) = \frac{\sigma}{\sqrt{2\ell_{\text{min}} + 1}} ,
\]

where \(\sigma^2 \equiv \sum (d_m - \bar{d})^2 / (2\ell_{\text{min}})\) is the sample variance. The Z-scores in the figure are simply defined by \(Z = \bar{d} / \sigma(\bar{d})\) for each multipole bin.

There are hints of deviations in the ranges around \(\ell \approx 70\) and \(\ell \approx 230\). Significance levels are 2.5\(\sigma\) for the former and 3.1\(\sigma\) for the latter. For the latter signal, the significance is slightly larger for the Planck map. The other multipole ranges are consistent with the zero mean hypothesis.

### IV. DISCUSSION

#### A. Instrument Noise

The formulation described in the previous section ignores the effect of the instrument noise. Although instrument noise that is expected to have a zero mean would not bias the test of the zero mean hypothesis, it degrades the statistical power. To demonstrate how the instrument noise of the WMAP and Planck could affect the test of the zero
mean hypothesis, we construct the variable \( d_m \) from the expected noise values for the sky in the WMAP and Planck SMICA (Spectral Matching Independent Component Analysis) maps and show the results in Fig. 3 for the multipole range of \( \ell \approx 213-256 \). As expected, the noise in the Planck map are negligibly small compared with the signal for this angular scale, because of its high angular resolution. On the other hand, noise can contribute up to 40% for the WMAP case, and this might be a reason for a smaller S/N from the WMAP than that from Planck at this scale (see Fig. 3). Thus we did not explore the test at smaller scales for the WMAP map, though we did test the hypothesis in the multipole range of \( \ell \approx 257-300 \) for the Planck maps.

B. Foreground

Another issue we have to address is the foreground. One disadvantage of the simple and clear method described in this paper is that it must utilize an axisymmetric mask to simplify the convolution of the mask as in Eq. (7), while the foreground is, of course, not axisymmetric. To estimate how the foreground has contaminated the results in the previous section, we examine the same analysis but with more extensive and more aggressive masks that cut the region in the galactic latitude \( |b| \leq 20^\circ \pm 5^\circ \). The results are shown in Fig. 1. Overall, we find mutually consistent results. In fact, for the eighth bin (\( \ell \approx 213 – 256 \)), the significance remains the same even for \( |b| \leq 25^\circ \) although the standard deviation \( \sigma \) becomes larger because of the smaller analysed sky area. This is consistent with what the previous work found with the WMAP seven-year map [18].

The same analysis is also done using other types of foreground-reduced maps from Planck as the SMICA map, namely, SEVEM (internal template fitting) and NLIC (linear internal combination in a needlet space) maps. These maps are generated through processes completely different from those of SMICA maps and thus have different weights to both the frequencies and multipoles.

Table I summarizes the probabilities of supporting the null hypothesis for different sky cuts and different foreground-
FIG. 2: Deviations from the zero mean hypothesis from WMAP nine-year data (red dashed) and Planck SMICA (black solid) maps. The vertical error bars are ±σ, while the horizontal error bars are the width of the binning. The WMAP data point at the 9th bin is not shown here because large noise contamination is expected on such small scales.

FIG. 3: Histograms of stochastic variable \( d_m \) from the signal plus noise (red; solid line) and expected noise only (blue; dashed line), normalized by the sample variance for the bin of \( 213 \leq \ell \leq 256 \). The top panel is for the WMAP data and the bottom is for Planck (SMICA).

reduced maps. We find that all the different foreground-reduced maps from Planck give consistent, almost indistinguishable results. The WMAP and Planck data are also consistent with each other, suggesting that instrumental and scanning effects that may cause apparent violations of statistical homogeneity are negligible. At the eighth bin (\( \ell \approx 213 - 256 \)) the signal is slightly reduced for the WMAP map, but the small signal may be attributable to the instrument noise as discussed above.

An analysis has been made in Kashino et al. [18], where we found an anomalously large deviation from the zero mean hypothesis at the multipole range \( \ell \approx 221 - 240 \) using Monte-Carlo simulations with the WMAP maps. The deviation was as large as 99.93% confidence level, regardless of the different frequency maps and different masks. The multipole ranges that show deviations from the zero mean hypothesis are consistent between Kashino et al. [18] and the results presented in this paper, although the methods used are completely independent from each other.

C. Look-Elsewhere Effect

Finally, let us evaluate the significance as a whole to draw a conclusion against the null hypothesis. Because we have tested nine multipole bins for Planck maps (eight for the WMAP), we should take into account the effect whether apparent anomalies are found just because of statistical outliers. The effect is often called the look-elsewhere effect.
| bin | ±20° cut |
|-----|----------|
| ℓ_{min} | ℓ_{max} | SMICA | SEVEM | NLIC | WMAP |
| 19   | 38       | 56.1% | 56.2% | 56.3% | 65.3% |
| 39   | 60       | 13.1% | 13.3% | 13.2% | 11.2% |
| 87   | 112      | 58.8% | 53.1% | 53.6% | 42.0% |
| 113  | 142      | 46.2% | 47.8% | 47.8% | 52.9% |
| 143  | 176      | 95.8% | 99.6% | 99.7% | 87.2% |
| 177  | 212      | 5.61% | 6.11% | 6.13% | 4.33% |
| 213  | 256      | 0.192%| 0.213%| 0.175%| 1.00% |
| 256  | 300      | 58.8% | 64.1% | 63.1% | —    |

Stouffer’s Z 1.74 1.33 1.38 2.39

TABLE I: Probabilities of supporting the null hypothesis (p values) that the CMB fluctuations have a zero mean, for different maps.

| bin | ±15° | ±20° | ±25° |
|-----|------|------|------|
| ℓ_{min} | ℓ_{max} | SMICA | SEVEM | NLIC | WMAP |
| 19   | 38   | 59.6% | 56.1% | 52.7% |
| 39   | 60   | 13.1% | 13.1% | 12.9% |
| 87   | 112  | 1.28% | 1.38% | 1.88% |
| 113  | 142  | 63.8% | 58.8% | 51.6% |
| 143  | 176  | 40.1% | 46.2% | 74.1% |
| 177  | 212  | 99.7% | 95.8% | 92.0% |
| 213  | 256  | 0.511%| 0.192%| 0.193%|
| 256  | 300  | 84.9% | 58.8% | 28.0% |

TABLE II: Probabilities of supporting the null hypothesis (p values) that the CMB fluctuations have a zero mean, for different sky cuts.

In order to take this effect into account, we combine p-values by calculating the Stouffer’s weighted Z (Liptak-Stouffer method), which is defined as

\[ Z = \sum_{i=1}^{n} w_i Z_i \sqrt{\sum_{i=1}^{n} w_i^2}. \]  

(14)

Here \( Z_i \) is the so-called Z-score defined by \( Z_i = \Phi^{-1}(1 - p_i) \), where \( \Phi \) is the standard normal cumulative distribution function and \( w_i \) is the number of degrees of freedom for the \( i \)-th bin. The combined variable \( Z \), which follows the standard normal distribution if the common hypothesis is true, reflects the fact that we have done multiple tests for a common hypothesis. From the p-values listed in Table I, we find the values of Stouffer’s weighted Z as

\[ Z = 1.74 \text{ (SMICA), } 2.39 \text{ (WMAP).} \]  

(15)

Therefore we may conclude that the zero mean hypothesis is consistent with observational data as a whole. The significance is reduced for the Planck SMICA map compared with the WMAP result, because the mean at the ninth bin which has the largest number of degrees of freedom is consistent with the null hypothesis.

There is the possibility to perform additional tests to confirm whether or not the anomalous deviations from the zero mean hypothesis found here are just statistical fluctuations due to a particular realization of the Universe. A straightforward test is to make use of full-sky CMB polarization data that will soon be released from the Planck collaboration. Although the polarization anisotropies are made from common curvature fluctuations, their transfer functions do not completely coincide with those of temperature anisotropies, and thus they will lend additional statistical power. Another test is to look into large-scale structure data, which offers an independent probe for
primordial fluctuations [21–23]. The comoving scale that corresponds to the multipole range of $\ell \approx 213 - 256$ is approximately $k \approx 0.015 - 0.018 \text{ Mpc}^{-1}$, which is at the edge of the current galaxy survey by BOSS [24] and will be within reach in future galaxy surveys, such as Euclid [25], LSST [26], SKA [31] and others. Interesting ideas have been discussed in Refs. [22, 27–30], which include arguments that cosmic star formation histories and the kinetic Sunyaev-Zel’dovich effect can be used to probe inside our past light cone and thus they become powerful tools to probe into the cosmic homogeneity.

Before concluding, we would like to comment on the connection with studies on non-Gaussianity in the CMB. In analyses of higher-order statistics, such as the bispectrum, the zero mean condition has been implicitly assumed and one estimates a correlation of the form $\langle a_{\ell m} a_{\ell' m'} a_{\ell'' m''} \rangle$. Consider a case where the mean of $a_{\ell m}$ was not zero but the bispectrum was zero around the mean; it is expected that the three point correlation of the above form would have an amplitude on the order of

$$\langle a_{\ell m} a_{\ell' m'} a_{\ell'' m''} \rangle \sim C_\ell \langle a_{\ell m} \rangle.$$  \hspace{1cm} (16)

Therefore, constraints on non-Gaussianity using the bispectrum in the literature could be used to put constraints on the mean of the spherical harmonic coefficients when this is the case.

V. CONCLUSION

We have tested one working cosmological hypothesis, which states that cosmological perturbations have a zero ensemble mean, using the latest CMB temperature anisotropy maps from the WMAP and Planck satellites. We find evidence against the zero mean hypothesis in two particular ranges of multipoles, with significance levels of $2.5\sigma$ at $\ell \approx 61 - 86$ and $3.1\sigma$ at $\ell \approx 213 - 256$. However, in the present analysis, we conclude that the zero mean hypothesis is consistent with the current observational data on the basis of the Stouffer’s weighted-Z statistics, which takes into account multiple testing. The zero mean hypothesis can be further tested by future CMB polarization data that will be available soon from Planck satellite.

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