ON NON-VANISHING OF THE FOURIER COEFFICIENTS OF
PRIMITIVE FORMS

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Abstract. In this semi-expository article, we discuss about the non-vanishing of
the Fourier coefficients of primitive forms. We shall make a note of a discrepancy
in the statement of [5, Lemma 2.2].

1. Introduction

In 1947, Lehmer conjectured that Ramanujan’s tau function \( \tau(n) \) is non-vanishing
for all \( n \). In [6], he proved that the smallest \( n \) for which \( \tau(n) \neq 0 \) must be a prime
and showed that \( \tau(n) \neq 0 \) for all \( n < 33, 16, 799 \). It is well-known that the Fourier
coefficients of Ramanujan’s Delta function \( \Delta(z) \) are in fact \( \tau(n)(n \in \mathbb{N}) \). Note that
\( \Delta(z) \) is a cuspidal Hecke eigenform of weight 12 and level 1. It is a natural question
to ask if a similar phenomenon continue to hold for cusp forms of higher weight and
higher level.

In this semi-expository article, we study the non-vanishing of the Fourier coeffi-
cients of primitive forms of any weight and any level. We take this opportunity to
make a correction in the statement of [5, Lemma 2.2].

2. Preliminary

In this section, we shall define modular forms and recall some basic facts about
them. For more details, we refer the reader to consult [3], [7].

2.1. Congruence subgroups. The modular group \( SL_2(\mathbb{Z}) \) is defined by
\[
SL_2(\mathbb{Z}) := \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}.
\]
For any \( N \in \mathbb{N} \), we shall define a subgroup of \( SL_2(\mathbb{Z}) \) by
\[
\Gamma(N) = \left\{ \gamma \in SL_2(\mathbb{Z}) \mid \gamma \equiv \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \pmod{N} \right\}.
\]
Definition 2.1. We say that a subgroup \( \Gamma \) of \( SL_2(\mathbb{Z}) \) is a congruence subgroup, if \( \Gamma \)
contains \( \Gamma(N) \) for some \( N \in \mathbb{N} \).

In this theory, the following congruence subgroups play an important role
\[
\Gamma_1(N) = \left\{ \gamma \in SL_2(\mathbb{Z}) \mid \gamma \equiv \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right) \pmod{N} \right\},
\]
\[
\Gamma_0(N) = \left\{ \gamma \in SL_2(\mathbb{Z}) \mid \gamma \equiv \left( \begin{array}{cc} 1 & 0 \\ 0 & * \end{array} \right) \pmod{N} \right\} \quad \text{for any } N \in \mathbb{N}.
\]
The subgroup \( \Gamma(N) \) is called the principal congruence subgroup of \( SL_2(\mathbb{Z}) \). Note
that \( \Gamma(N) \leq \Gamma_1(N) \leq \Gamma_0(N) \leq SL_2(\mathbb{Z}) \), and \( \Gamma(1) = \Gamma_1(1) = \Gamma_0(1) = SL_2(\mathbb{Z}) \).

The modular group \( SL_2(\mathbb{Z}) \) acts on the complex upper half plane \( \mathcal{H} = \{ \tau \in \mathbb{C} \mid \Im(\tau) > 0 \} \) via
\[
\gamma \tau = \frac{a\tau + b}{c\tau + d}.
\]

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where $\tau \in \mathcal{H}$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$. For more details, please refer to [3 §1.2].

2.2. Modular forms. In this section, we shall define modular forms and recall some results related to them.

Let $X$ be the space of all complex valued holomorphic functions on $\mathcal{H}$. We can define an action of $\text{SL}_2(\mathbb{Z})$ on $X$ by using the action of $\text{SL}_2(\mathbb{Z})$ on $\mathcal{H}$ as follows. For any $k \in \mathbb{N}$, $f \in X$ and $\gamma \in \text{SL}_2(\mathbb{Z})$, we define the slash operator

$$(f|_{k}\gamma)(\tau) := (c\tau + d)^{-k}f(\gamma\tau), \quad \tau \in \mathcal{H},$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Now, we define the notion of modular forms for any congruence subgroup $\Gamma$ of $\text{SL}_2(\mathbb{Z})$.

**Definition 2.2.** Let $\Gamma$ be a congruence subgroup of $\text{SL}_2(\mathbb{Z})$. A function $f \in X$ is said to be a modular form of weight $k$ with respect to $\Gamma$ if

1. $f|_{k}\gamma = f, \forall \gamma \in \Gamma$,
2. $f|_{k}\alpha$ is holomorphic at $\infty$, $\forall \alpha \in \text{SL}_2(\mathbb{Z})$.

**Remark 2.3.** Note that one needs to verify condition (2) only for the representatives of distinct cosets of $\Gamma$ in $\text{SL}_2(\mathbb{Z})$.

Now, we explain the meaning of $f$ being holomorphic at $\infty$. From condition (1), it is clear that then $f$ will be $h\mathbb{Z}$-periodic, where $h$ is the smallest integer such that $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \Gamma$ (such $h$ exists since $\Gamma(N \leq \Gamma)$). This implies that there exists a function $g : D' \to \mathbb{C}$, where $D'$ is unit puncture disk, such that $f(\tau) = g(q_h)$ for all $\tau \in \mathcal{H}$, where $q_h = e^{2\pi i h}$. It is clear that, the function $g$ is holomorphic on $D'$, since $f$ is so on $\mathcal{H}$. The function $f$ is said to be holomorphic at $\infty$ if $g$ extends holomorphically to $q = 0$. Similarly, one can define the meaning of $f|_{k}\alpha$ being holomorphic at $\infty$. For more details, please refer to [3 §1.1, §1.2].

We denote the space of all modular forms of weight $k$ and level $\Gamma$ by $M_k(\Gamma)$.

2.3. Fourier expansion. Let $f \in M_k(\Gamma)$. Let $h$ be the smallest integer such that $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \Gamma$. Since $f$ is holomorphic at $\infty$, then $f$ has a Fourier expansion

$$f(\tau) = \sum_{n=0}^{\infty} a_f(n)q_h^n, \quad q_h = e^{2\pi i h}$$

for $\tau \in \mathcal{H}$.

**Definition 2.4.** Let $f \in M_k(\Gamma)$. We say that $f$ is a cusp form if $a_f|_{k}\alpha(0) = 0$ for all $\alpha \in \text{SL}_2(\mathbb{Z})$. We denote the space of all cusp forms of weight $k$ and level $\Gamma$ by $S_k(\Gamma)$.

Note that $M_k(\Gamma), S_k(\Gamma)$ are vector spaces over $\mathbb{C}$. By [3 Theorem 3.5.1 and Theorem 3.6.1], these are in fact finite dimensional vector spaces over $\mathbb{C}$. Now, we shall give some examples of modular forms and cusp forms.

**Example 2.5.** For any $k \geq 2$, we define the Eisenstein series of weight $2k$

$$G_{2k}(\tau) = \sum_{(c,d) \in \mathbb{Z}^2 - \{(0,0)\}} \frac{1}{(c\tau + d)^{2k}} \in M_{2k}(\text{SL}_2(\mathbb{Z})).$$
It is easy to check that $G_{2k}$ is a modular form of weight $2k$ and level 1 (cf. [3 Page 4]). The Fourier expansion of $G_{2k}$ at $\infty$ is given by

$$G_{2k}(\tau) = 2\zeta(2k) + 2\frac{(2\pi i)^{2k}}{(2k-1)!} \sum_{n=1}^{\infty} \sigma_{2k-1}(n)q^n, \quad k \geq 1, \quad (2.1)$$

where $\sigma_{2k-1}(n) = \sum_{m|n, m > 0} m^{2k-1}$. The normalized Eisenstein series is defined by $E_{2k}(\tau) := \frac{G_{2k}(\tau)}{2\zeta(2k)}$. Therefore, the Fourier expansion of $E_{2k}$ at $\infty$ is given by

$$E_{2k}(\tau) = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n)q^n,$$

where $B_k$’s are the Bernoulli numbers (cf. [3 Page 10]).

**Example 2.6.** From the dimensions of $S_k(\text{SL}_2(\mathbb{Z}))$, one can see that 12 is the least integer for which there is a non-zero cusp form for $\text{SL}_2(\mathbb{Z})$. Moreover, dimension of $S_{12}(\text{SL}_2(\mathbb{Z}))$ is 1 and it is spanned by

$$\Delta(z) = (60G_4(z))^3 - 27(140G_6(z))^2 \in S_{12}(\text{SL}_2(\mathbb{Z})), \quad z \in \mathcal{H}.$$ 

The product formula for $\Delta(z)$ is given by $\Delta(z) = q \prod_{n \geq 1} (1 - q^n)^{24} = \sum_{n \geq 1} \tau(n)q^n,$ where $q = e^{2\pi iz}$.

**Example 2.7** ([3], Example 2.28). For $N \in \{2, 3, 5, 11\}$, $(\Delta(z)/\Delta(Nz))^{1/(N+1)} \in S_{24/(N+1)}(\Gamma_0(N))$. Moreover, the space $S_{24/(N+1)}(\Gamma_0(N))$ is one dimensional and it is spanned by $(\Delta(z)/\Delta(Nz))^{1/(N+1)}$.

### 2.4. Modular forms with character.

A Dirichlet character modulo $N$ is a group homomorphism $\chi : (\mathbb{Z}/N\mathbb{Z})^* \rightarrow \mathbb{C}^*$.

**Definition 2.8.** The space of all modular forms of weight $k$ level $N$ with character $\chi$ is defined by

$$M_k(N, \chi) = \{f \in M_k(\Gamma_1(N))| f|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \chi(d)f, \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)\}.$$ 

The space $M_k(\Gamma_1(N))$ decomposes as

$$M_k(\Gamma_1(N)) = \bigoplus_{\chi} M_k(N, \chi),$$

where $\chi$ varies over all Dirichlet characters of $(\mathbb{Z}/N\mathbb{Z})^*$ such that $\chi(-1) = (-1)^k$ (cf. [7 Lemma 4.3.1]). Similarly one can define the space of cusp forms of weight $k$ level $N$ with character $\chi$ and they are denoted by $S_k(N, \chi)$. One can easily check that $S_k(N, \chi) = S_k(\Gamma_1(N)) \cap M_k(N, \chi)$. Moreover, a similar decomposition holds as well, i.e.,

$$S_k(\Gamma_1(N)) = \bigoplus_{\chi} S_k(N, \chi),$$

where $\chi$ varies over all Dirichlet characters of $(\mathbb{Z}/N\mathbb{Z})^*$ with $\chi(-1) = (-1)^k$ (cf. [7 Lemma 4.3.1]).
Example 2.9 (Poincaré series). Let $\Gamma_{\infty} = \{(\frac{1}{2}, b) \mid b \in \mathbb{Z}\}$, and $\chi$ be any Dirichlet character modulo $N$. For $m \geq 1$, we define
\[ P_m(z) := \sum_{\gamma = (a \ b \ c \ d) \in \Gamma_{\infty} \backslash \Gamma_0(N)} \chi(\gamma) \frac{1}{(cz + d)^k} \exp(2\pi imc) \]
for any integer $k \geq 2$. By [4, Proposition 14.1], $P_m(z) \in S_k(N, \chi)$.

Now we will define two types of operators on the space of modular forms (resp., cusp forms). They are known as Hecke operators.

2.5. Hecke operators. Let $M_k(\Gamma_1(N))$ be a space of modular forms of weight $k$, level $N$. For any $(n, N) = 1$, we define the **diamond operator**
\[ \langle n \rangle : M_k(\Gamma_1(N)) \longrightarrow M_k(\Gamma_1(N)) \]
as
\[ \langle n \rangle f := f|_k \alpha, \text{ for any } \alpha = (a b c d) \in \Gamma_0(N) \text{ with } \delta \equiv n \text{ (mod } N). \]
We can also extend the definition of diamond operator to $\mathbb{N}$ via $\langle n \rangle = 0$ if $(n, N) > 1$. Observe that for any character $\chi : (\mathbb{Z}/N\mathbb{Z})^* \longrightarrow \mathbb{C}^*$,
\[ M_k(N, \chi) = \{ f \in M_k(\Gamma_1(N)) \mid \langle n \rangle f = \chi(n)f, \forall n \in (\mathbb{Z}/N\mathbb{Z})^* \}. \]
Note that, the diamond operator acts trivially on $M_k(\Gamma_0(N))$, since $M_k(\Gamma_0(N)) = M_k(N, \chi_N)$, where $\chi_N$ is the trivial character modulo $N$.

Now, we will define the second type of **Hecke operator** for any prime $p$, and they are denoted by $T_p$. If $f(\tau) = \sum_{n=0}^{\infty} a_f(n)q^n \in M_k(\Gamma_1(N))$, then
\[ (T_p f)(\tau) = \sum_{n=0}^{\infty} a_f(np)q^n + \chi_N(p)p^{k-1} \sum_{n=0}^{\infty} a_{(n\delta)} f(n)q^{np} \in M_k(\Gamma_1(N)). \]
Similarly, one can also defined the action of $T_p$ on $M_k(N, \chi)$ as follows: If $f(\tau) = \sum_{n=0}^{\infty} a_f(n)q^n \in M_k(N, \chi)$, then
\[ (T_p f)(\tau) = \sum_{n=0}^{\infty} a_f(np)q^n + \chi(p)p^{k-1} \sum_{n=0}^{\infty} a_f(n)p^{np} \in M_k(N, \chi), \]
In fact, for $n \in \mathbb{N}$, one can define the Hecke operators $T_n$ as follows:
\begin{enumerate}
  \item For any prime $p$ and $r \geq 2$ we define $T_{p^r} = T_pT_{p^{r-1}} - p^{k-1}\chi(p)T_{p^{r-2}}$.
  \item For $n = p_1^{e_1} \ldots p_k^{e_k}$ we define $T_n = T_{p_1^{e_1}} \ldots T_{p_k^{e_k}}$.
\end{enumerate}
One can check that, any two primes $p \neq q$, $T_p T_q = T_q T_p$. In fact, the Hecke operators respects the spaces $S_k(N, \chi)$ and $S_k(\Gamma_0(N))$. For more details, we refer the reader to [3, §5.3].

2.6. Petersson inner product. To study the space of cusp forms $S_k(\Gamma_1(N))$ further, we make it into an inner product space. In order to do so, we need to define an inner product on the space of cusp forms.

The **hyperbolic measure** on the upper half plane is defined by
\[ d\mu(\tau) := \frac{dx dy}{y^2}, \quad \tau = x + iy \in \mathfrak{H}. \]
For any congruence subgroup $\Gamma \leq \text{SL}_2(\mathbb{Z})$, the **Petersson inner product**
\[ \langle , \rangle_{\Gamma} : S_k(\Gamma) \times S_k(\Gamma) \longrightarrow \mathbb{C} \]
is given by
\[ \langle f, g \rangle_\Gamma = \frac{1}{V_\Gamma} \int_{\Gamma \backslash \mathbf{H}} f(\tau) \overline{g(\tau)} (\text{Im}(\tau))^k d\mu(\tau), \]
where \( V_\Gamma = \int_{\Gamma \backslash \mathbf{H}} d\mu(\tau). \)

This inner product is linear in \( f \), conjugate linear in \( g \), Hermitian symmetric and positive definite. By [3, Theorem 5.5.3], the Hecke operators \( \langle n \rangle \) and \( T_n \) are normal operators for \((n, N) = 1\). By [3, Theorem 5.5.4], we have that

**Theorem 2.10.** The space \( S_k(\Gamma_1(N)) \) has an orthogonal basis of simultaneous eigenforms for the Hecke operators \( \{\langle n \rangle, T_n : (n, N) = 1\} \).

Now, we shall introduce the theory of old forms and new forms. This in fact leads to define the notion of primitive forms. (cf. [3, §5.4] for more discussion on this).

### 2.7. Old forms and New forms.

For \( d \mid N \), we define the mapping
\[ i_d : (S_k(\Gamma_1(Nd^{-1})))^2 \rightarrow S_k(\Gamma_1(N)) \]
by
\[ (f, g) \rightarrow f + g|_k \left( \begin{smallmatrix} d & 0 \\ 0 & 1 \end{smallmatrix} \right). \]

The space of **old forms** is defined by
\[ S_k(\Gamma_1(N))^\text{old} = \sum_{p \mid N} i_p((S_k(\Gamma_1(Np^{-1})))^2). \]

The space of **new forms** (denote by \( S_k(\Gamma_1(N))^\text{new} \)) is defined to be the orthogonal complement of \( S_k(\Gamma_1(N))^\text{old} \) with respect to the Petersson inner product. By [3, Proposition 5.6.2], we see that the spaces \( S_k(\Gamma_1(N))^\text{old} \) and \( S_k(\Gamma_1(N))^\text{new} \) are stable under the action of \( T_n \) and \( \langle n \rangle \) for all \( n \in \mathbb{N} \).

**Definition 2.11.** A **primitive form** is a normalized eigenform in \( f \in S_k(\Gamma_1(N))^\text{new} \), i.e., \( f \) is an eigenform for the Hecke operators \( T_n, \langle n \rangle \) for all \( n \in \mathbb{N} \), and \( a_f(1) = 1 \).

By [3, Theorem 5.8.2], the set of primitive forms in the space \( S_k(\Gamma_1(N))^\text{new} \) forms an orthogonal basis. Each such primitive form lies in an eigen space \( S_k(N, \chi) \) for an unique character \( \chi \). In fact, its Fourier coefficients are its \( T_n \)-eigenvalues.

**Note 2.12.** When we say that \( f \in S_k(N, \chi) \) is a primitive form of weight \( k \), level \( N \), with character \( \chi \), actually we mean \( f \in S_k(\Gamma_1(N))^\text{new} \) is a primitive form and it belongs the eigenspace \( S_k(N, \chi) \).

**Proposition 2.13** ([3], Proposition 5.8.5). Let \( f = \sum_{n=1}^\infty a_f(n)q^n \in S_k(N, \chi) \). Then \( f \) is a normalized eigenform if and only if its Fourier coefficients satisfy the following relations

1. \( a_f(1) = 1 \),
2. \( a_f(m)a_f(n) = a_f(m)a_f(n) \) if \( (m, n) = 1 \),
3. \( a_f(p^r) = a_f(p)a_f(p^{r-1}) - p^k-1 \chi(p)a_f(p^{r-2}) \), for all prime \( p \) and \( r \geq 2 \).

For more details on this content, please refer to [3, §5.7, §5.8].
3. Classical modular forms

Recall that, Lehmer proved that the smallest \( n \) for which \( \tau(n) = 0 \) must be a prime. We are interested in studying a similar question for the Fourier coefficients of primitive forms of higher weight and higher level. Let \( f = \sum_{n=1}^{\infty} a_f(n)q^n \in S_k(N, \chi) \) be a primitive form of even weight \( k \), level \( N \), with character \( \chi \).

Suppose that \( a_f(n) = 0 \) for some \( n = \prod_{i} p_i^{r_i} \geq 1 \). By Proposition 2.13, we see that \( a_f(p_i^{r_i}) = 0 \) for some prime \( p_i \). In this section, we shall explore the relation between the vanishing (resp., non-vanishing) of \( a_f(p) \) and \( a_f(p^r) \) for \( r \geq 2 \). We begin this discussion with a lemma of Kowalski, Robert, and Wu (see [5, Lemma 2.2]).

**Proposition 3.1.** Let \( f = \sum_{n=1}^{\infty} a_f(n)q^n \in S_k(N, \chi) \) be a primitive form of even weight \( k \), level \( N \), with character \( \chi \). There exists an integer \( M_f \geq 1 \), such that for any prime \( p \nmid M_f \), either \( a_f(p) = 0 \) or \( a_f(p^r) \neq 0 \) for all \( r \geq 1 \).

**Proof.** If \( p \mid N \) then \( a_f(p^r) = a_f(p^r)^\chi \) for any \( r \geq 1 \), so in this case the conclusion holds trivially. Let \( p \) be a prime number such that \( p \nmid N \). If \( a_f(p) = 0 \), then there is nothing prove. Suppose that \( a_f(p) \neq 0 \) but \( a_f(p^r) = 0 \) for some \( r \geq 2 \). Since \( f \) is a primitive form, then by Hecke relations, we have

\[
a_f(p^{m+1}) = a_f(p)a_f(p^m) - \chi(p)p^{k-1}a_f(p^{m-1})
\]

for any \( m \in \mathbb{N} \). These relations can be re-interpreted as

\[
\sum_{r=0}^{\infty} a_f(p^r)X^r = \frac{1}{1 - a_f(p)X + \chi(p)p^{k-1}X^2}.
\] (3.1)

Suppose that

\[
1 - a_f(p)X + \chi(p)p^{k-1}X^2 = (1 - \alpha(p)X)(1 - \beta(p)X).
\] (3.2)

By comparing the coefficients, we get that

\[
\alpha(p) + \beta(p) = a_f(p) \quad \text{and} \quad \alpha(p)\beta(p) = \chi(p)p^{k-1} \neq 0,
\]

since \( p \nmid N \) and hence \( \chi(p) \neq 0 \). If \( \alpha(p) = \beta(p) \), then

\[
a_f(p^r) = (t+1)\alpha(p)^t \neq 0,
\] (3.3)

for any \( t \geq 2 \) and this cannot happen. Therefore, \( \alpha(p) \neq \beta(p) \). Then, by induction, we have the following

\[
a_f(p^r) = \frac{\alpha(p)^{t+1} - \beta(p)^{t+1}}{\alpha(p) - \beta(p)}.
\]

for any \( t \geq 2 \). Recall that \( a_f(p^r) = 0 \) for some \( r \geq 2 \). Therefore,

\[
a_f(p^r) = 0 \quad \text{if and only if} \quad \left( \frac{\alpha(p)}{\beta(p)} \right)^{r+1} = 1,
\] (3.4)

which implies that the ratio \( \frac{\alpha(p)}{\beta(p)} \) is a \( (r+1) \)-th root of unity. Since \( a_f(p) \neq 0 \), we get that \( \alpha(p) = \zeta \beta(p) \) where \( \zeta \) is a root of unity and \( \zeta \neq -1 \). By the product relation, we get that \( \alpha(p)^2 = \zeta \chi(p)p^{k-1} \), hence \( \alpha(p) = \pm \gamma p^{(k-1)/2} \), where \( \gamma^2 = \zeta \chi(p) \). Therefore,

\[
a_f(p) = (1 + \zeta^{-1})\alpha(p) = \pm \gamma(1 + \zeta^{-1})p^{(k-1)/2} \neq 0.
\]
In particular, \( \gamma(1 + \zeta^{-1})p^{(k-1)/2} \in \mathbb{Q}(f) \), where \( \mathbb{Q}(f) \) is the number field generated by the Fourier coefficients of \( f \) and by the values of \( \chi \). Since \( k \) is even, we have

\[
\gamma(1 + \zeta^{-1})\sqrt{p} \in \mathbb{Q}(f).
\]

We have that the number of such primes \( p \) are finite, since \( \mathbb{Q}(f) \) is a number field.

Take \( M_f \) to be the product of all such primes \( p \). Thus, for any prime \( p \mid M_f \), we have either \( a_f(p) = 0 \) or \( a_f(p^r) \neq 0 \) for all \( r \geq 1 \).

**Corollary 3.2.** Let \( f, M_f \) be as in the above Proposition. Then the smallest \( m \in \mathbb{N} \) with \( (m, M_f) = 1 \) with \( a_f(m) = 0 \) is a prime.

If \( M_f = 1 \), then the corollary is exactly the generalization of Lehmer’s result that that the smallest \( n \) for which \( \tau(n) = 0 \) must be a prime. Now, this leads to the question of calculating \( M_f \) for \( f \). In the second part of [5, Lemma 2.2], it was stated as follows:

**Proposition 3.3.** Let \( f, M_f \) be as in Proposition 3.1. If the character \( f \) is trivial and the Fourier coefficients of \( f \) are integers, then one can take \( M_f = N \).

However, we are able to produce examples which contradicts this statement.

**Example 3.4.** Let \( E \) be an elliptic curve defined by the minimal Weierstrass equation \( y^2 + y = x^3 - x \). The Cremona label for \( E \) is 37a1. Let \( f_E \) denote the primitive form (of weight 2 and level 37) associated to \( E \) by the modularity theorem. The Fourier expansion of \( f_E \) is given by

\[
f_E(q) = \sum_{n=1}^{\infty} a_f(n)q^n = q - 2q^2 - 3q^3 + 2q^4 - 2q^5 + 6q^6 - q^7 + 6q^9 + O(q^{10}).
\]

Note that \( (2, 37) = 1 \) and \( a_f(2) \) is non-zero but \( a_f(8) = 0 \).

**Example 3.5.** Let \( E \) be an elliptic curve defined by the minimal Weierstrass equation \( y^2 + xy + y = x^3 - x^2 \). The Cremona label for \( E \) is 53a1. Let \( f_E \) denote the primitive form (of weight 2 and level 53) associated to \( E \) by the modularity theorem. The Fourier expansion of \( f_E \) is given by

\[
f_E(q) = \sum_{n=1}^{\infty} a_f(n)q^n = q - q^2 - 3q^3 - q^4 + 3q^6 - 4q^7 + 3q^8 + 6q^9 + O(q^{10}).
\]

Note that \( (3, 53) = 1 \) and \( a_f(3) \) is non-zero but a simple calculation using the relations among the Fourier coefficients shows that \( a_f(3^5) = 0 \).

For the convenience of the reader, we shall recall their proof of Proposition 3.3.

**Proof.** Let \( p \) be a prime number such that \( p \mid N \). If \( a_f(p) = 0 \), then there is nothing prove. Suppose \( a_f(p) \neq 0 \) but \( a_f(p^r) = 0 \) for some \( r \geq 2 \). Arguing as in Proposition 3.1, the argument is valid till (3.5). After that, they wished to show that (3.5) does not hold for any prime \( p \mid N \).

By (3.2), (3.4), we get that \( \frac{a_f(3)}{a_f(3)} = \zeta \) is a root of unity in a quadratic extension of \( \mathbb{Q} \), hence \( \zeta \in \{-1, \pm i, \pm \omega_3, \pm \omega_3^2\} \). All those except \( \zeta = -1 \) contradict the fact that \( f \) has integer coefficients by simple considerations such as the following, for \( \zeta = \omega_3 \) say: we have \( \omega_3(p^2) = \omega_3p^{k-1}, \gamma = \pm \omega_3\omega_3^{k-1} \) and \( \lambda_f(p) = (1 + \omega_3^{-1})\gamma = \pm (1 + \omega_3^{-1})\omega_3^{k-1} \in \mathbb{Z} \). Therefore, (3.5) does not hold for any prime \( p \mid N \). \( \square \)
In the last part of the above proof, when we calculated the expression in (3.5) for \( \zeta \neq \pm 1 \), it seem to hold for \( p = 2 \) (resp., \( p = 3 \)) with some special values of \( a_f(2) \) (resp., \( a_f(3) \)). In the next proposition, we have calculated the optimal value of \( M_f \) and the correct version of Proposition 3.3 is

**Proposition 3.6.** Let \( f, M_f \) be as in Proposition 3.4. If the character \( \chi \) is trivial and the Fourier coefficients of \( f \) are integers, then \( M_f \) can be so chosen that \( (M_f, N) = 1 \) and \( M_f \mid 6 \).

**Proof.** If \( p \mid N \) then \( a_f(p^r) = a_f(p)^r \) for any \( r \geq 1 \), so in this case the conclusion of Proposition 3.1 holds trivially. Hence, the number \( M_f \) is relatively prime to \( N \).

If \( p \not\mid N \), we argue as in the proof of Proposition 3.3 till the last step. Now, we compute (3.5) for all values of \( \zeta \) to prove our proposition. Let \( \omega_n \) denote \( e^{2\pi i/n} \) for any \( n \in \mathbb{N} \).

1. The root of unity \( \zeta \) cannot be 1 because of (3.3).
2. The root of unity \( \zeta \) cannot be \(-1 \) because \( 0 \neq a_f(p) = \alpha(p) + \beta(p) \).
3. If \( \zeta = \omega_3 \), then \( \alpha(p)^2 = \omega_3 p^{k-1} \Rightarrow \alpha(p) = \pm \omega_3^{2/k} p^{k-1} \). This implies that \( a_f(p) = \pm (1 + \omega_3^2) \omega_3^2 p^{k-1} = \mp \omega_3^{k-1} \notin \mathbb{Z} \). For \( \zeta = \omega_3 \), we will get the same conclusion.
4. If \( \zeta = i \), then \( \alpha(p)^2 = ip^{k-1} \Rightarrow \alpha(p) = \pm \omega_8 p^{k-1} \Rightarrow a_f(p) = \pm (1 - i) \omega_8 p^{k-1} = \mp 2 p^{k-1} \). This implies that
   \[
   \sqrt{2} p^{k-1} \in \mathbb{Z} \iff p = 2,
   \]
   in which case \( a_f(2) = \pm 2^{k/2} \). For \( \zeta = -i \), we will get the same conclusion.
5. If \( \zeta = -\omega_3 \), then \( \alpha(p)^2 = -\omega_3 p^{k-1} \Rightarrow \alpha(p) = \pm \sqrt{-2} p^{k-1} \Rightarrow a_f(p) = \pm (1 - \sqrt{-2}) \omega_3^{-2} p^{k-1} = \pm \sqrt{3} p^{k-1} \). This implies that
   \[
   \sqrt{3} p^{k-1} \in \mathbb{Z} \iff p = 3,
   \]
   in which case \( a_f(3) = \pm 3^{k/2} \). If \( \zeta = -\omega_3^2 \), then we will get same conclusion.

This case by case analysis would imply that \( M_f \) is a divisor of 6. This means that the possible values of \( M_f \) are 1, 2, 3, 6. \( \square \)

For any prime \( p \), \( \chi_p^o \) denote the trivial character on \((\mathbb{Z}/p\mathbb{Z})^*\), i.e., for any \( N \in \mathbb{N} \), we have

\[
\chi_p^o(N) := \begin{cases} 0 & \text{if } p \mid N, \\ 1 & \text{if } p \not\mid N. \end{cases}
\]

Based on the proof of the above proposition, we can re-interpret the above result as follows:

**Lemma 3.7.** Let \( f, M_f \) be as in Proposition 3.6. Then \( M_f \) can be taken to be \( 2^{\chi_2^o(N)} 3^{\chi_3^o(N)} \). Further if

- 2 \mid \( M_f \), \( a_f(2) \neq \pm 2^{k/2} \), then 2 can be dropped from \( M_f \), i.e., \( M_f \) can be taken to be \( 2^{\chi_2^o(N)} 3^{\chi_3^o(N)} \),
- 3 \mid \( M_f \), \( a_f(3) \neq \pm 3^{k/2} \), then 3 can be dropped from \( M_f \), i.e., \( M_f \) can be taken to be \( 2^{\chi_2^o(N)} 3^{\chi_3^o(N)} \),
- 6 \mid \( M_f \), \( a_f(p) \neq \pm p^{k/2} \) (for \( p = 2, 3 \)), then 6 can be dropped from \( M_f \), i.e., \( M_f \) can be taken to be 1.
Note that the above lemma gives an optimal \( M_f \) for which Proposition 3.6 continues to hold. The following corollaries describes the nature of the first vanishing of Fourier coefficients of primitive forms of higher weight \( k \) and higher level \( N \).

**Corollary 3.8.** Let \( f = \sum_{n=1}^{\infty} a_f(n)q^n \in S_k(\Gamma_0(N)) \) be a primitive form of even weight \( k \) and level \( N \) with \( a_f(n) \in \mathbb{Z} \). Let \( M_f \) be as in Lemma 3.7. Then the smallest \( n \in \mathbb{N} \) with \( (n, M_f) = 1 \) and \( a_f(n) = 0 \) is prime.

**Proof.** Let \( n \) be the smallest integer with \( (n, M_f) = 1 \) such that \( a_f(n) = 0 \). Since \( f \) is a primitive form, we know that the Fourier coefficients of \( f \) satisfy
\[
a_f(n_1n_2) = a_f(n_1)a_f(n_2) \quad \text{if} \quad (n_1, n_2) = 1.
\]
This forces that \( n = p^r \), where \( p \) is a prime with \( (p, M_f) = 1 \). By Proposition 3.6, we get that \( r = 1 \). Therefore \( n \) has to be a prime. \( \square \)

The following two corollaries can be thought of as a generalization of the result of Lehmer which states that the smallest \( n \) for which \( \tau(n) = 0 \) must be a prime.

**Corollary 3.9.** Let \( f = \sum_{n=0}^{\infty} a_f(n)q^n \in S_k(\Gamma_0(N)) \) be a primitive form of even weight \( k \), level \( N \) with \( a_f(n) \in \mathbb{Z} \). If \( 6 \) divides \( N \), then the smallest \( n \) for which \( a_f(n) = 0 \) is a prime.

**Proof.** Since \( M_f \mid 6 \), and \( 6 \mid N \), we have that \( M_f \mid N \). Since \( (M_f, N) = 1 \), we have that \( M_f = 1 \). By Corollary 3.8, the result follows. \( \square \)

In order to get a similar conclusion as above for cusp forms when \( 6 \not\mid N \), e.g., for \( \Delta \)-function, we need to impose some conditions on \( a_f(2), a_f(3) \), which is the content of the following Corollary. It follows from Lemma 3.7 and coincides with [10, Proposition 4.2].

**Corollary 3.10.** Let \( f = \sum_{n=0}^{\infty} a_f(n)q^n \in S_k(\Gamma_0(N)) \) be a primitive form of even weight \( k \), level \( N \) with \( a_f(n) \in \mathbb{Z} \). Suppose \( a_f(2) \neq \pm 2^\tau \) and \( a_f(3) \neq \pm 3^\tau \). Then the smallest \( n \) for which \( a_f(n) = 0 \) is a prime.

**Proof.** We know that \( M_f \mid 6 \) and \( (M_f, N) = 1 \). By Lemma 3.7, it follows that \( M_f \) can be improved to 1. Therefore, the result follows by Corollary 3.8. \( \square \)

## 4. Hilbert modular forms

There is a generalization of Proposition 3.1 available in the context of Hilbert modular forms. In fact, we used this generalization to study the simultaneous non-vanishing of Fourier coefficients of distinct primitive forms at powers of prime ideals (cf. [2]). We shall state that generalization in this section.

Let \( K \) be a totally real number field of odd degree \( n \) and \( \mathbb{P} \) denote the set of all prime ideals of \( \mathcal{O}_K \) with odd inertia degree. Let \( \mathbb{P} \) denote the set of all prime ideals of \( \mathcal{O}_K \).

Let \( f \) be a primitive form over \( K \) of level \( \mathfrak{c} \), with character \( \chi \) and weight \( 2k = (2k_1, \ldots, 2k_n) \). Let \( 2k_0 \) denote the maximum of \( \{2k_1, \ldots, 2k_n\} \). For each integral ideal \( \mathfrak{m} \subseteq \mathcal{O}_K \), let \( C(\mathfrak{m}, f) \) denote the Fourier coefficients of \( f \) at \( \mathfrak{m} \).

Now, we state the result which is analogous to Proposition 3.1 for \( f \).

**Proposition 4.1.** Let \( f \) be a primitive form over \( K \) of level \( \mathfrak{c} \), with character \( \chi \) and weight \( 2k \). Then there exists an integer \( M_f \geq 1 \) with \( N(\mathfrak{c}) \mid M_f \) such that for any
prime \( p \not| M_f \) and for any prime ideal \( \mathfrak{p} \in \mathbb{P} \) over \( p \), we have either \( C(\mathfrak{p}, \mathfrak{f}) = 0 \) or \( C(\mathfrak{p}^r, \mathfrak{f}) \neq 0 \) for all \( r \geq 1 \).

**Proof.** Let \( p \) be a prime number such that \( p \not| N(\mathfrak{c}) \). Let \( \mathfrak{p} \in \mathbb{P} \) be a prime ideal of \( \mathcal{O}_K \) over \( p \) and \( \mathfrak{p} \not| \mathfrak{c} \). If \( C(\mathfrak{p}, \mathfrak{f}) = 0 \), then there is nothing prove. If \( C(\mathfrak{p}, \mathfrak{f}) \neq 0 \), then we need to show that \( C(\mathfrak{p}^r, \mathfrak{f}) \neq 0 \) for all \( r \geq 2 \), except for finitely many prime ideals \( \mathfrak{p} \in \mathbb{P} \).

Suppose that \( C(\mathfrak{p}, \mathfrak{f}) \neq 0 \) but \( C(\mathfrak{p}^r, \mathfrak{f}) = 0 \) for some \( r \geq 2 \). Since \( \mathfrak{f} \) is a primitive form, then by Hecke relations, we have

\[
C(\mathfrak{p}^{m+1}, \mathfrak{f}) = C(\mathfrak{p}, \mathfrak{f})C(\mathfrak{p}^m, \mathfrak{f}) - \chi(\mathfrak{p})N(\mathfrak{p})^{2k_0-1}C(\mathfrak{p}^{m-1}, \mathfrak{f}).
\]

These relations can be re-interpreted as

\[
\sum_{r=0}^{\infty} C(\mathfrak{p}^r, \mathfrak{f})X^r = \frac{1}{1 - C(\mathfrak{p}, \mathfrak{f})X + \chi(\mathfrak{p})N(\mathfrak{p})^{2k_0-1}X^2}.
\]  \( \text{(4.1)} \)

Suppose that

\[
1 - C(\mathfrak{p}, \mathfrak{f})X + \chi(\mathfrak{p})N(\mathfrak{p})^{2k_0-1}X^2 = (1 - \alpha(\mathfrak{p})X)(1 - \beta(\mathfrak{p})X).
\]

By comparing the coefficients, we get that

\[
\alpha(\mathfrak{p}) + \beta(\mathfrak{p}) = C(\mathfrak{p}, \mathfrak{f}) \quad \text{and} \quad \alpha(\mathfrak{p})\beta(\mathfrak{p}) = \chi(\mathfrak{p})N(\mathfrak{p})^{2k_0-1} \neq 0,
\]

since \( \mathfrak{p} \not| \mathfrak{c} \) and hence \( \chi(\mathfrak{p}) \neq 0 \). If \( \alpha(\mathfrak{p}) = \beta(\mathfrak{p}) \), then

\[
C(\mathfrak{p}^r, \mathfrak{f}) = (r + 1)\alpha(\mathfrak{p})^r \neq 0,
\]

which cannot happen for any \( r \geq 2 \). So, \( \alpha(\mathfrak{p}) \) cannot be equal to \( \beta(\mathfrak{p}) \). Then by induction, for any \( r \geq 2 \), we have the following

\[
C(\mathfrak{p}^r, \mathfrak{f}) = \frac{\alpha(\mathfrak{p})^{r+1} - \beta(\mathfrak{p})^{r+1}}{\alpha(\mathfrak{p}) - \beta(\mathfrak{p})}.
\]

In this case, we have

\[
C(\mathfrak{p}^r, \mathfrak{f}) = 0 \quad \text{if and only if} \quad \left( \frac{\alpha(\mathfrak{p})}{\beta(\mathfrak{p})} \right)^{r+1} = 1,
\]

which implies that the ratio \( \frac{\alpha(\mathfrak{p})}{\beta(\mathfrak{p})} \) is a root of unity. Since \( C(\mathfrak{p}, \mathfrak{f}) \neq 0 \), we get that \( \alpha(\mathfrak{p}) = \zeta\beta(\mathfrak{p}) \) where \( \zeta \) is a root of unity and \( \zeta \neq -1 \). By the product relation, we get that \( \alpha(\mathfrak{p})^2 = \zeta\chi(\mathfrak{p})N(\mathfrak{p})^{2k_0-1} \), hence \( \alpha(\mathfrak{p}) = \pm \gamma N(\mathfrak{p})^{(2k_0-1)/2} \), where \( \gamma^2 = \zeta\chi(\mathfrak{p}) \). Therefore,

\[
C(\mathfrak{p}, \mathfrak{f}) = (1 + \zeta^{-1})\alpha(\mathfrak{p}) = \pm \gamma(1 + \zeta^{-1})N(\mathfrak{p})^{(2k_0-1)/2} \neq 0.
\]

In particular, \( \mathbb{Q}(\gamma(1 + \zeta^{-1})N(\mathfrak{p})^{(2k_0-1)/2}) \subseteq \mathbb{Q}(\mathfrak{f}) \), where \( \mathbb{Q}(\mathfrak{f}) \) is the field generated by \( \{C(\mathfrak{m}, \mathfrak{f})\}_{\mathfrak{m} \in \mathcal{O}_K} \) and by the values of the character \( \chi \). Since \( \mathfrak{p} \in \mathbb{P} \), \( N(\mathfrak{p}) = p^f \), where \( f \in \mathbb{N} \) odd. Hence, we have

\[
\mathbb{Q}(\gamma(1 + \zeta^{-1})p^{(2k_0-1)/2}) \subseteq \mathbb{Q}(\mathfrak{f}).
\]  \( \text{(4.2)} \)

Since \( 2k_0 - 1, f \) are odd, we have that

\[
\mathbb{Q}(\gamma(1 + \zeta^{-1})\sqrt{p}) \subseteq \mathbb{Q}(\mathfrak{f}).
\]  \( \text{(4.3)} \)

By [9] Proposition 2.8, the field \( \mathbb{Q}(\mathfrak{f}) \) is a number field. Hence, the number of such primes \( p \) are finite. Take \( M_f \) to be the product of all such primes \( p \) and \( N(\mathfrak{c}) \).
Thus, for any prime \( p \nmid M_f \) and for any prime ideal \( p \in \mathcal{P} \) over \( p \), we have either \( C(p, f) = 0 \) or \( C(p^r, f) \neq 0 \) for all \( r \geq 1 \).

We end this article with the following statement:

**Lemma 4.2.** Let \( f \) and \( K \) be as in Proposition 4.1. Further, if \( K \) is Galois over \( \mathbb{Q} \), then there exists an integer \( M_f \geq 1 \) with \( N(c) | M_f \) such that for any prime \( p \nmid M_f \) and for any prime ideal \( p \in \mathcal{P} \) over \( p \), we have either \( C(p, f) = 0 \) or \( C(p^r, f) \neq 0 \) for all \( r \geq 1 \).

We note that in a recent work of Bhand, Gun and Rath (cf. [1, Theorem 2]), they have computed the lower bounds of the Weil heights of \( C(p^r, f) \), when non-zero, for prime ideals \( p \) away from an ideal \( M \). In particular, the above lemma is a consequence of their Theorem.

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