Solvability graphs of finite groups

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Abstract

Let G be a finite non-solvable group with solvable radical Sol(G). The solvability graph Γs(G) of G is a graph with vertex set G \ Sol(G) and two distinct vertices u and v are adjacent if and only if ⟨u, v⟩ is solvable. We show that Γs(G) is not a star graph, a tree, an n-partite graph for any positive integer n ≥ 2 and not a regular graph for any non-solvable finite group G. We compute the girth of Γs(G) and derive a lower bound of the clique number of Γs(G). We prove the non-existence of finite non-solvable groups whose solvability graphs are planar, toroidal, double-toroidal, triple-toroidal or projective. We conclude the paper by obtaining a relation between Γs(G) and the solvability degree of G.

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1. Introduction

Let G be a finite group and u ∈ G. The solvabilizer of u, denoted by SolG(u), is the set given by {v ∈ G : ⟨u, v⟩ is solvable}. Note that the centralizer C_G(u) := {v ∈ G : uv = vu} is a subset of SolG(u) and hence the center Z(G) ⊆ SolG(u) for all u ∈ G. By [21, Proposition 2.13], |C_G(u)| divides |SolG(u)| for all u ∈ G though SolG(u) is not a subgroup of G in general. A group G is called a S-group if SolG(u) is a subgroup of G for all u ∈ G. A finite group G is a S-group if and only if it is solvable (see [21, Proposition 2.22]). Many other properties of SolG(u) can be found in [21]. We write Sol(G) = {u ∈ G : ⟨u, v⟩ is solvable for all v ∈ G}. It is easy to see that Sol(G) = u∈G SolG(u). Also, Sol(G) is the solvable radical of G (see [18]). The solvability graph of a finite non-solvable group G is a simple undirected graph whose vertex set is G \ Sol(G), and two vertices u and v are adjacent if ⟨u, v⟩ is a solvable. We write Γs(G) to denote this graph. It is worth mentioning that Γs(G) is the complement of the non-solvable graph of G considered in [4,21] and extension of commuting and nilpotent graphs of finite groups that are studied extensively in [1–3, 5, 6, 9–11, 13–16, 25, 26]. It is worth mentioning that the study of commuting graphs of finite groups is originated from a question posed by Erdös [23].

In this paper, we show that Γs(G) is not a star graph, a tree, an n-partite graph for any positive integer n ≥ 2 and not a regular graph for any non-solvable finite group G. In Section 2, we also show that the girth of Γs(G) is 3 and the clique number of Γs(G) is

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greater than or equal to 4. In Section 3, we first show that for a given non-negative integer $k$, there are at the most finitely many finite non-solvable groups whose soluble graph have genus $k$. We also show that there is no finite non-solvable group, whose soluble graph is planar, toroidal, double-toroidal, triple-toroidal or projective. We conclude the paper by obtaining a relation between $\Gamma_\text{s}(G)$ and $P_4(G)$ in Section 4, where $P_4(G)$ is the probability that a randomly chosen pair of elements of $G$ generate a soluble group (see [20]).

The reader may refer to [27] and [28] for various standard graph theoretic terminologies. For any subset $X$ of the vertex set of a graph $\Gamma$, we write $\Gamma[X]$ to denote the induced subgraph of $\Gamma$ on $X$. The girth of $\Gamma$ is the minimum of the lengths of all cycles in $\Gamma$, and is denoted by $\text{girth}(\Gamma)$. We write $\omega(\Gamma)$ to denote the clique number of $\Gamma$ which is the least upper bound of the sizes of all the cliques of $\Gamma$. The smallest non-negative integer $k$ is called the genus of a graph $\Gamma$ if $\Gamma$ can be embedded on the surface obtained by attaching $k$ handles to a sphere. Let $\gamma(\Gamma)$ be the genus of $\Gamma$. Then, it is clear that $\gamma(\Gamma) \geq \gamma(\Gamma_0)$ for any subgraph $\Gamma_0$ of $\Gamma$. Let $K_n$ be the complete graph on $n$ vertices and $mK_n$ the disjoint union of $m$ copies of $K_n$. It was proved in [7, Corollary 1] that $\gamma(\Gamma) \geq \gamma(K_m) + \gamma(K_n)$ if $\Gamma$ has two disjoint subgraphs isomorphic to $K_m$ and $K_n$. Also, by [28, Theorem 6-38] we have

$$\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil \text{ if } n \geq 3. \quad (1.1)$$

A graph $\Gamma$ is called planar, toroidal, double-toroidal and triple-toroidal if $\gamma(\Gamma) = 0, 1, 2$ and 3 respectively.

Let $N_k$ be the connected sum of $k$ projective planes. A simple graph which can be embedded in $N_k$ but not in $N_{k-1}$, is called a graph of crosscap $k$. The notation $\bar{\gamma}(\Gamma)$ stand for the crosscap of a graph $\Gamma$. It is easy to see that $\bar{\gamma}(\Gamma) \geq \gamma(\Gamma_0)$ for any subgraph $\Gamma_0$ of $\Gamma$. It was shown in [8] that

$$\bar{\gamma}(K_n) = \begin{cases} \lceil \frac{1}{6}(n-3)(n-4) \rceil & \text{if } n \geq 3 \text{ and } n \neq 7, \\ 3 & \text{if } n = 7. \end{cases} \quad (1.2)$$

A graph $\Gamma$ is called a projective graph if $\bar{\gamma}(\Gamma) = 1$. It is worth mentioning that $2K_5$ is not projective graph (see [17]).

2. Graph realization

We begin with the following lemma.

**Lemma 2.1.** For every $u \in G \setminus \text{Sol}(G)$ we have

$$\text{deg}(u) = |\text{Sol}_G(u)| - |\text{Sol}(G)| - 1.$$

**Proof.** Note that $\text{deg}(u)$ represents the number of vertices from $G \setminus \text{Sol}(G)$ which are adjacent to $u$. Since $u \in \text{Sol}_G(u)$, therefore $|\text{Sol}_G(u)| - 1$ represents the number of vertices which are adjacent to $u$. Since we are excluding $\text{Sol}(G)$ from the vertex set therefore $\text{deg}(u) = |\text{Sol}_G(u)| - |\text{Sol}(G)| - 1$. \hfill $\square$

**Proposition 2.2.** $\Gamma_\text{s}(G)$ is not a star.

**Proof.** Suppose for a contradiction $\Gamma_\text{s}(G)$ is a star. Let $|G| - |\text{Sol}(G)| = n$. Then there exists $u \in G \setminus \text{Sol}(G)$ such that $\text{deg}(u) = n - 1$. Therefore, by Lemma 2.1, $|\text{Sol}_G(u)| = |G|$. This gives $u \in \text{Sol}(G)$, a contradiction. Hence, the result follows. \hfill $\square$

**Proposition 2.3.** $\Gamma_\text{s}(G)$ is not complete bipartite.

**Proof.** Let $\Gamma_\text{s}(G)$ be complete bipartite. Suppose that $A_1$ and $A_2$ are parts of the bipartition. Then, by Proposition 2.2, $|A_1| \geq 2$ and $|A_2| \geq 2$. Let $u \in A_1, v \in A_2$. If $|\{u, v\} \setminus \text{Sol}(G)| \geq 2$, then there exists $y \in \{u, v\} \setminus \text{Sol}(G)$ such that $y \not\in A_1$ and $y \not\in A_2$. But then $y \notin A_1$ and $y \notin A_2$, a contradiction.
It follows that \(|\langle u, v \rangle \text{Sol}(G) \setminus \text{Sol}(G)| = 2\). In particular, \(\text{Sol}(G) = 1\) and \(\langle u, v \rangle\) is cyclic of order 3 or \(|\text{Sol}(G)| = 2\) and \(v = uz\) for \(z\) an involution in \(\text{Sol}(G)\). Now the neighbours of \(u \in A_1\) is just \(u^2 \in A_2\) or \(uz\) in the respective cases. Hence \(|A_2| = |A_1| = 1\), a contradiction. Hence, the result follows. \(\square\)

Following similar arguments as in the proof of Proposition 2.3 we get the following result.

**Proposition 2.4.** \(\Gamma_s(G)\) is not complete \(n\)-partite.

**Proposition 2.5.** For any finite non-solvable group \(G\), \(\Gamma_s(G)\) has no isolated vertex.

**Proof.** Suppose \(x\) is an isolated vertex of \(\Gamma_s(G)\). Then \(|\text{Sol}(G)| = 1\); otherwise \(x\) is adjacent to \(xz\) for any \(z \in \text{Sol}(G) \setminus \{1\}\). Thus it follows that \(o(x) = 2\); otherwise \(x\) is adjacent to \(x^2\). Let \(y \in G\). Then \(\langle x, x^y \rangle\) is dihedral and so \(x = x^9\) as \(x\) is isolated. Hence \(x \in Z(G)\) and so \(x \in Z(G) \leq \text{Sol}(G)\), a contradiction. Hence, \(\Gamma_s(G)\) has no isolated vertex. \(\square\)

The following lemma is useful in proving the next two results as well as some results in subsequent sections.

**Lemma 2.6.** Let \(G\) be a finite non-solvable group. Then there exist \(x \in G\) such that \(x, x^2 \notin \text{Sol}(G)\).

**Proof.** Suppose that for all \(x \in G\), we have \(x^2 \in \text{Sol}(G)\). Therefore, \(G/\text{Sol}(G)\) is elementary abelian and hence solvable. Also, \(\text{Sol}(G)\) is solvable. It follows that \(G\) is solvable, a contradiction. Hence, the result follows. \(\square\)

**Theorem 2.7.** Let \(G\) be a finite non-solvable group. Then \(\text{girth}(\Gamma_s(G)) = 3\).

**Proof.** Suppose for a contradiction that \(\Gamma_s(G)\) has no 3-cycle. Let \(x \in G\) such that \(x, x^2 \notin \text{Sol}(G)\) (by Lemma 2.6). Suppose \(|\text{Sol}(G)| \geq 2\). Let \(z \in \text{Sol}(G), z \neq 1\), then \(x, x^2\) and \(xz\) form a 3-cycle, which is a contradiction. Thus \(|\text{Sol}(G)| = 1\). In this case, every element of \(G\) has order 2 or 3; otherwise, \(\{x, x^2, x^3\}\) forms a 3-cycle in \(\Gamma_s(G)\) for all \(x \in G\) with \(o(x) > 3\). Therefore, \(|G| = 2^m3^n\) for some non-negative integers \(m\) and \(n\). By Burnside’s Theorem, it follows that \(G\) is solvable; a contradiction. Hence, \(\text{girth}(\Gamma_s(G)) = 3\). \(\square\)

**Theorem 2.8.** Let \(G\) be a finite non-solvable group. Then \(\omega(\Gamma_s(G)) \geq 4\).

**Proof.** Suppose for a contradiction that \(G\) is a finite non-solvable group with \(\omega(\Gamma_s(G)) \leq 3\). Let \(x \in G \setminus \text{Sol}(G)\) such that \(x^2 \notin \text{Sol}(G)\) according to Lemma 2.6. Suppose \(|\text{Sol}(G)| \geq 2\). Let \(z \in \text{Sol}(G), z \neq 1\), then \(\{x, x^2, xz, x^2z\}\) is a clique which is a contradiction. Thus \(|\text{Sol}(G)| = 1\). In this case every element of \(G \setminus \text{Sol}(G)\) has order 2, 3 or 4 otherwise \(\{x, x^2, x^3, x^4\}\) is a clique with \(o(x) > 4\), which is a contradiction. Therefore \(|G| = 2^m3^n\) where \(m, n\) are non-negative integers. Again, by Burnside’s Theorem, it follows that \(G\) is solvable; a contradiction. This completes the proof. \(\square\)

As a consequence of Theorem 2.7 and Theorem 2.8 we have the following corollary.

**Corollary 2.9.** The solvable graph of a finite non-solvable group is not a tree.

We conclude this section with the following result.

**Proposition 2.10.** \(\Gamma_s(G)\) is not regular.

**Proof.** Follows from [21, Corollary 3.17], noting the fact that a graph is regular if and only if its complement is regular. \(\square\)
3. Genus and diameter

We begin this section with the following useful lemma.

**Lemma 3.1.** Let $G$ be a finite group and $H$ a solvable subgroup of $G$. Then $\langle H, \text{Sol}(G) \rangle$ is a solvable subgroup of $G$.

**Proposition 3.2.** Let $G$ be a finite non-solvable group such that $\gamma(\Gamma_s(G)) = m$.

(a) If $S$ is a nonempty subset of $G \setminus \text{Sol}(G)$ such that $\langle x, y \rangle$ is solvable for all $x, y \in S$, then $|S| \leq \left\lfloor \frac{7 + \sqrt{1 + 48m}}{2} \right\rfloor$.

(b) $|\text{Sol}(G)| \leq \frac{1}{1-t} \left( \frac{7 + \sqrt{1 + 48m}}{2} \right)^t$, where $t = \max\{o(x\text{Sol}(G)) \mid x \text{Sol}(G) \in G/\text{Sol}(G)\}$.

(c) If $H$ is a solvable subgroup of $G$, then $|H| \leq \left( \frac{7 + \sqrt{1 + 48m}}{2} \right)^t |H \cap \text{Sol}(G)|$.

**Proof.** We have $\Gamma_s(G)[S] \cong K_{|S|}$ and $\gamma(K_{|S|}) = \gamma(\Gamma_s(G)[S]) \leq \gamma(\Gamma_s(G))$. Therefore, if $m = 0$ then $\gamma(K_{|S|}) = 0$. This gives $|S| \leq 4$, otherwise $K_{|S|}$ will have a subgraph $K_5$ having genus 1. If $m > 0$ then, by Heawood’s formula [27, Theorem 6.3.25], we have

$$|S| = \omega(\Gamma_s(G)[S]) \leq \omega(\Gamma_s(G)) \leq \chi(\Gamma_s(G)) \leq \left\lfloor \frac{7 + \sqrt{1 + 48m}}{2} \right\rfloor$$

where $\chi(\Gamma_s(G))$ is the chromatic number of $\Gamma_s(G)$. Hence part (a) follows.

Part (b) follows from Lemma 3.1 and part (a) considering $S = \sum_{i=1}^{t-1} y^i \text{Sol}(G)$, where $y \in G \setminus \text{Sol}(G)$ such that $o(y\text{Sol}(G)) = t$.

Part (c) follows from part (a) noting that $H = (H \setminus \text{Sol}(G)) \cup (H \cap \text{Sol}(G))$. □

**Theorem 3.3.** Let $G$ be a finite non-solvable group. Then $|G|$ is bounded above by a function of $\gamma(\Gamma_s(G))$.

**Proof.** Let $\gamma(\Gamma_s(G)) = m$ and $h_m = \left\lfloor \frac{7 + \sqrt{1 + 48m}}{2} \right\rfloor$. By Lemma 3.1, we have $\Gamma_s(G)[x \text{Sol}(G)] \cong K_{|\text{Sol}(G)|}$, where $x \in G \setminus \text{Sol}(G)$. Therefore by Proposition 3.2(a), $|\text{Sol}(G)| \leq h_m$.

Let $P$ be a Sylow $p$-subgroup of $G$ for any prime $p$ dividing $|G|$ having order $p^n$ for some positive integer $n$. Then $P$ is a solvable. Therefore, by Proposition 3.2(c), we have $|P| \leq h_m + |\text{Sol}(G)| \leq 2h_m$. Hence, $|G| < (2h_m)^h_m$ noting that the number of primes less than $2h_m$ is at most $h_m$. This completes the proof. □

As an immediate consequence of Theorem 3.3 we have the following corollary.

**Corollary 3.4.** Let $n$ be a non-negative integer. Then there are at the most finitely many finite non-solvable groups $G$ such that $\gamma(\Gamma_s(G)) = n$.

The following two lemmas are essential in proving the main results of this section.

**Lemma 3.5.** [24, Lemma 3.4] Let $G$ be a finite group.

(a) If $|G| = 7m$ and the Sylow 7-subgroup is normal in $G$, then $G$ has an abelian subgroup of order at least 14 or $|G| \leq 42$.

(b) If $|G| = 9m$, where 3 $\nmid m$ and the Sylow 3-subgroup is normal in $G$, then $G$ has an abelian subgroup of order at least 18 or $|G| \leq 72$.

**Lemma 3.6.** If $G$ is a non-solvable group of order not exceeding 120 then $\Gamma_s(G)$ has a subgraph isomorphic to $K_{11}$ and $\gamma(\Gamma_s(G)) \geq 5$.

**Proof.** If $G$ is a non-solvable group and $|G| \leq 120$ then $G$ is isomorphic to $A_5$, $A_5 \times \mathbb{Z}_2$, $S_5$ or $SL(2,5)$. Note that $|\text{Sol}(A_5)| = |\text{Sol}(S_5)| = 1$ and $|\text{Sol}(A_5 \times \mathbb{Z}_2)| = |\text{Sol}(SL(2,5))| = 2$. Also, $A_5$ has a solvable subgroup of order 12 and $S_5$, $A_5 \times \mathbb{Z}_2$, $SL(2,5)$ have solvable subgroups of order 24. It follows that $\Gamma_s(G)$ has a subgraph isomorphic to $K_{11}$. Therefore, by (1.1), $\gamma(\Gamma_s(G)) \geq \gamma(K_{11}) = 5$. □
The solvable graph of a finite non-solvable group is neither planar, toroidal, double-toroidal nor triple-toroidal.

**Proof.** Let $G$ be a finite non-solvable group. Note that it is enough to show $\gamma(\Gamma_s(G)) \geq 4$ to complete the proof. Suppose that $\gamma(\Gamma_s(G)) \leq 3$. Let $x \in G \setminus \text{Sol}(G)$ such that $x^2 \not\in \text{Sol}(G)$. Such element exists by Lemma 2.6. Since any two elements of the set $A = x \text{Sol}(G) \cup x^2 \text{Sol}(G)$ generate a solvable group, by Proposition 3.2(a), we have $|\text{Sol}(G)| = |A| \leq \frac{7 + \sqrt{1 + 4 \cdot 3 \cdot 53}}{2} = 9$. Thus $|\text{Sol}(G)| \leq 4$. Let $p$ be a prime divisor of $|G|$ and $P$ is a Sylow $p$-subgroup of $G$. Since $P$ is solvable, by Proposition 3.2(e), we get $|P| \leq 9 + |P \cap \text{Sol}(G)| \leq 13$. If $|P| = 11$ or $13$ then $|P \cap \text{Sol}(G)| = 1$. Therefore, $\Gamma_s(G)[P \setminus \text{Sol}(G)] \cong K_{10}$ or $K_{12}$. Using (1.1), we get $\gamma(\Gamma_s(G)[P \setminus \text{Sol}(G)]) = 4$ or $6$. Therefore, $\gamma(\Gamma_s(G)) \geq \gamma(\Gamma_s(G)[P \setminus \text{Sol}(G)]) \geq 4$, a contradiction. Thus $|P| \leq 9$ and hence $p \leq 7$. This shows that $|G|$ divides $2^5 \cdot 3^2 \cdot 5 \cdot 7$.

We consider the following cases.

**Case 1.** $|\text{Sol}(G)| = 4$.

If $H$ is a Sylow $p$-subgroup of $G$ where $p = 5$ or $7$ then $\langle H, \text{Sol}(G) \rangle$ is solvable since $H$ is solvable (by Lemma 3.1). We have $|H \cap \text{Sol}(G)| = 1$ and $|\langle H, \text{Sol}(G) \rangle| = 20, 28$ according as $p = 5, 7$ respectively. Therefore $\Gamma_s(G)[\langle H, \text{Sol}(G) \rangle \setminus \text{Sol}(G)] \cong K_{16}$ or $K_{24}$. By (1.1) we get $\gamma(\Gamma_s(G)) \geq \gamma(\Gamma_s(G)[\langle H, \text{Sol}(G) \rangle \setminus \text{Sol}(G)]) \geq 13$, which is a contradiction.

Thus $|G|$ is a divisor of $72$. Therefore, by Lemma 3.6 we have $\gamma(\Gamma_s(G)) \geq 5$, a contradiction.

**Case 2.** $|\text{Sol}(G)| = 3$.

If $H$ is a Sylow $p$-subgroup of $G$ where $p = 5$ or $7$ then $\langle H, \text{Sol}(G) \rangle$ is solvable. We have $|H \cap \text{Sol}(G)| = 1$ and $|\langle H, \text{Sol}(G) \rangle| = 15, 21$ according as $p = 5, 7$ respectively. Therefore $\Gamma_s(G)[\langle H, \text{Sol}(G) \rangle \setminus \text{Sol}(G)] \cong K_{12}$ or $K_{18}$. By (1.1) we get $\gamma(\Gamma_s(G)) \geq \gamma(\Gamma_s(G)[\langle H, \text{Sol}(G) \rangle \setminus \text{Sol}(G)]) \geq 6$, which is a contradiction.

Thus $|G|$ is a divisor of $72$. Therefore, by Lemma 3.6 we have $\gamma(\Gamma_s(G)) \geq 5$, a contradiction.

**Case 3.** $|\text{Sol}(G)| = 2$.

If $H$ is a Sylow $7$-subgroup of $G$ then $\langle H, \text{Sol}(G) \rangle$ is solvable. We have $|H \cap \text{Sol}(G)| = 1$ and $|\langle H, \text{Sol}(G) \rangle| = 14$. So, $\Gamma_s(G)[\langle H, \text{Sol}(G) \rangle \setminus \text{Sol}(G)] \cong K_{12}$. By (1.1) we get $\gamma(\Gamma_s(G)) \geq \gamma(\Gamma_s(G)[\langle H, \text{Sol}(G) \rangle \setminus \text{Sol}(G)]) \geq 6$, which is a contradiction. Let $K$ be a Sylow $3$-subgroup of $G$. If $|K| = 9$ then $\langle K, \text{Sol}(G) \rangle$ is solvable since $K$ is solvable (by Lemma 3.1). We have $|K \cap \text{Sol}(G)| = 1$ and $|\langle K, \text{Sol}(G) \rangle| = 18$. So, $\Gamma_s(G)[\langle K, \text{Sol}(G) \rangle \setminus \text{Sol}(G)] \cong K_{16}$. By (1.1) we get $\gamma(\Gamma_s(G)) \geq \gamma(\Gamma_s(G)[\langle K, \text{Sol}(G) \rangle \setminus \text{Sol}(G)]) \geq 13$, which is a contradiction.

Thus $|G|$ is a divisor of $120$. Therefore, by Lemma 3.6 we have $\gamma(\Gamma_s(G)) \geq 5$, a contradiction.

**Case 4.** $|\text{Sol}(G)| = 1$.

In this case, first we shall show that $7 \nmid |G|$. On the contrary, assume that $7 \mid |G|$. Let $n$ be the number of Sylow $7$-subgroups of $G$. Then $n \mid 2^3 \cdot 3^2$. and $n \equiv 1 \mod 7$. If $n \neq 1$ then $n \geq 8$. Let $H_1, \ldots, H_8$ be the eight distinct Sylow $7$-subgroups of $G$. Then the induced subgraphs $\Gamma_s(G)[H_i \setminus \text{Sol}(G)]$ for each $1 \leq i \leq 8$ contribute $\gamma(\Gamma_s(G)[H_i \setminus \text{Sol}(G)]) = 1$ to the genus of $\Gamma_s(G)$. Thus

$$\gamma(\Gamma_s(G)) \geq \sum_{i=1}^{8} \gamma(\Gamma_s(G)[H_i \setminus \text{Sol}(G)]) = 8,$$

a contradiction. Therefore, Sylow $7$-subgroup of $G$ is unique and hence normal. Since we have started with a non-solvable group, by Lemma 3.5, it follows that $G$ has an abelian subgroup of order at least $14$. Therefore, by (1.1) we have $\gamma(\Gamma_s(G)) \geq \gamma(K_{13}) = 8$, a contradiction. Hence, $|G|$ is a divisor of $2^3 \cdot 3^2 \cdot 5$.

Now, we shall show that $9 \nmid |G|$. Assume that, on the contrary, $9 \mid |G|$. If Sylow $3$-subgroup of $G$ is not normal in $G$, then the number of Sylow $3$-subgroups is greater than
or equal to 4. Let $H_1, H_2, H_3$ be the three Sylow 3-subgroups of $G$. Then the induced subgraph $\Gamma_S(G)[H_1 \setminus \text{Sol}(G)] \cong K_8$ and so it contributes $\gamma(\Gamma_S(G)[H_1 \setminus \text{Sol}(G)]) = 2$ to the genus of $\Gamma_S(G)$. If $|H_1 \cap H_2| = 1$, then the induced subgraph $\Gamma_S(G)[H_2 \setminus \text{Sol}(G)] \cong K_8$ and so it contributes +2 to the genus $\Gamma_S(G)$. Thus

$$\gamma(\Gamma_S(G)) \geq \gamma(\Gamma_S(G)[(H_1 \cup H_2) \setminus \text{Sol}(G)]) = 4$$

which is a contradiction. So assume that $|H_1 \cap H_2| = 3$. Similarly $|H_1 \cap H_3| = 3$ and $|H_2 \cap H_3| = 3$. Let $M = H_2 \setminus H_1$. Then $|M| = 6$. Also note that if $L = H_1 \cup H_2$ and $K = H_3 \setminus L$, then $|K| \geq 4$. Also $H_1 \cap M = H_1 \cap K = M \cap K = 0$.

If $|K| \geq 5$ then $H_1$ contribute +2 to genus of $\Gamma_S(G)$, $M$ and $K$ each contribute +1 to genus of $\Gamma_S(G)$. Hence genus of $\Gamma_S(G)$ is greater than or equal to 4, a contradiction.

Assume that $|K| = 4$. In this case $|M \cap H_3| = 2$. Let $x \in M \cap H_3$. Then $H_1$ contribute +2 to genus of $\Gamma_S(G)$, $M \setminus \{x\}$ and $K \cup \{x\}$ each contribute +1 to genus of $\Gamma_S(G)$. Hence genus of $\Gamma_S(G)$ is greater than or equal to 4, a contradiction.

These show that Sylow 3-subgroup of $G$ is unique and hence normal in $G$. Therefore, by Lemma 3.5 and Lemma 3.6, $G$ has an abelian subgroup $A$ of order at least 18. Hence,

$$\gamma(\Gamma_S(G)) \geq \gamma(\Gamma_S(G)[A \setminus \text{Sol}(G)]) \geq \gamma(K_17) = 16$$

which is a contradiction. The above theorem gives that $\gamma(\Gamma_S(G)) \geq 4$. Usually, genera of solvable graphs of finite non-solvable groups are very large. For example, if $G$ is the smallest non-solvable group $A_5$ then $\Gamma_S(G)$ has 59 vertices and 571 edges. Also $\gamma(\Gamma_S(G)) \geq 571/6 - 59/2 + 1 = 68$ (follows from [28, Corollary 6–14]). The following theorem shows that the crosscap number of the solvable graph of a finite non-solvable group is greater than 1.

**Proposition 3.8.** The solvable graph of a finite non-solvable group is not projective.

**Proof.** Suppose $G$ is a finite non-solvable group whose solvable graph is projective. Note that if $\Gamma_S(G)$ has a subgraph isomorphic to $K_n$ then, by (1.2), we must have $n \leq 6$. Let $x \in G$, such that $x, x^2 \notin \text{Sol}(G)$. Then

$$\Gamma_S(G)[x \text{Sol}(G) \cup x^2 \text{Sol}(G)] \cong K_{2|\text{Sol}(G)|}.$$ 

Therefore, $2|\text{Sol}(G)| \leq 6$ and hence $|\text{Sol}(G)| \leq 3$.

Let $p | |G|$ be a prime and $P$ be a Sylow $p$-subgroup of $G$. Then $\Gamma_S(G)[P \setminus \text{Sol}(G)] \cong K_{|P|/|\text{Sol}(G)|}$ since $P$ is solvable. Therefore, $|P \setminus \text{Sol}(G)| = |P| - |P \cap \text{Sol}(G)| \leq 6$ and hence $|P| \leq 9$. This shows that $|G|$ is a divisor of $2^3 \cdot 3^3 \cdot 5 \cdot 7$.

If $7 | |G|$ then Sylow 7-subgroup of $G$ is unique and hence normal in $G$; otherwise, let $H$ and $K$ be two Sylow 7-subgroups of $G$. Then $|H \cap K| = |H \cap \text{Sol}(G)| = |K \cap \text{Sol}(G)| = 1$. Therefore, $\Gamma_S(G)[(H \cup K) \setminus \text{Sol}(G)]$ has a subgraph isomorphic to $2K_5$. Hence, $\Gamma_S(G)$ has a subgraph isomorphic to $2K_5$, which is a contradiction. Similarly, if $9 | |G|$, then the Sylow 3-subgroup of $G$ is normal in $G$. Therefore, by Lemma 3.5, it follows that $|G| \leq 72$ or $|G|$ is a divisor of $2^3 \cdot 3^3 \cdot 5 \cdot 7$. In the both cases, by Lemma 3.6, $\Gamma_S(G)$ has complete subgraphs isomorphic to $K_{11}$, which is a contradiction. This completes the proof.  

We conclude this section, by an observation and a couple of problems regarding the diameter and connectedness of $\Gamma_S(G)$. Using the following programme in GAP [29], we see that the solvable graph of the groups $A_5, S_5, A_5 \times \mathbb{Z}_2, SL(2, 5), PSL(3, 2)$ and $GL(2, 4)$ are connected with diameter 2. The solvable graphs of $S_6$ and $A_6$ are connected with diameters greater than 2.
g:=PSL(3,2);
sol:=RadicalGroup(g);
L:=[];
gsol:=Difference(g,sol);
for x in gsol do
  AddSet(L,[x]);
  for y in Difference(gsol,L) do
    if IsSolvable(Subgroup(g,[x,y]))=true then
      break;
      fi;
    i:=0;
    for z in gsol do
      if IsSolvable(Subgroup(g,[x,z]))=true and
        IsSolvable(Subgroup(g,[z,y]))=true
      then
        i:=1;
        break;
      fi;
    od;
    if i=0 then
      Print("Diameter>2");
      Print(x," ",y);
    fi;
  od;
od;

In this connection, we have the following problems.

**Problem 3.1.** Is $\Gamma_s(G)$ connected for any finite non-solvable group $G$?

**Problem 3.2.** Is there any finite bound for the diameter of $\Gamma_s(G)$ when $\Gamma_s(G)$ is connected?

4. Relations with solvability degree

The solvability degree of a finite group $G$ is defined by the following ratio

$$P_s(G) := \frac{|\{(u, v) \in G \times G : \langle u, v \rangle \text{ is solvable}\}|}{|G|^2}.$$ 

Using the solvability criterion (see [12, Section 1]),

"A finite group is solvable if and only if every pair of its elements generates a solvable group"

for finite groups we have $G$ is solvable if and only if its solvability degree is 1. It was shown in [20, Theorem A] that $P_s(G) \leq \frac{11}{19}$ for any finite non-solvable group $G$. In this section, we study a few properties of $P_s(G)$ and derive a connection between $P_s(G)$ and $\Gamma_s(G)$ for finite non-solvable groups $G$. We begin with the following lemma.

**Lemma 4.1.** Let $G$ be a finite group. Then $P_s(G) = \frac{1}{|G|^2} \sum_{u \in G} |\text{Sol}_G(u)|$.

**Proof.** Let $S = \{(u, v) \in G \times G : \langle u, v \rangle \text{ is solvable}\}$. Then

$$S = \bigcup_{u \in G} \left( \{u\} \times \{v \in G : \langle u, v \rangle \text{ is solvable}\} \right) = \bigcup_{u \in G} \{u\} \times \text{Sol}_G(u).$$

Therefore, $|S| = \sum_{u \in G} |\text{Sol}_G(u)|$. Hence, the result follows. \qed
**Corollary 4.2.** $|G|P_s(G)$ is an integer for any finite group $G$.

**Proof.** By Proposition 2.16 of [21] we have that $|G|$ divides $\sum_{u \in G} |\text{Sol}_G(u)|$. Hence, the result follows from Lemma 4.1. \qed

We have the following lower bound for $P_s(G)$.

**Theorem 4.3.** For any finite group $G$,

$$P_s(G) \geq \frac{|\text{Sol}(G)|}{|G|} + \frac{2(|G| - |\text{Sol}(G)|)}{|G|^2}.$$  

**Proof.** By Lemma 4.1, we have

$$|G|^2P_s(G) = \sum_{u \in \text{Sol}(G)} |\text{Sol}_G(u)| + \sum_{u \in G \setminus \text{Sol}(G)} |\text{Sol}_G(u)|$$

$$= |G||\text{Sol}(G)| + \sum_{u \in G \setminus \text{Sol}(G)} |\text{Sol}_G(u)|.$$  

By Proposition 2.13 of [21], $|C_G(u)|$ is a divisor of $|\text{Sol}_G(u)|$ for all $u \in G$ where $C_G(u) = \{v \in G : uv = vu\}$, the centralizer of $u \in G$. Since $|C_G(u)| \geq 2$ for all $u \in G$ we have $|\text{Sol}_G(u)| \geq 2$ for all $u \in G$. Therefore

$$\sum_{u \in G \setminus \text{Sol}(G)} |\text{Sol}_G(u)| \geq 2(|G| - |\text{Sol}(G)|).$$

Hence, the result follows from (4.1). \qed

The following theorem shows that $P_s(G) > \text{Pr}(G)$ for any finite non-solvable group where $\text{Pr}(G)$ is the commuting probability of $G$ (see [19]).

**Theorem 4.4.** Let $G$ be a finite group. Then $P_s(G) \geq \text{Pr}(G)$ with equality if and only if $G$ is a solvable group.

**Proof.** The result follows from Lemma 4.1 and the fact that $\text{Pr}(G) = \frac{1}{|G|^2} \sum_{u \in G} |C_G(u)|$ noting that $C_G(u) \subseteq \text{Sol}_G(u)$ and so $|\text{Sol}_G(u)| \geq |C_G(u)|$ for all $u \in G$.

The equality holds if and only if $C_G(u) = \text{Sol}_G(u)$ for all $u \in G$, that is $\text{Sol}_G(u)$ is a subgroup of $G$ for all $u \in G$. Hence, by Proposition 2.22 of [21], the equality holds if and only if $G$ is solvable. \qed

Let $|E(\Gamma_s(G))|$ be the number of edges of the non-solvable graph $\Gamma_s(G)$ of $G$. The following theorem gives a relation between $P_s(G)$ and $|E(\Gamma_s(G))|$.

**Theorem 4.5.** Let $G$ be a finite non-solvable group. Then

$$2|E(\Gamma_s(G))| = |G|^2P_s(G) + |\text{Sol}(G)|^2 + |\text{Sol}(G)| - |G|(2|\text{Sol}(G)| + 1).$$

**Proof.** We have

$$2|E(\Gamma_s(G))| = |\{(x, y) \in (G \setminus \text{Sol}(G)) \times (G \setminus \text{Sol}(G)) : \langle x, y \rangle \text{ is solvable}\}| - |G| + |\text{Sol}(G)|.$$  

Also

$$S = \{(x, y) \in G \times G : \langle x, y \rangle \text{ is solvable}\} = \text{Sol}(G) \times \text{Sol}(G) \cup \text{Sol}(G) \times (G \setminus \text{Sol}(G)) \cup (G \setminus \text{Sol}(G)) \times \text{Sol}(G) \cup \{(x, y) \in (G \setminus \text{Sol}(G)) \times (G \setminus \text{Sol}(G)) : \langle x, y \rangle \text{ is solvable}\}.$$  

Therefore

$$|S| = |\text{Sol}(G)|^2 + 2|\text{Sol}(G)||(|G| - |\text{Sol}(G)|) + 2|E(\Gamma_s(G))| + |G| - |\text{Sol}(G)|$$

$$\implies |G|^2P_s(G) = |G|(2|\text{Sol}(G)| + 1) - |\text{Sol}(G)|^2 - |\text{Sol}(G)| + 2|E(\Gamma_s(G))|.$$  

Hence, the result follows. \qed
We conclude this paper noting that lower bounds for $|E(\Gamma_s(G))|$ can be obtained from Theorem 4.5 using the lower bounds given in Theorem 4.3, Theorem 4.4 and the lower bounds for $\Pr(G)$ obtained in [22].

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