Research Article

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Positive solutions for parametric 

\((p(z), q(z))\)-equations

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Abstract: We consider a parametric elliptic equation driven by the anisotropic \((p, q)\)-Laplacian. The reaction is superlinear. We prove a “bifurcation-type” theorem describing the change in the set of positive solutions as the parameter \(\lambda\) moves in \((0, +\infty)\).

Keywords: anisotropic regularity, anisotropic maximum principle, positive solutions, minimal positive solution, superlinear reaction

MSC 2020: 35J20, 35J70

1 Introduction

Let \(\Omega \subset \mathbb{R}^N\) be a bounded domain with a \(C^2\)-boundary \(\partial \Omega\). We study the following parametric anisotropic \((p, q)\)-equation:

\[
\begin{cases}
- \Delta_{p(z)} u(z) - \Delta_{q(z)} u(z) = \lambda f(z, u(z)) & \text{in } \Omega, \\
u_{\partial \Omega} = 0, u > 0, \lambda > 0.
\end{cases}
\]

\((P_\lambda)\)

In this problem, the exponents \(p\) and \(q\) are Lipschitz continuous on \(\overline{\Omega}\), that is, \(p, q \in C^0(\overline{\Omega})\) and 

\[1 < q = \min q \leq q = \max q < p = \min p \leq p = \max p.
\]

By \(\Delta_{p(z)}\) (respectively \(\Delta_{q(z)}\)) we denote the \(p(z)\)-Laplacian (respectively the \(q(z)\)-Laplacian) defined by 

\[
\Delta_{p(z)} u = \text{div}(|Du|^{p(z)-2}Du) \quad \forall u \in W^{1,p(z)}_0(\Omega)
\]

(respectively \(\Delta_{q(z)} u = \text{div}(|Du|^{q(z)-2}Du) \quad \forall u \in W^{1,q(z)}_0(\Omega)\)).

In the reaction (right hand side of \((P_\lambda)\)), \(f(z, x)\) is a Carathéodory function (that is, for all \(x \in \mathbb{R}, z \mapsto f(z, x)\) is measurable and for almost all \(z \in \Omega, x \mapsto f(z, x)\) is continuous), which is \((p, -1)\)-superlinear in the \(x\)-variable, but need not satisfy the Ambrosetti-Rabinowitz condition which is common in problems with superlinear reactions. Also, \(\lambda > 0\) is a parameter. We are looking for positive solutions of \((P_\lambda)\). More precisely, our aim is to determine the precise dependence on the parameter \(\lambda > 0\) of the set of positive solutions. We prove a bifurcation-type result, which establishes the existence of a critical parameter value \(\lambda^* > 0\) such that

- for all \(\lambda \in (0, \lambda^*)\) problem \((P_\lambda)\) has at least two positive solutions;
- for \(\lambda = \lambda^*\) problem \((P_{\lambda^*})\) has at least one positive solution;
- for all \(\lambda > \lambda^*\) there are no positive solutions for problem \((P_{\lambda})\).

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Our work here extends those of Gasiński-Papageorgiou [1,2], who studied parametric equations driven by the isotropic $p$-Laplacian with a $(p - 1)$-superlinear reaction. Nonlinear, nonparametric superlinear equations were also considered by Mugnai-Papageorgiou [3], Papageorgiou-Rădulescu [4], Papageorgiou-Scapellato [5] (isotropic problems) and Gasiński-Papageorgiou [6], Papageorgiou-Rădulescu-Repovš [7], Papageorgiou-Vetro [8] (anisotropic problems). They prove multiplicity results, producing also nodal (that is, sign changing) solutions. Also, we mention the relevant studies of Bahrouni-Rădulescu-Repovš [9] (existence of infinitely many solutions for anisotropic Dirichlet problems), Papageorgiou-Vetro-Vetro [10] (produce a continuous part of the spectrum for the Robin $(p, q)$-Laplacian), Vetro [11] (dealing with the asymptotic properties of the solutions of nonhomogeneous parametric isotropic equations), Vetro [12] (existence of a solution of an anisotropic Dirichlet problem), Vetro-Vetro [13] (a three-solution theorem for $(p, q)$-equations) and Vetro [14] (an infinity of solutions for isotropic $(p, q)$-equations).

Equations with variable exponents arise in many physical models. We refer to the book of Růžička [15] for such meaningful examples. The analysis of such problems requires the use of Lebesgue and Sobolev spaces with variable exponents. A comprehensive presentation of such spaces can be found in the book of Diening-Harjulehto-Hästö-Růžička [16] (see also the survey paper of Harjulehto-Hästö-Lê-Nuortio [17]). Various parametric boundary value problems with variable exponents can be found in the book of Rădulescu-Repovš [18]. Finally, we mention that we encounter $(p, q)$-equations (both isotropic and anisotropic), in many problems of mathematical physics. We refer to the studies of Bahrouni-Rădulescu-Repovš [19] (transonic flow problems), Benci-D’Avenia-Fortunato-Pisani [20] (quantum physics), Cherfils-Ilyasov [21] (reaction-diffusion systems) and Zhikov [22] (elasticity theory). We also mention the two informative survey papers by Marano-Mosconi [23] (isotropic problems) and Rădulescu [24] (isotropic and anisotropic problems).

## 2 Mathematical background – hypotheses

Let $M(\Omega)$ be the space of measurable functions $u : \Omega \to \mathbb{R}$. We identify two such functions that differ only on a set of zero Lebesgue measures. Also, let

$$E_i = \{r \in C(\overline{\Omega}) : 1 < r = \min r \} .$$

In the sequel given $r \in C(\overline{\Omega})$, we define

$$r. = \min_{\overline{\Omega}} q \quad \text{and} \quad r. = \max_{\overline{\Omega}} q .$$

Given $r \in E_i$, the variable exponent Lebesgue space $L^{r}(\Omega)$ is defined as follows:

$$L^{r}(\Omega) = \left\{ u \in M(\Omega) : \int_{\Omega} |u|^{r(z)} \, dz < +\infty \right\} .$$

This space is equipped with the so-called “Luxemburg norm” defined by

$$||u||_{r(z)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \left( \frac{|u|}{\lambda} \right)^{r(z)} \, dz \leq 1 \right\} .$$

Furnished with this norm, the space $L^{r}(\Omega)$ becomes a separable, reflexive (in fact, uniformly convex) Banach space. Let $r' \in E_i$ be defined by $\frac{1}{r(z)} + \frac{1}{r'(z)} = 1$. We know that $L^{r}(\Omega)^* = L^{r'}(\Omega)$ and we have the following Hölder-type inequality:

$$\int_{\Omega} |uh| \, dz \leq \left( \frac{1}{L} + \frac{1}{r'} \right) ||u||_{r(z)} ||h||_{r'(z)} \quad \forall u \in L^{r(z)}, h \in L^{r'(z)}(\Omega) .$$
If \( r_1, r_2 \in E_1 \) and \( r_1(z) < r_2(z) \) for all \( z \in \Omega \), then the embedding \( L^{r_2}(\Omega) \subseteq L^{r_2}(\Omega) \) is continuous.

Using the variable exponent Lebesgue spaces, we can define in the usual way the variable exponent Sobolev spaces. So, if \( r \in E_1 \), then we define

\[
W^{1,r(z)}(\Omega) = \{ u \in L^{r(z)}(\Omega) : |Du| \in L^{r(z)}(\Omega) \}
\]

(where the gradient \( Du \) is understood in the weak sense). This space is equipped with the following norm:

\[
\|u\|_{r(z)} = \|u\|_{r(z)} + \|Du\|_{r(z)}.
\]

In the sequel for notational simplicity, we write \( \|Du\|_{r(z)} = \|Du\|_{r(z)} \). Suppose that \( r \in E_1 \) is Lipschitz continuous (that is, \( r \in E_1 \cap C^{0,1}(\Omega) \)). Then we define

\[
W_0^{1,r(z)}(\Omega) = C_0^\infty(\Omega)^{1,r(z)}.
\]

Both \( W^{1,r(z)}(\Omega) \) and \( W_0^{1,r(z)}(\Omega) \) are separable, reflexive (in fact uniformly convex) Banach spaces.

For the space \( W_0^{1,r(z)}(\Omega) \), the Poincaré inequality holds, namely

\[
\|u\|_{1,r(z)} \leq \|Du\|_{r(z)} \quad \forall u \in W_0^{1,r(z)}(\Omega),
\]

for some \( c_0 > 0 \). So, on \( W_0^{1,r(z)}(\Omega) \) (recall that \( r \in E_1 \cap C^{0,1}(\Omega) \)), we can consider the following equivalent norm:

\[
\|u\|_{1,r(z)} = \|Du\|_{r(z)} \quad \forall u \in W_0^{1,r(z)}(\Omega).
\]

For \( r \in E_1 \), the critical Sobolev exponent corresponding to \( r \) is defined by

\[
r^*(z) = \begin{cases} \frac{N - r(z)}{N - r(z)} & \text{if } r(z) < N, \\ +\infty & \text{if } N \leq r(z). \end{cases}
\]

Suppose that \( r \in E_1 \cap C^{0,1}(\Omega) \), \( p \in E_1 \), \( p, < N \) and \( 1 < p(z) \leq r^*(z) \) (respectively \( 1 < p(z) < r^*(z) \)) for all \( z \in \Omega \). We have

\[
W_0^{1,r(z)}(\Omega) \subseteq L^{p(z)}(\Omega) \quad \text{continuously}
\]

(respectively: compactly).

Useful in the analysis of these variable exponent spaces is the following modular function:

\[
\varrho_r(u) = \int_\Omega |u|^{r(z)} \, dz \quad \forall u \in L^{r(z)}(\Omega),
\]

with \( r \in E_1 \). We write \( \varrho_r(Du) = \varrho_r(|Du|) \).

There is a close relation between this modular function and the norm. We assume \( r \in E_1 \).

**Proposition 2.1.**

(a) \( \|u\|_{r(z)} = \lambda \Rightarrow \varrho_r\left(\frac{u}{\lambda}\right) = 1 \) for all \( u \in L^{r(z)}(\Omega), \ u \neq 0 \).

(b) \( \|u\|_{r(z)} < 1 \) (resp. \( 1, > 1 \)) \( \Rightarrow \varrho_r(u) < 1 \) (resp. \( 1, > 1 \)).

(c) \( \|u\|_{r(z)} < 1 \Rightarrow \|u\|_{r(z)}^{r^*(z)} \leq \varrho_r(u) \leq \|u\|_{r(z)}^{r(z)} \).

(d) \( \|u\|_{r(z)} > 1 \Rightarrow \|u\|_{r(z)}^{r^*(z)} \leq \varrho_r(u) \leq \|u\|_{r(z)}^{r(z)} \).

(e) \( \|u_n\|_{r(z)} \to 0 \Rightarrow \varrho_r(u_n) \to 0 \).

(f) \( \|u_n\|_{r(z)} \to +\infty \Rightarrow \varrho_r(u_n) \to +\infty \).

More details can be found in the book of Diening-Harjulehto-Hästö-Růžička [16].

Consider the map \( A_{r(z)} : W_0^{1,r(z)}(\Omega) \to W_0^{1,r(z)}(\Omega)^* = W^{-1,r(z)}(\Omega) \) defined by

\[
\langle A_{r(z)}(u), h \rangle = \int_\Omega |Du|^{r(z)-2}(Du,h)_K \, dz \quad \forall u, h \in W_0^{1,r(z)}(\Omega).
\]
This map has the following properties (see Gasiński-Papageorgiou [6, Proposition 2.5] and Rădulescu-Repovš [18, p. 40]).

**Proposition 2.2.** The map \( A_{r(z)} : W_{0}^{1,r(z)}(\Omega) \to W_{0}^{1,r(z)}(\Omega)^{\ast} \) is bounded (that is, maps bounded sets to bounded sets), continuous, strictly monotone (hence maximal monotone too) and of type \((S)_{0}\), that is, “\( u_{n} \xrightarrow{w} u \) in \( W_{0}^{1,r(z)}(\Omega) \) and \( \limsup_{n \to +\infty} \langle A_{r(z)}(u_{n}), u_{n} - u \rangle \leq 0 \), imply that \( u_{n} \rightharpoonup u \) in \( W_{0}^{1,r(z)}(\Omega) \).”

In addition to the variable exponent spaces, we will also use the Banach space \( C_{0}^{0}(\Omega) = \{ u \in C(\Omega) : u|_{\partial \Omega} = 0 \} \).

This is an ordered Banach space with positive (order) cone

\[ C_{+} = \{ u \in C_{0}^{0}(\Omega) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega} \}. \]

This cone has a nonempty interior given by

\[ \text{int } C_{+} = \left\{ u \in C_{+} : u > 0, \frac{\partial u}{\partial n} \bigg|_{\partial \Omega} < 0 \right\}, \]

with \( n \) being the outward unit normal on \( \partial \Omega \).

Given \( u, v \in W^{1,r(z)}(\Omega) \) with \( u \leq v \), we define

\[ |u, v| = \{ h \in W_{0}^{1,r(z)}(\Omega) : u(z) \leq h(z) \leq r(z) \text{ for a.a. } z \in \Omega \}, \]

\[ [u] = \{ h \in W_{0}^{1,r(z)}(\Omega) : u(z) \leq h(z) \text{ for a.a. } z \in \Omega \}. \]

If \( h_{1}, h_{2} : \Omega \to \mathbb{R} \) are measurable functions, then we write \( h_{1} < h_{2} \), if for every compact set \( K \subseteq \Omega \), we have \( 0 < c_{K} \leq h_{2}(z) - h_{1}(z) \) for almost all \( z \in K \). Evidently, if \( h_{1}, h_{2} \in C(\Omega) \) and \( h_{2}(z) < h_{1}(z) \) for all \( z \in \Omega \), then \( h_{1} < h_{2} \).

A set \( S \subseteq W_{0}^{1,r(z)}(\Omega) \) is said to be “downward directed,” if for \( u_{1}, u_{2} \in S \), we can find \( u \in S \) such that \( u \leq u_{1}, u \leq u_{2} \).

By \( |\cdot|_{N} \) we denote the Lebesgue measure on \( \mathbb{R}^{N} \) and by \( \| \cdot \| \) the norm of \( W^{1,r(z)}(\Omega) \).

In the sequel for notational economy, by \( \| \cdot \| \) we denote the norm of the Sobolev space \( W_{0}^{1,r(z)}(\Omega) \). Recall that

\[ \| u \| = \| Du \|_{r(z)} \quad \forall u \in W^{1,r(z)}(\Omega). \]

If \( X \) is a Banach space and \( \varphi \in C(X) \), then we set

\[ K_{\varphi} = \{ u \in X : \varphi'(u) = 0 \} \]

(the critical set of \( \varphi \)). We say that \( \varphi \) satisfies the “Cerami condition,” if the following property holds:

“Every sequence \( \{ u_{n} \}_{n=1}^{\infty} \subseteq X \) such that \( \{ \varphi(u_{n}) \}_{n=1}^{\infty} \subseteq \mathbb{R} \) is bounded and

\[ (1 + \| u_{n} \|_{X}) \varphi'(u_{n}) \to 0 \quad \text{in } X^{\ast} \quad \text{as } n \to +\infty, \]

admits a strongly convergent subsequence.”

Now we introduce the hypotheses on the data problem \((P_{0})\).

\( H_{0} : p, q \in E_{1} \cap C^{0}([\Omega], \mathbb{R}) \), \( q_{i} < p \cdot \)

\( H_{1} : f : \Omega \times \mathbb{R} \to \mathbb{R} \) is a Carathéodory function such that \( f(z, 0) = 0 \) for a.a. \( z \in \Omega \) and

(i) \( f(z, x) \leq a(z)(1 + x^{r(z)-1}) \) for a.a. \( z \in \Omega \), all \( x \geq 0 \), with \( a \in L^{\infty}(\Omega) \) and \( p_{i} < r < p^{\ast}(z) \) for all \( z \in \overline{\Omega} \);

(ii) if \( F(z, x) = \int_{0}^{x} f(z, s)ds \), then

\[ \lim_{x \to +\infty} \frac{F(z, x)}{x^{p(z)}} = +\infty \quad \text{uniformly for a.a. } z \in \Omega; \]
(iii) if $\sigma(z, x) = f(z, x)x - p, F(z, x)$, then there exists $\eta \in L^1(\Omega)$ such that 

$$\sigma(z, x) \leq \sigma(z, y) + \eta(z) \text{ for a.a. } z \in \Omega, \text{ all } 0 \leq x \leq y;$$

(iv) for every $s > 0$, there exists $m_s > 0$ such that 

$$f(z, x) \geq m_s > 0 \text{ for a.a. } z \in \Omega, \text{ all } x \geq s,$$

and 

$$\lim_{x \to 0^+} \frac{f(z, x)}{x^q - 1} = +\infty \text{ uniformly for a.a. } z \in \Omega;$$

(v) for every $q > 0$, there exists $\xi_q > 0$ such that for a.a. $z \in \Omega$, the function $x \mapsto f(z, x) + \xi_q x^{p(z)-1}$ is nondecreasing on $[0, q]$.

Remark 2.3. Since we want to find positive solutions and the aforementioned hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0, +\infty)$, without any loss of generality, we may assume that 

$$f(z, x) = 0 \text{ for a.a. } z \in \Omega, \text{ all } x \leq 0. \quad (2.1)$$

Hypotheses $H_0(iii)$, (iii) imply that $f(z, \cdot)$ is $(p - 1)$-superlinear. Usually in the literature, such problems are treated using the well-known Ambrosetti-Rabinowitz condition (see Ambrosetti-Rabinowitz [25]). Here instead we use the less restrictive condition $H_0(iii)$, which is an extension of a condition used by Li-Yang [26]. This quasimonotonicity condition on the function $\sigma(z, \cdot)$ is equivalent to saying that there exists $M > 0$ such that for a.a. $z \in \Omega$, the quotient function $x \mapsto \frac{f(z, x)}{x^{p(z)-1}}$ is nondecreasing on $[M, +\infty)$. This superlinearity condition incorporates in our framework superlinear nonlinearities with “slower” growth near $+\infty$. For example, consider the following function:

$$f(z, x) = \begin{cases} x^{\tau(z)-1} & \text{if } 0 \leq x \leq 1, \\ x^{\mu(z)-1} \ln x + x^{\mu(z)-1} & \text{if } 1 < x \end{cases}$$

(see (2.1)), with $\tau, \mu \in E_1$ and $\tau, < q, \mu(z) \leq p(z)$ for all $z \in \Omega$. This function satisfies hypotheses $H_0$, but fails to satisfy the Ambrosetti-Rabinowitz condition.

We introduce the following two sets:

$$\mathcal{L} = \{ \lambda > 0 : \text{problem } (P_\lambda) \text{ admits a positive solution}\},$$

$$S_\lambda = \text{the set of positive solutions of } (P_\lambda).$$

Also, we set 

$$\lambda^* = \sup \mathcal{L}.$$

3 Positive solutions

We start by showing that the set of admissible parameters $\mathcal{L}$ is nonempty. Also, we determine the regularity properties of the elements in $S_\lambda$.

Proposition 3.1. If hypotheses $H_0, H_0(i)$, (iv) hold, then $\mathcal{L} \neq \emptyset$ and for every $\lambda > 0$, $S_\lambda \subseteq \text{int } C_r$.

Proof. We consider the following auxiliary Dirichlet problem:

$$\begin{cases} -\Delta_{p(z)} u(z) - \Delta_{q(z)} u(z) = 1 & \text{in } \Omega, \\ u|_{\partial \Omega} = 0. \end{cases} \quad (3.1)$$
The operator $u \mapsto A_{p(z)}(u) + A_{q(z)}(u)$ from $W^{1,p(z)}_0(\Omega)$ into $W^{1,p(z)}_0(\Omega)^*$ is continuous, strictly monotone (hence maximal monotone too) (see Proposition 2.2) and coercive. So, it is surjective (see Gasiński-Papageorgiou [27, Corollary 3.2.31, p. 319]). Hence, we can find $\bar{u} \in W^{1,p(z)}_0(\Omega)$, $\bar{u} \geq 0$, $\bar{u} \neq 0$ such that

$$A_{p(z)}(\bar{u}) + A_{q(z)}(\bar{u}) = 1 \text{ in } W^{1,p(z)}_0(\Omega)^*.$$  

The strict monotonicity of the operator implies that this solution is unique. So, $\bar{u}$ is the unique positive solution of (3.1). Theorem 4.1 of Fan-Zhao [28] implies that $\bar{u} \in L^\infty(\Omega)$. Then from Fukagai-Narukawa [29, Lemma 3.3] (see also Tan-Fang [30, Corollary 3.1] and Lieberman [31] for the corresponding isotropic regularity theory), we have that $\bar{u} \in C^{1,a}(\Omega) = C^{1,a}(\Omega) \cap C^1(\Omega)$ with $a \in (0, 1)$. Hence, $\bar{u} \in C_+ \setminus \{0\}$. From the anisotropic maximum principle (see Zhang [32]), we obtain that $\bar{u} \in \mathbb{R}$. Then from Fukagai-Narukawa [29, Lemma 3.3] (see also Tan-Fang [30, Corollary 3.1] and Lieberman [31] for the corresponding isotropic regularity theory), we have that $\bar{u} \in W^{1,p(z)}_0(\Omega)$.

Let $m = \|f(\cdot, \bar{u}(\cdot))\|_{\infty}$ (see hypothesis $H\text{iv}$) and choose $\lambda_0 > 0$ such that $\lambda_0 m \leq 1$. We have

$$-\Delta_{p(z)} u - \Delta_{q(z)} u \geq \lambda f(z, u) \text{ in } \Omega,$$

for all $\lambda \in (0, \lambda_0)$. We introduce the Carathéodory function $\hat{g}(z, x)$ defined by

$$\hat{g}(z, x) = \begin{cases} f(z, x') & \text{if } x \leq \bar{u}(z), \\ f(z, \bar{u}(z)) & \text{if } \bar{u}(z) < x. \end{cases}$$

We set

$$\hat{G}(z, x) = \int_0^x \hat{g}(z, s) \, ds$$

and for all $\lambda \in (0, \lambda_0)$ consider the $C^1$-functional $\hat{\phi}_\lambda : W^{1,p(z)}_0(\Omega) \to \mathbb{R}$ defined by

$$\hat{\phi}_\lambda(u) = \int_\Omega \frac{1}{p(z)} |Du|^{p(z)} \, dz + \int_\Omega \frac{1}{q(z)} |Du|^{q(z)} \, dz - \int_\Omega \lambda \hat{G}(z, u) \, dz \quad \forall u \in W^{1,p(z)}_0(\Omega).$$

From (3.3) and Proposition 2.1, it is clear that $\hat{\phi}_\lambda$ is coercive. Also, the anisotropic Sobolev embedding theorem implies that $\hat{\phi}_\lambda$ is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find $u_\delta \in W^{1,p(z)}_0(\Omega)$ such that

$$\hat{\phi}_\lambda(u_\delta) = \min_{u \in W^{1,p(z)}_0(\Omega)} \hat{\phi}_\lambda(u).$$

Hypothesis $H\text{iv}$ implies that given any $\theta > 0$, we can find $\delta = \delta(\theta) \in (0, 1)$ such that

$$F(z, x) \geq \frac{\theta}{q} x^q \quad \text{for a.a. } z \in \Omega, \quad 0 \leq x \leq \delta.$$  

Let $u \in \text{int } C_+$. Since $\bar{u} \in \text{int } C_+$, using Proposition 4.1.22 of Papageorgiou-Rădulescu-Repovš [33, p. 274], we can find $t \in (0, 1)$ small such that

$$tu(z) \leq \min\{\bar{u}(z), \delta\} \text{ for all } z \in \bar{\Omega}. $$

From (3.3), (3.5) and (3.6) and since $t \in (0, 1)$, we have

$$\hat{\phi}_\lambda(tu) \leq \frac{t^q}{q} \left( \varphi_p(Du) + \varphi_q(Du) - \theta \|u\|_V^q \right).$$

Since $\theta > 0$ is arbitrary, choosing $\theta > 0$ big from the aforementioned inequality, we infer that

$$\hat{\phi}_\lambda(tu) < 0,$$

so

$$\hat{\phi}_\lambda(u_\delta) < 0 = \hat{\phi}_\lambda(0)$$

(see (3.4)) and thus $u_\delta \neq 0$. 

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From (3.4), we have
\[ \hat{\phi}_\lambda^*(u_\lambda) = 0, \]
so
\[ \langle A_{p(z)}(u_\lambda), h \rangle + \langle A_{q(z)}(u_\lambda), h \rangle = \int_\Omega \lambda \hat{g}(z, u_\lambda) h dz \quad \forall h \in W_0^{1,p(z)}(\Omega). \] (3.7)

We test (3.7) with \( h = -u_\lambda^* \in W^{1,p(z)}(\Omega) \) and obtain
\[ \varrho_\lambda(Du_\lambda) + \varrho(Du_\lambda) = 0, \]
so \( u_\lambda \geq 0, \ u_\lambda \neq 0 \) (see Proposition (2.1)).

Next in (3.7) we choose \( h = (u_\lambda - \bar{u})^* \in W_0^{1,p(z)}(\Omega) \). We have
\[
\langle A_{p(z)}(u_\lambda), (u_\lambda - \bar{u})^* \rangle + \langle A_{q(z)}(u_\lambda), (u_\lambda - \bar{u})^* \rangle
\]
\[ = \int_\Omega A_f(z, \bar{u})(u_\lambda - \bar{u})' dz \leq \langle A_{p(z)}(\bar{u}), (u_\lambda - \bar{u})' \rangle + \langle A_{q(z)}(\bar{u}), (u_\lambda - \bar{u})' \rangle \]
(see (3.3) and (3.2)), so
\[ u_\lambda \leq \bar{u}. \]

So, we have proved that
\[ u_\lambda \in [0, \bar{u}], \ u_\lambda \neq 0. \] (3.8)

From (3.8), (3.3) and (3.7), it follows that \( u_\lambda \) is a positive solution of \( (P_\lambda) \). As before using the anisotropic regularity theorem (see Fan-Zhao [28], Fukagai-Narukawa [29]) and the anisotropic maximum principle (see Zhang [32]), we obtain that \( u_\lambda \in \text{int} \ C_\gamma \).

Therefore, we conclude that
\[ (0, \lambda_0] \subseteq L \neq \emptyset \]
and
\[ S_\lambda \subseteq \text{int} \ C_\gamma, \ \lambda > 0. \]

Next, we show that \( L \) is connected.

**Proposition 3.2.** If hypotheses \( H_0, H(f), (iv) \) hold, \( \lambda \in L \) and \( 0 < \mu < \lambda \), then \( \mu \in L \).

**Proof.** Since \( \lambda \in L \), we can find \( u_\lambda \in S_\lambda \subseteq \text{int} \ C_\gamma \) (see Proposition 3.1). We introduce the Carathéodory function \( \hat{g} \) defined by
\[ \hat{g}(z, x) = \begin{cases} f(z, x^*) & \text{if } x \leq u_\lambda(z), \\ f(z, u_\lambda(z)) & \text{if } u_\lambda(z) < x. \end{cases} \] (3.9)

We set
\[ \hat{G}(z, x) = \int_0^x \hat{g}(z, s) ds \]
and consider the \( C^1 \)-functional \( \overline{\phi}_\mu : W_0^{1,p(z)}(\Omega) \to \mathbb{R} \) defined by
\[ \overline{\phi}_\mu(u) = \int_\Omega \frac{1}{p(z)} |D u|^{p(z)} dz + \int_\Omega \frac{1}{q(z)} |D u|^{q(z)} dz - \int_\Omega \mu \hat{g}(z, u) dz \quad \forall u \in W_0^{1,p(z)}(\Omega). \]
On account of (3.9) and Proposition 2.1, $\tilde{\phi}_\mu$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $u_\mu \in W^{1,p(z)}(\Omega)$ such that

$$\tilde{\phi}_\mu(u_\mu) = \min_{u \in W^{1,p(z)}(\Omega)} \tilde{\phi}_\mu(u) < 0 = \tilde{\phi}_\mu(0)$$

(see the proof of Proposition 3.1), thus $u_\mu \neq 0$. We have

$$\tilde{\phi}_\mu'(u_\mu) = 0$$

and from this as in the proof of Proposition 3.1, using (3.9), we obtain

$$u_\mu \in [0, u_1], \quad u_\mu \neq 0,$$

so

$$u_\mu \in S_\mu \subseteq \text{int} \, C,$$

(see (3.9) and Proposition 3.1), hence $\mu \in \mathcal{L}$. □

A byproduct of the above proof is the following monotonicity property of the solution multifunction $\lambda \mapsto S_\lambda$.

**Corollary 3.3.** If hypotheses $H_0$, $H_i (i, \text{iv})$ hold, $\lambda \in \mathcal{L}$, $u_\lambda \in S_\lambda \subseteq \text{int} \, C$, and $0 < \mu < \lambda$, then $\mu \in \mathcal{L}$ and we can find $u_\mu \in S_\mu \subseteq \text{int} \, C$, such that $u_\mu \in u_\lambda$.

We can improve this monotonicity property, if we bring in the picture hypothesis $H_i (v)$.

**Proposition 3.4.** If hypotheses $H_0$, $H_i (i, \text{iv}, \text{v})$ hold, $\lambda \in \mathcal{L}$, $u_\lambda \in S_\lambda \subseteq \text{int} \, C$, and $0 < \mu < \lambda$, then $\mu \in \mathcal{L}$ and there exists $u_\mu \in S_\mu \subseteq \text{int} \, C$, such that

$$u_\lambda - u_\mu \in \text{int} \, C.$$

**Proof.** From Corollary 3.3, we already know that $\mu \in \mathcal{L}$ and that there exists $u_\mu \in S_\mu \subseteq \text{int} \, C$, such that $u_\mu \in u_\lambda$. Let $q = \|u_\lambda\|_{\infty}$ and let $\tilde{\xi}_0 > 0$ be as postulated by hypothesis $H_i (v)$. We have

$$-\Delta_{p(z)} u_\mu - \Delta_{q(z)} u_\mu + \tilde{\xi}_0 u_\mu^{p(z)-1} = mf(z, u_\mu) + \tilde{\xi}_0 u_\mu^{p(z)-1} - (\lambda - \mu)f(z, u_\mu)$$

$$\leqslant \tilde{\xi}_0 f(z, u_\lambda) + \tilde{\xi}_0 u_\lambda^{p(z)-1}(\lambda - \mu)f(z, u_\mu) \leqslant -\Delta_{p(z)} u_\lambda - \lambda f(z, u_\lambda) + \tilde{\xi}_0 u_\lambda^{p(z)-1}$$

(3.10)

(see hypothesis $H_i (v)$). Since $u_\mu \in \text{int} \, C$, on account of hypothesis $H_i (\text{iv})$, we have that

$$0 < (\lambda - \mu)f(\cdot, u_\mu(\cdot)).$$

Then from (3.10) and Proposition 2.4 of Papageorgiou-Rădulescu-Repovš [7], we conclude that $u_\lambda - u_\mu \in \text{int} \, C$. □

Next for every $\lambda \in \mathcal{L}$, we will produce a smallest (minimal) positive solution for problem $(P_\lambda)$. To this end, we need some preparation.

Hypotheses $H_i (i, \text{iv})$ imply that given $\beta > 0$, we can find $c_0 = c_0(\beta) > 0$ such that

$$f(z, x) \geqslant \beta x^{q(z)-1} - c_0 x^{r(z)-1} \text{ for a.a. } z \in \Omega, \text{ all } x \geqslant 0.$$  \hfill (3.11)

Motivated from this unilateral growth estimate for $f(z, \cdot)$, we consider the following auxiliary Dirichlet problem:

$$\begin{cases}
-\Delta_{p(z)} u(z) - \Delta_{q(z)} u(z) = \lambda(\beta u(z)^{q(z)-1} - c_0 u(z)^{r(z)-1}) \text{ in } \Omega, \\
u_{\lambda|\Omega} = 0, u > 0, \lambda > 0.
\end{cases}$$  \hfill (Q_\lambda)
Proposition 3.5. For every \( \lambda > 0 \), we can choose \( \beta = \beta(\lambda) > 0 \) big such that \( Q_\lambda \) has a unique positive solution \( \tilde{u}_\lambda \in \text{int } C_+^t \).

Proof. First we show the existence of a positive solution. To this end, we consider the \( C^1 \)-functional \( \sigma_\lambda : W_{0}^{1,p(z)}(\Omega) \to \mathbb{R} \) defined by

\[
\sigma_\lambda(u) = \int_\Omega \frac{1}{p(z)} |Du|^{p(z)} \, dz + \int_\Omega \frac{1}{q(z)} |Du|^{q(z)} \, dz + \frac{\lambda \beta}{r} ||u||_r^r - \frac{\lambda \beta}{q} ||u||_q^q \quad \forall u \in W_{0}^{1,p(z)}(\Omega).
\]

Since \( q < q(z) < p(z) \leq p_r < r \) for all \( z \in \overline{\Omega} \) (see hypothesis \( H_0 \)), we see that \( \sigma_\lambda \) is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find \( \tilde{u}_\lambda \in W_{0}^{1,p(z)}(\Omega) \) such that

\[
\sigma_\lambda(\tilde{u}_\lambda) = \min_{u \in W_{0}^{1,p(z)}(\Omega)} \sigma_\lambda(u). \tag{3.12}
\]

Consider \( u \in C_+ \) with \( ||u||_{\infty} \leq 1 \). For \( t \in (0, 1) \), we have

\[
\sigma_\lambda(tu) \leq \frac{t^p}{p} q_\rho(Du) + \frac{t^q}{q} (q_\beta(Du) - \lambda \beta ||u||_q^q) + \frac{\lambda \beta}{r} ||u||_r^r \leq \frac{t^q}{q} (q_\rho(Du) + q_\beta(Du) - \lambda \beta ||u||_q^q) + \frac{\lambda \beta}{r} ||u||_r^r.
\]

Recall that \( \beta > 0 \) is arbitrary. So, we choose \( \beta_0 > \frac{q_\beta(Du) + q_\beta(Du)}{\lambda ||u||_q^q} \) and obtain

\[
\sigma_\lambda(\tilde{u}_\lambda) < 0 = \sigma_\lambda(0)
\]

(see (3.12)), hence \( \tilde{u}_\lambda \neq 0 \).

From (3.12), we have

\[
\sigma_\lambda^t(\tilde{u}_\lambda) = 0,
\]

so

\[
\langle A_{p(z)}(\tilde{u}_\lambda), h \rangle + \langle A_{q(z)}(\tilde{u}_\lambda), h \rangle = \lambda \beta \int_\Omega (\tilde{u}_\lambda)^{q-1} h dz - \lambda c_1 \int_\Omega (\tilde{u}_\lambda)^{r-1} h dz \quad \forall h \in W_{0}^{1,p(z)}(\Omega). \tag{3.13}
\]

In (3.13), we choose \( h = -\tilde{u}^r \in W_{0}^{1,p(z)}(\Omega) \) and obtain

\[
q_\rho(D\tilde{u}_\lambda) + q_\beta(D\tilde{u}_\lambda) = 0,
\]

so \( \tilde{u}_\lambda \geq 0 \), \( \tilde{u}_\lambda \neq 0 \) (see Proposition 2.1).

Then from (3.13) we infer that \( \tilde{u}_\lambda \) is a positive solution of \( Q_\lambda \). Moreover, as before the anisotropic regularity theory and the anisotropic maximum principle imply

\[
\tilde{u}_\lambda \in \text{int } C_+^t. \tag{3.14}
\]

Let \( \tilde{v}_\lambda \in W_{0}^{1,p(z)}(\Omega) \) be another positive solution of \( Q_\lambda \). Again we have

\[
\tilde{v}_\lambda \in \text{int } C_+. \tag{3.15}
\]

We consider the integral functional \( j : L^1(\Omega) \to \mathbb{R} = \mathbb{R} \cup \{+\infty\} \) defined by

\[
j(u) = \begin{cases} 
\int_\Omega \frac{q}{p(z)} |Du|^{p(z)} \, dz + \int_\Omega \frac{q}{q(z)} |Du|^{q(z)} \, dz & \text{if } u \geq 0, \text{int } W_{0}^{1,p(z)}(\Omega), \\
+\infty & \text{otherwise}.
\end{cases}
\]
From Theorem 2.2 of Takáč-Giacomoni [34], we have that $j$ is convex. Let

$$\text{dom } j = \{ u \in L^1(\Omega) : j(u) < +\infty \}$$

(the effective domain of $j$). From (3.14), (3.15) and Proposition 4.1.22 of Papageorgiou-Rădulescu-Repovš [33, p. 274], we have

$$\tilde{u}_\lambda \in L^{\infty}(\Omega), \quad \tilde{v}_\lambda \in L^{\infty}(\Omega).$$

Let $h \in C^1(\Omega)$ with $|h|^{\frac{p}{q}} \in W_0^{1,p(z)}(\Omega)$. For $t \in (0, 1)$ small, we have

$$\tilde{u}_\lambda^t + th \in \text{dom } j \quad \text{and} \quad \tilde{v}_\lambda^t + th \in \text{dom } j.$$

Choose $h = \tilde{u}_\lambda^t - \tilde{v}_\lambda^t$. Evidently,

$$h \in C^1(\Omega) \quad \text{and} \quad |h| \leq \tilde{u}_\lambda^t + \tilde{v}_\lambda^t.$$

We have

$$h|^{\frac{p}{q}} \leq \tilde{u}_\lambda + \tilde{v}_\lambda,$$

so $|h|^{\frac{p}{q}} \in W_0^{1,p(z)}(\Omega)$.

Then on account of the convexity of $j$, it is Gâteaux differentiable at $\tilde{u}_\lambda^t$ and at $\tilde{v}_\lambda^t$ in the direction $h = \tilde{u}_\lambda^t - \tilde{v}_\lambda^t$. Moreover, we have (see also Takáč-Giacomoni [34])

$$j'(\tilde{u}_\lambda^t)(h) = \int_{\Omega} \frac{-\Delta_{p(z)} \tilde{u}_\lambda^t - \Delta_{q(z)} \tilde{u}_\lambda^t}{\tilde{u}_\lambda^{t-1}} h dz,$$

$$j'(\tilde{v}_\lambda^t)(h) = \int_{\Omega} \frac{-\Delta_{p(z)} \tilde{v}_\lambda^t - \Delta_{q(z)} \tilde{v}_\lambda^t}{\tilde{v}_\lambda^{t-1}} h dz.$$

The convexity of $j$ implies the monotonicity of $j'$. Hence,

$$0 \leq \lambda \int_{\Omega} (\tilde{u}_\lambda^{t-q} - \tilde{v}_\lambda^{t-q})(\tilde{v}_\lambda^t - \tilde{u}_\lambda^t) dz \leq 0$$

(recall that $q < r$), so

$$\tilde{u}_\lambda = \tilde{v}_\lambda,$$

thus $\tilde{u}_\lambda \in \text{int } C$, is the unique positive solution of $(Q_\lambda)$. \hfill \Box

Using $\tilde{u}_\lambda \in \text{int } C$, from Proposition 3.5, we can have a lower bound for the elements of $S_\lambda$.

**Proposition 3.6.** If hypotheses $H_0$, $H_i(t)$, (iv), (v) hold and $\lambda \in \mathcal{L}$, then $\tilde{u}_\lambda \leq u$ for all $u \in S_\lambda$.

**Proof.** Let $u \in S_\lambda$. We introduce the Carathéodory function $k(z, x)$ defined by

$$k(z, x) = \begin{cases} 
\beta(x)^{q-1} - a(x)^{q-1} & \text{if } x \leq u(z), \\
\beta u(z)^{q-1} - a u(z)^{q-1} & \text{if } u(z) < x.
\end{cases}$$

(3.16)

We set

$$K(z, x) = \int_0^x k(z, s) ds$$

and consider the $C^1$-functional $\gamma_\lambda : W_0^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by
\[ y_\lambda(u) = \int_\Omega \frac{1}{p(z)} |Du|^{p(z)} \, dz + \int_\Omega \frac{1}{q(z)} |Du|^{q(z)} \, dz - \int_\Omega \lambda K(z, u) \, dz \quad \forall u \in W_0^{1,p(z)}(\Omega). \]

From Proposition 2.1 and (3.16) it is clear that \( y_\lambda \) is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find \( \bar{u}_\lambda \in W_0^{1,p(z)}(\Omega) \) such that
\[ y_\lambda(\bar{u}_\lambda) = \min_{u \in W_0^{1,p(z)}(\Omega)} y_\lambda(u). \] (3.17)

As before (see the proof of Proposition 3.5), we have
\[ y_\lambda(\bar{u}_\lambda) < 0 = y_\lambda(0), \]
so \( \bar{u}_\lambda \neq 0 \).

From (3.17), we have
\[ y_\lambda'(\bar{u}_\lambda) = 0, \]
so
\[ \langle A_{p(z)}(\bar{u}_\lambda), h \rangle + \langle A_{q(z)}(\bar{u}_\lambda), h \rangle = \lambda \int_\Omega k(z, \bar{u}_\lambda) \, hdz \quad \forall h \in W_0^{1,p(z)}(\Omega). \] (3.18)

We test (3.18) with \( h = -\bar{u}_\lambda \in W_0^{1,p(z)}(\Omega) \) and obtain
\[ b_{p(z)}(Du) + b_{q(z)}(Du) = 0 \]
(see (3.16)), so \( \bar{u}_\lambda \geq 0, \bar{u}_\lambda \neq 0 \).

Next in (3.18) we choose \( h = (\bar{u}_\lambda - u)^+ \in W_0^{1,p(z)}(\Omega) \). Then
\[ \langle A_{p(z)}(\bar{u}_\lambda), (\bar{u}_\lambda - u)^+ \rangle + \langle A_{q(z)}(\bar{u}_\lambda), (\bar{u}_\lambda - u)^+ \rangle = \lambda \int_\Omega k(z, \bar{u}_\lambda) (\bar{u}_\lambda - u)^+ \, dz \leq \int_\Omega Af(z, \bar{u}_\lambda)(\bar{u}_\lambda - u)^+ \, dz \]
\[ = \langle A_{p(z)}(u), (\bar{u}_\lambda - u)^+ \rangle + \langle A_{q(z)}(u), (\bar{u}_\lambda - u)^+ \rangle \]
(see (3.16), (3.11) and use the fact that \( u \in S_\delta \)), so \( \bar{u}_\lambda \leq u \) (see Proposition 2.2).

So, we have proved that
\[ \bar{u}_\lambda \in [0,u], \quad \bar{u}_\lambda \neq 0. \] (3.19)

From (3.19), (3.16) and (3.18), it follows that \( \bar{u}_\lambda \) is a positive solution of \((Q_\lambda)\), hence \( \bar{u}_\lambda = \bar{u}_\lambda \) (see Proposition (3.6)). We conclude that \( \bar{u}_\lambda \leq u \) for all \( u \in S_\delta \). \( \square \)

Now we are ready to produce the minimal positive solution of problem \((P_\lambda)\), \( \lambda \in \mathcal{L} \).

**Proposition 3.7.** If hypotheses \( H_0, H_i(i), (iv), (v) \) hold and \( \lambda \in \mathcal{L} \), then problem \((P_\lambda)\) admits a smallest positive solution \( u_\lambda^* \in S_\delta \subseteq \text{int} \, C_i \) (that is, \( u_\lambda^* \leq u \) for all \( u \in S_\delta \)).

**Proof.** From Papageorgiou-Rădulescu-Repovš [35] (proof of Proposition 7; see also Filippakis-Papageorgiou [36]), we know that \( S_\delta \) is downward directed. So, by Lemma 3.10 of Hu-Papageorgiou [37, p. 178], we can find a decreasing sequence \( \{u_n\}_{n=1} \subseteq S_\delta \) such that
\[ \inf_{n \geq 1} S_\delta = \inf_{n \geq 1} u_n \] (3.20)
and
\[ \bar{u}_\lambda \leq u_0 \leq u_1 \quad \forall n \in \mathbb{N} \] (3.21)
(see Proposition 3.6). We have
\[ \langle A_{p(z)}(u_n), h \rangle + \langle A_{q(z)}(u_n), h \rangle = \int_\Omega Af(z, u_n) \, hdz \quad \forall h \in W_0^{1,p(z)}(\Omega), n \in \mathbb{N}. \] (3.22)
In (3.22), we use \( h = u_n \in W_0^{1,p(z)}(\Omega) \). From (3.21), hypothesis \( H_i \) and Proposition 2.1, it follows that the sequence \( \{u_n\}_{n=1}^\infty \subseteq W_0^{1,p(z)}(\Omega) \) is bounded.

So, we may assume that

\[
u_n \rightharpoonup u_1^* \text{ in } W_0^{1,p(z)}(\Omega) \quad \text{and} \quad u_n \to u_1^* \text{ in } L^{p(z)}(\Omega).
\]  (3.23)

We test (3.22) with \( h = u_n - u_1^* \in W_0^{1,p(z)}(\Omega) \), pass to the limit as \( n \to +\infty \) and use (3.23). We obtain

\[
\limsup_{n \to +\infty} \langle A_{p(z)}(u_n), u_n - u_1^* \rangle + \langle A_{q(z)}(u_1^*), u_n - u_1^* \rangle \leq 0
\]

(since \( A_{q(z)} \) is monotone), thus

\[
\limsup_{n \to +\infty} \langle A_{p(z)}(u_n), u_n - u_1^* \rangle \leq 0
\]

(see (3.23)) and hence

\[
u_n \to u_1^* \text{ in } W_0^{1,p(z)}(\Omega)
\]  (3.24)

(see Proposition 2.2).

Then passing to the limit as \( n \to +\infty \) in (3.22) and using (3.24) and (3.21), we obtain

\[
\langle A_{p(z)}(u_1^*), h \rangle + \langle A_{q(z)}(u_1^*), h \rangle = \int_{\Omega} \lambda f(z, u_1^*) h \, dz \quad \forall h \in W_0^{1,p(z)}(\Omega),
\]

so

\[
\bar{u}_1 \leq u_1^*
\]

and hence

\[
u_1^* \in S_1 \subseteq \text{int } C., \quad \inf C_1 = \inf S_1.
\] \[\square\]

We consider the map \( \lambda \mapsto u_1^* \) from \( \mathcal{L} \) into \( C_0^1(\bar{\Omega}) \).

**Proposition 3.8.** If hypotheses \( H_0, H_i, \) (iv), (v) hold, then the map \( \lambda \mapsto u_1^* \) from \( \mathcal{L} \) into \( C_0^1(\bar{\Omega}) \) is

(a) strictly increasing (that is, if \( 0 < \mu < \lambda \in \mathcal{L} \), then \( u_1^* - u_1^* \in \text{int } C_i \));

(b) left continuous.

**Proof.** (a) Suppose that \( 0 < \mu < \lambda \in \mathcal{L} \). Let \( u_1^* \in S_{\lambda} \subseteq \text{int } C_i \) be the minimal solution of problem \( (P_\mu) \) (see Proposition 3.7). According to Proposition 3.4, we can find \( u_1^* \in S_{\mu} \subseteq \text{int } C_i \) such that

\[
u_1^* - u_1^* \in \text{int } C_i,
\]

so

\[
u_1^* - u_1^* \in \text{int } C_i
\]

and hence the map \( \lambda \mapsto u_1^* \) is strictly increasing.

(b) Let \( \lambda_n \to \lambda \) with \( \lambda \in \mathcal{L} \). Let \( u_1^* = u_1^* \in \text{int } C_i \) for all \( n \in \mathbb{N} \). From part (a) and hypothesis \( H_i \), we see that the sequence \( \{u_1^*\}_{n=1}^\infty \subseteq W_0^{1,p(z)}(\Omega) \) is bounded.

Then from the anisotropic regularity theory (see Fukagai-Narukawa [29] and Tan-Fang [30]), we can find \( \alpha \in (0, 1) \) and \( c_0 > 0 \) such that
\[ u_*^n \in C_0^{1,a}(\overline{\Omega}), \|u_*^n\|_{C^{1,a}(\overline{\Omega})} \leq c_q, \quad \forall n \in \mathbb{N}. \]

Exploiting the compactness of the embedding \( C_0^{1,a}(\overline{\Omega}) \subseteq C^1(\overline{\Omega}) \), we have
\[ u_*^n \rightharpoonup u_1^* \text{ in } C^1(\overline{\Omega}). \] (3.25)

Evidently \( u_1^* \in S_i \). If \( u_1^* \neq u_1^* \), then we can find \( z_0 \in \Omega \) such that
\[ u_1^*(z_0) < u_1^*(z_0), \]
so
\[ u_1^*(z_0) < u_*^n(z_0) \quad \forall n \geq n_0 \]
(see (3.25)). This contradicts part (a). So, the map \( \lambda \mapsto u_1^* \) is left continuous. \( \square \)

So far, we only know that \( \mathcal{L} \) is nonempty and connected. We do not know if it is bounded or not. The next proposition shows that \( \mathcal{L} \) is bounded. In what follows, by \( \varphi_\lambda : W_0^{1,p(z)}(\Omega) \to \mathbb{R} \) we denote the energy (Euler) functional of problem \( (P_\lambda) \) defined by
\[ \varphi_\lambda(u) = \int_\Omega \frac{1}{p(z)}|Du|^{p(z)} \, dz + \int_\Omega \frac{1}{q(z)}|Du|^{q(z)} \, dz - \int_\Omega \lambda F(z,u) \, dz \quad \forall u \in W_0^{1,p(z)}(\Omega). \]

**Proposition 3.9.** If hypotheses \( H_0, H_1 \) hold, then \( \lambda^* < +\infty \).

**Proof.** We argue by contradiction. So, suppose that \( \lambda^* = +\infty \) (that is, \( \mathcal{L} = (0, +\infty) \)). Let \( \{\lambda_n\}_{n=1}^\infty \subseteq \mathcal{L} \) be such that \( \lambda_n \nearrow +\infty \). Then on account of Proposition 3.8 and hypothesis \( H_1(ii) \), we can find a nondecreasing sequence \( u_n \in S_{\lambda_n} \subseteq \text{int } C \), for \( n \in \mathbb{N} \) such that
\[ \varphi_{\lambda_n}(u_n) \leq c_5 \quad \forall n \in \mathbb{N}, \] (3.26)
for some \( c_5 > 0 \) and
\[ \varphi_{\lambda_n}'(u_n) = 0 \quad \forall n \in \mathbb{N}. \] (3.27)

From (3.27), we have
\[ \langle A_{p(z)}(u_n), h \rangle + \langle A_{q(z)}(u_n), h \rangle = \lambda_n \int_\Omega f(z, u_n) \, hdz \quad \forall h \in W_0^{1,p(z)}(\Omega). \] (3.28)

We test (3.28) with \( h = u_n \in W_0^{1,p(z)}(\Omega) \). Then
\[ -\varphi_p(Du_n) - \varphi_q(Du_n) + \lambda_n \int_\Omega f(z, u_n) u_n \, dz = 0 \quad \forall n \in \mathbb{N}. \] (3.29)

Also from (3.26), we have
\[ \int_\Omega \frac{1}{p(z)}|Du_n|^{p(z)} \, dz + \int_\Omega \frac{1}{q(z)}|Du_n|^{q(z)} \, dz - \lambda_n \int_\Omega F(z, u_n) \, dz \leq c_5 \quad \forall n \in \mathbb{N}, \]
so
\[ \frac{1}{p(z)}(\varphi_p(Du_n) + \varphi_q(Du_n)) - \lambda_n \int_\Omega F(z, u_n) \, dz \leq c_5 \quad \forall n \in \mathbb{N}, \]
thus
\[ \varphi_p(Du_n) + \varphi_q(Du_n) - \lambda_n \int_\Omega p(z) F(z, u_n) \, dz \leq p(c_5) \quad \forall n \in \mathbb{N}. \] (3.30)
Adding (3.29) and (3.30), we obtain
\[
\lambda_n \int_\Omega \sigma(z, u_n) \, dz \leq p, c_5 \quad \forall n \in \mathbb{N}
\]
so
\[
\int_\Omega \sigma(z, u_n) \, dz \leq \frac{p \cdot c_5}{\lambda_n} \quad \forall n \in \mathbb{N}.
\]  

(3.31)

Suppose that the sequence \( \{u_n\}_{n \geq 1} \subseteq W^{1,p(z)}_0(\Omega) \) is not bounded. We may assume that
\[
\|u_n\| \to +\infty \quad \text{as} \quad n \to +\infty.
\]  

(3.32)

We set \( y_n = \frac{u_n}{\|u_n\|} \) for \( n \in \mathbb{N} \). Then \( \|y_n\| = 1 \), \( y_n \geq 0 \) for all \( n \in \mathbb{N} \). We may assume that
\[
y_n \rightharpoonup w y \quad \text{in} \quad W^{1,p(z)}_0(\Omega) \quad \text{and} \quad y_n \to y \quad \text{in} \quad L^{p(z)}(\Omega), y \geq 0.
\]  

(3.33)

First suppose that \( y \neq 0 \). Let \( \hat{\Omega} = \{ y > 0 \} \). Then \( \hat{\Omega}|_\Omega > 0 \) (see (3.33)) and \( u_n(z) \to +\infty \) for almost all \( z \in \hat{\Omega} \).

On account of hypothesis \( H_2(ii) \), we have
\[
\frac{F(z, u_n(z))}{\|u_n\|^p} = \frac{F(z, u_n(z))}{u_n(z)^p} y_n(z)^p \to +\infty \quad \text{for a.a.} \quad z \in \hat{\Omega}.
\]

Then by Fatou’s lemma, we have
\[
\lim_{n \to +\infty} \int_\hat{\Omega} \frac{F(z, u_n)}{\|u_n\|^p} \, dz = +\infty.
\]  

(3.34)

Hypotheses \( H_2(i), (ii) \) imply that we can find \( c_6 > 0 \) such that
\[
\frac{F(z, x)}{x^p} \geq -c_6 \quad \text{for a.a.} \quad z \in \Omega, \quad \text{all} \quad x \geq 0.
\]  

(3.35)

We have
\[
\int_\Omega \frac{F(z, u_n)}{\|u_n\|^p} \, dz = \int_\Omega \frac{F(z, u_n)}{\|u_n\|^p} \, dz + \int_\Omega \frac{F(z, u_n)}{\|u_n\|^p} \, dz \geq \int_\Omega \frac{F(z, u_n)}{\|u_n\|^p} \, dz - c_7 \quad \forall n \in \mathbb{N}
\]
for some \( c_7 > 0 \) (see (3.35)), so
\[
\lim_{n \to +\infty} \int_\Omega \frac{F(z, u_n)}{\|u_n\|^p} \, dz = +\infty
\]  

(3.36).

(see (3.36)). From (3.29), we have
\[
- \int_\Omega \frac{1}{\|u_n\|^p - q(z)} |Dy_n|^{q(z)} - \int_\Omega \frac{1}{\|u_n\|^p - q(z)} |Dy_n|^{q(z)} + \lambda_n \int_\Omega f(z, u_n) u_n \, dz = 0 \quad \forall n \in \mathbb{N},
\]
so
\[
\lambda_n \int_\Omega \frac{f(z, u_n) u_n}{\|u_n\|^p} \, dz \leq c_8 \quad \forall n \in \mathbb{N},
\]
for some \( c_8 > 0 \) (see (3.32), recall that \( q(z) < p(z) \leq p \), for all \( z \in \hat{\Omega} \)), thus
\[
\lambda_n \int_\Omega \frac{p F(z, u_n)}{\|u_n\|^p} \, dz - \lambda_n \|\eta\|_1 \leq c_8 \quad \forall n \in \mathbb{N}
\]
(see hypothesis \( H_2(iv) \) and recall that \( u_n > 0 \), hence
\[
\lambda_n \int_\Omega \frac{p F(z, u_n)}{\|u_n\|^p} \, dz \leq \lambda_n \|\eta\|_1 \leq c_8 \quad \forall n \in \mathbb{N}
\]

\[ \int_\Omega p_i F(z, u_n) \, dz \leq \frac{c_0}{\lambda_n} + \| \eta \|_i \quad \forall n \in \mathbb{N}. \]  \hfill (3.37)

Comparing (3.36) and (3.37), we have a contradiction.

Next suppose that \( y = 0 \). We consider the \( C^1 \)-functional \( \phi^*_\lambda : W^{1,p,p}(\Omega) \to \mathbb{R} \) defined by

\[ \phi^*_\lambda(u) = \frac{1}{p_i} q_p(Du) - \lambda \int_\Omega F(z, u) \, dz \quad \forall u \in W^{1,p,p}(\Omega). \]

Evidently, we have

\[ \phi^*_\lambda \leq \phi^*_\lambda \quad \forall \lambda > 0. \]  \hfill (3.38)

Let \( \theta_n(t) = \phi^*_\lambda(t u_n) \) for all \( t \in [0, 1] \), all \( n \in \mathbb{N} \). We can find \( t_n \in [0, 1] \) such that

\[ \theta_n(t_n) = \max_{0 \leq t \leq 1} \theta_n(t). \]

Let \( \beta \geq 1 \) and set

\[ v_n(z) = (2\beta)^{p_i - 1} y_n(z) \quad \forall n \in \mathbb{N}. \]

Clearly, we have

\[ v_n \to 0 \quad \text{in } L^{p(z)}(\Omega) \]

(see (3.33) and recall that \( y = 0 \)), so

\[ \int_\Omega F(z, v_n) \, dz \to 0 \quad \text{as } n \to +\infty. \]  \hfill (3.39)

From (3.32), we see that we can find \( n_0 \in \mathbb{N} \) such that

\[ (2\beta)^{p_i - 1} \frac{1}{\| u_n \|} \leq 1 \quad \forall n \geq n_0, z \in \overline{\Omega}. \]

It follows that

\[ \theta_n(t_n) \geq \theta_n \left( \frac{(2\beta)^{p_i - 1}}{\| u_n \|} \right) \quad \forall n \geq n_0, z \in \overline{\Omega}, \]

so

\[ \phi^*_\lambda(t_n u_n) \geq \phi^*_\lambda \left( \frac{(2\beta)^{p_i - 1}}{\| u_n \|} \right) = \phi^*_\lambda(v_n) \quad \forall n \geq n_0, \]

thus

\[ \phi^*_\lambda(t_n u_n) \geq \frac{2\beta}{p_i} q_p(Dv_n) - \int_\Omega F(z, v_n) \, dz \quad \forall n \geq n_0, \]

and hence

\[ \phi^*_\lambda(t_n u_n) \geq \frac{\beta}{p_i} \quad \forall n \geq n_1 \geq n_0 \]  \hfill (3.40)

(see (3.39) and Proposition 2.1(a)).

Since \( \beta \geq 1 \) is arbitrary, from (3.40) we infer that

\[ \phi^*_\lambda(t_n u_n) \to +\infty \quad \text{as } n \to +\infty. \]  \hfill (3.41)
We have

\[ 0 \leq t_n u_n \leq u_n \quad \forall n \in \mathbb{N}, \]

so

\[ \sigma(z, t_n u_n) \leq \sigma(z, u_n) + \eta(z) \quad \text{for a.a. } z \in \Omega, \quad \forall n \in \mathbb{N} \]

(see hypothesis \( H_1(iii) \)), so

\[ \int_\Omega \sigma(z, t_n u_n) dz \leq \int_\Omega \sigma(z, u_n) dz + \|\eta\|_1 \leq c_0 \quad \forall n \in \mathbb{N} \tag{3.42} \]

for some \( c_0 > 0 \) (see (3.31)). We know that

\[ \varphi_{\lambda_n}^*(0) = 0 \quad \text{and} \quad \varphi_{\lambda_n}^*(u_n) \leq c_5 \quad \forall n \in \mathbb{N} \tag{3.43} \]

(see (3.26) and (3.38)). Then from (3.41) it follows that \( t_n \in (0, 1) \) for all \( n \geq n_2 \). Therefore, we can say that

\[ 0 = t_n \frac{d}{dt} \varphi_{\lambda_n}^*(t_n u_n) |_{t=t_n}, \]

so

\[ \langle (\varphi_{\lambda_n}^*)' (t_n u_n), t_n u_n \rangle = 0 \]

(by the chain rule), thus

\[ \theta_p(D(t_n u_n)) + \theta_q(D(t_n u_n)) - \lambda_n \int_\Omega f(z, t_n u_n)(t_n u_n) dz = 0 \quad \forall n \geq n_2 \]

and hence

\[ p, \varphi_{\lambda_n}^*(t_n u_n) \leq c_9 \quad \forall n \geq n_2 \tag{3.44} \]

(see (3.42)).

We compare (3.41) and (3.44) and have a contradiction. This proves that the sequence \( \{u_n\}_{n \geq 1} \subseteq W^{1, p(z)}_0(\Omega) \) is bounded. Recall that

\[ A_{p(z)}(u_n) + A_{q(z)}(u_n) = \lambda_n N_f(u_n) \quad \text{in } W^{1, p(z)}_0(\Omega)^* \quad \forall n \in \mathbb{N}, \]

with \( N_f(u_n)(\cdot) = f(\cdot, u_n(\cdot)) \) (the Nemytskii map corresponding to \( f \)). From Proposition 2.2, it follows that

\[ \lambda_n \|N_f(u_n)\|_{\ast} \leq c_{10} \quad \forall n \in \mathbb{N} \]

for some \( c_{10} > 0 \). Since \( u_n \geq u_1 \in \text{int } C_r \), on account of hypothesis \( H_1(iv) \) and since \( \lambda_n \to +\infty \), we have

\[ \lambda_n \|N_f(u_n)\|_{\ast} \to +\infty, \]

a contradiction. This proves that \( \lambda^* < +\infty \). \( \square \)

According to Proposition 3.9, we have

\[ (0, \lambda^*) \subseteq \mathcal{L} \subseteq (0, \lambda^*]. \]

**Proposition 3.10.** If hypotheses \( H_0, H_1 \) hold and \( \lambda \in (0, \lambda^*) \), then problem \( (P_0) \) has at least two positive solutions

\[ u_0, \hat{u} \in \text{int } C_r, \quad u_0 \leq \hat{u}, \quad u_0 \neq \hat{u}. \]

**Proof.** Let \( \lambda, \theta \in (0, \lambda^*), \lambda < \theta \). We have \( \lambda, \theta \in \mathcal{L} \). We can find \( u_0 \in S_{\theta} \subseteq \text{int } C_r \) and \( u_0 \in S_{\lambda} \subseteq \text{int } C_r \) such that

\[ u_0 - u_0 \in \text{int } C_r \]
We introduce the Carathéodory function \( g(z, x) \) defined by
\[
\begin{aligned}
g(z, x) &= \begin{cases} 
f(z, u_0(z)) & \text{if } x \leq u_0(z), \\
f(z, x) & \text{if } u_0(z) < x.
\end{cases}
\end{aligned}
\] (3.45)

We set
\[
G(z, x) = \int_0^x g(z, s) \, ds
\]
and consider the \( C^1 \)-functional \( \psi_A : W^{1,p(z)}(\Omega) \to \mathbb{R} \) defined by
\[
\psi_A(u) = \int_\Omega \frac{1}{p(z)} |Du|^{p(z)} \, dz + \int_\Omega \frac{1}{q(z)} |Du|^{q(z)} \, dz - \lambda \int_\Omega G(z, u) \, dz \quad \forall u \in W^{1,p(z)}(\Omega).
\]

Using (3.45) and the anisotropic regularity theory, we obtain
\[
K_{\psi_0} \subseteq [u_0] \cap \text{int } C_+.
\] (3.46)

We introduce the following truncation of \( g(z, \cdot) \)
\[
\bar{g}(z, x) = \begin{cases} 
g(z, x) & \text{if } x \leq u_0(z), \\
g(z, u_0(z)) & \text{if } u_0(z) < x.
\end{cases}
\] (3.47)

This is a Carathéodory function. We set
\[
\bar{G}(z, x) = \int_0^x \bar{g}(z, s) \, ds
\]
and consider the \( C^1 \)-functional \( \hat{\psi}_A : W^{1,p(z)}(\Omega) \to \mathbb{R} \) defined by
\[
\hat{\psi}_A(u) = \int_\Omega \frac{1}{p(z)} |Du|^{p(z)} \, dz + \int_\Omega \frac{1}{q(z)} |Du|^{q(z)} \, dz - \lambda \int_\Omega \bar{G}(z, u) \, dz \quad \forall u \in W^{1,p(z)}(\Omega).
\]

For this functional, we have that
\[
K_{\hat{\psi}_0} \subseteq [u_0, u_0] \cap \text{int } C_+.
\] (3.48)

We may assume that
\[
K_{\hat{\psi}_0} \cap [u_0, u_0] = [u_0].
\] (3.49)

Otherwise, on account of (3.46) and (3.45), we see that we already have a second positive smooth solution bigger than \( u_0 \) and so we are done.

The functional \( \hat{\psi}_0 \) is coercive (see Proposition 2.1 and (3.47)). Also, it is sequentially weakly lower semicontinuous. So, we can find \( \hat{u}_0 \in W^{1,p(z)}_0(\Omega) \) such that
\[
\hat{\psi}_A(\hat{u}_0) = \min_{u \in W^{1,p(z)}_0(\Omega)} \hat{\psi}_A(u),
\]
so \( \hat{u}_0 \in K_{\hat{\psi}_0} \subseteq [u_0, u_0] \cap \text{int } C_+ \) (see (3.33)).

Note that
\[
\hat{\psi}_A^{\tau}[u_0, u_0] = \hat{\psi}_A^{\tau}[u_0, u_0]
\]
(see (3.45) and (3.47)). So, it follows that \( \hat{u}_0 = u_0 \) (see (3.49)).

Since \( u_0 - u_0 \in \text{int } C_+ \), we see that \( u_0 \) is a local \( C_1^0(\Omega) \)-minimizer of \( \psi_A \).
\( u_0 \) is a local \( W^{1,p(z)}_0(\Omega) \)-minimizer of \( \psi_\lambda \) (3.50)

(see Gasiński-Papageorgiou [6] and Tan-Fang [30]).

From (3.46), we see that we may assume that \( K_{\psi_\lambda} \) is finite (otherwise we already have infinity of positive smooth solutions bigger than \( u_0 \) and so we are done). Then on account of (3.50) and using Theorem 5.7.6 of Papageorgiou-Rădulescu-Repovš [33, p. 449], we can find \( q \in (0, 1) \) small such that

\[
\psi_\lambda(u_0) < \inf \{ \psi_\lambda(u) : \|u - u_0\| = 0 \} = m_1. \tag{3.51}
\]

Also, if \( u \in \text{int} \, C_\epsilon \), then from (3.45) and hypothesis \( H(i) \) we have that

\[
\psi_\lambda(tu) \to -\infty \quad \text{as} \quad t \to +\infty. \tag{3.52}
\]

**Claim.** \( \psi_\lambda \) satisfies the Cerami condition.

Consider a sequence \( \{u_n\}_{n \geq 1} \subseteq W^{1,p(z)}_0(\Omega) \) such that

\[
|\psi_\lambda(u_n)| \leq c_1 \quad \forall n \in \mathbb{N}, \tag{3.53}
\]

for some \( c_1 > 0 \), so

\[
(1 + \|u_n\|) \psi_\lambda'(u_n) \to 0 \quad \text{in} \quad W^{1,p(z)}_0(\Omega)^* \quad \text{as} \quad n \to +\infty. \tag{3.54}
\]

From (3.54), we have

\[
\left| \langle A_{p(z)}(u_n), h \rangle + \langle A_{q(z)}(u_n), h \rangle - \lambda \int_\Omega g(z, u_n)hdz \right| \leq \frac{\varepsilon_n \|h\|}{1 + \|u_n\|} \quad \forall h \in W^{1,p(z)}_0(\Omega), \tag{3.55}
\]

with \( \varepsilon_n \to 0^+ \). In (3.55), we use \( h = -u_n^- \in W^{1,p(z)}_0(\Omega) \) and obtain

\[
q_p(Du_n^-) + q_q(Du_n^-) \leq c_2 \quad \forall n \in \mathbb{N},
\]

for some \( c_2 > 0 \) (see (3.45)), so

the sequence \( \{u_n^-\}_{n \geq 1} \subseteq W^{1,p(z)}_0(\Omega) \) is bounded (3.56)

(see Proposition 2.1).

Next in (3.55) we choose \( h = u_n^+ \in W^{1,p(z)}_0(\Omega) \). Then

\[
-q_p(Du_n^+) - q_q(Du_n^+) + \lambda \int_\Omega g(z, u_n^+)u_n^+dz \leq \varepsilon_n \quad \forall n \in \mathbb{N},
\]

so

\[
-q_p(Du_n^+) - q_q(Du_n^+) + \lambda \int_\Omega f(z, u_n^+)u_n^+dz \leq c_3 \quad \forall n \in \mathbb{N}, \tag{3.57}
\]

for some \( c_3 > 0 \).

From (3.53), (3.56) and (3.45), we have

\[
q_p(Du_n^+) + q_q(Du_n^+) - \lambda \int_\Omega p, F(z, u_n^+)dz \leq c_4 \quad \forall n \in \mathbb{N}, \tag{3.58}
\]

for some \( c_4 > 0 \).

We add (3.57) and (3.58) and obtain

\[
\lambda \int_\Omega \sigma(z, u_n^+)dz \leq c_5 \quad \forall n \in \mathbb{N}, \tag{3.59}
\]

for some \( c_5 > 0 \).
Using (3.59) and reasoning as in the proof of Proposition 3.9 (see the part of the proof after (3.31) up to (3.44)), we obtain that

\[ \text{the sequence } \{u_n^*\} \subseteq W_0^{1,p(z)}(\Omega) \text{ is bounded.} \]  

(3.60)

Then (3.50) and (3.60) imply that

\[ \text{the sequence } \{u_n\} \subseteq W_0^{1,p(z)}(\Omega) \text{ is bounded.} \]

So, we may assume that

\[ u_n \overset{w}{\rightharpoonup} u \text{ in } W_0^{1,p(z)}(\Omega) \text{ and } u_n \to u \text{ in } L'(\Omega) \text{ as } n \to +\infty. \]  

(3.61)

In (3.55), we test with \( h = u_n - u \in W_0^{1,p(z)}(\Omega) \) and pass to the limit as \( n \to +\infty \). As in the proof of Proposition 3.7, we obtain

\[ u_n \to u \text{ in } W_0^{1,p(z)}(\Omega) \text{ as } n \to +\infty \]

(see (3.24)), so \( \psi_1 \) satisfies the Cerami condition. This proves the Claim.

Then (3.51), (3.52) and the Claim permit the use of the mountain pass theorem and find \( \hat{u} \in W_0^{1,p(z)}(\Omega) \) such that

\[ \hat{u} \in K_{\psi_1} \subseteq [u_0] \cap \text{int } C, \text{ and } m_\lambda \leq \psi_1(\hat{u}) \]  

(3.62)

(see (3.46) and (3.51)).

From (3.62), (3.51) and (3.45), we conclude that \( \hat{u} \in \text{int } C \) is a positive solution of \((P)\), \( u_0 \leq \hat{u} \), \( u_0 \neq \hat{u} \).

\[ \square \]

It remains to decide what happens with critical parameter value \( \lambda^* < +\infty \).

**Proposition 3.11.** If hypotheses \( H_0, H_1 \) hold, then \( \lambda^* \in \mathcal{L} \).

**Proof.** Let \( \lambda_n \in (0, \lambda^*) \), \( n \in \mathbb{N} \) be such that \( \lambda_n \nearrow \lambda^* \). We can find \( u_n \in S_{\lambda_n} \subseteq \text{int } C \), nondecreasing such that

\[ \varphi_{\lambda_n}(u_n) \leq \alpha_{16} \quad \forall n \in \mathbb{N}, \]  

(3.63)

for some \( \alpha_{16} > 0 \), so

\[ \varphi_{\lambda_n}'(u_n) = 0 \quad \forall n \in \mathbb{N}. \]  

(3.64)

Using (3.63), (3.64) as in the proof of Proposition 3.9, first we obtain that the sequence \( \{u_n\}_{n=1}^\infty \subseteq W_0^{1,p(z)}(\Omega) \) is bounded and then via Proposition 2.2, at least for a subsequence, we have

\[ u_n \to u^* \text{ in } W_0^{1,p(z)}(\Omega). \]  

(3.65)

From (3.64) and (3.65), in the limit as \( n \to +\infty \), we obtain

\[ \langle A_{p(z)}(u^*), h \rangle + \langle A_{q(z)}(u^*), h \rangle = \lambda^* \int_\Omega f(z, u^*) h \, dz \quad \forall h \in W_0^{1,p(z)}(\Omega), \]

so \( u_1 \leq u^* \). Therefore, \( u^* \in S_{\lambda^*} \subseteq \text{int } C \) and so \( \lambda^* \in \mathcal{L} \).

\[ \square \]

We conclude that

\[ \mathcal{L} = (0, \lambda^*]. \]

So, summarizing our findings for problem \((P)\), we can state the following bifurcation-type theorem.
Theorem 3.12. If hypotheses $H_0$, $H_1$ hold, then there exists $\lambda^* > 0$ such that

(a) for all $\lambda \in (0, \lambda^*)$ problem $(P_\lambda)$ has at least two positive solutions

$$u_0, \tilde{u} \in \text{int } C_\lambda, u_0 \leq \tilde{u}, u_0 \neq \tilde{u};$$

(b) for $\lambda = \lambda^*$ problem $(P_\lambda)$ has at least one positive solution

$$u^* \in \text{int } C_{\lambda^*};$$

(c) for all $\lambda > \lambda^*$ problem $(P_\lambda)$ has no positive solutions;

(d) for all $\lambda \in (0, \lambda^*)$ problem $(P_\lambda)$ has a smallest (minimal) positive solution $u^*_\lambda \in \text{int } C_{\lambda^*}$ and the map $\lambda \mapsto u^*_\lambda$ from $\mathcal{L} = (0, \lambda^*)$ into $C^{\lambda}_{\lambda^*}(\Omega)$ is strictly increasing and left continuous.

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