SINGULAR LEVI-FLAT HYPERSURFACES IN COMPLEX PROJECTIVE SPACE INDUCED BY CURVES IN THE GRASSMANNIAN

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Abstract. Let \( H \subset \mathbb{P}^n \) be a real-analytic subvariety of codimension one induced by a real-analytic curve in the Grassmannian \( G(n+1,n) \). We show such an \( H \) is Levi-flat, the closure of its smooth points is a union of complex hyperplanes, and its singular set is either of dimension \( 2n - 2 \) or dimension \( 2n - 4 \). If the singular set is of dimension \( 2n - 4 \), then we show the hypersurface is algebraic and the Levi-foliation extends to a singular holomorphic foliation of \( \mathbb{P}^n \) with a meromorphic (rational of degree 1) first integral. In fact, in this case, \( H \) is in some sense simply a complex cone over an algebraic curve in \( \mathbb{P}^1 \). Similarly if \( H \) has a degenerate singularity, then \( H \) is also algebraic. If the dimension of the singular set is \( 2n - 2 \) and is nondegenerate, we show by construction that the hypersurface need not be algebraic nor semialgebraic. In doing so, we construct a Levi-flat real-analytic subvariety in \( \mathbb{P}^2 \) of real codimension 1 with compact leaves that is not contained in any proper real-algebraic subvariety of \( \mathbb{P}^2 \). Therefore a straightforward analogue of Chow’s theorem for Levi-flat hypersurfaces does not hold.

1. Introduction

Let \( \mathbb{P}^n \) be the \( n \)-dimensional complex projective space. Chow’s Theorem [6] tells us that any complex analytic subvariety of \( \mathbb{P}^n \) is algebraic. We naturally ask if Chow’s Theorem extends to singular Levi-flat hypersurfaces. A Levi-flat hypersurface is, after all, essentially a family of complex submanifolds. In \( \mathbb{P}^1 \) the answer is trivially no. In \( \mathbb{P}^n, n \geq 2 \), the question is significantly more difficult. To this end we study real-analytic subvarieties induced by a curve in the Grassmannian \( G(n+1,n) \), possibly the simplest examples of singular Levi-flat hypersurfaces in \( \mathbb{P}^n \). We show that while for certain hypersurfaces of this type Chow’s theorem holds, it does not hold in general.

To be precise, a singular Levi-flat hypersurface of \( \mathbb{P}^n \) is a closed real-analytic subvariety \( H \subset \mathbb{P}^n \) of real codimension 1 that is Levi-flat at every smooth point of top dimension: At every smooth point of top dimension there exist local holomorphic coordinates \( z \) such that \( H \) is given by

\[
\text{Im } z_1 = 0.
\]  
(1)

For any fixed \( t \in \mathbb{R} \), \( z_1 = t \) defines a complex hypersurface, giving a foliation at the smooth points of \( H \), called the Levi-foliation. We say a leaf of this foliation is compact if its closure in \( \mathbb{P}^n \) is compact, and we consider its closure to be the leaf itself. A compact leaf is a complex analytic subvariety of \( \mathbb{P}^n \), and hence by Chow’s theorem algebraic.

Singular Levi-flat hypersurfaces (subvarieties) have been the subject of much recent interest. Such hypersurfaces were first studied by Bedford [1] and Burns and Gong [4]. The problem of studying the singular set is connected to the problem of extending the Levi-foliation...
to a singular holomorphic foliation of a neighbourhood or more generally to a holomorphic k-web, see Bruneau [2]3, Fernández-Pérez [7], Cerveau and Lins Neto [3], Shafikov and Sukhov [13], and the author [11]. In projective space, these hypersurfaces have been studied by the author [10], Fernández-Pérez [8], and Fernández-Pérez, Mol and Rosas [9]. A related question is the nonexistence of smooth Levi-flat hypersurfaces in \(\mathbb{P}^n\), a line of research that began with the paper of Lins Neto [12] in the real-analytic case for \(n \geq 3\). The technique of Lins Neto centers on extending the foliation to all of \(\mathbb{P}^n\).

The first instinct to construct a singular Levi-flat hypersurface in \(\mathbb{P}^n\) is to take a real one dimensional curve in the Grassmannian \(G(n+1,n)\), which is the set of complex hyperplanes in \(\mathbb{P}^n\). If the induced set is a subvariety, we have a Levi-flat hypersurface. One might think it should be trivial to take a nonalgebraic real curve and obtain a nonalgebraic hypersurface \(H \subset \mathbb{P}^n\), but in general the induced set \(H\) fails to be a subvariety. In general \(H\) fails to be even a semianalytic set. We will construct such an example in \(\mathbb{P}^2\). But first, we will characterize those hypersurfaces induced by a curve in the Grassmannian.

We say a real subvariety \(H \subset \mathbb{P}^n\) is induced by a real-analytic curve in \(G(n+1,n)\) if there exists a one-dimensional (possibly not closed) real-analytic curve in \(G(n+1,n)\) such that all the corresponding planes are subsets of \(H\), and such that \(H\) is the smallest such subvariety.

Define \(H_s\) to be the set of singular points of \(H\), that is the points where \(H\) fails to be a real-analytic submanifold. Define \(H^*\) to be the set of points near which \(H\) is a real-analytic submanifold of dimension \(2n-1\). Note that the topological closure \(\overline{H^*}\) can be strictly smaller than \(H\), see the example below.

**Theorem 1.1.** Suppose \(H \subset \mathbb{P}^n\) is an irreducible real-analytic subvariety of real codimension one induced by a real-analytic curve in \(G(n+1,n)\). Then

(i) \(H\) is Levi-flat,
(ii) \(\overline{H^*}\) is a union of complex hyperplanes,
(iii) \(\dim H_s = 2n-2\) or \(\dim H_s = 2n-4\).

If \(\dim H_s = 2n-4\), then there exist homogeneous coordinates \([z_0, \ldots, z_n]\) such that

(i) the set \(H_s\) is given by \(z_1 = z_2 = 0\),
(ii) \(H\) is defined by a real bihomogeneous polynomial equation \(\rho(z_1, z_2, \bar{z}_1, \bar{z}_2) = 0\),
(iii) the holomorphic foliation given by the one-form \(z_2dz_1 - z_1dz_2\) extends the Levi-foliation of \(H^*\).

The function \(\frac{z_2}{z_1}\) is the rational first integral of the foliation given by \(z_2dz_1 - z_1dz_2\). In this case, in the \((z_1, z_2)\) space, \(H\) is simply a cone over a nonsingular algebraic curve. By a bihomogeneous polynomial (of bidegree \((j, k))\), we mean a polynomial \(p\) such that \(p(tz, \bar{t}z) = t^j\bar{t}^k p(z, \bar{z})\) for all \(t \in \mathbb{C}\). Note that homogeneous coordinates on \(\mathbb{P}^n\) are given up to an automorphism of \(\mathbb{P}^n\), i.e. an invertible linear map of \(\mathbb{C}^{n+1}\).

A point \(p \in H_s\) is degenerate if infinitely many leaves of the Levi-foliation of \(H\) pass through \(p\). Notice that if \(\dim H_s = 2n-4\), then every point in \(H_s\) is degenerate. It turns out a degenerate singularity always arises from a nondegenerate algebraic Levi-flat in lower dimension, where we consider a real curve in \(\mathbb{P}^1\) to be Levi-flat by definition. Clearly a curve in \(\mathbb{P}^1\) has no degenerate singularities.

**Theorem 1.2.** Suppose \(H \subset \mathbb{P}^n\) is an irreducible real-analytic subvariety of real codimension one induced by a real-analytic curve in \(G(n+1,n)\), further suppose \(H_s\) contains a degenerate singular point. Then there exists an integer \(1 \leq \ell \leq n\), homogeneous coordinates \([z_0, \ldots, z_n]\), and a bihomogeneous real polynomial \(\rho(z_0, \ldots, z_\ell, \bar{z}_0, \ldots, \bar{z}_\ell)\), such that
(i) $H$ is the zero set of $\rho$ (in particular, $H$ is algebraic),
(ii) $\rho$ defines a Levi-flat algebraic hypersurface $X \subset \mathbb{P}^{\ell}$ with no degenerate singularities
induced by a curve in $G(\ell + 1, \ell)$.

Theorem 1.2 implies that a nonalgebraic $H$ has no degenerate singularities.

Example 1.3 (Whitney umbrella with complex lines). Let us discuss a crucial algebraic
example in $\mathbb{C}^2$ (and hence in $\mathbb{P}^2$) showing, among other things, that $H$ itself may not be a
union of complex hyperplanes, and that the Levi-foliation need not extend. Let $(z, w) \in \mathbb{C}^2$
be our coordinates and write $z = x + iy$ and $w = s + it$. Consider the hypersurface $H$ given by
\[ sy^2 - txy - t^2 = 0. \] 
(2)
The hypersurface is Levi-flat with the nonsingular hypersurface part foliated by the complex
lines given by $w = cz + c^2$, where $c \in \mathbb{R}$. In fact, setting $z = \xi$ and $w = c\xi + c^2$ gives a
$H^*$ as an image of $\mathbb{C} \times \mathbb{R}$. This irreducible hypersurface has a lower-dimensional totally-real
component given by $\{t = 0, y = 0\}$, “half of it” (for $s$ sufficiently negative) sticking out of the
hypersurface as the “umbrella handle.” In the complement of $\{y = 0\}$, the hypersurface $H$
can be given by the graph as $s = \frac{txy + t^2}{y^2}$ or $s = x\left(\frac{t}{y}\right) + \left(\frac{t}{y}\right)^2$. The set $\{t = 0, y = 0, s < \frac{-x^2}{4}\}$
is not in the closure of $H^*$. Therefore, not all of $H$ is a union of complex lines; only the
closure $\overline{H^*}$ is such a union of lines, as is claimed in Theorem 1.1. Notice that when $x = 0$, we obtain the standard Whitney umbrella in $\mathbb{R}^3$ [14].

Figure 1. The Whitney Umbrella in $\mathbb{R}^3$.

The singular set is precisely $\{t = 0, y = 0, s \geq \frac{-x^2}{4}\}$, a set of real dimension $2n - 2 = 2$.
The singular set is a generic manifold at points arbitrarily near the origin, and so the Levi-
foliation does not extend as a singular holomorphic foliation to a neighbourhood of the origin.
The Levi-foliation does extend as a holomorphic 2-web given by $dw^2 + z\, dz\, dw - w\, dz^2 = 0$.
Its multivalued first integral is obtained by applying the quadratic formula to $w = cz + c^2$.
See [2][13].

See [10] for an example of an umbrella-type subvariety composed of complex lines where
the “umbrella handle” is a complex line.

Example 1.4. A simple but illustrative example to consider is the hypersurface $H$ in $\mathbb{P}^2$
given in homogeneous coordinates $[z_0, z_1, z_2]$ as
\[ |z_1|^2 = |z_2|^2. \] 
(3)
The singularity is the set \( \{ z_1 = z_2 = 0 \} \); that is, a single point in \( \mathbb{P}^2 \), which is a degenerate singularity. The singular holomorphic foliation extending the Levi-foliation is given by the one-form \( z_2 dz_1 - z_1 dz_2 \) as claimed, and \( H \) is the cone over a circle in \( \mathbb{P}^1 \).

One of the primary motivations of this paper is to find a nonalgebraic singular Levi-flat hypersurface in projective space. The author proved an analogue of Chow’s theorem for Levi-flat hypersurfaces under extra hypotheses in [10]: A Levi-flat subvariety whose Levi-foliation extends as a singular holomorphic foliation at each point and has infinitely many compact leaves is contained in a real-algebraic subvariety. It should be noted that a real-algebraic Levi-flat automatically has compact leaves.

**Theorem 1.5.** There exists a real-analytic singular Levi-flat hypersurface \( H \subset \mathbb{P}^2 \) with all leaves complex hyperplanes, such that \( H \) is not contained in any proper real-algebraic subvariety of \( \mathbb{P}^2 \).

Therefore, Chow’s theorem does not hold for Levi-flat hypersurfaces without extra assumptions. Also, the hypotheses in [10] are not unnecessary. In particular, the nonalgebraic hypersurface we construct has compact leaves, but the Levi-foliation does not extend at certain points of its singular set.

We also construct an example of a real-analytic singular Levi-flat hypersurface \( H \) that is semialgebraic, but not algebraic. That is, we construct an irreducible algebraic Levi-flat hypersurface globally reducible into two components as a real-analytic subvariety.

The author would like to acknowledge Xianghong Gong, who originally asked him about existence of nonalgebraic Levi-flats several years ago. This paper, I hope, gives a satisfactory answer. The author would also like to thank Arturo Fernández-Pérez for many discussions on the topic, and several very useful suggestions and corrections to the manuscript.

2. **Real-analytic subvarieties induced by curves in the Grassmannian**

A real-analytic subvariety \( H \subset \mathbb{P}^n \) is said to be *algebraic* if it is the zero set of a bihomogeneous polynomial. A set is said to be *semialgebraic*, if it is defined by a finite set of algebraic equalities and inequalities. A real-analytic subvariety can be semialgebraic, but not algebraic; a real-analytic algebraic subvariety can be irreducible as an algebraic subvariety, but reducible as a real-analytic subvariety.

Let us fix some notation. Let \( \sigma : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n \) be the natural projection. If \( X \) is a real-analytic subvariety of \( \mathbb{P}^n \), then let \( \tau(X) \subset \mathbb{C}^{n+1} \) be the set of points \( z \in \mathbb{C}^{n+1} \) such that \( \sigma(z) \in X \) or \( z = 0 \). We say \( X \subset \mathbb{C}^{n+1} \) is a complex cone if \( z \in X \) implies \( \lambda z \in X \) for all \( \lambda \in \mathbb{C} \). If \( X \) is a complex cone we write \( \sigma(X) \) for the induced set in \( \mathbb{P}^n \).

While \( X \subset \mathbb{P}^n \) may be a real subvariety, the set \( \tau(X) \) need not be a subvariety. This is precisely why Chow’s theorem fails in the real-analytic case. It is straightforward to see, via the standard proof of Chow’s theorem (see e.g. [10]), that if \( \tau(X) \) is a variety, then \( X \) is algebraic. If \( \tau(X) \) is semianalytic (in particular contained in a real subvariety of the same dimension), then \( X \) is at least semialgebraic.

Let us start with the more straightforward results claimed. First, we have the following special case of a well-known and easy observation.

**Proposition 2.1.** Suppose \( H \subset \mathbb{P}^n \) is an irreducible real-analytic subvariety of real codimension one induced by a real-analytic curve in \( G(n + 1, n) \). Then \( 2n - 4 \leq \dim H_s \leq 2n - 2 \).
Proof. A smooth Levi-flat hypersurface has through every point a unique non-singular complex hypersurface as a submanifold. Therefore the points where two leaves meet must be in the singularity. Any two complex hyperplanes in \( \mathbb{P}^n \) meet along a set of complex dimension \( n-2 \) or real dimension \( 2n-4 \). \( \square \)

The first two claims of Theorem 1.1 follow from the following lemma.

**Lemma 2.2.** Suppose \( H \subset \mathbb{P}^n \) is an irreducible real-analytic subvariety of real codimension 1, induced by a real-analytic curve in \( G(n+1, n) \). Then \( \overline{H^*} \) is a union of complex hyperplanes.

**Proof.** By assumption there is at least some point in \( H^* \) near which \( H \) is locally a union of complex hyperplanes.

Let us prove a local version first. Suppose \( H \) is locally irreducible at \( p \in H \), and without loss of generality assume \( p = 0 \). Let us also suppose for now that \( H \) is only defined near 0. And suppose there is some \( q \in H^* \) near which \( H \) is locally a union of complex hyperplanes.

Suppose \( H \) is defined by a real analytic function \( r(z, \bar{z}) \) near 0. The function \( r \) complexifies to an irreducible \( r(z, w) \), holomorphic in a neighbourhood of the origin in \( \mathbb{C}^n \times \mathbb{C}^n \). The equation \( r(z, w) = 0 \) defines a complex subvariety \( H \) of codimension 1 in \( 2n \) dimensions irreducible at the origin.

Let \( (H^*)' = \{ z \in H^* : dr(z) \neq 0 \} \). The set \( (H^*)' \) is open and connected in \( H^* \). For \( q \in (H^*)' \), the real form \( \theta = i(\partial r - \bar{\partial} r) \) defines the space \( T_q^{(1,0)} H \oplus T_q^{(0,1)} H \). This form complexifies and also defines a (complex) codimension 1 subspace \( T_q \) on the tangent space of \( H \) at smooth points.

The Hessian (the full real Hessian) \( Hr \) of \( r \) vanishes on \( T_q^{(1,0)} H \oplus T_q^{(0,1)} H \) for some point \( q \in (H^*)' \) where \( H \) is a union of complex hyperplanes. \( Hr \) also complexifies (the complexification is the full Hessian of the complexified \( r \)). The complexified \( Hr \) is identically zero on the vectors in \( T \) at all points in \( H \) near \( (q, \bar{q}) \). By identity \( Hr \) vanishes on all the vectors in \( T \) at all the smooth points of \( H \).

Then at all points \( q \) of \( (H^*)' \), the Hessian \( Hr \) also vanishes on \( T_q^{(1,0)} H \oplus T_q^{(0,1)} H \). As \( (H^*)' \) is dense in \( H^* \), we obtain the conclusion at all points of \( H^* \). That means the leaves near all points of \( H^* \) are complex hyperplanes. The limit set of complex planes is a union of complex planes, and hence \( \overline{H^*} \) is a union of complex planes.

The global result now follows as any two points in a global subvariety \( H \subset \mathbb{P}^n \) are connected via a finite sequence of locally irreducible subvarieties (as \( H \) is globally irreducible and compact). \( \square \)

We end the section by noting that while not every real-analytic curve in \( G(n+1, n) \) induces a real subvariety in \( \mathbb{P}^n \), the converse does hold.

**Proposition 2.3.** Suppose \( X \subset \mathbb{P}^n \) is a real-analytic subvariety. Then the set \( C \subset G(n+1, n) \) of complex hyperplanes contained in \( X \) is a real-analytic subvariety of \( G(n+1, n) \).

**Proof.** Suppose \( W \) is a hyperplane in \( G(n+1, n) \) passing through a point \( p \in X \). Without loss of generality, let \( p = 0 \) in some coordinate patch on \( \mathbb{P}^n \), and suppose \( W \) is given by \( z_1 = 0 \). The hyperplanes in \( G(n+1, n) \) in a neighbourhood of \( W \) are given by the equation

\[
z_1 = a_0 + a_2z_2 + \cdots + a_nz_n
\]

for \( a_j \) small. The corresponding hyperplane is contained in \( X \) if as a function of \( z_2, \ldots, z_n \),

\[
\rho(a_0 + a_2z_2 + \cdots + a_nz_n, z_2, \ldots, z_n) \equiv 0
\]
for all real-analytic \( \rho \) that vanish on \( X \) near \( p \). Expanding as a series in \( z_2, \ldots, z_n \), and setting its coefficients to 0, we obtain analytic equations for \( a_0, a_2, \ldots, a_n \), and the result follows.

The proposition implies we can assume the curve \( C \) inducing \( H \) is a closed subvariety, and furthermore we can assume \( t \) \( C \) gives all the complex hyperplanes in \( H \). We will assume so from now on.

3. Degenerate singularities

To prove Theorem 1.2, it is enough to prove the following lemma. The lemma will also be useful in the proof of Theorem 1.1.

**Lemma 3.1.** Suppose \( H \subset \mathbb{P}^n \) is an irreducible real-analytic subvariety of real codimension one induced by a real-analytic curve in \( G(n+1,n) \) and suppose \( H \) has a degenerate singularity. Then there exist homogeneous coordinates \([z_0, \ldots, z_n]\) such that \( H \) has an real polynomial defining function depending on \( z_0 \) through \( z_{n-1} \) only.

To obtain Theorem 1.2 that is, to obtain an \( X \) with no nondegenerate singularities, we simply repeatedly apply the lemma, noting a Levi-flat in \( \mathbb{P}^1 \) is simply a real curve and cannot have degenerate singularities. Then for the \( \ell \) in the theorem there exists an \( X \subset \mathbb{P}^\ell \) defined by the polynomial in \( z_0 \) through \( z_\ell \), such that \( \tau(H) = \tau(X) \times \mathbb{C}^{n-\ell} \). To prove the lemma, it is enough to show that in inhomogeneous coordinates where the origin is the degenerate singularity, \( H \) is a complex cone defined by a bihomogeneous polynomial, since then we have eliminated one variable from the defining equation and we have shown algebraicity.

**Proof.** Let us work in inhomogeneous coordinates defined by \( z_0 = 1 \), and suppose the degenerate point is the origin. We proceed as in the standard proof of Chow’s theorem. Let \( \rho \) be a real-analytic defining function for \( H \) near the origin. Decompose \( \rho \) into bihomogeneous components as

\[ \rho(z, \bar{z}) = \sum_{jk} \rho_{jk}(z, \bar{z}). \]  

(6)

As all the leaves of \( H \) are complex hyperplanes, then infinitely many of these hyperplanes pass through the origin. The smallest real subvariety \( \tilde{H} \) defined in a neighbourhood of the origin containing all these complex hyperplanes (again only in a neighbourhood of the origin) is of real codimension 1, and therefore it is one (or more) of the irreducible components of \( H \) at the origin. Remember, \( H \) could a priori be locally reducible at the origin.

Let \( V \) be the union of the hyperplanes through the origin contained in \( \tilde{H} \) near the origin. Again if \( z \) in \( V \), then \( tz \in V \) for all \( t \in \mathbb{C} \), and so

\[ 0 = \rho(tz, t\bar{z}) = \sum_{jk} t^j \bar{t}^k \rho_{jk}(z, \bar{z}). \]  

(7)

So for each \( j \) and \( k \), \( \rho_{jk}(z, \bar{z}) = 0 \) for \( z \in V \). As \( \tilde{H} \) is the smallest real subvariety containing \( V \) near the origin, we must have \( \rho_{jk}(z, \bar{z}) = 0 \) for \( z \in \tilde{H} \). As \( \rho \) is a defining function for \( H \), then \( \tilde{H} \) must be equal to \( H \) near the origin, and \( H \) is given by polynomial equations, so \( H \) is algebraic. The defining equations are bihomogeneous, and therefore if \( z \in H \), then \( tz \in H \) for all \( t \in \mathbb{C} \), in other words \( H \) is a cone. As these equations are polynomials and \( H \) is a cone, they define \( H \) exactly, globally on \( \mathbb{P}^n \). As \( H \) is a real hypersurface, only one real bihomogeneous polynomial is needed.
Finally the lemma follows as $H$ is a complex cone over the origin and hence defines an $X \subset \mathbb{P}^{n-1}$. Then in the right coordinates $\tau(H) = \tau(X) \times \mathbb{C}$. The lemma follows. \qed

We remark the lemma implies that to study hypersurfaces induced by curves in the Grassmannian, it is enough to study those that have no degenerate singularities.

4. Extending the foliation when the singularity is small

**Lemma 4.1.** Let $C \subset G(n+1, n)$, $n \geq 2$, be an irreducible real-analytic subvariety of real dimension one. Let $S \subset \mathbb{P}^n$ be the set of points where two or more complex hyperplanes in $C$ intersect. If $\dim_{\mathbb{R}} S \leq 2n - 3$, then after an automorphism of $\mathbb{P}^n$, every hyperplane in $C$ is given by an equation of the form

$$0 = a_1z_1 + a_2z_2,$$

and the set $S$ is given by the equations $z_1 = z_2 = 0$. In particular, $\dim_{\mathbb{R}} S = 2n - 4$.

**Proof.** Let $z_0, z_1, \ldots, z_n$ be the homogeneous coordinates on $\mathbb{P}^n$. An automorphism of $\mathbb{P}^n$ is an invertible linear transformation in $n + 1$ variables. Pick two distinct points on $C$, we will parametrize the curve $C$ near these two points. We will work in inhomogeneous coordinates given by $z_0 = 1$. Without loss of generality, after a change of coordinates, we can assume the two hyperplanes are given by $z_1 = 0$ and $z_2 = 0$. Our parametrization of hyperplanes in $C$ near these two points can be given as

$$z_1 = a_0(s) + a_2(s)z_2 + a_3(s)z_3 + \cdots + a_n(s)z_n,$$

$$z_2 = b_0(t) + b_1(t)z_1 + b_3(t)z_3 + \cdots + b_n(t)z_n.$$  \(8\) \(9\)

The functions $a_k$ and $b_k$ are real-analytic functions of one real variable defined near the origin and vanishing at 0.

The hypotheses of the lemma claim the intersection of these two families of planes is of dimension less than or equal to $2n - 3$. Furthermore we can assume that as a vector valued function, the $a$ and the $b$ is non-constant. That is, not all $a_k$ are identically zero, and similarly not all $b_k$ are identically zero. We will show, it must then be true that $a_0$ and $a_3, \ldots, a_n$ must be identically zero, and $b_0$ and $b_3, \ldots, b_n$ must be identically zero. That is, only $a_2$ and $b_1$ are allowed to vary (they must vary in fact). In this case, the lemma holds.

Let us suppose for contradiction the above assertion is not true. That is, after a linear change of variables in $z_0, z_3, \ldots, z_n$, we can assume $b_0$ and $a_0$ are not both identically zero. Furthermore, we can assume it is $a_0$ that is not identically zero. If $b_0$ is still identically zero, then $b_3$ through $b_n$ are identically zero as well and in that case $b_1$ is not identically zero.

Solving for $z_1$ and $z_2$, we obtain a piece of $S$ parametrized by $s, t$, and $z_3, \ldots, z_n$ as

$$z_1 = \frac{a_0(s) + a_2(s)(b_0(t) + b_3(t)z_3 + \cdots + b_n(t)z_n) + a_3(s)z_3 + \cdots + a_n(s)z_n}{1 - a_2(s)b_1(t)},$$

$$z_2 = \frac{b_0(t) + b_1(t)(a_0(s) + a_3(s)z_3 + \cdots + a_n(s)z_n) + b_3(t)z_3 + \cdots + b_n(t)z_n}{1 - a_2(s)b_1(t)}.$$  \(11\) \(12\)

where $z_3, \ldots, z_n$ simply parametrize themselves. As dimension of the image is less than or equal to $2n - 3$, it must be true when we set $z_3$ through $z_n$ to zero, the rank of the resulting
mapping (rank of the derivative)

\[ z_1 = \frac{a_0(s) + a_2(s)b_0(t)}{1 - a_2(s)b_1(t)}, \]
\[ z_2 = \frac{b_0(t) + b_1(t)a_0(s)}{1 - a_2(s)b_1(t)}, \]

must be of rank strictly less than 2 at all points. Clearly this is possible if both \( a_0 \) and \( b_0 \) are both identically 0, however, we supposed for contradiction \( a_0 \) is not identically zero.

If neither \( a_0 \) nor \( b_0 \) is identically zero, then the mapping is a finite map. That is, solving \( a_0(s) + a_2(s)b_0(t) = 0 \), and \( b_0(t) + b_1(t)a_0(s) = 0 \) for \( s \) and \( t \) near zero implies that both \( a_0 \) and \( b_0 \) must be zero. A finite map has generically full rank, that is 2, which contradicts our assumption on rank.

The final case to check is when \( b_0 \) is identically zero. The mapping becomes \( a_0(s) \)

and \( \frac{b_1(t)a_0(s)}{1 - a_2(s)b_1(t)} \), and that is easily seen to have derivative or rank 2 for generic points \( s, t \) near the origin if \( b_1 \) is not identically zero. By assumption, \( b_1 \) is not identically zero.

By irreducibility of \( C \) we find that if \( S \) is contained in all complex hyperplanes in some open set in \( C \), then \( S \) is contained in all complex hyperplanes in \( C \).

The following lemma proves the remainder of the claims in Theorem 1.1.

**Lemma 4.2.** Suppose \( H \subset \mathbb{P}^n \) is an irreducible real-analytic subvariety of real codimension one induced by a real-analytic curve in \( G(n+1, n) \) and suppose \( \dim H_s \leq 2n - 3 \). Then after an automorphism of \( \mathbb{P}^n \), there exist homogeneous coordinates \([z_0, \ldots, z_n]\) such that

1. \( H \) has a real polynomial defining function depending on \( z_1 \) and \( z_2 \) only.
2. The holomorphic foliation given by \( z_2dz_1 - z_1dz_2 \) extends the Levi-foliation of \( H^* \).
3. The set \( H_s \) is given by \( z_1 = z_2 = 0 \) and so in particular \( \dim \mathbb{R} H_s = 2n - 4 \).

**Proof.** Let \( C \subset G(n+1, n) \) be the subvariety defined by all complex hyperplanes in \( H \). Let \( C_1 \) be an irreducible one-dimensional component of \( C \). Use Lemma [4.1] to find the coordinates such that all hyperplanes in \( C_1 \) are given by linear functions of \( z_1 \) and \( z_2 \). In particular all the hyperplanes pass through the origin in the inhomogeneous coordinates given by \( z_0 = 1 \).

The origin is a degenerate singularity and therefore by Lemma [3.1] \( H \) is algebraic and defined by a bihomogeneous polynomial.

The foliation is given by the one-form \( z_2dz_1 - z_1dz_2 \), with the meromorphic first integral \( \frac{z_1}{z_2} \), as the leaves of this foliation are precisely complex hyperplanes given by linear functions of \( z_1 \) and \( z_2 \) only. This foliation then must extend the Levi-foliation of \( H \).

Let us now work in homogeneous coordinates \( z_0, \ldots, z_n \). The cone \( \tau(H) \) is an algebraic subvariety in \( \mathbb{C}^{n+1} \). By fixing \( z_0 \), and \( z_3, \ldots, z_n \) at a generic value we obtain a real polynomial in \( z_1 \) and \( z_2 \) that vanishes on a set of real dimension 3 in the \((z_1, z_2)\)-space. Since it must vanish on the hyperplanes in \( H \) it vanishes on \( H \). We hence obtain our defining polynomial depending on \( z_1 \) and \( z_2 \) only. Hence \( H \) is a complex cone over a curve in \( \mathbb{P}^1 \).

The singular set of \( H \) is therefore given either by the singular set of the foliation, which is precisely \( z_1 = z_2 = 0 \), or the singular set of the curve in \( \mathbb{P}^1 \). In either case the singular set is a complex subvariety, and hence of real dimension \( 2n - 2 \) or dimension \( 2n - 4 \). By assumption then, it must be of dimension \( 2n - 2 \). So the curve in \( \mathbb{P}^1 \) must be nonsingular and the singularity must be precisely the set \( \{z_1 = z_2 = 0\} \). □
5. The Construction of a Nonalgebraic Hypersurface

In this section we construct a real-analytic singular Levi-flat hypersurface $H \subset \mathbb{P}^2$ with all leaves compact, such that $H$ is not contained in any proper real-algebraic subvariety of $\mathbb{P}^2$. That is, the only bihomogeneous polynomial vanishing on all of $\tau(H)$ is identically zero. The $H$ we construct will be induced by a nonalgebraic real-analytic curve in $G(3, 2)$.

**Proof of Theorem 1.5.** Let $X \subset \mathbb{R}^2$ be a connected compact real-analytic curve not contained in any proper real-algebraic subvariety of $\mathbb{R}^2$. That is, any bihomogeneous polynomial $p(x, y)$ vanishing on $X$, vanishes identically. Suppose $a, b: \mathbb{R} \to X$ are two real valued real-analytic functions of a real variable $t$ such that the image of $(a(t), b(t))$ is all of $X$. We also assume $X$ has no singularities, and so we assume the derivatives $a'(t)$ and $b'(t)$ do not both vanish at any $t \in \mathbb{R}$.

Define a complex cone $\tilde{H}$ as

$$
\tilde{H} = \{(z_0, z_1, z_2) \in \mathbb{C}^3 : z_0 = z_1 x + z_2 y \text{ where } (x, y) \in X\} \cup \{z \in \mathbb{C}^3 : z_1 \bar{z}_2 = \bar{z}_1 z_2\}. \quad (15)
$$

Clearly $\tilde{H}$ is a complex cone and so $H = \sigma(\tilde{H})$ is a well-defined subset of $\mathbb{P}^2$. If $\tilde{H} \setminus \{0\}$ is a real-analytic subvariety of $\mathbb{C}^3 \setminus \{0\}$, then $H$ is a real-analytic subvariety of $\mathbb{P}^2$. Assuming $\tilde{H}$ is a real-analytic subvariety away from the origin then $H$ is Levi-flat, as $\tilde{H}$ is a union complex hyperplanes it is clearly Levi-flat. In fact, $H$ is induced by a nonsingular real-analytic curve in $G(3, 2)$.

It is therefore left to show $\tilde{H}$ is a subvariety away from the origin. We write

$$
\begin{bmatrix}
z_0 \\
z_0
\end{bmatrix} =
\begin{bmatrix}
1 & z_2 \\
\bar{z}_1 & \bar{z}_2
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}.
$$

If the matrix is invertible we solve for $x$ and $y$ in terms of $z$. Outside the set

$$
\Delta = \{z : z_1 \bar{z}_2 - \bar{z}_1 z_2 = 0\}, \quad (17)
$$

the set $\tilde{H}$ is an image of a real-analytic submanifold under a real-analytic diffeomorphism, and therefore a subvariety.

Let us take a point where $z_1 \bar{z}_2 - \bar{z}_1 z_2 = 0$. Call this point $p = (z_0^0, z_2^0)$. The complex values $z_1^0$ and $z_2^0$ lie on the same line through the origin. After moving all of $\mathbb{C}^3$ by a diagonal unitary we assume $(z_1^0, z_2^0)$ are real. As they are not both zero, we find a $2 \times 2$ real orthogonal matrix $U$ that takes the point $(1, 0)$ to $(z_1^0, z_2^0)$. Write $A(t) = (a(t), b(t))$. Then near $p$ and outside of $\Delta$, the set $\tilde{H}$ is the image of

$$
(\xi_1, \xi_2, t) \mapsto ((U \xi \cdot A(t)), U \xi) = ((\xi \cdot U^t A(t)), U \xi).
$$

Again by moving around (post-composing the above mapping) by an orthogonal mapping we assume $z_1^0 = 1$ and $z_2^0 = 0$.

We now work in non-homogeneous coordinates in $\mathbb{C}^2$ where we set $z_1 = 1$ and write $z_2 = \zeta$. The set $\Delta$ is the set where $\zeta$ is real valued. It is enough to consider the image of the map

$$
(\zeta, t) \mapsto (a(t) + \zeta b(t), \zeta)
$$

for $\zeta \in \mathbb{C}$ near the origin and $t \in \mathbb{R}$. The question is local and so it is enough to consider $t$ near the origin. We must show the image union $\Delta$ is a real-analytic subvariety.
We complexify by letting $\tilde{\zeta} = \omega$ and $\tilde{t} = s$, treating $t$ as a complex variable. As neither $a$ nor $b$ vanish identically, the complexified mapping (the coefficients of $a$ and $b$ are real)

$$(\zeta, t, \omega, s) \mapsto (a(t) + \zeta b(t), \zeta, a(s) + \omega b(s), \omega)$$

(20)

is a finite map near the origin. Hence it is a proper holomorphic mapping between two neighbourhoods of the origin. By proper mapping theorem this map takes the set $s = t$ to a complex analytic subvariety of a neighbourhood of the origin. We restrict to the “diagonal,” to what this image is intersected with the original $\mathbb{C}^2$ before complexification:

$$\zeta = \overline{\omega}, \quad a(t) + \zeta b(t) = a(\overline{t}) + zb(\overline{t}).$$

(21)

We substitute $\zeta = \overline{\omega}$ and $s = t$ into the right hand side of the second equation: $a(s) + \omega b(s) = a(\overline{t}) + zb(\overline{t})$. So

$$(a(t) - a(\overline{t})) + \zeta (b(t) - b(\overline{t})) = 0.$$  

(22)

As $a(t) - a(\overline{t}) = a(t) - \overline{a(t)}$ and $b(t) - b(\overline{t}) = b(t) - \overline{b(t)}$ are purely imaginary for all $t \in \mathbb{C}$, either $a(t) - a(\overline{t})$ and $b(t) - b(\overline{t})$ are both zero, or $\zeta$ is real.

The derivatives of $a$ and $b$ do not both vanish, so without loss of generality $b'(0) \neq 0$. Then $b(t) - b(\overline{t}) = 0$ implies $t$ is real. Therefore the image of the complexified mapping lies in $\mathbb{C}^2$ (on the diagonal), if either $t$ is real or if $\zeta$ is real. A point where $t$ is real is in the image of $(a(t) + \zeta b(t), \zeta)$ for a real $t$ (and therefore in $\tilde{H}$). A point where $\zeta$ is real is in $\Delta$ and therefore in $\tilde{H}$. Either way the image of $(a(t) + \zeta b(t), \zeta)$ for a real $t$ together with $\Delta$ is a subvariety. Therefore, $\tilde{H}$ is a subvariety.

To see that $\tilde{H}$ is not algebraic, fix $z_1 = z_1^0$ and $z_2 = z_2^0$ with different arguments. Then the set $\tilde{H}$ is given by $z_0 = z_0^0 a(t) + z_0^0 b(t)$, and this is a linear image of $X$ under a nonsingular mapping and therefore any polynomial vanishing on $\tilde{H}$ must vanish on $(z_0, z_1^0, z_2^0)$ for all $z_0$. By varying over different $(z_1^0, z_2^0)$ we obtain that the polynomial vanishes identically.

Note that not all of $\Delta$ was needed, it was only used to cover the points where the image of the map could change depending on what neighbourhood of the complexified variables we took in the local argument above. In fact $\Delta$ is an entire Levi-flat hypersurface. We only need to take the irreducible component of $\tilde{H}$ that includes $\tilde{H} \setminus \Delta$. Then $\tilde{H}$ is precisely the set $\{z : z_0 = z_1 x + z_2 y\}$ where $(x, y) \in X$ together with a subset of $\Delta$. This fact is key in the next result.

**Corollary 5.1.** There exists a singular Levi-flat hypersurface $H \subset \mathbb{P}^2$ that is a semialgebraic set, but is not real-algebraic. That is, there exists a real-algebraic Levi-flat subvariety of codimension 1 in $\mathbb{P}^2$ that is irreducible as a real-algebraic subvariety, but reducible as a real-analytic subvariety into two components of codimension 1.

**Proof.** The key is to find an algebraic curve with such properties in $\mathbb{R}^2$. For example, the curve given by

$$x(x-1)(x-2)(x-3) + y^2 = 0$$

(23)

has two components as a real-analytic subvariety, but is irreducible as a real-algebraic subvariety. Taking only one of those components we construct our $\tilde{H}$. This is clearly a proper subset of the algebraic set that contains $\tilde{H}$, which must contain both components. To construct the full algebraic set we simply solve for $x$ and $y$ in terms of rational functions of $z$ and $\bar{z}$, plug into the equation $x(x-1)(x-2)(x-3) + y^2 = 0$ and clear denominators. □
The proof that the construction in this note produces a real-analytic subvariety depends on the dimension $n = 2$. It is reasonable to conjecture that a similar construction ought to produce a subvariety in higher dimension.

Finally, let us remark why the Levi-foliation does not extend for hypersurfaces constructed above, a fact already clear from the theorem in [10], but let us identify these points explicitly. Without loss of generality and possibly translating $X$, we assume that for all small enough $x \in \mathbb{R}$, there exist at least two distinct $y \in \mathbb{R}$ such that $(x, y) \in X$. Given any real $s_2$, then for all sufficiently small real $s_0$, there are at least two distinct leaves of the hypersurface $H$ passing through $[s_0, 1, s_2]$. These points form a generic totally-real submanifold, biholomorphic to $\mathbb{R}^2$. This set would have to lie in the singular set of any holomorphic foliation tangent to $H$ (that is, extending the Levi-foliation) near $[0, 1, 0]$, which is impossible, as the singular set of a holomorphic foliation is a complex analytic subvariety.

References

[1] Eric Bedford, Holomorphic continuation of smooth functions over Levi-flat hypersurfaces, Trans. Amer. Math. Soc. 232 (1977), 323–341. MR0481100
[2] Marco Brunella, Singular Levi-flat hypersurfaces and codimension one foliations, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 6 (2007), no. 4, 661–672. MR2394414
[3] , Some remarks on meromorphic first integrals, Enseign. Math. (2) 58 (2012), no. 3-4, 315–324, DOI 10.4171/LEM/58-3-3. MR3058602
[4] Daniel Burns and Xianghong Gong, Singular Levi-flat real analytic hypersurfaces, Amer. J. Math. 121 (1999), no. 1, 23–53. MR1704996
[5] D. Cerveau and A. Lins Neto, Local Levi-flat hypersurfaces invariants by a codimension one holomorphic foliation, Amer. J. Math. 133 (2011), no. 3, 677–716, DOI 10.1353/ajm.2011.0018. MR2808329
[6] Wei-Liang Chow, On compact complex analytic varieties, Amer. J. Math. 71 (1949), 893–914. MR0033093
[7] Arturo Fernández-Pérez, On Levi-flat hypersurfaces with generic real singular set, J. Geom. Anal. 23 (2013), no. 4, 2020–2033, DOI 10.1007/s12220-012-9317-1. MR3107688
[8] Arturo Fernández-Pérez, Levi-Flat Hypersurfaces Tangent to Projective Foliations, Journal of Geometric Analysis, to appear, DOI 10.1007/s12220-013-9404-y, (to appear in print).
[9] Arturo Fernández-Pérez, Rogério Mol, and Rudy Rosas, On the dynamics of foliations in $\mathbb{P}^n$ tangent to Levi-flat hypersurfaces. preprint arXiv:1403.4802
[10] Jiří Lebl, Algebraic Levi-flat hypersurfaces in complex projective space, J. Geom. Anal. 22 (2012), no. 2, 410–432, DOI 10.1007/s12220-010-9201-9. MR2891732
[11] , Singular set of a Levi-flat hypersurface is Levi-flat, Math. Ann. 355 (2013), no. 3, 1177–1199, DOI 10.1007/s00208-012-0821-1. MR3020158
[12] Alcides Lins Neto, A note on projective Levi flats and minimal sets of algebraic foliations, Ann. Inst. Fourier (Grenoble) 49 (1999), no. 4, 1369–1385. MR1703092
[13] Rasul Shafikov and Alexandre Sukhov, Germs of singular Levi-flat hypersurfaces and holomorphic foliations. preprint arXiv:1405.4274.
[14] Hassler Whitney, The general type of singularity of a set of $2n - 1$ smooth functions of $n$ variables, Duke Math. J. 10 (1943), 161–172. MR0007784

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