The Mazur Intersection Property and Farthest Points

Pradipta Bandyopadhyay*

Abstract

K. S. Lau had shown that a reflexive Banach space has the Mazur Intersection Property (MIP) if and only if every closed bounded convex set is the closed convex hull of its farthest points.

In this work, we show that in general this latter property is equivalent to a property stronger than the MIP. As corollaries, we recapture the result of Lau and characterize the $w^*$-MIP in dual of RNP spaces.

Introduction

We work with only real Banach spaces. The notations are standard. Any unexplained terminology can be found in [4, 7].

Definition 1 For a closed and bounded set $K$ in a Banach space $X$, define

(i) $r_K(x) = \sup \{\|z - x\| : z \in K\}$, $x \in X$. $r_K$, called the farthest distance map, is a Lipschitz continuous convex function.

(ii) $Q_K(x) = \{z \in K : \|z - x\| = r_K(x)\}$, $x \in X$, the set of points farthest from $x$.

(iii) $D(K) = \{x \in X : Q_K(x) \neq \emptyset\}$.

(iv) $b(K) = \cup\{Q_K(x) : x \in D(K)\}$ is the set of farthest points of $K$.

(v) for $x \in X$ and $\alpha > 0$, let $C(K, x, \alpha) = \{z \in K : \|z - x\| > r_K(x) - \alpha\}$. $C(K, x, \alpha)$ will be called a crescent of $K$ determined by $x$ and $\alpha$.

Call a closed and bounded set $K$ densely remotal if $D(K)$ is norm dense in $X$ and almost remotal if $D(K)$ is generic, i.e., contains a dense $G_\delta$ in $X$. $K$ has the Property (R) if any crescent of $K$ contains a farthest point of $K$.

*Stat–Math Division, Indian Statistical Institute, 203, B. T. Road, Calcutta 700 035, INDIA, e-mail : pradipta@isical.ernet.in

AMS(MOS) Subject Classification (1980) : 46B20, 46B22

Keywords and Phrases : Mazur Intersection Property, Farthest points, Densely Remotal Sets, Crescents, Property (R).
Definition 2 Call a set \( K \subseteq X \) admissible if it is the intersection of closed balls containing it. Let \( F \) be a norming subspace of \( X^* \) (i.e., \( \|x\| = \sup \langle x, f \rangle \) for all \( x \in X \)). Let us denote the \( \sigma(X, F) \) topology on \( X \) simply by \( \sigma \). Then \( B(X) \), the closed unit ball of \( X \), is \( \sigma \)-closed and hence any admissible set is \( \sigma \)-closed, bounded (in norm) and convex. We say that \( X \) has \( F \)-MIP if the converse holds, i.e., every bounded, \( \sigma \)-closed convex set in \( X \) is admissible. This property was introduced in [1] generalizing the Mazur Intersection Property (MIP) (i.e., when \( F = X^* \)) and the w*-MIP (i.e., when \( X = Y^* \) and \( F = Y \)) (see [4, 8] for various characterizations and related results).

K. S. Lau [11] had shown that in a reflexive space the MIP is equivalent to the following:

*Every closed bounded convex set is the closed convex hull of its farthest points.*

In this work, we show that in a Banach space \( X \), every \( \sigma \)-closed bounded convex set is the \( \sigma \)-closed convex hull of its farthest points if and only if \( X \) has the \( F \)-MIP and every \( \sigma \)-closed bounded convex set in \( X \) has the Property (\( R \)). As corollaries, we recapture Lau’s result and characterize the w*-MIP in dual of spaces with the Radon-Nikodým Property (RNP) using a result of Deville and Zizler [5].

Some of the results presented here was first observed in the author’s Ph. D. Thesis [2] written under the supervision of Prof. A. K. Roy.

1 Main Results

The following Lemma is an easy consequence of the fact that the class of admissible sets is closed under arbitrary intersection.

**Lemma 1** Let \( X \) be a Banach space and \( F \) be a norming subspace of \( X^* \). Let \( K \subseteq X \) be \( \sigma \)-closed, bounded and convex. If for all \( l > 0 \), the set \( K_l = K + lB(X) \) is admissible, then so is \( K \).

**Remark 1** It follows that given any \( \sigma \)-closed, bounded convex inadmissible set \( K \subseteq X \), there exists an inadmissible set of the form \( K_l \), which has non-empty norm interior. This improves [11, Lemma 3.2].

**Question 1** Is the converse true?

**Theorem 2** Let \( X \) be a Banach space and \( F \) be a norming subspace of \( X^* \). The following statements are equivalent:
(a) Any $\sigma$-closed, bounded convex set in $X$ is the $\sigma$-closed convex hull of its farthest points.

(b) (i) $X$ has the F-MIP, and

(ii) Any $\sigma$-closed, bounded convex set in $X$ has the Property (R).

Proof: (a) $\implies$ (b) We first prove (i) following Lau’s lead [11]. Suppose $X$ lacks the F-MIP. By Remark [1], there exists a $\sigma$-closed, bounded convex set $K \subseteq X$ with $\text{int}(K) \neq \emptyset$ that is not admissible.

Let $M = \cap \{B : B$ closed ball containing $K\}$ and let $x_0 \in M \setminus K$ and $y_0 \in \text{int}(K) \subseteq \text{int}(M)$. Since $M \setminus K$ is open in $M$, there exists $0 < \ell < 1$ such that $z_0 = \ell x_0 + (1 - \ell)y_0 \in M \setminus K$. Note that $z_0 \in \text{int}(M)$, and hence, so is any point of the form

\[(*) \quad \alpha z_0 + (1 - \alpha)x, \quad \alpha \in (0, 1], \quad x \in K.
\]

Let $K_1 = \text{co}(K \cup \{z_0\})$. Then $K_1$ is $\sigma$-closed, bounded and convex. The proof will be complete once we show that $b(K_1) \subseteq K$.

Let $x \in X$. Then $B = \{z \in X : \|z - x\| \leq r_K(x)\}$ is a closed ball containing $K$, and so contains $M$. Since each point of the form $(*)$ is in $\text{int}(M)$, it is in $\text{int}(B)$, i.e., its distance from $x$ is strictly less than $r_K(x)$. Note that $r_K(x) \leq r_{K_1}(x) \leq r_M(x) = r_K(x)$. Thus, $Q_{K_1}(x) \subseteq K$. Since $x \in X$ was arbitrary, $b(K_1) \subseteq K$, contradicting (a).

Now, if there exists a $\sigma$-closed, bounded convex set $K$ and a crescent of $K$ that is disjoint from $b(K)$, then it is disjoint from $\text{co}^* b(K)$ as well. Hence $\text{co}^* b(K) \neq K$. This proves (ii).

(b) $\implies$ (a) Let $K$ be a $\sigma$-closed, bounded convex set in $X$. Let $L = \text{co}^*(b(K))$. Clearly, $L \subseteq K$. Suppose there exists $x \in K \setminus L$. Since $X$ has the F-MIP, there exists a crescent $C$ of $K$ disjoint from $L$. Since $K$ has the Property (R), $C \cap b(K) \neq \emptyset$. But, of course, $b(K) \subseteq L$.

Lemma 3 Let $K \subseteq X$ be a bounded set. Let $x \in X$ and $\alpha > 0$ be given. Then there exists $\varepsilon > 0$ such that for any $y \in X$ with $\|x - y\| < \varepsilon$, there exists $\beta > 0$ such that $C(K, y, \beta) \subseteq C(K, x, \alpha)$.

Proof: Take $0 < \varepsilon < \alpha/2$ and $0 < \beta < \alpha - 2\varepsilon$.

Proposition 4 Any densely remotal set has Property (R).

Proof: If $D(K)$ is dense in $X$ and $C(K, x, \alpha)$ is any crescent of $K$, then, by Lemma 3, there exists $y \in D(K)$ and $\beta > 0$ such that $C(K, y, \beta) \subseteq C(K, x, \alpha)$. Clearly, $Q_K(y) \subseteq C(K, x, \alpha)$. Thus, $K$ has the Property (R).

In the following Lemma, we collect some known results that identify some important classes of sets with the Property (R).

Lemma 5 (a) [11], Theorem 2.3 Any weakly compact set is almost remotal with respect to any equivalent norm.

(b) [3], Proposition 3 If $X$ has the RNP, every $w^*$-compact set in $X^*$ is almost remotal with respect to any equivalent dual norm.
Remark 2 Proposition 1 of [5] gives an example of a $w^*$-compact convex subset of $\ell^1$ that lacks farthest points. Thus, some additional assumptions are necessary even for Property ($R$).

Combining this with Theorem 2, we get

**Theorem 6** If $X$ has the RNP, then $X^*$ has the $w^*$-MIP if and only if every $w^*$-compact convex set in $X^*$ is the $w^*$-closed convex hull of its farthest points.

**Corollary 7** [11] If $X$ is reflexive, then $X$ has the MIP if and only if every closed bounded convex set in $X$ is the closed convex hull of its farthest points.

In both the above cases, the additional assumption on $X$ already implies condition (b)(ii) of Theorem 2 and hence the equivalence. However, in the corollary below, this is not so direct.

Recall (from [9]) that a space has the IP$_{f,\infty}$ if every family of closed balls with empty intersection contain a finite subfamily with empty intersection. For example, any space $X$ that is 1-complemented in a dual space, in particular any dual space, has the IP$_{f,\infty}$.

**Corollary 8** If $X$ has the IP$_{f,\infty}$, then $X$ has the MIP if and only if every closed bounded convex set in $X$ is the closed convex hull of its farthest points.

**Proof:** Sufficiency follows from Theorem 2. Conversely, if $X$ has both the MIP and IP$_{f,\infty}$, then it must be reflexive [9, Theorem VIII.5].

**Remark 3** (a) From the known characterizations it is easily seen that if $X$ has the MIP then $X^{**}$ has the $w^*$-MIP. And it is a long standing open question whether then $X$ is also Asplund, or equivalently, $X^*$ has the RNP (see [8]). If the answer to this question is yes, then by Theorem 6, so is that to the following

**Question 2** If $X$ has the MIP, is every $w^*$-compact convex set in $X^{**}$ the $w^*$-closed convex hull of its farthest points?

(b) The proof of (b) $\Rightarrow$ (a) in Theorem 2 combined with Lemma 3, also shows that if every weakly compact or compact convex set is admissible then each such set is the closed convex hull of its farthest points. Is the converse true? Clearly, a similar proof will not work unless the space is reflexive or finite dimensional. Now, can the specific form of $K_1$ as in Lemma 1 be utilized to prove the converse?

(c) We see from the last three results that in some cases the condition (a) of Theorem 2 becomes equivalent to the $F$-MIP. Is this generally true? We do not know the answer for arbitrary $F$. For example, we do not know whether Theorem 2 holds even without the RNP. However, the answer is negative for $F = X^*$ as the following example shows.
Example 1 There is a space $X$ with a Fréchet differentiable norm, and hence with MIP, and a closed bounded convex set $K$ in $X$ that lacks farthest points.

Proof: Notice that if the norm is strictly convex (respectively, locally uniformly convex), any farthest point of a closed bounded convex set is also an extreme (resp. denting) point. So, if every closed bounded convex set admits farthest points, then the space must necessarily have the Krein-Milman Property (KMP) (resp. the RNP) (see [4] or [7] for details). However, the space $c_0$, which does not have the KMP, admits a strictly convex Fréchet differentiable norm (see e.g., [6]).

Remark 4 Observe that since $c_0$ is Asplund, Theorem 8 shows that when equipped with the bidual norm of the above, every $w^*$-compact convex set in $\ell^\infty$ is the $w^*$-closed convex hull of its farthest points. Thus, there is closed bounded convex set $K \subseteq c_0$, such that no farthest point of $\tilde{K}$, the $w^*$-closure of the canonical image of $K$ in $X^{**}$, comes from $K$.

This gives rise to two very natural questions.

Question 3 (a) If $X$ has both RNP and MIP, does every closed bounded convex set in $X$ have the Property (R)?

(b) Can the condition (b)(ii) of Theorem 2 be replaced by the weaker condition that every $\sigma$-closed, bounded convex set in $X$ has farthest points?

Remark 5 The example in Proposition 1 of [8] shows also that RNP alone is not enough to ensure even existence of farthest points. But then, the space there does not even have the $w^*$-MIP. In fact, the set under consideration itself is not admissible. Indeed, the answer to the first question is likely to be affirmative.

As evidence, observe that

(a) if $z \in K$ is farthest from $x \in X$, then it is nearest from any point on the line $[x, z]$ extended beyond $z$ (this is just triangle inequality, and was observed in [10]), and

(b) if $X$ has both RNP and MIP, any crescent of any closed bounded convex set $K$ in $X$ contains a nearest point of $K$. This is because in spaces with RNP, any closed bounded convex set is the closed convex hull of its nearest points [3, Theorem 8.3].

One thus possibly needs to characterize farthest points among nearest points.

References

[1] P. Bandyopadhyay, The Mazur Intersection Property for Families of Closed Bounded Convex Sets in Banach Spaces, Colloq. Math. LIII (1992), 45–56.
[2] P. Bandyopadhyay, *The Mazur Intersection Property in Banach Spaces and Related Topics*, Ph. D. Thesis, Indian Statistical Institute, Calcutta (February, 1991).

[3] J. M. Borwein and S. Fitzpatrick, *Existence of Nearest Points in Banach Spaces*, Can. J. Math. 41 (1989), 702–720.

[4] R. D. Bourgin, *Geometric Aspects of Convex Sets with the Radon-Nikodým Property*, Lecture Notes in Math., No. 993, Springer-Verlag (1983).

[5] R. Deville and V. Zizler, *Farthest Points in W∗-Compact Sets*, Bull. Austral. Math. Soc. 38 (1988), 433–439.

[6] J. Diestel, *Geometry of Banach Spaces — Selected Topics*, Lecture Notes in Math., No. 485, Springer-Verlag (1975).

[7] J. Diestel and J. J. Uhl, Jr., *Vector Measures*, Math. Surveys 15, Amer. Math. Soc., Providence, R. I. (1977).

[8] J. R. Giles, D. A. Gregory and B. Sims, *Characterisation of Normed Linear Spaces with the Mazur’s Intersection Property*, Bull. Austral. Math. Soc. 18 (1978), 105–123.

[9] Giles Godefroy and N. Kalton, *The Ball Topology and Its Application*, Contemporary Math., 85 (1989).

[10] S. Fitzpatrick, *Metric Projections and the Differentiability of Distance Functions*, Bull. Austral. Math. Soc. 22 (1980), 291–312.

[11] K.-S. Lau, *Farthest Points in Weakly Compact Sets*, Israel J. Math. 22 (1975), 168–174.