NEW CLASS OF QUANTUM DEFORMATIONS OF $D = 4$
EUCLIDEAN SUPERSYMMETRIES *

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We present the class of deformations of simple Euclidean superalgebra, which
describe the supersymmetrization of some Lie algebraic noncommutativity of
$D = 4$ Euclidean space-time. The presented deformations are generated by the
supertwists. We provide new explicit formulae for a chosen twisted $D = 4$ Eu-
clidean Hopf superalgebra and describe the corresponding quantum covariant
deformation of chiral Euclidean superspace.

Keywords: Euclidean SUSY; Quantum Deformations; Drinfeld twist;Pseudo-
conjugation; Chiral superspace.

1. Introduction
In the applications of quantum deformations to fundamental interaction
theories it should be taken into considerations that supersymmetry is one
of the main basic building blocks of the string and M-theory. We recall that
the classification of deformations of $D = 4$ Poincaré symmetries is almost
complete$^{1,2}$ (see also Ref. 3,4), but systematic studies of supersymmetric
extensions of deformed Poincare algebras quite recently were considered$^{5–7}$
. In this note we shall deal with the superextensions of quantum $D = 4$

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Euclidean symmetries providing the Lie-algebraic structure of corresponding Euclidean space-time commutators. We add that Euclidean theories are often very useful as intermediate step in the calculations in Minkowski space.

Historically first and the most known example of Lie-algebraic space-time deformations are the $\kappa-$deformations of Minkowski space-time\textsuperscript{8–10}

\[
[\widehat{x}_\mu, \widehat{x}_\nu] = \frac{i}{\kappa}(a_\mu \widehat{x}_\nu - a_\nu \widehat{x}_\mu)
\]  

(1)

where $a_\mu = (1, 0, 0, 0)$ corresponds to the standard (time-like) $\kappa$-deformation\textsuperscript{11} and if $a_\mu = (1, 1, 0, 0)$ we get the light-cone $\kappa$-deformation.\textsuperscript{12}

In this short paper we would like to consider an example of $D = 4$ Euclidean supersymmetries, which in its bosonic sector provides the Lie-algebraic deformation of $D = 4$ space-time sector. This choice of deformation follows from our recent paper\textsuperscript{7} where we did present the Euclidean counterparts of Zakrzewski list of $D = 4$ Poincare classical $r$-matrices\textsuperscript{2} and classified their supersymmetrizations in the case of simple chiral (holomorphic) Euclidean superalgebras\textsuperscript{a}.

Plan of our presentation is the following. In section 2 we deliver the list of Euclidean $D = 4$ classical $r$-matrices and their supersymmetrizations presented in Ref. 7. We show that the conjugation and pseudoconjugation of Euclidean superalgebras are related with quaternionic structure of fundamental $O(3)$ and $O(4)$ spinors. In section 3 we give explicit formulae for particular $D = 4$ Euclidean quantum twisted superalgebra, which leads to the Lie-algebraic deformation of space-time sector. Further we provide the deformation of corresponding Euclidean chiral $N = (\frac{1}{2}, 0)$ superspace and indicate the link with Seiberg $N = \frac{1}{2}$ deformation of superspace. In section 4 we present short outlook.

\textsuperscript{a}Simple chiral (holomorphic) $D = 4$ Euclidean superalgebra with four complex supercharges is usually denoted as $N = (\frac{1}{2}, \frac{1}{2})$ Euclidean SUSY; by Hermitean conjugation of supercharges one obtains simple antichiral (antiholomorphic) $N = (\frac{1}{2}, \frac{1}{2})$ superalgebra. The lowest dimensional real Euclidean $N = (1, 1)$ superalgebra is described by eight complex supercharges satisfying $SU(2)$-Majorana reality conditions and it contains $N = (\frac{1}{2}, \frac{1}{2})$ chiral and antichiral sectors.
2. $D = 4$ Euclidean twisted deformations and their $N = \frac{1}{2}$ superextensions

2.1. Euclidean twisted deformations

Let us recall $D = 4$ Euclidean Lie algebra $E(4) = o(4) \times P_{4}$, where six Euclidean rotation generators $M_{\mu\nu} = -M_{\nu\mu} \in o(4)$ ($\mu, \nu = 0, 1, 2, 3$) and Euclidean four-momenta $P_{\mu} \in P_{4}$ satisfy the relations

$$[M_{\mu\nu}, M_{\lambda\rho}] = i(\delta_{\nu\lambda}M_{\mu\rho} - \delta_{\rho\mu}M_{\lambda\nu} - \delta_{\mu\lambda}M_{\rho\nu} + \delta_{\rho\nu}M_{\mu\lambda})$$  \hspace{1cm} (2)

$$[M_{\mu\nu}, P_{\rho}] = i(\delta_{\rho\mu}P_{\nu} - \delta_{\rho\nu}P_{\mu}), \quad [P_{\mu}, P_{\nu}] = 0.$$  \hspace{1cm} (3)

The generators $(M_{\mu\nu}, P_{\mu})$ are Hermitean. We recall that from relations (2)-(3) we can obtained real $D = 4$ Poincaré algebra $P(3, 1) = o(3, 1) \times \tilde{P}_{4}$ ($(\tilde{M}_{\mu\nu} \in o(3, 1), \tilde{P}_{\mu} \in \tilde{P}_{4}$ where $\mu, \nu = 0, 1, 2, 3$) after the substitution ($i, j = 1, 2, 3$)

$$\tilde{M}_{ij} = M_{ij}, \quad \tilde{M}_{i0} = -iM_{i0}, \quad \tilde{P}_{i} = P_{i}, \quad \tilde{P}_{0} = -iP_{0}. \hspace{1cm} (4)$$

Strictly speaking it means that one can introduce, on a complex algebra generated by the relations (2)-(3), two non-isomorphic real structures (i.e. $\star$-conjugation operations): the Euclidean one with respect to which the Euclidean generators $(M_{\mu\nu}, P_{\mu})$ are Hermitean (i.e. selfconjugated) and the Poincaré one with $(\tilde{M}_{\mu\nu}, \tilde{P}_{\mu})$ Hermitean. Both conjugations (real forms) coincide on the generators $(M_{ij}, P_{k})$.

In Ref. 7 we observed that all the real Poincaré $r$-matrices introduced by S. Zakrzewski in Ref. 2 describe equally well the classical $r$-matrices for complex $D = 4$ Euclidean Lie algebra $o(4; \mathbb{C}) \times \tilde{P}_{4}^{C}$ which describes also $D = 4$ Poincaré algebra. It appears that out of the Zakrzewski list with 21 classical $r$-matrices only 8 admits the reality conditions leading to $D = 4$ Euclidean real classical $r$-matrices. The list of such matrices is the following (we use $M_{i} = \frac{1}{2} \epsilon_{ijk} M_{jk}$, $N_{i} = M_{0i}$, $P_{3} = P_{0} \pm P_{3}$)

$$r_1 = i\xi N_{3} \land M_{3} + \zeta_{1} P_{+} \land P_{-} + \zeta_{2} P_{1} \land P_{2}$$

$$r_2 = \xi P_{0} \land M_{3} + r_{8}, \quad r_{3} = \xi P_{3} \land N_{3} + r_{8}, \quad r_{4} = \xi P_{1} \land M_{3} + r_{8},$$

$$r_{5} = \xi P_{+} \land M_{3} + r_{8}, \quad r_{6} = \zeta_{1} P_{1} \land P_{+},$$

$$r_{7} = \zeta_{2} P_{1} \land P_{2}, \quad r_{8} = \zeta_{1} P_{0} \land P_{3} + \zeta_{2} P_{1} \land P_{2}, \hspace{1cm} (5)$$

where the parameters $\xi, \zeta_1, \zeta_2$ are real.

It is interesting to deduce from (5) that the classical $r$-matrix describing standard (Poincaré)light-cone $\kappa$-deformation does not have its Euclidean counterparts.
2.2. Quaternionic structure of Euclidean spinors, Euclidean supercharges conjugation and pseudoconjugation.

The supercharges of Euclidean supersymmetry are described by fundamental $O(4)$ spinors, which are $O(4)$ Clifford algebra modules. Because $o(4) = o(3) \oplus o(3)$ let us consider firstly $o(3)$ spinors. Following the list of Clifford modules and fundamental spinors for orthogonal groups they are described by single real quaternion $q = q_0 + q_r e_r$, where $e_r$ describe the quaternionic units ($e_r e_s = -\delta_{rs} + \epsilon_{rst} e_t$) and $q_0, q_r$ are real. The irreducible fundamental representation of spinorial covering $SO(3) = Sp(1) \equiv U(1; H) = SU(2)$ is described by real unit quaternions $q q = q_0^2 + q_r^2 = 1$, $(q = q_0 - q_r e_r)$. We recall that in physics one uses rather complex realization of quaternions. In order to define a complex realization $q$ of real quaternions we should firstly introduce matrix realizations $e_r = -i \sigma_r$, where $\sigma_r$ are $2 \times 2$ Pauli matrices and one obtains unit quaternions as describing $2 \times 2 SU(2)$ matrices

$$U = \begin{pmatrix} q_0 - i q_3 & -q_2 - i q_1 \\ q_2 - i q_1 & q_0 + i q_3 \end{pmatrix} = \begin{pmatrix} z_1 - z_2^* \\ z_2^* \end{pmatrix} \in C^2$$

(6)

where $z_1 = q_0 - i q_3$, $z_2 = q_2 - i q_1$ and $\det U = |z_1|^2 + |z_2|^2 = 1$. Here $z^*$ denotes a complex conjugation of $z \in C$. The one-component quaternionic spinor $q$ can be represented as two-component complex spinor, described by the first column of the relation (6)

$$q = \begin{pmatrix} q_0 - i q_3 \\ q_2 - i q_1 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in C^2$$

(7)

The quaternionic nature of complex spinors (7) is exhibited by the presence of quaternionic pseudoconjugation ($\epsilon \equiv i \sigma_2$)$^{15,16}$

$$(q^H)_\alpha = \epsilon_{\alpha \beta} q^*_\beta \quad (q^H)^H = -q$$

(8)

which commutes with $SU(2) \simeq SO(3)$ rotations, i.e. $(Uq)^H = Uq^H$. The condition (8) is a complex realization of the quaternionic map $q \to q^H = -qe_2$ b, because the group $U(1; H) = SU(2)$ acts on $q$ from the left, its commutativity with the mapping $q \to q^H$ is obvious. In order to interpret the relation (8) in a quaternionic framework one can deduce from (7) that complex conjugation $q \to q^*$ represents the quaternionic map $q \to -e_2 q e_2$, and one gets $q^H = -qe_2 = -e_2(-e_2 q e_2)$, where in complex realization (8) we inserted $e_2 = -i \sigma_2$.

b This quaternionic map as well as its complex counterpart given by (8) is not unique. One can replace $e_2 \to \alpha_i e_i$ (with $\Sigma |\alpha_i|^2 = 1$) in $q^H = -qe_2$. 
The Euclidean supercharges for general $D = 4$ Euclidean $(n, m)$ superalgebra ($n, m$ are half-integers) are described by $n$ left chiral $SU_L(2)$ spinors denoted by $Q^a_L$ ($a = 1, \ldots, n$, $\alpha = 1, 2$) and $m$ right chiral $SU_R(2)$ spinors denoted by $Q^b_R$ ($b = 1, \ldots, m$). In order to obtain the simplest $D = 4$ Euclidean chiral (holomorphic) $N = (\frac{1}{2}, \frac{1}{2})$ superalgebra we need a doublet of complex supercharges $(Q_{L\alpha}, Q_{R\beta})$ which enters into the basic SUSY relation (see e.g. Ref. 17 where is denoted $Q_{L\alpha} \equiv Q_{\alpha}; Q_{R\alpha} \equiv Q_{\alpha}$)

\[
\{Q_{L\alpha}, Q_{R\beta}\} = 2(\sigma^E_\mu)_{\alpha\beta} P^\mu \\
\{Q_{L\alpha}, Q_{L\beta}\} = \{Q_{R\alpha}, Q_{R\beta}\} = 0
\] (9, 10)

where $Q_{L\alpha}, Q_{R\beta}$ transforms respectively under $SU_L(2), SU_R(2)$ and $P^\mu$ are complexified fourmomenta. Further

\[
(\sigma^E_\mu) = (iI_2, \sigma_i)
\] (11)

describe $D = 4$ Euclidean Pauli matrices which transform under $D = 4$ Euclidean rotations $SO(4) = SU_L(2) \otimes SU_R(2)$ as Euclidean fourvector $(U_{L,R} \in SU_L,R(2))$

\[
U_L \sigma^E_\mu U^T_R = A^\nu_\mu \sigma^E_\nu
\] (12)

where real $4 \times 4$ matrices $A^\nu_\mu$ represent $SO(4)$ four-vector rotations. This is due to the fact that the Euclidean Pauli matrices (11) satisfy the constraints

\[
(\sigma^E_\mu)^* = \epsilon \sigma^E_\mu \epsilon
\] (13)

while for $SU(2)$ matrices (6) one has $U^* = -\epsilon U \epsilon$ instead. This can be used to embed Euclidean space-time onto quaternions. We point out that the superalgebra (9-10) cannot be constrained by any reality condition, because the supercharges $Q_{L\alpha}, Q_{R\beta}$ transform under two independent $SU(2)$ groups.

From physical point of view it would be desirable to obtain in the superalgebra (9) the real (Hermitean) fourmomenta $P^\mu$. For that purpose one can adopt the quaternionic pseudoconjugation (8) to the Euclidean supercharges in the following two ways:

i) One introduces an abstract inner $SU(2)$--invariant graded antiautomorphism of fourth order $(\tau^2 = -1 \Rightarrow \tau^4 = 1)$

\[
\tau(Q_{L\alpha}) = \epsilon_{\alpha\beta} Q_{L\beta} \quad \tau(Q_{R\alpha}) = \epsilon_{\alpha\beta} Q_{R\beta} \tag{14}
\]

\footnote{It is well known that it does not exist $SU(2)$--invariant antiautomorphism of second order (conjugation) in two dimensions, i.e. matrices (6) cannot be made real at the same time by change of basis in $\mathbb{C}^2$.}
which satisfies the antilinearity property: \( \tau(zQ) = \bar{z}\tau(Q) \) and extends the Euclidean Hermitian conjugation from the bosonic sector into the fermionic.

The pseudoconjugation (14) was introduced by Nahm, Rittenberg, Scheinert,\textsuperscript{18} Berezin, Tolstoy\textsuperscript{19} and Manin\textsuperscript{20} (see also\textsuperscript{21,22}). Now one can check explicitly (taking into account (14) as well as the graded antihomomorphism property: \( \tau(Q_1Q_2) = -\tau(Q_2)\tau(Q_1) \)) that the relation (9-10) remains invariant under pseudoconjugation (14) if the Euclidean fourmomentum generators are real.

ii) one can assume that the supercharges \( Q_{L\alpha}, Q_{R\beta} \) admit well defined Hermitian conjugation which is an antilinear involutive outer map provided linearly independent conjugated antichiral generators \( Q^+_{L\alpha} \equiv Q_{L\dot{\alpha}}, Q^+_{R\beta} \equiv Q_{R\dot{\beta}} \). The generators \( Q^+_{L\dot{\alpha}}, Q^+_{R\dot{\beta}} \) form a basis of complex-conjugate antichiral (antiholomorphic) \( N = (\frac{1}{2}, \frac{1}{2}) \) superalgebra obtained by Hermitian conjugation of relations (9-10). One can introduce the following outer antilinear antiautomorphism of fourth order (i.e. pseudoconjugation) of \( N = (\frac{1}{2}, \frac{1}{2}) \textsuperscript{23,24} \)

\[
\tau(Q_{L\alpha}) = \epsilon_{\alpha\beta}Q^+_{L\beta} \quad \tau(Q_{R\alpha}) = \epsilon_{\alpha\beta}Q^+_{R\beta}
\]  

which maps the chiral \( N = (\frac{1}{2}, \frac{1}{2}) \) Euclidean superalgebra into antichiral one\textsuperscript{9}. If we use the relation (13) one can show easily that both superalgebras will have the same algebraic form (9-10) if the fourmomenta generators \( P_\mu \) are Hermitean (real). The pseudoconjugation (15) has the form of quaternionic pseudoconjugation (8), applied to two-component complex spinorial supercharges.

2.3. **Twisted quantum deformations od \( D = 4 \) \( N = (\frac{1}{2}, \frac{1}{2}) \) holomorphic Euclidean superalgebra.**

In this subsection we shall consider the Euclidean superalgebra (9-10), invariant under the pseudoconjugation (14) what implies that the fourmomenta \( P_\mu \) are real and Euclidean.

We shall study further the superextensions of the classical \( D = 4 \) Euclidean \( r \)-matrices \( r_A \) \( (A = 1, \ldots, 8) \) listed in formulae (5)

\[
r_A \rightarrow r_A^{SUSY} = r_A + s_A
\]  

\textsuperscript{9}The pseudoconjugation property \( \tau^2 = -1 \) follows if \( (Q^*)^+ = (Q^+)^* \) what implies that the antichiral supercharges \( Q^+_{L\alpha}, Q^+_{R\beta} \) are mapped into \( Q_{L\alpha}, Q_{R\beta} \) by the formula with the same algebraic form (15).
where \( s_A \) is the odd part of the supersymmetric \( r \)-matrix which is bilinear in \( Q_{La} \) and \( Q_{R\beta} \). The general complex formula for \( s_A \) is the following (we define \( Q \lor Q' = Q \otimes Q' + Q' \otimes Q \))

\[
s_A = s_A^{(1)\alpha\beta} Q_{La} \lor Q_{L\beta} + s_A^{(2)\alpha\beta} Q_{La} \lor Q_{R\beta} + s_A^{(3)\alpha\beta} Q_{Ra} \lor Q_{R\beta}
\]  

(17)

If we substitute (11) into the graded classical \( YB \) equation and use the \( D = 4 \) Euclidean superalgebra relations \( M_i^\pm = \frac{1}{2}(M_i \mp N_i) \)

\[
[M_i^+, Q_{La}] = -\frac{1}{2}(\sigma_i)_\alpha^\beta Q_{L\beta} \\
[M_i^-, Q_{La}] = [M_i^+, Q_{Ra}] = 0
\]

\[
[M_i^-, Q_{Ra}] = \frac{1}{2}(\sigma_i)_\alpha^\beta Q_{R\beta};
\]  

(18)

one gets that the classical \( r \)-matrices \( r_A \) (see (5)) for \( A = 1, ..., 5 \) do have a common superextension \( s \equiv s_A \) given by the formula

\[
s = \eta Q_{L2} \lor Q_{L1}
\]  

(19)

and for \( A = 6, 7, 8 \) we obtain another three-parameter superextension

\[
\tilde{s} = \eta^{\alpha\beta} Q_{La} \lor Q_{L\beta}
\]  

(20)

It should be added that the formulae (19-20) after the replacement \( Q_{La} \rightarrow Q_{Ra} \) provide right-handed chiral extensions of \( r \)-matrices (5), which as well describe the supersymmetrization of (16) satisfying the supersymmetric \( YB \) equation.

The quantum deformations described by classical \( r \)-matrices \( r_6, r_7, r_8 \) describe the canonical twist deformations, which leads to \( \mathbb{C} \)-number commutator of quantum space-time coordinates and deforms Grassmann superspace variables into the generators of \( o(4) \) Clifford algebra. More complicated are the the deformations described by \( r_1, ..., r_5 \). The first, denoted by \( r_1 \), generates the quadratic deformation of Euclidean space-time algebra and the remaining four \( r_2, r_3, r_4, r_5 \) lead to Lie-algebraic formulae for non-commutative Euclidean space-time algebra. In next paragraph we shall describe in detail the deformation described by supersymmetric classical \( r \)-matrix \( r_2^{SUSY} \) (see (5), (16) and (19)).

3. \( D = 4 \) Euclidean Hopf algebra providing Lie-algebraic deformation of the bosonic Euclidean space sector.

We shall consider now in detail quantum twisted Euclidean supersymmetry generated by the classical \( r \)-matrix \( r_2 \) (see (5)), and supersymmetrically
extended by the term (19). The algebraic sector of Euclidean twisted superalgebra remains undeformed (see (3) but the coproducts are modified by the similarity transformation

$$\Delta = F^{-1} \Delta_0 F$$

(21)

where $\Delta_0$ describes the undeformed primitive coproduct. The twist factor $F \in U(E_1) \otimes U(E_4)$ corresponding to $r_2$ looks as follows\(^a\). Here $\xi, \zeta_1, \zeta_2$ and $\eta$ are real deformation parameters in order to assure invariance ("unitarity") under the Euclidean Hermitean conjugation in the bosonic sector and pseudoconjugation (14) in the fermionic one. We get

$$F = \exp(i\xi M_3 \wedge P_0) \exp(i\zeta_1 P_3 \wedge P_0 + i\zeta_2 P_2 \wedge P_1 + \eta Q_1 \vee Q_2)$$

(22)

The fourmomentum coproducts are the following

$$\Delta(P_0) = \Delta_0(P_0) = 1 \otimes P_0 + P_0 \otimes 1$$

$$\Delta(P_j) = P_j \vee \cos(\xi P_0) + \sin(\xi P_0) \wedge \epsilon_{3jk} P_k, \quad j, k = 1, 2, 3$$

(23)

The coproducts for the "space" rotation generators are

$$\Delta(M_j) = M_j \vee \cos(\xi P_0) + \sin(\xi P_0) \wedge \epsilon_{3jk} M_k + \delta_{j3}\{\Delta_0(M_3) - M_3 \vee \cos(\xi P_0)\}$$

$$- \zeta_2 \{\epsilon_{1j} P_k \cos(\xi P_0) \wedge P_2 \cos(\xi P_0) - \epsilon_{1j} P_k \sin(\xi P_0) \vee P_1 \cos(\xi P_0)\}$$

$$+ P_1 \cos(\xi P_0) \wedge \epsilon_{2j} P_k \cos(\xi P_0) - P_2 \cos(\xi P_0) \vee \epsilon_{2j} P_k \sin(\xi P_0)\}$$

$$- \zeta_1 \{\epsilon_{3j} P_k \vee P_0 \cos(\xi P_0) + P_0 \sin(\xi P_0) \wedge P_j\} + \frac{\eta}{2} R_j$$

(24)

where $R_j$ is a shortcut for the following supersymmetric contribution

$$R_j = (\sigma_j)^{\alpha} \{Q_\beta \cos(\frac{\xi}{2} P_0) \vee Q_1 \cos(\frac{\xi}{2} P_0) + i Q_\beta \sin(\frac{\xi}{2} P_0) \wedge Q_1 \cos(\frac{\xi}{2} P_0)\}$$

$$+ (\delta_{j3} \delta_{j2} - i \epsilon_{3jk}(\sigma_k)^{\alpha}) \{i Q_\beta \cos(\frac{\xi}{2} P_0) \wedge Q_1 \sin(\frac{\xi}{2} P_0)\} + Q_\beta \sin(\frac{\xi}{2} P_0) \vee Q_1 \sin(\frac{\xi}{2} P_0)\} + (\sigma_j)^{\alpha} \{Q_2 \cos(\frac{\xi}{2} P_0) \vee Q_\beta \cos(\frac{\xi}{2} P_0) +$$

$$+ i Q_2 \cos(\frac{\xi}{2} P_0) \wedge Q_\beta \cos(\frac{\xi}{2} P_0)\} - (\delta_{j3} \delta_{j1} - i \epsilon_{3jk}(\sigma_k)^{\alpha}) \times$$

$$\times \{i Q_2 \sin(\frac{\xi}{2} P_0) \wedge Q_\beta \cos(\frac{\xi}{2} P_0) + Q_2 \sin(\frac{\xi}{2} P_0) \vee Q_\beta \sin(\frac{\xi}{2} P_0)\}.\]$$

\(^a\)In this section we shall drop the index $L$, i.e. $Q_L \alpha \rightarrow Q_\alpha$.\]
The coproducts for Euclidean "boosts" are the following \((j = 1, 2, 3)\)
\[
\Delta(N_j) = N_j \lor \cos(\xi P_0) + \sin(\xi P_0) \land \epsilon_{3jk} N_l - \xi\{\cos(\xi P_0) \land P_j
\]
\[+ \sin(\xi P_0) \land \epsilon_{3jk} P_k + \zeta_2 \delta_{1j} \{P_0 \cos(\xi P_0) \lor P_1 + P_0 \sin(\xi P_0) \land P_2\}
\]
\[\zeta_1 \{\epsilon_{3jk} P_k \cos(\xi P_0) \land P_3 \sin(\xi P_0) + P_0 \cos(\xi P_0) P_3 \land P_j \cos(\xi P_0)\}
\]
\[- \zeta_2 \delta_{1j} \{P_0 \cos(\xi P_0) \lor P_2 - P_0 \sin(\xi P_0) \land P_1\} + \frac{\eta}{2} R_j \quad (25)
\]

One can observe that for the generators \(M^\pm_j\) (see (18)) the \(R_j\) (supersymmetric) contribution appears only in the antichiral sector \(M^-_j\) while the chiral one remains bosonic. In more explicit form one, e.g., gets
\[
R_1 = Q_1 \cos(\frac{\xi}{2} P_0) \lor Q_1 \cos(\frac{\xi}{2} P_0) + Q_2 \cos(\frac{\xi}{2} P_0) \lor Q_2 \cos(\frac{\xi}{2} P_0)
\]
\[- Q_1 \sin(\frac{\xi}{2} P_0) \lor Q_1 \sin(\frac{\xi}{2} P_0) - Q_2 \sin(\frac{\xi}{2} P_0) \lor Q_2 \sin(\frac{\xi}{2} P_0)
\]
\[+ 2i Q_1 \sin(\frac{\xi}{2} P_0) \land Q_1 \cos(\frac{\xi}{2} P_0) - 2i Q_2 \sin(\frac{\xi}{2} P_0) \land Q_2 \cos(\frac{\xi}{2} P_0)
\]
\[
R_3 = i Q_2 \cos(\frac{\xi}{2} P_0) \land Q_1 \sin(\frac{\xi}{2} P_0) + i Q_1 \cos(\frac{\xi}{2} P_0) \land Q_2 \sin(\frac{\xi}{2} P_0)
\]
\[- i Q_1 \sin(\frac{\xi}{2} P_0) \land Q_2 \cos(\frac{\xi}{2} P_0) - i Q_2 \sin(\frac{\xi}{2} P_0) \land Q_1 \cos(\frac{\xi}{2} P_0)
\]

For the supercharges we obtain \((Q_{\alpha a} \rightarrow \tilde{Q}_{\alpha})\)
\[
\Delta(Q_{\alpha}) = Q_{\alpha} \lor \cos(\frac{\xi}{2} P_0) + i(\sigma_3)^{\alpha\beta} \tilde{Q}_{\beta} \land \sin(\frac{\xi}{2} P_0) \quad (26)
\]
\[
\Delta(\tilde{Q}_{\alpha}) = \cos(\frac{\xi}{2} P_0) \lor \tilde{Q}_{\alpha} + i \sin(\frac{\xi}{2} P_0) \land (\sigma_3)^{\beta\alpha} \tilde{Q}_{\beta} + 4\eta S_{\alpha(12)} \quad (27)
\]

where \(S_{\alpha(12)} = \frac{1}{2} (S_{\alpha 12} + S_{\alpha 21})\) denotes the symmetrization and
\[
S_{\alpha 12} = Q_1 \land \cos(\frac{\xi}{2} P_0)\{\{\sigma_2\}_{2\alpha} P_0 + (\sigma_3)_{2\alpha} P_3\}
\]
\[+ i(\sigma_3)_{1\alpha} \tilde{Q}_{\beta} \lor \sin(\frac{\xi}{2} P_0)\{\{\sigma_2\}_{2\alpha} P_0 + (\sigma_3)_{2\alpha} P_3\}
\]
\[+ Q_1 \cos(\xi P_0) \land \cos(\frac{\xi}{2} P_0)\{\sum_{i=1,2} (\sigma_1)_{2\alpha} P_i\}
\]
\[+ i(\sigma_3)_{1\alpha} \tilde{Q}_{\beta} \cos(\xi P_0) \lor \sin(\frac{\xi}{2} P_0)\{\sum_{i=1,2} (\sigma_1)_{2\alpha} P_i\}
\]
\[+ Q_1 \sin(\xi P_0) \lor \cos(\frac{\xi}{2} P_0)\{\sum_{i=1,2} (\sigma_1)_{2\alpha} \epsilon_{3ik} P_k\}
\]
\[+ i(\sigma_3)_{1\alpha} \tilde{Q}_{\beta} \sin(\xi P_0) \land \sin(\frac{\xi}{2} P_0)\{\sum_{i=1,2} (\sigma_1)_{2\alpha} \epsilon_{3ik} P_k\}
In particular, calculates

\[ S_{1\,21} = iQ_2 \wedge \exp(-\frac{i\xi}{2} P_0) \{ P_0 - iP_3 \} \]
\[ S_{2\,12} = iQ_1 \wedge \exp(\frac{i\xi}{2} P_0) \{ P_0 + iP_3 \} \]

Now due to unitarity of the twist (22) one has the following reality property

\[ \Delta(x^*) = \Delta(x)^* \]

for the coproduct, where \( x^* = x \) in the bosonic sector \((M_{\mu\nu}, P_\rho)\) and \( Q_\alpha^* = \tau(Q_\alpha) \) (see (14)) for the fermionic one. Indeed, it can be checked that \( F_j^* = F_j \) and \( S_{1\,21}^* = S_{2\,12} \), while \( S_{2\,12}^* = -S_{1\,21} \). Similarly, \( S_{1\,12}^* = S_{2\,21} \) \((S_{2\,21} = -S_{1\,12})\).

In what follows we shall use Mandelstam realization (see Ref. 7) of undeformed \( N = (\frac{1}{2}, \frac{1}{2}) \) Euclidean superalgebra on left chiral Euclidean superspace \((x^\mu, \theta^\alpha)\), where \( x^\mu \) are real Euclidean space-time coordinates and \( \theta^L_\alpha = \theta^\alpha \) describes (left) chiral Grassmannian spinors transforming under \( SU_L(2) \) \((\partial_\mu \equiv \partial_{x^\mu}, \partial_\alpha \equiv \partial_{\theta^\alpha})\)

\[ P_\mu = -i\partial_\mu, \quad M_{\mu\nu} = ix_{[\mu}[\partial_{\nu]} + \frac{1}{2}(\theta \sigma_{\mu\nu})^\alpha \partial_\alpha \]
\[ Q_\alpha = i\partial_\alpha, \quad \tilde{Q}_\beta = 2\theta^\alpha (\sigma^\alpha)^{\alpha\beta} \partial_\mu \]

Further we insert (21) into the \( \ast \)-product formula of chiral Euclidean fields \( \Phi(x_\mu, \theta_\alpha) \)

\[ \Phi(x_\mu, \theta_\alpha) \ast \Psi(x_\mu, \theta_\alpha) := m[F^{-1} \triangleright \Phi(x_\mu, \theta_\alpha) \otimes \Psi(x_\mu, \theta_\alpha)] \]

If \( \triangleright \) describes in (24) the realization (23) of Euclidean superalgebra generators occurring in \( F^{-1} \) one obtains the formulae for deformed superspace relations

\[ [x^\mu, x^{\nu}]_\ast = 2i(\zeta_1 \delta^{[\mu}_{3} \delta^{\nu]}_0 + \zeta_2 \delta^{[\mu}_{2} \delta^{\nu]}_0) + 2i\xi(\delta^{[\mu}_{3} \delta^\nu]_0 x^1 + \delta^{[\mu}_0 \delta^\nu]_1 x^2) \]
\[ [x^\mu, \theta^\alpha]_\ast = \xi \delta^\alpha_0 (\sigma_3) \theta^\beta \]
\[ \{\theta^\alpha, \theta^\beta\} = -2\eta \delta^{(\alpha}_1 \delta^\beta_2) \]

We observe that all deformation parameters are real and dimensionfull. The first part of relation (31) obtained by putting \( \xi = 0 \), together with the relation (33), provides canonical type of deformation, while the second part of (31) together with (32) \((\xi \neq 0)\) leads to space-time deformation of Lie-algebraic type. We add that the relations (32-33) are invariant under conjugation if we extend the pseudoconjugation (14) to the Grassmannian
coordinates $\theta^\alpha$, provided that the Euclidean spacetime coordinates $x^\mu$ are Hermitian.

We recall that first deformation of chiral $N = (\frac{1}{2}, \frac{1}{2})$ Euclidean supersymmetries was proposed by Seiberg\textsuperscript{26} with primary deformation introduced a priori by the canonical modification (33) of the anti-commutativity of Grassmann variables $\theta^\alpha$. In such a framework the quantum-deformed supersymmetries do not appear, but the anticommutators of supercharges are modified and it appears that half of the considered Euclidean supersymmetries are broken explicitly\textsuperscript{4}.

4. Outlook

In this lecture we presented mainly the results of our paper Ref. 7 but we also provided new results in Section 3, where the explicit coproducts for twisted $D = 4$, $N = (\frac{1}{2}, \frac{1}{2})$ Euclidean chiral Hopf superalgebra are written down. This Hopf superalgebra as well as the corresponding superspace do satisfy the Euclidean reality condition provided the fermionic sector undergoes the pseudoconjugation (14). It shows that the case of $N = (\frac{1}{2}, \frac{1}{2})$ Euclidean SUSY corresponds to $N = 1$ Poincaré supersymmetry, but as follows from Sect. 2.2, it can be also obtained from Poincaré case after doubling of supercharges (in $N = 1$ Poincaré superalgebra we have four real supercharges, and in $N = (\frac{1}{2}, \frac{1}{2})$ Euclidean case four complex). This passage from Poincaré to Euclidean SUSY is consistent with Osterwalder-Schrader analytic continuation procedure\textsuperscript{27} from Poincaré invariant theories in Minkowski space into Euclidean invariant theories in Euclidean $D = 4$ space, where the doubling of fermions is emphasized. It should be stressed that imposing the pseudoconjugation (14) or (15) cannot be used for the reduction of degrees of freedom of $N = (\frac{1}{2}, \frac{1}{2})$ supercharges.

The similarity between Euclidean and Poincaré case can be achieved however in the cases of $N = (1, 1)$ Euclidean and $N = 2$ Poincaré supersymmetries, in both cases described by four complex or eight real supercharges. This property follows from the introduction for $N = (1, 1)$ Euclidean SUSY the conjugation $\# (\#^2 = 1)$, which can be used for imposing the reality constraints. These reality constraints are called $SU(2)$–Majorana condition\textsuperscript{15} and are the consequences of quaternionic structure of $D = 4$ Euclidean space, as we demonstrate below.

In order to introduce the reality condition consistent with the quaternionic structure one has to consider a pair of quaternions $q^a$ ($a = 1, 2$)

\textsuperscript{4} Such explicit breaking has been also called $N = \frac{1}{2}$ breaking.\textsuperscript{26}
and define the quaternionic conjugation \((q^a)^\# = ((q^a)^\#)^\# = q^a\) in terms of quaternionic pseudoconjugation (8) as follows
\[
(q^a)^\# = -\epsilon^{ab}(q^b)^H \quad \Leftrightarrow \quad (q^a_\alpha)^\# = \epsilon^{ab} \epsilon_{\alpha\beta} q^b_{\beta} \tag{34}
\]
The quaternionic reality condition \((SU(2)\text{ Majorana reality condition})\) takes the form
\[
q^a = (q^a)^\# \quad \Leftrightarrow \quad q^a_\alpha = \epsilon^{ab} \epsilon_{\alpha\beta} q^b_{\beta} \tag{35}
\]
If we wish to obtain real Euclidean \(N = (1,1)\) superalgebra with real four-momenta one should introduce the pairs of left and right chiral supercharges \((Q^a_{L\alpha}, Q^a_{R\alpha}), a = 1,2\) which satisfy the superalgebraic counterpart of the reality condition (35).
\[
Q^a_{L\alpha} = \epsilon^{ab} \epsilon_{\alpha\beta} (Q^b_{L\beta})^\dagger \quad Q^a_{R\alpha} = \epsilon^{ab} \epsilon_{\alpha\beta} (Q^b_{R\beta})^\dagger \tag{36}
\]
In our next paper we shall consider the real \(N = (1,1)\) Euclidean superalgebra and its quantum twist deformations. It should be added that by considering only the doublets of supercharges \(Q^a_{L\alpha}\) \((Q^a_{R\alpha})\) we obtain \(N = (2,0)\) chiral \((N = (0,2)\) antichiral\) Euclidean superalgebras which can be mapped into each other by the \(N = 2\) extensions of the pseudoconjugations (15).

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