Low-energy next-to-leading contributions to the effective action in $\mathcal{N} = 4$ SYM theory

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Abstract

Using formulation of $\mathcal{N} = 4$ SYM theory in terms of $\mathcal{N} = 1$ superfields superfields we construct the derivative expansion of the one-loop $\mathcal{N} = 4$ SYM effective action in background fields corresponding to constant Abelian strength $F_{mn}$ and constant hypermultiplet. Any term of the effective action derivative expansion can be rewritten in terms of $\mathcal{N} = 2$ superfields. The action is manifestly $\mathcal{N} = 2$ supersymmetric but on-shell hidden $\mathcal{N} = 2$ supersymmetry is violated. We propose a procedure which allows to restore the hidden $\mathcal{N} = 2$ invariance.

1 Introduction

The exact low-energy quantum dynamics of $\mathcal{N} = 4$ SYM theory in $\mathcal{N} = 2$ vector multiplet sector is mastered by the non-holomorphic effective potential $\mathcal{H}(\mathcal{W}, \bar{\mathcal{W}}) = \frac{1}{(4\pi)^2} \ln \mathcal{W} \ln \bar{\mathcal{W}}$, depending on $\mathcal{N} = 2$ strengths $\mathcal{W}, \bar{\mathcal{W}}$ (see Refs. [1, 2]). This result can be obtained entirely on the symmetry grounds from the requirements of scale independence and $R$-invariance up to a numerical factor [1, 3] as well as by direct quantum field theory calculations (see e.g. [4]) using various formulation of the model ($\mathcal{N} = 1$ superspace, $\mathcal{N} = 2$ harmonic and $\mathcal{N} = 2$ projective superspaces).

Recently, the complete exact low-energy effective action containing the dependence both on $\mathcal{N} = 2$ gauge superfields and hypermultiplets has been discovered [5]. The additional hypermultiplet-dependent contributions containing the on-shell $\mathcal{W}, \bar{\mathcal{W}}$ and the hypermultiplet $q^{ia}$ superfields have been obtained on a purely algebraic ground and in the harmonic supergraph calculations in the complete on-shell $\mathcal{N} = 4$ supersymmetric form

$$L_q = c \left\{ (X - 1) \frac{\ln(1 - X)}{X} + [\text{Li}_2(X) - 1] \right\}, \quad X = -\frac{q^{ia} q_{ia}}{\mathcal{W} \bar{\mathcal{W}}}, \quad (1)$$

where $\text{Li}_2(X)$ is the Euler dilogarithm function and $c$ is a constant (see the details and denotations in Refs. [5]).

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In the present work we briefly discuss the problem of derivation of the subleading terms in the effective action, depending on all fields of $\mathcal{N} = 4$ supermultiplet and representation of these terms in complete $\mathcal{N} = 4$ supersymmetric form. This allows to construct the derivative expansion of the one-loop effective Lagrangian $\mathcal{L}_{\text{eff}}$ depending both on $\mathcal{N} = 2$ gauge background superfields, their spinor derivatives up to some order and hypermultiplet background superfields.

2 The background field method

The simplest $\mathcal{N} = 1$ background, which allows fulfilling one-loop calculations, belongs to the Cartan subalgebra of the gauge group $SU(2)$ spontaneously broken down to $U(1)$ and constant space-time background hypermultiplet $q^a$

$$\mathcal{W} = \Phi = \text{const}, \quad D^a_i \mathcal{W} = \lambda^i_\alpha = \text{const}, \quad q^a = \text{const},$$

(2)

where $\mathcal{W}, q^a$ are $\mathcal{N} = 2$ superfields carrying on-shell components of the $\mathcal{N} = 4$ vector multiplet.

The action of $\mathcal{N} = 4$ SYM model is formulated in terms of $\mathcal{N} = 1$ superspace as follows \cite{6}

$$S = \frac{1}{g^2} \text{tr} \left\{ \int d^4 x d^2 \theta W^2 + \int d^4 x d^4 \theta \Phi_i e^V \Phi^i e^{-V} + \frac{1}{3!} \int d^4 x d^2 \theta i c_{ijk} \Phi^i [\Phi^j, \Phi^k] + \frac{1}{3!} \int d^4 x d^2 \theta i c^{ijk} \bar{\Phi}_i [\bar{\Phi}_j, \bar{\Phi}_k] \right\}.$$ 

(3)

All superfields here are taken in the adjoint representation of the gauge group. In addition to the manifest $\mathcal{N} = 1$ supersymmetry and $SU(3)$ symmetry on the $i,j,k,...$ indices of $\Phi$ and $\bar{\Phi}$, the action has the hidden global supersymmetries (see e.g. \cite{7}).

We define one-loop effective action $\Gamma$ depending on the background superfields (2) by a path integral over quantum fields in the standard form

$$e^{i\Gamma} = \int \mathcal{D}v \mathcal{D}\varphi \mathcal{D}c \mathcal{D}c' \mathcal{D}\bar{c} \mathcal{D}\bar{c}' e^{i(S_{(2)} + S_{\text{FP}})},$$

(4)

where $S_{(2)}$ is a quadratic in quantum fields part of the classical action including a gauge-fixing condition and $S_{\text{FP}}$ is a corresponding ghost action. The main technical tool for the $\mathcal{N} = 1$-superfield calculations is the background covariant gauge-fixing $S_{\text{GF}} = \frac{1}{g^2} \int d^8 z (F^A F^A)$, with the convenient conditions for the quantum superfields $v$ and $\varphi$

$$\tilde{F}^A = \nabla^2 v^A + \left[ \frac{1}{\Box_+} \nabla^2 \varphi^i, \Phi_i \right]^A, \quad F^A = \bar{\nabla}^2 v^A + \left[ \frac{1}{\Box_-} \bar{\nabla}^2 \bar{\varphi}_i, \Phi^i \right]^A,$$

(5)

where $\Box_+, \Box_-$ are the standard notations for Laplace-like operators in the $\mathcal{N} = 1$ superspace. The gauge fixing functions (5) can be considered as a superfield form of so-called $R_x$-gauges (see Refs. \cite{8}). It should be noted that the used gauge fixing doesn’t preserve the hidden $\mathcal{N} = 2$ supersymmetries. In gauge theories not all hidden symmetries of the classical action can be maintained manifestly in the quantization procedure, e.g.: on-shell
supersymmetry, (super)conformal symmetry (see current status of problem in Ref.[9]). According to the analysis given in Ref.[10] the problem of keeping the rigid symmetries manifest at the quantum level is essentially equivalent to finding covariant gauge conditions. In the case of conformal symmetry as well as hidden supersymmetries of $\mathcal{N} = 4$ SYM theory in $\mathcal{N} = 1$ or $\mathcal{N} = 2$ harmonic superspace such gauge conditions do not exist. As a sequence, any rigid transformations has to be accompanied by a field-dependent non-local gauge transformation in order to restore the gauge slice.

3 One-loop effective action expansion.

The whole one-loop contribution to the effective action (4) has an extremely simple form and is determined only by vector loop contribution

$$\Gamma = \frac{i}{2} \sum_{I<J} \text{Tr} \ln(\mathcal{O}_V - M)_{IJ},$$

because ghost and hypermultiplet contributions mutually cancel each other. The details of calculation are given in Ref. [7]. Such a functional trace has been already calculated by different ways for models with one chiral background superfield (see [7, 11] and reference therein). The difference between the theory with and without hypermultiplets consists in the replacement the value $M = \bar{\Phi} \Phi$ with the $R$-symmetry group invariant $M = (\bar{\Phi} \Phi + \bar{Q} Q + \bar{Q} \bar{Q})$.

The trace (6) can be written as a power expansion of dimensionless combinations $\Psi$, $\bar{\Psi}$ in vector and hypermultiplet superfields, where

$$\bar{\Psi}^2 = \frac{1}{M^2} \nabla^2 \bar{W}^2, \quad \Psi^2 = \frac{1}{M^2} \nabla^2 W^2.$$

In the constant field approximation this expansion can be summed to the following expression for the whole one-loop effective action (see details in [4]):

$$\Gamma = \frac{1}{8\pi^2} \int d^8z \int_0^\infty dt e^{-t} \frac{W^2 \bar{W}^2}{M^2} \omega(t \Psi, t \bar{\Psi}),$$

As a result, we see that the only difference between the effective actions with and without the hypermultiplet background is stipulated by $M = (\phi \tilde{\phi} + f^a f_a)$, where $\phi, \tilde{\phi}$ and $f^a$ are physical bosonic fields of the $\mathcal{N} = 2$ vector multiplet and hypermultiplet. In component form, the closed relation for one-loop effective action (8) has natural Schwinger-type expansion over $F^2/M^2$ powers which doesn’t include $F^6$ term that is a property of $\mathcal{N} = 4$ SYM theory [11]. The expansion of the function $\omega$ defined in (8) (see [4]) induces the effective action (8) expansion in powers of $\Psi^2, \bar{\Psi}^2$ as follows

$$\Gamma = \Gamma(0) + \Gamma(2) + \Gamma(3) + \cdots,$$

where the term $\Gamma(n)$ in the bosonic sector corresponds to $\Gamma(n) \sim F^{4+2n}/M^{2+2n}$ and contains terms $c_{m,l} \Psi^{2m} \bar{\Psi}^{2l}$ with $m + l = n$.

As it was shown in Ref. [11, 12] any $\Gamma(n)$ term can be reconstructed to $\mathcal{N} = 2$ form. In particular, one can obtain an expression

$$\Gamma(0) = \frac{1}{(4\pi)^2} \int d^{12}z \left( \ln \mathcal{W} \ln \bar{\mathcal{W}} + \sum_{k=1}^\infty \frac{1}{k^2(k+1)} \chi^k \right),$$
where \( X \) was defined in (1). The second term in (10) can be transformed to the form (1). We see that the expression (10) is just the effective Lagrangian (1) found in [12]. Direct analysis also leads to the following expression for \( \Gamma(2) \) (\( \sim F^8 \)) in (9):

\[
\Gamma(2) = \frac{1}{2 \cdot 5! \cdot (4\pi)^2} \int d^{12}z \, \Phi^2 \Phi^2 \left( \frac{1}{(1-X)^2} + \frac{4}{(1-X)} + \frac{6X - 4}{X^3} \ln(1-X) + 4 \frac{X - 1}{X^2} \right),
\]

where \( \Phi^2 = \frac{1}{W^2} D^4 \ln \bar{W} \) is \( N = 2 \) scalar. Unfortunately, we can not guarantee that the reconstructed effective action will be invariant under the undeformed hidden \( N = 2 \) supersymmetry.

4 Construction of proper \( N = 4 \) supersymmetric effective action

All obtained \( N = 2 \) supersymmetric contributions should not be invariant under the undeformed hidden \( N = 2 \) supersymmetry transformation because of the background choice [2] and the gauge-fixing procedure [5]. The proper \( N = 4 \) calculations should take into account vector derivatives along with hypermultiplet derivatives. It is obvious that in order to obtain \( N = 4 \) supersymmetric contributions from the ones given in the previous section, we have to add to each term in the derivative expansion of (8) some extra terms containing fields \( \lambda = W \) of the vector multiplet, which are presented in the effective action [8], as well as fields \( \psi = Dq \) of the hypermultiplet, which are absent in our calculations because of the used background.

Let consider the on-shell \( N = 4 \) supersymmetric effective action which is described by manifestly \( N = 2 \) supersymmetric effective Lagrangian depending on \( \mathcal{W}, \bar{\mathcal{W}}, \) their spinor derivatives, \( q^+ \) and spinor derivatives of \( q^+ \). The superfield effective Lagrangian have to be dimensionless and chargeless. The dimensional quantities \( D^- q^+, (D^- q^+)^2, \) ... can be compensated by \( N = 2 \) strengths and their spinor derivatives. Hence, any contribution \( \Gamma(n) \) to the effective Lagrangian must be a finite order polynomial in derivatives \( D^- q^+ \) with the dimensionless coefficients \( g_{n,k}(X) \) and some polynomial \( P_{n,k}(D^i \mathcal{W}, D^i \bar{\mathcal{W}}, \mathcal{W}, \bar{\mathcal{W}}) \).

Symbolically it can be written as follows

\[
\Gamma(n) = \Gamma(n;0) + \Gamma(n;1) + \cdots + \Gamma(n;k), \quad \Gamma(n;m) = g_{n,m}(X) P_{n,m}(D^i \mathcal{W}, D^i \bar{\mathcal{W}}, \mathcal{W}, \bar{\mathcal{W}})(D^- q^+)^m,
\]

where \( m \) corresponds to the power of the derivatives \( D^- q^+ \) and \( \bar{D}^- q^+ \). If it is possible for some fixed \( n \), the polynomials [12] under undeformed hidden \( N = 2 \) transformation should transform as

\[
\Gamma(n;m) \rightarrow \Gamma(n;m) \oplus \Gamma(n;m+1).
\]

The analysis of invariance is greatly simplified when we calculate only the first \( (D^+ q^- \)-independent) term in the expansion [12]. One can examine that the obtained by direct calculation part of \( \Gamma_2 \) term is non-invariant under hidden \( N = 2 \) transformation. The transformation structure [13] allows to construct \( N = 4 \) supersymmetric on-shell term \( \Gamma_{(2,0)} \) because the transformation of the other terms \( \Gamma_{(2,k)}, k > 0 \) doesn’t affect on it.
Let’s suppose that $\Gamma_{(2,0)}$ can be rewritten as

$$
\Gamma_{(2,0)} = \frac{1}{2(4\pi)^2} \int d^{12}z I, \quad I = \sum_{n=0}^{\infty} I_n = \sum_{n=0}^{\infty} c_n \Psi^2 \bar{\Psi}^2 \left( \frac{-2q^a + q^-}{\mathcal{W} \bar{\mathcal{W}}} \right)^n.
$$

We rewrite all in terms of $\mathcal{N} = 2$ harmonic superspace and trace transformations evoked by parameters $\epsilon^{\alpha a}$. Saving only terms which can make contribution to the term $\Gamma_{(2,0)}$ one can find the variation of the general term

$$
\delta I_n = \delta^{(w)} I_n + \delta^{(q)} I_n = I_n \left[ -\frac{(n+2)(n+6)\delta \mathcal{W}}{(n+4)} + c_n \Psi^2 \bar{\Psi}^2 \left( \frac{-2q^b + q^-}{\mathcal{W} \bar{\mathcal{W}}} \right)^{n-1} \left[ -4n q^a + \delta q_a \right] \right].
$$

The chain of cancellation between variations $\delta^{(w)} I_{n+1}$ and $\delta^{(q)} I_n$ will occur when the recursion condition is satisfied

$$
c_n = c_{n-1} \frac{(n+1)(n+5)}{n(n+3)} \Rightarrow c_n = \frac{1}{6 \cdot 5!} (n+5)(n+4)(n+1).
$$

Summing the series (14), we find the proper leading part $\Gamma_{(2,0)}$ in the expansion (12) of on-shell $\mathcal{N} = 4$ supersymmetric $F^8$-term in the closed form

$$
\Gamma_{(2,0)} = \frac{1}{72} \frac{1}{(4\pi)^2} \int d^{12}z d\mathcal{W} \left( \Psi^2 \bar{\Psi}^2 \frac{1 - X + \frac{3}{16} X^2}{(1 - X)^4} \right).
$$

Thus, the leading bosonic part of complete on-shell $\mathcal{N} = 4$ supersymmetric extension of $F^8$ invariant is finally established.

## 5 Summary

We have studied the one-loop effective action in $\mathcal{N} = 4$ SYM theory, depending on $\mathcal{N} = 2$ vector multiplet and hypermultiplet fields. The calculations of superfield functional determinants are done on specific $\mathcal{N} = 1$ superfield background corresponding to constant Abelian strength $F_{mn}$ and constant hypermultiplet fields. The effective action depending on all fields of $\mathcal{N} = 4$ vector multiplet is restored on the base of calculations only in quantum $\mathcal{N} = 1$ vector multiplet sector by functional arguments replacement (see (6) and (8)). Obtained results are presented in a manifest $\mathcal{N} = 2$ supersymmetric form. The complete $\mathcal{N} = 4$ supersymmetric low-energy effective action, which has been discovered in [5], has been obtained. All terms (except the leading one) in derivative expansion of the effective action are not invariant under hidden $\mathcal{N} = 2$ supersymmetry transformations. We have considered the first subleading term in expansion of the effective action in $\mathcal{N} = 2$ vector multiplet sector ($F^8$-term written via $\mathcal{N} = 2$ superconformal invariants depending on strengths $\mathcal{W}, \bar{\mathcal{W}}$ and their spinor derivatives [4]) and proved that it can be completed up to on-shell $\mathcal{N} = 4$ supersymmetric form by the hypermultiplet dependent terms and presented as polynomial in hypermultiplet spinor derivatives. The first leading term of this polynomial, which depends on hypermultiplet but does not depend on its derivatives, is given in explicit form (17).
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