On a Fractional Nirenberg problem involving the square root of the Laplacian on $S^3$

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Abstract

In this paper, we are devoted to establishing the compactness and existence results of the solutions to the fractional Nirenberg problem for $n = 3$, $\sigma = 1/2$, when the prescribing $\sigma$-curvature function satisfies the $(n - 2\sigma)$-flatness condition near its critical points. The compactness results are new and optimal. In addition, we obtain a degree-counting formula of all solutions. From our results, we can know where blow up occur. Moreover, for any finite distinct points, the sequence of solutions that blow up precisely at these points can be constructed. We extend the results of Li in [26, CPAM, 1996] from the local problem to nonlocal cases.

Key words: Fractional Laplacian, Nirenberg problem, Blow up analysis.
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1 Introduction

Great attention has been focused on the study of fractional and nonlocal operators of elliptic type, both for pure mathematical research and in view of concrete real-world applications. This type of operator arises in a quite natural way in many different contexts, such as, the thin obstacle problem, optimization, phase transitions, minimal surfaces, materials science, water waves, population dynamics, geophysical fluid dynamics and mathematical finance. For more details and applications, see [3, 7, 8, 14, 18, 21, 22, 28, 34] and references therein.

In this paper, we are concerned with the Nirenberg’s problem in the fractional setting which constitutes in itself a branch in geometric analysis. We first introduce

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the Nirenberg problem. Let \((S^n, g_0)\) be the standard \(n\)-sphere. The Nirenberg problem is the following: which function \(K\) on \(S^2\) is the Gauss curvature of a metric \(g\) on \(S^2\) conformally equivalent to \(g_0\)? If we write \(g = e^v g_0\), this problem is equivalent to finding a function \(v\) on \(S^2\) to solving

\[-\Delta_{g_0} v + 1 = K(x)e^{2v} \quad \text{on } S^2,\]

where \(\Delta_{g_0}\) denotes the Laplace-Beltrami operator associated with the metric \(g_0\).

Naturally one may ask a similar question in higher dimensional case, namely which function \(K\) on \(S^n (n \geq 3)\) is the scalar curvature of a metric \(g\) on \(S^n\) conformally equivalent to \(g_0\)? If we write \(g = v^{4/(n-2)} g_0\), this problem is equivalent to finding a function \(v\) on \(S^n\) which satisfies the following equation:

\[-\Delta_{g_0} v + c(n) R_0 v = c(n) K(x)v^{\frac{n+2}{n-2}} \quad \text{on } S^n,\]

where \(c(n) = (n-2)/(4(n-1))\), \(R_0 = n(n-1)\) is the scalar curvature of \(g_0\).

It is well known that a necessary condition for solving (1.1) or (1.2) is that \(K\) should be positive somewhere. Kazdan and Warner [23] obtained another necessary condition for the existence of solutions by exploiting the center dilation conformal transformations of \(S^n\).

The first significant result on the Nirenberg problem was made by Koutroufiotis [24], which established the existence of the solutions to (1.1) by assuming that \(K\) is an antipodally symmetric function which close to 1. Morse [29] proved the existence of antipodally symmetric solutions to (1.1) for all antipodally symmetric functions \(K\) which are positive somewhere. Later on, Chang and Yang [11] further extended this existence result to the case of \(K\) without any symmetry assumption. In addition, Bahri and Coron [6] gave a sufficient condition for existence of the solutions to (1.2) in dimension \(n = 3\) by assuming that \(K(x)\) has only nondegenerate critical points. As for the compactness of all solutions in dimensions \(n = 2, 3\), Chang et al. [10], Han [19], and Schoen and Zhang [33] proved that a sequence of solutions cannot blow up at more than one point.

Li [25, 26] established the compactness and existence results for (1.2) by characterizing the flatness order of \(K(x)\) near its critical points with \((*)_\beta\) conditions. More precisely, the cases of \(\beta > n-2\) and \(\beta = n-2\) are given in [25] and [26], respectively. In these two papers, the compactness result is very different from the previous low-dimensional case. In fact, when \(n = 2\) or \(n = 3\), a sequence of solutions to the Nirenberg problem cannot blow up at more than one point. However, if \(n > 3\), there could be blow up at many points, which considerably complicates the study of the problem.

The linear operators defined on left-hand side of (1.1) and (1.2) are called the conformal Laplacian associated to the metric \(g_0\) and are denoted as \(P^0_1\). For any Riemannian manifold \((M, g)\), let \(R_g\) be the scalar curvature of \((M, g)\), and the conformal Laplacian be defined as \(P^g_1 = -\Delta_g + \frac{n-2}{4(n-1)} R_g\). The Paneitz operator \(P^g_2\) is another
conformal invariant operator, which was discovered by Paneitz [30]. Graham et al. [16] generalized the operators $P^g_1$ and $P^g_2$ to a sequence of integer order conformally covariant elliptic operators $P^g_k$ for $k \in \{1, 2, \cdots \}$ if $n$ is odd, and $k \in \{1, \cdots, n/2 \}$ if $n$ is even. Furthermore, Peterson [31] constructed an intrinsically defined conformally covariant pseudo-differential operator of arbitrary real number order. Graham and Zworski [17] introduced a mesomorphic family of conformally invariant operators on the conformal infinity of asymptotically hyperbolic manifolds. Chang and González [9] proved that the operator $P^g_σ$ of non-integer order $σ \in (0, n/2)$ can be defined as a Dirichlet-to-Neumann operator of a conformally compact Einstein manifold by using localization method in [8]. This lead naturally to a fractional order curvature $R^g_σ := P^g_σ(1)$, which will be called $σ$-curvature in this paper. The fractional operators $P^g_σ$ and their associated fractional order curvatures $P^g_σ(1)$ have been the subject of many studies, for instance, see [1, 2, 13, 12, 21, 22, 27].

As in the Nirenberg problem associated to $P^g_1$, the question of prescribing $σ$-curvature can be formulated as fractional Nirenberg problem as follows: which function $K$ on $S^n$ is the $σ$-curvature of a metric $g$ on $S^n$ conformally equivalent to $g_0$? If we denote $g = v^{4/(n-2σ)}g_0$, this problem can be expressed as finding the solution of the following nonlinear equation with critical exponent:

\[
P^g_σ(v) = c(n, σ)Kv^{\frac{n+4σ}{n-2σ}} \quad \text{on} \ S^n, \tag{1.3}
\]

where $c(n, σ) = \Gamma(\frac{n}{2} + σ)/\Gamma(\frac{n}{2} - σ)$, $K$ is a function defined on $S^n$,

\[
P^g_σ = \frac{\Gamma(B + \frac{1}{2} + σ)}{\Gamma(B + \frac{1}{2} - σ)} \sqrt{-Δg_0 + \left(\frac{n-1}{2}\right)^2},
\]

and $Γ$ is the Gamma function. In what follows, $P^g_σ$ is simply written as $P_σ$.

Let $K \in C^{1,1}(S^n)$ be a positive function and $β$ is a positive constant, we say that $K$ satisfies the flatness condition $(*)_β$ if for every critical point $ξ_0$ of $K$, in some geodesic normal coordinates $\{y_1, \cdots, y_n\}$ centered at $ξ_0$, there exists a small neighborhood $Ω$ of $0$ and $a_j(ξ_0) \neq 0$, $\sum_{j=1}^{n} a_j(ξ_0) \neq 0$, such that

\[
K(y) = K(0) + \sum_{j=1}^{n} a_j(ξ_0)|y_j|^β + R(y) \quad \text{in} \ Ω,
\]

where

\[
\sum_{s=0}^{[β]} |\nabla^s R(y)||y|^{-β-s} \rightarrow 0 \quad \text{as} \ y \rightarrow 0,
\]

here $\nabla^s$ denotes all possible derivatives of order $s$ and $[β]$ is the integer part of $β$.

For $0 < σ < 1$, Jin et al. [21, 22] proved the existence of the solutions to (1.3) and derived some compactness properties when $K$ satisfies the $(*)_β$ condition with the flatness order $β \in (n - 2σ, n)$, by using the approach based on approximation
of the solutions to (1.3) by a blow up subcritical method. Since their conclusions is valid only when the flatness order $\beta > n - 2\sigma$, some very interesting functions $K$ are excluded. In fact, note that an important class of functions, which is worth including in the results of existence and compactness for (1.3), are the Morse functions with only nondegenerate critical points. Such functions satisfy the $(\ast)_2$ condition.

By using a self-contained approach, the description of lack of compactness and the existence results of the solutions to (1.3) were given by Abdelhedi et al. [2] when $\beta \in (1, n - 2\sigma]$, and by Chthiou and Abdelhedi [13] when $\beta \in [n - 2\sigma, n]$. However, under the assumption of the flatness order $\beta = n - 2\sigma$, which is called the critical flatness condition in this paper, the precise compactness result and the degree-counting formula of the solutions to (1.3) is unknown. Therefore, it is natural to study the compactness results when the prescribing curvature function $K$ satisfy the critical flatness condition. When $\sigma = 1$ and $K$ satisfy the critical flatness condition, namely $\beta = n - 2$, the compactness and existence results of the solutions to (1.2) was obtained by Li [26].

What we consider here is the case when the prescribing $\sigma$-curvature function satisfy the critical flatness order $\beta = n - 2\sigma = 2$, which include an important class of functions, for instance the Morse functions. In addition, we can establish the optimal compactness result and give a degree-counting formula of all solutions to (1.3) in this case. In this paper, we study the equation (1.3) when $n = 3$ and $\sigma = 1/2$. Especially, from our results, we show that a sequence of solutions to (1.3) can blow up at more than one point and for any finite distinct points on $S^3$, we can construct a sequence of solutions to (1.3) that blow up precisely at these points. In a forthcoming paper, we deal with the higher order case, i.e., $n = 2\sigma + 2$ for any $1 < \sigma < n/2$.

Before state our results, we introduce some definitions and notations.

For $\sigma \in (0, 1)$, the fractional Laplacian is a nonlocal pseudo-differential operator, taking the form:

$$\begin{align*}
(-\Delta)^\sigma u(x) : = C(n, \sigma) P. V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2\sigma}} dy &= C(n, \sigma) \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x - y|^{n+2\sigma}} dy, \quad x \in \mathbb{R}^n, \\
\end{align*}
(1.4)$$

where $B_\varepsilon(x)$ is the ball centered at $x \in \mathbb{R}^n$ with radius $\varepsilon$. Here $P. V.$ is a commonly used abbreviation for “in the principal value sense” and $C(n, \sigma)$ is a dimensional constant that depends on $n$ and $\sigma$, precisely given by

$$C(n, \sigma) := \left( \int_{\mathbb{R}^n} \frac{1 - \cos \zeta_1}{|\zeta|^{n+2\sigma}} d\zeta \right)^{-1}$$

with $\zeta = (\zeta_1, \zeta')$, $\zeta' \in \mathbb{R}^{n-1}$.

The singular integral given in (1.4) can be written as a weighted second-order...
differential quotient as follows (see [15, Lemma 3.2]):

\[ (-\Delta)^{\sigma} u(x) := -\frac{1}{2} C(n, \sigma) \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2\sigma}} \, dy, \quad x \in \mathbb{R}^n. \]

This operator is well defined in \( \mathcal{S} \), the Schwartz space of rapidly decreasing \( C^\infty \) function in \( \mathbb{R}^n \), and it can be equivalently defined in terms of the Fourier transform:

\[ (-\Delta)^{\sigma} u(x) := \mathcal{F}^{-1}(|\xi|^{2\sigma} (\mathcal{F} u)(\xi))(x), \quad x \in \mathbb{R}^n. \]

where \( \mathcal{F} \) denotes the Fourier transform operator.

Let \( \dot{H}^\sigma(\mathbb{R}^n) \) denote the closure of the set \( C^\infty_c(\mathbb{R}^n) \) of compactly supported smooth functions under the norm \( \| u \|_{\dot{H}^\sigma(\mathbb{R}^n)} = \| |\xi|^\sigma \mathcal{F}(u)(\xi) \|_{L^2(\mathbb{R}^n)} \).

For any \( u \in \dot{H}^\sigma(\mathbb{R}^n) \), we set

\[ U(x, t) = \mathcal{P}_\sigma[u] := \int_{\mathbb{R}^n} \mathcal{P}_\sigma(x - \xi, t) u(\xi) d\xi, \quad (x, t) \in \mathbb{R}^{n+1} := \mathbb{R}^n \times (0, \infty), \quad (1.5) \]

where

\[ \mathcal{P}_\sigma(x, t) = \beta(n, \sigma) \frac{t^{2\sigma}}{(|x|^2 + t^2)^{(n+2\sigma)/2}}, \]

with a constant \( \beta(n, \sigma) \) such that \( \int_{\mathbb{R}^n} \mathcal{P}_\sigma(x, 1) dx = 1 \). Let us denote that for any open set \( D \subset \mathbb{R}^{n+1}_+ \), the space \( L^2(t^{1-2\sigma}, D) \) is the Banach space endowed with the norm

\[ \| V \|_{L^2(t^{1-2\sigma}, D)} := \left( \int_D t^{1-2\sigma} V^2 \, dX \right)^{1/2} < \infty, \]

for any \( V \in L^2(t^{1-2\sigma}, D) \). Then the above \( U(x, t) \in L^2(t^{1-2\sigma}, K) \) for any compact set \( K \) in \( \mathbb{R}^{n+1}_+ \), \( \nabla U(x, t) \in L^2(t^{1-2\sigma}, \mathbb{R}^{n+1}) \) and \( U(x, t) \in C^\infty(\mathbb{R}^{n+1}_+) \).

By the celebrated work by Caffarelli and Silvestre (see [8]), one can find that \( U(x, t) \) satisfies

\[ \text{div}(t^{1-2\sigma} \nabla U) = 0 \quad \text{in} \ \mathbb{R}^{n+1}_+, \]

\[ \| \nabla U \|_{L^2(t^{1-2\sigma}, \mathbb{R}^{n+1}_+)} = N_\sigma \| u \|_{\dot{H}^\sigma(\mathbb{R}^n)}, \]

and

\[ -\lim_{t \to 0} t^{1-2\sigma} \partial_t U(x, t) = N_\sigma (-\Delta)^{\sigma} u(x) \quad \text{in} \ \mathbb{R}^n \]

in the distribution sense, where \( N_\sigma = 2^{1-2\sigma} \Gamma(1 - \sigma)/\Gamma(\sigma) \). Here one refer \( U(x, t) = \mathcal{P}_\sigma[u] \) in (1.5) as the extension of \( u \in \dot{H}^\sigma(\mathbb{R}^n) \).
Let $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) be a domain, $\tau_i \geq 0$, $i = 1, 2, \cdots$, satisfy $\lim_{i \to \infty} \tau_i = 0$, $p_i = (n+2\sigma)/(n-2\sigma) - \tau_i$, and $K_i \in C^{1,1}(\Omega)$ satisfy, for some constants $A_1, A_2 > 0$,

$$1/A_1 \leq K_i(x) \leq A_1 \quad \text{for all } x \in \Omega, \quad \|K_i\|_{C^{1,1}(\Omega)} \leq A_2. \quad (1.6)$$

Let $u_i \in L^\infty(\Omega) \cap H^\sigma(\mathbb{R}^n)$ with $u_i \geq 0$ in $\mathbb{R}^n$ satisfy

$$(-\Delta)^\sigma u_i = c(n, \sigma)K_i u_i^p \quad \text{in } \Omega, \quad (1.7)$$

where $c(n, \sigma)$ is as in (1.3).

**Definition 1.1.** Suppose that $\{K_i\}$ satisfies (1.6) and $\{u_i\}$ satisfies (1.7). A point $\overline{y} \in \Omega$ is called a blow up point of $\{u_i\}$ if there exists a sequence $y_i$ tending to $\overline{y}$ such that $u_i(y_i) \to \infty$.

**Definition 1.2.** A blow up point $\overline{y} \in \Omega$ is called an isolated blow up point of $\{u_i\}$ if there exist $0 < \overline{r} < \text{dist}(\overline{y}, \Omega)$, $\overline{C} > 0$, and a sequence $y_i$ tending to $\overline{y}$, such that $y_i$ is a local maximum point of $u_i$, $u_i(y_i) \to \infty$ and

$$u_i(y) \leq \overline{C}|y - y_i|^{-2\sigma(p_i - 1)} \quad \text{for all } y \in B_{\overline{r}}(y_i). \quad (1.8)$$

Let $y_i \to \overline{y}$ be an isolated blow up point of $\{u_i\}$, and define, for $r > 0$,

$$\overline{u}_i(r) := \frac{1}{|\partial B_r(y_i)|} \int_{\partial B_r(y_i)} u_i \quad \text{and} \quad \overline{w}_i(r) := r^{2\sigma/(p_i - 1)}\overline{u}_i(r).$$

**Definition 1.3.** A point $y_i \to \overline{y} \in \Omega$ is called an isolated simple blow up point if $y_i \to \overline{y}$ is an isolated blow up point such that for some $\rho > 0$ (independent of $i$), $\overline{w}_i$ has precisely one critical point in $(0, \rho)$ for large $i$.

For $K \in C^2(\mathbb{S}^3)$, we introduce the following notation:

$$\mathcal{K} = \{q \in \mathbb{S}^3 : \nabla g_0 K(q) = 0\},$$

$$\mathcal{K}^+ = \{q \in \mathbb{S}^3 : \nabla g_0 K(q) = 0, \Delta g_0 K(q) > 0\},$$

$$\mathcal{K}^- = \{q \in \mathbb{S}^3 : \nabla g_0 K(q) = 0, \Delta g_0 K(q) < 0\},$$

$$\mathcal{M}_K = \{v \in C^2(\mathbb{S}^3) : v \text{ satisfies } (1.3)\}. \quad (1.9)$$

For any $k \in \mathbb{N}_+$ distinct points $q^{(1)}, \cdots, q^{(k)} \in \mathcal{K} \setminus \mathcal{K}^+$, the following $k \times k$ real symmetric matrix $M$ is defined by, for $i, j = 1, \cdots, k$,

$$M_{ii} = -\frac{\Delta g_0 K(q^{(i)})}{K(q^{(i)})^3},$$

$$M_{ij} = -6\frac{G_{q^{(i)}}(q^{(j)})}{K(q^{(i)})K(q^{(j)})}, \quad i \neq j, \quad (1.10)$$
where
\[ G_{q^{(i)}}(q^{(j)}) = \frac{1}{1 - \cos d(q^{(i)}, q^{(j)})} \] (1.11)
is the Green’s function of \( P_\sigma \) on \( S^3 \), and \( d(\cdot, \cdot) \) denotes the geodesic distance. Let \( \mu(M) \) denote the smallest eigenvalue of \( M \), and when \( k = 1 \),
\[ \mu(M) = M = -\frac{\Delta_{g_0} K(q^{(1)})}{K(q^{(1)})^3}. \]

Now we are going to present our first result about characterization of blow up behavior of the solutions, which is:

**Theorem 1.1.** Let \( K \in C^2(S^3) \) be a positive function and \( \mathcal{K}, \mathcal{K}^-, \mathcal{K}^+ \) be as in (1.9). Let \( p_i \) satisfy \( p_i \leq 2, p_i \to 2, \tau_i = 2 - p_i, K_i \in C^2(S^3) \) satisfy \( K_i \to K \) in \( C^2(S^3) \), and \( v_i \in C^2(S^3) \) satisfy
\[ P_\sigma(v_i) = K_i v_i^{p_i}, \quad v_i > 0 \text{ on } S^3, \] (1.12)
and
\[ \lim_{i \to \infty} \max_{S^3} v_i = \infty. \]

Then there exists a constant \( \delta^* > 0 \) depending only on \( \min_{S^3} K \) and \( \|K\|_{C^2(S^3)} \), such that after passing to a subsequence,

(i) \( \{v_i\} \) (still denote the subsequence by \( \{v_i\} \) ) has only isolated simple blow up points \( q^{(1)}, \ldots, q^{(k)} \in \mathcal{K} \setminus \mathcal{K}^+ \) (\( k \geq 1 \)) with \( |q^{(j)} - q^{(0)}| \geq \delta^*, \forall j \neq \ell, \) and \( \mu(M(q^{(1)}, \ldots, q^{(k)})) \geq 0 \). Furthermore, \( q^{(1)}, \ldots, q^{(k)} \in \mathcal{K}^- \) if \( k \geq 2 \).

(ii) Let \( q^{(1)}, \ldots, q^{(k)} \) be as in (i), and \( q^{(j)}_i \) be the local maximum of \( v_i \) with \( q^{(j)}_i \to q^{(j)}, \) we have
\[ \lambda_j := K(q^{(j)})^{-1} \lim_{i \to \infty} v_i(q_i^{(1)})(v_i(q_i^{(j)}))^{-1} \in (0, \infty), \] (1.13)
\[ \mu^{(j)} := \lim_{i \to \infty} \tau_i v_i(q_i^{(j)})^2 \in [0, \infty). \] (1.14)

(iii) Let \( \lambda_j, \mu^{(j)}, j = 1, \ldots, k \) be as in (ii), then when \( k = 1 \),
\[ \mu^{(1)} = -4\frac{\Delta_{g_0} K(q^{(1)})}{K(q^{(1)})^3}, \] (1.15)
when \( k \geq 2, \)
\[ \sum_{\ell=1}^{k} M_{ij}(q^{(1)}, \ldots, q^{(k)}) \lambda_\ell = \lambda_j \mu^{(j)}, \quad \forall j : 1 \leq j \leq k. \] (1.16)
(iv) \( \mu^{(j)} \in (0, \infty), \forall j = 1, \cdots, k \), if and only if \( \mu(M(q^{(1)}, \cdots, q^{(k)})) > 0 \).

In what follows, we define

\[
\mathcal{A} = \{ K \in C^2(S^3) : K > 0 \text{ on } S^3, \Delta_{g_0} K \neq 0 \text{ on } \mathcal{H}, \mu(M(q^{(1)}, \cdots, q^{(k)})) \neq 0, \forall q^{(1)}, \cdots, q^{(k)} \in \mathcal{H}^-, k \geq 2 \};
\]

and

\[
C^2(S^3)^+ := \{ K \in C^2(S^3) : K > 0 \text{ on } S^3 \}.
\]

It is obvious that \( \mathcal{A} \) is open in \( C^2(S^3) \) and \( \mathcal{A} \) is dense in \( C^2(S^3)^+ \) with respect to the \( C^2 \) norm.

We will introduce an integer-valued continuous function Index: \( \mathcal{A} \to \mathbb{Z} \), which has an explicit formula for \( K \in \mathcal{A} \) being a Morse function.

**Definition 1.4.** We define \( \text{Index}: \mathcal{A} \to \mathbb{Z} \) by the following properties:

(i) For any Morse function \( K \in \mathcal{A} \) with \( K^- = \{ q^{(1)}, \cdots, q^{(s)} \} \), we define

\[
\text{Index}(K) = -1 + \sum_{k=1}^{s} \sum_{1 \leq i_1 < \cdots < i_k \leq s} (-1)^{k-1 + \sum_{j=1}^{k} i(j)} \mu(M(q^{(i_1)}, \cdots, q^{(i_k)})) > 0,
\]

where \( i(q^{(i)}) \) denotes the Morse index of \( K \) at \( q^{(i)} \).

(ii) \( \text{Index}: \mathcal{A} \to \mathbb{Z} \) is continuous with respect to the \( C^2(S^3) \) norm of \( \mathcal{A} \) and hence is locally constant.

**Remark 1.1.** The existence and uniqueness of the Index mapping follows from Theorem 1.2 and the proof of Theorem 1.3 below.

Our second result is about the compactness of the solutions when \( K \in \mathcal{A} \), which is:

**Theorem 1.2.** Let \( \mathcal{A} \) be as in (1.17) and \( K \in \mathcal{A} \). Then there exists a constant \( C = C(K) > 0 \), such that for any \( K_i \to K \) in \( C^2(S^3) \), and any \( v_i \in \mathcal{M}_{K_i} \), we have

\[
1/C \leq \liminf_{i \to \infty} (\min_{S^3} v_i) \leq \limsup_{i \to \infty} (\max_{S^3} v_i) \leq C.
\]

Furthermore, for any \( \alpha \in (0, 1) \), there exists a constant \( R = R(K, \alpha) > 0 \), such that for any \( v \in \mathcal{M}_K \), we have

\[
1/R < v(x) < R, \quad \forall x \in S^3 \quad \text{and} \quad \|v\|_{C^{2, \alpha}(S^3)} < R,
\]

where \( \mathcal{M}_{K_i} \) and \( \mathcal{M}_K \) are as in (1.9).
For any given $0 < \alpha < 1$, $R > 0$, we define
\[
\mathcal{O}_R := \{v \in C^{2,\alpha}(\mathbb{S}^3) : 1/R < v < R, \|v\|_{C^{2,\alpha}(\mathbb{S}^3)} < R\}.
\] (1.20)

Our third result is about degree-counting formula and the existence of the solutions to (1.3), which is:

**Theorem 1.3.** Let $\mathcal{A}$ be as in (1.17), $K \in \mathcal{A}$ and $\text{Index}(K)$ be as in Definition 1.4. Then for any $\alpha \in (0, 1)$, there exists a constant $R_0 = R_0(K, \alpha)$, such that for all $R > R_0$, we have
\[
\text{deg}_{C^{2,\alpha}}(v - P^{-1}(Kv^2), \mathcal{O}_R, 0) = \text{Index}(K),
\] (1.21)
where $\text{deg}_{C^{2,\alpha}}$ denotes the Leray-Schauder degree in $C^{2,\alpha}(\mathbb{S}^3)$.

Furthermore, if $\text{Index}(K) \neq 0$, then (1.3) has at least one solution.

**Remark 1.2.** It follows from Theorem 1.1 that when $K \in \mathcal{A}$, the solutions to (1.3) belong to $\mathcal{O}_R$ for some $R > 0$. We call the left-hand side of (1.21) the total degree of the solutions to the fractional equation. From Theorem 1.3, the total degree is $\text{Index}(K)$.

For any finite subset $\mathcal{R} \subset \mathbb{S}^3$, we use $\sharp \mathcal{R}$ to denote the number of elements in the set $\mathcal{R}$. Let us now state a corollary of Theorem 1.3, which is:

**Corollary 1.1.** Let $\mathcal{A}$ be as in (1.17) and $K \in \mathcal{A}$ be a Morse function satisfying $\sharp \mathcal{K}^- \leq 1$ or for any distinct $P, Q \in \mathcal{K}^-$,
\[
\Delta_{g_0}K(P)\Delta_{g_0}K(Q) < 9K(P)K(Q).
\] (1.22)
Then for any $\alpha \in (0, 1)$, there exists a constant $C = C(K, \alpha) > 0$, such that for all solutions $v$ of (1.3), we have
\[
1/C < v(x) < C, \quad \forall x \in \mathbb{S}^3, \quad \|v\|_{C^{2,\alpha}(\mathbb{S}^3)} < C,
\]
and for all $R \geq C$,
\[
\text{deg}_{C^{2,\alpha}}(v - P^{-1}(Kv^2), \mathcal{O}_R, 0) = \text{Index}(K) = -1 + \sum_{\nabla_{g_0}K(q_0) = 0 \atop \Delta_{g_0}K(q_0) < 0} (-1)^{i(q_0)},
\]
where $i(q_0)$ denotes the Morse index of $K$ at $q_0$.

Furthermore, if
\[
\sum_{\nabla_{g_0}K(q_0) = 0 \atop \Delta_{g_0}K(q_0) < 0} (-1)^{i(q_0)} \neq 1,
\]
then (1.3) has at least one solution.
Our fourth result is about the blow up behavior of the solutions when the $\sigma$-curvature function $K \notin \mathcal{A}$, which is:

**Theorem 1.4.** Let $\mathcal{A}$ be as in (1.17) and $C^2(S^3)^+$ be as in (1.18). Then for any $K \in C^2(S^3)^+ \setminus \mathcal{A} = \partial \mathcal{A}$, there exists $K_i \to K$ in $C^2(S^3)$ and $v_i \in \mathcal{M}_{K_i}$, such that

$$\lim_{i \to \infty} (\max_{S^3} v_i) = \infty, \quad \lim_{i \to \infty} (\min_{S^3} v_i) = 0,$$

(1.23)

where $\mathcal{M}_{K_i}$ is as in (1.9).

From Remark 1.2, the total degree of solutions to (1.3) strongly depend on the sign of the smallest eigenvalue of $M(q^{(1)}, \cdots, q^{(k)})$, which plays a role in counting the total degree of solutions and in the compactness result. In fact, the points $q^{(1)}, \cdots, q^{(k)}$ for which $\mu(M(q^{(1)}, \cdots, q^{(k)}))$ is positive characterize the so-called asymptotic in the theory of critical points at infinity developed by Bahri [4, 6]. For instance, considering a continuous family of functions $K_t$ ($0 \leq t \leq 1$), the total degree changes when the smallest eigenvalue of $M_t(q^{(1)}, \cdots, q^{(k)})$ crosses zero while it remains unchanged when other eigenvalues cross zero.

It follows from Theorem 1.4 that when $K \notin \mathcal{A}$, the solutions to (1.3) may blow up. A natural question is where the blow up occur? The following results present the accurate location of the blow up.

For any $K \in C^2(S^3)^+$, we first define

$$\mathcal{H}(K) = \{(q^{(1)}, \cdots, q^{(k)}): k \geq 1, q^{(j)} \in \mathcal{H} \setminus \mathcal{H}^+, \forall j: 1 \leq j \leq k, q^{(j)} \neq q^{(\ell)}, \forall j \neq \ell, \mu(M(q^{(1)}, \cdots, q^{(k)})) = 0\}. \quad (1.24)$$

Our fifth result is about the location of blowing up when $K \notin \mathcal{A}$, which is:

**Theorem 1.5.** Let $\mathcal{A}$ be as in (1.17) and $C^2(S^3)^+$ be as in (1.18). For a given function $K \in C^2(S^3)^+ \setminus \mathcal{A}$, we have the following results:

(i) For any $K_i \to K$ in $C^2(S^3)$, and $v_i \in \mathcal{M}_{K_i}$ with $\max_{S^3} v_i \to \infty$, then for some $(q^{(1)}, \cdots, q^{(k)}) \in \mathcal{H}(K)$, $\{v_i\}$ (after passing to a subsequence) blows up at precisely the $k$ points.

(ii) For any $(q^{(1)}, \cdots, q^{(k)}) \in \mathcal{H}(K)$, there exists $K_i \to K$ in $C^2(S^3)$, $v_i \in \mathcal{M}_{K_i}$, such that $\{v_i\}$ blows up at precisely the $k$ points.

**Corollary 1.2.** For any $k \in \mathbb{N}_+$ distinct points $q^{(1)}, \cdots, q^{(k)} \in S^3$, there exists a sequence of Morse functions $\{K_i\} \subset \mathcal{A}$, such that for some $v_i \in \mathcal{M}_{K_i}$, $\{v_i\}$ blows up at precisely the $k$ points.

When further characterizing the blow up behavior of the solution to (1.3) (see Section 3 below), we mainly use the Pohozaev type identity (see Proposition 2.1 below) and its property (see Proposition 2.2 below) to judge the sign of the Laplace
of the prescribing curvature function at the blow up point. Due to the limitation of
the form of the Pohozaev type identity, the method in this paper is only effective for
the case $n - 2\sigma = 2$. In a forthcoming paper, we deal with the higher order case, i.e.,
n = 2\sigma + 2$ for any $1 < \sigma < n/2$.

The paper is organized as follows: In section 2, we recall some known results on
blow up analysis of the fractional Nirenberg problem obtained by Jin-Li-Xiong [21].

In section 3, our main task is to prove Theorem 1.1 and Theorem 1.2. By using
the method of subcritical approximation, we obtain Theorem 1.1, which further char-
acterizes the blow up points for solutions to (1.3). More precisely, we consider the
subcritical equation with $\tau > 0$ small:

$$P_\sigma v = Kv^{2-\tau}, \quad v > 0 \text{ on } S^3. \quad (1.25)$$

Then we use Theorem 1.1 and some results in [21] to prove Theorem 1.2.

Section 4 is devoted to proving the Theorems 1.3, 1.4, and 1.5. Firstly, we give the
definition of $\Sigma_\tau = \Sigma_\tau(P_1, \ldots, P_k)$, for $P_1, \ldots, P_k \in \mathcal{X}^-$ with $\mu(M(P_1, \ldots, P_k)) > 0$. Then by using Theorem 1.1 and some results in [21], we obtain that for $\tau > 0$
very small, the solutions to (1.25) either stay bounded or stay in one of the $\Sigma_\tau$ (see
Proposition 4.1 below). Furthermore, we obtain the $H^\sigma$ topological degree of the
solutions to (1.25) on $\Sigma_\tau$ (see Theorem 4.1 below). It follows from the above results
that for all $0 < \tau < 2$, the $H^\sigma$ total degree of the solutions to (1.25) is equal to $-1$ (see
Proposition 4.6 below). Then we can conclude that $H^\sigma$ topological degree of those
solutions to (1.25) which remain bounded as $\tau$ tends to zero is equal to Index($K$).

Some well-known results in degree theory imply that the $H^\sigma$ degree contribution
above is equal to the $C^{2,\alpha}$ topological degree of those bounded solutions to (1.25).

Thus, we prove Theorem 1.3. Furthermore, we complete the proof of Theorem 1.4
by using the degree-counting formula and perturbing the function $K$ near its critical
point. In the end, using Theorem 1.1 and the idea of the proof of Theorem 1.4, we
prove Theorem 1.5.

In the Appendix, we provide some useful technical results and elementary esti-
mates.

Finally, we make some conventions on notation. Let $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0, +\infty)$. For
$X = (x, t) \in \mathbb{R}^{n+1}$ and $R \geq 0$, the symbol $B_R(X)$ denotes the balls in $\mathbb{R}^{n+1}$ with
radius $R$ and center $X$, and $B_R^+(X) := B_R^+(X) \cap \mathbb{R}^{n+1}_+$. The symbol $B_R(x)$ denotes the
ball in $\mathbb{R}^n$ with radius $R$ and center $x$. We also write $B_R, B_R^+, B_R$ for $B_R(0), B_R^+(0),$
$B_R(0)$, respectively. We always denote by $C$ a positive constant which is independent
of the main parameters, but it may vary from line to line.

2 Quick review of some known facts

In this section, we review some results about the blow up analysis of the fractional
Nirenberg problem obtained in Jin-Li-Xiong [21].
Let $\sigma \in (0, 1)$, $u_i \in C^2(\Omega) \cap \dot{H}^\sigma(\mathbb{R}^n)$ with $u_i \geq 0$ in $\mathbb{R}^n$ satisfy (1.7) with $K_i$ satisfying (1.6). Let $U_i(x, t)$ be the extension of $u_i$ as in (1.5), we have

$$
\begin{align*}
\left\{ \begin{array}{ll}
\text{div} \left( t^{1-2\sigma} \nabla U_i \right) = 0 & \quad \text{in } \mathbb{R}^{n+1}_+, \\
- \lim_{t \to 0} t^{1-2\sigma} \partial_t U_i(x, t) = c_0 K_i(x) U_i(x, 0)^p & \quad \text{for any } x \in \Omega,
\end{array} \right.
\end{align*}
$$

(2.1)

where $c_0 = 2^{1-2\sigma} c(n, \sigma) \Gamma(1 - \sigma) / \Gamma(\sigma)$.

We say that $U \in H(|t|^{1-2\sigma}, D)$ if $U \in L^2(|t|^{1-2\sigma}, D)$, and its weak derivatives $\nabla U$ exist and belong to $L^2(|t|^{1-2\sigma}, D)$. The norm of $U$ in $H(|t|^{1-2\sigma}, D)$ is given by

$$
\| U \|_{H(|t|^{1-2\sigma}, D)} := \left( \int_D |t|^{1-2\sigma} U^2 \, dX + \int_D |t|^{1-2\sigma} |\nabla U|^2 \, dX \right)^{1/2}.
$$

In the following, for a domain $D \subset \mathbb{R}^{n+1}$ with boundary $\partial D$, we denote by $\partial D$ the interior of $\overline{D} \cap \partial \mathbb{R}^{n+1}$ in $\mathbb{R}^n = \partial \mathbb{R}^{n+1}$, and we set $\partial'' D = \partial D \setminus \partial D$.

**Proposition 2.1** (Pohozaev type identity). Suppose that $K \in C^1(B_{2R})$. Let $U \in H(t^{1-2\sigma}, B^{+}_{2R})$ with $U \geq 0$ in $B^{+}_{2R}$ be a weak solution of

$$
\begin{align*}
\left\{ \begin{array}{ll}
\text{div} \left( t^{1-2\sigma} \nabla U \right) = 0 & \quad \text{in } B^{+}_{2R}, \\
- \lim_{t \to 0} t^{1-2\sigma} \partial_t U(x, t) = K(x) U^p(x, 0) & \quad \text{on } \partial B^{+}_{2R},
\end{array} \right.
\end{align*}
$$

where $p > 0$. Then

$$
\int_{\partial'' B^{+}_{R}} B'(X, U, \nabla U, R, \sigma) + \int_{\partial'' B^{+}_{R}} t^{1-2\sigma} B''(X, U, \nabla U, R, \sigma) = 0,
$$

where

$$
B'(X, U, \nabla U, R, \sigma) = \frac{n - 2\sigma}{2} KU^{p+1} + \langle X, \nabla U \rangle KU^p,
$$

$$
B''(X, U, \nabla U, R, \sigma) = \frac{n - 2\sigma}{2} U \frac{\partial U}{\partial \nu} - \frac{R}{2} |\nabla U|^2 + R \left| \frac{\partial U}{\partial \nu} \right|^2.
$$

(2.2)

**Proposition 2.2.** Let $M \in \mathbb{R}$ and $\alpha(X)$ be some differentiable function near the origin with $\alpha(0) = 0$. Then for $U(X) = |X|^{2\sigma-n} + M + \alpha(X)$, we have

$$
\lim_{\delta \to 0} \int_{\partial'' B^{+}_{\delta}} t^{1-2\sigma} B''(X, U, \nabla U, \delta, \sigma) = -\frac{(n - 2\sigma)^2}{2} M |S^{n-1}| B(n/2, 1 - \sigma),
$$

(2.3)

where $B(\cdot, \cdot)$ is the Beta function.

**Proof.** Since $U(X) = |X|^{2\sigma-n} + M + \alpha(X)$, we have

$$
\nabla U(X) = (2\sigma - n) \delta^{2\sigma-n-2} X + \nabla \alpha(X) \quad \text{on } \partial'' B^{+}_{\delta},
$$

12
and 
\[
\frac{\partial U}{\partial \nu} = \nabla U \cdot \nu = (2\sigma - n)\delta^{2\sigma-n-1} + \frac{\nabla \alpha(X) \cdot X}{\delta} \quad \text{on } \partial''B^+_{\delta}.
\]

It follows that 
\[
|\nabla U|^2 = (2\sigma - n)^2\delta^{4\sigma-2n-2} + 2(2\sigma - n)\delta^{2\sigma-n-2}\nabla \alpha(X) \cdot X + |\nabla \alpha(X)|^2,
\]
and 
\[
\left|\frac{\partial U}{\partial \nu}\right|^2 = (2\sigma - n)^2\delta^{4\sigma-2n-2} + 2(2\sigma - n)\delta^{2\sigma-n-2}\nabla \alpha(X) \cdot X + |\nabla \alpha(X) \cdot X|^2\delta^{-2}
\]
on \partial''B^+_{\delta}. Substituting the above results into (2.2), we can easily obtain
\[
\lim_{\delta \to 0} \int_{\partial''B^+_{\delta}} t^{1-2\sigma} B''(X, U, \nabla U, \delta, \sigma) = \lim_{\delta \to 0} -\frac{(2\sigma - n)^2}{2} M \delta^{2\sigma-n-1} \int_{\partial''B^+_{\delta}} t^{1-2\sigma}
\]
\[
= -\frac{(2\sigma - n)^2}{2} M \int_{\partial''B^+_{\delta}} s^{1-2\sigma}
\]
\[
= -\frac{(n - 2\sigma)^2}{4} M |S^{n-1}|B(n/2, 1 - \sigma)|.
\]

Proposition 2.2 follows from the above. \qed

**Proposition 2.3.** Suppose that for all \( \varepsilon \in (0, 1) \), \( U \in H(t^{1-2\sigma}, B^+_1 \setminus \overline{B^+_\varepsilon}) \) with \( U > 0 \) in \( B^+_1 \setminus \overline{B^+_\varepsilon} \) is a weak solution of
\[
\begin{cases}
\text{div } (t^{1-2\sigma} \nabla U) = 0 & \text{in } B^+_1 \setminus \overline{B^+_\varepsilon}, \\
-\lim_{t \to 0} t^{1-2\sigma} \partial_t U(x, t) = 0 & \text{on } B_1 \setminus \overline{B^+_\varepsilon}.
\end{cases}
\]

Then
\[
U(X) = A|X|^{2\sigma-n} + \mathcal{W}(X),
\]
where \( A \) is a nonnegative constant and \( \mathcal{W} \in H(t^{1-2\sigma}, B^+_1) \) satisfies
\[
\begin{cases}
\text{div } (t^{1-2\sigma} \nabla \mathcal{W}) = 0, & \text{in } B^+_1, \\
-\lim_{t \to 0} t^{1-2\sigma} \partial_t \mathcal{W}(x, t) = 0 & \text{on } B_1.
\end{cases}
\]

**Proposition 2.4.** Suppose that \( u_i \in C^2(\Omega) \cap \dot{H}^\sigma(\mathbb{R}^n) \) with \( u_i \geq 0 \) in \( \mathbb{R}^n \) satisfies (1.7) with \( K_i \) satisfying (1.6), and \( y_i \to 0 \) is an isolated blow up point of \( \{u_i\} \), i.e., for some positive constants \( A_3 \) and \( \tau \) independent of \( i \),
\[
|y - y_i|^{2\sigma/(p_i-1)} u_i(y) \leq A_3 \quad \text{for all } y \in B_\tau \subset \Omega.
\]
(2.4)
Denote $U_i = \mathcal{P}_\sigma[u_i]$ and $Y_i = (y_i, 0)$. Then for any $0 < r < \tau/3$, we have the following Harnack inequality:

$$
\sup_{B^+_r(Y_i) \setminus B^+_{r/2}(Y_i)} U_i \leq C \inf_{B^+_r(Y_i) \setminus B^+_{r/2}(Y_i)} U_i,
$$

where $C$ is a positive constant depending only on $n, \sigma, A_3, \tau$, and $\sup_i \|K_i\|_{L^\infty(B_r(y_i))}$.

**Proposition 2.5.** Under the hypotheses of Proposition 2.4, then for any $R_i \to \infty$ and $\varepsilon_i \to 0^+$, we have, after passing to a subsequence (still denoted as $\{u_i\}$, $\{y_i\}$, etc.),

$$
\|m_i^{-1} u_i (m_i^{-(p_i-1)/2\sigma} \cdot + y_i) - (1 + k_i \cdot |2|^{(2\sigma-n)/2}) \|_{C^2(B_{2R_i}(0))} \leq \varepsilon_i,
$$

$$
R_i m_i^{-(p_i-1)/2\sigma} \to 0 \quad \text{as } i \to \infty,
$$

where $m_i = u_i(y_i)$ and $k_i = K_i(y_i)^{1/\sigma}/4$.

**Proposition 2.6.** Under the hypotheses of Proposition 2.4, and in addition that $y_i \to 0$ is also an isolated simple blow up point with constant $\rho$, we have

$$
\tau_i = O(u_i(y_i)^{-2/(n-2\sigma)+o(1)}) \quad \text{and} \quad u_i(y_i)^{\tau_i} = 1 + o(1).
$$

Moreover,

$$
u_i(y) \leq C u_i^{-1}(y_i) |y - y_i|^{2\sigma-n} \quad \text{for all } |y - y_i| \leq 1.
$$

**Proposition 2.7.** Under the hypotheses of Proposition 2.4, we have

$$
\int_{|y - y_i| \leq r_i} |y - y_i|^s u_i(y)^{p_i+1} = \begin{cases} 
O(u_i(y_i)^{-2s/(n-2\sigma)}), & -n < s < n, \\
O(u_i(y_i)^{-2n/(n-2\sigma)} \log u_i(y_i)), & s = n, \\
o(u_i(y_i)^{-2n/(n-2\sigma)}), & s > n, 
\end{cases}
$$

and

$$
\int_{r_i < |y - y_i| \leq 1} |y - y_i|^s u_i(y)^{p_i+1} = \begin{cases} 
o(u_i(y_i)^{-2s/(n-2\sigma)}), & -n < s < n, \\
O(u_i(y_i)^{-2n/(n-2\sigma)} \log u_i(y_i)), & s = n, \\
O(u_i(y_i)^{-2n/(n-2\sigma)}), & s > n. 
\end{cases}
$$

### 3 Compactness of solutions and characterization of blow up behavior

In this section, our main task is to prove Theorem 1.1 and Theorem 1.2. We first give the proof of Theorem 1.1, which further characterizes the blow up points for solutions to (1.3) and plays a key role in proving Theorem 1.2. Recall the definitions of the matrix $M$ given in (1.10) and its smallest eigenvalue $\mu(M)$.
Proof of Theorem 1.1. From Jin-Li-Xiong [21, Theorem 5.3], there exists a constant $\delta^* > 0$ depending only on $\min_{\mathcal{K}} K$ and $\|K\|_{C^2(\mathcal{K})}$ such that $\{v_1\}$ has only isolated simple blow up points $q^{(1)}, \ldots, q^{(k)} \in \mathcal{K}$ ($k \geq 1$) with $|q^{(j)} - q^{(\ell)}| \geq \delta^*$ ($j \neq \ell$).

Under the stereographic projection $F$ with $q^{(j)}$ being the south pole:

$$F : \mathbb{R}^3 \rightarrow S^3 \setminus \{-q^{(j)}\}, \quad y \mapsto \left(\frac{2y}{1 + |y|^2}, \frac{|y|^2 - 1}{|y|^2 + 1}\right),$$

the Eq. (1.12) is transformed to

$$(-\Delta)^\sigma u_i(y) = \tilde{K}_i(y)H_i(y)^\tau u_i(y)^{p_i}, \quad y \in \mathbb{R}^3,$$  \hspace{1cm} (3.1)

where

$$u_i(y) = \left(\frac{2}{1 + |y|^2}\right)v_i(F(y)), \quad \tilde{K}_i(y) = K_i(F(y)), \quad H_i(y) = \frac{2}{1 + |y|^2}. \hspace{1cm} (3.2)$$

Since 0 is an isolated simple blow up point of $u_i$. Let $U_i(Y), Y := (y, t) \in \mathbb{R}^4_+$, be the extension of $u_i(y)$ and satisfy

$$\begin{cases}
\text{div} \left( t^{1-2\sigma} \nabla U_i \right) = 0 & \text{in } \mathbb{R}_+^4, \\
-\lim_{t \to 0} t^{1-2\sigma} \partial_t U_i(y, t) = \tilde{K}_i(y)H_i(y)^\tau u_i(y)^{p_i} & \text{for } y \in \mathbb{R}^3. \hspace{1cm} (3.3)
\end{cases}$$

Propositions 2.4, 2.6, 2.3, and elliptic theory together imply that

$$U_i(Y^{(j)})U_i(Y) \rightarrow \mathcal{H}^{(j)}(Y) := \mathcal{A}^{(j)}|Y|^{-2} + \mathcal{W}^{(j)}(Y) \quad \text{in } C^2_{\text{loc}}(\mathbb{R}^4_+ \setminus \cup_{\ell=1}^k Y^{(\ell)}), \hspace{1cm} (3.4)$$

where $\mathcal{A}^{(j)} > 0$ is a constant and $\mathcal{W}^{(j)}(Y)$ satisfies

$$\begin{cases}
\text{div} \left( t^{1-2\sigma} \nabla \mathcal{W}^{(j)} \right) = 0 & \text{in } \mathbb{R}_+^4, \\
-\lim_{t \to 0} t^{1-2\sigma} \partial_t \mathcal{W}^{(j)}(y, t) = 0 & \text{for } y \in \mathbb{R}^3. \hspace{1cm} (3.5)
\end{cases}$$

It follows from the maximum principle and the Harnack inequality that

$$\tilde{\mathcal{W}}^{(j)}(Y) \equiv 0 \quad \text{if } k = 1, \quad \tilde{\mathcal{W}}^{(j)}(Y) > 0 \quad \text{if } k \geq 2. \hspace{1cm} (3.6)$$

Let’s next calculate $\mathcal{A}^{(j)}$. Multiplying (3.3) by $U_i(Y^{(j)})$ and integrating by parts on $B^+_1$ leads to

$$0 = \int_{B^+_1} U_i(Y^{(j)}) \text{div}(\nabla U_i)$$

$$= \int_{B^+_1} u_i(y^{(j)}_i) \tilde{K}_i(y)H_i(y)^\tau u_i(y)^{p_i} + \int_{\partial B^+_1} \frac{\partial}{\partial \nu} (U_i(Y^{(j)})U_i) =: I_1 + I_2. \hspace{1cm} (3.7)$$
Let $R_i$ be given in Proposition 2.5, and
\[ m_{ij} := u_i(y_i^{(j)}), \quad r_i := R_i(m_{ij})^{-(p_i-1)}. \]  \hspace{1cm} (3.8)

For $I_1$, from Propositions 2.5 and 2.6 we have
\[
I_1 = \int_{|y - y_i^{(j)}| \leq r_i} m_{ij} \tilde{K}_i(y) u_i(y)^{p_i} + \int_{|y - y_i^{(j)}| > r_i \cap \{|y| < 1\}} m_{ij} \tilde{K}_i(y) H_i(y) u_i(y)^{p_i} + O\left(\tau_i \int_{|y - y_i^{(j)}| \leq r_i} m_{ij} \tilde{K}_i(y) u_i(y)^{p_i}\right)
= m_{ij}^2 \tilde{K}_i(0) + O(|y|) \int_{|x| \leq R_i} (m_{ij}^{-1} u_i(m_{ij}^{-(p_i-1)} x + y_i^{(j)}))^{p_i}
= -2\pi |S^2| \tilde{K}_i(q^{(j)})^{-2} + o(1). \]  \hspace{1cm} (3.9)

For $I_2$, it follows from (3.4) and (3.5) that
\[
\lim_{i \to \infty} I_2 = \int_{\partial B_i^+} \frac{\partial}{\partial \nu}(A^{(j)}|Y|^{-2} + W^{(j)}(Y)) = \int_{\partial B_i^+} -2A^{(j)} = -\frac{\pi |S^2| A^{(j)}}{2}. \hspace{1cm} (3.10)
\]

By (3.7), (3.9), and (3.10), we conclude that $A^{(j)} = 4K(q^{(j)})^{-2}$.

From (3.4), we have
\[
U_i(Y_i^{(j)}) U_i(Y) \to H^{(j)}(Y) := 4K(q^{(j)})^{-2}|Y|^{-2} + W^{(j)}(Y) \quad \text{in } C^2_{\text{loc}}(\mathbb{R}^4 \setminus \{x = 1 Y^{(\ell)}\}),
\]
and
\[
u_i(y_i^{(j)}) u_i(y) \to h^{(j)}(y)
:= 4K(q^{(j)})^{-2}|y|^{-2} + W^{(j)}(y) \quad \text{in } C^2_{\text{loc}}(\mathbb{R}^3 \setminus \{x = 1 y^{(\ell)}\}), \hspace{1cm} (3.11)
\]
where $W^{(j)}(y) := W^{(j)}(y, 0)$.

By (3.2) and $y_i^{(j)} \to 0$ as $i \to \infty$, we have
\[
\lim_{i \to \infty} v_i(q_i^{(j)}) v_i(q) = \frac{1}{4} \lim_{i \to \infty} (1 + |y|^2) u_i(y_i^{(j)}) u_i(y),
\]
combining with (3.11), it easy to see that for $q \neq q^{(j)}$ and close to $q^{(j)}$,
\[
\lim_{i \to \infty} v_i(q_i^{(j)}) v_i(q) = 2G_{q^{(j)}}(q) K(q^{(j)})^{-2} + \tilde{W}^{(j)}(q) \quad \text{in } C^2_{\text{loc}}(S^3 \setminus \{x = 1 q^{(\ell)}\}), \hspace{1cm} (3.12)
\]
where $\tilde{W}^{(j)}(q)$ is some regular function on $S^3 \setminus \bigcup_{\ell \neq j} q^{(\ell)}$ satisfying $P_{q} \tilde{W}^{(j)} = 0$, and
$G_{q^{(j)}}(q)$ is the Green function defined as in (1.11).

When $k \geq 2$, taking into account the contribution of all the poles, we deduce
\[
\lim_{i \to \infty} v_i(q_i^{(j)}) v_i(q) = 2 \frac{G_{q^{(j)}}(q)}{K(q^{(j)})^2} + 2 \sum_{\ell \neq j} \lim_{i \to \infty} \frac{v_i(q_i^{(j)})}{v_i(q_i^{(\ell)})} \frac{G_{q^{(j)}}(q)}{K(q^{(j)})^2} \quad \text{in } C^2_{\text{loc}}(S^3 \setminus \{x = 1 q^{(\ell)}\}). \hspace{1cm} (3.13)
\]
In fact, subtracting all the poles from the limit function, we obtain a regular function \( \widetilde{W}_0 : \mathbb{S}^3 \to \mathbb{R} \) such that \( P_r \widetilde{W}_0 = 0 \) on \( \mathbb{S}^3 \), so it must be \( \widetilde{W}_0 \equiv 0 \).

Using (3.13), we have, for \( |y| > 0 \) small,

\[
h^{(j)}(y) = \frac{4}{K(q^{(j)})} + 8 \sum_{\ell \neq j} \lim_{i \to \infty} \frac{v_i(q^{(j)})}{v_i(q^{(\ell)})} \frac{G_{q^{(\ell)}}(q^{(j)})}{K(q^{(\ell)})} + O(|y|). \tag{3.14}
\]

The conclusion obtained from the above is easy to see that (1.14) is true and (1.13) can be obtained from Proposition 2.6. We have proved Part (ii).

Before stating the result to be proved, we give the following estimates (3.15) and (3.16). Using [21, Lemmas 4.13 and 4.14], we obtain

\[
|\nabla K_i(y^{(j)}_i)| = O(u_i(y^{(j)}_i)^{-1}), \quad \tau_i = O(u_i(y^{(j)}_i)^{-2}),
\]

and from Propositions 2.6, 2.6, and 2.7, we get, for sufficiently small \( \delta > 0 \),

\[
\sum_{j=1}^{3} \left| \int_{B_{\delta}} x_j u_i(y + y^{(j)}_i)^{p_i + 1} \right| = o(u_i(y^{(j)}_i)^{-1}),
\]

\[
\sum_{j \neq \ell} \left| \int_{B_{\delta}} x_j x_{\ell} u_i(y + y^{(j)}_i)^{p_i + 1} \right| = o(u_i(y^{(j)}_i)^{-2}),
\]

\[
\int_{\partial B_{\delta}} u_i(y + y^{(j)}_i)^{p_i + 1} = O(u_i(y^{(j)}_i)^{-p_i - 1}),
\]

\[
\lim_{i \to \infty} u_i(y^{(j)}_i)^{2} \int_{B_{\delta}} |y|^{2} u_i(y + y^{(j)}_i)^{p_i + 1} = 6\pi |\mathbb{S}^2| K(q^{(j)})^{-5}.
\]

Now we give the proof only for the last formula in (3.16). Let \( m_{ij} \) and \( r_i \) be as in (3.8). Applying Propositions 2.5, 2.6, and 2.7, we have

\[
m_{ij}^{2} \int_{|y| \leq \delta} |y|^{2} u_i(y + y^{(j)}_i)^{p_i + 1}
\]

\[
= m_{ij}^{2} \int_{|y| \leq r_i} |y|^{2} u_i(y + y^{(j)}_i)^{p_i + 1} + m_{ij}^{2} \int_{r_i < |y| \leq \delta} |y|^{2} u_i(y + y^{(j)}_i)^{p_i + 1}
\]

\[
= m_{ij}^{2} (2 - p_i) \int_{|x| \leq r_i} |x|^{2} \left( m_{ij}^{-1} u_i(m_{ij}^{-2(p_i - 1)} x + y^{(j)}_i) \right)^{p_i + 1}
\]

\[
+ m_{ij}^{2} \int_{r_i \leq |x - y^{(j)}_i| \leq \delta} |x - y^{(j)}_i|^{2} u_i(x)^{p_i + 1}
\]

\[
= 6\pi |\mathbb{S}^2| K(q^{(j)})^{-5} + o(1).
\]

For any \( 0 < \delta < 1 \), combining (3.16) with Proposition 2.7, we can obtain

\[
\tau_i^{2} \int_{B_{\delta}} \tilde{K}_i(y + y^{(j)}_i) H_i(y + y^{(j)}_i) \tau_i u_i(y + y^{(j)}_i)^{p_i + 1} \leq C u_i(y^{(j)}_i)^{-4} = o(u_i(y^{(j)}_i)^{-2}),
\]
\[ \tau_i \int_{B_s} \langle y, \nabla (\tilde{K}_i(y + y_i^{(j)})H_i(y + y_i^{(j)})\tau_i) \rangle u_i(y + y_i^{(j)})^{p_i + 1} = o(u_i(y_i^{(j)})^{-2}), \]

and

\[ \tau_i \int_{\partial B_s} \tilde{K}_i(y + y_i^{(j)})H_i(y + y_i^{(j)})\tau_i u_i(y + y_i^{(j)})^{p_i + 1} = o(u_i(y_i^{(j)})^{-2}). \]

Then using Proposition 2.7 again, we have

\[ \frac{\tau_i}{3} \int_{B_s} \tilde{K}_i(y + y_i^{(j)})(H_i(y + y_i^{(j)})\tau_i - 1)u_i(y + y_i^{(j)})^{p_i + 1} \]

\[ \leq C\tau_i^2 \int_{B_s} u_i(y)^{p_i + 1} = o(u_i(y_i^{(j)})^{-2}). \]  \hspace{1cm} (3.17)

The above estimates, Proposition 2.1, and (3.16) yield, for any \(0 < \delta < 1\),

\[ \int_{\partial B_s} B'(Y, U_i(Y + Y_i^{(j)}), \nabla U_i(Y + Y_i^{(j)}), \delta, \sigma) \]

\[ = \int_{B_s} \tilde{K}_i(y + y_i^{(j)})H_i(y + y_i^{(j)})\tau_i u_i(y + y_i^{(j)})^{p_i + 1} \]

\[ + \int_{B_s} \langle y, \nabla u_i(y + y_i^{(j)}) \rangle \tilde{K}_i(y + y_i^{(j)})H_i(y + y_i^{(j)})\tau_i u_i(y + y_i^{(j)})^{p_i} \]

\[ = -\frac{\tau_i}{3} \int_{B_s} \tilde{K}_i(y + y_i^{(j)})H_i(y + y_i^{(j)})\tau_i u_i(y + y_i^{(j)})^{p_i + 1} \]

\[ - \frac{1}{3} \int_{B_s} \langle y, \nabla (\tilde{K}_i(y + y_i^{(j)})H_i(y + y_i^{(j)})\tau_i) \rangle u_i(y + y_i^{(j)})^{p_i + 1} \]

\[ + \frac{\delta}{3} \int_{\partial B_s} \tilde{K}_i(y + y_i^{(j)})H_i(y + y_i^{(j)})\tau_i u_i(y + y_i^{(j)})^{p_i + 1} \]

\[ + o(u_i(y_i^{(j)})^{-2}) \]

\[ =: \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 + o(u_i(y_i^{(j)})^{-2}), \]  \hspace{1cm} (3.18)

where in the first equality, we take advantage of the fact that the Taylor expansion:

\[ \frac{1}{p_i + 1} = \frac{1}{3 - \tau_i} = \frac{1}{3(1 - \tau_i/3)} = \frac{1}{3} \left( 1 + \frac{\tau_i}{3} + O(\tau_i^2) \right). \]

It follows from (3.16), (3.17), Proposition 2.5, and Proposition 2.7 that

\[ \mathcal{J}_1 = -\frac{\tau_i}{3} \int_{B_s} \tilde{K}_i(y + y_i^{(j)})u_i(y + y_i^{(j)})^{p_i + 1} + o(m_{ij}^{-2}) \]

\[ = -\frac{\tau_i}{3} \frac{2^3}{K(q^{(j)})^2} \int_{\mathbb{R}^3} \frac{1}{(1 + |z|^2)^3} + o(m_{ij}^{-2}) \]

\[ = -\frac{\pi |S^2|}{6} \frac{\tau_i}{K(q^{(j)})^2} + o(m_{ij}^{-2}). \]  \hspace{1cm} (3.19)
Applying Proposition 2.7 and (3.16), we conclude that

\[ J_2 = -\frac{1}{3} \int_{B_{\delta}} \langle y, \nabla (\tilde{K}_i(y + y_i^{(j)})) \tilde{H}_i(y + y_i^{(j)}) \rangle u_i(y + y_i^{(j)})^{p_i + 1} + o(m_i^{-2}) \]

\[ = -\frac{1}{3} \int_{B_{\delta}} \langle y, \nabla (\tilde{K}_i(y + y_i^{(j)})) \tilde{H}_i(y + y_i^{(j)}) \rangle u_i(y + y_i^{(j)})^{p_i + 1} \]

\[ - \frac{1}{3} \int_{B_{\delta}} \langle y, \tilde{K}_i(y + y_i^{(j)}) \nabla (\tilde{H}_i(y + y_i^{(j)}) \rangle u_i(y + y_i^{(j)})^{p_i + 1} + o(m_i^{-2}) \]

\[ = -\frac{1}{3} \sum_{\ell} \int_{B_{\delta}} y_{\ell} \frac{\partial \tilde{K}_i}{\partial y_{\ell}} (y + y_i^{(j)}) u_i(y + y_i^{(j)})^{p_i + 1} + o(m_i^{-2}) \]

\[ = -\frac{1}{3} \int_{B_{\delta}} y \cdot \nabla \tilde{K}_i(y_i^{(j)}) u_i(y + y_i^{(j)})^{p_i + 1} \]

\[ = -\frac{1}{3} \sum_{\ell, m} \int_{B_{\delta}} y_{\ell} y_{m} \frac{\partial^2 \tilde{K}_i}{\partial y_{\ell} \partial y_m} (y_i^{(j)}) u_i(y + y_i^{(j)})^{p_i + 1} + o(m_i^{-2}) \]

\[ = -\frac{1}{9} \Delta \tilde{K}(0) \int_{B_{\delta}} |y|^2 u_i(y + y_i^{(j)})^{p_i + 1} + o(m_i^{-2}) \]

\[ = -\frac{4}{9} \Delta_{g_0} K(q^{(j)}) \int_{B_{\delta}} |y|^2 u_i(y + y_i^{(j)})^{p_i + 1} + o(m_i^{-2}), \quad (3.20) \]

where we used the definition of the Laplace-Beltrami operator in the last equality. By (3.16), we get

\[ J_3 = \frac{\delta}{3} \int_{\partial B_{\delta}} \tilde{K}_i(y + y_i^{(j)}) \tilde{H}_i(y + y_i^{(j)})^{\tau_i} u_i(y + y_i^{(j)})^{p_i + 1} = o(m_i^{-2}). \quad (3.21) \]

It follows from Proposition 2.1 and (3.18)–(3.21) that

\[ \int_{\partial B_{\delta}^+} B''(Y, U_i(Y + Y_i^{(j)}), \nabla U_i(Y + Y_i^{(j)}), \delta, \sigma) \]

\[ = -\int_{\partial B_{\delta}^+} B'(Y, U_i(Y + Y_i^{(j)}), \nabla U_i(Y + Y_i^{(j)}), \delta, \sigma) \]

\[ = \frac{\pi |S|^2}{6} \frac{\tau_i}{K(q^{(j)})^2} + \frac{4}{9} \Delta_{g_0} K(q^{(j)}) \int_{B_{\delta}} |y|^2 u_i(y + y_i^{(j)})^{p_i + 1} + o(m_i^{-2}). \quad (3.22) \]

By (3.2) and the definition of \( \mu^{(j)} \), we have

\[ \mu^{(j)} = \lim_{i \to \infty} \tau_i u_i(q_i^{(j)})^2 = \lim_{i \to \infty} \frac{1}{4} \tau_i u_i(y_i^{(j)})^2. \]

Thus, multiplying (3.22) by \( U_i(Y_i^{(j)})^2 \) and sending \( i \) to \( \infty \), and using Proposition 2.1
and (3.16), we conclude that
\[
\int_{\partial \Omega^+} B''(Y, \mathcal{H}^{(j)}(Y + Y_i^{(j)}), \nabla \mathcal{H}^{(j)}(Y + Y_i^{(j)}), \delta, \sigma) \frac{8\pi |S^2| \Delta g_0 K(q^{(j)})}{3K(q^{(j)})^5} + 2\pi |S^2| \mu^{(j)} \frac{1}{3K(q^{(j)})^2}.
\]
Let \( \delta \to 0 \), it follows from Proposition 2.2 that
\[
\frac{8\pi |S^2| \Delta g_0 K(q^{(j)})}{3K(q^{(j)})^5} + 2\pi |S^2| \mu^{(j)} \frac{1}{3K(q^{(j)})^2} = -\frac{2\pi |S^2| W^{(j)}(0)}{K(q^{(j)})^2}. \tag{3.23}
\]
Consequently, we have \( q^{(j)} \in \mathcal{K} \setminus \mathcal{K}^+ \), \( \forall 1 \leq j \leq k \), and when \( k \geq 2 \), \( q^{(j)} \in \mathcal{K}^- \), \( \forall 1 \leq j \leq k \).

It is easy to see that (1.15) follows from (3.6) and (3.23) when \( k = 1 \). When \( k \geq 2 \), by (3.14) we have
\[
W^{(j)}(0) = 8 \sum_{\ell \neq j} \frac{\lambda_{\ell}}{\lambda_j} G_{q^{(\ell)}}(q^{(j)}) = \frac{\Delta g_0 K(q^{(j)})}{K(q^{(j)})^2} \lambda_j = \frac{1}{4} \lambda_j \mu^{(j)}. \tag{3.24}
\]
Substituting (3.24) into (3.23), we get
\[
-6 \sum_{\ell \neq j} \frac{G_{q^{(\ell)}}(q^{(j)})}{K(q^{(j)})^2} \lambda_{\ell} - \frac{\Delta g_0 K(q^{(j)})}{K(q^{(j)})^2} \lambda_j = \frac{1}{4} \lambda_j \mu^{(j)}.
\]
We have established (1.16) and thus verified Part (iii).

We claim that there exists some
\[
\eta = (\eta_1, \ldots, \eta_k) \neq 0 \quad \text{with} \quad \eta_\ell \geq 0, \quad \forall \ell = 1, \ldots, k, \tag{3.25}
\]
such that
\[
\sum_{\ell=1}^k M_{\ell j}(q^{(1)}, \ldots, q^{(k)}) \eta_\ell = \mu(M) \eta_j, \quad \forall \ j = 1, \ldots, k.
\]
Indeed, choose \( \Lambda > \max_i M_{ii} \), then the matrix \( \Lambda I - M \) is a positive matrix (see [20] for the definition), where \( I \) denotes the unit matrix. The claim follows from [20, Theorem 8.2.2].

Multiplying (1.16) by \( \eta_j \) and summing over \( j \), and using Part (ii) and (3.25), we have
\[
\mu(M) \sum_j \lambda_j \eta_j = \sum_{\ell,j} M_{\ell j} \lambda_\ell \eta_j = \frac{1}{4} \sum_j \lambda_j \eta_j \mu^{(j)} \geq 0. \tag{3.26}
\]
It follows that \( \mu(M) \geq 0 \). We have verified Part (i). Part (iv) follows from Parts (i)–(iii). The proof of Theorem 1.1 is completed. \( \square \)
Using Theorem 1.1, we can give the proof of Theorem 1.2.

**Proof of Theorem 1.2.** We first prove the upper bounds in (1.19). Suppose this assertion of the theorem is false. Then we can find that there exists $K_i \to K$ in $C^2(S^3)$ such that $\max_{S^3} v_i \to \infty$ for some $v_i \in \mathcal{M}_K$. Theorem 1.1 shows that $\{v_i\}$ has only isolated simple blow up points $\{q^{(1)}, \ldots, q^{(k)}\}$. It follows from [21, Theorem 5.5] that $k > 1$. Using Part (i) of Theorem 1.1, we obtain $q^{(1)}, \ldots, q^{(k)} \in \mathcal{K}^-$.

Applying Theorem 1.1 with $\tau_i = 0$, we deduce that $q^{(1)}, \ldots, q^{(k)} \in \mathcal{K}^-$ and for all $1 \leq j \leq k$, $\sum_{\ell=1}^{k} M_{\ell j} \lambda_{\ell} = 0$, where $\lambda_{\ell} > 0$, $\ell = 1, \ldots, k$.

Analysis similar to that in the proof of Theorem 1.1 shows that $\mu(M)$ has at least one nonnegative eigenvector $\eta = (\eta_1, \ldots, \eta_k)$ as in (3.25), then we have

$$\mu(M) \sum_j \lambda_j \eta_j = \sum_{\ell,j} M_{\ell j} \lambda_{\ell} \eta_j = 0.$$ 

It follows that $\mu(M) = 0$. This leads to a contradiction with $K \in \mathcal{A}$. Then by the Harnack inequality in [21, Lemma 4.3] and Schauder estimates in [21, Theorem 2.11], we complete the proof of Theorem 1.2. $\square$

**4 The existence results on $S^3$**

In this section, we first prove Theorem 1.3, which is about the degree-counting formula and the existence of the solutions. Before that, we prove that as $\tau \to 0^+$, the solutions to the subcritical equation (see (4.1) below) either stay bounded and converge to the solutions to critical equations (1.3) in $C^2$ norm or become unbounded and blow up at finite points.

Then by using Theorem 1.3 and perturbing the prescribing function near its critical point, we can know exactly where the blow up occur when $K \notin \mathcal{A}$. From Theorem 1.1 and the proof of Theorem 1.4, we show Theorem 1.5 holds.

**4.1 On the case of subcritical equations**

In this subsection, we consider the following subcritical equation:

$$P_\sigma v = K v^{2-\tau} \quad \text{on } S^3,$$

where $K \in C^2(S^3)$ and $\tau > 0$.

Denote the $H^\sigma(S^3)$ inner product and norm by

$$\langle u, v \rangle = \int_{S^3} (P_\sigma u) v, \quad \|u\|_\sigma = \sqrt{\langle u, u \rangle}.$$ 

The Euler-Lagrange functional associated with (4.1) is

$$I_\tau(v) = \frac{1}{2} \int_{S^3} (P_\sigma v) v - \frac{1}{3 - \tau} \int_{S^3} K |v|^{3-\tau}, \quad \forall v \in H^\sigma(S^3).$$ (4.2)
Definition 4.1. Let $K \in C^2(S^3)$, $\mathcal{K}^-$ be as in (1.9) and $k \in \mathbb{N}_+$. Let $\overline{P}_1, \ldots, \overline{P}_k \in \mathcal{K}^-$ be the critical points of $K$ with $\mu(M(\overline{P}_1, \ldots, \overline{P}_k)) > 0$ and $\varepsilon_0 > 0$ be sufficiently small. Define

$\Omega_{\varepsilon_0} = \Omega_{\varepsilon_0}(\overline{P}_1, \ldots, \overline{P}_k)$

$= \{(\alpha, t, P) \in \mathbb{R}^k_+ \times \mathbb{R}^k_+ \times (S^3)^k : |\alpha_i - 1/K(P_i)| < \varepsilon_0, t_i > 1/\varepsilon_0, |P_i - \overline{P}_i| < \varepsilon_0, 1 \leq i \leq k\}$.

It is well known that for $P \in S^3$ and $t > 0$,

$$\delta_{P,t}(x) := \frac{t}{1 + \frac{t^2 - 1}{2}(1 - \cos d(x, P))}, \quad \forall x \in S^3$$

(4.3)

is a family of positive solutions to

$$P_\sigma v = v^2, \quad v > 0 \quad \text{on} \quad S^3,$$

(4.4)

where $d(\cdot, \cdot)$ is the distance induced by the standard metric of $S^3$.

Using the idea introduced in [2] and [5], we have the following lemma.

Lemma 4.1. Let $\varepsilon_0$ be sufficiently small and $\Omega_{\varepsilon_0} = \Omega_{\varepsilon_0}(\overline{P}_1, \ldots, \overline{P}_k)$ be as in Definition 4.1. For any $u \in H^\sigma(S^3)$ satisfying the inequality

$$\left\| u - \sum_{i=1}^k \tilde{\alpha}_i \delta_{P_i, t_i} \right\|_\sigma < \varepsilon_0$$

for some $(\tilde{\alpha}, \tilde{t}, \tilde{P}) \in \Omega_{\varepsilon_0}/2$, then there exists a unique $(\alpha, t, P) \in \Omega_{\varepsilon_0}$ such that

$$u = \sum_{i=1}^k \alpha_i \delta_{P_i, t_i} + v,$$

with $v$ satisfies

$$\langle v, \delta_{P_i, t_i} \rangle = \left\langle v, \frac{\partial \delta_{P_i, t_i}}{\partial P_i^{(0)}} \right\rangle = \left\langle v, \frac{\partial \delta_{P_i, t_i}}{\partial t_i} \right\rangle = 0,$$

(4.5)

where $\frac{\partial}{\partial P_i^{(0)}}$ denotes the corresponding derivatives.

We denote the set of $v \in H^\sigma(S^3)$ satisfying (4.5) by $E_{P,t}$. In what follows, we work in some orthonormal basis near $\{\overline{P}_1, \ldots, \overline{P}_k\}$.

Definition 4.2. Let $A$ be sufficiently large, $\varepsilon_0, \nu_0 > 0$ be sufficiently small, $k \in \mathbb{N}_+$, and $\Omega_{\varepsilon_0}/2 = \Omega_{\varepsilon_0}/2(\overline{P}_1, \ldots, \overline{P}_k)$ be as in Definition 4.1. Define

$$\Sigma_{\tau}(\overline{P}_1, \ldots, \overline{P}_k)$$

$= \{(\alpha, t, P, v) \in \Omega_{\varepsilon_0}/2 \times H^\sigma(S^3) : |P_i - \overline{P}_i| < \tau^{1/2} |\log \tau|, A^{-1} \tau^{-1/2} < t_i < A\tau^{-1/2}, v \in E_{P,t}, \|v\| < \nu_0\}$.

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Without confusion we use the same notation for

\[ \Sigma_\tau(\mathcal{P}_1, \cdots, \mathcal{P}_k) = \left\{ u = \sum_{i=1}^{k} \alpha_i \delta_{\mathcal{P}_i}, v : (\alpha, t, P, v) \in \Sigma_\tau \right\} \subset H^\sigma(S^3). \]

**Remark 4.1.** Due to Theorem 1.2, we only need to prove Theorem 1.3 for \( K \in \mathcal{A} \) being a Morse function. Once this is achieved, we also prove that the Index as in Definition 1.4 is well defined on \( \mathcal{A} \).

Using blow up analysis, we first give the necessary conditions on blowing up solutions to (4.1) when \( \tau \) tends to 0.

**Proposition 4.1.** Let \( K \in \mathcal{A} \) be a Morse function and \( \mathcal{K}^- \) be as in (1.9). Then for any \( \alpha \in (0, 1) \), there exists some positive constants \( \varepsilon_0, \nu_0 \ll 1 \), and \( A, R \gg 1 \) depending only on \( K \), such that when \( \tau > 0 \) is sufficiently small, for all \( u \) satisfying \( u \in H^\sigma(S^3) \), \( u > 0 \), \( I'_\tau(u) = 0 \), we have

\[ u \in \mathcal{O}_R \cup \{ \bigcup_{k \geq 1} \cup \mathcal{P}_i \in \mathcal{K}^-, \mu(M(\mathcal{P}_i)) > 0 \} \Sigma_\tau(\mathcal{P}_1, \cdots, \mathcal{P}_k) \}, \]

where \( I'_\tau(u) \) is as in (4.1), \( \mathcal{O}_R \) is as in (1.20) and \( \Sigma_\tau(\mathcal{P}_1, \cdots, \mathcal{P}_k) \) is as in (4.6).

**Proof.** For any \( \tau > 0 \) sufficiently small, let \( u_\tau \in H^\sigma(S^3) \), \( u_\tau > 0 \) be a critical point of \( I_\tau(u) \). If \( u_\tau \) is uniformly bounded, then there exists a \( R > 0 \) such that \( u_\tau \in \mathcal{O}_R \), and the proof is now completed. If not, there exists \( \tau_1 \to 0 \) such that \( u_{\tau_1} \to \infty \). It follows from Theorem 1.1 and \( K \in \mathcal{A} \) that there exists a constant \( \delta^* > 0 \) such that \( \{ u_{\tau_1} \} \) has only isolated simple blow up points \( q^{(1)}, \cdots, q^{(k)} \in \mathcal{K}^- \), with \( |q^{(j)} - q^{(\ell)}| \geq \delta^*, \forall j \neq \ell \), and \( \mu(q^{(1)}, \cdots, q^{(k)}) > 0 \). Then Proposition 4.1 can be deduced from Propositions 2.3, 2.4, 2.6, and elliptic theory. \( \square \)

Now we are going to show that if \( K \in \mathcal{A} \) is a Morse function, one can construct solutions highly concentrating at arbitrary points \( q^{(1)}, \cdots, q^{(k)} \in \mathcal{K}^- \) provided \( \mu(M(q^{(1)}, \cdots, q^{(k)})) > 0 \).

**Theorem 4.1.** Let \( K \in \mathcal{A} \) be a Morse function and \( \mathcal{K}^- \) be as in (1.9). Let \( \tau, \varepsilon_0, \nu_0 > 0 \) be sufficiently small, \( A > 0 \) be sufficiently large and \( k \in \mathbb{N}_+ \). Then for any \( \mathcal{P}_1, \cdots, \mathcal{P}_k \in \mathcal{K}^- \) satisfying \( \mu(M(\mathcal{P}_1, \cdots, \mathcal{P}_k)) > 0 \), we have

\[ \deg_{H^\sigma} \left( u - P^{-1}_\sigma(K|u|^{1-\tau}u), \Sigma_\tau(\mathcal{P}_1, \cdots, \mathcal{P}_k), 0 \right) = (-1)^k + \sum_{i=1}^{k} i(\mathcal{P}_i), \]

where \( \deg_{H^\sigma} \) denotes the Leray-Schauder degree in \( H^\sigma(S^3) \), and \( i(\mathcal{P}_i) \) is the Morse index of \( K \) at \( \mathcal{P}_i \).

The following conclusion is needed for proving Theorem 4.1.
Proposition 4.2. Under the assumptions of the Theorem 4.1, in addition that $\Sigma_\tau(\mathcal{P}_1, \ldots, \mathcal{P}_k)$ is as in (4.7) and $E_{P,t}$ is as in (4.5) for the given $(\alpha, t, P)$. Then there exists a unique minimizer $v = v_\tau(\alpha, t, P) \in E_{P,t}$ of $I_\tau(\sum_{i=1}^k \alpha_i \delta_{P_i,t_i} + v)$ with respect to $\{v \in E_{P,t} : \|v\|_\sigma < \nu_0\}$. Furthermore, there exists a constant $C$ independent of $\tau$ such that

$$\|v\|_\sigma \leq C \sum_{i=1}^k |\nabla K(P_i)|^{1/2} + C \tau \log \tau \leq C \tau \log \tau.$$  

Proof. For $(\alpha, t, P, v) \in \Sigma_\tau(\mathcal{P}_1, \ldots, \mathcal{P}_k)$, which is simply written as $\Sigma_\tau$, it follows from (4.5) that

$$I_\tau\left(\sum_{i=1}^k \alpha_i \delta_{P_i,t_i} + v\right)$$

$$= \frac{1}{2} \sum_{i=1}^k \alpha_i^2 \int_{S^3} \delta_{P_i,t_i}^3 + 2 \sum_{i \neq j} \alpha_i \alpha_j \int_{S^3} \delta_{P_i,t_i}^2 \delta_{P_j,t_j} + \frac{1}{2} \int_{S^3} (P_\sigma v) v$$

$$+ \frac{1}{3 - \tau} \int_{S^3} K \left| \sum_{i=1}^k \alpha_i \delta_{P_i,t_i} + v \right|^{3 - \tau}. \tag{4.8}$$

Using Lemma A.1 and (A.9), we have,

$$I_\tau\left(\sum_{i=1}^k \alpha_i \delta_{P_i,t_i} + v\right)$$

$$= \left|\frac{S^3}{2} \sum_{i=1}^k \alpha_i^2 + \sum_{i<j} \alpha_i \alpha_j \int_{S^3} \delta_{P_i,t_i}^2 \delta_{P_j,t_j}\right.$$

$$- \frac{1}{3 - \tau} \int_{S^3} K \left(\sum_{i=1}^k \alpha_i \delta_{P_i,t_i}\right)^{3 - \tau}\left. - \int_{S^3} K \left(\sum_{i=1}^k \alpha_i \delta_{P_i,t_i}\right)^{2 - \tau} v \right.$$  

$$+ \frac{1}{2} \int_{S^3} (P_\sigma v) v - \frac{2 - \tau}{2} \int_{S^3} \left(\sum_{i=1}^k \alpha_i \delta_{P_i,t_i}\right)^{1 - \tau} v^2 + V(\tau, \alpha, t, P, v),$$

where $|V(\tau, \alpha, t, P, v)| \leq C \|v\|_\sigma^{3 - \tau}$ and $C$ depends only on $K, \nu_0$, and $A$.

For $\varphi, v \in E_{P,t}$, set

$$f_\tau(v) = - \int_{S^3} K \left(\sum_{i=1}^k \alpha_i \delta_{P_i,t_i}\right)^{2 - \tau} v, \tag{4.9}$$

$$Q_\tau(v, \varphi) = \frac{1}{2} \int_{S^3} (P_\sigma v) \varphi - \frac{2 - \tau}{2} \int_{S^3} K \left(\sum_{i=1}^k \alpha_i \delta_{P_i,t_i}\right)^{1 - \tau} v \varphi, \tag{4.10}$$

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\[ Q_0(v, \varphi) = \frac{1}{2} \int_{\mathbb{R}^3} (P_\sigma v) \varphi - \int_{\mathbb{R}^3} \left( \sum_{i=1}^k \alpha_i \delta_{P_i, t_i} \right) v \varphi. \] (4.11)

A direct calculation gives, for all \( \varphi \in E_{P,t} \),
\[ I'_\tau \left( \sum_{i=1}^k \alpha_i \delta_{P_i, t_i} + v \right) \varphi = f_\tau (\varphi) + 2Q_\tau (v, \varphi) + \langle V_v (\tau, \alpha, t, P, v), \varphi \rangle, \] (4.12)

where \( V_v \) is some function satisfying \( \| V_v (\tau, \alpha, t, P, v) \| \leq C \| v \|^{2-\tau}_\sigma \).

Since \( f_\tau \) is a continuous linear functional over \( E_{P,t} \), there exists a unique \( \tilde{f}_\tau \in E_{P,t} \) such that
\[ f_\tau (\varphi) = \langle \tilde{f}_\tau, \varphi \rangle, \forall \varphi \in E_{P,t}. \] (4.13)

It is proved in [27] that there exists a constant \( \delta_0 > 0 \) (independent of \( \tau \)) such that
\[ Q_0 (v, v) \geq \delta_0 \| v \|_\sigma^2, \forall (\alpha, t, P, v) \in \Sigma_\tau. \]

We choose \( \varepsilon_0 \) sufficiently small from the beginning. Using some elementary estimates as in Appendix, we have, for \( \tau > 0 \) small,
\[ Q_\tau (v, v) \geq \frac{\delta_0}{2} \| v \|_\sigma^2, \forall (\alpha, t, P, v) \in \Sigma_\tau. \] (4.14)

Thus, there exists a unique symmetric continuous and coercive operator \( \tilde{Q}_\tau \) from \( E_{P,t} \) onto itself such that, for any \( \varphi \in E_{P,t} \),
\[ Q_\tau (v, \varphi) = \langle \tilde{Q}_\tau v, \varphi \rangle. \] (4.15)

Using these notations, (4.12), (4.13), and (4.15), we have
\[ I'_\tau \left( \sum_{i=1}^k \alpha_i \delta_{P_i, t_i} + v \right) = \tilde{f}_\tau + 2\tilde{Q}_\tau v + V_v (\tau, \alpha, t, P, v). \] (4.16)

There is an equivalence between the existence of minimizer \( \overline{v}_\tau \) and
\[ \tilde{f}_\tau + 2\tilde{Q}_\tau v + V_v (\tau, \alpha, t, P, v) = 0, \quad v \in E_{P,t}. \] (4.17)

As in [27, 32], by the implicit function theorem, there exists a unique \( v_\tau \in E_{P,t} \) with \( \| v \|_\sigma < \nu_0 \) satisfying (4.17) and
\[ \| \overline{v}_\sigma \| \leq C \| \tilde{f}_\tau \|_\sigma. \] (4.18)
Thus, we only need to estimate \( \| \tilde{f}_\tau \|_\sigma \). From Lemma A.3, (A.12), and (A.17), we can obtain

\[
f_\tau(v) = -\int_{\mathbb{S}^3} K \left( \sum_{i=1}^{k} \alpha_i^{2-\tau} \delta_{P_i,t_i}^2 \right) v + O \left( \sum_{i \neq j} \int_{\mathbb{S}^3} \delta_{P_i,t_i} \delta_{P_j,t_j} |v| \right)
\]

\[
= -\int_{\mathbb{S}^3} (K - K(P_i)) \sum_{i=1}^{k} \alpha_i^{2-\tau} \delta_{P_i,t_i}^2 v + O \left( \sum_{i=1}^{k} \int_{\mathbb{S}^3} \delta_{P_i,t_i}^{2-\tau} - \delta_{P_i,t_i}^2 |v| \right)
\]

\[
+ O \left( \sum_{i \neq j} \| \delta_{P_i,t_i} \delta_{P_j,t_j} \|_{L^{3/2}(\mathbb{S}^3)} \|v\|_\sigma \right)
\]

\[
= O \left( \sum_{i=1}^{k} |\nabla g_i K(P_i)| \int_{\mathbb{S}^3} |P - P_i| \delta_{P_i,t_i}^2 |v| \right) + O \left( \sum_{i=1}^{k} \int_{\mathbb{S}^3} |P - P_i|^2 \delta_{P_i,t_i}^2 |v| \right)
\]

\[
+ O(\tau |\log \tau||v||_\sigma),
\]

where \(|P - P_i|\) represents the distance between two points \(P\) and \(P_i\) after through a stereographic projection.

Using (A.17) again, we have, for all \((\alpha, t, P, v)\in \Sigma_\tau, \)

\[
|f_\tau(v)| \leq C \left\{ \tau^{1/2} \sum_{i=1}^{k} |\nabla K(P_i)| + \tau + \tau |\log \tau| \right\} \|v\|_\sigma
\]

\[
\leq C \tau |\log \tau||v||_\sigma,
\]

(4.19)

this, combining (4.13) and (4.18), we complete the proof. \(\square\)

**Proposition 4.3.** Under the assumptions of Theorem 4.1, in addition that \(\Sigma_\tau(\overline{P}_1, \ldots, \overline{P}_k)\) is as in (4.7). Then for any \((\alpha, t, P, v)\in \Sigma_\tau(\overline{P}_1, \ldots, \overline{P}_k)\), there exists \(V_\alpha(\tau, \alpha, t, P, v)\) such that

\[
\frac{\partial}{\partial \alpha_i} I_{\tau} \left( \sum_{j=1}^{k} \alpha_j \delta_{P_j,t_j} + v \right) = -|\mathbb{S}^3| \beta_i + V_\alpha(\tau, \alpha, t, P, v),
\]

where \(\beta_i := \alpha_i - 1/K(P_i), i = 1, \ldots, k.\) Furthermore, let \(\overline{v}\) be as in Proposition 4.2, then

\[
\frac{\partial}{\partial \alpha_i} I_{\tau} \left( \sum_{j=1}^{k} \alpha_j \delta_{P_j,t_j} + \overline{v} \right) = -|\mathbb{S}^3| \beta_i + O(\|\beta\|^2 + \tau |\log \tau|).
\]
Proof. From (A.9), (A.11), Lemma A.2, Lemma A.3, and (A.12), we have

\[ \frac{\partial}{\partial \alpha_i} I_\tau \left( \sum_{j=1}^{k_\tau} \alpha_j \delta_{P_j, t_j} + v \right) \]

\[ = \alpha_i \int_{S^3} \delta_{P_i, t_i}^2 + \frac{1}{2} \sum_{j \neq i} \alpha_j \int_{S^3} \delta_{P_j, t_j}^2 \delta_{P_i, t_i} \]

\[ - \int_{S^3} K \left| \sum_{j=1}^{k_\tau} \alpha_j \delta_{P_j, t_j} + v \right|^{1-\tau} \left( \sum_{j=1}^{k_\tau} \alpha_j \delta_{P_j, t_j} + v \right) \delta_{P_i, t_i} \]

\[ = |S^3| \alpha_i - \int_{S^3} K \left( \sum_{j=1}^{k_\tau} \alpha_j \delta_{P_j, t_j}^{3-\tau} \right) \delta_{P_i, t_i} - (2 - \tau) \int_{S^3} K \left( \alpha_i^{1-\tau} \delta_{P_i, t_i}^{3-\tau} \right) \delta_{P_i, t_i} v \]

\[ + O(\tau | \log \tau |) + O(\tau) + O(\|v\|^{2-\tau}_\sigma). \]

By using (A.17), we obtain

\[ \int_{S^3} K \alpha_i^2 \delta_{P_i, t_i}^{3-\tau} = \int_{S^3} K(P_i) \alpha_i^2 \delta_{P_i, t_i}^{3-\tau} - \int_{S^3} (K(P) - K(P_i)) \alpha_i^2 \delta_{P_i, t_i}^{3-\tau} \]

\[ = \int_{S^3} K(P_i) \alpha_i^2 \delta_{P_i, t_i}^{3-\tau} + O(\tau). \] (4.20)

Similarly, by (4.5), (A.15), (4.6), and (A.17), we have

\[ \int_{S^3} K \alpha_i \delta_{P_i, t_i}^{2-\tau} v \]

\[ = \int_{S^3} K(P_i) \alpha_i \delta_{P_i, t_i}^2 v + \int_{S^3} (K(P) - K(P_i)) \alpha_i \delta_{P_i, t_i}^2 v + O(\tau | \log \tau | \|v\|_\sigma) \]

\[ = O(\tau | \log \tau |) + O(\|v\|^{2}_\sigma). \] (4.21)

It follows from the fact \(|\alpha_i^{2-\tau} - \alpha_i^2| = O(\tau), (4.20), (4.21), (A.2), and (A.20)|\) that

\[ \frac{\partial}{\partial \alpha_i} I_\tau \left( \sum_{j=1}^{k_\tau} \alpha_j \delta_{P_j, t_j} + v \right) \]

\[ = |S^3| \alpha_i - \int_{S^3} K \alpha_i^{2} \delta_{P_i, t_i}^{3-\tau} - 2 \int_{S^3} K \alpha_i \delta_{P_i, t_i}^{2-\tau} v + O(\tau | \log \tau |) + O(\|v\|^{2-\tau}_\sigma) \]

\[ = - \beta_i \int_{S^3} \delta_{P_i, t_i}^2 + O(|\beta|^2) + O(\tau | \log \tau |) + O(\|v\|^{2-\tau}_\sigma) \]

\[ = - |S^3| \beta_i + O(|\beta|^2) + O(\tau | \log \tau |) + O(\|v\|^{2-\tau}_\sigma). \]

Hence

\[ \frac{\partial}{\partial \alpha_i} I_\tau \left( \sum_{j=1}^{k_\tau} \alpha_j \delta_{P_j, t_j} + v \right) = - |S^3| \beta_i + V_{\alpha_i}(\tau, \alpha, t, P, v), \] (4.22)
where
\[ V_\alpha(\tau, \alpha, t, P, v) = O(|\beta|^2) + O(\tau |\log \tau|) + O(\|v\|^2_{\sigma}). \]
Combining with Proposition 4.2, we get
\[
\frac{\partial}{\partial \alpha_i} I_\tau \left( \sum_{j=1}^k \alpha_j \delta_{P_j, t_j} + v \right) = -|\mathcal{S}^3| \beta_i + V_\alpha(\tau, \alpha, t, P, v) = -|\mathcal{S}^3| \beta_i + O(|\beta|^2 + \tau |\log \tau|). \tag{4.23}
\]
Proposition 4.3 follows from the above. □

**Proposition 4.4.** Under the assumptions of Theorem 4.1, in addition that \( \Sigma_\tau(\overline{P}_1, \ldots, \overline{P}_k) \) is as in (4.7). Then for any \((\alpha, t, P, v) \in \Sigma_\tau(\overline{P}_1, \ldots, \overline{P}_k)\), there exists \( V_{t_i}(\tau, \alpha, t, P, v) \) such that
\[
\frac{\partial}{\partial t_i} I_\tau \left( \sum_{j=1}^k \alpha_j \delta_{P_j, t_j} + v \right) = \frac{\Gamma_3}{K(P_i)^2 t_i} \tau + \frac{\Gamma_4 \Delta_{g_0} K(P_i)}{K(P_i)^3 t_i^3} + \sum_{j \neq i} \frac{\Gamma_5 G_{P_i}(P_j)}{K(P_i)K(P_j) t_i^2 t_j} + V_{t_i}(\tau, \alpha, t, P, v),
\]
where
\[ V_{t_i}(\tau, \alpha, t, P, v) = O(|\beta|^{3/2}) + O(\tau \|v\|_\sigma) + O(\tau^{1/2} \|v\|_{2-\tau}) + O(\tau^{3/2} |\log \tau|), \]
\( \Gamma_3, \Gamma_4, \Gamma_5 \) are positive constants, and \( G_{P_i}(P_j) \) is as in (1.11).

**Proof.** Using (4.8), Lemma A.2, (A.10), Hölder inequality, and Sobolev embedding, we have,
\[
\frac{\partial}{\partial t_i} I_\tau \left( \sum_{j=1}^k \alpha_j \delta_{P_j, t_j} + v \right) = \frac{1}{2} \sum_{j \neq i} \alpha_i \alpha_j \frac{\partial}{\partial t_i} \int_{\mathcal{S}^3} \delta_{P_j, t_j} \delta_{P_i, t_i} - \int_{\mathcal{S}^3} K\left( \sum_{j=1}^k \alpha_j \delta_{P_j, t_j} \right)^{2-\tau} \alpha_i \frac{\partial \delta_{P_i, t_i}}{\partial t_i}
\]
\[
- (2 - \tau) \int_{\mathcal{S}^3} K\left( \sum_{j=1}^k \alpha_j \delta_{P_j, t_j} \right)^{1-\tau} v \alpha_i \frac{\partial \delta_{P_i, t_i}}{\partial t_i} + O \left( \|v\|_{2-\tau} \left\| \frac{\partial \delta_{P_i, t_i}}{\partial t_i} \right\|_\sigma \right). \tag{4.24}
\]
By (4.5), we have
\[
\int_{\mathcal{S}^3} \delta_{P_i, t_i} \frac{\partial \delta_{P_i, t_i}}{\partial t_i} v = \frac{1}{2} \int_{\mathcal{S}^3} v \frac{\partial}{\partial t_i} (P_{\sigma} \delta_{P_i, t_i}) = \frac{1}{2} \frac{\partial}{\partial t_i} \langle v, \delta_{P_i, t_i} \rangle = \frac{1}{2} \langle v, \frac{\partial \delta_{P_i, t_i}}{\partial t_i} \rangle = 0. \tag{4.25}
\]
It follows from (4.25), (A.15), (A.10), and (A.17) that

\[
\left| \int_{S^3} K \delta_{P_i,t_i}^{1-\tau} \frac{\partial \delta_{P_i,t_i}}{\partial t_i} v \right|
\]

\[
= \left| \int_{S^3} (K - K(P_i)) \delta_{P_i,t_i} \frac{\partial \delta_{P_i,t_i}}{\partial t_i} v + \int_{S^3} K(\delta_{P_i,t_i}^{1-\tau} - \delta_{P_i,t_i}) \frac{\partial \delta_{P_i,t_i}}{\partial t_i} v \right|
\]

\[
\leq C\tau^{1/2} \log \tau \int_{S^3} |P - P_i| \delta_{P_i,t_i} \frac{\partial \delta_{P_i,t_i}}{\partial t_i} v + O\left(\|\delta_{P_i,t_i}^{1-\tau} - \delta_{P_i,t_i}\|_{L^3(S^3)} \left\| \frac{\partial \delta_{P_i,t_i}}{\partial t_i} \right\|_\sigma \|v\|_\sigma \right)
\]

\[
\leq (\tau^{1/2} \log \tau) O\left(\|P - P_i\|_{L^3(S^3)} \left\| \frac{\partial \delta_{P_i,t_i}}{\partial t_i} \right\|_\sigma \|v\|_\sigma \right) + O(\tau^{3/2} \|v\|_\sigma)
\]

\[
\leq C\tau \|v\|_\sigma,
\]

this, and (A.13) yields

\[
\left| \int_{S^3} K \left( \sum_{j=1}^k \delta_{P_j,t_j} \right)^{1-\tau} \frac{\partial \delta_{P_i,t_i}}{\partial t_i} v \right|
\]

\[
\leq \int_{S^3} K \alpha_i^{1-\tau} \delta_{P_i,t_i}^{1-\tau} \left| \frac{\partial \delta_{P_i,t_i}}{\partial t_i} \right| + C \sum_{j \neq i} \int_{S^3} \delta_{P_j,t_j}^{1-\tau} \left| \frac{\partial \delta_{P_i,t_i}}{\partial t_i} \right| |v|
\]

\[
\leq C\tau \|v\|_\sigma + O\left(\sum_{j \neq i} \left\| \delta_{P_j,t_j}^{1-\tau} \left| \frac{\partial \delta_{P_j,t_j}}{\partial t_i} \right| \right\|_{L^{3/2}(S^3)} \|v\|_\sigma \right)
\]

\[
\leq C\tau \|v\|_\sigma.
\] (4.26)

Using (4.26) and Lemma A.3, we obtain

\[
\frac{\partial}{\partial t_i} I_{\tau} \left( \sum_{j=1}^k \alpha_j \delta_{P_j,t_j} + v \right)
\]

\[
= \frac{1}{2} \sum_{j \neq i} \alpha_i \alpha_j \frac{\partial}{\partial t_i} \int_{S^3} \delta_{P_j,t_j}^{2} \delta_{P_i,t_i} - \int_{S^3} K \left( \sum_{j=1}^k \alpha_j \delta_{P_j,t_j} \right) \frac{1}{\alpha_i} \frac{\partial \delta_{P_i,t_i}}{\partial t_i}
\]

\[
+ O(\tau \|v\|_\sigma) + O(\tau^{1/2} \|v\|^{2-\tau}_\sigma)
\]

\[
= \frac{1}{2} \sum_{j \neq i} \alpha_i \alpha_j \frac{\partial}{\partial t_i} \int_{S^3} \delta_{P_j,t_j} \delta_{P_i,t_i} - \int_{S^3} K \alpha_j^2 \delta_{P_i,t_i} \frac{\partial \delta_{P_i,t_i}}{\partial t_i}
\]

\[
- \sum_{j \neq i} \int_{S^3} \alpha_i K \alpha_j^2 \delta_{P_j,t_j} \frac{\partial \delta_{P_i,t_i}}{\partial t_i}
\]

\[
+ O(\tau \|v\|_\sigma) + O(\tau^{1/2} \|v\|^{2-\tau}_\sigma) + O(\tau^{3/2} \log \tau).
\]
We have used the following facts. By (A.12), (A.13), (A.10), and (A.21), we have

\[
\sum_{j \neq \ell} \int_{\mathbb{S}^3} \delta_{P_{i,t_i}}^{1-\tau} \frac{\partial \delta_{P_{i,t_i}}}{\partial t_i} - \frac{1}{2} \int_{\mathbb{S}^3} \sum_{j=1}^{k} \alpha_j \frac{\partial}{\partial t_i} \left( \sum_{j \neq 1} \alpha_j \delta_{P_{j,t_j}} + v \right)
\]

\[
= \int_{\mathbb{S}^3} \left( \sum_{j=i} \delta_{P_{i,t_i}} \frac{\partial \delta_{P_{i,t_i}}}{\partial t_i} + \int_{\mathbb{S}^3} \delta_{P_{i,t_i}} \frac{\partial \delta_{P_{i,t_i}}}{\partial t_i} - \frac{1}{2} \int_{\mathbb{S}^3} \sum_{j=1}^{k} \alpha_j \frac{\partial}{\partial t_i} \left( \sum_{j \neq 1} \alpha_j \delta_{P_{j,t_j}} + v \right) \right)
\]

\[
= O(\sum_{j\neq i} \frac{\partial}{\partial t_i} \int_{\mathbb{S}^3} \delta_{P_{i,t_i}}^{2-\tau} + O(\tau^{5/2} |\log \tau|) + O(\left( \frac{\partial}{\partial t_i} \int_{\mathbb{S}^3} \delta_{P_{i,t_i}}^{2-\tau} \right) + O(\tau^2). (4.27)
\]

By (A.18), we have

\[
- \int_{\mathbb{S}^3} K\alpha_i \delta_{P_{i,t_i}}^{2-\tau} \frac{\partial \delta_{P_{i,t_i}}}{\partial t_i}
\]

\[
= - \frac{1}{3 - \tau} K(P_i) \frac{\partial}{\partial t_i} \int_{\mathbb{S}^3} \delta_{P_{i,t_i}}^{2-\tau} - \frac{2}{3(3 - \tau)} \Delta g_{0} K(P_i) \frac{\partial}{\partial t_i} \int_{\mathbb{S}^3} |P - P_i|^2 \alpha_i \delta_{P_{i,t_i}}^{2-\tau} + O(\tau^2). (4.28)
\]

Let

\[
\mathcal{E} = O(\tau \|v\|_\sigma) + O(\tau^{1/2} \|v\|_\sigma^{2-\tau}) + O(\tau^{3/2} |\log \tau|),
\]

then, by (A.7), (A.8), and (4.28), we have

\[
\frac{\partial}{\partial t_i} \mathcal{I}_{\tau} \left( \sum_{j=1}^{k} \alpha_j \delta_{P_{j,t_j}} + v \right)
\]

\[
= \frac{1}{2} \sum_{j\neq i} \alpha_i \alpha_j \frac{\partial}{\partial t_i} \int_{\mathbb{S}^3} \delta_{P_{j,t_j}} \delta_{P_{i,t_i}} - \int_{\mathbb{S}^3} K\alpha_i \delta_{P_{i,t_i}}^{2-\tau} \frac{\partial \delta_{P_{i,t_i}}}{\partial t_i}.
\]
\[ -\alpha_i \sum_{j \neq i} K(P_j)\alpha_j^2 \frac{\partial}{\partial t_i} \int_{\mathbb{S}^3} \delta_{P_i, t_j}^2 + \mathcal{E} \]

\[ = \sum_{j \neq i} \left\{ \frac{1}{2} \alpha_i \alpha_j - \alpha_i \alpha_j^2 K(P_j) \right\} \frac{\partial}{\partial t_i} \int_{\mathbb{S}^3} \delta_{P_i, t_j}^2 + \mathcal{E} \]

\[ = -\frac{1}{3 - \tau} \alpha_i^3 K(P_i) \frac{\partial}{\partial t_i} \int_{\mathbb{S}^3} \delta_{P_i, t_i}^{3-\tau} + \mathcal{E} \]

\[ = -\frac{2}{3(3 - \tau)} \alpha_i^3 \Delta_{g_0} K(P_i) \frac{\partial}{\partial t_i} \int_{\mathbb{S}^3} |P - P_i|^{2\delta_{P_i, t_i}^{3-\tau} + \mathcal{E}} + O(|\beta|^{3/2}). \]

It follows from (A.3), (A.7), and (A.8) that

\[ \frac{\partial}{\partial t_i} I_\tau \left( \sum_{j=1}^k \alpha_j \delta_{P_i, t_j} + v \right) \]

\[ = -\frac{\Gamma_3}{K(P_i)^2 t_i} + \frac{\Gamma_4 \Delta_{g_0} K(P_i)}{K(P_i)^3 t_i^3} \]

\[ + \sum_{j \neq i} \frac{\Gamma_5 G(P_j)}{K(P_j) K(P_i)} \frac{1}{t_i^2 t_j} + V_t(\tau, \alpha, t, P, v), \quad (4.30) \]

where

\[ V_t(\tau, \alpha, t, P, v) = O(|\beta|^{3/2}) + O(\|v\|_\sigma) + O(\tau^{1/2}\|v\|_\sigma^{2-\tau}) + O(\tau^{3/2}|\log \tau|), \]

\[ \Gamma_3 = \frac{4}{3}\pi|\mathbb{S}^2|, \quad \Gamma_4 = \frac{2}{3}\pi|\mathbb{S}^2|, \quad \Gamma_5 = 2\pi|\mathbb{S}^2|. \]

Proposition 4.4 follows from the above. \( \square \)

**Proposition 4.5.** Under the assumptions of Theorem 4.1, in addition that \( \Sigma_\tau (P_1, \ldots, P_k) \) is as in (4.7). Then for any \((\alpha, t, P, v) \in \Sigma_\tau(P_1, \ldots, P_k)\), there exists a constant \( \nu_1 > 0 \) independent of \( \tau \) and a vector \( V(P_\tau(\tau, \alpha, t, P, v), \) such that

\[ \frac{\partial}{\partial P_i} I_\tau \left( \sum_{j=1}^k \alpha_j \delta_{P_i, t_j} + v \right) = -\Gamma_6 \nabla_{g_0} K(P_i) + V_P(\tau, \alpha, t, P, v), \]

where \( \Gamma_6 \geq \nu_1 > 0 \) is a constant, and

\[ V_P(\tau, \alpha, t, P, v) = O(\tau^{1/2}) + O(\|v\|_\sigma) + O(\tau^{-1/2}\|v\|_\sigma^{2-\tau}). \]
Proof. Using Lemma A.2, we have

\[
\frac{\partial}{\partial P_i} I_\tau \left( \sum_{j=1}^k \alpha_j \delta_{P_j, t_j} + v \right)
\]

\[
= \frac{1}{2} \sum_{j \neq i} \alpha_i \alpha_j \frac{\partial}{\partial P_i} \int_{S^3} \alpha_j \delta_{P_j, t_j} \delta_{P_i, t_i}
\]

\[
- \int_{S^3} K \left| \sum_{j=1}^k \alpha_j \delta_{P_j, t_j} + v \right|^{1-\tau} \left( \sum_{j=1}^k \alpha_j \delta_{P_j, t_j} + v \right) \alpha_i \frac{\partial \delta_{P_i, t_i}}{\partial P_i}
\]

\[
= \frac{1}{2} \sum_{j \neq i} \alpha_i \alpha_j \frac{\partial}{\partial P_i} \int_{S^3} \alpha_j \delta_{P_j, t_j} \delta_{P_i, t_i}
\]

\[
- \int_{S^3} K \left( \sum_{j=1}^k \alpha_j \delta_{P_j, t_j} \right)^{2-\tau} \alpha_i \frac{\partial \delta_{P_i, t_i}}{\partial P_i}
\]

\[
- (2 - \tau) \int_{S^3} K \left( \sum_{j=1}^k \alpha_j \delta_{P_j, t_j} \right)^{1-\tau} v \alpha_i \frac{\partial \delta_{P_i, t_i}}{\partial P_i}
\]

\[
+ O \left( \|v\|^{2-\tau} \left\| \frac{\partial \delta_{P_i, t_i}}{\partial P_i} \right\|_\sigma \right).
\]

(4.31)

By (4.5), we have

\[
\int_{S^3} \delta_{P_i, t_i} \frac{\partial \delta_{P_i, t_i}}{\partial P_i} v = \frac{1}{2} \frac{\partial}{\partial P_i} \int_{S^3} \delta_{P_i, t_i}^2 v = \frac{1}{2} \left\langle \frac{\partial \delta_{P_i, t_i}}{\partial P_i}, v \right\rangle = 0.
\]

(4.32)

It follows from (A.19), (A.23), (4.32), and (A.15) that

\[
\int_{S^3} K \left( \sum_{j=1}^k \alpha_j \delta_{P_j, t_j} \right)^{1-\tau} v \alpha_i \frac{\partial \delta_{P_i, t_i}}{\partial P_i}
\]

\[
= \int_{S^3} K(\alpha_i^{1-\tau} \delta_{P_i, t_i}) \alpha_i \frac{\partial \delta_{P_i, t_i}}{\partial P_i} v + O \left( \sum_{j \neq i} \int_{S^3} \delta_{P_j, t_j}^{1-\tau} \left\| \frac{\partial \delta_{P_i, t_i}}{\partial P_i} \right\|_\sigma \right)
\]

\[
= K(P_i) \alpha_i^{2-\tau} \int_{S^3} \delta_{P_i, t_i}^{1-\tau} \frac{\partial \delta_{P_i, t_i}}{\partial P_i} v + O(\|v\|_\sigma)
\]

\[
= O(\|v\|_\sigma).
\]

(4.33)

Then Lemma A.3, (4.33), (A.9), and (A.20) yields

\[
\frac{\partial}{\partial P_i} I_\tau \left( \sum_{j=1}^k \alpha_j \delta_{P_j, t_j} + v \right)
\]

\[
= \frac{1}{2} \sum_{j \neq i} \alpha_i \alpha_j \frac{\partial}{\partial P_i} \int_{S^3} \delta_{P_j, t_j}^{2} \delta_{P_i, t_i}
\]

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Thus, by \((\alpha, \beta)\) and \(\frac{\partial \delta_{P_1,t_i}}{\partial P_i}\), let
\[
F = \text{stereographic projection}
\]
Then we have
\[
F(0) = P_i
\]
and
\[
\frac{\partial \delta_{P_1,t_i}}{\partial P_i} = \text{stereographic projection coordinates,}
\]

\[
\alpha_i \int_{S^3} \frac{K(\alpha_i \delta_{P_1,t_i})}{\partial P_i} - \sum_{j \neq i} \alpha_i \int_{S^3} \frac{K(\alpha_j \delta_{P_1,t_i})}{\partial P_i}
\]

\[
+ O(\|v\|_\sigma) + O(\tau^{-1/2} \|v\|^{2-\tau}_\sigma)
\]

\[
= O\left(\sum_{j \neq i} \frac{\partial}{\partial P_i} \int_{S^3} \delta_{P_1,t_i} \delta_{P_1,t_i}\right)
\]

\[
- \alpha_i^3 \int_{S^3} \frac{K \delta_{P_1,t_i}^{2-\tau}}{\partial P_i} + O\left(\tau \int_{S^3} \delta_{P_1,t_i}^{2-\tau} \frac{\partial \delta_{P_1,t_i}}{\partial P_i}\right)
\]

\[
+ O\left(\sum_{j \neq i} \int_{S^3} \frac{\partial}{\partial P_i} \int_{S^3} \delta_{P_1,t_i}^{2-\tau}\right)
\]

\[
= - \alpha_i^3 \int_{S^3} \frac{K \delta_{P_1,t_i}^{2-\tau}}{\partial P_i} + O(\tau^{1/2}) + O(\|v\|_\sigma) + O(\tau^{-1/2} \|v\|^{2-\tau}_\sigma)
\]

Thus, by \((A.20)\),
\[
\frac{\partial}{\partial P_i} \left(\sum_{j=1}^k \alpha_j \delta_{P_j,t_j} + v\right)
\]

\[
= - \alpha_i^3 \int_{S^3} (K(P) - K(P_i)) \frac{\partial \delta_{P_1,t_i}^{2-\tau}}{\partial P_i} - \alpha_i^3 \int_{S^3} K(P_i) \frac{\partial \delta_{P_1,t_i}^{2-\tau}}{\partial P_i}
\]

\[
+ O(\tau^{1/2}) + O(\|v\|_\sigma) + O(\tau^{-1/2} \|v\|^{2-\tau}_\sigma)
\]

\[
= - \Gamma_6 \nabla g_0 K(P_i) + V_{P_i}(\tau, \alpha, t, P, v),
\]

where
\[
\Gamma_6(\tau, \alpha, t, P, v) \geq \nu_1 > 0 \quad \text{with } \nu_1 \text{ independent of } \tau, \quad (4.34)
\]

and
\[
V_{P_i}(\tau, \alpha, t, P, v) = O(\tau^{1/2}) + O(\|v\|_\sigma) + O(\tau^{-1/2} \|v\|^{2-\tau}_\sigma). \quad (4.35)
\]

The existence of \(\nu_1\) is proved below. In fact, let \(P_i\) be the south pole and make a stereographic projection \(F\) to the equatorial plane of \(S^3\) with \(y = (y^{(1)}, y^{(2)}, y^{(3)})\) as the stereographic projection coordinates, let \(\widetilde{K} = K(F(y))\) and \(|J_F| := (2/(1+|y|^2))^3\). Then we have \(F(0) = P_i\) and
\[
\alpha_i^3 \int_{S^3} (K(P) - K(P_i)) \frac{\partial \delta_{P_1,t_i}^{2-\tau}}{\partial P_i}
\]

\[
= \alpha_i^3 \int_{B^3} t_i y(\widetilde{K}(y) - \widetilde{K}(0)) \omega_{0,t_i}^2 (|J_F|^{1/3} \omega_{0,t_i}^{-1})^7
\]

\[
= : \mathcal{L} = (\mathcal{L}^{(1)}, \mathcal{L}^{(2)}, \mathcal{L}^{(3)}),
\]

where \(\omega_{0,t_i} := 2t_i/(1 + t_i^2|y|^2)\) is the solution of
\[
(-\Delta)^{1/2} \omega_{0,t_i} = \omega_{0,t_i}^2 \quad \text{on } \mathbb{R}^3.
\]
For $j = 1, 2, 3$, we have

$$L^{(j)} = -\alpha_i^3 \int_{\mathbb{R}^3} t_i y^{(j)}(\bar{K}(y) - \bar{K}(0))\omega_{0,t_i}^4(|J_F|^{1/3}\omega_{0,t_i}^{-1})^\tau$$

$$= \alpha_i^3 \int_{\mathbb{R}^3} t_i y^{(j)}(\nabla \bar{K}(0) \cdot y + O(|y|^2))\omega_{0,t_i}^4(|J_F|^{1/3}\omega_{0,t_i}^{-1})^\tau$$

$$- \frac{1}{3} \alpha_i^3 \frac{\partial \bar{K}}{\partial y^{(j)}}(0) \int_{\mathbb{R}^3} t_i |y|^2 \omega_{0,t_i}^4(|J_F|^{1/3}\omega_{0,t_i}^{-1})^\tau,$$

thus,

$$L = \nabla \bar{K}(0) \left\{ \frac{\alpha_i^3}{3} \int_{\mathbb{R}^3} t_i |y|^2 \omega_{0,t_i}^4(|J_F|^{1/3}\omega_{0,t_i}^{-1})^\tau + O(\tau^{1/2}) \right\}$$

$$= \nabla \varphi_0 K_1(\tau) \frac{2\alpha_i^3}{3} \left\{ \int_{\mathbb{R}^3} t_i |y|^2 \omega_{0,t_i}^4(|J_F|^{1/3}\omega_{0,t_i}^{-1})^\tau + O(\tau^{1/2}) \right\}.$$

It follows from $t_i^{1-\tau} \leq (|J_F|^{1/3}\omega_{0,t_i}^{-1})^\tau \leq t_i^\tau$ that

$$\int_{\mathbb{R}^3} t_i |y|^2 \omega_{0,t_i}^4(|J_F|^{1/3}\omega_{0,t_i}^{-1})^\tau \geq t_i^{1-\tau} \int_{\mathbb{R}^3} t_i |y|^2 \omega_{0,t_i}^4 \to \int_{\mathbb{R}^3} \frac{|y|^2}{(1 + |y|^2)^3},$$

as $\tau \to 0$. This ensures the existence of $\nu_i$. We have proved Proposition 4.5.

We now apply Propositions 4.2, 4.3, 4.4, 4.5 and construct a family of homotopy Id+compact operators to obtain the degree-counting formula of the solutions to the subcritical equation (4.1) on $\Sigma_\tau(\overline{P}_1, \cdots, \overline{P}_k)$.

**Proof of Theorem 4.1.** Given $\tau > 0$ and $K \in \mathcal{A}$, let $\mathcal{X}^-$ be as in (1.9) and $\Sigma_\tau(\overline{P}_1, \cdots, \overline{P}_k)$ be as in (4.6) for the given $\overline{P}_1, \cdots, \overline{P}_k \in \mathcal{X}^-$. For $u = \sum_{i=1}^k \alpha_i \delta_{P_i,t_i} + v \in \Sigma_\tau(\overline{P}_1, \cdots, \overline{P}_k)$, we have

$$T_u H^\sigma(S^3) = E_{P,t} \bigoplus \text{span}\{\delta_{P_i,t_i}, \frac{\partial \delta_{P_i,t_i}}{\partial t_i}, \frac{\partial \delta_{P_i,t_i}}{\partial P_i}\}.$$

Since $I^\tau_r(u) \in T_u H^\sigma(S^3)$, there exist $\xi \in E_{P,t}$, $\eta \in \text{span}\{\delta_{P_i,t_i}, \frac{\partial \delta_{P_i,t_i}}{\partial t_i}, \frac{\partial \delta_{P_i,t_i}}{\partial P_i}\}$ such that

$$I^\tau_r(u) = \xi + \eta.$$

From (4.12), we obtain, for all $\varphi \in E_{P,t}$,

$$\langle \xi, \varphi \rangle = I^\tau_r(u) \varphi = f^\tau_r(\varphi) + 2Q^r(v, \varphi) + \langle V^r(\varphi, \alpha, t, P, v), \varphi \rangle,$$

where $\|V^r(\varphi, \alpha, t, P, v)\|_\sigma \leq C\|v\|^{2-\tau}_\sigma$. Replacing $\varphi$ by $v$ in (4.36) and using (4.14), we have

$$\|\xi\|_\sigma \geq \delta_0 \|v\|_\sigma - \|f^\tau_r\|_\sigma - O(\|v\|^{2-\tau}_\sigma) \geq \frac{\delta_0}{2} \|v\|_\sigma - \|f^\tau_r\|,$$

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where \( \delta_0 \) is as in (4.14).
Let \( \beta = (\beta_1, \ldots, \beta_k) \), \( \beta_i = \alpha_i - 1/K(P_i) \) be as in Proposition 4.3, we define
\[
\hat{\Sigma}_\tau = \left\{ u = \sum_{i=1}^{k} \alpha_i \delta_{P_i,t_i} + v \in \Sigma_\tau(P_1, \ldots, P_k) : \|v\|_\sigma < \tau |\log \tau|^3, |\beta| < \tau |\log \tau|^2 \right\}.
\]
It follows from Proposition 4.2 and (4.23) that
\[
I_\tau'(u) \neq 0, \quad \forall u \in \Sigma_\tau(P_1, \ldots, P_k) \setminus \hat{\Sigma}_\tau.
\]
For \( u = \sum_{i=1}^{k} \alpha_i \delta_{P_i,t_i} + v \in \hat{\Sigma}_\tau \), by using (4.8), we have
\[
\langle \eta, \delta_{P_i,t_i} \rangle = I_\tau'(u) \delta_{P_i,t_i}
\]
\[
= \alpha_i \int_{\mathbb{S}^3} \delta_{P_i,t_i}^3 + \frac{1}{2} \sum_{j \neq i} \alpha_j \int_{\mathbb{S}^3} \delta_{P_i,t_i}^2 \delta_{P_j,t_j}
\]
\[
- \int_{\mathbb{S}^3} K \left| \sum_{j=1}^{k} \alpha_j \delta_{P_j,t_j} + v \right|^{1-\tau} \left( \sum_{j=1}^{k} \alpha_j \delta_{P_j,t_j} + v \right) \delta_{P_i,t_i}
\]
\[
= \frac{\partial}{\partial \alpha_i} I_\tau \left( \sum_{j=1}^{k} \alpha_j \delta_{P_j,t_j} + v \right)
\]
\[
= - |\mathbb{S}^3| \beta_i + V_o(\tau, \alpha, t, P, v), \quad (4.37)
\]
where \( V_o \) satisfying
\[
V_o(\tau, \alpha, t, P, v) = O(|\beta|^2) + O(\tau |\log \tau|) + O(\|v\|^{2-\tau})
\]
\[
\leq C \left( |\beta|^2 + \tau |\log \tau| \right).
\]
It follows from (4.24) and (4.30) that
\[
\langle \eta, \frac{\partial \delta_{P_i,t_i}}{\partial t_i} \rangle = I_\tau'(u) \frac{\partial \delta_{P_i,t_i}}{\partial t_i}
\]
\[
= \frac{1}{2} \sum_{j \neq i} \alpha_j \frac{\partial}{\partial t_i} \int_{\mathbb{S}^3} \delta_{P_j,t_j}^2 \delta_{P_i,t_i}
\]
\[
- \int_{\mathbb{S}^3} K \left| \sum_{j=1}^{k} \alpha_j \delta_{P_j,t_j} + v \right|^{1-\tau} \left( \sum_{j=1}^{k} \alpha_j \delta_{P_j,t_j} + v \right) \frac{\partial \delta_{P_i,t_i}}{\partial t_i}
\]
\[
= \frac{1}{\alpha_i} \frac{\partial}{\partial t_i} I_\tau \left( \sum_{j=1}^{k} \alpha_j \delta_{P_j,t_j} + v \right)
\]
\[
= \frac{1}{\alpha_i} \left\{ \frac{\Gamma_3}{K(P_i)^2} \tau + \frac{\Gamma_4 \Delta_9 \alpha K(P_i)}{K(P_i)^3} \frac{1}{t_i^2} \right. \]
\[
+ \sum_{j \neq i} \frac{\Gamma_5 G(P_j)}{K(P_i) K(P_j)} \frac{1}{t_i^2 t_j} + V_o(\tau, \alpha, t, P, v) \right\}, \quad (4.38)
\]
where $|V_{t_i}(\tau, \alpha, t, P, v)| = O(\tau^{3/2}|\log \tau|)$.

Applying (4.31) and (4.35), we obtain
\[
\left\langle \eta, \frac{\partial \delta_{P,t_i}}{\partial t_i} \right\rangle = \left( \alpha_i \right) \frac{\partial}{\partial P_i} \left( \sum_{i=1}^{k} \alpha_i \delta_{P,t_i} + v \right) = \frac{1}{\alpha_i} \left\{ -\Gamma_3 \nabla_{\gamma_0} K(P_i) + V_{P_i}(\tau, \alpha, t, P, v) \right\},
\]
with $V_{P_i}$ satisfying $|V_{P_i}(\tau, \alpha, t, P, v)| \leq C\tau^{1/2}$.

Under the conditions (4.36)–(4.39) stated above, we define a family of operators on $\tilde{\Sigma}_{\tau}$ as follows: for $u = \sum_{i=1}^{k} \alpha_i \delta_{P,t_i} + v \in \tilde{\Sigma}_{\tau}$ given above,
\[
X_{\theta}(u) := \xi(\theta(u)) + \eta(\theta(u)), \quad 0 \leq \theta \leq 1,
\]
where for any $\varphi \in E_{P,t_i}$,
\[
\langle \xi(\theta), \varphi \rangle := \theta f(\tau, \varphi) + (1 - \theta)\langle v, \phi \rangle + 2\theta Q_{\tau}(\varphi, v) + \theta Q(\tau, \alpha, t, P, v), \quad (4.40)
\]
and
\[
\langle \eta(\theta), \delta_{P,t_i} \rangle := \left\{ \frac{1}{\alpha_i} + (1 - \theta) \right\} \left\{ \frac{\Gamma_3}{K(P_i)} \right\} \left\{ \frac{\nabla_{\gamma_0} K(P_i)}{K(P_i)} \right\} \frac{1}{\alpha_i} \nabla_{\gamma_0}(\theta) = \left\{ \frac{1}{\alpha_i} + \frac{\theta}{\alpha_i} \right\} \left\{ \frac{\nabla_{\gamma_0} K(P_i)}{K(P_i)} \right\} \frac{1}{\alpha_i} \nabla_{\gamma_0}(\theta), \quad (4.41)
\]
where $P_i(\theta)$ is the short geodesic trajectory on $S^3$ with $P_i(0) = P_{t_i}, P_i(1) = P_i$.

Obviously, $X_1 = I_i(\tau, \varphi, \xi + \eta)$. From Sobolev compact embedding theorem and the explicit forms of $V_{\tau}, V_{\alpha, t}, V_{t_i}, V_{P_i}$, we conclude that $I_i(\tau, \varphi, \xi + \eta)$ is of the form Id+compact on $\tilde{\Sigma}_{\tau}$. Since $\xi_{\alpha/2}$ in the definition of $\tilde{\Sigma}_{\tau}$ is a finite dimensional submanifold of $H^s(\xi_{\alpha/2})$, we easily obtain from (4.40) and (4.41) that $X_\theta$ (0 \leq \theta \leq 1) is the form Id+compact. Furthermore, we have $X_\theta \neq 0$ on $\tilde{\Sigma}_{\tau}, \forall 0 \leq \theta \leq 1$. In fact, for a given $u = \sum_{i=1}^{k} \alpha_i \delta_{P,t_i} + v \in \partial \tilde{\Sigma}_{\tau}$, we obtain $\xi \neq 0$ by using (4.36) and (4.19). When $\theta = 0, \xi_0 = v \neq 0$. It follows from (4.40) that $\xi_0 \neq 0, \forall 0 < \theta < 1$.

By the homotopy invariance of the Leray-Schauder degree, we have
\[
\deg_{H^s}(X_1, \tilde{\Sigma}_{\tau}, 0) = \deg_{H^s}(X_0, \tilde{\Sigma}_{\tau}, 0), \quad (4.42)
\]
From (4.40) and (4.41), we can obtain, for \( u = \sum_{i=1}^{k} \alpha_{i} \delta_{P_{i}, t_{i}} + v \in \tilde{\Sigma}_{\tau} \),

\[
X_{0}(u) = \xi_{0}(u) + \eta_{0}(u),
\]

where \( \xi_{0} \in E_{P,t}, \eta_{0} \in \text{span}\{\delta_{P_{i}, t_{i}} : \frac{\partial \delta_{P_{i}, t_{i}}}{\partial t_{i}}, \frac{\partial \delta_{P_{i}, t_{i}}}{\partial P_{i}}\} \) satisfy

\[
\langle \xi_{0}, \varphi \rangle = \langle v, \varphi \rangle,
\]

\[
\langle \eta_{0}, \delta_{P_{i}, t_{i}} \rangle = -2\pi |S^{3}|(\alpha_{i} - K(P_{i})^{-1}),
\]

\[
\langle \eta_{0}, \frac{\partial \delta_{P_{i}, t_{i}}}{\partial t_{i}} \rangle = \frac{\Gamma_{3}}{K(P_{i})^{2} t_{i}} + \frac{\Gamma_{4} S_{2}}{K(P_{i})^{3}} \frac{1}{t_{i}^{3}} + \sum_{j \neq i} \frac{\Gamma_{5} S_{2}}{K(P_{i})^{2} K(P_{j})^{2}} \frac{1}{t_{i} t_{j} t_{j}^{2}},
\]

\[\tag{4.43}\]

\[
\langle \eta_{0}, \frac{\partial \delta_{P_{i}, t_{i}}}{\partial P_{i}} \rangle = -\nabla_{g_{0}} K(P_{i}).
\]

Recalling the definition of \( M(P_{1}, \ldots, P_{k}) \), which is simply written as \( (M_{ij}) \). By (4.44), we can easily get

\[
X_{0}(u) = 0 \quad \text{on} \ \tilde{\Sigma}_{\tau},
\]

if and only if

\[
\alpha_{i} = K(P_{i})^{-1}, \quad P_{i} = P_{i}, \quad v = 0,
\]

\[\tag{4.44}\]

\[\frac{4}{K(P_{i})^{2} t_{i}} \frac{\tau}{t_{i}^{3}} - \left( M_{ii} \frac{1}{t_{i}^{3}} + \sum_{j=1}^{k} M_{ij} \frac{1}{t_{i} t_{j}^{2}} \right) = 0.
\]

For any \((s_{1}, \ldots, s_{k}) \in \mathbb{R}^{k}\), \( s_{i} > 0, \ i = 1, \ldots, k \), we define

\[
F(s_{1}, \ldots, s_{k}) := -\sum_{j=1}^{k} \left( \frac{4\tau}{K(P_{j})^{2}} \log s_{j} \right) + \frac{1}{2} \sum_{i=1}^{k} \left( M_{ii} s_{i}^{2} + \sum_{j=1}^{k} M_{ij} s_{i} s_{j} \right),
\]

and for \( t_{i} = s_{i}^{-1} \),

\[\hat{F}(t_{1}, \ldots, t_{k}) := F(s_{1}, \ldots, s_{k}).\]

The derivative with respect to \( t_{i} \) is

\[
\frac{\partial \hat{F}}{\partial t_{i}}(t_{1}, \ldots, t_{k}) = \frac{4}{K(P_{i})^{2} t_{i}} \frac{\tau}{t_{i}^{3}} - \left( M_{ii} \frac{1}{t_{i}^{3}} + \sum_{j=1}^{k} M_{ij} \frac{1}{t_{i} t_{j}^{2}} \right),
\]

combining this and (4.43), we have

\[
\langle \eta_{0}, \frac{\partial \delta_{P_{i}, t_{i}}}{\partial t_{i}} \rangle = \frac{\pi |S^{2}|}{3} \frac{\partial \hat{F}}{\partial t_{i}}(t_{1}, \ldots, t_{k}).
\]

It is obvious that \( \nabla \hat{F}(t_{1}, \ldots, t_{k}) = 0 \) if and only if \( \nabla F(s_{1}, \ldots, s_{k}) = 0 \). A trivial verification shows that \( F(s_{1}, \ldots, s_{k}) \) is a strictly convex function, and having
a unique critical point in the first quadrant. It follows that \( \hat{F}(t_1, \cdots, t_k) \) has unique critical point in the first quadrant with Morse index zero. Hence \( X_0 \) has precisely one nondegenerate zero in \( \hat{\Sigma}_\tau \). Furthermore, by (4.44) we can easily obtain

\[
\deg_{H^\sigma} (X_0, \hat{\Sigma}_\tau, 0) = (-1)^{k + \sum_{i=1}^k j(\mathcal{P}_i)}.
\] (4.45)

Combining (4.45) and (4.42), we complete the proof of Theorem 4.1. \( \square \)

Recall the definition of \( \mathcal{O}_R \) in (1.20). For \( \delta > 0 \) suitably small, define

\[
\mathcal{O}_{R, \delta} := \{ u \in H^\sigma (S^3) : \inf_{\omega \in \mathcal{O}_R} \| u - \omega \|_\sigma < \delta \}.
\] (4.46)

**Proposition 4.6.** Let \( K \in \mathcal{A} \) be a Morse function and \( 0 < \tau_0 < \tau \leq 4/(n-2\sigma) - \tau_0 \). Then there exist some constants \( C_0 > 0, \delta_0 > 0 \) depending only on \( \tau_0, \min_{S^3} K \), and the modulo of the continuity of \( K \), such that

\[
\{ u \in H^\sigma (S^3) : u > 0 \text{ a.e., } I'_\tau (u) = 0 \} \subset \mathcal{O}_{C_0, \delta_0}.
\] (4.47)

Furthermore, we have \( I'_\tau (u) \neq 0 \) on \( \partial \mathcal{O}_{C_0, \delta_0} \) and

\[
\deg_{H^\sigma} (u - P_{\sigma}^{-1}(K|u|^{1-\tau} u), \mathcal{O}_{C_0, \delta_0}, 0) = -1.
\] (4.48)

**Proof.** From Proposition 4.1, we know that for \( \tau > 0 \) small there exists some suitable value of \( \nu_0, A, R \) such that \( u \) satisfying \( u \in H^\sigma (S^3), u > 0 \text{ a.e., } I'_\tau (u) = 0 \) are either in \( \mathcal{O}_R \) or in some \( \Sigma_\tau (q^{(1)}, \cdots, q^{(k)}) \). Combining (4.6), (4.5), (A.1), and (A.9), we conclude that there exists some positive constants \( C_0 \) and \( \delta_0 \) such that (4.47) holds.

For \( K^*(x) = x^{(4)} + 2, x = (x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)}) \in S^3 \subset \mathbb{R}^4 \), we consider \( K_t = tK + (1-t)K^* \). By the homotopy invariance of the Leray-Schauder degree, we only need to establish (4.48) for \( K^* \) and \( \tau \) very small. It is easy to see that \( K^* \in \mathcal{A} \) is a Morse function. The proof of (4.48) is straightforward by the Kazdan-Warner condition and Theorem 4.1. \( \square \)

### 4.2 The proof of the Theorems 1.3 and 1.4

Using Theorem 4.1 and Proposition 4.6, we next prove Theorem 1.3.

**Proof of Theorem 1.3.** Using Theorem 1.2 and the homotopy invariance of the Leray-Schauder degree, for \( \tau > 0 \) sufficiently small, we obtain that there exist a constant \( R \) such that,

\[
\deg_{C^2, \sigma} (u - P_{\sigma}^{-1}(Ku^2), \mathcal{O}_R, 0) = \deg_{C^2, \sigma} (u - P_{\sigma}^{-1}(K|u|^{1-\tau}u), \mathcal{O}_R, 0).
\] (4.49)

For \( C_0 \gg R, 0 < \delta_1 \ll \delta_0 \), and \( \tau_0 \) be given by Proposition 4.6. Using (4.48), Proposition 4.1, (4.7), and the excision property of the degree, we have

\[
\deg_{H^\sigma} (u - P_{\sigma}^{-1}(K|u|^{1-\tau}u), \mathcal{O}_{R, \delta_1}, 0) = \text{Index}(K).
\] (4.50)
As in the proof of Proposition 4.6, one can check that there are no critical points of $I_\tau$ in $\overline{\mathcal{O}_{R,\delta_1}} \setminus \mathcal{O}_R$. Using the same proof idea as Li [25, Theorem B.2], we can easily get

$$
\deg_{C^2,\partial}(u - P_\sigma^{-1}K(|u|^{-\tau}u), \mathcal{O}_R, 0) = \deg_{B\partial}(u - P_\sigma^{-1}K(|u|^{-\tau}u), \mathcal{O}_{R,\delta_1}, 0). \quad (4.51)
$$

It follows from (4.49)–(4.51) that for $R > C$, (1.21) is proved. Theorem 1.3 follows from the above. □

Using the theory of linear algebra, we give the proof of Corollary 1.1.

**Proof of the Corollary 1.1.** If $\sharp \mathcal{K}^- = 1$, from the proof of Theorem 1.3, we can easily obtain the conclusion. If for any distinct $P, Q \in \mathcal{K}^-, \Delta_{g_0}K(P)\Delta_{g_0}K(Q) < 9K(P)K(Q)$, we claim that there is no integer $k \geq 2$ such that $q^{(1)}, \cdots, q^{(k)} \in \mathcal{K}^-$, $\mu(M(q^{(1)}, \cdots, q^{(k)})) > 0$.

In fact, for any distinct $q^{(1)}, \cdots, q^{(k)} \in \mathcal{K}^-$, $k \geq 2$, $\mu(M(q^{(1)}, \cdots, q^{(k)})) > 0$ if and only if $M(q^{(1)}, \cdots, q^{(k)})$ is a positive definite matrix. By (1.22), we have the fact that 2-order principle minor determinant strictly less than zero. Therefore, we proved the claim. Obviously, Corollary 1.1 follows from the claim. □

We next prove Theorem 1.4.

**Proof of the Theorem 1.4.** Since the Morse functions in $C^2(\mathbb{S}^3)^+ \setminus \mathcal{A} = \partial \mathcal{A}$ are dense in $\partial \mathcal{A}$, without loss of generality we consider the case that $K \in \partial \mathcal{A}$ is a Morse function. First recall the definition of $\mathcal{K}$ and $\mathcal{K}^+$, we can assume here $\mathcal{K} \setminus \mathcal{K}^+ = \{q^{(1)}, \cdots, q^{(m)}\}, m \in \mathbb{N}_+$. From the definition of $\mathcal{A}$ and $K \in \partial \mathcal{A}$, we know that there exists $1 \leq i_1 < \cdots < i_k \leq m$, $k \geq 1$, such that

$$
\mu(M(q^{(i_1)}, \cdots, q^{(i_k)})) = 0. \quad (4.52)
$$

By perturbing the function $K$ near its some critical points to change the Hessian matrix of $K$ at these points, we obtain a sequence of $K_\ell$ satisfying: $K_\ell \to K$ in $C^2(\mathbb{S}^3)^+$ as $\ell \to \infty$; $K_\ell$ are identically the same as $K$ except in some small balls and have the same critical points with the same Morse index; there is only one such $(i_1, \cdots, i_k)$ such that (4.52) is true for any $\ell$. Refer to the perturbation method as in the proof of Li [26, Theorem 0.8] for more details. Using the same $C^2$ perturbation method for $K_\ell$, we can obtain a smooth, one-parameter family of Morse functions $\{K_{\ell,t}\} \ (-1 \leq t \leq 1)$ with the following properties:

(a) $K_{\ell,t} \ (-1 \leq t \leq 1)$ are identically the same as $K_\ell$ except in some small balls around $q^{(i_1)}, \cdots, q^{(i_k)}$ and $K_0 = K_\ell$. $K_{\ell,t}$ have the same critical points with the same Morse index for any $-1 \leq t \leq 1$.

(b) $\mu(M_{\ell,t}(q^{(j_1)}, \cdots, q^{(j_s)}))$ have the same sign for $-1 < t < 1$ for any $1 \leq j_1 < \cdots < j_s \leq m, (j_1, \cdots, j_s) \neq (i_1, \cdots, i_k)$, $\mu(M_{\ell,t}(q^{(i_1)}, \cdots, q^{(i_k)})) < 0$ for $-1 < t < 0$, and $\mu(M_{\ell,t}(q^{(i_1)}, \cdots, q^{(i_k)})) > 0$ for $0 < t < 1$.
It is easily seen that \( K_{\ell,t} \in \mathcal{A} \) when \( t \neq 0 \). From the definition of Index, we have

\[
\text{Index}(K_1) = \text{Index}(K_{-1}) + (-1)^{k-1+\sum_{j=1}^{k} i(q^{(j)})},
\]

thus, \( \text{Index}(K_1) \neq \text{Index}(K_{-1}) \). By the homotopy invariance of the Leray-Schauder degree, there exists \( t_i \to 0 \) and \( v_{\ell,i} \in \mathcal{M}_{K_{\ell,t_i}} \), such that

\[
\lim_{i \to \infty} \|v_{\ell,i}\|_{C^2(\mathbb{S}^3)} = \infty \quad \text{or} \quad \lim_{i \to \infty} \min_{\mathbb{S}^3} v_{\ell,i} = 0.
\]

Combining the Harnack inequality in [21, Lemma 4.3] and Schauder estimates in [21, Theorem 2.11], we deduce that (1.23) holds. It follows from Theorem 1.2, \( K_{\ell,t} \in \mathcal{A}(t \neq 0) \) and Theorem 1.1 that \( K_{\ell,t_i} \to K_\ell \) and \( \{v_{\ell,i}\} \) blows up exactly at \( k \) points \( q^{(i_1)}, \ldots, q^{(i_k)} \).

From the above, we know that there exists a sequence of \( K_i \to K \) in \( C^2(\mathbb{S}^3) \), \( v_i \in \mathcal{M}_K \), such that \( \{v_i\} \) blows up at precisely the \( k \) points \( q^{(i_1)}, \ldots, q^{(i_k)} \). We have thus proved Theorem 1.4.

**Proof of Theorem 1.5.** By using Theorem 1.1 we can prove the Part (i) of Theorem 1.5. The Part (ii) of Theorem 1.5 is similar to the proof of Theorem 1.4, we omit it here. \( \square \)

**A Appendix**

In this appendix, we provide some elementary calculations which have been used in the proof of Theorem 1.3.

**Lemma A.1.** Let \( \alpha \geq 2 \), there exists a positive constant \( C \) depending only on \( \alpha \) such that, for any \( a \geq 0, b \in \mathbb{R} \),

\[
|a + b|^{\alpha-1}(a + b) - a^\alpha - \alpha a^{\alpha-1}b - \frac{\alpha(\alpha - 1)}{2} a^{\alpha-2}b^2 \leq C \left(|b|^\alpha + a^\gamma |b|^\alpha - \gamma \right),
\]

where \( \gamma = \max\{0, \alpha - 3\} \).

**Lemma A.2.** Let \( 1 < \beta < 2 \), there exists a universal positive constant \( C \) such that, for any \( a > 0, b \in \mathbb{R} \),

\[
|a + b|^{\beta-1}(a + b) - a^\beta - \beta a^{\beta-1}b \leq C|b|^{\beta}.
\]

**Lemma A.3.** Let \( \beta > 1 \) and \( k \in \mathbb{N}_+ \), there exists a constant \( C \), such that for any \( (a_1, \ldots, a_k) \in \mathbb{R}^k \),

\[
\left|\left(\sum_{i=1}^{k} a_i\right)^\beta - \sum_{i=1}^{k} a_i^\beta\right| \leq C \sum_{i \neq j} |a_i|^{\beta-1}|a_j|.
\]
Lemma A.4. Let $\varepsilon_0, \tau > 0$ be suitably small and $A > 0$ be suitably large. Let $A^{-1} \tau^{-1/2} < t_1, t_2 < A \tau^{-1/2}$, $P_1, P_2 \in S^3$, $|P_1 - P_2| \geq \varepsilon_0$, $\delta_{P, t_1}$ be as in (4.3) and $G_{P_1}(P_2)$ be as in (1.11) ($|P_1 - P_2|$ represents the distance between two points $P_1$ and $P_2$ after through a stereographic projection). Then, we have,

\[
\int_{S^3} \delta_{P_1, t_1}^2 \delta_{P_2, t_2} = 4\pi |S^2| \frac{G_{P_1}(P_2)}{t_1 t_2} + O(\tau^{3/2}),
\]  

(A.1)

\[
\int_{S^3} \delta_{P_1, t_1}^2 \delta_{P_2, t_2} = O(\tau),
\]  

(A.2)

\[
\frac{\partial}{\partial t_1} \int_{S^3} \delta_{P_1, t_1}^2 \delta_{P_2, t_2} = -4\pi |S^2| \frac{G_{P_1}(P_2)}{t_1^2} + O(\tau^2),
\]  

(A.3)

\[
\frac{\partial}{\partial t_1} \int_{S^3} \delta_{P_2, t_2}^2 \delta_{P_1, t_1} = \frac{\partial}{\partial t_1} \int_{S^3} \delta_{P_2, t_2}^2 \delta_{P_1, t_1} + O(\tau^{5/2} |\log \tau|),
\]  

(A.4)

\[
\frac{\partial}{\partial t_1} \int_{S^3} \delta_{P_2, t_2}^2 \delta_{P_1, t_1} = \frac{\partial}{\partial t_1} \int_{S^3} \delta_{P_2, t_2}^2 \delta_{P_1, t_1} + O(\tau^{5/2} |\log \tau|),
\]  

(A.5)

\[
\int_{S^3} |P - P_1|^2 \delta_{P_1, t_1}^3 = \frac{1}{2 t_1} \frac{3\pi}{2} |S^2| + O(\tau^2 |\log \tau|),
\]  

(A.6)

\[
\frac{\partial}{\partial t_1} \int_{S^3} \delta_{P_1, t_1}^3 = -\frac{\tau}{t_1^2} \frac{\pi}{2} |S^2| + O(\tau^{5/2} |\log \tau|),
\]  

(A.7)

\[
\frac{\partial}{\partial t_1} \int_{S^3} |P - P_1|^2 \delta_{P_1, t_1}^3 = -\frac{3\pi}{t_1^3} |S^2| + O(\tau^{5/2} |\log \tau|).
\]  

(A.8)

Lemma A.5. Under the hypotheses of Lemma A.4, in addition that $\Gamma_1, \Gamma_2$ are positive constants independent of $\tau$. Then, we have,

\[
\langle \delta_{P_1, t_1}, \delta_{P_1, t_1} \rangle = |S^3|, \quad \langle \frac{\partial \delta_{P_1, t_1}}{\partial P_1^{(t)}}, \frac{\partial \delta_{P_1, t_1}}{\partial P_1^{(t)}} \rangle = \Gamma_1 t_1^2,
\]  

(A.9)

\[
\langle \frac{\partial \delta_{P_1, t_1}}{\partial t_1}, \frac{\partial \delta_{P_1, t_1}}{\partial t_1} \rangle = \Gamma_2 t_1^{-2},
\]  

(A.10)

\[
\langle \delta_{P_1, t_1}, \delta_{P_2, t_2} \rangle = O(\tau),
\]  

(A.11)
\[
\| \delta_{P_1,t_1}^{1-\tau} \delta_{P_2,t_2} \|_{L^{3/2}(S^3)} = O(\tau |\log \tau|), \quad \text{(A.12)}
\]

\[
\left\| \delta_{P_1,t_1}^{1-\tau} \frac{\partial \delta_{P_2,t_2}}{\partial t_1} \right\|_{L^{3/2}(S^3)} = O(\tau^{3/2} |\log \tau|), \quad \text{(A.13)}
\]

\[
\left\| \delta_{P_1,t_1}^{1-\tau} \delta_{P_2,t_2} \frac{\partial \delta_{P_3,t_3}}{\partial t_1} \right\|_{L^1(S^3)} = O(\tau^{3/2} |\log \tau|), \quad \text{(A.14)}
\]

\[
\| \delta_{P_1,t_1}^{2-\tau} - \delta_{P_2,t_1}^{2} \|_{L^{3/2}(S^3)} = O(\tau |\log \tau|), \quad \text{(A.15)}
\]

\[
\| \delta_{P_1,t_1}^{1-\tau} - \delta_{P_1,t_1} \|_{L^3(S^3)} = O(\tau |\log \tau|), \quad \text{(A.16)}
\]

\[
\| P - P_1 \|_{L^{3/2}(S^3)} = O(\tau^{1/2}), \quad \text{(A.17)}
\]

\[
\| P - P_1 \|_{L^{3/2}(S^3)} = O(\tau), \quad \text{(A.18)}
\]

Lemma A.6. Let \( \varepsilon_0, \tau, A \) be as in Lemma A.4, \( P_1, P_2, P_3 \in S^3 \) satisfy \( |P_i - P_j| \geq \varepsilon_0 \), \( i \neq j \), and \( A^{-1} \tau^{-1/2} < t_1, t_2, t_3 \leq A \tau^{-1/2} \). Then, we have,

\[
\left| \frac{\partial}{\partial P_1} \int_{S^3} \delta_{P_1,t_1}^{3-\tau} \right| = O(\tau^{1/2}), \quad \left| \frac{\partial}{\partial P_1} \int_{S^3} \frac{\partial \delta_{P_1,t_1}}{\partial t_1} \right|_{L^{3/2}(S^3)} = O(\tau^{1/2} |\log \tau|), \quad \text{(A.19)}
\]

\[
\left| \frac{\partial}{\partial P_1} \int_{S^3} \delta_{P_1,t_1}^{2-\tau} \right| = O(\tau^{1/2}), \quad \left| \frac{\partial}{\partial P_1} \int_{S^3} \frac{\partial \delta_{P_1,t_1}}{\partial t_1} \right| = O(\tau^{1/2}), \quad \text{(A.20)}
\]

\[
\left| \frac{\partial}{\partial P_1} \int_{S^3} \delta_{P_1,t_1}^{1-\tau} \frac{\partial \delta_{P_1,t_1}}{\partial t_1} \right|_{L^1(S^3)} = O(\tau^{1/2} |\log \tau|). \quad \text{(A.21)}
\]

Lemma A.7. In addition to the hypotheses of Lemma A.4, we assume that \( K \in C^1(S^3) \). Then

\[
\frac{\partial}{\partial t_1} \int_{S^3} (K(P) - K(P_2)) \delta_{P_2,t_2}^{1-\tau} \delta_{P_1,t_1} = O(\tau^2). \quad \text{(A.22)}
\]

Lemma A.8. In addition to the hypotheses of Lemma A.7, we assume that \( v \in E_{P_1,t_1} \). Then

\[
\int_{S^3} (K(P) - K(P_1)) \delta_{P_1,t_1}^{1-\tau} \frac{\partial \delta_{P_1,t_1}}{\partial P_1} v = O(\tau^{1/2} |\log \tau| ||v||_\sigma). \quad \text{(A.23)}
\]
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