Momentum section on Courant algebroid and constrained Hamiltonian mechanics

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Abstract

We propose a generalization of the momentum map on a symplectic manifold with a Lie algebra action to a Courant algebroid structure. The theory of a momentum section on a Lie algebroid is generalized to the theory compatible with a Courant algebroid. As an example, we identify the momentum section in a constrained Hamiltonian mechanics with Courant algebroid symmetry. Moreover, we construct cohomological formulations by considering the BFV and BV formalism of this Hamiltonian system. The Weil algebra for this structure is constructed.

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1 Introduction

Interesting structures that generalize a Lie group and a Lie algebra in Poisson geometry are a Lie algebroid and a Lie groupoid. (for example, see a textbook [38]) As a further generalization, a Courant algebroid [37] which is a 2-categorical generalization of a Lie algebroid appears in the analysis of the Dirac structure [12], and a generalization of a Lie bialgebra and Poisson Lie group theory, generalized geometry [23, 19], a topological sigma model [24, 44], T-duality in string theory [11], etc.

A geometric structure was discovered as compatibility conditions of gauge theories and Hamiltonian mechanics with a Lie algebroid structure [40, 32, 33, 27]. A moment(um) map theory is a fundamental theory in symplectic geometry inspired by the mechanics with a Lie group action. Recently, Blohmann and Weinstein [7] have proposed the equivalent structure as a generalization of a momentum map and a Hamiltonian $G$-space with a Lie algebra action to the Lie algebroid setting, inspired by analysis of the Hamiltonian formalism of the general relativity [8]. They are called a momentum section and a Hamiltonian Lie algebroid. The author gave new examples in the constrained Hamiltonian mechanics and sigma models, and generalized a momentum section theory to a pre-multisymplectic manifold [26]. There are related papers [16, 9].

In this paper, we generalize a momentum section theory on a pre-symplectic manifold compatible with a Lie algebroid to a Courant algebroid. We obtain conditions similar to a Lie algebroid, however geometric quantities of the Lie algebroid are replaced to ones of the Courant algebroid. We define a Hamiltonian Courant algebroid as a generalization of the Hamiltonian $G$-space.

We show that a simple constrained Hamiltonian mechanics system is an example of our proposal. The compatibility condition of the constrained Hamiltonian mechanics on a cotangent bundle $T^*M$ with a Courant algebroid $E$ over a base manifold $M$ gives existence of a momentum section. Essentially, the 0-th order term with respect to the momentum $p$ in the constraint is identified to a momentum section. The conditions are realized as Poisson brackets of constraints and the Hamiltonian. This example is a very natural physical system, thus, a momentum section can be understood as a natural geometric structure in physical theories.

In the preceding sections, we consider cohomological realizations of a momentum section
and a Hamiltonian Courant algebroid, which give clear interpretations of complicated conditions. We consider the BFV formalism and the BV formalism of the above Hamiltonian mechanics. The BRST-BFV charge functional $S_{BFV}$ in the BFV formalism, and the BV action functional $S_{BV}$ in the BV formalism are the Hamiltonian for homological vector fields $Q$ such that $Q^2 = 0$, which are differentials of complexes. In the BFV formalism, Poisson brackets of two fundamental functions, the BFV charge function $S_{BFV}$ and the BFV Hamiltonian $H_{BFV}$, are equivalent to the consistency conditions of a Hamiltonian Courant algebroid. In the BV formalism, the BV bracket (an odd Poisson bracket) of the BV action functional $S_{BV}$ is equivalent to the consistency conditions. As applications, we construct the Weil model and the Cartan model of a Hamiltonian Courant algebroid to apply the equivariant cohomology theory.

This paper is organized as follows. In Section 2, we give definitions of a momentum section on a Courant algebroid and consider some examples. In Section 3, We show that a momentum section appears in a constrained Hamiltonian mechanics with a Courant algebroid structure. In Section 4, we consider the BFV formalism of the Hamiltonian mechanics in Section 3. In Section 5, the BV formalism is constructed based on the FHGD formulation In Section 6, the Weil algebra and the Cartan model are discussed. Section 7 is devoted to discussion and outlook. In Appendix, some formulas are summarized.

Note: After this article has been completed, we were informed that Hancharuk and Strobl considered a Lie 2-algebroid extension of the same constrained mechanics, which is different but complementary generalization from our results. See [21].

2 Courant algebroid and momentum section

2.1 Courant algebroid and $Q$-manifold

We summarize basic properties of a Courant algebroid used in this paper.

**Definition 2.1 (Courant algebroid)** [37] Let $E$ be a vector bundle over a manifold $M$. A Courant algebroid is a quadruple $(E, [-, -]_D, \rho, \langle -, - \rangle)$ where $[-, -]_D$ is a bilinear bracket on $\Gamma(E)$ called the Dorfman bracket, $\rho : E \to TM$ is a bundle map called the anchor map, and an inner product $\langle -, - \rangle$ is a non-degenerate bilinear form on $\Gamma(E)$. They satisfy the
following axioms for any $e_i \in \Gamma(E)$ and $f \in C^\infty(M)$:

1. The bracket $[-,-]_D$ satisfies the Leibniz identity

$$[e_1, [e_2, e_3]_D] = [[e_1, e_2]_D, e_3]_D + [e_2, [e_1, e_3]_D]_D.$$  \hspace{1cm} (1)

2. $\rho([e_1, e_2]_D) = [\rho(e_1), \rho(e_2)]$.

3. $[e_1, fe_2]_D = f[e_1, e_2]_D + (\rho(e_1) \cdot f)e_2$.

4. $[e, e]_D = \frac{1}{2}D(e, e)$.

5. $\rho(e_1) \cdot \langle e_2, e_3 \rangle = \langle [e_1, e_2]_D, e_3 \rangle + \langle e_2, [e_1, e_3]_D \rangle$.

Here $D$ is a generalized exterior derivative on $\Gamma(E)$ defined by $\langle Df, x \rangle = \frac{1}{2}\rho(x)f$.

**Example 2.1** Let $M$ be a smooth manifold and take the direct product bundle $TM \oplus T^*M$. For $X + \xi, Y + \eta \in \Gamma(TM \oplus T^*M)$, we define the inner product,

$$\langle X + \xi, Y + \eta \rangle = \iota_X \eta + \iota_Y \xi.$$  \hspace{1cm} (2)

the anchor map

$$\rho(X + \xi)f = Xf,$$  \hspace{1cm} (3)

for $f \in C^\infty(M)$, and the Dorfman bracket,

$$[X + \xi, Y + \eta]_D = [X, Y] + (X \xi - d\iota_Y \xi + \iota_X \iota_Y h),$$  \hspace{1cm} (4)

for a closed 3-form $h \in \Omega^3(M)$. Then, $(TM \oplus T^*M, \langle -,- \rangle, \rho, [-,-]_D)$ is a Courant algebroid. This Courant algebroid is called the standard Courant algebroid with $h$-flux.

In order to describe a Courant algebroid as a Q-manifold [15], we consider a shifted vector bundle, $E[1]$, which is a graded manifold with a coordinate of the fiber shifted by 1. A section on $E^*$ is identified as a function on a graded manifold $E[1]$. One can refer to some references of mathematics of a graded manifold, Q-manifold related to a Courant algebroid [13, 10, 25].

Even and odd local coordinates $(x^i, \eta^a)$ on $E[1]$ are introduced, where $x^i$ is a coordinate on $M$ and $\eta^a$ is a basis of the fiber of degree 1. We denote a fiber metric on $E^*$ by $k^{ab} = \langle \eta^a, \eta^b \rangle$.
A Q-manifold structure for a Courant algebroid is defined on a graded cotangent bundle $T^*[2]E[1]$. Canonical conjugates of $x^i$ and $\eta^a$ are denoted by $p_i$, and $k_{ab}\eta^b$, where we identify $E$ and $E^*$ by the inner product $\langle -, - \rangle$. We have coordinates $(x^i, p_i, \eta^a)$ of degree $(0, 2, 1)$.

The homological vector field $Q$ for a Courant algebroid on $T^*[2]E[1]$ is given by

$$Q = \rho^i_a(x)\eta^a \frac{\partial}{\partial x^i} + \left( \rho^i_a(x)p_i + \frac{1}{2} f_{acd}\eta^c \eta^d \right) k^{ab} \frac{\partial}{\partial \eta^b} + \left( \partial_i \rho^j_a(x)p_j \eta^a + \frac{1}{3!} \partial_i f_{abc} \eta^a \eta^b \eta^c \right) \frac{\partial}{\partial p_i},$$

where $\rho(e_a) = \rho^i_a(x)\partial_i$ for a basis $e_a$ of $E$, and $f_{abc}$ is a structure function of the Dorfman bracket satisfying $[e_a, e_b]_D = f_{abc}k^{cd}e_d$. $Q$ is a vector field of degree 1 such that $Q^2 = 0$ if and only if $E$ is a Courant algebroid, which define the Courant algebroid differential $E_d$ on the complex in the space $C^\infty(T^*[2]E[1])$.

### 2.2 Momentum section and Hamiltonian Courant algebroid

In this section, we propose a momentum section of the Courant algebroid and a Hamiltonian Courant algebroid as a generalization of the paper [7].

Let $M$ be a pre-symplectic manifold with a pre-symplectic form $B \in \Omega^2(M)$, i.e., a closed 2-form which is not necessarily nondegenerate. We consider a Courant algebroid $(E, [-,-]_D, \rho, \langle -,- \rangle)$ over a pre-symplectic manifold $(M, B)$.

We introduce a connection (a linear connection) on $E$, i.e., a covariant derivative $D : \Gamma(E) \to \Gamma(E \otimes T^*M)$, satisfying $D(fe) = fDe + df \otimes e$ for a section $e \in \Gamma(E)$ and a function $f \in C^\infty(M)$. The connection is extended to $\Gamma(M, \wedge^*T^*M \otimes E)$ as a degree 1 operator.

An $E^*$-valued 1-form $\gamma \in \Omega^1(M, E^*)$ is defined by

$$\langle \gamma(v), e \rangle = -B(v, \rho(e)),$$

for all sections $e \in \Gamma(E)$ and vector fields $v \in \mathfrak{X}(M)$. The following two conditions are introduced for $E$ on a pre-symplectic manifold $(M, B)$.

(H2) A section $\mu \in \Gamma(E^*)$ is a $D$-momentum section if it satisfies

$$D\mu = \gamma,$$

(H2) and (H3) correspond to the number in [7]. (H1) is discussed later.
(H3) A $D$-momentum section $\mu$ is *bracket-compatible* if it satisfies

$$E^d(\mu_1, \mu_2) = -\langle \gamma(\rho(\mu_1)), \mu_2 \rangle,$$

for all sections $\mu_1, \mu_2 \in \Gamma(E)$. A $D$-momentum section on a Lie algebroid has formally the same equation as Equation (8) however the $E$-differential $E^d$ is different. In Equation (8), $E^d$ is the Courant algebroid differential induced from $Q$ in (5). Especially, note that

$$E^d\mu(f) = \rho(\mu^*)(f),$$

for $f \in C^\infty(M)$ is automatically satisfied for the Courant algebroid differential $E^d$. Here the dual momentum section $\mu^* \in \Gamma(E)$ is defined by $\mu^*(e) = \langle \mu, e \rangle$.

A Hamiltonian Courant algebroid is defined as follows.

**Definition 2.2** A Courant algebroid $E$ with a connection $D$ and a section $\mu \in \Gamma(E^*)$ is called *weakly Hamiltonian* if (H2) is satisfied.

**Definition 2.3** A Courant algebroid $E$ with a connection $D$ and a section $\mu \in \Gamma(E^*)$ is called *Hamiltonian* if (H2) and (H3) are satisfied.

We can add the following condition corresponding to (H1).

**(H1):** $E$ is *presymplectically anchored with respect to $D$* if $D\gamma = 0$.

Here $D$ is a dual connection on $E^*$ defined by $d\langle \mu, e \rangle = \langle D\mu, e \rangle + \langle \mu, De \rangle$, for all sections $\mu \in \Gamma(E^*)$ and $e \in \Gamma(E)$.

The condition (H1) is regarded as a flatness condition of the connection $D$ since $D\gamma = D^2\mu$.

In this paper, we do not assume the condition (H1) for the existence of a momentum section. (H1) can be imposed as an extra condition.

### 2.3 Examples

#### 2.3.1 Trivial Lie algebra bundle: momentum map

A momentum section is a generalization of a momentum map on a symplectic manifold with a Lie group action. Definitions of a momentum section (H2) and (H3) reduce to definitions of a momentum map if a Courant algebroid $E$ is a trivial bundle of a Lie algebra with an inner product $\langle -, - \rangle$. 

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Suppose $B$ is nondegenerate, i.e., $B$ is a symplectic form. Consider an action Lie algebroid on the trivial bundle $E = M \times \mathfrak{g}$. It means that an infinitesimal Lie algebra action is given by a bundle map $\rho : \mathfrak{g} \times M \rightarrow TM$, such that

$$[\rho(e_1), \rho(e_2)] = \rho([e_1, e_2]).$$

(10)

The bracket in left hand side is a Lie bracket of vector fields. An action Lie algebroid is regraded as a special case of a Courant algebroid if a Lie algebra has an inner product. In this case, we can take a zero connection, $D = d$, and axioms of a momentum section reduce to the following equations.

(H2) A section $\mu \in \Gamma(M \times \mathfrak{g}^*)$ is regarded as a map $\mu : M \rightarrow \mathfrak{g}^*$. Then, Equation (7) is

$$d \mu(e) = \iota_{\rho(e)} B.$$  

(11)

A map $\mu$ is a Hamiltonian for the vector field $\rho(e)$.

(H3) We substitute Equation (11) to the condition (H3), i.e., Equations (8) and (9). Equation (9) is trivial, and Equation (8) becomes

$$\text{ad}^*_{e_1} \mu(e_2) = \langle \mu, [e_1, e_2]\rangle.$$  

(12)

for $e_1, e_2 \in \mathfrak{g}$. This means that $\mu$ is $\mathfrak{g}$-equivariant. Note that in this case, Equation (11) automatically leads (H1) since $d^2 = 0$:

(H1)

$$d \gamma = d^2 \mu = 0.$$  

(13)

Independent conditions are (11) and (12). It shows that $\mu$ is an infinitesimally equivariant momentum map.

2.3.2 Standard Courant algebroid

We consider the standard Courant algebroid with $H$-flux in Example 2.1. The homological vector field (5) is

$$Q = \eta^i \frac{\partial}{\partial x^i} + \left(p_i + \frac{1}{2} h_{ijk} \eta^j \eta^k \right) \frac{\partial}{\partial \xi_k} + \frac{1}{3!} \partial_j h_{ijk} \eta^j \eta^k \eta^l \frac{\partial}{\partial p_i},$$

(14)
where \( \eta^a \) is decomposed to degree 1 coordinates \( \eta^a = (\eta^i, \xi_i) \) of the fiber coordinates of \( T[1]M \) and \( T^*[1]M \), and we can take the metric,

\[
 k = \begin{pmatrix}
 0 & 1 \\
 1 & 0 \\
\end{pmatrix}.
\]

(15)

A momentum section \( \mu \in \Gamma(T^*M \oplus TM) \) is decomposed to \( \mu = \nu + Z \), where \( \nu \in \Omega(M) \) and \( Z \in \mathfrak{X}(M) \).

The condition (H2), Equation (17) is written as

\[
 DZ = 0, \\
 D\nu = -B.
\]

(16) (17)

where \( D \) is an affine connection on \( TM \) and the corresponding induced connection on \( T^*M \).

The condition (H3), Equation (18) reduces to

\[
 (d\nu + \iota_Z h)(X, Y) = -B(X, Y),
\]

(18)

for all \( X, Y \in \mathfrak{X}(M) \).

### 3 Constrained Hamiltonian mechanics

We consider a nontrivial example of a momentum section of a Courant algebroid: a constrained Hamiltonian mechanics system on the cotangent bundle \( T^*M \) over a smooth manifold \( M \) consistent with the Courant algebroid \( E \).

Take local coordinates \( (x^i, p_i) \) on the cotangent bundle \( T^*M \). The constraint Hamiltonian mechanics has two functions on a cotangent bundle (a phase space in physical terminology), a Hamiltonian \( H \) and constraints \( G_a \), which satisfy Poisson brackets,

\[
 \{H, G_a\} = \sigma^b_a G_b, \\
 \{G_a, G_b\} = C^c_{ab} G_c.
\]

(19) (20)

\( H = H(x, p) \) is a function on \( T^*M \) and \( G_a = G_a(x, p) \) is a function on \( T^*M \) taking a value on \( E^* \). \( \sigma^b_a = \sigma^b_a(x, p) \) and \( C^c_{ab} = C^c_{ab}(x, p) \) are local functions on the cotangent bundle. The above equations (19) and (20) are a realization of actions of groups or groupoids in the mechanics. In this paper, we consider an action of the Courant algebroid, thus, it is assumed that \( G_a \) has the Courant algebroid structure.
3.1 Courant algebroid structure on constraints

First, in this section, we consider the simplest $G_a$ with a Courant algebroid structure.

A cotangent bundle has the canonical symplectic form,

$$\omega_{\text{can}} = dx^i \wedge dp_i,$$

which gives the canonical Poisson bracket,

$$\{x^i, p_j\} = \delta^i_j.$$ (22)

Suppose that $G_a$ is linear with respect to the momentum $p_i$. We can assume that

$$G_a = \rho_i^a(x)p_i,$$ (23)

where $\rho_i^a$ is a local function of $x$. We denote this function by $\rho$ since later, we identify the function as the local coordinate expression of the anchor map. We impose the following Poisson brackets,

$$\{G_a, G_b\} = C_{ab}^c G_c,$$ (24)

with some function $C_{ab}^c(x)$. Equation (24) means that $G_a$ consists of a Lie algebra under the Poisson bracket. Then, $G_a$ is called first class constraints. Note that from order counting of $p$, $C_{ab}^c(x)$ is a zeroth order with respect to $p$ and must only depend on $x$. Referring to the identity $\rho([e_1, e_2]_D) = [\rho(e_1), \rho(e_2)]$ of the Courant algebroid corresponding to Equation (170), the straightforward calculation gives that Equation (24) is satisfied if a coefficient function $\rho_i^a(x)$ is the local coordinate expression of the anchor map $\rho$ and $C_{ab}^c = f_{cab} k^{cd}$. Therefore, $G_a$ satisfies the following Poisson bracket,

$$\{G_a, G_b\} = k^{cd} f_{cab} G_c.$$ (25)

Another consistency check of this choice is needed. The Jacobi identity $\{G_a, \{G_b, G_c\}\} + \text{cyclic}(abc) = 0$ must be satisfied. The Jacobi identity is proved using identities (169)–(171) of the Courant algebroid. Thus the Poisson algebra of $G_a$ is consistent with the Courant algebroid structure on $E$.

Note that the converse claim is not true. In fact, suppose a function $G_a$ in (23) without any assumption of a structure on the vector bundle $E$. We only need one identity (170).
to satisfy Equation \((24)\). The Jacobi identity \(\{G_a, \{G_b, G_c\}\} + \text{Cycl}(abc) = 0\) impose the equation for \(\rho^i_a, k^{ab}\) and \(f_{abc}\):

\[
\rho^i_e \partial_i f_{dab} + k^{ef} f_{eab}f_{cdf} + (abc \text{ cyclic}) = N_{abcd},
\]

(26)

where \(\partial_i = \frac{\partial}{\partial x^i}\), and the right hand side is an arbitrary \((abc)\)-antisymmetric tensor \(N_{abcd}(x)\) such that \(\rho^i_k k^{ed} N_{abcd} = 0\). There are ambiguities to choose a tensor \(N_{abcd}\) for the Jacobi identity. The simplest solution is \(N_{abcd} = 0\), which give a Lie algebroid structure on the vector bundle \(E\). This case was analyzed in \[27\]. We took another solution,

\[
N_{abcd} = \rho^i_d \partial_i f_{abc}
\]

(27)

under a Courant algebroid structure on \(E\) because we can prove that

\[
\rho^i_e k^{ed} N_{abcd} = \rho^i_e k^{ed} \rho^i_d \partial_i f_{abc} = 0,
\]

(28)

from the identity \([169]\) and \([171]\).

There are possibilities of other solutions. The summary is as follows.

**Theorem 3.1** Let \(E\) be a Courant algebroid. Then, constraints \((23)\) satisfy the Poisson bracket \((25)\) and the Jacobi identity.

\(G_a = \rho^i_a(x)p_i\) is the first class constraint if \(E\) is a Courant algebroid.

### 3.2 Free Hamiltonian

In this section, we introduce a Hamiltonian in the mechanics and discuss consistency with a Courant algebroid.

We consider the simple free Hamiltonian,

\[
H = \frac{1}{2} g^{ij}(x)p_ip_j,
\]

(29)

where \(g^{ij}(x)\) is a symmetric tensor which is identified an inverse of a metric \(g \in \Gamma(T^*M \otimes T^*M)\) on \(M\). Suppose that the Hamiltonian has a symmetry generated by \(G_a\), i.e., the following Poisson bracket is imposed,

\[
\{H, G_a\} = \sigma^b_a G_b,
\]

(30)
where $\sigma^b_a = \sigma^b_a(x,p)$ is some function of $x$ and $p$. From the order counting of $p$, a function $\sigma^b_a$ must be a linear function of $p$. Thus we can assume that
\[
\{ H, G_a \} = -\Gamma^b_{ai}(x)p_iG_b.
\] (31)
where $\Gamma^b_{ai}(x)$ is a function of $x$. Equations (24) and (31) must be covariant under transformations of the fiber of $E$ using a transition function $M^a_b$. This requires that $\Gamma^b_{ai} = g_{ij}\Gamma^j_a$ transforms as a connection 1-form on $E$,
\[
\Gamma' = M\Gamma M^{-1} + dMM^{-1}.
\] (32)

We denote the Courant algebroid connection defined by $\Gamma^b_{ai}$ by $D : \Gamma(E) \to \Gamma(E \otimes T^*M)$. Substituting Equations (23) and (29) to (31), we obtain the condition for the metric $g$,
\[
^E D g = 0,
\] (33)
where $^E D$ is an $E$-connection $^E D : \Gamma(TM) \to \Gamma(TM \otimes E^*)$ on $TM$ defined by
\[
^E D_v e := L_{\rho(e)}v + \rho(D_v e),
\] (34)
where $v \in \mathfrak{X}(M)$ is a vector field and $e \in \Gamma(E)$ is a section of $E$. $L$ is the Lie derivative. An $E$-connection is extended to a covariant derivative on the tensor product space of $TM$ and $T^*M$ similar to a normal connection. Especially, $^E D$ in Equation (33) is the induced $E$-connection on the space $T^*M \otimes T^*M$. In summary, we obtain the following theorem.

**Theorem 3.2** Let $E$ be a Courant algebroid. Then, Equation (30) is satisfied if and only if Equation (33) is satisfied.

This theorem is the compatibility condition with a metric $g$ and the mechanics with a Courant algebroid structure.

### 3.3 Inhomogeneous generalization and momentum section

We generalize the mechanics in the previous section to the system with constraints and Hamiltonian inhomogeneous with respect to the order of $p$. A momentum section appears in the mechanics.
We generalize the constraint $G_a$ in addition to the zeroth order term of $p$,

$$G_a = \rho_a^i(x)p_i + \alpha_a(x), \quad (35)$$

where $\alpha_a$ is a local section of $E^*$. The Hamiltonian is also generalized to inhomogeneous one as follows,

$$H = \frac{1}{2}g^{ij}(x)p_ip_j + \beta^i(x)p_i + V(x), \quad (36)$$

where $\beta = \beta^i\partial_i \in \mathfrak{X}(M)$ is a vector field on $M$ and $V \in C^\infty(M)$.

The second term $\beta^i(x)p_i$ in the Hamiltonian is absorbed by redefining the conjugate momentum $p_i$ as $p'_i = p_i + A_i$, where $A_i = g_{ij}\beta^j$ is a 1-form. The 1st order term of $p$ in the Hamiltonian is absorbed to other terms as expected,

$$H = \frac{1}{2}g^{ij}(x)p'_ip'_j + V'(x), \quad (37)$$

where

$$V'(x) = V(x) - \frac{1}{2}g^{ij}A_iA_j. \quad (38)$$

The symplectic form is changed to

$$\omega = dp'_i \wedge dx^i + \frac{1}{2}B_{ij}(x)dx^i \wedge dx^j, \quad (39)$$

with $B = dA \in \Omega^2(M)$. Obviously, $B$ is closed, thus it defines a pre-symplectic form on $M$. The Poisson bracket of $p$’s becomes

$$\{p'_i, p'_j\} = -B_{ij}. \quad (40)$$

Using $p'$, $G_a$ is written as

$$G_a = \rho_a^i(x)p'_i + \mu_a, \quad (41)$$

where $\mu_a = \alpha_a - \rho_a^iA_i$ is a section of $E^*$.

In order to construct consistent constrained mechanics, we require that $G_a$ satisfies the same equation as Equation (24),

$$\{G_a, G_b\} = k^{cd}f_{cab}G_d. \quad (42)$$
which means that $G_a$ is the first class constraint. This condition imposes the following conditions to $\mu$

$$
E \mathcal{d}\mu(e_1, e_2) = \rho^*(B)(e_1, e_2),
$$

(43)

for $e_i \in \Gamma(E)$ in addition to the Courant algebroid structure in subsection 3.1. Here $\rho^*$ is the induced map of the anchor to $\Omega^*(M)$, mapping ordinary differential forms to $E$-differential forms, $\rho^*(B) = \frac{1}{2} B_{ij} \rho_a^i \rho_b^j q^a q^b \in \Gamma(\wedge^2 E^*)$. If we define $\gamma \in \Omega^1(M, E^*)$ as

$$
\langle \gamma(v), e \rangle = -B(v, \rho(e)),
$$

(44)

for all $v \in \mathfrak{X}(M)$ and $e \in \Gamma(E)$, Equation (43) is written as

$$
E \mathcal{d}\mu(e_1, e_2) = -\langle \gamma(\rho(e_1)), e_2 \rangle,
$$

(45)

which is the same equation as (8), i.e., (H3) in the definition of the momentum section. Equation (4) is automatically satisfied.

Thus, we obtain the following result.

**Theorem 3.3** Assume $G_a$ in Equation (41) under a Courant algebroid structure. Then Equation (42) gives the identity (45), i.e. the condition (H3).

Next, we require the same equation as Equation (30) for the Poisson bracket of $H$ and $G_a$,

$$
\{H, G_a\} = \sigma^b_a G_b,
$$

(46)

i.e., $G_a$ is required to generate a symmetry of the Hamiltonian. Here the coefficient function is generalized to an inhomogeneous function $\sigma^b_a(x) = -\Gamma^b_a(x)p_i - \tau^b_a(x)$, where $\tau$ is a local endomorphism of sections of $E$. If we require $G_a$ is a global section of $E^*$, a gauge transformation must transform $\Gamma$ and $\tau$ as

$$
\Gamma' = M \Gamma M^{-1} + dMM^{-1},
$$

(47)

$$
\tau' = M \tau M^{-1} + \iota_\beta (dMM^{-1}),
$$

(48)

where $M^a_b$ is a transition function. Thus, $\Gamma^b_a = \Gamma^b_{aj} dx^j$ transform as a connection 1-form on $E$, and $\tau = \iota_\beta \Gamma = g(\Gamma, A)$. Thus

$$
\{H, G_a\} = -g^{ij} \Gamma^b_{aj} p_i G_b,
$$

(49)
We compute identities for coefficient functions by substituting concrete expressions of \( H, G_a \) and \( \sigma \) \((35), (36)\) and \((49)\) to Equation \((49)\). The \( p^2 \) order of Equation \((49)\) gives Equation \((33)\), which is the same consistency condition for the metric in the homogeneous case. The \( p^1 \) order of Equation \((49)\) gives a new condition for \( \mu \),

\[
D\mu = \gamma,
\]

which is the condition \((H2)\) in the definition of momentum section. The 0-th order of \( p \) gives a condition for the potential function \( V' \) as

\[
E^dV' = 0,
\]

which is independent of other data including the momentum section.

**Theorem 3.4** Assume inhomogeneous \( G_a \) and \( H \) in Equation \((35)\) and \((36)\) under a Courant algebroid structure. Then Equation \((49)\) gives the identities \((33), (50)\) and \((51)\). Especially, the Poisson bracket \((49)\) gives the condition \((H2)\).

We summarize all results of the constrained Hamiltonian system in this section.

**Proposition 3.5** Let \( E \) be a Courant algebroid over \( M \). The inhomogeneous constrained Hamiltonian system on \( T^*M \) with the constraint \((35)\) and the Hamiltonian \((36)\). If we require consistency with a Courant algebroid \( E \) over \( M \), we obtain Poisson brackets \((12)\) and \((19)\) with a connection 1-form \( \Gamma^b_a \), and

\[
E^Dg = 0, \tag{52}
\]

\[
\tau = g(\Gamma, A), \tag{53}
\]

\[
E^dV' = 0, \tag{54}
\]

Moreover, \( \mu = \alpha - \iota_{\rho}A \) is a bracket compatible \( D \)-momentum section with respect to the presymplectic structure defined by \( B = dA \). Here \( A = \iota_{\beta}g \in \Omega^1(M) \) and \( D \) is a connection induced from \( \Gamma^b_a \).

### 4 BFV and BV description: BFV formalism

In order to formulate cohomological description of the momentum section and the Hamiltonian Courant algebroid, we consider the BFV and BV formalism \([2, 3, 4, 5]\) of the constrained
Hamiltonian mechanics in Section 3. In the Lie algebroid case, the BFV and BV formalism were analyzed in the paper [27].

In this section, we consider the BFV formalism. The classical BFV formalism is defined on a graded manifold $T^*[2]E[1]$, where $T^*[2]M$ is an original phase space of the Hamiltonian mechanics and $E[1]$ is a vector bundle whose fiber degree is shifted by 1. The BFV data consist of even and odd functions $H_{BFV}$ and $S_{BFV}$ satisfying

\begin{align}
\{S_{BFV}, S_{BFV}\} &= 0, \\
\{S_{BFV}, H_{BFV}\} &= 0, \\
\{H_{BFV}, H_{BFV}\} &= 0,
\end{align}

where $\{-,-\}$ is a Poisson bracket. Note that Equation (55) is not a trivial equation since $S_{BFV}$ is an odd function though Equation (57) is trivial. Two functions are constructed to be equivalent to the original Hamiltonian mechanics.

### 4.1 BFV of homogeneous constrained Hamiltonian mechanics

We consider the homogeneous constrained Hamiltonian mechanics in Section 3.1 and 3.2.

We introduce odd coordinates $\eta^a$ of degree one on the fiber of $E[1]$ in addition to local coordinates $(x^i, p_i)$ of the cotangent bundle $T^*M$. Degree of $(x^i, p_i)$ is assigned by $(0, 2)$. The dual bundle $E^*[1]$ is identified $E[1]$ by using the fiber metric $\langle - , - \rangle$. Since $T^*[2]E[1]$ is the cotangent bundle, the canonical graded symplectic form is introduced,

$$\omega_{BFV} = dx^i \wedge dp_i + \frac{1}{2} k_{ab} d\eta^a \wedge d\eta^b,$$

as an extension of the canonical symplectic form in the phase space of the mechanics. The canonical Poisson brackets are

\begin{align}
\{x^i, p_j\} &= \delta^i_j, \\
\{\eta^a, \eta^b\} &= k^{ab}, \tag{60}
\end{align}

and other Poisson brackets vanish. The BFV charge $S_{BFV}$ is nothing but the Chevalley-Eilenberg operator of the Courant algebroid complex, which is the Hamiltonian function of the homological function $Q$ in Equation (5). The formula gives

$$S_{BFV} = \eta^a G_a - \frac{1}{3!} f_{abc} \eta^a \eta^b \eta^c.$$

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where the constraint \( G_a \) is given in (25). We can easily check that \( S_{BFV} \) satisfies \( E \mathcal{d} = Q = \{ S_{BFV}, - \} \), and Equation (55), which is equivalent to \( Q^2 = 0 \). Since \( G_a \) is homogeneous with respect to \( p_i \) and a degree 2 function, \( S_{BFV} \) is a homogeneous function of degree 3.

Next, we determine the BFV Hamiltonian \( H_{BFV} \) by solving Equation (56). \( H_{BFV} \) is expanded by \( \eta^a \) as

\[
H_{BFV} = \sum_{I=0}^{\infty} H_{BFV}^{(I)},
\]

where \( H_{BFV}^{(I)} \) is the \( 2I \)-th order term. Each term is even order of \( \eta^a \) since \( H_{BFV} \) is an even function.

\[
H_{BFV}^{(I)} = \frac{1}{(2I)!} \eta^{a_1} \cdots \eta^{a_{2I}} H_{BFV,a_1 \cdots a_{2I}}(x,p).
\]

The 0-th part is fixed to the classical Hamiltonian \( H_{BFV}^{(0)} = H \). Higher terms are determined recursively by substituting (61), (62) and (63) to Equation (56), i.e., \( \{ S_{BFV}, H_{BFV} \} = 0 \).

Since degree of the classical Hamiltonian \( H \) is 4, the BFV Hamiltonian \( H_{BFV} \) must be a degree 4 function from homogeneity. Thus, the BFV Hamiltonian is expanded as

\[
\begin{align*}
H_{BFV} &= H_{BFV}^{(0)} + H_{BFV}^{(1)} + H_{BFV}^{(2)}, \\
H_{BFV}^{(0)} &= H_c(x,p) = \frac{1}{2} g^{ij}(x) p_i p_j, \\
H_{BFV}^{(1)} &= \frac{1}{2} H_{ab}^{(1)}(x) p_i \eta^a \eta^b, \\
H_{BFV}^{(2)} &= \frac{1}{4!} H_{abcd}^{(2)}(x) \eta^a \eta^b \eta^c \eta^d, \\
H_{BFV}^{(I)} &= 0, \quad \text{for } I \geq 3.
\end{align*}
\]

Substituting this expression to (56), we obtain

\[
H_{BFV}^{(1)} = \frac{1}{2} \eta^a \eta^b g^{ij} \Gamma_{ab} p_j,
\]

which depends on the connection \( \Gamma \), where \( \Gamma_{abj} \equiv \Gamma^c_{aj} k_{cb} \).

In order to solve \( H_{BFV}^{(2)} \), we covariantize the conjugate momentum \( p_i \) as

\[
p^\nabla_i = p_i + \frac{1}{2} \Gamma_{ab} \eta^a \eta^b.
\]
In fact, $p_i^\nabla$ transforms covariantly under the super diffeomorphism on $E[1]$. Poisson brackets of covariantized canonical coordinates are

\begin{align}
\{x^i, p_j^\nabla\} &= \delta^i_j, \quad (71) \\
\{p_i^\nabla, \eta^a\} &= \Gamma^a_{bj} \eta^b, \quad (72) \\
\{\eta^a, \eta^b\} &= k^{ab}, \quad (73) \\
\{p_i^\nabla, p_j^\nabla\} &= -\frac{1}{2} R^b_{ijab} k^{bc} \eta^a \eta^c, \quad (74)
\end{align}

where $R^b_{ijab}$ is the curvature with respect to $\Gamma^b_{ai}$. See Appendix A.2 for the concrete definition. If we introduce the basis of $E$, $e_a$, it is natural that the Poisson bracket is defined as

\begin{equation}
\{p_i^\nabla, e_a\} := -\Gamma^a_{ai} e_b, \quad (75)
\end{equation}

from $p_i^\nabla$ is the Hamiltonian for $-D_i$ and the definition of a connection,

\begin{equation}
D_i e_a = \Gamma^b_{ai} e_b, \quad (76)
\end{equation}

Considering $\eta = \eta^a e_a$, the Poisson bracket (72) is written in the covariant form,

\begin{equation}
\{p_i^\nabla, \eta\} = 0. \quad (77)
\end{equation}

The BFV charge function is also covariantized as

\begin{equation}
S_{BFV}^* = \eta^a \rho^i_a p_i^\nabla - \frac{1}{3!} T_{abc} \eta^a \eta^b \eta^c, \quad (78)
\end{equation}

where

\begin{equation}
T_{abc} = f_{abc} - \rho^i_a \Gamma_{bc}^i, \quad (79)
\end{equation}

is the $E$-torsion (the Gualtieri torsion) on the Courant algebroid. An $E$-torsion $T \in \Gamma(\wedge^3 E^*)$ on a Courant algebroid $E$ is defined by [19]

\begin{equation}
T(e_1, e_2, e_3) := -\frac{1}{2} (E_{e_1} e_2 - E_{e_2} e_1, e_3) + \frac{1}{3} ([e_1, e_2]_C, e_3) + (123 \text{ cyclic}), \quad (80)
\end{equation}

for $e_I \in \Gamma(E)$. Using the covariant coordinate $p_i^\nabla$, $H^{(1)}$ is absorbed to $H^{(0)}$. The BFV Hamiltonian is simplified to

\begin{equation}
H_{BFV} = H^{\nabla(0)} + H^{\nabla(2)}, \quad (81)
\end{equation}
where

\[
H^{(0)} = \frac{1}{2} \gamma^{ij} p_i \nabla p_j,
\]

(82)

\[
H^{(2)} = \frac{1}{4!} U_{abcd}(x) \eta^a \eta^b \eta^c \eta^d.
\]

(83)

Here \( U \in \Gamma(\wedge^4 E^*) \) is an \( E \)-4-form on \( M \). Substituting Equation (81) to (56) we obtain Equation of \( U \),

\[
\langle \rho v, U \rangle (e_1, e_2, e_3) = \langle g^{-1}(S(e_1, e_2), v), e_3 \rangle,
\]

(84)

\[
EDU + \langle T, U \rangle = 0,
\]

(85)

where \( S_{cab} \) is the basic curvature \( S \in \Gamma(T^*M \otimes E \otimes \wedge^2 E^*) \) defined by

\[
S = DT + 2\text{Alt}(\iota_R R),
\]

(86)

where \( \text{Alt} \) denotes an antisymmetrization over \( E^* \otimes E^* \). The second term is a kind of Bianchi identity of \( U \). If the basic curvature \( S \) of \( E \) is zero, we can take the simplest solution \( U = 0 \). Even if \( S \neq 0 \), we can obtain solutions of the BFV Hamiltonian \( H_{BFV} \) by solving Equations (102) and (103) under the geometric condition. We summarize the result.

**Theorem 4.1** There exists a solution of the BFV equations satisfying (55)–(57) if there exists \( U \in \Gamma(\wedge^4 E^*) \) satisfying

\[
\langle \rho v, U \rangle (e_1, e_2, e_3) = \langle g^{-1}(S(e_1, e_2), v), e_3 \rangle,
\]

(87)

\[
EDU + \langle T, U \rangle = 0,
\]

(88)

where \( v \in \mathfrak{X}(M) \) and \( e_1 \in \Gamma(E) \).

We call a Courant algebroid \( E \) with connection \( D \) a **Cartan Courant algebroid** if its basic curvature \( S = 0 \) as in the case of a Lie algebroid [6].

**Corollary 4.2** There exists a solution of the BFV equations satisfying (55)–(57) if a Courant algebroid is the Cartan Courant algebroid.

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4.2 BFV in inhomogeneous system and momentum section

We consider a generalization to inhomogeneous constraints and Hamiltonian in Section 3.3.

First we replace $p_i$ to $p_i = p'_i - A_i$ similar to the change of basis in Section 3.3. Then, the BFV symplectic form is changed to

$$\omega_{BFV} = dx^i \wedge dp'_i + \frac{1}{2} B_{ij} dx^i \wedge dx^j + \frac{1}{2} k_{ab} d\eta^a \wedge d\eta^b.$$  (89)

The Poisson bracket of $\{p'_i, p'_j\}$ is deformed as

$$\{p'_i, p'_j\} = -B_{ij}. \quad (90)$$

The BFV charge $S_{BFV}$ is formally the same equation (61) as in the homogeneous case,

$$S_{BFV} = \eta^a G_a - \frac{1}{3!} f_{abc} \eta^a \eta^b \eta^c,$$  (91)

since the Poisson bracket of $G_a$ is the same equation (49). However, the constraint $G_a$ is given by inhomogeneous Equation (41) including a momentum section $\mu$,

$$G_a = \rho^i_a(x)p'_i + \mu_a.$$  (92)

Thus the covariantized BFV function is deformed by the momentum section $\mu$,

$$S_{BFV} = \rho^i_a \eta^a p'_i - \frac{1}{3!} T_{abc} \eta^a \eta^b \eta^c + \mu \eta^a.$$  (93)

Straight calculation shows that under the Courant algebroid structure, The second order term in $\{S_{BFV}, S_{BFV}\} = 0$ is equivalent to the condition (H3) in the definition of the momentum section. We expand the constraint $G_a = G_{2a} + G_{0a}$ by degree,

$$G_{2a} = \rho^i_a(x)p'_i, \quad G_{0a} = \mu_a.$$  (94)

The corresponding expansion of $S_{BFV}$ by degree is

$$S_{BFV_3} = \eta^a G_{2a} - \frac{1}{3!} f_{abc} \eta^a \eta^b \eta^c; \quad S_{BFV_1} = \eta^a G_{0a}.$$  (95, 96)

$S_{BFV_1}$ is the term of the momentum section. In the term of BFV formulation, the condition (H3) is written as

$$\{S_{BFV_3}, S_{BFV_1}\} = -\frac{1}{2} \{S_{BFV_3}, S_{BFV_3}\}.$$

(97)
The BFV Hamiltonian $H_{BFV}$ is also inhomogeneous since the classical Hamiltonian $H_{cl}$ is Equation (37),

$$H^{(0)}_{BFV} = H_{cl} = \frac{1}{2} g^{ij}(x)p_i' p_j' + V'(x).$$ (98)

The solution of Equation \{$S_{BFV}, H_{BFV}$\} = 0 is obtained by deforming $p_i$ to $p_i'$ in the solution $H_{BFV}$ for the homogeneous case. We fix the deformed inhomogeneous BFV Hamiltonian as

$$H_{BFV} = H^{(0)} + V'(x) + H^{(2)},$$ (99)

where

$$H^{(0)} = \frac{1}{2} g^{ij} p_i' \nabla p_j',$$ (100)

$$H^{(2)} = \frac{1}{4!} U_{abcd}(x) \eta^a \eta^b \eta^c \eta^d.$$ (101)

Substituting inhomogeneous BFV functional (91) and Hamiltonian (99) to Equation (56), we obtain conditions.

$U$ is independent of a momentum section $\mu$ and additional inhomogeneous terms. The condition for $U$ is the same as homogeneous Hamiltonian case.

**Theorem 4.3** There exists a solution of the BFV equations in inhomogeneous constrained Hamiltonian mechanics satisfying (55)–(57) if there exists $U \in \Gamma(\wedge^4 E^*)$ satisfying

$$\langle \rho v, U \rangle(e_1, e_2, e_3) = \langle g^{-1}(S(e_1, e_2), v), e_3 \rangle,$$ (102)

$$E D U + \langle T, U \rangle = 0,$$ (103)

where $v \in \mathfrak{x}(M)$ and $e_I \in \Gamma(E)$.

Suppose a solution $U$. For other geometric quantities, we obtain the equivalent solution as in Theorem 3.4 i.e., $E D g = 0$, $E dV' = 0$ and $\mu = \alpha - \iota_A$ satisfies the condition (H2).

We can summarize the BFV formalism.

**Theorem 4.4** Suppose a solution $U$ in Theorem 4.3 and $E D g = 0$ and $E dV' = 0$. Then, the BFV formalism with $S_{BFV}$ and $H_{BFV}$ of the inhomogeneous constrained Hamiltonian mechanics is equivalent to the conditions (H2) and (H3) of a momentum section on a Courant algebroid.
We expand $H_{BFV}$ by degree,

\begin{align}
H_{BFV} &= H_{BFV4} + H_{BFV0}, \\
H_{BFV4} &= H^{\nabla(0)} + H^{\nabla(2)}, \\
H_{BFV4} &= V'(x).
\end{align}

From the expansion of $S_{BFV}$ and $H_{BFV}$ with respect to degree, the condition (h2) of the momentum section is written as

\begin{equation}
\{S_{BFV1}, H_{BFV4}\} = \{S_{BFV3}, H_{BFV4}\}.
\end{equation}

## 5 BV-FHGD formalism

The classical BV formalism is equivalent to a Lagrangian mechanics. The BV formalism is defined on a graded manifold $M_{BV} = T^*[-1] M_{BRST}$, and consists of a nondegenerate odd Poisson bracket $(-, -)$ called the BV bracket and one even function $S_{BV}$ satisfying

\begin{equation}
(S_{BV}, S_{BV}) = 0.
\end{equation}

The BV bracket is defined by an odd symplectic form $\omega_{BV}$.

An action functional in the Lagrangian formalism corresponding to the constrained Hamiltonian mechanics \cite{25,30} is \cite{15,13}

\begin{equation}
S_{cl} = \int_\mathbb{R} dt (p_i \dot{x}^i - H + \lambda^a G_a),
\end{equation}

where $\dot{x}$ is the time derivative of $x$ and $\lambda^a$ is a Lagrange multiplier. In fact, the Legendre transformation of canonical conjugates gives the Hamiltonian $H$ and the variation with respect to $\lambda^a$ gives the constraint equation $G_a \approx 0$.

Though there is a normal procedure to construct a BV bracket and a BV action functional \cite{22}, in this paper, we use the Fisch-Henneaux-Grigoriev-Damgaard (FHGD) method \cite{22,17,18} since the procedure is simpler than the normal BV construction. The FHGD method is the formula to construct the BV symplectic form and the BV functional from the BFV data. See also \cite{28}.

In this section, the method and formulas are briefly explained.
Let \( z^I(t) \) denote fields in the BFV formalism including the ghosts. The BFV data are then given by a BFV symplectic form \( \omega_{BFV} \), an odd function \( S_{BFV}(\sigma) \), and an even function \( H_{BFV} \). The BFV symplectic form can be written as

\[
\omega_{BFV} = \omega_{IJ}(z) dz^I \wedge dz^J,
\]

where \( \omega_{IJ}(z) \) is a nondegenerate, graded-antisymmetric matrix. Likewise, the BFV functional \( S_{BFV}(z) \) and the BFV Hamiltonian \( H_{BFV}(z) \) are functions of \( z \).

The FHGD procedure in the formulation of Grigoriev and Damgaard, then works as follows. First, for each coordinate \( z^I(t) \) one introduces a superpartner field \( w^I(t) \). In addition, it is convenient to introduce a superpartner coordinate \( \theta \) corresponding to time \( t \). This permits one to introduce a superfield \( Z^I(t, \theta) \) by means of

\[
Z^I(t, \theta) = z^I(t) + \theta w^I(t).
\]

The BV symplectic form just becomes a super extension of the BFV symplectic form (110),

\[
\omega_{BV} = \int_{T[1]|R} d\theta dt \, \omega_{IJ}(Z) \delta Z^I \wedge \delta Z^J,
\]

where \( \delta \) denotes the de Rham differential on the extended BV space of fields.

Assume that the BFV symplectic is exact, \( \omega_{BFV} = -\delta \vartheta_{BFV} \) for some local 1-form, \( \vartheta_{BFV}(z) dz^I \).

Then one can define the BV action functional \( S_{BV} \) as follows:

\[
S_{BV} := \int_{T[1]|R} d\theta dt \, \vartheta^I(Z) dZ^I - \int_{T[1]|R} d\theta dt \, (S_{BFV}(Z) + \theta^0 H_{BFV}(Z)).
\]

Here \( d \equiv \theta \frac{d}{dt} \) can be viewed as the de Rham differential on the line \( \mathbb{R} \) or the corresponding odd and nilpotent vector field on its super extension \( T[1]|\mathbb{R} \). After integrating out the odd variable \( \theta^0 \), (114) becomes a functional for the fields on \( \mathbb{R} \). It satisfies the equation, \( (S_{BV}, S_{BV}) = 0 \) from equations (55)–(57). Here \((-,-)\) is the BV bracket induced from the BV symplectic form (??).
5.1 BV for homogeneous constrained system

First we consider the homogeneous Hamiltonian system in Section 3.1 and 3.2.

We apply the above procedure to the constrained Hamiltonian mechanics. The parameter space of time is extended to the super time space $T^*[1\mathbb{R}]$ by introducing an odd super time $\theta$. All the coordinates $(x^i, p_i, \eta^a)$ on the phase space of the BFV formalism $T^*[2E[1]]$ is extended to super coordinates $(X^i, P_i, Y^a)$, where

$$X^i(t, \theta) = x^i - \theta p^i,$$

$$P_i(t, \theta) = p_i + \theta x^*_i,$$

$$Y^a(t, \theta) = \eta^a - \theta \eta^*_a,$$

where $p^*_i$ and $x^*_i$ are odd coordinates and $\eta^*_a$ is an even coordinate. The BV phase space is the mapping space

$$\text{Map}(T^*[1\mathbb{R}], T^*[2E[1]]).$$

The BV symplectic form is the super extension of the BFV symplectic form (58),

$$\omega_{BV} = \int_{T^*[1\mathbb{R}]} d\theta dt \left( \delta X^i \wedge \delta P_i + \frac{1}{2} k_{ab}(x) \delta Y^a \wedge \delta Y^b \right)$$

$$= \int_{\mathbb{R}} dt \left( \delta x^i \wedge \delta x^*_i + \delta p_i \wedge \delta p^*_i + k_{ab}(x) \delta \eta^a \wedge \delta \eta^b - \frac{1}{2} \partial_i k_{ab}(x) p^*_i \eta^a \wedge \delta \eta^b \right).$$

If we assume that $k_{ab}$ is a constant for simplicity, $\omega_{BFV}$ is an exact form. Then, the BV action functional $S_{BV}$ is constructed using the formula (114) as follows.

$$S_{BV} = \int_{T^*[1\mathbb{R}]} d\theta dt \left[ P_i \dot{X}^i - \frac{1}{2} k_{ab}(x) \delta Y^a \delta Y^b - (S_{BFV}(X, P, Y) + \theta H_{BFV}(X, P, Y)) \right]$$

$$= \int_{\mathbb{R}} dt \left[ \dot{p}_i \dot{x}^i - \frac{1}{2} k_{ab}(x) \delta \eta^a \delta \eta^b + \lambda^a G_a + x^*_i \dot{p}^i_a(x) \eta^a - p^*_i \partial_i \dot{p}^i_a(x) p_j \eta^a - \frac{1}{2} f_{abc}(x) \lambda^a \eta^b \eta^c - \frac{1}{3!} \partial_i f_{abc}(x) p^*_i \eta^a \eta^b \eta^c - \frac{1}{2} g^{ij}(x) \dot{p}_i \dot{p}_j - \frac{1}{2} g^{ij}(x) \Gamma_{ab} \eta^a \eta^b \right],$$

where $\frac{d}{dt}$ is the superderivative and $\eta^*_a = \lambda$. Here we substitute the concrete BFV functional (61) and the BFV Hamiltonian (62). If all antifields and ghosts are zero, $x^*_i = p^*_i = \eta^a = 0$, Equation (123) reduces to the expected classical action (109).
Note that the BV action functional (123) is not necessarily the same as the BV action functional constructed in the traditional BV procedure from the gauge transformations. Ambiguity of the FHGD formalism was discussed in [28].

An interesting feature is that if we set \( x_i^* = p_i^* = \lambda = 0 \), the \( S_{BV} \) gives a supersymmetric mechanics with a Courant algebroid structure. The 'supersymmetry' is generated by \( Q = (S_{BV}, -) \). Further analysis of the super mechanics leaves in future research.

### 5.2 BV for inhomogeneous constrained system

We apply the FHGD method to the BFV formalism for the inhomogeneous Hamiltonian mechanics in Section (4.2).

\( p_i \) is deformed to \( p'_i = p_i + A_i \). The BFV symplectic form \( \omega_{BFV} \), the BFV functional \( S_{BFV} \) and the BFV Hamiltonian \( H_{BFV} \) are deformed to (89), (91) and (99), respectively.

The parameter superspace is the same as in the homogeneous case. Super coordinates on the BV phase space Map(\( T[1]\mathbb{R} \), \( T^*[2]E[1] \)) are \( (X^i, p'_i, \eta^a) \) which correspond to coordinates \( (x^i, p'_i, \eta^a) \) on the phase space of the BFV formalism. They are expanded as

The BV symplectic form is the super extension of the BFV symplectic form (58),

\[
\omega_{BV} = \int_{T[1]\mathbb{R}} dt \left( \delta x^i \wedge \delta p'_i + \frac{1}{2} B_{ij}(X) \delta x^i \wedge \delta x^j + \frac{1}{2} k_{ab}(X) \delta Y^a \wedge \delta Y^b \right)
\]

For the symplectic form \( \omega_{BFV} \), the Liouville 1-form \( \vartheta \) such that \( \omega_{BV} = -d\vartheta \) is locally written as

\[
\vartheta_{BFV} = p'_i dx^i - A_i(x) dx^i - \frac{1}{2} k_{ab} \eta^a d\eta^b.
\]

Finally, the BV action functional \( S_{BV} \) is obtained from the formula in the FHGD formulation by substituting concrete expressions of the BFV functional (61) and the BFV Hamiltonian.
\[
S_{BV} = \int_{T[1]\mathbb{R}} d\theta dt [P'_i dX^i - A_i(X) dX^i - \frac{1}{2} k_{ab} Y^a dY^b - (S_{BFV}(X, P', Y) + \theta H_{BFV}(X, P', Y))] \\
= \int_{T[1]\Sigma} dt \left[ p'_i \dot{x} - \frac{1}{2} k_{ab} \eta^a \eta^b \right] + \int_{T[1]\Sigma} d\theta dt d\sigma B(x) \\
\quad + \int_{T[1]\mathbb{R}} d\theta dt \left[ \lambda^a (\bar{\rho}^a p'_i + \mu_a) + x^*_i \rho^a(x) \eta^a - p'^i (\partial_i \rho^a(x)p'_j + \partial_i \mu_a(x)) \eta^a \\
- \frac{1}{2} f_{abc}(x) \lambda^a \eta^b \eta^c - \frac{1}{3!} \partial_i f_{abc}(x) p^{i\alpha} \eta^a \eta^b \eta^c - \frac{1}{2} g^{ij}(x) p'_i p'_j - V'(x) \\
- \frac{1}{2} g^{ij}(x) \Gamma_{ab} \rho^a \eta^b \eta^c - \frac{1}{8} g^{ij}(x) \Gamma_{abc} \eta^a \eta^b \eta^c \eta^d - \frac{1}{4!} U_{abcd}(x) \eta^a \eta^b \eta^c \eta^d \right], \quad (123)
\]
where \( \Sigma \) is a two dimensional manifold such that \( \mathbb{R} = \partial \Sigma \) with a local coordinate \((t, \sigma)\). The second term in (122) gives the WZ term on \( T[1]\Sigma \) since \( B = dA \). If all antifields and ghosts are zero, \( x^*_i = p^*_i = \eta^a = 0 \), Equation (123) reduces to the classical action (109).

6 Weil algebra

The Weil algebra is a model for the equivariant cohomology based on a graded algebra We refer to Kalkman’s formulation [29, 30] and the textbook [20].

We construct a Weil algebra from a \( Q \)-manifold. A Weil algebra on a \( Q \)-manifold has been discussed in [41, 1]. We need only the homological vector field \( Q \), however, we consider a \( QP \)-manifold by considering a graded symplectic structure. We can easily connect the Weil algebra with the BFV and BV formalism since the both BFV and BV formalism have \( QP \)-manifolds.

6.1 Weil model on \( Q \)-manifold

A graded manifold \( \mathcal{M} \) is called an \( N \)-manifold if a graded manifold is a nonnegatively graded.

**Definition 6.1** A \( QP \)-manifold (differential graded symplectic manifold) is a \( N \)-manifold \( \mathcal{M} \) with a graded symplectic form \( \omega \) endowed with a degree 1 homological vector field \( Q \) such that \( L_Q \omega = 0 \), where \( L_Q \) is a graded Lie derivative. [45].

A graded symplectic form gives a graded Poisson bracket \( \{-, -\} \). For any \( QP \) manifold of positive degree, there exists a Hamiltonian function \( \Theta \in C^\infty(\mathcal{M}) \) of \( Q \) with respect to the
graded Poisson bracket \(-, -\),
\[
Q = \{\Theta, -\}. \tag{124}
\]

Then, the homological condition of \(Q\), \(Q^2 = 0\), implies that \(\Theta\) is a solution of the equation
\[
\{\Theta, \Theta\} = 0. \tag{125}
\]

A Weil algebra \((W, d, \iota, L)\) is a graded algebra \(W\) with three derivations \(d, \iota, L\). We construct a Weil algebra from a QP-manifold \(\mathcal{M}\). The space of a Weil algebra is \(W = C^\infty(T[1]\mathcal{M})\). Three operations \((d, \iota, L)\) are are derivations of degree \((1, -1, 0)\) on \(W\).

Take a local coordinate \((e^a, \theta^a)\) on \(T[1]\mathcal{M}\) of degree \(|\theta| = |e| + 1\). We denote the superderivative of an element \(e \in C^\infty(\mathcal{M})\) by \(\delta e = \theta^a \frac{\partial e}{\partial e^a}\) called a tangent vector along \(e\). Note that \(\delta e\) is a linear function of the tangent direction of \(T[1]\mathcal{M}\).

For functions \(e, e_1, e_2 \in C^\infty(\mathcal{M})\) on a base graded manifold, we define \(d, \iota, L : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})\) as follows,
\[
d_e := F_e + Qe = \delta e + Qe. \tag{126}
\]
\[
\iota_{e_1}(e_2) := \{e_1, e_2\}, \tag{127}
\]
\[
L_{e_1}(e_2) := \{\{e_1, \Theta\}, e_2\}. \tag{128}
\]

From the definition, \(d, \iota, L\) is of degree \(1, -1, 0\), respectively. For a basis \(e^a\) on \(\mathcal{M}\), Equation (126) is \(de^a = \theta^a + Qe^a\). \(e\) is regraded as a 'connection' and \(F_e = de - Qe\) is a 'curvature'.

Three operations \(d, \iota, L\) on the tangent direction are defined as follows. We require that the 'differential' \(d\) satisfies \(d^2 = 0\), and the 'Lie derivative' \(L_e\) satisfies the Cartan magic formula, \(L_e = \iota_e d - (-1)^{|e|} dt_e = \iota_e d + (-1)^{|e|} dt_e\), respectively, where \(|\iota_e| = |e| - 1\) is degree of \(\iota_e\). From Equation (128) on the basis and \(d^2 = 0\), we obtain Equation,
\[
d\theta^a = -dQe^a = -\delta Qe^a. \tag{129}
\]

since \(Q^2 = 0\) and \(F_e = (d - Q)e = \delta e\). From the requirement \(L_e = \iota_e d + (-1)^{|e|} dt_e\), the following definitions of \(\iota\theta^a\) and \(L\theta^a\) are obtained,
\[
\iota_{e_1}\theta^a = -(-1)^{|e_1|}(d - Q)\iota_{e_1}e^a = -(-1)^{|e_1|}\delta \iota_{e_1}e^a, \tag{130}
\]
\[
L_{e_1}\theta^a = (-1)^{|e_1|}(d - Q)L_{e_1}e^a = (-1)^{|e_1|}\delta L_{e_1}e^a. \tag{131}
\]

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Equations (126)–(128) and (129)–(131) fix all operations of \((d, \iota, L)\) on elements \(W = C^\infty(T[1]M)\), thus we obtain the Weil algebra from a QP-manifold.

We summarize useful formulas calculated the above definitions of operations \((d, \iota, L)\). For the element \(F_e = \delta_e\), the following formulas are obtained.

\[ dF_e = -dQe = -F_Qe, \quad (132) \]

since \(Q^2 = 0\). From the Cartan magic formula, we obtain

\[ \iota_{e_1} F_{e_2} = -(-1)^{|e_1|} F_{\iota_{e_1} e_2}, \quad (133) \]
\[ L_{e_1} F_{e_2} = -(-1)^{|e_1|} F_{L_{e_1} e_2}, \quad (134) \]

Operations \((d, \iota, L)\) consist of a graded Lie algebra with a graded Lie bracket \([-,-]\) as follows,

\[ [\iota_{e_1}, \iota_{e_2}] = \iota_{[e_1, e_2]}, \quad (135) \]
\[ [\iota_{e_1}, L_{e_2}] = \iota_{[L_{e_1} e_2]}, \quad (136) \]
\[ [L_{e_1}, L_{e_2}] = L_{[L_{e_1} e_2]}, \quad (137) \]

\([e_1, e_2]_D := L_{e_1} e_2 = \{\{e_1, \Theta\}, e_2\}\) gives a bilinear bracket. In fact, since this bracket satisfies the Leibniz identity \([1]\), but not necessarily skew symmetric, it is the Dorfman bracket \([31]\).

Thus we can also write (136) and (137) as

\[ [\iota_{e_1}, L_{e_2}] = \iota_{[e_1, e_2]_D}, \quad (138) \]
\[ [L_{e_1}, L_{e_2}] = L_{[e_1, e_2]_D}. \quad (139) \]

**Definition 6.2** We call the graded Lie algebra acting on the space \(W = C^\infty(T[1]M)\) generated by \((d, \iota_e, L_e)\) defined by Equations (126) – (131) a Weil algebra for a graded (QP-)manifold \(M\).

Since \(\omega\) is symplectic, \(\iota_e = \{e, -\}\) is nondegenerate, and thus the Weil algebra satisfies the property corresponding to a locally free action called \textit{type (C)} in [20].

**Definition 6.3** Let \(\varphi \in W\).

1. If \(\iota_e \varphi = 0\) for any \(e \in C^\infty(M)\), \(\varphi\) is called \textit{horizontal}. \(W_{\text{hor}} = \{\varphi \in W | \iota_e \varphi = 0\}\).

2. If \(\varphi\) satisfies \(L_e \varphi = \iota_e \varphi = 0\) for any \(e \in C^\infty(M)\), it is called \textit{basic}. \(W_{\text{bas}} = \{\varphi \in W | L_e \varphi = \iota_e \varphi = 0\}\).
6.2 Cartan Model

Let $B$ be a (graded) $\mathcal{M}$-module. $\mathcal{M}$ acts on $B$ as infinitesimal actions, i.e., operations $(d, \iota, L)$. The space of the Cartan model is $W \otimes B$. Three operations on $W \otimes B$ are given by

\begin{align*}
L_e &= L_e \otimes 1 + 1 \otimes L_e, \\
\iota_e &= \iota_e \otimes 1 + 1 \otimes \iota_e, \\
d &= d \otimes 1 + 1 \otimes d.
\end{align*}

$(W \otimes B)_{\text{hor}} = \{ \varphi \in W \otimes B | \iota \varphi = 0 \}$ is the space of horizontal functions and $(W \otimes B)_{\text{bas}} = \{ \varphi \in W \otimes B | \nu \varphi = L \varphi = 0 \}$ is the space of basic functions.

We can construct the generator of the Mathai-Quillen map from the space of the Weil model to the space of the Cartan model \[20\]. It is defined by

\[ \phi = \exp (h \otimes \iota). \]

Here $h$ is an operator such that $[h, \iota] \varphi = -\varphi$, for a horizontal function $\varphi \in W$. The computation gives $d \varphi - hL \varphi = 0$. Thus, we obtain the equivariant differential $d_C$ such that $d_C^2 = 0$ for the Cartan model as

\[ d_C = \phi (d \otimes 1 + 1 \otimes d) \phi^{-1} = 1 \otimes d - (d - Q) e^a \otimes \iota_e^a = 1 \otimes d - \theta^a \otimes \iota_e^a. \]

We have the following equivalence of the Weil model and the Cartan model. Let $S$ be the horizontal subspace of $W$ and $C_G(B) = (S \times B)^G$ is an invariant subspace of the action of $G$ whose infinitesimal action is defined by $\mathcal{M}$. We consider the basic subspace, $(W \otimes B)_{bas}$. On this space the Mathai-Quillen isomorphism shows equivalence of the cohomologies of the Weil model and the Cartan model $\phi$ maps $(W \otimes B)_{bas}$ into $C_G(B)$ and $d$ to $d_C$. Thus we obtain following equivalence of equivariant cohomology computed from the Weil model and the Cartan model.

\[ H^*((W \otimes B)_{bas}, d) = H^*(C_G(B), d_C). \]

\[ \text{§} \] For $G$, the integration of the graded manifold $T^*[2]E[1]$ and a Courant algebroid to a group-like object has been analyzed in \[36\ \[42\]. A Lie rackoid is also proposed as an integration of a Courant algebroid \[34\ \[35\].
6.3 Weil model and Cartan model for Courant algebroid

In this section, we compute concrete formulas. For the Courant algebroid, we consider the QP-manifold for the Courant algebra $\mathcal{M} = T^*[2]E[1]$ with the homological vector field $Q$. The homological function $\Theta$ is $S_{BFV}$ in Equation (61).

The Weil model is defined on the graded algebra $W = C^\infty(T[1]T^*[2]E[1])$.

For local coordinates $(x^i, \eta^a, p_i)$ on $T^*[2]E[1]$, ones on the tangent direction $T[1]$ are $(F_{x^i}, F_{\eta^a}, F_{p_i})$ of degree $(1, 2, 3)$.

Substituting Equation (61) to the definitions (126)–(128), we obtain the Weil differential $d$ is

\begin{align}
\text{d}x^i &= F_{x^i}^i - \rho^i_a \eta^a, \\
\text{d}\eta^a &= F_{\eta^a}^a + k^{ab} \rho^i_b F_{p_i}^a + k^{ab} \Gamma_{bcd}^i \eta^b \eta^d - \frac{1}{2} k^{ab} T_{bcd}^i \eta^b \eta^d, \\
\text{d}p_i^\nabla &= F_{p_i}^\nabla + D\rho^j_a \eta^a p_i^\nabla - \frac{1}{3!} S_d^{abc} \eta^a \eta^b \eta^c. \tag{148}
\end{align}

Note that the Poisson bracket of the basis $e_a$ on $E$ with $p_i^\nabla$ is not zero as (76), $e_a$ has a nontrivial transformation under $Q$. $\eta = \eta^a e_a$ covariantly transforms under operations of the Weil algebra. For $F_x = F_{x^i}^i \partial_i, F_\eta = F_{\eta^a}^a e_a, F_p = F_{p_i} d x^i$, we obtain covariant formulas,

\begin{align}
\text{d}x &= F_x - \rho(\eta), \tag{149} \\
\text{d}\eta &= F_\eta + \iota_{\rho^\ast} p^\nabla - T^* (\eta, \eta), \tag{150} \\
\text{d}p^\nabla &= F_{p^\nabla} + p^\nabla (D \rho(\eta)) - S^* (\eta, \eta), \tag{151}
\end{align}

where $x = x^i \partial_i, \eta = \eta^a e_a, p^\nabla = p^\nabla dx^i$. $\rho^\ast \in \Gamma(TM \oplus E), T^* \in \Gamma(\wedge^2 E \oplus E)$ and $S^* \in \Gamma(T^* M \oplus \wedge^3 E^*)$ are defined by $\langle \rho^\ast, e \rangle = \rho(e), \langle T^*, e \rangle (-, -) = T(-, -, e)$ and $S^*(-, -, e) = \langle S, -, -, e \rangle$ for $e \in \Gamma(E)$.

For the tangent direction, we obtain the Weil differential by acting $d$ to Equations (146)–(148),

\begin{align}
\text{d}F_x &= F_x \rho(\eta) + \rho(F_\eta), \tag{152} \\
\text{d}F_\eta^a &= F_x D \left[ - \iota_{\rho^\ast} p^\nabla + T^* (\eta, \eta) \right] - \iota_{\rho^\ast} F_p + T^* (F_\eta, \eta), \tag{153} \\
\text{d}F_{p^\nabla} &= \iota_{F_x} \left[ - p^\nabla (D \rho^\ast (\eta)) + S^* (\eta, \eta, \eta) \right] \\
&\quad + \iota_{D \rho(\eta)} F_{p^\nabla} + \left[ - p^\nabla (D \rho(F_\eta)) + S^* (F_\eta, \eta, \eta) \right]. \tag{154}
\end{align}
\(F_{xD}\) is a covariantized vector field given by \(F_{xD} = F_x^i D_i\). Important formulas from the general theory are \(F_x^i\), \(F_{\eta}^a\), and \(F_{pi}\) are horizontal,

\[
\iota_\epsilon F_x^i = \iota_\epsilon F_{\eta}^a = \iota_\epsilon F_{pi} = 0. \tag{155}
\]

For the Cartan model, we consider the basic subspace,

\[
(W \otimes B)_{bas} = \{ \gamma \in W \otimes B | L_\varphi = \iota_\varphi = 0 \}. \tag{156}
\]

The equivariant differential on the Cartan model on the basic subspace is

\[
dC = 1 \otimes d - F_{pi} \otimes \iota_p^i + F_x^i \otimes \iota_x^i - \frac{1}{2} k_{ab} F_{\eta}^a \otimes \iota_{\eta}^b. \tag{157}
\]

### 6.4 Weil model and Cartan model with momentum section

We choose the inhomogeneous BFV functional \(S_{BFV}\) in Equation (91) as a homological function \(\Theta\). The Weil model in Subsection 6.3 is deformed by the momentum section term. The Weil differential \(d\) is deformed to

\[
d'x^i = dx^i, \tag{158}
\]

\[
d'\eta^a = d\eta^a + k_{ab} \mu_b, \tag{159}
\]

\[
d'p_{\nabla}^i = dp_{\nabla}^i + (D_i \mu_a - \gamma_a) \eta^a. \tag{160}
\]

The coordinate independent form is

\[
d'x = dx, \tag{161}
\]

\[
d'\eta = d\eta + \mu^*, \tag{162}
\]

\[
d'p_{\nabla} = dp_{\nabla} + D\mu(\eta) - \gamma(\eta), \tag{163}
\]

where \(\mu^* \in \Gamma(E)\) is defined by \(\mu(e) = (\mu^*, e)\). By the straightforward calculation, we obtain the deformations of the Weil differentials for the tangent direction,

\[
d'F_x = dF_x + \rho(\mu^*), \tag{164}
\]

\[
d'F_{\eta} = dF_{\eta} - \iota_{\varphi^*}(D\mu - \gamma)(\eta) + T^*(\mu^*, \eta), \tag{165}
\]

\[
d'F_{pi} = dF_{pi} + \iota_{D\rho(\eta)}(D\mu - \gamma)(\eta) - \iota_{D\rho(\mu^*)} p_{\nabla} + S^*(\mu^*, \eta, \eta). \tag{166}
\]

The Weil algebra has been constructed from a QP-manifold. If \(\mu\) is a momentum section, \(Q^2 = 0\) is satisfied. Thus we obtain the general formula of the Weil algebra.
Theorem 6.4 If $\mu$ is a momentum section, $F_x^i$, $F_\eta^a$ and $F_{pi}$ are horizontal,

$$\iota_e F_x^i = \iota_e F_\eta^a = \iota_e F_{pi} = 0.$$  \hfill (167)

The equivariant differential on the Cartan model on the basic subspace is given by

$$d_C = 1 \otimes d - F_{pi} \otimes \iota_p^i + F_x^i \otimes \iota_x^i - \frac{1}{2} k_{ab} F_\eta^a \otimes \iota_\eta^b.$$  \hfill (168)

7 Conclusion and discussion

We have defined a momentum section on a pre-symplectic manifold with a Courant algebroid and a Hamiltonian Courant algebroid. They are a generalization of the momentum map theory on a pre-symplectic manifold with a Lie algebra action.

We analyzed the constrained Hamiltonian mechanics as a concrete nontrivial example. We checked that conditions of the momentum section are equivalent to the consistency conditions of the Hamiltonian mechanics. The BFV and BV formalisms of this mechanics are constructed and a momentum section is described as terms of inhomogeneous degree of the BFV function, the BFV Hamiltonian and the BV functional. The BFV and BV formalisms give cohomological formulations of momentum sections. As an application, we consider the Weil algebra to realize this structure.

An important application of the momentum map theory is the symplectic reduction [39]. Several proposal of 'Groupoid' objects for the Courant algebroid has been in [36, 42, 34, 35]. Since we have constructed a momentum section theory for a Courant algebroid, if a 'groupoid' object $\mathcal{G}$ for the Courant algebroid acts on a symplectic manifold $M$, the quotient space $M/\mathcal{G}$ should be a symplectic manifold. We need more analysis since the 'groupoid' object is more complicated than a momentum map case for a Lie group.

Another important but related application is the equivariant cohomology. The theory is strongly connected to quantizations of physical theories. If we can obtain a localization formula similar to the Duistermaat-Heckman formula [14] for our setting, we will be able to apply it to quantum theories with Courant algebroid structures.
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A Local coordinate expression

A.1 Courant algebroid

Let $E$ be a Courant algebroid over a smooth manifold with the inner product $\langle - , - \rangle$, the anchor map $\rho : E \rightarrow TM$, and the Dorfman bracket $[- , -]_D$.

A local coordinate on $M$ is $x^i$, and a basis of the fiber of $E$ is $e_a$. We denote $k_{ab} = \langle e_a, e_b \rangle$, $\rho(e_a) = \rho^i_a \partial_i$, and $[e_a, e_b]_D = k^{cd}f_{dab}e_d$.

Local coordinate expression of conditions of the Courant algebroid are

\[
k^{ab}\rho^i_a\rho^j_b = 0, \tag{169}
\]

\[
\rho^i_a \partial_j \rho^j_b - \rho^i_a \partial_j \rho^j_b + k^{ef}f_{dab} = 0, \tag{170}
\]

\[
\rho^i_j \partial_j f_{abc} - \rho^i_j \partial_j f_{bcd} - \rho^i_j \partial_j f_{dab} + k^{ef}f_{eab}f_{cdf} + k^{ef}f_{eac}f_{dbf} + k^{ef}f_{ead}f_{bcf} = 0. \tag{171}
\]

A.2 Connection, curvature and torsion

Let $D : \Gamma(E) \rightarrow \Gamma(E \times T^*M)$ be a connection and $\Gamma^b_a = \Gamma^b_{ai}dx^i$ be a connection 1-form of $D$.

The covariant derivatives with respect to $D$ is for $\mu^a e_a \in \Gamma(E)$ and $\mu_a e^a \in \Gamma(E^*)$ are

\[
D_i \mu^a = \partial_i \mu^a + \Gamma^b_{ai} \mu^b, \tag{172}
\]

\[
D_i \mu_a = \partial_i \mu_a - \Gamma^b_{ai} \mu_b. \tag{173}
\]

The $E$-covariant derivatives are

\[
E D_a \mu^b = \rho^i_a (\partial_i \mu^b + \Gamma^b_{ci} \mu^c), \tag{174}
\]

\[
E D_a \mu_b = \rho^i_a (\partial_i \mu_b - \Gamma^c_{bi} \mu_c). \tag{175}
\]
Local coordinate expressions for the curvature, the $E$-torsion, and the basic curvature are

$$R^b_{ija} = \partial_i \Gamma^b_{aj} - \partial_j \Gamma^b_{ai} - \Gamma^c_{ai} \Gamma^b_{cj} + \Gamma^c_{aj} \Gamma^b_{ci},$$  \hspace{1cm} (176)

$$T_{abc} = f_{abc} - \frac{1}{2}(\rho^i_a \Gamma^d_{bi} k_{dc} + \text{Cycl}(abc))$$

$$= f_{abc} - (\rho^i_a \Gamma^d_{bci} + \text{Cycl}(abc)), \hspace{1cm} (177)$$

$$S^c_{jab} = D_j T^c_{ab} + \rho^i_a R^c_{ijb} - \rho^i_b R^a_{ijb},$$  \hspace{1cm} (178)

where $D_j$ is the covariant derivative on $\Gamma(E \otimes \wedge^2 E^*)$.

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