The Hodge Conjecture is equivalent to a statement about conditions under which a complex vector bundle on a smooth complex projective variety (stably) admits a holomorphic structure. In the case of abelian four-folds, recent work in gauge theory suggests an approach using Spin(7) instantons. I advertise a class of examples due to Mumford where this approach could be tested. I construct explicit smooth vector bundles whose Chern characters are given Hodge classes - an instanton connection on these bundles would endow them with a holomorphic structure and thus prove that these classes are algebraic. I use complex multiplication to exhibit Cayley cycles representing the given Hodge classes. What is missing is an appropriate glueing theorem.
1. Introduction

Let $X$ be a smooth complex projective variety of dimension $n$, and $c$ a rational $(p,p)$ cohomology class ($0 < p < n$). The Hodge Conjecture is that

$H$: there exist finitely many (reduced, irreducible) $p$-dimensional subvarieties $Y_i$ and rational numbers $a_i$ such that $c = \sum_i a_i [Y_i]$, where $[Y_i]$ is the (rational) cohomology class dual to $Y_i$. That is, $c$ is dual to a rational algebraic cycle.

This is equivalent to

$V$: there exists a holomorphic vector bundle $E$ such that its Chern character $ch(E)$ is equal to a rational multiple of $c$ modulo (classes of) rational algebraic cycles.

The second statement implies the first because the Chern character of a holomorphic (and therefore algebraic) bundle factors through the Chow ring of algebraic varieties. The converse also holds (albeit, using resolution of singularities, Grothendieck-Riemann-Roch and an induction on dimension).

Let $X, c$ be as above. By a theorem of Atiyah-Hirzebruch ([A-H], page 19), the Chern character map $ch: K^0(X) \otimes \mathbb{Q} \rightarrow H_{\text{even}}^*(X, \mathbb{Q})$ is a bijection, where $K^0(X)$ is the Grothendieck group of (topological/smooth) vector bundles on $X$. Thus we are assured of the existence of a smooth bundle $E$ and and an integer $n > 0$ such that $ch(E) = \text{rank}(E) + nc$. A possible strategy to show that a given class $c$ is algebraic suggests itself – find a suitable such bundle $E$ and then exhibit a holomorphic structure on it. This note is written to argue that recent progress in mathematical gauge theory, and in particular the work of G. Tian and C. Lewis, makes this worth pursuing, at least in the case of certain abelian four-folds. Such an approach to the Hodge Conjecture for the case of Calabi-Yau four-folds is surely known to the experts, but I have only been able to locate some coy references.

Before proceeding, let us note that the known “easy” cases of the Hodge conjecture are proved essentially by the above method. First, given an integral class $c \in H^2(X, \mathbb{Z})$, a smooth hermitian line bundle $L$ exists with (first) Chern class equal to $c$. Given any real 2-form $\Omega$ representing $c$ there exists an unitary connection on $L$ with curvature $-2\pi i \Omega$. If $c$ is a $(1,1)$ class, it can be represented by an $\Omega$ which is $(1,1)$. The corresponding connection defines a holomorphic structure on $L$. If $c$ is an integral $(n-1, n-1)$ class, the strong Lefshetz theorem exhibits the dual class as a rational linear combination of complete intersections.

Most of the results that follow can be verified by rather easy computations which I either only sketch or omit altogether.

2. Mumford’s examples

We consider Hodge classes on certain abelian four-folds. These examples are due to Mumford ([P]).
Let $P = ax^4 + bx^2 + cx + d$ be an irreducible polynomial with rational coefficients and all roots $x_1, x_2, x_3, x_4$ real. We will suppose that the roots are numbered such that $x_1 > x_2 > x_3 > x_4$. Let $L_1/Q$ be the splitting field $L_1 = Q[x_1, x_2, x_3, x_4] \subset \mathbb{R}$. We suppose that $P$ is chosen such that the Galois group is $S_4$. This is equivalent to demanding that $[L_1 : Q] = 24$. We set $L \equiv L_1[i]$. This is a Galois extension of $Q$, with Galois group $S_4 \times \{e, \rho\}$, where $\rho$ is complex conjugation.

Consider a cube, with vertices labeled as in the figure:

Let $G$ denote the group of symmetries of the cube. We have the exact sequence:

$$1 \rightarrow \{e, \rho\} \rightarrow G \rightarrow S_4 \rightarrow 1$$

where now $\rho$ denotes inversion, and $S_4$ is the group of permutations of the four diagonals. Splitting this, identifying $S_4$ with (special orthogonal) rotations implementing the corresponding permutation of diagonals. we get an identification

$$G \sim S_4 \times \{e, \rho\} = \text{Gal}(L/Q)$$

Let $H$ denote the stabiliser of the vertex 1, $F$ the corresponding fixed field, and $\varphi_1 : F \rightarrow L \rightarrow \mathbb{C}$ the corresponding embedding. The left cosets of $H$ can be identified with the vertices of the cube, as well as embeddings of $F$ in $\mathbb{C}$. We label the latter $\varphi_j, \varphi_j \bar{j}$ ($j = 1, 2, 3, 4$).

Note that the field $F$ is invariant under complex conjugation, which therefore acts on it with fixed field $F_1$. Clearly, $F_1 = Q[x_1]$. We set

$$D = (x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_3 - x_4)$$

Given our ordering of the roots, $D > 0$. Note that $iD \in F$, $F = F_1[iD]$, and $\Delta \equiv D^2$ is a rational number. We will assume that (after multiplying all the $x_i$ by a common natural number if necessary) $\Delta$ is an integer (and so $D$ is an algebraic integer). We will repeatedly use the fact that the Galois conjugates of $iD \in F$ are given by

$$\phi_j(iD) = -(-1)^j iD$$

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(1)
Let $V = F \otimes_{\mathbb{Q}} \mathbb{R}$ and $\Lambda = o_F$, the ring of integers of $F$. Then $\Lambda \subset V$ is a lattice in the eight-dimensional real vector space $V$, and $V/\Lambda$ is a real 8-torus. The embeddings $\varphi_i : F \to \mathbb{C}$ induce $\mathbb{R}$-linear maps $z_i : V \to \mathbb{C}$, such that $z = (z_1, z_2, z_3, z_4)$ is an isomorphism of $\mathbb{R}$-vector spaces $V \to \mathbb{C}^4$. We let $X$ denote the complex manifold $V/\Lambda$ obtained thus. Note that if $a \in o_F$, multiplication by $a$ is a $\mathbb{Q}$-linear map $F \to F$ which induces a $\mathbb{R}$-linear map $V \to V$ taking the lattice $\Lambda$ to itself. If $z(a) = (a_1, a_2, a_3, a_4)$, and $u \in V$ with $z(u) = (z_1, z_2, z_3, z_4)$ we also have $z(au) = (a_1z_1, a_2z_2, a_3z_3, a_4z_4)$, so that we see that this induces an analytic map (in fact an isogeny) $X \to X$. In other words, $o_F$ acts on $X$ by “complex multiplication”.

As a complex torus, $X$ is certainly Kähler, and we shall see below that it is algebraic. What is relevant for our purposes is that it is possible to describe explicitly the Hodge decomposition as well as the rational structure of the complex cohomology of $X$. Let $T$ (for “top”) denote the set of indices $\{1, 2, 3, 4\}$ and $B$ (for “bottom”) the indices $\{\bar{1}, \bar{2}, \bar{3}, \bar{4}\}$. (The corresponding vertices are denoted 1b, etc. in the figure.)

**Proposition 2.1.** A basis of $H^{p,q}$ is labeled by subsets $P \subset T$, $Q \subset B$, with $|P| = p$, and $|Q| = q$, and given by the translation-invariant forms $dz^Pd\bar{z}^Q$, where for example, if $P = \{i, j\}$, with $i < j$ we set $dz^P = dz_i \wedge dz_j$, and if $Q = \{\bar{i}, \bar{j}\}$ (again with $i < j$), we set $d\bar{z}^Q = d\bar{z}_i \wedge d\bar{z}_j$.

A basis of the rational cohomology $H^r_\mathbb{Q}$ is labelled by pairs $(R, \chi)$ where

- $R$ is an orbit of $G$ in the set of sequences $\mu \equiv (\mu_1, \ldots, \mu_r)$ of distinct elements in $T \cup B$, and
- $\chi$ runs over a $\mathbb{Q}$-basis of $H_R$, the space of $G$-equivariant maps $R \to L$, satisfying

$$\chi(\mu_{\sigma(1)}, \ldots, \mu_{\sigma(r)}) = \text{sign}(\sigma)\chi(\mu_1, \ldots, \mu_r)$$

for any permutation $\sigma$ such that $\mu, \mu_{\sigma} \in R$.

The corresponding classes are given by the forms

$$\sum_{\mu \in R} \chi(\mu)dz^\mu$$

We use the notation $dz^\mu = dz_{\mu_1} \wedge \cdots \wedge dz_{\mu_r}$, with the convention that $dz_{\bar{1}} = d\bar{z}_1$, etc.

It is useful to note the following

**Lemma 2.2.** Given $R$, the $\mathbb{Q}$-dimension of $H_R$ is $|R|/r!$.

Note that if $r = 2p$, a rational class as above is of type $(p, p)$ iff the orbit consists of sequences with elements equally divided between the top and bottom faces of the cube. In particular, the rational $(1, 1)$ classes correspond to the $G$-orbit of the sequence $(1, \bar{1})$. Since in this case $H_R$ has dimension 4, we see that the Neron-Severi group has rank 4.

Consider now the orbit of the sequence $(1, 3, 2, \bar{4})$. This corresponds to a two-dimensional space $M$ of rational $(2, 2)$ classes, which have the property that these are not products of rational $(1, 1)$ classes. It is easy to check that but for these, the rational $(2, 2)$ classes are generated by rational $(1, 1)$ classes.
**Proposition 2.3.** A $\mathbb{Q}$-basis of $\mathcal{M}$ is given by the classes

- $M_1 = i(dz_1\bar{d}z_2d\bar{z}_3d\bar{z}_4 - d\bar{z}_1dz_2d\bar{z}_3d\bar{z}_4)$
- $M_2 = D(dz_1\bar{d}z_2d\bar{z}_3d\bar{z}_4 + d\bar{z}_1dz_2d\bar{z}_3d\bar{z}_4)$

So the Hodge conjecture in this case would be that: **these classes are algebraic.**

We will use complex multiplication in an essential way later; here I illustrate its use by showing how it can be used to halve our work. Consider multiplication by the algebraic integer $a = 1 + i\mathbb{D} \in \mathfrak{o}_F$. This induces a (covering) map $\pi_a : X \to X$ and one easily computes:

$$\pi_a^* M_1 = ((1 - \Delta)^2 - 4\Delta)M_1 - 4(1 - \Delta)M_2$$

$$\pi_a^* M_2 = ((1 - \Delta)^2 - 4\Delta)M_2 + 4(1 - \Delta)\Delta M_1$$

This proves

**Proposition 2.4.** Algebraicity of either of the $M_i$'s implies that of the other.

Before moving on, we find a positive rational $(1,1)$ form $\omega$ on $X$, which will show that it is projective. Let $\mu_1 \in F_1$ (to be chosen in a moment) and consider the form

$$\omega = \frac{i\mathbb{D}}{\Delta} (\mu_1 dz_1d\bar{z}_1 - \mu_2 dz_2d\bar{z}_2 + \mu_3 dz_3d\bar{z}_3 - \mu_4 dz_4d\bar{z}_4)$$

where $\mu_i$ are Galois conjugates. Clearly this is a rational $(1,1)$ form, and it will be positive provided $(-1)^{j+1}\mu_j > 0$. For example, we can take $\mu_1 = (x_1 - x_2)(x_1 - x_3)(x_1 - x_4)$, and we will do so. With this choice the holomorphic four-form $\theta = dz_1d\bar{z}_2d\bar{z}_3dz_4$ satisfies

$$\frac{\omega^4}{4!} = \theta \wedge \bar{\theta}$$

3. Expressing $M_1$ in terms of Chern characters

What follows is the result of much trial and error and computations using *Mathematica*. Consider the $G$-orbit of $(1,3)$. The corresponding subspace of $H^3_\mathbb{Q}$ is spanned by the classes of

$$A_1 = a_{13}(x_1 - x_3)dz_1dz_3 + ....$$

where $a_{13}$ belongs to the fixed field of the subgroup of $G$ that leaves the set of vertices $\{1,3\}$ invariant. We introduce the notation

$$T_a = a_{13}a_{21}(x_1 - x_3)(x_2 - x_4) - a_{12}a_{34}(x_1 - x_2)(x_3 - x_4) + a_{13}a_{32}(x_1 - x_4)(x_3 - x_2)$$

Squaring $A_1$, we get

$$A_1^2 = 2a_{13}a_{24}(x_1 - x_3)(x_2 - x_4)dz_1dz_3dz_2d\bar{z}_4 + ..$$

$$+ 2a_{12}a_{13}(x_1 - x_2)(x_1 - x_3)d\bar{z}_1dz_2dz_1dz_3 + ..$$

$$+ 2a_{12}a_{21}(x_1 - x_2)(x_2 - x_1)d\bar{z}_1d\bar{z}_2dz_2dz_1 + ..$$

$$+ 2T_a dz_1dz_3d\bar{z}_2d\bar{z}_4 + ..$$
If we make the replacement $a_{13} \rightsquigarrow i\mathbb{D}a_{13}$, we get a class $A_2$, such that

$$A_2^2/\Delta = 2a_{13}a_{24}(x_1 - x_3)(x_2 - x_4)dz_1dz_3dz_2dz_4 + ..$$

$$+ 2a_{12}a_{13}(x_1 - x_2)(x_1 - x_3)dz_1dz_2dz_3dz_4 + ..$$

$$+ 2a_{12}a_{21}(x_1 - x_2)(x_2 - x_1)dz_1dz_2dz_3dz_4 - ..$$

On the other hand, substituting $a_{13} \rightsquigarrow (1 + i\mathbb{D})a_{13}$, we get the class $A_3$, whose square satisfies

$$A_3^2/(1 + \Delta) = 2a_{13}a_{24}(x_1 - x_3)(x_2 - x_4)dz_1dz_3dz_2dz_4 + ..$$

$$+ 2a_{12}a_{13}(x_1 - x_2)(x_1 - x_3)dz_1dz_2dz_3dz_4 + ..$$

$$+ 2a_{12}a_{21}(x_1 - x_2)(x_2 - x_1)dz_1dz_2dz_3dz_4 + ..$$

$$+ (\frac{1 - \Delta + 2i\mathbb{D}}{1 + \Delta} - 2T_adz_1dz_2dz_3dz_4 + ..)$$

Suppose now that the classes $A_i$ are integral. (This is easily arranged by clearing denominators.) Let $L_i$ $(i = 1, 2, 3)$ be the line bundle with Chern class $A_i$.

**Proposition 3.1.** Let $V_i = L_i \oplus L_i^{-1}$, $i = 1, 2, 3$. Then

$$ch(V_1^\Delta \ominus V_2) = 4\Delta(T_adz_1dz_3dz_2dz_4 + ..)$$

$$ch(V_1^{(1+\Delta)} \ominus V_3) = 4(\Delta - i\mathbb{D})(T_adz_1dz_3dz_2dz_4 + ..)$$

where the equality is modulo (rational) 0- and 8-forms.

We have the freedom to choose the coefficient $a_{13}$, which by Galois covariance determines the other coefficients, and hence the above classes. We now make the choice

$$a_{13} = h_3$$

where for later use we introduce the notation

$$h_2 = (x_1x_2 + x_3x_4)$$

(4)

$$h_3 = (x_1x_3 + x_2x_4)$$

$$h_4 = (x_1x_4 + x_2x_3)$$

Then $T_a = -2\mathbb{D}$, and we get

**Theorem 3.2.** With the above choice,

$$ch(V_1^\Delta \ominus V_2) = 8\Delta M_2$$

$$ch(V_1^{(1+\Delta)} \ominus V_3) = 8\Delta M_2 - 8\Delta M_1$$

where the equality is modulo (rational) 0- and 8-forms.

The virtual bundles $V_1^\Delta \ominus V_2$ and $V_1^{(1+\Delta)} \ominus V_3$ have the properties: $c_1 = 0$, and $c_2 \wedge \omega^2 = 0$, where $\omega$ is the rational Kähler class defined at the end of §2. (This is because the $M_i$, as can be easily seen, are orthogonal to $\omega$.) This will not do for reasons to do with the Bogomolov inequality, but this can be fixed because of a minor miracle:
Proposition 3.3. With the above choices,

\[ A^2_1 \wedge \omega = -2i\Delta \frac{1}{\mu_4} dz_1 d\bar{z}_1 dz_2 d\bar{z}_2 dz_3 d\bar{z}_3 + \ldots \]

In particular, \( A^2_1 \wedge \omega \) is a (rational) (3,3) form; also

\[ A^2_1 \wedge \omega^2 = 0 \]

this also hold for \( A_2 \) and \( A_3 \).

We will suppose that \( k\omega \) (for some positive integer \( k \)) is an integral class, and let \( L_{k\omega} \) denote a (holomorphic, in fact ample) line bundle with this Chern class.

Theorem 3.4. Let \( \hat{V}_1 = L_1 \otimes L_{k\omega} \oplus L^{-1}_1 \otimes L_{k\omega} \), and \( \hat{V}_i = V_i \), \( i = 2, 3 \). Then

\[
\text{ch}(\hat{V}_1^{(1+\Delta)} \oplus \hat{V}_3) = 8\Delta M_2 - 8\Delta M_1 + k^2 (1 + \Delta) \omega^2
\]

where the equality is modulo (rational) 0-, (3,3)- and 8-forms.

In particular, these bundles satisfy the "Bogomolov inequality" ((5) below). We use the quote marks since we are not (yet!) talking of holomorphic bundles.

4. Spin(7) instantons

In this section we recall the definition of Spin(7) instantons ([B-K-S], [T]), specialised to the case of a Kähler four-folds \( X \) with trivial canonical bundle \( K_X \). We fix a Ricci-flat Kähler form \( \omega \), and let \( \theta \) denote a trivialisation of \( K_X \) satisfying (3). We define a (complex antilinear) endomorphism \( \star : \Omega^{(0,2)} \to \Omega^{(0,2)} \), by

\[ |\alpha|^2 \theta = \alpha \wedge \star \alpha \]

We have \( \star^2 = 1 \), so we can decompose the bundle into a self-dual and anti-self-dual part:

\[ \Omega^{(0,2)} = \Omega_+^{(0,2)} \oplus \Omega_-^{(0,2)} \]

Let \( E \) be a hermitian \( (C^\infty) \) vector bundle on \( X \). A Spin(7) instanton is a hermitian connection \( A \) on \( E \), whose curvature \( F \) satisfies

\[ F_+^{(0,2)} = 0, \quad \Lambda F = 0 \]

Here \( \Lambda \) denotes as usual contraction with the Kähler form. A crucial point is the following ([L]):

Proposition 4.1. \[ \|F_-^{(0,2)}\|^2 = \int Tr(F \wedge F) \wedge \theta \]

In particular, if the invariant on the right vanishes, a Spin(7) instanton is equivalent to a holomorphic structure on \( E \) together with a Hermite-Einstein connection. Clearly, such a bundle would be polystable, and hence (or directly from the Hermite-Einstein condition) satisfy the Bogomolov inequality:

\[ c_2(E) \omega^2 \geq \frac{r-1}{2r} c_1(E)^2 \omega^2 \]
where \( r \) denotes the rank of \( E \).

Now a \( \text{Spin}(7) \) instanton connection on our bundles would have a curvature of type \((1, 1)\) by the above Proposition. (Since the virtual bundles have positive ranks, we are justified, up to some non-canonical choices, in dropping the qualifier “virtual”.) There are two possible approaches to the construction of such a connection.

1. Exhibit an instanton by glueing.
2. The fact that the bundles are exhibited as a difference of two vector bundles, each of which is in turn a sum of explicit line bundles, suggests the use of monads, possibly combined with a twistor construction. This would involve a matrix of sections of line bundles.

I comment on the first approach where I have made some progress. For the second approach to proceed, it turns out to be necessary to consider non-rational Kähler forms on \( X \).

5. Calibrations; Cayley submanifolds

It is known by the results of G. Tian ([T]) that when instantons degenerate the curvature concentrates on Cayley currents of an appropriate calibration. Conversely, C. Lewis [L] shows how (in one particular case) one can construct an instanton by glueing around a suitable Cayley submanifold. (See also [B].) We define these terms below, and then exhibit some relevant Cayley cycles that arise in our context. (References are [H-L], and [J]; but we follow the conventions of [T].)

**Definition 5.1.** Let \( M \) be a Riemannian manifold. A closed \( l \)-form \( \phi \) is said to be a **calibration** if for every oriented tangent \( l \)-plane \( \xi \), we have

\[
\phi|_\xi \leq \text{vol}_\xi
\]

where \( \text{vol}_\xi \) is the (Riemannian) volume form. Given a calibration \( \phi \), an oriented submanifold \( N \) is said to be **calibrated** if \( \phi \) restricts to \( N \) as the Riemannian volume form.

It is easy to see that a calibrated submanifold is minimal. Two examples are relevant. First, if \( M \) is Kähler, with Kähler form \( \omega \), for any integer \( p \geq 1 \), the form \( \omega^p/p! \) is a calibration, and the calibrated submanifolds are precisely the complex submanifolds.

The case that concerns us is that of a four-fold \( X \) with trivial canonical bundle \( K_X \). We fix an integral Ricci-flat Kähler form \( \omega \), and let \( \theta \) denote a trivialisation of \( K_X \) with normalisation as in (3). Then \( 4 \text{Re}(\theta) \) is a second calibration, and the calibrated submanifolds are called **Special Lagrangian submanifolds**. There is a “linear combination” of the two, defined by the form

\[
\Omega = \frac{w^2}{2} + 4 \text{Re}(\theta)
\]

which defines the **Cayley calibration**. The corresponding calibrated manifolds are called **Cayley manifolds**. Any smooth complex surface (on which the second term will restrict to zero) or any
Special Lagrangian submanifold (on which the first term will vanish) furnish examples. In fact, the Cayley cycles we deal with will be of the latter kind.

Cayley manifolds are not easy to find. We will use the following result (Proposition 8.4.8 of [J]):

**Proposition 5.2.** Let $X$ be as above, and $\sigma : X \to X$ an anti-holomorphic isometric involution such that $\sigma^0 = \theta$. Then the fixed point set is a Special Lagrangian submanifold.

We return to the constructions of our paper. Recall that the field $F$ is invariant under complex conjugation, which therefore acts on it with fixed field $F_1$. This induces an involution $\sigma_1 : V \to V$ such that $z(\sigma_1(u)) = \bar{z}(u)$, where, if $z = (z_1, z_2, z_3, z_4)$, we set $\bar{z} = (\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4)$.

The induced involution $\sigma_1 : X \to X$ has fixed locus which we will denote $Y$. Note that $\sigma$ satisfies the conditions of the previous Proposition and therefore $Y$ is Special Lagrangian.

**Theorem 5.3.** There exist (rational) Cayley cycles representing the Hodge classes $M_i$.

**Proof.** Recall the isogeny $\pi_a : X \to X$, given by multiplication by the algebraic integer $a = 1 + iD$.

It is easy to check

$$\pi_a^* \omega = (1 + \Delta)\omega$$
$$\pi_a^* \theta = (1 + \Delta)^2 \theta$$

We will also need a second isogeny $\pi_b$, where $b = iD$, which satisfies

$$\pi_b^* \omega = \Delta \omega$$
$$\pi_b^* \theta = \Delta^2 \theta$$

These equations guarantee the maps $\pi_a, \pi_b$ take Cayley cycles to Cayley cycles (possibly introducing singularities.)

We have the following table giving the action of the above isogenies on four-forms of various types (all the forms in the list are eigenvectors):

| Form             | eigenvalue of $\pi_a^*$ | eigenvalue of $\pi_b^*$ | “multiplicity” |
|------------------|-------------------------|-------------------------|---------------|
| $dz_1dz_2dz_3dz_4$ | $(1 + \Delta)^2$        | $\Delta^2$              | 2 × 1         |
| $dz_1d\bar{z}_2d\bar{z}_3dz_4$ | $(1 + \Delta)^2$        | $\Delta^2$              | 2 × 8         |
| $d\bar{z}_1d\bar{z}_2dz_3d\bar{z}_4$ | $(1 + \Delta)(1 - iD)^2$ | $-\Delta^2$             | 2 × 4         |
| $d\bar{z}_1d\bar{z}_2dz_3d\bar{z}_4$ | $(1 + \Delta)(1 - iD)^2$ | $-\Delta^2$             | 2 × 4         |
| $d\bar{z}_1d\bar{z}_2d\bar{z}_3d\bar{z}_4$ | $(1 + \Delta)^2$        | $\Delta^2$              | 6             |
| $d\bar{z}_1d\bar{z}_2dz_3d\bar{z}_4$ | $(1 + \Delta)(1 - iD)^2$ | $-\Delta^2$             | 2 × 12        |
| $d\bar{z}_1d\bar{z}_2d\bar{z}_3d\bar{z}_4$ | $(1 + \Delta)^2$        | $\Delta^2$              | 4             |
| $d\bar{z}_1d\bar{z}_2dz_3d\bar{z}_4$ | $(1 + iD)^4$            | $\Delta^2$              | 1             |
| $d\bar{z}_1d\bar{z}_2d\bar{z}_3d\bar{z}_4$ | $(1 - iD)^4$            | $\Delta^2$              | 1             |

(We list only forms of type (4,0), (3,1) and (2,2), omitting types that are related to the ones in the list by conjugation. The term “multiplicity” refers to the number of forms of a given type, not the multiplicity of eigenvalues.)

Consider the operator

$$\Phi_a = (\pi_a^* - (1 + \Delta)^2)(\pi_b^* + \Delta^2)$$
From the list it follows that the space $\mathcal{M} \otimes \mathbb{Q} \mathbb{C}$ (spanned by the $M_i$) is the sum of the eigenspaces of $\Phi_a$ corresponding to the non-zero eigenvalues. We have (using (2))

$$\Phi_a^* M_1 = -8\Delta^2 [2\Delta M_1 + (1 - \Delta) M_2]$$
$$\Phi_a^* M_2 = -8\Delta^2 [-(1 - \Delta) \Delta M_1 + 2\Delta M_2]$$

Next, note that the Cayley cycle $Y$ defined above satisfies

$$< Y, M_2 > = 2\mathbb{D}\delta$$
$$< Y, M_1 > = 0$$

Here $<,>$ denotes the integration pairing of cycles and forms, and $\delta$ denotes the co-volume of the lattice $\mathfrak{O}_{F_1} \subset F_1 \otimes \mathbb{Q} \mathbb{R}$. By standard facts in algebraic number theory, $\delta$ is a rational multiple of $\mathbb{D}$; so the above pairings are rational, as they had better be.

We now consider the Cayley cycle

$$C_a = (\pi_a - (1 + \Delta)^2)(\pi_b + \Delta^2)Y$$

By construction $C_a$ is orthogonal to all the forms in the above list except the $M_i$. Its pairings with these are as follows:

$$< C_a, M_2 > = -32\Delta^3 \mathbb{D}\delta$$
$$< C_a, M_1 > = -16\Delta^2 (1 - \Delta) \mathbb{D}\delta$$

Let now $\bar{a} = (1 - i\mathbb{D})$, and repeat the above construction with operators $\Phi_{\bar{a}}$, etc.

$$\Phi_{\bar{a}}^* M_1 = -8\Delta^2 [2\Delta M_1 - (1 - \Delta) M_2]$$
$$\Phi_{\bar{a}}^* M_2 = -8\Delta^2 [(1 - \Delta) \Delta M_1 + 2\Delta M_2]$$

This gives a cycle $C_{\bar{a}}$ satisfying

$$< C_{\bar{a}}, M_2 > = -32\Delta^3 \mathbb{D}\delta$$
$$< C_{\bar{a}}, M_1 > = 16\Delta^2 (1 - \Delta) \mathbb{D}\delta$$

Clearly the theorem is proved. □

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