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ON JOURDAIN’S PRINCIPLE

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Abstract—This paper reexamines the effect of the validity, or the lack thereof, of the commutation rule for kinematically possible, or admissible, and virtual variations, i.e. \( \delta(\cdot) \) vs \( \delta d(\cdot) \), on the form of Jourdain’s principle (JP), in general quasivariables. It is shown that JP is independent of any such rule. A similar independence holds in the derivation of the equations of motion from Lagrange’s principle.

1. INTRODUCTION

In recent years the differential variational principle of Jourdain (JP), originally formulated by him in 1909 [1,2] as an “intermediate” principle between those of Lagrange and Gauss/Gibbs, and for long regarded by some as nothing more than an academic curiosity, has reemerged as a very versatile and powerful tool for the treatment of both finite and impulsive motion of constrained mechanical systems [3–8]. Unfortunately, and inexplicably, this principle seems to be very little known among English-speaking engineers; its brief statement in Kane and Levinson [9] does not correspond to the classical principle as encountered in the literature.

The formulation of JP assumes, explicitly or implicitly, the satisfaction of the commutation rule

\[
\delta(\dot{r}) - \dot{\delta}(r) = 0, \tag{1a}
\]

or

\[
\frac{d}{dr} (\delta r) = \delta \left( \frac{dr}{dt} \right) = \delta v, \tag{1b}
\]

where \( r = \) inertial position vector of a typical particle \( P \), of the system under discussion \( S \), \( v = \) inertial velocity of \( P \), \( \delta(\cdot) \) and \( \delta(\cdot) \) = kinematically possible and virtual inertial differentials of \( (\cdot) \). One solidly gets the impression that equations (1) are necessary not only for JP, but for the subsequent derivation of the correct equations of motion. This, however, is in sharp contrast with the corresponding Lagrange’s Principle (LP). In the latter, as Hamel [10,11] et al., have clearly demonstrated, the derivation of the equations of motion from LP and from its derivative Central Equation (CE—the “Zentralgleichung” of Heun and Hamel, which is a slight transformation of LP to first-order differential expressions), is completely independent of any particular assumptions about

\[
\delta(\dot{r}) - \dot{\delta}(r) \quad \text{or} \quad (\delta r)' - \delta v. \tag{2}
\]

The following question then naturally arises: is JP also independent of commutation rules, like (1)? The purpose of this paper is to reexamine the theoretical foundations of JP and establish the necessity, or lack thereof, of the commutation rule (1). Section 2 summarizes the theory of LP and JP, while Section 3 investigates in detail the Jourdain form of the commutation rule. The results are summarized in the concluding Section 4.

2. THE PRINCIPLES OF LAGRANGE AND JOURDAIN

Consider a material system \( S \) subject to constraints that are bilateral, ideal, and linear in the (particle and/or system) velocities; these constraints may be genuinely nonintegrable (nonholonomic), or they may be integrable (holonomic) disguised in kinematical, or
differential, or velocity form. A typical particle \( P \) of \( S \) obeys Newton’s “second law”
\[
d m \, a = d F + d R, \tag{3}
\]
where \( d m \) = mass of \( P \), \( a \) = inertial acceleration of \( P \), i.e. \( d^2 r / dt^2 \), \( r \) = inertial position vector and \( t \) = time; \( d F \) = total impressed force on \( P \), and \( d R \) = total constraint reaction force on \( P \).

The derivation of reactionless equations of motion is based on Lagrange’s Principle (LP):
\[
\delta \{ W' - \mathbf{S}(d \mathbf{r} \cdot \mathbf{a} - d \mathbf{F}) \} = 0, \tag{4}
\]
which, due to (3), takes the customary Lagrangean form:
\[
\mathbf{S}[(d m \, a - d \mathbf{F}) \cdot \delta \mathbf{r}] - 0, \tag{5}
\]
where \( \delta \mathbf{r} \) represents the (inertial) virtual displacement of \( P \) (i.e. under \( dt \to 0 \)), and
\[
\mathbf{S}(\cdots)
\]
denotes material summation over all the elements (particles) of \( S \) (as in Stieltjes’ integral). For details see Hamel [11, pp. 361-363].

Next taking the total differential of both sides of (5), with respect to time, as that equation holds for some time interval, and since
\[
\mathbf{S}(\cdots) \text{ and } \frac{d}{dt}(\cdots) \equiv (\cdots)' \text{ commute},
\]
we find:
\[
\mathbf{S}\left[ \frac{d}{dt}(d m \, a - d \mathbf{F}) \cdot \delta \mathbf{r} \right] + \mathbf{S}\left[ (d m \, a - d \mathbf{F}) \cdot \frac{d}{dt}(\delta \mathbf{r}) \right] - 0. \tag{6}
\]
This leads to the following differential variational principle: the reactionless (or purely kinetic) equations of motion of \( S \) follow from the differential variational equation
\[
\mathbf{S}[(d m \, a - d \mathbf{F}) \cdot (\delta \mathbf{r})'] = 0, \tag{7}
\]
under the (fixed time and position) constraints
\[
\delta t = 0 \quad \text{(as in LP), and} \quad \delta \mathbf{r} = 0. \tag{8}
\]
On the other hand, Jourdain’s principle (JP), in its original enunciation by Jourdain [1], states that the (same) equations of motion of \( S \) follow from the variational principle
\[
\mathbf{S}[(d m \, a - d \mathbf{F}) \cdot (\delta v)] = 0, \tag{9}
\]
where
\[
v = \frac{d \mathbf{r}}{dt} = \dot{\mathbf{r}} \quad (= \text{inertial velocity of } P),
\]
under the constraints (8).

Clearly if we assume the commutation rule
\[
\frac{d}{dt}(\delta \mathbf{r}) = \delta (d \mathbf{r}), \quad \text{or} \quad \frac{d}{dt}(\delta \mathbf{r}) = \delta (\frac{d \mathbf{r}}{dt}) = \delta v. \tag{10}
\]
then the principles (7) and (9) coincide, under the set of assumptions (8). The question is: can this equivalence happen under all circumstances, i.e. without any special assumptions like (10), that is, even if
\[
\delta^* \mathbf{r} = (\delta \mathbf{r})' - \delta (\dot{\mathbf{r}}) \neq 0.
\]
The answer to this requires utilization of the most general expression for \( \delta^* \mathbf{r} \). We now turn our attention to this problem.
3. THE TRANSITIVITY (OR COMMUTATIVITY) RELATIONS

In terms of the fundamental, generally nonholonomic, particle and system basis vectors \( \{a_i, i = m + 1, \ldots, n; m = \text{number of velocity constraints}, n = \text{number of (initially independent) coordinates} q \text{— see e.g. [10–12]} \}, \) the particle virtual displacement \( \delta r \) is expressed as the following linear combination:

\[
\delta r = \sum_{i=1}^{n} e_i \delta q_i = \sum_{i} a_i \delta \theta_i \quad (i = m + 1, \ldots, n; a_i = a_i(t, q)),
\]

where

\[
\delta \theta = (\delta \theta_1 = 0, \ldots, \delta \theta_m = 0; \delta \theta_{m+1}, \ldots, \delta \theta_n) = \text{virtual independent system quasicoordinate variations},
\]

\[
e_i = \partial r / \partial q_i \quad (i = 1, \ldots, n), \quad \delta \theta_i = \sum_{i=1}^{n} a_i \delta q_i,
\]

\[
o_i = d\theta_i / dt = \text{independent system quasivelocities},
\]

\[
\omega_i = \sum_{i=1}^{n} a_{ii} \dot{q}_i + a_i \quad (a_{ii}, a_i: \text{known functions of} \ q, t).
\]

The first of (11) may also be viewed as the definition of the \( \{a_i\} \).

Since the \((n - m)\) vectors \( \{a_i\} \) are independent, the (second) Jourdain constraint (8) yields

\[
\delta r = 0 \rightarrow \delta \theta_i = 0. \tag{12}
\]

Differentiating the first of (11) in time, while utilizing (12), results in

\[
(\delta r)' = \sum_i (\dot{a}_i \delta \theta + a_i(\delta \theta_i)')
\]

\[
= \sum_i a_i(\delta \theta_i)' = \sum_i \left( \frac{\partial v}{\partial \omega_i} \right)(\delta \theta_i)' . \tag{13}
\]

If we are to obtain something nontrivial from (6–8), then we should not assume that from (12) it follows that \( (\delta r)' = 0, \) and/or \( (\delta \theta_i)' = 0. \)

Let’s now take for algebraic simplicity, but without loss of generality, the case of scleronomic systems, i.e. stationary constraints. Then \( a_i \) [last of (11)] \( \rightarrow 0, \) and

\[
v = \sum_i e_i \dot{q}_i = \sum_i a_i \omega_i. \tag{14}
\]

Varying this à la Jourdain, from now on to be denoted by \( \delta'(\cdots), \) yields

\[
\delta'v = \sum_i ((\delta' a_i) \omega_i + a_i(\delta' \omega_i)) = \sum_i a_i(\delta' \omega_i)
\]

[since for linear velocity constraints \( a_i = a_i(t, q) \rightarrow \delta' a_i = 0]. \tag{15}
\]

Therefore, from (13), (15), we obtain

\[
(\delta r)' - \delta'v = \sum_i a_i ((\delta \theta_i)' - \delta' \omega_i), \quad \left[ = \sum_i e_i ((\delta q_i)' - \delta \dot{q}_i) \right],
\]

or

\[
(\delta r)' = \delta'v + \sum_i a_i \delta^* \theta_i, \tag{16}
\]

\[
\delta^* \theta_i = \frac{d}{dt} (\delta \theta_i) - \delta' \omega_i \neq 0, \tag{17}
\]

despite (12).
Remark: Equation (16), or
\[ \delta^* r = \sum \delta^* \theta_i, \]
simply states that \( \delta^* r = (\delta r)' - \delta' (r) = (\delta r)' - \delta' v \) lies in the virtual plane of that particle at the "point" \((q, t)\). Inserting now (16) into \( d/dt(LP) \rightarrow \) equation (7), yields
\[
S[(d\mathbf{m} - d\mathbf{F}) \cdot \delta' v] + S \left[ (d\mathbf{m} - d\mathbf{F}) \cdot \left( \sum_i a_i (\delta^* \theta_i) \right) \right] = 0,
\]
or
\[
S[(d\mathbf{m} - d\mathbf{F}) \cdot \delta' v] + \sum_i \{ S[(d\mathbf{m} - d\mathbf{F}) \cdot a_i] \} \delta^* \theta_i = 0,
\]
from which—and this is the key step in the entire discussion—due to LP in quasivariables (see e.g. [11,12]), i.e.
\[
S[(d\mathbf{m} - d\mathbf{F}) \cdot a_i] = 0,
\]
we obtain
\[
S[(d\mathbf{m} - d\mathbf{F}) \cdot \delta' v] = 0,
\]
under
\[
\delta t = 0, \text{ and } \delta r = 0,
\]
but not \( \delta^* r' = 0 \); i.e. the original Jourdain formulation. In short, the derivation of JP from LP does not depend on the equality of \( d(\delta r) \) with \( \delta (dr) \); or, condition (10) is sufficient but not necessary for it.

Finally, substituting (15) into (20), and since the \((n-m)\delta' \omega_i\) are independent, reproduces the "raw" form of the fundamental kinetic (reactionless) equations (19):
\[
S[(d\mathbf{m} - d\mathbf{F}) \cdot a_i] = S(d\mathbf{F} \cdot a_i).
\]
Further transformation of (19,21) to system variables leads to such equations of motion as those by Appell, Maggi, and Boltzmann-Hamel (see e.g. [13]).

Remarks

(1) Equation (16) can also result from the general transitivity equations specialized for the Jourdain variation (8,12). The starting point is
\[
(\delta \theta_k)' - \delta \omega_k = \sum_i A_{ki} \{(\delta \theta_i)' - \delta \omega_i\} + \sum_i \gamma_i^{\alpha k} \omega_i \delta \theta_i,
\]
where the three-index (Hamel) symbols \( \gamma_i^{\alpha k} \) are defined by (22); for notation and relevant theory see, e.g. [11, pp. 243, 475–477, equations (15a); 14].

Inverting (22) we obtain
\[
(\delta \theta_k)' - \delta \omega_k = \sum_i A_{ki} \{(\delta \theta_i)' - \delta \omega_i\} - \sum_{l,r,s} A_{kl} \gamma_l^{\alpha r} \omega_s \delta \theta_r,
\]
where \( \{A_{\alpha} \} \) and \( \{a_{\alpha} \} \) are inverse matrices.

Therefore, we have, successively,
\[
(\delta r)' - \delta v = \sum_k e_k \{(\delta q_k)' - \delta (\dot{q}_k)\}
\]
\[
= \sum_k e_k \left\{ \sum_l A_{lk} \{(\delta \theta_l)' - \delta \omega_l\} \right\} - \sum_k e_k \left\{ \sum_{l,r,s} A_{kl} \gamma_l^{\alpha r} \omega_s \delta \theta_r \right\}
\]
\[
= \sum_k (e_k A_{\alpha}) \delta^* \theta_i - \sum_{k,l,r,s} (e_k A_{\alpha}) \gamma_l^{\alpha r} \omega_s \delta \theta_r,
\]
\[
= \sum_l a_l \delta^* \theta_i - \sum_{l,r,s} a_l \gamma_l^{\alpha r} \omega_s \delta \theta_r, \quad \text{ (since: } a_l = \sum_k A_{kl} e_k) \]
(24)
But in a constrained system
\[ \delta \theta_1, \ldots, \delta \theta_m = 0, \]  
while from the Jourdain requirement (8,12)
\[ \delta \theta_m, \ldots, \delta \theta_n = 0; \]  
i.e. \( \{\delta \theta_r; r = 1, \ldots, n (\& n + 1)\} = 0 \), and (25) reduces here to
\[ \delta^* r = \sum_{i} a_i \delta^* \theta_i \rightarrow \sum_{i=m+1}^{n} a_i \delta^* \theta_i, \]  
as before.

(2) If we had assumed that \( \delta^* r = 0 \), i.e.
\[ d(\delta^* r) = \delta (dr), \]  
then (28) and (23), under the Jourdain constraints, would have led us to
\[ \delta^* \theta_i = 0, \quad \text{or} \quad d(\delta \theta_i) = \delta (d \theta_i), \]  
and
\[ \delta^* q_k = \sum_{i} A_{ki} \delta^* \theta_i = 0, \]  
or
\[ d(\delta q_i) = \delta (d q_i), \]  
and vice versa.

This is something that holds here due to the Jourdain variation.

In general, as (22,23) show, without the Jourdain constraints either \( \delta^* q_i = 0 \) or \( \delta^* \theta_i = 0 \) holds true; but not both.

(3) One can easily envision similar extensions of the above reasoning to the higher-order differential variational principles of Gauss, and Mangeron-Deleanu. For their basic theory see, e.g. [13, pp. 237–258].

4. CONCLUSIONS

We have demonstrated that Jourdain's Principle results naturally from the (total) time differentiation of Lagrange's Principle, and that like it, produces the correct equations of motion independently of any commutation assumptions regarding the \( d(\cdots) \) and \( \delta(\cdots) \) variations.

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