THE ROHLIN INVARIANT AND \( \mathbb{Z}/2 \)-VALUED INVARIANTS OF HOMOLOGY SPHERES

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ABSTRACT. In this paper we prove that the Rohlin invariant is the unique invariant inducing a homomorphism on the Torelli group. Using this result we generalize the construction of invariants of homology 3-spheres from families of trivial 2-cocycles on the Torelli group given by Pitsch to include invariants with values on an abelian group with 2-torsion.

1. Introduction

Let \( \Sigma_g \) be an oriented surface of genus \( g \) standardly embedded in the oriented 3-sphere \( S^3 \). Denote by \( \Sigma_{g,1} \) the complement of the interior of a small disk embedded in \( \Sigma_g \). The surface \( \Sigma_g \) separates \( S^3 \) into two genus \( g \) handlebodies \( S^3 = \mathcal{H}_g \cup -\mathcal{H}_g \) with opposite induced orientation. Denote by \( \mathcal{M}_{g,1} \) the mapping class group of \( \Sigma_{g,1} \), i.e. the group of isotopy classes of orientation-preserving diffeomorphisms of \( \Sigma_g \) which are the identity on the disk modulo isotopies which again fix that small disk point-wise. The embedding of \( \Sigma_g \) in \( S^3 \) determines three natural subgroups of \( \mathcal{M}_{g,1} \): the subgroup \( \mathcal{B}_{g,1} \) of mapping classes that extends to the inner handlebody \( \mathcal{H}_g \), the subgroup \( \mathcal{A}_{g,1} \) of mapping classes that extends to the outer handlebody \( -\mathcal{H}_g \), and their intersection \( \mathcal{A}\mathcal{B}_{g,1} \).

By the theory of Heegaard splittings we know that any element in \( S^3 \), the set of diffeomorphism classes of compact, closed and oriented smooth homology 3-spheres, can be obtained by cutting \( S^3 \) along \( \Sigma_g \) for some \( g \) and glueing back the two handlebodies by some element of the Torelli group \( \mathcal{T}_{g,1} \), which is the group formed by the elements of the mapping class group \( \mathcal{M}_{g,1} \) that act trivially on the first homology group of the surface \( \Sigma_g \).

In [10], Pitsch proved that the lack of injectivity of this construction is controlled by the subgroups \( \mathcal{T}\mathcal{B}_{g,1} = \mathcal{T}_{g,1} \cap \mathcal{B}_{g,1} \), \( \mathcal{T}\mathcal{A}_{g,1} = \mathcal{T}_{g,1} \cap \mathcal{A}_{g,1} \), and the conjugation by elements of the group \( \mathcal{A}\mathcal{B}_{g,1} \). More precisely the injection \( \Sigma_{g,1} \hookrightarrow \Sigma_{g+1,1} \) induces a natural injective stabilization map \( \mathcal{T}_{g,1} \hookrightarrow \mathcal{T}_{g+1,1} \), which is compatible with the definitions of the above subgroups and one gets a well-defined bijective map:

\[
\lim_{g \to \infty} (\mathcal{T}\mathcal{A}_{g,1}/\mathcal{T}\mathcal{B}_{g,1})_{\mathcal{A}\mathcal{B}_{g,1}} \sim S^3. \tag{1.1}
\]

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As a consequence of this bijection, every invariant \( F \) of homology 3-spheres with values in an arbitrary abelian group \( A \) can be viewed as a family of functions \( \{F_g\}_g \) from the Torelli groups \( T_{g,1} \) to the abelian group \( A \) satisfying the following properties:

i) \( F_{g+1}(x) = F_g(x) \) for every \( x \in T_{g,1} \),

ii) \( F_g(\xi_a x \xi_b) = F_g(x) \) for every \( x \in T_{g,1}, \xi_a \in TA_{g,1}, \xi_b \in TB_{g,1} \),

iii) \( F_g(\phi x \phi^{-1}) = F_g(x) \) for every \( x \in T_{g,1}, \phi \in AB_{g,1} \).

Using this reformulation of invariants of homology 3-spheres in terms of family of functions on the Torelli group, in the aforementioned paper Pitsch gave a tool to construct invariants of integral homology 3-spheres, with values in any abelian group without 2-torsion from a family of trivial 2-cocycles on the Torelli group \( T_{g,1} \) with an \( AB_{g,1} \)-invariant trivialization and satisfying the following conditions:

(1) The 2-cocycles \( \{C_g\}_g \) are compatible with the stabilization map,

(2) The 2-cocycles \( \{C_g\}_g \) are invariant under conjugation by elements in \( AB_{g,1} \),

(3) If \( \phi \in TA_{g,1} \) or \( \psi \in TB_{g,1} \) then \( C_g(\phi, \psi) = 0 \).

In this paper we generalize this construction of invariants of homology 3-spheres from trivial 2-cocycles to include values in an abelian group with 2-torsion.

The main difficulty to generalize this construction is the existence of non-zero homomorphisms on the Torelli group that satisfy the aforementioned properties i)-iii) and therefore reassemble to invariants of homology 3-spheres. Nevertheless we show that the homomorphisms induced by the Rohlin invariant are the unique ones with such properties.

**Theorem 1.1.** Let \( A \) be an abelian group and \( A_2 \) the subgroup of 2-torsion elements. Provided \( g \geq 4 \), the BCJ-homomorphism \( \sigma : T_{g,1} \to B_3 \) composed with the projection \( \pi_g : B_3 \to B_0 \cong \mathbb{Z}/2 \) and the injection \( \varepsilon^x : B_0 \to A_2 \) defined by sending 1 to \( x \), induces an isomorphism

\[
\Lambda : A_2 \longrightarrow Hom(T_{g,1}, A)_{AB_{g,1}},
\]

\[
x \mapsto \mu_g^x = \varepsilon^x \circ \pi_g \circ \sigma.
\]

Moreover the family of homomorphisms \( \{\mu_g^x\}_g \) reassemble into the Rohlin invariant.

This result allows us to generalize the construction of invariants of homology 3-spheres given by Pitsch in [10] to include values on an abelian group with 2-torsion.

**Theorem 1.2.** Let \( A \) be an abelian group and \( A_2 \) the subgroup of 2-torsion elements. For each \( x \in A_2 \) and \( g \geq 4 \), a family of cocycles \( \{C_g\}_{g \geq 3} \) on the Torelli group \( T_{g,1} \) satisfying conditions (1)-(3) provides a compatible family of trivializations \( F_g + \mu_g^x : T_{g,1} \to A \) that reassemble into an invariant of integral homology 3-spheres

\[
\lim_{g \to \infty} F_g + \mu_g^x : S^3 \to A
\]

if and only if the following two conditions hold:

(i) The associated cohomology classes \( [C_g] \in H^2(T_{g,1}; A) \) are trivial,

(ii) The associated torsors \( \rho(C_g) \in H^1(AB_{g,1}, Hom(T_{g,1}, A)) \) are trivial.
Outlines of this work

In Section 2 we review some definitions about the mapping class group, the symplectic representation, the Boolean algebra, the Birman-Craggs-Johnson-homomorphism and handlebodies. We also compute the $GL_g(\mathbb{Z})$-coinvariants of the Boolean algebras of degree 2, 3 and give a basic 3-dimensional topology lemma about handlebodies. In Section 3 we recall the definitions of contractible bounding pair twists and the Luft group. Moreover, we exhibit some results about the handlebody subgroup $B_{g,1}$ and the Luft-Torelli group $LT B_{g,1}$. In Section 4 we prove that $H_1(T_{g,1}; \mathbb{Z})_{AB_{g,1}}$ is isomorphic to $\mathbb{Z}/2$, and we also show that $H_1(T B_{g,1}; \mathbb{Z})_{AB_{g,1}}$ and $H_1(T A_{g,1}; \mathbb{Z})_{AB_{g,1}}$ are zero. At the end of this section, using the aforementioned computations, we prove Theorem 1.1. Finally, in Section 5 we give the proof of Theorem 1.2.

2. Preliminaries

2.1. The Symplectic representation. For a given integer $g \geq 1$ consider the basis \{a_1, \ldots, a_g, b_1, \ldots, b_g\} of $H_1(\Sigma_{g,1}; \mathbb{Z})$ given by the homology class of the respective curves \{a_1, \ldots, a_g, \beta_1, \ldots, \beta_g\} depicted in Fig. 1. Transverse intersection of oriented paths on $\Sigma_{g,1}$ induces a symplectic form $\omega$ on $H_1(\Sigma_{g,1}; \mathbb{Z})$, with $\omega(b_i, a_i) = -\omega(a_i, b_i) = 1$ and zero otherwise. Moreover, both sets of displayed homology classes \{a_i \mid 1 \leq i \leq g\} and \{b_i \mid 1 \leq i \leq g\} form a symplectic basis, and in particular generate supplementary transverse Lagrangians A and B. As a symplectic space we write $H_1(\Sigma_{g,1}; \mathbb{Z}) = A \oplus B$.

The symplectic form $\omega$ is preserved by the natural action of the mapping class group on the first homology of $\Sigma_{g,1}$ and gives rise to the symplectic representation

$$
\Psi : \mathcal{M}_{g,1} \longrightarrow Sp_{2g} (\mathbb{Z}),
$$

which is known to be a surjective map (cf. [1]). In particular, by [10] Lemma 3, the action of $AB_{g,1}$ on $H_1(\Sigma_{g,1}; \mathbb{Z})$ factors through $GL_g(\mathbb{Z})$, where an element $G \in GL_g (\mathbb{Z})$ acts on $H_1(\Sigma_{g,1}; \mathbb{Z})$ via the action of the matrix \begin{pmatrix} G & 0 \\ 0 & G^{-1} \end{pmatrix} \in Sp_{2g} (\mathbb{Z}).

To understand better $\Psi$ we show what it does to Dehn twists: given $k \in \mathbb{Z}$ and two oriented simple closed curves $x$ and $y$ in $\Sigma_{g,1}$ with respective homology classes $[x]$ and $[y]$, the action of the Dehn twist $T^k_x$ on $H_1(\Sigma_{g,1}; \mathbb{Z})$ is given by:

$$
\Psi(T_x^k)([y]) = [y] + k \omega([y], [x])[x].
$$
To avoid too heavy notation in all subsequent sections we will sometimes abbreviate $H := H_1(\Sigma_{g,1}; \mathbb{Z})$ and $H_2 := H_1(\Sigma_{g,1}; \mathbb{Z}/2)$.

2.2. The Boolean algebra and the BCJ-homomorphism. The Boolean polynomial algebra $\mathcal{B} = \mathcal{B}_g$ is a $\mathbb{Z}/2$-algebra (with unit 1) with a generator $x$ for each $x \in H_1(\Sigma_{g,1}; \mathbb{Z}/2)$ and subject to the relations:

(a) $x + y = x + y + \omega(x, y) \mod 2$,
(b) $x^2 = x$.

The relation (b) implies that $p^2 = p$ for any $p \in \mathcal{B}$ and also that if $\{e_i \mid i \in \{1, \ldots, 2g\}\}$ is a basis for $H_1(\Sigma_{g,1}; \mathbb{Z}/2)$ then the set of all monomials $e_{i_1}e_{i_2} \ldots e_{i_r}$ with $0 \leq r \leq 2g$ and $1 \leq i_1 < i_2 < \ldots < i_r \leq 2g$ is a $\mathbb{Z}/2$-basis for $\mathcal{B}$. Denote by $\mathcal{B}_k = \mathcal{B}_g^k$ the subspace generated by all monomials of “degree” $\leq k$.

In [4], Johnson constructed a surjective homomorphism $\sigma : T_{g,1} \to \mathcal{B}_3$, called the Birman-Craggs-Johnson homomorphism (abbreviated BCJ-homomorphism), which may be described as follows. Consider the $\mathbb{Z}/2$-basis of $\mathcal{B}_3$ given by $\{1, a_i, b_i, a_i a_j, b_i b_j, a_i b_j, a_i b_i, a_i a_j a_k, b_i b_j b_k, a_i a_j b_k, a_i b_j b_k, a_i b_i b_j, a_i a_j a_k\}$, where $i,j,k \in \{1, \ldots, g\}$ are pairwise distinct. Consider the curves depicted in the following figure:

![Figure 2. A bounding simple closed curve and a bounding pair in $\Sigma_{g,1}$.](image)

The BCJ-homomorphism is given on a BP-map $T_\beta T_{\beta'}^{-1}$ by (cf. [4, Lemma 12b])

$$\sigma(T_\beta T_{\beta'}^{-1}) = \sum_{i=1}^{k} \tau_i \bar{d}_i (\tau + \bar{1}),$$

where $e$ is the homology class of $\beta$, and $\{c_i, d_i\}$ is a symplectic basis of a subsurface $\Sigma_{k,1}$ of $\Sigma_{g,1}$ with boundary component $\gamma$, such that $\gamma \cup \beta \cup \beta'$ is the boundary of a subsurface with genus zero in $\Sigma_{g,1}$. And on a Dehn twist about a bounding simple closed curve $\gamma$ of genus $k$, by (cf. [4, Lemma 12a])

$$\sigma(T_\gamma) = \sum_{i=1}^{k} \tau_i \bar{d}_i,$$

where $\{c_i, d_i\}$ is the symplectic basis of the subsurface of genus $k$ bounded by $\gamma$.

By [4, Lemma 13], the BCJ-homomorphism $\sigma$ is $\mathcal{M}_{g,1}$-equivariant. The action of $\mathcal{M}_{g,1}$ on $\mathcal{B}_3$ factors through $Sp_{2g}(\mathbb{Z})$ via the symplectic representation $\Psi : \mathcal{M}_{g,1} \to Sp_{2g}(\mathbb{Z})$.
where $Sp_{2g}(\mathbb{Z})$ acts on $\mathcal{B}_3$ via its action on $H_1(\Sigma_{g,1}, \mathbb{Z})$ modulo 2. In other words, for a given $f \in \mathcal{M}_{g,1}$ and three elements $z_i, z_j, z_k$ of $H_1(\Sigma_{g,1}, \mathbb{Z}/2)$, the action is given by:

$$f \cdot (z_i z_j z_k) = \Psi(f) \cdot z_i \Psi(f) \cdot z_j \Psi(f) \cdot z_k.$$

Lemma 2.1. Provided $g \geq 3$, the isomorphisms $(\mathcal{B}_2)_{GL_g(\mathbb{Z})} \cong (\mathcal{T}, \mathcal{M}_1) \cong (\mathbb{Z}/2)^2$ hold, and provided $g \geq 4$ the isomorphism $(\mathcal{B}_3)_{GL_g(\mathbb{Z})} \cong (\mathcal{T}) = \mathbb{Z}/2$ also holds.

Proof. Since the action of $GL_g(\mathbb{Z}) \subset Sp_{2g}(\mathbb{Z})$ on the Boolean algebra $\mathcal{B}_3$ does not increase the degree, we will compute coinvariants bottom up. Denote by $E_{ij}$ the matrix of size $g \times g$ with 1 at position $(i, j)$ and zero elsewhere. Keep in mind that the symmetric group $S_g \subset GL_g(\mathbb{Z})$ acts on $H$ by permuting the indices of the generating set $\{a_i, b_i\}_{1 \leq i \leq g}$.

In degree 0 we only have one element $\mathcal{T}$, and it is clearly invariant.

In degree 1 the coinvariant module is generated by the elements $\mathcal{M}_1, \mathcal{B}_1$.

Apply the element $G = Id_g + E_{21}$, to get:

$$G \cdot \mathcal{M}_1 = \mathcal{M}_1 + \mathcal{B}_1.$$

So in the coinvariants quotient $\mathcal{M}_2 = 0$, but this is the class of $\mathcal{M}_1$.

The same computation shows that in the quotient $\mathcal{B}_1 = 0$.

In degree 2 the coinvariants module is generated by the products $\mathcal{M}_2, \mathcal{B}_2, \mathcal{M}_1 \mathcal{B}_1, \mathcal{B}_1 \mathcal{B}_2$.

We proceed now as before.

- Acting by $G = Id_g + E_{32}$ on $\mathcal{M}_2$ gives:

$$G \cdot \mathcal{M}_2 = \mathcal{M}_2 + \mathcal{B}_2 + \mathcal{M}_1 \mathcal{B}_1.$$

So in the coinvariants quotient $\mathcal{M}_3 = 0$, but this is the class of $\mathcal{M}_2$.

The same holds true for $\mathcal{B}_1 \mathcal{B}_2$.

- Acting by $G = Id_g + E_{21}$ on $\mathcal{B}_2$ gives:

$$G \cdot \mathcal{B}_2 = (\mathcal{M}_2 + \mathcal{B}_2) \mathcal{B}_1 = -\mathcal{M}_2 \mathcal{B}_1 - \mathcal{B}_2 \mathcal{B}_1.$$

Since in the coinvariants quotient $\mathcal{B}_1 \mathcal{B}_1 = 0$, we get that $\mathcal{B}_2 \mathcal{B}_1 = 0$ in the coinvariants quotient. But this is the class of $\mathcal{B}_2 \mathcal{B}_1$.

Thus we are left with $\mathcal{M}_1 \mathcal{B}_1$. We show that this element is not zero in the coinvariants module: Consider the $GL_g(\mathbb{Z})$-invariant homomorphism $p : \mathcal{B}_2 \to \mathcal{B}_2 / \mathcal{B}_1 \xrightarrow{\cong} \mathbb{Z}/2$, where $\mathcal{B}$ sends an element $\mathcal{M}$ of $\mathcal{B}_2 / \mathcal{B}_1$ to $\omega(x, y)$ mod 2. Applying this homomorphism to $\mathcal{M}_1 \mathcal{B}_1$ we get that $p(\mathcal{M}_1 \mathcal{B}_1) = 1 \neq 0$ and therefore $\mathcal{M}_1 \mathcal{B}_1 \neq 0$.

All this shows that $(\mathcal{B}_2)_{GL_g(\mathbb{Z})} \cong (\mathcal{T}, \mathcal{M}_1) \cong (\mathbb{Z}/2)^2$.

In degree 3 the coinvariants module is generated by the products $\mathcal{M}_3, \mathcal{B}_3, \mathcal{M}_1 \mathcal{M}_2, \mathcal{M}_1 \mathcal{B}_2, \mathcal{M}_2 \mathcal{B}_1, \mathcal{M}_2 \mathcal{B}_3, \mathcal{B}_1 \mathcal{B}_2, \mathcal{B}_1 \mathcal{B}_3, \mathcal{B}_2 \mathcal{B}_3$.

Here is where we need $g \geq 4$. 

• Acting by $G = \text{Id}_g + E_{41}$ on $\theta_1 \bar{\theta}_2 \bar{\theta}_3$ gives

$$G \cdot \theta_1 \bar{\theta}_2 \bar{\theta}_3 = (a_1 + a_3) \bar{a}_2 \bar{a}_3 = \theta_1 \bar{a}_2 \bar{a}_3 + a_3 \bar{a}_2 \bar{a}_3.$$ 

So in the coinvariants quotient $\theta_4 \bar{\theta}_2 \bar{\theta}_3 = 0 = \theta_1 \bar{a}_2 \bar{a}_3$. The action by the same element shows in the same way that in the coinvariants quotient $\theta_1 \bar{\theta}_2 \bar{b}_3 = 0$, $\bar{\theta}_1 \bar{\theta}_2 \bar{b}_3 = 0$, and similarly $\theta_1 \bar{b}_1 \bar{b}_2 = 0 = \theta_1 \bar{b}_2 \bar{b}_3 = \bar{b}_1 \bar{b}_2 \bar{b}_3$.

• Acting by $G = \text{Id}_g + E_{21}$ on $\theta_1 \theta_2 \bar{\theta}_2$, whose transpose-inverse is $\text{Id}_g - E_{12}$, we have:

$$G \cdot \theta_1 \theta_2 \bar{\theta}_2 = (a_1 + a_2) \theta_2 (\bar{b}_2 - b_1) = \theta_1 \theta_2 \bar{\theta}_2 + \theta_1 \bar{\theta}_2 \theta_1 + \theta_2 \bar{\theta}_2 - \theta_2 \bar{\theta}_2.$$ 

So in the coinvariants quotient $-\theta_1 \theta_2 \bar{\theta}_1 + \theta_2 \bar{\theta}_2 - \theta_2 \bar{\theta}_2 = 0$, but we already know that in the quotient $\theta_1 \theta_2 \bar{\theta}_1 = 0 = \theta_2 \bar{\theta}_2$, so finally $\theta_2 \bar{\theta}_2 = 0 = \theta_1 \bar{\theta}_1$.

All this shows that $(\mathcal{B}_3)_{GL_2(\mathbb{Z})}$ is $\mathbb{Z}/2$, generated by $\bar{T}$.

\[\square\]

2.3. Handlebodies. Let $B_1, \ldots, B_n$ be a collection of closed 3-balls and $D_1, \ldots, D_m$, $D'_1, \ldots, D'_m$ be a collection of pairwise disjoint disks in $\bigcup \partial B_i$. For each $1 \leq i \leq m$, consider a homeomorphism $\phi_i : D_i \to D'_i$. Denote $H$ the result of gluing along $\phi_1$, then gluing along $\phi_2$, and so on. We say that $H$ is a handlebody if after the final gluing if $H$ is connected. Equivalently, we say that a 3-manifold with boundary $H$ is a handlebody if there exists a collection $\{D_1, \ldots, D_m\}$ of properly embedded essential disks (i.e. properly embedded disk whose boundary does not bound a disk in the boundary of $H$) such that the complement of a regular neighbourhood of $\bigcup D_i$ is a collection of balls. Such family of disks is called a system of disks of $H$ and we say that this system of disks is minimal if its complement is connected. The existence of such family of disks is ensured by [6, Lemma 2.2].

We now state a basic 3-dimensional topology lemma that is a consequence of the loop theorem (cf. [3, Theorem 3.1]) and of which we will omit the proof.

Lemma 2.2. Let $D_\beta$, $D_{\beta'}$ be two essential proper embedded disks in $\mathcal{H}_g$ with respective boundaries $\beta$, $\beta'$ such that the union of $\beta$ and $\beta'$ bounds a subsurface of $\Sigma_{g,1}$ with two boundary components. There exist $g - 1$ essential proper embedded disks $D_{\beta_1}, \ldots, D_{\beta_{g-1}}$ in $\mathcal{H}_g$, with boundaries $\beta_1, \ldots, \beta_{g-1}$ respectively, such that

$$\text{Int}(\mathcal{H}_g) - N \left( D_\beta \cup D_{\beta'} \cup D_{\beta_1} \cup \ldots \cup D_{\beta_{g-1}} \right)$$

is the disjoint union of two open 3-balls.

3. The Luft Group and CBP-twists

Denote by $\mathcal{L}_{g,1}$ the kernel of the map $B_{g,1} \to \text{Aut}(\pi_1(\mathcal{H}_g))$, the Luft group, which was identified by Luft in [17] as the “twist group” of the handlebody $\mathcal{H}_g$, and by $\mathcal{LTB}_{g,1}$ the intersection $\mathcal{L}_{g,1} \cap TB_{g,1}$, the Luft-Torelli group. In [10], Pitsch characterized $\mathcal{LTB}_{g,1}$ as the group generated by contractible bounding pair twists (abbreviated CBP-twists). A CBP-twist is a map of the form $T_\beta T_{\beta'}^{-1}$, where $\beta$ and $\beta'$ are two homologous non-isotopic and disjoint simple closed curves on $\Sigma_{g,1}$ such that each one is not null-homologous in $H_1(\Sigma_{g,1}; \mathbb{Z})$ and bounds a properly embedded disk in $\mathcal{H}_g$. In all what follows we refer as
CBP-twists of genus $k$ to those CBP-twists $T_{\beta}T_{\beta'}^{-1}$ such that the union of $\beta$ and $\beta'$ bounds a subsurface of $\Sigma_{g,1}$ of genus $k$ with two boundary components.

Consider the short exact sequence given in [10, Lemma 1],

$$1 \longrightarrow TB_{g,1} \longrightarrow B_{g,1} \xrightarrow{\Psi} GL_g(\mathbb{Z}) \ltimes S_g(\mathbb{Z}) \longrightarrow 1.$$  

Restricting this short exact sequence to the Luft group $\mathcal{L}_{g,1}$, we get that

**Proposition 3.1.** There is a short exact sequence

$$1 \longrightarrow LTB_{g,1} \longrightarrow \mathcal{L}_{g,1} \xrightarrow{\Psi} S_g(\mathbb{Z}) \longrightarrow 1. \quad (3.1)$$

**Proof.** Notice that $\mathcal{L}_{g,1}$ is contained in the kernel of $B_{g,1} \rightarrow \text{Aut}(H_1(\mathcal{H}_g)) = GL_g(\mathbb{Z})$. Then $\Psi(\mathcal{L}_{g,1}) \subset S_g(\mathbb{Z})$. Now we prove that $\Psi : \mathcal{L}_{g,1} \rightarrow S_g(\mathbb{Z})$ is surjective. Recall that $S_g(\mathbb{Z})$ is generated by the family of matrices $\{E_{ii} \mid 1 \leq i \leq g\} \cup \{SE_{ij} \mid 1 \leq i < j \leq g\}$, where $E_{ij}$ denotes the matrix with 1 at the position $(i, j)$ and 0’s elsewhere, and $SE_{ij} = E_{ij} + E_{ji}$ for $i \neq j$. Thus it is enough to find a preimage for each $E_{ii}$ and $SE_{ij}$. Consider $\beta_i$, $\beta_j$ and $\gamma_{ij}$ the curves given in the following figure:

![Figure 3. Curves involved in the lift of $SE_{ij}$ to $\mathcal{L}_{g,1}$.](image)

Notice that the curves $\beta_k$, $\gamma_{ij}$ are contractible in $\mathcal{H}_g$. Then $T_{\beta_k}, T_{\beta_i}, T_{\gamma_{ij}}, T_{\beta_j}^{-1}$ are elements of the Luft group $\mathcal{L}_{g,1}$ with the following images through the symplectic representation:

$$\Psi(T_{\beta_k}^{-1}) = \begin{pmatrix} Id & 0 \\ E_{kk} & Id \end{pmatrix}, \quad \Psi(T_{\beta_i}^{-1}T_{\gamma_{ij}}T_{\beta_j}^{-1}) = \begin{pmatrix} Id & 0 \\ SE_{ij} & Id \end{pmatrix}. \quad \Box$$

**Proposition 3.2.** For every $h \in B_{g,1}$, there exist elements $l \in \mathcal{L}_{g,1}$, $f \in \mathcal{A}B_{g,1}$ and $\xi_b \in TB_{g,1}$ such that $h = \xi_bfl$, i.e.

$$B_{g,1} = TB_{g,1} \cdot \mathcal{A}B_{g,1} \cdot \mathcal{L}_{g,1}. \quad (3.2)$$

**Proof.** Consider the short exact sequence

$$1 \longrightarrow TB_{g,1} \longrightarrow B_{g,1} \xrightarrow{\Psi} GL_g(\mathbb{Z}) \ltimes S_g(\mathbb{Z}) \longrightarrow 1. \quad (3.2)$$

By [10, Lemma 3] and Proposition 3.1 we know that $\Psi(\mathcal{A}B_{g,1}) \cong GL_g(\mathbb{Z})$ and $\Psi(\mathcal{L}_{g,1}) \cong S_g(\mathbb{Z})$. Then, given $h \in B_{g,1}$ there exist $f \in \mathcal{A}B_{g,1}$, $l \in \mathcal{L}_{g,1}$ such that $\Psi(h) = \Psi(fl)$, and by the short exact sequence (3.2), there exists an element $\xi_b \in TB_{g,1}$ such that $h = \xi_bfl$. 

Proposition 3.3. The group $B_{g,1}$ acts transitively on CBP-twists of a given genus.

Proof. Let $T_{\beta}T_{\beta'}^{-1}$ be CBP-twists of genus $k$ on $\Sigma_{g,1}$. We prove that there exists $\psi \in B_{g,1}$ such that $\psi(\beta) = \zeta$, $\psi(\beta') = \zeta'$ getting that

$$\psi(T_{\beta}T_{\beta'}^{-1})\psi^{-1} = T_{\psi(\beta)}T_{\psi(\beta')}^{-1} = T_{\zeta}T_{\zeta'}^{-1}.$$ 

Since $T_{\beta}T_{\beta'}^{-1}$, $T_{\zeta}T_{\zeta'}^{-1}$ are CBP-twists of genus $k$, there exist properly embedded disks $D_\beta, D_{\beta'}, D_\zeta, D_{\zeta'}$ in $\mathcal{H}_g$ with respective boundaries $\beta, \beta', \zeta, \zeta'$. Then, by Lemma 2.2 there exist $g - 1$ essential proper embedded disks $D_{\beta_1}, \ldots, D_{\beta_{g-1}}$ (resp. $D_{\zeta_1}, \ldots, D_{\zeta_{g-1}}$) in $\mathcal{H}_g$, with boundaries $\beta_1, \ldots, \beta_{g-1}$ (resp. $\zeta_1, \ldots, \zeta_{g-1}$), such that

$$\text{Int}(\mathcal{H}_g) - N (D_\beta \cup D_{\beta'} \cup D_{\beta_1} \cup \ldots \cup D_{\beta_{g-1}})$$

(resp. $\text{Int}(\mathcal{H}_g) - N (D_\zeta \cup D_{\zeta'} \cup D_{\zeta_1} \cup \ldots \cup D_{\zeta_{g-1}})$)

is the disjoint union of two open 3-balls.

Since $T_{\beta}T_{\beta'}^{-1}$, $T_{\zeta}T_{\zeta'}^{-1}$ are BP-maps of the same genus, by the change of coordinates principle (cf. 2.1 Section 1.3), there is a homeomorphism $\phi$ from $\Sigma_{g,1}$ to $\Sigma_{g,1}$ sending $\{\beta, \beta', \beta_1, \ldots, \beta_{g-1}\}$ to $\{\zeta, \zeta', \zeta_1, \ldots, \zeta_{g-1}\}$ respectively. By 6.1 Lemma 2.9, $\phi$ extends to a homeomorphism $\psi$ on $\mathcal{H}_g$ and therefore $\phi \in B_{g,1}$. □

Proposition 3.4. Every CBP-twist of genus $k$ is a product of $k$ CBP-twists of genus 1.

Proof. Let $T_{\beta}T_{\beta'}^{-1}$ be a CBP-twist of genus $k$. Consider the following simple closed curves in the standarly embedded surface $\Sigma_{g,1}$:

![Figure 4](image-url)

Observe that $T_{\zeta}T_{\zeta'}^{-1}$ is a CBP-twist of genus $k$ and that for $i = 0, \ldots, k - 1$, the maps $T_{\zeta_i}T_{\zeta_{i+1}}^{-1}$ are CBP-twists of genus 1, where $\zeta_0 = \zeta$, $\zeta_k = \zeta'$. By Proposition 3.3 there exists an element $h \in B_{g,1}$ such that $T_{\beta}T_{\beta'}^{-1} = hT_{\zeta}T_{\zeta'}^{-1}h^{-1}$. Therefore,

$$T_{\beta}T_{\beta'}^{-1} = hT_{\zeta}T_{\zeta'}^{-1}h^{-1} = (hT_{\zeta_0}T_{\zeta_1}^{-1}h^{-1})(hT_{\zeta_1}T_{\zeta_2}^{-1}h^{-1}) \cdots (hT_{\zeta_{k-1}}T_{\zeta_k}^{-1}h^{-1}) = (T_{h(\zeta_0)}T_{h(\zeta_1)}^{-1})(T_{h(\zeta_1)}T_{h(\zeta_2)}^{-1}) \cdots (T_{h(\zeta_{k-1})}T_{h(\zeta_k)}^{-1}).$$
Since $T_{G_i}T_{G_i+1}^{-1}$ is a CBP-twists of genus 1 for $i = 0, \ldots, k - 1$, and $h \in B_{g,1}$, the element $T_{h(G_i)}T_{h(G_i+1)}^{-1}$ is also a CBP-twists of genus 1 for $i = 0, \ldots, k - 1$. □

Remark 3.1. A posteriori we found that Proposition 3.4 was obtained independently by Omori in [9].

4. INVARIANTS OF INTEGRAL HOMOLOGY 3-Spheres

Throughout this section we set $A$ an abelian group and $A_2$ the subgroup of 2-torsion elements of $A$. Consider an $A$-valued invariant of homology 3-spheres $F : S^3 \to A$. By [10] Theorem 1] there is a bijection

$$\lim_{g \to \infty} (T\mathcal{A}_{g,1} \setminus T\mathcal{B}_{g,1})_{AB_{g,1}} \sim S^3. \quad (4.1)$$

Precomposing an invariant $F$ with the canonical maps $T_{g,1} \to \lim_{g \to \infty} T_{g,1} \sim S^3$ we get a family of maps $\{F_g\}_g$ with $F_g : T_{g,1} \to A$ satisfying the following properties:

i) $F_{g+1}(x) = F_g(x)$ for every $x \in T_{g,1}$,
ii) $F_g(\xi_a x \xi_b) = F_g(x)$ for every $x \in T_{g,1}$, $\xi_a \in T\mathcal{A}_{g,1}$, $\xi_b \in T\mathcal{B}_{g,1}$,
iii) $F_g(\phi x \phi^{-1}) = F_g(x)$ for every $x \in T_{g,1}$, $\phi \in AB_{g,1}$.

Because of property i), without loss of generality we can assume $g \geq 4$, this avoids having to deal with some peculiarities in the homology of low genus mapping class groups.

Using this framework we prove that modulo a multiplicative constant $x \in A_2$ there is only one family of homomorphisms $\{F_g\}_g$ satisfying the aforementioned properties and that this family reassembles to the Rohlin invariant. For such purpose we have to understand the following three groups:

$$H_1(T_{g,1}; Z)_{AB_{g,1}}, \quad H_1(T\mathcal{A}_{g,1}; Z)_{AB_{g,1}}, \quad H_1(T\mathcal{B}_{g,1}; Z)_{AB_{g,1}}.$$

Proposition 4.1. For a given integer $g \geq 4$, the Birman Craggs Johnson homomorphism $\sigma : T_{g,1} \to \mathcal{B}_3$ composed with the projection $\mathcal{B}_3 \to \mathcal{B}_0$ induces an isomorphism

$$H_1(T_{g,1}; Z)_{AB_{g,1}} \cong \mathbb{Z}/2.$$

Proof. By the fundamental result of Johnson [5], we have an extension:

$$0 \longrightarrow \mathcal{B}_2 \longrightarrow H_1(T_{g,1}; Z) \longrightarrow \Lambda^3 H \longrightarrow 0.$$

All the maps appeared in the short exact sequence above are $\mathcal{M}_{g,1}$-equivariant, where the action of $\mathcal{M}_{g,1}$ on the above three groups is through the symplectic action on $H$. In particular, taking $AB_{g,1}$-coinvariants we get an exact sequence:

$$(\mathcal{B}_2)_{GL_g(Z)} \longrightarrow H_1(T_{g,1}; Z)_{GL_g(Z)} \longrightarrow (\Lambda^3 H)_{GL_g(Z)} \longrightarrow 0. \quad (4.2)$$

First, observe that $-Id \in GL_g(Z)$ acts by multiplication by $-1$ on $H$ and hence on $\Lambda^3 H$. Therefore, for any $w \in (\Lambda^3 H)_{GL_g(Z)}$, $-w = w$ and hence there are isomorphisms

$$(\Lambda^3 H)_{GL_g(Z)} \cong (\Lambda^3 H)_{GL_g(Z)} \cong (\mathcal{B}_3/\mathcal{B}_2)_{GL_g(Z)}.$$
By Lemma 2.1 this last group is zero, and by the exact sequence (4.2) the $\mathbb{Z}/2$-vector space $(\mathfrak{B}_2)_{GL_g}(\mathbb{Z})$ surjects onto $H_1(\mathcal{T}_g,\mathbb{Z})_{GL_g}(\mathbb{Z})$ getting that all elements of this last group have $2$-torsion. Therefore by [3, Theorem 1] the BCJ homomorphism $\sigma : \mathcal{T}_g,\mathbb{Z}$ induces an isomorphism $H_1(\mathcal{T}_g,\mathbb{Z})_{GL_g}(\mathbb{Z}) \cong (\mathfrak{B}_3)_{GL_g}(\mathbb{Z})$ and we conclude by Lemma 2.1.

**Lemma 4.1.** Provided $g \geq 4$, the groups $H_1(\mathcal{T}B_g,\mathbb{Z})_{AB_g}$ and $H_1(\mathcal{T}A_g,\mathbb{Z})_{AB_g}$ are zero.

**Proof.** We only give the proof for $\mathcal{T}B_g$, the other case is similar.

Denote by $IA$ the kernel of the natural map $Aut(\pi_1(\mathcal{H}_g)) \to GL_g(\mathbb{Z})$. Consider the following short exact sequence:

$$1 \longrightarrow \mathcal{LT}B_g \longrightarrow \mathcal{T}B_g \longrightarrow IA \longrightarrow 1. \quad (4.3)$$

Taking $AB_g$-coinvariants on the associated 5-term exact sequence, we get another exact sequence

$$H_1(\mathcal{LT}B_g,\mathbb{Z})_{AB_g} \longrightarrow H_1(\mathcal{T}B_g,\mathbb{Z})_{AB_g} \longrightarrow H_1(IA,\mathbb{Z})_{AB_g} \longrightarrow 0,$$

and we conclude by Lemma 4.2 and Lemma 4.3.

**Lemma 4.2.** For a given integer $g \geq 3$, the group $(H_1(IA,\mathbb{Z}))_{AB_g}$ is zero.

**Proof.** By [7, Corollary 2.1] the action of $B_g$ on the fundamental group of the inner handlebody $\mathcal{H}_g$ induces a surjective map $B_g \to Aut(\pi_1(\mathcal{H}_g))$. Indeed, the restriction of this map to $AB_g$ also gives a surjective map $AB_g \to Aut(\pi_1(\mathcal{H}_g))$ (cf. the paragraph after Lemma 2 in [10]). Therefore we have an isomorphism

$$(H_1(IA,\mathbb{Z}))_{AB_g} \cong (H_1(IA,\mathbb{Z}))_{Aut(\pi_1(\mathcal{H}_g))}.$$

According to Magnus [8], for the given generators $\alpha_1, \cdots, \alpha_g$ of $\pi_1(\mathcal{H}_g)$, the group $IA$ is normally generated as a subgroup of $Aut(\pi_1(\mathcal{H}_g))$ by the automorphism $K_{12}$ given by $K_{12}(\alpha_1) = \alpha_2 \alpha_1 \alpha_2^{-1}$ and $K_{12}(\alpha_i) = \alpha_i$ for $i \geq 2$. Then it is enough to show that $K_{12}$ is equivalent to zero.

Consider $f \in Aut(\pi_1(\mathcal{H}_g))$ given by $f(\alpha_3) = \alpha_3 \alpha_2$ and $f(\alpha_i) = \alpha_i$ for $i \neq 3$, with inverse $f^{-1} \in Aut(\pi_1(\mathcal{H}_g))$ given by $f^{-1}(\alpha_3) = \alpha_3 \alpha_2^{-1}$ and $f^{-1}(\alpha_i) = \alpha_i$ for $i \neq 3$, and take the element $K_{13} \in IA$ given by $K_{13}(\alpha_1) = \alpha_3 \alpha_1 \alpha_3^{-1}$ and $K_{13}(\alpha_i) = \alpha_i$ for $i \geq 2$. Observe that

$$fK_{13}f^{-1}(\alpha_1) = \alpha_3 \alpha_2 \alpha_1 \alpha_2^{-1} \alpha_3^{-1} \quad \text{and} \quad fK_{13}f^{-1}(\alpha_i) = \alpha_i \quad \text{for } i \geq 2.$$

Consequently, $fK_{13}f^{-1} = K_{12}K_{13}$ and the following equation holds

$$K_{13} = fK_{13}f^{-1} = K_{12}K_{13} = K_{12} + K_{13}.$$

Therefore $K_{12}$ is equivalent to zero.

**Lemma 4.3.** For a given integer $g \geq 4$, the group $(H_1(\mathcal{LT}B_g,\mathbb{Z}))_{\mathcal{LT}B_g,\mathbb{Z}}$ is zero.

**Proof.** By Proposition 3.3 the group $\mathcal{LT}B_g$ is generated by CBP-twists of genus 1. Then it is enough to show that all CBP-twists of genus 1 are equivalent to zero. Next we
divide the proof in two steps. In the first step we show that all CBP-twists of genus 1 are equivalent, and in the second step we show that all CBP-twists of genus 1 are equivalent to zero.

**Step 1.** We show that all CBP-twists of genus 1 are equivalent. Consider the CBP-twist $T_\beta T_{\beta'}^{-1} \in LT B_{g,1}$ of genus 1 depicted in the following figure:

![Figure 5](image1.png)

**Figure 5.** A contractible bounding pair of genus 1 in $\Sigma_{g,1}$.

By Proposition 3.3, for every CBP-twist of genus 1, $T_\nu T_{\nu'}^{-1}$, on $\Sigma_{g,1}$ there exists an element $h \in B_{g,1}$ such that $T_\nu T_{\nu'}^{-1} = h T_\beta T_{\beta'}^{-1} h^{-1}$, and by Proposition 3.2, there exist elements $l \in L_{g,1}$, $f \in AB_{g,1}$ and $\xi_b \in T B_{g,1}$ such that $h = \xi_b f l$.

Therefore in the coinvariant module we get that

$$T_\nu T_{\nu'}^{-1} = h T_\beta T_{\beta'}^{-1} h^{-1} = \xi_b f l T_\beta T_{\beta'}^{-1} l^{-1} f^{-1} \xi_b^{-1} = l T_\beta T_{\beta'}^{-1} l^{-1}. \quad (4.4)$$

Take the set-theoretic cross-section $s : S_g(\mathbb{Z}) \to L_{g,1}$ of $\Psi : L_{g,1} \to S_g(\mathbb{Z})$, i.e. a function $s : S_g(\mathbb{Z}) \to L_{g,1}$ such that $\Psi \circ s = id$ not necessarily being an isomorphism, given by

$$s(E_{ij}) = T_{\beta_i}^{-1}, \quad s(SE_{ij}) = \begin{cases} T_{\gamma_i}^{-1} T_{\gamma_i'}^{-1} & \text{for } i = 1 \\ T_{\beta_i}^{-1} T_{\gamma_i}^{-1} T_{\beta_j}^{-1} & \text{for } i \geq 2, \end{cases}$$

where the curves $\beta_i$, $\gamma_{ij}$, $\gamma_{ij}'$ are given in the following picture:

![Figure 6](image2.png)

**Figure 6.** Curves involved in the set-theoretic cross-section $s$.

By the short exact sequence (3.1), given an element $l \in L_{g,1}$, there exists an element $\xi_b \in LT B_{g,1}$ such that $l = \xi_b s(\Psi(l))$. Then, by Eq. (4.4), in the coinvariant module, we
have that
\[ T_\nu T_{\nu'}^{-1} = lT_\beta T_{\beta'}^{-1}l^{-1} = \xi_b s(\Psi(l))T_\beta T_{\beta'}^{-1} s(\Psi(l))^{-1} \xi_b^{-1} \]
\[ = s(\Psi(l))T_\beta T_{\beta'}^{-1} s(\Psi(l))^{-1}. \quad (4.5) \]
Now observe that \( s(\Psi(l)) \) is a product of the following elements:
\[ \{ T_\nu \mid i \geq 2 \}, \quad \{ T_{\nu'} \mid 2 \leq i < j \}, \quad \{ T_{\beta} \mid 1 \leq i \leq g \}. \]
Since the curves \( \gamma_{12}, \\{ \gamma'_{ij} \mid 2 \leq i < j \}, \\{ \beta_i \mid 1 \leq i \leq g \} \)
are disjoint with the curves \( \beta, \beta', \{ \gamma_{1j} \mid j \geq 3 \} \), the geometric intersection number between a curve of the family (4.6) and a curve of the family (4.7) is zero. Therefore, the elements of the family of Dehn twists \( T_\gamma \), \( T_{\gamma'} \), \( \{ T_{\beta} \mid 1 \leq i \leq g \} \)
commutes with the elements of the family of Dehn twists \( T_\beta, T_{\beta'}, \{ T_{\gamma_{1j}} \mid j \geq 3 \} \).
Furthermore, the elements of the family \( \{ T_{\gamma_{1j}} \mid j \geq 3 \} \) commute between them because the curves of the family \( \{ \gamma_{1j} \mid j \geq 3 \} \) are pairwise disjoint. Therefore,
\[ s(\Psi(l))T_\beta T_{\beta'}^{-1} s(\Psi(l))^{-1} = (T_{\gamma_{13}} \cdots T_{\gamma_{1g}})T_\beta T_{\beta'}^{-1} (T_{\gamma_{13}} \cdots T_{\gamma_{1g}})^{-1}, \quad (4.10) \]
for some \( x_3, \ldots, x_g \in \mathbb{Z} \). And as a consequence of the Eqs. (4.5) and (4.10) we get the following equation:
\[ T_\nu T_{\nu'}^{-1} = (T_{\gamma_{13}} \cdots T_{\gamma_{1g}})T_\beta T_{\beta'}^{-1} (T_{\gamma_{13}} \cdots T_{\gamma_{1g}})^{-1} = T_\beta T_{\beta'}^{-1} (T_{\gamma_{13}} \cdots T_{\gamma_{1g}})(\beta'). \quad (4.11) \]
Next, we prove that in the coinvariant module,
\[ T_{\beta} T_{\beta'}^{-1} = T_\beta T_{\beta'}^{-1} + \sum_{i=3}^{g} x_i (T_\beta T_{\beta'}^{-1} - T_\beta T_{\beta'}^{-1}(\beta')). \]
Consider the curves \( \{ \gamma_{1j}, \gamma'_{1j} \mid 3 \leq j \geq g \} \) given in the following picture:

**Figure 7.** Curves involved in the definition of elements of \( AB_{g,1} \).
Fix an integer $j$ with $3 \leq j \leq g$. Consider $\beta''_j = (T^{x(j+1)}_{\gamma_{13}} \cdots T^{x_g}_{\gamma_{13}}) (\beta')$ for $3 \leq j \leq g - 1$ and $\beta''_g = \beta'$. For $k \geq 1$ we have that

$$T_{\beta'} T^{-1}_{\gamma_{13}} (\beta''_j) = T_{\beta'} T^{-1}_{\gamma_{13}} + T_{\beta'} T^{-1}_{\gamma_{13}} (\beta') + T_{\beta'} T^{-1}_{\gamma_{13}} (\beta''_j).$$

Since $T_{\gamma_{13}} T^{-1}_{\gamma_{13}} \in \mathcal{A} \mathcal{B}_{g,1}$ for $3 \leq j \leq g$, conjugating by $T_{\gamma_{13}} T^{-1}_{\gamma_{13}}$ the last two terms of the above equation, in the coinvariant module, we get that

$$T_{\beta'} T^{-1}_{\gamma_{13}} (\beta''_j) = T_{\beta'} T^{-1}_{\gamma_{13}} + T_{\beta'} T^{-1}_{\gamma_{13}} (\beta') + T_{\beta'} T^{-1}_{\gamma_{13}} (\beta''_j).$$

Applying Eq. (4.12) from $k = x_j$ to $k = 1$, we obtain that

$$T_{\beta'} T^{-1}_{\gamma_{13}} (\beta''_j) = x_j T_{\beta'} T^{-1}_{\gamma_{13}} (\beta') + T_{\beta'} T^{-1}_{\gamma_{13}} (\beta''_j) = x_j (T_{\beta'} T^{-1}_{\gamma_{13}} (\beta') + T_{\beta'} T^{-1}_{\gamma_{13}} (\beta'').$$

Applying recursively Eq. (4.13) from $j = 3$ to $j = g$, we get the following formula:

$$T_{\beta'} T^{-1}_{\gamma_{13}} (\beta''_j) = T_{\beta'} T^{-1}_{\gamma_{13}} + \sum_{i=3}^{g} x_i (T_{\beta'} T^{-1}_{\gamma_{13}} (\beta') + T_{\beta'} T^{-1}_{\gamma_{13}} (\beta'')).$$

In sequel we prove that for $3 \leq k \leq g$, in the coinvariant module,

$$T_{\beta'} T^{-1}_{\gamma_{13}} = T_{\beta'} T^{-1}_{\gamma_{13}} (\beta').$$

Consider the element $f_k \in \mathcal{A} \mathcal{B}_{g,1}$ given by the half twist of the shaded ball depicted in the following figure, that exchanges the holes 3 and $k$.

![Figure 8. Half twist $f_k$ exchanging holes 3 and $k$.](image)

Since $f_k$ leaves $\beta$, $\beta'$ invariant and sends $\gamma_{1k}$ to $\gamma_{13}$, for $3 \leq k \leq g$, in the coinvariant module, we have that

$$T_{\beta'} T^{-1}_{\gamma_{1k}} (\beta') = T_{\gamma_{1k}} T_{\beta'} T^{-1}_{\gamma_{1k}} = f_k T_{\gamma_{1k}} f^{-1}_k T_{\beta'} T^{-1}_{\gamma_{1k}} f^{-1}_k = T_{\gamma_{13}} T_{\beta'} T^{-1}_{\gamma_{13}} = T_{\beta'} T^{-1}_{\gamma_{13}} (\beta').$$

Therefore it is enough to show that in the coinvariant module,

$$T_{\beta'} T^{-1}_{\gamma_{13}} = T_{\beta'} T^{-1}_{\gamma_{13}} (\beta').$$
Since $\beta_1, \beta_3$ are disjoint with $\beta, \beta', \gamma_{13}$, we have that
\[ T_{\beta}T_{\gamma_{13}'}^{-1}(\beta) = T_{\gamma_{13}}T_{\beta}T_{\beta'}^{-1}T_{\gamma_{13}}^{-1} = (T_{\beta_1}^{-1}T_{\gamma_{13}}T_{\beta_3})T_{\beta}T_{\beta'}^{-1}(T_{\beta_1}^{-1}T_{\gamma_{13}}T_{\beta_3})^{-1}. \]

Now take $f \in AB_{g,1}$ given by $f = T_{\alpha_3}T_{\eta_{34}}T_{\beta_4}$, where $\alpha_3, \eta_{34}, \beta_4$ are the curves on $\Sigma_{g,1}$ given in the following picture:

![Diagram of curves](image)

**Figure 9.** Curves involved in the definition of $f \in AB_{g,1}$.

Since $\alpha_3, \eta_{34}, \beta_4$ do not intersect either of $\beta, \beta'$, the element $f$ commutes with $T_{\beta}T_{\beta'}^{-1}$ and in the coinvariant module we have that
\[ T_{\beta}T_{\gamma_{13}'}^{-1}(\beta) = (f(T_{\beta_1}^{-1}T_{\gamma_{13}}T_{\beta_3})f^{-1})T_{\beta}T_{\beta'}^{-1}(f(T_{\beta_1}^{-1}T_{\gamma_{13}}T_{\beta_3})f^{-1})^{-1}. \]  
(4.16)

Observe that
\[
\Psi(f(T_{\beta_1}^{-1}T_{\gamma_{13}}T_{\beta_3})f^{-1}) = \Psi(f)\Psi(T_{\beta_1}^{-1}T_{\gamma_{13}}T_{\beta_3})\Psi(f^{-1}),
\]
\[
\Psi(T_{\beta_1}^{-1}T_{\gamma_{13}}T_{\beta_3}) = \Psi(T_{\alpha_3}^{-1}T_{\eta_{34}}T_{\beta_4}).
\]
\[
\Psi(T_{\gamma_{13}}T_{\beta_3}) = \Psi(T_{\gamma_{13}}T_{\beta_3})^{-1} = \left( 0 \text{Id} \right).
\]
\[
\Psi(f) = \Psi(T_{\alpha_3}^{-1}T_{\eta_{34}}T_{\beta_4}) = \left( Id - E_{34} \quad 0 \right),
\]
\[
\Psi(f^{-1}) = \Psi(f)^{-1} = \left( Id + E_{34} \quad 0 \right).
\]

A direct computation shows that
\[
\Psi(f(T_{\beta_1}^{-1}T_{\gamma_{13}}T_{\beta_3})f^{-1}) = \Psi((T_{\beta_1}^{-1}T_{\gamma_{13}}T_{\beta_3})(T_{\beta_1}^{-1}T_{\gamma_{13}}T_{\beta_3})^{-1}).
\]

Then, by the short exact sequence (3.11), there is an element $\xi_b \in LTB_{g,1}$ such that
\[ f(T_{\beta_1}^{-1}T_{\gamma_{13}}T_{\beta_3})f^{-1} = \xi_b(T_{\beta_1}^{-1}T_{\gamma_{13}}T_{\beta_3})(T_{\beta_1}^{-1}T_{\gamma_{13}}T_{\beta_3})^{-1}. \]  
(4.17)

Since $T_{\beta_1}, T_{\beta_3}, T_{\beta_4}$ commute with $T_{\gamma_{13}}, T_{\gamma_{14}}, T_{\beta}, T_{\beta'}$, and $f$ commutes with $T_{\beta}, T_{\beta'}$, by the Eqs. (4.16) and (4.17), in the coinvariant module, we get that
\[ T_{\beta}T_{\gamma_{13}'}^{-1}(\beta) = (T_{\gamma_{13}}^{-1}T_{\gamma_{14}}T_{\beta}T_{\beta'}^{-1}(T_{\gamma_{13}}T_{\gamma_{14}})^{-1} = T_{\beta}T_{\gamma_{13}}T_{\gamma_{14}}(\beta'). \]  
(4.18)
Notice that we have the following equation:

\[ T_\beta T_{\gamma_{13}}^{-1} T_{\gamma_{14}} (\beta') = T_\beta T_{\beta'}^{-1} + T_{\beta'} T_{\gamma_{13}}^{-1} (\beta) + T_{\gamma_{13}}^{-1} (\beta) T_{\gamma_{13}}^{-1} T_{\gamma_{14}} (\beta'). \]

Conjugating the last two terms by \( T_{\gamma_{13}}^{-1} T_{\gamma_{14}} (\beta') \in AB_{g,1} \), in the coinvariant module,

\[ T_\beta T_{\gamma_{13}}^{-1} T_{\gamma_{14}} (\beta') = T_\beta T_{\beta'}^{-1} + T_{\gamma_{13}}^{-1} (\beta') T_\beta^{-1} + T_\beta T_{\gamma_{14}}^{-1} (\beta'), \]

and conjugating the last term by \( f_4 \in AB_{g,1} \), in the coinvariant module,

\[ T_\beta T_{\gamma_{13}}^{-1} T_{\gamma_{14}} (\beta') = T_\beta T_{\beta'}^{-1} - T_\beta T_{\gamma_{14}}^{-1} (\beta') + T_\beta T_{\gamma_{13}}^{-1} (\beta'). \quad (4.19) \]

Hence, by the Eqs. (4.18) and (4.19),

\[ T_\beta T_{\gamma_{13}}^{-1} T_{\gamma_{14}} (\beta') = T_\beta T_{\beta'}^{-1} - T_\beta T_{\gamma_{14}}^{-1} (\beta') + T_\beta T_{\gamma_{13}}^{-1} (\beta'). \]

Then in the coinvariant module \( T_\beta T_{\beta'}^{-1} = T_\beta T_{\gamma_{14}}^{-1} (\beta') \) and by the Eqs. (4.13), (4.14) and (4.11), \( T_\beta T_{\gamma_{14}}^{-1} (\beta') \) is equivalent to \( T_\beta T_{\beta'}^{-1} \), and hence all CBP-twists of genus one are equivalent.

**Step 2.** We show that all CBP-twists of genus 1 are equivalent to zero using Step 1 and the lantern relation.

First of all notice that all CBP-twists of genus one have 2-torsion since by Step 1, \( T_\nu T_{\nu'}^{-1} = T_{\nu'} T_{\nu'}^{-1} \). Then, any CBP-twists of genus two is equivalent to zero since by Proposition 3.4, any CBP-twists of genus two is a product of two CBP-twists of genus one.

Consider the following curves in the standardly embedded surface \( \Sigma_{g,1} \):

![Figure 10. Two lantern configurations embedded in \( \Sigma_{g,1} \).](image-url)

Observe that for \( i = 1, 2, 3 \), \( T_\zeta T_{\zeta_i}^{-1} \) are CBP-twists of genus 1, \( T_\xi T_{\xi_i}^{-1} \) are CBP-twists of genus 2. Consider the following lantern relations:
(T_{\xi_2^{-1}} T_{\xi_1^{-1}})(T_{\xi_3^{-1}} T_{\xi_2^{-1}}) = T_\gamma, \quad (T_{\xi_1^{-1}} T_{\xi_2^{-1}})(T_{\xi_3^{-1}} T_{\xi_2^{-1}}) = T_\gamma.

Figure 11. Lantern configurations with the induced lantern relations.

Putting these relations together we get that

\[(T_{\xi_1^{-1}} T_{\xi_2^{-1}}) = (T_{\xi_2^{-1}} T_{\xi_1^{-1}})(T_{\xi_3^{-1}} T_{\xi_2^{-1}}) = (T_{\xi_2^{-1}} T_{\xi_1^{-1}})(T_{\xi_3^{-1}} T_{\xi_2^{-1}})(T_{\xi_1^{-1}} T_{\xi_2^{-1}}).
\]

Since all CBP-twists of genus one are equivalent and all CBP-twists of genus two are equivalent to zero, \(T_{\xi_1^{-1}} T_{\xi_2^{-1}}\) is equivalent to zero, and we conclude by Step 1. \(\square\)

**Proof of Theorem 1.1**. Let \(A\) be an abelian group, recall that we denote by \(A_2\) the subgroup of 2-torsion elements of \(A\). For a given integer \(g \geq 4\), consider the BCJ-homomorphism \(\sigma : T_{g,1} \to \mathcal{B}_3\), the projection \(\pi_g : \mathcal{B}_3 \to \mathcal{B}_0 \cong \mathbb{Z}/2\) and the injection \(\varepsilon^x : \mathcal{B}_0 \to A_2\) defined by sending \(\overline{1}\) to \(x \in A_2\). By Proposition 4.1, composing the pull-back of the aforementioned homomorphisms we get the following sequence of isomorphisms:

\[
A_2 \xrightarrow{(\varepsilon^x)^*} \text{Hom}(\mathbb{Z}/2, A) \xrightarrow{\sigma^* \circ \pi^*} \text{Hom}(H_1(T_{g,1}; \mathbb{Z}), \mathcal{A}_{\mathcal{B}_g,1}, A) = \text{Hom}(T_{g,1}, A)^{\mathcal{A}_{\mathcal{B}_g,1}}.
\]

Therefore we get an isomorphism

\[
\lambda : A_2 \xrightarrow{\sim} \text{Hom}(T_{g,1}, A)^{\mathcal{A}_{\mathcal{B}_g,1}}
\]

\[
x \mapsto \mu^x_g = \varepsilon^x \circ \pi_g \circ \sigma.
\]

We show that the family of homomorphisms \(\{\mu^x_g\}_g\) reassemble into the Rohklin invariant. By the constructions of \(\sigma, \pi_g\) and \(\varepsilon^x\), these maps are compatible with the stabilization map and then the compositions of these maps \(\{\mu^x_g\}_g\) are also compatible with the stabilization map. By Lemma 1.1, the \(\mathcal{A}_{\mathcal{B}_g,1}\)-invariant homomorphisms \(\{\mu^x_g\}_g\) are zero on \(T \mathcal{A}_{g,1}, T \mathcal{B}_{g,1}\). Then, by the bijection 4.1, the family of homomorphism \(\{\mu^x_g\}_g\) reassemble into an invariant of integral homology 3-spheres. Besides, precomposing the Rohlin invariant \(R : S^3 \to \mathbb{Z}/2\) with the bijection 4.1 we get a family of homomorphisms \(\{R_g\}_g\) with \(R_g \in \text{Hom}(T_{g,1}; \mathbb{Z}/2)^{\mathcal{A}_{\mathcal{B}_g,1}}\). Since there is only one non-zero element in \(\text{Hom}(T_{g,1}; \mathbb{Z}/2)^{\mathcal{A}_{\mathcal{B}_g,1}}\), by Proposition 4.1, \(\pi_g \circ \sigma\) and \(R_g\) must coincide. Therefore, \(\mu^x_g\) and \(\varepsilon^x \circ R_g\) must coincide too.
5. Application

As we learnt from [10], for a given invariant of integral homology 3-spheres \( F : S^3 \rightarrow A \) there is an associated family of trivial 2-cocycles \( \{ C_g \} \) on the Torelli group which measure the failure of the maps \( \{ F_g \} \) to be homomorphisms of groups,

\[
C_g : T_{g,1} \times T_{g,1} \rightarrow A \\
(\phi, \psi) \mapsto F_g(\phi) + F_g(\psi) - F_g(\phi \psi).
\]

Since \( F \) is an invariant, this family of 2-cocycles inherits the following properties:

1. The 2-cocycles \( \{ C_g \} \) are compatible with the stabilization map,
2. The 2-cocycles \( \{ C_g \} \) are invariant under conjugation by elements in \( AB_{g,1} \),
3. If \( \phi \in TA_{g,1} \) or \( \psi \in TB_{g,1} \) then \( C_g(\phi, \psi) = 0 \).

In the sequel, we show that for a given family of trivial 2-cocycles satisfying the conditions (1)-(3) and with a \( AB_{g,1} \)-invariant trivialization, this family induces \( A_2 \)-valued invariants of integral homology 3-spheres.

First we show two other ways of expressing the condition about the existence of a \( AB_{g,1} \)-invariant trivialization. Consider \( Q_{C_g} \) the set of all trivializations of the 2-cocycle \( C_g \):

\[
Q_{C_g} = \{ q : T_{g,1} \rightarrow A \mid q(\phi) + q(\psi) - q(\phi \psi) = C_g(\phi, \psi) \}.
\]

Recall that any two trivializations of a given 2-cocycle differ by an element of \( Hom(T_{g,1}, A) \). As the cocycle \( C_g \) is invariant under conjugations by \( AB_{g,1} \), this group acts on \( Q_{C_g} \) via its conjugation action on the Torelli group and it confers the set \( Q_{C_g} \) the structure of an affine set over the abelian group \( Hom(T_{g,1}, A) \). Then the existence of an \( AB_{g,1} \)-invariant trivialization is equivalent to the existence of a fixed point for the action of \( AB_{g,1} \in Q_{C_g} \).

This condition can be also seen as a cohomological condition: choose an arbitrary element \( q \in Q_{C_g} \) and define a map as follows

\[
\rho_q : AB_{g,1} \rightarrow Hom(T_{g,1}, A) \\
\phi \mapsto \phi \cdot q - q.
\]

A direct computation shows that \( \rho_q \) is a derivation and the difference \( \rho_q - \rho_{q'} \) for two elements in \( Q_{C_g} \) is a principal derivation. Therefore there is a well-defined cohomology class

\[
\rho(C_g) \in H^1(AB_{g,1}; Hom(T_{g,1}, A))
\]

called the torsor of the cocycle \( C_g \).

**Proposition 5.1.** The natural action of \( AB_{g,1} \) on \( Q_{C_g} \) admits a fixed point if and only if the associated torsor \( \rho(C_g) \) is trivial.

Finally, we finish this section with the proof of the main theorem of this paper.
Proof of Theorem 1.2. Suppose that for every $g \geq 4$ there is a fixed point $q_g$ of $QC_g$ for the action of $AB_{g,1}$ on $QC_g$. Since every pair of $AB_{g,1}$-invariant trivializations differ by an $AB_{g,1}$-invariant homomorphism, by Theorem 1.1 the fixed points are $q_g + \mu^x_g$ with $x \in A_2$.

By construction, the family $\{\mu^x_g\}_g$ is compatible with the stabilization map. Then, given two different fixed points $q_g, q'_g$ of $QC_g$ for the action of $AB_{g,1}$, we have that the following equation holds:

$$q_g|_{T_{g-1,1}} - q'_g|_{T_{g-1,1}} = (q_g - q'_g)|_{T_{g-1,1}} = \mu^x_g|_{T_{g-1,1}} = \mu^x_{g-1}.$$ 

Therefore the restriction of the trivializations of $QC_g$ to $T_{g-1,1}$ give us a bijection between the fixed points of $QC_g$ for the action of $AB_{g,1}$ and the fixed points of $QC_{g-1}$ for the action of $AB_{g-1,1}$. Consequently, for a given $AB_{g,1}$-invariant trivialization $q_g$ and each $x \in A_2$, we get a well-defined map

$$q + \mu^x = \lim_{g \to \infty} q_g + \mu^x_g : \lim_{g \to \infty} T_{g,1} \to A.$$

These are the only candidates to be $A$-valued invariants of integral homology 3-spheres with associated family of 2-cocycles $\{C_g\}_g$. For these maps to be invariants, since they are already $AB_{g,1}$-invariant, we only have to prove that they are well-defined on the double cosets $T_{g,1}/T_{g-1,1}$.

From property (3) of our cocycle we have that $\forall \phi \in T_{g,1}$, $\forall \psi_a \in TA_{g,1}$ and $\forall \psi_b \in TB_{g,1}$,

$$(q_g + \mu^x_g)(\phi) - (q_g + \mu^x_g)(\phi\psi_b) = -(q_g + \mu^x_g)(\psi_b),$$

$$(q_g + \mu^x_g)(\phi) - (q_g + \mu^x_g)(\psi_a\phi) = -(q_g + \mu^x_g)(\psi_a).$$  

(5.1)

(5.2)

In particular, taking $\phi \in TB_{g,1}$ in (5.1) and $\phi \in TA_{g,1}$ in (5.2), we get that $q_g + \mu^x_g$ are $AB_{g,1}$-invariant homomorphisms on $TA_{g,1}$ and $TB_{g,1}$ and by Lemma 4.1 the maps $q_g + \mu^x_g$ are zero on these last two groups. Therefore the maps $q_g + \mu^x_g$ are well-defined on the double coset $TA_{g,1}/T_{g-1,1}/TB_{g,1}$.

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