Spin glass approach to the 2-distance minimal dominating set problem

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November 19, 2019

Abstract

The L-distance minimal dominating set (MDS) problem is widely applied in various types of dominating set problems. Recently, we studied the regular dominating set problem using the cavity method and developed two algorithms (belief propagation decimation (BPD) algorithm and survey propagation decimation (SPD) algorithm) to obtain the solution of a given graph, which provide a very good estimation of the minimal dominating size. Now, we have developed spin glass theory to study the 2-distance MDS problem. First, We found that the Belief Propagation equation does not converge when the inverse temperature is greater than a certain threshold value on the regular random network and ER random network. Second, the entropy density of the Replica Symmetry population dynamics has the transition point at the finite inverse temperature on the regular random graph when the node degree is from 3 to 9, and on the ER random network when the node degree is from 4.2 to 10.4; there is no entropy transition point (or $\beta = \infty$) in the other circumstance. Third, the results of the belief propagation algorithm were the same as those of replica symmetry theory, and the results of the BPD algorithm were better than those of the greedy heuristic algorithm.

Keywords: 2-distance minimal dominating set, belief propagation, ER random graph, regular random graph, belief propagation decimation.

I Introduction

Consider simple network W formed by N nodes and M undirected links, where each link connects two different nodes. There is one set $\gamma$. If any node of the network belongs to this set, or at least one neighbor or one quasi (2-distance)
neighbor node belongs to $\gamma$, then this set is called the 2-distance minimal dominating set (2DMDS) of the given network $W$.

Many new types of dominating set problem have been studied by mathematicians and computer scientists in recent years. Deepak Sehrawat et al. studied the double dominating set \[1,5\]. Sangram K. Jena, Ramesh K. Jallu, and Gautam K. Das studied the generalized liar’s dominating set \[2\] called the distance-$d$ $(m,l)$-liar’s dominating set, which is a subset $D \subseteq V$ such that (i) each vertex in $V$ is distance-$d$ dominated by at least $m$ vertices in $D$, and (ii) each pair of distinct vertices in $V$ is distance-$d$ dominated by at least $l$ vertices in $D$, where $m < l$. They proved that it is NP-complete. Khaled M. Alzoubi, Peng-Jun Wan, and Ophir Frieder studied weakly connected dominating sets \[3\], where if set $D$ is a weakly connected dominating set, then (i) $D$ is a minimal dominating set, and (ii) the nodes in $D$ are connected at least in 2-distance, and the authors provided two algorithms to solve this problem. Jie Wu et al. studied the extended dominating set \[4\] using cooperative communication, in which each node was covered by either one neighbor or several 2-distance neighbors. They proposed several heuristic algorithms to construct a minimal extended dominating set. Ioannis Lamprou et al. proposed budgeted connected domination and budgeted edge-vertex domination \[6\]. The budgeted connected dominating set has applications particularly in ad hoc wireless networks. The budgeted edge-vertex dominating set considers cases in which resources must be positioned on the links of a network to dominate network nodes. Tanveer Iqbal et al. studied the 2-distance paired domination of the flower graph \[7\], which is a 2-distance dominating set, and its induced subgraph has a perfect matching. Firouz Beggas et al. studied the $[1,2]$-dominating set \[8\], in which every vertex not in the $[1,2]$-dominating set has at least one and at most two neighbors in this set. $(\sigma, \rho)$ dominating set \[9\], convex dominating set \[10\], $k$-tuple dominating set \[11\], directed edge dominating set \[12\], and $k$-connected $m$-dominating sets \[13\]. The dominating set problem applied to more new scientific and technical field \[14–23\].

Although there are many types of dominating set, almost of them consist of three types of dominating set: regular dominating set, connected dominating set, and L-distance dominating set. There has been some work purely on the L-distance dominating set problem \[24,25\] and the connected dominating set problem \[26,27\]. The statistical physics of spin glass systems has been widely applied to optimization problems, such as the minimal vertex cover problem \[28,29\], the minimum feedback vertex set problem \[30,31\], and to satisfiability problems, such as K-SAT \[32–34\] and XOR-SAT \[35–36\], and the minimal dominating set (MDS) problem \[37–40\]. Recently, we have used statistical physics to study the regular minimal dominating set problem. We introduced belief propagation decimation (BPD), warning propagation, and survey propagation decimation algorithms to obtain the minimal dominating set. We found that our algorithms were very close to the optimal solution and the speed was very fast. The solution space has condensation transition and cluster transition on the undirected regular random (RR) graph, but it has only one transition on the undirected Erdos– Renyi (ER) random graph and directed both (ER random
and RR) graph. In this paper, we continue to use statistical physics to study the L-distance minimal dominating set (LDMDS). The ground-state energy appears when the entropy density is equal to zero on the undirected ER random graph, and the entropy always is positive when mean degree is not between 4.2 to 10.4. The ground-state energy still appears when the entropy is equal to zero on the undirected RR random graph, but the entropy density only has phase transition in the range of 3 to 9. We used three algorithms, population dynamics, BPD, and the greedy heuristic algorithm, to calculate the 2DMDS. We found that the population dynamics and BPD results were always better than those of the greedy heuristic algorithm on the single ER and RR random graphs.

This paper is organized as follows: In Section 2, we introduce replica symmetry (RS) theory for the 2DMDS problem and present the belief propagation (BP) equation and the corresponding thermodynamic quantities. In Section 3, we introduce the BPD algorithm and greedy algorithm for the 2DMDS problem, and derive the BP equation and marginal probability equation for the different vertex state conditions. We also construct the proper BPD process to estimate the 2DMDS. In Section 4, we draw conclusions and summarize our results.

II Replica symmetry

In this section, we introduce mean field theory for the 2DMDS problem. The energy function of the 2DMDS problem cannot be written in the standard manner, but we can write the partition function in the standard manner. Depending on the RS mean field theory of statistical physics, we can write partition function $Z$ as

$$Z = \sum_{\xi} \prod_{i \in W} e^{-\beta \delta_{c_i}^0} [1 - (1 - \delta_{c_i}^0) \prod_{j \in \partial_i} (1 - \delta_{c_j}^{c_i} - 1) - \Theta(\sum_{j \in \partial_i} (\delta_{c_i}^{c_j} + 2 \delta_{c_j}^{c_i} - 1) - 1)],$$

where $\xi \equiv (c_1, c_2, \ldots, c_n)$ denotes one of the possible configurations, $c_i = 0$ if node $i$ is occupied, $c_i = 1$ if node $i$ is not be occupied but at least one neighbor is occupied, and $c_i = 2$ if node $i$ is not occupied and no neighbor is occupied but at least one neighbor is in state $c_i = 1$, $\beta$ denotes the inverse temperature, and $\partial_i$ denotes the neighbor nodes of node $i$. The partition function therefore only takes into account all the 2DMDS.

RS mean field theories, such as the Bethe–Peierls approximation [41] and partition function expansion [42, 43], can solve the above spin glass model. These two theories obtain the same results, but the Bethe–Peierls approximation theory equation is easier to read; thus, we introduce the Bethe–Peierls approximation equation. We set cavity message $p_{i \rightarrow j}^{(c_i, c_j)}$ on each edge, and the message must satisfy
where 2-distance quasi neighbor. 

\[ p_{k \to i}^{(c_i, c_j)} - (1 - \delta^0_{c_i})(\delta^0_{c_j} + \delta^0_{c_j} + 1) \prod_{k \in \partial \setminus j \cap A} \sum p_{k \to i}^{(c_i, c_j)} \]

which is called the BP equation, where the Kronecker symbol \( \delta^m_n = 1 \) if \( m = n \) and \( \delta^m_n = 0 \) otherwise. Cavity message \( p_{i \to j}^{(c_i, c_j)} \) represents the joint probability that node \( i \) is in occupation state \( c_i \) and its adjacent node \( j \) is in occupation state \( c_j \) when the constraint of node \( j \) is not considered. If node \( i \) is in state \( c_i = 0 \), then it requests that the neighbor nodes only take state \( c_k = 0 \) or \( c_k = 1 \), and state \( c_k = 2 \) is forbidden. If node \( i \) is in state \( c_i = 1 \), then it requests that the neighbor nodes take any state \( c_k = 0 \), \( c_k = 1 \), or \( c_k = 2 \), but at least one neighbor must be occupied. If node \( i \) is in state \( c_i = 2 \), then it requests that the neighbor nodes only take state \( c_k = 1 \) or \( c_k = 2 \), but at least one 2-distance quasi neighbor must be occupied, and state \( c_k = 0 \) is forbidden. Set \( A \) represents the possible states of \( c_k \). Marginal probability \( p^i_{c} \) of node \( i \) is expressed as

\[
p^i_{c} = \frac{e^{-\beta \delta^0_{c_i}} \prod_{j \in \partial \setminus c_j \in A} \sum p_{j \to i}^{(c_j, c_i)} - (1 - \delta^0_{c_i})(\delta^0_{c_j} + \delta^0_{c_j} + 1) \prod_{j \in \partial \setminus c_j \geq c_i} p_{j \to i}^{(c_j, c_i)}}{\sum_{c_i} e^{-\beta \delta^0_{c_i}} \prod_{j \in \partial \setminus c_j \in A} \sum p_{j \to i}^{(c_j, c_i)} - (1 - \delta^0_{c_i})(\delta^0_{c_j} + \delta^0_{c_j} + 1) \prod_{j \in \partial \setminus c_j \geq c_i} p_{j \to i}^{(c_j, c_i)}}. \tag{3}
\]

Messages \( p_{j \to i}^{(c_j, c_i)} \) are converged messages, that is, the marginal probability is calculated after the BP equation converges. \( p^i_{c} \) denotes the probability that node \( i \) is covered, \( p^i_{1} \) denotes the probability that node \( i \) has at least one covered neighbor, and \( p^i_{2} \) denotes the probability that node \( i \) has at least one covered 2-distance quasi neighbor.

Finally, the free energy can be calculated using mean field theory:

\[
F_0 = \sum_{i=1}^{N} F_i - \sum_{(i,j) = 1}^{M} F_{(i,j)}, \tag{4}
\]

where

\[
F_i = -\frac{1}{\beta} \ln \left[ \sum_{c_i} \frac{e^{-\beta \delta^0_{c_i}} \prod_{j \in \partial \setminus c_j \in A} \sum p_{j \to i}^{(c_j, c_i)} - (1 - \delta^0_{c_i})(\delta^0_{c_j} + \delta^0_{c_j} + 1) \prod_{j \in \partial \setminus c_j \geq c_i} p_{j \to i}^{(c_j, c_i)}}{\sum_{c_i} e^{-\beta \delta^0_{c_i}} \prod_{j \in \partial \setminus c_j \in A} \sum p_{j \to i}^{(c_j, c_i)} - (1 - \delta^0_{c_i})(\delta^0_{c_j} + \delta^0_{c_j} + 1) \prod_{j \in \partial \setminus c_j \geq c_i} p_{j \to i}^{(c_j, c_i)}} \right] \tag{5}
\]

\[
F_{(i,j)} = -\frac{1}{\beta} \ln \left[ \sum_{c_i, c_j \in A} p_{i \to j}^{(c_i, c_j)} p_{j \to i}^{(c_j, c_i)} \right], \tag{6}
\]

where \( F_i \) denotes the free energy of function node \( i \) and \( F_{(i,j)} \) denotes the free energy of edge \( (i, j) \). The BP equation is iterated until it converges to
one stable point, and then mean free energy $f \equiv F/N$ and energy density $\omega = 1/N \sum_i p_i^0$ are calculated using (3) and (4). The entropy density is calculated as $s = \beta(\omega - f)$.

Figure 1 shows that the BP equation cannot converge when the inverse temperature is greater than 9.7 on the ER random graph, which mean connectivity equals to four. The entropy density is always positive and the change rate becomes increasingly smaller as the inverse temperature increases, so the entropy density reaches the transition point when the inverse temperature is extremely high. Because the BP equations cannot converge when the inverse temperature is greater than some threshold value both on the ER random graph and RR random graph, the ground-state energy is obtained using data fitting techniques. MATLAB interpolant curve fitting is used to determine the ground-state energy. We found that the more data were fitted, the better the effect of fitting.

Figure 2 shows that the BP equation can still converge at the transition point of the entropy density on the RR graph when the vertex degree is from two to nine, so we did not need to fit data in this range. The difference in the entropy density between the ER random network and RR network indicates that the solution spaces of the networks have essential differences.
Figure 2: RS and BP results for the 2DMDS problem on the RR random graph with mean connectivity $c = 3$ and $N = 10^3$ using BP and population dynamics. In the A,B,C graphs, the x-axis denotes the inverse temperature $\beta$ and the y-axis denotes the thermodynamic quantities. In the graph D, the x-axis denotes the energy density and the y-axis denotes the entropy density.

III Belief propagation decimation algorithm and greedy algorithm

In this paper, we use two algorithms to determine the solution of the given graph: the greedy algorithm and BPD algorithm. The greedy algorithm very fast, but it does not guarantee good results such as BPD. The BPD algorithm is not as fast as the greedy algorithm, but it always provides a good estimation for the 2DMDS problem.

III.1 Belief Propagation Decimation

If node $i$ is unobserved (it is empty and all the neighbor and 2-distance quasi neighbor nodes are not occupied), then output message $p_{i\rightarrow j}$ on the link $(i,j)$ between node $j$ and node $i$ is updated according to Eq.(2). By contrast, if node $i$ is empty but observed and it has at least one occupied neighbor node, that is, $c_i = 1$, then this node presents no restriction to the states of all its unoccupied neighbors. For such a node $i$, it has no opportunity to take $c_i = 2$, and the output message $p_{i\rightarrow j}$ on the link $(i,j)$ is then updated according to
For node $i(c_i = 1)$, if at least one neighbor node $j$ is covered, then it sends a message to node $i$ as $p_{j \rightarrow i}^{(0,0)} = p_{j \rightarrow i}^{(0,1)} = 0.5$. It leads $p_{j \rightarrow i}^{(0,1)} + p_{j \rightarrow i}^{(1,1)} + p_{j \rightarrow i}^{(2,1)} = p_{j \rightarrow i}^{(0,1)}$, so the constraints of node $i$ to all the other neighbor nodes are automatically removed. The marginal probability is calculated by

$$p_i^{(c_i, c_j)} = \frac{e^{-\beta \delta_{c_i}^0} (1 - \delta_{c_j}^2)}{\sum_{c_i, c_j} e^{-\beta \delta_{c_i}^0} (1 - \delta_{c_j}^2) \prod_{k \in \partial_i \setminus j} \sum_{c_k \in A} p_k^{(c_k, c_i)}} \prod_{k \in \partial_i \setminus j} \sum_{c_k \in A} p_k^{(c_k, c_i)}. \quad (7)$$

If node $i$ is empty but observed (it has no adjacent occupied node, but it has at least one occupied 2-distance quasi neighbor node), this node then presents no restriction to the occupation states of all its unoccupied 2-distance quasi neighbors. For such a node $i$, output message $p_{i \rightarrow j}$ on the link $(i, j)$ is then updated according to

$$p_i^{(c_i, c_j)} = \frac{e^{-\beta \delta_{c_i}^0} \prod_{k \in \partial_i \setminus j} \sum_{c_k \in A} p_k^{(c_k, c_i)} - (1 - \delta_{c_i}^0 - \delta_{c_j}^2) (\delta_{c_j}^0 + \delta_{c_j}^0 + 1) \prod_{k \in \partial_i \setminus j} \sum_{c_k \in A} p_k^{(c_k, c_i)}}{\sum_{c_i, c_j} e^{-\beta \delta_{c_i}^0} \prod_{k \in \partial_i \setminus j} \sum_{c_k \in A} p_k^{(c_k, c_i)} - (1 - \delta_{c_i}^0 - \delta_{c_j}^2) (\delta_{c_j}^0 + \delta_{c_j}^0 + 1) \prod_{k \in \partial_i \setminus j} \sum_{c_k \in A} p_k^{(c_k, c_i)}}. \quad (9)$$

For node $i(c_i = 2)$, if at least one neighbor node $j$ takes state $c_j = 1$, then it sends a message to node $i$ as $p_{j \rightarrow i}^{(2,1)} = p_{j \rightarrow i}^{(2,2)} = 0$. It leads $p_{j \rightarrow i}^{(1,2)} + p_{j \rightarrow i}^{(2,2)} = p_{j \rightarrow i}^{(1,2)}$, so the constraints of node $i$ to all the other neighbor nodes are automatically removed. The marginal probability is calculated by

$$p_i^{c_i} = \frac{e^{-\beta \delta_{c_i}^0} \prod_{j \in \partial_i \setminus c_i} \sum_{c_j \in A} p_j^{(c_j, c_i)} - (1 - \delta_{c_i}^0 - \delta_{c_j}^0) \prod_{j \in \partial_i \setminus c_i} \sum_{c_j \in A} p_j^{(c_j, c_i)}}{\sum_{c_i, c_j} e^{-\beta \delta_{c_i}^0} \prod_{j \in \partial_i \setminus c_i} \sum_{c_j \in A} p_j^{(c_j, c_i)} - (1 - \delta_{c_i}^0 - \delta_{c_j}^0) \prod_{j \in \partial_i \setminus c_i} \sum_{c_j \in A} p_j^{(c_j, c_i)}}. \quad (10)$$

We implement the BPD algorithm as follows:

1. Input network $W$, set all the nodes to be unobserved, and set all the cavity messages $p_{i \rightarrow j}^{(c_i, c_j)}$ to be uniform messages. Set inverse temperature $\beta$ to be sufficiently large (depending on the at most convergence inverse temperature).
2. Then iterate the BP equation using Eq.(2) until it converges to one stable point. Finally, compute the occupation probability of each node $i$ using Eq.(3).
3. Cover the small fraction $\gamma$ (e.g., $\gamma = 0.01$) of unfixed nodes that have the highest covering probabilities.
4. Update the state of all the uncovered nodes as follows: if node $i$ is uncovered
and has at least one neighbor that takes state $c_i = 0$, then it takes state $c_i = 1$, and if node $i$ is unobserved and has at least one neighbor that takes state $c_i = 1$, then it takes state $c_i = 2$.

(4) Fix the observed node’s state, that is, if all the neighbor nodes of observed node $c_i = 1$ are covered or in the state $c_j = 1$, then fix the state of node $i$ to $c_i = 1$. If all the neighbor nodes of observed node $c_i = 2$ are $c_j = 1$, then fix the state of node $i$ to $c_i = 2$.

(5) If network $W$ still contains unobserved nodes, then perform the BP equation using Eqs. (2), (7), or (9). Calculate the marginal probability using Eqs. (3), (8), or (10), depending on the state of node $i$. Repeat operations (2)–(4) until all nodes are observed.

III.2 Greedy

We can develop very simple greedy algorithm in the literature to solve the 2DMDS problem approximately, which is based on the concept of the node’s general impact. The general impact of unoccupied node $i$ equals the sum of the impact of all the neighbor nodes that are not occupied. The impact of unoccupied node $i$ equals the number of nodes that will be observed by occupying $i$. Starting from input network $W$ with all the nodes unobserved, the greedy algorithm uniformly selects at random node $i$ from the subset of nodes with the highest general impact and fixes its occupation state to $c_i = 0$. Then all the neighbor nodes and the 2-distance quasi neighbor nodes of $i$ are observed. If there are still unobserved nodes in the network, then the impact and general impact value for each of the unoccupied nodes is updated and the greedy occupying process is repeated until all the nodes are observed. This pure greedy algorithm is very easy to implement and very fast. We found that it typically reaches a true 2-distance MDS when the input network contains more edges.

The results of the greedy algorithm for the ER random network and RR random network are compared with the results of the greedy algorithm in Figs. 3 and 4. The BPD algorithm outperformed the greedy algorithm, and provided results that were very close to those of the RS theory on both the ER and RR random graphs. The results of RS on the RR random network show that the RS theory only considered the optimistic graph for any vertex degree. For example, any graph of the RR random network whose vertex degree equaled two included different types of cycles if the graph only contained one cycle (Fig. 5A) or many cycles (Fig. 5D). If the graph only contained five nodes or five times nodes, then its 2DMDS equaled the results of the RS theory. If the graph only contained one cycle (Fig. 5A), then the 2DMDS equaled $N/5$. If it only contained many triangle cycles (Fig. 5B), then the 2DMDS equaled $N/3$, $N/4$ for rectangle cycles (Fig. 5C), and $N/5$ for the pentagon cycles (Fig. 5D). Finally, we can derive a general equation of the 2DMDS for the cycles of size $M$ as $\left[\frac{N}{M}\right] \times (\left[\frac{N}{5}\right] + \min(1, \text{mod}(\frac{N}{5}))) + (\text{mod}(\frac{N}{5}) + \min(1, \text{mod}(\frac{N}{5})))$, where $[\ ]$ represents the operation of taking integers and $\text{mod}(\ )$ represents the operation of taking remainders. Because the RR single graph contains diverse cycles, the
energy density is greater than the results of the RS theory. However, it is almost equal to the results of the RS theory when we average over many single graphs.

IV Discussion

In this paper, we proposed two heuristic algorithms (a greedy-impact local algorithm and a BPD message-passing algorithm) and presented an RS mean field theory for solving the network 2-distance dominating set problem algorithmically and theoretically. We found that the BP and RS algorithm may lead to an entropy transition in the RR network when the mean degree is from 3 to 9 (see Fig. 2), and in the ER random network when the mean degree is from 4.2 to 10.4; however, it does not occur in the other circumstance (see Fig. 1). This is because the solution space of the 2DMDs on the both ER and RR networks have different structure in the different mean degree. We will use the one-step RS breaking theory to study the solution space of the 2DMDs problem. Our numerical results shown in Figs. 3 and 4 suggest that the greedy algorithm and BPD algorithm can construct a near-optimal 2DMDs for random networks.

A great deal of theoretical work remains to be studied. A direct extension of our work is to consider the 2DMDs problem of the directed network. We will work on the directed 2DMDs problem as soon as possible. A more challenging and common problem in the dominating set is the connected dominating set
problem. We will use spin glass theory \cite{25} to study both the minimal connected dominating set problem and 2-distance minimal connected dominating set problem. We will also study the various types of dominating set problem (e.g., double dominating set \cite{1,5}, liar’s dominating set \cite{2}, and extended dominating set \cite{4}) using spin glass theory.

V Acknowledgement

Yusupjan Habibulla thanks Prof. Haijun Zhou for helpful discussions and guidance. This research was supported by the doctoral startup fund of Xinjiang University of China (grant number 208-61357) and partially supported by the National Natural Science Foundation of China (grant numbers 11465019, 11664040, 11765021,11705279 and 61662078). We thank Maxine Garcia, PhD, from Liwen Bianji, Edanz Group China (www.liwenbianji.cn/ac) for editing the English text of a draft of this manuscript.

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Figure 5: Different structure of the RR graph, where the vertex degree equals two. (A) The graph only contains one large cycle: yellow nodes represent the covered nodes and blue nodes represent the observed nodes. One node can observe five neighbors and 2-distance quasi neighbors (including itself). (B) The graph only contains many triangle cycles: we can cover any one node in the triangle cycle to observe all the nodes in this cycle. (C) The graph only contains many rectangle cycles: we can cover any one node in the rectangle cycle to observe all the nodes in this cycle. (D) The graph only contains many pentagon cycles: we can cover any one node in the pentagon cycle to observe all the nodes in this cycle.
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