A Lichnerowicz theorem for singular foliations using blow-up groupoids

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Abstract

We introduce a blow-up construction of a smooth manifold along the singular leaves of an arbitrary singular foliation in the sense of Stefan and Sussmann. We also define a blow-up groupoid of the holonomy groupoid defined by Androulidakis and Skandalis. This answers a question of Debord and Skandalis.

As an application, we use this groupoid to obtain a Lichnerowicz theorem for singular foliations.

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Introduction

Let $M$ be a smooth manifold. One of the basic questions in Riemannian geometry is to determine if $M$ admits a Riemannian metric with positive scalar curvature. According to Gromov [Gro20], there is generally two methods to attack this problem. The first involves minimal hypersurfaces and the second involves the Dirac operator. We are concerned in this article with the second method.

The starting point of the second method is the following theorem

**Theorem 0.1 (Lichnerowicz [Lic63]).** Let $M$ be a compact spin manifold. If $M$ admits a metric with positive scalar curvature, then the Â-genus $\hat{A}(M)$ vanishes.

This theorem can be seen as a corollary of the Atiyah-Singer index theorem. In fact the theorem follows from the following

a) Let $D$ be the Dirac operator on $M$ defined by Atiyah-Singer. Then Lichnerowicz formula implies that the analytic index $\text{Ind}_a(D)$ of $D$ vanishes.
b) By Atiyah-Singer index theorem \( \text{Ind}_a(D) = \text{Ind}_t(D) \), where \( \text{Ind}_t(D) \) is the topological index. Furthermore Atiyah and Singer showed that \( \text{Ind}_t(D) = \hat{A}(M) \).

Lichnerowicz theorem was generalised by many authors. In this article we are interested in Connes’s generalisation.

**Theorem 0.2** (Connes [Con86]). Let \( M \) be a compact smooth manifold, \( F \subseteq TM \) a regular spin foliation, \( D \) the longitudinal Dirac operator. Then

a) If \( M \) admits a Riemannian metric which has positive scalar curvature on the leaves of \( F \), then the analytic index \( \text{Ind}_a(D) \) of \( D \) vanishes. Here the analytic index is an element of \( K_0(C^*_r F) \) the \( K \)-theory of the \( C^* \)-algebra of \( F \).

b) If \( \text{Ind}_a(D) = 0 \) and \( M \) is oriented, then for any \( y \in \mathcal{R} \),

\[
\langle y \cup \hat{A}(F), [M] \rangle = 0,
\]

where \( \mathcal{R} \) is the subring of \( H^*(M) \) generated by the Pontryagin classes of \( TM/F \) and \( [M] \) is the fundamental class and \( \hat{A}(F) \) is the \( \hat{A} \) class of \( F \). In particular \( \hat{A}(M) = 0 \).

Let us examine each part.

- Theorem 0.2a is proved by following the strategy used by Lichnerowicz. More precisely by Connes-Skandalis [CSS83], one defines the longitudinal Dirac operator. Then the longitudinal Lichnerowicz formula (which is just a parametrised version of Lichnerowicz formula) imply that the analytic index of the longitudinal Dirac operator vanishes as an element of \( K_0(C^*_r F) \).

- Theorem 0.2b is an immediate consequence of the injectivity of the Baum-Connes map for \( F \). Unfortunately, the Baum-Connes map fails in general to be injective [HLS02]. Nevertheless using cyclic cohomology and Kasparov’s Dirac dual Dirac method [Kas88], Connes shows that there exists enough group homeomorphisms \( K_0(C^*_r F) \to \mathbb{R} \) to deduce Part b without supposing the injectivity of the Baum-Connes map.

The main goal of this article is to extend Theorem 0.2 to singular foliations. We use the definition of singular foliations due to Stefan [Ste74] and Sussmann [Sus73]. A singular foliation \( F \) is a locally finitely generated involutive submodule of the \( C^\infty(M, \mathbb{R}) \)-module of real compactly supported vector fields on \( M \).

Let \( x \in M \). In [AS09], the authors define two fibers of \( F \) at \( x \).

- The first fiber is

\[
F_x := \{ X(x) : X \in F \}.
\]

- The second fiber is

\[
\mathcal{F}_x = \mathcal{F}/I_x \mathcal{F},
\]

where \( I_x \subseteq C^\infty(M, \mathbb{R}) \) is the ideal of smooth functions vanishing at \( x \). Since \( \mathcal{F} \) is locally finitely generated one deduces that \( \mathcal{F}_x \) is a finite dimensional vector space.
Clearly one has a surjective linear map

\[ \mathcal{F}_x \to F_x. \]  

We now mention the main difficulty one faces in trying to extend Theorem 0.2.a to singular foliations. The functions \( x \to \dim(F_x) \) and \( x \to \dim(F_x) \) aren’t continuous (they are continuous if and only if \( \mathcal{F} \) is a regular foliation). So if one naively defines an orientation of \( \mathcal{F} \) as an orientation of \( F_x \) or \( F_x \) for each \( x \in M \), then it isn’t clear how to state continuity of such an orientation as orientation behaves poorly with jumps in dimension. Of course this issue remains if one wishes to define a spin structure on \( \mathcal{F} \).

We resolve this difficulty by introducing a topological blow-up construction. The starting point is the following proposition

**Proposition 0.3** ([AS09]). Let \( M_{\text{reg}} \) be of set of \( x \in M \) such that the map \( 0.1 \) is an isomorphism. Then

a) \( M_{\text{reg}} \) is an open dense subset of \( M \).

b) On \( M_{\text{reg}} \), the function \( x \to \dim(F_x) = \dim(F_x) \) is a continuous function.

**Blow-up space.** The idea is then to introduce a topological blow-up of \( M \) along the complement of \( M_{\text{reg}} \). If \( x \in M \), then we define a subset \( \text{Blup}(\mathcal{F})_x \) of the Grassmannian manifold \( \text{Gr}(F^*_x) \) of subspaces of \( F^*_x \) as follows. A subspace \( V \subseteq F^*_x \) belongs to \( \text{Blup}(\mathcal{F})_x \) if and only if there exists a sequence \( x_n \in M_{\text{reg}} \) such that

\[ x_n \to x, \quad F^*_{x_n} \to V, \]

where the second convergence is taken inside a Grassmannian manifold of big enough dimension (constructed using any local generators of \( \mathcal{F} \) around \( x \)). The convergence doesn’t depend on the choice of the ambient Grassmannian manifold.

Let

\[ \text{Blup}(\mathcal{F}) = \sqcup_{x \in M} \text{Blup}(\mathcal{F})_x. \]

In other words we replace each point \( x \) of \( M \) with \( \text{Blup}(\mathcal{F})_x \). Since for \( x \in M_{\text{reg}} \), \( \text{Blup}(\mathcal{F})_x = (F^*_x) \), the space \( \text{Blup}(\mathcal{F}) \) only differs from \( M \) on \( M^c_{\text{reg}} \). The set \( \text{Blup}(\mathcal{F}) \) is naturally equipped with a topology (defined by embedding \( \text{Blup}(\mathcal{F}) \) inside \( M \times \text{Gr}(\mathbb{R}^k) \) for \( k \) big enough). The space \( \text{Blup}(\mathcal{F}) \) is our definition of the blow-up of \( M \) along \( M^c_{\text{reg}} \).

We remark that if \( V \subseteq M \) is any submanifold, \( \mathcal{F} \) the module of vector fields vanishing on \( V \), then \( M_{\text{reg}} = M \setminus V \) and the space \( \text{Blup}(\mathcal{F}) \) is equal to the classical blow-up from differential geometry of \( M \) along \( V \).

The space \( \text{Blup}(\mathcal{F}) \) has the following properties.

- \( \text{Blup}(\mathcal{F}) \) is compact if \( M \) is compact.

- On \( \text{Blup}(\mathcal{F}) \), one has a natural vector bundle denoted \( T\mathcal{F} \) whose fiber over \( V \in \text{Blup}(\mathcal{F})_x \) is equal to \( V^* \). We define an orientation and spin structure on \( \mathcal{F} \) as an orientation and a spin structure on \( T\mathcal{F} \) respectively.

- The space \( \text{Blup}(\mathcal{F}) \) fails in general to be a smooth manifold. Nevertheless it is naturally a closed subset of a smooth manifold. In particular the ring of smooth functions on \( \text{Blup}(\mathcal{F}) \) is well defined.
**Blow-up groupoid.** The fibers of $\mathcal{F}$ can be integrated.

- The integral of $F_x$ is an immersed submanifold $L_x \subseteq M$. This result is due to Stefan and Sussmann.
- The fiber $F_x$ can also be integrated to a manifold denoted $L_x$. But since $\dim(F_x)$ can be strictly bigger than $\dim(M)$, the integral isn’t an immersed submanifold of $M$ in general. Nevertheless one has a natural map $L_x \to L_x$ whose differential is the map $\partial_{\mu}$. This result is due to Debord [Deb13].

Recall that in [AS09], the authors define a holonomy groupoid $\mathcal{H}(\mathcal{F}) \rightrightarrows M$ of $\mathcal{F}$, which has the property that $\mathcal{H}(\mathcal{F})_x = L_x$, the integral of $F_x$. This groupoid is longitudinally smooth but in general it has a very ill-behaved topology. As the vector spaces $F_x$ (here it is seen as the Lie algebroid of $\mathcal{H}(\mathcal{F})$) have varying dimension, it isn’t clear how to define the longitudinal Dirac operator using the groupoid $\mathcal{H}(\mathcal{F})$.

We overcome this difficulty by defining a groupoid $\mathcal{H}^{blup}(\mathcal{F})$ which we call the **holonomy blow-up groupoid** whose space of objects is $\text{Blup}(\mathcal{F})$. It has the following properties

a) The groupoid $\mathcal{H}^{blup}(\mathcal{F})$ is longitudinally smooth. The bundle $T\mathcal{F}$ is the Lie algebroid of $\mathcal{H}^{blup}(\mathcal{F})$. Since the Lie algebroid is a vector bundle, we can define the longitudinal Dirac operator on $\mathcal{H}^{blup}(\mathcal{F})$. We deduce

**Theorem 0.4.** Let $M$ be a compact smooth manifold, $\mathcal{F}$ a singular foliation, $\mathcal{H}^{blup}(\mathcal{F}) \rightrightarrows \text{Blup}(\mathcal{F})$ the blow-up groupoid, $T\mathcal{F}$ its Lie algebroid. Then if

(a) $T\mathcal{F}$ is spin (our definition of $\mathcal{F}$ being spin).

(b) there exists a Euclidean metric $g$ on $\mathcal{F}_{|\text{reg}}$ on $M_{\text{reg}}$ such that

i. $g$ extends to a Euclidean metric on $T\mathcal{F}$

ii. There exists $\epsilon > 0$ such that for each $x \in M_{\text{reg}}$, the scalar curvature of $g$ on $L_x = L_x$ is $\geq \epsilon$.

Then the analytic index of the longitudinal Dirac operator on $\mathcal{H}^{blup}(\mathcal{F})$ vanishes as an element of $K(C^*_r\mathcal{H}^{blup}(\mathcal{F}))$.

One can possibly use the Baum-Connes conjecture for $\mathcal{H}^{blup}(\mathcal{F})$ or cyclic cohomology to deduce topological obstructions from the vanishing of the analytic index of the longitudinal Dirac operator. We will maybe explore such questions in a future paper.

We remark that the Baum-Connes assembly map for $\mathcal{H}^{blup}(\mathcal{F})$ is well defined by the work of Baum-Higson-Connes [BCH94]. This isn’t the case for the groupoid $\mathcal{H}(\mathcal{F})$. In fact the Baum-Connes assembly map for $\mathcal{H}(\mathcal{F})$ introduced in [AS19] is only defined under some mild hypothesis on $\mathcal{H}(\mathcal{F})$.

b) Even though $\mathcal{H}^{blup}(\mathcal{F})$ isn’t a smooth manifold, locally it is a closed subset of a smooth manifold. In particular the ring of smooth functions on $\mathcal{H}^{blup}(\mathcal{F})$ is well defined.
c) The relation between the groupoids $\mathcal{H}(\mathcal{F})$ and $\mathcal{H}_{blup}(\mathcal{F})$ is that there exists a natural $C^*$-homomorphism

$$f : C^*_{\text{max}} \mathcal{H}(\mathcal{F}) \to C^*_{\text{max}} \mathcal{H}_{blup}(\mathcal{F}).$$

In fact we define this map at the level of pre algebras, in which case one have the stronger statement. If $\mathcal{A}$ denotes the dense $*$-subalgebra of $C^*\mathcal{H}(\mathcal{F})$ defined in [AS09], then $f$ sends $\mathcal{A}$ to smooth functions on $\mathcal{H}_{blup}(\mathcal{F})$.

The construction of $\mathcal{H}_{blup}(\mathcal{F})$ and $f$ answers [DS19] Question 5 on Page 28.

Finally, since the groupoid $\mathcal{H}_{blup}(\mathcal{F})$ is a locally compact locally metrizable groupoid which is very close to being a Lie groupoid. The standard theory of Lie groupoid apply to it. In particular one can talk about

a) Baum-Connes conjecture for singular foliations as the Baum Connes conjecture for $\mathcal{H}_{blup}(\mathcal{F})$.

b) Novikov conjecture for singular foliations. One can also define the longitudinal signature operator of $\mathcal{F}$ as a differential operator on $\mathcal{H}_{blup}(\mathcal{F})$.

Hence one can define the signature of singular foliations (as element of $K(C^*_r \mathcal{H}_{blup}(\mathcal{F}))$)

c) amenability condition for singular foliation as simply the amenability of $\mathcal{H}_{blup}(\mathcal{F})$. Results of Tu [Tu99] apply to $\mathcal{H}_{blup}(\mathcal{F})$. Thus one obtains that the Baum-Connes conjecture for amenable singular foliations holds.

d) index theory and ellipticity for differential operators longitudinal to singular foliations.

The article is organised as follows

a) In Section 1 the spaces $\text{Blup}(\mathcal{F})$ and $\mathcal{H}_{blup}(\mathcal{F})$ are constructed in the case of foliations coming from groups actions.

b) In Section 2 the spaces $\text{Blup}(\mathcal{F})$ and $\mathcal{H}_{blup}(\mathcal{F})$ are constructed in the general case.

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1 Blow-up construction

1.1 Regular points and blow-up space

Regular points. Let $G$ be a connected Lie group acting smoothly on a smooth manifold $M$. The derivative of the action gives the anchor map

$$\#: \mathfrak{g} \to \Gamma(TM),$$
where \( g \) is the Lie algebra of \( G \), \( \Gamma(TM) \) the space of smooth sections of \( TM \). The anchor map satisfies the identity

\[
\#([X,Y]) = [\#(X), \#(Y)], \quad X,Y \in g.
\]

Let \( x \in M \). We will denote by

\[ \#_x : g \to T_x M \]

the composition of \( \# \) with the evaluation map at \( x \). Clearly \( \#_x(g) \) is the tangent space of the orbit \( G \cdot x \) at \( x \). Let

\[ \mathfrak{h}_x = \ker(\#_x). \]

We define \( p_x \subseteq g \) to be the set of \( X \in g \) such that there exists \( k \in \mathbb{N} \) and \( f_i \in C^\infty(M, \mathbb{R}), Y_i \in \mathfrak{g} \) for \( i = 1, \ldots, k \) such that

\[
\#(X) = \sum_{i=1}^k f_i \#(Y_i), \quad f_i(x) = 0 \quad \forall i.
\]  

The quotient \( g/p_x \) will be denoted by \( g_x \).

We remark that

- \( p_x, \mathfrak{h}_x \) are Lie subalgebras of \( g \)
- \( p_x \) is an ideal in \( \mathfrak{h}_x \).

One has a natural surjective linear map

\[ g_x \to \#_x(g). \]  

**Definition 1.1.**
- A point \( x \) is called regular if the map (1.2) is an isomorphism. Equivalently \( p_x = \mathfrak{h}_x \).
- The set of regular points is denoted \( M_{\text{reg}} \).

**Proposition 1.2.**

a) The set of regular points \( M_{\text{reg}} \) is an open dense subset of \( M \).

b) The function \( x \to \dim(g_x) \) is upper semi-continuous, or equivalently \( x \to \dim(p_x) \) is lower semi-continuous.

c) The function \( x \to \dim(\#_x(g)) \) is lower semi-continuous, or equivalently \( x \to \dim(\mathfrak{h}_x) \) is upper semi-continuous.

d) The bundle \( \sqcup_{x \in M_{\text{reg}}} \mathfrak{h}_x \) is a vector subbundle of the trivial bundle \( M_{\text{reg}} \times g \) on \( M_{\text{reg}} \).

Proposition 1.2 is due to Androulidakis and Skandalis, see [AS09, Proposition 1.2]. We remark that the set \( M_{\text{reg}} \) is \( G \)-invariant.
**Blow-up space.** Let $\text{Gr}(g)$ be the Grassmannian manifold of linear subspaces of $g$. For $x \in M$, we define the set $\text{Blup}(G \times M)_x \subseteq \text{Gr}(g)$ to be the set of subspaces $V \subseteq g$ such that there exists a sequence $x_n \in M_{\text{reg}}$ such that $x_n \to x$ and $h_{x_n} \to V \in \text{Gr}(g)$.

Here are a few immediate properties of the $\text{Blup}(G \times M)$.

**Proposition 1.3.** Let $x \in M$, $V \in \text{Blup}(G \times M)_x$.

a) The set $\text{Blup}(G \times M)_x$ is a nonempty closed subset of $\text{Gr}(g)$.

b) One has $p_x \subseteq V \subseteq h_x$.

c) The vector subspace $V$ is a Lie subalgebra of $g$.

d) If $g \in G$, then $\text{Ad}(g)V \in \text{Blup}(G \times M)_{gx}$.

e) If $x \in M_{\text{reg}}$, then $\text{Blup}(G \times M)_x = \{h_x\} = \{p_x\}$.

**Proof.**

a) Since the Grassmannian space $\text{Gr}(g)$ is compact, it follows that $\text{Blup}(G \times M)_x$ is nonempty. Closeness is obvious.

b) Let $V \in \text{Blup}(G \times M)_x$. By definition there exists $x_n \in M_{\text{reg}}$ such that $x_n \to x$ and $h_{x_n} \to V$. Let $X \in p_x$. Let $f_i, Y_i$ be as in Equation 1.1. It follows that

$$X - \sum_{i=1}^k f_i(x_n)Y_i \in h_{x_n}.$$ 

Since $X - \sum_{i=1}^k f_i(x_n)Y_i \xrightarrow{n \to \infty} X$, one deduces that $X \in V$. Hence $p_x \subseteq V$.

The inclusion $V \subseteq h_x$ is obvious.

c) Since $V$ is the limit of Lie subalgebras, one deduces that $V$ is a Lie subalgebra.

d) Let $V \in \text{Blup}(G \times M)_x$, $x_n \in M_{\text{reg}}$ such that $x_n \to x$ and $h_{x_n} \to V$. Since $\text{Ad}(g)h_{x_n} = h_{gx_n}$, and $gx_n \to gx$. One obtains that $\lim_{n \to x} \text{Ad}(g)h_{x_n} = \text{Ad}(g)V \in \text{Blup}(G \times M)_{gx}$.

e) By Proposition 1.2, the set $M_{\text{reg}}$ is open set and on it $(h_y = p_y)_{y \in M_{\text{reg}}}$ form a vector bundle. The result is then immediate.

**Remark 1.4.** By Property c, $\text{Blup}(G \times M)_x$ can be treated as a subset of $\text{Gr}(g_x)$. This is the point of view that will be used for general singular foliations. In this section we will continue to treat $\text{Blup}(G \times M)_x$ as subset of $\text{Gr}(g)$.

**Definition 1.5.** We define the blow-up space of $G \times M$ to be the set

$$\text{Blup}(G \times M) := \{(V, x) : x \in M, V \in \text{Blup}(G \times M)_x\}$$

with the subspace topology from the inclusion $\text{Blup}(G \times M) \hookrightarrow \text{Gr}(g) \times M$.

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1In the introduction, we used the set $\{V \subseteq g^* : V \in \text{Blup}(G \times M)\}$ instead of $\text{Blup}(G \times M)$. We use the definition given here as it statements simpler to read/write.
Examples 1.6.  

a) Consider the group $SL_n$ acting on $\mathbb{R}^n$. For $x \in \mathbb{R}^n \setminus \{0\}$, let

$$\ker(x) := \{ A \in sl_n : A(x) = 0 \}.$$ 

One sees that

$$p_x = \begin{cases} \ker(x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}, \quad h_x = \begin{cases} \ker(x) & \text{if } x \neq 0 \\ sl_n & \text{if } x = 0 \end{cases}$$

It follows that $M_{\text{reg}} = \mathbb{R}^n \setminus \{0\}$. One can easily see that

$$\text{Blup}(G \ltimes M)_x = \begin{cases} \{ \ker(x) \} & \text{if } x \neq 0 \\ \{ \ker(y) : y \in \mathbb{R}^n \setminus \{0\} \} & \text{if } x = 0. \end{cases}$$

Hence $\text{Blup}(G \ltimes M)_0$ can be identified with the projective space $\mathbb{P}^{n-1}$. The space $\text{Blup}(G \ltimes M)$ is then the usual blow-up of $\mathbb{R}^n$ at $0$.

b) Consider $G = \mathbb{R}$ acting on $M = \mathbb{R}$ by the flow of the vector field $X = \rho(x)\frac{\partial}{\partial x}$, where $\rho$ is any smooth function such $\rho > 0$ on $]-1,1[$ and $\text{supp}(\rho) = [-1,1]$. One easily sees that

$$\text{Blup}(G \ltimes M)_x = \begin{cases} \{ 0 \} & \text{if } x \in ]-1,1[ \\ \{ \mathbb{R} \} & \text{if } x \in ]-\infty,-1[ \cup [1,\infty[ \cup [-1,1] \\ \{ 0,\mathbb{R} \} & \text{if } x \in \{-1,1\}. \end{cases}$$

It follows that $\text{Blup}(G \ltimes M)$ can be identified with $]-\infty,-1[ \cup [-1,1] \cup [1,\infty[$ with the three intervals disjoint by doubling the endpoints $-1$ and $1$. It follows that in general $\text{Blup}(G \ltimes M)$ isn’t connected.

Remark 1.7. The above construction can be generalised as follows. Let $M$ be a smooth manifold, $E, F \to M$ vector bundles $\phi : E \to F$ a vector bundle morphism. For $x \in M$, we define $A_x = \ker(\phi_x)$ and $B_x \subseteq E_x$ the subspace of $e \in E_x$ such that there exists $X \in \Gamma(E)$, $f \in C^\infty(M,\mathbb{R})$, $y \in \Gamma(E)$ such that

$$\phi(X) = \sum_i f_i(y_i), \quad f_i(x) = 0 \forall i, \quad X(x) = e.$$ 

Then one defines $M_{\text{reg}} = \{ x \in M : A_x = B_x \}$. It is straightforward to see that $M_{\text{reg}}$ is an open dense set. One can then proceed to define a blow-up space $\text{Blup}(\phi)$ similarly to the construction of $\text{Blup}(G \ltimes M)$.

1.2 Longitudinal smoothness

If $V \subseteq \mathfrak{g}$ is a Lie algebra, then we denote by $\exp(V)$ the connected immersed Lie subgroup of $G$ whose Lie algebra is $V$.

Theorem 1.8. Let $x \in M$, $V \in \text{Blup}(G \ltimes M)_x$. Then $\exp(V)$ is a closed Lie subgroup of $G$.

Before giving the proof we remark that

- the subgroup $\exp(\mathfrak{h}_x)$ is closed because $\exp(\mathfrak{h}_x)$ is the connected component at the identity of $\{ g \in G : gx = x \}$. 

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the subgroup $\exp(p_x)$ is also closed. This is a special case of a theorem of Debord [Deb13].

Theorem 1.8 as well as the closeness of $\exp(p_x)$ are corollaries of the periodic bounding lemma.

**Lemma 1.9** (Periodic bounding lemma [AR67, Ozo72]). Let $M$ be a smooth manifold, $X$ a compactly supported vector field, then there exists $\epsilon > 0$ such that for any $x \in M$ either $X(x) = 0$ or $\phi^t_X(x) \neq x$ for all $t < \epsilon$, where $\phi^t_X(\cdot)$ is the flow of $X$.

**Proof of Theorem 1.8** We will follow closely the proof in [Deb13], but since the foliation in this case has a global bi-submersion ($G \rightarrow M$) the proof is easier to write.

Let $x_n \in M_{\text{reg}}$ be a sequence such that $x_n \to x, h_{x_n} \to V$. We will denote by $\bar{x} = \{x_n : n \in \mathbb{N}\} \cup \{x\}$ the closed subset of $M$.

**Lemma 1.10.** Let $g \in \exp(V)$. Then there exists a continuous function $\phi : G \times \bar{x} \to G$ such that

- for every $y \in \bar{x}, h \in G \phi(h, y)y = hy$.
- for every $h \in G, \phi(h, x) = hg$

**Proof.** Suppose that $g = g_1 g_2$ with $g_1, g_2 \in \exp(V)$ and we proved the lemma for $g_1$ and $g_2$. If $\phi_1$ and $\phi_2$ are the corresponding functions, then the lemma holds for $g$ with

$$\phi(h, y) = \phi_2(\phi_1(h, y), y).$$

By the definition of $\exp(V)$, we can write $g = \exp(X_1) \cdots \exp(X_k)$ for some $X_i \in V$. It follows that it is enough to prove the lemma for $g = \exp(X)$ with $X \in V$. Let $X_n \in h_{x_n}$ such that $X_n \to X$. Then we define $\phi$ by the formula

$$\phi(h, y) = \begin{cases} \exp(X_n) & \text{if } y = x_n \\ \exp(X) & \text{if } y = x \end{cases}$$

We fix $S \subseteq \mathfrak{g}$ a transverse linear subspace of $p_x$. We also fix any norm on $S$.

**Lemma 1.11.** Let $K \subseteq M$ be a compact neighbourhood of $x$. Then there exists $\eta > 0$ such that for every $y \in K$ and $X \in S$ if $\exp(X)y = y$, then either $X \in h_y$ or $\|X\| \geq \eta$.

**Lemma 1.11** is a parametrised version of the periodic bounding lemma, see [Deb13] the proof of Proposition 1.11.

**Lemma 1.12.** There exists $\eta > 0$ such that for any $X \in S$ if $\exp(X) \in \exp(V)$ and $\|X\| < \eta$ then $X \in S \cap V$.

**Proof.** By [AS09] Proposition 2.10], there exists a smooth function

$$\log : G \times M \to S$$

defined in a neighbourhood of $(e, x)$ such that

$$\exp(\log(g, y))y = g y.$$
Furthermore the proof of [AS09, Proposition 2.10] shows that we can further suppose that \( \log(\exp(X), x) = X \) for all \( X \in S \).

Let \( \eta_1 \) be such that if \( X \in S \) with \( \|X\| < \eta_1 \), then \( (\exp(X), x) \in \text{dom}(\phi) \). Let \( \eta_2 \) be the constant obtained from Lemma 1.11 applied to \( K = \bar{x} \). Then for any \( X \in S \) such that \( \exp(X) \in \exp(V) \) and \( \|X\| < \min(\eta_1, \eta_2) \). Let \( \phi : G \times \bar{x} \to G \)
be a continuous function obtained by applying Lemma 1.10 to \( \exp(X) \). Then for \( n \) big enough,
\[
\exp\left(\log(\phi(e, x_n), x_n)\right) x_n = x_n.
\]

By Lemma 1.11 it follows that for \( n \) big enough \( \log(\phi(e, x_n), x_n) \in \mathfrak{h}_{x_n} \). Hence
\[
\log(\phi(e, x), x) = \lim_{n \to \infty} \log(\phi(e, x_n), x_n) \in V.
\]

Since \( X = \log(\phi(e, x), x) \), the result follows.

Suppose now that \( v_n \in \exp(V) \) is a sequence that converges to \( v \in G \). By multiplying by \( v_m^{-1} \) for \( m \) big enough we can suppose that \( v_n, v \) are in a small enough neighbourhood of the identity. The map
\[
S \times \mathfrak{p}_x \to G, \quad (X, Y) \to \exp(X)\exp(Y)
\]
is a local diffeomorphism at \((0, 0)\). Therefore we can suppose that
\[
v_n = \exp(X_n)\exp(Y_n)
\]
with \( X_n, Y_n \in \mathfrak{p}_x \) furthermore \( \|X_n\|, \|Y_n\| \leq \frac{\eta_2}{2} \). By going to a subsequence we can suppose that \( X_n, Y_n \) converge to \( X, Y \) respectively. By Proposition 1.3.b, \( \exp(X_n) = v_n \exp(-Y_n) \in \exp(V) \). By Lemma 1.11, \( X_n \in V \). Hence \( X \in V \). So \( v = \exp(X)\exp(Y) \) is in \( \exp(V) \).

Notice that we proved the following stronger statement

**Theorem 1.13.** Let \( x \in M \), \( V \in \text{Blup}(G \times M)_x \), \( S \subseteq \mathfrak{g} \) a subspace transverse to \( V \). Then there exists an open neighbourhood \( 0 \in S' \subseteq S \), and an open neighbourhood \( (V, x) \in L \subseteq \text{Blup}(G \times M) \) such that for any \( (W, y) \in L \), the map
\[
\exp : S' \to G/\exp(W), \quad X \to \exp(X)\exp(W)
\]
is an embedding.

### 1.3 Blow-up Groupoid

In this section we define a groupoid \( \mathcal{H}\text{blup}(G \times M) \supseteq \text{Blup}(G \times M) \). We define the set \( \mathcal{H}\text{blup}(G \times M) \) as the set of all left cosets of \( \exp(V) \) for \( V \in \text{Blup}(G \times M)_x \), \( x \in M \). By Proposition 1.3.d, the set of left cosets agree with the set of right cosets. Hence we define
\[
\mathcal{H}\text{blup}(G \times M) = \{(g\exp(V), x) : x \in M, V \in \text{Blup}(G \times M)_x\} = \{(\exp(V)g, x) : x \in M, V \in \text{Blup}(G \times M)_{gx}\}
\]
The groupoid structure is given by

\[ s(\exp(V), x) = (V, x) \]
\[ r(\exp(V), x) = (\operatorname{Ad}(g)V, gx) \]
\[ (\exp(Ad(h)V), hx) \cdot (\exp(V), x) = (\operatorname{g}h\exp(V), x) \]
\[ (\exp(V), x)^{-1} = (\exp(V)^{-1}\operatorname{Ad}(g)V), gx) = (\exp(V)g^{-1}, gx) \]

The topology is defined as follows. A sequence (or a net) \((g_n, \exp(V_n), x_n)\) converges to \((\exp(V), x)\) if

- \(x_n \to x\) and \(V_n \to V \in \operatorname{Gr}(g)\)
- there exists \(y_n \in \exp(V_n), y \in \exp(V)\) such that \(g_n y_n \to gy\).

This topology is metrizable as follows. Let \(d_M\) a metric on \(M\), \(d_G\) a metric on \(G\), \(d_g\) a metric on \(g\). Then we define

\[ d((\exp(V), x), (\exp(W), y)) = d_M(x, y) + d_G(\exp(V), \exp(W)) + d_g(V, W). \]

The following theorem is then straightforward to check.

**Theorem 1.14.** The groupoid \(\mathcal{H}\operatorname{blup}(G \times M)\) is a metrizable locally compact topological groupoid.

Let \((V, x) \in \operatorname{Blup}(G \times M)\). Then it is clear that \(\mathcal{H}\operatorname{blup}(G \times M)_{(V, x)} \cong G/\exp(V)\). It follows from Theorem 1.13 that the groupoid \(\mathcal{H}\operatorname{blup}(G \times M)\) is longitudinally smooth.

**Smooth structure** If \(N, L\) is a smooth manifold \(A \subseteq N\), \(f : A \to L\) is a continuous function, then we say that \(f\) is smooth if \(f\) admits a smooth extension to an open neighbourhood of \(A\).

In general the spaces \(\mathcal{H}\operatorname{blup}(G \times M)\), \(\operatorname{Blup}(G \times M)\) aren’t smooth manifolds. Nevertheless they can be canonically regarded as closed subsets of smooth manifolds. For \(\operatorname{Blup}(G \times M)\), this is clear as it is defined as a closed subset of \(\operatorname{Gr}(g) \times M\).

For \(\mathcal{H}\operatorname{blup}(G \times M)\), we proceed as follows. Let \((\exp(V), x) \in \mathcal{H}\operatorname{blup}(G \times M)\). Consider \(S \subseteq g\) be any linear subspace transverse to \(V\). We then define

\[ \psi : S \times \operatorname{Blup}(G \times M) \to \mathcal{H}\operatorname{blup}(G \times M) \]
\[ (X, (W, y)) \to ((\exp(X))\exp(W), y) \]

Theorem 1.13 implies that \(\psi\) is a local homeomorphism around \((0, (V, x))\) onto an open neighbourhood of \((\exp(V), x)\). Furthermore if one choose a different base point \((g'\exp(V'), x')\) and another transversal \(S'\) and if \(\psi'\) is the function associated then the transition function \(\psi^{-1} \circ \psi'\) is smooth on its domain of definition.

Let \(L\) is a smooth manifold and \(f : \mathcal{H}\operatorname{blup}(G \times M) \to L\) is a continuous function. Then we say that \(f\) is smooth if for any \(\psi\) as above \(f \circ \psi\) is smooth. It follows that the ring \(C^\infty(\mathcal{H}\operatorname{blup}(G \times M))\) is well defined.

We remark that all the structural maps of \(\mathcal{H}\operatorname{blup}(G \times M)\) are easily seen to be smooth in the above sense.
1.4 Convolution algebra and densities

Consider the inclusion $i : \text{Blup}(G \times M) \to \text{Gr}(g) \times M$. Let $E$ be the canonical vector bundle on $\text{Gr}(g) \times M$ whose fiber at $(V, x)$ is equal to $V$, $F$ the trivial vector bundle on $\text{Gr}(g) \times M$ with fiber $g$. Then it is easy to see that $i^*(F/E)$ is equal to the Lie algebroid of $\mathcal{H}\text{blup}(G \times M)$. We will denote $i^*(F/E)$ by $T_F$.

We will use densities to define convolution algebra following [Con82]. If $E \to M$ is a vector bundle, then we use $|\Lambda|^\alpha E \to M$ for the bundle of $\alpha$-densities on $E$. We will denote by $\Gamma_c(M, E)$ the space of smooth compactly supported sections of $E$.

In what follows we use the construction of the $C^\ast$-algebras of groupoids using densities [Vas01]. One defines a map

$$\int : \Gamma_c(G \times M, |\Lambda|^1 g) \to \Gamma_c(\mathcal{H}\text{blup}(G \times M), s^*|\Lambda|^\frac{1}{2}TF \otimes r^*|\Lambda|^\frac{1}{2}TF)$$

$$\int(f)(\exp(V), x) = \int_{\exp(V)} f(g \cdot X, x) dX$$

We remark that by writing $|\Lambda|^1 g = |\Lambda|^1 V \otimes |\Lambda|^1 \mathfrak{g}$, one deduces that the integral is well defined with values in $|\Lambda|^1 \mathfrak{g}$ which is the fiber of $s^*|\Lambda|^\frac{1}{2}TF \otimes r^*|\Lambda|^\frac{1}{2}TF$ at $(gV, x)$. A straightforward computation shows that $\int$ is a $*$-algebra homomorphism. Therefore it extends to a $C^\ast$-algebra homomorphism

$$\int : G \times_{\text{max}} C_0(M) \to C^\ast_{\text{max}} \mathcal{H}\text{blup}(G \times M).$$

We remark that the map $\int$ isn’t defined for the reduced $C^\ast$-algebra in general. For example if $M = \{pt\}$, then the right hand side is $\mathbb{C}$, and $\int : C^\ast_{\text{max}} G \to \mathbb{C}$ is just the trivial representation.

2 Extension to general singular foliations

In this Section, we will follow the notation of [AS09]. Recall that if $G$ is a Lie group acting on $M$, then one has a singular foliation $\mathcal{F}$ on $M$ defined by

$$\mathcal{F} = \{ \sum_{i=1}^k f_i \#(X_i) : k \in \mathbb{N}, f_i \in C^\infty_c(M, \mathbb{R}), X_i \in \mathfrak{g} \}.$$

In Section 1 we defined the blow-up space and blow-up groupoid for foliations coming from group actions. The construction in the general case is a straightforward modification using the language of bi-submersions and morphisms of bi-submersions developed in [AS09].

2.1 Blow-up space

Let $\mathcal{F}$ be a singular foliation on $M$, see [AS09] Definition 1.1]. We denote by $\#_x$ the natural linear map $\mathcal{F}_x \to F_x$. We remark that the Lie bracket of vector fields endows $\mathfrak{h}_x := \ker(\#_x)$ with the structure of a Lie algebra.

As in the introduction, let $M_{\text{reg}}$ be the set of $x \in M$ such that $\#_x$ is an isomorphism.
Let $x \in M$. If $X \in \mathcal{F}$, then we will denote by $[X]_x \in \mathcal{F}/I_x \mathcal{F}$ the class of $X$. Let $X_1, \ldots, X_k \in \mathcal{F}$ such that $[X_1]_x, \ldots, [X_k]_x$ form a basis of $\mathcal{F}_x$. By [AS09] Proposition 1.5.a, there exists a neighbourhood $U$ of $x$ such that $\mathcal{F}$ restricted to $U$ is generated by $X_1, \ldots, X_k$. In particular if $y \in U$, then $[X_1]_y, \ldots, [X_k]_y$ generate $\mathcal{F}_y$. Let $\phi_y$ be the surjective linear map defined by

$$\phi_y : \mathcal{F}_x \to \mathcal{F}_y, \quad \phi_y([X_i]_x) = [X_i]_y, \quad \forall i \in \{1, \ldots, k\}.$$  

**Definition 2.1.** The set $\text{Blup}(\mathcal{F})_x$ denotes the set of vector subspaces $V \in \mathcal{F}_x$ such that there exists a sequence $x_n \in M_{\text{reg}}$ such that $x_n \to x$ and $\ker(\phi_{x_n}) \to V$, where the latter convergence is in $\text{Gr}(\mathcal{F}_x)$.

**Proposition 2.2.** Let $x \in M$, $V \in \text{Blup}(\mathcal{F})_x$.

a) The set $\text{Blup}(\mathcal{F})_x$ is a non-empty closed subset of $\text{Gr}(\mathcal{F}_x)$ which doesn’t depend on the choice of $X_1, \ldots, X_k$.

b) One has $V \subseteq \mathfrak{h}_x$.

c) The vector subspace $V$ is a Lie subalgebra of $\mathfrak{h}_x$.

d) If $x \in M_{\text{reg}}$, then $\text{Blup}(\mathcal{F})_x = \{\{0\}\}$.

**Proof.**

a) We will only prove the independence of the choice of $X_1, \ldots, X_k$.

The rest is proved similarly to Proposition [3]. Let $Y_1, \ldots, Y_k \in \mathcal{F}$ such that $[Y_1]_x, \ldots, [Y_k]_x$ for a basis of $\mathcal{F}_x$. Suppose $U$ is small enough that $\mathcal{F}$ restricted to $U$ is generated by $X_1, \ldots, X_k$ and is also generated by $Y_1, \ldots, Y_k$. It follows that there exists smooth functions $f_{ij} \in C^\infty(U, \mathbb{R})$ such that

$$X_i = \sum_{j=1}^k f_{ij} Y_j, \quad \text{(on } U).$$

Since both $[X_1]_x, \ldots, [X_k]_x$ and $[Y_1]_x, \ldots, [Y_k]_x$ form a basis of $\mathcal{F}_x$, it follows that the matrix $(f_{ij}(x))_{1 \leq i, j \leq k}$ is invertible. We can suppose $U$ is small enough that for each $y \in U$, the matrix $(f_{ij}(y))_{1 \leq i, j \leq k}$ is invertible.

Let $(f^{ij}(y))$ be the inverse, $\psi_y : \mathcal{F}_x \to \mathcal{F}_y$ be the linear map defined by $\psi_y([Y_i]_x) = [Y_i]_y$, $L_y : \mathcal{F}_x \to \mathcal{F}_x$ be the linear map defined by

$$L_y([X_i]_x) = \sum_{j,l=1}^k f_{ij}(y) f^{kl}(x) [X_l]_x.$$

A direct computation shows that $\psi_y \circ L_y = \phi_y$. Since $L_y$ converges to the identity as $y \to x$, the result follows.

b) This follows directly from the definition.

c) Since $\mathcal{F}$ is closed under Lie brackets, there exists smooth functions $f_{ij}^l \in C^\infty(U, \mathbb{R})$ such that

$$[X_i, X_j] = \sum_{l=1}^k f_{ij}^l X_l, \quad \text{(on } U).$$

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Let $A, B \in V$ with
\[
A = \sum_{i=1}^{k} a_{i}[X_{i}], \quad B = \sum_{i=1}^{k} b_{i}[X_{i}].
\]

By definition there exists $x_{n} \in M_{\text{reg}}$ such that $x_{n} \to x$ and $\ker(\phi_{x_{n}}) \to V$.
Hence there exists sequences $a_{i}^{n}, \cdots, a_{k}^{n}, b_{1}^{n}, \cdots, b_{k}^{n}$ such that $a_{i}^{n} \xrightarrow{n \to \infty} a_{i}, b_{i}^{n} \xrightarrow{n \to \infty} b_{i}$ for all $i$ and $\sum_{i=1}^{k} a_{i}^{n}[X_{i}]_{x} \in \ker(\phi_{x_{n}})$ and $\sum_{i=1}^{k} b_{i}^{n}[X_{i}]_{x} \in \ker(\phi_{x_{n}})$. In other words
\[
\sum_{i=1}^{k} a_{i}^{n}X_{i}(x_{n}) = \sum_{i=1}^{k} b_{i}^{n}X_{i}(x_{n}) = 0.
\]
Hence
\[
[\sum_{i=1}^{k} a_{i}^{n}X_{i}, \sum_{i=1}^{k} b_{i}^{n}X_{i}](x_{n}) = 0.
\]
Since $x_{n} \in M_{\text{reg}}$, one deduces that
\[
\sum_{i,j,l=1}^{k} a_{i}^{n}b_{j}^{n}f_{ij}^{l}(x)[X_{l}]_{x} \in \ker(\phi_{x_{n}}).
\]
Hence $\sum_{i,j,l=1}^{k} a_{i}^{n}b_{j}^{n}f_{ij}^{l}(x)[X_{l}]_{x} \in V$. The result follows.

d) Since $b_{x} = 0$ if $x \in M_{\text{reg}}$, the result follows.

Let $\phi$ be an automorphism of $\mathcal{F}$. This means that $\phi : M \to M$ is a diffeomorphism such that if $X \in \mathcal{F}$, then $\phi_{*}(X) \in \mathcal{F}$. For $x \in M$, one has a linear map $\phi_{*} : \mathcal{T}_{x} \to \mathcal{T}_{\phi(x)}$ given by $\phi_{*}([X]_{x}) = [\phi_{*}(X)]_{\phi(x)}$. The following follows directly from Proposition 2.2.a.

**Proposition 2.3.** If $x \in M$, $V \in \text{Blup}(\mathcal{F})_{x}$, then $\phi_{*}(V) \in \text{Blup}(\mathcal{F})_{\phi(x)}$.

**Topology.** Let $\text{Blup}(\mathcal{F}) = \{(V, x) : x \in M, V \in \text{Blup}(\mathcal{F})_{x}\}$. We equip $\text{Blup}(\mathcal{F})$ with the topology such that

- the natural projection $\pi : \text{Blup}(\mathcal{F}) \to M$ is continuous
- if $x \in M$, $U \subseteq M$ an open neighbourhood of $x \in M$, $X_{1}, \cdots, X_{k} \in \mathcal{F}$ as above, then the inclusion
  \[
  \pi^{-1}(U) \cap \text{Blup}(\mathcal{F}) \to \text{Gr}(\mathcal{F}_{x}) \times U, \quad (V, y) \to (\phi_{y}^{-1}(V), y)
  \]
  is an embedding.

It is straightforward to check that the topology on $\text{Blup}(\mathcal{F})$ is well defined.

**Examples 2.4.** a) Let $N \subseteq M$ be a smooth submanifold, $\mathcal{F}$ the module of compactly supported vector fields on $M$ which vanish on $N$. Then one sees that
\[
F_{x} = \begin{cases} 
T_{x}M & \text{if } x \notin N, \\
0 & \text{if } x \in N,
\end{cases}
\]
\[
\mathcal{F}_{x} = \begin{cases} 
T_{x}M & \text{if } x \notin N, \\
\text{Hom}(T_{x}M, T_{x}N) & \text{if } x \in N.
\end{cases}
\]
Hence $M_{\text{reg}} = N^c$. A direct computation shows that for $x \in N$,
\[
\text{Blup}(\mathcal{F})_x = \{ \ker (v) : v \in T_x M / T_x N, v \neq 0 \},
\]
where $\ker (v) = \{ L \in \text{Hom}(T_x M / T_x N) : L(v) = 0 \}$. Hence $\text{Blup}(\mathcal{F})_x$ can be identified with the projective space $\mathbb{P}(T_x M / T_x N)$. Furthermore the space $\text{Blup}(\mathcal{F})$ is then homeomorphic to the classical blow-up of $M$ along $N$.

b) Consider $M = \mathbb{R}^2 \times \mathbb{R}$ with variables $(x, y, t)$. Let $X = \partial_x$, $Y = x \partial_y$,
\[
Z = \partial_y.
\]
\[
\mathcal{F} = \langle tX, t^2Y, t^3Z \rangle.
\]

Then a direct computation shows that
\[
F_{(x,y,t)}(\partial_x \partial_x \mathbb{R} \partial_y \mathbb{R}, s) = \begin{cases} 
\partial_x + \partial_y & \text{if } t \neq 0 \\
0 & \text{if } t = 0.
\end{cases}
\]

On the other hand one has
\[
\mathcal{F}_{(x,y,t)} = \begin{cases} 
\mathbb{R}[tX](x,y,t) \oplus \mathbb{R}[t^3Z](x,y,t) & \text{if } t \neq 0 \\
\mathbb{R}[tX](x,y,t) \oplus \mathbb{R}[t^2Y](x,y,t) & \text{if } x \neq 0 \\
\mathbb{R}[tX](x,y,t) \oplus \mathbb{R}[t^2Y](x,y,t) \oplus \mathbb{R}[t^3Z](x,y,t) & \text{if } x = t = 0.
\end{cases}
\]

One has $\mathcal{F}_{(x,y,0)} = \mathfrak{h}(x,y,0)$ is a Lie algebra. If $x \neq 0$, then Lie algebra is commutative, but if $x = 0$ it is the Heisenberg Lie algebra.

Clearly $M_{\text{reg}} = \mathbb{R}^2 \times \mathbb{R}^*$. One can check that in this case
\[
\text{Blup}(\mathcal{F})_{(x,y,t)} = \begin{cases} 
\{0\} & \text{if } t \neq 0 \\
\{0\} & \text{if } x \neq 0 \\
\mathbb{R}[t^2Y](0,y,0) - \lambda [t^3Z](0,y,0) : \lambda \in \mathbb{R} & \text{if } x = t = 0.
\end{cases}
\]

### 2.2 Blow-up groupoid.

We restrict ourselves to the minimal atlas ([AS09](#) Definition 3.1 and Examples 3.4). Let $(U, r_U, s_U)$ be a bi-submersion ([AS09](#) Definition 2.1), $u \in U$. It follows from the definition of bi-submersions that the linear map
\[
d_u s_U : \ker (dr_U)_u \rightarrow F_{s_U(u)}
\]
lifts to a surjective linear map
\[
\vartheta_u s_U : \ker (dr_U)_u \rightarrow F_{s_U(u)}.
\]
To see this, let $v \in \ker (dr_U)_u$. Then let $X \in \Gamma(\ker (dr_U))$ be any vector field such that $X(u) = v$. By the definition of a bi-submersion, there exists $f_i \in C^\infty(U, \mathbb{R})$, $Y_i \in \mathcal{F}$ such that $ds_U(X) = \sum f_i Y_i \circ s_U$. We then define
\[
\vartheta_u s_U(v) := \sum f_i(u)[Y_i]_{s_U(u)} \in F_{s_U(u)}.
\]
It is straightforward to check that this is well defined linear map.

Let \( x \in M, V \in \text{Blup}(F)_x \). In what follows we denote \( s_{U_x}^{-1}(x) \) by \( U_x \) and \( r_{U_x}^{-1}(x) \) by \( U^x \). The vector spaces \( (\mathfrak{d}_u s_{U_x}^{-1}(V))_{\mathfrak{d}U_x} \) form a vector subbundle of \( \ker(\mathfrak{d}r_{U_x})_{U_x} \), which we denote by \( \mathfrak{d}s_{U_x}^{-1}(V) \). By Proposition 2.2.b it follows that
\[
\mathfrak{d}s_{U_x}^{-1}(V) \subseteq \ker(\mathfrak{d}s_{U_x})_{U_x} = T U_x.
\]
By Proposition 2.2.c, \( \mathfrak{d}s_{U_x}^{-1}(V) \) is a regular foliation in the manifold \( U_x \).

Let \( u \in U_x \). Then we denote by \( u \cdot \exp(V) \) the leaf of \( \mathfrak{d}s_{U_x}^{-1}(V) \) passing through \( u \) and by \( U_x / \exp(V) \) the quotient space of \( U_x \) by the foliation \( \mathfrak{d}s_{U_x}^{-1}(V) \) with the quotient topology.

We remark that since \( \mathfrak{d}s_{U_x}^{-1}(V) \subseteq \ker(\mathfrak{d}r_{U_x}) \), it follows that \( r_{U_x}(u \cdot \exp(V)) = \tau_{U}(u) \).

**Theorem 2.5.** The leaves of the foliation \( \mathfrak{d}s_{U_x}^{-1}(V) \) are all embedded submanifolds and the quotient space is a smooth manifold.

The proof of Theorem 2.5 is a direct modification of the proof of Theorem 1.8 where one replaces \( G \times M \) with the bi-submersion \( U \) and \( \phi \) with morphisms of bi-submersions, see [Deb13].

In fact the periodic bounding lemma implies the following refinement of Theorem 2.5.

**Theorem 2.6.** Let \( u \in U_x \), \( T \subseteq U \) a smooth transversal of \( \mathfrak{d}s_{U_x}^{-1}(V) \) at \( u \), in other words
\[
T \cdot T \oplus \mathfrak{d}s_{U_x}^{-1}(V)_u = T U_x.
\]
Then there exists a neighbourhood \( u \in T' \subseteq T \) and an neighbourhood \( L \) of \( (V, x) \) in \( \text{Blup}(F) \) such that

a) if \( u' \in T' \), \( (W, y) \in L \) such that \( u' \in U_y \), then \( T' \) is transverse to \( \mathfrak{d}s_{U_y}^{-1}(W) \) at \( u' \).

b) for every \( (W, y) \in L \), the map \( T' \cap U_y \to U_y / \exp(W) \) is a homeomorphism onto its open image.

**Sketch of the Proof of Theorem 2.5 and Theorem 2.6.** Let

- \( u_0 \in U \) be fixed and suppose for simplicity that \( u_0 \) represents the identity in \( H(F), x = u_0 \).
- \( X_1, \cdots, X_n \in F \) such that \( [X_1]_x, \cdots, [X_n]_x \) form a basis of \( F_x \)
- \( \phi_y : F_x \to F_y \) the linear map \( \phi([X_1]_x) = [X_i]_y \).
- \( x_n \in M_{\text{reg}} \) such that \( x_n \to x \) and \( \ker(\phi(x_n)) \to V \).
- \( \bar{x} = \{x_n : n \in N \} \sqcup \{x\} \) with the subspace topology from \( M \) and \( U_x = s_{U_x}^{-1}(\bar{x}) \).

The proof then proceeds exactly as the proof of Theorem 1.8.

**Lemma 2.7.** Let \( u \in u_0 \cdot \exp(V) \). Then there exists \( f : U_x \to U_x \) such that

- \( su \circ f = su \) and \( \tau_u \circ f = \tau_u \).
\begin{itemize}
  \item $f(u_0) = u$.
\end{itemize}

The proof of Lemma 2.7 is the same as that of Lemma 1.10. Let $W \subseteq M \times \mathbb{R}^k$ be the minimal bi-submersion constructed in [AS09, Proposition 2.10.a] using the vectors fields $X_1, \cdots, X_k$.

**Lemma 2.8.** Let $K$ be a compact neighbourhood of $x$. There exists $\eta > 0$ such that if $(y, t) \in W$ such that $r_W(y, t) = y$ and $\|t\| < \eta$ and $y \in K$, then $\sum t_i[X_i]_x \in \ker(\phi_y)$.

Lemma 2.8 is a parametrised periodic bounding lemma.

Let
\begin{itemize}
  \item $L$ be a bi-section of $U$ at $u_0$ carrying the identity, see [AS09, Section 2.2],
  \item $\tilde{X}_1, \cdots, \tilde{X}_k$ vector fields on $U$ such that $\tilde{X}_i \in \ker(ds_U)$ and $dr_U(\tilde{X}_i) = X_i \circ r_U$. See the proof of [AS09, Proposition 2.10.b] for the existence.
\end{itemize}

By the definition of $u_0$ being the identity element in $\mathcal{H}(F)$, there exists a morphism of bi-submersions

$$\log : U \to W$$

defined in a neighbourhood of $u_0$ which sends $u_0$ to $(x, 0)$. Furthermore the construction of the map log in [AS09, Proposition 2.10.b] shows that one can suppose that for any $t_1, \cdots, t_k \in \mathbb{R}$, $l \in L$, the following holds whenever it is defined

$$\log(\psi^{t_1}_{\tilde{X}_1}(\cdots(\psi^{t_k}_{\tilde{X}_k}(l)))) = (s_U(l), t_1, \cdots, t_k).$$

Here $\psi$ is the flow.

**Lemma 2.9.** There exists $\eta > 0$ such that for any $t \in \mathbb{R}^k$ with $\|t\| < \eta$. If

$$\psi^{t_1}_{\tilde{X}_1}(\cdots(\psi^{t_k}_{\tilde{X}_k}(u_0))) \in u_0 \cdot \exp(V),$$

then $\sum t_i[X_i]_x \in V$

Proof. We choose $\eta$ similarly to Lemma 1.12. Let $f$ be a map given by Lemma 2.7 applied to $\psi^{t_1}_{\tilde{X}_1}(\cdots(\psi^{t_k}_{\tilde{X}_k}(u_0)))$. Then since $L$ is a bi-section for $n$ big enough there exists a unique $l_n \in L$ such that $s_U(l_n) = r_U(l_n) = x_n$. Then one has

$$\log(f(l_n)) = (x_n, p_n)$$

for some $p_n \in \mathbb{R}^k$. Since $r_U(\log(f(l_n))) = r_U(f(l_n)) = r_n(l_n) = x_n$ it follows from Lemma 2.8 that $\sum p_{ni}[X_i]_x \in \ker(\phi_{x_n})$, where $p_{ni}$ are the components of $p_n$. Since $\log(f(l_n)) \to (x, t)$, one deduces that $p_n \to t$. Hence $\sum t_i[X_i]_x = \lim_n \sum p_{ni}[X_i]_x \in V$.

It is then straightforward to adapt the rest of the proof of Theorem 1.8 to show Theorem 2.5 and Theorem 2.6.

**Blow-up groupoid.**

**Proposition 2.10.** Let $(U, r_U, s_U), (U', r_{U'}, s_{U'})$ be bi-submersions, $f : U \to U'$ a morphism of bi-submersions, $u \in U$, $V \in \Blup(F)_{s_U(u)}$. Then $f(u \cdot \exp(V)) \subseteq f(u) \cdot \exp(V)$.
The range of \( p \) of \( W \)

We now define the structural maps. The source map is given by \( L \) defined in a neighbourhood of \( \phi \), and follows from Proposition 2.10. Notice that this immediately implies that \( V = V' \) and \( s_U(u) = s_{U'}(u) \) and \( r_U(u) = r_{U'}(u) \). This is an equivalence relation by [AS09, Corollary 2.11].

We remark that \((U, u \cdot \exp(V), s_U(u))\) is equivalent to \((U, u' \cdot \exp(V), s_U(u'))\) if \( u' \in u \cdot \exp(V) \) by simply taking \( f \) to be the identity.

\textbf{Source and range maps.} We now define the structural maps. The source map is given by

\[
s_{\text{blup}(F)}(U, u \cdot \exp(V), s_U(u)) = (V, s_U(u)) \in \text{Blup}(F).
\]

The range of \((U, u \cdot \exp(V), s_U(u))\) is defined as follows. Let \( L \) be any bi-section of \( \text{U} \) at \( u \). If \( \phi : \text{dom}(\phi) \subseteq M \rightarrow M \) is an automorphism of \( F \) associated to \( L \) defined in a neighbourhood of \( s_U(u) \). Recall that by the definition of \( \phi \), \( \phi(s_U(u)) = r_U(u) \). We define

\[
r_{\text{blup}(F)}(U, u \cdot \exp(V), s_U(u)) = (\phi_*(V), r_U(u)).
\]

\textbf{Proposition 2.11.} The range map \( r_{\text{blup}(F)} \) is well defined.

\textit{Proof.} \( \star \) If \( L, L' \) are two bi-sections at \( u \). Then by [AS09, Proof of Corollary 2.11], there exists a morphism of bi-submersions \( f : U \rightarrow U \) (locally defined around \( u \)) such that \( f(L) = L' \). The independence of the choice of \( L \) then follows from Proposition 2.10.

\( \star \) If \( f : U \rightarrow U' \) is a morphism of bi-submersion, \( L \subseteq U \) is a bi-section, then \( f(L) \) is also a bi-section, see [AS09, Proof of Corollary 2.11]. It follows that \( r_{\text{blup}(F)}(U, u \cdot \exp(V), s_U(u)) \) only depends on the equivalence class of \((U, u \cdot \exp(V), s_U(u))\).

Here is a more conceptual way to define the range map. One can define

\[
\partial_u r_U : \ker(d_{s_U})_u \rightarrow F_{r_U(u)}
\]

just like the map \( \partial s_U \). One then has

\begin{equation*}
r_{\text{blup}(F)}(U, u \cdot \exp(V), s_U(u)) = (\partial_U(\partial s_U^{-1}(V)), r_U(u)).
\end{equation*}

\footnote{For \( \text{blup}(F) \) to be a set, one has to restrict to bi-submersions which are subsets of \( \mathbb{R}^\infty \).}
**Product and inverse.** Let \((U, u \cdot \exp(V), s_{U}(u)) \) be elements of \(\mathcal{Hblup}(\mathcal{F})\) such that

\[
s_{\mathcal{Hblup}(\mathcal{F})}(U', u' \cdot \exp(V'), s_{U'}(u')) = r_{\mathcal{Hblup}(\mathcal{F})}(U, u \cdot \exp(V), s_{U}(u)).
\]

This implies that \(s_{U'}(u') = r_{U}(u)\). Hence \((u', u) \in U' \circ U\) ([AS09, Proposition 2.4]). We define

\[
(U', u' \cdot \exp(V'), s_{U'}(u')) \cdot (U, u \cdot \exp(V), s_{U}(u)) = (U' \circ U, (u', u) \cdot \exp(V), s_{U}(u)).
\]

For the inverse, we define

\[
(U, u \cdot \exp(V), s_{U}(u))^{-1} = (U^{-1}, u \cdot \exp(\partial_{U} \partial_{U}^{-1}(V)), r_{U}(u)).
\]

\(U^{-1}\) is the inverse bi-submersion of \(U\), see [AS09, Proposition 2.4].

**Identity.** Let \(x \in M, X_{1}, \cdots, X_{k} \in \mathcal{F}\) such that \([X_{1}]_{x}, \cdots, [X_{k}]_{x}\) form a basis of \(\mathcal{F}_{x}\). Let \(U \subseteq M \times \mathbb{R}^{k}\) the bi-submersion given by [AS09, Proposition 2.10.a]. Then the identity at \((V, x)\) is equal to

\[
(U, (x, 0) \cdot \exp(V), x).
\]

It is straightforward to check that \(\mathcal{Hblup}(\mathcal{F}) \rightrightarrows \text{Blup}(\mathcal{F})\) is an algebraic groupoid.

We remark that if \((V, x) \in \text{Blup}(\mathcal{F})\), then \(\mathcal{Hblup}(\mathcal{F})(V, x)\) can be naturally identified with \(\mathcal{H}(\mathcal{F})_{x}/\exp(V)\). Since if \(U\) is a minimal bi-submersion \((\dim(U) = \dim(M) + \dim(\mathcal{F}_{s_{U}(u)}))\), then \(U_{x}\) is an open subset of \(\mathcal{H}(\mathcal{F})_{x}\). It follows from Theorem 2.4 that \(\mathcal{Hblup}(\mathcal{F})\) is longitudinally smooth. Its algebroid vector bundle \(\mathfrak{Hblup}(\mathcal{F})\) is the vector bundle over \(\text{Blup}(\mathcal{F})\) whose fiber at \((V, x)\) is equal to \(\mathfrak{T}_{x}\).

**Examples 2.12.** We follow the notation of Examples 2.4

a) In this Example, the groupoids \(\mathcal{H}(\mathcal{F})\) is given by

\[
N^{c} \times N^{c} \sqcup_{x \in N} \text{Hom}(\frac{T_{x}M}{T_{x}N}, T_{x}M),
\]

where the group structure on \(\text{Hom}(\frac{T_{x}M}{T_{x}N}, T_{x}N)\) is the commutative structure from addition of linear maps. This description of \(\mathcal{H}(\mathcal{F})\) follows from [AZ13, Thm. 4.1].

On the other hand a straightforward computation shows that

\[
\mathcal{Hblup}(\mathcal{F}) = N^{c} \times N^{c} \sqcup_{x \in N} \left\{((X, Y) \in \frac{T_{x}M \times T_{x}M}{T_{x}N} : X, Y \neq 0)/\mathbb{R}^{*}\right\},
\]

where the \(\mathbb{R}^{*}\)-action is given by \(\lambda \cdot (X, Y) = (\lambda X, \lambda Y)\). We remark that \(\mathcal{Hblup}(\mathcal{F})\) is naturally an open subset of the blow-up of \(M \times M\) along the diagonal \(N\). This groupoid appears [DS17].

b) Again by [AZ13, Thm. 4.1], one deduces that \(\mathcal{H}(\mathcal{F})\) is equal to

\[
((\mathbb{R}^{2} \times \mathbb{R}^{*}) \times (\mathbb{R}^{2} \times \mathbb{R}^{*})) \sqcup_{(x, y, 0) \in \mathbb{R}^{*} \times \mathbb{R} \times \{0\}} \mathbb{R}^{2} \sqcup_{(y, 0, 0) \in \{0\} \times \mathbb{R} \times \{0\}} \mathbb{R}^{3},
\]

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Theorem 2.6 implies that \( \psi \) defined using \( d \) we obtain independence of the topology from \( \psi \) defined using \( u \).

One can define the vector bundle only need them to be defined locally around \( u \) at \( d \).

Proposition 2.13. Let \( (U, u \cdot \exp(V), s_U) \) be an element of \( \mathcal{H}blup(F) \). Let \( T, L \) be as in Theorem 2.6. Then we define a map

\[
\psi : T \times_{s_U} L \to \mathcal{H}blup(F) \quad (t, (W, y)) \to (U, t \cdot \exp(W), y).
\]

Theorem 2.6 implies that \( \psi \) is injective. We define the topology on \( \mathcal{H}blup(F) \) by declaring the image of \( \psi \) an open set and \( \psi \) an embedding.

**Proposition 2.13.** The topology on \( \mathcal{H}blup(F) \) is well defined. The space \( \mathcal{H}blup(F) \) is locally compact locally metrizable.

**Proof.** We prove well definedness of the topology. This means that if one chooses a different \( \psi \) with different data \((T, L, (U, u \cdot \exp(V), s_U))\) the transition function is continuous. Let \( U' \) be a bi-submersion, \( f : U \to U' \) a morphism of bi-submersions with \( f \) a submersion. One has

\[
\partial_u s_U = (\partial_{f(u)} s_{U'}) \circ (d_u f).
\]

Equation 2.1 implies that \( \ker(df)_{|U \cap (U')} \subseteq \mathcal{H}blup^{-1}(V) \). Since \( T \) is transversal to \( \mathcal{H}blup^{-1}(V) \), we obtain that after possibly reducing \( T \) to a small neighbourhood of \( u \in T \), \( f(T) \) is a smooth manifold and \( f : T \to f(T') \) is a diffeomorphism at \( u \). By Equation 2.1, \( f(T) \) is transversal to \( \mathcal{H}blup^{-1}(V) \). It is then clear that \( \psi \) defined using \((T, L, (U, u \cdot \exp(V), s_U))\) is compatible with the one defined using \((f(T), L, (U', f(u) \cdot \exp(V), s_U))\). By [AS09] the proof of Corollary 2.11, we obtain independence of the topology from \( U \).

We now prove independence of \( T \). Let \( x \in M, W \subseteq H \) a subspace. One can define the vector bundle \( \mathcal{H}blup^{-1}(W) \subseteq T U_x \). We choose any basis of \( W, X_1, \cdots, X_n \). We choose any lifts of \( X_1, \cdots, X_n \) to sections \( \bar{X}_1, \cdots, \bar{X}_n \in \mathcal{H}blup^{-1}(W) \). One can choose \( \tilde{X}_t \) to depend smoothly on \( X_1, \cdots, X_n \) because we only need them to be defined locally around \( u \). We then complete them with \( X_{n+1}, \cdots, X_k \) of basis of \( \ker(\partial) \). Consider the map

\[
\mathbb{R}_k \times T_x \to U_x, \quad ((t_1, \cdots, t_k), t) \mapsto \phi_{\bar{X}_1}^t (\cdots (\phi_{\bar{X}_k}^{t_k} (t)) \cdots),
\]

where \( \phi \) means the flow. By hypothesis on \( T \) for \( W \) close enough to \( V \), the map 2.2 is a local diffeomorphism at \( (0, u) \). It is clear that the transition map for different choices of \( T \) is simply a composition of a map of the form 2.2 and its inverse. This proves that the topology is well defined.

The rest of assertions follow from the fact that \( \psi \) is an embedding.

We remark that \( \mathcal{H}blup(F) \) isn’t necessarily Hausdorff because if \( F \) is regular then \( \mathcal{H}blup(F) \) is then the holonomy groupoid \( \mathcal{H}(F) \) of \( F \) which fails in general to be Hausdorff.

Notice that in the proof proof of Proposition 2.13, the transition functions are smooth as in Section 1.3. Notice that all the structural maps are smooth.
2.3 Integration

Let $U$ be a bi-submersion, $(\Omega^2 U)_u = |A|^{\frac{2}{n}} \ker(ds_u) \otimes |A|^{\frac{2}{n}} \ker(dr_U) u$, see [AS09] Section 4.

Let $u \in U, V \in \text{Blup}(\mathcal{F})_{sv(u)}$. Consider the exact sequence

$$0 \to \mathcal{D}S_{U}^{-1}(V) \to \ker(dr_U) \to \frac{\mathcal{F}_{sv(u)}}{V} \to 0.$$

It follows that we have a canonical isomorphism

$$|A|^{\frac{2}{n}} \frac{\mathcal{F}_{sv(u)}}{V} \otimes |A|^{\frac{2}{n}} \mathcal{D}S_{U}^{-1}(V) u \simeq |A|^{\frac{2}{n}} \ker(ds_u) u.$$

We also have the short exact sequence

$$0 \to \mathcal{D}S_{U}^{-1}(V) u \to \ker(ds_U) \to \frac{\mathcal{F}_{sv(u)}}{\mathcal{D}r_U(\mathcal{D}S_{U}^{-1}(V))} \to 0.$$

Hence a canonical isomorphism

$$|A|^{\frac{2}{n}} \frac{\mathcal{F}_{sv(u)}}{\mathcal{D}r_U(\mathcal{D}S_{U}^{-1}(V))} \otimes |A|^{\frac{2}{n}} \mathcal{D}S_{U}^{-1}(V) u \simeq |A|^{\frac{2}{n}} \ker(ds_u) u.$$

Taking the tensor product of the two isomorphisms we get

$$|A|^{\frac{2}{n}} \mathcal{F}_{sv(u)} \otimes |A|^{\frac{2}{n}} \mathcal{D}S_{U}^{-1}(V) u \simeq (\Omega^2 U) u.$$

Let $\mathcal{H}\text{Blup(\mathcal{F})}$ be the Lie algebroid vector bundle of $\text{Blup(\mathcal{F})}$. Then let $\Omega^2 \mathcal{H}\text{Blup(\mathcal{F})}$ be the vector bundle whose fiber at $u$ is equal to

$$\left(\Omega^2 \mathcal{H}\text{Blup(\mathcal{F})}\right)_u = |A|^{\frac{2}{n}} \mathcal{S}_{\text{Blup(\mathcal{F})}} \mathcal{H}\text{Blup(\mathcal{F})} \otimes |A|^{\frac{2}{n}} \mathcal{H}\text{Blup(\mathcal{F})} = |A|^{\frac{2}{n}} \mathcal{F}_{sv(u)} \otimes |A|^{\frac{2}{n}} \mathcal{D}S_{U}^{-1}(V) u.$$

Let $f \in \Gamma_\times(U, \Omega^2 U), u \in U$. Then since the tangent bundle of $u \cdot \exp(U)$ is equal to $|A|^{\frac{2}{n}} \mathcal{D}r_U^{-1}(V)$, we get that the following integral is well defined

$$\int_{u \cdot \exp(U)} f \in \left(\Omega^2 \mathcal{H}\text{Blup(\mathcal{F})}\right)_u.$$

Let $\mathcal{A}$ be the algebra defined in [AS09] Section 4.3. We have thus defined a map

$$\int : \mathcal{A} \to \Gamma_\times(\mathcal{H}\text{Blup(\mathcal{F})}, \Omega^2 \mathcal{H}\text{Blup(\mathcal{F})}).$$

It is straightforward to check that this map is a $*$-algebra homomorphism. Hence it passes to the completion

$$\int : C^*_{\max} \mathcal{F} \to C^*_{\max} \mathcal{H}\text{Blup(\mathcal{F})}.$$
2.4 Lichnerowicz theorem for smooth groupoids

Let $G$ be a Lie groupoid such that $G^0$ is compact, $\mathfrak{A}G$ its Lie algebroid. We suppose that $\mathfrak{A}G$ is equipped with a Euclidean metric and is spin. It follows that one can construct the longitudinal Dirac operator on $G$ (with respect to the foliation $\ker(ds_G)$). Hence the theory developed in [NWX99, Vas06] applies to the operator $D$. In particular one obtains the following theorem

**Theorem 2.14.** If $\mathfrak{A}G$ admits a Euclidean metric such that the induced metric on $G_x$ has positive scalar curvature for every $x \in G^0$. Then the analytic index of the Dirac operator $\text{Ind}_a(D) \in K_0(C_\ast^* G)$ vanishes.

The theory developed in [NWX99, Vas06] is done for Lie groupoids but it also applies to groupoids which are longitudinally smooth, locally metrizable, and whose Lie algebroid forms a vector bundle. In particular it applies to $\mathcal{H}\text{blup}(\mathcal{F})$. Since on $M_{\text{reg}}$, $\mathcal{H}\text{blup}(\mathcal{F})$ is equal to the holonomy groupoid of the regular foliation $\mathcal{F}_{\text{reg}}$, one obtains Theorem 0.4

References

[AR67] Ralph Abraham and Joel Robbin. *Transversal mappings and flows*. W. A. Benjamin, Inc., New York-Amsterdam, 1967. An appendix by Al Kelley.

[AS09] Iakovos Androulidakis and Georges Skandalis. The holonomy groupoid of a singular foliation. *J. Reine Angew. Math.*, 626:1–37, 2009.

[AS19] Iakovos Androulidakis and Georges Skandalis. A baum–connes conjecture for singular foliations. *Annals of K-Theory*, 4(4):561–620, Dec 2019.

[AZ13] Iakovos Androulidakis and Marco Zambon. Smoothness of holonomy covers for singular foliations and essential isotropy. *Math. Z.*, 275(3-4):921–951, 2013.

[BCH94] Paul Baum, Alain Connes, and Nigel Higson. Classifying space for proper actions and K-theory of group $C^*$-algebras. In *C*-algebras: 1943–1993 (San Antonio, TX, 1993), volume 167 of *Contemp. Math.*, pages 240–291. Amer. Math. Soc., Providence, RI, 1994.

[Con82] A. Connes. A survey of foliations and operator algebras. In *Operator algebras and applications, Part I (Kingston, Ont., 1980)*, volume 38 of *Proc. Sympos. Pure Math.*, pages 521–628. Amer. Math. Soc., Providence, R.I., 1982.

[Con86] A. Connes. Cyclic cohomology and the transverse fundamental class of a foliation. In *Geometric methods in operator algebras (Kyoto, 1983)*, volume 123 of *Pitman Res. Notes Math. Ser.*, pages 52–144. Longman Sci. Tech., Harlow, 1986.

[CS84] A Connes and G Skandalis. The longitudinal index theorem for foliations. *Publ. Res. Inst. Math. Sci.*, 20(6):1139–1183, 1984.
[Deb13] Claire Debord. Longitudinal smoothness of the holonomy groupoid. *C. R. Math. Acad. Sci. Paris*, 351(15-16):613–616, 2013.

[DS17] Claire Debord and Georges Skandalis. Blowup constructions for lie groupoids and a boutet de monvel type calculus, 2017.

[DS19] Claire Debord and Georges Skandalis. Lie groupoids, pseudodifferential calculus and index theory, 2019.

[Gro20] Misha Gromov. Four lectures on scalar curvature, 2020.

[HLS02] N. Higson, V. Lafforgue, and G. Skandalis. Counterexamples to the Baum-Connes conjecture. *Geom. Funct. Anal.*, 12(2):330–354, 2002.

[Kas88] G. G. Kasparov. Equivariant $KK$-theory and the Novikov conjecture. *Invent. Math.*, 91(1):147–201, 1988.

[Lic63] André Lichnerowicz. Spineurs harmoniques. *C. R. Acad. Sci. Paris*, 257:7–9, 1963.

[NWX99] Victor Nistor, Alan Weinstein, and Ping Xu. Pseudodifferential operators on differential groupoids. *Pacific J. Math.*, 189(1):117–152, 1999.

[Ozo72] V. Ozols. Critical points of the length of a Killing vector field. *J. Differential Geometry*, 7:143–148, 1972.

[Ste74] P Stefan. Accessible sets, orbits, and foliations with singularities. *Proc. London Math. Soc. (3)*, 29:699–713, 1974.

[Sus73] Héctor J Sussmann. Orbits of families of vector fields and integrability of distributions. *Trans. Amer. Math. Soc.*, 180:171–188, 1973.

[Tu99] Jean-Louis Tu. La conjecture de Baum-Connes pour les feuilletages moyennables. *K-Theory*, 17(3):215–264, 1999.

[Vas01] S. Vassout. Feuilletages et résidu non commutatif longitudinal, 2001.

[Vas06] Stéphane Vassout. Unbounded pseudodifferential calculus on Lie groupoids. *J. Funct. Anal.*, 236(1):161–200, 2006.