Implicit Regularization in Deep Learning
May Not Be Explainable by Norms

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Abstract

Mathematically characterizing the implicit regularization induced by gradient-based optimization is a longstanding pursuit in the theory of deep learning. A widespread hope is that a characterization based on minimization of norms may apply, and a standard test-bed for studying this prospect is matrix factorization (matrix completion via linear neural networks). It is an open question whether norms can explain the implicit regularization in matrix factorization. The current paper resolves this open question in the negative, by proving that there exist natural matrix factorization problems on which the implicit regularization drives all norms (and quasi-norms) towards infinity. Our results suggest that, rather than perceiving the implicit regularization via norms, a potentially more useful interpretation is minimization of rank. We demonstrate empirically that this interpretation extends to a certain class of non-linear neural networks, and hypothesize that it may be key to explaining generalization in deep learning.

1 Introduction

A central mystery in deep learning is the ability of neural networks to generalize when having far more learnable parameters than training examples. This generalization takes place even in the absence of any explicit regularization (see [72]), thus a view by which gradient-based optimization induces an implicit regularization has arisen (see, e.g., [55]). Mathematically characterizing this implicit regularization is regarded as a major open problem in the theory of deep learning (cf. [57]). A widespread hope (initially articulated in [56]) is that a characterization based on minimization of norms (or quasi-norms) may apply. Namely, it is known that for linear regression, gradient-based optimization converges to solution with minimal $\ell_2$ norm (see for example Section 5 in [72]), and the hope is that this result can carry over to neural networks if we allow $\ell_2$ norm to be replaced by a different (possibly architecture- and optimizer-dependent) norm (or quasi-norm).

A standard test-bed for studying implicit regularization in deep learning is matrix completion (cf. [30]): given a randomly chosen subset of entries from an unknown matrix $W^*$, the task is to recover the unseen entries. This may be viewed as a prediction problem, where each entry in $W^*$ stands for a data point: observed entries constitute the training set, and the average reconstruction error over the unobserved entries is the test error, quantifying generalization. Fitting the observed entries is obviously an underdetermined problem with multiple solutions. However, an extensive body of work (see [23] for a survey) has shown that if $W^*$ is low-rank, certain technical assumptions (e.g. “incoherence”) are satisfied and sufficiently many entries are observed, then various algorithms

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1 A quasi-norm $\| \cdot \|$ on a vector space $V$ is a function from $V$ to $\mathbb{R}_{\geq 0}$ that satisfies the same axioms as a norm, except for the triangle inequality $\forall v_1, v_2 \in V : \|v_1 + v_2\| \leq \|v_1\| + \|v_2\|$, which is replaced by the weaker requirement $\exists c \geq 1 \ s.t. \ \forall v_1, v_2 \in V : \|v_1 + v_2\| \leq c \cdot (\|v_1\| + \|v_2\|)$. 

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One may try to solve matrix completion using shallow neural networks. A natural approach, matrix factorization, boils down to parameterizing the solution as a product of two matrices — \( W = W_2 W_1 \) — and optimizing the resulting (non-convex) objective for fitting observations. Formally, this can be viewed as training a depth 2 linear neural network. It is possible to explicitly constrain the rank of the produced solution by limiting the shared dimension of \( W_1 \) and \( W_2 \). However, Gunasekar et al. have shown in (30) that in practice, even when the rank is unconstrained, running gradient descent with small learning rate (step size) and initialization close to the origin (zero) tends to produce low-rank solutions, and thus allows accurate recovery if \( W^* \) is low-rank. Accordingly, they conjectured that the implicit regularization in matrix factorization boils down to minimization of nuclear norm:

**Conjecture 1** (from (30), informally stated). With small enough learning rate and initialization close enough to the origin, gradient descent on a full-dimensional matrix factorization converges to a minimal nuclear norm solution.

In a subsequent work — (6) — Arora et al. considered deep matrix factorization, obtained by adding depth to the setting studied in (30). Namely, they considered solving matrix completion by training a depth \( L \) linear neural network, i.e. by running gradient descent on the parameterization \( W = W_L W_{L-1} \cdots W_1 \), with \( L \in \mathbb{N} \) arbitrary (and the dimensions of \( \{ W_i \}_{i=1} \) set such that rank is unconstrained). It was empirically shown that deeper matrix factorizations (larger \( L \)) yield more accurate recovery when \( W^* \) is low-rank. Moreover, it was conjectured that the implicit regularization, for any depth \( L \geq 2 \), can not be described as minimization of a mathematical norm (or quasi-norm):

**Conjecture 2** (based on (6), informally stated). Given a (shallow or deep) matrix factorization, for any norm (or quasi-norm) \( \| \cdot \| \), there exists a set of observed entries with which small learning rate and initialization close to the origin can not ensure convergence of gradient descent to a minimal (in terms of \( \| \cdot \| \)) solution.

Conjectures 1 and 2 contrast each other, and more broadly, represent opposing perspectives on the question of whether norms may be able to explain implicit regularization in deep learning. In this paper, we resolve the tension between the two conjectures by affirming the latter. In particular, we prove that there exist natural matrix completion problems where fitting observations via gradient descent on a depth \( L \geq 2 \) matrix factorization leads — with probability 0.5 or more over (arbitrarily small) random initialization — all norms (and quasi-norms) to grow towards infinity, while the rank essentially decreases towards its minimum. This result is in fact stronger than that suggested by Conjecture 2, in the sense that: (i) not only is each norm (or quasi-norm) disqualified by some setting, but there are actually settings that jointly disqualify all norms (and quasi-norms); and (ii) not only are norms (and quasi-norms) not necessarily minimized, but they can grow towards infinity. We corroborate the analysis with empirical demonstrations.

Our findings imply that, rather than viewing implicit regularization in (shallow or deep) matrix factorization as minimizing a norm (or quasi-norm), a potentially more useful interpretation is minimization of rank. As a step towards assessing the generality of this interpretation, we empirically explore an extension of matrix factorization to tensor factorization. Our experiments show that in analogy with matrix factorization, gradient descent on a tensor factorization tends to produce solutions with low rank, where rank is defined in the context of tensors. Similarly to how matrix factorization corresponds to a linear neural network whose input-output mapping is represented by a matrix, it is known (see (19)) that tensor factorization corresponds to a convolutional arithmetic circuit (certain type of non-linear neural network) whose input-output mapping is represented by a tensor. We thus obtain a second exemplar of a neural network architecture whose implicit regularization strives to lower a notion of rank for its input-output mapping. This leads us to believe that the phenomenon may be general, and formalizing notions of rank for input-output mappings of contemporary models may be key to explaining generalization in deep learning.

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2 The nuclear norm (also known as trace norm) of a matrix is the sum of its singular values, regarded as a convex relaxation of rank.

3 For the sake of this paper, tensors can be thought of as \( N \)-dimensional arrays, with \( N \in \mathbb{N} \) arbitrary (matrices correspond to the special case \( N = 2 \)).

4 The rank of a tensor is the minimal number of summands required to express it, where each summand is an outer product between vectors.
The remainder of the paper is organized as follows. Section 2 reviews related work. Section 3 presents the deep matrix factorization model. Section 4 delivers our analysis, showing that its implicit regularization can drive all norms to infinity. Experiments, with both the analyzed setting and tensor factorization, are given in Section 5. Finally, Section 6 summarizes.

2 Related work

Theoretical analysis of implicit regularization in deep learning is a highly active area of research. Our work extends the bulk of literature concerning mathematical characterization of the implicit regularization induced by gradient-based optimization. Existing characterizations focus on different aspects of learning, for example: dynamics of optimization; curvature (“flatness”) of obtained minima; frequency spectrum of learned input-output mappings; invariant quantities throughout training; and statistical properties imported from data.

A ubiquitous approach, arguably more prevalent than the aforementioned, is to demonstrate that learned input-output mappings minimize some notion of norm, or analogously, maximize some notion of margin. Works along this line have treated: linear (single-layer) predictors; normalized linear models; certain polynomially parameterized linear models; homogeneous (and sum of homogeneous) models; ultra-wide neural networks; linear neural networks with a single output; and matrix factorization — the subject of our inquiry.

Matrix factorization is perhaps the most extensively studied model in the context of implicit regularization induced by non-convex gradient-based optimization. It corresponds to linear neural networks with multiple inputs and outputs, typically trained to recover low-rank linear mappings. The literature on matrix factorization for low-rank matrix recovery is far too broad to cover here — we refer to a recent survey, while mentioning that the technique is often attributed to. Notable works proving successful recovery of a low-rank matrix via matrix factorization trained by gradient descent with no explicit regularization are. Of these, can be viewed as affirming Conjecture (from) for a certain special case. has affirmed Conjecture under different assumptions, but nonetheless argued empirically that it does not hold true in general, in resonance with Conjecture (from). To the best of our knowledge, no theoretical support for the latter was provided prior to its proof in this paper. We note that the proof relies on technical results derived in (restated in Appendix as Lemmas and respectively).

3 Deep matrix factorization

Suppose we would like to complete a $d$-by-$d'$ matrix based on a set of observations $\{b_{i,j} \in \mathbb{R}\}_{(i,j) \in \Omega}$, where $\Omega \subset \{1, 2, \ldots, d\} \times \{1, 2, \ldots, d'\}$. A standard (underdetermined) loss function for the task is:

$$\ell : \mathbb{R}^{d \times d'} \to \mathbb{R}_{\geq 0}, \quad \ell(W) = \frac{1}{2} \sum_{(i,j) \in \Omega} ((W)_{i,j} - b_{i,j})^2.$$ (1)

Employing a depth $L$ matrix factorization, with hidden dimensions $d_1, d_2, \ldots, d_{L-1} \in \mathbb{N}$, amounts to optimizing the overparameterized objective:

$$\phi(W_1, W_2, \ldots, W_L) := \ell(W_{L-1}) = \frac{1}{2} \sum_{(i,j) \in \Omega} ((W_{L-1})_{i,j} - b_{i,j})^2,$$ (2)

where $W_l \in \mathbb{R}^{d_l \times d_{l-1}}$, $l = 1, 2, \ldots, L$, with $d_L := d$, $d_0 := d'$, and:

$$W_{L-1} := W_L W_{L-2} \cdots W_1.$$ (3)

referred to as the product matrix of the factorization. Our interest lies on the implicit regularization of gradient descent, i.e. on the type of product matrices (Equation 1) it will find when applied to the overparameterized objective (Equation 2). Accordingly, and in line with prior work (cf. 30 6), we focus on the case in which the search space is unconstrained, meaning $\min\{d_l\}_{l=0}^L = \min\{d_0, d_L\}$ (rank is not limited by the parameterization).

5 As opposed to works studying the relation between implicit regularization and generalization (cf. 55 57 22), or ones analyzing other sources of implicit regularization such as dropout (e.g. 69 3).
As a theoretical surrogate for gradient descent with small learning rate and near-zero initialization, similarly to \cite{30} and \cite{6} (as well as other works analyzing linear neural networks, e.g. \cite{64,4,45,5}), we study gradient flow (gradient descent with infinitesimally small learning rate)\footnote{A technical subtlety of optimization in continuous time is that in principle, it is possible to asymptote (diverge to infinity) after finite time. In such a case, the asymptote is regarded as the end of optimization, and time tending to infinity ($t \to \infty$) is interpreted as tending towards that point.},

\[
\dot{W}_i(t) := \frac{d}{dt} W_i(t) = -\frac{\partial}{\partial W_i} \phi(W_1(t), W_2(t), \ldots, W_L(t)) , \quad t \geq 0 , \quad l = 1, 2, \ldots, L , \tag{4}
\]

and assume balancedness at initialization, i.e.:

\[
W_{l+1}(0) = W_l(0) W_l(0)^\top , \quad l = 1, 2, \ldots, L - 1 . \tag{5}
\]

In particular, when considering random initialization, we assume that \(\{W_l(0)\}_{l=1}^L = \) are drawn from a joint probability distribution by which Equation (5) holds almost surely. This is an idealization of standard random near-zero initializations, e.g. Xavier (\cite{27}) and He (\cite{56}), by which Equation (5) holds approximately with high probability (note that the equation holds exactly in the standard “residual” setting of identity initialization — cf. \cite{53,54}).

The condition of balanced initialization (Equation (5)) played an important role in the analysis of [4], facilitating derivation of a differential equation governing the product matrix of a linear neural network (see Lemma 1 in Appendix B.2). It was shown in [4] empirically (and will be demonstrated again in Section 5) that there is an excellent match between the theoretical predictions of gradient flow with balanced initialization, and its practical realization via gradient descent with small learning rate and near-zero initialization. Other works (e.g. [5, 29]) have supported this match theoretically.

Formally stated, Conjecture 1 from \cite{30} treats the case \(L = 2\), where the product matrix \(W_{L:1}\) (Equation (3)) holds \(\alpha \cdot W_{init}\) at initialization, \(W_{init}\) being a fixed arbitrary full-rank matrix and \(\alpha\) a varying positive scalar. Taking time to infinity \((t \to \infty)\) and then initialization size to zero \((\alpha \to 0^+)\), the conjecture postulates that if the limit product matrix \(\bar{W}_{L:1} := \lim_{\alpha \to 0^+} \lim_{t \to \infty} W_{L:1}\) exists and is a global optimum for the loss \(\ell(.)\) (Equation (1)), i.e. \(\ell(\bar{W}_{L:1}) = 0\), then it will be a global optimum with minimal nuclear norm, meaning \(\bar{W}_{L:1} \in \text{argmin}_{\Omega(\bar{W})=0} \|\bar{W}\|_{\text{nuclear}}\). In contrast to Conjecture 1 Conjecture 2 from \cite{6} can be interpreted as saying that for any depth \(L \geq 2\) and any norm or quasi-norm \(\|\cdot\|\), there exist observations \(\{b_{i,j}\}_{(i,j) \in \Omega}\) for which global optimization of loss \(\lim_{\alpha \to 0^+} \lim_{t \to \infty} \ell(W_{L:1}) = 0\) does not imply minimization of \(\|\cdot\|\) (i.e. we may have \(\lim_{\alpha \to 0^+} \lim_{t \to \infty} \|W_{L:1}\| \neq \min_{\Omega(\bar{W})=0} \|\bar{W}\|\)). Due to technical subtleties (for example the requirement of Conjecture 1 that a double limit of the product matrix with respect to time and initialization size exists), Conjectures 1 and 2 are not necessarily contradictory. However, they are in direct opposition in terms of the stances they represent — one supports the prospect of norms being able to explain implicit regularization in matrix factorization, and the other does not. The current paper seeks a resolution.

4 Implicit regularization can drive all norms to infinity

In this section we prove that for matrix factorization of depth \(L \geq 2\), there exist observations \(\{b_{i,j}\}_{(i,j) \in \Omega}\) with which optimizing the overparameterized objective (Equation 4) via gradient flow (Equations 4 and 5) leads — with probability 0.5 or more over random ("symmetric") initialization — all norms and quasi-norms of the product matrix (Equation 3) to grow towards infinity, while its rank essentially decreases towards minimum. By this we not only affirm Conjecture 2 but in fact go beyond it in the following sense: (i) the conjecture allows chosen observations to depend on the norm or quasi-norm under consideration, while we show that the same set of observations can apply jointly to all norms and quasi-norms; and (ii) the conjecture requires norms and quasi-norms to be larger than minimal, while we establish growth towards infinity.

For simplicity of presentation, the current section delivers our construction and analysis in the setting \(d = d^2 = 2\) (i.e. 2-by-2 matrix completion) — extension to different dimensions is straightforward (see Appendix A). We begin (Subsection 4.1) by introducing our chosen observations \(\{b_{i,j}\}_{(i,j) \in \Omega}\) and discussing their properties. Subsequently (Subsection 4.2), we show that with these observations, \(\lim_{t \to \infty} W_{init}\) applied to symmetric matrix factorization and positive definite \(W_{init}\), but it is claimed thereafter that affirming the conjecture would imply the same for the asymmetric setting considered in this paper. We also note that the conjecture is stated in the context of matrix sensing, thus in particular applies to matrix completion (a special case).
decreasing loss often increases all norms and quasi-norms while lowering rank. Minimization of loss is treated thereafter (Subsection 4.3). Finally (Subsection 4.4), robustness of our construction to perturbations is established.

4.1 A simple matrix completion problem

Consider the problem of completing a 2-by-2 matrix based on the following observations:

\[
\Omega = \{(1, 2), (2, 1), (2, 2)\}, \quad b_{1,2} = 1, \quad b_{2,1} = 1, \quad b_{2,2} = 0.
\]

The solution set for this problem (i.e. the set of matrices obtaining zero loss) is:

\[
\mathcal{S} = \{ W \in \mathbb{R}^{2,2} : (W)_{1,2} = 1, (W)_{2,1} = 1, (W)_{2,2} = 0 \}.
\]

Proposition 1 below states that minimizing a norm or quasi-norm along \( W \in \mathcal{S} \) requires confining \((W)_{1,1}\) to a bounded interval, which for Schatten-\( p \) (quasi-)norms (in particular for nuclear, Frobenius and spectral norms) is simply the singleton \( \{0\} \).

**Proposition 1.** For any norm or quasi-norm over matrices \( \| \cdot \| \) and any \( \epsilon > 0 \), there exists a bounded interval \( I_{\| \cdot \|, \epsilon} \subset \mathbb{R} \) such that if \( W \in \mathcal{S} \) is an \( \epsilon \)-minimizer of \( \| \cdot \| \) (i.e. \( \| W \| \leq \inf_{W' \in \mathcal{S}} \| W' \| + \epsilon \)) then necessarily \((W)_{1,1} \in I_{\| \cdot \|, \epsilon}\). If \( \| \cdot \| \) is a Schatten-\( p \) (quasi-)norm, then in addition \( W \in \mathcal{S} \) minimizes \( \| \cdot \| \) (i.e. \( \| W \| = \inf_{W' \in \mathcal{S}} \| W' \| \)) if and only if \((W)_{1,1} = 0\).

Proof sketch (for complete proof see Appendix B.3). The (weakened) triangle inequality allows us to lower bound \( \| \cdot \| \) by \((W)_{1,1}\), (up to multiplicative and additive constants). Thus, the set of \((W)_{1,1}\) values corresponding to \( \epsilon \)-minimizers must be bounded. If \( \| \cdot \| \) is a Schatten-\( p \) (quasi-)norm, a straightforward analysis shows it is monotonically increasing with respect to \((W)_{1,1}\), implying it is minimized if and only if \((W)_{1,1} = 0\).

In addition to norms and quasi-norms, we are also interested in the evolution of rank throughout optimization of a deep matrix factorization. More specifically, we are interested in the prospect of rank being implicitly minimized, as demonstrated empirically in [30, 3]. The discrete nature of rank renders its direct analysis unfavorable from a dynamical perspective (the rank of a matrix implies little about its proximity to low-rank), thus we consider the following surrogate measures: (i) effective rank (Definition 1 below; from [63]) — a continuous extension of rank used for numerical analyses; and (ii) distance from infimal rank (Definition 2 below) — (Frobenius) distance from the minimal rank that a given set of matrices may approach. According to Proposition 2 below, these measures independently imply that, although all solutions to our matrix completion problem — i.e. all \( W \in \mathcal{S} \) (see Equation (4)) — have rank 2, it is possible to essentially minimize the rank to 1 by taking \((W)_{1,1} \to \infty\). Recalling Proposition 1, we conclude that in our setting, there is a direct contradiction between minimizing norms or quasi-norms and minimizing rank — the former requires confinement to some bounded interval, whereas the latter demands divergence towards infinity. This is the critical feature of our construction, allowing us to deem whether the implicit regularization in deep matrix factorization favors norms (or quasi-norms) over rank or vice versa.

**Definition 1.** (from [63]) The effective rank of a matrix \( 0 \neq W \in \mathbb{R}^{d,d} \) with singular values \( \{\sigma_r(W)\}_{r=1}^{\min\{d,d'\}} \) is defined to be \( \text{erank}(W) := \exp(\{H(\rho_1(W), \rho_2(W), \ldots, \rho_{\min\{d,d'\}}(W))\}) \), where \( \{\rho_r(W) := \sigma_r(W)/\sum_{r'=1}^{\min\{d,d'\}} \sigma_{r'}(W)\}_{r=1}^{\min\{d,d'\}} \) is a distribution induced by the singular values, and \( H(\rho_1(W), \rho_2(W), \ldots, \rho_{\min\{d,d'\}}(W)) := -\sum_{r=1}^{\min\{d,d'\}} \rho_r(W) \cdot \ln \rho_r(W) \) is the (Shannon) entropy (by convention \( 0 \cdot \ln 0 = 0 \)).

**Definition 2.** For a matrix space \( \mathbb{R}^{d,d} \), we denote by \( D(\mathcal{S}, \mathcal{S}') \) the (Frobenius) distance between two sets \( \mathcal{S}, \mathcal{S}' \subset \mathbb{R}^{d,d} \) (i.e. \( D(\mathcal{S}, \mathcal{S}') := \inf\{\| W - W' \|_{F_{\rho_o}} : W \in \mathcal{S}, W' \in \mathcal{S}'\} \)), by \( D(W, \mathcal{S}') \) the distance between a matrix \( W \in \mathbb{R}^{d,d} \) and the set \( \mathcal{S}' \) (i.e. \( D(W, \mathcal{S}') := \inf\{\| W - W' \|_{F_{\rho_o}} : W' \in \mathcal{S}'\} \)), and by \( \mathcal{M}_r \), for \( r = 0, 1, \ldots, \min\{d,d'\} \), the set of matrices with rank \( r \) or less.

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8 For \( p \in (0, \infty] \), the Schatten-\( p \) (quasi-)norm of a matrix \( W \in \mathbb{R}^{d,d} \) with singular values \( \{\sigma_r(W)\}_{r=1}^{\min\{d,d'\}} \) is defined as \( (\sum_{r=1}^{\min\{d,d'\}} \sigma_r^p(W))^{1/p} \) if \( p < \infty \) and as \( \max\{\sigma_r(W)\}_{r=1}^{\min\{d,d'\}} \) if \( p = \infty \). It is a norm if \( p \geq 1 \) and a quasi-norm if \( p < 1 \). Notable special cases are nuclear (trace), Frobenius and spectral norms, corresponding to \( p = 1, 2 \) and \( \infty \) respectively.
On the other hand: An immediate consequence of Theorem 1 is that, if the product matrix (Equation (3)) has positive quasi-norms and singular values of $W$, Theorem 1. Suppose we complete the observations in Equation 4. Correspondingly, the infimal rank (Definition 2) of $W$ is 1, and the distance of $W \in S$ from infimal rank is maximized when $(W)_{1,1} = 0$, monotonically decreasing to 0 as $|(W)_{1,1}|$ grows.

Proof sketch (for complete proof see Appendix B.4). Analyzing $\sigma_1(W) \geq \sigma_2(W) \geq 0$ — singular values of $W \in S$ — reveals that: (i) $\sigma_1(W)$ attains a minimal value of 1 when $(W)_{1,1} = 0$, monotonically increasing to $\infty$ as $|(W)_{1,1}|$ grows; and (ii) $\sigma_2(W)$ attains a maximal value of 1 when $(W)_{1,1} = 0$, monotonically decreasing to 0 as $|(W)_{1,1}|$ grows. The results for effective rank, infimal rank and distance from infimal rank readily follow from this characterization.

4.2 Decreasing loss increases norms

Consider the process of solving our matrix completion problem (Subsection 4.1) with gradient flow over a depth $L \geq 2$ matrix factorization (Section 3). Theorem 1 below states that if the product matrix (Equation (3)) has positive determinant at initialization, lowering the loss leads norms and quasi-norms to increase, while the rank essentially decreases.

Theorem 1. Suppose we complete the observations in Equation 4 by employing a depth $L \geq 2$ matrix factorization, i.e. by minimizing the overparameterized objective (Equation 5) via gradient flow (Equations 6 and 5). Denote by $W_{L,1}(t)$ the product matrix (Equation (3)) at time $t \geq 0$ of optimization, and by $\ell(t) := \ell(W_{L,1}(t))$ the corresponding loss (Equation (1)). Assume that $\det(W_{L,1}(0)) > 0$. Then, for any norm or quasi-norm over matrices $\|\cdot\|$: $\|W_{L,1}(t)\| \geq a_{\|\cdot\|} \cdot \frac{1}{\sqrt{\ell(t)}} - b_{\|\cdot\|}$, $t \geq 0$, (8)

where $b_{\|\cdot\|} := \max \{\sqrt{2}a_{\|\cdot\|}, 8c_1^2, \max_{i,j \in \{1,2\}} \|e_i e_j^\top\|\}$, $a_{\|\cdot\|} := \|e_1 e_1^\top\|/(\sqrt{2}c_{\|\cdot\|})$, the vectors $e_1, e_2 \in \mathbb{R}^2$ form the standard basis, and $c_{\|\cdot\|} \geq 1$ is a constant with which $\|\cdot\|$ satisfies the weakened triangle inequality (see Footnote 7). On the other hand:

$$\text{erank}(W_{L,1}(t)) \leq \inf_{W' \in S} \text{erank}(W') + \frac{2\sqrt{\text{erank}(S)}}{\sqrt{\text{det}(W_{L,1}(0))}} \cdot \frac{1}{\sqrt{\ell(t)}}$$, $t \geq 0$, (9)

$$D(W_{L,1}(t), \mathcal{M}_{\text{irank}(S)}) \leq 3\sqrt{2} \cdot \frac{1}{\sqrt{\ell(t)}}$$, $t \geq 0$, (10)

where $\text{erank}(\cdot)$ stands for effective rank (Definition 7), and $D(\cdot, \mathcal{M}_{\text{irank}(S)})$ represents distance from the infimal rank (Definition 2) of the solution set $S$ (Equation (7)).

Proof sketch (for complete proof see Appendix B.5). Using a dynamical characterization from [6] for the singular values of the product matrix (restated in Appendix B as Lemma 2), we show that the latter’s determinant does not change sign, i.e. it remains positive. This allows us to lower bound $\|W_{L,1}(t)\|_1$ by $1/\sqrt{\ell(t)}$ (up to multiplicative and additive constants). Relating $\|W_{L,1}(1,t)\|$ to (quasi-)norms and singular values of $W_{L,1}(t)$ then leads to the desired bounds.

An immediate consequence of Theorem 1 is that, if the product matrix (Equation (3)) has positive determinant at initialization, convergence to zero loss leads all norms and quasi-norms to grow to infinity, while the rank is essentially minimized. This is formalized in Corollary 1 below.

Corollary 1. Under the conditions of Theorem 1 global optimization of loss, i.e. $\lim_{t \to \infty} \ell(t) = 0$, implies that for any norm or quasi-norm over matrices $\|\cdot\|$: $\lim_{t \to \infty} \|W_{L,1}(t)\|_1 = \infty$,

where $W_{L,1}(t)$ is the product matrix of the deep factorization (Equation (3)) at time $t$ of optimization. On the other hand:

$$\lim_{t \to \infty} \text{erank}(W_{L,1}(t)) = \inf_{W' \in S} \text{erank}(W')$$, $\lim_{t \to \infty} D(W_{L,1}(t), \mathcal{M}_{\text{irank}(S)}) = 0$.

9 As stated in Section 3 we consider full-dimensional factorizations, in this case meaning that hidden dimensions $d_1, d_2, \ldots, d_{L-1}$ are all greater than or equal to 2.
where \( \text{erank}(\cdot) \) stands for effective rank (Definition 1), and \( D(\cdot, \mathcal{M}_{\text{irank}(S)}) \) represents distance from the infimal rank (Definition 2) of the solution set \( S \) (Equation (7)).

**Proof.** Taking the limit \( \ell(t) \to 0 \) in the bounds given by Theorem 1 establishes the results. \( \square \)

Theorem 1 and Corollary 1 imply that in our setting (Subsection 4.1), where minimizing norms (or quasi-norms) and minimizing rank contradict each other, the implicit regularization of deep matrix factorization is willing to completely give up on the former in favor of the latter, at least on the condition that the product matrix (Equation (3)) has positive determinant at initialization. How probable is this condition? By Proposition 5 below, it holds with probability 0.5 if the product matrix is initialized by any one of a wide array of common distributions, including matrix Gaussian distribution with zero mean and independent entries, and a product of such. We note that rescaling (multiplying by \( \alpha > 0 \)) initialization does not change sign of product matrix’s determinant, therefore as postulated by Conjecture 2, initialization close to the origin (along with small learning rate) can not ensure convergence to solution with minimal norm or quasi-norm.

**Proposition 3.** If \( W \in \mathbb{R}^{d,d} \) is a random matrix whose entries are drawn independently from continuous distributions, each symmetric about the origin, then \( \text{Pr} \left( \text{det}(W) > 0 \right) = \text{Pr} \left( \text{det}(W) < 0 \right) = 0.5 \). Furthermore, for \( L \in \mathbb{N} \), if \( W_1, W_2, \ldots, W_L \in \mathbb{R}^{d,d} \) are random matrices drawn independently from continuous distributions, and there exists \( l \in \{1, 2, \ldots, L\} \) with \( \text{Pr} \left( \text{det}(W_l) > 0 \right) = 0.5 \), then \( \text{Pr} \left( \text{det}(W_L, W_{L-1}, \ldots, W_1) > 0 \right) = \text{Pr} \left( \text{det}(W_L, W_{L-1}, \ldots, W_1) < 0 \right) = 0.5 \).

**Proof sketch (for complete proof see Appendix B.6).** Multiplying a row of \( W \) by \(-1\) keeps its distribution intact while flipping the sign of its determinant. This implies \( \text{Pr} \left( \text{det}(W) > 0 \right) = \text{Pr} \left( \text{det}(W) < 0 \right) < 0.5 \). The first result then follows from the fact that a matrix drawn from a continuous distribution is almost surely non-singular. The second result is an outcome of the same fact, as well as multiplicativity of determinant and the law of total probability. \( \square \)

### 4.3 Convergence to zero loss

It is customary in the theory of deep learning (cf. [30, 32, 6]) to distinguish between implicit regularization — which concerns the type of solutions found in training — and the complementary question of whether training loss is globally optimized. We supplement our implicit regularization analysis (Subsection 4.2) by addressing this complementary question in two ways: (i) in Section 5 we empirically demonstrate that on the matrix completion problem we analyze (Subsection 4.1), gradient descent over deep matrix factorizations (Section 3) indeed drives training loss towards global optimum, i.e. towards zero; and (ii) in Proposition 4 below we theoretically establish convergence to zero loss for the special case of depth 2 and scaled identity initialization (treatment of larger depths and additional initializations is left for future work). We note that when combined with Corollary 1 Proposition 4 affirms that in the latter special case, all norms and quasi-norms indeed grow to infinity while rank is essentially minimized.

**Proposition 4.** Consider the setting of Theorem 1 in the special case of depth \( L = 2 \) and initial product matrix (Equation (3)) \( W_{L,1}(0) = \alpha \cdot I \), where \( I \) stands for the identity matrix and \( \alpha \in (0, 1] \). Under these conditions \( \lim_{t \to \infty} \ell(t) = 0 \), i.e. training loss is globally optimized.

**Proof sketch (for complete proof see Appendix B.7).** We first establish that the product matrix is positive definite for all \( t \). This simplifies a dynamical characterization from [4] (restated in Appendix B as Lemma 1), yielding lucid differential equations governing the entries of the product matrix. Careful analysis of these equations then completes the proof. \( \square \)

### 4.4 Robustness to perturbations

Our analysis (Subsection 4.2) has shown that when applying a deep matrix factorization (Section 3) to the matrix completion problem defined in Subsection 4.1, if the product matrix (Equation (3)) has positive determinant at initialization — a condition that holds with probability 0.5 under the wide range of common distributions — then training loss is globally optimized. However, what happens if the matrix is perturbed? One might imagine that training loss will not remain globally optimized. We thus turn our attention to the question of whether training loss is globally optimized. We supplement our implicit regularization analysis of these equations then completes the proof.
variety of random distributions specified by Proposition 3—then the implicit regularization drives all norms and quasi-norms towards infinity, while rank is essentially driven towards its minimum. A natural question is how common this phenomenon is, and in particular, to what extent does it persist if the observed entries we defined (Equation 6) are perturbed. Theorem 2 below generalizes Theorem 1 (from Subsection 4.2) to the case of arbitrary non-zero values for the off-diagonal observations \( b_{1,2}, b_{2,1} \), and an arbitrary value for the diagonal observation \( b_{2,2} \). In this generalization, the assumption (from Theorem 1) of the product matrix’s determinant at initialization being positive is modified to an assumption of it having the same sign as \( b_{1,2} \cdot b_{2,1} \) (the probability of which is also 0.5 under the random distributions covered by Proposition 3). Conditioned on the modified assumption, the smaller \(|b_{2,2}|\) is compared to \(|b_{1,2} \cdot b_{2,1}|\), the higher the implicit regularization is guaranteed to drive norms and quasi-norms, and the lower it is guaranteed to essentially drive the rank. Two immediate implications of Theorem 2 are: (i) if the diagonal observation is unperturbed \((b_{2,2} = 0)\), the off-diagonal ones \((b_{1,2}, b_{2,1})\) can take on any non-zero values, and the phenomenon of implicit regularization driving norms and quasi-norms towards infinity (while essentially driving rank towards its minimum) will persist; and (ii) this phenomenon gracefully recedes as the diagonal observation is perturbed away from zero.

We demonstrate these implications empirically in Section 5.

**Theorem 2.** Consider the setting of Theorem 1 subject to the following changes: (i) the observations from Equation 6 are generalized to:

\[ \Omega = \{(1,2), (2,1), (2, 2)\}, \quad b_{1,2} = z \in \mathbb{R} \setminus \{0\}, \quad b_{2,1} = z' \in \mathbb{R} \setminus \{0\}, \quad b_{2,2} = \epsilon \in \mathbb{R}, \]

leading to the following solution set in place of that from Equation 7:

\[ \tilde{S} = \{ W \in \mathbb{R}^{2,2} : (W)_{1,2} = z, (W)_{2,1} = z', (W)_{2,2} = \epsilon \} ; \]

and (ii) the assumption \( \det(W_{L:1}(0)) > 0 \) is generalized to \( \text{sign}(\det(W_{L:1}(0))) = \text{sign}(z \cdot z') \), where \( W_{L:1}(t) \) denotes the product matrix (Equation 5) at time \( t \geq 0 \) of optimization. Under these conditions, for any norm or quasi-norm over matrices \( \| \cdot \| \):

\[ \| W_{L:1}(t) \| \geq a_{\| \cdot \|} \cdot \frac{|z| \cdot |z'|}{|\epsilon| + \sqrt{2}(t)} - b_{\| \cdot \|}, \quad t \geq 0, \]

where \( b_{\| \cdot \|} := \max \{ a_{\| \cdot \|} |z| \cdot |z'| / (|\epsilon| + \min\{|z|, |z'\}|) \}, \quad a_{\| \cdot \|} := \max\{|z|, |z'|, |\epsilon|\} \max_{i,j \in \{1,2\}} \| e_i e_j^\top \| \}, \quad a_{\| \cdot \|} := \max\{|z|, |z'|, |\epsilon|\} \max_{i,j \in \{1,2\}} \| e_i e_j^\top \| \}

\[ D(W_{L:1}(t), \mathcal{M}_{\text{rank}(\tilde{S})}) \leq 4 |\epsilon| + \left( 4 + \frac{\sqrt{|z| \cdot |z'|}}{\min\{|z|, |z'\}|} \right) \sqrt{2}(t), \quad t \geq 0, \]

where \( \text{rank}(\cdot) \) stands for effective rank (Definition 1), and \( D(\cdot, \mathcal{M}_{\text{rank}(\tilde{S})}) \) represents distance from the infimal rank (Definition 2) of the solution set \( \tilde{S} \).

**Proof sketch (for complete proof see Appendix B.3).** The proof follows a line similar to that of Theorem 1 with slightly more involved derivations.

## 5 Experiments

This section presents our empirical evaluations. We begin in Subsection 5.1 with deep matrix factorization (Section 3) applied to the setting we analyzed (Section 4). Then, we turn to Subsection 5.2 and experiment with an extension to tensor (multi-dimensional array) factorization. For brevity, many of the details behind our implementation are deferred to Appendix C.

### 5.1 Analyzed setting

In [30], Gunasekar et al. experimented with matrix factorization, arriving at Conjecture 1. In the following work [6], Arora et al. empirically evaluated additional settings, ultimately arguing against Conjecture 1 and raising Conjecture 2. Our analysis (Section 4) affirmed Conjecture 2 by providing a
Figure 1: Implicit regularization in matrix factorization can drive all norms (and quasi-norms) towards infinity. For the matrix completion problem defined in Subsection 4.1, our analysis (Subsection 4.2) implies that with small learning rate and initialization close to the origin, when the product matrix (Equation (3)) is initialized to have positive determinant, gradient descent on a matrix factorization leads absolute value of unobserved entry to increase as loss decreases, i.e., as observations are fit (which in turn means norms and quasi-norms increase).

This is demonstrated in the plots above, which for representative runs, show absolute value of unobserved entry as a function of the loss (Equation (1)), with iteration number encoded by color. Each plot corresponds to a different depth for the matrix factorization, and presents runs with varying configurations of learning rate and initialization (abbreviated as “lr” and “init”, respectively). Both balanced (Equation (5)) and unbalanced (layer-wise independent) random initializations were evaluated (former is marked by “(b)”).

Independently for every depth, runs were iteratively carried out, with both learning rate and standard deviation for initialization decreased towards each run, until the point where further reduction did not yield a noticeable change (presented runs are those from the last iterations of this process). Notice that depth, balancedness, and small learning rate and initialization, all contribute to the examined effect (absolute value of unobserved entry increasing as loss decreases), with depth being most significant. Notice also that all runs initially follow the same curve, differing from one another in the point at which they divert (enter a phase where examined effect is lesser). Theoretical investigation of these phenomena is left for future work. For further details, and a similar experiment with perturbed observations, see Appendix C.

5.2 From matrix to tensor factorization

At the heart of our analysis (Section 4) lies a matrix completion problem whose solution set (Equation (7)) entails a direct contradiction between minimizing norms (or quasi-norms) and minimizing rank. We have shown that on this problem, gradient descent over (shallow or deep) matrix factorization is willing to completely give up on the former in favor of the latter. This suggests that, rather than viewing implicit regularization in matrix factorization through the lens of norms (or quasi-norms), a potentially more useful interpretation is minimization of rank. Indeed, while global minimization of rank is in the worst case computationally hard (cf. [62]), it has been shown in [6] (theoretically as well as empirically) that the dynamics of gradient descent over matrix factorization promote sparsity of singular values, and thus they may be interpreted as searching for low rank locally. As a step towards assessing the generality of this interpretation, we empirically explore an extension of matrix factorization to tensor factorization.

In the context of matrix completion, (depth 2) matrix factorization amounts to optimizing the loss in Equation (1) by applying gradient descent to the parameterization \( W = \sum_{r=1}^{R} w_r \otimes w'_r \), where \( R \in \mathbb{N} \).

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12 The reader is referred to [43] and [33] for an introduction to tensor factorizations.
Figure 2: Gradient descent over tensor factorization exhibits an implicit regularization towards low (tensor) rank. Plots above report results of tensor completion experiments, comparing: (i) minimization of loss (Equation (16)) via gradient descent over tensor factorization (Equation (17), with $R$ large enough for expressing any tensor) starting from (small) random initialization (method is abbreviated as “tf”); against (ii) trivial baseline that matches observations while holding zeros in unobserved locations — equivalent to minimizing loss via gradient descent over linear parameterization (i.e. directly over $W$) starting from zero initialization (hence this method is referred to as “linear”). Each pair of plots corresponds to a randomly drawn low rank ground truth tensor, from which multiple sets of observations varying in size were randomly chosen. The ground truth tensors corresponding to left and right pairs both have rank 1 (for results obtained with additional ground truth ranks see Figure 3 in Appendix C.1), with sizes 8-by-8-by-8 (order 3) and 8-by-8-by-8-by-8 (order 4) respectively. The plots in each pair show reconstruction errors (Frobenius distance from ground truth) and ranks (numerically estimated) of final solutions as a function of the number of observations in the task, with error bars spanning interquartile range (25th to 75th percentiles) over multiple trials (differing in random seed for initialization), and markers showing median. For gradient descent over tensor factorization, we employed an adaptive learning rate scheme to reduce run times (see Appendix C.2 for details), and iteratively ran with decreasing standard deviation for initialization, until the point at which further reduction did not yield a noticeable change (presented results are those from the last iterations of this process, with the corresponding standard deviations annotated by “init”). Notice that gradient descent over tensor factorization indeed exhibits an implicit tendency towards low rank (leading to accurate reconstruction of low rank ground truth tensors), and that this tendency is stronger with smaller initialization. For further details and experiments see Appendix C.

is a predetermined constant, $\otimes$ stands for outer product, and $\{w_r \in \mathbb{R}^d\}_{r=1}^R, \{w'_r \in \mathbb{R}^{d'}\}_{r=1}^R$ are the optimized parameters. The minimal $R$ required for this parameterization to be able to express a given $\bar{W} \in \mathbb{R}^{d,d}$ is precisely the latter’s rank. Implicit regularization towards low rank means that even when $R$ is large enough for expressing any matrix (i.e. $R \geq \min\{d, d'\}$), solutions expressible (or approximable) with small $R$ tend to be learned.

A generalization of the above is obtained by switching from matrices (tensors of order 2) to tensors of arbitrary order $N \in \mathbb{N}$. This gives rise to a tensor completion problem, with corresponding loss:

$$\ell : \mathbb{R}^{d_1,d_2,\ldots,d_N} \to \mathbb{R}_ \geq 0,$$

$$\ell(\bar{W}) = \frac{1}{2} \sum_{(i_1,i_2,\ldots,i_N) \in \Omega} ((W)_{i_1,i_2,\ldots,i_N} - b_{i_1,i_2,\ldots,i_N})^2,$$

where $\{b_{i_1,i_2,\ldots,i_N} \in \mathbb{R} \} \times \Omega \subset \{1, 2, \ldots, d_1\} \times \{1, 2, \ldots, d_2\} \times \cdots \times \{1, 2, \ldots, d_N\}$, stands for the set of observed entries. One may employ a tensor factorization by minimizing the loss in Equation (16) via gradient descent over the parameterization:

$$\bar{W} = \sum_{r=1}^R w^{(1)}_r \otimes w^{(2)}_r \otimes \cdots \otimes w^{(N)}_r, \quad w^{(n)}_r \in \mathbb{R}^{d_n}, \quad r = 1, 2, \ldots, R, \quad n = 1, 2, \ldots, N,$$

where again, $R \in \mathbb{N}$ is a predetermined constant, $\otimes$ stands for outer product, and $\{w^{(n)}_r\}_{r=1}^R \in \mathbb{R}^{d_1,d_2,\ldots,d_N}$ are the optimized parameters. In analogy with the matrix case, the minimal $R$ required for this parameterization to be able to express a given $\bar{W} \in \mathbb{R}^{d_1,d_2,\ldots,d_N}$ is defined to be the latter’s (tensor) rank. An implicit regularization towards low rank here would mean that even when $R$ is large enough for expressing any tensor, solutions expressible (or approximable) with small $R$ tend to be learned.

Figure 2 displays results of tensor completion experiments, in which tensor factorization (optimization of loss in Equation (16)) is applied to
As discussed in Section 1, matrix completion can be seen as a prediction problem, and tensor factorization as its solution with a convolutional arithmetic circuit — see Figure 3. Convolutional arithmetic circuits form a class of non-linear neural networks that has been studied extensively in theory (cf. [19, 16, 17, 20, 65, 55, 21, 7, 47]), and has also demonstrated promising results in practice (see [15, 18, 66]). Analogously to how the input-output mapping of a linear neural network is naturally represented by a matrix, that of a convolutional arithmetic circuit admits a natural representation as a tensor. Our experiments (Figure 2) showed that (at least in some settings) when learned via gradient descent, this tensor tends to have low rank. We thus obtain a second exemplar of a neural network architecture whose implicit regularization strives to lower a notion of rank for its input-output mapping. This leads us to believe that the phenomenon may be general, and formalizing notions of rank for input-output mappings of contemporary models may be key to explaining generalization in deep learning.

6 Summary

The extent to which norms (and quasi-norms) can explain the implicit regularization induced by gradient-based optimization is a central question in the theory of deep learning. A standard test-bed for its study is matrix factorization — matrix completion via linear neural networks trained by gradient descent — which in practice tends to produce low rank solutions. It is an open problem whether the implicit regularization in matrix factorization can be characterized as minimization of a norm (or quasi-norm) — Conjecture \[1\] from [30] supports this supposition, whereas Conjecture \[2\]

\[\text{out} = \sum_{i=1}^{N} \text{pool}(r)\]

Figure 3: Tensor factorizations correspond to convolutional arithmetic circuits (class of non-linear neural networks studied extensively), analogously to how matrix factorizations correspond to linear neural networks. Specifically, the tensor factorization in Equation (17) corresponds to the convolutional arithmetic circuit illustrated above (illustration assumes \(d_1 = d_2 = \ldots = d_N = d\) to avoid clutter). The input to the network is a tuple \((x_1, x_2, \ldots, x_N) \in \{1, 2, \ldots, d_1\} \times \{1, 2, \ldots, d_2\} \times \cdots \times \{1, 2, \ldots, d_N\}\), represented via one-hot vectors \((x_1, x_2, \ldots, x_N) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \times \cdots \times \mathbb{R}^{d_N}\). These vectors are processed by a hidden layer comprising: (i) locally connected linear operator with \(R\) channels, the \(r\)’th one computing inner products against filters \((w_r^{(1)}, w_r^{(2)}, \ldots, w_r^{(N)}) \in \mathbb{R}^{R_1} \times \mathbb{R}^{R_2} \times \cdots \times \mathbb{R}^{R_N}\) (this operator is referred to as “1×1 conv”, appealing to the case of weight sharing, i.e. \(w_r^{(1)} = w_r^{(2)} = \ldots = w_r^{(N)}\); followed by (ii) global pooling computing products of all activations in each channel. The result of the hidden layer is then reduced through summation to a scalar — output of the network. Overall, given input tuple \((x_1, x_2, \ldots, x_N)\), the network outputs \((\mathcal{W})_{i_1,i_2,\ldots,i_N}\) drawn from a low-rank ground truth tensor. As can be seen in terms of both reconstruction error (distance from ground truth tensor) and (tensor) rank of the produced solutions, tensor factorizations indeed exhibit an implicit regularization towards low rank. The phenomenon thus goes beyond the special case of matrix (order 2) tensor factorization. Theoretically supporting this finding is regarded as a promising avenue for future research.

\[\text{conv}(n,r) = (w_r^{(n)} x_n)\]

\[\text{pool}(r) = \prod_{i=1}^{N} \text{conv}(n,r)\]

\[\text{out} = \sum_{i=1}^{N} \text{pool}(r)\]

\[\text{hidden layer}\]

\[\text{1x1 conv}\]

\[\text{global pooling}\]

\[\text{sum (output)}\]
from [6] opposes it. We presented a simple (and robust to perturbations) matrix completion setting for which, with probability 0.5 or more over random initialization of gradient descent, the implicit regularization in matrix factorization provably drives all norms (and quasi-norms) to grow towards infinity, while rank is essentially minimized. This affirms Conjecture 2 and although it does not formally refute Conjecture 1 (the latter’s technical assumptions are not satisfied when norms grow to infinity), we believe that in essence our result implies that norm (or quasi-norm) minimization cannot explain implicit regularization in matrix factorization, let alone in deep learning altogether.

The crux behind the matrix completion setting we defined is that its solution set entails a direct contradiction between minimizing norms (or quasi-norms) and minimizing rank. The fact that the former is given up on in favor of the latter suggests that, rather than viewing implicit regularization in matrix factorization through the lens of norms (or quasi-norms), a potentially more useful interpretation is minimization of rank. As a step towards assessing the generality of this interpretation, we experimented with an extension of matrix factorization to tensor factorization, and found that it too exhibits an implicit regularization towards low rank, where rank is defined in the context of tensors. Similarly to how matrix factorization corresponds to a linear neural network whose input-output mapping is represented by a matrix, tensor factorization corresponds to a convolutional arithmetic circuit (certain type of non-linear neural network) whose input-output mapping is represented by a tensor. We thus obtain a second exemplar of a neural network architecture whose implicit regularization strives to lower a notion of rank for its input-output mapping. Theoretical investigation of the implicit regularization in tensor factorization is regarded as a promising direction for future research. More broadly, we believe that neural networks minimizing notions of rank for their input-output mappings may be a general phenomenon, and hypothesize that formalizing such notions in the context of contemporary models may be key to explaining generalization in deep learning.

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A Extension to different matrix dimensions

In this appendix we outline an extension of the construction and analysis given in Section 4 to matrix dimensions beyond 2-by-2. The extension is not unique, but rather one simple option out of many.

Beginning with square matrices, for $2 \leq d \in \mathbb{N}$, consider completion of a $d$-by-$d$ matrix based on the following observations:

$$\Omega = \{1, \ldots, d\} \times \{1, \ldots, d\} \setminus \{(1, 1)\},$$

$$b_{i,j} = \begin{cases} 1 & \text{if } i = j \geq 3 \text{ or } (i, j) \in \{(1, 2), (2, 1)\} \\ 0 & \text{otherwise} \end{cases}, \text{ for } (i, j) \in \Omega, \quad (18)$$

where as in Section 3 $\Omega$ represents the set of observed locations, and $\{b_{i,j} \in \mathbb{R}\}_{(i,j) \in \Omega}$ the corresponding set of observed values. The solution set for this problem (i.e. the set of matrices zeroing the loss in Equation (1)) is:

$$S_d := \left\{ \begin{pmatrix} w_{1,1} & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \in \mathbb{R}^{d,d} : w_{1,1} \in \mathbb{R} \right\}. \quad (19)$$

Observing $S_d$, while comparing to the solution set $S$ in our original construction (Equation (1)), we see that the former has a 2-by-2 block diagonal structure, with the top-left block holding the latter, and the bottom-right block set to identity. This implies that $d - 2$ of the singular values along $S_d$ are fixed to one, and the remaining two are identical to the singular values along $S$. Results analogous to Propositions 1 and 2 can therefore easily be proven. Since the determinant along $S_d$ is bounded below and away from zero (it is equal to $-1$), approaching $S_d$ while having positive determinant necessarily means that absolute value of unobserved entry (i.e. of the entry in location $(1, 1)$) grows towards infinity. Combining this with the fact that the product matrix (Equation (4)) of a depth $L \geq 2$ matrix factorization maintains the sign of its determinant (see Lemma 3 in Appendix B), results analogous to Theorem 1 and Corollary 1 in Appendix B may readily be established. That is, one may show that, with probability 0.5 or more over random initialization, gradient descent drives all norms (and quasi-norms) towards infinity, while essentially driving rank towards its minimum.

Moving on to the rectangular case, for $2 \leq d, d' \in \mathbb{N}$, consider completion of a $d$-by-$d'$ matrix based on the same observations as in Equation (18), but with additional zero observations such that only the entry in location $(1, 1)$ is unobserved. Assuming $d \leq d'$ without loss of generality, we have that the singular values along the solution set for this problem are the same as those along $S_d$ (Equation (19)). Moreover, if a matrix factorization applied to this problem is initialized such that its product matrix holds zeros in columns $d + 1$ to $d'$, then a dynamical characterization from [4] (restated as Lemma 4 in Appendix B), along with the structure of the loss (Equation (1)), ensure the leftmost $d$-by-$d$ submatrix of the product matrix evolves precisely as in the square case discussed above, while the remaining columns $(d + 1$ to $d')$ stay at zero. Results thus carry over from the square to the rectangular case.

B Deferred proofs

B.1 Preliminaries

We define a few notational conventions before moving on to the proofs. For $N \in \mathbb{N}$, let $[N]$ denote the set of indices $\{1, \ldots, N\}$. Let $\{e_i\}_{i=1}^d \subset \mathbb{R}^d$ be the standard basis vectors with 1 in their $i$th coordinate and 0 elsewhere. The singular values of a matrix $W \in \mathbb{R}^{d,d'}$ are denoted by $\sigma_1(W) \geq \ldots \geq \sigma_{\min\{d,d'\}}(W)$, and accordingly the eigenvalues of a symmetric square $d \times d$ matrix by $\lambda_1(W) \geq \ldots \geq \lambda_d(W)$. We let $\|W\|_{p,*}$, for $p \in (0, \infty]$, be the Schatten-$p$ quasi-norm of a matrix $W \in \mathbb{R}^{d,d'}$. Since norms are a special case of quasi-norms, when giving results applicable to both, only the latter is explicitly treated. The order $k$ derivative of a function $f(t)$ is denoted by $f^{(k)}(t)$, with $f^{(0)}(t) := f(t)$ by convention. For consistency with differential equations literature, we also denote the first order derivative with respect to time by $\dot{f}(t)$. When clear from context, we often omit the time index $t$. 

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B.2 Useful lemmas

B.2.1 Deep matrix factorization

For completeness, we include the following result from [4], which characterizes the evolution of the product matrix under gradient flow on a deep matrix factorization:

**Lemma 1** (adaptation of Theorem 1 in [4]). Let \( \ell : \mathbb{R}^{d,d'} \rightarrow \mathbb{R}_{\geq 0} \) be an analytic loss, overparameterized by a deep matrix factorization:

\[
\phi(W_1, \ldots, W_L) = \ell(W_L W_{L-1} \cdots W_1).
\]

Suppose we run gradient flow over the factorization:

\[
\dot{W}_t(t) := \frac{\partial}{\partial W_t} \phi(W_1(t), \ldots, W_L(t)), \quad t \geq 0, t = 1, \ldots, L,
\]

with a balanced initialization, i.e.:

\[
W_{l+1}(0)^\top W_{l+1}(0) = W_l(0) W_l(0)^\top, \quad l = 1, \ldots, L - 1.
\]

Then, the product matrix \( W_{L:1}(t) = W_L(t) \cdots W_1(t) \) obeys the following dynamics:

\[
W_{L:1}(t) = -\sum_{l=1}^{L} \left[ W_{L:1}(t) W_{L:1}(t)^\top \right]^\frac{l-1}{2} \cdot \nabla \ell(W_{L:1}(t)) \cdot \left[ W_{L:1}(t)^\top W_{L:1}(t) \right]^\frac{L-l}{2},
\]

where \([\cdot]^\beta, \beta \in \mathbb{R}_{\geq 0}\) stands for a power operator defined over positive semidefinite matrices (with \(\beta = 0\) yielding identity by definition).\(^{16}\)

Additionally, recall from [6] the following characterization for the singular values of \( W_{L:1}(t) \):

**Lemma 2** (adaptation of Lemma 1 and Theorem 3 in [6]). Consider the setting of Lemma 1 for depth \(2 \leq L \in \mathbb{N}\). Then, there exist analytical functions \(\{\sigma_r : [0, \infty) \rightarrow \mathbb{R}\}_{r=1}^{\min\{d,d'\}}, \{u_r : [0, \infty) \rightarrow \mathbb{R}^{d} \}_{r=1}^{\min\{d,d'\}}\) and \(\{v_r : [0, \infty) \rightarrow \mathbb{R}^{d'} \}_{r=1}^{\min\{d,d'\}}\) such that:

\[
\sigma_r(t) \geq 0, \quad u_r(t)^\top u_{r'}(t) = v_r(t)^\top v_{r'}(t) = \begin{cases} 1, & r = r' \\ 0, & r \neq r' \end{cases}, \quad t \geq 0, \quad r, r' \in \min\{d,d'\},
\]

\[
W_{L:1}(t) = \sum_{r=1}^{\min\{d,d'\}} \sigma_r(t) u_r(t)^\top v_r(t),
\]

i.e. \(\sigma_r(t)\) are the singular values of \(W_{L:1}(t)\), and \(u_r(t), v_r(t)\) are corresponding left and right (respectively) singular vectors. Furthermore, the singular values \(\sigma_r(t)\) evolve by:

\[
\dot{\sigma}_r(t) = -L \cdot (\sigma_r^2(t))^{1-1/L} \cdot \langle \nabla \ell(W_{L:1}(t)), u_r(t)^\top v_r(t) \rangle, \quad r = 1, \ldots, \min\{d,d'\}. \quad (20)
\]

We rely on this result to establish that for square product matrices the sign of \(\det(W_{L:1}(t))\) does not change throughout time.

**Lemma 3.** Consider the setting of Lemma 1 with depth \(2 \leq L \in \mathbb{N}\) and \(d = d'\). Then, the determinant of \(W_{L:1}(t)\) has the same sign as its initial value \(\det(W_{L:1}(0))\). That is, \(\det(W_{L:1}(t))\) is identically zero if \(\det(W_{L:1}(0)) = 0\), is positive if \(\det(W_{L:1}(0)) > 0\), and is negative if \(\det(W_{L:1}(0)) < 0\).

**Proof.** We prove an analogous claim for the singular values of \(W_{L:1}(t)\), from which the lemma readily follows. That is, for \(r \in [d]\), the singular value \(\sigma_r(t)\) is identically zero if \(\sigma_r(0) = 0\), and is positive if \(\sigma_r(0) > 0\).

For conciseness, define \(g(t) := -L \cdot \langle \nabla \ell(W_{L:1}(t)), u_r(t)^\top v_r(t) \rangle\). Invoking Lemma 2, let us solve the differential equation for \(\sigma_r(t)\). If \(L = 2\), the solution to Equation (20) is \(\sigma_r(t) = \)

---

\(^{16}\) An infinitely differentiable function \(f : D \rightarrow \mathbb{R}\) is analytic if at every \(x \in D\) its Taylor series converges to it on some neighborhood of \(x\) (see [44] for further details). Specifically, the square loss considered (Equation (1)) is analytic.

\(^{17}\) As discussed in Section [3], mounting empirical and theoretical evidence suggest a close match between the predictions of gradient flow with balanced initialization, and its practical realization via gradient descent with small learning rate and near-zero initialization (cf. [4, 5, 39]). It was recently argued in [60] that certain aspects of balancedness do not transfer from gradient flow to gradient descent. However, the definitions in [60] deviate from the conventional ones, hence its conclusions are not applicable to standard settings.

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We will also make use of the following lemma: we have:

\[ \text{Analytic functions are closed under summation, multiplication, and composition. The analyticity of } f(t) \text{ thus implies that } \phi(\cdot) \text{ (Equation (2)) is analytic as well. From Theorem 1.1 in [37], it then follows that under gradient flow (Equation (4)) } W_1(t), W_2(t), \ldots, W_L(t), W_{L:1}(t) \text{ and } \nabla \ell(W_{L:1}(t)) \text{ are analytic functions of } t. \]

\[ \text{Proof.} \]

Let \( f, g : [0, \infty) \to \mathbb{R} \) be real analytic functions (see Footnote 17) such that \( f^{(k)}(0) = g^{(k)}(0) \) for all \( k \in \mathbb{N} \cup \{0\} \). Then, \( f(t) = g(t) \) for all \( t \geq 0 \).

\[ \text{Proof.} \]

Define the function \( h(t) := f(t) - g(t) \). Since analytic functions are closed under subtraction, \( h \) is analytic as well. An analytic function with all zero derivatives at a point is constant on the corresponding connected component. Noticing that \( h^{(k)}(0) = 0 \) for all \( k \in \mathbb{N} \cup \{0\} \), we may conclude that \( h(t) = 0 \) and \( f(t) = g(t) \) for all \( t \geq 0 \).

\[ \square \]
Lemma 7. Let $A, B \in \mathbb{R}^{d,d}$, and suppose $B$ is positive semidefinite. Then,

$$\text{Tr}(A^\top BA) \geq \lambda_1(B) \cdot \sigma_d(A)^2.$$ 

Proof. The matrix $A^\top BA$ is positive semidefinite since for all $y \in \mathbb{R}^d$ we have

$$y^\top A^\top BA y = (Ay)^\top B(Ay) \geq 0.$$ 

Therefore, $\text{Tr}(A^\top BA) \geq \lambda_1(A^\top BA)$. Let $B = ODO^\top$ be the orthogonal eigenvalue decomposition of $B$, i.e. $O \in \mathbb{R}^{d,d}$ is an orthogonal matrix with columns $\{o_i\}_{i=1}^d$, and $D \in \mathbb{R}^{d,d}$ is diagonal holding the non-negative eigenvalues of $B$. Additionally, let $A = U\Sigma V^\top$ be the singular value decomposition of $A$, where $U, V \in \mathbb{R}^{d,d}$ are orthogonal matrices and $\Sigma \in \mathbb{R}^{d,d}$ is diagonal holding the singular values of $A$. For any unit vector (with respect to the $\ell_2$ norm) $y \in \mathbb{R}^d$ it holds that:

$$y^\top A^\top BA y = \sum_{i=1}^d \lambda_i(B)(o_i^\top Ay)^2 \geq \lambda_1(B)(o_1^\top Ay)^2.$$ 

Replacing $A$ with its singular value decomposition and choosing $y = VU^\top o_1$:

$$\lambda_1(B)(o_1^\top Ay)^2 = \lambda_1(B)(o_1^\top U\Sigma U^\top o_1)^2.$$ 

Recalling that for any unit vector the quadratic form of a symmetric matrix is bounded by the maximal and minimal eigenvalues completes the proof:

$$\text{Tr}(A^\top BA) \geq \lambda_1(A^\top BA) \geq \lambda_1(B)(o_1^\top U\Sigma U^\top o_1)^2 \geq \lambda_1(B) \cdot \sigma_d(A)^2.$$ 

Lemma 8. Let $g : [0, \infty) \to \mathbb{R}$ be a continuously differentiable function, and fix some $t > 0$. If $g(t) < g(0)$, then for any $a \in (g(t), g(0)]$ there exists $t_a \in [0, t)$ such that $g(t_a) = a$ and $g'(t_a) \leq 0$. Similarly, if $g(t) > g(0)$, then for any $a \in [g(0), g(t))$ there exists $t_a \in [0, t)$ such that $g(t_a) = a$ and $g'(t_a) \geq 0$.

Proof. Let $t > 0$ be such that $g(t) < g(0)$, and fix some $a \in (g(t), g(0)]$. Define $t_a := \max\{t' : t' \leq t \text{ and } g(t') = a\}$. Continuity of $g$, along with the intermediate value theorem, imply that $t_a$ is well defined (maximum of a closed non-empty set bounded from above). Assume by contradiction that $g(t_a) > 0$. Then, $g$ is monotonically increasing on some neighborhood of $t_a$. Thus, by the intermediate value theorem, there exists $t' \in (t_a, t)$ such that $g(t') = a$, in contradiction to the definition of $t_a$. An identical argument establishes the analogous result for the case $g(t) > g(0)$.

Lemma 9. Let $g : [0, \infty) \to \mathbb{R}$ be a non-negative differentiable function. Assume there exist constants $a, b > 0$ such that $\int_{t=t_0}^{t_1} g(t')dt' \leq a$ and $\dot{g}(t) \leq b$ for all $t \geq 0$. Then, $\lim_{t \to \infty} g(t) = 0$.

Proof. By way of contradiction let us assume that $g(t)$ does not converge to 0. Let $\varepsilon > 0$ be such that for all $M > 0$ there exists $t > M$ with $g(t) > \varepsilon$. We claim that for all $M, \varepsilon' > 0$ there exists $t > M$ such that $g(t) < \varepsilon'$. Otherwise, we have a contradiction to the bound on the integral of $g$. Combined with our assumption, this means that for all $M > 0$ we can find an interval $[t_1, t_2]$, with $t_3 > M$, where $g(t)$ transitions from $\frac{\varepsilon}{2}$ to $\varepsilon$. We now examine one such interval. Formally, for $t_0$ with $g(t_0) < \frac{\varepsilon}{2}$, we define:

$$t_2 := \min\{t \mid t \geq t_0 \text{ and } g(t) = \varepsilon\}, \quad t_1 := \max\{t \mid t \leq t_2 \text{ and } g(t) = \varepsilon/2\}.$$ 

Due to the fact that $g$ is continuous, $t_2$ and $t_1$ are well defined as they are the minimum and maximum, respectively, of closed non-empty sets bounded from below and above, respectively. Furthermore, notice that $t_0 < t_1 < t_2$. From the mean value theorem and the bound on the derivative of $g$ we have $t_2 - t_1 \geq \varepsilon/2b$. Since $g(t) \geq \varepsilon/2$ over the interval $[t_1, t_2]$, this gives us $\int_{t'=t_1}^{t_2} g(t')dt' \geq \varepsilon^2/4b$. Recall there are infinitely many such occurrences, implying that $\int_{t'=0}^{\infty} g(t')dt' = \infty$, in contradiction to the bound on the integral.
B.3 Proof of Proposition 1

For a quasi-norm $\|\cdot\|$, the weakened triangle inequality (see Footnote [1]) implies that there exists a constant $c_{\|\cdot\|} \geq 1$ for which

$$\|W\| \geq \frac{1}{c_{\|\cdot\|}} \|(W)_{1,1}e_1e_1^\top\| - \|W - (W)_{1,1}e_1e_1^\top\|,$$

$$= \|(W)_{1,1}\| \left(\frac{\|e_1e_1^\top\|}{c_{\|\cdot\|}} \right) - \|e_2e_1^\top + e_1e_2^\top\|,$$

(21)

for any $W \in S$. Fix some $\epsilon > 0$ and define $M_{\|\cdot\|,\epsilon} := \{(W)_{1,1} \in \mathbb{R} : \|W\| \leq \inf_{W' \in S} \|W'\| + \epsilon, \ W \in S\}$, the set of $(W)_{1,1}$ values corresponding to $\epsilon$-minimizers of $\|\cdot\|$. The first part of the proposition thus boils down to showing $M_{\|\cdot\|,\epsilon}$ is bounded. By Equation (21), there exist a $C > 0$ such that $(W)_{1,1} > C$ means $\|W\| > \inf_{W' \in S} \|W'\| + \epsilon$. Hence, $M_{\|\cdot\|,\epsilon} \subseteq I_{\|\cdot\|,\epsilon} := [-C, C]$.

If in addition $\|\cdot\|$ is a Schatten-$p$ quasi-norm for $p \in (0, \infty]$, we now show that $W$ is its minimizer over $S$ if and only if $(W)_{1,1} = 0$. Let $W_x \in S$ denote the solution matrix with $(W_x)_{1,1} = x$ for $x \in \mathbb{R}$. The singular values of an arbitrary such $W_x$ are:

$$\{\sigma_1(W_x), \sigma_2(W_x)\} = \left\{\left(\frac{x + \sqrt{x^2 + 4}}{2}\right), \left(\frac{x - \sqrt{x^2 + 4}}{2}\right)\right\}.$$

(22)

Starting with $p = \infty$, the corresponding norm is the spectral norm $\|W_x\|_{S_{\infty}} := \sigma_1(W_x)$. When $x = 0$, we have that $\sigma_1(W_0) = 1$. If $x > 0$, then $\sigma_1(W_x) = (x + \sqrt{x^2 + 4}) / 2 > 1$. Similarly, if $x < 0$, then $\sigma_1(W_x) = (-x + \sqrt{x^2 + 4}) / 2 > 1$. Therefore, $\|W_x\|_{S_{\infty}}$ attains its minimal value of 1 if and only if $x = 0$.

Moving to the case of $p \in (0, \infty)$, the corresponding quasi-norm is $\|W_x\|_{S_p} := (\sigma_1(W_x)^p + \sigma_2(W_x)^p)^{\frac{1}{p}}$. We now examine $\|W_x\|_{S_p}^p$ for $x > 0$:

$$\|W_x\|_{S_p}^p = \left(\frac{x + \sqrt{x^2 + 4}}{2}\right)^p + \left(\frac{-x + \sqrt{x^2 + 4}}{2}\right)^p.$$

Differentiating with respect to $x$, we arrive at:

$$\frac{p}{2^p} \left( (x + \sqrt{x^2 + 4})^{p-1} \left(1 + \frac{x}{\sqrt{x^2 + 4}}\right) + (x - \sqrt{x^2 + 4})^{p-1} \left(-1 + \frac{x}{\sqrt{x^2 + 4}}\right) \right)$$

$$> \frac{p}{2^p} \left( (x + \sqrt{x^2 + 4})^{p-1} - (x - \sqrt{x^2 + 4})^{p-1} \right)$$

$$> 0,$$

where in the first transition we used the fact that both $(x + \sqrt{x^2 + 4})^{p-1}$ and $(x - \sqrt{x^2 + 4})^{p-1}$ are positive (as well as $x$). It then directly follows that $\|W_x\|_{S_p}^p$ and thus $\|W_x\|_{S_p}$ are monotonically increasing with respect to $x$ on $(0, \infty)$.

Similar arguments show that for negative $x$ the Schatten-$p$ quasi-norm of $W_x$ is monotonically decreasing with respect to $x$, implying that $\|W_x\|_{S_p}$ is minimized if and only if $x = 0$.\[19]\]

B.4 Proof of Proposition 2

As in the proof of Proposition 1 (Appendix B.3), we denote by $W_x \in S$ the solution matrix with $(W_x)_{1,1} = x$. We begin by analyzing the behavior of $\sigma_1(W_x)$ and $\sigma_2(W_x)$ with respect to $x$. When $x = 0$ the singular values are simply $\sigma_1(W_0) = \sigma_2(W_0) = 1$. When $x$ is positive, the singular values may be written as:

$$\sigma_1(W_x) = \frac{x + \sqrt{x^2 + 4}}{2}, \quad \sigma_2(W_x) = \frac{-x + \sqrt{x^2 + 4}}{2}.$$

\[19\]The claim relies on the fact that the Schatten-$p$ quasi-norm of $W_x$ is continuous with respect to $x$ for all $p \in (0, \infty)$. We note, however, that quasi-norms in general may be discontinuous.
We are now in a position to obtain the desired results for effective and infimal ranks. The effective rank $\sigma_1(W_x)$ can be upper bounded separately. Multiplying by $x$, taking the derivative with respect to $t$, and applying the bounds from Equation (23) completes the proof:

$$\frac{d}{dx}\sigma_1(W_x) = \frac{1}{2} + \frac{x}{2\sqrt{x^2 + 4}}, \quad \frac{d}{dx}\sigma_2(W_x) = -\frac{1}{2} + \frac{x}{2\sqrt{x^2 + 4}}.$$  

Since $x > 0$, we have that $d/dx\sigma_1(W_x) > 0$ and $d/dx\sigma_2(W_x) < 0$. In other words, $\sigma_1(W_x)$ is monotonically increasing, while $\sigma_2(W_x)$ is monotonically decreasing when $x > 0$. It can easily be verified that $\sigma_1(W_x)$ and $\sigma_2(W_x)$ are even functions of $x$, i.e., $\sigma_1(W_x) = \sigma_1(W_{-x})$ and $\sigma_2(W_x) = \sigma_2(W_{-x})$. It then follows that $\sigma_1(W_x)$ is monotonically decreasing (conversely $\sigma_2(W_x)$ is monotonically increasing) when $x < 0$. Noticing that $\lim_{x\to\infty}\sigma_1(W_x) = \infty$ and $\lim_{x\to\infty}\sigma_2(W_x) = 0$ (accordingly $\lim_{x\to\infty}\sigma_1(W_x) = \infty$ and $\lim_{x\to\infty}\sigma_2(W_x) = 0$), we have a characterization of the behavior of $\sigma_1(W_x)$ and $\sigma_2(W_x)$.

We are now in a position to obtain the desired results for effective and infimal ranks. The effective rank (Definition 1) of $W_x$ can be written as

$$\text{erank}(W_x) = \exp \left\{ H\left(\frac{\sigma_1(W_x)}{\sigma_1(W_x) + \sigma_2(W_x)}, \frac{\sigma_2(W_x)}{\sigma_1(W_x) + \sigma_2(W_x)}\right) \right\}.$$  

The binary entropy function is bounded by $\ln(2)$, hence, the effective rank over $S$ is bounded by 2. This upper bound is attained at $x = 0$. According to the singular values analysis, when $|x| \to \infty$ we have that $\rho_1(W_x)$ monotonically increases towards 1, starting from the value $\rho_1(W_0) = \frac{1}{2}$. Noticing that this implies the entropy function and effective rank monotonically decrease towards 0 and 1, respectively, completes the effective rank analysis.

Next, we analyze the infimal rank of $S$ and the distance of $W_x$ from that infimal rank. The distance of $W_x$ from $M_2$ is $D(W_x, M_2) = \sigma_2(W_x)$. Since $\lim_{x\to\infty}\sigma_2(W_x) = 0$, we have $D(S, M_1) = 0$. Clearly $D(S, M_0) > 0$, leading to the conclusion that the infimal rank of $S$ is 1. Finally, the analysis of $\sigma_2(W_x)$ directly implies that the distance of $W_x$ from the infimal rank of $S$ is maximized when $x = 0$, monotonically tending to 0 as $|x| \to \infty$.

\begin{proof}

We begin by deriving loss-dependent bounds for $|w_{1,1}|$, $\sigma_1(W_{L:1})$ and $\sigma_2(W_{L:1})$. Writing the loss explicitly:

$$\ell(W_{L:1}) = \frac{1}{2} \left[ (w_{1,2} - 1)^2 + (w_{2,1} - 1)^2 + w_{2,2}^2 \right],$$

we can upper bound each of the non-negative terms separately. Multiplying by 2 and taking the square root of both sides yields:

$$|w_{2,2}| \leq \sqrt{2\ell(W_{L:1})}, \quad |w_{1,2} - 1| \leq \sqrt{2\ell(W_{L:1})}, \quad |w_{2,1} - 1| \leq \sqrt{2\ell(W_{L:1})}. \quad (23)$$

The following lemma characterizes the relation between $|w_{1,1}|$ and the loss.

**Lemma 10.** Suppose $\ell(W_{L:1}) < \frac{1}{2}$. Then:

$$|w_{1,1}| > \frac{(1 - \sqrt{2\ell(W_{L:1})})^2}{\sqrt{2\ell(W_{L:1})}} = \frac{1}{\sqrt{2\ell(W_{L:1})}} - 2 + \sqrt{2\ell(W_{L:1})}. \quad (24)$$

\begin{proof}

From Lemma 3 the determinant of $W_{L:1}$ does not change signs and remains positive, i.e.,

$$\det(W_{L:1}) = w_{1,1}w_{2,2} - w_{1,2}w_{2,1} > 0.$$ \quad (24)

Under the assumption that $\ell(W_{L:1}) < \frac{1}{2}$, both $w_{1,2}$ and $w_{2,1}$ are positive and lie inside the open interval $(0, 2)$. Since the determinant is positive, $w_{2,2} \neq 0$ and $w_{1,1}w_{2,2} > 0$ must hold. Rearranging Equation (24), we may therefore write $|w_{1,1}w_{2,2}| > w_{1,2}w_{2,1}$. Dividing both sides by $|w_{2,2}|$ and applying the bounds from Equation (23) completes the proof:

$$|w_{1,1}| > \frac{(1 - \sqrt{2\ell(W_{L:1})})^2}{\sqrt{2\ell(W_{L:1})}} = \frac{1}{\sqrt{2\ell(W_{L:1})}} - 2 + \sqrt{2\ell(W_{L:1})}. \quad \square$$

\end{proof}

\end{proof}
Furthermore, if \( \ell \) We turn to lower bound the quasi-norm of the product matrix. It holds that:

\[
\sigma_1(W_{L:1}) \geq |w_{1,1}| - \sqrt{2\ell(W_{L:1})}, \quad \sigma_2(W_{L:1}) \leq 3\sqrt{2\ell(W_{L:1})}.
\]  

(25)

Furthermore, if \( \ell(W_{L:1}) < \frac{1}{2} \), then:

\[
\sigma_1(W_{L:1}) \geq \frac{1}{\sqrt{2\ell(W_{L:1})}} - 2.
\]  

(26)

**Proof.** Define \( W_S := \begin{pmatrix} w_{1,1} & 1 \\ 1 & 0 \end{pmatrix} \), the orthogonal projection of \( W_{L:1} \) onto the solution set \( S \). By Corollary 8.6.2 in [29] we have that:

\[
|\sigma_i(W_{L:1}) - \sigma_i(W_S)| \leq \|W_{L:1} - W_S\|_F = \sqrt{2\ell(W_{L:1})}, \ i = 1, 2.
\]  

(27)

One can easily verify that \( W_S \) is a symmetric indefinite matrix with eigenvalues

\[
\{\lambda_1(W_S), \lambda_2(W_S)\} = \left\{ \left( w_{1,1} + \frac{\sqrt{w_{1,1}^2 + 4}}{2}, \ (w_{1,1} - \frac{\sqrt{w_{1,1}^2 + 4}}{2} \right) \right\}.
\]

Suppose that \( w_{1,1} \geq 0 \). We thus have:

\[
\sigma_1(W_S) = \max_{i=1,2} |\lambda_i(W_S)| = \frac{w_{1,1} + \sqrt{w_{1,1}^2 + 4}}{2} \geq |w_{1,1}|,
\]

and

\[
\sigma_2(W_S) = \min_{i=1,2} |\lambda_i(W_S)|
\]

\[
= \sqrt{\frac{w_{1,1}^2 + 4 - w_{1,1}}{2}}
\]

\[
= \frac{2}{\sqrt{2w_{1,1}^2 + 4 + w_{1,1}}}
\]

\[
\leq \frac{2}{2 + w_{1,1}},
\]

where in the third transition we made use of the identity \( a - b = \frac{a^2 - b^2}{a+b} \). If \( \ell(W_{L:1}) \geq \frac{1}{2} \), it holds that \( \sigma_2(W_S) \leq 2/(2 + w_{1,1}) \leq 1 \leq 2\sqrt{2\ell(W_{L:1})} \). Otherwise, we may apply the lower bound on \( w_{1,1} \) (Lemma 10) and conclude that \( \sigma_2(W_S) \leq 2\sqrt{2\ell(W_{L:1})} \) for any loss value. Having established that \( \sigma_1(W_S) \geq |w_{1,1}| \) and \( \sigma_2(W_S) \leq 2\sqrt{2\ell(W_{L:1})} \), Equation (27) completes the proof of Equation (25). It remains to see that if \( \ell(W_{L:1}) < \frac{1}{2} \), from the lower bound on \( w_{1,1} \) (Lemma 10), Equation (26) immediately follows.

By similar arguments, Equations (25) and (26) hold for \( w_{1,1} < 0 \) as well. \( \square \)

### B.5.1 Proof of Equation (8) (lower bound for quasi-norm)

We turn to lower bound the quasi-norm of the product matrix. It holds that:

\[
\|W_{L:1}\| \geq \frac{1}{c_{\|\cdot\|}} \|w_{1,1} e_1 e_1^T\| - \|W_{L:1} - w_{1,1} e_1^T\|,
\]  

(28)

where \( c_{\|\cdot\|} \geq 1 \) is a constant for which \( \|\cdot\| \) satisfies the weakened triangle inequality (see Footnote 1). We now assume that \( \ell(W_{L:1}) < \frac{1}{2} \). Later this assumption will be lifted, providing a bound that
holds for all loss values. Subsequent applications of the weakened triangle inequality, together with homogeneity of \( \| \cdot \| \) and the bounds on the entries of \( W_{L:1} \) (Equation (23)), give:

\[
\| W_{L:1} - w_{1,1}e_{1}^T \| \leq c_{\| \cdot \|} |w_{2,1}| \| e_{2}^T e_{1}^T \| + c_{\| \cdot \|}^2 (|w_{2,1}| \| e_{2}^T e_{1}^T \| + |w_{1,1}| \| e_{1}^T e_{2}^T \|)
\]

\[
\leq c_{\| \cdot \|} \sqrt{2\ell(W_{L:1})} \| e_{2}^T e_{1}^T \| + c_{\| \cdot \|}^2 \left(1 + \sqrt{2\ell(W_{L:1})}\right) (\| e_{2}^T e_{1}^T \| + \| e_{1}^T e_{2}^T \|)
\]

\[
\leq c_{\| \cdot \|} \| e_{2}^T e_{1}^T \| + 2c_{\| \cdot \|}^2 (\| e_{2}^T e_{1}^T \| + \| e_{1}^T e_{2}^T \|)
\]

\[
\leq 2c_{\| \cdot \|}^2 (\| e_{2}^T e_{1}^T \| + \| e_{2}^T e_{1}^T \| + \| e_{1}^T e_{2}^T \|) .
\]

Plugging the inequality above and the lower bound on \( |w_{1,1}| \) (Lemma 10) into Equation (28), we have:

\[
\| W_{L:1} \| \geq \frac{\| e_{1}^T e_{1}^T \|}{c_{\| \cdot \|}} \left(1 - \frac{2 + \sqrt{2\ell(W_{L:1})}}{c_{\| \cdot \|}} \right) - 2c_{\| \cdot \|}^2 (\| e_{2}^T e_{1}^T \| + \| e_{2}^T e_{1}^T \| + \| e_{1}^T e_{2}^T \|)
\]

\[
\geq \frac{\| e_{1}^T e_{1}^T \|}{c_{\| \cdot \|}} \left(1 - 2 \frac{\| e_{1}^T e_{1}^T \|}{c_{\| \cdot \|}} \right) - 2c_{\| \cdot \|}^2 (\| e_{2}^T e_{1}^T \| + \| e_{2}^T e_{1}^T \| + \| e_{1}^T e_{2}^T \|)
\]

\[
\geq \frac{\| e_{1}^T e_{1}^T \|}{c_{\| \cdot \|}} \left(1 - 2c_{\| \cdot \|}^2 \right) (\| e_{1}^T e_{1}^T \| + \| e_{2}^T e_{1}^T \| + \| e_{2}^T e_{1}^T \| + \| e_{1}^T e_{2}^T \|) .
\]

Since \( \| W_{L:1} \| \) is trivially lower bounded by zero, defining the constants

\[
a_{\| \cdot \|} := \frac{\| e_{1}^T e_{1}^T \|}{\sqrt{2c_{\| \cdot \|}}} , \quad b_{\| \cdot \|} := \max \left\{ \sqrt{2a_{\| \cdot \|}}, 8c_{\| \cdot \|}^2 \max_{i,j \in \{1,2\}} \| e_{i}^T e_{j}^T \| \right\},
\]

allows us, on the one hand, to arrive at a bound of the form:

\[
\| W_{L:1} \| \geq a_{\| \cdot \|} \cdot \frac{1}{\sqrt{\ell(W_{L:1})}} - b_{\| \cdot \|} ,
\]

and on the other hand, to lift our previous assumption on the loss: when \( \ell(W_{L:1}) \geq \frac{1}{2} \) the bound is vacuous, i.e. non-positive and trivially holds. Noticing this is exactly Equation (8) (recall we omitted the time index \( t \)), concludes the first part of the proof.

**B.5.2 Proof of Equation (9) (upper bound for effective rank)**

During the following effective rank (Definition 1) analysis we operate under the assumption of \( \ell(W_{L:1}) < \frac{1}{2} \). We later remove this assumption, delivering a bound that holds for all loss values. Making use of the obtained bounds on \( \sigma_1(W_{L:1}) \) and \( \sigma_2(W_{L:1}) \) (Lemma 11) we arrive at:

\[
\rho_1(W_{L:1}) = \frac{\sigma_1(W_{L:1})}{\sigma_1(W_{L:1}) + \sigma_2(W_{L:1})}
\]

\[
\geq \frac{\sigma_1(W_{L:1})}{\sigma_1(W_{L:1}) + 3\sqrt{2\ell(W_{L:1})}}
\]

\[
= 1 - \frac{3\sqrt{2\ell(W_{L:1})}}{\sigma_1(W_{L:1}) + 3\sqrt{2\ell(W_{L:1})}}
\]

\[
\geq 1 - \frac{3\sqrt{2\ell(W_{L:1})}}{\sqrt{\ell(W_{L:1})} - 2 + 3\sqrt{2\ell(W_{L:1})}}
\]

\[
= 1 - \frac{6\ell(W_{L:1})}{6\ell(W_{L:1}) - 2\sqrt{2\ell(W_{L:1})} + 1}.
\]

Given our assumption on the loss, we have \( 1 - 2\sqrt{2\ell(W_{L:1})} \geq \frac{1}{2} \) and thus

\[
\rho_2(W_{L:1}) = 1 - \rho_1(W_{L:1}) \leq \frac{6\ell(W_{L:1})}{6\ell(W_{L:1}) - \frac{2}{2}} \leq 12\ell(W_{L:1}).
\]

(29)
Let \( h (\rho_2(W_{L,1})) := -\rho_2(W_{L,1}) \cdot \ln (\rho_2(W_{L,1})) - (1 - \rho_2(W_{L,1})) \cdot \ln (1 - \rho_2(W_{L,1})) \) denote the binary entropy function, and recall that the effective rank of \( W_{L,1} \) is defined to be \( \text{erank}(W_{L,1}) := \exp \{ h (\rho_2(W_{L,1})) \} \). The exponent function is convex and therefore upper bounded on the interval \([0, \ln(2)]\) by the linear function that intersects it at these points. Formally, for \( x \in [0, \ln(2)] \) it holds that \( \exp(x) \leq 1 + \frac{1}{\ln(2)} \cdot h (\rho_2(W_{L,1})) \).

By Lemma 5 we have that \( h (\rho_2(W_{L,1})) \leq 2 \sqrt{\rho_2(W_{L,1})} \). Combined with Equation (29), since \( \inf_{W' \in S} \text{erank}(W') = 1 \) (Proposition 2), this leads to:

\[
\text{erank}(W_{L,1}) \leq \inf_{W' \in S} \text{erank}(W') + \frac{2\sqrt{12}}{\ln(2)} \cdot \sqrt{\ell(W_{L,1})}.
\]

Recall that the time index \( t \) is omitted, and the result holds for all \( t \geq 0 \), i.e. this is exactly Equation (9).

To remove our assumption on the loss, notice that when \( \ell(t) \geq \frac{1}{12} \) the bound is trivial as the right-hand side is greater than 2, which is the maximal effective rank (for a \( 2 \times 2 \) matrix).

**B.5.3 Proof of Equation (10) (upper bound for distance from infimal rank)**

According to Proposition 2 the infimal rank of \( S \) is 1. The quantity we seek to upper bound is therefore \( D(W_{L,1}(t), M_1) = \sigma_2(W_{L,1}(t)) \). By Equation (25) in Lemma 11 for all \( t \geq 0 \) we have

\[
D(W_{L,1}(t), M_1) \leq 3\sqrt{2} \cdot \sqrt{\ell(t)},
\]

completing the proof.

**B.6 Proof of Proposition 3**

Define \( W_{-1} \) to be the matrix obtained from \( W \) by multiplying its first row by \(-1\). On the one hand, symmetry around the origin implies that \( W_{-1} \) and \( W \) follow the same distribution. On the other hand, \( \det(W_{-1}) = -\det(W) \). Due to the fact that the set of matrices with zero determinant has probability 0 under continuous distributions (see, e.g., Remark 2.5 in [33]), we may conclude \( \Pr(\det(W) > 0) = \Pr(\det(W) < 0) = 0.5 \).

For \( W_1, W_2, \ldots, W_L \) random matrices drawn independently, let \( l \in [L] \) be the index such that \( \Pr(\det(W_l) > 0) = 0.5 \). Since \( \Pr(\det(W_{l'}) = 0) = 0 \) for any \( l' \in [L] \), the proof readily follows from determinant multiplicativity and the law of total probability:

\[
\Pr(\det(W_L W_{L-1} \cdots W_1) > 0) = \Pr(\det(W_l) > 0) \cdot \Pr(\Pi_{i \neq l} \det(W_i) > 0) \\
+ \Pr(\det(W_l) < 0) \cdot \Pr(\Pi_{i \neq l} \det(W_i) < 0) \\
= \frac{1}{2} [\Pr(\Pi_{i \neq l} \det(W_i) > 0) + \Pr(\Pi_{i \neq l} \det(W_i) < 0)] \\
= 0.5.
\]

An identical computation yields \( \Pr(\det(W_L W_{L-1} \cdots W_1) < 0) = 0.5 \).

**B.7 Proof of Proposition 4**

The proof makes use of the following lemma, proven in Subsection 3.7.1.

**Lemma 12.** Consider the setting of Theorem (arbitrary depth \( L \in \mathbb{N} \)) in the special case of an initial product matrix \( W_{L,1}(0) = \alpha \cdot I \), where \( I \) stands for identity matrix and \( \alpha \in (0,1] \). Then, \( W_{L,1}(t) \) is positive definite for all \( t \geq 0 \).

With Lemma 12 in place, we may derive the exact differential equations governing the product matrix in our setting of depth \( L = 2 \). Then, a detailed analysis of the dynamics will yield convergence of the loss to global minimum, i.e. \( \lim_{t \to \infty} \ell(t) = 0 \). As usual, we omit the time index \( t \) when stating results for all \( t \), or when clear from the context.
According to Lemma 12, the product matrix $W_{L:1}$ is symmetric and positive definite. Thus, we may write the loss and its gradient with respect to $W_{L:1}$ as:

$$\ell(W_{L:1}) = \frac{1}{2} \left[ w_{1,2}^2 + 2(w_{1,2} - 1)^2 \right], \quad \nabla \ell(W_{L:1}) = \begin{pmatrix} 0 & w_{1,2} - 1 \\ w_{1,2} - 1 & -2 \end{pmatrix},$$

where $\{w_{i,j}\}_{i,j \in [2]}$ are the entries of $W_{L:1}$. Since the factors $W_1$ and $W_2$ are balanced at initialization (Equation (5)), the differential equation governing the product matrix (Lemma 11) for depth $L = 2$ gives:

$$\dot{W}_{L:1} = -\left[ W_{L:1} W_{L:1}^\top \right]^{\frac{1}{2}} \cdot \nabla \ell(W_{L:1}) - \nabla \ell(W_{L:1}) \cdot \left[ W_{L:1} W_{L:1}^\top \right]^{\frac{1}{2}}$$

where the transition is by positive definiteness of $W_{L:1}$. Writing the differential equation of each entry separately, we have:

$$\begin{align*}
\dot{w}_{1,1} &= 2w_{1,2}(1 - w_{1,2}), \\
\dot{w}_{2,2} &= 2w_{1,2}(1 - w_{1,2}) - 2w_{2,2}, \\
\dot{w}_{1,2} &= w_{2,2}(1 - 2w_{1,2}) + w_{1,1}(1 - w_{1,2}).
\end{align*}$$

Let us characterize the behavior of these entries throughout time.

**Lemma 13.** The following holds for all $t \geq 0$:

1. $w_{1,1} > 0$ and is monotonically non-decreasing.
2. $0 \leq w_{1,2} \leq 1$.
3. $0 < w_{2,2} \leq 1$.

**Proof.** Since $W_{L:1}$ is positive definite, it follows that $w_{1,1}$ and $w_{2,2}$ are positive. Examining the behavior of $w_{1,2}$ (Equation (32)): on the one hand, when $w_{1,2} = 0$ then $\dot{w}_{1,2} = w_{2,2} + w_{1,1} > 0$, and on the other hand, when $w_{1,2} = 1$ then $\dot{w}_{1,2} = -w_{2,2} < 0$. Because $w_{1,2}$ is initialized at 0, it stays in the interval $[0, 1]$. Otherwise, by Lemma 8, we have a contradiction to the positivity of $w_{1,2}$ when $w_{1,2} = 0$ or its negativity when $w_{1,2} = 1$. Similarly, if $w_{2,2} > \frac{1}{2}$ we have $\dot{w}_{2,2} < 2w_{1,2}(1 - w_{1,2}) - \frac{1}{2} \leq 0$. Since at initialization $w_{2,2}(0) = \alpha \leq 1$, by Lemma 8 it will not go above 1. Lastly, since $w_{1,2}$ is in the interval $[0, 1]$, it holds that $\dot{w}_{1,1} \geq 0$, i.e. $w_{1,1}$ is monotonically non-decreasing. \qed

We turn our focus to the derivative of the loss with respect to $t$:

$$\frac{d}{dt} \ell(W_{L:1}) = \langle \nabla \ell(W_{L:1}), \dot{W}_{L:1} \rangle.$$ 

Plugging in Equation (31) and recalling the fact that $\langle A, B \rangle = \text{Tr}(A^\top B)$ for matrices $A, B$ of the same size:

$$\frac{d}{dt} \ell(W_{L:1}) = -\text{Tr}(\nabla \ell(W_{L:1})^\top W_{L:1} \nabla \ell(W_{L:1})) - \text{Tr}(\nabla \ell(W_{L:1})^\top \nabla \ell(W_{L:1}) W_{L:1}).$$

From the cyclic property of the trace operator and symmetry of $\nabla \ell(W_{L:1})$ (Equation (30)), we arrive at the following expression:

$$\frac{d}{dt} \ell(W_{L:1}) = -2 \text{Tr}(\nabla \ell(W_{L:1}) W_{L:1} \nabla \ell(W_{L:1})).$$

Notice that since $\nabla \ell(W_{L:1}) W_{L:1} \nabla \ell(W_{L:1})$ is positive semidefinite the trace is non-negative and $\frac{d}{dt} \ell(W_{L:1}) \leq 0$. That is, the loss is monotonically non-increasing throughout time. Invoking Lemma 12 we can upper bound the derivative by:

$$\frac{d}{dt} \ell(W_{L:1}) \leq -2\lambda_1(W_{L:1}) \cdot \sigma_2(\nabla \ell(W_{L:1}))^2,$$

where $\lambda_1(W_{L:1})$ is the maximal eigenvalue of $W_{L:1}$ and $\sigma_2(\nabla \ell(W_{L:1}))$ is the minimal singular value of $\nabla \ell(W_{L:1})$. The maximal eigenvalue of a symmetric matrix is greater than its diagonal entries.

25
Therefore, $\lambda_1(W_{L:1}) \geq w_{1,1}$. Since $w_{1,1}$ is initialized at $\alpha > 0$, and by Lemma 13 is monotonically non-decreasing, we have $\lambda_1(W_{L:1}) \geq \alpha$. Writing the eigenvalues of $\nabla \ell(W_{L:1})$ explicitly:

$$
\lambda_1(\nabla \ell(W_{L:1})) = \frac{w_{2,2} + \sqrt{w_{2,2}^2 + 4(1 - w_{1,2})^2}}{2},
$$

$$
\lambda_2(\nabla \ell(W_{L:1})) = \frac{w_{2,2} - \sqrt{w_{2,2}^2 + 4(1 - w_{1,2})^2}}{2},
$$

we can see that, since $w_{2,2}$ is positive (Lemma 13), $\sigma_2(\nabla \ell(W_{L:1})) = \min_{i=1,2} |\lambda_i(\nabla \ell(W_{L:1}))| = (\sqrt{w_{2,2}^2 + 4(1 - w_{1,2})^2} - w_{2,2})/2$. Applying the identity $a - b = \frac{a^2 - b^2}{a + b}$ and the bounds on $w_{2,2}$ and $w_{1,2}$ (Lemma 13),

$$
\sigma_2(\nabla \ell(W_{L:1})) = \frac{2(1 - w_{1,2})^2}{\sqrt{w_{2,2}^2 + 4(1 - w_{1,2})^2} + w_{2,2}} \geq \frac{2(1 - w_{1,2})^2}{\sqrt{1 + 4(1 - w_{1,2})^2} + 1} \geq \frac{2(1 - w_{1,2})^2}{2|1 - w_{1,2}| + 2} \geq \frac{1}{2} (1 - w_{1,2})^2,
$$

where in the penultimate transition we bounded the square root of a sum by the sum of square roots. Returning to Equation (33) we have:

$$
\frac{d}{dt} \ell(W_{L:1}) \leq -b(1 - w_{1,2})^4,
$$

for $b = \frac{1}{2} \alpha$. We are now in a position to prove that $w_{1,2} \to 1$ as $t$ tends to infinity. Integrating both sides with respect to time:

$$
\ell(W_{L:1}(t)) - \ell(W_{L:1}(0)) \leq -b \int_{t'=0}^t (1 - w_{1,2}(t'))^4 dt'.
$$

Since $\ell(W_{L:1}(t)) \geq 0$, by rearranging the inequality we may write:

$$
\int_{t'=0}^t (1 - w_{1,2}(t'))^4 dt' \leq \frac{\ell(W_{L:1}(0))}{b}.
$$

Going back to the differential equation of $\dot{w}_{1,2}$ (Equation (32)), by applying the bounds on $w_{1,2}$ and $w_{2,2}$ (Lemma 13) we have that $\dot{w}_{1,2} \geq -1$. Defining $g(t) := (1 - w_{1,2}(t))^4$, it then holds that $\dot{g}(t) \leq 4$. Since $g(\cdot)$ is non-negative and has an upper bounded integral and derivative, from Lemma 1 we can conclude that $\lim_{t \to \infty} g(t) = 0$ and $\lim_{t \to \infty} w_{1,2}(t) = 1$.

Because $\ell(W_{L:1}(t))$ is monotonically non-increasing, we need only show that for each $\epsilon > 0$ there exists $t_\epsilon > 0$ such that $\ell(W_{L:1}(t_\epsilon)) < \epsilon$. Having already established that $w_{1,2}(t)$ converges to 1, this amounts to finding a large enough $t_\epsilon$ for which $w_{2,2}(t_\epsilon)$ is sufficiently close to 0. Fix some $\epsilon > 0$ and let $\hat{t} > 0$ be such that for all $t \geq \hat{t}$ the following holds:

$$
2(1 - w_{1,2}(t))^2 < \epsilon, \quad 2w_{2,2}(t)(1 - w_{1,2}(t)) < \epsilon.
$$

Such $\hat{t}$ exists since all terms above converge to 0. Returning to the differential equation of $\dot{w}_{2,2}$ (Equation (32)):

$$
\dot{w}_{2,2}(t) < \epsilon - 2w_{2,2}(t)^2.
$$

Recalling that $w_{2,2}(t) > 0$ (Lemma 13), it follows that there exists $t_\epsilon \geq \hat{t}$ with $w_{2,2}(t_\epsilon) > -\epsilon$ (otherwise $w_{2,2}(t)$ goes to $-\infty$ as $t \to \infty$, in contradiction to the positivity of $w_{2,2}(t)$). For the above $t_\epsilon$, by rearranging the terms in Equation (35) we achieve $w_{2,2}(t_\epsilon) < \sqrt{\epsilon}$. Finally, combined with Equation (34), the result readily follows:

$$
\ell(W_{L:1}(t_\epsilon)) = \frac{1}{2} [w_{2,2}(t_\epsilon)^2 + 2(w_{1,2}(t_\epsilon) - 1)]^2 < \epsilon,
$$

concluding the proof. 

\[\square\]
B.7.1 Proof of Lemma 12

The proof proceeds as follows. We initially consider initializations where \( W(0) \) form a symmetric factorization of \( W(0) \) (Definition 5), and show that this ensures the product matrix stays symmetric. Then, we establish that for every balanced initial factors (Equation 35) with a positive definite product matrix there exist alternative balanced factors such that: (i) the initial product matrix is the same; and (ii) the factors form a symmetric factorization of the product matrix. Since the product matrices for the original and the constructed initializations obey the exact same dynamics (Lemma 1), the proof concludes.

**Definition 3.** We say that the matrices \( W_1, W_2, \ldots, W_L \in \mathbb{R}^{d,d} \) form a symmetric factorization of \( W \) if \( W = W_L W_{L-1} \cdots W_1 \) and

\[
W_l = W_{L-l+1}^T, \quad l \in \{1, \ldots, \lfloor L/2 \rfloor + 1\}.
\]

A straightforward result is that matrices with a symmetric factorization are symmetric themselves.

**Lemma 14.** If a matrix \( W \in \mathbb{R}^{d,d} \) has a symmetric factorization, then it is symmetric.

**Proof.** Let \( W_1, W_2, \ldots, W_L \in \mathbb{R}^{d,d} \) form a symmetric factorization of \( W \). It directly follows that

\[
W = W_L W_{L-1} \cdots W_1 = W_1^T \cdots W_L^T W_L = W^T.
\]

By Lemma 4, \( W_1(t), \ldots, W_L(t), W_{L:1}(t) \) and \( \nabla \ell(W_{L:1}(t)) \) are analytic, and hence infinitely differentiable, with respect to \( t \). Lemmas 15 and 16 below thus establish that if \( W(0), \ldots, W(L) \) form a symmetric factorization of \( W_{L:1}(0) \), then the product matrix stays symmetric for all \( t \).

**Lemma 15.** Under the setting of Lemma 12 assume that the matrices \( W_1(0), \ldots, W(L) \) form a symmetric factorization of \( W_{L:1}(0) \) (Definition 5). Then, for all \( k \in \mathbb{N} \cup \{0\} \):

\[
W_{L:1}^{(k)}(0) = W_{L:1}^{(k)}(0)^T,
\]

and

\[
W_l^{(k)}(0) = W_l^{(k)}(0)^T, \quad l \in \{1, \ldots, \lfloor L/2 \rfloor + 1\}.
\]

**Proof.** Before delving into the proof, we introduce notation to admit a compact representation of matrix products. For \( 1 \leq a \leq b \leq L \) and matrices \( W_1, \ldots, W_L \), for which the product \( W_L W_{L-1} \cdots W_1 \) is defined, denote:

\[
\prod_{r=a}^{b} W_r := W_b \cdots W_a,
\]

\[
\prod_{r=a}^{b} W_r^T := W_a^T \cdots W_b^T.
\]

By definition, if \( a > b \), then both \( \prod_{r=a}^{b} W_r \) and \( \prod_{r=a}^{b} W_r^T \) are identity matrices with size dependent on the context.

We proceed to the proof, which is done by induction over \( k \). For \( k = 0 \), the claim holds directly from the initialization assumption and Lemma 14. For \( k \in \mathbb{N} \), suppose the claim is true for all \( m \in \mathbb{N} \cup \{0\} \) with \( m < k \). We begin by showing Equation 56 holds for \( k \). In turn, this will lead to Equation 56 holding as well. For \( l \in [L] \), the dynamics of \( W_l(t) \) under gradient flow are

\[
W_l^{(1)}(t) = - \frac{\partial}{\partial W_l} \phi(W_1(t), W_2(t), \ldots, W_L(t)) = - \prod_{r=l+1}^{L} W_r(t)^T \cdot G(t) \cdot \prod_{r=1}^{l-1} W_r(t)^T,
\]

where \( G(t) := \nabla \ell(W_{L:1}(t)) \) denotes the loss gradient with respect to \( W_{L:1} \) at time \( t \). We can explicitly write the \( k \)'th \( (k \geq 1) \) derivative with respect to \( t \) of each \( W_l(t) \) using the product rule for higher order derivatives:

\[
W_l^{(k)}(t) = - \sum_{i_1, \ldots, i_L} \binom{k-1}{i_1, \ldots, i_L} \prod_{r=l+1}^{L} W_r^{(i_r)}(t)^T \cdot G^{(i_r)}(t) \cdot \prod_{r=1}^{l-1} W_r^{(i_r)}(t)^T,
\]

where.

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With Equation (39) in place, we can conclude the proof by showing 
\[ \sum_{i_1, \ldots, i_L} \left( \sum_{r=1}^{L-l} W_r^{(i_r)}(t) \cdot G^{(i_i)}(t)^\top \cdot \prod_{l=1}^{r} W_r^{(i_r)}(t) \right)^{m} \]

That is, 
\[ \sum_{i_1, \ldots, i_L} \left( \sum_{r=1}^{L-l} W_r^{(i_r)}(t) \cdot G^{(i_i)}(t)^\top \cdot \prod_{l=1}^{r} W_r^{(i_r)}(t) \right)^{m} \]

Turning our attention to \( G(t) \), we may write it explicitly as:
\[ G(t) = \nabla \ell(W_{L;1}(t)) = \left( \begin{array}{cc} 0 & w_{1,2}(t) - 1 \\ w_{2,1}(t) - 1 & w_{2,2}(t) \end{array} \right), \]
where \( \{w_{i,j}(t)\}_{i,j \in [2]} \) are the entries of \( W_{L;1}(t) \). For \( m < k \), note that when \( W_{L;1}^{(m)}(t) \) is symmetric so is \( G^{(m)}(t) \). With this in hand, the inductive assumption (Equation (36)) implies that \( G^{(m)}(0) \) is symmetric (for all \( m < k \)). Combined with Equation (37) (for \( m < k \), from the inductive assumption), we may write Equation (38) for \( t = 0 \) as:
\[ W^{(0)}_{L;1} = - \sum_{i_1, \ldots, i_L} \left( \sum_{r=1}^{L-l} W_r^{(i_r)}(0) \cdot G^{(i_i)}(0)^\top \cdot \prod_{l=1}^{r} W_r^{(i_r)}(0)^\top \right) \]

Reordering the sum according to \( h_r := i_{L-r+1} \) and noticing that \( (k-1)_{i_1, \ldots, i_L} = (k-1)_{i_1, \ldots, i_L} \), we conclude:
\[ W^{(0)}_{L;1} = - \sum_{h_1, \ldots, h_L} \left( \sum_{r=1}^{L-l} W_r^{(h_r)}(0) \cdot G^{(h_{L-r+1})}(0)^\top \cdot \prod_{l=1}^{r} W_r^{(h_r)}(0)^\top \right) \]

That is,
\[ W^{(0)}_{L;1} = W^{(0)}_{L-l+1}, \]
proving Equation (37).

It remains to show that \( W^{(k)}_{L;1}(0) \) is symmetric. Similarly to before, we take the \( k \)’th derivative of \( W_{L;1}(t) := W_L(t) \cdots W_1(t) \) using the product rule:
\[ W^{(k)}_{L;1}(t) = \sum_{i_1, \ldots, i_L} \left( \sum_{r=1}^{L} W_r^{(i_r)}(t)^\top \cdot \prod_{l=1}^{r} W_r^{(i_r)}(t) \right)^{k} \]
where \( \sum_{i=1}^{L} i_k = k \) and \( (k)_{i_1, \ldots, i_L} = k!(i_1! \cdots i_L!) \) for \( i_1, \ldots, i_L \in \{0, 1, \ldots, k\} \). For convenience, we denote \( B_{i_1, \ldots, i_L}(t) := \left( \sum_{r=1}^{L} W_r^{(i_r)}(t)^\top \cdot \prod_{l=1}^{r} W_r^{(i_r)}(t) \right)^{k} \). Pairing up elements in the sum with indices \( (i_1, \ldots, i_L) \) that are a reverse order of each other, i.e. \( (i_1, \ldots, i_L) \) is paired with \( (i_L, \ldots, i_1) \):
\[ W^{(k)}_{L;1}(t) = \sum_{i_1, \ldots, i_L} \frac{1}{2} \left( B_{i_1, \ldots, i_L}(t) + B_{i_L, \ldots, i_1}(t) \right) \]

With Equation (39) in place, we can conclude the proof by showing \( W^{(k)}_{L;1}(0) \) is a sum of symmetric matrices. By the inductive assumption for Equation (37), which was established in the first part of the proof for \( k \) as well, we have:
\[ B_{i_1, \ldots, i_L}(0) = B_{i_L, \ldots, i_1}(0)^\top, \]
for each \( (i_1, \ldots, i_L) \). Therefore, the matrix \( B_{i_1, \ldots, i_L}(0) + B_{i_L, \ldots, i_1}(0) \) is symmetric. Plugging Equation (40) into Equation (39) with \( t = 0 \), we arrive at a representation of \( W^{(k)}_{L;1}(0) \) as a sum of symmetric matrices. Thus, \( W^{(k)}_{L;1}(0) \) is symmetric, completing the proof. □

**Lemma 16.** Under the setting of Lemma [2] assume that the matrices \( W_1(0), \ldots, W_L(0) \) form a symmetric factorization of \( W_{L;1}(0) \) (Definition [3]). Then, \( W_{L;1}(t) \) is symmetric for all \( t \geq 0 \).

**Proof.** By Lemma [15] and Lemma [9] we may conclude that for all \( t \geq 0 \):
\[ W_l(t) = W_{L-l+1}(t)^\top, \quad l \in \{1, \ldots, [L/2] + 1\}. \]
In words, \( W_1(t), \ldots, W_L(t) \) form a symmetric factorization of \( W_{L;1}(t) \), and therefore \( W_{L;1}(t) \) is symmetric (Lemma [13]). □
Going back to the setting of Lemma 12 — initialization is balanced (Equation (5)), but does not necessarily comprise a symmetric factorization — we show that here too the product matrix remains symmetric throughout optimization. To do so, we first construct a factorization of $W_{L:1}(0)$ that is both balanced and symmetric, for which Lemma 16 ensures the product matrix stays symmetric throughout optimization. We then prove that the trajectories of the product matrix for the original and the modified initializations coincide.

Recall that $W_{L:1}(0) = \alpha \cdot I$ and define $\bar{W}_i(0) := \alpha \frac{z}{w} \cdot I$ for $l \in [L]$. It is easily verified that:

- $W_{L:1}(0) = W_L(0) \cdots W_1(0)$.
- $\bar{W}_i(0) = W_{L-l+1}(0)^\top$ for $l \in [L]$.
- $\bar{W}_{l+1}(0)^\top \bar{W}_{l+1}(0) = \bar{W}_i(0)W_i(0)^\top$ for $l \in [L-1]$.

Meaning, $\bar{W}_1(0), \ldots, \bar{W}_L(0)$ are balanced, and form a symmetric factorization of $W_{L:1}(0)$. Suppose the factors $W_1(t), \ldots, W_L(t)$ follow the gradient flow dynamics, with initial values $W_1(0), \ldots, W_L(0)$, and let $\bar{W}_{L:1}(t) := \bar{W}_L(t) \cdots \bar{W}_1(t)$ be the induced product matrix. From Lemma 16 it follows that $\bar{W}_{L:1}(t)$ is symmetric for all $t \geq 0$.

As characterized in [4] (restated as Lemma 1), if the initial factors are balanced, the product matrix trajectory depends only on its initial value $W_{L:1}(0)$. Since both the original and modified initializations are balanced and have the same product matrix, they lead to the exact same trajectory. Thus, $W_{L:1}(t) = \bar{W}_{L:1}(t)$ for all $t \geq 0$, and specifically, $W_{L:1}(t)$ is symmetric.

The last step is to see that $W_{L:1}(t)$ is not only symmetric, but positive definite as well. Since its initial value $W_{L:1}(0)$ is positive definite, it suffices to show that its eigenvalues do not change sign. By Lemma 3 the determinant of $W_{L:1}(t)$ is positive for all $t$. Specifically, the product matrix does not have zero eigenvalues. Recalling that $W_{L:1}(t)$ is an analytic function of $t$ (Lemma 4), Theorem 6.1 in [42] implies that its eigenvalues are continuous in $t$. Therefore, they can not change sign, as that would require them to pass through zero, concluding the proof.

B.8 Proof of Theorem 2

The proof follows a similar line to that of Theorem 1 (Appendix B.5), where the differences mostly stem from the fact that the solution set $S$ (Equation (12)) is not confined to symmetric matrices, as opposed to the original $S$ (Equation (7)), slightly complicating the computation of singular values. For the sake of the proof, as mentioned in Appendix B.1, we omit the time index $t$ when stating results for all $t \geq 0$, or when clear from context. We also let $\{w_{i,j}\}_{i,j \in [2]}$ denote the entries of the product matrix $W_{L:1}$.

We begin by deriving loss-dependent bounds for $|w_{1,1}|, \sigma_1(W_{L:1})$ and $\sigma_2(W_{L:1})$. The entries of $W_{L:1}$ can be trivially bounded by the loss as follows:

$$|w_{2,2} - \epsilon| \leq \sqrt{2\ell(W_{L:1})}, \quad |w_{1,2} - z| \leq \sqrt{2\ell(W_{L:1})}, \quad |w_{2,1} - z'| \leq \sqrt{2\ell(W_{L:1})}.$$  (41)

Lemma 17 below, analogous to Lemma 10 from the proof of Theorem 1 characterizes the relation between $|w_{1,1}|$ and the loss.

**Lemma 17.** Suppose $\ell(W_{L:1}) < \min\{z^2/2, z'^2/2\}$. Then:

$$|w_{1,1}| > \frac{(z - \sqrt{2\ell(W_{L:1})})(z' - \sqrt{2\ell(W_{L:1})})}{|\epsilon| + \sqrt{2\ell(W_{L:1})}} \geq \frac{|z| \cdot |z'|}{|\epsilon| + \sqrt{2\ell(W_{L:1})}} - (|z| + |z'|).$$

**Proof.** According to Lemma 3 the determinant of $W_{L:1}$ does not change sign, i.e. it remains equal to $\text{sign}(z \cdot z')$ (the initial sign assumed). Under the assumption that $\ell(W_{L:1}) < \min\{z^2/2, z'^2/2\}$, both $w_{1,2}$ and $w_{2,1}$ have the same signs as $z$ and $z'$, respectively, implying that $w_{2,2} \neq 0$ (otherwise we have a contradiction to the sign of the product matrix determinant). If $z \cdot z' > 0$, the determinant is positive as well, and it holds that $w_{1,1}w_{2,2} > w_{1,2}w_{2,1} > 0$. Otherwise, if $z \cdot z' < 0$ we have $w_{1,1}w_{2,2} < w_{1,2}w_{2,1} < 0$. Putting it together we may write $|w_{1,1}w_{2,2}| > |w_{1,2}w_{2,1}|$. Dividing by $|w_{2,2}|$ and applying the bounds from Equation (41) then completes the proof:

$$|w_{1,1}| > \frac{(z - \sqrt{2\ell(W_{L:1})})(z' - \sqrt{2\ell(W_{L:1})})}{|\epsilon| + \sqrt{2\ell(W_{L:1})}} \geq \frac{|z| \cdot |z'|}{|\epsilon| + \sqrt{2\ell(W_{L:1})}} - (|z| + |z'|).$$

$\square$
Furthermore, if \( \ell \) is compared to \( |w_{1,1}| \), the higher \( \|w_{1,1}\| \) will be driven when the loss is minimized. With Lemma 17 in place, we are now able to bound the singular values of \( W_{L,1} \).

**Lemma 18.** The singular values of \( W_{L,1} \) fulfill:

\[
\begin{align*}
\sigma_1(W_{L,1}) & \geq \frac{1}{\sqrt{2}} \cdot |w_{1,1}| - \sqrt{2 \ell(W_{L,1})}, \\
\sigma_2(W_{L,1}) & \leq 4|\epsilon| + \left( 4 + \sqrt{|z| \cdot |z'|} / \min(|z|, |z'|) \right) \sqrt{2 \ell(W_{L,1})}.
\end{align*}
\]  \( \text{(42)} \)

Furthermore, if \( \ell(W_{L,1}) < \min \{ z^2 / 2, z'^2 / 2 \} \), the bound on \( \sigma_2(W_{L,1}) \) may be simplified:

\[
\sigma_2(W_{L,1}) \leq 4|\epsilon| + 4\sqrt{2 \ell(W_{L,1})}.
\]  \( \text{(43)} \)

**Proof.** Define \( W_{\tilde{S}} := \begin{pmatrix} w_{1,1} & z' \\ z & \epsilon \end{pmatrix} \), the orthogonal projection of \( W_{L,1} \) onto the solution set \( \tilde{S} \). From Corollary 8.6.2 in [29] we know that:

\[
|\sigma_i(W_{L,1}) - \sigma_i(W_{\tilde{S}})| \leq \|W_{L,1} - W_{\tilde{S}}\|_F = \sqrt{2 \ell(W_{L,1})}, \quad i = 1, 2.
\]  \( \text{(44)} \)

This means that any bound on the singular values of \( W_{\tilde{S}} \) can be transferred to those of \( W_{L,1} \) (up to an additive loss-dependent term). It is straightforwardly verified that the squared singular values of \( W_{\tilde{S}} \) are

\[
\begin{align*}
\sigma_1^2(W_{\tilde{S}}) &= \frac{1}{2} \left( w_{1,1}^2 + z^2 + z'^2 + \epsilon^2 + \sqrt{(w_{1,1}^2 + z^2 + z'^2 + \epsilon^2)^2 - 4 (w_{1,1} \epsilon - z z')^2} \right), \\
\sigma_2^2(W_{\tilde{S}}) &= \frac{1}{2} \left( w_{1,1}^2 + z^2 + z'^2 + \epsilon^2 - \sqrt{(w_{1,1}^2 + z^2 + z'^2 + \epsilon^2)^2 - 4 (w_{1,1} \epsilon - z z')^2} \right).
\end{align*}
\]  \( \text{(45)} \)

Note that the term inside the square roots is non-negative for all \( w_{1,1}, z, z', \epsilon \). Since all elements in the expression for \( \sigma_1^2(W_{\tilde{S}}) \) are non-negative, we have \( \sigma_1(W_{\tilde{S}}) \geq (1/\sqrt{2}) \cdot |w_{1,1}| \). Combining this with Equation (44) completes the lower bound for \( \sigma_1(W_{L,1}) \).

Next, let \( W_{\tilde{S}_0} := \begin{pmatrix} w_{1,1} & z' \\ z & 0 \end{pmatrix} \) be the matrix obtained by replacing the bottom-right entry of \( W_{\tilde{S}} \) by 0. Replacing \( \epsilon \) with 0 in Equation (45) and applying the identity \( a - b = \frac{a^2 - b^2}{a + b} \), we get:

\[
\sigma_2^2(W_{\tilde{S}_0}) = \frac{2 z^2 z'^2}{w_{1,1}^2 + z^2 + z'^2 + \sqrt{(w_{1,1}^2 + z^2 + z'^2)^2 - 4 z^2 z'^2}} \leq \frac{2 z^2 z'^2}{w_{1,1}^2 + z^2 + z'^2}.
\]  \( \text{(46)} \)

We initially prove Equation (43) holds in the case where \( \ell(W_{L,1}) < \min \{ z^2 / 2, z'^2 / 2 \} \). By lifting said assumption we then show that the bound on \( \sigma_2(W_{L,1}) \) in Equation (42) holds for any loss value. Under the assumption that \( \ell(W_{L,1}) < \min \{ z^2 / 2, z'^2 / 2 \} \), taking the square root of both sides in Equation (46), we arrive at the following bound:

\[
\sigma_2(W_{\tilde{S}_0}) \leq \sqrt{2} \cdot \frac{|z| \cdot |z'|}{w_{1,1}^2 + z^2 + z'^2} \leq \sqrt{6} \cdot \frac{|z| \cdot |z'|}{|w_{1,1}| + |z| + |z'|} \leq \sqrt{6} \cdot \frac{|z| \cdot |z'|}{|z| + |z'|} \leq 3 \left( |\epsilon| + \sqrt{2 \ell(W_{L,1})} \right),
\]
where in the second transition we applied the inequality \( \sqrt{w_{1,1}^2 + z^2 + z'^2} \geq (|w_{1,1}| + |z| + |z'|)/\sqrt{3}, \) and in the third made use of the bound on \( |w_{1,1}| \) (Lemma 17). Applying Corollary 8.6.2 from [29] twice, once for the matrices \( W_{L,1} \) and \( W_S \), and another for \( W_S \) and \( W_{S_0} \), we have:

\[
\sigma_2(W_{L,1}) = 3 \left( ||e|| + \sqrt{2\ell(W_{L,1})} \right) + ||e|| + \sqrt{2\ell(W_{L,1})} = 4 \left( ||e|| + \sqrt{2\ell(W_{L,1})} \right),
\]

achieving the desired result from Equation (43). It remains to see that the bound on \( \sigma_2(W_{L,1}) \) in Equation (42) holds regardless of the loss value. When \( \ell(W_{L,1}) < \min \{ z'^2/2, z''/2 \} \) it obviously holds since it is only looser than the bound already obtained under this assumption. Otherwise, going back to Equation (46), it can be seen that

\[ \sigma_2^2(W_{S_0}) \leq \frac{2z^2z'^2}{(z - z')^2 + 2|z| \cdot |z'|} \leq |z| \cdot |z'|. \]

Thus, \( \sigma_2(W_{S_0}) \leq \sqrt{|z| \cdot |z'|} \). Following the same procedure as before (applying Corollary 8.6.2 from [29]), combined with the fact that \( \ell(W_{L,1}) \geq \min \{ z'^2/2, z''/2 \} \) concludes the proof:

\[
\sigma_2(W_{L,1}) \leq \sqrt{|z| \cdot |z'| + ||e|| + 2\ell(W_{L,1})} \\
\leq \frac{\sqrt{|z| \cdot |z'|}}{\min\{|z|, |z'|\}} \cdot \sqrt{2\ell(W_{L,1})} + ||e|| + \sqrt{2\ell(W_{L,1})} \\
\leq 4 ||e|| + \left( 4 + \frac{\sqrt{|z| \cdot |z'|}}{\min\{|z|, |z'|\}} \right) \sqrt{2\ell(W_{L,1})}.
\]

\[\square\]

**B.8.1 Proof of Equation (13) (lower bound for quasi-norm)**

Turning our attention to \( \|W_{L,1}\| \), following the same steps as in the proof of Theorem 1 (Appendix B.3.1) will lead to a generalized bound. By the triangle inequality:

\[
\|W_{L,1}\| \geq \frac{1}{c_{\|\cdot\|}} \left( \|w_{1,1}e_1e_1^T\| - \|W_{L,1} - w_{1,1}e_1e_1^T\| \right), \quad (47)
\]

where \( c_{\|\cdot\|} \geq 1 \) is a constant with which \( \|\cdot\| \) satisfies the weakened triangle inequality (see Footnote 1). Let us initially assume that \( \ell(W_{L,1}) < \min\{z'^2/2, z''/2\} \). We later lift this assumption, delivering a bound that holds for all loss values. Invoking Equation (41) we may bound the negative term in Equation (47) as follows:

\[
\|W_{L,1} - w_{1,1}e_1e_1^T\| \leq c_{\|\cdot\|} \|w_{2,1}\| \left( \|e_2e_2^T\| + c_{\|\cdot\|} \left( \|w_{1,2}\| \|e_2e_1^T\| + \|w_{1,2}\| \|e_1e_2^T\| \right) \right) \\
\leq 3c_{\|\cdot\|} \left( \max\{|z|, |z'|, ||e||\} + \sqrt{2\ell(W_{L,1})} \right) \max_{(i,j)\neq(1,1)} \|e_ie_j^T\| \\
\leq 6c_{\|\cdot\|} \max\{|z|, |z'|, ||e||\} \cdot \max_{i,j\in\{1,2\}, (i,j)\neq(1,1)} \|e_ie_j^T\|,
\]

Returning to Equation (47), applying the inequality above and the bound on \( |w_{1,1}| \) (Lemma 17) we have:

\[
\|W_{L,1}\| \geq \frac{\|e_1e_1^T\|}{c_{\|\cdot\|}} \left( \frac{|z| \cdot |z'|}{||e|| + \sqrt{2\ell(W_{L,1})}} - |z| \cdot |z'| - 6c_{\|\cdot\|} \max\{|z|, |z'|, ||e||\} \cdot \max_{(i,j)\neq(1,1)} \|e_ie_j^T\| \right) \\
\geq \frac{\|e_1e_1^T\|}{c_{\|\cdot\|}} \cdot \frac{|z| \cdot |z'|}{||e|| + \sqrt{2\ell(W_{L,1})}} - 8c_{\|\cdot\|} \max\{|z|, |z'|, ||e||\} \cdot \max_{i,j\in\{1,2\}} \|e_ie_j^T\|. 
\]

Since \( \|W_{L,1}\| \) is trivially lower bounded by zero, defining the constants

\[
a_{\|\cdot\|} := \frac{\|e_1e_1^T\|}{c_{\|\cdot\|}}, \quad b_{\|\cdot\|} := \max\left\{ \frac{a_{\|\cdot\|} \cdot |z| \cdot |z'|}{||e|| + \min\{|z|, |z'|\}}, 8c_{\|\cdot\|} \max\{|z|, |z'|, ||e||\} \cdot \max_{i,j\in\{1,2\}} \|e_ie_j^T\| \right\},
\]

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Additionally, under our assumption that \( \ell(W_{t:1}) \geq \min\{z^2/2, z'^2/2\} \), the bound is non-positive and trivially holds. Noticing this is exactly Equation (13) (recall we omitted the time index \( t \)), concludes this part of the proof.

B.8.2 Proof of Equation (14) (upper bound for effective rank)

Derivation of the upper bound for effective rank (Definition 1) is initially done under the assumption that \( \ell(W_{t:1}) < \min\{z^2/8, z'^2/8\} \). We then remove this assumption, establishing a bound that holds for all loss values.

The bounds on \( \sigma_1(W_{t:1}) \) and \( \sigma_2(W_{t:1}) \) in Lemma 17 give:

\[
\rho_1(W_{t:1}) = \frac{\sigma_1(W_{t:1})}{\sigma_1(W_{t:1}) + \sigma_2(W_{t:1})} \\
\geq \frac{\sigma_1(W_{t:1})}{\sigma_1(W_{t:1}) + 4(|\epsilon| + \sqrt{2\ell(W_{t:1})})} \\
= 1 - \frac{4(|\epsilon| + \sqrt{2\ell(W_{t:1})})}{\sigma_1(W_{t:1}) + 4(|\epsilon| + \sqrt{2\ell(W_{t:1})})} \\
\geq 1 - \frac{1}{\sqrt{2}} \cdot |w_{1,1}| + 4|\epsilon| + 3\sqrt{2\ell(W_{t:1})} \\
\geq 1 - \frac{4\sqrt{2}(|\epsilon| + \sqrt{2\ell(W_{t:1})})}{|w_{1,1}|}.
\]

Additionally, under our assumption that \( \ell(W_{t:1}) < \min\{z^2/8, z'^2/8\} \), the bound on \( |w_{1,1}| \) in Lemma 17 can be simplified to:

\[
|w_{1,1}| \geq \frac{(|z| - \sqrt{2\ell(W_{t:1})})(|z'| - \sqrt{2\ell(W_{t:1})})}{|\epsilon| + \sqrt{2\ell(W_{t:1})}} \geq \frac{\min\{|z|, |z'|\}^2}{4(|\epsilon| + \sqrt{2\ell(W_{t:1})})}.
\]

Combining the last two inequalities we have:

\[
\rho_2(W_{t:1}) = 1 - \rho_1(W_{t:1}) \leq \frac{16\sqrt{2}(|\epsilon| + \sqrt{2\ell(W_{t:1})}^2)}{\min\{|z|, |z'|\}^2}.
\]

It is now possible to see that, in accordance with Subsection 4.4, the smaller \(|\epsilon|\) is compared to \(\min\{|z|, |z'|\}\), the closer to zero \(\rho_2(W_{t:1})\) becomes as the loss is minimized. Let \( h(\rho_2(W_{t:1})) := -\rho_2(W_{t:1}) \cdot \ln(\rho_2(W_{t:1})) - (1 - \rho_2(W_{t:1})) \cdot \ln(1 - \rho_2(W_{t:1})) \) denote the binary entropy function, and recall that the effective rank of the product matrix defined to be \( \text{erank}(W_{t:1}) := \exp(h(\rho_2(W_{t:1}))) \). As in the proof of Theorem 1 (Appendix B.5.2), we may bound the exponent on the interval \([0, \ln(2)]\) by the linear function intersecting it at these points. That is,

\[
\text{erank}(W_{t:1}) \leq 1 + \frac{1}{\ln(2)} \cdot h(\rho_2(W_{t:1})).
\]

From Lemma 5 it holds that \( h(\rho_2(W_{t:1})) \leq 2\sqrt{\rho_2(W_{t:1})} \). Plugging this into the inequality above leads to:

\[
\text{erank}(W_{t:1}) \leq 1 + \frac{8 \cdot 2^{\frac{1}{2}}}{\ln(2) \cdot \min\{|z|, |z'|\}} \cdot (|\epsilon| + \sqrt{2\ell(W_{t:1})}) \\
\leq 1 + \frac{16}{\min\{|z|, |z'|\}} \cdot (|\epsilon| + \sqrt{2\ell(W_{t:1})}).
\]
where the second transition is a slight simplification of the constants \(2^{1/4}/\ln(2) < 2\). As will be shown below, \(\inf_{W' \in \mathcal{S}} \text{rank}(W') = 1\). We may thus conclude:

\[
\text{rank}(W_{L,1}) \leq \inf_{W' \in \mathcal{S}} \text{rank}(W') + \frac{16}{\min \{|z|, |z'|\}} \cdot (|\epsilon| + \sqrt{2\ell(W_{L,1})}) .
\]

Notice that when \(\ell(W_{L,1}) \geq \min\{z^2/8, z'^2/8\}\) the inequality trivially holds since the right-hand side is greater than \(2\) (the maximal effective rank for a \(2 \times 2\) matrix). This establishes Equation 14 (time index is omitted).

It remains to prove that \(\inf_{W' \in \mathcal{S}} \text{rank}(W') = 1\). If \(\epsilon \neq 0\), it is trivial since there exists \(W' \in \tilde{\mathcal{S}}\) with \(\text{rank}(W') = 1\), meaning \(\sigma_2(W') = 0\) and \(\text{rank}(W') = 1\). If \(\epsilon = 0\), examining the squared singular values of \(W' \in \tilde{\mathcal{S}}\) (Equation 45 with \((W')_{1,1}\) in place of \(w_{1,1}\)) reveals that \(\lim_{(W')_{1,1} \to \infty} \sigma_2(W') = 0\), while \(\lim_{(W')_{1,1} \to \infty} \sigma_1(W') = \infty\). Thus, there exists a matrix in \(\tilde{\mathcal{S}}\) with effective rank arbitrarily close to 1. Since the effective rank of any matrix is at least 1, this implies that \(\inf_{W' \in \mathcal{S}} \text{rank}(W') = 1\).

### B.8.3 Proof of Equation 15 (upper bound for distance from infimal rank)

We claim that the infimal rank (Definition 2) of \(\tilde{\mathcal{S}}\) is 1. Since \(z, z' \neq 0\), it cannot be 0. If \(\epsilon \neq 0\), our claim is trivial since there exists \(W' \in \tilde{\mathcal{S}}\) with \(\text{rank}(W') = 1\). Otherwise, inspecting the squared singular values of a matrix \(W' \in \tilde{\mathcal{S}}\) (Equation 45 with \((W')_{1,1}\) in place of \(w_{1,1}\)), we can see that, when \(\epsilon = 0\), taking \((W')_{1,1}\) to infinity drives the minimal singular value towards zero \((\lim_{(W')_{1,1} \to \infty} \sigma_2(W') = 0)\). Hence, the distance of \(\tilde{\mathcal{S}}\) from the set of matrices with rank 1 or less is 0 in this case as well.

The distance of the product matrix from the infimal rank of \(\tilde{\mathcal{S}}\) is therefore \(D(W_{L,1}(t), \mathcal{M}_1) = \sigma_2(W_{L,1}(t))\). From Lemma 18 we have

\[
D(W_{L,1}(t), \mathcal{M}_1) \leq 4|\epsilon| + \left(4 + \frac{\sqrt{|z| \cdot |z'|}}{\min \{|z|, |z'|\}}\right) \sqrt{2\ell(t)} ,
\]

for all \(t \geq 0\), concluding the proof.

### C Further experiments and implementation details

#### C.1 Further experiments

Figure 3 supplements Figure 1 from Subsection 5.1 by demonstrating empirically that the phenomenon of implicit regularization in matrix factorization driving all norms (and quasi-norms) towards infinity is robust to perturbations, as established theoretically in Subsection 4.2. Figure 5 supplements Figure 2 from Subsection 5.2 further demonstrating that gradient descent over tensor factorization exhibits an implicit regularization towards low (tensor) rank.

#### C.2 Implementation details

We provide a full description of the technical details omitted from the experiments section (Section 5). Source code for reproducing our results and figures, based on the PyTorch framework ([59]), can be found at https://github.com/noamrazin/imp_reg_dl_not_norms.

##### C.2.1 Deep matrix factorization (Figures 1 and 4)

In all experiments, gradient descent with a fixed learning rate was run until the loss reached a value lower than \(10^{-4}\) (or 5 \(\cdot\) 10^{6} iterations elapsed). Both balanced (Equation 5) and unbalanced (layer-wise independent) random initializations were calibrated according to a desired standard deviation \(\sigma > 0\) for the entries of the initial product matrix. In the unbalanced initialization, each weight was sampled independently from a zero-mean Gaussian distribution with standard deviation \((\sigma^2/2^L-1)^{1/2L}\), where \(L\) stands for factorization depth. For the balanced initialization, we used Procedure 1 from [5], based on a Gaussian distribution with independent entries, zero mean and
We conducted our analysis by the minimal \( \gamma \) with \( \text{(i)} \) if the diagonal observation \( b_{1.2} \) is unperturbed (stays at zero), the off-diagonal ones \( b_{1.2}, b_{2.1} \) can take on any non-zero values, and as long as the product matrix's determinant at initialization has the same sign as \( b_{1.2} \), absolute value of unobserved entry will grow to infinity; and \( \text{(ii)} \) the extent to which absolute value of unobserved entry grows gracefully recedes as \( b_{2.2} \) is perturbed away from zero. This is demonstrated in the plots above, which for representative runs, show absolute value of unobserved entry as a function of the loss (Equation 1), with iteration number encoded by color. Each plot corresponds to a different assignment for \( b_{1.2}, b_{2.1} \), and presents runs with varying values for \( b_{2.2} \). Presented runs were obtained with a depth 3 matrix factorization initialized randomly by an unbalanced (layer-wise independent) distribution, with the latter’s standard deviation and learning rate for gradient descent set to the smallest values used for Figure 1 (all other settings we evaluated produced similar results). For further details see Appendix C.2.

The experiments in Figure 4 were carried out with the smallest values reported for depth 3 in Figure 1 (i.e. learning rate and standard deviation of \( 5 \cdot 10^{-3} \) and \( 10^{-6} \), respectively).

C.2.2 Tensor factorization (Figures 2 and 5)

Plots begin at the smallest sample size for which stable results were obtained, and end when all entries but one are observed. Specifically, experiments with sample sizes \( \{50, 100, 150, \ldots, 400, 450, 511\} \) and \( \{100, 500, 1000, 1500, \ldots, 3000, 3500, 4005\} \) were conducted for rank-1 ground truth tensors of orders 3 and 4, respectively. In the case of rank-3 ground truth tensors, plots start from 3 times as many observations (i.e. minimal sample sizes of 150 and 300 for orders 3 and 4, respectively). Gradient descent was initialized by independently sampling all weights from a zero-mean Gaussian distribution, and run until the training loss reached a value lower than \( 10^{-6} \) (or \( 10^{5} \) iterations elapsed). We conducted 5 trials (differing in random seed for initialization) for each standard deviation of initialization from the set \( \{10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}\} \). To allow efficient experimentation, we employed an adaptive learning rate scheme, where at each iteration a base learning rate of \( 10^{-2} \) is divided by the square root of an exponential moving average of gradient norms squared. That is, for base learning rate \( \eta = 10^{-2} \) and weighted average coefficient \( \beta = 0.99 \), at iteration \( t \) we set the learning rate \( \eta_t = \eta / (\sqrt{\gamma_t / (1-\beta^t)} + 10^{-6}) \) for \( \gamma_t = \beta \cdot \gamma_{t-1} + (1-\beta) \cdot \sum_{k=1}^{t} \|\partial w / \partial w\|_2 \cdot (V(t))_k \|^2 \), with \( \gamma_0 = 0 \). We emphasize that only the learning rate is altered, without any modification to the direction of the gradient. Comparisons between the adaptive learning rate scheme and a fixed small learning rate showed no noticeable impact to the end result, with significant difference in run times.

When referring to tensor rank, we consider the classic CP-rank (see [43]). While exact tensor rank estimation is known to be computationally hard in the worst case (35), in practice, a standard way of measuring is by the minimal \( R' \) for which the Alternating Least Squares (ALS) algorithm achieves reconstruction error below a certain threshold (see [43] for further details). We follow this...
Figure 5: Gradient descent over tensor factorization exhibits an implicit regularization towards low (tensor) rank. This figure is identical to Figure 2 except that the experiments it portrays had ground truth tensors of rank 3 (instead of 1). For further details see caption of Figure 2 as well as Appendix C.2.

In practice, defining a threshold of mean square error below $10^{-6}$. Sampling a ground truth rank-$R$ tensor $\mathcal{W}^* \in \mathbb{R}^{d_1, d_2, \ldots, d_N}$ is done by computing:

$$\mathcal{W}^* = \sum_{r=1}^{R} w_r^{*(1)} \otimes w_r^{*(2)} \otimes \cdots \otimes w_r^{*(N)}$$

where $\{w_r^{*(n)}\}_{r=1}^{R} \in \mathbb{R}^{d_n}$, $r = 1, 2, \ldots, R$, $n = 1, 2, \ldots, N$, are drawn independently from the standard normal distribution. After every draw, we estimate the rank of the obtained tensor to affirm it is indeed $R$ (the sampling procedure ensures only that it is at most $R$). For convenience, we subsequently normalize the ground truth tensor to be of unit Frobenius norm.