Quantum tunneling time of a Bose-Einstein condensate traversing through a laser-induced potential barrier

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We theoretically study the effect of atomic nonlinearity on the tunneling time in the case of an atomic Bose-Einstein condensate (BEC) traversing the laser-induced potential barrier. The atomic nonlinearity is controlled to appear only in the region of the barrier by employing the Feshbach resonance technique to tune interatomic interaction in the tunneling process. Numerical simulation shows that the atomic nonlinear effect dramatically changes the tunneling behavior of the BEC matter wave packet, and results in the violation of Hartman effect and the occurrence of negative tunneling time.

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Quantum tunneling of a wave packet through a potential barrier is one of the fundamental topic in quantum physics [1, 3]. The issue of tunneling time has attracted a lot of attention for decades since it was first put forward by Condon [4]. In the early 1960’s, Hartman predicted that the tunneling time becomes independent of barrier length for thick enough barriers, ultimately resulting in unbounded tunneling velocities [5]. Such a phenomenon, termed as the Hartman effect later on, seems to imply the superluminal velocities inside the barriers and leads to a wide interest in many different fields [6, 8].

Mathematically, the quantum tunneling is governed by the Schrödinger equation, which is a linear equation describing the quantum wave nature of a single particle. So far the Hartman effect or related topics studied in the literature are limited to the single-particle linear case. After mid-1990’s, there are significant advancements in the realization of atomic Bose-Einstein condensate (BEC), a macroscopic quantum mechanical wave packet with nonlinear behavior due to interatomic interactions. Such a macroscopic coherent matter wave packet of BEC opens a new window to study the nonlinear quantum dynamics governed by the nonlinear Schrödinger equation or the mean field Gross-Pitaevskii (GP) equation [10]. No doubt, tunneling of a BEC matter wave packet would exhibit different behaviors compared with the single particle tunneling in the linear quantum mechanics. In fact, there already exist many studies on BEC tunneling through different kinds of potentials in the literature [11, 13]. However, to our knowledge, the effect of non-linear interparticle interaction on the Hartman effect has not been explored.

In this brief report, we theoretically investigate how the interatomic interaction affects the tunneling time in the case of a coherent BEC wave packet traversing through a potential barrier created by laser beam. We consider a BEC wave packet confined in a quasi-one-dimensional atomic waveguide to traverse a potential barrier created by a far blue-detuned laser beam, which is shown in Fig. 1. Suppose that the quasi-one-dimensional atomic waveguide is transversely confined by a harmonic trapping potential, and the dynamics of atomic BEC can be described by the 3D nonlinear Schrödinger equation

\[ i\hbar \frac{\partial \Psi}{\partial t} = \left( -\frac{\hbar^2 \nabla^2}{2m} + V(x) + \frac{1}{2} m \omega_{\perp} r_{\perp}^2 + gN |\Psi|^2 \right) \Psi, \]  

where \( \Psi \) is the normalized macroscopic wave function of Bose Einstein condensates, \( r_{\perp} = (y, z) \), \( m \) the atomic mass, and \( \omega_{\perp} \) the trapping frequency in the radial(transverse) direction. \( g = 4\pi\hbar^2 a_s/m \) describes the interatomic interaction with \( a_s \) being the atomic s-wave scattering length. \( N \) is the total number of atoms in the condensate, and \( V(x) \) is the potential barrier created by the blue-detuned laser beam.

When transverse confinement is very strong, the transverse motion of BEC atoms may be considered to remain in the ground state. As a result, we can approximate the total wave function of the BEC as

\[ \Psi(x, r, t) \approx \psi(x, t) \varphi_{\perp}(r) e^{-i\omega_{\perp} t}, \]  

where \( \varphi_{\perp}(r) \) is the transverse ground state wave function of the BEC, which can approximately be replaced by a

FIG. 1: (Color online) Schematic diagram of a BEC wave packet traversing a blue detuning laser beam in an atomic waveguide.
Gaussian function in the weak nonlinear limit

\[ \varphi_\perp (r) = \frac{1}{\sqrt{\pi a_\perp}} \exp \left( -\frac{r^2}{2a_\perp^2} \right), \]

with \( a_\perp = \sqrt{\hbar/ma_\perp} \) being the ground state length of harmonic trapping potential. The longitudinal wave function \( \psi(x, t) \) is governed by the effective 1D nonlinear Schrödinger equation:

\[ i\frac{\partial \psi}{\partial t} = -\frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} + V(x) \psi + g |\psi|^2 \psi. \]

In Eq. (4), we express \( x \) in units of \( a_\perp \), \( t \) in units of \( \omega_\perp^{-1} \), the dimensionless nonlinear interaction \( g = 2a_s N/a_\perp \), and the dimensionless wave function \( \psi = \psi/\sqrt{a_\perp} \). \( V(x) \) is the potential barrier experienced by the BEC wave packet and written in the form

\[ V(x) = V_0 f(x), \]

where \( V_0 \) is the peak value of the potential normalized by \( \hbar \omega_\perp \) and \( f(x) \) is the barrier profile, which is controlled by the laser beam.

Now we have obtained a general dimensionless equation (4) to describe the BEC wave packet transmitting through a barrier. As pointed out in Ref. [14], in which both the barrier and nonlinear interaction are set constant. To cover different experimental situations, it is convenient to introduce such scaling transformation in numerical simulation. In this paper, we set \( \omega_\perp \approx 2\pi \times 100 \text{ Hz} \) and \( \eta = 10 \), then for \(^{87}\text{Rb} \) atoms, \( a_\perp \approx 1 \mu\text{m} \). In principle, Eq. (4) can describe a BEC wave packet transmitting through an arbitrary barrier. To simplify the problem, without loss of physical feature, in this paper we just consider a rectangular barrier case. Such a rectangular barrier can be approximately created by a super-Gaussian laser beam with a large enough order [15–17]. Therefore the barrier profile \( f(x) \) has the form

\[ f(x) = \begin{cases} 1, & -\frac{L}{2} < x < \frac{L}{2}, \\ 0, & x < -\frac{L}{2} \text{ or } x > \frac{L}{2}. \end{cases} \]

In the following, we simulate finite wave packets traversing through the rectangular barrier via the split operator method [21]. Assume the normalized initial wave packet is Gaussian

\[ \psi(x, 0) = \frac{1}{\sqrt{\sqrt{\pi} \Delta x}} \exp \left( -\frac{(x-x_0)^2}{2\Delta x^2} + ik_0 (x-x_0) \right), \]

where \( x_0, \Delta x \) and \( k_0 \) are the initial center position, the initial half width and the initial center momentum of the wave packet, respectively. From Eq. (4) we can find that the interatomic interaction will induce the self-phase modulation (SPM) [18, 19] of matter wave packet. The SPM happens in both free region and potential region and causes confusion of the nonlinear effect on the tunneling process inside the barrier with the SPM process outside the barrier region. To avoid this confusion, we shall eliminate the nonlinear interaction outside of the potential region, i.e., make the \( s \)-wave scattering length \( a_s \) vanishing outside of the potential region. In principle, this can be realized by employing the Feshbach resonance technique to tune the atomic scattering length via a spatially varying magnetic field [20]. For spatially varying magnetic field, the scattering length \( a_s (x) = a_{s0} (1 - \Delta B/(B(x) - B_0)) \), where \( a_{s0} \) is the background scattering length, \( \Delta B \) the resonance width, \( B_0 \) the magnetic field of resonance, \( B(x) \) the magnetic field. By tuning the magnetic field, one can control the atomic nonlinearity with positive sign (corresponding to repulsive interatomic interaction), negative one (attractive interaction) and zero (no interaction). In our work we set the spatial profile of the scattering length is consistent with that of the barrier, namely, \( g(x) = g_0 f(x) \). Considering the finite width of initial atomic BEC wave packet used in experiment, we set \( \Delta x = 50 \) in all the following simulations, corresponding to a half-width 500 \( \mu\text{m} \) for a BEC wave packet composed of \(^{87}\text{Rb} \) atoms.

With the above assumptions, we now investigate the tunneling of the BEC wave packet through the barrier with atomic nonlinearity. In Fig. 2, we present the transmitted wave packets with different nonlinear interaction at time \( t = 440 \), at which the transmitted wave packets just emerges from the barrier. Meanwhile we also show the freely propagating wave packet as reference, for which both the barrier and nonlinear interaction are set to zero. From the top panel, one observes that the transmitted wave packet with negative nonlinearity slightly lags behind the reference one. The middle panel is the linear case, where the nonlinear interaction is neglected. In contrast to the negative nonlinear case, the transmitted wave packet is ahead of the reference one due to the
finite-width effect of incident wave packet, as has been pointed out by previous works\cite{22,24}. In the bottom panel, we find the transmitted wave packet with positive nonlinearity is far ahead of the reference one, which is quite different from other two cases. The results in Fig. 2 show that the nonlinear interaction indeed, as excepted, affects the tunneling process of a BEC matter wave packet through the barrier. Below we explain the physics behind the numerical results in detail.

We consider the transmitted spectra in the momentum space. Due to the SPM-induced spectral broadening as well as the filtering effect of the barrier\cite{3,25}, the central momenta in the transmitted spectra exhibit great difference in the linear, positive and negative nonlinear cases, as shown in Fig. 3. The filtering of the barrier takes effect on both the linear and nonlinear cases, while the spectral nonlinear broadening only occurs in the nonlinear cases. The combination of filtering effect and nonlinear broadening leads to a large transmitted central momentum in the positive nonlinear case. Therefore, with the same barrier parameters and incident wave number, the BEC wave packet with repulsive interatomic interaction tends to spend less time in the barrier. However, we note that the transmitted central momenta in the negative nonlinear and linear cases are less than that in the positive nonlinear case. This can also be well understood from the view of point below. In fact, when the interatomic interaction exists, the atomic nonlinearity modifies the potential barrier and the atoms “see” an effective potential

\[
V_{eff} = V_0 f(x) + g(x)|\psi(x,t)|^2. \tag{8}
\]

For a repulsive interaction, the nonlinear term has a positive sign. Consequently, the height of the linear barrier is raised, and vice versa for an attractive interaction. In terms of the Hartman’s calculation\cite{3}, the wave packet traverses faster through a higher barrier than a lower one with the same and enough wide barrier width. Therefore the transmitted wave packet with negative or positive nonlinearity spend more or less time than the linear one in the barrier.

Now we turn to the quantitative calculation for the tunneling time of BEC wave packet through the laser-induced barrier. For a finite wave packet transmitting through a barrier, we can not use the usual stationary phase method to find the transmission time of the transmitted wave packet, because the condition for the stationary phase method is not satisfied in this situation. However, the widely used time-of-flight method is applicable to measure the transmission time of transmitted wave packet\cite{26}. The method is described as follows. Assume that the incident wave packet is placed at the position \(x(0) = -x_0(x_0 > 0)\) at the time \(t = 0\), somewhere to the left of the barrier, and let it move to the right with initial momentum \(k_0\). After transmitting through the barrier the position of transmitted wave packet is located in \(x(t_T)\) at the time \(t = t_T\). Due to the effect of barrier and nonlinear interaction, the transmitted wave packet is usually deformed. In this case, the appropriate way to describe the center of the transmitted wave packet is to define the expected position as \(x(t_T) = \int_{x > 0} x |\psi(x,t)|^2 \, dx / \int_{x > 0} |\psi(x,t)|^2 \, dx\). So one gets the tunneling time

\[
\Delta t = t_T - \frac{x_0 - L/2}{k_0} - \frac{x(t_T) - L/2}{k_0}, \tag{9}
\]

where \(\bar{k}_0\) is the transmitted central momentum, defined as

\[
\bar{k}_0 = \int k |\psi_T(k)|^2 \, dk, \tag{10}
\]

with \(\psi_T(k)\) the momentum distribution of transmitted wave packet.

\[\text{FIG. 3: The corresponding momentum spectra of transmitted wave packets in Fig. 2. Solid curve stands for the case without nonlinearity, and dashed and dash-dotted curves for the cases with positive and negative nonlinearity, respectively.}\]

\[\text{FIG. 4: Tunneling time as a function of the nonlinear interaction strength. Other parameters are } k_0 = 0.6, V_0 = 1 \text{ and } L = 6.\]

First, we study the dependence of tunneling time on the nonlinear interaction. Fig. 4 plots the behavior of tunneling time via nonlinear interaction strength. Obviously the tunneling time descends with the nonlinear interaction strength increasing. This means that...
the stronger the repulsive interatomic interaction is, the faster the BEC wave packet traverses through the barrier. It is interesting to note that the tunneling time is negative for large positive nonlinearity. The negative tunneling time implies that the transmitted wave packet exits from the barrier just before the incident wave packet arrives at the barrier. Such a phenomenon has also been found and discussed in the literatures for different situations [28, 29]. Here the negative tunneling time is due to the repulsive interatomic interaction, and completely a nonlinear quantum mechanical phenomenon for an atomic Bose-Einstein condensate.

![Graph showing Tunneling time as a function of the barrier width](image)

**FIG. 5:** Tunneling time as a function of the barrier width with nonlinear strength $g_0 = 0$ (Solid line), $g_0 = 5$ (dashed line) and $g_0 = -2$ (dash-dotted line). Other parameters are $k_0 = 0.6$ and $V_0 = 1$.

Now, we examine the dependence of tunneling time on the barrier width as shown in Fig. 5. We find that the tunneling time in linear and positive nonlinear cases decreases, while increases in negative nonlinear case, with the barrier width increasing. These results imply that the Hartman effect is violated in both nonlinear and linear cases. For the linear case, such a violation is due to the finite width of the incident wave packet. This has been touched in many theoretical and experimental works [22–24].

In summary, we have numerically studied the effect of nonlinear interatomic interaction on the quantum tunneling time of a BEC matter wave packet through a laser-induced barrier. Analysis shows that both the sign and the strength of nonlinear interaction significantly affect the tunneling time. As a result, the so-called Hartman effect in the linear quantum mechanics could be violated in the nonlinear quantum mechanics with a macroscopic matter wave packet of Bose-Einstein condensate.

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