Steady-State Two Atom Entanglement in a Pumped Cavity Enhanced by Nonlinear Mirrors

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Abstract

We demonstrate the steady-state entanglement of two two-level atoms inside a pumped cavity with photon leakage through a nonlinear mirror and through spontaneous decay, and show that the entanglement is enhanced by the presence of a nonlinear mirror. Our model assumes the vacuum Rabi splitting of the dressed states of the system to be much larger than any of the decay parameters of the system. We also discuss how the dressed states of the system offer us intuition as to where the entanglement lies in the state space spanned by the system, and allow us the optimize the system.

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I. INTRODUCTION

In an earlier paper [1] we showed that two two-level atoms placed in a pumped high-Q cavity with normal weakly lossy mirrors will not have sustained entanglement in the case of closed two-level systems, but may have such entanglement in the case of open two-level systems. Our treatment included cavity leakage and spontaneous emission. It assumed that the Q-factor of the cavity was large enough so that the vacuum Rabi splitting was larger than any of the decay rates of the system. This allowed us to express the system in terms of the atom-field dressed state picture. The coherence of the system was built into the dressed states allowing us to treat the effects of cavity losses and spontaneous emission in terms of simple incoherent transitions between these dressed states. The advantage of expressing the system in the dressed state basis is that we can immediately see within which manifold the entanglement between the atoms lies, and tailor our system to maximize the population in that manifold.

Due to experimental advances in atomic traps and cavity QED [2, 3, 4, 5, 6, 7] it is within our technological limits to trap and manipulate individual atoms inside a microcavity to study their entanglement behavior. As a result, more attention is directed towards the entanglement of atoms and fields within these cavities [8, 9, 10, 11, 12, 13]. More specifically, to determine the conditions in which atoms can get entangled in these system, and to characterize the states of these entangled atoms. One of the interesting things to note in open systems, such as the one we describe in this paper, is that it allows the possibility of steady-state entanglement [10, 12] without the assistance of post-selection.

The intuition gained from the dressed-state formalism suggested that the addition of a nonlinear mirror to the cavity might allow the production of steady-state entanglement even in the case of closed two-level systems. In this paper, we will demonstrate the possibility of steady-state entanglement between two atoms by employing a nonlinear mirror to construct the cavity. The nonlinearity we are interested in is the reverse-saturated absorption (RSA) property of the mirror [14, 15, 16] which offers a larger cavity photon loss at greater intra-cavity field intensities. The RSA mirror effectively changes the photon number distribution in the cavity which will preferentially sustain the low photon number states which in turn means a larger population in the lower manifolds of the dressed states of the system we consider. This concentration of population in the lower manifolds yields the entanglement
between the two atoms in the cavity.

Here we will investigate the steady-state entanglement of two two-level atoms inside a pumped high-Q cavity with a nonlinear mirror. We express the state of the two atoms as a mixed state in the dressed state basis in which the weighting factors are determined by constructing rate equations governing the steady-state population in each of the dressed states. This is justified by assuming that the vacuum Rabi splitting is much larger than any decay parameter of the system \[1\]. We will show that only the \( n = 1 \) manifold of the dressed states contribute to the entanglement of the system, and how the employment of a nonlinear mirror can help generate entanglement between the two two-level atoms inside the cavity.

II. MODEL SYSTEM: TWO TWO-LEVEL ATOMS INSIDE A CAVITY

The system we are considering consists of two atoms, each with energy structure shown in Fig.(1), in a cavity which is externally pumped on resonance with the atomic transition and the cavity. The system can lose energy through cavity leakage or through spontaneous emission of the atoms. However, unlike ref.\[1\], we will take the output mirror to be a nonlinear mirror such that the power transmission coefficient, \( K \), is a function of the number of photons in the cavity.

\[ \Pi \rightarrow \begin{array}{ccc}
|1\rangle & \omega & \Gamma \\
|2\rangle & & \\
\end{array} \rightarrow K \]

FIG. 1: Two two-level atoms inside a cavity

We assume the cavity has a high Q-factor such that the vacuum Rabi splitting is much larger than the spontaneous decay rate and the cavity leakage rate. The large Rabi splitting justifies the use of the rate equations with respect to the dressed states for our system.

Since we wish to characterize this system using rate equations, we will need the dressed states of the closed system, and calculate the transition rates between these dressed states. From this we will construct the rate equation and determine the mixed state density matrix of the two atoms in the cavity.
The Hamiltonian of the closed system in the interaction picture is

\[ \hat{H} = g_1 \hat{\sigma}^{(1)}_{12} \hat{a}^\dagger + g_2 \hat{\sigma}^{(2)}_{12} \hat{a}^\dagger + H.C. \]

where \( \hat{\sigma}^{(n)}_{ij} \) is the atomic transition operator for the \( n^{th} \) atom and \( \hat{a}^\dagger \) is the field creation operator. For the sake of simplicity we will assume the coupling constants, \( g_i \), to be the same for each atom. In the case in which there is only one excitation in the system there are three essential states,

\[ |11; 1\rangle, |12; 0\rangle, |21; 0\rangle, \]

and three dressed states,

\[ |\chi_o\rangle = \frac{1}{\sqrt{2}} (|12; 0\rangle - |21; 0\rangle), \]

\[ |\chi_+\rangle = \frac{1}{\sqrt{2}} |11; 1\rangle + \frac{1}{2} (|12; 0\rangle + |21; 0\rangle), \]

\[ |\chi_-\rangle = \frac{1}{\sqrt{2}} |11; 1\rangle - \frac{1}{2} (|12; 0\rangle + |21; 0\rangle). \]

Here \( |ab;c\rangle = |a\rangle_1 \otimes |b\rangle_2 \otimes |c\rangle \) indicates the first atom is in state \( |a\rangle \), the second atom in state \( |b\rangle \), and the field in state \( |c\rangle \).

The \( n \geq 2 \) excitation of the system will have a different set of dressed states since there are four essential states in this case. The four essential states for \( n \geq 2 \) are,

\[ |11; n\rangle, |12; n - 1\rangle, |21; n - 1\rangle, |22; n - 2\rangle. \]

The dressed states are then given by,

\[ |\phi_o^n\rangle = \frac{1}{\sqrt{2}} (|12; n - 1\rangle - |21; n - 1\rangle), \]

\[ |\phi_+^n\rangle = \frac{1}{\sqrt{2}} (|11; n\rangle - |22; n - 2\rangle), \]

\[ |\phi_-^n\rangle = \frac{1}{2} (|11; n\rangle + |12; n - 1\rangle + |21; n - 1\rangle + |22; n - 2\rangle), \]
\[ |\phi^n_\pm\rangle = \frac{1}{2} (|11; n\rangle - |12; n - 1\rangle - |21; n - 1\rangle + |22; n - 2\rangle). \] (10)

Using the prescription described in ref. [1], we can obtain the rate equations governing the population in the \( n = 0, \ n = 1, \) and \( n = 2 \) dressed state as,

\[
\begin{align*}
\frac{dP_s}{dt} &= \Gamma P_{s1} + \frac{K(1)}{2} P_{s1} - \Pi P_g, \\
\frac{dP_{s1}}{dt} &= -\Gamma P_{s1} - \frac{K(1)}{2} P_{s1} + \Pi P_g + \frac{3}{2} \Gamma P_{s2} + \Gamma P_{\varphi',2} + \frac{1}{4} (K(2) + 2K(1)) P_{s2} + \frac{1}{2} K(2) P_{\varphi',2} - \Pi P_{s1}, \\
\frac{dP_{s2}}{dt} &= -\frac{3}{2} \Gamma P_{s2} - \frac{1}{4} (K(2) + 2K(1)) P_{s2} + \frac{3}{4} \Pi P_{s1}, \\
\frac{dP_{\varphi',2}}{dt} &= -\Gamma P_{\varphi',2} - \frac{1}{2} K(2) P_{\varphi',2} + \frac{1}{4} \Pi P_{s1}, \\
\frac{dP_o}{dt} &= 0, \\
\frac{dP_{o'}}{dt} &= 0,
\end{align*}
\] (11)

with

\[
\begin{align*}
P_{s1} &= P_+ + P_-, \\
P_{s2} &= P_{+,2} + P_{-,2},
\end{align*}
\] (12)

where \( P_g \) is the population of the ground state of the system, \( P_\pm \) is the population in the \( |\chi_\pm\rangle \) dressed states, \( P_0 \) is the population in the \( |\chi_o\rangle \) dressed state, \( P_{\pm,n} \) is the population in \( |\phi^n_\pm\rangle \), \( P_{o,n} \) is the population in the \( |\phi^n_o\rangle \), and \( P_{\varphi',n} \) is the population in \( |\phi^n_o'\rangle \). The Einstein A coefficient of the \( 2 \rightarrow 1 \) transition of the single atom in free space is given by \( \Gamma \), the single photon pumping rate inside the cavity is given by \( \Pi \), and the power transmission coefficient of the cavity output mirror as a function of the number of photons in the cavity is given by \( K(n_p) \). Here we assume that there is no population initially in the \( |\chi_o\rangle \) and \( |\phi^n_o\rangle \) dressed states, and because these states do not couple to any other states, they will not accumulate any population at later times.

As mentioned in ref. [1], truncating the rate equation to \( n = 2 \) will over-estimate the entanglement content between the two atoms since there will be some population beyond the \( n = 2 \) manifold that does not directly decay down to the \( n = 1 \) manifold. To correct for this, we will go one step further to obtain \( P_{s3} \) and \( P_{\varphi',3} \) in order to determine what fraction of the \( n \geq 2 \) population lies in the \( n = 2 \) manifold. To simplify the equations, we will assume \( K(n_p) = K(2) \) for \( n_p \geq 2 \), and express \( K(2) \) as \( K(2) = \eta K(1) = \eta K \). Here \( \eta \) is the measure of nonlinearity in the mirror since it tells us how much more (or less) the cavity transmits...
depending on the intra-cavity field intensity. The equations for \( P_{s3} \) and \( P_{o',3} \) are then given by,

\[
\frac{dP_{s3}}{dt} = -\frac{3}{2} \Gamma P_{s3} - \frac{1}{4} K (3 \eta + 1) P_{s3} + \Pi P_{s2},
\]

\[
\frac{dP_{o',3}}{dt} = -\Gamma P_{o',3} - \frac{1}{2} K (\eta + 1) P_{o',3} + \Pi P_{o',2}.
\]

(13)

Solving the above equations in the steady-state we get,

\[
P_{s3} = \frac{\Pi}{\frac{3}{2} \Gamma + \frac{3}{4} K (3 \eta + 1)} P_{s2}
\]

\[
P_{o',3} = \frac{\Pi}{\frac{1}{2} \Gamma + \frac{1}{2} K (\eta + 1)} P_{o',2}.
\]

(14)

To determine the fraction of population in the \( n = 2 \) manifold within the \( n \geq 2 \) manifolds we have to solve the equations,

\[
P_{s2} + P_{s3} = 1
\]

\[
P_{o',2} + P_{o',3} = 1.
\]

(15)

Substituting the expressions for \( P_{s3} \) and \( P_{o',3} \) in the above equations we get,

\[
P_{s2} = \alpha = \frac{6 \Gamma + K (3 \eta + 1)}{6 \Gamma + 4 \Pi + K (3 \eta + 1)}
\]

\[
P_{o',2} = \beta = \frac{2 \Gamma + K (\eta + 1)}{2 (\Gamma + \Pi) + K (\eta + 1)}.
\]

(16)

This suggests that the rate equations for the dressed states should be modified to,

\[
\frac{dP_s}{dt} = \Gamma P_{s1} + \frac{\eta}{2} P_{s1} - \Pi P_g,
\]

\[
\frac{dP_s}{dt} = -\Gamma P_{s1} - \frac{3}{2} \alpha P_{s2} + \beta \Gamma P_{o',2} + \frac{1}{4} \alpha (\eta + 2) K P_{s2} + \frac{1}{2} \beta \eta K P_{o',2} - \Pi P_{s1},
\]

\[
\frac{dP_{o',2}}{dt} = \frac{3}{2} \alpha \Gamma P_{s2} - \frac{1}{4} \alpha (\eta + 2) K P_{s2} + \frac{3}{4} \Pi P_{s1},
\]

\[
\frac{dP_{o',2}}{dt} = -\beta \Gamma P_{o',2} - \frac{1}{2} \beta \eta K P_{o',2} + \frac{1}{4} \Pi P_{s1},
\]

(17)

where we have replaced \( P_{s2} \) and \( P_{o',2} \) by \( \alpha P_{s2} \) and \( \beta P_{o',2} \) to reflect the true population decay of the \( n = 2 \) manifold.

The steady-state solution to these rate equations is,

\[
P_g = \mathcal{N} \beta \alpha (2 \Gamma + K)(2 \Gamma + \eta K)(6 \Gamma + (2 + \eta) K),
\]

\[
P_{s1} = \mathcal{N} 2 \Pi \beta \alpha (2 \Gamma + \eta K)(6 \Gamma + (2 + \eta) K),
\]

\[
P_{s2} = \mathcal{N} 6 \Pi^2 \beta (2 \Gamma + \eta K),
\]

\[
P_{o',2} = \mathcal{N} \Pi^2 \alpha (6 \Gamma + (2 + \eta) K),
\]

(18)
where we have defined the normalization constant,

\[ N = 6\beta\eta\Pi^2K + 12\beta\Gamma\Pi^2 + 6\Pi^2\Gamma\alpha + \Pi^2\alpha\eta K + 2\Pi^2\alpha K + 24\alpha\beta\Gamma^3 + 16\alpha\beta\eta\Gamma^2K + 20\alpha\beta\Gamma^2K + 2\alpha\beta\Gamma\eta K^2 + 12\alpha\beta\Gamma\eta K^2 + 4\alpha\beta\Gamma K^2 + \alpha\beta\eta^2 K^3 + 2\alpha\beta\eta K^3 + 24\alpha\beta\Pi^2 + 16\alpha\beta\eta\Pi\Gamma K + 8\alpha\beta\Pi\eta^2 K^2 + 4\alpha\beta\Pi\eta K^2. \] (19)

We now want to investigate the entanglement between the two atoms in the cavity. To do this we trace out the field component of the density matrix and obtain the reduced density matrix of just the two atoms. The reduced density matrix of the two atoms is given by,

\[ \hat{\rho}_{\text{atoms}} = P_g\hat{\rho}_g + P_s\hat{\rho}_s + P_o\hat{\rho}_o \] (20)

where

\[ \hat{\rho}_g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \hat{\rho}_s = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \hat{\rho}_{o'} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}. \] (21)

In order to calculate the entanglement content we employ Wootters’ concurrence [19] which is defined as,

\[ C = \max(\sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}, 0) \] (22)

where \(\lambda_i\) are the eigenvalues, in descending order of value, of the matrix \(\rho \bar{\rho} = (\sigma_y \otimes \sigma_y)\rho^* (\sigma_y \otimes \sigma_y)\).

The concurrence of \(\hat{\rho}_{\text{atoms}}\) is given by,

\[ C_{\text{atoms}} = \max \left( \frac{1}{2}(P_{s1} + P_{s2}) - \frac{1}{2} \left[ (P_{s2} + 2P_{o',2})(2P_{o',2} + P_{s2} + 4P_g + 2P_{s1}) \right]^{\frac{1}{2}}, 0 \right). \] (23)

We have shown in ref. [1] that in the case of a linear mirror (\(\eta = 1\)) there is no combination of parameters which yields a nonzero concurrence. Would an \(\eta \neq 1\) yield a nonzero concurrence? To answer this question, first we note that the three dressed \(|\phi_{\nu}\rangle, |\phi_{\pi}\rangle\), and, \(|\phi_{n}\rangle\) have no entanglement (i.e \(C = 0\)) between the atoms. The only manifold in the dressed
state picture which offers a nonzero entanglement between the atoms is the $n = 1$ manifold dressed states $|\chi_{\pm}\rangle$ with $C = \frac{1}{2}$. Therefore, it stands to reason that we would want to put as much population in the $n = 1$ manifold as possible. In order to put more population in the $n = 1$ manifold we would require $\eta > 1$. This means that the cavity experiences a larger loss of photons for higher intra-cavity intensities. One possible way to do this would be to incorporate a reverse saturable absorber in the cavity \[14,15,16\].

We can see in Fig.(2) how the entanglement between the two atoms is affected by the pump rate and the photon leakage rate (both expressed in units of $\Gamma$) with different values of $\eta$. It is clear that as we increase the value of $\eta$, the maximum value of the plot rises. We start seeing a nonzero value of concurrence at $\eta \approx 7.746$. However, it seems that one would need $\eta > 10$ to be able to see entanglement between the atoms for any realistic system. This would mean that we require a nonlinearity such that a two-photon state of the cavity will decay at a rate which is ten times faster than that of a single photon state of the cavity.

FIG. 2: Plot of $C$ against $\Pi$ and $K$ (in units of $\Gamma$), $\eta=10$. 
FIG. 3: Plot of $C$ against $\Pi$ and $K$ (in units of $\Gamma$), $\eta=12$.

III. CONCLUSION

We have derived the steady-state reduced density matrix for two spontaneously decaying two-level atoms inside a high-Q cavity which is pumped and experiences photon leakage through a RSA mirror. In our model we assumed that the vacuum Rabi splitting is much larger than any decay parameter in the system which allows us to express the density matrix of the system as a mixture of the dressed states of the system with a weighting factor that is determined by the rate equations of these dressed states. We show that the atoms in the system can get entangled in the steady-state by choosing $\eta$, the nonlinearity parameter, to be greater than one. Therefore, employing reverse saturable mediums can, in principle, entangle two two-level atoms inside a high-Q cavity in our model.

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