Deviation equations
in spaces with affine connection

Bozhidar Z. Iliev * † ‡
Sawa S. Manoff § ¶ ∥

Short title: Deviation equations in spaces with affine connection

Basic ideas: → 1980–1983
Initial typeset: → 1983
Secondary typeset: → November 2–23, 2005
Last update: → December 14, 2005
Produced: → June 22, 2018

Published as JINR Communication P2-83-897: JINR, Dubna, 1983 (In Russian)
Translation from Russian and new typeset: Bozhidar Z. Iliev

Subject Classes:
General relativity, Differential geometry

2001 MSC numbers: | 2003 PACS numbers:
53B05, 83C99, 53B50 | 02.40.Sf, 04.90.+e

Key-Words:
Deviation equations
Spaces with affine (linear) connection, Spaces with torsion

*aLaboratory of Mathematical Modeling in Physics, Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, Boul. Tzarigradsko chaussée 72, 1784 Sofia, Bulgaria
†E-mail address: bozho@inrne.bas.bg
‡URL: [http://theo.inrne.bas.bg/~bozho/](http://theo.inrne.bas.bg/~bozho/)
§Laboratory of Solitons, coherence and geometry, Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, Boul. Tzarigradsko chaussée 72, 1784 Sofia, Bulgaria
¶Prof. Sawa Manoff passed away on May 27, 2005
∥URL: [http://theo.inrne.bas.bg/elpart/SavaManov/smanoff.html](http://theo.inrne.bas.bg/elpart/SavaManov/smanoff.html)
Abstract

Connections between Lie derivatives and the deviation equation has been investigated in spaces $L_n$ with affine connection. The deviation equations of the geodesics as well as deviation equations of non-geodesics trajectories have been obtained on this base. This is done via imposing certain conditions on the Lie derivatives with respect to the tangential vector of the basic trajectory.
0. Introduction

In the last years, the deviation equations find a broad application in the study of many mathematical and physical problems in gravitational physics and astrophysics [1–3]. Several versions of these equations are known in Riemannian spaces $V_n$. In [1], on the base of Lie derivatives, a general method is proposed for derivation of deviation equations in $V_n$.

The purpose of the present investigation is in the application of the methods in [1] to spaces (manifolds) with affine connection $L_n$. The approach proposed for finding deviation equations in $L_n$ is illustrated on a number of examples.

The so-called generalized deviation equation in spaces with affine connection is considered in Sect. I. In Sect. II a physical interpretation of that equation is presented. Sect. III contains examples of additional conditions restricting the form of the generalized deviation equation. Particular examples of deviation equations are given in Sect. II. In the Appendix, Sect. IV is recalled the notion of Lie derivative of the connection coefficients of a linear connection [4].

1. The generalize deviation equation in spaces with affine connection

1.1. The considerations that follow are based on the identity (see the Appendix, section A)

$\mathcal{L}_\xi \Gamma^k_{ij} = \xi^k_{[ij]} - R^k_{ijl} \xi^l - (T^k_{ij})_{[l]}$.  \hfill (1.1)

Here all quantities are evaluated at a point $x \in L_n$ and the following notation is introduced: $\mathcal{L}_\xi$ is the Lie derivative operator [4] along a $C^2$ vector field $\xi = \xi^i E_i$; the Latin indices $i, j, k, \ldots$ run from 1 to $n \in \mathbb{N}$ and summation over indices repeated on different levels is understood over the whole range of their values; $\{E_i\}$ is a frame in the tangent bundle $T(L_n)$, i.e. $E_i = A^\alpha_i \partial_\alpha$, where the Greek indices $\alpha, \beta, \cdots = 1, \ldots, n$ number a coordinate frame $\{\partial_\alpha := \frac{\partial}{\partial x^\alpha}\}$ and $A = [A^\alpha_i]$ is a non-degenerate matrix-valued function; $\Gamma^i_{ij}$ are the coefficients of the affine connection of $L_n$ in the frame $\{E_i\}$ and are generally non-symmetric in their subscripts; $\xi^k_{[ij]} := \xi^k_{ij} := (\xi^k_i)_{[j]}$ with the suffix “$[j]$” denoting the action of the covariant derivation operator along the basic vector field $E_j$ relative to the connection coefficients $\Gamma^i_{ij}$; respectively with the suffix “$i$” it will be denoted the action of $E_i$ on the components of objects over $L_n$ considered as scalar functions, for instance, we have $\xi^i_{[j]} = \xi^i_{j} + \Gamma^i_{kij} \xi^k$ with $\xi^i_{[j]} := E_i(\xi^k)$:

$R^i_{jkl} := -2\Gamma^i_{[j][k,l]} - 2^m_{j[k} \Gamma^i_{m][l]} - \Gamma^i_{j[m} C^{m}_{kl}$

are the components of the curvature tensor in $\{E_i\}$, where

$B_{[ij]} := \frac{1}{2}(B_{ij} - B_{ji}) \quad B_{[ij][kl]} := \frac{1}{2}(B_{ijkl} - B_{ijlk})$

$C^{i}_{jk} := -2A^i_j A^0_{[j,k]} \quad [A^i] := [A^j]^{-1}$;

and

$T^i_{jk} := -2\Gamma^i_{[j][k]} - C^i_{jk}$

are the components of the torsion tensor of $L_n$ relative to $\{E_i\}$.

1.2. Let $u = u^i E_i$ be arbitrary contravariant $C^1$ vector field. Projecting (1.1) on $u^i$ and $u^j$ and taking into account the equation $u^i u^j \xi_{ij} = u^j (\xi^k_i u^k)_{ij} - \xi^k_{ij} u^k$, we get the relation

$u^j (u^i \xi^k_{ij})_{ij} = R^k_{ij} u^j u^i \xi^k + \xi^k_{ij} (u^j u^k)_{ij} + u^j u^i (T^k_{ij})_{[l]} + u^j u^i \mathcal{L}_\xi \Gamma^k_{ij}$. \hfill (1.2)

If $u$ is a $C^2$ vector field, then

$u^i u^j \mathcal{L}_\xi \Gamma^k_{ij} = \mathcal{L}_\xi (u^k_i u^j) - u^i (\mathcal{L}_\xi u^k)_{ij} - u^k_i \mathcal{L} u^i$
and, consequently, (1.2) can be rewritten as
\[ u^j(u^i \xi^k)_{ij} = R^k_{ij} u^j u^i \xi^l + \xi^k(u^j u^i) + u^i u^j(T^k_{ij} \xi^l)_{ij} + \mathcal{L}_\xi(u^k u^i) - u^i(\mathcal{L}_\xi u^k)_{ij} - u^k_{ij} \mathcal{L} u^i. \] (1.2)

1.3. Let \( x(s) = (x^1(s), \ldots, x^n(s)) \) be a \( C^2 \) path with parameter \( s \) in \( L_n \) and the vector field \( u \) be chosen such that
\[ u^\alpha \bigg|_{x=x(s)} = \frac{dx^\alpha(s)}{ds} := u^\alpha(s) = u^\alpha, \] (1.3)
i.e. \( u(s) \) to be the vector tangent to \( x(s) \) at the parameter values \( s \).

Denote by \( \nabla_i \) the covariant derivative operator of tensor field components; for example
\[ \nabla_i \xi^k = \xi^k_i. \] Then \( \frac{D}{ds} := u^i \nabla_i = u^\alpha \nabla_\alpha \) is the covariant derivative along \( x(s) \) with respect to \( s \).

Using the operator \( \hat{D}/ds \), we can rewrite (1.2) and (1.2a) along \( x(s) \) respectively as
\begin{align*}
\frac{\hat{D}^2 \xi^k}{ds^2} &= R^k_{ijl} u^j u^l \xi^l + \xi^k_j F^j + u^i \frac{\hat{D}(T^k_{ij} \xi^l)}{ds} + u^i u^j \mathcal{L}_\xi \Gamma^k_{ij}, \\
\frac{\hat{D}^2 \xi^k}{ds^2} &= R^k_{ijl} u^j u^l \xi^l + \xi^k_j F^j + u^i \frac{\hat{D}(T^k_{ij} \xi^l)}{ds} + \mathcal{L}_\xi F^k - \frac{\hat{D}(\mathcal{L}_\xi u^k)}{ds} - u^k_{ij} \mathcal{L}_\xi u^i,
\end{align*}
(1.4)
where
\[ \hat{D} := \hat{D}/ds. \]

The equation (1.4) or its equivalent version (1.4a) will be called generalized deviation equation in spaces with affine connection \( L_n \) in the frame \( \{E_i\} \). By its essence equation (1.4) is an identity, which is a corollary of the definitions of the quantities and operators entering into it. It can be regarded as an equation relative to some of the quantities in it only if it is presupposed that between them some connection exists. So, rigorously speaking, by a deviation equation we shall understand the identity (1.4) together with some additional condition(s) imposed on the quantities entering in it. Namely the variety of the possible additional conditions is the cause for the existence of different forms (modifications) of deviation equations; see [1] in a case of a Riemannian manifold \( V_n \). It should be mentioned, the additional conditions are sometimes useful to be considered as first integrals of the deviation determine.

Remark 1.1. In a coordinate/frame-independent language, the equations (1.4) and (1.4a) respectively read:
\begin{align*}
\frac{D^2 \xi}{ds^2} &= \hat{R}(u, \xi) u + C^1_1(F \otimes D \xi) + \frac{D(\hat{T}(F, \xi))}{ds} - \hat{T}(F, \xi) + \mathcal{L}_\xi \Gamma(u, u), \\
\frac{D^2 \xi}{ds^2} &= \hat{R}(u, \xi) u + C^1_1(F \otimes D \xi) + \frac{D(\hat{T}(F, \xi))}{ds} - \hat{T}(F, \xi) + \mathcal{L}_\xi F - \frac{D(\mathcal{L}_\xi u)}{ds} - C^2_1((Du) \otimes (\mathcal{L}_\xi u)).
\end{align*}
(1.5)
Here we have introduce the following notation: \[ \nabla_i := u^i \nabla_i \] with \( \nabla_i = \nabla_{E_i} \) is the covariant derivative along \( E_i \) (e.g. \( \nabla_i u = (\hat{\nabla}_i u^k)E_k \)); \( \xi = \xi^k E_k, U = u^k E_k, F = F^k E_k; \hat{R}(u, \xi) = [\nabla_u, \nabla_\xi] - \nabla_{[u, \xi]}, \) \( \mathcal{L}_\xi \) is the torsion operator in \( L_n; \mathcal{L}_\xi u = (\mathcal{L}_\xi u^k)E_k = [\xi, u] = -\mathcal{L}_u \xi. \)
Remark 1.2. Substituting in (1.4) the equality
\[ u^j \Gamma^k_{ji} = \mathcal{L}_\xi u^k_i - \left[ \mathcal{L}_\xi \nabla_i \right] u^k \]
we get the following form of the generalized deviation equation:
\[ \frac{D^2 \xi^k}{ds^2} = R_{ijkl} u^i u^j \xi^l + \xi^k \mathcal{T}^\ell_{ij} \frac{\bar{D}(\mathcal{T}(u, \xi))}{ds} + u^i \{ \mathcal{L}_\xi (u^k_i) - \left( \mathcal{L}_\xi u^i \right)_i \} \]
(1.4b)
or, in coordinate/frame independent notation,
\[ \frac{D^2 \xi}{ds^2} = \dot{R}(u, \xi)u + C_1^1 (F \otimes \mathcal{D}_\xi) + \frac{D(\bar{T}(F, \xi))}{ds} - \bar{T}(F, \xi) - [\nabla_u, \mathcal{L}_\xi]u. \]
(1.5b)

2. Physical interpretation of the deviation equation

2.1. In this section, when interpreting physically the deviation equation, we shall restrict ourselves to infinitesimal deviation vector with components \( \xi^i \) defined below via (2.1a), which means that we shall have in mind the so-called local deviation equation. The cause for that is in the non-local problems arising when one tries to derive non-local deviation equations, which are connected with the comparison of tensors defined at non-infinitesimally near points in \( L_n \). These problems are out of the range of this investigation and do not have a unique and global solution at present. However, there are arguments that indicate the validity in the nonlocal case of the physical interpretation of the deviation equation presented below.

2.2. Consider two point-like particles 1 and 2 moving along the \( C^2 \) paths \( x_1(\tau_1) \) and \( x_2(\tau_2) \), respectively, in \( L_n \) with parameters \( \tau_1 \) and \( \tau_2 \), i.e. \( x_1(\tau_1) \) and \( x_2(\tau_2) \) are their trajectories. Assume the behaviour of the particles is “observed” by a particle (observer) with \( C^2 \) trajectory \( x_0(\tau) \), the basic trajectory, with parameter \( \tau \). It will be supposed that the mappings \( x_0, x_1 \) and \( x_2 \) are injective and invertible on the sets of their ranges and that \( \tau \neq \text{const} \), i.e. \( d\tau \neq 0 \).

Mathematically the “observation” process means existence of mappings \( \phi_1 \) and \( \phi_2 \) such that, when the observer is at a point \( x_0(\tau) \) and the observed particles at \( x_1(\tau_1) \) and \( x_2(\tau_2) \), we have \( x_1(\tau_1) = \phi_1(x_0(\tau)) \) and \( x_2(\tau_2) = \phi_2(x_0(\tau)) \). This means that, when the observer “knows” its own position in \( L_n \), he can find the positions of the observed particles.

Let us put:
\[ u^\alpha := \frac{dx_0^\alpha(\tau)}{d\tau}, \quad u := u^\alpha \partial_\alpha = u^i E_i, \quad F := \frac{Du}{d\tau} = u^i \nabla_i u \]  
(2.1)
\[ \xi^\alpha := x_2^\alpha(\tau_2) - x_1^\alpha(\tau_1) = \phi_2^\alpha(x_0(\tau)) - \phi_1^\alpha(x_0(\tau)) = \xi^\alpha(\tau) \]  
(2.1a)
\[ \xi := \xi^\alpha \partial_\alpha = \xi^k E_k \]  
(2.1b)
\[ V := \frac{D\xi}{d\tau} = u^k \nabla_k \xi, \quad V^i = \frac{D\xi^i}{d\tau} = \frac{d\xi^i}{d\tau} + \Gamma^i_{jk} \xi^j u^k. \]  
(2.1c)

The contravariant vector \( \xi \) will be called the deviation vector. Evidently, it describes the relative position of the particle 2 with respect to the particle 1 (as seen from a point on the basic trajectory). The vector \( V \) is physically interpreted as a relative velocity of the particle 2 with respect to particle 1 as seen from the basic trajectory whose parameter \( \tau \) is interpreted as observer’s “proper” time.

The deviation equation in the form (1.5b) for the deviation vector \( \xi \) reads
\[ \frac{D^2 \xi}{d\tau^2} = \frac{DV}{d\tau} = \bar{R}(u, \xi)u + C_1^1 (F \otimes (D\xi)) + \frac{\bar{T}(u, \xi)}{d\tau} \]
\[ - \bar{T}(F, \xi) + \mathcal{L}_\xi F - \frac{D(L\xi u)}{d\tau} - C_1^1 ((Du) \otimes (L\xi u)). \]
(2.2)
It is clear, with respect to a fixed observer, the vector $\frac{DV}{dt}$ is the relative acceleration between the particle 2 relative to the particle 1. Therefore the generalized deviation equation gives the relative acceleration between the observed particles as a function of the space $L_n$, i.e. $\bar{R}$, $\bar{T}$, and $\Gamma_{jk}^i$, the trajectory of the observer ($\tau, u$ and $F$), and the relative motion of the observed particles ($\xi$ and $V$). It should be recalled, together with (2.2), in any particular case, one should consider also a suitable additional condition(s).

Notice, any type of restrictions on the trajectories considered can be regarded as additional conditions to the deviation equation.

3. Examples of additional conditions defining deviation equations

3.1. The symmetries of $L_n$ as a source of additional conditions.

If the space $L_n$ admits some symmetries, this immediately entails definite additional conditions in the generalized deviation equation [1, 5]. This is so because, when deriving the deviation equation, we have used only quantities characterizing $L_n$ without imposing on them any restrictions.

To illustrate the above, we present bellow four examples. The mathematical derivation of the corresponding additional conditions in [4–7] for Riemannian space $V_n$ and for $L_n$ space it can be obtained in a way similar to the one pointed in these references.

(i) If $L_n$ admits geodesic mappings, which map the geodesics of $L_n$ in geodesics if $L_n$, we have the equations

$$\mathcal{L}_\xi \Gamma_{jk}^i = 0 \quad (3.1)$$

where

$$\Gamma_{jk}^i := \Gamma_{(jk)}^i - \frac{2}{n + 1} (\delta_j^i \Gamma_{(k)}^l (j) k) \quad (3.1a)$$

are the Thomas projective parameters [4] (symmetrization is performed over the indices included in parentheses, $B_{(ij)} = \frac{1}{2} (B_{ij} + B_{ji})$);

(ii) If $L_n$ admits affine transformations, which preserve the the parallelism of the tangent vectors, this leads to the conditions

$$\mathcal{L}_\xi \Gamma_{jk}^i = 0. \quad (3.2)$$

(iii) If in $L_n$ a metric with local components $g_{ij} (= g_{ji})$ is given and $L_n$ admits symmetries, i.e. mappings $L_n \rightarrow L_n$ preserving the distances defined by the metric, than the conditions

$$\mathcal{L}_\xi g_{ij} = 0 \quad (3.3)$$

act as additional conditions to the generalized deviation equation.

(iv) If in $L_n$ a metric $g_{ij}$ is given and the space admits conformal transformations, which preserve the angles between the tangent vectors, then there exists a function $\Phi: L_n \rightarrow \mathbb{R}$ such that

$$\mathcal{L}_\xi g_{ij} = 2\Phi g_{ij}. \quad (3.4)$$

If the metric tensor is non-degenerate, $\det[g_{ij}] \neq 0, \infty$, the equation (3.3) can be rewritten as

$$\mathcal{L}_\xi (|g|^{-1/n} g_{ij}) = 0 \quad (3.4a)$$
with \( g := \det \{ g_{ij} \} \) and \( n \) being the dimension of the \( L_n \)-space.

### 3.2. Particular realizations of \( L_n \) as examples of additional conditions.

As additional conditions in the deviation equation can play role relations that connect structures over \( L_n \) or defining new objects over \( L_n \) and their possible connections with the already existing ones. Examples of this type are presented in the following list:

(i) \( L_n \)-spaces without torsion,
\[
T^{jk}_i = 0 \quad (\hat{T} = 0). \tag{3.5}
\]

(ii) \( p \)-recurrent, \( p \in \mathbb{N}, L_n \)-spaces,
\[
R^{i}_{jk|l\ldots i\ldots p} = R^{i}_{jkl} \cdot A_{i\ldots i\ldots p} \tag{3.6}
\]
with \( A_{i\ldots i\ldots p} \) being the components of a tensor.

(iii) Affine (locally flat) \( L_n \)-spaces: for every point \( x_0 \in L_n \), there exists its neighborhood \( U \) such that
\[
R^{i}_{jk|l\mid U} = 0 \quad (\hat{R}|_U = 0) \tag{3.7}
\]
or, equivalently, their is a frame \( \mathcal{E}_{j0} \) such that
\[
\Gamma^{i0}_{j0k0}|_U = 0. \tag{3.7a}
\]

(iv) Equiaffine (generally with torsion) space: \( L_n \)-space with symmetric Ricci tensor,
\[
R_{ij} = R_{ji} \quad (R_{ij} := R^i_{jik}). \tag{3.8}
\]

(v) A metrical \( L_n \)-space with metric tensor with components \( g_{ij} \) and additional vector \( w_i \) that satisfy the semi-metrical transport condition,
\[
g_{ij|k} = w_k \cdot g_{ij}. \tag{3.9}
\]
In particular, of this kind are the Weil spaces in which the metric is nondegenerate and the inverse metric tensor \( g^{ij} \) is defined via
\[
g^{ik}g_{kj} = \delta^i_j. \tag{3.10}
\]

(vi) Einstein spaces (generally with torsion): metrical \( L_n \)-spaces in which, for some scalar function \( f \),
\[
R_{ij} = f \cdot g_{ij}. \tag{3.11}
\]

(vii) Riemannian manifolds (generally with torsion): equiaffine Weil spaces with metrical transport:
\[
g_{ij|k} = 0 \quad (g = \det \{ g_{ij} \} \neq 0, \infty; \ R_{ij} = R_{ji}). \tag{3.12}
\]

(viii) Conformal-Euclidean spaces (generally with torsion): Weil spaces in which exists a tensor with components \( P_{ij} \) such that
\[
R^i_{jk|l} = 2(\delta^i_k P_{lj} - \delta^i_j P_{kl} - g_{jk} P_{lm} g^{ml})|_{[kl]} \tag{3.13}
\]
\[
\nabla_{[i} P_{j]k} = 0. \tag{3.13a}
\]

For \( n \geq 3 \), from (3.13) follow (3.13a) and
\[
P_{ij} = \frac{1}{n-2} \left( R_{ij} + \frac{2}{n} R_{[ij]} - \frac{1}{2(n-1)} (R_{kl} g^{lk}) g_{ij} \right). \tag{3.14}
\]
Anyone of the conditions i–vii presented above, or a collection of them (if they are compatible), together with the identity (1.4) define a particular deviation equation in the corresponding spaces.

3.3. Other additional conditions.

The study of concrete problems, connected with deviation equations, is based on mathematical or physical reasonings and leads to appropriate additional conditions. Examples of such conditions and their interpretation in coordinate frames are presented in [1], where Riemannian spaces without torsion are considered. Without repeating this reference, below we present some examples of additional conditions and the corresponding to them deviation equations (1.5) or (1.5a) in spaces $L_n$ with affine connection.

(i) $F = 0$ ($x(s)$ is a geodesic path in $L_n$):

$$\frac{D^2 \xi}{ds^2} = \hat{R}(u, \xi) + \frac{D}{ds} (\hat{T}(u, \xi) - \mathcal{L}_\xi u) - C^2_1((Du) \otimes (\mathcal{L}_\xi u)).$$

(ii) $\mathcal{L}_\xi u = 0$:

$$\frac{D^2 \xi}{ds^2} = \hat{R}(u, \xi)u + C^1_1(F \otimes (D\xi)) + \frac{D}{ds}(T(u, \xi)) - \hat{T}(F, \xi) + \mathcal{L}_\xi F.$$

(iii) $u^i|_k = 0$ and hence $Du = 0$ and $F = 0$:

$$\frac{D^2 \xi}{ds^2} = \mathcal{L}_\xi u.$$

(iv) $\mathcal{L}_\xi F = -F$:

$$\frac{D^2 \xi}{ds^2} = \hat{R}(u, \xi)u + C^1_1(F \otimes (D\xi)) - F + \frac{D}{ds}(T(u, \xi) - \mathcal{L}_\xi u) - \hat{T}(F, \xi) - C^2_1((Du) \otimes (\mathcal{L}_\xi u)).$$

(v) $u = \xi$ and hence $\mathcal{L}_\xi u = [\xi, u] = 0$:

$$\frac{D^2 \xi}{ds^2} = \mathcal{L}_\xi u = 0.$$

(vi) $u^i u^j \mathcal{L}_\xi \Gamma^k_{ij} = -(F^k + \xi^k F^j)$:

$$\frac{D}{ds} \left( u + \frac{D \xi}{ds} \right) = \hat{R}(u, \xi)u + \frac{D}{ds}(T(u, \xi)) - \hat{T}(F, \xi).$$

4. Examples of deviation equations in $L_n$

4.1. Let $x(s, q)$, $s, q \in \mathbb{R}$, be a 2-parameter $C^2$ family of curves in a space $L_n$; rigorously speaking [3], the parameter $q$ should be replaced with $n - 1$ independent parameters $q_1, \ldots, q_{n-1}$ whose employment does not change essentially the next considerations.

Define the vectors $u$ and $v$ as tangent to respectively the $s$-curves and $q$-curves,

$$u := u^i E_i = u^\alpha \partial_\alpha$$

$$v := v^i E_i = v^\alpha \partial_\alpha$$

Let $\delta q$ be an infinitesimal constant. The vector

$$\xi := (\delta q)v$$

(4.2)
Hence the deviation equation takes the form

$$0 = \frac{\partial^2 x^\alpha(s, q)}{\partial q \partial s} - \frac{\partial^2 x^\alpha(s, q)}{\partial s \partial q} = \frac{\partial u^\alpha}{\partial s} - \frac{\partial v^\alpha}{\partial q} = L_u u^\alpha - L_v v^\alpha. \quad (4.3)$$

From (12) and (13), we derive

$$L_\xi u = -L_u \xi = 0 \quad (4.4)$$

which can be rewritten as

$$\frac{D\xi}{ds} = C^1_l (\xi \otimes (Du)) - \hat{T}(\xi, u). \quad (4.5)$$

Hence the deviation equation takes the form

$$\frac{D^2 \xi}{ds^2} = \hat{R}(u, \xi)u + C^1_l (\xi \otimes (DF + T)) \quad (4.6)$$

with

$$T := T^k_i E_k \otimes E^i \quad T^k_i = -u^m \nabla_m (T^k_i u^i) + u^j T^k_j (u^i_\| - T^i_m u^m). \quad (4.6)$$

The traditional method for obtaining this particular deviation equation consists in the application of the operator \( \frac{D}{\xi} = u^k \nabla_k \) to (4.5) and a suitable changes in the result obtained.

4.2. Consider two free point particles with \( C^2 \) trajectories \( x_0(r) \) and \( x_1(r_1) \); the former will be interpreted as an observer and the latter one as observed particle. The freedom means that the trajectories are geodesics which implies the equation

$$F^i = u^k_\| u^k = A^i_{\alpha} \left( \frac{d^2 x^\alpha_0(r)}{dr^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta_0}{dr} \frac{dx^\gamma_0}{dr} \right) = 0 \quad (4.7)$$

$$L_\xi u = 0. \quad (4.8)$$

The condition (4.7) defines the geodesic \( x_0(r) \), while (4.8), which is equivalent to (4.1), characterizes the existence of the mapping \( \phi_2 \) (see section 2); the condition (4.7) can also be considered as obtained via the method in the previous example (where the curves \( x_0(r) \) and \( x_1(r_1) \), belonging to the family \( x(s) \) for \( s = r, r_1 \), are studied (for \( q = \text{const} = q_0 \), the curves \( x(s = r, q_0) \) are geodesics).

Now the deviation equation takes the following form (see point 3.3, case 2 above):

$$\frac{DV}{dr} = \hat{R}(u, \xi)u + C^1_l (\xi \otimes T), \quad (4.9)$$

where

$$V := \frac{D\xi}{dr} \quad T := T^k_i E_k \quad T^k_i = -T^k_{(j)\|n} u^j n^m + u^j T^k_j (u^i_\| - T^i_m u^m). \quad (4.10)$$

From (4.9), we conclude that the relative acceleration between two free particles (“tidal” acceleration) has two sources: the curvature tensor and the torsion tensor. For that reason, in a \( L_n \)-space is possible a situation which cannot arise in the \( V_\mu \)-spaces (without torsion): the torsion can ‘compensate’ the curvature so that the tidal acceleration will vanish, \( \frac{DV}{dr} = 0 \), along the whole base trajectory, on some its parts, or at some single point on it. Consequently, if one can experimentally prove that in a non-flat spacetime region is realized one of these possibilities, this possible fact will mean that in the region mentioned the torsion does not vanish (as \( L_n \) is used as a spacetime model).
4.3. Let an observer and observed particle have arbitrary \( C^2 \) trajectories \( x_0(r) \) and \( x_1(r_1) \), respectively. Let us define

\[
\begin{align*}
  w &:= \frac{dr_1}{dr} (\neq 0, \infty) \quad u^\alpha := \frac{dx_0^\alpha(r)}{dr} \quad u := u^\alpha \partial_\alpha = u^k E_k \\
  v^\alpha &:= \frac{dx_1^\alpha(r_1)}{dr_1} \quad v := v^\alpha \partial_\alpha = v^k E_k.
\end{align*}
\]

As \( x_0(r) \) and \( x_1(r_1) \) are, by definition, bijective and invertible functions of \( r \) and \( r_1 \), respectively, we can write

\[
v = \frac{1}{w} (u + V) \quad V := \frac{D\xi}{dr}.
\]

The mapping \( x_0(r) \mapsto x_1(r_1) = x_0(r) + \xi(r) \) deforms the vector \( F^k := u^k_\dot{u}^l \) and the connection \( \Gamma^i_{jk} \) in respectively

\[
\begin{align*}
  'F^i &= F^i + \varepsilon \mathcal{L}_\xi F^i = F^i + \mathcal{L}_\xi F^i \\
  '\Gamma^i_{jk} &= \Gamma^i_{jk} + \varepsilon \mathcal{L}_\xi \Gamma^i_{jk} = \Gamma^i_{jk} + \mathcal{L}_\xi \Gamma^i_{jk}
\end{align*}
\]

A natural additional condition for the deviation equation, in the particular case, can be obtained by requiring \( 'F^i \) to describe the change of \( v^i \) on the observed trajectory, viz.

\[
'F^i = v^j \nabla_j v^i = v^j (v^i + '\Gamma^i_{jk} v^k).
\]

This means that the mapping \( x_0(r) \mapsto x_1(r_1) \) drags/deforms all structures on \( x_0(r) \) and the links between them to similar ones but defined on the observed trajectory \( x_1(r_1) \).

One can prove that a necessary and sufficient condition for the validity of (4.15) is

\[
\mathcal{L}_\xi F^i = -F^i + \frac{1}{w^2} \left\{ F^i - (u^i + V^i) \left[ V^j (\ln w)_j \frac{D\ln w}{dr} \right] + V^j (u^j + V^j)_j + u^j V^i_j + (u^j + V^j)(u^k + V^k) \mathcal{L}_\xi \Gamma^i_{jk} \right\}.
\]

In a similarly way one can derive other additional condition [1] if, instead of \( v^\alpha := \frac{dx_1^\alpha(r_1)}{dr_1} \), one defines \( v^\alpha := \frac{1}{w^2} u^\alpha \) and requires the fulfillment of (4.15). In this case, the transformation \( x_0(r) \mapsto x_1(r_1) \) maps/deforms the basic trajectory in such a way that the tangent vectors to basic and deformed (observed) curves are proportional.

The authors are thankful to N. A. Charnikov for discussions.

A. Appendix: Lie derivatives of the coefficients \( \Gamma^i_{jk} \) of a connection in \( L_n \) in arbitrary frame

The Lie derivatives of the coefficients \( \Gamma^i_{jk} \) of an affine connection in \( L_n \) are defined in a coordinated frame, e.g., in [4]. Below only the case of an arbitrary frame \( \{E_i\} \) will be considered. Recall, under a frame change \( \{E_i\} \mapsto \{E'_i := A^i_j E_j \} \), with \( A := [A^i_j] \) being a nondegenerate matrix-valued function, the \( \Gamma \)’s transform according to

\[
\Gamma^i_{jk} \mapsto \Gamma'^i_{jk'} = A^i_j A^j_{k'} \Gamma^i_{jk} + A^i_{j'} A^j_k \Gamma^i_{jk'},
\]

where \([A^i_j] := A^{-1}\).

Let us consider in \( L_n \) an infinitesimal point transformation

\[
x \mapsto \bar{x}
\]
with coordinate representation
\[ \tilde{x}^\alpha = x^\alpha + \varepsilon \xi^\alpha(x^\beta). \]
(A.3)

Here \( \varepsilon \) is an infinitesimal parameter and \( \xi^\alpha \) are the components of \( C^2 \) vector field. The frame \( \{ \tilde{E}_i = A^\lambda_i E_i \} \) with
\[ A^j_i = \delta^j_i (\delta^k_j - \varepsilon \Sigma^j_k) \quad \Sigma^j_k := \xi^j_k + C^j_{kl} \xi^l, \quad \xi^l := A^l_\alpha \xi^\alpha \]
is called dragged by the point transformation (A.2) (cf. [7]). The so-defined quantities satisfy the equations
\[ \mathcal{L}_\xi E_i = -\Sigma^k_i E_k \quad \delta^j_i E_j = E_i + \varepsilon \mathcal{L}_\xi E_i = \delta^j_i A^k_j E_k. \]
(A.5)

Equation (A.4) implies
\[ A^j_k = \delta^j_k (\delta^k_j + \varepsilon \Sigma^k_j), \]
(A.6)

where here and henceforth the terms of order \( \varepsilon^2 \) and more are neglected.

By definition, the dragged (deformed) by (A.2) coefficients \( \Gamma^i_{jk} \) are (cf. [4])
\[ \Gamma^i_{jk}(x) = A^i_j A^j_k \delta^k_p A^p_r A^r_q \Gamma^p_q(x) + A^i_j(x) A^j_k(x). \]
(A.7)

Inserting (A.4) and (A.6) into the last expression and using that
\[ \Gamma^p_{qr}(\tilde{x}) = \Gamma^p_{qr}(x) + \varepsilon \xi^m \Gamma^p_{qr,m}(x), \]
(A.8)
we get
\[ \Gamma^i_{jk} = \Gamma^i_{jk} + \varepsilon (-\Gamma^i_{jk} \Sigma^i_n + \Gamma^i_{nk} \Sigma^i_j + \Gamma^i_{jn} \Sigma^i_k + \Gamma^i_{jk,n} \xi^i). \]
(A.9)

The Lie derivative of \( \Gamma^i_{jk} \) with respect to \( \xi \) is [4]
\[ \mathcal{L}_\xi \Gamma^i_{jk} := \lim_{\varepsilon \to 0} \frac{\Gamma^i_{jk} - \Gamma^i_{jk}}{\varepsilon}, \]
(A.10)
from where one immediately finds that (see (A.9))
\[ \mathcal{L}_\xi \Gamma^i_{jk} = -\Gamma^i_{jk} \Sigma^i_n + \Gamma^i_{nk} \Sigma^i_j + \Gamma^i_{jn} \Sigma^i_k + \Gamma^i_{jk,n} \xi^i. \]
(A.11)

If the partial derivatives in (A.11) are replace via covarient ones \( \big( \xi^i_k = \gamma^i_k \xi^k \big) \), one gets the equality (1.1) after appropriate calculations.

References

[1] Sawa S. Manov. The deviation equations and Lie derivatives in Riemannian spaces. JINR Communication P2-12026, Dubna, 1978. In Russian.

[2] C. W. Misner, K. S. Thorne, and J. A. Wheeler. Gravitation. W. H. Freeman and Company, San Francisco, 1973.

[3] N. V. Mitskevich. Physical fields in general theory of relativity. Nauka, Moscow, 1969. (In Russian).

[4] K. Yano. The theory of Lie derivatives and its applications. North-Holland Publ. Co., Amsterdam, 1957.

[5] A. P. Norden. Spaces with affine connection. Nauka, Moscow, second edition, 1976. (In Russian).

[6] N. S. Sinyukov. Geodesic mappings of Riemannian manifolds. Nauka, Moscow, 1979. In Russian.

[7] J. A. Schouten. Tensor analysis for physicists. Clarendon Press, Oxford, 1951.