PAIRS OF MODULES AND DETERMINANTAL ISOLATED SINGULARITIES

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Abstract. We continue the development of the study of the equisingularity of isolated singularities, in the determinantal case.

1. Introduction

In an earlier paper the first author introduced a framework for studying the equisingularity of families of isolated singularities using the multiplicity of pairs of modules and their polar curves ([9]).

In this paper we specialize to the case of maximal rank determinantal singularities and use the framework to study equisingularity problems there. With the additional hypotheses the framework yields invariants which control the Whitney equisingularity type whose definition is independent of the family under study.

A determinantal singularity is one which is given by the minors of a matrix, where the singularity so defined has the expected dimension. A determinantal singularity is a maximal rank singularity if the order of the minors is maximal for the given matrix. This type of singularity contains all ICIS singularities and is a next step in the program of studying all singularities.

We compute one of the key invariants of [9] as a sum of intersection numbers of modules naturally associated with the singularity; these intersection numbers in turn are the colengths of a collection of ideals. Using this we give a formula for the codimension $d$ polar multiplicity of the singularity in terms of the multiplicity of the pair and an alternating sum of colengths of ideals.

In the framework mentioned above, in the case of isolated singularities, a main role was played by a pair of modules, $(M, N)$. The module $M$ was the Jacobian module of the singularity, and the module $N$ consisted of elements in the integral closure of $M$ except perhaps at the singular point. Extending these ideas to the total space of the family then restricting to fibers presented some technical obstructions to obtaining family independent invariants in general. The direction in the study of this case was motivated by the following philosophy. In studying the equisingularity of a family of isolated singularities, fix in advance a component of the base space of the versal deformation, from which the given family is induced. This can be done explicitly or implicitly as in our first case where we restrict to determinantal deformations. This has the effect of fixing the generic fiber of the versal deformation to which all members of our family can be deformed. Invariants associated with the geometry/topology of this general member should
provide important information about the singularities in the original family. The infinitesimal deformations of $X$ induced from the fixed component should give the elements of $N$.

In general, examples are known where the isolated singularities can be smoothed in two different ways with differing topologies. In these cases, of course, the base of the versal deformation space is not irreducible. This makes it impossible to define invariants of $X$ based on the smoothing of $X$ if the smoothing is not well defined.

In the second section we introduce background material from theory of integral closure of modules, and describe the framework developed in (9) for proving equisingularity results. In the third section we introduce the pair of modules we use to control equisingularity. One is the Jacobian module of the singularity, the other is related to the infinitesimal deformations of the singularity. We denote this module by $N$. To calculate the multiplicity of the polar curve of $N$ in a deformation to a smoothing, it is necessary describe Projan $\mathcal{R}(N)$. We do this by showing an equivalence between Projan $\mathcal{R}(N)$ and a modification of $X$ based on the presentation matrix of the singularity. This equivalence then gives a decomposition of the multiplicity of the polar of $N$ as a sum of intersection numbers of generic plane sections with the exceptional fiber of the modification. In section four we compute these intersection numbers as the alternating sum of intersections of modules which depend only on the presentation matrix, and give an example of a computation for a family of space curves.

In section 5 we review various equisingularity conditions and describe the consequences for these conditions based on section 4.

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2. The theory of the Integral closure of modules

Let $(X, x)$ be a germ of a complex analytic space and $X$ a small representative of the germ, and let $\mathcal{O}_X$ denote the structure sheaf on a complex analytic space $X$

**Definition 2.1.** Suppose $(X, x)$ is the germ of a complex analytic space, $M$ a submodule of $\mathcal{O}_{X,x}^p$. Then $h \in \mathcal{O}_{X,x}^p$ is in the integral closure of $M$, denoted $\overline{M}$, if for all analytic $\phi : (\mathbb{C}, 0) \to (X, x)$, $h \circ \phi \in (\phi^*M)\mathcal{O}_1$. If $M$ is a submodule of $N$ and $\overline{M} = \overline{N}$ we say that $M$ is a reduction of $N$.

To check the definition it suffices to check along a finite number of curves whose generic point is in the Zariski open subset of $X$ along which $M$ has maximal rank. (Cf. [6].)

If a module $M$ has finite colength in $\mathcal{O}_{X,x}^p$, it is possible to attach a number to the module, its Buchsbaum-Rim multiplicity, $e(M, \mathcal{O}_{X,x}^p)$. We can also define the multiplicity $e(M, N)$ of a pair of modules $M \subset N$, $M$ of finite colength in $N$, as well, even if $N$ does not have finite colength in $\mathcal{O}_X^p$.

We recall how to construct the multiplicity of a pair of modules using the approach of Kleiman and Thorup [14]. Given a submodule $M$ of a free $\mathcal{O}_{X,x}^p$
module $F$ of rank $p$, we can associate a subalgebra $R(M)$ of the symmetric $\mathcal{O}_{X,x}$ algebra on $p$ generators. This is known as the Rees algebra of $M$. If $(m_1, \ldots, m_p)$ is an element of $M$ then $\sum \overline{m_i} T_i$ is the corresponding element of $R(M)$. Then $Projan(R(M))$, the projective analytic spectrum of $R(M)$ is the closure of the projectivized row spaces of $M$ at points where the rank of a matrix of generators of $M$ is maximal. Denote the projection to $X^d$ by $c$. If $M$ is a submodule of $N$ or $h$ is a section of $N$, then $h$ and $M$ generate ideals on $Projan R(N)$; denote them by $\rho(h)$ and $\rho(M)$. If we can express $h$ in terms of a set of generators $\{n_i\}$ of $N$ as $\sum g_i n_i$, then in the chart in which $T_1 \neq 0$, we can express a generator of $\rho(h)$ by $\sum g_i T_i/T_1$. Having defined the ideal sheaf $\rho(M)$, we blow it up.

On the blow up $B_{\rho(M)}(Projan R(N))$ we have two tautological bundles. One is the pullback of the bundle on $Projan R(N)$. The other comes from $Projan R(M)$. Denote the corresponding Chern classes by $c_M$ and $c_N$, and denote the exceptional divisor by $D_{M,N}$. Suppose the generic rank of $N$ (and hence of $M$) is $r$.

Then the multiplicity of a pair of modules $M, N$ is:

\[ e(M,N) = \sum_{j=0}^{d+r-2} \int D_{M,N} \cdot c_M^{d+r-2-j} \cdot c_N^j. \]

Kleiman and Thorup show that this multiplicity is well defined at $x \in X$ as long as $\mathcal{M} = \mathcal{N}$ on a deleted neighborhood of $x$. This condition implies that $D_{M,N}$ lies in the fiber over $x$, hence is compact. Notice that when $N = F$ and $M$ has finite colength in $F$ then $e(M,N)$ is the Buchsbaum-Rim multiplicity $e(M, \mathcal{O}_{X,x}^p)$. There is a fundamental result due to Kleiman and Thorup, the principle of additivity [14], which states that given a sequence of $\mathcal{O}_{X,x}$-modules $M \subset N \subset P$ such that the multiplicity of the pairs is well defined, then

\[ e(M,P) = e(M,N) + e(N,P). \]

Also if $\mathcal{M} = \mathcal{N}$ then $e(M,N) = 0$ and the converse also holds if $X$ is equidimensional. Combining these two results we get that $\mathcal{M} = \mathcal{N}$ if and only if $e(M,N) = e(N,P)$. These results will be used in Section 5.

In studying the geometry of singular spaces, it is natural to study pairs of modules. In dealing with non-isolated singularities, the modules that describe the geometry have non-finite colength, so their multiplicity is not defined. Instead, it is possible to define a decreasing sequence of modules, each with finite colength inside its predecessor, when restricted to a suitable complementary plane. Each pair controls the geometry in a particular codimension.

We also need the notion of the polar varieties of $M$. The polar variety of codimension $l$ of $M$ in $X$, denoted $\Gamma_l(M)$, is constructed by intersecting $Projan R(M)$ with $X \times H_{g+l-1}$ where $H_{g+l-1}$ is a general plane of codimension $g+l-1$, then projecting to $X$.

Setup: We suppose we have families of modules $M \subset N$, $M$ and $N$ submodules of a free module $F$ of rank $p$ on an equidimensional family of spaces with equidimensional fibers $X^{d+k}$, $X$ a family over a smooth base.
We assume that the generic rank of $M, N$ is $r \leq p$. Let $P(M)$ denote $\text{Projan} \mathcal{R}(M)$. We will be interested in computing, as we move from the special point $0$ to a generic point, the change in the multiplicity of the pair $(M, N)$, denoted $\Delta(e(M, N))$. We will assume that the integral closures of $M$ and $N$ agree off a set $C$ of dimension $k$ which is finite over $Y$, and assume we are working on a sufficiently small neighborhood of the origin, so that every component of $C$ contains the origin in its closure. Then $e(M, N, y)$ is the sum of the multiplicities of the pair at all points in the fiber of $C$ over $y$, and $\Delta(e(M, N))$ is the change in this number from $0$ to a generic value of $y$.

Let $C(M)$ denote the locus of points where $M$ is not free, i.e., the points where the rank of $M$ is less than $g$, $C(\text{Projan} \mathcal{R}(M))$ its inverse image under $\pi_M$.

We now state the Multiplicity Polar Theorem. The proof in the ideal case appears in [9]; the general proof appears in [10].

**Theorem 2.2.** (Multiplicity Polar Theorem) Suppose in the above setup we have that $M = N$ off a set $C$ of dimension $k$ which is finite over $Y$. Suppose further that $C(\text{Projan} \mathcal{R}(M))(0) = C(\text{Projan} \mathcal{R}(M(0)))$ except possibly at the points which project to $0 \in \mathcal{X}(0)$. Then, for $y$ a generic point of $Y$,

$$\Delta(e(M, N)) = \text{mult}_y \Gamma_d(M) - \text{mult}_y \Gamma_d(N)$$

where $\mathcal{X}(0)$ is the fiber over $0$ of the family $\mathcal{X}^{d+k}$, $C(\text{Projan} \mathcal{R}(M))(0)$ is the fiber of $C(\text{Projan} \mathcal{R}(M))$ over $0$ and $M(0)$ is the restriction of the module $M$ to $\mathcal{X}(0)$.

One of application of the Multiplicity Polar theorem we will need pertains to the intersection multiplicity of two modules as defined by Serre ([22]). Given modules $M_1 \subset F_1$ and $M_2 \subset F_2$, $F_i$ free $\mathcal{O}_{\mathcal{X},x}$ modules of rank $p_i$ as above, Serre’s intersection number is the alternating sum of the lengths of the $\text{Tor}^i(F^{p_1}/M_1, F^{p_2}/M_2)$. (Cf. [12] for all of the necessary hypotheses for this number to be defined.) If the local ring of $X$ is Cohen Macaulay, then we can hope to calculate this number as a length. The two candidates are $e(M_1, \mathcal{O}_{C(M_2),x})$ and $e(M_2, \mathcal{O}_{C(M_1),x})$. If the local ring of $X$ is Cohen Macaulay, then these multiplicities are the colength of the ideal of maximal minors of each module ([12] cor 2.4). In [12] Theorem 2.3 and Corollary 2.5 it is shown that these numbers are equal and both are equal to Serre’s intersection number.

We now discuss the framework for addressing equisingularity problems we will use in this paper.

We are interested in equisingularity conditions which are equivalent to the inclusion of the partial derivatives of map germ with respect to the parameter values in the integral closure of a module. The conditions which can be studied in this way include the Whitney conditions, and Thom’s $A_f$ condition.

Given the total space of a family of spaces and the module, the inclusion conditions depend on the polar varieties of the module.
Suppose we have a family of spaces parametrized by a smooth $Y^k$; then in [9] it is shown that if the codimension $d$ polar variety of $M$ is empty, and $h$ is generically in the integral closure of $M$, then it is in the integral closure of $M$. The multiplicity polar theorem shows that to ensure that the polar variety of codimension $d$ is empty, i.e., has multiplicity 0 over $Y$, it suffices that $e(M(0), N(0)) + \text{mult}_y \Gamma_d(N)$ is the same as the multiplicity of $e(M(y), N(y))$ for a generic value of $Y$.

In this setting it then becomes important to see what the correct choice of $N$ is, and to control its codimension $d$ polar variety. This is the topic of the next section.

### 3. The determinantal normal module

In [9], in applications, given a module $M$, the pair considered was $(M, N)$ where $N = H_{d-1}(M)$; $H_{d-1}(M)$ is the module of elements which were in the integral closure of $M$ off a subset of codimension at least $d$. If $M$ is the Jacobian module, then there is a link between $H_{d-1}(M)$ and the infinitesimal deformations of $X$.

Let $I$ be the ideal of $O_X$ defining $X^d$; denote the set of all elements $h \in I$ such that the partial derivatives of $h$ are in $I$ by $\int I$. Note that we can identify $O_X^p$ with its dual $\text{hom}(O_X, O_X)$. If $I$ has $p$ generators, we have the following short exact sequence of $O_X$ modules.

$$0 \rightarrow R \rightarrow O_X^p \rightarrow I/I^2 \rightarrow 0$$

Here $R$ is the module of relations. Denote the map to $I/I^2$ by $j$. This gives the injection

$$0 \rightarrow \text{hom}(I/I^2, O_X) \rightarrow \text{hom}(O_X^p, O_X).$$

So, we can identify elements in the image of this last inclusion with their preimages. Note that each partial derivative operator defines an element of $\text{hom}(I/I^2, O_X)$. Denote the submodule of $\text{hom}(I/I^2, O_X)$ generated by the elements defined by the partial derivative operators by $D$.

**Proposition 3.1.** With the identification of $O_X^p$ with $\text{hom}(O_X^p, O_X)$, the module $JM(X)$ is the image of $D$ under the inclusion of $\text{hom}(I/I^2, O_X)$ in $\text{hom}(O_X^p, O_X)$, and $H_0(JM(X))$ is the image of $\text{hom}(I/\int I, O_X)$.

**Proof.** Cf. [8] Proposition 5.1.

If we assume $I$ is radical then in fact we have the module $H_0(JM(X))$ is the image of the normal module, $\text{hom}(I/I^2, O_X)$ for then the inclusion of $\text{hom}(I/\int I, O_X)$ into $\text{hom}(I/I^2, O_X)$ is an isomorphism. (Cf. [18] lemma 1.15.)

It is not hard to see, that if the singular locus of $X^{d+k}$ has codimension $d$, then $H_0(JM(X)) = H_{d-1}(JM(X))$.

If $X, 0$ is the germ of a Cohen-Macauley (CM) subvariety of $\mathbb{C}^q, 0$ of codimension 2, then its ideal is generated by the maximal minors of an $(n + 1) \times n$ matrix with entries in $O_q$. Let $M_X$ denote the presentation matrix of $X$, $M_{i,j}$, the $(i,j)$ entry of $M_X$. We have a description of the normal module in terms of the presentation matrix.
Proposition 3.2. The normal module of $X$ is given by

$$\text{Mat}(n+1, n; \mathcal{O}_q)/\text{Im}(g)$$

where $g$ is the map

$$\text{Mat}(n+1, n+1; \mathcal{O}_q) \oplus \text{Mat}(n, n; \mathcal{O}_q) \xrightarrow{\delta} \text{Mat}(n+1, n; \mathcal{O}_q)$$

mapping $(A, B) \mapsto AM + MB$.

Proof. [3] lemma 2.6. \qed

If $X$ is determinantal then by considering only deformations of $X$ which arise as deformations of the presentation matrix, $M_X$, we define $N_D(X)$, the determinantal normal module. If $\mathcal{D}$ has an isolated singularity since $N_D(X)$ sits between $JM(X)$ and $H_0(JM(X))$, it is clear that $JM(X)$ has finite co-length inside $N_D(X)$.

It is not hard to describe a set of generators for $N_D(X)$. If the size of $M_X$ is $(n+k) \times n$, let $\Delta(l_1, \ldots, l_k)$ denote the maximal minor of $M_X$ obtained by deleting rows $(l_1, \ldots, l_k)$. (We assume $l_i < l_{i+1}$ for $1 \leq i \leq k - 1$.) Let $\delta_{i,j}$ be the $(n+k) \times n$ matrix with 1 in the $(i, j)$ entry other entries 0. Consider the deformation of $M_X$ given by $M_X + t\delta_{i,j}$. Each such deformation gives a column in a matrix of generators for $N_D(X)$, so there are $(n+k)n$ columns in a matrix of generators for $N_D(X)$. The $(l_1, \ldots, l_k)$ entry of the $(i, j)$ column is gotten by taking the $(l_1, \ldots, l_k)$ minors of $M_X + t\delta_{i,j}$, and taking the linear part of this in $t$. This entry is denoted by $m_{i,j}(l_1, \ldots, l_k)$ and is the cofactor of $M_{i,j}$ in the expansion of $\Delta(l_1, \ldots, l_k)$. Of course it is 0 if $i = l_k$ for some $k$.

If we have a family $\mathcal{X}$ of determinantal singularities defined by deforming the entries of a presentation matrix, then is is clear that $N_D(\mathcal{X})$ specializes to members of the family, and to sub-deformations.

Following Frübis-Krüger, given $X \subset \mathbb{C}^q$ with presentation matrix $M_X$, we can consider the group action $G$ on $M_X$ given by multiplication of $M_X$ with invertible matrices on the left and right, and compositions with coordinate changes on $\mathbb{C}^q$. The action of this group is well understood, and its extended tangent space in the sense of Mather is quotient module of $N_D(X)$ by $JM(X)$, which has finite co-length if $X$ has an isolated singularity. The theory of these actions then shows that $M_X$ is finitely determined in the sense that perturbing the entries of $M_X$ by a sufficiently high power of the maximal ideal doesn’t change $X$ up to a coordinate change of $\mathbb{C}^q$. Further, $M_X$ has a deformation $M_X$ with smooth finite dimensional base $Y$ such that any deformation of $X$ defined by deforming the entries of $M_X$ can be induced from $M_X$. It follows that any two smoothings of $X$ defined by deforming the entries of $M_X$ are homeomorphic.

In the next section we want to calculate the multiplicity of the polar of $N_D(X)$ in the case where the generic fiber of the deformation is smooth, so we want to describe $\text{Proj}(R(N_D(X)))$. The description will give will apply equally to $\text{Proj}(R(N_D(X)))$.

Since the generators of $N_D(X)$ are in one to one correspondence with the entries of $M_X$, we know that

$$\text{Proj}(R(N_D(X))) \cong \mathcal{O}_X[T_{i,j}]/I, \quad 1 \leq i \leq n+k, 1 \leq j \leq n,$$
where \( I \) is the ideal of relations between the \( T_{i,j} \) under the map which sends \( T_{i,j} \) to the \( i,j \) generator of \( \mathcal{R}(N_D(X)) \).

**Lemma 3.3.** \( I \) contains the entries of the matrices \( [T_{i,j}]^t M_X \) and \( M_X [T_{i,j}]^t \).

**Proof.** We can view the map from \( \text{Mat}(n + 1, n; \mathcal{O}_q) \) to the normal module in the proof of lemma 2.6 of \( \mathfrak{X} \), as a map from \( \mathcal{O}_X[T_{i,j}] \) to \( \mathcal{R}(N) \). So we can use the image of \( g \) to find relations in this case by translating to \( \mathcal{O}_X[T_{i,j}] \).

The analogous map from \( \text{Mat}(n + k, n; \mathcal{O}_q) \) to \( N_D(X) \), still carries the image of \( g \) to the trivial deformations, hence to zero in \( N_D(X) \). So the image of \( g \) still gives elements of \( I \). We claim these elements are entries of the matrices \( [T_{i,j}]^t M_X \) and \( M_X [T_{i,j}]^t \). We trace through the relations between these two settings to show this. Suppose \( \delta_{i,j} \) is the matrix in \( \text{Mat}(n + k, n + k; \mathcal{O}_q) \) with 1 in the \((i,j)\) entry and zero elsewhere. Then the matrix \( \delta_{i,j} M_X \) has the \( i\)-th row as the only non-zero row with entries \((m_{j,r})\). This gives the element \( \sum_{r=1}^n m_{j,r} T_{i,r} \) in \( I \) in \( \mathcal{O}_X[T_{i,j}] \). In turn this is \( (M_X [T_{i,j}]^t)_j,i \). The computation for \( [T_{i,j}]^t M_X \) is similar.

**Remark 3.4.** The lemma means that the entries of \( [T_{i,j}]^t M_X \) = 0 and \( M_X [T_{i,j}]^t \) = 0 are some of the equations of \( \text{Projan}(\mathcal{R}(N_D(X))) \). If we work at a point \( x \) where \( X \) is smooth, hence \( M_X \) has rank \( n-1 \), then the values of the rows and columns of \( [T_{i,j}] \) must all be in the kernel of \( M_X(x) \) and \( M_X(x) \) respectively. This implies that the entries of \( [T_{i,j}] \) give a matrix of rank 1.

We will use the remark in the proof of the next theorem which will allow us to decompose the computation of \( \text{mult}_g \Gamma_d(N_D(X)) \) into manageable pieces.

There are two interesting transforms we can make of \( X \) using \( M_X \). Let \( X_{n-1} \) denote the set of points of \( X \) where \( M_X \) has rank \( n-1 \); we define:

\[
X_M := \{(x, l_1, l_2) | x \in X_{n-1}, l_1 \in \mathbb{P}(\ker(M_X^t(x))), l_2 \in \mathbb{P}(\ker(M_X(x)))\}
\]

\[
X_T := \{(x, l) | x \in X_{n-1}, l \in \mathbb{P}(\ker(M_X(x)))\}
\]

where \( \mathbb{P}(\ker(M_X(x))) \) is the projectivization of the kernel of \( M_X(x) \). Hence \( X_M \) is contained in \( X \times \mathbb{P}^{n+k-1} \times \mathbb{P}^{n-1} \), while \( X_T \subset X \times \mathbb{P}^{n-1} \). The transform \( X_T \) is known as the Tjurina transform.

There is a third transform of the ambient space \( \mathbb{C}^{d+k+1} \),

\[
\mathbb{C}^{d+k+1}_T := \{(x, l) | x \in \mathbb{C}^{d+k+1}, l \in \mathbb{P}(\ker(M_X^t(x)))\}.
\]

Hence \( \mathbb{C}^{d+k+1}_T \) is contained in \( \mathbb{C}^{d+k+1} \times \mathbb{P}^{n+k-1} \).

We will alter the presentation matrix \( M \) by dropping rows; this will induce new transforms of the three types defined above.

**Theorem 3.5.** Suppose \( X \) is a maximal rank reduced determinantal singularity, with \( X_{n-1} \) dense in every component of \( X \), then \( \text{Projan} \mathcal{R}(N_D(X)) \) is isomorphic to \( X_M \) as sets.

**Proof.** Since both sets are defined by the closures of the points over \( X_{n-1} \) it suffices to work on this set. Let \( (S_1, \ldots, S_{n+k}) \) be coordinates on \( \mathbb{P}^{n+k-1} \) and \( (T_1, \ldots, T_n) \) be coordinates on \( \mathbb{P}^{n-1} \). Consider the Veronese embedding \( \mathbb{P}^{n+k-1} \times \mathbb{P}^{n-1} \) to \( \mathbb{P}^{(n+k)(n-1)} \) given by \( (S_1, \ldots, S_{n+k}) \times (T_1, \ldots, T_n) \) maps to \( T_{i,j} = S_T \). This embedding sends the points of \( X_M \) over \( x \in X_{n-1} \)
to matrices of rank 1 whose rows are multiples of a fixed non-zero kernel vector of $M_X(x)$ (hence the matrix has rank 1) and whose columns are in the kernel of $M'(x)$. The set of such matrices is a subvariety of dimension $(n - 1) + (n + k - 1)$. The fiber of $X_M$ over $x$ by the remark maps to points containing the fiber of $\text{Projan} \mathcal{R}(N_D(X))$. Further the fiber dimension of $X_M$ is clearly $k$ as the kernel of $M'(x)$ has dimension $k + 1$. Meanwhile the fiber dimension of $\text{Projan} \mathcal{R}(N_D(X))$ is one less than the rank of the Jacobian module of $X$, which is the expected codimension of $X$, so the fiber dimension is $k$ also. Since the image of the fiber of $X_M$ over $x$ is irreducible as is the fiber of $\text{Projan} \mathcal{R}(N_D(X))$, they are the same. Hence the closure of the image of $X_M$ is the same as $\text{Projan} \mathcal{R}(N_D(X))$.  

In the next section we will use this theorem to compute the degree over the base $Y^1$ of the polar variety of dimension 1 of $N_D(X)$, where $X$ is the total space of the deformation, and a generic fiber is smooth. As we have seen $\text{Projan} \mathcal{R}(N_D(X))$ is a subset of $X \times \mathbb{P}^{n(n+k)-1}$. The hyperplane class on this space denoted $h$ is represented by $X \times H$, where $H$ is a hyperplane in $\mathbb{P}^{n(n+k)-1}$. The hyperplane classes on $X \times \mathbb{P}^{n-1}$ and $X \times \mathbb{P}^{n+k-1}$ are denoted by $h_2$ and $h_1$ respectively. As classes, the pullback of $h$ to $X \times \mathbb{P}^{n+k-1} \times \mathbb{P}^{n-1}$ by the Veronese $V$ is $h_1 + h_2$. Denote the fiber over the origin in $X$ of $\text{Projan} \mathcal{R}(N_D(X))$ by $E$. If $N_D(X)$ was an ideal, this would be the fiber of the exceptional divisor of the blow-up of $X$ by $N_D(X)$.

**Theorem 3.6.** Suppose $X^{d+1}$ is a maximum rank determinantal singularity which is a smoothing of a maximal rank determinantal singularity $X^d$, with smooth base $Y^1 = \mathbb{C}^1$. The degree of the polar variety of $\text{Projan} \mathcal{R}(N_D(X))$ over $Y$ at the origin, $\Gamma_d(N_D(X))$ is $(h_1 + h_2)^{d+k} \cdot E$.

**Proof.** By definition the degree of $\Gamma_d(N_D(X))$ over $Y$ at the origin is the degree of the projection to $Y$ at the origin of $\Gamma_d(N_D(X))$. In turn, $\Gamma_d(N_D(X)) = \pi_X(\text{Projan} \mathcal{R}(N_D(X))) \cap h^{d+k}$. We can assume the representative of $h^{d+k}$ chosen so that it is transverse to all components of $E$ of maximal dimension which is $d + \text{cod}(X) - 1 = d + k$. Each point of intersection contributes 1 to the degree, so the degree is $V^*(h^{d+k}) \cdot V^*E = (h_1 + h_2)^{d+k} \cdot V^*E$.  

Define $\Gamma_{i,j}(N_D(X))$ to be $\pi_X(X_M \cap h_1^i h_2^j)$. We call these the mixed polars of type $(i, j)$ of $N_D(X)$. Denote the degree of this mixed polar by $h_1^i h_2^j$.

**Corollary 3.7.** The degree of $\Gamma_d(N_D(X))$ over $Y^1$ is

$$\sum_{i=0}^{d+k} \binom{d+k}{i} h_1^i h_2^{d+k-i}.$$ 

**Proof.** The reasoning is similar to the proof of theorem-the degree of the $\pi_X(X_M \cap h_1^i h_2^j)$ over $Y$ is $h_1^i h_2^j \cdot V^*E$.

These degrees will be computed in the next section.
4. Computing the Degrees of the Mixed Polars and Applications to Equisingularity

We define the terms we use in our formula, then prove the formulas subject to some genericity assumptions. Throughout we assume the size of $M_X$ is $(n + k) \times n$, dropping the subscript on $M_X$.

Let $M_{d+k+i}^t$ denote the submodule of $R^r$ where $r = (n + k) - (d + k + l) = n - d - l$, whose matrix of generators consists of the first $n - d - l$ rows of $M$; $R$ is the ring of either $X, 0$ or $X, 0$. (The context tells which.) In the formulas, $l$ will be $j - i$ where $i$ is the exponent of $h_2$ and $j$ is an index. Because $i$ is the exponent of $h_2$, we must have $0 \leq i \leq \min \{d, n - 1\}$. Notice that $i \leq d$ implies $n - d + i$, the number of rows if $j = 0$, is less than or equal to the number of columns. In order for the number of rows to be non-zero for $j > 0$, we need for $n - d + i - j > 0$ or $j \leq n - d + i - 1$.

Let $M_{d+k-i,j}^{c,i}$ denote the submodule of $R^e$ where $r_c = n + k - j - 1$, whose matrix of generators consists of the last $n - i$ columns of $M$ and the first $n - d + i - j - 1$ rows of $M$ as well as the last $d + k - i$ rows of $M$. Notice that if $j = 0$, then $M_{d+k-i,j}^{c,i}$ has $n + k - 1$ rows in all. Incrementing $j$ by 1 drops another row. Sometimes, in the last term in our sums $j = k$; in this case $M_{d+k-i,j}^{c,i} = 0$. Otherwise, we assume $j \leq i + k - 1$ to ensure there are at least as many rows as columns in the matrix of generators.

The assumption in these constructions is that we are using the coordinate hyperplanes as the representatives of the powers of the $h_i$. As we shall see, dropping the last $d + k - i + j$ rows of $M$ is the effect of setting the last $d + k - i + j$ of the $S_l$ to 0. In a similar way, dropping the first $i$ columns is the effect of setting the first $i$ of the $T_l$ to 0.

The genericity conditions we need are of the following type: for the presentation matrix, after row and column operations, representing the $h_i^T$ by coordinate hyperplanes, then $(h_1 + h_2)^{d+k} \cdot E$ has the expected dimension and $(h_1 + h_2)^{d+k+1} \cdot E$ also has the expected dimension 0 in the case of a one parameter smoothing. There are submatrices of $M$ that appear in our formula, and they have the analogues of the $(h_1 + h_2)^{d+k+1} \cdot E$. We also ask these intersections have the expected dimension.

To give an overview of how the computation goes, we look at one of the formulas we will prove:

$$h_1^{d+k} \cdot E = \sum_{j=0}^{\min(n-d-1,k)} (-1)^j M_{d+k+j} \cdot M_{d+k+j}^{c,0}.$$

The terms on the right hand side are both the colength of the ideal generated by the maximal minors of the presentation matrices of the modules, and the degree over $Y$ of the projection to $Y$ of the curves defined by those maximal minors on the total space of the smoothing.

The curves defined by each term are determinantal by genericity; with the exception of the last term of the sum, they fall into two types. The term with $j = 0$ contains $\Gamma_{d+k,0}$ and a type shared with $M_{d+k+1} \cdot M_{d+k+1}^{c,0}$. In turn, the second type of curve for $M_{d+k+1} \cdot M_{d+k+1}^{c,0}$ is shared with the next term in the sum. The last term in the sum can have two possible
forms. If $k < n - d - 1$, then the presentation matrix of $M_{d+k,k-1}^{c,0}$, which is part of the pentultimate term, is a square matrix, and the curves associated with the last term ($j = k$) are defined by the maximal minors of $M_{d+k+k}$ alone by our convention. These curves are shared with the previous term. If $0 < n - d - 1 \leq k$, then $n + k - (d + k + (n - d - 1)) = 1$ so the presentation matrix of $M_{k+n-1}$ has 1 row, and this term has only one type of curve, shared with the previous term. It is possible that $n - d - 1$ may be negative or 0. (For example, if $n = 2$.) In this case, $\Gamma_{d+k,0}$ is empty. The genericity conditions ensure that the sum is telescoping, and its value is the degree of $\Gamma_{d+k,0}$ over $Y$.

For the computation of the degree, it is crucial that the sets we work with are determinantal; this is checked by the next lemma.

**Lemma 4.1.** Suppose $M$ is an $(n + k) \times n$ matrix and the coordinates on $\mathbb{P}^{n-1}$ and $\mathbb{P}^{n+k-1}$ are generic in the sense of the above paragraph. Then for each pair of modules $(M_{d+k+j-i}, M_{d+k+i,j}^{c,i})$ the co-supports of the modules on either $\mathbb{C}^{d+k+1}$ or $\mathbb{C}^{d+k+1} \times Y$ are determinantal. Further, the co-supports of either module in the local ring of the co-support of the other is determinantal. Hence the ideal of maximal minors of both matrix of generators defines a set of dimension 1 finite over $Y$.

**Proof.** The co-support of either $(M_{d+k+j-i}, M_{d+k+i,j}^{c,i})$ is defined by the maximal minors of the matrix of generators. The matrix of generators of $M_{d+k+j-i}$ has size $(n-d+i-j) \times n$, so the vanishing of the ideal of minors has expected codimension $n - (n - d + i - j) + 1 = d - i + j + 1$. Meanwhile $M_{d+k-i,j}^{c,i}$ has a matrix of generators of size $(n+k-j-1) \times (n-i)$, so the vanishing of the ideal of minors has expected codimension $n + k - j - 1 - (n - i) + 1 = k + i - j$, so the expected codimension of the intersection is $(d-i+j+1)+(k+i-j) = d+k+1$ which is the dimension of the ambient space of $X$. The genericity hypotheses ensure that the expected codimensions are realized; since the ideals defining the co-supports are ideals of minors, the result follows.  

Given a pair of modules $(M, N)$ denote by $J(M, N)$ the ideal of maximal minors of their matrix of generators. Next we show that the first pair of modules we look at in each sum actually contains the desired polar varieties as components of the co-support of the pair.

**Lemma 4.2.** $V(J(M_{d+k-i}, M_{d+k-i,0}^{c,i}))$ contains $\Gamma_{d+k-i,i}$ as a union of components.

**Proof.** The expected dimension of $\Gamma_{d+k-i,i}$ is 1; by the previous lemma this is true for $V(J(M_{d+k-i}, M_{d+k-i,0}^{c,i}))$ as well, so it suffices to prove that $\Gamma_{d+k-i,i} \subseteq V(J(M_{d+k-i}, M_{d+k-i,0}^{c,i}))$. Suppose $x \in \Gamma_{d+k-i,i}$, assume the standard coordinates on the projective spaces are generic. Then $M(x)$, the presentation matrix with entries evaluated at $x$, satisfies two conditions.

1) $\ker(M^t(x))$ contains $l \subseteq \{0 = S_{n+k} = \cdots = S_{n-d+i+1}\}$.

A basis for this space is $\{e_1, \ldots, e_{n-d+i}\}$ which has codimension $d+k-i$ in $\mathbb{C}^{n+k}$. This condition implies $x \in V(J(M_{d+k-i}))$.
2) \( \ker M(x) \), contains \( I \subset \{ 0 = S_1 = \cdots = S_l \} \). This implies that the last \( n-i \) columns of \( M(x) \) are linearly dependent. In turn, this implies \( x \in V(J(M_{d+k-i,0}^c)) \).

In the next lemma we will show that \( V(J(M_{d+k-i+j}, M_{d+k-i,j}^c)) \) contains two types of components. We now describe what these components are for this pair of modules. Consider the presentation matrix \( M \), and delete \( j \) rows by deleting rows \( n-d+i, n-d+i-1, \ldots, n-d+i-(j-1) \) – denote the new presentation matrix by \( M(j) \). Associated to \( M(j) \), we have \( \Gamma_{d+k-j-i,j}(M(j)) \). The effect of dropping \( j \) rows is to subtract \( j \) from \( k \). Notice that if \( j = 0 \), then \( M(0) = M \) and \( \Gamma_{d+k-j-i,j}(M(j)) = \Gamma_{d+k-i,j}(M) \). In general we have:

**Lemma 4.3.** \( V(J(M_{d+k-i+j}, M_{d+k-i,j}^c)) \) consists of two types of components, \( \Gamma_{d+k-j-i,j}(M(j)) \) and \( \Gamma_{d+k-(j+1)-i,j}(M(j+1)) \), for \( j \leq \min(n-d+i-1,k+i) \), \( 0 \leq i \leq d \), and no curve belongs to both types. In addition if \( j = \min(n-d+i-1,k+i) \), then \( \Gamma_{d+k-(j+1)-i,j}(M(j+1)) \) is empty.

**Proof.** Suppose \( j \neq \min(n-d+i-1,k+i) \). By genericity we can assume that \( \Gamma_{d+k-(j+1)-i,j}(M(j+1)) \) is empty. The proof that \( \Gamma_{d+k-j-i,j}(M(j)) \) is contained in \( V(J(M_{d+k-i+j}, M_{d+k-i,j}^c)) \) is similar to the proof of the previous lemma. Consider the matrix of generators for \( M_{d+k-i+j} \). This has \( n-d+i-j \) rows and \( n \) columns. There are two cases –either the first \( n-d+i-(j+1) \) rows are linearly independent or they are not. If they are, then all of the first \( n-d+i-j \) rows are in the span of these, since \( x \in V(J(M_{d+k-i+j})) \).

Then \( x \in V(J(M_{d+k-i,j}^c)) \) implies \( x \in V(J(M_{d+k-i,j}^c)) \), which implies \( x \in \Gamma_{d+k-j-i,j}(M(j)) \). If the first \( n-d+i-(j+1) \) rows are linearly dependent, then \( x \notin V(J(M_{d+k-i,j-1})) \); if it were genericity would be violated because \( x \in \Gamma_{d+k-(j+1)-i,j}(M(j+1)) \).

Suppose \( j = \min(n-d+i-1,k+i) \), say \( j = n-d+i-1 \); then the matrix of generators for \( M_{d+k-i+j} \) consists of a single row, so the first \( n-d+i-(j+1) \) rows are linearly independent so there is only one case.

If \( j = k+i \leq n-d+i-1 \), then the matrix of generators for \( M_{d+k-i+j} \) has \( n-(k+d) \) columns, while the matrix of generators of \( M_{d+k-i,j}^c \) has \( n-i \) columns, and fewer than \( n-i \) rows, hence \( J(M_{d+k-i,j}^c) = 0 \), so again there is one kind of component.

**Theorem 4.4.** We have

\[
h_1^{d+k-i}h_2^i \cdot E = \sum_{j=0}^{\min(n-d+i-1,i+k)} (-1)^j M_{d+k-j-i}^c \cdot M_{d+k-i,j}^c.
\]

**Proof.** If we fix \( i \) then by lemma 4.3, we know that each term is the sum of two numbers, one of which is a summand of the previous term except for the first term and last terms. Since this is an alternating sum the common terms cancel. The first term in the sum by lemma 4.2 is the sum of two numbers, one of which is \( h_1^{d+k+1-i}h_2^i \cdot E \), and the other is a summand in the
second term, so it cancels. Again by lemma 4.3, the last term in the sum
consists of a single summand shared with the previous term. □

Corollary 4.5. Assuming that coordinates are chosen generically,

\[ \text{multy}(\Gamma_d(N_D(X))) = \]
\[ \sum_{i=0}^{\min(d,n-1)} \sum_{j=0}^{\min(n-d+i-1,i+k-1)} (-1)^j \binom{d+k}{i} M_{d+k+j-i} \cdot M_{d+i,j}^{c,i}. \]

Proof. This follows from the previous corollary and Cor. 3.7. □

Corollary 4.6. Let \( s \) denote the minimum of \( \{n-d+i-1, i+k\} \). Assuming
that coordinates are chosen generically,

\[ \text{multy}(\Gamma_d(JM_s(X))) = \]
\[ e(JM(X), N_D(X)) + \sum_{i=0}^{\min(d,n-1)} \sum_{j=0}^{s} (-1)^j \binom{d+k}{i} M_{d+k+j-i} \cdot M_{d+i,j}^{c,i}. \]

Proof. This follows from the previous corollary and the multiplicity polar
theorem. □

Corollary 4.7. Let \( s \) denote the minimum of \( \{n-d+i-1, i+k\} \). Assuming
that coordinates are chosen generically, and \( H \) is not a limiting tangent
hyperplane to \( X \) at the origin,

\[ (-1)^d \chi(X_s) + (-1)^{d-1} \chi((X \cap H)s) = \]
\[ e(JM(X), N_D(X)) + \sum_{i=0}^{\min(d,n-1)} \sum_{j=0}^{s} (-1)^j \binom{d+k}{i} M_{d+k+j-i} \cdot M_{d+i,j}^{c,i}. \]

Proof. This follows from the previous corollary and the fact that

\[ \text{multy}(\Gamma_d(JM_s(X))) = (-1)^d \chi(X_s) + (-1)^{d-1} \chi((X \cap H)s). \]

(Cf. [9] p 130, [11].) □

From this formula, in the usual way we can get a formula for the Euler
characteristic of a smoothing of a maximal rank determinantal singularity,
for the given presentation matrix. If we take hyperplane slices of \( X \) all of
the above modules specialize. Let \( J_q \) denote the Jacobian module of \( X \cap l_q \)
where \( l_q \) is a generic plane of codimension \( q \), let \( N_q \) denote \( N_D(X \cap l_q) \),
\( M(q)_i \) and \( M(q)^{c,i}_{s,t} \) the modules corresponding to the \( M_i \) and \( M^{c,i}_{s,t} \). Then
we get:

Corollary 4.8. Let \( t \) the minimum of \( \{d-q,n-1\} \), and \( s \) denote the
minimum of \( \{n-d+i-1, i+k\} \). Assuming that coordinates and hyperplanes
are chosen generically,

\[ (-1)^d \chi(X_s) = \sum_{q=0}^{d} (-1)^q e(J_q, N_q) + \]
\[ \sum_{q=0}^{d} \sum_{i=0}^{t} \sum_{j=0}^{s} (-1)^{q+j} \binom{d-q+k}{i} M(q)_{d-q+k+j-i} \cdot M(q)^{c,i}_{d-q+k-i,j}. \]
Proof. This follows from the previous corollary by adding the terms from the corresponding formulas, thereby creating a telescoping sum. □

Now we would like to give some examples.
In the first example, we calculate the multiplicity over $Y$ of the relative polar curve of a smoothing of the family of space curves $X_l$. This will entail calculating the multiplicity of the pair $(JM(X_l), N_D(X_l))$. This calculation will be facilitated by a theorem of Steven Kleiman (Private Communication).

We will also need the multiplicity over $Y$ of the polar curve of $N_D(X_l)$.

The singularities $X_l$ are defined by the minors of

$$M_{X_l} = \begin{bmatrix} z & x \\ y & z \\ x^l & y \end{bmatrix}$$

We assume $l - 1$ is not divisible by 3.

For the formula for the multiplicity over $Y$ of the polar curve of $N_D(X_l)$, we have $n = 2$, $k = 1$, $d = 1$, so Corollary 3.7 becomes

$$\sum_{i=0}^{2} \binom{2}{i} h_1^i h_2^{2-i} = h_1^2 + 2h_1h_2 + h_2^2.$$

By Theorem 4.4 we have:

$h_1^2 = M_{d+k} \cdot M_{d+k,0}^{c,0} = M_2 \cdot M_{2,0}^{c,0}$

$h_1h_2 = M_{d+k-1} \cdot M_{d+k-1,0}^{c,1} - M_{d+k} \cdot M_{d+k-1,1}^{c,1} = M_1 \cdot M_{1,0}^{c,1} - M_2 \cdot M_{1,1}^{c,1}$

In this example our matrix is generic enough; no operations on rows and columns are needed.

We have $M_2 \cdot M_{2,0}^{c,0}$ is the colength of the elements of the first row and the determinant of the last two rows which is the colength of $(x, z, y^2 - xz)$ which is 2.

Denoting the $(i, j)$ entry of $M$ by $a_{i,j}$, we have $M_1 \cdot M_{1,0}^{c,1}$ is the colength of the ideal generated by $(a_{1,1}a_{2,2} - a_{1,2}a_{2,1}, a_{1,2}, a_{3,2}) = (a_{1,1}a_{2,2}, a_{1,2}, a_{3,2}) = (z^2, x, y)$ which is 2. We also have $M_2 \cdot M_{1,1}^{c,1}$ is the colength of the ideal generated by $(a_{1,1}, a_{1,2}, a_{3,2}) = (z, x, y)$, which is 1. this means that for all $l$ the multiplicity over $Y$ of the polar curve of $N_D(X_l) = 2 \cdot (2 - 1) + 2 = 4$, which is $m + 1$ since $m$ for these curves is 3.

Notice that in general we see that the ideal $(a_{1,1}a_{2,2}, a_{1,2}, a_{3,2})$ has two groups of components; those of $(a_{1,1}, a_{1,2}, a_{3,2})$ and of $(a_{2,2}, a_{1,2}, a_{3,2})$. It is those of the second type that make up the $\Gamma_{1,1}$. Those of the first type are subtracted in the second term.

We must now calculate $e(JM(X_l), N_D(X_l))$. We will show it is $2l - 2$.

Tracing through the connection between the deformations of the presentation matrix and the elements of $N_D$, we get that a matrix of generators $M[N_D]$, of $N_D$ is:

$$M[N_D] = \begin{bmatrix} z & -x & 0 & -y & z & 0 \\ 0 & y & -z & 0 & -x^l & y \\ y & 0 & -x & 0 & -x^l & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$
Because $X$ is weighted homogeneous with weights $(3, 2l + 1, l + 2)$, $X$ has a paramterization by $r(t) = (t^3, t^{2l+1}, t^{l+2})$. For our computations it will be convenient to work over $O_1$ by pulling back our modules by $r^*$.  

Pulling back the matrix of generators $M[N_D]$, we obtain:

$$r^*(M[N_D]) = \begin{bmatrix} t^{l+2} & -t^3 & 0 & -t^{2l+1} & t^{l+2} & 0 \\ 0 & t^{2l+1} & -t^{l+2} & 0 & -t^{3l} & t^{2l+1} \\ t^{2l+1} & 0 & -t^3 & -t^{3l} & 0 & t^{l+2} \end{bmatrix}.$$  

As this matrix has generic rank 2, $r^*(N_D)$ can be generated freely by 2 generators, so a matrix of generators $R_N$ of $r^*(N_D)$ with a minimal number of columns is

$$R_N = \begin{bmatrix} -t^3 \\ t^{2l+1} \\ 0 \end{bmatrix}.  $$

A calculation shows that $r^*(JM(X))$ is generated by the columns of:

$$R_{JM} = \begin{bmatrix} -t^3 & 2t^{l+2} \\ 2t^{2l+1} & -t^{3l} \\ t^{l+2} & t^{2l+1} \end{bmatrix}. $$

Note that

$$R_{JM} = R_N \begin{bmatrix} 1 & -2t^{l-1} \\ -t^{l-1} & -t^{2l-2} \end{bmatrix}. $$

Since $r^*(N_D)$ is freely generated, $e(r^*(JM(X)), r^*(N_D))$ is the colength of the submodule of $O_1^2$ generated by the columns of the $2 \times 2$ matrix in the last equation. This is the colength of the determinant of the matrix of generators which is $2l - 2$ as claimed. By the result of Kleiman mentioned above, $e(r^*(JM(X)), r^*(N_D)) = e(JM(X), N_D)$.  

By results of Watanabe et al [17], $\delta(X_l) = l$. Hence the Milnor number is $2\delta = 2l$, so $e(JM(X_l), N_D(X_l)) = \mu - 2$, and so $e(JM(X_l), N_D(X_l))$ plus the multiplicity over $Y$ of the polar curve of $N_D(X_l)$ is $\mu - 2 + m + 1 = \mu + m - 1$, which is the multiplicity over $Y$ of the relative polar curve of a smoothing of the family of space curves $X_l$ as predicted.

5. Consequences for equisingularity–the Whitney conditions, $W_f$, $A_l$ and $A_f$ conditions.

In this section we will apply the results from previous sections to provide criteria for various equisingularity conditions. We first review the conditions we will study, and recall some basic properties.

The conditions we will study are concerned with the relation between tangent planes plane to a stratum and limiting tangent hyperplanes from higher-dimensional strata, so we need a way of measuring distance between linear spaces. Suppose $A, B$ are linear subspaces at the origin in $\mathbb{C}^N$, then define the distance from $A$ to $B$ as:

$$\text{dist}(A, B) = \sup_{\|u\| = 1} \|u, v\|. \quad v \in B - \{0\}$$

$$\text{dist}(A, B) = \sup_{\|u\| = 1} \|u\| \|v\|. \quad v \in A - \{0\}$$
In the applications $B$ is the “big” space and $A$ the “small” space. (Note that $\text{dist}(A, B)$ is not in general the same as $\text{dist}(B, A)$.)

We recall Verdier’s condition $W$. Suppose $Y \subset X$, where $X, Y$ are strata in a stratification of an analytic space, and $\text{dist}(TY_0, TX_x) \leq C\text{dist}(x, Y)$ for all $x$ close to $Y$. Then the pair $(X, Y)$ satisfies Verdier’s condition $W$ at $0 \in Y$.

The Whitney conditions are also used to describe the incidence relation of two strata; however Teissier proved condition $W$ is equivalent to these two Whitney conditions in the complex analytic case [21], V.1.2, so we will use the two terms interchangeably.

The following result connects theory of integral closure and the condition $W$.

**Proposition 5.1.** ([6], Theorem 2.5) Suppose $(X^{d+k}, 0) \subset (\mathbb{C}^{n+k}, 0)$ is an equidimensional complex analytic set, $X = F^{-1}(0)$, $Y$ a smooth subset of $X$, $F : \mathbb{C}^{n+k} \to \mathbb{C}^p$ coordinates chosen so that $\mathbb{C}^k \times 0 = Y$, $m_Y = (z_1, \ldots, z_n)$ denoting the ideal defining $Y$, $F$ defines $X$ with reduced structure. Then, $\frac{\partial F}{\partial \nu} \in m_n\Gamma(M(F))$ for all tangent vectors $\frac{\partial}{\partial \nu}$ to $Y$ if and only if $W$ holds for $(X_0, Y)$.

**Proof.** See ([6], Theorem 2.5) \qed

If $f$ is the germ of a complex analytic mapping defined on the closure of a stratum $X$, then it is useful to have a notion of condition $W$ relative to $f$; this is obtained from the above definition of $W$ by replacing the tangent space of $X$ by the tangent space of the fiber of $f$ at points where $f$ is a submersion onto its image. The $W_f$ condition holds for the pair $(X, Y)$ when the new condition holds; when it holds with some unspecified exponent on the term $\text{dist}(x, Y)$, we say that Thom’s condition $A_f$ holds. (It can be shown using the theory of integral closure, that if every limiting tangent plane to the fibers of $f$ contains $TY_y$, along $Y$, then the distance condition holds with some exponent.)

If we have a maximal rank isolated determinantal singularity $X$, then the invariants of this section take the form $e(M, N_D) + \text{mult}_Y\Gamma(N_D)$, where $Y$ is the base of the smoothing defined by the matrix defining our singularity. In this first part, $M$ will be the Jacobian module of $X$ or $mJM(F)$. We denote the sum $e(M, N_D) + \text{mult}_Y\Gamma(N_D)$ by $e_f(M, N_D)$; by the results of the previous section, this is depends only on the presentation matrix of $X$, hence is independent of any family deforming $X$ induced from a deformation of the presentation matrix.

Here is our result for condition $W$. The result is a modification of theorem 5.7 of [9] to fit the context of this paper. Since in our situation we assume that $Y$ is in the cosupport of $JM(F)$, and $m_YJM(F) = JM(F)$ off $Y$, it follows that $e(m_yJM(F_y), N_X, (z, y)) = e(JM(F_y, N_X, (z, y)))$, $y \neq 0$.

**Proposition 5.2.** Suppose $(X^{d+k}, 0) \subset (\mathbb{C}^{n+k}, 0)$, $X = F^{-1}(0)$, $F : \mathbb{C}^{n+k} \to \mathbb{C}^p$, $Y$ a smooth subset of $X$, coordinates chosen so that $\mathbb{C}^k \times 0 = Y$, $F$ induced from a deformation of the presentation matrix of $X_0$, $X$ equidimensional with equidimensional fibers, of expected dimension.
A) Suppose $X_y$ are isolated, maximal rank determinantal singularities, suppose the singular set of $X$ is $Y$. Suppose $e_\Gamma(m_y JM(F_y), N_D(y))$ is independent of $y$. Then the union of the singular points of $X_y$ is $Y$, and the pair of strata $(X - Y, Y)$ satisfies condition $W$.

B) Suppose $\Sigma(X)$ is $Y$ and the pair $(X - Y, Y)$ satisfies condition $W$. Then $e_\Gamma(m_y JM(F_y), N_D(y))$ is independent of $y$.

Proof. Again note that $JM(F) \in H_0(JM_z(F))$.

Now we prove A). We can embed the family in a restricted versal unfolding with smooth base $\tilde{Y}$. Consider the polar variety of $JM_z(F)$ of dimension $l$, and the degree of its projection to $\tilde{Y}^l$ along points of $Y$. The hypothesis on $e_\Gamma$ implies by the multiplicity polar theorem that this degree is constant over $Y$. In turn this implies that the polar variety over $Y$ does not split, hence the polar of the original deformation is empty. This then implies that the PSID holds.

Since $W$ holds generically, by the PSID, it follows that

$$JM_Y(F) \subset m_Y JM_z(F).$$

This implies that $JM(F) \subset JM_z(F)$. Hence the union of the singular points of $F_y$ which is the cosupport of $JM_z(F)$ is equal to the cosupport of $JM(F)$ which is $Y$. Then the inclusion $JM_Y(F) \subset m_Y JM_z(F)$ implies $W$ for $(X - Y, Y)$. (Cf [6])

Now we prove B). $W$ implies $JM_Y(F) \subset m_Y JM_z(F)$ which implies that $m_Y JM(F) = m_Y JM_z(F)$. We know by [21] that condition $W$ implies that the fiber dimension of the exceptional divisor of $B_{m_Y}(C(X))$ over each point of $Y$ is as small as possible. The integral closure condition $m_Y JM(F) = m_Y JM_z(F)$ implies that the same is true for $B_{m_Y}(\text{Proj}(JM_z(F)))$. This implies that the polar of $m_Y JM_z(F)$ is empty, hence by the multiplicity polar formula the invariant $e_\Gamma(m_y JM(F_y), N_D(y))$ is independent of $y$.

There is a nice geometric interpretation of the number $e(mJM(F_y))$ which we now describe. We denote the multiplicity of the polar variety of $X_y, 0$ of dimension $i$ by $m^i(X_y)$.

**Theorem 5.3.** Suppose $N$ any submodule in $O^\mathbb{R}_{X_y,0}$ such that $\dim \mathbb{C} N/JM(F_y) < \infty$, then

$$e(mJM(F_y), N) = e(JM(F_y), N) + \sum_{i=1}^d \binom{n-1}{i} m^i(X_y).$$

**Proof.** This is exactly the content of the formula in Theorem 9.8 (i) p.221 [14].

**Corollary 5.4.** We have:

$$e_\Gamma(mJM(F_y), N_D(y)) = e_\Gamma(JM(F_y), N_D(y)) + \sum_{i=1}^d \binom{n-1}{i} m^i(X_y).$$

**Proof.** Simply add $\text{mult}_Y \Gamma_d(N_D)$ to both sides of the formula of the last theorem.
On earlier work on ICIS it was possible to drop the hypothesis $S(X) = Y$ in part A) of theorem 5.2. There since $m_d(X^d(y))$, was non-zero at any singular point, the independence of $m_d(X^d(y))$ from $y$ prevented the singular locus from splitting. In order to strengthen 5.2A, we would like to know when $e_1(JM(F_1), N_D)$ is non-zero. As a first step we prove a result about when $\text{multy}_{d} \Gamma_d(N_D)$ is non-zero.

**Proposition 5.5.** Assume the presentation matrix of $X^d \subset \mathbb{C}^d$ has size $(n + k, n)$, all of whose entries are zero at zero. Then the polar curve of $N_D$ is non-empty if and only if $d + k + 1 = q \leq 2n + k - 1$.

**Proof.** We assume $X$ is a smoothing of $X$. Then $X$ is itself an isolated determinantal singularity. The non-emptiness of the polar curve is equivalent to the dimension of the fiber of $X_M$ over the origin having maximal dimension $q - 1$. This is maximal because the dimension of $X_M$ is $q$. The fiber of $X_M$ over the origin is inside $\mathbb{P}^{n+k-1} \times \mathbb{P}^{n-1}$ Thus a necessary condition for the fiber over the origin to be of dimension $q - 1$ is for $q - 1 \leq (n + k - 1) + (n - 1) = 2n + k - 2$ or $q \leq 2n + k - 1$.

Assume $q \leq 2n + k - 1$. We must show the fiber over the origin has dimension $q - 1$.

Consider the set of equations of form $(T_1, \ldots, T_n)[M]^t = 0$ and of form $(S_1, \ldots, S_{n+k})[M] = 0$, where $(T_1, \ldots, T_n)$ (resp. $(S_1, \ldots, S_{n+k})$) are coordinates on $\mathbb{P}^{n-1}$ (resp. $\mathbb{P}^{n+k-1}$). Off $0 \times \mathbb{P}^{n+k-1} \times \mathbb{P}^{n-1}$, these are a set of defining equations for $X_M$. Denote by $V$ the zero set of these equations. Locally at points of $0 \times \mathbb{P}^{n+k-1} \times \mathbb{P}^{n-1}$, we can cut down on the number of equations needed. To illustrate the idea, suppose we are working at $(1,0,\ldots,0) \times (1,0,\ldots,0)$. The equations of the first type become $(1,\ldots,T_n)[M]^t = 0$. If the equations hold at a point then the first column of $[M]$ must be in the span of the other columns. So it suffices to use only $n - 1$ equations of the form $(S_1,\ldots,S_{n+k})[M] = 0$ to cut out the vanishing of the original set of equations. Locally then we use $(n + k) + (n - 1)$ equations to cut out $V$. This implies that each component of $V$ must have codimension at most $2n + k - 1$ or dimension at least $[(q + 1) + (n - 1) + (n + k - 1)] - (2n + k - 1) = q$. If $q - 1 = 2n + k - 2$, then although $0 \times \mathbb{P}^{n+k-1} \times \mathbb{P}^{n-1}$ lies in $V$, it cannot be a component of $V$, hence is in $X_M$, and the dimension of the fiber over $0$ of $X_M$ is maximal.

So we may suppose $q - 1 < 2n + k - 2$. In this case, since $X$ has an isolated singularity, $V$ has two components $0 \times \mathbb{P}^{n+k-1} \times \mathbb{P}^{n-1}$ and $X_M$. By Grothendieck’s connectedness theorem ([13], XIII2.1) it follows that the two components intersect in a set of dimension $q - 1$, which finishes the proof.

**Corollary 5.6.** Assume the presentation matrix of $X^d \subset \mathbb{C}^d$ has size $(n + k, n)$, all of whose entries are zero at zero. Then $m_d(X^d)$ is non-zero if $q \leq 2n + k - 1$.

**Proof.** If the polar of $N_D$ is non-empty, since $e(JM(X), N_D(0))$ is non-negative, and $m_d(X^d)$ is the sum of these two terms, then $m_d(X^d)$ is non-zero.

The next example shows the necessity for the hypothesis about the singular set of our family.
Damon-Pike [2] looked at invariants connected with isolated singularities in the case of $2 \times 3$ matrices. They viewed the matrices as maps $F$ from $\mathbb{C}^p \to \text{Hom}(\mathbb{C}^3, \mathbb{C}^2)$. Their techniques allowed them to calculate the reduced Euler characteristic $b_3 - b_2$ of the smoothings for many determinantal singularities of this type with $p = 5$. In particular, they showed that the reduced Euler characteristic was $-1$ for the smoothings of the singularities associated with the following maps:

$$F_k = \begin{bmatrix} w & y & x \\ z & w & y + v^k \end{bmatrix} \quad 1 \leq k \leq 5.$$ 

Denote the associated singularity by $X_k$. We take a generic linear section of $X_k$ by restricting $F_k$ to a generic hyperplane and considering the associated singularities of the restricted map. If we do this by taking $v = 0$ for example, we get a surface singularity of type associated with the map:

$$G = \begin{bmatrix} w & y & x \\ z & w & y \end{bmatrix} \quad 1 \leq k \leq 5.$$ 

The Tjurina number, $\tau$, of this is 2 ([4]); Pereira and Ruas ([19]) showed that for this type of singularity, the Milnor number of the smoothing, and the only non-zero Betti number appearing in the reduced Euler characteristic, satisfies $\tau = \mu + 1$. So here $\mu = 1$.

Since $m_3(X^d) = (-1)^d \chi(H_*(X_0)) + (-1)^{d-1} \chi(H_*(X \cap H)_0)$, we get $m_3(X_k) = b_3 - b_2 + \mu = 0$.

So for these singularities the relative Jacobian module of the smoothing has no polar curve.

Now consider the family of sets $X_t$ defined by the family of maps:

$$F_{(2,3)} = \begin{bmatrix} w & y & x \\ z & w & y + t v^3 \end{bmatrix}.$$ 

For each value of $t$, $m_3(X_t) = 0$. Further, all of these singularities have equivalent hyperplane slices; this immediately means that the multiplicity and the multiplicity of the codimension 1 polars of the $X_t$ are the same, as these multiplicities are invariant under slicing by generic hyperplanes. It also implies that the multiplicity of the family of polar curves of the $X_t$ is constant as well. We will recall this. Consider the union of the polar curves; when we intersect this set with a generic hyperplane which contains the parameter axis, we get the parameter axis and a curve. This curve is the relative polar curve of the total space of $X \cap H$. It is non-empty if and only the multiplicity of the union of curves changes with the parameter. In our situation, the relative polar curve of $X \cap H$ is empty because $X \cap H$ is a family of analytically equivalent surfaces. (More than we need.) Hence the multiplicity of the polar curves of $X_t$ are constant.

However, the singular locus of the total space is given by the zeros of the entries of $F_{(2,3)}$. These are $x = y = w = z = 0, v^2(t + v) = 0$, so the locus splits at $t = 0$. Thus $X - Y$ is not a stratum in this case, although $(X - S(X), Y)$ is a pair of strata which satisfy the Whitney conditions at the origin, as there is no polar curve, and the other polar multiplicities are constant. For this type of example, a necessary and sufficient condition to
prevent the splitting is to ask that the multiplicity of the ideal of entries of
the $F_t$ be independent of $t$.

In the examples, on the list of Damon and Pike, the ideal of entries is
generated by 5 elements; hence the above independence of parameter is
necessary and sufficient for no splitting of the singular set of a family.

We can use Corollary 5.6 to strengthen 5.2A when the hypotheses of 5.6
apply.

**Proposition 5.7.** Suppose in Theorem 5.2A, $d = 1$ or $d = 2$. If the invari-
ant $e_V(m_yJM(F_y), N_D(y))$ is independent of $y$, then $X - Y$ is smooth, and
the pair of strata $(X - Y, Y)$ satisfies condition $W$.

**Proof.** If $d = 2$, then for any $n$, a maximal rank determinantal singular-
ity $V$ with presentation matrix of size $(n, n + k)$ has the property that
$e_V(JM(V), N_D(V))$ is positive. Thus the independence of the invariant
$e_V(m_yJM(F_y), N_D(y))$ from parameter implies that splitting cannot oc-
cur.

We now turn to the two relative conditions $A_f$ and $W_f$.

The result for $A_f$ is a re-tuning of theorem 5.6 of [9] for our situation.

**Theorem 5.8.** Suppose $(X^{d+k}, 0) \subset (\mathbb{C}^{n+k}, 0)$, is a determinantal singu-
lar-ity with presentation matrix $M$.

$X = G^{-1}(0)$, $G : \mathbb{C}^{n+k} \to \mathbb{C}^p$, $Y$ a smooth subset of $X$, coordinates
chosen so that $\mathbb{C}^k \times 0 = Y$, $X$ equidimensional with equidimensional fibers
of the expected dimension, $X$ reduced.

Suppose $F : (X, Y) \to \mathbb{C}$, $F \in m^2_F$, $Z = F^{-1}(0)$.

A) Suppose $X_y$ and $Z_y$ are isolated singularities, suppose the singular set
of $F$ is $Y$. Suppose $e_V(JM(G_y; F_y), O_{n+k} \oplus N_D(y))$ is independent of $y$.
Then the union of the singular points of $F_y$ is $Y$, and the pair of strata
$(X - Y, Y)$ satisfies Thom’s $A_F$ condition.

B) Suppose $\Sigma(F)$ is $Y$ or is empty, and the pair $(X - Y, Y)$ satisfies
Thom’s $A_F$ condition. Then $e_V(JM(G_y; F_y), O_{n+k} \oplus N_D(y))$ is independent
of $y$.

**Proof.** The condition that $X_y$ and $Z_y$ are isolated singularities implies that
the integral closure of $JM(G_y; F_y)$ contains $O_{n+k} \oplus N_D(y)$ for all $y$, so the
multiplicity of the pair $(JM(G_y; F_y), O_{n+k} \oplus N_D(y))$ is well defined. For
the module $O_{n+k} \oplus N_D$, $\text{Projan } R(O_{n+k} \oplus N_D)$ is the join of a point with
$\text{Projan } R(N_D)$ in $\mathbb{P}^n$. This implies that the polar variety of $O_{n+k} \oplus N_D$ of
dimension $K$ is the same as the polar variety of dimension $k$ of $N_D$. With
this observation the proof of Theorem 5.6 in [9] goes through.

The hypothesis $F \in m^2_F$ is technical, and is used to ensure the components
of the relative conormal over $Y$ have maximal dimension. Without this
hypothesis it seems necessary to assume that $W_A$ holds for the pair $(X -
Y, Y)$.

The result and proof for $W_F$ is similar.

**Theorem 5.9.** Suppose $(X^{d+k}, 0) \subset (\mathbb{C}^{n+k}, 0)$, is a determinantal singu-
lar-ity with presentation matrix $M$. 
\[ X = G^{-1}(0), \quad G : \mathbb{C}^{n+k} \to \mathbb{C}^p, \quad Y \text{ a smooth subset of } X, \text{ coordinates chosen so that } \mathbb{C}^k \times 0 = Y, \text{ } X \text{ equidimensional with equidimensional fibers of the expected dimension, } X \text{ reduced.} \]

Suppose \( F : (X, Y) \to (\mathbb{C}, 0), \ Z = F^{-1}(0). \)

\( A) \) Suppose \( X_y \) and \( Z_y \) are isolated singularities, suppose the singular set of \( F \) is \( Y \). Suppose \( e_\Gamma (\mathfrak{m}_Y \mathcal{M}(G_y; F_y), \mathcal{O}_{n+k} \oplus N_D(y)) \) is independent of \( y \). Then the union of the singular points of \( F_y \) is \( Y \), and the pair of strata \( (X - Y, Y) \) satisfies the \( W_F \) condition.

\( B) \) Suppose \( \Sigma(F) \) is \( Y \) or is empty, and the pair \( (X - Y, Y) \) satisfies the \( W_F \) condition. Then \( e_\Gamma (\mathfrak{m}_Y \mathcal{M}(G_y; F_y), \mathcal{O}_{n+k} \oplus N_D(y)) \) is independent of \( y \).

\textbf{Proof.} Note that we can drop the hypothesis \( F \in \mathfrak{m}_Y^2 \). This is because the construction of \( \text{Projan} \mathcal{R}(\mathfrak{m}_Y \mathcal{M}_z (G; F)) \) involves blowing up by the pullback of \( \mathfrak{m}_Y \) to \( \text{Projan} \mathcal{R}(\mathcal{M}_z (G; F)) \); this has the effect of ensuring that all components of the inverse image of \( Y \) have maximal dimension. Then the proof is similar to that of Corollary 4.14 of \[11\], keeping in mind the remarks made in the proof of the last theorem. \( \square \)

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