A note: the relativistic transformations for the optical constants of media

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The Lorentz transformations for the optical constants (electric permittivity, magnetic permeability and index of refraction) of moving media are considered.

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1. Introduction

Some novel vacuum effects associated with the dynamical quantities (such as energy, spin and momentum) of zero-point fluctuation fields have been considered in the literature. These effects include the Casimir effect (related to the vacuum energy and mode distribution structure) \([1,2]\), magnetoelastic birefringences of the quantum vacuum \([3]\), vacuum-induced Berry’s phase of spinning particles (related to the quantum vacuum fluctuation) \([4]\) and quantum-vacuum geometric phase of zero-point field \([5,6]\).

As to the effect associated with the momentum of vacuum zero-point fields, more recently, Feigel has considered the quantum vacuum contribution to the momentum of electromagnetic media \([7]\). We think that the relativistic transformations for the optical constants (electric permittivity and magnetic permeability) of the medium should be taken into account when writing the Lagrangian of the moving electromagnetic system. It can be shown that the effect arising from such a transformation will also provide a quantum vacuum contribution to the velocity of media, in addition to the one derived by Feigel himself \([7]\). This Note presents the relativistic transformations for the optical constants of moving media.

2. The relativistic transformations for magnetic permeability and electric permittivity

The transformations of \(\mathbf{E}, \mathbf{B}, \mathbf{H}\) and \(\mathbf{D}\). Assume that an inertial frame of reference moves at velocity \(v\) relative to a rest system in which an electromagnetic medium (homogeneous and isotropic) is fixed. In the following, all the physical quantities with a prime refer to those in the moving frame while the ones without a prime refer to the quantities in the rest frame. According to SR, the relativistic transformation for the electric field strength \(\mathbf{E}\) and the magnetic induction \(\mathbf{B}\) is of the form

\[
\begin{align*}
\mathbf{E}' &= \gamma (\mathbf{E} + \mathbf{v} \times \mathbf{B}), \\
\mathbf{B}' &= \gamma (\mathbf{B} - \frac{\mathbf{v}}{c^2} \mathbf{v} \times \mathbf{E}),
\end{align*}
\]

and the relativistic transformation for the magnetic field strength \(\mathbf{H}\) and the electric displacement vector \(\mathbf{D}\) takes the following form

\[
\begin{align*}
\mathbf{H}' &= \gamma (\mathbf{H} - \mathbf{v} \times \mathbf{D}), \\
\mathbf{D}' &= \gamma (\mathbf{D} + \frac{\mathbf{v}}{c^2} \mathbf{v} \times \mathbf{H}),
\end{align*}
\]

where \(\gamma\) denotes the relativistic factor. These two sets of equations will be employed to derive the transformation relationships of the optical constants between different frames of reference.

The transformation of magnetic permeability. According to the relation \(\mu' \mathbf{H}' = \gamma (\mathbf{H} - \mathbf{v} \times \mathbf{D})\), one can obtain

\[\mu' \mathbf{H}' = \gamma \mu' (\mathbf{H} - \epsilon \mathbf{v} \times \mathbf{E}) .\]  (3)

In the meanwhile, the relation \(\mathbf{B}' = \gamma (\mathbf{B} - \frac{\mathbf{v}}{c^2} \mathbf{v} \times \mathbf{E})\) can be rewritten as

\[\mu' \mathbf{H}' = \gamma \left(\mu H - \frac{1}{c^2} \mathbf{v} \times \mathbf{E}\right) .\]  (4)

Thus, it follows from Eqs. (3) and (4) that

\[\mu' (\mathbf{H} - \epsilon \mathbf{v} \times \mathbf{E}) = \mu \mathbf{H} - \frac{1}{c^2} \mathbf{v} \times \mathbf{E}.\]  (5)

Here, for convenience and simplicity, we consider a time-harmonic electromagnetic wave propagating along the relative velocity \(v\) between the two inertial frames. The wave vector of such a wave is pointed opposite to the direction of \(v\). So, the set of vectors \((\mathbf{v}, \mathbf{H}, \mathbf{E})\) form a right-handed system. In this sense, the direction of the term \(-\epsilon \mathbf{v} \times \mathbf{E}\) in (5) is parallel to that of \(\mathbf{H}\). If \(v, H, E\) are the magnitudes of \(\mathbf{v}, \mathbf{H}, \mathbf{E}\), respectively, then the magnitude of the term \(-\epsilon \mathbf{v} \times \mathbf{E}\) is \(\epsilon v E\). Thus we have the relation

\[\mu' (H + \epsilon v E) = \mu H + \frac{1}{c^2} v E,\]  (6)

and consequently obtain

\[\mu' = \frac{\mu H + \frac{1}{c^2} v E}{H + \epsilon v E} .\]  (7)
By substituting the relation \( E = \sqrt{\frac{\mu}{\varepsilon}} H \) for the time-harmonic wave into Eq. (7), one can obtain the relativistic transformation of the magnetic permeability, i.e.,

\[
\mu' = \sqrt{\frac{\mu}{\varepsilon}} \left( \frac{\sqrt{\frac{\mu_r}{\mu_0}} + \frac{u}{c}}{1 + \sqrt{\frac{\mu_r}{\mu_0}} \frac{u}{c}} \right),
\]

or

\[
\mu' = \sqrt{\frac{\mu_r}{\varepsilon_r}} \left( \frac{\sqrt{\frac{\mu_r}{\mu_0}} + \frac{u}{c}}{1 + \sqrt{\frac{\mu_r}{\mu_0}} \frac{u}{c}} \right) = \sqrt{\frac{\mu_r}{\varepsilon_r}} \tanh \left( \sqrt{\frac{\mu_r}{\mu_0}} \frac{u}{c} \right),
\]

(8)

where \( \mu' = \mu', \mu_r = \mu_0 \), and \( \varepsilon_r = \varepsilon_0 \).

The transformation of electric permittivity

The relation \( D' = \gamma (D + \frac{1}{c^2} v \times H) \) can be rewritten as

\[
\epsilon' E' = \gamma \left( \epsilon E + \frac{1}{c^2} v \times H \right).
\]

According to the relation \( E' = \gamma (E + v \times B) \), one has

\[
\epsilon' E' = \gamma \epsilon' (E + \mu v \times H).
\]

Thus the following relation is derived

\[
\epsilon' (E + \mu v H) = \gamma \epsilon E + \frac{1}{c^2} v H
\]

(10)

which can be rewritten in the form

\[
\epsilon' (E + \mu v H) = \gamma \epsilon E + \frac{1}{c^2} v H
\]

under the condition that the set of vectors \((v, H, E)\) form a right-handed system. So, the permittivity observed in the moving frame is

\[
\epsilon' = \frac{\gamma \epsilon E + \frac{1}{c^2} v H}{E + \mu v H}.
\]

Insertion of the relation \( E = \sqrt{\frac{\mu}{\varepsilon}} H \) into Eq. (14) yields

\[
\epsilon' = \sqrt{\frac{\varepsilon}{\mu}} \left( \frac{\sqrt{\frac{\mu}{\varepsilon}} + \frac{u}{c}}{1 + \sqrt{\frac{\mu}{\varepsilon}} \frac{u}{c}} \right),
\]

or

\[
\epsilon' = \sqrt{\frac{\varepsilon_r}{\mu_r}} \left( \frac{\sqrt{\frac{\mu_r}{\mu_0}} + \frac{u}{c}}{1 + \sqrt{\frac{\mu_r}{\mu_0}} \frac{u}{c}} \right) = \sqrt{\frac{\varepsilon_r}{\mu_r}} \tanh \left( \sqrt{\frac{\mu_r}{\mu_0}} \frac{u}{c} \right),
\]

(15)

where \( \epsilon'_r (\equiv \epsilon'_r) \) denotes the relative permittivity of the medium observed in the moving frame.

It should be noted that here the relativistic transformations for the permeability and permittivity refer only to those along the directions perpendicular to the wave vector (or \( v \)). The components of the permeability and permittivity along the wave vector (or \( v \)) does not alter under the Lorentz transformation.

3. Discussions

The trivial cases

It follows from Eqs. (9) and (16) that when the relative velocity \( u \) between the two frames vanishes, the permeability and permittivity

\[
\mu_r' = \mu_r, \quad \epsilon_r' = \epsilon_r.
\]

(16)

It is a trivial case deserving no consideration. In addition, for a vacuum medium in which the relative permeability and permittivity equal the unity, i.e., \( \mu_r = \epsilon_r = 1 \), Eqs. (9) and (16) give

\[
\mu'_r = 1, \quad \epsilon'_r = 1.
\]

(17)

This means that both the permeability and the permittivity of the vacuum medium are invariant under the Lorentz transformation (which is in connection with the principle of invariance of light speed). In this sense, vacuum can be considered a Lorentz-invariance medium, all the optical constants of which measured by different observers who move at relative velocities with respect to each other are correspondingly the same.

Optical refractive index

The optical refractive index of the medium observed from the moving frame and the rest frame are expressed by

\[
n' = \sqrt{\epsilon' \mu'} = \sqrt{\epsilon_n \mu_n} = \sqrt{\gamma \epsilon \mu} = \sqrt{\gamma \epsilon \mu} = \sqrt{\epsilon \mu},
\]

(18)

where \( \gamma = \frac{c}{\sqrt{1 - \frac{u^2}{c^2}}} \) is the velocity factor. According to Eqs. (9) and (16), the Lorentz transformation for the optical refractive index is

\[
n' = \frac{n + \frac{u}{c}}{1 + n \frac{u}{c}} = \tanh \left( n + \frac{u}{c} \right).
\]

(19)

Note that here the transformation for the refractive index refers only to that of the directions perpendicular to the wave vector (or \( v \)).

Fizeau’s experiment

The phase velocity of a light in a moving medium (or seen from a moving frame) is defined as \( u'_p = \frac{c}{n} \). It follows from the expression (19) that the phase velocity \( u'_p \) is

\[
u_p' = \frac{c}{n} \left( \frac{1 + \frac{nu}{c}}{1 + \frac{nu}{c}} \right).
\]

(20)

It can be expanded up to the second order in \( \frac{u}{c} \), and the result is

\[
u_p' = \frac{c}{n} \left( 1 - \frac{1}{n^2} \right) v - \frac{c}{n} \frac{v^2}{c^2} + O \left( \frac{(v/c)^3}{c} \right).
\]

(21)
It was shown in Fizeau’s experiment that the speed of light in a moving medium is \( u_p = \frac{x}{n} + (1 - \frac{n^2}{c^2}) v \). So, Fizeau’s experimental result is just an effect of the relativistic transformation of the optical refractive index of the moving medium.

**Superposition principle of phase velocity** The expression (20) for the phase velocity \( u'_p \) can be rewritten

\[
u'_p = \frac{u_p + v}{1 + \frac{v}{c}} = c \tan \left( \frac{u_p + v}{c} \right)
\]

(22)

with \( u_p = \frac{x}{n} \). This, therefore, means that the phase velocity of light also agrees with the traditional superposition principle of velocity in SR.

The Lorentz invariance of characteristic impedance of media It follows from the expressions (8) and (15) that

\[
\sqrt{\frac{\mu'}{\epsilon}} = \sqrt{\frac{\mu}{\epsilon}}.
\]

(23)

This, therefore, implies that the characteristic impedance of the medium is a Lorentz scalar, and that the ratio of \( E \) to \( H \) for a time-harmonic wave is an invariant.

4. **Doppler’s effect in a moving medium**

Suppose we have a regular anisotropic medium with the rest refractive index tensor \( n \), which is moving relative to an inertial frame \( K \) with speed \( v \) in the arbitrary directions. The rest refractive index tensor \( n \) can be written as \( n = \text{diag} [n_1, n_2, n_3] \) with \( n_i > 0 \), \( i = 1, 2, 3 \). The wave vector of the electromagnetic wave under consideration and the relative velocity \( v \) between the two frames are chosen to be antiparallel (seen from the inertial frame \( K \)). Thus the wave vector of a propagating wave with the frequency \( \omega \) reads \( k = \frac{-\omega}{c} (n_1, n_2, n_3) \) measured by the observer fixed at this medium. In this sense, one can define a 3-D vector \( n = (n_1, n_2, n_3) \), and the wave vector \( k \) may be rewritten as \( k = -n \frac{\dot{x}}{c} \). Now we analyze the phase \( \omega t - k \cdot x \) of the above time-harmonic electromagnetic wave under the following Lorentz transformation

\[
x' = \gamma (x - vt), \quad t' = \gamma \left( t - \frac{v \cdot x}{c^2} \right),
\]

(24)

where \((x', t')\) and \((x, t)\) respectively denote the spacetime coordinates of the initial frame \( K \) and the system of moving medium, the spatial origins of which coincide when \( t = t' = 0 \). Here the relativistic factor \( \gamma = \left( 1 - \frac{v^2}{c^2} \right)^{-\frac{1}{2}} \).

Thus by using the transformation (23), the phase \( \omega t - k \cdot x \) observed inside the moving medium may be rewritten as the following form by using the spacetime coordinates of \( K \)

\[
\omega t - k \cdot x = \gamma \omega \left( 1 + \frac{n \cdot v}{c} \right) t' - \gamma \omega \left( -\frac{n}{c} - \frac{v}{c^2} \right) \cdot x',
\]

(25)

the term on the right-handed side of which is just the expression for the wave phase in the initial frame \( K \). Hence, the frequency \( \omega' \) and the wave vector \( k' \) of the observed wave we measure in the initial frame \( K \) are given

\[
\omega' = \gamma \omega \left( 1 + \frac{n \cdot v}{c} \right), \quad k' = \gamma \omega \left( -\frac{n}{c} - \frac{v}{c^2} \right),
\]

(26)

respectively.

To gain some insight into the meanings of the expression (25), let us consider the special case of a boost (of the Lorentz transformation (23)) in the \( \dot{x}_1 \)-direction, in which the medium velocity relative to \( K \) along the positive \( \dot{x}_1 \)-direction is \( v \). In the meanwhile, we assume that the wave vector of the electromagnetic wave is also parallel to the negative \( \dot{x}_1 \)-direction. The modulus of wave vector seen from \( K \) is

\[
k' = \gamma \omega \left( \frac{n}{c} + \frac{v}{c^2} \right).
\]

(27)

If the rest refractive index of the medium in the \( \dot{x}_1 \)-direction is \( n \), its (moving) refractive index in the same direction measured by the observer fixed at the initial frame \( K \) is of the form

\[
n' = \frac{ck'}{\omega'} = n + \frac{v}{1 + \frac{n}{c}}
\]

(28)

which is a relativistic formula for the addition of “refractive indices” (\( \frac{v}{c} \) provides an effective index of refraction). Obviously, Eq. (27) is in agreement with Eq. (19).

Eq. (25) shows that the mathematical expression for Doppler’s effect in a moving medium is

\[
\omega' = \frac{1 + \frac{n}{c} \frac{v}{c^2} \omega}{\sqrt{1 - \frac{v^2}{c^2}}}.
\]

(29)

For the vacuum medium \((n = 1)\), Eq. (28) is reduced to the one familiar to us, i.e.,

\[
\omega' = \frac{c + v}{c - v} \omega,
\]

(30)

which is an expression for the longitudinal Doppler’s effect.

Recently, a kind of artificial media called left-handed materials which possess a negative index of refraction attract attention of many investigators [8]. Consider a left-handed medium, the index of refraction of which is \( n = -1 \). In this case, Eq. (28) is \( \omega' = \sqrt{\frac{c}{c-v}} \omega \). Compared with (29), the effect here is shown to be the reversal of Doppler’s shift. Similarly, the reversal of Cerenkov radiation will also arise in such a negative refractive index material.

5. **On the quantum vacuum contribution to the momentum of media**

The linear dispersion relation in a moving electromagnetic medium According to the transformation relationships \( \mathbf{E}' = \gamma (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \) and \( \mathbf{D}' = \gamma (\mathbf{D} + \frac{1}{c} \mathbf{v} \times \mathbf{H}) \),
we can obtain $\epsilon'(E + v \times B) = (D + \frac{1}{\epsilon'} v \times H)$. Thus we have

$$D' = \epsilon' E + \left(\frac{\mu'}{\epsilon' \mu} - \frac{1}{\epsilon'^2}\right) v \times B. \quad (30)$$

In the same fashion, from $H' = \gamma(H - v \times D)$ and $B' = \gamma(B - \frac{1}{\epsilon'} v \times E)$, we have $\mu'(H - v \times D) = B - \frac{1}{\epsilon'} v \times E$, and in consequence obtain

$$B = \mu'H + \left(\epsilon \mu' - \frac{1}{\epsilon} \right) E \times v. \quad (31)$$

Note that here Eqs. (30) and (31) are not the same to the following dispersion relations

$$\begin{align*}
D &= \epsilon E + \left(\frac{\mu}{\epsilon^2} - \frac{1}{\epsilon^2}\right) v \times B, \\
B &= \mu H + \left(\epsilon \mu - \frac{1}{\epsilon}\right) E \times v 
\end{align*} \quad (32)$$

derived by Feigel [7]. In Eqs. (32), the relativistic transformation is, however, not taken into account for $\epsilon$ and $\mu$.

**Up to the first order in $\frac{v}{c}$ for the permittivity and permeability of moving medium** In order to treat the quantum vacuum contribution to the momentum of media, Feigel considered the first-order approximation (in $\frac{v}{c}$) of the Lagrangian of the moving electromagnetic system, but did not consider the first-order (in $\frac{v}{c}$) contribution of $\epsilon$ and $\mu$. The following expressions

$$\begin{align*}
\epsilon' &= \epsilon + \sqrt{\frac{\mu}{\epsilon}} (1 - \epsilon \mu) \frac{v}{c} + O\left(\frac{v^2}{c^2}\right), \\
\mu' &= \mu + \sqrt{\frac{\epsilon}{\mu}} (1 - \epsilon \mu) \frac{v}{c} + O\left(\frac{v^2}{c^2}\right) 
\end{align*} \quad (33)$$

show that the first-order approximation of the optical constants may also contribute to the momentum of the medium and deserve consideration.

**Lagrangian of moving electromagnetic system** Here, for the convenient comparison with the result obtained by Feigel [7], we adopt the unit system used in Feigel’s paper. So, the relativistic transformations of $\epsilon$ and $\mu$ take the form

$$\begin{align*}
\epsilon' &= \sqrt{\frac{\epsilon}{\mu}} \left(\frac{v}{c} \sqrt{\frac{\epsilon}{\mu}} - \frac{\mu}{\epsilon}\right), \\
\mu' &= \sqrt{\frac{\mu}{\epsilon}} \left(\frac{v}{c} \sqrt{\frac{\mu}{\epsilon}} - \frac{\epsilon}{\mu}\right) 
\end{align*} \quad (34)$$

In the case of magnetoelectrics, as stated by Feigel, a term $\frac{1}{\mu} B \cdot \hat{\chi}^T E$ must be added to the Lagrangian of the moving electromagnetic system [7]. In a moving magnetoelectric medium, such a term can be rewritten as

$$\begin{align*}
\frac{1}{\mu} B \cdot \hat{\chi}^T E \rightarrow \\
\frac{1}{\mu} B \cdot \hat{\chi}^T E + \frac{1}{\mu c} [B \cdot \hat{\chi}^T (v \times B) + (E \times v) \cdot \hat{\chi}^T E] \\
+ \frac{1}{\mu c} v \left(\sqrt{\frac{\epsilon}{\mu}} - \frac{1}{\sqrt{\epsilon}}\right) B \cdot \hat{\chi}^T E, \quad (35)
\end{align*}$$

Note that compared with the result derived by Feigel, the expression $\frac{1}{\mu c} v \left(\sqrt{\frac{\epsilon}{\mu}} - \frac{1}{\sqrt{\epsilon}}\right) B \cdot \hat{\chi}^T E$ in (35) is a new term, which arises from the relativistic transformation of $\mu$. By using the relations $B \cdot \hat{\chi}^T (v \times B) = (v \times B) \cdot \hat{\chi} B = v \cdot (B \times (\hat{\chi} B))$ and $(E \times v) \cdot \hat{\chi}^T E = \hat{\chi}^T E \cdot (E \times v) = -v \cdot (E \times (\hat{\chi} E))$, one can rewrite the term $B \cdot \hat{\chi}^T (v \times B) + (E \times v) \cdot \hat{\chi}^T E$ in Eq. (35) as $v \cdot [(B \times (\hat{\chi} B)) - v \cdot (E \times (\hat{\chi} E))]$ [7]. Thus, assuming that the velocity $v$ of the medium is along the $z$-direction, i.e., $v = vz$ with $\hat{z}$ being a unit vector, the Lagrangian of the moving medium is thus of the form

$$L_{ME} = L_{FM} + \int \frac{d^3x}{4\pi} \frac{1}{\mu} B \cdot \hat{\chi}^T E$$

$$+ \frac{1}{\mu c} \int \frac{d^3x}{4\pi} v \cdot \left[\{B \times (\hat{\chi} B)] - [E \times (\hat{\chi} E)]\right]$$

$$+ \frac{1}{\mu c} \int \frac{d^3x}{4\pi} v \cdot \hat{z} \left(\sqrt{\frac{\epsilon}{\mu}} - \frac{1}{\sqrt{\epsilon}}\right) B \cdot \hat{\chi}^T E. \quad (37)$$

Compared with Eq. (19) in Feigel’s paper [7], the final expression on the right-handed side of Eq. (37) in this Note is a new term. According to the Lagrangian equation (Liquid’s equation) of motion [7], one can arrive at the following equation

$$\rho v^2 (z) = \frac{1}{4\pi \mu c} [(\epsilon \mu - 1) E \times B + E \times (\hat{\chi}^T E) - B \times (\hat{\chi} B)]$$

$$- \frac{1}{4\pi \mu c} \left(\sqrt{\epsilon \mu} - \frac{1}{\sqrt{\epsilon \mu}}\right) B \cdot \hat{\chi} E z. \quad (38)$$

Note that the final expression on the right-handed side of Eq. (38) is a new quantum vacuum contribution to the momentum of media, which has not yet been taken into consideration in Feigel’s work [7].

[1] H.B.G. Casimir, Proc. K. Ned. Akad. Wet. 51, 793 (1948).
[2] S.K. Lamoreaux, Phys. Rev. Lett. 78, 5 (1997).
[3] G.L.J.A. Rikken and C. Rizzo, Phys. Rev. A 63, 012107 (2000).
[4] I. Fuentes-Guridi, A. Carollo, S. Bose, and V. Vedral, Phys. Rev. Lett. 89, 220404 (2002).
[5] J.Q. Shen and L.H. Ma, Phys. Lett. A 308, 355 (2003).
[6] J.Q. Shen, J. Opt. B: Quantum Semiclass. Opt. 6, L13 (2004).
[7] A. Feigel, Phys. Rev. Lett. 92, 020404 (2004).
[8] J.B. Pendry, A.J. Holden, D.J. Robbins, and W.J. Stewart, IEEE Trans. Microwave Theory Tech. 47, 2075 (1999) and references therein.