Extra Skolem Difference Mean Labeling of Various Graphs

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ABSTRACT

Let graph $G=(V(G),E(G))$ attains a Skolem difference mean labeling with $p$ vertices and $q$ edges is said to be an extra Skolem difference mean labeling of graph $G$ if all the labels of the vertices are odd. The graph which attains an extra Skolem difference mean labeling is called an extra Skolem difference mean graph. We obtain an extra Skolem difference mean labeling for Comb graph, Twig of a path $P_n$, $H$ graph of a path $P_n$, $K_{1,2}^*(1,n)$ graph, $K_{1,3}^*(1,n)$ graph, $m$- Join of $H_n$, $P_n\odot K_{(1,m)}$ graph, $HSS(P_n)$ graph, $H\odot [mK_1]^1$-graph of a path $P_n$.

KEYWORDS

Extra Skolem difference mean labeling, Comb graph, Twig of a path $P_n$, $K_{1,2}^*(1,n)$ graph, $K_{1,3}^*(1,n)$ graph

1. Introduction

We consider finite, connected and undirected graph. We consider graph $G$ having set of vertices $V(G)$ and set of edges $E(G)$. An excellence reference on this subject is the survey by J. A. Gallian [4]. Skolem difference mean labeling was introduced by K. Murugan and A. Subramanian in [6]. Selvi, Ramya and Jeyanthi [7] define an extra Skolem difference mean labeling of graphs. We refer Gross and Yellen [5], for all kinds of definitions and notations.

Definition: Let $G = (V(G),E(G))$ be a graph with $p$ vertices and $q$ edges. An injective mapping $f: V \rightarrow \{1,2,3,...,p+q\}$ is called a Skolem difference mean labeling if which includes a bijective edge labeling $f^*: E \rightarrow \{1,2,3,...,q\}$ defined by $f^*(e = uv) = \frac{|f(u)−f(v)|+1}{2}$, if $|f(u)−f(v)|$ is even otherwise $\frac{|f(u)−f(v)|+1}{2}$, if $|f(u)−f(v)|$ is odd and the graph is called a Skolem difference mean graph.[1]

Definition: - Let graph $G = (V(G),E(G))$ attains a Skolem difference mean labeling with $p$ vertices and $q$ edges is said to be an extra Skolem difference mean labeling of graph $G$ if all the labels of the vertices are odd. The graph which attains an extra Skolem difference mean labeling is called an extra Skolem difference mean graph.[2]

SOME EXISTING RESULTS:

- The graph $T < K_{1,n_{1}}; K_{1,n_{2}}; ... K_{1,n_{m}} >$ is an extra Skolem difference mean graph.[7]
- Path is an extra Skolem difference mean graph.[7]
- The graph $T < K_{1,n}; K_{1,n}; ... K_{1,n} >$ is an extra Skolem difference mean graph.[7].
- The graph $T < K_{1,n_{1}}; K_{1,n_{2}}; ... K_{1,n_{m−1}}; K_{1,n_{m}} >$ is an extra Skolem difference mean graph.[7]
- $T_{p}$—tree is an extra Skolem difference mean graph. [2]
- $<T\odot K_{1,n}>$ is an extra Skolem difference mean graph. [2]
- The Caterpillar graph is an extra Skolem difference mean graph. [2]
- The graph $S_{m,n}$ is an extra Skolem difference mean graph.[2]
• Star is an extra Skolem difference mean graph. [3]
• \( B(m,n) \) is an extra Skolem difference mean graph. [3]
• \( B(m,n,k) \) is an extra Skolem difference mean graph. [3]
• Coconut tree \( CT(m,n) \) is an extra Skolem difference mean graph. [3]
• \( F - \) tree \( FP_n \) is an extra Skolem difference mean graph. [3]
• \( Y - \) tree is an extra Skolem difference mean graph. [3]

**Definition:** The **Comb** \( P_n \odot K_1 \) is the graph obtained from a path \( P_n \) by attaching a pendant edge to each vertex of the path. [1]

**Definition:** The graph obtained from a path \( P_n \) by attaching exactly two pendant edges to each internal vertex of the path is called the **Twig** graph. [11]

**Definition:** The **H graph** of path \( P_n \) is obtained from two copies of \( P_n \) with vertices \( u_1, u_2, ..., u_n \) & \( v_1, v_2, ..., v_n \) by joining the vertices \( u_{\frac{n}{2}+1} \) & \( v_{\frac{n}{2}+1} \) by an edge if \( n \) is odd and the vertices \( u_{\frac{n}{2}+1} \) & \( v_{\frac{n}{2}} \) if \( n \) is even. [10]

**Definition:** \( K_{1,2} \ast K_{1,n} \) is the graph obtained from \( K_{1,2} \) by attaching root of a star \( K_{1,n} \) at each pendant vertex of \( K_{1,2} \).

**Definition:** \( K_{1,3} \ast K_{1,n} \) is the graph obtained from \( K_{1,3} \) by attaching root of a star \( K_{1,n} \) at each pendant vertex of \( K_{1,3} \). [1]

**Definition:** The **corona** \( G_1 \odot G_2 \) of two graphs \( G_1 \) & \( G_2 \) is defined as the graph \( G \) obtained by taking one copy of \( G_1 \) (which has \( p_1 \) vertices) and \( p_1 \) copies of \( G_2 \) and joining the \( i^{th} \) vertex of \( G_1 \) to every vertex in the \( i^{th} \) copy of \( G_2 \). [10]

**Definition:** \( m - \) **Joins of H graph** is a graph where each of \( H \) graph denoted by \( H_{n_1} \) by an edge \( e_1 \) with \( H \) graph denoted by \( H_{n_2} \), \( H \) graph denoted by \( H_{n_3} \) by an edge \( e_2 \) with \( H \) graph denoted by \( H_{n_3} \) and so on with \( H \) graph denoted by \( H_{n_m-1} \) by an edge \( e_{m-1} \) with \( H \) graph denoted by \( H_{n_m} \) such that \( n_1 = n_2 = n_m \). [9]

**Definition:** Let \( G \) be a graph. A graph obtained from \( G \) by replacing each edge \( e_i \) by a \( H \) graph in such a way that the ends of \( e_i \) are merged with a pendant vertex in \( P_2 \) and pendant vertex in \( P'_{2} \) is called **H super subdivision of G** is denoted by \( HSS(G) \), where the \( H \) graph is a tree on 6 vertices in which exactly two vertices of degree 3. [8]

### 2. Main Results

**Theorem 2.1.1** The Comb \( P_n \odot K_1 \) graph is an extra Skolem difference mean graph.

**Proof:** Let \( G = P_n \odot K_1 \) with \( V(G) = \{u_k, v_k : 1 \leq k \leq n\} \) and

\[ E(G) = \{(u_k u_{k+1}) : 1 \leq k \leq n - 1\} \cup \{(u_k v_k) : 1 \leq k \leq n\}. \]

\[ \therefore |V(G)| = 2n \text{ and } |E(G)| = 2n - 1. \]

Define \( f : V(G) \rightarrow \{1,2,3, ...,4n - 1\} \),

\[ f(u_{2k-1}) = 4n + 3 - 4k \quad 1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor \]

\[ f(u_{2k}) = 4k - 1 \quad 1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor \]

\[ f(v_{2k-1}) = 4k - 3 \quad 1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor \]

\[ f(v_{2k}) = 4n + 1 - 4k \quad 1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor \]

We define edge function \( f^* : E(G) \rightarrow \{1,2,3, ...,2n - 1\} \) as follows.

\[ f^*(u_k u_{k+1}) = 2n - 2k \quad 1 \leq k \leq n - 1 \]

\[ f^*(u_k v_k) = 2n + 1 - 2k \quad 1 \leq k \leq n \]

Which is bijective function. Hence Comb \( P_n \odot K_1 \) is an extra Skolem difference mean graph.

**Illustration:** An extra Skolem difference mean labeling of \( P_5 \odot K_1 \) is shown in Figure-1.
**Theorem 2.1.2** twig obtained from path $TW(P_n)$ is an extra Skolem difference mean graph $\forall n \geq 3$.

**Proof:** Let $G = TW(P_n)$

$V(G) = \{u_k : 1 \leq k \leq n\} \cup \{v_k, w_k : 1 \leq k \leq n-2\}$

$E(G) = \{(u_k u_{k+1}) : 1 \leq k \leq n-1\} \cup \{(u_{k+1} v_k) : 1 \leq k \leq n-2\} \cup \{(u_{k+1} w_k) : 1 \leq k \leq n-2\}$

So, $|V(G)| = 3n - 4$ & $|E(G)| = 3n - 5$.

Define $f: V(G) \to \{1, 2, 3, \ldots, 6n - 9\}$,

**Case 1** $n$ is even

$f(u_{2k-1}) = 2k - 1 \quad 1 \leq k \leq \frac{n}{2}$

$f(u_{2k}) = 6n - 7 - 2k \quad 1 \leq k \leq \frac{n}{2}$

$f(v_{n-2k-1}) = 5n - 7 - 2k \quad 1 \leq k \leq \frac{n-2}{2}$

$f(v_{n-2k}) = n - 1 + 2k \quad 1 \leq k \leq \frac{n-2}{2}$

$f(w_{2k-1}) = 2n - 3 + 2k \quad 1 \leq k \leq \frac{n-2}{2}$

$f(w_{2k}) = 4n - 5 - 2k \quad 1 \leq k \leq \frac{n-2}{2}$

**Case 2** $n$ is odd

$f(u_{2k-1}) = 2k - 1 \quad 1 \leq k \leq \frac{n+1}{2}$

$f(u_{2k}) = 6n - 7 - 2k \quad 1 \leq k \leq \frac{n-1}{2}$

$f(v_{n-2k}) = 5n - 6 - 2k \quad 1 \leq k \leq \frac{n-1}{2}$

$f(v_{n-2k-1}) = n + 2k \quad 1 \leq k \leq \frac{n-3}{2}$

$f(w_{2k-1}) = 2n - 3 + 2k \quad 1 \leq k \leq \frac{n-1}{2}$

$f(w_{2k}) = 4n - 5 - 2k \quad 1 \leq k \leq \frac{n-3}{2}$

For both cases we define following edge function, $f^*: E(G) \to \{1, 2, 3, \ldots, 3n - 5\}$ as follows.

$f^*(u_ku_{k+1}) = 3n - 4 - k \quad 1 \leq k \leq n - 1$

$f^*(u_{k+1}v_k) = n - 1 - k \quad 1 \leq k \leq n - 2$

$f^*(u_{k+1}w_k) = 2n - 3 - k \quad 1 \leq k \leq n - 2$

Which is bijective function. So, $TW(P_n)$ is an extra Skolem difference Mean graph.

**Illustration:** An extra Skolem difference Mean labeling of $TW(P_5)$ Figure-2.
Theorem 2.1.3 \( H \) – graph of a path \( P_n \) is an extra Skolem difference mean graph.

Proof: Let \( G \) be a \( H \) graph of path \( P_n \), let \( P_n \) be the path \( u_1, u_2, ..., u_n \). We can obtain \( H \) – graph by considering two copies of \( P_n \).

Hence we have \( V(G) = \{ u_k, v_k: 1 \leq k \leq n \} \) and

\[
E(G) = \{(u_k, u_{k+1}): 1 \leq k \leq n - 1\} \cup \{(v_k, v_{k+1}): 1 \leq k \leq n - 1\} : n \text{ is odd}\) OR

\[
E(G) = \{(u_k, u_{k+1}): 1 \leq k \leq n - 1\} \cup \{(v_k, v_{k+1}): 1 \leq k \leq n - 1\} : n \text{ is even}.\)

\[\therefore |V(G)| = 2n \text{ & } |E(G)| = 2n - 1.\]

Define \( f: V(G) \to \{1, 2, 3, ... , 4n - 1\} \).

Case: 1 If \( n \) is odd

\[
f(u_{2k-1}) = 2k - 1 \quad 1 \leq k \leq \frac{n+1}{2}
\]
\[
f(u_{2k}) = 4n + 1 - 2k \quad 1 \leq k \leq \frac{n-1}{2}
\]
\[
f(v_{2k-1}) = 3n + 2 - 2k \quad 1 \leq k \leq \frac{n+1}{2}
\]
\[
f(v_{2k}) = n + 2k \quad 1 \leq k \leq \frac{n-1}{2}
\]

Case: 2 If \( n \) is even

\[
f(u_{2k-1}) = 2k - 1 \quad 1 \leq k \leq \frac{n}{2}
\]
\[
f(u_{2k}) = 4n + 1 - 2k \quad 1 \leq k \leq \frac{n}{2}
\]
\[
f(v_{2k-1}) = n - 1 + 2k \quad 1 \leq k \leq \frac{n}{2}
\]
\[
f(v_{2k}) = 3n + 1 - 2k \quad 1 \leq k \leq \frac{n}{2}
\]

We define following edge function \( f^*: E(G) \to \{1, 2, 3, ..., 2n - 1\} \) as follows.

\[
f^*(u_{k+1}u_k) = 2n - k \quad 1 \leq k \leq n - 1
\]
\[
f^*(u_{k}v_k) = n - k \quad 1 \leq k \leq n
\]
\[
f^*(\frac{u_{k+1}v_{k+1}}{2}) = n, \text{ n is odd}
\]
\[
f^*(\frac{u_{k+1}v_{k}}{2}) = n, \text{ n is even}
\]

Which is bijective function.

Hence \( H \) – graph of a path \( P_n \) graph is an extra Skolem difference mean graph.

Illustration: An extra Skolem difference mean labeling of \( H \) graph of a path \( P_5 \) Figure-3.
Theorem 2.1.4 The graph $K_{1,2} * K_{1,n}$ is an extra Skolem difference mean graph for all $n \geq 2$.

Proof: Let $G = K_{1,2} * K_{1,n}$ with $V(G) = \{u, v, w, u_k, v_k : 1 \leq k \leq n\}$

and $E(G) = \{uw, vw, uu_k, vv_k : 1 \leq k \leq n\}$.

Hence $|V(G)| = 2n + 3$ and $|E(G)| = 2n + 2$

Define $f : V(G) \to \{1, 2, 3, \ldots, 4n + 5\}$ by

$f(u) = 1$
$f(v) = 3$
$f(w) = 2n + 5$
$f(u_k) = 2n + 5 + 2k \quad 1 \leq k \leq n$
$f(v_k) = 3 + 2k \quad 1 \leq k \leq n$

We define following edge function $f^* : E(G) \to \{1, 2, 3, \ldots, 2n\}$ as follows.

$f^*(uw) = n + 2 + k \quad 1 \leq k \leq n$
$f^*(vw) = n + 1$
$f^*(uw) = n + 2$

Thus the induced edge labels are distinct from $1, 2, 3, \ldots, 2n + 2$. Hence the graph $K_{1,2} * K_{1,n}$ is an extra Skolem difference mean graph.

Illustration: An extra Skolem difference mean labeling of $K_{1,2} * K_{1,5}$ is shown in Figure-4.
**Theorem 2.1.5** The graph $K_{1,3} * K_{1,n}$ is an extra Skolem difference mean graph for all $n \geq 2$.

**Proof:** Let $G = K_{1,3} * K_{1,n}$ with $V(G) = \{x, u, v, w, u_k, v_k, w_k : 1 \leq k \leq n\}$ and $E(G) = \{ux, xv, xw, uu_k, vv_k, ww_k : 1 \leq k \leq n\}$.

Hence $|V(G)| = 3n + 4$ and $|E(G)| = 3n + 3$.

Define $f : V(G) \rightarrow \{1, 2, 3, \ldots, 6n + 7\}$ by

$f(u) = 1$
$f(v) = 3$
$f(w) = 2n + 5$
$f(x) = 4n + 7$
$f(u_k) = 6n + 9 - 2k \quad 1 \leq k \leq n$
$f(v_k) = 4n + 7 - 2k \quad 1 \leq k \leq n$
$f(w_k) = 2n + 5 - 2k \quad 1 \leq k \leq n$.

We define following edge function $f^* : E(G) \rightarrow \{1, 2, 3, \ldots, 3n + 3\}$ by

$f^*(uw_k) = 3n - k + 4 \quad 1 \leq k \leq n$
$f^*(vw_k) = 2n - k + 2 \quad 1 \leq k \leq n$
$f^*(ww_k) = k \quad 1 \leq k \leq n$
$f^*(wx) = 2n + 3$
$f^*(xv) = 2n + 2$
$f^*(xw) = n + 1$.

Thus, the induced edge labels are distinct from $1, 2, 3, \ldots, 3n + 3$. Hence the graph $K_{1,3} * K_{1,n}$ is an extra Skolem difference mean graph.

**Illustration:** An extra Skolem difference mean labeling of $K_{1,3} * K_{1,6}$ is shown in Figure-5.

![Figure-5 $K_{1,3} * K_{1,6}$](image-url)
Theorem 2.1.6 $m$-Join of $H_n$ is an extra Skolem difference mean graph.

Proof: Let $G = m$-Join of $H_n$

\[ V(G) = \{ u_k, v_k : 1 \leq k \leq n \} \cup \{ u'_k, v'_k : 1 \leq k \leq n \} \cup \ldots \cup \{ u'_m, v'_m : 1 \leq k \leq n \} \]

\[ E(G) = \{ u_ku_{k+1} : 1 \leq k \leq n - 1 \} \cup \{ u'_{k}u'_{k+1} : 1 \leq k \leq n - 1 \} \cup \ldots \cup \{ u'_m, u'_{m+1} : 1 \leq k \leq n - 1 \} \cup \{ v_{m+1}, v_{m+2} : 1 \leq k \leq n - 1 \} \cup \ldots \cup \{ v_{2n}, v_{2n+1} : 1 \leq k \leq n - 1 \} \cup \{ u_m, u'_{m+1} : 1 \leq k \leq n - 1 \} \cup \ldots \cup \{ u'_m, u'_{m+1} : 1 \leq k \leq n - 1 \} \cup \{ v_{2n}, v_{2n+1} : 1 \leq k \leq n - 1 \} \cup \ldots \cup \{ v_{2n}, v_{2n+1} : 1 \leq k \leq n - 1 \} \]

So, $|V(G)| = 2mn + 2n$ & $|E(G)| = 2mn + 2n - 1$.

Define $f: V(G) \cup E(G) \rightarrow \{ 1, 2, 3, \ldots, 4mn + 4n - 1 \}$ as follows.

**Case 1**: $n$ is even.

\[ f(u_{2k-1}) = 2k - 1 \quad 1 \leq k \leq \frac{n}{2} \]

\[ f(u_{2k}) = 4mn + 4n + 1 - 2k \quad 1 \leq k \leq \frac{n}{2} \]

\[ f(v_{2k-1}) = n - 1 + 2k \quad 1 \leq k \leq \frac{n}{2} \]

\[ f(v_{2k}) = 4mn + 3n + 1 - 2k \quad 1 \leq k \leq \frac{n}{2} \]

\[ f(u'_{2k-1}) = 2n - 1 + 2k \quad 1 \leq k \leq \frac{n}{2} \]

\[ f(u'_{2k}) = 4mn + 2n + 1 - 2k \quad 1 \leq k \leq \frac{n}{2} \]

\[ f(v'_{2k-1}) = 3n - 1 + 2k \quad 1 \leq k \leq \frac{n}{2} \]

\[ f(v'_{2k}) = 4mn + n + 1 - 2k \quad 1 \leq k \leq \frac{n}{2} \]

\[ f(u''_{2k-1}) = 4n - 1 + 2k \quad 1 \leq k \leq \frac{n}{2} \]

\[ f(u''_{2k}) = 4mn + 1 - 2k \quad 1 \leq k \leq \frac{n}{2} \]

\[ f(v''_{2k-1}) = 5n - 1 + 2k \quad 1 \leq k \leq \frac{n}{2} \]

\[ f(v''_{2k}) = 4mn - n + 1 - 2k \quad 1 \leq k \leq \frac{n}{2} \]

**Case 2**: $n$ is odd.

\[ f(u_{2k-1}) = 2k - 1 \quad 1 \leq k \leq \frac{n+1}{2} \]

\[ f(u_{2k}) = 2mn + 4n + 1 - 2k \quad 1 \leq k \leq \frac{n}{2} \]

\[ f(v_{2k-1}) = 2mn + n - 1 + 2k \quad 1 \leq k \leq \frac{n}{2} \]

\[ f(v_{2k}) = 2mn + 3n + 1 - 2k \quad 1 \leq k \leq \frac{n}{2} \]
\[ f(u_{2k}) = 4mn + 4n + 1 - 2k \quad 1 \leq k \leq \frac{n-1}{2} \]
\[ f(v_{2k-1}) = 4mn + 3n + 2 - 2k \quad 1 \leq k \leq \frac{n+1}{2} \]
\[ f(v_{2k}) = n + 2k \quad 1 \leq k \leq \frac{n-1}{2} \]
\[ f(u'_{2k-1}) = 2n - 1 + 2k \quad 1 \leq k \leq \frac{n+1}{2} \]
\[ f(u'_{2k}) = 4mn + 2n + 1 - 2k \quad 1 \leq k \leq \frac{n-1}{2} \]
\[ f(v'_{2k-1}) = 4mn + n + 2 - 2k \quad 1 \leq k \leq \frac{n+1}{2} \]
\[ f(v'_{2k}) = 3n + 2k \quad 1 \leq k \leq \frac{n-1}{2} \]
\[ f(u''_{2k-1}) = 4n - 1 + 2k \quad 1 \leq k \leq \frac{n+1}{2} \]
\[ f(u''_{2k}) = 4mn + 1 - 2k \quad 1 \leq k \leq \frac{n-1}{2} \]
\[ f(v''_{2k-1}) = 4mn - n + 2 - 2k \quad 1 \leq k \leq \frac{n+1}{2} \]
\[ f(v''_{2k}) = 5n + 2k \quad 1 \leq k \leq \frac{n-1}{2} \]
\[ \cdots \]
\[ f(u''_{m-2k-1}) = 2mn - 1 + 2k \quad 1 \leq k \leq \frac{n+1}{2} \]
\[ f(u''_{m-2k}) = 2mn + 4n + 1 - 2k \quad 1 \leq k \leq \frac{n-1}{2} \]
\[ f(v''_{m-2k-1}) = 2mn + 3n + 2 - 2k \quad 1 \leq k \leq \frac{n+1}{2} \]
\[ f(v''_{m-2k}) = 2mn + n + 2k \quad 1 \leq k \leq \frac{n-1}{2} \]

For this we define following edge function, \( f^*: E(G) \to \{1, 2, 3, \ldots, 2mn + 2n - 1\} \) by
\[ f^*(u_k u_{k+1}) = 2mn + 2n - k \quad 1 \leq k \leq n - 1 \]
\[ f^*(v_{n+1} v_{n+1}) = 2mn + n \quad n \text{ odd} \]
\[ f^*(u_{n+1} v_{n+1}) = 2mn + n \quad n \text{ even} \]
\[ f^*(v_{k+1} v_{k+1}) = 2mn + n - k \quad 1 \leq k \leq n - 1 \]
\[ f^*(v_{n+1} u'_1) = 2mn \]
\[ f^*(u'_k u'_{k+1}) = 2mn - k \quad 1 \leq k \leq n - 1 \]
\[ f^*(u'_{n+1} v'_{n+1}) = 2mn - n \quad n \text{ odd} \]
\[ f^*(u'_{n+1} v'_{n}) = 2mn - n \quad n \text{ even} \]
\[ f^*(v'_{k+1} v'_{k+1}) = 2mn - n - k \quad 1 \leq k \leq n - 1 \]
\[ f^*(v'_{n} u''_1) = 2mn - 2n \]
\[ \cdots \]
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\[ f^*(v_{m-1}^n u_{m-1}^n) = 2n \]
\[ f^*(u_{m-k}^m u_{k+1}^m) = 2n - k \quad 1 \leq k \leq n - 1 \]
\[ f^*(u_{n+1}^m u_{m+1}^n) = n, n \text{ odd} \]
\[ f^*(u_{n+1}^m u_{m+1}^n) = n, n \text{ even} \]
\[ f^*(v_{m-k}^m v_{k+1}^m) = n - k \quad 1 \leq k \leq n - 1 \]

Thus the induced edge labels are distinct from 1, 2, 3, ..., 2mn + 2n - 1. Hence the graph \( m \)-Join of \( H_n \) is an extra Skolem difference mean graph.

**Illustration:** An extra Skolem difference mean labeling of \( 2 \)-Join of \( H_4 \) graph is shown in Figure-6.

**Figure-6 2-Join of \( H_4 \)**

**Theorem 2.1.7** \( P_n \odot K_{1,m} \) is an extra Skolem difference Mean graph

**Proof:** Let \( G = P_n \odot K_{1,m} \)

\( V(G) = \{ u_k, u'_k, u''_k, ..., u_m^k : 1 \leq k \leq n \} \)
\( E(G) = \{ (u_k u_{k+1}) : 1 \leq k \leq n - 1 \} \cup \{ (u_k u'_k) : 1 \leq k \leq n \} \cup \{ (u_k u''_k) : 1 \leq k \leq n \} \cup ... \cup \{ (u_k u_m^k) : 1 \leq k \leq n \} \)

So, \( |V(G)| = n + nm \) & \( |E(G)| = n + nm - 1 \).

Define \( f: V(G) \to \{ 1, 2, 3, ..., 2n + 2nm - 1 \} \).

\[ f(u_{2k-1}) = 2n + 2nm + 2m + 1 - (2m + 2)k \quad 1 \leq k \leq \left\lceil \frac{n}{2} \right\rceil \]
\[ f(u_{2k}) = (2 + 2m)k - 1 \quad 1 \leq k \leq \left\lceil \frac{n}{2} \right\rceil \]
\[ f(u'_{2k-1}) = (2 + 2m)k - (2m + 1) \quad 1 \leq k \leq \left\lceil \frac{n}{2} \right\rceil \]
\[ f(u'_{2k}) = 2n + 2nm + 2m - 1 - (2m + 2)k \quad 1 \leq k \leq \left\lceil \frac{n}{2} \right\rceil \]
\[ f(u''_{2k-1}) = (2 + 2m)k - (2m - 1) \quad 1 \leq k \leq \left\lceil \frac{n}{2} \right\rceil \]
\[ f(u''_{2k}) = 2n + 2nm + 2m - 3 - (2m + 2)k \quad 1 \leq k \leq \left\lceil \frac{n}{2} \right\rceil \]

...
\[
f(u_{2k}) = 2n + 2nm + 1 - (2m + 2)k \quad 1 \leq k \leq \lfloor \frac{n}{2} \rfloor
\]

For this we define following edge function, \( f^*: E(G) \rightarrow \{1,2,3,..., n + nm - 1\} \) by

\[
f^*(u_k u_{k+1}) = n + nm - (1 + m)k \quad 1 \leq k \leq n - 1
\]

\[
f^*(u_k u'_k) = n + nm + m - (1 + m)k \quad 1 \leq k \leq n
\]

\[
f^*(u_k u''_k) = n + nm + m - 1 - (1 + m)k \quad 1 \leq k \leq n
\]

Thus the induced edge labels are distinct from 1,2,3,..., \( n + nm - 1 \). Hence the graph \( P_n \odot K_{1,m} \) is an extra Skolem difference mean graph.

**Illustration:** An extra Skolem difference mean labeling of \( P_4 \odot K_{1,3} \) is shown in Figure-7.

**Theorem 2.1.8** The \( H \) – super subdivision of a path \( HSS(P_n) \) is an extra Skolem difference mean graph.

**Proof:** Let \( G = HSS(P_n) \)

\[
V(G) = \{ u_k, u_k^{(1)}(k+1), u_k^{(2)}(k+1), u_k^{(2)}(k+1) : 1 \leq k \leq n - 1 \} \cup \{ u_n \}
\]

\[
E(G) = \{ u_k u_k^{(1)}(k+1), u_k^{(1)}(k+1), u_k^{(2)}(k+1), u_k^{(1)}(k+1), u_k^{(2)}(k+1), u_k^{(2)}(k+1) : 1 \leq k \leq n - 1 \}
\]

So, \( |V(G)| = 5n - 4 \) & \( |E(G)| = 5n - 5 \).

Define \( f: V(G) \cup E(G) \rightarrow \{1,2,3,..., 10n - 9\} \) as follows.

**Case:**\( 1 \) \( n \) is even.

\[
f(u_{2k-1}) = 10k - 9 \quad 1 \leq k \leq \lfloor \frac{n}{2} \rfloor
\]

\[
f(u_{2k}) = 10n - 3 - 10k \quad 1 \leq k \leq \lfloor \frac{n}{2} \rfloor
\]

\[
f(u_{2k}(2k-1)) = 10n + 1 - 10k \quad 1 \leq k \leq \lfloor \frac{n}{2} \rfloor
\]

\[
f(u_{2k}(2k-1)) = 10k - 5 \quad 1 \leq k \leq \lfloor \frac{n}{2} \rfloor
\]

\[
f(u_{2k}(2k+1)) = 10k - 3 \quad 1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor
\]

\[
f(u_{2k+1}(2k)) = 10n - 7 - 10k \quad 1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor
\]
Extra Skolem Difference Mean Labeling of Various Graphs

Theorem 2.1.9 $H \odot mK_1$—graph of a path $P_n$ is a Difference perfect square cordial graph.

**Proof:** Let $G = H \odot mK_1$, graph of a path $P_n$.

$V(G) = \{u_k, v_k, u_{ki}, v_{ki}; 1 \leq k \leq n, 1 \leq i \leq m\},$

$E(G) = \{u_{k+1}u_k; 1 \leq k \leq n - 1\} \cup \{v_{k+1}v_k; 1 \leq k \leq n - 1\} \cup \{u_{ki}u_{ki}; 1 \leq k \leq n, 1 \leq i \leq m\} \cup \{v_{ki}v_{ki}; 1 \leq k \leq n, 1 \leq i \leq m\} \cup \{u_{ki}v_{ki}; n \text{ odd}\}$ OR

$E(G) = \{u_{k+1}u_k; 1 \leq k \leq n - 1\} \cup \{v_{k+1}v_k; 1 \leq k \leq n - 1\} \cup \{u_{ki}u_{ki}; 1 \leq k \leq n, 1 \leq i \leq m\} \cup \{v_{ki}v_{ki}; 1 \leq k \leq n, 1 \leq i \leq m\} \cup \{u_{ki}v_{ki}; n \text{ even}\}$
So, \(|V(G)| = 2n + 2nm \ & \ |E(G)| = 2n + 2nm - 1. \\
Define a function \(f: V(G) \to \{1,2,3,\ldots, 4n + 4nm - 1\}\) as,

**Case 1**: \(n\) is even.

\[
\begin{align*}
    f(u_{2k}) &= (2 + 2m)k - 1 & \quad & 1 \leq k \leq \frac{2n}{2m+1} \\
    f(u_{2k+1}) &= 4n + 4nm + 2m - 1 - (2 + 2m)k & \quad & 1 \leq k \leq \frac{2n}{2m+1} \\
    f(u_{2k+2}) &= 4n + 4nm + 2m - 3 - (2 + 2m)k & \quad & 1 \leq k \leq \frac{2n}{2m+1} \\
    f(u_{2k+3}) &= 4n + 4nm + 2m - 5 - (2 + 2m)k & \quad & 1 \leq k \leq \frac{2n}{2m+1} \\
    \cdots
\end{align*}
\]

\[
\begin{align*}
    f(u_{2k-3}) &= (2 + 2m)(k - 3) & \quad & 1 \leq k \leq \frac{2n}{2m+1} \\
    f(u_{2k-2}) &= 4n + 4nm + 2m - 1 - (2 + 2m)(k - 3) & \quad & 1 \leq k \leq \frac{2n}{2m+1} \\
    f(u_{2k-1}) &= 4n + 4nm + 2m - 3 - (2 + 2m)(k - 3) & \quad & 1 \leq k \leq \frac{2n}{2m+1} \\
    f(u_{2k}) &= 4n + 4nm + 2m - 5 - (2 + 2m)(k - 3) & \quad & 1 \leq k \leq \frac{2n}{2m+1} \\
    \cdots
\end{align*}
\]

**Case 2**: \(n\) is odd.

\[
\begin{align*}
    f(u_{2k}) &= (2 + 2m)k - 1 & \quad & 1 \leq k \leq \frac{n+1}{2} \\
    f(u_{2k-1}) &= 4n + 4nm + 2m + 1 - (2 + 2m)k & \quad & 1 \leq k \leq \frac{n+1}{2} \\
    f(u_{2k-2}) &= 4n + 4nm + 2m + 3 - (2 + 2m)k & \quad & 1 \leq k \leq \frac{n+1}{2} \\
    f(u_{2k-3}) &= 4n + 4nm + 2m + 5 - (2 + 2m)k & \quad & 1 \leq k \leq \frac{n+1}{2} \\
    \cdots
\end{align*}
\]
Extra Skolem Difference Mean Labeling of Various Graphs

Illustration: An extra Skolem difference mean labeling of \( H \odot 2K_1 \) graph of a path \( P_5 \) is shown in Figure-9.
3. Conclusion

In this paper we obtain an extra Skolem difference mean labeling for Comb graph, Twig of a path $P_n$, $H$ graph of a path $P_n$, $K_{1,2} \ast K_{1,n}$ graph, $K_{1,3} \ast K_{1,n}$ graph, $m –$ Join of $H_n$, $P_n \odot K_{1,m}$ graph, $HSS(P_n)$ graph, $H \odot mK_1$ –graph of a path $P_n$. We can discuss more similar results for various graphs.

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