Criteria for integer and modulo 2 embeddability of graphs to surfaces *†

A. I. Bikeev ‡

November 17, 2021

Contents

1 Introduction and main results 1

2 Proofs 4

1 Introduction and main results

The study of graph drawings on 2-surfaces is an active area of mathematical research. Surveying these studies is beyond the scope of the present paper; see Remark 1.3 for results most closely related to ours.

Our main results are criteria for $\mathbb{Z}_2$-embeddability and $\mathbb{Z}$-embeddability (see definitions below) of graphs to surfaces (Theorems 1.1 and 1.4). See Remarks 1.2 and 1.5 for applications, comments and relations to other results.

In this paper we use the following conventions and notations. Let $K$ be a graph and $M$ be a 2-dimensional surface. Denote by $V = V(K)$ the set of vertices of graph $K$. Denote by $E = E(K)$ the set of edges of graph $K$. We work in the piecewise-linear (PL) category. We shorten ‘2-dimensional surface’ to ‘2-surface’.

A general position PL map $f : K \to M$ is called a $\mathbb{Z}_2$-(almost) embedding (a.k.a. Hanani-Tutte drawing) if $|f\sigma \cap f\tau|$ is even for any pair $\sigma, \tau$ of non-adjacent (a.k.a. independent) edges.

Definitions of $S_g$ and $M_m$. Denote by $S_g$ (see Fig. 1 left) the union of a disk and $2g$ non-twisted ribbons $\lambda_1, \ldots, \lambda_{2g}$ such that the ribbons $\lambda_{2i-1}$ and $\lambda_{2i}$ interlace for each $i \in [g]$ and the other pairs of the ribbons do not interlace. This $S_g$ is homeomorphic to the sphere with $g$ handles and a hole. $\mathbb{Z}_2$-embeddability of a graph to the sphere with $g$ handles is equivalent to $\mathbb{Z}_2$-embeddability of the graph to $S_g$.

Denote by $M_m$ disk with $m$ Möbius bands (see Fig. 1 right), i.e., the union of a disk and $m$ twisted pairwise non-interlacing ribbons $\mu_1, \ldots, \mu_m$. This $M_m$ is homeomorphic to

---

*Supported by the Russian Foundation for Basic Research Grant No. 19-01-00169. I am grateful for many useful discussions to A. Skopenkov, R. Fulek, J. Kynčl, A. Kliaczk and E. Kogan.
†We borrowed Remark 1.2, Remark 1.3, and definitions of $\mathbb{Z}_2$- and $\mathbb{Z}$- embeddings, compatibility (modulo 2) and algebraic intersection number from [KS21].
‡bikeev99@mail.ru, Moscow Institute of Physics and Technology
A symmetric matrix with $\mathbb{Z}_2$-entries is

- **even** (a.k.a. alternate) if its diagonal contains only zeros;
- **odd** (a.k.a. non-alternate) if its diagonal contains at least one entry 1.

The graph $K$ is called **compatible modulo 2** to a symmetric matrix $A$ of size $|E| \times |E|$ with $\mathbb{Z}_2$-entries if there is a general position PL map $f : K \to \mathbb{R}^2$ such that

\[(C_2) \quad A_{\sigma,\tau} = |f_\sigma \cap f_\tau|_2 \quad \text{for any non-adjacent edges } \sigma, \tau \text{ of } K.\]

Clearly, compatibility modulo 2 is algorithmically decidable.

**Theorem 1.1.** (a) (Fulek–Kynčl) A graph $K$ has a $\mathbb{Z}_2$-embedding to $S_g$ if and only if $K$ is compatible modulo 2 to some even matrix $A$ such that $\text{rk } A \leq 2g$.

(b) A graph $K$ has a $\mathbb{Z}_2$-embedding to $M_m$ if and only if $K$ is compatible modulo 2 to some odd matrix $A$ such that $\text{rk } A \leq m$.

**Remark 1.2.** (a) Theorem 1.1.a is proved in [FK19, Proposition 10 and Corollary 11 for $g_0$] ($\implies$) and in a private communication by R. Fulek, using ideas of [SS13, §2] ($\iff$). Our proof is similar to the Fulek-Kynčl proof. The main difference is that using homology groups allows us to use a well-known algebraic result (Lemma 2.1 below). Presumably the Fulek-Kynčl proof of Theorem 1.1.a can also be generalized to Theorem 1.4.

Theorem 1.1.b is easily implied by Theorem 1.1.a and [SS13, Lemma 3]. See also [KS21, Lemma 2.4.2].

Theorem 1.1.b is different from [FK19, Proposition 10 and Corollary 11 for $e g_0$], which is the implication ($\implies$) of the following result that can be proved similarly to Theorem 1.1.

A graph $K$ has a $\mathbb{Z}_2$-embedding to some connected surface of Euler characteristic $e$ if and only if there is a matrix $A$ such that $\text{rk } A \leq 2 - e$ and $K$ is compatible modulo 2 to $A$.

(b) The following result is implied by Theorem 1.1. There are an algorithm for checking $\mathbb{Z}_2$-embeddability of graphs to $S_g$ and an algorithm for checking $\mathbb{Z}_2$-embeddability of graphs to $M_m$.

This result is known, although it was not stated explicitly in the literature. The result follows because the property of a graph admitting a $\mathbb{Z}_2$-embedding to a fixed 2-surface is preserved under taking graph minors (i.e., under deleting of an edge or contracting of an edge). Therefore by the Robertson-Seymour graph minor theorem [RS04] there exists a finitely many forbidden minors characterizing such a property. Hence there exists a polynomial time algorithm (because we can test in a polynomial time if a fixed graph is a minor of a given graph [KKR]). This result is non-constructive, i.e. we only know that an algorithm
exists, but the algorithm itself would be in practice even worse than exponential (‘galactic’ [GA]). This is so because even for the graphs embeddable into the torus the set of all the forbidden minors is not known. The current proof of Theorem 1.1 together with [KS21, Lemma 2.3.2] gives a practical algorithm for small \( m \) and \( g \).

(c) Theorem 1.1 reduces \( \mathbb{Z}_2 \)-embeddability to finding minimal rank of ‘partial matrix’ (and to related problems); this is extensively studied in computer science, see e.g., [Ko21] and survey [NKS].

(d) Puncturing a 2-surface (more precisely, deleting an open 2-disk whose closure is a closed 2-disk) does not change \( \mathbb{Z}_2 \)-embeddability of graphs there. So it suffices to study \( \mathbb{Z}_2 \)-embeddability to a connected 2-surface whose boundary is the circle. By classification of 2-surfaces, any such 2-surface is homeomorphic to \( S_g \) or to \( M_m \).

(e) The expression \( y^T_E y_r = y^T_r y_r \) from the proof of Theorem 1.1.b appeared in [SS13, §2.2, equality (3)], [FK19, §3.1, equality (1)].

(f) The expression \( y^T H_{2,g} y_r \) from the proof of Theorem 1.1 appeared in [PT19, equality (7) in §3]. The common ideas and methods of the proofs in the current paper and the proofs in [PT19] may be covered by classical arguments and ideas and methods of [FK19], [SS13]. In particular, our constructions of \( \mathbb{Z}_2 \)- and \( \mathbb{Z} \)-embeddings from matrices uses known construction of a map inducing given homomorphism in homology. We do not use cohomological arguments as opposed to [Ha69], [Jo02] and [PT19].

**Remark 1.3** (Closely related known results). (a) Some proofs of non-planarity of \( K_5 \) and \( K_{3,3} \) actually show that these graphs are not \( \mathbb{Z}_2 \)-embeddable to the plane, see e.g. survey [Sk18, §1.4]. By [FK19, Theorem 1] if \( K_{m,n} \) has a \( \mathbb{Z}_2 \)-embedding to the sphere with \( g \) handles (or, equivalently, \( S_g \)), then \( g \geq \frac{(m-2)(n-2) - m - 3}{4} \). Hence if \( K_{2n} \) has a \( \mathbb{Z}_2 \)-embedding to the sphere with \( g \) handles, then \( g \geq \frac{(n-3)^2}{4} \).

(b) Let \( M \) be either the plane or the torus or the Möbius band. If a graph has a \( \mathbb{Z}_2 \)-embedding to \( M \), then the graph has an embedding into \( M \). For the plane this is the (strong) Hanani-Tutte Theorem (see e.g. survey [Sk18, Theorem 1.5.3] and the references therein). For the torus this is proved in [FPS], and for the Möbius band in [PSS], [CKP+].

(c) There is a graph having a \( \mathbb{Z}_2 \)-embedding to the sphere with 4 handles but not an embedding into the sphere with 4 handles. [FK17]

(d) Theorem 1.1 is related to the van Kampen-Shapiro-Wu criterion for embeddability of \( k \)-complexes into \( \mathbb{R}^{2k} \) and to the Paták-Tancer criterion for embeddability of \( k \)-complexes into \( 2k \)-manifolds (as explained in [KS21, §1.3]). See [PT19], [Ha69, Theorem 1 and Corollaries 6, 7, 8], [Jo02] and [KS21] for higher dimensional analogues of results of this paper.

Take an orientation on \( S_g \). Assume that \( f : K \to S_g \) is a general position PL map.

Then preimages \( y_1, y_2 \in K \) of any double point \( y \in S_g \) lie in the interiors of edges. By general position, \( f \) is ‘linear’ on some neighborhood \( U_j \) of \( y_j \) for each \( j = 1, 2 \). Given orientation on the edges, we can take a basis of 2 vectors formed by oriented \( fU_1, fU_2 \). The intersection sign of \( y \) is the sign \( \pm 1 \) of the basis. The algebraic intersection number \( f\sigma \cdot f\tau \in \mathbb{Z} \) for non-adjacent oriented edges \( \sigma, \tau \) is defined as the sum of the intersection signs of all intersection points from \( f\sigma \cap f\tau \).

An example of a \( \mathbb{Z}_2 \)-embedding which is not a \( \mathbb{Z} \)-embedding is shown in Fig. 2, right.

The map \( f \) is called a \( \mathbb{Z} \)-(almost) embedding if \( f\sigma \cdot f\tau = 0 \) for any pair \( \sigma, \tau \) of non-adjacent edges. (The sign of \( f\sigma \cdot f\tau \) depends on an arbitrary choice of orientations for \( \sigma, \tau \) and on the order of \( \sigma, \tau \), but the condition \( f\sigma \cdot f\tau = 0 \) does not.)
An integer analogue of Theorem 1.1.a is obtained by replacing $\mathbb{Z}_2$ by $\mathbb{Z}$ and the modulo 2 intersection number by the algebraic ('integer') intersection number. Graph $K$ is called compatible to a skew-symmetric matrix $A$ of size $|E| \times |E|$ with $\mathbb{Z}$-entries if there is a general position PL map $f : K \to \mathbb{R}^2$ such that for some collection of orientations on edges of $K$ we have

\[(C)\quad A_{\sigma,\tau} = f\sigma \cdot f\tau \quad \text{for any non-adjacent } k\text{-faces } \sigma, \tau \text{ of } K.\]

It is not clear whether compatibility is algorithmically decidable.

Denote by $\text{rk}_\mathbb{Q} A$ the rank over $\mathbb{Q}$ of matrix $A$ with $\mathbb{Z}$-entries.

**Theorem 1.4.** A graph $K$ has a $\mathbb{Z}$-embedding to $S_g$ if and only if $K$ is compatible to a skew-symmetric matrix $A$ such that $\text{rk}_\mathbb{Q} A \leq 2g$.

**Remark 1.5.** (a) Analogues of Remark 1.2.bcdef, Remark 1.3.d hold for $\mathbb{Z}$-embeddability.

(b) Our statement of Theorem 1.4 (as well as Theorem 1.1) is different from statements of [PT19, Theorem 1, Corollary 3, Theorem 4]. The main difference is that we use the property of compatibility (modulo 2), while the statements of [PT19, Theorem 1, Corollary 3, Theorem 4] use cohomological terms. Also Theorems 1.1 and 1.4 are formulated only for graphs in 2-surfaces, while [PT19] concerns $k$-complexes in $2k$-manifolds, where $k \geq 1$ in [PT19, Theorem 1, Corollary 3] and $k \geq 3$ in [PT19, Theorem 4].

\section{Proofs}

Let $\pi : S_g \to \mathbb{R}^2$ be the standard map (i.e., drawing with self-intersections), see Fig. 1.

**Proof of the implication ($\implies$) of Theorem 1.1.a.** Let $h : K \to S_g$ be a general position PL $\mathbb{Z}_2$-embedding. Take a disk $D' \subset D$. We can assume that $hv \in D$ for any $v \in V$.

For any edge $\sigma$ of graph $K$ take

- a polygonal line $\bar{\sigma}$ in the disk $D$ joining the ends of $h\sigma$;
- a polygonal cycle $\tilde{\sigma} := h\sigma \cup \bar{\sigma}$, see Fig. 3.

We can assume that the polygonal cycles $\tilde{\sigma}$ and $\tilde{\tau}$ are in general position for any distinct $\sigma, \tau \in E(K)$.

Take a map $f = \pi \circ h$. We can assume that $f$ is a general position PL map.

Define a matrix $A$ with $\mathbb{Z}_2$-entries by the formula $A_{\sigma,\sigma} = 0, A_{\sigma,\tau} \equiv |\tilde{\sigma} \cap \tilde{\tau}| \mod 2$ for any distinct $\sigma, \tau \in E$.

The graph $K$ is compatible modulo 2 to $A$ because for any non-adjacent edges $\sigma, \tau$ of graph $K$ we have

$$|f\sigma \cap f\tau| = |f\sigma \cap f\tau|_2 + |h\sigma \cap h\tau|_2 = |f\sigma \cap f\tau|_2 + |h\sigma \cap h\tau|_2$$
\[
= |f\sigma \cap f\tau|_2 + |\sigma \cap \pi\tau|_2 + |h\sigma \cap \pi\tau|_2 + |\sigma \cap h\tau|_2 + |\hat{\sigma} \cap \hat{\tau}|_2 \overset{(3)}{=} \\
= |(\pi \circ h)\sigma \cap (\pi \circ h)\tau|_2 + |\pi\sigma \cap \pi\tau|_2 + |(\pi \circ h)\sigma \cap \pi\tau|_2 + |\pi\sigma \cap (\pi \circ h)\tau|_2 + |\check{\sigma} \cap \check{\tau}|_2 \overset{(4)}{=} \\
= |\pi\hat{\sigma} \cap \pi\hat{\tau}|_2 + |\hat{\sigma} \cap \hat{\tau}|_2 \overset{(5)}{=} |\hat{\sigma} \cap \hat{\tau}|_2 \overset{(6)}{=} A_{\sigma,\tau}.
\]

Here
- the first equality holds because the map $h$ is $\mathbb{Z}_2$-embedding;
- the second equality follows from the equality $\hat{\sigma} = \sigma \cup h\sigma$;
- the third equality follows from the equality $f = \pi \circ h$ because the map $\pi|_D$ is injective;
- the fourth equality follows from the equality $\pi\hat{\sigma} = \pi\sigma \cup (\pi \circ h)\sigma$;
- the fifth equality holds because the modulo 2 intersection number of two polygonal cycles $\pi\hat{\sigma}$ and $\pi\hat{\tau}$ in the plane is even.
- the sixth equality holds by definition of $A$.

The matrix $A$ is even by definition. Take a basis in $H_1(S_g;\mathbb{Z}_2)$ whose elements correspond to the ribbons $\lambda_1, \ldots, \lambda_{2g}$. Then the matrix of the modulo 2 intersection form of $S_g$ in the basis is $H_2, g$. For any $\sigma \in E$ let $y_{\sigma}$ be the coordinate vector of the cycle $\hat{\sigma}$ modulo 2 in the basis. The matrix $A$ is the Gramian matrix of the set $\{y_\sigma | \sigma \in E\}$. By the following well-known lemma we have $rk A \leq 2g$.

\begin{lemma}
Let $v_1, v_2, \ldots, v_n$ be vectors in some $d$-dimensional linear space over a field with a bilinear symmetric product $(\cdot, \cdot)$. Let $G$ be the Gramian matrix of $v_1, v_2, \ldots, v_n$. Then $rk G \leq d$.
\end{lemma}

\begin{proof}
Without loss of generality we can assume that $v_1, v_2, \ldots, v_k$ is a basis of the linear span of vectors $v_1, v_2, \ldots, v_n, k \leq d$. Then for each $t \in [n]$ there exist $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that $v_t = \sum_{i=1}^k \alpha_i v_i$. Then $\langle v_t, v_s \rangle = \left( \sum_{i=1}^k \alpha_i v_i, v_s \right) = \sum_{i=1}^k \alpha_i \langle v_i, v_s \rangle$ for any $s$. Hence for each $t$ the row $((v_t, v_1), (v_t, v_2), \ldots (v_t, v_n))$ of matrix $G$ is a linear combination of those rows of the matrix $G$ that correspond to vectors $v_1, v_2, \ldots, v_k$. Hence $rk G \leq k \leq d$.
\end{proof}

The statement of the following simple algebraic result is appeared in a discussion with A. Skopenkov.
Lemma 2.2. Let $Y$ be a matrix of size $m \times n$ with $\mathbb{Z}_2$-entries such that $Y^T Y$ is even. Then $\text{rk}(Y^T Y) \leq m - 1$.

Proof. The matrix $Y^T Y$ is the Gramian matrix of the set $\{b_i | i \in [n]\}$ of the columns of the matrix $Y$. Let us prove that the vectors $b_i, i \in [n]$ are elements of some $(m - 1)$-dimensional linear space over $\mathbb{Z}_2$. Then by Lemma 2.1 we have $\text{rk}(Y^T Y) \leq m - 1$.

For any $i \in [n]$ of the matrix $Y$ we have $b_i^T b_i \equiv (Y^T Y)_{ii} \equiv 0 \mod 2$ because $Y^T Y$ is even. Therefore $b_i$ contains even number of 1-entries.

For any $j \in [m - 1]$ take a vector $b_j'$ of size $m$ such that $(b_j')_m = (b_j')_j = 1$ and any other entry of $b_j'$ is 0. Then any vector $v$ of size $m$ with $\mathbb{Z}_2$-entries containing exactly two 1-entries is a linear combination of the vectors $b_j', j \in [m - 1]$. Then any vector $v$ of size $m$ with $\mathbb{Z}_2$-entries containing even number 1-entries is a linear combination of the vectors $b_j', j \in [m - 1]$. Therefore vectors $b_i, i \in [n]$ are elements of some $(m - 1)$-dimensional linear space over $\mathbb{Z}_2$. 

Proof of the implication $(\Longrightarrow)$ of Theorem 1.1.b. The proof can be obtained from the proof of the implication $(\Longrightarrow)$ of Theorem 1.1.a without the last paragraph by the following changes.

Replace $S_y$ by $M_m$. Replace $\lambda_k$ by $\mu_k$. Replace $2g$ by $m$. Replace $H_{2,g}$ by the identity matrix of size $m$. Replace the paragraph with the definition of matrix $A$ by the following argument.

Take a matrix $Y$ of size $m \times |E|$ such that the vectors $y_\sigma, \sigma \in E$ are the columns of $Y$.

If the matrix $Y^T Y$ is odd, then take a matrix $A = Y^T Y$. By Lemma 2.1 we have $\text{rk} A \leq m$.

If the matrix $Y^T Y$ is even, then by Lemma 2.2 we have $\text{rk}(Y^T Y) \leq m - 1$. Denote by $A$ the matrix obtained from $Y^T Y$ by replacing the entry $(Y^T Y)_{11}$ by 1. Then $\text{rk} A \leq m$ and $A$ is odd.

Proof of the implication $(\Longrightarrow)$ of Theorem 1.4. Let $h$ be a general position PL $\mathbb{Z}$-embedding of graph $K$ to $S_y$. Take some collection of orientations on edges of $K$. We can assume that $hv \in D$ for any $v \in V$. For any oriented edge $\sigma$ of graph $K$ take

- a polygonal line $\sigma$ in the disk $D$ joining the ends of the oriented polygonal line $h\sigma$;
- an orientation on $\overline{\sigma}$ such that polygonal cycle $\hat{\sigma} := h\sigma \cup \overline{\sigma}$ is an oriented polygonal cycle, see Fig. 3.

We can assume that $\hat{\sigma}$ and $\hat{\tau}$ are in general position for any distinct $\sigma, \tau \in E(K)$.

Take a map $f = \pi \circ h$. We can assume that $f$ is a general position PL map.

Define a matrix $A$ with $\mathbb{Z}$-entries by the formula $A_{\sigma,\sigma} = 0, A_{\sigma,\tau} = -\hat{\sigma} \cdot \hat{\tau}$ for any distinct $\sigma, \tau \in E$.

The graph $K$ is compatible to $A$ because for any non-adjacent edges $\sigma, \tau$ of graph $K$ we have

$$f\sigma \cdot f\tau \overset{(1)}{=} f\sigma \cdot f\tau - h\sigma \cdot h\tau \overset{(2)}{=} f\sigma \cdot f\tau + \overline{\sigma} \cdot \overline{\tau} + h\sigma \cdot \overline{\tau} + \overline{\sigma} \cdot h\tau - \hat{\sigma} \cdot \hat{\tau} \overset{(3)}{=} (\pi \circ h)\sigma \cdot (\pi \circ h)\tau + \pi\overline{\sigma} \cdot \pi\overline{\tau} + (\pi \circ h)\sigma \cdot \pi\overline{\tau} + \pi\overline{\sigma} \cdot (\pi \circ h)\tau - \hat{\sigma} \cdot \hat{\tau} \overset{(4)}{=} \pi\overline{\sigma} \cdot \pi\overline{\tau} - \hat{\sigma} \cdot \hat{\tau} \overset{(5)}{=} -\hat{\sigma} \cdot \hat{\tau} \overset{(6)}{=} A_{\sigma,\tau}.$$
• the first equality holds because the map $h$ is $\mathbb{Z}$-embedding;
• the second equality follow from the equality $\hat{\sigma} = \sigma \cup h\sigma$;
• the third equality follows from the definition of the map $f$ because the map $\pi|_{\sigma}$ is injective;
• the fourth equality follows from the equality $\pi\hat{\sigma} = \pi\sigma \cup (\pi \circ h)\sigma$;
• the fifth equality holds because the integer intersection number of two oriented polygonal cycles $\pi\hat{\sigma}$ and $\pi\hat{\tau}$ in the plane is zero.
• the sixth equality holds by the definition of $A$.

The matrix $A$ is skew-symmetric by definition. Take a basis in $\mathbb{Z}$-module $H_1(S_g; \mathbb{Z})$ whose elements correspond to the ribbons $\lambda_1, \ldots, \lambda_{2g}$ such that the matrix of the integer intersection form of $S_g$ in the basis is $-H_g$. For any $\sigma \in E$ let $y_{\sigma}$ be the coordinate vector of the cycle $\hat{\sigma}$. The matrix $A$ is the Gramian matrix of the set $\{y_{\sigma}|\sigma \in E\}$. Consider the matrix $A$ as a matrix over $\mathbb{Q}$. Consider the vector $y_{\sigma}$ as a vector over $\mathbb{Q}$. By Lemma 2.1 we have $rk_{\mathbb{Q}} A \leq 2g$.

$\square$

The following algebraic lemma is proved in [Al, Theorem 3].

**Lemma 2.3.** For each even matrix $A$ of size $n \times n$ with $\mathbb{Z}_2$-entries its rank is even and there is a matrix $Y$ of size $rk A \times n$ such that $A = Y^T H_{2,g} Y$.

**Proof of the implication** ($\Longleftrightarrow$) of **Theorem 1.1.a.** Take $f$ and $A$ from the definition of compatibility modulo 2.

We may assume that $rk A = 2g$, because $\mathbb{Z}_2$-embeddability of a graph to $S_g$ follows from $\mathbb{Z}_2$-embeddability of the graph to $S_{g-1}$. Apply Lemma 2.3 to the matrix $A$ and $n = |E|$. We get that the number $rk A$ is even and there is a matrix $Y$ of size $2g \times E$ such that $A = Y^T H_{2,g} Y$. Denote by $y_{\sigma}$ the corresponding column of matrix $Y$.

Take a disk $D' \subset D$. We may assume that $fK \subset D'$. Take any $\sigma \in E$.

Take a cycle $\tilde{\sigma} \subset S_g \setminus D'$ (see Fig. 4) such that

(\tilde{\sigma}1) the polygonal line $\tilde{\sigma}$ passes exactly once through the ribbon $\lambda_k$ if $y_{\sigma,k} = 1$ and does not pass through the ribbon $\lambda_k$ otherwise.

Take a polygonal line $l_{\sigma}$ joining a point in the cycle $\tilde{\sigma}$ and a point in the polygonal line $f\sigma$ such that

(1) $l_{\sigma} \cap f\tau = \emptyset$ for any distinct $\sigma, \tau \in E(K)$;
(2) $l_{\sigma} \cap \tilde{\tau} = \emptyset$ for any distinct $\sigma, \tau \in E(K)$.

Take a general position PL map $h : K \rightarrow S_g$ obtained from $f$ by connected summation of $f|_{\sigma}$ and $\tilde{\sigma}$ along the polygonal line $l_{\sigma}$ for any $\sigma \in E$. We may assume that the following property holds by the properties (1)-(2).

(h) $h\sigma \cap h\tau \subset (\tilde{\sigma} \cap \tilde{\tau}) \cup (f\sigma \cap f\tau)$ for any $\sigma, \tau \in E$ such that $\sigma \neq \tau$.

By the property ($\tilde{\sigma}1$) there is a basis in $H_1(S_g, \mathbb{Z}_2)$ such that

(b1) $\tilde{\sigma}$ represents the homology class with coordinate vector $y_{\sigma}$ in this basis for any $\sigma \in E$;
(b2) the matrix $H_{2,g}$ is the matrix of the modulo 2 intersection form in this basis.

Then $h$ is a $\mathbb{Z}_2$-embedding, because for any non-adjacent edges $\sigma, \tau$ we have

$$|h\sigma \cap h\tau|_2 \overset{(1)}{=} |(\tilde{\sigma} \cap \tilde{\tau})|_2 + |f\sigma \cap f\tau|_2 \overset{(2)}{=} y_{\sigma}^T H_{2,g} y_{\tau} + |f\sigma \cap f\tau|_2 \overset{(3)}{=} 0.$$  

Here
• the first equality follows from the property (h).
Figure 4: The polygonal line $h\sigma$

- the second equality follows from the properties (b1)-(b2);
- the third equality follows from the definitions of compatibility modulo 2 and the vectors $y_\sigma, \sigma \in E$. □

The following algebraic lemma is proved in [MW69, Theorem 1].

**Lemma 2.4.** For each odd matrix $A$ of size $n \times n$ with $\mathbb{Z}_2$-entries there is a matrix $Y$ of size $\operatorname{rk} A \times n$ such that $A = Y^T Y$.

**Proof of the implication** ($\iff$) **of Theorem 1.1.b.** The proof can be obtained from the proof of the implication ($\implies$) of Theorem 1.1.a by the following changes.

Replace Lemma 2.3 by Lemma 2.4. Replace $S_g$ by $M_m$. Replace $\lambda_k$ by $\mu_k$. Replace $2g$ by $m$. Replace $H_{2g}$ by the identity matrix of size $m$. □

The following algebraic lemma easily follows from [Bo66, Chapter IX, §5, Theorem 1].

**Lemma 2.5.** For each skew-symmetric matrix $A$ of size $n \times n$ with $\mathbb{Z}$-entries its rank over $\mathbb{Q}$ is even and there is a matrix $B$ of size $\operatorname{rk}_\mathbb{Q} A \times n$ with $\mathbb{Z}$-entries such that $A = B^T H_{\operatorname{rk}_\mathbb{Q} A} B$.

**Proof of the implication** ($\iff$) **of Theorem 1.4.** We may assume that $\operatorname{rk} A = 2g$, because $\mathbb{Z}$-embeddability of a graph to $S_g$ follows from $\mathbb{Z}$-embeddability of the graph to $S_{g-1}$. Apply Lemma 2.5 to the matrix $A$ and $n = |E|$. We get that the number $\operatorname{rk} A$ is even and there is a matrix $Y$ of size $2g \times E$ such that $A = Y^T H_g Y$. Denote by $y_\sigma$ the corresponding column of matrix $Y$. Then for any non-adjacent edges $\sigma, \tau$ we have

$$y_\sigma^T H_g y_\tau = f\sigma \cdot f\tau$$

Take an arbitrary collection of orientations on the edges of the graph $K$. Take corresponding orientations on the polygonal lines $f\sigma$ for any $\sigma \in E$. For any $i \in [2g]$ take an
oriented polygonal line $\gamma_i$ passing in the middle of the ribbon $\lambda_i$ and joining two points on the boundary of $D$.

Take a disk $D' \subset D$. We may assume that $fK \subset D'$. For any $\sigma \in E$ take an oriented cycle $\tilde{\sigma} \subset S_g \setminus D'$ (see Fig. 4) such that

- $(\tilde{\sigma}1')$ the oriented polygonal line $\tilde{\sigma}$ passes through the ribbon $\lambda_k$ in the positive direction (i.e. in the direction of the oriented polygonal line $\gamma_k$) $|y_{\sigma,k}|$ times if $y_{\sigma,k} > 0$;

- $(\tilde{\sigma}2')$ the oriented polygonal line $\tilde{\sigma}$ passes through the ribbon $\lambda_k$ in the negative direction $|y_{\sigma,k}|$ times if $y_{\sigma,k} < 0$;

- $(\tilde{\sigma}3')$ the oriented polygonal line $\tilde{\sigma}$ does not pass through the ribbon $\lambda_k$ if $y_{\sigma,k} = 0$.

Take a polygonal line $l_{\sigma}$ joining a point in the cycle $\tilde{\sigma}$ and a point in the polygonal line $f_{\sigma}$ such that the properties (l1)-(l2) from the proof of the implication ($\Leftarrow$) of Theorem 1.1.a hold.

Take a general position map $h : K \to S_g$ obtained from $f$ by connected summation of $f|_{\sigma}$ and $\tilde{\sigma}$ along the polygonal line $l_{\sigma}$ for any $\sigma \in E$. We may assume that the property (h) from the proof of the implication ($\Leftarrow$) of Theorem 1.1.a hold by the properties (l1)-(l2).

By the properties $\tilde{\sigma}1'$-$\tilde{\sigma}3'$ there is a basis in $\mathbb{Z}$-module on $H_1(S_g, \mathbb{Z})$ such that

- (b1') $\tilde{\sigma}$ represents the homology class with coordinate vector $y_{\sigma}$ in this basis for any $\sigma \in E$;

- (b2') the matrix $-H_g$ is the matrix of the integer intersection form in this basis.

Then the map $h$ is a $\mathbb{Z}$-embedding, because for any non-adjacent edges $\sigma, \tau$ we have

$$h\sigma \cdot h\tau \overset{(1)}{=} \tilde{\sigma} \cdot \tilde{\tau} + f\sigma \cap f\tau \overset{(2)}{=} -y_{\sigma}^T H_g y_{\tau} + f\sigma \cap f\tau \overset{(3)}{=} 0.$$ 

Here the proof of the equalities (1)-(3) is the same as the proof of the congruences (1)-(3) in the proof of the implication ($\Leftarrow$) of Theorem 1.1.a.

References

[Al] A. Adrian Albert. Symmetric and alternate matrices in an arbitrary field, I. Trans. Amer. Math. Soc., (1938) 43(3):386–436.

[Bo66] N. Bourbaki. Elements of mathematics, Algebra. Translation from French. Moscow, Science, (1966).

[CKP+] E. Colin de Verdière, V. Kaluza, P. Paták, Z. Patáková and M. Tancer. A direct proof of the strong Hanani-Tutte theorem on the projective plane. Journal of Graph Algorithms and Applications, 21:5 (2017) 939–981.

[FK17] R. Fulek, J. Kynčl, Counterexample to an Extension of the Hanani-Tutte Theorem on the Surface of Genus 4, Combinatorica, 39 (2019) 1267–1279. arXiv:1709.00508

[FK19] R. Fulek, J. Kynčl. $\mathbb{Z}_2$-genus of graphs and minimum rank of partial symmetric matrices, 35th Intern. Symp. on Comp. Geom. (SoCG 2019), Article No. 39, (2019) 39:139:16. https://drops.dagstuhl.de/opus/volltexte/2019/10443/pdf/LIPIcs-SoCG-2019-39.pdf. arXiv:1903.08637

[FPS] R. Fulek, M.J. Pelsmajer, M. Schaefer. Strong Hanani-Tutte for the Torus, arXiv:2009.01683.

[GA] *. https://en.wikipedia.org/wiki/Galactic_algorithm
Books, surveys and expository papers in this list are marked by the stars.