DERIVATIONS AND ALBERTI REPRESENTATIONS

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Abstract. We relate generalized Lebesgue decompositions of measures along curve fragments (“Alberti representations”) and Weaver derivations. This correspondence leads to a geometric characterization of the local norm on the Weaver cotangent bundle of a metric measure space \((X, \mu)\): the local norm of a form \(df\) “sees” how fast \(f\) grows on curve fragments “seen” by \(\mu\). This implies a new characterization of differentiability spaces in terms of the \(\mu\)-a.e. equality of the local norm of \(df\) and the local Lipschitz constant of \(f\). As a consequence, the “Lip-lip” inequality of Keith must be an equality. We also provide dimensional bounds for the module of derivations in terms of the Assouad dimension of \(X\).

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1. Introduction

Overview. The aim of this paper is to relate two structures associated with metric measure spaces: Alberti representations and Weaver derivations. The main results establish a correspondence between these objects that respects local invariants, such as the dimension and local norms. These results are then applied to further investigate the structure of differentiability spaces in the sense of Cheeger-Keith [Che99, Kei04]. Further applications will appear in a sequel to this work.

Alberti representations. In this Subsection we give an informal account of Alberti representations. This tool has played an important rôle in the proof of the rank-one property of BV functions [Alb93], in understanding the structure of measures which satisfy the conclusion of the classical Rademacher’s Theorem [ACP10, AM], and in describing the structure of measures in differentiability spaces [Bat12]. We start with a description in vague terms and refer the reader to [Bat12] and Subsection 2.1 for further details.

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An **Alberti representation** of a Radon measure $\mu$ is a generalized Lebesgue decomposition of $\mu$ in terms of rectifiable measures supported on path fragments; a **path fragment** in $X$ is a biLipschitz map $\gamma : K \to X$ where $K \subset \mathbb{R}$ is compact with positive Lebesgue measure. Denoting the set of fragments in $X$ by $\text{Frag}(X)$, an Alberti representation of $\mu$ takes the form:

$$
\mu = \int_{\text{Frag}(X)} \nu_{\gamma} \, dP(\gamma)
$$

where $P$ is a regular Borel probability measure on $\text{Frag}(X)$ and $\nu_{\gamma} \ll \mathcal{H}^1_{\gamma}$; here $\mathcal{H}^1_{\gamma}$ denotes the 1-dimensional Hausdorff measure on the image of $\gamma$. Note that in general it is necessary to work with path fragments instead of Lipschitz paths because the space $X$ on which $\mu$ is defined might lack any rectifiable curve.

**Example 1.2.** A simple example of an Alberti representation is offered by Fubini’s Theorem. Let $\mathcal{H}^n$ denote the Lebesgue measure in $\mathbb{R}^n$ and, for $y \in [0, 1]^{n-1}$, let

$$
\psi(y) : [0, 1] \to \text{Frag}([0, 1]^n)
$$

$$
t \mapsto y + te_n,
$$

$e_n$ denoting the last vector in the standard orthonormal basis of $\mathbb{R}^n$. Then $P = \psi_\# \mathcal{H}^{n-1}[0, 1]^{n-1}$ is a regular Borel probability measure on $\text{Frag}([0, 1]^n)$; if we let $\nu_{\gamma} = \mathcal{H}^1_{\gamma}$, by Fubini’s Theorem

$$
\mathcal{H}^n[0, 1]^n = \int_{\text{Frag}([0, 1]^n)} \nu_{\gamma} \, dP(\gamma)
$$

and so $(P, \nu)$ is an Alberti representation of $\mathcal{H}^n[0, 1]^n$.

**Weaver derivations.** In this Subsection we give an informal account of Weaver derivations, hereafter simply called derivations, and refer the reader to [Wea99, Wea00] and Subsection 2.2 for more details. Note that we need a less general setting than that of Weaver because we deal with Radon measures on separable metric spaces.

We will denote by $\text{Lip}(X)$ the space of real-valued Lipschitz functions defined on $X$ and by $\text{Lip}_b(X)$ the real algebra of real-valued bounded Lipschitz functions. The algebra $\text{Lip}_b(X)$ is a dual Banach space\(^1\) and therefore has a well-defined weak* topology: a sequence $f_n \to f$ in the weak* topology if and only if the global Lipschitz constants of the $f_n$ are uniformly bounded and $f_n \to f$ pointwise.

**Derivations** can be regarded as **measurable vector fields** in metric spaces allowing one to take partial derivatives of Lipschitz functions once a background measure $\mu$ has been established. More precisely, a **derivation** is a weak* continuous bounded linear map $D : \text{Lip}_b(X) \to L^\infty(\mu)$ which satisfies the product rule $D(fg) = fDg + gDf$. The set of derivations forms an $L^\infty(\mu)$-module $X(\mu)$.

**Example 1.5.** Many examples of derivations can be found in [Wea00, Sec. 5]. Considering $\mathbb{R}^n$, the partial derivatives $\frac{\partial}{\partial x_i}$ induce derivations with respect to the Lebesgue measure $\mathcal{H}^n$ because of Rademacher’s Theorem; this example generalizes to Lipschitz manifolds [Wea00, Subsec. 5.B]. Similarly, for an $m$-rectifiable set $M \subset \mathbb{R}^n$, $X(\mathcal{H}^m \llcorner M)$ can be identified with the module of bounded measurable sections of approximate tangent spaces [Wea00, Thm. 38].

\(^1\)With a unique predual
Matching Alberti representations and Derivations. We now start to describe the results obtained in this paper. Note that we will assume metric spaces to be separable. The first step is to use Alberti representations to construct derivations. We introduce the notion of Lipschitz and biLipschitz Alberti representations: an Alberti representation \( \mathcal{A} = (P, \nu) \) is called \( C \)-Lipschitz (resp. \( (C, D) \)-biLipschitz) if \( P \) gives full-measure to the set of \( C \)-Lipschitz (resp. \( (C, D) \)-biLipschitz) fragments. In Subsection 3.1 we show that (Theorem 3.11):

- To a Lipschitz Alberti representation \( \mathcal{A} \) of \( \mu \), one can associate a derivation \( D_{\mathcal{A}} \in \mathcal{X}(\mu) \) by using duality and by taking an average of derivatives along fragments (3.12).
- Denoting by \( \text{Alb}_{\text{sub}}(\mu) \) the set of Lipschitz Alberti representations of some measure of the form \( \mu_{LS} \) for \( S \subseteq X \) Borel, we obtain a map

\[
\text{Der} : \text{Alb}_{\text{sub}}(\mu) \to \mathcal{X}(\mu).
\]

Note that the previous construction produces a wealth of derivations; in fact, Subsection 2.1 provides a standard criterion (Theorem 2.64) which allows to produce Alberti representations which are \((1, 1 + \varepsilon)\)-biLipschitz: this is an improvement on the treatment in [Bat12] and will play an important rôle in the rest of the paper. We also point out that the choice of \( \text{Alb}_{\text{sub}}(\mu) \) reflects the fact that \( \mathcal{X}(\mu) \) depends only on the measure class of \( \mu \): i.e. if \( \mu_1 \ll \mu_2 \) and \( \mu_2 \ll \mu_1 \), \( \mathcal{X}(\mu_1) = \mathcal{X}(\mu_2) \).

We next relate the notion of algebraic independence in \( \mathcal{X}(\mu) \) to a notion of independence for Alberti representations introduced in [Bat12]: if \( f : X \to \mathbb{R}^n \) is Lipschitz and \( \mathcal{C} \) is an \( n \)-dimensional cone field, i.e. a Borel map on \( X \) which takes values in the set of cones in \( \mathbb{R}^n \) (see Definition 2.8), an Alberti representation \( \mathcal{A} = (P, \nu) \) is in the \( f \)-direction of \( \mathcal{C} \) if, for \( P \)-a.e. fragment \( \gamma \) and \( L^1_X \text{dom} \gamma \)-a.e. point \( t \), \( (f \circ \gamma)'(t) \in \mathcal{C}(\gamma(t)) \); cone fields \( \{C_i\}_{i=1}^k \) are independent if for each \( x \in X \), choosing \( v_i \in C_i(x) \setminus \{0\} \), the \( \{v_i\}_{i=1}^k \) are linearly independent. In Subsection 3.1 we show that (Theorem 3.24):

- If \( \mathcal{A} \) is in the \( f \)-direction of \( \mathcal{C} \), then \( D_{\mathcal{A}} f \in \mathcal{C} \).
- Alberti representations \( \{A_i\}_{i=1}^k \) in the \( f \)-directions of independent cone fields \( \{C_i\}_{i=1}^k \) generate independent derivations \( \{D_{\mathcal{A}_i}\}_{i=1}^k \).

Furthermore, our construction relates to a notion of speed for Alberti representations introduced in [Bat12]: if \( f : X \to \mathbb{R} \) is Lipschitz we say that an Alberti representation \( \mathcal{A} = (P, \nu) \) has \( f \)-speed \( \geq \delta \) if, for \( P \)-a.e. \( \gamma \) and \( L^1_X \text{dom} \gamma \)-a.e. point \( t \), \( (f \circ \gamma)'(t) \geq \delta \text{md} \gamma(t) \), where \( \text{md} \gamma \) denotes the metric differential of \( \gamma \) (Definition 2.12). Using an averaging process (3.30), we associate an effective speed \( \sigma_{\mathcal{A}} \) to \( \mathcal{A} \) and show that:

- If \( \mathcal{A} \) has \( f \)-speed \( \geq \delta \), then \( D_{\mathcal{A}} f \geq \sigma_{\mathcal{A}} \delta \) (Theorem 3.31).

We then address questions related to the injectivity and surjectivity of Der. The map Der is very far from being injective: as an illustration of this fact, we show that (Lemma 3.41):

- Given a nondegenerate compact interval \( I \subseteq \mathbb{R} \), an Alberti representation \( \mathcal{A} \) can be replaced by a new representation \( \mathcal{A}' \), whose probability is concentrated on fragments with domain inside \( I \), and such that \( D_{\mathcal{A}} = D_{\mathcal{A}'} \).
• Properties like the Lipschitz/biLipschitz constant, the speed and the direction are preserved in replacing $\mathcal{A}$ by $\mathcal{A}'$.

To study the surjectivity of Der, we use derivations to produce Alberti representations, the intuition being that independent derivations can be used to produce Alberti representations in the directions of independent cone fields. The starting point is the observation that the independence of derivations $\{D_i\}_{i=1}^k \subset X(\mu \underline{U})$, up to taking a Borel partition of $U$ and linear combinations of the $D_i$, is detected by pseudodual Lipschitz functions $\{g_i\}_{i=1}^k \subset \text{Lip}_b(X)$ such that $D_i g_j = \delta_{i,j} \chi_U$ (by Corollary 2.121). To deal with Borel partitions, one is led to introduce the restriction of $\mathcal{A} = (P, \nu)$ to a Borel set $U$: $\mathcal{A} \underline{U} = (P, \nu \underline{U})$ [Bat12, pg. 6]. In Subsection 3.2 we show:

• If the $\{D_i\}_{i=1}^k \subset X(\mu \underline{U})$ have pseudodual Lipschitz functions $\{g_i\}_{i=1}^k \subset \text{Lip}_b(X)$, letting $g = \langle g_i \rangle_{i=1}^k$, for any constant $k$-dimensional cone field $\mathcal{C}$, it is possible to obtain a $(1, 1 + \varepsilon)$-biLipschitz Alberti representation of $\mu \underline{U}$ in the $g$-direction of $\mathcal{C}(w, \alpha)$ with almost optimal (3.61) $(w, g)$-speed (Theorem 3.60).

• If the $\{D_i\}_{i=1}^k \subset X(\mu \underline{U})$ are independent, passing to a Borel partition $U = \bigcup_{\alpha} U_\alpha$, there are Lipschitz functions $f_\alpha$ such that, for each $k$-dimensional cone field $\mathcal{C}$, there is an Alberti representation $\mathcal{A}$ of $\mu$ with $\mathcal{A} \underline{U}_\alpha$ in the $f_\alpha$-direction of $\mathcal{C}$ (Corollary 3.92).

• If $f : X \rightarrow \mathbb{R}^k$ and $\mu$ admits an Alberti representation in the $f$-direction of $k$ independent cone fields, then for each $k$-dimensional cone field $\mathcal{C}$, $\mu$ admits an Alberti representation in the $f$-direction of $\mathcal{C}$ (Corollary 3.94).

Corollary 3.94 is saying that there cannot be gaps in the directions accessible by curve fragments. This result was obtained, for differentiability spaces and constant cone fields, in [Bat12, Sec. 9] using porosity techniques. The method employed here is more general and employs some “soft” Functional Analysis.

In Subsection 3.2 we finally show:

• If $X(\mu)$ is finitely generated, then Der is surjective (Theorem 3.116).

• In the general case, $\text{Der(Alb}_{\text{sub}}(\mu))$ is weak* dense in $X(\mu)$ (Theorem 3.96).

The dual module $\mathcal{E}(\mu)$ of $X(\mu)$ can be regarded as the $L^\infty(\mu)$-module of differential forms because each $f \in \text{Lip}_b(X)$ gives rise to a form $df \in \mathcal{E}(\mu)$. The modules $X(\mu)$ and $\mathcal{E}(\mu)$ admit local norms $\cdot|_{X(\mu),\text{loc}}$ and $\cdot|_{\mathcal{E}(\mu),\text{loc}}$, which can be thought of as families $\{\cdot|_{x,\mu}\}_{x \in X}$ of pointwise norms from which one can reconstruct the global norms by taking the essential supremum. In Subsection 3.3 we obtain a geometric characterization of $\cdot|_{\mathcal{E}(\mu),\text{loc}}$, which plays a central rôle in our characterization of differentiability spaces:

• For $U$ Borel, $f \in \text{Lip}_b(X)$ and $\alpha > 0$, if $|df|_{\mathcal{E}(\mu \underline{U}),\text{loc}} \approx \alpha$, then there is an Alberti representation $\mathcal{A} = (P, \nu)$ of $\mu \underline{U}$ with $P$ concentrated on the fragments where $(f \circ \gamma)' \approx \alpha$ and $\gamma$ (this is an informal restatement of Theorem 3.141).

This research was motivated by the work of Bate [Bat12] on differentiability spaces: Bate obtained an intrinsic characterization of differentiability spaces in terms of Alberti representations which implies that these spaces have a rich structure of curve fragments. We started our investigation with the vague question of what happens in spaces which, despite not being differentiability spaces, still have
a rich curve structure. In particular, our results apply to quite a general class of metric measure spaces; examples are:

- Spaces \((X_{\text{lack}}, \mu_{\text{lack}})\) which either lack any rectifiable curve or where \(\mu_{\text{lack}}\) does not admit any Alberti representation: in this case \(\mathcal{X}(\mu_{\text{lack}}) = \{0\}\).
- Spaces which are \(k\)-rectifiable \((X_{k\text{-rect}}, \mathcal{H}^k)\).
- Products \((X_{\text{lack}} \times X_{k\text{-rect}}, \mu_{\text{lack}} \times \mathcal{H}^k)\) and quotients of such products, for example Laakso spaces and the non-doubling Laakso spaces of [Wea00] and [BS11].
- Differentiability spaces.
- Carnot groups equipped with a Radon measure \(\mu\): in this case \(\mathcal{X}(\mu)\) is always finitely generated and the number of generators is at most the dimension of the horizontal distribution.
- Spaces which have rectifiable fragments in infinitely many directions, for example the Hilbert cubes in \(l^p\) and \(c_0\) considered in [Wea00].

Structure of differentiability spaces. We now describe the implications of the present work for the theory of differentiability spaces, first recalling the current knowledge about these spaces. For more details, we refer to Subsection 2.3.

We first recall notions of finite dimensionality for measures on metric spaces:

- A measure \(\mu\) on \(X\) is doubling (with constant \(C\)) if, for all pairs \((x, r) \in X \times (0, \text{diam} X]\),
  \[
  \mu(B(x, r/2)) \geq C \mu(B(x, r)).
  \]  
  (1.7)

- If (1.7) holds for \(\mu\)-a.e. \(x\) for \(r \in (0, r(x)]\), the measure \(\mu\) is called asymptotically doubling (with constant \(C\)).
- If there are disjoint Borel sets \(X_\alpha\) with \(\mu(X \setminus \bigcup_\alpha X_\alpha) = 0\) and the measure \(\mu\downharpoonright X_\alpha\) is doubling on \(X_\alpha\), \(\mu\) is called \(\sigma\)-asymptotically doubling.

We now recall the definitions of infinitesimal Lipschitz constants and differentiability for Lipschitz functions. For a real-valued Lipschitz function \(f\), the variation of \(f\) at \(x\) at scale \(r\) is \(\sup_{y \in B(x, r)} |f(x) - f(y)|/r\); the lower and upper limits of the variation as \(r \searrow 0\) are denoted by \(\ell f(x)\) and \(\mathcal{L} f(x)\). Following Cheeger [Che99], given \(f \in \text{Lip}(X)\) and \(g : X \to \mathbb{R}^k\) Lipschitz, we say that \(f\) is differentiable at \(x\) with respect to \(g\) with derivative \((a_i)_{i=1}^k \in \mathbb{R}^k\), if

\[
\mathcal{L} \left( f - \sum_{i=1}^k a_i g_i \right)(x) = 0.
\]  
  (1.8)

A differentiability space \((X, \mu)\) is a metric measure space for which there are an integer \(N\) and countably many charts

\[
\{(U_\alpha, \psi_\alpha)\}_\alpha \subset \{\text{Borel subsets of } X\} \times \{\text{Lipschitz maps } f : X \to \mathbb{R}^Q, (Q \leq N)\}
\]  
  (1.9)

such that each \(f \in \text{Lip}(X)\) is \(\mu\)-a.e. differentiable, with a unique derivative, on \(U_\alpha\) with respect to the \(\psi_\alpha\); the number of components \(N_\alpha\) of \(\psi_\alpha\) is called the dimension of the chart \((U_\alpha, \psi_\alpha)\) and the \(l\)-th component of the derivative, the partial derivative,
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is denoted by $\partial f/\partial \psi^l_\alpha$. We will require in the definition of the charts that there is a constant $C_\alpha$ such that, for each $l$,

$$\|\partial f/\partial \psi^l_\alpha\|_{L^\infty(\mu \mathcal{U}_\alpha)} \leq C_\alpha L(f),$$

where $L(f)$ denotes the global Lipschitz constant of $f$; in this case one obtains linear bounded operators $\partial/\partial \psi^l_\alpha : \text{Lip}(X) \to L^\infty(\mu \mathcal{U}_\alpha)$ which are called partial derivative operators. The requirement (1.10) is not assumed by all authors in the definition of a differentiability space; however, it can always be satisfied by partitioning the charts. For example, considering the differentiability space $(\mathbb{R}, L^1)$ and following our terminology, one would not consider $((1, \infty), 1/(|x| + 1))$ a chart; however, on writing $(1, \infty) = \bigcup_{n=1}^{\infty} (n, n+1]$, one would consider the $\{(n,n+1), 1/(|x| + 1)\}_n$ charts. The smallest value of $N$ in (1.9) is called the differentiability dimension of $(X, \mu)$.

The notions of a differentiability space and of measurable charts that we have presented have originated in Cheeger’s work [Che99]. Later Keith [Kei04] clarified some aspects of this construction: a nice exposition can be found in [KM11].

We will now present two sufficient conditions for a metric measure space to be a differentiability space. A $(C, \tau, p)$-PI space is a doubling metric measure space with constant $C$ supporting a $p$-Poincaré inequality ([Hei01]) with constant $\tau$. In [Che99] Cheeger showed that:

**Theorem 1.11.** If $(X, \mu)$ is a $(C, \tau, p)$-PI-space, then it is a differentiability space with differentiability dimension $\leq N(C, \tau)$. Moreover, for each Lipschitz function $f$, the equality $\ell f = Ef$ holds $\mu$-a.e.

In [Kei04] Keith was able to relax the assumptions of Cheeger:

**Theorem 1.12.** If $(X, \mu)$ is a metric measure space with a $C$-doubling measure and there is a constant $\tau > 0$ such that, for each real-valued Lipschitz function $f$,

$$\tau f(x) \geq Ef(x) \quad (\text{for } \mu\text{-a.e. } x),$$

then $(X, \mu)$ is a differentiability space with differentiability dimension $\leq N(C, \tau)$.

The inequality (1.13) is sometimes called the Lip-lip inequality.

Theorems 1.11 and 1.12 can be proved under the relaxed assumption that $\mu$ is asymptotically doubling. Moreover, in [BS11] Bate and Speight showed:

**Theorem 1.14.** If $(X, \mu)$ is a differentiability space, then $\mu$ is $\sigma$-asymptotically doubling.

**Example 1.15.** Note that one cannot conclude that $\mu$ is asymptotically doubling, i.e. that the local doubling constant is uniformly bounded. For example, consider a compact metric measure space with $\mu$ finite which is obtained by gluing together, along some geodesics, countably many (rescalings of) Laakso spaces with Hausdorff dimensions tending to $\infty$; this construction produces a differentiability space but the constant $C$ in (1.7) is not uniformly bounded.

To deal with the case in which there is only a local bound on the differentiability dimension we introduce the following terminology: a $\sigma$-differentiability space $(X, \mu)$ is a complete separable metric measure space, such that, there are disjoint Borel sets $X_\alpha$ with $\mu(X \setminus \bigcup \alpha X_\alpha) = 0$ and such that $(X_\alpha, \mu \res X_\alpha)$ is a differentiability

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5for example, it is used in [Che99] but not in [Kei04]
space. In [Bat12] Bate obtained the following characterization of $\sigma$-differentiability spaces:

**Theorem 1.16.** A metric measure space $(X, \mu)$ is a $\sigma$-differentiability space if and only if:

1. The measure $\mu$ is $\sigma$-asymptotically doubling.
2. There is a Borel map $\tau : X \to (0, \infty)$ such that, for each real-valued Lipschitz function $f$, the measure $\mu$ admits an Alberti representation with $f$-speed $\geq \ell f/\tau$.

Note that Theorem 1.16 implies that a weaker form of the Lip-lip inequality, where $\tau$ is allowed to be a Borel function, must hold in a $\sigma$-differentiability space. We significantly strengthen this result showing that, in a $\sigma$-differentiability space, the equality $\ell f = \ell f$ holds $\mu$-a.e. Our proof relies on the geometric characterization of $|\cdot|_{E(\mu),\text{loc}}$; another proof will appear in [CKS].

The work [Bat12] is a partial extension to metric measure spaces of a structure theory for null sets in Euclidean spaces developed by Alberti-Csörnyei-Preiss (we refer the reader to the expository papers [ACP05, ACP10]; for the structure of measures we refer the reader to [AM]). In particular, Alberti-Csörnyei-Preiss found a collection of sets which must be annihilated by measures satisfying the conclusion of the classical Rademacher’s Theorem.

Our characterization of $\sigma$-differentiability spaces uses a similar idea: we introduce the collection $\text{Gap}(X)$ of Borel subsets of $X$: $S \in \text{Gap}(X)$ if there are a Lipschitz function $f$ and constants $\alpha > \beta \geq 0$, such that, for each Radon measure $\mu'$,

$$|df|_{E(\mu'),\text{loc}} \leq \beta < \alpha \leq \ell f \quad (\mu'-a.e. \text{ on } S).$$

In Subsection 4.2 we show that $(X, \mu)$ is a $\sigma$-differentiability space if and only if any of the following equivalent conditions holds:

- Every $S \in \text{Gap}(X)$ is $\mu$-null (Theorem 4.7).
- The measure $\mu$ is $\sigma$-asymptotically doubling and, for each real-valued Lipschitz function $f$, the following equalities hold $\mu$-a.e.:

$$|df|_{E(\mu),\text{loc}} = \ell f = \ell f \quad ($\text{Theorem 4.9}$).$$

There is also a connection between differentiability and derivations. We show that:

**Theorem 1.19.** A metric measure space $(X, \mu)$ is a differentiability space with dimension $\leq N$ if and only if $\mu$ is $\sigma$-asymptotically doubling and there are a conformal factor $\lambda \in L^\infty(\mu)$ and derivations $\{D_i\}_{i=1}^N \subset \text{Der}(\mu)$ such that, for each $f \in \text{Lip}_b(X)$,

$$\lambda(x) \max_i |D_if(x)| \geq \ell f(x) \quad \text{for } \mu\text{-a.e. } x.$$ 

In [Sch], motivated by [Gon12], the author showed that (1.20) gives sufficient conditions for the existence of a differentiable structure$^6$; in [Sch] the author also proved a partial converse: if $(X, \mu)$ is a differentiability space with $\mu$ doubling and if the partial derivative operators are derivations, then (1.20) holds. Thus Theorem 1.19 follows from [Sch] because we provide two different proofs that the partial derivative operators are derivations: the first proof uses [Bat12] and can be found.

$^6$In [Sch] the measure $\mu$ was assumed doubling. However, taking a Borel partition, it suffices to assume that $\mu$ is $\sigma$-asymptotically doubling.
in Subsection 4.1; the second proofs uses (1.18) and can be found at the end of Subsection 4.2. We include both proofs to illustrate the difference between Bate’s characterization of $\sigma$-differentiability spaces and our quantitative description. In [Bat12] the existence of Alberti representations in independent directions is the starting point of a characterization of $\sigma$-differentiability spaces. On the other hand, we obtain independent Alberti representations as a consequence of the quantitative relation (1.18).

A important consequence of Theorem 1.19 is that in Theorems 1.11 and 1.12 one can replace the bound $N(C, \tau)$ by the Assouad dimension, removing the dependence on $\tau$ (Corollary 4.6).

**Technical tools.** In this Subsection we give an overview of four technical tools used in this paper.

The first tool is an approximation scheme for Lipschitz functions in the weak* topology. The intuition is that if a set $S$ is $\text{Frag}(X, f, \delta)$-null (Definitions 2.50 and 3.62), i.e. does not contain fragments where $(f \circ \gamma)'(t) \geq \delta \text{md} \gamma(t)$, then $f \in \text{Lip}_b(X)$ can be approximated by Lipschitz functions which have Lipschitz constant at most $\delta$ in sufficiently small balls centred on $S'$, where $S' \subset S$ has full measure in $S$. We prove an approximation scheme, Theorem 3.66, which takes into account also the direction of the fragments. We state here a simplified version which is sufficient for the results on differentiability spaces.

**Theorem 1.21.** Let $X$ be a compact metric space, $f : X \to \mathbb{R}$ $L$-Lipschitz and $S \subset X$ compact. Let $\mu$ be a Radon measure on $X$. Assume that $S$ is $\text{Frag}(X, f, \delta)$-null. Then there are $\max(L, \delta)$-Lipschitz functions $g_k \xrightarrow{\text{w*}} f$ with $g_k \mu$-a.e. locally $\delta$-Lipschitz on $S$.

The motivation to prove Theorem 3.66 came from reading [Bat12, Subsec. 6.1]: the author observed that Bate’s construction can be used to produce an approximation scheme for Lipschitz functions with $L f = 0$ (flat) on $S$. In the author’s opinion, [Bat12, Subsec. 6.1] is a metric space version of a construction sketched in [ACP10, Defn. 1.14]. However, the approximation scheme based on [Bat12, Subsec. 6.1] could be used only to prove part of the results presented here: the major obstacle is that the approximation works only for functions which are flat on $S$. The problem stems from the presence of a potential term $\delta \mathcal{H}^1$ in the $Q$-functional introduced by Bate. However, the discussion in [ACP10, Sec. 3] provides evidence suggesting that it is possible to produce a finer approximation scheme. The starting point of [ACP10, Sec. 3] is [ACP10, Thm. 2.4]: the author found a proof of that result written in A. Marchese’s PhD thesis and, starting from it, elaborated Theorem 3.66.

The passage from Euclidean spaces to general metric spaces uses an abstract nonsense construction of a cylinder which is a geodesic metric space containing the graph of the function to be approximated and extra degrees of freedom to deform it. The starting point, as in [Bat12, Subsec. 6.1], is a Kuratowski embedding in $l^\infty$.

The second technical tool is a decomposition of $\mathcal{X}(\mu)$ into free modules. The problem stems from the fact that $L^\infty(\mu)$ is not an integral domain and thus the notion of linear independence of derivations behaves quite differently than in a vector space. In [Sch] the author introduced the following concept of finite dimensionality:

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7To appear in [AM]
Definition 1.22. The module $\mathcal{X}(\mu U)$ is said to have index $\leq N$ if any linearly independent (over $L^\infty(\mu U)$) set of derivations in it has cardinality at most $N$. If, for any $\mu$-measurable $U$, the module $\mathcal{X}(\mu U)$ has index $\leq N$, we say that $\mathcal{X}(\mu)$ has index locally bounded by $N$.

Under this assumption the author [Sch] obtained the following decomposition result.

Theorem 1.23. Suppose that $\mathcal{X}(\mu)$ has index locally bounded by $N$. Then there is a uniform upper bound on the number of Lipschitz functions which are differentiable on a set $S$ such that, if $\mu(S_i) > 0$, then $\mathcal{X}(\mu|S_i)$ is free of rank $i$. A basis of $\mathcal{X}(\mu|X_i)$ will be called a local basis of derivations.

Theorem 1.23 should compared with [Wea00, Thm. 10]; a posteriori the two approaches turn out to be equivalent; however, in the author’s opinion, a local dimensional bound and a decomposition into free modules are more natural in the study of differentiability spaces.

The third technical tool is to relate Alberti representations and blow-ups of metric spaces. A blow-up of a metric space $X$ at a point $p$ is a (complete) pointed metric space $(Y, q)$ which is a pointed Gromov-Hausdorff limit of a sequence $(\frac{1}{t_n}X, p)$ where $t_n \searrow 0$: the notation $\frac{1}{t_n}X$ means that the metric on $X$ is rescaled by $1/t_n$; the class of blow-ups of $X$ at $p$ is denoted by $\text{Tan}(X, p)$. A blow-up of a Lipschitz function $f : X \to \mathbb{R}^Q$ at a point $p$ is is a triple $(Y, q, g)$ with $(\frac{1}{t_n}X, p) \to (Y, q, g) \in \text{Tan}(X, p)$, $g : X \to \mathbb{R}^Q$ Lipschitz and such that the rescalings $(f - f(p))/t_n : \frac{1}{t_n}X \to \mathbb{R}^Q$ converge to $g$; the class of blow-ups of $f$ at $p$ is denoted by $\text{Tan}(X, p, f)$. The general existence of blow-ups requires the notion of ultralimits: we simplified the treatment assuming that $X$ has finite Assouad dimension ([MT10]) because this assumption is not restrictive in the theory of differentiability spaces.

In Subsection 5.2 we show that:
- If $f : X \to \mathbb{R}^N$ and $\mu$ admits Alberti representations in the $f$-directions of independent cone fields, for $\mu$-a.e. $p$ all blow-ups $(Y, q, g) \in \text{Tan}(X, p, f)$ are such that $g : Y \to \mathbb{R}^N$ is surjective (Theorem 5.93).
- If $X$ has Assouad dimension $D$, then $\mathcal{X}(\mu)$ has index locally bounded by $D$ (Corollary 5.136).

Note that Corollary 5.136 improves [Gon12, Lem. 1.10] by giving an explicit bound equal to the Assouad dimension.

The motivation to prove Theorem 5.93 came from a construction of separated nets in [Bat12, Sec. 8] and the work of Keith [Kei04] which uses blow-ups to study spaces satisfying a Lip-lip inequality.

The fourth tool is a construction of independent Lipschitz functions, Theorem 4.13. Lipschitz functions $\{\psi_1, \ldots, \psi_n\}$ are independent on a set $S$ if, for each $x \in S$, the map $\mathbb{R}^n \ni (a_i) \mapsto \mathcal{L}(\sum_{i=1}^n a_i \psi_i)(x)$ is a norm. The property of being a differentiability space can be reformulated as a finite dimensionality statement: there is a uniform upper bound on the number of Lipschitz functions which are independent on a positive measure set.

In the case of Euclidean spaces, instead of constructing independent functions on a set $S$, it is more natural to construct Lipschitz functions which are not differentiable on $S$. In the case of $\mathbb{R}$, there is a classical construction of Zahorski
([Zah46]); for $\mathbb{R}^n$, a generalized construction is announced in [ACP10, Thm. 1.15]; in the case in which one relaxes the conclusion to nondifferentiability $\mu$-a.e. on $S^g$, the construction is simplified as it can handled independently on different parts of $S$ using a truncation principle (Lemma 5.138). Recently, Alberti-Marchese [AM] strengthened this approach showing that the set of Lipschitz functions which are nondifferentiable on a large part of $S$ is comeagre in a suitable metric space of Lipschitz functions.

In [Bat12, Sec. 4] Bate produces a construction of nondifferentiable Lipschitz functions for measures on metric spaces; the approach is similar to the Euclidean case but requires the tool of **structured charts** introduced in [BS11]. Theorem 4.13 is a reformulation of this result in the language of independent functions; the author thinks this is useful because: [1] it is technically simpler avoiding a discussion of structured charts; [2] it fits more naturally with the approaches of Cheeger and Keith. The author wonders whether it is possible to do a construction of independent functions for sets in metric spaces.

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**List of Symbols**

| Symbol | Description |
|--------|-------------|
| $\text{Alb}_{\text{sub}}(\mu)$ | family of Lipschitz Alberti representations of measures of the form $\mu L^S$ |
| $\mathcal{A}U$ | restriction of $\mathcal{A}$ to $U$ |
| $\text{Der}$ | map associating to an Alberti representation $\mathcal{A}$ a derivation $D_\mathcal{A}$ |
| $D_\mathcal{A}$ | derivation associated to the Alberti representation $\mathcal{A}$ |
| $\mathcal{B}^\infty(X)$ | bounded real-valued Borel functions defined on $X$ |
| $\mathcal{X}(\mu)$ | module of derivations |
| $| \cdot |_{\mathcal{X}(\mu),\text{loc}}$ | local norm on $\mathcal{X}(\mu)$ |
| $\mathcal{E}(\mu)$ | module of forms |
| $| \cdot |_{\mathcal{E}(\mu),\text{loc}}$ | local norm on $\mathcal{E}(\mu)$ |
| $\mathcal{C}(w,\alpha)$ | cone / cone field of direction $w$ and angle $\alpha$ |
| $\text{Frag}(X)$ | parametrized biLipchitz curve fragments in $X$ |
| $\text{md}$ | metric differential |
| $L(f)$ | global Lipschitz constant |
| $\text{Lip}_b(X)$ | bounded real-valued Lipschitz functions defined on $X$ |
| $\text{Lip}(X)$ | real-valued Lipschitz functions defined on $X$ |
| $\mathcal{L}f(x)$ | pointwise upper Lipschitz constant |
| $\ell f(x)$ | pointwise lower Lipschitz constant |
| $\chi_U$ | indicator function of $U$ |
| $\mathcal{H}^m$ | $m$-dimensional Hausdorff measure |
| $\mu L^A$ | restriction of the measure $\mu$ on $A$ |
| $M(Z)$ | finite Radon measures on $Z$ |
| $P(Z)$ | finite Borel probability measures on $Z$ |

\[^9\text{“Nondifferentiability for measures”}\]
2. Preliminaries

2.1. Alberti representations. The goal of this Subsection is to define Alberti representations precisely and prove Theorem 2.64, which can be regarded as a standard criterion to produce Alberti representations. The treatment has been expanded a bit to address what seem to be a couple of little gaps in [Bat12]:

(1) In [Bat12, Lem. 5.2] Alberti representations are produced with $P$ a probability measure on 1-rectifiable measures, instead of fragments (point addressed in Lemma 2.56).

(2) In [Bat12, Sec. 6] Alberti representations are produced with $P$ defined on $\text{Frag}(S)$ where $S$ is a Banach space containing $X$ (point addressed in Theorem 2.14).

We start by defining paths and fragments. Throughout this Section $X$ will denote a Polish metric space.

Definition 2.1. Given a metric space $Y$, let $\text{Haus}(Y)$ denote the topological space of compact sets with the Vietoris topology, which is induced by the Hausdorff distance. If $Y$ is compact, $\text{Haus}(Y)$ is compact and if $Y$ is Polish, $\text{Haus}(Y)$ is Polish. We introduce the set of path fragments:

\begin{equation}
\text{Frag}(X) = \{ \gamma : K \to X : \gamma \text{ biLipschitz}, K \subset \mathbb{R} \text{ compact}, L^1(K) > 0 \};
\end{equation}

which is identified with a subspace of $\text{Haus}(\mathbb{R} \times X)$ via the map $\gamma \mapsto \text{Graph}\gamma$. Given a nondegenerate compact interval $I \subset \mathbb{R}$, we denote by $\text{Frag}(X,I)$ the subset of fragments $\gamma$ with $\text{dom}\gamma \subset I$.

In order to define Alberti representations precisely, we need to recall some facts from measure theory. Let $Z$ denote a locally compact metric space, $M(Z)$ the Banach space of finite (signed) Borel measures on $Z$ and $P(Z) \subset M(Z)$ the subset of probability measures. It might be useful to recall that finite Borel measures on metric spaces are regular and that a finite Borel measure on a Polish space is Radon [Bog07, Thm. 7.1.7]. The Banach space $M(Z)$ is also a dual Banach space; in the definition of $M(Z)$-valued Borel maps we will consider on $M(Z)$ the weak* topology. In particular, given a metric space $Y$, a map $\psi : Y \to M(Z)$, $\psi$ is Borel if and only if for each $g \in C_c(Z)$\(^{10}\) the map $y \mapsto \int_Z g\,d\psi(y)$ is Borel (compare [Alb93, Rem. 1.1]).

We will use the following result ([Alb93, Defn. 1.2]):

Lemma 2.3. Let $Y$ be a separable locally compact topological space, and $\lambda$ a locally finite Borel measure on $Y$. Let $\psi : Y \to M(Z)$ be Borel and, denoting by $\|\psi(y)\|_{M(Z)}$ the norm of $\psi(y)$, assume that

\begin{equation}
\int_Y \|\psi(y)\|_{M(Z)}\,d\lambda(y) < \infty;
\end{equation}

then the integral

\begin{equation}
\mu = \int_Y \psi(y)\,d\lambda(y)
\end{equation}

exists and defines an element of $M(Z)$. More precisely, for a Borel set $A \subset Z$,

\begin{equation}
\mu(A) = \int_Y \psi(y)(A)\,d\lambda(y).
\end{equation}

\(^{10}\)Real-valued continuous functions with compact support
We can now state precisely the definition of an Alberti representation; note that condition (4) has been added for technical convenience (compare the proof of Lemma 3.1).

**Definition 2.7.** Let $\mu$ be a Radon measure on a metric space $X$; an Alberti representation of $\mu$ is a pair $(P, \nu)$:

1. The measure $P$ is in $P(\text{Frag}(X))$.
2. The map $\nu : \text{Frag}(X) \to M(X)$ is Borel and $\nu \ll \mathcal{H}^1$, the one-dimensional Hausdorff measure associated to the image of $\gamma$.
3. The measure $\mu$ can be represented as $\mu = \int_{\text{Frag}(X)} \nu_\gamma \, dP(\gamma)$.
4. For each Borel set $A \subset X$ and for all real numbers $a \geq b$, the map $\gamma \mapsto \nu_\gamma (A \cap \gamma(\text{dom} \gamma \cap [a, b]))$ is Borel.

We now define precisely the notions of Euclidean cones and metric differential employed in the Introduction to introduce the notions of direction and speed.

**Definition 2.8.** Let $\alpha \in (0, \pi/2)$, $w \in S^{q-1}$; the open cone $C(w, \alpha) \subset \mathbb{R}^q$ with axis $w$ and opening angle $\alpha$ is:

$$(2.9) \quad C(w, \alpha) = \{u \in \mathbb{R}^q : \tan \alpha \langle w, u \rangle > \|w^\perp u\|_2\}.$$  

The set of open cones is given the topology of $S^{q-1} \times (0, \pi/2)$.

**Remark 2.10.** Note that if $u \in C(w, \alpha) \cap S^{q-1}$,

$$(2.11) \quad \|u - w\|_2 \leq (1 - \cos \alpha) + \sin \alpha.$$  

We recall the definition of metric differential, which is an adaptation of [AT04, Defn. 4.1.2] (compare [AK00, Sec. 3]):

**Definition 2.12.** Given $\gamma \in \text{Frag}(X)$, the metric differential $\text{md} \gamma(t)$ of $\gamma$ at $t \in \text{dom} \gamma$ is the limit

$$(2.13) \quad \lim_{\text{dom} \gamma \ni t' \to t} \frac{d(\gamma(t'), \gamma(t))}{|t' - t|}$$

whenever it exists.$^{11}$

We will abbreviate a set of conditions on the Lipschitz/biLipschitz constant, speed and direction by (COND).

If $X$ is a metric space and $C \subset X$ with $\mu$ a Radon measure on $C$, there are a priori two different notions of Alberti representations $(P, \nu)$ depending on whether $P \in P(\text{Frag}(X))$ or $P \in P(\text{Frag}(C))$; in the former case we will say that the Alberti representation is defined on $\text{Frag}(X)$. We will prove Theorem 2.14 which produces an Alberti representation defined on $\text{Frag}(C)$ given an Alberti representation defined on $\text{Frag}(X)$ and preserves a set (COND).

**Theorem 2.14.** Let $C$ be compact subset of $X$. Suppose that the Radon measure $\mu$, with support contained in $C$, admits an Alberti representation $(P, \nu)$ defined on $\text{Frag}(X)$ and satisfying (COND). Then $\mu$ admits an Alberti representation $(P', \nu')$ defined on $\text{Frag}(C)$ and satisfying (COND).

The proof of Theorem 2.14 requires some preparation. We start introducing some subsets of fragments.

---

$^{11}$We convene that if $t$ is an isolated point of $\text{dom} \gamma$, the limit does not exist
**Definition 2.15.** Let \( C \subset X \) and \( n \in \mathbb{N} \). We define:

\[
\text{Frag}(X, C) = \{ \gamma \in \text{Frag}(X) : \gamma^{-1}(C) \text{ has positive Lebesgue measure} \}.
\]

\[
\text{Frag}_n(X) = \{ \gamma \in \text{Frag}(X) : \text{dom } \gamma \subset [-n, n], \gamma \text{ is } \left(\frac{1}{n}, n\right)\text{-biLipschitz}, \text{ and } \mathcal{L}^1(\text{dom } \gamma) \geq \frac{1}{n} \};
\]

\[
\text{Frag}_n(X, C) = \{ \gamma \in \text{Frag}_n(X) : \mathcal{L}^1(\gamma^{-1}(C)) \geq \frac{1}{n} \}.
\]

**Lemma 2.19.** The sets \( \text{Frag}_n(X) \) and \( \text{Frag}_n(X, C) \) are closed in \( \text{Haus}(\mathbb{R} \times X) \). If \( C \subset X \) is compact, then \( \text{Frag}_n(C) \) is compact.

**Proof.** The subset of \( \text{Haus}(\mathbb{R} \times X) \) consisting of graphs of \( (\frac{1}{n}, n)\) biLipschitz maps \( \gamma : K \subset \mathbb{R} \to X \) is closed. Also the set of those compact sets \( K \subset \mathbb{R} \times X \), whose projection \( \pi_\mathbb{R}(K) \) on \( \mathbb{R} \) satisfies \( \pi_\mathbb{R}(K) \subset [-n, n] \), is closed. Consider a sequence of compact sets \( \{K_k\} \subset \text{Haus}(\mathbb{R}) \) and assume that for each \( k \) one has \( \mathcal{L}^1(K_k) \geq \frac{1}{n} \); then, if the \( K_k \) converge to \( K \), we have \( \mathcal{L}^1(K) \geq \frac{1}{n} \). We thus conclude that the set \( \text{Frag}_n(X) \) is closed in \( \text{Haus}(\mathbb{R} \times X) \). The compactness of \( \text{Frag}_n(C) \) follows from Ascoli-Arzelà. We now show that \( \text{Frag}_n(X, C) \) is a closed subset of \( \text{Frag}_n(X) \); suppose that the sequence of fragments \( \{\gamma_k\} \subset \text{Frag}_n(X, C) \) converges to \( \gamma \); by passing to a subsequence we can also assume that the compact sets \( \gamma_k^{-1}(C) \) converge to a compact set \( K \subset \text{dom } \gamma \). We then have \( \mathcal{L}^1(K) \geq \frac{1}{n} \) and \( \gamma(K) \subset C \), showing that \( \text{Frag}_n(X, C) \) is closed.

An immediate consequence of Lemma 2.19 is:

**Lemma 2.20.** The sets \( \text{Frag}(X) \) and \( \text{Frag}(X, C) \) are of class \( F_\sigma \) in \( \text{Haus}(\mathbb{R} \times X) \). If \( C \subset X \) is compact, then \( \text{Frag}(C) \) is of class \( K_\sigma \) in \( \text{Haus}(\mathbb{R} \times X) \).

The key tool to prove Theorem 2.14 is the following Lemma.

**Lemma 2.21.** If \( X \) is a complete separable metric space and if \( C \subset X \) is compact, the restriction map

\[
\text{Red}_C : \text{Frag}(X, C) \to \text{Frag}(C)
\]

\[
\gamma \mapsto \gamma|\gamma^{-1}(C)
\]

is Borel.

**Proof.** Note that \( \text{Red}_C \) maps \( \text{Frag}_n(X, C) \) to \( \text{Frag}_n(C) \) so by Lemma 2.20 it suffices to show that the restriction \( \text{Red}_{C,n} = \text{Red}_C|\text{Frag}_n(X, C) \) is Borel. By [Bog07, Lem. 6.2.5] it suffices to show that for each \( \psi \in C(\text{Frag}_n(C)) \)\textsuperscript{12}, the map \( \psi \circ \text{Red}_{C,n} \) is Borel.

If \( (t, x) \in \mathbb{R} \times C \), let \( d_{(t,x)}(K) \) denote the distance from the set \( K \) to the point \( (t, x) \). Note that \( d_{(t,x)} \) is Lipschitz: for each \( \varepsilon > 0 \) if \( (t_1, x_1) \in K \) is a closest point to \( (x, t) \), there is a point \( (t_2, x_2) \in K' \) with

\[
d((t_1, x_1), (t_2, x_2)) \leq d_H(K, K') + \varepsilon,
\]

which implies

\[
d_{(t,x)}(K') \leq d_{(t,x)}(K) + d_H(K, K').
\]

\textsuperscript{12}Set of real-valued continuous functions
Note that these functions separate points in Haus(\(\mathbb{R} \times X\)). As \(\text{Frag}_n(C)\) is compact, by the Stone-Weierstraß Theorem [Rud76, Thm. 7.32] the unital subalgebra of \(C(\text{Frag}_n(C))\) generated by the functions \(\{d(t,x)\}\) is dense. In particular, any \(\psi \in C(\text{Frag}_n(C))\) is a uniform limit of polynomials in the \(\{d(t,x)\}\). Therefore, it suffices to show that \(d(t,x) \circ \text{Red}_{C,n}\) is Borel.

We show that \(d(t,x) \circ \text{Red}_{C,n}\) is lower semicontinuous: assume that \(\gamma_m \rightarrow \gamma\) in \(\text{Frag}_n(X, C)\) and that along a subsequence \(m_k\) one has

\[
(2.25) \quad d(t,x)\left(\text{Red}_{C,n}(\gamma_{m_k})\right) \leq \varepsilon.
\]

It is then possible to find points \(t_{m_k} \in \text{dom} \gamma_{m_k}\) with \(\gamma_{m_k}(t_{m_k}) = x_{m_k} \in C\) and \(d((t_{m_k}, x_{m_k}), (t, x)) \leq \varepsilon\). By passing to a subsequence we can assume that \(t_{m_k} \rightarrow \bar{t} \in \text{dom} \gamma, x_{m_k} \rightarrow \bar{x} \in C\) and \(\bar{x} = \gamma(\bar{t})\). As \(d((\bar{t}, \bar{x}), (t, x)) \leq \varepsilon\),

\[
(2.26) \quad d(t,x)\left(\text{Red}_{C,n}(\gamma)\right) \leq \liminf_{m \rightarrow \infty} d(t,x)\left(\text{Red}_{C,n}(\gamma_m)\right).
\]

□

The second step in the proof of Theorem 2.14 is a simplification of the description of Alberti representations which relies on the following map \(\Psi\).

**Lemma 2.27.** Let \(X\) be a Polish space and consider the map

\[
(2.28) \quad \Psi : \text{Frag}(X) \rightarrow M(X) \quad \gamma \mapsto \gamma^\# (L_1^1 \setminus \text{dom} \gamma) = \Psi_\gamma;
\]

then \(\Psi\) is Borel and, if \(A \subset X\) is Borel and \([a,b] \subset \mathbb{R}\), the map

\[
(2.29) \quad \gamma \mapsto \Psi_\gamma(A \cap \gamma(\text{dom} \gamma \cap [a,b]))
\]

is Borel.

**Proof.** It suffices to show that if \(g\) is a real-valued continuous function on \(X\), the map

\[
(2.30) \quad \Psi_{g,a,b} : \gamma \mapsto \int_a^b g \circ \gamma(t) \chi_{\text{dom} \gamma}(t) \, dt
\]

is Borel. Without loss of generality we can assume that \(g\) is nonnegative, in which case we will show that \(\Psi_{g,a,b}\) is upper semicontinuous. Assuming that \(\gamma_n \rightarrow \gamma\) in \(\text{Frag}(X)\), we have:

\[
(2.31) \quad \limsup_{n \rightarrow \infty} (g \circ \gamma_n \chi_{\text{dom} \gamma_n}) \leq g \circ \gamma \chi_{\text{dom} \gamma};
\]

in fact, either a point \(t\) belongs to infinitely many of the \(\text{dom} \gamma_n\), in which case \(t \in \text{dom} \gamma\) and \(\gamma_n(t) \rightarrow \gamma(t)\), or eventually the point \(t\) does not belong to \(\text{dom} \gamma_n\), in which case the left hand side of (2.31) is 0. By the reverse Fatou Lemma:

\[
(2.32) \quad \limsup_{n \rightarrow \infty} \int_a^b g \circ \gamma_n \chi_{\text{dom} \gamma_n} \, dt \leq \int_a^b g \circ \gamma \chi_{\text{dom} \gamma} \, dt,
\]

which is the upper semicontinuity of \(\Psi_{g,a,b}\).

□

We now obtain a simplification of the description of Alberti representations: one can assume that \(\nu_\gamma = g \Psi_\gamma\), where \(g\) is a Borel function on \(X\).
Lemma 2.33. Let $X$ be a compact metric space with a Radon measure $\mu$ admitting an Alberti representation $(P, \nu)$ satisfying (COND). Then there is an Alberti representation $A' = (P', \nu')$ which satisfies (COND) and with $\nu' = g \Psi_\gamma$, where $g$ is Borel on $X$.

Proof. By Lemma 2.20 we can find disjoint Borel sets $F_n \subset \text{Frag}_n(X)$ with $\text{Frag}(X) = \bigcup_n F_n$. Note that if $\gamma \in F_n$, then we have the bound $\|\Psi_\gamma\| \leq 2n$. By Lemma 2.3 the integral
\begin{equation}
\tilde{\mu} = \sum_n \int_{F_n} \frac{1}{2n} \Psi_\gamma dP(\gamma)
\end{equation}
defines a finite Radon measure on $X$ with total mass at most 1. If $\tilde{\mu}(A) = 0$, then for $P$-a.e. $\gamma$, $\Psi_\gamma(A) = 0$ which implies $\nu_\gamma(A) = 0$ as $H^1_\gamma \ll \Psi_\gamma$. Introducing the measure
\begin{equation}
\tilde{P} = \sum_n \frac{1}{2n} P_{\text{Frag}} F_n,
\end{equation}
we obtain a finite Borel measure on $\text{Frag}(X)$ and we have
\begin{equation}
\tilde{\mu} = \int_{\text{Frag}(X)} \Psi_\gamma d\tilde{P}(\gamma).
\end{equation}
As $\tilde{\mu} \gg \mu$, by the Radon-Nikodym Theorem [Rud87, Thm. 6.10], there is a Borel function $\tilde{g}$ on $X$ with $\mu = \tilde{g} \tilde{\mu}$. Then
\begin{equation}
\mu = \int_{\text{Frag}(X)} \tilde{g} \Psi_\gamma d\tilde{P}
\end{equation}
and the result follows letting
\begin{align}
&g = \tilde{g} \tilde{P}(\text{Frag}(X)); \\
&\nu' = g \Psi_\gamma; \\
&P' = \frac{1}{\tilde{P}(\text{Frag}(X))} \tilde{P}.
\end{align}
From the way in which $\tilde{P}$ was obtained from $P$, we conclude that if the original Alberti representation $(P, \nu)$ satisfied (COND), so does the new one $(P', \nu')$. \hfill \Box

We can now prove Theorem 2.14.

Proof of Theorem 2.14. By the previous Lemma we can assume $\nu_\gamma = g \Psi_\gamma$. Note that as
\begin{equation}
\mu(X \setminus C) = 0,
\end{equation}
for $P$-a.e. $\gamma \in \text{Frag}(X) \setminus \text{Frag}(X, C)$,
\begin{equation}
g \Psi_\gamma = 0.
\end{equation}
In particular, replacing $P$ by
\begin{equation}
\frac{P_{\text{Frag}}(X,C)}{P(\text{Frag}(X,C))},
\end{equation}
we can assume that
\begin{equation}
\mu(X \setminus \text{Frag}(X, C)) = 0.
\end{equation}
Then for a Borel, 
\[
\mu(A) = \int_{\text{Frag}(X, C)} dP(\gamma) \int_X g\chi_A d\Psi_\gamma \\
= \int_{\text{Frag}(X, C)} dP(\gamma) \int_X g\chi_A \chi_C d\Psi_\gamma \quad \text{(by (2.41))} \\
= \int_{\text{Frag}(X, C)} dP(\gamma) \int_X g\chi_A d\Psi(\text{Red}_C(\gamma)) \\
= \int_{\text{Frag}(C)} d\text{Red}_C P(\gamma) \int_X g\chi_A d\Psi_\gamma \quad \text{(by pushing forward)}
\]
where in the last step we used that \text{Red}_C is Borel by Lemma 2.21. Letting \( P' = \text{Red}_C P \) and \( \nu' = g\Psi_\gamma \), \((P', \nu')\) is an Alberti representation of \( \mu \) defined on \( \text{Frag}(C) \) (using again Lemma 2.3). From the way in which \( P' \) was obtained from \( P \), if \((P, \nu)\) satisfied (COND), so does \((P', \nu')\). □

The following is a gluing principle for Alberti representations.

**Theorem 2.46.** Let \( \mu \) be a Radon measure on \( Z \) and \( U \subset Z \) Borel. Assume that there are disjoint Borel sets \( U_n \subset U \) isometrically embedded in some Polish spaces \( Y_n (U_n \hookrightarrow Y_n) \) with \( \mu|U_n \) admitting an Alberti representation defined on \( \text{Frag}(Y_n) \) and satisfying (COND). Then \( \mu|U \) admits an Alberti representation defined on \( \text{Frag}(Z) \) and satisfying (COND).

**Proof.** Using that \( \mu \) is Radon we can find disjoint compact subsets \( \{C_\alpha\} \) such that for each \( \alpha \) there is an \( n_\alpha \) with \( C_\alpha \subset U_{n_\alpha} \) and

\[
\mu \left( U \setminus \bigcup_\alpha C_\alpha \right) = 0.
\]

As \( \mu|U_{n_\alpha} \) admits an Alberti representation \( (P_{n_\alpha}, \nu_{n_\alpha}) \) defined on \( \text{Frag}(Y_{n_\alpha}) \) and satisfying (COND), so does \( \mu|C_\alpha \) by letting \( P'_\alpha = P_{n_\alpha} \) and \( \nu'_\alpha = \nu_{n_\alpha} \text{Red}_{C_\alpha} P_{n_\alpha} \). Then by Theorem 2.14, \( \mu|C_\alpha \) admits an Alberti representation \( (P_\alpha, \nu_\alpha) \) defined on \( \text{Frag}(C_\alpha) \) and satisfying (COND). Observing that the \( \text{Frag}(C_\alpha) \) are disjoint Borel subsets of \( \text{Frag}(Z) \), we obtain an Alberti representation \( (P, \nu) \) of \( \mu|U \) defined on \( \text{Frag}(Z) \) and satisfying (COND) by letting

\[
P = \sum_{\alpha = 1}^{\infty} 2^{-\alpha} P_\alpha, \\
\nu = \sum_{\alpha = 1}^{\infty} 2^{\alpha} \nu_\alpha.
\]

□

We now introduce the key concept to build Alberti representations in terms of nullity of a set with respect to a family of fragments.

**Definition 2.50.** If \( G \subset \text{Frag}(X) \), a Borel \( S \subset X \) is said to be \( G \)-null if for each \( \gamma \in G, \beta(1, \gamma)(S) = 0 \). In case \( G \) is the family of fragments satisfying a set of conditions (COND), we say that \( S \) is (COND)-null.
We now prove Lemma 2.56 which produces biLipchitz Alberti representations in Banach spaces. Throughout the remainder of this Subsection $\mathcal{K}$ denotes a closed convex compact subset of $S^*$, where $S$ is a separable Banach space. This is motivated by the fact that any separable metric space $X$ can be isometrically embedded in $l^\infty$ via a Kuratowski embedding. The closed convex hull $\mathcal{K}$ of $X$ in $S^*$ is compact by [Rud91, Thm. 3.25].

**Definition 2.51.** For positive $\varepsilon$ and $\tau$, let $QG(\varepsilon, \tau) \subset \text{Frag}(S^*)$ denote the compact subset

\[
QG(\varepsilon, \tau) = \{ \gamma : [0, \tau] \to \mathcal{K} : \forall t, s \in [0, \tau], |t - s| \leq \|\gamma(t) - \gamma(s)\|_{S^*} \leq (1 + \varepsilon)|t - s| \}.
\]

**Lemma 2.53.** Let $\Psi : QG(\varepsilon, \tau) \to P(\mathcal{K})$ be defined by

\[
\Psi_\gamma = \frac{1}{\tau} \gamma_1 \mathcal{L}^1[0, \tau];
\]

then $\Psi$ is continuous.

**Proof.** Being $QG(\varepsilon, \tau)$ and $P(\mathcal{K})$ metric spaces, it suffices to check continuity for sequences. Assume that $\gamma_n \to \gamma$ in $QG(\varepsilon, \tau)$ and that $f$ is continuous with compact support in $\mathcal{K}$. Then, as $f \circ \gamma_n \to f \circ \gamma$ uniformly,

\[
\lim_{n \to \infty} \frac{1}{\tau} \int_0^\tau f \circ \gamma_n \, d\mathcal{L}^1 = \frac{1}{\tau} \int_0^\tau f \circ \gamma \, d\mathcal{L}^1.
\]

\[\square\]

**Lemma 2.56.** Let $G \subset QG(\varepsilon, \tau)$ be compact and $\bar{H}_G$ denote the closed convex hull of $\Psi(G)$ in $M(\mathcal{K})$. Let $\mu$ be a Radon measure on $\mathcal{K}$. Then

\[
\mu = \mu_G + \mu_{FG},
\]

with $\mu_G \ll \rho_G$ and $\rho_G \in \bar{H}_G$; furthermore, there are a Borel regular probability measure $P_G$ on $G$ and a Borel function $f_G$ on $\mathcal{K}$ such that

\[
\rho_G = \int_G \Psi_\gamma \, dP_G,
\]

\[
\mu_G = \int_G f_G \Psi_\gamma \, dP_G.
\]

The set $F_G$ is a $G$-null $F_\sigma$.

**Proof.** The space $M(\mathcal{K})$ in the weak* topology is a Fréchet space and so by [Rud91, Thm. 3.25] the closed convex hull of a compact subset is compact, so $\bar{H}_G$ is compact. By Rainwater’s Lemma ([Rud08, Thm. 9.4.4], [Rain69]) we can find $\mu_G, \rho_G \in \bar{H}_G$ and $F_G$ such that:

1. The measure $\mu$ decomposes as $\mu = \mu_G + \mu_{FG}$.
2. The measure $\mu_G$ satisfies $\mu_G \ll \rho_G$.
3. The set $F_G$ is $\Psi(G)$-null in the sense that, for each $\gamma \in G$, $\Psi_\gamma(F_G) = 0$.

Note that (3) implies that $F_G$ is $G$-null because for $\gamma \in G$, $\Psi_\gamma$ and $\mathcal{K}^1_\gamma$ are absolutely continuous one with respect to the other. By [Rud91, Thm. 3.28] there is a regular Borel probability measure $\pi_G$ supported on $\Psi(G)$ and such that

\[
\rho_G = \int_{\Psi(G)} \sigma \, d\pi_G(\sigma).
\]
As $\Psi$ is continuous and $\mathcal{G}$ is compact, by [Bog07, Thm. 6.9.7] there is a Borel $\tilde{\mathcal{G}} \subset \mathcal{G}$ with $\Psi(\tilde{\mathcal{G}}) = \Psi(\mathcal{G})$ and such that $\Psi$ is injective on $\tilde{\mathcal{G}}$ with a Borel inverse

(2.61) \hspace{1cm} \tilde{\Psi}^{-1} : \Psi(\tilde{\mathcal{G}}) \to \mathcal{G};

in particular,

\begin{align*}
\rho_{\tilde{\mathcal{G}}} &= \int_{\Psi(\tilde{\mathcal{G}})} \sigma \, d\pi_{\tilde{\mathcal{G}}}(\sigma) \\
(2.62) &= \int_{\Psi(\tilde{\mathcal{G}})} \Psi(\tilde{\Psi}^{-1}(\sigma)) \, d\pi_{\mathcal{G}}(\sigma) \\
&= \int_{\mathcal{G}} \Psi_{\gamma} \, d\left(\tilde{\Psi}_{\tilde{\mathcal{G}}}^{-1}\pi_{\tilde{\mathcal{G}}}(\gamma)\right).
\end{align*}

If $f\mathcal{G}$ is a Borel representative of the Radon-Nikodym derivative of $\mu_{\mathcal{G}}$ with respect to $\rho_{\tilde{\mathcal{G}}}$,

(2.63) \hspace{1cm} \mu_{\mathcal{G}} = \int_{\tilde{\mathcal{G}}} f_{\mathcal{G}} \Psi_{\gamma} \, dP_{\mathcal{G}}.

\[\square\]

The following theorem is the main tool we will use to produce Alberti representations.

**Theorem 2.64.** Let $X$ be a separable metric space and $\mu$ a Radon measure on $X$. Let $\Omega$ denote the set of fragments in the $f$-direction of $\mathcal{C}(v, \alpha)$ with $g$-speed $> \delta^{13}$. Then the following are equivalent:

1. The measure $\mu$ admits an Alberti representation in the $f$-direction of $\mathcal{C}(v, \alpha)$ with $g$-speed $> \delta$.
2. For each $K \subset X$ compact and $\Omega$-null, $\mu(K) = 0$.
3. For each $\varepsilon > 0$, $\mu$ admits a $(1, 1 + \varepsilon)$-biLipschitz Alberti representation in the $f$-direction of $\mathcal{C}(v, \alpha)$ with $g$-speed $> \delta$.

**Proof.** We show that (1) implies (2). Let $\mathcal{A} = (P, \nu)$ with $P$ giving full measure to a Borel set $\Xi$ of fragments in the $f$-direction of $\mathcal{C}(v, \alpha)$ with $g$-speed $> \delta$. If $K \subset X$ is $\Omega$-null, it is $\Xi$-null, so

(2.65) \hspace{1cm} \mu(K) = \int_{\Xi} \nu_{\gamma}(K) \, dP(\gamma) = 0.

We show that (2) implies (3) first assuming that $X$ is compact and $v$, $\alpha$ and $\delta$ are constant. We rescale $f : X \to \mathbb{R}^q$ and $g$ to be 1-Lipschitz, embedd $\text{Graph}(f,g)$ isometrically in $X \times \mathbb{R}^q \times \mathbb{R} = S^*$ with norm

(2.66) \hspace{1cm} \|\langle \xi, v, t \rangle\|_{L^\infty \times \mathbb{R}^q \times \mathbb{R}} = \max(\|\xi\|_{L^\infty}, \|v\|_2, |t|),

and take the convex hull $\chi$ of the resulting set.

---

13 We require a strict inequality
Having chosen strictly decreasing sequences \( \{\tau_n\}, \{\eta_n\} \) converging to 0, we let
\[
(2.67) \quad G_n = \{ \gamma \in \text{QG}(\varepsilon, \tau_n) : \forall t, s \in [0, \tau_n], \quad 
\sgn(t-s)(f \circ \gamma(t) - f \circ \gamma(s)) \in \tilde{C}(v, \alpha - \eta_n) \\
\text{and} \quad \sgn(t-s)(g \circ \gamma(t) - g \circ \gamma(s)) \geq (\delta + \eta_n)|t-s| \},
\]
which is a compact subset of \( \text{QG}(\varepsilon, \tau_n) \). We then apply Lemma 2.56 repeatedly; having obtained
\[
(2.68) \quad \mu = \mu G_1 + \mu \mathcal{L} \gamma_1,
\]
we apply Lemma 2.56 to \( \mu \mathcal{L} \gamma_1 \) to obtain a decomposition
\[
(2.69) \quad \mu = \mu G_1 + \mu G_2 + \mu \mathcal{L} \gamma_1,2
\]
where \( \mu G_1 \) and \( \mu G_2 \) are concentrated on disjoint \( G_\delta \) sets, and where \( \mathcal{L} \gamma_1,2 \) is an intersection of two \( \gamma \) sets and is \( G_1, G_2 \)-null. Iterating,
\[
(2.70) \quad \mu = \sum_{n=1}^{\infty} \mu G_n + \mu \mathcal{L} \gamma,
\]
where \( F \) is an \( \gamma \) which is \( G_n \)-null for every \( n \).

We now show that \( \mu(F) = 0 \) using condition (2). The following observation (Ob1) will be used repeatedly: if \( \gamma \in \text{Frag}(\mathcal{X}) \) and the \( \{K_n\} \subset \text{dom} \gamma \) are (countably many) compact sets with
\[
(2.71) \quad \mathcal{L}^1(\text{dom} \gamma \setminus \bigcup_{\alpha} K_\alpha) = 0
\]
and \( \mathcal{H}^1 |_{K_\alpha}(F) = 0 \), then \( \mathcal{H}^1 \gamma(F) = 0 \).

If \( \gamma \) is a fragment in the \( f \)-direction of \( \mathcal{C}(v, \alpha) \) with \( g \)-speed \( > \delta \), by (Ob1) and Egorov and Lusin’s Theorems [Bog07, Thms. 7.1.12 and 7.1.13], letting \( I_\gamma = [\text{min} \gamma, \text{max} \gamma] \) we can assume that \( \forall t, s \in \text{dom} \gamma \)
\[
(2.72) \quad \|f \circ \gamma(t) - f \circ \gamma(s) - w(t-s)\| \leq \rho|t-s|;
\]
\[
(2.73) \quad (l - \rho)|t-s| \leq \|\gamma(t) - \gamma(s)\|_{S^*} \leq (L + \rho)|t-s|;
\]
\[
(2.74) \quad \sgn(t-s)(g \circ \gamma(t) - g \circ \gamma(s)) \geq (\delta + \eta - \rho)\|\gamma(t) - \gamma(s)\|_{S^*},
\]
where \( \eta > 0, w \in \mathcal{C}(v, \alpha - \eta), \frac{L}{T} < 1 + \varepsilon \) and \( \rho \) is small enough to be chosen later. Moreover, by the Lebesgue’s differentiation theorem we can also assume that \( \forall t, s \in I_\gamma \) with at least one of them in \( \text{dom} \gamma \),
\[
(2.75) \quad \mathcal{L}^1(\text{dom} \gamma \cap [t, s]) \geq (1-\rho)|t-s|.
\]
Let \( \{(u_n, v_n)\} \) be the components of \( I_\gamma \setminus \text{dom} \gamma \). On each component \( (u_n, v_n) \) we extend \( \gamma \) using the fact that \( \mathcal{X} \) is convex
\[
(2.76) \quad \gamma(t) = \frac{t-u_n}{v_n-u_n} \gamma(v_n) + \frac{v_n-t}{v_n-u_n} \gamma(u_n);
\]
if \( (s, t) \subset (u_n, v_n) \),
\[
(2.77) \quad \gamma(t) - \gamma(s) = \frac{t-s}{v_n-u_n}(\gamma(v_n) - \gamma(u_n))
\]
so that (2.73) remains true; in particular, the bound on the metric derivative implies that the extension is \((L + \rho)\)-Lipschitz. Similarly,

\begin{align}
(2.78) \quad f \circ \gamma(t) - f \circ \gamma(s) &= \frac{t-s}{v_n - u_n} (f \circ \gamma(v_n) - f \circ \gamma(u_n)) \\
(2.79) \quad g \circ \gamma(t) - g \circ \gamma(s) &= \frac{t-s}{v_n - u_n} (g \circ \gamma(v_n) - g \circ \gamma(u_n))
\end{align}

so that (2.72) and (2.74) remain true. If \(t, s \in I_\gamma \setminus \text{dom } \gamma\) but in different components, there are \(s', t' \in \text{dom } \gamma\):

\begin{align}
(2.80) \quad s \leq s' \leq t' \leq t
\end{align}

and each of \((s, s')\) and \((t', t)\) is contained in a component of \(I_\gamma \setminus \text{dom } \gamma\). In particular,

\begin{align}
(2.81) \quad |s - s'| \leq \rho |s' - t| \leq \rho |s - t|; \\
(2.82) \quad |t' - t| \leq \rho |s - t'| \leq \rho |s - t|.
\end{align}

Therefore, (2.72), (2.73) and (2.74) generalize to

\begin{align}
(2.83) \quad \|f \circ \gamma(t) - f \circ \gamma(s) - w(t-s)\|_2 &\leq 2\rho(L + \rho + \|v\|) |t-s|; \\
(2.84) \quad |(l-\rho)(1-2\rho) - 2(L + \rho)\rho| \|t-s\| \leq \|\gamma(t) - \gamma(s)\|_{s^*} \leq [L + \rho(1 + 2(L + \rho))] |t-s|;
\end{align}

\begin{align}
(2.85) \quad \text{sgn}(t-s) (g \circ \gamma(t) - g \circ \gamma(s)) \geq \left(\delta + \eta - \rho - \frac{L + \delta + \eta}{(l-\rho)(1-2\rho) - 2(L + \rho)\rho}\right) \|\gamma(t) - \gamma(s)\|_{s^*}.
\end{align}

If \(\eta' \in (0, \eta)\), for \(\rho\) sufficiently small we have:

\begin{align}
(2.86) \quad B(w, 2\rho(L + \rho + \|v\|)) &\in \mathcal{C}(v, \alpha - \eta'); \\
(2.87) \quad \frac{L + \rho(1 + 2(L + \rho))}{(l-\rho)(1-2\rho) - 2(L + \rho)\rho} &\leq 1 + \varepsilon; \\
(2.88) \quad \left(\delta + \eta - \rho - 2\rho\frac{L + \delta + \eta}{(l-\rho)(1-2\rho) - 2(L + \rho)\rho}\right) &\geq \delta + \eta'.
\end{align}

We can now reparametrize \(\gamma\) to be \((1, 1 + \varepsilon)\)-biLipschitz, subdivide the domain and choose \(n\) sufficiently large so that \(\gamma \in \mathcal{G}_n\). Thus, \(\mathcal{H}^1(\gamma(F)) = 0\). This implies that

\begin{align}
(2.89) \quad \mu = \sum_{n=1}^{\infty} \mu_{\mathcal{G}_n} \quad \text{(on account of condition (2))}.
\end{align}

As the measures \(\mu_{\mathcal{G}_n}\) are concentrated on pairwise disjoint sets, by Theorem 2.46 we obtain a representation \(\mathcal{A}' = (P', \nu')\) of \(\mu\) in the \(f\)-direction of \(\mathcal{C}(v, \alpha)\) with \(g\)-speed \(\geq \delta\) with \(P' \in P(\text{Frag}(\mathcal{X}))\); Theorem 2.14 gives a representation \(\mathcal{A} = (P, \nu)\) with \(P \in P(\text{Frag}(X))\).

The case in which \(X\) is not compact and \(v, \alpha\) and \(\delta\) are not constant is treated by using Egorov and Lusin’s Theorems to find Borel functions \((v_n, \alpha_n, \delta_n)\) such that:

- There are disjoint compacts \(\{K_{n,j}\}_j\) with \(\mu \left( X \setminus \bigcup_j K_{n,j} \right) = 0\) and the \((v_n, \alpha_n, \delta_n)\) are constant on each \(K_{n,j}\).
The functions $\delta_n \searrow \delta$ pointwise.

The cone fields $C(v_n, \alpha_n) \nearrow C(v, \alpha)$ pointwise.

One then applies the construction for constant cone fields and speeds on the $K_{n,j}$ recursively to obtain $\mu = \sum_{n,j} \mu_{n,j}$ where the measures $\{\mu_{n,j}\}_{n,j}$ have pairwise disjoint supports and each $\mu_{n,j}$ has an Alberti representation in the $f$-direction of $C(v_n, \alpha_n)$ with $g$-speed $> \delta_n$. These representations are then glued via Theorem 2.46 to give a representation of $\mu$.

That (3) implies (1) is immediate from the definitions.

2.2. Derivations and $L^\infty$-modules. In this Subsection we recall some facts about derivations and $L^\infty$-modules. Recall that the Banach space $L^\infty(\mu)$ is a real Banach algebra, and, in particular, a ring. We will denote by $L^\infty_+(\mu)$ the set of nonnegative elements $f$:

\begin{equation}
\mu \{ x \in X : f(x) < 0 \} = 0.
\end{equation}

The map absolute value

\begin{equation}
L^\infty(\mu) \xrightarrow{|\cdot|} L^\infty_+(\mu)
\end{equation}

\begin{equation}
f \mapsto |f|
\end{equation}

is sort of an $L^\infty$-valued seminorm:

(1) For all $c_1, c_2 \in \mathbb{R}$, and for all $f_1, f_2 \in L^\infty(\mu)$,

\begin{equation}
|c_1 f_1 + c_2 f_2| \leq |c_1| |f_1| + |c_2| |f_2|.
\end{equation}

(2) For all $f_1, f_2 \in L^\infty(\mu)$,

\begin{equation}
|f_1 f_2| = |f_1||f_2|.
\end{equation}

note also that $L^\infty_+(\mu) = \{ f : f = |f| \}$. In particular, $\|f\|_{L^\infty(\mu)} = \|f\|_{L^\infty(\mu)}$.

An $L^\infty(\mu)$-module $M$ is a Banach space $M$ which is also an $L^\infty(\mu)$-module (with the usual boundedness requirement on the $L^\infty(\mu)$-action). An algebraic submodule $M' \subset M$ is a called a submodule if it is closed. Algebraic concepts like direct sum, linear independence, free module and basis of a free module, extend immediately. Given $S \subset M$ the linear span of $S$ will be denoted by $\text{Span}_{L^\infty(\mu)}(S)$. The dual hom $(M, L^\infty(\mu))$ of an $L^\infty(\mu)$-module $M$ is the $L^\infty(\mu)$-module of bounded module homomorphisms

\begin{equation}
f : M \to L^\infty(\mu).
\end{equation}

Among $L^\infty(\mu)$-modules a special rôle is played by $L^\infty(\mu)$-normed modules.

**Definition 2.95.** An $L^\infty(\mu)$-module $M$ is said to be an $L^\infty(\mu)$-normed module if there is a map

\begin{equation}
| \cdot |_{M,\text{loc}} : M \to L^\infty_+(\mu)
\end{equation}

such that:

(1) The function $| \cdot |_{M,\text{loc}}$ is a “seminorm”: $\forall c_1, c_2 \in \mathbb{R}, \forall m_1, m_2 \in M$,

\begin{equation}
|c_1 m_1 + c_2 m_2| \leq |c_1| |m_1| + |c_2| |m_2|.
\end{equation}

(2) For each $\lambda \in L^\infty(\mu)$ and each $m \in M$,

\begin{equation}
|\lambda m|_{M,\text{loc}} = |\lambda| |m|_{M,\text{loc}}.
\end{equation}
(3) The local seminorm $|·|_{M, \text{loc}}$ can be used to reconstruct the norm of any $m \in M$:

$$\|m\|_M = \|m|_{M, \text{loc}}\|_{L^\infty(\mu)}.$$  

However, if the module $M$ is a Banach space $L^\infty(\mu)$-normed module. The ring $L^\infty(\mu)$ has many idempotents $\chi_U \in L^\infty_+(\mu)$ (characteristic function of the $\mu$-measurable set $U$), which we will call projections as they give direct sum decompositions

$$L^\infty(\mu) = \chi_U L^\infty(\mu) \oplus (1 - \chi_U) L^\infty(\mu),$$

and by Lipschitz functions defined on $X$ and by $L^\infty(\mu)$-module if and only if

$$\|m\|_M = \max(||\chi_U m||_M, ||(1 - \chi_U)m||_M);$$

in particular if the $U_\alpha$ are disjoint and $\mu(X \setminus \bigcup U_\alpha) = 0$,

$$\|m\|_M = \sup_\alpha \|\chi_{U_\alpha} m\|_M.$$  

A simple and important criterion ([PW98, Thm. 9]) for an $L^\infty(\mu)$-module $M$ to be an $L^\infty(\mu)$-normed module is the following:

**Lemma 2.102.** An $L^\infty(\mu)$-module $M$ is an $L^\infty(\mu)$-normed module if and only if

$$\forall U \text{-measurable and } \forall m \in M:$$

$$\|m\|_M = \max(||\chi_U m||_M, ||(1 - \chi_U)m||_M);$$

in particular if the $U_\alpha$ are disjoint and $\mu(X \setminus \bigcup U_\alpha) = 0$,

$$\|m\|_M = \sup_\alpha \|\chi_{U_\alpha} m\|_M.$$  

**Example 2.105.** While $L^\infty(\mu)$ is an $L^\infty(\mu)$-normed module, the previous Lemma implies that for $p \in [1, \infty)$, the $L^p(\mu)$ are not $L^\infty(\mu)$-normed modules.

**Lemma 2.106.** If $M$ is an $L^\infty(\mu)$-module, $\text{hom}(M, L^\infty(\mu))$ is an $L^\infty(\mu)$-normed module.

**Proof.** Given $\xi \in \text{hom}(M, L^\infty(\mu))$ and $\varepsilon > 0$, choose $m \in M$ with $\|m\|_M = 1$ and

$$\|\xi\|_{\text{hom}(M, L^\infty(\mu))} \leq \|\xi(m)\|_{L^\infty(\mu)} + \varepsilon.$$  

As $L^\infty(\mu)$ is an $L^\infty(\mu)$-normed module, $\forall U \text{-measurable,}$

$$\|\xi(m)\|_{L^\infty(\mu)} = \max(||\chi_U \xi(m)||_{L^\infty(\mu)}, ||(1 - \chi_U)\xi(m)||_{L^\infty(\mu)})$$

$$\leq \max(||\chi_U \xi||_{\text{hom}(M, L^\infty(\mu))}, ||(1 - \chi_U)\xi||_{\text{hom}(M, L^\infty(\mu))}).$$

**Remark 2.109.** Actually, the previous argument shows that $\mathcal{B}L(X,N)$ (bounded linear maps from the Banach space $X$ to $N$) is an $L^\infty(\mu)$-normed module whenever $N$ is too.

For $L^\infty(\mu)$-normed modules there is an analogue of the Hahn-Banach Theorem [Wea00, Thm. 5]:

**Lemma 2.110.** Let $M'$ be an algebraic submodule of the $L^\infty(\mu)$-normed module $M$ and $\Phi' \in \text{hom}(M', L^\infty(\mu))$ with norm $\leq c$. Then there is a $\Phi \in \text{hom}(M, L^\infty(\mu))$ extending $\Phi'$ with norm $\leq c$.

We now introduce derivations, but first recall some facts about Lipschitz functions. We will denote by $\text{Lip}(X)$ the space of real-valued Lipschitz functions defined on $X$ and by $\text{Lip}_b(X)$ the real algebra of real-valued bounded Lipschitz functions
defined on $X$. For $f \in \text{Lip}(X)$ its global Lipschitz constant will be denoted by $L(f)$. For $f \in \text{Lip}_b(X)$ we define the norm

$$\|f\|_{\text{Lip}_b(X)} = \max(\|f\|_{\text{Lip}(X)}, L(f)).$$

This gives $(\text{Lip}_b(X), \|\cdot\|_{\text{Lip}_b(X)})$ the structure of a Banach algebra, compare [Wea99, sec. 4.1]. An important property of Lip$_b(X)$ is that it is a dual Banach space and it has a unique predual. For more information we refer the reader to [Wea99, chap. 2]. For the scope of the present work, the most useful topology on Lip$_b(X)$ is the weak$^*$ topology. As we are assuming $X$ separable, by Krein-Šmulian, on bounded subsets of Lip$_b(X)$ the weak$^*$ topology will be metrizable; it turns out that a sequence $f_n \rightharpoonup f$ if and only if $f_n \to f$ pointwise and if $\sup_n L(f_n) < \infty$.

In the sequel, especially in Section 3, we will sometimes need to consider functions which are Lipschitz with respect to a different (pseudo)-distance $d'$. If $f \in \text{Lip}(X)$ is $L$-Lipschitz with respect to $d'$, we will say that $f$ is $(1, d')$-Lipschitz.

**Definition 2.112.** A derivation $D : \text{Lip}_b(X) \to L^\infty(\mu)$ is a weak$^*$ continuous, bounded linear map satisfying the product rule:

$$D(fg) = fDg + gDf.$$

Note that the notion of derivation depends on the measure $\mu$ and that the product rule implies that $Df = 0$ if $f$ is constant. The collection of all derivations $\mathcal{X}(\mu)$ is an $L^\infty(\mu)$-normed module. Moreover, any $D \in \mathcal{X}(\mu\mathcal{U})$ gives rise to an element of $\mathcal{X}(\mu)$ by extending $Df$ to be 0 on the complement of $U$. In this way, $\mathcal{X}(\mu\mathcal{U})$ can be naturally identified with the submodule $\chi_U \mathcal{X}(\mu)$ of $\mathcal{X}(\mu)$. Derivations are local in the following sense [Wea00, Lem. 27]:

**Lemma 2.114.** If $U$ is $\mu$-measurable and if $f, g \in \text{Lip}_b(X)$ agree on $U$, then $\forall D \in \mathcal{X}(\mu), \chi_U Df = \chi_U Dg$.

**Remark 2.115.** This locality property allows to extend derivations to Lipschitz functions. Considering countably many disjoint compact sets $K_\alpha$ with uniformly bounded diametre and points $c_\alpha \in K_\alpha$, with $\mu(X \setminus \bigcup_\alpha K_\alpha) = 0$, we define for $f \in \text{Lip}(X)$

$$Df = \sum_\alpha \chi_{K_\alpha} D\left( \min \left( \max \left( f - f(c_\alpha), -\max_{K_\alpha} |f - f(c_\alpha)| \right) \right), \max_{K_\alpha} |f - f(c_\alpha)| \right)$$

which agrees with the previous definition for $f \in \text{Lip}_b(X)$, using locality. Observe that we used the norm of $f - f(c_\alpha)$ in Lip$_b(K_\alpha)$ is bounded in terms of $L(f)$ and the diametre of $K_\alpha$. Again using locality, it is possible to show that the extension does not depend on the choice of the $K_\alpha$ or the point $c_\alpha$.

**Definition 2.117.** The dual $L^\infty(\mu)$-normed module of $\mathcal{X}(\mu)$ will be denoted by $\mathcal{E}(\mu)$ and called module of forms. For $f \in \text{Lip}_b(X)$ (or Lip$(X)$) let $df \in \mathcal{E}(\mu)$ be defined by

$$\forall D \in \mathcal{X}(\mu), \quad (df, D) = Df;$$

the map

$$d : \text{Lip}_b(X) \to \mathcal{E}(\mu)$$

$$f \mapsto df$$

Note that this implies uniform convergence on compact subsets.
is linear, satisfies the product rule
\[(2.120) \quad d(fg) = fdg + gdf \]
and is weak* continuous.

The following result is also used repeatedly in the following. Proofs can be found in [Gon12, Sch].

**Corollary 2.121.** Let \( \{D_i\}_{i=1}^N \subset \mathcal{X}(\mu U) \) be linearly independent, where \( U \) is Borel. Then there are disjoint Borel subsets \( V_\alpha \subset U \) with \( \mu(U \setminus \bigcup \alpha V_\alpha) = 0 \) such that, for each \( \alpha \), there are 1-Lipschitz functions \( \{g_{i,\alpha}\}_{i=1}^N \) and derivations \( \{D_{\alpha,i}\}_{i=1}^N \) in the linear span of \( \{\chi_{V_\alpha}, D_i\}_{i=1}^N \), satisfying
\[(2.122) \quad D_{\alpha,i}g_{\alpha,i} = \delta_{ij}\chi_{V_\alpha}.
\]

**Remark 2.123.** The space \( \mathcal{BL}(\text{Lip}_b(X), L^\infty(\mu)) \) of bounded linear maps
\[(2.124) \quad \text{Lip}_b(X) \rightarrow L^\infty(\mu)
\]
is a dual Banach space. In fact, let \( B_1 \) denote the unit ball in \( \text{Lip}_b(X) \) and consider the Banach space
\[(2.125) \quad l^1(B_1; L^1(\mu)) = \left\{ f : S \rightarrow L^1(\mu) : \sum_{s \in S} \|f(s)\|_{L^1(\mu)} < \infty \right\};
\]
then \( \mathcal{BL}(\text{Lip}_b(X), L^\infty(\mu)) \) can be identified with a subspace of the dual of the generalized \( l^1 \)-space: \( l^1(B_1; L^1(\mu)) \); on \( \mathcal{X}(\mu) \subset \mathcal{BL}(\text{Lip}_b(X), L^\infty(\mu)) \) we can consider the corresponding weak* topology; in concrete terms, this is the coarsest topology making continuous all the seminorms \( \{\nu_\xi\}_{\xi \in \Omega(B_1; L^1(\mu))} \):
\[(2.126) \quad \nu_\xi(D) = \left| \sum_{f \in B_1} \int \xi(f) Df d\mu \right|;
\]
a basis for the weak* topology consists of the sets
\[(2.127) \quad \Omega(\xi_1, \ldots, \xi_k; D_0; \varepsilon) = \{ D \in \mathcal{X}(\mu) : \nu_{\xi_i}(D - D_0) < \varepsilon (\forall i \in \{1, \ldots, k\}) \}
\]
where \( \{\xi_1, \ldots, \xi_k\} \subset l^1(B_1; L^1(\mu)) \), \( D_0 \in \mathcal{X}(\mu) \), \( \varepsilon > 0 \).

### 2.3. Differentiability spaces

In this Subsection we recall some facts about differentiability spaces. We first recall the definition of the infinitesimal Lipschitz constants of \( f \in \text{Lip}(X) \).

**Definition 2.128.** For \( f \in \text{Lip}(X) \) we define the **lower and upper variations** of \( f \) at \( x \) below scale \( r \) by
\[(2.129) \quad \ell f(x, r) = \sup_{s \leq r} \sup_{y \in B(x,s)} \frac{|f(x) - f(y)|}{s},
\]
\[(2.130) \quad \ell f(x, r) = \inf_{s \leq r} \sup_{y \in B(x,s)} \frac{|f(x) - f(y)|}{s}.
\]
The (“big” and “small”) infinitesimal Lipschitz constants of \( f \) at \( x \) are defined by
\[(2.131) \quad \ell f(x) = \inf_{r \geq 0} \ell f(x, r),
\]
\[(2.132) \quad \ell f(x) = \sup_{r \geq 0} \ell f(x, r).
\]
The functions $\mathcal{L}f(\cdot, r)$, $\mathcal{L}f(\cdot, r)$, $\mathcal{L}f$ and $\ell f$ are Borel.

The key notion for generalizing the classical Rademacher’s Theorem to metric spaces is that of infinitesimal independence.

**Definition 2.133.** Let $\{f_i\}_{i=1}^n \subset \text{Lip}(X)$, then

$$
\Phi \cdot, \{f_i\}_{i=1}^n : \mathbb{R}^n \to [0, \infty)
$$

defines a seminorm on $\mathbb{R}^n$; the functions $\{f_i\}_{i=1}^n$ are called **infinitesimally independent on a $\mu$-measurable** $A \subset X$ if $\Phi \cdot, \{f_i\}_{i=1}^n$ is a norm for $\mu$-a.e. $x \in A$.

We can now define differentiability spaces.

**Definition 2.135.** A metric measure space $(X, \mu)$ is called a **differentiability space** if there is a uniform bound $N$ on the number of Lipschitz functions that can be infinitesimally independent on a positive measure set. In this case there are (countably many) $\mu$-measurable $\{X_\alpha\}$ with $X = \bigcup X_\alpha$ such that:

1. For each $\alpha$ there is $\{x^j_\alpha\}_{j=1}^{N_\alpha} \subset \text{Lip}(X)$ such that $\forall f \in \text{Lip}(X)$ there are unique $\left\{\frac{\partial f}{\partial x^j_\alpha}\right\}_{j=1}^{N_\alpha} \subset L^\infty(\mu L X_\alpha)$ such that

$$
\mathcal{L} \left( f - \sum_{j=1}^n \frac{\partial f}{\partial x^j_\alpha}(x)x^j_\alpha \right)(x) = 0 \quad \text{for } \mu\text{-a.e. } x \in X_\alpha.
$$

2. The $N_\alpha$ are uniformly bounded by $N$ and the minimal value of $N$ is called the **differentiability dimension**.

Each $(X_\alpha, \{x^j_\alpha\}_{j=1}^{N_\alpha})$ is called a **chart**, the $\{x^j_\alpha\}_{j=1}^{N_\alpha}$ are called **chart functions**, and the maps:

$$
\frac{\partial}{\partial x^j_\alpha} : \text{Lip}(X) \to L^\infty(\mu L X_\alpha)
$$

are called **partial derivative operators**; the representatives $\frac{\partial f}{\partial x^j_\alpha}$ can be taken to be Borel if $X_\alpha$ is Borel.

Note that differentiability spaces are usually assumed to be Polish.

### 3. Derivations and Alberti representations

#### 3.1. From Alberti representations to Derivations

In this Subsection we associate derivations to Alberti representations; the fundamental construction is provided by Theorem 3.11; its proof uses the following Lemma to check the measurability of certain functions.

**Lemma 3.1.** Suppose that $X$ is Polish and $\mathcal{A} = (P, \nu)$ is an Alberti representation. Then for each $(f, g) \in \text{Lip}(X) \times \mathcal{B}\infty(X)$ the map:

$$
H_{f, g} : \text{Frag}(X) \to \mathbb{R}
$$

$$
\gamma \mapsto \int_{\text{dom } \gamma} (f \circ \gamma)'(t) (g \circ \gamma)(t) d(\gamma^{-1} \nu)(t)
$$
is Borel.

**Proof.** We start by showing that $H_{f, 1}$ is Borel. Without loss of generality we can assume that $f$ is 1-Lipschitz and just show that the restriction $H_{f, 1}|_{\text{Frag}_n(X)}$ is Borel. Recall that the set

$$\text{Lip}_{b, n}(\mathbb{R}) = \{ \psi \in \text{Lip}_b(\mathbb{R}) : L(\psi) \leq n \}$$

is closed in $\text{Lip}_b(\mathbb{R})$ and is a Polish space if we consider the weak* topology. Note that the set

$$\text{Ext}_n = \{ (\gamma, \psi) \in \text{Frag}_n(X) \times \text{Lip}_{b, n}(\mathbb{R}) : \psi \text{ extends } f \circ \gamma \}$$

is closed in $\text{Frag}_n(X) \times \text{Lip}_{b, n}(\mathbb{R})$. For each $\gamma$, by taking a McShane extension of $f \circ \gamma$, one concludes that the section $(\text{Ext}_n)_{\gamma}$ is non-empty. Moreover, by Ascoli-Arzelà each section $(\text{Ext}_n)_{\gamma}$ is compact. By the Lusin-Novikov uniformization Theorem [Kec95, Thm. 18.10], the set $\text{Ext}_n$ admits a Borel uniformization and thus there is a Borel uniformizing function:

$$F_n : \text{Frag}_n(X) \to \text{Lip}_{b, n}(\mathbb{R})$$

with $(\gamma, F_n(\gamma)) \in \text{Ext}_n$. To avoid a cumbersome notation for the value of $F_n(\gamma)$ at a point $t$, we will also write $F_n, \gamma$ to denote $F_n(\gamma)$. We now show that the function $G_{f, 1} : \text{Frag}_n(X) \to \mathbb{R}$ defined by:

$$G_{f, 1}(\gamma) = \int_{\mathbb{R}} F_{n, \gamma}(t) d(\gamma^{-1}d\nu_{\gamma})(t)$$

is Borel. For $k \in \mathbb{N}$ we let $\Delta^k$ denote the collection of closed dyadic intervals in $\mathbb{R}$ of the form $[\frac{m}{2^k}, \frac{m+1}{2^k}]$ for $m \in \mathbb{Z}$. Given an interval $I \in \Delta^k$ we denote by $a_I, b_I$ its extremes so that $[a_I, b_I] = I$. Consider the maps $G_{f, 1; k} : \text{Frag}_n(X) \to \mathbb{R}$ defined by:

$$G_{f, 1; k}(\gamma) = \sum_{I \in \Delta^k} \int_I F_{n, \gamma}(a_I) d(\gamma^{-1}d\nu_{\gamma})(t)$$

as the measure $\gamma^{-1}d\nu_{\gamma}$ is finite and $F_{n, \gamma}$ is Lipschitz, $\lim_{k \to \infty} G_{f, 1; k}(\gamma) = G_{f, 1}(\gamma)$. Note also that

$$\int_{\mathbb{R}} F_{n, \gamma}(a_I) d(\gamma^{-1}d\nu_{\gamma})(t) = F_{n, \gamma}(a_I) \nu_{\gamma}(\gamma(\text{dom } \gamma \cap [a_I, b_I]));$$

as the map $\gamma \mapsto F_{n, \gamma}(a_I)$ is Borel because it is the composition of $F_{n, \gamma}$ with the evaluation at the point $a_I$, and as the map $\gamma \mapsto \nu_{\gamma}(\gamma(\text{dom } \gamma \cap [a_I, b_I]))$ is Borel by condition (4) in the Definition 2.7 of Alberti representations, we conclude that the maps $G_{f, 1; k}$ and $G_{f, 1}$ are Borel. We now fix $k \in \mathbb{N}$ and note that also the map

$$\gamma \mapsto \int_{\mathbb{R}} F_{n, \gamma} \left( t + \frac{1}{k} \right) d(\gamma^{-1}d\nu_{\gamma})(t)$$

is Borel. We thus conclude that the function $H_{f, 1; k} : \text{Frag}_n(X) \to \mathbb{R}$ defined by:

$$H_{f, 1; k}(\gamma) = \int_{\mathbb{R}} \frac{F_{n, \gamma}(t + \frac{1}{k}) - F_{n, \gamma}(t)}{1/k} d(\gamma^{-1}d\nu_{\gamma})(t),$$

is Borel. Note that for $L^1$-a.e. $t$ we have $\lim_{k \to \infty} \frac{F_{n, \gamma}(t + \frac{1}{k}) - F_{n, \gamma}(t)}{1/k} = F_{\nu_{\gamma}}(t)$, which agrees with $(f \circ \gamma)'(t)$ for $\gamma^{-1}d\nu_{\gamma}$-a.e. $t$. As $H_{f, 1}(\gamma) = \lim_{k \to \infty} H_{f, 1; k}(\gamma)$, we conclude that $H_{f, 1}$ is Borel. For a Borel $A \subset X$, the previous argument can be applied to
the Alberti representation \((P, \nu \ll A)\) of \(\mu \ll A\) to conclude that \(H_{f, \chi} A\) is Borel. As each \(g \in B^\infty(X)\) is a pointwise limit of uniformly bounded simple functions, we conclude that \(H_{f, g}\) is Borel.

**Theorem 3.11.** If \(A = (P, \nu)\) is a C-Lipschitz Alberti representation, the formula

\[
\int_X g D_A f \, d\mu = \int_{\text{Frag}(X)} dP(\gamma) \int_{\text{dom} \gamma} (f \circ \gamma)'(t) g \circ \gamma(t) d(\gamma^{-1} \nu_\gamma)(t)
\]

\[
= \int_{\text{Frag}(X)} dP(\gamma) \int_X g \partial_\gamma f \, d\nu_\gamma (g \in L^1(\mu) \cap B^\infty(X)),
\]

where

\[
\partial_\gamma f(x) = \begin{cases} (f \circ \gamma)'(\gamma^{-1}(x)) & \text{if } x \in \text{im} \gamma \text{ and the derivative exists}, \\ 0 & \text{otherwise}, \end{cases}
\]

defines a derivation \(D_A \in X(\mu)\) with \(\|D\|_{X(\mu)} \leq C\).

Denoting by \(\text{Alb}(\mu)\) the set of Alberti representations of \(\mu\) which are Lipschitz and letting

\[
\text{Alb}_{\text{sub}}(\mu) = \bigcup \{\text{Alb}(\mu \ll S) : S \subset X \text{ Borel}\},
\]

we obtain a map:

\[
\text{Der} : \text{Alb}_{\text{sub}}(\mu) \to X(\mu)
\]

\(A \mapsto D_A\).

**Proof.** Let \(A = (P, \nu)\) be a C-Lipschitz Alberti representation of \(\mu\); considering a C-Lipschitz fragment \(\gamma\), from the estimate:

\[
|(f \circ \gamma)'(t)| \leq L(f) \, \text{md} \gamma(t) \leq C L(f),
\]

we obtain

\[
\left| \int_X g D_A f \, d\mu \right| \leq C L(f) \|g\|_{L^1(\mu)};
\]

in particular, we conclude that \(D_A : \text{Lip}_b(X) \to L^\infty(\mu)\) is a linear operator with norm bounded by \(C\).

That \(D_A\) satisfies the product rule follows from a direct computation.

We show that \(D_A\) is weak* continuous; let \(f_n \overset{w^*}{\rightharpoonup} f\) in \(\text{Lip}_b(X)\), and consider a bounded Borel function \(g \in L^1(\mu)\). Now, \(\frac{d}{dt}\) is a derivation of \(\mathbb{R}\) with respect to the Lebesgue measure; as \(\gamma^{-1} \nu_\gamma\) is absolutely continuous with respect to the Lebesgue measure, the function

\[
\frac{d\gamma^{-1} \nu_\gamma}{d\text{Leb}} g
\]

is integrable, and so

\[
\lim_{n \to \infty} \int_{\text{dom} \gamma} (f_n \circ \gamma)'(t) g \circ \gamma(t) \, d(\gamma^{-1} \nu_\gamma)(t) = \int_{\text{dom} \gamma} (f \circ \gamma)'(t) g \circ \gamma(t) \, d(\gamma^{-1} \nu_\gamma)(t).
\]
Assume that \( L(f_n), L(f) \leq L \) and let

\[
(3.20) \quad h_n(\gamma) = \int_{\text{dom } \gamma} (f_n \circ \gamma)'(t) g \circ \gamma(t) d(\gamma^{-1} \nu_\gamma)(t),
\]

\[
(3.21) \quad h(\gamma) = \int_{\text{dom } \gamma} (f \circ \gamma)'(t) g \circ \gamma(t) d(\gamma^{-1} \nu_\gamma)(t),
\]

\[
(3.22) \quad H(\gamma) = CL \int_{\text{dom } \gamma} |g| \circ \gamma(t) d(\gamma^{-1} \nu_\gamma)(t);
\]

note that \( h_n \to h \) pointwise, \( |h_n|, |h| \leq H \). The map \( H \) is Borel by Definition 2.7 of Alberti representations, and the maps \( h_n, h \) and \( H \) are Borel by Lemma 3.1; we thus conclude by the Lebesgue’s dominated convergence Theorem that

\[
(3.23) \quad \lim_{n \to \infty} \int_{\text{Frag}(X)} dP(\gamma) \int_{\text{dom } \gamma} (f_n \circ \gamma)'(t) g \circ \gamma(t) d(\gamma^{-1} \nu_\gamma)(t)
= \int_{\text{Frag}(X)} dP(\gamma) \int_{\text{dom } \gamma} (f \circ \gamma)'(t) g \circ \gamma(t) d(\gamma^{-1} \nu_\gamma)(t),
\]

showing weak* continuity.

The definition of \( \text{Der} \) is well-posed because \( \mathcal{X}(\mu \mathcal{L} \mathcal{S}) \) can be canonically identified with \( \chi_\mathcal{S} \mathcal{X}(\mu) \).

We now describe what happens if \( \mathcal{A} \) is in the \( f \)-direction of some cone field.

**Theorem 3.24.** Suppose \( \mathcal{A} = (P, \nu) \in \text{Alb}(\mu) \) is in the \( f \)-direction of \( \mathcal{C}(v, \alpha) \); then

\[
(3.25) \quad D_A f(x) \in \mathcal{C}(v(x), \alpha(x)) \quad (\text{for } \mu\text{-a.e. } x).
\]

In particular, if the Alberti representations \( \{\mathcal{A}_k\}_{k=1}^K \) of \( \mu \) are in the \( f \)-directions of independent cone fields \( \{\mathcal{C}(v_k, \alpha_k)\}_{k=1}^K \), the derivations \( \{D_{A_k}\}_{k=1}^K \) are independent.

**Proof.** Note that if \( U \subset X \) is Borel, \( D_A \chi_U = \chi_U D_A \); therefore we can assume that \( \mu \) is finite with \( \mu(X) > 0 \) and tan \( \alpha \in L^1(\mu) \). If \( \varphi \in \mathcal{B}^\infty(X) \) is nonnegative with \( \int \varphi d\mu > 0 \),

\[
(3.26) \quad \int \varphi \tan \alpha \langle v, D_A f \rangle \, d\mu = \int_{\text{Frag}(X)} dP(\gamma) \int \varphi \tan \alpha \langle v, \partial_\gamma f \rangle \, d\nu_\gamma
\]

\[
> \int_{\text{Frag}(X)} dP(\gamma) \int \varphi \|\pi_v^{\perp} \partial_\gamma f\|_2 \, d\nu_\gamma,
\]

were we used that \( \mu(X) > 0 \) and \( \int \varphi d\mu > 0 \). On the other hand, \( \forall \varepsilon > 0 \) there are countably many \( (K_j, w_j) \) such that

(1) The \( K_j \subset X \) are disjoint compact sets and \( \mu \left( X \setminus \bigcup_j K_j \right) = 0 \).
(2) Each \( w_j \) is a unit vector field orthogonal to \( v \) on \( K_j \).
(3) The inequality \( \|\pi_v^{\perp} D_A f\|_2 \leq \langle w_j, D_A f \rangle + \varepsilon \) holds on \( K_j \).
Thus
\[ \int \varphi \| \pi_\nu^+ D_A f \|_2^2 \, d\mu \leq \sum_j \int_{K_j} \varphi(w_j, D_A f) \, d\mu + \max \varphi \varepsilon \mu(X) \]
\[ = \sum_j \int_{\text{Frag}(X)} dP(\gamma) \int_{K_j} \varphi(w_j, \partial_\gamma f) \, d\nu_\gamma + \max \varphi \varepsilon \mu(X) \]
\[ \leq \int_{\text{Frag}(X)} dP(\gamma) \int \varphi \| \pi_\nu^+ \partial_\gamma f \|_2^2 \, d\nu_\gamma + \max \varphi \varepsilon \mu(X); \]

letting \( \varepsilon \downarrow 0 \) and combining (3.26) and (3.27)
\[ \int \varphi \left( \tan \alpha(v, D_A f) - \| \pi_\nu^+ D_A f \|_2 \right) \, d\mu > 0, \]
which implies the result. \( \square \)

We now introduce the notion of effective speed of an Alberti representation.

**Definition 3.29.** Let \( \mathcal{A} = (P, \nu) \in \text{Alb}(\mu) \) an Alberti representation and define the measure
\[ \Sigma_A = \int_{\text{Frag}(X)} m \, \gamma \circ \gamma^{-1} \, \nu_\gamma \, dP(\gamma); \]
an effective speed of \( \mathcal{A} \) is a Borel representative \( \sigma_A \) of the Radon-Nikodym derivative of \( \Sigma_A \) with respect to \( \mu \).

We now describe what happens if one knows a lower bound of the \( f \)-speed of \( \mathcal{A} \).

**Theorem 3.31.** If \( \mathcal{A} \in \text{Alb}(\mu) \) has \( f \)-speed \( \geq \delta \), then
\[ D_A f(x) \geq \delta(x) \sigma_A(x) \quad (\text{for } \mu\text{-a.e. } x). \]

**Proof.** Let \( \varphi \in L^1(\mu) \) nonnegative; then
\[ \int \varphi D_A f \, d\mu = \int dP(\gamma) \int \varphi \partial_\gamma f \, d\nu_\gamma \]
\[ \geq \int dP(\gamma) \int \varphi \delta \, d(m \gamma \circ \gamma^{-1} \nu_\gamma) \]
\[ = \int \varphi \delta \sigma_A \, d\mu, \]
from which the result follows. \( \square \)

As we deal with parametrized fragments, it is useful to know how Alberti representations are affected by affine reparametrizations of the fragments.

**Lemma 3.34.** For \( a \in \mathbb{R} \setminus \{0\} \) and \( b \in \mathbb{R} \) let \( \tau_{a,b} : \mathbb{R} \to \mathbb{R} \) denote the homeomorphism
\[ \tau_{a,b}(x) = ax + b; \]
then
\[ \tau_{a,b}^\sharp : \text{Frag}(X) \to \text{Frag}(X) \]
\[ \gamma \mapsto \gamma \circ \tau_{a,b}, \]
is a homeomorphism.
If $\mu \mathcal{L}A$ admits an Alberti representation $\mathcal{A} = (P, \nu)$ then
\[(3.37) \quad \tau_{a,b}^\sharp A = \left( (\tau_{a,b}^\sharp)_t P, \nu \circ (\tau_{a,b}^\sharp)^{-1} \right)\]
is an Alberti representation and, moreover, if

1. The Alberti representation $\mathcal{A}$ is C-Lipschitz ($(C, D)$-biLipschitz) then $\tau_{a,b}^\sharp A$ is $C[a]$-Lipschitz ($(C[a], D[a])$-biLipschitz).
2. If $\mathcal{A}$ has $f$-speed $\geq \delta$ then $\tau_{a,b}^\sharp A$ has $\text{sgn} \cdot f$-speed $\geq \delta$.
3. If $\mathcal{A}$ is in the $f$-direction of $\mathcal{C}(v, \alpha)$, then $\tau_{a,b}^\sharp A$ is in the $f$-direction of $\mathcal{C}(\text{sgn} \cdot v, \alpha)$.

Finally, if $\mathcal{A}$ is Lipschitz, $D_{\tau_{a,b}^\sharp A} = aD_A$.

\textbf{Proof.} Note that $\tau_{a,b}^\sharp$ is $\max(|a|, \frac{1}{|a|})$-Lipschitz and with inverse $\tau_{1/a,-b/a}$; if $B$ is Borel, from $\gamma \mapsto \nu_\gamma(B)$ being Borel, it follows that $\gamma \mapsto \nu((\tau_{a,b}^\sharp)^{-1}\gamma)(B)$ is Borel. Then
\[(3.38) \quad \mu \mathcal{L}A(B) = \int_{\text{Frag}(X)} \nu(\gamma(B)) dP(\gamma) = \int_{\text{Frag}(X)} \nu((\tau_{a,b}^\sharp)^{-1}\gamma)(B) d(\tau_{a,b}^\sharp)_t P(\gamma),\]
which shows that $\tau_{a,b}^\sharp A$ is an Alberti representation of $\mu$. Points (1)–(3) follow observing that $(\tau_{a,b}^\sharp)_t^\lambda P$ is supported on $\tau_{a,b}^\sharp(\text{spt} P)$. Finally,
\[(3.39) \quad \int D_{\tau_{a,b}^\sharp A} f g d\mu = \int_{\text{Frag}(X)} d(\tau_{a,b}^\sharp)_t P(\gamma) \int_{\text{dom} \gamma} (f \circ \gamma)(t)(g \circ \gamma)(t)\]
\[\times d \left( \gamma^{-1}_t \nu \left( (\tau_{a,b}^\sharp)^{-1} \gamma \right) \right)(t)\]
\[= \int_{\text{Frag}(X)} dP(\tilde{\gamma}) \int_{\text{dom} \gamma} (f \circ \tilde{\gamma} \circ \tau_{a,b})(t)(g \circ \tilde{\gamma} \circ \tau_{a,b})(t)\]
\[\times d \left( \left( \gamma \circ \tau_{a,b} \right)^{-1} \nu(\tilde{\gamma}) \right)(t),\]
where $\gamma = \tau_{a,b}^\sharp \tilde{\gamma}$; then
\[(3.40) \quad \int D_{\tau_{a,b}^\sharp A} f g d\mu = \int_{\text{Frag}(X)} dP(\tilde{\gamma}) \int_{\text{dom} \gamma} a(f \circ \tilde{\gamma})' \circ \tau_{a,b}(t)(g \circ \tilde{\gamma} \circ \tau_{a,b})(t)\]
\[\times d \left( \tau_{a,b}^\sharp \tilde{\gamma}^{-1} \nu(\tilde{\gamma}) \right)(t)\]
\[= \int_{\text{Frag}(X)} dP(\tilde{\gamma}) \int_{\text{dom} \tilde{\gamma}} a(f \circ \tilde{\gamma})'(t)(g \circ \tilde{\gamma})(t) d \left( \tilde{\gamma}^{-1} \nu(\tilde{\gamma}) \right)(t)\]
\[= \int aD_A f g d\mu.\]

To illustrate the lack of injectivity of Der we provide a useful reparametrization result which allows to use fragments with domain contained in a prescribed interval.

\textbf{Lemma 3.41.} Let $\mathcal{A} \in \text{Alb}(\mu \mathcal{L}S)$; then there is $\mathcal{A}' = (P', \nu') \in \text{Alb}(\mu \mathcal{L}S)$ with
\[(3.42) \quad P' (\text{Frag}(X) \setminus \text{Frag}(X, I)) = 0\]
and such that

1. We have the identity $D_{\mathcal{A}'} = D_{\mathcal{A}}$. 

If \( A \) satisfies a set of conditions (COND), so does \( A' \).

**Proof.** We can assume that \( A = (P, \nu) \in \text{Alb}(\mu) \) and that \( X \) is compact. Recall that \( \text{Haus}(\mathbb{R}) \) denotes the set of nonempty compact subsets of \( \mathbb{R} \) with the Vietoris topology. The maps:
\[
\max, \min : \text{Haus}(\mathbb{R}) \to \mathbb{R} \quad K \mapsto \max_{x \in K} x, \min_{x \in K} x
\]
are continuous. In particular,
\[
\Delta_1 : \text{Haus}(\mathbb{R}) \to \text{Haus}(\mathbb{R}) \quad K \mapsto K \cap [\min(K), \min(K) + \mathcal{L}^1(I)]
\]
is continuous. For \( n > 1 \) let \( O_n \) denote the open set
\[
O_n = \{ K \in \text{Haus}(\mathbb{R}) : \max(K) - \min(K) > (n - 1) \mathcal{L}^1(I) \}
\]
and \( \Delta_n \) the continuous map
\[
\Delta_n : O_n \to \text{Haus}(X) \quad K \mapsto \Delta_1 \left( K \setminus \bigcup_{k=1}^{n-1} \Delta_k(K) \right).
\]
Note that the set
\[
F_i = \{ \gamma \in \text{Frag}(X) : \mathcal{L}^1(\Delta_i(\text{dom } \gamma)) > 0 \}
\]
is Borel and the map \( R_i : F_i \to \text{Frag}(X, I) \)
\[
\gamma \mapsto \tau_{1, \min(I) - \min(\text{dom } \gamma), \sharp} |_{\Delta_i(\text{dom } \gamma)}
\]
is Borel.

The following discussion applies only for those \( i \) for which \( P(F_i) > 0 \). We apply the Disintegration Theorem [Fre06, 452O] for
\[
R_i : \left( F_i, \frac{P \uplus F_i}{P(F_i)} \right) \to \left( \text{Frag}(X, I), \frac{R_i \uplus P(F_i)}{P(F_i)} \right)
\]
which is, in the terminology of [Fre06], an **inverse-measure-preserving** function. By [Fre06, 434K(b)] \( \text{Haus}(\mathbb{R} \times X) \), being Polish, is a Radon measure space and, as \( F_i \) is a Borel subset of \( \text{Haus}(\mathbb{R} \times X) \), it is also a Radon measure space by [Fre06, 434F(c)]. On the other hand, \( (\text{Frag}(X, I), P_i) \) is **strictly localizable** in the terminology of [Fre06] because the measure \( P_i \) is finite. The Disintegration Theorem yields Radon probability measures \( \{ \pi_{\gamma} \}_{\gamma \in \text{Frag}(X, I)} \) on \( \text{Frag}(X) \) such that
\[
\frac{P \uplus F_i}{P(F_i)} = \int_{\text{Frag}(X, I)} \pi_{\gamma} \, dP_i(\gamma)
\]
\[
\pi_{\gamma} \left( R_i^{-1}(\{ \gamma \}) \right) = 1 \quad \text{(for } P_i\text{-a.e. } \gamma\text{)}.
\]
Consider the weakly measurable maps \( \tilde{\nu}_i : \text{Frag}(X) \to M(X) \)
\[
\tilde{\nu}_i(\gamma) = \begin{cases} 0 & \text{if } \gamma \notin F_i; \\ P(F_i) \nu_{\gamma X} |_{\Delta_i(\text{dom } \gamma)} & \text{otherwise}; \end{cases}
\]

(3.50)
\[
\frac{P \uplus F_i}{P(F_i)} = \int_{\text{Frag}(X, I)} \pi_{\gamma} \, dP_i(\gamma)
\]
(3.51)
\[
\pi_{\gamma} \left( R_i^{-1}(\{ \gamma \}) \right) = 1 \quad \text{(for } P_i\text{-a.e. } \gamma\text{)}.
\]
Consider the weakly measurable maps \( \tilde{\nu}_i : \text{Frag}(X) \to M(X) \)
\[
\tilde{\nu}_i(\gamma) = \begin{cases} 0 & \text{if } \gamma \notin F_i; \\ P(F_i) \nu_{\gamma X} |_{\Delta_i(\text{dom } \gamma)} & \text{otherwise}; \end{cases}
\]

(3.52)
and the measures

$$(3.53)\quad \mu_i = \frac{1}{\mathcal{P}(F_i)} \int_{F_i} \tilde{\nu}_i(\gamma) \, d\mathcal{P}(\gamma);$$

if we let $\nu_i : \text{Frag}(X, I) \to M(X)$ be given by

$$(3.54)\quad \nu_i(\gamma) = \int_{\text{Frag}(X)} \tilde{\nu}_i(\tilde{\gamma}) \, d\pi(\tilde{\gamma}),$$

$\mathcal{A}_i = (P_i, \nu_i)$ is an Alberti representation of $\mu_i$ supported on the closure of $\text{im} R_i$. In particular, if $\mathcal{A}$ satisfies (COND), so does $\mathcal{A}_i$. The derivation $D_i \in \mathcal{X}(\mu_i)$ given by

$$(3.55)\quad \int gD_i f \, d\mu_i = \frac{1}{\mathcal{P}(F_i)} \int_{F_i} d\mathcal{P}(\gamma) \int g\partial_t f \tilde{\nu}_i(\gamma)$$

is naturally identified with an element of $\mathcal{X}(\mu)$ because $\mu_i \ll \mu$; in particular, $D_{\mathcal{A}} = \sum_i D_i$. The calculation

$$(3.56)\quad \frac{1}{\mathcal{P}(F_i)} \int_{F_i} d\mathcal{P}(\tilde{\gamma}) \int_{\text{dom} \tilde{\gamma}} (f \circ \tilde{\gamma})' (t) (g \circ \tilde{\gamma})(t) d \left( \tilde{\gamma}^{-1} \tilde{\nu}_i(\tilde{\gamma}) \right)(t)$$

$$= \int_{\text{Frag}(X, I)} dP_i(\gamma) \int_{\text{Frag}(X)} d\pi(\gamma) \int_{\text{dom} \gamma} (f \circ \gamma)'(t) (g \circ \gamma)(t) d \left( \gamma^{-1} \nu_i(\gamma) \right)(t)$$

$$= \int_{\text{Frag}(X, I)} dP_i(\gamma) \int_{\text{dom} \gamma} (f \circ \gamma)'(t) (g \circ \gamma)(t) d \left( \gamma^{-1} \nu_i(\gamma) \right)(t)$$

$$= \int gD_{\mathcal{A}_i} f \, d\mu_i$$

shows that $D_i = D_{\mathcal{A}_i}$. Let $P'$ the probability measure on $\text{Frag}(X, I)$ given by

$$(3.57)\quad P' = \sum_i 2^{-i} P_i$$

and $\varphi_i$ a Borel representative of the Radon-Nikodym derivative of $P_i$ with respect to $P'$; let

$$(3.58)\quad \nu' : \text{Frag}(X, I) \to M(X)$$

$$\gamma \mapsto \sum_i \nu_i(\gamma) \varphi_i(\gamma);$$

then $\mathcal{A}' = (P', \nu')$ is an Alberti representation of $\sum_i \mu_i = \mu$ and, if $\mathcal{A}$ satisfies (COND), so does $\mathcal{A}'$. Moreover,

$$(3.59)\quad D_{\mathcal{A}'} = \sum_i D_{\mathcal{A}_i} = \sum_i D_i = D_{\mathcal{A}}.$$

□
3.2. From Derivations to Alberti representations. The goal of this Subsection is to prove Theorems 3.60, 3.96 and 3.116 and Corollaries 3.92 and 3.94. Recall that we deal with separable metric spaces.

**Theorem 3.60.** Let $X$ be a metric space and $\mu$ a Radon measure on $X$. Consider a Borel $V \subset X$, derivations $\{D_1, \ldots , D_k\} \subset X(\mu)$ and a Lipschitz function $g : X \to \mathbb{R}^k$ such that $D_i g_j = \delta_{i,j} X_V$. Then for each $\varepsilon > 0$, unit vector $w \in \mathbb{S}^{k-1}$, angle $\alpha \in (0, \pi/2)$ and speed parameter $\sigma \in (0,1)$, the measure $\mu_{LV}$ admits a $(1, 1 + \varepsilon)$-biLipschitz Alberti representation in the $g$-direction of $C(w, \alpha)$ with $(w, g)$-speed

\[
\frac{\sigma}{|D_w|_{X(\mu_{LV})}, \text{loc} + (1 - \sigma)}.
\]

The proof of Theorem 3.60 relies on an approximation scheme for Lipschitz functions, Theorem 3.66. We state the relevant definitions and the approximation scheme here, and defer the proof to Subsection 5.1. We define the following classes of fragments:

**Definition 3.62.** For $\delta > 0$ and $f : X \to \mathbb{R}$ Lipschitz we define:

\[
\text{Frag}(X, f, \delta) = \{ \gamma \in \text{Frag}(X) : (f \circ \gamma)'(t) \geq \delta \text{ mod } \gamma(t) \text{ for } L^1\text{-a.e. } t \in \text{dom } \gamma \};
\]

For $\delta > 0$, $f : X \to \mathbb{R}^q$ Lipschitz, $w \in \mathbb{S}^{q-1}$ and $\alpha \in (0, \pi/2)$, we define:

\[
\text{Frag}(X, f, \delta, w, \alpha) = \{ \gamma \in \text{Frag}(X, (w, f), \delta) : (f \circ \gamma)'(t) \in C(w, \alpha) \text{ for } L^1\text{-a.e. } t \in \text{dom } \gamma \}.
\]

**Definition 3.65.** Let $f : X \to \mathbb{R}$ be a Lipschitz function, $S \subset X$ Borel and $\mu$ a Radon measure on $X$. We say that $f$ is locally $\delta$-Lipschitz on $S$ if for each $x \in S$ there is an $r > 0$ such that the restriction $f|B(x, r)$ is $\delta$-Lipschitz. Finally, we say that $f$ is $\mu$-a.e. locally $\delta$-Lipschitz on $S$ if for $\mu$-a.e. $x \in S$ there is an $r > 0$ such that the restriction $f|B(x, r)$ is $\delta$-Lipschitz.

**Theorem 3.66.** Let $X$ be a compact metric space, $f : X \to \mathbb{R}^q$ $L$-Lipschitz and $S \subset X$ compact. Let $\mu$ be a Radon measure on $X$. Assume that $S$ is $\text{Frag}(X, f, \delta, w, \alpha)$-null. Denoting by $d_{\delta, \alpha}$, $\bar{d}_{\delta, \alpha}$ the distances

\[
d_{\delta, \alpha}(x, y) = \delta d(x, y) + \cot \alpha \| \pi_w f(x) - \pi_w f(y) \|_2, \\
\bar{d}_{\delta, \alpha} = \max(\sqrt{q}Ld(\cdot, \cdot), d_{\delta, \alpha}),
\]

there are $(1, \bar{d}_{\delta, \alpha})$-Lipschitz maps $g_k \overset{w^*}{\to} (w, f)$ with $g_k$ $\mu$-a.e. locally $(1, \bar{d}_{\delta, \alpha})$-Lipschitz on $S$.

The proof of Theorem 3.60 relies on the following technical Lemmas 3.68 and 3.76.

**Lemma 3.68.** Let $X$ be a separable metric space and $\mu$ a Radon measure on $X$. Consider a Borel $S \subset X$, a Lipschitz function $f : X \to \mathbb{R}$, a derivation $D \in X(\mu)$, and a (pseudo)-distance function $d'$ on $X$ satisfying $d' \leq Cd$, where $d$ denotes the metric on $X$. Assume that $f|S$, regarded as a function defined on the metric space $(S, d)$, is $\mu_{S}$-a.e. locally $(1, d')$-Lipschitz and that, for some $c \geq 0$,

\[
\chi_S |Dd'(\cdot, p)| \leq c \quad (\forall p \in S);
\]
then
\[ \chi_S |Df| \leq c. \]

**Proof.** As \( \mu \) is Radon, it suffices to show (3.70) when \( S \) is replaced by a compact subset \( K \). Fix a ball \( B \subset X \) centred on \( K \) such that \( f \) is \((1, d')\)-Lipschitz in \( B \cap K \). Let \( N_\eta \) a finite \( \eta \)-net in \( B \cap K \) (which is compact) and \( f_\eta \) the McShane extension of \( f \) on \( \overline{N} \) to \( X \):
\[ f_\eta(x) = \min_{p \in N_\eta} (f(p) + d'(x, p)); \]

note that \( f_\eta \) is \((C, d')\)-Lipschitz. Let
\[ \hat{\eta}(x) = \min \left( \max \left( f_\eta(x), -\sup_B |f_\eta| \right), \sup_B |f_\eta| \right), \]

so that \( \hat{\eta} \in \text{Lip}_b(X) \) and \( \hat{\eta} = f_\eta \) on \( B \). Choose a sequence \( \eta_n \searrow 0 \). As \( X \) is separable, the predual of \( X \) is separable (compare the description of the predual using molecules given in [Wea99, Subec. 2.2]) and thus the weak* topology on \( \text{Lip}_b(X) \) is metrizable on bounded subsets of \( \text{Lip}_b(X) \); by Banach-Alaoglu, the sequence \( \hat{\eta}_n \) has a convergent subsequence \( \hat{\eta}_n \xrightarrow{\text{weak*}} \hat{\eta} \) in \( \text{Lip}_b(X) \). Note that \( \hat{\eta} = f \) on \( B \cap K \). For a fixed \( \eta \) there are finitely many closed sets \( C_p \subset \overline{B} \cap K \) (\( p \in N \)) which cover \( B \cap K \) and such that, for each \( x \in C_p \),
\[ f_\eta(x) = f(p) + d'(x, p); \]

this implies, by Lemma 2.114,
\[ \chi_{C_\mu \cap K} |D\hat{\eta}| = \chi_{C_\mu \cap K} |D\hat{\eta}| \leq c; \]
thus
\[ \left\| \chi_{K} |D\hat{\eta}| \right\|_{L^\infty(p\text{-}\mathcal{B})} \leq c. \]
Using lower semicontinuity of the norm under weak* convergence and Lemma 2.114, it follows that
\[ \left\| \chi_{K} |D\hat{\eta}| \right\|_{L^\infty(p\text{-}\mathcal{B})} = \left\| \chi_{K} |Df| \right\|_{L^\infty(p\text{-}\mathcal{B})} \leq c \]
which implies (3.70). \( \square \)

**Lemma 3.76.** Let \( X \) be a separable metric space and \( \mu \) a Radon measure on \( X \). Consider a Lipschitz function \( g : X \to \mathbb{R}^k \), a derivation \( D \in \mathcal{X}(\mu) \) and a Borel \( S \subset X \). Assume that \( S \) is \( \text{Frag}(X, g, \delta, w, \alpha)\)-null, where \( w \in \mathbb{S}^{k-1} \) and \( \alpha \in (0, \pi/2) \), and that, for each \( u \in \mathbb{S}^{k-1} \) orthogonal to \( w \), one has
\[ \chi_S |D(u, g)| \leq \varepsilon |D|_{\mathcal{X}(\mu),\text{loc}}; \]
then one has
\[ \chi_S |D(w, g)| \leq (\delta + (k - 1)\varepsilon \cot \alpha) |D|_{\mathcal{X}(\mu),\text{loc}}. \]

In particular, if the derivations \( \{D_1, \ldots, D_k\} \subset \mathcal{X}(\mu) \) satisfy \( D_ig_j = \delta_{i,j} \mu\text{-a.e.,} \) letting \( D_w = \sum_{i=1}^k w_iD_i \), one has
\[ \chi_S |D_w(w, g)| \leq \delta |D_w|_{\mathcal{X}(\mu L_S),\text{loc}}. \]

\(^{15}\)This argument is a version of Ascoli-Arzelà in disguise.
Proof. The second part, (3.79), follows from the first, (3.78) as \(D_w\) annihilates \(\langle u, g \rangle\) for \(u\) orthogonal to \(w\). As \(\mu\) is Radon, it suffices to show (3.78) for \(S\) replaced by a compact subset \(K\).

Consider the metric space \((K, d)\) and choose an orthonormal basis \(B\) of the plane orthogonal to \(w\); let

\[
d'(x, y) = \delta d(x, y) + \cot \alpha \sum_{u \in B} |\langle u, g(x) - g(y) \rangle|,
\]

\[
\tilde{d}(x, y) = \max(\sqrt{\kappa}L(g)d(x, y), d'(x, y)).
\]

As \(K\) is \(\text{Frag}(K, g, \delta, w, \alpha)\)-null, by Theorem 3.66 there are functions \(f_n\)

1. Which are \((1, \tilde{d}')\)-Lipschitz and \(\mu\)-\(K\)-a.e. locally \((1, \tilde{d}')\)-Lipschitz on \(K\).
2. Which converge weak* to \(\langle w, g \rangle\) in \(\text{Lip}_b(K)\).

Reasoning as in the proof of Lemma 3.68, we can take extensions \(\tilde{f}_n \in \text{Lip}_b(X)\) of the \(f_n\) with \(\tilde{f}_n \xrightarrow{w^*} \tilde{f}^{16}\) and \(\tilde{f} = f\) on \(K\). Let \(p \in K\); as \(d(\cdot, p)\) is \(1\)-Lipschitz,

\[
|Dd(\cdot, p)| \leq |D|_{X(\mu), \text{loc}};
\]
on the other hand, by [Wea99, Lem. 7.2.2]\(^{17}\)

\[
|D|\langle u, g(\cdot) - g(p) \rangle \leq |D\langle u, g(\cdot) - g(p) \rangle|;
\]

but by hypothesis,

\[
\chi_K |D\langle u, g(\cdot) - g(p) \rangle| \leq \varepsilon |D|_{X(\mu), \text{loc}};
\]

putting together (3.81), (3.82) and (3.83):

\[
\chi_K |Dd'(\cdot, p)| \leq (\delta + (k - 1)\varepsilon \cot \alpha) |D|_{X(\mu), \text{loc}}.
\]

By Lemma 3.68 applied to the \(\tilde{f}_n\),

\[
\chi_K \left|D\tilde{f}_n\right| \leq (\delta + (k - 1)\varepsilon \cot \alpha) |D|_{X(\mu), \text{loc}};
\]

by lower semicontinuity of the norm under weak* convergence,

\[
\chi_K \left|D\tilde{f}\right| \leq (\delta + (k - 1)\varepsilon \cot \alpha) |D|_{X(\mu), \text{loc}};
\]

finally by Lemma 2.114,

\[
\chi_K |Df| \leq (\delta + (k - 1)\varepsilon \cot \alpha) |D|_{X(\mu), \text{loc}}.
\]

\[16\]After passing to a subsequence

\[17\]This is a consequence of weak* continuity and the classical approximation of functions in \(\text{Lip}_b([0, 1])\) by polynomials
we let \( \delta_{\beta} = \frac{1}{\lVert \chi_K D_w \rVert_{X(\mu)}} - \eta_1 \) and \( \alpha' \in (0, \alpha) \). We want to show that if a Borel \( S \subset K \) is \( \text{Frag}(K, g, \delta_{\beta}, w, \alpha') \)-null, then it is \( \mu \)-null. Taking \( X = K \) in Lemma 3.76, we get

\[
\chi_S = \chi_SD_w(w, g) \leq \delta_{\beta} \frac{D_w}{\chi(\mu), \text{loc}} \leq \delta_{\beta} \frac{\chi_K D_w}{\chi(\mu)} \\
\leq 1 - \eta_1 \frac{\chi_K D_w}{\chi(\mu)} < 1,
\]

which implies \( \mu(S) = 0 \). By Theorem 2.64 the measure \( \mu \mathcal{L} K \) admits an Alberti representation in the \( g \)-direction of \( \mathcal{C}(v, \alpha) \) with

\[
\langle w, g \rangle \text{-speed} \geq \frac{1}{\lVert \chi_K D_w \rVert_{X(\mu)}} - \eta_1 \\
\geq \frac{1}{\inf_{x \in K} |D_w|_{X(\mu), \text{loc}}(x) + \eta_0} - \eta_1 \\
\geq \sup_{x \in K} \frac{|D_w|_{X(\mu), \text{loc}}(x)}{\sigma} (1 - \sigma).
\]

Corollary 3.92. Let \( V \) be Borel and assume that \( X(\mu \mathcal{L} V) \) contains \( k \)-independent derivations. Then there is a Borel partition \( V = \bigcup_\alpha V_\alpha \) and, for each \( \alpha \), there is a 1-Lipschitz function \( f_\alpha : V \to \mathbb{R}^k \) such that, for each \( \varepsilon > 0 \) and for all Borel maps \( w : X \to \mathbb{S}^{k-1} \) and \( \theta : X \to (0, \pi/2) \), the measure \( \mu \) admits a \((1, 1 + \varepsilon)\)-biLipschitz Alberti representation \( \mathcal{A} \) with \( \mathcal{A} \mathcal{L} V_\alpha \) in the \( f_\alpha \)-direction of the cone field \( \mathcal{C}(w, \theta) \).

In particular, the measure \( \mu \mathcal{L} V_\alpha \) admits Alberti representations in the \( f_\alpha \)-directions of independent cone fields \( \mathcal{C}_1, \cdots, \mathcal{C}_k \).

Proof. Applying Corollary 2.121 one finds disjoint Borel sets \( \{ V_\alpha \} \) with \( V = \bigcup_\alpha V_\alpha \) and such that, for each \( \alpha \), there are 1-Lipschitz functions \( f_\alpha : X \to \mathbb{R}^k \) and derivations \( \{ D_{\alpha,1}, \cdots, D_{\alpha,k} \} \subset X(\mu) \) with

\[
D_{\alpha,i} f_{\alpha,j} = \chi_{V_\alpha} \delta_{i,j}.
\]

In the case in which \( w \) and \( \theta \) are constant, the result follows applying Theorem 3.60. If \( w \) and \( \theta \) are not constant, one can find disjoint compact sets \( \{ C_{\alpha,\beta} \} \beta \):

1. Each \( C_{\alpha,\beta} \) is a subset of \( V_\alpha \).
2. We have the identity \( \mu \left( V_\alpha \setminus \bigcup_\beta C_{\alpha,\beta} \right) = 0 \).
3. For each \( \beta \) there are a unit vector \( w_\beta \in \mathbb{S}^{k-1} \) and an angle \( \theta_\beta \in (0, \pi/2) \) with \( C(w_\beta, \theta_\beta) \subset C(w(x), \theta(x)) \) for each \( x \in C_{\alpha,\beta} \).

One then applies Theorem 2.46 to glue together the Alberti representations of the measures \( \mu \mathcal{C}_{\alpha,\beta} \) in the \( f_\alpha \)-directions of the cone fields \( \mathcal{C}(w_\beta, \theta_\beta) \).

Corollary 3.94. Let \( f : X \to \mathbb{R}^k \) be Lipschitz and \( \mu \) a Radon measure on \( X \) admitting Alberti representations \( A_1, \cdots, A_k \) in the \( f \)-directions of independent cone fields \( \mathcal{C}_1, \cdots, \mathcal{C}_k \). Then, for each \( \varepsilon > 0 \) and for all Borel maps \( w : X \to \mathbb{S}^{k-1} \) and \( \theta : X \to (0, \pi/2) \), the measure \( \mu \) admits a \((1, 1 + \varepsilon)\)-biLipschitz Alberti representation in the \( f \)-direction of \( \mathcal{C}(w, \theta) \).
Proof. By Theorem 2.64 it is possible to assume that the Alberti representations are biLipschitz. According to Theorem 3.24 the corresponding derivations \( D_A \) are independent. By the gluing principle for Alberti representations, Theorem 2.46, it suffices to prove the statement for \( w \) and \( \theta \) constant and to show that there is a Borel partition \( \text{spt} \mu = \bigcup U_\alpha \) with each \( \mu \triangledown U_\alpha \) admitting an Alberti representation in the \( f \)-direction of \( C(w, \theta) \). As the cone fields \( C_i \) are independent, the matrix \((D_A,f_j)_{i,j=1}^k\) is invertible \( \mu \)-a.e. This allows to pass to a Borel partition \( \text{spt} \mu = \bigcup U_\alpha \) such that, for each \( U_\alpha \), there are derivations \( D_{\alpha,i} \) with \( D_{\alpha,i} f_j = \delta_i^j \) \( \mu \triangledown U_\alpha \)-a.e. This partition is necessary because the inverse of \((D_A,f_j)_{i,j=1}^k\) does not need to have its entries in \( L^\infty(\mu) \). However, there are disjoint Borel sets \( U_\alpha \) with \( \mu(\text{spt} \mu \setminus \bigcup U_\alpha) = 0 \) and such that on each \( U_\alpha \) the inverse of \((D_A,f_j)_{i,j=1}^k\) has its entries in \( L^\infty(\mu) \). In fact, the entries of \((D_A,f_j)_{i,j=1}^k\) are in \( L^\infty(\mu) \) and one needs a lower bound on its determinant on each \( U_\alpha \), for example letting

\[
U_n = \left\{ x : \left| \det (D_A,f_j)_{i,j=1}^k \right| \leq \frac{1}{n+1} \right\}.
\]

The result now follows by applying Theorem 3.60. \( \square \)

**Theorem 3.96.** The set \( \text{Der}(\text{Alb}_{\text{sub}}(\mu)) \) is weak* dense in \( \mathcal{X}(\mu) \).

The proof of Theorem 3.96 relies on the following technical Lemma.

**Lemma 3.97.** Consider a \( 1 \)-Lipschitz function \( F : X \to \mathbb{R}^N \), a derivation \( D_0 \in \mathcal{X}(\mu) \), a compact \( K \subset X \) and a vector \( V = \lambda w \) where \( \lambda > 0 \) and \( w \in S^{N-1} \), suppose that for \( 0 < s_1 < s_2 \) and \( \varepsilon > 0 \)

\[
|D_0|_{\mathcal{X}(\mu),\text{loc}}(x) \in (s_1, s_2) \quad (\forall x \in K)
\]

\[
\sup_{x \in K} \|D_0 F(x) - V\|_2 < \varepsilon,
\]

then, for each \( \alpha \in (0, \frac{\pi}{2}) \), the measure \( \mu \upharpoonright K \) admits a \( C \)-Lipschitz Alberti representation \( A' \) with

\[
\|D_{A'}\|_{\mathcal{X}(\mu \upharpoonright K)} \leq C \leq (1 + \varepsilon) \left( (1 + \varepsilon)s_2 + \varepsilon(N - 1) \frac{s_2}{s_1} \cot \alpha + 2\varepsilon \right),
\]

\[
\|D_{A'} F - V\|_2 \leq (\varepsilon + \sin \alpha + 1 - \cos \alpha) \left( (1 + \varepsilon)s_2 + \varepsilon(N - 1) \frac{s_2}{s_1} \cot \alpha + 2\varepsilon \right).
\]

**Proof.** Note that if \( u \perp w \), one has

\[
\chi_K |D_0 \langle u, F \rangle| = \chi_K |\langle u, D_0 F - V \rangle| < \varepsilon;
\]

supposing that \( S \subset K \) is Borel and \( \text{Frag}(X,F,\delta,w,\alpha)\)-null, Lemma 3.76 implies

\[
\chi_S |D_0 \langle w, F \rangle| \leq \left( \delta + (N - 1) \frac{\varepsilon \cot \alpha}{s_1} \right) |D_0|_{\mathcal{X}(\mu),\text{loc}};
\]

on the other hand,

\[
\chi_K |D_0 \langle w, F \rangle| \geq \chi_K \langle w, V \rangle - \chi_K |\langle w, D_0 F - V \rangle| \geq \chi_K (\lambda - \varepsilon).
\]

This implies

\[
\delta \geq \frac{\lambda - \varepsilon}{s_2} - (N - 1) \frac{\varepsilon \cot \alpha}{s_1};
\]
thus the measure $\mu|_K$ admits a $(1, 1 + \varepsilon)$-biLipschitz Alberti representation $A = (P, \nu)$ in the $F$-direction of $C(w, \alpha)$ with $\langle w, F \rangle$-speed $\geq \delta_0$ where
\[ \delta_0 = \frac{\lambda - 2\varepsilon}{s_2} - (N - 1) \frac{\varepsilon \cos \alpha}{s_1}. \]
Thus for $P$-a.e. $\gamma$ and for $\gamma^{-1} \mu$-a.e. $t$, one has
\[ \| (F \circ \gamma)'(t) - w \|_2 \leq \left\| (F \circ \gamma)'(t) - \frac{(F \circ \gamma)'(t)}{\| (F \circ \gamma)'(t) \|_2} \right\|_2 + \left\| \frac{(F \circ \gamma)'(t)}{\| (F \circ \gamma)'(t) \|_2} - w \right\|_2 \]
\[ \leq \| (F \circ \gamma)'(t) \|_2 - 1 + \sin \alpha + 1 - \cos \alpha \]
\[ \leq \varepsilon + \sin \alpha + 1 - \cos \alpha; \]
note that this implies
\[ \| D_A F - w \|_2 \leq \varepsilon + \sin \alpha + 1 - \cos \alpha; \]
on the other hand, from
\[ \delta_0 \leq \langle w, (F \circ \gamma)'(t) \rangle \leq 1 + \varepsilon, \]
one gets
\[ \lambda \leq (1 + \varepsilon) s_2 + \varepsilon (N - 1) \frac{s_2}{s_1} \cos \alpha + 2 \varepsilon. \]
If $A' = \tau_{\lambda, \theta}\gamma_A$ Lemma 3.34 implies (3.99). \hfill \Box

Proof of Theorem 3.96. It suffices to show that $\forall \Omega(\xi_1, \ldots, \xi_k; D_0; \varepsilon)$ there is $A \in \text{Alb}_{\text{sub}}(\mu)$:
\[ D_A \in \Omega(\xi_1, \ldots, \xi_k; D_0; \varepsilon). \]
Note that as the $\xi_i$ satisfy $\sum_{f \in B_1} \| \xi_i(f) \|_{L^1(\mu)} < \infty$, there are $\{(f_i, g_i)\}_{i=1}^N \subset \text{Lip}_b(X) \times L^1(\mu)$ and $\varepsilon_1 > 0$ such that if
\[ \| D \|_{\mathcal{X}(\mu)} \leq 2 \| D_0 \|_{\mathcal{X}(\mu)}, \]
\[ \left| \int g_i (D - D_0) f_i \, d\mu \right| < \varepsilon_1 \quad (\forall i), \]
then $D \in \Omega(\xi_1, \ldots, \xi_k; D_0; \varepsilon)$. Let $F = (f_i)_{i=1}^N$ and, for $D \in \mathcal{X}(\mu)$, $DF = (D f_i)_{i=1}^N$; note that by possibly shrinking $\varepsilon_1$ we can assume that $F : X \to \mathbb{R}^N$ is 1-Lipschitz. Moreover, as we need to produce an Alberti representation for a Borel subset of $S \subset X$ where $|D_0|_{\mathcal{X}(\mu), \text{loc}} > 0$ and $\| D_0 F \|_2 > 0$, we can assume $S = X$. Using Egorov’s Theorem we find countably many $\{(K_j, \varepsilon_{2,j})\}$ with $K_j \subset X$ compact and $\varepsilon_{2,j} \in (0, \infty)$ such that if (3.110) holds and
\[ \sup_{x \in K_j} \| D F(x_j) - D_0 F(x_j) \|_2 < \varepsilon_{2,j} \quad (\forall j), \]
then (3.111) holds. Note that we can also assume that
\[ \inf_{x \in K_j} |D_0|_{\mathcal{X}(\mu), \text{loc}}(x) > 0 \]
and
\[ \inf_{x \in K_j} \| D_0 F(x) \|_2 > 0. \]
Suppose that $D$ and $A$ are Alberti representations $A(3.119)$ and $(3.110)$ and $(3.113)$ hold for $D$. Suppose that Theorem 3.116.

Using again Egorov’s Theorem, we can further partition the $K_j$ so that there are $(W_j, \varepsilon_{3,j}, \alpha_j) \in (\mathbb{R}^N_1 \setminus \{0\}) \times (0, \frac{\pi}{2}) \times (0, \frac{\pi}{2})$ such that:

1. The supremum $\sup_{x \in K_j} \|D_0 F(x_j) - W_j\|_2 < \varepsilon_{3,j}$.
2. For each $x \in K_j$ $|D_0|_{X(\mu), \text{loc}} \in (s_{1,j}, s_{2,j}) \subset (0, \infty)$ with $\frac{s_{2,j}}{s_{1,j}} \leq (1 + \varepsilon_{3,j})$.
3. We have the inequality

\[
(1 + \varepsilon_{3,j}) \left(1 + \varepsilon_{3,j}\right) s_{2,j} + \varepsilon_{3,j} \left(N_1 - 1\right) \frac{s_{2,j}}{s_{1,j}} \cot \alpha_j + 2\varepsilon_{3,j} \leq 2\|D_0\|_{X(\mu)}.
\]

4. We have the inequality $2\|D_0\|_{X(\mu)} (\varepsilon_{3,j} + \sin \alpha_j + 1 - \cos \alpha_j) < \frac{\varepsilon_{3,j}}{2}$. Using Lemma 3.97 we obtain an Alberti representation $A$ such that $(3.110)$ and $(3.113)$ hold for $D_A$. Using the gluing principle Theorem 2.46 we obtain $A \in \text{Alb}_{\text{sub}}(\mu)$ such that $3.109$ holds.

**Theorem 3.116.** Suppose that $X(\mu)$ is finitely generated and $D \in X(\mu)$; let

\[
S = \left\{ x \in X : |D|_{X(\mu), \text{loc}} > 0 \right\};
\]

then $D \in \text{Der}(\text{Alb}(\mu L S))$.

The proof of Theorem 3.116 relies on the following technical Lemma.

**Lemma 3.118.** Suppose $A \in \text{Alb}(\mu L S)$ and $U \subset X$ Borel. Then there is $A' \in \text{Alb}(\mu L S)$ with $D_A = \chi_U D_A$. Moreover,

1. If $A$ is $C$-Lipschitz $(C(D), \text{biLipschitz})$, so is $A'$.
2. If $A$ has $f$-speed $\geq \delta$ or is in the $f$-direction of $C(w, \alpha)$, so does $A' \cup U$.

**Proof.** Without loss of generality, $A = (P, \nu) \in \text{Alb}(\mu)$. Using Lemma 3.41, we find an Alberti representation $A_1 = (P_1, \nu_1) \in \text{Alb}(\mu L U^c)$ with

\[
P_1 (\text{Frag}(X, [0, 1])) = 1
\]

and $D_{A_1} = \chi_{U^c} D_A$. Moreover, if $A$ satisfies a set of conditions (COND), so does $A_1$. Similarly, combining Lemmas 3.41 and 3.34, we can find $A_2 = (P_2, \nu_2) \in \text{Alb}(\mu L U^c)$, which is an Alberti representation of $\mathcal{A} U U^c$ with

\[
P_2 (\text{Frag}(X, [2, 3])) = 1
\]

and

\[
D_{A_2} = D_{\tau_{-1,0} A_1} = -\chi_{U^c} D_A;
\]

note that conditions on the Lipschitz or biLipschitz constants satisfied by $A$ are also satisfied by $A_2$. Using Lemma 3.41 we find an Alberti representation $A_3 = (P_3, \nu_3) \in \text{Alb}(\mu L U)$ with

\[
P_3 (\text{Frag}(X, [4, 5])) = 1
\]

and $D_{A_3} = \chi_U D_A$; note that if $A$ satisfies (COND), so does $A_3$. We now let

\[
P' = \frac{1}{4} (P_1 + P_2) + \frac{1}{2} P_3;
\]

\[
\nu'(\gamma) = \begin{cases} 
2\nu_2(\gamma) & \text{if } \gamma \in \text{Frag}(X, [0, 1]) \\
2\nu_2(\gamma) & \text{if } \gamma \in \text{Frag}(X, [2, 3]) \\
2\nu_3(\gamma) & \text{if } \gamma \in \text{Frag}(X, [4, 5]) 
\end{cases}
\]
note that $\mathcal{A}' = (P', \nu') \in \text{Alb}(\mu)$ and
\begin{equation}
D_{\mathcal{A}'} = \frac{1}{2}D_A + \frac{1}{2}D_{A_2} + D_{A_3} = \chi U D_A.
\end{equation}

\[ \square \]

Proof of Theorem 3.116. Without loss of generality we can assume that $S = X$, $X(\mu)$ is free on $\{D_{A_i}\}_{i=1}^k$ where $A_i \in \text{Alb}(\mu)$ and
\begin{equation}
D = \sum_{i=1}^k \lambda_i D_{A_i},
\end{equation}
where the $\{\lambda_i\}_{i=1}^k \subset B^\infty(X)$ are nonnegative. The result now follows from the following Claims:

(CL1): If $\{A_j\}_{j=1}^m \subset \text{Alb}(\mu)$, $\exists A' \in \text{Alb}(\mu)$ with
\begin{equation}
D_{A'} = \sum_{j=1}^m D_{A_j}.
\end{equation}

To show (CL1), Lemma 3.41 gives $\{A'_{\dashv} = (P_{\dashv}, \nu_{\dashv})\}_{j=1}^m$ with $D_{A'_{\dashv}} = D_{A_j}$ and
\begin{equation}
P_j (\text{Frag}(X, [2(j-1), 2(j-1)+1])) = 1;
\end{equation}
if we let
\begin{equation}
P_{\oplus} = \frac{1}{m} \sum_{j=1}^m P_j,
\end{equation}
\begin{equation}
\nu_{\oplus}(\gamma) = \begin{cases} 
\nu_j(\gamma) & \text{if } \gamma \in \text{Frag}(X, [2(j-1), 2(j-1)+1]) \\
0 & \text{otherwise,}
\end{cases}
\end{equation}
$\mathcal{A}_{\oplus} = (P_{\oplus}, \nu_{\oplus}) \in \text{Alb}(\mu)$ and
\begin{equation}
D_{\mathcal{A}_{\oplus}} = \frac{1}{m} \sum_{j=1}^m D_{A_j};
\end{equation}
thus it suffices to take $\mathcal{A}' = \tau_{m,0}^* \mathcal{A}_{\oplus}$. To show (CL2), let $2^{<N}$ denote the set of finite strings on $\{0, 1\}$:
\begin{equation}
2^{<N} = \{ \alpha = (a_1, \ldots, a_m) : m \in \mathbb{N}, a_i \in \{0, 1\} \};
\end{equation}
the length of $\alpha \in 2^{<N}$ will be denoted by $|\alpha|$, and the $k$-th entry by $\alpha(k)$. Suppose that the range of $\lambda$ lies in $[0, M)$ and let
\begin{equation}
\Delta : 2^{<N} \rightarrow [0, M)
\end{equation}
\begin{equation}
\alpha \mapsto \sum_{k=1}^{|\alpha|} \frac{\alpha(k)}{2^k} M;
\end{equation}
let $I(\alpha)$ denote the unique subinterval $[a, b]$ of the dyadic subdivision of $[0, M)$ in $2^{|\alpha|}$ intervals which contains $\alpha$. If we let $U_\alpha$ denote the Borel set
\begin{equation}
U_\alpha = \{ x \in X : \lambda(x) \in I(\alpha) \},
\end{equation}
we can write
\begin{equation}
\lambda = \sum_{n=1}^{\infty} \frac{M}{2^n} \sum_{|\alpha| = n} \alpha(n) \chi_{U_n}.
\end{equation}
By Lemma 3.118 we can find $A_n = (P_n, \nu_n) \in \text{Alb}(\mu)$ with
\begin{equation}
D_{A_n} = M \sum_{|\alpha| = n} \alpha(n) \chi_{U_n} D_{\lambda}
\end{equation}
and
\begin{equation}
P_n \left( \text{Frag}(X, [2(n-1), 2(n-1) + 1]) \right) = 1;
\end{equation}
if we let
\begin{equation}
P' = \sum_{n} 2^{-n} P_n \quad \text{and}
\end{equation}
\begin{equation}
\nu'_n(\gamma) = \begin{cases} 
\nu_n(\gamma) & \text{if } \gamma \in \text{Frag}(X, [2(n-1), 2(n-1) + 1]) \\
0 & \text{otherwise},
\end{cases}
\end{equation}
$A' = (P', \nu') \in \text{Alb}(\mu)$ and $D_{A'} = \lambda D_{A}$.

3.3. Geometric characterization of $| \cdot |_{\text{ε}(\mu), \text{loc}}$. In this subsection we prove Theorem 3.141 which gives an intrinsic and geometric characterization of the Weaver norm on $X(\mu)$. We also show that the inequality $\| Df \|_{L^\infty(\mu)} \leq \| D \|_{X(\mu)} L(f)$ localizes (Lemma 3.148): $L(f)$ can be replaced by $\L(f)$.

**Theorem 3.141.** Let $X$ be a compact metric space, $f : X \to \mathbb{R}$ Lipschitz, $S \subset X$ Borel, $\mu$ a Radon measure on $X$. Then $|df|_{\text{ε}(\mu), \text{loc}} \leq \alpha$ if and only if for each $\varepsilon > 0$ there is an $S_\varepsilon \subset S$ which is Borel, $\text{Frag}(X, f, \alpha + \varepsilon)$-null, and satisfies $\mu(S \setminus S_\varepsilon) = 0$. In particular,
\begin{equation}
\| df \|_{\text{ε}(\mu)} = \sup \left\{ \| Df \|_{L^\infty(\mu)} : D \in \text{Der}(\text{Alb}_{\text{sub}}(\mu)) \text{ and } \| D \|_{X(\mu)} \leq 1 \right\}.
\end{equation}

**Proof.** Necessity is proven by contrapositive. Assume that for some $\varepsilon > 0$ $S$ does not contain a full $\mu$-measure subset which is $\text{Frag}(X, f, \alpha + \varepsilon)$-null. Inspection of the proof of 2.64 shows that $\forall \eta > 0$
\begin{equation}
\mu = \mu \L G_\eta + \mu \L F_\eta,
\end{equation}
where $G_\eta$, $F_\eta$ are complementary, $F_\eta$ being a $\text{Frag}(X, f, \alpha + \varepsilon)$-null $F_{\sigma \delta}$ and $\mu \L G_\eta$ admitting a $(1, 1 + \eta)$-bLipschitz Alberti representation with $f$-speed $\geq \alpha + \varepsilon$. By assumption, $F_\eta \cap S$ cannot be a full $\mu$-measure subset of $S$, so $\mu(G_\eta \cap S) > 0$. If $D_\eta$ denotes the derivation associated to that Alberti representation,
\begin{equation}
\| D_\eta \|_{\chi(\mu)} \leq 1 + \eta
\end{equation}
and
\begin{equation}
\chi_{G_\eta} D_\eta f \geq \alpha + \varepsilon
\end{equation}
so that
\begin{equation}
|df|_{\mu \L G_\eta, \text{loc}} \geq \frac{\alpha + \varepsilon}{1 + \eta} > \alpha
\end{equation}
for $\eta$ sufficiently small.

---

\textsuperscript{18}With a uniform bound on the Lipschitz constant
Sufficiency is proven via the approximation scheme Theorem 1.21, reducing to the case in which $S$ is compact. By replacing $S$ by $\bigcap_n S_n^{\perp}$, we can assume that $S$ is, $\forall n, \text{Frag}(X, f, a+\frac{1}{n})$-null; then $\exists g_n$ $\max (L(f), a+\frac{1}{n})$-Lipschitz and $\mu$-a.e. $(a+\frac{1}{n})$-Lipschitz on $S$ with $\|g_n - f\|_{\infty} \leq \frac{1}{n}$. Thus, $g_n \overset{w^*}{\to} f$ so that $dg_n \overset{w^*}{\to} df$ and by lower semicontinuity,
\begin{equation}
\|df\|_{L^\infty(\mu \ll S)} \leq \liminf_{n \to \infty} \|dg_n\|_{L^\infty(\mu \ll S)} \leq \alpha.
\end{equation}
\[\square\]

We also present a proof of the following useful result.

**Lemma 3.148.** Let $X$ be a metric space, $\mu$ a Radon measure on $X$, $f \in \operatorname{Lip}_b(X)$ and $D \in \mathcal{X}(\mu)$; then
\begin{equation}
|Df| \leq |D|_{\mathcal{X}(\mu), \text{loc}} \mu f.
\end{equation}

**Proof.** For all $\varepsilon_0, \varepsilon_1 > 0$, using Egorov and Lusin's Theorems, there are triples $(K_\alpha, \lambda_\alpha, r_\alpha)$ such that:
1. The $K_\alpha$ are disjoint compact with $\mu(X \setminus \bigcup\alpha K_\alpha) = 0$ and $\mu(K_\alpha) > 0$.
2. The constants $\lambda_\alpha$ are nonnegative.
3. The constants $r_\alpha$ are positive and $K_\alpha$ has diameter less than $r_\alpha$.
4. For each $x \in K_\alpha$ and each $r \in (0, r_\alpha]$, 
\begin{equation}
\mathcal{L} f(x, r) \in (\lambda_\alpha - \varepsilon_0, \lambda_\alpha).
\end{equation}
5. The local norm $|D|_{\mathcal{X}(\mu \ll K_\alpha), \text{loc}}$ lies in $\left(\|D\|_{\mathcal{X}(\mu \ll K_\alpha)} - \varepsilon_1, \|D\|_{\mathcal{X}(\mu \ll K_\alpha)}\right]$. 

By point (3) $f$ is $(\lambda_\alpha)$-Lipschitz on $K_\alpha$; let $F$ denote a McShane extension of $f|K_\alpha$ with $L(F) \leq \lambda_\alpha$. Then
\begin{equation}
\|Df\|_{L^\infty(\mu \ll K_\alpha)} = \|DF\|_{L^\infty(\mu \ll K_\alpha)} \leq \lambda_\alpha \|D\|_{\mathcal{X}(\mu \ll K_\alpha)};
\end{equation}
thus
\begin{equation}
\chi_{K_\alpha} |Df| \leq \lambda_\alpha \left(|D|_{\mathcal{X}(\mu \ll K_\alpha), \text{loc}} + \varepsilon_1\right) < (\mathcal{L} f + \varepsilon_0) \left(|D|_{\mathcal{X}(\mu \ll K_\alpha), \text{loc}} + \varepsilon_1\right).
\end{equation}

As $\sum_\alpha \chi_{K_\alpha} = 1$ in $L^\infty(\mu)$,\(^\text{19}\)
\begin{equation}
|Df| < (\mathcal{L} f + \varepsilon_0) \left(|D|_{\mathcal{X}(\mu), \text{loc}} + \varepsilon_1\right).
\end{equation}

Letting $\varepsilon_0, \varepsilon_1 \searrow 0$ completes the proof. \[\square\]

4. Structure of differentiability spaces

4.1. Differentiability spaces and derivations. In this Subsection we make a first application of derivations to study differentiability spaces. The goal is to prove Lemma 4.1 and Corollary 4.6. We give a first proof of Lemma 4.1 which depends on the results in [Bat12]; another proof, which relies on Theorem 4.7 and [Sch], can be found at the end of Subsection 4.2.

**Lemma 4.1.** Let $(U, x)$ be a differentiability chart, with $U$ Borel, in a differentiability space $(X, \mu)$. Then the partial derivative operators $\frac{\partial}{\partial x^j}$ are derivations.

\(^{19}\)The convergence of the series is in the weak* sense
Proof. Let $n$ be the dimension of the chart $(U, x)$; we first show that there are disjoint Borel sets $\{U_\alpha\}$ with $U_\alpha \subset U$ and $\mu(U \setminus \bigcup \alpha U_\alpha) = 0$, and such that, for each $\alpha$, it is possible to find $g_{\alpha, k}^i \in L^\infty(\mu U_\alpha)$ ($i, k = 1, \cdots, n$) and derivations associated to Alberti representations $\{A_i\}_{i=1}^n$, such that

$$\sum_{i=1}^n g_{\alpha, k}^i D_{A_i} U_\alpha, x^j = \delta^j_k \chi U_\alpha;$$

in fact, because of [Sch, Lem. 6.1], in a differentiability space derivations are completely determined by their action on the chart functions and so (4.2) represents $\chi U_\alpha \frac{\partial}{\partial x^j}$. Note that even though in [Sch, Sec. 6] the measure $\mu$ is assumed doubling, this is not restrictive by Theorem 1.14. By [Bat12, Thm. 6.6] we know that $\mu U$ admits Alberti representations in the $x$-directions of independent cone fields $\{C_i\}_{i=1}^n$. Using Corollary 3.92, we obtain $(1,3/2)$-bLipschitz Alberti representations $\{A_i\}_{i=1}^n$ in the $x$-directions of cone fields $C(e_i, \theta)$ where $\{e_i\}_{i=1}^n$ is the standard Euclidean basis and $\theta$ is sufficiently small to ensure that the cone fields $C(e_i, \theta)$ are independent. In particular, letting $M = (D_{A_i} x^j)_{i,j=1}^n$, and choosing Borel representatives for the entries of $M$, the bounded Borel function $\det M$ is different from zero on a $\mu$-full measure subset. In particular, one can choose disjoint Borel subsets $U_\alpha \subset U$ with $\mu(U \setminus \bigcup \alpha U_\alpha) = 0$, and such that, on each $U_\alpha$ the determinant $\det M$ is uniformly bounded from below by $\delta_\alpha > 0$. Inverting the matrix $M$ one concludes that there are $g_{\alpha, k}^i \in L^\infty(\mu U_\alpha)$ such that (4.2) holds.

The definition of chart implies that the partial derivative operators $\frac{\partial}{\partial x^j}$ are bounded linear maps from $\text{Lip}_b(X) \to L^\infty(\mu U)$ that satisfy the product rule. In order to show weak* continuity, suppose that $f_t \xrightarrow{w^*} f$ in $\text{Lip}_b(X)$ and that $h \in L^1(\mu U)$. For each $\varepsilon > 0$ there are finitely many $\{U_\alpha\}_{s=1}^{N(\varepsilon)}$ such that

$$\sup_t \left| \int_{U \setminus \bigcup_{s=1}^{N(\varepsilon)} U_\alpha} h \frac{\partial f_t}{\partial x^j} \, d\mu \right| \leq \varepsilon;$$

$$\left| \int_{U \setminus \bigcup_{s=1}^{N(\varepsilon)} U_\alpha} h \frac{\partial f}{\partial x^j} \, d\mu \right| \leq \varepsilon.$$

Combining equations (4.3) and (4.4) with the fact that $\sum_{s=1}^{N(\varepsilon)} \chi U_\alpha \frac{\partial}{\partial x^j}$ is a derivation, we obtain

$$\lim_{t \to \infty} \int_U h \frac{\partial f_t}{\partial x^j} \, d\mu = \int_U h \frac{\partial f}{\partial x^j} \, d\mu,$$

which shows that $\frac{\partial}{\partial x^j}$ is weak* continuous. □

We now prove the dimensional bound Corollary 4.6 which significantly strengthens previous bounds on differentiability dimensions. In fact, we remove from Theorems 1.11, 1.12 and 1.16 the dependence on $\tau$ from the bound on the differentiability dimension.

**Corollary 4.6.** Suppose $(X, \mu)$ is a $\sigma$-differentiability space with $\mu$ doubling, then $(X, \mu)$ is a differentiability space and the dimension of the measurable differentiable structure is at most the Assouad dimension of $X$.

**Proof.** It follows by applying Lemma 5.136 and Lemma 4.1. □
4.2. Characterization of differentiability spaces. In this Subsection we obtain a new characterization of differentiability spaces. The goal is to prove Theorems 4.7 and 4.9. Throughout this section the metric space $X$ is assumed to be Polish.

Theorem 4.7. The metric measure space $(X, \mu)$ is a $\sigma$-differentiability space if and only if
\begin{equation}
\mu(S) = 0 \quad (\forall S \in \text{Gap}(X)).
\end{equation}

Theorem 4.9. The metric measure space $(X, \mu)$ is a $\sigma$-differentiability space if and only if $\mu$ is $\sigma$-asymptotically doubling and one of the following equivalent conditions holds:

1. For each $f \in \text{Lip}(X)$
\begin{equation}
|df|_{\mathcal{L}(\mu), \text{loc}} (x) = \mathcal{L}f(x) \quad (\text{for } \mu\text{-a.e. } x).
\end{equation}

2. For each $f \in \text{Lip}(X)$, denoting by $S_f$ the Borel set
\begin{equation}
\{ x \in X : \mathcal{L}f(x) > 0 \},
\end{equation}
for all $\varepsilon, \sigma \in (0, 1)$ the measure $\mu\upharpoonright S_f$ admits a $(1, 1 + \varepsilon)$-biLipschitz Alberti representation with $f$-speed $\geq \sigma \mathcal{L}f$.

3. For each $f \in \text{Lip}(X)$,
\begin{equation}
\mathcal{L}f(x) = \ell f(x) \quad (\text{for } \mu\text{-a.e. } x).
\end{equation}

The proofs of Theorems 4.7 and 4.9 require some preparation. We will use Theorem 4.13, the proof of which is deferred to Subsection 5.3.

Theorem 4.13. Let $S \subset X$ be a Borel set and assume that for some $\delta_0 \in (0, 1]$ and for each $m \in \mathbb{N}$ there is an $L$-Lipschitz function $f_m$ such that:

1. There is a $\rho_m \in (0, \frac{1}{m})$ such that if $B$ is a ball of radius $\rho_m$ centred at some point of $S$, the Lipschitz constant of the restriction $f_m|_B$ is at most $\frac{1}{m}$.

2. For each $x \in S$ there is $y \in B(x, \frac{1}{m})$ such that
\begin{equation}
|f_m(x) - f_m(y)| \geq \delta_0 d(x, y) > 0.
\end{equation}

Then for all $(M, \alpha) \in \mathbb{N} \times (0, \min \left(\frac{1}{2}, \sqrt{\frac{\alpha^2/L}{1 + 2^{-6} \alpha}}, \frac{1}{2} \delta_0 \right))$ there are a Borel subset $S' \subset S$ and Lipschitz functions $\{ \psi_0, \ldots, \psi_{M-1} \}$ such that:

- The set $S \setminus S'$ is $\mu$-null.
- The $\psi_i$ have Lipschitz constant bounded by
\begin{equation}
3 \left( L + \alpha + \frac{\alpha^2/L}{1 - \alpha^2/L} (1 + 2^{-6} \alpha) \right).
\end{equation}

- For each $x \in S'$:
\begin{equation}
\mathcal{L} \left( \sum_{i=0}^{M-1} \lambda_i \psi_i \right) (x) \geq \left( \delta_0 - \frac{\alpha^2/L}{1 - \alpha^2/L} - \alpha \right) \max_{i=0, \ldots, M-1} |\lambda_i|.
\end{equation}

As a first step, we prove a Lemma that produces flat functions as in the hypothesis of Theorem 4.13.

Lemma 4.17. Let $X$ be a compact metric space with finite Assouad dimension, $f : X \to \mathbb{R}$ Lipschitz, $S \subset X$ Borel, $\mu$ a Radon measure on $X$. Suppose that there are $\alpha$ and $\beta$ such that $\alpha > \beta \geq 0$ and
\begin{equation}
\inf_{x \in S} \mathcal{L}f(x) \geq \alpha > \beta \geq |df|_{\mathcal{L}(\mu\upharpoonright S), \text{loc}};
\end{equation}
then for each \((m, \delta, \gamma, \varepsilon) \in \mathbb{N} \times (0, 1) \times (0, \alpha - \beta) \times (0, 1)\), there is a triple \((K, r_m, h)\):

1. The set \(K\) is compact with \(\mu(S \setminus K) \leq \varepsilon\).
2. The function \(h\) is \(3L(f)\)-Lipschitz and \(r_m > 0\).
3. For each \(x \in K\) there is \(y:\)

\[
0 < d(x, y) \leq \frac{1}{m} \quad \text{and} \quad |h(x) - h(y)| \geq \gamma d(x, y).
\]

(4.19)

4. For each \(x \in K\), \(L(h|B(x, r_m)) \leq \delta\).

In the case in which \(\beta = 0\) the assumption on the finite Assouad dimension is not needed.

Proof. Without loss of generality we can assume that \(S\) is compact and, for each \(\eta_0 > 0\), \(\text{Frag}(X, f, \beta + \eta_0)\)-null. Thus, by Theorem 1.21 there are functions \(g_n : X \to \mathbb{R}\) which are

\[
L_0 = \max (L(f), \beta + \eta_0)\)-Lipschitz,
\]

and such that \(g_n \overset{w^*}{\to} f\) and \(g_n\) is \(\mu\)-a.e. locally \((\beta + \eta_0)\)-Lipschitz on \(S\).

The part of the argument starting here is the only one that requires that the Assouad dimension of the space is finite and is not needed if \(\beta = 0\). Note that \(dg_n \overset{w^*}{\to} df\) in \(\mathcal{E}(\mu)\). By finiteness of the Assouad dimension, by Lemma 5.136 and Theorem 1.23, there are finitely many disjoint Borel \(\{X_\alpha\}\) with \(\mu(X_\alpha) > 0\), \(\mu(X \setminus \bigcup_\alpha X_\alpha) = 0\) and \(\mathcal{X}(\mu \mathbb{L}X_\alpha)\) free of rank \(N_\alpha\). Note that

\[
\mathcal{X}(\mu) = \bigoplus_\alpha \mathcal{X}(\mu \mathbb{L}X_\alpha)
\]

and Lemma 2.110 implies that \(\mathcal{E}(\mu \mathbb{L}X_\alpha)\) is also free of rank \(N_\alpha\). In particular, we have

\[
\mathcal{E}(\mu) = \bigoplus_\alpha \mathcal{E}(\mu \mathbb{L}X_\alpha)
\]

and for each \(\alpha\) one can find a basis of \(\mathcal{X}(\mu \mathbb{L}X_\alpha)\), \(\{D_{\alpha, i}\}_{i=1}^{N_\alpha}\), and a constant \(C_\alpha > 0\): for each \(\omega \in \mathcal{E}(\mu \mathbb{L}X_\alpha)\),

\[
|\omega|_{\mathcal{E}(\mu \mathbb{L}X_\alpha), \text{loc}} \leq C_\alpha \left( \sum_{i=1}^{N_\alpha} |\langle D_{\alpha, i}, \omega \rangle|^2 \right)^{1/2}.
\]

(4.23)

We let \(N = \max_\alpha N_\alpha\), \(C = \max_\alpha C_\alpha\) and \(\{e_i\}_{i=1}^{N_\alpha}\) the standard basis of \(\mathbb{R}^{N_\alpha}\). Consider the map

\[
\iota : \mathcal{E}(\mu) \to L^2(\mu, \mathbb{R}^N)
\]

\[
\omega \mapsto \sum_{\alpha} \sum_{i=1}^{N_\alpha} \chi_{X_\alpha}(D_{\alpha, i}, \omega) e_i;
\]

then

\[
|\omega|_{\mathcal{E}(\mu), \text{loc}} \leq C|\iota(\omega)|_{L^2} \in L^\infty(\mu)
\]

(4.25)
and \( \iota(d\hat{g}_n) \xrightarrow{w-L^2} \iota(df) \). By Mazur’s Lemma there are tail-convex combinations
\[
(4.26) \quad \hat{g}_n = \sum_{k=n}^{M(n)} t_k g_k
\]
with \( \iota(\hat{g}_n) \xrightarrow{L^2} \iota(df) \). Note that the \( \hat{g}_n \) are \( L_0 \)-Lipschitz and \( \mu \)-a.e. locally \((\beta + \eta_0)\)-Lipschitz on \( S \). By passing to a subsequence we can assume that \( \iota(\hat{g}_n) \to \iota(df) \) \( \mu \)-a.e. Using Egorov’s Theorem, for each \( \eta_1, \eta_2 > 0 \) there are a compact \( K_0 \subset S \) and \( n_0 \in \mathbb{N} \):
\[
(4.27) \quad \mu(S \setminus K_0) \leq \eta_1,
(4.28) \quad \sup_{x \in K_0} |\iota(\hat{g}_{n_0})(x) - \iota(df)(x)| \leq \frac{\eta_2}{C},
\]
which implies
\[
(4.29) \quad |d\hat{g}_{n_0} - df|_{L^1(\mu|_{K_0}), loc} \leq \eta_2.
\]
This is the end of the part of the argument that requires that the Assouad dimension of the space is finite.

As \( \mathcal{L} \) behaves like a seminorm,
\[
(4.30) \quad \inf_{x \in S} \mathcal{L}(f - \hat{g}_{n_0})(x) \geq \inf_{x \in S} \mathcal{L}(f(x) - \sup_{x \in S} \mathcal{L}\hat{g}_{n_0}(x) \geq \alpha - \beta - \eta_0;
\]
using the Borel measurability of \( \mathcal{L}(f - \hat{g}_{n_0}) \) and Egorov and Lusin’s Theorems, for all positive \( \eta_3, \eta_4 \) and for each natural \( m \), there are a compact \( K_1 \subset K_0 \), and constants \( \rho_0 \geq \rho_1 > 0 \) such that:

1. We have the inequalities:
\[
(4.31) \quad \rho_1 \leq \rho_0 < \frac{1}{m},
(4.32) \quad \mu(K_0 \setminus K_1) \leq \eta_3.
\]

2. For each \( x \in K_1 \) there is \( y \in X : d(x, y) \in [\rho_1, \rho_0] \) and
\[
(4.33) \quad |(f - \hat{g}_{n_0})(x) - (f - \hat{g}_{n_0})(y)| \leq (\alpha - \beta - \eta_0 - \eta_4)d(x, y).
\]
Note that \( f - \hat{g}_{n_0} \) is
\[
(4.34) \quad L_1 = (\mathcal{L}(f) + L_0)\text{-Lipschitz}.
\]
Note also that for each \( \eta_5 > 0 \) the compact \( K_0 \) is \( \text{Frag}(X, f - \hat{g}_{n_0}, \eta_2 + \eta_5) \)-null. Applying again Theorem 1.21 we can find functions \( h_n : X \to \mathbb{R} \) which are
\[
(4.35) \quad L_2 = \max(L_1, \eta_2 + \eta_5)\text{-Lipschitz},
\]
with \( h_n \xrightarrow{w^*} f - \hat{g}_{n_0} \) and which are \( \mu \)-a.e. locally \((\eta_2 + \eta_5)\)-Lipschitz on \( K_0 \). Thus, for all \( \eta_6, \eta_7 > 0 \), there are \( n_1 \in \mathbb{N} \), a compact \( K_2 \subset K_1 \) and constants \( \rho_1 \geq \rho_2 > 0 \):
\[
(4.36) \quad \mu(K_1 \setminus K_2) \leq \eta_6,
(4.37) \quad \|h_{n_1} - (f - \hat{g}_{n_0})\|_{\infty} \leq \eta_7 \rho_1,
\]
\( ^{20} \text{Weak convergence in } L^2(\mu, \mathbb{R})^N \)
and for each \( x \in K_2, L(h|B(x, \rho_2)) \) does not exceed \( \eta_2 + \eta_5 \). Let \( x \in K_2 \); then there is \( y \in X \) with \( d(x, y) \in [\rho_1, \rho_0] \) such that (4.33) holds. Therefore,

\[
|h_{n_1}(x) - h_{n_1}(y)| \geq |(f - \hat{g}_{n_0})(x) - (f - \hat{g}_{n_0})(y)| - 2\|h_{n_1} - (f - \hat{g}_{n_0})\|_{\infty} \\
\geq (\alpha - \beta - \eta_0 - \eta_4 - 2\eta) d(x, y).
\]

We now require the constants \( \eta_i \) to satisfy

\[
\eta_1 + \eta_3 + \eta_6 \leq \varepsilon \\
\eta_2 + \eta_5 \leq \min(\delta, L(f)) \\
\gamma \leq (\alpha - \beta - \eta_0 - \eta_4 - 2\eta) \\
\beta + \eta_0 \leq 2\beta;
\]
as a consequence, \( L_2 \leq 3L(f) \). We let \( K = K_2, h = h_{n_1} \) and \( r_m = \rho_2 \).

To see how porosity interacts with differentiability we will need to consider the pointwise Lipschitz constants of a function with respect to a given subset.

**Definition 4.43.** Let \( Y \subset X, f : X \to \mathbb{R} \) Lipschitz. For \( y \in Y \), the **pointwise upper and lower Lipschitz constants of \( f|_Y \) at \( y \)** will be denoted by \( L_Y f(y) \) and \( \ell_Y f(y) \). Note that

\[
L_Y f(y) \leq L f(y) \\
\ell_Y f(y) \leq \ell f(y).
\]

We now introduce the classes of subsets used to characterize differentiability.

**Definition 4.46.** Given a complete metric space \( X \) and a Radon measure \( \mu \) on \( X \), we define by \( \text{Gap}(\mu, X) \) the class of Borel subsets \( V \subset X \) such that there are \( \alpha > \beta \geq 0 \) and a Lipschitz function \( f : X \to \mathbb{R} \) with

\[
\inf_{x \in V} L f(x) \geq \alpha \quad \text{and} \quad |df|_{E(\mu L V), \text{loc}} \leq \beta.
\]

Note that the inequality involving \( |df|_{E(\mu L V), \text{loc}} \) holds in \( L^\infty(\mu L V) \). We define by \( \text{Gap}_0(\mu, X) \) the class of Borel subsets \( V \subset X \) such that there are \( \alpha > 0 \) and a Lipschitz function \( f : X \to \mathbb{R} \) with

\[
\inf_{x \in V} L f(x) \geq \alpha \quad \text{and} \quad |df|_{E(\mu L V), \text{loc}} = 0.
\]

Note that \( \text{Gap}_0(\mu, X) \subset \text{Gap}(\mu, X) \) and if \( Y \subset X \) is Borel, applying McShane’s Lemma and (4.44),

\[
\text{Gap}(\mu L Y, Y) \subset \text{Gap}(\mu, X) \\
\text{Gap}_0(\mu L Y, Y) \subset \text{Gap}_0(\mu, X).
\]

We finally let

\[
\text{Gap}(X) = \bigcap_{\mu \text{ Radon}} \text{Gap}(\mu, X) \\
\text{Gap}_0(X) = \bigcap_{\mu \text{ Radon}} \text{Gap}_0(\mu, X).
\]

Note that \( \text{Gap}_0(X) \subset \text{Gap}(X) \) and if \( Y \subset X \) is Borel,

\[
\text{Gap}(Y) \subset \text{Gap}(X) \\
\text{Gap}_0(Y) \subset \text{Gap}_0(X).
\]
Even though we found useful to work with $\text{Gap}(\mu, X)$ sets, the following Lemma shows that one can just work with $\text{Gap}(X)$ sets.

**Lemma 4.55.** If $S \in \text{Gap}(\mu, X)$ with $\mu(S) > 0$, then there is a Borel $S' \subset S$ with $\mu(S \setminus S') = 0$ and $S' \in \text{Gap}(X)$. The same conclusion holds replacing $\text{Gap}$ with $\text{Gap}_0$.

**Proof.** Let $(f, \alpha, \beta)$ as in the defining property of Gap. By Theorem 3.141, for $n$ sufficiently large, there is a Frag$(X, f, \beta + \frac{1}{n})$-null Borel $S_n \subset S$ such that $\mu(S \setminus S_n) = 0$. Let $S' = \bigcap_n S_n$ so that $S'$ is a full $\mu$-measure Borel subset of $S$ which is Frag$(X, f, \beta + \frac{1}{n})$-null for each $n$. Thus, Theorem 3.141 implies that

\[
|df|_{\mathcal{L}(\mu, S')}, \text{loc} \leq \beta
\]

for all Radon measures $\nu$. As

\[
\inf_{x \in S'} \mathcal{L} f(x) \geq \alpha,
\]

$S' \in \text{Gap}(X)$. \hfill $\square$

We now use Lemma 4.17 to construct independent Lipschitz functions.

**Lemma 4.58.** Let $X$ be a compact metric space with finite Assouad dimension, $f : X \to \mathbb{R}$ Lipschitz, $S \in \text{Gap}(\mu, X)$ with $\mu(S) > 0$. Then for all $\varepsilon > 0$ and $M \in \mathbb{N}$, there is a Borel $S' \subset S$ with $\mu(S \setminus S') \leq \varepsilon$ and there are 1-Lipschitz functions $\{\psi_0, \cdots, \psi_{M-1}\}$ which are infinitesimally independent on $S'$.

In the case in which $S \in \text{Gap}_0(\mu, X)$ the assumption on the Assouad dimension is not needed.

**Proof.** Let $(f, \alpha, \beta)$ as in the defining property of Gap. By Lemma 4.17, for each $m \in \mathbb{N}$ and $\varepsilon_1^{(m)} > 0$, there are $(K_m, \rho_m, h_m)$:

1. The set $K_m$ is compact with $\mu(S \setminus K_m) \leq \varepsilon_1^{(m)}$.
2. The function $h_m$ is $3\mathcal{L}(f)$-Lipschitz and $\rho_m > 0$.
3. For each $x \in K_m$ there is $y$:
   \[
   0 < d(x, y) \leq \frac{1}{m} \quad \text{and} \quad |h_m(x) - h_m(y)| \geq \frac{\alpha}{2} d(x, y).
   \]
4. For each $x \in K_m$, $\mathcal{L}(h_m | B(x, \rho_m)) \leq \frac{1}{m}$.

Let $K = \bigcap_m K_m$ so that

\[
\mu(S \setminus K) \leq \sum_m \varepsilon_1^{(m)};
\]

choosing the $\varepsilon_1^{(m)}$ sufficiently small, we can ensure $\mu(S \setminus K) \leq \varepsilon$. Applying Theorem 4.13 to $K$ we construct the functions $\{\psi_0, \cdots, \psi_{M-1}\}$. \hfill $\square$

We now recall some facts about porosity and refer the reader to the survey [Zaj05] for more information.

**Definition 4.61.** For $Y \subset X$ and $c > 0$ we say that $Y$ is $c$-porous at $y$ if there is a sequence $y_n \to y$ with

\[
0 < cd(y_n, y) < \text{dist}(\{y_n\}, Y).
\]

For $y \in Y$ we let

\[
W_c(y, Y) = \{y' \in Y : 0 < cd(y', y) < \text{dist}(\{y'\}, Y)\};
\]
thus $c$-porosity of $Y$ at $y$ is equivalent to:

\[(4.64) \quad W_c(y, Y) \cap B(y, r) \neq \emptyset \quad (\forall r > 0).\]

If $Y$ is $c$-porous at each $y \in Y$, $Y$ is called $c$-**porous**.

**Remark 4.65.** Given a $c$-porous set $Y$, one might wonder if there is also a Borel porous subset of $X$. By [MMPZ03, Lem. 1.4] it follows that if $Y$ is $c$-porous, for each $\varepsilon > 0$ there is $\tilde{Y}$, a $G_\delta$ set, with $Y \subseteq \tilde{Y}$ and $\tilde{Y}$ $(c - \varepsilon)$-porous.

**Remark 4.66.** Porosity is related to the relationship between $\mathcal{L}_Y f$ and $\mathcal{L} f$. For example, assume that $Y$ is $c$-porous at $y$ and let $f = \text{dist}(Y, \cdot)$.

Then

\[(4.67) \quad \mathcal{L} f(y) \geq \limsup_{n \to \infty} \frac{\text{dist}(\{y_n\}, Y)}{d(y_n, y)} > c;\]

but $\mathcal{L}_Y f(y) = 0$ as $f$ is identically zero on $Y$. On the other hand, if for all $c \in (0, \frac{1}{2})$ the set $Y$ is not $c$-porous at $y \in Y$, then, for each Lipschitz function $f$,

\[(4.68) \quad \mathcal{L}_Y f(y) = \mathcal{L} f(y).\]

To see this, choose $x_n \to y$, with $d(x_n, y) > 0$ and

\[(4.69) \quad \mathcal{L} f(y) = \lim_{n \to \infty} \frac{|f(y) - f(x_n)|}{d(y, x_n)};\]

as $Y$ is not $c$-porous at $y$, $\exists r_c > 0$ such that

\[(4.70) \quad \text{dist}(B(y, r), Y) \leq cr \quad (\forall r \in (0, r_c]).\]

So for $n$ large enough we find $y_0 \neq y_n$ with

\[(4.71) \quad d(x_n, y_n) \leq cd(x_n, y);\]

this implies

\[(4.72) \quad \mathcal{L}_Y f(y) \geq \limsup_{n \to \infty} \frac{|f(y) - f(y_n)|}{d(y, y_n)} \geq \limsup_{n \to \infty} \frac{|f(y) - f(x_n)| - c \mathcal{L}(f)d(y, x_n)}{(1 + c)d(x_n, y)} \geq \frac{1}{1 + c} (\mathcal{L} f(y) - c \mathcal{L}(f)).\]

Letting $c \to 0$ we obtain (4.68).

We now show that porous sets are of class $\text{Gap}_0(X)$.

**Lemma 4.73.** If $S \subset X$ is $c$-porous and Borel, then $S \in \text{Gap}_0(X)$.

**Proof.** Let $\nu$ be a Radon measure on $X$. Let $f = \text{dist}(S, \cdot)$. From (4.67) it follows that

\[(4.74) \quad \inf_{x \in S} \mathcal{L} f(x) \geq c.\]

If $\nu(S) = 0$ we have, trivially, $|df|_{\mathcal{E}(\nu \ll \mathcal{L} S), \text{loc}} = 0$. If $\nu(S) > 0$ we note that $\forall D \in \mathcal{X}(\nu)$, by locality of derivations, as $f = 0$ on $S$, $\chi_s Df = 0$. This implies

\[(4.75) \quad |df|_{\mathcal{E}(\nu \ll \mathcal{L} S), \text{loc}} = 0.\]

We need the following technical Lemma because our approximation schemes are designed to work in compact spaces.
Lemma 4.76. If $K \subset X$ is compact and $c$-porous, then there is a compact $Y$ with $K \subset Y \subset X$ and such that $K$ is \( \frac{2c}{3} \)-porous in $Y$.

Proof. For $m \in \mathbb{N}$ define
\[
\psi_m : K \to \left( 0, \frac{1}{m} \right]
\]  
(4.77)
\[x \mapsto \sup \left\{ d(x, y) : y \in W_c(x, K) \cap B(x, \frac{1}{m}) \right\} ;
\]
note that $\psi_m$ is lower-semicontinuous as \( \{ x \in K : \psi_m(x) > r \} \) is open. In fact, choose $y \in W_c(x, K)$ with $d(x, y) > r$. For $x' \in K$ sufficiently close to $x$, $d(x', y) > r$ and $y \in W_c(x', K)$. Let $r_m = \min_K \psi_m$. As $K$ is compact, we can choose a finite \( \frac{2c}{3} \)-net $N_m \subset K$. For $x \in N_m$ choose
\[
w_m(x) \in W_c(x, K) \cap B(x, \frac{1}{m})
\]  
(4.78)
with $d(x, w_m(x)) > \frac{2r_m}{3}$. Let $W_m = \bigcup_{x \in N_m} \{ w_m(x) \}$. Let $x' \in K$; choose $x \in N_m$ with $d(x, x') \leq \frac{2r_m}{3}$. Then
\[d(x', w_m(x)) > \frac{r_m}{3}
\]  
(4.79)
and
\[d(w_m(x), x') \leq d(x, w_m(x)) + d(x, x')
\]  
(4.80)
\[< d(x, w_m(x)) + \frac{1}{2} d(x, w_m(x))
\]
\[= \frac{3}{2} d(x, w_m(x)).
\]
This implies that
\[
\frac{2c}{3} d(w_m(x), x') < cd(x, w_m(x)) < d(w_m(x), K);
\]  
(4.81)
thus $w_m(x) \in W_c(x', K)$ and $K$ is \( \frac{2c}{3} \)-porous in
\[Z = K \cup \bigcup_m W_m.
\]  
(4.82)
We let $Y$ be the closure of $Z$ in $X$ and show that $Y$ is compact by showing that $Z$ is totally bounded. Let $\varepsilon > 0$. By compactness of $K$, finitely many balls $B_1, \ldots, B_M$ cover $K$; As $B_1 \cup \cdots \cup B_M$ is an open neighbourhood of $K$ and as $W_m$ lies in a \( \frac{1}{m} \)-neighbourhood of $K$, there is $m_0$ such that $m \geq m_0$ implies
\[W_m \subset B_1 \cup \cdots \cup B_M;
\]  
(4.83)
note that $\bigcup_{m < m_0} W_m$ is finite so finitely many balls of radius $\varepsilon$ are needed to cover it. \( \square \)

We now prove Theorem 4.7.

Proof of Theorem 4.7. We first prove necessity. Suppose that $S$ is a Borel $c$-porous subset of $X$ with $\mu(S) > 0$. Then there is a compact $K \subset S$ with $\mu(K) > 0$ and, by Lemma 4.76, a compact $Y \supset K$ in which $K$ is \( \frac{2c}{3} \)-porous. By Lemma 4.73 $K \in \text{Gap}_0(Y)$ and so, by Lemma 4.58 and as $(Y, \mu_{LY})$ is a $\sigma$-differentiability
space, \( \mu(K) = 0 \), yielding a contradiction. This implies that \( \mu(S) = 0 \). As observed in [BS11], [MMPZ03, Thm. 3.6] implies that if \( \mu \) annihilates porous sets, then \( \mu \) is asymptotically doubling. We can therefore find disjoint compact sets \( K_\alpha \) with \( \mu(K_\alpha) > 0 \), \( \mu(X \setminus \bigcup_\alpha K_\alpha) = 0 \) and \( \mu L K_\alpha \) doubling on \( K_\alpha \).

We now show that on a full \( \mu \)-measure Borel subset of \( K_\alpha \), the set \( K_\alpha \) is not \( c \)-porous for any \( c \in (0, \frac{1}{2}) \). Let

\[
P_\alpha = \left\{ x \in K_\alpha : \exists c \in \left(0, \frac{1}{2}\right) : K_\alpha \text{ is } c\text{-porous at } x \right\};
\]

Note that

\[
O_{c, \alpha}(r) = \{ x \in X : \exists y \in B(x, r) : \text{dist}(K_\alpha, \{ y \}) > cd(x, y) > 0 \}
\]

is open; then

\[
P_\alpha = \bigcup_{c \in \mathbb{Q} \cap (0, \frac{1}{2})} \bigcap_{n \in \mathbb{N}} \bigcup_{r \in (0, \frac{1}{2})} O_{c, \alpha}(r) \cap K_\alpha
\]

is a \( G_\delta \) and is a countable union of the porous sets

\[
\bigcap_{n \in \mathbb{N}} \bigcup_{r \in (0, \frac{1}{2})} O_{c, \alpha}(r) \cap K_\alpha \quad (c \in \mathbb{Q} \cap (0, \frac{1}{2})).
\]

Thus \( \mu(P_\alpha) = 0 \).

Let \( S \in \text{Gap}(X) \) and let \((f, \alpha, \beta)\) be as in the defining property of Gap. Then (4.68) implies that if \( S \cap (K_\alpha \setminus P_\alpha) \neq \emptyset \), then \( S \cap (K_\alpha \setminus P_\alpha) \in \text{Gap}(K_\alpha) \). But by Lemma 4.58, \((K_\alpha, \mu L K_\alpha)\) being a \( \sigma \)-differentiability space, \( \mu(S \cap K_\alpha) = 0 \) so that \( \mu(S) = 0 \).

We now prove sufficiency. If \( S \) is Borel and \( c \)-porous, \( S \in \text{Gap}_b(X) \) by Lemma 4.73, so \( S \) is \( \mu \)-null by hypothesis. Then, as in the necessity argument, \( \mu \) is \( \sigma \)-asymptotically doubling and we find disjoint compact sets \( K_\alpha \) with \( \mu(K_\alpha) > 0 \), \( \mu(X \setminus \bigcup_\alpha K_\alpha) = 0 \) and \( \mu L K_\alpha \) doubling on \( K_\alpha \); by Corollary 5.136 and Theorem 1.23 we can partition the \( K_\alpha \) and assume that the module \( X(\mu L K_\alpha) \) is free of rank \( N_\alpha \). As in the necessity argument, \( P_\alpha \) is \( \mu \)-null so it suffices to show, by (4.68), that each \((K_\alpha, \mu L K_\alpha)\) is a differentiability space in order to conclude that \((X, \mu)\) is a \( \sigma \)-differentiability space.

Let \( S \in \text{Gap}(\mu L K_\alpha, K_\alpha) \subset \text{Gap}(\mu, X) \); by Lemma 4.55 it contains a \( \mu \)-full measure Borel subset of type Gap(X), so \( \mu L K_\alpha(S) = 0 \). By definition of Gap sets, this implies that for every \( f : K_\alpha \to \mathbb{R} \)

\[
|df|_{\varepsilon(\mu L K_\alpha), \text{loc}}(x) = \mathcal{L} f(x) \quad (\text{for } \mu L K_\alpha\text{-a.e. } x);
\]

by Corollary 2.121 up to furthering partitioning the \( K_\alpha \), we can assume that there is a basis \( \{ D_\alpha,i \}_{i=1}^{N_\alpha} \) of \( X(\mu L K_\alpha) \) and there are 1-Lipschitz functions \( \{ g_{\alpha,j} \}_{j=1}^{N_\alpha} \) with

\[
D_{\alpha,i} g_{\alpha,j} = \delta_{ij} \chi_{K_\alpha}.
\]

Because of (4.88), for each \( \varepsilon > 0 \) there are disjoint Borel \( \{ U_{\alpha,\beta} \} \) which are subsets of \( K_\alpha \), which satisfy \( \mu \left( K_\alpha \setminus \bigcup_\beta U_{\alpha,\beta} \right) = 0 \), and such that there are derivations

\[22] K_\alpha, \text{ being closed, is a } G_\delta \]
\{D_{\alpha,\beta}\}$ with 
\begin{align}
|D_{\alpha,\beta}|_{X(\mu \mathcal{L}_{\alpha,\beta}), \text{loc}} &= 1 \\
D_{\alpha,\beta} f &\geq \mathcal{L} f - \varepsilon \quad \text{on } U_{\alpha,\beta},
\end{align}

moreover, there are $\{\lambda_{i,\alpha,\beta}\}_{i=1}^{N_\alpha} \subset \mathcal{B}^\infty(K_\alpha)$ with 
\begin{equation}
D_{\alpha,\beta} = \sum_{i=1}^{N_\alpha} \lambda_{i,\alpha,\beta} D_{\alpha,i};
\end{equation}
evaluating on the $g_{\alpha,j}$ gives 
\begin{equation}
||\lambda_{i,\alpha,\beta}||_{L^\infty(\mu \mathcal{K}_\alpha)} \leq 1,
\end{equation}
so that 
\begin{equation}
|D_{\alpha,\beta} f| \leq N_\alpha \max_{i=1,\ldots,N_\alpha} |D_{\alpha,i} f|.
\end{equation}
Thus 
\begin{equation}
\mathcal{L} f(x) \leq N_\alpha \max_{i=1,\ldots,N_\alpha} |D_{\alpha,i} f|(x) \quad (\text{for } \mu \mathcal{L} K_\alpha\text{-a.e. } x),
\end{equation}
and [Sch, Thm. 2.148] shows that $(K_\alpha, \mu \mathcal{L} K_\alpha)$ is a differentiability space. \hfill \Box

We now prove Theorem 4.9.

\textbf{Proof of Theorem 4.9.} The characterization in terms of (1) is immediate because of Theorem 4.7, in particular of (4.88). The characterization in terms of (2) follows from (1) and Theorem 3.141. The characterization in terms of (3) is an immediate consequence of (2). \hfill \Box

We now give an alternative proof of Theorem 4.1 which relies on Theorem 4.7.

\textbf{Alternative proof of Theorem 4.1.} Let $(U, x)$ be an $n$-dimensional chart for the differentiability space $(X, \mu)$; then (4.95) shows that it is possible to find derivations $\{D_i\}_{i=1}^n \subset X(\mu \mathcal{L} U)$ such that, for each $f \in \text{Lip}_b(X)$,
\begin{equation}
\mathcal{L} f(x) \leq n \max_{i=1,\ldots,n} |D_i f|(x)
\end{equation}
holds for $\mu \mathcal{L} U\text{-a.e. } x$. We can then apply [Sch, Lem. 6.20] to conclude that the partial derivative operators $\frac{\partial}{\partial x}$ are derivations; note that even though in [Sch] the measure $\mu$ is assumed doubling, this property of $\mu$ is used only to ensure that the Lebesgue’s Differentiation Theorem holds for the measure $\mu$. However, the Lebesgue Differentiation Theorem holds for asymptotically doubling measures and so it is possible to apply the results in [Sch]. \hfill \Box

\section{Technical tools}

\subsection{An approximation scheme.} The goal of this Subsection is to prove Theorem 3.66, whose proof requires some preparation. The first step is the geometric construction of a cylinder. For this we need to compare the notion of miliity for fragments in different spaces. We introduce the following auxiliary definition:
**Definition 5.1.** For $\delta > 0$, metric spaces $X$ and $W$, Lipschitz functions $G : X \to W$ and $f : X \to \mathbb{R}^q$, we define

\[
\text{Frag}(X, f, G, \delta, w, \alpha) = \left\{ \gamma \in \text{Frag}(X) : ((w, f) \circ \gamma)'(t) \geq \delta \text{md } G \circ \gamma(t) \right. \\
\left. \text{and } (f \circ \gamma)'(t) \in C(w, \alpha) \text{ for } L^1\text{-a.e. } t \in \text{dom } \gamma \right\}.
\]

**Lemma 5.3.** Assume that $\psi : X \to Y$ is an isometric embedding and $S \subset X$. Then $S$ is $\text{Frag}(X, f, G, \delta, w, \alpha)$-null if and only if

\[
\text{Frag}(Y, \tilde{f}, \tilde{G}, \delta, w, \alpha)-null
\]

where $\tilde{f}$ and $\tilde{G}$ are any Lipschitz extensions of $f \circ \psi^{-1}$, $G \circ \psi^{-1}$.

**Proof.** Necessity is proven by contrapositive, assuming that there is a

\[
\gamma \in \text{Frag}(Y, \tilde{f}, \tilde{G}, \delta, w, \alpha)
\]

with $\mathcal{H}^1(\gamma \cap \psi(S)) > 0$. It is then possible to find a compact $K' \subset \text{dom } \gamma$ with $\gamma(K') \subset \psi(S)$ and $\mathcal{H}^1((\gamma|K') \cap \psi(S)) > 0$. Let $\tilde{\gamma} = \psi^{-1} \circ (\gamma|K')$. Then

\[
(f \circ \tilde{\gamma})'(t) = (\tilde{f} \circ \gamma)'(t)
\]

\[
\text{md } \tilde{\gamma}(t) = \text{md } \gamma(t)
\]

\[
\text{md } \tilde{G} \circ \gamma(t) = \text{md } G \circ \gamma(t)
\]

at any $t \in K'$ which is a Lebesgue density point for $K'$. In particular,

\[
\text{Frag}(X, f, G, \delta, w, \alpha)
\]

and $\mathcal{H}^1(\tilde{\gamma} \cap S) > 0$.

Sufficiency is proven by observing that the previous part of the argument allows to identify $\text{Frag}(X, f, G, \delta, w, \alpha)$ with $\text{Frag}(\psi(X), \tilde{f}, \tilde{G}, \delta, w, \alpha)$ via $\gamma \mapsto \psi \circ \gamma$ because the metric differential and the derivative of a Lipschitz function along a fragment are determined, at a point $t$, by the behaviour of the fragment on a subset for which $t$ is a Lebesgue density point. Sufficiency then follows because $\text{Frag}(X, f, G, \delta, w, \alpha) \subset \text{Frag}(Y, \tilde{f}, \tilde{G}, \delta, w, \alpha)$. \qed

We now introduce the definition of cylinder.

**Definition 5.10.** Let $X$ be a compact metric space and $M > 0$. The cylinder $\text{Cyl}(X, M)$ is the compact metric space $X \times [0, M]$ with metric

\[
d((x_1, t_1), (x_2, t_2)) = \max(d(x_1, x_2), |t_1 - t_2|).
\]

The projection on the base $X$ will be denoted by $\pi$ and the projection on the axis $[0, M]$ by $\tau$.

**Remark 5.12.** Note that if $X$ is geodesic, $\text{Cyl}(X, M)$ is geodesic. If

\[
\gamma : [0, d(x_1, x_2)] \to X
\]
denotes a unit speed geodesic joining $x_1$ to $x_2$, then
\begin{equation}
\sigma : [0, d((x_1, t_1), (x_2, t_2))] \rightarrow X
\end{equation}
\begin{equation}
s \mapsto \left( \gamma\left( \frac{sd(x_1, x_2)}{d((x_1, t_1), (x_2, t_2))} \right), \tau, t_1 + \frac{s(t_2 - t_1)}{d((x_1, t_1), (x_2, t_2))} \right)
\end{equation}

is a unit speed geodesic joining $(x_1, t_1)$ to $(x_2, t_2)$.

We now reduce the general approximation problem to the cylindrical case.

**Lemma 5.16.** If $f : X \rightarrow \mathbb{R}^q$ is Lipschitz and $S \subset X$ is compact and
\begin{equation}
\text{Frag}(X, f, \delta, w, \alpha)-\text{null},
\end{equation}
we can assume that:

1. The space $X$ is the cylinder $\text{Cyl}(Y, M)$ for $Y = Z \times Q$, where $Z$ is a compact geodesic metric space, $Q$ is a compact rectangle in $\mathbb{R}^{q-1}$ and $M \leq \text{diam} X$.
2. If $\tilde{\tau} : Y \rightarrow Q$ is the projection, $(\pi_w \circ f, (w, \tilde{f})) = (\tilde{\tau}, \tau)$.
3. If $\pi_Z : Y \rightarrow Z$ is the projection, $S$ is Frag($\text{Cyl}(Y, M), (\tilde{\tau}, \tau), \pi_Z, \delta, e_q, \alpha$)-null.

Points of $\text{Cyl}(Y, M)$ will be denoted by $(z, v, t)$ where $z \in Z$, $v \in Q$ and $t \in [0, M]$.

**Proof.** Considering a Kuratowski embedding of $X$ in $l^\infty$ we can assume that $X$ lies in $l^\infty$. Let $Z$ denote the closed convex hull of $X$ in $l^\infty$ which is compact ([Rud91, Thm. 3.25]). Moreover, postcomposing $f$ with an orthogonal transformation $A \in \text{O}(q)$, we can assume that the vector $w = e_q$, where $\{e_1, \ldots, e_q\}$ is the standard Euclidean basis of $\mathbb{R}^q$. Note that this does not affect the global Lipschitz constant of $f$ with respect to the $l^2$-norm $\| \cdot \|_2$.

Replace $f$ by taking a McShane extension on $Z$ of each component $f_i$: this does not affect the Frag($X, f, \delta, w, \alpha$)-nullity of $S$ even though it might increase the global Lipschitz constant of $f$ by a factor $\sqrt{q}$. Postcomposing $f$ with an affine transformation $\lambda x + b$, where $(\lambda, b) \in (0, \infty) \times \mathbb{R}^q$, we can assume that:

- The function $f$ is 1-Lipschitz with respect to the norm $\| \cdot \|_2$.
- The minimum of $f_i$ is 0.

Note that postcomposing affects $\delta$ but not $\alpha$; furthermore, considering a geodesic from a point where the minimum is attained to a point where the maximum is attained, all the values in $[0, \text{max } f_i]$ are assumed by $f_i$.

Let $M = \text{max } f_i \leq \text{diam } Z = \text{diam } X$ and
\begin{equation}
Q = [0, \text{max } f_1] \times \cdots \times [0, \text{max } f_{N-1}] \subset \mathbb{R}^{q-1}
\end{equation}
with the metric induced by the norm $\| \cdot \|_2$. Let $Y = Z \times Q$ with the metric
\begin{equation}
d((z_1, v_1), (z_2, v_2)) = \max (d(z_1, z_2), \|v_1 - v_2\|_2).
\end{equation}

Let $\tilde{\tau}$ denote the projection on the $Q$ factor. The metric space $Y$ is geodesic because if
\begin{equation}
\gamma : [0, d(z_1, z_2)] \rightarrow Z
\end{equation}
denotes a unit speed geodesic joining \( z_1 \) to \( z_2 \), then
\[
\sigma : [0, d((z_1, v_1), (z_2, v_2))] \rightarrow X
\]
\[
(5.21)
\]
\[
s \mapsto \left( \gamma \left( \frac{sd(z_1, z_2)}{d((z_1, v_1), (z_2, v_2))}, v_1 + \frac{s(v_2 - v_1)}{d((z_1, v_1), (z_2, v_2))} \right) \right)
\]
\[
(5.22)
\]
is a unit speed geodesic joining \((z_1, v_1)\) to \((z_2, v_2)\). We will replace \( X \) by \( Y \).

Considering the isometric embedding
\[
\psi : Z \rightarrow \text{Cyl}(Y, M)
\]
\[
(5.23)
\]
z \mapsto ((z, f_1(z), \ldots, f_q(z)), f_q(z)),
\]
we conclude that \( \tilde{\tau} \) extends \((f_1, \ldots, f_{q-1}) \circ \psi^{-1}\) and \( \tau \) extends \( f_q \circ \psi^{-1} \); noting that for \( \gamma \in \text{Frag}(Z) \)
\[
md \gamma = md (\pi_Z \circ \psi) \circ \gamma,
\]
we conclude from Lemma 5.3 that \( \psi(S) \) is \( \text{Frag} (\text{Cyl}(Y, M), (\tilde{\tau}, \tau), \pi_Z, \delta, e_q, \alpha)\)-null. The Radon measure \( \mu \) on \( X \) is replaced by the pushforward \( \psi_\mu \).

The next step is to cover \( S \) by \textit{thin} strips.

**Definition 5.25.** Given a Lipschitz function \( f : Y \rightarrow [0, M] \) and \( h > 0 \) we define the open \textbf{strip of width} \( h \) \textit{above} \( f \) by
\[
\text{St}(f, h) = \{(y, t) \in \text{Cyl}(Y, M) : t \in (f(y), f(y) + h)\}.
\]
\[
(5.26)
\]
\[
\text{St}(f, h)
\]
is an open set. The \textbf{lower and upper hypersurfaces bounding} \( \text{St}(f, h) \) are the closed sets:
\[
\partial_- \text{St}(f, h) = \{(y, t) \in \text{Cyl}(Y, M) : t = f(y)\};
\]
\[
(5.27)
\]
\[
\partial_+ \text{St}(f, h) = \{(y, t) \in \text{Cyl}(Y, M) : t = f(y) + h\}.
\]
\[
(5.28)
\]

**Lemma 5.29.** Assume that the compact \( S \subset \text{Cyl}(Y, M) \) is \( \text{Frag}(\text{Cyl}(Y, M), (\tilde{\tau}, \tau), \pi_Z, \delta, e_q, \alpha)\)-null; then for each \( n \in \mathbb{N} \) the set \( S \) can be covered by \( M(n) \) open strips
\[
\left\{ \text{St} \left( f_i, \frac{\delta + \cot \alpha + 1}{n} \right) \right\}_{i=1}^{M(n)}
\]
\[
(5.30)
\]
where the \( f_i \) are 1-Lipschitz with respect to the distance
\[
d_{d, \alpha}((z_1, v_1, t_1), (z_2, v_2, t_2)) = \delta \max (d(z_1, z_2), |t_1 - t_2|) + \cot \alpha \|v_1 - v_2\|_2,
\]
and
\[
\lim_{n \rightarrow \infty} \frac{M(n)}{n} = 0.
\]
\[
(5.31)
\]

**Proof.** Let \( N_Y \) be a \( \frac{1}{n} \)-net in \( Y \): a maximal set of points in \( Y \) which are separated by a distance \( \geq \frac{1}{n} \). By compactness of \( Y \), the net \( N_Y \) is finite. Let \( N_{[0, M]} \) be a \( \frac{1}{n} \)-net in \([0, M]\), which is finite by compactness. Then \( N_{\text{Cyl}(Y, M)} = N_Y \times N_{[0, M]} \) is a \( \frac{1}{n} \)-net in \( \text{Cyl}(Y, M) \). Note that the open balls of radius \( \frac{1}{n} \) centred on the points \( N_{\text{Cyl}(Y, M)} \) cover \( \text{Cyl}(Y, M) \). We now consider the set of those points in \( N_{\text{Cyl}(Y, M)} \) such that the corresponding balls intersect \( S \):
\[
N_{\text{Cyl}(Y, M)}^S = \left\{ (y, t) \in N_{\text{Cyl}(Y, M)} : B((y, t), \frac{1}{n}) \cap S \neq \emptyset \right\}.
\]
\[
(5.33)
\]
On $N^S_{\text{Cyl}(Y,M)}$ define the partial order $\preceq$:
\begin{equation}
(z_1,v_1,t_1) \preceq (z_2,v_2,t_2) \iff t_2 - t_1 \geq \delta d(z_1,z_2)
\end{equation}
and $(t_2-t_1) \tan \alpha \geq \|v_1-v_2\|_2$;
the binary relation $\preceq$ is reflexive; it is antisymmetric because if $(z_1,v_1,t_1) \preceq (z_2,v_2,t_2)$ and $(z_2,v_2,t_2) \preceq (z_1,v_1,t_1)$, then $d(z_1,z_2) = 0$, $\|v_1-v_2\|_2 = 0$ and $t_2-t_1 = 0$. Transitivity follows from the triangle inequality:
\begin{align}
& t_2 - t_1 \geq \delta d(z_1, z_2) \quad \text{and} \quad (t_2-t_1) \tan \alpha \geq \|v_1-v_2\|_2 \\
& t_3 - t_2 \geq \delta d(z_2, z_3) \quad \text{and} \quad (t_3-t_2) \tan \alpha \geq \|v_2-v_3\|_2
\end{align}

imply
\begin{align}
& t_3 - t_1 \geq \delta d(z_1, z_2) + \delta d(z_2, z_3) \geq \delta d(z_1, z_3), \\
& (t_3-t_1) \tan \alpha \geq \|v_1-v_2\|_2 + \|v_2-v_3\|_2 \geq \|v_1-v_3\|_2.
\end{align}

Let $M(n)$ denote the length of a maximal chain in $(N^S_{\text{Cyl}(Y,M)}, \preceq)^{23}$. If
\begin{equation}
\lim_{n \to \infty} \frac{M(n)}{n} \neq 0,
\end{equation}
there are naturals $n \to \infty$ with $M(n) \geq Cn$ for $C > 0$. Let $(z_i^{(n)}, v_i^{(n)}, t_i^{(n)})_{i=1}^{M(n)}$ denote a maximal chain with the $t_i^{(n)}$ in increasing order. We want to construct a biLipschitz path $\gamma_n : [0, M] \to \text{Cyl}(Y,M)$:
\begin{itemize}
  \item On $[t_i^{(n)}, t_{i+1}^{(n)}]$, $\gamma_n$ is defined as a constant speed geodesic joining $(z_i^{(n)}, v_i^{(n)}, t_i^{(n)})$ to $(z_{i+1}^{(n)}, v_{i+1}^{(n)}, t_{i+1}^{(n)})$.
  \item On $[0, t_1^{(n)}]$, $\gamma_n$ is defined as a constant speed geodesic from $(z_1^{(n)}, v_1^{(n)}, 0)$ to $(z_1^{(n)}, v_1^{(n)}, t_1^{(n)})$.
  \item On $[t_{M(n)}^{(n)}, M)$, $\gamma_n$ is defined as a constant speed geodesic from $(z_{M(n)}^{(n)}, v_{M(n)}^{(n)}, t_{M(n)}^{(n)})$ to $(z_{M(n)}^{(n)}, v_{M(n)}^{(n)}, M)$.
\end{itemize}

Note that
\begin{equation}
1 \leq \frac{d((z_i^{(n)}, v_i^{(n)}, t_i^{(n)}), (z_{i+1}^{(n)}, v_{i+1}^{(n)}, t_{i+1}^{(n)}))}{t_{i+1}^{(n)} - t_i^{(n)}} \leq \max \left( \frac{1}{\delta}, \tan \alpha \right);
\end{equation}
this implies that $\text{md } \gamma_n \in [1, \max \left( \frac{1}{\delta}, \tan \alpha \right)]$. Moreover, $(\tau \circ \gamma_n)' = 1$ $L^1$-a.e. implying that $\gamma_n$ is $(1, \max \left( \frac{1}{\delta}, \tan \alpha \right))$-biLipschitz. Furthermore, as $\pi_Z \circ \gamma_n$ is $\frac{1}{\delta}$-Lipschitz and $\tilde{\tau} \circ \gamma_n$ is tan $\alpha$-Lipschitz, $\gamma_n \in \text{Path(Cyl}(Y,M), (\tilde{\tau}, \tau), \pi_Z, \delta, \epsilon_q, \alpha)$. By Ascoli-Arzelà we can assume that the $\gamma_n$ converge uniformly to a
\begin{equation}
\left( 1, \max \left( \frac{1}{\delta}, \tan \alpha \right) \right) - \text{biLipschitz } \gamma : [0, M] \to \text{Cyl}(Y,M).
\end{equation}

Note that $(\tau \circ \gamma)' = 1$ $L^1$-a.e., that $\pi_Z \circ \gamma$ is $\frac{1}{\delta}$-Lipschitz and that $\tilde{\tau} \circ \gamma$ is tan $\alpha$-Lipschitz; this implies that that $\gamma \in \text{Path(Cyl}(Y,M), (\tilde{\tau}, \tau), \pi_Z, \delta, \epsilon_q, \alpha)$. Moreover,
\begin{equation}
\gamma_n \left( \left[ t_i^{(n)} - \frac{\min(\delta, \cot \alpha)}{4n}, t_i^{(n)} + \frac{\min(\delta, \cot \alpha)}{4n} \right] \right) \subseteq B((z_i^{(n)}, v_i^{(n)}, t_i^{(n)}), \frac{1}{n})
\end{equation}

\textsuperscript{23}Note that the nets depend on $n$
and the intervals \( [t_{i_1}^{(n)} - \min(\delta, \cot \alpha), t_{i_2}^{(n)} + \min(\delta, \cot \alpha)] \) are disjoint\(^{24}\); in particular, there is a compact \( K_n \subset [0, M] \) with

\[
L^1(K_n) \geq \frac{\min(\delta, \cot \alpha)(M(n) - 2)}{2n} \geq \frac{(C - \frac{1}{n}) \min(\delta, \cot \alpha)}{2}
\]

on which \( d(\gamma_n, S) \leq \frac{\delta}{2n} \). We can pass to a subsequence such that the \( K_n \) converge to a compact \( K \). Regularity of the Lebesgue measure will imply that

\[
L^1(K) \geq \limsup_{n \to \infty} L^1(K_n)
\]

because any \( \varepsilon \)-neighbourhood of \( K \) will eventually contain the \( K_n \). But on \( K \) \( d(\gamma, S) = 0 \), contradicting that \( S \) is \( \text{Frag(Cyl(Y,M),} (\tilde{\tau}, \tau), \pi_Z, \delta, e_{\pi}, \alpha) \)-null.

By Mirsky’s Lemma (dual to Dilworth’s Lemma, [Mir71]), there are \( M(n) \) antichains \( A_1, \ldots, A_{M(n)} \) covering \( \mathcal{N}_{\gamma} \). Recall that in an antichain no two elements are comparable with respect to the order. So if \( (z, v, t), (z', v', t') \in A_p \), either

\[
|t - t'| \leq \delta d(z, z') \quad \text{or} \quad |t - t'| \leq \cot \alpha \|v - v'\|_2;
\]

in particular, \( A_p \) can be regarded as the graph of a function \( g_p : \pi_Z \times \tilde{\tau}(A_p) \to \mathbb{R} \) which is 1-Lipschitz with respect to the distance \( d_{\delta, \alpha} \); these functions can be extended to 1-Lipschitz functions \( g_p : Y \to \mathbb{R} \). If \( (z_i^{(n)}, v_i^{(n)}, t_i^{(n)}) \in A_p \) and if \( (z, v, t) \in B((z_i^{(n)}, v_i^{(n)}, t_i^{(n)}), \frac{1}{n}) \), then

\[
|g_p(z, v) - t| \leq |g_p(z, v) - g_p(z_i^{(n)}, v_i^{(n)})| + |g_p(z_i^{(n)}, v_i^{(n)}) - t_i^{(n)}| + |t_i^{(n)} - t|
\]

\[
\leq \delta d(z, z_i^{(n)}) + \cot \alpha \|v - v_i^{(n)}\|_2 + |t_i^{(n)} - t| < \frac{\delta + \cot \alpha + 1}{n};
\]

in particular,

\[
S \subset \bigcup_{p=1}^{M(n)} \text{St} \left( g_p \frac{\delta + \cot \alpha + 1}{n}, 2 \frac{\delta + \cot \alpha + 1}{n} \right).
\]

We now rearrange the strips in increasing order.

**Lemma 5.48.** Given open strips \( \{\text{St} (f_i, h)\}_{i=1}^{N} \) \( (h > 0) \) where the \( f_i \) are 1-Lipschitz with respect to the distance \( d_{\delta, \alpha} \), there are 1-Lipschitz functions (with respect to \( d_{\delta, \alpha} \)) \( \{\tilde{f}_i\}_{i=1}^{N} \) such that

\[
\tilde{f}_1 \leq \tilde{f}_2 \leq \cdots \leq \tilde{f}_N,
\]

and

\[
\bigcup_{i=1}^{N} \text{St} (f_i, h) = \bigcup_{i=1}^{N} \text{St} \left( \tilde{f}_i, h \right).
\]

\(^{24}\)For \( i = 1, M(n) \) one might have to consider one-sided intervals
Proof. We argue by induction on \( N \), for \( N = 1 \) there being nothing to prove. We now assume the result true for \( N \) and prove it for \( N+1 \). By the inductive hypothesis we can assume that the first \( N \) strips are defined by functions

\[
(5.51) \quad f_1 \leq f_2 \leq \cdots \leq f_N;
\]

the sorting is now done with respect to \( f_{N+1} \). Let \( F_1 = \min(f_1, f_{N+1}) \) and \( G_1 = \max(f_1, f_{N+1}) \). For consistency we will also let \( G_0 = f_{N+1} \) and for \( 1 < i \leq N \) we will define \( F_i = \min(f_i, G_{i-1}) \) and \( G_i = \max(f_i, G_{i-1}) \). We will finally let \( F_{N+1} = G_N \). We will show that the desired strips are defined by the functions \( F_i \).

Note that taking maxima and minima of 1-Lipschitz functions produce 1-Lipschitz functions. We show that \( F_i(x) \) is nondecreasing in \( i \) for \( i \in \{1, \cdots, N+1\} \). In fact,

\[
(5.52) \quad F_i = \min(f_i, G_{i-1}) \leq \min(f_{i+1}, G_{i-1}) \leq \min(f_{i+1}, G_i) = F_{i+1}.
\]

Similarly,

\[
(5.53) \quad F_N = \min(f_N, G_{N-1}) \leq G_N = F_{N+1}.
\]

We want to show that

\[
(5.54) \quad \text{St}(F_i, h) \subset \bigcup_{i=1}^{N+1} \text{St}(f_i, h);
\]

assume \((y, t) \in \text{St}(F_i, h)\). If \( F_i(y) = f_i(y) \) then \((y, t) \in \text{St}(f_i, h)\). If this is not the case, then \( F_i(y) = G_{i-1}(y) \) (recall that \( G_0 = f_{N+1} \) so that if \( i = 1 \) then we have \((y, t) \in \text{St}(f_{N+1}, h)\)). If \( G_{i-1}(y) = f_{i-1}(y) \) then \((y, t) \in \text{St}(f_{i-1}, h)\). Otherwise \( G_{i-1}(y) = G_{i-2}(y) \) and one continues to argue in a similar way till one either stops at some \( G_{i-k}(y) = f_{i-k}(y) \) or at \( G_0(y) \). We want to show that

\[
(5.55) \quad \text{St}(f_i, h) \subset \bigcup_{i=1}^{N+1} \text{St}(F_i, h);
\]

the argument is similar to the one above but it is better to distinguish between the cases \( i \in \{1, \cdots, N\} \) and \( i = N + 1 \). Assume that \( 1 \leq i \leq N \) and \((y, t) \in \text{St}(f_i, h)\). From the definition of \( F_i \) and \( G_i \) we see that either \( f_i(y) = F_i(y) \) or \( f_i(y) = G_i(y) \). In the first case \((y, t) \in \text{St}(F_i, h)\). In the second case one observes that either \( G_i(y) = G_{i+1}(y) \) or \( G_i(y) = G_{i+1}(y) \). In the first case \((y, t) \in \text{St}(F_{i+1}, h)\). Otherwise one continues to argue in a similar way till one either stops at some \( G_{i+k}(y) = F_{i+k}(y) \) or at \( G_N(y) = F_{N+1}(y) \). If \((y, t) \in \text{St}(f_{N+1}, h)\) then either \( f_{N+1}(y) = F_i(y) \) or \( f_{N+1}(y) = G_i(y) \). In the first case \((y, t) \in \text{St}(F_i, h)\). In the second case one keeps arguing as above.

The next step is to make the strips disjoint to induce an order structure on the set of strips. The cost to pay is to make the strips slightly bigger and allow to cover a full measure subset of \( S \), but not necessarily the whole of \( S \).

**Lemma 5.56.** Assume that \( S \subset \bigcup_{i=1}^{N} \text{St}(f_i, h) \) (\( h > 0 \)) where the functions \( f_i \) are \((1, d_{i,0})\)-Lipschitz. Then there are \((1, d_{i,0})\)-Lipschitz functions \( \{g_i\}_{i=1}^{N} \) and \( \lambda_i \in (1, \frac{2}{3}) \) such that

\[
(5.57) \quad g_1 \leq g_2 \leq \cdots \leq g_N;
\]

\[
(5.58) \quad \text{St}(g_i, \lambda_i h) \cap \text{St}(g_j, \lambda_j h) = \emptyset \quad \text{for } i \neq j,
\]
and

\[(5.59) \quad \mu \left(S \setminus \bigcup_{i=1}^{N} \text{St} (g_i, \lambda_i h)\right) = 0.\]

**Proof.** Because of the previous Lemma we can assume that for each \(y\) the map \(i \mapsto f_i(y)\) is nondecreasing in \(i\). As \(\mu\) is a finite measure and \((1, \frac{3}{2})\) is uncountable, it is possible to find \(\lambda_1 \in (1, \frac{3}{2})\) such that

\[(5.60) \quad \mu \left(\partial_+ \text{St} (f_1, \lambda_1 h)\right) = 0.\]

Let \(g_1 = f_1\) so that \(\text{St} (f_1, h) \subset \text{St} (g_1, \lambda_1 h)\). For \(j \in \{2, \cdots, N\}\) we let \(g_j = \max(g_{j-1} + \lambda_{j-1} h, f_j)\) and choose \(\lambda_j \in (1, \frac{3}{2})\) such that

\[(5.61) \quad \mu \left(\partial_+ \text{St} (g_j, \lambda_j h)\right) = 0;\]

this is possible because \(\mu\) is a finite measure and \((1, \frac{3}{2})\) is uncountable. We want now to show that

\[(5.62) \quad \text{St} (f_j, h) \subset \bigcup_{i=1}^{j} \text{St} (g_i, \lambda_i h) \cup \bigcup_{i=1}^{j-1} \partial_+ \text{St} (g_i, \lambda_i h).\]

If \(f_j(y) \geq g_{j-1}(y) + \lambda_{j-1} h\) then \(g_j(y) = f_j(y)\) so that

\[(5.63) \quad (f_j(y), f_j(y) + h) \subset (g_{j-1}(y), g_{j-1} + \lambda_{j-1} h);\]

otherwise \(f_j(y) < g_{j-1}(y) + \lambda_{j-1} h\) which implies \(g_j(y) = g_{j-1}(y) + \lambda_{j-1} h\). If \(f_j(y) \geq g_{j-1}(y)\) we conclude that

\[(5.64) \quad (f_j(y), f_j(y) + h) \subset (g_{j-1}(y), g_{j-1} + \lambda_{j-1} h) \cup \{g_{j-1}(y) + \lambda_{j-1} h\} \cup (g_j(y), g_j(y) + \lambda_j h);\]

if \(f_j(y) < g_{j-1}(y)\) then \(f_{j-1}(y) < g_{j-1}(y)\) which implies \(g_{j-1}(y) = g_{j-2}(y) + \lambda_{j-2} h\). If \(f_j(y) \geq g_{j-2}(y)\) we get, similarly to the previous equation,

\[(5.65) \quad (f_j(y), f_j(y) + h) \subset (g_{j-2}(y), g_{j-2} + \lambda_{j-2} h) \cup \{g_{j-2} + \lambda_{j-2} h\} \cup (g_{j-1}(y), g_{j-1} + \lambda_{j-1} h);\]

one can continue to argue inductively by decreasing the indices till

\[(5.66) \quad g_k(y) \leq f_j(y) < g_k(y) + \lambda_k h = g_{k+1}(y);\]

in that case,

\[(5.67) \quad (f_j(y), f_j(y) + h) \subset (g_k(y), g_k(y) + \lambda_k h) \cup \{g_k(y) + \lambda_k h\} \cup (g_{k+1}(y), g_{k+1}(y) + \lambda_{k+1} h);\]

this argument shows that (5.62) holds. The result now follows, noting also that by definition of the functions \(\{g_j\}\), for each \(y\) the map \(j \mapsto g_j(y)\) is nondecreasing in \(j\). \(\square\)

We can now prove Theorem 3.66.

**Proof of Theorem 3.66.** By Lemma 5.16 we can assume that we are in the cylindrical case so that we will approximate \(\tau\); in particular, we have to show that we will obtain approximations which are globally 1-Lipschitz with respect to the distance:

\[(5.68) \quad D ((z_1, v_1, t_1), (z_2, v_2, t_2)) = \max (|t_1 - t_2|, d_{\delta, \alpha} ((z_1, v_1), (z_2, v_2))).\]
By Lemma 5.56 we can “µ-essentially” cover \( S \) by \( M(n) \) disjoint strips

\[
\left\{ \text{St} \left( f_i, 2\lambda_i \frac{\delta + \cot \alpha + 1}{n} \right) \right\}_{i=1}^{M(n)},
\]

where the functions \( f_i \) are 1-Lipschitz with respect to the distance \( d_{\delta, \alpha} \). One can define a natural total order relation between the hypersurfaces bounding the open strips:

\[
\partial_{-} \text{St} \left( f_i, 2\lambda_i \frac{\delta + \cot \alpha + 1}{n} \right) \prec \partial_{+} \text{St} \left( f_i, 2\lambda_i \frac{\delta + \cot \alpha + 1}{n} \right)
\]

\[
\prec \partial_{-} \text{St} \left( f_{i+1}, 2\lambda_{i+1} \frac{\delta + \cot \alpha + 1}{n} \right) \prec \partial_{+} \text{St} \left( f_{i+1}, 2\lambda_{i+1} \frac{\delta + \cot \alpha + 1}{n} \right).
\]

The approximations are defined integrating over lines issuing from the base of the cylinder:

\[
\tau_n(z, v, t) = \int_{0}^{t} \chi \tau_n(z, v, s) \, ds;
\]

given points \( (z_1, v_1, t_1), (z_2, v_2, t_2) \in \text{Cyl}(Y, M) \), we say that they are separated by a hypersurface \( \partial_{\pm} \text{St} \left( f_j, 2\lambda_j \frac{\delta + \cot \alpha + 1}{n} \right) \) if

\[
\begin{pmatrix}
  f_j(y_1) + 2\lambda_j \frac{\delta + \cot \alpha + 1}{n} - t_1
  \\
  f_j(y_2) + 2\lambda_j \frac{\delta + \cot \alpha + 1}{n} - t_2
\end{pmatrix} \leq 0.
\]

If the two points are not separated by a hypersurface, there are 4 possibilities:

1. They lie in a strip \( \text{St} \left( f_j, 2\lambda_j \frac{\delta + \cot \alpha + 1}{n} \right) \).
2. They lie between two strips \( \text{St} \left( f_j, 2\lambda_j \frac{\delta + \cot \alpha + 1}{n} \right), \text{St} \left( f_{j+1}, 2\lambda_{j+1} \frac{\delta + \cot \alpha + 1}{n} \right) \)

in the sense that

\[
t_i - f_j(z_i, v_i) + 2\lambda_j \frac{\delta + \cot \alpha + 1}{n} > 0 \quad \text{and} \quad t_i - f_{j+1}(z_i, v_i) < 0 \quad \text{for} \ i = 1, 2.
\]

3. They both lie below the first strip:

\[
t_i - f_1(z_i, v_i) < 0 \quad \text{for} \ i = 1, 2.
\]

4. They both lie above the last strip:

\[
t_i - f_{M(n)}(z_i, v_i) - 2\lambda_{M(n)} \frac{\delta + \cot \alpha + 1}{n} > 0 \quad \text{for} \ i = 1, 2.
\]

We define the constants \( \eta_j \):

\[
\eta_j = \sum_{i=1}^{j-1} 2\lambda_i \frac{\delta + \cot \alpha + 1}{n}.
\]

In case (1) we have

\[
\tau_n(z_i, v_i, t_i) = f_j(y_i) - \eta_{j-1}
\]

which implies

\[
|\tau_n(z_1, v_1, t_1) - \tau_n(z_2, v_2, t_2)| \leq d_{\delta, \alpha}( (z_1, v_1), (z_2, v_2) ).
\]
This also implies that the function $\tau_n$ is $\mu$-a.e. locally 1-Lipschitz on $S$ with respect to the distance $d_{\delta,\alpha}$. In case (2) we have

$$\tau_n(z_1, v_1, t_1) = t_1 - \eta_j$$

which implies

$$|\tau_n(z_1, v_1, t_1) - \tau_n(z_2, v_2, t_2)| \leq |t_1 - t_2|.$$  

Cases (3) and (4) are treated like case (2).

Suppose now that the points $(z_1, v_1, t_1), (z_2, v_2, t_2) \in \text{Cyl}(Y, M)$ are separated by a hypersurface. We can choose the hypersurface minimal with respect to the order $\prec$ and there are two possibilities:

1. They are separated by a hypersurface $\partial_{-}\text{St} \left(f_j, 2\gamma_j \frac{\delta + \cot \alpha + 1}{\alpha} \right)$.
2. They are separated by a hypersurface $\partial_{+}\text{St} \left(f_j, 2\gamma_j \frac{\delta + \cot \alpha + 1}{\alpha} \right)$.

We consider just case (1) and assume that $t_1 \leq f_j(z_1, v_1)$ and $t_2 \geq f_j(z_2, v_2)$. The assumption of minimality on the separating strip implies that

$$\tau_n(z_1, v_1, t_1) = t_1 - \eta_{j-1};$$  

$$\tau_n(z_2, v_2, t_2) \in [f_j(z_2, v_2) - \eta_{j-1}, t_2 - \eta_{j-1}];$$

in particular,

$$\tau_n(z_2, v_2, t_2) \leq t_2 - \eta_{j-1} = t_2 - t_1 + t_1 - \eta_{j-1} \leq |t_2 - t_1| + \tau_n(z_1, v_1, t_1);$$

moreover,

$$\tau_n(z_1, v_1, t_1) = t_1 - \eta_{j-1} \leq f_j(z_1, v_1) - \eta_{j-1}$$

$$= f_j(z_2, v_2) - \eta_{j-1} + f_j(z_1, v_1) - f_j(z_2, v_2)$$

$$\leq \tau_n(z_2, v_2, t_2) + d_{\delta,\alpha}(z_1, z_2).$$

This shows that the function $\tau_n$ is 1-Lipschitz with respect to the distance $D$.

We finally observe that

$$\|\tau - \tau_n\|_{\infty} \leq \eta_M(n) \leq 3(1 + \delta + \cot \alpha) \frac{M(n)}{n} = O(1/n);$$

choosing a sequence $n_k \to \infty$ and letting $g_k = \tau_{n_k}$ we conclude that $g_k \overset{w}{\to} \tau$. \qed

5.2. Dimensional bounds and tangent cones. The goal of this Subsection is to prove Theorem 5.93 and Corollary 5.136. We first recall the definition of blow-ups of Lipschitz functions on metric spaces, following the approach of isometrically embedding all the pointed metric spaces in a common proper metric space [Kei03, Sec. 2.2].

**Definition 5.86.** A blow-up of a metric space $X$ at a point $p$ is a complete pointed metric space $(Y, q)$ such that there is a sequence $(t_n > 0)_n$ with $t_n \to 0$ and $\left( \frac{1}{t_n} X, p \right) \to (Y, q)$ in the Gromov-Hausdorff sense. The class of blow-ups at $p$ is denoted by $\text{Tan}(X, p)$.

To show the existence of blow-ups the following notion of finite dimensionality for metric spaces is useful.
**Definition 5.87.** A metric space $X$ is **doubling** if there is a constant $C$ such that every set of diameter $\leq N$ can be covered by at most $C$ sets of diameter $\leq N/2$. By induction it follows that $X$ admits a covering function $C(\varepsilon) = C\varepsilon^{-D}$ where any set of diameter $\leq N$ can be covered by at most $C(\varepsilon)$ sets of diameter $\leq \varepsilon N$. The minimal exponent $D$ is called the **Assouad dimension** of $X$.

**Remark 5.88.** If $X$ is doubling, given a sequence $(t_n > 0)_n$ with $t_n \to 0$, it is possible to find a subsequence $(t_{nk})$ such that the sequence $(\frac{1}{t_{nk}} X, p)$ converges in the Gromov-Hausdorff sense. In this case the spaces $(\frac{1}{t_{nk}} X, p)$ and $(Y, q)$ can be isometrically embedded into a proper metric space $(Z, z)$ so that, for each $R > 0$,

$$\lim_{k \to \infty} \sup_{y \in B(z, R) \cap Y} \text{dist}(\frac{1}{t_{nk}} X, \{y\}) = 0 \quad (5.89)$$

$$\lim_{k \to \infty} \sup_{x \in B(z, R) \cap \frac{1}{t_{nk}} X} \text{dist}(Y, \{x\}) = 0 \quad (5.90)$$

In particular, any point $q' \in Y$ can be approximated by a sequence $p'_{nk} \in \frac{1}{t_{nk}} X$ such that $p'_{nk} \to q'$ in $Z$.

We now define blow-ups of Lipschitz functions.

**Definition 5.91.** A **blow-up** of a Lipschitz function $f : X \to \mathbb{R}^Q$ at a point $p$ is a triple $(Y, q, g)$ where:

1. The space $(Y, q) \in \text{Tan}(X, p)$ and is a limit realized by a sequence $(t_n)_n$ of scaling factors.
2. The function $g : Y \to \mathbb{R}^Q$ is Lipschitz with $g(q) = 0$.
3. If $p'_n$ approximates $q'$,

$$\lim_{n \to \infty} \frac{f(p'_n) - f(p)}{t_n} = g(q') \quad (5.92)$$

The class of all blow-ups of $f$ at $p$ will be denoted by $\text{Tan}(X, p, f)$. By an Ascoli-Arzelà argument, if $X$ is doubling, $\text{Tan}(X, p, f) \neq \emptyset$.

We now prove Theorem 5.93: the assumption on completeness of $\mu$ is required to ensure that Suslin sets are $\mu$-measurable. Note that any Radon measure can be extended so that sets in the $\sigma$-algebra generated by Suslin sets are measurable.

**Theorem 5.93.** Let $\mu$ be a complete Radon measure on a metric space $X$ which has finite Assouad dimension $D$. Consider a Lipschitz function $f : X \to \mathbb{R}^N$, points $\{v_i\}_{i=1}^N \subset S^{N-1}$, and constants $\{\alpha_i\}_{i=1}^N \subset (0, \pi/2)$ and $\delta > 0$. Suppose that $\mu$ admits Lipschitz Alberti representations $\{A_i\}_{i=1}^N$ such that:

1. The Alberti representation $A_i$ is in the $f$-direction of $C(v_i, \alpha_i)$ with $\langle v_i, f \rangle$-speed $\geq \delta$.
2. For some $\theta > 0$ the cone fields $C(v_i, \alpha_i + \theta)$ are independent.

Then there is a $\mu$-full measure Borel subset $U \subset X$ such that for each $p \in U$ and for each blow up $(Y, q, g) \in \text{Tan}(X, p, f)$, the function $g : Y \to \mathbb{R}^N$ is surjective.

---

25 Mapping basepoints to basepoints
Proof. We assume that the Alberti representations \( \{A_i\}_{i=1}^N \) are \( C \)-Lipschitz. Given a \( C \)-Lipschitz \( A \) in the \( f \)-direction of a cone field \( C(v,\alpha) \) with \( \langle v, f \rangle \)-speed \( \geq \delta \), we define the set \( \text{Frag}(p, R, \varepsilon, A) \) of fragments \( \gamma \) that satisfy the following conditions at \( p \):

1. The fragment \( \gamma \) is a \( C \)-Lipschitz path fragment in the \( f \)-direction of \( C(v,\alpha) \) with \( f \)-speed \( \geq \delta \), and such that \( \mathcal{L}^1(\text{dom} \gamma) > 0 \) and \( 0 \) is a density point of \( \text{dom} \gamma \).

2. The metric differential \( \text{md} \gamma(0) \) and the derivative \( (f \circ \gamma)'(0) \in C(v,\alpha) \) exist, \( \gamma(0) = p \) and

\[
|(f \circ \gamma)'(0)| \geq \delta \text{md} \gamma(0).
\]

3. For each \( r \in (0, R) \) one has \( \mathcal{L}^1(\text{dom} \gamma \cap B(0, r)) \geq 2r(1-\varepsilon) \).

4. For each \( t \in \text{dom} \gamma \cap B(0, r) \) one has

\[
|f(\gamma(t)) - f(p) - (f \circ \gamma)'(0)t| \leq \varepsilon|t|.
\]

5. For all \( t, s \in \text{dom} \gamma \cap B(0, r) \) one has\(^{26}\)

\[
|d(\gamma(t), \gamma(s)) - \text{md} \gamma(0)|t-s| \leq \varepsilon(|t| + |s|).
\]

Let \( F(R_1, \varepsilon_1, A_1) = \{p \in U : \text{Frag}(p, R_1, \varepsilon_1, A_1) \neq \emptyset\} \); this set is Suslin and, as \( \mu \) admits the Alberti representation \( A_1 \), as \( R_1 \searrow 0 \) the sets \( \{F(R_1, \varepsilon_1, A_1)\} \) increase to a full measure subset of \( U \). By the Jankoff measurable selection principle \([\text{Bog07, Thm. 6.9.1}]\) there is a selection function \( \Gamma(R_1, \varepsilon_1, A_1) \) associating to \( p \in F(R_1, \varepsilon_1, A_1) \) a fragment satisfying the conditions (1)–(5) above. This choice is measurable in the sense that

\[
\varphi(R_1, \varepsilon_1, A_1)(p) = (f \circ \Gamma(R_1, \varepsilon_1, A_1))'(0)
\]

\[
\psi(R_1, \varepsilon_1, A_1)(p) = \text{md} \Gamma(R_1, \varepsilon_1, A_1)(0)
\]

are measurable functions\(^{27}\). By Lusin's Theorem \([\text{Bog07, Thm. 7.1.13}]\) there are compact sets

\[
F_c(R_1, \varepsilon_1, A_1; \tau_1) \subset F(R_1, \varepsilon_1, A_1)
\]

such that

1. for all \( p, q \in F_c(R_1, \varepsilon_1, A_1; \tau_1) \) with \( d(p, q) \leq C\tau_1 \) one has

\[
|\varphi(R_1, \varepsilon_1, A_1)(p) - \varphi(R_1, \varepsilon_1, A_1)(q)| < \varepsilon_1,
\]

\[
|\psi(R_1, \varepsilon_1, A_1)(p) - \psi(R_1, \varepsilon_1, A_1)(q)| < \varepsilon_1;
\]

2. as \( \tau_1 \to 0 \)

\[
\mu(F(R_1, \varepsilon_1, A_1) \setminus F_c(R_1, \varepsilon_1, A_1; \tau_1)) \to 0.
\]

The construction proceeds inductively in the following way: assuming for \( k < N \) that

\[
F(R_1, \varepsilon_1, A_1; \tau_1; \cdots; R_k, \varepsilon_k, A_k),
\]

\[
\Gamma(R_1, \varepsilon_1, A_1; \tau_1; \cdots; R_k, \varepsilon_k, A_k),
\]

and

\[
F_c(R_1, \varepsilon_1, A_1; \tau_1; \cdots; R_k, \varepsilon_k, A_k; \tau_k)
\]

\(^{26}\)This is the approximate continuity of the metric differential ([AK00, Thm. 3.3])

\(^{27}\)With respect to the \( \sigma \)-algebra generated by Suslin sets
have been constructed, let

\[(5.106)\quad F(R_1, \varepsilon_1, A_1; \tau_1; \cdots; R_k, \varepsilon_k, A_k; \tau_k; R_{k+1}, \varepsilon_{k+1}, A_{k+1})\]

be the set of those

\[(5.107)\quad p \in F_c(R_1, \varepsilon_1, A_1; \tau_1; \cdots; R_k, \varepsilon_k, A_k; \tau_k) \cap F(R_{k+1}, \varepsilon_{k+1}, A_{k+1})\]

such that \(\exists \gamma \in \text{Frag}(p, R_{k+1}, \varepsilon_{k+1}, A_{k+1})\):

\[(5.108)\quad \forall r \leq R_{k+1} \quad \mathcal{L}^1(\gamma^{-1}(F_c(R_1, \varepsilon_1, A_1; \tau_1; \cdots; R_k, \varepsilon_k, A_k; \tau_k)) \cap B(0, r)) \geq 2r(1 - \varepsilon_{k+1}).\]

These sets are Suslin measurable and as \(R_{k+1} \to 0\), increase to a full measure subset of \(F_c(R_1, \varepsilon_1, A_1; \tau_1; \cdots; R_k, \varepsilon_k, A_k; \tau_k)\). By Jankoff measurable selection principle there is a selection function \(\Gamma(R_{k+1}, \varepsilon_{k+1}, A_{k+1})\) associating to

\[(5.109)\quad p \in F(R_1, \varepsilon_1, A_1; \tau_1; \cdots; R_k, \varepsilon_k, A_k; \tau_k; R_{k+1}, \varepsilon_{k+1}, A_{k+1})\]

a fragment satisfying the conditions above. This choice is measurable in the sense that

\[(5.110)\quad \varphi(R_{k+1}, \varepsilon_{k+1}, A_{k+1})(p) = (f \circ \Gamma(R_{k+1}, \varepsilon_{k+1}, A_{k+1}))(0)\]

\[(5.111)\quad \psi(R_{k+1}, \varepsilon_{k+1}, A_{k+1})(p) = \text{md} \Gamma(R_{k+1}, \varepsilon_{k+1}, A_{k+1})(0)\]

are measurable functions. By Lusin’s theorem there are compact sets

\[(5.112)\quad F_c(R_1, \varepsilon_1, A_1; \tau_1; \cdots; R_k, \varepsilon_k, A_k; \tau_k; R_{k+1}, \varepsilon_{k+1}, A_{k+1}; \tau_{k+1}) \subset F(R_1, \varepsilon_1, A_1; \tau_1; \cdots; R_k, \varepsilon_k, A_k; \tau_k; R_{k+1}, \varepsilon_{k+1}, A_{k+1}).\]

\[\forall d(p, q) \leq C \tau_{k+1},\]

\[(5.113)\quad |\varphi(R_{k+1}, \varepsilon_{k+1}, A_{k+1})(p) - \varphi(R_{k+1}, \varepsilon_{k+1}, A_{k+1})(q)| < \varepsilon_{k+1};\]

\[(5.114)\quad |\psi(R_{k+1}, \varepsilon_{k+1}, A_{k+1})(p) - \psi(R_{k+1}, \varepsilon_{k+1}, A_{k+1})(q)| < \varepsilon_{k+1};\]

as \(\tau_{k+1} \to 0\) we can assume that

\[(5.115)\quad F_c(R_1, \varepsilon_1, A_1; \tau_1; \cdots; R_k, \varepsilon_k, A_k; \tau_k; R_{k+1}, \varepsilon_{k+1}, A_{k+1}; \tau_{k+1})\]

increases to a full measure subset of

\[(5.116)\quad F(R_1, \varepsilon_1, A_1; \tau_1; \cdots; R_k, \varepsilon_k, A_k; \tau_k; R_{k+1}, \varepsilon_{k+1}, A_{k+1}).\]

Having fixed \(c > 0\), it is possible to choose \((R_h^{(i)}), (\tau_h^{(i)})\) such that:

- The sequences \((R_h^{(i)}), (\tau_h^{(i)})\) are decreasing in both \(h \in \mathbb{N}\) and \(i \in \{1, \cdots, N\}\).
- One has that \(\lim_{h \to \infty} R_h^{(i)} = \lim_{h \to \infty} \tau_h^{(i)} = 0\).
- Letting

\[(5.117)\quad F_h = F_c(R_h^{(1)}, \frac{1}{h}, A_1; \tau_h^{(1)}; \cdots; R_h^{(N)}, \frac{N}{h}, A_N; \tau_h^{(N)}),\]

the Borel set \(V = \bigcap_h F_h\) satisfies \(\mu(X \setminus V) < c\).

We will simplify the notation for selectors and derivatives writing \(\Gamma_h^{(i)}, \varphi_h^{(i)}, \psi_h^{(i)}\).

Let \(p \in V\) and

\[(5.118)\quad \left(\frac{1}{t_h} X, p, \frac{f - f(p)}{t_h}\right) \to (Y, q, g);\]
consider a subsequence \( t_h \) such that \( \lim_{h \to \infty} \frac{\tau(N)}{t_h} = \infty \). By passing to further subsequences we can assume that:

- The \( \varphi^{(i)}(p) \) converge to \( w_i \in C(v_i, \alpha_i+\theta) \setminus B(0, \delta) \), implying that the \( \{w_i\}_{i=1}^N \) are independent.
- The \( \psi^{(i)}(p) \) converge to \( \delta_i \in [\delta, C] \).

In particular, the functions

\[
\Gamma_h^{(N)}(p) : \text{dom} \Gamma_h^{(N)}(p) \to X
\]

are \( C \)-Lipschitz and we define

\[
\tilde{\Gamma}_h^{(N)}(p) : \frac{1}{t_h} \text{dom} \Gamma_h^{(N)}(p) \to \frac{1}{t_h} X
\]

by

\[
\tilde{\Gamma}_h^{(N)}(p)(s) = \Gamma_h^{(N)}(p)(t_h \cdot s).
\]

From property (3) we obtain

\[
\mathcal{L}^1(\text{dom} \tilde{\Gamma}_h^{(N)} \cap B(0, \frac{\tau(N)}{t_h})) \geq 2 \frac{\tau(N)}{t_h} (1 - \frac{1}{h}).
\]

The functions \( \tilde{\Gamma}_h^{(N)}(p) \) are still \( C \)-Lipschitz with respect to the rescaled metrics and \( \tilde{\Gamma}_h^{(N)}(p)(0) = p \) and by a variant of Ascoli-Arzelà we get a \( C \)-Lipschitz

\[
\tilde{\Gamma}_\infty^{(N)} : \mathbb{R} \to Y
\]

with \( \tilde{\Gamma}_\infty^{(N)}(0) = p \). For \( s \in \mathbb{R} \) we choose \( s_h \in \text{dom} \tilde{\Gamma}_h^{(N)}(p) \) converging to \( s \) and observe that by (4)

\[
\left| \frac{f \circ \tilde{\Gamma}_h^{(N)}(p)(s_h) - f(p) - \varphi^{(N)}(p) \cdot s_h}{s_h} \right| \leq \frac{|s_h|}{h}
\]

which implies

\[
g \circ \tilde{\Gamma}_\infty^{(N)}(s) = w_N \cdot s.
\]

A similar argument involving the metric derivative and \( \psi^{(N)}(p) \) shows that

\[
d(\tilde{\Gamma}_\infty^{(N)}(s), \tilde{\Gamma}_\infty^{(N)}(s')) = \delta_N |s - s'|.
\]

If \( s \in \mathbb{R} \), because of (5.108), there are \( s_h \in \text{dom} \tilde{\Gamma}_h^{(N)} \) converging to \( s \) such that

\[
\Gamma_h^{(N-1)}(\Gamma_h^{(N)}(p)(s_h))
\]

is defined. Let

\[
\tilde{\Gamma}_h^{(N-1)} : \frac{1}{t_h} \text{dom} \Gamma_h^{(N-1)}(\Gamma_h^{(N)}(p)(s_h)) \to \frac{1}{t_h} X
\]

be defined by

\[
\tilde{\Gamma}_h^{(N-1)}(\sigma) = \Gamma_h^{(N-1)}(\Gamma_h^{(N)}(p)(s_h)) (t_h \cdot \sigma).
\]

These functions are \( C \)-Lipschitz with respect to the rescaled metrics and a variant of Ascoli-Arzelà yields a \( C \)-Lipschitz

\[
\tilde{\Gamma}_\infty^{(N-1)} : \mathbb{R} \to Y
\]
with $\hat{\Gamma}^{(N-1)}(0) = \hat{\Gamma}^{(N)}(s)$. There is an analogue of (5.124) where $\varphi_h^{(N)}(p)$ is replaced by $\varphi_h^{(N-1)}\left(\hat{\Gamma}^{(N)}_h(p)(s_h)\right)$ but for $h$ sufficiently large

$$\left|\varphi_h^{(N-1)}\left(\hat{\Gamma}^{(N)}_h(p)(s_h)\right) - \varphi_h^{(N-1)}(p)\right| < \frac{1}{h}$$

so

$$g \circ \hat{\Gamma}^{(N-1)}(\sigma) - g \circ \hat{\Gamma}^{(N-1)}(0) = w_{N-1} \cdot \sigma.$$  

A similar argument involving the metric derivative shows that

$$d(\hat{\Gamma}^{(N-1)}(s), \hat{\Gamma}^{(N-1)}(s')) = \delta_{N-1} |s - s'|.$$  

Continuing inductively we conclude that $\exists \hat{\Gamma} : \mathbb{R} \to Y$ such that $\hat{\Gamma}(0) = q$ and $\forall(0 \leq k \leq N-1) \forall s \in \mathbb{R}^k \hat{\gamma}(t) = \hat{\Gamma}((0, t, s))$ is a $\delta_{N-k}$-constant speed geodesic with

$$g \circ \gamma(t) - g \circ \gamma(0) = w_{N-k} \cdot t.$$  

Being the $w_i$ independent, $g$ is surjective.

We need a Lemma relating the Assouad dimension of a space to that of a blow-up [MT10, Prop. 6.1.5]:

**Lemma 5.134.** If $X$ has Assouad dimension $\leq D$ and if $(Y, q) \in \text{Tan}(X, p)$, then $Y$ has Assouad dimension $\leq D$.

The following Lemma provides a lower bound on the Assouad dimension.

**Lemma 5.135.** If $Y$ has Assouad dimension $\leq D$ (or Hausdorff dimension $\leq D$) and if there is a surjective Lipschitz map $g : Y \to \mathbb{R}^N$, then $D \geq N$.

**Proof.** The argument in [Hei01, Subsec. 8.7] shows that the Assouad dimension of $Y$ is at least its Hausdorff dimension. Now Lipschitz maps do not increase the Hausdorff dimension so the Hausdorff dimension of $g(Y)$ is at least the Hausdorff dimension of $\mathbb{R}^N$. $\square$

We can now prove Corollary 5.136.

**Corollary 5.136.** If $X$ is either a metric space with Assouad dimension $D$ and if $\mu$ is a Radon measure, then $X(\mu)$ has index locally bounded by $D$.

**Proof.** We can reduce to the hypothesis of Theorem 5.93 because of Corollary 3.92; we then apply Lemmas 5.134, 5.135. $\square$

**Remark 5.137.** Note that using ultralimits one can replace in Corollary 5.136 the assumption on the Assouad dimension with a uniform upper bound $D$ on the Hausdorff dimension of the blow-ups of $X$.

5.3. Construction of independent Lipschitz functions. The goal of this Subsection is to prove Theorem 4.13. We will use the following truncation Lemma ([Bat12, Lem. 4.1]).

**Lemma 5.138.** Let $\varepsilon \in (0, h/4)$ and assume that $S \subset X$ is Borel and $f : X \to \mathbb{R}$ is $L$-Lipschitz. Then there is an $L$-Lipschitz function $g$, constructed from $f$, such that:

1. The function $g$ satisfies $0 \leq g \leq h$.
2. The function $g$ is supported in $B(S, \frac{2h}{L})$. 
Proof of Theorem 4.13. Choose \( m_1 \) such that \( \frac{1}{m_1} < \frac{\alpha^2}{2L} \) and let \( h_1 = \frac{\alpha^2}{4} \) and \( \varepsilon_1 = \frac{L}{m_1} \). We use Lemma 5.138 to find \( g_1 \) and \( S_1 \subset S \) such that the following holds:

1. We have the inequalities \( \mu(S_1) \geq \left(1 - \frac{4\varepsilon}{h}\right) \mu(S) \).
2. The function \( g_1 \) is supported in \( B(S, \frac{2\rho}{L}) = B(S, \frac{\alpha^2}{2L}) \) and for each \( x, y \in B(S, \frac{\alpha^2}{L}) \)
   \[ |f_m(x) - f_m(y)| \geq |g_1(x) - g_1(y)|. \]
3. If \( B \) is a ball of radius \( \rho_{m_1} \), centred at some point of \( S \), the Lipschitz constant of \( g_1 | B \) is at most \( \frac{1}{m_1} \).
4. As \( \frac{\varepsilon_1}{L} = \frac{1}{m_1} \), for each \( x \in S_1 \) there is \( x_1 \in B(x, \frac{1}{m_1}) \) with
   \[ |g_1(x) - g_1(x_1)| \geq \delta_0 d(x, x_1) > 0. \]

We define \( g_{k+1} \) inductively in the following way. Having chosen \( m_{k+1} \) such that

\[ \frac{1}{m_{k+1}} \leq \left(\frac{\alpha^2}{L}\right)^{k} 2^{-(k+1)+4k} \rho_{m_k}, \]

we let \( h_{k+1} = \frac{\alpha^2}{4} \rho_{m_k} \) and \( \varepsilon_{k+1} = \frac{L}{m_{k+1}} \). We use Lemma 5.138 to find \( g_{k+1} \) and \( S_{k+1} \subset S \) such that the following holds:

1. We have the inequalities \( \mu(S_{k+1}) \geq \left(1 - \frac{4\varepsilon_{k+1}}{L}\right) \mu(S) \geq \left(1 - \frac{\rho_{m_{k+1}}}{2\varepsilon_{k+1}}\right) \mu(S). \)
2. The function \( g_{k+1} \) is supported in \( B(S, \frac{2h_{k+1}}{L}) = B(S, \frac{\alpha^2}{2L}) \) and for each \( x, y \in B(S, \frac{\alpha^2}{L}) \)
   \[ |f_{m_{k+1}}(x) - f_{m_{k+1}}(y)| \geq |g_{k+1}(x) - g_{k+1}(y)|. \]
3. If \( B \) is a ball of radius \( \rho_{m_{k+1}} \), centred at some point of \( S \), the Lipschitz constant of \( g_{k+1} | B \) is at most \( \frac{1}{m_{k+1}} \).
4. As \( \frac{\varepsilon_{k+1}}{L} = \frac{1}{m_{k+1}} \), for each \( x \in S_{k+1} \) there is \( x_{k+1} \in B(x, \frac{1}{m_{k+1}}) \) with
   \[ |g_{k+1}(x) - g_{k+1}(x_{k+1})| \geq \delta_0 d(x, x_{k+1}) > 0. \]

Note that from the definition of \( m_k \) one can verify by induction that

\[ \frac{1}{m_k} \leq \left(\frac{\alpha^2}{L}\right)^k 2^{-(k+1)+4k}, \]

note also that

\[ h_{s+1} = \frac{\alpha^2}{4} \rho_{m_s} \leq \frac{\alpha^2}{4} \frac{1}{m_s}, \]
\[ h_{s+k} = \frac{\alpha^2}{4} \left(\frac{\alpha^2}{L}\right)^{k-1} 2^{-(s+5)} \rho_{m_s} \text{ (here } k > 1); \]
so that
\[ \sum_{k \geq s+1} h_k \leq \frac{\alpha^2}{4} \left( 1 + 2^{-(s+5)} \frac{\alpha^2/L}{1 - \alpha^2/L} \right) \rho_{m_s}. \]

This implies that if \( \sum h_k \to \infty \), then \( \varphi = \sum h_k < \infty \) defines a continuous function.

We show that \( \varphi \) is Lipschitz with Lipschitz constant
\[ 3 \left( L + \alpha + \frac{\alpha^2/L}{1 - \alpha^2/L} \left( 1 + 2^{-6\alpha} \right) \right). \]

We want to bound \( |\varphi(x) - \varphi(y)| \). In the first case, we assume \( x \in S \). If \( y \neq x \) assume there is some \( s \) such that \( d(x,y) \in (\frac{\alpha}{2}\rho_{m_s}, \rho_{m_s}] \). For \( t \leq s \) \( y \in B(x, \rho_{m_s}) \) so
\[ |g_t(x) - g_t(x_t)| \leq \frac{1}{m_t} d(x,y); \]
in particular, from the estimate for \( \frac{1}{m_k} \) we deduce that
\[ \sum_k \frac{1}{m_k} \leq \frac{\alpha^2/L}{1 - \alpha^2/L}; \]
so that
\[ \sum_{t \leq s} |g_t(x) - g_t(x_t)| \leq \frac{\alpha^2/L}{1 - \alpha^2/L} d(x,y). \]

For the second estimate we use the bound on \( h_t \):
\[ \sum_{t \geq s+1} |g_t(x) - g_t(x_t)| \leq 2 \sum_{t \geq s+1} h_t \]
\[ \leq \frac{\alpha^2}{2} \left( 1 + 2^{-(s+5)} \frac{\alpha^2/L}{1 - \alpha^2/L} \right) \rho_{m_s} \]
\[ \leq \alpha \left( 1 + 2^{-(s+5)} \frac{\alpha^2/L}{1 - \alpha^2/L} \right) d(x,y). \]

We therefore get the bound
\[ |\varphi(x) - \varphi(y)| \leq \left( \alpha + \frac{\alpha^2/L}{1 - \alpha^2/L} (1 + 2^{-6\alpha}) \right) d(x,y). \]

The other possibility is that \( d(x,y) \in (\rho_{m_{s+1}}, \frac{\alpha}{2}\rho_{m_s}] \) for some \( s \) or \( d(x,y) > \rho_{m_s} \).

In this case we can estimate as above except that for \( g_{s+1} \) we must use the full Lipschitz constant \( L \). We therefore have:
\[ |\varphi(x) - \varphi(y)| \leq \left( L + \alpha + \frac{\alpha^2/L}{1 - \alpha^2/L} (1 + 2^{-6\alpha}) \right) d(x,y). \]

In the second case \( x, y \notin S \). We use that the functions \( g_s \) are supported on a neighbourhood of \( S \). Without loss of generality we can assume \( d(x, S) \leq d(y, S) \).

Now, for some \( s \) we have \( d(x,S) \in [\frac{\alpha^2}{4\pi \rho_{m_{s+1}}}, \frac{\alpha^2}{4\pi \rho_{m_s}}] \) (for \( s = 0 \) we let \( \rho_{m_0} = 1 \)) or \( d(x,S) \geq \frac{\alpha^2}{4\pi} \). So for \( t \geq s+2 \) we have that \( g_t(x) = g_t(y) = 0 \) and in the last case \( \varphi(x) = \varphi(y) = 0 \) so there is nothing to prove. We assume that \( d(x,y) < \frac{\alpha^2}{4\pi \rho_{m_s}} \).

Then the hypothesis on \( \alpha \) implies that \( \alpha^2/L < 1 \) so that \( x, y \) belong to a ball centred
on \( S \) of radius at most \( \rho_{m_\epsilon} \). For \( t \leq s \) the Lipschitz constant of \( g_t \) is then at most \( \frac{1}{m_\epsilon} \) implying

\[
(5.158) \quad \sum_{t \leq s} |g_t(x) - g_t(y)| \leq \frac{\alpha^2/L}{1 - \alpha^2/L} d(x,y).
\]

We therefore have:

\[
(5.159) \quad |\varphi(x) - \varphi(y)| \leq \left(L + \frac{\alpha^2/L}{1 - \alpha^2/L}\right) d(x,y).
\]

We assume \( d(x,y) \geq \frac{\alpha^2}{2\alpha} \rho_{m_\epsilon} \). We can find \( \tilde{x} \in S \) such that \( d(x,\tilde{x}) \leq \frac{\alpha^2}{2\alpha} \rho_{m_\epsilon} \), so that

\[
(5.160) \quad |\varphi(x) - \varphi(\tilde{x})| \leq \left(L + \alpha + \frac{\alpha^2/L}{1 - \alpha^2/L} (1 + 2^{-\epsilon})\right) d(x,\tilde{x})
\]

\[
(5.161) \quad |\varphi(y) - \varphi(\tilde{x})| \leq \left(L + \alpha + \frac{\alpha^2/L}{1 - \alpha^2/L} (1 + 2^{-\epsilon})\right) d(y,\tilde{x})
\]

Note now that \( d(\tilde{x},y) \leq 2d(x,y) \) and \( d(\tilde{x},x) \leq d(x,y) \). Therefore:

\[
(5.162) \quad |\varphi(x) - \varphi(y)| \leq 3 \left(L + \alpha + \frac{\alpha^2/L}{1 - \alpha^2/L} (1 + 2^{-\epsilon})\right) d(x,y).
\]

We now pass to the construction of \( M \) independent functions. We let

\[
(5.163) \quad \psi_j = \sum_{k \equiv j \mod M} g_k
\]

and

\[
(5.164) \quad \tilde{S}_j = \bigcap_{k \geq k} \bigcup_{s \equiv j \mod M} S_s
\]

which is a full measure Borel subset of \( S \). In particular \( S' = \bigcap_{j} \tilde{S}_j \) is a full measure Borel subset of \( S \). Let \( \tilde{x} \in S' \) and assume that \(|\lambda_0| \geq \max_{0 \leq i \leq M-1}\) |\lambda_i|\). For each \( k \) there is an \( n_k \geq k \) with \( x \in S_{n_k} \) and \( n_k \equiv 0 \mod M \). We can then find a point \( x_{n_k} \) such that

\[
(5.165) \quad |g_{n_k}(x) - g_{n_k}(x_{n_k})| \geq \delta_0 d(x,x_{n_k}) > 0.
\]

Then

\[
(5.166) \quad \left| \sum_{j=0}^{M-1} \lambda_j \psi_j(x) - \lambda_j \psi_j(x_{n_k}) \right| \geq |\lambda_0| \left( \delta_0 d(x,x_{n_k}) - \sum_{t \neq n_k} |g_t(x) - g_t(x_{n_k})| \right).
\]

The terms for \( t > n_k \) can be bound using (5.155) to get:

\[
(5.167) \quad \sum_{t > n_k} |g_t(x) - g_t(x_{n_k})| \leq \alpha \left(1 + 2^{-\epsilon(n_k+5)} \frac{\alpha^2/L}{1 - \alpha^2/L}\right) d(x,x_{n_k}).
\]

Note that for \( t < n_k \), \( d(x,x_k) < \rho_{m_\epsilon} \) so we have the following bound on the terms for \( t < n_k \):

\[
(5.168) \quad \sum_{t < n_k} |g_t(x) - g_t(x_{n_k})| \leq \frac{\alpha^2/L}{1 - \alpha^2/L} d(x,x_{n_k}).
\]
Letting $k \nearrow \infty$ we conclude that

$$\mathcal{L} \left( \sum_{i=0}^{M-1} \lambda_i \psi_i \right) (x) \geq \left( \delta_0 - \frac{\alpha^2/L}{1 - \alpha^2/L} - \alpha \right) |\lambda_0|.$$

\[ \square \]

**Remark 5.170.** Consider what happens to $\varphi_\alpha$ as $\alpha \searrow 0$: we get a full measure Borel subset $S' \subset S$ with

$$\inf_{x \in S'} \mathcal{L} \varphi_\alpha (x) \geq \left( \delta_0 - \frac{\alpha^2/L}{1 - \alpha^2/L} - \alpha \right)$$

and if $x \in S'$ and $r \in \left( \frac{m}{2}, \rho_{m_s}, \rho_{m_s} \right]$,

$$\text{Var} \varphi_\alpha (x, r) \leq \left( \alpha + \frac{\alpha^2/L}{1 - \alpha^2/L} (1 + 2^{-6} \alpha) \right);$$

in particular, on a full measure subset of $S$ the Lip-lip inequality (1.13) is violated.

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