AN APPROACH TOWARD SUPERSYMMETRIC CLUSTER ALGEBRAS

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Abstract. In this paper we propose the notion of cluster superalgebras which is a supersymmetric version of the classical cluster algebras introduced by Fomin and Zelevinsky. We show that the symplectic-orthogonal supergroup \( SpO(2|1) \) admits a cluster superalgebra structure and as a consequence of this, we deduce that the supercommutative superalgebra generated by all the entries of a superfrieze is a cluster superalgebra. We also show that the coordinate superalgebra of the super Grassmannian \( G(2|0; 4|1) \) of chiral conformal superspace (that is, \( (2|0) \) planes inside the superspace \( \mathbb{C}^{4|1} \)) is a quotient of a cluster superalgebra.

1. Motivation behind the notion of cluster superalgebras

The study of cluster algebras was initiated by Fomin and Zelevinsky in 2001 \[5\]. Cluster algebras are commutative rings with a set of distinguished generators called cluster variables. These algebras are different from usual algebras in the sense that they are not presented at the outset by a complete set of generators and relations. Instead, an initial seed consisting of initial cluster variables and an exchange matrix is given and then using an iterative process called mutation, the rest of the cluster variables are generated. Examples of cluster algebras include the homogeneous coordinate rings of any Grassmannian, and the algebra of regular functions on 2-frieze pattern.

The initial motivation behind study of these algebras was to provide an algebraic and combinatorial framework for Lusztig’s work on canonical bases but now the study of cluster algebras has gone far beyond that initial motivation. Cluster algebras now have connections to string theory, Poisson geometry, algebraic geometry, combinatorics, representation theory and Teichmuller theory. Cluster algebras provide a unifying algebraic and combinatorial framework for a wide variety of phenomenon in above mentioned settings. Interestingly, the mutation rule proposed by Fomin and Zelevinsky came up naturally in the Witten-Seiberg duality in string theory. Recently, it has been suggested that the geometric ideas coming from quantum field theory lead to a natural extension of the theory of cluster algebras.

The biggest stumbling block to the applicability of cluster algebras to quiver gauge theory is that it is almost impossible to characterize when every dual theory obtained by successive applications of Seiberg-like dualities has enough quadratic terms in the superpotential that, after integrating out the chiral multiplets involved, no oriented 2-cycles are left. So, one of our motivations behind the introduction of the notion of cluster superalgebras is to deal with the situation when we have oriented 2-cycles.

The study of symmetry has always been the central idea in mathematics and physics. In the theory of spin, we deal with rotational symmetry. Poincare symmetry is crucial in understanding of the classification of elementary particles. Similarly, permutation symmetry plays an important role in dealing with the systems of identical particles. But when one deals with both the bosons and the fermions, then to formulate the symmetry arising in this situation, the ordinary Lie group theory is insufficient. Superalgebras were introduced by physicists to provide an algebraic framework for describing boson-fermion symmetry. The boson-fermion symmetry

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is called supersymmetry and it holds the key to unified field theory. See [15] for more details on supersymmetry.

A super vector space \( V \) is a vector space that is \( \mathbb{Z}_2 \)-graded, that is, it has a decomposition \( V = V_0 \oplus V_1 \) with \( 0, 1 \in \mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z} \). The elements of \( V_0 \) are called the even (or bosonic) elements and the elements of \( V_1 \) are called the odd (or fermionic) elements. The elements in \( V_0 \cup V_1 \) are called homogeneous and their parity, denoted by \( p \), is defined to be 0 or 1 according as they are even or odd. The morphisms in the category of super vector spaces are linear maps which preserve the gradings.

A superalgebra \( A \) is an associative algebra with an identity element (which is necessarily an even element) such that the multiplication map \( A \otimes A \rightarrow A \) is a morphism in the category of super vector spaces. This is the same as requiring \( p(ab) = p(a) + p(b) \) for any two homogeneous elements \( a \) and \( b \) in \( A \). A superalgebra \( A \) is called supercommutative if \( ab = (-1)^{p(a)p(b)}ba \), for all (homogeneous) \( a, b \in A \). This means in a supercommutative superalgebra, odd elements anticommute with each other, that is, \( ab = -ba \) for any two odd elements \( a, b \in A \), whereas even elements commute with any other element (even or odd). We refer the reader to [10] and [12] for further details on superalgebras.

The study of scattering amplitudes is crucial to our understanding of quantum field theory. Scattering amplitudes are complicated functions of the helicities and momenta of the external particles. But to visually interpret them in an easier manner, one may label particles involved by \( \{1, 2, \ldots, n\} \) and the interaction of particles involved could be associated with a permutation of \( \{1, 2, \ldots, n\} \). One of the ways to represent scattering amplitudes is an on-shell diagram. See Arkani-Hamed et al [1] for more details on scattering amplitudes and on-shell diagrams.

Consider the following two four-particle scattering amplitudes. Let us label the particles involved as indicated below and assume that rays bend to the right of black vertices and to the left of white vertices.

The above on-shell diagrams denote the following two permutations of scattering of four particles, respectively \(
\begin{pmatrix}
1 & 2 & 3 & 4 \\
\downarrow & \downarrow & \downarrow & \downarrow
\end{pmatrix},
\begin{pmatrix}
1 & 2 & 3 & 4 \\
\downarrow & \downarrow & \downarrow & \downarrow
\end{pmatrix}
\) . Let \( \mathcal{N} \) denote the supersymmetry supercharge. Note that \( \mathcal{N} = 0 \) represents only bosons and \( \mathcal{N} > 0 \) represents bosons with fermions. The latter case is also known as supersymmetry. The number of fermions and type of Lagrangian determine the value of \( \mathcal{N} \). The dual of a bipartite graph in the physics of \( \mathcal{N} = 4 \) supersymmetry is called a quiver. We associate a quiver with a planar bipartite graph by taking a vertex for each face and for each edge in the bipartite graph, we draw an arrow in this quiver in such a way that it sees the white vertex in left as depicted below.
The complicated on-shell diagrams are related with complicated quiver gauge theories and it makes sense to be able to deal with situations when we have loops and oriented 2-cycles in quiver to model more natural physical systems.

Recently, Ovsienko [14] has made an attempt to define cluster superalgebras. He considers an extension of a quiver by adding odd vertices and makes it an oriented hypergraph. Note that in an oriented hypergraph, an arrow can connect any number of vertices. The main limitations of his approach are:

(1) absence of arrows between odd vertices.
(2) lack of exchange relations for the odd variables.
(3) hypergraphs restrict the visual representation.

The objective behind this paper is to propose a notion of cluster superalgebras which is a natural supersymmetric analogue of classical cluster algebras and provides some interesting geometric examples. Surprisingly, although our definition of cluster superalgebra is quite different from that of [14], we are able to prove some results similar to [14] in our setting too. But the most important contribution of this approach is that we are able to get an exciting new geometric example of cluster superalgebras. In particular, we show that the coordinate superalgebra of the super Grassmannian $G(2|0; 4|1)$ of $(2|0)$ planes inside the superspace $\mathbb{C}^{4|1}$ is a quotient of a cluster superalgebra. This super Grassmannian $G(2|0; 4|1)$ is called the chiral conformal superspace. The elements of the coordinate superalgebra of super Grassmannian $G(2|0; 4|1)$ are identified with chiral superfields. It may be noted here that chiral superfields appear naturally in supersymmetric theories and so we expect that our results will have some far reaching consequences. Among other examples, we show that the symplectic-orthogonal supergroup $SpO(2|1)$ admits a cluster superalgebra structure and as a consequence of this, we deduce that the supercommutative superalgebra generated by all the entries of a superfrieze is a cluster superalgebra.

2. Recollections from cluster algebras

In this section we recall the basics of cluster algebras. Let $\mathbb{P}$ be a multiplicative abelian group generated by $y_1, \ldots, y_n$ with an auxiliary addition that is commutative, associative and distributive with respect to multiplication. Consider the group ring $\mathbb{Z}\mathbb{P}$. This is the ring of Laurent polynomials in $y_1, \ldots, y_n$. Since $\mathbb{Z}\mathbb{P}$ is a domain, we may consider its field of fractions $\mathbb{Q}\mathbb{P}$. We set our ambient field $\mathcal{F} = \mathbb{Q}\mathbb{P}(x_1, \ldots, x_m)$, the field of rational functions in $x_1, \ldots, x_m$ with coefficients in $\mathbb{Q}\mathbb{P}$. For simplicity, we take $\mathbb{P}$ to be the trivial group and so we have $\mathcal{F} = \mathbb{Q}(x_1, \ldots, x_m)$. Consider the initial seed $(X, B)$ where $X = \{x_1, \ldots, x_m\}$ and $B = [b_{ij}]$ is an $m \times m$ skew-symmetrizable integer matrix (that is, there exists a diagonal integer matrix $D$ such that $DB$ is a skew-symmetric matrix). The set $X = \{x_1, \ldots, x_m\}$ is called the initial cluster and each $x_i$ in this set is called an initial cluster variable. There is a mutation in the direction of each initial cluster variable $x_k$ which is denoted by $\mu_k$. The mutation is defined as $\mu_k(X, B) = (X', B')$ with $X' = \{x_1', \ldots, x_m'\}$ and $B' = [b_{ij}']$, where
\[ x'_j = x_j, \quad j \neq k \]

\[ x'_k = \frac{1}{x_k} \left[ \left( \prod_{b_{ik} > 0} x_i^{b_{ik}} \right) + \left( \prod_{b_{ik} < 0} x_i^{-b_{ik}} \right) \right] \]

and

\[ b'_{ij} = -b_{ij} \text{ if } i = k \text{ or } j = k, \text{ otherwise} \]

\[ b'_{ij} = b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2}. \]

Let \( \mathcal{X} \) be the set of all cluster variables obtained by applying a finite sequence of mutations from \((X, B)\). The cluster algebra \( A(X, B) \) is defined to be a \( \mathbb{Z} \)-subalgebra of \( \mathcal{F} \) generated by \( \mathcal{X} \).

Note that in the special case when \( B \) is a skew-symmetric integer matrix, \( B \) can be encoded by a connected quiver \( Q \) which has no loops and no 2-cycles and then mutation of \( B \) can be suitably rephrased in terms of mutation of \( Q \) as follows:

We write \( \mu_k(Q) = Q' \), where \( Q' \) is obtained from \( Q \) by keeping the same vertices and changing arrows by the following rule:

(i) If there is a path \( x_i \to x_k \to x_j \), add an arrow \( x_i \to x_j \) for each distinct path.

(ii) Reverse all arrows incident at \( x_k \).

(iii) Delete any 2-cycle produced in the process.

For more details on cluster algebras, we refer the reader to [5], [6], [7] and [8].

3. Definition of cluster superalgebras

Consider an initial seed \((X|Y, Q)\) where \( X = \{x_1, \ldots, x_m\} \) is the set of variables that commute with each other and \( Y = \{y_1, \ldots, y_n\} \) is the set of Grassman (or odd) variables that anticommute with each other. These variables \( x_1, \ldots, x_m, y_1, \ldots, y_n \) are called initial supercluster variables. Let \( Q \) be a colored quiver with \( m+n \) vertices which are labeled as \( x_1, \ldots, x_m, y_1, \ldots, y_n \). We will call the vertices labeled with \( x_1, \ldots, x_m \) as even vertices and the vertices labeled with \( y_1, \ldots, y_n \) as odd vertices. There may be two types of even (odd) variables: exchangeable even (odd) variables and frozen even (odd) variables. The variables that get transformed into another variable under the mutation map, which will be defined below, are called exchangeable whereas those which remain unaffected under the mutation map are called frozen. Just as in the case of classical cluster algebras, we do not allow 2-cycles between two even variables or two odd variables. However, we allow 2-cycles between an even vertex and an odd vertex. Just as in the case of classical cluster algebras, we do not allow loops on even vertices however, we do allow loops on odd vertices. We will call such a quiver a superquiver.

Throughout this paper, if a superquiver has odd vertices we will restrict ourselves to those superquivers that satisfy at least one of the properties below for each even vertex:

**C1.** If \( x_i \) is a mutable even vertex such that there exists \( k, l \) with \( k \neq l \) such that \( y_k \to x_i \to y_l \) and \( y_k \not\to y_l \) for odd vertices \( y_k \) and \( y_l \) and \( x_j \) is an even vertex adjacent to \( x_i \), then \( x_j \) is a frozen vertex.

**C2.** If \( x_i \) is a mutable even vertex and there exists indices \( j, k \) with \( j \neq k \) such that \( y_j \to x_i \to y_k \), then either

(i) \( y_j \to y_k \); or

(ii) \( y_k \to x_i \to y_j \).
3.1. **Mutation.** We first define Fomin-Zelevinsky type mutation for our setting. We define two types of mutations: an even mutation and an odd mutation. We will denote the even mutation in the direction of an exchangeable vertex \( x_k \) as \( \mu_k \) and odd mutation in the direction of an exchangeable vertex \( y_k \) as \( \eta_k \).

3.1.1. **Even Mutation.** We define the even mutation in the direction of vertex \( x_k \) as

\[
\mu_k(x_1, \cdots, x_m, y_1, \cdots, y_n, Q) = (\mu_k(x_1), \cdots, \mu_k(x_m), \mu_k(y_1), \cdots, \mu_k(y_n), \mu_k(Q))
\]

where

\[
\begin{align*}
\mu_k(y_i) &= y_i & \text{for each } i \\
\mu_k(x_i) &= x_i & \text{for each } i \neq k
\end{align*}
\]

\[
\mu_k(x_k) = \frac{1}{x_k} \left[ (-1)^u \left( \prod_{x_i \rightarrow x_k} x_i \right) + (-1)^v \left( \prod_{x_k \rightarrow x_j} x_j \right) + \left( \sum_{y_i \rightarrow x_k \rightarrow y_j} y_i y_j \right) \right]
\]

where \( u \) is the total number of loops on all odd vertices which have arrows between them and even vertices \( x_i \) with arrow \( x_i \rightarrow x_k \), \( v \) is the total number of loops on all odd vertices which have arrows between them and even vertices \( x_j \) with arrow \( x_k \rightarrow x_j \). In the expression above, both conditions \( y_i \rightarrow x_k \rightarrow y_j \) and \( y_i \rightarrow y_j \) must be satisfied and we consider the multiplicity in the sense that if there are \( a \) arrows \( y_i \rightarrow x_k \), \( b \) arrows \( x_k \rightarrow y_j \), then it will contribute \( a b y_i y_j \) in the sum provided that \( y_i \not\rightarrow y_j \).

The new superquiver \( \mu_k(Q) \) is obtained from \( Q \) by modifying vertices in view of the above mentioned exchange rules and changing arrows as follows:

(i) If there is a path \( x_i \rightarrow x_k \rightarrow x_j \), add an arrow \( x_i \rightarrow x_j \) for each distinct path.

(ii) Reverse all arrows connecting \( x_k \) to another even vertex.

(iii) Delete any 2-cycle produced between two even variables in the process.

3.1.2. **Odd Mutation.** We define the odd mutation in the direction of an exchangeable odd vertex \( y_i \) as

\[
\eta_i(x_1, \cdots, x_m, y_1, \cdots, y_n, Q) = (\eta_i(x_1), \cdots, \eta_i(x_m), \eta_i(y_1), \cdots, \eta_i(y_n), \eta_i(Q))
\]

where

\[
\begin{align*}
\eta_i(y_j) &= y_j, & j \neq i \\
\eta_i(y_i) &= \delta(y_i) y_i + \left( \prod_{x_k \rightarrow y_i} \frac{1}{x_k} \right) \left[ \left( \sum_{y_i \rightarrow y_j} y_j \right) \left( \prod_{y_i \rightarrow x_j \rightarrow y_j} x_j \right) + \left( \sum_{y_j \rightarrow y_i} y_j \right) \left( \prod_{y_j \rightarrow x_i} x_i \right) \right] \\
\eta_i(x_k) &= x_k & \text{for each } k
\end{align*}
\]

Here \( \delta(y_i) = 1 \) if there is no arrow between \( y_i \) and another odd vertex and \( \delta(y_i) = 0 \) otherwise.

In the expressions \( \left( \sum_{y_i \rightarrow y_j} y_j \right) \) and \( \left( \sum_{y_j \rightarrow y_i} y_j \right) \) above, we consider only for \( i \neq j \).

The new superquiver \( \eta_i(Q) \) is obtained from \( Q \) by modifying vertices in view of the above mentioned exchange rules and changing arrows as follows:
(i) If there is a path \( y_k \rightarrow y_i \rightarrow y_j \) for \( k \neq i \neq j \), then add an arrow \( y_k \rightarrow y_j \) for each distinct path.
(ii) Reverse all arrows incident on \( y_i \).
(iii) Delete any 2-cycle produced between two odd variables in the process.

**Note:** The empty product will be considered to be 1 and the empty sum will be considered to be zero.

It is not difficult to see that the even mutation is involutive, that is, \( \mu^2_k = 1 \). So, if there are \( m' \) number of exchangeable even variables, then the exchange pattern for even vertices is an \( m' \)-regular tree. Also, we have \( \eta_k^2(y) = \eta_k(y) \) for each \( y \in Y \) and each element in \( \eta_k^2(Y) \) can be generated by \( X, Y \) and \( \eta_k(Y) \).

Now, we proceed to define cluster superalgebras.

**Definition 3.1.** Let \( X_{\text{even}} \) be the set of all supercluster even variables that can be obtained by applying a sequence of even mutations to the initial seed \((X|Y,Q)\) and \( X_{\text{odd}} \) be the set of all supercluster odd variables that can be obtained by applying a sequence of odd mutations to the initial seed \((X|Y,Q)\). Then the cluster superalgebra \( C_K(X|Y,Q) \) over a field \( K \) (of characteristic different from 2) is defined to be the supercommutative \( K \)-superalgebra generated by \( X_{\text{even}} \cup X_{\text{odd}} \).

**Remark 3.2.** If there are no Grassman variables, then even mutation is exactly the same as Fomin-Zelevinsky mutation for classical cluster algebras.

**Remark 3.3.** Implicitly in the definition of cluster superalgebras, a sequence of mutations consisting of both even and odd mutations is not allowed when constructing a cluster superalgebra. However, a sequence of mutations consisting of both even and odd mutations is allowed in the mutation of a superquiver.

### 3.2. Laurent Phenomenon.

In the case of classical cluster algebras Fomin and Zelevinsky proved that if \( x_1, \ldots, x_m \) are initial cluster variables and \( u \) is any cluster variable, then

\[
u = \frac{f(x_1, \ldots, x_m)}{x_1^{d_1} \cdots x_m^{d_m}},
\]

where \( f \in \mathbb{Z}[x_1, \ldots, x_m] \) and \( d_i \in \mathbb{Z} \). This is known as Laurent phenomenon of cluster algebra. It has been recently proved in [9] and [11] that the coefficients in these Laurent polynomials are positive. Next, we show that the supersymmetric analogue of the Laurent phenomenon holds. We begin with the following lemma.

**Lemma 3.4.** Let \( x_1, \ldots, x_n \) be initial even cluster variables and define the algebra \( S \) as \( \mathbb{K}[\varepsilon_1, \ldots, \varepsilon_n]/ < \varepsilon_1^2 - 1, \ldots, \varepsilon_n^2 - 1 > \). For a sign-skew-symmetric matrix \( B = [b_{ij}] \), define

\[
u_k(x_k) = \frac{1}{x_k} \left[ \prod_{b_{ik} > 0} (\varepsilon_i x_i)^{b_{ik}} + \prod_{b_{ik} < 0} (\varepsilon_i x_i)^{-b_{ik}} \right]
\]

and

\[
u_k(z_k) = \frac{1}{z_k} \left[ \prod_{b_{ik} > 0} z_i^{b_{ik}} + \prod_{b_{ik} < 0} z_i^{-b_{ik}} \right]
\]

and define the mutation of \( B \) as usual. Let \( x_i(t) \) be the cluster at seed \( t \). Then \( x_k(t) = \varepsilon_k z_k(t) |_{z_i = \varepsilon_i x_i} \) for every \( i \).
Proof. We follow by induction on the distance between \( t \) and \( t_0 \).

\[
t_0 \quad \bullet \quad \cdots \quad \bullet \quad \cdots \quad t \quad \mu_{k} \quad t'
\]

We have that \( z_k(t') = \mu_k(z_k) = \frac{1}{z_k(t)} \left[ \left( \prod_{b_{lk}<0} z^{-b_{lk}}_{li} \right) + \left( \prod_{b_{lk}>0} z^{b_{lk}}_{li} \right) \right] \). Now since \( \epsilon_i^2 = 1 \) for all \( i \),

\[
\epsilon_k z_k(t')|_{z_i = \epsilon_i x_i} = \epsilon_k \frac{1}{z_k x_k(t)} \left[ \left( \prod_{b_{lk}<0} \frac{x_i}{\epsilon_i} \right)^{b_{lk}} + \left( \prod_{b_{lk}>0} \frac{x_i}{\epsilon_i} \right)^{-b_{lk}} \right] = \mu_k(x_k(t)) = x_k(t').
\]

\( \Box \)

Note that formula (3.5) is how even mutation is defined in the case that \( y_j y_j \) does not contribute to the sum in the numerator of \( \mu_i(x_i) \) for \( i \in \{1, \ldots, n\} \) when \( \epsilon_i \in \{1, -1\} \) for all \( i \).

**Corollary 3.5.** Assume condition (C2). For \( \epsilon_i \in \{1, -1\} \), \( x_k(t) \) is a Laurent polynomial in the initial even cluster variables since \( z_k(t) \) is Laurent by Laurent Phenomenon of classical cluster algebras.

**Theorem 3.6.** After any sequence of iterated even mutations, \( \mu_{i_k} \circ \cdots \circ \mu_{i_1}(x_i) \) is a Laurent polynomial in the initial supercluster variables.

**Proof.** If there is no term of the form \( y_j y_j \) contributing in \( \mu_i(x_i) \) then we are in classical cluster algebra set up where we already know that the Laurent Phenomenon holds. But if there is a term of the form \( y_j y_j \) contributing then the Condition (C1) guarantees that \( \mu_i \circ \mu_j \circ \mu_i(x_i) = x_i \) as the variable \( x_j \) is frozen now and this again causes Laurent Phenomenon to follow because \( \mu_i(x_i) \) is always a Laurent Polynomial in the initial cluster variables. Condition (C2) ensures \( y_j y_j \) will never be a term in numerator of \( \mu_i(x_i) \) which leads to the Laurent Phenomenon by Corollary 3.5. Note the requirement in (C2) that \( j \neq k \). This is since if \( j = k \) then the term \( y_j^2 \) would be contributed in the sum in the numerator of \( \mu_i(x_i) \), but \( y_j^2 = 0 \) so it causes no issues allowing \( j = k \). If neither of the hypotheses to (C1) or (C2) are satisfied, then \( y_j y_j \) will not contribute to the sum in \( \mu_i(x_i) \) and Laurent Phenomenon follows again from Corollary 3.5. \( \Box \)

Note that in the view of the above, the cluster superalgebra may be viewed as a subalgebra of \( \mathbb{K}[x_1^\pm 1, \ldots, x_m^\pm 1] \otimes \mathbb{K}[y_1, \ldots, y_n] \) and hence in the expression for even or odd mutations, \( x_k^{-1} \) makes sense. Note that \( \mathbb{K}[y_1, \ldots, y_n] \) is the Grassman algebra.

**Example 3.7.**

\[
Q_1 = \quad \begin{array}{c} \cup \end{array} \quad y_1 \quad \begin{array}{c} \cup \end{array} \quad y_2
\]

\[
\begin{array}{c} \downarrow \end{array} \quad x_1 \quad \cdots \quad \begin{array}{c} \downarrow \end{array} \quad x_2 \quad \begin{array}{c} \downarrow \end{array} \quad x_3
\]

\[
Q_2 = \quad z_1 \quad \begin{array}{c} \downarrow \end{array} \quad z_2 \quad \begin{array}{c} \leftarrow \end{array} \quad z_3
\]

Here, set \( (\epsilon_1, \epsilon_2, \epsilon_3) = (1, -1, -1) \). In general, \( \epsilon_i = (-1)^{u_i} \) where \( u_i \) is the total number of loops on odd vertices adjacent to \( x_i \). On superquiver \( Q_1 \) condition (C2) is satisfied and we apply rules for even mutation. On quiver \( Q_2 \) we apply mutation as in classical cluster algebras. We can see that

\[
\mu_1(x_1) = \frac{1 - x_2}{x_1}, \quad \mu_2 \circ \mu_1(x_2) = \frac{1 - x_2 - x_1 x_3}{x_1 x_2}
\]

and \( \mu_1 \circ \mu_2 \circ \mu_1(x_1) = \frac{x_1 x_3 - 1}{x_2} \).

Also,

\[
\mu_1(z_1) = \frac{1 + z_2}{z_1}, \quad \mu_2 \circ \mu_1(z_2) = \frac{1 + z_2 + z_1 z_3}{z_1 z_2}
\]
and $\mu_1 \circ \mu_2 \circ \mu_1(z_1) = \frac{z_1 z_2 + 1}{z_2}$.

Now one can see and verify by computation that $x_k(t) = \varepsilon_k z_k(t)|_{z_i = \varepsilon_i x_i}$.

4. Combinatorial geometric model of even and odd mutations

Consider a bipartite planar graph $B$ given as follows:

We associate a quiver $Q = Q(B)$ with the bipartite graph $B$ in the following manner. Take a vertex for each face and for each edge in the graph $B$, we draw an arrow in this quiver in such a way that it sees the white vertex in left. So we get the following quiver:

If we do the “flip” move on $B$ (which swaps white and black vertices), we get the new bipartite graph $B'$ as

Clearly the quiver $Q' = Q(B')$ associated to this bipartite graph is
Note that if we take vertices 1, 3, 5 as even vertices and 2, 4 as odd, then under our definition of even and odd mutations we have $\eta_4 \circ \eta_2 \circ \mu_1(Q) = Q'$. This shows that the "flip" move of planar bipartite graphs provides a geometric combinatorial model for a sequence of even and odd mutations.

Let us see in some examples how do these even and odd mutations work.

**Example 4.1.** $X = \{x_1\}$ and $Y = \{y_1, y_2\}$ with the superquiver $Q$ as follows and all vertices mutable.

(4.1)

We have:

(4.2)

\[
\mu_1(x_1) = \frac{2}{x_1}, \quad \eta_1(y_1) = y_1, \quad \eta_2(y_2) = y_2
\]

**Example 4.2.** $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2, y_3\}$ with the superquiver $Q$ as follows, $x_1$ frozen, and all other vertices mutable.

(4.3)

We have:

(4.4)

\[
\mu_2(x_2) = \frac{1}{x_2}(1 + x_1 + y_3y_2 + y_3y_1), \quad \mu_2(x_1) = x_1
\]

(4.5)

\[
\eta_1(y_1) = \frac{y_2}{x_1}, \quad \eta_2(y_2) = y_1x_2, \quad \eta_3(y_3) = y_3
\]
Example 4.3. $X = \{ x_1, x_2 \}$ and $Y = \{ y_1, y_2 \}$ with the superquiver $Q$ as follows, $x_2$ frozen, and all other vertices mutable.

\[
Q = \begin{array}{c}
\vdots \\
\end{array}
\]

(4.6)

\[\mu_1(x_1) = \frac{1 + x_2 + 2y_2y_1}{x_1}\]

(4.7)

\[\eta_1(y_1) = 2y_2x_2, \eta_2(y_2) = 2y_1\]

(4.8)

Example 4.4. $X = \{ x_1, x_2 \}$ and $Y = \{ y_1, y_2 \}$ with the superquiver $Q$ as follows and all vertices mutable.

\[
Q = \begin{array}{c}
\vdots \\
\end{array}
\]

(4.9)

We have:

\[\mu_1(x_1) = \frac{1}{x_1}(1 + x_2), \mu_2(x_2) = \frac{1 - x_1}{x_2}\]

(4.10)

\[\eta_1(y_1) = y_1, \eta_2(y_2) = y_2\]

(4.11)

5. Examples of Cluster Superalgebras

5.1. Symplectic-orthogonal supergroup. A supermatrix over a superalgebra $A$ is a matrix $M = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right)$, where the matrices $A, D$ have even entries and they are of sizes $m \times m$ and $n \times n$, respectively. The matrices $B, C$ have odd entries and are of sizes $m \times n, n \times m$, respectively.

The general linear supergroup $GL(m|n)$ is the group of all invertible supermatrices $M$ of size $(m + n) \times (m + n)$, i.e. supermatrices $M$ such that the superdeterminant $sdet(M) = det(A - BD^{-1}C)det(D^{-1})$ is invertible in $A$. Superdeterminant is also known as Berizinian.

It is not difficult to see that $M$ is invertible if and only if both $A$ and $D$ are invertible.

Let $A^t$ denote transpose of a matrix $A$ and $M^{st}$ denote the supertranspose of a supermatrix $M$. Then

\[
M^{st} = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right)^{st} = \left( \begin{array}{cc} A^t & C^t \\ -B^t & D^t \end{array} \right)
\]
Let us write
\[
J_m = \begin{pmatrix} 0 & -I_m \\ -I_m & 0 \end{pmatrix}, \quad K_{2k+1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -I_k \\ 0 & -I_k & 0 \end{pmatrix}, \quad K_{2k} = \begin{pmatrix} 0 & -I_k \\ -I_k & 0 \end{pmatrix}
\]
and
\[
J_{m,n} = \text{diag}(J_m, K_n).
\]
The symplectic-orthogonal supergroup \(SpO(2m|n)\) is defined via its functor of points. Let \(R = R_0 \oplus R_1\) be a superring. The set of \(R\)-points of the symplectic-orthogonal supergroup \(SpO(2m|n)\) consists of \((2m + n) \times (2m + n)\) supermatrices \(M\) with entries in \(R\) such that \(sdet(M)\) is invertible in \(R\) and \(M^* J_{m,n} M = J_{m,n}\).

**Theorem 5.1.** The symplectic-orthogonal supergroup \(SpO(2|1)\) over any superring \(R = R_0 \oplus R_1\) admits a cluster superalgebra structure.

**Proof.** Let \(R = R_0 \oplus R_1\) be a superring. The set of \(R\)-points of the symplectic-orthogonal supergroup \(SpO(2|1)\) consists of supermatrices
\[
\begin{pmatrix} a & b & \gamma \\ c & d & \delta \\ \alpha & \beta & e \end{pmatrix}
\]
such that
\[
(5.1) \quad ad = 1 + bc + \alpha \beta, e = 1 + \alpha \beta, \gamma = a \beta - b \alpha, \delta = c \beta - d \alpha.
\]
The elements \(a, b, c, d, e \in R_0\) and \(\alpha, \beta, \gamma, \delta \in R_1\). Note that the elements \(a, b, c, d, \alpha, \beta\) generate the symplectic-orthogonal superalgebra \(SpO(2|1)\).

Choose \(a, b, c, \in R_0\) and \(\alpha, \beta \in R_1\). Consider the initial seed \((X|Y)\) with \(X = \{a, b, c\}\) where \(b, c\) are frozen and \(Y = \{\alpha, \beta\}\) and consider the following superquiver \(Q\):

We have
\[
(5.2) \quad \mu_a(a) = \frac{1}{a} [bc + 1 + \alpha \beta]
\]
Set \(\mu_a(a) = d\). Then \(ad = 1 + bc + \alpha \beta\). This gives us the first relation of the Equation 5.3.

Note that \(\mu^2_a(a) = a\), so iterating even mutations does not produce more new exchangeable even variables other than \(a\) and \(d\), thus \(X_{\text{even}} = \{a, b, c, d\}\).

Next, we show that every odd variable can be generated by \(\{a, \alpha, \beta\}\). Indeed, iterated odd mutations may produce three more distinct superquivers:

By (3.3), for all the four quivers, we must have \(\eta_{\alpha'}(\alpha') = \beta'\) or \(a \beta'\), and \(\eta_{\beta'}(\beta') = a \alpha'\) or \(a \alpha\); thus the new odd cluster variables are generated by \(a, \alpha', \beta'\). Therefore all odd cluster variables are generated by \(a, \alpha, \beta\) by induction. In fact, a simple computation shows
\[
X_{\text{odd}} = \{a^i \alpha \mid i \geq 0\} \cup \{a^i \beta \mid i \geq 0\}.
\]
This shows that the symplectic-orthogonal supergroup \(SpO(2|1)\) over any superring \(R = R_0 \oplus R_1\) admits a cluster superalgebra structure. \(\square\)
We do not know whether the more general result that the symplectic-orthogonal supergroup \( \text{SpO}(2m|n) \) over any superring \( R = R_0 \oplus R_1 \) admits a cluster superalgebra structure is true or not.

For \( m = 1 \) and \( n = 2 \), we have

**Theorem 5.2.** The symplectic-orthogonal supergroup \( \text{SpO}(2|2) \) over any superring \( R = R_0 \oplus R_1 \) is quotient of a cluster superalgebra.

**Proof.** The symplectic-orthogonal supergroup \( \text{SpO}(2|2) \) over a superring \( R = R_0 \oplus R_1 \) consists of supermatrices

\[
M = \begin{pmatrix}
a & b & \gamma_1 & \gamma_2 \\
c & d & \delta_1 & \delta_2 \\
\alpha_1 & \beta_1 & e_1 & e_2 \\
\alpha_2 & \beta_2 & e_3 & e_4 \\
\end{pmatrix}
\]

such that \( \text{sdet}(M) \) is invertible in \( R \) and

\[
M^{st} J_{1,2} M = J_{1,2}.
\]

The above condition gives us the following set of equations

\[
\begin{align*}
ad & = 1 + bc - \alpha_1 \beta_2 - \alpha_2 \beta_1 \\
e_1 e_4 + e_2 e_3 & = 1 - \gamma_1 \delta_2 - \gamma_2 \delta_1 \\
e_1 e_3 & = -\gamma_1 \delta_1 \\
e_2 e_4 & = -\gamma_2 \delta_2 \\
-c\gamma_1 + a\delta_1 & = e_1 \alpha_2 + e_3 \alpha_1 \\
-c\gamma_2 + a\delta_2 & = e_2 \alpha_2 + e_4 \alpha_1 \\
-d\gamma_1 + b\delta_1 & = e_1 \beta_2 + e_3 \beta_1 \\
-d\gamma_2 + b\delta_2 & = e_2 \beta_2 + e_4 \beta_1
\end{align*}
\]

Choose \( a, b, c, e_1, e_2, e_3 \in R_0 \) and \( \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2 \in R_1 \).

Consider the initial seed \( (X|Y) \) with \( X = \{a, b, c, e_1, e_2, e_3\} \) where \( b, c, e_2, e_3 \) are frozen and \( Y = \{\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2\} \) and consider the following superquiver:

\[
\begin{array}{c}
\beta_1 \quad \alpha_1 \\
\beta_2 \\
b \\
\alpha_2 \\
e_2 \\
e_1 \\
\delta_2 \\
\delta_1 \quad \gamma_1 \quad \gamma_2 \\
\end{array}
\]

Mutating along the directions of vertices \( a \) and \( e_1 \), we get

\[
\mu_a(a) = \frac{1}{a}[bc + 1 + \beta_2 \alpha_1 + \beta_1 \alpha_2]
\]

\[
\mu_{e_1}(e_1) = \frac{1}{e_1}[1 - e_2 e_3 + \delta_2 \gamma_1 + \delta_1 \gamma_2]
\]
Set \( \mu_a(a) = d \) and \( \mu_{e_1}(e_1) = e_4 \).

This shows that the supergroup \( SpO(2|2) \) is a quotient of the cluster superalgebra \( C_\mathbb{K}(X|Y, Q) \).

5.2. Super Grassmannian. A superspace \( X_a = (X, \mathcal{O}_X) \) consists of a topological space \( X \) endowed with a sheaf of superalgebras \( \mathcal{O}_X \) such that the stalk at any point \( x \in X \), denoted by \( \mathcal{O}_{X,x} \), is a local superalgebra. In particular, the superspace \( \mathbb{C}^{4|1} \) is the complexified Minkowski space \( \mathbb{C}^4 \) endowed with the following sheaf of superalgebras. For any open subset \( U \) of \( \mathbb{C}^4 \),

\[
\mathcal{O}_{\mathbb{C}^{4|1}}(U) = C^\infty(U) \otimes \mathbb{C}[y_1]
\]

where \( C^\infty(U) \) is the algebra of regular functions on \( U \) and \( \mathbb{C}[y_1] \) is the Grassman algebra generated by \( y_1 \).

The super Grassmannian \( G(r|s; m|n) \) is the supermanifold of \( (r, s) \)-dimensional supervector subspaces \( U \) of a given \( (m, n) \)-dimensional supervector space \( V = V_0 \oplus V_1 \), that is, \( U \subset V \), \( \dim(V_0) = m \), \( \dim(V_1) = n \), \( \dim(U \cap V_0) = r \) and \( \dim(U \cap V_1) = s \). For more details on supergeometry we refer the reader to [12] and [15].

We consider the super Grassmannian \( G(2|0; 4|1) \) of \( (2|0) \) planes inside the superspace \( \mathbb{C}^{4|1} \). This super Grassmannian is called chiral conformal superspace. Note that the supervariety of super Grassmannian is, in general, not a projective supervariety, contrary to the classical setting. However, the particular supervariety of super Grassmannian \( G(2|0; 4|1) \) is a projective supervariety with the Grassmannian superalgebra \( O(G(2|0; 4|1)) \) given in terms of generators and relations as (see [2])

\[
O(G(2|0; 4|1)) = \frac{\mathbb{C}[q_{ij}, \lambda_k, a_{55}]}{I_P}
\]

where \( q_{ij}, a_{55} \) are even elements, \( \lambda_k \) are odd elements and \( I_P \) is the two-sided ideal generated by the super Plucker relations given below

\[
q_{12}q_{34} - q_{13}q_{24} + q_{14}q_{23} = 0
\]

\[
q_{ij}\lambda_k - q_{ik}\lambda_j + q_{jk}\lambda_i = 0, \quad 1 \leq i < j < k \leq 4
\]

\[
\lambda_i\lambda_j = a_{55}q_{ij}, \quad 1 \leq i < j \leq 4
\]

\[
\lambda_i a_{55} = 0, \quad 1 \leq i \leq 4
\]

Consider the initial seed \( (X|Y) \), where \( X = \{q_{12}, q_{14}, q_{23}, q_{24}, q_{34}, a_{55}\} \) with \( q_{24} \) being exchangeable and others frozen; \( Y = \{\lambda_1, \lambda_2, \lambda_4\} \) with \( \lambda_2 \) being exchangeable and others frozen. Consider the following superquiver \( Q \).

Mutating along even vertex \( q_{24} \), we get

\[
\mu_{q_{24}}(q_{24}) = \frac{1}{q_{24}}[q_{12}q_{34} + q_{23}q_{14}].
\]
Mutating along odd vertex $\lambda_2$, we get
\[
\eta_2(\lambda_2) = \frac{1}{q_{14}} [\lambda_4 q_{12} + \lambda_1 q_{24}].
\]

Setting $\eta_2(\lambda_2) = \lambda_3$, we get
\[
q_{12} \lambda_4 - q_{14} \lambda_3 + q_{24} \lambda_1 = 0.
\]

Thus, we have

**Theorem 5.3.** The coordinate superalgebra of the super Grassmannian $G(2|0; 4|1)$ is the quotient of the cluster superalgebra $C_K(X|Y, Q)$. More precisely, the coordinate superalgebra of the super Grassmannian $G(2|0; 4|1)$ is $C_K(X|Y, Q)/\mathcal{I}$ where
\[
\mathcal{I} = \langle \lambda_1 \lambda_2 - a_{55} q_{12}, \lambda_1 \lambda_3 - a_{55} q_{13}, \lambda_1 \lambda_4 - a_{55} q_{14}, \lambda_2 \lambda_3 - a_{55} q_{23}, \lambda_2 \lambda_4 - a_{55} q_{24}, \lambda_3 \lambda_4 - a_{55} q_{34}, \lambda_1 a_{55}, \lambda_2 a_{55}, \lambda_3 a_{55}, \lambda_4 a_{55} \rangle.
\]

Note that it is not much of a setback not being able to obtain the relations in $\mathcal{I}$ as these are not in the spirit of mutation in classical cluster algebras.

6. Finite mutation type superquiver

In the theory of cluster algebras, a quiver $Q$ is said to be of finite mutation type if its mutation-equivalence class is finite. It is known that a connected quiver $Q$ with at least three vertices is of finite mutation type if and only if it comes from the triangulation of a surface or it is mutation-equivalent to one of the exceptional types $E_6, E_7, E_8, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}, E_7^{(1,1)}, E_8^{(1,1)}, X_6, X_7$.

We adapt the notion of mutation-equivalence for classical cluster algebras in the case of cluster superalgebras as follows:

**Definition 6.1.** A superquiver $Q'$ is said to be mutation-equivalent to another superquiver $Q$ if there exists mutations $\sigma_1, \ldots, \sigma_r$ with each $\sigma_i$ being either an even mutation or an odd mutation such that $\sigma_r \circ \cdots \circ \sigma_1(Q) = Q'$. As in the case of classical cluster algebras, we will denote by $\text{Mut}(Q)$ the set of all superquivers mutation-equivalent to a superquiver $Q$.

**Definition 6.2.** A superquiver $Q$ is said to be of finite mutation type if the number of superquivers $Q'$ that are mutation-equivalent to $Q$ is finite.

In the theorem below, we characterize superquivers that are of finite mutation type.

**Theorem 6.3.** Let $Q$ be a superquiver and denote by $Q_X$ the full subquiver of $Q$ obtained by removing all odd vertices of $Q$ and let $Q_Y$ be the full subquiver of $Q$ obtained by removing all even vertices of $Q$. Then $Q$ is of finite mutation type if and only if $Q_X$ and $Q_Y$ are of finite mutation type in the classical sense.

**Proof.** First note that on $Q_X$ and $Q_Y$, our notion of superquiver mutation coincides with the classical quiver mutation. Clearly $|\text{Mut}(Q)| \geq |\text{Mut}(Q_X)| \cdot |\text{Mut}(Q_Y)|$, so if $Q$ is of finite mutation type then $Q_X$ and $Q_Y$ are of finite mutation type. Now suppose $Q_X$ and $Q_Y$ are of finite mutation type. Let $|X| = m$, $|Y| = n$, $|\text{Mut}(Q_X)| = r$, and $|\text{Mut}(Q_Y)| = s$. In $Q$, each vertex in $Q_Y$ can have an arrow between at most $m$ vertices in $Q_X$. There are four possible cases of arrows between a vertex $x_k$ in $Q_X$ and a vertex $y_j$ in $Q_Y$ in $Q$:

1. There are no arrows between $x_k$ and $y_j$.
2. There are an equal number of arrows $x_k \to y_j$ as there are $y_j \to x_k$.
3. There are $a$ number of arrows $x_k \to y_j$ and $b$ number of arrows $y_j \to x_k$ with $a > b$.
4. There are $a$ number of arrows $x_k \to y_j$ and $b$ number of arrows $y_j \to x_k$ with $a < b$.
Applying mutation to an odd vertex satisfying (1) maintains condition (1) on that odd vertex and applying mutation to an odd vertex satisfying (2) maintains condition (2) on that odd vertex. Applying mutation to an odd vertex satisfying (3) changes it to satisfying condition (4) on that odd vertex with \(a\) becoming \(b\) and \(b\) becoming \(a\). Applying mutation to an odd vertex satisfying (4) changes it to satisfying condition (3) on that odd vertex with \(a\) becoming \(b\) and \(b\) becoming \(a\). Thus (1) – (4) encompass all possibilities of connections between an even and odd vertex after applying mutation. Since for each of the \(r\) possible quivers in \(\text{Mut}(Q_x)\) we can have \(s\) possible orientations for \(Q_y\) and there are 4 possible ways an even and odd vertex can have arrows between then and each of the \(m\) even vertices can have arrows between at most \(n\) odd vertices, we have that \(|\text{Mut}(Q)| < rs \cdot 4^mn\). Thus \(Q\) is of finite mutation type. □

Note the strict inequality due to it being impossible for a connection between a vertex in \(Q_x\) and a vertex in \(Q_y\) to attain every state (1) – (4) through mutation. As previously discussed, the only time a connection between a vertex in \(Q_x\) and a vertex in \(Q_y\) can change is when mutating at an odd vertex satisfying (3) or (4). Thus if \(Q_x\) and \(Q_y\) are finite mutation type, we can refine the bound on the size of the mutation equivalence class as \(|\text{Mut}(Q)| \leq rs \cdot 2^n\) since for each of the \(n\) odd variables \(y_j\) there are two possible configurations for the set of arrows between \(y_j\) and even variables.

**Definition 6.4.** A cluster superalgebra \(C \_K(X\vert Y, Q)\) is said to be of finite type if the number of supercluster variables is finite.

Cluster algebras of finite type are completely characterized in [6] and their classification, quite surprisingly, turns out to be identical to the Cartan-Killing classification of semisimple Lie algebras and finite root systems. It is known that a cluster algebra \(A(X, Q)\) is of finite type if and only if \(Q\) is mutation equivalent to Dynkin diagram of type \(A, D\) or \(E\).

We propose the following problem for further development of the notion of cluster superalgebras.

**Problem 6.5.** Characterize cluster superalgebras that are of finite type.

### 7. Limitations and extensions

One of the limitations of our study is that we have considered only those superquivers in this paper that satisfy at least one of the conditions (C1) and (C2). The reason behind imposing this restriction is that if we consider an arbitrary superquiver and follow our even and odd mutations, then the Laurent phenomenon fails to hold. We illustrate this in the example below where conditions (C1) and (C2) are both violated. Consider the initial seed \((X\vert Y, Q)\) with \(X = \{x_1, x_2\}, Y = \{y_1, y_2\}\) with \(x_1\) and \(x_2\) mutable and superquiver \(Q\) given as:

\[
\begin{array}{c}
\xymatrix{ x_1 & \ar[r] & x_2 \\
& y_1 & \ar[l] & y_2 & \ar[u] }\end{array}
\]

The even mutation \(\mu_1\) gives \(\mu_1\{x_1, x_2, y_1, y_2, Q\} = \{x'_1, x_2, y_1, y_2, \mu_1(Q)\}\) where \(x'_1 = \frac{1 + x_2}{x_1}\) and \(\mu_1(Q)\) is

\[
\begin{array}{c}
\xymatrix{ x'_1 & \ar[r] & x_2 \\
& y_1 & \ar[l] & y_2 & \ar[u] }\end{array}
\]

(7.2)
Next, when we apply $\mu_2$ we have

$$\mu_2 \circ \mu_1 \{x_1, x_2, y_1, y_2, Q\} = \{x'_1, x'_2, y_1, y_2, \mu_2 \circ \mu_1(Q)\}$$

where

$$x'_2 = \frac{1 + x_2 + x_1(1 + y_1y_2)}{x_1x_2}$$

and $\mu_2 \circ \mu_1(Q)$ is

$$\begin{align*}
&x'_1 \\
&\downarrow \\
&x'_2 \\
&\downarrow \\
&y_1 \\
&\downarrow \\
&y_2
\end{align*}$$

(7.3)

Now, we apply mutation $\mu_1$ again and this gives us

$$\mu_1 \circ \mu_2 \circ \mu_1 \{x_1, x_2y_1, y_2, Q\} = \{x''_1, x'_2, y_1, y_2, \mu_1 \circ \mu_2 \circ \mu_1(Q)\}$$

where

$$x''_1 = \frac{1 + x_2 + x_1(1 + y_1y_2) + x_1x_2}{x_2(1 + x_2)}$$

and $\mu_1 \circ \mu_2 \circ \mu_1(Q)$ is

$$\begin{align*}
&x''_1 \\
&\downarrow \\
&x'_2 \\
&\downarrow \\
&y_1 \\
&\downarrow \\
&y_2
\end{align*}$$

(7.4)

Note that $x''_1$ obtained above is not a Laurent polynomial in initial cluster variables $x_1, x_2, y_1, y_2$ and this shows that the Laurent phenomenon fails to hold in the case.

To be able to extend this study to any superquiver, one needs some kind of analogue of allowed and forbidden mutations as in [14]. For example, in the case of an arbitrary superquiver if we allow any sequence of even mutations where no two mutations are in the same direction, then our definition of cluster superalgebras clearly extends to arbitrary superquivers and the Laurent phenomenon holds in this case. As a consequence of this extension, we are able to show that the supercommutative superalgebra generated by all the entries of a superfrieze is a subalgebra of a cluster superalgebra. Note that this result has been obtained in [14] too and a suitable adaptation of his argument works in our approach as well.
Definition 7.1. A superfrieze, or a supersymmetric frieze pattern over a superring \( \mathcal{R} = \mathcal{R}_0 \oplus \mathcal{R}_1 \) has been defined in [13] as the following array:

\[
\begin{array}{cccccccc}
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 1 & 1 & 1 & \cdots \\
\varphi_{0,0} & \varphi_{1,1} & \varphi_{2,2} & \cdots \\
f_{0,0} & f_{1,1} & f_{2,2} & \cdots \\
f_{-1,0} & f_{0,1} & f_{1,2} & \cdots \\
f_{2-m,1} & f_{0,m-1} & f_{1,m} & \cdots \\
\cdots & \varphi_{2-m,2} & \cdots & \varphi_{0,m} & \varphi_{1,m+1} & \varphi_{3-m,3} & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\end{array}
\]

where \( f_{i,j} \in \mathcal{R}_0 \) and \( \varphi_{i,j} \in \mathcal{R}_1 \), and where every elementary diamond:

\[
\begin{pmatrix}
A & B \\
\Xi & \Psi \\
C & D
\end{pmatrix}
\]

satisfies the following conditions:

\[
\begin{align*}
AD - BC &= 1 + \Sigma \Xi, \\
B \Phi - A \Psi &= \Xi, \\
B \Sigma - D \Xi &= \Psi,
\end{align*}
\]

that we call the frieze rule.

The number of even rows between the rows of 1’s is called the width of the superfrieze.

The last two equations of (7.5) are equivalent to

\[
\begin{align*}
A \Sigma - C \Xi &= \Phi, \\
D \Phi - C \Psi &= \Sigma.
\end{align*}
\]

Note also that these equations also imply \( \Xi \Sigma = \Phi \Psi \), so that the first equation can also be written as follows: \( AD - BC = 1 - \Phi \Psi \).

As observed in [14], one can associate an elementary diamond with every element of \( \text{SpO}(2|1) \) using the following formula:

\[
\begin{pmatrix}
a & b & \gamma \\
c & d & \delta \\
\alpha & \beta & e
\end{pmatrix}
\leftrightarrow
\begin{pmatrix}
-a & \gamma \\
\alpha & -c \\
-\beta & \delta \\
\end{pmatrix}
\]
so that the relations \([5.3]\) and \([7.5]\) coincide.

We are able to obtain \([14, \text{Theorem 3}]\) with an almost identical proof by considering the following extension of the standard Dynkin quiver of type \(A_n\) with a set of \(n\) even vertices \(X = \{x_1, \ldots, x_n\}\) and a set of \(n+1\) odd vertices \(Y = \{y_1, \ldots, y_{n+1}\}\):

\[
\begin{array}{cccccccc}
  & & & & & & & \\
  & y_1 & & y_2 & & y_3 & & \cdots & y_n & & y_{n+1} \\
 & x_1 & & x_2 & & x_3 & & \cdots & x_n & \\
\end{array}
\]

Let \(\tilde{Q}\) denote the superquiver in \([7.6]\) and consider \(C_K(X|Y, \tilde{Q})\) with \(K\) a field of characteristic different from 2. We have the following:

**Theorem 7.2.** The supercommutative superalgebra generated by all the entries of a superfrieze of width \(n\) is a subalgebra of the cluster superalgebra \(C_K(X|Y, \tilde{Q})\).

**Proof.** Choose the following entries of the superfrieze on parallel diagonals:

\[
\begin{array}{cccccccc}
  1 & & & & & & & 1 \\
  & & * & y_1 & * & y'_1 & & \cdots & \cdots & \cdots \\
  & & * & y_2 & * & y'_2 & & \cdots & \cdots & \cdots \\
  & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  & & * & y_n & * & y'_n & & \cdots & \cdots & \cdots \\
  & & * & x'_1 & & x'_1 & & \cdots & \cdots & \cdots \\
  & & * & x'_2 & & x'_2 & & \cdots & \cdots & \cdots \\
  & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  & & * & x'_{n-1} & & x'_{n-1} & & \cdots & \cdots & \cdots \\
  & & * & x'_n & & x'_n & & \cdots & \cdots & \cdots \\
  & & 1 & & 1 & & \cdots & \cdots & \cdots & \cdots \\
\end{array}
\]

All entries of the superfrieze are determined by \(x_1, \ldots, x_n, y_1, \ldots, y_{n+1}\) and hence these can be taken as initial coordinates. Note that we are done if we can show that the superfrieze entries \(x'_1, \ldots, x'_n, y'_1, \ldots, y'_{n+1} \in C_K(X|Y, \tilde{Q})\). By \([7.5]\) we have that

\[
(x_k x'_k) = 1 + x_{k+1} x'_{k-1} + y_{k+1} y_k
\]

Now perform even mutations on \(\tilde{Q}\) at vertices \(x_1, \) then \(x_2\), until \(x_n\). After the first \(k-1\) mutations we obtain the following quiver:

\[
\begin{array}{cccccccc}
  & & & & & & & \\
  & y_1 & & y_2 & & y_3 & & \cdots & y_k & & y_{k+1} & & \cdots \\
 & x'_1 & & x'_2 & & x'_3 & & \cdots & x'_{k-1} & & x'_k & & x'_{k+1} & & \cdots \\
\end{array}
\]

Mutating at vertex \(x_k\) we obtain the following from our rule for even mutation as in \([3.2]\):

\[
(x_k x'_k) = 1 + x_{k+1} x'_{k-1} + y_{k+1} y_k
\]

This coincides with \([7.7]\). This shows that the entries \(x'_1, \ldots, x'_n\) from the superfrieze are the same as the supercluster variables \(x_1, \ldots, x'_n\) obtained by iterated even mutations at consecutive even vertices in \(\tilde{Q}\). We now consider the odd entries of the superfrieze \(y'_1, \ldots, y'_{n+1}\). By the third equality of \([7.5]\) we have that

\[
y'_1 = y_2 - x'_1 y_1,
\]
so clearly \( y'_1 \in \mathcal{C}_K(X|Y, \tilde{Q}) \). Lemma 2.5.3 of [13] implies the following for odd entries of the superfrieze for all \( k \):
\[
y'_k = y'_{k-1} - y_1 x'_k.
\]
As \( y'_k \) is a linear combination of \( y'_{k-1}, y_1 \), and \( x'_k \) for all \( k \), it follows that \( y'_k \in \mathcal{C}_K(X|Y, \tilde{Q}) \) for all \( k \) since it has already been established that \( y'_1 \in \mathcal{C}_K(X|Y, \tilde{Q}) \). By a similar argument, this holds for all parallel diagonals and by the relations in (7.5) it can be established that all entries of the superfrieze are contained in \( \mathcal{C}_K(X|Y, \tilde{Q}) \).

We close the paper with the remark that it would be useful to come up with a complete characterization of allowed mutations so that we may extend the theory to arbitrary superquivers.

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