Nonlinear model order reduction of continuous-time image reconstruction systems

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Abstract: As an image reconstruction method for computed tomography, continuous-time nonlinear dynamical systems were proposed. However, as the nonlinear dynamical systems are very large-scale, the numerical analysis is costly and the reconstructed images are not easily obtained. Thus, we apply nonlinear model order reduction algorithm to the nonlinear dynamical systems, based on proper orthogonal decomposition. For the continuous-time image reconstruction systems, stability analysis of the equilibria is extremely important, because the reconstructed images are obtained by the equilibria. Hence, stability analysis of the reduced-order systems is presented, in which the equilibria are proved to be asymptotically stable. In the numerical examples, robustness and efficacy of the proposed reduced-order systems will be demonstrated.

Key Words: computed tomography, nonlinear dynamical systems, nonlinear model order reduction, proper orthogonal decomposition, stability analysis

1. Introduction

X-ray computed tomography (CT) apparatus is widely used in clinics to determine whether there is a tumor inside human body. CT apparatus yields tomographic images reconstructed from projection data acquired with X-ray sensors.

The basic problem in CT is defined by a rectangular linear system with projection data and sparse projection operator that is obtained from discrete Radon transform. A filtered back-projection (FBP) method [1, 2], which is a transform method, is widely used in conventional CT apparatus. This method generates a high-quality image only when sampling theorem is satisfied for the number of projections to a full scan of 360°. To be more specific, the FBP method generates low-quality image with streak artifacts if the number of projections is insufficient. To obtain a high-quality image without artifacts, a large amount of projection data is required, which implies high X-ray doses so that it is not preferable.

In [3, 4], a continuous-time dynamical system is proposed to reconstruct images. Compared with the FBP method, this method can produce a high-quality image even from a small amount of projection data.
data. This method also has an advantage for image reconstruction, in which image without negative pixels is generated. Thus, the dynamical system has a positive constraint. As other methods that can generate high-quality images even from a small number of projection data, iterative methods such as algebraic reconstruction techniques (ART) [2] and the expectation-maximization (EM) method [7] are known. Although the computational costs of these methods are lower than that of the continuous-time dynamical system, they do not necessarily generate images with nonnegative pixels. In [5], a box constraint is given to the continuous-time dynamical system in order to reconstruct binary images. Moreover, this method is effectively extended to gray-scale images when the maximum pixel value is defined a priori [6].

Although the continuous-time image reconstruction systems, which are categorized into nonlinear dynamical systems, have useful image processing properties, analyzing the dynamics is not easy because the systems are very large-scale. To overcome the difficulty, we intend to use nonlinear model order reduction strategies [9, 10]. As model reduction algorithms, trajectory piecewise-linear approach (TPWL) [10] and proper orthogonal decomposition (POD) [9] are known. The reduced-order model in TPWL is expressed by weighted sum of time-invariant systems and this method is efficiently applicable to nonlinear dynamical systems with complex nonlinear functions. On the other hand, POD is effective for nonlinear dynamical systems with low-order nonlinearity such as bilinear systems. If nonlinear functions have a complex structure, function evaluations in the reduced-order models ruin the effect of the dimension reduction. Thus, many researcher paid attention to efficient nonlinear function evaluations [11–13]. POD discrete empirical interpolation (POD-DEIM) provides reduced-order systems with efficient nonlinear function evaluations [14]. This method attains efficient simulation under acceptable error. Moreover, the error estimations of POD-DEIM are explicitly given on general continuous- and discrete-time nonlinear dynamical system settings [15].

In this paper, nonlinear model order reduction of continuous-time image reconstruction systems is presented based on POD. In the image reconstruction systems, as the nonlinear terms are second polynomials or cross terms of two variables, POD provides an accurate and compact reduced-order model. For the image reconstruction systems, stability of the equilibria is extremely important, because the reconstructed images are obtained by the steady-state solutions or the equilibria. Thus, stability analysis of the equilibria is presented in [3–6], in which the equilibria of the nonlinear dynamical systems are proved to be asymptotically stable. Here, modifying Galerkin procedure used in POD, we provide reduced-order systems, the equilibria of which are guaranteed to be asymptotically stable. Moreover, the error estimations of the reduced-order systems are provided, following the derivation of POD-DEIM [15]. Although DEIM is not used in the proposed method, the error estimations have similar forms with POD-DEIM. This is due to the modification of Galerkin procedure stated above. Furthermore, the equilibrium points of the reduced-order systems obtained from our reduction method are calculated by solving a least-squares equation without necessity of analyzing the transient behavior of the reduced-order systems. Therefore, the reconstructed images are efficiently obtained after model reduction.

In the numerical examples, comparing with the reduced-order systems using Galerkin procedure without modification, we demonstrate that the reduced-order systems proposed in this paper are more robust than the systems with the original Galerkin procedure. Moreover, we will show that the acceleration effects of using the reduced-order systems provided is extremely high.

2. Continuous-time image reconstruction systems

The basic problem for CT is to find a solution vector \( \mathbf{x}^* \in \mathbb{R}^N_+ \) of the rectangular linear system:

\[
Ax^* = y, \tag{1}
\]

where \( A \in \mathbb{R}^{P \times N}_+ \), which is a sparse matrix obtained from discrete Radon transform, corresponds to a projection operator, and \( y \in \mathbb{R}^P_+ \) represents projection data acquired from X-ray detectors of CT apparatus. The symbols \( P, N, \) and \( \mathbb{R}_+ \) correspond to the number of projections, the number of pixels, and positive real space, respectively. When the square matrix \( A^T A \) is nonsingular, the least-squares
solution of (1) is expressed as \((A^TA)^{-1}A^Ty\) when \(P > N\). If \(P < N\), the least-squares solution is expressed as \(A^T(AA^T)^{-1}y\) on a premise of nonsingular \(AA^T\).

A least-squares optimization problem is defined to find a positive solution to the linear system (1) as

\[
\min_{x \in \mathbb{R}^N_+} V_r(x),
\]

\[
V_r(x) := \frac{1}{2} \|Ax - y\|^2_2,
\]

where \(\|\cdot\|_2\) expresses the Euclidean norm for a vector.

As an approach finding a positive solution of (1), the continuous-time dynamical system is introduced \([3, 4]\) as

\[
dx_p(t) = -\text{diag}(x_p(t)) A^T(Ax_p(t) - y),
\]

where \(\text{diag}(x_p(t))\) is an \(N \times N\) diagonal matrix whose diagonal elements are those of \(x_p(t)\). In \([3]\), it is proven that the dynamical system has a positive solution as the following lemma.

**Lemma 1** \([3]\) If the initial solution \(x_p(0) \in \mathbb{R}^N_+\) in (3), then \(x_p(t) \in \mathbb{R}^N_+\) for \(t \in \mathbb{R}_+\).

Equation (3) is a dynamical system with positive constraint and the objective or energy function \(V_r(x_p(t))\) becomes a Lyapunov one of (3). The following lemma guarantees asymptotic stable of the equilibrium point.

**Lemma 2** \([3]\) If the initial solution \(x_p(0) \in \mathbb{R}^N_+\) in (3), \(V_r(x_p(t))\) is monotonically decreasing. If there exists an equilibrium point as \(x^* \in \mathbb{R}^N_+\), then \(x^*\) is asymptotically stable and a unique least-squares solution of (1).

Additionally, the following lemma excludes the zero equilibrium from all the possible ones.

**Lemma 3** \([3]\) The zero equilibrium of (3) is unstable.

In \([4]\), generalized Kullback-Leibler (GKL) divergence of two nonnegative vectors \(\alpha\) and \(\beta\) is introduced to obtain a Lyapunov function as

\[
\text{GKL}(\alpha, \beta) = \sum_j \beta_j \log \frac{\beta_j}{\alpha_j} + \alpha_j - \beta_j,
\]

where \(\alpha_j\) and \(\beta_j\) are the \(j\)-th elements of \(\alpha\) and \(\beta\), respectively. Moreover, GKL divergence is extended to stability analysis of the continuous-time system with box constraint:

\[
dx_b(t) = -\text{diag}(x_b(t)) \text{diag}(1 - x_b(t)) A^T(Ax_b(t) - y),
\]

where \(1 \in \mathbb{R}^N\) is a vector whose elements are all ones. The following lemmas are obtained in \([5]\), where \(\Omega^N = (0, 1)^N\) is an \(N\)-dimensional open hypercube with each side \((0, 1)\).

**Lemma 4** \([5]\) If the initial solution \(x_b(0) \in \Omega^N\) in (5), then \(x_b(t) \in \Omega^N\) for \(t \in \mathbb{R}_+\).

**Lemma 5** \([5]\) If the initial solution \(x_b(0) \in \Omega^N\) in (5), a Lyapunov function is obtained using GKL divergence. If there exists an equilibrium point as \(x^* \in \Omega^N\), then \(x^*\) is asymptotically stable and a unique least-squares solution of (1).

**Lemma 6** \([5]\) The zero and \(1\) equilibria of (5) are unstable.

### 3. Nonlinear model order reduction by POD

The linear rectangular system solution of (1) is obtained by finding the equilibrium point of the nonlinear dynamical system (3) or (5). However, many steps of numerical integration are required for obtaining the equilibrium point. Then, we would obtain approximate solutions while looking for
a compromise between calculation time and reconstructed image quality. In this case, model order reduction of nonlinear dynamical system becomes a powerful tool. Model order reduction generates a lower order model than the original system. When the system is linear, Krylov subspace provides a projection matrix $V \in \mathbb{R}^{N \times M}$ that has orthonormal columns. As $M$ is much smaller than $N$ in general, congruence transforms of coefficient matrices provide a reduced-order dynamical system. This concept can be also applied to nonlinear dynamical systems.

Assume that a projection matrix $V$ is obtained by a method. The state vector $x_p(t)$ of (3) is approximated by $x_p(t) = V \hat{x}_p(t)$. Then, the nonlinear dynamical system with positive constraint is approximated as

$$V \frac{d\hat{x}_p(t)}{dt} = - \text{diag}(V \hat{x}_p(t)) A^\top (AV \hat{x}_p(t) - y).$$

(6)

Since $V$ is full column rank, we can uniquely determine the solution $\hat{x}_p(t)$ of (6) by multiplying the matrix $V^\top$ from the left of both sides of (6) as

$$\frac{d\hat{x}_b(t)}{dt} = -V^\top \text{diag} (V \hat{x}_b(t)) (1 - V \hat{x}_b(t)) A^\top (AV \hat{x}_b(t) - y),$$

(7)

which is the reduced-order dynamical system with positive constraint. Since the order of (7) is smaller than (3), we can analyze the behavior of (7) efficiently.

Similarly to the positive constrained case, the nonlinear dynamical system with box constraint is approximated as

$$V \frac{d\hat{x}_b(t)}{dt} = - \text{diag}(V \hat{x}_b(t)) \text{diag} (1 - V \hat{x}_b(t)) A^\top (AV \hat{x}_b(t) - y).$$

(8)

We can uniquely determine the solution $\hat{x}_b(t)$ of (8) as

$$\frac{d\hat{x}_b(t)}{dt} = -V^\top \text{diag} (V \hat{x}_b(t)) \text{diag} (1 - V \hat{x}_b(t)) A^\top (AV \hat{x}_b(t) - y),$$

(9)

which is the reduced-order dynamical system with box constraint.

Consider how to obtain the projection matrix $V$. Assume that simulation data as $Y = [x_p^0, x_p^1, \ldots, x_p^{m-1}]$ or $[x_b^0, x_b^1, \ldots, x_b^{m-1}]$, in which $x_p^k$ and $x_b^k$ are the state vectors of (3) and (5) obtained at the $k$-th time step, respectively. POD was proposed as a method for computational fluid dynamics and it extracts proper modes associated with partial differential equations. In the snapshot POD method, eigenvalue decomposition of covariant matrix $YY^\top$ is applied to calculate the proper modes. As the order of $YY^\top$ is high and matrix $Y$ is a dense matrix, eigenvalue decomposition of $YY^\top$ is costly. Thus, singular value decomposition (SVD) of $Y$ is used instead of eigenvalue decomposition. All singular values are not required for model order reduction and some largest ones are necessary. Thus, MATLAB function svds or ARPACK [8] can be used efficiently. In these codes, implicit restarted Arnoldi method is used. This method requires the matrix-vector product $YY^\top v$ for SVD of $Y \in \mathbb{R}^{N \times m}$ and $Y^\top Y v$ for SVD of $Y^\top \in \mathbb{R}^{m \times N}$. Thus, if $N > m$, SVD can be efficiently applied as $Y^\top = U \Sigma V^T$, where $U$ and $V$ are orthonormal matrices and $\Sigma$ is a diagonal matrix, the diagonal entries of which are singular values.

Expressions such as (7) and (9) are called Galerkin procedure. For the image reconstruction systems, we aim to find the equilibria. However, as orthonormal matrix $V$ is separately determined, $A^\top (AV \hat{x}_b(t) - y) = 0$ having a solution $\hat{x}_b(t)$ is not guaranteed. For example, if $y = Ax^*$, vector $V \hat{x}_b(t) - x^*$ must be the kernel of $A^\top A$. However, as $M$, which is the number of columns of $V$, is smaller than $N$, $V \hat{x}_b(t) - x^*$ cannot be necessarily the kernel of $A^\top A$. Therefore, we introduce a weighted matrix to the reduced-order systems (7) and (9). Equation (6) is improved as

$$V \frac{d\hat{x}_p(t)}{dt} = - \text{diag}(V \hat{x}_p(t)) V V^\top A^\top (AV \hat{x}_p(t) - y).$$

(10)

Then, we can uniquely determine the solution $\hat{x}_p(t)$ of (10) as

$$\frac{d\hat{x}_p(t)}{dt} = -V^\top \text{diag} (V \hat{x}_p(t)) V V^\top A^\top (AV \hat{x}_p(t) - y).$$

(11)
The merit of using (11) is to obtain the equilibrium point by solving the least-squares equation:
\[ V^\top A^\top (AV \hat{x}_p(t) - y) = 0. \] (12)

Even if \( A^\top A \) is singular, this does not mean that \( V^\top A^\top AV \) is also singular. Since a large matrix \( A^\top A \) is contracted by congruence transform with \( V \) and \( V^\top A^\top AV \) is a small square matrix, it must become a nonsingular matrix. Thus, the equilibrium point of (11) is the unique solution of (12). Note that least-squares methods based on Gram–Schmidt orthogonalization cannot be directly applied to solving this problem, because the solution of (1) must be constrained to be positive or bounded. In this case, we can use a least-squares method with positive constraint [18], which is implemented on a MATLAB function \texttt{lsqnonneg}. However, the MATLAB function is not applicable to a large-scale problem that we need for the image reconstruction. In the proposed method, using the dynamics of the original system, the least-squares solutions are constrained to be positive or bounded, because the projection matrix \( V \) associated with (12) reflects positivity or boundedness of the snapshot matrix \( Y \) which is made from the solution vectors of the nonlinear dynamics. Since the snapshot matrix is obtained by using an explicit numerical integration method, the proposed method can be applied to a large-scale problem that \texttt{lsqnonneg} cannot be used. As a result, the dynamics of the reduced-order models may not be necessary for image reconstruction application, but theoretical and numerical considerations of the reduced-order models are interesting from an academic viewpoint. Thus, we provide these considerations in the rest of this paper.

Similarly to the positive constrained case, the nonlinear dynamical system with box constraint is approximated as
\[ V \frac{d\hat{x}_b(t)}{dt} = -\text{diag}(V \hat{x}_b(t)) \text{diag} (1 - V \hat{x}_b(t)) V V^\top A^\top (AV \hat{x}_b(t) - y). \] (13)

Then, the solution \( \hat{x}_b(t) \) of (13) is expressed as
\[ \frac{d\hat{x}_b(t)}{dt} = -V^\top \text{diag}(V \hat{x}_b(t)) \text{diag} (1 - V \hat{x}_b(t)) V V^\top A^\top (AV \hat{x}_b(t) - y), \] (14)
which is the reduced-order system with box constraint. The equilibrium point of (14) is also obtained by solving the similar equation with (12).

We provide stability analysis of the equilibria of the reduced-order systems in the rest of this section. **Lemmas 1** and **4** also hold for the reduced-order systems as the following theorem.

**Theorem 1** If the initial solution \( V \hat{x}_p(0) \in \mathbb{R}^N_+ \) in (10), then \( V \hat{x}_p(t) \in \mathbb{R}^N_+ \) for \( t \in \mathbb{R}_+ \). Also, if the initial solution \( V \hat{x}_b(0) \in \Omega \) in (13), then \( V \hat{x}_b(t) \in \Omega \) for \( t \in \mathbb{R}_+ \).

**Proof:** When the \( i \)-th component of \( V \hat{x}_p(t) \) is zero, the \( i \)-th component of \( dV \hat{x}_p(t)/dt \) is also zero and it does not change anymore. Thus, the solution of (10) never passes through each axis in the solution space, if \( V \hat{x}_p(0) \in \mathbb{R}^N_+ \). Similarly, when the \( i \)-the component of \( V \hat{x}_b(t) \) is zero or one, the \( i \)-th component of \( dV \hat{x}_b(t)/dt \) is zero. Thus, the solution \( V \hat{x}_b(t) \) is confined in \( \Omega^N \) for \( t \in \mathbb{R}_+ \).

**Lemmas 2** and **5** also hold for the reduced-order systems, i.e., the equilibria of the reduced-order systems are asymptotically stable.

**Theorem 2** If the initial solution \( V \hat{x}_p(0) \in \mathbb{R}^N_+ \) in (10), \( V_r(V \hat{x}_p(t)) \) is monotonically decreasing. If there exists an equilibrium point as \( V \hat{x}_p^\dagger \in \mathbb{R}^N_+ \), then \( V \hat{x}_p^\dagger \) is asymptotically stable. Also, if the initial solution \( V \hat{x}_b(0) \in \Omega^N \) in (13), \( V_r(V \hat{x}_b(t)) \) is monotonically decreasing. If there exists an equilibrium point as \( V \hat{x}_b^\dagger \in \Omega^N \), then \( V \hat{x}_b^\dagger \) is asymptotically stable.

**Proof:** With (11) and (14), the time derivatives of \( V_r(V \hat{x}_p(t)) \) and \( V_r(V \hat{x}_b(t)) \) are respectively written by

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\[
\frac{dV_r(\hat{x}_p(t))}{dt} = \frac{d\hat{x}_p(t)}{dt}^T V^T A^T (AV\hat{x}_p(t) - y) \\
= -(AV\hat{x}_p(t) - y)^T AVV^T \text{diag}(V\hat{x}_p(t))VV^T A^T (AV\hat{x}_p(t) - y), \\
\frac{dV_r(\hat{x}_b(t))}{dt} = \frac{d\hat{x}_b(t)}{dt}^T V^T A^T (AV\hat{x}_b(t) - y) \\
= -(AV\hat{x}_b(t) - y)^T AVV^T \text{diag}(1 - V\hat{x}_b(t))VV^T A^T (AV\hat{x}_b(t) - y).
\] (15)

From Theorem 1, \(\text{diag}(V\hat{x}_p(t)), \text{diag}(V\hat{x}_b(t)),\) and \(\text{diag}(1 - V\hat{x}_b(t))\) are all positive definite. Thus, \(V_r(\hat{x}_p(t))\) and \(V_r(\hat{x}_b(t))\) are monotonically decreasing. If the equilibrium point obtained by solving (12) satisfies \(\hat{x}_b^1 \in \mathbb{R}^N_+\), then \(\hat{x}_b^1\) is asymptotically stable. Also, if \(\hat{x}_b^1 \in \Omega^N\), then \(\hat{x}_b^1\) is also asymptotically stable.

**Lemma 3 and 6** do not hold for the reduced-order systems. Rewriting (10) by \(d\hat{x}_p(t)/dt = f_p(\hat{x}_p(t))\), we obtain
\[
\left. \frac{\partial f_p(\hat{x}_p(t))}{\partial \hat{x}_p(t)} \right|_{\hat{x}_p(t)=0} = V^T \text{diag}(VV^T A^T y) V.
\] (16)

This matrix is not always positive definite, which means that the zero equilibrium point is not always unstable. Also, rewriting (13) by \(d\hat{x}_b(t)/dt = f_b(\hat{x}_b(t))\), we obtain
\[
\left. \frac{\partial f_b(\hat{x}_b(t))}{\partial \hat{x}_b(t)} \right|_{\hat{x}_b(t)=0} = V^T \text{diag}(VV^T A^T y) V, \\
\left. \frac{\partial f_b(\hat{x}_b(t))}{\partial \hat{x}_b(t)} \right|_{\hat{x}_b(t)=1} = V^T \text{diag}(VV^T A^T (A1 - y)) V.
\] (17)

These matrices are not always positive definite, which implies that the zero and 1 equilibria of the reduced-order system with box constraint are not always unstable.

### 4. Error estimations

Error estimations of reduced-order systems obtained by POD-DEIM [14] are presented in [15]. The estimations can be applied to the reduced-order systems (7) and (9). However, we cannot use them directly to the reduced-order systems (11) and (14), since the Galerkin procedure is modified. In this section, the error estimations are obtained for (11) and (14), in which two settings are considered as the previous work, i.e., the error estimations for settings of the continuous-time dynamical system and the discrete-time one are presented. For a simplicity, the 2-norm of \(x\) is expressed as \(||x||\), dropping the suffix 2.

#### 4.1 Continuous-time dynamical system setting

Write an initial value problem with respect to (3) and (5) as
\[
\dot{x}(t) = F(t, x(t)), \quad x(0) = x_0.
\] (18)

Following the notations [15], the reduced-order system is expressed as
\[
\dot{\hat{x}}(t) = \hat{F}(t, \hat{x}(t)), \quad \hat{x}(0) = V^T x_0,
\] (19)

where
\[
\hat{F}(t, \hat{x}(t)) = V^T \hat{F}(t, V \hat{x}(t)), \\
\hat{F}(t, x(t)) = \begin{cases} -\text{diag}(x(t))VV^T A^T (Ax(t) - y) & \text{for } (11), \\
-\text{diag}(x(t))\text{diag}(1 - x(t))VV^T A^T (Ax(t) - y) & \text{for } (14). \end{cases}
\]

Consider the error between the original and the reduced-order system solutions as
The error can be also written as
\[ e(t) = \rho(t) - \theta(t), \] (21)
where \( \rho(t) = x(t) - \mathbf{VV}^\top x(t) \) and \( \theta(t) = \mathbf{VV}^\top x(t) - \dot{\mathbf{x}}(t) \). Note that \( \theta(0) = 0 \). Also, note that \( \rho(t)^\top \theta(t) = 0 \) implies \( ||e(t)||^2 = ||\rho(t)||^2 + ||\theta(t)||^2 \). Define \( \dot{\theta}(t) := \mathbf{V}^\top \theta(t) \). Then, \( \theta(t) := \mathbf{V} \dot{\theta}(t) \), which implies \( ||\theta(t)|| = ||\dot{\theta}(t)|| \).

Consider that
\[ \mathbf{V}^\top \dot{\mathbf{x}}(t) = \tilde{\mathbf{F}} \left( t, \mathbf{V}^\top x(t) \right) + \tilde{\mathbf{r}}(t), \] (22)
\[ \dot{\mathbf{x}}(t) = \tilde{\mathbf{F}} \left( t, \dot{\mathbf{x}}(t) \right), \] (23)
where
\[ \tilde{\mathbf{r}}(t) = \mathbf{V}^\top \mathbf{F} \left( t, x(t) \right) - \tilde{\mathbf{F}} \left( t, \mathbf{V}^\top x(t) \right). \]

Then, we obtain \( \ddot{\theta}(t) = \mathbf{V}^\top \dot{\mathbf{x}}(t) - \dot{\mathbf{x}}(t) = \tilde{\mathbf{F}} \left( t, \mathbf{V}^\top x(t) \right) - \tilde{\mathbf{F}} \left( t, \dot{\mathbf{x}}(t) \right) + \tilde{\mathbf{r}}(t) \). Using these expressions, the following inequality is obtained [15]:
\[ \frac{d}{dt} ||\dot{\theta}(t)|| \leq M[\tilde{\mathbf{F}}] ||\dot{\theta}(t)|| + ||\tilde{\mathbf{r}}(t)||, \] (24)
where
\[ M[\tilde{\mathbf{F}}] = \sup_{\mathbf{V}^\top x(t) \neq \dot{\mathbf{x}}(t)} \frac{\langle \dot{\theta}(t), \tilde{\mathbf{F}} \left( t, \mathbf{V}^\top x(t) \right) - \tilde{\mathbf{F}} \left( t, \dot{\mathbf{x}}(t) \right) \rangle}{||\dot{\theta}(t)||^2} \]
is the logarithmic Lipschitz constant. As \( ||\theta(t)|| = ||\dot{\theta}(t)|| \) and \( ||\dot{\theta}(0)|| = 0 \), we obtain
\[ ||\theta(t)|| \leq e^{M[\tilde{\mathbf{F}}]t} ||\theta(0)|| + \int_0^t e^{M[\tilde{\mathbf{F}}](t-\tau)} ||\tilde{\mathbf{r}}(\tau)|| d\tau = \int_0^t e^{M[\tilde{\mathbf{F}}](t-\tau)} ||\tilde{\mathbf{r}}(\tau)|| d\tau. \] (25)

Next, we rewrite \( \tilde{\mathbf{r}} \) as
\[ \tilde{\mathbf{r}}(t) = \mathbf{V}^\top \mathbf{F} \left( t, x(t) \right) - \tilde{\mathbf{F}} \left( t, \mathbf{V}^\top x(t) \right) \]
\[ = \mathbf{V}^\top \left\{ \mathbf{F} \left( t, x(t) \right) - \tilde{\mathbf{F}} \left( t, \mathbf{V}^\top x(t) \right) \right\} + \mathbf{V}^\top \tilde{\mathbf{F}} \left( t, x(t) \right) \] (26)
\[ = \mathbf{V}^\top \left\{ \mathbf{F} \left( t, x(t) \right) - \tilde{\mathbf{F}} \left( t, x(t) \right) \right\} + \mathbf{V}^\top \tilde{\mathbf{F}} \left( t, x(t) \right). \]
Note that the first term of the last line of (26) vanishes when (3) and (5) are used, i.e., the error estimations of the reduced-order systems (11) and (14) have a similar form with POD-DEIM, even though an approximate nonlinear function evaluation is not used as DEIM. Since the following relation holds,
\[ \mathbf{F} \left( t, x(t) \right) - \tilde{\mathbf{F}} \left( t, x(t) \right) = \left\{ \begin{array}{ll} - \text{diag} \left( x(t) \right) (I - \mathbf{VV}^\top) \mathbf{A}^\top (\mathbf{Ax}(t) - y) & \text{for (11)} \, , \\
- \text{diag} \left( x(t) \right) \text{diag} (1 - x(t)) (I - \mathbf{VV}^\top) \mathbf{A}^\top (\mathbf{Ax}(t) - y) & \text{for (14)} \, . \\
\end{array} \right. \]
we can put \( ||\mathbf{F} \left( t, x(t) \right) - \tilde{\mathbf{F}} \left( t, x(t) \right)|| \leq \beta \), where
\[ \beta = \left\{ \begin{array}{ll} ||\text{diag} \left( x(t) \right)|| \cdot ||\mathbf{A}^\top|| \cdot ||(\mathbf{Ax}(t) - y)|| & \text{for (11)} \, , \\
||\text{diag} \left( x(t) \right)|| \cdot ||(1 - x(t))|| \cdot ||\mathbf{A}^\top|| \cdot ||\mathbf{Ax}(t) - y|| & \text{for (14)} \, . \\
\end{array} \right. \] (27)
Note that since \( I - \mathbf{VV}^\top \) is an orthogonal projector, \( ||I - \mathbf{VV}^\top|| = 1 \), which is used to derive (27).

The Lipschitz continuity implies that \( ||\tilde{\mathbf{F}} \left( t, x(t) \right) - \tilde{\mathbf{F}} \left( t, \mathbf{VV}^\top x(t) \right)|| \leq L_f ||x(t) - \mathbf{VV}^\top x(t)|| = L_f ||p(t)|| \). Thus, from (26), we obtain
\[ ||\tilde{r}(t)|| \leq \alpha ||p(t)|| + \beta, \] (28)
where $\alpha = L_f$. For $t \in [0, T]$, we define

$$
\tilde{q}_M(t) := \int_0^t e^{2M[F](t-\tau)}d\tau = \begin{cases} 
\frac{1}{2M[F]}(e^{2M[F]t} - 1), & M[\hat{F}] \neq 0 \\
M[\hat{F}] = 0.
\end{cases}
$$

Then, the relation

$$
\|\theta(t)\|^2 \leq \tilde{q}_M(t) \int_0^t \|\rho(\tau)\|^2d\tau \leq 2\tilde{q}_M(T) \left( \alpha^2 \int_0^T \|\rho(\tau)\|^2d\tau + \beta^2T \right)
$$

holds with Cauchy-Schwarz inequality. The error metric of POD is known [16] as $\int_0^T \|\rho(t)\|^2dt = \int_0^T \|x(t) - VV^\top x(t)\|^2dt = \sum_{t=k+1}^{t} \lambda_{t}^\infty$, where $\lambda_{t}^\infty$ is related to square of singular value of the nonlinear snapshot matrix $Y$. Therefore, the accumulated error of (20) is bounded as

$$
\int_0^T \|e(t)\|^2dt = \int_0^T \|\theta(t)\|^2dt + \int_0^T \|\rho(\tau)\|^2dt \leq C(T) \left( \sum_{t=k+1}^{t} \lambda_{t}^\infty + T \right),
$$

where $C(T) = \{ 1 + 2T\tilde{q}_M(T)\alpha^2, 2T\tilde{q}_M(T)\beta^2 \}$. As a result, the error metric (30) has a similar form to that for POD-DEIM, even though discrete empirical interpolation is not used in the proposed method.

### 4.2 Discrete-time dynamical system setting

Almost model order reduction algorithms assume that original systems are analyzed with an implicit numerical integration. However, the dynamical behavior of the continuous-time image reconstruction systems cannot be efficiently analyzed by using an implicit formula, since the matrix $A^\top A$ is almost dense and the Jacobian matrix is almost also dense. Thus, an explicit numerical integration method should be used to obtain the snapshot matrix, considering sparse patterns of matrix $A$. In this subsection, we provide the error estimation on discrete-time dynamical system setting.

Using the forward Euler method, the discrete-time dynamics of the original and reduced systems are respectively expressed by

$$
\frac{x_j - x_{j-1}}{\Delta t} = F(t_{j-1}, x_{j-1}), \quad \frac{\hat{x}_j - \hat{x}_{j-1}}{\Delta t} = \hat{F}(t_{j-1}, \hat{x}_{j-1}).
$$

where $j = 1, 2, \ldots, n_t$, and $n_t = T/\Delta t$. In (31), $x_j$ and $\hat{x}_j$ are approximations of $x(t_j)$ and $\hat{x}(t_j)$ at $t_j = j\Delta t$, respectively.

The error between the original solution and one obtained via a reduced-order system is defined as

$$
e_j = x_j - V\hat{x}_j = \rho_j + \theta_j,
$$

where $\rho_j = x_j - VV^\top x_j$, $\theta_j = VV^\top x_j - V\hat{x}_j$, and $j = 0, 1, \ldots, n_t - 1$.

Define $\hat{\theta}_j := V^\top \hat{\theta}_j$. From (31), we have $V^\top(x_j - x_{j-1})/\Delta t = V^\top F(t_{j-1}, x_{j-1})$. Then, the following relations are obtained.

$$
V^\top \left( \frac{x_j - x_{j-1}}{\Delta t} \right) = \hat{F}(t_{j-1}, V^\top x_{j-1}) + \hat{r}_{j-1},
$$

$$
\frac{\hat{x}_j - \hat{x}_{j-1}}{\Delta t} = \hat{F}(t_{j-1}, \hat{x}_{j-1}),
$$

where $\hat{r}_{j-1} = V^\top F(t_{j-1}, x_{j-1}) - \hat{F}(t_{j-1}, V^\top x_{j-1})$. Then, considering $\hat{\theta}_j = V^\top x_j - \hat{x}_j$, we obtain $(\hat{\theta}_j - \hat{\theta}_{j-1})/\Delta t = \hat{F}(t_{j-1}, V^\top x_{j-1}) - \hat{F}(t_{j-1}, \hat{x}_{j-1}) + \hat{r}_{j-1}$. Thus,
\[
\frac{||\dot{\theta}_j|| - ||\dot{\theta}_{j-1}||}{\Delta t} \leq \frac{1}{\Delta t} \left( \frac{\langle \dot{\theta}_j, \dot{\theta}_j - \dot{\theta}_{j-1} \rangle}{||\dot{\theta}_j - \dot{\theta}_{j-1}||} \right) \\
= \frac{1}{||\dot{\theta}_j - \dot{\theta}_{j-1}||} \langle \dot{\theta}_j, \dot{\theta}_j - \dot{\theta}_{j-1} \rangle \\
= \frac{1}{||\dot{\theta}_j - \dot{\theta}_{j-1}||} \langle \dot{\theta}_j, \hat{F} (t_{j-1}, V^T x_{j-1}) - \hat{F} (t_{j-1}, \bar{x}_{j-1}) + \hat{r}_{j-1} \rangle \\
\leq M[\hat{F}][||\dot{\theta}_j - \dot{\theta}_{j-1}|| + ||\hat{r}_{j-1}||].
\] (35)

Using \(||\dot{\theta}_j|| = ||\dot{\theta}_j||\) and \(\zeta := 1 + \Delta t M[\hat{F}]\), the following inequality is obtained.

\[
||\dot{\theta}_j|| \leq \zeta ||\dot{\theta}_{j-1}|| + ||\hat{r}_{j-1}|| \Delta t = \zeta ||\dot{\theta}_0|| + \Delta t \sum_{i=1}^{j} \zeta^{t-1} ||\hat{r}_{j-i}|| \\
\leq \Delta t \left( q_{j-1} \sum_{i=0}^{j-1} ||\hat{r}_i||^2 \right)^{1/2},
\] (36)

where \(q_{j-1} = \sum_{i=0}^{j-1} \zeta^{2i}\).

Next, we rewrite \(\hat{r}_i\) as

\[
\hat{r}_l = V^T F (t_l, x_l) - \hat{F} (t_l, V^T x_l) \\
= V^T \left( F (t_l, x_l) - \hat{F} (t_l, V V^T x_l) \right) \\
= V^T \left( F (t_l, x_l) - \hat{F} (t_l, x_l) \right) + V^T \left( \hat{F} (t_l, x_l) - \hat{F} (t_l, V V^T x_l) \right).
\] (37)

Note that the first term of the last line of (37) vanishes when (3) and (5) are used, i.e., the error estimations of the reduced-order systems (11) and (14) have a similar form with POD-DEIM, even though an approximate nonlinear function evaluation is not used as DEIM. Since the following relation holds,

\[
F (t_l, x_l) - \hat{F} (t_l, x_l) = \begin{cases} 
- \text{diag} (x_l) \left( I - V V^T \right) A^T (Ax_l - y) & \text{for (11)}, \\
- \text{diag} (x_l) \text{diag} (I - V V^T) A^T (Ax_l - y) & \text{for (14)},
\end{cases}
\]

the norm is approximated as \(||F (t_l, x_l) - \hat{F} (t_l, x_l)|| \leq \beta\), where \(\beta\) is obtained from (27) at \(x(t) = x_l\).

Then, we have

\[
\hat{r}_l \leq \alpha ||\rho_l|| + \beta,
\] (38)

where \(\alpha = L_f\), which is a Lipschitz constant. From (36) and (38), we have

\[
||\dot{\theta}_j||^2 \leq \Delta t^2 q_{j-1} \sum_{i=0}^{j-1} ||\hat{r}_i||^2 \leq 2\Delta t^2 q \left( \alpha^2 \sum_{i=0}^{j-1} ||\rho_i||^2 + \beta^2 \sum_{i=0}^{j-1} \Delta t \right),
\] (39)

where \(q = \sum_{i=0}^{n-1} \zeta^{2i} \leq \sum_{i=0}^{\infty} \zeta^{2i} = 1/(1 - 2\Delta t M[\hat{F}] - \Delta t^2 M[\hat{F}]^2)\).

Multiplying \(\Delta t\) to both sides of (39) and summing (39) over \(j = 0, 1, \ldots, n_t - 1\), we have

\[
\sum_{j=0}^{n_t-1} ||\theta_j||^2 \Delta t \leq \sum_{j=0}^{n_t-1} 2\Delta t^2 q \left( \alpha^2 \sum_{i=0}^{j-1} ||\rho_i||^2 \Delta t + \beta^2 \sum_{i=0}^{j-1} \Delta t \right) \\
\leq \sum_{j=0}^{n_t-1} 2\Delta t^2 q \left( \alpha^2 \sum_{i=0}^{n_t-1} ||\rho_i||^2 \Delta t + \beta^2 \sum_{i=0}^{n_t-1} \Delta t \right) \\
= 2\Delta t q T \left( \alpha^2 \sum_{i=k+1}^{n} \lambda_i^\infty + \beta^2 T \right),
\] (40)
where \( T = n_t \Delta t \) and \( \sum_{j=0}^{n_t-1} \Vert \rho_j \Vert^2 \Delta t = \sum_{i=k+1}^r \lambda_i^\infty \). Therefore, the accumulated error of (32) is bounded as
\[
\sum_{j=0}^{n_t-1} \Vert e_j \Vert^2 \Delta t \leq \left( 1 + 2qT \alpha^2 \right) \sum_{l=k+1}^r \lambda_l^\infty + 2\Delta tq\beta^2 T^2 \\
\leq \bar{C}(T) \left( \sum_{l=k+1}^r \lambda_l^\infty + T \right),
\]
(41)
where \( \bar{C}(T) = \max \{ 1 + 2\Delta tqT\alpha^2, 2\Delta tqT\beta^2 \} \). Inequality (41) corresponds to (30) in the continuous-time dynamical system setting. Moreover, the error norm \( \Vert e_j \Vert \) is uniformly bounded as
\[
\Vert e_j \Vert^2 \leq \tilde{c} \left( \sum_{l=k+1}^r \lambda_l^\infty + T \right),
\]
(42)
where \( \tilde{c} = \max \{ 1/T + \tilde{q}\alpha^2, \tilde{q}\beta^2 \} \) and \( \tilde{q} = 2\Delta tq = 1/(\| M[\hat{F}] \| - \Delta t M[\hat{F}]^2/2) \).

5. Experimental results

5.1 Preliminaries for image reconstructions

We used Shepp-Logan phantom image, which imitates a skull, as shown in Fig. 1. Intensities of 0 and 1 correspond to pixel values of 0 (black) and 255 (white), respectively. The number of virtual X-ray detectors was set to 367 in accordance with the “radon” function in MATLAB®. We also set the angular range of projections to \([0, 180)\)° and the number of projection directions as 100, i.e., we assumed that an X-ray was exposed every 1.8°. In this case, \( P = 9,500 \) and \( N = 4,096 \) for the 64 \( \times \) 64 image, \( P = 36,700 \) and \( N = 16,384 \) for the 128 \( \times \) 128 image, and \( P = 36,700 \) and \( N = 65,536 \) for the 256 \( \times \) 256 image.

The computer used in our experiments had an Intel® Xeon® E5-1620 v2 CPU with 3.70 GHz and a main memory of 32.0 GB. The version of MATLAB® that we used was R2020a.

Fig. 1. Original phantom image.

5.2 Comparisons between (7) and (11)

The reduced-order system (7) obtained by conventional Galerkin procedure was compared with the modified one (11). For the 64 \( \times \) 64 image, the snapshot matrix \( \mathbf{Y} \) was produced by 100 or 50,000 steps of forward Euler method with time step size \( 1.0 \times 10^{-5} \). SVD was applied to \( \mathbf{Y}^\top \) as \( \mathbf{Y}^\top = \mathbf{U} \Sigma \mathbf{V}^\top \), where \( \mathbf{V} \in \mathbb{R}^{4,096 \times M} \). To evaluate performances of the reduced-order models, we used peak signal to noise ratio (PSNR) as follows:
\[
\text{PSNR} = 10 \log_{10} \frac{255^2 N}{\| \mathbf{x} - \mathbf{x}^* \|_2^2},
\]
where \( \mathbf{x} \) and \( \mathbf{x}^* \) are approximate and true solutions.

Table I shows PSNR values of reconstructed images obtained by (7) and (11). When 100 forward Euler steps were taken, both the reduced-order systems show almost same performance. However,
Table I. PSNR values of reconstructed images obtained by the reduced-order systems with positive constraint. (a) PSNR values obtained by (7). (b) PSNR values obtained by (11). In these tables, N implies that forward Euler method diverged.

| (a) | (b) |
|-----|-----|
| M \ steps | 100 | 50,000 | M \ steps | 100 | 50,000 |
| 1 | 12.936 | 13.396 | 1 | 12.942 | 13.397 |
| 3 | 13.198 | 15.737 | 3 | 13.200 | 15.730 |
| 5 | 13.198 | N | 5 | 13.201 | 16.797 |
| 10 | 13.199 | N | 10 | 13.200 | 16.797 |

when 50,000 steps were taken, the forward Euler method to (7) diverged. This means that the Lipschitz continuity associated with (38) to discrete-time system does not hold for (7). On the other hand, even though more steps than 50,000 were taken for (11), the dynamics never diverged. Therefore, the reduced-order system based on (11) is more robust than that on (7).

5.3 Image reconstruction results

A fixed time step size numerical integration method such as forward Euler method is useful to see theoretical aspects of reduced-order modeling as provided in Sect. 4.2. However, it is not always suitable from practical points of view. So, we used a MATLAB function ode45 [19, 20] in which the options ‘RelTol’ = 10\(^{-3}\) and ‘AbsTol’ = 10\(^{-3}\), which are respectively acceptable relative and absolute errors, were given\(^1\). Tables II and III show the reconstruction performances obtained by the reduced-order systems with positive and box constraints, respectively. In these tables, snapshot, pod, and ls imply making the snapshot matrix \(Y\), solving the reduced-order systems by ode45 again, and obtaining the least-squares solution (12) with orthonormal matrix. Moreover, ns, tran, svd, and sol imply the number of time points considered, computational time with MATLAB functions ode45 and svds, respectively, and computational time for obtaining the least-squares solution. For SVD, 15 orthonormal vectors were extracted. Further, tmax is analysis time for nonlinear dynamical systems.

The longer the nonlinear dynamics is analyzed, the higher-quality image we could obtain as shown in Tables II and III. Moreover, we show the acceleration effects of simulation with reduced-order

Table II. Image reconstruction performances obtained by reduced-order systems with positive constraint. Here, snapshot, pod, and ls imply making the snapshot matrix \(Y\), solving the reduced-order system, and obtaining the least-squares solution (12). Moreover, ns, tran, svd, and PSNR imply the number of time points considered, computational time with MATLAB functions ode45 and svds, respectively, computational time for obtaining the least-squares solution, and peak signal to noise ratio. (a) tmax = 1. (b) tmax = 10.

| (a) | (b) |
|-----|-----|
| size | snapshot | pod | ls | size | snapshot | pod | ls |
| ns | tran [s] | svd [s] | PSNR | ns | tran [s] | PSNR | sol [s] | PSNR |
| 64 | 1,257 | 4.20 | 0.28 | 35.8 | 1,273 | 0.18 | 39.6 | 0.04 | 40.9 |
| 128 | 2,481 | 34.10 | 1.95 | 34.8 | 2,489 | 1.00 | 36.3 | 0.16 | 36.9 |
| 256 | 4,969 | 338.85 | 13.77 | 32.8 | 4,977 | 5.75 | 33.2 | 0.66 | 33.4 |
| ns | tran [s] | svd [s] | PSNR | ns | tran [s] | PSNR | sol [s] | PSNR |
| 64 | 12,325 | 42.02 | 2.54 | 45.8 | 12,281 | 1.66 | 48.3 | 0.04 | 48.9 |
| 128 | 24,561 | 347.24 | 18.66 | 39.9 | 24,429 | 8.93 | 41.0 | 0.16 | 41.5 |
| 256 | 49,461 | 6,577.9 | 1,077.4 | 34.5 | 49,337 | 59.6 | 34.8 | 0.72 | 35.1 |

\(^1\) ode45 is based on Runge-Kutta (4,5) formula and the accuracy belongs to middle class. This function has variable time step size and gets fully error controlled.
Table III. Image reconstruction performances obtained by reduced-order systems with box constraint. (a) $t_{\text{max}} = 1$. (b) $t_{\text{max}} = 10$.

| size | snapshot | pod | ls |
|------|----------|-----|----|
|      | ns | tran | svd | PSNR | ns | tran | PSNR | sol | PSNR |
| 64   | 757 | 2.56 | 0.16 | 31.9 | 769 | 0.12 | 31.8 | 0.04 | 37.3 |
| 128  | 1,445 | 19.73 | 1.09 | 35.4 | 1,465 | 0.60 | 35.0 | 0.16 | 38.5 |
| 256  | 2,833 | 186.85 | 8.61 | 35.6 | 2,861 | 3.92 | 35.1 | 0.64 | 37.5 |

Table IV. Acceleration effects of solving reduced-order systems using \texttt{ode45} and obtaining least-squares solution versus solving the original systems using \texttt{ode45} to obtain snapshot matrix. (a) Positive constrained case. (b) Box constrained case.

| size | tmax = 1 | tmax = 10 |
|------|----------|-----------|
|      | snapshot/pod | snapshot/ls | snapshot/pod | snapshot/ls |
| 64   | 23.4 | 99.8 | 25.3 | 965.1 |
| 128  | 34.1 | 216.4 | 38.9 | 2,183.3 |
| 256  | 59.0 | 517.1 | 110.3 | 9,186.3 |

system and obtaining the least-squares solution to the simulation without reduction as shown in Table IV. From this table, solving the least-squares solution, we can obtain high-quality reconstructed images more efficiently than using numerical integration to the reduced-order systems. In Table IV, model order reduction is more effective for the positive constrained case than for the box constrained one. However, the computation time in the box constrained case is faster than that in the positive constrained one and the image quality in the box constrained case is almost better than that in the positive constrained one. Thus, we should use the image reconstruction system with box constraint.

For $t_{\text{max}} = 1$, we show the reconstructed images obtained by the nonlinear dynamical systems with positive or box constraint as shown in Figs. 2 and 3, respectively. We cannot recognize differences between these images visually and all images almost identify the original image shown in Fig. 1. This means that the reduced-order modeling of continuous-time image reconstruction systems is satisfactory.

6. Conclusion

Nonlinear model order reduction of continuous-time image reconstruction systems is presented based on POD, in which Galerkin procedure is modified so that the steady-state solutions of reduced-order systems are directly calculated. We provide the error estimations on continuous- and discrete-time dynamical system settings. As a result, we show that the estimations have similar forms with POD-DEIM, although discrete empirical interpolation is not used. In the numerical results, we illustrate
that the proposed method is more robust than model order reduction without modification of Galerkin procedure. Moreover, we show that the reconstructed image is efficiently obtained by solving the least-squares equation. From the reconstructed image quality and the computational efficiency, we suggest that the nonlinear dynamical system with box constraint is more preferable than that with positive constraint.

However, the proposed model order reduction is input specific, because this method is based on POD. To provide high-quality reconstructed images to many types of input images, deep learning strategy [17] may be necessary. We will introduce this concept to our work.

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