ASSET PRICING IN AN IMPERFECT WORLD

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Abstract. In a model with no given probability measure, we consider asset pricing in the presence of frictions and other imperfections and characterize the property of coherent pricing, a notion related to (but much weaker than) the no arbitrage property. We show that prices are coherent if and only if the set of pricing measures is non-empty, i.e. if pricing by expectation is possible. We then obtain a decomposition of coherent prices highlighting the role of bubbles. Eventually we show that under very weak conditions the coherent pricing of options allows for a very clear representation which allows, as in Breeden and Litzenberger [7], to extract the implied probability.

1. Introduction

In this paper we study asset pricing for economies in which trading is prone to a wide variety of restrictions and costs and in which, in addition, agents need not possess the degree of sophistication required in order to assess uncertain outcomes via a probability function – no matter how general we interpret this concept. Yet we obtain an exact relationship between prices and integrals and a representation of option prices in terms of the implied probability.

The assumption of an exogenously given probability contributes substantially to traditional financial models. In the first place, it sets the ambient space with quantities being defined up to null sets rather than as a detailed list of characteristics, as in Arrow’s notion of a contingent good. Second, in continuous time working with null sets is essential for a mathematically sound definition of the process of gains from trade and, more generally, for the construction of a rich enough set of admissible trading strategies so as to guarantee the opportunity of hedging many different derivatives. But even more importantly, the assumption of a given probability is crucial for the core principle of modern financial theory, i.e. risk neutral pricing. Although many an author inclines to believe that this basic principle rests on the simple tenet asserting that markets populated by rational economic agents cannot admit arbitrage opportunities, the proof of this claim, the fundamental theorem of asset pricing, has long been a challenge for mathematical economists, from Kreps [26] to Delbaen and Schachermayer [14]. In fact it requires a much more stringent condition than absence of arbitrage in which probability is needed to induce an appropriate topology.

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A convincing amount of empirical and experimental observations seems, however, to document attitudes of investors towards uncertainty which are deeply at odds with probabilistic rationality. Moreover, some frequently observed facts, as reported e.g. by Lamont and Thaler in [27] and [28], or as embodied in the high ratio of option prices violating some basic no arbitrage condition, are often hard to reconcile with classical financial theory. Eventually, the tradition of subjective probability suggests that probabilities should be regarded more as the outcome of choice than as an input to it. We take these considerations as the starting motivation for our choice in favor of an asset pricing model with no probability assumption, a step first made in [8] and then by a few papers which include [34] and [39].

Once this crucial choice is made a natural possibility to explore is whether it is possible to retain the full fledged mathematical power of traditional models by detecting some reference probability that may employed with no additional assumption and without affecting the underlying economic model. This is clearly the case with a countable state space, in which any strictly positive probability may be introduced at no modeling cost. With a general state space, however, this is hardly possible as any probability would introduce many more null sets than the empty one, thus distorting the original financial model. Following this path it is then unavoidable to replace the assumption of a given probability with one guaranteeing some countable structure of the state space, as done in [34] where the state space is a complete separable metric space and assets payoff are taken to be continuous.

We take here a radically different and fully general approach with no mathematical assumption, neither topological nor measure theoretic, to start with. Our \textit{a priori} is rather a partial order providing a qualitative description of how uncertain outcomes are ranked by agents. We propose a reasonable set of axioms which cover several possible situations of interest, including the classical model which embeds thus in our framework as a particularly interesting special case, another one being that introduced in [8] in which agents base their choice on a class of negligible events. We show in Theorem 1 that a partial order satisfying our axioms may also be interpreted in terms of a corresponding coherent risk measure.

Another distinguishing feature of this paper is a rather general description of the market mechanism. Not only we consider bid/ask spreads and fixed trading costs but we also allow for several additional restrictions to trade. First, agents may be prevented from forming portfolios of arbitrarily large size – it may thus not be possible to run money pumps, in case they exist. Arbitrage phenomena have as a consequence a minor impact on economic equilibrium and the no arbitrage principle loses part of its appeal. Second, not all portfolios may be shorted, and not just as a result of trading restrictions. In the presence of credit risk, taking long or short positions should be considered as two separate investments, given that the implicit level of risk depends on the reliability of the investor playing the short side. Third, we do not identify assets with their payoff.

\footnote{See [8] and references therein for a brief survey of the experimental psychology literature and its relation to finance.}

\footnote{See the paper of Vovk [39] for a general introduction to this topic and for further references.}
so that it is possible to have two assets promising the same payoff at different prices. Fourth, the opportunity to invest in some asset may be available only if combined with other assets, e.g. with some collateral. The impact of margins when shorting options, an empirical fact accommodated in our model, has rarely been considered in asset pricing. Eventually, we don’t assume the existence of a riskless asset but rather of a numéraire whose non negative payoff is used as the discount factor.

In contrast with the basic principles of neoclassical economics, the choice of the numéraire, given the absence of a reference probability, is non neutral and an arbitrage opportunity arising with a given numéraire may no longer be such with a different discount factor.

In the framework outlined above we discuss three distinct notions of market rationality: coherence, efficiency and absence of arbitrage opportunities. We obtain in Theorem 3 a characterization of coherent prices in terms of a set of pricing measures. Our result is near in spirit to that obtained in the pioneering work of Jouini and Kallal [23] (in an \( L^2(\mathbb{P}) \) setting) but departs from it in several ways. First, pricing measures do not apply but to claims with limited discounted losses; second, pricing measures are just finitely additive. We obtain in Theorem 4 exact necessary and sufficient conditions for countable additivity which justify the conclusion that this additional property should be regarded more as a mathematical artifact than an economic implication. In Theorem 5 we decompose coherent prices highlighting the role of bubbles. It is also possible to represent pricing measures with a capacity, similarly to what assumed in the work of Chateauneuf et al. [11]. The connection with capacities is also explored in Cerreia-Vio-glio et al. [10].

Of course, over the years several authors have investigated restrictions to trading similar to those considered here. Leaving aside the microstructure literature, in which transaction costs are the heart of the matter, the first papers have been those of Bensaid et al. [3] and, most of all, of Jouini and Kallal [23]. More recent papers include Bouchard [6], Napp [32] (who first models trading restrictions via closed convex cones), Jouini and Napp [24] (who describe investments as cash flows with convex cone constraints and assume no numéraire), Kabanov and Stricker [25] (who consider very general forms of costs) and Schachermayer [36]. With no claim to completeness, one should also mention the work of Amihud and Mendelson [1], Prisman [33] and, most recently, Roux [35].

This paper is structured as follows. In section 2 we describe markets and introduce the partial order needed to rank uncertain outcomes. In section 3 we discuss the properties of coherence, efficiency and of absence of arbitrage and, in the following section 4 we obtain an explicit characterization of coherent prices in terms of pricing measures. In section 5 we develop a decomposition of coherent asset prices emphasizing the existence of bubbles. We then consider option markets in section 6 where we prove a general representation for prices of convex derivatives, involving bubbles and an implicit pricing measure. Some indications for applied work are also given. Auxiliary results and some of the proofs will be found in the Appendix.

1.1. **Notation.** Throughout the paper we adopt the following mathematical symbols and conventions. \( \mathbb{F}(X) \) denotes the collection of real-valued functions on some space \( X \) and \( \mathbb{F}_0(X) \) designates
those \( f \in \mathfrak{F}(X) \) whose support \( \{ x \in X : f(x) \neq 0 \} \) is a finite set. If \( f \in \mathfrak{F}(X) \) and if \(-X \subset X\), the symbol \( f^c \) will be used to denote the conjugate of \( f \) defined as \( f^c(x) = -f(-x) \quad x \in X \). We set conventionally \( 0/0 = 0, \sup \emptyset = -\infty, \inf \emptyset = \infty \) and \( \sum \emptyset = 0 \). \( \mathbb{R} \) denotes the extended real numbers.

When a given set \( \Omega \) (resp. a family \( \mathcal{A} \) of subsets of some set \( \Omega \)) is given, \( ba \) (resp. \( ba(\mathcal{A}) \)) denotes the family of bounded, finitely additive set functions defined on all subsets of \( \Omega \) (resp. on all sets in \( \mathcal{A} \)). \( \mathcal{F}(\mathcal{A}) \) denotes the family of \( \mathcal{A} \) simple functions. When \( \mu \in ba \) and \( f \in \mathfrak{F}(\Omega) \) we define its integral as

\[
\int f \, d\mu = \lim_n \int [(f^+ \wedge n) - (f^- \wedge n)] \, d\mu
\]

if such limit exists in \( \mathbb{R} \), or else \( \int f \, d\mu = \infty \). It is easily seen that this definition coincides with [17, III.2.17] whenever \( f \) is \( \mu \) integrable, i.e. \( f \in L^1(\mu) \).

2. Markets, Prices, Investors

Assets traded on the market are identified with a “ticker”, \( \alpha \in \mathfrak{A} \), and are associated with a corresponding payoff, \( X(\alpha) \). The latter is modeled simply as a function on some given space \( \Omega \), i.e. an element of \( \mathfrak{F}(\Omega) \). As discussed in the Introduction, no mathematical structure is imposed on the set of traded payoffs. Although it is natural to interpret \( \Omega \) as the sample space and \( X(\alpha) \) as a random quantity (which makes our model look intrinsically static) we may as well choose \( \Omega = S \times \mathbb{R}_+ \), with \( S \) the sample space and \( \mathbb{R}_+ \) the time domain – and thus give to our construction a full fledged dynamical structure.

2.1. Trading Strategies. Investors trade claims by taking a finite number of either long or short positions, in respect of the restrictions imposed by the market. A trading rule is then just an element of the space \( \mathfrak{F}_0(\mathfrak{A}) \). The trading rule which consists solely of one unit of the claim \( \alpha \in \mathfrak{A} \) will be denoted by \( \delta_\alpha \). To each trading rule \( \theta \) corresponds the final gain

\[
X(\theta) = \sum_{\alpha \in \mathfrak{A}} \theta(\alpha) X(\alpha)
\]

Of course, \( X(\delta_\alpha) = X(\alpha) \).

Inspired by real markets, one may imagine several restrictions to asset trading, further to the constraints of respecting the balance of budget and of forming finite portfolios. These include short selling prohibitions or margin requirements and others that ultimately aim at enforcing some form of bound to losses. The symbol \( \Theta \), that denotes hereafter the set of all admissible trades, specifies all restrictions to asset trading. We assume the following:

**Assumption 1.** \( \Theta \) is a convex subset of \( \mathfrak{F}_0(\mathfrak{A}) \) containing the origin.

Under Assumption 1 investors need not be permitted to take positions of either sign, long or short. This is consistent with the restriction to losses recalled above. Moreover, investors may encounter restrictions in the choice of the scale of the investment. On this point we depart significantly from
much of the literature on asset pricing with or without transaction costs, see e.g. [23], [25] or [31].

A possible relaxation of this restriction (on which we shall return) is to allow \( \lambda \theta \in \Theta \) for all \( \lambda \geq 0 \) whenever \( \theta \in \Theta \) satisfies \( X(\theta) \geq 0 \). A major implication of this is that arbitrage opportunities, when available, may not have a disruptive impact on market equilibrium and the no arbitrage principle, as a consequence, may no longer be crucial. Eventually, we do not require that \( \delta \alpha \in \Theta \), i.e. that each asset may be traded individually due, e.g., to the requirement of putting up margins when taking positions on derivative markets.

2.2. Prices and Costs. For each \( \alpha \in \mathfrak{A} \) we denote by \( q^a(\alpha) \) and \( q^b(\alpha) \) its ask and bid price respectively.

**Assumption 2.** The functions \( q^a, q^b \in \mathfrak{F}(\mathfrak{A}) \) are such that \( q^a(\alpha) \geq q^b(\alpha) \) for all \( \alpha \in \mathfrak{A} \).

We highlight that in our model financial prices are not defined as functions of the asset payoff but depend rather of its name. This apparently innocuous detail makes our approach compatible with some pricing anomalies reporting that the trading of one same asset at different market locations or simply under different names, may produce different prices (see the examples on close-end funds or of twin stocks reported in Lamont and Thaler [28]).

In order to form a given trading strategy \( \theta \in \Theta \) an investor pays an ask price for each long position and earns a bid price for each short one. The corresponding cost amounts thus to

\[
q(\theta) = \sum_{\alpha \in \mathfrak{A}} \left[ \theta(\alpha)^+ q^a(\alpha) - \theta(\alpha)^- q^b(\alpha) \right] \quad \theta \in \Theta
\]

It is clear that \( q(\delta \alpha) = q^a(\alpha) \) and, if \( -\delta \alpha \in \Theta \), that \( q^c(\delta \alpha) = -q(-\delta \alpha) = q^b(\alpha) \).

The basic assumption behind (3) is that each position in a portfolio is priced separately. This is in accordance with trading anonymity prevailing in specialist markets but is perhaps not an adequate description of OTC trading. Options markets, on which we shall focus in the last sections, are quite well represented by (3) at least for orders which fall below the size limits of the market maker. Large orders, instead, are in general processed on a separate track and the price is set *ad hoc*.

Market frictions include, further to the bid/ask spread, also some fixed costs, such as brokerage fees, in the form of a sunk payment due to have access to the market. Given that on each market investors may trade more than one asset, we may thus think of markets as a partition \( \mathfrak{M} \) of subsets of \( \mathfrak{A} \) and for each \( M \in \mathfrak{M} \) we designate by \( c(M) \geq 0 \) the corresponding fixed cost. For example, all options on a given underlying are traded on the same market, independently of the strike or maturity so that any option strategy will involve the same fees. Thus the fixed cost associated with an investment strategy is

\[
c(\theta) = \sum_{\{ M \in \mathfrak{M} : \sup_{\alpha \in M} |\theta(\alpha)| > 0 \}} c(M) \quad \theta \in \Theta
\]

A realistic modeling of fixed trading costs turns out to be quite difficult due to their extremely various nature. (4) is just one possible model.

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3 As a matter of fact transaction fees tend to be stepwise increasing with the order size rather then fixed.
The total cost associated with a trading strategy $\theta$ amounts to

$$t(\theta) = q(\theta) + c(\theta) \quad \theta \in \Theta$$

It is easy to deduce from (3) and (5) some elementary properties:

**Lemma 1.** The functional $q \in F(\Theta)$ defined in (3) is (i) positively homogeneous, (ii) subadditive and (iii) such that

$$q(f + g) = q(f) + q(g) \quad \text{for all } f, g \in \Theta \text{ with } fg \geq 0$$

The functional $t \in F(\Theta)$ defined in (5) is subadditive and satisfies $t(0) = 0$.

In many a paper on asset pricing with frictions, starting with the seminal paper of Jouini and Kallal [23], subadditivity is the only distinguishing feature characterizing the existence of bid/ask spreads. Another paper following this choice is that of Luttmer [31]. In Chateauneuf et al. [11], the pricing functional is represented via a capacity and is then not only subadditive but even comonotonic, a property somehow akin to (6). In these papers an explicit description of the costs of trading is omitted (a remarkable exception is [25]) and in so doing, we claim, one risks to miss important details of the price mechanism and to mix up effects that may actually originate from different sources, e.g. bounded rationality or restrictions to market participation. We will refer to property (6) as anonymity and, although in the following Theorems 3 through 5 this plays no role, it will be important when dealing with option prices for which, we believe, it is perfectly adequate.

2.3. **The Numéraire Asset.** Financial models (with the noteworthy exception of [24]) commonly assume the existence of a riskless asset, often interpreted as a bond, that may equally well serve the purpose of borrowing or lending. This assumption plays three distinct roles. First, it enables agents to move wealth back and forth in time in a safe way and thus provides a firm basis to define the present value of future wealth. Second, it allows to identify explicitly the numéraire of the economy, removing the arbitrariness that arises whenever several assets may play that same role. Third, if the investment in the bond is unrestricted then this asset plays a residual role in portfolio models, guaranteeing the effectiveness of portfolio constraints.

This assumption is however not only strongly counterfactual but more troublesome than it appears at first sight. First, if the bond is not fully free of risk but just evolves in a predictable way (as is often the case in continuous time models) then the role of the discounting asset is no longer neutral as its implicit risk entwines with the one originating from the underlying asset. El Karoui and Ravanelli [18] discuss this point at length and show that risk measures are affected by discounting in a significant way, when the discount factor is risky. Moreover, in equilibrium models, such as those considering the role of noise trading (see [15] or [37]), the riskless nature of the bond may not survive Walras law unless its elasticity of supply is infinite. Eventually if investors are prone to credit risk one should consider borrowing and lending as two different financial contracts, given that the final payoff is ultimately a function of the reliability of the two intervening parts.

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4See the criticisms to this assumption made by Lowenstein and Willard [30].
We summarize the preceding discussion in the following:

**Assumption 3.** There exists \( \alpha_0 \in \mathfrak{A} \) such that (i) \( 1 \geq X(\alpha_0) > 0 \), (ii) if \( \theta \in \Theta \) then \( \theta + \lambda \delta_{\alpha_0} \in \Theta \) if and only if \( \lambda \geq 0 \), (iii) \( c(\delta_{\alpha_0}) = 0 \), (iv) \( q(\alpha_0) > 0 \).

We shall refer to \( \alpha_0 \) as the *numéraire* asset and simplify \( \delta_{\alpha_0} \), \( X(\alpha_0) \) and \( q(\alpha_0) \) as \( \delta_0 \), \( X_0 \) and \( q_0 \).

Assumption 3 (i) is fairly general to allow for virtually all sorts of dynamics but it excludes the occurrence of default with no recovery value. If one visualizes the *numéraire* asset as government bonds one may perhaps consider this restriction not too far from reality, given that in market economies government bonds have always been redeemed at some positive value. Alternatively, identifying the *numéraire* asset with the bank account one may argue, likewise, that the bank account is in most countries assisted by some form of deposit insurance, maybe just in the form of the role of lender of last resort played by the Central Bank. As in the real world, in our model investors are unrestricted in deciding the amount to invest in the *numéraire* asset, but they cannot take negative positions as this would more appropriately be considered a different asset, as argued above. Property (iii) may perhaps be seen as the outcome of competition among banks, while (iv) justifies referring to \( \alpha_0 \) as the *numéraire*.

We define normalized payoffs as

\[
\bar{X}(\theta) = \frac{X(\theta)}{X_0}
\]

2.4. **Stochastic ordering.** An agent’s decision to invest in a given trading strategy \( \theta \in \Theta \) is motivated, we assume, by the payoff \( X(\theta) \) that it generates. However, a full description of this quantity for each possible future state \( \omega \in \Omega \) is not necessarily a correct model of choice as economic agents often do not regard future outcomes as *functions* but rather as *equivalence classes*. This is clearly the case in expected utility theory and, more generally, in all probability models. Equivalence, however, is not only the outcome of an accurate probabilistic assessment – as the classical model implicitly suggests – but it often emerges from the inability of individuals to fully compare events or from their attitude to focus attention on scenarios selectively, a fact often documented in empirical decision theory and experimental psychology.

These remarks suggest to treat stochastic order as an explicit a priori of our model and to model it via a binary relation \( \geq_* \) on \( \mathfrak{F}(\Omega) \). We assume to this end:

**Assumption 4.** The binary relation \( \geq_* \) on \( \mathfrak{F}(\Omega) \) is reflexive, transitive and satisfies:

- (TRIV) \( 0 \geq_* 1 \);
- (CONE) \( f_i \geq g_i \) and \( a_i \in \mathbb{R}_+ \) for \( i = 1, 2 \), imply \( a_1 f_1 + a_2 f_2 \geq_* a_1 g_1 + a_2 g_2 \);
- (CERT) \( f \geq 0 \) implies \( f \geq_* 0 \);
- (APPR) \( f + 2^{-n} \geq_* 0 \) for \( n = 1, 2, \ldots \) implies \( f \geq_* 0 \);
- (REST) \( f \geq_* 0 \) and \( A \subset \Omega \) imply \( f1_A \geq_* 0 \).

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5 We reserve the notation \( f \geq g \), \( f \lor g \), \( f \land g \) or \(|f|\) to pointwise ordering.
(TRIV) prevents trivial situations in which all bounded functions are equivalent to 0. (CONE) guarantees that non negative elements define a convex cone, a crucial property when forming portfolios. (CERT) states that the order does not contradict certainty; (APPR) establishes, more interestingly, that a quantity that may be approximated by one considered as non negative by an amount which is arbitrarily small for all practical purposes should itself be considered as non negative. Eventually, (REST) implies that non negativity is a global assessment and is preserved when passing to subsets.

We also write \( f >_* g \) (resp. \( f =_* g \)) when \( f \geq_* g \) but \( g \nleq_* f \) (resp. \( g \geq_* f \)) and define

\[
(8) 
 f_* = \sup\{\alpha \in \mathbb{R} : f \geq_* \alpha\} \quad \text{and} \quad f^* = -(-f)_*
\]

A partial order on \( \mathcal{F}(\Omega) \) satisfying Assumption 4 will be referred to as a regular stochastic order. The following are examples such order.

**Example 1** (Certainty and probability). Define \( f \succeq_0 g \) and \( f \succeq_P g \) to mean \( f \geq g \) and \( P(f \geq g) = 1 \) if, given a (countably additive) probability \( P \). Both \( \succeq_0 \) and \( \succeq_P \) are regular stochastic orders, the first often being referred to as zero-th order stochastic dominance. First order stochastic dominance is not a regular stochastic order as it fails to satisfy (CONV). Our setting therefore covers the case of certainty and probabilistic sophistication.

**Example 2** (Qualitative probability). In his pioneering work, de Finetti [13] introduced the idea of modeling the qualitative judgment “event A is more likely than B” as a binary relation, \( A \succeq B \) satisfying the axioms: (a) completeness, (b) transitivity, (c) \( \Omega \succeq A \succeq \emptyset \) for all events \( A \) and (d) if \( C \cap (A \cup B) = \emptyset \) then \( A \succeq B \) if and only if \( A \cup C \succeq B \cup C \). One may define

\[
(9) \quad f \succeq_{d F} g \quad \text{if and only if} \quad \{f - g \leq -\eta\} \succeq \emptyset \quad \text{for all} \quad \eta > 0
\]

It is easily seen that, adding the axiom (e) \( \emptyset \nsubseteq \Omega \), then \( \succeq_{d F} \) is a regular stochastic order exactly because the collection \( \{A \subset \Omega : A \succeq \emptyset\} \) is an ideal of subsets of \( \Omega \). A generalization of this idea was introduced in [8] where an ideal \( \mathcal{N} \) of so-called negligible events (not including \( \Omega \)) was taken as a primitive and a corresponding order \( \succeq_{\mathcal{N}} \) was defined as in (9), i.e.

\[
(10) \quad f \succeq_{\mathcal{N}} g \quad \text{if and only if} \quad \{f - g \leq -\eta\} \in \mathcal{N} \quad \text{for all} \quad \eta > 0
\]

Regular stochastic orders induced by ideals of sets can be characterized as follows:

**Lemma 2.** A regular stochastic order \( \succeq \) is induced by an ideal of subsets of \( \Omega \) if and only if there exists a weak* compact set \( \mathbb{P}_\succeq \subset \mathbb{P} \) such that \( \sup\{\mu(A) : \mu \in \mathbb{P}_\succeq\} \in \{0, 1\} \) for all \( A \subset \Omega \) and

\[
(11) \quad f \succeq g \quad \text{is equivalent to} \quad \inf_{\mu \in \mathbb{P}_\succeq} \int (f - g) d\mu \geq 0
\]

**Proof.** Assume that \( \succeq \) is a regular stochastic order induced by an ideal \( \mathcal{N} \) and let

\[
(12) \quad \mathbb{P}_\succeq = \{\mu \in \mathbb{P} : \mu(A) = 0 \text{ for all } A \in \mathcal{N}\}
\]
Suppose that \( \mu(A) > 0 \) for some \( A \subset \Omega \) and \( \mu \in \mathbb{P}_\geq \). Then \( \mu_A \), the conditioning of \( \mu \) to \( A \), is again an element of \( \mathbb{P}_\geq \) and \( \mu_A(A) = 1 \). Thus \( \sup \{ \mu(A) : \mu \in \mathbb{P}_\geq \} \in \{0,1\} \) for all \( A \subset \Omega \). Take \( A \notin \mathcal{N} \).

The set \( \{ a1_A : a \geq 0 \} \) is a convex cone and \( a1_A \geq 1 \) would imply \( 0 \geq 1_A \), a contradiction. By Theorem 8 there exists \( \mu \in \mathbb{P}_\geq \) such that \( \mu(A^c) = 0 \). This proves that \( \sup \{ \mu(A) : \mu \in \mathbb{P}_\geq \} = 1 \) if and only if \( A \notin \mathcal{N} \). Suppose that \( f \geq 0 \). Fix \( \eta > 0 \) and observe that \( -\eta 1_{\{ f < -\eta \}} \geq f 1_{\{ f < -\eta \}} \geq 0 \), by (CERT) and (REST). It follows from (CONV) that \( \{ f < -\eta \} \in \mathcal{N} \) so that

\[
\inf_{\mu \in \mathbb{P}_\geq} \int f d\mu = \inf_{\mu \in \mathbb{P}_\geq} \int_{\{ f < -\eta \}} f d\mu \geq -\eta
\]

and thus that \( \inf_{\mu \in \mathbb{P}_\geq} \int f d\mu \geq 0 \). Conversely, let \( \inf_{\mu \in \mathbb{P}_\geq} \int f d\mu \geq 0 \). If \( \sup_{\mu \in \mathbb{P}_\geq} \mu(f < -\eta) = 1 \) for some \( \eta > 0 \), then, by weak* compactness there would exist \( \mu_0 \in \mathbb{P}_\geq \) such that \( \mu_0(f < -\eta) = 1 \) and so \( \int d\mu_0 \leq -\eta \), a contradiction. Thus necessarily \( \sup_{\mu \in \mathbb{P}_\geq} \mu(f < -\eta) = 0 \), i.e. \( \{ f < -\eta \} \in \mathcal{N} \) so that \( f \geq 0 \). Eventually, if \( \mathbb{P}_\geq \) has the above properties and (11) is taken as a definition of \( \geq \), define \( \mathcal{N} = \{ A \subset \Omega : \sup_{\mu \in \mathbb{P}_\geq} \mu(A) = 0 \} \). It is easily seen that \( \mathcal{N} \) is an ideal of subsets of \( \Omega \) and that \( \Omega \notin \mathcal{N} \). Moreover, if \( \inf_{\mu \in \mathbb{P}_\geq} \int (f - g) d\mu \geq 0 \) it must be that \( \sup_{\mu \in \mathbb{P}_\geq} \mu(f - g \leq -\eta) = 0 \) for all \( \eta > 0 \) and so \( \geq \) is indeed a regular stochastic order induced by an ideal.

Despite the natural interpretation of \( \geq \) as an element of subjective choice, it may well be interpreted in more applied terms. Let us recall [18 Definition 3.1] that a map \( \rho : \mathfrak{F}(\Omega) \to \mathbb{R} \) is a coherent, cash subadditive risk measure whenever \( \rho \) is (a) positively homogeneous, (b) subadditive, (c) inversely monotone (i.e. \( f \leq g \) implies \( \rho(f) \geq \rho(g) \)) and (d) cash-subadditive, (i.e. \( \rho(f + \alpha) \geq \rho(f) - \alpha \) when \( \alpha \in \mathbb{R}_+ \)). If, in addition, \( \rho \) satisfies (e) \( \rho(f) = \rho(f \land 0) \), then we refer to it as a loss measure. It may be easily proved that if \( \rho \) is a coherent, cash subadditive risk measure, then \( \hat{\rho}(f) = \rho(f \land 0) \) is a coherent, cash subadditive loss measure.

**Theorem 1.** \( \geq \) is a regular stochastic order if and only if there exists a coherent, cash subadditive loss measure \( \rho \) such that \( \rho(-1) > 0 \) and

\[
(13) \quad f \geq g \quad \text{if and only if} \quad \rho(f - g) \leq 0 \quad \text{for all} \quad f, g \in \mathfrak{F}(\Omega)
\]

**Proof.** Let \( \geq \) be a regular stochastic order. Define

\[
(14) \quad \sigma_{\geq}(f) = \inf \{ \beta \in \mathbb{R} : (f \land 0) + \beta \geq 0 \}
\]

\( \sigma_{\geq} \) is subadditive and positively homogeneous by (CONE) and inversely monotone by (CERT). \( \sigma_{\geq}(-1) > 0 \) is obvious given (TRIV). To prove cash subadditivity we may restrict to the case of \( f \in \mathfrak{F}(\Omega) \) and \( a > 0 \) such that \( \sigma_{\geq}(f + a) < \infty \) i.e. such that \( ((f + a) \land 0) + \beta \geq 0 \) for some \( \beta \in \mathbb{R} \).

But then

\[
f \land 0 + (a + \beta) \geq f \land 0 + a + (a + \beta)
\]

so that, by (CERT), \( a + \beta \geq \sigma_{\geq}(f) \) i.e. \( \sigma_{\geq}(f + a) \geq \sigma_{\geq}(f) - a \). To see that \( \geq \) is related with \( \sigma_{\geq} \) via (13), observe that \( \sigma_{\geq}(f) \leq 0 \) is equivalent to \( (f \land 0) + 2^{-n} \geq 0 \) i.e., by (APPR), to \( (f \land 0) \geq 0 \) and this in turn to \( f \geq 0 \) by (CERT) and (REST).
Conversely, assume that $\rho$ is a coherent, cash subadditive loss measure and define $\succeq$ via (13).

Suppose that $f_i \succeq g_i$ and that $a_i \in \mathbb{R}_+$ for $i = 1, 2$. Then,

$$\rho(a_1(f_1 - g_1) + a_2(f_2 - g_2)) \leq a_1\rho(f_1 - g_1) + a_2\rho(f_2 - g_2) \leq 0$$

so that $a_1f_1 + a_2f_2 \succeq a_1g_1 + a_2g_2$ and (CONE) holds. (CERT) is clear while (REST) follows from

$$\rho(f \mathbf{1}_A) = \rho(\mathbf{1}_A(f \land 0)) \leq \rho(f \land 0) = \rho(f)$$

Eventually, by $\rho(f + 2^{-n}) \geq \rho(f) - 2^{-n}$ we conclude that $f + 2^{-n} \succeq 0$ for all $n$ implies $f \succeq 0$. □

We will write

$$\mathcal{N}_* = \{A \subset \Omega : 0 \geq_* \mathbf{1}_A\}, \quad \mathfrak{B}_* = \{f \in \mathfrak{F}(\Omega) : \eta \geq_* |f| \text{ for some } \eta > 0\}$$

(15)

and, when $\mathcal{A}$ is an algebra of subsets of $\Omega$ containing $\mathcal{N}_*$,

$$(16) \quad \mathbb{P}_* = \{\mu \in \mathbb{P} : \mu(A) = 0 \text{ for all } A \in \mathcal{N}_*\}$$

(16)

A noteworthy property of the regular stochastic order $\geq_*$ is the following:

$$(18) \quad f \geq_* 0 \text{ and } b \in \mathfrak{B}_+ \quad \text{ imply } \quad fb \geq_* 0$$

a fact that follows from (REST) when $b$ is simple and extends to the more general case by (APPR) and uniform convergence.

Property (18) and Assumption 3(i) have an interesting economic implication, namely that $\bar{X} \geq_* 0$ implies $X \geq_* 0$ but the converse need not be true. This special feature of our model highlights the role of discounting in the overall level of risk. The statement $X \geq_* 0$, in fact, does not exclude losses but rather implies $\{X < -\eta\} \in \mathcal{N}_*$ for every $\eta > 0$, i.e. that losses may be considered as arbitrarily small. The statement $\bar{X} \geq_* 0$ means, on the other hand, that losses from $X$ may be hedged by investing an arbitrarily small amount in the numé raire asset. If the numé raire does not guarantee a minimum payoff, the losses associated with $\theta$, although small, may require a potentially unbounded amount of such asset in order to be hedged. The problem arises whenever losses from a portfolio occur jointly with a drop in the value for $X_0$, as is often the case during financial crises. The risk management aspects of the choice of the numé raire are also discussed by El-Karoui and Ravanelli [18] and Filipovic [19].

3. Coherence, Efficiency and Arbitrage

The basis of financial economics is the tenet that markets populated by rational agents do not permit arbitrage opportunities. However, if there is agreement on this general statement, its translation into a convenient mathematical notion is much less uncontroversial. Definitions vary from one another mainly for the ambient space adopted and, since Harrison and Kreps [20], the
choice has traditionally been some $L^p(P)$ space, for a given exogenous probability measure $P$. We rather propose here the following definitions:

**Definition 1.** A functional $\phi \in \mathcal{F}(\Theta)$ is said to be coherent (with the no arbitrage principle) if $\bar{X}(\theta) \geq 0$ implies $\phi(\theta) \geq 0$; $\phi$ is said to be efficient if $\theta, \theta' \in \Theta$ and $\bar{X}(\theta) \geq \bar{X}(\theta')$ imply $\phi(\theta) \geq \phi(\theta')$. Moreover, $\theta \in \Theta$ is an arbitrage opportunity for $\phi$ if

\begin{equation}
\bar{X}(\theta) \geq 0 \quad \text{but} \quad \phi(\theta) \leq 0
\end{equation}

and at least one of the two inequalities is strict.

Although with a linear pricing rule and no restriction to short selling, coherence and efficiency are equivalent properties, in the more general case treated here coherence is weaker than efficiency nor does it guarantee absence of arbitrage per se. Coherent pricing does not exclude that an investment which yields a strictly positive (discounted) payoff is sold for free. Coherence is thus a rather weak and basic financial property and we shall investigate it in depth.

The inequality $\bar{X}(\theta) \geq 0$ may be rephrased in terms of the minimal margin $M_*(\theta)$ to be invested in the numéraire asset in order to hedge losses away (if possible). Formally,

\begin{equation}
M_*(\theta) = \inf \{ \eta > 0 : \bar{X}(\theta + \eta \delta_0) \geq 0 \} = -(\bar{X}(\theta)_* \wedge 0)
\end{equation}

By Assumption 3(ii), $\theta + M_*(\theta) \delta_0 \in \Theta$ if and only if $M_*(\theta) < \infty$ or, equivalently, if $\theta$ belongs to the set of hedgeable strategies

\begin{equation}
\Theta_* = \{ \theta : \bar{X}(\theta)_* > -\infty \}
\end{equation}

In fact regulated markets do not allow investors to enter positions with unlimited potential losses so that $\Theta_*$ is often considered as the set of all reasonable investment strategies – see [14] where a condition akin to $\theta \in \Theta_*$ is the basis for the concept of free lunch with vanishing risk.

Observe that $M_*(\theta + \alpha \delta_0) \geq M_*(\theta) - \alpha$ although the basic intuition used by El-Karoui and Ravanelli to justify cash subadditivity (namely that the discount factor is less than 1, see [18, p. 568]) does not apply as we do not impose $X_0 \geq 1$. Moreover, $M_*(\theta + M_*(\theta) \delta_0) = 0$.

**Theorem 2.** Let $t$ be the total cost functional $t$ defined in (5). Then,

(i) $t$ is coherent if and only if

\begin{equation}
t(\theta) + q_0 M_*(\theta) \geq 0 \quad \text{for all} \quad \theta \in \Theta_*
\end{equation}

(ii) if $t$ is convex then it is coherent if and only if there exists $\mu \in \mathcal{P}_*$ such that

\begin{equation}
t(\theta) \geq q_0 \int (\bar{X}(\theta) \wedge 0) \, d\mu \quad \text{for all} \quad \theta \in \Theta_*
\end{equation}

(iii) $t$ admits no arbitrage opportunity if and only if it satisfies

\begin{equation}
t(\theta) + q_0 M_*(\theta) > 0 \quad \text{for all} \quad \theta \in \Theta_* \text{ such that } \bar{X}(\theta)_* + M_*(\theta) > 0
\end{equation}
Proof. In studying coherence we are obviously entitled to restrict attention to \( \Theta_\ast \). For each \( \theta \in \Theta_\ast \), let \( \theta' = \theta + M_\ast(\theta)\delta_0 \in \Theta \). By Theorem 1 and Assumption 3(iii),
\[
(25) \quad \bar{X}(\theta') = \bar{X}(\theta) + M_\ast(\theta) \geq \ast \quad \text{and} \quad t(\theta') = t(\theta) + q_0M_\ast(\theta) \quad \bar{X}(\theta')^* = \bar{X}(\theta)^* + M_\ast(\theta)
\]

(i). If \( t \) is coherent then, \( 0 \leq t(\theta') = t(\theta) + M_\ast(\theta)q_0 \) and (22) holds. If, conversely, \( \bar{X}(\theta) \geq \ast \), i.e. \( M_\ast(\theta) = 0 \), then \( (22) \) implies \( t(\theta) \geq 0 \) so that \( t \) is coherent.

(ii). Assume that \( t \) is convex. Consider the sets \( \mathcal{H}_0 = \{ (\bar{X}(\theta) \wedge 0) q_0 - t(\theta) : \theta \in \Theta \} \) and
\[
\mathcal{H} = \{ f \in \mathcal{F}(\Omega) : f^- \in \mathcal{B}, \lambda h \geq f \text{ for some } h \in \mathcal{H}_0, \text{ and } \lambda \geq 0 \}
\]
If \( \lambda_1, \ldots, \lambda_N > 0 \) and \( \theta^1, \ldots, \theta^N \in \Theta \) then \( \sum_{n=1}^N \lambda_n [(\bar{X}(\theta^0) \wedge 0) q_0 - t(\theta^0)] \leq \lambda [(\bar{X}(\theta) \wedge 0) q_0 - t(\theta)] \) with \( \lambda = \sum_{n=1}^N \lambda_n \) and \( \theta = \sum_{n=1}^N (\lambda_n/\lambda) \theta^0 \in \Theta \). Moreover, \( h_n \geq f_n \) for \( n = 1, \ldots, N \) implies \( \sum_{n=1}^N h_n \geq \sum_{n=1}^N f_n \). Thus \( \mathcal{H} \) is a convex cone of uniformly lower bounded functions which, by (22), contains no element \( f \geq 1 \). By Theorem 8 in the Appendix there exists \( \mu \in \mathbb{P}_\ast \) such that
\[
\mathcal{H} \subset L^1(\mu) \quad \text{and} \quad \sup_{f \in \mathcal{H}} \int f d\mu \leq 0
\]
Moreover, if \( \theta \in \Theta_\ast \) and \( f = (\bar{X}(\theta) \wedge 0) q_0 - t(\theta) \) then
\[
0 \geq \int_{\{\bar{X}(\theta) \bar{X}(\theta) \wedge 0 \}} f d\mu = q_0 \int_{\{\bar{X}(\theta) \bar{X}(\theta) \wedge 0 \}} (\bar{X}(\theta) \wedge 0) d\mu - t(\theta) = q_0 \int (\bar{X}(\theta) \wedge 0) d\mu - t(\theta)
\]
which proves the direct implication. The converse follows from the inequality
\[
\int (\bar{X}(\theta) \wedge 0) d\mu = \int_{\bar{X}(\theta) \geq \bar{X}(\theta) \wedge 0} (\bar{X}(\theta) \wedge 0) d\mu \geq (\bar{X}(\theta) \ast) - \eta
\]
\( q_0 > 0 \) and (22).

(iii). If \( \bar{X}(\theta)^\ast + M_\ast(\theta) > 0 \) then, by (25), \( \theta' \) is an arbitrage opportunity unless \( t(\theta) + q_0M_\ast(\theta) > 0 \). (24) is thus necessary for absence of arbitrage. Conversely, choose \( \theta \in \Theta_\ast \) and fix \( \varepsilon \geq 0 \) such that \( \bar{X}(\theta)^\ast + M_\ast(\theta) + \varepsilon > 0 \). By assumption, \( \theta_\varepsilon = \theta + [\varepsilon + M_\ast(\theta)]\delta_0 \in \Theta \) and \( \bar{X}(\theta_\varepsilon)^\ast + M_\ast(\theta_\varepsilon) > 0 \). If (24) holds then
\[
0 < t(\theta_\varepsilon) + q_0M_\ast(\theta_\varepsilon) = t(\theta) + [\varepsilon + M_\ast(\theta)]q_0
\]
so that \( t(\theta) + M_\ast(\theta)q_0 > -\varepsilon q_0 \) for all \( \varepsilon > 0 \). Thus \( t(\theta) + M_\ast(\theta)q_0 \geq 0 \) for all \( \theta \in \Theta_\ast \) and \( t \) is coherent. If \( \bar{X}(\theta)^\ast \geq 0 \) and \( \bar{X}(\theta) > 0 \), then (24) implies \( t(\theta) = t(\theta) + M_\ast(\theta)q_0 \geq 0 \) so that \( t \) admits no arbitrage opportunity.\( \square \)

The representation (23), although quite manageable, relies crucially on convexity, a key property which is hard to justify based on the available empirical evidence which indicates, contrariwise, that fixed costs increase less than proportionally. This conclusion suggests that a more interesting representation may require to focus on the pricing functional, not including fixed costs.
4. Coherent Pricing

Since the early work of Bensaid et al.\cite{3} it is known that many properties of asset prices are revealed by the super hedging functional and our model is no exception. We adapt this concept in defining the following extended real valued functional:

\[
\pi(f) = \inf \left\{ \lambda q(\theta) : \lambda \tilde{X}(\theta) \geq f, \lambda \geq 0, \theta \in \Theta \right\} \quad f \in \mathfrak{H}(\Omega)
\]

Clearly, \(\pi(1) \leq q_0\) and \(\pi^c(1) \geq 0\); if, in addition, \(q\) is coherent, then \(\pi(0) = 0\) and \(\pi^c(f) \leq \pi(f)\) for all \(f \in \mathfrak{H}(\Omega)\) – see Lemma 5. But even assuming coherence we cannot exclude the somehow abnormal situations \(\pi(1) = 0\) and \(\pi^c(1) = 0\) (see Example 4 below). In particular:

**Lemma 3.** \(\pi^c(1) = 0\) if and only if \(q(\theta) \geq 0\) for every \(\theta \in \Theta_+\).

Define

\[
\mathcal{K} = \{ f \in \mathfrak{H}(\Omega) : \pi(|f|) < \infty \} \quad \text{and} \quad \mathcal{K}_+ = \{ f \in \mathcal{K} : f \geq -\infty \}
\]

The set \(\mathcal{K}\) plays in this paper the role of the ambient space and it is interesting to remark that its definition is entirely market based and does not require any mathematical structure.

The following is the most important result of this section.

**Theorem 3.** The price functional \(q\) is coherent if and only if for each \(h \in \mathfrak{B}_+\) the set

\[
\mathcal{M} = \left\{ m \in \mathfrak{B}_+ : \mathcal{K} \subset L(m) \quad \text{and} \quad \int f dm \leq \pi(f) \quad \text{for all} \quad f \in \mathcal{K}_+ \right\}
\]

contains an element \(m_h\) such that \(\int hdm_h = \pi(h)\).

**Proof.** By Lemma 5, if \(q\) is coherent then the space \(\mathcal{K}\) is a vector sublattice of \(\mathfrak{H}(\Omega)\) containing \(\mathfrak{B}_+\) and \(\pi a \geq_{\mathcal{K}}\)-monotone, positively homogeneous and subadditive functional on \(\mathcal{K}\). Fix \(h \in \mathfrak{B}_+\) and consider the set \(C_h = \{ \lambda h : 0 \leq \lambda \leq 1 \}\). By Theorem 9 there is a positive linear functional \(\beta_h\) on \(\mathcal{K}\) vanishing on \(\mathfrak{B}_+\) and \(m_h \in \mathfrak{B}_+\) such that \(\mathcal{K} \subset L^1(m_h)\) and \(\pi(f) \geq \beta_h(f) + \int f dm_h\) for all \(f \in \mathcal{K}\) and such that \(\pi(h) = \beta_h(h) + \int hdm_h = \int hdm_h\). Suppose that \(g \in \mathcal{K}_+\). Then, \(g^- \in \mathfrak{B}_+\) and thus

\[
\pi(g) \geq \beta_h(g) + \int g dm_h = \beta_h(g^+) + \int g dm_h \geq \int g dm_h
\]

so that \(m_h \in \mathcal{M}\). Conversely, if \(m \in \mathcal{M}\) and \(\tilde{X}(\theta)_+ \geq 0\) then (28) implies

\[
q(\theta) \geq \pi(\tilde{X}(\theta)) \geq \int \tilde{X}(\theta) dm = \int \tilde{X}(\theta)1_{\{\tilde{X}(\theta)_+ \geq \tilde{X}(\theta)_- - \varepsilon\}} dm \geq [\tilde{X}(\theta)_+ - \varepsilon]m(\Omega) \geq -\varepsilon m(\Omega)
\]

for every \(\varepsilon > 0\) so that \(q\) is coherent. \(\square\)

We refer to \(\mathcal{M}\) as the set of pricing measures. It corresponds to the set of equivalent martingale measures in traditional models.\cite{5}

As in other papers in this field, Theorem 3 asserts that a coherent price system is consistent with risk neutral pricing, i.e. with a pricing rule appropriate for a market free of imperfections and

\footnote{Although the notion of equivalence has no meaning here.}
thus supports the view expressed in the microstructure literature that the bid and ask prices are set starting from a consensus price. However, one should remark that prices may be identified with integrals only for strategies in $\Theta$, as claims not included in $\mathcal{K}$ may not be integrable at all. This is a consequence of not defining the ambient space exogenously. Moreover it should be noted that pricing measures are just finitely additive but in addition are defined on all subsets of $\Omega$ rather than a given algebra $\mathcal{A}$. This last remark is relevant for the definition of market completeness which may be here given with no reference to an artificial family of sets.

Given the exclusive emphasis of the literature on countably additive pricing measures, we characterize next this special property.

**Theorem 4.** Let $\pi^c(1) > 0$ and $\mathcal{A}$ be an algebra including $\mathcal{N}_\pi$. The following are equivalent:

(i) there exists $0 \neq \mu \in \mathcal{M}$ such that $\mu$ is countably additive in restriction to $\mathcal{A}$;
(ii) there exists $P \in \mathbb{P}_\pi(\mathcal{A})$ countably additive and such that for any sequence $(f_n)_{n \in \mathbb{N}}$ in $\mathcal{I}(\mathcal{A})$,
$$\limsup_n \pi(f_n) \leq 0 \implies \liminf_n \int f_n dP \leq 0;$$
(iii) there exists $P \in \mathbb{P}_\pi(\mathcal{A})$ countably additive and such that for any sequence $(f_n)_{n \in \mathbb{N}}$ in $\mathcal{I}(\mathcal{A})^+$,
$$\lim_n \int f_n dP = 0 \implies \lim_n \pi^c(f_n) = 0.$$

*Proof.* (ii)$\Rightarrow$(i). Choose $0 \neq m_0 \in \mathcal{M}$ to be countably additive on $\mathcal{A}$ and write $P$ for the restriction to $\mathcal{A}$ of $m_0/\|m_0\|$. Then it is obvious that $\int f dP \leq \pi(f)/\|m_0\|$ for each $f \in \mathcal{I}(\mathcal{A})$ so that (ii) holds. (i)$\Rightarrow$(ii). Fix $P$ as in (ii) and, assuming that (ii) fails, pick a sequence $(h_n)_{n \in \mathbb{N}}$ in $\mathcal{I}(\mathcal{A})^+$ which converges to 0 in $L^1(P)$ but such that $\inf_n \pi^c(h_n) > \delta > 0$. Write $f_n = 1 - h_n\pi^c(h_n)$. Then, $f_n \in \mathcal{I}(\mathcal{A})$ converges in $L^1(P)$ to 1 while $\pi(f_n) \leq \pi(1) + \pi(-h_n)\pi^c(h_n) = 0$, so that (ii) fails. (i)$\Rightarrow$(iii). Let now $P$ be as in (ii) and suppose that no $m \in \mathcal{M}$ satisfies $m \ll P$ in restriction to $\mathcal{A}$. For each $m \in \mathcal{M}$ we may then construct a sequence $(F_m(n))_{n \in \mathbb{N}}$ in $\mathcal{A}$ such that $\lim_n P(F_m(n)) = 0 < \delta(m) \equiv \inf m(F_m(m))$. Upon choosing $n$ sufficiently large and setting $h_n(m) = 1_{F_n(m)}(\delta(m))^{-1}$ we obtain $\int h_n(m) dP < 2^{-n}$ while $\int h_n(m) dm \geq 1$. Let $\mathcal{H}_n = \{h \in \mathcal{I}(\mathcal{A})^+ : \int hdP < 2^{-n}\}$. Then
$$\inf_{m \in \mathcal{M}} \sup_{h \in \mathcal{H}_n} \int h dm \geq 1$$

Observe that $\mathcal{M}$ is convex and weak* compact and that $\mathcal{H}_n$ is convex. By the minimax Theorem of Sion [38, Corollary 3.3], there exists then $h_n \in \mathcal{H}_n$ such that
$$\pi^c(h_n) = \inf_{m \in \mathcal{M}} \int h_n dm \geq 1/2$$

The sequence $(h_n)_{n \in \mathbb{N}}$ so obtained contradicts (ii).

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7 Theorem 4 may be established without assuming $\pi^c(1) > 0$ upon replacing $\pi$ with
$$\pi_\varepsilon(b) = \sup \left\{ \int b dm : m \in \mathcal{M}, m(\Omega) \geq \varepsilon \right\} \varepsilon^{-1} \quad b \in \mathbb{B}_*$$

but the statement would be less clear to interpret.
Theorem 4 contributes to clarifying that the existence of a countable additive pricing measure is equivalent to some form of continuity of market prices with respect to the \( L^1(P) \) topology, an extremely unlikely property in the absence of an \textit{ad hoc} assumption.

Traditionally, the existence of a countably additive pricing measure is obtained after imposing the \textit{No-Free-Lunch} condition introduced by Kreps [26] which however requires the choice of \( L^p(P) \) as the ambient space for some given probability \( P \). We adapt from [23, Definition 2.1] the following:

\textbf{Definition 2.} Financial markets are said to satisfy the \textit{(NFL) condition} if there exists \( P \in \mathbb{P}(\mathcal{A}) \) countably additive such that for each sequence \( (x_n)_{n \in \mathbb{N}} \) in \( \mathbb{R} \) converging to some \( x \geq 0 \) and all sequence \( (f_n)_{n \in \mathbb{N}} \) in \( \mathcal{I}(\mathcal{A}) \) converging to \( f \) in \( L^1(P) \) and such that \( \pi(1)x_n + \pi(f_n) \leq 0 \) one has \( \int f dP \leq -x \).

If \( P \) is obtained from some \( 0 \neq m \in \mathcal{M} \) which is countably additive in restriction to \( \mathcal{A} \) and the sequences \( (x_n)_{n \in \mathbb{N}} \) and \( (f_n)_{n \in \mathbb{N}} \) are as in Definition 2, then

\[ 0 \geq \lim_n \{ \pi(1)x_n + \|m\| \int f_n dP \} = \pi(1)x + \|m\| \int f dP \]

so that \( \int f dP \leq -\frac{\pi(1)}{\|m\|} x \leq -x \). Conversely, the sequence \( (f_n)_{n \in \mathbb{N}} \) in \( \mathcal{I}(\mathcal{A}) \) constructed to prove the implication (22) \( \Rightarrow \) (23) in Theorem 4 is such that \( f_n \) converges to 1 in \( L^1(P) \) while \( x_n = -\pi(f_n) \geq 0 \), contradicting \( (NFL) \). This proves that

\textbf{Corollary 1.} Under the conditions of Theorem 4, there exists \( 0 \neq \mu \in \mathcal{M} \) which is countably additive in restriction to \( \mathcal{A} \) if and only if \( (NFL) \) holds.

Theorem 4 also provides some insight, suggesting cases in which \( \mathcal{M} \) may admit no countably additive elements.

\textbf{Example 3.} Let \( \Omega \) be a separable metric space and \( \mathcal{A} \) its Borel \( \sigma \) algebra. Assume that there exists an increasing net \( \{N_\alpha\}_{\alpha \in \mathcal{A}} \) in \( N_* \) with \( N_\alpha \) open and \( \Omega = \bigcup_\alpha N_\alpha \). This is the case, e.g., if each \( \omega \in \Omega \) admits a neighborhood contained in \( N_* \). Fix \( P \in \mathbb{P}(\mathcal{A}) \) countably additive. By [5, Proposition 7.2.2],

\[ 1 = \lim_\alpha P(N_\alpha) = \lim_k P(N_k) \]

for some suitable sequence \( (N_k)_{k \in \mathbb{N}} \) from \( \{N_\alpha\}_{\alpha \in \mathcal{A}} \). Set \( f_k = 1_{N_k^c} \). Then, \( \lim_k \int f_k dP = 0 \) while

\[ \pi^c(f_k) = \inf_{m \in \mathcal{M}} m(N_k^c) = \inf_{m \in \mathcal{M}} m(\Omega) = \pi^c(1) \]

so that condition (ii) of Theorem 4 fails. No pricing measure is then countably additive outside of the special case \( \pi^c(1) = 0 \). Actually, decomposing each \( m \in \mathcal{M} \) as \( m = m^c + m^\perp \), with \( m^c \) countably additive and \( m^\perp \) purely finitely additive (see [17, III.7.8]), and exploiting the inclusion \( \mathcal{M} \subset ba_* \), we conclude that in the case of this example all pricing measures are purely finitely additive.

The special situation illustrated in Example 3 highlights that countable additivity of the pricing measures may not only fail but actually contrast with coherence if the partial order \( \geq_* \) is an \textit{a priori} of the model.
Based on the results of the preceding section, we develop here some decompositions of coherent price functionals which highlight the role of asset bubbles.

**Theorem 5.** The price $q$ is coherent if and only if the set $\mathcal{M}$ of pricing measures is the unique non empty, convex, weak$^*$ compact subset of $\mathcal{B}_*$ admitting the decomposition

$$\pi(f) = \beta(f) + \sup_{m \in \mathcal{M}} \int f dm \quad \text{for all } f \in \mathcal{K}$$

where $\beta : \mathcal{K} \to \mathbb{R}$ vanishes on $\mathcal{B}_*$.

**Proof.** Indeed if $q$ is coherent then $\mathcal{M}$ is a non empty, convex and weak$^*$ compact subset of $\mathcal{B}_*$, by Lemma 6; moreover, if $m \in \mathcal{M}$ and $f \in \mathcal{K}$ then, by (28), $|\int f dm| \leq \int |f| dm \leq \pi(|f|) < \infty$ so that (29) may be regarded as an implicit definition of $\beta$. Observe that from (28) and Lemma 6 we obtain

$$\inf_{\phi \in \Phi(\pi)} \phi(f) \leq \pi(f) - \sup_{m \in \mathcal{M}} \int f dm$$

$$= \beta(f)$$

$$= \sup_{\phi \in \Phi(\pi)} \phi(f) - \sup_{m \in \mathcal{M}} \int f dm$$

$$\leq \sup_{\phi \in \Phi(\pi)} \phi^+(f) + \sup_{\mu \in \mathcal{H}} \int f d\mu - \sup_{m \in \mathcal{M}} \int f dm$$

$$= \sup_{\phi \in \Phi(\pi)} \phi^+(f)$$

(30)

Given that $\sup_{\phi \in \Phi(\pi)} \phi^+(f) = 0$ for all $f \in \mathcal{B}_*$, as we showed in the proof of Theorem 8, we conclude that $\beta$ vanishes on $\mathcal{B}_*$. This proves existence. To show uniqueness, suppose that $\bar{\beta}$ and $\mathcal{H}$ is another pair with the same properties of $\beta$ and $\mathcal{M}$ and for which the decomposition (29) holds. If $\mu \in \mathcal{H} \setminus \mathcal{M}$, then there exists $f \in \mathcal{B}$ such that $\sup_{m \in \mathcal{M}} \int f dm \geq \int f d\mu > \sup_{m \in \mathcal{M}} \int f dm$ but $\bar{\beta}(f) = \beta(f) = 0$, a contradiction of (29). To show that (29) is sufficient for $q$ to be coherent, let $f \in \mathcal{K}$. Then, $f^- \in \mathcal{B}_*$ and thus $\beta(f) = \beta(f^+) \geq 0$ and thus

$$\pi(f) \geq \sup_{m \in \mathcal{M}} \int f dm \geq f_* \sup_{m \in \mathcal{M}} \|m\| = f_* \pi(1)$$

Therefore, if $\bar{X}(\theta) \geq 0$ for some $\theta \in \Theta$ then $q(\theta) \geq \pi(\bar{X}(\theta)) \geq 0$ and $q$ is coherent. \qed

For each $m \in \mathcal{M}$ the quantity $\int \bar{X}(\theta) dm$ is rightfully interpreted as the fundamental value of the portfolio $\theta$ given $m$. In order to overcome the arbitrariness implicit in having a multiplicity of possible pricing measures and obtain an unambiguous definition, it is correct to identify the fundamental value of $\theta$ with the quantity

$$\sup_{m \in \mathcal{M}} \int \bar{X}(\theta) dm$$

(31)
Of course, the supremum of a family of integrals may be represented as the Choquet integral with respect to a supermodular capacity having \( \mathcal{M} \) as its core. Differently from classical asset pricing formulas, the fundamental value is not linear here, due to transaction costs. It could be interpreted as the maximum price paid for \( \theta \) in an economy identical with the one considered above but with no transaction costs. The main point is not only the multiplicity of pricing measures, which would be prevalent even in economies with incomplete financial markets, but rather the fact that the intervening expectations do not agree on the set of traded payoffs so that the integral appearing in (31) is not invariant with respect to the choice of \( m \in \mathcal{M} \).

In general, deviations of prices from fundamental values are interpreted in the literature as evidence of the existence of bubbles. See [12], [22] or [29] for examples of models dealing with bubbles in continuous time. In so doing, however, inefficiency phenomena and the potential contribution of asset bubbles to an efficient pricing are mixed together.

Inefficiency is measured by the quantity \( q(\theta) - \pi(\bar{X}(\theta)) \). The empirical literature typically reports a relatively large number of violations, e.g., of the PUT/CALL parity, by which, say, a CALL option may be replaced by a less costly synthetic constructed using the corresponding PUT, future and riskless asset. Luttmer [31], takes this mispricing as the sole source of subadditivity. For a coherent price system inefficiencies are a consequence of the restrictions which prevent investors to exploit them to obtain immediate profits. Empirical explanations, such as those invoked by Lamont and Thaler [27], draw attention on the fixed costs of trading which impair the arbitrage profits emerging from considering prices only. However, even fixed transaction costs would play virtually no role if investors were not somehow constrained in their ability to either take short positions or in choosing the scale for their investments arbitrarily large.

We deduce from (29) that, even in the absence of market inefficiencies and with only two dates, prices may differ from fundamental values by a bubble component, \( \beta \), interpreted as the price of the tail part of the asset discounted payoff. We base this interpretation on the inequality

\[
|\beta(\bar{X}(\theta))| \leq \lim_{n} \left\{ \pi \left( (\bar{X}(\theta)^{+} - n)^{+} \right) + \pi \left( (\bar{X}(\theta)^{-} - n)^{+} \right) \right\}
\]

By (32), \( \beta(\bar{X}(\theta)) \) is rightfully viewed as the component of the price of \( \theta \) which only depends on the event \( \{|\bar{X}(\theta)| \geq n\} \) for all \( n \in \mathbb{N} \), i.e. on the extreme fluctuations of the portfolio discounted payoff. Observe that necessarily the price of the numéraire and of other derivatives written on it, such as futures and options, admits no bubbles. (32) suggests in addition that, like in other models, bubbles are related to the limit of the price of a CALL option as the strike price increases to infinity. This finding is consistent with similar conclusions linking the existence of asset bubbles to some mispricings of options (see [12] and [21]). Assuming some form of monotone continuity of the pricing functional, as in [11], excludes the existence of bubbles.

The following example illustrates the economic role of bubbles in a special case.

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8 The use of capacities in finance was introduced by Chateauneuf et al. [11, Theorem 1.1] precisely with the aim of modeling transaction costs. In their paper, however, this representation is an assumption (see also [10]).
Example 4 (Efficient Bubbles). Consider a market on which $\bar{X}(\theta) \geq 0$ for all $\theta \in \Theta$ and assume that $q(\theta) = X(\theta)^* - X(\theta)_*$. The price function is clearly subadditive and positively homogeneous. Given that $q(\theta) \geq 0$ for all $\theta \in \Theta$ it is coherent too – although it will be inefficient in general.

From Lemma 3 we know that $\pi(1) = 0$. Assume that $X_0,*, = 0$ and $X_0^* = 1$ and that, for each $n \in \mathbb{N}$ there exists $\theta_n \in \Theta$ with $X(\theta_n) = a + X_02^{-n}$ with $a > 0$. Then, $\bar{X}(\theta_n) \geq a + 2^{-n}$. On the other hand $X(\theta_n)* = a$ and $X(\theta_n)^* = a + 2^{-n}$. But then $q_0 = 1$ while

\[
\pi(1) = \inf_{\bar{X}(\theta) > 0} q(\theta)/\bar{X}(\theta)_* \leq \inf_n (1 + a2^n)^{-1} = 0
\]

Thus the numéraire is priced inefficiently and, from (29), the only possible non null efficient price is a bubble.

6. Option Pricing

In this section we apply our preceding results to option pricing, under the only assumption that options are traded anonymously (i.e. that (6) holds) and at non negative prices. $X > 0$ will be hereafter the payoff of a given underlying and $K(X)$ the set of strike prices (including $k = 0$) of all CALL options written on it. The ticker of each of these options and the corresponding strategy, price and payoff will be indicated by $\alpha_X(k)$, $\theta_X(k)$, $q_X(k)$ and $X(k)$ respectively. Define also

\[
\mathfrak{A}_X = \{\alpha_X(k) : k \in K(X)\}
\]

\[
\Theta_X = \{\theta \in \Theta : \theta \geq 0 \text{ and } \theta(\alpha) = 0 \text{ whenever } \alpha \notin \mathfrak{A}_X\}
\]

\[
\pi_X(h) = \inf \left\{ \lambda q(\theta) : \frac{X(\theta)}{X \land 1} \geq h, \lambda > 0, \theta \in \Theta_X \right\} \quad h \in \mathfrak{F}(\Omega)
\]

and

\[
\mathcal{K}_X = \{h \in \mathfrak{F}(\Omega) : \pi_X(|h|) < \infty\}
\]

Observe that the restriction of $q$ to $\Theta_X$ is coherent, given our assumption of non negative prices.

On the other hand, the change of numéraire implicit in (35) entails a different concept of efficiency. In particular we shall say that options are priced efficiently if

\[
q_X(k) = \pi_X \left( \frac{X(k)}{X \land 1} \right) \quad k \in K(X)
\]

The criterion adopted in (37) is indeed quite weak as, for example, it does not involve PUT options nor Futures or short positions. This is desirable since the larger the set of derivatives involved the more likely is it that efficiency may fail. For example, the PUT/CALL parity is well known to generate a large number of violations as well as the lower bound for CALL options.

Define the set

\[
\Gamma = \left\{ f \in \mathfrak{F}(\mathbb{R}_+) : f \geq 0 = f(0), \text{ } f \text{ convex, } \lim_{n \to \infty} f(n)/n < \infty \right\}
\]

\[\text{Correa et al. [10] construct a pricing model for markets which are assumed to satisfy the PUT/CALL parity.}\]

\[\text{The limit appearing in (38) exists by convexity.}\]
To start discussing the issue of options efficiency, denote by $J(X) \subset K(X)$ the subset of strike prices $k$ possessing the following property

\[(39) \quad q_X(k) \leq \inf \{aq_X(k_1) + (1-a)q_X(k_2) : k_1, k_2 \in K(X), 0 \leq a \leq 1, \, ak_1 + (1-a)k_2 \leq k \}\]
i.e. which satisfy the butterfly spread condition. One should remark that in the present setting this is not an arbitrage restriction.

For reasons of technical convenience, in the rest of this section we shall adopt the following

**Assumption 5.** $X^* < \infty$.

We turn now to the issue of derivatives hedging.

**Theorem 6.** Assume that $(X^{1\{X \leq j\}})^* = j < X^*$ for each $j \in J(X)$. For each $g \in \Gamma$ there exists $\theta_X(g) \in \Theta_X$ such that $q(\theta_X(g)) = \pi_X(g(X)/X \land 1)$. Moreover: (i) if $g_1, g_2 \in \Gamma$ then $\theta_X(g_1 + g_2) = \theta_X(g_1) + \theta_X(g_2)$, (ii) if $j \in J(X)$ and $g(x) = (x - j)^+$ then $\theta_X(g) = \theta_X(j)$.

**Proof.** The existence claim is proved in Lemma [8] in the Appendix where the explicit composition of $\theta_X(g)$ is described, see [62]. From it we deduce (i). (ii) follows upon setting $g(x) = (x - j)^+$ in [64].

We deduce from Theorem 6 that an option is priced efficiently if and only if its strike price is included in $J(X)$.

The following is the most important result of the paper.

**Theorem 7.** Assume that $(X^{1\{X \leq x\}})^* = x \land X^*$ when $x \geq 0$. Let $G = \{g_t : t \in \mathbb{R}_+\} \subset \Gamma$ satisfy

\[(40) \quad g_t \leq ag_{t_1} + (1-a)g_{t_2} \quad \text{whenever} \quad 0 \leq a \leq 1 \quad \text{and} \quad t \geq at_1 + (1-a)t_2\]

There exist $\beta^G(X) \geq 0$ and $\nu^G_X \in \text{cag}(\mathcal{B}(\mathbb{R}_+))_+$ such that

\[(41) \quad \pi_X(g_t(X)/X \land 1) = \beta^G(X) + \int_t^\infty \nu^G_X(x > z)dz \quad \text{for all} \quad t \geq 0\]

**Proof.** The function $t \to q^G_X(t) = \pi_X(g_t(X)/X \land 1) : \mathbb{R}_+ \to \mathbb{R}_+$ is clearly decreasing and convex and thus satisfies (39). Replace the original option market with one in which all strikes $0 \leq t \leq X^*$ are traded at the fictitious prices $q^G_X(t)$ and define $\pi^G_X$ exactly as in (35) after such replacement. By construction, all option prices are efficient, i.e. $J^G(X) = [0, X^*]$. Moreover, if $a_1, \ldots, a_N \geq 0$ and $t_1, \ldots, t_N \in [0, X^*]$ then

\[
\pi^G_X \left( \frac{\sum_{n=1}^N a_n X(t_n)}{X \land 1} \right) = \sum_{n=1}^N a_n \pi^G_X \left( \frac{X(t_n)}{X \land 1} \right) = \sum_{n=1}^N a_n q^G_X(t_n)
\]

This follows clearly from Theorem 6 if one tries to hedge the payoff $\sum_{n=1}^N a_n X(t_n)/X \land 1$ with a finite set of options whose strikes include $t_1, \ldots, t_N$. Write

\[
\mathcal{H}^G_X = \{f \in \mathcal{F}(\Omega) : \pi^G_X(|f|) < \infty \} \quad \text{and} \quad \mathcal{G}^G_X = \left\{ \frac{\sum_{0 \leq t < X^*} a(t)X(t)}{X \land 1} : a \in \mathcal{F}_0([0, X^*])_+ \right\}
\]
endowed with the partial order \( \geq_* \) and observe that \( \mathcal{B}_* \subset \mathcal{K}_X^G \).

By Theorem [9] we obtain a \( \geq_* \) positive, linear functional \( \phi_X^G : \mathcal{K}_X^G \to \mathbb{R} \) such that \( \phi_X^G \leq \pi_X^G \) and that \( \phi_X^G = \pi_X^G \) in restriction to \( \mathcal{G}_X^G \). Define

\[
F^G(t) = \phi_X^G \left( \frac{X(t)}{X \wedge 1} \right) \quad t \geq 0
\]

Of course, \( F^G(t) = q_X^G(t) \); in addition, it is decreasing and convex. By a standard result on convex functions, we may write

\[
F^G(t_2) = F^G(t_1) + \int_{t_1}^{t_2} f^G(t) dt \quad 0 < t_1 < t_2
\]

where, for definiteness, we take \( f^G(t) \) to be the right derivative of \( F^G \) for \( t \in \mathbb{R}_+ \). Suppose that \( \{u \geq X > t\} \in \mathcal{N}_* \) for some \( 0 \leq t < u \) and fix \( 0 < h \leq (u - t)/2 \). There is then a negligible set outside of which each of the options with strike prices \( t, t + h, u - h, u \) expires in the money if and only if all the others do. In other words

\[
\frac{X(t) + X(u)}{X \wedge 1} = \frac{X(t + h) + X(u - h)}{X \wedge 1}
\]

from which it follows

\[
F^G(t) + F^G(u) = \phi_X^G \left( \frac{X(t) + X(u)}{X \wedge 1} \right) = \phi_X^G \left( \frac{X(t + h) + X(u - h)}{X \wedge 1} \right) = F^G(t + h) + F^G(u - h)
\]

Thus,

\[
\frac{F^G(u) - F^G(u - h)}{h} = \frac{F^G(t + h) - F^G(t)}{h}
\]

i.e. the left derivative of \( F^G \) at \( u \) and the right derivative of \( F^G \) at \( t \) coincide. There exists then a set \( D \subset \mathbb{R}_+ \) with \( \mathbb{R}_+ \setminus D \) at most countable and such that \( \{X > u\} \triangle N_1 = \{X > t\} \triangle N_2 \) for \( t, u \in D \) and \( N_1, N_2 \in \mathcal{N}_* \) imply \( f(t) = f(u) \). It is therefore possible to define a positive set function \( \lambda_0^G \) on the collection \( \mathcal{A}_0(X) \) of subsets of \( \Omega \) formed by \( \emptyset, \Omega \) and all sets of the form \( \{X > t\} \triangle N \) with \( t \in \mathbb{R}_+ \) and \( N \in \mathcal{N}_* \) implicitly by letting \( \lambda_0^G(\emptyset) = -f^G(0) \), \( \lambda_0^G(\Omega) = 0 \) and

\[
\lambda_0^G(\{X > t\} \triangle N) = \sup_{\{u \in D : u \geq t\}} -f^G(u) \quad t \in \mathbb{R}_+, \ N \in \mathcal{N}_*
\]

To see that this definition is well taken, observe that, if there is \( t_\infty \in D \) such that \( \{X > t_\infty\} = \emptyset \) then \( F^G(t_\infty + h) = F^G(t_\infty) \) and so \( f^G(t_\infty) = 0 \). Likewise, if \( \{X > t_0\} = \Omega \) for some \( t_0 \in D \), then \( \{X > t_0\} = \{X > 0\} \) so that, as seen above, \( f^G(t_0) \) coincides with \( f^G(0) \). If either \( t_0 \) or \( t_\infty \) do not exist, then one can choose the corresponding value of \( \lambda_0^G \) arbitrarily. Since the elements of \( \mathcal{A}_0(X) \) are linearly ordered by inclusion it follows that if \( A_i = \{X > t_i\} \triangle N_i \) and \( N_i \in \mathcal{N}_* \) for \( i = 1, 2 \) with \( t_1 \geq t_2 \) then

\[
\lambda_0^G(A_1) + \lambda_0^G(A_2) = \lambda_0^G(\{X > t_1\} \triangle N_1) + \lambda_0^G(\{X > t_2\} \triangle N_2)
\]

\[
= \lambda_0^G(\{X > t_1\} \cap \{X > t_2\}) + \lambda_0^G(\{X > t_1\} \cup \{X > t_2\})
\]

\[
= \lambda_0^G(A_1 \cap A_2) + \lambda_0^G(A_1 \cup A_2)
\]
as \( \{X > t_1\} \triangle (A_1 \cap A_2), \{X > t_2\} \triangle (A_1 \cup A_2) \subset N_1 \cup N_2 \in \mathcal{N}_+ \). It follows from [11, Theorems 3.1.6 and 3.2.10], that there exists a unique extension \( \lambda^G \in ba(\mathcal{A}(X))_+ \) of \( \lambda^G_0 \) to the algebra \( \mathcal{A}(X) \) generated by \( \mathcal{A}_0(X) \) and thus such that \( \lambda^G(N) = 0 \) when \( N \in \mathcal{N}_+ \). Let

\[
\beta^G_X = \lim_{k \to \infty} q^G_X(k)
\]

Then we obtain from (42)

\[
\beta^G_X = F^G(k) - \int_k^\infty \lambda^G(X > t) dt \quad k \geq 0
\]

To eventually get (41), write

\[
\mathcal{A} = \{ A \subset \mathbb{R}_+ : X^{-1}(A) \in \mathcal{A}(X) \}
\]

It is clear that \( \mathcal{A} \) is an algebra containing the algebra \( \mathcal{A}(\mathbb{R}_+) \) generated by the left open intervals of \( \mathbb{R}_+ \). Define then \( \lambda^G_X \in ba(\mathcal{A}(\mathbb{R}_+)) \) by letting \( \lambda^G_X(A) = \lambda^G(X \in A) \) and observe from (42) that

\[
\int_{\mathbb{R}_+} \lambda^G_X(x > t) dt = -\int_{\mathbb{R}_+} f(t) dt \leq F^G(0) \quad \text{so that } \lim_t \lambda^G_X((t, \infty)) = 0.
\]

Exploiting standard rules of the Lebesgue integral and integration by parts we obtain

\[
\int_{\mathbb{R}_+} \lambda^G(X > t) dt = \int_{\mathbb{R}_+} \lambda^G_X(x > t) dt = \int_{\mathbb{R}_+} xd\lambda^G_X(x) = \int x d\lambda^G_X(x)
\]

as \( \lambda^G_X(x < -t) = 0 \) for all \( t \geq 0 \). It follows from [16, Lemma 2, p. 191] that, uniquely associated with \( \lambda^G_X \) is its conventional companion \( \nu^G_X \in ca(\mathcal{A}(\mathbb{R}_+))_+ \) with the property that

\[
\int h(x) d\lambda^G_X = \int h(x) d\nu^G_X
\]

for any continuous function \( h : \mathbb{R} \to \mathbb{R} \) for which either integral is well defined. The extension from \( \mathcal{A}(\mathbb{R}_+) \) to the generated \( \sigma \) algebra \( \mathcal{B}(\mathbb{R}_+) \) is standard. Thus the representation (41) is implicit in \( \Theta^G_X \) being priced efficiently. Suppose that \( \beta^G_X \geq 0 \) and \( \nu^G_X \in ca(\mathcal{B}(\mathbb{R}_+))_+ \) is another pair for which the representation (41) holds. Then,

\[
\beta^G_X - \bar{\beta}^G_X = \int_k^\infty [\nu^G_X(x > t) - \nu^G_X(x > t)] dt \quad \text{for all } k \geq 0
\]

which implies \( \beta^G_X = \bar{\beta}^G_X \).

Observe that, by standard rules,

\[
\int_t^\infty \nu^G_X(x > z) dz = \int (x - t)^+ d\nu^G_X(x)
\]

Thus (41), represents the price of the \( G \) derivatives as the sum of a bubble part and the fundamental value. By (41) the term \( \beta^G(X) \) represents the (fictitious) option price as the strike approaches infinity and contributes to explaining the overpricing of deeply out of the money CALL’s often documented empirically in some form of the smile effect.

The most important implications of Theorem 7 regard the empirical analysis of option markets. In this perspective one should start noting that the CALL function \( q^G_X \), although not a quoted price, is entirely market based and may be computed explicitly, once the collection \( G \) has been chosen. Statistical estimation of the CALL function, to the contrary, follows from some optimal statistical criterion and does not guarantee a direct market interpretation. In principle one could make the
choice of the collection $G$ sample based and study whether the estimate $q_X^G(t)$ possesses reasonable statistical properties. This implicitly suggests a new non parametric empirical strategy.

A second fact arising from (41) is the representation of option prices via a countably additive probability $\nu_X^G$ implicit in option prices. Although $\nu_X^G$ will generally depend on $G$, (41) makes it possible, even in a model with minimal mathematical structure as the one developed here, to run the classical exercise of Breeden and Litzenberger [7] and Banz and Miller [2] by computing

$$\nu^G_X(x > t) = -\frac{dd^G_X(k)}{dk} \bigg|_{k=t} \quad \text{for all } t \geq 0$$

(47)

Remark that in the model of Black and Scholes (47) translates into the classical formula

$$\nu_X^{BS}(x > k) = e^{-rT}\Phi(d_2) \quad \text{with} \quad d_2 = \frac{\ln(S_0/k) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

(48)

so that $\|\nu_X^{BS}\| = \exp(-rT)$. In more general traditional models, the CALL lower bound, $q_X \geq q_X^G(k) \geq q_X - kq_0$ implies the inequality $\|\nu_X\| \leq q_0 \leq 1$. However, this inequality cannot be deduced from arbitrage arguments when the numéraire asset is not free of risk. In particular, we do not have any a priori bound to impose on the norm of $\nu_X$. If, however, we assume in addition $X \geq \eta$, we then obtain for $G = \{(x-t)^+ : t \in \mathbb{R}_+\}$

$$\|\nu_X^G\| = \nu_X^G(x > 0) = \lim_{k \to 0} \frac{q_X^G(0) - q_X^G(k)}{k} \leq \pi_X(1) \leq \frac{q_X(0)}{\eta}$$

**APPENDIX A. AUXILIARY RESULTS**

Let us start with two general results.

**Theorem 8.** Let $\succeq$ be a partial order on $\mathfrak{F}(\Omega)$ satisfying Assumption [4] and $\Gamma \subset \mathfrak{F}(\Omega)$ a convex cone. There is no $f \in \Gamma$ with $f \succeq 1$ if and only if there exists $m \in \mathbb{P}_\succeq = \{\mu \in \mathbb{P} : \mu(N) = 0 \text{ when } 0 \succeq 1\}$ such that

$$\Gamma_\succeq = \{f \in \Gamma : f \succeq a \text{ for some } a \in \mathbb{R}\} \subset L^1(m) \quad \text{and} \quad \sup_{f \in \Gamma_\succeq} \int f dm \leq 0$$

(49)

**Proof.** Define the following collection

$$\Gamma_1 = \left\{ g \in \mathfrak{F}(\Omega) : g^- \in \mathfrak{B}, \sum_{i=1}^{I} f_i 1_{N_i^c} \succeq g \text{ for some } f_1, \ldots, f_I \in \Gamma_\succeq, N_1, \ldots, N_I \in N_\succeq \right\}$$

Observe that $\Gamma_1$ is a convex cone of lower bounded functions on $\Omega$. Suppose that $\Gamma_1$ contains a sure win, i.e., an element $g \geq 1$. Then there exist $f_1, \ldots, f_I \in \Gamma_\succeq$ and $N_1, \ldots, N_I \in N_\succeq$ such that $\sum_{i=1}^{I} f_i 1_{N_i^c} \succeq g$. However, remarking that $N = \bigcup_i N_i \in N_\succeq$ (so that $\alpha 1_N \succeq 0$ for any $\alpha \in \mathbb{R}$)

$$\sum_{i=1}^{I} f_i \succeq 1 + \sum_{i} f_i 1_{N_i} \succeq 1 + \min_{i}(f_i \succeq 1) 1_N \succeq 1$$

contradicting $\sum_{i=1}^{I} f_i \in \Gamma$. By [9] Proposition 1, there exists $m \in \mathbb{P}$ such that

$$\Gamma_1 \subset L^1(m) \quad \text{and} \quad \sup_{g \in \Gamma_1} \int g dm \leq 0$$
$N \in \mathcal{N}_\geq$ implies $0 \geq 1_N$ so that $1_N \in \Gamma_1$, since $0 \in \Gamma_\geq$. Therefore $m(N) \leq 0$ and $m \in \mathbb{P}_\geq$. On the other hand $f \in \Gamma_\geq$ implies $f_n = f1_{\{f \geq 2^{-n}\}} \in \Gamma_1$ so that $\int |f|dm = \int |f_n|dm < \infty$ and $f \int fdm = \int f_n\int dm \leq 0$. For the converse, suppose that $f \in \Gamma$ and $f \geq 1$. Then $\{f \leq 1/2\} \in \mathcal{N}_\geq$ and $f \in \Gamma_\geq$ so that $0 \geq \int fdm = \int \{f > 1/2\} fdm \geq 1/2$, a contradiction. □

**Lemma 4.** Let $\mathcal{L} \subset \mathcal{F}(\Omega)$ be a vector lattice containing $\mathfrak{B}$. Each positive linear functional $\phi$ on $\mathcal{L}$ admits the decomposition

$$\phi(f) = \phi^\perp(f) + \int f dm_\phi \quad f \in \mathcal{L}$$

where $m_\phi \in ba$ and $\phi^\perp$ is a positive linear functional vanishing on $\mathfrak{B}$.

**Proof.** See [9, Theorem 1]. □

**Theorem 9.** Let $\mathcal{L} \subset \mathcal{F}(\Omega)$ be a vector lattice containing $\mathfrak{B}_+$, $C \subset \mathcal{L}$ a convex set containing the origin and $\gamma : \mathcal{F}(\mathcal{L})$ be $\geq_*$-monotone, subadditive and positively homogeneous. Then

$$\gamma \left( \sum_{n=1}^{N} f_n \right) = \sum_{n=1}^{N} \gamma(f_n) \quad f_1, \ldots, f_N \in C$$

if and only if there exist (i) a positive linear functional $\beta$ on $\mathcal{L}$ vanishing on $\mathfrak{B}_+$ and (ii) $m \in ba_{*,+}$ such that $\mathcal{L} \subset L^1(m)$

$$\gamma(h) \geq \beta(h) + \int h dm \quad \text{and} \quad \gamma(f) = \beta(f) + \int f dm \quad \text{for all } h \in \mathcal{L}, \ f \in C$$

**Proof.** (51) holds on $C$ if and only if it holds over the whole convex cone generated by $C$, by positive homogeneity and the inclusion $0 \in C$. Let $f_1, \ldots, f_N, g_1, \ldots, g_K \in C$ and $\lambda_1, \ldots, \lambda_N, \alpha_1, \ldots, \alpha_K \in \mathbb{R}$ be such that $\sum_{k=1}^{K} \alpha_k g_k =_* \sum_{n=1}^{N} \lambda_n f_n$. Then, $\sum_{k=1}^{K} \alpha_k^2 g_k + \sum_{n=1}^{N} \lambda_n^2 f_n =_* \sum_{k=1}^{K} \alpha_k g_k + \sum_{n=1}^{N} \lambda_n f_n$. By (51) and $\geq_*$ monotonicity

$$\sum_{k=1}^{K} \alpha_k^2 \gamma(g_k) + \sum_{n=1}^{N} \lambda_n^2 \gamma(f_n) = \gamma \left( \sum_{k=1}^{K} \alpha_k^2 g_k + \sum_{n=1}^{N} \lambda_n^2 f_n \right) \leq \gamma \left( \sum_{k=1}^{K} \alpha_k g_k + \sum_{n=1}^{N} \lambda_n f_n \right) = \sum_{k=1}^{K} \alpha_k \gamma(g_k) + \sum_{n=1}^{N} \lambda_n \gamma(f_n)$$

i.e. $\sum_{k=1}^{K} \alpha_k \gamma(g_k) = \sum_{n=1}^{N} \lambda_n \gamma(f_n)$. Thus the quantity

$$\phi_0 \left( \sum_{n=1}^{N} \lambda_n f_n \right) = \sum_{n=1}^{N} \lambda_n \gamma(f_n) \quad f_1, \ldots, f_N \in C, \ \lambda_1, \ldots, \lambda_N \in \mathbb{R}$$
Implicitly defines a linear functional on the linear span Lin(C) of C. It is easy to conclude from (51) and subadditivity that
\[ \phi_0 \left( \sum_{n=1}^{N} \lambda_n f_n \right) = \phi_0 \left( \sum_{n=1}^{N} \lambda_n^+ f_n \right) - \phi_0 \left( \sum_{n=1}^{N} \lambda_n^- f_n \right) \]
\[ = \gamma \left( \sum_{n=1}^{N} \lambda_n^+ f_n \right) - \gamma \left( \sum_{n=1}^{N} \lambda_n^- f_n \right) \]
\[ \leq \gamma \left( \sum_{n=1}^{N} \lambda_n f_n \right) \]

and thus that \( \phi_0 \leq \gamma \) on Lin(C). By Hahn Banach we may thus find an extension \( \phi \) of \( \phi_0 \) to the whole of \( \mathcal{L} \) such that \( \phi \leq \pi \). Given that \( \gamma \) is \( \geq_{\ast} \)-monotone and positive homogeneous we conclude that \( \phi \) is positive and, by Lemma 4, that it admits the decomposition (50). Write \( \beta = \phi^\perp \) and \( m = m_\phi \). If \( N \in \mathcal{N}_* \), then \( 1_N = \ast \) so that \( 0 = \phi(1_N) = \beta(1_N) + m(N) = m(N) \) i.e. \( m \in ba_{\ast} \). Likewise, if \( g \in \mathcal{L} \) then \( 0 \leq \beta(|g|1_N) = \phi(|g|1_N) \leq \gamma(|g|1_N) \leq 0 \) so that \( \beta \) vanishes on \( \mathcal{B}_\ast \), as claimed. The converse is obvious.

**Appendix B. Proofs**

**Proof of Lemma 7** In any lattice \( X \) the operation \( x \to x^- \) is subadditive, that is \( (x+y)^- \leq x^- + y^- \). Thus, if \( \mathcal{X} = \{X(\alpha) : \alpha \in \mathcal{A} \} \) and \( f, g \in \mathcal{F}_0(\mathcal{X}) \)
\[ q(f + g) = \sum_{X \in \mathcal{X}} \left\{ [(f + g)(X)]^+ a(X) - [(f + g)(X)]^- b(X) \right\} \]
\[ = \sum_{X \in \mathcal{X}} \left\{ (f + g)(X)a(X) + (f + g)(X)^-(a(X) - b(X)) \right\} \]
\[ \leq \sum_{X \in \mathcal{X}} \left\{ (f + g)(X)a(X) + (f(X)^- + g(X)^-)(a(X) - b(X)) \right\} \]
\[ = q(f) + q(g) \]
Positive homogeneity is clear. Suppose now that \( f, g \in \mathcal{F}_0(X) \) satisfy \( fg \geq 0 \) that is \( f(X) \) and \( g(X) \) have the same sign for all \( X \in \mathcal{X} \). It is then obvious that \( (f + g)(X)^- = f(X)^- + g(X)^- \) from which the claim follows.

**Lemma 5.** The functional \( \pi : \mathcal{F}(\Omega) \to \mathbb{R} \) is \( \geq_{\ast} \)-monotone, positively homogeneous and satisfies
\[ \pi(X(\theta)) \leq q(\theta) \quad \theta \in \Theta \quad \text{and} \quad \pi(f + g) \leq \pi(f) + \pi(g) \]
for all \( f, g \in \mathcal{F}(\Omega) \) for which the sum \( \pi(f) + \pi(g) \) is defined. Moreover, the following properties are equivalent: (i) \( q \) is coherent, (ii) \( \pi(0) = 0 \), (iii) \( \pi^c(1) \leq q_0 \) and (iv)
\[ |\pi(b)| < \infty \quad \text{and} \quad \pi(f^\ast) \geq \pi(f + b) \geq \pi(f) + \pi^c(b^\ast) \quad \text{for all} \quad f \in \mathcal{F}(\Omega), \ b \in \mathcal{B}_\ast \]

**Proof.** Monotonicity, positive homogeneity and the first part of (53) are obvious properties of \( \pi \). Assume that \( f, g \in \mathcal{F}(\Omega) \) are such that \( \pi(f) + \pi(g) \) is a well defined element of \( \mathbb{R} \). Thus if, say, \( \pi(f) =
of (54) follows from (53). Conversely, by (54) we conclude that 

\[ \lambda \]

then there exist \( \pi \) such that \( \pi \geq 0 \) and \( \theta_f, \theta_g \in \Theta \) such that \( \lambda_f \bar{X}(\theta_f) \geq_\ast f \) and \( \lambda_g \bar{X}(\theta_g) \geq_\ast g \) so that \( \lambda(\bar{X}(\theta_f') + \bar{X}(\theta_g')) \geq_\ast f + g \), with \( \lambda = \lambda_f + \lambda_g \) and \( \theta_f' = \theta_f \lambda_f / \lambda \) and \( \theta_g' = \theta_g \lambda_g / \lambda \) (with the convention 0/0 = 0). Given that, by Assumption 1 \( \theta = \theta_f' + \theta_g' \in \Theta \) we conclude that

\[ \pi(f + g) \leq \lambda q(\theta) \leq \lambda(q(\theta_f') + q(\theta_g')) = \lambda_f q(\theta_f) + \lambda_g q(\theta_g) \]

and, the inequality being true for all \( \lambda_f, \lambda_f \) and \( \theta_f, \theta_g \) as above, the second half of (53) follows. (53) also implies \( \pi(0) \leq 0 \). It is then clear that (ii) is equivalent to (i). If \( \theta \in \Theta \) and \( \lambda \geq 0 \) are such that \( \lambda \bar{X}(\theta) \geq_\ast -1 \) then \((1 + \lambda) \bar{X} \left( \frac{\lambda \theta + \delta_0}{1 + \lambda} \right) \geq_\ast 0 \) so that

\[ \pi(0) \leq (1 + \lambda) q \left( \frac{\lambda \theta + \delta_0}{1 + \lambda} \right) \leq \lambda q(\theta) + q_0 \]

We thus conclude that \( q_0 \geq \pi(0) + \pi^c(1) \) and so that \( \pi(0) = 0 \) implies \( \pi^c(1) \leq q_0 \). If \( b \in \mathfrak{B}_* \), then (53) implies \( |b|^* \pi(-1) \leq |b| q(\pi) \leq |b|^* \pi(1) \) so that from (iii) we deduce \( |\pi(b)| \leq \infty \). But then the sums \( \pi(f) + \pi(b) \) and \( \pi(f + b) + \pi(-b) \) are well defined for each \( f \in \mathfrak{F}(\Omega) \) and the second half of (54) follows from (53). Conversely, by (54) we conclude that \( \pi(0) = n \pi(0) \) for each \( n \in \mathbb{N} \) and \( \pi(0) \in \mathbb{R} \) so that \( \pi(0) = 0 \).

**Proof of Lemma 3** If \( \pi^c(1) = 0 \) and \( \bar{X}(\theta) \geq_\ast 0 \) then \( q(\theta) \geq \pi(-1) \). If \( \bar{X}(\theta) \ast < 0 \) then \( \bar{X}(\theta)/|\bar{X}(\theta)| \geq_\ast -1 \) and so \( q(\theta)/|\bar{X}(\theta)| \geq_\ast 0 \). Conversely, \( \lambda \bar{X}(\theta) \geq_\ast -1 \) and \( \lambda > 0 \) imply \( \theta \in \Theta_* \) and thus \( q(\theta) \geq 0 \) so that \( \pi(-1) \geq 0 \).

Denote by

\[ \Phi(\pi) = \{ \phi \in \mathfrak{F}(\mathcal{K}) : \phi \text{ positive, linear and such that } \phi \leq \pi \} \]

Adopting the notation of Lemma 1 we can also write

\[ \mathcal{M}(\pi) = \{ m_\phi : \phi \in \Phi(\pi) \} \quad \text{and} \quad \Phi^\perp(\pi) = \{ \phi^\perp : \phi \in \Phi(\pi) \} \]

**Lemma 6.** If \( q \) is coherent then the set \( \mathcal{M} \) defined in (28) is non empty, convex and weak* compact subset of \( ba_+ \). Moreover, \( \mathcal{M} = \mathcal{M}(\pi) \) (see (56)).

**Proof.** If \( q \) is coherent, \( \mathcal{M} \) is non empty by Theorem 3\(^1\). By (28), \( \mathcal{M}(\pi) \subset \mathcal{M} \). Thus, we only need to prove that \( \mathcal{M} \) is closed in the weak* topology of \( ba \) and that \( \mathcal{M} \subset \mathcal{M}(\pi) \). Let \( m_0 \) be an element of the closure of \( \mathcal{M} \) and \( f \in \mathcal{K} \). Then \( m_0 \in ba_{*,+} \) and

\[
\int (|f| \wedge n) dm_0 \leq \sup_{m \in \mathcal{M}} \int (|f| \wedge n) dm = \sup_{m \in \mathcal{M}} \int |f| dm \leq \pi(|f|)
\]

so that the sequence \( (|f| \wedge n)_{n \in \mathbb{N}} \) is Cauchy in \( L^1(m_0) \). Moreover, for all \( c > 0 \)

\[
v(m_0)(|f| > c + |f| \wedge n) \leq v(m_0)(|f| > c + n) \leq \frac{1}{c + n} \int (|f| \wedge (c + n)) dm_0 \leq \frac{\pi(|f|)}{c + n}
\]

\(^1\)By \( v(m) \) we total variation of \( m \) as defined in [17, III.1.9].
Lemma 7. Let Assumption 5 hold. Let $m$ and the fact that $\sup m$ and (57) is proved. (58) follows from (59).

Likewise, given that $\mathfrak{m}$ and so that $\phi = \beta$.
Moreover,

Proof. Positivity of $\beta$ follows from (30) and the fact that $\phi^\perp(f) \geq 0$ for each $f \in \mathcal{X}_*$ and $\phi \in \Phi(\pi)$. Moreover,

$$\sup_{m \in \mathcal{M}} \int f \, dm = \sup_{m \in \mathcal{M}} \lim_{n} \int (f \land -n) \, dm \leq \lim_{n} \sup_{m \in \mathcal{M}} \int (f \lor -n) \, dm \leq \lim_{n} \pi(f \lor -n)$$

and (57) is proved. (58) follows from

$$\beta(f) = \pi(f) - \sup_{m \in \mathcal{M}} \int f \, dm = \pi(f) - \sup_{m \in \mathcal{M}} \lim_{n} \int (f \land n) \, dm = \pi(f) - \lim_{n} \sup_{m \in \mathcal{M}} \int (f \land n) \, dm$$

and the fact that $\sup_{m \in \mathcal{M}} \int (f \land n) \, dm = \pi(f \land n)$ whenever $f \in \mathcal{X}_*$.

In the next results write $J(X) = \{0 = j_0 < j_1 < \ldots < j_I\}$ and $j_{I+1} = X^*$.

Corollary 2. The functional $\beta$ defined in (29) is positive and satisfies

$$- \lim_{n} \{\pi(f) - \pi(f \land n)\} \leq \beta(f) \leq \lim_{n} \{\pi(f) - \pi(f \land n)\} \quad f \in \mathcal{X}$$

and

$$\beta(f) = \lim_{n} \{\pi(f) - \pi(f \land n)\} \quad f \in \mathcal{X}_*$$

Proof. Positivity of $\beta$ follows from (30) and the fact that $\phi^\perp(f) \geq 0$ for each $f \in \mathcal{X}_*$ and $\phi \in \Phi(\pi)$. Moreover,

$$\sup_{m \in \mathcal{M}} \int f \, dm = \sup_{m \in \mathcal{M}} \lim_{n} \int (f \land -n) \, dm \leq \lim_{n} \sup_{m \in \mathcal{M}} \int (f \lor -n) \, dm \leq \lim_{n} \pi(f \lor -n)$$

and (57) is proved. (58) follows from

$$\beta(f) = \pi(f) - \sup_{m \in \mathcal{M}} \int f \, dm = \pi(f) - \sup_{m \in \mathcal{M}} \lim_{n} \int (f \land n) \, dm = \pi(f) - \lim_{n} \sup_{m \in \mathcal{M}} \int (f \land n) \, dm$$

and the fact that $\sup_{m \in \mathcal{M}} \int (f \land n) \, dm = \pi(f \land n)$ whenever $f \in \mathcal{X}_*$.

In the next results write $J(X) = \{0 = j_0 < j_1 < \ldots < j_I\}$ and $j_{I+1} = X^*$.

Lemma 7. Let Assumption 5 hold. Let $g \in \Gamma$, $F(X) = \sum_{i=1}^{I} \alpha_i X(j_i)$. Then,

$$\frac{F(X)}{X \land 1} \geq \frac{g(X)}{X \land 1} \quad \text{if and only if} \quad F(j_i) \geq g(j_i) \quad i = 1, \ldots, I + 1$$
Lemma 8. Let Assumption 5 hold, choose \( g \) as in (61) may be taken to be of the form

\[
(62) \quad \begin{cases} g(j_1) \\ g(j_2) \\ \vdots \\ g(j_I) \\ g(j_{I+1}) \end{cases}
\]

and

\[
(63) \quad D = \begin{bmatrix} (j_1 - j_0) & 0 & \ldots & 0 \\ (j_2 - j_0) & (j_2 - j_1) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (j_I - j_0) & (j_I - j_1) & \ldots & 0 \\ (j_{I+1} - j_0) & (j_{I+1} - j_1) & \ldots & (j_{I+1} - j_I) \end{bmatrix}
\]

The program

\[
(64) \quad \min_{\{\lambda \in \Theta_X, \; \lambda > 0\}} \lambda q(\theta) \text{ subject to } \lambda \frac{X(\theta)}{X \land 1} \geq g(X)
\]

is solved by the vector

\[
(65) \quad \theta_X(g) = \sum_{i=0}^{I} w[i+1] \delta_X(j_i) \in \Theta_X
\]

Proof. If \( F(j_i) < g(j_i) - \varepsilon \) for some \( \varepsilon > 0 \) and \( i = 1, \ldots, I+1 \) then by continuity there exists \( (j_i - j_{i-1})/2 \geq \eta > 0 \) such that \( f < g - \varepsilon \) in restriction to the set \( A_i = \{k_i - \eta < X \leq k_i\} \). By Assumption 5, \( A_i \notin \mathcal{N}_r \), moreover, \( X' \geq (j_0 + j_{i-1})/2 > 0 \) on \( A_i \) so that \( F(X')/(X' \land 1) \geq g(X)/(X \land 1) \) is contradicted.

Conversely, if \( F(j_i) \geq g(j_i) \) holds for \( j = 1, \ldots, I+1 \), then, given that \( F(0) = g(0) = 0 \), that \( g \) is convex and \( f \) piecewise linear, we conclude that \( F(x) \geq g(x) \) for all \( 0 \leq x \leq X^* \) and so that \( f(X) \geq g(X) \) and thus \( F(X)/(X \land 1) \geq g(X)/(X \land 1) \).

Theorem 8. Let Assumption 5 hold, choose \( g \in \Gamma \). Write

\[
(66) \quad g = \begin{bmatrix} g(j_1) \\ g(j_2) \\ \vdots \\ g(j_I) \\ g(j_{I+1}) \end{bmatrix}, \quad D = \begin{bmatrix} (j_1 - j_0) & 0 & \ldots & 0 \\ (j_2 - j_0) & (j_2 - j_1) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (j_I - j_0) & (j_I - j_1) & \ldots & 0 \\ (j_{I+1} - j_0) & (j_{I+1} - j_1) & \ldots & (j_{I+1} - j_I) \end{bmatrix}
\]

The program

\[
(67) \quad \min_{\{\lambda \in \Theta_X, \; \lambda > 0\}} \lambda q(\theta) \text{ subject to } \lambda \frac{X(\theta)}{X \land 1} \geq g(X)
\]

is solved by the vector

\[
(68) \quad \theta_X(g) = \sum_{i=0}^{I} w[i+1] \delta_X(j_i) \in \Theta_X
\]
that $\lambda X(\theta) \triangleright_\pi q(X)$

$$
\lambda q(\theta) \geq \min_{\{a \in \mathbb{R}_+^{f+1} : Da \geq f\}} q^T a = \min_{\{a \in \mathbb{R}_+^{f+1} : Da \geq f\}} b^T Da \geq b^T g = q^T w = q(\theta X(g))
$$

\[\square\]

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