Two Resolutions of the Margin Loan Pricing Puzzle

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Abstract

This paper supplies two possible resolutions of Fortune’s (2000) margin-loan pricing puzzle. Fortune (2000) noted that the margin loan interest rates charged by stock brokers are very high in relation to the actual (low) credit risk and the cost of funds. If we live in the Black-Scholes world, the brokers are presumably making arbitrage profits by shorting dynamically precise amounts of their clients’ portfolios.

First, we extend Fortune’s (2000) application of Merton’s (1974) no-arbitrage approach to allow for brokers that can only revise their hedges finitely many times during the term of the loan. We show that extremely small differences in the revision frequency can easily explain the observed variation in margin loan pricing. In fact, four additional revisions per three-day period serve to explain all of the currently observed heterogeneity.

Second, we study monopolistic (or oligopolistic) margin loan pricing by brokers whose clients are continuous-time Kelly gamblers. The broker solves a general stochastic control problem that yields simple and pleasant formulas for the optimal interest rate and the net interest margin. If the author owned a brokerage, he would charge an interest rate of \((r + \nu)/2 - \sigma^2/4\), where \(r\) is the cost of funds, \(\nu\) is the compound-annual growth rate of the S&P 500 index, and \(\sigma\) is the volatility.

Keywords: Margin loans, Arbitrage pricing, Super-hedging, Interest margin, Continuous-time Kelly rule

JEL Classification: C44, D42, D43, D53, D81, E43, G11, G13, G24

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1 Introduction

Anyone who has used a significant amount of margin debt is well aware that stock brokers in the United States differ widely in the interest rates they charge their clients on such loans. In addition to this heterogeneity, Fortune (2000) noted the puzzling fact that margin rates are very high in relation to the actual (low) credit risk and the cost of funds. For one thing, U.S. law caps the initial loan-to-value ratio on such debt at 50%. For another, U.S. brokerage customers who use margin debt will (in the aggregate) hold very liquid, high-quality collateral (e.g. the market portfolio). Even if market fluctuations cause some accounts to have negative equity, in practice customers will often respond to margin calls by depositing additional funds into the account. The broker’s low risk is underscored by the fact that there is an organized market for funding for such loans. As of this writing (September 2018) the “broker call rate” is $r = 3.75\%$, a mere 86 basis points above the 5-year Treasury yield.

Not only is the credit risk low, but certainly brokers also have the wherewithal to hedge some or all of it away, albeit at the cost of lower expected profits. In the Black-Scholes (1973) world, assuming that the client’s portfolio follows a geometric Brownian motion, the broker could eliminate risk by shorting a precise, continuously revised amount of the client’s holdings. The cost of delta-hedging would eat into the net interest margin, but the broker would be guaranteed a riskless profit without using any of his own capital. Fortune (2000) showed that, even assuming that stocks follow jump diffusions with very high volatility, Merton’s (1974) no-arbitrage analysis fails to rationalize the observed margin rates.

In the Black-Scholes world, if the broker can continuously monitor the client’s portfolio for solvency, then the no-arbitrage axiom implies that the price of margin debt must equal the cost of funds. Because of the continuous sample paths, there is
no default risk, as the broker can liquidate an account the instant its equity equals zero (or some other threshold). For example, Interactive Brokers (which offers the lowest available margin rates) behaves in just this way. Thus, a risk premium \( R > r \) obtains only if the portfolio is unmonitored over some fixed loan term, \( T \). Outside of major panics, the practical default risk comes from leveraged portfolios that are held overnight or over the weekend. Hence, a reasonable value of \( T \) should not exceed three calendar days.

Section 2 extends Fortune’s (2000) analysis to model brokers that can only revise their hedges finitely many times over the loan term. Instead of exact, continuous hedging, the broker is now assumed to super-hedge (or super-replicate) his liability on a “filled-in” binomial lattice at \( N \) discrete points in time. This framework accommodates very general price dynamics, as the gross-return \( S(t + \Delta t)/S(t) \) may be distributed over \([d, u]\) in any manner whatsoever. A super-hedge, as defined by Bensaid, Lesne, Pages, and Scheinkman (1992), is a trading strategy, together with an initial deposit of money, that guarantees to make no loss for all possible market behavior. The super-hedging cost (or super-hedging price) of a contingent claim is the lowest possible monetary deposit that (together with some special trading strategy) produces cash flows that dominate the derivative payoff. In our particular problem, as \( N \) increases, the broker is able to lower the rate it charges on margin loans, while still being able to guarantee no loss. As \( N \to \infty \), the broker’s short position converges to the one specified by the Black-Scholes (\( \Delta \)-hedging) strategy, and the broker’s margin rate converges to the rate studied by Fortune (2000). We show that for a loan term of \( T = 3 \) days, the only distinguishing feature of the lowest-cost broker is that it is able to revise its short position an additional four times. The difference between 15 and 19 revisions is enough to explain all the heterogeneity observed among U.S. investment brokerages.
Section 3 studies the optimal behavior of a broker whose customers are continuous-time Kelly gamblers (Luenberger 1998). By contrast to the Markowitz (1952) mean-variance theory of investing, a Kelly (1956) gambler eschews the tangency portfolio (of maximum Sharpe ratio) in exchange for an asymptotically dominant trading strategy that has the maximum expected continuously-compounded (read: logarithmic) growth rate. It is well known (Breiman 1961) that in the long-run, with probability approaching 1, the Kelly rule outperforms any “essentially different” strategy by an exponential factor. A Kelly gambler who uses leverage will happen to maintain a certain (fixed) loan-to-value ratio at all times, which ratio depends on the quality of the available investment opportunities. Since the Kelly gamblers at a given brokerage will hold asymptotically 100% of the wealth, they will also owe 100% of all margin debt in the limit. Thus, we assume that the Kelly gambler’s broker acts as a monopolist over margin loans in the context of a permanent, infinite-horizon interaction. The broker solves a general stochastic control problem that yields simple and pleasant formulas for the optimal interest rate and the net interest margin.

2 Arbitrage pricing

We start by describing Fortune’s (2000) no-arbitrage method, which is a straight application of Merton (1974). Suppose that a client borrows $D$ dollars at $t = 0$ to finance the purchase of a single share of stock for $S_0$. The initial account equity is $E_0 = S_0 - D$. The stock price $S_t$ is assumed to follow a geometric Brownian motion

\[ \frac{dS_t}{S_t} = \mu dt + \sigma dW_t, \]  

(1)
where $\mu$ is the drift, $\sigma$ is the volatility, and $W_t$ is a unit Brownian motion. Assume that the risk-free rate (“broker call rate”) is $r$, and that the broker charges interest at a continuously-compounded rate of $R$ per year. At time $t$, the client’s equity is $E_t = S_t - De^{T}$. If the account is continuously monitored for solvency, then the broker can liquidate the account the instant that $E_t = 0$. Thus, under continuous monitoring, no-arbitrage considerations dictate that $R = r$.

To rationalize a margin rate $R > r$, we must assume that the broker will not check for solvency until some future time $T$. If $E_T \geq 0$, then the debt of $De^{RT}$ dollars will be paid in full. Otherwise, the client will abandon the account, leaving the broker with collateral worth $S_T$. Assuming that the broker borrowed the initial $D$ dollars on the money market at the risk-free rate, the broker’s final profit (loss) is

$$\pi_T = \min(S_T, De^{RT}) - De^{rT}. \quad (2)$$

Using the fact that $\max(x, y) + \min(x, y) = x + y$, and rearranging, we get

$$\pi_T = S_T + De^{RT} - De^{rT} - \max(S_T, De^{RT}) = S_T - De^{rT} - \max(S_T - De^{RT}, 0). \quad (3)$$

Thus, the client has in effect exchanged his initial equity $E_0$ for a call on one share of the stock, at the strike price $K = De^{RT}$. To rule out arbitrage opportunities, the expected present value of $\pi_T$ at $t = 0$ with respect to the risk-neutral measure must be zero:

$$E_0 = S_0 - D = C(S_0, 0; De^{RT}), \quad (4)$$

where $C$ is the time-0 price of a European call maturing at $T$ with strike price $K = De^{RT}$. Thus, no-arbitrage dictates the sensible requirement that the initial exchange (of equity $E_0$ for the call) be actuarially fair. Fortune (2000) notes that observed
interest rates on margin debt are far higher than no-arbitrage considerations would seem to dictate. However, he does not specify the particular horizon $T$ that was used in his study. As Figure 1 illustrates, in order to rationalize the current margin rates charged by different brokers in (May) 2018, we must admit horizons of up to three years. In reality, these brokers check for solvency on a weekly or even daily basis. We have used the sample values $r = 0.035, \sigma = 0.4, S_0 = 100, D = 50$. Note that an initial loan-to-value ratio of $D/S_0 = 50\%$ is the maximum allowed by U.S. law.

### 2.1 Super-hedging on a “filled-in” binomial lattice

If we live in the Black-Scholes (1973) world, then we must conclude that that either:

1. retail brokers are making large arbitrage profits on margin loans, or
2. these

![Figure 1: The implied time between solvency checks, for different margin rates. $r = 0.035, \sigma = 0.4, S_0 = 100, D = 50$. (The observed margin rates are from May 2018)](image-url)
brokers wait months or years before issuing margin calls. Neither explanation is particularly satisfactory. And in reality, the situation is even more favorable to the brokers than the model would suggest, because many clients will deposit additional funds instead of abandoning their insolvent accounts. In the United States, of course, defaulting on margin debt results in adverse legal action and negative credit reports.

However, we will show below that seemingly minor differences in brokers’ abilities to hedge their risks can result in substantial differences in the margin rates they can safely charge. To illustrate these effects, we consider a “filled-in” binomial lattice model of stock dynamics. We assume the broker subdivides the interval \([0, T]\) into \(N\) subintervals of length \(\Delta t = T/N\) each. We match the binomial lattice parameters \(u, d\) to the geometric Brownian motion in the standard way (Luenberger 1998): let \(u = e^{\sigma \sqrt{\Delta t}}\) and \(d = 1/u\). Unlike the usual binomial \(\{u, d\}\) lattice, we will allow the price relative \(S(t + \Delta t)/S(t)\) to take on any value between \(d\) and \(u\). Apart from the restriction that \(S(t + \Delta t)/S(t) \in [d, u]\), we will make no additional assumptions on the price process \(S_t\). For instance, it would be permissible for \(S(t + \Delta t)/S(t)\) to have a uniform distribution (or any other distribution) over the interval \([d, u]\).

Following Ritchken and Kuo (1988), the no-arbitrage price of a call on this filled-in lattice cannot exceed the Cox-Ross-Rubinstein (1979) price

\[
\sum_{j=0}^{N} \binom{N}{j} q^j (1 - q)^{N-j} \max(S_0 u^j d^{N-j} - K),
\]

where \(K\) is the strike price and \(q\) is the risk-neutral probability. Using this upper bound, the present value at \(t = 0\) of the broker’s arbitrage profit (loss) is at least

\[
S_0 - D - \sum_{j=0}^{N} \binom{N}{j} q^j (1 - q)^{N-j} \max(S_0 u^{2j-N} - De^{RT}, 0),
\]
where \( q = \frac{(e^{r\Delta t} - d)}{(u - d)} \) is the risk-neutral probability of an uptick. As \( N \to \infty \), this number converges (Cox and Rubinstein 1985) to \( S_0 - D - C(S_0, 0; K) \), where \( C(S_0, 0; K) \) is the Black-Scholes (1973) price. The highest rational price \( R \) that can be charged by a broker with the ability to make \( N \) revisions over \([0, T]\) is characterized by the equation

\[
S_0 - D = \sum_{j=0}^{N} \binom{N}{j} q^j (1 - q)^{N-j} \max(S_0 u^{2j-N} - D e^{RT}, 0). \tag{7}
\]

For each possible interest rate \( R > r \), let \( N(R) \) be the smallest number of revisions for which the broker can guarantee an arbitrage profit. Figure 2 plots \( N(R) \) for \( r = 0.035, \sigma = 0.4, T = 3/365, S_0 = 100, \) and \( D = 50 \). Note well that for a three-
day margin loan, very small changes in the frequency with which the broker is able to revise his (super) hedge will generate wide discrepancies in the margin rates the broker can profitably charge. Four additional revisions within a three-day period are enough to explain the difference between one broker who charges 4 percent and another who must charge 8 or 9 percent in order to guarantee no loss. To reiterate, we find that four additional revisions per three-day period suffice to explain all of the currently observed heterogeneity in margin loan pricing.

3 Margin loans to Kelly gamblers

3.1 Instantaneous demand

We take up the general stock market with $n$ correlated stocks $i \in \{1, \ldots, n\}$ in geometric Brownian motion, where the price $S_{it}$ of stock $i$ evolves according to

$$dS_{it}/S_{it} = \mu_i \, dt + \sigma_i \, dW_{it}.$$  (8)

$\mu_i$ and $\sigma_i$ are the drift and volatility, respectively, of stock $i$, and $W_{1t}, \ldots, W_{nt}$ are correlated unit Brownian motions. We let $\rho_{ij} = \text{Corr}(dW_{it}, dW_{jt})$ and $\sigma_{ij} = \rho_{ij} \sigma_i \sigma_j = \text{Cov}(dS_{it}/S_{it}, dS_{jt}/S_{jt})/dt$. $\Sigma = [\sigma_{ij}]_{n \times n}$ is the covariance matrix of instantaneous returns per unit time. Let $r_L$ be the interest rate the broker charges on margin loans, and let $r$ be the cost of funds (the money-market rate).

We assume that the broker’s sole customer is a Kelly (1956) gambler who follows a constant rebalancing rule $b = (b_1, \ldots, b_n)' \in \mathbb{R}^n$. This means that, at every instant $t$, the trader bets the fraction $b_i$ of his wealth on stock $i$. We let $V_t(b)$ denote the trader’s wealth at $t$. If $\sum_{i=1}^{n} b_i \geq 1$, then the trader has taken a margin loan of $\left(\sum_{i=1}^{n} b_i - 1\right) \cdot V_t(b)$.
dollars. If $\sum_{i=1}^{n} b_i \leq 1$, then the trader has cash deposit of $(1 - \sum_{i=1}^{n} b_i)V_t(b)$ dollars. If $b_i < 0$, then the trader has a short position in stock $i$. The gambler must continuously rebalance his portfolio so as to maintain the target allocation $b$. At instant $t$, he holds $b_i V_t(b)/S_{it}$ shares of stock $i$. Thus, the client’s wealth follows the geometric Brownian motion

$$dV_t/V_t = \left\{ \sum_{i=1}^{n} b_i \mu_i - \max \left( \sum_{i=1}^{n} b_i - 1, 0 \right) r_L + \max \left( 1 - \sum_{i=1}^{n} b_i, 0 \right) r \right\} dt$$

$$+ \sum_{i=1}^{n} b_i \sigma_i dW_{it}. \quad (9)$$

Note that there is no default risk, since a geometric Brownian motion is always positive. Money that is borrowed at $t$ gets repaid with interest at $t + dt$, and then the trader borrows again. Letting $B = \sum_{i=1}^{n} b_i$ and $\mu = (\mu_1, ..., \mu_n)'$, we get the concise expression

$$dV_t/V_t = \alpha dt + \sum_{i=1}^{n} b_i \sigma_i dW_{it}, \quad (10)$$

where $\alpha = \mu'b - (B - 1)^+ r_L + (1 - B)^+ r$ is the drift of the gambler’s wealth, and $x^+ \triangleq \max(x, 0)$ denotes the positive part of $x$. Applying Itô’s Lemma for several diffusion processes (Wilmott 2001), we obtain

$$V_t = V_0 \cdot \exp \left\{ (\alpha - b'\Sigma b/2) t + \sum_{i=1}^{n} b_i \sigma_i W_{it} \right\}. \quad (11)$$

The trader’s asymptotic growth rate is

$$\lim_{t \to \infty} \frac{\log(V_t/V_0)}{t} = \lim_{t \to \infty} \left\{ \alpha - b'\Sigma b/2 + \frac{1}{t} \sum_{i=1}^{n} b_i \sigma_i W_{it} \right\} = \alpha - b'\Sigma b/2. \quad (12)$$
Definition 1. The **Kelly rule** (or **growth-optimal rebalancing rule**) is

\[ b^* = \arg \max_{b \in \mathbb{R}^n} \{ \alpha(b) - b'\Sigma b/2 \}. \] (13)

The maximum asymptotic growth rate is called the **Kelly growth rate**.

**Proposition 1.** The growth rate \( \Gamma(b) = \alpha(b) - b'\Sigma b/2 \) is concave over \( \mathbb{R}^n \).

**Proof.** Since \( \Sigma \) is positive semidefinite, the term \( \mu'\Sigma - b'\Sigma b/2 \) is concave. Thus, it suffices to show that \((1-B)^+r - (B-1)^+r_L\) is concave in \( B \), since the linear transformation \( B = \sum_{i=1}^n b_i \) will preserve the concavity. To achieve this, we add and subtract \((1-B)^+r_L\) to get

\[
(1-B)^+(r-r_L) + [(1-B)^+ - (B-1)^+]r_L = (1-B)^+(r-r_L) + (1-B)r_L. \] (14)

Since \( r-r_L < 0 \) and \((1-B)^+\) is convex in \( B \), we have the sum of a concave function and a linear function, and the result follows. \( \square \)

Figure 3 gives a typical plot of \((1-B)^+r - (B-1)^+r_L\). Figure 4 gives a typical plot of the growth rate \( \Gamma(b) \) for leveraged bets on a single high-quality stock or index.

**Proposition 2.** If the trader uses margin debt, his rebalancing rule will be \( b^* = \Sigma^{-1}(\mu - r_L 1) \), where \( 1 \) is an \( n \times 1 \) vector of ones. If the trader holds positive cash balances, his bets will be \( b^* = \Sigma^{-1}(\mu - r 1) \). If the trader’s net cash position is zero, he will use the portfolio \( b^* = \Sigma^{-1}(\mu - \lambda 1) \), where \( \lambda = (1'\Sigma^{-1}\mu - 1)/(1'\Sigma^{-1}1) \).

**Proposition 3.** Let \( \lambda = (1'\Sigma^{-1}\mu - 1)/(1'\Sigma^{-1}1) \). The trader will use margin debt if and only if \( r_L < \lambda \). The trader will hold a positive cash balance if and only if \( \lambda < r_D \). The trader’s net cash position will be zero if and only if \( r_D \leq \lambda \leq r_L \).
Figure 3: Plot of $(1 - B)^+ r - (B - 1)^+ r_L$ for $r = 0.01$ and $r_L = 0.06$.

**Corollary 1.** At time $t$, the trader’s demand for margin debt is

$$q = V_t(b)[I'\Sigma^{-1}\mu - 1 - (I'\Sigma^{-1}I)r_L] = V_t(b)(I'\Sigma^{-1}I)(\lambda - r_L) = C - Dr_L,$$

where $C = V_t(b)(I'\Sigma^{-1}\mu - 1)$ and $D = V_t(b)I'\Sigma^{-1}I$.

**Corollary 2.** The elasticity of the continuous-time Kelly gambler’s demand for margin debt is given by

$$\epsilon = \frac{r_L}{\lambda - r_L},$$

where $\lambda$ is the shadow price of margin debt.
Figure 4: Growth rates for different bets $b$ on one stock, $\sigma = 0.2, \nu = 0.09, \mu = \nu + \sigma^2/2, r_L = 0.03$. The kink is at $b = 1$, and the Kelly rule is $b^* = 2$. Overbetting ($b > 2$) is insane, since it entails a lower growth rate and more risk.
3.2 Monopoly pricing

We assume that the broker acts monopolistically, with instantaneous profit

$$\pi(r_L) = D(\lambda - r_L)(r_L - r).$$  \hspace{1cm} (17)$$

Differentiating, we have the first-order condition

$$\lambda - r_L - (r_L - r) = 0.$$  \hspace{1cm} (18)$$

Thus, the monopoly price of margin debt (for clients who are continuous-time Kelly gamblers) is

$$r_L = \frac{r + \lambda}{2} = \frac{r + (1'\Sigma^{-1}\mu - 1)/1'\Sigma^{-1}1}{2}. \hspace{1cm} (19)$$

**Example 1.** For a single stock with drift $\mu$, volatility $\sigma$, and growth rate $\nu = \mu - \sigma^2/2$, the monopoly price of margin debt is

$$r_L^* = \frac{r + \mu - \sigma^2}{2} = \frac{r + \nu}{2} - \sigma^2/4.$$  \hspace{1cm} (20)$$

and the net interest margin is $r_L - r = (\nu - r)/2 - \sigma^2/4$. Thus, the interest rate is increasing in $r$ but the net interest margin is decreasing in $r$. As the asset becomes more attractive (higher growth rate and less volatility), the interest rate increases.

**Example 2.** For the values $r = 0.035, \nu = 0.09, \sigma = 0.25$, a monopolistic broker should charge 4.7% margin interest.

**Example 3.** Assume there are two stocks with correlation $\rho = 0.5$, both having the the compound-growth rate $\nu = 0.09$ and volatility $\sigma = 0.25$, with the money market rate being $r = 0.035$. The broker should charge 5.47% interest on margin loans.
3.3 Cournot pricing

More generally, we can consider margin loans supplied by $N$ oligopolistic brokers in Cournot competition. The inverse aggregate demand curve is $r_L = \lambda - Q/D$, where $Q = \sum_{i=1}^{N} q_i$ is the aggregate quantity of margin loans, and $q_i$ is broker $i$’s quantity. Broker $i$’s profit is

$$\pi_i(q_1, ..., q_N) = q_i(\lambda - r - q_i/D - Q_{-i}/D),$$

(21)

where $Q_{-i} = \sum_{j \neq i} q_j$ is the aggregate quantity supplied by $i$’s competitors. Broker $i$’s first-order condition is

$$\lambda - r - Q/D - q_i/D = 0.$$  

(22)

In the (symmetric) Cournot equilibrium, all brokers supply the same quantity $q_i \equiv q$, and we have $Q = Nq$. Thus, we have

$$q = \frac{D(\lambda - r)}{N + 1}$$

$$Q = \frac{ND(\lambda - r)}{N + 1}$$

$$r_L = \frac{Nr + \lambda}{N + 1}$$

$$r_L - r = \frac{\lambda - r}{N + 1}$$

$$\Pi = ND\left(\frac{\lambda - r}{N + 1}\right)^2,$$

(23)

where $\Pi$ is the aggregate profit and $r_L - r$ is the net interest margin.
3.4 Risk-averse broker with an infinite horizon

In this subsection, we model the broker as a risk-averse monopolist with an infinite horizon. We let $e^{-\beta t}$ denote the broker’s discount factor, and we let $\pi_t$ denote the instantaneous rate of profit per unit time. Thus, the broker picks an interest rate $r_L$ that solves

$$\max_{r_L \in (r, \lambda)} E_0 \left\{ \int_0^{\infty} e^{-\beta t} \log \pi_t \, dt \right\},$$

where $\pi_t = V_t(b)(1')\Sigma^{-1}(1)(\lambda - r_L)(r_L - r)$ and $b = \Sigma^{-1}(\mu - r_L 1)$ is the client’s continuously-rebalanced portfolio. After simplification and monotonic transformation, the (concave) objective function becomes

$$U(r_L) = \beta \log \left[ (\lambda - r_L)(r_L - r) \right] + \alpha - b'\Sigma b/2$$

$$= \beta \log \left[ (\lambda - r_L)(r_L - r) \right] + r_L + \frac{1}{2}(\mu - r_L 1)'\Sigma^{-1}(\mu - r_L 1).$$

(25)

The broker’s first-order condition is

$$1'\Sigma^{-1}(1)(\lambda - r_L)^2(r_L - r) = 2\beta \left( \frac{r + \lambda}{2} - r_L \right).$$

(26)

**Proposition 4.** If the broker is a risk-averse monopolist with an infinite horizon, then it always sets a margin rate that lower than the instantaneous monopoly price $(r + \lambda)/2$.

*Proof.* Since the left-hand side of the first-order condition (26) is positive, the right-hand side must be positive as well. This yields $r_L < (r + \lambda)/2$. 

**Proposition 5.** The broker’s margin rate is strictly increasing in the discount rate $(dr_L/d\beta > 0)$. As $\beta \to +\infty$, the broker’s margin rate $r_L^*(\beta)$ converges to the in-
stantaneous monopoly rate \((r + \lambda)/2\). As \(\beta \to 0^+\), the margin rate approaches the money-market rate \(r\).

**Proof.** Differentiating the broker’s first-order condition implicitly with respect to \(\beta\), we find

\[
\frac{dr_L}{d\beta} \left( \frac{1}{r_L - r} - \frac{2}{\lambda - r_L} + \frac{2}{r + \lambda - 2r_L} \right) = \frac{1}{\beta}.
\]

The expression in parentheses is positive: viz, let \(x = r_L - r, y = \lambda - r_L\), where \(y - x\) is positive. We must show that

\[
\frac{1}{x} - \frac{2}{y} + \frac{2}{(y - x)} > 0,
\]

which is equivalent to \(2x^2 + y(y - x) > 0\), which is true. Now, as \(\beta \to +\infty\), the left-hand side of (26) is a bounded quantity. Thus, the right-hand side must remain bounded as well. The only way to avoid contradiction is for \(r_L \to (r + \lambda)/2\). Similarly, as \(\beta \to 0^+\), the right-hand side tends to 0. Thus, we must have \(r_L \to \lambda\) or \(r_L \to r\). However, since \(r_L\) is less than the midpoint of the interval \((r, \lambda)\), we must have \(r_L \to r\). \(\square\)

**Corollary 3.** For any observed margin rate \(r_L < (r + \lambda)/2\) charged by a given broker, there is a unique discount rate \(\beta\) that rationalizes \(r_L\).

Figure 5 plots \(r^*_L\) against \(\beta\) for the case of a single stock with \(\sigma = 25\%\) annual volatility that grows at a compound rate of \(\nu = 10\%\) per year. The cost of funds is assumed to be \(r = 3.5\%\).

### 3.5 The general stochastic control problem

In this subsection, we formulate and solve the general version of the broker’s stochastic control problem. In so doing, we show that the restriction to constant interest rate
Figure 5: Margin rate versus broker’s personal discount rate: $n = 1, r = 0.035, \nu = 0.1, \sigma = 0.25$. 
policies entails no loss of generality. We now allow the broker’s instantaneous interest rate \( r_L = r_L(V_t) \) to depend on the customer’s wealth \( V_t \). Thus, the broker chooses a feedback-control policy \( r_L(V_t) \) to solve

\[
\max_{r_L(\cdot)} E_0 \left\{ \int_0^\infty e^{-\beta t} \log \left[ V_t(\lambda - r_L)(r_L - r) \right] dt \right\},
\]

(29)

under the transition law

\[
dV_t/V_t = \{ r_L + (\mu - r_L 1)\Sigma^{-1}(\mu - r_L 1) \} dt + \sum_{i=1}^n b_i \sigma_i dW_{it},
\]

(30)

where \( b = \Sigma^{-1}(\mu - r_L 1) \) is the customer’s portfolio. Letting \( J = J(V) \) denote the broker’s maximum value function, we have the HJB equation (Kamien and Schwartz 1981)

\[
\beta J(V) = \max_{r_L} \left\{ \log \left[ V(\lambda - r_L)(r_L - r) \right] \\
+ [r_L + (\mu - r_L 1)\Sigma^{-1}(\mu - r_L 1)] V J'(V) + \frac{1}{2}(\mu - r_L 1)'\Sigma^{-1}(\mu - r_L 1)V^2 J''(V) \right\}.
\]

(31)

We make the guess \( J(V) = c_1 + c_2 \log V \), where \( c_1 \) and \( c_2 \) are undetermined coefficients. After simplification, this turns the HJB equation into

\[
\beta c_1 + (\beta c_2 - 1) \log V = \max_{r_L} \left\{ \log \left[ (\lambda - r_L)(r_L - r) \right] + c_2 \left[ r_L + \frac{1}{2}(\mu - r_L 1)'\Sigma^{-1}(\mu - r_L 1) \right] \right\}.
\]

(32)
Since the right-hand side of the HJB equation does not depend on $V$, we must have $c_2 = 1/\beta$. Thus, we get

$$c_1 = \max_{r_L} \left\{ \frac{1}{\beta} \log \left[ (\lambda - r_L)(r_L - r) \right] + \frac{1}{\beta^2} \left[ r_L + \frac{1}{2} (\mu - r_L) \Sigma^{-1} (\mu - r_L 1) \right] \right\}. \tag{33}$$

In spite of its complicated form, we have succeeded in expressing $c_1$ solely in terms of the model parameters $\beta, \lambda, r, \mu,$ and $\Sigma$. The maximizer $r^*_L$, which is independent of $V$, is characterized by the first-order condition (26) that we already derived. For these particular coefficients $c_1$ and $c_2$, substitution of $J = c_1 + c_2 \log V$ turns the HJB equation into an identity. This proves that the broker’s optimal feedback-control policy is a constant interest rate $r^*_L$, a rate that solves the maximization problem (33) above.

4 Conclusion

This paper supplied two possible resolutions of Fortune’s (2000) margin-loan pricing puzzle. Fortune (2000) pointed out that the observed margin interest rates charged by stock brokers are very high in relation to the actual (low) credit risk and the cost of funds. In the Black-Scholes (1973) world, the broker could eliminate risk by continuously shorting a dynamically precise amount of the customer’s portfolio, earning substantial arbitrage profits on the margin loan.

First, we extended Fortune’s application of Merton’s (1974) no-arbitrage approach to allow for brokers who can only revise their hedges finitely many times over the term of the loan. We concluded that very small differences in the revision frequency (say, four extra revisions per 3-day period) can easily explain the wide discrepancies in observed margin loan interest rates.
Next, we studied monopolistic (or oligopolistic) margin loan pricing by brokers whose customers are continuous-time Kelly gamblers (Luenberger 1998). This is a sensible assumption, since continuous-time Kelly gamblers will asymptotically “crowd out” the other brokerage customers, holding (in the limit) all of the customer wealth and shouldering all of the margin debt. The continuous-time Kelly gambler’s instantaneous elasticity of demand for margin loans is \( \epsilon = r_L/(\lambda - r_L) \), where \( \lambda = (1'\Sigma^{-1}\mu - 1)/(1'\Sigma^{-1}1) \) is the shadow price of a one-dollar margin loan. Here, \( \mu \) denotes the drift vector of the stock market, \( \Sigma \) is the covariance of instantaneous returns per unit time, and \( r_L \) is the (continuously-compounded) annual interest rate charged by the broker. We found that the instantaneous monopoly price of margin debt is \((r + \lambda)/2\), the midpoint of the broker’s cost of funds and the customer’s shadow value. More generally, under Cournot competition with \( N \) brokers, the correct interest rate is the convex combination \((Nr + \lambda)/(N + 1)\) and the net interest margin is \((\lambda - r)/(N + 1)\).

Finally, we modelled the broker as a patient, risk-averse monopolist (with log utility) on an infinite time-horizon. This led to the cubic equation

\[
1'\Sigma^{-1}1(\lambda - r_L)^2(r_L - r) = 2\beta\left(\frac{r + \lambda}{2} - r_L\right),
\]

which uniquely characterizes the optimum interest rate \( r_L \). Under these assumptions, we found that the broker always charges a price that is lower than the instantaneous monopoly price \((r + \lambda)/2\). However, as the broker becomes increasingly impatient (and his discount rate \( \beta \) tends to \(+\infty\)), his margin rate increases monotonically to the instantaneous monopoly rate. As the broker gets more patient (\( \beta \to 0^+ \)), his price converges to the money-market rate \( r \). On account of the unique correspondence between the broker’s discount rate \( \beta \) and its margin rate \( r_L \), we were able to uniquely
rationalize heterogeneous observed pricing behavior \( r_L \in [r, (r + \lambda)/2] \), on the basis of the brokers’ various levels of impatience to book current profits. To close the paper, we showed that the broker gets no benefit from conditioning his interest rate \( r_L = r_L(V_t) \) on the customer’s wealth \( V_t \). We formulated and solved the general version of the broker’s stochastic control problem, finding that the optimal feedback-control policy \( r_L(V_t) \) is to set a constant interest rate \( r^*_L \), a rate which is characterized by (34).

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