Covariant formulation of electrodynamics in isotropic media

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Abstract

The equations of electromagnetic fields in a medium is usually written in the rest frame of the medium. We outline a method of generalizing the discussion to arbitrary inertial frames. In the discussion, we also include the possibility that the medium is optically active, a possibility that is often overlooked in discussions of electromagnetic fields in a medium.

1 Aim of the paper

Covariant formulation of electrodynamics in the vacuum is a subject that appears in standard textbooks of electrodynamics [1]. Electrodynamics in material media is also a standard subject, but is usually not discussed in a manifestly covariant manner. A medium of course provides a preferred frame for the discussion, and this frame is used for the standard formulations. However, a preferred frame does not preclude us from discussing the subject from other frames. When we discuss the decay of a particle, for example, there is a preferred frame, viz. the rest frame of the particle; but it does not mean that we cannot ask what would be the lifetime in any other frame. Similarly, for the case of electrodynamics, we might be interested about problems with a moving medium, and it is mandatory to obtain a Lorentz-covariant formulation of electrodynamics in a medium in order to discuss such subjects.

There have been attempts [2, 3] at general co-ordinate invariance, something that would be consistent with the general theory of relativity. This is a considerably involved topic, and will not be touched on in this paper. We keep our attention on a formulation consistent with the special theory of relativity, i.e., a Lorentz-invariant formulation. Surely, there are previous attempts in this direction as well. Some of them have been aimed at a quantum field theoretic formulations of the problem [4, 5]. They usually employ the scalar and vector potentials, encapsulated in the 4-vector potential $A^\mu$, to formulate the problem. There are also other attempts [6] where the response equations do not always look manifestly Lorentz-invariant, although covariance can be proved with a certain amount of effort.
Our aim in this article is twofold. First, we note that the presence of the medium implies an extra 4-vector in the problem, viz. the velocity 4-vector of the center of mass of the medium, $u^\mu$. While this 4-vector has been extensively used in the quantum field theoretic formulations of electrodynamics in a medium [4][5][7], to our knowledge it has not been used in textbook-level formulation of classical electrodynamics. We use only gauge-invariant objects in the discussion, except where $A^\mu$ is absolutely indispensable — viz., in the Lagrangian formulation, for writing the interaction of the field with the sources of the field.

Secondly, we include the description of natural optical activity in our formalism. Hardly any textbook on electrodynamics discusses this phenomenon. In one exception that does [8], tensorial responses in the medium seems essential to the explanation. However, even isotropic systems (like a sugar solution) show natural optical activity, so tensorial response functions should not be essential for describing the phenomenon. In the context of quantum field theoretical formulation, it was shown [9] that the most general linear response of an isotropic medium contains three response functions, i.e., one more above the dielectric function and the magnetic permeability. Later, the need for this extra function was demonstrated in the classical formulation, using 3-dimensional vectors for the electric and magnetic fields [10]. Here, we show how this extra constant can be accommodated in a completely covariant formulation.

Throughout, we use the Heaviside-Lorenz system of electromagnetic units, which is the most suitable system for a relativistic formulation.

2 Summary of covariant electrodynamics in the vacuum

We assume that the reader is familiar with the 4-dimensional covariant formulation of classical electrodynamics in the vacuum. We are presenting the key results here [1] in order to set up the notation and to aid the discussion of subsequent sections.

In the covariant formulation, the electromagnetic field is represented by a rank-2 anti-symmetric tensor $F_{\mu\nu}$, where the Greek indices are spacetime indices, 0 being the temporal direction and 1,2,3 the spatial directions. This tensor can be written as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$  \hspace{1cm} (1)

where $A_\mu$ is the 4-vector for electromagnetic potentials, and

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu}.$$  \hspace{1cm} (2)

The sources of the electromagnetic field are summarized by a 4-vector $J^\mu$ that is a function of position and time, just as $F_{\mu\nu}$ is. The relation is through the differential equation which can be written as

$$\partial_\mu F^{\mu\nu} = \frac{1}{c} J^\nu$$  \hspace{1cm} (3a)

by suitably adjusting a constant factor in the definition for $J^\mu$. Written in 3-dimensional vector notation, this equation contains two of the Maxwell equations, in particular the inhomogeneous
ones. The other two, i.e., the homogeneous equations, can be written in the 4-dimensional notation as

\[ \partial_{\mu} \tilde{F}^{\mu\nu} = 0, \quad (3b) \]

where \( \tilde{F}^{\mu\nu} \) is called the dual of the field tensor \( F_{\mu\nu} \), defined as

\[ \tilde{F}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\lambda\rho} F_{\lambda\rho} \quad (4) \]

with the help of the completely antisymmetric Levi-Civita symbol in 4-dimensional spacetime. While Eq. (3b) follows from the definitions in Eqs. (1) and (4), the inhomogeneous equations of Eq. (3a) can be derived from a Lagrangian density

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{c} J^{\mu} A_{\mu}. \quad (5) \]

The Euler-Lagrange equation, treating the \( A^{\mu} \)'s as the fields, are

\[ \partial_\alpha \left( \frac{\partial \mathcal{L}}{\partial (\partial_\alpha A_\beta)} \right) = \frac{\partial \mathcal{L}}{\partial A_\beta}. \quad (6) \]

When one uses the derivative rules

\[ \frac{\partial (\partial_\mu A_\nu)}{\partial (\partial_\alpha A_\beta)} = \delta^\alpha_\mu \delta^\beta_\nu, \quad (7a) \]
\[ \frac{\partial A_\nu}{\partial A_\beta} = \delta^\beta_\nu, \quad (7b) \]

one obtains Eq. (3a) in a straightforward manner.

3 Medium with a linear response

A medium contains charged particles. Thus, in any electromagnetic problem in a medium, there are two kinds of sources of the electromagnetic fields. If all these charges and currents, along with externally placed sources, are taken into account, the equations given in Sec. 2 are still valid. However, they are not convenient, or maybe even impossible, to use in practical problems, because of the difficulty of accounting for all charges and currents of the particles that constitute in the medium itself. The usual escape route is to parametrize the charges and currents bound within the medium by some quantities and set up the equations with the free charges and currents, over which an experimenter can have any kind of handle. We define the 3-vectors \( \vec{D} \) and \( \vec{H} \) whose sources are the free charges and currents, and write the inhomogeneous Maxwell equation in terms of them. These vectors are assumed to be linearly related to the electric and magnetic fields, \( \vec{E} \) and \( \vec{B} \):

\[ \vec{D} = \epsilon \vec{E}, \quad \vec{H} = \frac{1}{\mu} \vec{B}. \quad (8) \]
We will assume these linear relations throughout this article. As explained in Sec. 1, we need a covariant formulation for discussing problems with a moving medium. The problem is that, the covariant formulation in the vacuum uses the tensor $F_{\mu\nu}$, and we cannot say that in a medium we have an antisymmetric tensor that is proportional to $F_{\mu\nu}$ and which depends only on the free charges and currents. In other words, we cannot simply write an equation

$$\partial_{\mu} G^{\mu\nu} = \frac{1}{c} J_{\nu}^{\text{free}},$$

and claim that $G^{\mu\nu}$ is proportional to $F_{\mu\nu}$, because that way one would obtain just one parameter connecting the two tensors, whereas Eq. (8) contains two parameters of proportionality, or two response functions.

However, there is at least another 4-vector connected to the physical description of the problem. We are talking about a medium which can be in motion in the frame that we choose. If the medium, as a whole, moves with a velocity $\vec{v}$ in the frame of reference, we can define its dimensionless velocity 4-vector,

$$u^{\mu} = \frac{1}{\sqrt{1 - v^2/c^2}} \left\{ 1, \frac{\vec{v}}{c} \right\},$$

which satisfies the relation

$$u^{\mu} u_{\mu} = 1.$$  \hspace{1cm} (11)

We can use this vector to formulate the equations of electrodynamics in a medium.

Immediately, we notice that we can use this 4-vector and the electromagnetic field tensor to define two new 4-vectors [5,11]:

$$E^{\mu} = F^{\mu\nu} u_{\nu},$$

$$B^{\mu} = -\tilde{F}^{\mu\nu} u_{\nu} = \frac{1}{2} \varepsilon^{\mu\nu\lambda\rho} u_{\nu} F_{\lambda\rho}.$$  \hspace{1cm} (12a, 12b)

We have adjusted the signs in these equations in such a way that, in the rest frame of the medium, where the only non-zero component of $u^{\mu}$ is the time component, the components of these two 4-vectors are given by

$$E^{\mu} \overset{v=0}{\rightarrow} \{ 0, \vec{E} \},$$

$$B^{\mu} \overset{v=0}{\rightarrow} \{ 0, \vec{B} \},$$

so that $E^{\mu}$ and $B^{\mu}$ can be called the electric field 4-vector and magnetic field 4-vector respectively. Of course, it has to be acknowledged that the sign on the right side of Eq. (13b) depends on the choice of the components of the Levi-Civita symbol. Our convention has been described in the Appendix. Note also that

$$u^{\mu} E_{\mu} = 0, \quad u^{\mu} B_{\mu} = 0,$$

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relations which follow easily from Eq. (12) because of the antisymmetry of $F_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$.

It seems that the two 4-vectors $E^\mu$ and $B^\mu$ contain all the information that is there in $F_{\mu\nu}$. That is indeed true. If fact, just as these 4-vectors can be determined from the field tensor, the field tensor can also be reconstructed from these two 4-vectors. The relation is

$$F_{\mu\nu} = E^\mu u^\nu - E^\nu u^\mu + \varepsilon^{\mu\nu\lambda\rho} B_\lambda u_\rho .$$

One can easily check, using Eq. (14), that Eq. (12) follows from this expression. It is also instructive to write the Maxwell equations using the electric field and the magnetic field 4-vectors. Clearly, contracting Eqs. (3a) and (3b) with $u_\nu$, one obtains

$$\frac{\partial}{\partial \mu} E^\mu = \frac{1}{c} J^\nu u_\nu ,$$
$$\frac{\partial}{\partial \mu} B^\mu = 0 .$$

The other two equations will read

$$\varepsilon^{\mu\nu\lambda\rho} u_\nu \partial_\lambda E_\rho - u^\alpha \partial_\alpha B^\mu = 0 ,$$
$$\varepsilon^{\mu\nu\lambda\rho} u_\nu \partial_\lambda B_\rho + u^\alpha \partial_\alpha E^\mu = - \frac{1}{c} (J^\mu - u^\mu u^\nu J_\nu) .$$

In the form given in Eq. (15), $F_{\mu\nu}$ is the sum of two different antisymmetric tensors — one composed of the electric 4-vector only, and the other composed of the magnetic 4-vector only. We can now see how one might define an antisymmetric tensor containing two constitutive parameters. We can put two Lorentz-invariant parameters to go with the two antisymmetric tensors to define

$$G_{\mu\nu} = \varepsilon (E^\mu u^\nu - E^\nu u^\mu) + \frac{1}{\mu} \varepsilon^{\mu\nu\lambda\rho} B_\lambda u_\rho .$$

Alternatively, we can define the 4-vectors

$$D^\nu = \varepsilon E^\nu ,$$
$$H^\nu = \frac{1}{\mu} B^\nu ,$$

and define

$$G_{\mu\nu} = D^\mu u^\nu - D^\nu u^\mu + \varepsilon^{\mu\nu\lambda\rho} H_\lambda u_\rho .$$

The two definitions are obviously equivalent. The inhomogeneous Maxwell equations involving the free sources is given by Eq. (9). The objects $\varepsilon$ and $\mu$ are the response functions.

It is well-known that relations such as those in Eq. (18) are strictly valid in terms of the Fourier transforms of the corresponding fields. The point is that, the demand of linear response does not preclude terms with extra derivatives in the definitions of $D^\nu$ and $H^\nu$. In the Fourier space, however, all these derivatives turn into functions of the wave 4-vector $k^\mu$. Thus, Eq. (18) really says that the Fourier transform of 4-vectors $D^\nu$ and $H^\nu$ bear linear relations to the
Fourier transforms of $E^\nu$ and $B^\nu$, and in each case the proportionality factor can be a function of $k^\mu$.

Earlier, we said that $\epsilon$ and $\mu$ are invariant. Now we are saying that they are functions of $k^\mu$. Taken together, the two statements mean that $\epsilon$ and $\mu$ can depend only on Lorentz invariant quantities that can be constructed from the wave vector. Using the medium 4-vector $u^\mu$, we see that there are two such invariants, $k^\mu u_\mu$ and $k^\mu k_\mu$. We can define the independent variables to be

$$\omega = k^\mu u_\mu , \quad K = \sqrt{(k \cdot u)^2 - k^\mu k_\mu} .$$

In the rest frame of the medium, $\omega$ becomes the frequency and $K$ becomes the magnitude of the wave 3-vector. Thus, the response functions $\epsilon$ and $\mu$ can be functions of the frequency and wavenumber, the latter being the magnitude of the 3-vector $\vec{k}$. This is what is expected in an isotropic medium. If the medium is not isotropic, there will be other vectors associated with the medium, and one will be able to construct more invariants.

4 The Lagrangian

We now want to see how we might obtain Eq. (9) from a Lagrangian. Replacing $F^\mu\nu$ by $G^\mu\nu$ in Eq. (5) would not do, for the simple reason that it would yield Euler-Lagrange equations containing quadratic combinations of the response functions. However, we can try

$$\mathcal{L} = -\frac{1}{4} F^\mu\nu G^\nu\mu - \frac{1}{c} j^\mu_{\text{free}} A_\mu .$$

(21)

Note that Eq. (7a) gives

$$\frac{\partial (E_\mu)}{\partial (\partial_\alpha A_\beta)} = - \delta^{\alpha}_{\mu} u^\beta + \delta^{\beta}_{\mu} u^\alpha ,$$

(22a)

$$\frac{\partial (B_\mu)}{\partial (\partial_\alpha A_\beta)} = - \varepsilon^{\mu\alpha\beta} u_\nu .$$

(22b)

Using the expression for $F^\mu\nu$ and $G^\mu\nu$ from Eqs. (1) and (17) and using the derivatives from Eq. (22), it is easy to verify that one obtains Eq. (9) as the Euler-Lagrange equation.

There is another way of looking at the Lagrangian density of Eq. (21) which might offer new insight. Using Eqs. (11) and (14), one finds

$$(u^\nu E^\nu - u^\nu E^\mu)(u_\mu E_\nu - u_\nu E_\mu) = 2 E^\nu E_\nu .$$

(23)

Similarly, using Eq. (A.3), it is easy to show that

$$(\varepsilon^{\mu\nu\lambda\rho} u_\lambda B_\rho)(\varepsilon_{\mu\nu\alpha\beta} u^\alpha B^\beta) = -2 B^\rho B_\rho ,$$

(24)

whereas, because of the complete antisymmetry of the Levi-Civita symbol,

$$(u_\mu E_\nu - u_\nu E_\mu) \varepsilon^{\mu\nu\lambda\rho} u_\lambda B_\rho = 0 .$$

(25)
This exercise shows that we can write the Lagrangian density of Eq. (21) in the alternative form

\[ \mathcal{L} = -\frac{1}{2} \left( \epsilon E^\alpha E_\alpha - \frac{1}{\mu} B^\alpha B_\alpha \right) - \frac{1}{c} J^\mu_{\text{free}} A_\mu. \]  

(26)

It is obvious that if we put \( \epsilon = \mu = 1 \), then this expression reduces to the Lagrangian density in the vacuum. This is an equivalent way of defining the response functions.

5 Stress-energy-momentum tensor of the field

The stress-energy-momentum tensor (or SEM tensor for the sake of brevity) in the vacuum is given by

\[ T^{\mu\nu} = -\eta_{\lambda\rho} F^{\mu\lambda} F_{\nu\rho} + \frac{1}{4} \eta^{\mu\nu} F_{\lambda\rho} F^{\lambda\rho}. \]  

(27)

In a medium, since we expect to obtain an expression that is linear in the response functions, we should guess that the appropriate form should be

\[ T^{\mu\nu} = -\eta_{\lambda\rho} F^{\mu\lambda} G_{\nu\rho} + \frac{1}{4} \eta^{\mu\nu} G_{\lambda\rho} G^{\lambda\rho}. \]  

(28)

Recalling Eqs. (15) and (19), this result can be cast in the form

\[ T^{\mu\nu} = -(E^\mu D^\nu + H^\mu B^\nu) - u^\mu u^\nu \left( E^\alpha D_\alpha + H^\alpha B_\alpha \right) \]

\[ -\left( \varepsilon^{\mu\alpha\beta} u^\nu u_\alpha D_\beta + \varepsilon^{\nu\alpha\beta} u^\mu u_\alpha E_\beta \right) + \frac{1}{2} \eta^{\mu\nu} \left( E^\alpha D_\alpha + H^\alpha B_\alpha \right). \]  

(29)

It is easy to see that, in the rest frame of the medium, this tensor has the components

\[ T^{00} = -\frac{1}{2} \left( E^\alpha D_\alpha + H^\alpha B_\alpha \right) = \frac{1}{2} \left( \tilde{E} \cdot \tilde{D} + \tilde{H} \cdot \tilde{B} \right), \]  

(30a)

\[ T^{0i} = -\varepsilon^{ij0k} E_j H_k = \left( \tilde{E} \times \tilde{H} \right)^i, \]  

(30b)

\[ T^{i0} = -\varepsilon^{ij0k} D_j B_k = \left( \tilde{D} \times \tilde{B} \right)^i, \]  

(30c)

\[ T^{ij} = -(E^i D^j + H^i B^j) + \frac{1}{2} \delta^{ij} \left( \tilde{E} \cdot \tilde{D} + \tilde{H} \cdot \tilde{B} \right), \]  

(30d)

which are the expected results, available in textbooks.

6 Optical activity

Optical activity, and its connection with the Maxwell equations, is a subject that is not discussed in most textbooks on Electrodynamics. In one exception where it is discussed [8], the property is shown to be connected with the tensorial structure of the dielectric function. However, it
was pointed out a while ago that it is not necessary to go beyond the numerical response functions, as are appropriate for an isotropic medium, in order to accommodate an explanation of natural optical activity. For this, it is important to realize that mimicking the form for \( F_{\mu\nu} \) given in Eq. (1) to write down the expression for \( G_{\mu\nu} \) in Eq. (17) does not give the most general form for \( G_{\mu\nu} \) for an isotropic medium. It was shown in these papers that the most general form for \( G_{\mu\nu} \) should contain another response function, which can explain the phenomenon of natural optical activity.

In the covariant formulation that we have been discussing, there have been several hints that something is not being considered. For example, if we look at Eq. (26), we see a term containing \( E^\alpha E_\alpha \) and a term containing \( B^\alpha B_\alpha \), but no term containing \( E^\alpha B_\alpha \). Equivalently, we can look at Eq. (17) and ask ourselves, why have we not included terms with the tensors

\[
\Gamma_{E}^{\mu\nu} = \varepsilon^{\mu\nu\lambda\rho} E_\lambda u_\rho,
\]

\[
\Gamma_{B}^{\mu\nu} = B^\mu u^\nu - B^\nu u^\mu,
\]

which are obtained by interchanging the roles of the 4-vectors \( E^\mu \) and \( B^\mu \) in the two antisymmetric tensors that appear there?

It seems that there is a conflict between the two statements of incompleteness that we just put forward. In the Lagrangian formulation, there seems to be just one term missing, and we can rectify the incompleteness by writing

\[
\mathcal{L} = -\frac{1}{2} \left( \varepsilon E^\alpha E_\alpha - \frac{1}{\mu} B^\alpha B_\alpha - \zeta E^\alpha B_\alpha \right) - \frac{1}{c} J^\mu_{\text{free}} A_\mu.
\]

But if we look at Eq. (31), it seems that it calls for two extra coefficients to be introduced in the definition of \( G_{\mu\nu} \), one for each of the tensors defined there. The resolution of this apparent contradiction will be discussed shortly.

Meanwhile, let us proceed with Eq. (32). The Euler-Lagrange equation from this Lagrangian can be easily derived by using Eq. (7), and the result is given by Eq. (9), where

\[
G_{\mu\nu} = \varepsilon (E^\mu u^\nu - E^\nu u^\mu) + \frac{1}{\mu} \varepsilon^{\mu\nu\lambda\rho} B_\lambda u_\rho + \frac{1}{2} \zeta \left( B^\mu u^\nu - B^\nu u^\mu + \varepsilon^{\mu\nu\alpha\beta} E_\alpha u_\beta \right).
\]

As seen here, the expression that multiplies \( \zeta \) is \( \Gamma_{E}^{\mu\nu} + \Gamma_{B}^{\mu\nu} \). If two response functions were allowed corresponding to the two tensors shown in Eq. (31), that would have meant that we could add another term proportional to \( \Gamma_{E}^{\mu\nu} - \Gamma_{B}^{\mu\nu} \) in this equation. But, starting from the definition of Eq. (4), it is easy to see that Eq. (15) implies

\[
\tilde{\Gamma}_{\mu\nu} = -B^\mu u^\nu + B^\nu u^\mu + \varepsilon^{\mu\nu\alpha\beta} E_\alpha u_\beta = \Gamma_{E}^{\mu\nu} - \Gamma_{B}^{\mu\nu}.
\]

Adding a term proportional to it in the expression for \( G_{\mu\nu} \) would not make any difference in the Euler-Lagrange equation, because the dual tensor satisfies Eq. (3b). Such a term would be irrelevant. This is why we can add only one extra response function, not two. Another way of saying it is that Eq. (3b) means

\[
\partial_\mu \Gamma_{E}^{\mu\nu} = \partial_\mu \Gamma_{B}^{\mu\nu},
\]
which is the same as Eq. (16c). Thus, we can use only one of the two tensors $\Gamma^\mu\nu_E$ and $\Gamma^\mu\nu_M$ in the expression for $G^\mu\nu$, augmenting the coefficient from $\frac{1}{2}\zeta$ to $\zeta$.

There is another interesting point about the $\zeta$-term. If we take Eq. (A.4) given in the Appendix with $u^\mu$ in place of the arbitrary 4-vector, and contract with $F^\mu\nu \tilde{F}_\mu^\nu$, we obtain the identity

$$E^\alpha B_\alpha = -\frac{1}{4}F^\mu\nu \tilde{F}_\mu^\nu. \quad (36)$$

However, we also know that

$$\begin{align*}
F^\mu\nu \tilde{F}_\mu^\nu &= \frac{1}{2}\varepsilon_{\mu\nu\lambda\rho} F^\mu\nu F^\lambda\rho = 2\varepsilon_{\mu\nu\lambda\rho} (\partial^\mu A^\nu)(\partial^\lambda A^\rho) \\
&= \partial^\mu (2\varepsilon_{\mu\nu\lambda\rho} A^\nu \partial^\lambda A^\rho), \quad (37)
\end{align*}$$

a total derivative. Such terms in a Lagrangian do not contribute to the Euler-Lagrange equations.

But obviously the $\zeta$-term makes a difference in the Euler-Lagrange equation. This apparent contradiction implies that $\zeta$ cannot be a constant. In the momentum space, it should depend on $\omega$ and $K$, introduced in Eq. (20). This conclusion was reached from the non-relativistic treatment as well [10].

The effect of the $\zeta$-term can be easily understood by considering the parity transformation property of the different terms in either the Lagrangian of Eq. (32) or in the tensor defined in Eq. (33). It is easily seen that under parity,

$$\begin{align*}
E^\mu \xrightarrow{\text{parity}} P^\mu E^\nu, \\
B^\mu \xrightarrow{\text{parity}} -P^\mu B^\nu,
\end{align*} \quad (39a)$$

where

$$P^\mu = \text{diag} \left( +1, -1, -1, -1 \right). \quad (40)$$

Thus, the terms involving $\epsilon$ and $\mu$ are parity invariant, whereas the terms involving $\zeta$ are not. Because of this, the presence of a non-zero value of $\zeta$ implies different dispersion relations for the right-circular and left-circular polarizations of electromagnetic waves in the medium [9][10], which is the root cause for optical activity.

Finally, we might wonder about the form of the SEM tensor in presence of optical activity. Since $G^\mu\nu$ contains the activity constant $\zeta$, it is naively expected that when this expression in substituted into Eq. (28), we will see some $\zeta$-dependence in the SEM tensor. However, once the substitution is made, it is seen that the $\zeta$-dependent terms all cancel out and we obtain exactly Eq. (29).

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Appendix

Here we collect a few results related to the 4-dimensional Levi-Civita symbol. We define it as

\[ \varepsilon_{\mu\nu\lambda\rho} = \begin{cases} +1 & \text{if the indices form an even permutation of 0,1,2,3,} \\ -1 & \text{if the indices form an even permutation of 0,1,2,3,} \\ 0 & \text{otherwise.} \end{cases} \] (A.1)

The usual rules of raising and lowering indices would then imply

\[ \varepsilon^{\mu\nu\lambda\rho} = \begin{cases} -1 & \text{if the indices form an even permutation of 0,1,2,3,} \\ +1 & \text{if the indices form an even permutation of 0,1,2,3,} \\ 0 & \text{otherwise.} \end{cases} \] (A.2)

We have used the expression obtained by contracting two indices of a pair of Levi-Civita symbols:

\[ \varepsilon^{\mu\nu\lambda\rho} \varepsilon_{\mu\nu\alpha\beta} = -2(\delta_\alpha^\lambda \delta_\beta^\rho - \delta_\beta^\lambda \delta_\alpha^\rho). \] (A.3)

Another important identity, used in the text, involves an arbitrary 4-vector \( V^\mu \) and states that

\[ \varepsilon^{\mu\nu\lambda\rho} V^\alpha - \varepsilon^{\alpha\nu\lambda\rho} V^\mu - \varepsilon^{\mu\alpha\lambda\rho} V^\nu - \varepsilon^{\mu\nu\alpha\rho} V^\lambda - \varepsilon^{\mu\nu\lambda\alpha} V^\rho = 0. \] (A.4)

The proof is very simple. It can be checked easily that the left side of the equation is completely antisymmetric in the interchange of any pair of indices, and is therefore a rank-5 antisymmetric tensor. Since five indices cannot be antisymmetrized in a 4-dimensional geometry, the left side must vanish.

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