HEAT KERNELS OF GENERALIZED DEGENERATE SCHRODINGER OPERATORS AND HARDY SPACES

THE ANH BUI, TAN DUC DO, AND NGUYEN NGOC TRONG

Abstract. Let \( L = -\frac{1}{w} \text{div}(A \nabla u) + \mu \) be the generalized degenerate Schrödinger operator in \( L_w^2(\mathbb{R}^d) \) with \( d \geq 3 \) with suitable weight \( w \) and measure \( \mu \). The main aim of this paper is threefold. First, we obtain an upper bound for the fundamental solution of the operator \( L \). Secondly, we prove some estimates for the heat kernel of \( L \) including an upper bound, the Hölder continuity and a comparison estimate. Finally, we apply the results to study the maximal function characterization for the Hardy spaces associated to the critical function generated by the operator \( L \).

1. Introduction

Consider the generalized degenerate Schrödinger operator of the form

\[ L = -\frac{1}{w} \text{div}(A \nabla u) + \mu \]

in \( L_w^2(\mathbb{R}^d) \) with \( d \geq 3 \). Here \( w, A \) and \( \mu \) satisfy the following conditions:

- The coefficient matrix \( A \) is real symmetric with measurable entries. Furthermore there exists a constant \( \Lambda \geq 1 \) such that
  \[ \Lambda^{-1} w(x)|\xi|^2 \leq A(x) \xi \cdot \xi \leq \Lambda w(x)|\xi|^2 \]
  for a.e. \( x \in \mathbb{R}^d \) and for all \( \xi \in \mathbb{C}^d \).
- The weight \( w \in A_2 \). See below for the precise definition of the class \( A_2 \). Moreover, we also assume that \( w \in RD_\beta \) with some \( \beta > 2 \), i.e., there exists a \( C > 0 \) such that
  \[ w(B(x, tr)) \geq C t^\beta w(B(x, r)) \]
  for all \( t > 1 \) and \( x \in \mathbb{R}^d \).
- \( \mu \) is a positive Radon measure satisfying the following conditions:
  (i) There exist \( C_0 > 0 \) and \( \delta > 0 \) such that
  \[ \frac{r^2}{w(B(x, r))} \pi(B(x, r)) \leq C_0 \left( \frac{r}{R} \right) \delta \frac{R^2}{w(B(x, R))} \pi(B(x, R)) \]
  for all \( x \in \mathbb{R}^d \) and \( R > r > 0 \), where \( d\pi = w \, d\mu \).
  (ii) There exists a \( C_1 > 0 \) such that
  \[ \pi(B(x, 2r)) \leq C_1 \left( \pi(B(x, r)) + \frac{w(B(x, r))}{r^2} \right) \]
  for all \( x \in \mathbb{R}^d \) and \( r > 0 \), where \( d\pi = w \, d\mu \).

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A precise description of $L$ via form method is given in Section 3.1. Let $w \in L^1_{\text{loc}}(\mathbb{R}^d)$ be a nonnegative locally integrable function. For $p \in (1, \infty)$, we say that $w \in A_p$ if
\[
\left( \frac{1}{|B|} \int_B w(x) \, dx \right) \left( \frac{1}{|B|} \int_B w(x)^{1-p'} \, dx \right)^{p-1} \leq C
\]
for all balls $B \subset \mathbb{R}^d$.

Note that if $w \in A_p$ then
\[
w(\lambda B) \leq \lambda^{d/p} w(B)
\]
for all balls $B$ and $\lambda \geq 1$.

We would like to summarize the body of research regarding the generalized degenerate Schrödinger operator of the form (1).

(a) The well-known form of the operator $L$ is when $w = 1$, $A(x) = I$ and $\mu(x) = V(x) \, dx$ with $V$ satisfying a reverse Hölder’s inequality, i.e.,
\[
\left( \frac{1}{|B|} V(x)^q \, dx \right)^{1/q} \leq \frac{1}{|B|} V(x) \, dx
\]
for all balls $B$ with $q \geq d/2$. Such Schrödinger operators were introduced in [18] of which the estimates for the fundamental solution of $L$ and the Riesz transforms were investigated. The theory of new Hardy spaces associated to the operator $L$ was treated in [9, 10, 3].

(b) The degenerate elliptic operators corresponding to $L$ of the form (1) with $\mu = 0$ were studied in [12] in which Fabes, Kenig and Serapioni proved some results on local regularity of degenerate elliptic operators on domains. The Hardy spaces $H^1_L$ associated to the degenerate Schrödinger with $\mu(x) = V(x) \, dw(x)$, where $V(x)$ satisfies a reverse Hölder’s inequality were obtained in [11].

(c) The study of Schrödinger operators of the form (1) with $\mu$ being a Radon measure is less well-known. The case $w = 1$ and $A = I$ was first introduced by Shen [18]. Note that in this case the measure $\mu$ contains the class of measures satisfying scale–invariant Kato condition. See [18, p.522]. In this work, Shen proved the bounds for the fundamental solutions of $L$ and the boundedness of the Riesz transforms and the imaginary powers of $L$. The Hardy spaces $H^1_L$ related to these operators have been studied recently in [21].

Motivated by the above body of work, this paper will work with the generalized degenerate Schrödinger operators of the form (1). We would like to describe our main results.

(a) First, we prove an upper bound for the fundamental solution to the operator $L$. See Theorem 1.1. In a particular case when $w = 1$ and $A = I$ our result is in line with that in [18]. Note that our method can be modified to obtain the exponential decay in the upper bound, but we do not aim to pursue this problem since the polynomial decay in the upper bound is enough for our purpose.

(b) Secondly, we obtain some estimates for the heat kernel of the semigroup $e^{-tL}$ generated by $L$. The results include the upper bound, the Hölder continuity estimates and the comparison estimates. See Theorem 1.2. It emphasizes that our estimates recover known estimates in the corresponding particular case of $L$ such as Schrödinger operator on $\mathbb{R}^d$ (see [9]), degenerate Schrödinger operator (see [11]) and the generalized Schrödinger operator on $\mathbb{R}^d$ (see [21]).
The last result relates to the theory of Hardy spaces associated to differential operators. The theory of Hardy spaces adapted to general operators was initially introduced by [1]. Then it has become an interesting topic in harmonic analysis and has attracted a great deal of attention. See for example [8, 15, 14] and the references therein. In the last result, we apply the findings on the heat kernel of \( L \) to prove the maximal function characterization to the Hardy spaces associated to \( L \). Let us remind that the maximal function characterization of the Hardy spaces has a long history. The maximal function characterizations for the classical Hardy spaces were obtained in [4, 5]. Then the results were extended to new Hardy spaces \( H^p \) associated to Schrödinger operators \( L \) in various settings. See [9, 10, 23, 21, 11]. In Theorem 1.6 we prove the maximal function for the new Hardy spaces \( H^p \) for \( p \leq 1 \). We note that our result not only recovers known results in [9, 10, 23, 21, 11], but also extends those to the range \( p < 1 \).

To formulate our main results the following notion plays a fundamental role. For all \( x \in \mathbb{R}^d \) define

\[
\rho_w(x, \mu) := \frac{1}{m_w(x, \mu)} := \sup \left\{ r > 0 : \frac{r^2}{w(B(x, r))} \pi(B(x, r)) \leq C_1 \right\},
\]

where \( C_1 \) is given by (5), which is called the critical function.

Our first main result on the upper bound of the fundamental solution of \( L \) is as follows.

**Theorem 1.1.** Let \( \Gamma_\mu(x, \cdot) \) be the fundamental solution of \( L \). Then for every \( k \in \mathbb{N} \) there exists a \( C = C(k) > 0 \) such that

\[
0 \leq \Gamma_\mu(x, y) \leq \frac{C}{(1 + |x - y| m_w(x, \mu))^k w(B(x, |x - y|))} |x - y|^2
\]

for all \( x, y \in \mathbb{R}^d \) such that \( x \neq y \).

The next result will be on the heat kernel estimates of \( L \). Before coming to details, we recall the principle part of \( L \) given by

\[
L_0 u = -\frac{1}{w} \text{div}(A \nabla u).
\]

Denote by \( h_t(\cdot, \cdot) \) and \( k_t(\cdot, \cdot) \) the kernels of \( e^{-tL_0} \) and \( e^{-tL} \) for \( t > 0 \), respectively. Then we have the following.

**Theorem 1.2.** Let \( k_t(\cdot, \cdot) \) be the kernel of \( e^{-tL} \). Then we have:

(i) There exists a \( c > 0 \) such that for all \( N \geq 0 \) there exists a \( C = C(N) > 0 \) satisfying

\[
0 \leq k_t(x, y) \leq \frac{C}{w(B(x, \sqrt{t}))} \exp\left(-\frac{|x - y|^2}{ct}\right) \left(1 + \frac{\sqrt{t}}{\rho_w(x, \mu)} + \frac{\sqrt{t}}{\rho_w(y, \mu)}\right)^{-N}
\]

for all \( x, y \in \mathbb{R}^d \) and \( t > 0 \).

(ii) For any \( 0 < \theta < \min[\delta, \gamma] \), there exists a \( C > 0 \) such that

\[
|k_t(x, y) - k_t(x, \overline{y})| \leq \frac{C}{w(B(x, \sqrt{t}))} \frac{1}{\sqrt{t}} \exp\left(-\frac{|x - y|^2}{ct}\right)
\]

for all \( t > 0 \) and \( |y - \overline{y}| < \sqrt{t} \), where \( \delta \) is the constant in [4] and \( \gamma \) is the constant in Proposition 4.1 (ii).
We now move on to the final main result regarding new Hardy spaces. In order to present this result, we introduce new local Hardy spaces associated to critical functions \( \rho \) and \( \varepsilon \). In what follows, for \( w \in \Lambda_2 \) we define \( q_w = \inf \{ p \in (1, \infty) : w \in A_p \} \). It is well-known that \( q_w < 2 \) (see [19]), and we set \( n = q_w d \).

**Definition 1.3.** Let \( \rho_w \) be the critical function defined by (7). Let \( p \in (\frac{n-1}{n-1}, 1], q \in [1, \infty) \cap (p, \infty) \) and \( \varepsilon \in (0, 1) \). A function \( a \) is called a \( (p, q, \rho_w, \varepsilon) \)-atom associated to the ball \( B(x_0, r) \) if

(i) \( \sup \{ \mathbb{A} \subset B(x_0, r) \text{ and } r \leq \rho_w(x_0, \mu) \} \);

(ii) \( \| a \|_{L^q_w (\mathbb{R}^d)} \leq w(B(x_0, r))^{1/q - 1/p} \);

(iii) \( \int a(x) \mathrm{d} w(x) = 0 \) if \( r < \varepsilon \rho_w(x_0, \mu)/4 \).

For the sake of convenience, when \( \varepsilon = 1 \) we shall write \( (p, q, \rho_w) \) atom instead of \( (p, q, \rho_w, \varepsilon) \)-atom.

**Definition 1.4.** Let \( \rho_w \) be the critical function defined by (7). Let \( p \in (\frac{n-1}{n-1}, 1], q \in [1, \infty) \cap (p, \infty) \) and \( \varepsilon \in (0, 1) \). We say that \( f = \sum \lambda_j a_j \) is an atomic \( (p, q, \rho_w, \varepsilon) \)-representation if \( \{ \lambda_j \}_{j=0}^\infty \in \ell^p \), each \( a_j \) is a \( (p, q, \rho_w, \varepsilon) \)-atom, and the sum converges in \( L^2_w (\mathbb{R}^d) \). The space \( h^{p,q}_{w,\rho_w,e} (\mathbb{R}^d, w) \) is then defined as the completion of

\[
\left\{ f \in L^2_w (\mathbb{R}) : f \text{ has an atomic } (p, q, \rho_w, \varepsilon)\text{-representation} \right\},
\]

with the norm given by

\[
\| f \|_{h^{p,q}_{w,\rho_w,e} (\mathbb{R}^d, w)} = \inf \left\{ \left( \sum |\lambda_j|^p \right)^{1/p} : f = \sum \lambda_j a_j \text{ is an atomic } (p, q, \rho_w, \varepsilon)\text{-representation} \right\}.
\]

In the particular case \( \varepsilon = 1 \) we write \( h^{p,q}_{w,\rho_w} (\mathbb{R}^d, w) \) instead of \( h^{p,q}_{w,\rho_w,e} (\mathbb{R}^d, w) \).

**Definition 1.5.** Let \( \mathcal{M} \) be defined in (1). For \( p \in (0, 1] \), the Hardy space \( H^p_w (\mathbb{R}^d, w) \) is defined as a completion of the set

\[
\left\{ f \in L^2_w (\mathbb{R}^d) : \mathcal{M}_t f \in L^p_w (\mathbb{R}^d) \right\}
\]

under the norm

\[
\| f \|_{H^p_w (\mathbb{R}^d, w)} = \| \mathcal{M}_t f \|_{L^p_w (\mathbb{R}^d)},
\]

where

\[
\mathcal{M}_t f = \sup_{t>0} |e^{-t} f|.
\]

The last main result of this paper is the following:

**Theorem 1.6.** Let \( \theta = \min \{ \delta, \gamma \} \), where \( \delta \) is the constant in (4) and \( \gamma \) is the constant in Proposition 4.2 (ii). For all \( p \in (\frac{n}{n-1}, 1] \) and \( q \in [1, \infty) \cap (p, \infty) \) we have

\[
h^{p,q}_{w,\rho_w} (\mathbb{R}^d, w) \equiv H^p_w (\mathbb{R}^d, w).
\]
The organization of the paper is as follows. Section 2 will establish some basic properties for the critical function for the later uses. The proofs of Theorem 1.1, Theorem 1.2 and Theorem 1.6 will be given in Section 3, Section 4 and Section 5, respectively.

**Notations.** Throughout the paper the following set of notation is used without mentioning. Set \( \mathbb{N} = \{0, 1, 2, 3, \ldots \} \) and \( \mathbb{N}^* = \{1, 2, 3, \ldots \} \). Given an \( j \in \mathbb{N} \) and a ball \( B = B(x, r) \), we let \( 2^j B = B(x, 2^j r) \), \( S_0(B) = B \) and \( S_j(B) = 2^j B \setminus 2^{j-1} B \) if \( j \geq 1 \). For all \( a, b \in \mathbb{R} \), \( a \land b = \min(a, b) \) and \( a \lor b = \max(a, b) \).

For all ball \( B \subset \mathbb{R}^d \) we write \( w(B) := \int_B w \). The constants \( C \) and \( c \) are always assumed to be positive and independent of the main parameters whose values change from line to line. For any two functions \( f \) and \( g \), we write \( f \leq g \) and \( f \sim g \) to mean \( f \leq Cg \) and \( cg \leq f \leq Cg \) respectively. Given a \( p \in [1, \infty) \), the conjugate index of \( p \) is denoted by \( p' \). If \( f \) is defined on \( \mathbb{R}^d \times \mathbb{R}^d \), the gradient with respect to the first variable of \( f \) (if it exists) is written as \( \nabla_1 f \) and this notation is generalized to higher orders in an obvious manner. We write \( L^2(\mathbb{R}^d) \) to mean the space of square-integrable functions with respect to the Lebesgue measure. In a weighted setting of Lebesgue spaces we will use the notation \( L^2_0(\mathbb{R}^d) = L^2(\mathbb{R}^d, dw) \).

2. The critical function \( \rho(\cdot, \mu) \) and some properties

This section will prove some basic properties for the critical function \( \rho_w(\cdot, \mu) \). These properties are a corner stone in the proofs of main results.

**Proposition 2.1.** Let \( \rho_w(\cdot, \mu) \) be the function defined in (7). Then we have the following.

(i) The function \( \rho_w(\cdot, \mu) \) is well-defined, i.e., \( \rho_w(x, \mu) \in (0, \infty) \) for every \( x \in \mathbb{R}^d \).

(ii) For every \( x \in \mathbb{R}^d \) one has
\[
\frac{w(B(x, r))}{r^2} \leq \pi(B(x, r)) \leq C_1 \frac{w(B(x, r))}{r^2},
\]
with \( r = \rho_w(x, \mu) \).

(iii) If \( |x - y| \leq \rho_w(x, \mu) \), then \( \rho_w(x, \mu) \sim \rho_w(y, \mu) \).

(iv) There exist \( k_0 > 0 \) and \( C > 1 \) such that
\[
C^{-1} m_w(y, \mu) \left(1 + |x - y| m_w(y, \mu)\right)^{-k_0/(k_0 + 1)} \leq m_w(x, \mu) \leq C m_w(y, \mu) \left(1 + |x - y| m_w(y, \mu)\right)^{k_0}
\]
for all \( x, y \in \mathbb{R}^d \).

**Proof.** Let \( x, y \in \mathbb{R}^d \), \( r = \rho_w(x, \mu) \) and \( R = \rho_w(y, \mu) \).

(i) It follows from (9) that
\[
\lim_{t \to 0} \frac{t^2}{w(B(x, t))} \pi(B(x, t)) = 0 \quad \text{and} \quad \lim_{t \to \infty} \frac{t^2}{w(B(x, t))} \pi(B(x, t)) = \infty.
\]

This, in combination with (4), implies \( \rho_w(x, \mu) \in (0, \infty) \).

(ii) By definition we have
\[
\pi(B(x, r)) = \lim_{t \to r} \pi(B(x, t)) \leq C_1 \frac{w(B(x, r))}{r^2}.
\]

Also
\[
2^{\beta - 2} C_1 \frac{w(B(x, r))}{r^2} \leq C_1 \frac{w(B(x, 2r))}{4r^2} \leq \pi(B(x, 2r)) \leq C_1 \left( \pi(B(x, r)) + \frac{w(B(x, r))}{r^2} \right),
\]
where we used (3) in the first step, the definition of $\rho_w$ in the second step and (5) in the last step. From this we deduce that
\[
\pi(B(x, r)) \geq \frac{w(B(x, r))}{r^2}.
\]

(iii) Suppose that $|x - y| < C r$ for some $C > 0$. Then $B(y, r) \subset B(x, (C + 1) r)$. Using (5) and (ii) we obtain
\[
\pi(B(x, (C + 1) r)) \leq \pi(B(x, r)) \leq \frac{w(B(x, r))}{r^2}.
\]
Consequently, it follows from (4) that
\[
\frac{(tr)^2}{w(B(y, tr))} \pi(B(y, tr)) \leq C_0 t^\delta \frac{r^2}{w(B(y, r))} \pi(B(y, r)) \leq t^\delta \frac{r^2}{w(B(x, r))} \pi(B(x, (C + 1) r)) \leq t^\delta < C_1,
\]
where $t$ is chosen to be sufficiently small. Therefore $R \geq tr$ by definition, where we recall that $R = \rho_w(y, \mu)$. Note that this in turn implies $|x - y| \leq R$. By swapping the roles of $x$ and $y$ in the above argument, we then obtain $R \leq r$.

(iv) The case $|x - y| < R$ follows from (iii). So we assume that $|x - y| \geq R$. Let $j \in \mathbb{N}^*$ be such that $2^{j-1} R \leq |x - y| \leq 2^j R$. Then $B(x, R) \subset B(y, (2^j + 1) R)$. It follows from (ii) and (5) that
\[
\pi(B(x, R)) \leq (C_1 + 2^{2d-2} j) \frac{w(B(y, R))}{R^2} \leq (C_1 + 2^{2d-2} j) (1 + 2^j)^d \frac{w(B(x, R))}{R^2} \leq (1 + 2^j)^b \frac{w(B(x, R))}{R^2},
\]
where $b := d + \log_2(C_1 + 2^{2d-2})$. Using (4),
\[
\frac{(tR)^2}{w(B(x, tR))} \pi(B(x, tR)) \leq C_0 t^\delta \frac{R^2}{w(B(x, R))} \pi(B(x, R)) \leq t^\delta (1 + 2^j)^b < C_1,
\]
where we choose $t = \left( \frac{C_1}{2} (1 + 2^j)^{-b} \right)^{1/\delta}$. So the definition of $\rho_w$ gives $r \geq t R$ or equivalently
\[
(11) \quad m_w(x, \mu) \leq \frac{m_w(y, \mu)}{t} \sim m_w(y, \mu) \left( 1 + |x - y| m_w(y, \mu) \right)^{k_0},
\]
where $k_0 := b/\delta$.

For the remaining inequality, using (11) we obtain that
\[
1 + |x - y| m_w(x, \mu) \leq \left( 1 + |x - y| m_w(y, \mu) \right)^{k_0 + 1}.
\]
With this in mind we apply (11) again to obtain
\[
m_w(y, \mu) \geq m_w(x, \mu) \left( 1 + |x - y| m_w(x, \mu) \right)^{-k_0/(k_0 + 1)}.
\]

The proof is complete.
Lemma 2.2. There exist constants $N_0 \in \mathbb{N}$ and $C > 0$ such that

\[ \frac{R^2}{w(B(x, R))} \pi(B(x, R)) \leq C \left( \frac{R}{\rho_w(x, \mu)} \right)^{N_0} \]

for all $x \in \mathbb{R}^d$ and $R > 0$ such that $R \geq \rho_w(x, \mu)$.

**Proof.** Let $x \in \mathbb{R}^d$ and $r_0 = \rho_w(x, \mu)$. By the definition of $\rho_w(x, \mu)$,

\[ \pi(B(x, r_0)) \geq C_1 \frac{w(B(x, r))}{r^2} \]

for all $r \geq r_0$.

Therefore, (5) implies that $\pi$ is a doubling measure on all balls $B(x, r)$ with $r \geq r_0$, i.e.,

\[ \pi(B(x, 2r)) \leq \pi(B(x, r)) \]

for all $r \geq r_0 = \rho_w(x, \mu)$.

Let $j_0 \in \mathbb{N}$ be such that $2^{j_0} r_0 \leq R < 2^{j_0+1} r_0$. Then

\[ \pi(B(x, R)) \leq C_1^{j_0+1} \pi(B(x, r_0)) \leq C_1^j \frac{w(B(x, r_0))}{r_0^2}, \]

where we used Proposition 2.1 (ii) in the last step.

Hence,

\[ \frac{R^2}{w(B(x, R))} \pi(B(x, R)) \leq C_1^{j_0} \frac{R^2}{r_0^2} \frac{w(B(x, r_0))}{w(B(x, R))} \leq \left( \frac{R}{r_0} \right)^{\log_2 C_1}. \]

This justifies our claim. \(\square\)

**Lemma 2.3.** Let $x \in \mathbb{R}^d$ and $t > 0$. If $\sqrt{t} \leq \alpha \rho_w(x, \mu)$ for some $\alpha > 0$, then there exists a $C = C(\alpha) > 0$ such that

\[ \int_{\mathbb{R}^d} \frac{1}{w(B(x \land y, \sqrt{t}))} \exp \left( - \frac{|x - y|^2}{ct} \right) \, d\pi(y) \leq \frac{C}{t} \left( \frac{\sqrt{t}}{\rho_w(x, \mu)} \right)^{\delta}, \]

where $w(B(x \land y, \sqrt{t})) = \min\{w(B(x, \sqrt{t})), w(B(y, \sqrt{t}))\}$ and $\delta$ is the constant in (4).

If $\sqrt{t} \geq \alpha \rho_w(x, \mu)$ for some $\alpha > 0$, then there exists a $C = C(\alpha) > 0$ such that

\[ \int_{\mathbb{R}^d} \frac{1}{w(B(x \land y, \sqrt{t}))} \exp \left( - \frac{|x - y|^2}{ct} \right) \, d\pi(y) \leq \frac{C}{t} \left( \frac{\sqrt{t}}{\rho_w(x, \mu)} \right)^{N_0}, \]

where $N_0$ is the constant in Lemma 2.2.

**Proof.** We first prove (12). Setting $r = \rho_w(x, \mu)$, $B = B(x, \sqrt{t})$ and taking $j_0 \in \mathbb{N}$ such that $2^{j_0} \sqrt{t} \leq r < 2^{j_0+1} \sqrt{t}$, then we have

\[ \int_{\mathbb{R}^d} \frac{1}{w(B(x \land y, \sqrt{t}))} \exp \left( - \frac{|x - y|^2}{ct} \right) \, d\pi(y) \]

\[ = \sum_{j=0}^{\infty} \int_{S_j(B)} \frac{1}{w(B(x \land y, \sqrt{t}))} \exp \left( - \frac{|x - y|^2}{ct} \right) \, d\pi(y) \]

\[ =: \sum_{j=0}^{j_0} E_j + \sum_{j=j_0+1}^{\infty} E_j. \]
For $j = 0, 1, \ldots, j_0$ we have
\begin{equation}
E_j \lesssim e^{-c_{2j}} \frac{\pi(2^j B)}{w(2^j B)} \frac{w(2^j B)}{w(B(x \wedge y, \sqrt{t}))} = e^{-c_{2j}} \pi(2^j B) \frac{w(2^j B)}{w(2^j B)} \frac{w(2^j B)}{w(B(x \wedge y, \sqrt{t}))}
\end{equation}
\begin{align}
\lesssim & \quad 2^{2j} e^{-c_{2j}} \frac{\pi(2^j B)}{w(2^j B)} \\
\lesssim & \quad e^{-c_{2j}} \frac{\pi(2^j B)}{w(2^j B)}
\end{align}

where in the second inequality we used (6).

Note that for $j = 0, 1, \ldots, j_0$ one has $2^j \sqrt{t} \leq r = \rho_w(x, \mu)$. Hence, owing to (4) and Proposition 2.1, we have
\begin{equation}
\pi(2^j B) \lesssim \left(\frac{2^j \sqrt{t}}{r} \right)^{\delta} \frac{1}{2^{2j} t} \frac{\rho_B(B(x, r))}{w(B(x, r))} \sim \left(\frac{2^j \sqrt{t}}{r} \right)^{\delta} \frac{1}{2^{2j} t}.
\end{equation}

Plugging this into the estimate of $E_j$, we can simplify that
\begin{equation}
E_j \lesssim e^{-c_{2j}} \frac{1}{t} \left(\frac{\sqrt{t}}{r} \right)^{\delta} = e^{-c_{2j}} \frac{1}{t} \left(\frac{\sqrt{t}}{\rho_w(x, \mu)} \right)^{\delta}
\end{equation}

for all $j \leq j_0$, which implies
\begin{equation}
\sum_{j=0}^{j_0} E_j \lesssim \frac{1}{t} \left(\frac{\sqrt{t}}{\rho_w(x, \mu)} \right)^{\delta}.
\end{equation}

For all $j > j_0$, similarly to (15) we have
\begin{equation}
E_j \lesssim e^{-c_{2j}} \frac{\pi(2^j B)}{w(2^j B)} \quad \text{and} \quad 2^j \sqrt{t} \geq r = \rho_w(x, \rho).
\end{equation}

Applying Lemma 2.2 and the fact that $\sqrt{t} \leq Cr$,
\begin{equation}
E_j \lesssim e^{-c_{2j}} \left(\frac{2^j \sqrt{t}}{r} \right)^{N_0} \lesssim e^{-c_{2j}} \left(2^{j_0} \right)^{N_0} \frac{1}{t} \lesssim e^{-c_{2j}} \frac{1}{t},
\end{equation}
which implies
\begin{equation}
\sum_{j>j_0} E_j \lesssim e^{-c_{2j_0}} \frac{1}{t} \lesssim 2^{-j_0 \delta} \frac{1}{t}.
\end{equation}

This, along with the fact that $2^{-j_0} \sim \sqrt{t}/r$, yields that
\begin{equation}
\sum_{j>j_0} E_j \lesssim \frac{C'}{t} \left(\frac{\sqrt{t}}{\rho_w(x, \mu)} \right)^{\delta}.
\end{equation}

Collecting this estimate and the estimates (16) and (14) we deduce to the desired estimate (12).

The proof of (13) can be done similarly. Hence, we omit the details.

This completes our proof. \hfill \square

We end this section with the following useful lemma regarding a covering result of a family of balls whose radii are equal to the values of the critical function at their centers.
Lemma 2.4. There exist a sequence \( (x_j)_{j \in \mathbb{N}} \subset \mathbb{R}^d \) and a family of functions \( (\psi_j)_{j \in \mathbb{N}} \) such that the following hold.

(i) \( \bigcup_{j \in \mathbb{N}} B_j = \mathbb{R}^d \), where \( \rho_j = \rho_w(x_j, \mu) \) and \( B_j = B(x_j, \rho_j) \) for all \( j \in \mathbb{N} \).

(ii) For all \( \tau \geq 1 \) there exist constants \( C, \zeta_0 > 0 \) such that
\[
\sum_{j \in \mathbb{N}} X_B(x_j, \tau \rho_j) \leq C \tau^{\zeta_0}.
\]

(iii) \( \text{supp} \psi_j \subset B(x_j, \rho_j) \) and \( 0 \leq \psi_j \leq 1 \).

(iv) \( |\nabla \psi_j(x)| \leq 1/\rho_j \) for all \( x, y \in \mathbb{R}^d \).

(v) \( \sum_{j \in \mathbb{N}} \psi_j = 1 \).

Proof. We note that \( \rho_w(\cdot, \mu) \) acquires all the properties analogous to those of \( \rho(\cdot, \mu) \) given in \cite{18}. Hence the proof for this lemma is done verbatim as in \cite{18}. Proof of Lemma 3.3].

3. Upper bounds for the fundamental solution \( \Gamma_\mu(x, y) \)

This section is devoted to the proof of Theorem 3.1. To do this, we first establish some solution/subsolution estimates for the equation \( (L + i \tau)u = f \). Before coming to the details we need to set up the formal definition of the operator \( L \).

In what follows we denote
\[
W^{1,2}_{w,\text{loc}}(\mathbb{R}^d) := \{ u \in L^2_{w,\text{loc}}(\mathbb{R}^d) : \partial_j u \in L^2_{w,\text{loc}}(\mathbb{R}^d) \text{ for all } j \in \{1, \ldots, d\} \}.
\]

3.1. The formal definition of \( L \). We first recall the Poincaré’s inequality in \cite{16} Lemma 5.

Proposition 3.1. Let \( x_0 \in \mathbb{R}^d \), \( R > 0 \) and \( B = B(x_0, R) \). Then
\[
\int_B \int_B |\phi(x) - \phi(y)|^2 \, dw(x) \, dw(y) \leq C R^2 w(B) \int_B |\nabla \phi(x)|^2 \, dw(x)
\]
for all \( \phi \in C^1(B) \).

We now state the following result which plays a key role in the construction of the formal definition of the operator \( L \).

Proposition 3.2. Let \( u \in W^{1,2}_{w,\text{loc}}(\mathbb{R}^d) \) such that \( \nabla u \in L^2_w(\mathbb{R}^d) \). Then the following hold.

(a) If \( u \in L^2_w(\mathbb{R}^d) \) then \( m_{w}(\cdot, \mu) u \in L^2_w(\mathbb{R}^d) \) and
\[
\int_{\mathbb{R}^d} |u|^2 m_w(\cdot, \mu)^2 \, dw \leq \int_{\mathbb{R}^d} |\nabla u|^2 \, dw + \int_{\mathbb{R}^d} |u|^2 \, d\tau.
\]

(b) If \( m_w(\cdot, \mu) u \in L^2_w(\mathbb{R}^d) \) then \( u \in L^2_w(\mathbb{R}^d) \) and
\[
\int_{\mathbb{R}^d} |u|^2 \, d\tau \leq \int_{\mathbb{R}^d} |\nabla u|^2 \, dw + \int_{\mathbb{R}^d} |u|^2 m_w(\cdot, \mu)^2 \, dw.
\]

Proof. We prove (a) only. The proof for (b) is done analogously.

Let \( x_0 \in \mathbb{R}^d \) and \( r_0 = \rho_w(x_0, \mu) \). Set \( B = B(x_0, r_0) \). By Proposition 2.1 (ii) we have
\[
I := \int_B \left( \frac{w(B)}{r_0^2} \wedge \pi(B) \right) |u|^2 \, dw \geq \frac{w(B)}{r_0^2} \int_B |u|^2 \, dw.
\]
Also it follows from Proposition 3.1 that
\[
I \leq \int_B \int_B \frac{1}{r_0} |u(x) - u(y)|^2 \, dw(x) \, dw(y) + w(B) \int_B |u(y)|^2 \, d\pi(y)
\]
\[
\leq w(B) \left( \int_B |\nabla u(x)|^2 \, dw(x) + \int_B |u(x)|^2 \, d\pi(x) \right).
\]

Hence
\[
\frac{1}{r_0} \int_B |u|^2 \, dw \leq \int_B |\nabla u|^2 \, dw + \int_B |u|^2 \, d\pi,
\]
or equivalently
\[
\int_B |u|^2 \, m_w(\cdot, \mu)^2 \, dw \leq \int_B |\nabla u|^2 \, dw + \int_B |u|^2 \, d\pi,
\]
as \(m_w(x, \mu) \sim 1/r_0\) for all \(x \in B\) by Proposition 2.1(iii).

Hence, let \(\{B_j\}_{j \in \mathbb{N}}\) be the family of balls in Lemma 2.4. Then we have
\[
\int_{B_j} |u|^2 \, m_w(\cdot, \mu)^2 \, dw \leq \int_{B_j} |\nabla u|^2 \, dw + \int_{B_j} |u|^2 \, d\pi
\]
for each \(j \in \mathbb{N}\).

Summing over all \(j \in \mathbb{N}\) and using (i) and (ii) of Lemma 2.4, we arrive at the conclusion. \(\square\)

The following result is a direct consequence of Proposition 3.2.

**Corollary 3.3.** Let
\[
H := \left\{ u \in W^{1,2}_{w,\text{loc}}(\mathbb{R}^d) : \nabla u \in L^2_w(\mathbb{R}^d) \text{ and } m_w(\cdot, \mu) u \in L^2_w(\mathbb{R}^d) \right\}
\]
be equipped with the norm
\[
\|u\|_H = \int_{\mathbb{R}^d} |\nabla u|^2 \, dw + \int_{\mathbb{R}^d} m_w(\cdot, \mu)^2 |u|^2 \, dw,
\]
and
\[
H' := \left\{ u \in W^{1,2}_{w,\text{loc}}(\mathbb{R}^d) : \nabla u \in L^2_w(\mathbb{R}^d) \text{ and } u \in L^2_w(\mathbb{R}^d, d\mu) \right\}
\]
be equipped with the norm
\[
\|u\|_{H'} = \int_{\mathbb{R}^d} |\nabla u|^2 \, dw + \int_{\mathbb{R}^d} |u|^2 \, w \, d\mu.
\]

Then \(H = H'\) with equivalent norms.

Moreover, \(H\) is a Hilbert space (with respect to the induced inner product).

Consider the quadratic form
\[
a(u, v) = \int_{\mathbb{R}^d} \nabla u \cdot \nabla v \, dx + \int_{\mathbb{R}^d} u \, v \, d\pi
\]
on the domain
\[
D(a) = H \cap L^2_{w,\text{loc}}(\mathbb{R}^d) = \left\{ u \in W^{1,2}_w(\mathbb{R}^d) : u \in L^2_{\pi}(\mathbb{R}^d) \right\},
\]
Lemma 3.6. The form $a$ for all $u \in D$ we endow $D(a)$ with the graph norm
$$||u||_{D(a)} = a(u, u) + ||u||_{W^{1,2}(\mathbb{R}^d)}$$
for all $u \in D(a)$. It follows from (2) and Corollary 3.3 that
$$||u||_{D(a)} \sim \int_{\mathbb{R}^d} |\nabla u|^2 \, dw + \int_{\mathbb{R}^d} |u|^2 \, m_{w, \cdot, \mu} \, dw + \int_{\mathbb{R}^d} |u|^2 \, dw$$
for all $u \in D(a)$. It is easy to see that $a$ is positive and symmetric. We will show in addition that $a$ is also densely defined and closed.

We need the following auxiliary result. In what follows define
$$W^{1,2}_{w,0}(B) := (C_c^\infty(B), || \cdot ||_{W^{1,2}}).$$

Lemma 3.4. Let $B \subset \mathbb{R}^d$ be a ball. Then the embedding $W^{1,2}_{w,0}(B) \hookrightarrow L^2(B, \, d\mu)$ is continuous.

Proof. Let $\{x_i\}_{i \in \mathbb{N}}$ and $\{\psi_i\}_{i \in \mathbb{N}}$ be as in Lemma 2.4. Since $B$ is compact we can cover it by a finite number of balls $B_j := B(x_j, \rho_j)$. Without loss of generality assume that $B \subset \cup_{j=1}^{j_0} B_j$ for some $j_0 \in \mathbb{N}^\ast$.

Therefore using Proposition 3.2, one has
$$\int_B |u|^2 \, d\mu \leq \int_B |\nabla u|^2 \, dw + \int_B |u|^2 \, m_{w, \cdot, \mu} \, dw$$
$$\leq \int_B |\nabla u|^2 \, dw + \sum_{j=1}^{j_0} \int_{B \cap B_j} |u|^2 \, m_{w, \cdot, \mu} \, dw$$
$$\leq \int_B |\nabla u|^2 \, dw + \sum_{j=1}^{j_0} m_{w}(x_j, \mu) \int_{B \cap B_j} |u|^2 \, dw$$
$$\leq \left(1 + \sum_{j=1}^{j_0} m_{w}(x_j, \mu)\right) ||u||_{W^{1,2}_{w,0}(B)} < \infty$$
for all $u \in W^{1,2}_{w,0}(B)$, where we used Proposition 3.2(iii) in the third step.

This verifies our claim. □

Lemma 3.5. The space $C_c^\infty(\mathbb{R}^d)$ is a form core for $a$. Consequently, $a$ is densely defined in $L^2_w(\mathbb{R}^d)$.

Proof. Let $f \in D(a)$. By multiplying $f$ with a cut-off function when necessary, we may assume that $\text{supp} \, f \subset B$ and $f \in W^{1,2}_{w,0}(B)$ for some ball $B \subset \mathbb{R}^d$. So there exists a sequence $(f_j)_{j \in \mathbb{N}} \subset C_c^\infty(B)$ such that $\lim_{j \to \infty} f_j = f$ in $W^{1,2}_{w,0}(B)$. By Lemma 3.4 we also have that $\lim_{j \to \infty} f_j = f$ in $L^2(B, d\mu)$. Hence $\lim_{j \to \infty} f_j = f$ in $D(a)$. To finish note that $C_c^\infty(\mathbb{R}^d)$ is dense in $L^2_w(\mathbb{R}^d)$ by [17, Theorem 1.1]. □

Lemma 3.6. The form $a$ is closed in $L^2_w(\mathbb{R}^d)$.
Proof. Let \( \{f_j\}_{j \in \mathbb{N}} \subset D(a) \) be a Cauchy sequence. Then \( \{f_j\}_{j \in \mathbb{N}} \) is Cauchy in \( W^{1,2}_w(\mathbb{R}^d) \) and \( L^2_w(\mathbb{R}^d) \). Hence there exist functions \( u \in W^{1,2}_w(\mathbb{R}^d) \) and \( f \in L^2_w(\mathbb{R}^d) \) such that \( \lim_{j \to \infty} f_j = u \) in \( W^{1,2}_w(\mathbb{R}^d) \) and \( \lim_{j \to \infty} f_j = f \) in \( L^2_w(\mathbb{R}^d) \). By using a subsequence if necessary we may conclude that \( \lim_{j \to \infty} f_j = u \) a.e. in \( \mathbb{R}^d \). Hence \( u = f \). It follows that \( u \in D(a) \) and \( \lim_{j \to \infty} f_j = u \) in \( D(a) \). \( \Box \)

We are ready to give the formal definition of the operator \( L \). From Lemmas 3.5 and 3.6, there exists a unique self-adjoint operator
\[
Lu := -\frac{1}{w} \text{div}(A \nabla u) + \mu u
\]
on the domain
\[
D(L) = \{ u \in D(a) : Lu \in L^2_w(\mathbb{R}^d) \}
\]
such that
\[
a(u, v) = \langle Lu, v \rangle_{L^2_w(\mathbb{R}^d)}
\]
for all \( u \in D(L) \) and \( v \in D(a) \).

3.2. Some estimates on solutions to the equation \( L_0 u = f \). Define
\[
L_0 = -\frac{1}{w} \text{div}(A \nabla).
\]
Let \( \Gamma_0(\cdot, \cdot) \) be its fundamental solution in \( \mathbb{R}^d \).

Definition 3.7. Let \( \Omega \subset \mathbb{R}^d \) be open. Let \( u \in W^{1,2}_{w,\text{loc}}(\Omega) \) and \( f \in L^1_{w,\text{loc}}(\Omega) \). Then \( u \) is called a weak solution of \( L_0 u = f \) in \( \Omega \) if
\[
\int_{\Omega} A \nabla u \cdot \nabla \psi \, dx = \int_{\Omega} f \psi \, dw
\]
for all \( \psi \in C^1_c(\Omega) \), where \( dw = w \, dx \).

Definition 3.8. Let \( \Omega \subset \mathbb{R}^d \) be open and \( u \in W^{1,2}_{w,\text{loc}}(\Omega) \). Then \( u \) is called a sub-solution of \( L_0 \) in \( \Omega \) if
\[
\int_{\Omega} A \nabla u \cdot \nabla \psi \, dx \leq 0
\]
for all non-negative function \( \psi \in C^1_c(\Omega) \).

The following two estimates are taken from [16, Lemmas 8 and 7] respectively.

Lemma 3.9. Let \( x \in \mathbb{R}^d, R > 0 \). Let \( u \) be a non-negative sub-solution of \( L_0 \) in \( B(x, 2R) \). Then for all \( \sigma \in (0, 1) \) there exists a constant \( C = C(\sigma) \) such that
\[
\sup_{B(x, \sigma R)} u \leq C \frac{1}{w(B(x, R))} \int_{B(x, R)} u \, dw.
\]

Proposition 3.10. There exists a \( C > 0 \) such that
\[
0 \leq \Gamma_0(x,y) \leq C \frac{|x-y|^2}{w(B(x,|x-y|))}
\]
for all \( x, y \in \mathbb{R}^d \).
3.3. Existence of solutions/subsolutions.

**Definition 3.11.** Let $\Omega \subset \mathbb{R}^d$ be open. Let $u \in W^{1,2}_{w,\text{loc}}(\Omega)$ and $f \in L^1_{w,\text{loc}}(\Omega)$. Then $u$ is called a weak solution of $Lu = f$ in $\Omega$ if

$$\int_{\Omega} A \nabla u \cdot \nabla \psi \, dx + \int_{\Omega} u \psi \, d\pi = \int_{\Omega} f \psi \, dw$$

for all $\psi \in C^1_c(\Omega)$, where we remind that $d\pi = w \, d\mu$ and $dw = w \, dx$.

**Definition 3.12.** Let $\Omega \subset \mathbb{R}^d$ be open and $u \in W^{1,2}_{w,\text{loc}}(\Omega)$. Then $u$ is called a sub-solution of $L$ in $\Omega$ if

$$\int_{\Omega} A \nabla u \cdot \nabla \psi \, dx + \int_{\Omega} u \psi \, d\pi \leq 0$$

for all non-negative function $\psi \in C^1_c(\Omega)$.

**Proposition 3.13.** Let $f \in L^1_{w,\text{loc}}(\mathbb{R}^d)$ be such that $m_{w,\mu}^{-1} f(\cdot) \in L^2_{w,\text{loc}}(\mathbb{R}^d)$. Then $Lu = f$ has a unique weak solution $u_f \in H$, where $H$ is defined by (17).

**Proof.** This is immediate from the Lax-Milgram theorem. \hfill \square

Schwartz kernel theorem now ensures that there exists a unique distributional $\Gamma_{\mu}(\cdot, \cdot)$ such that the representation

$$u_f(x) = \int_{\mathbb{R}^d} \Gamma_{\mu}(x, y) f(y) \, dw(y)$$

holds for a.e. $x \in \mathbb{R}^d$, $f \in L^2_{w,\text{loc}}(\mathbb{R}^d)$, where $u_f$ is as in Proposition 3.13. Such a $\Gamma_{\mu}(\cdot, \cdot)$ in fact enjoys further properties as stated in Proposition 3.20 below.

Recall that

$$L_0 = -\frac{1}{w} \text{div}(A \nabla).$$

**Lemma 3.14.** Let $u \in W^{1,2}_{w,\text{loc}}(\Omega)$ be a sub-solution of $Lu = 0$ in $\Omega$. Then $u^+ := u \vee 0$ is a sub-solution of $L_0$ in $\Omega$.

**Proof.** Let $\phi \in C^1_c(\Omega)$ be positive and set

$$\psi = \phi \times \frac{u^+}{u^+ + \epsilon} \in C^1_c(\Omega)$$

for each $\epsilon > 0$. Since $u$ is a sub-solution of $Lu = f$ in $\Omega$, it follows that

(18) \quad \int_{\Omega} A \nabla u \cdot \nabla \psi \, dx + \int_{\Omega} u \psi \, d\pi \leq 0.

Observe that the left-hand side equals to

$$\int_{\Omega} A \nabla u \cdot (\nabla \phi) \frac{u^+}{u^+ + \epsilon} \, dx + \int_{\Omega} A \nabla u \cdot (\nabla u^+) \frac{\epsilon \phi}{(u^+ + \epsilon)^2} \, dx + \int_{\Omega} u \phi \frac{u^+}{u^+ + \epsilon} \, d\pi$$

$$= \int_{\{u > 0\}} A \nabla u \cdot (\nabla \phi) \frac{u}{u + \epsilon} \, dx + \int_{\{u > 0\}} A \nabla u \cdot (\nabla u) \frac{\epsilon \phi}{(u + \epsilon)^2} \, dx + \int_{\{u > 0\}} \phi \frac{u^2}{u + \epsilon} \, d\pi$$
for all $\epsilon > 0$, where we used

$$
\nabla(u^+) = \begin{cases} 
\nabla u & \text{on } [u > 0], \\
0 & \text{otherwise},
\end{cases}
$$

(cf. [13, Lemma 7.6]) and the definition of $u^+$.

By taking the limits on both sides of (18) as $\epsilon \to 0$ we obtain

$$
\int_{\Omega} A \nabla u^+ \cdot \nabla \phi \, dx = \int_{[u > 0]} A \nabla u \cdot \nabla \phi \, dx \leq \int_{[u > 0]} A \nabla u \cdot \nabla \phi \, dx + \int_{[u > 0]} \phi \, du = 0.
$$

Since $0 \leq \phi \in C^1_c(\Omega)$ is arbitrary, a density argument justifies the claim. □

**Lemma 3.15.** Let $u \in W^{1,2}_{w,loc}(\Omega)$ be a weak solution of $Lu = 0$ in $\Omega$. Then $u^+$ and $|u|$ are sub-solutions of $L_0$ in $\Omega$.

**Proof.** By hypothesis, $u$ and $-u$ are sub-solutions of $Lu = f$ in $\Omega$. An application of Lemma 3.14 yields that $u^+$ and $-u^+ := (-u)^+$ are sub-solutions of $L_0$ in $\Omega$. Hence $|u| = |u^+| + |u^-|$ is also a sub-solution of $L_0$ in $\Omega$. □

**Lemma 3.16.** Let $u \in W^{1,2}_{w,loc}(\mathbb{R}^d)$ and $f \in L^1_{w,loc}(\mathbb{R}^d)$. Suppose that $f \geq 0$, $u$ is a weak solution of $Lu = f$ in $\mathbb{R}^d$, and

$$
\lim_{R \to \infty} \sup_{|x| = R} \frac{1}{w(B(x, R/2))} \int_{B(x, R/2)} |u| \, dw = 0.
$$

Then $u \geq 0$ in $\mathbb{R}^d$.

**Proof.** Since $f \geq 0$ we deduce that $-u$ is a sub-solution in $\mathbb{R}^d$. It follows from Lemma 3.14 that $u^-$ is a sub-solution of $L_0$ in $\mathbb{R}^d$. The maximum principle in [12, Theorem 2.3.8] now implies

$$
\sup_{B(0, R)} u^- \leq \sup_{\partial B(0, R)} u^-.
$$

However, according to Lemma 3.9 for all $x \in \partial B(0, R)$ we have

$$
u^-(x) \leq \frac{1}{w(B(x, R/2))} \int_{B(x, R/2)} u^- \, dw \leq \frac{1}{w(B(x, R/2))} \int_{B(x, R/2)} |u| \, dw \to 0
$$
as $R \to \infty$ by hypothesis. Hence $u^- = 0$ in $\mathbb{R}^d$. This in turn implies $u = u^+ \geq 0$. □

**Lemma 3.17.** Let $R > 0$. Then there exists a $C > 0$ such that

$$
0 < R^6 w(B(0, 1)) \leq C w(B(x, R/2))
$$

for all $x \in \mathbb{R}^d$ such that $R \leq |x| \leq 2R$.

**Proof.** Since $B(x, R/2) \subset B(0, 3R)$, we also have

$$
w(B(x, R/2)) \geq \left( \frac{|B(x, R/2)|}{|B(0, 3R)|} \right)^{2d} w(B(0, 3R)) \geq R^6 w(B(0, 1)) > 0,
$$

where we used (0) and (3) in the first and second steps, respectively. □
Proposition 3.18. Let \( f \in L^2_{w,c}(\mathbb{R}^d) \) be positive. Let \( u = u_f \), where \( u_f \) is given by Proposition 3.13. Then
\[
0 \leq u(x) \leq \int_{\mathbb{R}^d} \Gamma_0(x, y) f(y) \, dw(y),
\]
where \( \Gamma_0 \) is the fundamental solution of \( L_0 \) in \( \mathbb{R}^d \).

Proof. Let \( R > 0 \). It follows from Proposition 2.1(iv) that
\[
(20) \quad \frac{1}{R^2} \int_{R/2 \leq |x| \leq 2R} |u|^2 \, dw \leq \frac{1}{R^2} \left( \frac{1}{w(B(x, R/2))} \right) \left( \int_{B(x, R/2)} |u|^2 \, dw \right)^{1/2}
\]
where \( H \) is given by (17).

This leads to
\[
\lim_{R \to \infty} \sup_{|x| = R} \frac{1}{w(B(x, R/2))} \int_{B(x, R/2)} |u| \, dw \leq \lim_{R \to \infty} \sup_{|x| = R} \left( \frac{R^2}{w(B(x, R/2))} \frac{1}{R^2} \left( \int_{B(x, R/2)} |u| \, dw \right)^{1/2} \right)\]
\[
\leq \lim_{R \to \infty} \sup_{|x| = R} \left( \frac{1}{w(B(0, 1))} \frac{1}{R^2} \left( \int_{B(0, R/2)} |u| \, dw \right)^{1/2} \right)\]
\[
= 0,
\]
where we used Lemma 3.17 and the fact that \( \beta > 2 \) in the second-to-last step. This verifies (19). Therefore \( u \geq 0 \) by Lemma 3.16.

For the remaining inequality set
\[
v(x) = \int_{\mathbb{R}^d} \Gamma_0(x, y) f(y) \, dw(y).
\]
Then \( v \in W^{1,2}_{w, loc}(\mathbb{R}^d) \), \( L_0 v = f \) in \( \mathbb{R}^d \) and \( v \geq 0 \) (cf. [6, Theorem 1.3]). Also \( u - v \) is a sub-solution of \( Lu = 0 \) in \( \mathbb{R}^d \). Lemma 3.14 now implies that \((u - v)^+\) is a sub-solution of \( L_0 \) in \( \mathbb{R}^d \). Next we use the maximal principle in [12, Theorem 2.3.8] and Lemma 3.15 to derive
\[
\sup_{B(0,R)} (u - v)^+ \leq \sup_{\partial B(0,R)} (u - v)^+ \leq \frac{1}{w(B(0, R))} \left( \int_{R/2 \leq |x| \leq 2R} |u(x)| + |v(x)| \right) \, dw(x).
\]
Note that Proposition 3.10 implies \( v(x) = O\left( \frac{R^2}{w(B(x, R))} \right) \) for all \( x \in \mathbb{R}^d \) such that \( R/2 \leq |x| \leq 2R \). This together with (20) gives \((u - v)^+ = 0\) in \( \mathbb{R}^d \). Hence \( u \leq v \) in \( \mathbb{R}^d \). \( \square \)

We end this subsection with a domination property of the fundamental solutions of \( L + i\tau \) for \( \tau \in \mathbb{R} \).

Proposition 3.19. Let \( \tau \in \mathbb{R} \) and \( \Gamma_\mu(\cdot, \cdot, \tau) \) be the fundamental solution of \( L + i\tau \). Then there exists a \( C = C(d) > 0 \) such that
\[
|\Gamma_\mu(x, y, \tau)| \leq \Gamma_\mu(x, y) \leq C \Gamma_0(x, y)
\]
for all \( x, y \in \mathbb{R}^d \).
Proof. Let $\tau \in \mathbb{R}$ and $x, y \in \mathbb{R}^d$. Using Lemma 3.18 and Lebesgue differentiation theorem (see for example [20, Theorem]) we easily obtain

$$\Gamma_{\mu}(x, y) \leq C \Gamma_0(x, y).$$

For the first inequality, using functional calculus one has

$$\Gamma_{\mu}(x, y, \tau) = \int_0^{\infty} e^{-it\tau} k_t(x, y) \, dt,$$

where we recall that $k_t(x, y)$ is the heat kernel of $L$.

Since $k_t(x, y) \geq 0$ (see (22) below), we have

$$|\Gamma_{\mu}(x, y, \tau)| \leq \int_0^{\infty} k_t(x, y) \, dt = \Gamma_{\mu}(x, y).$$

The claim now follows. \qed

Proposition 3.20. The following statements hold.

(i) $\Gamma_{\mu}(x, \cdot) \in L^p_{\text{loc}}(\mathbb{R}^d)$ for all $p \in (0, \frac{d}{d-2})$ and $x \in \mathbb{R}^d$.

(ii) For all $f \in L^2_{w,c}(\mathbb{R}^d)$ the function

$$u(\cdot) = \int_{\mathbb{R}^d} \Gamma_{\mu}(\cdot, y) f(y) \, dw(y)$$

is the unique weak solution of $Lu = f$ in $\mathbb{R}^d$.

Proof. By [6, Theorem 1.3(v)] we know that $\Gamma_0(x, \cdot) \in L^p_{\text{loc}}(\mathbb{R}^d)$ for all $p \in (0, \frac{d}{d-2})$ and $x \in \mathbb{R}^d$. The two statements then follow immediately from Propositions 3.13 and 3.19. \qed

3.4. Upper bounds for solutions. We first prove a Caccioppoli’s inequality for solutions to the equation $(L + i\tau)u = 0$.

Lemma 3.21. Let $\tau \in \mathbb{R}$, $x_0 \in \mathbb{R}^d$, $R > 0$ and $B = B(x_0, R)$. Let $u$ be a solution of $Lu + i\tau u = 0$ in $B$. Then for every $\sigma \in (0, 1)$ there exists a $C > 0$ such that

$$\int_B |\nabla u|^2 \, dw + \int_{\sigma B} |u|^2 \, d\pi + \int_{\sigma B} |\tau| |u|^2 \, dw \leq C \frac{R^2}{\sigma^2} \int_B |u|^2 \, dw.$$

Proof. Let $\eta \in C^\infty_0(B)$ be such that

$$\eta \geq 0, \quad \eta|_{\partial B} = 1 \quad \text{and} \quad |\nabla \eta| \leq \frac{1}{(1-\sigma) R}.$$

Using $\eta^2 \eta$ as a test function, we have

$$\int_B \eta^2 \Lambda \nabla u \cdot \nabla \eta \, dw + \int_B \eta^2 |u|^2 \, d\pi + i\tau \int_B \eta^2 |u|^2 \, dw = -2 \int_B \eta \Lambda \nabla u \cdot \nabla \eta \, dw.$$
Consequently,

\[
\Lambda^{-1} \int_{\sigma B} \eta^2 |\nabla u|^2 \, dw + \int_{\sigma B} \eta^2 |u|^2 \, d\tau + |\tau| \int_{\sigma B} |u|^2 \, dw \\
\leq \left| \int_B \eta^2 \Lambda \nabla u \cdot \nabla \Pi \, dw + \int_B \eta^2 |u|^2 \, d\tau + i \tau \int_B \eta^2 |u|^2 \, dw \right| \\
= 2 \left| \int_B \eta \Lambda \nabla u \cdot \nabla \eta \, dw \right| \\
\leq 2 \Lambda \int_B |\eta \nabla u| |\Pi \nabla \eta| \, dw \\
\leq \varepsilon \int_B \eta^2 |\nabla u|^2 \, dw + \frac{\Lambda^2}{\varepsilon} \int_B |u|^2 |\nabla \eta|^2 \, dw \\
\leq \varepsilon \int_B \eta^2 |\nabla u|^2 \, dw + \frac{\Lambda^2}{\varepsilon (1 - \sigma)^2 R^2} \int_B u^2 \, dw
\]

for all \( \varepsilon > 0 \).

Choosing a sufficiently small \( \varepsilon \) in the above inequality, our claim is justified. \( \Box \)

**Lemma 3.22.** Let \( x_0 \in \mathbb{R}^d \), \( R > 0 \) and \( B = B(x_0, R) \). Let \( u \) be a solution of \( Lu + i \tau u = 0 \) in \( 4B \). Then for all \( k \in \mathbb{N} \) there exists a \( C > 0 \) such that

\[
\sup_B |u| \leq \frac{C}{(1 + R \sqrt{\tau})^k (1 + R m_w(x_0, \mu))} \left( \frac{1}{w(2B)} \int_{2B} |u|^2 \, dw \right)^{1/2}.
\]

**Proof.** Let \( k \in \mathbb{N} \), \( B = B(x_0, R) \) and \( B_k = (1 + 2^{-k} \lfloor k_0 + 1/2 \rfloor)B \), where \( \lfloor \alpha \rfloor \) denotes the smallest integer greater than \( \alpha \) for every \( \alpha \in \mathbb{R} \).

To begin with, we will show that \( |u|^2 \) is a sub-solution of \( L_0 \) in \( 4B \) (in the sense of Definition 3.8). Let \( \psi \in C_c^1(4B) \) be non-negative. Direct calculations give

\[
\int_{4B} \Lambda \nabla (|u|^2) \cdot \nabla \psi \, dx = \int_{4B} \Lambda \nabla (u \Pi) \cdot \nabla \psi \, dx = \int_{4B} \Lambda \left( \Pi \nabla u + u \nabla \Pi \right) \cdot \nabla \psi \, dx \\
= 2 \Re \int_{4B} \Pi \Lambda \nabla u \cdot \nabla \psi \, dx,
\]

where \( \Re z \) denotes the real part of a complex number \( z \).

On the other hand, by using \( \Pi \psi \) as a test function, the hypothesis gives

\[
\int_{4B} \psi \Lambda \nabla u \cdot \nabla \Pi \, dx + \int_{4B} \Pi \Lambda \nabla u \cdot \nabla \psi \, dx + \int_{4B} |u|^2 \psi \, d\tau + i \tau \int_{4B} |u|^2 \, dw = 0.
\]

Now we take the real parts of both sides to derive

\[
\int_{4B} \psi \Lambda \nabla u \cdot \nabla \Pi \, dx + \Re \int_{4B} \Pi \Lambda \nabla u \cdot \nabla \psi \, dx + \int_{4B} |u|^2 \psi \, d\tau = 0,
\]
or equivalently,

\[ \Re \int_{4B} \nabla u \cdot \nabla \psi \, dx = -\int_{4B} \psi \nabla u \cdot \nabla \nabla \psi \, dx - \int_{4B} |u|^2 \psi \, d\tau \]

\[ \leq -\Lambda^{-1} \int_{4B} \psi |\nabla u|^2 \, dw - \int_{4B} |u|^2 \psi \, d\tau \leq 0, \]

where we used (2) in the second step.

Hence

\[ \int_{4B} \nabla (|u|^2) \cdot \nabla \psi \, dx \leq 0 \]

and so \(|u|^2\) is a sub-solution of \(L_0\) in \(4B\).

With this in mind we next apply Lemma 3.9 to obtain

\[ \sup_B |u| \leq \left( \frac{1}{w(B_k)} \int_{B_k} |u|^2 \, dw \right)^{1/2}. \]

Hence the claim is clear if \(k = 0\).

Next suppose \(k \geq 1\). By Lemma 3.21 one has

\[ \int_{B_k} |\nabla u|^2 \, dw + \int_{B_k} |u|^2 \, d\tau \leq \frac{1}{R^2} \int_{B_{k-1}} |u|^2 \, dw. \]

Next let \(\eta \in C_0^\infty(B_{k-1})\) be such that

\[ \eta|_{B_k} = 1, \quad \eta \leq 1 \quad \text{and} \quad |\nabla \eta| \leq \frac{1}{R}. \]

Applying Proposition 3.2 to \(u \eta\) yields

\[ \int_{B_k} m_w(\cdot, \mu)^2 |u|^2 \, dw \leq \int_{B_{k-1}} |\nabla u|^2 \, dw + \int_{B_{k-1}} |u|^2 \, d\tau + \frac{1}{R^2} \int_{B_{k-1}} |u|^2 \, dw, \]

which in turn implies

\[ \int_{B_k} m_w(\cdot, \mu)^2 |u|^2 \, dw \leq \frac{1}{R^2} \int_{B_{k-1}} |u|^2 \, dw. \]

Combining this with Proposition 2.1(iv) we yield

\[ \int_{B_k} |u|^2 \, dw \leq \frac{1}{(1 + R m_w(x_0, \mu))^{2/(k_0 + 1)}} \int_{B_{k-1}} |u|^2 \, dw. \]

Iterating the above estimate \(k \lceil (k_0 + 1)/2 \rceil\) times and using Lemma 3.9 we arrive at

\[ \int_B |u|^2 \, dw \leq \frac{C}{(1 + R m_w(x_0, \mu))^k} \left( \frac{1}{w(2B)} \int_{2B} |u|^2 \, dw \right)^{1/2}. \]

Similar arguments together with Lemma 3.21 also gives

\[ \int_B |u|^2 \, dw \leq \frac{C}{(1 + R \sqrt{\tau})^k} \left( \frac{1}{w(2B)} \int_{2B} |u|^2 \, dw \right)^{1/2}. \]

To conclude we combine these two estimates and yield the claim. \(\square\)
Proposition 3.23. Let \( \tau \in \mathbb{R} \). Let \( \Gamma_{\mu}(x,y,\tau) \) be the fundamental solution of \( L + i\tau \) in \( \mathbb{R}^d \). Then for every \( k \in \mathbb{N} \) there exists a \( C = C(k) > 0 \) such that

\[
|\Gamma_{\mu}(x,y,\tau)| \leq \frac{C}{(1 + |x - y| \sqrt{\tau})^k (1 + |x - y| m_w(x, \mu))^k w(B(x, |x - y|))}
\]

for all \( x, y \in \mathbb{R}^d \) such that \( x \neq y \).

**Proof.** Let \( x, y \in \mathbb{R}^d \) be such that \( x \neq y \). Set \( R = |x - y| \) and \( B = B(x, R/8) \). By Proposition 3.10,

\[
\Gamma_0(z,y) \lesssim \frac{|z - y|^2}{w(B(z, |z - y|))}
\]

for all \( z \in 2B \).

Applying (6),

\[
\Gamma_0(z,y) \lesssim \frac{|z - y|^2}{w(B,z,8R)} R^2 \frac{w(B(z,8R))}{w(B(z,|z-y|)) w(B,z,8R))}
\]

\[
\lesssim \left( \frac{|z - y|}{R} \right)^{2d-2} \frac{R^2}{w(8B)}
\]

(21)

\[
\lesssim \frac{R^2}{w(B)}
\]

for all \( z \in 4B \).

Next observe that \( u(\cdot) := \Gamma_{\mu}(\cdot, y, \tau) \) is a weak solution of \( (L + i\tau)u = 0 \) in \( 4B \). It follows that for all \( k \in \mathbb{N} \) one has

\[
\sup_B |u| \lesssim \frac{1}{(1 + R m_w(x, \mu))^k} \left( \frac{1}{w(2B)} \int_{2B} |u|^2 \, dw \right)^{1/2}
\]

\[
\lesssim \frac{1}{(1 + R m_w(x, \mu))^k} \left( \frac{1}{w(2B)} \int_{2B} |\Gamma_0(\cdot, y)|^2 \, dw \right)^{1/2}
\]

\[
\lesssim \frac{1}{(1 + R m_w(x, \mu))^k} \frac{R^2}{w(B)}
\]

where we used Lemma 3.22 in the first step, Proposition 3.19 in the second step as well as (21) in the last step. \( \square \)

**Proof of Theorem 1.1.** We note that \( \Gamma_{\mu}(\cdot, \cdot) \geq 0 \) follows from Proposition 3.19. The upper bound (8) for \( \Gamma_{\mu}(\cdot, \cdot) \) is obtained by setting \( \tau = 0 \) in Proposition 3.23. \( \square \)

4. **Estimates on heat kernel \( k_t(x, y) \)**

This section is dedicated to proving Theorem 1.2.

We recall some estimates on the kernel \( h_t(\cdot, \cdot) \) of \( e^{-tL_0} \). See for example [11, Section 3].

**Proposition 4.1.** The following properties hold for the kernel \( h_t(x, y) \) of \( e^{-tL_0} \).
(i) There exist constants \( C, c > 0 \) such that
\[
\frac{1}{w(B(x, \sqrt{t}))} \exp \left( -\frac{|x - y|^2}{ct} \right) \leq h_t(x, y) \leq \frac{1}{w(B(x, \sqrt{t}))} \exp \left( -\frac{|x - y|^2}{ct} \right)
\]
for all \( x, y \in \mathbb{R}^d \) and \( t > 0 \).

(ii) There exist constants \( C > 0 \) and \( \gamma \in (0, 1] \) such that
\[
|h_t(x, y) - h_t(x, z)| \lesssim \frac{1}{w(B(x, \sqrt{t}))} \left( \frac{|y - z|}{\sqrt{t}} \right)^\gamma \left[ \exp \left( -\frac{|x - y|^2}{ct} \right) + \exp \left( -\frac{|x - z|^2}{ct} \right) \right]
\]
for all \( x, y \in \mathbb{R}^d \) and \( t > 0 \).

(iii) There exists a constant \( C > 0 \) such that for all \( k \in \mathbb{N} \) there exists a \( c > 0 \) satisfying
\[
|\partial^k_t h_t(x, y)| \leq \frac{c}{t^k w(B(x, \sqrt{t}))} \exp \left( -\frac{|x - y|^2}{ct} \right)
\]
for all \( x, y \in \mathbb{R}^d \) and \( t > 0 \).

(iv) For every \( x \in \mathbb{R}^d \),
\[
\int_{\mathbb{R}^d} h_t(x, y) \, dw(y) = 1.
\]

Since \( \mu \) is a non-negative Radon measure, a perturbation formula asserts that
\[
0 \leq k_t(x, y) \leq h_t(x, y) \lesssim \frac{1}{w(B(x, \sqrt{t}))} \exp \left( -\frac{|x - y|^2}{ct} \right)
\]
for all \( x, y \in \mathbb{R}^d \) and all \( t > 0 \).

We are now ready to give the proof of Theorem 1.2 (i).

**Proof of Theorem 1.2 (i).** In what follows, let \( x, y \in \mathbb{R}^d \) and \( t > 0 \). We divide the proof into two cases.

**Case I:** \( |x - y| \geq \sqrt{t} \).

By functional calculus,
\[
k_t(x, y) = C \int_{\mathbb{R}} e^{i\tau \Gamma_\mu(x, y, \tau)} \, d\tau.
\]

Hence Theorem 1.1 gives
\[
k_t(x, y) \lesssim \frac{1}{(1 + \sqrt{t} m_w(x, \mu))^k w(B(x, \sqrt{t}))}
\]
for all \( k \in \mathbb{N} \). Combining this with (22) and Proposition 4.1 (i), we have
\[
k_t(x, y) \lesssim \frac{1}{(1 + \sqrt{t} m_w(x, \mu))^k w(B(x, \sqrt{t}))} \exp \left( -\frac{|x - y|^2}{ct} \right)
\]
for all \( k \in \mathbb{N} \).

**Case II:** \( |x - y| < \sqrt{t} \).

The semigroup \( \{e^{-tL}\}_{t>0} \) can be extended to a holomorphic contraction semigroup on \( L^2_w(\mathbb{R}^d) \). Therefore for all \( k \in \mathbb{N} \) there exists a \( C > 0 \) such that
\[
\|\partial^k_t e^{-tL}\|_{L^2_w(\mathbb{R}^d) \to L^2_w(\mathbb{R}^d)} \leq \frac{C}{t^k}.
\]
Observe that
\[(\partial_t k_t)(x,y) = (\partial_t e^{-tL})(k_t(\cdot,y))(x)\]
Consequently, we obtain
\[\| (\partial_t k_t)(x,\cdot) \|_{L^2_w(\mathbb{R}^d)} \lesssim \frac{1}{t} \| k_t(x,\cdot) \|_{L^2_w(\mathbb{R}^d)} \lesssim \frac{1}{t w(B(x, \sqrt{t}))^{1/2}},\]
where we used (22) and Proposition 4.1(i) in the last step. Using Schwartz’s inequality,
\[| (\partial_t k_t)(x,y) | \lesssim \frac{1}{t w(B(x, \sqrt{t}))^{1/2}} w(B(y, \sqrt{t}))^{1/2}.\]
Next we estimate \( k_t \) as follows
\[k_t(x,y) = \int_{\mathbb{R}^d} \Gamma_t(x,z) (\partial_t k_t)(z,y) \, d\nu(z)\]
\[\lesssim \int_{\mathbb{R}^d} \frac{1}{(1 + |x-z| m_w(x,\mu))^N w(B(x,|x-z|)) t w(B(y, \sqrt{t}))^{1/2}} w(B(z, \sqrt{t}))^{1/2} \, d\nu(z)\]
\[= \sum_{j \in \mathbb{N}} \int_{2^j B \setminus 2^{j-1} B} \cdots + \sum_{j \in -\mathbb{N}^*} \int_{2^j B \setminus 2^{j-1} B} \cdots =: I + II\]
for all \( N > 0 \), where \( B := B(x, \rho w(x,\mu)) \).
Next we estimate each term separately.
**Term I:** We have
\[I \lesssim \sum_{j \in \mathbb{N}} \int_{2^j B \setminus 2^{j-1} B} \frac{1}{(1 + 2^j (1 + |x-z| m_w(x,\mu))^N) w(B(x,2^j \rho w(x,\mu)))} \frac{1}{t w(B(y, \sqrt{t}))^{1/2}} (1 + 2^j \rho w(x,\mu)/\sqrt{t})^{2d} \, d\nu(z)\]
\[\lesssim \left( \frac{\rho w(x,\mu)}{\sqrt{t}} \right)^2 \frac{1}{w(B(x, \sqrt{t}))^{1/2}} w(B(y, \sqrt{t}))^{1/2} \left( 1 + \frac{2^j \rho w(x,\mu)}{\sqrt{t}} \right)^{2d} \, d\nu(z)\]
\[\times \sum_{j \in \mathbb{N}} \int_{2^j B \setminus 2^{j-1} B} \frac{1}{w(B(x,2^j \rho w(x,\mu)))} \left( 1 + \frac{\rho w(x,\mu)}{\sqrt{t}} \right)^{2d} \, d\nu(z)\]
\[\lesssim \left( \frac{\rho w(x,\mu)}{\sqrt{t}} \right)^2 \frac{1}{w(B(x, \sqrt{t}))^{1/2}} w(B(y, \sqrt{t}))^{1/2} \]
where \( N \) is chosen large enough and we use the facts that \( \sqrt{t} m_w(x,\mu) \geq 1 \), \( |x-z| \sim 2^j \rho w(x,\mu) \) and
\[w(B(x, \sqrt{t})) \lesssim w(B(z, \sqrt{t})) \left( 1 + \frac{|x-z|}{\sqrt{t}} \right)^{2d} \]due to the doubling property (6) of \( w \).
Term II: Observe that in this case $|x-z| < 2 \rho_w(x, \mu) < 2 \sqrt{t}$. So $w(B(x, \sqrt{t})) \sim w(B(z, \sqrt{t}))$ and we have

$$II \lesssim \sum_{j \in \mathbb{N}} \int_{2^{j}B \setminus 2^{j-1}B} \frac{2^j \rho_w(x, \mu)^2}{w(B(x, 2^j \rho_w(x, \mu)))} \times \frac{1}{tw(B(x, \sqrt{t}))^{1/2}w(B(y, \sqrt{t}))^{1/2}} \, dw(z)$$

$$\lesssim \left( \frac{\rho_w(x, \mu)}{\sqrt{t}} \right)^2 \frac{1}{w(B(x, \sqrt{t}))^{1/2}w(B(y, \sqrt{t}))^{1/2}}.$$  

In sum we have proved that

$$k_t(x, y) \lesssim \left( \frac{\rho_w(x, \mu)}{\sqrt{t}} \right)^2 \frac{1}{w(B(x, \sqrt{t}))^{1/2}w(B(y, \sqrt{t}))^{1/2}}.$$  

Also note that when $|x-y| < \sqrt{t}$ and $\sqrt{t} m_w(x, \mu) \geq 1$, Proposition 2.1(iii) implies

$$\left( \sqrt{t} m_w(x, \mu) \right)^{-1} \leq (1 + \sqrt{t} m_w(y, \mu))^{-1/(k_0+1)}.$$  

Keeping in mind these two estimates, we now invoke the symmetry of $k_t$ and use Proposition 4.1 to obtain

$$k_t(x, y) \lesssim \frac{1}{w(B(x, \sqrt{t}))^{1/2}w(B(y, \sqrt{t}))^{1/2}} \exp \left( -\frac{|x-y|^2}{ct} \right) \times (1 + \sqrt{t} m_w(x, \mu))^{-1/(k_0+1)} (1 + \sqrt{t} m_w(y, \mu))^{-1/(k_0+1)}.$$  

In turn this better estimate of $k_t$ (compared to (23)) implies a better estimate of $\partial_1 k_t$ (compared to (24)). Particularly one has

$$||\partial_1 k_t||_{L^1} \lesssim \frac{1}{tw(B(x, \sqrt{t}))^{1/2}w(B(y, \sqrt{t}))^{1/2}} \times (1 + \sqrt{t} m_w(x, \mu))^{-N} (1 + \sqrt{t} m_w(y, \mu))^{-N}$$

and

$$||\partial_1 k_t||_{L^1} \lesssim \frac{1}{tw(B(x, \sqrt{t}))^{1/2}w(B(y, \sqrt{t}))^{1/2}} \times (1 + \sqrt{t} m_w(x, \mu))^{-N} (1 + \sqrt{t} m_w(y, \mu))^{-N}.$$
Thus the claim follows after applying the estimate
\[
\frac{1}{w(B(y, \sqrt{t}))} \exp \left( - \frac{|x-y|^2}{ct} \right) \leq \frac{1}{w(B(x, \sqrt{t}))} \exp \left( - \frac{|x-y|^2}{c't} \right).
\]
This finishes the proof of Theorem 1.2 (i). \qed

The proof of the item (iii) in Theorem 1.2 will be given below.

Proof of Theorem 1.2 (iii): Let \( t > 0 \) and \( x, y \in \mathbb{R}^d \). If \( t \geq \rho_w(x, \mu)^2 \) the claim follows at once from Theorem 1.2. Hence we need only to prove the theorem assuming that \( t < \rho_w(x, \mu)^2 \).

By Duhamel’s formula we have
\[
h_t(x, y) - k_t(x, y) = \int_0^t \int_{\mathbb{R}^d} h_s(x, u) k_{t-s}(u, y) d\pi(u) ds.
\]

It follows from Proposition 4.1 (i) that
\[
q_t(x, y) \leq \int_0^t \int_{\mathbb{R}^d} \frac{1}{w(B(x, \sqrt{s}))} e^{-\frac{|x-u|^2}{cs}} k_{t-s}(u, y) d\pi(u) ds =: I + II,
\]
where
\[
I := \int_{t/2}^t \int_{\mathbb{R}^d} \frac{1}{w(B(x, \sqrt{s}))} e^{-\frac{|x-u|^2}{cs}} k_{t-s}(u, y) d\pi(u) ds \quad \text{and}
\]
\[
II := \int_0^{t/2} \int_{\mathbb{R}^d} \frac{1}{w(B(x, \sqrt{s}))} e^{-\frac{|x-u|^2}{cs}} k_{t-s}(u, y) d\pi(u) ds.
\]

Next we estimate each term separately.

Term I: The Gaussian upper bound of the kernel \( k_t(\cdot, \cdot) \) allows us to get the bound
\[
I \leq \int_{t/2}^t \int_{\mathbb{R}^d} \frac{1}{w(B(x, \sqrt{s})) w(B(y, \sqrt{t-s}))} e^{-\frac{|x-u|^2}{cs}} e^{-\frac{|y-u|^2}{ct(t-s)}} d\pi(u) ds
\]
\[
\leq \int_0^{t/2} \int_{\mathbb{R}^d} \frac{1}{w(B(x, \sqrt{s})) w(B(y, \sqrt{t}))} e^{-\frac{|x-u|^2}{cs}} e^{-\frac{|y-u|^2}{ct}} d\pi(u) ds.
\]

Note that
\[
e^{-\frac{|x-u|^2}{cs}} e^{-\frac{|y-u|^2}{ct}} \leq e^{-\frac{|x-u|^2}{ct}} e^{-\frac{|y-u|^2}{ct}} \leq e^{-\frac{|x-y|^2}{ct}}.
\]
Consequently,

\[
I \lesssim \frac{1}{w(B(y, \sqrt{t}))} \exp \left( -\frac{|x-y|^2}{ct} \right) \int_0^{t/2} \frac{1}{w(R)} \int_{B(x, \sqrt{s})} e^{\frac{|x-u|^2}{c't}} d\pi(u) ds \\
\lesssim \frac{1}{w(B(y, \sqrt{t}))} \exp \left( -\frac{|x-y|^2}{ct} \right) \int_0^{t/2} \frac{1}{s} \left( \frac{\sqrt{s}}{\rho_w(x, \mu)} \right)^\delta ds \\
\sim \frac{1}{w(B(y, \sqrt{t}))} \exp \left( -\frac{|x-y|^2}{ct} \right) \left( \frac{\sqrt{t}}{\rho_w(x, \mu)} \right)^\delta \\
\lesssim \frac{1}{w(B(x, \sqrt{t}))} \exp \left( -\frac{|x-y|^2}{ct} \right) \left( \frac{\sqrt{t}}{\rho_w(x, \mu)} \right)^\delta,
\]

where we used Lemma 2.3 in the second step and the fact that

\[
\frac{1}{w(B(y, \sqrt{t}))} \exp \left( -\frac{|x-y|^2}{ct} \right) \lesssim \frac{1}{w(B(x, \sqrt{t}))} \exp \left( -\frac{|x-y|^2}{c't} \right)
\]

in the last step.

**Term II:** Using a change of variables we can rewrite II as follows:

\[
\int_0^{t/2} \frac{1}{w(R)} \int_{B(x, \sqrt{t-s})} w(B(y, \sqrt{s})) e^{\frac{|x-u|^2}{c(t-s)}} e^{\frac{|y-u|^2}{cs}} d\pi(u) ds.
\]

Arguing similarly to I we conclude that

\[
II \lesssim \frac{1}{w(B(x, \sqrt{t}))} \exp \left( -\frac{|x-y|^2}{ct} \right) \left( \frac{\sqrt{t}}{\rho_w(y, \mu)} \right)^\delta.
\]

On the other hand, from Proposition 2.3(iii) and the fact that \( t < \rho_w(x, \mu)^2 \),

\[
\frac{1}{\rho_w(y, \mu)} \leq \frac{1}{\rho_w(x, \mu)} \left( 1 + \frac{|x-y|}{\rho_w(x, \mu)} \right)^{k_0} \leq \frac{1}{\rho_w(x, \mu)} \left( 1 + \frac{|x-y|}{t} \right)^{k_0},
\]

which implies that

\[
\left( \frac{\sqrt{t}}{\rho_w(x, \mu)} \right)^\delta \exp \left( -\frac{|x-y|^2}{ct} \right) \leq \left( \frac{\sqrt{t}}{\rho_w(y, \mu)} \right)^\delta \exp \left( -\frac{|x-y|^2}{c't} \right).
\]

Consequently,

\[
II \lesssim \frac{1}{w(B(x, \sqrt{t}))} \exp \left( -\frac{|x-y|^2}{ct} \right) \left( \frac{\sqrt{t}}{\rho_w(x, \mu)} \right)^\delta.
\]

Combining the estimates for I and II, we arrive at the claim.

This completes our proof. \( \square \)

In order to prove Theorem 1.2(ii), we need some technical ingredients. Set

\[
q_t(x, y) = h_t(x, y) - k_t(x, y)
\]

for all \( x, y \in \mathbb{R}^d \) and \( t > 0 \). We have the following estimates on \( q_t(x, y) \).
Proposition 4.2. For any $0 < \theta < \min(\gamma, \delta)$ there exist $C$ and $c > 0$ so that

$$|q_t(x, y) - q_t(x, y)| \leq C \min \left\{ \left( \frac{|x - y|}{\rho_w(y, \mu)} \right)^\theta, \left( \frac{|x - z|}{\sqrt{t}} \right)^\theta \right\} \frac{1}{w(B(x, \sqrt{t}))} \exp \left( -\frac{|x - y|^2}{ct} \right)$$

for all $t > 0$, $|x - y| < |x - y|/4$ and $|x - z| < \rho_w(x, \mu)$.

Proof. By Duhamel’s formula, we have

$$q_t(x, y) = \int_0^t \int_{\mathbb{R}^d} (h_s(x, z) - h_s(x, y)) k_{t-s}(y, z) d\pi(z) ds$$

We now take care of $I_1$ first. To do this we write

$$I_1 = \int_0^{t/2} \int_{B(x, |x-y|/2)} \ldots + \int_0^{t/2} \int_{B(x, |x-y|/2)^c} \ldots := I_{11} + I_{12}.$$ 

Note that for $z \in B(x, |x-y|/2), |z-y| \sim |x-y|$. Applying Proposition 4.1(ii), Theorem 1.2 and using the fact that $t-s \sim t$ for $s \in (0, t/2)$, we can bound the term $I_{11}$ as follows:

$$I_{11} \leq \int_0^{t/2} \int_{B(x, |x-y|/2)} \left( \frac{|x - z|}{\sqrt{s}} \right)^\theta \frac{1}{w(B(z, \sqrt{s}))} \left[ \exp \left( -\frac{|x - z|^2}{cs} \right) + \exp \left( -\frac{|x - z|^2}{cs} \right) \right]$$

$$\times \frac{1}{w(B(y, \sqrt{t}))} \exp \left( -\frac{|x - y|^2}{ct} \right) \left( 1 + \frac{\sqrt{t}}{\rho_w(y, \mu)} \right)^{-\theta \rho_w(y, \mu)} d\pi(z) ds$$

$$\leq \int_0^{\rho_w(y, \mu)^2} \int_{B(x, 2|x-y|)} \ldots + \int_0^{\rho_w(y, \mu)^2} \int_{B(x, 2|x-y|)^c} \ldots := J_1 + J_2.$$ 

Note that $\rho_w(x, \mu) \sim \rho_w(x, \mu)$ for $|x - y| \leq \rho_w(x, \mu)$. This, together with Lemma 2.3 and $\delta > 0$, gives

$$J_1 \leq \left( \frac{|x - z|}{\rho_w(y, \mu)} \right)^\theta \frac{1}{w(B(y, \sqrt{t}))} \exp \left( -\frac{|x - y|^2}{ct} \right) \left( 1 + \frac{\sqrt{t}}{\rho_w(y, \mu)} \right)^{-\theta \rho_w(y, \mu)} (\frac{|x - z|}{\rho_w(y, \mu)})^\delta ds \frac{\delta}{s}.$$ 

Owing to Proposition 2.1

$$J_1 \leq \left( \frac{|x - z|}{\rho_w(y, \mu)} \right)^\theta \frac{1}{w(B(y, \sqrt{t}))} \exp \left( -\frac{|x - y|^2}{ct} \right) \left( 1 + \frac{\sqrt{t}}{\rho_w(y, \mu)} \right)^{\theta \rho_w(y, \mu)} \left( 1 + \frac{\sqrt{t}}{\rho_w(y, \mu)} \right)^{-\theta \rho_w(y, \mu)}.$$ 

Using the inequality

$$\left( 1 + \frac{|x - y|}{\rho_w(y, \mu)} \right)^{\theta \rho_w(y, \mu)} \left( 1 + \frac{\sqrt{t}}{\rho_w(y, \mu)} \right)^{-\theta \rho_w(y, \mu)} \leq \left( 1 + \frac{|x - y|}{\sqrt{t}} \right)^{\theta \rho_w(y, \mu)},$$

we obtain that

$$J_1 \leq \left( \frac{|x - z|}{\rho_w(y, \mu)} \right)^\theta \frac{1}{w(B(y, \sqrt{t}))} \exp \left( -\frac{|x - y|^2}{ct} \right) \left( 1 + \frac{\sqrt{t}}{\rho_w(y, \mu)} \right)^{\theta \rho_w(y, \mu)} \left( 1 + \frac{\sqrt{t}}{\rho_w(y, \mu)} \right)^{-\theta}.$$ 

$$\leq \left( \frac{|x - z|}{\rho_w(y, \mu)} \right)^\theta \frac{1}{w(B(y, \sqrt{t}))} \exp \left( -\frac{|x - y|^2}{ct} \right) \left( 1 + \frac{\sqrt{t}}{\rho_w(y, \mu)} \right)^{-\theta}.$$
Similarly, by Lemma 2.3 and $N > N_0 > \delta > \theta$, we have

\[
J_2 \leq \frac{1}{w(B(y, \sqrt{t}))} \exp \left( -\frac{|x - y|^2}{ct} \right) \left( 1 + \frac{\sqrt{t}}{\rho_w(y, \mu)} \right)^{-N} \int_{\rho_w(x, \mu)}^{t/2} \left( \frac{|x - \xi|}{\rho_w(x, \mu)} \right)^{\theta} \left( \frac{\sqrt{t}}{\rho_w(x, \mu)} \right)^{N_0 \theta} ds
\]

\[
\leq \frac{1}{w(B(y, \sqrt{t}))} \exp \left( -\frac{|x - y|^2}{ct} \right) \left( \frac{|x - \xi|}{\rho_w(x, \mu)} \right)^{\theta} \left( \frac{\sqrt{t}}{\rho_w(x, \mu)} \right)^{N_0 \theta} (1 + \frac{\sqrt{t}}{\rho_w(x, \mu)})^{-N}
\]

Applying Proposition 2.1,

\[
J_2 \leq \frac{1}{w(B(y, \sqrt{t}))} \exp \left( -\frac{|x - y|^2}{ct} \right) \left( \frac{|x - \xi|}{\rho_w(x, \mu)} \right)^{\theta} \left( 1 + \frac{|x - y|}{\rho_w(y, \mu)} \right)^{(N_0 - \theta)k_0} \left( 1 + \frac{\sqrt{t}}{\rho_w(y, \mu)} \right)^{-N}
\]

Using the following inequality

\[
\left( 1 + \frac{|x - y|}{\rho_w(y, \mu)} \right)^{(N_0 - \theta)k_0} \left( 1 + \frac{\sqrt{t}}{\rho_w(y, \mu)} \right)^{-N} \leq \left( 1 + \frac{|x - y|}{\sqrt{t}} \right)^{(N_0 - \theta)k_0},
\]

and taking $N = (N_0 - \theta)k_0 + \theta$, we obtain

\[
J_2 \leq \left( \frac{|x - \xi|}{\rho_w(x, \mu)} \right)^{\theta} \frac{1}{w(B(y, \sqrt{t}))} \exp \left( -\frac{|x - y|^2}{ct} \right) \left( 1 + \frac{|x - y|}{\rho_w(y, \mu)} \right)^{(N_0 - \theta)k_0} \left( 1 + \frac{\sqrt{t}}{\rho_w(y, \mu)} \right)^{-\theta}
\]

\[
\leq \left( \frac{|x - \xi|}{\rho_w(x, \mu)} \right)^{\theta} \frac{1}{w(B(y, \sqrt{t}))} \exp \left( -\frac{|x - y|^2}{c't} \right) \left( 1 + \frac{\sqrt{t}}{\rho_w(y, \mu)} \right)^{-\theta}.
\]

Consequently,

\[
I_{11} \leq \left( \frac{|x - \xi|}{\rho_w(y, \mu)} \right)^{\theta} \frac{1}{w(B(y, \sqrt{t}))} \exp \left( -\frac{|x - y|^2}{c't} \right) \left( 1 + \frac{\sqrt{t}}{\rho_w(y, \mu)} \right)^{-\theta}.
\]

Arguing similarly we obtain

\[
I_{12} \leq \left( \frac{|x - \xi|}{\rho_w(y, \mu)} \right)^{\theta} \frac{1}{w(B(y, \sqrt{t}))} \exp \left( -\frac{|x - y|^2}{c't} \right) \left( 1 + \frac{\sqrt{t}}{\rho_w(y, \mu)} \right)^{-\theta}.
\]

Taking estimates $I_{11}$ and $I_{12}$ into account we conclude that

\[
I_1 \leq \left( \frac{|x - \xi|}{\rho_w(y, \mu)} \right)^{\theta} \frac{1}{w(B(y, \sqrt{t}))} \exp \left( -\frac{|x - y|^2}{ct} \right) \left( 1 + \frac{\sqrt{t}}{\rho_w(y, \mu)} \right)^{-\theta}
\]

\[
\leq \min \left\{ \left( \frac{|x - \xi|}{\sqrt{t}} \right)^{\theta}, \left( \frac{|x - \xi|}{\rho_w(y, \mu)} \right)^{\theta} \right\} \frac{1}{w(B(y, \sqrt{t}))} \exp \left( -\frac{|x - y|^2}{ct} \right),
\]

where in the last inequality we used

\[
\left( \frac{|x - \xi|}{\rho_w(y, \mu)} \right)^{\theta} \left( 1 + \frac{\sqrt{t}}{\rho_w(y, \mu)} \right)^{-\theta} \leq \min \left\{ \left( \frac{|x - \xi|}{\sqrt{t}} \right)^{\theta}, \left( \frac{|x - \xi|}{\rho_w(y, \mu)} \right)^{\theta} \right\}.
\]

It remains to take care of the term $I_2$. By a change of variable we can rewrite

\[
I_2 = \int_0^{t/2} \int_{\mathbb{R}^d} (h_{t-s}(x, z) - h_{t-s}(\xi, z))k_s(z, y) d\pi(z) ds.
\]
By Proposition 4.1 (ii), Theorem 1.2 and the fact that \( t - s \sim t \) for \( s \in (0, t/2] \),

\[
I_2 \leq \int_0^{t/2} \int_{\mathbb{R}^d} \left( \frac{|x - z|}{\sqrt{t}} \right)^\theta \frac{1}{w(B(z, \sqrt{t}))} \exp \left( -\frac{|x - z|^2}{ct} \right) \\
\times \frac{1}{w(B(y, \sqrt{s}))} \exp \left( -\frac{|z - y|^2}{cs} \right) \left( 1 + \frac{\sqrt{s}}{\rho_w(y, \mu)} \right)^{-N} d\pi(z) ds \\
+ \int_0^{t/2} \int_{\mathbb{R}^d} \left( \frac{|x - z|}{\sqrt{t}} \right)^\theta \frac{1}{w(B(z, \sqrt{t}))} \exp \left( -\frac{|x - z|^2}{ct} \right) \\
\times \frac{1}{w(B(y, \sqrt{s}))} \exp \left( -\frac{|z - y|^2}{cs} \right) \left( 1 + \frac{\sqrt{s}}{\rho_w(y, \mu)} \right)^{-N} d\pi(z) ds \\
=: I_{21} + I_{22},
\]

where \( N > 0 \) will be fixed later.

Note that for \( s \in (0, t/2] \) we have

\[
\exp \left( -\frac{|x - z|^2}{ct} \right) \exp \left( -\frac{|z - y|^2}{cs} \right) \leq \exp \left( -\frac{|x - y|^2}{c't} \right) \exp \left( -\frac{|z - y|^2}{c''s} \right).
\]

Inserting this into the expression of \( I_{21} \) we obtain

\[
I_{21} \leq \left( \frac{|x - x|}{\sqrt{t}} \right)^\theta \frac{1}{w(B(x, \sqrt{t}))} \exp \left( -\frac{|x - y|^2}{c't} \right) \\
\times \int_0^{t/2} \int_{\mathbb{R}^d} \frac{1}{w(B(y, \sqrt{s}))} \exp \left( -\frac{|z - y|^2}{c''s} \right) \left( 1 + \frac{\sqrt{s}}{\rho_w(y, \mu)} \right)^{-N} d\pi(z) ds.
\]

If \( t/2 > \rho_w(y, \mu) \), then by Lemma 2.3 we have

\[
\int_0^{t/2} \int_{\mathbb{R}^d} \frac{1}{w(B(y, \sqrt{s}))} \exp \left( -\frac{|z - y|^2}{c''s} \right) \left( 1 + \frac{\sqrt{s}}{\rho_w(y, \mu)} \right)^{-N} d\pi(z) ds \\
\leq \int_0^{\rho_w(y, \mu)^2} \left( \frac{\sqrt{s}}{\rho_w(y, \mu)} \right) \delta ds + \int_{\rho_w(y, \mu)^2}^{\infty} \left( \frac{\sqrt{s}}{\rho_w(y, \mu)} \right)^N \left( \frac{\sqrt{s}}{\rho_w(y, \mu)} \right)^{-N} ds \\
\leq 1.
\]

Hence,

\[
I_{21} \leq \left( \frac{|x - x|}{\sqrt{t}} \right)^\theta \frac{1}{w(B(x, \sqrt{t}))} \exp \left( -\frac{|x - y|^2}{c't} \right) \\
\leq \min \left\{ \left( \frac{|x - x|}{\sqrt{t}} \right)^\theta, \left( \frac{|x - x|}{\rho_w(y, \mu)} \right)^\theta \right\} \frac{1}{w(B(x, \sqrt{t}))} \exp \left( -\frac{|x - y|^2}{c't} \right).
\]
If \( t/2 < \rho_w(y, \mu) \), taking \( N = \delta - \theta \) then by Lemma 2.3 we obtain
\[
I_{21} \leq \left( \frac{|x - \overline{x}|}{\sqrt{t}} \right)^\theta \frac{1}{w(B(x, \sqrt{t}))} \exp \left( -\frac{|x - y|^2}{ct} \right) \left( \frac{\sqrt{s}}{\rho_w(y, \mu)} \right)^\delta \left( \frac{\sqrt{s}}{\rho_w(y, \mu)} \right)^{-\delta + \theta} ds.
\]
By a similar argument, we also have
\[
I_{22} \leq \frac{1}{w(B(x, \sqrt{t}))} \exp \left( -\frac{|x - y|^2}{ct} \right) \frac{1}{w(B(y, \sqrt{t}))} \exp \left( -\frac{|x - y|^2}{ct} \right),
\]
where in the third inequality we used the fact that \( |x - y| \sim |x - y| \).

This completes our proof. \( \square \)

We are now ready to give the proof for Theorem 1.2 (ii).

**Proof of Theorem 1.2 (ii):** Due to the kernel bound in (i) of Theorem 1.2 we may assume that \( |y - \overline{y}| < \sqrt{t}/4 \). We now consider 2 cases.

**Case 1:** \(|y - \overline{y}| < |x - y|/4\).

If \( |y - \overline{y}| < \rho_w(y, \mu) \), then using the estimates in Proposition 4.2 and Proposition 4.1 we obtain (9).

Otherwise, if \( |y - \overline{y}| \geq \rho_w(y, \mu) \), then applying (i) of Theorem 1.2
\[
(25) \quad \|k_t(x, y) - k_t(x, \overline{y})\| \leq \frac{C}{w(B(x, \sqrt{t}))} \exp \left( -\frac{|x - y|^2}{ct} \right) \left[ \left( \frac{\rho_w(y, \mu)}{\sqrt{t}} \right)^\theta + \left( \frac{\rho_w(\overline{y}, \mu)}{\sqrt{t}} \right)^\theta \right].
\]

On the other hand, by Proposition 2.3
\[
\rho_w(\overline{y}, \mu) \leq C\rho_w(y, \mu) \left( 1 + \frac{|y - \overline{y}|}{\rho_w(\overline{y}, \mu)} \right)^{k_0 + \tau}.
\]

This, along with \( |y - \overline{y}| \geq \rho_w(y, \mu) \), implies that \( |y - \overline{y}| \geq \rho_w(\overline{y}, \mu) \). Therefore, it follows from (25) that
\[
\|k_t(x, y) - k_t(x, \overline{y})\| \leq \left( \frac{|y - \overline{y}|}{\sqrt{t}} \right)^\theta \frac{C}{w(B(x, \sqrt{t}))} \exp \left( -\frac{|x - y|^2}{ct} \right),
\]
which proves (9).

**Case 2:** \(|y - \overline{y}| \geq |x - y|/4\).
We borrow some ideas in [10] to write
\[
    k_t(x, y) - k_t(x, y') = \int_{\mathbb{R}^d} k_{t/2}(x, z) \left[ k_{t/2}(z, y) - k_{t/2}(z, y') \right] \, dw(z)
\]
\[
    = \int_{\|y-z\| \geq 4|y-y'|} \ldots + \int_{\|y-z\| < 4|y-y'|} \ldots
\]
\[
    = E_1 + E_2.
\]

We can apply the estimate in Case 1 to dominate the term \(E_1\) by
\[
    \left( \frac{|y-y'|}{\sqrt{t}} \right)^\theta \int_{\mathbb{R}^d} \frac{1}{w(B(y, \sqrt{t})))} \exp \left( - \frac{|x-z|^2}{ct} \right) \, dw(z).
\]

Owing the Gaussian upper bound of \(k_t\) we further obtain
\[
    |E_1| \leq \left( \frac{|y-y'|}{\sqrt{t}} \right)^\theta \int_{\mathbb{R}^d} \frac{1}{w(B(x, \sqrt{t})))} \exp \left( - \frac{|x-z|^2}{ct} \right) \frac{1}{w(B(y, \sqrt{t})))} \exp \left( - \frac{|z-y|^2}{ct} \right) \, dw(z).
\]

By using the following inequality
\[
    \exp \left( - \frac{|x-z|^2}{ct} \right) \exp \left( - \frac{|z-y|^2}{ct} \right) \leq \exp \left( - \frac{|x-y|^2}{ct'} \right) \exp \left( - \frac{|z-y|^2}{ct''} \right),
\]
we arrive at
\[
    |E_1| \leq \left( \frac{|y-y'|}{\sqrt{t}} \right)^\theta \frac{1}{w(B(x, \sqrt{t})))} \exp \left( - \frac{|x-y|^2}{ct'} \right) \int_{\mathbb{R}^d} \frac{1}{w(B(y, \sqrt{t})))} \exp \left( - \frac{|z-y|^2}{ct''} \right) \, dw(z)
\]
\[
    \leq \left( \frac{|y-y'|}{\sqrt{t}} \right)^\theta \frac{1}{w(B(x, \sqrt{t})))} \exp \left( - \frac{|x-y|^2}{ct'} \right).
\]

It remains to evaluate the term \(E_2\). By invoking the Gaussian upper bound of \(k_t\), we have
\[
    |E_2| \leq \int_{B(y,4|y-y'|)} \frac{1}{w(B(x, \sqrt{t})))} \exp \left( - \frac{|x-z|^2}{ct} \right) \frac{1}{w(B(y, \sqrt{t})))} \exp \left( - \frac{|z-y|^2}{ct} \right) \, dw(z)
\]
\[
    + \int_{B(y,4|y-y'|)} \frac{1}{w(B(x, \sqrt{t})))} \exp \left( - \frac{|x-z|^2}{ct} \right) \frac{1}{w(B(y, \sqrt{t})))} \exp \left( - \frac{|z-y|^2}{ct} \right) \, dw(z)
\]
\[
    \leq \int_{B(y,4|y-y'|)} \frac{1}{w(B(x, \sqrt{t})))} \exp \left( - \frac{|x-y|^2}{ct'} \right) \frac{1}{w(B(y, \sqrt{t})))} \, dw(z)
\]
\[
    + \int_{B(y,4|y-y'|)} \frac{1}{w(B(x, \sqrt{t})))} \exp \left( - \frac{|x-y|^2}{ct'} \right) \frac{1}{w(B(y, \sqrt{t})))} \, dw(z),
\]

where in the last inequality we used
\[
    \exp \left( - \frac{|x-z|^2}{ct} \right) \exp \left( - \frac{|z-y|^2}{ct} \right) \leq \exp \left( - \frac{|x-y|^2}{ct'} \right)
\]
and
\[
    \exp \left( - \frac{|x-z|^2}{ct} \right) \exp \left( - \frac{|z-y|^2}{ct} \right) \leq \exp \left( - \frac{|x-y|^2}{ct''} \right).
\]

Since \(|y-y'| < \sqrt{t}, w(B(y, \sqrt{t})) \sim w(B(y, \sqrt{t}))\) and
\[
    \exp \left( - \frac{|x-y|^2}{ct} \right) \sim \exp \left( - \frac{|x-y|^2}{ct} \right),
\]
We thus obtain

\[ |E_2| \leq \int_{B(y,4|y|-y)} \frac{1}{w(B(x, \sqrt{t}))} \exp \left( -\frac{|x-y|^2}{c't} \right) \frac{1}{w(B(y, \sqrt{t}))} dw(z) \]

\[ = \frac{1}{w(B(x, \sqrt{t}))} \exp \left( -\frac{|x-y|^2}{c't} \right) \frac{1}{w(B(y, \sqrt{t}))} \int_{B(y,4|y|-y)} \]

\[ \leq \left( \frac{|y-y|}{\sqrt{t}} \right)^{\beta} \frac{1}{w(B(x, \sqrt{t}))} \exp \left( -\frac{|x-y|^2}{c't} \right) \]

\[ \leq \left( \frac{|y-y|}{\sqrt{t}} \right)^{\beta} \frac{1}{w(B(x, \sqrt{t}))} \exp \left( -\frac{|x-y|^2}{c't} \right), \]

where in the third inequality we used (3) and in the last inequality we used \( \beta \geq 2 > 0 \).

This completes our proof. \( \square \)

5. Maximal function characterization for Hardy spaces \( h_{\ell,at}^{p,q}(\mathbb{R}^d, w) \)

This section is dedicated to proving Theorem 1.6.

5.1. Local Hardy spaces. We recall the notion of atomic Hardy spaces in [22].

**Definition 5.1.** Let \( p \in \left( \frac{n}{n+1}, 1 \right], q \in [1, \infty] \cap (p, \infty] \) and \( \ell > 0 \). A function \( \alpha \) is called a local \((p, q)_\ell\)-atom associated to the ball \( B(x_0, r) \) if

(i) \( \text{supp } \alpha \subset B(x_0, r); \)

(ii) \( \|\alpha\|_{L^q(\mathbb{R}^d)_w} \leq w(B(x_0, r))^{1/q - 1/p}; \)

(iii) \( \int \alpha dw = 0 \) if \( r < \ell. \)

The local Hardy spaces are defined as follows.

**Definition 5.2.** Let \( p \in \left( \frac{n}{n+1}, 1 \right], q \in [1, \infty] \cap (p, \infty] \) and \( \ell > 0 \). The local Hardy space \( h_{\ell,at}^{p,q}(\mathbb{R}^d, w) \) is defined to be the completion of the set of all \( f = \sum_{j} \lambda_j \alpha_j \) in \( L^p_w(\mathbb{R}^d) \) under the norm

\[ \|f\|_{h_{\ell,at}^{p,q}(\mathbb{R}^d, w)} = \inf \left\{ \sum_{j} |\lambda_j|^p : f = \sum_{j} \lambda_j \alpha_j \right\}, \]

where \( \{\alpha_j\}_{j \in \mathbb{N}} \) are local \((p, q)_\ell\)-atoms and \( \{\lambda_j\}_{j \in \mathbb{N}} \subset \mathbb{C} \) such that \( \sum_j |\lambda_j|^p < \infty \).

It was proved in [23] that \( h_{\ell,at}^{p,q}(\mathbb{R}^d, w) = h_{\ell,at}^{p,r}(\mathbb{R}^d, w) \) for all \( \frac{n}{n+1} < p \leq 1 \), \( q, r \in [1, \infty] \cap (p, \infty] \) and \( \ell > 0 \). For this reason, we define the local Hardy spaces \( h_{\ell}^{p}(\mathbb{R}^d, w) \) with \( \frac{n}{n+1} < p \leq 1 \) and \( \ell > 0 \) to be any space \( h_{\ell,at}^{p,r}(\mathbb{R}^d, w) \) with \( q \in [1, \infty] \cap (p, \infty] \).

We recall the following result in [2] Theorem 2.10.

**Theorem 5.3.** Let \( \frac{n}{n+1} < p \leq 1 \), \( q \in [1, \infty] \cap (p, \infty] \) and \( \ell \in \mathbb{R} \). If \( f \in h_{\ell}^{p}(\mathbb{R}^d, w) \) is supported in a ball \( B \) with radius of \( r_B \geq \ell \), then there exist a number \( c_0 \), a sequence of numbers \( \{\lambda_j\}_{j \in \mathbb{N}} \), and a sequence \( \{\alpha_j\}_{j \in \mathbb{N}} \) of local \((p, q)_\ell\)-atoms such that for each \( j \), \( \alpha_j \) is supported in \( c_0 B \) such that \( f = \sum_{j=1}^{\infty} \lambda_j \alpha_j \) and

\[ \|f\|_{h_{\ell}^{p}(\mathbb{R}^d, w)} \sim \sum_{j=1}^{\infty} |\lambda_j|^p. \]

If \( f \in h_{\ell}^{p}(\mathbb{R}^d, w) \cap C(\mathbb{R}^d) \), then the statement is also true with \( q = \infty \).
5.2. Some estimates on Hardy spaces $h_{at,ρω}^{p,q}$. Let $\{B_j\}_{j∈N}$ and $\{ψ_j\}_{j∈N}$ be families of balls and functions in Lemma 2.4. From Proposition 2.1 there exists a $C_{pω}$ such that

$$C_{pω} \leq ρω(y, μ) ≤ C_{pω} \leq ρω(x, μ) \quad \text{whenever} \quad |x - y| < ρω(x, μ).$$

We define $B^+ = 4c_0 B$, where $c_0$ is the constant in Theorem 5.3.

We first prove the following result which gives a localized maximal function estimate.

**Lemma 5.4.** Let $\frac{n+θ}{n+δ} < p ≤ 1$ with $0 < θ < min{δ, γ}$ and $q ∈ (p, ∞] ∩ [1, ∞]$, where $δ$ is the constant in (4) and $γ$ is the constant in Proposition 4.1(ii). Then there exists a $C > 0$ such that for any $0 < θ ≤ 1$, we have

$$\left\| \sup_{0 < t ≤ ρω(x, µ)} |e^{-tL_0}(fψ_j)(x)| \right\|_{L^p_ω(\mathbb{R}^d)}^{p} ≤ C \gamma^θ \sum_{j ∈ I_j} \|fψ_j\|_{h_{at,ρω, c}^{p,q}}^{p,q}(\mathbb{R}^d)$$

for all $f ∈ h_{at,ρω, c}^{p,q}(\mathbb{R}^d)$. 

**Proof.** From Corollary 3.4 in [2], we have

$$\left\| \sup_{0 < t ≤ ρω(x, µ)} |e^{-tL_0}(fψ_j)(x)| \right\|_{L^p_ω(\mathbb{R}^d)}^{p} ≤ C \gamma^θ \sum_{j ∈ I_j} \|fψ_j\|_{h_{at,ρω, c}^{p,q}}^{p,q}(\mathbb{R}^d)$$

for each $j ∈ N$, where $I_j = \{i ∈ N : B_i ∩ B_j ≠ \emptyset\}$. From Proposition 2.4 it is easy to see that the cardinality of $I_j$ is uniformly bounded by a constant for every $j ∈ N$. Therefore, summing the above inequality for all $j ∈ N$ we obtain the desired estimate.

This completes our proof.

The following two results are just direct consequences of Lemma 3.5 and Theorem 3.1 in [2].

**Lemma 5.5.** Let $\frac{n+θ}{n+δ} < p ≤ 1$ with $0 < θ < min{δ, γ}$ and $q ∈ (p, ∞] ∩ [1, ∞]$. Then, for any $0 < θ ≤ 1$, we have

$$\left\| \sum_{j ∈ N} \sup_{0 < t ≤ ρω(x, µ)} |ψ_j(x)e^{-tL_0}f(x) - e^{-tL_0}(fψ_j)(x)| \right\|_{L^p_ω(\mathbb{R}^d)}^{p} ≤ e^{θp} \|f\|_{h_{at,ρω, c}^{p,q}}^{p,q}(\mathbb{R}^d, w)$$

for all $f ∈ h_{at,ρω, c}^{p,q}(\mathbb{R}^d, w)$.

**Theorem 5.6.** Let $\frac{n+θ}{n+δ} < p ≤ 1$ with $0 < θ < min{δ, γ}$ and $q ∈ [1, ∞] ∩ (p, ∞]$. Then, for any $f ∈ L^q_ω(\mathbb{R}^d)$, we have

$$\left\| \sup_{0 < t ≤ ρω(x, µ)} |e^{-tL_0}f(x)| \right\|_{L^q_ω(\mathbb{R}^d)}^{q} \sim \|f\|_{h_{at,ρω}^{p,q}}^{p,q}(\mathbb{R}^d, w).$$

5.3. Proof of Theorem 1.6. We now give the proof for Theorem 1.6. In order to do this, we split the proof into 2 steps.

**Step 1:** $h_{at,ρω}^{p,q}(\mathbb{R}^d, w) \hookrightarrow H_{L}^{p}(\mathbb{R}^d, w)$.

**Step 2:** $H_{L}^{p}(\mathbb{R}^d, w) \hookrightarrow h_{at,ρω}^{p,q}(\mathbb{R}^d, w)$.
Proof of Step 1. We first prove the continuous embedding $h_{at,ρ}^{p,q}(\mathbb{R}^d, w) \hookrightarrow H_{L_w}^{p}(\mathbb{R}^d, w)$. Since the space $L^2(\mathbb{R}^d)$ is dense in both $h_{at,ρ}^{p,q}(\mathbb{R}^d, w)$ and $H_{L_w}^{p}(\mathbb{R}^d, w)$, it suffices to show that $h_{at,ρ}^{p,q}(\mathbb{R}^d) \cap H_{L_w}^{p}(\mathbb{R}^d, w)$. Since $\mathcal{M}_L$ is dominated by the Hardy–Littlewood maximal function $\mathcal{M}$ (see for example [7]), $\mathcal{M}_L$ is bounded on $L^2_{w}(\mathbb{R}^d)$. Therefore, it suffices to prove that

$$\tag{26} ||\mathcal{M}_L a||_{L_w^p} \leq C$$

for all $(p, q, ρ)$-atoms associated to balls $B = B(x_0, r)$.

To prove (26), we first write

$$||\mathcal{M}_L a||_{L_w^p} \leq ||\mathcal{M}_L a||_{L_w^p(4B)} + ||\mathcal{M}_L a||_{L_w^p(\mathbb{R}^d \setminus 4B)} := I_1 + I_2.$$ 

Using Hölder’s inequality and the $L^p_w$-boundedness of $\mathcal{M}_L$, we can dominate $I_1$ by a constant. So, it remains to consider the contribution of $I_2$. To do this, we consider two cases.

Case 1: $ρ_w(x_0, µ)/4 \leq r \leq ρ_w(x_0, µ)$.

Using Theorem 1.2 with $N = 1$, we have

$$I_2 \leq \int_{\mathbb{R}^d \setminus 4B} \sup_{t > 0} \left[ \int_B \frac{1}{w(B(y, \sqrt{t})]} \exp \left( - \frac{|x-y|^2}{ct} \right) \left( 1 + \frac{\sqrt{t}}{ρ_w(y, µ)} \right)^{-1} |a(y)| dw(y) \right] \, dw(x)$$

$$\leq \int_{\mathbb{R}^d \setminus 4B} \sup_{t > 0} \left[ \int_B \frac{1}{w(B(y, |x-y|) |x-y|)} \exp \left( - \frac{|x-y|^2}{ct} \right) \left( 1 + \frac{\sqrt{t}}{ρ_w(y, µ)} \right)^{-1} |a(y)| dw(y) \right] \, dw(x).$$

Since $y \in B(x_0, r)$ with $r \sim ρ_w(x_0, µ)$, by Proposition 2.1, $ρ_w(y, µ) \sim r$. Hence,

$$\exp \left( - \frac{|x-y|^2}{ct} \right) \left( 1 + \frac{\sqrt{t}}{ρ_w(y, µ)} \right)^{-1} \leq \left( \frac{r}{|x-y|} \right).$$

Consequently,

$$I_2 \leq \int_{\mathbb{R}^d \setminus 4B} \left[ \int_B \frac{1}{w(B(y, |x-y|) |x-y|)} \frac{r}{|x-y|} |a(y)| dw(y) \right]^p \, dw(x)$$

$$\sim \int_{\mathbb{R}^d \setminus 4B} \left[ \int_B \frac{1}{w(B(x_0, |x-y_0|) |x-y_0|)} \frac{r}{|x-y_0|} |a(y)| dw(y) \right]^p \, dw(x)$$

$$\leq ||a||_{L_q^1} \int_{\mathbb{R}^d \setminus 4B} \left[ \frac{1}{w(B(x_0, |x-y_0|) |x-y_0|)} \frac{r}{|x-y_0|} \right]^p \, dw(x)$$

$$\leq w(B)^{p-1} \int_{\mathbb{R}^d \setminus 4B} \left[ \frac{1}{w(B(x_0, |x-y_0|) |x-y_0|)} \frac{r}{|x-y_0|} \right]^p \, dw(x).$$

By a simple calculation, we come up with

$$\int_{\mathbb{R}^d \setminus 4B} \left[ \frac{1}{w(B(x_0, |x-y_0|) |x-y_0|)} \frac{r}{|x-y_0|} \right]^p \, dw(x) \leq w(B)^{1-p}$$

as long as $\frac{n}{n+1} < p \leq 1$.

It follows that $I_2 \leq 1$. Hence, (26) is proved.

Case 2: $r < ρ_w(x_0, µ)/4$.

Observe that

$$I_2 \leq \int_{\mathbb{R}^d \setminus 4B} \sup_{0 \leq t \leq r^2} \left[ \int_B k_t(x, y) a(y) dw(y) \right]^p \, dw(x) + \int_{\mathbb{R}^d \setminus 4B} \sup_{t \geq 4r^2} \left[ \int_B k_t(x, y) a(y) dw(y) \right]^p \, dw(x)$$

$$= I_{21} + I_{22}.$$
By Theorem 1.2, 

\[ I_{21} \leq \int_{\mathbb{R}^d} \sup_{0 < t < 4r^2} \left[ \int_B \frac{1}{w(B(y, |x - y|))} \left| a(y) \right| dw(y) \right] \frac{t}{|x - y|} |a(y)| dw(y) \right]^{\frac{p}{n}} dw(x). \]

Arguing similarly to the estimate of \( I_2 \) in Case 1, we have 

\[ I_{21} \leq 1. \]

To take care of \( I_{22} \) we use the cancellation property of \( a \) to arrive at 

\[ I_{22} = \int_{\mathbb{R}^d} \sup_{t \geq 4r^2} \left| \int_{B} \left[ k_t(x, y) - p_t(x, x_0) \right] a(y) dw(y) \right|^{\frac{p}{n}} dw(x). \]

Owing to Theorem 1.2, 

\[ I_{22} \leq \int_{\mathbb{R}^d} \sup_{t \geq 4r^2} \left| \int_{B} \left( \frac{y - x_0}{|y - x|} \right)^{\frac{p}{n}} \frac{1}{w(B(y, |x - y|))} \exp \left( - \frac{|x - y|^2}{ct} \right) |a(y)| dw(y) \right|^{\frac{p}{n}} dw(x) \]

\[ \leq \int_{\mathbb{R}^d} \sup_{t \geq 4r^2} \left| \int_{B} \left( \frac{r}{|x - x_0|} \right)^{\frac{p}{n}} \frac{1}{w(B(x_0, |x - x_0|))} |a(y)| dw(y) \right|^{\frac{p}{n}} dw(x). \]

At this stage, employing the argument used in the estimate of \( I_2 \) in Case 1, we also obtain 

\[ I_{22} \leq 1, \]

provided that \( p > n/(n + 0). \)

Therefore, this completes the proof of Step 1. \( \square \)

In order to prove Step 2, we need the following estimates.

**Lemma 5.7.** Let \( \frac{n}{n + 0} < p \leq 1 \) and \( q \in [p, \infty) \cap [1, \infty]. \) Then there exists a \( \kappa > 0 \) such that for any \( 0 < \epsilon \leq 1, \) we have 

\[ \left\| \sup_{0 < t \leq \epsilon \rho_w (x, \mu)} \left| (e^{-tL} - e^{-tL_0}) f(x) \right| \right\|_{L^p_w(\mathbb{R}^d)} \leq e^\kappa \left\| f \right\|_{L^p_w(\mathbb{R}^d, w)} \]

for all \( f \in h_{\mathcal{P}, \rho_w, \epsilon}(\mathbb{R}^d, w). \)

**Proof.** Observe that 

\[ \sup_{0 < t \leq \epsilon \rho_w (x, \mu)} \left| (e^{-tL} - e^{-tL_0}) f(x) \right| \leq \sup_{t > 0} \left| (e^{-tL} - e^{-tL_0}) f(x) \right| \leq M f(x), \]

where \( M \) is the Hardy–Littlewood maximal function.

It follows that the operator 

\[ f \mapsto \sup_{0 < t \leq \epsilon \rho_w (x, \mu)} \left| (e^{-tL} - e^{-tL_0}) f(x) \right| \]
is bounded on $L^p_w(\mathbb{R}^d)$. Hence, it suffices to prove (27) for all $(p, q, \rho, \mu, \epsilon)$ atoms. Let $\alpha$ be $(p, q, \rho, \mu, \epsilon)$ atom associated to a ball $B := B(x_0, r)$. We write

$$
\left\| \sup_{0 < t \leq [\epsilon \rho_w(x, \mu)]^2} \left\| (e^{-tl} - e^{-tl_0} f(x)) \right\|_{L^p_w(\mathbb{R}^d)} \right\|^p \leq \left\| \sup_{0 < t \leq [\epsilon \rho_w(x, \mu)]^2} \left\| (e^{-tl} - e^{-tl_0} f(x)) \right\|_{L^p_w(\mathbb{R}^d)} \right\|^p + \left\| \sup_{0 < t \leq [\epsilon \rho_w(x, \mu)]^2} \left\| (e^{-tl} - e^{-tl_0} f(x)) \right\|_{L^p_w(\mathbb{R}^d \setminus B)} \right\|^p
$$

= $I_1 + I_2$.

Using (10), Hölder’s inequality and the $L^p_w$-boundedness of the Hardy-Littlewood maximal function $M$, we get that

$$
I_1 \leq \int_{4B} \left[ \sup_{0 < t \leq [\epsilon \rho_w(x, \mu)]^2} \left( \frac{\sqrt{t}}{\rho_w(x, \mu)} \right) \right]^\delta \int_B \frac{1}{w(B(x, \sqrt{t}))} \exp \left( - \frac{|x-y|^2}{ct} \right) |a(y)| d\mu(w(y))^p w(x)
$$

\leq e^{p\delta} w(B)^{1-p/q} \int_{4B} \left[ \sup_{0 < t \leq [\epsilon \rho_w(x, \mu)]^2} \left( \frac{\sqrt{t}}{\rho_w(x, \mu)} \right) \right]^\delta \int_B \frac{1}{w(B(x, \sqrt{t}))} \exp \left( - \frac{|x-y|^2}{ct} \right) |a(y)| d\mu(w(y))^q w(x)^{p/q} \leq e^{p\delta}.

In order to take care of $I_2$, we consider the following two cases.

**Case 1:** $\epsilon \rho_w(x_0, \mu) / 4 \leq \epsilon \rho_w(x_0, \mu)$.

By (10) again,

$$
I_2 = \int_{\mathbb{R}^d \setminus 4B} \left[ \sup_{0 < t \leq [\epsilon \rho_w(x, \mu)]^2} \left( \frac{\sqrt{t}}{\rho_w(x, \mu)} \right) \right]^\delta \int_B \frac{1}{w(B(x_0, |x-x_0|))} \exp \left( - \frac{|x-x_0|^2}{c[\epsilon \rho_w(x, \mu)]^2} \right) w(x)^p w(x)
$$

\leq e^{p\delta} \|a\|_{L^1} \int_{\mathbb{R}^d \setminus 4B} \left[ \frac{1}{w(B(x_0, |x-x_0|))} \exp \left( - \frac{|x-x_0|^2}{c[\epsilon \rho_w(x, \mu)]^2} \right) w(x)^p w(x)
$$

By Proposition 2.1, we have

$$
\frac{\epsilon \rho_w(x, \mu)}{|x-x_0|} \leq \frac{\epsilon \rho_w(x_0, \mu)}{|x-x_0|} \left( 1 + \frac{|x-x_0|}{\epsilon \rho_w(x, \mu)} \right)^{\frac{k_0}{e\rho_w(x, \mu)}} \leq \frac{\epsilon \rho_w(x_0, \mu)}{|x-x_0|} \left( 1 + \frac{|x-x_0|}{\epsilon \rho_w(x_0, \mu)} \right)^{\frac{k_0}{e\rho_w(x_0, \mu)}} \leq \min \left\{ \frac{\epsilon \rho_w(x_0, \mu)}{|x-x_0|}, \frac{\epsilon \rho_w(x_0, \mu)}{|x-x_0|} \right\} = \{ \epsilon \rho_w(x_0, \mu) / |x-x_0| \} = \{ \epsilon \rho_w(x_0, \mu) / |x-x_0| \} \}
$$

which implies that for any $N > 0$ there exists a $C_N$ such that

$$
\exp \left( - \frac{|x-x_0|^2}{c[\epsilon \rho_w(x, \mu)]^2} \right) \leq C_N \left( \frac{\epsilon \rho_w(x_0, \mu)}{|x-x_0|} \right)^N.
$$
Inserting this into the above bound of $I_2$, we obtain, for $N > n(1 - p)/p$, that

$$I_2 \leq e^{pB} w(B)^{p-1} \int_{\mathbb{R}^d \setminus 4B} \left[ \frac{1}{w(B(x_0, |x - x_0|))} \left( \frac{\epsilon \rho_w(x_0, \mu)}{|x - x_0|} \right)^N \right]^p dw(x)$$

$$\leq e^{pB} w(B)^{p-1} \int_{\mathbb{R}^d \setminus 4B} \left[ \frac{1}{w(B(x_0, |x - x_0|))} \left( \frac{r}{|x - x_0|} \right)^N \right]^p dw(x)$$

$$\leq e^{pB}.$$

**Case 2:** $r < \epsilon \rho_w(x_0, \mu)/4$.

In this situation, since $\int \alpha(y) dw(y) = 0$, we have

$$I_2 = \int_{\mathbb{R}^d \setminus 4B} \sup_{0 < t \leq (\epsilon \rho_w(x_0, \mu))^2} \left[ \int_B (q_L(x, y) - q_L(x, x_0)) \alpha(y) dw(y) \right]^p dw(x).$$

Owing to Proposition 4.2

$$I_2 = \int_{\mathbb{R}^d \setminus 4B} \left[ \sup_{0 < t \leq (\epsilon \rho_w(x_0, \mu))^2} \left( \frac{r}{\rho_w(x, \mu)} \right)^\theta \frac{1}{w(B(x_0, |x - x_0|))} \exp \left( - \frac{|x - y|^2}{ct} \right) \alpha(y) dw(y) \right]^p dw(x)$$

$$\leq \int_{\mathbb{R}^d \setminus 4B} \left[ \left( \frac{r}{\rho_w(x, \mu)} \right)^\theta \frac{1}{w(B(x_0, |x - x_0|))} \exp \left( - \frac{|x - x_0|^2}{c(\epsilon \rho_w(x, \mu))^2} \right) \alpha(y) dw(y) \right]^p dw(x)$$

$$\leq \int_{\mathbb{R}^d \setminus 4B} \left[ \left( \frac{er}{|x - x_0|} \right)^\theta \frac{1}{w(B(x_0, |x - x_0|))} \alpha(y) dw(y) \right]^p dw(x)$$

$$\leq e^{\theta p},$$

as long as $p > n/(n + \theta)$. This completes our proof.

We are ready for the proof of Step 2.

**Proof of Step 2:** Observe that for fixed numbers $\epsilon_1, \epsilon_2 \in (0, 1)$, there exists a $C = C(\epsilon_1, \epsilon_2)$ such that

$$C^{-1} \| \cdot \|_{h_{\ast, \rho_w, \epsilon_1}^{p, q}(\mathbb{R}^d, w)} \leq \| \cdot \|_{h_{\ast, \rho_w, \epsilon_2}^{p, q}(\mathbb{R}^d, w)} \leq C \| \cdot \|_{h_{\ast, \rho_w, \epsilon_0}^{p, q}(\mathbb{R}^d, w)}.$$

For this reason, we need only to prove that there exists an $\epsilon_0 \in (0, 1]$ so that

$$\| f \|_{h_{\ast, \rho_w, \epsilon_0}^{p, q}(\mathbb{R}^d, w)} \leq \| f \|_{H^p_{\ast}(\mathbb{R}^d, w)} \cap L^2_w(\mathbb{R}^d).$$

Let $\{B_j\}_{j \in \mathbb{N}}$ and $\{\psi_j\}_{j \in \mathbb{N}}$ be families of balls and functions in Lemma 2.4. For each $j \in \mathbb{N}$, $f\psi_j$ is supported in the ball $B_j = B(x_j, \rho_w(x_j, \mu))$ and $f\psi_j \in L^2_w(\mathbb{R}^d)$, which implies that $f \in h^p_{\ast}(\mathbb{R}^d, w)$ with $\ell = \rho_w(x_j, \mu)$. Applying Theorem 5.3 we can decompose $f\psi_j$ into an atomic $(p, q, \rho_w, \epsilon)$-representation with $(p, q, \rho_w, \epsilon)$-atoms supported in $B_j^\ast$. Moreover we have, from Proposition 2.1 the existence of $\Lambda_0$ so that

$$\Lambda_0^{-1} \rho_w(x_j, \mu) \leq \rho_w(x_j, \mu) \leq \Lambda_0 \rho_w(x_j, \mu) \quad \text{for all } x \in B_j^\ast \text{ and all } j \in \mathbb{N}.$$
This, in combination with Theorem 5.6 yields
\[
\sum_{j \in \mathbb{N}} \| \psi_j f \|^p_{H^{1,p,q}_{\alpha_1,\rho_w,\epsilon}(\mathbb{R}^d)} \leq \sum_{j \in \mathbb{N}} \left\| \sup_{0 < t < [c_{\rho_w}(x_j,\mu)]^2} |e^{-tL_0} (\psi_j f)| \right\|^p_{L^p_w(\mathbb{R}^d)}
\]
\[
\leq \sum_{j \in \mathbb{N}} \left\| \sup_{0 < t < [c_{\rho_w}(x_j,\mu)]^2} |e^{-tL_0} (\psi_j f)| \right\|^p_{L^p_w(B^*_j)} + \sum_{j \in \mathbb{N}} \left\| \sup_{0 < t < [c_{\rho_w}(x_j,\mu)]^2} |e^{-tL_0} (\psi_j f)| \right\|^p_{L^p_w(\mathbb{R}^d \setminus B^*_j)}.
\]

By Lemma 5.4
\[
\sum_{j \in \mathbb{N}} \left\| \sup_{0 < t < [c_{\rho_w}(x_j,\mu)]^2} |e^{-tL_0} (\psi_j f)| \right\|^p_{L^p_w(\mathbb{R}^d \setminus B^*_j)} \leq \epsilon^6 \sum_{j \in \mathbb{N}} \| \psi_j f \|^p_{H^{1,p,q}_{\alpha_1,\rho_w,\epsilon}(\mathbb{R}^d \setminus B^*_j)}.
\]

By taking $\epsilon$ small enough, from these two estimates we infer
\[
\sum_{j \in \mathbb{N}} \| \psi_j f \|^p_{H^{1,p,q}_{\alpha_1,\rho_w,\epsilon}(\mathbb{R}^d)} \leq \sum_{j \in \mathbb{N}} \left\| \sup_{0 < t < [c_{\rho_w}(x_j,\mu)]^2} |e^{-tL_0} (\psi_j f)| \right\|^p_{L^p_w(B^*_j)}.
\]

From Proposition 2.1 we can find $\bar{\alpha}_0$ such that
\[
\rho_w(x,\mu) \leq \bar{\alpha}_0 \rho_w(x_j,\mu)
\]
for all $x \in B^*_j$.

Hence, setting $\bar{\epsilon} = \bar{\alpha}_0 \epsilon$, then we have
\[
\sum_{j \in \mathbb{N}} \| \psi_j f \|^p_{H^{1,p,q}_{\alpha_1,\rho_w,\epsilon}(\mathbb{R}^d)} \leq \sum_{j \in \mathbb{N}} \left\| \sup_{0 < t < [c_{\rho_w}(\cdot,\mu)]^2} |e^{-tL_0} (\psi_j f)(\cdot)| \right\|^p_{L^p_w(\mathbb{R}^d)},
\]
which implies, thanks to Lemma 2.4 that
\[
\| f \|^p_{H^{1,p,q}_{\alpha_1,\rho_w,\epsilon}(\mathbb{R}^d)} \leq \sum_{j \in \mathbb{N}} \left\| \sup_{0 < t < [c_{\rho_w}(\cdot,\mu)]^2} |e^{-tL_0} (\psi_j f)(\cdot)| \right\|^p_{L^p_w(\mathbb{R}^d)}.
\]

Therefore,
\[
\| f \|^p_{H^{1,p,q}_{\alpha_1,\rho_w,\epsilon}(\mathbb{R}^d)} \leq \sum_{j \in \mathbb{N}} \left\| \sup_{0 < t < [c_{\rho_w}(\cdot,\mu)]^2} |e^{-tL_0} (\psi_j f)(\cdot) - \psi_j(\cdot) e^{-tL_0} f(\cdot)| \right\|^p_{L^p_w(\mathbb{R}^d)}
\]
\[
+ \sum_{j \in \mathbb{N}} \left\| \sup_{0 < t < [c_{\rho_w}(\cdot,\mu)]^2} |\psi_j(\cdot) [e^{-tL_0} - e^{-tL}] f(\cdot)| \right\|^p_{L^p_w(\mathbb{R}^d)}
\]
\[
+ \sum_{j \in \mathbb{N}} \left\| \sup_{0 < t < [c_{\rho_w}(\cdot,\mu)]^2} |\psi_j(\cdot) e^{-tL} f(\cdot)| \right\|^p_{L^p_w(\mathbb{R}^d)}.
\]

We now estimate the terms on the RHS of the inequality above.

First, using Lemma 2.4
\[
\sum_{j \in \mathbb{N}} \left\| \sup_{0 < t < [c_{\rho_w}(\cdot,\mu)]^2} |e^{-tL_0} (\psi_j f)(\cdot) - \psi_j(\cdot) e^{-tL_0} f(\cdot)| \right\|^p_{L^p_w(\mathbb{R}^d)}
\]
\[
\leq \sum_{j \in \mathbb{N}} \left\| \sup_{0 < t < [c_{\rho_w}(\cdot,\mu)]^2} |e^{-tL_0} (\psi_j f)(\cdot) - \psi_j(\cdot) e^{-tL_0} f(\cdot)| \right\|^p_{L^p_w(\mathbb{R}^d)}
\]
\[
\leq \epsilon^6 \| f \|^p_{H^{1,p,q}_{\alpha_1,\rho_w,\epsilon}(\mathbb{R}^d)},
\]

where in the last step we used Lemma 5.5.
Secondly, by Lemma 5.7
\[
\sum_j \left\| \sup_{0<t<\epsilon \rho(w,\mu)} |h \psi_j(\cdot) (e^{-tL_0} - e^{-tL}) f(\cdot) | \right\|_{L_p(B_j^*)}^p \leq \sum_j \left\| \sup_{0<t<\epsilon \rho(w,\mu)} |e^{-tL_0} - e^{-tL}| f | \right\|_{L_p(R^d)}^p
\leq \epsilon^p \left\| f \right\|_{h_{b,q,\epsilon}^{p,q}(R^d,w)}^p.
\]

Finally, by Proposition 2.4 again,
\[
\sum_j \left\| \sup_{0<t<\epsilon \rho(w,\mu)} |h \psi_j(\cdot) e^{-tL} f(\cdot) | \right\|_{L_p(B_j^*)}^p \leq \sum_j \left\| \sup_{0<t<\epsilon \rho(w,\mu)} |e^{-tL} f | \right\|_{L_p(R^d)}^p.
\]

Consequently,
\[
\left\| f \right\|_{h_{b,q,\epsilon}^{p,q}(R^d,w)}^p \leq \epsilon^p \left\| f \right\|_{h_{b,q}^{p,q}(R^d,w)}^p + \left\| \sup_{0<t<\epsilon \rho(w,\mu)} |e^{-tL} f(\cdot) | \right\|_{L_p(R^d)}^p \leq \epsilon^p \left\| f \right\|_{h_{b,q,\epsilon}^{p,q}(R^d,w)}^p + \left\| \sup_{t>0} |e^{-tL} f(\cdot) | \right\|_{L_p(R^d)}^p.
\]

Taking \( \epsilon \) small enough,
\[
\left\| f \right\|_{h_{b,q,\epsilon}^{p,q}(R^d,w)}^p \leq \left\| \sup_{t>0} |e^{-tL} f(\cdot) | \right\|_{L_p(R^d)}^p.
\]

This completes our proof. \( \square \)

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