Arc length preserving approximation of circular arcs by Pythagorean-hodograph curves of degree seven

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Abstract

In this paper interpolation of two planar points, corresponding tangent directions and curvatures with Pythagorean-hodograph (PH) curves of degree seven preserving an arc length is considered. A general approach using complex representation of PH curves is presented and a detailed analysis of the problem for data arising from a circular arc is provided. In the case of several solutions some criteria for the selection of the most appropriate one are described and an asymptotic analysis is given. Several numerical examples are included which confirm theoretical results.

Keywords: geometric interpolation, circular arc, arc length, Pythagorean-hodograph curve, solution selection

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1. Introduction

Interpolation of local planar geometric data, such as points, tangent directions and curvatures, by parametric polynomial curves is a standard problem in Computer Aided Geometric Design (CAGD) and a common way to construct parametric objects from given discrete data. If such interpolants are joined together, they form geometrically continuous splines of order \( k \) (or \( G_k \) continuous splines), where \( k \) depends on the type of the interpolated data (\( k = 0 \) if only positions of points are given, \( k = 1 \) if in addition also tangent directions are provided, etc.). For a detailed survey of geometric interpolation methods the reader is referred to Hoschek and Lasser (1993) or Farin et al. (2002). However, there are not many results concerning also the interpolation of some global geometric data, such as the arc length. This becomes extremely important when some global shape control is needed or methods relying on optimization of curve energies, such as bending energy, are to be developed. It turned out that there is a specific class of curves which are of great help to solve such kind of problems, the so called polynomial Pythagorean-hodograph (PH) curves introduced in Farouki and Sakkalis (1990) and comprehensively described in Farouki (2008). They are unique among polynomial curves possessing polynomial arc length function. This implies several nice properties which will be explained in detail later. Some recent results concerning interpolation of \( G^1 \) data by PH quintic curves preserving an arc length are in Farouki (2016) and they confirm the advantage of PH curves if interpolation of global geometric data is considered. The author studied

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the interpolation of two points together with the corresponding tangent directions and a prescribed arc length. A detailed analysis of the interpolation problem was done and a simple algorithm relaying basically only on the solution of the quadratic equation was described. A special type of $G^2$ data interpolation by PH curves was considered also in [Farouki (2014)], but the specification of an arc length was not considered. In this paper we intend to extend recent results to the interpolation of $G^2$ data and an arc length by PH curves of degree 7. As already guessed in [Farouki (2016)], it is hard to believe that this problem will possess such simple solution as in the quintic case. We will confirm this prediction after specifying a general problem and later concentrate only on interpolation of particular type of data, i.e. the one arising from a circular arc which will be addressed as circular arc data. It should be noted that even in this case there is no relevant literature available. Most of the approximation techniques namely consider the interpolation of local geometric data only and use the remaining parameters to minimize the distance between the interpolant and the circular arc, to minimize the deviation of the curvature, etc. The results of this type can be found in Dokken et al. (1990), Goldapp (1991), Lyche and Mørken (1994), Mørken (1995), Ahn and Kim (1997), Kim and Ahn (2007), Jaklič et al. (2007), Jaklič et al. (2013), Kovač and Zagar (2016), Jaklič (2016), Jaklič and Kozak (2018), Knez and Zagar (2018), Vavpetič and Zagar (2019), Ahn (2019), Vavpetič (2020) and Vavpetič and Zagar (2021), if we mention just the most important and recent ones. Although the proposed algorithms provide good approximations of circular arcs if the Hausdorff distance is considered as a measure of the error, they do not include an arc length in interpolation data. Our approach intend to fill this gap and provide interpolants which possess required arc length and remain small Hausdorff distance.

The paper is organized as follows. In Section 2 some basic properties of complex representation of PH curves is given. Special attention is given to PH curves of degree 7 which are presented in detail and all quantities needed for a solution of the interpolation problem are derived. In the next section the arc length preserving interpolation of $G^2$ data is considered and the system of nonlinear equations for the general case is presented. Two examples of such interpolation are given showing that the solution of the problem highly depends on prescribed data. In Section 4 the interpolation of circular arc data in canonical position is considered. The system of nonlinear equations derived for the general case is simplified and a new, simpler system of two nonlinear equations involving the angle $\alpha$ arising from the circular arc as a parameter are provided. A detailed analysis of the solvability is done. Two cases are considered. For the first the absence of real solutions is confirmed, and for the second one the existence of two real solutions is first proved for any $\alpha \in (0, \pi/2]$. A simple (numerical) procedure to check the existence of four admissible solutions is described which works well for several particular practically important values $\alpha$ tested. In Section 5 the existence of four solutions in confirmed for any $\alpha$ small enough. and asymptotic expansions of solutions are provided. In Section 6 some criteria for the selection of the most appropriate solution are described and in the next section several numerical examples together with the numerical confirmation of the approximation order are given. The paper is concluded by Section 8.

2. Preliminaries

Planar PH curves are an important subclass of planar parametric polynomial curves. A regular planar parametric polynomial curve $p : [0, 1] \to \mathbb{R}^2$ is a PH curve if $\|p'\|$ is a polynomial, where $\| \cdot \|$ denotes the standard Euclidean norm on $\mathbb{R}^2$. This characterization implies several important geometric properties of PH curves (Farouki (2008)): rational unit tangent, normal, curvature and
offset, ... Furthermore, the arc length function of a PH curve is polynomial. All these properties make them useful for interpolation of local geometric data as well as for the interpolation of global geometric quantities, such as an arc length. Let \( p = (x, y)^T \) be a PH curve of degree \( n \) where \( x' \) and \( y' \) are relatively prime polynomials. It is known that polynomials \( x' \) and \( y' \) can be expressed in terms of two polynomials \( u \) and \( v \) as \( x' = u - v^{2} \) and \( y' = 2uv \) which implies \( x'^{2} + y'^{2} = (u^{2} + v^{2})^{2} \) and consequently the arc length function is a polynomial \( u^{2} + v^{2} \). It is often better to use a complex representation of PH curves (Farouki (1994)). If \( p' = w^{2} \) is a PH curve \( p \) of degree \( n \). Furthermore, its unit tangent vector \( g \), the curvature \( \kappa \) and the arc length \( s \) are given by

\[
g(t) = \frac{w^{2}(t)}{\sigma(t)}, \quad \kappa(t) = 2\frac{\text{Im}(\bar{w}(t)w'(t))}{\sigma^{2}(t)}, \quad s(t) = \int_{0}^{t} \sigma(\tau) d\tau,
\]

where \( \sigma = |w|^{2} \). Since we will consider PH curves of degree 7, we shall start by a complex cubic polynomial \( w \) with complex Bernstein coefficients \( w_{k} = u_{k} + i v_{k}, k = 0, 1, 2, 3 \). Integration of its square results in a PH curve \( p \) of degree 7, which can be written in Bernstein-Bézier form as

\[
p(t) = \sum_{k=0}^{7} B_{k}^{7} p_{k},
\]

where

\[
\begin{align*}
p_{1} &= p_{0} + \frac{1}{7} w_{0}^{2}, & \quad p_{2} &= p_{1} + \frac{1}{7} w_{0} w_{1}, & \quad p_{3} &= p_{2} + \frac{1}{7} 13 w_{1}^{2} + 2 w_{0} w_{2}, \\
p_{4} &= p_{3} + \frac{1}{7} 19 w_{1} w_{2} + w_{0} w_{3}, \\
p_{5} &= p_{4} + \frac{1}{7} 13 w_{2}^{2} + 2 w_{1} w_{3}, & \quad p_{6} &= p_{5} + \frac{1}{7} w_{2} w_{3}, & \quad p_{7} &= p_{6} + \frac{1}{7} w_{3}^{2},
\end{align*}
\]

and \( p_{0} \) is a free complex integration constant. By (1) and (2) we obviously have

\[
g(0) = \left( \frac{w_{0}}{|w_{0}|} \right)^{2}, \quad g(1) = \left( \frac{w_{3}}{|w_{3}|} \right)^{2},
\]

and by using some basic properties of \( w \) we also get

\[
\kappa(0) = 6 \text{Im} \left( \frac{\bar{w}_{0} w_{1}}{|w_{0}|^{4}} \right), \quad \kappa(1) = -6 \text{Im} \left( \frac{\bar{w}_{3} w_{2}}{|w_{3}|^{4}} \right).
\]

Furthermore, the total arc length \( L \) of \( p \) is

\[
L = \int_{0}^{1} |w(\tau)|^{2} d\tau = \frac{1}{7} \left( |w_{0}|^{2} + 2 \text{Re}(w_{0} \bar{w}_{1}) + \frac{1}{5} \text{Re}(w_{0} \bar{w}_{2}) + \frac{1}{10} \text{Re}(w_{0} \bar{w}_{3}) + \frac{3}{5} |w_{1}|^{2}
\right.
\]

\[
+ \frac{9}{10} \text{Re}(w_{1} \bar{w}_{2}) + \frac{2}{5} \text{Re}(w_{1} \bar{w}_{3}) + \frac{3}{5} |w_{2}|^{2} + \text{Re}(w_{0} \bar{w}_{1}) + |w_{3}|^{2}
\)]

These results will now be used in the following section where an interpolation problem of general \( G^{2} \) data by PH curves of degree 7 with a prescribed arc length will be considered.
3. Arc length preserving interpolation of $G^2$ data

A construction of parametric polynomial curves is usually based on interpolation of particular geometric data arising from practical observations, such as point positions, tangent directions, curvatures, etc. As it was already mentioned in the previous section, in some problems also the interpolation of global geometric data, such as an arc length, is required. We shall follow the approach in Farouki (2016), where the author considered the problem of $G^1$ data interpolation by PH quintics of prescribed arc length. The problem can be extended to $G^2$ data interpolation, but the degree of the interpolating PH curve must be elevated to 7, since PH quintic curves do not possess enough free parameters.

Let us assume the complex representation and suppose we want to interpolate two given end points $q_0, q_1$, their corresponding tangent directions $g_0, g_1$, curvatures $\kappa_0$ and $\kappa_1$. Furthermore, we require that the resulting interpolant has a fixed arc length, say $L > ||q_1 - q_0||$. Since we are looking for an interpolant $p$ among PH curves of degree 7, there are 10 free parameters involved (8 parameters arising from the complex Bernstein coefficients (1) and two of them from a complex integration constant $p_0$). The interpolation conditions provide 9 scalar equations. The remaining parameter could be used for shape control or for optimization of some geometric property, but this would definitely lead to a challenging optimization process. In order to avoid it, we will use the approach from Farouki (2016), i.e., we shall assume equal lengths of the tangents of $p$ at the boundary points. This reduces the number of involved free parameters by one and gives some hope that the interpolant is fully determined already by given geometric data. The assumption is not too restrictive and it is quite natural since it ensures symmetric solutions for symmetric data.

In Farouki (2016), the author considered the reduction of data to the canonical form, which has previously been used also in Farouki and Neff (1995). The idea is to consider a new coordinate system which should simplify the analysis of the problem as much as possible. Following the above mentioned references, we can assume that the given data is of the form $q_0 = 0$, $q_1 = 1$, $g_0 = \exp(i \theta_0)$, $g_1 = \exp(i \theta_1)$ and $L > 1$, where $\theta_0, \theta_1 \in (-\pi, \pi]$. The original data are transformed to the canonical one by an appropriate translation, rotation and scaling. Moreover, the obtained interpolant is finally pulled back to the original coordinate system by inverse transformations. Thus, if the above canonical data are assumed, the interpolation conditions become

$$\frac{1}{7} \left( w_0^2 + w_0 w_1 + \frac{3w_1^2 + 2w_0 w_2}{5} + \frac{9w_1 w_2 + w_0 w_3}{10} + \frac{3w_2^2 + 2w_1 w_3}{5} + w_2 w_3 + w_3^2 \right) - 1 = 0,$$

$$w_0 - d \exp \left( \frac{i}{2} \theta_0 \right) = 0, \quad w_3 - d \exp \left( \frac{i}{2} \theta_1 \right) = 0,$$

$$6 \text{Im} \left( \frac{w_0 w_1}{|w_0|^4} \right) - \kappa_0 = 0, \quad 6 \text{Im} \left( \frac{w_3 w_2}{|w_3|^4} \right) + \kappa_1 = 0,$$

$$\frac{1}{7} \left( |w_0|^2 + \text{Re} (w_0 \overline{w}_1) + \frac{2}{5} \text{Re} (w_0 \overline{w}_2) + \frac{1}{10} \text{Re} (w_0 \overline{w}_3) + \frac{3}{5} |w_1|^2 + \frac{9}{10} \text{Re} (w_1 \overline{w}_2) + \frac{2}{5} \text{Re} (w_1 \overline{w}_3) + \frac{3}{5} |w_2|^2 + \text{Re} (w_0 \overline{w}_1) + |w_3|^2 \right) - L = 0.$$

The first equation arises from the interpolation of two points, the second and the third one from the interpolation of tangent directions, the next two ensure prescribed curvatures and the last one prescribes the arc length $L$. 

Let us write $w_1 = u_1 + i v_1$, $w_2 = u_2 + i v_2$, $c_i = \cos(\theta_i/2)$, $s_i = \sin(\theta_i/2)$, $i = 0, 1$, replace the above equations by their appropriate linear combinations and use some basic trigonometric identities. This leads to

$$
6u_1^2 + 9u_1u_2 + 6v_2^2 + (10c_0^2 + 10c_1^2 + c_0c_1) d^2 + 10d (u_1c_0 + u_2c_1) + 4d (u_1c_1 + u_2c_0) - 35(L + 1) = 0,
$$

$$
6v_1^2 + 9v_1v_2 + 6v_2^2 + (10s_0^2 + 10s_1^2 + s_0s_1) d^2 + 10d (v_1s_0 + v_2s_1) + 4d (v_1s_1 + v_2s_0) - 35(L - 1) = 0,
$$

$$
\kappa_0d^3 + 6s_0u_1 - 6c_0v_1 = 0, \quad \kappa_1d^3 - 6s_1u_2 + 6c_1v_2 = 0, \quad (4)
$$

$$
12u_1v_1 + 9u_2v_1 + 9u_1v_2 + 12u_2v_2 + (s_0(20c_0 + c_1) + s_1(c_0 + 20c_1)) d^2 + 2 ((5v_1 + 2v_2)c_0 + (2v_1 + 5v_2)c_1 + (5u_1 + 2u_2)s_0 + (2u_1 + 5u_2)s_1) d = 0.
$$

Since the equations arising from $G^2$ conditions are linear in $u_1$, $u_2$, $v_1$ and $v_2$, some further reduction of the system (4) is definitely possible. However, the analysis of the solvability for general data seems to be extremely complicated as already guessed in Farouki (2016). To justify this, let us consider two particular simple examples which show that the existence of a solution heavily depends on data. Assume the data $\theta_0 = \pi/2$, $\theta_1 = -\pi/2$, $\kappa_0 = \kappa_1 = 0$ and $L = 9/8$. Plugging corresponding constants in (4) and doing some manipulations with equations reveal that $v_1 = u_1$ and $v_2 = -u_2$. This further implies that either $u_2 = u_1$ or $u_2 = -u_1 - 5\sqrt{2}/6 d$. If $u_2 = u_1$, we end up with two biquadratic equations for $u_1$ and $d$, namely

$$
d^2 + 8\sqrt{2}d u_1 + 18u_1^2 - 70 = 0, \quad 80d^2 + 80\sqrt{2}d u_1 + 96u_1^2 - 315 = 0.
$$

They actually represent a hyperbola and an ellipse shown in Fig. 1 (the first figure on the left) and it is easy to show that they do not intersect. Similarly we can check that also for $u_2 = -u_1 - 5\sqrt{2}/6 d$ we do not have a solution (a hyperbola and an ellipse in Fig. 1 (the second figure from the left)). For the second example consider the same data as in the previous one, except that $L = 2$. Similar procedure as before leads to $v_1 = u_1$, $v_2 = -u_2$ and and again to $u_2 = u_1$ or $u_2 = -u_1 - 5\sqrt{2}/6 d$. If $u_2 = u_1$, we end up with two biquadratic equations for $u_1$ and $d$, namely

$$
19d^2 + 12\sqrt{2}d u_1 + 6u_1^2 - 70 = 0, \quad 3d^2 + 4\sqrt{2}d u_1 + 6u_1^2 - 30 = 0,
$$

again representing a hyperbola and an ellipse shown in Fig. 1 (the third figure from the left). It is clearly seen that they intersect in four points, thus the system of nonlinear equations (4) has four solutions. The case $u_2 = -u_1 - 5\sqrt{2}/6 d$ again implies a hyperbola and an ellipse with no intersections (Fig. 1 (the last figure in the row)). It is clear that changing also the angles $\theta_i$ and curvatures $\kappa_i$, $i = 0, 1$, would imply even more complicate examples with solutions relying heavily on the data.

4. Interpolation of circular arc data

The analysis of the system of nonlinear equations determining PH curve of degree 7, which interpolates given $G^2$ data with a prescribed arc length, is in general highly nontrivial task as can be seen from the examples in the previous section. In this section we shall consider the same interpolation problem, but for circular arc data, i.e., the data arising from a circular arc. Even if this simplification is considered, there seems to be no results available in the literature, so the problem is worth studying. A similar problem was considered in Farouki et al. (2021) where the authors considered the arc length preserving approximations of monotone clothoid segments.
Let the data be sampled from the circular arc with its inner angle equal to $2\alpha$. The canonical position and some elementary geometry imply $\alpha = \theta_0 = -\theta_1$, $\kappa_0 = \kappa_1 = -2\sin \alpha$, $L = \alpha \csc \alpha$, the radius of the arc equals to $1/(2\sin \alpha)$ and its centre is $(1/2, -1/2 \cot \alpha)^T$. Note that in the following we will use two additional standard trigonometric functions $\csc = 1/ \sin$ and $\sec = 1/ \cos$. We shall further assume that $0 < \alpha \leq \pi/2$, since for practical applications it is enough to construct good approximations of circular arcs up to the semicircle. It also seems that similar analysis as in the following could be done for $\pi/2 < \alpha < \pi$, too.

The circular data are first used to determine constants in the nonlinear system (4). The third and the fourth equation are then solved on $u_1$ and $v_2$, i.e.,

$$v_1 = \frac{1}{3} \tan \left( \frac{\alpha}{2} \right) \left( 3u_1 - 2d^3 \cos \left( \frac{\alpha}{2} \right) \right), \quad v_2 = \frac{1}{3} \tan \left( \frac{\alpha}{2} \right) \left( 3u_2 - 2d^3 \cos \left( \frac{\alpha}{2} \right) \right).$$  

(5)

Combining this with the fifth equation leads to

$$(u_1 - u_2) \left( 6(u_1 + u_2) - d(d^2 - 10) \cos \left( \frac{\alpha}{2} \right) \right) = 0.$$  

We obviously have two possibilities, $u_2 = -u_1 + \frac{1}{6}d(d^2 - 10) \cos \left( \frac{\alpha}{2} \right)$ or $u_2 = u_1$. Note that (4) implies that if $(d, u_1, v_1, u_2, v_2)$ is a solution, then also $(-d, -u_1, -v_1, -u_2, -v_2)$ is a solution which, by (3), provides the same interpolant $p$, so it is enough to consider solutions with $d > 0$ only.

4.1. The case $u_2 = -u_1 + \frac{1}{6}d(d^2 - 10) \cos \left( \frac{\alpha}{2} \right)$

We will prove that there are no real solutions in this case. Some particular linear combinations of the first two equations in (4) lead to two nonlinear equations

$$18(4 \cos \alpha - 3)u_1^2 + 3d(d^2 - 10) \left( \cos \left( \frac{\alpha}{2} \right) - 2 \cos \left( \frac{3\alpha}{2} \right) \right) u_1$$
$$+ \cos^2 \left( \frac{\alpha}{2} \right) (-3d^6 + 18d^4 - 34d^2 - 420 + 4d^2(d^4 - 6d^2 + 30) \cos \alpha) = 0,$$  

(6)

$$36 \left( 3 \sin \left( \frac{3\alpha}{2} \right) - 11 \sin \left( \frac{\alpha}{2} \right) \right) u_1^2 + 6 \sin \left( \frac{\alpha}{2} \right) \left( 5 \cos \left( \frac{\alpha}{2} \right) - 3 \cos \left( \frac{3\alpha}{2} \right) \right) d(d^2 - 10) u_1$$
$$+ (840\alpha - 8d^2(d^4 - 6d^2 + 30) \sin \alpha + d^2(3d^4 - 18d^2 + 34) \sin(2\alpha)) \cos \left( \frac{\alpha}{2} \right) = 0.$$
Fortunately, the resultant of the polynomials on the left hand side of \([6]\) with respect to \(u_1\) simplifies to
\[
r(d) = (16 \sin^3 \alpha d^2 + 30((3 \cos \alpha - 4) \sin \alpha + \alpha(4 \cos \alpha - 3)))^2
\]
and the candidates for solutions of \([6]\) with positive \(d\) are positive zeros of \(r\).

**Lemma 1.** For \(\alpha \in (0, \pi/2]\), function \(r\) has precisely one (double) positive zero
\[
d_1 = \frac{\sqrt{30}}{4} \sqrt{\frac{\alpha(3 - 4 \cos \alpha) + (4 - 3 \cos \alpha) \sin \alpha}{\sin^3 \alpha}} > \frac{5}{2}.
\]

**Proof.** By \([7]\) (double) zeros of \(r\) are \(\pm d_1\). The result of the lemma will follow if we prove that \(f(\alpha) > 0\) on \((0, \pi/2]\), where
\[
f(\alpha) = \alpha(3 - 4 \cos \alpha) + (4 - 3 \cos \alpha) \sin \alpha - \frac{10}{3} \sin^3 \alpha.
\]
Quite clearly \(f(0) = 0\) and \(f'(\alpha) = \sin(4\alpha + 6 \sin \alpha - 5 \sin 2\alpha) \geq 6 \sin^2 \alpha (1 - \cos \alpha) > 0\), where we have used the known fact that \(\alpha > \sin \alpha\) for \(\alpha > 0\). Consequently \(f\) is positive on \((0, \pi/2]\) and the proof is completed. \(\square\)

Note that the second equation in \([6]\) is quadratic in \(u_1\) with the discriminant
\[
504 \left(2 \cos \left(\frac{\alpha}{2}\right) + 6 \sin \left(\frac{\alpha}{2}\right) \sin \alpha\right) \sin \left(\frac{\alpha}{2}\right) f_\alpha(d),
\]
where
\[
f_\alpha(d) = 960\alpha + -8d^2(d^4 - 4d^2 + 20) \sin \alpha + d^2(3d^4 - 12d^2 - 4) \sin 2\alpha.
\]

**Lemma 2.** Function \(f_\alpha\) is negative on \([5/2, \infty)\) for all \(\alpha \in (0, \pi/2]\).

**Proof.** The idea of the proof is similar as in the proof of Lemma 1. Let
\[
g(\alpha) = f_\alpha(5/2) = 960\alpha - \frac{13625 \sin \alpha}{8} + \frac{15275}{64} \sin 2\alpha.
\]
Obviously \(g(0) = 0\) and \(g'(\alpha) = \frac{5}{32}(6110 \cos^2 \alpha - 10900 \cos \alpha + 3089)\). By solving a simple quadratic equation one can conclude that \(\alpha_0 \approx 1.2069\) is the unique zero of \(g'\) on \((0, \pi/2]\) implying the local minimum \(g(\alpha_0) \approx -274.2089\). Since also \(g(\pi/2) \approx -195.1605\), function \(g\) must be negative on \((0, \pi/2]\). Furthermore, zeros of \(f'_{\alpha}\) are \(d_1 = 0\),
\[
d_{2,3} = \pm \sqrt{\frac{16 - 12 \cos \alpha - h(\alpha)}{12 - 9 \cos \alpha}}, \quad d_{4,5} = \pm \sqrt{\frac{16 - 12 \cos \alpha + h(\alpha)}{12 - 9 \cos \alpha}},
\]
where
\[
h(\alpha) = \sqrt{-614 + 288 \cos \alpha + 90 \cos 2\alpha}.
\]
Obviously \(h^2 < 0\), and \(d_{2,3,4,5}\) are complex thus \(f_\alpha\) must be monotone on \([5/2, \infty)\). Since \(f''_{\alpha}(0) = -16 \sin \alpha(20 \sin \alpha + \cos \alpha) < 0\), \(f_\alpha\) is decreasing which together with \(f_\alpha(5/2) < 0\) implies the result of the lemma. \(\square\)

From Lemma 1, Lemma 2 and \([8]\) now follows that the system of nonlinear equations \([6]\) has no real solutions for any \(\alpha \in (0, \pi/2]\).
4.2. The case $u_1 = u_2$

Let us consider now the symmetric case, i.e., $u_1 = u_2$. Equations (5) imply $v_2 = -v_1$ and some particular linear combinations of the first two equations in (4) lead to

$$3(1 + \cos \alpha) d^2 + 8 \cos \left(\frac{\alpha}{2}\right) d u_1 + 6 u_1^2 - 10(1 + \alpha \csc \alpha) = 0,$$

where

$$4d^6 - 24d^4 + 57d^2 - 12 \sec \left(\frac{\alpha}{2}\right) d(d^2 - 3)u_1 + 9 \sec^2 \left(\frac{\alpha}{2}\right) u_1^2 + 105 \csc^2 \left(\frac{\alpha}{2}\right)(1 - \alpha \csc \alpha) = 0,$$

the system of two nonlinear equations for $d$ and $u_1$. The first equation again represents an ellipse, while the second one is much more complicated algebraic curve of degree 6 (see Fig. 2). But it is quadratic in $u_1$ and we can detect its two branches

$$u_1(d) = \frac{1}{3} \cos \left(\frac{\alpha}{2}\right) \left( 2d^3 - 6d \pm \sqrt{21} \sqrt{5 \csc^2 \left(\frac{\alpha}{2}\right)(\alpha \csc \alpha - 1) - d^2} \right),$$

(11)

(the plus sign is the black solid part of the curve and the minus sign is the black dashed part in Fig. 2). It is also easy to see that it is a closed curve with $d \in [-d_{\max}, d_{\max}]$ where

$$d_{\max} = \csc \left(\frac{\alpha}{2}\right) \sqrt{5(\alpha \csc \alpha - 1)}.$$

(12)

**Lemma 3.** For $\alpha \in (0, \pi/2]$ the inequality $d_{\max} > \sqrt{10/3}$ holds true.

**Proof.** It is easy to see that the inequality from the lemma is equivalent to $f(\alpha) > 0$, where $f(\alpha) = 3(\alpha - \sin \alpha) - \sin \alpha(1 - \cos \alpha)$. Since $f(0) = 0$ and $f'(\alpha) = 8 \sin^4 \left(\frac{\alpha}{2}\right) > 0$, the proof of the lemma is complete. □

The following lemma guarantees the existence of the solution of the considered interpolation problem.

**Lemma 4.** The system of nonlinear equations (9) and (10) has at least two real solutions with $d > 0$ for any $\alpha \in (0, \pi/2]$.

**Proof.** During the proof we will refer to Fig. 2. Consider the $(d, u_1)$ plane. Let $u_{1e}$ and $u_{1a}$ be the positive intersections of (9) and (10) with $d = 0$, respectively, and similarly, let $d_e$ and $d_a$ be the positive intersections of the same curves with $u_1 = 0$, respectively. If we show that

$$(u_{1a} - u_{1e})(d_a - d_e) < 0,$$

(13)

then curves must intersect in the first quadrant of the chosen coordinate system. But due to the symmetric properties they must also intersect in the fourth quadrant and the result of the lemma will follow.

Quite clearly, for (13) it is enough to see $u_{1a} > u_{1e}$ and $d_e > d_a$. Let us start by proving the first inequality. If $d = 0$, then (9) and (11) imply

$$u_{1e} = \sqrt{\frac{5}{3} \sqrt{\alpha \csc \alpha + 1}}, \quad u_{1a} = \frac{35}{3} \cot \left(\frac{\alpha}{2}\right) \sqrt{\alpha \csc \alpha - 1}.$$
Some straightforward calculations reveal that $u_{1a} > u_{1e}$ is equivalent to $f(\alpha) := \cos \alpha (4\alpha - 3 \sin \alpha) - 4 \sin \alpha + 3\alpha > 0$ on $(0, \pi/2]$. Since $f'(\alpha) = 2 \sin \alpha(3 \sin \alpha - 2\alpha)$, $f$ is strictly increasing on $(0, \alpha^*)$ and strictly decreasing on $(\alpha^*, \pi/2]$, where $\alpha^* \in (\pi/4, \pi/2)$. But $f(0) = 0$, $f(\pi/2) = (3\pi - 8)/2 > 0$, and the conclusion $f > 0$ on $(0, \pi/2]$ follows.

For the second inequality observe that $d_{max} > d_{a}$ and it is enough to see that $d_{e} \geq d_{max}$. Inserting $u_{1} = 0$ in (9) and considering (12) lead us to show that $g(\alpha) := \cos \alpha \sin \alpha + 2 \sin \alpha - 2\alpha \cos \alpha - \alpha \geq 0$ on $(0, \pi/2]$. But this follows immediately from $g(0) = 0$ and $g'(\alpha) = 2 \sin \alpha(\alpha - \sin \alpha) > 0$ on $(0, \pi/2]$.

**Remark 1.** Identical proof can be done for the case $\alpha \in (0, \alpha_{max})$, where $\alpha_{max} \approx 2.0682$ is the first positive zero of $f$ defined in the proof of the previous lemma.

From the previous lemma it actually follows that the considered system of nonlinear equations has an even number of solutions with positive $d$ for any $\alpha \in (0, \pi/2]$. Numerical examples reveal that there might be four of them as indicated in Fig. 2. However, it seems quite difficult to prove the existence of other two solutions ($d_{1}$ and $d_{2}$ on Fig. 2) in general, since they tend to each other for small $\alpha$ and they disappear for $\alpha > \alpha_{crit} \approx 2.2337$, i.e., the critical value of $\alpha$ which is a solution of (9), (10) their zero Jacobian (see Fig. 3). In the following, we will provide an easy (numerical) procedure to check the existence of four solutions with $d > 0$ for a particular $\alpha \in (0, \pi/2]$ and prove their existence for $\alpha$ small enough. The PH interpolant of degree seven arising from the positive solution $d_{j}$, $j = 1, 2, 3, 4$, where $d_{1} < d_{2} < d_{3} < d_{4}$, will be denoted by $p_{j}$.

Let us first transform the system of nonlinear equations (9) and (10) to a more appropriate one for the analysis. This can be done by using a Gröebner basis (Adams and Loustaunau (1994)) with respect to a particular ordering of the unknowns. The system of nonlinear equations then reads as
Figure 3: Four solutions with positive $d$ for $\alpha < \alpha_{\text{crit}} \approx 2.2337$ (left) which degenerate to three of them for $\alpha = \alpha_{\text{crit}}$ (middle) and transform to two solutions for $\alpha > \alpha_{\text{crit}}$ (right).

\[ p_1(d) = -32 \sin^6 \alpha d^{12} + 256 \sin^6 \alpha d^{10} - 1184 \sin^6 \alpha d^8 \\
- 96 \sin^3 \alpha (-40 \alpha + 9 \sin \alpha + 20 \sin 2\alpha + 7 \sin 3\alpha - 30 \alpha \cos \alpha) d^6 \\
+ 96 \sin^3 \alpha (-160 \alpha + 99 \sin \alpha + 80 \sin 2\alpha + 7 \sin 3\alpha - 120 \alpha \cos \alpha) d^4 \\
+ 13440(\alpha - \sin \alpha) \sin^5 \alpha \csc^2 \left(\frac{\alpha}{2}\right) d^2 \\
- 1800(6 \alpha + 8 \alpha \cos \alpha - 2 \sin \alpha (3 \cos \alpha + 4))^2 = 0, \] (14)

\[ p_2(u_1, d) = 24d (d^2 - 2) \sec \left(\frac{\alpha}{2}\right) u_1 + 210 \csc^2 \left(\frac{\alpha}{2}\right) (\alpha \csc \alpha - 1) \\
+ \sec^2 \left(\frac{\alpha}{2}\right) (-30 \alpha \csc \alpha + 9d^2 \cos \alpha + 9d^2 - 30) - 8d^6 + 48d^4 - 114d^2 = 0. \] (15)

Let us analyze the polynomial $p_1$ first. Since it is even of degree 12, we can reduce its degree to 6 by introducing $p(x) = p_1(\sqrt{x})$. A computer algebra system reveals that

\[ p(x) = -32q(x)\alpha^6 + O(\alpha^8), \quad q(x) = x^6 - 8x^5 + 37x^4 - 134x^3 + 284x^2 - 280x + 100, \] (16)

and real zeros of $q$ are

\[ x_1 = x_2 = 1 \quad x_3 \approx 2.1842, \quad x_4 \approx 3.2872. \] (17)

Note that $x_3$ and $x_4$ can also be written in radicals since they are zeros of some quartic polynomial but expressions are too complicated to be given here explicitly.

**Lemma 5.** For any $\alpha \in (0, \pi/2]$, the polynomial $p$ has at most four positive zeros on $(0, d_{\text{max}}^2)$. If $p(y_i), i = 0, 1, \ldots, 4,$ where $y_0 = 0, y_1 = 1, y_2 = x_3, y_3 = x_4$ and $y_4 = d_{\text{max}}^2$, are of alternating signs, then $p$ has precisely four positive zeros.

**Proof.** First observe that the fourth derivative of $p$ is of particularly simple form, namely $p^{(iv)}(x) = -768 \sin^6 \alpha (15x^2 - 40x + 37)$. Quite clearly it is negative and consequently $p$ has at most four real zeros. By Lemma 3 and (17) we have $y_0 < y_1 < y_2 < y_3 < y_4$ and $p(y_i)p(y_{i+1}) < 0, i = 0, 1, 2, 3,$ implies precisely four zeros due to the continuity of $p$. □
We are now ready to prove the main theorem of this paper.

**Theorem 1.** If \( \alpha \in (0, \pi/2] \) then the number of real solutions of the nonlinear system (9), (10) with \( d > 0 \) is the same as the number of positive zeros of \( p \).

**Proof.** Since nonlinear system (9), (10) is equivalent to the system (14), (15), the only candidates for real solutions with \( d > 0 \) are, by Lemma 5, positive zeros of \( p \). Thus we have to prove that each positive zero of \( p \) implies the unique real solution of the system (14), (15). Let \( z \) be a positive zero of \( p \). Then \( d_z = \sqrt{z} \) is a positive zero of \( p_1 \) and the solution of \( p_2(u_1, d_z) = 0 \) on \( u_1 \) provides the desired solution of the system of nonlinear equations. But \( p_2(u_1, \cdot) \) is a linear polynomial and it remains to prove that its leading coefficient does not vanish at \( d_z \). It is equivalent to verifying that \( z \neq 0, 2, \) or equivalently

\[
p(0) = -7200f_1(\alpha)^2 < 0, \quad p(2) = -32f_2(\alpha)^2 < 0,
\]

where

\[
f_1(\alpha) = \cos \alpha (4\alpha - 3 \sin \alpha) - 4 \sin \alpha + 3\alpha,
\]

\[
f_2(\alpha) = 45\alpha - 2 \sin^3 \alpha - 66 \sin \alpha + 60\alpha \cos \alpha + 6 \sin \alpha \cos^2 \alpha - 45 \sin \alpha \cos \alpha.
\]

Thus it is enough to show that \( f_1, f_2 > 0 \) on \((0, \pi/2]\). The inequality \( f_1 > 0 \) follows from the fact that \( f_1 = f \) from the proof of Lemma 4. To confirm \( f_2 > 0 \), observe that \( f_2(0) = 0 \) and \( f_2'(\alpha) = -6 \sin \alpha f_3(\alpha) \), where \( f_3(\alpha) = 10\alpha - 15 \sin \alpha + 4 \sin \alpha \cos \alpha \). Since \( f_3'(\alpha) = 6 - 15 \cos \alpha + 8 \sin^2 \alpha, f_3' \) has precisely one zero on \([0, \pi/2]\) and consequently \( f_3 \) has at most two zeros there. Since \( f_3(0) = 0, f_3(\alpha) = -\alpha + \mathcal{O}(\alpha^3) \) and \( f_3(\pi/2) = 5(\pi - 3) > 0, f_3 \) has the unique zero \( \alpha_0 \in (0, \pi/2] \). Thus \( f_2 \) is increasing on \((0, \alpha_0]\) and decreasing on \((\alpha_0, \pi/2]\). Since \( f_2(0) = 0 \) and \( f_2(\pi/2) = (45\pi - 136)/2 > 0 \), function \( f_2 \) must be positive on \((0, \pi/2]\) and the result of the theorem follows. \( \square \)

Previous theorem provides an efficient and easy way to check the number polynomial parametric approximants interpolating \( G^2 \) data and an arc length arising from a circular arc given by an inner angle \( 2\alpha \). For some practically important angles \( \alpha \), such as \( \alpha = \pi/2, \pi/3, \pi/4, \pi/8, \ldots \), the direct application of Lemma 5 confirmed the existence of four zeros of \( p \), except for \( \alpha = \pi/2 \), where we had to replace \( y_2 = x_3 \) by \( y_2 = 2 \). A direct formal proof that precisely four solutions exist for any \( \alpha \in (0, \pi/2] \) seems to be quite a difficult task, since the analysis of symbolic expressions involving combinations of algebraic and trigonometric terms in Lemma 5 would be needed. However, if \( \alpha \) is small enough, the expansion (16) enables us to prove the existence of four solutions in general. This will be done in the following section.

5. **Asymptotic analysis**

Let us now consider \( \alpha \) small enough. Using (16) and considering some additional terms in the expansion, we get

\[
p(y_0) = -3200\alpha^6 + \mathcal{O}(\alpha^8), \quad p(y_1) = 64\alpha^{10} + \mathcal{O}(\alpha^{12}), \quad p(y_2) \approx -52.3867\alpha^8 + \mathcal{O}(\alpha^{10}),
\]

\[
p(y_3) \approx 2292.89\alpha^8 + \mathcal{O}(\alpha^{10}), \quad p(y_4) = -\frac{156800}{729}\alpha^6 + \mathcal{O}(\alpha^8).
\]
Consequently, \( p \) has four positive zeros by Lemma 5. The leading terms constants can be written also in a closed form, thus their numerical values can be computed with arbitrary precision.

In the following we will find asymptotic expansions of positive zeros of \( p \) which provide asymptotic expansions of real solutions of the system of nonlinear equations (9), (10). Let \( z_i, i = 1, 2, 3, 4 \), be a positive zero of \( p \). Then (16) suggests the expansion of \( z_i \) as

\[
z_i = x_i + \sum_{j=1}^{\infty} c_{i,j} \alpha^j, \quad i = 1, 2, 3, 4.
\]

Constants \( c_{i,j} \) can now be found as a solution of the system of equations for \( c_{i,j} \) arising from the condition that terms in the expansion of \( p(z_i) \) vanish for all \( \alpha \). Let us demonstrate the procedure for \( i = 2 \), since the solution \( z_2 \) will later turn out as the most appropriate one. The expansion of \( p(z_2) \) reads as

\[
p(z_2) = -1248 c_{2,1}^2 \alpha^8 + 64 c_{2,1} (23 c_{2,1}^2 - 39 c_{2,2} - 2) \alpha^9 - 32 (12 c_{2,1}^4 - 3 (46 c_{2,2} + 9) c_{2,1}^2 + 78 c_{2,3} c_{2,1} + 39 c_{2,2}^2 + 4 c_{2,2} - 2) \alpha^{10} + \ldots
\]

The requirement that the coefficients at \( \alpha^j, j = 8, 9, 10 \), vanish, leads to the triangular system of nonlinear equations with solutions \( c_{2,1} = 0, c_{2,2} = (-2 \pm \sqrt{82})/39 \). Since \( z_2 > 1 \), we must take \( c_2 = (-2 + \sqrt{82})/39 \). Considering more terms in the expansion, we can similarly compute additional constants \( c_{2,j} \) but we will skip the details. Recall that \( d_2 = \sqrt{z_2} \), so the asymptotic expansion of \( d_2 \) is

\[
d_2 = 1 - \frac{1}{78} \left(2 - \sqrt{82}\right) \alpha^2 + \frac{(37966 + 10579 \sqrt{82})}{19456632} \alpha^4 + \ldots \quad (18)
\]

Together with (15) we get the asymptotic expansions

\[
u_{1,2} = u_{2,2} = 1 + \frac{1}{312} \left(73 - 4 \sqrt{82}\right) \alpha^2 - \frac{(3071515 - 636632 \sqrt{82})}{311306112} \alpha^4 + \ldots \quad (19)
\]

and finally from (5) also

\[
v_{1,2} = -v_{2,2} = \frac{\alpha}{6} + \frac{(371 - 36 \sqrt{82})}{1872} \alpha^3 - \frac{(8660963 - 1179080 \sqrt{82})}{103768704} \alpha^5 + \ldots \quad (20)
\]

Similarly we compute asymptotic expansions for other three solutions \( d_1, d_3, d_4 \) and consequently also expansions for \( u_{1,j}, u_{2,j}, v_{1,j} \) and \( v_{2,j}, j = 1, 3, 4 \). Either in non-asymptotic or in asymptotic approach we obtain several solutions. In the next section we will provide some suggestions how to choose the most appropriate one.

6. Solution selection

Multiple solutions are regularly observed fact when one is dealing with interpolation by PH curves. Usually, some of them are more appropriate for applications (without undesirable loops, e.g.) than the other ones. This was observed already in the early papers dealing with interpolation by PH curves (Albrecht and Farouki (1996), Farouki and Neff (1995)). There are several suggestions how to choose the most appropriate solution, but non of them can be considered as a universal one.
Quite standard measures of fairness is the absolute rotation index (Farouki 2008, p. 532), which is defined as

\[ R_{\text{abs}} = \int_{0}^{1} |\kappa(t)||p'(t)|\,dt. \] (21)

It was successfully used in Farouki (2016) to identify more appropriate solutions. We can use the same criterion here for the general interpolation of \( G^2 \) data. However, for the circular arc data it seems reasonable to observe the deviation of the curvature of the interpolant from the (constant) curvature of the corresponding circular arc in \( L^2 \) norm. Since the curvature of the circular arc in the chosen canonical position is \(-2\sin\alpha\), the error becomes

\[ E_\kappa = \int_{0}^{1} (\kappa(t) + 2\sin\alpha)^2\,dt. \] (22)

Note that the (numerical) evaluation of \( E_\kappa \) for a PH curve is particularly simple since its curvature \( \kappa \) is a rational function.

For the asymptotic case explained in the previous section, it is promising to choose the solution which provides a curve with the best approximation properties, such as the minimal Hausdorff distance. Since the approximation of a circular arc is considered, one can use the radial distance \( d_{\text{rad}} \) as the error measure (Degen (1992)). It is a special type of parametric distance considered in Lyche and Mørken (1994) and later in Jaklič and Kozak (2018) where the authors have proved that it actually coincides with the Hausdorff distance in the case of circular arc approximation. Let \( p = (p_x, p_y)^T \) be a PH curve of degree seven approximating the circular arc \( c \) given by some small inner angle \( 2\alpha \) in the canonical position. Then the radial distance is defined as

\[ d_{\text{rad}}(p; \alpha) = \max_{t \in [0,1]} \left| \sqrt{\left( \frac{p_x(t) - \frac{1}{2}}{\cos \alpha} \right)^2 + \left( \frac{p_y(t) + \frac{1}{2}}{\cos \alpha} \right)^2} - \frac{1}{2\sin\alpha} \right|, \]

i.e., the distance between the point \( p(t) \) and the intersection of the line passing through the centre of the circular arc \((1/2, -1/2\cot\alpha)^T\) and \( p(t) \) with the circular arc. Using the asymptotic expansions (18)–(20) we can derive

\[ d_{\text{rad}}(p; \alpha) = c\alpha^r + \mathcal{O}(\alpha^{r+1}), \] (23)

where \( c \) is some positive constant and \( r \in \mathbb{N} \) is the asymptotic approximation order. Since \( p \) interpolates two points, two tangent directions, two curvatures and an arc length, the expected approximation order is 7. Indeed, for the solution \( d = d_2 \) which implies the PH interpolant \( p_2 \) we have

\[ d_{\text{rad}}(p_2; \alpha) = \frac{47773 - 5264\sqrt{82}}{318898944}\alpha^7 + \mathcal{O}(\alpha^8) \approx 3.3068 \times 10^{-7}\alpha^7 + \mathcal{O}(\alpha^9), \] (24)

while \( d_1, d_3 \) and \( d_4 \) imply interpolants \( p_1, p_3 \) and \( p_4 \), respectively, with inferior leading term constant or much lower approximation order. More precisely,

\[ d_{\text{rad}}(p_1; \alpha) = \frac{47773 + 5264\sqrt{82}}{318898944}\alpha^7 + \mathcal{O}(\alpha^8) \approx 2.9928 \times 10^{-4}\alpha^7 + \mathcal{O}(\alpha^9), \] \[ d_{\text{rad}}(p_3; \alpha) = 0.0173\alpha + \mathcal{O}(\alpha^3), \quad d_{\text{rad}}(p_4; \alpha) = 0.1246\alpha + \mathcal{O}(\alpha^3). \] (25)
7. Numerical examples

In all numerical examples canonical data will be considered. Since we know that there are always several solutions of the problem, we can find them numerically either by applying the continuation method (Allgower and Georg (1990)) or by using Lemma 5 which provides excellent starting values for an iterative algorithm (such as Newton-Raphson method) to find positive real zeros of \( p \). One of the selection criteria (21) (for general data), (22) (for general circular arc data) or (23) (for circular data with \( \alpha \) small enough) is then used to identify the most pleasant interpolant.

Although we analysed the solvability of the problem only for circular data, we shall first present some examples confirming that the interpolation method can be successfully applied for general data, too.

First, let us consider the data

\[ \theta_0 = \pi/2, \quad \theta_1 = -\pi/4, \quad \kappa_0 = -1, \quad \kappa_1 = 2, \quad L = 1.75. \]  

(27)

The problem has two solutions and they are shown in Fig. 4. The one without loops has the absolute rotation index (21) approximately 3.01, and for the one with loops we have \( R_{abs} \approx 10.43 \). In this case the shape measure (21) clearly rejects it.

![Figure 4: Two solution curves interpolating data (27).](image)

Consider now similar data as in the previous example with parallel tangent directions

\[ \theta_0 = \theta_1 = \pi/3, \quad \kappa_0 = -1, \quad \kappa_1 = 2, \quad L = 1.5. \]  

(28)

There are again two interpolants which are plotted in Fig. 5. In this case both of them have visually pleasant shape and its difficult to prefer one of them. This is also confirmed by the absolute rotation index, which is approximately 5.08 for the left one and 6.14 for the right one.

As the last example of general data let us consider

\[ \theta_0 = \pi/2, \quad \theta_1 = -\pi/2, \quad \kappa_0 = \kappa_1 = -8, \quad L = 1.21106. \]  

(29)

Again two formally admissible interpolants exist. They are shown in Fig. 6. The data is actually taken from the elliptic arc \( 4(x - 1/2)^2 + 16y^2 = 1 \). It is clearly seen that the first approximant follows the ellipse (the elliptic arc is gray dashed and almost identical as the interpolant), while the other one takes quite different shape. The absolute rotation indices are 3.14 (the analytical value of the rotation index of the considered elliptic arc is \( \pi \)) and 5.77. The second one is not so much bigger since the curve is still visually pleasant.
For the last two examples let us take the data from the circular arc. Suppose first that
\[ \alpha := \theta_0 = -\theta_1 = \pi/2, \quad \kappa_0 = \kappa_1 = -2, \quad L = \pi/2. \]  

We know from Lemma 5 that four admissible solutions exist. The error (22) is taken as the selection criterion. Its corresponding values for the approximants \( p_i \) arising from the positive solutions \( d_i, \ i = 1, 2, 3, 4 \), are \( 4.2527 \times 10^{-2}, \ 8.6586 \times 10^{-8}, \ 2.4235 \times 10^{6} \) and \( 34.0648 \), respectively. The approximant \( p_2 \) corresponding to \( d_2 \approx 1.2756 \) is clearly the most appropriate, which is confirmed also in Fig. 7. The Hausdorff distance of the chosen interpolating curve and the circular arc is approximately \( 1.2850 \times 10^{-5} \) and it is attained at the middle of the arc. Thus the constructed PH curve of degree 7 can be considered as a very accurate approximation of the semicircle preserving the arc length. For the \( G^2 \) approximation of the whole circle just consider the spline approximant build by the constructed interpolant and its rotation.

For the last example the data from the circular arc with \( \alpha > \pi/2 \) will be considered. Let
\[ \alpha := \theta_0 = -\theta_1 = 5\pi/6, \quad \kappa_0 = \kappa_1 = -1, \quad L = 5\pi/3. \]  

The system of nonlinear equations (9) and (10) has only two admissible solutions. According to (22), the first one is clearly rejected, since the error is \( E_\kappa \approx 61.3568 \), much higher that the error of the second one \( E_\kappa \approx 9.0995 \times 10^{-6} \). This is evidently confirmed also in Fig. 8 where approximants together with their curvature profiles are shown. The Hausdorff distance of the second interpolant and the circular arcs is \( 1.6607 \times 10^{-3} \), which is less than \( 0.2\% \) relatively to the radius. Note that we could take even greater value of \( \alpha \). Numerical examples confirm admissible solutions of the system of nonlinear equations for any \( \alpha < \pi \), i.e., for the data arising from circular arcs up to almost the whole circle.

Let us conclude this section by numerical evaluation of the approximation orders (24)–(26). For the sake of simplicity we will consider just interpolants \( p_2 \) and \( p_3 \). They are computed for \( \alpha_n = \pi/2^n \),
Figure 7: Plots of semicircle approximants $p_i$, $i = 1, 2, 3, 4$, interpolating data (30) (top) together with corresponding curvature profiles (bottom). The gray horizontal line is the curvature of the approximated semicircle. Note that $p_3$ possesses two almost invisible tiny loops.

Figure 8: Plots of interpolants of data (31) (the first and the second on the left) and corresponding curvature profiles. Dotted lines are curvatures of the circular arc from which the data were taken.

$n = 1, 2, \ldots, 5$, and the corresponding Hausdorff errors $e_n$ are determined. From $e_n \approx c_\alpha r_n$ one easily concludes that $r \approx \log(e_n/e_{n+1})/\log 2$. Numerical results are collected in Tab. [1] and they confirm theoretical values established in the previous section.

8. Closure

PH curves of degree seven are promising object for interpolation of $G^2$ local data and preserving an arc length. Since for general data the problem turns out to be quite complicated, a relaxation to the circular arc data was done and a detailed analysis provided. It turned out that the above mentioned curves provide an excellent approximants of circular arcs and the preserve a prescribed arc length. An algorithm for the construction of such curves was provided. It basically requires just solving an algebraic equation of degree six. An asymptotic analysis reveals that the approximation order is seven.

For the future work it would be nice to do some progress in studying the interpolation of general data. This requires some deeper analysis of general system of nonlinear equations [4]. Another approach to solve the same problem would be using the PH quintic biarcs, a generalization of cubic biarcs studied already in an early paper by Farouki and Peters (1996).
| $\alpha$ | $d_{rad}(p_2; \alpha)$ | $r$ | $d_{rad}(p_3; \alpha)$ | $r$ |
|--------|----------------|----|----------------|----|
| $\pi/2$ | $1.2850 \times 10^{-5}$ | $-$ | $1.3865 \times 10^{-2}$ | $-$ |
| $\pi/4$ | $6.8517 \times 10^{-8}$ | 7.55 | $1.3143 \times 10^{-2}$ | 0.08 |
| $\pi/8$ | $4.9016 \times 10^{-10}$ | 7.13 | $6.7687 \times 10^{-3}$ | 0.96 |
| $\pi/16$ | $3.7474 \times 10^{-12}$ | 7.03 | $3.3944 \times 10^{-3}$ | 1.00 |
| $\pi/32$ | $2.9119 \times 10^{-14}$ | 7.01 | $1.6980 \times 10^{-3}$ | 1.00 |

Table 1: Radial distances and estimated approximation orders for $G^2$ interpolants $p_2$ (the second and the third column) and $p_3$ (the fourth and the fifth column).

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