Directional mechanical response in the bulk of topological metamaterials

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Keywords: topology, isostatic lattice, mechanical response

Abstract

Mechanical metamaterials are those structures designed to convey force and motion in novel and desirable ways. Recently, Kane and Lubensky showed that lattices at the point of marginal mechanical stability (Maxwell lattices) possess a topological invariant that describes the distribution of floppy, zero-energy edge modes. Here, we show that applying force at a point in the bulk of these lattices generates a directional mechanical response, in which stress or strain is induced only on one side of the force. This provides both a bulk metric for mechanical polarization and a design principle to convey stresses and strains towards or away from parts of the structure. We also characterize the effects of removing bonds on the material’s structure and floppy modes, establishing a relationship between edge modes and bulk response.

1. Introduction

Mechanical metamaterials have engineered structures that imbue them with novel mechanical properties. Such structures can be auxetic (negative Poisson’s ratio) [1], pentamode (fluid-like vanishing of most elastic moduli) [2, 3] or even possess negative compressibilities [4]. The control of mechanical and acoustic response permits programmable elastic response [5], cloaking [6, 7] and soft robotics [8]. Even beyond metamaterials, the mechanical response of natural and conventionally engineered materials is determined by their structures.

Of particular interest are Maxwell lattices [9] which exist at the isostatic point, possessing the minimal number of bonds necessary to support external force, just as do naturally occurring systems such as fiber networks [10] and jammed packings [11]. Each missing bond at the edge of a finite Maxwell lattice generates a floppy mode in which particles may move without stretching bonds. Kane and Lubensky [12] revealed that Maxwell lattices possess a nontrivial topological polarization vector \( \mathbf{P} \) determined by the structure of their bulk vibrational modes and that, remarkably, the bonds removed at one edge can generate zero modes on the far edge. Subsequently, it has been shown that this topological polarization couples to line-like [13] and point-like [14] defects to create new ways of distributing force and motion, controlling both where lattices deform freely and when they buckle irreversibly.

However, while the effects of the topological polarization on edges and interfaces are now apparent, whether it has a signature in the bulk has been less clear. Continuum elastic analyses [9, 12] of topologically polarized systems reveal stress/strain relationships that respect spatial inversion symmetry, and quite general considerations of reciprocity in linear mechanical response [15, 16] suggest that the system must deform in all directions in response to an applied force. In contrast to this expected behavior, we show that when one considers both particles and bonds, topologically polarized lattices have entirely directional bulk response that echoes their exotic edge modes. Local forces applied to a polarized lattice generate either stress (for generic forces) or strain (for forces applied only along bonds) on only one side of the applied force. This directionality is a powerful source of static mechanical isolation, so that topological polarization may be engineered to protect part of the system. As we will see, external forces applied at one point may generate stresses or strains at a second that are orders of magnitude greater than propagation in the other direction. The directional response functions serve to fully characterize the behavior of the system, describing its response to externally-imposed forces and...
deformations of any spatial distribution (through superposition). It also describes the system’s response to changes in the lengths of bonds, and can incorporate missing and/or additional bonds.

This conceptual framework relies largely on the topological structure of the linear map between the system’s degrees of freedom and its energetic interactions, and as such extends beyond any particular system. Topological polarization and other topological states have been considered in mechanical systems consisting of Hookean springs [12], rigid frames [14, 17], beams [13], rotors [12, 18], origami/kirigami sheets [19], geared cogs [20], masses with three-body interactions [21] and gyroscopes [22–24]. Kane and Lubensky mapped their systems onto topological insulators in the BDI class and considerations of symmetry and dimension show that other classes of topological insulators also have mechanical analogs [25]. Because topology is robust against disorder, does not require fine-tuning and controls mechanical response, it has been suggested as a mechanism that arises in natural biological structures [26–28]. Related topological states with nontrivial edge modes arise in a variety of other classical systems, including photonics [29], magnetic systems [30–32], classical electrical circuits [33, 34], acoustic resonators [35], active fluid [36, 37] and nonequilibrium stochastic biological systems [38].

While our present analysis focuses on linear response, topological polarization is tightly connected to nonlinear behavior. In one dimension, the zero modes that are localized in the linear regime give rise to both zero-energy solitons [18] and to highly nonreciprocal mechanical response [39]. In two dimensions (or more), the bulk elastic response of a Maxwell lattice [9, 40, 41] contains a characteristic mode, the ‘Guest mode’, which deforms the lattice uniformly without stretching any bonds. Finally, the linear response of topological lattices that determines the tension at interfaces extends to the nonlinear regime in which the beams at those interfaces begin to buckle [18]. Indeed, while the directionality is not apparent in the linear finite-frequency response, this nonlinear response suggests that the directionality of the systems may extend to how energy is deposited into or drawn out of the system and how waves are conveyed across it [42].

The rest of our paper is arranged as follows. Section 2 describes the general response (Green’s) functions of mechanical lattices. Section 3 discusses the implications of topological polarization. Sections 4 and 5 consider the particular examples of 1D chains and 2D kagome lattices, respectively. Section 6 associates zero modes with missing bonds in the bulk and edge. In section 7 we discuss the implications of our work.

### 2. Mechanical response functions in Maxwell lattices

We primarily consider systems of the form shown in figure 1, consisting of periodic lattices of particles in $d$ dimensions connected by bonds that we treat as simple Hookean springs, generally not distinguishing between ‘bonds’ and ‘springs.’ For simplicity, we take masses and Hookean spring moduli to be unity, although the zero modes obtained (distinct from the force-bearing equilibrium configurations) do not depend on the specific energy cost associated with stretching and compressing the bonds. Restricting ourselves to small strains, we can
linearly map the vector of site displacements $\mathbf{u}$ onto the vector of bond extensions $\mathbf{e}$:

$$
\mathbf{e} = \mathbf{R}\mathbf{u}.
$$

(1)

$\mathbf{R}$, which we refer to as the rigidity matrix (also ‘compatibility’ or ‘kinetic’ matrix), is determined by the sites’ equilibrium positions and bond structure, as described in appendix A. Because the tensions of our springs are proportionate to their extensions and generate force along the bond directions, we also have the vector of forces $\mathbf{f}$ given by

$$
\mathbf{f} = \mathbf{R}^T\mathbf{e}.
$$

(2)

Combining these relationships leads to the energy functional $E$ and to the relationship between forces and displacements, both in terms of the dynamical matrix $\mathbf{D} = \mathbf{R}^T\mathbf{R}$:

$$
\mathbf{f} = \mathbf{D}\mathbf{u},
$$

(3)

$$
E = \frac{1}{2}\mathbf{u}^T\mathbf{D}\mathbf{u}.
$$

(4)

The above relations are a set of linear maps between the space of particle displacements and the space of bond extensions as represented in figure 2. Linear zero modes, whose number we denote by $N_0$, are those displacements, such as uniform translations, which do not stretch any bonds and thus lie in the nullspace of $\mathbf{R}$. Sets of bond tensions or elongations which do not generate net forces similarly lie in the nullspace of $\mathbf{R}^T$; these ‘self stresses’ number $N_s$. The relationships of equations (1) and (2) then lead to the following index theorem, given in terms of the system’s numbers of degrees of freedom ($N_{\text{dof}}$) and constraints ($N_{\text{con}}$) [9, 40]:

$$
N_{\text{dof}} - N_{\text{con}} = N_0 - N_s.
$$

(5)

In particular, for Maxwell lattices zero modes may be generated only in conjunction with self stresses or by removing bond constraints.

The response of such a mechanical structure to external forces would seem to be entirely reciprocal, even when the structures themselves break spatial inversion symmetry. Consider a pair of external forces and resulting deformations of a structure satisfying $\mathbf{f}_{L(R)} = \mathbf{D}\mathbf{u}_{L(R)}$, with $\mathbf{f}_L$ localized in an area the left of $\mathbf{f}_R$. From the symmetry of the dynamical matrix, it follows that

$$
\mathbf{f}_L \cdot \mathbf{u}_R = \mathbf{f}_R \cdot \mathbf{u}_L.
$$

(6)

This result, well-known in the engineering community as the Maxwell–Betti reciprocity theorem [15, 16], seems to indicate that, intuitively, if a force at one point generates a response at a second then a force applied at the second point generates a response at the first. However, of course, $\mathbf{f}_R \cdot \mathbf{u}_L = 0$ does not strictly require that $\mathbf{f}_L$, $\mathbf{u}_L$ have no spatial overlap. In particular, Maxwell lattices can respond to external forces with local deformations that only rotate rather than stretching bonds. Hence, forces applied along the lengths of bonds (stretching rather than rotating them) generate responses that automatically satisfy equation (6) ($\mathbf{f}_L \cdot \mathbf{u}_R = \mathbf{f}_R \cdot \mathbf{u}_L = 0$) regardless of their spatial distribution. In particular, it is possible for the response to lie entirely to one side of an external force, and thus to be directional even while respecting reciprocity.
We then consider two ways in which the periodic structure may be altered: applying external forces and *swelling* certain bonds. We use ‘swelling’ to refer to any static or quasistatic process that changes the equilibrium length of a bond, including fixed length disorder, dynamic fluctuations, external forces, or mechanical actuation (the concept of altering equilibrium lengths is implicit in self stresses, wherein tension is induced in bonds without changing their lengths). Although such analysis may also be applied to disordered systems, we now restrict ourselves to periodic structures described by $d$ lattice vectors in a $d$-dimensional space (for some systems, such as origami, these two numbers are not equal, though this does not affect the analysis). The concept of the response of a lattice to external forces, the lattice Green’s function, is well-developed (see, e.g., [43, 44]). Unlike previous studies of response in spring networks, the present work follows the concept of Kane and Lubensky in taking the ‘square root’ of the Green’s function and examining the effect of bonds on sites and vice-versa, thereby revealing topologically-nontrivial behavior. In contrast, conventional Green’s functions consider responses such as displacements of sites resulting from forces on sites.

The position of crystal cells in the lattice is indexed by a discrete vectorial index $\mathbf{n}$ and, in reciprocal space, a wavenumber $\mathbf{q}$. Treating our wavenumber as a continuous variable, as is valid even over short distances in large systems, leads to maps of the form

$$e = R(q)u,$$

etc. Now, $u$ is a vector consisting of the Cartesian components of the displacements of each particle within a single crystal cell, such that $u_n = \exp(i\mathbf{q} \cdot \mathbf{n})u$. Inverting the above relationship provides us with the response function of the lattice to the swelling of a single bond, $e_n = e_{\mathbf{h} \cdot \mathbf{n}}$:

$$R^{-1,g}(\mathbf{n} - \mathbf{n}_i) = \frac{1}{(2\pi)^d} \int_{-\infty}^{\infty} d^d \mathbf{q} \exp(i\mathbf{q}(\mathbf{n} - \mathbf{n}_i)) R^{-1,g}(\mathbf{q}).$$

We may similarly invert equations (2) and (3) to relate $u$ to $f$ via $D^{-1,g}$ and $e$ to $f$ via $(R^T)^{-1,g}$. Here, $R^{-1,g}$ denotes the Moore–Penrose generalized pseudo-inverse, which may be readily generated from singular-value decomposition even for non-square matrices [45, 46] corresponding to non-Maxwell systems. This set of site displacements satisfies the linear map of equation (1) when possible and more generally is the best possible solution in the sense of minimizing the magnitude of the additional extensions, $R\mathbf{u} - e$, and hence the energy of the lattice.

For some systems which lack translational and rotational invariance, such as the rotor system considered in section 4, the rigidity matrix is fully invertible and thus it is possible to generate any set of bond extensions from some combination of site modes (rotor rotations). However, for the Maxwell spring networks considered here, the only zero modes are $d$ uniform translations present at $q = 0$, from which the index theorem of equation (5) implies $d$ self stresses at $q = 0$. Thus, the rigidity matrix maps the $N_{\text{ dof}}$ modes of particle displacements not onto the equally-sized space of bond extensions but onto a subspace of dimension $N_{\text{ dof}} - d$. Similarly, sets of bond tensions can generate any set of forces on the sites for which the net force on the system vanishes in each of its $d$ components. Because of this, it is not generally possible for the lattice to adjust itself perfectly to swelling a single bond and no others. Instead, for the deformed kagome lattices of section 5 we will use as sources $e$, particular swellings of combinations of $d + 1$ bonds within a crystal cell chosen so as not to couple to the global self stresses ($e_{\mathbf{s}} \cdot e_{\mathbf{h} \cdot \mathbf{n}} = 0$). In the analogous problem of external forces, one must similarly avoid coupling to global translations, which is achieved simply by ensuring that the net force on the lattice is zero. In this way, for the Maxwell lattices we may establish a one-to-one correspondence between the $N_{\text{ dof}} - d$ displacement modes that include no net translations and the like number of bond extensions that do not couple to self stresses. Even this issue does not arise for sources occurring at finite wavenumber, as in section 6.

Finally, we note that the form of the response function obtained in equation (8) depends on how we divide sites and bonds up into crystal cells, a *gauge choice* detailed in appendix A, but that such choices also redefine the extensions and displacements such that physical observables are gauge-independent.

### 3. Topological polarization and directional response

As discussed in appendix B, a Maxwell lattice always possesses some number of zero modes set by the number of bonds in a unit cell: some of them simply grow without bound, rendering them inconsistent with fixed or periodic boundary conditions. For example, for the 1D chain considered in the next section, the sole zero mode occurs at complex wavenumber such that $\exp(i\alpha) = b/a$ and hence grows or shrinks as one moves across the lattice. For a finite lattice with open boundaries a *zero* mode that, e.g., is growing as one moves to the right is exponentially localized on the right edge. Kane and Lubensky [12] discovered the remarkable result that these edge modes are not determined solely by the local coordination number (i.e., by how many bonds were removed) but also by the topological polarization, the winding of the phase of $R(q)$ across the principal directions of the Brillouin Zone:
with $G_j$ the $j$th vector of the reciprocal lattice. This beautiful relationship means that, once gauge and boundary conditions are fixed \[12\], this polarization, the winding of the bulk finite-energy modes, determines the placement of the edge zero modes. Generically, \( R(q) \) may vanish for some $q$ (beyond the trivial zero modes present $q = 0$) and $P^T$ in fact may differ at different wavenumbers \[47\], but we do not address this case in the present work, instead focusing on lattices in which $P_j^T$ has a constant value for any set of loops across the Brillouin Zone.

We focus on the particular case in which every zero mode of a lattice shrinks (or grows) along a lattice direction $a_j$. We call this special case of polarization 'fully polarized'. All zero modes in these lattices have $\text{Im}(q_j) > (\leq) 0$, as shown in figure 3. Thus, by considering the poles of $R(q)^{-1}$ as shown in appendix B, we find that when a bond is swollen in a fully polarized lattice displacements of the polarized lattice will be localized on only the unpolarized side. Repeating the analysis for applied forces and noting that for every zero mode of $R(q)$ there is a self stress of $-R(q)^T$, we see that similarly, when a force is applied in a fully polarized lattice bonds will be stressed on only the polarized side, as shown in figure 1.

### 4. One-dimensional topological chains

We now illustrate the general result of the preceding section by considering the simplest topological response function. We consider the same 1D topological chain of rotors used by Kane and Lubensky \[12\], depicted in figure 4(a). The extension of a bond is related to the displacement of two rotor heads via

$$e_n = au_n - bu_{n-1}. \quad (10)$$

One may show that the relative contributions of the two modes to the extension of a spring is related via simple geometry to the equilibrium angular distance of the rotors to the vertical $\theta$, and the distance between the rotor pivots $d$, relative to the lengths of the rotors (up to an overall rescaling of extensions) by

$$a = d \cos \theta - 2 \sin^2 \theta, \quad (11)$$

$$b = d \cos \theta + 2 \sin^2 \theta. \quad (12)$$

In particular, note that when a rotor is made parallel to one of the bonds one of the coefficients vanishes, while vertical rotors have $a = b$, equivalent to a simple, non-topological spring system.
Beyond the rotor system, this represents the most general uniform one-dimensional mapping from a mode $u$ to a set of energetically costly deformations $e$, also describing for example an origami chain \[19\]. We now consider the effect of swelling a single bond, i.e. requiring $d = e_n$. For such a simple system, we may for the finite case explicitly invert equation (10) and for the infinite case perform a contour integration (see appendix C for details), the latter resulting in

$$u(n) = e_i \left( \frac{b}{a} \right)^{a-n_i} \text{sign}(n - n_i) \Theta[(n - n_i)(a - b)].$$ (13)

We see that the displacements are exponentially localized on only one side of the swollen bond. That side is determined by the location of the zero mode $\exp(iq) = b/a$ in the complex plane, which also determines the topological polarization. The resultant mode is shown in figure 4, in good agreement with the results obtained from direct calculation of systems with periodic boundary conditions and only a handful of sites $N \approx 20$. Indeed, this form of the mode holds when $(b/a)^N \ll 1$ or $\gg 1$, when the mode vanishes before the boundary. For the case $a = b$, as in a simple chain of balls and springs, the swelling of a single spring results in displacements that do not diminish with distance. In all other cases, the bulk mode of equation (13) is precisely (up to rescaling) the zero edge mode obtained by Kane and Lubensky [12]: allowing a single bond in a periodic system to swell or contract freely permits a mode wherein the site to the right (left) experiences a missing bond to its left (right), precisely the condition experienced by a system with open boundaries. Just as the topological polarization allows a single deformation mode to grow in one direction towards the edge, it allows one to grow towards the swollen bond.

5. Generalized kagome lattice

We now consider a two-dimensional, translationally invariant Maxwell lattice, the generalized kagome lattice. The crystal cell consists of three sites connected via six bonds to each other and to three neighboring cells, as shown in figure 1. When we vary the positions of the sites in the basis (in general changing the lengths of the bonds between them) such that pairs of bonds are made parallel to one another, the topological polarization changes abruptly from one integer value to another. However, there is no simple, transparent geometrical signature of finite, as opposed to zero, values of the topological polarization defined in equation (9). Barring
direct calculation, topological polarization may be inferred either through an imbalance of boundary modes or through the bulk directional response we consider here.

Unlike the 1D chain of section 4, the kagome lattice possesses the general features of a Maxwell lattice: a non-scalar rigidity matrix, multiple lattice directions and translational zero modes and attendant states of self stress. In considering the response to the swelling of a bond, the first two features preclude a simple analytic expression as obtained in equation (10), but are still readily addressed numerically. The self stresses, however, require additional consideration.

When we swell a single bond, the pseudoinversion of equation (8) generates a spatially extended projection into the two global self stresses. Instead, to exemplify the fully-polarized result of figures 5(e) and (f), we swell three bonds in a combination selected to cancel out the contribution to the two self-stresses, choosing

\[ e_r = a_1 e_1 + a_2 e_2 + a_3 e_3 \]

such that

\[ e_r \cdot e_r = 1, \]

\[ e_r \cdot e_{SS}^{(1)} = 0, \]

Figure 5. The response of unpolarized (top row) and polarized (bottom row) generalized kagome lattices to swelling bonds (left two columns) and applied forces (right two columns). Hexagons are unit cells with bonds and sites not shown; blue color indicates intensity of displacements while red color indicates intensity of bond extensions/compressions. The central red-highlighted cell has bond swellings externally induced, while blue-highlighted cells have forces applied. Induced swellings are chosen to avoid self stresses as described in the main text so that when bonds are swollen in (b), (f) no tension is induced in other cells. The principal effect of topological polarization is visible in (e), where sites move on only one side of swollen bonds and (h), where bonds stretch only on one side of applied forces. In contrast, the response in (g) is present to some degree in all directions, while in (f) it is equally absent in every direction.
The result is that, as shown in figures 5(b) and (f), none of the remaining bonds in the lattice are extended. Hence, we see the effect of full topological polarization described in the previous section: when bonds are extended sites shift position only to the left (figure 5(e)) whereas when forces are applied to sites bonds are extended and compressed only to the right (figures 1 and 5(h)).

Consideration of topological polarization for the lattice of figure 5(e) indicates that a cell located at \( n_1 \mathbf{a}_1 + n_2 \mathbf{a}_2 \) can only have a zero mode resulting from a swollen bond at \( n_s \mathbf{a}_1 \) when \( n_1 < n_s \). However, we may just as easily consider the basis in which the location of the same cell is given by \( 1 \mathbf{a}_2 + a_1 \mathbf{a}_1 \). In general, the polarization in this new basis cannot be inferred from the polarization in the previous basis. For the lattice in question, the polarization is also \( 1,0 \) in this new basis, providing new information that gives the additional restriction that zero modes propagate only to those cells for which \( n_2 + n_1 < n_s^2 + n_1^2 \), restricting the mode not to one half of the plane but rather to one third, as shown figure 5(e). Hence, topological polarization in the original basis does not fully determine the bulk response of the kagome lattice: the analogous phenomenon in terms of the edge modes would be the observation that the number of edge modes along a cut parallel to \( \mathbf{a}_2 - \mathbf{a}_1 \) is not determined solely by the topological polarization along the two original lattice directions.

In figure 6, we examine more closely a range of lattices passing from sets of parameters well in the unpolarized region to those well in the polarized region. This family of lattices has constant bond lengths, and in fact a particular lattice may be transformed by inducing a uniform twisting of the angle of triangular tiles relative to one another (the Guest mode) through a critical value \( \theta_c \) [17]. As one approaches the critical value from the polarized side, one of the modes begins to delocalize, leading to displacements further and further from the source. Passing through the critical value, the mode begins to re-localize not on the original side of the source but rather on the other. Although for finite size lattices there remains some displacement on the side disallowed by polarization, we see in figure 6(b) that for larger lattices the mode intensity on that side becomes exponentially small even very close to the transition. In figure 6(c) we see that in the polarized region, even close to the transition, the total displacement integrated over the entire transverse direction decreases exponentially as one moves along the polarization direction. Thus, for large systems a wide range of sets of parameters in the polarized region generate qualitatively similar directional response.

Finally, we consider polarized lattices with quenched random bond disorder, as shown in figure 7. By superposition, the same methodology applied to dynamical swelling of a single bond applies to the quenched length disorder. In order to show the effects of polarization clearly, we consider disorder applied to several bonds...
chosen at random, with most bonds held to their uniform values. Unlike for the trios of bonds chosen previously, these random bonds do in fact couple to the uniform self stresses, $e_{ss}$. For a single source, this effect decreases with system size, but for a finite density of disorder, it leads to a finite density of stress in the bonds (red inset). Because the Maxwell lattice has only the uniform states of self stress, the stresses resulting are uniform and entirely delocalized from the changes in bond length, in contrast to fully rigid systems in which stresses are localized to sources. Along with these uniform stresses, there are local displacements in the direction indicated by topological polarization. This principle may be utilized to direct distortions induced by bond disorder away from protected regions via topological polarization.

6. Bond modes and edge modes

As noted in section 4, the mode associated with swelling a bond in the periodic rotor chain resembles the mode on an edge of a chain with open boundary conditions, which in fact reflects a more general phenomenon. We have obtained modes which swell certain designated bonds and no others; these must necessarily be the zero modes of a lattice in which the designated bonds are removed rather than swollen. In the 1D chain, this simply means that removing the bond between the $N$th site and the first site generates an open system with a zero mode on the left or right edge, and this is the same as the mode associated with the swelling of that same bond, which occurs wholly on either its right or left. Associating this displacement mode with a particular swelling or missing bond, we refer to it as a ‘bond mode’.

In translationally-invariant systems, the picture is complicated by the zero modes and self stresses. Our method of choosing a combination of a few bonds at a time to avoid the self stresses does not generate a convenient basis. However, we may consider a bond vector $e$ at a particular wavenumber $q = 0$. Then, for general lattices (i.e., ones without extra self stresses resulting from redundant bonds) $R(q)$ is invertible and we may define a particular bond mode $u = R^{-1}(q) e$. Similarly, we may define ‘force modes’, sets of bond elongations resulting from forces on certain sites: $e = R'(q)^{-1} f$.

From the set of bond modes, we may now make a connection to the edge modes first associated with the topological winding number. Consider two sets of bond elongations, $e(n) = e_1^n \delta_{n_m} \exp(iq_1 n_2)$, where the two bonds are those cut by a line drawn along the second lattice direction, as shown in figure 8(a). Our response function of equation (8) tells us that for $q = 0$ the modes resulting from these sets of bond swellings are exponentially localized either to the left or to the right of $n_i^l$, with the number on either side determined by the topological polarization. As in the 1D chain, these are also the modes on the right and left edges, respectively, of a large system with open boundaries. That is, we may use our bulk topological response function, which sees zero modes shifting from one side of a source to the other, to recapitulate the modes resultant from missing edge bonds that were first associated with the topological winding number, those shown in figure 8(b). This allows us to associate each edge mode to one of the missing bonds in the crystal cell (as in the 1D chain), or more generally with a linear combination of the same, as in figure 8(a).

The directional response of the bulk visible in figures 5(c) and (g) also explicates the edge modes of the lattices. In (e) the mode that stretches only to the left of a swollen bond also is a zero mode on the right edge of a finite lattice, and the lack of any such modes stretching to the right reflects the lack of zero modes on the left edge of such a lattice. Similarly, the swollen bonds only on the right of an applied force in (g) reveal that a force on the
left edge will stretch bonds and cost energy, while one on the right will displace sites without energetic cost. This distinction has been shown \cite{17} to increase the edge stiffness exponentially.

Finally, we note that our bond swelling mode bears a striking resemblance to the soft modes observed in the same class of lattices when subject to a defect with a Burger’s vector aligned in a certain way with the topological polarization \cite{14}. Our systems differ in that bond lengths are changed or forces applied, rather than bond structures actually being changed, and our modes differ as well in that they are precisely zero energy rather than soft. However, there is a reason that Maxwell lattices in particular must respond to different sources in similar manners. Unlike fully rigid lattices, which have extensive numbers of self stresses capable of balancing forces, Maxwell lattices have only the special sub-extensive modes for which $\det R (q) = 0$ or $\det R^T (-q) = 0$. Away from the source, the response of Maxwell lattices must consist of modes drawn from this relatively limited class.

7. Discussion

We have shown that the physical implications of the topological winding number identified by Kane and Lubensky for Maxwell lattices extend beyond the edge modes to the bulk directional response of the lattice to swelling bonds. Similarly, analogously to the edge self stresses, the bulk response to applied forces in fully polarized lattices is for tension to be generated in bonds on only one side of the force. This counter-intuitive result is possible because the site displacements and bond extensions exist in separate, though closely linked, vector spaces (or, in Kane and Lubensky’s formulation, in distinct subspaces of the combined vector space $(\mathbf{u}, \mathbf{e})$). In contrast, for response functions of Maxwell lattices, just as those of conventional elastic systems, the relationship between the forces on sites and their displacements is reciprocal rather than directional.

We have considered the linear, zero-frequency local modes associated with topologically polarized Maxwell lattices. There are, however, a rich variety of other sets of zero modes present in Maxwell lattices. Weyl lattices possess isolated pairs of zero modes at $q = \pm q_m$, \cite{47, 48} while other lattices possess curved lines of zero modes in the Brillouin Zone \cite{17, 19, 49, 50}. Generalized pyrochlore lattices, 3D isostatic systems, possess Weyl lines \cite{51}, curved lines of zero modes. General Maxwell lattices transition from ‘shear-dominant’ states with two lattice directions with vanishing speed of sound to ‘dilation-dominant’ states without these soft directions \cite{17}.
Any of these may display novel mechanical response apparent in their linear Green’s functions. In particular, because topological polarization occurs at finite complex wavenumber it does not appear to influence long-wavelength elasticity in a direct fashion; in contrast, many of these other systems possess a number of zero modes at small wavenumbers. Beyond the Maxwell lattices studied here, there are a host of other topological mechanical systems, characterized by varying dimensionality and symmetry [25], whose electronic analogs are known to have topological Green’s functions [52]. An examination of some or all of their response functions may yield significant results. Indeed, the ‘Maxwell’ condition is not limited to mechanical systems like ball-and-spring lattices or origami. It is, rather, a statement that the two coupled vector spaces, displacements \( \mathbf{u} \) and extensions \( \mathbf{e} \), are of the same size. Any such systems with periodicity and without inversion symmetry may possess similarly directional response functions.

Our formulation linking topological polarization to the directionality of the displacements resulting from applied stresses also suggests a definition of mechanical polarization extending beyond regular, isostatic lattices. It has already been shown that even in regular lattices topological polarization can become wavenumber-dependent [47] and can acquire, e.g., half-integer values in the original formulation as the crystalline symmetry of the lattice is reduced [17]. Self stresses in jammed packings have been shown to have a length scale that diverges as the isostatic point is approached from above [53–55], consistent with the infinitely long-range self stresses in our crystal lattices. Other isostatic systems without any lattice symmetry should have significant response to bond swelling, and one could examine rigidity percolation models representing fibrous systems [56], disordered versions of triangulated origami [19], etc. Perhaps most intriguingly, an ‘allosteric’ algorithm has recently been implemented in disordered isostatic bond networks to alter their structure in order to maximize some fitness criterion [57, 58]—polarization could serve as just such a criterion, raising the question of whether disordered lattices could achieve the same level of directionality as periodic Maxwell lattices. They have already been shown to have edge modes with similar exponential decay [59, 60].

Extending our linear analysis to nonlinear deformations is of clear interest. Initial examinations of two-dimensional systems suggest that the linear mode extends in a regular way to somewhat nonlinear deformations [17, 57]. However, the nonlinear extension edge mode of the 1D rotor chain has been revealed to be a soliton [18, 61], meaning that finite bond swellings can generate and drive modes that spatially separate from the swelling itself. Conversely, origami chains [19] are governed by the same linear map, equation (10), as the rotors but do not possess a soliton, demonstrating that linear analysis cannot predict nonlinear results conclusively. Intriguingly, recent work has shown that nonlinear deformations of systems.

Acknowledgments

The author is grateful to Bryan G Chen, Michael Lawler, Xiaoming Mao and Jim Sethna for helpful conversations. The author gratefully acknowledges support from the the ICAM postdoctoral fellowship, the Bethe/KIC Fellowship, and the National Science Foundation Grant No. NSF DMR-1308089.

Appendix A. Characterization of system via rigidity and compatibility matrices

In this appendix we describe how a linear mechanical system, and in particular a periodic one, can be described in terms of a set of linear maps.

We consider as our archetypal system a network of particles connected via Hookean springs. Consider two such particles initially located at \( \mathbf{r}_1, \mathbf{r}_2 \) and undergoing small, linear displacements \( \mathbf{u}_1, \mathbf{u}_2 \). Only the component of the displacement that projects along the direction pointing from one particle to the other, \( \hat{r}_{12} \), contributes to the spring extension:

\[
e_{12} = \hat{r}_{12} \cdot (\mathbf{u}_2 - \mathbf{u}_1).
\]

We can combine all of our displacements into a single vector describing the total configuration of our system, \( \mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{N_{\text{par}}}) \), and our displacements via \( \mathbf{e} = (e_1, e_2, \ldots, e_{N_{\text{par}}}) \). In this way, all of the above relationships may be expressed in terms of a single linear map, the rigidity (also ‘compatibility’ or ‘kinetic’) matrix:

\[
\mathbf{e} = \mathbf{R} \mathbf{u}.
\]

Similarly, we may determine the forces on particles from their spring tensions. For springs with unit spring constant, the tension is simply the (negative) extension and so the force on the first site, as defined above, is

\[
\mathbf{f}_1 = e_{12} \hat{r}_{12}.
\]

As before, we can combine these into a single map between bond extensions and forces on sites, known as the equilibrium matrix. However, because of the above relationships the equilibrium matrix is simply the transpose of the rigidity matrix:
\( f = R^T e. \)

Combined, these give us the full dynamics. In terms of the dynamical matrix, \( D = R^T R \), the energy is

\[
E = \frac{1}{2} e^2 = \frac{1}{2} (R u)^2 = \frac{1}{2} u^T D u.
\]

And the relationship between forces and displacements is simply

\[
f = D u.
\]

We now wish to introduce periodic structure, with a lattice subject to periodic boundary conditions running \( N_j \) sites along lattice vector \( a_j \) in the \( j \)th direction. As such, the \( n \)th particle in the \( n \)th cell is:

\[
r_n^a = b_n^a + \sum_j n_j a_j.
\]

Assuming periodic modes of the form \( u_n = u \exp(iq \cdot n) \) with \( n_j = 0, 1, \ldots, N_j - 1 \) and \( q_j = 0, 1, \ldots, N_j - 1 \) means that our rigidity matrix above will become a function of wavenumber, since springs connect to particles in different cells. The wavenumber is flipped when relating forces to extensions, and \( R \rightarrow R_q, R^T \rightarrow R_q^T \).

The advantage of expressing the systems mechanical relations in this way (the same advantage present for many local, linear relationships) is that it diagonalizes our relations, such that \( u_q = R_q e_q \) rather than \( u_q = \sum_q R_q q e_q \). In this way, a large system is reduced to a large number of wavenumber-dependent small systems. Thus, we may invert the above relationships to determine the displacements resulting from a set of forces or of bond extensions, or the extensions resulting from forces:

\[
u_n = \frac{1}{\prod_j n_j} \sum_{q \neq 0} e^{i(q \cdot n)} R_q^{-1} \delta e_q,
\]

\[
u_n = \frac{1}{\prod_j n_j} \sum_{q \neq 0} e^{i(q \cdot n)} D_q^{-1} \delta f_q,
\]

\[
e_n = \frac{1}{\prod_j n_j} \sum_{q \neq 0} e^{i(q \cdot n)} (R_q^T)^{-1} \delta f_q.
\]

Here, the \( '−q \)' indicates a Moore–Penrose generalized pseudo-inverse. It is easily calculated in terms of the singular value decomposition, even for non-square matrices. It gives either the actual solution to the original linear relationships or the solution that minimizes the 'error'.

We now consider a large system, \( N_j \gg 1 \). This does not, however, imply that the distance from source to target, \( n - n_s \), is large. The relationships then become

\[
u(n) = \frac{1}{(2\pi)^d} \int_{-\infty}^{\infty} d^d q e^{i(q \cdot n)} R^{-1} \delta \left( \Phi_q \right) e_q,
\]

\[
u(n) = \frac{1}{(2\pi)^d} \int_{-\infty}^{\infty} d^d q e^{i(q \cdot n)} D^{-1} \delta \left( \Phi_q \right) f_q,
\]

\[
e(n) = \frac{1}{(2\pi)^d} \int_{-\infty}^{\infty} d^d q e^{i(q \cdot n)} (R_q^T)^{-1} \delta \left( \Phi_q \right) f_q.
\]

In defining our periodic structure, we have some freedom in including bonds (or sites) in certain cells. We might include a bond that connects a site with a site in another cell to its right in either the first or the second cell, leading alternately to

\[
e_{12} = \hat{r}_{12} \cdot (u_2 - e^{-i\theta} u_1),
\]

\[
e_{21} = \hat{r}_{12} \cdot (e^{i\theta} u_2 - u_1).
\]

This gauge choice leads to a \( q \)-dependent rescaling of extensions, displacements, and the rigidity matrix of the form

\[
u(q) \rightarrow \Phi(q) \nu(q),
\]

\[
e(q) \rightarrow \Phi(q) e(q),
\]

\[
R(q) \rightarrow \Phi(q) R(q) \Phi^{-1}(q).
\]
Appendix B. Topology, zero modes and response functions

In this appendix, we show how the direction of the response to a point source is determined by the topological polarization. To begin, we review how this polarization is related to the edge modes.

Consider a spring in crystal cell \( n \) connecting one of that cell’s sites to a site in the cell whose first index is one higher. Based on our mode, this implies a spring extension of the form

\[
e_{12}(q) = \tilde{r}_{12}(e^{i\phi}u_{2}(q) - u_{1}(q)).
\]

(35)

In this way, we see that the rigidity matrix’s elements are simply powers of \( \{z_{j}\} \equiv \{e^{i\phi}\} \). These complex numbers must have unit magnitude to satisfy periodicity or normalization. However, for finite lattices with open boundary conditions, missing or swollen bonds periodicity is broken and \( |z_{j}| \neq 1 \) modes become relevant. The condition for such zero modes is \( \det(R) = 0 \), leading to a polynomial in \( \{z_{j}\} \) whose order is set by the number of bonds between unit cells in various lattice directions.

In this way, we may readily obtain the modes existing on edges with missing bonds parallel to the \( a_{j} \) lattice direction. We merely choose particular values of the remaining \( z_{-j} \) (in particular, we can use all \( |z_{-j}| = 1 \) modes as a basis of the lattice) and solve for the values of \( z_{j} \) that satisfy \( \det(R) = 0 \). The fundamental theorem of algebra requires that we obtain one such mode (on one edge for \( |z_{j}| < 1 \), on the other for \( |z_{j}| > 1 \) or in the bulk for \( |z_{j}| = 1 \)) for each missing bond, ensuring that each edge mode permitted by the index theorem given in the main text.

Although obtaining the exact complex wavenumber requires solving the above algebraic equation, we can determine how many zero modes rest on which edge purely by considering the winding number of \( \det(R) = 0 \) across the bulk modes. The result is a topological polarization

\[
P^{T}_{j} = -\frac{1}{2\pi i} \int_{q_{j} \rightarrow q_{j} + G_{j}} \frac{\partial}{\partial q_{j}} \log \det[R(q)],
\]

(36)

which gives the number of zero modes on the \(+a_{j}\) edge (and the opposite on the \(-a_{j}\) edge) in excess of that indicated by the local coordination number. This formulation follows from the Argument Principle of complex analysis, which in turn depends on the number of poles enclosed by the contour along \( |z_{j}| = 1 \). Those poles depend on how we choose to assign bonds to crystal cells (a gauge choice); our above result applies to a symmetrical choice and would otherwise need to be shifted.

Now, though, we can set aside all notion of an edge and simply consider evaluating the response functions of the previous appendix, which similarly require an integral around \(-\pi < \phi \leq \pi \) or around the contour \( |z_{j}| = 1 \). This is evaluated using the contour of the second figure of the main text. Ignoring the contributions at \( q_{j} = 0 \) (which are dealt with in detail in the main text), the response comes from the poles of \( R^{-1}(q) \), which are the zeroes of \( R(q) \) (the elements of the inverse of an \( n \times n \) matrix are combinations of products of \( n - 1 \) elements of the original matrix divided by its determinant).

Depending on which side of the source we wish to calculate the response, it is limited to zeroes at \( |z_{j}| > ( < ) 1 \) (for \( n_{j} < ( > ) n_{j}' \)). Hence, the bulk response depends on the same non-periodic zero modes as the edge modes and the topological polarization. In particular, a lattice that is fully polarized in the sense of, e.g., \( |z_{j}| > 1 \) for all its modes, will have displacements responding to a swollen bond only on the \(-a_{j}\) side of the bond, growing until they reach the bond and then vanishing.

The analysis is similar for the equilibrium matrix, \( R^{T}(-q) \), which has zeroes at \( z_{j} \rightarrow z_{j}^{-1} \), revealing that in the above case spring extensions can only be induced on the \( a_{j} \) side of forces applied to sites. However, in this case displacements are generated as well, as indeed is the only way to induce spring extensions absent externally-imposed swelling. Similarly, the dynamical matrix \( D(q) = R^{T}(-q)R^{T}(q) \) has pairs of zero modes at \( z_{j} \) and self stresses at \( z_{j}^{-1} \), both of which correspond to states of mechanical equilibrium.

Appendix C. The one-dimensional chain

In this appendix, we present a more detailed derivation of the mechanical response of the 1D topological chain, whose rigidity matrix is given by the relation between extensions \( e_{n} \) and site modes \( u_{n} \)

\[
e_{n} = au_{n} - bu_{n-1}.
\]

(37)

More generally, this represents a one-dimensional lattice system with the most general uniform linear mapping from a mode \( u \) to a set of energetically costly deformations \( e \). Requiring that no such deformations occur leads to a zero mode of the form

\[
u_{n} = \left( \frac{b}{a} \right)^{n-1} u_{0}.
\]

(38)
For a system such as a simple ball and spring chain, which we term 'critical', $a = b$ and there is a uniform zero mode, in that case a mode of translation. More generally, though, this is an edge mode, exponentially decaying as one moves left (right) for $b/a > 1$ ($b/a < 1$). Such an edge mode is possible in a finite system consisting of $N$ modes and $N - 1$ interactions between them. We, however, consider systems subject to periodic boundary conditions, in which no such zero modes exist and the above equation is invertible. The result, as may be verified by direct algebraic evaluation, is

$$u_n = \frac{1}{\alpha} \sum_{n'=1}^{N} \sum_{n''=1}^{N} \alpha^{n'-n''} \alpha^{n''} \epsilon_{n''}. \quad (39)$$

This complete inversion, which is impossible for the general case in which zero modes yet exist, allows us to construct the explicit response of the system to a localized source $\epsilon = \epsilon_{n'}$. Such a source may be thought of as changing the length of a particular spring, either by altering its equilibrium length or by applying a force. The resultant displacement is

$$u_n = \frac{c_\epsilon}{a} \left( \frac{b}{a} \right)^{n-n_0} \begin{cases} \frac{1}{1 - (b/a)^n} & n \geq n_0, \\ \frac{1}{(a/b)^n - 1} & n < n_0. \end{cases} \quad (40)$$

Although this direct method of solution is perfectly satisfactory when it exists, it does not illustrate the topological nature of the system, which lies in the winding of its phonon bands. To see this, we work in reciprocal space, where our rigidity matrix is simply $R(q) = a - b \exp(-2\pi i q)$. Using the orthonormal basis of reciprocal space, the mode resultant from the local bond swelling is

$$u_n = \frac{1}{N} \sum_{q} \exp[2\pi i q (n-n_0) / N] \left[ a - b \exp(-2\pi i q / N) \right] \epsilon_{q}. \quad (41)$$

In the limit of large system sizes, this becomes

$$u_n = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp[i(q(n-n_0)) / a - b \exp(-iq)] d\epsilon_{q}. \quad (42)$$

This is the simplest possible form for our contour integration technology, leading to a mode

$$u_n = \frac{c_\epsilon}{a} \left( \frac{b}{a} \right)^{n-n_0} \text{sign}(n-n_0) \Theta[(n-n_0)(a-b)]. \quad (43)$$

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