Behavior of Solutions to An Initial Boundary Value Problem for a Hyperbolic System With Relaxation

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Abstract: The behavior of solutions to an initial boundary value problem for a hyperbolic system with relaxation is studied when the relaxation parameter is small, by using the method of Fourier Series and the energy method.

1 Introduction

In this paper, we consider the initial boundary value problem for the following hyperbolic system with relaxation:

\[ u_t + v_x = 0 \]  \hspace{1cm} (1.1)

\[ v_t + a^2 u_x = \frac{bu - v}{\epsilon} \] \hspace{1cm} (1.2)

with the boundary condition:

\[ u(0,t) = u(1,t) = 0, \] \hspace{1cm} (1.3)

and the initial condition:

\[ u(x,0) = f(x) \] \hspace{1cm} (1.4)

\[ v(x,0) = g(x) \] \hspace{1cm} (1.5)

where \( a > 0 \) and \( b \) are constants, and \( \epsilon > 0 \) is the relaxation parameter. We assume that

\[ |b| < a, \] \hspace{1cm} (1.6)

such that the so called “subcharacteristic condition” is satisfied. System (1.1) can be transform to:

\[ u_{tt} - a^2 u_{xx} + \frac{1}{\epsilon} (bu_x + u_t) = 0 \] \hspace{1cm} (1.7)

with the boundary and initial conditions:

\[ u(0,t) = u(1,t) = 0, \] \hspace{1cm} (1.8)

and the initial condition:

\[ u(x,0) = f(x), \quad u_t(x,0) = -g'(x). \] \hspace{1cm} (1.9)

In order to match the initial and boundary conditions, we require that

\[ f(0) = f(1), \quad g'(0) = g'(1) = 0. \] \hspace{1cm} (1.10)

Formally, as \( \epsilon \to 0 \), system (1.1) is relaxed to the equilibrium

\[ u_t + bu_x = 0, \] \hspace{1cm} (1.11)

\[ v = bu. \] \hspace{1cm} (1.12)
System \((1.1)\) is related to a general relaxation system of Jin-Xin model \([3]\), for which the asymptotic behavior as the relaxation parameter tends to zero of solutions to the initial value problem was discussed in \([1, 5, 6, 7]\). The asymptotic behavior as the relaxation parameter tends to zero for the initial boundary problems in a quarter plane in \((x, t)\) was discussed in \([8, 9, 10]\) with one boundary \(x = 0\). In this paper, we are interested in the behavior of solutions in a \((x, t)\)-strip, \(0 \leq x \leq 1, \ t \geq 0\). It should be noted that the Fourier-Laplace transformation is used in \([9]\) to study the problem in a quarter plane. For the problem in a strip studied in this paper, we have two boundaries, \(x = 0\) and \(x = 1\). This is the main difference of the problem studied in this paper, compared with those in a quarter plane. For example, the Fourier-Laplace transformation is not applicable to our problem any more. Instead, we use the Fourier series method.

The boundary layer behavior is a main concern of this paper, which is shown by the Fourier series solution.

Relaxation phenomena are important in many physical situations. For more background, please refer to \([2, 4]\).

2 Fourier Series Solution.

2.1 Solution Formula

**Theorem 2.1.** The Fourier series solution \(u(x, t) = \sum_{n=1}^{\infty} T_n(t)X_n(x)\) of problem \((1.7), (1.8)\) and \((1.9)\) is given by:

\[
u(x, t) = \sum_{n=1}^{k} e^{\frac{b}{2a^{2}\pi x}} \sin(n\pi x) \left( c_n e^{\alpha_n^{-t}} + d_n e^{\alpha_n^{+t}} \right) + \sum_{n=k+1}^{\infty} e^{\frac{b}{2a^{2}\pi x}} \sin(n\pi x) e^{-\frac{1}{2\epsilon} t} \left[ c_n \cos(\beta_n t) + d_n \sin(\beta_n t) \right]
\]

where \(k = \lfloor \sqrt{\frac{a^{2}-b^{2}}{4a^{2}\pi^{2}}} \rfloor\),

\[
c_n = \int_{0}^{1} f(x) e^{\frac{-b}{2a^{2}\pi x}} \sin(n\pi x) dx - \frac{2\epsilon}{\sqrt{(1 - \frac{b^{2}}{a^{2}})} - 4a^{2}n^{2}\epsilon^{2}\pi^{2}} \int_{0}^{1} \left( \frac{f(x)}{2\epsilon} - g'(x) \right) e^{-\frac{b}{2a^{2}\pi x}} \sin(n\pi x) dx,
\]

\[
d_n = \int_{0}^{1} f(x) e^{\frac{-b}{2a^{2}\pi x}} \sin(n\pi x) dx + \frac{2\epsilon}{\sqrt{(1 - \frac{b^{2}}{a^{2}})} - 4a^{2}n^{2}\epsilon^{2}\pi^{2}} \int_{0}^{1} \left( \frac{f(x)}{2\epsilon} - g'(x) \right) e^{-\frac{b}{2a^{2}\pi x}} \sin(n\pi x) dx
\]

\[
\alpha_{n\pm} = \frac{-1 \pm \sqrt{1 - \frac{b^{2}}{a^{2}}} - 4a^{2}n^{2}\epsilon^{2}\pi^{2}}{2\epsilon}
\]

(2.1)
for $n \leq k$, and

$$c_n = 2 \int_0^1 f(x) e^{-\frac{b}{2\pi x}} \sin(n\pi x) \, dx \quad (2.5)$$

$$d_n = \frac{4\epsilon}{\sqrt{(\frac{b^2}{a^2} - 1) + 4a^2n^2\epsilon^2\pi^2}} \int_0^1 \left( \frac{f(x)}{2\epsilon} - g'(x) \right) e^{-\frac{b}{2\pi x}} \sin(n\pi x) \, dx, \quad (2.6)$$

$$\beta_n = \frac{\sqrt{(\frac{b^2}{a^2} - 1) + 4a^2n^2\epsilon^2\pi^2}}{2\epsilon} \quad (2.7)$$

for $n \geq k + 1$.

ii) If $\sqrt{\frac{a^2-b^2}{4\epsilon^2a^4\pi^2}}$ is an integer, let $k = \sqrt{\frac{a^2-b^2}{4\epsilon^2a^4\pi^2}}$. Then

$$u(x,t) = \sum_{n=1}^{k-1} e^{\frac{b}{2\pi x}} \sin(n\pi x) \left( c_n e^{\alpha_n t} + d_n e^{\gamma_n t} \right) + 2e^{-\frac{b}{2\pi t}} \int_0^1 f(x) \sin(k\pi x) \, dx + \int_0^1 \frac{-g'(x)}{e^{\frac{b}{2\pi x}}} \sin(k\pi x) \, dx e^{-\frac{b}{2\pi t}} + e^{-\frac{b}{2\pi t}} \sum_{n=k+1}^{\infty} e^{\frac{b}{2\pi x}} \sin(n\pi x) \left[ c_n \cos(\beta_n t) + d_n \sin(\beta_n t) \right] \quad (2.8)$$

where $c_n, d_n$ and $\beta_n$ are the same as in case i).

**Proof**

For the initial boundary value problem (1.7), (1.8) and (1.9), we use the separation of variables and let

$$u(x,t) = X(x)T(t). \quad (2.9)$$

Substitute this in (1.7) to get

$$T''(t)X(x) - a^2T(t)X''(x) + \frac{1}{\epsilon} (bT(t)X'(x) + T'(t)X(x)) = 0.$$ 

Thus,

$$\frac{T''(t)}{T(t)} - a^2 \frac{X''(x)}{X(x)} + \frac{b}{\epsilon} \frac{X'(x)}{X(x)} + \frac{1}{\epsilon} \frac{T'(t)}{T(t)} = 0. \quad (2.10)$$

Reorganize this and let

$$\frac{T''(t)}{T(t)} + \frac{1}{\epsilon} \frac{T'(t)}{T(t)} = a^2 \frac{X''(x)}{X(x)} - \frac{b}{\epsilon} \frac{X'(x)}{X(x)} = \lambda \quad (2.10)$$

$\lambda$ is a constant.

From the left side of (2.10), we get an ODE of $T(t)$

$$T''(t) + \frac{1}{\epsilon} T'(t) - \lambda T(t) = 0 \quad (2.11)$$

and the right side turns out to be an ODE of $X(x)$

$$a^2X''(x) - \frac{b}{\epsilon}X'(x) - \lambda X(x) = 0 \quad (2.12)$$
with boundary condition

\[ X(0) = X(1) = 0. \quad (2.13) \]

The characteristic equation for (2.12) is

\[ a^2 \alpha^2 - \frac{b}{\epsilon} \alpha - \lambda = 0. \quad (2.14) \]

Then we have the following cases:

Case 1: \( \Delta > 0 \)

\[ \left( -\frac{b}{\epsilon} \right)^2 + 4a^2 \lambda > 0 \]

\[ \lambda > -\frac{b^2}{4a^2 \epsilon^2} \]

then (2.14) has two roots \( \alpha_{\pm} \) given by (2.4), and

\[ X(x) = C_1 e^{\alpha_{-}x} + C_2 e^{\alpha_{+}x}, \]

where \( C_1 \) and \( C_2 \) are constants. By (2.13), we have

\[ X(0) = C_1 + C_2 = 0, X(1) = C_1 e^{\alpha_{-}} + C_2 e^{\alpha_{+}} = 0 \]

\( \Rightarrow C_1 = C_2 = 0. \)

Case 2: \( \Delta = 0 \)

\[ \left( -\frac{b}{\epsilon} \right)^2 + 4a^2 \lambda = 0 \]

\[ \lambda = -\frac{b^2}{4a^2 \epsilon^2} \]

then (2.14) has only one root

\[ \alpha = \frac{b}{2a^2 \epsilon} \]

\[ X(x) = C_1 e^{\frac{b^2 x}{2a^2 \epsilon}} + C_2 x e^{\frac{b^2 x}{2a^2 \epsilon}}, \]

for some constants \( C_1 \) and \( C_2 \). Just like the Case 1, we get \( C_1 = C_2 = 0. \)

Case 3: \( \Delta < 0 \)

\[ \left( -\frac{b}{\epsilon} \right)^2 + 4a^2 \lambda < 0 \]

\[ \lambda < -\frac{b^2}{4a^2 \epsilon^2} \]

then (2.14) has two complex roots

\[ \alpha = \frac{b}{2a^2 \epsilon} \pm i \sqrt{\frac{-(\frac{b}{\epsilon})^2 - 4a^2 \lambda}{2a^2}} \]

\[ X(x) = e^{\frac{b}{2a^2 \epsilon} x} \left( C_1 \cos \left( \frac{\sqrt{-(\frac{b}{\epsilon})^2 - 4a^2 \lambda}}{2a^2} x \right) + C_2 \sin \left( \frac{\sqrt{-(\frac{b}{\epsilon})^2 - 4a^2 \lambda}}{2a^2} x \right) \right) \]

\[ X(0) = 0 \Rightarrow C_1 = 0 \]
\[
X(1) = C_2 e^{\frac{\epsilon}{2a^2}} \sin \left( \frac{\sqrt{-\left(\frac{b^2}{a^2}\right)^2 - 4a^2\lambda}}{2a^2} \right) = 0
\]

\[
\Rightarrow \sin \left( \frac{\sqrt{-\left(\frac{b^2}{a^2}\right)^2 - 4a^2\lambda}}{2a^2} \right) = 0 \Rightarrow \sqrt{-\left(\frac{b^2}{a^2}\right)^2 - 4a^2\lambda} = n\pi, n = 1, 2, 3, \ldots
\]

\[
\Rightarrow \lambda_n = -a^2n^2\pi^2 - \frac{b^2}{4a^2\epsilon^2}.
\]

Therefore, only Case 3 fits the condition, so

\[
X_n = e^{\frac{\epsilon}{2a^2} x} \sin (n\pi x) \quad (2.15)
\]

Next, solve (2.11) with

\[
\lambda_n = -a^2n^2\pi^2 - \frac{b^2}{4a^2\epsilon^2}
\]

its characteristic equation is:

\[
\alpha^2 + \frac{1}{\epsilon} \alpha + \left( a^2n^2\pi^2 + \frac{b^2}{4a^2\epsilon^2} \right) = 0 \quad (2.16)
\]

If \( \Delta < 0 \),

\[
T_n(t) = e^{-\frac{1}{2\epsilon} t} \left[ c_n \cos \left( \frac{\sqrt{\left(\frac{b^2}{a^2}\right) - 1} + 4a^2n^2\epsilon^2\pi^2}{2\epsilon} t \right) + d_n \sin \left( \frac{\sqrt{\left(\frac{b^2}{a^2}\right) - 1} + 4a^2n^2\epsilon^2\pi^2}{2\epsilon} t \right) \right] \quad (2.17)
\]

If \( \Delta > 0 \), two roots of (2.16) are

\[
\alpha_{n\pm} = \frac{-1 \pm \sqrt{(1 - \frac{b^2}{a^2}) - 4a^2n^2\epsilon^2\pi^2}}{2\epsilon}
\]

\[
T_n(t) = c_n e^{\alpha_{n-} t} + d_n e^{\alpha_{n+} t} \quad (2.18)
\]

We have the following cases:

Case 1: \( \sqrt{\frac{a^2-b^2}{4\epsilon^2a^2\pi^2}} \) is not an integer
when \( n \leq \left\lfloor \sqrt{\frac{a^2-b^2}{4\epsilon^2 a^2 \pi^2}} \right\rfloor, \Delta > 0 \)

\[
T_n(t) = (2.18)
\]

When \( n > \left\lfloor \sqrt{\frac{a^2-b^2}{4\epsilon^2 a^2 \pi^2}} \right\rfloor, \Delta < 0 \)

\[
T_n(t) = (2.17)
\]

Case 2: \( \sqrt{\frac{a^2-b^2}{4\epsilon^2a^2\pi^2}} \) is an integer
when \( n < \sqrt{\frac{a^2-b^2}{4\epsilon^2 a^2 \pi^2}}, \Delta > 0 \)

\[
T_n(t) = (2.18)
\]
when \( n = \sqrt{\frac{a^2 - b^2}{4\varepsilon^2a^2\pi^2}} \), \( \Delta = 0 \)

\[
T_n(t) = c_n e^{-\frac{t}{\pi}} + d_n t e^{-\frac{t}{\pi}}
\]

when \( n > \sqrt{\frac{a^2 - b^2}{4\varepsilon^2a^2\pi^2}} \), \( \Delta < 0 \)

\[
T_n(t) = (2.17)
\]

For case 1, let \( k = [\sqrt{\frac{a^2 - b^2}{4\varepsilon^2a^2\pi^2}}] \)

\[
u(x,t) = \sum_{n=1}^{\infty} X_n(x) T_n(t)
\]

With initial condition (1.4),

\[
u(x,0) = e^{\frac{b}{2\varepsilon}x} \sum_{n=1}^{k} (c_n + d_n) \sin(n\pi x) + e^{\frac{b}{2\varepsilon}x} \sum_{n=k+1}^{\infty} c_n \sin(n\pi x) = f(x)
\]

We have

\[
c_n + d_n = 2 \int_0^1 \frac{f(x)}{e^{\frac{b}{2\varepsilon}x}} \sin(n\pi x) dx \quad n = 1, 2, 3...k \quad (2.19)
\]

\[
c_n = 2 \int_0^1 \frac{f(x)}{e^{\frac{b}{2\varepsilon}x}} \sin(n\pi x) dx \quad n = k + 1, k + 2, k + 3... \quad (2.20)
\]

Let

\[
\beta_n = \sqrt{\left(\frac{b^2}{a^2} - 1\right) + 4a^2n^2\varepsilon^2n^2} \quad (2.21)
\]

and substitute

\[
\cos(\beta_nt) = \frac{e^{i\beta_nt} + e^{-i\beta_nt}}{2} \quad (2.22)
\]

\[
\sin(\beta_nt) = \frac{e^{i\beta_nt} - e^{-i\beta_nt}}{2i} \quad (2.23)
\]

\[
u(x,0) \text{ can be written as}
\]

\[
u(x,0) = e^{\frac{b}{2\varepsilon}x} \sum_{n=1}^{k} (c_n + d_n) \sin(n\pi x)
\]

\[
+ e^{\frac{b}{2\varepsilon}x} \sum_{n=k+1}^{\infty} \left[ \frac{c_n - id_n}{2} e^{i\beta_nt} + \frac{c_n + id_n}{2} e^{-i\beta_nt} \right] e^{-\frac{1}{2\varepsilon}t} \quad (2.24)
\]

With initial condition (1.5),

\[
u_t(x,0) = e^{\frac{b}{2\varepsilon}x} \sum_{n=1}^{k} (c_n\alpha_{n-} + d_n\alpha_{n+}) \sin(n\pi x)
\]

\[
+ e^{\frac{b}{2\varepsilon}x} \sum_{n=k+1}^{\infty} \left[ \frac{c_n - id_n}{2} (i\beta_n - \frac{1}{2\varepsilon}) + \frac{c_n + id_n}{2} (-i\beta_n - \frac{1}{2\varepsilon}) \right] \quad (2.25)
\]

\[
= -g'(x)
\]
We have
\[ c_n \alpha_n^- + d_n \alpha_n^+ = 2 \int_0^1 -g'(x)e^{-\frac{b}{2\pi \epsilon} x} \sin(n\pi x)dx \quad n \leq k \] (2.26)
\[ \frac{c_n - id_n}{2} (i\beta_n - \frac{1}{2\epsilon}) + \frac{c_n + id_n}{2} (-i\beta_n - \frac{1}{2\epsilon}) \]
\[ = 2 \int_0^1 -g'(x)e^{-\frac{b}{2\pi \epsilon} x} \sin(n\pi x)dx \quad n \geq k + 1 \] (2.27)

This implies
\[ -\frac{1}{2\epsilon} c_n + d_n \beta_n = 2 \int_0^1 -g'(x)e^{-\frac{b}{2\pi \epsilon} x} \sin(n\pi x)dx, \quad n \geq k + 1 \] (2.28)

When \( n \leq k \), we obtain \( c_n \) and \( d_n \) given by (2.2) and (2.5) from (2.19) and (2.26). When \( n \geq k + 1 \), the formula for \( c_n \) and \( d_n \) follows from (2.20) and (2.28), and (2.41) is proved.

Case ii) in Theorem 2.2 can be shown similarly.

### 2.2 Analysis of the Solutions of Fourier Series

We prove the following Theorem

**Theorem 2.2.** For the solution given in Theorem 2.2, let \( u_n(x,t) = T_n(t)X_n(x) =: A_n(x,t)\sin(n\pi x) \). Then when \( \epsilon \) is sufficiently small

i) For \( b > 0 \),
\[ |A_1(x,t)| \leq A \epsilon \exp \left( \frac{b}{2a^2\epsilon} \left( x - \frac{a^2}{b} \left( 1 - \sqrt{1 - \frac{b^2}{a^2} - 4a^2\epsilon^2\pi^2} \right) t \right) \right), \] (2.29)

for small \( \epsilon \), and for \( k = \lfloor \sqrt{\frac{a^2 - b^2}{4a^2\pi^2}} \rfloor \), \( m \geq 1 \), and \( \epsilon \leq \frac{\delta}{m} \) for some small \( \delta > 0 \),
\[ |A_{k-m}(x,t)| \leq \frac{B \sqrt{\epsilon}}{\sqrt{m}} \exp \left( \frac{b}{2a^2\epsilon} \left( x - \frac{a^2}{b} \left( 1 - \sqrt{1 - \frac{b^2}{a^2} - 4a^2\epsilon^2(k-m)^2\pi^2} \right) t \right) \right), \] (2.30)
\[ |A_{k+m}(x,t)| \leq \frac{C \sqrt{\epsilon}}{\sqrt{m}} \exp \left( \frac{b}{2a^2\epsilon} \left( x - \frac{a^2}{b} t \right) \right), \] (2.31)

for \( 0 \leq x \leq 1 \) and \( t > 0 \), where \( A \), \( B \) and \( C \) are constants independent of \( \epsilon \).

**Remark 2.3.** The case for \( b < 0 \) can be discussed similarly, by replacing \( x \) by \( 1 - x \).

**Remark 2.4.** \( A_n \) \( (n \geq 1) \) are the amplitudes of Fourier modes. The case for \( b < 0 \) can be discussed similarly, by replacing \( x \) by \( 1 - x \). For \( b > 0 \), by (2.29), and (2.30), we have, for \( n < k \), that \( A_n(x,t) \to 0 \) as \( \epsilon \to 0 \) for \( x - \frac{a^2}{b} \left( 1 - \sqrt{1 - \frac{b^2}{a^2}} \right) t \) < 0.

For \( n > k \), we have that \( A_n(x,t) \to 0 \) as \( \epsilon \to 0 \) for \( x - \frac{a^2}{b} t \) < 0.
Proof of Theorem 2.2. For \( n = 1 \), note that
\[
\alpha_{1+} = -1 \pm \sqrt{(1 - \frac{\nu^2}{\sigma^2}) - 4a^2e^2\pi^2} \over 2\epsilon, \tag{2.32}
\]
\[
c_1 = \int_0^1 f(x)e^{-\frac{b}{2\pi \sigma^2}x} \sin(\pi x)dx - \frac{2\epsilon}{\sqrt{(1 - \frac{\nu^2}{\sigma^2}) - 4a^2e^2\pi^2}} \int_0^1 \left( \frac{f(x)}{2\epsilon} - g'(x) \right) e^{-\frac{b}{2\pi \sigma^2}x} \sin(\pi x)dx, \tag{2.33}
\]
\[
d_1 = \int_0^1 f(x)e^{-\frac{b}{2\pi \sigma^2}x} \sin(\pi x)dx + \frac{2\epsilon}{\sqrt{(1 - \frac{\nu^2}{\sigma^2}) - 4a^2e^2\pi^2}} \int_0^1 \left( \frac{f(x)}{2\epsilon} - g'(x) \right) e^{-\frac{b}{2\pi \sigma^2}x} \sin(\pi x)dx, \tag{2.34}
\]
\[
u_1(x,t) = e^{\frac{b}{2\pi \sigma^2}x} \sin(\pi x) \left( c_1e^{\alpha_{1-t}} + d_1e^{\alpha_{1+t}} \right) = A_1(x,t) \sin(\pi x). \tag{2.35}
\]
Apparently,
\[
|A_1|(x,t) \leq (|c_1| + |d_1|)e^{\frac{b}{2\pi \sigma^2}x + \alpha_{1-t}}, \tag{2.36}
\]
for \( 0 \leq x \leq 1 \) and \( t > 0 \). We estimate \( c_1 \) and \( d_1 \) as follows,
\[
\left| \int_0^1 f(x)e^{-\frac{b}{2\pi \sigma^2}x} \sin(\pi x)dx \right|
\leq \max_{x \in [0,1]} |f(x)| \int_0^1 e^{-\frac{b}{2\pi \sigma^2}x}dx \leq \frac{2a^2\epsilon}{b} \max_{x \in [0,1]} |f(x)|, \tag{2.37}
\]
\[
\left| \frac{2\epsilon}{\sqrt{(1 - \frac{\nu^2}{\sigma^2}) - 4a^2e^2\pi^2}} \int_0^1 \left( \frac{f(x)}{2\epsilon} - g'(x) \right) e^{-\frac{b}{2\pi \sigma^2}x} \sin(\pi x)dx \right|
\leq \frac{\sqrt{2}}{\sqrt{(1 - \frac{\nu^2}{\sigma^2})}} \int_0^1 |f(x)|e^{-\frac{b}{2\pi \sigma^2}x}dx
\leq \frac{\sqrt{2}}{\sqrt{(1 - \frac{\nu^2}{\sigma^2})}} \frac{2a^2\epsilon}{b} \max_{x \in [0,1]} |f(x)|, \tag{2.38}
\]
\[
\left| \frac{2\epsilon}{\sqrt{(1 - \frac{\nu^2}{\sigma^2}) - 4a^2e^2\pi^2}} \int_0^1 g'(x)e^{-\frac{b}{2\pi \sigma^2}x} \sin(\pi x)dx \right|
\leq \frac{\sqrt{2}}{\sqrt{(1 - \frac{\nu^2}{\sigma^2})}} \frac{2a^2\epsilon}{b} \max_{x \in [0,1]} |g'(x)|, \tag{2.39}
\]
for small \( \epsilon \). Obviously,
\[
\alpha_{1-} \leq -1 + \sqrt{(1 - \frac{\nu^2}{\sigma^2})} \over 2\epsilon. \tag{2.40}
\]
(2.29) follows from the above estimates then.
\[ u(x, t) = \sum_{n=1}^{k} e^{\frac{b}{2\pi^2 x^2} \sin(n\pi x)} \left( c_n e^{\alpha_n t} + d_n e^{\alpha_{n+1} t} \right) + \sum_{n=k+1}^{\infty} e^{\frac{b}{2\pi^2 x^2} \sin(n\pi x)} e^{-\frac{t}{4\pi^2}} \left[ c_n \cos(\beta_n t) + d_n \sin(\beta_n t) \right] \quad (2.41) \]

where \( k = \lfloor \sqrt{\frac{a^2-b^2}{4\pi^2 \pi^2}} \rfloor \),

\[ u_1 = \sin(\pi x) \left( c_1 e^{\frac{b}{2\pi^2 x^2} -\frac{1+1/2}{\pi^2} t} + d_1 e^{\frac{b}{2\pi^2 x^2} +\frac{1+1/2}{\pi^2} t} \right) \]

\[ \frac{b}{2a^2 \epsilon} x - 1 + \sqrt{\frac{1-b^2}{a^2 \epsilon} t} = \frac{b}{2a^2 \epsilon} \left[ x - \frac{a^2}{b} \left( 1 + \sqrt{1 - \frac{b^2}{a^2}} \right) t \right] \]

\[ \frac{b}{2a^2 \epsilon} x + 1 - \sqrt{\frac{1-b^2}{a^2 \epsilon} t} = \frac{b}{2a^2 \epsilon} \left[ x - \frac{a^2}{b} \left( 1 - \sqrt{1 - \frac{b^2}{a^2}} \right) t \right] \]

\[ c_1 = \int_0^1 f(x) e^{-\frac{b}{2\pi^2 x^2} \sin(\pi x)} dx - \frac{2\epsilon}{\sqrt{1 - \frac{b^2}{a^2}}} \int_0^1 \left( \frac{f(x)}{2\epsilon} - g'(x) \right) e^{-\frac{b}{2\pi^2 x^2} \sin(\pi x)} dx, \quad (2.42) \]

\[ d_n = \int_0^1 f(x) e^{-\frac{b}{2\pi^2 x^2} \sin(n\pi x)} dx + \frac{2\epsilon}{\sqrt{1 - \frac{b^2}{a^2}}} \int_0^1 \left( \frac{f(x)}{2\epsilon} - g'(x) \right) e^{-\frac{b}{2\pi^2 x^2} \sin(\pi x)} dx \quad (2.43) \]

For \( n = k - m, m \geq 1 \), we estimate \( c_{k-m} \) and \( d_{k-m} \) as follows.

\[ 1 - \frac{b^2}{a^2} - 4a^2(k-m)^2 \epsilon^2 \pi^2 \]

\[ = 1 - \frac{b^2}{a^2} - 4a^2 \epsilon^2 \pi^2 \left( \left\lfloor \frac{\sqrt{a^2-b^2}}{2a^2 \pi \epsilon} \right\rfloor^2 - 2\left\lfloor \frac{\sqrt{a^2-b^2}}{2a^2 \pi \epsilon} \right\rfloor m + m^2 \right) \]

\[ \geq 8a^2 \epsilon^2 \pi^2 m \left\lfloor \frac{\sqrt{a^2-b^2}}{2a^2 \pi \epsilon} \right\rfloor - 4a^2 \epsilon^2 \pi^2 m^2 \]

\[ \geq 2\epsilon \pi m \sqrt{a^2-b^2} - 4a^2 \epsilon^2 \pi^2 m^2 \]

\[ \geq \pi m \epsilon \sqrt{a^2-b^2}, \]

if \( \epsilon \leq \frac{\sqrt{a^2-b^2}}{4\pi a^2 \pi^2} \). Hence

\[ |c_{k-m}| + |d_{k-m}| \leq \frac{4 \epsilon^2}{b} \max_{x \in [0,1]} |f(x)| + \frac{4 \epsilon}{\sqrt{\pi m \sqrt{a^2-b^2}}} \max_{x \in [0,1]} (|f(x)| + \epsilon |g'(x)|), \]

if \( \epsilon \leq \frac{\delta}{m} \) for some small \( \delta \). This proves (2.30), (2.31) can be proved similarly.
2.3 Case for \( b = 0 \)

When \( b = 0 \), the equilibrium equation (1.11) becomes \( u_t = 0 \) to which we denote the solution by \( u^c(x, t) \) which is independent of \( t \). Then we have

\[
u^c(x, t) = f(x) = \sum_{n=0}^{\infty} a_n \sin(nx) \quad \text{where} \quad a_n = 2 \int_0^1 f(x) \sin(x) dx.
\]

Denote the solution for the problem (1.7), (1.8) and (1.9) by \( u(x, t) = \sum_{n=1}^{\infty} A_n(x, t) \sin(n \pi x) \). We will show that for any fixed \( n < k \), \( A_n(x, t) - a_n \to 0 \) as \( \epsilon \to 0 \) for \( t > 0 \).

From (2.41), when \( b = 0 \), we have \( k = \lfloor \frac{1}{2 \pi \alpha} \rfloor \),

\[
u(x, t) = \sum_{n=1}^{k} \sin(n \pi x) \left( c_n e^{\alpha_{n-} t} + d_n e^{\alpha_{n+} t} \right)
\]

\[
+ \sum_{n=k+1}^{\infty} \sin(n \pi x) e^{-\frac{\epsilon}{\pi}} \left[ c_n \cos(\beta_n t) + d_n \sin(\beta_n t) \right] \quad (2.44)
\]

where, for \( n \leq k \),

\[
\alpha_{n\pm} = -1 \pm \sqrt{1 - 4a^2 n^2 \epsilon^2 \pi^2}
\]

\[
c_n = \int_0^1 f(x) \sin(n \pi x) dx - \frac{2 \epsilon}{\sqrt{1 - 4a^2 n^2 \epsilon^2 \pi^2}} \int_0^1 \left( \frac{f(x)}{2 \epsilon} - g'(x) \right) \sin(n \pi x) dx,
\]

\[
d_n = \int_0^1 f(x) \sin(n \pi x) dx + \frac{2 \epsilon}{\sqrt{1 - 4a^2 n^2 \epsilon^2 \pi^2}} \int_0^1 \left( \frac{f(x)}{2 \epsilon} - g'(x) \right) \sin(n \pi x) dx
\]

and for \( n > k \),

\[
c_n = 2 \int_0^1 f(x) e^{-\frac{b}{2 \pi \alpha^2} x} \sin(n \pi x) dx \quad (2.47)
\]

\[
d_n = \frac{4 \epsilon}{\sqrt{1 - 4a^2 n^2 \epsilon^2 \pi^2}} \int_0^1 \left( \frac{f(x)}{2 \epsilon} - g'(x) \right) \sin(n \pi x) dx \quad (2.48)
\]

Obviously, For any fixed \( n < k \) and \( t > 0 \), \( e^{\alpha_{n-} t} \to 0 \) as \( \epsilon \to 0 \).

We denote

\[
u(x, t) = \sum_{n=1}^{\infty} A_n(x, t) \sin(n \pi x).
\]

When \( n \leq k \), Let \( w_n = \frac{4 \epsilon}{\sqrt{1 - 4a^2 n^2 \epsilon^2 \pi^2}} \int_0^1 \left( \frac{f(x)}{2 \epsilon} - g'(x) \right) \sin(n \pi x) dx \), Then

\[
c_n = \frac{a_n - w_n}{2} \quad (2.49)
\]

\[
d_n = \frac{a_n + w_n}{2} \quad (2.50)
\]

Do Taylor expansion to \( \alpha_{n+} \) to get

\[
\alpha_{n+} = -1 + \sqrt{1 - 4a^2 n^2 \epsilon^2 \pi^2} = -1 + 1 - \frac{3a^2 n^2 \epsilon^2 \pi^2}{2} + n^4 O(\epsilon^4) = -a^2 n^2 \pi^2 \epsilon + n^4 O(\epsilon^3)
\]
Furthermore we have
\[ e^{\alpha_n t} = 1 - a^2 n^2 \pi^2 t + n^4 t O(\epsilon^3) \]
and
\[ w_n = \frac{4\epsilon}{\sqrt{1 - 4a^2 n^2 \pi^2}} \int_0^1 \left( \frac{f(x)}{2\epsilon} - g'(x) \right) \sin(n\pi x) dx \]
\[ = 2 \int_0^1 (f(x) - 2\epsilon g'(x)) \sin(n\pi x) dx \frac{1}{\sqrt{1 - 4a^2 n^2 \pi^2}} \]
By Taylor expansion, it is easy to show that
\[ w_n = a_n + O(\epsilon) + n^2 O(\epsilon^2), \]
for any fixed \( n < k \), as \( \epsilon \to 0 \).
Therefore, when \( \epsilon \to 0 \),
\[ d_n e^{\alpha_n t} - a_n \]
\[ = a_n + w_n e^{\alpha_n t} - a_n \]
\[ = a_n + a_n + O(\epsilon) + n^2 O(\epsilon^2) \left( 1 - a^2 n^2 \pi^2 t + n^4 t O(\epsilon^3) \right) - a_n \]
\[ = O(\epsilon) + n^2 (1 + t) O(\epsilon^2). \]
Since \( e^{\alpha_n t} \to 0 \) as \( \epsilon \to 0 \) for \( t > 0 \) and \( n < k \), we have \( A_n(x, t) - a_n \to 0 \) as \( \epsilon \to 0 \) for any fixed \( n < k \) and \( t > 0 \). The case \( n = k + m \) for \( m \geq 1 \) can be analysed as for the case when \( b > 0 \).

3 Analysis by the energy method

3.1 The case when \( b = 0 \)

In the case of \( b = 0 \), the equilibrium equation (1.11) becomes
\[ \ddot{u} = 0. \]
With the initial value \( \ddot{u}(x, 0) = f(x) \), then we have \( \ddot{u}(x, t) = f(x), \; x \in [0, 1], \; t \geq 0. \)
Let \( u \) be the smooth solution of (1.7), (1.8) and (1.9). Set
\[ w = u - \ddot{u} \]
(3.1)
Then \( w \) is a solution to the following initial boundary value problem:
\[ \left\{ \begin{array}{l}
\ddot{w} - a^2 \dddot{w} - a^2 \dddot{w} + \frac{1}{\epsilon} \dddot{w} = 0, \; 0 \leq x \leq 1, \; t > 0, \\
\dot{w}(0, t) = \dot{w}(1, t) = 0, \\
\dot{w}(x, 0) = 0, \; \dddot{w}(x, 0) = -g'(x). \end{array} \right. \]
(3.2)

Theorem 3.1. Let \( w \) be the solution to problem \( (3.2) \). It then holds that
\[ \int_0^1 w^2(x, t) dx + \int_0^t \int_0^1 w^2_t(x, s) dx ds \leq C \epsilon \left( \int_0^1 (g'(x))^2 dx + t \int_0^1 (f'(x))^2 + (f''(x))^2 dx \right), \]
(3.3)
for \( 0 < \epsilon < 1/4 \) and \( t > 0 \), where \( C \) is a constant only depending on \( a \).
Proof. Multiply (3.2) with \( w_t \) and \( w \) respectively, and integrate the resulting equations by parts over \([0,1]\), and use the boundary conditions to get

\[
\frac{d}{dt} \int_0^1 \left( \frac{w^2}{2\epsilon} + w w_t \right) (x,t) dx - \int_0^1 w_t^2 (x,t) dx + a^2 \int_0^1 w_x^2 (x,t) dx = - \int_0^1 a^2 \bar{u}_x w_x (x,t) dx
\]

(3.4)

By the Cauchy-Schwarz inequality, we have

\[
\int_0^1 |a^2 \bar{u}_x w_t| dx \leq \frac{1}{2} \int_0^1 (w_t^2 + \frac{a^2}{2} w_{xx}^2) dx
\]

\[
\int_0^1 a^2 |\bar{u}_x w_x| dx \leq \int_0^1 (\frac{a^2}{2} w_x^2 + \frac{a^2}{2} \bar{u}_{xx}^2) dx.
\]

Therefore,

\[
\frac{d}{dt} \int_0^1 \left( \frac{1}{2\epsilon} \frac{1}{2} w^2 + w w_t + \frac{a^2}{2} w_t^2 \right) dx + \frac{1}{\epsilon} \int_0^1 w_t^2 dx + a^2 \int_0^1 w_x^2 dx
\]

\[
= - \int_0^1 a^2 \bar{u}_x w_x dx + \int_0^1 a^2 \bar{u}_{xx} w_t dx
\]

\[
\leq a^2 \int_0^1 \bar{u}_x^2 dx + \frac{a^2}{2} \int_0^1 w_x^2 dx + \int_0^1 \left( \frac{1}{2} w_t^2 + \frac{a^2}{2} \bar{u}_{xx}^2 \right) dx.
\]

Hence,

\[
\frac{d}{dt} \int_0^1 \left( \frac{1}{2\epsilon} \frac{1}{2} w^2 + w w_t + \frac{a^2}{2} w_t^2 \right) dx + \frac{1}{\epsilon} \int_0^1 w_t^2 dx + a^2 \int_0^1 w_x^2 dx
\]

(3.6)

\[
\leq \frac{a^2}{2} \int_0^1 \bar{u}_x^2 dx + \frac{a^2}{2} \int_0^1 \bar{u}_{xx}^2 dx.
\]

Noting that

\[
\frac{1}{2\epsilon} \frac{1}{2} w^2 + w w_t + \frac{a^2}{2} w_t^2 \geq (\frac{1}{2\epsilon} - 1) w^2 + \frac{1}{4} w_t^2,
\]

\[
-(w^2 + \frac{1}{4} w_t^2) \leq w w_t
\]

\[
\leq w^2 + \frac{1}{4} w_t^2
\]

Integrating (3.6) to get

\[
\int_0^1 (\frac{1}{2\epsilon} - 1) w^2 + \frac{3}{4} w_t^2 + \frac{a^2}{2} w_t^2 (x,t) dx + (\frac{1}{2\epsilon} - 3) \int_0^1 \int_0^t w_t^2 (x,s) dx ds + \frac{a^2}{2} \int_0^t \int_0^1 w_x^2 (x,s) dx ds
\]

\[
\leq \int_0^1 (\frac{1}{2\epsilon} + 1) w^2 + \frac{3}{4} w_t^2 + \frac{a^2}{2} w_t^2 (x,0) dx + \int_0^t \int_0^1 \left( \frac{a^2}{2} \bar{u}_x^2 + \frac{a^4}{2} \bar{u}_{xx}^2 \right) (x,s) dx ds.
\]

By the initial condition \( w(x,0) = w_x(x,0) = 0 \) and \( w_t(x,0) = -g'(x) \), we have

\[
\int_0^1 \left( \frac{1}{2\epsilon} - 1 \right) w_t^2 (x,t) dx + (\frac{1}{2\epsilon} - 3) \int_0^t \int_0^1 w_t^2 dx ds
\]

\[
\leq \int_0^t \int_0^1 \frac{3}{4} (g'(x))^2 (x,0) + \int_0^t \int_0^1 \frac{a^2}{2} (\bar{u}_x^2 + \frac{a^4}{2} \bar{u}_{xx}^2) (x,s) dx ds.
\]

This proves (3.3) and complete the proof of Theorem 3.1.
3.2 The case for $b < 0$

In this subsection, we present the analysis for the case when $b < 0$ by using the boundary layer profile and the energy method. When $b < 0$, the boundary layer occurs at the boundary $x = 0$. The case for $b > 0$ can be handled similarly for which the boundary layer occurs at $x = 1$. Denote the solution at equilibrium as $u_e(x,t)$, then it satisfies the following equations:

\[
\begin{align*}
\partial_t u_e + b \partial_x u_e &= 0, \\
u_e(1,t) &= 0, \\
u_e(x,0) &= f(x).
\end{align*}
\]  

Solving for $u_e$, we have:

\[
u_e(x,t) = \begin{cases} 
  f(x - bt) & x \leq 1 + bt \\
  0 & x > 1 + bt.
\end{cases}
\]

The solution $u_e$ is illustrated in the above figures. Take $a = 1$ for convenience, we may write the problem (1.7), (1.8) and (1.9) as

\[
\begin{align*}
u_{tt} - \nu_{xx} + \frac{1}{\epsilon}(\nu_t + bu_x) &= 0, \\
u(0,t) &= \nu(1,t) = 0, \\
u(x,0) &= f(x), \\
u(0,x) &= -g'(x),
\end{align*}
\]

where we have used $u_e$ to indicate the dependence of the solution on $\epsilon$.

Boundary layer expansion:

\[
u(x,t) = \nu(x,t) + U_0(y,t) + \epsilon U_1(y,t) + w(x,t),
\]
where \( y = \frac{x}{\epsilon} \).

Plug (3.15) in (3.11), we obtain:
\[
\partial^2_t u - \partial^2_x u + \frac{1}{\epsilon}(\partial_t u + b \partial_x u) +
\]
\[
\partial^2_t U_0 - \frac{1}{\epsilon^2} \partial^2_y U_0 + \frac{1}{\epsilon}(\partial_t U_0 + \frac{b}{\epsilon} \partial_x U_0) +
\]
\[
\epsilon [\partial^2_t U_1 - \frac{1}{\epsilon^2} \partial^2_y U_1 + \frac{1}{\epsilon}(\partial_t U_1 + \frac{b}{\epsilon} \partial_y U_1)] +
\]
\[
\partial^2_t w - \partial^2_x w + \frac{1}{\epsilon}(\partial_t w + b \partial_x w) = 0.
\]

To make the \( O(\epsilon^{-2}) \) and \( O(\epsilon^{-1}) \) order 0, we have:
\[
O(\epsilon^{-2}) : -\partial^2_y U_0 + b \partial_y U_0 = 0, \quad (3.17)
\]
\[
O(\epsilon^{-1}) : \partial_t U_0 - \partial^2_y U_1 + b \partial_y U_1 = 0. \quad (3.18)
\]

After solving for (3.17) and (3.18), we take
\[
U_0(y, t) = c(t)e^{by}, \quad (3.19)
\]
\[
U_1(y, t) = \frac{c'(t)}{b} \left[(y - \frac{1}{b})e^{by} + \frac{1}{b}\right] + \frac{d(t)}{b}(e^{by} - 1). \quad (3.20)
\]

Then (3.16) becomes:
\[
\partial^2_t u - \partial^2_x u + \partial^2_t U_0 + \epsilon \partial^2_t U_1 +
\]
\[
\partial_t U_1 + \partial^2_t w - \partial^2_x w + \frac{1}{\epsilon}(\partial_t w + b \partial_x w) = 0. \quad (3.21)
\]

Recall that
\[
w = u - U_0 - \epsilon U_1 - u. \quad (3.22)
\]

We want \( w(0, t) = w(1, t) = 0 \). Since \( u(0, t) = u(1, t) = 0 \) and
\[
u(1, t) = 0 \quad (3.23)
\]
\[
u(0, t) = \begin{cases} 
  f(-bt) & t \leq -\frac{1}{b} \\
  0 & t > -\frac{1}{b} 
\end{cases} \quad (3.24)
\]

To make \( w(0, t) = w(1, t) = 0 \), the following equations must be satisfied:
\[
(U_0 + \epsilon U_1)(0, t) = -u(0, t) \quad (3.25)
\]
\[
(U_0 + \epsilon U_1)(1, t) = 0 \quad (3.26)
\]

Solving (3.25) and (3.26)
\[
c(t) = \begin{cases} 
  -f(-bt) & t \leq -\frac{1}{b} \\
  0 & t > -\frac{1}{b} 
\end{cases} \quad (3.27)
\]
\[ d(t) = \frac{bc(t) e^{\frac{b}{\epsilon}} + c'(t)\left[\left(\frac{1}{\epsilon} - \frac{1}{b}\right)e^{\frac{b}{\epsilon}} + \frac{1}{b}\right]}{1 - e^{\frac{b}{\epsilon}}} \]  

(3.28)

Also, we can see that

\[ \lim_{\epsilon \to 0} d(t) = \frac{c'(t)}{b} \]  

(3.29)

Consider (3.21), now we have:

\[ \partial_t^2 w - \partial_x^2 w + \frac{1}{\epsilon} (\partial_t w + b \partial_x w) + (\epsilon \partial_t^2 u^e - \partial_x^2 u^e) + (\epsilon \partial_t^2 U_1 + \partial_t U_1 + \partial_x^2 U_0) = 0 \]  

(3.30)

with boundary condition:

\[ w(0, t) = w(1, t) = 0 \]  

(3.31)

Also, for the initial condition:

\[ w(x, 0) = u^e(x, 0) - u^e(x, 0) - U_0 \left( \frac{x}{\epsilon}, 0 \right) - \epsilon U_1 \left( \frac{x}{\epsilon}, 0 \right) \]  

\[ = -\epsilon U_1 \left( \frac{x}{\epsilon}, 0 \right) \]  

(3.32)

Therefore by (3.20), (3.28), (3.31) we have

\[ w(x, 0) = -\epsilon U_1 \left( \frac{x}{\epsilon}, 0 \right), \]  

(3.33)

\[ w_t(x, 0) = u_t^e - u_t^e - U_t - U_{0t}. \]  

(3.34)

From the definitions of \( u^e, U_1 \) and \( U_0 \), it is easy to see that

\[ |w(x, 0)| \leq C \epsilon |f'(0)|, \]  

(3.35)

\[ |w_t(x, 0)| \leq C \epsilon (|f'(0)| + |f''(0)|) + C(|g'(x)| + |b||f'(x)|) \]  

(3.36)

\[ |w_x(x, 0)| \leq C |f'(0)|, \]  

(3.37)

where and in the following, we use \( C \) to denote a generic constant independent of \( \epsilon \).

**Theorem 3.2.** Suppose that \( f \in C^3([0, 1]) \) and \( f'(0) = 1 \). Let \( w \) be the solution to problem (3.30), (3.31) and (3.34). Then it holds that

\[
\int_0^1 w^2(x, t)dx + \int_0^t \int_0^1 w^2dxds \\
\leq C \epsilon^2 e^{2t} \left( \int_0^1 (g'(x))^2 + b^2 |f'(x)|^2)dx + (|f'(0)|^2 + |f''(0)|^2) \right) \\
+ C \epsilon t e^{2t} \max_{[0,1]} \sum_{i=1}^3 |f^{(i)}(x)|^2.
\]  

(3.38)

**Remark 3.3.** It is easy to verify that \( \int_0^1 U_0^2(x, t)dx \leq O(1) \epsilon \) and \( \int_0^1 \epsilon^2 U_1^2(x, t)dx \leq O(1) \epsilon^2 \). It follows from (3.38) that, for any fixed \( t > 0 \), \( \int_0^1 (u^e - u^e)^2(x, t)dx \) converges to zero in the order of \( \epsilon \) as \( \epsilon \to 0 \).
Proof of Theorem 3.2
Multiply (3.30) by $w_t$, then integrate both sides on $(0,1)$ with respect to $x$:

$$\frac{d}{dt} \int_0^1 \left( \frac{w_t^2}{2} + \frac{w_x^2}{2} \right) dx + \frac{1}{\epsilon} \int_0^1 w_t^2 dx + \frac{b}{\epsilon} \int_0^1 w_x w_t dx = - \int_0^1 (\partial_t^2 u^\epsilon - \partial_x^2 u^\epsilon) w_t dx - \int_0^1 (\epsilon \partial_t^2 U_1 + \partial_t U_1 + \partial_x^2 U_0) w_t dx. \quad (3.39)$$

Similarly, multiply (3.30) by $w$, then integrate both sides on $(0,1)$ with respect to $x$:

$$\frac{d}{dt} \int_0^1 (ww_t + \frac{1}{2\epsilon} w^2) dx + \int_0^1 (w_x^2 - w_t^2) dx = - \int_0^1 (\partial_t^2 u^\epsilon - \partial_x^2 u^\epsilon) w dx - \int_0^1 (\epsilon \partial_t^2 U_1 + \partial_t U_1 + \partial_x^2 U_0) w dx. \quad (3.40)$$

Denote

$$G(x,t) = \left[ (\partial_t^2 u^\epsilon(x,t) - \partial_x^2 u^\epsilon(x,t)) \right] + \left[ \epsilon \partial_t^2 U_1(\frac{x}{\epsilon},t) + \partial_t U_1(\frac{x}{\epsilon},t) + \partial_x^2 U_0(\frac{x}{\epsilon},t) \right]. \quad (3.41)$$

Then (3.39) + (3.40) $\times k$ for a constant $k$ to be determined later yields that

$$\frac{d}{dt} \int_0^1 \left( \frac{w_t^2 + w_x^2}{2} + kw_t + \frac{kw^2}{2\epsilon} \right) dx + \int_0^1 \left( \frac{1}{\epsilon} - k \right) w_t^2 dx + \frac{b}{\epsilon} \int_0^1 w_x w_t dx + k \int_0^1 w_x^2 dx = - \int_0^1 Gw_t dx - k \int_0^1 Gw dx. \quad (3.42)$$

The left hand side of (3.42) can be written as:

$$L.H.S = \frac{d}{dt} \int_0^1 \left\{ \frac{1}{2} [(w_t + kw)^2 + \frac{(k - k^2)w^2}{\epsilon}] + \frac{1}{2} w_x^2 \right\} dx + \int_0^1 k(w_x + \frac{k}{2\epsilon k} w_t)^2 + (\frac{1}{\epsilon} - k - \frac{b^2}{4\epsilon^2 k}) w_t^2 dx. \quad (3.43)$$

In order to have positive coefficients, we have:

$$\frac{k}{\epsilon} - k^2 > 0, \quad (3.44)$$
$$\frac{1}{\epsilon} - k - \frac{b^2}{4\epsilon^2 k} > 0, \quad (3.45)$$
$$k > 0. \quad (3.46)$$

Solving for these conditions, we obtain:

$$k \in \left( \frac{1 - \sqrt{1 - b^2}}{2\epsilon}, \frac{1 + \sqrt{1 - b^2}}{2\epsilon} \right). \quad (3.47)$$

Thus we take

$$k = \frac{1}{2\epsilon}. \quad (3.48)$$
Then (3.42) becomes:

\[
\frac{1}{2} \frac{d}{dt} \int_0^1 \left[ (w_t + w) + \frac{1}{4\epsilon^2} w^2 + w_x^2 \right] dx + \int_0^1 \left[ \frac{1}{2\epsilon} (w_x + bw_t)^2 + \frac{1-b^2}{2\epsilon} w_t^2 \right] dx = -\int_0^1 (Gw_t + \frac{Gw}{2\epsilon}) dx. \tag{3.49}
\]

Integrate both sides of (3.49) on (0, t) with respect to t:

\[
\frac{1}{2} \int_0^1 \left[ (w_t + w) + \frac{1}{4\epsilon^2} w^2 + w_x^2 \right] dx dt + \int_0^t \int_0^1 \left[ \frac{1}{2\epsilon} (w_x + bw_t)^2 + \frac{1-b^2}{2\epsilon} w_t^2 \right] dx ds = \frac{1}{2} \int_0^1 \left[ (w_t + w) + \frac{1}{4\epsilon^2} w^2 + w_x^2 \right] dx + \int_0^t \int_0^1 (Gw_t + \frac{Gw}{2\epsilon}) dx ds. \tag{3.50}
\]

By the Cauchy-Schwarz inequality:

\[
|\int_0^t \int_0^1 Gw_t dx ds| \leq \frac{1-b^2}{4\epsilon} \int_0^t \int_0^1 w_t^2 dx ds + \frac{\epsilon}{1-b^2} \int_0^t \int_0^1 G^2 dx ds. \tag{3.51}
\]

Then (3.50) becomes

\[
\frac{1}{2} \int_0^1 \left[ (w_t + w) + \frac{1}{4\epsilon^2} w^2 + w_x^2 \right] dx dt + \int_0^t \int_0^1 \left[ \frac{1}{2\epsilon} (w_x + bw_t)^2 + \frac{1-b^2}{4\epsilon} w_t^2 \right] dx ds \\
\leq \frac{1}{2} \int_0^1 \left[ (w_t + w) + \frac{1}{4\epsilon^2} w^2 + w_x^2 \right] dx + \int_0^t \int_0^1 \int_0^1 Gw dx ds + \frac{\epsilon}{1-b^2} \int_0^t \int_0^1 G^2 dx ds. \tag{3.52}
\]

With the initial conditions (3.33), (3.34), we have

\[
\frac{1}{2} \int_0^1 \left[ (w_t + w) + \frac{1}{4\epsilon^2} w^2 + w_x^2 \right] dx dt + \int_0^t \int_0^1 \left[ \frac{1}{2\epsilon} (w_x + bw_t)^2 + \frac{1-b^2}{4\epsilon} w_t^2 \right] dx ds \\
\leq C \int_0^1 \left( w_t^2 + \frac{1}{\epsilon^2} w^2 + w_x^2 \right) dx + \int_0^t \int_0^1 \int_0^1 Gw dx ds + \frac{\epsilon}{1-b^2} \int_0^t \int_0^1 G^2 dx ds. \tag{3.53}
\]

Again, by Cauchy-Schwarz Inequality,

\[
\left| \int_0^t \int_0^1 Gw dx ds \right| \leq \frac{1}{4\epsilon} \int_0^t \int_0^1 (w^2 + G^2) dx ds. \tag{3.54}
\]

From (3.54) and (3.53), we have

\[
\frac{1}{8\epsilon^2} \int_0^1 w^2(x, t) dx + \frac{1}{4\epsilon} \int_0^t \int_0^1 w^2 dx ds + \frac{1}{2\epsilon} \int_0^t \int_0^1 G^2 dx ds \tag{3.55}
\]

for small \( \epsilon \).
From the definition of $G$, we can see that,
\[
\int_0^1 G^2(x,t)dx \leq C \max_{[0,1]} \sum_{i=1}^3 |f^{(i)}(x)|^2.
\] (3.56)

Denote
\[
F(t) = \int_0^t \int_0^1 w^2dxds.
\] (3.57)

In view of (3.35), and (3.56), (3.55) becomes
\[
F'(t) - 2\epsilon F(t)
\leq C\epsilon^2 \int_0^1 (w_t^2 + \frac{1}{\epsilon^2} w^2 + w_x^2)(x,0)dx + 4\epsilon \int_0^t \int_0^1 G^2dxds
\leq C\epsilon^2 \left( \int_0^1 ((g'(x))^2 + b^2|f'(x)|^2)dx + (|f'(0)|^2 + |f''(0)|^2) \right)
+ C\epsilon t \max_{[0,1]} \sum_{i=1}^3 |f^{(i)}(x)|^2.
\] (3.58)

for small $\epsilon$.

Multiply this by $e^{-2\epsilon t}$ on both sides and integrate with respect to $t$, we obtain,
\[
F(t)
\leq C\epsilon^2 e^{-2\epsilon t} \left( \int_0^1 ((g'(x))^2 + b^2|f'(x)|^2)dx + (|f'(0)|^2 + |f''(0)|^2) \right)
+ C\epsilon e^{-2\epsilon t} \max_{[0,1]} \sum_{i=1}^3 |f^{(i)}(x)|^2.
\] (3.59)

This proves (3.38).

4 Numerical Results

We obtained the results from the Fourier solutions with $f(x) = \sin(\pi x)$, $g'(x) = -\pi \sin(\pi x)$, $a = 2$ and $b = 1$. 

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