A gradient enhanced transversely isotropic damage plasticity model for rock - formulation and comparison of different approaches

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Abstract
The nonlinear mechanical behavior of intact rock and rock mass is characterized by irreversible deformation, associated with strain hardening, strain softening and degradation of stiffness. In the case of stratified rock types like shales or phyllites, the mechanical behavior additionally depends on the loading direction with respect to the orientation of the principal material directions, resulting in inherent anisotropic material behavior, or in many cases more specifically in transversely isotropic material behavior. On the basis of a critical review of several approaches for modeling nonlinear anisotropic material behavior, proposed in the literature, the linear mapping of the Cauchy stress tensor into a fictitious isotropic configuration, originally proposed by Boehler [1], was identified as the best choice for the application in 3D finite element simulations. Within this framework, an isotropic damage plasticity model for rock, proposed by Unteregger et al. [2], extended to inherent transversely isotropic behavior is presented. It is implemented into a FE-program by means of the return mapping algorithm and it is regularized by an over-nonlocal implicit gradient enhancement for ensuring mesh-insensitive results. For validation, numerical simulations of triaxial compression tests on Tournemire shale, documented in Niandou et al. [3], are performed at both the integration point level and the structural level. The performance of the constitutive model is assessed by its ability to predict the complex direction-dependent mechanical behavior of transversely isotropic rock and the failure modes observed in triaxial compression tests with different confining pressures and different inclination angles of the stratification planes with respect to the direction of axial loading.

KEYWORDS
anisotropy, constitutive modeling, damage plasticity model, localization, rock mechanics, transverse isotropy

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Tunnel advance is a very demanding task in engineering practice. For the mathematical description of the material behavior of the surrounding rock mass, isotropic, linear-elastic perfectly-plastic material laws are frequently used in numerical simulations of tunneling. These material laws are based on the failure hypothesis according to Hoek et al.\textsuperscript{4}. In the latter, discontinuities in a rock mass are considered in a simplified manner by reducing the strength parameters of intact rock by an empirical approach derived from a rock mass classification system. However, compared to the real mechanical behavior of rock mass, some essential effects of discontinuities are neglected by the respective simplifications. For example, plastic deformation in the pre-peak region of the stress-strain curves as well as strain softening in the post-peak domain, accompanied by degradation of the material stiffness, are ignored. In addition, linear-elastic behavior is incorrectly predicted for stress paths with dominating hydrostatic compression, although it is well known that in this case the material behavior depends on the hydrostatic pressure.

These shortcomings significantly influence the prediction of the deformation and stress state in numerical simulations of tunnel advance. In order to overcome the mentioned shortcomings, an isotropic damage plasticity model for rock and rock mass, denoted as rock damage plasticity (RDP) model, was developed by Unteregger et al.\textsuperscript{2}. In the RDP model the mentioned effects are considered by coupling plasticity theory with damage mechanics, the latter for describing the post-peak degradation of strength and stiffness. The flow potential and the formulations of hardening and softening, originally proposed by Grassl and Jirásek\textsuperscript{5}, were modified for intact rock and extended for rock mass by introducing empirical reduction factors according to Hoek et al.\textsuperscript{4}. For ensuring mesh-independent results in finite element simulations involving softening rock mass behavior, the RDP model was extended by a regularization technique based on an implicit gradient enhancement by Schreter et al.\textsuperscript{6}.

The mentioned models for intact rock and rock mass are still restricted to rock types with isotropic material behavior. However, depending on the origin of a particular rock, primary and secondary rock structures result in inherent anisotropic material behavior of intact rock and rock mass both at the micro- and macro-level, which have a significant impact on the deformation and stress state in the respective intact rock or rock mass. Especially in a stratified rock mass, for example Innsbruck quartz phyllite or Tournemire shale, often the special cases of anisotropic material behavior, i.e., orthotropic or transversely isotropic material behavior, are encountered.

In addition to the inherent anisotropic behavior, directional dependencies can also occur due to load-induced micro-cracking commonly known as induced anisotropy. Since the focus of the present publication is on the representation of inherent anisotropy, an overview on different approaches for considering the latter in constitutive models is provided in the following.

The oldest approach for incorporating anisotropic material behavior in elasto-plastic models is based on the formulation of an anisotropic yield criterion. Departing from the failure criterion by von Mises, Hill\textsuperscript{7} proposed the first anisotropic yield function for describing the influence of Lüder’s bands in metals. For this purpose, the isotropic criterion was extended by six material parameters representing anisotropic behavior. Based on Hill’s criterion, Pariseau\textsuperscript{8} presented an extension for anisotropic rocks and Tsai and Wu\textsuperscript{9} proposed a generalized anisotropic criterion for fiber-reinforced composites.

Another approach for modeling stratified materials relies on the ubiquitous joint concept, which is based on the Single Plane of Weakness Theory by Jaeger.\textsuperscript{10} In the original ubiquitous joint (OUJ) model, introduced by Goodman,\textsuperscript{11} joints are modeled as cracks with a given direction, distributed (or smeared) throughout the rock. In this way, a stratified material is substituted by a homogeneous anisotropic continuum. Failure may occur in the rock matrix, along the joints, or in both the rock matrix and the joints. Since in the OUJ model only strength anisotropy is considered, isotropic linear-elastic material behavior is assumed up to failure. To overcome this shortcoming, the OUJ model was further developed to the modified ubiquitous joint (MUJ) model by Ismael and Konietzky.\textsuperscript{12} In the finite element program Plaxis\textsuperscript{13} a jointed rock model is implemented following the ubiquitous joint concept. It is characterized by assuming transversely isotropic linear-elastic material behavior of the intact rock matrix and thus potential failure in the rock matrix is neglected. Failure is modeled by sliding of the rock along up to three stratification planes according to the Mohr-Coulomb failure criterion. However, the ubiquitous joint models presented so far, only consider linear-elastic perfectly-plastic material behavior, i.e., both hardening and softening behavior, which significantly influence the mechanical behavior, are neglected. For considering strain hardening in the pre-peak regime of the stress-strain relation and softening in the post-peak regime in ubiquitous joint models, the so-called subiquitous joint (SUJ) models, were developed. The Transubi model, presented by Ismael et al.\textsuperscript{14} belongs to this category. In this model a bilinear Mohr-Coulomb yield surface with tension cut-off is employed for both the intact rock matrix and the stratification planes.
A further possibility of modeling inherent anisotropic behavior was published by Pietruszczak and Mroz\textsuperscript{15} and Gao et al.\textsuperscript{16}. It relies on the generalization of isotropic yield functions by so-called scalar anisotropy parameters. These parameters are postulated by means of a microstructure tensor, which takes into account the influence of the inclination of the stratification planes on the material parameters. Based on Pietruszczak’s approach, Chen et al.\textsuperscript{17,18} and Parisio et al.\textsuperscript{19} proposed coupled elasto-plastic damage models.

A different continuum-based approach for the extension of isotropic failure criteria to anisotropic behavior was presented by Boehler,\textsuperscript{1} wherein anisotropic behavior is described via a linear mapping tensor depending on the microstructure of the material. With the latter, an anisotropic failure criterion can be formulated in terms of mixed invariants of the stress tensor and a mapping tensor. In Boehler and Sawczuk,\textsuperscript{20} this approach was used for an extension of the isotropic Mohr-Coulomb failure criterion for both, transversely isotropic and orthotropic materials and Crook et al.\textsuperscript{21} developed an orthotropic elasto-plastic material model for weak shales by extending the Modified Cam Clay (MCC) model. Departing from the MCC model, Semnani et al.\textsuperscript{22} also used a linear mapping tensor for deriving a transversely isotropic plasticity model to investigate the combined effects of inherent anisotropy and temperature on strain localizations.

An alternative approach for considering both, inherent and induced anisotropic material behavior, are micro-plane models. So far, they were mainly used for concrete (Bažant et al.,\textsuperscript{23} Zreid and Kaliske\textsuperscript{24}) and only few applications for anisotropic rock are available in the literature (Li et al.\textsuperscript{25}). Contrary to standard continuum models, the constitutive relations are formulated on integration planes of various orientations, representing the discretized surface of a unit hemisphere. For this purpose, the strain tensor is projected into the integration planes and the stress tensor is computed from the stress vectors acting on those integration planes by numerical integration. A similar concept is pursued by multi-laminate models (Zienkiewicz and Pande,\textsuperscript{26} Schweiger et al.\textsuperscript{27}). In contrast to the micro-plane models, in the latter the stress tensor is projected into the integration planes. However, a general drawback of those models is the significantly higher computational effort compared to standard continuum based models, rendering both models computationally very expensive for large-scale simulations. For the sake of completeness, in addition to the mentioned continuum-based approaches, for modeling rock mass with single distinct stratification planes discontinuous approaches based on the discrete element method are mentioned, as discussed in Barton\textsuperscript{28} and Wittke.\textsuperscript{29} However, they require a high computational effort and they are usually limited by the lack of information on the structure of the discontinuities.

For the application of damage plasticity models in large-scale finite element simulations, a stable and robust formulation and implementation of the constitutive model is crucial for ensuring convergence. Hence, a damage plasticity model for transversely isotropic materials fulfilling these requirements will be presented in this paper.

For this purpose, first a review of popular approaches for considering anisotropic behavior of elasto-plastic materials is presented and main advantages and disadvantages are identified. Subsequently, the isotropic rock damage plasticity model by Unteregger et al.\textsuperscript{2} is extended to transversely isotropic behavior. The capabilities of the new model are demonstrated by numerical simulations of triaxial compression tests on Tournemire Shale, documented by Niandou et al.,\textsuperscript{3} and a comparison of the numerical results with the respective experimental data.

The remainder of the paper is organized as follows: In Section 2, three popular approaches for describing transversely isotropic material behavior, i.e., the ubiquitous joint concept, the introduction of scalar anisotropy parameters and the linear mapping into a fictitious isotropic configuration, are critically reviewed and examined. To this end, a linear-elastic perfectly-plastic model with the simple Drucker-Prager yield criterion is extended accordingly. Based on this comparison in Section 3 the most promising approach is chosen for enhancing an existing isotropic rock damage plasticity model to account for transversely isotropic material behavior. In Section 4 the results of numerical simulations of triaxial compression tests on Tournemire shale on both the material point level and the structural level will be presented, thereby demonstrating the capabilities of the selected model. Finally, in Section 5 the paper is closed with a summary and a discussion on recommended future research activities.

2 A CRITICAL REVIEW ON MODELING TRANVERSELY ISOTROPIC MATERIAL BEHAVIOR WITHIN THE FRAMEWORK OF PLASTICITY THEORY

This section is devoted to the review of three popular approaches for extending an existing isotropic elasto-plastic model to a transversely isotropic elasto-plastic model, i.e., (i) the ubiquitous joint concept, (ii) the introduction of scalar anisotropy parameters and (iii) the projection of the Cauchy stress tensor into a mapped isotropic
configuration. To this end, the approaches (i)–(iii) will be briefly reviewed, and the key features will be subsequently assessed based on the extension of a linear-elastic perfectly-plastic model with the Drucker-Prager yield criterion to transversely isotropic behavior, serving as simple yet illustrative representatives of transversely isotropic elasto-plastic constitutive models.

2.1 Preliminaries

The focus is on transversely isotropic constitutive behavior based on coupling linear elasticity and plasticity theory. Therein, it is convenient to express the stress-strain relation in the principal material directions

\[
\sigma'_mn = C^{t(\iota)}_{mn pq} \varepsilon'_pq = C^{t(\iota)}_{mn pq} \left( \varepsilon'_pq - \varepsilon^{(p)}_{pq} \right),
\]

where the superscript prime (\(\cdot)\)' designates variables formulated in the local coordinate system aligned with the principal material directions (cf. Figure 1).

\(\sigma'\) describes the nominal Cauchy stress tensor, \(\varepsilon'\) the total strain tensor, \(\varepsilon'^{(e)}\) the elastic strain tensor and \(\varepsilon'^{(p)}\) the plastic strain tensor, considering the additive split of the total strain into an elastic and a plastic part. \(C^{t(\iota)}\) denotes the fourth-order transversely isotropic elastic stiffness tensor, reading in Voigt-notation as

\[
\begin{bmatrix}
\frac{1}{E_1} & -\nu_{12} & -\nu_{12} & 0 & 0 & 0 \\
-\frac{\nu_{12}}{E_2} & \frac{1}{E_2} & -\nu_{23} & 0 & 0 & 0 \\
-\frac{\nu_{12}}{E_2} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_2} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{G_{12}} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{G_{12}} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{2(1 + \nu_{23})}{E_2} \\
\end{bmatrix}^{-1},
\]

FIGURE 1 Sketch of the global and local orthogonal bases, \(e^{(\iota)}\) and \(e^{(\iota)'}\), \(i = 1, 2, 3\), respectively, for a transversely isotropic material, with \(e^{(1)}\) defining the axis of material symmetry and \(e^{(2)}\) and \(e^{(3)}\) the plane of isotropic material behavior.
with the Young’s modulus $E_1$ in $x'_1$-direction, the Poisson’s ratio $\nu_{12}$ and the shear modulus $G_{12}$ in planes containing the axis of material symmetry $e^{(1)}$ and the Young’s modulus $E_2$ and the Poisson’s ratio $\nu_{23}$ both in the $x'_2x'_3$-plane of isotropic material behavior.

The stress and the strain tensor, defined in the local coordinate system, are transformed into the global coordinate system according to

$$\varepsilon_{kl} = a_{pk} a_{ql} \varepsilon'_{pq}$$  \hspace{1cm} (3a)$$

and

$$\sigma_{ij} = a_{mi} a_{nj} \sigma'_{mn}$$  \hspace{1cm} (3b)$$

by means of the orthogonal coordinate transformation matrix $a_{ij} = \cos \angle(e^{(i)}, e^{(j)})$, with the entries representing the direction cosines between the base vectors of the local and the global coordinate system. (3a) and (3b) are applied in an analogous manner for the transformation of the elastic and plastic strain tensors. Rearrangement of (3a) and insertion of (1) and (3a) into (3b) yields the stress-strain relation formulated with respect to the global coordinate system

$$\sigma_{ij} = a_{mi} a_{nj} a_{pk} a_{ql} C^{(l)}_{ijkl} \left( \varepsilon_{kl} - \varepsilon^{(p)}_{kl} \right).$$  \hspace{1cm} (4)$$

$C^{(l)}$ represents the fourth-order transversely isotropic elastic stiffness tensor in the global coordinate system. The stress tensor (4), referred to the global coordinate system, enters the equilibrium equations.

### 2.2 Ubiquitous joint models

In ubiquitous joint models, two different failure criteria are combined, one representing failure of the rock matrix (“non-sliding mode”), denoted as $f(r)$, and the other one failure along the stratification planes (“sliding mode”), denoted as $f^{(sp)}$. The latter is defined in terms of the traction vector $t_i^{(n)} = \sigma_{ij} n_j$ acting on an infinitesimal area of the stratification planes with the normal vector $n$, leading to the incorporation of direction-dependent behavior. For introducing multiple failure criteria in ubiquitous joint models, the constitutive framework is extended to multi-surface plasticity as described by Simo. According to the latter, the set of admissible yield functions is defined as

$$\mathcal{J}_{all} = \left\{ \xi \in \{r, sp\} \mid f^{(\xi)} = 0 \right\}$$  \hspace{1cm} (5)$$

with the subset of active yield functions

$$\mathcal{J}_{act} = \left\{ \xi \in \mathcal{J}_{all} \mid f^{(\xi)} = 0 \right\}.$$  \hspace{1cm} (6)$$

According to the ubiquitous joint concept, an arbitrary number of stratification planes can be considered, whereas for transversely-isotropic materials one joint set is sufficient.

For the evolution of the plastic strain, the non-associated flow rule

$$\dot{\varepsilon}^{(p)}_{kl} = \sum_{\kappa \in \mathcal{J}_{act}} \dot{\lambda}^{(\kappa)} \cdot \frac{\partial g^{(\kappa)}(\sigma)}{\partial \sigma_{kl}}$$  \hspace{1cm} (7)$$

is employed with $\dot{\varepsilon}^{(p)}$ denoting the plastic strain rate, $\kappa$ the subset of active yield criteria, $g^{(\kappa)}(\sigma)$ the plastic potential function and $\dot{\lambda}^{(\kappa)}$ the corresponding consistency parameter. In the following, the isotropic Drucker-Prager yield
\[ f^{(r)}(\mathbf{\sigma}) = \sqrt{s_{ij}s_{ij}} + \frac{\mu}{\sqrt{3}} I_1 - \sqrt{2} \tau_y \]  

(8)

is used for the rock matrix with \( s \) and \( I_1 \) denoting the deviatoric part and the first invariant of the Cauchy stress tensor, respectively, and \( \tau_y \) and \( \mu \) the cohesive and frictional material parameters. A non-associated plastic potential function

\[ g^{(r)}(\mathbf{\sigma}) = \sqrt{s_{ij}s_{ij}} + \frac{\gamma}{\sqrt{3}} I_1 \]  

(9)

is assumed for determining the plastic strain rate (7), with \( \gamma \) representing a dilatancy parameter. According to the ubiquitous joint concept, transversely isotropic material behavior is taken into account by an additional yield criterion for modeling failure due to sliding along the stratification planes. For this purpose, the Mohr-Coulomb-type friction law

\[ f^{(sp)}(\mathbf{\sigma}, n) = \left( \sum_k \left| (\delta_{kl} - n_k n_l) \sigma_{ij} n_j \right|^2 \right)^{1/2} + \sigma_{ij} n_j n_i \tan(\phi) - c \]  

(10)

is considered. Therein, \( \delta_{kl} \) denotes the Kronecker delta, \( n \) the normal vector to the stratification planes, \( \sigma_{nt} \) and \( \sigma_{nn} \) the shear and the normal component of the stress vector \( f^{(n)} \) acting on the latter, and \( \phi \) and \( c \) the friction angle and the cohesion, respectively. The plastic potential function in (7) is defined for the stratification planes as

\[ g^{(sp)}(\mathbf{\sigma}, n) = \sigma_{nt} + \sigma_{nn} \tan(\psi) \]  

(11)

with \( \psi \) denoting the dilatancy angle.

### 2.3 Introduction of scalar anisotropy parameters

As a commonly employed representative for capturing inherent anisotropy in plasticity models by the direction-dependent formulation of strength-related material parameters, the framework by Pietruszczak and Mroz\(^{15}\) is selected for this review. The layered fabric of a material is taken into account by a second-order microstructure tensor

\[ M_{ij} = m_1 e_i^{(1)} e_j^{(1)} + m_2 e_i^{(2)} e_j^{(2)} + m_3 e_i^{(3)} e_j^{(3)}, \]  

(12)

formulated with respect to the principal material axes, and \( m_1, m_2 \) and \( m_3 \) denoting the respective principal values. To take into account the influence of the loading direction with respect to the principal material directions, a generalized unit loading vector \( l \) (cf. Figure 2) is proposed as

\[ l_i = \frac{1}{\sqrt{\sigma_{pq} \sigma_{pq}}} \left( \sum_k \left| \sigma_{ij} e_k^{(j)} \right|^2 \right)^{1/2} = \frac{||f^{(n)}||}{||\sigma||}. \]  

(13)

Based on (12) and (13), a scalar anisotropy parameter \( \eta \) is formulated as projection of the microstructure tensor \( M \) in the direction of the generalized unit loading vector

\[ \eta = M_{ij} l_i l_j. \]  

(14)
It is convenient to split the microstructure tensor into a spherical part $\eta_0 = M_{kk}/3$ and a deviatoric part $D$, i.e.,

$$M_{ij} = \frac{1}{3} M_{kk} \delta_{ij} + D_{ij} = \eta_0 \delta_{ij} + D_{ij}.$$  \hfill (15)

Inserting (15) into (14) yields the final representation of the scalar anisotropy parameter $\eta$

$$\eta = \eta_0 \left( 1 + \frac{1}{\eta_0} D_{ij} l_i l_j \right) = \eta_0 \left( 1 + \tilde{D}_{ij} l_i l_j \right)$$  \hfill (16)

with $\tilde{D}_{ij} = D_{ij}/\eta_0$. Thus, anisotropic behavior is incorporated into an existing isotropic model by replacing the direction-independent material parameter $\eta_0$ by $\eta$. Since in case of isotropic material behavior the principal values of the microstructure tensor $M$ are equal, the deviatoric part of the microstructure tensor $D$ vanishes and (16) degenerates to $\eta = \eta_0$. For the present case of transversely isotropic material behavior with the stratification planes corresponding to the $x'_2x'_3$-plane according to Figure 1, the principal values $m_2$ and $m_3$ of the microstructure tensor are equal, i.e., $m_2 = m_3 = m$. Thus, the deviatoric part of the microstructure tensor can be expressed in terms of one independent parameter $d$, fulfilling the condition $D_{ij} = 0$:

$$D_{ij} = \begin{bmatrix} -2d & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & d \end{bmatrix}.$$  \hfill (17)

According to (16), only a monotonic increase or decrease of a material parameter in terms of the loading direction can be modeled. This contradicts, for example, the direction-dependent behavior of the uniaxial compressive strength of layered rocks. Hence, Pietruszczak and Mroz\textsuperscript{31} proposed an extension of (16) by taking into account also higher-order terms up to degree $n + 1$

$$\eta = \eta_0 \left( 1 + \tilde{D}_{ij} l_i l_j + q_1 \cdot (\tilde{D}_{ij} l_i l_j)^2 + q_2 \cdot (\tilde{D}_{ij} l_i l_j)^3 + \cdots + q_n \cdot (\tilde{D}_{ij} l_i l_j)^{n+1} \right)$$  \hfill (18)

with $q_i, i = 1, \ldots, n,$ representing additional material parameters.

For extending the Drucker-Prager yield criterion to transversely isotropic behavior, a direction-dependent friction parameter $\mu$ is derived from (18). Replacing $\eta$ and $\eta_0$ in (18) by $\mu$ and $\mu_0$, and taking $n = 1$, yields the transversely isotropic
friction parameter

\[ \mu = \mu_0 \left( 1 + D_{ij} l_i l_j + q \cdot (D_{ij} l_i l_j)^2 \right). \]  (19)

It is characterized by two additional parameters, that is, \( d \) in \( \bar{D} \) and \( q \), which replaces \( q_1 \) in (18). Inserting (19) into (8) leads to the transversely isotropic Drucker-Prager yield function

\[ f(\sigma) = \sqrt{s_{ij} s_{ij}} + \frac{\mu_0 \left( 1 + D_{ij} l_i l_j + q \cdot (D_{ij} l_i l_j)^2 \right)}{\sqrt{3}} I_1 - \sqrt{2} \tau_y, \]  (20)

with \( \mu_0 \) as an additional material parameter. In case of isotropic rock, the last two terms on the right hand side of (19) are zero and, thus, \( \mu \) degenerates to \( \mu_0 \) representing the isotropic friction parameter. Furthermore, it is assumed that the cohesion \( \tau_y \) remains direction-independent. The transversely isotropic plastic potential function \( g(\sigma) \) is obtained by omitting the last term on the right hand side in (20) and by replacing \( \mu_0 \) by the dilatancy parameter \( \gamma_0 \):

\[ g(\sigma) = \sqrt{s_{ij} s_{ij}} + \frac{\gamma_0 \left( 1 + D_{ij} l_i l_j + q \cdot (D_{ij} l_i l_j)^2 \right)}{\sqrt{3}} I_1. \]  (21)

### 2.4 Stress projection into a fictitious isotropic configuration by means of a microstructure tensor

A further approach for considering transversely isotropic elasto-plastic material behavior, based on the notion of a fictitious isotropic configuration, was proposed by Boehler\(^1\) and applied, for example, in Semnani et al.,\(^2\) Borja et al.\(^3\) and Bryant.\(^3\) Based on a fourth-order super-symmetric linear mapping tensor \( \mathbb{P} \), the Cauchy stress tensor \( \sigma \) is projected to a mapped stress tensor \( \sigma^* \) as follows:

\[ \sigma^*_{ij} = \mathbb{P}_{ijkl} \sigma_{kl}. \]  (22)

Variables defined in the fictitious isotropic configuration are designated by a superscript asterisk (*)\(^*\). \( \mathbb{P} \) is defined as

\[ \mathbb{P}_{ijkl} = c_1 \cdot (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) + \frac{c_2}{2} \cdot (\phi_{ik} \phi_{jl} + \phi_{il} \phi_{jk}) + \frac{c_3}{4} \cdot (\delta_{ik} \phi_{jl} + \delta_{il} \phi_{jk} + \phi_{ik} \delta_{jl} + \phi_{il} \delta_{jk}), \]  (23)

incorporating transversely isotropic material behavior by means of the microstructure tensor \( \phi \) with the components

\[ \phi_{ij} = n_i n_j, \]  (24)

determined from the normal vector to the stratification planes \( n \) (cf. Figure 1). An expression of the mapping tensor \( \mathbb{P} \) in terms of the Walpole algebra is provided in Appendix A. The constants \( c_1, c_2 \) and \( c_3 \) denote parameters for considering transversal isotropy. According to Semnani et al.\(^2\) a constraint for the latter parameters is required for restricting the otherwise infinite number of parameter combinations leading to an identical mechanical response to a unique set. In the present paper this constraint is chosen as

\[ c_1 + c_2 + c_3 = 1. \]  (25)

A detailed description of the constraint (25) together with an illustrative example is given in Appendix B. Following the assumption of Boehler,\(^1\) the anisotropic yield function \( f \) in terms of the stress tensor \( \sigma \) is replaced by an isotropic yield function.
FIGURE 3 Example of applying the mapping tensor $P$ to a transversely isotropic material: (A) uniaxial stress $\sigma_{33}$, (B) resulting multiaxial stress state $\sigma^*$ in the fictitious isotropic configuration, (C) resulting strain state

function $f^*$ in terms of the mapped stress tensor $\sigma^*$:

$$f(\sigma) = f^*(\sigma^*) = f^*(P : \sigma).$$

(26)

As shown, for example, in Betten, the objectivity condition

$$f^*(P_{ijkl}, \sigma_{kl}) = f^*(\hat{a}_{im} \hat{a}_{jn} \hat{a}_{kp} \hat{a}_{iq} P_{mnpq}, \hat{a}_{kp} \hat{a}_{iq} \sigma_{pq})$$

(27)

is met for arbitrary orthogonal transformation matrices $\hat{a}_{ij}$.

In Figure 3, the stress tensor for a transversely isotropic continuum with stratification planes, defined by the normal vector $n = [0 \ 1/\sqrt{2} \ 1/\sqrt{2}]^T$, subjected to uniaxial tension in the loading direction $x_3$, is projected into the fictitious isotropic configuration by the mapping tensor $P$. Applying (22) to the respective stress tensor with the only non-zero component $\sigma_{33}$ results in the stress state shown in Figure 3B. In the mapped state, in addition to the two axial stress components, also shear stresses are present in the $x_2x_3$-plane. However, since the normal vector lies within the $x_2x_3$-plane, the components $\sigma^*_{12}, \sigma^*_{13}$, and $\sigma^*_{11}$ are zero. The resulting strain tensor, illustrated in Figure 3C, is obtained in two ways: (i) either from the original uniaxial stress state acting in the transversely isotropic material or (ii) from the mapped stress tensor $\sigma^*$ acting in an equivalent isotropic material.

According to (26) the transversely isotropic Drucker-Prager yield function is simply expressed as

$$f^*(\sigma^*) = \sqrt{s^*_{ij} s^*_{ij}} + \frac{\mu}{\sqrt{3}} I_1^* - \sqrt{2} \tau_y$$

(28)

with the first invariant of the mapped stress tensor $I_1^* = P_{mnkl} \sigma_{kl}$ and the deviatoric part of the mapped stress tensor $s^*_{ij} = P_{ijkl}^\text{dev} \sigma_{kl}$, calculated by the deviatoric part of the mapping tensor $P_{ijkl}^\text{dev} = P_{ijkl} - \frac{1}{3} \delta_{ij} P_{mnkl}$. Compared to the isotropic yield function, only the stress invariants are replaced by the respective invariants of $\sigma^*$. Applying the chain rule to the non-associated plastic strain rate (7) gives

$$\dot{\varepsilon}_{kl}^{(p)} = \lambda \cdot \frac{\partial g^*(\sigma^*)}{\partial \sigma^*_{kl}} \cdot \frac{\partial \sigma^*_{ij}}{\partial \sigma_{kl}} = \lambda P_{ijkl} \frac{\partial g^*(\sigma^*)}{\partial \sigma^*_{ij}}$$

(29)
Table 1  Parameter sets for the three approaches

|                      | Ubiquitous Joint Model | Pietruszczak’s approach | Boehler’s approach |
|----------------------|------------------------|-------------------------|--------------------|
| $\mu$ (-)            | $\tau_y$ (MPa)         | $c$ (MPa)               | $\phi$ (°)         |
| 0.577                | 12.02                  | 6                       | 27                 |
|                      | $\tau_y$ (MPa)         | $d$ (-)                 | $q$ (-)            |
| 0.577                | 7.07                   | 0.289                   | 4                  |
|                      | $\tau_y$ (MPa)         | $c_1$ (-)               | $c_3$ (-)          |
| 0.577                | 15.64                  | 1.3                     | 1.4                |

FIGURE 4  Uniaxial compressive strength in terms of the inclination angle of the stratification planes $\beta$: experimental data (taken from Pietruszczak and Mroz\textsuperscript{31}) and numerical results predicted by the three selected approaches

with

$$g^*(\sigma^*) = \sqrt{s_{ij} s_{ij}^T + \frac{\gamma}{\sqrt{3}} I_1^p}$$

(30)

defining the plastic potential function of the transversely isotropic Drucker-Prager model.

2.5  Evaluation of the selected approaches for considering transversely isotropic material behavior

The performance of the reviewed approaches for considering transversely isotropic material behavior, i.e., (i) the Ubiquitous Joint Concept (cf. Section 2.2), (ii) Pietruszczak’s approach (cf. Section 2.3) and (iii) Boehler’s approach (cf. Section 2.4) is evaluated on the example of the presented extensions of the Drucker-Prager yield criterion. Based on this comparison the most suitable approach is selected for developing a gradient enhanced transversely isotropic rock damage plasticity model in Section 3. At first, the ability of the three approaches to reproduce the direction-dependent uniaxial compressive strength of a transversely isotropic rock is investigated on the basis of the experimental data taken from Pietruszczak and Mroz\textsuperscript{31} for Tournemire shale, which can be viewed as a typical representative of layered rocks.

For this purpose, the material parameters for the three selected models, summarized in Table 1, are determined from the test data on Tournemire shale aiming at the best fit approximation of the direction-dependent uniaxial compressive strength using a nonlinear least-squares optimization algorithm. Figure 4 depicts a comparison of the predicted uniaxial compressive strength with the measured one in terms of the inclination angle $\beta$ of the stratification planes with respect to the $x_1x_2$-plane. From Figure 4 follows that the approaches by Boehler and Pietruszczak yield similar results with only slight deviations between 0° and 15° and between 75° and 90°. Compared to the experimental results, the minimum value
of the compressive strength is represented well at a slightly different corresponding inclination angle of $\beta = 45^\circ$. In contrast, the Ubiquitous Joint Model yields an entirely different behavior: Whereas the experimentally determined minimum value of the uniaxial compressive strength is perfectly matched, for a wide range of inclination angles, i.e., from $0^\circ$ to $40^\circ$ and from $77^\circ$ to $90^\circ$, the uniaxial compressive strength remains constant, i.e., direction-independent, corresponding to the isotropic strength of the rock matrix. The value of the constant compressive strength for a quite large variation of $\beta$ is determined from the material parameters of the rock matrix $\mu$ and $\tau_y$. The range for which the latter becomes active is influenced by both, the parameters of the rock matrix and the parameters $c$ and $\varphi$ of the yield function (10) for sliding failure along the stratification planes, which only becomes active if high shear stresses in the stratification planes are present or high tensile stresses perpendicular to the latter occur. This leads to an overestimation of the uniaxial compressive strength for inclination angles between $15^\circ$ and $40^\circ$ by a factor of up to 1.4. In other words, the uniaxial compressive strength is overestimated for a wide range of inclination angles, for which the isotropic failure criterion for the rock matrix is dominant and the uniaxial compressive strength is independent of the inclination of the stratification planes. This shortcoming could be remedied by formulating an anisotropic failure criterion for the rock matrix, however at the expense of losing the simplicity of the Ubiquitous Joint Model.

For the evaluation of the strength of a transversely isotropic rock, subjected to multi-axial stress states, predicted by the three selected modeling approaches, sections through the yield surfaces in the principal stress space obtained for different orientations of the stratification planes are plotted in Figure 5. In Figure 5 A-C, sections through the yield surface are plotted in the $\sigma_1 \sigma_2$-plane at $\sigma_3 = 0$ for the three selected inclination angles $\alpha = 0^\circ$, $30^\circ$ and $90^\circ$ with respect to the $x_1 \cdot x_3$-plane. For biaxial stress states the transversely isotropic yield surfaces for $\alpha = 0^\circ$ and $\alpha = 90^\circ$ are mirror-symmetric with respect to the equibiaxial-line $\sigma_1 = \sigma_2$ for all three approaches (cf. Figure 5A and C). In Figure 5 D-F, sections through the yield surface in the Rendulic plane are shown for the stratification planes inclined by the angle $\beta$ with respect to the
$x_1x_2$-plane. They represent axisymmetric stress states characterized by $\sigma_1 = \sigma_2$, occurring, e.g., in conventional triaxial compression tests.

Summarizing, from Figure 5 follows that the *Ubiquitous Joint Concept* offers a simple approach for introducing transversely isotropic (and in general anisotropic) plastic behavior by adding failure criteria for considering sliding along stratification planes, complementing to the one of the isotropic matrix. Thus, a physical interpretation of the additionally introduced parameters and consequently the direct measurement of the latter in experiments is possible. Nevertheless, from Figure 4 to Figure 5 follows that for a wide range of stress states the failure criterion of the rock matrix is dominant, leading to an over- or underestimation of the direction-dependent strength.

One of the main advantages of *Pietruszczak*’s approach is its simplicity: An isotropic yield criterion can be used for introducing transverse isotropy with the only difference that the material parameters are replaced by direction-dependent ones. Furthermore, for certain sets of material parameters, a direct fit of the anisotropic distribution function might be possible. However, according to Figure 5 some regions of the yield surface exhibit non-convexity, which is traced back to the unintended stress-dependence in addition to the direction-dependence of the material parameters, which is also discussed in Fuchs. Thus, in view of applying a transversely isotropic constitutive model in finite element simulations, in which arbitrary stress paths may occur, this approach is questionable.

According to *Boehler*’s approach, anisotropic behavior is considered by projecting the physical stress into a mapped stress, entering the isotropic failure criterion. This way, a direction-dependent strength for arbitrary multiaxial stress states is obtained. Two additional independent material parameters $c_1$ and $c_3$ are introduced for describing the influence of transversely isotropic behavior. However, they cannot be directly measured, they rather have to be determined by means of inverse parameter identification, explained in detail in Section 3.4.

Hence, for extending an existing isotropic yield criterion to transverse isotropy within the theory of plasticity, Boehler’s approach, based on the projection of the Cauchy stress to a mapped stress, is considered superior to both the Ubiquitous Joint Model and Pietruszczak’s approach. Thus, the gradient enhanced transversely isotropic rock damage plasticity model, to be presented in the following section, will be based on Boehler’s approach.

### 3 TRANSVERSELY ISOTROPIC ROCK DAMAGE PLASTICITY MODEL

For modeling both the distinct direction-dependent and the highly nonlinear material behavior of layered rock, in the following a transversely isotropic rock damage plasticity (TI-RDP) model is proposed. It is based on a combination of (i) the isotropic rock damage plasticity model by Unteregger et al. for representing the highly nonlinear material behavior of isotropic intact rock up to failure, and (ii) the approach by Boehler (cf. Section 2.4) for capturing the transversely isotropic response.

#### 3.1 Stress-strain relation

The TI-RDP model is formulated based on the theory of plasticity in the effective stress space combined with the theory of continuum damage mechanics. To this end, (4) is reformulated by replacing the Cauchy stress tensor $\sigma$ by the effective stress tensor $\bar{\sigma}$ (force per undamaged area) leading to the effective stress-strain relation expressed in terms of the global coordinate system

$$\bar{\sigma}_{ij} = C_{ijk\ell}^{(ii)} \left( \varepsilon_{k\ell} - \varepsilon_{k\ell}^{(p)} \right). \quad (31)$$

The nominal Cauchy stress tensor $\sigma$ (force per total area) is linked to the effective stress tensor by

$$\sigma_{ij} = (1 - \omega)\bar{\sigma}_{ij} \quad (32)$$

with the scalar isotropic damage variable $\omega$ ranging from 0 (undamaged material) to 1 (completely damaged material).
3.2 Plasticity formulation

For describing transversely isotropic elasto-plastic behavior, by analogy to (22) the effective stress tensor \( \bar{\sigma} \) is projected to a mapped effective stress tensor \( \bar{\sigma}^* \). Hence, the Haigh-Westergaard coordinates

\[
\bar{\sigma}_m^* = \frac{\bar{I}_1}{3}, \quad \bar{\rho}^* = \sqrt{2 \frac{J_2}{J_1}}, \quad \bar{\theta}^* = \frac{1}{3} \arccos \left( \frac{3 \sqrt{3}}{2 \sqrt{J_3}} \right)
\]

of the mapped stress tensor \( \bar{\sigma}^* \), i.e., the mean stress \( \bar{\sigma}_m^* \), the deviatoric radius \( \bar{\rho}^* \) and the Lode angle \( \bar{\theta}^* \) replace the respective quantities in the yield function of the original isotropic RDP model, proposed by Unteregger et al.\(^2\). Thus, according to (26) the equivalent isotropic yield function \( f^* \) reads

\[
f^*(\bar{\sigma}^*, q_h(\alpha_p)) = \left( 1 - q_h(\alpha_p) \right) f_{cu}^2 \left( \bar{\sigma}_m^* + \frac{\bar{\rho}^*}{\sqrt{6}} \right)^2 + \sqrt{\frac{3}{2}} \frac{\bar{\rho}^*}{f_{cu}^*} m_0^* \left( \bar{\sigma}_m^* + r(\bar{\theta}^*, e) \frac{\bar{\rho}^*}{\sqrt{6}} \right) - q_h^2(\alpha_p).
\]

The equivalent isotropic yield surface exhibits a cap for predominant hydrostatic loading of the mapped stress tensor and evolves depending on a stress-like hardening variable \(0 \leq q_h \leq 1\). The parameters \( f_{cu}^* \) and \( m_0^* \) represent the equivalent isotropic uniaxial compressive strength and the friction parameter, respectively, and \( r(\bar{\theta}^*, e) \) designates the Willam-Warnke function\(^3\) for controlling the shape of the yield function in deviatoric planes with the eccentricity parameter \( e \) ranging between 0.5 (triangular shape) and \( < e \leq 1 \) (circular shape) for ensuring a convex shape. In case of uniaxial compression perpendicular to the stratification planes, characterized in Voigt-notation by \( \bar{\sigma}^* = [-\sigma 0 0 0 0 0]^T \), the equivalent isotropic yield surface reduces to \( f^* = (\sigma/f_{cu}^*)^2 - q_h^2 \). It can be seen that \( q_h \) controls the uniaxial compressive stress normalized by the uniaxial compressive strength.

The evolution of the plastic strain is governed by the non-associated flow rule (29) with the non-associated plastic potential function of the original isotropic rock damage plasticity model, however, formulated in terms of the mapped effective stress tensor:

\[
g^*(\bar{\sigma}^*, q_h(\alpha_p)) = \left( 1 - q_h(\alpha_p) \right) f_{cu}^* \left( \bar{\sigma}_m^* + \frac{\bar{\rho}^*}{\sqrt{6}} \right)^2 + \sqrt{\frac{3}{2}} \frac{\bar{\rho}^*}{f_{cu}^*} m_1^* \left( \bar{\sigma}_m^* + m_2^* \frac{\bar{\rho}^*}{\sqrt{6}} \right).
\]

For obtaining a realistic volumetric plastic strain response for rock, two equivalent isotropic dilatancy parameters \( m_{g1}^* \) and \( m_{g2}^* \) are introduced. Considering (25), \( m_{g1}^* \) can be determined from compression tests, loaded perpendicular to the stratification planes and triaxial compression tests, loaded in the principal material directions described in Unteregger,\(^3\) whereas \( m_{g2}^* \) is a dependent parameter.

The stress-like hardening variable

\[
q_h(\alpha_p) = \begin{cases} 
\frac{f_{cy}^*}{f_{cu}^*} + \left( 1 - \frac{f_{cy}^*}{f_{cu}^*} \right) \alpha_p \left( \alpha_p^2 - 3 \alpha_p + 3 \right) & \alpha_p < 1 \\
1 & \alpha_p \geq 1 
\end{cases}
\]

with \( f_{cy}^* \) denoting the equivalent isotropic compressive yield stress, is formulated in terms of the strain-like internal variable \( \alpha_p \). The initial yield surface is obtained for \( \alpha_p = 0 \), with increasing values of \( \alpha_p \) the yield surface is expanding until at \( \alpha_p = 1 \) the ultimate strength surface is attained, corresponding to the smooth version of the Hoek-Brown failure criterion for intact rock, proposed by Menétrey and Willam.\(^3\)

In Figure 6 the evolution of the yield surface according to (34) is illustrated during strain hardening in the principal effective stress space for three different inclination angles of the stratification planes \( \beta = 0^\circ, 45^\circ \) and \( 90^\circ \), respectively. Since the stratification planes are rotated by \( \beta \) around the \( x_2 \)-axis, the uniaxial strength for loading in \( x_2 \)-direction does
FIGURE 6  Evolving yield surface of the transversely isotropic rock damage plasticity (TI-RDP) model for three different inclination angles of the stratification planes: \( \beta = 0^\circ \) (left), \( \beta = 45^\circ \) (center) and \( \beta = 90^\circ \) (right), each for the three different values of the hardening variable: \( q_h = 0 \) (yellow), \( q_h = 1/2 \) (grey) and \( q_h = 1 \) (blue)

not depend on \( \beta \). For \( \beta = 0^\circ \), the uniaxial strength in the \( x_1x_2 \) - plane is constant for arbitrarily chosen directions in this plane, as the latter represents the stratification planes. In contrast, the uniaxial strength in \( x_3 \) - direction, perpendicular to the stratification planes, differs from the one in the stratification planes. Rotating the transversely isotropic rock by \( 90^\circ \) around the \( x_2 \)-axis results in the yield surface for \( \beta = 90^\circ \). For \( \beta = 45^\circ \), the axes \( x_1 \) and \( x_3 \) are equally inclined with respect to the stratification planes, hence, the respective uniaxial strengths for loading in the directions of the respective axes are equal but differ from the uniaxial strength for loading in \( x_2 \)-direction. Note that due to the transversely isotropic behavior the apex of the initial yield surface is not located on the hydrostatic axis anymore.

In (36) \( \alpha_p \) follows from the hardening law

\[
\dot{\alpha}_p = \dot{\lambda} h^+ (\sigma^*, \alpha_p),
\]

with the hardening function

\[
h^+ (\sigma^*, q_h(\alpha_p)) = \frac{1}{x_h^+(\mathbb{H} : \sigma)} \left( 1 + \frac{3}{\beta^* + f_{cu}^*} \cdot 10^{-8} \cos^2 \left( \frac{3\beta^*}{2} \right) \right)^{1/2} \left[ \frac{\partial g^+ (\sigma^*, q_h(\alpha_p))}{\partial \sigma^*} \right],
\]

expressed in terms of the mapped stress tensor. It is obtained by replacing the effective stress in the isotropic RDP model by the mapped effective stress and controls the degree of strength mobilization with increasing plastic strains represented by the norm of the derivative of the plastic potential function with respect to the stress tensor. The term in the parentheses in (38) represents the Lode angle dependence factor \( L_f \). For mapped stress states located at the compressive meridian (\( \delta^* = \pi / 3 \)), \( L_f = 1 \) and for mapped stress states located at the tensile meridian (\( \delta^* = 0 \)), \( L_f = 4 \). In case of mapped hydrostatic stress states (\( \delta^* = 0 \)) the dependence on the Lode angle vanishes, i.e., \( L_f = 1 \). The equivalent hardening ductility measure

\[
x_h^+(\mathbb{H} : \sigma) = \begin{cases} E_h \exp \left( \frac{R_h(\mathbb{H} : \sigma)}{F_h} \right) + D_h & R_h(\mathbb{H} : \sigma) < 0 \\ A_h - (A_h - B_h) \exp \left( -\frac{R_h(\mathbb{H} : \sigma)}{C_h} \right) & R_h(\mathbb{H} : \sigma) \geq 0 \end{cases},
\]

controls the ductility in the hardening regime. \( A_h, B_h, C_h \) and \( D_h \) are independent hardening parameters and

\[
E_h = B_h - D_h, \quad F_h = (B_h - D_h) \frac{C_h}{A_h - B_h}
\]

are parameters, depending on the former. The transition point

\[
R_h(\mathbb{H} : \sigma) = -\text{tr}(\mathbb{H} : \sigma)/f_{cu}^* - G_h,
\]
Inverse of the hardening ductility measure $x_h^*(\mathbb{H} : \sigma)^{-1}$ for three different values of the hardening parameter $H_h$, i.e., $H_h = 0.05$ (dashed line), $H_h = 1.0$ (solid line), and $H_h = 2.0$ (dashdotted line) (left) and the stress-strain curves for uniaxial compression in $x_3$-direction (cf. Figure 6) for $\beta = 0^\circ$, $\beta = 45^\circ$, and $\beta = 90^\circ$ and the same values of $H_h$ (right).

which depends on the orientation of the stratification planes and the effective stress, ensures a more ductile pre-peak response in compression than in tension similarly to the original isotropic RDP model by Unteregger et al. $R_h$ depends on the model parameter $G_h \leq 1/3$ and, particularly for the TI-RDP model, on a modified mapping tensor

$$H_{ijkl} = \frac{1}{2} \cdot (H_h \cdot (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) + (1 - H_h) \cdot (\phi_{ik} \phi_{jl} + \phi_{il} \phi_{jk}))$$

(42)

for projecting the effective stress tensor into a mapped stress tensor $\mathbb{H} : \sigma$.

In (42) $H_h$ is an additional anisotropy parameter. The modified projection tensor $\mathbb{H}$ governs the direction-dependence of the ductility measure $x_h^*$, which is required for replicating a realistic hardening behavior observed in experiments on transversely isotropic rocks. For the special case of isotropic material behavior $H_h = 1$, $\mathbb{H} : \sigma$ degenerates to $\sigma$ yielding the original formulation for the isotropic RDP model.

In Figure 7 (left), the impact of the hardening parameter $H_h$ in (42) on the equivalent isotropic hardening ductility measure $x_h^*$ is illustrated in terms of the inclination angle $\beta$ of the stratification planes for a given uniaxial stress $\sigma = [0 \ 0 \ -25 \text{MPa} \ 0 \ 0 \ 0]^T$, serving as a representative for an arbitrary compressive stress $\sigma_{33}$.

For $H_h = 0.05$ the inverse of the hardening ductility measure increases monotonically with increasing inclination angles $\beta$, whereas for $H_h = 1.0$ a constant hardening ductility measure is obtained, i.e., $x_h^*(\mathbb{H} : \sigma)^{-1}$ is independent of the orientation of the material. For $H_h = 2.0$, $x_h^*(\mathbb{H} : \sigma)^{-1}$ decreases with increasing $\beta$. In Figure 7 (right) the impact of $H_h$ on the stress-strain relation is depicted for a uniaxial compression test, loaded in $x_3$-direction, for three different values of the inclination angle $\beta$.

For inclination angles of $\beta \neq 0^\circ$, the influence of values $H_h < 1$ on the stress-strain curves is manifested in a more brittle hardening behavior and a more ductile hardening behavior for $H_h > 1$. It is also shown, that due to the formulation of a modified mapping tensor $\mathbb{H}$ the stress-strain curve for loading perpendicular to the stratification planes, i.e., $\beta = 0^\circ$, is not affected by the parameter $H_h$. Note that the uniaxial compressive strength is not influenced by $H_h$ and its dependence on the loading direction with respect to the stratification planes is only controlled by the parameters $c_1$ and $c_3$.

### 3.3 Damage formulation

Damage is initiated when the stress-like hardening variable $q_h(\alpha_p)$ attains its maximum value of 1. The damage evolution law of the TI-RDP model is assumed as an exponential relation

$$\omega(\alpha_d) = 1 - \exp \left( \frac{-\alpha_d}{\varepsilon_f^*} \right),$$

(43)
controlled by the strain-like softening variable $\alpha_d$ and the equivalent isotropic softening modulus $\varepsilon^i$. The rate of the strain-like softening variable

$$\dot{\alpha}_d = \begin{cases} 0 & \alpha_p < 1 \\ \frac{\dot{\varepsilon}^{(p)}_{\text{vol}}}{x_s(\dot{\varepsilon}^{(p)})} & \alpha_p \geq 1 \end{cases}$$  \hfill (44)

is given in terms of the volumetric plastic strain rate $\dot{\varepsilon}^{(p)}_{\text{vol}} = \text{tr}(\dot{\varepsilon}^{(p)})$ and the softening ductility measure $x_s$. According to (29), the plastic strain rate is dependent on the mapping tensor $P$. Hence, the evolution of the strain-like softening variable $\alpha_d$ and consequently the evolution of the damage parameter $\omega$ is influenced by the direction-dependent material behavior.

The softening ductility measure, reading as

$$x_s(\dot{\varepsilon}^{(p)}) = 1 + A_s \left( R_s(\dot{\varepsilon}^{(p)}) \right)^{B_s}$$  \hfill (45)

accounts for the influence of multi-axial stress states on the softening material behavior with $A_s$ and $B_s$ denoting two model parameters to be calibrated from uniaxial compression tests, loaded perpendicular to the stratification planes, and triaxial compression tests, loaded in the principal material directions, with default values suggested in Unteregger et al. as $A_s = 15$ and $B_s = 2$.

$$R_s(\dot{\varepsilon}^{(p)}) = \frac{\dot{\varepsilon}^{(p)}_{\text{vol}}}{\dot{\varepsilon}^{(p)}}$$  \hfill (46)

represents an auxiliary variable calculated from the compressive part of the volumetric plastic strain rate

$$\dot{\varepsilon}^{(p)}_{\text{vol}} = \sum_{i=1}^{3} < -\dot{\varepsilon}_i^{(p)} >$$  \hfill (47)

with $\dot{\varepsilon}_i^{(p)}$ denoting the principal components of the plastic strain rate.

A detailed description of the numerical implementation of the TI-RDP model, i.e., the stress-update algorithm, is given in Appendix C.

### 3.4 Proposal for determining the material parameters from experimental data

For determining the parameters entering the plasticity formulation, the global effective stress tensor $\tilde{\sigma}$ is transformed into the local coordinate system aligned with the principal material directions as shown in Figure 1. Thus, the mapping (22) together with (23) between the local stress tensor $\sigma'$ and the mapped stress tensor is expressed in Voigt-notation as

$$\begin{bmatrix} \tilde{\sigma}_{11}^* \\ \tilde{\sigma}_{22}^* \\ \tilde{\sigma}_{33}^* \\ \tilde{\sigma}_{12}^* \\ \tilde{\sigma}_{13}^* \\ \tilde{\sigma}_{23}^* \end{bmatrix} = \begin{bmatrix} c_1 + c_2 + c_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_1 + \frac{c_2}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_1 + \frac{c_3}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_1 \end{bmatrix} \begin{bmatrix} \sigma'_{11} \\ \sigma'_{22} \\ \sigma'_{33} \\ \sigma'_{12} \\ \sigma'_{13} \\ \sigma'_{23} \end{bmatrix},$$  \hfill (48)

i.e., the mapping tensor $P$ boils down to a diagonal tensor. Five different loading scenarios are now considered:
(1) For uniaxial loading **perpendicular** to the stratification planes, i.e., uniaxial loading in $x'_1$-direction, the only non-zero effective stress component is $\sigma'_{11}$. According to (48) together with the constraint (25) the only non-zero mapped effective stress component is $\sigma^*_1 = \sigma'_{11}$. Hence, in the limit state of $\sigma^*_1$ attaining the equivalent isotropic uniaxial compressive strength $f^*_c$, the latter corresponds to the uniaxial compressive strength $f^{(1)}_c$ of a layered rock, loaded perpendicular to the stratification planes, with the superscript $(1)$ indicating the loading direction, i.e., from $\sigma^*_1 = \sigma'_{11}$ follows $f^*_c = f^{(1)}_c$. The same holds for the equivalent isotropic uniaxial compressive yield stress $f^*_y = f^{(1)}_y$ as well as the equivalent isotropic uniaxial tensile strength $f^*_t$.

(2) For uniaxial loading **parallel** to the stratification planes, let us consider, e.g., loading in the direction of the $x'_2$-axis. The effective stress component $\sigma'_{22}$ yields the only non-zero mapped effective stress component $\sigma^*_2 = c_1 \sigma'_{22}$. Thus, the limit state of attaining the uniaxial compressive strength parallel to the stratification planes is characterized by $\sigma'_{22} = f^{(2)}_c$ with the superscript $(2)$ denoting an arbitrarily chosen direction in the isotropic plane. Hence, the equivalent isotropic uniaxial compressive strength is obtained as $f^*_c = c_1 f^{(2)}_c$. Combining the equivalent isotropic uniaxial compressive strength $f^*_c = f^{(1)}_c$ from the previous loading scenario (1) with the present one yields the anisotropy parameter

$$c_1 = \frac{f^{(1)}_c}{f^{(2)}_c}. \quad (49)$$

Thus, $c_1$ represents the ratio of the uniaxial compressive strengths in directions perpendicular and parallel to the stratification planes.

(3) For simple shearing **in the isotropic stratification planes** (i.e., in the $x'_2x'_3$-plane), characterized by the only non-zero effective stress component $\sigma'_{23}$, the mapped stress component $\sigma^*_2 = c_1 \sigma'_{23}$ is obtained from (48). The limit state is described by attaining the shear strength $\sigma^*_2 = f^{(23)}_s$ in the respective plane, yielding the equivalent isotropic shear strength $f^*_s = c_1 f^{(23)}_s$.

(4) For simple shearing in planes **perpendicular** to the stratification planes let us consider, e.g., shearing in the $x'_1x'_2$-plane. The only non-zero effective stress component is $\sigma'_{12}$, which according to (48) is related to the mapped effective stress component $\sigma^*_1 = (c_1 + c_3/2) \sigma'_{12}$. The limit state is obtained by attaining the shear strength $\sigma^*_2 = f^{(12)}_s$ in the respective plane, corresponding to the equivalent isotropic shear strength $f^*_s = (c_1 + c_3/2) f^{(12)}_s$. Combining the equivalent isotropic shear strength $f^*_s = c_1 f^{(23)}_s$ from the previous loading scenario (3) with the present one yields the anisotropy parameter

$$c_3 = 2 \frac{f^{(1)}_c}{f^{(2)}_c} \left( \frac{f^{(23)}_s}{f^{(12)}_s} - 1 \right). \quad (50)$$

Finally, with $c_1$ and $c_3$ at hand, $c_2$ is obtained from the constraint (25) as

$$c_2 = 1 - c_1 - c_3. \quad (51)$$

(5) For equibiaxial compressive loading **in the stratification planes**, the only non-zero effective stress components are $\sigma'_{22} = \sigma'_{33}$. Hence, from (48) follows $\sigma^*_2 = c_1 \sigma'_{22} = \sigma^*_3$. Thus, together with (49) the equivalent isotropic biaxial compressive strength $f^*_b$ is obtained as

$$f^*_b = c_1 f^{(23)}_b = \frac{f^{(1)}_c}{f^{(2)}_c} f^{(23)}_b, \quad (52)$$

with $f^{(23)}_b$ denoting the biaxial compressive strength in the isotropic stratification planes.

Summarizing, the parameters $f^*_c, f^*_y, f^*_t, f^*_b, f^{(1)}_c, c_1$ and $c_3$ can be calibrated by means of experiments comprising uniaxial compression and tension tests as well as simple shear tests, characterized by loading directions perpendicular and parallel.
to the stratification planes and biaxial compression tests with biaxial loading in the stratification planes. According to the formulations for the eccentricity parameter $e$ and the isotropic friction parameter $m_0$, proposed by Grassl and Jirásek, the eccentricity parameter of the equivalent isotropic yield surface can be calculated as

$$e = \frac{1 + \epsilon}{2 - \epsilon}, \quad \text{with} \quad \epsilon = \frac{f_{tu}^2 - f_{tu}^2}{f_{bu}^2 - f_{tu}^2} = \frac{f_{tu}^2}{f_{tu}^2 + f_{tu}^2} \left( \frac{f_{cu}^2}{f_{bu}^2} \right)^2 - \frac{f_{tu}^2}{f_{tu}^2}$$

and the equivalent isotropic friction parameter can be determined as

$$m_0^* = 3 \frac{f_{cu}^2 - f_{tu}^2}{f_{cu}^2 + f_{tu}^2} \frac{e}{e + 1}$$

for the TI-RDP model. In the absence of respective experimental data, i.e., $f_{cu}^1, f_{cu}^2, f_{tu}^1$ and $f_{bu}^2$, Unteregger et al. suggested a default value for $e = 0.51$. In this case, the equivalent isotropic friction parameter $m_0^*$ needs to be determined via inverse parameter identification.

For calibrating the equivalent isotropic softening modulus $\varepsilon_f$ loading scenario (1) is considered with uniaxial tension perpendicular to the stratification planes. In this case, due to the formulation of the plastic potential function (35), the only non-zero plastic strain rate component controlling the evolution of $\alpha_d$ in (44) is $\dot{\varepsilon}_{\text{II}}^{(p)}$. Thus, together with the constraint (25), the equivalent softening modulus $\varepsilon_f^*$ corresponds to the softening modulus for uniaxial loading perpendicular to the stratification planes $\varepsilon_f^{(1)}$. The latter can be calibrated by the mode I fracture energy $G_f^{(1)}$ for loading perpendicular to the stratification planes.

4 | NUMERICAL STUDY

The aim of the numerical study is to evaluate the performance of the transversely isotropic rock damage plasticity model to predict the direction-dependent, highly nonlinear mechanical behavior of layered intact rock. For this purpose triaxial compression tests, documented in the paper by Niandou et al. on specimens of Tournemire shale, are considered. In the respective experimental program cylindrical specimens, characterized by a height of 75 mm and a diameter of 37 mm, were mounted in a triaxial test apparatus. The specimens were tested at different confining pressures $\sigma_0$ and inclination angles $\beta$ of the stratification planes in a displacement controlled manner by monotonically increasing the axial displacement at the top surface of the specimens. Most of the experimentally determined stress-strain curves for confining pressures of $\sigma_0 = 5, 10, 30, 40$ and 50 MPa and inclination angles of $\beta = 0^\circ, 45^\circ$ and $90^\circ$ are documented in Niandou et al.

The capability of the transversely isotropic rock damage plasticity model to reproduce the mechanical behavior of the triaxial compression tests is evaluated by (i) single element tests and (ii) three-dimensional structural finite element simulations. The latter were carried out for capturing the complex, non-uniform deformation pattern of the specimens due to strain localization in the post-peak regime.

4.1 | Parameter identification

The experimental results for two different confining pressures, i.e., $\sigma_0 = 5$ MPa and $\sigma_0 = 40$ MPa, and three different inclination angles of the stratification planes, i.e., $\beta = 0^\circ, 45^\circ$ and $90^\circ$, are used for the identification of material parameters. The remaining experimental results are used for validation. An overview of the experimental tests is given in Table 2, in which the abbreviations PI, V and N.A indicate, if the respective test is used for parameter identification or validation or if the respective test results are not available in Niandou et al.

As the tests for determining the material parameters of the TI-RDP model according to the approach, proposed in subsection 3.4, are not available in the test series of Niandou et al., the parameters listed in Table 3, are identified as follows:
Table 2 Overview of the tests, documented in Niandou et al., classified by the applied confining pressure $\sigma_0$ and the inclination angle of the stratification planes $\beta$, indicating if the respective test is used for parameter identification (PI), validation (V) or if the test results are not available (N.A.)

| Confining pressure $\sigma_0$ | 5 MPa | 10 MPa | 30 MPa | 40 MPa | 50 MPa |
|-----------------------------|-------|--------|--------|--------|--------|
| $\beta = 0^\circ$           | PI    | V      | V      | PI     | V      |
| $\beta = 45^\circ$          | PI    | N.A.   | N.A.   | PI     | V      |
| $\beta = 90^\circ$          | PI    | N.A.   | V      | PI     | V      |

Table 3 Material parameters for the TI-RDP model, identified from the experimental data for Tournemire shale (Niandou et al.)

| $E_1$ (MPa) | $E_2$ (MPa) | $\nu_{12}$ | $\nu_{23}$ | $G_{12}$ (MPa) | $f^{(1)}_{cu}$ (MPa) | $m^*_g$ | $e$ | $m^*_g$ | $f^{(1)}_{cy}$ (MPa) |
|------------|-------------|-------------|-------------|-----------------|----------------------|--------|-----|--------|---------------------|
| 7000       | 18000       | 0.2         | 0.25        | 4000            | 42.54                | 4.487  | 0.51| 4.487  | 22.25               |
| $A_h$      | $B_h$       | $C_h$       | $D_h$       | $G_h$           | $H_h$                | $c_1$  | $c_3$| $A_s$  | $B_s$               |
| (−)        | (−)         | (−)         | (−)         | (−)             | (−)                  | (−)    | (−) | (−)    | (−)                 |
| 0.022      | 0.01        | 1           | $10^{-6}$   | 0.0             | 0.01                 | 0.897  | 0.696| 15      | 2                   |

Figure 8 Computed distribution of the maximum deviatoric stress $\sigma_{dev} = \sigma_{33} - \sigma_0$ in terms of the confining pressure $\sigma_0$ and the inclination angle of the stratification planes $\beta$, compared to the experimental data by Niandou et al.

1. Based on the ultimate strength surface, i.e., the yield surface (34) at $q_h = 1$, the parameters $f^{(1)}_{cu}$ and $m^*_g$ as well as the two independent anisotropy parameters $c_1$ and $c_3$ are identified for a best fit with measured strength values using a least-squares optimization algorithm. The distribution of the maximum deviatoric stress $\sigma_{dev} = \sigma_{33} - \sigma_0$ in terms of the confining pressure $\sigma_0$ and the inclination angle $\beta$, computed from the already identified material parameters, is compared in Figure 8 with the experimental data by Niandou et al. The experimental data, plotted as orange dots, was used for parameter identification, whereas the test data, plotted as blue dots, served for validation.

2. The elastic parameters $E_1, E_2, \nu_{12}, \nu_{23}$ and $G_{12}$ and the parameters $f^{(1)}_{cy}, m^*_g, A_h, B_h, C_h$ and $H_h$ are calibrated for a best fit with the measured stress-strain curves from the tests conducted at the confining pressures $\sigma_0 = 5$ MPa and $\sigma_0 = 40$ MPa, again using a least-squares optimization approach. For the remaining parameters $D_h, G^*_h, A_s, B_s$ and $e$ the default values suggested by Unteregger et al. are employed. Figure 9 contains a comparison of the computed stress-strain curves, used for parameter identification, with the respective experimental data.

4.2 Single element tests

The TI-RDP model is validated by computing the stress-strain curves for the tests, which were not used for parameter identification, on the basis of the material parameters, summarized in Table 3. To this end, a single 3D finite element with linear shape functions, resulting in a uniform stress and strain state throughout the specimen, is used. Figure 10 depicts a comparison of the computed stress-strain diagrams with the respective experimental data.
From Figure 10 follows that by the TI-RDP model the triaxial compressive strength is predicted well for different confining pressures and loading directions with respect to the stratification planes and the computed stress-strain curves are in good agreement with the experimental data.

As a non-uniform strain distribution, which typically develops in the post-peak regime, cannot be captured by the employed single finite element with a uniform strain distribution, in Figures 9 and 10 the computed stress-strain curves are only shown for the pre-peak regime.

### 4.3 Three-dimensional finite element simulations

The application of the TI-RDP model for predicting the complex deformation states of specimens in the post-peak region of tests due to the formation of shear bands, is demonstrated by 3D finite element simulations of the triaxial compression tests, carried out by Niandou et al.\(^3\).

The geometry of the FE-model together with the prescribed boundary and loading conditions is illustrated in Figure 11. Since the stratification planes are inclined with respect to the horizontal plane by the angle \(\beta\), defined as a rotation around the \(x_2\)-axis, failure modes symmetric with respect to the \(x_1x_3\)-plane are expected. Hence, the respective symmetry is exploited by discretizing only one half of the specimen by a structured mesh, consisting of reduced integrated 20-node quadrilateral finite elements with an element size of 2.5 mm, and applying the respective boundary conditions in the plane of symmetry. Following the test procedure, in the first step of the numerical simulations the confining pressure is applied, followed by application of the axial loading in the subsequent steps by imposing a monotonically increasing uniform vertical displacement of the top surface of the specimen. To trigger localized failure, slightly reduced values for \(f^{(1)}_{cy}\) and \(f^{(1)}_{cu}\) are assigned to some elements at the center of the specimen, as indicated in Figure 11.

The respective load-displacement curves are computed in a mesh-insensitive manner by means of an implicit over-nonlocal gradient enhanced regularization of the TI-RDP model, following the approach described in Schreter et al.\(^6\) for the isotropic damage plasticity rock model. For this purpose, the internal strain-like softening variable \(\alpha_d\) is related to its nonlocal counterpart \(\tilde{\alpha}_d\) by the screened Poisson equation

\[
\alpha_d = \tilde{\alpha}_d - l^2 \nabla^2 \tilde{\alpha}_d ,
\]

with \(l\) denoting an internal length scale parameter. The discretized weak form of (55) together with the discretized weak form of the equilibrium equations results in a coupled system of equations, which is solved for the nodal displacements and the nodal values of the nonlocal field \(\tilde{\alpha}_d\) (cf. Schreter et al.\(^6\)).

In order to overcome the spurious localization of the plastic strain, an over-nonlocal softening variable \(\hat{\alpha}_d\) is defined according to Vermeer and Brinkgreve\(^40\) as

\[
\hat{\alpha}_d = m \tilde{\alpha}_d + (1-m)\alpha_d
\]
FIGURE 10  Stress-strain curves (deviatoric stress versus axial strain) for triaxial compression tests, used for validation of the TI-RDP model: experimental results by Niandou et al.\textsuperscript{3} (discrete data points) and numerical results (continuous lines)

FIGURE 11  Sketch of the geometry and boundary conditions of the FE-model representing the triaxial compression tests by Niandou et al.\textsuperscript{3}
FIGURE 12 Computed load-displacement curves and contour plots of the magnitude of the displacement vector (scale factor 1) in an advanced stage of localized deformation of the triaxial compression tests for the specimens with inclination angles of the stratification planes of $\beta = 0^\circ$, $45^\circ$ and $90^\circ$ and confining pressures of $\sigma_0 = 10$, 30 and 50 MPa

with $m$ denoting a weighting parameter for controlling the regularization. In the context of an over-nonlocal approach $m > 1$ is chosen. The damage variable $\omega$ is finally calculated by replacing the local variable $\alpha_d$ in (43) by the over-nonlocal variable $\hat{\alpha}_d$

$$\omega(\alpha_d, \hat{\alpha}_d) = 1 - \exp\left(-\frac{\hat{\alpha}_d(\alpha_d, \hat{\alpha}_d)}{\varepsilon_f^*}\right). \quad (57)$$

For the FE-simulations the additional parameters for the over-nonlocal implicit gradient enhanced formulation are chosen to $l = 5$ mm, $m = 1.05$ and $\varepsilon_f^* = 4.75 \cdot 10^{-4}$.

Apart from the notion of a mapped stress tensor in the TI-RDP model, the theoretical framework for the damage formulation of the proposed model is identical to the one of the isotropic RDP model (Schreter et al.\textsuperscript{6}). Therein, it has been already confirmed that complex shear failure modes can be predicted in a mesh-insensitive manner using the gradient enhanced formulation. For the sake of conciseness, an additional mesh refinement study based on the TI-RDP model is omitted.

In Figure 12 the computed load-displacement curves are plotted for the specimens with the inclination angles of the stratification planes $\beta = 0^\circ$, $45^\circ$ and $90^\circ$ and the three confining pressures $\sigma_0 = 10$, 30 and 50 MPa. In the respective
diagrams $\Delta F$ denotes the axial compression force, acting on a specimen in addition to the confining pressure. The predicted influence of the confining pressure on the structural behavior of the specimens is summarized as follows: with increasing confining pressure the peak value of the axial force and the corresponding displacement are increasing. A similar trend applies to the post-peak region, for which an increasingly ductile behavior is predicted for higher values of the confining pressure. The influence of the inclination angle of the stratification planes on the load-displacement curves is manifested by an increase of the initial stiffness with increasing values of the inclination angle, accompanied by an increasingly brittle behavior of the specimens.

For all specimens the peak values of the deviatoric stress, computed from the peak values of the axial force $\Delta F$ and the initial cross-sectional area $A_0 = 1075 \text{ mm}^2$ of the specimen, are in good agreement with the respective peak values, documented in Niandou et al. $^3$ and plotted in Figures 9 and 10.

The axial strain documented in Niandou et al. $^3$ was measured by strain gauges attached to the surface of the specimens halfway between the top and bottom surface. As the strain is localizing in a shear band, a non-uniform strain distribution develops. Thus, the top displacement of a specimen cannot be calculated from a single measured strain value and compared to the computed one. The computed deformation patterns of the specimens, plotted in Figure 12 for an advanced stage of localization of the displacements, can be compared to the failure modes sketched in Figure 20 in Niandou et al. $^3$. Good agreement can be stated for computed and observed failure modes.

5 | CONCLUSION AND OUTLOOK

In this paper, the extension of an isotropic rock damage plasticity (RDP) model to transversely isotropic material behavior was presented. It is denoted as transversely isotropic rock damage plasticity (TI-RDP) model.

After a review of approaches for modeling elasto-plastic anisotropic material behavior a detailed comparison of three popular approaches for considering transversely isotropic material behavior, comprising (i) the Ubiquitous Joint Concept, (ii) the introduction of scalar anisotropy parameters and (iii) the linear mapping of the Cauchy stress tensor into a fictitious isotropic configuration revealed the superiority of the latter approach which originally was proposed by Boehler. $^1$ In this approach shortcomings of the other two approaches are not present and, moreover, it is well suited for a stable and robust implementation into a FE-program on the basis of a return mapping algorithm. Hence, a quadratic rate of asymptotic convergence at the structural level is ensured, which is essential for applications in large-scale finite element simulations.

Two approaches for determining the material parameters of the TI-RDP model were proposed. In the first one they are determined from standard tests on rock samples with different orientations of the stratification planes with respect to the loading direction. Alternatively, inverse parameter identification procedures can be employed. The latter approach was demonstrated on the example of test results on Tournemire shale proposed by Niandou et al. $^3$, as some of the required results from standard tests were not available.

The TI-RDP model was validated by numerical simulations of triaxial compression tests on Tournemire shale, which were not used for parameter identification. They comprised tests on rock samples at different confining pressures and different orientations of the stratification planes with respect to the direction of axial loading (Niandou et al. $^3$). In a first step the stress-strain curves were simulated numerically up to the peak stress at the integration point level. As in the post-peak region strain softening, resulting in the formation of shear bands, is observed, 3D FE-simulations of the rock samples were performed at the structural level. For ensuring mesh-insensitive results an implicit over-nonlocal gradient enhanced regularization of the TI-RDP model was employed according to Schreter et al. $^6$.

The numerical simulations at the integration point level and at the structural level confirmed the capability of the TI-RDP model of representing the nonlinear behavior of transversely isotropic rock both in the pre-peak and post-peak domain. The predicted response of the triaxial tests up to failure corresponds well to the observed behavior in the tests, documented in Niandou et al. $^3$. In particular, for advanced stages of localization of the displacements good agreement of the predicted deformation patterns of the specimens with the failure modes reported in Niandou et al. $^3$ can be stated. This holds for different confining pressures as well as different inclination angles of the stratification planes with respect to the direction of axial loading.

Summarizing, the transversely isotropic rock damage plasticity model is well suited for modeling the inherent direction-dependent highly nonlinear behavior of stratified rock types in a realistic manner. The successful application to 3D numerical simulations of triaxial compression tests pave the way to the application of the constitutive model in 2D and large-scale 3D finite element simulations of tunnel advance in transversely isotropic rock mass and numerical simulations of anchor pull-out tests in stratified rock mass.
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DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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**APPENDIX A: MAPPING TENSOR **\( \mathbf{P} \) **EXPRESSED IN TERMS OF THE WALPOLE ALGEBRA**

As proposed in Semnani et al., the mapping of the Cauchy stress tensor into the fictitious isotropic configuration can be written as

\[
\sigma^* = \mathbf{P} : \sigma = c_1 \sigma + c_2 (\phi \cdot \sigma \cdot \phi) + \frac{c_3}{2} (\sigma \cdot \phi + \phi \cdot \sigma),
\]

(A.1)

using the anisotropy parameters \( c_1, c_2 \) and \( c_3 \). Following Walpole and Bryant et al., the mapping tensor \( \mathbf{P} \) can also be reformulated in terms of the Walpole algebra

\[
\mathbf{P} = \left( \frac{c_1 + c_2 + c_3}{2} \right) \mathbf{E}^{(1)} + \frac{c_1}{\beta} \left( \mathbf{E}^{(2)} + \mathbf{F} \right) + \frac{c_1 + c_3/2}{\gamma} \mathbf{G},
\]

(A.2)
with the anisotropy parameters $\alpha$, $\beta$ and $\gamma$ and the sub-tensors
\[
E^{(1)} = \phi_{ij}\phi_{kl}
\]
\[
E^{(2)} = \frac{1}{2} \chi_{ij}\chi_{kl}
\]
\[
F = \frac{1}{2} (\chi_{ik}\chi_{jl} + \chi_{il}\chi_{jk} - \chi_{ij}\chi_{kl})
\]
\[
G = \frac{1}{2} (\phi_{ik}\chi_{jl} + \phi_{il}\chi_{jk} + \chi_{ik}\phi_{jl} + \chi_{il}\phi_{jk})
\]

which are defined by the microstructure tensor $\phi_{ij} = n_i n_j$ (second rank projection tensor onto the direction of transverse isotropy) and its orthogonal tensor $\chi_{ij} = \delta_{ij} - n_i n_j$ (second rank projection tensor onto the plane of isotropy)

**APPENDIX B: EXPLANATION OF THE CONSTRAINT (25)**

For better understanding, the constraint (25) for the anisotropy parameters is explained on the example of the equivalent yield function (34) of the TI-RDP model. In case of uniaxial compression perpendicular to the stratification planes, the yield function simplifies to

\[
f^s = \left( \frac{\alpha \cdot \sigma_{11}}{f_{cu}^s} \right)^2 - q_h^2 = \left( \frac{\sigma_{11}}{f_{cu}^{(1)}} \right)^2 - q_h^2
\]

(B.1)

It can be seen, that the first term is affected by the material parameters $\alpha = (c_1 + c_2 + c_3)$ and $f_{cu}^s$. That is why the same material response can be obtained for different combinations of the parameters $c_1$, $c_2$, and $f_{cu}^s$. Thus, $\alpha$ is set to 1 by the constraint (25) and the axial behavior perpendicular to the stratification planes corresponds to the fictitious isotropic behavior, described by the parameters of the yield surface. Hence, the equivalent isotropic uniaxial compressive strength $f_{cu}^s$ corresponds to the uniaxial compressive strength $f_{cu}^{(1)}$ perpendicular to the stratification planes.

In case of uniaxial compression parallel to the stratification planes, the yield function simplifies to

\[
f^s = \left( \frac{c_1 \cdot \sigma_{22}}{f_{cu}^s} \right)^2 - q_h^2 = \left( \frac{\sigma_{22}}{f_{cu}^{(2)}} \right)^2 - q_h^2
\]

(B.2)

Considering $c_1 = f_{cu}^{(1)} / f_{cu}^{(2)}$ and $f_{cu}^s = f_{cu}^{(1)}$ according to (49) it can be seen, that in this case the uniaxial compressive strength parallel to the stratification planes $f_{cu}^{(2)}$ enters the yield function.

**APPENDIX C: STRESS-UPDATE ALGORITHM**

The constitutive rate equations of the TI-RDP model are integrated numerically by employing an implicit (backward) EULER algorithm. For this purpose, an elastic effective trial stress tensor $\dot{\sigma}^{(n)}_{tr} = \dot{\sigma}^{(n-1)}_{tr} + C^{(i)} : \Delta \dot{\varepsilon}^{(n)}$ is calculated from the prescribed increment $\Delta \varepsilon^{(n)}$ of the total strain tensor at the current (pseudo-)time step $t^{(n)}$. If $\dot{\sigma}^{(n)}_{tr}$ satisfies the yield condition $f^s \leq 0$ (34), it is an admissible solution. However, if it violates the yield condition (34), the KUHN-TUCKER constraints are not satisfied and the following algebraic equation system has to be solved for the unknowns $\dot{\sigma}^{(n)}$, $\alpha_p^{(n)}$, and $\Delta \lambda^{(n)}$:

\[
R_2\left(\dot{\sigma}^{(n)}, \alpha_p^{(n)}, \Delta \lambda^{(n)}\right) = \dot{\sigma}^{(n,\text{tr})} - \dot{\sigma}^{(n)} - \Delta \lambda^{(n)} : C^{(i)} : P = 0,
\]

(C.1a)

\[
R_3\left(\dot{\sigma}^{(n)}, \alpha_p^{(n)}, \Delta \lambda^{(n)}\right) = \alpha_p^{(n-1)} - \alpha_p^{(n)} + \Delta \lambda^{(n)} : h^s\left(\dot{\sigma}^{(n)}, \alpha_p^{(n)}\right) = 0,
\]

(C.1b)

\[
f^s\left(\dot{\sigma}^{(n)}, q_h\left(\alpha_p^{(n)}\right)\right) = 0.
\]

(C.1c)
The latter is solved by employing an iterative Newton-Raphson solution algorithm. For this purpose, the unknowns are collected in a solution vector \( \mathbf{x}^{(n)} = [\mathbf{\sigma}^{(n)}, \alpha_p^{(n)}, \Delta \lambda^{(n)}]^T \) at the timestep \( t^{(n)} \), which is updated in the \( i \)-th iteration step of the solution procedure by \( \mathbf{x}^{(n, i)} = \mathbf{x}^{(n, i-1)} + \delta \mathbf{x}^{(n, i)} \). The iterative corrections of the unknowns \( \delta \mathbf{x}^{(n, i)} \) are calculated by solving the linear equation system

\[
\begin{bmatrix}
K_{\sigma \sigma}^{(n, i-1)} & K_{\sigma \alpha_p}^{(n, i-1)} & K_{\sigma \Delta \lambda}^{(n, i-1)} \\
K_{\alpha_p \sigma}^{(n, i-1)} & K_{\alpha_p \alpha_p}^{(n, i-1)} & K_{\alpha_p \Delta \lambda}^{(n, i-1)} \\
K_{\Delta \lambda \sigma}^{(n, i-1)} & K_{\Delta \lambda \alpha_p}^{(n, i-1)} & K_{\Delta \lambda \Delta \lambda}^{(n, i-1)}
\end{bmatrix}
\begin{bmatrix}
\delta \mathbf{\sigma}^{(n, i)} \\
\delta \alpha_p^{(n, i)} \\
\delta \Delta \lambda^{(n, i)}
\end{bmatrix}
= \begin{bmatrix}
-R_{\sigma} \mathbf{\sigma}^{(n, i-1)}, \alpha_p^{(n, i-1)}, \Delta \lambda^{(n, i-1)} \\
-R_{\alpha_p} (\sigma^{(n, i-1)}, \alpha_p^{(n, i-1)}, \Delta \lambda^{(n, i-1)}) \\
-f^* (\sigma^{(n, i-1)}, q_h^{(n, i-1)})
\end{bmatrix}, \tag{C.2}
\]

with the derivatives

\[
K_{\sigma \sigma}^{(n, i-1)} = \frac{\partial R_{\sigma}}{\partial \sigma} \Big|_{\mathbf{x}^{(n, i-1)}} = -I - \Delta \lambda^{(n, i-1)} \cdot C^{(i)} : \mathbf{P} : \frac{\partial^2 g^*}{\partial \sigma \partial \sigma^*} (\sigma^{(n, i-1)}, \alpha_p^{(n, i-1)}), \tag{C.3}
\]

\[
K_{\sigma \alpha_p}^{(n, i-1)} = \frac{\partial R_{\sigma}}{\partial \alpha_p} \Big|_{\mathbf{x}^{(n, i-1)}} = -\Delta \lambda^{(n, i-1)} \cdot C^{(i)} : \mathbf{P} : \frac{\partial^2 g^*}{\partial \sigma \partial \alpha_p} (\sigma^{(n, i-1)}, \alpha_p^{(n, i-1)}), \tag{C.4}
\]

\[
K_{\sigma \Delta \lambda}^{(n, i-1)} = \frac{\partial R_{\sigma}}{\partial \Delta \lambda} \Big|_{\mathbf{x}^{(n, i-1)}} = C^{(i)} : \mathbf{P} : \frac{\partial g^*}{\partial \sigma} (\sigma^{(n, i-1)}, \alpha_p^{(n, i-1)}), \tag{C.5}
\]

\[
K_{\alpha_p \sigma}^{(n, i-1)} = \frac{\partial R_{\sigma}}{\partial \alpha_p} \Big|_{\mathbf{x}^{(n, i-1)}} = \Delta \lambda^{(n, i-1)} \cdot \frac{\partial h^*}{\partial \alpha_p} (\sigma^{(n, i-1)}, \alpha_p^{(n, i-1)}), \tag{C.6}
\]

\[
K_{\alpha_p \alpha_p}^{(n, i-1)} = \frac{\partial R_{\alpha_p}}{\partial \alpha_p} \Big|_{\mathbf{x}^{(n, i-1)}} = h^* (\sigma^{(n, i-1)}, \alpha_p^{(n, i-1)}), \tag{C.7}
\]

\[
K_{\alpha_p \Delta \lambda}^{(n, i-1)} = \frac{\partial R_{\alpha_p}}{\partial \Delta \lambda} \Big|_{\mathbf{x}^{(n, i-1)}} = \frac{\partial f^*}{\partial \alpha_p} (\sigma^{(n, i-1)}, \alpha_p^{(n, i-1)}), \tag{C.8}
\]

\[
K_{\Delta \lambda \sigma}^{(n, i-1)} = \frac{\partial f^*}{\partial \sigma} \Big|_{\sigma^{(n, i-1)}, \alpha_p^{(n, i-1)}}, \tag{C.9}
\]

\[
K_{\Delta \lambda \alpha_p}^{(n, i-1)} = \frac{\partial f^*}{\partial \alpha_p} \Big|_{\sigma^{(n, i-1)}, \alpha_p^{(n, i-1)}} \quad \text{and} \tag{C.10}
\]

\[
K_{\Delta \lambda \Delta \lambda}^{(n, i-1)} = \frac{\partial f^*}{\partial \Delta \lambda} \Big|_{\sigma^{(n, i-1)}, \alpha_p^{(n, i-1)}} = 0. \tag{C.11}
\]

If damage is initiated (\( \alpha_p^{(n)} = 1 \)), the local strain-like softening variable is calculated by the converged incremental plastic strain tensor \( \Delta \varepsilon_p^{(n)} \) and the incremental volumetric plastic strain \( \Delta \varepsilon_{p, \text{vol}}^{(n)} \) as

\[
\alpha_d^{(n)} = \alpha_d^{(n-1)} + \frac{\Delta \varepsilon_{p, \text{vol}}^{(n)}}{\chi_s (\Delta \varepsilon_p^{(n)})}. \tag{C.12}
\]
Finally, the damage variable $\omega^{(n)}$ is calculated by the local and the prescribed non-local softening variable, i.e., $\alpha_d^{(n)}$ and $\tilde{\alpha}_d^{(n)}$, respectively, by

$$\omega^{(n)} = 1 - \exp \left( -\tilde{\alpha}_d^{(n)} \frac{\alpha_d^{(n)}}{\varepsilon_i^*} \right), \quad (C.13)$$

and the effective stress tensor is converted to the nominal stress tensor $\sigma^{(n)}$ by

$$\sigma^{(n)} = (1 - \omega^{(n)}) \tilde{\sigma}^{(n)} \quad (C.14)$$

which enters the equilibrium equation.