Inverse Spectral Problems for Sturm-Liouville Operators on Hedgehog-type Graphs with General Matching Conditions

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Abstract. Boundary value problems on hedgehog-type graphs for Sturm-Liouville differential operators with general matching conditions are studied. We investigate inverse spectral problems of recovering the coefficients of the differential equation from the spectral data. For this inverse problem we prove a uniqueness theorem and provide a procedure for constructing its solution.

Key words: Hedgehog-type graphs, differential operators, inverse spectral problems

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1. Introduction.

1.1. We study an inverse spectral problem for Sturm-Liouville differential operators on so-called hedgehog-type graphs with general matching conditions in the interior vertices. Inverse spectral problems consist in recovering operators from their spectral characteristics. The main results on inverse spectral problems for Sturm-Liouville operators on an interval are presented in the monograph [1] and other works. Differential operators on graphs (networks, trees) often appear in natural sciences and engineering (see [2] and the references therein). Most of the results in this direction are devoted to direct problems of studying properties of the spectrum and the root functions for operators on graphs. Inverse spectral problems, because of their nonlinearity, are more difficult to investigate, and nowadays there exists only a small number of papers in this area. In particular, inverse spectral problems of recovering coefficients of differential operators on trees (i.e., on graphs without cycles) were solved in [3-4].

Inverse problems for Sturm-Liouville operators on graphs with a cycle were studied in [5] and other papers but only in the case of so-called standard matching conditions.

In the present paper we consider Sturm-Liouville operators on hedgehog-type graphs with generalized matching conditions. This class of matching conditions appears in applications and produces new qualitative difficulties in investigating nonlinear inverse coefficient problems. For studying this class of inverse problems we develop the ideas of the method of spectral mappings [6]. We prove a uniqueness theorem for this class of nonlinear inverse problems, and provide a constructive procedure for their solution. In order to construct the solution, we solve, in particular, an important auxiliary inverse problem for a quasi-periodic boundary value problem on the cycle with discontinuity conditions in interior points. The obtained results are natural generalizations of the well-known results on inverse problems for differential operators on an interval and on graphs with standard matching conditions.

1.2. Consider a compact graph $G$ in $\mathbb{R}^m$ with the set of edges $\mathcal{E} = \{e_0, \ldots, e_r\}$, where $e_0$ is a cycle, $\mathcal{E}' = \{e_1, \ldots, e_r\}$ are boundary edges. Let $\{v_1, \ldots, v_{r+N}\}$ be the set of vertices, where $V = \{v_1, \ldots, v_r\}$, $v_k \in e_k$, are boundary vertices, and $U = \{v_{r+1}, \ldots, v_{r+N}\}$ are internal vertices. $U = \mathcal{E}' \cup e_0$. The cycle $e_0$ consists of $N$ parts: $e_0 = \bigcup_{k=1}^N e_{r+k}$, $e_{r+k} = [v_{r+k}, v_{r+k+1}]$, $k = 1, N$, $v_{r+N+1} := v_{r+1}$. Each boundary edge $e_j$, $j = 1, r$ has the initial point at $v_j$, and the end point in the set $U$. The set $\mathcal{E}'$ consists of $N$ groups of edges: $\mathcal{E}' = \mathcal{E}_1 \cup \ldots \cup \mathcal{E}_N$, $\mathcal{E}' \cap e_0 = e_{r+k}$. Let $r_k$ be the number of edges in $\mathcal{E}_k$; hence $r = r_1 + \ldots + r_N$. Denote $m_0 = 1$, $m_k = r_1 + \ldots + r_k$, $k = 1, N$. Then $\mathcal{E}_k = \{e_j\}$, $j = m_{k-1} + 1, m_k$, $v_{r+k} = \bigcap_{j=m_{k-1}+1}^{m_k} e_j$, $k = 1, N$. Thus, the boundary edge $e_j \in \mathcal{E}_k$ can be viewed as the segment $[v_j, v_{r+k}]$.

Let $T_j$ be the length of the edge $e_j$, $j = 1, r + N$, and let $T := T_{r+1} + \ldots + T_{r+N}$ be the length of the cycle $e_0$. Put $b_0 = 0$, $b_k = T_{r+1} + \ldots + T_{r+k}$, $k = 1, N$. Then $b_N = T$.

Each edge $e_j$, $j = 1, r + N$ is parameterized by the parameter $x_j \in [0, T_j]$, and $x_j = 0$ corresponds to the vertex $v_j$. The whole cycle $e_0$ is parameterized by the parameter $x \in [0, T]$, where $x = x_{r+j} + b_{j-1}$ for $x_{r+j} \in [0, T_{r+j}]$, $j = 1, N$. 


An integrable function $Y$ on $G$ may be represented as $Y = \{y_j\}_{j=1,r=N}$, where the function $y_j(x_j)$, $x_j \in [0, T_j]$, is defined on the edge $e_j$. The function $y(x) = y_{r+j}(x_{r+j})$, $j = 1, N$. Let $Q = \{q_j\}_{j=1,r=N}$ be an integrable real-valued function on $G$; $Q$ is called the potential. The function $q(x)$, $x \in [0, T]$ is defined by $q(x) = q_{r+j}(x_{r+j})$, $j = 1, r + N$. Denote $U_j(Y) := y_j(0) - h_j y_j(0)$, $j = 1, r + N$, $U_{r+N} := U_{r+1}$, where $h_j$ are real numbers. Consider the following differential equation on $G$:

$$-y''(x_j) + q_j(x_j)y_j(x_j) = \lambda y_j(x_j), \quad x_j \in [0, T_j], \quad j = 1, r + N,$$

the functions $y_j, y'_j$, $j = 1, r + N$, are absolutely continuous on $[0, T_j]$ and satisfy the following matching conditions in each internal vertex $v_{\mu + 1}$, $\mu = r + 1, r + N$:

$$y_{\mu + 1}(0) = \alpha_j y_j(T_j) \quad \text{for all } e_j \in E_{\mu - r + 1}, \quad U_{\mu + 1}(Y) = \sum_{e_j \in E_{\mu - r + 1}} \beta_j y'_j(T_j), \quad (2)$$

$$y_{r + N + 1} := y_{r + 1}, \quad h_{r + N + 1} := h_{r + 1}, \quad E_{N + 1} := E_1, \quad E_{\mu - r + 1} := E_{\mu - r + 1} \cup E_{\mu},$$

where $\alpha_j$ and $b_j$ are real numbers, and $\alpha_j \beta_j \neq 0$. For definiteness, let $\alpha_j \beta_j > 0$. The matching conditions (2) are a generalization of the standard matching conditions (see [5]), where $\alpha_j = \beta_j = 1$, $h_j = 0$.

Let us consider the boundary value problem $B_0$ on $G$ for equation (1) with the matching conditions (2) and with the following boundary conditions at the boundary vertices $v_1, \ldots, v_r$:

$$U_j(Y) = 0, \quad j = 1, r.$$

Denote by $\Lambda_0 = \{\lambda_{n_0}\}_{n \geq 0}$ the eigenvalues (counting with multiplicities) of $B_0$. Moreover, we also consider the boundary value problems $B_{\nu_1, \ldots, \nu_p}$, $p = 1, r$, $1 \leq \nu_1 < \ldots < \nu_p \leq r$ for equation (1) with the matching conditions (2) and with the boundary conditions

$$y_k(0) = 0, \quad k = \nu_1, \ldots, \nu_p, \quad U_j(Y) = 0, \quad j = 1, r, \quad j \neq \nu_1, \ldots, \nu_p.$$ Denote by $\Lambda_{\nu_1, \ldots, \nu_p} := \{\lambda_{n, \nu_1, \ldots, \nu_p}\}_{n \geq 0}$ the eigenvalues (counting with multiplicities) of $B_{\nu_1, \ldots, \nu_p}$.

It will be shown in Section 2 that an important role for solving inverse problems on graphs is played by an auxiliary quasi-periodic boundary value problem on the cycle with discontinuity conditions in interior points. The parameters of this auxiliary problem depend on the parameters of $B_0$. More precisely, let us introduce real numbers $\gamma_j, \eta_j$, $(j = 1, N - 1)$, $h, \alpha, \beta$:

$$\gamma_j = \sqrt{\frac{\alpha_h}{\beta_h}}, \quad h_j = \gamma_j h_{r + j + 1}, \quad j = 1, N - 1, \quad h = h_{r + 1},$$

$$\alpha = \alpha_{r + N} \prod_{j=1}^{N-1} \frac{\gamma_j}{\beta_j}, \quad \beta = \beta_{r + j} \prod_{j=1}^{N-1} \gamma_j.$$ Clearly, $\alpha \beta > 0$, $\gamma_j > 0$, $j = 1, N - 1$. Using these parameters we consider the following quasi-periodic discontinuity boundary value problem $B$ on the cycle $e_0$:

$$-y''(x) + q(x)y(x) = \lambda y(x), \quad x \in (0, T), \quad (4)$$

$$y(0) = \alpha y(T), \quad y'(0) - h y(0) = \beta y'(T), \quad (5)$$

$$y(b_j + 0) = \gamma_j y(b_j - 0), \quad y'(b_j + 0) - \gamma_j^{-1} y'(b_j - 0) + \eta_j y(b_j - 0), \quad j = 1, N - 1, \quad (6)$$

$$0 < b_1 < \ldots < b_{N-1} < b_N = T.$$ Let $S(x, \lambda)$ and $C(x, \lambda)$ be solutions of equation (4) satisfying discontinuity conditions (6) and the initial conditions $S(0, \lambda) = C'(0, \lambda) = 0$, $S'(0, \lambda) = C(0, \lambda) = 1$. Put $\varphi(x, \lambda) = \gamma^{-1} \eta y(b_j - 0), \quad j = 1, N - 1, \quad (6)$
$C(x, \lambda) + hS(x, \lambda)$. Eigenvalues $\{\lambda_n\}_{n \geq 1}$ of $B$ coincide with zeros of the characteristic function
\[ a(\lambda) = \alpha \varphi(T, \lambda) + \beta S'(T, \lambda) - (1 + \alpha \beta). \]  
(7)

Put $d(\lambda) := S(T, \lambda)$, $Q(\lambda) = \alpha \varphi(T, \lambda) - \beta S'(T, \lambda)$. All zeros $\{z_n\}_{n \geq 1}$ of the entire function $d(\lambda)$ are simple, i.e. $d(z_n) \neq 0$, where $d(\lambda) := \frac{d}{d\lambda} d(\lambda)$. Denote $M_n = -d_1(z_n)/d(z_n)$, where $d_1(\lambda) := C(T, \lambda)$. The sequence $\{M_n\}_{n \geq 1}$ is called the Weyl sequence. Put $\omega_n = \text{sign} Q(z_n)$, $\Omega = \{\omega_n\}_{n \geq 1}$.

We choose and fix one edge $e_i \in E_i$ from each block $E_i$, $i = 1, N$, i.e. $m_{i-1} + 1 \leq \xi_i \leq m_i$. Denote by $\xi := \{k : k = \xi_1, \ldots, \xi_N\}$ the set of indices $\xi_i$, $i = 1, N$. Let $\alpha_j$ and $\beta_j$, $j = 1, r + N$, be known a priori. The inverse problem is formulated as follows.

**Inverse problem 1.** Given $2^N + r - N$ spectra $\Lambda_j$, $j = 0, r$, $\Lambda_{\nu_1, \ldots, \nu_p}$, $p = 2, N$, $1 \leq \nu_1 < \ldots < \nu_p \leq r$, $\nu_j \in \xi$, and $\Omega$, construct the potential $Q$ on $G$ and $H := [h_j]_{j=1}^{r+N}$.

Obviously, in general it is not possible to recover also all coefficients $\alpha_j$ and $\beta_j$. Note that this inverse problem is a generalization of the classical inverse problems for Sturm-Liouville operators on an interval or on graphs.

Let us formulate the uniqueness theorem for the solution of Inverse Problem 1. For this purpose together with $q$ we consider a potential $\tilde{q}$. Everywhere below if a symbol $\alpha$ denotes an object related to $q$, then $\tilde{\alpha}$ will denote the analogous object related to $\tilde{q}$.

**Theorem 1.** If $\Lambda_j = \tilde{\Lambda}_j$, $j = 0, r$, $\Lambda_{\nu_1, \ldots, \nu_p} = \Lambda_{\nu_1, \ldots, \nu_p}$, $p = 2, N$, $1 \leq \nu_1 < \ldots < \nu_p \leq r$, $\nu_j \in \xi$, and $\Omega = \tilde{\Omega}$, then $Q = \tilde{Q}$ and $H = \tilde{H}$.

We will also provide there a constructive procedure for the solution of Inverse Problem 1.

**2. Solution of the inverse problem.**

2.1. Let $S_j(x_j, \lambda)$, $C_j(x_j, \lambda)$, $j = 1, r + N$, $x_j \in [0, T_j]$, be the solutions of equation (1) on the edge $e_j$ with the initial conditions $S_j(0, \lambda) = C_j' (0, \lambda) = 0$, $S_j'(0, \lambda) = C_j(0, \lambda) = 1$. Put $\varphi_j(x, \lambda) = C_j (x_j, \lambda) + h_j S_j (x_j, \lambda)$. Clearly, $\langle \varphi_j (x, \lambda), S_j(x_j, \lambda) \rangle \equiv 1$, where $(y, z) := y' - y'\zeta$ is the Wronskian of $y$ and $z$.

Let here and below $\lambda = \rho^2$, $\tau := 1 + \rho^2 \geq 0$, $\Pi := \{\rho : \tau \geq 0\}$, $\Pi_\delta := \{\rho : \arg \rho \in [\delta, \pi - \delta]\}$, $\delta \in (0, \pi/2)$. For $|\rho| \to \infty$, $\rho \in \Pi_\delta$ the following relations hold (see [7]):

\[ a(\lambda) = \frac{(\alpha + \beta)\xi}{2} e^{-i \rho T}[1], \quad d(\lambda) = -\frac{\xi}{2i \rho} e^{-i \rho T}[1], \quad \xi := \prod_{j=1}^{N-1} \xi_j^+, \]

\[ a(\lambda) = O(e^{\rho T}), \quad d(\lambda) = O(\rho^{-1} e^{\rho T}), \quad |\rho| \to \infty, \quad \rho \in \Pi. \]

(9)

Fix $k = \frac{1}{1, r}$. Let $\Phi_k = \{\Phi_{kj}\}_{j=1}^{r+N}$, be the solution of equation (1) satisfying (2) and the boundary conditions

\[ U_j(\Phi_k) = \delta_{jk}, \quad j = 1, r, \]

where $\delta_{jk}$ is the Kronecker symbol. Denote $M_k(\lambda) := \Phi_{kj}(0, \lambda)$, $k = \frac{1}{1, r}$. The function $M_k(\lambda)$ is called the Weyl function with respect to the boundary vertex $v_k$. Clearly,

\[ \Phi_{kk}(x_k, \lambda) = S_k(x_k, \lambda) + M_k(\lambda) \varphi_k (x_k, \lambda), \quad x_k \in [0, T_k], \quad k = \frac{1}{1, r}, \]

\[ \langle \varphi_k (x, \lambda), \Phi_{kk}(x_k, \lambda) \rangle \equiv 1. \]

(12)

Denote $M_{1j}^k(\lambda) := \Phi_{kj}(0, \lambda)$, $M_{0j}^k(\lambda) := \Phi_{kj}'(0, \lambda)$. Then

\[ \Phi_{kj}(x_j, \lambda) = M_{0j}^k(\lambda) S_j(x_j, \lambda) + M_{1j}^k(\lambda) \varphi_j(x_j, \lambda), \quad x_j \in [0, T_j], \quad j = \frac{1}{1, r + N}, \quad k = \frac{1}{1, r}. \]

(13)

In particular, $M_{00}^k(\lambda) = 1$, $M_{10}^k(\lambda) = M_k(\lambda)$. Substituting (13) into (2) and (10) we obtain a linear algebraic system $D_k$ with respect to $M_{1j}^k(\lambda)$, $\nu = 0, 1$, $j = \frac{1}{1, r + N}$. The determinant $\Delta_0(\lambda)$ of $D_k$ does not depend on $k$ and has the form

\[ \Delta_0(\lambda) = \sigma(\lambda) \left( a_0(\lambda) + \sum_{k=1}^{N} \sum_{1 \leq \mu_1 < \ldots < \mu_k \leq N} a_{\mu_1 \ldots \mu_k}(\lambda) \prod_{i=1}^{k} \left( \sum_{\xi_j \in \xi_{\mu_i}} \Omega_j(\lambda) \right) \right), \]

(14)
\[
\sigma(\lambda) = \prod_{j=1}^{r}(\alpha_j \varphi_j(T_j, \lambda)), \quad \Omega_j(\lambda) = \frac{\beta_j \varphi_j'(T_j, \lambda)}{\alpha_j \varphi_j(T_j, \lambda)}, \quad a_0(\lambda) = a(\lambda), \quad a_1(\lambda) = \alpha d(\lambda). \quad (15)
\]

We note that the coefficients \(a_0(\lambda)\) and \(a_{\mu_1,\ldots,\mu_k}(\lambda)\) in (14) depend only on \(S_j^{(\nu)}(T_j, \lambda)\) and \(C_j^{(\nu)}(T_j, \lambda)\), for \(j = r+1, \ldots, r+N\). We do not need concrete formulae for the other coefficients \(a_{\mu_1,\ldots,\mu_k}(\lambda)\). The function \(\Delta(\lambda)\) is entire in \(\lambda\) of order \(1/2\), and its zeros coincide with the eigenvalues of the boundary value problem \(B_0\). The function \(\Delta(\lambda)\) is called the characteristic function for the boundary value problem \(B_0\). Let \(\Delta_{\nu_1,\ldots,\nu_p}(\lambda)\), \(p = 1, r\), \(1 \leq \nu_1 < \ldots < \nu_p \leq r\), be the function obtained from \(\Delta(\lambda)\) by the replacement of \(\varphi_j^{(\nu)}(T_j, \lambda)\) with \(S_j^{(\nu)}(T_j, \lambda)\) for \(j = \nu_1, \ldots, \nu_p\), \(\nu = 0, 1\). More precisely,

\[
\Delta_{\nu_1,\ldots,\nu_p}(\lambda) = \sigma_{\nu_1,\ldots,\nu_p}(\lambda) \left( a_0(\lambda) + \sum_{k=1}^{N} \sum_{1 \leq \mu_1 < \ldots < \mu_k \leq N} a_{\mu_1,\ldots,\mu_k}(\lambda) \right) \prod_{j=1}^{k} \left( \sum_{e_j \in E_{\mu_j}, j \neq \nu_1, \ldots, \nu_p} \Omega_j(\lambda) + \sum_{e_j \in E_{\mu_j}, j = \nu_1, \ldots, \nu_p} \Omega^0_j(\lambda) \right), \quad (16)
\]

\[
\sigma_{\nu_1,\ldots,\nu_p}(\lambda) = \prod_{j=1, j \neq \nu_1, \ldots, \nu_p}^{r} (\alpha_j \varphi_j(T_j, \lambda)) \prod_{j=\nu_1, \ldots, \nu_p}^{r} (\alpha_j S_j(T_j, \lambda)), \quad \Omega^0_j(\lambda) = \frac{\beta_j S_j'(T_j, \lambda)}{\alpha_j S_j(T_j, \lambda)}. \quad (17)
\]

The function \(\Delta_{\nu_1,\ldots,\nu_p}(\lambda)\) is entire in \(\lambda\) of order \(1/2\), and its zeros coincide with the eigenvalues of the boundary value problem \(B_{\nu_1,\ldots,\nu_p}\). The function \(\Delta_{\nu_1,\ldots,\nu_p}(\lambda)\) is called the characteristic function for the boundary value problem \(B_{\nu_1,\ldots,\nu_p}\).

Solving the algebraic system \(D_k\) we get by Cramer’s rule: \(M^s_j(\lambda) = \Delta^{s}_j(\lambda)/\Delta(\lambda)\), \(s = 0, 1, j = 1, r + N\), where the determinant \(\Delta^{s}_j(\lambda)\) is obtained from \(\Delta(\lambda)\) by the replacement of the column which corresponds to \(M^s_j(\lambda)\) with the column of free terms. In particular,

\[
M_k(\lambda) = -\Delta(\lambda)/\Delta(\lambda), \quad k = 1, r. \quad (18)
\]

2.2. It is known (see [8]) that for each fixed \(j = 1, r + N\), on the edge \(e_j\), there exists a fundamental system of solutions of equation (1) \(\{e_{j1}(x_j, \rho), e_{j2}(x_j, \rho)\}\), \(x_j \in [0, T_j]\), \(\rho \in \Pi, |\rho| \geq \rho^*\) with the properties:

1) the functions \(e_{j1}^{(\nu)}(x_j, \rho), \nu = 0, 1\), are continuous for \(x_j \in [0, T_j]\), \(\rho \in \Pi, |\rho| \geq \rho^*\);
2) for each \(x_j \in [0, T_j]\), the functions \(e_{j1}^{(\nu)}(x_j, \rho), \nu = 0, 1\), are analytic for \(\operatorname{Im} \rho > 0, |\rho| > \rho^*\);
3) uniformly in \(x_j \in [0, T_j]\), the following asymptotical formulae hold

\[
e_{j1}^{(\nu)}(x_j, \rho) = (i \rho)^\nu \exp(i \rho x_j)[1], \quad e_{j2}^{(\nu)}(x_j, \rho) = (-i \rho)^\nu \exp(-i \rho x_j)[1], \quad \rho \in \Pi, |\rho| \to \infty, \quad (19)
\]

where \([1] = 1 + O(\rho^{-1})\). Fix \(k = 1, r\). One has

\[
\Phi_{kj}(x_j, \lambda) = A_{kj}^{(\nu)}(\rho) e_{j1}(x_j, \rho) + A_{kj}^{(\nu)}(\rho) e_{j2}(x_j, \rho), \quad x_j \in [0, T_j], \quad j = 1, r + N. \quad (20)
\]

Substituting (20) into (2) and (10) we obtain a linear algebraic system \(D_k^{\nu}\) with respect to \(A_{kj}^{(\nu)}(\rho)\), \(\nu = 0, 1, j = 1, r + N\). The determinant \(\delta(\rho)\) of \(D_k^{\nu}\) does not depend on \(k\), and has the form

\[
\delta(\rho) = \left( \delta_0 + O\left( \frac{1}{\rho} \right) \right) \rho^{r+N} \exp \left( -i \rho \sum_{j=1}^{r+N} T_j \right), \quad (21)
\]

where \(\delta_0\) is the determinant obtained from \(\delta(\rho)\) by the replacement of \(e_{j1}^{(\nu)}(0, \rho), e_{j2}^{(\nu)}(T_j, \rho), e_{j2}^{(\nu)}(0, \rho), e_{j2}^{(\nu)}(T_j, \rho)\) and \(h_j\) with \(1, 0, (-1)^\nu, (-1)^\nu\) and 0, respectively. We assume that \(\delta_0 \neq 0\). This condition is called the regularity condition for matching. Differential operators on \(G\) which do not satisfy the regularity condition, possess qualitatively different properties in
In particular, \( \alpha \) matching conditions we have in this paper; they require a separate investigation. We note that for classical Kirchhoff’s boundary value problem

\[
q \omega \text{ the potential } \\
\text{obviously. Solving the algebraic system } D_k^h \\
\text{and using (19)-(21) we get for each fixed } x_k \in [0, T_k] \ :
\]

\[
\Phi_k^{(\nu)}(x_k, \lambda) = (i \rho)^{\nu-1} \exp(i \rho \omega x_k)[1], \\
\rho \in \Pi_\delta, |\rho| \to \infty.
\]  

(22)

In particular, \( M_k(\lambda) = (i \rho)^{-1}[1], \rho \in \Pi_\delta, |\rho| \to \infty. \)

Moreover, uniformly in \( x_k \in [0, T_k], \)

\[
\varphi_k^{(\nu)}(x_k, \lambda) = \frac{1}{2} ((i \rho)^{\nu} \exp(i \rho \omega x_k)[1] + (-i \rho)^{\nu} \exp(-i \rho \omega x_k)[1]), \\
\rho \in \Pi, |\rho| \to \infty.
\]  

(23)

Using (14), (23), (8) and (9), by the well-known method, one can obtain the following properties of the characteristic function \( \Delta_0(\lambda) \) and the eigenvalues \( \Lambda_0 \) of the boundary value problem \( B_0 \).

1. For \( \rho \in \Pi, |\rho| \to \infty \) : \( \Delta_0(\lambda) = O(\exp(\tau \sum_{j=1}^{\tau+N} T_j)). \)

2. There exist \( h > 0, C_h > 0 \) such that \( |\Delta_0(\lambda)| \geq C_h \exp(\tau \sum_{j=1}^{\tau+N} T_j) \) for \( \tau \geq h \). Hence, the eigenvalues \( \lambda_{n0} = \rho_{n0}^2 \) lie in the domain \( 0 \leq \tau < h. \)

3. The number \( N_\xi \) of zeros of \( \Delta_0(\lambda) \) in the rectangle \( \Lambda_\xi = \{ \rho : \tau \in [0, h], \Re \rho \in [\xi, \xi + 1] \} \) is bounded with respect to \( \xi. \)

4. For \( n \to \infty, \rho_{n0} = \rho_{n0}^0 + O(1/\rho_{n0}^0), \) where \( \lambda_{n0}^0 = (\rho_{n0}^0)^2 \) are the eigenvalues of the boundary value problem \( B_0 \) with \( Q = 0 \) and \( H = 0. \)

The characteristic functions \( \Delta_{\nu_1, \ldots, \nu_p}(\lambda) \) have similar properties. In particular, for \( \rho \in \Pi, |\rho| \to \infty, \Delta_{\nu_1, \ldots, \nu_p}(\lambda) = O(|\rho|^{-p} \exp(\tau \sum_{j=1}^{\tau+N} T_j)). \)

Using the properties of the characteristic functions and Hadamard’s factorization theorem, one gets that the specification of the spectrum \( \Lambda_0 \) uniquely determines the characteristic function \( \Delta_0(\lambda) \), i.e. if \( \Lambda_0 = \Lambda^0_0 \), then \( \Delta_0(\lambda) \equiv \tilde{\Delta}_0(\lambda) \). Analogously, if \( \Lambda_{\nu_1, \ldots, \nu_p} = \Lambda_{\nu_1, \ldots, \nu_p}^0 \), then \( \Delta_{\nu_1, \ldots, \nu_p}(\lambda) \equiv \tilde{\Delta}_{\nu_1, \ldots, \nu_p}(\lambda) \). The characteristic functions can be constructed as the corresponding infinite products (see [1] for details).

2.3. Fix \( k = \overline{1, r} \), and consider the following auxiliary inverse problem on the edge \( e_k \), which is called IP(k).

**IP(k).** Given two spectra \( \Lambda_0 \) and \( \Lambda_k \), construct \( q_k(x_k), x_k \in [0, T_k] \), and \( h_k \).

In IP(k) we construct the potential only on the edge \( e_k \), but the spectra bring a global information from the whole graph. In other words, IP(k) is not a local inverse problem related to the edge \( e_k \). Let us prove the uniqueness theorem for the solution of IP(k).

**Theorem 2.** Fix \( k = \overline{1, r} \). If \( \Lambda_0 = \tilde{\Lambda}_0 \) and \( \Lambda_k = \tilde{\Lambda}_k \), then \( q_k(x_k) = \tilde{q}_k(x_k) \), a.e. on \([0, T_k] \), and \( h_k = \tilde{h}_k \). Thus, the specification of two spectra \( \Lambda_0 \) and \( \Lambda_k \) uniquely determines the potential \( q_k \) on the edge \( e_k \), and the coefficient \( h_k \).

**Proof.** Since \( \Lambda_0 = \tilde{\Lambda}_0, \Lambda_k = \tilde{\Lambda}_k \), it follows that \( \Delta_0(\lambda) \equiv \tilde{\Delta}_0(\lambda), \Delta_k(\lambda) \equiv \tilde{\Delta}_k(\lambda) \), and according to (18),

\[
M_k(\lambda) = \tilde{M}_k(\lambda).
\]  

(24)

Consider the functions

\[
P_{1s}^k(x_k, \lambda) = (-1)^{s-1} \left( \varphi_k^{(s)}(x_k, \lambda) \Phi_k^{(2-s)}(x_k, \lambda) - \hat{\varphi}_k^{(2-s)}(x_k, \lambda) \Phi_k^{(s)}(x_k, \lambda) \right), \quad s = 1, 2.
\]  

(25)

Using (12) we calculate

\[
\varphi_k^{(s)}(x_k, \lambda) = P_{11}^k(x_k, \lambda) \hat{\varphi}_k^{(s)}(x_k, \lambda) + P_{12}^k(x_k, \lambda) \varphi_k^{(s)}(x_k, \lambda).
\]  

(26)

It follows from (23), (24) and (26) that

\[
P_{1s}^k(x_k, \lambda) = \delta_{1s} + O(\rho^{-1}), \quad \rho \in \Pi_\delta, |\rho| \to \infty, x_k \in (0, T_k].
\]  

(27)
According to (11) and (25),
\[
P_{1s}^k(x_k, \lambda) = (-1)^{s-1} \left( (\varphi_k(x_k, \lambda)\tilde{\varphi}_k^{(2-s)}(x_k, \lambda) - \tilde{\varphi}_k^{(2-s)}(x_k, \lambda)S_k(x_k, \lambda) \right)
+ (M_k(\lambda) - \tilde{M}_k(\lambda))\varphi_k(x_k, \lambda)\tilde{\varphi}_k^{(2-s)}(x_k, \lambda)).
\]

It follows from (24) that for each fixed \( x_k \), the functions \( P_{1s}^k(x_k, \lambda) \) are entire in \( \lambda \) of order 1/2. Together with (27) this yields \( P_{11}^k(x_k, \lambda) \equiv 1, \ P_{12}^k(x_k, \lambda) \equiv 0 \). Substituting these relations into (26) we get \( \varphi_k(x_k, \lambda) \equiv \tilde{\varphi}_k(x_k, \lambda) \) for all \( x_k \) and \( \lambda \), and consequently, \( q_k(x_k) = \tilde{q}_k(x_k) \) a.e. on \([0, T_k]\), \( h_k = \tilde{h}_k \). Theorem 2 is proved.

Using the method of spectral mappings [6] for the Sturm-Liouville operator on the edge \( e_k \) one can get a constructive procedure for finding \( q_k \) and \( h_k \).

Now we study the following auxiliary inverse problem on the cycle \( e_0 \), which is called IP(0). Consider the boundary value problem \( B \) of the form (4)-(6), where the parameters of \( B_0 \) are defined by (3), and \( \alpha, \beta \) are known.

**IP(0).** Given \( a(\lambda), d(\lambda) \) and \( \Omega \), construct \( q(x), x \in [0, T], h, \gamma_j \) and \( \eta_j, j = \overline{1, N-1} \).

This inverse problem is a generalization of the classical periodic inverse problem. Moreover, for the standard matching conditions ( \( \alpha_j = \beta_j = 1, h_j = 0 \) ), IP(0) coincides with the classical periodic inverse problem.

This inverse problem IP(0) was solved in [7], where the following theorem is established.

**Theorem 3.** The specification \( a(\lambda), d(\lambda) \) and \( \Omega \) uniquely determines \( q(x), h, \gamma_j \) and \( \eta_j, j = \overline{1, N-1} \). The solution of IP(0) can be found by the following algorithm.

**Algorithm 1.**

1) Find zeros \( \{z_n\}_{n \geq 1} \) of the entire function \( d(\lambda) \).
2) Calculate \( M_n = -d_k(z_n)/d(z_n) \).
3) Find \( \tilde{d}(z_n) \).
4) Calculate the Weyl sequence \( \{M_n\}_{n \geq 1} \) via \( M_n = -d_k(z_n)/d(z_n) \).
5) Find \( \tilde{d}(z_n) \).
6) From the given data \( \{z_n, M_n\}_{n \geq 1} \) construct \( q(x), \gamma_j, \eta_j, j = \overline{1, N-1} \), by solving the inverse Dirichlet problem with discontinuities inside the interval (see [9]).
7) Find \( S(T, \lambda), S'(T, \lambda) \) and \( C(T, \lambda) \).
8) Calculate \( h \), using (7).

2.4. Let us go on to the solution of Inverse problem 1. Firstly, we give the proof of Theorem 1. Assume that \( \Lambda_k = \Lambda_{k}, k = \overline{0, r}, \Lambda_{\nu_1, \ldots, \nu_p} = \Lambda_{\nu_1, \ldots, \nu_p} \), \( p = 2, \overline{N}, 1 \leq \nu_1 < \ldots < \nu_p \leq r \), \( \nu_j \in \xi \), and \( \Omega = \tilde{\Omega} \). Then one has

\[
\Delta_k(\lambda) \equiv \tilde{\Delta}_k(\lambda), \quad k = \overline{0, r},
\]

\[
\Delta_{\nu_1, \ldots, \nu_p}(\lambda) \equiv \tilde{\Delta}_{\nu_1, \ldots, \nu_p}(\lambda), \quad p = 2, \overline{N}, 1 \leq \nu_1 < \ldots < \nu_p \leq r, \nu_j \in \xi.
\]

Moreover, according to (3), \( \gamma_j = \tilde{\gamma}_j \), \( j = \overline{1, N-1} \), and \( \alpha = \tilde{\alpha}, \beta = \tilde{\beta} \). Using Theorem 2, we get \( q_k(x_k) = \tilde{q}_k(x_k) \) a.e. on \([0, T_k]\) and \( h_k = \tilde{h}_k \), \( k = \overline{1, r}, \) and consequently,

\[
C_k(x_k, \lambda) \equiv \tilde{C}_k(x_k, \lambda), \quad S_k(x_k, \lambda) \equiv \tilde{S}_k(x_k, \lambda), \quad \varphi_k(x_k, \lambda) \equiv \tilde{\varphi}_k(x_k, \lambda), \quad k = \overline{1, r}.
\]

By virtue of (15), (17) and (28) one has

\[
\sigma(\lambda) \equiv \tilde{\sigma}(\lambda), \quad \sigma_{\nu_1, \ldots, \nu_p}(\lambda) \equiv \tilde{\sigma}_{\nu_1, \ldots, \nu_p}(\lambda), \quad \Omega_j(\lambda) \equiv \tilde{\Omega}_j(\lambda), \quad \Omega^0_j(\lambda) \equiv \tilde{\Omega}^0_j(\lambda), \quad j = \overline{1, r}.
\]

Using (14) and (16), we obtain, in particular, \( a_0(\lambda) = \tilde{a}(\lambda), a_1(\lambda) = \tilde{a}_1(\lambda) \). In view of (15), this yields \( a(\lambda) = \tilde{a}(\lambda), d(\lambda) = \tilde{d}(\lambda) \). It follows from Theorem 3 that \( q_k(x_k) = \tilde{q}_k(x_k) \) a.e. on \([0, T_k]\) and consequently, \( h_k = \tilde{h}_k \), \( k = \overline{1, r}, \).
on \([0, T_k]\), \(k = r + 1, r + N\), and \(h = \tilde{h}, \eta_j = \tilde{\eta}_j, j = 1, N - 1\). Taking (3) into account, we get \(H = \tilde{H}\). Theorem 1 is proved.

The solution of Inverse problem 1 can be constructed by the following algorithm.

**Algorithm 2.** Given \(\Lambda_k, k = 0, r\); \(\Lambda_{\nu_1}, \ldots, \Lambda_{\nu_p}\), \(p = 2, N\), \(1 \leq \nu_1 < \ldots < \nu_p \leq r\), \(\nu_j \in \xi\), and \(\Omega\).

1) Construct \(\Delta_k(\lambda)\) and \(\Delta_{\nu_1}, \ldots, \Delta_{\nu_p}(\lambda)\).
2) Calculate \(\gamma_j, j = 1, N - 1\), \(\alpha\) and \(\beta\), using (3).
3) For each fixed \(k = 1, r\), solve the inverse problem \(IP(k)\) and find \(q_k(x_k), x_k \in [0, T_k]\) on the edge \(e_k\) and \(h_k\).
4) For each fixed \(k = 1, r\), construct \(C_k(x_k, \lambda), S_k(x_k, \lambda)\) and \(\varphi_k(x_k, \lambda), x_k \in [0, T_k]\).
5) Calculate \(a(\lambda)\) and \(d(\lambda)\), using (14), (15) and (16).
6) From the given \(a(\lambda), d(\lambda)\) and \(\Omega\), construct \(q_k(x_k), [0, T_k]\), \(k = r + 1, r + N\), \(h\) and \(\eta_j, j = 1, N - 1\).
7) Find \(H\), using (3).

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