A class of integrable lattices and KP hierarchy

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Abstract. We introduce a class of integrable $l$-field first-order lattices together with corresponding Lax equations. These lattices may be represented as consistency condition for auxiliary linear systems defined on sequences of formal dressing operators. This construction provides simple way to build lattice Miura transformations between one-field lattice and $l$-field ($l \geq 2$) ones. We show that the lattices pertained to above class is in some sense compatible with KP flows and define the chains of constrained KP Lax operators.

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1. Introduction

In recent times, the Kadomtsev–Petviashvili (KP) hierarchy and its possible reductions has drawn much attention because of vast variety of applications to different branches of physics. Recall that in Sato framework the KP hierarchy is given by Lax equation

\[ \partial_p Q = \left[ (Q^p)_+, Q \right] \]  

on first-order pseudo-differential operator (ΨDO) \( Q = \partial + \sum_{k=1}^{\infty} U_k(t) \partial^{-k} \). Equivalent form of Lax equation (1) is Sato–Wilson equation

\[ \partial_p \hat{w} = -\left( \hat{w} \partial_p \hat{w}^{-1} \right)_{\hat{w}} = \left( \hat{w} \partial_p \hat{w}^{-1} \right)_{\hat{w}} - \hat{w} \partial_p \]

on a formal dressing operator \( \hat{w} \) being defined through \( Q = \hat{w} \partial \hat{w}^{-1} \). Another objects associated with \( Q \) are formal Backer–Akhiezer wave function \( \psi \) and its conjugate \( \psi^* \) defined by \( \psi(t, z) = \hat{w} \exp(\xi(t, z)) \) and \( \psi^*(t, z) = \hat{w}^{s-1} \exp(-\xi(t, z)) \), respectively, with \( \xi(t, z) = \sum_{p=1}^{\infty} t_p z^p \).

The very important observation is that the KP wave functions satisfy the bilinear identity \( \text{res}_z \psi(t, z) \psi^*(t', z) = 0 \) (3) providing description of KP hierarchy in terms of Hirota’s bilinear equations on \( \tau \)-function. After introducing \( \tau(t) \) through

\[ \psi(t, z) = \frac{\tau(t - [z^{-1}])}{\tau(t)} \exp(\xi(t, z)), \quad \psi^*(t, z) = \frac{\tau(t + [z^{-1}])}{\tau(t)} \exp(-\xi(t, z)) \]

the identity (3) becomes

\[ \text{res}_z \tau(t - [z^{-1}]) \tau(t' + [z^{-1}]) \exp(\xi(t - t', z)) = 0 \]

with \( [z^{-1}] = (1/z, 1/(2z^2), 1/(3z^3), ...) \).

In present article we suggest construction which provides relationship between some class of multi-field lattices and chains of constrained KP Lax operators. Recently constrained KP (cKP) hierarchies gained a lot of interest because of its strong relationship with multi-matrix models in non-perturbative string theory \([4]\), \([5]\). As is known cKP hierarchies contain a number of interesting, from the physical point of view, integrable nonlinear evolution equations whose applications range from hydrodynamics to modern theories in high-energy elementary particle physics. Important property of cKP hierarchies is the existence of discrete symmetry structure defined by successive Darboux – Bäcklund transformations of suitable Lax operators \([5]\). The integrable lattices play key rôle in this situation and serve as a source of discrete symmetries for evolution type systems \([6]\), \([7]\), \([8]\) “gluing” a copies of the equations of cKP. In particular it is well known relationship between Toda and Volterra lattices and some class of cKP hierarchies.
We exhibit in this paper a class of integrable \( l \)-field lattices. They are defined as consistency condition of some auxiliary linear systems. In fact these auxiliary systems define compatible pairs of shifts \( s_1 \) and \( s_2 \) on sequences of dressing operators \( \{ \hat{w}_i, \in \mathbb{Z} \} \). This construction naturally provides building of infinite sets of constrained Lax operators “glued” together by compatible pair of similarity gauge transformations. The important property of the lattices that they are consistent in a sense with KP flows given by Lax equation (1) or Sato–Wilson equation (2).

This paper in many respects was influenced by the work \[2\] where the system

\[
(\partial + u_i)\hat{w}_i = \hat{w}_{i+1}\partial,
\]

\[
\partial_p\hat{w}_i = -(Q^p_i)\hat{w}_i, \quad \text{where} \quad Q_i = \hat{w}_i\partial_p\hat{w}_i^{-1}
\]

\[
\partial_p u_i = (Q^p_{i+1})(\partial + u_i) - (\partial + u_i)(Q^p_i), \quad i \in \mathbb{Z}
\]

was introduced and referred to as modified KP hierarchy. It was shown there that modified KP hierarchy in fact is equivalent to discrete KP (1-Toda lattice) hierarchy. Lax operators \( Q_i \) in this case are unconstrained and only connected with each other by similarity gauge transformations. Our construction provides building of sequences of constrained KP copies. As we have mentioned above this is achieved by imposing on the sequences \( \{ \hat{w}_i, i \in \mathbb{Z} \} \) two compatible auxiliary linear constraints. A compatibility of the latter is guaranteed by equations of integrable lattices. Another important inspiration of this work is Krichever’s rational reductions of the KP hierarchy \[3\] (or constrained KP, as these commonly called).

A scheme of the paper is as follows. In section 2 we present integrable chains. In section 3 we show how one can construct Miura lattice transformations and provide the reader by some examples. Also we write down Miura transformations in explicit form for two-field lattices. In section 4 we show compatibility of the lattices with KP flows.

2. Integrable lattices

Main objective of this section is to define a class of an integrable lattices. The term “integrable” in our context means only that given lattice admits Lax representation. We will consider first-order differential-difference systems on finite number of fields \( a_k(i, x) \) being functions of discrete variable \( i \in \mathbb{Z} \) and continuous variable \( x \in \mathbb{R}^1 \).

Before proceeding further let us give some remarks concerning notations which will be used in the following. The unknown functions (fields) depend on spatial variable \( x \in \mathbb{R}^1 \) and some evolution parameters \( t_p \). We use short-hand notation \( \hat{f} \) to denote infinite collection of independent variables \( (x, t_2, t_3, ...) \). The symbols \( \partial \) and \( \partial_p \) stand for derivation with respect to \( x \) and \( t_p \), respectively. Let \( A \) be a \( \Psi \)DO \( A = \sum_{i=-\infty}^{N} a_i(x)\partial^i \),
of order $N$. The subscripts $+ \text{ and } -$ mean, respectively, differential and integral part of $A$. We write $\partial_p A$ to denote derivative of $\Psi DO A$ with respect to $t_p$ (not a product).

For an arbitrary pair of integers $n \in \mathbb{N}$ and $m \leq n - 1$ we define infinite collection of first-order differential operators

$$H_i = \partial - \sum_{k=1}^{n} a_0(i + k - 1, x), \quad i \in \mathbb{Z},$$

and $\Psi DO$’s

$$G_i = \partial + \sum_{k=1}^{m} a_0(i - k, x) + \sum_{k=1}^{l-1} a_k(i + m, x) H_{i-k_n}^{-1} \ldots H_{i-2n}^{-1} H_{i-n}^{-1},$$

for $m \leq -1$ and

$$G_i = \partial - \sum_{k=1}^{m} a_0(i + k - 1, x) + \sum_{k=1}^{l-1} a_k(i + m, x) H_{i-k_n}^{-1} \ldots H_{i-2n}^{-1} H_{i-n}^{-1},$$

for $m = 0, \ldots, n - 1$. Notice that definition of $H_i$’s and $G_i$’s involves a finite number of fields \{a_0(i, x), a_1(i, x), \ldots, a_{l-1}(i, x)\}.

Let us define following auxiliary equations on infinite collection of dressing operators \{\hat{w}_i, i \in \mathbb{Z}\}:

$$G_i \hat{w}_i = \hat{w}_{i-m} \partial, \quad H_i \hat{w}_i = \hat{w}_{i-n} \partial$$

Obviously the latter can be rewritten in terms of BA functions as

$$G_i \psi_i = z\psi_{i+m}, \quad H_i \psi_i = z\psi_{i+n}.$$  \hspace{1cm} (6)

The linear system (6) (or (3)) is overdetermined but one can show that the compatibility conditions of (3) are well-determined system of equations for the fields $a_k(i, x)$.

Formally, consistency condition of (6) is given by

$$G_{i+n} H_i = H_{i+m} G_i$$  \hspace{1cm} (7)

or equivalently as $H_{i+m}^{-1} G_{i+n} = G_i H_i^{-1}$. The relation (7) is not convenient for further calculations. The technical observation which is helpful in this situation is that (3) can be rewritten in terms of $(L, A)$-pair

$$L(\psi_i) = z\psi_i, \quad \psi'_i = A(\psi_i)$$

with $L$ and $A$ being difference operators acting on the space of sequences of BA functions \{\psi_i, \ i \in \mathbb{Z}\} as

$$L(\psi_i) = z\psi_{i+n-m} + \left(\sum_{s=1}^{n-m} a_0(i + s - 1)\right) \psi_{i-m} + \sum_{j=1}^{l-1} a_j(i) \psi_{i-j_n},$$

$$A(\psi_i) = z\psi_{i+n} + \left(\sum_{s=1}^{n} a_0(i + s - 1)\right) \psi_i.$$  \hspace{1cm} (8)
Then consistency condition of (6) are expressed in a form of the Lax equation

\[ L' = [A, L] = AL - LA. \]  

(9)

One can show that the latter holds if the functions \( a_k(i, x) \) satisfy \( l \)-field lattice

\[
\sum_{s=1}^{n-m} a'_0(i + s - 1) = \sum_{s=1}^{n-m} a_0(i + s - 1) \\
\times \left( \sum_{s=1}^{n} a_0(i + s - 1) - \sum_{s=1}^{n} a_0(i + s - m - 1) \right) \\
+ a_1(i + n) - a_1(i),
\]

(10)

Here it is understood that \( a_l(i, x) = 0 \).

**Example 2.1** Consider the case \( n = 1, m = -1, l = 2 \). The system (10) becomes

\[
a'_0(i) + a'_0(i + 1) \\
= (a_0(i) + a_0(i + 1))(a_0(i) - a_0(i + 1)) + a_1(i + 1) - a_1(i),
\]

\[ a'_1(i) = 0. \]

So, actually we have in this case one-field lattice†

\[
r'_i + r'_{i+1} = r_i^2 - r_{i+1}^2 + \nu_i,
\]

(11)

with \( \nu_i = a_1(i + 1) - a_1(i) \) being some constants. As is known the lattice (11) describes elementary Darboux transformation for Schrödinger operator \( L = \partial^2 - q(x) \). An interesting property of the lattice (11) is that it reduces to Painlevé transcendents \( P_4 \) and \( P_5 \) due to imposing periodicity conditions

\[
r_{i+N} = r_i, \ \nu_{i+N} = \nu_i
\]

for \( N = 3 \) and \( N = 4 \), respectively [9].

**Example 2.2** In the case \( m = 0, n = 1, l \geq 2 \) we obtain well known generalized Toda systems

\[
a'_0(i) = a_1(i + 1) - a_1(i),
\]

\[
a'_k(i) = a_k(i)(a_0(i) - a_0(i - k))
\]

\[ + a_{k+1}(i + 1) - a_{k+1}(i), \ \ k = 1, ..., l - 1. \]  

(12)

† For one-field lattices we use notation \( a_0(i) = r_i \).
In particular if \( l = 2 \) we obtain ordinary Toda lattice in polynomial form
\[
a'_0(i) = a_1(i + 1) - a_1(i),
\]
\[
a'_1(i) = a_1(i) (a_0(i) - a_0(i - 1)).
\]
Defining \( u_i \) by relation \( a_0(i) = -u'_i \) and \( a_1(i) = \exp(u_{i-1} - u_i) \) we arrive at more familiar exponential form of the Toda lattice \( u'_i = e^{u_{i-1} - u_i} - e^{u_i - u_{i+1}}. \)

**Example 2.3** Let \( m = n - 1, \ l = 1, n \geq 2. \) This choice corresponds to Bogoyavlenskii lattices.

\[
r'_i = r_i \left( \sum_{k=1}^{n-1} r_{i+k} - \sum_{k=1}^{n-1} r_{i-k} \right).
\]

### 3. Miura lattice transformations

Representation of the chains as consistency condition of an auxiliary linear equations allows us to construct in simple and algorithmical way Miura mapping which connect solutions of one-field lattices with solutions of corresponding \( l \)-field \( (l \geq 2) \) ones.

Firstly notice that \( F_i = G_{i+(l-1)n}H_{i+(l-2)n}...H_{i+n}H_i \) is \( l \)-order differential operator. As consequence of (13) we obtain
\[
F_i \psi_i = z^l \psi_{i+(l-1)n+m}, \quad H_i \psi_i = z \psi_{i+n}.
\]
Here it is important to notice that (13) in fact is equivalent to (13). Indeed one can express \( G_i \) as \( G_i = F_{i-(l-1)n}H_{i-(l-2)n}...H_{i-2n}H_{i-n} \) and obtain (13) as consequence of (13).

Define two integers
\[
\overline{n} = ln, \quad \overline{m} = (l - 1)n + m.
\]
It is evident that \( \overline{n} \geq 2 \) and \( \overline{m} < \overline{n} \). Let us identify \( \psi_i = \overline{\psi}_i \), where \( \overline{\psi}_i \) is BA functions being determined by auxiliary linear system
\[
\overline{G}_i \overline{\psi}_i = z^l \overline{\psi}_{i+(l-1)n+m}, \quad \overline{H}_i \overline{\psi}_i = z \overline{\psi}_{i+n},
\]
with \( \overline{H}_i = \partial - \sum_{k=1}^{\overline{n}} r_{i+k-1} \) and
\[
\overline{G}_i = \begin{cases} \partial + \sum_{k=1}^{\overline{m}} r_{i-k}, & \text{for } \overline{m} \leq -1, \\ \partial - \sum_{k=1}^{\overline{m}} r_{i+k-1}, & \text{for } \overline{m} \geq 1. \end{cases}
\]

Consistency condition of auxiliary equations is equivalent to one-field lattice
\[
\sum_{s=1}^{\overline{n}-\overline{m}} r'_{i+s-1} = \sum_{s=1}^{\overline{n}-\overline{m}} r_{i+s-1} \left( \sum_{s=1}^{\overline{n}} r_{i+s-1} - \sum_{s=1}^{\overline{n}} r_{i+s-\overline{m}-1} \right).
\]
As consequence of auxiliary equations (17) we have
\[ F_i \psi_i = z^l \psi_i + l, \]
\[ H_i \psi_i = z \psi_i + n, \]
(19)
with \( F_i = G_{i-l} \cdots G_{i-m} G_i \). Comparing (19) with (15) we arrive at the following identification
\[ F_i = F_{li}, \quad H_i = H_{li}. \]
(20)

Let us exhibit some examples of Miura transformations calculated by using (20).

**Example 3.1** Take, for example \( \pi = 2 \) and \( \pi = 1 \). Solving (16) gives
\[ n = 1, \quad m = 0 \quad \text{and} \quad l = 2. \]
In this case we derive well known relations
\[ a_0(i) = r_{2i} + r_{2i+1}, \quad a_1(i) = r_{2i-1} r_{2i}, \]
(21)
defining a mapping of solutions of the Volterra lattice
\[ r^\prime_i = r_i(r_{i+1} - r_{i-1}) \]
(22)
into solutions of the Toda lattice (13) [11].

**Example 3.2** For the system
\[ a'_0(i) = a_1(i + 1) - a_1(i), \]
\[ a'_1(i) = a_1(i) (a_0(i) - a_0(i - 1)) + a_2(i + 1) - a_2(i), \]
\[ a'_2(i) = a_2(i) (a_0(i) - a_0(i - 2)) \]
we obtain following Miura transformation
\[ a_0(i) = r_{3i} + r_{3i+1} + r_{3i+2}, \]
\[ a_1(i) = r_{3i-2} r_{3i} + r_{3i-1} r_{3i} + r_{3i-1} r_{3i+1}, \]
\[ a_2(i) = r_{3i-4} r_{3i-2} r_{3i} \]
The latter relates (23) to Bogoyavlenskii lattice \( r^\prime_i = r_i (r_{i+2} + r_{i+1} - r_{i-1} - r_{i-2}) \).

¿From (16) follows that the same one-field lattice is connected by Miura transformations, generally speaking, with a number of \( l \)-field ones. It is obvious that the number of such lattices is defined by the fact how many divisors of \( \pi \) are among \( l = 2, \ldots, \pi \).

**Example 3.3** Consider Bogoyavlenskii lattice
\[ r^\prime_i = r_i(r_{i+3} + r_{i+2} + r_{i+1} - r_{i-1} - r_{i-2} - r_{i-3}) \]
(24)
corresponding to the choice \( \pi = 4 \) and \( \pi = 3 \) in (18). Equations (16) in this case have two solutions: \( n = 2, \quad m = 1, \quad l = 2 \) and \( n = 1, \quad m = 0, \quad l = 4 \). For the first solution of (16) we obtain Miura transformation
\[ a_0(i) = r_{2i} + r_{2i+1}, \quad a_1(i) = r_{2i-3} r_{2i} \]
relating (24) to two-field system

\[ a'_0(i) = a_0(i)(a_0(i + 1) - a_0(i - 1)) + a_1(i + 2) - a_1(i), \]

\[ a'_1(i) = a_1(i)(a_0(i + 1) + a_0(i) - a_0(i - 2) - a_0(i - 3)). \]

Second solution of (16) corresponds to generalized Toda lattice (12) in the case \( l = 4 \), i.e.

\[ a'_0(i) = a_1(i + 1) - a_1(i), \]

\[ a'_1(i) = a_1(i)(a_0(i) - a_0(i - 1)) + a_2(i + 1) - a_2(i), \]

\[ a'_2(i) = a_2(i)(a_0(i) - a_0(i - 2)) + a_3(i + 1) - a_3(i), \]

\[ a'_3(i) = a_3(i)(a_0(i) - a_0(i - 3)). \]

Miura transformation in this case is given by

\[ a_0(i) = r_{4i} + r_{4i+1} + r_{4i+2} + r_{4i+3}, \]

\[ a_1(i) = r_{4i-3r_{4i}} + r_{4i-2r_{4i+1}} + r_{4i-1r_{4i+2}} + r_{4i-2r_{4i} + r_{4i-1r_{4i}} + r_{4i-1r_{4i+1}}}, \]

\[ a_2(i) = r_{4i-6r_{4i}} + r_{4i-5r_{4i-3r_{4i}}} + r_{4i-5r_{4i-2r_{4i}}} + r_{4i-5r_{4i-2r_{4i+1}}}, \]

\[ a_3(i) = r_{4i-9r_{4i}} + r_{4i-6r_{4i-3r_{4i}}}. \]

Let us exhibit results of calculations of Miura transformations for two-field systems. It can be written in unique form

\[ a_0(i) = r_{2i} + r_{2i+1}, \quad a_1(i) = \sum_{s=1}^{n-m} r_{2i+s-m-1} \cdot \sum_{s=1}^{n-m} r_{2i+s-1} \]

Notice that the systems corresponding to \( m \leq -1 \) and \( n = |m| \) are excluded from consideration since we have \( \overline{m} = 0 \). In fact we deal in this situation with one-field lattices since \( a'_1(i) = 0 \).

4. The chains of KP Lax operators

The relations (7) play key rôle for defining Lax operators \( Q_i \) connected with each other by compatible pair of similarity transformations. The subscript \( i \in \mathbb{Z} \) can be interpreted as discrete evolution parameter. The principal problem naturally raised here is to define equations for the fields \( a_k(i) = a_k(i, L) \) which guarantee the compatibility of the mappings with respect to \( i \) with \( t_p \)-flows given by Lax equation (1) or equivalently by Sato-Wilson equation (2).
Proposition. By virtue (4), Lax operators are connected with each other by two invertible compatible gauge transformations

\[ Q_{i+m} = G_i Q_i G_i^{-1}, \]  
\[ Q_{i+n} = H_i Q_i H_i^{-1}. \]  

(25) (26)

**Proof.** By virtue (4), we have

\[ Q_{i+m} = \hat{w}_{i+m} \partial \hat{w}_{i+m}^{-1} = (G_i \hat{w}_i \partial^{-1}) \partial (\hat{w}_i^{-1} G_i) \]

\[ = G_i \hat{w}_i \partial \hat{w}_i^{-1} G_i^{-1} = G_i Q_i G_i^{-1}. \]  

(27)

The similar calculations are needed to prove (26). The mapping \( Q_i \to Q_i = Q_{i+m} \) we denote as \( s_1 \), while \( s_2 \) stands for transformation \( Q_i \to \overline{Q}_i = Q_{i+n} \). The compatibility of \( s_1 \) and \( s_2 \) also follows from (7). Indeed, we obtain

\[ Q_{i+n+m} = G_{i+n} Q_{i+n} G_{i+n}^{-1} = G_{i+n} H_i Q_i H_i^{-1} G_{i+n}^{-1} \]

\[ = H_{i+m} G_i Q_i G_i^{-1} H_{i+m}^{-1} = H_{i+m} Q_{i+m} H_{i+m}^{-1}. \]

So we can write \( s_1 \circ s_2 = s_2 \circ s_1 \). The inverse maps \( s_1^{-1} \) and \( s_2^{-1} \) are well defined by the formulas \( Q_{i-m} = G_{i-m}^{-1} Q_i G_{i-m} \) and \( Q_{i-n} = H_{i-n}^{-1} Q_i H_{i-n} \). □

Let \( p \) and \( q \) are relatively prime integers such that \( pn = qm \). Without loss of generality we can thought that \( p \geq 0 \). It is obvious that relation \( s_1^q = s_2^q \) holds. Indeed the left- and right-hand side of this relation correspond to the same mapping \( Q_i \to Q_{i+pm} = Q_{i+qm} \). Let us summarize the statements above in the following theorem.

**Theorem.** Let the collection \( \{a_0(i), a_1(i), ..., a_{l-1}(i)\} \) solves equations of the lattice (14) Then by virtue (4) the set of Lax operators \( \{Q_i = \hat{w}_i \partial \hat{w}_i^{-1}, i \in \mathbb{Z}\} \) admits the action of discrete group \( G \) with pair of generators \( s_1 \) and \( s_2 \) realized as gauge transformations (28) and (29). In addition the group elements \( s_1 \) and \( s_2 \) are restricted by relations \( s_1 \circ s_2 = s_2 \circ s_1 \) and \( s_1^q = s_2^q \) where \( p \) and \( q \) are co-prime integers such that \( pn = qm \), \( p \geq 0 \).

Let us now suppose that each \( Q_i \) solves evolution equations of KP hierarchy (1). Differentiating the left- and right-hand sides of auxiliary equations (4) with respect to \( t_p \) by virtue (2) yields evolution equations

\[ \partial_p G_i = (Q_{i+m})_+ G_i - G_i (Q_i^p)_+, \]  
\[ (28) \]

\[ \partial_p H_i = (Q_{i+m})_+ H_i - H_i (Q_i^p)_+. \]  
\[ (29) \]
Our next goal is to show that the pair of equations (28) and (29) is properly defined and consistent. To prove correctness of definition of (29), standard arguments are needed. One rewrite (26) as \( Q_{i+n}H_i = H_iQ_i \). From this follows
\[
Q_{i+n}^p H_i = H_i Q_i^p
\]
for arbitrary \( p \in \mathbb{N} \). By virtue of this relation one can write
\[
(Q_{i+n}^p H_i - H_i(Q_i^p)_+) = -H_i(Q_{i+n}^p)_- + (Q_i^p)_- H_i.
\] (30)
The right-hand side of (31) is zero-order \( \Psi DO \), while the left-hand side of (30) is purely differential operator. So one conclude that the expression \((Q_{i+n}^p H_i - H_i(Q_i^p)_+)\) is differential operator of zero-order or simply function.

More complicated situation with (28) since \( G_i \)'s, generally speaking, are \( \Psi DO \)'s of special form. However in this situation one can use equivalent auxiliary system (17), with \( F_i \) being, as we have mentioned above, purely differential operator of \( l \)-order. Evolution equation on \( F_i \) follows from (28) and (29) and looks as
\[
\partial_p F_i = (Q_{i+m+(l-1)n}^p F_i + F_i - F_i(Q_i^p)_+) \quad \text{(31)}
\]
while the relation
\[
Q_{i+m+(l-1)n} = F_i^1 Q_i F_i^{-1} \quad \text{(32)}
\]
is valid. Apparently the gauge transformation (32) corresponds to the group element \( s_1 \circ s_1^{l-1} \in G \). By using the same arguments as for \( H_i \)'s one can easily prove that the right-hand side of (31) is \((l-1)\)-order differential operator.

It remains to prove correctness of simultaneous definition of (28) and (29). To do this, it is enough to show that differentiating the left- and right-hand sides of (7) with respect to \( t_p \), by virtue of (28) and (29) gives identity. It is straightforward calculation. Let us exhibit some examples of evolution equations (28) and (29) for \( t_2 \)-flows.

**Example 4.1** Consider the case \( n = 2, m = 1, l = 1 \) corresponding to Volterra lattice. \( t_2 \) flow for operators \( G_i = \partial - r_i \) and \( H_i = \partial - r_i - r_{i+1} \) is defined by evolution equation
\[
\partial_2 r_i = (r_i' + r_i^2 + 2r_{i-1}r_i)' \quad \text{(33)}
\]
By virtue of \( r_i' = r_i(r_i+1 - r_{i-1}) \) from (33) one obtain higher counterpart of Volterra lattice
\[
\partial_2 r_i = r_i(r_i+1 r_{i+2} - r_{i-1}r_{i-2} + r_{i+1}^2 - r_{i-1}^2 + r_i r_{i+1} - r_i r_{i-1}).
\]

**Example 4.2** In the case \( n = 3, m = 1, l = 1 \) we obtain
\[
\partial_2 r_i = (r_i' + r_i^2 + r_{i-2}r_{i-1} + r_{i-2}r_i + r_{i-1}r_i)' \quad \text{(34)}
\]
Notice that if we introduce variables \( s_i = r_i + r_{i+1} \) then by virtue
\[
r_i' + r_{i+1}' = (r_i + r_{i+1})(r_{i+2} - r_{i-1})
\]
the equation
\[ \partial_2 s_i = s_i(s_{i+2} s_{i+1} - s_{i-1} s_{i-2}). \] (35)
holds. Notice that (35) can be obtained as consequence of
\[ \partial_2 r_i = s_i s_{i+1} - s_{i-2} s_{i-1}. \]
By straightforward calculations one can check that \( x \)- and \( t_2 \)-flows commute.

Situation in the above example is generalized as follows. The lattice (35) come into well known class of the integrable ones [10]
\[ \partial_t s_i = s_i \left( \prod_{k=1}^{n-1} s_{i+k} - \prod_{k=1}^{n-1} s_{i-k} \right), \quad n \geq 2. \] (36)

**Remark.** It is known that any of the systems (36) can be interpreted as well as Bogoyavlenskii lattices (14) as discrete variant of the Korteweg–de Vries equation [10].

Let \( s_i = \sum_{i=1}^{n-1} r_{i+k-1} \). Then one-field lattices corresponding to the choice \( m = 1, n \geq 2 \) can be written as \( s'_i = s_i(r_{i+n-1} - r_{i-1}). \) Notice that (36) is consequence of equation
\[ \partial_t r_i = \prod_{k=0}^{n-1} s_{i-k+1} - \prod_{k=0}^{n-1} s_{i-k}. \]
Now by straightforward calculations one can check the commutativity of \( x \) and \( t \) flows.

Let us show that the systems (36) can be interpreted as restrictions of 1-Toda lattice flows on corresponding invariant manifolds. In the case \( m = 1, n \geq 2, l = 1 \) eigenvalue problem \( L(\psi_i) = z \psi_i \) takes on the form
\[ z \psi_{i+n-1} + s_i \psi_{i-1} = z \psi_i. \] (37)
In the following it is convenient to define new wave functions by relation \( \varphi_i = z^i \psi_i \). In terms of \( \varphi_i \)'s auxiliary equation (37) have following form:
\[ \varphi_{i+n-1} + s_i z^{n-1} \varphi_{i-1} = z^{n-1} \varphi_i. \] (38)
Step-by-step one can expand the left-hand side of equation (38) to obtain eigenvalue problem \( M(\varphi_i) = z^{n-1} \varphi_i \), where
\[ M = E^{n-1} + \sum_{j=1}^{\infty} \left( \prod_{k=1}^{j} s_{i-j+1} \right) E^{n-j-1} \] (39)
with \( E \) being an operator of elementary shift \( E(\eta_i) = \eta_{i+1} \).

For each \( n \geq 2 \) define \( A_n = M_+ \), where subscript + stands for projection on positive part of \( M \). The difference operator \( A_n \) is used to define auxiliary evolution equation \( \partial_\varphi_i / \partial t = A_n(\varphi_i) \). It is easy to verify that the latter is consistent with (38) provided that equations (36) satisfy. From this one can conclude that the Lax equation \( \partial_t M = [A_n, M] \) is equivalent to the system (36).
Let us explain how above is relevant to 1-Toda lattice hierarchy \cite{1}. For Lax operator
\[ Q = E + \sum_{k \geq 0} q_k(i)E^{-k} \]
one defines restriction \( Q^{n-1} = M \), where \( M \) is in \cite{2}. This implies that some algebraic constraints on coefficients \( q_k(i) \) must be imposed. For example in the simplest case of Volterra lattice \((n = 2)\) these constraints are given by
\[ q_k(i) = s_{i-k}...s_{i-1}s_i = q_0(i-k)...q_0(i-1)q_0(i), \ k \geq 1. \]

To conclude this section, let us consider one example which illustrate relationship between a class of the lattices \cite{10} and other known lattices. In the case \( n = 1, m = -1, l = 3 \) the equations \cite{10} take on the form
\[
\begin{align*}
  a_0'(i) + a_0'(i+1) &= a_0^2(i) - a_0^2(i+1) + a_1(i+1) - a_1(i), \\
  a_1'(i) &= a_2(i+1) - a_2(i), \\
  a_2'(i) &= a_2(i)(a_0(i) - a_0(i-1))
\end{align*}
\] (40)

Using \cite{20} one calculates Miura transformation between \cite{10} and \cite{34} to obtain
\[
\begin{align*}
  a_0(i) &= r_{3i} + r_{3i+1} + r_{3i+2}, \\
  a_1(i) &= (r_{3i-1} + r_{3i})(r_{3i} + r_{3i+1}) \\
  &\quad + (r_{3i} + r_{3i+1})(r_{3i+1} + r_{3i+2}) + (r_{3i+1} + r_{3i+2})(r_{3i+2} + r_{3i+3}), \\
  a_2(i) &= (r_{3i-2} + r_{3i-1})(r_{3i-1} + r_{3i})(r_{3i} + r_{3i+1}).
\end{align*}
\] (41)

Higher counterpart of \cite{10} is nothing but Blaszak–Marciniak lattice \cite{13}
\[
\begin{align*}
  \partial_2p_i &= u_{i+2} - u_i, \\
  \partial_2v_i &= p_iu_{i+1} - u_ip_{i-1}, \\
  \partial_2u_i &= u_i(v_i - v_{i-1}),
\end{align*}
\] (42)

with \( \partial_2a_0(i) = u_{i+1} - u_i \). Here we denote \( p_i = a_0(i) + a_0(i+1), v_i = a_1(i), u_i = a_2(i) \).

From \cite{11} we easy obtain Miura transformation
\[
\begin{align*}
  p_i &= s_{3i} + s_{3i+2} + s_{3i+4}, \\
  v_i &= s_{3i-1}s_{3i} + s_{3i}s_{3i+1} + s_{3i+1}s_{3i+2}, \\
  u_i &= s_{3i-1}s_{3i-1}s_{3i},
\end{align*}
\]

between \cite{42} and \cite{35}. 

5. Conclusion

Starting from auxiliary linear equations (3), we have defined a class of integrable first-order $l$-field lattices. The main feature shared by the latter is in some sense compatibility with KP flows. Taking this fact into account, one can exploit natural idea to search for solutions to the above lattices and its higher counterparts in terms of the KP $\tau$-functions. In future we are going to continue our activity in this direction. It will be of interest to investigate such a questions as lattice Bäcklund transformations and nonlinear superposition formulas.

The results of the paper might be of potential interest for investigation of constrained KP hierarchies. We believe that any integrable chains (up to Miura transformations) presented in the paper underlie some differential integrable hierarchies with $s_1$ and $s_2$ being discrete symmetries. Activity in this direction is in the paper [4] (see also [14]) where we have constructed modified version of Krichever’s rational reductions of KP hierarchy. Moreover, the discrete symmetries $s_1$ and $s_2$ in this case as was shown in [4] correspond to one-field lattices with $m \in \mathbb{N}$.

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