Topological fluids and FQH edge dynamics

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In this letter, we map the Chern-Simons-Ginzburg-Landau theory into a hydrodynamic action with topological terms, written in terms of the auxiliary Clebsch variables. The hydrodynamic equations we obtain are first order in gradients with an additional constitutive equation known as Hall constraint. In the absence of external fields the first order hydrodynamics allows for two possible tangential boundary conditions at the hard wall interface: no-slip or no-stress. The no-slip condition means that the fluid sticks to the wall, i.e., the tangential velocity must vanish at the boundary. The no-stress condition, on the other hand, allows the fluid to slip at the wall, as long as the flow does not generate tangential forces at the boundary. We show that the anomaly inflow mechanism is fundamentally incompatible with the no-slip condition. In order to obtain the tangential no-stress condition within variational principle, we are required to add a chiral boson action at the boundary which is coupled non-linearly to the edge density. This non-linear edge coupling is unusual from the Chern-Simons perspective; nevertheless it arises naturally from the requirement of vanishing tangential stress at the boundary. Upon coupling to the external gauge fields, this edge action leads to the correct anomaly equation. In fact, this action provides a natural starting point to study fluid aspects of the FQH state beyond topological quantum field theories and, equivalently, the edge dynamics beyond the chiral Luttinger liquid theory.

Introduction: The ground state of an interacting many-body system, under particular conditions, can lead to fluid dynamic behavior. These quantum fluids are inherently dissipationless, since they flow with zero shear and bulk viscosities, and are often dubbed superfluids. Helium-II is certainly the best known example of a superfluid [1]; however, both superconductivity [2] and the fractional quantum Hall (FQH) effect can also be understood as quantum fluid phenomena [3]. In fact, the dynamics of FQH states with filling factor \( \nu = \frac{k+1}{2k} (k \in \mathbb{N}) \) has been expressed in terms of continuity and Euler equations together with an additional constraint [3-4]. This “Hall constraint” relates the charge density to the fluid vorticity and the external magnetic field. These hydrodynamic equations, along with the Hall constraint, were derived starting from the Chern-Simons-Ginzburg-Landau (CSGL) theory, which models the FQH states as composite bosons coupled to a Chern-Simons gauge field, usually referred to as the flux attachment procedure [5-7].

The connection between the FQH state and fluid dynamics was made at the level of equations of motion. However, it is well known that ideal fluid equations can be derived from a variational principle written in terms of auxiliary fields known as Clebsch potentials. From this viewpoint, a natural question is whether or not the FQH superfluid dynamics can also be obtained from such a hydrodynamic action. In this letter, we derive this variational principle in terms of Clebsch potentials starting from the CSGL theory. In fact, the action obtained here first appears in a different context in Ref. [8] and is equivalent to an Eulerian fluid action (written in terms of Clebsch variables) with additional topological terms. The resulting hydrodynamic equations, however, rely on the velocity flow parametrization in terms of the Clebsch potentials. We define velocity such that the resulting stress tensor is first order in gradient expansion [9]. The CSGL action in the first order formalism not only gives the equations of motion, but the well-posedness of the variational problem also supplies the boundary conditions for the FQH edge dynamics.

In the first order hydrodynamics, the usual boundary conditions for fluids confined within rigid walls are the no-penetration condition, which states that the normal component of the velocity has to vanish at the boundary, together with either the no-slip or the no-stress boundary condition. The no-slip condition means that the fluid sticks to the wall, i.e., the tangential velocity must vanish at the boundary. The no-stress condition, on the other hand, allows the fluid to slip at the wall, as long as the flow does not generate tangential forces at the boundary. Since both boundary conditions perform no work, we can use the variational principle to systematically derive them for the FQH edge physics. Typically, for classical fluids, the choice between no-stress and no-slip conditions requires the details of the fluid interface. For the case of the FQH state, the presence of a tangential electric field leads to charge non-conservation at the edge via the gauge anomaly. We show that the no-penetration condition is modified by the anomaly inflow, with the tangential electric field driving the normal current, resulting in charge accumulation at the boundary. This charge is subsequently forced to flow along the edge by the same tangential electric field, which is in contradiction with the no-slip condition.

Unlike the no-slip condition, the no-stress condition cannot be obtained directly from boundary variations of the hydrodynamic fields and requires an additional auxiliary field at the edge. The boundary action for this
field resembles a chiral boson coupled non-linearly to the boundary density \[ \mathbf{10} \] and its equations of motion provide the chiral dynamics of the edge charge density. This equation of motion is indeed the no-tangent stress condition in disguise. In the presence of the tangential electric field, this chiral dynamics at the edge morphs into the anomaly equation and appropriately modifies the no-stress condition.

The CSGL action with hard-wall boundary has been studied previously in Refs. \[ \mathbf{11-13} \], where the linearized edge dynamics is derived in the presence of a uniform and constant magnetic field and in the absence of an electric field. However, the constitutive relations in these works are not first order in gradients. In \[ \mathbf{11} \], the authors neglected the quantum pressure, leading to an ideal fluid dynamics with a Hall constraint. Within this approximation, the only possible boundary condition is the no-penetration condition. On the other hand, in Refs. \[ \mathbf{12-13} \], while the quantum pressure is retained, the authors impose the vanishing of the fluid density at the boundary. This zero edge density condition can be obtained within our first order framework; however, we show that it is incompatible with the anomaly inflow mechanism.

In summary, we show using first order formalism for fluid dynamics that not every boundary condition is compatible with the anomaly inflow mechanism for the FQH state. Further, the edge dynamics of the FQH state arises from the requirement of no-tangent stress condition in the absence of electric field. Finally, our work paves a way to systematically derive non-linear edge dynamics of the FQH state by applying known fluid dynamical approaches such as multi-scale analysis \[ \mathbf{14} \].

**Topological Fluid Action from CSGL Theory:**

The CSGL theory is comprised of a charged bosonic matter field \( \Phi \) coupled to an external electromagnetic field \( (A_0, A_i) \) as well as a statistical Chern-Simons gauge field \( (a_0, a_i) \), with \( i = 1, 2 \). Its action can be written as

\[
S = -\int dt \int d^2x \left[ \frac{\hbar}{2m_e} |D_\mu \Phi|^2 - \frac{\hbar^2}{2m_*} |D_\mu \Phi|^2 - V(\Phi) \right. \\
\left. - \frac{\hbar v}{4\pi} \epsilon^{ij} \left( a_0 \partial_i a_j - a_i \partial_j a_0 \right) \right],
\]

where \( D_\mu = \partial_\mu + i \left( \frac{\hbar}{\pi} A_\mu + a_\mu \right) \), \( \mu = 0, 1, 2 \), is the covariant derivative with \( e \) and \( m_* \) being the charge and effective mass of the electron quasiparticle, respectively.

In order to map the CSGL theory into a bulk hydrodynamic action, we write \( \Phi = \sqrt{n} e^{i\varphi} \), where the Madelung variables \( n \) and \( \theta \) define the condensate density and the superfluid phase, respectively. Further, we write the gauge fields in the Clebsch form \( a_i = \frac{e}{2\pi} A_i = \alpha \partial_i \beta \). (Recall that an Abelian gauge field in 3 dimensions has the Clebsch parametrization \( d\varphi + \alpha d\beta \), see \[ \mathbf{13} \] for a general discussion. We use this for the spatial components here, absorbing \( \varphi \) into the phase \( \theta \).) In terms of these variables, the action \[ \mathbf{1} \] can be written as,

\[
S = -\int dt \int d^2x \left[ \hbar n \partial_\mu \varphi + \frac{\hbar^2}{2m_*} \left( n \partial_\mu \theta + \alpha \partial_\mu \beta \right)^2 + \frac{(\partial_t n)^2}{4n} \right. \\
\left. + V(n) - \frac{\hbar v}{2\pi} \epsilon^{ij} \partial_\mu \alpha \partial_j \beta - \frac{\hbar v}{2\pi} \left( \alpha \partial_\mu \beta - \frac{e A_i}{2\hbar} \right) \epsilon^{ij} E_j \right. \\
\left. + \left( \hbar a_0 + e A_0 \right) \left( n - \frac{\nu e B}{2\pi \hbar} + \frac{\nu}{2\pi} \epsilon^{ij} \alpha \partial_j \beta \right) + \frac{\nu e^2}{4\pi \hbar} A_0 B \right],
\]

up to total derivatives, which do not contribute to the equations of motion. Notice that \( a_0 \) is just a Lagrange multiplier, imposing the Hall constraint

\[
n - \frac{\nu e B}{2\pi \hbar} + \frac{\nu}{2\pi} \epsilon^{ij} \partial_\mu \alpha \partial_j \beta = 0.
\]

Therefore, we can use the equation of motion for \( a_0 \), i.e., the Hall constraint \[ \mathbf{4} \], to drop the terms proportional to \( A_0 \) in the action and rewrite \( \epsilon^{ij} \partial_\mu \alpha \partial_j \beta \) in terms of the fluid density and magnetic field. Without loss of generality, we neglect terms from the action that do not contribute to the bulk equations of motion. These terms will be either total derivative terms or gauge terms that do not involve any dynamical fields \[ \mathbf{10} \].

The fluid velocity is not yet fixed and must be identified in terms of the field content of the action. We choose the fluid velocity such that the resulting stress tensor is of the first order in gradients of density and velocity fields. Accordingly, we define the velocity field as

\[
v_i = \frac{\hbar}{m_*} \left( \partial_i \theta + \alpha \partial_i \beta + \frac{e_i}{2m} \partial_i n \right).
\]

The advantage of choosing the first order formalism will be apparent when we discuss boundary conditions. With the definition Eq. (4), and adding the total derivative terms \( \hbar^2 \epsilon^{ij} \partial_\mu |n(\partial_\mu \theta + \alpha \partial_i \beta)|/(2m_* \) and \( (\nu e/2\pi)(B \partial_\theta + \epsilon^{ij} \partial_i \theta E_j) \), the action in Eq. (2) can be written as,

\[
S = -\int dt \int d^2x \left\{ \frac{\hbar}{\nu e B} (\partial_\mu \theta + \alpha \partial_\mu \beta) \\
+ H + \hbar a_0 \left( n - \frac{\nu e B}{2\pi \hbar} + \frac{\nu}{2\pi} \epsilon^{ij} \partial_\mu \alpha \partial_j \beta \right) \right. \\
\left. - \frac{\hbar v}{2\pi} \left( \partial_\mu \theta + \alpha \partial_\mu \beta \right) \epsilon^{ij} E_j \right\, ,
\]

\[
H = \hbar \left[ n v_i (\partial_i \theta + \alpha \partial_i \beta) + \frac{\epsilon^{ij} \partial_\mu \alpha \partial_j \beta}{2} \right] - \frac{m_*}{2} n v_i^2 \\
+ V(n) + \frac{\pi \hbar^2}{vm_*} n^2 - \frac{\hbar}{2m_*} eB.
\]

In these expressions, \( v_i \) is to be treated as an independent variable whose equation of motion will reproduce the definition Eq. (4). The last term in \( H \) cancels the ground state energy of the fluid coming from the kinetic term, since \( eB/m_* \) is nothing but the cyclotron frequency \( (\omega_B) \), and the second to last term is an effective contact
where the stress tensor is given by
\[ \tau_{ij} = \frac{\nu e B}{2\pi R^2} \left( \partial_i \theta + \alpha \partial_j \beta \right). \]

In terms of the polarization the Hall constraint becomes
\[ n = \frac{\nu e B}{2\pi R^2} + \frac{1}{\epsilon} \partial_i P^i, \]
therefore, \( \nu e B/(2\pi R^2) \) can be identified with the electron free charge.

**Hydrodynamic equations:** It is not hard to see that the hydrodynamic equations obtained from the action \( S_0 \) are
\[ \partial_t n + \partial_j (n v^j) = 0, \]
\[ \frac{eB}{m_*} - \frac{2\pi R^2}{\nu m_*} n - \frac{\hbar}{2m_*} \partial_i \left( \frac{\partial^j n}{n} \right) = \epsilon^{ij} \partial_j v, \]
\[ \partial_i v_i + \vec{v}^j \partial_j v_i = \frac{\partial_j T^j_i}{m_* n} - \frac{e}{m_*} (E_i + B \epsilon^{ij} v^j) - \frac{\hbar e}{2m_*^2} \partial_i B, \]
where the stress tensor is given by
\[ T^j_i = \left[ V - n V' - \frac{\pi h^2 n^2}{\nu m_*} \right] \delta^j_i - \frac{\hbar n}{2} (\epsilon_{ik} \partial^k v^j + \epsilon^{jk} \partial_j v_k). \]

As anticipated, the velocity defined as in Eq. \( (1) \) leads to a first order stress tensor. The first term in Eq. \( (12) \) is the fluid pressure, whereas the second one has the form of the odd viscosity part of the stress tensor. Note that the value of this “odd viscosity” is negative \( \hbar n/2 \) and does not depend on the filling factor. There is no reason to expect that the hydrodynamic odd viscosity, as obtained here by mapping the CSGL theory to a first order hydrodynamic system should coincide with the Hall viscosity coming from the adiabatic curvature of the space of modular transformations \([17, 20]\). The stress tensor of the hydrodynamic theory need not be the same as the stress tensor of the CSGL theory, since the mapping from the CSGL theory to the hydrodynamics action \( (5) \) involves certain metric-dependent identifications, for example, via the constraint \( (6) \). A proper definition of the Hall viscosity will require coupling the action \( (4) \) to a strain rate or a time-dependent metric \([21, 23]\) before mapping it to a hydrodynamic system.

**Boundary conditions:** The variational principle \( (5) \) provides a more natural starting point to study the fractional quantum Hall fluid confined in a domain, since the boundary conditions are written in terms of hydrodynamic quantities. Here, we focus on boundary conditions that can be explicitly obtained by the variation of the fluid action. For simplicity, we consider the Hall fluid to be confined in the lower half-plane, that is, \( y \leq 0 \). It is straightforward to generalize this discussion for any rigid wall domain.

As discussed in the introduction, in the absence of electric fields, one should expect the boundary conditions at a hard wall to be the no-penetration together with either no-slip or no-tangent stress boundary conditions. In fact, starting from the action \( (5) \), we see that boundary variation of the fluid action. For simplicity, we consider the Hall fluid to be confined in the lower half-plane, that is, \( y \leq 0 \). It is straightforward to generalize this discussion for any rigid wall domain.

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a canonical reduction. This means that $a_0 = 0$ can be obtained as a gauge choice even in the bulk, leading to Eq. (15) even without the variational procedure.

Note that the boundary condition $\hat{n} = 0$, used in the Refs. [12][13], can be obtained by adding the boundary action $-\frac{h}{2} \int dt dx \, \hat{n} \hat{v}_x$ to Eq. (16). Nevertheless, the condensate density vanishing at the wall is obviously incompatible to the anomaly inflow, that is, Eq. (14).

Equation (14) corresponds to an influx of charged particles into the sample edge when a tangential electric field is applied. The accumulated charge is then forced to flow along the edge by the same tangential electric field, however, the no-slip condition forbids any surface charge current. From Eq. (14), we see that the fluid density cannot be singular at the boundary. A pillbox calculation, of integrating Eq. (9) in the domain displayed in Fig. 1 and taking the limit of the width of the rectangle going to zero, shows that the no-slip condition is in contradiction with the continuity equation. We end up with

$$\frac{\nu e}{2\pi \hbar} \int_{x_0}^{x_1} \hat{E}_x \, dx = 0, \quad (16)$$

which is only satisfied if $\hat{E}_x = 0$. Therefore, we conclude that the no-slip condition is incompatible with the anomaly inflow mechanism.

Having ruled out the no-slip boundary condition, we now investigate if the no-tangent stress boundary condition is consistent with the gauge anomaly. In the absence of electric field, $\hat{T}_{yx} = 0$ is obtained from a variational principle through the introduction of an auxiliary boundary field ($\phi$). This is the case because $\hat{T}_{yx} = 0$ can be rewritten as a dynamical equation involving velocity and densities at the boundary, that is,

$$\hat{T}_{yx} = \frac{\hbar n}{2} \left( \partial_y v_y - \partial_x v_x \right) \bigg|_{y=0} = -\hbar \sqrt{n} \left[ \partial_x (\sqrt{n}) + \partial_x (\sqrt{n} \hat{v}_x) \right], \quad (17)$$

where we used the no-penetration condition $\hat{v}_y = 0$ in the last line. This dynamical equation (17) is indeed the equation of motion for $\phi$. The appropriate boundary action can be obtained by taking the hard wall limit, i.e., $h(x,t) = 0$ of the free-surface action considered in Ref. [10]. The resulting edge action is of the form

$$S_{edge} = -\frac{\hbar}{2} \int dt dx \partial_t \phi (2\sqrt{n} - \partial_x \phi). \quad (18)$$

As pointed out in Ref. [10], this boundary action is of the form of a chiral boson that is non-linearly coupled to the edge density $n$. In the following, we show that the gauged version of the action (18) is indeed consistent with the anomaly inflow mechanism. The natural gauging of (18) leads to the action

$$S_{edge} = \frac{\hbar}{2} \int dt dx \left( \partial_t \phi + \frac{\nu e}{2\pi \hbar} A_0 \right) \left( \partial_x \phi + \frac{\nu e}{2\pi \hbar} A_x - 2\sqrt{n} \right). \quad (19)$$

The addition of $S_{edge}$ preserves the Eqs. (14) and (15), but replaces the the no-slip condition (13) and (15), with the following dynamical equation for the boundary fields,

$$\partial_t \phi + \frac{\nu e}{2\pi \hbar} A_0 = -\sqrt{n} \hat{v}_x, \quad (20)$$

$$\partial_x (\sqrt{n}) + \partial_x (\sqrt{n} \hat{v}_x) = -\frac{\nu e}{4\pi \hbar} \hat{E}_x. \quad (21)$$

Here, Eq. (21) is the equation of motion for $\phi$, simplified using Eq. (20). It may be viewed as just the anomaly equation with the identification

$$\bar{\rho} = -e \sqrt{n} \quad \text{and} \quad \bar{I} = -e \sqrt{n} \hat{v}_x, \quad (22)$$

where we interpret $\bar{\rho}$ as the boundary charge and $\bar{I}$ as the edge current. In fact, the factor $\nu e/(2\pi \hbar)$ in the gauge coupling was chosen to reproduce the anomaly equation (21). Moreover, Eq. (20) together with the identification (22) provides us the bosonized expression for the edge current, that is,

$$\bar{I} = e \left( \partial_x \phi + \frac{\nu e}{2\pi \hbar} A_0 \right). \quad (23)$$

The appearance of $\sqrt{n}$ in (22) may seem a bit unusual. Notice that such factors would emerge if we consider the charge and current densities as obtained by integrating the bulk values over a thin boundary layer of thickness $1/\sqrt{n}$; i.e., $\bar{\rho} = \int dy \nu \sim \tilde{n} (1/\sqrt{n}) \sim \sqrt{n}$, $\bar{I} = \int dy \nu \hat{v}_x \sim \tilde{n} \hat{v}_x (1/\sqrt{n}) \sim \sqrt{n} \hat{v}_x$. Therefore, we consider the expressions in (22) as indicative of a boundary layer of thickness $\sim 1/\sqrt{n}$.

Finally, upon combining Eqs. (9) and (21), we note that the anomaly induces tangential forces on the wall in the presence of electric field, i.e.,

$$\frac{\hbar n}{2} \left( \partial_y v_y - \partial_x v_x \right) \bigg|_{y=0} = \frac{\nu e}{4\pi} \hat{E}_x \left( \sqrt{n} + \frac{\partial_y n}{n} \right) \bigg|_{y=0}, \quad (24)$$

**Discussion and Outlook** : We have shown that the CSGL theory can be mapped to fluid dynamics with a specific choice of the possible topological terms. As in any theory, requiring a well-defined variational formulation selects the allowed set of boundary conditions. We show that the no-slip boundary condition is ruled out by the charge inflow due to the anomaly. By augmenting the action with a boundary term for the edge modes, one can obtain the appropriately modified boundary conditions. While our analysis has been at the level of fluid dynamics, a systematic derivation of the fully nonlinear dynamics of the edge states for, say, the FQH states would be ideal in completing the circle of arguments.

A second interesting point relates to the fact that, in view of the Hall constraint (3), the polarization provides a complete set of observables. This suggests an alternate description with an action directly in terms of $P^i$, whose
Poisson brackets are proportional to the Hall conductivity [34], that is,

\[ \{ P^i(x), P^j(y) \} = -\frac{e^2}{2\pi\hbar}\epsilon^{ij}\delta(x-y). \quad (25) \]

Finally, we remark that known examples of quantum fluids can be characterized by the nature of the topological terms in the action. A general question of interest is whether the classification of topological terms can yield a complete set of possible quantum fluids. We leave these matters to future work.

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The fundamental Poisson brackets are given by

\{P^i(x), P^j(y)\} = \left(\frac{e\nu}{2\pi}\right)^2 \epsilon^{ijl} \partial_k \alpha \partial_l \beta \frac{\delta(x - y)}{\rho - \bar{\rho}}

\quad = -\frac{\nu e^2}{2\pi \hbar} \epsilon^{ij} \delta(x - y) \tag{29}

where we used the Hall constraint in going to the second line of this equation.

From the fundamental brackets we can also obtain

\{(\rho - \bar{\rho})(x), P^j(y)\} = -\frac{e\nu}{2\pi} \epsilon^{ij} \frac{\partial}{\partial x^i} \delta(x - y) \tag{30}

From Eq. (29) we can also obtain

\hbar e \{\partial_k P^k(x), P^j(y)\} = -\frac{e\nu}{2\pi} \epsilon^{ij} \frac{\partial}{\partial x^i} \delta(x - y) \tag{31}

Comparison of these two equations shows that it is possible to impose the constraint (8) in the strong sense for the set of variables \((\rho - \bar{\rho}, P^i)\).

Since the density \(n\) can be written in terms of \(P^i\), the configuration space of the fluid is completely determined by the polarization field. Further, since the constraint can be imposed strongly, we may seek an action in terms of \(P^i\) which directly leads to the Poisson brackets (29).

The relevant Poisson brackets involving \(n\) can be worked out from (29) by defining \(n\) in terms of \(\partial_k P^k\) as in (8). Such an alternate description is indeed provided by the action (1), after rewriting the Chern-Simons gauge field as \(a_i = \frac{e\nu}{2\pi} \epsilon_{ij} P^j - \partial_i \theta - \frac{\nu e}{2\pi} A_i\). The result is

\[ S = \int dt d^2 x \left[ \frac{\pi \hbar}{e^2\nu} \epsilon_{ij} P^i \partial_j P^j - H \right] \tag{32} \]

The first term in this action leads to (29) treating \(P^i\) as independent dynamical variables, and all factors of \(n\) in \(H\) are to be interpreted as \((\partial_k P^k - e^2\nu B/2\pi\hbar)/\epsilon\), as in (8). It would be interesting to explore this version of the action further.