Unified Framework of Mean-Field Formulations for Optimal Multi-period Mean-Variance Portfolio Selection

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Abstract

When a dynamic optimization problem is not decomposable by a stage-wise backward recursion, it is nonseparable in the sense of dynamic programming. The classical dynamic programming-based optimal stochastic control methods would fail in such nonseparable situations as the principle of optimality no longer applies. Among these notorious nonseparable problems, the dynamic mean-variance portfolio selection formulation had posed a great challenge to our research community until recently. Different from the existing literature that invokes embedding schemes and auxiliary parametric formulations to solve the dynamic mean-variance portfolio selection formulation, we propose in this paper a novel mean-field framework that offers a more efficient modeling tool and a more accurate solution scheme in tackling directly the issue of nonseparability and deriving the optimal policies analytically for the multi-period mean-variance-type portfolio selection problems.

KEY WORDS: Stochastic optimal control; mean-field formulation; multi-period portfolio selection; multi-period mean-variance formulation; intertemporal restrictions; risk control over bankruptcy.
I. INTRODUCTION

The mean-field type of optimal stochastic control models deals with problems in which either the system dynamics or the objective functional, or both, involve the states as well as the expected values of the states. The past few years have witnessed an increasing number of successful applications of the mean-field formulation, including mean-field type of stochastic control formulations, in various fields of science, engineering, financial management, and economics. Although the research in this direction has been well developed for continuous-time control problems, it lacks progress in both theoretical investigation and applications in discrete-time problems. The current work in this paper aims to employ the mean-field formulation to cope with seemingly non-tractable nonseparability in discrete-time portfolio selection problems. In particular, we revisit three challenging, yet practically important, portfolio selection models over a finite-time investment horizon (see Li and Ng [19], Costa and Nabholz [11], Zhu et al. [31]), reformulate them as discrete-time linear-quadratic control problems of a mean-field type, and derive their optimal strategies with improved solution qualities.

Since Markowitz [21] published his seminal work on mean-variance portfolio selection sixty years ago, the mean-risk portfolio selection framework has become one of the most significant ingredients in the modern financial theory. An important yet essential research theme under the mean-risk portfolio selection framework is to strike a balance between achieving a high mean of the investment return and minimizing the corresponding risk. If we adopt the variance of the terminal wealth as a risk measure for investment, we have the following mathematical formulation of the classical static mean-variance models,

$$\text{(MV}_s\text{)} \quad \max \ E(x_1) - \omega \text{Var}(x_1),$$

s.t. \( x_1 = x_0 + u_0 \cdot S_0, \)

where \( x_t \) is the wealth at time \( t \), \( u_t \) is the portfolio strategy at time \( t \), \( S_t \) is the random return at time \( t \), \( x_0 + u_0 \cdot S_0 \) denotes the random terminal wealth \( x_1 \) from applying strategy \( u_0 \) in the market with initial wealth \( x_0 \), and \( \omega > 0 \) denotes the trade-off between the two conflicting objectives of maximizing expected return and minimizing the risk. Letting the parameter \( \omega \) vary in \( (0, \infty) \) yields the efficient frontier in the mean-variance plane of the terminal wealth. The optimal portfolio strategy and solution scheme of \( \text{(MV}_s\text{)} \) can be found in Merton [23] when shorting is allowed and in Markowitz [21] when shorting is prohibited.
However, the extension of Markowitz’s static version of mean-variance portfolio selection to its dynamic version was blocked for almost four decades until recently. Let us consider the following abstract form for the dynamic mean-variance portfolio selection problem,

\[
(MV(\omega)) \quad \max_u \mathbb{E}(x_T) - \omega \text{Var}(x_T),
\]

\[
\text{s.t. } x_T = x_0 + \{u_t \cdot S_t\}_{t=0}^{T-1}.
\]

where \(x_0 + \{u_t \cdot S_t\}_{t=0}^{T-1}\) denotes the random terminal wealth \(x_T\) from applying strategy \(\{u_t\}_{t=1}^{T-1}\) in the market with initial wealth \(x_0\). Due to that the variance term is nonlinear with respect to the expected wealth, it does not satisfy the smoothing property, i.e.,

\[
\text{Var(Var(\cdot | \mathcal{F}_i) | \mathcal{F}_j))} \neq \text{Var(\cdot | \mathcal{F}_j)}, \quad \forall \ i > j,
\]

where \(\mathcal{F}_j\) is the information set available at time \(j\) and \(\mathcal{F}_{j-1} \subset \mathcal{F}_j\). Problem \((MV(\omega))\) is thus \textit{nonseparable} in the sense of dynamic programming as its objective function cannot be decomposed by a stage-wise backward recursion and thus does not satisfy the principle of optimality. Therefore, all the traditional dynamic programming-based optimal stochastic control solution methods no longer apply in such nonseparable situations.

We now briefly summarize the main approaches in the current literature in overcoming the difficulty resulted from the nonseparability. Adopting an embedding scheme, Li and Ng [19] and Zhou and Li [30] considered the following family of auxiliary problems, \(A(\omega, \lambda)\), parameterized in \(\lambda\),

\[
A(\omega, \lambda) \quad \min_u \mathbb{E}(\omega x_T^2 - \lambda x_T),
\]

\[
\text{s.t. } x_T = x_0 + \{u_t \cdot S_t\}_{t=0}^{T-1}.
\]

Note that problem \(A(\omega, \lambda)\) is a separable linear-quadratic stochastic control (LQSC) formulation and can be thus solved analytically. Li and Ng [19] and Zhou and Li [30] derived the optimal policy to the primal nonseparable problem \((MV(\omega))\) via identifying optimal parameter \(\lambda^*\) under which the optimal policy to \(A(\omega, \lambda^*)\) also solves \((MV(\omega))\). The embedding scheme has been also extended to multi-period mean-variance model with intertemporal restrictions (see Costa and Nabholz [11]), multi-period mean-variance model in a stochastic market whose evolution is governed by a Markovian chain (see Çelikyurt and Özekici [6]), a generalized mean-variance
model with risk control over bankruptcy (see Zhu et al. [31]), and dynamic mean-variance asset-liability management (see Leippold et al. [18], Chiu and Li [10], Chen and Yang [9]).

By introducing an auxiliary variable $d$ and an equality constraint $\mathbb{E}(x_T) = d$ for the expected terminal wealth, Li et al. [20] paved the road to study the following slightly modified, albeit equivalent, version of $(MV(\omega))$ (we omit the no-shorting constraint here and focus on the model itself),

$$(MV(d)) \min_u \text{Var}(x_T) = \mathbb{E}(x_T - d)^2,$$

s.t. $\mathbb{E}(x_T) = d$,

$$x_T = x_0 + \{u_t \cdot S_t\}_{t=0}^{T-1}.$$  

Introducing a Lagrangian multiplier $\lambda$ and applying Lagrangian relaxation to $(MV(d))$ give rise to the following LQSC problem,

$$(L(\lambda)) \min \mathbb{E}(x_T - d)^2 - \lambda \mathbb{E}(x_T - d),$$

s.t. $x_T = x_0 + \{u_t \cdot S_t\}_{t=0}^{T-1}.$

The optimal policy of $(MV(d))$ can be obtained by maximizing the dual function $L(\lambda)$ over all Lagrangian multiplier $\lambda \in \mathbb{R}$. In fact, the Lagrangian problem $(L(\lambda))$ can be further written as the following LQSC problem,

$$(MVH(m)) \min \mathbb{E}(x_T - m)^2,$$

s.t. $x_T = x_0 + \{u_t \cdot S_t\}_{t=0}^{T-1},$

where $m = d + \lambda/2$. Problem $(MVH(m))$ is a special mean-variance hedging problem, in which an investor hedges the target $m$ by his/her portfolio under a quadratic objective function. It has been well studied and can be solved by LQSC theory (see Li et al. [20]), martingale/convex duality theory (see Schweizer [26], Xia and Yan [28]) and sequential regression method (see Černý and Kallsen [7]).

In all the literature mentioned above, a static optimization procedure is always necessary to identify an optimal parameter in the parameterized auxiliary problem $\mathcal{A}(\omega, \lambda)$, $(L(\lambda))$ or $(MVH(m))$. Actually, based on the pure geometric structure of $(MV(\omega))$, Sun and Wang [27]
proved that the optimal terminal wealth \( x_T^* \) takes the following form,

\[
x_T^* = x_0 + \frac{1}{2\omega} \mathbb{E}(1 - \{\varphi^*_t \cdot S_t\}_{t=0}^{T-1}) \{\varphi^*_1 \cdot S_t\}_{t=0}^{T-1},
\]

where \( \varphi^* \) is the policy of the following particular mean-variance hedging problem,

\[
(MVH(1)) \quad \min \mathbb{E}(x_T - 1)^2,
\]

s.t. \( x_T = \{\varphi_t \cdot S_t\}_{t=0}^{T-1} \).

All the above approaches attempt to embed the “nontractable” nonseparable mean-variance portfolio selection problem into a family of tractable LQSC problems. Although these transformations seem necessary, one meaningful yet challenging question emerges naturally: Are we able to directly tackle the above nonseparable dynamic mean-variance problems (without introducing an auxiliary problem)?

The mean-variance problem is in fact a special case of the mean-field type problems where both the underlying dynamic system and the objective functional involve state processes as well as their expected values (hence the name mean-field). This critical feature differentiates the mean-variance problem from standard stochastic control problems. The theory of the mean-field stochastic differential equation can be traced back to Kac [17] who presented the McKean-Vlasov stochastic differential equation motivated by a stochastic toy model for the Vlasov kinetic equation of plasma. Since then, the research on related topics and their applications has become a notable and serious endeavor among researchers in applied probability and optimal stochastic controls, particularly in financial engineering. This new direction, however, requires new analytical tools and solution techniques. For instance, in a recent research on mean-field forward stochastic LQ optimal control problems, Yong [29] introduced a system of two Riccati equations as a solution scheme. Representative works in mean-field include, but not limited to, Mckean [22], Dawson [13], Chan [8], Buckdahn et al. [5], Borkar and Kumar [3], Crisan and Xiong [12], Andersson and Djehiche [2], Buckdahn et al. [4], Meyer-Brandis et al. [24], Nourian et al. [25], Yong [29] and Adlakha and Johari [1], Gomes et al. [15], Iyer et al. [16] in mean-field games. Despite active research efforts on mean-field in recent years, the topic of multi-period models in discrete-time remains a relatively unexplored subject where the mean-field modeling scheme has not yet been applied. Note that, for the continuous-time portfolio selection, the stochastic process is often assumed to be driven by Brownian motions. One can
therefore directly transfer the corresponding dynamic stochastic control problem into classical constrained partial differential equations by Itô lemma and the variational method. When we cope with discrete-time portfolio selection, we aspire to achieve some more prominent distribution-free results as attained in Li and Ng [19]. Hence, these powerful mathematical tools suitable for continuous-time cases no longer apply to discrete-time problems.

In this paper, we will develop a unified framework of mean-field formulations to investigate three multi-period mean-variance models in the literature: the classical multi-period mean-variance model in Li and Ng [19], the multi-period mean-variance model with intertemporal restrictions in Costa and Nabholz [11], and the generalized mean-variance model with risk control over bankruptcy in Zhu et al. [31]. We demonstrate that the mean-field approach represents a new promising way in dealing with nonseparable stochastic control problems related to the mean-variance formulations and even improves solution quality of some existing results in the literature.

II. MEAN-FIELD FORMULATIONS FOR MULTI-PERIOD MEAN-VARIANCE PORTFOLIO SELECTION

We consider in this paper a capital market consisting of one riskless asset and n risky assets within a time horizon $T$. Let $s_t (>1)$ be a given deterministic return of the riskless asset at period $t$, and $e_t = [e_1^t, \cdots, e_n^t]'$, the vector of random returns of the $n$ risky assets at period $t$, be defined over the probability space $(\Omega, \mathcal{F}, P)$. We assume that vectors $e_t, t = 0, 1, \cdots, T-1$, are statistically independent and the only information known about the random return vector $e_t$ is its first two unconditional moments, its mean $\mathbb{E}(e_t) = [\mathbb{E}(e_1^t), \cdots, \mathbb{E}(e_n^t)]'$ and its $n \times n$ positive definite covariance $\text{Cov}(e_t) = \mathbb{E}(e_t e_t') - \mathbb{E}(e_t) \mathbb{E}(e_t') = [\{\sigma_{i,j}\}]$. From the above assumptions, we have

\[
\begin{pmatrix}
  s_t^2 & s_t \mathbb{E}(e_t') \\
  s_t \mathbb{E}(e_t) & \mathbb{E}(e_t e_t')
\end{pmatrix} \succ 0.
\]

We further define the excess return vector of risky assets $P_t = [P_1^t, \cdots, P_n^t]'$ as $[(e_1^t-s_t), \cdots, (e_n^t-s_t)]'$. The following is then true for $t = 0, 1, \cdots, T-1$:

\[
\begin{pmatrix}
  s_t^2 & s_t \mathbb{E}(P_t') \\
  s_t \mathbb{E}(P_t) & \mathbb{E}(P_t P_t')
\end{pmatrix} =
\begin{pmatrix}
  1 & 0' \\
  -1 & I
\end{pmatrix}
\begin{pmatrix}
  s_t^2 & s_t \mathbb{E}(e_t') \\
  s_t \mathbb{E}(e_t) & \mathbb{E}(e_t e_t')
\end{pmatrix}
\begin{pmatrix}
  1 & -1' \\
  0 & I
\end{pmatrix} \succ 0,
\]
where $1$ and $0$ are the $n$-dimensional all-one and all-zero vectors, respectively, and $I$ is the $n \times n$ identity matrix, which further implies, for $t = 0, 1, \ldots, T-1$, $\mathbb{E}(P_t P'_t) > 0$ and $s_t^2(1-B_t) > 0$, where $B_t \overset{\Delta}{=} \mathbb{E}(P_t')\mathbb{E}^{-1}(P_t P'_t)\mathbb{E}(P_t)$.

An investor joins the market at the beginning of period 0 with an initial wealth $x_0$. He/she allocates $x_0$ among the riskless asset and $n$ risky assets at the beginning of period 0 and reallocates his/her wealth at the beginning of each of the following $(T-1)$ consecutive periods. Let $x_t$ be the wealth of the investor at the beginning of period $t$, and $u_i^t$, $i = 1, 2, \ldots, n$, be the amount invested in the $i$-th risky asset at period $t$. Then, $x_t - \sum_{i=1}^n u_i^t$ is the amount invested in the riskless asset at period $t$. We denote the information set at the beginning of period $t$, $t = 1, 2, \ldots, T-1$, as $\mathcal{F}_t = \sigma(P_0, P_1, \ldots, P_{t-1})$ and the trivial $\sigma$-algebra over $\Omega$ as $\mathcal{F}_0$. Therefore, $\mathbb{E}(\cdot|\mathcal{F}_0)$ is just the unconditional expectation $\mathbb{E}(\cdot)$. We confine all admissible investment strategies to be $\mathcal{F}_t$-measurable Markov controls, i.e., $u_t \in \mathcal{F}_t$. Then, $P_t$ and $u_t$ are independent, $\{x_t\}$ is an adapted Markovian process and $\mathcal{F}_t = \sigma(x_t)$.

The conventional multi-period mean-variance model is to seek the best strategy, $u^*_t = [(u_1^t)^*, (u_2^t)^*, \ldots, (u_n^t)^*]'$, $t = 0, 1, \ldots, T-1$, which is the optimizer of the following stochastic discrete-time optimal control problem,

$$
(MMV) \quad \max_{u} \mathbb{E}(x_T) - \omega_T \text{Var}(x_T),
$$

subject to

$$
x_{t+1} = \sum_{i=1}^n c_i^t u_i^t + \left( x_t - \sum_{i=1}^n u_i^t \right) s_t \\
= s_t x_t + P_t' u_t, \quad t = 0, 1, \ldots, T-1,
$$

where $\omega_T > 0$ is the trade-off parameter between mean and variance of the terminal wealth.

The multi-period mean-variance model with intertemporal restrictions is to find the optimal control of the following problem,

$$
(MMV - IR) \quad \max_{u} \sum_{t \in T_\alpha} \alpha_t \left[ \ell_t \mathbb{E}(x_t) - \rho_t \text{Var}(x_t) \right],
$$

subject to

$$
\{x(t), u_t\} \text{ satisfies equation (1)},
$$

where $I_\alpha = \{\tau_1, \ldots, \tau_\alpha\}$ with $\tau_\alpha = T$ being the set of time instances on which the investor evaluates the performance of the portfolio, $\alpha_t \ell_t$ and $\alpha_t \rho_t > 0$ are the time-$t$ weights of the mean and the variance in the objective functional, respectively. In particular, if we choose $I_\alpha = \{T\}$, $\alpha_T \ell_T = 1$ and $\alpha_T \rho_T = \omega_T > 0$, $(MMV - IR)$ reduces to the conventional multi-period

\[ \text{control of the following problem,} \]

\[ \text{subject to} \]

\[ \{x(t), u_t\} \text{ satisfies equation (1)}, \]

where $I_\alpha = \{\tau_1, \ldots, \tau_\alpha\}$ with $\tau_\alpha = T$ being the set of time instances on which the investor evaluates the performance of the portfolio, $\alpha_t \ell_t$ and $\alpha_t \rho_t > 0$ are the time-$t$ weights of the mean and the variance in the objective functional, respectively. In particular, if we choose $I_\alpha = \{T\}$, $\alpha_T \ell_T = 1$ and $\alpha_T \rho_T = \omega_T > 0$, $(MMV - IR)$ reduces to the conventional multi-period
mean-variance portfolio selection model (MMV) studied in Li and Ng [19]. If \( I_\alpha \) contains time instances other than \( T \), (MMV – IR) is the multi-period portfolio selection problem with intertemporal restrictions considered in Costa and Nabholz [11]. Without loss of generality, we let \( I_\alpha \) include all time instants from 0 to \( T \), while setting some \( \alpha_t = \ell_t = \rho_t = 0 \) for these time instances which do not need to be evaluated.

The dynamic mean-variance portfolio selection model with risk control over bankruptcy is formulated as the following problem,

\[
(MMV - B) \quad \max \mathbb{E}(x_T) - \omega_T \text{Var}(x_T), \\
\text{s.t.} \{x(t), u_t\} \text{ satisfies equation (1)}, \\
P(x_t \leq b_t) \leq a_t, \ t = 1, \cdots, T - 1,
\]

where \( b_t \) is the disaster level and \( a_t \) is the acceptable maximum probability of bankruptcy set by the investor. As this probabilistically constrained dynamic portfolio selection problem is hard to solve directly, we replace \( P(x_t \leq b_t) \) by its upper bound \( \frac{\text{Var}(x_t)}{(\mathbb{E}(x_t) - b_t)^2} \) using Tchebycheff inequality as proposed in Zhu et al. [31], resulting in the following generalized mean-variance (GMV) model (see Zhu et al. [31]),

\[
(GMV) \quad \max \mathbb{E}(x_T) - \omega_T \text{Var}(x_T), \\
\text{s.t.} \{x(t), u_t\} \text{ satisfies equation (1)}, \\
\text{Var}(x_t) \leq a_t (\mathbb{E}(x_t) - b_t)^2, \ t = 1, 2, \cdots, T - 1,
\]

which only requires information of the first and second moments of \( \{P_t\} \). The optimal solution to (GMV) is feasible in (MMV – B), thus serving as an approximated solution to (MMV – B).

To solve (GMV), we consider first the following Lagrangian maximization problem,

\[
(L(\omega)) \quad \max \mathbb{E}(x_T) - \omega_T \text{Var}(x_T) \\
- \sum_{t=1}^{T-1} \omega_t [\text{Var}(x_t) - a_t (\mathbb{E}(x_t) - b_t)^2], \\
\text{s.t.} \{x(t), u_t\} \text{ satisfies equation (1)},
\]

where \( \omega = (\omega_1, \omega_2, \cdots, \omega_{T-1})' \in \mathbb{R}^{T-1}_+ \) is the vector of Lagrangian multipliers, and carry out next the dual search over \( \omega \).
We now build up the mean-field formulations for problems \((MMV - IR)\) and \((L(\omega))\), respectively. For \(t = 0, 1, \cdots, T - 1\), the evolution of the expectation of the wealth dynamics specified in (1) can be presented as

\[
\begin{align*}
\mathbb{E}(x_{t+1}) &= s_t \mathbb{E}(x_t) + \mathbb{E}(P'_t)\mathbb{E}(u_t), \\
\mathbb{E}(x_0) &= x_0,
\end{align*}
\]

(2)
due to the independence between \(P_t\) and \(u_t\). Combining (1) and (2) yields the following for \(t = 0, 1, \cdots, T - 1\),

\[
\begin{align*}
x_{t+1} - \mathbb{E}(x_{t+1}) &= s_t (x_t - \mathbb{E}(x_t)) + P'_t u_t - \mathbb{E}(P'_t)\mathbb{E}(u_t) \\
&= s_t (x_t - \mathbb{E}(x_t)) + P'_t (u_t - \mathbb{E}(u_t)) + (P'_t - \mathbb{E}(P'_t))\mathbb{E}(u_t), \\
x_0 - \mathbb{E}(x_0) &= 0.
\end{align*}
\]

(3)

What we are actually doing here is to enlarge the state space \((x_t)\) into \((\mathbb{E}(x_t), x_t - \mathbb{E}(x_t))\) and the control space \((u_t)\) into \((\mathbb{E}(u_t), u_t - \mathbb{E}(u_t))\). Although the two control vectors \(\mathbb{E}(u_t)\) and \(u_t - \mathbb{E}(u_t)\) can be decided independently at time \(t\), they should be chosen such that

\[
\mathbb{E}(u_t - \mathbb{E}(u_t)) = 0, \quad t = 0, 1, \cdots, T - 1.
\]

We also confine admissible investment strategies \((\mathbb{E}(u_t), u_t - \mathbb{E}(u_t))\) to be \(\mathcal{F}_t\)-measurable Markov controls.

The problem \((MMV - IR)\) can be now reformulated as the following mean-filed type of linear quadratic optimal stochastic control problem,

\[
(MMV - MF) \quad \max \sum_{t=1}^{T} \alpha_t \left\{ \ell_t \mathbb{E}(x_t) - \rho_t \mathbb{E}\left[(x_t - \mathbb{E}(x_t))^2\right] \right\},
\]

s.t. \(\mathbb{E}(x_t)\) satisfies equation (2),

\[
x_{t+1} - \mathbb{E}(x_{t+1})\text{ satisfies equation (3)},
\]

\[
\mathbb{E}(u_t - \mathbb{E}(u_t)) = 0, \quad t = 0, 1, \cdots, T - 1.
\]
Similarly, problem \((L(\omega))\) can be reexpressed as the following mean-field type formulation,

\[
(L - MF(\omega)) \max \mathbb{E}(x_T) - \omega_T\mathbb{E}[(x_T - \mathbb{E}(x_T))^2] \\
- \sum_{t=1}^{T-1} \omega_t \left\{ \mathbb{E}[(x_t - \mathbb{E}(x_t))^2] - a_t(\mathbb{E}(x_t) - b_t)^2 \right\},
\]

s.t. \(\mathbb{E}(x_t)\) satisfies equation (2),

\[x_{t+1} - \mathbb{E}(x_{t+1})\] satisfies equation (3),

\[\mathbb{E}(u_t - \mathbb{E}(u_t)) = 0, \ t = 0, 1, \ldots, T - 1.\]

Adopting a mean-field formulation translates an originally nonseparable unconstrained stochastic control problem into a separable, albeit constrained, linear quadratic optimal stochastic control problem. More specifically, under the mean-field formulation, the mean-variance type objective becomes separable in the expanded state space \((\mathbb{E}(x_t), x_t - \mathbb{E}(x_t))\), which enables us to apply dynamic programming. In addition to a dimension increase of the state space, a technical difficulty induced by this transformation is the introduction of a nonconventional linear constraint in expectation for the two control vectors \(\mathbb{E}(u_t)\) and \(u_t - \mathbb{E}(u_t)\), which requires caution when applying dynamic programming. Note that, when the constraint \(\mathbb{E}(u_t - \mathbb{E}(u_t)) = 0\) is satisfied for all \(t\), the two states \(\mathbb{E}(x_t)\) and \(x_t - \mathbb{E}(x_t)\) satisfy the relation \(\mathbb{E}(x_t - \mathbb{E}(x_t)) = 0\) for all \(t\) due to (3).

III. Optimal Policies for Multi-period Mean-Variance Portfolio Selection with and without Intertemporal Restrictions

In this section, we reconsider the classical multi-period mean-variance model in Li and Ng [19] and the multi-period mean-variance model with intertemporal restrictions, \((MMV - IR)\), in Costa and Nabholz [11] under a mean-field formulation framework. Before deriving the explicit optimal strategies of these problems by dynamic programming, we introduce three useful lemmas first.

**Lemma 1 (Sherman-Morrison formula):** Suppose that \(A\) is an invertible square matrix and \(\mu\) and \(\nu\) are two given vectors. If

\[1 + \nu' A^{-1} \mu \neq 0,\]

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then the following holds,
\[(A + \mu'\nu')^{-1} = A^{-1} - \frac{A^{-1}\mu'\nu'A^{-1}}{1 + \nu'\nu A^{-1}}.\]

**Lemma 2:** Let \(B_t = \mathbb{E}(P_t')\mathbb{E}^{-1}(P_tP_t')\mathbb{E}(P_t)\). Then
\[
\left[\mathbb{E}(P_tP_t') - \mathbb{E}(P_t)\mathbb{E}(P_t')\right]^{-1}\mathbb{E}(P_t) = \frac{\mathbb{E}^{-1}(P_tP_t')\mathbb{E}(P_t)}{1 - B_t}.
\]

**Proof.** Applying Sherman-Morrison formula gives rise to the following,
\[
\mathbb{E}(P_tP_t') - \mathbb{E}(P_t)\mathbb{E}(P_t') = \frac{\mathbb{E}^{-1}(P_tP_t')\mathbb{E}(P_t)}{1 - B_t}.
\]

We proceed now to the third lemma which reveals a useful result based on dynamic programming. Let us consider the following general separable multi-period control problem,
\[
\max \mathbb{E} \left[ \sum_{t=0}^{T-1} h_t(x_t, v_t) + h_T(x_T) \right],
\]
subject to \(x_{t+1} = f(x_t, v_t), \quad t = 0, 1, \cdots, T - 1,\)
where \(x_t\) denotes the state, \(v_t\) denotes the control, \(f(x_t, v_t)\) represents the state dynamics and \(h_t(x_t, v_t)\), the running benefit of period \(t\) in the objective function, is assumed to be concave in \(v_t\). Based on the principle of optimality in dynamic programming, the optimal control at time \(t\) can be obtained by solving the following recursion,
\[
v_t^* = \arg\max_{v_t} \left\{ \mathbb{E}[J_{t+1}(x_{t+1}; v_0, v_1, \cdots, v_t)|\mathcal{F}_t] + h_t(x_t, v_t) \right\}, \quad (4)
\]
where \(J_{t+1}(x_{t+1}; v_0, v_1, \cdots, v_t)\) is the benefit-to-go function defined as follows,
\[
J_{t+1}(x_{t+1}; v_0, v_1, \cdots, v_t) = \max_{v_{t+1}, \cdots, v_T} \mathbb{E} \left[ \sum_{j=t+1}^{T-1} h_j(x_j, v_j) + h_T(x_T) \bigg| \mathcal{F}_{t+1} \right],
\]
with \(\mathcal{F}_t\) being the information set at time \(t\) and \((v_0, v_1, \cdots, v_t)\) being the control sequence before time \(t + 1\). Now let us assume that the benefit-to-go function at time \(t + 1\) can be decomposed into the following special form,
\[
\mathbb{E}[J_{t+1}(x_{t+1}; v_0, v_1, \cdots, v_t)|\mathcal{F}_t] = G_{t}^1(x_t; v_0, v_1, \cdots, v_t) + G_{t}^2(x_t; v_0, v_1, \cdots, v_t),
\]
with $\mathbb{E}[G_t^2(x_t; v_0, v_1, \cdots, v_t)|\mathcal{F}_0] = 0$. A key question is whether we are able to simplify the recursion in (4) under such a circumstance. The following lemma offers a positive answer to this question. More specifically, the lemma proves that $G_t^2$ does not appear in the expression of the benefit-to-go function at time $t$ and can be removed from the dynamic programming recursion.

**Lemma 3:** Assume that

$$
\mathbb{E}[J_{t+1}(x_{t+1}; v_0, v_1, \cdots, v_t)|\mathcal{F}_t] = G_t^1(x_t; v_0, v_1, \cdots, v_t) + G_t^2(x_t; v_0, v_1, \cdots, v_t),
$$

where $\mathbb{E}[G_t^2(x_t; v_0, v_1, \cdots, v_t)|\mathcal{F}_0] = 0$ holds for any admissible $(v_0, v_1, \cdots, v_t)$. Then

$$
v_t^* = \arg \max_{v_t} \{ G_t^1(x_t; v_0, v_1, \cdots, v_t) + h_t(x_t, v_t) \}, \quad t = 0, 1, \cdots, T - 1,
$$

$$
J_0(x_0) = \max_{v_0, \cdots, v_T} \left\{ \mathbb{E}[G_t^1(x_t; v_0, v_1, \cdots, v_t)|\mathcal{F}_0] + \sum_{j=0}^{t} \mathbb{E}[h_j(x_j, v_j)|\mathcal{F}_0] \right\},
$$

i.e., $G_t^1(x_t; v_0, v_1, \cdots, v_t^*) + h_t(x_t, v_t^*)$ can be regarded as the benefit-to-go function at time $t$.

Proof. Based on the principle of optimality of dynamic programming, the optimal control subsequence up to time $t$ can be determined by maximizing the summation of the running benefits up to time $t$ and the benefit-to-go function at time $t + 1,

$$(v_0^*, v_1^*, \cdots, v_t^*) = \arg \max_{v_0, \cdots, v_t} \left\{ \mathbb{E}[J_{t+1}(x_{t+1}; v_0, v_1, \cdots, v_t)|\mathcal{F}_0] + \sum_{j=0}^{t} \mathbb{E}[h_j(x_j, v_j)|\mathcal{F}_0] \right\}.
$$

By the smoothing property of the expectation, i.e., $\mathbb{E}(\mathbb{E}(\cdot|\mathcal{F}_i)|\mathcal{F}_j) = \mathbb{E}(\cdot|\mathcal{F}_j), \forall i > j$, we can simplify the above expression to the following,

$$(v_0^*, v_1^*, \cdots, v_t^*)$$

$$= \arg \max_{v_0, \cdots, v_t} \left\{ \mathbb{E}[\mathbb{E}[J_{t+1}(x_{t+1}; v_0, v_1, \cdots, v_t)|\mathcal{F}_t]|\mathcal{F}_0] + \sum_{j=0}^{t} \mathbb{E}[h_j(x_j, v_j)|\mathcal{F}_0] \right\}$$

$$= \arg \max_{v_0, \cdots, v_t} \left\{ \mathbb{E}[G_t^1(x_t; v_0, v_1, \cdots, v_t) + G_t^2(x_t; v_0, v_1, \cdots, v_t)|\mathcal{F}_0] + \sum_{j=0}^{t} \mathbb{E}[h_j(x_j, v_j)|\mathcal{F}_0] \right\}$$

$$= \arg \max_{v_0, \cdots, v_t} \left\{ \mathbb{E}[G_t^1(x_t; v_0, v_1, \cdots, v_t)|\mathcal{F}_0] + \sum_{j=0}^{t} \mathbb{E}[h_j(x_j, v_j)|\mathcal{F}_0] \right\}$$

$$= \arg \max_{v_0, \cdots, v_t} \left\{ \mathbb{E}[\cdots \mathbb{E}[G_t^1(x_t; v_0, v_1, \cdots, v_t) + h_t(x_t, v_t)|\mathcal{F}_{t-1}] + h_{t-1}(x_{t-1}, v_{t-1})|\mathcal{F}_{t-2}] \cdots |\mathcal{F}_0] \right.$$

$$+ h_0(x_0, v_0) \right\}.$$
where the assumption of $\mathbb{E}[G_t^2(x_t; v_0, v_1, \cdots, v_t)|F_0] = 0$ is used in deriving the third equality, which further implies the following based on dynamic backward recursion,

$$v_t^* = \arg\max_{v_t} \{G_t^1(x_t; v_0, v_1, \cdots, v_t) + h_t(x_t, v_t)\}.$$ 

Since $\mathbb{E}[G_t^2(x_t; v_0, v_1, \cdots, v_t)|F_0] = 0$ holds for any admissible $(v_0, v_1, \cdots, v_t)$, we have

$$J_0(x_0) = \max_{v_0, \cdots, v_T} \left\{ \mathbb{E}[J_{t+1}(x_{t+1}; v_0, v_1, \cdots, v_t)|F_0] + \sum_{j=0}^{t} \mathbb{E}[h_j(x_j, v_j)|F_0] \right\}$$

$$= \max_{v_0, \cdots, v_T} \left\{ \mathbb{E}[G_t^1(x_t; v_0, v_1, \cdots, v_t)|F_0] + \sum_{j=0}^{t} \mathbb{E}[h_j(x_j, v_j)|F_0] \right\}.$$ 

\[\square\]

**Remark 1:** If $h_t(x_t, v_t) = h_t(x_t)$, i.e., $h_t$ is independent of control $v_t$, the conclusion of Lemma 3 can be expressed as follows,

$$v_t^* = \arg\max_{v_t} G_t^1(x_t; v_0, v_1, \cdots, v_T), \quad t = 0, 1, \cdots, T - 1,$$

$$J_0(x_0) = \max_{v_0, \cdots, v_T} \left\{ \mathbb{E}[G_t^1(x_t; v_0, v_1, \cdots, v_T)|F_0] + \sum_{j=0}^{t} \mathbb{E}[h_j(x_j)|F_0] \right\},$$

i.e., $G_t^1(x_t; v_0, v_1, \cdots, v_t^*) + h_t(x_t)$ can be regarded as the benefit-to-go function at time $t$.

Before presenting our main proposition, we still need to define two deterministic sequences $\{p_t\}$ and $\{q_t\}$ which satisfy the following backward recursions,

$$\begin{cases} p_t = \alpha_t \ell_t + s_t^2 (1 - B_t) p_{t+1}, & t = 0, 1, \cdots, T - 1, \\ p_T = \alpha_T \ell_T, & \end{cases}$$

$$\begin{cases} q_t = \alpha_t \ell_t + s_t q_{t+1}, & t = 0, 1, \cdots, T - 1, \\ q_T = \alpha_T \ell_T, & \end{cases}$$

for $t = T - 1, T - 2, \cdots, 1$. We also set $\prod_{\emptyset}(\cdot) = 1$ and $\sum_{\emptyset}(\cdot) = 0$ for the convenience.

**Proposition 1:** The optimal strategy of problem $(MMV - MF)$ is given by

$$u_t^* - \mathbb{E}(u_t^*) = -s_t (x_t - \mathbb{E}(x_t)) \mathbb{E}^{-1}(P_t \ell_t^*) \mathbb{E}(P_t),$$

$$\mathbb{E}(u_t^*) = \frac{q_{t+1} \mathbb{E}^{-1}(P_t \ell_t^*) \mathbb{E}(P_t)}{2p_{t+1} - B_t},$$

for $t = 0, 1, \cdots, T - 1$, where the optimal expected wealth level is

$$\mathbb{E}(x_t) = x_0 \prod_{k=0}^{t-1} s_k + \sum_{j=0}^{t-1} \frac{q_{j+1} B_j}{2p_{j+1}(1 - B_j)} \prod_{\ell=j+1}^{t-1} s_{\ell}.$$
Proof. We first prove that, with information set $\mathcal{F}_t$, i.e., with knowledge of $(\mathbb{E}(x_t), x_t - \mathbb{E}(x_t))$, we have the following expression,

$$
J_t(\mathbb{E}(x_t), x_t - \mathbb{E}(x_t)) = -p_t(x_t - \mathbb{E}(x_t))^2 + q_t\mathbb{E}(x_t) + \sum_{j=t}^{T-1} \frac{q_{j+1}^2}{4p_{j+1}} \frac{B_j}{1 - B_j},
$$

as the benefit-to-go function at time $t$.

When $t = T$, expression (7) is obvious. Assume that we have expression (7) as the benefit-to-go function at time $t + 1$. We prove that expression (7) still holds for the benefit-to-go function at time $t$. For given information set $\mathcal{F}_t$, i.e., $(\mathbb{E}(x_t), x_t - \mathbb{E}(x_t))$, the recursive equation reads as

$$
J_t(\mathbb{E}(x_t), x_t - \mathbb{E}(x_t))
= -\alpha_pt(x_t - \mathbb{E}(x_t))^2 + \alpha_t\ell_t\mathbb{E}(x_t) + \max_{(\mathbb{E}(u_t), u_t - \mathbb{E}(u_t))} \mathbb{E}[J_{t+1}(\mathbb{E}(x_{t+1}), x_{t+1} - \mathbb{E}(x_{t+1})) | \mathcal{F}_t].
$$

Based on (2) and (3), we deduce

$$
\mathbb{E}[J_{t+1}(\mathbb{E}(x_{t+1}), x_{t+1} - \mathbb{E}(x_{t+1})) | \mathcal{F}_t]
= \mathbb{E}[-p_{t+1}(x_{t+1} - \mathbb{E}(x_{t+1}))^2 + q_{t+1}\mathbb{E}(x_{t+1}) | \mathcal{F}_t] + \sum_{j=t+1}^{T-1} \frac{q_{j+1}^2}{4p_{j+1}} \frac{B_j}{1 - B_j}
$$

$$
= -p_{t+1}\mathbb{E}\left[s_t^2(x_t - \mathbb{E}(x_t))^2 + \left(P'_t(u_t - \mathbb{E}(u_t))\right)^2 + \left(\mathbb{E}(u'_t)(P_t - \mathbb{E}(P_t))\right)^2\right]
+ 2s_t(x_t - \mathbb{E}(x_t))P'_t(u_t - \mathbb{E}(u_t)) + 2s_t(x_t - \mathbb{E}(x_t))(P'_t - \mathbb{E}(P'_t))\mathbb{E}(u_t)
$$

$$
+ 2(u_t - \mathbb{E}(u_t))'P_t(P'_t - \mathbb{E}(P'_t))\mathbb{E}(u_t) | \mathcal{F}_t] + q_{t+1}[s_t\mathbb{E}(x_t) + \mathbb{E}(P'_t)\mathbb{E}(u_t)] + \sum_{j=t+1}^{T-1} \frac{q_{j+1}^2}{4p_{j+1}} \frac{B_j}{1 - B_j}.
$$

Since both $u_t - \mathbb{E}(u_t)$ and $\mathbb{E}(u_t)$ are $\mathcal{F}_t$-measurable and $P_t$ is independent to $\mathcal{F}_t$, we have

$$
\mathbb{E}\left[\left(P'_t(u_t - \mathbb{E}(u_t))\right)^2 | \mathcal{F}_t\right] = (u_t - \mathbb{E}(u_t))^2\mathbb{E}(P'_t)\mathbb{E}(u_t),
$$

$$
\mathbb{E}\left[\left(\mathbb{E}(u'_t)(P_t - \mathbb{E}(P_t))\right)^2 | \mathcal{F}_t\right] = \mathbb{E}(u'_t)^2(\mathbb{E}(P'_t) - \mathbb{E}(P_t)\mathbb{E}(P'_t))\mathbb{E}(u_t),
$$

$$
\mathbb{E}\left[2s_t(x_t - \mathbb{E}(x_t))P'_t(u_t - \mathbb{E}(u_t)) | \mathcal{F}_t\right] = 2s_t(x_t - \mathbb{E}(x_t))\mathbb{E}(P'_t)(u_t - \mathbb{E}(u_t)),
$$

$$
\mathbb{E}\left[2s_t(x_t - \mathbb{E}(x_t))(P'_t - \mathbb{E}(P'_t))\mathbb{E}(u_t) | \mathcal{F}_t\right] = 0,
$$

$$
\mathbb{E}\left[2(u_t - \mathbb{E}(u_t))'P_t(P'_t - \mathbb{E}(P'_t))\mathbb{E}(u_t) | \mathcal{F}_t\right] = 2(u_t - \mathbb{E}(u_t))'\mathbb{E}(P'_t)\mathbb{E}(u_t).
$$
which further implies,
\[
\mathbb{E}[J_{t+1}(\mathbb{E}(x_{t+1}), x_{t+1} - \mathbb{E}(x_{t+1})) | \mathcal{F}_t]
\]
\[
= - p_{t+1} \left[ s_t^2 (x_t - \mathbb{E}(x_t))^2 + \left( u_t - \mathbb{E}(u_t) \right) \left( \mathbb{E}(P_t P'_t) (u_t - \mathbb{E}(u_t)) \right) \right] + q_{t+1} \mathbb{E}(P'_t) \mathbb{E}(u_t) + s_t q_{t+1} \mathbb{E}(x_t) + \sum_{j=t+1}^{T-1} \frac{q_{j+1}^2 B_j}{4p_{j+1}} \frac{B_j}{1 - B_j}
\]
\[
- 2p_{t+1} \left( u_t - \mathbb{E}(u_t) \right) \left( \mathbb{E}(P_t P'_t) - \mathbb{E}(P_t) \mathbb{E}(P'_t) \right) \mathbb{E}(u_t)
\]
\[
= G_t^1(\mathbb{E}(x_t), x_t - \mathbb{E}(x_t); \mathbb{E}(u_t), u_t - \mathbb{E}(u_t)) + G_t^2(\mathbb{E}(x_t), x_t - \mathbb{E}(x_t); \mathbb{E}(u_t), u_t - \mathbb{E}(u_t)),
\]
where
\[
G_t^1(\mathbb{E}(x_t), x_t - \mathbb{E}(x_t); \mathbb{E}(u_t), u_t - \mathbb{E}(u_t))
\]
\[
= - p_{t+1} \left[ s_t^2 (x_t - \mathbb{E}(x_t))^2 + \left( u_t - \mathbb{E}(u_t) \right) \left( \mathbb{E}(P_t P'_t) (u_t - \mathbb{E}(u_t)) \right) \right] + q_{t+1} \mathbb{E}(P'_t) \mathbb{E}(u_t) + s_t q_{t+1} \mathbb{E}(x_t) + \sum_{j=t+1}^{T-1} \frac{q_{j+1}^2 B_j}{4p_{j+1}} \frac{B_j}{1 - B_j}
\]
\[
G_t^2(\mathbb{E}(x_t), x_t - \mathbb{E}(x_t); \mathbb{E}(u_t), u_t - \mathbb{E}(u_t)) = -2p_{t+1} \left( u_t - \mathbb{E}(u_t) \right) \left( \mathbb{E}(P_t P'_t) - \mathbb{E}(P_t) \mathbb{E}(P'_t) \right) \mathbb{E}(u_t).
\]
Note that any admissible strategy of \((MMV-MF), (\mathbb{E}(u_t), u_t-\mathbb{E}(u_t))\) satisfies \(\mathbb{E}(u_t-\mathbb{E}(u_t)) = 0\), which implies
\[
\mathbb{E} \left[ G_t^2(\mathbb{E}(x_t), x_t - \mathbb{E}(x_t); \mathbb{E}(u_t), u_t - \mathbb{E}(u_t)) \big| \mathcal{F}_0 \right]
\]
\[
= -2p_{t+1} \mathbb{E} \left[ (u_t - \mathbb{E}(u_t)) \left( \mathbb{E}(P_t P'_t) - \mathbb{E}(P_t) \mathbb{E}(P'_t) \right) \mathbb{E}(u_t) \big| \mathcal{F}_0 \right] = 0.
\]
Using Lemma 3 and Remark 1, we get
\[
(\mathbb{E}(u_t^*)^\ast, u_t^* - \mathbb{E}(u_t^*)) = \arg \max G_t^1(\mathbb{E}(x_t), x_t - \mathbb{E}(x_t); \mathbb{E}(u_t), u_t - \mathbb{E}(u_t)).
\]
By means of Lemma 2, we deduce

\[ G_t^1(\mathbb{E}(x_t), x_t - \mathbb{E}(x_t); \mathbb{E}(u_t), u_t - \mathbb{E}(u_t)) \]

\[ = -p_{t+1} \left\{ s_t^2(1 - B_t)(x_t - \mathbb{E}(x_t))^2 + \left[ (u_t - \mathbb{E}(u_t)) + s_t(x_t - \mathbb{E}(x_t)) \right] E^{-1}(P_tP'_t)E(P'_t) \right\} \]

\[ \cdot \mathbb{E}(P_tP'_t) \left[ (u_t - \mathbb{E}(u_t)) + s_t(x_t - \mathbb{E}(x_t)) \right] E^{-1}(P_tP'_t)E(P'_t) \}

\[ - p_{t+1} \left[ \mathbb{E}(u_t) - \frac{q_{t+1}}{2p_{t+1}} E^{-1}(P_tP'_t)E(P'_t) \right] \left( (P_tP'_t) - \mathbb{E}(P_t)E(P'_t) \right) \]

\[ \cdot \left[ \mathbb{E}(u_t) - \frac{q_{t+1} E^{-1}(P_tP'_t)E(P'_t)}{2p_{t+1}} \right] + \frac{q_{t+1} B_t}{4p_{t+1}} \frac{1}{1 - B_t} + s_t q_{t+1} \mathbb{E}(x_t) + \sum_{j=t+1}^{T-1} \frac{q_j^2}{4p_j} \frac{1}{1 - B_j}. \]

We first identify optimal \((\mathbb{E}(u_t^*), u_t^* - \mathbb{E}(u_t))\) by maximizing \(G_t^1\) without considering the linear constraint \(\mathbb{E}(u_t - \mathbb{E}(u_t)) = 0\), and verify then the derived optimal strategy satisfies this constraint automatically. More specifically, maximizing \(G_t^1\) yields

\[ u_t^* - \mathbb{E}(u_t^*) = -s_t(x_t - \mathbb{E}(x_t)) E^{-1}(P_tP'_t)E(P_t), \]

\[ \mathbb{E}(u_t^*) = \frac{q_{t+1} E^{-1}(P_tP'_t)E(P_t)}{2p_{t+1}}. \]

Therefore, based on Remark 1, we have

\[ G_t^1(\mathbb{E}(x_t), x_t - \mathbb{E}(x_t); \mathbb{E}(u_t^*), u_t^* - \mathbb{E}(u_t^*)) = -\alpha_t \rho_t (x_t - \mathbb{E}(x_t))^2 + \alpha_t \ell_t \mathbb{E}(x_t) \]

\[ = -p_t(x_t - \mathbb{E}(x_t))^2 + q_t \mathbb{E}(x_t) + \sum_{j=t}^{T-1} \frac{q_{j+1} B_j}{4p_{j+1}} \frac{1}{1 - B_j} \]

as the benefit-to-go function at time \(t\).

Substituting the optimal expected portfolio policy in (6) into the dynamics of the expected wealth in (2), we further deduce the following recursive relationship of the optimal expected wealth level,

\[ \mathbb{E}(x_{t+1}) = s_t \mathbb{E}(x_t) + \frac{q_{t+1}}{2p_{t+1}} \cdot \frac{B_t}{1 - B_t}, \]

which implies

\[ \mathbb{E}(x_t) = x_0 \prod_{k=0}^{t-1} s_k + \sum_{j=0}^{t-1} \frac{q_{j+1} B_j}{2p_{j+1}} \cdot \frac{1}{1 - B_j} \cdot \prod_{\ell=j+1}^{t-1} s_\ell. \]

Finally, we show that this optimal strategy satisfies the linear constraint. At time 0, \(\mathbb{E}(u_0^* - \mathbb{E}(u_0^*)) = 0\) is obvious due to \(x_0 = \mathbb{E}(x_0)\). Then, according to the dynamic system (3), we have
\[\mathbb{E}(x_1 - \mathbb{E}(x_1)) = 0,\] which further implies \[\mathbb{E}(u_t^* - \mathbb{E}(u_t^*)) = 0.\] Repeating this argument, we have \[\mathbb{E}(u_t^* - \mathbb{E}(u_t^*)) = 0\] holds for all \(t.\) \(\square\)

The optimal strategy obtained in Proposition 1 covers the exiting results in the literature on this subject as its special cases.

Case 1: Let \(I_\alpha = \{T\}, \) \(\alpha_T\ell_T = 1\) and \(\alpha_T\rho_T = \omega_T > 0.\) Then, we have
\[p_t = \omega_T \prod_{j=t}^{T-1} s_j^2 (1 - B_j), \quad q_t = \prod_{j=t}^{T-1} s_j,\]
which further implies
\[\mathbb{E}(x_t) = \prod_{k=0}^{t-1} s_k x_0 + \frac{1}{2\omega_T} \prod_{k=t}^{T-1} s_k^{-1} \sum_{j=0}^{t-1} B_j \prod_{t=j}^{T-1} (1 - B_t)^{-1}\]
\[= \prod_{k=0}^{t-1} s_k x_0 + \frac{1}{2\omega_T} \prod_{k=t}^{T-1} s_k^{-1} \frac{1 - \prod_{k_0=0}^{t-1} (1 - B_k)}{\prod_{k=0}^{T-1} (1 - B_k)},\]
\[\mathbb{E}(x_T) = \prod_{k=0}^{T-1} s_k x_0 + \frac{1}{2\omega_T} \cdot \frac{1 - \prod_{k=0}^{T-1} (1 - B_k)}{\prod_{k=0}^{T-1} (1 - B_k)}.\]
Therefore, we have
\[u_t^* = -s_t \mathbb{E}^{-1}(\mathbf{p}_t^* \mathbf{p}_t) \mathbb{E}(\mathbf{p}_t^* \mathbf{p}_t) x_t + s_t \mathbb{E}^{-1}(\mathbf{p}_t^* \mathbf{p}_t) \mathbb{E}(\mathbf{p}_t^* \mathbb{E}(x_t)) + \mathbb{E}(u_t^*)\]
\[= -s_t \mathbb{E}^{-1}(\mathbf{p}_t^* \mathbf{p}_t) \mathbb{E}(\mathbf{p}_t^* \mathbf{p}_t) x_t + \mathbb{E}^{-1}(\mathbf{p}_t^* \mathbf{p}_t) \mathbb{E}(\mathbf{p}_t^* \mathbf{p}_t) \left[ x_0 \prod_{k=0}^{T-1} s_k + \frac{1}{2\omega_T} \prod_{k=0}^{T-1} (1 - B_k) \right] \prod_{k=t+1}^{T-1} s_k^{-1},\]
which is the optimal portfolio strategy obtained in Li and Ng [19].

Substituting (6) and (8) to dynamics (3) yields
\[\mathbb{E}(x_{t+1} - \mathbb{E}(x_{t+1})) = s_t^2 (1 - B_t) \mathbb{E}(x_t - \mathbb{E}(x_t))^2 + \frac{1}{\prod_{k=t+1}^{T-1} s_k^2 (1 - B_k)} \cdot \frac{B_t}{4\omega_T^2 \prod_{k=t}^{T-1} (1 - B_j)},\]
which further implies
\[\mathbb{E}(x_T - \mathbb{E}(x_T))^2 = \sum_{j=0}^{T-1} \prod_{k=j+1}^{T-1} s_k^2 (1 - B_k) \frac{1}{\prod_{k=j+1}^{T-1} s_k^2 (1 - B_k)} \cdot \frac{B_j}{4\omega_T^2 \prod_{k=j}^{T-1} (1 - B_j)} \]
\[= \frac{1}{4\omega_T^2} \sum_{j=0}^{T-1} B_j \prod_{k=j}^{T-1} (1 - B_k)^{-1}\]
\[= \frac{1 - \prod_{k=0}^{T-1} (1 - B_k)}{4\omega_T^2 \prod_{k=0}^{T-1} (1 - B_k)}.\]
Thus, the efficient frontier is given by
\[
\text{Var}(x_T) = \mathbb{E}(x_T - \mathbb{E}(x_T))^2 = \frac{\prod_{k=0}^{T-1}(1 - B_k)}{1 - \prod_{k=0}^{T-1}(1 - B_k)} \left( \mathbb{E}(x_T) - x_0 \prod_{k=0}^{T-1} s_k \right)^2 \text{ for } \mathbb{E}(x_T) \geq x_0 \prod_{k=0}^{T-1} s_k,
\]
which is the same as the efficient frontier established in Li and Ng [19].

Case 2: Let \( I_\alpha = \{\tau_1, \ldots, \tau_\alpha\} \) with \( \tau_\alpha = T \). Then we have the optimal portfolio strategy as follows,
\[
u^*_t = -s_t x_t \mathbb{E}^{-1}(P_t P'_t) \mathbb{E}(P_t) + s_t \mathbb{E}(x_t) \mathbb{E}^{-1}(P_t P'_t) \mathbb{E}(P_t) + \frac{q_{t+1}}{2p_{t+1}} \mathbb{E}^{-1}(P_t P'_t) \mathbb{E}(P_t) \left( 1 - B_t \right),
\]
where
\[
\begin{align*}
p_t &= \alpha_t \rho_t + s_t^2 (1 - B_t) \rho_{t+1}, \quad t = \tau_i, \\
p_t &= s_t^2 (1 - B_t) \rho_{t+1}, \quad \tau_i < t < \tau_{i-1}, \\
p_T &= \alpha_T \rho_T,
\end{align*}
\]
and
\[
\mathbb{E}(x_{t+1}) = s_t \mathbb{E}(x_t) + \frac{q_{t+1}}{2p_{t+1}} \frac{B_t}{1 - B_t},
\]
which are the same as the results developed in Costa and Nabholz [11]. Note that Costa and Nabholz originally studied a market consisting of all risky assets in their investigation. When we introduce a riskless asset into the market, parameters \( G_t, S_t, A_t \) and \( D_t \) defined in (22), (23), (28) and (29), respectively, in Costa and Nabholz [11] are modified to
\[
G_t = -2p_{r_1}, \quad S_t = -q_{r_1}, \quad A_t = \prod_{k=\tau_i}^{\tau_{i+1}-1} s_k, \quad D_t = \frac{1 - \prod_{k=\tau_i}^{\tau_{i+1}-1}(1 - B_k)}{\prod_{k=\tau_i}^{\tau_{i+1}-1}(1 - B_k)} \cdot \frac{q_{\tau_{i+1}}}{2p_{r_{i+1}}}.
\]

IV. GENERALIZED MEAN-VARIANCE STRATEGY WITH RISK CONTROL OVER BANKRUPTCY

In this section, we reconsider \((GMV)\), the generalized mean-variance model with risk control over bankruptcy in Zhu et al. [31]. Under the mean-field framework, we first solve problem \((L - MF(\omega))\). For \( t = T - 1, T - 2, \ldots, 1 \), we define the following three deterministic sequences, \( \tilde{p}_t, \eta_t \) and \( \xi_t \), by the following recursions,
\[
\begin{align*}
\tilde{p}_t &= \omega_t + s_t^2 (1 - B_t) \tilde{p}_{t+1}, \\
\tilde{p}_T &= \omega_T, \\
\eta_t &= \omega_t a_t + s_t^2 \xi_{t+1} + \eta_{t+1}, \\
\eta_T &= 0, \\
\xi_t &= -\omega_t a_t b_t + s_t \xi_{t+1} \xi_{t+1}, \\
\xi_T &= \frac{1}{2},
\end{align*}
\]
where we have Lagrangian multiplier \( \omega_t \geq 0 \) and
\[
\xi_{t+1} = \frac{\tilde{p}_{t+1}(1 - B_t)}{\tilde{p}_{t+1}(1 - B_t) - \eta_{t+1} B_t},
\]
Lemma 4: Suppose that \( \hat{p}_{t+1}(1 - B_t) - \eta_{t+1}B_t \neq 0 \) holds. Then

\[
[\hat{p}_{t+1}\mathbb{E}(P_t P'_t) - (\hat{p}_{t+1} + \eta_{t+1})\mathbb{E}(P_t)\mathbb{E}(P'_t)]^{-1}\mathbb{E}(P_t) = \frac{\mathbb{E}^{-1}(P_t P'_t)\mathbb{E}(P'_t)}{\hat{p}_{t+1}(1 - B_t) - \eta_{t+1}B_t}.
\]

Proof. Applying Sherman-Morrison formula (Lemma 1) yields

\[
[\hat{p}_{t+1}\mathbb{E}(P_t P'_t) - (\hat{p}_{t+1} + \eta_{t+1})\mathbb{E}(P_t)\mathbb{E}(P'_t)]^{-1}\mathbb{E}(P_t)
= \frac{\mathbb{E}^{-1}(P_t P'_t)\mathbb{E}(P'_t)}{1 - \hat{p}_{t+1}(\hat{p}_{t+1} + \eta_{t+1})\mathbb{E}(P'_t)\mathbb{E}^{-1}(P_t P'_t)\mathbb{E}(P_t)}
= \frac{\mathbb{E}^{-1}(P_t P'_t)\mathbb{E}(P'_t)}{\hat{p}_{t+1}(1 - B_t) - \eta_{t+1}B_t}.
\]

\( \Box \)

Proposition 2: The optimal strategy of problem \((L - MF(\omega))\) is given by

\[
\mathbf{u}^*_t - \mathbb{E}(\mathbf{u}^*_t) = -s_t(x_t - \mathbb{E}(x_t))\mathbb{E}^{-1}(P_t P'_t)\mathbb{E}(P_t),
\]

\[
\mathbb{E}(\mathbf{u}^*_t) = \frac{\xi_{t+1} + \eta_{t+1} s_t \mathbb{E}(x_t)}{\hat{p}_{t+1}(1 - B_t) - \eta_{t+1} B_t} \mathbb{E}^{-1}(P_t P'_t)\mathbb{E}(P_t),
\]

where the optimal expected wealth level \( \mathbb{E}(x_t) \) evolves according to

\[
\mathbb{E}(x_t) = x_0 \prod_{k=0}^{t-1} \zeta_{k+1}s_k + \sum_{j=0}^{t-1} \frac{\xi_{j+1}B_j}{\hat{p}_{j+1}(1 - B_j) - \eta_{j+1} B_j} \prod_{\ell=j+1}^{t-1} \zeta_{\ell+1}s_\ell.
\]

Moreover, the optimal objective function of \((L - MF(\omega))\) is

\[
H(\omega) = \eta_1 s_0^2 x_0^2 + 2\xi_1 s_0 x_0 + \sum_{j=0}^{T-1} \left[ \frac{\xi_{j+1}^2 B_j}{\hat{p}_{j+1}(1 - B_j) - \eta_{j+1} B_j} + \omega_j a_j b_j^2 \right],
\]

with \( \omega_0 = a_0 = b_0 = 0 \).

Proof. We first prove that for information set \( \mathcal{F}_t \), we have the following expression,

\[
J_t(\mathbb{E}(x_t), x_t - \mathbb{E}(x_t))
= -\hat{p}_t(x_t - \mathbb{E}(x_t))^2 + \eta_t(\mathbb{E}(x_t))^2 + 2\xi_t \mathbb{E}(x_t) + \sum_{j=t}^{T-1} \left[ \frac{\xi_{j+1}^2 B_j}{\hat{p}_{j+1}(1 - B_j) - \eta_{j+1} B_j} + \omega_j a_j b_j^2 \right],
\]

as the benefit-to-go function at time \( t \).

When \( t = T \), expression (14) is obvious. Assume that expression (14) holds at time \( t+1 \) as the benefit-to-go function. We show that expression (14) still holds for the benefit-to-go function at
time $t$. For given information set $\mathcal{F}_t$, i.e., $(\mathbb{E}(x_t), x_t - \mathbb{E}(x_t))$, applying the dynamic programming recursive equation yields

$$J_t(\mathbb{E}(x_t), x_t - \mathbb{E}(x_t)) = -\omega_t(x_t - \mathbb{E}(x_t))^2 + \omega_t a_t(\mathbb{E}(x_t))^2 - 2\omega_t a_t b_t \mathbb{E}(x_t) + \omega_t a_t b_t^2 + \max_{(\mathbb{E}(u_t), u_t - \mathbb{E}(u_t))} \mathbb{E} \left[ J_{t+1}(\mathbb{E}(x_{t+1}), x_{t+1} - \mathbb{E}(x_{t+1})) | \mathcal{F}_t \right].$$

Based on (2) and (3), we have

$$\mathbb{E} \left[ J_{t+1}(\mathbb{E}(x_{t+1}), x_{t+1} - \mathbb{E}(x_{t+1})) | \mathcal{F}_t \right] = \mathbb{E} \left[ -\bar{p}_{t+1}(x_{t+1} - \mathbb{E}(x_{t+1}))^2 + \eta_{t+1}(\mathbb{E}(x_{t+1}))^2 + 2\xi_{t+1}\mathbb{E}(x_{t+1}) | \mathcal{F}_t \right] + \sum_{j=t+1}^{T-1} \left[ \frac{\xi_{j+1}B_j}{\bar{p}_{j+1}(1 - B_j) - \eta_{j+1}B_j} + \omega_j a_j b_j^2 \right]$$

Similar to the proof of Proposition 1, we have

$$\mathbb{E} \left[ J_{t+1}(\mathbb{E}(x_{t+1}), x_{t+1} - \mathbb{E}(x_{t+1})) | \mathcal{F}_t \right] = -\bar{p}_{t+1} \left[ s_t^2(x_t - \mathbb{E}(x_t))^2 + (u_t - \mathbb{E}(u_t))^2 \right] \mathbb{E}(\mathbf{p}_t' \mathbf{p}_t' \mathbb{E}(x_t) - \mathbb{E}(u_t)) + 2s_t(x_t - \mathbb{E}(x_t)) \mathbb{E}(\mathbf{p}_t' \mathbb{E}(u_t)) - \mathbb{E}(u_t) \left[ \bar{p}_{t+1} \mathbb{E}(\mathbf{p}_t \mathbf{p}_t') - (\bar{p}_{t+1} + \eta_{t+1})\mathbb{E}(\mathbf{p}_t)\mathbb{E}(\mathbf{p}_t') \right] \mathbb{E}(u_t) + (2\xi_{t+1} + 2\eta_{t+1} s_t) \mathbb{E}(x_t) \mathbb{E}(\mathbf{p}_t') \mathbb{E}(u_t) + \eta_{t+1} s_t^2 (\mathbb{E}(x_t))^2 + 2\xi_{t+1} s_t \mathbb{E}(x_t) + \sum_{j=t+1}^{T-1} \left[ \frac{\xi_{j+1}B_j}{\bar{p}_{j+1}(1 - B_j) - \eta_{j+1}B_j} + \omega_j a_j b_j^2 \right] - 2\bar{p}_{t+1}(u_t - \mathbb{E}(u_t))^2 \left( \mathbb{E}(\mathbf{p}_t \mathbf{p}_t') - \mathbb{E}(\mathbf{p}_t)\mathbb{E}(\mathbf{p}_t') \right) \mathbb{E}(u_t) = G_t(\mathbb{E}(x_t), x_t - \mathbb{E}(x_t); \mathbb{E}(u_t), u_t - \mathbb{E}(u_t)) + G_t(\mathbb{E}(x_t), x_t - \mathbb{E}(x_t); \mathbb{E}(u_t), u_t - \mathbb{E}(u_t)).
where

\[ G_1^t(E(x_t), x_t - E(x_t); E(u_t), u_t - E(u_t)) \]
\[ = - \tilde{p}_{t+1} \left[ s_t^2 (x_t - E(x_t))^2 + (u_t - E(u_t))^t P_t P_t' (u_t - E(u_t)) \right. \]
\[ + 2s_t (x_t - E(x_t))^t P_t' (u_t - E(u_t)) \] \[ - E(u_t)^t \left[ \tilde{p}_{t+1} E(P_t P_t') - (\tilde{p}_{t+1} + \eta_{t+1}) E(P_t) E(P_t') \right] E(u_t) \]
\[ + \left. (2 \xi_{t+1} + 2 \eta_{t+1} s_t E(x_t))^t P_t' E(u_t) + \eta_{t+1} s_t^2 (E(x_t))^2 + 2 \xi_{t+1} s_t E(x_t) \right] \]
\[ + \sum_{j=t+1}^{T-1} \left[ \frac{\xi_{j+1} B_j}{\tilde{p}_{j+1} (1 - B_j) - \eta_{j+1} B_j} + \omega_j a_j b_j^2 \right]. \]

\[ G_2^t(E(x_t), x_t - E(x_t); E(u_t), u_t - E(u_t)) = -2 \tilde{p}_{t+1} (u_t - E(u_t))^t P_t P_t' (u_t - E(u_t)) \]
\[ \left( E(P_t P_t') - E(P_t) E(P_t') \right) E(u_t). \]

Note that any admissible control of \( (L - MF(\omega)) \), \( E(u_t), u_t - E(u_t) \) satisfies \( E(u_t - E(u_t)) = 0 \), which implies

\[ E\left[ G_2^t(E(x_t), x_t - E(x_t); E(u_t), u_t - E(u_t)) | F_0 \right] = 0. \]

Using Lemma 3 and Remark 1 gives rise to

\[ (E(u_t^*), u_t^* - E(u_t^*)) = \arg \max G_1^t(E(x_t), x_t - E(x_t); E(u_t), u_t - E(u_t)). \]

By means of Lemma 4, we deduce

\[ G_1^t(E(x_t), x_t - E(x_t); E(u_t), u_t - E(u_t)) \]
\[ = - \tilde{p}_{t+1} E \left\{ s_t^2 (1 - B_t) (x_t - E(x_t))^2 + \left[ (u_t - E(u_t)) + s_t (x_t - E(x_t))^t \right] \left[ E(P_t P_t') \right. \]
\[ \cdot \left. \left[ (u_t - E(u_t)) + s_t (x_t - E(x_t))^t \right] E^{-1}(P_t P_t') E(P_t') \right\} \]
\[ - E(u_t)^t \left[ \frac{\xi_{t+1} + \eta_{t+1} s_t E(x_t)}{\tilde{p}_{t+1} (1 - B_t) - \eta_{t+1} B_t} E^{-1}(P_t P_t') E(P_t') \right] \]
\[ \cdot \left[ E(u_t)^t - \frac{\xi_{t+1} + \eta_{t+1} s_t E(x_t)}{\tilde{p}_{t+1} (1 - B_t) - \eta_{t+1} B_t} E^{-1}(P_t P_t') E(P_t') \right] \]
\[ \cdot \left[ \frac{\xi_{t+1} + \eta_{t+1} s_t E(x_t)}{\tilde{p}_{t+1} (1 - B_t) - \eta_{t+1} B_t} E^{-1}(P_t P_t') E(P_t') \right] \]
\[ + \eta_{t+1} s_t^2 (E(x_t))^2 + 2 \xi_{t+1} s_t E(x_t) + \sum_{j=t+1}^{T-1} \left[ \frac{\xi_{j+1} B_j}{\tilde{p}_{j+1} (1 - B_j) - \eta_{j+1} B_j} + \omega_j a_j b_j^2 \right]. \]

We first identify \( (E(u_t^*), u_t^* - E(u_t^*)) \) by maximizing the above \( G_1^t \) without considering the linear constraint \( E(u_t - E(u_t)) = 0 \), and verify then the derived optimal strategy satisfies this constraint.
automatically. More specifically, maximizing $G_t^1$ gives rise to the following optimal pair,

$$u_t^* - E(u_t^*) = -s_t (x_t - E(x_t)) E^{-1}(P_t P'_t) E(P_t),$$

$$E(u_t^*) = \frac{\xi_{t+1} + \eta_{t+1}s_t E(x_t)}{p_{t+1}(1 - B_t) - \eta_{t+1}B_t} E^{-1}(P_t P'_t) E(P_t).$$

Based on Remark 1, we can find

$$G_t^1(E(x_t), x_t - E(x_t); E(u_t^*), u_t^* - E(u_t^*))$$

$$- \omega_t (x_t - E(x_t))^2 + \omega_t a_t (E(x_t))^2 - 2 \omega_t a_t b_t E(x_t) + \omega_t a_t b_t^2$$

$$= - \bar{p}_t (E(x_t))^2 + \eta_t (E(x_t))^2 + 2 \xi_t E(x_t) + \sum_{j=t}^{T-1} \left[ \frac{\xi_{j+1}B_j}{\bar{p}_{j+1}(1 - B_j) - \eta_{j+1}B_j} + \omega_j a_j b_j^2 \right]$$

as the benefit-to-go function at time $t$.

Substituting the optimal expected portfolio policy in (11) into the dynamics of the expected wealth in (2) gives rise to

$$E(x_{t+1}) = \zeta_{t+1}s_t E(x_t) + \frac{\xi_{t+1}B_t}{p_{t+1}(1 - B_t) - \eta_{t+1}B_t},$$

which implies

$$E(x_t) = x_0 \prod_{k=0}^{t-1} \zeta_{k+1}s_k + \sum_{j=0}^{t-1} \frac{\xi_{j+1}B_j}{\bar{p}_{j+1}(1 - B_j) - \eta_{j+1}B_j} \cdot \prod_{\ell=j+1}^{t-1} \zeta_{\ell+1}s_\ell.$$  

Noting that $\omega_0 = a_0 = b_0 = 0$, then the expression of the optimal objective function of $(L - MF(\omega))$ is obvious.

Finally, we show that this optimal strategy satisfies the linear constraint. At time 0, $E(u_0^* - E(u_0^*)) = 0$ is obvious due to $x_0 = E(x_0)$. Then, according to the dynamic system (3), we have $E(x_1 - E(x_1)) = 0$, which further implies $E(u_1^* - E(u_1^*)) = 0$. Repeating this argument, we have $E(u_t^* - E(u_t^*)) = 0$ holds for all $t$.

Substituting the explicit optimal strategy specified in (10) and (11) to the dynamics in (3) yields

$$E(x_{t+1} - E(x_{t+1}))^2 = s_t^2(1 - B_t) E(x_t - E(x_t))^2 + \frac{(\xi_{t+1} + \eta_{t+1}s_t E(x_t))^2}{(\bar{p}_{t+1}(1 - B_t) - \eta_{t+1}B_t)^2} (B_t - B_t^2),$$

which leads to the following expression of the variance of the optimal wealth level,

$$\text{Var}(x_t) = E(x_t - E(x_t))^2 = \sum_{j=0}^{t-1} \frac{(\xi_{j+1} + \eta_{j+1}s_j E(x_j))^2}{(\bar{p}_{j+1}(1 - B_j) - \eta_{j+1}B_j)^2} \cdot (B_j - B_j^2) \cdot \prod_{\ell=j+1}^{t-1} s_\ell^2(1 - B_\ell).$$
The remaining task of solving \((GMV)\) is to derive the optimal Lagrangian multiplier vector \(\omega^*\) through

\[
\omega^* = \arg \min_{\omega \in \mathbb{R}^{T+1}} H(\omega).
\] (15)

Due to the derived explicit form of \(H(\omega)\) and the convexity of \(H(\omega)\) (see [31]), we can find optimal \(\omega^*\) easily by any local search algorithms, including steepest descent algorithm and interior point algorithm.

One issue is to ensure the satisfaction of the condition of \(\bar{p}_{t+1}(1 - B_t) - \eta_{t+1}B_t \neq 0\) in Lemma 4, along with any descent searching path when minimizing \(H(\omega)\). Assume that this condition is first violated at time \(t\) in (14). Then, \(\bar{p}_{t+1}(1 - B_t) - \eta_{t+1}B_t = 0\) can only lead to \(J_t(E(x_t), x_t - E(x_t)) = +\infty\), which further implies \(H(\omega) = +\infty\). Thus, any descent algorithm will not direct the search direction such that this condition is violated.

In comparison, Zhu et al. [31] analyzed the Lagrangian problem \((L(\omega))\) via the embedding scheme and were unable to obtain an analytical form of the optimal objective value function \(H(\omega)\). Thus, they invoked a prime-dual iterative algorithm to identify optimal Lagrangian multiplier vector \(\omega^*\). Briefly speaking, in each iteration of the prime-dual iterative algorithm, for a given \(\omega\), a system of linear equations needs to be solved first to get the embedding parameter vector \(\lambda(\omega)\), then the optimal policy of \(L(\omega)\) and a feasible decent direction are computed, and finally a line search along the feasible decent direction is carried out to determine the optimal step-size (Please refer to the detailed description of the algorithm on page 453 in [31]). In summary, our new mean-field formulation clearly, yet powerfully, offers a more efficient and more accurate solution scheme in solving \((GMV)\).

**Example 1:** We consider an example of constructing a pension fund consisting of S&P 500 (SP), the index of Emerging Market (EM), Small Stock (MS) of U.S market and a bank account. Based on the data provided in Elton et al. [14], Table I presents the expected values, variances and correlation coefficients of the annual return rates of these three indices.

Thus, for any time \(t\), we have

\[
E(P_t) = \begin{bmatrix} 0.09 \\ 0.11 \\ 0.12 \end{bmatrix}, \quad \text{Cov}(P_t) = \begin{bmatrix} 0.0342 & 0.0355 & 0.0351 \\ 0.0355 & 0.0900 & 0.0540 \\ 0.0351 & 0.0540 & 0.0576 \end{bmatrix}, \quad \text{E}(P_tP_t^T) = \begin{bmatrix} 0.0423 & 0.0454 & 0.0459 \\ 0.0454 & 0.1021 & 0.0672 \\ 0.0459 & 0.0672 & 0.0720 \end{bmatrix}.
\]


Table I
Data for the Asset Allocation Example

|            | SP  | EM  | MS  |
|------------|-----|-----|-----|
| Expected Return | 14% | 16% | 17% |
| Variance     | 18.5% | 30% | 24% |
| Correlation coefficient | 1  | 0.64 | 0.79 |
| SP           | 1   | 0.64 | 0.79 |
| EM           | 1   | 0.75 |     |
| MS           |     | 1    |     |

We consider problem \((GMV)\), the generalized mean-variance model with risk control over bankruptcy problem with 5 time periods and annual risk free rate 5\% \((s_t = 1.05)\). Assume that the investor has initial wealth \(x_0 = 1\) and trade-off parameter \(\omega_5 = 1\). The disaster level and the acceptable maximum probability of bankruptcy are set at \(b_t = 0\) and \(a_t = 0.10\), respectively, for \(t = 1, 2, 3, 4\).

To identify the optimal multiplier vector \(\omega^*\) through (15), we adopt interior point algorithm, which is an available option of Matlab optimization function “fmincon” and obtain

\[
\omega^* = [0, 0.3068, 0.1832, 0]^T.
\]

We further have

\[
\begin{bmatrix}
\bar{p}_1 \\
\bar{p}_2 \\
\bar{p}_3 \\
\bar{p}_4 \\
\bar{p}_5
\end{bmatrix} =
\begin{bmatrix}
0.9654 \\
1.1148 \\
0.9331 \\
0.8659 \\
1
\end{bmatrix},
\begin{bmatrix}
\eta_1 \\
\eta_2 \\
\eta_3 \\
\eta_4 \\
\eta_5
\end{bmatrix} =
\begin{bmatrix}
0.0569 \\
0.0510 \\
0.0183 \\
0 \\
0
\end{bmatrix},
\begin{bmatrix}
\xi_1 \\
\xi_2 \\
\xi_3 \\
\xi_4 \\
\xi_5
\end{bmatrix} =
\begin{bmatrix}
0.6188 \\
0.5819 \\
0.5513 \\
0.5250 \\
0.5
\end{bmatrix},
\begin{bmatrix}
\zeta_1 \\
\zeta_2 \\
\zeta_3 \\
\zeta_4 \\
\zeta_5
\end{bmatrix} =
\begin{bmatrix}
1.0164 \\
1.0127 \\
1.0054 \\
1.0000 \\
1
\end{bmatrix}.
\]

The optimal expected wealth levels can be then expressed by

\[
\mathbb{E}(x_0) = 1, \ \mathbb{E}(x_1) = 1.2452, \ \mathbb{E}(x_2) = 1.4684, \ \mathbb{E}(x_3) = 1.7124, \ \mathbb{E}(x_4) = 1.9636, \ \mathbb{E}(x_5) = 2.1984.
\]
Therefore, according to Proposition 2, the optimal strategy of (GMV) is specified as follows,

\[
\begin{align*}
    u_0^* &= (-1.05x_0 + 1.9596)K, \\
    u_1^* &= (-1.05x_1 + 2.0575)K, \\
    u_2^* &= (-1.05x_2 + 2.3368)K, \\
    u_3^* &= (-1.05x_3 + 2.5699)K, \\
    u_4^* &= (-1.05x_4 + 2.6984)K,
\end{align*}
\]

where

\[
K = E^{-1}(P_tP_t')E(v_t) = \begin{bmatrix}
1.0580 \\
-0.1207 \\
1.1052
\end{bmatrix}.
\]

Finally, the variances of the optimal wealth levels are given as

\[
\begin{align*}
    \text{Var}(x_1) &= 0.0946, \\
    \text{Var}(x_2) &= 0.1418, \\
    \text{Var}(x_3) &= 0.2095, \\
    \text{Var}(x_4) &= 0.2819, \\
    \text{Var}(x_5) &= 0.3124.
\end{align*}
\]

Table II presents a computational comparison between the approach in Zhu et al. [31] and ours, in which the combinations of two trade-off parameters, \(\omega_5 = 0.5\) and \(\omega_5 = 1\), and two initial points, \(\bar{\omega} = [0.5, 0.5, 0.5, 0.5]'\) and \(\hat{\omega} = [0, 1, 1, 0]'\), are considered. It is clear from Table II that, although the approach in Zhu et al. [31] is very sensitive to the initial point of \(\omega\), our method, in general, consumes much less computational time than the approach in Zhu et al. [31].

In the primal-dual iteration algorithm of [31], the explicit form of \(H(\omega)\) is not known. Thus, the search direction derived by their numerical scheme may deviate significantly from the optimal search direction offered by gradient descent, Newton’s Method or Quasi-Newton method. Furthermore, a line search along this “near-optimal” search direction does not guarantee the convergence of the algorithm. To achieve the convergence of the prime-dual iteration algorithm in [31], different tolerance levels for the line search are needed to be figured out for different cases.

Varying the trade-off parameter \(\omega_5\) from 0 to \(+\infty\) in (GMV) yields the efficient frontier of this example presented in Figure 1 as the dash dot line. For a comparison purpose, the efficient frontier of the classical five-period mean-variance model is also plotted as the solid curve in the figure.


### TABLE II

**Computational Comparison**

| Time (s) | Artificial benchmark | $H(\omega^*)$ |
|----------|----------------------|---------------|
| $\bar{\omega}, \overline{\omega}$ | 9.58 | [0, 0.3707, 0.4101, 0.3967]$^{\dagger}$ | 2.060769840 |
| $\bar{\omega}, \overline{\omega}$ | 2.28 | [0, 0.3709, 0.4099, 0.3968]$^{\dagger}$ | 2.060769838 |
| $\hat{\omega}, \hat{\omega}$ | 1.52 | [0, 0.3066, 0.1832, 0]$^{\dagger}$ | 1.823323029 |
| $\hat{\omega}, \hat{\omega}$ | 1.08 | [0, 0.3065, 0.1832, 0]$^{\dagger}$ | 1.823323030 |

| Approach in this Paper | Time (s) | $\omega^*$ | $H(\omega^*)$ |
|-------------------------|----------|----------|---------------|
| $\bar{\omega}, \overline{\omega}$ | 0.54 | [0, 0.3709, 0.4099, 0.3968]$^{\dagger}$ | 2.060769838 |
| $\bar{\omega}, \overline{\omega}$ | 0.55 | [0, 0.3709, 0.4099, 0.3968]$^{\dagger}$ | 2.060769838 |
| $\hat{\omega}, \hat{\omega}$ | 0.54 | [0, 0.3066, 0.1832, 0]$^{\dagger}$ | 1.823323029 |
| $\hat{\omega}, \hat{\omega}$ | 0.54 | [0, 0.3066, 0.1832, 0]$^{\dagger}$ | 1.823323030 |

**Fig. 1.** Efficient frontiers of $(MV)$ and $(GMV)$

### V. Conclusions

Although the nonseparable multi-period mean-variance problem and its various variants have been solved in the literature via embedding scheme, Lagrangian formulation or mean-variance hedging problem, we may not be able to derive optimal value functions of these transformed problems.
problems analytically when some constraints are attached to the problem setting. Hence, we may need to invoke some numerical algorithms to compute the corresponding best auxiliary parameter or Lagrangian parameter. In this paper, we adopt the mean-field formulation, as a more efficient means, to directly tackle the nonseparable multi-period mean-variance portfolio selection model, multi-period mean-variance model with intertemporal restrictions, and generalized mean-variance model with risk control over bankruptcy. Under this newly proposed framework of mean-field formulations, we are capable of deriving analytical solutions for all these problems, thus improving the solution quality and facilitating the solution process. Furthermore, the mean-field framework seems to provide us a promising new platform to tackle some long-standing challenging research topics of practical importance when considering market frictions in real world, such as portfolio constraints, management fees and transaction costs.

Although we confine our investigation in this paper to situations where random return vectors are assumed to be statistically independent among different periods, the results can be readily extended to situations where random return vectors are correlated, which are referred as stochastic opportunity set in the literature. This extension can be achieved based on the concept of the so-called opportunity-neutral measure introduced by Černý and Kallsen [7]. More specifically, the opportunity-neutral measure $P^*$ defined on $(\Omega, \mathcal{F})$ can be expressed by

$$\frac{dP^*}{dP} = \prod_{t=1}^{T} \frac{L_t}{\mathbb{E}_{t-1}[L_t]},$$

where $L_T = 1$ and $L_t = \mathbb{E}_t(L_{t+1}(1 - \mathbb{E}_t(L_{t+1}P_t')\mathbb{E}_t^{-1}(L_{t+1}P_tP_t'))P_t), t = T - 1, \ldots, 1,$ in our notations. The optimal strategy of problem $(MMV)$ under correlated $P$ is then just the strategy derived under $P^*$ using the results in this paper (as the return vectors under $P^*$ were independent) (See Theorem 8.7 of [7] for a rigorous proof). It is worth to mention that this probability measure change only provides a technical approach for deriving the optimal strategy and under $P^*$, the return vectors may not become independent (See Section 7 of [7] for more detailed discussion).

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