INTERNAL OBSERVABILITY OF THE WAVE EQUATION
IN A TRIANGULAR DOMAIN

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ABSTRACT. We investigate the internal observability of the wave equation with Dirichlet boundary conditions in a triangular domain. More precisely, the domain taken into exam is the half of the equilateral triangle. Our approach is based on Fourier analysis and on tessellation theory: by means of a suitable tiling of the rectangle, we extend earlier observability results in the rectangle to the case of a triangular domain. The paper includes a general result relating problems in general domains to their tiles, and a discussion of the triangular case. As an application, we provide an estimation of the observation time when the observed domain is composed by three strips with a common side to the edges of the triangle.

1. Introduction

We consider the problem

\[
\begin{cases}
  u_{tt} - \Delta u = 0 & \text{in } \mathbb{R} \times \Omega \\
  u = 0 & \text{on } \mathbb{R} \times \partial \Omega \\
  u(t,0) = u_0, \ u_t(t,0) = u_1 & \text{in } \Omega
\end{cases}
\]

where \( \Omega \) is a bounded open domain of \( \mathbb{R}^2 \) that can be tiled by the open triangle \( T \) whose vertices are \((0,0), (1/\sqrt{3},0)\) and \((0,1)\). More precisely, we use the symbol \( cl(\Omega) \) to denote the closure of a set \( \Omega \) and we say that an open set \( \Omega_1 \) tiles \( \Omega_2 \) if there exist a finite number \( N \) of rigid transformations \( K_1, \ldots, K_N \) such that

\[
cl(\Omega_2) = \bigcup_{h=1}^{N} K_h cl(\Omega_2).
\]

and such that \( K_h(\Omega_1) \cap K_j(\Omega_1) = \emptyset \) for all \( h \neq j \). The triangle \( T \) is the half of an equilateral triangle of side \( 2/\sqrt{3} \), and it tiles the rectangle

\[
R := (0, \sqrt{3}) \times (0,1).
\]

In particular, we have that the rectangle \( R \) can be tiled by \( T \) by means of 6 rigid transformations \( K_1, \ldots, K_6 \): to keep the discussion at an introductory

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level, we postpone the explicit definition of the $K_h$’s to Section 3, however such tiling is depicted in Figure 1.

As it is well known, a complete orthonormal base for $L^2(\mathcal{R})$ is given by the eigenfunctions of $-\Delta$ in $H^1_0(\mathcal{R})$

$$\tau_k := \sin(\pi k_1 x_1/\sqrt{3}) \sin(\pi k_2 x_2), \quad \text{where} \quad k = (k_1, k_2), \quad k_1, k_2 \in \mathbb{N}$$

and the associated eigenvalues are $\gamma_k = \frac{k_1^2}{3} + \frac{k_2^2}{3}$. In [29], a folding technique (that we recall in detail in Section 3) is used to derive from $\{\tau_k\}$ an orthogonal base $\{e_k\}$ of $L^2(\mathcal{T})$ formed by the eigenfunctions of $-\Delta$ in $H^1_0(\mathcal{T})$. In particular, $\{e_k\} \subset \text{span}\{\tau_k\}$ and $\{e_k\}$ and $\{\tau_k\}$ share the same eigenvalues $\gamma_k$.

The explicit knowledge of a eigenspace for $H^1_0(\mathcal{T})$ allows us to set the problem (1.1) (with $\Omega = \mathcal{T}$) in the framework of Fourier analysis. Our goal is to exploit the deep relation between the eigenfunctions for $H^1_0(\mathcal{R})$ and those of $H^1_0(\mathcal{T})$ in order to extend known observability results for $\mathcal{R}$ to $\mathcal{T}$.

In particular, we are interested in the internal observability of (1.1), i.e., in the validity of the estimates

$$\|u_0\|_{L^2(\Omega)}^2 + \|u_1\|_{H^{-1}(\Omega)}^2 \leq \mathcal{A} \int_0^T \int_{\Omega_0} |u(t,x)|^2 dx$$

where $\Omega_0$ is a subset of $\Omega$ and $T$ is sufficiently large. Here and in the sequel $A \asymp B$ means $c_1 A \leq B \leq c_2 A$ with some constants $c_1$ and $c_2$ which are independent from $A$ and $B$. When we need to stress the dependence of these estimates on the couple of constants $c = (c_1, c_2)$, we write $A \asymp c B$. Also by writing $A \leq_c B$ we mean the inequality $c A \leq B$ while the expression $A \geq_c B$ denotes $c A \geq B$. 

![Figure 1. The tiling of $\mathcal{R}$ with $\mathcal{T}$. Note that $K_1$ is the identity map, hence $K_1(\mathcal{T}) = \mathcal{T}$.](image-url)
1.1. **Statement of the main results.** We begin by introducing a few notations. Let \( \{ e_k \} \subset H^1_0(\Omega) \cap L^2(\Omega) \) be an orthonormal base of \( L^2(\Omega) \) formed by eigenvalues of \( -\Delta \) and let \( \{ \gamma_k \} \) be the associated eigenvalues. Denote by \( D^s(\Omega) \) the completion of \( \{ e_k \} \) with respect to the Euclidean norm

\[
\left\| \sum_{k \in \mathbb{Z}^2} c_k \gamma_k \right\|_s := \left( \sum_{k \in \mathbb{Z}^2} |\gamma_k|^2 |c_k|^2 \right)^{1/2}.
\]

Identifying \( L^2(\Omega) \) with its dual we have

\[
D^0(\Omega) = L^2(\Omega), \quad D^{-1}(\Omega) = H^{-1}(\Omega).
\]

We have

**Theorem 1.1.** Let \( \overline{u} \) be the solution of

\[
\begin{cases}
\overline{u}_{tt} - \Delta \overline{u} = 0 & \text{on } \mathbb{R} \times \mathbb{R} \\
\overline{u} = 0 & \text{in } \mathbb{R} \times \partial \mathbb{R} \\
\overline{u}(t,0) = \overline{u}_0, \quad u_t(t,0) = \overline{u}_1 & \text{in } \mathbb{R},
\end{cases}
\]

let \( \mathcal{S} \) be a subset of \( \mathcal{R} \) and assume that there exists a constant \( T_{\mathcal{S}} \geq 0 \) such that if \( T > T_{\mathcal{S}} \) then there exists a couple of constants \( c = (c_1, c_2) \) such that \( \overline{u} \) satisfies

\[
\|u_0\|^2_0 + \|u_1\|_{-1}^2 \preceq c \int_0^T \int_{\mathcal{S}} |\overline{u}(t,x)|^2 \, dx
\]

for all \( (u_0, u_1) \in D^0(\mathcal{R}) \times D^{-1}(\mathcal{R}) \). Moreover let \( u \) be the solution \( u \) of

\[
\begin{cases}
u_{tt} - \Delta u = 0 & \text{on } \mathbb{R} \times \mathcal{T} \\
u = 0 & \text{in } \mathbb{R} \times \partial \mathcal{T} \\
u(t,0) = u_0, \quad u_t(t,0) = u_1 & \text{in } \mathcal{T},
\end{cases}
\]

and set

\[
\mathcal{S} := \bigcup_{h=1}^{6} (K_{h}^{-1} \mathcal{S} \cap \mathcal{T})
\]

Then for each \( T > T_{\mathcal{S}} \) the solution \( u \) satisfies

\[
\|u_0\|^2_0 + \|u_1\|_{-1}^2 \preceq c \int_0^T \int_{\mathcal{S}} |u(t,x)|^2 \, dx
\]

for all \( (u_0, u_1) \in D^0(\mathcal{T}) \times D^{-1}(\mathcal{T}) \).

The result also holds by replacing every occurrence of \( \preceq c \) with \( \leq c \) or \( \geq c \).

We point out that the time of observability \( T_{\mathcal{S}} \) stated in Theorem 1.1 as well as the couple \( c \) of constants in the estimates (1.3) and (1.5), are the same for both the domains \( \mathcal{R} \) and \( \mathcal{T} \). Also note that in Section 3 we prove a slightly stronger version of Theorem 1.1 that is Theorem 3.5: its precise statement requires some technicalities that we chose to avoid here, however we may anticipate to the reader that the assumption on initial data
Figure 2. The triangle $\mathcal{T}$ and, in gray, the observation domain $S_\alpha$ with $\alpha = 0.125$.

$(\overline{u}_0, \overline{u}_1) \in D^0(\mathbb{R}) \times D^{-1}(\mathbb{R})$ can be weakened by replacing $D^0(\mathbb{R}) \times D^{-1}(\mathbb{R})$ with an appropriate subspace.

Now we state the second main result of the present paper: its proof strongly relies on Theorem 1.1 and on a couple of technical lemmas that can be found in Section 4.

**Theorem 1.2** (Observability on strips along the edges of $\mathcal{T}$). Let $\alpha \in (0, 1/(3 + \sqrt{3}])$. Set $r_\alpha := 1 - \alpha(3 + \sqrt{3})$,

$$S_\alpha := \mathcal{T} \setminus c(\alpha, \mathcal{T} + (\alpha, \alpha)),$$

$$t_\alpha := \inf_{k \in \mathbb{N}} \int_0^\alpha \sin^2(\pi k x / \sqrt{3}) dx$$

and

$$T_\alpha := 8 \sqrt{5 \pi / 3} t_\alpha.$$

If $u$ is the solution of (1.4), then for every $T > T_\alpha$

$$\|u_0\|_0^2 + \|u_1\|_{-1}^2 \leq c_\alpha \int_0^T \int_{S_\alpha} |u(t, x)|^2 dx.$$

for all $(u_0, u_1) \in D^0(\mathcal{T}) \times D^{-1}(\mathcal{T})$ with

$$c_\alpha := \frac{T}{\pi} \left( \frac{t_\alpha}{\sqrt{3}} - \frac{40}{3T^2} \right) > 0.$$

**Remark 1.3.** If $\alpha \leq 1/(3 + \sqrt{3}) \simeq 0.211$ then $S_\alpha$ can be equivalently viewed as the intersection between $\mathcal{T}$ and the union of three open strips $s_1(\alpha), s_2(\alpha)$ and $s_3(\alpha)$ of width equal to $\alpha$, each of which has a common side with an edge of $\mathcal{T}$, see Figure 2. If $\alpha = 1/(3 + \sqrt{3})$ then we are setting as domain of observation $\mathcal{T} \setminus \{(\alpha, \alpha)\}$: note that in this case the point $(\alpha, \alpha)$ is the incenter of $\mathcal{T}$. Finally if $\alpha \geq 1/(3 + \sqrt{3})$ then the union of $s_1(\alpha), s_2(\alpha)$ and $s_3(\alpha)$ covers $\mathcal{T}$.
Proof. Let
\[ \bar{S}_\alpha := [(0, \alpha) \times (0, 1)] \cup [(0, \sqrt{3}) \times (0, \alpha)]. \]
Using a result in [15], we show in Lemma 4.2 that if \( T > T_\alpha \) then the solution \( \bar{u} \) of (1.2) satisfies (1.3) for all initial data \( (\bar{u}_0, \bar{u}_1) \in D^0(\mathbb{R}) \times D^{-1}(\mathbb{R}) \). Then, by Theorem 1.1 setting
\[ S'_\alpha := \bigcup_{h=1}^6 (K_h^{-1} \bar{S}_\alpha \cap \mathcal{T}) \]
we have that if \( u \) is the solution of (1.4) then \( T > T_\alpha \) implies
\[ \|u_0\|_0^2 + \|u_1\|_{-1}^2 \leq c_\alpha \int_0^T \int_{S'_\alpha} |u(t, x)|^2 dx. \]
for all \( (u_0, u_1) \in D^0(\mathcal{T}) \times D^{-1}(\mathcal{T}). \) The claim hence follows by showing that \( S_\alpha = S'_\alpha \) for all \( \alpha \geq 0 \), which is proved in Lemma 4.3. □

1.2. Organization of the paper. In Section 2, we consider a generic domain \( \Omega_1 \) tiling a larger domain \( \Omega_2 \): we establish a result, Theorem 2.7, relating the observability properties of wave equation on \( \Omega_2 \) and on its tile \( \Omega_1 \). In Section 3, we specialize this result to the case in which \( \Omega_1 \) is the triangle \( \mathcal{T} \) and the tiled domain \( \Omega_2 \) is the rectangle \( \mathcal{R} \): this is the core of the proof of Theorem 1.1. Finally Section 4 is devoted to the proof of Theorem 1.2.

2. An observability result on tilings

The goal of this section is to state an equivalence between an observability problem on a domain \( \Omega_1 \) and an observability problem on a larger domain \( \Omega_2 \), under the assumption that \( \Omega_1 \) tiles \( \Omega_2 \). We begin with some definitions.

**Definition 2.1** (Tilings, foldings and prolongations). Let \( \Omega_1 \) and \( \Omega_2 \) be two open bounded subsets of \( \mathbb{R}^2 \). We say that \( \Omega_1 \) tiles \( \Omega_2 \) if there exists a set \( \{K_h\}_{h=1}^N \) rigid transformations of \( \mathbb{R}^2 \) such that
\[ \text{cl}(\Omega_2) = \bigcup_{h=1}^N \text{cl}(K_h(\Omega_1)). \]
and such that \( K_h(\Omega_1) \cap K_j(\Omega_1) = \emptyset \) for all \( h \neq j \).

Let \( (\Omega_1, \{K_h\}_{h=1}^N) \) be a tiling of \( \Omega_2 \) and \( \delta = (\delta_1, \ldots, \delta_N) \in \{-1, 1\}^N \). The prolongation with coefficients \( \delta \) of a function \( u : \Omega_1 \to \mathbb{R} \) to \( \Omega_2 \) is the function \( P_\delta u : \Omega_2 \to \mathbb{R} \)
\[ P_\delta u(K_h x) = \delta_h u(x) \text{ for each } h = 1, \ldots, N. \]
The folding with coefficients \( \delta \) of a function \( \bar{u} : \Omega_2 \to \mathbb{R} \) is the function \( F_\delta \bar{u} : \Omega_1 \to \mathbb{R} \)
\[ F_\delta \bar{u}(x) = \frac{1}{N^2} \sum_{h=1}^N \delta_h \bar{u}(K_h x) \text{ for each } h = 1, \ldots, N. \]
A tiling $(\Omega_1, K_h)$ of $\Omega_2$ is admissible if there exists $\delta \in \{-1, 1\}^N$ such that

\begin{equation}
F_\delta \varphi \in H^1_0(\Omega_1) \quad \forall \varphi \in H^1_0(\Omega_2).
\end{equation}

under scripts and we simply write $P$ and $F$.

**Example 2.1.** We show in Lemma 3.1 below that the tiling of $\mathcal{R}$ with $T$ depicted in Figure 1 is admissible, in particular (2.1) holds with $\delta = (1, -1, 1, -1, 1)$.

On the other hand the tiling of $\mathcal{R}' := (0, 1/\sqrt{3}) \times (0, 1)$ given by the transformations $K'_1 := \text{id}$ and

\begin{equation*}
K'_2 : (x_1, x_2) \mapsto -(x_1, x_2) + (1/\sqrt{3}, 1),
\end{equation*}

see Figure 3, is not admissible. Let indeed $v_1 := (1/\sqrt{3}, 0)$, $v_2 := (0, 1)$ and $x_\lambda := \lambda v_1 + (1 - \lambda)v_2$ with $\lambda \in (0, 1)$. Then $x_\lambda \in \partial T$ and

\begin{equation*}
K_2(x_\lambda) = x_{1-\lambda}
\end{equation*}

Therefore it suffices to choose $\varphi \in H^1_0(\mathcal{R})$ such that $\varphi(x_\lambda) \neq \pm \varphi(x_{1-\lambda})$ to obtain

\begin{equation*}
F_\delta \varphi(x_\lambda) = \delta_1 \varphi(x_\lambda) + \delta_2 \varphi(x_{1-\lambda}) \neq 0
\end{equation*}

for all $\delta_1, \delta_2 \in \{-1, 1\}$. Consequently $F_\delta \varphi \notin H^1_0(T)$ for all $\delta \in \{-1, 1\}^2$.

**Remark 2.2.** We borrowed the notion of prolongation and folding from [29]: while our definition of $P_\delta$ is exactly as it is given in [29], we introduced a normalizing term $1/N^2$ in the definition of $F_\delta$ in order to enlighten the notations. Note that the following equality holds:

\begin{equation}
F_\delta(P_\delta u) = \frac{1}{N} u
\end{equation}
for all $u : \Omega_1 \to \mathbb{R}$.

Also remark that we shall need to prolong and fold also functions $u : \mathbb{R} \times \Omega_1 \to \mathbb{R}$ and $\bar{u} : \mathbb{R} \times \Omega_2 \to \mathbb{R}$, in this case the definition of $P$ and $F$ naturally extends by applying the transformations $K_h$’s to the spatial variables $x$. For instance if $u : \mathbb{R} \times \Omega_1 \to \mathbb{R}$ then its prolongation to $\mathbb{R} \times \Omega_2$ reads

$$P_\delta u(t, K_h x) = \delta_h u(t, x).$$

We want to establish a relation between solutions of a wave equation with Dirichlet boundary conditions and their prolongation. To this end we introduce the notations

$$P_\delta L^2(\Omega_1) := \{ P_\delta u \mid u \in L^2(\Omega_1) \},$$

$$P_\delta H^1_0(\Omega_1) := \{ P_\delta u \mid u \in H^1_0(\Omega_1) \}$$

and

$$P_\delta H^{-1}(\Omega_1) := \{ P_\delta u \mid u \in H^{-1}(\Omega_1) \}.$$

Note that $P_\delta L^2(\Omega_1) \subset L^2(\Omega_2)$, $P_\delta H^1_0(\Omega_1) \subset H^1_0(\Omega_2)$ and $P_\delta H^{-1}(\Omega_1) \subset H^{-1}(\Omega_1)$.

All results below hold under the following assumptions on the domains $\Omega_1, \Omega_2$ and on a base $\{ e_k \}$ for $L^2(\Omega_1)$:

**Assumption 1.** $(\Omega_1, \{ K_h \}_{h=1}^N)$ is an admissible tiling of $\Omega_2$.

**Assumption 2.** $\{ e_k \}$ is a base of eigenvectors of $-\Delta$ in $H^1_0(\Omega_1)$, it is defined on $\Omega_1 \cup \Omega_2$ and there exists $\delta \in \{-1, 1\}^N$ such that

$$P_\delta(e_k|_{\Omega_1}) = e_k|_{\Omega_2}$$

for each $k \in \mathbb{N}$.

**Remark 2.3** (Some remarks on Assumption 2). We note that Assumption 2 can be equivalently stated as

$$e_k(K_h x) = \delta_h e_k(x) \quad \text{for all } x \in \Omega_1, \ h = 1, \ldots, N, \ k \in \mathbb{N}.$$  

(2.3)

Indeed, by definition of prolongation and noting $\delta_h^2 \equiv 1$, we have

$$e_k(K_h x) = \delta_h^2 e_k(K_h x) = \delta_h P_\delta e_k(x) = \delta_h e_k(x)$$

for every $x \in \Omega_1$, $h = 1, \ldots, N$ and $k \in \mathbb{N}$.

Also remark that, in view of (2.2), Assumption 2 also implies

$$F_\delta e_k = \frac{1}{N} e_k.$$  

(2.4)

**Example 2.4.** Let $\Omega_1 = (0, \pi)^2$ and $\Omega_2 = (0, 2\pi)^2$. Consider the transformations of $\mathbb{R}^2$

$$K_1 := \text{id}, \quad K_2 : (x_1, x_2) \mapsto (-x_1 + 2\pi, x_2),$$

$$K_3 : (x_1, x_2) \mapsto (x_1, -x_2 + 2\pi), \quad K_4 : (x_1, x_2) \mapsto -(x_1, x_2) + (2\pi, 2\pi).$$

Then $\{ \Omega_1, \{ K_h \}_{h=1}^4 \}$ is a tiling for $\Omega_2$. In particular, Assumption 2 is satisfied: indeed setting $\delta = (1, -1, -1, 1)$ we have for each $\varphi \in H^1_0(\Omega_2)$

$$F_\delta \varphi(x) = 0 \quad \forall x \in \partial \Omega_1.$$
Also note that the functions
\[ e_k(x) := \sin(k_1 x_1) \sin(k_2 x_2) \quad k = (k_1, k_2) \in \mathbb{N}^2 \]
satisfy Assumption 2, indeed they are a base for \( L^2(\Omega_1) \) composed by eigenfunctions of \(-\Delta\) in \( H^1_0(\Omega_1) \) and
\[ e_k(K_h x) := \delta_h e_k(x) \]
for all \( x \in \mathbb{R}^2, h = 1, \ldots, 4 \) and \( k \in \mathbb{N}^2 \). The space \( \mathcal{P}_\delta L^2(\Omega_1) \) in this case coincides with the space of so-called \((2,2)\)-cyclic functions, i.e., functions in \( L^2(\Omega_2) \) which are odd with respect to both axes \( x_1 = \pi \) and \( x_2 = \pi \). We refer to [17] for some results on observability of wave equation with \((p,q)\)-cyclic initial data.

Our starting point is to show that, under Assumptions 1 and 2, the base of eigenfunctions \( \{e_k\} \) is also a base of eigenfunctions also for an appropriate subspace of \( L^2(\Omega_2) \), and to compute the associated coefficients.

**Lemma 2.5.** Let \( \Omega_1, \Omega_2 \) and \( \{e_k\} \) satisfy Assumption 1 and Assumption 2. Then \( \{e_k\} \subset H^1_0(\Omega_2) \) and it is also a complete base for \( \mathcal{P}_\delta L^2(\Omega_1) \) formed by eigenfunctions of \(-\Delta\) in \( \mathcal{P}_\delta H^1_0(\Omega) \).

In particular, for every \( k \in \mathbb{N} \), if \( u_k \) is the coefficient of \( u \in L^2(\Omega_1) \) (with respect to \( e_k \)) then \( Nu_k \) is the coefficient of \( \mathcal{P}_\delta u \).

**Proof.** The proof is organized two steps.

Claim 1: \( \{e_k\} \) is a set of eigenfunctions of \(-\Delta\) in \( H^1_0(\Omega_2) \). Extending a result given in [29], we need to show that, under Assumption 1 and Assumption 2 if \( e_k \in H^1_0(\Omega_1) \) is a solution of the boundary value problem
\[
\int_{\Omega_1} \nabla e_k \nabla \varphi dx = \int_{\Omega_1} \gamma_k e_k \varphi dx \quad \forall \varphi \in H^1_0(\Omega_1)
\]
for some \( \gamma_k \in \mathbb{R} \), then \( e_k \) is also solution of the boundary value problem on \( \Omega_2 \)
\[
\int_{\Omega_2} \nabla e_k \nabla \varphi dx = \int_{\Omega_2} \gamma_k e_k \varphi dx \quad \forall \varphi \in H^1_0(\Omega_2).
\]
Now, recall from Assumption 1 that if \( \phi \in H^1_0(\Omega_2) \) then \( F_\delta \phi \in H^1_0(\Omega_1) \). Then it follows again from Assumption 1 and from Assumption 2 (in particular by recalling that \( K_h \)'s are isometries and (2.3)) that for all \( \phi \in H^1_0(\Omega_2) \)

\[
\int_{\Omega_2} \nabla e_k(x) \nabla \phi(x) \, dx = \int_{\bigcup_{h=1}^N K_h \Omega_1} \nabla e_k(x) \nabla \phi(x) \, dx
\]

\[
= \sum_{h=1}^N \int_{\Omega_1} \nabla e_k(K_h x) \nabla \phi(K_h x) \, dx = \int_{\Omega_1} \nabla e_k(x) \sum_{h=1}^N \delta_h \nabla \phi(K_h x) \, dx
\]

\[
= \int_{\Omega_1} \nabla e_k(x) \nabla F_\delta \phi(x) \, dx = \int_{\Omega_1} \gamma_k e_k(x) F_\delta \phi(x) \, dx
\]

\[
= \int_{\Omega_2} \gamma_k e_k(x) \phi(x) \, dx.
\]

and this completes the proof of Claim 1.

Claim 2: completeness of \( \{e_k\} \) and computation of coefficients By Assumption 1 and Assumption 2 and by recalling \( \delta^2_h = 1 \) for each \( h = 1, \ldots, N \), we have

\[
\int_{\Omega_2} \mathcal{P}_\delta u(x) e_k(x) \, dx = \int_{\Omega_2} \mathcal{P}_\delta u(x) \mathcal{P}_\delta e_k(x) \, dx
\]

\[
= \sum_{h=1}^N \int_{K_h \Omega_1} \mathcal{P}_\delta u(x) \mathcal{P} e_k(x) \, dx
\]

\[
= \sum_{h=1}^N \int_{K_h \Omega_1} \delta^2_h u(K_h x) e_k(K_h x) \, dx
\]

\[
= \sum_{h=1}^N \int_{\Omega_1} u(x) e_k(x) \, dx = N \int_{\Omega_1} u(x) e_k(x) \, dx,
\]

where the second to last equality holds because \( K_h \)'s are rigid transformations. Then we may deduce two facts: first if \( \{u_k\} \) are the coefficients of \( u \in L^2(\Omega_1) \) then \( \{N u_k\} \) are coefficients of \( \mathcal{P}_\delta u \). Secondly, \( \{e_k\} \) is a complete base for \( \mathcal{P}_\delta L^2(\Omega_1) \), indeed if the coefficients of \( \mathcal{P}_\delta u \) are identically null, then also the coefficients of \( u \) are identically null: since \( \{e_k\} \) is complete for \( \Omega_1 \) then \( u \equiv 0 \) and, consequently, \( \mathcal{P}_\delta u \equiv 0 \), as well. \( \Omega_2 \) and, consequently, \( \partial \Omega_2 \subset \bigcup_{h=1}^N K_h(\partial \Omega_1) \).

Next result establishes a relation between solutions of wave equations on tiles and their prolongations.

**Lemma 2.6.** Let \( \Omega_1, \Omega_2 \) and \( \{e_k\} \) satisfy Assumption 1 and Assumption 2. Let \( u \) be the solution of

\[
\begin{align*}
\begin{cases}
u_{tt} - \Delta u & = 0 & \text{in } \mathbb{R} \times \Omega_1 \\
u & = 0 & \text{on } \mathbb{R} \times \partial \Omega_1 \\
u(t, 0) = u_0, \ u_t(t, 0) = u_1 & \text{in } \Omega_1 
\end{cases}
\end{align*}
\]

(2.5)
Then \( u \) is well defined in \( \Omega_1 \cup \Omega_2 \) and \( \overline{u} = Nu|_{\Omega_2} \) is the solution of

\[
\begin{aligned}
\overline{u}_{tt} - \Delta \overline{u} &= 0 & \text{on } \mathbb{R} \times \Omega_2 \\
\overline{u} &= 0 & \text{in } \mathbb{R} \times \partial \Omega_2 \\
\overline{u}(t,0) &= \mathcal{P}_\delta u_0, \quad \overline{u}(t,0) = \mathcal{P}_\delta u_1 & \text{in } \Omega_2
\end{aligned}
\]  

(2.6)

Conversely, if \( \bar{u} \) is the solution of (2.6) then \( \mathcal{F}_\delta \bar{u} \) is the solution of (2.5) and for every \( h = 1, \ldots, N \)

\[
\mathcal{F}_\delta \bar{u}(t,x) = \frac{\delta_h}{N} \bar{u}(t,K_h x)
\]

for each \( x \in \Omega_1 \).

Proof. Let \( \{\gamma_k\} \) be the sequence of eigenvalues associated to \( \{e_k\} \) and set \( \omega_k = \sqrt{\gamma_k} \), for every \( k \in \mathbb{N} \). Expanding \( u(t,x) \) with respect to \( e_k \) we obtain

\[
u(t,x) = \sum_{k=1}^{\infty} (a_k e^{i \omega_k t} + b_k e^{-i \omega_k t}) e_k(x)
\]

with \( a_k \) and \( b_k \) depending only the coefficients \( c_k \) and \( d_k \) of \( u_0 \) and \( u_1 \) with respect to \( \{e_k\} \). In particular \( a_k + b_k = c_k \) and \( a_k - b_k = -id_k/\omega_k \). We then have that the natural domain of \( u \) coincides with the one of \( \{e_k\} \)'s, hence it is included in \( \Omega_1 \cup \Omega_2 \). By Lemma 2.5 the coefficients of \( \mathcal{P}_\delta u_0 \) and \( \mathcal{P}_\delta u_1 \) are \( Nc_k \) and \( Nd_k \), respectively. Then it is immediate to verify that

\[
Nu(t,x) = \sum_{k=1}^{\infty} (Na_k e^{i \omega_k t} + Nb_k e^{-i \omega_k t}) e_k(x)
\]

is the solution of (2.6).

Now, let

\[
\bar{u}(t,x) = \sum_{k=1}^{\infty} (\bar{a}_k e^{i \omega_k t} + \bar{b}_k e^{-i \omega_k t}) e_k(x)
\]

be the solution of (2.6), and note that, by the reasoning above, setting \( a_k := \frac{1}{N} \bar{a}_k \) and \( b_k := \frac{1}{N} \bar{b}_k \) we have that

\[
u(t,x) := \sum_{k=1}^{\infty} (a_k e^{i \omega_k t} + b_k e^{-i \omega_k t}) e_k(x) = \frac{1}{N} \bar{u}(t,x)
\]

is the solution of (2.5). Hence to prove that \( u(t,x) = \mathcal{F}_\delta \bar{u}(t,x) \) it suffices to note that by Assumption 1 (see in particular (2.4))

\[
\mathcal{F}_\delta \bar{u}(t,x) = \sum_{k=1}^{\infty} (\bar{a}_k e^{i \omega_k t} + \bar{b}_k e^{-i \omega_k t}) \mathcal{F}_\delta e_k(x)
\]

\[
= \frac{1}{N} \sum_{k=1}^{\infty} (\bar{a}_k e^{i \omega_k t} + \bar{b}_k e^{-i \omega_k t}) e_k(x) = \frac{1}{N} \bar{u}(t,x).
\]
Finally, we show (2.7): for each \( h = 1, \ldots, N \) we have
\[
\tilde{u}(t, x) = \delta_h^2 \tilde{u}(t, x) = \delta_h \sum_{k=1}^{\infty} (\tilde{a}_k e^{i\omega_k t} + \tilde{b}_k e^{-i\omega_k t}) \delta_h e_k(x) = \sum_{k=1}^{\infty} (\tilde{a}_k e^{i\omega_k t} + \tilde{b}_k e^{-i\omega_k t}) e_k(K_h x) = \delta_h \tilde{u}(t, K_h x)
\]
and this concludes the proof. (2.6).

We are now in position to state the main result of this section, that bridges observability of tiles with their prolongations.

**Theorem 2.7.** Let \( \Omega_1, \Omega_2 \) and \( \{e_k\} \) satisfy Assumption 1 and Assumption 2. Let \( u \) be the solution of

\[
\begin{aligned}
\begin{cases}
     u_{tt} - \Delta u = 0 & \text{on } \mathbb{R} \times \Omega_1 \\
     u = 0 & \text{in } \mathbb{R} \times \partial \Omega_1 \\
     u(t, 0) = u_0, \quad u_t(t, 0) = u_1 & \text{in } \Omega_1
\end{cases}
\end{aligned}
\tag{2.8}
\]

with \( u_0, u_1 \in D^0(\Omega_1) \times D^{-1}(\Omega_1) \) and let \( \overline{u} \) be the solution of

\[
\begin{aligned}
\begin{cases}
     u_{tt} - \Delta u = 0 & \text{on } \mathbb{R} \times \Omega_2 \\
     u = 0 & \text{in } \mathbb{R} \times \partial \Omega_2 \\
     u(t, 0) = \mathcal{P}_\delta u_0, \quad u_t(t, 0) = \mathcal{P}_\delta u_1 & \text{in } \Omega_2.
\end{cases}
\end{aligned}
\tag{2.9}
\]

Also let \( \tilde{S} \subset \Omega_2 \) and define

\[
S := \bigcup_{h=1}^{N} K_h^{-1} \tilde{S} \cap \Omega_1.
\]

Then for every \( T > 0 \) and for every couple \( c = (c_1, c_2) \) of positive constants, the inequalities

\[
\|u_0\|_0^2 + \|u_1\|_{-1}^2 \asymp c_1 \int_0^T \int_S |u(t, x)|^2 \, dx.
\tag{2.10}
\]

hold if and only if

\[
\|\mathcal{P}_\delta u_0\|_0^2 + \|\mathcal{P}_\delta u_1\|_{-1}^2 \asymp c_2 \int_0^T \int_{\tilde{S}} |\overline{u}(t, x)|^2 \, dx.
\tag{2.11}
\]

**Proof.** By Lemma 2.6, \( u \) and \( \overline{u} \) satisfy

\[
u(t, x) = \frac{\delta_h}{N} \overline{u}(t, K_h x) \quad \text{for all } h = 1, \ldots, N.
\]

Since \( \Omega_1 \) tiles \( \Omega_2 \), then setting \( S_h := K_h^{-1} \tilde{S} \cap \Omega_1 \) we have \( S = \bigcup_{h=1}^{N} S_h \) and \( \overline{S} = \bigcup_{h=1}^{N} K_h S_h \), and that these unions are disjoint. Hence, also recalling
\( |\delta_h| \equiv 1 \) we have
\[
\int_I \int_S |\overline{u}(t,x)|^2 dx = \sum_{h=1}^N \int_I \int_{K_hS_h} |\overline{u}(t,x)|^2 dx = \sum_{h=1}^N \int_I \int_{S_h} |\overline{u}(t,K_hx)|^2 dx
\]
\[
= N^2 \sum_{h=1}^N \int_I \int_{S_h} |\delta_h N \overline{u}(t,K_hx)|^2 dx
\]
\[
= N^2 \sum_{h=1}^N \int_I \int_{S_h} |u(t,x)|^2 dx = N^2 \int_I \int_S |u(t,x)|^2 dx
\]
Finally, by Lemma 2.5
\[
\|P_{\delta u_0}\|_0^2 = N^2 \|u_0\|_0^2 \quad \text{and} \quad \|P_{\delta u_1}\|_{-1}^2 = N^2 \|u_1\|_{-1}^2.
\]
and this implies the equivalence between (2.10) and (2.11). \( \square \)

3. Proof of Theorem 1.1

The proof of Theorem 1.1 is based on the application of Theorem 2.7 to the particular case
\( \Omega_1 = \mathcal{T} \quad \text{and} \quad \Omega_2 = \mathcal{R} \).
2/\( \sqrt{3} \) as well as side 2/\( \sqrt{3} \). wave equation in \( \mathcal{T} \) bridge the well-established solutions in the rectangle \( \mathcal{R} \) to the ones with domain equal to the rhombus or the hexagon.
We then need to admissibly tile \( \mathcal{R} \) with \( \mathcal{T} \) and a base \( \{e_k\} \) formed by the eigenfunctions of \( -\Delta \) in \( H_0^1(\mathcal{T}) \) satisfying Assumption 2. Such ingredients are provided in [29]: in order to introduce them we need some notations. We consider the Pauli matrix
\[
\sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]
and the rotation matrix
\[
R_\alpha := \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix}
\]
where \( \alpha := \pi/3 \). Now let \( v_1 := (0, 1/\sqrt{3}) \) and \( v_2 := (0, 1) \) be two of the three vertices of \( \mathcal{T} \) and define the transformations from \( \mathbb{R}^2 \) onto itself
\[
\begin{align*}
K_1 &= \text{id;} & K_4 : x \mapsto -R_\alpha(x - v_2) + 3v_1 \\
K_2 : x \mapsto -R_\alpha(x - v_2) + v_2; & K_5 : x \mapsto -R_\alpha(x - v_2) + 3v_1 + v_2 \\
K_3 : x \mapsto R_\alpha(x - v_2) + v_2; & K_6 : x \mapsto -x + 3v_1 + v_2
\end{align*}
\]
(3.1)
Figure 4. The tiling of $\mathcal{R}$ with $\mathcal{T}$, the gray areas correspond to negative $\delta_h$’s.

and note $(\mathcal{T}, \{K_h\}_{h=1}^6)$ is a tiling for $\mathcal{R}$. Indeed

$$cl(\mathcal{R}) = \bigcup_{h=1}^6 K_h cl(\mathcal{T}),$$

and the sets $K_h \mathcal{T}$, for $h = 1, \ldots, 6$, do not overlap – see Figure 4 and [29].

We set

$$\delta := (1, -1, 1, 1, -1, 1).$$

and, in next result, we prove that $\mathcal{T}$ admissibly tiles $\mathcal{R}$.

Lemma 3.1. $(\mathcal{T}, \{K_h\}_{h=1}^6)$ is an admissible tiling of $\mathcal{R}$.

Proof. We want to show that if $\varphi \in H^1_0(\mathcal{R})$ then $\mathcal{F}_\delta \varphi \in H^1_0(\mathcal{T})$. To this end let $v_0 := (0, 0)$, $v_1 := (1/\sqrt{3}, 0)$ and $v_2 := (0, 1)$ be the vertices of $\mathcal{T}$ and define

$$x^\lambda_{ij} := \lambda v_i + (1 - \lambda)v_j,$$

so that $\partial \mathcal{T} = \{x^\lambda_{ij} : \lambda \in [0, 1], 0 \leq i < j \leq 2\}$. By a direct computation, for all $\lambda \in [0, 1]$

$$K_1(x^\lambda_{01}), K_6(x^\lambda_{01}) \in \partial \mathcal{R},$$

$$K_2(x^\lambda_{01}) = K_4(x^\lambda_{01}),$$

and

$$K_3(x^\lambda_{02}) = K_5(x^\lambda_{02}).$$

Since $\varphi \in H^1_0(\mathcal{R})$ then $\mathcal{F}_\delta \varphi(x^\lambda_{01}) = 0$. Similarly, for all $\lambda \in [0, 1]$

$$K_1(x^\lambda_{02}), K_6(x^\lambda_{02}) \in \partial \mathcal{R},$$

$$K_2(x^\lambda_{02}) = K_3(x^\lambda_{02}),$$

and

$$K_4(x^\lambda_{02}) = K_5(x^\lambda_{02})$$

therefore $\mathcal{F}_\delta \varphi(x^\lambda_{02}) = 0$ for all $\lambda \in [0, 1]$. Finally for all $\lambda \in [0, 1]$

$$K_3(x^\lambda_{12}), K_4(x^\lambda_{12}) \in \partial \mathcal{R},$$
\[ K_1(x_{12}) = K_2(x_{12}), \]

and

\[ K_5(x_{12}) = K_6(x_{12}) \]

therefore we get also in this case \( F_\delta \varphi(x_{12}) = 0 \) for all \( \lambda \in [0, 1] \) and we may conclude that \( F_\delta \varphi \in H^0_0(T) \). □

**Remark 3.2.** Lemma 3.1 was remarked in [29, p.312], but to the best of our knowledge, this is the first time an explicit proof is provided.

Now, consider the eigenfunctions of \(-\Delta\) in \( H^1_0(R)\):

\[ \bar{e}_k(x_1, x_2) := \sin(\pi k_1 \frac{x_1}{\sqrt{3}}) \sin(\pi k_2 x_2), \quad k = (k_1, k_2) \in \mathbb{N}^2. \]

We finally define for every \( k \in \mathbb{N}^2 \)

(3.3)

\[ e_k(x) := N^2 F_\delta \bar{e}_k = \sum_{h=1}^{6} \delta_h \bar{e}_k(K_h x). \]

Next result, proved in [29], states that Assumption 2 is satisfied by \( \{e_k\} \).

**Lemma 3.3.** The set of functions \( \{e_k\} \) defined in (3.3) is a complete orthogonal base for \( T \) formed by the eigenfunction of \(-\Delta\) in \( H^1_0(T) \). Furthermore \( \mathcal{P}_\delta e_k(x) = e_k(x) \).

**Remark 3.4.** For each \( k \in \mathbb{N}^2 \), the eigenfunctions \( e_k \) and \( \bar{e}_k \) share the same eigenvalue \( \gamma_k = \pi^2 (\frac{k_1^2}{3} + k_2^2) \), see [29].

Next gives access to classical results on observability of rectangular membranes for the study of triangular domains.

**Theorem 3.5.** Let \( S \subset R \) be such that the solution \( \bar{u} \) of

(3.4)

\[
\begin{cases}
  u_{tt} - \Delta u = 0 & \text{on } \mathbb{R} \times \mathbb{R} \\
  u = 0 & \text{in } \mathbb{R} \times \partial \mathbb{R} \\
  u(t, 0) = \mathcal{P}_\delta u_0, \; u(t, 0) = \mathcal{P}_\delta u_1 & \text{in } \mathbb{R}.
\end{cases}
\]

satisfies for some \( T > 0 \) and some couple of positive constants \( c = (c_1, c_2) \)

(3.5)

\[
\|\mathcal{P}_\delta u_0\|_2^2 + \|\mathcal{P}_\delta u_1\|_{2-1}^2 \lesssim_c \int_0^T \int_S |\bar{u}(t, x)|^2 dx
\]

for all \( (u_0, u_1) \in D^0(T) \times D^{-1}(T) \). Moreover let

\[ S := \bigcup_{h=1}^6 (K_h^{-1} S \cap T) \]

Then the solution \( u \) of

(3.6)

\[
\begin{cases}
  u_{tt} - \Delta u = 0 & \text{on } \mathbb{R} \times T \\
  u = 0 & \text{in } \mathbb{R} \times \partial T \\
  u(t, 0) = u_0, \; u(t, 0) = u_1 & \text{in } T.
\end{cases}
\]
satisfies
\begin{equation}
\|u_0\|_0^2 + \|u_1\|_{-1}^2 \leq_c \int_0^T \int_S |u(t,x)|^2 dx.
\end{equation}
for all \((u_0, u_1) \in D^0(T) \times D^{-1}(T)\).

\textbf{Proof.} Since \(T, R\) and \(\{\epsilon_k\}\) satisfy Assumption 1 and Assumption 2, then the claim follows by a direct application of Theorem 2.7 with \(\Omega_1 = T\) and \(\Omega_2 = R\). \(\square\)

We conclude this section by showing that Theorem 1.1 is a direct consequence of Theorem 3.5:

\textbf{Proof of Theorem 1.1.} By Lemma 2.5, if \((u_0, u_1) \in D^0(T) \times D^{-1}(T)\) then \((P_\delta u_0, P_\delta u_1) \in D^0(R) \times D^{-1}(R)\). The claim hence follows by Theorem 3.5.

\(\square\)

4. PROOF OF THEOREM 1.2

In this Section we keep all the notations used in Section 3. So that for instance \(\{K_h\}\) are the transformations given in (3.1) and \(\delta = (1, -1, 1, -1, 1)\). Also recall the definitions

\[ S_\alpha := T \setminus cl(r_\alpha T + (\alpha, \alpha)) \]

and

\[ \overline{S}_\alpha := [(0, \sqrt{3}) \times (0, \alpha)] \cup [(0, \alpha) \times (0, 1)] \]

As mentioned in the Introduction, to prove Theorem 1.2 we need two auxiliary results: Lemma 4.2, which is an internal observability result on rectangles, and Lemma 4.3, which is a geometric result characterizing \(S_\alpha\).

We begin by recalling the following:

\textbf{Proposition 4.1 (18, Proposition 1.1).} Let \(\Omega = (0, \ell_1) \times (0, \ell_2)\), let \(J_1 \subset (0, \ell_1)\) and \(J_2 \subset (0, \ell_2)\) and define

\[ \overline{\Xi} := [J_1 \times (0, \ell_2)] \cup [(0, \ell_1) \times J_2] \]

Also define the positive constants
\[ t_1 := \inf_{k \in \mathbb{N}} \int_{J_1} \sin \left( \frac{\pi k x}{\ell_1} \right) dx, \quad t_2 := \inf_{k \in \mathbb{N}} \int_{J_2} \sin \left( \frac{\pi k x}{\ell_2} \right) dx. \]

and
\[ m := \min \left\{ \frac{t_1}{\ell_1}, \frac{t_2}{\ell_2} \right\} \]

If \(T > 0\) satisfies the condition
\begin{equation}
\frac{(\ell_1^2 + \ell_2^2)(\ell_1^4 + \ell_2^4)}{T^2 \ell_1^4 \ell_2^4} < m
\end{equation}
then the solutions \(u\) of (1.1) satisfy the estimate
\begin{equation}
\|u_0\|_0^2 + \|u_1\|_{-1} \leq_c \int_0^T \int_S |u(t,x)|^2 dx dt
\end{equation}
for all \((u_0, u_1) \in D^1(\Omega) \times D^0(\Omega)\), with

\begin{equation}
(4.3) \quad c := \frac{T}{\pi} \left( m - \frac{(\ell_1^2 + \ell_2^2)(\ell_1^4 + \ell_2^4)}{T^2 \ell_1^2 \ell_2^2} \right) > 0.
\end{equation}

We apply Proposition 4.1 to prove

Lemma 4.2. Let \(T > T_\alpha\). Then the solutions \(u\) of (1.2) satisfy the estimate

\begin{equation}
(4.4) \quad |u_0|_0^2 + |u_1|_{-1}^2 \leq c_\alpha \int_0^T \int_{\bar{S}_\alpha} |u(t, x)|^2 \, dx \, dt
\end{equation}

for all \((u_0, u_1) \in D^0(\mathcal{R}) \times D^{-1}(\mathcal{R})\).

Proof. We apply Proposition 4.1 to \(\Omega = \mathcal{R}\), so that \(\ell_1 = \sqrt{3}\) and \(\ell_2 = 1\), and to \(\bar{S} = \bar{S}_\alpha\) so that \(J_1 = J_2 = (0, \alpha)\). Recall from the statement of Theorem 1.2 the definition

\[ t_\alpha := \inf_{k \in \mathbb{N}} \int_0^\alpha \sin^2\left(\pi k x / \sqrt{3}\right) \, dx. \]

Moreover let

\[ t'_\alpha := \inf_{k \in \mathbb{N}} \int_0^\alpha \sin (\pi k x) \, dx \]

and

\[ m_\alpha := \min\{t_\alpha / \sqrt{3}, t'_\alpha\}. \]

With a little algebraic manipulation of (4.1) we have that if

\[ T > \bar{T}_\alpha := 8 \sqrt{\frac{5}{\sqrt{3}} m_\alpha} \]

then

\begin{equation}
(4.5) \quad |u_0|_0^2 + |u_1|_{-1}^2 \leq \bar{\tau}_\alpha \int_0^T \int_{\bar{S}_\alpha} |u(t, x)|^2 \, dx \, dt
\end{equation}

for all \((u_0, u_1) \in D^0(\mathcal{R}) \times D^{-1}(\mathcal{R})\), with

\[ \bar{\tau}_\alpha := \frac{T}{\pi} \left( m_\alpha - \frac{40}{3T^2} \right). \]

Since

\[ \frac{t_\alpha}{\sqrt{3}} = \inf_{k \in \mathbb{N}} \frac{1}{\sqrt{3}} \int_0^\alpha \sin^2 \left( \frac{\pi k x}{\sqrt{3}} \right) \, dx \]

\[ = \inf_{k \in \mathbb{N}} \int_0^{\alpha / \sqrt{3}} \sin^2 (\pi k x) \, dx \leq t'_\alpha, \]

then \(m_\alpha = \frac{t_\alpha}{\sqrt{3}}\). Hence \(T_\alpha = \bar{T}_\alpha\) and \(c_\alpha = \bar{c}_\alpha\) and this concludes the proof.

\(\square\)

Finally we prove
Lemma 4.3. If $\alpha \in (0, 1/(3 + \sqrt{3})]$ then

$$S_\alpha = \bigcup_{h=1}^{6} K_h^{-1} S_\alpha \cap T.$$ 

Proof. First of all we recall $cl(R) = \bigcup_{h=1}^{6} K_h(cl(T))$. Then $cl(T) = \bigcap_{h=1}^{6} K_h^{-1}(cl(R))$ and

$$(4.6) \quad T_\alpha := r_\alpha cl(T) + (\alpha, \alpha) = \bigcap_{h=1}^{6} r_\alpha K_h^{-1}(cl(R)) + (\alpha, \alpha)$$

Also recall that $K_h's$ are affine maps. Then for each $a \in \mathbb{R}$, $b \in \mathbb{R}^2$

$$K_h(ax + b) = aK_h(x) + K_h(b) - K_h(0) \quad \forall x \in \mathbb{R}^2, \ h = 1, \ldots, 6.$$ 

In particular for all $h = 1, \ldots, 6$

$$(4.7) \quad K_h(r_\alpha K_h^{-1} cl(R) + (\alpha, \alpha)) = r_\alpha cl(R) + K_h(\alpha, \alpha) - r_\alpha K_h(0)$$

By a direct computation, the assumption $\alpha \geq 0$ implies

$$(4.8) \quad K_h(\alpha, \alpha) - r_\alpha K_h(0) \geq (\alpha, \alpha), \quad \forall h = 1, \ldots, 6$$

where vector inequalities are meant componentwise. Since the complement set $\overline{S}_\alpha$ of $S_\alpha$ satisfies

$$(4.8) \quad \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1, x_2 \geq \alpha\} \subset \overline{S}_\alpha,$$

$$(4.7) \quad \text{and} \quad (4.8) \quad \text{imply}$$

$$K_h(r_\alpha K_h^{-1} cl(R) + (\alpha, \alpha)) \subset \overline{S}_\alpha \quad \forall h = 1, \ldots, 6.$$ 

Therefore

$$r_\alpha K_h^{-1} cl(R) + (\alpha, \alpha) \subset K_h^{-1} \overline{S}_\alpha \quad \forall h = 1, \ldots, 6$$

and, consequently,

$$T_\alpha = \bigcap_{h=1}^{6} r_\alpha K_h^{-1}(cl(R)) + (\alpha, \alpha) \subset \bigcap_{h=1}^{6} K_h^{-1}(\overline{S}_\alpha)$$

By the definition of $S_\alpha$ we then have

$$S_\alpha = T_\alpha^c \cap T \supseteq \left( \bigcap_{h=1}^{6} K_h^{-1}(\overline{S}_\alpha^c) \right)^c \cap T = \bigcup_{h=1}^{6} K_h^{-1} S_\alpha \cap T.$$ 

To prove the other inclusion, note that if

$$(x_1, x_2) \in T_\alpha^c = co\{(0, 0), (r_\alpha/\sqrt{3}, 0), (0, r_\alpha)\}^c + (\alpha, \alpha)$$

then either $x_1 \leq \alpha$, $x_2 \leq \alpha$ or $x_2 \geq -\sqrt{3}x_1 + 1 - 2\alpha$. Hence

$$T \setminus T_\alpha = A_1 \cup A_2$$
with
\[
A_1 := \{(x_1, x_2) \in T \mid x_1 \leq \alpha \text{ or } x_2 \leq \alpha \}
\]
\[
= S_\alpha \cap T = K_1^{-1}(S_\alpha) \cap T \subset \bigcup_{h=1}^{6} K_h^{-1} S_\alpha \cap T
\]
and
\[
A_2 := \{(x_1, x_2) \in T \mid x_2 \geq -\sqrt{3}x_1 + 1 - 2\alpha \}
\]
\[
= K_4^{-1}(S_\alpha) \cap T \subset \bigcup_{h=1}^{6} K_h^{-1} S_\alpha \cap T.
\]

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