Horizon surface gravity as 2D geodesic expansion

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Abstract

The surface gravity of any Killing horizon, in any spacetime dimension, can be interpreted as a local, two-dimensional expansion rate seen by freely falling observers when they cross the horizon. Any two-dimensional congruence of geodesics invariant under the Killing flow can be used to define this expansion, provided that the observers have unit Killing energy.

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1. Introduction

A Killing horizon is a null hypersurface whose null generators are flow lines of a Killing field. At each point of a Killing horizon, a scalar quantity called the surface gravity is defined. Its usual definition refers only to the properties of the Killing vector (and the metric)\textsuperscript{3}. The main purpose of this paper is to indicate how it can be defined instead as the rate of expansion of space in the direction of the Killing frame as viewed by freely falling observers crossing the horizon. This is analogous to cosmological expansion, but it involves only one spatial direction, and no preferred observer is selected. Any freely falling observer with unit Killing energy will determine the same expansion at the horizon, which is equal to the surface gravity.

This investigation was initially motivated by the expression for surface gravity in Painlevé–Gullstrand type coordinates. Consider for example the two-dimensional line element

\[ ds^2 = dt^2 - (dx - v(x) \, dt)^2, \]

which has a Killing vector $\partial_t$ and a Killing horizon where $v(x) = \pm 1$. The surface gravity turns out to be given by the gradient $dv/dx$ evaluated at the horizon. This may be interpreted as the fractional rate of expansion of the flow defined by the geodesics satisfying $dx = v \, dt$, in the following sense. These geodesics are orthogonal to the constant $t$ lines, on which $dx$ measures proper length. The proper distance $\delta x$ between neighboring geodesics therefore satisfies

\textsuperscript{3} The geometric concepts used in this paper are explained in most modern textbooks on general relativity. See, for example, [1].
\[
\frac{d(\ln \delta x)}{dt} = \frac{\delta v}{\delta x} = \frac{d\nu}{dx}.
\]
Since the proper time along these geodesics is just \( dt \), this is the fractional rate of change with respect to proper time. Here we formulate what amounts to a covariant version of this relation between surface gravity and two-dimensional expansion, and we demonstrate its universal applicability to any Killing horizon in any spacetime dimension, with the observer moving in any direction.

Aside from the geometric perspective it offers, another motivation for this paper is that the 2D expansion field, defined both on and off the horizon, may turn out to be physically relevant in settings where a local preferred frame is present. Condensed matter black hole analogs \([2–4]\), theories with a dynamical ‘aether’\([5–7]\), and Lorentz violating dispersion \([3]\) or dissipation \([8]\) are examples of such settings.

2. Surface gravity

There are several ways of defining the surface gravity \( \kappa \) using only the Killing vector \( \chi^a \) and the metric. For the present purposes, the most convenient is via the relation

\[
\nabla_a \chi^2 = -2\kappa \chi_a,
\]
(2)

evaluated at the horizon. The gradient of \( \chi^2 \) is normal to the horizon, since this function is constant (equal to zero) on the horizon. The Killing vector is also normal to the horizon, since it lies along the null tangent direction which is always the normal direction to a null hypersurface. These vectors are therefore parallel, so \( \kappa \) is well defined by (2). Note however that the value of \( \kappa \) depends on the normalization of the Killing vector, which is not determined by the symmetry alone. Typically the norm is fixed to be unity on the worldline of some Killing observer, for example at infinity for an asymptotically flat black hole spacetime.

We can use (2) to find the surface gravity for the metric (1) with respect to the Killing vector \( \partial_t \). (In this example the Killing vector has a unit norm where \( v(x) = 0 \).) The \( x \) component of (2) yields \( \kappa = (-1/2) g_{tt,x}/g_{xx} = v,x \).

3. 2D expansion

The expansion of a congruence of curves is defined by the divergence \( \theta = \nabla_a u^a \) of the unit tangent vector field \( u^a \), where \( \nabla_a \) is the covariant derivative operator. Here we will apply this notion to two-dimensional timelike congruences in a spacetime of any dimension. The 2D surface generated by such a congruence has an induced metric and induced covariant derivative operator \( D_a \), obtained from \( \nabla_a \) by restricting the derivative to directions in the surface and projecting the Levi-Civita connection onto the surface. In terms of the surface projector \( h^{a}{}_{b} \), the 2D expansion is given by

\[
\theta_{2D} \equiv D_a u^a = h^{a}{}_{b} \nabla_a u^b.
\]
(3)

The 2D congruences of interest here are composed of timelike geodesics and are invariant under the flow of a Killing vector \( \chi^a \). A timelike vector \( u^a \) at a single point \( p \) uniquely determines such a congruence, namely, the geodesic through \( p \) with tangent \( u^a \), together with the image of this geodesic along the Killing flow. The Lie derivative \( \mathcal{L}_\chi u = [\chi, u] \) of the resulting 2D \( u^a \) field vanishes by construction. Thus the derivatives of \( u^a \) and \( \chi \) are related by

\[
\chi^a \nabla_a u^b = u^a \nabla_a \chi^b.
\]
(4)

Note that the congruence defined in this way, given \( u^a \) at a single point, is independent of the normalization of the Killing vector.

The tangent plane to the 2D congruence at each point is spanned by \( u^a \) and an orthogonal, unit spacelike vector \( s^a \) in terms of which the orthogonal projector is simply given by
Expressing $s^a$ in terms of $u^a$ and $\chi^a$ as
\begin{equation}
   s^a = (\chi \cdot s)^{-1} ((\chi \cdot u) u^a - \chi^a),
\end{equation}
and inserting (5) into the expression (3), the terms involving contractions with $u$ vanish either because of the geodesic equation or because of the normalization of $u$, leaving just
\begin{equation}
   \theta_{2D} = - (\chi \cdot s)^{-2} \chi^b \nabla_{a} u^b.
\end{equation}
Using the Lie dragging condition (4), and $\chi^2 = (\chi \cdot u)^2 - (\chi \cdot s)^2$, (7) becomes
\begin{equation}
   \theta_{2D} = \frac{1}{2} u^a \nabla_{a} \ln \chi^2 - \frac{1}{(\chi \cdot u)^2}.
\end{equation}

This expression for the expansion holds for the 2D geodesic Killing congruence generated by any unit timelike vector. It is independent of the overall scale of the Killing vector, and its value generally depends on the direction of the unit vector.

It is worth noting that the 2D expansion is also equal to the spatial derivative of the velocity relative to the Killing frame, generalizing the coordinate expression $dv/dx$ for the expansion of the flow in (1). Indeed, using the decomposition
\begin{equation}
   \chi^a = (\chi \cdot u) (u^a - vs^a),
\end{equation}
where $v = (\chi \cdot s)/(\chi \cdot u)$ is the velocity of $u^a$ relative to the Killing frame, (3) becomes
\begin{equation}
   \theta_{2D} = - s^a s^b \nabla_{a} u^b
   = - s^a s^b \nabla_{a} ((u \cdot \chi)^{-1} \chi^b + vs^b)
   = s^a \nabla_{a} v.
\end{equation}
(When $u^a$ is chosen tangent to the free-fall trajectory $dx = v(x) \, dt$ in the coordinate system of (1), $v$ in (9) coincides with $v(x)$, and the final expression in (10) reads $dv/dx$.) In the last step we used (i) the fact that the derivative of $u \cdot \chi$ along $s^a$ vanishes (since it is constant along each geodesic and along the Killing flow), (ii) Killing’s equation $\nabla_{a} \chi_{b} = 0$, and (iii) $s_a s^a = -1$. Moreover, using (9) and the Killing symmetry $\chi^a \nabla_{a} v = 0$, this result may be expressed in terms of the fractional rate of change of $v$ respect to proper time,
\begin{equation}
   \theta_{2D} = u^a \nabla_{a} \ln v.
\end{equation}

4. 2D expansion and surface gravity

On the horizon, $\chi^2$ in the denominator of (8) vanishes, hence
\begin{equation}
   \theta^\text{hor}_{2D} = - \frac{1}{2} u^a \nabla_{a} \chi^2 - \frac{1}{(\chi \cdot u)^2}.\end{equation}
Using (2) this yields
\begin{equation}
   \kappa = (u \cdot \chi) \theta^\text{hor}_{2D}.
\end{equation}
The 2D expansion on the horizon is thus equal to the surface gravity, provided that $\chi^a$ is normalized such that $u^a$ has unit Killing energy, i.e. $u \cdot \chi = 1$. It is curious that all dependence on $u^a$ except for the Killing energy drops out of the 2D expansion when evaluated on the horizon.

If a normalization of the Killing vector is fixed, then at each point of the horizon there is a $(D-2)$-parameter family of unit timelike $u^a$ with unit Killing energy, $D$ being the dimension of spacetime. For all of the corresponding observers the 2D expansion is equal to the surface gravity. Alternatively, for any observer, if the Killing field is normalized so the observer has unit Killing energy, the surface gravity is equal to the 2D expansion.
5. Examples

We now illustrate the central result of this paper with four examples. First, for an asymptotically flat rotating black hole in four dimensions, the horizon-generating Killing vector is \( \chi = \partial_t + \Omega H \partial_\phi \), where \( \partial_t \) is the time translation and \( \partial_\phi \) is the axial rotation. If \( \partial_t \) is normalized to unity at spatial infinity as usual, then geodesics that fall from rest at infinity with zero angular momentum have \( u \cdot \partial_t = 1 \) and \( u \cdot \partial_\phi = 0 \), so they have unit Killing energy, \( u \cdot \chi = 1 \). When they fall across the horizon such observers measure a 2D expansion that is precisely the surface gravity of \( \chi \). But these are not the only observers with unit Killing energy. At each point there is a two-parameter family of such observers, all of whom measure the same 2D expansion at the horizon for their associated geodesic Killing congruence. The 4-velocities of these observers are related to each other by Lorentz transformations (in the tangent space) that leave the Killing vector fixed. On the horizon these are null rotations, while outside they are spatial rotations or boosts, depending on whether \( \chi \) is timelike or spacelike.

Our second example is the Rindler horizon in the 2D Minkowski spacetime, generated by the boost Killing vector \( x \partial_t + t \partial_x \), where \( t \) and \( x \) are Minkowski coordinates. Let us consider the geodesic Killing congruence determined by the unit timelike vector \( u = \partial_t \) located at the point \( (t, x) = (0, x_0) \). The fiducial geodesic through this point with this tangent vector is a straight line in the \( t \) direction, while the geodesics obtained by dragging along the Killing flow are straight lines tangent at the other points along the hyperbola \( x^2 - t^2 = x_0^2 \) (see figure 1). For this congruence we have \( \theta^{2D}_{\text{hor}} = \partial_t u' + \partial_x u' \), which at the horizon-crossing point \( (x_0, x_0) \) evaluates to \( \theta^{2D}_{\text{hor}} = 1/x_0 \). The Killing energy of this congruence is given by \( u \cdot (x \partial_t + t \partial_x) = x_0 \), so according to (13) the surface gravity is then \( \kappa = 1 \), as can of course also be verified directly from (2) using just the Killing vector. This is dimensionless, since the Killing vector normalized as above has dimensions of length. If we instead normalize the Killing vector as \( (1/x_0)(x \partial_t + t \partial_x) \), so that it is dimensionless and \( u \) has unit Killing energy, then the surface gravity is equal to the 2D expansion at the horizon, \( \kappa = 1/x_0 \).
For the third example, we use the concept of the 2D expansion to illuminate the sense in which a Rindler horizon with non-zero surface gravity is an $M \to \infty$ limit of Schwarzschild black hole horizons. A black hole of mass $M$ has a surface gravity $\kappa = 1/4M$ with respect to the Killing vector normalized to unity at spatial infinity. If we simply take the limit of this quantity as the mass is increased, holding fixed the norm of the Killing field at infinity, the surface gravity goes to zero. Equivalently, the 2D expansion $\theta_{2D}^{\text{hor}}$ measured by free fall observers dropped from rest at infinity goes to zero in this limit. To approach the Rindler limit with a non-zero 2D expansion, we can instead drop the free fall observers from a fixed proper distance $d_0$ above the horizon, measured on a spatial slice orthogonal to the Killing vector. According to (13) this yields $\theta_{2D}^{\text{hor}} = 1/4M \sqrt{1 - 2M/r_0}$, where $r_0(M, d_0)$ is the radial coordinate corresponding to the proper distance $d_0$. If the Killing vector is normalized at the drop point, these observers have unit Killing energy and therefore $\theta_{2D}^{\text{hor}}$ is equal to the surface gravity. If $M$ is much larger than $d_0$, one finds $r_0 \approx 2M + d_0^2/8M$. In the infinite mass limit $\theta_{2D}^{\text{hor}}$ thus tends to $1/d_0$. This is precisely the result, discussed in the previous example, for the case when the observers are ‘dropped’ from a proper distance $d_0$ above a Rindler horizon.

As a final example, we consider the 2D-expansion in a region where the Killing vector is spacelike. In this case the timelike geodesics can be chosen orthogonal to the Killing vector, $u \cdot \chi = 0$, so the local expansion (8) takes the form

$$\theta_{2D} = u^a \nabla_a \ln |\chi|,$$

where $|\chi| = (-\chi^a \chi_a)^{1/2}$ is the norm of the Killing vector. Since neighboring geodesics in the congruence are connected by a fixed Killing parameter, they are separated by a proper distance $\delta L$ proportional to $|\chi|$. Therefore the expansion is also equal to $u^a \nabla_a (\ln \delta L)$, which is the fractional rate of expansion in the Killing direction. In a homogeneous, isotropic cosmological metric this 2D expansion is nothing but the Hubble constant, provided $u^a$ is orthogonal to all of the spatial Killing vectors. In the presence of anisotropy this yields the different expansion rates along the different Killing directions. The construction can even be applied to the Killing vector $\partial_t$ in the ergoregion of the Kerr metric, where it yields a local notion of 2D Killing expansion, given a choice of $u^a$.

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References

[1] Wald R M 1984 General Relativity (Chicago, IL: University of Chicago Press)
[2] Unruh W G 1981 Experimental black hole evaporation Phys. Rev. Lett. 46 1351
[3] Jacobson T 1999 Trans-Planckian redshifts and the substance of the space-time river Prog. Theor. Phys. Suppl. 136 1 (Preprint hep-th/0001085)
[4] Barcelo C, Liberati S and Visser M 2005 Analogue gravity Living Rev. Rel. 8 12 (Preprint gr-qc/0505065) http://relativity.livingreviews.org/Articles/lrr-2005-12
[5] Jacobson T 2008 Einstein–Aether gravity: a status report PoS(QG-Ph)020 (Preprint arXiv:0801.1547)
[6] Mukohyama S 2005 Black holes in the ghost condensate Phys. Rev. D 71 104019 (Preprint hep-th/0502189)
[7] Dubovsky S, Tseytlin P and Zaldarriaga M 2007 Bumpy black holes from spontaneous Lorentz violation J. High Energy Phys. JHEP11(2007)083 (Preprint arXiv:0706.3288)
[8] Parentani R 2007 Constructing QFT’s wherein Lorentz invariance is broken by dissipative effects in the UV PoS(QG-Ph)031 (Preprint arXiv:0709.3943)