Some nonexistence results for space–time fractional Schrödinger equations without gauge invariance

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To Professor J. A. Tenreiro Machado, in Memoriam

Abstract
In this paper, we consider the Cauchy problem in \( \mathbb{R}^N \), \( N \geq 1 \), for semi-linear Schrödinger equations with space–time fractional derivatives. We discuss the nonexistence of global \( L^1 \) or \( L^2 \) weak solutions in the subcritical and critical cases under some conditions on the initial data and the nonlinear term. Furthermore, the nonexistence of local \( L^1 \) or \( L^2 \) weak solutions in the supercritical case are studied.

Keywords Schrödinger equations (primary) · Fractional derivatives and integrals · Test function method · Nonexistence of global solution

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1 Introduction

In this paper, we consider the problem

$$\begin{cases}
i^\alpha c D_{0+}^\alpha u - (-\Delta)^{\beta/2} u = \lambda |u|^p, & (t, x) \in (0, T) \times \mathbb{R}^N, \\
u(x, 0) = \varepsilon u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1}$$

where $u$ is a complex-valued unknown function of $(t, x)$, $0 < \alpha < 1$, $0 < \beta < 2$, $N \geq 1$, $p > 1$, $T > 0$, $\varepsilon > 0$, $\lambda \in \mathbb{C} \setminus \{0\}$, and $i^\alpha$ is the principal part of $i^\alpha$, i.e.

$$i^\alpha = \cos \left( \frac{\alpha \pi}{2} \right) + i \cos \left( \frac{\alpha \pi}{2} \right),$$

$c D_{0+}^\alpha$ is the Caputo fractional derivative and $(-\Delta)^{\beta/2} : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is the fractional Laplacian which can be defined by a pointwise representation as given in Definition 6.

Different fractional generalizations of the Schrödinger equation appeared in the literature: The classical Schrödinger equations with nongauge power nonlinearity, i.e. (1) with $\alpha = 1$ and $\beta = 2$, has been studied by Ikeda and Wakasugi [8] and Ikeda and Inui [9, 10], the spatial fractional Schrödinger equation involving fractional order space derivatives, i.e. (1) with $\alpha = 1$ and $\beta \in (0, 2)$, has been investigated in [4, 5, 14–16], the fractional temporal Schrödinger equation involving a fractional time derivative, i.e. $\alpha \in (0, 1)$ and $\beta = 2$, has been studied in [19, 20, 25], the semirelativistic Schrödinger equation with nongauge invariant power nonlinearity, i.e. (1) with $\alpha = 1$ and $\beta = 1/2$, got interest by Fujiwara [6], Inui [11], Fujiwara and Ozawa [7], and the spatio-temporal fractional Schrödinger equation with both time and space fractional derivatives attracted the attention of [2, 22].

The expected critical exponent can be determined by the following scaling argument: If $u(x, t)$ is a solution of (1) with initial data $u_0$, then

$$v(t, x) = \gamma^{\frac{\beta}{p-1}} u(\gamma^{\alpha/\beta} t, \gamma x),$$

for all $\gamma > 0$, is also a solution of (1) with initial data $v_0(x) = v(0, x) = \gamma^{\frac{\beta}{p-1}} u_0(\gamma x)$, for all $x \in \mathbb{R}^N$. We choose $p = p_s$ such that we get an invariant $H^s$-norm of the initial data:

$$\|v_0\|_{H^s} = \gamma^{\frac{\beta}{p-1} - \frac{N-2s}{2}} \|u_0\|_{H^s} = \|u_0\|_{H^s};$$

this happens if and only if

$$p = p_s = 1 + \frac{2\beta}{N - 2s}.$$ 

Therefore, the case $p = p_s$ is called $H^s$-critical case; the case $p < p_s$ (resp. $p > p_s$) is called $H^s$-subcritical case (resp. $H^s$-supercritical case). On the other hand, the Fujita critical exponent for the corresponding heat equation with fractional Laplacian is

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\[ p_F = 1 + \frac{\beta}{N}. \]

Our main goal is to study the nonexistence of \( L^1 \) or \( L^2 \) global weak solutions under the condition \( p \leq p_s \) or \( p \leq p_F \) as well as the nonexistence of \( L^1 \) or \( L^2 \) local weak solutions under the condition that \( p > p_s \) or \( p > p_F \) (see e.g. [10]), using the test function method (see e.g. [24]) or a fractional differential equation approach (i.e. construct a fractional differential equation for a new function and using comparison principle). The local existence for (1) is expected in the \( H^s \)-subcritical case, but this is not our case. We refer the reader to [8, Appendix] by using the Strichartz estimates recently studied by Lee [17].

Let

\[ X_T = \{ \varphi \in C([0, \infty), H^\beta(\mathbb{R}^N)) \cap C^1([0, \infty), L^2(\mathbb{R}^N)) ; \text{supp}\ \varphi \subset Q_T, \varphi \text{ is } \mathbb{R} \text{-valued} \}, \]

and

\[ Y_T = \{ \varphi \in C([0, \infty), H^\beta(\mathbb{R}^N)) \cap C^1([0, \infty), L^\infty(\mathbb{R}^N)) ; \text{supp}\ \varphi \subset Q_T, \varphi \text{ is } \mathbb{R} \text{-valued} \}, \]

where \( Q_T := [0, T] \times \mathbb{R}^N \) and the homogeneous fractional Sobolev space \( H^\beta(\mathbb{R}^n) \), \( \beta \in (0, 2) \) is defined by

\[ H^\beta(\mathbb{R}^n) = \begin{cases} \{u \in L^2(\mathbb{R}^n); (-\Delta)^{\beta/2}u \in L^2(\mathbb{R}^n)\}, & \text{if } \beta \in (0, 1), \\ H^1(\mathbb{R}^n), & \text{if } \beta = 1, \\ \{u \in H^1(\mathbb{R}^n); (-\Delta)^{\beta/2}u \in L^2(\mathbb{R}^n)\}, & \text{if } \beta \in (1, 2), \end{cases} \]

endowed with the norm

\[ \|u\|_{H^\beta(\mathbb{R}^n)} = \begin{cases} \|u\|_{L^2(\mathbb{R}^n)} + \|(-\Delta)^{\beta/2}u\|_{L^2(\mathbb{R}^n)}, & \text{if } \beta \in (0, 1), \\ \|u\|_{L^2(\mathbb{R}^n)} + \|\nabla u\|_{L^2(\mathbb{R}^n)}, & \text{if } \beta = 1, \\ \|u\|_{L^2(\mathbb{R}^n)} + \|\nabla u\|_{L^2(\mathbb{R}^n)} + \|(-\Delta)^{\beta/2}u\|_{L^2(\mathbb{R}^n)}, & \text{if } \beta \in (1, 2). \end{cases} \]

**Definition 1** [\( L^2 \)-weak solution] Let \( u_0 \in L^2(\mathbb{R}^N) \) and \( T > 0 \). We say that \( u \) is an \( L^2 \)-weak solution of (1) if

\[ u \in L^1((0, T), L^2(\mathbb{R}^N)) \cap L^p((0, T), L^{2p}(\mathbb{R}^N)), \]

and

\[ \begin{aligned} &\lambda \int_{Q_T} |u|^p \varphi(t, x) \, dt \, dx + \varepsilon i^{\alpha} \int_{Q_T} u_0(x)^c D^\alpha_{t|x} \varphi(t, x) \, dt \, dx \\ &\quad = i^\alpha \int_{Q_T} u^c D^\alpha_{t|x} \varphi \, dt \, dx - \int_{Q_T} u(-\Delta)^{\beta/2} \varphi(t, x) \, dt \, dx, \quad (2) \end{aligned} \]
holds for all $\varphi \in X_T$. We denote the lifespan for the $L^2$-weak solution by

$$T_w(\varepsilon) := \sup\{T \in (0, \infty); \text{ there exists a unique } L^2\text{-weak solution } u \text{ to } (1.1)\}.$$  

Moreover, if $T > 0$ can be arbitrary chosen, i.e. $T_w(\varepsilon) = \infty$, then $u$ is called a global $L^2$-weak solution of (1).

**Definition 2** ($L^1$-weak solution) Let $u_0 \in L^1(\mathbb{R}^N)$ and $T > 0$. We say that $u$ is an $L^1$-weak solution of (1) if $u, |u|^p \in L^1((0, T), L^1(\mathbb{R}^N))$ and

$$\lambda \int_{Q_T} |u|^p \varphi(t, x) \, dt \, dx + \varepsilon i^\alpha \int_{Q_T} u_0(x)^c D_1^\alpha |t \varphi(t, x) \, dt \, dx$$

$$= i^\alpha \int_{Q_T} u^c D_1^\alpha |t \varphi(t, x) \, dt \, dx - \int_{Q_T} u(-\Delta)^{\beta/2} \varphi(t, x) \, dt \, dx,$$  

(3)

holds for all $\varphi \in Y_T$. We denote the lifespan for the $L^1$-weak solution by

$$T_w(\varepsilon) := \sup\{T \in (0, \infty); \text{ there exists a unique } L^1\text{-weak solution } u \text{ to } (1.1)\}.$$  

Moreover, if $T > 0$ can be arbitrary chosen, i.e. $T_w(\varepsilon) = \infty$, then $u$ is called a global $L^1$-weak solution to (1).

## 2 Preliminaries

**Definition 3** (Absolutely continuous functions) ([21, Chap. 1])

A function $g : [a, b] \rightarrow \mathbb{R}$ with $a, b \in \mathbb{R}$, is absolutely continuous if and only if there exists a Lebesgue summable function $\psi \in L^1(a, b)$ such that

$$g(t) = g(a) + \int_a^t \psi(s) \, ds, \quad \text{for all } t \in [a, b].$$

The space of these functions is denoted by $AC[a, b]$.

**Definition 4** (Riemann–Liouville fractional integrals) ([21, Chap. 1])

Let $g \in L^1(0, T)$ with $T > 0$. The Riemann–Liouville left- and right-sided fractional integrals of order $\sigma \in (0, 1)$ are, respectively, defined by

$$I_0^\sigma g(t) := \frac{1}{\Gamma(\sigma)} \int_0^t (t-s)^{-(1-\sigma)} g(s) \, ds, \quad t > 0,$$

and

$$I_T^\sigma g(t) := \frac{1}{\Gamma(\sigma)} \int_t^T (s-t)^{-(1-\sigma)} g(s) \, ds, \quad t < T,$$

where $\Gamma$ is the Euler gamma function.
Definition 5  (Caputo fractional derivatives) ([21, Chap. 1])
Let \( f \in AC[0, T] \) with \( T > 0 \). The Caputo left- and right-sided fractional derivatives of order \( \delta \in (0, 1) \) exists almost everywhere on \([0, T]\) and defined, respectively, by
\[
^{c}D_{0^+}^\delta f(t) := \frac{d}{dt} I_{0^+}^{1-\delta} [f(t) - f(0)] = I_{0^+}^{1-\delta} [f'(t)], \quad t > 0,
\]
and
\[
^{c}D_{T^-}^\delta f(t) := -\frac{d}{dt} I_{T^-}^{1-\delta} [f(t) - f(T)] = -I_{T^-}^{1-\delta} [f'(t)], \quad t < T.
\]

Lemma 1  ([12, Lemma 2.22, p. 96])
Let \( 0 < \delta < 1 \) and \( T > 0 \). If \( f \in AC[0, T] \) or \( f \in C^1[0, T] \), then
\[
I_{0^+}^\delta \frac{d}{dt} I_{0^+}^{1-\delta} f(t) = f(t) - f(0). \tag{4}
\]

Given \( T > 0 \), let us define the function \( w : [0, T] \to \mathbb{R} \) by the following formula:
\[
w(t) = (1 - t/T)^\eta \quad \text{for all } 0 \leq t \leq T, \tag{5}
\]
where \( \eta \gg 1 \). Later on, we need the following properties concerning the function \( w \).

Lemma 2  ([12, Property 2.16, p. 95])
Let \( T > 0 \), \( \eta > \alpha - 1 \), and \( 0 < \alpha < 1 \). For all \( t \in [0, T] \), we have
\[
^{c}D_{t}^\alpha w(t) = \frac{\Gamma(\eta + 1)}{\Gamma(\eta + 1 - \alpha)} T^{-\alpha} (1 - t/T)^{\eta-\alpha}. \tag{6}
\]

Lemma 3  Let \( T > 0 \), \( 0 < \alpha < 1 \), \( \eta > \alpha p/(p - 1) - 1 \), and \( p > 1 \). Then, we have
\[
\int_0^T (w(t))^{-\frac{1}{p-1}} \left|^{c}D_{t}^\alpha w(t)\right|^\frac{p}{p-1} dt = C_1 T^{1-\alpha} \frac{p}{p-1}, \tag{7}
\]
and
\[
\int_0^T \left|^{c}D_{t}^\alpha w(t)\right| dt = C_2 T^{1-\alpha}, \tag{8}
\]
where
\[
C_1 = \frac{1}{\eta + 1 - \alpha} \left[ \frac{\Gamma(\eta + 1)}{\Gamma(\eta + 1 - \alpha)} \right]^{\frac{p}{p-1}}, \quad \text{and} \quad C_2 = \frac{\Gamma(\eta + 1)}{\Gamma(\eta + 2 - \alpha)}.
\]
Proof Let us start by proving (7). Using Lemma 2, we have
\[
\int_0^T (w(t))^{-\frac{1}{p-1}}|c D_0^\alpha w(t)|^{\frac{p}{p-1}} dt
= \left[ \frac{\Gamma(\eta + 1)}{\Gamma(\eta + 1 - \alpha)} \right]^{-\frac{p}{p-1}} T^{-\alpha} \int_0^T (w(t))^{-\frac{1}{p-1}} (w(t))^{\frac{\alpha}{p-1}} dt
= \left[ \frac{\Gamma(\eta + 1)}{\Gamma(\eta + 1 - \alpha)} \right]^{\frac{p}{p-1}} T^{-\alpha} \int_0^T (1-t/T)^{\eta-\alpha} dt
= \left[ \frac{\Gamma(\eta + 1)}{\Gamma(\eta + 1 - \alpha)} \right]^{\frac{p}{p-1}} T^{1-\alpha} \int_0^1 (1-s)^{\eta-\alpha} ds = C \frac{T^{1-\alpha}}{p-1}.
\]
Similarly, we get (8).

Lemma 4 Let \( T > 0, 0 < \alpha < 1, p > 1, A, B \geq 0, \) and \( v \in C^1([0, T], \mathbb{R}) \) satisfying the following fractional differential inequality
\[
\frac{c}{p} D_0^\alpha v(t) \geq B \left[ |v(t)|^p - A \right], \quad t \in [0, T),
\]
such that \( v(0) > A \frac{1}{p}. \) Then \( v(t) \geq A \frac{1}{p} \) for all \( t \in [0, T). \)

Proof Fixing \( T_1 \in (0, T) \), we show that \( v(t) \geq A \frac{1}{p} \) for any \( t \in (0, T_1). \) Then, since \( T_1 \) is arbitrary, the claim follows. Let us start by defining \( T^* = \inf \{ t > 0; \ v(t) \geq A \frac{1}{p} \} \).

Since \( v \) is continuous and \( v(0) > A \frac{1}{p}, \) we have \( T^* > 0. \) We claim \( T^* = T_1. \) Otherwise, we have \( v(t) > A \frac{1}{p} \) for all \( t \in (0, T^*) \) such that \( v(T^*) = A \frac{1}{p}; \) this implies, in particular, that
\[
F(t, v(t)) := B \left[ |v(t)|^p - A \right] \geq 0, \quad \text{for all } t \in [0, T^*].
\]

On the other hand, since the right hand side of (9) is continuous on \([0, T_1]\) and \( v \in C^1([0, T_1]), \) applying the Riemann–Liouville fractional integral \( I_0^\alpha \) to (9) on \([0, T_1]\) and using (4), we get
\[
A^{\frac{1}{p}} = v(T^*) = v(0) + \frac{1}{\Gamma(\alpha)} \int_0^{T^*} (T^* - s)^{-(1-\alpha)} F(s, v(s)) ds \geq v(0) > A^{\frac{1}{p}},
\]
where we have used (10); contradiction. This completes the proof.

Using [18, Proposition 4.6] and applying the same argument as in the proof of Lemma 4, one can define the function \( g \in C([0, T_b], \mathbb{R}^+) \) which is the unique solution of
\[
\begin{align*}
& c D_0^\alpha g(t) = B g^p(t), \quad t \in [0, T_b), \\
g(0) & > 0,
\end{align*}
\]
where \( T_b \) is the maximal time of existence.

**Proposition 1** (Fractional differential inequalities) 
Let \( T_b > 0 \) be the blow-time of the solution of (11), and let \( T > T_b, \) \( 0 < \alpha < 1, \) \( p > 1, \) \( B > 0, \) and \( f \in C^1([0, T), \mathbb{R}) \) be a nonnegative solution of the following fractional differential inequality

\[
\begin{aligned}
\left\{ \begin{array}{ll}
\frac{\partial}{\partial t} t^\alpha f(t) \geq B f^p(t), & t \in [0, T), \\
f(0) > 0. & 
\end{array} \right.
\end{aligned}
\] (12)

Then \( f \) blows up at \( T_b, \) i.e. \( \lim_{t \to T_b^-} f(t) = +\infty. \) Moreover, the following upper and lower bound of \( T_b \) are also given

\[
T_L \leq T_b \leq T_U,
\] (13)

where

\[
T_U := \left( \frac{\Gamma(1 + \alpha)}{B (f(0))^{p-1} H(p, \alpha)} \right)^{1/\alpha} \quad \text{and} \quad T_L := \left( \frac{\Gamma(1 + \alpha)}{B (f(0))^{p-1} G(p)} \right)^{1/\alpha},
\]

with

\[
G(p) = \min \left( 2^p, \frac{p^p}{(p-1)^{p-1}} \right), \quad H(p, \alpha) = \max \left( p - 1, 2 - \frac{p\alpha}{p-1} \right).
\] (14)

**Proof** Applying [3, Theorem 5.1], we conclude that the solution \( g \) of (11) is an increasing function and

\[
\lim_{t \to T_b^-} g(t) = +\infty.
\]

On the other hand, by taking \( g(0) = f(0), \) applying [18, Theorem 4.10] and using (11), (12), we conclude that

\[
f(t) \geq g(t) \geq 0,
\]

this implies that

\[
\lim_{t \to T_b^-} f(t) = +\infty.
\]

Moreover, using [3, Theorem 5.2], we get (13)

**Definition 6** ([13, 23]) Let \( s \in (0, 1) \) and \( X \) be a suitable set of functions defined on \( \mathbb{R}^N. \) The fractional Laplacian \( (-\Delta)^s \) in \( \mathbb{R}^N \) is a non-local operator defined as the following singular integral

\[
(-\Delta)^s : v \in X \mapsto (-\Delta)^s v(x) := C_{N,s} \ p.v. \int_{\mathbb{R}^N} \frac{v(x) - v(y)}{|x - y|^{N+2s}} dy,
\]
as long as the right-hand side exists, \( p.v. \) stands for Cauchy’s principal value, and 
\[ C_{N,s} := \frac{4^{s} \Gamma \left( \frac{N}{2} + s \right)}{\pi^{\frac{N}{2}} \Gamma(-s)} \]
is a normalization constant.

**Lemma 5** ([1, Lemma 2.3]) Let 
\[ \langle x \rangle := (1 + |x|^2)^{1/2} \text{ for all } x \in \mathbb{R}^N. \]
Let \( s \in (0, 1) \) and \( \phi : \mathbb{R}^N \to \mathbb{R} \) be a function defined by 
\[ \phi(x) = \langle x \rangle^{-q}, \text{ where } n < q \leq N + 2s. \]
Then, \( \phi \in H^{2s}(\mathbb{R}^N) \) and the following estimate holds:
\[ \left| (-\Delta)^s \phi(x) \right| \leq C_{N,q} \phi(x), \text{ for all } x \in \mathbb{R}^N, \quad C_{N,q} = C(s, N, q) > 0. \quad (15) \]

**Lemma 6** ([1, Lemma 2.4]) Let \( s \in (0, 1) \), and let \( \psi \) be a smooth function satisfying 
\( \partial^2_x \psi \in L^\infty(\mathbb{R}^N) \). For any \( R > 0 \), let \( \psi_R \) be a function defined by
\[ \psi_R(x) := \psi(x/R) \text{ for all } x \in \mathbb{R}^N. \]
Then, \( (-\Delta)^s \psi_R \) satisfies the following scaling properties:
\[ (-\Delta)^s \psi_R(x) = R^{-2s}((-\Delta)^s \psi)(x/R), \text{ for all } x \in \mathbb{R}^N. \]

**Lemma 7** Let \( s \in (0, 1) \), \( R > 0 \) and \( p > 1 \). Then, the following estimate holds
\[ \int_{\mathbb{R}^N} (\phi_R(x))^{-\frac{1}{p-1}} \left| (-\Delta)^s \phi_R(x) \right|^{\frac{p}{p-1}} dx \leq C_3 R^{-\frac{2sp}{p-1} + N}, \]
where \( C_3 = (C_{N,q})^{p/(p-1)} A_0 > 0 \), \( A_0 \) is defined below, \( \phi_R(x) := \phi(x/R) \), and \( \phi \) is given in Lemma 5.

**Proof** If \( 0 < s < 1 \), then using the change of variable \( \tilde{x} = x/R \) and Lemma 6 we have 
\[ (-\Delta)^s \phi_R(x) = R^{-2s}(-\Delta)^s \phi(\tilde{x}). \]
Therefore, by Lemma 5 we conclude that
\[ \int_{\mathbb{R}^N} (\phi_R(x))^{-\frac{1}{p-1}} \left| (-\Delta)^s \phi_R(x) \right|^{\frac{p}{p-1}} dx \leq (C_{N,q})^{\frac{p}{p-1}} R^{-\frac{2sp}{p-1} + N} \int_{\mathbb{R}^N} \phi(\tilde{x}) d\tilde{x} \]
\[ = (C_{N,q})^{\frac{p}{p-1}} A_0 R^{-\frac{2sp}{p-1} + N}, \]
where
\[ A_0 = \int_{\mathbb{R}^N} \phi(x) dx > 0. \]

### 3 Non-existence of global \( L^1 \)-weak solution in the case \( p \leq p_F \)

To state our first result, we set
\[ \lambda = \lambda_1 + i\lambda_2, \quad u_0 = g + ih, \]

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where $\lambda_i \in \mathbb{R}$ ($i = 0, 1$) and $g$ and $h$ are real-valued functions; the real and imaginary parts of $i^\alpha u_0$ can be written, respectively, as

$$G_1(x) = \cos \left( \frac{\alpha \pi}{2} \right) g(x) - \sin \left( \frac{\alpha \pi}{2} \right) h(x), \quad \text{and}$$

$$G_2(x) = \cos \left( \frac{\alpha \pi}{2} \right) h(x) + \sin \left( \frac{\alpha \pi}{2} \right) g(x).$$

**Theorem 1** (Non-existence of global $L^1$-weak solution in the case $p \leq p_F$)

Let $0 < \alpha < 1$, $0 < \beta < 2$, $N \geq 1$, $\varepsilon = 1$.

1. If $1 < p < 1 + \frac{\beta}{N} = p_F$, and $u_0 \in L^1(\mathbb{R}^N)$ satisfies

$$\lambda_1 \int_{\mathbb{R}^N} G_1(x) \, dx > 0 \quad \text{or} \quad \lambda_2 \int_{\mathbb{R}^N} G_2(x) \, dx > 0,$$

then problem (1) admits no global $L^1$-weak solution.

2. If $p = p_F$, and $u_0 \in L^2(\mathbb{R}^N)$ satisfies

$$|\lambda_1|^{2-p} \lambda_1 \int_{\mathbb{R}^N} G_1(x) \, dx > C_0 A_0 \quad \text{or} \quad |\lambda_1|^{2-p} \lambda_2 \int_{\mathbb{R}^N} G_2(x) \, dx > C_0 A_0,$$

where $A_0 = \int_{\mathbb{R}^N} \langle x \rangle^{-N-\beta} \, dx$, and $C_0$ is defined in (25), then problem (1) admits no global $L^1$-weak solution.

**Proof** We argue by contradiction. Suppose that $u$ is a global weak solution to (1), then

$$\lambda \int_{Q_T} |u|^p \varphi(t, x) \, dt \, dx + i^\alpha \int_{Q_T} u_0(x) \overline{D_t^{\alpha} \varphi(t, x)} \, dt \, dx$$

$$= i^\alpha \int_{Q_T} u \cdot \overline{D_t^{\alpha} \varphi} \, dt \, dx - \int_{Q_T} u (-\Delta)^{\beta/2} \varphi(t, x) \, dt \, dx,$$

for all $T > 0$ and all $\varphi \in Y_T$. In order to get a non-negativity in the left hand side of (17), we consider four cases:

**Case I:** If $\lambda_1 > 0$, then $\int_{\mathbb{R}^N} G_1 \, dx > 0$, therefore by taking the real part (Re) of the both sides of (17), we get:

$$\lambda_1 \int_{Q_T} |u|^p \varphi(t, x) \, dt \, dx + \int_{Q_T} G_1(x) \overline{D_t^{\alpha} \varphi(t, x)} \, dt \, dx$$

$$= \int_{Q_T} \text{Re}(i^\alpha u) \overline{D_t^{\alpha} \varphi} \, dt \, dx - \int_{Q_T} \text{Re}(u (-\Delta)^{\beta/2} \varphi(t, x)) \, dt \, dx.$$

**Case II:** If $\lambda_1 < 0$, then $\int_{\mathbb{R}^N} G_1 \, dx < 0$ therefore by taking (-Re) of the both sides of (17) we get:

$$(-\lambda_1) \int_{Q_T} |u|^p \varphi(t, x) \, dt \, dx - \int_{Q_T} G_1(x) \overline{D_t^{\alpha} \varphi(t, x)} \, dt \, dx.$$

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\[ = - \int_{Q_T} \text{Re}(i^\alpha u)^c D_{_{1|T}}^{\alpha} \varphi \, dt \, dx + \int_{Q_T} \text{Re}(u)(-\Delta)^{\beta/2} \varphi(t, x) \, dt \, dx. \]

**Case III:** If \( \lambda_2 > 0 \), then \( \int_{\mathbb{R}^N} G_2 \, dx > 0 \), therefore by taking the imaginary part (Im) of the both sides of (17), we get:

\[
\lambda_2 \int_{Q_T} |u|^p \varphi(t, x) \, dt \, dx + \int_{Q_T} G_2(x)^c D_{_{1|T}}^{\alpha} \varphi(t, x) \, dt \, dx
\]

\[
= \int_{Q_T} \text{Im}(i^\alpha u)^c D_{_{1|T}}^{\alpha} \varphi \, dt \, dx - \int_{Q_T} \text{Im}(u)(-\Delta)^{\beta/2} \varphi(t, x) \, dt \, dx.
\]

**Case IV:** If \( \lambda_2 < 0 \), then \( \int_{\mathbb{R}^N} G_2 \, dx < 0 \), therefore by taking (-Im) of the both sides of (17), we get:

\[
(-\lambda_2) \int_{Q_T} |u|^p \varphi(t, x) \, dt \, dx - \int_{Q_T} G_2(x)^c D_{_{1|T}}^{\alpha} \varphi(t, x) \, dt \, dx
\]

\[
= - \int_{Q_T} \text{Im}(i^\alpha u)^c D_{_{1|T}}^{\alpha} \varphi \, dt \, dx + \int_{Q_T} \text{Im}(u)(-\Delta)^{\beta/2} \varphi(t, x) \, dt \, dx.
\]

Then we only consider the Case I, since the other cases can be treated in the same way, by assuming \( \lambda_1 > 0 \), \( u_0 \in L^1(\mathbb{R}^N) \) and

\[
\int_{\mathbb{R}^N} G_1(x) \, dx > 0. \quad (18)
\]

Thus we have

\[
\lambda_1 \int_{Q_T} |u|^p \varphi(t, x) \, dt \, dx + \int_{Q_T} G_1(x)^c D_{_{1|T}}^{\alpha} \varphi(t, x) \, dt \, dx
\]

\[
\leq \int_{Q_T} \left| \cos \left( \frac{\alpha \pi}{2} \right) \text{Re} - \sin \left( \frac{\alpha \pi}{2} \right) \text{Im}(u) \right| \left| c D_{_{1|T}}^{\alpha} \varphi(t, x) \right| \, dt \, dx
\]

\[
+ \int_{Q_T} |\text{Re}(u)| \left| (-\Delta)^{\beta/2} \varphi(t, x) \right| \, dt \, dx
\]

\[
\leq 2 \int_{Q_T} |u| \left| c D_{_{1|T}}^{\alpha} \varphi(t, x) \right| \, dt \, dx + \int_{Q_T} |u| \left| (-\Delta)^{\beta/2} \varphi(t, x) \right| \, dt \, dx, \quad (19)
\]

all \( \varphi \in Y_T \). Using the \( \epsilon \)-Young inequality

\[
ab \leq \epsilon a^p + C_\epsilon b^{\frac{p}{p-1}}, \quad \text{for all } \epsilon > 0, \, a, \, b \geq 0, \quad C_\epsilon = \frac{(p - 1)(p\epsilon)^{-\frac{1}{p-1}}}{p} \quad (20)
\]

we get

\[
2 \int_{Q_T} |u| \left| c D_{_{1|T}}^{\alpha} \varphi(t, x) \right| \, dt \, dx
\]
\[ = \int_{Q_T} |u|^{\frac{1}{p}} \varphi^{-\frac{1}{p}} \left| c \, D_{t}^{\alpha} \varphi(t, x) \right| \, dt \, dx \]
\[ \leq \varepsilon \int_{Q_T} |u|^{p} \varphi(t, x) \, dt \, dx + \int_{Q_T} \varphi^{-\frac{1}{p-1}} \left| c \, D_{t}^{\alpha} \varphi(t, x) \right|^{\frac{p}{p-1}} \, dt \, dx, \quad (21) \]
where
\[ C_4 = 2^{\frac{p}{p-1}} C_\varepsilon = \frac{2^{\frac{p}{p-1}} (p-1)(p\varepsilon)^{-\frac{1}{p-1}}}{p}. \]

Similarly,
\[ \int_{Q_T} |u| \left| (-\Delta)^{\beta/2} \varphi(t, x) \right| \, dt \, dx \]
\[ \leq \varepsilon \int_{Q_T} |u|^{p} \varphi(t, x) \, dt \, dx + \int_{Q_T} \varphi^{-\frac{1}{p-1}} \left| (-\Delta)^{\beta/2} \varphi(t, x) \right|^{\frac{p}{p-1}} \, dt \, dx, \quad (22) \]
where
\[ C_5 = C_\varepsilon = \frac{(p-1)(p\varepsilon)^{-\frac{1}{p-1}}}{p}. \]

Combining (21)–(22) with (19), we obtain
\[ (\lambda_1 - 2\varepsilon) \int_{Q_T} |u|^{p} \varphi(t, x) \, dt \, dx + \int_{Q_T} G_1(x)^c \, D_{t}^{\alpha} \varphi(t, x) \, dt \, dx \]
\[ \leq C_4 \int_{Q_T} \varphi^{-\frac{1}{p-1}} \left| c \, D_{t}^{\alpha} \varphi(t, x) \right|^{\frac{p}{p-1}} \, dt \, dx \]
\[ + C_5 \int_{Q_T} \varphi^{-\frac{1}{p-1}} \left| (-\Delta)^{\beta/2} \varphi(t, x) \right|^{\frac{p}{p-1}} \, dt \, dx, \quad (23) \]
which implies, by taking \( \varepsilon \leq \lambda_1/2 \), that
\[ \int_{Q_T} G_1(x)^c \, D_{t}^{\alpha} \varphi(t, x) \, dt \, dx \]
\[ \leq C_4 \int_{Q_T} \varphi^{-\frac{1}{p-1}} \left| c \, D_{t}^{\alpha} \varphi(t, x) \right|^{\frac{p}{p-1}} \, dt \, dx \]
\[ + C_5 \int_{Q_T} \varphi^{-\frac{1}{p-1}} \left| (-\Delta)^{\beta/2} \varphi(t, x) \right|^{\frac{p}{p-1}} \, dt \, dx, \quad (23) \]
all \( \varphi \in X_T \). At this stage, we take the test function
\[ \varphi(t, x) := \varphi_R(x)w(t), \]
with \( \phi_R(x) := \phi(x/R) \), \( R > 0 \), where \( \phi(x) \) and \( w(t) \) are defined in Sect. 2 with \( s = \beta/2 \) and \( q = N + \beta \). Therefore, from (23) we obtain

\[
\int_{\mathbb{R}^N} G_1(x) \phi_R(x) \, dx \left. \int_0^T c D_{I,T}^\alpha w(t) \, dt \right|_{t = T} \leq C_4 \int_{\mathbb{R}^N} \phi_R(x) \, dx \int_0^T (w(t))^{-\frac{1}{p-1}} \left| c D_{I,T}^\alpha w(t) \right|^\frac{p}{p-1} \, dt + C_5 \int_0^T w(t) \, dt \int_{\mathbb{R}^N} (\phi_R(x))^{-\frac{1}{p-1}} \left| (-\Delta)^{\beta/2} \phi_R(x) \right|^\frac{p}{p-1} \, dx.
\]

As

\[
\int_{\mathbb{R}^N} \phi_R(x) \, dx = \int_{\mathbb{R}^N} \phi(\tilde{x}) R^N \, d\tilde{x} = A_0 R^N, \quad \text{and} \quad \int_0^T w(t) \, dt = \frac{T}{\eta + 1},
\]

so, using Lemmas 3 and 7 with \( s = \beta/2 \) and \( \eta > \alpha p/(p - 1) - 1 \), we obtain

\[
C_2 T^{1-\alpha} \int_{\mathbb{R}^N} G_1(x) \phi_R(x) \, dx \leq C_6 R^N T^{1-\alpha} \frac{\eta}{p-1} + C_7 T R^\frac{\eta}{p-1} + N,
\]

where

\[
C_6 = C_1 C_4 A_0, \quad \text{and} \quad C_7 = \frac{C_3 C_5}{\eta + 1}.
\]

Choosing \( R = T^{\alpha/\beta} \), we get

\[
\int_{\mathbb{R}^N} G_1(x) \phi(x/T^{\alpha/\beta}) \, dx \leq C_8 T^\frac{N}{p} \left[ \frac{1}{\eta - 1} \right],
\]

(24)

where

\[
C_8 = \frac{1}{C_2} \max\{C_6, C_7\}.
\]

By taking, e.g., \( \varepsilon = \lambda_1/2 \), \( C_8 \) can be written as

\[
C_8 = C_0 A_0 \lambda_1^{-\frac{1}{p-1}},
\]

where

\[
C_0 = \frac{2^\frac{1}{p-1}}{p^{\frac{\eta}{p-1}} C_2} \max \left\{ C_1 2^\frac{p}{p-1}, \left( \frac{C_{N,N+\beta}}{\eta + 1} \right)^\frac{\eta}{p-1} \right\}.
\]

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If \( p < 1 + \frac{\beta}{N} \), then \( \frac{N}{p} - \frac{1}{p-1} < 0 \). As \( G_1 \in L^1(\mathbb{R}^N) \), letting \( T \to \infty \) and using the dominated convergence theorem we derive

\[
\int_{\mathbb{R}^N} G_1(x) \, dx \leq 0,
\]
a contraction with (18).

If \( p = 1 + \frac{\beta}{N} \), using again the same argument, we arrive at

\[
\int_{\mathbb{R}^N} G_1(x) \, dx \leq C_0 A_0 \lambda_1^{\frac{1}{p-1}},
\]
which is a contradiction.

**Remark 1** We note that the regularity of \( u_0 \) is not so important in Theorem 1, in fact, we can replace \( u_0 \in L^1(\mathbb{R}^N) \) by \( u_0 \in L^2(\mathbb{R}^N) \) and we get a nonexistence of global \( L^2 \)-weak solution. In this case, to ensure the existence of the conditions on \( G_1 \) and \( G_2 \), we need also to assume that \( G_1 \) or \( G_2 \) are in \( L^1(\mathbb{R}^N) \).

### 4 Non-existence of global \( L^2 \)-weak solution in \( L^2 \)-subcritical case for small data

**Theorem 2** (Non-existence for global \( L^2 \)-weak solution in \( L^2 \)-subcritical case and for small data) Let \( 0 < \alpha < 1, \ 0 < \beta < 2, \ N \geq 1, \ \varepsilon > 0 \). Let \( u_0 \in H^s(\mathbb{R}^N), \ s \geq 0, \) and \( u \) be an \( L^2 \)-weak solution on \([0, T_w(\varepsilon))\). We assume that \( 1 < p < 1 + 2\beta/N \) and \( u_0 \) satisfies

\[
\lambda_1 \ G_1(x) \ \text{or} \ \lambda_2 \ G_2(x) \geq \begin{cases} 
|x|^{-k}, & \text{if } |x| > 1, \\
0, & \text{if } |x| \leq 1,
\end{cases}
\]

where \( N/2 < k < \frac{\beta}{p-1} \). Then, \( u \) is not global, i.e. \( T_w(\varepsilon) < \infty \). More precisely, there exists a constant \( \varepsilon_0 > 0 \) such that

\[
T_w(\varepsilon) \leq \begin{cases} 
B_0 \varepsilon^{\frac{1}{\kappa_0}}, & \text{if } \varepsilon \in (0, \varepsilon_0), \\
1, & \text{if } \varepsilon \in [\varepsilon_0, \infty),
\end{cases}
\]

where \( \kappa_0 = \frac{1}{p-1} - \frac{k}{p} > 0 \) and

\[
B_0 = \left( C_0(k + \beta)\omega_N^{-1} 2^{\frac{N+\beta}{2}} A_0 \lambda_1^{\frac{p-2}{2} \frac{1}{\alpha_0}} \right),
\]

with \( \omega_N \) stands for the \((N - 1)\)-dimensional surface measure of the unit sphere.
Proof. Repeating the same calculations as in the proof of Theorem 1, by taking here $\varepsilon \neq 1$, and assuming only

$$\lambda_1 > 0 \text{ and } G_1(x) \geq \begin{cases} \lambda_1^{-1} |x|^{-k}, & \text{if } |x| > 1, \\ 0, & \text{if } |x| \leq 1, \end{cases}$$

(the other cases can be treated similarly). From (24), we obtain

$$\varepsilon \int_{\mathbb{R}^N} G_1(x) \phi(x/T^{\alpha/\beta}) \, dx \leq C_0 A_0 \lambda_1^{1-1/p} T^{\alpha \left[ \frac{N}{p} + \frac{1}{p-1} \right]},$$

for all $T \in (0, T_w(\varepsilon))$. On the other hand,

$$\varepsilon \int_{\mathbb{R}^N} G_1(x) \phi(x/T^{\alpha/\beta}) \, dx = \varepsilon T^{\alpha N} \int_{\mathbb{R}^N} G_1(yT^{\alpha/\beta}) \phi(y) \, dy$$

$$\geq \lambda_1^{-1} \varepsilon T^{\alpha(N-k)} \int_{|y| > T^{\alpha/\beta}} |y|^{-k} \phi(y) \, dy$$

$$= \lambda_1^{-1} \varepsilon T^{\alpha(N-k)} K(T),$$

where

$$K(T) := \int_{|y| > T^{\alpha/\beta}} |y|^{-k} \phi(y) \, dy.$$

Therefore, from (28), we arrive at

$$\varepsilon K(T) \leq C_0 A_0 \lambda_1^{1-1/p} T^{\alpha \left[ \frac{k}{p} - \frac{1}{p-1} \right]}, \text{ for all } 0 < T < T_w(\varepsilon).$$

(29)

It remains to estimate from below the last inequality.

First, let $\varepsilon_0 = B_0^{\alpha k_0}$, then

$$T_w(\varepsilon) \leq 1,$$

for all $\varepsilon \geq \varepsilon_0$. Indeed, suppose on the contrary that there exists $\varepsilon \geq \varepsilon_0$ such that $T_w(\varepsilon) > 1$. Applying (29) with $\tau \in (1, T_w(\varepsilon))$, we obtain

$$\varepsilon K(\tau) \leq C_0 A_0 \lambda_1^{1-1/p} \tau^{\alpha \left[ \frac{k}{p} - \frac{1}{p-1} \right]}, \text{ for all } 1 < \tau < T_w(\varepsilon).$$

(30)

Using the fact that

$$|y| \leq (1 + |y|^2)^{1/2} \leq \sqrt{2} |y|, \text{ for all } |y| > 1,$$

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we have
\[
\frac{\omega_N}{(k + \beta)2^{\frac{N+\beta}{2}}} = 2^{-\frac{N+\beta}{2}} \int_{|y|>1} |y|^{-k-N-\beta} \, dy \\
\leq K(1) \\
\leq \int_{|y|>1} |y|^{-k-N-\beta} \, dy \\
= \frac{\omega_N}{(k + \beta)}.
\]

Whereupon,
\[
K(\tau) \geq K(1) \geq \frac{\omega_N}{(k + \beta)2^{\frac{N+\beta}{2}}}, \quad \text{for all } 1 < \tau < T_w(\epsilon). \tag{31}
\]

Combining (30) and (31), we obtain
\[
\epsilon \leq (k + \beta)\omega_N^{-1}2^{\frac{N+\beta}{2}} C_0 A_0 \lambda_1^{\frac{p-2}{p-\tau}} \tau^\alpha \left[\frac{1}{\beta} - \frac{1}{p-1}\right],
\]
i.e.
\[
\tau \leq B_0 \epsilon^{-\frac{1}{\alpha x_0}}, \quad \text{for all } 1 < \tau < T_w(\epsilon).
\]

Letting \( \tau \to T_w(\epsilon) \), we get
\[
T_w(\epsilon) \leq B_0 \epsilon^{-\frac{1}{\alpha x_0}} \leq B_0 \epsilon_0^{-\frac{1}{\alpha x_0}} = 1,
\]
a contradiction. Therefore, \( T_w(\epsilon) \leq 1 \), for all \( \epsilon \geq \epsilon_0 \).

On the other hand, suppose \( \epsilon < \epsilon_0 \). If \( T_w(\epsilon) \leq 1 \), it follows that
\[
T_w(\epsilon) \leq 1 = B_0 \epsilon_0^{-\frac{1}{\alpha x_0}} \leq B_0 \epsilon^{-\frac{1}{\alpha x_0}}.
\]

Hence, it is sufficient to consider \( T_w(\epsilon) > 1 \). By the above argument, we get again
\[
T_w(\epsilon) \leq B_0 \epsilon^{-\frac{1}{\alpha x_0}}.
\]

This completes the proof.

**Remark 2** We note that the condition \( k > \frac{N}{2} \) in Theorem 2 is necessary to ensure the existence of at least an \( H^s \)-function \( u_0 \) satisfying (32), for all \( s \geq 0 \).
5 Non-existence of global $L^2$-weak solution for large data

**Theorem 3** (Non-existence of global $L^2$-weak solution for $p > 1$ and large data)

Let $0 < \alpha < 1$, $0 < \beta < 2$, $N \geq 1$, $\varepsilon > 0$, and $p > 1$. Let $u_0 \in H^s(\mathbb{R}^N)$, $s \geq 0$, and $u$ be an $L^2$-weak solution on $[0, T_w(\varepsilon))$. We assume that $u_0$ satisfies

$$\lambda_1 G_1(x) \text{ or } \lambda_2 G_2(x) \geq \begin{cases} |x|^{-k}, & \text{if } |x| \leq 1, \\ 0, & \text{if } |x| > 1, \end{cases}$$

where $k < \min \left\{ \frac{N}{2} - s, \frac{\beta}{p-1} \right\}$. Then, there exists a constant $\varepsilon_1 > 0$ such that for any $\varepsilon > \varepsilon_1$, $u$ is not global, i.e. $T_w(\varepsilon) < \infty$. More precisely,

$$T_w(\varepsilon) \leq \overline{C} \varepsilon^{\frac{1}{\alpha \kappa_0}},$$

for all $\varepsilon > \varepsilon_1$, where $\kappa_0 = \frac{1}{p-1} - \frac{k}{\beta} > 0$ and

$$\overline{C} = \left( C_0 (N - k) \omega_N^{-1} 2^{\frac{N+\beta}{p-1}} A_0 \lambda_1^{\frac{p-2}{p-1}} \right)^{\frac{1}{\alpha \kappa_0}}.$$

**Proof** Repeating the same calculations as in the proof of Theorem 1, by taking here $\varepsilon \neq 1$, and considering only the case

$$\lambda_1 > 0 \text{ and } G_1(x) \geq \begin{cases} \lambda_1^{-1} |x|^{-k}, & \text{if } |x| \leq 1, \\ 0, & \text{if } |x| > 1, \end{cases}$$

as the other cases can be treated similarly. From (24), we obtain

$$\varepsilon \int_{\mathbb{R}^N} G_1(x) \phi(x/T^{\alpha/\beta}) \, dx \leq C_0 A_0 \lambda_1^{-\frac{1}{p-1}} T^{\alpha \left[ \frac{N}{p} + 1 - \frac{p}{p-1} \right]},$$

for all $T \in (0, T_w(\varepsilon))$. On the other hand,

$$\varepsilon \int_{\mathbb{R}^N} G_1(x) \phi(x/T^{\alpha/\beta}) \, dx = \varepsilon T^{\frac{aN}{p}} \int_{\mathbb{R}^N} G_1(yT^{\alpha/\beta}) \phi(y) \, dy \geq \lambda_1^{-1} \varepsilon T^{\frac{a(N-k)}{p}} \int_{|y| \leq T^{\frac{aN}{p}}} |y|^{-k} \phi(y) \, dy \geq \lambda_1^{-1} \varepsilon T^{\frac{a(N-k)}{p}} L(T),$$

where

$$L(T) := \int_{|y| \leq T^{\frac{aN}{p}}} |y|^{-k} \phi(y) \, dy.$$
Therefore, from (33), we arrive at
\[ \varepsilon L(T) \leq C_0 A_0 \lambda_1^{\frac{p-2}{p-1}} T^\alpha \left[ \frac{k}{p-1} \right], \text{ for all } 0 < T < T_w(\varepsilon). \] (34)

It remains to estimate from below the last inequality.

We claim that there exists a constant \( \varepsilon_1 > 0 \) such that for any \( \varepsilon > \varepsilon_1 \),
\[ T_w(\varepsilon) \leq 1. \] (35)

Indeed, suppose on the contrary that for all \( \varepsilon_1 > 0 \), there exists \( \varepsilon > \varepsilon_1 \) such that \( T_w(\varepsilon) > 1 \). Applying (34) with \( T = 1 \), we have
\[ \varepsilon L(1) \leq C_0 A_0 \lambda_1^{\frac{p-2}{p-1}}. \] (36)

Using the fact that \( k < N \), and
\[ \frac{1}{2^{N+\beta}} \leq \phi(y) \leq 1, \text{ for all } 0 \leq |y| \leq 1, \]
it is easy to check that
\[ \frac{\omega_N}{(N-k)2^{\frac{N+\beta}{2}}} \leq L(1) \leq \frac{\omega_N}{(N-k)}. \] (37)

Combining (36) and (37), we obtain
\[ \varepsilon \leq (N-k)\omega_N^{-1} 2^{N+\beta} C_0 A_0 \lambda_1^{\frac{p-2}{p-1}} =: \varepsilon_1; \]
contradiction. Thus the claim is proved.

Therefore, for all \( T < T_w(\varepsilon) \leq 1 \), we have
\[ L(T) \geq \int_{|y| \leq 1} |y|^{-k} \phi(y) \, dy = L(1) \geq \frac{\omega_N}{(N-k)2^{\frac{N+\beta}{2}}}, \]
which implies, using again (34),
\[ \varepsilon \frac{\omega_N}{(N-k)2^{\frac{N+\beta}{2}}} \leq C_0 A_0 \lambda_1^{\frac{p-2}{p-1}} T^\alpha \left[ \frac{k}{p-1} \right], \text{ for all } 0 < T < T_w(\varepsilon), \]
i.e.
\[ T \leq \frac{1}{C_0 \varepsilon} \frac{1}{\lambda_1^{\frac{1}{p-1}}}, \text{ for all } 0 < T < T_w(\varepsilon). \]

Since \( T \) is arbitrary in \((0, T_w(\varepsilon))\), the proof is completed by letting \( T \to T_w(\varepsilon) \).
Remark 3 In Theorem 3, it is sufficient to just consider the case $p \leq 1 + 2\beta/(N - 2s)$, because the other case $p > 1 + 2\beta/(N - 2s)$ is proved below in Sect. 6, (non local implies non global existence), and in this case we take $k < \frac{N}{2} - s \left(\leq \frac{\beta}{p-1}\right)$.

Remark 4 We note that the condition $k < \frac{N}{2} - s$ in Theorem 3 is necessary to ensure the existence of at least an $H^s$-function $u_0$ satisfying (32).

6 Nonexistence of local $L^2$-weak solution in $H^s$-supercritical case

Theorem 4 (Non-existence of local $L^2$-weak solution in $H^s$-supercritical case) Let $0 < \alpha < 1, 0 < \beta < 2, N \geq 1, \varepsilon > 0$, and $p > 1 + 2\beta/(N - 2s)$. Assume $u_0 \in H^s(\mathbb{R}^N), 0 \leq s < N/2$, such that $u_0$ satisfies (32) with $\beta/(p - 1) < k < N/2 - s$. Then there is no local $L^2$-weak solution of (1).

Proof Suppose that there exists an $L^2$-weak solution $u$ on $[0, T)$ for some $0 < T < T_w(\varepsilon)$. Repeating the same proof of Theorem 3, we have

$$
\varepsilon L(\tau) \leq C_0 A_0 \lambda_1^{\frac{p-2}{p-1}} \tau^{\alpha \left[\frac{k}{\beta} - \frac{1}{p-1}\right]}, \quad \text{for all } 0 < \tau < T.
$$

For all $\tau < 1$, we have

$$
L(\tau) \geq \int_{|y| \leq 1} |y|^{-k} \phi(y) dy = L(1) \geq \frac{\omega_N}{(N - k)2^{N+\beta}},
$$

whereupon

$$
\varepsilon \frac{\omega_N}{(N - k)2^{N+\beta}} \leq C_0 A_0 \lambda_1^{\frac{p-2}{p-1}} \tau^{\alpha \left[\frac{k}{\beta} - \frac{1}{p-1}\right]}, \quad \text{for all } 0 < \tau < \min\{1, T\},
$$

i.e.

$$
\varepsilon \leq (N - k)\omega_N^{-1} 2^{\frac{N+\beta}{2}} C_0 A_0 \lambda_1^{\frac{p-2}{p-1}} \tau^{\alpha \left[\frac{k}{\beta} - \frac{1}{p-1}\right]}, \quad \text{for all } 0 < \tau < \min\{1, T\}.
$$

As $\beta/(p - 1) < k$, we have $k/\beta - 1/(p - 1) > 0$. Therefore, taking $\tau \to 0^+$, we obtain $\varepsilon = 0$; contradiction. This completes the proof.

7 Nonexistence of local $L^1$-weak solution in the case $p > p_F$

Theorem 5 (Non-existence of local $L^1$-weak solution in the supercritical case) Let $0 < \alpha < 1, 0 < \beta < 2, N \geq 1, \varepsilon > 0$, and $p > 1 + \beta/N = p_F$. Assume $u_0 \in L^1(\mathbb{R}^N)$ and satisfying (32) with $\beta/(p - 1) < k < N$. Then there is no local $L^1$-weak solution of (1).
Some nonexistence results for space–time fractional...

Proof Suppose that there exists an $L^1$-weak solution $u$ on $[0, T)$ for some $0 < T < T_w(\varepsilon)$. Applying the proof of Theorem 4, step by step. The only difference is the condition $k < N$ instead of $k < N/2 - s$, which is required to ensure that there exists an $L^1$-function $u_0$ satisfying (32).

8 Nonexistence of global $L^2$-weak solution: New approach

Theorem 6 (Nonexistence for global $L^2$-weak solution: New approach) Let $0 < \alpha < 1$, $0 < \beta < 2$, $N \geq 1$, $p > 1$, $T > 0$, and

$$X(T) = C([0, T), L^2(\mathbb{R}^N)) \cap C^1([0, T), H^{-\frac{\beta}{2}}(\mathbb{R}^N)) \cap L^\infty((0, T), L^p(\mathbb{R}^N)).$$

Assume $u_0 \in L^2(\mathbb{R}^N)$ and satisfies

$$M_R(0) > C_{N, p, \beta, \gamma} R^{N - \frac{\beta}{p - 1}},$$

for some $R > 0$ and $\gamma \in \mathbb{C}$ satisfying $\text{Re}(\gamma \lambda) > 0$, where

$$M_R(0) = \text{Re} \left( i^\alpha \gamma \int_{\mathbb{R}^N} u(0, x) \phi_R(x) \, dx \right),$$

with $\phi_R(x) := \phi(x/R)$, $R > 0$, ($\phi(x)$ is defined in Sect. 2 with $q = N + \beta$), and

$$C_{N, p, \beta, \lambda, \gamma}^p = 2^{1/2} (\text{Re}(\gamma \lambda))^{-\frac{p}{p - 1}} |\gamma|^{\frac{p}{p - 1}} A^p_0 \left( C_{N, N + \beta} \right)^{\frac{p}{p - 1}}.$$

Then there is no distributional solution $u \in X(T)$, with $T > T_b$, for (1), where [see (13)]

$$T_b \sim \left( \frac{R^{N(p-1)} \Gamma(1 + \alpha)}{D_{N, p, \beta, \lambda, \gamma} \left[ M_R(0) - C_{N, p, \beta, \gamma} R^{N - \frac{\beta}{p - 1}} \right]^{p-1}} \right)^{1/\alpha},$$

and

$$D_{N, p, \beta, \lambda, \gamma} = 2^{-1} \text{Re}(\gamma \lambda) |\gamma|^{-p} A_0^{-(p-1)}.$$

Proof Suppose, on the contrary, that there exists a distributional solution $u \in X(T)$ with $T > T_b$. Let

$$M_R(t) = \text{Re} \left( i^\alpha \gamma \int_{\mathbb{R}^N} u(t, x) \phi_R(x) \, dx \right).$$
By Lemmas 5 and 6, we have

\[
\mathcal{D}^a_{t=0} M_R(t) = \text{Re} \left( \gamma \int_{\mathbb{R}^N} i^a c \mathcal{D}^a_{t=0} u(t, x) \phi_R(x) \, dx \right)
\]

\[
= \text{Re} \left( \gamma \lambda \int_{\mathbb{R}^N} |u(t, x)|^p \phi_R(x) \, dx \right.
\]

\[
+ R^{-\beta} \text{Re} \left( \gamma \int_{\mathbb{R}^N} u(t, x) \left( (-\Delta)^{\beta/2} \phi \right) (x / R) \, dx \right)
\]

\[
\geq \text{Re} \left( \gamma \lambda \int_{\mathbb{R}^N} |u(t, x)|^p \phi(x / R) \, dx \right.
\]

\[
-C_{N,N+\beta} R^{-\beta} |\gamma| \int_{\mathbb{R}^N} |u(t, x)| \phi(x / R) \, dx.
\] (40)

In order to get a differential inequality, we start by estimating the second term in the right hand side of (40). Using 1/2-Young’s inequality (20), we obtain

\[
C_{N,N+\beta} R^{-\beta} |\gamma| \int_{\mathbb{R}^N} |u(t, x)| \phi(x / R) \, dx
\]

\[
= \int_{\mathbb{R}^N} |u(t, x)| \left[ \text{Re} \left( \gamma \lambda \phi(x / R) \right) \right]^{1/2} C_{N,N+\beta} R^{-\beta} |\gamma|
\]

\[
\left[ \text{Re} \left( \gamma \lambda \right) \right]^{-1/2} \left[ \phi(x / R) \right]^{p-1/p} \, dx
\]

\[
\leq \frac{1}{2} \text{Re} \left( \gamma \lambda \right) \int_{\mathbb{R}^N} |u(t, x)|^p \phi(x / R) \, dx
\]

\[
+C_{1/2} |\gamma|^{p-1/p} \left( C_{N,N+\beta} \right)^{p-1/p} R^{-\beta} \left( \text{Re} \left( \gamma \lambda \right) \right)^{1/p-1} \int_{\mathbb{R}^N} \phi(x / R) \, dx
\]

\[
= \frac{1}{2} \text{Re} \left( \gamma \lambda \right) \int_{\mathbb{R}^N} |u(t, x)|^p \phi(x / R) \, dx
\]

\[
+C_{1/2} A_0 |\gamma|^{p-1/p} \left( \text{Re} \left( \gamma \lambda \right) \right)^{1/p-1} \left( C_{N,N+\beta} \right)^{p-1/p} R^{N-\beta p/p-1},
\] (41)

where

\[
C_{1/2} = (p-1) p^{-1/p-1} 2^{1/p-1} \quad \text{and} \quad A_0 = \int_{\mathbb{R}^N} \phi(\tilde{x}) \, d\tilde{x}.
\]

On the other hand, by estimating the first term in the right hand side of (40) by using Hölder’s inequality, we get

\[
|M_R(t)| = \left| \text{Re} \left( i^a \gamma \int_{\mathbb{R}^N} u(t, x) \phi(x / R) \, dx \right) \right|
\]

\[
\leq |\gamma| \int_{\mathbb{R}^N} |u(t, x)| \phi(x / R) \, dx
\]

\[
= |\gamma| \int_{\mathbb{R}^N} |u(t, x)| \left( \phi(x / R) \right)^{1/p} \left( \phi(x / R) \right)^{p-1/p} \, dx
\]
\[
\leq |\gamma| \left( \int_{\mathbb{R}^N} |u(t,x)|^p \phi(x/R) \, dx \right)^{1/p} \left( \int_{\mathbb{R}^N} \phi(x/R) \, dx \right)^{p-1/p} \\
= |\gamma| A_0^{p-1/p} R^{N(p-1)/p} \left( \int_{\mathbb{R}^N} |u(t,x)|^p \phi(x/R) \, dx \right)^{1/p},
\]
i.e.
\[
\int_{\mathbb{R}^N} |u(t,x)|^p \phi(x/R) \, dx \geq |\gamma|^{-p} A_0^{-(p-1)} R^{-N(p-1)} |M_R(t)|^p.
\] (42)

Inserting (41)–(42) into (40), we conclude that
\[
^{c} D_{t|T}^\alpha M_R(t) \\
\geq 2^{-1} \text{Re} (\gamma \lambda) \int_{\mathbb{R}^N} |u(t,x)|^p \phi(x/R) \, dx \\
- C_{1/2} A_0 |\gamma|^{p-1/p} (\text{Re} (\gamma \lambda))^{-1/p-1} (C_{N,N+\beta})^{p-1/p} R^{N-\beta_p/p-1} \\
\geq 2^{-1} \text{Re} (\gamma \lambda) |\gamma|^{-p} A_0^{-(p-1)} R^{-N(p-1)} |M_R(t)|^p \\
- C_{1/2} A_0 |\gamma|^{p-1/p} (\text{Re} (\gamma \lambda))^{-1/p-1} (C_{N,N+\beta})^{p-1/p} R^{N-\beta_p/p-1} \\
= 2^{-1} \text{Re} (\gamma \lambda) |\gamma|^{-p} A_0^{-(p-1)} R^{-N(p-1)} \left[ |M_R(t)|^p - C_{N,p,\beta,\lambda,\gamma}^P R^P \left( N - \frac{\beta}{p-1} \right) \right],
\]
i.e.
\[
^{c} D_{t|T}^\alpha M_R(t) \geq D_{N,p,\beta,\lambda,\gamma} R^{-N(p-1)} \left[ |M_R(t)|^p - C_{N,p,\beta,\lambda,\gamma}^P R^P \left( N - \frac{\beta}{p-1} \right) \right].
\] (43)

Applying Lemma 4 and using (38), we conclude that
\[
M_R(t) \geq C_{N,p,\beta,\gamma} R^{N-\frac{\beta}{p-1}} > 0, \quad \text{for all } t \in [0, T),
\] (44)
which implies, by using (43) and the following elementary inequality
\[
a^p - b^p \geq (a - b)^p, \quad \text{for all } a > b \geq 0, \quad p > 1,
\]
that
\[
^{c} D_{t|T}^\alpha M_R(t) \geq D_{N,p,\beta,\lambda,\gamma} R^{-N(p-1)} \left[ M_R(t) - C_{N,p,\beta,\gamma} R^P \left( N - \frac{\beta}{p-1} \right) \right]^p.
\] (45)

Apply Proposition 1 and the fact that \(^{c} D_{t|T}^\alpha C = 0\), for any constant \(C > 0\), we infer that
\[
\lim_{t \to T_0} M_R(t) = +\infty.
\]
Since
\[ M_R(t) \leq \|u(t)\|_{L^\infty((0,T),L^2(\mathbb{R}^N))} \|\phi(\cdot / R)\|_{L^2(\mathbb{R}^N)} < \infty, \quad \text{for all } t \in [0, T), \]
we get a contradiction, and this completes the proof.

**Remark 5** Note that, from (14), we have
\[ H(p, \alpha) = \max\left(p - 1, 2 - \frac{p\alpha}{p-1}\right) \geq p - 1; \]
this implies that \( T_b \) can be chosen as
\[ T_b = \left( \frac{R^{N(p-1)} \Gamma(1 + \alpha)}{(p-1)D_{N,p,\beta,\lambda,\gamma} \left(M_R(0) - C_{N,p,\beta,\gamma} R^{N-\frac{\beta}{p-1}} \right)^{p-1}} \right)^{1/\alpha}, \]
which is the same blow-up time as in the ordinary differential equation when \( \alpha = 1 \).

**Corollary 1** (Theorem 1: New approach)
Let \( 0 < \alpha < 1, 0 < \beta < 2, N \geq 1, \gamma \in \mathbb{C}, \varepsilon = 1, p > 1 \). Assume that \( p < 1 + \beta/N \), and \( u_0 \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N) \) satisfies
\[ \text{Re}(\gamma \lambda) > 0 \quad \text{and} \quad \text{Re}\left(i^{\alpha} \int_{\mathbb{R}^N} u_0(x) \, dx\right) > 0. \quad (46) \]
Then there is no distributional solution \( u \in X(T) \) to (1) for sufficiently large \( T > 0 \).

**Proof** By (46), using the dominated convergence theorem, we conclude that
\[ \lim_{R \to \infty} M_R(0) = \text{Re}\left(i^{\alpha} \int_{\mathbb{R}^N} u_0(x) \, dx\right) > 0. \]
On the other hand, as \( p < 1 + \beta/N \),
\[ C_{N,p,\beta,\gamma} R^{N-\frac{\beta}{p-1}} \to 0, \quad \text{when } R \to \infty. \]
Therefore, there exists \( R_0 > 0 \) such that condition (38) is satisfied. Using Theorem 6, the proof is completed.

**Remark 6** Note that, by taking \( \gamma = \pm 1, \pm i \) in Corollary 1, condition (46) implies (16), which means that (46) is more general than (16). Therefore, in the subcritical case, Theorem 1 can be seen as a particular case of Corollary 1, but with different regularity.

**Corollary 2** (Theorem 2: New approach)
Let $0 < \alpha < 1$, $0 < \beta < 2$, $N \geq 1$, $\varepsilon > 0$, $\gamma \in \mathbb{C}$, and $p > 1$. Assume that $p < 1 + 2\beta/N$, and $u_0 \in H^s(\mathbb{R}^N)$, $s \geq 0$, satisfies

$$
\text{Re}(\gamma \lambda) > 0 \quad \text{and} \quad \text{Re}\left(i^\alpha \gamma u_0(x)\right) \geq \begin{cases} 
|x|^{-k}, & \text{if } |x| > 1, \\
0, & \text{if } |x| \leq 1,
\end{cases}
$$

(47)

where $N/2 < k < \frac{\beta}{p-1}$. Then, there exists a constant $\varepsilon_2 > 0$ such that for all $\varepsilon \in (0, \varepsilon_2]$, there is no distributional solution $u \in X(T)$ to (1) for sufficiently large $T > T_b$ with $T_b$ defined in (39). Moreover $T_b$ can be estimated as follows

$$
T_b \leq B_1 \varepsilon^{-\frac{1}{\alpha \kappa_1}},
$$

(48)

for all $\varepsilon \in (0, \varepsilon_2]$, where $\kappa_1 = \frac{1}{p-1} - \frac{\min(N,k)}{\beta} > 0$.

$$
B_1 = (p - 1)^{-1/\alpha} D_{N,p,\beta,\lambda,\gamma}^{-1/\alpha} \Gamma(1 + \alpha)^{1/\alpha} 2^{\frac{1}{\alpha \kappa_1}} (C_{N,p,\beta,\gamma}) \frac{\min(N,k)(p-1)}{\alpha^{\kappa_1}} I_1^{-\frac{1}{\alpha \kappa_1}},
$$

and

$$
I_1 := \begin{cases} 
2^{-N-\beta-1} \omega_N (N-k)^{-1} R^{N-k}, & \text{if } k < N, \\
2^{-N-\beta} \omega_N \int_1^2 r^N r^{-1-k} dr, & \text{if } k \geq N.
\end{cases}
$$

**Proof** In order to apply Theorem 6, we need to estimate $M_R(0)$ from below, for some $R > 0$. Let

$$
\varepsilon_2 = \begin{cases} 
I_1^{-1} C_{N,p,\beta,\gamma} 2^{1-\frac{\beta \kappa_1}{\alpha}} \frac{1}{\alpha \kappa_1}, & \text{if } k < N, \\
I_1^{-1} C_{N,p,\beta,\gamma} 2^{1-\beta \kappa_1}, & \text{if } k \geq N.
\end{cases}
$$

Let $\varepsilon \in (0, \varepsilon_2]$. We choose $R = R(\varepsilon)$ such that

$$
\begin{cases} 
R \geq 2^{1/(N-k)}, & \text{if } k < N, \\
R \geq 2, & \text{if } k \geq N.
\end{cases}
$$

(49)

Then, as $R^{N-k} - 1 \geq R^{N-k}/2$, when $k < N$, using (47), we have

$$
M_R(0) \geq \varepsilon \text{Re} \left( \gamma i^\alpha \int_{\mathbb{R}^N} u_0(x) \phi(x/R) dx \right)
$$

$$
\geq \varepsilon \int_{|x| \geq 1} |x|^{-k} \phi(x/R) dx
$$

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\[
\geq \varepsilon \int_{1 \leq |x| \leq R} |x|^{-k} \phi(x/R) \, dx \\
\geq \varepsilon 2^{-N-\beta} \int_{1 \leq |x| \leq R} |x|^{-k} \, dx \\
= \varepsilon 2^{-N-\beta} \omega_N \int_1^R r^{N-1-k} \, dr \\
\geq \varepsilon 2^{-N-\beta} \omega_N \left\{ \begin{array}{ll}
(N - k)^{-1}(R^{N-k} - 1), & \text{if } k < N, \\
\int_1^2 r^{N-1-k} \, dr, & \text{if } k \geq N,
\end{array} \right.
\geq \varepsilon I_1 R^{(N-k)+},
\]
with \((N - k)_+ = \max(N - k, 0)\). Therefore,
\[
M_R(0) - C_{N,p,\beta,\gamma} R^{N-\frac{\beta}{p-1}} \geq R^{(N-k)+} \left( \varepsilon I_1 - C_{N,p,\beta,\gamma} R^{-\beta \kappa_1} \right) \\
= R^{(N-k)+} \left( \frac{\varepsilon I_1}{2} \right) > 0,
\tag{50}
\]
where \(R\) is chosen to ensure the last equality, namely
\[
R = \left( \frac{2 C_{N,p,\beta,\gamma}}{\varepsilon I_1} \right)^{\frac{1}{\beta \kappa_1}}. \tag{51}
\]
It is clear, by our choice of \(\varepsilon_2\), that condition (49) is satisfied. Applying Theorem 6, we conclude that there is no solution \(u \in X(T)\) to (1) for all \(T > T_b\). Moreover, from (13), (39) and the fact that \(H(p, \alpha) \geq p - 1\), we obtain
\[
T_b \leq T_U \leq (p - 1)^{-1/\alpha} \left( \frac{R^{(p-1) \Gamma(1+\alpha)}}{D_{N,p,\beta,\lambda,\gamma} \left[ M_R(0) - C_{N,p,\beta,\gamma} R^{N-\frac{\beta}{p-1}} \right]^{p-1}} \right)^{1/\alpha}.
\]
Then, using (50) and (51), we conclude that
\[
T_b \leq B_1 \varepsilon^{-\frac{1}{\alpha \kappa_1}}.
\]
This complete the proof. \(\square\)

**Remark 7** Note that, \(\kappa_1 > \kappa_0\), this means that (27) is better than (48). Moreover, by taking \(\gamma = \pm 1, \pm i\) in Corollary 2, condition (47) implies (26), which means that (47) is more general than (26). Therefore, Theorem 2 can be seen as a particular case of Corollary 2, but with different regularity.
Corollary 3 (Theorem 3: New approach) Let $0 < \alpha < 1$, $0 < \beta < 2$, $N \geq 1$, $\varepsilon > 0$, $\gamma \in \mathbb{C}$, and $p > 1$. Assume that $u_0 \in H^s(\mathbb{R}^N)$, $s \geq 0$, satisfies

$$\text{Re}(\gamma \lambda) > 0 \quad \text{and} \quad \text{Re}(i^\alpha \gamma u_0(x)) \geq \begin{cases} |x|^{-k}, & \text{if } |x| \leq 1, \\ 0, & \text{if } |x| > 1, \end{cases}$$

(52)

where $k < \min\left\{ \frac{N}{2} - s, \frac{\beta}{p-1} \right\}$. Then, there exists a constant $\varepsilon_3 > 0$ such that for any $\varepsilon \geq \varepsilon_3$, there is no distributional solution $u \in X(T)$ to (1) for sufficiently large $T > T_b$ with $T_b$ is defined in (39). Moreover $T_b$ can be estimated as follows

$$T_b \leq B_2 \varepsilon^{-\frac{1}{\alpha \kappa_0}},$$

for all $\varepsilon \geq \varepsilon_3$, where $\kappa_0 = \frac{1}{p-1} - \frac{k}{\beta} > 0$,

$$B_2 = (p - 1)^{-1/\alpha} D_{N,p,\beta,\lambda,\gamma}^{-1/\alpha} \Gamma(1 + \alpha)^{1/\alpha} 2^{\frac{1}{\alpha \kappa_0}} (C_{N,p,\beta,\gamma})^{\frac{k(p-1)}{\alpha \kappa_0}} \omega N^{-\frac{1}{\alpha \kappa_0}},$$

and

$$I_2 := 2^{-N-\beta} \omega N (N - k)^{-1}.$$

Proof In order to apply Theorem 6, we need to estimate $M_R(0)$ from below, for some $R > 0$. Let

$$\varepsilon_3 = 2 I_2^{-1} C_{N,p,\beta,\gamma}.$$

Let $\varepsilon \geq \varepsilon_3$. We choose $R = R(\varepsilon) \leq 1$. Then, using (52), we have

$$M_R(0) \geq \varepsilon \text{Re} \left( \gamma i^\alpha \int_{\mathbb{R}^N} u_0(x) \phi(x/R) \, dx \right)$$

$$\geq \varepsilon \int_{|x| \leq 1} |x|^{-k} \phi(x/R) \, dx$$

$$\geq \varepsilon \int_{|x| \leq R} |x|^{-k} \phi(x/R) \, dx$$

$$\geq \varepsilon 2^{-N-\beta} \int_{|x| \leq R} |x|^{-k} \, dx$$

$$= \varepsilon 2^{-N-\beta} \omega N \int_0^R r^{N-1-k} \, dr$$

$$= \varepsilon 2^{-N-\beta} \omega N (N - k)^{-1} R^{N-k}$$

$$= \varepsilon I_2 R^{N-k}.$$
Therefore
\[ M_R(0) - C_{N, p, \beta, \gamma} R^{N - \frac{\beta}{p - 1}} \geq R^{N - k} \left( \varepsilon I_2 - C_{N, p, \beta, \gamma} R^{\beta \kappa_0} \right) \]
\[ = R^{N - k} \left( \frac{\varepsilon I_2}{2} \right) > 0, \tag{53} \]

where
\[ R = \left( \frac{2C_{N, p, \beta, \gamma}}{\varepsilon I_2} \right)^{\frac{1}{\beta \kappa_0}}. \tag{54} \]

It is clear, by our choice of \( \varepsilon_3 \), that \( R \leq 1 \). Applying Theorem 6, we conclude that there is no solution \( u \in X(T) \) of (1) for all \( T > T_b \). Moreover, from (13), (39) and \( H(p, \alpha) \geq p - 1 \), we obtain

\[ T_b \leq T_U \leq (p - 1)^{-1/\alpha} \left( \frac{R^{N(p-1)} \Gamma(1 + \alpha)}{D_{N, p, \beta, \lambda, \gamma} \left[ M_R(0) - C_{N, p, \beta, \gamma} R^{N - \frac{\beta}{p - 1}} \right]^{p-1}} \right)^{1/\alpha}. \]

Then, using (53) and (54), we conclude that

\[ T_b \leq B_2 \varepsilon^{-\frac{1}{\alpha \kappa_0}}. \]

This complete the proof.

**Remark 8** Note that, by taking \( \gamma = \pm 1, \pm i \) in Corollary 3, condition (52) implies (32), which means that (52) is more general than (32). Therefore, Theorem 3 can be seen as a particular case of Corollary 3, but with different regularity.

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**Declaration**

**Conflict of interest** The authors declare that they have no conflict of interest.

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