Free Fermion and Seiberg-Witten Differential
in Random Plane Partitions

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Abstract

A model of random plane partitions which describes five-dimensional $\mathcal{N} = 1$ supersymmetric $SU(N)$ Yang-Mills is studied. We compute the wave functions of fermions in this statistical model and investigate their thermodynamic limits or the semi-classical behaviors. These become of the WKB type at the thermodynamic limit. When the fermions are located at the main diagonal of the plane partition, their semi-classical wave functions are obtained in a universal form. We further show that by taking the four-dimensional limit the semi-classical wave functions turn to live on the Seiberg-Witten curve and that the classical action becomes precisely the integral of the Seiberg-Witten differential. When the fermions are located away from the main diagonal, the semi-classical wave functions depend on another continuous parameter. It is argued that they are related with the wave functions at the main diagonal by the renormalization group flow of the underlying gauge theory.

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1 Introduction

Recently it becomes possible \cite{1,2} to compute the exact partition functions of four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories. The celebrated Seiberg-Witten solutions \cite{3} of the gauge theories emerge \cite{2,4} through the statistical models of random partitions. In particular, Nekrasov and Okounkov show \cite{2} that the Seiberg-Witten geometry is realized as the thermodynamic limit or the semi-classical approximation of the statistical models.

In \cite{5} the authors consider the statistical model of random plane partitions relevant to describe five-dimensional $\mathcal{N} = 1$ supersymmetric $SU(N)$ Yang-Mills theory on $\mathbb{R}^4 \times S^1$ (more precisely plus the Chern-Simons term). This model has a smooth four-dimensional limit and gives rise to the exact partition function for four-dimensional $\mathcal{N} = 2$ supersymmetric Yang-Mills.

In this article we further develop the previous study. Random plane partitions are known \cite{6} to be described by two-dimensional free fermions ($2d$ conformal field theory). We compute the one-point functions of fermions in the statistical model and clarify their thermodynamic limits. We start Section 2 with a brief review of the model. The interpretation as $q$-deformed random partitions is emphasized. In Section 3 we introduce one-point functions of fermions which are located at the main diagonal of the plane partition. They turn out to have a statistical interpretation. They are expressed as the statistical sums of suitable wave functions associated with partitions. Asymptotics of these wave functions, which are relevant to obtain the thermodynamic limit of the one-point functions, are computed.

As the $q$-deformed random partitions, the model can be specified by the Boltzmann weights attached to partitions rather than plane partitions. For a very large partition, the Boltzmann weight is measured by the energy function. In Section 4 and Appendix A, we compute the energy function in a way different from \cite{2}. Our computations are based on the regularized density of the Maya diagram. The comparison with the known description using the profile is presented in Appendix B.

The thermodynamic limits or the semi-classical approximations of the fermion one-point functions at the main diagonal are proved in Section 5 to have a universal form in the sense that it is irrespective of detail of the minimizers of the energy function. They are given by the WKB type wave functions. In particular, at the four-dimensional limit, these semi-classical
one-point functions are shown to be living on the Seiberg-Witten curve of the gauge theory. They are the WKB type wave functions whose classical action is precisely the integral of $dS_{s,w}$ on the curve, where $dS_{s,w}$ is the Seiberg-Witten differential.

The diagonal slices of plane partitions are labelled by the discretized time $m$. The main diagonal slice is at $m = 0$. The discretized time becomes a continuous time $t$ at the thermodynamic limit. It is identified with a coordinate of the limit shape \[6\] of random plane partition. In Section 6 we consider the $U(1)$ theory and compute the semi-classical wave functions of fermions which are located away from the main diagonal. We then argue the role of the parameter $t$ in gauge theories.

## 2 A model of random plane partitions

A plane partition $\pi$ is an array of non-negative integers

\[
\pi_{11} \quad \pi_{12} \quad \pi_{13} \quad \cdots \\
\pi_{21} \quad \pi_{22} \quad \pi_{23} \quad \cdots \\
\pi_{31} \quad \pi_{32} \quad \pi_{33} \quad \cdots \\
\vdots \quad \vdots \quad \vdots \quad \vdots
\]  

(2.1)

satisfying $\pi_{ij} \geq \pi_{i+1,j}$ and $\pi_{ij} \geq \pi_{ij+1}$ for all $i, j \geq 1$. Plane partitions are identified with the three-dimensional Young diagrams as depicted in Figure 1-(a). The three-dimensional diagram $\pi$ is a set of unit cubes such that $\pi_{ij}$ cubes are stacked vertically on each $(i, j)$-element of $\pi$. The size of $\pi$ is $|\pi| \equiv \sum_{i,j \geq 1} \pi_{ij}$, which is the total number of cubes of the diagram. Each diagonal slice of $\pi$ becomes a partition, that is, a sequence of weakly decreasing non-negative integers. See Figure 1-(b). Let $\pi(m)$ be a partition along the $m$-th diagonal slice.

\[
\pi(m) = \begin{cases} 
(\pi_{1 \ m+1}, \pi_{2 \ m+2}, \pi_{3 \ m+3}, \cdots) & \text{for } m \geq 0 \\
(\pi_{-m+1 1}, \pi_{-m+2 2}, \pi_{-m+3 3}, \cdots) & \text{for } m \leq -1.
\end{cases}
\]  

(2.2)

In particular $\pi(0) = (\pi_{11}, \pi_{22}, \pi_{33}, \cdots)$ will be called the main diagonal partition. The series of partitions $\pi(m)$ satisfies the condition

\[
\cdots \prec \pi(-2) \prec \pi(-1) \prec \pi(0) \succ \pi(1) \succ \pi(2) \succ \cdots,
\]  

(2.3)
where $\mu \succ \nu$ means the following interlace relation between two partitions $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq 0)$ and $\nu = (\nu_1 \geq \nu_2 \geq \cdots \geq 0)$

$$\mu \succ \nu \iff \mu_1 \geq \nu_1 \geq \mu_2 \geq \nu_2 \geq \mu_3 \geq \cdots. \quad (2.4)$$

We have $|\pi| = \sum_{m=\infty}^{+\infty} |\pi(m)|$, where the size of a partition $\mu$ is also denoted by $|\mu| \equiv \sum_{i \geq 1} \mu_i$.

A model of random plane partitions which describes five-dimensional $\mathcal{N} = 1$ supersymmetric $SU(N)$ gauge theory is discussed in [5]. The partition function of the model is given by

$$Z_q^{SU(N)} \equiv \sum_{\pi} q^{|\pi|} Q^{|\pi(0)|} e^{V(\pi(0))}. \quad (2.5)$$

The Boltzmann weight consists of three parts. The first contribution comes from the energy of a plane partition $\pi$. The second contribution is a chemical potential for the main diagonal partition $\pi(0)$, and the third comes from the $N$-periodic potential [2] for the main diagonal partition. Let $\xi_r \in \mathbb{R}$ ($1 \leq r \leq N$) be such that $\sum_{r=1}^{N} \xi_r = 0$, and define the $N$-periodic
potential \( v \) on \( \mathbb{Z} \) by \( v(k) = \xi_{k+1} \mod N \). The potential \( V \) for a partition \( \mu \) is formally given \(^1\) by
\[
V(\mu) = \sum_{i=1}^{+\infty} v(\mu_i - i). \tag{2.6}
\]

To contact with the \( SU(N) \) gauge theory we also need to identify the indeterminates \( q \) and \( Q \) with the following field theory variables.
\[
q = e^{-\frac{2\pi}{N} k}, \quad Q = (2R\Lambda)^2, \tag{2.7}
\]
where \( R \) is the radius of \( S^1 \) in the fifth dimension, and \( \Lambda \) is the lambda parameter of the underlying four-dimensional field theory.

**Transfer matrix approach**

The transfer matrix approach \(^2\) allows us to express the random plane partitions (2.5) in terms of two-dimensional conformal field theory (2d free fermion system).

It is well known that partitions are realized as states of 2\(d\) free fermions by using the Maya diagrams. Let \( \psi(z) = \sum_{k \in \mathbb{Z}} \psi_k z^{-k} \) and \( \psi^*(z) = \sum_{k \in \mathbb{Z}} \psi_k^* z^{-k} \) be complex fermions with the anti-commutation relations
\[
\{\psi_k, \psi_l\} = \delta_{k+l,0}, \quad \{\psi_k, \psi^*_l\} = \{\psi_k^*, \psi^*_l\} = 0. \tag{2.8}
\]

Let \( \mu = (\mu_1, \mu_2, \cdots) \) be a partition. The Maya diagram \( \mu \) is a series of the strictly decreasing integers \( x_i(\mu) \equiv -i + \mu_i \), where \( i \in \mathbb{Z}_{\geq 1} \). The correspondence with the Young diagram is depicted in Figure 2. By using the Maya diagram the partition can be mapped to the following fermion state
\[
|\mu; n\rangle = \psi_{-x_1(\mu)-1-n} \psi_{-x_2(\mu)-1-n} \cdots \psi_{-x_l(\mu)-1-n} \psi_{-l(\mu)+1+n}^* \psi_{-l(\mu)+2+n}^* \cdots \psi_n^* |\emptyset; n\rangle, \tag{2.9}
\]
where \( l(\mu) \) is the length of \( \mu \), that is, the number of the non-zero \( \mu_i \). In (2.9) the state \( |\emptyset; n\rangle \) is the ground state of the charge \( n \) sector. It is defined by the conditions
\[
\psi_k |\emptyset; n\rangle = 0 \quad \text{for} \quad k \geq -n,
\]
\[
\psi_k^* |\emptyset; n\rangle = 0 \quad \text{for} \quad k \geq n + 1. \tag{2.10}
\]

We mainly consider the \( n = 0 \) sector in the below.

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\(^1\)See Appendix A for more information.
Figure 2: The correspondence between the Maya diagram and the Young diagram of $\mu = (7, 5, 4, 2, 2, 1)$. Elements of the Maya diagram are denoted by •.

The basic ingredient of the transfer matrix approach is the following evolution operator at a discretized time $m \in \mathbb{Z}$.

$$
\Gamma(m) \equiv \begin{cases} 
\exp\left(\sum_{k=1}^{+\infty} \frac{1}{k} q^{k(m+\frac{1}{2})} J_{-k}\right) & \text{for } m \geq 0 \\
\exp\left(\sum_{k=1}^{-\infty} \frac{1}{k} q^{-k(m+\frac{1}{2})} J_k\right) & \text{for } m \leq -1,
\end{cases}
$$

(2.11)

where $J_{\pm k}$ are the modes of the standard $U(1)$ current

$$
: \psi \psi : (z) = \sum_n J_n z^{-n-1}.
$$

(2.12)

Implications of the above operators in random plane partitions can be understood from their matrix elements: For $m \geq 0$,

$$
\langle \mu; 0 | \Gamma(m) | \nu; 0 \rangle = \begin{cases} 
q^{(m+\frac{1}{2})(|\mu|-|\nu|)} & \mu \succ \nu \\
0 & \text{otherwise}, 
\end{cases}
$$

(2.13)

and for $m \leq -1$,

$$
\langle \mu; 0 | \Gamma(m) | \nu; 0 \rangle = \begin{cases} 
q^{(m+\frac{1}{2})(|\mu|-|\nu|)} & \mu \prec \nu \\
0 & \text{otherwise}, 
\end{cases}
$$

(2.14)
It follows from (2.13) and (2.14) that the partition function is expressed as

\[ Z^\text{SU}(N)_q = \langle \emptyset; 0 | \left\{ \prod_{m \leq -1} \Gamma(m) \right\} Q^{J_0} \exp \left( \sum_{r=1}^{N} \xi_r J_0^{(r)} \right) \left\{ \prod_{m \geq 0} \Gamma(m) \right\} | \emptyset; 0 \rangle. \quad (2.15) \]

Here, \( J_0^{(r)} \) are the zero modes of the \( U(1) \) currents of \( N \)-component fermions \( \psi^{(r)} \) and \( \psi^{(r)*} \) which are realized \([7]\) by folding \( \psi \) and \( \psi^* \).

**q-deformed random partitions**

We can interpret the random plane partitions (2.5) as a model of random partitions. It is identified with a \( q \)-deformation of the random partitions \([2]\). (The deformation itself differs from \([2]\).) To see this, we rewrite the partition function by using the Schur functions. An insertion of the unity \( 1 = \sum_\mu |\mu; 0 \rangle \langle \mu; 0| \) factorizes (2.15) into

\[ Z^\text{SU}(N)_q = \sum_\mu Q^{\mu} e^{V(\mu)} \langle \emptyset; 0 | \prod_{m \leq -1} \Gamma(m) | \mu; 0 \rangle \langle \emptyset; 0 | \prod_{m \geq 0} \Gamma(m) | \emptyset; 0 \rangle. \quad (2.16) \]

The matrix elements in the above turn to be

\[ \langle \emptyset; 0 | \prod_{m \leq -1} \Gamma(m) | \mu; 0 \rangle = \langle \emptyset; 0 | \prod_{k=1}^{+\infty} \exp \left( \frac{1}{k} \sum_{i=1}^{+\infty} q^{k(i-\frac{1}{2})} J_k \right) | \mu; 0 \rangle \]

\[ = s_\mu(q^{\frac{1}{2}}, q^2, \cdots), \quad (2.17) \]

\[ \langle \mu; 0 | \prod_{m \geq 0} \Gamma(m) | \emptyset; 0 \rangle = \langle \mu; 0 | \prod_{k=1}^{+\infty} \exp \left( \frac{1}{k} \sum_{i=1}^{+\infty} q^{k(i-\frac{1}{2})} J_{-k} \right) | \emptyset; 0 \rangle \]

\[ = s_\mu(q^{\frac{1}{2}}, q^2, \cdots), \quad (2.18) \]

where \( s_\mu(q^{\frac{1}{2}}, q^2, \cdots) \) is the Schur function \( s_\mu(x_1, x_2, \cdots) \) specialized at \( x_i = q^{i-\frac{1}{2}} \) \( (i \geq 1) \). Therefore we obtain

\[ Z^\text{SU}(N)_q = \sum_\mu Q^{\mu} e^{V(\mu)} s_\mu(q^{-\rho})^2, \quad (2.19) \]

where the multiple index \( \rho \equiv (-\frac{1}{2}, -\frac{3}{2}, \cdots, -i + \frac{1}{2}, \cdots) \) is used. The expression (2.13) allows us to interpret (2.5) as a model of \( q \)-deformed random partitions. It is also clear from (2.16) that partitions \( \mu \) are the main diagonal partitions \( \pi(0) \).
The four-dimensional limit of the model is obtained by letting \( R \to 0 \) under the identification \((2.7)\). To take the limit the following product formula of the Schur function \([8]\) becomes useful.

\[
\begin{align*}
  s_{\mu}(q^{-\rho}) &= q^{n(\mu) + \frac{1}{2}|\mu|} \prod_{(i,j) \in \mu} \frac{1}{(1 - q^{h(i,j)})}, \quad (2.20)
\end{align*}
\]

where \( h(i, j) \) is the hook length of the box \((i, j)\) in the Young diagram, and \( n(\mu) \equiv \sum_{i \geq 1} (i - 1)\mu_i \).

By using \((2.20)\) we can see from \((2.19)\) that

\[
\lim_{R \to 0} Z^{SU(N)}_{q} = \sum_{\mu} \left( \frac{N \Lambda}{\hbar} \right)^{2|\mu|} e^{V(\mu)} \left( \frac{dim(\mu)}{|\mu|!} \right)^{2}. \quad (2.21)
\]

This is the model of random partitions \([2]\) which describes four-dimensional \( \mathcal{N} = 2 \) supersymmetric \( SU(N) \) Yang-Mills at the thermodynamic limit.

Thermodynamic limit of the model is achieved by letting \( \hbar \to 0 \). To see this, we consider the \( U(1) \) theory or the \( SU(N) \) theory with the periodic potential turned off. In such a situation the partition function can be computed by using the standard technique of 2d conformal field theory. It becomes

\[
Z^{U(1)}_{q} = \prod_{n=1}^{+\infty} (1 - Qq^n)^{-n}. \quad (2.22)
\]

The mean values of \( |\pi| \) and \( |\pi(0)| \) are respectively given by \( q \frac{\partial}{\partial q} \ln Z \) and \( Q \frac{\partial}{\partial Q} \ln Z \). It follows from \((2.22)\) that they behave \( \langle |\pi| \rangle = \mathcal{O}(\hbar^{-3}) \) and \( \langle |\pi(0)| \rangle = \mathcal{O}(\hbar^{-2}) \) as \( \hbar \to 0 \) (\( q \to 1 \)). Therefore a typical plane partition \( \pi \) near the limit \( \hbar \to 0 \) is a plane partition of order \( \hbar^{-3} \), and its main diagonal partition \( \pi(0) \) or \( \mu \) becomes a partition of order \( \hbar^{-2} \).
3 One-point functions at the main diagonal

In the present section and section 5 we investigate one-point functions of the free fermions located at the main diagonal of the plane partition $\pi$. From the transfer matrix viewpoint they are given by

$$\Psi^* \equiv \frac{1}{Z_{SU(N)}(\emptyset; -1)} \left\{ \prod_{m \leq -1} \Gamma(m) \right\} \psi^* \left( z \right) q^{L_0} Q e^{\sum_{r=1}^N \xi_r^J(r)} \left\{ \prod_{m \geq 0} \Gamma(m) \right\} |\emptyset; 0>,$$

(3.1)

$$\Psi \equiv \frac{1}{Z_{SU(N)}(\emptyset; +1)} \left\{ \prod_{m \leq -1} \Gamma(m) \right\} \psi \left( z \right) q^{-\frac{1}{2} L_0} Q e^{\sum_{r=1}^N \xi_r^J(r)} \left\{ \prod_{m \geq 0} \Gamma(m) \right\} |\emptyset; 0>,$$

(3.2)

where $q^{\frac{\pm L_0}{2}}$ are inserted for the later convenience of normalization.

The goal of our discussions is to clarify the thermodynamic limits of the above one-point functions. These are treated in section 5 by using the results of this section.

Statistical interpretation of fermion one-point functions

We regard the random plane partitions (2.5) as the $q$-deformed random partitions (2.19). The Boltzmann weight for a partition $\mu$ is given by $Q^{|\mu|} e^{V(\mu)} s_\mu(q^{-\rho})^2$. The above one-point functions admit to have a statistical interpretation. As we confirm subsequently, they can be expressed as the following statistical sums.

$$\Psi^* \left( z \right) = \frac{1}{Z_{SU(N)}^q(\emptyset)} \sum_\mu \chi^*_q \left( z \mid \mu \right) Q^{|\mu|} e^{V(\mu)} s_\mu(q^{-\rho})^2,$$

(3.3)

$$\Psi \left( z \right) = \frac{1}{Z_{SU(N)}^q(\emptyset)} \sum_\mu \chi_q \left( z \mid \mu \right) Q^{|\mu|} e^{V(\mu)} s_\mu(q^{-\rho})^2,$$

(3.4)

where $\chi^*_q(z \mid \mu)$ and $\chi_q(z \mid \mu)$ are thought as wave functions associated with each partition $\mu$. They are defined by

$$\chi^*_q(z \mid \mu) \equiv \sum_{i=1}^{+\infty} z^{-x_i(\mu)-1}(-1)^{i-1} q^{\frac{1}{2}|\mu|} \frac{s(\mu; i)(q^{-\rho})}{s_\mu(q^{-\rho})},$$

(3.5)

$$\chi_q(z \mid \mu) \equiv \sum_{i=1}^{+\infty} z^{-x_i(\mu)-1}(-1)^{i-1} q^{\frac{1}{2}|\mu|} \frac{s(\mu; i)(q^{\rho})}{s_\mu(q^{\rho})}.$$

(3.6)
Here \((\mu : i)\) is a partition obtained from \(\mu\) for each \(i \in \mathbb{Z}_{\geq 1}\). It is described in terms of the Maya diagram as follows.

\[
x_k(\mu : i) = \begin{cases} 
  x_k(\mu) + 1 & \text{for } 1 \leq k < i \\
  x_{k+1}(\mu) + 1 & \text{for } k \geq i.
\end{cases}
\]  

(3.7)

We can see from the above that \(|(\mu : i)| = |\mu| - x_i(\mu) - 1\). The partition conjugate to \(\mu\) is denoted by \(\tilde{\mu}\) in (3.6). The corresponding Young diagram is obtained by flipping the Young diagram \(\mu\) over its main diagonal. One can find a relation between \(\chi_q(z \mid \mu)\) and \(\chi_q^*(z \mid \mu)\). By comparing (3.5) with (3.6) we obtain

\[
\chi_q(z \mid \mu) = \chi_q^*(z \mid \tilde{\mu}).
\]  

(3.8)

Let us verify the descriptions (3.3) and (3.4). We first consider the one-point function \(\Psi^*_q(z)\). By the insertion of the unity \(1 = \sum_\mu |\mu; 0\rangle \langle \mu; 0|\) the following factorization of (3.1) is obtained.

\[
\frac{1}{Z_{SU(N)}^{q \lambda}} \sum_\mu Q^{|\mu|} e^{V(\mu)} q^{\frac{1}{2}|\mu|} \langle \theta; -1| \prod_{m \leq -1} \Gamma(m) \right \} \psi^*(z) |\mu; 0\rangle \langle \mu; 0| \prod_{m \geq 0} \Gamma(m)|\theta; 0\rangle.
\]  

(3.9)

This can be written down by using the Schur functions. The second component is just (2.18). In order to express the first component in terms of the Schur functions, it is convenient to describe the action of \(\psi^*(z)\) on the partition \(\mu\) explicitly. By using (2.9) we can see that the actions of \(\psi^*_k\) on the partition become

\[
\psi^*_k |\mu; 0\rangle = \begin{cases} 
  (-)^{i-1}|(\mu : i); -1\rangle & \text{if } k = x_i(\mu) + 1 \\
  0 & \text{otherwise},
\end{cases}
\]  

(3.10)

where \((\mu : i)\) is the partition given in (3.7). The above actions are translated to

\[
\psi^*(z) |\mu; 0\rangle = \sum_{i=1}^{+\infty} z^{-x_i(\mu)-1}(-)^{i-1}|(\mu : i); -1\rangle.
\]  

(3.11)

By using (3.11) and also taking (2.17) into account, the first component of (3.9) becomes

\[
\langle \theta; -1| \prod_{m \leq -1} \Gamma(m) \right \} \psi^*(z) |\mu; 0\rangle = \sum_{i=1}^{+\infty} z^{-x_i(\mu)-1}(-)^{i-1} s_{(\mu : i)}(q^{-\rho}).
\]  

(3.12)
The description (3.3) is obtained from (3.1) by plugging (3.12) and (2.18) into (3.9) and then arranging them noting the definition (3.5).

Nextly we consider the case of $\Psi_q(z)$. The description (3.4) can be obtained from (3.2) by taking the same steps as above. We first factorize (3.2) into

$$
\frac{1}{Z_q^{SU(N)}} \sum_{\mu} Q_{\mu} e^{V(\mu)} q^{\frac{|\mu|}{2}} \langle \emptyset; +1 | \prod_{m \leq -1} \Gamma(m) \psi(z) | \mu; 0 \rangle \langle \mu; 0 | \prod_{m \geq 0} \Gamma(m) | \emptyset; 0 \rangle. \tag{3.13}
$$

To write down the first component in terms of the Schur functions we compute the action of $\psi(z)$ on the partition $\mu$. The actions of $\psi_k$ on the partition can be read as

$$
\psi_k | \mu; 0 \rangle = \begin{cases} (-)^{i+x_i(\mu)} | (\mu : i); +1 \rangle & \text{if } k = x_i(\mu) \\ 0 & \text{otherwise}. \end{cases} \tag{3.14}
$$

These are translated to

$$
\psi(z) | \mu; 0 \rangle = \sum_{i=1}^{+\infty} z^{-x_i(\mu)-1} (-)^{i+x_i(\mu)} s_{\mu} (q^{-\rho}) \tag{3.15}
$$

Therefore the first component of (3.13) becomes

$$
\langle \emptyset; +1 | \prod_{m \leq -1} \Gamma(m) \psi(z) | \mu; 0 \rangle = \sum_{i=1}^{+\infty} z^{-x_i(\mu)-1} (-)^{i+x_i(\mu)} s_{\mu} (q^{-\rho}). \tag{3.16}
$$

The description (3.4) is obtained from (3.2) by plugging (3.16) and (2.18) into (3.13) and making use of the identity $s_{\mu} (q^{-\rho}) = (-)^{|\mu|} s_\mu (q^\rho)$.

**Large $|\mu|$ behaviors of wave functions associated with partitions**

Near the thermodynamic limit, which is achieved by letting $\hbar \to 0$ in the model, very large partitions dominate. Their size becomes of order $\hbar^{-2}$. We will clarify the asymptotics of $\chi_q^\ast (z | \mu)$ and $\chi_q (z | \mu)$ when $\mu$ is such a large partition. The Maya diagram of such a partition can be thought as a quantity of order $\hbar^{-1}$. For the description we conveniently rescale $x_i(\mu)$.

Let us introduce the order $\hbar^0$ quantities $u$ and $s$ by the following rescalings.

$$
x = \frac{N}{\hbar} u, \quad i = \frac{N}{\hbar} s, \tag{3.17}
$$

where $u \in \mathbb{R}$ and $s \in \mathbb{R}_{\geq 0}$. Correspondingly, the Maya diagram is scaled to a function $u(s | \mu)$ as

$$
x_i(\mu) = \frac{N}{\hbar} u(s | \mu) + O(\hbar^0). \tag{3.18}
$$
Notice that the Maya diagram is a strictly decreasing series satisfying the conditions that 
\( x_i(\mu) \geq -i \) for \( \forall i \in \mathbb{Z}_{\geq 1} \) and that \( x_i(\mu) = -i \) for sufficiently large \( i \). Therefore, the function 
\( u(s \mid \mu) \) is monotonically decreasing, and satisfies the conditions that \( u(s \mid \mu) \geq -s \) for \( \forall s \in \mathbb{R}_{\geq 0} \) and that \( u(s \mid \mu) \to -s \) as \( s \to +\infty \).

The large \( |\mu| \) behaviors of the wave functions can be computed from (3.5) and (3.6) by scaling \( \mu \) as (3.18). As the result, they turn out to be

\[
\chi_q^*(z \mid \mu) \approx \frac{N}{\hbar} \int_0^{+\infty} ds \ z^{-\frac{\mu}{\hbar}} u(s|\mu) \\
\times \exp \left\{ -\frac{RN}{2\hbar} u(s \mid \mu)^2 + \frac{N}{\hbar} \int_{-\infty}^{+\infty} dv \ \frac{ds(v \mid \mu)}{dv} \ln \left( \frac{\sinh R(u(s \mid \mu) - v)}{RA} \right) \right\},
\]

\[
(3.19)
\]

\[
\chi_q(z \mid \mu) \approx \frac{N}{\hbar} \int_0^{+\infty} ds \ z^{-\frac{\mu}{\hbar}} u(s|\bar{\mu}) \\
\times \exp \left\{ \frac{RN}{2\hbar} u(s \mid \bar{\mu})^2 - \frac{N}{\hbar} \int_{-\infty}^{+\infty} dv \ \frac{ds(v \mid \mu)}{dv} \ln \left( \frac{\sinh R(-u(s \mid \bar{\mu}) - v)}{RA} \right) \right\}.
\]

\[
(3.20)
\]

In the above we have used the density function \( \frac{d}{du} s(u \mid \mu) \). The function \( s(u \mid \mu) \) is the inverse of \( u(s \mid \mu) \), that is, \( u(s(v \mid \mu) \mid \mu) = v \). It becomes non-negative and weakly decreasing over \( \mathbb{R} \), satisfying the condition that \( s(u \mid \mu) \geq -u \). The asymptotic behaviors are \( s(u \mid \mu) \to -u \) as \( u \to -\infty \) and \( s(u \mid \mu) \to 0 \) as \( u \to +\infty \). Therefore, the density function \( \frac{d}{du} s(u \mid \mu) \) takes values in \([-1, 0]\), and satisfies the asymptotic conditions that \( \frac{d}{du} s(u \mid \mu) \to -1 \) as \( u \to -\infty \) and that \( \frac{d}{du} s(u \mid \mu) \to 0 \) as \( u \to +\infty \). We should note the density function in the proper sense \(^2\) is \(-\frac{d}{du} s(u \mid \mu)\) rather than \( \frac{d}{du} s(u \mid \mu) \). However, in this article, \( \frac{d}{du} s(u \mid \mu) \) is called the density function as well, unless it causes any confusion.

Before confirming the asymptotics (3.19) and (3.20), we shall describe their four-dimensional limits. The limits are obtained from (3.19) and (3.20) by letting \( R \to 0 \). They become

\[
\chi_{4d}^*(z \mid \mu) \equiv \lim_{R \to 0} \chi_q^*(z \mid \mu) \\
\approx \frac{N}{\hbar} \int_0^{+\infty} ds \ z^{-\frac{\mu}{\hbar}} u(s|\mu) \exp \left\{ \frac{N}{\hbar} \int_{-\infty}^{+\infty} dv \ \frac{ds(v \mid \mu)}{dv} \ln \left( \frac{u(s \mid \mu) - v}{\Lambda} \right) \right\},
\]

\[
(3.21)
\]

\(^2\)See (A.7) in Appendix A.
\[
\chi_{4d}(z \mid \mu) \equiv \lim_{R \to 0} \chi_q(z \mid \mu) \\
\approx \frac{N}{\hbar} \int_0^{+\infty} ds \, z^{-\frac{\omega}{\Lambda}(\bar{s} \bar{\mu})} \exp \left\{ \frac{N}{\hbar} \int_{-\infty}^{+\infty} dv \, ds(v \mid \mu) \ln \left( -u(s \mid \bar{\mu}) - v \right) \right\}. \\
(3.22)
\]

Let us derive the asymptotics \((3.19)\) and \((3.20)\). Consider the case of \(\chi_q^*(z \mid \mu)\). The following product formula of the (specialized) Schur function \([8]\) becomes useful to obtain the asymptotics.

\[
s_\mu(q^{-\rho}) = q^{-\frac{1}{2}\kappa(\mu)} \prod_{1 \leq i < j < +\infty} \left\{ \frac{q^{\frac{1}{2}(x_j(\mu) - x_i(\mu))} - q^{\frac{1}{2}(x_i(\mu) - x_j(\mu))}}{q^{\frac{1}{2}(j-i)} - q^{\frac{1}{2}(j-i)}} \right\},
\]

where \(\kappa(\mu) \equiv 2 \sum_{(i,j) \in \mu} (j - i).

We translate the Schur functions in \((3.5)\) into the infinite products by using the above formula. \(x_k(\mu : i)\) which appear in the product representation of \(s_{(\mu : i)}(q^{-\rho})\) can be converted to \(x_k(\mu)\) by using \((3.7)\). It turns out that almost all the products cancel between \(s_\mu(q^{-\rho})\) and \(s_{(\mu : i)}(q^{-\rho})\). Finally the ratio of the Schur functions in \((3.5)\) becomes as follows;

\[
\frac{s_{(\mu : i)}(q^{-\rho})}{s_\mu(q^{-\rho})} = q^{-\frac{1}{2}|\mu| + \frac{1}{2}(x_i(\mu) + 1)(x_i(\mu) + 2)} \prod_{1 \leq j \leq i - 1} \left( q^{\frac{1}{2}(j-i)} - q^{\frac{1}{2}(i-j)} \right)^{-1}
\]

\[
\times \prod_{j \neq i}^{+\infty} \left\{ \frac{q^{\frac{1}{2}(x_j(\mu) - x_i(\mu))} - q^{\frac{1}{2}(x_i(\mu) - x_j(\mu))}}{q^{\frac{1}{2}(j-i)} - q^{\frac{1}{2}(j-i)}} \right\}^{-1}. \hspace{1cm} (3.24)
\]

This gives the following expression of \(\chi_q^*(z \mid \mu)\);

\[
\chi_q^*(z \mid \mu) = \sum_{i = 1}^{+\infty} z^{-x_i(\mu) - 1} q^{\frac{1}{2}(x_i(\mu) + 1)(x_i(\mu) + 2)} \exp \left\{ \Phi_q(i \mid \mu) \right\}, \hspace{1cm} (3.25)
\]

where \(\Phi_q(i \mid \mu)\) is defined by

\[
\exp \left\{ \Phi_q(i \mid \mu) \right\} \equiv \prod_{j = 1}^{i-1} \left( q^{\frac{1}{2}(i-j)} - q^{\frac{1}{2}(j-i)} \right)^{-1} \prod_{j \neq i} \left\{ \frac{q^{\frac{1}{2}(x_j(\mu) - x_i(\mu))} - q^{\frac{1}{2}(x_i(\mu) - x_j(\mu))}}{q^{\frac{1}{2}(j-i)} - q^{\frac{1}{2}(j-i)}} \right\}^{-1}. \hspace{1cm} (3.26)
\]

The similar expression of \(\chi_q(z \mid \mu)\) is obtainable from \((3.25)\) by using \((3.8)\).

The large \(|\mu|\) behavior of \(\chi_q^*(z \mid \mu)\) can be extracted from \((3.25)\) by scaling \(x_i(\mu)\) as prescribed in \((3.18)\). Taking account of \(q = e^{-2\hbar/N}\) we first observe

\[
q^{\frac{1}{2}(x_i(\mu) + 1)(x_i(\mu) + 2)} = \exp \left\{ -\frac{RN}{2\hbar} u(s \mid \mu)^2 + O(h^0) \right\}, \hspace{1cm} (3.27)
\]

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where we put \( i = \frac{N}{N} \). The large \( |\mu| \) behavior of \( \Phi_q(i \mid \mu) \) can be computed as follows;

\[
\Phi_q(i \mid \mu) = \sum_{j=i+1}^{+\infty} \ln \left( q^{\frac{1}{2}(j-i)} - q^{\frac{1}{2}(j-i)} \right) - \sum_{j \neq i}^{+\infty} \ln \left( q^{\frac{1}{2}(x_j(\mu)-x_i(\mu))} - q^{\frac{1}{2}(x_i(\mu)-x_j(\mu))} \right)
\]

\[
= \frac{N}{\hbar} \int_0^{+\infty} dx \ln \left( e^{Rx} - e^{-Rx} \right) - \frac{N}{\hbar} \int_0^{+\infty} ds \ln \left( e^{R(u(s) \mid \mu) - u(s \mid \mu)} - e^{R(u(s) \mid \mu) - u(\tilde{u} \mid \mu)} \right) + O(h^0)
\]

\[
= \frac{N}{\hbar} \int_0^{+\infty} dx \ln \left( \frac{\sinh Rx}{R \Lambda} \right) - \frac{N}{\hbar} \int_0^{+\infty} ds \ln \left( \frac{\sinh R(u(s) \mid \mu) - u(s \mid \mu))}{R \Lambda} \right) + O(h^0),
\]

(3.28)

where the last expression enables us to make contact with the four-dimensional theory. By plugging (3.27) and (3.28) into (3.25), the large \( |\mu| \) behavior of \( \chi_q(z \mid \mu) \) can be read as

\[
\chi_q(z \mid \mu) \approx C \frac{N}{\hbar} \int_0^{+\infty} ds z^{-\frac{N}{N} u(s \mid \mu)} - u(s \mid \mu)) \exp \left\{ -\frac{RN}{2\hbar} u(s \mid \mu)^2 + \frac{N}{\hbar} \int_{-\infty}^{+\infty} dv ds(v \mid \mu) \ln \left( \frac{\sinh R(u(s) \mid \mu) - v)}{R \Lambda} \right) \right\},
\]

(3.29)

where \( C \equiv \exp \left\{ \frac{N}{\hbar} \int_0^{+\infty} dx \ln \left( \frac{\sinh Rx}{R \Lambda} \right) \right\} \). The constant \( C \) will be dropped out by absorbing it into \( \psi^*(z) \) as a wave function renormalization. The density function appears in the above by the change of variable, \( v = u(s \mid \mu) \). Thus we obtain the asymptotics (3.19).

The large \( |\mu| \) behavior of \( \chi_q(z \mid \mu) \) can be computed in the similar manner as above by noting the relation (3.8). It becomes

\[
\chi_q(z \mid \mu) \approx C^{-1} \frac{N}{\hbar} \int_0^{+\infty} ds z^{-\frac{N}{N} u(s \mid \mu)} - u(s \mid \mu)) \exp \left\{ \frac{RN}{2\hbar} u(s \mid \mu)^2 - \frac{N}{\hbar} \int_{-\infty}^{+\infty} dv ds(v \mid \mu) \ln \left( \frac{\sinh R(u(s) \mid \mu) - v)}{R \Lambda} \right) \right\}.
\]

(3.30)

The constant \( C^{-1} \) will be dropped out. In the above, the density function of \( \mu \) has been used instead of \( \tilde{\mu} \). These two are related by

\[
\frac{ds}{du}(u \mid \tilde{\mu}) = -1 + \frac{ds}{du}(-u \mid \mu).
\]

(3.31)
4 Energy functions of partition

The Boltzmann weight for a partition \( \mu \) is \( \mathcal{Q}^{[\mu]} e^{V(\mu)} s_{\mu}(q^{-\rho})^2 \). The Boltzmann weight for a very large partition can be measured by the so-called energy function. The technique used in the previous section is also relevant to obtain the energy function.

Let \( \mu \) be a partition of order \( \hbar^{-2} \). We rescale the Maya diagram \( x_i(\mu) \) to \( u(s | \mu) \) by (3.18).

The asymptotics of the Boltzmann weight follows from the large \( |\mu| \) behaviors of \( s_{\mu}(q^{-\rho}) \) and the \( N \)-periodic potential \( V(\mu) \). The large \( |\mu| \) behavior of \( s_{\mu}(q^{-\rho}) \) can be obtained by using (3.23). The computations analogous to the previous section give rise to

\[
\ln s_{\mu}(q^{-\rho})^2 = \frac{N^2}{\hbar^2} \int_0^{+\infty} dx \ln \left( \frac{\sinh Rx}{R\Lambda} \right)^2 - \frac{N^2}{\hbar^2} \int_{-\infty}^{+\infty} du \left( 1 + 2 \frac{ds(u | \mu)}{du} \right) \frac{R u^2}{2} \\
+ \frac{N^2}{\hbar^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} du dv \left( 1 + \frac{ds(u | \mu)}{du} \right) \frac{ds(v | \mu)}{dv} \ln \left( \frac{\sinh R(u - v)}{R\Lambda} \right)^2 \\
+ O(\hbar^{-1}). \tag{4.1}
\]

In order to make the asymptotics of the \( N \)-periodic potential be of order \( \hbar^{-2} \) (the same order as (4.1)), we should regard \( \xi_r \) in the potential as the order \( \hbar^{-1} \) quantities and rescale them as well. Let \( \zeta_r \) be the order \( \hbar^0 \) quantities defined by the rescalings

\[
\xi_r = \frac{\zeta_r}{\hbar}, \tag{4.2}
\]

where \( \sum_{r=1}^{N} \zeta_r = 0 \). In Appendix A, we compute the asymptotics of the \( N \)-periodic potential in a way different from [2]. Our computations are based on the regularized density. To carry out these computations we need to make the following assumption; the density functions are non-decreasing for the partitions dominating near the thermodynamic limit. As the result, we obtain

\[
V(\mu) = \frac{N}{\hbar^2} \sum_{r=1}^{N} \zeta_r \int_{u_r}^{u_{r-1}} du \, u \frac{d^2 s(u | \mu)}{du^2} + O(\hbar^{-1}), \tag{4.3}
\]

where \( u_r \) (\( 0 \leq r \leq N \)) are defined by the conditions

\[
\left. \frac{d}{du} s(u | \mu) \right|_{u=u_r} = -\frac{r}{N}. \tag{4.4}
\]
Our assumption ensures that $u_r$ exist and are ordered $u_r < u_{r-1}$ with $u_{0,N} = \pm \infty$.

By using (4.1) and (4.3) we can see that the Boltzmann weight asymptotes to

$$Q^{|\mu|} e^{V(\mu)} s_\mu(q^{-\rho})^2 \approx C' Q^{|\mu|} \exp \left\{-\frac{N^2}{\hbar^2} E[s(\cdot | \mu)] \right\},$$

(4.5)

where $E[s(\cdot | \mu)]$ is the energy function of $\mu$ defined by

$$E[s(\cdot | \mu)] \equiv \int_{-\infty}^{+\infty} du \left(1 + 2 \frac{ds(u | \mu)}{du} \right) \frac{R u^2}{2}$$

$$- \int \int_{-\infty < u < v < +\infty} dudv \left(1 + \frac{ds(u | \mu)}{du} \right) \frac{ds(v | \mu)}{dv} \ln \left(\frac{\sinh R(u-v)}{R\Lambda} \right)^2$$

$$- \frac{1}{N} \sum_{r=1}^{N} \zeta_r \int_{u_r}^{u_{r-1}} du \frac{d^2 s(u | \mu)}{du^2},$$

(4.6)

and $C' \equiv \exp \left\{ \frac{N^2}{\hbar^2} \int_0^{+\infty} dx \ln \left(\frac{\sinh R x}{R\Lambda} \right)^2 \right\}$. The constant $C'$ will be dropped out. At the four-dimensional limit the above energy function becomes

$$E_{4d}[s(\cdot | \mu)] \equiv \lim_{R \to 0} E[s(\cdot | \mu)]$$

$$= - \int \int_{-\infty < u < v < +\infty} dudv \left(1 + \frac{ds(u | \mu)}{du} \right) \frac{ds(v | \mu)}{dv} \ln \left(\frac{u-v}{\Lambda} \right)^2$$

$$- \frac{1}{N} \sum_{r=1}^{N} \zeta_r \int_{u_r}^{u_{r-1}} du \frac{d^2 s(u | \mu)}{du^2}.$$  

(4.7)

The energy functions (4.6) and (4.7) are the functionals of $s(u | \mu)$. It is also possible to express them by using the so-called profile of a partition $\mu$. In Appendix B, we present the descriptions in terms of the profile and thereby compare (4.6) and (4.7) with the energy functions used in [2]. The energy function (4.6) turns out to be different from that used [2] for a description of five-dimensional supersymmetric $SU(N)$ Yang-Mills. The difference originates in the existence of the Chern-Simons term of five-dimensional gauge theories. It is shown [5] that the random plane partitions (2.5) describe five-dimensional supersymmetric $SU(N)$ Yang-Mills

\[ \text{3The statistical model considered in this paper is related [9] to topological string on a non-compact toric Calabi-Yau threefold [10]. The precise correspondence is described in [5].} \]
with the Chern-Simons term at the thermodynamic limit. The Chern-Simons correction appears in (4.6) as the $u^2$ potential term. This term vanishes at the four-dimensional limit. Therefore the energy function (4.7) coincides with that used [2] for the description of four-dimensional $\mathcal{N} = 2$ supersymmetric $SU(N)$ Yang-Mills.

5 Semi-classical wave function and Seiberg-Witten differential

In this section we investigate the thermodynamic limits of the one-point functions (3.1) and (3.2). We use the asymptotics obtained in the previous sections. Near the thermodynamic limit (the $\hbar \to 0$ limit), partitions of order $\hbar^{-2}$ dominate in the statistical model. In order to describe the one-point functions at the thermodynamic limit, taking account of the expressions (3.3) and (3.4), we need to find out the classical configurations from among these dominant partitions. As described in (4.5) their Boltzmann weights are measured by the energy function $E [s(\cdot | \mu)]$. This means that the classical configurations are the minimizers of $E [s(\cdot)]$. Let us suppose $s^\star (u)$ be such a minimizer.

The thermodynamic limits or the semi-classical approximations of the fermion one-point functions prove to have a universal form in the sense that it is irrespective of detail of the minimizers. They turn out to be given by the following WKB type wave functions.

$$
\Psi^\star_q (z) \approx \exp \left\{ -\frac{N}{\hbar} \int^z u(z) \, d \ln z \right\},
$$

$$
\Psi_q (z) \approx \exp \left\{ \frac{N}{\hbar} \int^z u(z) \, d \ln z \right\}.
$$

In the above, the function $u(z)$ is determined by the equation

$$
\frac{dS(u)}{du} = N \ln z,
$$

where

$$
S(u) \equiv -\frac{RN}{2} u^2 + N \int_{-\infty}^{+\infty} dv \, \frac{ds^\star (v)}{dv} \ln \left( \frac{\sinh R(u - v)}{R \Lambda} \right).
$$
Let us confirm (5.1) and (5.2). By following the mode expansions of the complex fermions, we first write the fermion one-point functions as
\[
\Psi^\ast_q(z) = \sum_{j=-\infty}^{+\infty} \hat{\Psi}^\ast_q(j) z^{-j}, \quad \Psi_q(j) = \sum_{j=-\infty}^{+\infty} \hat{\Psi}_q(j) z^{j-1},
\]
where \(\hat{\Psi}^\ast_q(j)\) and \(\hat{\Psi}_q(j)\) are respectively the one-point functions of \(\psi_j^\ast\) and \(\psi_{-j}\). In other words they are given by the contour integrals
\[
\hat{\Psi}^\ast_q(j) = \oint \frac{dz}{2\pi i} z^{-j-1} \Psi^\ast_q(z), \quad \hat{\Psi}_q(j) = \oint \frac{dz}{2\pi i} z^{-j} \Psi_q(z),
\]
for each \(j \in \mathbb{Z}\).

Tracing the representations (3.3) and (3.4) these wave functions can be expressed as the following statistical sums;
\[
\hat{\Psi}^\ast_q(j) = \frac{1}{Z_{SU(N)}^q} \sum_{\mu} Q^{\vert \mu \vert} e^{V(\mu)} s_{\mu}(q^{-\rho})^2 \hat{\psi}^\ast_q(j \mid \mu), \quad (5.7)
\]
\[
\hat{\Psi}_q(j) = \frac{1}{Z_{SU(N)}^q} \sum_{\mu} Q^{\vert \mu \vert} e^{V(\mu)} s_{\mu}(q^{-\rho})^2 \hat{\psi}_q(j \mid \mu), \quad (5.8)
\]
where \(\hat{\psi}^\ast_q(j \mid \mu)\) and \(\hat{\psi}_q(j \mid \mu)\) are obtained from \(\chi^\ast_q(z \mid \mu)\) and \(\chi_q(z \mid \mu)\) in (3.3) and (3.4) by
\[
\hat{\psi}^\ast_q(j \mid \mu) = \oint \frac{dz}{2\pi i} z^{j-1} \chi^\ast_q(z \mid \mu), \quad (5.9)
\]
\[
\hat{\psi}_q(j \mid \mu) = \oint \frac{dz}{2\pi i} z^{-j} \chi_q(z \mid \mu). \quad (5.10)
\]

The main contributions to \(\chi^\ast_q(z \mid \mu)\) and \(\chi_q(z \mid \mu)\) come from \(\hat{\psi}^\ast_q(j \mid \mu)\) and \(\hat{\psi}_q(j \mid \mu)\) with \(j \sim 1/\hbar\) when \(\mu\) is a partition of order \(\hbar^{-2}\). These are seen from (3.10) and (3.11). It follows that \(\psi_j^\ast\) generically subtracts \(j\) boxes from the Young diagram and \(\psi_{-j}\) adds \(j\) boxes to it, when they act on the partition. In particular, \(\hat{\psi}^\ast_q(j \mid \mu)\) with \(j \sim 1/\hbar\) subtracts the same number of boxes from the partition. It is also similar to \(\hat{\psi}_q(j \mid \mu)\). We rescale \(j\) in (5.9) and (5.10) to the order \(\hbar^0\) quantity \(u\) by
\[
j = \frac{N}{h} u. \quad (5.11)
\]

Let us first consider the \(\hbar\)-expansions of \(\hat{\psi}^\ast_q(j \mid \mu)\) and \(\hat{\psi}_q(j \mid \mu)\), where we regard \(j = O(1/\hbar)\) and \(|\mu| = O(\hbar^{-2})\). The semi-classical terms of the \(\hbar\)-expansions can be obtained by plugging
(3.19) and (3.20) respectively into (5.9) and (5.10) with putting \( j = Nu/h \). They turn out to be

\[
\hat{\psi}^*_q(j | \mu) \approx \exp \left\{ \frac{1}{\hbar} S[u; s(\cdot | \mu)] \right\}
\]

(5.12)

\[
\hat{\psi}_q(j | \mu) \approx \exp \left\{ \frac{-1}{\hbar} S[u; s(\cdot | \mu)] \right\}
\]

(5.13)

where

\[
S[u; s(\cdot | \mu)] = -\frac{RN}{2} u^2 + N \int_{-\infty}^{+\infty} dv \frac{ds(v | \mu)}{dv} \ln \left( \frac{\sinh R(u - v)}{RA} \right).
\]

(5.14)

The semi-classical approximations of \( \hat{\Psi}^*_q(j) \) and \( \hat{\Psi}_q(j) \) can be obtained as follows. Let \( s_*(u) \) be a minimizer of the energy function \( E[s(\cdot)] \). The representations (5.7) and (5.8) will be helpful. Taking account of these statistical representations, their semi-classical approximations are given by (5.12) and (5.13) evaluated at the minimizer \( s_*(u) \). Thus we obtain

\[
\hat{\Psi}^*_q(j) \approx \exp \left\{ \frac{1}{\hbar} S(u) \right\},
\]

(5.15)

\[
\hat{\Psi}_q(j) \approx \exp \left\{ \frac{-1}{\hbar} S(u) \right\},
\]

(5.16)

where the classical action \( S(u) \) is read from (5.12) and (5.13) as

\[
S(u) = S[u; s_*(\cdot)]
\]

(5.17)

and given by (5.4).

The semi-classical approximations or the thermodynamic limits of \( \Psi^*_q(z) \) and \( \Psi_q(z) \) can be obtained from (5.15) and (5.16). Taking account of the scaling (5.11) we can replace the expressions (5.5) with the following integrals at the thermodynamic limit.

\[
\Psi^*_q(z) \approx \frac{N}{\hbar} \int_{-\infty}^{+\infty} du \exp \left\{ \frac{1}{\hbar} (-u \ln z^N + S(u)) \right\},
\]

(5.18)

\[
\Psi_q(z) \approx \frac{N}{\hbar} \int_{-\infty}^{+\infty} du \exp \left\{ \frac{1}{\hbar} (u \ln z^N - S(u)) \right\}.
\]

(5.19)

Since \( \hbar \) is very small, we can evaluate the above integrations by applying the saddle point method. Critical points of the exponents become solutions of the equation (5.3). Let \( u(z) \) be the critical point. The critical value can be obtained as follows

\[
u(z) \ln z^N - S(u(z)) = \int_{u(z)}^{u(z)} d\left( \frac{dS(u)}{du} \right) - \int_{u(z)}^{u(z)} \frac{dS(u)}{du} \]
\[
\int_{u(z)} u d \left( \frac{dS(u)}{du} \right) = N \int u(z) d \ln z.
\] (5.20)

Therefore, the saddle point method gives rise to the semi-classical wave functions of the one-point functions at the main diagonal as described in (5.1) and (5.2).

**Semi-classical wave function and Seiberg-Witten differential**

Roles of the semi-classical wave functions (5.1) and (5.2) in the Seiberg-Witten geometry are not obvious. In order to get some insight we look at the four-dimensional case. Let us first supplement the previous discussions by providing the four-dimensional limits of the relevant quantities. The four-dimensional limits of (5.12) and (5.13) become

\[
\hat{\psi}_{4d}^*(j | \mu) \equiv \lim_{R \to 0} \hat{\psi}_q^*(j | \mu) \approx \exp \left\{ \frac{1}{\hbar} S_{4d}[u; s(\cdot | \mu)] \right\},
\] (5.21)

\[
\hat{\psi}_{4d}(j | \mu) \equiv \lim_{R \to 0} \hat{\psi}_q^*(j | \mu) \approx \exp \left\{ -\frac{1}{\hbar} S_{4d}[u; s(\cdot | \mu)] \right\},
\] (5.22)

where

\[
S_{4d}[u; s(\cdot | \mu)] = \int_{-\infty}^{+\infty} dv ds(v | \mu) \ln \left( \frac{u - v}{\Lambda} \right).
\] (5.23)

Thus the minimizer \( s_*(u) \) of \( E_{4d}[s(\cdot)] \) leads to the following semi-classical wave functions.

\[
\hat{\Psi}_{4d}(j) \equiv \lim_{R \to 0} \hat{\Psi}_q(j) \approx \exp \left\{ \frac{1}{\hbar} S_{4d}(u) \right\},
\] (5.24)

\[
\hat{\Psi}_{4d}(j) \equiv \lim_{R \to 0} \hat{\Psi}_q(j) \approx \exp \left\{ -\frac{1}{\hbar} S_{4d}(u) \right\},
\] (5.25)

where

\[
S_{4d}(u) = S_{4d}[u; s_*(\cdot)].
\] (5.26)

We use the solution obtained by Nekrasov and Okounkov [2]. Let \( C_N \) be the spectral curve of the \( N \)-periodic Toda chain

\[
h + \frac{1}{\hbar} = \frac{P(x)}{\Lambda^N}.
\] (5.27)
where \( P(x) \) is the \( N \)-th order monic polynomial. The curve \( \mathcal{C}_N \) is identified with the Seiberg-Witten curve of \( \mathcal{N} = 2 \) supersymmetric \( SU(N) \) Yang-Mills. The solution of \( \mathcal{C}_N \) is expressed as

\[
\int_{-\infty}^{+\infty} dv \frac{d^2 s_4(v)}{dv^2} \frac{1}{x - v} = \frac{1}{N} \frac{d \ln h}{dx}.
\]

(5.28)

It follows from (5.28) that the differential \( dS_{4d}(x) \) equals to

\[
dS_{4d}(x) = dx \ln h.
\]

(5.29)

Therefore, the classical action (5.26) can be written as the following integral on \( \mathcal{C}_N \);

\[
S_{4d}(u) = \int u \, dx \ln h.
\]

(5.30)

The thermodynamic limits of the four-dimensional counterparts of \( \Psi^*_q(z) \) and \( \Psi_q(z) \) are described by

\[
\Psi^*_4(z) \equiv \lim_{R \to 0} \Psi^*_q(z) \approx \frac{N}{\hbar} \int_{-\infty}^{+\infty} du \exp \left\{ \frac{1}{\hbar} (-u \ln z^N + S_{4d}(u)) \right\},
\]

(5.31)

\[
\Psi_4(z) \equiv \lim_{R \to 0} \Psi_q(z) \approx \frac{N}{\hbar} \int_{-\infty}^{+\infty} du \exp \left\{ \frac{1}{\hbar} (u \ln z^N - S_{4d}(u)) \right\}.
\]

(5.32)

We can also apply the saddle point method to the above integrations. In the present case equation for the critical point becomes

\[
h = z^N.
\]

(5.33)

We then obtain the following semi-classical wave functions.

\[
\Psi^*_4(z) \approx \exp \left\{ -\frac{1}{\hbar} \int^z x \, d \ln h \right\},
\]

(5.34)

\[
\Psi_4(z) \approx \exp \left\{ \frac{1}{\hbar} \int^z x \, d \ln h \right\}.
\]

(5.35)

Recall that \( xd \ln h \) is nothing but the Seiberg-Witten differential \( dS_{s,w} \) for the \( SU(N) \) Yang-Mills. Therefore we have shown above that the semi-classical approximations or the thermodynamic limits of the fermion one-point functions are given by the WKB type wave functions whose classical action is the integral of \( dS_{s,w} \) on the curve \( \mathcal{C}_N \)

\[
\Psi^*_4(z) \approx \exp \left\{ -\frac{1}{\hbar} \int^z dS_{s,w} \right\},
\]

(5.36)
\[
\Psi_{4d}(z) \approx \exp \left\{ \frac{1}{\hbar} \int_{S_{\text{w}}} dz S_{\text{w}} \right\} .
\]

(5.37)

Meromorphic function \( h \) is the Floquet multiplier in the spectral analysis of the periodic Toda chain. Equation (5.33) shows that \( z \) is nothing but the spectral parameter of the associated linear problem. This implies a hidden relation between the above fermion wave functions and the Baker-Akhiezer functions of the Toda hierarchy. It is argued in [13] that the isospectral problem (the \( N \)-band solitons) is converted to an isomonodromy problem by imposing a homogeneity condition analogous to the renormalization group equation. The associated linear system turns to admit a WKB analysis (multiscale analysis of the isomonodromy problem), which gives the semi-classical wave functions (5.36) and (5.37).

6 One-point functions away from the main diagonal

\( \Gamma(m) \) in the transfer matrix is the hamiltonian operator at the discretized time \( m \). The discretized time becomes a continuous time \( t \) at the \( \hbar \to 0 \) limit. In fact, \( t \) is identified with a coordinate of the limit shape [6] of random plane partition. The limit shape is interpreted [9] as the mirror of the toric \( \mathbb{C}^3 \). Free fermions \( \psi^*(z) \) and \( \psi(z) \) have been proposed [14] as probe branes in the mirror. In this section we compute the semi-classical wave functions of the fermions located away from the main diagonal. We restrict to the \( U(1) \) case. In particular, we put \( q = e^{-2\hbar t} \). We compute these wave functions by applying the method developed in [6]. It turns out that \( t \) is identical to the scale parameter of the gauge theory.

One-point functions at the discretized time \( m_0 \in \mathbb{Z}_{\leq 0} \) are given by

\[
\Psi^*_q(z; m_0) \equiv \frac{1}{Z^U_q} \langle \emptyset; -1 | \left\{ \prod_{m=-\infty}^{m_0-1} \Gamma(m) \right\} \psi^*(z) \left\{ \prod_{m=m_0}^{-1} \Gamma(m) \right\} Q^{L_0} \left\{ \prod_{m=0}^{+\infty} \Gamma(m) \right\} |\emptyset; 0 \rangle ,
\]

(6.1)

\[
\Psi_q(z; m_0) \equiv \frac{1}{Z^U_q} \langle \emptyset; +1 | \left\{ \prod_{m=-\infty}^{m_0-1} \Gamma(m) \right\} \psi(z) \left\{ \prod_{m=m_0}^{-1} \Gamma(m) \right\} Q^{L_0} \left\{ \prod_{m=0}^{+\infty} \Gamma(m) \right\} |\emptyset; 0 \rangle .
\]

(6.2)

Note that one-point functions at a positive discretized time are defined in the same way as above, and that their computations become similar to those at the negative time time presented below.
The above one-point functions can be calculated exactly. Let us define the adjoint actions of $\Gamma(m)$ on the fermions as

$$ Ad\left(\Gamma(m)\right) \psi^*(z) = \Gamma(m) \psi^*(z) \Gamma(m)^{-1}, \quad Ad\left(\Gamma(m)\right) \psi(z) = \Gamma(m) \psi(z) \Gamma(m)^{-1}. \quad (6.3) $$

By using (6.3) we can evaluate (6.1) and (6.2) as follows;

$$ \Psi^*_{q}(z; m_0) = \langle \emptyset; -1 | Ad\left(\prod_{m=0}^{+\infty} \Gamma(m)^{-1}\right) Ad\left(\prod_{m=+\infty}^{m_0-1} \Gamma(m)\right) \psi^*(z) | \emptyset; 0 \rangle, $$

(6.4)

$$ \Psi_{q}(z; m_0) = \langle \emptyset; +1 | Ad\left(\prod_{m=0}^{+\infty} \Gamma(m)^{-1}\right) Ad\left(\prod_{m=+\infty}^{m_0-1} \Gamma(m)\right) \psi(z) | \emptyset; 0 \rangle. $$

(6.5)

The adjoint actions (6.3) are computed from (2.11). They become

$$ Ad\left(\Gamma(m)\right) \psi^*(z) = \begin{cases} (1 - q^{m+\frac{1}{2}}z^{-1}) \psi^*(z) & \text{for } m \geq 0 \\ (1 - q^{-m-\frac{1}{2}}z) \psi^*(z) & \text{for } m \leq -1, \end{cases} $$

(6.6)

$$ Ad\left(\Gamma(m)\right) \psi(z) = \begin{cases} (1 - q^{m+\frac{1}{2}}z^{-1})^{-1} \psi(z) & \text{for } m \geq 0 \\ (1 - q^{-m-\frac{1}{2}}z)^{-1} \psi(z) & \text{for } m \leq -1. \end{cases} $$

(6.7)

By plugging (6.6) and (6.7) into (6.4) and (6.5) we obtain

$$ \Psi^*_{q}(z; m_0) = \frac{(q^{-m_0+\frac{1}{2}}z; q)}{(q^2 Qz^{-1}; q)} \Psi^*_{q}(z; m_0), \quad \Psi_{q}(z; m_0) = \Psi^*_{q}(z; m_0)^{-1}, $$

(6.8)

where the infinite product $(a; q)_{\infty} \equiv \prod_{n=0}^{+\infty}(1 - aq^n)$ is used.

**Semi-classical wave functions**

Asymptotics of the higher modes of $\Psi^*_{q}(z; m_0)$ and $\Psi_{q}(z; m_0)$ can be studied as in accord with the previous discussion at the main diagonal. Let $j \in \mathbb{Z}$. We introduce

$$ \tilde{\Psi}^*_{q}(j; m_0) \equiv \oint \frac{dz}{2\pi i} z^{j-1} \Psi^*_{q}(z; m_0), $$

(6.9)

$$ \tilde{\Psi}_{q}(j; m_0) \equiv \oint \frac{dz}{2\pi i} z^{-j} \Psi_{q}(z; m_0). $$

(6.10)
To find out their asymptotics we need to rescale the integers \( j \) and \( m_0 \) appropriately. Guided by (5.11), we regard \( m_0 \) also of order \( \hbar^{-1} \) and introduce the order \( \hbar^0 \) quantities \( u \) and \( t \) by

\[
j = \frac{u}{\hbar}, \quad m_0 = \frac{t}{\hbar}, \tag{6.11}
\]

where \( u \in \mathbb{R} \) and \( t \in \mathbb{R}_{\leq 0} \). Let us consider the \( \hbar \)-expansions of \( \hat{\Psi}^*_q \left( \frac{u}{\hbar}; \frac{t}{\hbar} \right) \) and \( \hat{\Psi}_q \left( \frac{u}{\hbar}; \frac{t}{\hbar} \right) \) \( (q = e^{-2Rh}) \). They turn out to be written in the following form.

\[
\hat{\Psi}^*_q \left( \frac{u}{\hbar}; \frac{t}{\hbar} \right) = \oint \frac{dz}{2\pi i} \exp \left\{ \frac{1}{\hbar} S^{(0)}(z; (u, t)) + O(\hbar^0) \right\}, \tag{6.12}
\]

\[
\hat{\Psi}_q \left( \frac{u}{\hbar}; \frac{t}{\hbar} \right) = \oint \frac{dz}{2\pi i} \exp \left\{ -\frac{1}{\hbar} S^{(0)}(z; (u, t)) + O(\hbar^0) \right\}. \tag{6.13}
\]

The classical action \( S^{(0)}(z; (u, t)) \) in the above can be computed as follows. We first notice that the \( \hbar \)-expansion of \( \ln(a; q)_\infty \) reads as

\[
\ln(a; q)_\infty = \frac{1}{2Rh} \int_0^a dx \frac{\ln(1 - x)}{x} + O(\hbar^0). \tag{6.14}
\]

Then, (6.8) leads to

\[
\ln z^{j-1} \Psi^*_q(z; m_0)
\]

\[
= (j - 1) \ln z + \ln \left( q^{-m_0 + \frac{1}{2}z}; q \right)_\infty - \ln \left( q^{\frac{1}{2}Qz^{-1}}; q \right)_\infty
\]

\[
= \frac{1}{\hbar} \left\{ u \ln z + \frac{1}{2R} \int_0^z dx \frac{\ln(1 - e^{2Rt}x)}{x} - \frac{1}{2R} \int_0^{z^{-1}} dx \frac{\ln(1 - Qx)}{x} \right\} + O(\hbar^0). \tag{6.15}
\]

Therefore we obtain

\[
S^{(0)}(z; (u, t))
\]

\[
= u \ln z + \frac{1}{2R} \int_0^z dx \frac{\ln(1 - e^{2Rt}x)}{x} - \frac{1}{2R} \int_0^{z^{-1}} dx \frac{\ln(1 - Qx)}{x}. \tag{6.16}
\]

Since \( \hbar \) is very small, the saddle point method becomes applicable to the contour integrals in (6.12) and (6.13). The critical points of \( S^{(0)}(z; (u, t)) \) turn to be the solutions \( \alpha, \bar{\alpha} \) of the quadratic equation

\[
e^{2Rh} z^2 - (1 + e^{2Rt}Q - e^{-2Rt}) z + Q = 0. \tag{6.17}
\]
This allows us to write the critical points as
\[ \alpha, \bar{\alpha} = Q^{\frac{1}{2}} e^{-Rt \pm i \theta_*(u,t)}. \] (6.18)

The phase \( \theta_*(u, t) \) in (6.18) satisfies the equation
\[ \cos \theta_*(u, t) = R \Lambda_{\text{eff}}(t) + \frac{1 - e^{-2Ru}}{2 \Lambda_{\text{eff}}(t)} , \] (6.19)
where \( \Lambda_{\text{eff}}(t) \equiv \Lambda e^{Rt} \).

The saddle point method gives rise to the following semi-classical wave functions.
\[ \hat{\Psi}^*_q \left( \frac{u}{\hbar}, \frac{t}{\hbar} \right) \approx \exp \left\{ \frac{1}{\hbar} S^{(0)}(\alpha; (u, t)) \right\} , \] (6.20)
\[ \hat{\Psi}_q \left( \frac{u}{\hbar}, \frac{t}{\hbar} \right) \approx \exp \left\{ -\frac{1}{\hbar} S^{(0)}(\alpha; (u, t)) \right\} , \] (6.21)
where the critical point \( \alpha \) is taken for simplicity. The critical value \( S^{(0)}(\alpha; (u, t)) \) can be read easily up to the \( u \)-independent term since we have
\[ \frac{d}{du} S^{(0)}(\alpha; (u, t)) = \ln \alpha. \] (6.22)

The integration of (6.22) gives rise to
\[ S^{(0)}(\alpha; (u, t)) = -Ru + \frac{u^2}{2} \ln Q + i \int^u dx \theta_*(x, t) + u \text{-independent term}. \] (6.23)

**Family of curves**

The above semi-classical analysis has its interpretation in complex geometry. To explain this, it is convenient to start with the four-dimensional case. It follows from (6.19) that the phase \( \theta_*(u, t) \) becomes independent of \( t \) at the limit \( R \to 0 \). Equation (6.19) is translated to
\[ \cos \theta^{\text{ld}}_*(u) = \frac{u}{2 \Lambda}, \] (6.24)
where \( \theta^{\text{ld}}_*(u) \equiv \lim_{R \to 0} \theta_*(u, t) \). The phase \( \theta^{\text{ld}}_* \) can be thought as a meromorphic function on a Riemann surface. Let \( C \) be the curve (\( \mathbb{P}^1 \))
\[ y^2 = x^2 - 4 \Lambda^2, \] (6.25)
and \( h \) be the meromorphic function on \( C \)

\[
h = \frac{x + y}{2\Lambda}. \tag{6.26}
\]

Then we can write equation (6.24) as

\[
\theta^4_d(u) = \frac{1}{i} \ln h \bigg|_{x=u+i0}. \tag{6.27}
\]

The classical action becomes an integral of \( dx \ln h \) on the curve \( C \).

\[
i \int^u dx \theta^4_d(x) = \int^u dx \ln h. \tag{6.28}
\]

It is tempting to think \( x \) and \( \ln h \) in the above as the action-angle variables of the one-dimensional Toda chain. Note that the curve \( C \) can be expressed as

\[
h + \frac{1}{h} = \frac{x}{\Lambda}. \tag{6.29}
\]

This is the spectral curve of the infinite Toda chain.

We now move on to the five-dimensional case. For each \( t \in \mathbb{R}_{\leq 0} \), we associate the curve \( C_t \)

\[
y^2 = z^2 - 4\Lambda_{\text{eff}}^2(t), \tag{6.30}
\]

and the meromorphic function \( h_t \)

\[
h_t \equiv \frac{y + z}{2\Lambda_{\text{eff}}(t)}. \tag{6.31}
\]

The identification of \( u \) with \( z \) gets slightly involved in the five-dimensional theory. Let \( \mathbb{C}/\mathbb{Z} \) be the cylinder obtained by identifying \( x \) with \( x + \frac{\pi i}{R} n \). We will regard \( u \) as the real line of the cylinder. Note that the cylinder becomes \( \mathbb{C} \) as \( R \to 0 \). We introduce the holomorphic function on \( \mathbb{C}/\mathbb{Z} \) by

\[
z_t(x) \equiv 2R\Lambda_{\text{eff}}^2(t) + \frac{1 - e^{-2Rx}}{2R}. \tag{6.32}
\]

Then the equation (6.19) can be written as

\[
\theta^4(u, t) = \frac{1}{i} \ln h_t \bigg|_{z=z_t(u+i0)}. \tag{6.33}
\]
This shows that $\theta_*$ has the same form as the four-dimensional counterpart. Effect of the fifth dimension is encoded in $z_t(x)$ such that (6.33) smoothly reduces to (6.27) as $R \to 0$. In fact, we have $\lim_{R \to 0} z_t(x) = x$ and $\lim_{R \to 0} \Lambda_{\text{eff}}(t) = \Lambda$. Therefore (6.33) becomes (6.27) at the four-dimensional limit. The classical action becomes the following integral on the curve $C_t$.

$$i \int^u dx \theta_*(x,t) = \int^u dx \ln h_t(z_t(x)) = \int_{z_t(u+i0)}^{z_t(u-i0)} dz \left( \frac{dz_t}{dx} \right)^{-1} \ln h_t.$$

(6.34)

An exact solution of the five-dimensional theory can be obtained from the geometrical data (6.30), (6.31) and (6.32) at $t = 0$. It should be noted that the time dependence of these data is only via $\Lambda_{\text{eff}}(t)$. Recall that $\Lambda = \Lambda_{\text{eff}}(0)$ should be identified with the lambda parameter of the gauge theory. The standard dimensional argument shows that the renormalization group flow is realized effectively by scaling the lambda parameter although it is originally a RG invariant. We conjecturally identify the above time evolution of the system with the RG flow. Since $\Lambda_{\text{eff}}(t) = \Lambda e^{2Rt}$ ($t \leq 0$), it becomes zero at $t = -\infty$ and the curve (6.30) gets degenerate to $y^2 = z^2$. We expect that the geometrical data near $t = \pm \infty$, reflecting the holographic principle, describes the gauge theory in the perturbative regime \(^4\). This issue will be reported elsewhere [15] from the viewpoint of integrable systems.

**A Proof of equation (4.3)**

We first express the $N$-periodic potential (2.6) in a form relevant to study its asymptotics. It is convenient to consider partitions paired with the $U(1)$ charges. Let $(\mu, n)$ be the charged partition, where $\mu$ is a partition and $n$ is the $U(1)$ charge. The states $|\mu; n\rangle$ constitute bases of the Fock space of a single complex fermion. System of $N$ component fermions is realized [7] on this Fock space. Thereby any charged partition $(\mu, n)$ can be expressed in a unique way as a set of $N$ charged partitions $(\lambda^{(p)}, p_r)$ and vice versa. In terms of the Maya diagrams the

\(^4\)The analysis of the one-point functions at a positive time leads to the same geometrical data with a slight change of $\Lambda_{\text{eff}}(t)$. Namely, $\Lambda_{\text{eff}}(t) = \Lambda e^{-2Rt}$ for $t \geq 0$. We have $\lim_{t \to +\infty} \Lambda_{\text{eff}}(t) = 0$. Therefore $C_t$ becomes the same degenerate curve at $t = +\infty$. 

correspondence can be read as follows;

\[
\left\{ n + x_i(\mu) \right\}_{i \geq 1} = \bigcup_{r=1}^{N} \left\{ N(p_r + x_i(\lambda^{(r)})) + r - 1 \right\}_{i \geq 1}.
\]  

(A.1)

The periodic potential \( V(\mu) \) is shown \([2]\) to be \( \sum_{r=1}^{N} \xi_r p_r \), where \( p_r \) are the \( U(1) \) charges of the \( N \) charged partitions corresponding to \((\mu, 0)\).

For a charged partition \((\mu, n)\), we introduce the density \( \rho(x \mid \mu \, ; \, n) \) by

\[
\rho(x \mid \mu \, ; \, n) \equiv \sum_{i=1}^{+\infty} \delta(x - n - x_i(\mu)) - \sum_{i=1}^{+\infty} \delta(x + i),
\]  

(A.2)

In the neutral case we simply denote \( \rho(x \mid \mu) \equiv \rho(x \mid \mu \, ; \, 0) \). The density \( \rho(x \mid \mu \, ; \, n) \) is not sensitive to the \( U(1) \) charge since \( n \) could be absorbed into a shift of \( x \). It is convenient to modify the above density as

\[
\rho_{\text{reg}}(x \mid \mu \, ; \, n) = \sum_{i=1}^{+\infty} \delta(x - n - x_i(\mu)) - \sum_{i=1}^{+\infty} \delta(x + i),
\]  

(A.3)

where the subtraction is prescribed so that it satisfies

\[
\int_{-\infty}^{+\infty} dx \, \rho_{\text{reg}}(x \mid \mu \, ; \, n) = n.
\]  

(A.4)

The following expression of the \( N \)-periodic potential becomes important in the subsequent discussion.

\[
V(\mu) = \sum_{r=1}^{N} \xi_r \int_{-\infty}^{+\infty} dx \, \rho_{\text{reg}}(x \mid \lambda^{(r)} \, ; \, p_r),
\]  

(A.5)

where \((\lambda^{(r)}, p_r)\) are the charged partitions which describe \((\mu, 0)\). Note that \( \sum_{r=1}^{N} p_r = 0 \).

When \( \lambda^{(r)} \) are of order \( h^{-2} \) and \( p_r \) are of order \( h^{-1} \), we rescale the variables as

\[
x = \frac{u}{h}, \quad i_r = \frac{s}{h}, \quad p_r = \frac{\eta_r}{h}, \quad x_i(\lambda^{(r)}) = \frac{u(s \mid \lambda^{(r)})}{h} + O(h^0).
\]  

(A.6)

The asymptotics of the densities for \((\lambda^{(r)}, p_r)\) as \( h \to 0 \) can be computed in the similar manner as in the text. They become

\[
\rho(x \mid \lambda^{(r)} \, ; \, p_r) = -\frac{ds(u - \eta_r \mid \lambda^{(r)})}{du} + O(h^1),
\]  

(A.7)
\[ \rho_{\text{reg}}(x \mid \lambda^{(r)}; p_r) = -\left\{ \frac{ds(u - \eta_r \mid \lambda^{(r)})}{du} + \theta(-u) \right\} + O(h^1), \]  

(A.8)

where \( \theta(u) \) is the step function, that is, \( \theta(u) = 1 \) for \( u > 0 \), and \( 0 \) for \( u < 0 \).

Thanks to the correspondence (A.1) we can write \( \rho(x \mid \mu) \) and \( \rho_{\text{reg}}(x \mid \mu) \) as the superpositions of the densities for the \( N \) charged partitions in the following manner:

\[ \rho(x \mid \mu) = \frac{1}{N} \sum_{r=1}^{N} \rho \left( \frac{x - r + 1}{N} \mid \lambda^{(r)}; p_r \right), \]  

(A.9)

\[ \rho_{\text{reg}}(x \mid \mu) = \frac{1}{N} \sum_{r=1}^{N} \rho_{\text{reg}} \left( \frac{x - r + 1}{N} \mid \lambda^{(r)}; p_r \right). \]  

(A.10)

(A.1) also shows that the rescalings (3.17) and (3.18) are consistent with (A.6). Therefore, at the thermodynamic limit, the above relations turn to be

\[ \frac{ds(u \mid \mu)}{du} = \frac{1}{N} \sum_{r=1}^{N} \frac{ds(u - \eta_r \mid \lambda^{(r)})}{du}, \]  

(A.11)

where \( \sum_{r=1}^{N} \eta_r = 0 \).

\( \frac{d}{du}s(u \mid \mu) \) takes values in \([-1, 0]\) and asymptotes respectively to 0 as \( u \to +\infty \) and \(-1\) as \( u \to -\infty \). At this stage we impose a condition on partitions. In the below our consideration is restricted to a class of partitions satisfying the condition that \( \frac{d}{du}s(u \mid \mu) \) is non-decreasing. As shown in Appendix B, this is equivalent to say that the profile \( f(u \mid \mu) \) is convex. For such a partition \( \mu \), it follows from (A.4) and (A.8) that \( \frac{d}{du}s(u - \eta \mid \mu) + \theta(-u) \) has a compact support.

Let us compute the asymptotics of \( V(\mu) \) based on the expression (A.5). We rescale \( \xi_r \) in the potential to \( \zeta_r \) by (4.2). The asymptotics can be computed by using (A.8) as

\[ \sum_{r=1}^{N} \xi_r \int_{-\infty}^{+\infty} dx \rho_{\text{reg}} (x \mid \lambda^{(r)}; p_r) \]

\[ = -\frac{1}{h^2} \sum_{r=1}^{N} \zeta_r \int_{-\infty}^{+\infty} du \left\{ \frac{ds(u - \eta_r \mid \lambda^{(r)})}{du} + \theta(-u) \right\} + O(h^{-1}) \]

\[ = \frac{1}{h^2} \sum_{r=1}^{N} \zeta_r \int_{-\infty}^{+\infty} du u \frac{d^2s(u - \eta_r \mid \lambda^{(r)})}{du^2} + O(h^{-1}), \]  

(A.12)

where the last equality follows by the partial integration.
Figure 3: The graph of $\frac{d}{du}s(u \mid \mu)$ for $\eta_1 > \eta_2 > \cdots > \eta_N$, where $\eta_r$ are sufficiently separated from one another. $u_r$ ($0 \leq r \leq N$) are determined by (4.4).

Without losing generality it is enough to consider the case of $\eta_1 > \eta_2 > \cdots > \eta_N$. In addition, we suppose that $\eta_r$ are sufficiently separated from one another. The graph of $\frac{d}{du}s(u \mid \mu)$ is depicted in Figure 3. The relation (A.11) leads to the following equalities:

$$\int_{-\infty}^{+\infty} du \frac{d^2 s(u - \eta_r \mid \lambda^{(r)})}{du^2} = N \int_{u_r}^{u_{r-1}} du \frac{d^2 s(u \mid \mu)}{du^2},$$  \hspace{1cm} (A.13)

where $u_r$ ($0 \leq r \leq N$) are determined by the condition (4.4). By plugging (A.13) into (A.12) we obtain (4.3).
B Comparison between the energy functions of partitions

We express the energy functions (4.6) and (4.7) by using the (rescaled) profile of partition. Let \( \mu \) be a partition. The profile function \( f_\mu(x) (x \in \mathbb{R}) \) is defined \([2]\) by

\[
 f_\mu(x) \equiv |x| + \sum_{i=1}^{+\infty} \left\{ |x - x_i(\mu)| - |x - x_i(\mu)| - |x + i - 1| + |x + i| \right\}. \tag{B.1}
\]

The profile function becomes a quantity of order \( \hbar^{-1} \) when \( \mu \) is a partition of order \( \hbar^{-2} \). By the rescalings (3.17) and (3.18) it is translated to

\[
 f_\mu(x) = \frac{1}{\hbar} f(u | \mu) + O(\hbar^0), \tag{B.2}
\]

where the rescaled function \( f(u | \mu) \) is described by

\[
 f(u | \mu) = N|u| - 2N \int_0^{+\infty} ds \left\{ \theta(u - u(s | \mu)) - \theta(u + s) \right\}. \tag{B.3}
\]

(B.3) leads to the following relation with the density function.

\[
 \frac{df(u | \mu)}{du} = N \left( 1 + 2 \frac{ds(u | \mu)}{du} \right). \tag{B.4}
\]

We can rephrase the assumption made in Appendix A such that the rescaled profile functions for partitions dominating near the thermodynamic limit are convex.

The asymptotics (4.3) turns out to be the surface tension \([2]\). Let \( \sigma(y) \) be the concave and piecewise-linear function on \([-N,N]\) defined by \( \frac{d}{dy} \sigma(y) = \zeta_r \) for \( y \in [N - 2r, N - 2(r - 1)] \). It is a straightforward computation to see

\[
 N \sum_{r=1}^{N} \zeta_r \int_{u_r}^{u_{r-1}} du u \frac{d^2s(u | \mu)}{du^2} = -\frac{1}{2} \int_{-\infty}^{+\infty} du \sigma \left( \frac{df(u | \mu)}{du} \right). \tag{B.5}
\]

The energy function (4.6) can be rewritten as a functional of \( f(u | \mu) \) by using (B.4). It becomes

\[
 N^2 E[s(\cdot | \mu)] = \int_{-\infty}^{+\infty} du \frac{df(u | \mu)}{du} \frac{NRu^2}{2} \\
 + \frac{1}{4} \int_{-\infty}^{+\infty} du dv \left( N + \frac{df(u | \mu)}{du} \right) \left( N - \frac{df(v | \mu)}{dv} \right) \ln \left( \frac{\sinh R(u - v)}{RA} \right)^2 \\
 + \frac{1}{2} \int_{-\infty}^{+\infty} du \sigma \left( \frac{df(u | \mu)}{du} \right). \tag{B.6}
\]
Note that the above energy function is different from that used in [2] since the $u^2$ potential term does not appear there. While this, the four-dimensional limit (4.7) becomes

$$N^2 E_{4d} \left[ s(\cdot \mid \mu) \right] = \frac{1}{4} \int \int_{-\infty < u < v < +\infty} du dv \left( N + \frac{df(u \mid \mu)}{du} \right) \left( N - \frac{df(v \mid \mu)}{dv} \right) \ln \left( \frac{u - v}{\Lambda} \right)^2$$

$$+ \frac{1}{2} \int_{-\infty}^{+\infty} du \sigma \left( \frac{df(u \mid \mu)}{du} \right).$$

(B.7)

This coincides with the energy function used in [2]. It is shown there that the minimizer of (B.7) is described by the Seiberg-Witten geometry of four-dimensional $\mathcal{N} = 2$ supersymmetric $SU(N)$ Yang-Mills.

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