PRV FOR THE FUSION PRODUCT, THE CASE $\lambda \gg \mu$

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Abstract. Given a complex simple Lie algebra $g$ and a positive integer $\ell$, under the assumption $\lambda \gg \mu$, we show that irreducible representations of $g$ of the form $V(\lambda + w\mu)$, $w \in W$, with level at most $\ell$ appear in the fusion product of $V(\lambda)$ and $V(\mu)$. This verifies the existence of PRV components for the fusion product in this special case.

1. Introduction

Given a finite dimensional complex simple Lie algebra $g$, and two simple $g$-modules $V(\lambda)$ and $V(\mu)$, with dominant integral highest weights $\lambda$ and $\mu$ respectively, there is a ‘largest’ component in the decomposition of their tensor product $V(\lambda) \otimes V(\mu)$, due to Cartan, which is $V(\lambda + \mu)$ and occurs with multiplicity one. There is also a ‘smallest’ component, existence of which was proved by Parthasarathy, Rao and Varadarajan (PRV) in 1960s [11]; it is $V(\lambda + w_0\mu)$ and occurs again with multiplicity one. Here $w_0$ denotes the longest element in the Weyl group $W$ of $g$, and $\mathfrak{g}$ denotes the unique dominant element in the $W$–orbit of $\nu$.

The (classical) PRV conjecture, seeking the existence of components in the tensor product decomposition of a similar form, where the longest element $w_0$ above is now replaced by an arbitrary Weyl group element, was first established by Kumar [8] (a refinement in [9]) and Mathieu [10] independently. More precisely, it is shown that for any $w \in W$ the irreducible $g$-module $V(\lambda + w\mu)$ occurs with multiplicity at least one in $V(\lambda) \otimes V(\mu)$. Subsequently, weights of the form $\lambda + w\mu$ are called PRV weights, and the corresponding representations $V(\lambda + w\mu)$ are called PRV components of $V(\lambda) \otimes V(\mu)$. (See [7] for a historical background of the problem).

For a finite dimensional complex simple Lie algebra $g$ with the additional data of a positive integer $\ell$, one constructs the corresponding (untwisted) affine Lie algebra $\tilde{g}$ with its central element acting via the scalar $\ell$. Then, integrable irreducible representations of $\tilde{g}$ are parametrized by a finite set $P_\ell$. There is an associated product structure on two such simple $\tilde{g}$-modules of level $\ell$, called the fusion product.
It is then very natural to ask if the PRV components of level ℓ appear in the fusion product decomposition. For the affine Lie algebra corresponding to $\mathfrak{g} = \mathfrak{sl}_n$, this is true by works of Belkale on quantum Horn and saturation conjectures [3]. For the remaining simple Lie algebras the question is open in its full generality. In literature, this is sometimes referred as the quantum counterpart of the PRV conjecture.

Here we give an elementary algebraic proof of the existence of such PRV components for any simple Lie algebra $\mathfrak{g}$ under the assumption $\lambda \gg \mu$, by which we mean that all weights of $V(\mu)$ shifted by $\lambda$ lies in the dominant cone. Using the definition of the fusion product, formulated in terms of ‘truncated’ tensor product of finite dimensional representations of $\mathfrak{g}$ that are of level at most $\ell$, our main result is the following (Theorem 4.1 in the main text).

**Theorem 1.1.** Suppose $\lambda, \mu \in P_\ell$, and $\lambda \gg \mu$. For any $w \in W$, if $\lambda + w\mu$ is in $P_\ell$, then $V(\lambda + w\mu)$ occurs with multiplicity one in the fusion product of $V(\lambda)$ and $V(\mu)$.

Though this is a special case, it should be seen as supporting evidence in the direction of the quantum analogue of the PRV conjecture for any simple Lie algebra $\mathfrak{g}$.

Furthermore, for the same data $\mathfrak{g}$, $\ell$ and three dominant integral weights $\vec{\theta} = (\theta_1, \theta_1, \theta_3)$ in $P_3^\ell$ one constructs a vector bundle $\mathbb{V}_{\vec{\theta}}$ over the moduli space of 3-pointed stable rational curves, whose fiber over a point is called the space of conformal blocks [13]. Structure coefficients in the fusion product decomposition are intimately related to the dimensions of these spaces (cf. Section 3.1). Denote the dual of a weight $\nu$ by $\nu^*$; if further $\nu$ is in $P_\ell$ so is $\nu^*$. Then, our result in particular shows that, under the assumptions of the Theorem 1.1, the rank of the vector bundle $\mathbb{V}_{\vec{\theta}}$ with $\vec{\theta} = (\lambda, \mu, (\lambda + w\mu)^*)$ is equal to the rank of the invariants of the tensor product associated to the same triple (and both equal to one). For any triple $\vec{\theta} \in P_3^\ell$ this equality of ranks necessarily holds for $\ell$ beyond a critical level. We remark that although our assumption in Theorem 1.1 is restrictive on the distribution of weights, we do not impose such a bound on the level $\ell$.

To describe our results precisely, we set up the notation and recall basic definitions and concepts in Section 2. In Section 3 we recall the definition of the fusion product and its properties that is pertinent to the proof of the main result which is given in Section 4, where an explicit decomposition is also given at the end.
2. Preliminaries

The main reference for this section is [1].

Let $G$ be a connected, simply connected, simple affine algebraic group over $\mathbb{C}$. We fix a Borel subgroup $B$ of $G$ and a maximal torus $T \subset B$. Let $\mathfrak{h}, \mathfrak{b}$ and $\mathfrak{g}$ denote the Lie algebra of $T$, $B$ and $G$ respectively.

Let $R = R(\mathfrak{h}, \mathfrak{g}) \subset \mathfrak{h}^*$ be the root system; there is the root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus (\oplus_{\alpha \in R}\mathfrak{g}_\alpha)$. The choice of $\mathfrak{b}$ determines a set of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ of $R$, where $r$ is the rank of $G$. Let $\mathfrak{h}_R$ denote the real span of elements of $\mathfrak{h}$ dual to $\Delta$. For each root $\alpha$, denote by $H_\alpha$ the unique element of $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ such that $\alpha(H_\alpha) = 2$; it is called the coroot associated to the root $\alpha$. For any $\alpha \in R$, the Lie subalgebra $\mathfrak{g}_\alpha \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \oplus \mathfrak{g}_{-\alpha}$ is isomorphic as a Lie algebra to $\mathfrak{sl}_2$ and thus will be denoted by $\mathfrak{sl}_2(\alpha)$.

Let $\{\omega_i\}_{1 \leq i \leq r}$ be the set of fundamental weights, defined as the basis of $\mathfrak{h}^*$ dual to $\{H_{\alpha_i}\}_{1 \leq i \leq r}$. We denote the set of dominant weights by $P^+$, that is,

$$P^+ := \{\lambda \in \mathfrak{h}^* : \lambda(H_{\alpha_i}) \in \mathbb{Z}_{\geq 0} \forall \alpha_i \in \Delta\},$$

where $\mathbb{Z}_{\geq 0}$ is the set of nonnegative integers. The set $P^+$ parametrizes the set of isomorphism classes of all the finite dimensional irreducible representations of $\mathfrak{g}$. For $\lambda \in P^+$, let $V(\lambda)$ be the associated finite dimensional irreducible representations of $\mathfrak{g}$ with highest weight $\lambda$.

Let $\rho$ denote the sum of fundamental weights and $\theta$ the highest root of $\mathfrak{g}$. Define $\tilde{h}(\mathfrak{g}) := 1 + \rho(H_\theta)$, denoted in short by $\tilde{h}$; it is called the dual Coxeter number.

Let $(\cdot | \cdot)$ denote the Killing form on $\mathfrak{g}$ normalized such that $(H_\theta | H_\theta) = 2$. We will use the same notation for the restricted form on $\mathfrak{h}$, and the induced form on $\mathfrak{h}^*$. We introduce the Weyl group $W := N_G(T)/T$. The Killing form is positive definite on $\mathfrak{h}_R$, it induces the following canonical isomorphism

$$F : \mathfrak{h}_R^* \rightarrow \mathfrak{h}_R ; \ \alpha \mapsto (2/(H_\alpha | H_\alpha))H_\alpha$$

with $F^{-1}(H_\alpha) = (2/(\alpha | \alpha))\alpha$. Under the above identification, $W$ is thought as generated by elements $w_\alpha$ for $\alpha \in \Delta$; acting on $\mathfrak{h}_R^*$ as

$$w_\alpha \beta = \beta - \beta(H_\alpha)\alpha.$$ 

Let $w_0$ denote the longest element in the Weyl group. Then for any $\lambda \in P$, the dual of $\lambda$ denoted by $\lambda^*$ is $-w_0\lambda$. For $\lambda \in P^+, \lambda^*$ is again in $P^+$ and, moreover, $V(\lambda)^* \simeq V(\lambda^*)$. 

\[3\]
Let \( \tilde{g} = g \otimes \mathbb{C}((z)) \oplus \mathbb{C}[z] \) denote the (untwisted) affine Lie algebra associated to \( g \) over \( \mathbb{C}((z)) \) (where \( \mathbb{C}((z)) \) denotes the field of Laurent polynomials in one variable \( z \)), with the Lie bracket
\[
[x \otimes f, y \otimes g] = [x, y] \otimes fg + (x | y) \text{Res}_{z=0}(gdf) \cdot \mathbb{C}
\]
for \( x, y \in g \) and \( f, g \in \mathbb{C}((z)) \).

We fix a positive integer \( \ell \), called the level, and define the set
\[
P_\ell(G) := \left\{ \lambda \in P^+ : \lambda(H_\theta) \leq \ell \right\};
\]
it is finite and closed under taking the dual; we denote it in short by \( P_\ell \). For any \( \lambda \in P^+ \), the integer \( \lambda(H_\theta) \) is said to be the level of the representation \( V(\lambda) \).

It is well known from representation theory of affine Lie algebras that for each \( \lambda \in P_\ell \) there exists a unique (upto isomorphism) left \( \tilde{g} \)-module \( \mathcal{H}(\lambda) \), called the integrable highest weight \( \tilde{g} \)-module, with the central element \( C \) acting as \( \ell \cdot \text{Id} \) (see e. g. [6]).

3. Tensor product, fusion product and PRV components

Let \( \mathcal{R}(g) \) denote the ring of finite dimensional representations of \( g \); it is a free \( \mathbb{Z} \)-module with basis \( \{ V(\lambda) : \lambda \in P^+ \} \) and product structure
\[
V(\lambda) \otimes V(\mu) = \sum_{\nu \in P^+} m_{\lambda \mu}^{\nu} V(\nu),
\]
where \( m_{\lambda \mu}^{\nu} = \dim \text{Hom}_g(V(\lambda) \otimes V(\mu) \otimes V(\nu^*), \mathbb{C}) \), with \( \mathbb{C} \) thought as a trivial \( g \) module.

The classical Parthasarathy–Ranga Rao–Varadarajan (PRV) conjecture (already proven, see [8, 9] or [10]) is as follows:

**Theorem 3.1.** Let \( V(\lambda_1) \) and \( V(\lambda_2) \) be two finite dimensional irreducible \( g \)-modules with highest weights \( \lambda_1 \) and \( \lambda_2 \) respectively. Then, for any \( w \in W \) the irreducible \( g \)-module \( V(\lambda_1 + w\lambda_2) \) occurs with multiplicity at least one in \( V(\lambda_1) \otimes V(\lambda_2) \), where \( \lambda_1 + w\lambda_2 \) denotes the unique dominant element in the \( W \)-orbit of \( \lambda_1 + w\lambda_2 \).

Weights of the form \( \lambda_1 + w\lambda_2 \) are called PRV weights, and corresponding components \( V(\lambda_1 + w\lambda_2) \) are called PRV components of the tensor product \( V(\lambda_1) \otimes V(\lambda_2) \).

We are interested in a natural generalization of the above theorem for the decomposition of the fusion product of two integrable irreducible representations of \( \tilde{g} \) of level \( \ell \). There are various equivalent definitions of the fusion product (see for example [4]), we will use the one that allows us to formulate the problem in terms of a ‘truncated product’ of finite dimensional representations of \( g \) that have level at most \( \ell \).
3.1. **Definition of fusion product.** The fusion ring associated to $g$ and a fixed nonnegative integer $\ell$, $\mathcal{R}_\ell(g)$, is a free $\mathbb{Z}$-module with basis $\{V(\lambda) : \lambda \in P_\ell\}$. The ring structure, called the **fusion product**, is defined as follows:

$$V(\lambda) \otimes^F V(\mu) := \bigoplus_{\nu \in P_\ell^+} n_{\lambda,\mu}^\nu V(\nu),$$

where $n_{\lambda,\mu}^\nu$ is the dimension of

$$V_{p_\ell^1}^\dagger(\lambda, \nu, \mu^*) := \text{Hom}_g(V(\lambda) \otimes V(\mu) \otimes V(\nu^*), \mathbb{C}).$$

Above $g \otimes \mathcal{O}(\mathbb{P}^1 - \bar{p})$ denotes the Lie algebra of $g$-valued regular functions on the Riemann sphere punctured at 3 distinct points $\bar{p} = \{p_1, p_2, p_3\}$. We refer to [13] for the precise action of $g \otimes \mathcal{O}(\mathbb{P}^1 - \bar{p})$ on $\mathcal{H}(\lambda) \otimes \mathcal{H}(\mu) \otimes \mathcal{H}(\nu^*)$, above $\mathbb{C}$ is thought as a trivial module for the algebra. The space $V_{p_\ell^1}^\dagger(\lambda, \nu, \mu^*)$ is called the **space of conformal blocks** on $\mathbb{P}^1$ with three marked points $\bar{p}$ and weights $\lambda, \mu, \nu^*$ attached to them with central charge $\ell$; its dimension, that is $n_{\lambda,\mu}^\nu$, is given by the Verlinde formula.

By its definition the fusion product $\otimes^F$ is commutative; it is also associative as a result of the ‘factorization rules’ of [13]. Furthermore, the canonical map

$$V_{p_\ell^1}^\dagger(\lambda, \nu, \mu^*) \to \text{Hom}_g(V(\lambda) \otimes V(\mu) \otimes V(\nu^*), \mathbb{C})$$

induced from the natural inclusion $V(\lambda) \otimes V(\mu) \otimes V(\nu^*) \hookrightarrow \mathcal{H}(\lambda) \otimes \mathcal{H}(\mu) \otimes \mathcal{H}(\nu^*)$ is an injection [12]. In particular, for any $\ell \in \mathbb{Z}_{>0}$, the inequality

$$n_{\lambda,\mu}^\nu \leq m_{\lambda,\mu}^\nu$$

holds.

We introduce some additional notation.

The (finite) Weyl group $W$ of $G$ acts on $P$, and hence on $P_\mathbb{R} := P \otimes_{\mathbb{Z}} \mathbb{R}$. Let $W_\ell$ be the group of affine transformations of $P_\mathbb{R}$ generated by $W$ and the translation $\lambda \mapsto \lambda + (\ell + \tilde{h})\theta$. Then, $W_\ell$ is the semi-direct product of $W$ by the lattice $(\ell + \tilde{h})Q^\text{long}$, where $Q^\text{long}$ is the sublattice of $P$ generated by the long roots. It is well known that $W_\ell$ is a Coxeter group, in particular the length of an element is meaningful. For any root $\alpha \in R$ and $n \in \mathbb{Z}$, define the **affine wall**

$$H_{\alpha,n} = \{\lambda \in P_\mathbb{R} : (\lambda|\alpha) = n(\ell + \tilde{h})\}. $$

The closures of the connected components of $P_\mathbb{R} \setminus (\cup_{\alpha \in R, n \in \mathbb{Z}} H_{\alpha,n})$ are called the **alcoves**. Then, any alcove is a fundamental domain for the action of $W_\ell$. The **fundamental alcove** is by definition

$$A_\ell^0 = \{\lambda \in P_\mathbb{R} : \lambda(H_{\alpha_i}) \geq 0 \forall \alpha_i \in \Delta, \text{ and } \lambda(H_0) \leq \ell + \tilde{h}\}.$$
3.2. Definition of $\pi : \mathcal{R}(g) \to \mathcal{R}_\ell(g)$. Following Beauville [2], we define the $\mathbb{Z}$-linear map $\pi$ as follows:

$$
\pi(V(\lambda)) = \begin{cases} 
0, & \text{if } \lambda + \rho \text{ lies on an affine wall} \\
\varepsilon(w)V(\mu) & \text{otherwise}
\end{cases}
$$

where $\mu$ is the unique element of $P_\ell$, and $w \in W_\ell$ such that $\lambda + \rho = w(\mu + \rho)$. Here $\varepsilon(w)$ is the sign of the affine Weyl group element $w$. With the above definition, we recall the following (see Faltings [5]).

**Theorem 3.2.** The $\mathbb{Z}$-linear map $\pi : \mathcal{R}(g) \to \mathcal{R}_\ell(g)$ is an algebra homomorphism with respect to the fusion product on $R_\ell(g)$.

With the notation as above, the PRV conjecture for the fusion product is as follows:

**Conjecture 3.3.** Suppose $\lambda$ and $\mu$ are in $P_\ell$. Then, for any $w \in W_\ell$ the irreducible $g$-module $V(\lambda + w\mu)$ occurs with multiplicity at least one in $V(\lambda) \otimes^F V(\mu)$, where $\lambda + w\mu$ denotes the unique element in the $W_\ell$-orbit of $\lambda + w\mu$ lying in $P_\ell$.

For $g = sl_n$, the conjecture is true by works of Belkale on quantum Horn and saturation conjectures [3].

4. The case $\lambda \gg \mu$

Suppose $\lambda$ and $\mu$ are in $P_\ell$. Consider the weight space decomposition

$$
V(\mu) = \sum_{\nu \in \mathfrak{h}^*} m_{\mu}(\nu)V_{\nu}
$$

where $V_{\nu} = \{v \in V(\mu) : h \cdot v = \nu(h)v \text{ for all } h \in \mathfrak{h}\}$ and $m_{\mu}(\nu)$ is the dimension of the weight space $V_{\nu}$. Let $\Pi(\mu)$ denote the set of all weights of $V(\mu)$. We are looking at the special case $\lambda \gg \mu$, by that we mean $\lambda + \nu \in P^+$ for all $\nu \in \Pi(\mu)$. Thus, under the assumption $\lambda \gg \mu$, PRV components of the tensor product $V(\lambda) \otimes V(\mu)$ are precisely $\{V(\lambda + w\mu) = V(\lambda + w\mu), w \in W\}$.

We will prove the following formulation of the problem in this special case.

**Theorem 4.1.** Suppose $\lambda, \mu \in P_\ell$, and $\lambda \gg \mu$.

For any $w \in W$, if $\lambda + w\mu$ is in $P_\ell$, then $V(\lambda + w\mu)$ occurs with multiplicity one in $V(\lambda) \otimes^F V(\mu)$.

Consider the character homomorphism $\chi : \mathcal{R}(g) \to \mathbb{Z}[P]$ mapping $V \mapsto \sum (\dim(V_{\mu}))e^\mu$. For $\lambda$ dominant integral, denote $\chi(V(\lambda))$ in short
by \( \chi_\lambda \); it is given by the Weyl character formula

\[
\chi_\lambda = \frac{\sum_{w \in W} \varepsilon(w) e^{w(\lambda + \rho)}}{\sum_{w \in W} \varepsilon(w) e^{w \rho}}.
\]

Denote \( D := \sum_{w \in W} \varepsilon(w) e^{w \rho} \). Then, using the fact that the elements of \( \Pi(\mu) \) are permuted by \( W \) and \( \dim V_\nu = \dim V_{w \nu} \) for any \( w \in W \),

\[
\chi_\lambda \chi_\mu = \sum_{\nu \in \Pi(\mu)} m_\mu(\nu) e^{\nu D^{-1}} \sum_{w \in W} \varepsilon(w) e^{w(\lambda + \rho)} = D^{-1} \sum_{w \in W} \sum_{\nu \in \Pi(\mu)} m_\mu(\nu) \varepsilon(w) e^{w(\lambda + \nu + \rho)}
\]

Now interchanging the order of summation, as \( \lambda + \nu \) is dominant integral for all \( \nu \in \Pi(\mu) \), we may use the Weyl character formula again and conclude that

\[
\chi_\lambda \chi_\mu = \sum_{\nu \in \Pi(\mu)} m_\mu(\nu) D^{-1} \sum_{w \in W} \varepsilon(w) e^{w(\lambda + \nu + \rho)} = \sum_{\nu \in \Pi(\mu)} m_\mu(\nu) \chi_{\lambda + \nu}.
\]

The above expression is a special case of a formula attributed to Brauer and Kliymyk independently. Thus the tensor product decomposition of \( V(\lambda) \) and \( V(\mu) \) is very explicit when \( \lambda \gg \mu \), it is

\[
V(\lambda) \otimes V(\mu) = \sum_{\nu \in \Pi(\mu)} m_\mu(\nu) V(\lambda + \nu).
\]  

(2)

As any component of \( V(\lambda) \otimes V(\mu) \) has level at most \( 2\ell \), any highest weight \( \lambda + \nu \) of the decomposition in (2) that does not lie in \( P_\ell \) satisfies the inequality \( \ell + 1 \leq (\lambda + \nu)(H_\rho) \leq 2\ell \).

We start with grouping the terms on the right hand side of (2) as follows. We call

\[
S := \{ \nu \in \Pi(\mu) : \nu \text{ is not of the form } w \mu \text{ for any } w \in W \}.
\]

Then, using the fact that \( \dim V_\mu = \dim V_{w \mu} = 1 \), \((w \in W)\) in \( V(\mu) \),

\[
V(\lambda) \otimes V(\mu) = \sum_{\nu \in S} m_\mu(\nu) V(\lambda + \nu) + \sum_{w \in W} V(\lambda + w \mu).
\]

We further group the representations above in terms of their levels as

\[
V(\lambda) \otimes V(\mu) = \sum_{\{\nu \in S: \lambda + \nu \in P_\ell \}} m_\mu(\nu) V(\lambda + \nu) + \sum_{\{\nu \in S: (\lambda + \nu)(H_\rho) = \ell + 1 \}} m_\mu(\nu) V(\lambda + \nu) + \sum_{\ell + 1 < (\lambda + \nu)(H_\rho) \leq 2\ell} m_\mu(\nu) V(\lambda + \nu) + \sum_{\{w \in W: \lambda + w \mu \in P_\ell \}} V(\lambda + w \mu) + \sum_{\{w \in W: (\lambda + w \mu)(H_\rho) = \ell + 1 \}} V(\lambda + w \mu) + \sum_{\ell + 1 < (\lambda + w \mu)(H_\rho) \leq 2\ell} V(\lambda + w \mu). \tag{3}
\]
We now investigate the image of the tensor product decomposition as expressed in (3) under the homomorphism $\pi$ of Theorem 3.2.

Any representation $V(\xi)$ in (3) with highest weight $\xi$ satisfying $(\xi + \rho|\theta) = \ell + \tilde{h}$ (i.e. any representation $V(\xi)$ of level $\xi(H_\theta) = \ell + 1$) is mapped to zero under $\pi$. Now consider representations $V(\xi)$ in (3) whose level is not $\ell + 1$. As $A_\ell^o$ is a fundamental domain for the action of $W_\ell$, for any $\xi \in P_\ell^+$, there is a unique $\beta \in P_\ell$ and a $\tilde{w} \in W_\ell$ such that

$$\tilde{w} \cdot \xi := \tilde{w}(\xi + \rho) - \rho = \beta.$$  

Then, by definition of $\pi$, $\pi(V(\xi)) = \varepsilon(\tilde{w})V(\beta)$. (If $\xi$ is already in $P_\ell$, then $\beta = \xi$ and $\pi(V(\xi)) = V(\xi)$.)

Now consider the following element of $W_\ell$

$$s_{\theta,\ell+h}(\xi) := s_\theta(\xi) + (\ell + \tilde{h})\theta.$$  

Its shifted action is given by

$$s_{\theta,\ell+h} \cdot \xi = s_\theta(\xi) + (\ell + 1)\theta.$$  

It is known that there is a single length 1 element in $W_\ell/W$ for any simple Lie algebra $g$, which may be represented by the class of $s_{\theta,\ell+h}$. The following proposition reveals the significance of this distinguished element under the assumption $\lambda \gg \mu$.

**Proposition 4.2.** Suppose $\lambda, \mu \in P_\ell$ and $\lambda \gg \mu$.

For any $V(\xi)$ in the decomposition of $V(\lambda) \otimes V(\mu)$ with

$$\ell + 1 < \xi(H_\theta) \leq 2\ell,$$  

(4)

we have that $s_{\theta,\ell+h} \cdot \xi$ is in $P_\ell$.

Moreover, $V(s_{\theta,\ell+h} \cdot \xi)$ is not of the form $V(\lambda + w\mu)$ for any $w \in W$, that is, it is not a PRV component for the tensor product.

**Proof.** The level of $s_{\theta,\ell+h} \cdot \xi$ is given by

$$(s_{\theta,\ell+h} \cdot \xi)(H_\theta) = (\xi - \xi(H_\theta)\theta + (\ell + 1)\theta)(H_\theta) = -\xi(H_\theta) + 2(\ell + 1).$$

Using the above equation together with inequality (4), we get that

$$2 \leq (s_{\theta,\ell+h} \cdot \xi)(H_\theta) \leq \ell.$$  

Thus to prove that $s_{\theta,\ell+h} \cdot \xi$ is in $P_\ell$, it suffices to show that it is dominant. We will show this in two cases; though this separation of cases is not necessary, it makes the exposition clearer.

**Case 1:** Suppose $\xi$ is of the form $\xi = \lambda + w\mu$ ($w \in W$), satisfying inequality (4). We then necessarily have $(w\mu)(H_\theta) > 0$ (as $\lambda \in P_\ell$), and thus $w\mu - s_\theta(w\mu)$ will be a positive integer multiple of $\theta$. 

Now consider weights in $\Pi(\mu)$ of the form $w\mu + k\theta$ ($k \in \mathbb{Z}$). The subspace of $V(\mu)$ spanned by the weight spaces $V_{w\mu + k\theta}$ ($k \in \mathbb{Z}$) is invariant under $\mathfrak{sl}_2(\theta)$, thus the weights in $\Pi(\mu)$ of the form
\[
w\mu, w\mu - \theta, \cdots, w\mu - (w\mu)(H_\theta)\theta = s_\theta(w\mu)
\]
form a connected (uninterrupted) string. We will show that
\[
s_{\theta, \ell + h} \cdot (\lambda + w\mu) = \lambda + w\mu - (\lambda + w\mu)(H_\theta)\theta + (\ell + 1)\theta
\]
is in the connected string
\[
\lambda + w\mu, \lambda + w\mu - \theta, \cdots, \lambda + w\mu + k\theta, \cdots, \lambda + s_\theta(w\mu),
\]
which is obtained from (5) by adding $\lambda$ to each element. This will verify that $s_{\theta, \ell + h} \cdot (\lambda + w\mu)$ is in the dominant cone, as all weights in the string (6) are dominant by our assumption $\lambda \gg \mu$.

Now express $s_{\theta, \ell + h} \cdot (\lambda + w\mu) = \lambda + w\mu + n\theta$, where
\[
m = -(\lambda + w\mu)(H_\theta) + (\ell + 1).
\]
To show that $s_{\theta, \ell + h} \cdot (\lambda + w\mu)$ is in the connected string (6), it suffices to show that
\[
-(w\mu)(H_\theta) \leq m \leq 0.
\]
As $\lambda + w\mu$ satisfies inequality (4) we immediately get that $m < 0$. Also, as $\lambda \in P_\ell$, we have that $m = (\ell + 1 - \lambda(H_\theta)) - (w\mu)(H_\theta) > -(w\mu)(H_\theta)$. Thus both inequalities in (7) are satisfied, and in fact strictly.

Case 2: Suppose $\xi$ is of the form $\xi = \lambda + \nu$ with $\nu \in S$ satisfying inequality (4). As $\lambda \in P_\ell$, we again necessarily have $\nu(H_\theta) > 0$.

As in case 1, the weights in $\Pi(\mu)$ of the form $\nu + k\theta$ form the connected string (weights of any simple $\mathfrak{sl}_2(\theta)$ submodule of $V(\mu)$ containing $\nu$)
\[
\nu + q\theta, \cdots, \nu, \cdots, \nu - r\theta,
\]
where $r, q$ are nonnegative integers satisfying $r - q = \nu(H_\theta) > 0$. We will show that
\[
s_{\theta, \ell + h} \cdot (\lambda + \nu) = \lambda + \nu - (\lambda + \nu)(H_\theta)\theta + (\ell + 1)\theta
\]
is in the $\lambda$-shifted connected string
\[
\lambda + \nu + q\theta, \cdots, \lambda + \nu, \cdots, \lambda + \nu - r\theta.
\]
This will verify that $s_{\theta, \ell + h} \cdot (\lambda + \nu)$ is in the dominant cone, as all weights in the string (8) are dominant, again by the assumption $\lambda \gg \mu$.

Now express $s_{\theta, \ell + h} \cdot (\lambda + \nu) = \lambda + \nu + n\theta$ with $n = \ell + 1 - (\lambda + \nu)(H_\theta)$. To show that $s_{\theta, \ell + h} \cdot (\lambda + \nu)$ is in the connected string (8), it suffices to show that
\[
-r \leq n \leq q.
\]
As \( \lambda + \nu \) satisfies inequality (4), we have \( n < 0 \) and hence \( n < q \) as \( q \geq 0 \). Expressing \( r = q + \nu(\mathbf{H}_\theta) \), and using the fact that \( \lambda \in P_\ell \), we immediately get \(-r < n\). Thus both inequalities in (9) are (strictly) satisfied.

To complete the proof of the proposition what remains to show is that \( s_{\theta,l+\hat{h}} \cdot \xi \) is not of the form \( \lambda + w\mu \) for any \( w \in W \), and thus is not a PRV weight for the tensor product. We will now explain how this immediately follows from our calculations above revealing that inequalities (7) and (9) are strictly satisfied.

Using the explicit form of the tensor product decomposition (2), highest weights of PRV components appearing in \( V(\lambda) \otimes V(\mu) \) are in one to one correspondence with vertices of the convex hull of the set of weights of \( V(\mu) \) translated by \( \lambda \). For any such vertex \( v \), simply because it lies on the boundary of the \( \lambda \)-translated convex hull of \( \Pi(\mu) \), \( v - \lambda \in \Pi(\mu) \) is necessarily a weight at one of the end points of any root strings in \( \Pi(\mu) \) containing it. Then, to prove the claim it suffices to show that \( s_{\theta,l+\hat{h}} \cdot \xi \) does not lie at any of the end points of the \( \theta \)-string (as in equations (6) or (8)) that it is contained.

To show this, let us first suppose \( \eta = s_{\theta,l+\hat{h}} \cdot \xi \) and \( \xi = \lambda + w\mu \) \((w \in W)\) as in Case 1. Then, as \( \eta = \lambda + w\mu + m\theta \) lies on the string (6), the only possible vertex of the \( \lambda \)-shifted convex hull of \( \Pi(\mu) \) that \( \eta \) can be is \( \lambda + s_\theta(w\mu) = \lambda + w\mu - (w\mu)(\mathbf{H}_\theta)\theta \). But in Case 1 above, we showed that \( m > -(w\mu)(\mathbf{H}_\theta) \) which implies that \( \eta \neq \lambda + w\mu - (w\mu)(\mathbf{H}_\theta)\theta \), hence \( \eta \) is not a PRV weight. (As \( m < 0 \), \( \eta \) is clearly not equal to \( \lambda + w\mu \).)

Similarly, suppose \( \eta = s_{\theta,l+\hat{h}} \cdot \xi \) and \( \xi = \lambda + \nu \) with \( \nu \in S \) as in Case 2. Consider again the \( \theta \)-string (5) that \( \eta = \lambda + \nu + n\theta \) is an element of. The only possible vertices of the \( \lambda \)-shifted convex hull of the weights of \( V(\mu) \) that may be on that string would be the initial weight \( \lambda + \nu - r\theta \) or the terminal weight \( \lambda + \nu + q\theta \) (possibly neither). In Case 2 above we showed that \(-r < n < q\), which implies that \( \eta \) is neither of those weights, thus it is not a vertex, and hence cannot be a PRV weight. \( \square \)

The following result immediately follows from the definition of \( \pi \).

**Corollary 4.3.** Suppose \( \lambda, \mu \in P_\ell \) and \( \lambda \gg \mu \). For any \( V(\xi) \) in the decomposition of \( V(\lambda) \otimes V(\mu) \) with \( \ell + 1 < \xi(\mathbf{H}_\theta) \leq 2\ell \), we have \( \pi(V(\xi)) = -V(s_{\theta,l+\hat{h}} \cdot \xi) \).

By Corollary 4.3 and Theorem 3.2, we have the following very explicit expression for the fusion product of \( V(\lambda) \) and \( V(\mu) \) (as virtual modules)
whenever $\lambda \gg \mu$

$$V(\lambda) \otimes^F V(\mu) = \sum_{\nu \in S: \lambda + \nu \in P_\ell} m_\mu(\nu) V(\lambda + \nu) - \sum_{\nu \in S: \ell + 1 < (\lambda + \nu)(H_\theta) \leq 2\ell} m_\mu(\nu) V(s_{\theta,\ell+h} \cdot (\lambda + \nu))$$

$$+ \sum_{w \in W: \lambda + w\mu \in P_\ell} V(\lambda + w\mu) - \sum_{w \in W: \ell + 1 < (\lambda + w\mu)(H_\theta) \leq 2\ell} V(s_{\theta,\ell+h} \cdot (\lambda + w\mu)). \quad (10)$$

**Proof of Theorem 4.1.** By Proposition 4.2 none of the weights in decomposition (10) belonging to the set

$$\tilde{S} := \{s_{\theta,\ell+h} \cdot (\lambda + w\mu): \ell + 1 < (\lambda + w\mu)(H_\theta) \leq 2\ell, w \in W\} \cup$$

$$\{s_{\theta,\ell+h} \cdot (\lambda + \nu): \ell + 1 < (\lambda + \nu)(H_\theta) \leq 2\ell, \nu \in S\}$$

is of the form $\lambda + w\mu$ for any $w \in W$. Then, using equation (10) we get that the representations $\{V(\lambda + w\mu): \lambda + w\mu \in P_\ell\}$ are preserved in the fusion product decomposition, that is, no ‘cancellation’ by virtual modules $-V(\eta)$, $\eta \in \tilde{S}$. This proves the theorem.

More explicitly, for $\lambda \gg \mu$, by Theorem 3.2, inequality (1), and Theorem 4.1, we have

$$V(\lambda) \otimes^F V(\mu) = \sum_{\nu \in S: \lambda + \nu \in P_\ell} \tilde{m}_\mu(\nu) V(\lambda + \nu) + \sum_{w \in W: \lambda + w\mu \in P_\ell} V(\lambda + w\mu). \quad (11)$$

for some nonnegative integer $\tilde{m}_\mu(\nu) = m_\mu(\nu) - m_\mu(\beta)$, where $\beta \in \Pi(\mu)$ satisfies

$$s_{\theta,\ell+h}^{-1}(\lambda + \beta + \rho) - \lambda - \rho = \nu \quad \text{and} \quad \lambda + \beta \in P_\ell.$$ 

(If no such $\beta$ exists we take $m_\mu(\beta) = 0$.)

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