Exact solutions in Einstein-Yang-Mills-Dirac systems

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Abstract

We present exact solutions in Einstein-Yang-Mills-Dirac theories with gauge groups $SU(2)$ and $SU(4)$ in Robertson-Walker space-time $\mathbb{R} \times S^3$, which are symmetric under the action of the group $SO(4)$ of spatial rotations. Our approach is based on the dimensional reduction method for gauge and gravitational fields and relates symmetric solutions in EYMD theory to certain solutions of an effective dynamical system.

We interpret our solutions as cosmological solutions with an oscillating Yang-Mills field passing between topologically distinct vacua. The explicit form of the solution for spinor field shows that its energy changes the sign during the evolution of the Yang-Mills field from one vacuum to the other, which can be considered as production or annihilation of fermions.

Among the obtained solutions there is also a static sphaleron-like solution, which is a cosmological analogue of the first Bartnik-McKinnon solution in the presence of fermions.

1 Introduction

Exact solutions in gravity coupled to fields of different types have always attracted much attention. In particular, the last few years witnessed a great interest in solutions to EYM systems. There were found both numerical \cite{1, 2} and exact solutions with $SO(3)$ and $SO(4)$ groups of spacial symmetry \cite{3, 4}. The exact $SO(3)$-symmetric solutions turn out to be static and singular and are, in fact,
a generalization of Reissner-Nordström solutions to non-abelian gauge theories. The $SO(4)$-symmetric solutions correspond to the Robertson-Walker ansatz for the metric and are interpreted either as wormhole solutions in the euclidean domain [3] or as cosmological solutions for the radiation dominated universe [3].

There were attempts to accommodate spinor fields in this picture as well, but they usually amounted to considering spinor fields in the EYM background. For example, in [7, 8, 9] an anomalous fermion production in the EYM background was discussed.

The study of exact solutions in Einstein-Dirac systems also attracted much attention [10, 11]. It was found that the Robertson-Walker ansatz in such a system leads to the so-called ”ghost solutions” in ED systems, for which the Dirac field has a vanishing energy-momentum tensor.

In the present paper we consider a self-consistent EYMD system and find exact solutions for the case of the gauge groups $SU(2)$ and $SU(4)$. The group $SU(4)$ is the simplest gauge group which gives qualitatively new results in comparison with the group $SU(2)$. Our solutions describe the radiation dominated universe, in which an exchange of energy between the YM and Dirac fields takes place. They can be of interest for treating the dynamics of the early universe, because they are based on the energy-momentum tensor derived from the fundamental Lagrangians of particle physics, rather than on the phenomenological one. Our solutions also solve the problem of the ”ghost solutions” in ED systems.

To find these solutions we employ the dimensional reduction method for gravitational, gauge and spinor fields [12, 13], which enables us to relate symmetric solutions in EYMD system to certain solutions of an effective dynamical system and essentially simplifies the problem of finding symmetric solutions.

2 The effective action

We consider an $SO(4)$-symmetric Einstein-Yang-Mills-Dirac system with the standard action in space-time $M = \mathbb{R} \times S^3$:

$$S = S_E + S_{YM} + S_D,$$  \hspace{1cm} (1)

where

$$S_E = \frac{1}{16\pi\kappa} \int_M (R - \Lambda) \ dv,$$  \hspace{1cm} (2)

$$S_{YM} = \frac{1}{8g^2} \int_M \text{tr}(\hat{F}_{\alpha\beta}\hat{F}^{\alpha\beta}) \ dv,$$  \hspace{1cm} (3)

$$S_D = \int_M \frac{i}{2} \bar{\psi} \gamma^\alpha E_\alpha^\mu \left( \frac{\partial}{\partial x^\mu} - \frac{1}{8} \omega_{\mu\alpha\beta}[\gamma^\alpha, \gamma^\beta] + \delta(\hat{A}_\mu) \right) \psi \ dv + h.c.$$

Here $\Lambda$ denotes the cosmological constant, $R$ is the scalar curvature and $dv = \sqrt{|\det(g)|} d^4x$ is the canonical volume form corresponding to a metric $g$ on $M = \mathbb{R} \times S^3$. 

\( \mathbb{R} \times S^3 \). \( \hat{F} \) and \( \hat{A} \) are the Yang-Mills field strength and potential respectively, and the trace is taken in the adjoint representation of the Lie algebra of the gauge group. In the Dirac action, \( \psi \) denotes the spinor field, \( \{ E_\alpha = E_\alpha^\mu \partial_\mu \} \) is an orthonormal frame on \( M = \mathbb{R} \times S^3 \), \( \{ \gamma^\alpha \} \) are the gamma matrices and \( \delta \) is the representation of the gauge group in spinor space.

We identify \( S^3 \) with the group manifold of \( SU(2) \). Then the action \( \sigma \) of \( K = SO(4) \equiv (SU(2) \times SU(2))/\{ \mathbb{I}, -\mathbb{I} \} \) on \( SU(2) \) is given by

\[
\sigma((k_1, k_2), x) = k_1 x k_2^{-1}, \quad (k_1, k_2) \in SU(2) \times SU(2), \quad x \in SU(2). \tag{5}
\]

The isotropy group \( H \) at \( x = \mathbb{I}_{SU(2)} \) is isomorphic to \( SU(2) \) and is given by

\[
H = \{ (k, k) \in K, \quad k \in SU(2) \} \tag{6}.
\]

Further, we have the reductive decomposition \( K = H \oplus M \) of the Lie algebra \( K \) of \( K \), where

\[
\mathfrak{h} = \{ (X, X) ; \quad X \in su(2) \}, \tag{7}
\]

\[
\mathfrak{m} = \{ (X, -X) ; \quad X \in su(2) \}. \tag{8}
\]

Obviously \( S^3 = (SU(2) \times SU(2))/H \) is a symmetric space, and \( SO(4) \)-invariance is equivalent to invariance under \( SU(2) \times SU(2) \). The isotropy representation of \( H = SU(2) \) in \( \mathfrak{m} \) is \( 3 \), i.e. it is isomorphic to the adjoint representation.

Now we have to reduce the action of the original theory due to the \( SO(4) \)-symmetry. We begin with the action of the gravitational field.

The most general form of an \( SO(4) \)-invariant metric \( g \) on \( M \) is

\[
g = -N^2 dt^2 + a^2 d\Omega^2_{S^3}, \tag{9}
\]

where \( d\Omega^2_{S^3} \) is the standard metric on a 3-sphere of radius 2 resp. \(-1/2\) the Killing metric on \( SU(2) \equiv S^3 \). For the sake of the future convenience we assume \( N \) and \( a \) to have the dimension of length and \( t \) and \( d\Omega^2_{S^3} \) to be dimensionless.

An orthonormal coframe \( \{ \theta^\mu \} \) on \( M \) is given by

\[
\theta^0 = N dt, \quad \theta^i = a \vartheta^i, \quad i = 1, 2, 3, \tag{10}
\]

where \( \vartheta^i \sigma_i/(2i) \equiv \vartheta \) is the canonical left invariant 1-form on \( SU(2) \), \( \sigma_i \) being the Pauli matrices. In what follows we set \( \tau_k = \sigma_k/(2i) \) and denote by \( \eta_{i\alpha\beta} \) the Minkowski metric \( \eta = \text{diag}(-1, 1, 1, 1) \). It is a matter of simple calculations to find the components of the spin connection on \( M \):

\[
\omega_{0i} = -\omega_{i0} = -\dot{a} \frac{1}{aN} \delta_{il} \theta^l, \quad i = 1, 2, 3, \tag{11}
\]

\[
\omega_{ik} = -\frac{1}{2a} \varepsilon_{ikl} \theta^l, \quad i = 1, 2, 3.
\]
Using the standard formulae for the curvature, substituting it into (2) and integrating over \( S^3 \), one easily finds the reduced gravitational action

\[
S_E = \frac{16\pi^2}{16\pi\kappa}\int_{\mathbb{R}} a^3 N \left( \frac{3}{2a^2} + 6 \left( \frac{\dot{a}}{aN} \right)^2 - \frac{\dot{a}^2}{a^2} - \frac{6}{a^3} - \frac{\Lambda}{N} \right) dt ,
\]

where \( 16\pi^2 \) is the volume of \( SU(2) = S^3 \) with the standard metric \( d\Omega^2_{S^3} \). Omitting a complete divergence, we get the effective action

\[
S_E = \frac{16\pi^2}{16\pi\kappa}\int_{\mathbb{R}} \left( \frac{3}{2} aN - \Lambda a^3 N - 6\frac{1}{N} a^2 \right) dt ,
\]

which we consider as the reduced action of the gravitational field.

Next we turn to the gauge field action (3). An \( SO(4) \)-symmetric gauge potential \( \hat{A} \) on \( \mathbb{R} \times S^3 \) is in one-to-one correspondence with a triplet \( \{ \tau, A, \Phi \} \), where \( \tau \) is a homomorphism from the isotropy group \( H \) into the gauge group \( G \)

\[
\tau : H \to G ,
\]

\( A \) is a gauge potential on \( \mathbb{R} \) with values in the centralizer \( C_G(\tau(H)) \) of \( \tau(H) \) in \( G = \text{Lie}(G) \) and \( \Phi \) is a linear mapping

\[
\Phi : \mathbb{R} \to \mathfrak{M}^* \otimes \mathfrak{G}
\]

with

\[
\Phi \circ \text{Ad}(h) = \text{Ad}(\tau(h)) \circ \Phi , \quad \forall h \in H .
\]

Here \( \text{Ad}(h) \) is the restriction to \( H \) of the adjoint representation of \( K \) applied to \( \mathfrak{M} \) and \( \text{Ad}(\tau(h)) \) is the restriction to \( \tau(H) \) of the adjoint representation of \( G \) in \( \mathfrak{G} \). As we have already mentioned, we will consider gauge groups \( SU(2) \) and \( SU(4) \). In the first case the centralizer is trivial, and there is no reduced gauge potential \( A \), whereas in the second case there can be a nontrivial centralizer. Therefore, the case of the group \( SU(4) \) is more general, and we will carry out all the calculations for this case and then explain the difference from the case of \( SU(2) \).

Thus, we take the gauge group \( G = SU(4) \) and define the homomorphism \( \tau : H \to G \) by the decomposition of the fundamental representation \( 4 \) of \( SU(4) \) :

\[
4 \to (2,2) .
\]

i.e. we represent the space \( \mathbb{C}^4 \) by the tensor product \( \mathbb{C}^2 \otimes \mathbb{C}^2 \) and let the fundamental representation of \( \tau(H) \) act on the first factor and the fundamental representation of the centralizer \( C_G(\tau(H)) = SU(2) \) act on the second factor.
It is known that the adjoint representation of $sl(n)$ can be expressed in terms of its fundamental representation by \[ \text{ad} \, sl(n) = n^* \otimes n, \] (18)
where tilde means dropping out a one-dimensional trivial representation and $n^*$ denotes the contragradient representation, $t(x)^* = -t(x)^T$. Therefore we get
\[ \text{ad} \, su(4) = (2 \otimes 2)^* \otimes (2 \otimes 2) \rightarrow (3 \otimes 1) \oplus (3 \otimes 3) \oplus (1 \otimes 3). \] (19)
A basis in $su(4)$ adapted to this decomposition is
\[ H_k = \tau_k \otimes \mathbb{I} = \begin{pmatrix} \tau_k & 0 \\ 0 & \tau_k \end{pmatrix}, \] (20)
\[ P_{kA} = \tau_k \otimes \sigma_A, \] (21)
\[ H_A = \mathbb{I} \otimes \tau_A, \] (22)
where equation (20) explains our rule for evaluating the tensor product of matrices. It is easy to calculate the commutators of these generators, for instance
\[ [P_{kA}, P_{lB}] = \varepsilon_{klm} H_m \delta_{AB} + \varepsilon_{ABC} H_C \delta_{kl}. \] (23)
The constraint (16) means that the mapping $\Phi$ is an intertwining operator, which intertwines the isotropy representation 3 of $H = SU(2)$ in $\mathfrak{m}$ with the representation (19). We introduce basic intertwining operators from $\mathfrak{m}$ into $\mathfrak{g}$ defined by the relations
\[ I((\tau_k, -\tau_k)) = H_k, \] (24)
\[ I_A((\tau_k, -\tau_k)) = P_{kA}. \] (25)
Then for any $X = (X^k \tau_k) \in su(2)$ we have
\[ \Phi((X, -X)) = \xi I(X, -X) + \xi^A I_A(X, -X) \] (26)
\[ = X^k (\xi H_k + \xi^A P_{kA}) = X \otimes (\xi \mathbb{I} + \xi^A \sigma_A), \] (27)
where $\xi$ and $\xi^A, A = 1, 2, 3$ are real valued functions on $\mathbb{R}$. In what follows we denote the vector $(\xi^1, \xi^2, \xi^3)$ by $\tilde{\xi}$ and set $\hat{\xi} = \xi \mathbb{I} + \xi^A \sigma_A$ and $\tilde{\xi} = \xi^A \sigma_A$. The centralizer $\mathfrak{c}_{su(4)}(\tau'(su(2)))$ of $\tau'(su(2))$ in $\mathfrak{g} = su(4)$ is $su(2)$ and is spanned by the Lie algebra elements $H_A, A = 1, 2, 3$. The matrix $\hat{\xi}$ is in the representation $(1 + 3)$ of the centralizer $\mathfrak{c}$ and $i \hat{\xi} \in u(2)$.

The symmetric gauge potential $\hat{A}$ on $\mathbb{R} \times S^3$ can be easily expressed in terms of the matrix $\hat{\xi}$ and the canonical left invariant 1-form $\vartheta = \vartheta^i \tau_i$ on $SU(2) \equiv S^3$:
\[ \hat{A} = \frac{1}{2} \xi \otimes (\hat{\xi} \mathbb{I}) + \vartheta \otimes A_0 dt. \] (28)
Here $A_0$ is a function on $\mathbb{R}$ with values in $su(2)$, i.e. $A_0 = A_0^\tau A$. The term $I \otimes A_0 dt$ is the reduced gauge potential on $\mathbb{R}$, which can be gauged out, but we keep it for the moment, because it is necessary for deriving the equations of motion. It is not difficult to calculate the corresponding field strength $\hat{F}$ and to obtain the reduced Yang-Mills action, which takes the simple form

\begin{equation}
S_{YM} = \frac{16\pi^2}{8g^2} \int_{\mathbb{R}} \left( \frac{a}{2N} \xi^2 + \frac{a}{2N} (\xi + A_0 \times \vec{\xi})^2 \right) \frac{N}{8a} \left( (\xi^2 + (\vec{\xi})^2 - 1)^2 + 4\xi^2(\vec{\xi})^2 \right) dt ,
\end{equation}

where $\vec{A}_0$ is the vector $(A_1^0, A_2^0, A_3^0)$ and $\vec{A}_0 \times \vec{\xi}$ is the vector product of $\vec{A}_0$ and $\vec{\xi}$.

If the gauge group is $G = SU(2)$, the unique nontrivial homomorphism $\tau : H \rightarrow G$ can be defined by the identity mapping, i.e.

\begin{equation}
\tau((k, k)) = k , \quad k \in SU(2) .
\end{equation}

In this case the centralizer $\mathfrak{C}_\Theta(\tau(\mathfrak{F}))$ is trivial, and the intertwining operator is

\begin{equation}
\Phi((X, -X)) = \xi(t)X , \quad X \in su(2) \, , \, \xi(t) \in \mathbb{R} .
\end{equation}

It is clear that the gauge potential $\hat{A}$ on $\mathbb{R} \times S^3$ still has the form (28) with $\vec{\xi} = 0, A_0 = 0$. The reduced Yang-Mills action is also given by (29), provided one puts $\vec{\xi} = 0$ and rescales the coupling constant $g \rightarrow 2g$.

Next we have to reduce the action (4) for symmetric spinor field. We choose $\delta$ to be the fundamental representation, i.e. we can write the spinor $\psi$ as a $4 \times 4$ resp. $4 \times 2$ matrix on which an element $g$ of $SU(4)$ resp. $SU(2)$ acts via right multiplication by $g^{-1}$.

In accordance with our choice of the metric signature (3), we have

\begin{equation}
\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu} , \quad \eta^{\mu\nu} = \text{diag}(--++) , \quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 ,
\end{equation}

\begin{equation}
\gamma^0 = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} , \quad \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix} , \quad \gamma^5 = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix} , \quad i = 1, 2, 3 .
\end{equation}

The bispinor representation $\Delta(s)$ is defined by

\begin{equation}
\Delta(s)^{-1}\gamma^\mu\Delta(s) = \Lambda(s)^\mu_\rho\gamma^\rho ,
\end{equation}

where $s$ is an element of the group $Spin(1, 3)$ and $\Lambda$ is the covering homomorphism from $Spin(1, 3)$ onto $SO(1, 3)$. On Lie algebra level we get

\begin{equation}
[\Delta'(A), \gamma^\mu] = -A^\mu_\nu\gamma^\nu , \quad \Delta'(A) = \frac{1}{8}A^\mu_\nu[\gamma_\mu, \gamma^\nu] ,
\end{equation}
where \( A_{\mu\nu} = -A_{\nu\mu} \) is an element of \( so(1,3) \equiv spin(1,3) \equiv sl(2,\mathbb{C}) \).

An \( SO(4) \)-symmetric spinor field \( \psi \) on \( \mathbb{R} \times S^3 \) is in one-to-one correspondence with a matrix valued function \( \rho \) on \( \mathbb{R} \), which satisfies the condition

\[
(\delta'(\tau'(h)) + \Delta'(\lambda'(h))) \rho = 0 \quad , \forall h \in \mathfrak{h} .
\tag{36}
\]

Here \( \lambda' : \mathfrak{h} \rightarrow so(1,3) \) is the homomorphism induced by the isotropy representation, which can be calculated explicitly: \( \lambda'(\tau_a^b)_c = -\varepsilon_a^{bc} \). Therefore, if the gauge group \( G \) is \( SU(4) \) equation (36) reads

\[
\left( \frac{1}{4} \gamma^j \gamma^i \varepsilon_{ijk} + \delta' (H_k) \right) \rho = ((\tau_k \otimes \mathbb{I}) \rho - \rho (\tau_k \otimes \mathbb{I})) = 0 .
\tag{37}
\]

The general solution of this constraint equation is

\[
\rho = \left( \begin{array}{ll} u_1 \mathbb{I} & u_2 \mathbb{I} \\ v_1 \mathbb{I} & v_2 \mathbb{I} \end{array} \right) \otimes \left( \begin{array}{ll} u_1 & u_2 \\ v_1 & v_2 \end{array} \right) , \quad u_1, u_2, v_1, v_2 \in C^\infty(\mathbb{R}) ,
\tag{38}
\]

i.e. a symmetric spinor on \( \mathbb{R} \times S^3 \) is parameterized by two complex doublets \( u = (u_1, u_2)^T \) and \( v = (v_1, v_2)^T \), one for each chirality. We see from (38) that the reduced gauge group \( C = SU(2) \) acts on both doublets by the fundamental representation. Taking into account equations (11) and (28), it is a matter of simple calculations to get the reduced action

\[
S_D = 16\pi^2 \int_{\mathbb{R}}^{} a^3 N \left( \frac{i}{N}(\bar{u} \dot{u} - \dot{\bar{u}} \bar{u} + \bar{\bar{v}} \dot{\bar{v}} - \dot{\bar{v}} v) + \frac{1}{N} A_0 B (\bar{u} \sigma_B u + \bar{v} \sigma_B v) - \frac{3}{2a} (\bar{u} \hat{\xi} u - \bar{v} \hat{\xi} v) \right) dt .
\tag{39}
\]

If the gauge group \( G \) is \( SU(2) \), equation (39) reads

\[
((\tau_k \otimes \mathbb{I}) \rho - \rho \tau_k) = 0 ,
\tag{40}
\]

and we get

\[
\rho = \left( \begin{array}{l} u \mathbb{I} \\ v \mathbb{I} \end{array} \right) = \mathbb{I} \otimes \left( \begin{array}{l} u \\ v \end{array} \right) , \quad u, v \in C^\infty(\mathbb{R}) ,
\tag{41}
\]

i.e. a symmetric spinor on \( \mathbb{R} \times S^3 \) for \( G = SU(2) \) depends on two arbitrary complex functions \( u \) and \( v \), one for each chirality. The reduced action has the same form (39), if we put there \( \hat{\xi} = 0, A_0 = 0 \).

Now we can write down the reduced action of the coupled EYMD system. In
the case of the gauge group $SU(4)$ it has the form

$$S = S_E + S_{YM} + S_D$$

$$= 16\pi^2 \int_R \left\{ \frac{1}{16\pi\kappa} \left( \frac{3}{2} aN - \Lambda a^2 N - 6 \frac{a}{N} \dot{a}^2 \right) + \frac{24}{8g^2} \left( \frac{a}{2N} \xi^2 + \frac{a}{2N} (\dot{\xi} + \dot{A}_0 \times \ddot{\xi})^2 - \frac{N}{8a} ((\dot{\xi}^2 + (\ddot{\xi})^2 - 1)^2 + 4\xi^2 (\ddot{\xi}^2) \right) + \left( ia^3 (\ddot{u} \dot{\bar{u}} - \ddot{\bar{u}} u + \ddot{v} \dot{\bar{v}} - \ddot{\bar{v}} v) + a^3 A_0^B (\ddot{u} \sigma_B u + \ddot{v} \sigma_B v) - \frac{3a^2 N}{2} (\ddot{u} \dot{\bar{u}} u - \ddot{\bar{v}} v) \right) \right\} \, dt . \quad (42)$$

If we choose the gauge group to be $SU(2)$, we have to put $\ddot{\xi} = 0$, $A_0 = 0$ in this action, to rescale the coupling constant $g \rightarrow 2g$, and to take into account that the variables $u$ and $v$ are no longer isospinors, but ordinary functions.

### 3 The field equations and solutions

Variation of this action with respect to $a$, $N$, $\xi$, $\ddot{\xi}$, $u$, $v$ and $\ddot{u}$, $\ddot{v}$ gives us the field equations. When taken in the special gauge $A_0 = 0$, $a = N$ (the latter
condition means that \( t \) is now the conformal time, they have the form
\[
\frac{a}{\delta a} S = \frac{1}{16\pi\kappa} \left( \frac{3}{2} a^2 - 3\Lambda a^4 - 6a^2 + 12a\dot{a} \right) \\
+ \frac{3}{2g^2} \left( \dot{\xi}^2 + (\ddot{\xi})^2 + \frac{1}{4} \left( (\xi^2 + (\ddot{\xi})^2 - 1)^2 + 4\xi^2(\ddot{\xi})^2 \right) \right) \\
+ 3a^3 \left( i(\ddot{u}u - \dot{u}u + \ddot{v}v - \dot{v}v) - (\ddot{u}\dot{\xi}u - \ddot{v}\dot{\xi}v) \right) = 0, \quad (43)
\]
\[
\frac{a}{\delta N} S = \frac{1}{16\pi\kappa} \left( \frac{3}{2} a^2 - \Lambda a^4 + 6a^2 \right) \\
- \frac{3}{2g^2} \left( \dot{\xi}^2 + (\ddot{\xi})^2 + \frac{1}{4} \left( (\xi^2 + (\ddot{\xi})^2 - 1)^2 + 4\xi^2(\ddot{\xi})^2 \right) \right) \\
- \frac{3}{2} a^3 (\ddot{u}\dot{\xi}u - \ddot{v}\dot{\xi}v) = 0, \quad (44)
\]
\[
\frac{\delta}{\delta \xi} S = - \frac{3}{2g^2} \left( 2\ddot{\xi} + (\xi^2 + (\ddot{\xi})^2 - 1) + 2\xi(\ddot{\xi})^2 \right) \\
- \frac{3}{2} a^3 (\ddot{u}u - \ddot{v}v) = 0, \quad (45)
\]
\[
\frac{\delta}{\delta \xi^A} S = - \frac{3}{g^2} \left( \dot{\xi}^A + \frac{1}{2} \left( \xi^A(\xi^2 + (\ddot{\xi})^2 - 1) + 2\xi^2\xi^A \right) \right) \\
- \frac{3}{2} a^3 (\ddot{u}\sigma^A u - \ddot{v}\sigma^A v) = 0, \quad (46)
\]
\[
\frac{1}{2} a^{-\frac{3}{2}} \frac{\delta}{\delta \bar{u}} S = \frac{i}{4} \frac{d}{dt} a^\frac{3}{2} u - \frac{3}{4} \xi a^\frac{3}{2} u = 0, \\
\frac{1}{2} a^{-\frac{3}{2}} \frac{\delta}{\delta \bar{v}} S = \frac{i}{4} \frac{d}{dt} a^\frac{3}{2} \bar{u} + \frac{3}{4} a^\frac{3}{2} \bar{\xi} = 0, \quad (47)
\]
\[
\frac{1}{2} a^{-\frac{3}{2}} \frac{\delta}{\delta \bar{v}} S = \frac{i}{4} \frac{d}{dt} a^\frac{3}{2} v + \frac{3}{4} \xi \dot{a} v \dot{\xi} = 0, \\
\frac{1}{2} a^{-\frac{3}{2}} \frac{\delta}{\delta \bar{v}} S = \frac{i}{4} \frac{d}{dt} a^\frac{3}{2} \bar{v} - \frac{3}{4} a^\frac{3}{2} \bar{\xi} \dot{v} = 0. \quad (48)
\]

Variation with respect to \( A_0 \) gives a constraint
\[
\frac{\delta S}{\delta A_0^B} = \frac{3}{g^2} \left( \varepsilon_{BCD} \xi^C \xi^D \right) + a^3 (\ddot{u}\sigma_B u + \ddot{v}\sigma_B v) = 0, \quad (49)
\]
which means that the total isospin of the gauge and the spinor fields equals zero.

Now we will show that it is possible to find exact solutions to this system of equations. We begin with the simpler case of the gauge group \( SU(2) \). There is no constraint (49) in this case, and we also have to drop the equation (46), to put \( \bar{\xi} = 0 \) in the others and to rescale \( g \to 2g \).

The Dirac equations (47) and (48) give
\[
\bar{u}u = \frac{C_u}{a^3}, \quad \bar{v}v = \frac{C_v}{a^3} \quad (50)
\]
and

\[
\bar{u} \dot{u} - \bar{u} u = -\frac{3i}{2} \bar{\xi} u, \quad (51)
\]

\[
\bar{v} \dot{v} - \bar{v} v = \frac{3i}{2} \bar{\xi} v, \quad (52)
\]

where \(C_u\) and \(C_v\) are arbitrary positive constants. We will discuss the meaning of these constants later, here we assume that they are proportional to the number of fermions with positive resp. negative chirality on \(S^3\).

Now equations (43) and (44) simplify considerably. Their sum gives

\[
\frac{1}{16\pi\kappa} (3a^2 - 4\Lambda a^4 + 12a\ddot{a}) = 0. \quad (53)
\]

Multiplying this equation by \(\dot{a}/a\) and integrating we see that

\[
\frac{3}{2} a^2 - \Lambda a^4 + 6a^2 = E, \quad (54)
\]

where \(E\) is an arbitrary constant which has the meaning of the total energy of the system. Equation (54) is the standard Friedmann equation for the radiation dominated universe and has a simple analog in mechanics. We can consider \(a\) as the coordinate of a particle with mass 1 and energy \(E/12\) which moves in a potential

\[
W(a) = \frac{1}{8} a^2 - \frac{1}{12} \Lambda a^4. \quad (55)
\]

If \(\Lambda < \Lambda_E = \frac{9}{16E}\), then the motion will be periodical. This means that our solution describes a universe which first expands and then contracts, where \(a = 0\) corresponds to a singular metric in the beginning and the end. If \(\Lambda \geq \Lambda_E\), then the solution can be either static or can describe an expanding universe.

Next we turn to the YM equation (45) (we recall that we have rescaled \(g \rightarrow 2g\) in the case under consideration). Due to equation (50), it decouples from the equations for \(u\) and \(v\):

\[
\frac{3}{8g^2} \left(2\ddot{\xi} + (\xi(\xi^2 - 1))\right) + \frac{3}{2} (C_u - C_v) = 0. \quad (56)
\]

The first integral of this equation is

\[
\frac{3}{8g^2} \left(\xi^2 + \frac{1}{4}(\xi^2 - 1)^2\right) + \frac{3}{2} \xi (C_u - C_v) = \frac{E}{16\pi\kappa}, \quad (57)
\]

where the integration constant is due to equations (44) and (44).
Equation (57) also has an analogue in mechanics. A point particle with mass \( m_1 \), energy \( \frac{E g^2}{12 \pi \kappa} \) and coordinate \( \xi \) moves in a double-well potential

\[
V(\xi) = \frac{1}{8}(\xi^2 - 1)^2 + 2g^2 \xi (C_u - C_v) .
\]  

(58)

We can interpret the first term in (57) as the energy of the Yang-Mills field and the second term as the energy of the Dirac field due to the interaction with the gauge field. The equation describes an exchange of energy of the two fields, the coupling between the gauge and the spinor field being proportional to the difference of the numbers of left and right handed fermions \( C_u - C_v \). The exact solution of equation (57) is possible in terms of elliptic functions of the first kind [15].

We consider for instance the case, where the energy \( \frac{E g^2}{12 \pi \kappa} \) of the system is larger than the local maximum of the potential \( V(\xi) \). Then the system will move between the turning points defined by the real zeros of the polynomial

\[
\frac{E g^2}{12 \pi \kappa} - V(\xi).
\]

(59)

In the case under consideration we have two real zeros \( \alpha \) and \( \beta \), \( \alpha > 0 > \beta \), and two complex conjugated zeros \( m - in \) and \( m + in \), i.e.

\[
\frac{1}{8}(\alpha - \xi)(\xi - \beta)((\xi - m)^2 + n^2) = \frac{E g^2}{12 \pi \kappa} - V(\xi) = \frac{1}{2} \xi^2 .
\]

(60)

We have to solve the integral

\[
t(\xi) = \int_\beta^\xi \frac{dx}{\sqrt{\frac{E g^2}{6 \pi \kappa} - \frac{1}{4}(x^2 - 1)^2 - 4g^2 x (C_u - C_v)}} .
\]

(61)

Its exact solution is given by [15]

\[
t(\xi) = \frac{2}{\sqrt{pq}} F\left( \begin{array}{c} 2 \arccctg \sqrt{\frac{q(\alpha - \xi)}{p(\xi - \beta)}}, \frac{1}{2} \sqrt{\frac{-(p - q)^2 + (\alpha - \beta)^2}{pq}} \end{array} \right) ,
\]

(62)

where \( F \) is the elliptic integral of the first kind and \( p, q \) are defined by

\[
p^2 = (m - \alpha)^2 + n^2 \quad , \quad q^2 = (m - \beta)^2 + n^2 .
\]

(63)

This function can be inverted, and we can get \( \xi(t) \) expressed in terms of the Jacobi elliptic function \( \text{sinus amplitudinis} \).

With a given solution \( \xi \) we can solve the Dirac equations (47) and (48)

\[
u = \sqrt{C_v \over a^3} \exp \left\{ -i \int \frac{3}{4} \xi dt \right\} ,
\]

(64)

\[
u = \sqrt{C_v \over a^3} \exp \left\{ i \int \frac{3}{4} \xi dt \right\} .
\]

(65)
One easily checks that our solution fulfills the whole system of field equations.

Now we consider the case of gauge group $SU(4)$. The Dirac equations (47) and (48) again lead to (50), but instead of (51) and (52) we now have

$$\bar{u} \dot{u} - \dot{\bar{u}} u = -\frac{3i}{2} \bar{u} \dot{\xi} u, \quad (66)$$

$$\bar{v} \dot{v} - \dot{\bar{v}} v = \frac{3i}{2} \bar{v} \dot{\xi} v. \quad (67)$$

These equations are also sufficient to decouple the Friedmann equation from the Yang-Mills-Dirac equations, and we get again the equation (54) for the scale factor $a$.

Now we have to solve the Yang-Mills equations (45) and (46). We start with the discussion of the constraint (49). In what follows we restrict ourselves to the case, where

$$\varepsilon_{BCD} \xi^C \dot{\xi}^D = 0 \quad \text{for} \quad B = 1, 2, 3, \quad (68)$$

i.e. the isospins of the gauge field and the Dirac field vanish separately. This equation means that the angular momentum of the motion in the $\xi$-space equals zero, that is the motion goes along a straight line passing through the origin, and the vector $\xi$ is always proportional to a fixed vector $\xi_0$. We use the remaining gauge freedom to choose $\xi_0 = (0, 0, 1)$, i.e. $\dot{\xi} = \xi_3, \xi \in \mathbb{R}$. In this gauge we obtain from the Yang-Mills equations (46)

$$\bar{u} \sigma_A u - \bar{v} \sigma_A v = 0 \quad \text{for} \quad A = 1, 2. \quad (69)$$

Further we get from equations (49) and (68)

$$\bar{u} \sigma_B u + \bar{v} \sigma_B v = 0, \quad B = 1, 2, 3. \quad (70)$$

Hence, we have $\bar{u} \sigma_1 u = \bar{u} \sigma_2 u = 0$. These equations have two solutions:

1st case: $u = a^{-3/2} \alpha_1 w_+, v = a^{-3/2} \beta_1 w_-$ \quad (71)

2nd case: $u = a^{-3/2} \alpha_2 w_-, v = a^{-3/2} \beta_2 w_+$ \quad (72)

where $w_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $w_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ \quad (73)

and $\alpha_i, \beta_i \in \mathbb{C}$, $|\alpha_i| = |\beta_i|, i = 1, 2$. In particular, we obtain $C_u = C_v = |\alpha_i|^2$ in the i'th case. If $\xi = \xi_3 \equiv 0$, then the Yang-Mills equations (46) demand $u = v = 0$. Therefore, we consider only the nontrivial case, when $\xi \neq 0$. Hence we have two Yang-Mills equations (46) and (48), which now take the form:

$$\frac{3}{2g^2} \left(2 \ddot{\xi} + (\xi (\xi^2 + \xi^2 - 1) + 2 \xi \zeta^2)\right) = 0, \quad (74)$$

$$\frac{3}{2g^2} \left(2 \ddot{\zeta} + (\zeta (\xi^2 + \xi^2 - 1) + 2 \xi^2 \zeta)\right) + 3S = 0, \quad (75)$$
with \( S := C_u = |\alpha_1|^2 \) in the first and \( S := -C_u = -|\alpha_2|^2 \) in the second case.

To solve equations (74) and (75), we pass to new variables in accordance with
\[
\xi = \frac{1}{2}(x + y), \\
\zeta = \frac{1}{2}(x - y).
\]  

(76)

It is easy to check that the equations for \( x \) and \( y \) decouple and take the form
\[
\frac{3}{2g^2} \left( 2\dot{x} + x(x^2 - 1) \right) + 3S = 0, \\
\frac{3}{2g^2} \left( 2\dot{y} + y(y^2 - 1) \right) - 3S = 0.
\]

(77) (78)

The first integrals of these equations are
\[
\frac{3}{2g^2} \left( x^2 + \frac{1}{4}(x^2 - 1)^2 \right) + 3Sx = \frac{E_1}{8\pi\kappa}, \\
\frac{3}{2g^2} \left( y^2 + \frac{1}{4}(y^2 - 1)^2 \right) - 3Sy = \frac{E_2}{8\pi\kappa},
\]

where due to (44) and (54) the constants \( E \) fulfill
\[
E_1 + E_2 = E.
\]

(81)

These equations can also be solved exactly in terms of Jacobi elliptic functions, but unlike the case of the group \( SU(2) \), there will be two different periods of motion in \( x \) and \( y \).

We would like to note here that equations of the type (74), (75) with \( S = 0 \) for the Euclidean EYM system were first found in [5], but they were solved there only for the case either \( \xi = 0 \) or \( \zeta = 0 \).

Substituting the solutions for \( \xi \) and \( \zeta \) into equation (28) we get
\[
\hat{A} = \frac{1}{2} \theta \otimes \left( \hat{\xi} + \hat{\eta} \right) = \begin{pmatrix}
\frac{1+x}{2} \theta & 0 \\
0 & \frac{1+y}{2} \theta
\end{pmatrix},
\]

(82)

i.e. the gauge potential \( \hat{A} \) takes values only in an \( su(2) \oplus su(2) \) subalgebra of \( su(4) \), each \( su(2) \) part of the gauge potential being coupled to only one of the spinor fields \( u \) resp. \( v \).

If we have a solution to the system of equations (74) and (80) it is easy to integrate the Dirac equations (47) and (48). With given solutions \( x \) and \( y \) we obtain
\[
u = w_+ \sqrt{\frac{C_u}{a^2}} \exp \left\{ -i \int \frac{3}{4} x \, dt \right\}, \\
v = w_- \sqrt{\frac{C_u}{a^2}} \exp \left\{ i \int \frac{3}{4} y \, dt \right\}
\]

(83) (84)
in the first and

\[ u = \pm \sqrt{\frac{C_u}{a^3}} \exp \left\{ \pm i \int \frac{3}{4} \, y \, dt \right\}, \quad (85) \]

\[ v = \pm \sqrt{\frac{C_u}{a^3}} \exp \left\{ \mp i \int \frac{3}{4} \, x \, dt \right\}. \quad (86) \]

in the second case.

4 Discussion

In studying the self-consistent EYMD system we found that the evolution of the metric decouples from the remaining system and is described by the Friedmann equation for the radiation dominated universe. The same result for the case of EYM systems was obtained earlier in [5, 6], and solutions for the spinor field in this background were studied in [8, 16]. Unlike the latter solutions, our solutions take into account the back reaction of the spinor field on the YM field.

The Yang-Mills equations in the case \( G = SU(2) \) admit three static solutions: two minima and one local maximum of the potential

\[ V(\xi) = \frac{1}{8} (\xi^2 - 1)^2 + 2g^2 \xi (C_u - C_v). \quad (87) \]

Of course, these are solutions of the whole system only if the constant \( E \) is equal to \( V(\xi) \). If we have \( C_u = C_v \), we can interpret these extrema as two vacua (\( \xi = -1, \xi = +1 \)) with Chern-Simons numbers 0 and 1 and as a sphaleron-like solution (\( \xi = 0 \)) with Chern-Simons number 1/2 lying on top of the potential barrier between the vacua [16, 8]. Chiral spinor fields shift slightly the location of the extrema and the Chern-Simons index of the corresponding gauge field configurations, which is quite natural in the presence of matter fields [17, 18, 19].

By fine tuning the cosmological constant \( \Lambda \) and the energy \( E \), we can get a static sphaleron-like solution of the whole system. This solution corresponds to the local maxima of \( V(\xi) \) and \( W(a) \), see equations (58) and (59). Obviously, this solution has two unstable modes – one in the gravitational and one in the gauge field sector. This is another indication that the static solution is a cosmological analog of the first Bartnik-McKinnon solution [20, 21, 22].

At this point, we have to comment on the meaning of our classical spinor field. The problem of interpreting spinor fields in cosmology has been discussed for many years [10, 11], but so far no satisfactory solution has been found. It is clear that our solution describes just one energy level of the Dirac field in the EYM background, which is exactly the so called zero mode. If we assume that the Dirac field is normalized to unity, the influence of this field on the EYM system is negligible. Therefore, we suggest that the Dirac field of our solution
is normalized arbitrarily and is, in fact, an effective field describing fermionic matter with $SO(4)$-invariant energy-momentum tensor. This assumptions seems to be reasonable in the cosmological setting, because the energy levels of the Dirac field in the EYM background must be very dense, and replacing the contribution of fermions on the lowest levels by that of the zero mode level could be a good approximation.

Equation (44) is the $(0,0)$ component of the Einstein equations, and therefore we can interpret the constant $E$ as the total energy of the YMD system. On the other hand, equations (57) resp. (79) and (80) describe the exchange of energy between Yang-Mills and spinor field. This interpretation is supported by the solutions (64) and (65) for the spinor field in the case of the gauge group $SU(2)$: the momentary frequency resp. energy of the spinor field is given by the integrand in the exponent of the solutions and this is up to a factor $\xi$.

This observation also means that our solutions describe creation and annihilation of fermions. If the total energy of the YMD system is larger than the local maximum of the potential $V(\xi)$, the motion in the variable $\xi$, stemming from the YM field, will be periodical. When $\xi$ crosses the maximum of the potential $V$, it changes its sign. With the above interpretation we see that the energy of the spinor field also changes its sign. This is in accordance with the observations in [8, 9, 23], where it was shown that the spinor field has zero modes in the sphaleron-background and that moving between neighbouring vacua of the gauge field results in a shift of the energy level of the spinor field. We also obtain from our solution that the effect is opposite for the spinor field with opposite chirality. The corresponding violation of the fermion number can be calculated explicitly.

The classical $U(1)$ vector current of the Dirac field is given by

$$j = \bar{\psi} \gamma_\alpha \psi \theta^\alpha = \frac{C_u + C_v}{a^3} \theta^0.$$ (88)

This current is classically conserved, i.e. $d(*j) = 0$. Here the star denotes the Hodge star and $\{ \theta^\mu \}$ is the orthonormal coframe on $R \times S^3$, see equation (10). But if one considers chiral matter on quantum level, this current has an anomaly [24, 19]:

$$d(*j) = (1/16\pi^2) \text{tr}(\hat{F} \wedge \hat{F}).$$ (89)

Integrating (89) over $I \times S^3$, $I = [t_i, t_f]$, we get

$$N_F(t_f) - N_F(t_i) = (1/16\pi^2) \int_{I \times S^3} \text{tr}(\hat{F} \wedge \hat{F}) = \int_{S^3} Q(t_f) - \int_{S^3} Q(t_i),$$ (90)

where $Q$ is the Chern-Simons 3-form

$$Q = \frac{1}{16\pi^2} \text{tr}(\hat{A} \wedge d\hat{A} + \frac{1}{3} \hat{A} \wedge [\hat{A}, \hat{A}]).$$ (91)
and
\[ N_F(t) = \int_{\{t\} \times S^3} * j \] (92)
is the total fermion number at time \( t \). Hence, for chiral fermions one obtains on quantum level that the change in the fermion number is given by the difference of the topological charges of the gauge field at the two ends of the time interval \( I \). The topological charge
\[ q = \int_{S^3} Q \]
at any fixed \( t \) can be calculated from the formula for the symmetric potential \( \hat{A} \), which gives
\[ q(t) = \frac{1}{4}(2 + 3\xi(t) - \xi(t)^3). \]
If we pass from the configuration with \( \xi = -1 \) at \( t = t_i \) to the configuration with \( \xi = 1 \) at \( t = t_f \), the topological charge changes by 1. Setting in our solution one of the constants, \( C_u \) or \( C_v \), equal to zero, which implies that we have a chiral theory with either left handed or right handed fermions from the very beginning, we can interpret the corresponding energy level crossing of the Dirac field as creation or annihilation of fermions. We would like to emphasize that this process takes place in real time, in contrast to instanton like effects related to barrier penetration in Euclidean space time.

Finally, we shortly comment on the case of gauge group \( SU(4) \). Under assumption (68), we can completely solve the field equations. We get two spinor fields with opposite chirality and equal density (\( C_u = C_v \)), see equations (83) and (84), resp. (85) and (86). Comparing them to (82) we find that the Yang-Mills potential takes values only in an \( su(2) \oplus su(2) \) subalgebra of \( su(4) \) and, therefore, splits into two parts. Each part is coupled to one spinor field of a definite chirality. Thus, in some sense, we simply have a doubling of the solution for \( SU(2) \).

As it was mentioned above, equations (79) and (80) describe the exchange of energy between spinor and gauge field. Therefore, in the case of gauge group \( SU(4) \) we also have energy-level crossing in the evolution of the spinor field. But, in contrast to the \( SU(2) \)-case, we have - due to constraints (39) and (70) - only solutions with equal density of left and right handed fermions, i.e. we have no chiral solutions.

**Acknowledgements**

The authors are grateful to Yu. Kubyshin for fruitful discussions. One of the authors (I.V.) is grateful to the Center of Natural Sciences of the University of Leipzig for the warm hospitality extended to him during his stay in Leipzig. He also acknowledges partial support under the INTAS-93-1630-EXT project.
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