Communication, and concurrency with logic-based restriction inside a calculus of structures

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Abstract

It is well known that we can use structural proof theory to refine, or generalize, existing paradigmatic computational primitives, or to discover new ones. Under such a point of view we keep developing a programme whose goal is establishing a correspondence between proof-search of a logical system and computations in a process algebra. We give a purely logical account of a process algebra operation which strictly includes the behavior of restriction on actions we find in Milner CCS. This is possible inside a logical system in the Calculus of Structures of Deep Inference endowed with a self-dual quantifier. Using proof-search of cut-free proofs of such a logical system we show how to solve reachability problems in a process algebra that subsumes a significant fragment of Milner CCS.

1 Introduction

This is a work in structural proof-theory which builds on [1,4,5,6]. Broadly speaking we aim at using structural proof theory to study primitives of paradigmatic programming languages, and to give evidence that some are the natural ones, while others, which we might be used to think of as “given once for all”, can, in fact, be refined or generalized. In our case this means to keep developing the programme in [1] whose goal is establishing a correspondence between proof-search of a logical system, and computations in a process algebra. From [1], we already know that both (i) sequential composition of Milner CCS [3] gets modeled by the non commutative logical operator Seq of BV [2], which is the paradigmatic calculus of structures in Deep Inference, and (ii) parallel composition of Milner CCS gets modeled by the commutative logical operator Par of BV so that communication becomes logical annihilation. This is done under a logic-programming analogy. It says that the terms of a calculus C — which is a fragment of Milner CCS in the case of [1] — correspond to formulas of a logical system L — which is BV in the case of [1] —, and that computations inside C recast to searching cut-free proofs in L, as summarized in (1) here below.

| Paradigmatic calculus C | Logical system L |
|------------------------|-----------------|
| term                   | formula         |
| step of computation    | logical rule    |
| computation            | searching a cut-free proof |

(1)
Contributions. We show that in \cite{1} we can take BVQ \cite{4,5,6} for \mathcal{L}, and CCS_{spq} for \mathcal{C}. The system BVQ extends BV with a self-dual quantifier, while CCS_{spq} is introduced by this work (Section 6). The distinguishing aspect of CCS_{spq} is its operational semantics which subsumes the one of the fragment of Milner CCS that contains sequential, parallel, and restriction operators, and which we identify as CCS_{spr}. Specifically, the self-dual quantifier of CCS_{spq} allows to relax the operational semantics of the restriction operator in CCS_{spr} without getting to an inconsistent calculus of processes. This is a direct consequence of (the analogous of) the a cut-elimination property for BVQ \cite{4,5,6}.

The main step that allows to take BVQ for \mathcal{L}, and CCS_{spq} for \mathcal{C} is proving Soundness of BVQ with respect to CCS_{spq} (Section 8). The following example helps explaining what Soundness amounts to. Let us suppose we want to observe what the following judgment

\[ ((a.b.E) \mid (\overline{a}.F))_a \xrightarrow{b} (E \mid F)_a \]  

\( (2) \)

The process \(a.b.E\) can perform actions \(a\), and \(b\), in this order, before entering \(E\). The other process can perform \(\overline{a}\) before entering \(F\). In particular, \(a.b.E\), and \(\overline{a}.F\) internally communicate when simultaneously firing \(a\), and \(\overline{a}\). In any case, firing on \(a\), or \(\overline{a}\), would remain private because of the outermost restriction \(\cdot|_a\) which hides both \(a\), and \(\overline{a}\) to the environment.\(1\) The action \(b\) is always observable because \(b\) differs from \(a\). Of course, we might describe one of the possible dynamic evolutions of \((2)\) thanks to a suitable labeled transition system able to develop a derivation like \((3)\):

\[
\begin{array}{c}
\frac{a.b.E \xrightarrow{a} b.E}{(a.b.E) \mid (\overline{a}.F)} \xrightarrow{a} ((b.E) \mid F)_{ia} \xrightarrow{a \neq a} (E \mid F)_{ia} \\
\frac{a.b.E \xrightarrow{b} E}{(b.E) \mid F} \xrightarrow{b \neq a} (E \mid F)_{ia} \\
\end{array}
\]

\( (3) \)

Soundness says that instead of rewriting \(a.b.E\) to \(\overline{a}.F\), as in \((3)\), we can (i) compile the whole judgment \((a.b.E) \mid (\overline{a}.F))_{ia} \xrightarrow{b} (E \mid F)_{ia}\) to a structure, say \(R\), of BVQ, and (ii) search for a cut-free proof, say \(P\) of \(R\), and (iii) if \(P\) exists, then Soundness assures that \((2)\) holds. So, in general, Soundness recasts the reachability problem “Is it true that \(E \xrightarrow{a} F\)” to a problem of proof search. Noticeably, the Soundness we prove poses weaker constraints on the form of \(F\) than those ones we find in Soundness of \(1\). Specifically, only the silent process \(0\) can be the target of the reachability problem in \(11\). Here, \(F\) can belong to the set of simple processes which contains \(0\). Intuitively, every simple process different from \(0\) is normal with respect to internal communication, but is alive if we consider the external ones. Finally, from a technical standpoint, our proof of Soundness in neatly decomposed in steps that makes it reusable for further extensions of both BVQ, and CCS_{spr}.

Road map. Section 2 recalls BVQ and its symmetric version SBVQ mainly from \cite{6}. Section 3 is about two proof-theoretical properties of BVQ which were not proved in \cite{4,5,6} but which Soundness relies on. The first one says that every Tensor-free derivations of BVQ has at least corresponding standard one. The second one supplies sufficient conditions for a

\(1\)We write something related to Milner CCS. Indeed, hiding both \(a\), and \(\overline{a}\) in Milner CCS is \(\cdot|_a\overline{a}\).
structure of BVQ to be invertible, somewhat internalizing derivability of BVQ. Section 5 has the pedagogical aim of showing, with many examples, why the derivations of BVQ embody a computational meaning. Section 6 introduces CCS_{spq}, namely the process calculus that BVQ embodies. Section 7 first formalizes the connections between BVQ, and CCS_{spq}. Then it shows how computations inside the labeled transition system of CCS_{spq} recast to proof-search inside BVQ, justifying the need to prove Soundness. Section 8 proves Soundness, starting with a pedagogical overview of what proving it means. Section 9 points to future work, mainly focused on CCS_{spq}.

2 Recalling the systems SBVQ and BVQ

We briefly recall SBVQ, and BVQ from [6].

**Structures.** Let \( a, b, c, \ldots \) denote the elements of a countable set of positive propositional variables. Let \( \bar{a}, \bar{b}, \bar{c}, \ldots \) denote the elements of a countable set of negative propositional variables. The set of names, which we range over by \( l, m, n, \ldots \), contains both positive, and negative propositional variables, and nothing else. Let \( \circ \) be a constant, different from any name, which we call unit. The set of atoms contains both names and the unit, while the set of structures identifies formulas of SBV. Structures belong to the language of the grammar in (4).

\[
R ::= \circ \mid l \mid \bar{R} \mid (R \otimes R) \mid (R \cdot R) \mid [R \otimes R] \mid [R]_u \tag{4}
\]

We use \( R, T, U, V \) to range over structures, in which \( \bar{R} \) is a Not, \( (R \otimes T) \) is a CoPar, \( (R \cdot T) \) is a Seq, \( [R \otimes T] \) is a Par, and \( [R]_u \) is a self-dual quantifier Sdq, which comes with the proviso that \( a \) must be a positive atom. Namely, \( [R]_\bar{a} \) is not in the syntax. Sdq induces obvious notions of free, and bound names [6].

**Size of the structures.** The size \(|R|\) of \( R \) is the number of occurrences of atoms in \( R \) plus the number of occurrences of Sdq that effectively bind an atom. For example, \(|[a \otimes \bar{a}]| = |([a \otimes \bar{a}])_a| = 2\), while \(|[a \otimes \bar{a}])_\bar{a}| = 3\).

**(Structure) Contexts.** We denote them by \( S\{ \} \). A context is a structure with a single hole \( \{ \} \) in it. If \( S[R] \), then \( R \) is a substructure of \( S \). We shall tend to shorten \( S[[R \otimes U]] \) as \( S[R \otimes U] \) when \( [R \otimes U] \) fills the hole \( \{ \} \) of \( S\{ \} \) exactly.

**Congruence \( \approx \) on structures.** Structures are partitioned by the smallest congruence \( \approx \) we obtain as reflexive, symmetric, transitive and contextual closure of the relation \( \sim \) whose defining clauses are [5], through (21) here below.
**Fact 2.2 (Normalization to canonical structures)** Given a structure $R$: (i) negations can move inward to atoms, and, possibly, disappear, thanks to (5), . . . , (10), (ii) units can be removed thanks to (16), . . . , (18), and (iii) brackets can move rightward by (13), . . . , (15).

So, for every $R$ we can take the equivalent canonical structure which is either $\circ$, or different from $\circ$.

**The system SBVQ.** It contains the set of inference rules in (22) here below. Every rule has form $\frac{T}{R}$, name $\rho$, premise $T$, and conclusion $R$. 

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**Contextual closure** means that $S[R] \approx S[T]$ whenever $R \approx T$. Thanks to (21), we abbreviate $[\cdots [R]_{a_{1}} \cdots]_{a_{n}}$ as $[R]_{d}$, where we may also interpret $d$ as one of the permutations of $a_{1}, \ldots, a_{n}$.

---

**Canonical structures.** We inspire to the normal forms of (2) to define structures in canonical form inside SBVQ. Canonical structures will be used to define environment structures (Section 7, page 14). A structure $R$ is canonical when either it is the unit $\circ$, or the following four conditions hold: (i) the only negated structures appearing in $R$ are negative propositional variables, (ii) no unit $\circ$ appears in $R$, but at least one name occurs in it, (iii) the nesting of occurrences of Par, Tensor, Seq, and Sdq build a right-recursive syntax tree of $R$, and (iv) no occurrences of Sdq can be eliminated from $R$, while maintaining the equivalence.

**Example 2.1 (Canonical structures)** The structure $[(\overline{a} \otimes \overline{b}) \otimes [\overline{c}]_{\circ}]$ is not canonical, but it is equivalent to the canonical one $[a \otimes (\overline{b} \otimes [\overline{c}]_{\circ})]$, whose syntax tree is right-recursive. Other non canonical structures are $[a \otimes (\circ \otimes \overline{b})]$, and $[(a \otimes (\circ \otimes \overline{b})) \otimes (\circ \otimes \overline{b})]$, and $[\overline{a} \otimes (\overline{b} \otimes [\overline{c}]_{\circ})]$. The first two are equivalent to $(\overline{a} \otimes \overline{b})$ which, instead, is canonical. Finally, also $[a \otimes \circ]$ is not canonical, equivalent to the canonical one $a$.

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**Negation**

- $\overline{\circ} \sim \circ$ (5)
- $\overline{R} \sim R$ (6)
- $[[R \otimes T]] \sim ([R \otimes T])$ (7)
- $(R \otimes T) \sim ([R \otimes T])$ (8)
- $[[R \otimes T]_{\circ}] \sim ([R \otimes T]_{\circ})$ (9)
- $(R \otimes T) \sim ([R \otimes T]_{\circ})$ (10)

**Symmetry**

- $[R \otimes T] \sim [T \otimes R]$ (11)
- $(R \otimes T) \sim (T \otimes R)$ (12)

**Unit**

- $(\circ \otimes R) \sim R$ (16)
- $(\circ \otimes R) \sim (R \otimes \circ) \sim R$ (17)
- $[\circ \otimes R] \sim R$ (18)

**Associativity**

- $(R \otimes (T \otimes V)) \sim ((R \otimes T) \otimes V)$ (13)
- $(R \otimes (T \otimes V)) \sim ((R \otimes T) \otimes V)$ (14)
- $[R \otimes [T \otimes V]] \sim [[R \otimes T] \otimes V]$ (15)
- $[R \otimes [T \otimes V]] \sim [[R \otimes T] \otimes V]$ (16)
- $(\circ \otimes R) \sim R$ (17)
- $(\circ \otimes R) \sim (R \otimes \circ) \sim R$ (18)
- $[\circ \otimes R] \sim R$ (19)

- $[R]_{a} \sim R$ if $a \notin \text{fn}(R)$ (19)
- $[[R]_{a}]_{\circ} \sim [[R]_{a}]_{\circ}$ if $a \notin \text{fn}(R)$ (20)
- $[[R]_{a}]_{\circ} \sim [[R]_{a}]_{\circ}$ (21)
Derivations vs. proofs. A derivation in SBVQ is either a structure or an instance of the above rules or a sequence of two derivations. Both $\mathcal{D}$, and $\mathcal{E}$ will range over derivations. The topmost structure in a derivation is its premise. The bottommost is its conclusion. The length $|\mathcal{D}|$ of a derivation $\mathcal{D}$ is the number of rule instances in $\mathcal{D}$. A derivation $\mathcal{D}$ of a structure $R$ in $SBVQ$ from a structure $T$ in $SBVQ$, only using a subset $B \subseteq SBVQ$ is $\mathcal{D}|_{B}$. The equivalent space-saving form is $\mathcal{D}: T \vdash_{B} R$. The derivation $\mathcal{D}|_{B}$ is a proof whenever $T \approx \circ \mathcal{D}$. We denote it as $\mathcal{D}|_{B}; \mathcal{R} \vdash B R$ or $\mathcal{D}: T \vdash_{B} R$. Both $\mathcal{D}$, and $\mathcal{E}$ will range over proofs. In general, we shall drop $B$ when clear from the context. In a derivation, we write $\rho_{1}, \ldots, \rho_{m}$ to derive $R$ from $T$ with the help of $n_{1}, \ldots, n_{p}$ instances of $\mathcal{S}, \ldots, \mathcal{U}$. To avoid cluttering derivations, whenever possible, we shall tend to omit the use of negation axioms $\mathcal{S}, \ldots, \mathcal{U}$, associativity axioms $\mathcal{T}, \mathcal{V}, \mathcal{W}$, and symmetry axioms $\mathcal{X}, \mathcal{Y}$. This means we avoid writing all brackets, as in $[R \otimes [T \otimes U]]$, in favor of $[R \otimes T \otimes U]$, for example. Finally, if, for example, $q > 1$ instances of some axiom $(n)$ of $\mathcal{S}, \ldots, \mathcal{U}$ occurs among $n_{1}, \ldots, n_{p}$, then we write $(n)^{q}$.

Up and down fragments of SBVQ. The set $\{a|_{B}, s, q|_{B}, u|_{B}\}$ is the down fragment $BVQ$ of SBVQ. The up fragment is $\{a|_{B}, s, q|_{B}, u|_{B}\}$. So $s$ belongs to both.

Corollary 2.3 $[5][6]$ The up-fragment $\{a|, q|, u|\}$ of SBVQ is admissible for BVQ. This means that we can transform any proof $\mathcal{D}: \vdash_{SBVQ} \mathcal{R}$ into a proof $\mathcal{D}: \vdash_{BVQ} \mathcal{R}$ free of every occurrence of rules that belong to the up-fragment of SBVQ.

Remark 2.4 Thanks to Corollary 2.3, we shall always focus on the up-fragment $BVQ$ of SBVQ.

3 Standardization inside a fragment of BVQ

Taken a derivation $\mathcal{D}$ of BVQ, standardization reorganizes $\mathcal{D}$ into another derivation $\mathcal{E}$ with the same premise, and conclusion, as $\mathcal{D}$. The order of application of the instances of $a|_{B}$ in $\mathcal{E}$ satisfies a specific, given constraint which some examples illustrate. Standardization in BVQ is one of the properties we need to recast reachability problems in a suitable calculus of communicating, and concurrent processes, to proof-search inside (a fragment) of BVQ.

Example 3.1 (Standard derivations of BVQ) Both $(23)$, and $(24)$ here below are standard derivations of the same conclusion $[(a \otimes R) \otimes (b \otimes T) \otimes (a \otimes b)]$ from the same premise $[R \otimes T]$. 
Our goal is to show that we can transform a sufficiently large set of derivations in BVQ into standard ones. We start by supplying the main definitions.

**Right-contexts.** We rephrase, inductively, and extend to BVQ the namesake definition in \([1]\). The following grammar generates right-contexts which we denote as \(S\{\cdot\}^\cdot\).

\[
S\{\cdot\}^\cdot := \{\} \mid (S'\{\cdot\}^\cdot \circ R) \mid [S'\{\cdot\}^\cdot \circ R] \mid \langle S'\{\cdot\}^\cdot \circ R \rangle \\
[\langle R \circ S'\{\cdot\}^\cdot \rangle] \mid [\langle R \circ S'\{\cdot\}^\cdot \rangle]_L \\
[\langle S'\{\cdot\}^\cdot \rangle]_L
\]

**(25)**

**Example 3.3 (Right-contexts)** A right-context is \([a \in [b \in [\langle 1 \cdot \circ d \rangle]_L]]\]_. Instead, \([a \in [b \in [\not\circ 1 \cdot \circ d \rangle]_L]]\]_L is not.

**Left atomic interaction.** Recalling it from \([1]\), the left atomic interaction is:

\[
\text{ati}_L \quad \frac{S(\sigma)^\cdot}{S[a \in \sigma]^\cdot}
\]

**(26)**

**Example 3.4 (Some left atomic interaction instances)** Let three proofs of BVQ be given:

\[
\text{ati}_L \quad \frac{\sigma}{[b \in \overline{b}]} \quad \frac{[b \in \overline{b}]}{\langle \sigma \cdot [b \in \overline{b}] \rangle} \quad \frac{[a \in \overline{a}]}{[a \in \overline{a}]} \quad \frac{[a \in \overline{a}]}{\langle [a \in \overline{a}] \cdot [b \in \overline{b}] \rangle} \quad \frac{[a \in \overline{a}] \cdot \sigma}{\langle [a \in \overline{a}] \cdot [b \in \overline{b}] \rangle}
\]

**(27)**, **(28)**, **(29)**
The two occurrences of $a\iota_1$ in (27) can correctly be seen as two instances of $a\iota_\LL$, as outlined by (28). Instead, the occurrence of $a\iota_1$ in (29) cannot be seen as an instance of $a\iota_\LL$ as it occurs to the right of $\Seq$, namely in the context $(a \otimes \overline{a}) \ast \{\}$. This is the key observation that allows us to proceed.

**Fact 3.5** By definition, every occurrence of $a\iota_\LL$ is one of $a\iota_1$. The vice versa is false.

**Standard derivations of BVQ.** Let $R$, and $T$ be structures. A derivation $\mathcal{D} : T \vdash_{\BVQ} R$ is *standard* whenever all the atomic interactions that $\mathcal{D}$ contains can be labeled as $a\iota_\LL$. We notice that nothing forbids $T \approx \emptyset$.

### 3.1 Standardization

We reorganize derivations of $\{a\iota_\LL, a\iota_1, q\iota_1, u\iota_1\} \subset BVQ$ which operate on Tensor-free structures only.

**Tensor-free structures.** By definition, $R$ in BVQ is Tensor-free whenever it does not contain $(R_1 \otimes \cdots \otimes R_n)$, for any $R_1, \ldots, R_n$, and $n > 1$.

**Our goal** is to prove the following theorem, inspiring to the standardization in [1]:

**Theorem 3.6 (Standardization in $\{a\iota_\LL, a\iota_1, q\iota_1, u\iota_1\}$)** Let $T$, and $R$ be Tensor-free. For every $\mathcal{D} : T \vdash_{\BVQ} R$, there is a standard derivation $\mathcal{E} : T \vdash_{\BVQ} R$.

It proof relies on the coming lemmas, and proposition.

**Lemma 3.7 (Existence of $a\iota_\LL$)** The topmost instance of $a\iota_1$ in a proof $\mathcal{D} : \vdash_{\BVQ} R$ is always an instance of $a\iota_\LL$.

**Proof** Let $\mathcal{D}$ be $\frac{S[\iota]}{\iota_1 \in \{a \otimes \overline{a}\}}$ where $S[\iota]$ is the topmost occurrence of $\iota_1$, which cannot be relabeled as $a\iota_\LL$. By contraction, let us assume $S[\iota]$ be a non right-context, namely $S[\iota] \approx S'(T \ast S''[\iota])$ for some $S'[\iota], S''[\iota]$, and $T$ such that $T \neq \emptyset$. In this case, to let the names of $T$, and, maybe, those ones of $S''[\iota]$, to disappear from $\iota_1$ we would have $\frac{S'(T \ast S''[\iota])}{\iota_1}$ to apply at least one instance of $a\iota_1$ which would occur in $\mathcal{D}$, against our assumption on the position of $\iota_1$.

**Lemma 3.8 (Commuting conversions in $\{a\iota_\LL, a\iota_1, q\iota_1, u\iota_1\}$)** Let $R, T$, and $S[\iota]$ be Tensor-free. Also, let $\rho \in \{a\iota_\LL, q\iota_1, u\iota_1\}$. Finally, let $\mathcal{D}$ be $\frac{\mathcal{D}[\{a \otimes \overline{a}\}] \ast}{\iota_1}$, where $\iota_1$ is the topmost occurrence of $\iota_1$ which is not $a\iota_\LL$. Then, there is $\frac{T}{\mathcal{D}[\{a \otimes \overline{a}\}] \ast}$, where $\mathcal{D}$ are Tensor-free, and $\iota_1$ may be an instance of $a\iota_\LL$. 
Proof The proof is, first, by cases on \( \rho \), and, then, by cases on \( S[a \not\equiv \overline{a}]^\ast \). Fixed \( S[a \not\equiv \overline{a}]^\ast \), the proof is by cases on \( R \) which must contain a redex of \( \mathcal{a}l_i \), \( q_i \), or \( u_i \), that, after \( \mathcal{a}l_i \ast \), leads to the chosen \( S[a \not\equiv \overline{a}]^\ast \). (Appendix A)

\[
\begin{align*}
&T \\
&\vdash_{\mathcal{BVQ}} V \\
&\Downarrow \\
&U \\
&\Downarrow \\
&R \\
&\Downarrow \\
\end{align*}
\]

Proposition 3.9 (One-step standardization in \( \{\mathcal{a}l_i, \mathcal{a}l_i, q_i, u_i\} \)) Let \( \mathcal{a}l_i \ast \) be a derivation in \( \vdash_{\mathcal{BVQ}} \mathcal{D} \) \( \vdash_{\mathcal{BVQ}} \mathcal{D}' \). If \( \mathcal{a}l_i \ast \) is the topmost instance of \( \mathcal{a}l_i \). There exists a derivation \( \mathcal{E} : T \vdash_{\mathcal{BVQ}} \mathcal{D} \), if any, can be directly relabeled as \( \mathcal{a}l_i \ast \), leads \( \mathcal{a}l_i \ast \) is already an instance of \( \mathcal{a}l_i \), and we are done. Otherwise, we can apply Lemma 3.8 moving \( \mathcal{a}l_i \ast \) one step upward, getting to \( \mathcal{E} : T \vdash_{\mathcal{BVQ}} \mathcal{D} \), whose derivations contain \( \mathcal{a}l_i \ast \) is no more than \( n - 1 \) rules far from \( T \). An obvious inductive argument allows to conclude thanks to Lemma 3.7.

Proof of Theorem 3.6 Let \( X \mathcal{D} \) be the set of all instances of \( \mathcal{a}l_i \) in \( \mathcal{D} \), that can be directly seen as instances of \( \mathcal{a}l_i \), and \( Y \mathcal{D} \) the set of all other instances of \( \mathcal{a}l_i \) in \( \mathcal{D} \). If \( Y \mathcal{D} = \emptyset \) we are done because \( \mathcal{E} \) is \( \mathcal{D} \) where every instance of \( \mathcal{a}l_i \) in \( X \mathcal{D} \), if any, can be directly relabeled as \( \mathcal{a}l_i \). Otherwise, let us pick the topmost occurrence of \( \mathcal{a}l_i \) in \( \mathcal{D} \) out of \( Y \mathcal{D} \), and apply Proposition 3.9 to it. We get \( \mathcal{E} : T \vdash_{\mathcal{BVQ}} \mathcal{D} \), whose set \( Y \mathcal{E} \) is strictly smaller than \( Y \mathcal{D} \). An obvious inductive argument allows to conclude.

Standard fragment \( \mathcal{BVQ} \) of BVQ. After Theorem 3.6 it is sensible defining \( \mathcal{BVQ} \) as \( \{\mathcal{a}l_i, q_i, u_i\} \subset \mathcal{BVQ} \) whose derivations contain Tensor-free only structures.

4 Internalizing derivability of BVQ

Roughly, internalizing derivability in BVQ shows when we can “discharge assumptions”. It is another of the properties we need to recast reachability problems in a suitable calculus of communicating, and concurrent processes, to proof-search inside (a fragment) of BVQ. The internalization links to the notion of invertible structures.

Invertible, and co-invertible structures. We define them in (30) here below.

\[
T \text{ is invertible whenever } \vdash_{\mathcal{BVQ}} [T \not\equiv P] \text{ implies } \vdash_{\mathcal{BVQ}} T \vdash_{\mathcal{BVQ}} P, \text{ for every } T, \text{ and } P \quad (30)
\]

If \( T \) is invertible, then, by definition, \( \overline{T} \) is co-invertible.

Remark 4.1 Clearly, definition (30) here above omits the implication “If \( \mathcal{D} \vdash_{\mathcal{BVQ}} T \not\equiv P \), then \( \mathcal{D} \vdash_{\mathcal{BVQ}} [T \not\equiv P] \)” on purpose. It always holds because \( \mathcal{a}l_i \) is derivable in BVQ. Moreover, our invertible structures inspire to the namesake concept in [8].

8
The following proposition gives sufficient conditions for a structure to be invertible.

**Proposition 4.2 (A language of invertible structures)** The following grammar (31) generates invertible structures.

\[
T ::= \circ \ | \ [T] \ | \ (T \otimes T) \ | \ (T \cdot T) \ | \ [T]_n
\]
where \( n > 0 \), and, for every \( 1 \leq i, j \leq n \), if \( i \neq j \) then \( t_i \neq t_j \), (31)

**Proof** Let \( \mathcal{D} : \tau_{BVQ} [T \otimes P] \) be given with \( T \) in (31). We reason by induction on \( ||T \otimes P|| \), and we build \( \mathcal{D} \) of (34), proceeding by cases on \( T \). (Details in Appendix B.)

5 Intermezzo

We keep the content of this section at an intuitive level. We describe how structures of BVQ model terms in a language whose syntax is not formally identified yet, but which is related to the one of Milner CCS.

**Example 5.1 (Modeling internal communication inside BVQ)** Derivations of BVQ model internal communication if we look at structures of BVQ as they were terms of Milner CCS, as in (1). Let us focus on (32) here below.

\[
\begin{align*}
E & \vdash E \otimes F \\
\circ \cdot [E \otimes E] & \vdash a \cdot [a \cdot E] \otimes [E \otimes F] \\
\circ \cdot (a \cdot E) & \vdash (a \cdot E) \otimes (a \cdot F)
\end{align*}
\]

\[
E \cdot E \vdash a \cdot E \\
\begin{align*}
a \cdot E \cdot F & \vdash a \cdot E \cdot F \\
\pi & \vdash F \\
\end{align*}
\]

The instance of \( q_1 \) moves atoms \( a \), and \( \overline{a} \), one aside the other, and \( a i_1 \) annihilates them. Annihilation can be seen as an internal communication between the two components \( (a \cdot E) \), and \( (\overline{a} \cdot F) \) of the structure \( [(a \cdot E) \otimes (\overline{a} \cdot F)] \). The usual way to formalize such an internal communication is \( (33) \), derivation that belongs to the labeled transition system of Milner CCS. The sequential composition of \( (33) \) stands for \( \text{Seq} \), parallel composition for \( \text{Par} \), and both \( E \) and \( F \) in \( (32) \) are represented by corresponding processes \( E \) and \( F \) in \( (33) \).

**Example 5.2 (Modeling external communication inside BVQ)** Derivations of BVQ model external communication if we look at structures of BVQ as they were terms of Milner CCS, as in (1). Let us focus on (34) here below.

\[
\begin{align*}
E & \vdash E \otimes \circ \\
\circ \cdot [E \otimes \circ] & \vdash a \cdot [a \cdot E] \otimes [E \otimes \circ] \\
\circ \cdot (a \cdot E) & \vdash (a \cdot E) \otimes (a \cdot \circ)
\end{align*}
\]

\[
E \vdash a \cdot E \\
\begin{align*}
a \cdot E & \vdash a \cdot E \\
\pi & \vdash F \\
\end{align*}
\]

We look at \( [(a \cdot E) \otimes \overline{a}] \) as containing two sub-structures with different meaning. The structure \( (a \cdot E) \) corresponds to the process \( a \cdot E \). Instead, \( \overline{a} \) can be seen as an action of the context “around” \( (a \cdot E) \). This means that \( (34) \) formalizes Milner CCS derivation \( (33) \).

**Remark 5.3 (“Processes”, and “contexts” are first-citizens)** The structure \( [(a \cdot E) \otimes \overline{a}] \) is equivalent to \( [(a \cdot E) \otimes (\overline{a} \cdot \circ)] \) in (34). This highlights a first difference between modeling the
communication by means of (a sub-system of) BVQ, instead than with Milner CCS. This latter constantly separates terms from the contexts they interact with. Instead, the structures of BVQ make no difference, and represent contexts as first-citizens. Namely, choosing which structures are the “real processes”, and which are “contexts” is, somewhat, only matter of taste. Specifically, in our case, we could have said that \( \langle \pi \cdot \diamond \rangle \) represents the process \( \pi.0 \), instead than the context.

**Example 5.4 (Hiding communication)** Derivations in BVQ model hidden communications of Milner CCS thanks to Sdq. So, we strictly extend the correspondence between a DI system and Milner CCS, as given in [1]. We build on Example 5.2 placing an instance of Sdq around every of the two components of \( \langle a \cdot E \rangle \oplus \pi \) in (34).

![Diagram](image)

We can look at Sdq, which binds \( a_i \) and \( \overline{\alpha} \) as restricting the visibility of the communication. The derivation in the labeled transition system of Milner CCS that models (36) is (37).

**Example 5.5 (More freedom inside BVQ)** Inside \( \langle a \cdot E \rangle \oplus \langle \alpha \cdot (\overline{\beta} \cdot (\overline{\gamma} \cdot F)) \rangle \oplus \langle b \cdot (c \cdot F) \rangle \), of (38) among others, we can identify the “processes” \( G_1 \equiv \langle a \cdot E \rangle \), \( G_2 \equiv \langle \alpha \cdot (\overline{\beta} \cdot (\overline{\gamma} \cdot F)) \rangle \), \( G_3 \equiv (\overline{\beta} \cdot (\overline{\gamma} \cdot F)) \), and \( G_4 \equiv (b \cdot (c \cdot F)) \):

![Diagram](image)

The lowermost instance of \( q_i \) predisposes \( G_1 \), and \( G_2 \) to an interaction through \( a \), and \( \overline{a} \). However, only the instance of \( a_i \) makes the interaction effective. Before that, the instance of \( i_j \) identifies \( G_4 \) as the noation of \( G_3 \), and annihilates them in a whole. So, (38) suggests that modeling process computations inside BVQ may result more flexible than usual, because it introduces a notion of “negation of a process” which sounds as a higher-order ingredient of proof-search-as-computation.

### 6 Communication, and concurrency with logic restriction

The correspondences Section 5 highlights, justify the introduction of a calculus of processes which we identify as CCS_{spq}. Specifically, CCS_{spq} is a calculus of communicating, and concurrent processes, with a logic-based restriction, whose operational semantics is driven by the logical behavior of \( q_i \) rule.
Remark 6.1 (CCS_{spq} vs. Milner CCS) It will turn out that CCS_{spq} is not Milner CCS\[\cite{ref}. The concluding Section\[9\] will discuss on this.

Actions on terms of CCS_{spq}. Let $a, b, c, \ldots$ denote the elements of a countable set of names, and let $\overline{a}, \overline{b}, \overline{c}, \ldots$ denote the elements of a countable set of co-names. The set of labels, which we range over by $l, m, n$ contains both names, and co-names, and nothing else. Let $\epsilon$ be the silent, or perfect action, different from any name, and co-name. The (set of) sequences of actions contains equivalence classes defined on the language that \eqref{eq:sequences} yields:

$$s ::= \epsilon \mid l \mid \overline{s} \mid s; s$$ \tag{39}

By definition, the equivalence relation \eqref{eq:equiv} here below induces the congruence $\cong$ on \eqref{eq:sequences}.

$$\epsilon \sim \epsilon \quad a \sim a \quad s; s' \sim s; s' \quad \epsilon; s \sim s$$ \tag{40}

We shall use $\alpha, \beta, \gamma$ to range over the elements in the set of actions sequences.

Processes of CCS_{spq}. The terms of CCS_{spq}, i.e. processes, belong to the language of the grammar \eqref{eq:grammar} here below.

$$E ::= 0 \mid \overline{l}E \mid (E | E) \mid E|a$$ \tag{41}

We use $E, F, G, \text{ and } H$ to range over processes. The inactive process is $0$, the parallel composition of $E$, and $F$ is $E | F$. The sequential composition $\overline{l}E$ sets the occurrence of the action prefix $l$ before the occurrence of $E$. Logic restriction $E|a$ hides all, and only, the occurrences of $a$, and $\overline{a}$, inside $E$, which becomes invisible outside $E$.

Size of processes. The size $|E|$ of $E$ is the number of symbols of $E$.

Congruence on processes of CCS_{spq}. We partition the processes of CCS_{spq} up to the smallest congruence which, by abusing notation, we keep calling $\cong$, and which we obtain as reflexive, transitive, and contextual closure of the relation \eqref{eq:congruence} here below.

$$\overline{a} \sim a \quad E | 0 \sim E \quad E | F \sim E \quad E | (F | G) \sim (E | F) | G \quad E|a|E \sim E|a|E$$ \tag{42}

In \eqref{eq:congruence} (i) $E|a|E|b|$ denotes a standard clash-free substitution of $a$ for both $b$, and $\overline{a}$ in $E$ that we can define as usual, and (ii) fn($\cdot$) is the set of free-names of a term in CCS_{spq}, whose definition, again, is the obvious one. Namely, neither $a$, nor $\overline{a}$ belong to the set fn($E|a|$).

Labeled transition system of CCS_{spq}. Its rules are in \eqref{eq:rules}, and they justify why CCS_{spq} is not Milner CCS.
In (43), the rule $a$ implements external communication, by firing the action prefix $l$, as usual. The rule $c$ implements internal communication, annihilating two complementary actions. The rules $p_a$ and $p_b$ allow processes, one aside the other, to communicate, even when both are inside a logic restriction. This is a consequence of the logical nature of $\mathrm{Sdq}$, which binds names, and co-names, up to their renaming, indeed. The rule $\mathrm{ctx}$ leaves processes, one aside the other, to evolve independently. Finally, $\mathrm{rfl}$ makes the relation reflexive.

**Example 6.2 (Using the labeled transition system)** As a first example, we rewrite \((a.b.E \mid \overline{a}.F)_l\) to \((E \mid F)_l\), observing the action $b$, as follows:

\[
\begin{array}{c}
a \overset{a}{\longrightarrow} a.b.E \Rightarrow b.E \overset{\overline{a}}{\longrightarrow} \overline{a}.F \overset{\gamma}{\longrightarrow} F \\
\overset{\in}{(a.b.E)\mid \overline{a}.F}_l \overset{\gamma \neq a}{\longrightarrow} (b.E)\mid F\mid 0 \overset{\in \neq a}{\longrightarrow} (b.E)\mid E\mid F\mid 0 \\
\overset{\text{ctx}}{\longrightarrow} (b.E)\mid E\mid \overline{a}.F\mid F\mid 0 \\
\overset{\text{rfl}}{\longrightarrow} (b.E)\mid E\mid \overline{a}.F\mid F\mid 0 \\
\overset{\text{trm}}{\longrightarrow} (b.E)\mid E\mid \overline{a}.F\mid F\mid 0 \\
\end{array}
\]

As a second example, we show that the labeled transition system (43) allows some interaction which originates from the logical nature of $\mathrm{Sdq}$. In $\mathrm{CCS}_{\mathrm{spq}}$ we model that \((a.b.E)_{\text{la}} \mid \overline{a}.F)_{\text{la}}\) reduces to \((E \mid F)_{\text{la}}\), observing $b$, unlike in Milner CCS:

\[
\begin{array}{c}
a \overset{a}{\longrightarrow} a.b.E \Rightarrow b.E \overset{\overline{a}}{\longrightarrow} \overline{a}.F \overset{\gamma}{\longrightarrow} F \\
\overset{\in}{(a.b.E)\mid \overline{a}.F}_{\text{la}} \overset{\gamma \neq a}{\longrightarrow} (b.E)\mid F\mid 0 \overset{\in \neq a}{\longrightarrow} (b.E)\mid E\mid F\mid 0 \\
\overset{\text{ctx}}{\longrightarrow} (b.E)\mid E\mid \overline{a}.F\mid F\mid 0 \\
\overset{\text{rfl}}{\longrightarrow} (b.E)\mid E\mid \overline{a}.F\mid F\mid 0 \\
\overset{\text{trm}}{\longrightarrow} (b.E)\mid E\mid \overline{a}.F\mid F\mid 0 \\
\end{array}
\]

**Simple processes.** They are the last notion we introduce in this section. They are useful for technical reasons which Section 8 will make apparent. A process $E$ is a *simple process* whenever it satisfies two constraints. First, $E$ must belong to the language of (46):

\[
E ::= 0 \mid 1.0 \mid E \mid E_{\text{la}}
\]
Second, if $l_1, \ldots, l_n$ are all, and only, the action prefixes that occur in $E$, then $i \neq j$ implies $l_i \neq l_j$, for every $i, j \in \{1, \ldots, n\}$.

**Example 6.3 (Simple processes)** Some are in the following table.

| $(a.0) \mid (\overline{l}.0)$ | $(a.0) \mid ((a.0) \mid (\overline{l}.0))_{\langle a \rangle} \mid (b.0)_{\langle b \rangle} \mid (a.0) | ((a.0) \mid (\overline{c}.0))_{\langle c \rangle} \mid (b.0)_{\langle b \rangle} \mid (a.0) |

Both the second, and the third process are simple because they belong to $\{\text{6}\}$, and $a, b, \overline{c}$ is the list of their pairwise distinct action prefixes.

**Remark 6.4 (Aim, and nature of simple processes)** In coming Section 7 we shall intuitively show that simple processes play the role of results of computations when we use derivations of BVQ to compute what the labeled transition system in $\{\text{43}\}$ can, in fact, compute by itself.

### 7 How computing in $\text{CCS}_{spq}$ by means of BVQ

Given BVQ, and $\text{CCS}_{spq}$ we illustrate how transforming questions about the existence of computations of $\text{CCS}_{spq}$ into questions about proof-search inside the standard fragment BVQ of BVQ. Let $E$, and $F$, be two processes of $\text{CCS}_{spq}$, with $F$ simple. Let us assume we want to check $E \xrightarrow{l_1 :: \delta_1} F$. Next we highlight the main steps to answer such a question by answering a question about proof-search inside BVQ, without resuming to computations in the labeled transition system of $\text{CCS}_{spq}$.

To that purpose, this section has two parts. The first one formalizes the notions that makes the link between processes of $\text{CCS}_{spq}$, and structures of BVQ precise. The second part, i.e. Subsection 7.2 delineates the steps to transform one question into the other, eventually justifying also the need to prove the Soundness of BVQ — not BVQ — w.r.t. $\text{CCS}_{spq}$, in Section 8.

#### 7.1 Connecting $\text{CCS}_{spq}$, and BVQ

**Process structures.** They belong to the language of the grammar (47) here below, and, clearly, they are Tensor-free:

$$R ::= \circ \mid (\cdot R) \mid [R \equiv R] \mid [R]_a$$  \hspace{1cm} (47)

Like at page4 we range over variable names of process structures by $l, m, n, \overline{l}$.  

**Fact 7.1 (Process correspond to process structures)** Processes, and process structures isomorphically correspond thanks to the following isomorphism, so extending the correspondence in $\{\text{1}\}$ among CCS terms, and BV structures.

$$\langle \emptyset \rangle \mapsto \circ \hspace{1cm} \langle e. \! E \rangle \mapsto \langle \cdot \mid \! E \rangle$$

$$\langle a. \! \emptyset \rangle \mapsto a \hspace{1cm} \langle l. \! E \rangle \mapsto \langle l \mid \! E \rangle$$

$$\langle \overline{a} \rangle \mapsto \overline{a} \hspace{1cm} \langle E \mid \! F \rangle \mapsto \langle E \mid \! F \rangle$$

$$\langle E \mid \! l_1 \rangle \mapsto \langle E \mid \! \rangle \hspace{1cm} \langle \overline{E} \rangle_{\langle a \rangle}$$  \hspace{1cm} (48)
**Environment structures.** Let us recall Example (5.2). It shows that representing an external communication as a derivation of BVQ requires to assign a specific meaning to the structures in the conclusion of the derivation. One structure represents a process. The other one encodes the labels that model the sequence of messages between the process, and an environment. So, we need to identify the environment structures, namely the set of structures that can fairly represent the sequence of messages. By definition, we say that every environment structure is a canonical structure (page 4) that the following grammar (49) generates:

$$R ::= \circ \mid 1 \mid (1 \cdot R) \mid \lfloor (1 \cdot R) \rfloor$$ (49)

If different from $\circ$, we have to think of every environment structure as a list, possibly in the scope of some instance of $Sdq$, that we can consume from its leftmost component, onward.

**Example 7.2 (Environment structures)** Let $\overline{a}, a_1, b_1, b_2 \neq \circ$.

\(<a_1 \cdot \lfloor \overline{a} \cdot \lfloor (b_2 \cdot b_1) \rfloor \rfloor \rfloor \rangle \) example (50)
\(<a_1 \cdot \lfloor \overline{a} \cdot \lfloor (b_2 \cdot b_1) \rfloor \rfloor \rfloor \rangle \) example (51)
\(<a_1 \cdot \lfloor \circ \cdot \lfloor (b_2 \cdot b_1) \rfloor \rfloor \rfloor \rangle \) counterexample (52)
\(<a_1 \cdot \lfloor \circ \cdot \lfloor (b_2 \cdot b_1) \rfloor \rfloor \rfloor \rangle \) counterexample (53)

(52) is not an environment structure because $b_4$ does not occur in the structure. (53) is not an environment structure because $\circ$ occurs in it.

**Fact 7.3 (Environment structures map to sequences of actions)** The map (54) takes both an environment structure, and a set of atoms as arguments. The map transforms a given environment structure to a sequence of actions that may work as a label of transitions in (43).

\[
\begin{align*}
\llbracket \circ \rrbracket_X &\mapsto \epsilon \\
\llbracket 1 \rrbracket_X &\mapsto \epsilon \ (1 \in X) \\
\llbracket 1 \rrbracket_X &\mapsto \top \ (1 \notin X) \\
\llbracket R \rrbracket_X &\mapsto \llbracket R \rrbracket_{X, \delta \overline{a} \overline{b}} \\
\llbracket (P \cdot R) \rrbracket_X &\mapsto \llbracket P \rrbracket_X; \llbracket R \rrbracket_X
\end{align*}
\] (54)

Given an environment structure, the map yields the corresponding sequence, if its second argument is $\emptyset$.

**Example 7.4 (From an environment structure to actions)** Both $b_1$, and $b_2$ are internal actions of $\llbracket (a_1 \cdot (a_1 \cdot ((b_2 \cdot b_1) \ldots)) \rrbracket_\emptyset = a_1; \overline{a}_1; \epsilon; \epsilon \equiv a_1; \overline{a}_1$ in (51). Intuitively, if a variable name $l$ that occurs in a structure $E$ belongs to $X$ in $\llbracket E \rrbracket_X$, then $l$ gets mapped to $\epsilon$. The reason why $l$ is in $X$ is that $l$ is not a free name of $E$.

**Trivial derivations.** By definition, a derivation $\mathcal{D}$ of BVQ is trivial if (i) $\mathcal{D}$ only operates on Tensor-free structures, and (ii) $\mathcal{D}$ does not contain any occurrence of $a[i]$. All the others are non-trivial derivations.

**Example 7.5 (A trivial derivation)** It is in (55) here below.
By definition, no \( a_{ij} \) can exist in \( \mathcal{D} \). Let us assume an instance \( \frac{\{l \cdot (R' \otimes R'')\}}{[l \cdot R' \otimes R'']} \) exists in \( \mathcal{D} \). Since \( \mathcal{D} \) is Tensor-free, it must be \( T \approx \circ \) and we can eliminate such an \( s \). Let us assume one instance of \( a_{ij} \) exists in \( \mathcal{D} \). In general it would be \( \frac{\{l \cdot m \cdot (R' \otimes R'')\}}{[l \cdot R' \otimes (m \cdot R'')]} \) \((*)\), for some \( l, m, R', \) and \( R'' \). So, let us assume such a \((*)\) occurs in \( \mathcal{D} \) with \( l, m \approx \circ \). In absence of \( a_{ij} \), even though we might have \( l \approx m \), the structure \( \{l \otimes m\} \) could not disappear from \( \mathcal{D} \), namely from \( T \). Consequently, \( T \) could not be a process structure, against assumption.

**Simple structures.** This notion strengthens the idea that “trivial” stands for “no interactions”. A structure \( R \) is a *simple structure* if it satisfies two constraints. First, it must belong to the language of \((56)\).

| \( q_l \) | \( [(a \otimes \overline{a}) \cdot (b \otimes \overline{b}) \cdot (R \otimes T)]_{\mu} \) |
| \( \mu \) | \( [(a \otimes \overline{a}) \cdot (b \otimes \overline{b}) \cdot (R \otimes T)]_{\mu} \) |
| \( \mu \) | \( [(a \otimes \overline{a}) \cdot (b \otimes \overline{b}) \cdot (R \otimes T)]_{\mu} \) |
| \( \mu \) | \( [(a \otimes \overline{a}) \cdot (b \otimes \overline{b}) \cdot (R \otimes T)]_{\mu} \) |

Second, if \( l_1, \ldots, l_n \) are all, and only, the variable names that occur in \( R \), then \( i \neq j \) implies \( l_i \neq l_j \), for every \( i, j \in \{1, \ldots, n\} \).

**Fact 7.7 (Basic properties of simple structures)**

- Trivially, by definition, simple structures are co-invertible, because every of them is the negation of an invertible structure (Proposition 4.2)
- Simple structures are the logical counterpart of simple processes, thanks to the isomorphism \(48\).

**Example 7.8 (Simple structures)** The following table shows some instances of simple structures which correspond to the simple processes in Example \(63\).
We begin by introducing the following fact:

**Fact 7.6** (Simple structures) The derivation \( D : T \vdash B \) is trivial, and \( B = \{ u \} \), and \( R \) is simple as well.

**Proof** Fact 7.6 implies that the derivation \( D \) only contains instances of \( u \), and of very specific instances of \( q \). Both kinds of rules neither erase, nor introduce atoms, or new occurrences of \( seq \) in between \( R \), and \( T \). Let us assume that \( D \) effectively contains an instance of \( q \) with reduct \( \langle l \cdot R' \rangle \), for some \( l \), and \( R' \). Then, the occurrence of \( seq \) would occur in \( T \), as well, making it no simple, against our assumption. So, no occurrence of \( q \) exists in \( D \). This, of course, does not prevent the existence of \( \langle l \cdot R' \rangle \) along \( D \), and, in particular, inside \( R \). However, \( u \) could not eliminate it, and an occurrence of \( seq \) would be inside \( T \). In that case \( T \) could not be simple, against assumption. But if no occurrence of \( \langle l \cdot R' \rangle \) is inside \( D \), then our assumptions imply that \( R \) is a simple structure.

**7.2 Recasting labeled transitions to proof-search**

Once connected BVQ, and CCS_{sqq} as in the previous subsection, we get back to our initial reachability problem. Let us assume we want to check \( E \xrightarrow{\text{l} \cdot \cdots \cdot l} F \) in CCS_{sqq}, where \( F \) is a simple process. The following steps recast the problem of CCS_{sqq} into a problem of searching inside BVQ:

1. First we “compile” both \( E \) and \( F \) into process structures \( \langle E \rangle \) and \( \langle F \rangle \), where \( \langle F \rangle \) is forcefully simple. Then, we fix an \( R \) such that \( \llbracket R \rrbracket_0 = l_1 \cdots l_r \).

2. Second, it is sufficient to look for \( D : T \vdash \langle E \rangle \otimes \langle F \rangle \otimes R \) inside BVQ as the up-fragment of SBVQ is admissible for BVQ (Corollary 2.3 [6]).

3. Finally, if \( D \) of point 2 here above exists, we can conclude \( E \xrightarrow{\text{l} \cdot \cdots \cdot l} F \) in CCS_{sqq}.

Point 3 rests on some simple observations. The structure \( \llbracket F \rrbracket \) is invertible thanks to Fact 7.7. So, it exists \( D' : \llbracket F \rrbracket \vdash \text{BVQ} \llbracket E \rrbracket \otimes R \) where both \( \llbracket E \rrbracket \), and \( \llbracket F \rrbracket \) are Tensor-free because they are process structures. The same holds for \( R \) which is an environment structure. Consequently, every instance of \( s \) in \( D \), if any, can only be \( s \) \( \llbracket (R \otimes U) \otimes \circ \rrbracket \), and it can be erased. This
means that $\mathcal{D}$ only contains rules that belong to \{at, q, u\}. Standardization (Theorem 3.6), which applies to \{at, q, u\}, implies we can transform $\mathcal{D}$ in BVQ to a standard derivation $\mathcal{E}$ of BVQ. The only missing step is in the coming section. It shows that proof-search in BVQ is sound w.r.t. the computations of the labeled transition system defined for CCS_{spq}.

8 Soundness of BVQ \text{ w.r.t. } CCS_{spq}

The goal is proving Soundness whose formal statement is in Theorem 5.9 below. We remark that our statement generalizes the one in [1], and our proof pinpoints many of the details missing in [1].

Soundness relies on the notions “reduction of a non-trivial derivation”, and “environment structures that are consumed”, and needs some technical lemma.

Reduction of non-trivial, and standard derivations of BVQ. Let $R$, and $T$ be process structures. Let $\mathcal{D}$ be a non-trivial, and standard derivation $\mathcal{D} : T \vdash_{BVQ} S[a]$, where $S[a]$ is the lowermost occurrence of at in $\mathcal{D}$. The reduction of $\mathcal{D}$ is the derivation $\mathcal{E}$ of rules of BVQ that we get from $\mathcal{D}$ by (i) replacing $\circ$ for all occurrences of $a$, and $\mathcal{T}$ in $\mathcal{D}$, that eventually, form the redex of $\mathcal{E}$, and by (ii) eliminating all the fake instances of rules that the previous step may have created.

Fact 8.1 (Reduction preserves process structures) Let $R$, and $T$ be process structures. For every non-trivial, and standard derivation $\mathcal{D} : T \vdash_{BVQ} R$, its reduction $\mathcal{E} : T' \vdash_{BVQ} R'$ is such that both $R'$, and $T'$ are process structures. Moreover, $\mathcal{E}$ may not be non-trivial, namely, no at may remain in $\mathcal{E}$. However, if $\mathcal{E}$ is non-trivial, then it is standard.

Proof The first statement follows from the definition of process structures. If we erase any sub-structure from a given process structure, we still get a process structure which, at least, is $\circ$. Moreover, the lowermost instance of at disappears, after a reduction. So, if it was the only one, none remains. Finally, reduction does not alter the order of rules in $\mathcal{D}$.

Fact 8.2 (Preserving right-contexts) Let $\mathcal{D}$ be a trivial derivation $\mathcal{D} : S'[a] \vdash_{q,u} S[a]$, for some $S[\{}$, $S'[\{}$, and $a$.

1. If $S[a]$ is not a right-context, then $S'[a]$ cannot be a right-context as well.
2. If $S'[a]$ is a right-context, then $S[a]$ is a right-context as well.

Proof 1. If $S[a]$ is not a right-context, then it has form $S[a] \approx S_0 (R \cdot S_1[a])$, with $R \neq \circ$, for some $S_0[\{}$ and $S_1[\{}$. Seq is not commutative. So, going upward in $\mathcal{D}$, there is no hope to transform $S_0 (R \cdot S_1[a])$ into some $S_0' (S_1'[a] \cdot R')$ where the occurrence of $a$ in the first structure is the same occurrence as $a$ in the second one. Moreover, $[R \cdot T]$ is not derivable in $[q, u] \subset BVQ$. So, $S_0 (R \cdot S_1[a])$ cannot transform into some $S_0' (R' \cdot S_1'[a])$, going upward in $\mathcal{D}$.

2. By contraposition of the previous point [1].

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Proposition 8.3 (Process structures, trivial derivations, and right-contexts) Let \( R \) be a process structure, and \( \mathcal{D} \) be a trivial derivation \( \mathcal{D} : S[b \triangleleft b] \vdash_{\{q_1, u_1\}} R \), for some \( S \{ \} \), \( b \), and \( \overline{b} \). Then:

1. \( R \not\vdash_{\{\}b} \), and both \( b, \overline{b} \) occur in it.
2. The structure \( R \) is a right-context for both \( b \), and \( \overline{b} \). Namely, \( R \approx S' \{ b \} \), and \( R \approx S'' \{ \overline{b} \} \) for some \( S' \{ \} \), and \( S'' \{ \} \).
3. \( R \not\approx S'\langle \alpha + \lambda \rangle \), and \( R \not\approx S''\langle \alpha + \lambda \rangle \), for any \( S' \{ \} \), \( S'' \{ \} \), and \( \lambda \).
4. \( R \not\approx \{ S'[b] \} \equiv \{ S''[\overline{b}] \} \), with \( b \in \text{fn}(S'[b]) \), and \( R \not\approx \{ S'[b] \} \equiv \{ S''[\overline{b}] \} \), with \( b \in \text{fn}(S''[\overline{b}]) \), for any \( S' \{ \} \), \( \lambda \), and \( \overline{b} \).
5. Let \( \overline{d} \) be a, possibly empty, sequence of names. Let \( T \) be a process structure, possibly such that \( T \approx \circ \). Then \( R \approx \lfloor S'[b] \cap \lfloor S''[\overline{b}] \cap \lfloor T \rfloor \rfloor \), such that either (i) \( b \in \text{fn}(S'[b]) \), and \( \overline{b} \in \text{fn}(S''[\overline{b}]) \), or (ii) \( b \in \text{bn}(S'[b]) \), and \( \overline{b} \in \text{bn}(S''[\overline{b}]) \).
6. Let \( S'[b] \) be the one in Point (5) here above. If \( E \), and \( F \) are processes such that \( \lfloor E \rfloor = S'[b] \), and \( \lfloor F \rfloor = S'\{ \circ \} \), then \( E \dashv \vdash F \), where \( \vdash \) is \( S' \{ \} \), and \( \vdash \) for \( b \).
7. Let \( S'[b] \), and \( S''[\overline{b}] \) be the ones in Point (5) here above. If \( E, F, E', \), and \( F' \) are processes such that \( \lfloor E \rfloor = S'[b] \), \( \lfloor F \rfloor = S''[\overline{b}] \), \( \lfloor E' \rfloor = S'\{ \circ \} \), and \( \lfloor F' \rfloor = S''\{ \circ \} \), then \( E \dashv \vdash F \).

Proof Concerning point (1), since no rule of \( \mathcal{D} \) generates atoms both \( b \), and \( \overline{b} \) must already occur in \( R \).

Concerning point (2), we start from point (1), and we look at \( S[b \triangleleft b] \) by first “hiding” \( b \), which gives \( S_0[b] \equiv S[b \triangleleft b] \), for some \( S_0 \{ \} \), and then “hiding” \( \overline{b} \) yielding \( S_1[\overline{b}] \equiv S[b \triangleleft \overline{b}] \), for some \( S_1 \{ \} \). Then, we apply point (2) of Fact 8.2 to \( S_0[b] \). It implies that \( R \approx S' \{ \} \), and \( R \approx S'' \{ \} \). Analogously, point (2) on Fact 8.2 to \( S_1[\overline{b}] \) implies that \( R \approx S'' \{ \} \).

Point (3) directly follows from point (2).

Point (4) holds because, for example, \( b \) cannot enter the scope of \( \{ S''[\overline{b}] \} \).

Point (5) follows from (4).

Point (6) holds by proceeding inductively on \( \lfloor E \rfloor \), and by cases on the form of \( S' \{ \} \), or \( S'' \{ \} \), respectively. (Details, relative to \( S' \{ \} \), in Appendix (7).

Point (7) holds thanks to points (4), and (6), by proceeding inductively on \( \lfloor E \rfloor \), and by cases on the form of \( S' \{ \} \), and \( S'' \{ \} \). (Details in Appendix (7).

The coming theorem says that the absence of interactions, as in a trivial derivation, models non interacting transitions inside the labeled transition system of \( \text{CCS}_{\text{seq}} \). We include proof details here, and not in an Appendix, because this proof supplies the simplest technical account of what we shall do for proving soundness.

Theorem 8.4 (Trivial derivations model empty computations in labeled transition system) Let \( E \), and \( F \) be processes, with \( F \) simple. If \( \mathcal{D} : \lfloor F \rfloor \vdash_{\text{BVQ}} \lfloor E \rfloor \) is trivial — beware, not necessarily in \( \text{BVQ} \) — , then \( E \dashv \vdash F \).
Proof. Fact 7.9 implies that $\langle E \rangle$ is simple, like $\langle F \rangle$ is, and that $\mathcal{D}$ can only contain instances of $\mathcal{U}_1$, if any rule occurs. We proceed by induction on the number $n$ of instances of $\mathcal{U}_1$ in $\mathcal{D}$.

If $n = 0$, forcefully $\langle E \rangle \equiv \langle F \rangle$. We conclude by refl, i.e. $E \xrightarrow{} E$. Otherwise, the last rule of $\mathcal{D}$ is:

$$\begin{align*}
\text{S} \vdash \langle E' \rangle \not\equiv \langle E'' \rangle \ | \mathcal{U}_1 \\
\text{S} \vdash \langle E' \rangle \not\equiv \langle E'' \rangle\ | \mathcal{U}_1
\end{align*}$$

for some context $\text{S} \{ \}$, and processes $E'$, and $E''$, such that $\langle E \rangle \equiv \text{S} \{ \langle E' \rangle \not\equiv \langle E'' \rangle \} \ | \mathcal{U}_1$. We can proceed by cases on the form of $\mathcal{S} \{ \}$.

- Let $\mathcal{S} \{ \} \not\equiv \{ \}$. So, $E$ must be $E'|_{\mathcal{U}_1} | \ E''|_{\mathcal{U}_1}$, and we can write:

$$\begin{align*}
\text{E} | \ E'' & \xrightarrow{\text{E} | \ E''} \text{E} | \ E'' \equiv \text{E} | \ E'' | \mathcal{U}_1 \\
\text{E} | \ E'' & \xrightarrow{\text{E} | \ E''} (\text{E} | \ E'')|_{\mathcal{U}_1} \equiv (\text{E} | \ E'')|_{\mathcal{U}_1} \ (\text{E} | \ E'')|_{\mathcal{U}_1} \xrightarrow{\text{E} | \ E''} \ F
\end{align*}$$

where $\langle \text{E} | \ E'' \rangle|_{\mathcal{U}_1} \xrightarrow{\text{E} | \ E''} \ F$ holds by induction because $\langle \text{F} \rangle \not\equiv \langle \text{E} | \ E'' \rangle |_{\mathcal{U}_1}$ is shorter than $\mathcal{D}$.

- Let $\mathcal{S} \{ \} \not\equiv \{ \} \not\equiv T$. So, $E$ must be $E'|_{\mathcal{U}_1} | \ E''|_{\mathcal{U}_1} | \ F'$, with $\langle F' \rangle = T$. The case is analogous to the previous one, with the proviso that an instance of $\text{ctx}$ must precede the instance of $\mathcal{P}_1$. In particular, $(E' | \ E'')|_{\mathcal{U}_1} \xrightarrow{\text{E} | \ E''} \ F$ holds by induction because $\langle E' | \ E'' \rangle|_{\mathcal{U}_1} \not\equiv \langle F' \rangle$ is shorter than $\mathcal{D}$.

The third case $\mathcal{S} \{ \} \not\equiv \{ \} \not\equiv T$ that we could obtain by assuming $E = 1.E'$ cannot occur because $E$ would not be simple, against assumptions.

Remark 8.5 (Why do we define simple structures as such?) Theorem 8.4 would not hold if we used “process structures” in place of “simple structures”. Let us pretend, for a moment, that $F$ be any process structure, and not only a simple one, indeed. The bottommost rule in $\mathcal{D}$ might well be:

$$\begin{align*}
\langle | \{1\} \cdot (\langle E' \rangle \not\equiv \langle E'' \rangle)\rangle \\
\langle | \{1\} \cdot (\langle E' \rangle \not\equiv \langle E'' \rangle)\rangle
\end{align*}$$

for some $E'$, and $E''$, such that $E = E' | (1.E'')$. By induction, $1.(E' | E'') \xrightarrow{\text{E} | \ E''} F$. However, in the labeled transition system $\text{CCS}_{\text{seq}}$ we cannot deduce $E' | (1.E'') \not\equiv (1.E'')$ whenever $1$ occurs free in $E'$. So, as we did in the definition of simple processes, we must eliminate any occurrence of $\text{seq}$ structure.

Theorem 8.6 (Soundness w.r.t. internal communication) Let $E$, and $F$ be processes, with $F$ simple, and $E \not\equiv \circ$. Let $\mathcal{D}$ be the derivation $\mathcal{A} \vdash (\text{E} | \circ)$ which, besides being standard,

$$\begin{align*}
\text{E} | \circ \xrightarrow{\text{E} | \circ} (\text{E} | \circ) \\
\text{E} | \circ \xrightarrow{\text{E} | \circ} \text{E}
\end{align*}$$

we assume to be non-trivial, and such that $(\text{E} | \circ)$ is its lowestmost instance of $\mathcal{A}$. If, for some process $G$, the derivation $\mathcal{D} : (\langle F \rangle \not\equiv (\text{E} | \circ) \ | \mathcal{P}_1 \not\equiv \mathcal{U}_1 \not\equiv T)$ is the reduction of $\mathcal{D}$, then $E \xrightarrow{} G$.

Proof. The derivation $\mathcal{D}'$ satisfies the assumptions of Point 2 in Proposition 8.3 which implies $\langle E \rangle \equiv S'(b)^+$, and $\langle E \rangle \equiv S''(b)^+$, for some $S'(b)^+$, and $S''(b)^+$, which must be process structures. We proceed on the possible distinct forms that $\langle E \rangle$ can assume. Point 7 of Proposition 8.3 will help concluding. (Details in Appendix E.)
Environment structures that get consumed. Let \( T \) and \( U \) be process structures, and \( R \) be an environment structure. Let \( \mathcal{D} : U \vdash_{\text{BVQ}} [T \equiv R] \) which, since belongs to \( \text{BVQ} \), is standard. We say that \( \mathcal{D} \) consumes \( R \) if every atom of \( R \) eventually annihilates with an atom of \( T \) thanks to an instance of \( \text{at}_\perp \), so that none of them occurs in \( U \).

Example 8.7 (Consuming environment structures) Derivations that consume the environment structure \((\alpha \cdot b)\) that occurs in their conclusion are \([23]\), and \([24]\). If we consider only a part of \([24]\), as here below, we get a standard derivation that does not consume \((\alpha \cdot b)\):

\[
\begin{align*}
\text{at}_\perp & \quad [T \equiv (\overline{b} \cdot U) \equiv b] \\
\vdash \quad [([a \equiv \overline{a}] \cdot [T \equiv b]) \equiv (\overline{b} \cdot U)] \\
\hline
\alpha \quad [\alpha \cdot T) \equiv (\overline{b} \cdot U) \equiv (\overline{a} \cdot b)]
\end{align*}
\]

(57)

Theorem 8.8 (Soundness w.r.t. external communication) Let \( E \), and \( F \) be processes, and \( R \) be an environment structure. Let \( \mathcal{D} \) be a non-trivial, and standard derivation that assumes one of the following forms:

\[
\begin{align*}
\llbracket F \rrbracket & \quad \mathcal{D} \vdash_{\text{BVQ}} \llbracket \epsilon \rrbracket \\
\mathcal{D} \vdash_{\text{BVQ}} \llbracket S \rrbracket \quad \text{or} \quad \llbracket F \rrbracket & \quad \mathcal{D} \vdash_{\text{BVQ}} \llbracket \epsilon \rrbracket \\
\mathcal{D} \vdash_{\text{BVQ}} \llbracket \llbracket b \equiv \overline{b} \rrbracket \rrbracket & \quad \mathcal{D} \vdash_{\text{BVQ}} \llbracket \llbracket b \equiv \overline{b} \rrbracket \rrbracket
\end{align*}
\]

such that \((\ast)\) is its lowermost instance of \(\text{at}_\perp\), and \(\overline{b} \) in \(\llbracket b \equiv \overline{b} \rrbracket \) is the same occurrence of \(\overline{b} \) as the one in \(\llbracket (\overline{b} \cdot R) \rrbracket \). If \(\mathcal{D} : \llbracket F \rrbracket \vdash_{\text{BVQ}} \llbracket G \rrbracket \equiv R \) is the reduction of \(\mathcal{D} \), then \(E \xrightarrow{\cdot} G\) if \(b \in \text{bn}(E)\). Otherwise, if \(b \in \text{fn}(E)\), then \(E \xrightarrow{\cdot} G\).

Proof First, \(\mathcal{D} \) necessarily consumes \((\overline{b} \cdot R)\), or \([\llbracket (\overline{b} \cdot R) \rrbracket]_b\), in either cases. The reason is twofold. Being \(\llbracket F \rrbracket\) a simple structure implies it cannot contain any \(\text{Seq}\) structure which, instead, is one of the operators that can compose \((\overline{b} \cdot R)\), or \([\llbracket (\overline{b} \cdot R) \rrbracket]_b\). Moreover, no occurrence of \(b \) inside \(R \) can annihilate with the first occurrence of \(\overline{b} \) inside \(\llbracket (\overline{b} \cdot R) \rrbracket\), or \([\llbracket (\overline{b} \cdot R) \rrbracket]_b\).

Second, \(\mathcal{D} \) satisfies the assumptions of Proposition \([8.7]\). So, its Point \([2]\) applies to \([\llbracket E \rrbracket \equiv \llbracket (\overline{b} \cdot R) \rrbracket]\), and \([\llbracket E \rrbracket \equiv \llbracket (\overline{b} \cdot R) \rrbracket]_b\). Since \(\overline{b} \) occurs in \(\llbracket (\overline{b} \cdot R) \rrbracket\), for some \(S' \equiv \cdot\), it must be \(\llbracket E \rrbracket \equiv S'(b)\) in which the occurrence of \(b \) we outline is the one that annihilates the given \(\overline{b} \). We proceed on the possible forms that \(\llbracket E \rrbracket\) can assume, in relation with the form of \(R\). Point \([6]\) of Proposition \([8.4]\) will help concluding. (Details in Appendix \([\overline{16}]\))

Theorem 8.9 (Soundness) Let \( E \), and \( F \) be processes with \( F \) simple. For every standard \(\llbracket F \rrbracket\) derivation \(\mathcal{D} \), and every environment structure \(R\), if \(\mathcal{D} \vdash_{\text{BVQ}} \llbracket \epsilon \rrbracket\), and \(\mathcal{D} \) consumes \(R\), then \(\llbracket E \rrbracket \equiv R\)

\(E \xrightarrow{\text{IE}_\perp} F\).

Proof As a basic case we assume \(\llbracket E \rrbracket \equiv \epsilon\). This means that \(E \) is \(0\). Moreover, since \(\mathcal{D} \) consumes \(R\), and no atom exists in \(\llbracket E \rrbracket\) to annihilate atoms of \(R\), we must have \(\llbracket F \rrbracket \equiv \epsilon\), i.e. \(F \equiv 0\), and \(R \equiv \epsilon\). Since \(0 \xrightarrow{\cdot} 0\), thanks to \(\text{rfl}\), we are done.

Instead, if \(\llbracket E \rrbracket \not\equiv \epsilon\), in analogy with \([1]\), we proceed by induction on the number of rules in \(\mathcal{D}\), in relation with the two cases where \(R \equiv \epsilon\), or \(R \not\equiv \epsilon\).

Since \(\mathcal{D}\) is non-trivial, and standard, we can focus on its lowermost occurrence \((\ast)\) of \(\text{at}_\perp\). Let us assume the redex of \((\ast)\) be \([b \equiv \overline{b}]\). We can have the following cases.
• Let $R \approx \circ$, and $\mathcal{E} : \langle F \rangle \vdash_{BVQ} \langle G \rangle$ be the reduction of $\mathcal{D}$.

1. The first case is with $\mathcal{E}$ non-trivial. The inductive hypothesis holds on $\mathcal{E}$, and we get $G \xrightarrow{\epsilon + \in + \lambda} F$.

2. The second case is with $\mathcal{E}$ trivial, so we cannot apply the inductive hypothesis on $\mathcal{E}$. However, Theorem 8.4 holds on $\mathcal{E}$, and we get $G \xrightarrow{\epsilon} F$.

Finally, both $\mathcal{D}$ and $\mathcal{E}$ satisfy the assumptions of Theorem 8.6, so it implies $E \xrightarrow{\epsilon} G$, and the statement we are proving holds thanks to trn.

• Let $\circ \neq R \approx \langle b \cdot T \rangle_b$, for some environment structure $T$. Let $\mathcal{E} : \langle F \rangle \vdash_{BVQ} \langle G \rangle \approx \langle (\circ \cdot T) \rangle_b$ be the reduction of $\mathcal{D}$. Since $\langle (\circ \cdot T) \rangle_b$ is an environment structure, it is canonical, so, necessarily $\langle (\circ \cdot T) \rangle_b \approx [T]_b \approx T$ because $R \neq \fn(T)$. Hence, $\mathcal{E} : \langle F \rangle \vdash_{BVQ} \langle G \rangle \approx T$. Moreover, since $b$ disappears along $\mathcal{D}$, we forcefully have $b \in \fn(\langle E \rangle)$.

1. Let $\mathcal{E}$ be non-trivial. The inductive hypothesis holds on $\mathcal{E}$, implying $G \xrightarrow{\langle T \rangle_b} F$.

Moreover, $\mathcal{D}$ satisfies the assumptions of Theorem 8.8, which implies $E \xrightarrow{\epsilon} G$ also because, as we said, $b \in \fn(\langle E \rangle)$. So, the statement holds because $\langle T \rangle_{[b, \beta]} \equiv \epsilon; \langle T \rangle_{[b, \beta]} = \langle \bar{b} \rangle_{[b, \beta]}; \langle T \rangle_{[b, \beta]} = \langle \langle (\bar{b} \cdot \circ) \rangle_b \rangle_b$, and by trn we get $E \xrightarrow{\langle (\bar{b} \cdot \circ) \rangle_b} F$.

2. The second case is with $\mathcal{E}$ trivial, so we cannot apply the inductive hypothesis on $\mathcal{E}$. However, Theorem 8.4 holds on $\mathcal{E}$, and we get $G \xrightarrow{\epsilon} F$, which implies $T \approx \circ$. Indeed, if $T \neq \circ$, then $\mathcal{D}$ could not consume $T$. The reason is that being $\mathcal{E}$ a trivial derivation, it cannot contain any instance of $\alpha$]. But a $\mathcal{D}$ not consuming $T$, would mean $\mathcal{D}$ not consuming $R$, against assumption. Finally, Theorem 8.8 holds on $\mathcal{D}$, and implies $E \xrightarrow{\epsilon} G$; because, as we said, $b \in \fn(\langle E \rangle)$. So, the statement holds because $\langle \circ \rangle_{[b, \beta]} \equiv \epsilon; \langle \circ \rangle_{[b, \beta]} = \langle \bar{b} \rangle_{[b, \beta]}; \langle \circ \rangle_{[b, \beta]} = \langle \langle (\bar{b} \cdot \circ) \rangle_b \rangle_b$, and by trn we get $E \xrightarrow{\langle (\bar{b} \cdot \circ) \rangle_b} F$.

We could proceed in the same way when $\circ \neq R \approx \langle b \cdot T \rangle_b$.

• Let $\circ \neq R \approx \langle b \cdot T \rangle_b$. Then, both $\mathcal{E} : \langle F \rangle \vdash_{BVQ} \langle G \rangle \approx T$, and $b \in \fn(\langle E \rangle)$ for the reasons analogous to the ones given in the previous case.

1. The first case is with $\mathcal{E}$ non-trivial. The inductive hypothesis holds on $\mathcal{E}$, and we get $G \xrightarrow{\langle T \rangle_b} F$. Moreover, Theorem 8.8 holds on $\mathcal{D}$, and implies $E \xrightarrow{b} G$, because, as we said, $b \in \fn(\langle E \rangle)$. So, the statement holds because $\langle \bar{b} \rangle_b; \langle T \rangle_b = \langle \langle (\bar{b} \cdot \circ) \rangle_b \rangle_b$, and by trn we get $E \xrightarrow{\langle (\bar{b} \cdot \circ) \rangle_b} F$.

2. The second case is with $\mathcal{E}$ trivial, so we cannot apply the inductive hypothesis on $\mathcal{E}$. However, Theorem 8.4 holds on $\mathcal{E}$, and we get $G \xrightarrow{b} F$, which implies $T \approx \circ$ for reasons analogous to the ones given in the previous case. Moreover, Theorem 8.8 holds on $\mathcal{D}$, and implies $E \xrightarrow{b} G$, because, as we said, $b \in \fn(\langle E \rangle)$. So, the statement holds because $\langle \bar{b} \rangle_b \equiv \langle \bar{b} \rangle_b; \epsilon = \langle \bar{b} \rangle_b; \langle \circ \rangle_b = \langle \langle (\bar{b} \cdot \circ) \rangle_b \rangle_b$, and by trn we get $E \xrightarrow{\langle (\bar{b} \cdot \circ) \rangle_b} F$.

We could proceed in the same way when $\circ \neq R \approx \langle b \cdot T \rangle_b$. 

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8.1  An instance of the proof of Soundness

The derivation \( \text{\text{58}} \) is standard.

\[
\begin{align*}
\text{at}_{1}. & \quad \frac{[\langle E' \rangle \equiv \langle F' \rangle]}{[\langle a \equiv b \rangle \cdot [\langle E' \rangle \equiv \langle F' \rangle]]} \\
\text{at}_{1}. & \quad \frac{[\langle a \equiv b \rangle \cdot [\langle E' \rangle \equiv \langle F' \rangle]]}{[\langle b \equiv b \rangle \cdot [\langle E' \rangle \equiv \langle F' \rangle]]} \quad (\ast)
\end{align*}
\]

Hence, \( \text{\text{58}} \) is an instance of the assumption \( \mathcal{D} : \langle F \rangle \vdash_{\text{BQV}} \langle E \rangle \equiv R \) in Theorem \( \text{\text{8.9}} \) above. The structure \( [\langle a \cdot b \cdot \langle E' \rangle \equiv \langle F' \rangle \rangle \equiv \langle a \equiv \langle F' \rangle \rangle] \) in \( \text{\text{58}} \) plays the role of \( \langle E \rangle \), while \( \bar{b} \) corresponds to \( R \). Finally \( [\langle E' \rangle \equiv \langle F' \rangle] \) in \( \text{\text{58}} \) plays the role of \( \langle F \rangle \), for some process \( E' \) and \( F' \). By definition, \( E = ((a,b,E') \mid (a \equiv \langle F' \rangle))] \), and \( F = (E' \mid F')] \). Once identified the lowermost instance (\( \ast \)) of \( \text{at}_{1} \), we replace \( \circ \) for all those occurrences of atoms that, eventually, annihilate in (\( \ast \)). So, \( \text{\text{58}} \) becomes the structure \( \text{\text{59}} \) which is not a derivation because it contains fake instances of rules.

\[
\begin{align*}
\text{at}_{1}. & \quad \frac{[\langle E' \rangle \equiv \langle F' \rangle]}{[\langle b \equiv b \rangle \cdot [\langle E' \rangle \equiv \langle F' \rangle]]} \\
\text{at}_{1}. & \quad \frac{[\langle a \equiv a \rangle \cdot [\langle b \equiv b \rangle \cdot [\langle E' \rangle \equiv \langle F' \rangle \rangle \equiv \langle a \equiv \langle F' \rangle \rangle \rangle]}{[\langle b \equiv b \rangle \cdot [\langle E' \rangle \equiv \langle F' \rangle \rangle \equiv \langle a \equiv \langle F' \rangle \rangle \rangle] \equiv \langle a \equiv \langle F' \rangle \rangle \rangle} \quad (\text{\text{59}})
\end{align*}
\]

Removing all the fake rules, we get to \( \mathcal{D}' \) in \( \text{\text{60}} \):
The lowermost instance \((\ast)\) of \(\alpha\)\(\downarrow\) in (58) has disappeared from (60). The inductive argument on (60) implies \(((b,E') | F')\)\(\downarrow\) \(\longrightarrow\) \((E' | F')\)\(\downarrow\). Since we can prove:

\[
\begin{align*}
\mathcal{E}_a \downarrow \{ & a, b, E' \rightarrow a, b, E \} \mathcal{E}^0 \\
\mathcal{E}_b \downarrow \{ & a, b, E' \rightarrow b, E' \} \mathcal{E}^0 \\
\mathcal{E}_c \downarrow \{ & a, b, E' \rightarrow E' \} \mathcal{E}^0 \\
\mathcal{E}_d \downarrow \{ & a, b, E' \rightarrow E' \} \mathcal{E}^0 \\
\mathcal{E}_e \downarrow \{ & a, b, E' \rightarrow E' \} \mathcal{E}^0 \\
\mathcal{E}_f \downarrow \{ & a, b, E' \rightarrow E' \} \mathcal{E}^0
\end{align*}
\]

(61)

by transitivity, we conclude \(((a,b,E') | (\overline{F'}, F'))\)\(\downarrow\) \(\longrightarrow\) \((E' | F')\)\(\downarrow\).

9 Final discussion, and future work

This work shows that BVQ \([4][5][6]\), which we can consider as a minimal extension of BV \([2]\), is expressive enough to model concurrent and communicating computations, as expressed by the language CCS\(_{spr}\), whose logic-based restriction con hide actions to the environment in an unusual flexible way, as compared to the restriction of Milner CCS. The reason why, in various points, we have kept relating CCS\(_{spr}\) with a fragment of Milner CCS is twofold. First, we start from the programme of \([1]\), that shows the connections between BV and the smallest meaningful fragment of Milner CCS. Second, it is evident we can define BVQ\(^-\) as follows. We take \(BVQ \setminus \{u\}\) and we forbid clauses \((19)\), and \((20)\) on its structures. So defined, BVQ\(^-\) would be very close to the fragment of Milner CCS, which we have called CCS\(_{spr}\), and which only contains restriction, and both sequential, and parallel composition. The reason is that BVQ\(^-\) could simulate the two standard rules for restriction:

\[
\begin{align*}
\mathcal{E}_a \downarrow \{ & a, b, E' \rightarrow a, b, E \} \mathcal{E}^0 \\
\mathcal{E}_b \downarrow \{ & a, b, E' \rightarrow b, E' \} \mathcal{E}^0 \\
\mathcal{E}_c \downarrow \{ & a, b, E' \rightarrow E' \} \mathcal{E}^0 \\
\mathcal{E}_d \downarrow \{ & a, b, E' \rightarrow E' \} \mathcal{E}^0 \\
\mathcal{E}_e \downarrow \{ & a, b, E' \rightarrow E' \} \mathcal{E}^0 \\
\mathcal{E}_f \downarrow \{ & a, b, E' \rightarrow E' \} \mathcal{E}^0
\end{align*}
\]

(62)

but not the rules \(p\), and \(p\) in \([4][5]\). However, in fact, Sdq looks much closer to the hiding operator \((\nu a)E\) of \(\pi\)-calculus \([7]\). Clause \((21)\) “is” \((\nu a)(\nu b)E \approx (\nu b)(\nu a)E\). Clause \((19)\)
generalizes \((\nu a)0 \approx 0\). The instance:

\[
\begin{align*}
\mathcal{E}_a \downarrow \{ & E \rightarrow E' \} \mathcal{E}^0 \\
\mathcal{E}_b \downarrow \{ & E \rightarrow E' \} \mathcal{E}^0
\end{align*}
\]

(62)

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weakly corresponds to scope extrusion \((\nu a)(E \mid F) \approx (\nu a)E \mid F\) which holds, in both directions, whenever \(a\) is not free in \(F\). We postpone the study of semantics and of the relation between \(\text{CCSpq}\), and the corresponding fragment of \(\pi\)-calculus, to future work.

Further future work we see as interesting, is about the generalization of Soundness. We believe that a version of Soundness where no restriction to simple processes holds. The reason is twofold. First, thanks to the Splitting theorem of \(\text{BVQ}\) it is possible to prove that every proof of \(\text{BVQ}\) can be transformed in a standard proof of \(\text{BVQ}\). So, no need to restrict to Tensor-free derivations of \(\text{BVQ}\) exists to have standard proofs. Second, the reduction process looks working on standard proofs as well, and no obstacle seems to exist to the application of inductive arguments analogous to those ones we have used to prove our current Soundness.

We conclude with a remark on the “missing” Completeness. Our readers may have noticed the lack of any reference to a Completeness of \(\text{BVQ}\), w.r.t. \(\text{CCSpq}\). Completeness would say that \(\text{BVQ}\) has enough derivations to represent any computation in the labeled transition system of \(\text{CCSpq}\). Formally, it would amount to:

**Theorem 9.1 (Completeness of \(\text{BVQ}\))** For every process structure \(E\), and \(F\), if \([E]\xrightarrow{R}\emptyset[F]\), then \(\mathcal{D} : F \vdash_{\text{BVQ}} [E \otimes R]\).

Ideally, we leave the proof of Theorem 9.1 as an exercise. The system \(\text{BVQ}\) is so flexible that, proving it complete, amounts to show that every rule of \(\text{CCSpq}\) is derivable in \(\text{BVQ}\).

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A Proof of commuting conversions in \( \{ \text{at}_\Left, \text{ai}_\Left, \text{q}_\Left, \text{u}_\Left \} \)

(Lemma 3.8, page 7)

The proof is, first, by cases on \( \rho \), and, then, by cases on \( S[a \otimes \overline{a}] \). Fixed \( S[a \otimes \overline{a}] \), the proof is by cases on \( R \) which must contain a redex of \( \text{ai}_\Left, \text{q}_\Left, \) or \( \text{u}_\Left \), that, after \( \text{ai}_\Left^+ \), leads to the chosen \( S[a \otimes \overline{a}] \).

We start with \( \rho \equiv \text{ai}_\Left \).

- Let \( S[a \otimes \overline{a}] \Leftrightarrow [a \otimes \overline{a}] \). So, \( [a \otimes (\text{ai}_\Left \cdot [b \otimes \overline{b}])_b] \), and \( [a \otimes (b \otimes \overline{b})]_b \) are the most relevant forms of \( R \). Others can be \( [a \otimes (\text{ai}_\Left \cdot [b \otimes \overline{b}])_b] \), and \( ([a \otimes \overline{a}] \Leftrightarrow [b \otimes \overline{b}])_b \), and \( ([a \otimes \overline{a}] \Leftrightarrow [b \otimes \overline{b}])_b \).

We fully develop only the first case with \( R \equiv [a \otimes (\text{ai}_\Left \cdot [b \otimes \overline{b}])_b] \). In it the derivation

\[
\text{ai}_\Left \Downarrow \begin{array}{c} \text{ai}_\Left \Downarrow \end{array} [a \otimes (\text{ai}_\Left \cdot [b \otimes \overline{b}])_b] \quad \text{transforms to} \quad \begin{array}{c} \text{ai}_\Left \Downarrow \end{array} [a \otimes \overline{a}]_b \]

If, instead, \( S[a \otimes \overline{a}] \Leftrightarrow [a \otimes (\text{ai}_\Left \cdot [b \otimes \overline{b}])_b] \), then no instances of \( \text{u}_\Left \) are required, but only one of \( \text{q}_\Left \).

- Let \( S[ \{ c \} \equiv S'' \leftarrow \cdot U' \} \).
  - If \( R \equiv [S'[a \otimes \overline{a}] \equiv S''[b \otimes \overline{b}]] \), with \( U' \equiv S''[b \otimes \overline{b}] \), then
    \[
    \text{at}_\Left \Downarrow \begin{array}{c} \text{at}_\Left \Downarrow \end{array} \frac{[R' \equiv U'']}{S'[a \otimes \overline{a}] \equiv S''[b \otimes \overline{b}]} \quad \text{transforms to} \quad \begin{array}{c} \text{at}_\Left \Downarrow \end{array} \frac{[R' \equiv U'']}{S'[a \otimes \overline{a}] \equiv S''[b \otimes \overline{b}]} \]
    for some \( R' \), and \( U'' \).
  - If \( R \equiv [S'[a \otimes \overline{a}] \equiv U' \} \equiv [S''[b \otimes \overline{b}] \equiv U' \} \), then
    \[
    \text{at}_\Left \Downarrow \begin{array}{c} \text{at}_\Left \Downarrow \end{array} \frac{[S' \equiv S'' \equiv U \} \equiv U' \}}{[S'[a \otimes \overline{a}] \equiv S''[b \otimes \overline{b}]]} \]
    some \( S'' \{ c \} \), which is \( S''[b \otimes \overline{b}] \) with \( [b \otimes \overline{b}] \) replaced by \( \circ \), and \( R' \), transforms to
    \[
    \text{at}_\Left \Downarrow \begin{array}{c} \text{at}_\Left \Downarrow \end{array} \frac{[R' \equiv U' \} \equiv U' \}}{[S'[a \otimes \overline{a}] \equiv S''[b \otimes \overline{b}]]} \]
    by \( \circ \).

- Let \( S[ \{ c \} \equiv S'' \leftarrow \cdot U' \} \), where \( c \) may also coincide to \( a \), or \( b \). This case is analogous to the last point of the previous case, because \( S'[a \otimes \overline{a}] \equiv S''[b \otimes \overline{b}] \), for some \( S'' \{ c \} \).

- Let \( S[ \{ c \} \equiv S'' \leftarrow \cdot U' \} \).
  - If \( R \equiv \langle S'[a \otimes \overline{a}] \equiv S''[b \otimes \overline{b}] \rangle \), with \( U' \equiv S''[b \otimes \overline{b}] \), then
    \[
    \text{at}_\Left \Downarrow \begin{array}{c} \text{at}_\Left \Downarrow \end{array} \frac{\langle R' \equiv U' \rangle}{S'[a \otimes \overline{a}] \equiv S''[b \otimes \overline{b}]} \quad \text{transforms to} \quad \begin{array}{c} \text{at}_\Left \Downarrow \end{array} \frac{\langle R' \equiv U' \rangle}{S'[a \otimes \overline{a}] \equiv S''[b \otimes \overline{b}]} \]
    for some \( R' \), and \( U'' \).
  - If \( R \equiv \langle S'[a \otimes \overline{a}] \equiv U' \} \equiv \langle S''[b \otimes \overline{b}] \equiv U' \} \), then
    \[
    \text{at}_\Left \Downarrow \begin{array}{c} \text{at}_\Left \Downarrow \end{array} \frac{\langle S' \equiv S'' \equiv U \} \equiv U' \}}{\langle S'[a \otimes \overline{a}] \equiv S''[b \otimes \overline{b}]} \]
    \( S'' \{ c \} \), which is \( S''[b \otimes \overline{b}] \), with \( [b \otimes \overline{b}] \) replaced by \( \circ \), and \( R' \), transforms to
This proof rests on Shallow splitting of \([5, 6]\) whose statement we recall here.

Let \(\rho \equiv q_1\).

- Let \(S \vdash S'((U' \cdot S') \cdot (U'' \cdot U'''))\). Then \(R \approx S'((U' \cdot S''[a \otimes \overline{a}]) \cdot (U'' \cdot U'''))\).

- Let \(S \vdash S'((S'''[a \otimes \overline{a}]) \cdot (U'' \cdot U'''))\). This case is analogous to the previous one.

Finally, let \(\rho \equiv u_1\). Then \(u_1\) involves the redex of \(a_1\) whenever \(S \vdash S'[S''[\cdot ]a \otimes [U']_a]a\).

So, \(R \approx S'[[S''[a \otimes \overline{a}]]a \otimes [U']_a]\), and \(u_1\) transforms to \(S'[[S''[a \otimes \overline{a}]]a \otimes [U']_a]\).

\[\begin{align*}
\text{Proposition B.1 (Shallow Splitting)} & \quad \text{Let } R, T, \text{ and } P \text{ be structures, and } a \text{ be a name, and } \mathcal{P} \text{ be a proof of } \text{BVQ.} \\
1. & \quad \text{If } \mathcal{P} : \vdash_{\text{BVQ}} [(R \cdot T) \otimes P], \text{ then there are } \mathcal{D} : \vdash_{\text{BVQ}} ([P_1 \cdot P_2] \vdash_{\text{BVQ}} P), \text{ and } \mathcal{P}_1 : \vdash_{\text{BVQ}} [R \otimes P_1], \text{ and } \mathcal{P}_2 : \vdash_{\text{BVQ}} [T \otimes P_2], \text{ for some } P_1, \text{ and } P_2. \\
2. & \quad \text{If } \mathcal{P} : \vdash_{\text{BVQ}} [(R \otimes T) \otimes P], \text{ then there are } \mathcal{D} : \vdash_{\text{BVQ}} ([P_1 \otimes P_2] \vdash_{\text{BVQ}} P), \text{ and } \mathcal{P}_1 : \vdash_{\text{BVQ}} [R \otimes P_1], \text{ and } \mathcal{P}_2 : \vdash_{\text{BVQ}} [T \otimes P_2], \text{ for some } P_1, \text{ and } P_2. \\
3. & \quad \text{Let } \mathcal{D} : \vdash_{\text{BVQ}} [R \otimes P] \text{ with } R \approx [l_1 \otimes \cdots \otimes l_m], \text{ such that } i \neq j \text{ implies } i \neq i_j, \text{ for every } i, j \in \{1, \ldots, m\}, \text{ and } m > 0. \text{ Then, for every structure } R_0, \text{ and } R_1, \text{ if } R \approx [R_0 \otimes R_1], \text{ there exists } \mathcal{D} : \vdash_{\text{BVQ}} [R_0 \otimes P]. \\
4. & \quad \text{If } \mathcal{P} : \vdash_{\text{BVQ}} [R \otimes P], \text{ then there are } \mathcal{D} : \vdash_{\text{BVQ}} [T \otimes P], \text{ and } \mathcal{P}' : \vdash_{\text{BVQ}} [R \otimes T], \text{ for some } T.
\end{align*}\]

Now, we reason by induction on \([\overline{T} \otimes P]\), proceeding by cases on the form of \(T\).

As a first case we assume \(\overline{T} \approx \overline{a}\), and we cope with a base case. The assumption becomes \(\mathcal{P} : \vdash_{\text{BVQ}} [\overline{a} \otimes P]\) which is exactly:

\[
\begin{align*}
\overline{a} & \approx a \\
\mathcal{P} & \vdash_{\text{BVQ}} [\overline{a} \otimes P] \\
\overline{a} & \approx a \\
\mathcal{P} & \vdash_{\text{BVQ}} [a \otimes P] \\
\overline{a} & \approx a \\
\mathcal{P} & \vdash_{\text{BVQ}} [a \otimes P]
\end{align*}
\]
As a second case we assume \( \overline{T} \approx [\overline{a_1} \otimes \cdots \otimes \overline{a_m}] \), and we cope with another base case. The assumption becomes \( \mathcal{D} : \vdash [(\overline{a_1} \otimes \cdots \otimes \overline{a_m}) \otimes \overline{P}] \). We conclude by Point 5 of Shallow Splitting (Proposition B.1) which implies \((\overline{a_1} \otimes \cdots \otimes \overline{a_m}) \vdash_{BvQ} \overline{P}\).

As a third case we assume \( \overline{T} \approx (R_1 \otimes R_2) \). So, the assumption is \( \mathcal{D} : \vdash [(R_1 \otimes R_2) \otimes \overline{P}] \).

Point 2 of Shallow Splitting (Proposition B.1) implies \( \mathcal{D} : [P_1 \otimes P_2] \vdash P \), and \( \mathcal{D}_1 : \vdash [R_1 \otimes P_1] \), and \( \mathcal{D}_2 : \vdash [R_2 \otimes P_2] \), for some \( P_1, P_2 \).

Both \( R_1 \) and \( R_2 \) are invertible, and \([R_1 \otimes P_1] < (((R_1 \otimes R_2) \otimes P]) \), and \([R_2 \otimes P_2] < (((R_1 \otimes R_2) \otimes P]) \). So, the inductive hypothesis holds on \( \mathcal{D}_1 \), and \( \mathcal{D}_2 \). We get \( \mathcal{D}_1' : R_1 \vdash P_1 \), and \( \mathcal{D}_2' : R_2 \vdash P_2 \). We conclude by:

\[
\begin{array}{c}
\frac{(R_1 \otimes R_2)}{[R_1 \otimes R_2]} \\
\frac{[R_1 \otimes P_1]}{\mathcal{D}_1} \\
\frac{[P_1 \otimes P_2]}{\mathcal{D}_2} \\
\frac{\mathcal{D}_2' \vdash P}{\mathcal{D}_1' \vdash P} \\
\end{array}
\]

As a fourth case we assume \( \overline{T} \approx [R]_a \) such that, without loss of generality, \( a \in \text{bn}([R]_a) \).

So, the assumption is \( \mathcal{D} : \vdash [[R]_a \otimes \overline{P}] \).

Point 3 of Shallow Splitting (Proposition B.1) implies \( \mathcal{D} : [T]_a \vdash P \), and \( \mathcal{D} : \vdash [R \otimes T] \), for some \( T \).

Both \( R \) invertible, and \([R \otimes T] < [[R]_a \otimes P]\), imply the induction holds on \( \mathcal{D} \). We get \( \mathcal{D}' : \overline{R} \vdash T \).

So, we conclude that:

\[
\begin{array}{c}
\frac{[R]_a}{\mathcal{D}} \\
\frac{[T]_a}{\mathcal{D}} \\
\frac{\mathcal{D}' \vdash P}{\mathcal{D} \vdash P} \\
\end{array}
\]

C  Proving point 6 of Process structures, trivial derivations and right-contexts (Proposition 8.3 page 18)

The proof is by induction on the size of \( E \), proceeding by cases on the form of \( S'\{ \}^\prec \), which, by assumption, is a process structure, so it can assume only specific forms.

- The base case is \( S'\{ \}^\prec \approx \langle \{ \} \cdot U \rangle \), for some \( U \). So, \( S'\{ \}^\prec \approx \langle \cdot \cdot U \rangle \approx U \). Moreover, \( \langle E \rangle = \langle b \cdot U \rangle \) implies that \( E \) is \( b \cdot E' \) for some \( E' \) such that \( \langle E' \rangle = U \). Since we can prove:

\[
\begin{array}{c}
a \frac{a}{b} \quad b \frac{b}{b} \quad \frac{b}{b} \\
\end{array}
\]

we are done because \( \langle F \rangle = \langle \cdot \cdot U \rangle \approx U = \langle \langle E' \rangle \rangle \).

A first remark is that we cannot have \( S'\{ \}^\prec \approx \langle \cdot \rangle^\prec \cdot F \) with \( \cdot \rangle^\prec \cdot \approx \{ \} \). Otherwise \( S'\{ \}^\prec \) would not be a process structure.
A second remark is that \( U \approx \circ \) does not pose any problem. In such a case \( E \) is \( b.0 \), and we can write \( b.0 \rightarrow b.0 \).

- Let \( S'(\cdot) \vdash \{\mathcal{S}'(\cdot) \not\approx U\} \). The assumptions \( \{E\} = [\mathcal{S}'(b) \not\approx U] \), and \( \{F\} = [\mathcal{S}'(\circ) \not\approx U] \) imply that \( E \) is \( E' \mid E'' \), and \( F \) is \( F' \mid E'' \), for some \( E', E'' \), and \( F' \) such that \( \{E'\} = \mathcal{S}'(b) \), and \( \{F'\} = \mathcal{S}'(\circ) \), and \( \{E''\} = U \). We can prove:

  \[
  \begin{array}{c}
  \text{ext} \\
  E' \rightarrow F'
  \end{array}
  \]

  \[
  \begin{array}{c}
  E' \mid E'' \\
  \rightarrow F' \mid E''
  \end{array}
  \]

  because the premise holds thanks to the inductive hypotheses, also assuring the desired constraints on \( \mathcal{S}' \).

- Let \( S'(\cdot) \vdash \{\mathcal{S}'(\cdot) \not\approx U\} \). The assumptions \( \{E\} = [\mathcal{S}'(b) \not\approx U] \), and \( \{F\} = [\mathcal{S}'(\circ) \not\approx U] \) imply that \( E \) is \( E'_{ul} \), and \( F \) is \( F'_{ul} \), for some \( E' \), and \( F' \) such that \( \{E'\} = \mathcal{S}'(b) \), and \( \{F'\} = \mathcal{S}'(\circ) \). We can prove:

  \[
  \begin{array}{c}
  \rho \\
  E' \rightarrow F'
  \end{array}
  \]

  \[
  \begin{array}{c}
  E'_{ul} \\
  \rightarrow F'_{ul}
  \end{array}
  \]

  because the premise holds thanks to the inductive argument. Of course we choose \( \rho \), depending on \( a \). If \( a \equiv b \), then \( \rho \) must be \( p_{b} \), and \( \gamma' \equiv \epsilon \). Otherwise, if \( a \not\equiv b \), then \( \rho \) must be \( p_{a} \), and \( \gamma' \equiv 1 \).

Point (3) of this Proposition excludes any further case.

D Proving point (7) of Process structures, trivial derivations and right-contexts (Proposition 8.3, page 18)

The proof is by induction on the size of \( E \mid F \), proceeding by cases on the forms of \( S'(\cdot) \vdash \cdot \), and \( S''(\cdot) \vdash \cdot \), which, by assumption, are process structures, so they can assume only specific forms.

- The base case has \( S'(\cdot) \vdash \cdot \approx \{\cdot \not= U'\} \), and \( S''(\cdot) \vdash \cdot \approx \{\cdot \not= U''\} \), for some \( U' \), and \( U'' \) every of which may well be \( \mathcal{S}' \). So, \( S'(\circ) \approx \langle \circ \not= U' \rangle \), and \( S''(\circ) \approx \langle \circ \not= U'' \rangle \), and \( U' \), and \( U'' \) imply that \( E = b.E' \), and \( F = \overline{b}.E' \). We can write:

    \[
    \begin{array}{c}
    a \\
    b.E' \rightarrow E'
    \end{array}
    \begin{array}{c}
    a \\
    b.E' \rightarrow E'
    \end{array}
    \]

    \[
    \begin{array}{c}
    \epsilon \\
    (b.E') \rightarrow \overline{b}.E'
    \end{array}
    \begin{array}{c}
    \epsilon \\
    (b.E') \rightarrow \overline{b}.E'
    \end{array}
    \]

  We remark that neither \( S'(\cdot) \vdash \cdot \approx \{\cdot \not= U'\} \) with \( S'(\cdot) \vdash \cdot \approx \{\cdot \not= U''\} \) with \( S''(\cdot) \vdash \cdot \approx \{\cdot \not= U''\} \), nor \( S'(\cdot) \vdash \cdot \approx \{\cdot \not= U''\} \) with \( S'(\cdot) \vdash \cdot \approx \{\cdot \not= U''\} \), nor neither \( S'(\cdot) \vdash \cdot \approx \{\cdot \not= U''\} \) could be process structures.

- Let \( S'(\cdot) \vdash \cdot \approx \{\mathcal{S}'(\cdot) \not= U'\} \). So, \( S'(\circ) \approx \{\mathcal{S}'(\circ) \not= U'\} \). The assumptions \( \{E\} = [\mathcal{S}'(b) \not= U']\), and \( \{F\} = [\mathcal{S}'(\circ) \not= U']\) imply that \( E = G_1 \mid G_2 \), and \( E' = G'_1 \mid G_2' \) such that \( \{G_1\} = \mathcal{S}'(b) \), and \( \{G'_1\} = \mathcal{S}'(\circ) \), and \( \{G_2\} = U' \).
Let \( S'' \{ \cdot \} \approx [\tilde{S}'' \{ \cdot \} \bowtie U'' \} \). So, \( S'' \{ \cdot \} \approx [\tilde{S}'' \{ \cdot \} \bowtie U'' \} \). The assumptions 
\(< F > = [\tilde{S}'' \{ \cdot \} \bowtie U'' \} \), and 
\(< F' > = [\tilde{S}'' \{ \cdot \} \bowtie U'' \} \) imply that \( F = H_1 \cup H_2 \), and 
\( F' = H'_1 \cup H_2 \) such that 
\(< H_1 > = \tilde{S}'' \{ \cdot \} \), and 
\(< H'_1 > = \tilde{S}'' \{ \cdot \} \), and 
\(< H_2 > = U'' \}. We can prove:

\[
\begin{array}{c}
\text{ctx} \quad G_1 \mid H_1 \xrightarrow{\cdot} G_1' \mid H_1' \\
\text{ctx} \quad G_1 \mid H_1 \cup H_2 \xrightarrow{\cdot} G_1' \mid H_1' \cup H_2 \\
\end{array}
\]

The premise holds thanks to the inductive hypothesis because both \( G_1 \mid H_1 \) is smaller than \( G_1 \mid H_2 \).

Let \( S'' \{ \cdot \} \approx [\tilde{S}'' \{ \cdot \} \bowtie U'' \} \) with \( \tilde{S}'' \{ \cdot \} \approx \{ \} \). Otherwise \( S'' \{ \cdot \} \) could not be a process structure. So, \( S'' \{ \cdot \} \approx \langle \cdot \bowtie U'' \} \approx U'' \). The assumptions 
\(< F > = [\tilde{b} \bowtie U'' \} \), and 
\(< F' > = \langle \cdot \bowtie U'' \} \approx U'' \} \) imply that \( F = \tilde{b}, F' \). We can prove:

\[
\begin{array}{c}
\text{ctx} \quad G_1 \mid \tilde{b}, F' \rightarrow G_1' \mid F' \\
\end{array}
\]

The premise holds thanks to the inductive hypothesis because \( G_1 \mid (\tilde{b}, F') \) is smaller than \( G_1 \mid (\tilde{b}, F') \).

Let \( S'' \{ \cdot \} \approx [\tilde{S}'' \{ \cdot \} \bowtie U'' \} \), for any \( a \). So, \( S'' \{ \cdot \} \approx [\tilde{S}'' \{ \cdot \} \bowtie U'' \} \). The assumptions 
\(< F > = [\tilde{S}'' \{ \cdot \} \bowtie U'' \} \), and 
\(< F' > = [\tilde{S}'' \{ \cdot \} \bowtie U'' \} \) imply that \( F = H_{1|b} \), and 
\( F' = H'_{1|b} \), for some \( H, H' \) such that 
\(< H > = \tilde{S}'' \{ \cdot \} \), and 
\(< H' > = \tilde{S}'' \{ \cdot \} \). We can prove:

\[
\begin{array}{c}
\text{ctx} \quad G_1 \mid (H)_{1|b} \rightarrow G_1' \mid (H')_{1|b} \\
\end{array}
\]

The premise holds thanks to the inductive hypothesis because \( G_1 \mid (H)_{1|b} \) is smaller than \( G_1 \mid (H)_{1|b} \).

Let \( S' \{ \cdot \} \approx \langle \tilde{S}' \{ \cdot \} \bowtie U' \} \) with \( \tilde{S}' \{ \cdot \} \approx \{ \} \). Otherwise \( S' \{ \cdot \} \) could not be a process structure. So, \( S' \{ \cdot \} \approx \langle \cdot \bowtie U' \} \approx U' \}. The assumptions 
\(< E > = (b \bowtie U') \), and 
\(< E' > = \langle \cdot \bowtie U' \} \approx U' \} \) imply that \( E = b, E' \).

We already considered the case with \( S'' \{ \cdot \} \approx [\tilde{S}'' \{ \cdot \} \bowtie U'' \} \). It is enough to switch \( S' \{ \cdot \} \) and \( S'' \{ \cdot \} \).

Letting \( S'' \{ \cdot \} \approx \langle \tilde{S}'' \{ \cdot \} \bowtie U'' \} \), with \( \tilde{S}'' \{ \cdot \} \approx \{ \} \), otherwise \( S'' \{ \cdot \} \) could not be a process structure, becomes the base case, we started with.

Let \( S' \{ \cdot \} \approx [\tilde{S}' \{ \cdot \} \bowtie U' \} \), for any \( a \). So, \( S' \{ \cdot \} \approx [\tilde{S}' \{ \cdot \} \bowtie U' \} \) where, thanks to \(\{2\} \), we can always be in a situation such that \( a \) is different from every element in \( \text{fn}(\tilde{S}' \{ \cdot \} \bowtie U' \} \). The assumptions 
\(< F > = [\tilde{S}' \{ \cdot \} \bowtie U' \} \), and 
\(< F' > = [\tilde{S}' \{ \cdot \} \bowtie U' \} \) imply that \( F = H_{1|b} \), and 
\( F' = H'_{1|b} \), for some \( H, H' \) such that 
\(< H > = \tilde{S}' \{ \cdot \} \), and 
\(< H' > = \tilde{S}' \{ \cdot \} \). We can prove:

\[
\begin{array}{c}
\rho \quad b, E' \mid H \rightarrow E' \mid H' \\
\end{array}
\]

where \( \rho \) can be any between \( p_0 \) and \( p_a \). The premise holds thanks to the inductive hypothesis because \( b, E' \mid H \) is smaller than \( (b, E')_{1|b} \).
• Let $S'(\vdash a) \approx [S'(\vdash a)]_a$ for a given $a$. So, $S'(\vdash a) \approx [S'(\vdash a)]_a$. The assumptions $\langle E \rangle = [S'(b)]_a$, and $\langle E' \rangle = [S'(b)]_a$ imply that $E = G_{\langle E \rangle}$, and $E' = G'_{\langle E \rangle}$, for some $G$, and $G'$ such that $\langle G \rangle = S'(b)_a$, and $\langle G' \rangle = S'(b)_a$.

  - We already considered the case with $S''(\vdash a) \approx [S''(\vdash a)]_a$. It is enough to switch $S'(\vdash a)$ and $S''(\vdash a)$.
  - We already considered the case with $S''(\vdash a) \approx [S''(\vdash a)]_a$. It is enough to switch $S'(\vdash a)$ and $S''(\vdash a)$.
  - Let $S''(\vdash a) \approx [S''(\vdash a)]_a$, for any $c$. So, $S''(\vdash a) \approx [S''(\vdash a)]_a$. The assumptions $\langle F \rangle = [S''(b)]_a$, and $\langle F' \rangle = [S''(b)]_a$ imply that $F = H_{\langle F \rangle}$, and $F' = H'_{\langle F \rangle}$, for some $H$, and $H'$ such that $\langle H \rangle = S''(b)_a$, and $\langle H' \rangle = S''(b)_a$. We need to consider the following cases where (i) $\rho$ can be $p_i$, or $p_a$, and (ii) the premise of all the given derivations exists thanks to the inductive arguments we have used so far in this proof.

  * As a first case let $a \equiv c$, and $a, c \neq b$. We can prove:

    $$
    \begin{array}{c}
    G \mid H \xrightarrow{\rho} G' \mid H' \\
    G_{\langle H \rangle} \mid H_{\langle H \rangle} \xrightarrow{\rho} G'_{\langle H \rangle} \mid H'_{\langle H \rangle}
    \end{array}
    $$

    We can proceed in the same way also when $a, c \equiv b$, the derivation becoming:

    $$
    \begin{array}{c}
    G \mid H \xrightarrow{\rho} G' \mid H' \\
    G_{\langle b \rangle} \mid H_{\langle b \rangle} \xrightarrow{\rho} G'_{\langle b \rangle} \mid H'_{\langle b \rangle}
    \end{array}
    $$

  * As a third case let $a \equiv b$, and $c \neq b$, we can prove:

    $$
    \begin{array}{c}
    G(\langle b \rangle) \mid H_{\langle b \rangle} \xrightarrow{\rho} G'_{\langle b \rangle} \mid H'_{\langle b \rangle} \\
    G_{\langle b \rangle} \mid H_{\langle b \rangle} \equiv G_{\langle b \rangle} \mid H_{\langle b \rangle} \equiv G'_{\langle b \rangle} \mid H'_{\langle b \rangle}
    \end{array}
    $$

    where $d$ neither occurs in $G$, nor it occurs in $H_{\langle b \rangle}$ so that we can apply (12).

E Proof of Soundness w.r.t. internal communication (Theorem 8.6, page 19)

• As a base case, let $\langle E \rangle \approx [\langle b \mid \{ E' \} \otimes \langle \overline{b} \mid \{ E'' \} \rangle]$, for some process $E'$, and $E''$. So, $E = (b, E') \mid (\overline{b}, E'')$, and $S'(\vdash a) \approx \langle \{ E' \} \rangle$, and $S''(\vdash a) \approx \langle \{ E'' \} \rangle$. We can take $G$ to be $E' \mid E''$ because $\langle \{ b \mid \{ E' \} \otimes \langle \overline{b} \mid \{ E'' \} \rangle \rangle \approx [\langle E' \rangle \otimes \{ E'' \}]$. We can write:

$$
\begin{array}{c}
(a) \quad b, E' \xrightarrow{\rho} E' \\
(b) \quad \overline{b}, E'' \xrightarrow{\rho} E'
\end{array}
$$

• Let $\langle E \rangle \approx [S'(b)]_a \otimes [S''(b)]_a \otimes [E''']_a$, for some $E'''$, and $c$. We remark that $c$ is either different from $b$ in both $S'(b)_a$, and $S''(b)_a$, or it is equal to $b$ in both of them. Otherwise, we could not get to the premise of $\nu[a]_c$ in $\langle E \rangle$. So, $E$ is $E'_{\langle a \rangle} \mid E''_{\langle a \rangle} \mid E'''$, where $\langle E' \rangle \approx S'(b)_a$, and $\langle E'' \rangle \approx S''(b)_a$. We can take $G$ as $G'_{\langle a \rangle} \mid G''_{\langle a \rangle} \mid E'''$, where...
because \( \langle G \rangle \approx [S'\{\circ\}]_c \approx [S''\{\circ\}]_c \approx \langle E''' \rangle \), with \( \langle G' \rangle \approx S'\{\circ\} \), and \( \langle G'' \rangle \approx S''\{\circ\} \). We can write:

\[
\begin{array}{c}
\rho \\
\text{ctx}
\end{array}
\frac{
E' | E'' \quad \epsilon
}{G' | G''}
\]

where \( \rho \) can be \( \rho_\text{p}_a \), or \( \rho_\text{p}_i \). The premise follows from Point (7) of Proposition 8.3.

- Let \( \langle E \rangle \approx [[S'(b) \triangleright S''(b)] \triangleright \langle E'' \rangle]_c \), for some \( E'' \), and \( c \). So, \( E \) is \( \langle E' \div E'' \rangle \rangle \langle E''' \rangle \rangle \), where \( \langle E' \rangle \approx S'(b) \rangle \rangle \), and \( \langle E'' \rangle \approx S''(b) \rangle \rangle \). We can take \( G \) as \( \langle G' \div G'' \rangle \rangle \langle E''' \rangle \rangle \), because \( \langle G \rangle \approx [[S'(\circ) \triangleright S''(\circ) \triangleright \langle E'' \rangle]_c \), with \( \langle G' \rangle \approx S'(\circ) \), and \( \langle G'' \rangle \approx S''(\circ) \). We can write:

\[
\begin{array}{c}
\rho \\
\text{ctx}
\end{array}
\frac{
E' | E'' \quad \epsilon
}{G' | G''}
\]

\[
\begin{array}{c}
\rho \\
\text{ctx}
\end{array}
\frac{
E' | E'' | 0 \quad \epsilon
}{G' | G'' | 0}
\]

where \( \rho \) can be \( \rho_\text{p}_a \), or \( \rho_\text{p}_i \). The premise follows from Point (7) of Proposition 8.3.

Of course, if \( \langle E \rangle \approx [[S'(b) \triangleright S''(b)] \triangleright \langle E'' \rangle] \), for some \( E'' \), we can proceed as here above, dropping \( \rho \).

Assuming that (\( * \)) is the lowermost instance of \( \exists \) of \( \mathcal{D} \) excludes other cases that would impede getting to the premise of (\( * \)) itself in a trivial derivation like \( \mathcal{D} \) has to be.

### F Proof of Soundness w.r.t. external communication (Theorem 8.8, page 20)

We proceed on the possible forms that \( \langle E \rangle \) can assume, in relation with the form of \( R \). Point (6) of Proposition 8.3 will help concluding.

**First case.** We focus on \( \mathcal{D} \) concluding with \([\langle E \rangle \approx ([\mathcal{R} \circ R]_b] \). In the simplest case, Points (3), and (4) of Proposition 8.3 imply that either \( \langle E \rangle \approx [[S'(b)]_b \approx \langle E'' \rangle] \), or \( \langle E \rangle \approx \langle b \div \langle E'' \rangle \rangle_b \), for some \( E'' \), and \( S'\{\circ\} \), such that \( b \in \text{fn}(S'(b)].

1. Let \( \langle E \rangle \approx [[b \div \langle E'' \rangle]_b \). So, \( E \) is \( \langle b.E'' \rangle \rangle \). We can take \( G \) coinciding to \( E'' \), because \( [[b \div \langle E'' \rangle]_b \approx [[E'' \rangle]_b \). We can prove:

\[
\begin{array}{c}
\rho \\
\text{p}
\end{array}
\frac{
b \quad \epsilon
}{b.E''}
\]

\[
\begin{array}{c}
\rho \\
\text{p}
\end{array}
\frac{
b.E''_b \quad \epsilon
}{E''_b}
\]

2. Let \( \langle E \rangle \approx [[S'(b)]_b \approx \langle E'' \rangle] \). So, \( E \) is \( E'_b | E'' \) where \( \langle E' \rangle \approx S'(b) \rangle \). We can take \( G \) as \( G'_b | E'' \) where \( \langle G' \rangle \approx S'(\circ) \rangle \). We can prove:

\[
\begin{array}{c}
\rho \\
\text{ctx}
\end{array}
\frac{
E' \quad \epsilon
}{G'}
\]

\[
\begin{array}{c}
\rho \\
\text{ctx}
\end{array}
\frac{
E'_{b} \quad \epsilon
}{G'_{b}}
\]

\[
\begin{array}{c}
\rho \\
\text{ctx}
\end{array}
\frac{
E'_b | E'' \quad \epsilon
}{G'_b | E''}
\]

Point (6) of Proposition 8.3 implies that the premise holds.
In fact, the most general situations that Points (3), and (4) of Proposition 8.3 imply are:

$$\{E\} \approx \lceil \cdots \lceil S'\{b\} \rceil_{a_m} \cdots \rceil_{a_1} \equiv \{E'\} \right) \rceil_{a_m} \cdots \rceil_{a_1}$$

$$\{E\} \approx \lceil \cdots \lceil (b \bowtie \{E'\}) \rceil_{a_m} \cdots \rceil_{a_1}$$

where $a_i \neq a_j$, for every $1 \leq i, j \leq m$, and $b \equiv a_i$, for some $1 \leq i \leq m$. We can resume to the situation we have just developed in detail, by rearranging the occurrences of $Sd\sigma$, thanks to congruence (42).

**Second case.** Let us assume that $\mathcal{D}$ concludes with $R \equiv \langle b \cdot R' \rangle$. Points (3), and (4) of Proposition 8.3 imply either $\{E\} \approx \langle b \cdot \{E'\} \rangle$, or $\{E\} \approx \{S'(b) \bowtie \{E'\}\}$, where $b \in \text{fn}(S'(b))$. Both combinations are simple sub-cases of the previous ones, just developed in detail.