Homogenization problems for the compressible Navier–Stokes system in 2D perforated domains

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In this paper, we study the homogenization problems for stationary compressible Navier–Stokes system in a bounded 2D domain, where the domain is perforated with very tiny holes (or obstacles) whose diameters are much smaller than their mutual distances. We obtain that the process of homogenization doesn't change the motion of the fluids. From another point of view, we obtain the same system of equations in asymptotic limit. It is the first result of homogenization problem in 2D compressible case.

KEYWORDS
Bogovskii’s operator, homogenization, Navier–Stokes system, perforated domains

MSC CLASSIFICATION
35B27; 76M50; 76N06

1 | INTRODUCTION

Homogenization of Newtonian fluid in physical domains perforated by a large number of tiny holes plays an important role in fluid mechanics and has gain lots of interest. Viscous fluid flow passing a great many fixed solid obstacles is a situation frequently occurring in real applications referring to Sánchez-Palencia.\(^1\) Based on these applications, the models like stationary, viscous fluid flows represented by the standard Stokes or Navier–Stokes system of equations in porous medium could be of vital importance.

The typical diameter and mutual distance of these holes become the main factors in the asymptotic behavior of fluid flows in the regime where the number of holes tends to infinity and their size tends to zero. With the increasing number of holes, the limit motion of fluid flow approaches a state governed by certain homogenized equations which are homogeneous in form of different cases (without obstacles).

Allaire\(^2,3\) (or earlier results by Tartar\(^4\)) provided a systematical study of Stokes and Navier–Stokes system for three different circumstances where the holes are periodically distributed with different sizes. Roughly speaking, the asymptotic limit behavior is governed by Darcy’s law when the holes are of supercritical size; when the holes are of the critical size, the asymptotic limit behavior gives rise to Brinkman’s law; the subcritical size of holes makes no differences in the motion of the asymptotic limit—the limit problem coincides with the original system of equations.

Moreover, relevant results has been extended to evolutionary (time-dependent) incompressible Navier–Stokes system by Mikelić\(^5\) and Allaire\(^6\) and more recently by Feireisl et al.\(^7\) For evolutionary barotropic (compressible) Navier–Stokes system, Masmoudi\(^8\) obtained that the homogenization limit was governed by porous medium equation (Darcy’s law), and similar results for the full Navier–Stokes–Fourier system were obtained in Feireisl et al.\(^9\) Authors considered the case of small holes for compressible Navier–Stokes equations in previous studies,\(^10–12\) steady compressible Navier–Stokes–Fourier system in Lu and Pokorný,\(^13\) and the homogenized equations mentioned above remain the same asymptotic limit as
original ones. Additionally, the case of compressible fluid with moving rigid body in homogenization was studied by Bravin and Nečasová.\textsuperscript{14}

### 1.1 Problem formulation

Similar to the case in Diening et al\textsuperscript{10} and Feireisl and Lu,\textsuperscript{11} where the homogenization of 3D steady compressible Navier–Stokes equations is considered, we mainly concentrate on homogenization of compressible (isentropic) stationary Navier–Stokes system in 2D of the subcritical case, and we show that the asymptotic limit coincides with original one, where \( \varepsilon \) denotes mutual distance between holes and diameter of the holes is taken as \( a_\varepsilon = e^{-\sigma \varepsilon - \alpha} \) with \( \alpha > 2 \).

Considering a bounded domain \( \Omega \subset \mathbb{R}^2 \) of class \( C^2 \), we introduce a family of \( \varepsilon \)-dependent perforated domains \( \{ \Omega_\varepsilon \}_{\varepsilon > 0} \), where

\[
\Omega_\varepsilon = \Omega \setminus \bigcup_{k \in K_\varepsilon} T_{\varepsilon,k}, \quad K_\varepsilon := \{ k | \varepsilon \tilde{C}_k \subset \Omega \},
\]

(1.1)

where the sets \( T_{\varepsilon,k} \) represent holes or obstacles. Suppose we have the following property concerning the distribution of the holes:

\[
T_{\varepsilon,k} := x_{\varepsilon,k} + a_\varepsilon T_k \subset \subset B(x_{\varepsilon,k}, b_0 a_\varepsilon) \subset \subset \varepsilon C_k \subset \Omega,
\]

(1.2)

with

\[
C_k := (0, 1)^2 + k, \quad k \in \mathbb{Z}^2, \quad a_\varepsilon = e^{-\sigma \varepsilon - \alpha} \quad \text{for} \quad \alpha > 2.
\]

(1.3)

Here, \( x_{\varepsilon,k} \in T_{\varepsilon,k}, b_0 \) and \( \sigma \) is positive constant independent of \( \varepsilon \), for each \( k \), \( T_k \subset \mathbb{R}^2 \) is a simply connected bounded domain of class \( C^2 \), and \( B(x, r) \) denotes the open ball centered at \( x \) with radius \( r \) in \( \mathbb{R}^2 \). The diameter of each \( T_{\varepsilon,k} \) is of size \( O(a_\varepsilon) \), and their mutual distance is \( O(\varepsilon) \), where their total number \( |K_\varepsilon| \) can be estimated as

\[
|K_\varepsilon| \leq \frac{|\Omega|}{\varepsilon^2} (1 + o(1)).
\]

For convenience, we use the symbol \( L^q_0(\Omega) \) to denote the space of functions in \( L^q(\Omega) \) with zero integral mean

\[
L^q_0(\Omega) := \left\{ f \in L^q(\Omega) : \int_{\Omega} f \, dx = 0 \right\}.
\]

(1.4)

Then, we consider the following stationary (compressible) Navier–Stokes system in \( \Omega_\varepsilon \)

\[
\text{div}(\rho \mathbf{u}) = 0,
\]

(1.5)

\[
\text{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\rho) = \text{div} S(\nabla \mathbf{u}) + \rho \mathbf{f} + \mathbf{g}.
\]

(1.6)

\[
S(\nabla \mathbf{u}) = \mu \left( \nabla \mathbf{u} + \nabla \mathbf{u}^T - \frac{2}{3} (\text{div} \mathbf{u}) \mathbf{I} \right) + \eta (\text{div} \mathbf{u}) \mathbf{I}, \quad \mu > 0, \quad \eta \geq 0.
\]

(1.7)

Here, \( \rho \) is fluid mass density, \( \mathbf{u} \) is velocity field, \( p = p(\rho) \) is pressure, and \( S(\nabla \mathbf{u}) \) stands for Newtonian viscous stress tensor with constant viscosity coefficients \( \mu, \eta \).

In the spatial domain \( \Omega_\varepsilon \), the system is supplemented with standard no-slip boundary condition

\[
\mathbf{u} = 0 \quad \text{on} \quad \partial \Omega_\varepsilon.
\]

(1.8)

For sake of simplicity, we concentrate on the isentropic pressure–density state relationship:

\[
p(\rho) = a \rho^\gamma, \quad a > 0,
\]

(1.9)

where the adiabatic exponent \( \gamma \) will be specified as follows.

The motion of the fluid is driven by volume force \( \mathbf{f} \) and nonvolume force \( \mathbf{g} \), defined on the whole domain \( \Omega \) and independent of \( \varepsilon \), those are supposed to be uniformly bounded,

\[
\|\mathbf{f}\|_{L^\infty(\Omega; \mathbb{R}^2)} + \|\mathbf{g}\|_{L^\infty(\Omega; \mathbb{R}^2)} \leq C < \infty.
\]

(1.10)
Specifically, we use symbol $C$ to denote a generic bounded constant that may vary from line to line in the following contents, but it is independent of the parameters of the problem, in particular of $\varepsilon$. Furthermore, we use the symbol $\tilde{h}$ to denote the zero extension of $h$ in $R^2$, which means

$$
\tilde{h} = h \text{ in } \Omega, \quad \tilde{h} = 0 \text{ in } R^2 \setminus \Omega.
$$

(1.11)

To be consistent with its physical interpretation, the density $\rho$ is nonnegative, and we fix the total mass of the fluid to be

$$
0 < \inf_{0<\varepsilon<1} M_{\varepsilon} \leq M_{\varepsilon} = \int_{\Omega_{\varepsilon}} \rho \, dx \leq \sup_{0<\varepsilon<1} M_{\varepsilon} < \infty.
$$

(1.12)

In particular, the restriction operator introduced by Allaire can be used in a compatible way in 2D to construct the inverse of divergence—Bogovskii's operator (see Bogovskii and Galdi).

The paper is organized as follows. In Sections 1.1–1.3, we introduce the formulation of the problem and the definition of weak solutions and state our main results. Then, in Section 1.4, we introduce a restriction operator and construct the inverse of divergence—Bogovskii's operator that plays a crucial role in the proof of uniform bound to solution $[\rho, u]$. After that, uniform estimates are obtained via this operator in Section 2.1 to identify the asymptotic limit for Navier–Stokes system in perforated domains. In Sections 2.2–2.4, we give the convergence (or homogenization) process of 2D compressible Navier–Stokes system in perforated domains, which shows the homogenization process for stationary compressible Navier–Stokes equations in a perforated domain is not affected by obstacles and the limit problem coincides with original one. Finally, we obtain main results listed in Section 1.3. A special note is needed that 2D unsteady case remains open.

### 1.2 Weak solutions

We recall the definition of a finite energy weak solution to (1.5)–(1.8).\textsuperscript{17}

**Definition 1.1.** A couple of functions $[\rho, u]$ is said to be a finite energy weak solution of Navier–Stokes system (1.5)–(1.7) supplemented with conditions (1.8)–(1.12) in $\Omega_{\varepsilon}$ provided:

$$
\rho \geq 0 \text{ a.e. in } \Omega_{\varepsilon}, \quad \int_{\Omega_{\varepsilon}} \rho \, dx = M_{\varepsilon}, \quad \rho \in L^{\beta(\gamma)}(\Omega_{\varepsilon}) \text{ for some } \beta(\gamma) > \gamma, \quad u \in W^{1,2}_{\rho}(\Omega_{\varepsilon}; R^2);
$$

(1.13)

for any test functions $\psi \in C^\infty(\overline{\Omega}_{\varepsilon})$ and $\varphi \in C^\infty_c(\Omega_{\varepsilon}; R^2)$:

$$
\int_{\Omega_{\varepsilon}} \rho u \cdot \nabla \psi \, dx = 0;
$$

(1.14)

$$
\int_{\Omega_{\varepsilon}} \rho u \otimes u : \nabla \varphi + p(\rho)\text{div} \varphi - S(\nabla u) : \nabla \varphi + (\rho f + g) \cdot \varphi \, dx = 0;
$$

(1.15)

and the energy inequality

$$
\int_{\Omega_{\varepsilon}} S(\nabla u) : \nabla u \, dx \leq \int_{\Omega_{\varepsilon}} (\rho f + g) \cdot u \, dx
$$

(1.16)

holds.

**Remark 1.2.** A finite energy weak solution of Navier–Stokes system (1.5)–(1.7) in $\Omega$ is similar to it in $\Omega_{\varepsilon}$.

**Definition 1.3.** A finite energy weak solution $[\rho, u]$ (see Novotný et al.\textsuperscript{17}) is said to be a renormalized weak solution if

$$
\int_{R^2} b(\rho) u \cdot \nabla \psi + (b(\rho) - b'(\rho) \rho) \text{div} u \psi \, dx = 0,
$$

(1.17)

for any $\psi \in C^\infty_c(R^2)$, where $[\rho, u]$ were extended to be zero outside $\Omega_{\varepsilon}$, and any $b \in C^0([0, \infty)) \cap C^1((0, \infty))$ such that

$$
b'(s) \leq c s^{-\lambda_2} \text{ for } s \in (0, 1], \quad b'(s) \leq c s^{\lambda_1} \text{ for } s \in [1, \infty),
$$

(1.18)
with

\[ c > 0, \quad \lambda_0 < 1, -1 < \lambda_1 \leq \frac{\beta(\gamma)}{2} - 1. \]  

(1.19)

**Lemma 1.4.** By DiPerna–Lions’ transport theory,\(^{17,18}\) for any \( r \in L^\beta(\Omega), \beta \geq 2, \mathbf{v} \in W^{1,2}_0(\Omega), \) where \( \Omega \subset \mathbb{R}^2 \) is a bounded domain of class \( C^2, \) such that

\[ \text{div}(r \mathbf{v}) = 0 \text{ in } D'(\Omega), \]  

(1.20)

the renormalized equation

\[ \text{div}(b(r)\mathbf{v}) + (rb'(r) - b(r)) \text{ div } \mathbf{v} = 0, \text{ holds in } D'(\mathbb{R}^2), \]  

(1.21)

for any \( b \in C^0([0, \infty)) \cap C^1((0, \infty)) \) satisfying \( (1.18)–(1.19) \) provided \( r \) and \( \mathbf{v} \) have been extended to be zero outside \( \Omega. \)

**Remark 1.5.** Recall that the relevant values of adiabatic exponent are \( 1 \leq \gamma \leq 5/3, \) where the case \( \gamma = 1 \) corresponds to isothermal case while \( \gamma = 5/3 \) is the adiabatic exponent of monoatomic gas. Based on energy type arguments combined with refined pressure estimates, Lions\(^{19}\) proved the existence of weak solutions in range \( \gamma > 5/3 \) in 3D. Lions’ theory for the existence of weak solutions to homogeneous Dirichlet (no-slip) boundary conditions has been extended to physical range \( \gamma \leq 5/3 \) by several authors, see Březina and Novotný,\(^{20}\) Plotnikov and Sokolowski\(^{21}\) for \( \gamma > 3/2, \) Frehse et al.\(^{22}\) for \( \gamma > 4/3, \) and Plotnikov and Weigant\(^{23}\) for \( \gamma > 1. \) While in 2D with \( \gamma > 1, \) Lions\(^{19}\) has given the existence of weak solution \([\varrho_\varepsilon, \mathbf{u}_\varepsilon] \in [L^{2r}(\Omega_\varepsilon)] \times [H^1_0(\Omega, \mathbb{R}^2)], \) which establishes the following uniform norm estimates for the solution \([\varrho, \mathbf{u}] \) in \( \Omega. \) Then, we need to show the asymptotic limit of solutions \([\varrho_\varepsilon, \mathbf{u}_\varepsilon] \) of steady compressible Navier–Stokes system \( (1.5)–(1.8) \) in \( \Omega_\varepsilon \) coincide with the solution of the same system on homogeneous domain \( \Omega. \)

### 1.3 Main results

In this paper, we consider the case of optimal adiabatic exponent \( \gamma > 1 \) in pressure law \( (1.9), \) which is also an innovation in this article.

**Theorem 1.6.** Suppose the conditions \((1.9), (1.10), \) and \((1.12)\) are satisfied. Let \( \gamma > 1 \) and \( \alpha > 2 \) be given and \([\varrho_\varepsilon, \mathbf{u}_\varepsilon]_{0 < \varepsilon < 1} \) be a family of finite energy weak solutions to \((1.5)–(1.8)\) in \( \Omega_\varepsilon, \) where \( \mathbf{f} \) and \( \mathbf{g} \) satisfy \((1.10). \) Then, we have uniform estimates

\[ \sup_{0 < \varepsilon < 1} \left( \|\varrho_\varepsilon\|_{L^{2r}(\Omega_\varepsilon)} + \|\mathbf{u}_\varepsilon\|_{W^{1,2}_0(\Omega_\varepsilon, \mathbb{R}^2)} \right) \leq C < \infty. \]  

(1.22)

Moreover, up to a subtractions of subsequence, the zero extensions \([\tilde{\varrho}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon] \) to outside \( \Omega_\varepsilon \) satisfy

\[ \tilde{\varrho}_\varepsilon \to \varrho \text{ weakly in } L^{2r}(\Omega), \tilde{\mathbf{u}}_\varepsilon \to \mathbf{u} \text{ weakly in } W^{1,2}_0(\Omega; \mathbb{R}^2), \]  

(1.23)

where \([\varrho, \mathbf{u}] \) is a finite energy weak solution to the same system of Equations \((1.5)–(1.8)\) in limit domain \( \Omega. \)

### 1.4 Preliminaries

#### 1.4.1 Bogovskii’s operator

Our principal result concerns the construction of the inverse of divergence operator on a family of perforated domains \( \Omega_\varepsilon \rangle_{\varepsilon > 0} \), which is totally different from classical Bogovskii’s operator (since the perforated domains are not uniform Lipschitz domains).

**Proposition 1.7.** Let \( \{\Omega_\varepsilon\}_{\varepsilon > 0} \) be a family of domains with properties specified in Section 1.1. Then, there exists a linear operator

\[ B_\varepsilon : L^2_0(\Omega_\varepsilon) \to W^{1,2}_0(\Omega_\varepsilon; \mathbb{R}^2), \]  

such that for any \( f \in L^2_0(\Omega_\varepsilon), \)

\[ \text{div}(B_\varepsilon(f)) = f \text{ in } \Omega_\varepsilon, \quad \|B_\varepsilon(f)\|_{W^{1,2}_0(\Omega_\varepsilon; \mathbb{R}^2)} \leq C\|f\|_{L^2(\Omega_\varepsilon)}, \]  

(1.24)

for some constant \( C \) independent of \( \varepsilon. \)
The existence of Bogovskii’s operator on a fixed Lipschitz domain has been shown by several authors, such as Galdi\textsuperscript{16} or Acosta et al\textsuperscript{24} and Diening et al\textsuperscript{25} especially by Bogovskii\textsuperscript{12} and Kapitanski and Piletskas.\textsuperscript{26} Here, we need to establish the uniform estimate (1.24), where the constant $C$ is independent of $\varepsilon$. For the sake of readers, we briefly give a proof of Proposition 1.7. Referring to Diening et al\textsuperscript{10} and Feireisl and Lu,\textsuperscript{11} such Bogovskii’s operator is established in 3D case. Considering the significance of restriction operator in proving the existence of Bogovskii’s operator, we introduce the following restriction operator $R_\varepsilon$ analogously by employing restriction operator constructed by Allaire.\textsuperscript{2}

**Lemma 1.8.** For $\Omega_\varepsilon$ defined in Section 1.1, there exists a linear bounded Restriction operator $R_\varepsilon : W^{1,2}_0(\Omega; R^2) \rightarrow W^{1,2}_0(\Omega_\varepsilon; R^2)$ such that

$$u \in W^{1,2}_0(\Omega_\varepsilon; R^2) \Rightarrow R_\varepsilon(u) = u \quad \text{in } \Omega_\varepsilon,$$

$$\text{div } u = 0 \quad \text{in } \Omega \Rightarrow \text{div } R_\varepsilon(u) = 0 \quad \text{in } \Omega_\varepsilon,$$

$$\|R_\varepsilon(u)\|_{W^{1,2}_0(\Omega_\varepsilon; R^2)} \leq C\|u\|_{W^{1,2}_0(\Omega; R^2)}, \quad C \text{ independent of } \varepsilon.$$  

This restriction operator is constructed in the proof of Lemma 2.1 in Allaire\textsuperscript{2} in the following way:

$$B(x_k, b_1 \varepsilon) \subset \varepsilon C_k, \tilde{B}_{\varepsilon,k} = \tilde{B}(x_k, b_0 a_k) \subset B(x_k, b_1 \varepsilon).$$  

Let us introduce the following decomposition of cube $\varepsilon C$ with $k \in K_\varepsilon$:

$$\varepsilon C_k = T_{\varepsilon,n} \bigcup \tilde{E}_{\varepsilon,k} \bigcup \tilde{F}_{\varepsilon,k} \text{ with } E_{\varepsilon,k} := B(x_k, b_1 \varepsilon) \setminus T_{\varepsilon,k}, F_{\varepsilon,k} := (\varepsilon C_k) \setminus B(x_k, b_1 \varepsilon),$$

where $b_1 > 0$. For any $u \in W^{1,2}_0(\Omega; R^2)$, we can define $R_\varepsilon$ by

$$\begin{cases}
R_\varepsilon(u) = u & \text{on } \varepsilon C_k \cap \Omega, \text{ for } k \notin K_\varepsilon, \\
R_\varepsilon(u) = \mathbf{v}_{\varepsilon,k} & \text{on } F_{\varepsilon,k}, \text{ for } k \in K_\varepsilon,
\end{cases}$$

where $\mathbf{v}_{\varepsilon,k} \in W^{1,2}_0(\varepsilon C_k; R^2)$ satisfies

$$\nabla p_{\varepsilon,k} - \Delta \mathbf{v}_{\varepsilon,k} = -\Delta u \quad \text{in } E_{\varepsilon,k},$$

$$\text{div } \mathbf{v}_{\varepsilon,k} = \text{div } u + \frac{1}{|E_{\varepsilon,k}|} \int_{E_{\varepsilon,k}} \text{div } u \, dx \quad \text{in } E_{\varepsilon,k},$$

$$\mathbf{v}_{\varepsilon,k} = u \quad \text{on } \partial E_{\varepsilon,k} - \partial T_{\varepsilon,k}, \mathbf{v}_{\varepsilon,k} = 0 \quad \text{on } \partial T_{\varepsilon,k}.$$  

**Proof of Proposition 1.7.** For $f \in L^2_0(\Omega_\varepsilon)$, we consider the following zero extension of $f$

$$\hat{f} = f \quad \text{in } \Omega_\varepsilon, \quad \hat{f} = 0 \quad \text{on } \Omega \setminus \Omega_\varepsilon = \bigcup_{k \in K_\varepsilon} T_{\varepsilon,k}.$$  

By classical Bogovskii’s operator\textsuperscript{15} defined on domain $\Omega$, we obtain $u := B(\hat{f}) \in W^{1,2}_0(\Omega; R^2)$ satisfying

$$\text{div } u = \hat{f} \quad \text{in } \Omega, \quad \|u\|_{W^{1,2}_0(\Omega; R^2)} \leq C\|\hat{f}\|_{L^2(\Omega; R^2)} = C\|f\|_{L^2(\Omega; R^2)}$$

where $C$ depends only on $\Omega$. Furthermore, by (1.34), we have

$$\text{div } u = \hat{f} = 0 \quad \text{in } T_{\varepsilon,k}.$$  

Applying the restriction operator constructed in (1.30)–(1.33), we obtain

$$\text{div } \mathbf{v}_{\varepsilon,k} = \text{div } u = \hat{f} = 0 \quad \text{on } E_{\varepsilon,k}.$$
whenever \( \mathbf{u} \) satisfies (1.35). Moreover, we have \( R_{\varepsilon}(\mathbf{u}) = \mathbf{u} \) in \( \Omega \setminus \bigcup_{k \in K_{\varepsilon}} E_{\varepsilon,k} \). Combined with (1.30) and (1.37), we conclude that
\[
\text{div} \ R_{\varepsilon}(\mathbf{u}) = f \text{ in } \Omega_{\varepsilon}.
\] (1.38)

Now, we define
\[
B_{\varepsilon}(f) := R_{\varepsilon}(\mathbf{u}) = R_{\varepsilon}(B(\tilde{f})).
\] (1.39)

where \( B \) is the classical Bogovskii’s operator on \( \Omega \). It’s easy to check that
\[
\|B_{\varepsilon}(f)\|_{W^{1,2}(\Omega_{\varepsilon}; R^{2})} := \|R_{\varepsilon}(\mathbf{u})\|_{W^{1,2}(\Omega_{\varepsilon}; R^{2})} \leq C\|\mathbf{u}\|_{W^{1,2}(\Omega; R^{2})} = C\|B(\tilde{f})\|_{W^{1,2}(\Omega; R^{2})} \leq C\|\tilde{f}\|_{L^{2}(\Omega)} = C\|f\|_{L^{2}(\Omega_{\varepsilon})}.
\] (1.40)

Then, we proved Proposition 1.7. \( \square \)

2 | PROOF OF THEOREM 1.6

This section deals with the proof of Theorem 1.6. By Lemma 1.4, we first stress that the solution \([\varphi_{\varepsilon}, \mathbf{u}_{\varepsilon}]\) is also a renormalized weak solution as follows.

**Lemma 2.1.** We have
\[
\text{div}(\varphi_{\varepsilon}(\tilde{\mathbf{u}}_{\varepsilon})) = 0,
\]
\[
\text{div}(b(\varphi_{\varepsilon}(\tilde{\mathbf{u}}_{\varepsilon})) + (\varphi_{\varepsilon}b'(\varphi_{\varepsilon}) - b(\varphi_{\varepsilon})) \text{ div } \mathbf{u}_{\varepsilon} = 0, \text{ in } D'(R^{2}),
\]
for any \( b \in C^{0}([0, \infty)) \cap C^{1}((0, \infty)) \) satisfying (1.18)–(1.19), where \([\varphi_{\varepsilon}, \mathbf{u}_{\varepsilon}]\) denotes the functions \([\varphi_{\varepsilon}, \mathbf{u}_{\varepsilon}]\) extended to be zero outside \( \Omega_{\varepsilon} \).

2.1 | Uniform bounds

As shown in Lions,\(^{19}\) if space dimension \( D = 2 \), adiabatic exponent \( \gamma > 1 \), we have the existence of solution \([\varphi_{\varepsilon}, \mathbf{u}_{\varepsilon}] \in [L^{2r}(\Omega_{\varepsilon})] \times [H^{1}_{0}(\Omega_{\varepsilon})]^{2}\) for any fixed \( \varepsilon \). However, the classical estimates of their norms depend on Lipschitz character of domain \( \Omega_{\varepsilon} \), which goes to infinity as \( \varepsilon \to 0 \). In order to show the uniform estimates (1.22), we employ the uniform Bogovskii’s operator \( B_{\varepsilon} \) constructed in Section 1.4 to establish the independence of \( \varepsilon \in (0, 1) \) in (1.22).

By Korn’s inequality and Hölder’s inequality, the energy inequality (1.16) implies
\[
\|\nabla \mathbf{u}_{\varepsilon}\|_{L^{2}(\Omega_{\varepsilon}; R^{2})} \leq C \left( \|f\|_{L^{\infty}(\Omega_{\varepsilon}; R^{2})}\|\varphi_{\varepsilon}\|_{L^{2}(\Omega_{\varepsilon})} \|\mathbf{u}_{\varepsilon}\|_{L^{2}(\Omega_{\varepsilon}; R^{2})} + \|g\|_{L^{\infty}(\Omega_{\varepsilon}; R^{2})}\|\mathbf{u}_{\varepsilon}\|_{L^{2}(\Omega_{\varepsilon}; R^{2})} \right),
\] (2.1)
where \( \frac{1}{s} + \frac{1}{t} = 1 \) and \( s \to 1^{+}, t \to \infty^{-} \).

**Remark 2.2.** In Section 2.1 and Section 2.2, we frequently use \( \infty^{-} \) to denote a bounded real number arbitrarily close to positive infinity and \( x + (x-) \) denotes the real number arbitrarily close to \( x \) on the right (left) side.

Since \( \mathbf{u}_{\varepsilon} \in W^{1,2}_{0}(\Omega_{\varepsilon}; R^{2}) \) has zero trace on the boundary, the Sobolev embedding and Poincaré inequality imply
\[
\|\mathbf{u}_{\varepsilon}\|_{L^{2}(\Omega_{\varepsilon}; R^{2})} \leq \|\mathbf{u}_{\varepsilon}\|_{W^{1,2}_{0}(\Omega_{\varepsilon}; R^{2})} \leq C \|\nabla \mathbf{u}_{\varepsilon}\|_{L^{2}(\Omega_{\varepsilon}; R^{2})},
\] (2.2)
\[
\|\mathbf{u}_{\varepsilon}\|_{L^{2}(\Omega_{\varepsilon}; R^{2})} \leq \|\mathbf{u}_{\varepsilon}\|_{W^{1,2}_{0}(\Omega_{\varepsilon}; R^{2})} \leq C \|\nabla \mathbf{u}_{\varepsilon}\|_{L^{2}(\Omega_{\varepsilon}; R^{2})},
\] (2.3)
for some constant \( C \) independent of domain \( \Omega_{\varepsilon} \).

By estimates (2.1)–(2.3), we deduce
\[
\|\nabla \mathbf{u}_{\varepsilon}\|_{L^{2}(\Omega_{\varepsilon}; R^{2})} + \|\mathbf{u}_{\varepsilon}\|_{L^{2}(\Omega_{\varepsilon}; R^{2})} \leq C \left( \|f\|_{L^{\infty}(\Omega_{\varepsilon}; R^{2})}\|\varphi_{\varepsilon}\|_{L^{2}(\Omega_{\varepsilon})} + \|g\|_{L^{\infty}(\Omega_{\varepsilon}; R^{2})} \right) \leq C \left( \|\varphi_{\varepsilon}\|_{L^{2}(\Omega_{\varepsilon})} + 1 \right).
\] (2.4)
Let $B_\varepsilon$ be the operator introduced in Proposition 1.7, we introduce a test function

$$
\varphi := B_\varepsilon \left( \phi_\varepsilon - \frac{1}{|\Omega_\varepsilon|} \int_{\Omega_\varepsilon} d\phi_\varepsilon \right).
$$

(2.5)

Since Remark 1.5, we notice that

$$
\phi_\varepsilon \in L^{2\gamma}(\Omega_\varepsilon), \quad u_\varepsilon \in W^{1,2}_0(\Omega_\varepsilon; \mathbb{R}^2), \quad \text{for any fixed } \varepsilon.
$$

(2.6)

Then, by Proposition 1.7 and (2.5), we have

$$
\text{div } \varphi = \phi_\varepsilon \cdot \frac{1}{|\Omega_\varepsilon|} \int_{\Omega_\varepsilon} d\phi_\varepsilon \quad \text{in } \Omega_\varepsilon,
$$

(2.7)

and

$$
\|\varphi\|_{W^{1,2}_0(\Omega_\varepsilon; \mathbb{R}^2)} \leq C \|\phi_\varepsilon\|_{L^{2\gamma}(\Omega_\varepsilon)} \leq C \left( \|\phi_\varepsilon\|_{L^{2\gamma}(\Omega_\varepsilon)} + \|\phi_\varepsilon\|_{L^{2}(\Omega_\varepsilon)} \right)
$$

(2.8)

Taking $\varphi$ as a test function in weak formulation of momentum Equation (1.15) gives

$$
\int_{\Omega_\varepsilon} p(\phi_\varepsilon) \phi_\varepsilon \, dx = \sum_{j=1}^{4} I_j,
$$

(2.9)

with

$$
I_1 := \int_{\Omega_\varepsilon} p(\phi_\varepsilon) \, dx \frac{1}{|\Omega_\varepsilon|} \int_{\Omega_\varepsilon} d\phi_\varepsilon,
$$

$$
I_2 := \int_{\Omega_\varepsilon} \mu \nabla u_\varepsilon \cdot \nabla \varphi \, dx + \int_{\Omega_\varepsilon} \left( \frac{\mu}{\varepsilon^2} + \eta \right) \text{div } u_\varepsilon \cdot \nabla \varphi \, dx,
$$

$$
I_3 := -\int_{\Omega_\varepsilon} \phi_\varepsilon u_\varepsilon \otimes u_\varepsilon \cdot \nabla \varphi \, dx,
$$

$$
I_4 := -\int_{\Omega_\varepsilon} (\phi_\varepsilon f + g) \cdot \varphi \, dx.
$$

For $I_1$:

$$
I_1 := \int_{\Omega_\varepsilon} p(\phi_\varepsilon) \, dx \frac{1}{|\Omega_\varepsilon|} \int_{\Omega_\varepsilon} d\phi_\varepsilon = \frac{a}{|\Omega_\varepsilon|} \|\phi_\varepsilon\|_{L^{2\gamma}(\Omega_\varepsilon)}^{2\gamma} \leq \frac{a}{|\Omega_\varepsilon|} \|\phi_\varepsilon\|_{L^{2\gamma}(\Omega_\varepsilon)}^{2\gamma} \|\phi_\varepsilon\|_{L^{2\gamma}(\Omega_\varepsilon)}^{2\gamma} = \frac{aM_\varepsilon^{2\gamma\theta_1}}{|\Omega_\varepsilon|} \|\phi_\varepsilon\|_{L^{2\gamma}(\Omega_\varepsilon)}^{2\gamma(1-\theta_1)},
$$

(2.10)

where we used (1.12), Young’s inequality, and interpolations between Lebesgue spaces. $M_\varepsilon$ is the total mass, and the number $\theta_1$ satisfies

$$
0 < \theta_1 < 1 \text{ s.t. } \frac{1}{\gamma} = \frac{\theta_1}{1} + \frac{1-\theta_1}{2\gamma}.
$$

(2.11)

For $I_2$:

$$
I_2 \leq C \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon; \mathbb{R}^{2\gamma})} \|\nabla \varphi\|_{L^{2\gamma}(\Omega_\varepsilon; \mathbb{R}^{2\gamma})} \leq C \left( \|\phi_\varepsilon\|_{L^{2\gamma}(\Omega_\varepsilon)} + 1 \right) \|\phi_\varepsilon\|_{L^{2\gamma}(\Omega_\varepsilon)}^{2\gamma},
$$

(2.12)
where we used (1.12), (2.4), and (2.8). The number $0 < \theta_2 < 1$ is determined by

$$\frac{1}{s} = \frac{\theta_2}{1} + \frac{1 - \theta_2}{2\gamma}, \quad \frac{1}{s} = \frac{1}{t} = 1, \ s \to 1+, \ t \to \infty-. \quad (2.13)$$

which implies $\theta_2 = \frac{1 - \frac{1}{s}}{1 - \frac{2}{2\gamma}} \to 1^-(1 - \theta_2 \to 0^+)$ for $s \to 1+$ and any $\gamma \in (1, +\infty)$.

For $I_3$:

$$I_3 = -\int_{\Omega_0} \phi \cdot \nabla \varphi \, dx$$

$$\leq C \| \phi \|_{L^q(\Omega_0)} \| u \|_{L^2(\Omega_0; \mathbb{R}^n)}^2 \| \nabla \varphi \|_{L^2(\Omega_0, \mathbb{R}^n)}$$

$$\leq C \| \phi \|_{L^q(\Omega_0)} \| \partial_t \varphi \|_{L^2(\Omega_0)}^\theta \| \partial_t \varphi \|_{L^2(\Omega_0)}^{1-\theta} + C \| \partial_t \varphi \|_{L^2(\Omega_0)}$$

$$\leq C \| \phi \|_{L^2(\Omega_0)} \| \partial_t \varphi \|_{L^2(\Omega_0)} + C \| \partial_t \varphi \|_{L^2(\Omega_0)}^{2(1-\theta)}$$

$$\leq CM^\theta \| \phi \|_{L^2(\Omega_0)} + CM^\theta \| \phi \|_{L^2(\Omega_0)}^{\gamma + (1-\theta)} + CM^\theta \| \phi \|_{L^2(\Omega_0)}^{\gamma + (1-\theta)}$$

where the estimates are similar to $I_1, I_2$ and

$$0 < \theta_2, \theta_3 < 1 \text{ s.t. } \frac{1}{s} = \frac{\theta_2}{1} + \frac{1 - \theta_2}{2\gamma} \quad \text{and} \quad \frac{1}{s_1} = \frac{\theta_3}{1} + \frac{1 - \theta_3}{2\gamma}. \quad (2.15)$$

This implies

$$\frac{1}{s_1} - \frac{1}{s} = 0 + (1 - \theta_3 \to 1-) \text{ as } s_1 \to 2+, \ \gamma \to 1+. \quad (2.16)$$

For $I_4$:

$$I_4 = -\int_{\Omega_0} (\phi \cdot f + g) \cdot \varphi$$

$$\leq C \| \phi \|_{L^q(\Omega_0)} \| \varphi \|_{L^2(\Omega_0, \mathbb{R}^n)} + C \| \varphi \|_{L^2(\Omega_0, \mathbb{R}^n)}$$

$$\leq C(1 + \| \phi \|_{L^2(\Omega_0)}) \| \varphi \|_{L^2(\Omega_0, \mathbb{R}^n)}$$

where $s, t$, and $\theta_2$ are the same as above.

Summing up the estimates for $I_1$ to $I_4$ implies

$$\| \phi \|_{L^2(\Omega_0)}^{2\gamma} \leq C \left( 1 + \| \phi \|_{L^2(\Omega_0)}^{\beta_1(\gamma)} \right), \quad (2.18)$$

where

$$\beta_1(\gamma) = \max\{2\gamma(1 - \theta_1), \gamma + (1 - \theta_3), \gamma + (1 - \theta_2), \gamma, \gamma + (1 - \theta_3) + 2(1 - \theta_2) \}.$$
Thus, up to a subtracstion of subsequence,\
\[ \hat{\phi} \to \phi \text{ weakly in } L^{2r}(\Omega), \quad \hat{u} \to u \text{ weakly in } W^{1,2}_0(\Omega; \mathbb{R}^2). \] (2.22)

We obtained the uniform estimate (1.22) and the weak convergence in (1.23).

### 2.2 Equations in homogeneous domain

Now, we show that the couple \([\hat{\phi}, \hat{u}]\) solves the momentum equations as (1.6) in \(\Omega\) up to a small remainder.

**Lemma 2.3.** Under assumptions in Theorem 1.6, there holds

\[
div(\hat{\phi} \hat{u} \otimes \hat{u}) + \nabla p(\hat{\phi}) = div \mathbb{S}(\nabla \hat{u}) + \hat{\phi} f + g + r, \quad \text{in } D'(\Omega; \mathbb{R}^2),
\] (2.23)

where the distribution \(r\) is small and satisfies:

\[
|(r, \varphi)_{D'(\Omega; \mathbb{R}^2), D(\Omega; \mathbb{R}^2)}| \leq C \epsilon^{\delta_1}(\|\nabla \varphi\|_{L^2(\Omega; \mathbb{R}^2)} + \|\varphi\|_{L^\infty(\Omega; \mathbb{R}^2)}),
\] (2.24)

for any \(\varphi \in C_c^\infty(\Omega; \mathbb{R}^2)\) and \(\delta_1 := \frac{1}{2} \alpha - 1 > 0, t \to 2^+, \) bounded constant \(C > 0.\)

**Proof.** By the assumptions on distribution and size of holes in (1.2) and the definition of capacity, we know that the capacity of all holes have an upper bound estimate \((\text{Cap}_2(B(x, r)) \leq C |\log r|^{-1} \text{ in } 2D).\) Then, we can find a series of admissible functions \(g_\epsilon\) infinitely close to the capacity and satisfying the same upper bound estimate. Therefore, there exist cut-off functions (admissible functions) \(g_\epsilon \in C_c^\infty(\Omega)\) satisfying \(0 \leq g_\epsilon \leq 1\) and

\[
g_\epsilon = 1 \text{ in } \bigcup_{k \in K_\epsilon} T_{k, k}, \quad g_\epsilon = 0 \text{ in } \Omega \setminus \bigcup_{k \in K_\epsilon} B(x_{k, k}, \eta_0 a_\epsilon), \quad \|g_\epsilon\|_{W^{1,2}(\Omega; \mathbb{R}^2)}^2 \leq 2C \epsilon^{-2} |\log a_\epsilon|^{-1}. \] (2.25)

Then, for any \(\varphi \in C_c^\infty(\Omega; \mathbb{R}^2)\), we have

\[
I_\epsilon = \int_\Omega \hat{\phi} \hat{u} \otimes \hat{u} : \nabla \varphi + p(\hat{\phi}) \text{div } \varphi - \mathbb{S}(\nabla \hat{u}) : \nabla \varphi + \hat{\phi} f : \varphi + g : \varphi \, dx
\]

\[
= \int_\Omega (\hat{\phi} \hat{u} \otimes \hat{u} : \nabla (1 - g_\epsilon) \varphi + p(\hat{\phi}) \text{div } ((1 - g_\epsilon) \varphi) - \mathbb{S}(\nabla \hat{u}) : \nabla (1 - g_\epsilon) \varphi) + \hat{\phi} f : ((1 - g_\epsilon) \varphi) + g : (g_\epsilon (1 - g_\epsilon) \varphi)) \, dx + I_\epsilon,
\]

where we used the fact that \((1 - g_\epsilon) \varphi \in C_c^\infty(\Omega_\epsilon; \mathbb{R}^2)\) is a good test function for momentum equations (1.6) in \(\Omega_\epsilon\), and the quantity \(I_\epsilon\) has the form

\[
I_\epsilon := \sum_{j=1}^4 I_{j, \epsilon},
\] (2.26)

with

\[
I_{1, \epsilon} := \int_\Omega \hat{\phi} \hat{u} \otimes \hat{u} : (g_\epsilon \nabla \varphi) + \hat{\phi} \hat{u} \otimes \hat{u} : (\nabla g_\epsilon \otimes \varphi) \, dx,
\]

\[
I_{2, \epsilon} := \int_\Omega p(\hat{\phi}) g_\epsilon \text{div } \varphi + p(\hat{\phi}) \nabla g_\epsilon \cdot \varphi \, dx,
\]

\[
I_{3, \epsilon} := -\int_\Omega \mathbb{S}(\nabla \hat{u}) : (g_\epsilon \nabla \varphi) + \mathbb{S}(\nabla \hat{u}) : (\nabla g_\epsilon \otimes \varphi) \, dx,
\]

\[
I_{4, \epsilon} := \int_\Omega \hat{\phi} f : g_\epsilon \varphi + g : g_\epsilon \varphi \, dx.
\]
We now give estimates for $I_{j,x}$ ($j = 1, 2, 3, 4$) one by one. For $I_{1,x}$, direct calculation gives

$$
|I_{1,x}| := |\int_{\Omega} \partial_t \tilde{\mathbf{u}} \cdot (g \nabla \varphi) + \partial_t \tilde{\mathbf{u}} \cdot (\nabla g \cdot \varphi) \, dx |
\leq C \| \partial_t \|_{L^2(\Omega; L^2(\Omega))} \| \tilde{\mathbf{u}} \|_{L^2 \times L^2(\Omega)} \left( \| \nabla g \|_{L^1(\Omega; L^2(\Omega))} \| \nabla \varphi \|_{L^1(\Omega; L^2(\Omega))} + \| g \|_{L^1(\Omega; L^2(\Omega))} \| \nabla \varphi \|_{L^1(\Omega; L^2(\Omega))} \right) \tag{2.27}
$$



where

$$
a > 2, \quad \frac{2}{s'} + \frac{1}{t'} = \left( \frac{1}{2} - \frac{1}{2\gamma} \right), \quad 1 < \gamma < \infty, \quad \text{and } s', t' \to \infty-, \quad t \to 2+ \text{ as } \gamma \to 1+. \tag{2.28}
$$

For $I_{2,x}$ and $I_{3,x}$, similar to the estimate for $I_{1,x}$, we have

$$
|I_{2,x}| := \left| \int_{\Omega} \mathbf{p}(\partial_t) g \cdot \nabla \varphi + \mathbf{p}(\partial_t) \nabla g \cdot \varphi \, dx \right|
\leq C a \| \partial_t \|_{L^2(\Omega; L^2(\Omega))} \left( \| g \|_{L^1(\Omega; L^2(\Omega))} \| \nabla \varphi \|_{L^1(\Omega; L^2(\Omega))} + \| \nabla g \|_{L^1(\Omega; L^2(\Omega))} \| \varphi \|_{L^1(\Omega; L^2(\Omega))} \right) \tag{2.29}
$$



where

$$
a > 2, \quad 1 < \gamma < \infty, \quad \frac{1}{s} + \frac{1}{t} = \frac{1}{2}, \quad \text{and } s \to \infty-, \quad t \to 2+. \tag{2.30}
$$

For $I_{4,x}$, similar argument gives the following analogous estimate:

$$
|I_{4,x}| := \left| \int_{\Omega} \partial_t \mathbf{f} \cdot g \cdot \varphi + g \cdot \mathbf{f} \cdot \varphi \, dx \right|
\leq C \| \varphi \|_{L^1(\Omega; L^2(\Omega))} \left( \| \partial_t \|_{L^2(\Omega; L^2(\Omega))} \| \mathbf{f} \|_{L^1(\Omega; L^2(\Omega))} \| g \|_{L^1(\Omega; L^2(\Omega))} + \| \mathbf{f} \|_{L^1(\Omega; L^2(\Omega))} \| g \|_{L^1(\Omega; L^2(\Omega))} \right) \tag{2.31}
$$



Summing up the estimates in (2.27), (2.28), (2.29), and (2.31), we finally obtain

$$
I_\varepsilon := \sum_{j=1}^{4} I_{j,x} \leq C \varepsilon^{\frac{1}{2}a-1} \| \varphi \|_{L^1(\Omega; L^1(\Omega))} + \| \nabla \varphi \|_{L^1(\Omega; L^1(\Omega))}, \tag{2.32}
$$

where

$$
\delta_1 := \frac{1}{2} a - 1 > 0, \quad t \to 2+. \tag{2.33}
$$

Thus, we completed the proof of Lemma 2.3.
2.3 The limit equations

In this section, we deduce the limit equation for the couple \([\rho, \mathbf{u}]\) obtained in (2.22) which represent a finite energy renormalized weak solution of (1.5) to (1.8) in \(\Omega\). First of all, from (2.22), we have the following convergence:

\[
\tilde{\varepsilon}_\epsilon \rightarrow \varepsilon \text{ weakly in } L^2(\Omega), \quad \tilde{\mathbf{u}}_\epsilon \rightarrow \mathbf{u} \text{ weakly in } W^{1,2}_0(\Omega; \mathbb{R}^2).
\]  

(2.34)

Applying compact Sobolev embedding, we have

\[
\tilde{\mathbf{u}}_\epsilon \rightarrow \mathbf{u} \text{ strongly in } L^q(\Omega; \mathbb{R}^2) \text{ for any } 1 \leq q < \infty.
\]  

(2.35)

Then, the following weak convergence of nonlinear terms holds:

\[
\tilde{\varepsilon}_\epsilon \tilde{\mathbf{u}}_\epsilon \rightarrow \varepsilon \mathbf{u} \text{ weakly in } L^q(\Omega; \mathbb{R}^2) \text{ for any } 1 < q < 2\gamma,
\]

\[
\tilde{\varepsilon}_\epsilon \tilde{\mathbf{u}}_\epsilon \otimes \tilde{\mathbf{u}}_\epsilon \rightarrow \varepsilon \mathbf{u} \otimes \mathbf{u} \text{ weakly in } L^q(\Omega; \mathbb{R}^2 \times \mathbb{R}^2) \text{ for any } 1 < q < 2\gamma.
\]

(2.36)

Then, in Lemma 2.1 and (2.23), passing with \(\epsilon \rightarrow 0\) gives

\[
\text{div} (\varepsilon \mathbf{u}) = 0,
\]

\[
\text{div} (\varepsilon \mathbf{u} \otimes \mathbf{u}) + \nabla \overline{p(\varepsilon)} = \text{div} S(\nabla \mathbf{u}) + \varepsilon \mathbf{f} + \mathbf{g},
\]

in the sense of distribution in \(D'(\Omega)\), where \(\overline{p(\varepsilon)}\) is the weak limit of \(p(\tilde{\varepsilon}_\epsilon)\) in \(L^2(\Omega)\). Furthermore, by Lemma 1.4, \([\rho, \mathbf{u}]\) satisfies the renormalization equation

\[
\text{div} (b(\varepsilon) \mathbf{u}) + (\varepsilon b'(\varepsilon) - b(\varepsilon)) \text{div} \mathbf{u} = 0, \quad \text{in } D'(\mathbb{R}^2),
\]

(2.38)

where \(b \in C^0((0, \infty)) \cap C^1((0, \infty))\) satisfies (1.18)–(1.19). To finish the proof of Theorem 1.6, it suffices to show \(\overline{p(\varepsilon)} = p(\varepsilon)\), which we obtain in the following section.

2.4 Convergence of pressure term—end of the proof

Here, we introduce \(p(\varepsilon) - \left(\frac{4\mu}{3} + \eta\right) \text{div} \mathbf{u}\) called effective viscous flux, which possesses some weak compactness property specified in the following lemma, which take up great significance in the existence theory of weak solutions for compressible Navier–Stokes equations.

**Lemma 2.4.** Up to a subtractions of subsequence, there holds for any \(\psi \in C^\infty_c(\Omega)\):

\[
\lim_{\epsilon \rightarrow 0} \int_\Omega \psi \left(p(\tilde{\varepsilon}_\epsilon) - \left(\frac{4\mu}{3} + \eta\right) \text{div} \tilde{\mathbf{u}}_\epsilon\right) \tilde{\varepsilon}_\epsilon \, dx = \int_\Omega \psi \left(\overline{p(\varepsilon)} - \left(\frac{4\mu}{3} + \eta\right) \text{div} \mathbf{u}\right) \varepsilon \, dx.
\]

(2.39)

**Proof.** The main idea is to give proper test functions via taking advantage of Fourier multiplier and Riesz operators. Referring to Section 1.3.7.2 in Novotný and Stránská 17 or Section 10.16 in Feireisl and Novotný 27 for the definitions and properties used here of Fourier multiplier and Riesz operators, we choose proper test functions defined by

\[
\psi \nabla \Delta^{-1}(1_\Omega \tilde{\varepsilon}_\epsilon), \quad \psi \nabla \Delta^{-1}(1_\Omega \varepsilon),
\]

(2.40)

where \(\psi \in C^\infty_c(\Omega)\) and \(\Delta^{-1}\) is Fourier multiplier on \(\mathbb{R}^2\) with symbol \(-\frac{1}{|\xi|^2}\).

Observing that

\[
\nabla \nabla \Delta^{-1} = (R_{ij})_{1 \leq i, j \leq 2}
\]

are the classical Riesz operators, then for any \(f \in L^q(\mathbb{R}^2), \ 1 < q < \infty\), we have

\[
\|\nabla \nabla \Delta^{-1}(f)\|_{L^q(\mathbb{R}^2; \mathbb{R}^{2 \times 2})} \leq C \|f\|_{L^q(\mathbb{R}^2)}.
\]
By virtue of $(1_{\Omega} \mathcal{Q}_\varepsilon) \in L^{2\gamma}(R^2; R) \cap L^{1+}(R^2; R)$, owing to $2 < 2\gamma < \infty$, $1 < 1+ < \infty$, we have
\[
\|\nabla \Delta^{-1}(1_{\Omega} \mathcal{Q}_\varepsilon)\|_{L^{2\gamma}(R^2; R^2)} \leq C \|1_{\Omega} \mathcal{Q}_\varepsilon\|_{L^{2\gamma}(R^2; R)} \leq C.
\]
\[
\|\nabla \Delta^{-1}(1_{\Omega} \mathcal{Q}_\varepsilon)\|_{L^{1+}(R^2; R^2)} \leq C \|1_{\Omega} \mathcal{Q}_\varepsilon\|_{L^{1+}(R^2; R)} \leq C.
\]
That means $\nabla \Delta^{-1}(1_{\Omega} \mathcal{Q}_\varepsilon) \in L^{2\gamma}(R^2; R^2) \cap L^{1+}(R^2; R^2)$. Using embedding theorem of homogeneous Sobolev spaces in Novotný and Straskraba or Feireisl and Novotný, and by $2\gamma > 2+ > D = 2$, we have
\[
\nabla \Delta^{-1}(1_{\Omega} \mathcal{Q}_\varepsilon) \in W^{1,2\gamma}(R^2; R^2) \hookrightarrow L^\infty(R^2; R^2), \tag{2.41}
\]
which means
\[
\|\nabla \Delta^{-1}(1_{\Omega} \mathcal{Q}_\varepsilon)\|_{L^{\infty}(R^2; R^2)} \leq \|\nabla \Delta^{-1}(1_{\Omega} \mathcal{Q}_\varepsilon)\|_{L^{2\gamma}(R^2; R^2)} \leq C \|1_{\Omega} \mathcal{Q}_\varepsilon\|_{L^{2\gamma}(R^2; R)} \leq C. \tag{2.42}
\]
Again by embedding theorem in homogeneous Sobolev spaces, we have for any $f \in L^q(R^2)$, supp $f \subset \Omega$:
\[
\|\nabla \Delta^{-1}(f)\|_{L^p(R^2; R^2)} \leq C \|f\|_{L^q(R^2)}, \quad \frac{1}{q^*} = \frac{1}{q} - \frac{1}{2}, \quad \text{if } 1 < q < 2. \tag{2.43}
\]
since $1_{\Omega} \mathcal{Q}_\varepsilon \in L^{1+}(R^2; R) \cap L^\infty(R^2; R)$, by interpolation theorem between Lebesgue spaces, then we have $1_{\Omega} \mathcal{Q}_\varepsilon \in L^p(R^2; R)$, $1+ < p < 2\gamma$ and
\[
\|\nabla \Delta^{-1}(1_{\Omega} \mathcal{Q}_\varepsilon)\|_{L^p(R^2; R^2)} \leq C \|1_{\Omega} \mathcal{Q}_\varepsilon\|_{L^p(R^2; R)} \leq C.
\]
where $\frac{1}{q^*} = \frac{1}{q} - \frac{1}{2}, 1+ < q < 2, 2+ < q^* < \infty$. Combined with (2.41), we obtain
\[
\nabla \Delta^{-1}(1_{\Omega} \mathcal{Q}_\varepsilon) \in L^q(R^2; R^2), \quad 2+ \leq q^* \leq \infty.
\]
Then, by uniform estimate for $\mathcal{Q}_\varepsilon$ and its weak limit $\mathcal{Q}$ in (2.22) and the fact that $2\gamma > 2+$ under our assumption $\gamma > 1$,
\[
\|\nabla \left(\psi \nabla \Delta^{-1}(1_{\Omega} \mathcal{Q}_\varepsilon)\right)\|_{L^{2\gamma}(\Omega; R^{2\gamma})} + \|\nabla \left(\psi \nabla \Delta^{-1}(1_{\Omega} \mathcal{Q})\right)\|_{L^{2\gamma}(\Omega; R^{2\gamma})} \leq C. \tag{2.44}
\]
Since $\delta_1 = \frac{1}{2}\alpha - 1 > 0$ in Lemma 2.3, thus (2.24) and (2.42) imply
\[
|\langle r, \psi \nabla \Delta^{-1}(1_{\Omega} \mathcal{Q}_\varepsilon)\rangle_{D'((\Omega; R^3) \times (\Omega; R^3))}| \\
\leq C \varepsilon^{\delta_1} \left(\|\nabla \left(\psi \nabla \Delta^{-1}(1_{\Omega} \mathcal{Q}_\varepsilon)\right)\|_{L^{2\gamma}(\Omega; R^{2\gamma})} + \|\psi \nabla \Delta^{-1}(1_{\Omega} \mathcal{Q}_\varepsilon)\|_{L^{p}(\Omega; R^{p})}\right) \\
\leq C \varepsilon^{\delta_1} \left(\|\nabla \left(\psi \nabla \Delta^{-1}(1_{\Omega} \mathcal{Q}_\varepsilon)\right)\|_{L^{2\gamma}(\Omega; R^{2\gamma})} + \|\psi \nabla \Delta^{-1}(1_{\Omega} \mathcal{Q}_\varepsilon)\|_{W^{2\gamma}(\Omega; R^{2\gamma})}\right) \\
\leq C \varepsilon^{\delta_1}, \ v \rightarrow 2+,
\]
which goes to zero as $\varepsilon \rightarrow 0$.

Then, we choose $\psi \nabla \Delta^{-1}(1_{\Omega} \mathcal{Q}_\varepsilon)$ as a test function in weak formulation of Equation (2.23) and pass $\varepsilon \rightarrow 0$. Moreover, we choose $\psi \nabla \Delta^{-1}(1_{\Omega} \mathcal{Q})$ as a test function in weak formulation of (2.37)$_2$. Comparing the results of these two operations, through long and straightforward calculations, we finally get
\[
I := \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \psi \left( p(\mathcal{Q}_\varepsilon) - \left(\frac{4\mu}{3} + \eta \right) \text{div} \mathbf{u}_\varepsilon \right) \mathcal{Q}_\varepsilon \ dx - \int_{\Omega} \psi \left( p(\mathcal{Q}) - \left(\frac{4\mu}{3} + \eta \right) \text{div} \mathbf{u} \right) \mathcal{Q} \ dx \\
= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \mathcal{Q}_\varepsilon \mathbf{u}_\varepsilon \cdot \psi \mathbf{R}_{i,j}(1_{\Omega} \mathcal{Q}_\varepsilon) \ dx - \int_{\Omega} \mathbf{R}_{i,j}(\psi \mathcal{Q}) \ dx. \tag{2.45}
\]
In addition, choosing $1_{\Omega} \text{div} \Delta^{-1}(\psi \mathcal{Q}_\varepsilon \mathbf{u}_\varepsilon)$ as a test function in weak formulation of Lemma 2.1 with $b(\mathcal{Q}) = \mathcal{Q}$ and $1_{\Omega} \text{div} \Delta^{-1}(\psi \mathcal{Q} \mathbf{u})$ as a test function in weak formulation of (2.37)$_1$ yields
\[
\int_{\Omega} 1_{\Omega} \mathcal{Q}_\varepsilon \mathbf{u}_\varepsilon \cdot \psi \mathcal{Q}_\varepsilon \mathbf{R}_{i,j}(\mathcal{Q}_\varepsilon \mathbf{u}_\varepsilon) \ dx = 0, \quad \int_{\Omega} 1_{\Omega} \mathbf{R}_{i,j}(\psi \mathcal{Q} \mathbf{u}) \ dx = 0. \tag{2.46}
\]
Substituting (2.46) into (2.45) generates

\[
I = \lim_{\varepsilon \to 0} \int_\Omega \left( \tilde{\alpha}_e \tilde{\alpha}_e \psi R_{i,j}(1 \Omega \tilde{\alpha}_e) - 1 \Omega \tilde{\alpha}_e R_{i,j}(\psi \tilde{\alpha}_e \tilde{\alpha}_e) \right) \, dx - \int_\Omega \left( \theta \theta \psi R_{i,j}(1 \Omega \theta) - 1 \Omega \theta R_{i,j}(\psi \theta) \right) \, dx. \tag{2.47}
\]

Now, we introduce the following Lemma.

**Lemma 2.5.** Let \(1 < p, q < \infty\) satisfy

\[
\frac{1}{r} = \frac{1}{p} + \frac{1}{q} < 1.
\]

Suppose \(u_e \to u\) weakly in \(L^p(\mathbb{R}^2)\), \(v_e \to v\) weakly in \(L^q(\mathbb{R}^2)\), as \(\varepsilon \to 0\).

Then, for any \(1 \leq i, j \leq 2\):

\[
u_e R_{i,j}(v_e) - v_e R_{i,j}(u_e) \to u R_{i,j}(v) - v R_{i,j}(u) \text{ weakly in } L^r(\mathbb{R}^2).
\]

Critically, applying Lemma 2.5 (refer to the proof of Lemma 3.4 in Feireisl and Novotný\(^2\)) to (2.47) to obtain \(I \to 0(\text{as } \varepsilon \to 0)\) in (2.45), then convergence result (2.39) can be deduced. \(\square\)

A direct consequence of the compactness of effective viscous flux is as follows.

**Lemma 2.6.** We denote \(\bar{p}(\rho)\) as the weak limit of \(p(\tilde{\alpha}_e)\tilde{\alpha}_e\) in \(L^{\frac{2\gamma}{\gamma+1}}(\Omega)\). Then, we have \(\bar{p}(\rho)\) weakly in \(L^{\frac{2\gamma}{\gamma+1}}(\Omega)\).

**Proof.** In the beginning, we have

\[
2\gamma - (\gamma + 1) = \gamma - 1 > 0.
\]

Then, by (2.21), we obtain

\[
p(\tilde{\alpha}_e)\tilde{\alpha}_e \to \bar{p}(\rho) \text{ weakly in } L^{\frac{2\gamma}{\gamma+1}}(\Omega).
\]

Taking \(b(s) = s \log s\) in the renormalized equations in Lemma 2.1 and (2.38) yields

\[
div ((\tilde{\alpha}_e \log \tilde{\alpha}_e) \tilde{\alpha}_e) + \tilde{\alpha}_e \text{ div } \tilde{\alpha}_e = 0, \quad div ((\rho \log \rho) \theta) + \theta \text{ div } \theta = 0, \text{ in } D'(\Omega). \tag{2.48}
\]

Passing \(\varepsilon \to 0\) in the first equation of (2.48) gives

\[
div \left( \frac{\rho \log \rho}{\theta} \theta \right) \theta \text{ div } \theta = 0, \text{ in } D'(\Omega), \tag{2.49}
\]

where we used strong convergence of the velocity in (2.35) and

\[
\tilde{\alpha}_e \log \tilde{\alpha}_e \to \bar{\rho} \log \rho \text{ weakly in } L^q(\Omega) \text{ for any } q < 2\gamma,
\]

\[\tilde{\alpha}_e \text{ div } \tilde{\alpha}_e \to \bar{\rho} \text{ div } \theta \text{ weakly in } L^{\frac{2\gamma}{\gamma+1}}(\Omega). \tag{2.50}\]

Then, for any \(\psi \in C_0^\infty(\Omega), (2.49)\) and (2.50) imply

\[
(2.39)_{\text{left}} = \lim_{\varepsilon \to 0} \int_\Omega \psi \left( p(\tilde{\alpha}_e) - \left( \frac{4\mu}{3} + \eta \right) \text{ div } \tilde{\alpha}_e \right) \tilde{\alpha}_e \, dx = \int_\Omega \psi \bar{p}(\rho) \rho - \left( \frac{4\mu}{3} + \eta \right) \bar{\rho} \log \rho \theta \cdot \nabla \psi \, dx. \tag{2.51}
\]

Utilizing the second equation in (2.48), we obtain

\[
(2.39)_{\text{right}} = \int_\Omega \psi \left( \bar{p}(\rho) - \left( \frac{4\mu}{3} + \eta \right) \text{ div } \theta \right) \rho \, dx = \int_\Omega \psi \bar{p}(\rho) \rho - \left( \frac{4\mu}{3} + \eta \right) \bar{\rho} \log \rho \theta \cdot \nabla \psi \, dx. \tag{2.52}
\]
Assume the test functions \( \{\psi_n\}_{n \in \mathbb{Z}_+} \subset C_c^\infty(\Omega) \) such that

\[
\psi_n(x) = 0 \text{ if } d(x, \partial \Omega) < \frac{1}{n}, \quad \psi_n(x) = 1 \text{ if } d(x, \partial \Omega) > \frac{2}{n}, \quad \|\nabla \psi_n\|_{L^\infty(\Omega; \mathbb{R}^2)} \leq 2n.
\]

Then, for any \( q \in [1, \infty] \):

\[
\|1 - \psi_n\|_{L^q(\Omega)} \leq C n^{-\frac{1}{2}}, \quad \|\nabla \psi_n\|_{L^q(\Omega; \mathbb{R}^2)} \leq C n^{1 - \frac{1}{q}},
\]

and consequently,

\[
\|d(x, \partial \Omega)\nabla \psi_n\|_{L^q(\Omega; \mathbb{R}^2)} \leq C n^{-\frac{1}{2}}.
\]

Here, by Hardy's inequality, the velocity \( u \in W^{1,2}_0(\Omega; \mathbb{R}^2) \) implies

\[
\|d(x, \partial \Omega)^{-1} u\|_{L^q(\Omega; \mathbb{R}^2)} \leq C \|u\|_{W^{1,2}(\Omega; \mathbb{R}^2)} \leq C.
\]

Therefore,

\[
\int_{\Omega} \nabla \psi_n \cdot (\rho \log \rho) u \, dx \\ \leq \|d(x, \partial \Omega)\nabla \psi_n\|_{L^q(\Omega; \mathbb{R}^2)}\|\rho \log \rho\|_{L^{1/q}(\Omega)} \|d(x, \partial \Omega)^{-1} u\|_{L^q(\Omega; \mathbb{R}^2)} \\ \leq C n^{-1/10}.
\]

Similarly,

\[
\int_{\Omega} \nabla \psi_n \cdot (\rho \log \rho) u \, dx \leq C n^{-1/10}. \tag{2.54}
\]

Take \( \psi = \psi_n \) in (2.39) and pass to the limit \( n \to \infty \). Using (2.51)–(2.54), we deduce

\[
\int_{\Omega} \frac{p(\rho)\rho - \overline{p(\rho)}\rho}{\rho} \, dx = 0. \tag{2.55}
\]

By strict monotonicity of the mapping \( \rho \mapsto p(\rho) \), applying Theorem 10.19 in Feireisl and Novotný\(^{27}\) or Lemma 3.35 in Novotný and Straskraba\(^{17}\) implies

\[
\overline{p(\rho)} \rho \geq \overline{p(\rho)}\rho, \text{ a.e. in } \Omega.
\]

Together with (2.55), we deduce

\[
\overline{p(\rho)} \rho = \overline{p(\rho)}\rho, \text{ a.e. in } \Omega.
\]

Then, we complete the proof of Lemma 2.6. \( \square \)

By virtue of the monotonicity of \( p(\cdot) \), and using Theorem 10.19 in Feireisl and Novotný\(^{27}\) again, we obtain \( \overline{p(\rho)} = p(\rho) \). Hence, we finished the proof of Theorem 1.6.

For convenience, we recall Theorem 10.19 in Feireisl and Novotný\(^{27}\): Let \( I \subset \mathbb{R} \) be an interval, \( Q \subset \mathbb{R}^d \) be a domain, and \( P \) and \( G \) be nondecreasing functions in \( C(I) \). Let \( \{\theta_n\}_{n \in \mathbb{N}} \) be a sequence in \( L^1(Q; I) \) such that

\[
P(\theta_n) \to P(\theta), \quad G(\theta_n) \to G(\theta), \quad P(\theta_n)G(\theta_n) \to P(\theta)G(\theta), \text{ weakly in } L^1(Q).
\]

Then, the following properties hold:

(i) \( \overline{P(\theta)} \overline{G(\theta)} \leq \overline{P(\theta)G(\theta)} \).

(ii) If \( P \in C(R), G \in C(R), G(R) = R, \) \( G \) is strictly increasing, and \( \overline{P(\theta)} \overline{G(\theta)} = \overline{P(\theta)G(\theta)} \), then \( \overline{P(\rho)} = P(\rho)G^{-1}(\rho) \). If, in particular, \( G(z) = z \) is an identity function, then there holds \( \overline{P(\rho)} = P(\rho) \).

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CONFLICT OF INTEREST

The authors declare no conflict of interest in this paper.

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REFERENCES

1. Sánchez-Palencia E. Non-Homogeneous Media and Vibration Theory, Lecture Notes in Physics, vol. 127: Springer; 1980.
2. Allaire G. Homogenization of the Navier-Stokes equations in open sets perforated with tiny holes. I. Abstract framework, a volume distribution of holes. Arch Ration Mech Anal. 1990;113:209-259.
3. Allaire G. Homogenization of the Navier-Stokes equations in open sets perforated with tiny holes. II. Noncritical sizes of the holes for a volume distribution and a surface distribution of holes. Arch Ration Mech Anal. 1990;113:261-298.
4. Tartar L. Incompressible fluid flow in a porous medium: convergence of the homogenization process. In: Sánchez-Palencia E, ed. Non-Homogeneous Media and Vibration Theory. Springer; 1980. 368–377.
5. Mikelić A. Homogenization of nonstationary Navier-Stokes equations in a domain with a grained boundary. Ann Mat Pura Appl. 1991;158:167-179.
6. Allaire G. Homogenization of the Stokes flow in a connected porous medium. Asymptotic Anal. 1989;2:203-222.
7. Feireisl E, Namlyeyeva Y, Nečasová S. Homogenization of the evolutionary Navier-Stokes system. Manusc Math. 2016;149:251-274.
8. Masmoudi N. Homogenization of the compressible Navier-Stokes equations in a perforated domain with a volume distribution of holes. J Math Fluid Mech. 2016;18:1371-1406.
9. Lu Y, Schwarzacher S. Homogenization of stationary Navier-Stokes equations in domains with tiny holes. J Differ Equ. 2015;265(4):1371-1406.
10. Lu Y, Pokorný M. Homogenization of stationary Navier-Stokes-Fourier system in domains with tiny holes. J Differ Equ. 2021;275(8):463-492.
11. Bogovskii ME. Solution of some vector analysis problems connected with operators div and grad. Trudy Sem S L Soboleva. 1980;80:5-40.
12. Galdi GP. An Introduction to the Mathematical Theory of the Navier-Stokes Equations: Steady-State Problems. Springer Science and Business Media; 2011.
13. Plotnikov P, Sokolowski J. Compressible Navier-Stokes Equations, Theory and Shape Optimization. Birkhäuser Springer; 2013.
14. Frehse J, Steinhauer M, Weigant W. The Dirichlet problem for steady viscous compressible flow in three dimensions. J Math Pures Appl. 2012;97:85-97.
15. Plotnikov P, Weigant W. Steady 3D viscous compressible flows with adiabatic exponent γ ∈ (1, ∞). J Math Pures Appl. 2015;104:58-82.
16. Acosta G, Durrán RG, Muschietti MA. Solutions of the divergence operator on John domains. Adv Math. 2006;206:373-401.
17. Diening L, Ružička M, Schumacher K. A decomposition technique for John domains. Ann Acad Sci Fenn. 2010;35:87-114.
18. Kapitanski IV, Piletskas KI. Some problems of vector analysis. (Russian) Boundary value problems of mathematical physics and related problems in the theory of functions. Zap Nauchn Sem Lomi. 1984;38(38):65-85.
19. Feireisl E, Novotný A. Singular Limits in Thermodynamics of Viscous Fluids. Birkhäuser Verlag; 2009.
20. Feireisl E, Novotný A. On the existence of globally defined weak solutions to the Navier-Stokes equations of compressible isentropic fluids. J Math Fluid Mech. 2001;3:358-392.

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