REGULARIZATION OF BACKWARD TIME-FRACTIONAL PARABOLIC EQUATIONS BY SOBOLEV EQUATIONS METHOD

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Abstract: It is well-known that backward parabolic equations in which the initial condition is to be determined from a measurement at a later time instant is ill-posed. In this paper, a regularization method for backward time-fractional parabolic equations by the Sobolev equations method is investigated. Error estimates of Hölder type are obtained with \textit{a priori} and \textit{a posteriori} regularization parameter choice rules. The proposed regularization method results in a stable noniterative numerical scheme. The theoretical error estimates are confirmed with numerical tests for one and two dimensional equations.

Keywords: Backward time-fractional parabolic equations, Sobolev equations method, numerical implementation.

1 Introduction

Research on fractional differential equations has increased substantially in the last three decades. Many works have shown that fractional differential equations can be used to describe many complex dynamical processes which have memory. These dynamical processes have been observed in viscoelastic materials, heterogeneous media (such as soil, heterogeneous aquifer, underground fluid flow), turbulence diffusion in the atmosphere, plasma physics, etc., see, e.g., [4, 7, 9, 12, 14, 29] and the references therein. They can also be used to describe carrier transport in amorphous semiconductors with multiple trapping [12]. Another application is in image processing [22]. Fractional differential equations can be used to describe super-diffusion and sub-diffusion processes, unlike classical diffusion equations which describe Gaussian diffusion processes.

In this work, we consider a backward time-fractional parabolic equation. This problem can arise from the problem of determination of contaminant sources in underground fluid flow [9]. Since the first work [13] devoted to the backward time-fractional diffusion equation, several papers on backward time-fractional parabolic equations have been published. For the mollification method, see [21, 28]; for the non-local boundary value problem method, see [6, 24, 26, 27]; and for Tikhonov regularization, we refer the reader to [1, 22, 23, 24].

Let \(\gamma\) be a scalar in \((0, 1)\). Consider the following backward time-fractional parabolic equation:

\[
\begin{aligned}
\frac{\partial^{\gamma} u}{\partial t^{\gamma}} + Au &= 0, \quad 0 < t < T, \\
\|u(T) - f\| &\leq \varepsilon,
\end{aligned}
\]

(1.1)
where
\[
\frac{\partial^\gamma u}{\partial t^\gamma} := \frac{1}{\Gamma(1 - \gamma)} \int_0^t (t - s)^{-\gamma} \frac{\partial u(\cdot, s)}{\partial s} ds, \quad \gamma \in (0, 1),
\]
(1.2)
is the Caputo derivative \cite{12, 17}, with \(\Gamma(\cdot)\) being Euler’s Gamma function, and \(A: D(A) \subset H \to H\) is a self-adjoint closed operator on a Hilbert space \(H\). We assume that \(-A\) generates a compact contraction semi-group \(\{S(t)\}_{t \geq 0}\) on \(H\) and \(A\) admits an orthonormal eigenbasis \(\{\phi_i\}_{i \geq 1}\) in \(H\) associated with the eigenvalues \(\{\lambda_i\}_{i \geq 1}\) such that
\[
0 < \lambda_1 < \lambda_2 < \ldots, \quad \text{and} \quad \lim_{i \to +\infty} \lambda_i = +\infty.
\]

In this work we denote the inner product and norm in \(H\) by \(\langle \cdot, \cdot \rangle\) and \(\| \cdot \|\), respectively.

As an example of operator \(A\), consider the case \(H = L^2(\Omega)\), where \(\Omega\) is a bounded domain in \(\mathbb{R}^d, d \geq 1\), with a sufficiently smooth boundary \(\partial \Omega\). As usual, we denote by \(H^p(\Omega)\) the Sobolev spaces. Denote by \(H^1_0(\Omega) = \{u \in H^1(\Omega) : u_{\partial \Omega} = 0\}\). Then \(A\) can be chosen as:
\[
(Au)(x) := -\sum_{i=1}^{d} \frac{\partial}{\partial x_i} \left( \sum_{j=1}^{d} a_{ij}(x) \frac{\partial}{\partial x_j} u(x) \right) + c(x)u(x), x \in \Omega,
\]
with \(D(A) = H^2(\Omega) \cap H^1_0(\Omega)\). Here, we assume that \(a_{ij} = a_{ji} \in C^1(\overline{\Omega}), i, j = 1, \ldots, d; \ c \in C(\overline{\Omega}), \ c(x) \geq 0, \ \forall x \in \overline{\Omega}; \) and \(\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geq \nu \sum_{i=1}^{d} \xi_i^2, \ \forall x \in \overline{\Omega}\) for some \(\nu > 0\).

For \(p > 0\), we define
\[
D(A^p) := \left\{ \psi \in H : \sum_{n=1}^{\infty} \lambda_n^{2p} \langle \psi, \phi_n \rangle^2 < \infty \right\}
\]
and the norm
\[
\|\psi\|_p := \left( \sum_{n=1}^{\infty} \lambda_n^{2p} \langle \psi, \phi_n \rangle^2 \right)^{\frac{1}{2}}, \ \psi \in D(A^p).
\]

For operator \(A\) defined above, it has been shown that \(D(A^p) \subset H^{2p}(\Omega), \ p > 0, \) and \(D(A^{\frac{1}{2}}) = H^1_0(\Omega), \) see, e.g., \cite{19}.

The backward problem (1.1) is well-posed for \(t > 0\) and ill-posed for \(t = 0\), see \cite{6}. To solve this problem, we apply the Sobolev equations regularization method. Namely, we approximate the solution \(u\) of (1.1) by the solution \(v_\alpha\) of the following Sobolev equation:
\[
\begin{aligned}
\frac{\partial^\gamma v_\alpha}{\partial t^\gamma} + A_\alpha v_\alpha &= 0, \quad 0 < t < T, \\
v_\alpha(T) &= f,
\end{aligned}
\]
(1.3)
where \(\alpha > 0\) is a regularization parameter and operator \(A_\alpha\) will be specified later, see Definition 2.
The Sobolev equations method was introduced by Gajewski and Zacharias for regularizing integer-order \((\gamma = 1)\) parabolic equations backward in time [5]. Since then, the method has been investigated in a number of other works, see, e.g., [2, 3, 8, 10, 11, 15, 16, 18, 20]. Gajewski and Zacharias in [5], Huang and Quan in [8], Showalter in [20] used the Sobolev equations method without proving convergence rates. In the other works, error estimates have been obtained for \(a\ priori\) regularization parameter choice rules. A \(a\ posteriori\) parameter choice rules were not investigated in those works.

To our best knowledge, this work is the first one to investigate the use of the Sobolev equations method for regularizing the backward time-fractional parabolic equations (1.1) with \(\gamma \in (0, 1)\). Our theoretical results include optimal convergence rates and error estimates of H"older type for both \(a\ priori\) and \(a\ posteriori\) parameter choice rules. Indeed, for our \(a\ priori\) parameter choice rule, we obtain a convergence rate of order \(\frac{p}{p+1}\) for \(0 < p < b\) and of order \(\frac{b}{b+1}\) for \(p \geq b\) (see Theorem 2). For our \(a\ posteriori\) parameter choice rule, we obtain a convergence rate of order \(\frac{p}{p+1}\) for \(0 < p < b - 1\) and of order \(\frac{b}{b+1}\) for \(p \geq b - 1\) (see Theorem 3). This order of convergence was proved to be optimal in Hao et al. [6], in which the authors also obtained error estimates of the same order as in this paper using the method of non-local boundary value problem. Our error estimates are confirmed using numerical tests for problems in one and two dimensions. To obtain high convergence rates, we choose \(A_\alpha\) in a different way compared to that of Gajewski and Zacharias [5] and the above works. More precisely, we consider \(A_\alpha := A(I + \alpha A^b)^{-1}\), where \(b \geq 1\) is an arbitrary real number, whereas Gajewski and Zacharias [5] and the above works considered \(A_\alpha := A(I + \alpha A)^{-1}\). The error estimates presented in Section 3 show that the convergence rate is high when \(b\) is a large number.

Compared to other methods for solving problem (1.1), our orders of error estimate are higher than those of [21, 23, 24, 25, 26, 27]. Indeed, as pointed out in Remarks 2 and 4, our orders of error estimates are larger than 2/3 for appropriate values of \(p\) and \(b\), whereas the order of the error estimate in [25, 27] is not greater than 1/2 while that in [23, 24, 26] is not greater than 2/3 for all \(p > 0\) for their \(a\ priori\) parameter choice rule and is not greater than 1/2 for their \(a\ posteriori\) parameter choice rule.

The rest of the paper is organized as follow: in Section 2 we recall some basic definitions and present simple inequalities which are needed for proving the main results in this paper. In Section 3 we describe our regularization method with the error estimates, the proofs of which will be given in Section 4. Numerical implementation of the proposed regularization scheme and numerical tests are presented in Section 5. Finally, conclusions are given in Section 6.
2 Auxiliary results

Denote by $E_{\gamma,\beta}(z)$ the Mittag-Leffler function \[12, 17\]:

$$E_{\gamma,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\gamma + \beta)}, z \in \mathbb{C}, \gamma > 0, \beta \in \mathbb{R}. \quad (2.1)$$

**Definition 1.** \[6\] For $u_0 \in \mathbb{H}$, a function $u(t) : [0, T] \to \mathbb{H}$ is called a solution to problem

$$\begin{cases}
\frac{\partial^\gamma u}{\partial t^\gamma} + Au = 0, & 0 < t < T, \\
u(0) = u_0,
\end{cases} \quad (2.2)$$

if $u(t) \in C^1((0, T), \mathbb{H}) \cap C([0, T], \mathbb{H})$, $u(t) \in D(A)$ for all $t \in (0, T)$, and (2.2) holds.

**Theorem 1.** \[6\] Problem (2.2) admits a unique solution, which can be represented in the form:

$$u(t) = \sum_{n=1}^{\infty} E_{\gamma,1}(-\lambda_n t^\gamma) \langle u_0, \phi_n \rangle \phi_n. \quad (2.3)$$

**Lemma 1.** (Young’s inequality) If $a, b$ are nonnegative numbers and $m, n$ are positive numbers such that $\frac{1}{m} + \frac{1}{n} = 1$, then $ab \leq \frac{a^m}{m} + \frac{b^n}{n}$.

**Lemma 2.** \[25\] For any $\lambda_n$ satisfying $\lambda_n \geq \lambda_1 > 0$, there exist positive constants $\overline{C}_1$ and $\overline{C}_2$ depending on $\gamma$, $T$, $\lambda_1$ such that

$$\frac{\overline{C}_1}{\lambda_n} \leq E_{\gamma,1}(-\lambda_n T^\gamma) \leq \frac{\overline{C}_2}{\lambda_n}.$$

**Lemma 3.** \[13\] There exist positive constants $\overline{C}_3$, $\overline{C}_4$ and $\overline{C}_5$ depending on $\gamma$ such that

- $a)$ \(\frac{\overline{C}_3}{\Gamma(1 - \gamma)} \cdot \frac{1}{1 - x} \leq E_{\gamma,1}(x) \leq \frac{\overline{C}_4}{\Gamma(1 - \gamma)} \cdot \frac{1}{1 - x}, \text{ for all } x \leq 0.\)

- $b)$ \(|E_{\gamma,0}(x)| \leq \frac{\overline{C}_5}{\Gamma(-\gamma)} \frac{1}{1 - x}, \text{ for all } x \leq 0.\)

**Lemma 4.** \[13\], p. 1779 Let $\gamma \in (0, 1)$ and $t > 0$. We have

$$\frac{d}{ds} E_{\gamma,1}(st^\gamma) = \frac{1}{st^\gamma} E_{\gamma,0}(st^\gamma), \text{ for all } s \in \mathbb{R}, \; s \neq 0. \quad (2.4)$$

**Definition 2.** Let $b \geq 1$. For every $v \in \mathbb{H}$ having the expansion $v = \sum_{n=1}^{\infty} \langle v, \phi_n \rangle \phi_n$, we define

- $a)$ \(B_\alpha v := \sum_{n=1}^{\infty} \left( \frac{1}{1 + \alpha \lambda_n^b} \right) \langle v, \phi_n \rangle \phi_n,\)
\( b) \ A_\alpha v := \sum_{n=1}^{\infty} \lambda_{an} \langle v, \phi_n \rangle \phi_n, \)

where

\[
\lambda_{an} := \frac{\lambda_n}{1 + \alpha \lambda_n^b}, \quad n = 1, 2, \ldots.
\] (2.5)

Remark 1. For \( A_\alpha \) and \( B_\alpha \) given by Definition 2, we have \( A_\alpha = AB_\alpha \).

3 Error estimates for the regularized solution

In this section, we state error estimates for the regularization of problem (1.1) by the Sobolev equation (1.3). We propose a priori and a posteriori methods for choosing the regularization parameter \( \alpha \) which yield error estimates of Hölder type. The theoretical results of this paper are stated in Theorem 2 and Theorem 3 below.

3.1 A priori parameter choice rule

Theorem 2. For \( b \geq 1 \), problem (1.3) is well-posed. Moreover, if the solution \( u(t) \) of problem (1.1) satisfies

\[
\|u(0)\|_p \leq E, \quad p > 0, \quad E > \varepsilon, \quad (3.1)
\]

then the following statements hold:

(i) If \( 0 < p < b \), then with \( \alpha = \left( \frac{\varepsilon}{E} \right)^{\frac{b}{p+1}} \), there exists a constant \( C_1 \) such that

\[
\|u(0) - v_\alpha(0)\| \leq C_1 \varepsilon^{\frac{p}{p+1}} E^{\frac{1}{p+1}}.
\]

(ii) If \( p \geq b \), then with \( \alpha = \left( \frac{\varepsilon}{E} \right)^{\frac{b}{p+1}} \), there exists a constant \( C_2 \) such that

\[
\|u(0) - v_\alpha(0)\| \leq C_2 \varepsilon^{\frac{b}{b+1}} E^{\frac{1}{b+1}}.
\]

Remark 2. We note that \( \frac{p}{p+1} > \frac{2}{3} \) when \( p > 2 \) and \( \frac{b}{b+1} > \frac{2}{3} \) when \( b > 2 \). Therefore, the order of our error estimates is greater than \( \frac{2}{3} \) when \( 2 < p < b \) or \( p \geq b > 2 \).

3.2 A posteriori parameter choice rule

Theorem 3. Let \( b > 1 \) and \( B_\alpha \) be given by Definition 2. Assume that \( 0 < \varepsilon < \|f\| \) and \( \tau > 1 \) is a constant satisfying \( 0 < \tau \varepsilon \leq \|f\| \). Then, there exists a unique number \( \alpha_\varepsilon > 0 \) such that

\[
\|B_{\alpha_\varepsilon} f - f\| = \tau \varepsilon.
\] (3.2)
If the solution \( u(t) \) of problem (1.1) satisfies (3.1), then there exist constants \( C_3 \) and \( C_4 \) such that the following estimates hold:

i) If \( p < b - 1 \), then

\[
\| u(0) - v_{\alpha_0}(0) \| \leq C_3 \varepsilon^{\frac{p}{p+1}} E_\frac{1}{p+1}. 
\]

ii) If \( p \geq b - 1 \), then

\[
\| u(0) - v_{\alpha_0}(0) \| \leq C_4 \varepsilon^{\frac{p}{p+1}} E_\frac{1}{p+1} + \varepsilon^{\frac{b-1}{b}} E_\frac{1}{b}.
\]

Remark 3. We note that in case (i) of both Theorem 2 and Theorem 3, the convergence rate \( E_\frac{1}{p+1} \delta_\frac{p}{p+1} \) is of optimal order as pointed out in [6].

Remark 4. We note that \( \frac{p}{p+1} > \frac{2}{3} \) when \( p > 2 \) and \( \frac{b-1}{b} > \frac{2}{3} \) when \( b > 3 \). Therefore the order of our error estimates in Theorem 3 is greater than \( \frac{2}{3} \) when \( 2 < p < b - 1 \) or \( p \geq b > 3 \).

Remark 5. The authors in [2, 3, 8, 10, 11, 15, 16, 18, 20] applied the Sobolev equations method to regularizing integer order parabolic equations, i.e., with \( \gamma = 1 \). Those results were obtained only for the \textit{a priori} parameter choice rule and with \( b = 1 \). Here we not only obtain the optimal convergence rates for the backward time-fractional equation (1.1) for both \textit{a priori} and \textit{a posteriori} parameter choice rules, but also with an arbitrary positive constant \( b \geq 1 \).

4 Proofs of the main results

4.1 Proof of Theorem 2

First, we present some auxiliary results.

Lemma 5. Problem (1.3) admits a unique solution

\[
v_{\alpha}(t) = \sum_{n=1}^{\infty} \frac{E_{\gamma,1} (-\lambda_{an} t^\gamma) \langle f, \phi_n \rangle \phi_n}{E_{\gamma,1} (-\lambda_{an} T^\gamma)}, \quad \forall t \in [0, T],
\]

where \( \lambda_{an} = \frac{\lambda_n}{1 + \alpha \lambda_n^b} \). Furthermore, there exists a constant \( C_6 \) such that

\[
\| v_{\alpha}(t) \| \leq C_6 (1 + \alpha^{-\frac{1}{2}}) \| f \|, \quad \forall t \in [0, T].
\]

Proof. Formula (4.1) is obtained by direct calculations. From inequalities \( 0 \leq E_{\gamma,1} (-\lambda_{an} T^\gamma) \leq 1 \), it follows that

\[
\| v_{\alpha}(t) \|^2 \leq \sum_{n=1}^{\infty} \frac{\langle f, \phi_n \rangle^2}{[E_{\gamma,1} (-\lambda_{an} T^\gamma)]^2}.
\]
From Lemma 3 we obtain
\[ \|v_\alpha(t)\|^2 \leq \sum_{n=1}^{\infty} \left( \frac{\langle f, \phi_n \rangle}{C^{\alpha_n T^n}} \right)^2 \leq \left( \frac{\Gamma(1-\gamma)}{C} \right)^2 \sum_{n=1}^{\infty} (1 + \lambda_{\alpha_n T^n})^2 \langle f, \phi_n \rangle^2. \] (4.2)

For \( b > 1 \), we have
\[ 1 + \alpha \lambda_n^b \geq \frac{b - 1}{b} \cdot \frac{b}{b - 1} + \frac{1}{b} \left( \alpha^{1/b} \lambda_n \right)^b \geq \alpha^{1/b} \lambda_n, \]
or
\[ 1 + \alpha \lambda_n^b \geq \alpha^{1/b} \lambda_n \text{ for all } b \geq 1. \]

Therefore
\[ \lambda_{\alpha_n} := \frac{\lambda_n}{1 + \alpha \lambda_n^b} \leq \alpha^{-1/b}. \]

It follows from this inequality and (4.2) that there exists a constant \( \overline{C}_7 > 0 \) such that
\[ \|v_\alpha(t)\|^2 \leq \overline{C}_7 (1 + \alpha^{-1/b})^2 \|f\|^2. \]

The lemma is proved.

In the following, we denote by \( w_\alpha(t) \) the solution of problem
\[ \begin{cases} \frac{\partial^\gamma w_\alpha}{\partial t^\gamma} + A_\alpha w_\alpha = 0, & 0 < t < T, \\ w_\alpha(T) = u(T), \end{cases} \] (4.3)

**Lemma 6.** If \( w_\alpha(t) \) is the solution of problem (4.3) and \( v_\alpha(t) \) is the solution of problem (1.3) then
\[ \|v_\alpha(0) - w_\alpha(0)\| \leq \overline{C}_6 (1 + \alpha^{-1/b}) \varepsilon. \]

**Proof.** We see that \( w_\alpha(t) - v_\alpha(t) \) solves problem (1.3) with \( f \) being replaced by \( u(T) - f \).

Using Lemma 5 we have
\[ \|v_\alpha(0) - w_\alpha(0)\| \leq \overline{C}_6 (1 + \alpha^{-1/b}) \|u(T) - f\| \leq \overline{C}_6 (1 + \alpha^{-1/b}) \varepsilon. \]

The lemma is proved.

**Lemma 7.** If \( \|u(0)\|_p \leq E \) for some positive constants \( p, E > 0 \), then there exist constants \( \overline{C}_8 \) and \( \overline{C}_9 \) such that
\[ \|u(0) - w_\alpha(0)\|^2 \leq \begin{cases} \overline{C}_8 \alpha^{2p/b} E^2 & \text{if } p < b, \\ \overline{C}_9 (\alpha^2 E^2 + \alpha^{2p/b} E^2) & \text{if } p \geq b. \end{cases} \]
Proof. We have

\[
\|u(0) - w_\alpha(0)\|^2 = \sum_{n=1}^{\infty} \langle u(0) - w_\alpha(0), \phi_n \rangle^2
\]

\[
= \sum_{n=1}^{\infty} \left( \langle u(0), \phi_n \rangle - \frac{\langle u(T), \phi_n \rangle}{E_{\gamma,1}(-\lambda_{\alpha n} T^\gamma)} \right)^2
\]

\[
= \sum_{n=1}^{\infty} \left( \langle u(0), \phi_n \rangle - \frac{E_{\gamma,1}(-\lambda_{\alpha n} T^\gamma) \langle u(0), \phi_n \rangle}{E_{\gamma,1}(-\lambda_{\alpha n} T^\gamma)} \right)^2
\]

\[
= \sum_{n=1}^{\infty} \langle u(0), \phi_n \rangle^2 \left( 1 - \frac{E_{\gamma,1}(-\lambda_{\alpha n} T^\gamma)}{E_{\gamma,1}(-\lambda_{\alpha n} T^\gamma)} \right)^2
\]

\[
= \sum_{n=1}^{n_1-1} \lambda_n^{2p} \langle u(0), \phi_n \rangle^2 \left( \frac{E_{\gamma,1}(-\lambda_{\alpha n} T^\gamma) - E_{\gamma,1}(-\lambda_{n} T^\gamma)}{E_{\gamma,1}(-\lambda_{\alpha n} T^\gamma)} \right)^2 \lambda_n^{-2p}
\]

\[
+ \sum_{n=n_1}^{\infty} \lambda_n^{2p} \langle u(0), \phi_n \rangle^2 \left( 1 - \frac{E_{\gamma,1}(-\lambda_{\alpha n} T^\gamma)}{E_{\gamma,1}(-\lambda_{\alpha n} T^\gamma)} \right)^2 \lambda_n^{-2p},
\]  

where \( n_1 = \min\{n : \lambda_n \geq \alpha^{-1/b}\} \). Let \( h(s) = E_{\gamma,1}(s T^\gamma), s < 0 \), then \( h(s) \) is an increasing function. Since \(-\lambda_n \leq -\lambda_{\alpha n}\), we have \( E_{\gamma,1}(-\lambda_{\alpha n} T^\gamma) \leq E_{\gamma,1}(-\lambda_n T^\gamma) \).

Therefore

\[
\left( 1 - \frac{E_{\gamma,1}(-\lambda_{\alpha n} T^\gamma)}{E_{\gamma,1}(-\lambda_{\alpha n} T^\gamma)} \right)^2 \leq 1.
\]  

Hence,

\[
\sum_{n=n_1}^{\infty} \lambda_n^{2p} \langle u(0), \phi_n \rangle^2 \left( 1 - \frac{E_{\gamma,1}(-\lambda_{\alpha n} T^\gamma)}{E_{\gamma,1}(-\lambda_{\alpha n} T^\gamma)} \right)^2 \lambda_n^{-2p} \leq \sum_{n=n_1}^{\infty} \lambda_n^{2p} \langle u(0), \phi_n \rangle^2 \lambda_n^{-2p}
\]

\[
\leq \alpha^{2p/b} \sum_{n=n_1}^{\infty} \lambda_n^{2p} \langle u(0), \phi_n \rangle^2 
\]

\[
\leq \alpha^{2p/b} E^2.
\]

By Lemma 4, we have \( \frac{d}{ds} h(s) = \frac{1}{s T^\gamma} E_{\gamma,0}(s T^\gamma) \). Therefore, there exist constants \( \xi_n \in (-\lambda_n, -\lambda_{\alpha n}), n \geq 1 \), such that

\[
E_{\gamma,1}(-\lambda_{\alpha n} T^\gamma) - E_{\gamma,1}(-\lambda_n T^\gamma) = \frac{1}{\xi_n T^\gamma} E_{\gamma,0} (\xi_n T^\gamma) (\lambda_{\alpha n} - \lambda_n)
\]

\[
= -\frac{1}{\xi_n} E_{\gamma,0} (\xi_n T^\gamma) \frac{\alpha \lambda_n^{b+1}}{1 + \alpha \lambda_n^b}.
\]

From Lemma 3 there exists a constant \( C_{10} > 0 \) such that

\[
|E_{\gamma,1}(-\lambda_{\alpha n} T^\gamma) - E_{\gamma,1}(-\lambda_n T^\gamma)| \leq \frac{-C_{10}}{\xi_n (1 - \xi_n T^\gamma)} \frac{\alpha \lambda_n^{b+1}}{1 + \alpha \lambda_n^b}.
\]
On the other hand, from $\lambda_n \leq \alpha^{-1/b}$, $n \leq n_1 - 1$, it follows that $\lambda_{an} \geq \frac{\lambda_n}{2} > \frac{\lambda_1}{2}$. From Lemma 2 it follows that there exists a constant $\overline{C}_{11} > 0$ such that

$$\sum_{n=1}^{n_1-1} \lambda_n^{2p} \langle u(0), \phi_n \rangle^2 \left( \frac{E_{\gamma,1}(-\lambda_{an} T^n) - E_{\gamma,1}(-\lambda_n T^n)}{E_{\gamma,1}(-\lambda_{an} T^n)} \right)^2 \lambda_n^{-2p}$$

$$\leq \overline{C}_{11} \sum_{n=1}^{n_1-1} \lambda_n^{2p} \langle u(0), \phi_n \rangle^2 \left( \frac{1}{\xi_n^{2T^n} + \alpha \lambda_n^b} \right)^2 \lambda_n^{-2p}$$

$$= \overline{C}_{11} \sum_{n=1}^{n_1-1} \lambda_n^{2p} \langle u(0), \phi_n \rangle^2 \left( \frac{1}{\xi_n^{2T^n} + \alpha \lambda_n^b \lambda_n^2} \right)^2 .$$

Note that $\xi_n \in (-\lambda_n, -\lambda_{an})$. It follows that $\xi_n^2 \geq \lambda_n^2$. Therefore

$$\sum_{n=1}^{n_1-1} \lambda_n^{2p} \langle u(0), \phi_n \rangle^2 \left( \frac{E_{\gamma,1}(-\lambda_{an} T^n) - E_{\gamma,1}(-\lambda_n T^n)}{E_{\gamma,1}(-\lambda_{an} T^n)} \right)^2 \lambda_n^{-2p}$$

$$\leq \overline{C}_{11} \sum_{n=1}^{n_1-1} \lambda_n^{2p} \langle u(0), \phi_n \rangle^2 \left( \alpha \lambda_n^{b-p} \right)^2 .$$

(4.8)

If $p < b$ then $\lambda_n^{b-p} \leq \alpha^{(p-b)/b}$, for all $n \leq n_1 - 1$. We obtain

$$\sum_{n=1}^{n_1-1} \lambda_n^{2p} \langle u(0), \phi_n \rangle^2 \left( \frac{E_{\gamma,1}(-\lambda_{an} T^n) - E_{\gamma,1}(-\lambda_n T^n)}{E_{\gamma,1}(-\lambda_{an} T^n)} \right)^2 \lambda_n^{-2p}$$

$$\leq \overline{C}_{11} \alpha^{2p/b} \sum_{n=1}^{n_1-1} \lambda_n^{2p} \langle u(0), \phi_n \rangle^2 \leq \overline{C}_{11} \alpha^{2p/b} E^2 .$$

(4.9)

If $p \geq b$ then $\lambda_n^{b-p} \leq \lambda_1^{b-p}$, for all $n \leq n_1 - 1$. There exists a constant $\overline{C}_{12} > 0$ such that

$$\sum_{n=1}^{n_1-1} \lambda_n^{2p} \langle u(0), \phi_n \rangle^2 \left( \frac{E_{\gamma,1}(-\lambda_{an} T^n) - E_{\gamma,1}(-\lambda_n T^n)}{E_{\gamma,1}(-\lambda_{an} T^n)} \right)^2 \lambda_n^{-2p}$$

$$\leq \overline{C}_{12} \alpha^2 \sum_{n=1}^{n_1-1} \lambda_n^{2p} \langle u(0), \phi_n \rangle^2 \leq \overline{C}_{12} \alpha^2 E^2 .$$

(4.10)

From (4.4), (4.6), (4.9) and (4.10) we obtain the second estimate of the lemma. The proof is complete.
Now we are in a position to prove Theorem 2. We note that the well-posedness of problem (1.3) is implied from Lemma 5.

**Proof of part (i) of Theorem 2.**

If $p < b$, from Lemma 6 and Lemma 7 there exists a constant $\bar{C}_{13} > 0$ such that

$$
\| u(0) - v_\alpha(0) \| \leq \| u(0) - w_\alpha(0) \| + \| v_\alpha(0) - w_\alpha(0) \|
\leq \bar{C}_{13} \left( \alpha^{p/b} E + \alpha^{1/b} \varepsilon + \varepsilon \right).
$$

Choosing $\alpha = \left( \frac{\varepsilon}{E} \right)^{\frac{b}{p+b}}$, we have

$$
\| u(0) - v_\alpha(0) \| \leq \bar{C}_{13} \left( 2\varepsilon^{\frac{p}{p+b}} E^{\frac{1}{p+b}} + \varepsilon \right).
$$

For $E > \varepsilon$, we have $\varepsilon \leq \varepsilon^{\frac{p}{p+b}} E^{\frac{1}{p+b}}$. Hence, part (i) of Theorem 2 is proved.

**Proof of part (ii) of Theorem 2.**

If $p \geq b$, from Lemma 6 and Lemma 7 there exists a constant $\bar{C}_{14} > 0$ such that

$$
\| u(0) - v_\alpha(0) \| \leq \| u(0) - w_\alpha(0) \| + \| v_\alpha(0) - w_\alpha(0) \|
\leq \bar{C}_{14} (\alpha E + \alpha^{p/b} E + \alpha^{-1/b} \varepsilon + \varepsilon).
$$

Choosing $\alpha = \left( \frac{\varepsilon}{E} \right)^{\frac{b}{p+b}}$, we have

$$
\| u(0) - v_\alpha(0) \| \leq \bar{C}_{14} (2\varepsilon^{\frac{p}{p+b}} E^{\frac{1}{p+b}} + \varepsilon^{\frac{p}{p+b}} E^{1-\frac{p}{p+b}} + \varepsilon).
$$

For $E > \varepsilon$ and $p \geq b$, we have $\varepsilon \leq \varepsilon^{\frac{b}{b+p}} E^{\frac{1}{b+p}}$ and $\varepsilon^{\frac{p}{p+b}} E^{1-\frac{p}{p+b}} \leq \varepsilon^{\frac{b}{b+p}} E^{\frac{1}{b+p}}$. Hence, part (ii) of Theorem 2 is proved. Therefore, the proof of Theorem 2 is complete.

**4.2 Proof of Theorem 3**

First, we prove the following lemma.

**Lemma 8.** Set $\rho(\alpha) := \| B_\alpha f - f \|$ and suppose that $f \neq 0$. Then

a) $\rho$ is a continuous function,

b) $\lim_{\alpha \to 0^+} \rho(\alpha) = 0$,

c) $\lim_{\alpha \to \infty} \rho(\alpha) = \| f \|$,

d) $\rho$ is a strictly increasing function.

**Proof.** a) From Definition 2 we have

$$
\rho^2(\alpha) = \| B_\alpha f - f \|^2 = \sum_{n=1}^{\infty} \left( \frac{\langle f, \phi_n \rangle}{1 + \alpha \lambda_n^b} - \langle f, \phi_n \rangle \right)^2 = \sum_{n=1}^{\infty} \frac{\alpha^2 \lambda_n^{2b} \langle f, \phi_n \rangle^2}{(1 + \alpha \lambda_n^b)^2}.
$$
For $\alpha_0 > 0$, we have
\[
|\rho^2(\alpha) - \rho^2(\alpha_0)| = \left| \sum_{n=1}^{\infty} \langle f, \phi_n \rangle^2 \left( \frac{\alpha}{1 + \alpha \lambda_n^b} + \frac{\alpha_0}{1 + \alpha_0 \lambda_n^b} \right) \frac{(\alpha - \alpha_0) \lambda_n^{2b}}{(1 + \alpha \lambda_n^b)(1 + \alpha_0 \lambda_n^b)} \right|
\leq \frac{|\alpha - \alpha_0| (\alpha + \alpha_0)}{\alpha \alpha_0} \sum_{n=1}^{\infty} \langle f, \phi_n \rangle^2 = \frac{|\alpha - \alpha_0| (\alpha + \alpha_0)}{\alpha \alpha_0} \|f\|^2.
\]

Therefore, $\rho$ is a continuous function.

b) Let $\delta$ be an arbitrary positive number. Since $\|f\|^2 = \sum_{n=1}^{\infty} \langle f, \phi_n \rangle^2$, there exists a positive integer $n_\delta$ such that $\sum_{n=n_\delta+1}^{\infty} \langle f, \phi_n \rangle^2 < \frac{\delta^2}{2}$. For $0 < \alpha < \frac{\delta}{\sqrt{2\lambda_n^b}}$, we have
\[
\rho^2(\alpha) \leq \sum_{n=1}^{n_\delta} \frac{\alpha^2 \lambda_n^{2b} \langle f, \phi_n \rangle^2}{(1 + \alpha \lambda_n^b)^2} + \sum_{n=n_\delta+1}^{\infty} \langle f, \phi_n \rangle^2
\leq \alpha^2 \lambda_n^{2b} \sum_{n=1}^{n_\delta} \langle f, \phi_n \rangle^2 + \frac{\delta^2}{2}
\leq \alpha^2 \lambda_n^{2b} \|f\|^2 + \frac{\delta^2}{2} \leq \delta^2.
\]
This implies that $\lim_{\alpha \to 0^+} \rho(\alpha) = 0$.

c) From (4.11) we have $\rho(\alpha) < \|f\|$. Since $\lambda_n \geq \lambda_1$ for all $n \geq 1$, (4.11) also implies that
\[
\rho^2(\alpha) \geq \sum_{n=1}^{\infty} \frac{\alpha^2 \lambda_n^{2b} \langle f, \phi_n \rangle^2}{(1 + \alpha \lambda_n^b)^2}.
\]
Therefore, $\|f\| \geq \rho(\alpha) \geq \frac{\alpha^2 \lambda_n^{2b} \|f\|}{(1 + \alpha \lambda_n^b)^2}$. This implies that $\lim_{\alpha \to +\infty} \rho(\alpha) = \|f\|$.

d) Since function $\frac{\alpha \lambda_n^b}{1 + \alpha \lambda_n^b}$ is strictly increasing with respect to $\alpha > 0$, it follows from (4.11) that $\rho(\alpha)$ is strictly increasing if there exists a positive integer $n$ such that $\langle f, \phi_n \rangle^2 > 0$. This condition is true since $\|f\| > 0$. The lemma is proved.

Now we are in a position to prove Theorem 3.

Proof of part (i) of Theorem 3.

It follows from Lemma 8 that there exists a unique number $\alpha_\varepsilon > 0$ satisfying (3.2). From (4.4), (4.5) and (4.8), there exists a constant $C_{15} > 0$ such that
\[
\|u(0) - w_{\alpha_\varepsilon}(0)\|^2 \leq C_{15} \left( \sum_{n=1}^{n_{2-1}} \lambda_n^{2b} \langle u(0), \phi_n \rangle^2 (\alpha_\varepsilon \lambda_n^{b-2})^2 + \sum_{n=n_{2-1}}^{\infty} \langle u(0), \phi_n \rangle^2 \right).
\]
\[
\leq C_{15} \left( \sum_{n=1}^{n_2-1} \langle u(0), \phi_n \rangle^2 \alpha_\varepsilon^2 \lambda_n^{2b} + \sum_{n=n_2}^{\infty} \langle u(0), \phi_n \rangle^2 \right),
\]

where \( n_2 = \min\{n : \lambda_n \geq \alpha_\varepsilon^{-1/b} \} \). For \( n \leq n_2 - 1 \), we have \( \lambda_n < \alpha_\varepsilon^{-1/b} \). Using the Hölder inequality, we obtain

\[
\sum_{n=1}^{n_2-1} \langle u(0), \phi_n \rangle^2 \alpha_\varepsilon^2 \lambda_n^{2b} \leq \sum_{n=1}^{n_2-1} \langle u(0), \phi_n \rangle^2 \alpha_\varepsilon^2 \frac{2p}{2p+1} \lambda_n^{2b} \left( \frac{2p}{2p+1} \alpha_\varepsilon \lambda_n^{-1} \right)
\]

\[
= \sum_{n=1}^{n_2-1} \left( | \langle u(0), \phi_n \rangle |^2 \right)^{\frac{2p}{2p+1}} \left( | \langle u(0), \phi_n \rangle | \right)^{\frac{2p}{2p+1}} \left( \sum_{n=1}^{n_2-1} \langle u(0), \phi_n \rangle^2 \alpha_\varepsilon^2 \lambda_n^{2b} \right)^{\frac{1}{2p+1}}
\]

\[
\leq \left( \sum_{n=1}^{n_2-1} \langle u(0), \phi_n \rangle^2 \lambda_n^{2b} \right)^{\frac{1}{p+1}} \left( \sum_{n=1}^{n_2-1} \langle u(0), \phi_n \rangle^2 \lambda_n^{2b} \alpha_\varepsilon^2 \right)^{\frac{p}{p+1}}
\]

\[
\leq E^{\frac{2}{p+1}} \left( \sum_{n=1}^{n_2-1} \langle u(0), \phi_n \rangle^2 \alpha_\varepsilon^2 \lambda_n^{2b} \right)^{\frac{p}{p+1}}.
\]

On the other hand, since \( \lambda_n \geq \alpha_\varepsilon^{-1/b} \) for \( n \geq n_2 \), we have

\[
\sum_{n=n_2}^{\infty} \langle u(0), \phi_n \rangle^2 = \sum_{n=n_2}^{\infty} \left( | \langle u(0), \phi_n \rangle |^2 \right)^{\frac{2p}{2p+1}} \left( | \langle u(0), \phi_n \rangle | \right)^{\frac{2p}{2p+1}} \left( \sum_{n=n_2}^{\infty} \langle u(0), \phi_n \rangle^2 \lambda_n^{2b} \right)^{\frac{1}{p+1}}
\]

\[
\leq \left( \sum_{n=n_2}^{\infty} \langle u(0), \phi_n \rangle^2 \lambda_n^{2b} \right)^{\frac{1}{p+1}} \left( \sum_{n=n_2}^{\infty} \langle u(0), \phi_n \rangle^2 \lambda_n^{2b} \right)^{\frac{p}{p+1}}
\]

\[
\leq E^{\frac{2}{p+1}} \left( \sum_{n=n_2}^{\infty} \langle u(0), \phi_n \rangle^2 \lambda_n^{2b} \right)^{\frac{p}{p+1}}.
\]

Using Lemma 2, we have

\[
\tau \varepsilon = \| B_{\alpha_\varepsilon} f - f \| = \left\| \sum_{n=1}^{\infty} \alpha_\varepsilon \lambda_n^b \langle f, \phi_n \rangle \phi_n \right\|
\]

\[
= \left\| \sum_{n=1}^{\infty} \alpha_\varepsilon \lambda_n^b \langle u(T), \phi_n \rangle \phi_n \right\| - \left\| \sum_{n=1}^{\infty} \alpha_\varepsilon \lambda_n^b \langle u(T) - f, \phi_n \rangle \phi_n \right\|
\]

\[
\geq \sum_{n=1}^{\infty} \alpha_\varepsilon \lambda_n^b \left( -\lambda_n T \gamma \right) \langle u(0), \phi_n \rangle \phi_n - \left\| \sum_{n=1}^{\infty} \langle u(T) - f, \phi_n \rangle \phi_n \right\|
\]

\[
\geq \sum_{n=1}^{\infty} \alpha_\varepsilon \lambda_n^b \phi_n - \varepsilon.
\]

Using \( \lambda_n < \alpha_\varepsilon^{-1/b} \) for all \( n \leq n_2 - 1 \) again, we obtain

\[
(\tau + 1) \varepsilon \geq C_1 \left\| \sum_{n=1}^{n_2-1} \alpha_\varepsilon \lambda_n^{b-1} \langle u(0), \phi_n \rangle \phi_n \right\|
\]

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Similarly, for \( n \geq n_2 \), we have

\[
(\tau + 1)\varepsilon \geq \overline{C}_1 \left\| \sum_{n=n_2}^{\infty} \frac{\alpha \varepsilon \lambda_n^{b-1} \langle u(0), \phi_n \rangle}{1 + \alpha \varepsilon \lambda_n^b} \right\| \\
= \overline{C}_1 \left\| \sum_{n=n_2}^{\infty} \frac{\lambda_n^{-1} \alpha \varepsilon \lambda_n^b \langle u(0), \phi_n \rangle}{1 + \alpha \varepsilon \lambda_n^b} \right\| \\
\geq \frac{\overline{C}_1}{2} \left\| \sum_{n=n_2}^{\infty} \lambda_n^{-1} \langle u(0), \phi_n \rangle \phi_n \right\|
\]  

(4.16)

From (4.12)–(4.16), there exists a constant \( \overline{C}_{16} > 0 \) such that

\[
\| u(0) - w_{\alpha \varepsilon}(0) \|^2 \leq \overline{C}_{16} E^{2 \frac{1}{p+1}} \varepsilon^{2 \frac{p}{p+1}}.
\]  

(4.17)

Hence, there exists a constant \( \overline{C}_{17} > 0 \) such that

\[
\| u(0) - v_{\alpha \varepsilon}(0) \| \leq \| u(0) - w_{\alpha \varepsilon}(0) \| + \| v_{\alpha \varepsilon}(0) - w_{\alpha \varepsilon}(0) \| \\
\leq \overline{C}_{17} \left( E^{2 \frac{1}{p+1}} \varepsilon^{\frac{p}{p+1}} + \alpha^{-1/b} \varepsilon + \varepsilon \right).
\]  

(4.18)

It follows from Lemma 2 that

\[
\tau \varepsilon = \| B_{\alpha \varepsilon} f - f \| = \left\| \sum_{n=1}^{\infty} \frac{\alpha \varepsilon \lambda_n^b \langle f, \phi_n \rangle \phi_n}{1 + \alpha \varepsilon \lambda_n^b} \right\| \\
= \left\| \sum_{n=1}^{\infty} \frac{\alpha \varepsilon \lambda_n^b \langle u(T), \phi_n \rangle \phi_n}{1 + \alpha \varepsilon \lambda_n^b} - \sum_{n=1}^{\infty} \frac{\alpha \varepsilon \lambda_n^b \langle u(T) - f, \phi_n \rangle \phi_n}{1 + \alpha \varepsilon \lambda_n^b} \right\| \\
\leq \left\| \sum_{n=1}^{\infty} \frac{\alpha \varepsilon \lambda_n^b E_{\gamma,1} (-\lambda_n T^\gamma) \langle u(0), \phi_n \rangle \phi_n}{1 + \alpha \varepsilon \lambda_n^b} \right\| + \left\| \sum_{n=1}^{\infty} \langle u(T) - f, \phi_n \rangle \phi_n \right\| \\
\leq \overline{C}_2 \left\| \sum_{n=1}^{\infty} \frac{\alpha \varepsilon \lambda_n^{b-1} \langle u(0), \phi_n \rangle \phi_n}{1 + \alpha \varepsilon \lambda_n^b} \right\| + \varepsilon.
\]  

(4.19)

If \( 0 < p < b - 1 \), using Lemma 1, we get

\[
\alpha \varepsilon \lambda_n^b + 1 \geq \frac{b - p - 1}{b} \left( \alpha \varepsilon \lambda_n^{b-p-1} \right)^{\frac{b}{b-p-1}} + \frac{p + 1}{b} \frac{b}{1+b}.
\]  

(4.20)

From (4.19) and (4.20), we have

\[
(\tau - 1)\varepsilon \leq \overline{C}_2 \alpha \varepsilon^{\frac{p+1}{p+1}} \left\| \sum_{n=1}^{\infty} \lambda_n^b \langle u(0), \phi_n \rangle \phi_n \right\| \leq \overline{C}_2 \alpha \varepsilon^{\frac{p+1}{p+1}} E.
\]  

(4.21)
Hence, from (4.18) and (4.21) and $\varepsilon \leq \varepsilon^{p+1} E^{p+1}$, we arrive at the conclusion of part (i) of Theorem 3.

**Proof of part (ii) of Theorem 3**

If $p \geq b - 1$, then from (4.19) we have

\[(\tau - 1)\varepsilon \leq C_2 \alpha \varepsilon \left\| \sum_{n=1}^{\infty} \lambda_n^{b-1} \langle u(0), \phi_n \rangle \phi_n \right\| \]
\[\leq C_2 \lambda_1^{b-1-p} \alpha \varepsilon \left\| \sum_{n=1}^{\infty} \lambda_n^p \langle u(0), \phi_n \rangle \phi_n \right\| \]
\[\leq C_2 \lambda_1^{b-1-p} \alpha \varepsilon E. \quad (4.22)\]

The conclusion of Part (ii) of Theorem 3 is followed from (4.18) and (4.22). The proof is complete.

## 5 Numerical implementation and examples

In this section we discuss the numerical implementation of the proposed regularization method for problem (1.1) and present some numerical tests for one and two dimensional equations. To focus our discussion on the performance of the regularization method, we chose the operator $A$ in such a way that its eigenvalues and eigenfunctions are explicitly available. This choice avoids possible misleading results due to error in the calculation of the eigenvalues and eigenfunctions.

In our numerical implementation, given the eigenvalues and eigenfunctions of operator $A$, the data $f = u(T)$ was generated by solving the forward problem (2.2) using expansion (2.3). The Mittag-Leffler functions $E_{\gamma,1}(-\lambda_n t^\gamma)$ were computed using an implementation in Matlab by Roberto Garrappa which is available for download at https://www.mathworks.com/matlabcentral/fileexchange/48154-the-mittag-leffler-function.

We approximated the infinite series in (2.3) by the following sum

\[u(t) \approx \sum_{n=1}^{N_p} E_{\gamma,1}(-\lambda_n t^\gamma) \langle u_0, \phi_n \rangle \phi_n, \quad t > 0. \quad (5.1)\]

To simulate noisy data, we added an additive uniformly distributed random noise of $L^2$-norm $\varepsilon$ to $u(T)$ to obtain data $f$. Given $f$ and the parameters $E$, $b$, $p$, $\epsilon$, the algorithm for calculating $u(t), 0 \leq t < T$, includes two steps:

- **Step 1**: Calculate the regularization parameter $\alpha$ using either the *a priori* or *a posteriori* choice rules described in Theorems 2 and 3. In the *a posteriori* choice rule, $\alpha$ is found as the unique solution of (3.2). Here, we approximate operator $B_{\alpha} f$ by a finite sum.
Step 2: Calculate the regularized solution \( v_\alpha \) using the following approximation of the explicit formula (4.1)

\[
v_{\alpha,\varepsilon}(t) := \sum_{n=1}^{N_i} \frac{E_{\gamma,1}(-\lambda_{\alpha} t^\gamma) \langle f, \phi_n \rangle \phi_n}{E_{\gamma,1}(-\lambda_{\alpha} T^\gamma)}, \quad \forall t \in [0, T]. \tag{5.2}
\]

In general, the number of basis functions \( N_i \) used in the inverse problem is not necessary equal to the number of basis functions \( N_p \) used in the approximation of the solution of the forward problem. In fact, we have observed through our numerical tests that \( N_i \) may have to be chosen smaller than \( N_p \) to avoid numerical instabilities in the solution of the backward equation. It was also mentioned in [6] that \( N_i \) is another regularization parameter that should be carefully chosen along with \( \alpha \).

**Example 1:** In this example, we consider the one-dimensional problem

\[
\begin{align*}
\frac{\partial^\gamma u(x,t)}{\partial t^\gamma} &= \frac{\partial^2 u(x,t)}{\partial x^2}, \quad x \in (0, \pi), \quad t \in (0, T), \\
 u(0,t) &= u(\pi,t) = 0, \quad t \in (0, T), \\
 u(x,0) &= u_0(x) := \sin(x) + \sin(2x) + \sin(3x), \quad x \in [0, \pi].
\end{align*}
\tag{5.3}
\]

Using the method of separation of variables, it is easy to obtain the eigenvalues \( \lambda_n = n^2 \) and orthonormal eigenfunctions \( \phi_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx), \quad n = 1, 2, \ldots \). Moreover,

\[
\langle u_0, \phi_n \rangle = \begin{cases} 
\sqrt{\frac{\pi}{2}} = \frac{1}{\|\phi_n\|}, & n \leq 2 \\
0, & n > 3.
\end{cases}
\]

Therefore, the solution \( u(x,t) \) of (1.3) is given by

\[
u(x,t) = E_{\gamma,1}(-t^\gamma) \sin(x) + E_{\gamma,1}(-4t^\gamma) \sin(2x) + E_{\gamma,1}(-9t^\gamma) \sin(3x) \]

That means, (5.1) becomes a true equality for \( N_p = 3 \). In the backward problem, if the data is exact, only 3 terms in (5.2) are needed for calculating \( v_\alpha \) exactly. However, since we expect the data to be noisy, we chose \( N_i = 5 \). This choice seems to be optimal for all the one-dimensional examples we discuss in this paper. This choice was also considered in [6] in similar tests.

First, we analyzed the effect of parameter \( p \) on the performance of the proposed algorithm. To this end, we chose \( \gamma = 1/2, \quad T = 1, \quad b = 4 \), and considered 6 noise levels of 0.1\%, 0.2\%, 0.4\%, 0.8\%, 1.6\%, and 3.2\%. The relative \( L^2 \)-norm error at time \( t \) is defined by

\[
e_r(\varepsilon, t) := \frac{\|u(\cdot,t) - v_{\alpha,\varepsilon}(\cdot,t)\|}{\|u(\cdot,t)\|} \times 100(\%).
\]

Table 1 shows the relative \( L^2 \)-norm error at \( t = 0 \) for three values of \( p \): \( p = 1 \), \( p = 2 \), and \( p = 3 \). We can observe that the error generally decreases as the measured
| Noise  | 0.1% | 0.2% | 0.4% | 0.8% | 1.6% | 3.2% |
|--------|------|------|------|------|------|------|
| $p = 1$ | 0.08 | 0.29 | 0.86 | 4.91 | 4.46 |
| $p = 2$ | 0.07 | 0.28 | 0.75 | 2.30 | 4.47 |
| $p = 3$ | 0.06 | 0.22 | 0.42 | 0.89 | 1.72 | 4.07 |

Table 1: Example 1: Relative $L^2$-norm error $e_r(\varepsilon, 0)$ (%) at $t = 0$ for $\gamma = 1/2$, $b = 4$ and $p = 1, 2, 3$. Measured data was given at $T = 1$. The regularization parameter $\alpha$ was chosen using the $a$ priori rule. For small error levels, the error decreases with respect to $p$, which is consistent with the error estimates in Theorem 2.

error decreases. Moreover, the larger $p$, the smaller the reconstruction error. This is consistent with the error estimates in Theorem 2. Since the behavior is similar for the $a$ posteriori parameter choice rule, we do not present it here.

In Figure 1 we compared the reconstruction results with the exact initial solution $(u(x, t))$ at $t = 0.1$ and $t = 0$ for $p = 3$ and $b = 4$ at two noise levels: 2% and 5%. The figure shows quite accurate reconstructions of $u(x, t)$ at $t = 0$ for both a priori and a posteriori parameter choice rules. In the latter case, the parameter $\tau$ in (3.2) was chosen as $\tau = 1.05$. The reconstructions at $t = 0.1$ are more accurate, as expected because problem (1.1) is well-posed for $t > 0$. Qualitatively, the accuracy of our results is comparable to those presented in Figure 2 of [6].

Next, we considered the case with $\gamma = 3/4$. We also chose $p$ and $b$ as in the previous case. The results for noise levels of 2% and 5% are depicted in Figure 2. We also obtained reasonably accurate results for both cases. However, the accuracy of the case $\gamma = 1/2$ is higher near $t = 0$ as shown in Figure 3 in which the $L^2$-norm error profile with respect to time is shown for noise level of 5%. Figure 3 also shows that the both parameter choice rules produced comparable results for $\gamma = 1/2$ but the $a$ priori parameter choice rules gives more accurate than the $a$ posteriori parameter choice rule for $\gamma = 3/4$.

**Example 2:** In this example we test the algorithm for another one dimensional problem which is described by the same equation as in (5.3) but the initial condition is given by

$$u(x, 0) = u_0(x) := \begin{cases} x, & 0 \leq x < \pi/2, \\ \pi - x, & \pi/2 \leq x \leq \pi. \end{cases}$$

It is easy to verify that

$$\langle u_0, \varphi_n \rangle = \sqrt{\frac{2}{\pi}} \int_0^\pi u_0(x) \sin(nx) dx = \frac{2\sqrt{2}}{n^2\sqrt{\pi}} \sin\left(\frac{n\pi}{2}\right).$$

We approximated the solution of the forward problem (2.2) by (5.1) with $N_p = 30$. In solving the backward equation, we again chose $N_i = 5$ as in Example 1. We also chose $p = 3$ and $b = 4$.

The solution values of the backward problem at $t = 0.1$ and $t = 0$ are shown in Figure 4 for noise levels of 2% and 5%. The figure shows that the reconstruction looks
Figure 1: Reconstruction result for Example 1 for $\gamma = 1/2$, $p = 3$, $b = 4$. Left column: noise = 2%; right column: noise = 5%.
(a) Data, noise = 2%

(b) Data, noise = 5%

(c) a priori choice, noise = 2%

(d) a priori choice, noise = 5%

(e) a posteriori choice, noise = 2%

(f) a posteriori choice, noise = 5%

Figure 2: Reconstruction result for Example 1 for $\gamma = 3/4$, $p = 3$, $b = 4$. Left column: noise = 2%; right column: noise = 5%.
Figure 3: Distribution of relative $L^2$-norm error over time for Example 1 with noise level = 5%. a) $\gamma = \frac{1}{2}$; b) $\gamma = \frac{3}{4}$. We can see that the error near $t = 0$ is larger for larger $\gamma$. That means, the larger $\gamma$, the more ill-posed the backward problem.

very accurate for both parameter choice rules at $t = 0.1$ and reasonably good at $t = 0$. These results are also comparable to the results obtained in Figure 6 of [6]. Note that due to the diffusion process, the nonsmooth initial condition is smoothed out rapidly in time, making the reconstruction of the nonsmooth behavior really challenging. We also observe that the *a priori* parameter choice gave more accurate reconstructions of the initial condition $u_0$, in particular, near the point $x = \pi/2$ at which the initial condition is not smooth. One possible reason for this due to the approximation of operator $B_\alpha$ in calculating the regularization parameter $\alpha$ in the *a posteriori* choice, which may not result in the optimal value of $\alpha$.

**Example 3:** As the last numerical example, we tested the algorithm against the following two-dimensional problem

$$
\begin{align*}
\frac{\partial^n u(x, y, t)}{\partial t^n} &= \frac{\partial^2 u(x, y, t)}{\partial x^2} + \frac{\partial^2 u(x, y, t)}{\partial y^2}, \quad x \in (0, \pi), \ y \in (0, \pi), \ t \in (0, T),
\end{align*}
$$

$$
\begin{align*}
u(0, y, t) &= u(\pi, y, t) = 0, \quad y \in (0, \pi), \ t \in (0, T),
u(x, 0, t) &= u(x, \pi, t) = 0, \quad x \in (0, \pi), \ t \in (0, T),
u(x, y, 0) &= u_0(x, y) := \sin(x) \sin(y) + \sin(2x) \sin(y), \quad x \in [0, \pi], \ y \in [0, \pi].
\end{align*}
$$

(5.4)

For this problem, the eigenvalues and eigenfunctions are given by

$$
\lambda_{nm} = n^2 + m^2, \quad \phi_{nm}(x, y) = \frac{2 \sin(nx) \sin(my)}{\pi}, \quad n, m = 1, 2, \ldots
$$

The inner products $\langle u_0, \phi_{nm} \rangle$ are given by

$$
\langle u_0, \phi_{nm} \rangle = \begin{cases} 
\pi/2, & n = 1, 2 \text{ and } m = 1, \\
0, & \text{otherwise}.
\end{cases}
$$

The implementation of the algorithm for this problem was similar to the one-dimensional case, except that we had to flatten the matrices of the eigenvalues and eigenfunctions to
Figure 4: Reconstruction result for Example 2 for $\gamma = 1/2$, $p = 3$, $b = 4$. Left column: noise = 2%; right column: noise = 5%.
obtain one-dimensional arrays and then sorted them in the nondecreasing order. Note that in this example, there are repeated eigenvalues, but the corresponding eigenfunctions are not the same.

The reconstructions of the initial condition are shown in Figures 5-6 for two noise levels of 2% and 10% with \( p = 3, b = 4 \) and \( \gamma = 1/2 \). In this example, we observed that \( N_i = 10 \) is a good truncation number in solving the backward problem. The figures indicate that the initial condition was reconstructed accurately taking into account the noise levels in the measured data.

6 Conclusions

We regularized the backward time-fractional parabolic equations using the Sobolev equations method. We obtained optimal error estimates for the regularized solutions for both \textit{a priori} and \textit{a posteriori} regularization parameter choice rules. The theoretical error estimates were supported by numerical tests for one- and two-dimensional equations.

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Figure 5: Reconstruction of the initial condition in Example 3 for $\gamma = 1/2$, $p = 3$, $b = 4$ for noise level = 2% using the \textit{a priori} parameter choice rule (c) and \textit{a posteriori} parameter choice rule (e). The maximum error is about 2.5% and the $L^2$-norm error is about 1.6%. Both parameter choice rules produced almost the same results.
Figure 6: Reconstruction of the initial condition in Example 3 for $\gamma = 1/2$, $p = 3$, $b = 4$ for noise level = 10% using the \textit{a priori} parameter choice rule (c) and \textit{a posteriori} parameter choice rule (e). The maximum error is about 10% and the $L^2$-norm error is about 5%. Both parameter choice rules produced almost the same results.
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