A toy model for representing regular black holes at the black string style

Milko Estrada\textsuperscript{1,}\textsuperscript{*}

\textsuperscript{1}Departamento de Física, Universidad de Antofagasta

(Dated: June 24, 2022)

Abstract

We provide a way of representing a four dimensional regular black hole geometry at the black string style. We enunciate a list of constrains in order to that the complete five dimensional geometry to be regular. Following these constraints were constructed both the four and the five dimensional geometries.

The assumptions used to solve the equations of motion suggest a relation between the $4D$ and the $5D$ Newton constants, which coincides with relations previously showed in the literature. Furthermore, the $(\mu, \nu)$ components of the five dimensional equations of motion adopt the form of the four dimensional equations of motion. Also, the five dimensional conservation equation adopts the form of the four dimensional conservation equation.

At the origin the topology of the five dimensional geometry corresponds to the product between the four dimensional de–Sitter space–time and $S^1$ with $z$ compact. This latter differs from the Kaluza-Klein black string, where, at the origin the topology corresponds to the product between the Schwarzschild singularity and $R(S^1)$ for $z$ non compact (compact). The topology of the complete five dimensional geometry corresponds to $S^2 \times S^1$. At the infinity of the radial coordinate the topology corresponds to the product between Minkowski and $S^1$.

At the induced four dimensional geometry we compute the first law of thermodynamics with the correct values of temperature and entropy.

\textsuperscript{*}milko.estrada@ua.cl
I. INTRODUCTION

In recent years, several aspects of theoretical high energy physics have provided theoretical indications about the possible existence of extra dimensions in the nature. Examples of this latter are the string theory and higher dimensional black holes.

One of the first models that admits the presence of extra dimensions corresponds to the Kaluza–Klein model. The geometrical representation of this model corresponds to the product between the four dimensional Minkowsky space–time $M_4$ and a compact extra dimension. The simplest scheme of this model is a cylinder, where $M_4$ is located in the horizontal direction and the compact extra dimension is given by the boundary of a $S^1$ sphere of radius $R$. So, the topology corresponds to $M_4 \times S^1$. See figures [1](extracted of reference [1]) and [2](extracted of reference [2]).

Other models that admit the presence of extra dimensions corresponds to black string models. These latter can be viewed as the simplest representation of black hole solutions.
into an extra dimension. In these models, slices of $(D - 1)$ black holes are stretched along a compact extra dimension, see figure 3. So, its topology corresponds to $S^{D-2} \times S^1$.

In the simplified scheme, for a five dimensional black string, the radial coordinate of these slices are represented in the horizontal direction and the extra dimensional coordinate corresponds to the boundary of a $S^1$ sphere. This simplified scheme corresponds to the cylinder of figure 4. However, it is worth to mention that, strictly speaking, in the horizontal direction should appears the four dimensional slices instead its only radial components.

In this connection, the simplest version of black strings corresponds to the following line
element:

\[ ds^2_{5D} = ds^2_{4D} + dz^2. \]  

The line element (1) corresponds to the Kaluza–Klein black string, where \( ds^2_{4D} \) corresponds to the four dimensional Schwarzschild black hole. For a non compact extra dimension, \( z \), the topology corresponds to \( S^2 \times R \). In the case, where the extra coordinate \( z \) is compact, the schematic representation corresponds to figure 5 (extracted from reference [3]). In the horizontal direction is represented the radial coordinate of a \( S^2 \) sphere, whereas the extra coordinate is represented in the boundary of a \( S^1 \) sphere. So, the topology corresponds to \( S^2 \times S^1 \). In this figure, the region where \( r < r_0 \), being \( r_0 \) the Schwarzschild radius, is known as cylindrical event horizon. So, as claims reference [4], the horizon completely wraps the compact extra dimension.

It is worth to mention that, in equation (1), the \( z \) coordinate is such that the quantity \( dz \) has unit of length for non compact \( z \), and for \( z \) compact the re–scaling \( dz \rightarrow Rdz \) is realized, such that in the right side \( R = 1 \) represents an unitary compactification radius (with unit of length) and \( dz \) now is an angular quantity. So, as indicate reference [3], the angular coordinate along the compact direction (as for example \( z \in [-L/2, L/2] \)) is defined such that \( z \sim z + L \). This latter does not imply that the geometry must have mirror symmetry (\( z \rightarrow -z \)). We can see a deep discussion about the background metric for black objects and black string in reference [3]. See also [5], where are shown black string solutions with negative cosmological constant, where also it is claimed that the direction \( z \) is periodic with period \( L \), and where the solution also does not have mirror symmetry (just like the Kaluza Klein black string).
It is worth to stress that, following figures 1, 2 and 3, similar slices of the four dimensional surface (in the simplified schemes 4 and 5, similar radial points $r$ of a $S^2$ sphere) are replicated in different locations of the compact extra dimension $z$. In that respect, reference [6] claims that this black string solution shows translational symmetry along the extra coordinate direction.

The horizon topology of the Kaluza Klein black string differs from the usual $S^3$ topology of the five dimensional Schwarzschild-Tangherlini solution [7]. On the other hand, the topology of the central singularity corresponds to the product between the Schwarzschild singularity and $R (S^1)$ for a non compact (compact) extra coordinate.

It is worth to mention that in reference [8] was showed a set of analytic solutions of General Relativity with negative cosmological constant, which describe black strings and black p-branes solutions. Black string have been widely studied in literature for different contexts. For example solutions with axionic scalar fields in Horndeski gravity [9], in $f(R)$ gravity [10], Lovelock gravity [11] and Chern Simon [12]. Other examples of recent applications in references [9, 13–18].

From the ideas mentioned above we can deduce that black string solutions with a central singularity are easily constructed through the representation of a singular black hole solution into an extra dimension. However, in the literature there are also non singular black holes solutions that could be introduced into an extra dimension, which clearly would not lead to a central singularity. These latter solutions are known as regular black holes. In these models, instead the formation of a central singularity, it is formed a de–Sitter core, i.e, the solution behaves as a de–Sitter space time near the origin. Thus, all the curvature invariants are regular everywhere. One very interesting and intriguing model is the Hayward metric [19]. Using this metric, the formation of the de–Sitter core is associated to quantum fluctuations, where the energy density is of order of Planck units near the origin. This latter model is called Planck stars [20, 21]. Although these models are theoretical, in reference [22], using radio astronomy data, is conjectured that Planck Stars represent a speculative but realistic possibility of testing quantum gravity effects.

Thus, motivated by the number of applications of black strings, where are introduced singular black holes into an extra dimension, is of physical interest the introduction of regular black hole solutions also into an extra dimension at the black string style, which has been explained above. In this work, we will study this latter mentioned problem. For this, we
will use the Hayward metric as toy model. We will compute the five dimensional geometry and analyze the way of how this solution modifies locally the four dimensional geometry and the information of the energy momentum tensor. Furthermore, we will discuss the first law of thermodynamics under some assumptions below discussed.

II. THE MODEL

The Einstein field equation in 5D are given by:

\[ G^M_N + \Lambda^{5D} \delta^M_N = 8\pi \tilde{G}_{5D} T^M_N, \]

where \( M, N = 0, 1, 2, 3, 5 \), and where \( \tilde{G}_{5D} \) represents to the five dimensional Newton constant. Using Planck units, the \( D \) dimensional Newton constant has units of \([ \tilde{G}_D ] = \ell_p^{D-2} \) [23], where \( \ell_p \) corresponds to the Planck length. So, the units of \([ \tilde{G}_{5D} ] = \ell_p^3 \).

As we are interested in studying spherically symmetric and static structures, then the corresponding line element reads as

\[ ds^2_{5D} = W(z) \cdot ds^2_{4D} + dz^2 \]

\[ = \exp(-2A(z)) \cdot g^{(4D)}_{\mu\nu} dx^\mu dx^\nu + dz^2 \]

\[ = \exp(-2A(z)) \cdot (-f(r) dt^2 + f(r)^{-1} dr^2 + r^2 d\Omega^2) + dz^2 \]

with \( \mu, \nu = 0, 1, 2, 3 \) and where \( W \) is a function that depends only on the extra coordinate \( z \). Thus, the 4D geometry, \( ds^2_{4D} \), is deformed by the function \( W \). Thus, although similar slices of the four dimensional surface are replicated along the extra dimension \( z \), the 5D geometry no longer has translational symmetry along the \( z \) coordinate, as occurs for the Kaluza Klein black string [6].

A. Constraints for regularity of the geometry

For the line element (5), the corresponding expression for the Ricci scalar is:

\[ R_{5D} = -20(\dot{A})^2 + 8\ddot{A} + W^{-1} \tilde{R}_{4D}, \]

where the dot indicates derivation with respect to the extra coordinate \( z \), and where \( \tilde{R}_{4D} \) corresponds to the Ricci scalar of the four dimensional geometry whose line
element is $d\bar{s}^2_{4D}$ in equation (3):

$$\bar{R}_{4D} = -f'' - \frac{4}{r} f' + \frac{2}{r^2} (1 - f).$$

(7)

where $'$ represents derivation with respect to the radial coordinate ($r$).

On the other hand, the Kretschmann scalar is given by:

$$K_{5D} = 40(\ddot{A})^4 + 16(\dddot{A})^2 - 32\ddot{A}(\dot{A})^2 - 4W^{-1}(\dot{A})^2\bar{R}_{4D} + W^{-2}\bar{K}_{4D},$$

(8)

where the four dimensional Kretschmann scalar $\bar{K}_{4D}$, which also corresponds to the 4D geometry $d\bar{s}^2_{4D}$, is given by:

$$\bar{K}_{4D} = (f'')^2 + \frac{4}{r^2} (f')^2 + \frac{4}{r^4} (1 - f)^2.$$

(9)

Thus, in order to describe a regular geometry, i.e both invariants $R_{5D}$ and $K_{5D}$ must be regular, we will impose the following constraints:

1. The four dimensional geometry must be regular, therefore the four dimensional invariants, $\bar{R}_{4D}$ and $\bar{K}_{4D}$ must be regular.

2. The function $A(z)$ must be continuous in its first and second derivatives.

3. The function $A(z)$ must be such that the function $W(A(z))$ can not be zero.

**B. A form for the energy momentum tensor**

The energy–momentum tensor is taken to be

$$T^M_N = \text{diag}( - \rho(r,z), p_r(r,z), p_\theta(r,z), p_\phi(r,z), p_z(r,z)),$$

(10)

where $\rho$ denotes the energy density while, $p_r$, $p_\theta$, $p_\phi$ and $p_z$ are representing the pressure components. Examples in 5D, where the energy momentum tensor depends on the extra coordinate we can see in reference [13], and, where the energy momentum tensor depends on the radial coordinate in reference [24] for black strings surrounded by quintessence matter.

As was mentioned, in equation (3) the 4D geometry, $d\bar{s}^2_{4D}$, is deformed by the function $W$. In this toy model we propose a form for the energy momentum tensor, such that the radial dependency is deformed by the function $W$:
\[ T_M^N = F(W(z)) \cdot K \cdot \text{diag}[-\bar{\rho}(r), \bar{\rho}_r(r), \bar{\rho}_\theta(r), \bar{\rho}_\phi(r), \tilde{\rho}_z(r)], \tag{11} \]

where \( F(W(z)) \) represents a function dependent of the extra coordinate and \( K \) is a constant to set the units.

For convenience, the above energy–momentum tensor can be written as follows

\[ T_M^N = \delta_M^\mu \delta_N^\nu \cdot F(W(z)) \cdot K \cdot \tilde{T}_\nu^\mu + T_5^5, \tag{12} \]

where, as we will see below, after the equation \([29]\), \( \tilde{T}_\nu^\mu \) represents the energy–momentum tensor associated to the four dimensional geometry, \( d\bar{s}_{4D} \), and is given by

\[ \tilde{T}_\nu^\mu = \text{diag}(-\bar{\rho}(r), \bar{\rho}_r(r), \bar{\rho}_\theta(r), \bar{\rho}_\phi(r)) \tag{13} \]

Thus \( \bar{\rho}(r), \bar{\rho}_r(r), \bar{\rho}_\theta(r), \bar{\rho}_\phi(r) \) are representing the energy density and the \( r, \theta, \phi \) pressure components of the four dimensional geometry.

So, both the four dimensional geometry, \( d\bar{s}_{4D} \), as the energy momentum tensor corresponding to this latter, \( \tilde{T}_\nu^\mu \), are deformed by the \( W(z) \) function.

On the other hand \( T_5^5 \) corresponds to the energy–momentum tensor associated to the extra dimension

\[ T_5^5 = \delta_5^5 F(W(z)) K \tilde{\rho}_z(r). \tag{14} \]

where \( \tilde{\rho}_z(r) \) represents the radial dependency, which is a part of \( T_5^5 \).

Following \([25]\), the \( D \) dimensional energy momentum tensor has units of \( [T_N^M] = \ell_p^{-D} \), and then, \( [G_N^M] = [\tilde{G}_D \cdot T_N^M] = \ell_p^{-2} \). So, in equations \([11,12]\), the five dimensional energy momentum tensor has units of \( [\tilde{T}_N^M] = \ell_p^{-5} \), the four dimensional energy momentum tensor has units of \( [\tilde{T}_\nu^\mu] = \ell_p^{-4} \), and the constant \( [K] = \ell_p^{-1} \).

It is worth to mention that, due that, in our case the four dimensional geometry is stretched along the extra dimension, then the four dimensional matter given by \( \tilde{T}_\nu^\mu \) spread over the whole extra dimension. So, the value of the five dimensional energy momentum tensor varies for different values of \( z \) through the function \( F(W(z)) \).
C. Equations of motion

We will use the following ansatz for the $f(r)$ function:

$$f(r) = 1 - \frac{2 \tilde{G}_D m(r)}{r},$$

where $m(r)$ is the so-called mass function and, as mentioned above in the point [1] of subsection [II A], must be such that the four-dimensional invariants must be regular. As was mentioned, using Planck units, the $D$-dimensional Newton constant has units of $[\tilde{G}_D] = \ell_p^{D-2}$ [23], and thus, $[\tilde{G}_D] = \ell_p^2$. Following [25], $[m(r)] = \ell_p^{-1}$.

For the line element (5) and the energy momentum tensor (11), with the ansatz (15), using the constraints $T^0_0 = T^1_1$ and $T^2_2 = T^3_3$, the components of the Einstein equations (2) are:

$(t - t)$ and $(r - r)$ components:

$$6(\dot{A})^2 - 3\ddot{A} + \Lambda_{5D} - \frac{2}{r^2} W^{-1} \tilde{G}_D \frac{d m}{d r} = -8\pi \tilde{G}_5 K F(W) \tilde{\rho}$$

$(\theta - \theta)$ and $(\phi - \phi)$ components:

$$6(\dot{A})^2 - 3\ddot{A} + \Lambda_{5D} - \frac{1}{r} W^{-1} \tilde{G}_D \frac{d^2 m}{d r^2} = 8\pi \tilde{G}_5 K F(W) \tilde{\rho}_\theta$$

and the $(z - z)$ component:

$$6(\dot{A})^2 + \Lambda_{5D} - \frac{2}{r^2} W^{-1} \tilde{G}_D \frac{d m}{d r} - \frac{1}{r} W^{-1} \tilde{G}_D \frac{d^2 m}{d r^2} = 8\pi \tilde{G}_5 K F(W) \tilde{\rho}_5.$$  

To solve these equations we will use the following steps:

- In the $(z - z)$ component we assume that

$$6(\dot{A})^2 = -\Lambda_{5D}.$$  

From this latter equation it is direct to check that $\dot{A} = 0$. Thus, using the condition (19), the two first terms of the equation (16) and the three first terms of the equations (16) and (17) are equal to zero.

Thus, for determining the form of $A(z)$, solving the equation (19) is enough. Below we will discuss about the solution of the equation (19).
• From equation (16) it is direct to check that:

\[
\frac{2}{r^2} W^{-1} \tilde{G}_{4D} \frac{dm}{dr} = 8\pi \tilde{G}_{5D} K F(W) \tilde{\rho},
\]

(20)

thus, it is imposed that:

\[
F(W(z)) = W^{-1} = \exp(2A(z))
\]

(21)

and

\[
m(r) = 4\pi \int_0^r x^2 \tilde{\rho}(x) dx,
\]

(22)

Below will be described the form for the energy density \( \tilde{\rho} \).

• On the other hand, from equations (20), (21) and (22), it is direct to check that:

\[
\tilde{G}_{4D} = K \tilde{G}_{5D}.
\]

(23)

For simplicity we define the constant \( \ell = K^{-1} \), where \([\ell] = \ell_p\). So, the last equation can be rewritten as:

\[
\tilde{G}_{5D} = \tilde{G}_{4D} \ell.
\]

(24)

So, our assumptions suggest that the equation (24) represents a relation between the 4D and the 5D Newton constants. It is worthwhile mentioning that an in–depth study is required to demonstrate a relationship between the Newton constants. However, it is worth to mention that the relation (24) is consistent, for \( D = 5 \), with the relation described in the appendix (A) for higher dimensional scenarios, where the \( D \) dimensional Newton constant is equal to the 4D Newton constant multiplied by an extra dimensional volume–like \( \ell^{D-4} \), see equation (A3).

• From equation (17) it is direct to check that:

\[
- \frac{1}{r} \frac{d^2 m}{dr^2} = 8\pi \tilde{\rho}_\theta
\]

(25)

• The radial and tangential pressures are easily computed by the conditions \( T^{0}_0 = T^{1}_1 \), \( i.e \, -\tilde{\rho} = \tilde{\rho}_r \), and \( T^{2}_2 = T^{3}_3 \), \( i.e \, \tilde{\rho}_\theta = \tilde{\rho}_\phi \).

• From equation (18) it is direct to check that:

\[
- \tilde{\rho} + \tilde{\rho}_\theta = \tilde{p}_5.
\]

(26)
From equations (3) and (21), we see that the four dimensional geometry is modified by the function $W(z)$. Furthermore, from equations (21) and (12), we note that the function $W(z)$ also modifies the information of the radial dependence on the energy momentum tensor (13) and the energy momentum tensor along the extra dimension (14).

It is worth to mention that, given our ansatz (19), the three first terms of the $(\mu, \nu)$ components, equations (16) and (17), are suppressed. On the other hand, due to the equations (21) and (23), the $(\mu, \nu)$ components take the form:

(t – t) and (r – r) components:

$$-\frac{2}{r^2} \tilde{G}_{4D} \frac{dm}{dr} = -8\pi \tilde{G}_{4D} \tilde{\rho}$$  \hspace{1cm} (27)

(θ – θ) and (ϕ – ϕ) components:

$$-\frac{1}{r} \tilde{G}_{4D} \frac{d^2 m}{d\theta^2} = 8\pi \tilde{G}_{4D} \tilde{p}_\theta$$  \hspace{1cm} (28)

The left part of equations (27) and (28) have the form of the Einstein tensor, $\tilde{G}_\mu^\nu$, which corresponds to the 4D geometry $d\bar{s}^2_{4D}$. Thus, under our assumptions used for our toy model, the five dimensional equations of motion (16) and (17) adopt the form of the four dimensional equations of motion

$$\tilde{G}_\mu^\nu = 8\pi \tilde{G}_{4D} \bar{T}^\mu_\nu,$$  \hspace{1cm} (29)

which correspond to the 4D geometry $d\bar{s}^2_{4D}$. So, as was above mentioned, the tensor $\bar{T}^\mu_\nu$ can be associated with the four dimensional geometry $d\bar{s}^2_{4D}$.

On the other hand, for our energy momentum tensor (11), with the line element (5) and with the relation (21), and for our case where $-\tilde{\rho} = \bar{\rho}_r$ and $\bar{p}_\theta = \bar{p}_\phi$, the equation of conservation $\nabla_M T^M_N = 0$ is:

$$-\hat{F}(\bar{p}_5 + \tilde{\rho} - \bar{p}_\theta) + F\left(-\tilde{\rho} - \frac{2}{r} (\tilde{\rho} + \bar{p}_\theta)\right) = 0 \hspace{1cm} (30)$$

The inner part of the parentheses of the first term is equal to zero due to the relation (26). On the other hand, we will see below that the function $F(W(z)) \neq 0$. So, under our assumptions used for our toy model, the five dimensional conservation equation adopts the form:

$$-\tilde{\rho}' - \frac{2}{r} (\tilde{\rho} + \bar{p}_\theta) = 0,$$  \hspace{1cm} (31)

which coincides with the conservation equation for the four dimensional geometry, $ds^2_{4D}$, present in the equations (4) and (5).
D. The four dimensional geometry

We will choose the Hayward \[19\] model as example. This model has a de–Sitter core, and thus, both the Ricci and the Kretschmann are regular everywhere. This is consistent with the point \[1\] of section II A. Based in the generalization of the Hayward metric of reference \[25\] we write the energy density as:

\[
\bar{\rho} = \frac{3}{2\pi} \frac{LM^2}{(2LM + r^3)^2},
\]

(32)

where \(L > 0\) is a constant parameter of units \([L] = \ell_p^4\). \(M\) is the so–called mass parameter and has units of \([M] = \ell^{-1}\). So the four dimensional energy density has units of \([\bar{\rho}] = \ell_p^{-4}\).

Replacing into equation (22):

\[
m(r) = \frac{Mr^3}{2LM + r^3}.
\]

(33)

Here \(m(r)\) is the so–called mass function and has units of \([m(r)] = \ell_p^{-1}\). So, the factor \(2\tilde{G}_{4D}m(r)/r\) is dimensionless.

The mass function behaves as \(m(r)|_{r=0} \approx r^3/(2L)\) near the origin. Thus, from equation (15), the four dimensional geometry \(ds_{4D}^2\) behaves as a de–Siter space time near the origin, \(i.e.\) the four dimensional geometry has a de–Sitter core. Thus, at the origin the topology of the five dimensional geometry corresponds to the product between the four dimensional de–Sitter space–time and \(R(S^1)\) for \(z\) non compact (compact). This latter differs from the Kaluza-Klein black string, equation (1), where, at the origin the topology corresponds to the product between the Schwarzschild singularity and \(R(S^1)\) for \(z\) non compact (compact).

An analysis of the four dimensional Hayward geometry can be viewed in reference \[21\]. See also \[27–29\] for a thermodynamics analysis.

It is worth to stress that both the four dimensional energy density and the mass function ensure a well asymptotic behavior of the four dimensional geometry \[30\]:

\[
\lim_{r \to \infty} \bar{\rho} = 0
\]

(34)

\[
\lim_{r \to \infty} m(r) = \text{constant} = M.
\]

(35)

From equation (35), it is easy to check that in equation (25):

\[
\lim_{r \to \infty} \bar{\rho}_\theta = 0
\]

(36)

Thus, the four dimensional geometry is asymptotically flat. Below we will discuss the consequences of this latter in the complete five dimensional geometry.
E. Complete five dimensional geometry

We choose the following cosmological constant:

$$\Lambda_{5D} = -\frac{6}{l^2}, \quad (37)$$

where $l$ corresponds to the AdS radius when the space time represents to an AdS space. From equation (19) it is easy to check that the line element (3) is:

$$ds^2_{5D} = C \exp\left( \pm \frac{2}{l}z \right) \cdot ds^2_{4D} + dz^2. \quad (38)$$

where $ds^2_{4D}$ is given by the generalized Hayward space time described in the previous subsection and $C$ is a positive constant. Thus, the function $W$ is:

$$W(z) = \exp(-2A(z)) = C \exp\left( \pm \frac{2}{l}z \right). \quad (39)$$

So, the point 2 of section II A is satisfied.

In order to comply with the point 3 of section II A, for the regularity of the geometry, the function $A(z)$ must be such that the function $W(A(z))$ can not be zero. Thus, for the positive (negative) branch of equation (39), the extra coordinate must not reach the limit $z \to -\infty \ (z \to +\infty)$. So, in our model, the extra coordinate must be compact at the black string style. This type of compactification has been explained in detailed in the introduction. In this connection, in this work we consider $z \in [-\pi, \pi]$. The domain of the $z$ coordinate, bounded between $z \in [-\pi, \pi]$, also implies that the function $F = W^{-1} = C \exp\left( \mp \frac{2}{l}z \right)$ is not zero, as was required after the equation (30).

On the other hand, by inserting the components of the four dimensional energy momentum tensor computed in the previous section II D and the function (39) into the equation (11), are directly computed the components of the five dimensional energy momentum tensor.

From equations (34) and (36) it is direct to check that in equation (26):

$$\lim_{r \to \infty} \tilde{p}_5 = 0. \quad (40)$$

So, from equation (11), it is direct to check that the five dimensional energy momentum tensor tends to zero at the asymptote of the radial coordinate. Thus, due that the four dimensional geometry is asymptotically flat, our complete five dimensional geometry behaves
at this place as an Anti de–Sitter five dimensional space time. On the other hand, also at
the infinity of the radial coordinate the topology corresponds to the product between the
four dimensional Minkowski space–time and \( S^1 \).

III. A NOTE ABOUT THE THERMODYNAMICS

We are interested in a 4–dimensional toy model describing a regular black hole stretched
in an extra dimension at the black string style, as was described in the introduction.

Due that the extra coordinate \( z \) is compact also at the black string style, we assume that
the 4–dimensional geometry is stretched between \( z \in [−π, π] \). So we call \( z_0 \) to each point
that belongs to the domain of the extra coordinate.

It is direct to check that the induced metric, for all point \( z_0 \in z \), where is stretched the
4D geometry along the extra dimension, is given by \[31\]
\[
 h_{\mu\nu} = g_{\mu\nu}^{(4D)} W(z_0). \tag{41}
\]

In order to compute the first law of thermodynamics we will use the conditions \( h_{tt}(r_+, M, z_0) = 0 \) and \( \delta h_{tt}(r_+, M, z_0) = 0 \), where \( h_{tt} \) and \( r_+ \) represent the temporal component of the induced
metric and the horizon radius. These latter conditions can be viewed as constraints on the
evolution along the space parameters \[27, 32\]. The first law of thermodynamic is computed
on the horizon of the induced metric, which does not depend on the location \( z = z_0 \). From
equations (41) and (5), the temporal component of the induced metric is:

\[
 h_{tt} = -W(z_0)f(r)\delta_{tt}. \tag{42}
\]

Following the above mentioned conditions:

\[
 \delta W(z_0)f(r_+, M) + W(z_0)\delta f(r_+, M) = 0. \tag{43}
\]

We can see from equation (5) that the function \( f(r) \) is part of the four dimensional geometry,
in our case, a four dimensional regular black hole stretched along the extra coordinate, thus,
\( f(r_+, M) = 0 \). Thus, the first term is equal to zero. On the other hand, from the point 3 of
section II A we can see that \( W(z) \neq 0 \), therefore \( W(z_0) \neq 0 \). Thus, our problem is reduced
to solve \( \delta f(r_+, M) = 0 \):

\[
 0 = \frac{\partial f}{\partial r_+} dr_+ + \frac{\partial f}{\partial M} dM. \tag{44}
\]
Following equation (44), for our four dimensional solution (15), where the mass function is given by the equation (33), the first law takes the form:

\[ \frac{\partial m}{\partial M} dM = \left( \frac{1}{4\pi} f'|_{r=r_+} \right) d\left( \frac{4\pi r_+^2}{4} \right). \] (45)

The above equation can be rewritten as:

\[ du = Td\left( \frac{A}{4} \right), \] (46)

where \( A \) corresponds to the area of a three dimensional sphere and the temperature and entropy are easily computed as:

\[ T = \frac{1}{4\pi} f'|_{r=r_+}, \] (47)

\[ S = \frac{A}{4} = \pi r_+^2. \] (48)

Thus, our computed definition of entropy follows the area’s law and so the existence of extra dimensions does not modify the correct values of temperature and entropy at the induced four dimensional geometry. The term \( du \) corresponds to a local definition of the variation of the energy defined in [27, 30]. In this definition the factor \( dm/dM \) in equation (45) is always positive, and thus, the sign of the variation of \( du \) always coincides with the sign of the variation of the total energy \( dM \).

IV. CONCLUSION AND SUMMARIZE

Following the style of representation for black strings, described in the introduction, we have provided a way of representing a four dimensional regular black hole geometry along a compact extra dimension. Thus, unlike the Kaluza Klein black string, where the four dimensional geometry corresponds to the four dimensional Schwarzschild singular black hole (see equation (1)), in our case the four dimensional geometry corresponds to a regular black hole.

In our toy model, the four dimensional geometry is deformed by the function \( W(z) \), see equation (3). Thus, due to the action of this function \( W(z) \), in our model there is not five dimensional translational symmetry along the \( z \) coordinate, as occurs for the Kaluza Klein black string [6]. Based in this fact, we propose a form for the energy momentum tensor, such
that the four dimensional energy momentum tensor, \( T_{\mu}^{\nu} \) is also deformed by the function \( W \), see equation (12).

It is worth to mention that, due that, in our case the four dimensional geometry is stretched along the extra dimension, then the four dimensional matter given by \( T_{\mu}^{\nu} \) spread over the whole extra dimension. So, the value of the five dimensional energy momentum tensor varies for different values of \( z \) through the function \( F(W(z)) \).

In subsection (II A) we have found relations between the four and five dimensional invariants of curvature. Using these latter, we have enunciated a list of constrains in order to that the complete five dimensional geometry to be regular. Following these constraints were constructed both the four and the five dimensional geometries. As example of four dimensional geometry, we have used the Hayward metric as toy model. However, following our list of constrains, could be constructed another geometries using other models of four dimensional regular black holes with a de–Sitter core, see for example [33]. This could be analyzed in elsewhere.

The assumptions used for our toy model suggest that the equation (24) represents a relation between the 4D and the 5D Newton constants. It is worthwhile mentioning that an in–depth study is required to demonstrate a relation between the Newton constants. However, it is worth to mention that the relation (24) is consistent with the relation described in the appendix (A) for higher dimensional scenarios, where the \( D \) dimensional Newton constant is equal to the 4D Newton constant multiplied by an extra dimensional volume–like \( \ell^{D-4} \), see equation (A3).

Under our assumptions used for our toy model, the \((\mu, \nu)\) components of the five dimensional equations of motion adopt the form of the four dimensional equations of motion
\[
\mathcal{G}_{\nu}^{\mu} = 8\pi\mathcal{G}_{4D}T_{\nu}^{\mu},
\]
which correspond to the 4D geometry \( ds_{4D}^2 \). Also under our assumptions, the five dimensional conservation equation adopts the form of the conservation equation for the four dimensional geometry.

At the origin the topology of the five dimensional geometry corresponds to the product between the four dimensional de–Sitter space–time and \( S^1 \) with \( z \) compact. This latter differs from the Kaluza-Klein black string [34], equation (1), where, at the origin the topology corresponds to the product between the Schwarzschild singularity and \( R(S^1) \) for \( z \) non compact (compact).

The topology of the complete five dimensional geometry corresponds to \( S^2 \times S^1 \). For
the region where $r < r_+$, being $r_+$ the horizon radius, also there is presence of a cylindrical event horizon.

In order to comply with the point 3 of section II A, for the regularity of the geometry, the function $A(z)$ must be such that the function $W(A(z))$ can not be zero. Thus, for the positive (negative) branch of equation [39], the extra coordinate must not reach the limit $z \to -\infty$ ($z \to +\infty$). So, in our model, the extra coordinate must be compact at the black string style. This type of compactification have been explained in detailed in the introduction. In this connection, in this work we consider $z \in [-\pi, \pi]$.

The five dimensional energy momentum tensor tends to zero at the asymptote of the radial coordinate. Thus, due that the four dimensional geometry is asymptotically flat, our complete five dimensional geometry behaves at this place as an Anti de–Sitter five dimensional space time.

In our model the 4–dimensional geometry is stretched between $z \in [-\pi, \pi]$. So, we have called $z_0$ to each point that belongs to the domain of the extra coordinate. We have computed the induced metric for the points $z_0$. So, following local relations, based on the evolution along the space of parameters [27, 32], the first law of thermodynamics at the induced four dimensional geometry has the form of equation (46) and has the correct values of entropy and temperature.

It is worth to mention that small perturbations around black strings show that these latter are unstable [35] and this instability could be stabilized if the extra dimension is compactified to a scale smaller than a minimum value. This issue is outside of the scope of our work, however should be tested in a future work.

**Appendix A: About the relation between the 4D and 5D Newton constants**

In order to relate the 4D and 5D Newton constants, we will use the relations showed in reference [36] for the “heuristics of higher-dimensional gravity”. In this latter, for a $D$ dimensional space time, the effective four dimensional Planck scale $M_4$ is given by:

$$M_4^2 = M_D^{D-2} \ell^{D-4},$$  \hspace{1cm} (A1)

where $M_D$ represents the $D$ dimensional Planck scale and where $\ell$ has units of length. In this context, as indicates the reference [36], the Newton constant in $D$ dimensions is given
by:

\[ \bar{G}_D = \frac{1}{M_D^{D-2}}. \]  

(A2)

So, equation [A1] can be rewritten as:

\[ \bar{G}_D = \bar{G}_{4D}{\ell^D}. \]  

(A3)

Using Planck units the \( D \) dimensional Newton constant has units of \( \bar{G}_D = \ell_p^{D-2} \), where \( \ell_p \) corresponds to the Planck length. This also can be viewed from equation (A2), assuming that \( [M_D] = \ell_p^{-1} \). Also using Planck units, in equation (A1) \( [\ell] = \ell_p \).

From equation (A3), the five dimensional Newton constant is related as:

\[ \bar{G}_{5D} = \bar{G}_{4D}\ell. \]  

(A4)

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