Stationarity of the detrended time series of S&P500

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Abstract. Our study presents the analysis of stock market data of S&P500 before and after been detrended. The analysis is based on two types of returns, simple return and log-return respectively. Both of them are non-stationary time series. This means that their statistical distribution change over time. Consequently a detrended process is made to neutralize the non-stationary effects. The detrended process is obtained by decomposing the financial time series into a deterministic trend and random fluctuations. We present an alternative method on detrending time series based on the classical moving average (MA) models, where Kurtosis is used to determine the windows size. Then, the dentrending fluctuation analysis (DFA) is use to show that the detrended part is stationary. This is done by considering the autocorrelation of detrended price return and the power spectrum analysis of detrended price.

1 Introduction

The time series actually encountered in industry or business [1], financial markets [2, 3], genetics [4], neuroscience [5, 6], biomedicine [7, 8], ocean dynamics [9] and seismology [10] exhibit complex dynamic systems that generate non-stationary and non-linear time series. The main feature of these time series are the variance in their basic statistical properties. In other words, these are time series non-identically distributed throughout their full length T [11, 12, 13] and characterized by distribution whose tail decays like a power law— fat tailed distributions—. In stock markets some of these type of time series are simple price return [2, 13, 14], volatility of index price [15] and trading stock volume [16]. In the statistical analysis of financial markets the time series which were extensively analysed are the simple and natural-logarithm price return [10]. These two types of returns correspond to an arithmetical and logarithmically difference respectively. However, dealing with these two types of price returns involves working with non-stationary processes produced by a non-constant activity during a trading day and heterogeneity of market participants [17, 18, 2].

The non-stationary series described above are difficult to analyse because the process remains in a non-equilibrium stage and consequently their models do not
achieve optimal forecasting and control [1]. Consequently it is improbable to derive an accurate model that can be used to calculate just the probability of a future value lying between initial conditions.

We suggest a different approach, the non-stationary time-series is decomposed into a deterministic time series (trend) and a stochastic time series that satisfies stationarity. Previous research shows a combination of MA and DFA analysis as Detrending moving average (DMA) analysis. However, DMA was used only as a tool to quantify correlations [3]. This paper extend this method to test stationarity in time series which can be use for monofractal and multifractal time series.

This paper address the question on whether the time series of the S&P500 index can be decomposed into a deterministic trend and a stochastic time series. In Section 2 the simple price return and the log of price return and introduced and the time evolution of PDF of these two quantities are discussed. Section 3 presented a detrending method based on the optimisation of the moving windows via the kurtosis evaluation. In Section 4 we perform various statistics on the detrended price to show that the detrended part is stationary. The statistics are based on the Detrended Fluctuation Analysis (DFA) method described in Section 4.1 to show that the detrended price is multifractal with a Hurst exponent slightly higher than 0.5. In section 4.2 we show that the detrended time series is multifractal and stationary. Section 4.3 is a separated analysis of the series where we show that the power spectrum of the detrended time series related with the Fourier transformation with this autocorrelation according to Gao et. al [23] and Witt et. al. [39]. The conclusions and perspectives of this work are presented in Section 5.

2 Non-stationarity of Price return

We have analysed the Standard & Poor’s 500 stock market data during 22-year period from January 1996 to May 2018 with an interval span of 1 min. The Figure 1 shows the time series of this data identifying the closing milestones of the S&P 500. The green and red circles display the highest and lowest milestones respectively. The index value of S&P 500 has been increasing, despite some periods of decline.

The price return is obtained from the dynamics of price fluctuation \( I(t) \). It have been identified two different types of price return over a time scale \( t \). The first one corresponds to a successive price variation or price return \([24, 25, 26]\),

\[
X(t_0, t) = I(t_0 + t) - I(t_0) \tag{1}
\]

The second one is a logarithmic difference of stock market price \([27, 28, 29]\);

\[
Z(t_0, t) = \ln \left( \frac{I(t_0 + t)}{I(t_0)} \right) \tag{2}
\]

The log-return \( Z \) is more used in technical analysis because is close to the percentage price change. However, simple price return \( X \) is the most visible measurement of an investment considering external factors such as inflation.

For \( t = 1 \) the price return and log-price return receive the notation of \( X(t, 1) = \delta X \) and \( Z(t, 1) = \delta Z \) which correspond to simple price return and simple log-price return respectively. For any interval of time \( t_0 \), the price return \( X(t, t_0) \) and \( Z(t, t_0) \) can be obtained from the simple price return and simple log-price return \([30]\),

\[
X(t_0, t) = \sum_{i=1}^{t} I(t_0 + 1 + i) - I(t_0 + i) \\
Z(t_0, t) = \sum_{i=1}^{t} \ln \left( \frac{I(t_0 + 1 + i)}{I(t_0 + i)} \right) \tag{3}
\]
Price returns determine losses and gains. Its probability density function (PDF) allows to evaluate the mean return for different time intervals, evaluate risk and has been used to elaborate strategies. Furthermore, high and low closing milestones of S&P500 produce fat tail distributions and are examined by applying statistical methods. Figure 1 displays the representative closing milestones of the last 22 years of S&P500.

Fig. 1. Price fluctuation $I(t)$ of stock markets S&P500 in the period of 02/01/96 to 31/05/18 (22 years). Displaying highest and lowest closing milestones.

The Figures 2a, 2b and 2c display the comparison between $X(t)$ and $Z(t)$ considering two statistical properties; the height of the PDF's $P_{\text{max}}(t)$ and the drift $\mu(t)$ which correspond to the $X$ or $Z$ value of $P_{\text{max}}(t)$. The Figures 2b and 2c make evident that $X(t)$ and $Z(t)$ present a similar behavior on their drift $\mu(t)$ and growth rates $P_{\text{max}}(t)$, exhibiting three well-defined zones which are limited by the abrupt change of the slope $P_{\text{max}}(t)$ and $\mu(t)$. These zones are: strong super-diffusion, weak super-diffusion and near classical diffusion. The values of $P_{\text{max}}(t)$ and $\mu(t)$ clearly depend on time $t$. Consequently, both price returns are classified as a non-stationary time series. Although some econophysicists state that log-price return neutralize most of the non-stationary effect, the $\mu(t)$ is non-constant and affects the growth rate of the time evolution of price return.

The Fokker Planck equation (FPE) is used to describe the time evolution of the probability density function of the velocity of a particle under the influence of a force. Being the price return the particle to be analyzed, the FPE for the probability density function $P(X,t)$ will be defined as follows [Eq. (4)],

$$\frac{\partial P}{\partial t} = \frac{\partial^2 (D_2(x,t)P)}{\partial x^2} + \frac{\partial (D_1(x,t)P)}{\partial x}.$$  \hspace{1cm} (4)

The diffusion coefficient and the drift are represented by $D_2(x,t)$ and $D_1(x,t)$ respectively. On our previous study, “Q-Gaussian diffusion in stock markets”, the drift
Fig. 2. (a) PDF evolution of Price return of S&P500 in the period of 02/01/96 to 31/05/18 (22 years). (b) Close-up of Figure (a) to notice the time evolution of the peak that fully disappears after 78 min. This subplot displays a Power-Law relation between $P_{max}$ vs $t$, to evaluate the growth rate of the time. (c) Power-Law relation between $\mu(t)$ vs $t$, to evaluate the drift behaviour. These plots highlight the non-stationarity of price return.
was removed by subtracting the average value with a time window of one month [35].

Then, the governing equation for $1 < t < 3000 \text{ min}$ was reduced to $\frac{\partial P}{\partial t} = \frac{\partial^2 (D_2 \frac{\partial P}{\partial x})}{\partial x^2}$.

The diffusion coefficient of the Linear Fokker Planck Equation (LFPE) was derived,

$$D_2(x,t) = \frac{(3 - q)D^\frac{3}{2}}{\alpha C_q^{1-q} t^{\frac{-1}{\alpha}}} \left( 1 - (1 - q) \frac{x^2}{(Dt)^{\frac{1}{\alpha}}} \right),$$

where, the parameter $q$ is the q-Gaussian exponent, $\alpha$ and $D$ are derived by fitting the collapse data of price return and $C_q$ is the normalization constant

$$C_q = \sqrt{\frac{\pi}{q - 1}} \Gamma \left( \frac{3 - q}{2(q - 1)} \right) \Gamma \left( \frac{1}{q - 1} \right).$$

This equation is valid for for $t < 3000 \text{ min}$ where the drift is small enough and easy to remove. This paper proposes an efficient way to remove the drift by applying a detrending method by a decomposition of the index price $I(t)$ into a deterministic trend and a stochastic time series.

### 3 Detrended method

The aim of this section is to obtain a stationary detrended data. The trend represents the behavior of a set of data and determines if there is a particular pattern on the time series. The process used to construct the trend of the data is the moving average (MA). This process is constructed based on an average value within a time window size that is shifted forward until the end of the data set. The most appropriate $t_w$ must be a midpoint between two features. The $t_w$ needs to be long enough to ensure a deterministic trend and short enough that the detrended part must be stationary.

The method of MA has been applied in different studies with some variations to remove the effect of the drift for time series [3, 35]. These modifications are regarding how the time window will be shifted until the end of the data set—overlapped or non-overlapped. The second modification is the order degree $k$ for a polynomial fitting in each segment of time window [21]. Although, $k = 1$ is used in some studies to eliminate local trends, the original MA uses $k = 0$ as an unweighted mean. These modifications were tested in this analysis with the index price $I(t)$. The most remarkable difference is that by using overlapping windows we obtain a better smoothing trend avoiding jumps on the crossing points on consecutive windows. The results nonetheless do not display major differences in comparison with the original MA for larger $t_w$. Our analysis displayed that windows size $t_w$ is the key parameter to obtain a stationary detrended data. Consequently, subsection 3.1 refers to find the optimal $t_w$. Then, the original MA is applied considering the most appropriate time window 3.2.

#### 3.1 Optimal time window

We adopt the method of Xu et al. to obtain the optimal $t_w$ [36]. To achieve an optimum window, the data set of the distribution in each window $t_w$ should be close as possible to a Gaussian, but with vary in variance. Xu et al. applied the kurtosis definition on their method as a tool to determine $t_w$ [36]. The kurtosis is the fourth standardized moment, defined as the relation between central moments $\langle u^n \rangle$ of order $n$,

$$K = \frac{\langle u^4 \rangle}{\langle u^2 \rangle^2}. \quad (7)$$
where \( \langle \rangle \) denotes long-time average.

According to Xu et al., the optimal window size must satisfy the condition of \( K = 3 \) [37, 36]. It is common to compare the kurtosis of a distribution to this value, because the kurtosis of any univariate normal distribution is 3. Then, the time series with a total length \( N \) is split into non-overlapping time windows \( |n = N/t_w| \) as a quick test to find the most appropriate value of \( t_w \). Next, for a given \( t_w \) the kurtosis in \( j \)th window is calculated, where \( j = 1, 2, 3, 4, \ldots, n \). After having the values of kurtosis for all the windows, the average of the kurtosis of the \( n \) windows is calculated:

\[
\overline{K_{tw}} = \frac{1}{n} \sum_{j=1}^{n} K_{tw}(j),
\]

where \( K_{tw} \) denote the average over the windows. Figure 3 displays the results of \( K_{tw} \). The optimal window size is \( 13 \pm 1 \) months. Then, the final value assumed for the MA analysis is one year.

\[Fig.~3. \text{Optimal window size } t_w \text{ that will be use for the Moving Average Analysis of S&P500 price fluctuation. The intersection with } \overline{K} = 3 \text{ yields to choose a time window of } 13 \pm 1 \text{ months. Considering an integer unit for } t_w \text{, the final value assumed is } t_w = 12 \text{ months.}\]

### 3.2 Moving Average Analysis (MA)

The trend is constructed for the index price of S&P 500 \( I(t) \), where \( t = 1, 2, 3, \ldots, N \) and \( t_w = 12 \) months. This process is constructed based on a weighted sum or average. The limits of the sum will depend on the position of the moving window. We consider three special cases following the Gao et. al. method [3]:

- For \( t < \frac{t_w}{2} \)

\[
\hat{I}(t) = \frac{1}{t_w} \sum_{k=-[(t-1)]}^{[(t_w-1)/2]} I(t + k)
\]
Fig. 4. (a) Trend $\tilde{I}(t)$ after applied the MA analysis for a $t_w$ of 12 months for S&P 500 data (b) Detrend price $I^*(t)$ after subtract the trend display on (a).

For $\frac{t_w}{2} < t < N - \frac{t_w}{2}$

$$\tilde{I}(t) = \frac{1}{t_w} \sum_{k=-[(t_w-1)/2]}^{[(t_w-1)/2]} I(t+k)$$  \hspace{1cm} (10)

- For $t > N - \frac{t_w}{2}$

$$\tilde{I}(t) = \frac{1}{t_w} \sum_{k=-[(t_w-1)/2]}^{[N-t]} I(t+k)$$  \hspace{1cm} (11)

By removing the optimal trend from $I$, we obtain a stationarity fluctuating part $I^*$.

$$I^*(t) = I(t) - \tilde{I}(t)$$  \hspace{1cm} (12)

The Figure 4a displays the trend $\tilde{I}(t)$ after applied the MA analysis for a $t_w$ of 12 months, the trend obtained represent a general tendency of the Index price $I(t)$. The Figure 4b displays the detrended price as a result to subtract the value of the trend obtained. The next section explain the method to support that $I^*(t)$ is a stationary time series.
4 Stationarity of detrended price

The detrended fluctuation analysis (DFA) [19] is used as a tool to test stationarity. The DFA method comprises the integration of the return of detrended price. Where, the time series is divided into equal non-overlapping segments with a length of $s$, which represent the time scale of the detrended time series. Then, the “self-similarity parameter” or Hurst exponent $H$ is obtained by calculating the power-law relation between the “statistical functions” $F_q$ for each segment of the detrended data vs the time scale $s$ [4, 20, 21].

The stationarity is tested by applying two methods. The first is related to autocorrelation function $C$ and the second to the power spectral density $S$. Both of them are characterized by a power law which depend on the Hurst exponent [22, 23].

4.1 Detrended Fluctuation Analysis (DFA)

Detrended fluctuation Analysis method developed by Peng et al. has become a widely technique to determine self-affinity of a signal [4, 22]. The scaling exponent $h(q)$ is obtained by calculating appropriate statistical functions $F_q$ based on central moments on the time series. Here $h(q)$ is the generalized Hurst exponent and quantifies the tendency of a time series either to regress strongly to the mean or to cluster in a particular direction [4, 22, 38]. A stationarity time series has an autocorrelation function $C$ and power spectral density $P(f)$ characterized by a power law that depends on the Hurst exponent $H$ [22, 23, 39, 38, 40]. Consequently, a procedure for testing stationarity of time series can be developed based on the concepts previously explained. In the following steps we adopt this method to test the stationarity of the detrended price $I^*(t)$ time series. By determining the statistical functions $F_q$, the Hurst exponent $H$ is calculated.

- **Step 1**: For our case the ‘profile’ or time series to analyze is represented by the detrended price $I^*(t)$ obtained after apply Eq. 12.
- **Step 2**: The profile $I^*(t)$ is divided into non-overlapping segments $[N_s = N/s]$ with the same length $s$.
- **Step 3**: The variance for each of the segments $v = 1, 2, 3,...N_s$ is obtained by applying the following equation,

$$ F^2(v,s) = \frac{1}{s} \sum_{i=1}^{s} (I^*(v-1)s + i) - \overline{I}(v))^2, \quad (13) $$

where, $\overline{I}$ represents the mean for each segment segment $v$.
- **Step 4**: The $q$-th order fluctuation function is obtained. Considering different orders $q$ of the statistical moments.

$$ F_q(s) = \left\{ \frac{1}{N_s} \sum_{i=1}^{N_s} [F^2(v,s)]^{q/2} \right\}^{1/q} \quad (14) $$

For series that present long-range correlations, the following power law is obtained $F_q(s) \sim s^{h(q)}$. Figure 5 display the results of DFA analysis, where a power law between $F_q$ vs $s$ is observed. For monofractal time series $h(q)$ is independent of $q$ due to a constant scaling behaviour of variance $F^2(v,s)$ over all the segments [14, 22]. However, Figure 5 shows a dependence of $h(q)$ on $q$. In such cases, a modified DFA has to be applied. Following the procedure, we use a generalize method for a multifractal detrended fluctuation analysis (MF-DFA).
Fig. 5. Calculation of the statistical function $F_q(s)$ using Eq. (14). The function of $F_q$ vs $s$ are perfect power laws while $h(q)$ depend on $q$. This feature proves that the time series is multifractal.

For a standard multifractal analysis (MF-DFA) the scaling exponent $\tau(q)$ needs to be obtained. This exponent is the power law defined between the $q$th-order overall function $F_{\tau}$ vs $s$. Where, $F_{\tau}(s)$ corresponds to the generalization statistical function $F_q$, which is based on the following relationship,

$$F_{\tau}(s) = \left(\frac{N}{s}\right)^q \sum_{v=1}^{N/s} |p_s(v)|^q \sim s^{\tau(q)},$$

where $p_s(v)$ is known as the box probability in a standard multifractal formalism for our price detrended $I^*(t)$. It can be expressed in terms of simple detrended price return $X^*(t) = I^*(t + 1) - I^*(t)$ as well,

$$p_s(v) = \sum_{s=(v-1)s+1}^{vs} X_t = I^*(vs) - I^*((v-1)s)$$

And $\tau(q)$ represents the classical multifractal scaling exponent.

$$\tau(q) = qH - 1$$

The Figures 6a and 6b show the results of the generalized statistical function $F_{\tau}$ for negative and positive values of $q$ respectively. It is remarkable the power law relation and $q$ dependence that will be evaluated to classify $I^*(t)$ as a stationary time series.

The Figure 7 shows a summary of the results obtained from Eq. (15) and previously shown in Figure 6, where the scaling exponent is $0.501 \pm 0.041$.

4.2 Testing stationarity in time series

To test the stationarity in time series two methods has been applied. The first one is in context of the multifractal detrending fluctuation analysis (MF-DFA) [22, 33] and
Fig. 6. Calculation of the “generalized statistical functions” $F_\tau$ vs $s$ for negative (a) and positive (b) values of order q. The best fit displays a power-law $\tau(q) = qH - 1$, where $H = 0.501 \pm 0.041$. See Figure 7.
the second refers how Hurst exponent relates to power spectrum for stationary time series [39, 23].

![Figure 7](image7.png)

**Fig. 7.** Generalized scaled exponent \( \tau(q) \) obtained from Figure 6. The results are fitted to the power law \( \tau(q) = qH - 1 \) where \( H = 0.501 \pm 0.041 \).

![Figure 8](image8.png)

**Fig. 8.** Autocorrelation of the detrended price calculated as \( C(s) \equiv \langle x_t x_{t+s} \rangle \) and it scales as \( C(s) \equiv \langle x_t x_{t+s} \rangle \sim s^{-\gamma} \). For large \( s \) we prove that \( \gamma = 2 - 2H \) that agrees with stationarity.

In context of a multifractal series, a time series is stationary if \( h(q) = H \) for \( q = 2 \) and the autocorrelation function \( C(s) \equiv \langle x_t x_{t+s} \rangle \sim s^{-\gamma} \) is characterized by a power law with \( \gamma = 2 - 2H \) for \( s \gg 1 \). Here \( x_t \) represents the unitary increments for the detrended series. For our analysis, we have that \( x_t = X^*(t) \). The Hurst exponent is determined by observing Figures 5 and 7. In Figure 5 the Hurst exponent \( H \) is identified as \( h(2) = 0.502 \pm 0.008 \). From the other part this value is checked in figure 7 by applying the Eq. 17 \( H = 0.501 \pm 0.041 \). It is evident the agreement between these two values obtained for the Hurst exponent of \( H \sim 0.501 \). Then, the autocorrelation function \( C(s) \) of \( X^*(t) \) with a time average limit of one month presents a power law \( \gamma = 1.02 \pm 0.52 \). From this power law, the Hurst exponent is calculated as
\( H = 0.49 \pm 0.054 \). All the values obtained of Hurst exponent are compatible between each other with a small tolerance.

![Power-spectral density](image)

**Fig. 9.** Power-spectral density \( P(f) \) of the detrended price displays a power law of \( 2H + 1 \), which agrees to the stationarity of price detrended. The power spectral density is calculated as 

\[
P(f) = \int_0^\infty |I^*(t)|^2.
\]

The second concept to test stationarity relates to the power spectrum \( P(f) \) of the time series \( I^*(t) \) and the Fourier transformation of its correlation. Where \( P(f) \sim 1/f^{(2H+1)} \). From Figure 9 the value of the exponent is \( 2H + 1 = 2 \). Once again \( H \) lies on the range of \( H \sim 0.501 \pm 0.041 \).

After testing both concepts, we conclude that \( I^*(t) \) is a stationary time series.

5 Conclusions

In this paper we provided two convincing arguments to shows that detrended price is stationarity. The detrending is performed by optimising the Kurtosis in the moving averaged detrended data. The stationarity is validated by the relationship between the Hurst exponent \( H \) and the power laws of the autocorrelation \( C \) and power spectral density \( P(f) \). The Hurst exponent \( H \) is calculated by adopting the Dentrending Fluctuation Analysis (DFA).

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