The spin structure of the system of quasifree fermions having total angular momentum \( J = 1/2 \) is studied in a consistently covariant approach. Within this model the relations between the spin functions are obtained. Their particular cases are the sum rules Wanzura - Wilczek, Efremov - Leader - Teryaev, Burkhardt - Cottingham and also the expression for the Wanzura - Wilczek twist 2 term \( g_{WW}^2 \). With the use of the proton valence quark distributions as an input, the corresponding spin functions are obtained. The resulting structure functions \( g_1 \) and \( g_2 \) are well compatible with the experimental data.

### I. INTRODUCTION

The nucleon structure functions, both unpolarized and polarized, are the basic tools for understanding the nucleon internal structure in the language of the QCD. Precision measurements on the polarized structure functions have been completed only in the recent years [1] - [8]. These functions contain information which is a crucial complement to the structure functions obtained in the unpolarized deep inelastic scattering (DIS) experiments. At the same time the interpretation and understanding of polarized structure functions seem to be much more difficult than in the case of unpolarized ones. Actually, until now it is not clear how the nucleon spin is generated from the spins and orbital momenta of the quarks and gluons. For the present status and perspectives of the nucleon spin physics see [9] and citations therein. The more formal aspects of the polarized DIS are explained in [10] and [11].

The spin in general is a very delicate quantity, which requires correspondingly precise treatment. It has been argued, that for correct evaluation of the quark contribution to the nucleon spin it is necessary to take properly into account the intrinsic quark motion [12] - [22]. The necessity of the covariant formulation of the quark - parton model (QPM) for the spin functions has been pointed out in [23]. These requirements are not satisfied in the standard formulation of the QPM, which is currently used for analysis and interpretation of the experimental data.

In the paper [22] we have demonstrated the role of the intrinsic motion for the spin structure functions, using a simple model of the system quasifree fermions on mass shell. The basic requirement was a consistently covariant formulation of the task for the system of fermions, which are not static, being characterized by some momenta distribution in the frame of their center of mass. In the present paper we attempt to further develop this approach.

In section II we are introducing the spin structure functions \( g_1(x), g_2(x) \), the longitudinal and transversal net spin density distributions \( s_L(x), s_T(x) \), and the density of total angular momentum \( j(x) \). Then in Sec. IV we show how all of these functions are mutually related. Finally in Sec. V we apply the suggested approach, with some simplified assumptions, to the description of the proton spin structure and make a comparison with the experimental data on the \( g_1(x) \) and \( g_2(x) \). The last section is devoted to the short summary and conclusion.

### II. SPIN STRUCTURE FUNCTIONS AND SPIN DISTRIBUTIONS

In the previous paper [22] we have shown that the spin structure functions, related to the spherically symmetric target consisting of the three quasifree fermions (of spin 1/2), having resulting total angular momentum \( J = 1/2 \), can be written as:

\[
g_1 = \frac{1}{2} \int H(p_0) \left[ m + \frac{\nu}{|\mathbf{q}|} p_1 + \frac{\nu^2}{|\mathbf{q}|^2} \frac{p_1^2}{p_0 + m} + \left( 1 - \frac{\nu^2}{|\mathbf{q}|^2} \right) \frac{p_2^2/2}{p_0 + m} \right] \delta \left( \frac{p_0 \nu + p_1 |\mathbf{q}|}{M \nu} - x \right) \frac{d^3p}{p_0},
\]

\[
g_2 = \frac{1}{2} \int H(p_0) \left[ m + \frac{\nu}{|\mathbf{q}|} p_1 + \frac{\nu^2}{|\mathbf{q}|^2} \frac{p_1^2}{p_0 + m} + \left( 1 - \frac{\nu^2}{|\mathbf{q}|^2} \right) \frac{p_2^2/2}{p_0 + m} \right] \delta \left( \frac{p_0 \nu + p_1 |\mathbf{q}|}{M \nu} - x \right) \frac{d^3p}{p_0},
\]

\[
g_2 = \frac{1}{2} \int H(p_0) \left[ m + \frac{\nu}{|\mathbf{q}|} p_1 + \frac{\nu^2}{|\mathbf{q}|^2} \frac{p_1^2}{p_0 + m} + \left( 1 - \frac{\nu^2}{|\mathbf{q}|^2} \right) \frac{p_2^2/2}{p_0 + m} \right] \delta \left( \frac{p_0 \nu + p_1 |\mathbf{q}|}{M \nu} - x \right) \frac{d^3p}{p_0},
\]

*to be published in Phys. Rev. D
\[ g_2 = -\frac{1}{2} \frac{\nu}{|q|} \int H(p_0) \left( p_1 + \frac{\nu}{|q|} \frac{p_1^2 - p_2^2/2}{p_0 + m} \right) \delta \left( \frac{p_0 \nu + p_1 |q|}{M \nu} - x \right) d^3 p, \]  

(2)

where \( p_0 = \sqrt{m^2 + p^2} \), \( p_1^2 = p_2^2 + p_3^2 \), and \( H \) is the charge weighted distribution

\[ H(p_0) = \sum_{k=1}^{3} c_k^2 \Delta G_k(p_0). \]  

(3)

This distribution is constructed from the polarized distributions of individual fermions

\[ \Delta G_k(p_0) = G_{k,+1/2}(p_0) - G_{k,-1/2}(p_0), \]  

(4)

which satisfy

\[ \int G_k(p_0) d^3 p = 1; \quad G_k(p_0) \equiv G_{k,+1/2}(p_0) + G_{k,-1/2}(p_0), \]  

(5)

\[ \int \Delta G(p_0) d^3 p = 1; \quad \Delta G(p_0) = \sum_{k=1}^{3} \Delta G_k(p_0). \]  

(6)

Distributions \( G_k, \lambda(p_0) \) measure probability to find a fermion in the state

\[ u(p, \lambda n) = \frac{1}{\sqrt{N}} \left( \frac{\phi_{\lambda n}}{p_0 + m} \phi_{\lambda n} \right); \quad \frac{1}{2} n^\sigma \sigma \phi_{\lambda n} = \lambda \phi_{\lambda n}, \quad \lambda = \pm \frac{1}{2}, \]  

(7)

where the direction \( n \) coincides with the direction of target polarization \( J \) and a standard normalization is used:

\[ N = \frac{2p_0}{p_0 + m}, \quad \phi_{\lambda n}^\dagger \phi_{\lambda n} = 1. \]  

(8)

Now, let us try using the same system to calculate some spin distribution functions. In the first step we shall define these distributions in terms of the fermion momenta related to the target rest frame, then we shall show their representation in the variable \( x \).

The net spin density corresponding to the projection on the direction \( n' \) is defined as

\[ S(p, n, n') = \sum_{\lambda} G_{\lambda}(p_0) u^\dagger(p, \lambda n) (n' \Sigma) u(p, \lambda n); \quad \Sigma = \frac{1}{2} \left( \begin{array}{cc} \sigma & 0 \\ 0 & -\sigma \end{array} \right). \]  

(9)

One can verify, that this expression can be modified

\[ S(p, n, n') = \frac{1}{N} \sum_{\lambda} G_{\lambda}(p_0) \left( \lambda n' - \lambda n^\prime \frac{p^2}{(p_0 + m)^2} + \phi_{\lambda n}^\dagger \phi_{\lambda n} \phi_{\lambda n}^\prime \phi_{\lambda n} \right). \]  

(10)

We assume, that the beam direction is defined by the vector \( k = (|k|, 0, 0) \), then one can obtain the following particular cases of the distribution \( S_L \).

1) Longitudinal polarization in longitudinally polarized target, i.e. \( n = n' = (1, 0, 0) \), then

\[ p \sigma \cdot n' = \sum_{i=1}^{3} p_i^2 \sigma_i n_i' + \sum_{j \neq i} p_j \sigma_i n_j' = p_1^2 \sigma_1 + p_2 \sigma_1 + p_3 \sigma_3 \]  

(11)

and the relation (10) can be simplified

\[ S_L(p) = \frac{1}{2p_0} \Delta G(p_0) \left( m + \frac{p_1^2}{p_0 + m} \right). \]  

(12)

2) Transverse polarization in transversely polarized target, i.e. \( n = n' = (0, 1, 0) \), then
\[ \mathbf{p} \sigma \cdot \mathbf{n} = \sum_{i=1}^{3} p_i^2 \sigma_i n_i + \sum_{j \neq i} p_i p_j \sigma_i n_j = p_2^2 \sigma_2 + p_1 p_2 \sigma_1 + p_3 p_2 \sigma_3 \] (13)

and the relation (10) can be simplified

\[ S_T(p) = \frac{1}{2p_0} \Delta G(p_0) \left( m + \frac{p_2^2}{p_0 + m} \right). \] (14)

3) In a similar way, one can also obtain the polarizations \( S_T^L(p) \) and \( S_L^T(p) \), which are related to the density of longitudinal polarization in the transversely polarized target and vice versa. The density \( S_T^L(p) \) can be obtained from the relation (10) after inserting \( n = (0,1,0) \), \( n' = (1,0,0) \) and \( S_L^T(p) \) with \( n = (1,0,0) \), \( n' = (0,1,0) \) correspondingly. After some calculation similar to that for obtaining the relations (12) and (14), one gets

\[ S_L^T(p) = S_T^L(p) = \frac{1}{2p_0} \Delta G(p_0) \frac{p_1 p_2}{p_0 + m}. \] (15)

4) The density of total angular momentum can be defined as:

\[ J(p, n, n') = \sum_{\lambda} G_{\lambda}(p_0) u^\dagger(p, \lambda n) (n' j) u(p, \lambda n); \quad j_k = \Sigma_k + l_k = \frac{1}{2} \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} - i \varepsilon_{klm} p_l \frac{\partial}{\partial p_m}. \] \] (16)

One can verify, that after some calculation this expression can be simplified:

\[ J(p, n, n') = \frac{1}{2} n n' \Delta G(p_0). \] (17)

This result implies, that \( J \) has rotational symmetry, so there is no distinction between longitudinal and transversal density:

\[ J_L(p) = J_T(p) = \frac{1}{2} \Delta G(p_0) \equiv J(p), \quad J_T^L(p) = J_T^L(p) = 0. \] (18)

Further, is it possible to express the obtained distributions as the functions of \( x \) instead of \( p \)? For a simplification we shall from now on assume that

\[ Q^2 \gg 4M^2 x^2, \] (19)

which implies

\[ \left| \frac{q}{\nu} \right| = \sqrt{1 + 4M^2 x^2 / Q^2} \to 1. \] (20)

Then the \( \delta \) function term, which defines the transformation \( p \to x \) in Eqs. (7), (8) will be simplified

\[ \delta \left( \frac{p_0 \nu + p_1 |q|}{M \nu} - x \right) \to \delta \left( \frac{p_0 + p_1}{M} - x \right), \] (21)

in this limit the coordinate \( p_1 \) defines the beam direction. The spin structure functions (5), (6) are now simplified accordingly:

\[ g_1(x) = \frac{1}{2} \int H(p_0) \left( m + p_1 + \frac{p_1^2}{p_0 + m} \right) \delta \left( \frac{p_0 + p_1}{M} - x \right) \frac{d^3p}{p_0}, \] (22)

\[ g_2(x) = -\frac{1}{2} \int H(p_0) \left( p_1 + \frac{p_1^2 - p_2^2/2}{p_0 + m} \right) \delta \left( \frac{p_0 + p_1}{M} - x \right) \frac{d^3p}{p_0}, \] (23)

\[ g_1(x) + g_2(x) = \frac{1}{2} \int H(p_0) \left( m + \frac{p_1^2/2}{p_0 + m} \right) \delta \left( \frac{p_0 + p_1}{M} - x \right) \frac{d^3p}{p_0}. \] (24)
Apparently, the convolution defined by the $\delta$ function \cite{21} also gives the rule for transformation of the spin distributions expressed in the variable $p$ to the corresponding representation in the variable $x$:

$$ j(x) = \int J(p)\delta \left( \frac{p_0 + p_1}{M} - x \right) d^3p = \frac{1}{2} \int \Delta G(p_0)\delta \left( \frac{p_0 + p_1}{M} - x \right) d^3p, \tag{25} $$

$$ s_L(x) = \int S_L(p)\delta \left( \frac{p_0 + p_1}{M} - x \right) d^3p = \frac{1}{2} \int \Delta G(p_0) \left( m + \frac{p_1^2}{p_0 + m} \right) \delta \left( \frac{p_0 + p_1}{M} - x \right) \frac{d^3p}{p_0}, \tag{26} $$

$$ s_T(x) = \int S_T(p)\delta \left( \frac{p_0 + p_1}{M} - x \right) d^3p = \frac{1}{2} \int \Delta G(p_0) \left( m + \frac{p_1^2}{2(p_0 + m)} \right) \delta \left( \frac{p_0 + p_1}{M} - x \right) \frac{d^3p}{p_0}. \tag{27} $$

In the last integral we could replace $p_1^2$ by $p_T^2/2$ because of the axial symmetry. Further, it is obvious, that the densities $S_L^T$ and $S_L^L$, expressed in the variable $x$, vanish due to the symmetry:

$$ s_L^T(x) = s_L^L(x) = \frac{1}{2} \int \Delta G(p_0) \frac{p_1 p_2}{p_0 + m} \delta \left( \frac{p_0 + p_1}{M} - x \right) \frac{d^3p}{p_0} = 0. \tag{28} $$

What is the meaning of the integrals in the relations \cite{22} - \cite{27}? To simplify this question, let us assume the same shape of the distributions $G_k(p_0)$ for all the three fermions. Then the distributions $\Delta G$ and $H$ differ only by a constant factor, in which charges and polarizations of individual fermions are absorbed:

$$ H(p_0) = \kappa \Delta G(p_0); \quad \kappa = \sum_{k=1}^{3} c_k^2 \int \Delta G_k(p_0) d^3p. \tag{29} $$

Now, in agreement with the results obtained in \cite{22}, one can observe the following. The relation \cite{22} can be rewritten

$$ g_1(x) = \frac{\kappa}{2} \int \Delta G(p_0) \left( m + p_1 + \frac{p_1^2}{p_0 + m} \right) \delta \left( \frac{p_0 + p_1}{M} - x \right) \frac{d^3p}{p_0}, \tag{30} $$

and after integration over $x$ one gets

$$ \Gamma_1 = \int g_1(x) dx = \frac{\kappa}{2} \int \Delta G(p_0) \left( \frac{1}{3} + \frac{2m}{3p_0} \right) d^3p = \kappa \left\langle n \Sigma \right\rangle, \tag{31} $$

where $n$ is the direction of the target polarization and $\left\langle n \Sigma \right\rangle$ represents the resulting projection of the spins coming from the individual fermions. In a similar way one gets:

$$ \int j(x) dx = \frac{1}{2} \int \Delta G(p_0) d^3p = \frac{1}{2}, \tag{32} $$

i.e. the integral represents the resulting projection of the total angular momentum. Further, one can easily check:

$$ \int s_L(x) dx = \int s_T(x) dx = \frac{1}{2} \int \Delta G(p_0) \left( \frac{1}{3} + \frac{2m}{3p_0} \right) d^3p = \frac{\Gamma_1}{\kappa}. \tag{33} $$

Moreover, the following relation is valid:

$$ \kappa \cdot s_T(x) = g_1(x) + g_2(x). \tag{34} $$

Let us note, despite our assumption that the target consists of just the three fermions, the suggested approach is more general. Since the spin functions are always based on the differences like \cite{8}, all the resulting relations are equally valid for any target consisting of the fermions with $\Delta G_k(p_0) \neq 0; \quad k = 1, 2, 3$, which are embedded in another system with compensated spins: $\Delta G_k(p_0) = 0; \quad k = 4, 5, 6, ...$

Now we can summarize:

i) The function $j(x)$ measures the contribution of total angular momenta (spin + orbital momentum) of the constituent fermions to the target spin.
ii) The functions $s_{L(T)}(x)$ measure the net spin contribution of the constituent fermions to the spin of target with longitudinal (transversal) polarization. Apparently, one can calculate also the corresponding densities of the orbital momentum as

$$l_{L(T)} = J(x) - s_{L(T)}(x).$$

(iii) Obviously the functions $g_1(x), j(x), s_L(x)$ and $s_T(x)$ are equivalent in the case of the static fermions, where orbital momentum does not play any role. The considered distribution functions $\Delta G$ and $H$ have rotational symmetry in the target rest frame, which is a necessary condition for the target with spin $J = 1/2$. It follows, that meaning of our function $j(x)$ suggested above does not depend on the orientation of the target polarization (longitudinal or transversal) with respect to the beam direction. Let us point out, the last statement can be deduced only in the framework of the relativistically covariant description, in which the rotational symmetry of the target is properly taken into account. At the same time, let us note that in general, the function $g_1(x)$ is not equivalent to the measure of the longitudinal spin density $s_L(x)$, only their integrals over $x$ are equal (up to factor $\kappa$).

### III. RELATIONSHIP AMONG THE SPIN FUNCTIONS

Before next discussion we shall first prove the following proposition:

The functions $V_n(x)$ defined as

$$V_n(x) = \int H(p_0) \left( \frac{p_0}{M} \right)^n \delta \left( \frac{p_0 + p_1}{M} - x \right) d^3p; \quad p_0 = \sqrt{m^2 + p^2}$$

satisfy

$$\frac{V'_j(x)}{V'_k(x)} = \left( \frac{x}{2} + \frac{x_0^2}{2x} \right)^{j-k}; \quad x_0 = \frac{m}{M}$$

for any powers $j, k$ and function $H$, for which the integral (36) exists. Proof of the last relation is given in the Appendix A.

### A. Spin structure functions $g_1(x), g_2(x)$

With the use of the relations (36), (37), as shown in the Appendix B, one can rewrite the Eqs. (23) and (24) as

$$g_2(x) = -\frac{1}{2} \left[ a(x)V_0(x) + \int_x^1 b(x,y)V_0(y)dy \right]$$

$$g_1(x) + g_2(x) = \frac{1}{2} \left[ A(x)V_0(x) + \int_x^1 B(x,y)V_0(y)dy \right]$$

where

$$a(x) = 2x - x_0 \frac{x}{x^2 + x_0^2}, \quad b(x,y) = \left[ \frac{3(x + x_0)^2}{(y + x_0)^4} - \frac{3x^2 + 2xx_0 + x_0^2}{(y^2 + x_0^2)^2} \right] \frac{y^2 - x_0^2}{x_0},$$

$$A(x) = \frac{2xx_0}{x^2 + x_0^2}, \quad B(x,y) = \left[ \frac{(x + x_0)^2}{(y + x_0)^4} + \frac{x^2 - x_0^2}{(y^2 + x_0^2)^2} \right] \frac{y^2 - x_0^2}{x_0}.$$  

Now, one can easily check, that in the limit $x_0 \to 0$ the relations (38), (39) are simplified:

$$g_2(x) = -V_0(x) + \int_x^1 \left( 6 \frac{x^2}{y^3} - 2 \frac{x}{y^2} \right) V_0(y)dy,$$
\[ g_1(x) + g_2(x) = \int_x^1 \left( 2 \frac{x^2}{y^2} - \frac{x}{y^2} \right) V_0(y) dy. \] (43)

These relations imply

\[ g_1(x) = V_0(x) - \int_x^1 \left( 4 \frac{x^2}{y^2} - \frac{x}{y^2} \right) V_0(y) dy \] (44)

and

\[ [g_1(x) + g_2(x)]' = \int_x^1 \left( 4 \frac{x}{y^2} - \frac{1}{y^2} \right) V_0(y) dy - \frac{V_0(x)}{x}. \] (45)

The combining of the last two relations gives

\[ [g_1(x) + g_2(x)]' = -\frac{g_1(x)}{x} \] (46)

or

\[ g_2(x) = -g_1(x) + \int_x^1 \frac{g_1(y)}{y} dy, \] (47)

which is the known expression for Wanzura - Wilczek twist-2 term for \( g_2 \) approximation [24].

Can we now obtain a similar relation for the case \( x_0 > 0 \), i.e. for the massive fermions? Let us combine Eqs. (38), (39) to the form

\[ g_1(x) + \left( 1 + \frac{A(x)}{a(x)} \right) g_2(x) = \frac{1}{2} \int_x^1 \left( B(x, y) - \frac{A(x)}{a(x)} b(x, y) \right) V_0(y) dy \] (48)

and let us try to express differentiation of r.h.s. as a combination of \( g_1 \) and \( g_2 \):

\[ \left[ \frac{1}{2} \int_x^1 \left( B(x, y) - \frac{A(x)}{a(x)} b(x, y) \right) V_0(y) dy \right]' = c_1(x) g_1(x) + c_2(x) g_2(x). \] (49)

In the Appendix C it is shown, that after inserting \( g_1, g_2 \) from Eqs. (38), (39) this equation is solvable for \( c_1(x), c_2(x) \), then after comparing with Eq. (48) we get

\[ \left[ g_1(x) + \left( 1 + \frac{A(x)}{a(x)} \right) g_2(x) \right]' = c_1(x) g_1(x) + c_2(x) g_2(x), \] (50)

where

\[ c_1(x) = -\frac{x^2 + 4xx_0 + x_0^2}{(x^2 - x_0^2)(x + 2x_0)}, \quad c_2(x) = -\frac{x_0 \left( x^2 + xx_0 + 4x_0^2 \right)}{(x - x_0)(x^2 - x_0^2)(x + 2x_0)}. \] (51)

In the same Appendix it is shown, that Eq. (50) implies

\[ g_2(x) = -\frac{x - x_0}{x} g_1(x) + \frac{x(x + 2x_0)}{(x + x_0)^2} \int_x^1 \frac{y^2 - x_0^2}{y^2} g_1(y) dy; \] (52)

\[ g_1(x) = -\frac{x}{x - x_0} g_2(x) - \frac{x + 2x_0}{x^2 - x_0} \int_x^1 g_2(y) dy. \] (53)

One can check, that for \( x_0 \to 0 \) both the last relations are equivalent to Eq. (46).

Further, in the limit \( x_0 \to 0 \) one can also easily calculate the momenta of the spin structure functions \( g_1, g_2 \). If we define

\[ \langle x^a \rangle = \int_0^1 x^a V_0(x) dx, \] (54)
then after integrating by parts the following relation is obtained:

\[
\int_{0}^{1} x^{\alpha} \int_{x}^{1} \frac{V_{0}(y)}{y^{\beta}} \, dy = \frac{\langle x^{\alpha-\beta+1} \rangle}{\alpha + 1}.
\] (55)

Application of this relation in Eqs. (43), (42) then gives

\[
\int_{0}^{1} x^{\alpha} \left[ g_{1}(x) + g_{2}(x) \right] \, dx = \langle x^{\alpha} \rangle \frac{\alpha + 1}{(\alpha + 2)(\alpha + 3)}.
\] (56)

\[
\int_{0}^{1} x^{\alpha} g_{2}(x) \, dx = -\langle x^{\alpha} \rangle \frac{\alpha (\alpha + 1)}{(\alpha + 2)(\alpha + 3)}
\] (57)

for any \( \alpha \), for which the integrals exist. The last two relations imply

\[
\int_{0}^{1} x^{\alpha} \left[ \frac{\alpha}{\alpha + 1} g_{1}(x) + g_{2}(x) \right] \, dx = 0,
\] (58)

which for \( \alpha = 2, 4, 6, \ldots \) corresponds to the Wanzura - Wilczek sum rules [24]. Other special cases correspond to the Burkhardt - Cottingham (\( \alpha = 0 \)) and the Efremov - Leader - Teryaev (ELT, \( \alpha = 1 \)) sum rules [25], [26].

B. Spin distributions \( j(x), s_{L}(x) \) and \( s_{T}(x) \)

Now, let us try to find the relations among the spin functions \( j(x), s_{L}(x), s_{T}(x) \) and the structure functions \( g_{1}, g_{2} \). For this purpose we only slightly change the definitions (25), (26), (27) in which we replace the function \( \Delta G(p_{0}) \) [Eq. (6), the sum of spin contributions from individual fermions] by the function \( H(p_{0}) \) [Eq. (3), spin contributions are weighted by the charges, see also the paragraph involving Eq. (29)]. The new relations read:

\[
\begin{align*}
 j(x) &= \frac{1}{2} \int H(p_{0}) \delta \left( \frac{p_{0} + p_{1}}{M} - x \right) \, d^{3}p, \\
 s_{L}(x) &= \frac{1}{2} \int H(p_{0}) \left( m + \frac{p_{1}^{2}}{p_{0} + m} \right) \delta \left( \frac{p_{0} + p_{1}}{M} - x \right) \, d^{3}p, \\
 s_{T}(x) &= \frac{1}{2} \int H(p_{0}) \left( m + \frac{p_{1}^{2}}{2(p_{0} + m)} \right) \delta \left( \frac{p_{0} + p_{1}}{M} - x \right) \, d^{3}p.
\end{align*}
\] (59) (60) (61)

Now, using a standard notation

\[
g_{T}(x) = g_{1}(x) + g_{2}(x),
\] (62)

we get from the relations (24) and (61) the equivalence

\[
g_{T}(x) = s_{T}(x).
\] (63)

Comparison of the relations (24) and (69) implies, that

\[
j(x) = \frac{1}{2} V_{0}(x).
\] (64)

In the Appendix D we have shown, that the distribution \( V_{0}(y) \) can be extracted from the relations (38), (39), so we get:

\[
j(x) = \frac{x^{2} + x_{0}^{2}}{2x^{2}} g_{1}(x) + \frac{3x^{2} + 2xx_{0} + 3x_{0}^{2}}{2(x + x_{0})^{2}} \int_{x}^{1} \frac{y^{2} - x_{0}^{2}}{y^{3}} g_{1}(y) \, dy + \int_{x}^{1} \ln \left( \frac{x(y + x_{0})^{2}}{y(x + x_{0})^{2}} \right) \frac{y^{2} - x_{0}^{2}}{y^{3}} g_{1}(y) \, dy,
\] (65)
\[ j(x) = -\frac{x^2 + x_0^2}{2x(x-x_0)}g_2(x) - \frac{1}{2(x^2 - x_0^2)} \int_x^1 \frac{6y^2x - 2yx^2 + x^2x_0 + 2yx_0^2 - x_0^3}{y^2} g_2(y) dy. \] (66)

Then for \( x_0 \to 0 \) we have:

\[ j(x) = \frac{1}{2} g_1(x) + \frac{1}{2} \int_x^1 \left( 3 + 2 \ln \frac{y}{x} \right) \frac{g_1(y)}{y} dy. \] (67)

\[ j(x) = -\frac{1}{2} g_2(x) - \frac{1}{x} \int_x^1 (3y - x) \frac{g_2(y)}{y} dy. \] (68)

Further, comparison of the relations (67) and (68) implies

\[ s_L(x) = g_1(x) - \frac{1}{2} \int H(p_0)p_1\delta \left( \frac{p_0 + p_1}{M} - x \right) \frac{d^3p}{p_0}. \] (69)

The last integral can be expressed by means of the function \( V_0 \) (see Appendix E):

\[ s_L(x) = \frac{1}{2} x^2 + \frac{x_0^2}{x} g_1(x) + \frac{x - 3x_0}{2(x + x_0)} \int_x^1 \frac{y^2 - x_0^2}{y^3} g_1(y) dy + \int_x^1 \ln \left( \frac{x(y + x_0)^2}{y(x + x_0)^2} \right) \frac{y^2 - x_0^2}{y^3} g_1(y) dy. \] (70)

A similar procedure gives also

\[ s_L(x) = -\frac{1}{2} x^2 + \frac{x_0^2}{x} g_2(x) + \int_x^1 \left( \frac{2y - x_0}{2y^2} - \frac{x + 2x_0}{x^2 - x_0^2} \right) g_2(y) dy \] (71)

One can easily check, that for \( x_0 \to 0 \) it follows

\[ s_L(x) = \frac{1}{2} g_1(x) + \int_x^1 \left( \ln \frac{y}{x} - \frac{1}{2} \right) \frac{g_1(y)}{y} dy, \] (72)

\[ s_L(x) = -\frac{1}{2} g_2(x) + \int_x^1 \left( \frac{1}{y} - \frac{1}{x} \right) g_2(y) dy. \] (73)

Finally, combining relations (72), (73) and (71) one gets

\[ g_T(x) = \frac{x_0}{x} g_1(x) + \frac{x(x + 2x_0)}{(x + x_0)^2} \int_x^1 \frac{y^2 - x_0^2}{y^3} g_1(y) dy, \] (74)

\[ g_T(x) = -\frac{x_0}{x - x_0} g_2(x) - \frac{x + 2x_0}{x^2 - x_0^2} \int_x^1 g_2(y) dy. \] (75)

and for \( x_0 \to 0 \) it follows

\[ g_T(x) = \int_x^1 \frac{g_1(y)}{y} dy = -\frac{1}{x} \int_x^1 g_2(y) dy. \] (76)

In the end of this section let us point out, that the simple relations above, which define mutual transformations among the functions \( g_1(x), g_2(x), j(x), s_L(x) \) and \( s_T(x) \) are obtained on the assumption that the fermions have some fixed effective mass \( x_0 \). In a more general case, for example when the structure functions are related to the system of fermions with some effective mass spectrum \([21]\), such simple transformations do not exist.
Now let us try to apply the suggested approach to the description of the proton spin structure. For simplicity, as in [22], we assume:

1) Spin contribution from the sea of quark-antiquark pairs and gluons can be neglected. Then the three fermions in our approach correspond to the three proton valence quarks. So, in this scenario, the proton spin is generated only by the valence quarks. Let us remark, in the cited paper we have shown that this assumption does not contradict the experimental value $\Gamma_1$.

2) In accordance with the non-relativistic $SU(6)$ approach, the spin contribution of individual valence terms is given as

$$s_u = \frac{4}{3}, \quad s_d = -\frac{1}{3}. \quad (77)$$

Let us denote momenta distributions of the valence quarks in the target rest frame by symbols $h_u$ and $h_d$ with the normalization

$$\frac{1}{2} \int h_u(p_0)d^3p = \int h_d(p_0)d^3p = 1, \quad (78)$$

then the generic distribution (3) reads

$$H(p_0) = \sum e_q^2 \Delta h_q(p_0) = \left(\frac{2}{3}\right)^2 \frac{2}{3} h_u(p_0) - \left(\frac{1}{3}\right)^2 \frac{1}{3} h_d(p_0). \quad (79)$$

In the papers [19], [21], using a similar approach, we have studied also the unpolarized structure functions. In particular we have suggested, that the structure function $F_2$, can be in the limit (19), expressed as

$$F_2(x) = x^2 \int K(p_0) \frac{M}{p_0} \delta \left(\frac{p_0 + p_1}{M} - x\right) d^3p; \quad K(p_0) = \sum_q e_q^2 h_q(p_0), \quad (80)$$

where $h_q$ are distributions of the quarks with charges $e_q$. For the valence quarks one can write

$$F_2(x) = \frac{4}{9} x u_V(x) + \frac{1}{9} x d_V(x), \quad (81)$$

then the Eq. (80) can be split:

$$q_V(x) = x \int h_q(p_0) \frac{M}{p_0} \delta \left(\frac{p_0 + p_1}{M} - x\right) d^3p; \quad q = u, d. \quad (82)$$

In an accordance with the definition (36), in which $h_q$ is inserted instead of $H$, one can write

$$q_V(x) = x V_1^q(x), \quad (83)$$

then the relation (37) implies

$$V_0^q(x) = \frac{1}{2} \left( x + \frac{x_0^2}{x} \right) V_1^q(x) + \frac{1}{2} \int_x^1 \left( 1 - \frac{x_0^2}{y^2} \right) V_1^q(x)dy, \quad (84)$$

which after inserting from Eq. (83) gives

$$V_0^q(x) = \frac{1}{2} \left[ \left( 1 + \frac{x_0^2}{x^2} \right) q_V(x) + \int_x^1 \left( 1 - \frac{x_0^2}{y^2} \right) \frac{q_V(y)}{y} dy \right]. \quad (85)$$

Obviously, the function $V_0(x)$ generated by distribution (74) according to the definition (37) can be decomposed

$$V_0(x) = \frac{8}{27} V_0^u(x) - \frac{1}{27} V_0^d(x) \quad (86)$$

and if we define
\[ w_g(x) = \frac{8}{27} w_V(x) - \frac{1}{27} d_V(x), \]  
(87)

then one can check, that inserting \( V_0 \) from the relation (83) to the relations (88), (89) with the use of Eqs. (85), (87) gives:

\[ g_1(x) = \frac{1}{2} \left[ w_g(x) - 2 (x + x_0)^2 \int_{x}^{1} \frac{y - x_0}{(y + x_0)^3} \frac{w_g(y)}{y} \, dy \right], \]  
(88)

\[ g_2(x) = \frac{1}{2} \left[ -(1 - \frac{x_0}{x}) w_g(x) + 3 (x + x_0)^2 \int_{x}^{1} \frac{y - x_0}{(y + x_0)^3} \frac{w_g(y)}{y} \, dy \right]. \]  
(89)

Obviously, the structure functions can be split into the two parts, corresponding to \( u \) and \( d \) quarks

\[ g_j(x) = \left( \frac{2}{3} \right)^2 \frac{2}{3} g_j^u(x) - \left( \frac{1}{3} \right)^2 \frac{1}{3} g_j^d(x); \quad j = 1, 2, \]  
(90)

where the partial structure functions read:

\[ g_1^q(x) = \frac{1}{2} \left[ q_V(x) - 2 (x + x_0)^2 \int_{x}^{1} \frac{y - x_0}{(y + x_0)^3} \frac{q_V(y)}{y} \, dy \right], \]  
(91)

\[ g_2^q(x) = \frac{1}{2} \left[ -(1 - \frac{x_0}{x}) q_V(x) + 3 (x + x_0)^2 \int_{x}^{1} \frac{y - x_0}{(y + x_0)^3} \frac{q_V(y)}{y} \, dy \right]; \quad q = u, d. \]  
(92)

Now we can express the corresponding contributions of different quarks to the spin distribution functions. Apparently

\[ s^q_L(x) = g_1^q(x) + g_2^q(x) = \frac{1}{2} \left[ \frac{x_0}{x} q_V(x) + (x + x_0)^2 \int_{x}^{1} \frac{y - x_0}{(y + x_0)^3} \frac{q_V(y)}{y} \, dy \right]. \]  
(93)

Further, after inserting from the relations (86), (88), (89) to the Eq. (91) one easily gets

\[ s^q_L(x) = \frac{1}{4} \left[ \left( 1 + \frac{x_0^2}{x^2} \right) q_V(x) - 4 (x + x_0)^2 \int_{x}^{1} \frac{y - x_0}{(y + x_0)^3} \frac{q_V(y)}{y} \, dy + \int_{x}^{1} \left( 1 - \frac{x_0^2}{y^2} \right) \frac{q_V(y)}{y} \, dy \right]. \]  
(94)

Let us remark, the Eq. (91) is obtained from the generic distribution \( H(p_0) \), in its place we now have the distribution \( q_V(p_0) \). Similarly, a comparison of the relations (74) and (88), (89) gives

\[ j^q(x) = \frac{1}{4} \left[ \left( 1 + \frac{x_0^2}{x^2} \right) q_V(x) - \int_{x}^{1} \left( 1 - \frac{x_0^2}{y^2} \right) \frac{q_V(y)}{y} \, dy \right]. \]  
(95)

Now, the net complete spin distributions can be obtained by adding individual valence terms with the weights (77), taking into account their normalization (78). If we define

\[ w_s(x) = \frac{2}{3} w_V(x) - \frac{1}{3} d_V(x), \]  
(96)

then the complete spin distributions can be obtained from the relations (83) - (85), in which the distribution \( q_V \) is replaced by \( w_s \). Then for \( x_0 \to 0 \) we obtain:

\[ g_1(x) = \frac{1}{2} \left[ w_s(x) - 2 x^2 \int_{x}^{1} \frac{w_s(y)}{y^3} \, dy \right], \]  
(97)

\[ g_2(x) = \frac{1}{2} \left[ -w_s(x) + 3 x^2 \int_{x}^{1} \frac{w_s(y)}{y^3} \, dy \right], \]  
(98)
\[ s_T(x) = \frac{1}{2} x^2 \int_x^1 \frac{w_s(y)}{y^3} dy, \] 

(99)

\[ s_L(x) = \frac{1}{4} \left[ w_s(x) - 4x^2 \int_x^1 \frac{w_s(y)}{y^3} dy + \int_x^1 \frac{w_s(y)}{y} dy \right], \] 

(100)

\[ j(x) = \frac{1}{4} \left[ w_s(x) + \int_x^1 \frac{w_s(y)}{y} dy \right]. \] 

(101)

Let us note, if one assumes \( u_V(x) \approx 2dV(x) \), then the following substitution can be used:

\[ w_g(x) \approx \frac{5}{9} F^2_{vol} x, \quad w_s(x) \approx \frac{F^2_{vol} x}{x}. \] 

(102)

Further, let us make a remark to the normalization of the distributions above. To simplify this consideration, we assume the case \( x_0 \to 0 \). The relation \((83)\) implies

\[ \int_0^1 q_V(x) dx = \int_0^1 x V^q_1(x) dx. \] 

(103)

Since the relation \((37)\) implies

\[ V^q_1(x) = \frac{2}{x} V^q_0(x) - 2 \int_x^1 \frac{V^q_0(y)}{y^2} dy, \] 

(104)

then using the relation \((D7)\) one gets

\[ \int_0^1 x V^q_1(x) dx = \int_0^1 V^q_0(x) dx. \] 

(105)

From the definition

\[ V^q_0(x) = \int h_q(p_0) \delta \left( \frac{p_0 + p_1}{M} - x \right) d^3 p \] 

(106)

one obtains

\[ \int_0^1 V^q_0(x) dx = \int h_q(p_0) d^3 p. \] 

(107)

This relation combined with \((105)\) and \((103)\) gives

\[ \int_0^1 q_V(x) dx = \int h_q(p_0) d^3 p, \] 

(108)

which in an accordance with the normalization \((78)\) implies

\[ \frac{1}{2} \int_0^1 u_V(x) dx = \int_0^1 d_V(x) dx = 1. \] 

(109)

Now, one can also check the normalization of the functions \((97) - (101)\). Taking into account, that

\[ \int_0^1 w_s(x) dx = 1, \quad \int_0^1 w_g(x) dx = \frac{5}{9}, \] 

(110)

then after integration with the use of relation \((D7)\) one gets

\[ \int_0^1 j(x) dx = \frac{1}{2}, \quad \int_0^1 s_T(x) dx = \int_0^1 s_L(x) dx = \frac{1}{6}. \] 

(111)
The meaning of these integrals was discussed in Sec. 11. The first integral (112) represents the sum rule on the total proton angular momentum $J = l + s = 1/2$, which has been discussed e.g. in [27]. The integrals on the net spin contributions $s_T, s_L$ are correlated with the $\Gamma_1$, which reaches its minimal value for $x_0 \to 0$, as we have discussed in [23], see also Eq. (31) above.

So, the obtained formulas enable us to calculate the spin functions from the input, in which only the valence distributions are used. For simplicity we shall now consider only massless quarks ($x_0 \to 0$) and for the valence functions $w_{V}(x)$ and $d_{V}(x)$ we use the parameterization obtained (for $Q^2 = 4 GeV^2/c^2$) by the standard global analysis in [28]. In the Fig. 1a we have shown the result of our calculation for $g_1$ according to Eq. (67) together with the experimental data represented by the new parameterization of the world data on $g_1$ for $Q^2 = 4 GeV^2/c^2$.

The calculation qualitatively agrees well with the data, however it is apparent, that the data are above our curve. This can be connected first of all with our simplification for $x_0 \to 0$, when the $\Gamma_1$ is minimal. In accordance with Eq. (112) we obtain $\Gamma_1 = 0.093$, but experimentally $\Gamma_1^0 = 0.118 \pm 0.004(stat.) \pm 0.007(syst.)$ at $Q^2 = 5 GeV^2/c^2$. Just this difference is exposed in the figure. Further, in the Fig. 1b we have shown the $g_2$ according to Eq. (68) and the precision measurement recently published by the E155 Collaboration [3]. The agreement with the data is very good and it can suggest, that dependence of the function $g_2$ on the mass terms is rather weak. At least in our approach $\Gamma_1$ does depend on mass, but $\Gamma_2 = 0$ regardless of the mass. In the Fig. 2 the corresponding spin distributions $j, s_T, s_L$ are shown for whole proton and also separately for $u$ and $d$ valence quarks corresponding to the assumed $SU(6)$ symmetry, which gives the fractions (72). Figure 1b and the left part of Fig. 2 also demonstrate, that the $g_1(x)$ and $s_L(x)$ are not equivalent. Slightly different shape of the distributions on $s_T, s_L$ is due to variable $x$, in which longitudinal and transversal (in respect to the beam) quark momentum components are not involved in a symmetric way. Otherwise, for given direction of the proton polarization, quark spin density cannot depend on the direction, in which the probing beam is coming.

In the end, let us remark, that another possible effect, which can in our approach contribute to an underestimation of $\Gamma_1$ is connected with the assumption (73). For example, if one assumes full spin alignment of the $u$ valence quarks, then

$$s_u = 2, \quad s_d = -1$$

and instead of the generic distribution (67) one gets

$$w_g(x) = \frac{4}{9} w_{V}(x) - \frac{1}{9} d_{V}(x), \quad \int_0^1 w_g(x) dx = \frac{7}{9},$$

which implies $\Gamma_1 = 7/54 \approx 0.13$. Obviously, assuming isotopic symmetry, the same procedure can also be used for the neutron. For the isotopic counterparts of the compositions (73) and (113) one gets $\Gamma_1^0 = 0$ and $\Gamma_0^0 = -1/27 \approx -0.037$, respectively. The composition (113) gives the maximum value $\Gamma_1$ for proton and minimum for neutron.

V. SUMMARY AND CONCLUSION

With the use of a consistently covariant version of the naive QPM we have studied the spin structure functions together with the spin density distributions for the system of quasifree fermions having fixed effective mass $m/M$ and the total spin $J = 1/2$. The main results can be summarized as follows.

1. We have shown that the corresponding spin structure functions $g_1(x)$ and $g_2(x)$ are mutually connected by a simple transformation. At the limit $x_0 \to 0$ this transformation is identical to the Wanzura - Wilczek relation for the twist-2 term of the $g_2(x)$ approximation. At the same time for $x_0 \to 0$ the relations for the $n - th$ momenta of the structure functions have been obtained. Their particular cases are identical to the known sum rules: Wanzura - Wilczek ($n = 2, 4, 6...$), Efremov - Leader - Teryaev ($n = 1$) and Burkhardt - Cottingham ($n = 0$). Further, we have shown how the structure functions are connected with the net spin densities $s_L(x), s_T(x)$ and with the density of the total angular momentum $j(x)$.

2. The proposed approach has been applied to the description of the proton spin structure with the assumption that the proton spin is generated only by the spins and orbital momenta of the valence quarks. Apart from that we have assumed that the spin contributions from $u$ and $d$ valence quarks can be defined by the $SU(6)$ symmetry and for quark effective mass we used approximation $x_0 \to 0$. We have suggested, how one can in this approach obtain the spin functions from the valence quark distributions. Then as an input we have used parameterization of the valence
terms resulting from the standard global analysis. On this basis, without any other free parameter, the proton spin structure functions and related spin densities have been calculated. Comparison of the obtained structure functions $g_1(x)$ and $g_2(x)$ with the experimental measurement demonstrates that the suggested approach well reproduces the basic features of the data on the proton spin structure.

To conclude, the results presented in this paper and discussion in [22] suggest, that both the proton structure functions $g_1$ and $g_2$ have a simple and natural interpretation even in terms of a naive QPM, provided that the model is based on a consistently covariant formulation, which takes into account spheric symmetry connected with the constraint $J = 1/2$. This is not satisfied for the standard formulation of QPM, which is based on the one-dimensional kinematics related only to the preferred reference system (infinite momentum frame). As a result, there is e.g. the known fact, that the function $g_2(x)$ has no well-defined meaning in the standard naive QPM. In this case it is just a result of the simplified kinematics and not because of an absence of dynamics.

Acknowledgements I would like to thank Anatoli Efremov and Oleg Teryaev for many useful discussions and valuable comments.

APPENDIX A: PROOF OF THE RELATION (37)

The integral

$$F(x) = \int K(p_0)\delta \left( \frac{p_0 + p_1}{M} - x \right) d^3 p; \quad p_0 = \sqrt{m^2 + p^2}$$

after the substitution

$$p_2 = \sqrt{p_0^2 - p_1^2 - m^2} \sin \varphi, \quad p_3 = \sqrt{p_0^2 - p_1^2 - m^2} \cos \varphi$$

reads:

$$F(x) = 2\pi \int_{E_{\text{max}}}^{E_{\text{max}}} K(p_0)p_0 \left[ \int_{-H}^{H} \delta \left( \frac{p_0 + p_1}{M} - x \right) dp_1 \right] dp_0; \quad H = \sqrt{p_0^2 - m^2}.$$ (A3)

For given $x$ and $p_0$ the inner integral contributes only for

$$p_1 = Mx - p_0$$ (A4)

in the limits

$$-\sqrt{p_0^2 - m^2} \leq p_1 \leq \sqrt{p_0^2 - m^2}.$$ (A5)

One can check, that the last two conditions are compatible only for

$$p_0 \geq \xi \equiv \frac{Mx}{2} + \frac{m^2}{2Mx}.$$ (A6)

It follows, that Eq. (A3) can be simplified

$$F(x) = 2\pi M \int_{\xi}^{E_{\text{max}}} K(p_0)p_0 dp_0.$$ (A7)

According to the relation (A6) the parameter $\xi$ is a function of $x$ with a minimum at $x_0 = m/M$, so for the fixed $\xi$ there are the two roots of $x$,

$$x_\pm = \frac{\xi \pm \sqrt{\xi^2 - m^2}}{M},$$ (A8)

so the Eq. (A7) can be rewritten:

$$F\left( \frac{\xi \pm \sqrt{\xi^2 - m^2}}{M} \right) = 2\pi M \int_{\xi}^{E_{\text{max}}} K(p_0)p_0 dp_0.$$ (A9)
Then differentiation in respect to $\xi$ gives
\[
F'(\frac{\xi \pm \sqrt{\xi^2 - m^2}}{M}) \left( \frac{1}{M} \pm \frac{\xi}{M \sqrt{\xi^2 - m^2}} \right) = -2\pi MK(\xi)\xi
\] (A10)
and this relation can be, with the use of Eq. (A8), applied to the functions (9):
\[
V'_n(x_\pm)x_\pm = \mp 2\pi M^2 H(\xi)\xi \sqrt{\xi^2 - m^2} \left( \frac{\xi}{M} \right)^n,
\] (A11)
which with the use of the relation (A6) implies the relation (37).

**APPENDIX B: PROOF OF THE RELATIONS (38),(39)**

In the relations (22) - (24) one can, due to the $\delta-$ function, make the following substitutions. First,
\[
p_1 = Mx - p_0
\] (B1)
and then from the relation
\[
p_1^2 = p_0^2 - p_1^2 - m^2
\] (B2)
one gets
\[
p_1^2 = 2Mxp_0 - (M^2x^2 + m^2).
\] (B3)
Now the relation (24) can be rewritten:
\[
g_1(x) + g_2(x) = \frac{1}{2} \left[ x_0 V_{-1}(x) + \int H(p_0) \left( \frac{Mxp_0 - (M^2x^2 + m^2) / 2}{p_0 (p_0 + m)} \right) \delta \left( \frac{p_0 + p_1}{M} - x \right) d^3p \right].
\] (B4)
In the next step we expand the fraction
\[
\frac{1}{p_0 + m} = \frac{1}{p_0} \sum_{j=0}^{\infty} \left( -\frac{m}{p_0} \right)^j
\] (B5)
then the relation (B4) can be expressed in terms of the functions (6)
\[
g_1(x) + g_2(x) = \frac{1}{2} \left[ x_0 V_{-1}(x) - \frac{x}{x_0} \sum_{j=1}^{\infty} (-x_0)^j V_{-j}(x) - \frac{x^2 + x_0^2}{2x_0} \sum_{j=2}^{\infty} (-x_0)^j V_{-j}(x) \right]
\] (B6)
\[
= \frac{1}{2} \left[ (x + x_0) V_{-1}(x) - \frac{(x + x_0)^2}{2x_0^2} \sum_{j=2}^{\infty} (-x_0)^j V_{-j}(x) \right].
\]
Further, from the relation (B7) one obtains
\[
V_{-j}(x) = -\int_x^1 \left( \frac{2y}{y^2 + x_0^2} \right)^j V'_0(y) dy
\] (B7)
and because
\[
\sum_{j=2}^{\infty} \left( - \frac{2x_0y}{y^2 + x_0^2} \right)^j = \frac{4x_0^2y^2}{(y + x_0)^2(y^2 + x_0^2)}
\] (B8)
the relation (B6) can be modified
\[
g_1(x) + g_2(x) = \frac{1}{2} \left[ - (x + x_0) \int_x^1 \frac{2y}{y^2 + x_0^2} V_0'(y) dy + \frac{(x + x_0)^2}{2x_0^2} \int_x^1 \frac{4x_0^2y^2}{(y + x_0)^2 (y^2 + x_0^2)} V_0'(y) dy \right]. \quad (B9)
\]

Now, the integration by parts gives
\[
g_1(x) + g_2(x) = \frac{1}{2} \left[ (x + x_0) \frac{2x}{x^2 + x_0^2} - \frac{(x + x_0)^2}{2x_0^2} \frac{4x_0^2 x^2}{(x + x_0)^2 (x^2 + x_0^2)} \right] V_0(x) \quad (B10)
\]
and one can check, that after some modifications, the corresponding terms ahead of \( V_0 \) coincide with the functions \([\mathbb{1}]\). The relation \([\mathbb{5}]\) can be proved by the similar procedure, so we suggest only main steps:

\[
g_2(x) = -\frac{1}{2} \left[ xV_{-1}(x) - V_0(x) + \int H(p_0) \left( \frac{p_0^2 - 3Mx_0p_0 + (3M^2x^2 + m^2)/2}{p_0 (p_0 + m)} \right) \delta \left( \frac{p_0 + p_1}{M} - x \right) d^3p \right],
\]

\[
g_2(x) = -\frac{1}{2} \left[ xV_{-1}(x) - V_0(x) + \sum_{j=0}^{\infty} (-x_0)^j V_{-j}(x) + 3 \frac{x}{x_0} \sum_{j=1}^{\infty} (-x_0)^j V_{-j}(x) + \frac{3x^2 + x_0^2}{2x_0^2} \sum_{j=2}^{\infty} (-x_0)^j V_{-j}(x) \right]
\]

\[
= -\frac{1}{2} \left[ - (2x + x_0) V_{-1}(x) + \frac{3 (x + x_0)^2}{2x_0^2} \sum_{j=2}^{\infty} (-x_0)^j V_{-j}(x) \right],
\]

\[
g_2(x) = -\frac{1}{2} \left[ (2x + x_0) \int_x^1 \frac{2y}{y^2 + x_0^2} V_0'(y) dy - \frac{3 (x + x_0)^2}{2x_0^2} \int_x^1 \frac{4x_0^2y^2}{(y + x_0)^2 (y^2 + x_0^2)} V_0'(y) dy \right].
\]

Then, the integration by parts gives:
\[
g_2(x) = -\frac{1}{2} \left[ - (2x + x_0) \frac{2x}{x^2 + x_0^2} + \frac{3 (x + x_0)^2}{2x_0^2} \frac{4x_0^2 x^2}{(x + x_0)^2 (x^2 + x_0^2)} \right] V_0(x)
\]

\[
-\frac{1}{2} \left[ \int_x^1 \left( 2x + x_0 \right) \frac{2 \left( y^2 - x_0^2 \right)}{(y^2 + x_0^2)^2} - \frac{3 (x + x_0)^2}{2x_0^2} \frac{8x_0^2 y \left( y^3 - x_0^3 \right)}{(y^2 + x_0^2)^2 (y + x_0)^2} \right] V_0(y) dy \right]
\]
and one can check, that the corresponding terms ahead of \( V_0 \) coincide with the functions \([\mathbb{4}]\).

**APPENDIX C: PROOF OF THE RELATIONS [50] - [53]**

First, if we define the functions
\[
f_1(y) = \frac{y_0^2 - x_0^2}{(y + x_0)^2} x_0, \quad f_2(y) = \frac{y_0^2 - x_0^2}{(y^2 + x_0^2)^2} x_0,
\]
\[
f_3(x) = \frac{A(x)}{a(x)} = \frac{x_0}{x - x_0}, \quad f_4(x) = A(x) + a(x) = \frac{2x^2}{x^2 + x_0^2},
\]
then taking into account Eqs. \([\mathbb{3}]\) - \([\mathbb{4}]\), the Eq. \([\mathbb{43}]\) can be rewritten

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\[
\left[ -(x + x_0)^2 \int_x^1 f_1(y)V_0(y)dy + (x^2 - x_0^2) \int_x^1 f_2(y)V_0(y)dy \right]'
\]

\[
+ \left[ -f_3(x)3 (x + x_0)^2 \int_x^1 f_1(y)V_0(y)dy + f_3(x) \left( 3x^2 + 2xx_0 + x_0^2 \right) \int_x^1 f_2(y)V_0(y)dy \right]'
\]

\[
= c_1(x)f_4(x)V_0(x) + c_1(x)2 (x + x_0)^2 \int_x^1 f_1(y)V_0(y)dy - c_1(x)2 \left( x^2 + xx_0 + x_0^2 \right) \int_x^1 f_2(y)V_0(y)dy
\]

\[-c_2(x)a(x)V_0(x) - c_2(x)3 (x + x_0)^2 \int_x^1 f_1(y)V_0(y)dy + c_2(x) \left( 3x^2 + 2xx_0 + x_0^2 \right) \int_x^1 f_2(y)V_0(y)dy.
\]

After differentiating of the l.h.s. one gets

\[
-2 (x + x_0) \int_x^1 f_1(y)V_0(y)dy + 2x \int_x^1 f_2(y)V_0(y)dy
\]

\[-3x_0 \frac{x^2 - 2xx_0 - 3x_0^2}{(x - x_0)^2} \int_x^1 f_1(y)V_0(y)dy + 3x_0 \frac{x^2 - 2xx_0 - x_0^2}{(x - x_0)^2} \int_x^1 f_2(y)V_0(y)dy
\]

\[+(x + x_0)^2 f_1(x) - (x^2 - x_0) f_2(x) + f_3(x)3 (x + x_0)^2 f_1(x) - f_3(x) \left( 3x^2 + 2xx_0 + x_0^2 \right) f_2(x) \int_x^1 f_0(x)
\]

\[= c_1(x)\frac{2x^2}{x^2 + x_0} V_0(x) + c_1(x)2 (x + x_0)^2 \int_x^1 f_1(y)V_0(y)dy - c_1(x)2 \left( x^2 + xx_0 + x_0^2 \right) \int_x^1 f_2(y)V_0(y)dy
\]

\[-c_2(x)2x \frac{x - x_0}{x^2 + x_0} V_0(x) - c_2(x)3 (x + x_0)^2 \int_x^1 f_1(y)V_0(y)dy + c_2(x) \left( 3x^2 + 2xx_0 + x_0^2 \right) \int_x^1 f_2(y)V_0(y)dy.
\]

This equality contains the three linearly independent terms

\[
V_0(x), \quad \int_x^1 f_1(y)V_0(y)dy, \quad \int_x^1 f_2(y)V_0(y)dy
\]

and comparison of these terms on both the sides gives the equations:

\[
c_1(x)x - c_2(x)(x - x_0) = -\frac{x + 2x_0}{x + x_0}, \quad (C6)
\]

\[
2c_1(x) - 3c_2(x) = -\frac{2x^2 - xx_0 - 7x^2}{(x + x_0)(x - x_0)^2}, \quad (C7)
\]

\[
2c_1(x) \left( x^2 + xx_0 + x_0^2 \right) - c_2(x) \left( 3x^2 + 2xx_0 + x_0^2 \right) = -\frac{2x^3 - x^2x_0 - 4xx_0^2 - 3x_0^3}{(x - x_0)^2}. \quad (C8)
\]

One can check, that these three equations with the two unknown $c_1, c_2$ are dependent and solvable, giving the solution $[11]$. Now, we can solve differential equation $[50]$ for $g_1$ or $g_2$. Its homogenous version for $g_1$ reads:

\[
g_1'(x) + \frac{x^2 + 4xx_0 + x_0^2}{(x^2 - x_0^2)(x + 2x_0)}g_1(x) = 0, \quad (C9)
\]
which has the solution
\[ g_1(x) = C \frac{x^2 + 2x_0}{x^2 - x_0^2} \] (C10)
Nonhomogeneous term [the part of Eq. (50) involving \( g_2 \)] gives the equation for the function \( C(x) \)
\[ C'(x) \frac{x^2 + 2x_0}{x^2 - x_0^2} = -\frac{x_0 (x^2 + xx_0 + 4x_0^2)}{(x - x_0) (x^2 - x_0^2) (x + 2x_0)} g_2(x) - \left( \frac{x}{x - x_0} g_2(x) \right)' \] (C11)
which has the solution
\[ C(x) = -\frac{x + x_0}{x + 2x_0} g_2(x) - \int_x^1 g_2(y) dy \] (C12)
After inserting into Eq. (C10) one gets the relation (53). The inverse relation (52) can be proved in a similar way.

**APPENDIX D: PROOF OF THE RELATIONS (65), (66)**

The relations (38) and (39)
\[ g_2(x) = -\left[ \frac{x - x_0}{x^2 + x_0^2} V_0(x) + \int_x^1 \left( \frac{3 (x + x_0)^2}{(y + x_0)^4} - \frac{3x^2 + 2xx_0 + x_0^2}{(y + x_0^2)^4} \right) \frac{y^2 - x_0^2}{2x_0} V_0(y) dy \right] \]
\[ g_1(x) + g_2(x) = \left[ \frac{xx_0}{x^2 + x_0^2} V_0(x) + \int_x^1 \left( \frac{(x + x_0)^2}{(y + x_0)^4} + \frac{x^2 - x_0^2}{(y^2 + x_0^2)^4} \right) \frac{y^2 - x_0^2}{2x_0} V_0(y) dy \right] \]
can be combined in such a way that the second terms in the integrals cancel:
\[ (3x^2 + 2xx_0 + x_0^2) g_1(x) + 2 (x^2 + xx_0 + x_0^2) g_2(x) \]
\[ = (x + 2x_0) x V_0(x) - (x + 2x_0) (x + x_0)^2 \int_x^1 \frac{y^2 - x_0^2}{(y + x_0)^4} V_0(y) dy. \] (D1)

Further, this equation can be modified
\[ \frac{x}{(x + x_0)^2} V_0(x) - \int_x^1 \frac{y^2 - x_0^2}{(y + x_0)^4} V_0(y) dy = \frac{3x^2 + 2xx_0 + x_0^2}{(x + 2x_0) (x + x_0)^2} g_1(x) + \frac{2 (x^2 + xx_0 + x_0^2)}{(x + 2x_0) (x + x_0)^2} g_2(x), \]
which implies a differential equation for \( V_0(x) \):
\[ \left( \frac{x}{(x + x_0)^2} V_0(x) \right)' + \frac{x^2 - x_0^2}{(x + x_0)^2} V_0(x) = \left( \frac{3x^2 + 2xx_0 + x_0^2}{(x + 2x_0) (x + x_0)^2} g_1(x) + \frac{2 (x^2 + xx_0 + x_0^2)}{(x + 2x_0) (x + x_0)^2} g_2(x) \right)' \] (D2)
The corresponding homogenous equation
\[ \frac{x}{(x + x_0)^2} V_0'(x) = 0 \] (D3)
gives the solution
\[ V_0(x) = C \] (D4)
and for \( C(x) \) we have the equation
\[ \frac{x}{(x + x_0)^2} C'(x) = \left( \frac{3x^2 + 2xx_0 + x_0^2}{(x + 2x_0) (x + x_0)^2} g_1(x) + \frac{2 (x^2 + xx_0 + x_0^2)}{(x + 2x_0) (x + x_0)^2} g_2(x) \right)' \] (D5)
The solution reads:

\[ V_0(x) = C(x) = \frac{3x^2 + 2xx_0 + x_0^2}{x + 2x_0} g_1(x) + \frac{2 \left( x^2 + xx_0 + x_0^2 \right)}{(x + 2x_0) x} g_2(x) \]  

\[ + \int_x^1 \frac{(3y^2 + 2yx_0 + x_0^2) (y - x_0)}{(y + x_0) (y + 2x_0) y^2} g_1(y) dy + \int_x^1 \frac{2 (y^2 + yx_0 + x_0^2) (y - x_0)}{(y + x_0) (y + 2x_0) y^2} g_2(y) dy. \]

After inserting from Eq. (53) one gets

\[ V_0(x) = \frac{3x^2 + 2xx_0 + x_0^2}{x + 2x_0} g_1(x) + \frac{2 \left( x^2 + xx_0 + x_0^2 \right)}{(x + 2x_0) x} \left( \frac{x - x_0}{x} g_1(x) + \frac{x (x + 2x_0)}{(x + 2x_0) x} \int_x^1 \frac{y^2 - x_0^2}{y} g_1(y) dy \right) \]

\[ + \int_x^1 \frac{(3y^2 + 2yx_0 + x_0^2) (y - x_0)}{(y + x_0) (y + 2x_0) y^2} g_1(y) dy \]

\[ + \int_x^1 \frac{2 (y^2 + yx_0 + x_0^2) (y - x_0)}{(y + x_0) (y + 2x_0) y^2} \left( \frac{y - x_0}{y} g_1(y) + \frac{y (y + 2x_0)}{(y + 2x_0) y} \int_y^1 \frac{z^2 - x_0^2}{z^3} g_1(z) dz \right) dy, \]

The double integral is calculated according to the formula

\[ \int_x^1 a(y) \left( \int_y^1 b(z) dz \right) dy = \int_x^1 (A(y) - A(x)) b(y) dy; \quad A'(x) = a(x), \]  

\[ (D7) \]

then after collecting the corresponding terms with \( g_i \) we obtain with the use of Eq. (64) the relation (65). A similar procedure with inserting from Eq. (53) into Eq. (66) gives the relation (66).

**APPENDIX E: PROOF OF THE RELATIONS (70), (71)**

In the relation (69) one can, due to the \( \delta \) function, make the substitution \( p_1 = Mx - p_0 \). Then, using the definition (60), one obtains

\[ s_{L}(x) = g_1(x) + \frac{1}{2} V_0(x) - \frac{1}{2} x V_{-1}(x). \]  

\[ (E1) \]

Further, the relation (37) implies

\[ V_{-1}(x) = \frac{2x}{x^2 + x_0^2} V_0(x) - 2 \int_x^1 \frac{y^2 - x_0^2}{(y^2 + x_0^2)^2} V_0(y) dy, \]  

\[ (E2) \]

which after inserting into Eq. (E1) gives

\[ s_{L}(x) = g_1(x) - \frac{x^2 - x_0^2}{2 (x^2 + x_0^2)} V_0(x) + x \int_x^1 \frac{y^2 - x_0^2}{(y^2 + x_0^2)^2} V_0(y) dy. \]  

\[ (E3) \]

After inserting \( V_0 = 2j(x) \) and using the relation (55) we obtain

\[ s_{L}(x) = \frac{1}{2} \frac{x^2 + x_0^2}{x^2} g_1(x) - \frac{x^2 - x_0^2}{2 (x^2 + x_0^2)} \frac{3x^2 + 2xx_0 + 3x_0^2}{(x + x_0)^2} \int_x^1 \frac{y^2 - x_0^2}{y^3} g_1(y) dy \]

\[ - \frac{x^2 - x_0^2}{x^2 + x_0^2} \int_x^1 \ln \left( \frac{x (y + x_0)^2}{y (x + x_0)^2} \right) \frac{y^2 - x_0^2}{y^3} g_1(y) dy + x \int_x^1 \frac{y^2 - x_0^2}{(y^2 + x_0^2) y} g_1(y) dy. \]
\begin{equation}
+ x \int_{x}^{1} \frac{y^2 - x_0^2}{(y^2 + x_0^2)^2} \frac{3y^2 + 2yx_0 + 3x_0^2}{(y + x_0)^2} \int_{y}^{1} \frac{z^2 - x_0^2}{z^3} g_1(z) dz dy
\end{equation}

\begin{equation}
+ 2x \int_{x}^{1} \frac{y^2 - x_0^2}{(y^2 + x_0^2)^2} \int_{y}^{1} \ln \left( \frac{(y + x_0)^2}{z (y + x_0)^2} \right) \frac{z^2 - x_0^2}{z^3} g_1(z) dz dy.
\end{equation}

Then calculation of the double integrals with the use of relation (D7) and collecting the corresponding terms with \( g_1 \) give the relation (70).

Further, one can insert \( g_1 \) from the relation (63) into relation (70), then a similar procedure with double integrals gives the relation (71).
FIG. 1. Proton spin structure functions. Our calculation, which is represented by the full lines, is compared with the experimental data: dashed line \( g_1 \) and full circles \( g_2 \).

FIG. 2. Calculation of the spin densities of the valence quarks inside the proton (left) and separately the contributions from \( u \) and \( d \) quarks (right). Full lines represent the total angular momenta \( j \), dotted and dashed lines correspond to longitudinal and transversal densities \( s_L \) and \( s_T \).