PROOF OF CLOZEL’S FINITENESS CONJECTURE OF SPECIAL EXPONENTS: A REDUCTION STEP

CAIHUA LUO

Abstract. In this paper, via Casselman–Tadic’s Jacquet module machine, we establish a “product formula” of the cardinality of the Jordan–Holder series of generalized principal series. As a byproduct, we prove Clozel’s finiteness conjecture of the exponents of discrete series, i.e. the so-called special exponents under the assumption that it holds for the relative rank-one case.

INTRODUCTION

Around 1980s in [Clo85], in his first attempt to prove Howe’s finiteness conjecture, Clozel proposed the following beautiful finiteness conjecture of special exponents.

There is no harm to assume that $G$ is of compact center. For a discrete series representation $\pi$ of $G$, we denote its associated supercuspidal support to be $\sigma$ which is a supercuspidal representation of some Levi subgroup $M$ of $G$. Denote by $\omega_\sigma$ the unramified part of the central character of $\sigma$, i.e. $\omega_\sigma \in a^*_M, C$. Such a character is called a special exponent.

Clozel’s finiteness conjecture. The set of special exponents is finite.

With the help of Bernstein–Zelevinsky’s monumental work on the classification of admissible representations of $GL_n$ (cf. [BZ77, Ze80]) and the work of Bernstein–Deligne–Kazhdan–Vignéras [BDKV84], Clozel showed that Clozel’s finiteness conjecture holds for type A groups (cf. [Clo85, §6]). In view of another far-reaching work done by Moeglin–Tadic on the classification of discrete series of classical groups about 20 years later (cf. [MT02, Moe, Moe03]), Clozel’s conjecture follows from a direct check for classical groups. But those are rather big theories of the classification of discrete series which still is a longstanding open question for other groups. It is also quite inefficient to do the classification first, then to check whether or not Clozel’s conjecture holds. In other words, there must exist a much simple intuitive proof. Recently, inspired by the talk given by Maxim Gurevich on an irreducibility criterion of parabolic inductions of $GL_n$ which says that $\pi_1 \times \cdots \times \pi_t$ is irreducible iff $\pi_i \times \pi_j$ is irreducible for any $i \neq j$,

we suddenly realized that we can prove the conjecture in general given the fact that the criterion holds for unitary inductions. Unfortunately, this is not true for unitary inductions as there exists a much rich $R$-group theory in the sense of Knapp–Stein and Silberger (cf. [Key82, Gol94]). Fortunately, the failure encourages us to search the literature to check if there exists a simple criterion of the irreducibility of generalized principal series, and we find that Muller has a theorem which says exactly what we want for principal series, but her proof is kind of via the analysis of intertwining operators (cf. [Mul79]). With some efforts, we come up with the idea of using Casselman–Tadic’s Jacquet module machine which has been practiced extensively in our $D_4$-papers (cf. [Luo18d, Luo18c]), and it really works and opens a gate to attack Clozel’s conjecture as well. This is how the paper is cooked up.

We end the introduction by recalling briefly the structure of the paper. In Section 1, we prepare some necessary notation. In Section 2, we prove a product formula of the cardinality of the Jordan–Holder series of generalized principal series which is a key observation for later use, and it also sheds light on a new understanding of the generic irreducibility property of parabolic inductions. In 2010 Mathematics Subject Classification. 22E35.
Key words and phrases. Special exponents, Discrete series, Intertwining operator, Jacquet module, $R$-group, Covering groups.
Section 3, with the aide of the product formula in the previous section, a proof of Clozel’s finiteness conjecture is served.

1. PRELIMINARIES

Let $G$ be a connected reductive group defined over a non-archimedean local field $F$ of characteristic 0. Denote by $|\cdot|_F$ the absolute value, by $\mathfrak{m}$ the uniformizer and by $q$ the cardinality of the residue field of $F$. Fix a minimal parabolic subgroup $B = TU$ of $G$ with $T$ a minimal Levi subgroup and $U$ a maximal unipotent subgroup of $G$, and let $P = MN$ be a standard parabolic subgroup of $G$ with $M$ the Levi subgroup and $N$ the unipotent radical.

Let $X(M)_F$ be the group of $F$-rational characters of $M$, and set

$$ a_M = \text{Hom}(X(M)_F, \mathbb{R}), \quad a_{M,C}^* = a_M^* \otimes \mathbb{C}, $$

where

$$ a_M^* = X(M)_F \otimes \mathbb{R} $$

denotes the dual of $a_M$. Recall that the Harish-Chandra homomorphism $H_P : M \rightarrow a_M$ is defined by

$$ q^{-1}(x,H_P(m)) = |\chi(m)|_F $$

for all $\chi \in X(M)_F$.

Next, let $\Phi$ be the root system of $G$ with respect to $T$, and $\Delta$ be the set of simple roots determined by $U$. For $\alpha \in \Phi$, we denote by $\alpha^\vee$ the associated coroot, and by $w_\alpha$ the associated reflection in the Weyl group $W = W_G$ of $T$ in $G$ with

$$ W = N_G(T)/C_G(T) = \{w_\alpha : \alpha \in \Phi\}. $$

Denote by $w_0^G$ the longest Weyl element in $W$, and similarly by $w_0^M$ the longest Weyl element in the Weyl group $W_M$ of a Levi subgroup $M$.

Likewise, we denote by $\Phi_M$ (resp. $\Phi^L_M$) the reduced relative root system of $M$ in $G$ (resp. the Levi subgroup $L$), by $\Delta_M$ the set of relative simple roots determined by $N$ and by $W_M = N_G(M)/M$ (resp. $W^L_M$) the relative Weyl group of $M$ in $G$ (resp. $L$). In general, a relative reflection $w_{\alpha} := w_0^M w_0^M$ with respect to a relative root $\alpha$ does not preserve our Levi subgroup $M$. Denote by $\Phi^0_M$ (resp. $\Phi^{L,0}_M$) the set of those relative roots which contribute reflections in $W_M$ (reps. $W^L_M$). It is easy to see that $W_M$ preserves $\Phi_M$, and further $\Phi^{0}_M$ as well, as $\omega_{\alpha} = w_\alpha w_\alpha^{-1}$. Note that $W_M$ (resp. $W^{L,0}_M$) in general is larger than $W_0^M$ (resp. $W^{L,0}_M$) the one generated by those relative reflections in $G$ (resp. $L$). Denote by $\Phi_M(P)$ the set of reduced roots of $M$ in $P$.

Recall that the canonical pairing

$$ (\cdot, \cdot) : a^* \times a_M \rightarrow \mathbb{Z} $$

suggests that each $\alpha \in \Phi_M$ will enjoy a one parameter subgroup $H_{\alpha^\vee}(F^\times)$ of $M$ satisfying: for $x \in F^\times$ and $\beta \in a^*_M$,

$$ \beta(H_{\alpha^\vee}(x)) = x^{(\beta, \alpha^\vee)}. $$

**Parabolic induction and Jacquet module:** For $P = MN$ a parabolic subgroup of $G$ and an admissible representation $(\sigma, V_\sigma)$ (resp. $(\pi, V_\pi)$) of $M$ (resp. $G$), we have the following normalized parabolic induction of $P$ to $G$ which is a representation of $G$

$$ I_P^G(\sigma) = \text{Ind}_P^G(\sigma) := \{\text{smooth } f : G \rightarrow V_\sigma | f(nmg) = \delta_P(m)^{1/2}\sigma(m)f(g), \forall n \in N, m \in M \text{ and } g \in G\} $$

with $\delta_P$ stands for the modulus character of $P$, i.e., denote by $\mathfrak{n}$ the Lie algebra of $N$,

$$ \delta_P(nm) = |\det Ad_n(m)|_F, $$

and the normalized Jacquet module $J_M(\pi)$ with respect to $P$ which is a representation of $M$

$$ \pi_N := V/\{\pi(n)e - e : n \in N, e \in V_\pi\}. $$

Given an irreducible unitary admissible representation $\sigma$ of $M$ and $\nu \in a^*_M$, let $I(\nu, \sigma)$ be the representation of $G$ induced from $\sigma$ and $\nu$ as follows:

$$ I(\nu, \sigma) = \text{Ind}_P^G(\sigma \otimes q^{(\nu,H_P(-))}). $$
Denote by $JH(I^G_Q(\sigma))$ the set of Jordan–Holder constituents of the parabolic induction $I^G_Q(\sigma)$.

**R-group:** In [Mul79], for a principal series $I(\lambda)$ of $G$, she defines a subgroup $W_\lambda^1$ of the Weyl group $W$ governing the reducibility of the “unitary” part of principal series on the Levi level, which is indeed the Knapp–Stein $R$-group as follows (cf. [Win78, Key82]),

$$
\Phi_\lambda^0 := \{\alpha \in \Phi : \lambda_\alpha = 1d\},
W^0_\lambda := \{w_\alpha : \alpha \in \Phi^0_\lambda\},
W^1_\lambda := \{w \in W_\lambda : w.(\Phi^0_\lambda)^+ > 0\},
W_\lambda := \{w \in W : w.\lambda = \lambda\}.
$$

In view of [Wal03, Lemma I.1.8], one has

$$W_\lambda = W^0_\lambda \rtimes W^1_\lambda.$$

Following the Knapp–Stein $R$-group theory (cf. [Sil79]), we denote by $R_\lambda$ the subgroup $W^1_\lambda$.

Likewise, for generalized principal series $I^G_P(\sigma)$ (cf. [Luo18a]),

$$
\Phi^0_\sigma := \{\alpha \in \Phi^0_M : w_\alpha.\sigma = \sigma\},
W^0_\sigma := \{w_\alpha : \alpha \in \Phi^0_\sigma\},
W^1_\sigma := \{w \in W_\sigma : w.(\Phi^0_\sigma)^+ > 0\},
W_\sigma := \{w \in W_\sigma : w.\sigma = \sigma\}.
$$

Via [Wal03, Lemma I.1.8], we have

$$W_\sigma = W^0_\sigma \rtimes W^1_\sigma,$$
and we denote $R_\sigma$ to be $W^1_\sigma$ following tradition, but it is not the exact Knapp–Stein $R$-group in the sense of Silberger.

**Special exponent:** There is no harm to assume that $G$ is of compact center. For a discrete series representation $\pi$ of $G$, we denote its associated supercuspidal support to be $\sigma$ which is a supercuspidal representation of some Levi subgroup $M$ of $G$. Denote by $\omega_\sigma$ the unramified part of the central character of $\sigma$, i.e. $\omega_\sigma \in a^*_M \subset C$. Such a character is called a special exponent.

2. A PRODUCT FORMULA

In this section, we prove a key observation of the decomposition of parabolic induction which opens a gate to understand some of the classical results/conjectures, for example the generic irreducibility property of parabolic induction and Clozel’s finiteness conjecture of special exponents.

Recall that $G$ is a connected reductive group defined over $F$ with the set of simple roots $\Delta$, $P = MN$ is a standard parabolic subgroup of $G$ associated to $G_M \subset \Delta$ and $\sigma$ is a supercuspidal representation of $M$ (not necessary unitary), one forms a parabolic induction $I^G_P(\sigma)$. Then our “product formula” is designed to ask the following question

When does the reducibility of $I^G_P(\sigma)$ only happen on the Levi-level?

i.e.

$$ (*) \quad \text{What is a reasonable condition for the irreducibility of } I^G_P(\tau) \text{ for all } \tau \in JH(I^G_M(\sigma)) ?$$

The answer traces back to a beautiful theorem of Muller [Mul79], which provides a natural criterion of the irreducibility of principal series, and its generalized version for generalized principal series [Luo18a], using the Knapp–Stein $R$-group and relative rank-one irreducibility. As the irreducibility is governed by the Knapp–Stein $R$-group and the relative rank-one irreducibility, a natural candidate for $(*)$ is to assume that those governing conditions occur only on the Levi-level. To be more precise, let $Q = LV$ be a standard parabolic subgroup associated to $G_L$ with $G_M \subset G_L \subset \Delta$, then our working assumption for $(*)$ is as follows.

**Working Hypothesis:**

(i) (Rank-one reducibility) The relative rank-one reducibility only occurs within $L$, i.e.

$$I^M_{\alpha}(\sigma) \text{ is reducible only for some } \alpha \in \Phi^L_M, \text{ i.e. } \alpha \in \Phi^L_M \text{ (cf. [Cas95] Theorem 7.1.4)].}$$
(ii) \textbf{(R-group)} The \textit{R-group} $R_\sigma$ associated to $\sigma$ is a subgroup of $W^L_M$, i.e.
\[ R_\sigma \subset W^L_M. \]
Under those hypothesis, via Casselman–Tadić’s Jacquet module machine, the confirmation of \textit{(⋆)} results from the associativity property of intertwining operators and the following observation/fact (cf. [MW95, Cas95]).

(i) (Bernstein–Zelevinsky geometric lemma)
\[ J^M_P(I^G_P(\sigma)) = \sum_{w \in W^M} \sigma^w. \]
(ii) (Bruhat–Tits decomposition)
\[ (*) \quad W_M = W_M(L)W^L_M, \]
where $W_M(L)$ is defined as follows:
\[ W_M(L) := \{ w \in W_M : w.(\Phi_M^L)^+ > 0 \}/W^L_M, \]

here $W^L_M$ is defined as in [Luo18a, Lemma 2.1]), i.e.
\[ W^L_M := \{ w \in W_M : w.(\Phi_M^L)^+ > 0 \}. \]

To be precise, \textbf{Theorem 2.1.} \textit{Keep the notions as above. Under the above Working Hypothesis, we have the following cardinality equality}
\[ \#JH(I^G_P(\sigma)) = \#JH(I^G_{L,M}(\sigma)). \]

Before moving on to the proof, let us first prove the Bruhat-Tits decomposition, i.e. \textit{(⋆)}, in the following.

\textit{Proof of \textit{(⋆)}.} First note that (cf. [Luo18a] Lemma 2.1)
\[ W_M := W^0_M \rtimes W^1_M \]
and
\[ W^L_M := W^L_0 \rtimes W^L_1, \]

with
\[ W^1_M := \{ w \in W_M : w.\Phi^1_M > 0 \} = \{ w \in W_M : w.\Phi^0_M = (\Phi^0_M)^+ \}. \]

So
\[ W_M/W^L_M = \{ w \in W_M : w.\Phi^L_M > 0 \}/W^L_M. \]

\textit{Proof of Theorem 2.1.} Note that the decomposition of $I^G_P(\sigma)$ is a partition of $W_M$. Recall that $R_\sigma$ is in general not the exact $R$-group in the sense of Silberger, as it is defined by
\[ W_\sigma := W^0_\sigma \rtimes R_\sigma, \]
where
\[ W_\sigma := \{ w \in W_M : w.\sigma = \sigma \}, \]

and
\[ W^0_\sigma := \{ w_\alpha : w_\alpha.\sigma = \sigma, \alpha \in \Phi_M \}. \]

But reducibility coming from $W^0_\sigma$ has been taken care of by the relative rank-one reducibility condition which only occurs within $L$ by the assumption.

Therefore it reduces to show that the intertwining operator $A(w, \sigma)$ associated to $w \in W_M(L)$ is an isomorphism, i.e.
\[ A(w, \sigma) : I^G_P(\sigma) \xrightarrow{\sim} I^G_P(\sigma^w). \]

Recall that $A(w, \sigma)$ is defined as follows:
\[ J_{P,P^w}(\sigma^w) \circ \lambda(w) : I^G_P(\sigma) \rightarrow I^G_{P^w}(\sigma^w) \rightarrow I^G_P(\sigma^w). \]
Thus the above isomorphism claim follows from the associativity property of intertwining operators (cf. [Wal03 IV.3 (4)] or [Luo18b] Lemma 3.5), i.e.
$$J_{P|P'}(σ^w)J_{P|P'}(σ) = \prod j_α(σ)J_{P|P'}(σ),$$
where $α$ runs over
$$Φ_M(P) \cap Φ_M(F^-).$$
Note that for $α ∈ Φ^L_M - Φ^L_M^0$, the associated relative rank-one induction is always irreducible and $j_α(σ) ≠ 0, ∞$ (cf. [Sil80 Corollary 1.8]). Therefore one only needs to consider those $j_α(σ)$ with $α ∈ Φ^L_M^0$.
Notice also that
$$w.(Φ^L_M)^* > 0,$$ so
$$Φ_M(P) \cap Φ_M(F^-) \cap Φ^L_M^0 = ∅.$$
Thus $A(w, σ)$ is an isomorphism. □

Denote by $Θ_0$ the associated subset of $Δ$ which determines the parabolic subgroup $Q$ of $G$. Explicitly, we decompose $Θ_L = Θ_1 \cup ⋯ \cup Θ_t$ into irreducible pieces, and accordingly $Θ_M = Θ^M_1 \cup ⋯ \cup Θ^M_t$. Assume that $R_σ$ decomposes into $R_σ = R_1 × ⋯ × R_t$ with respect to the decomposition of $Θ_L$, and a similar decomposition pattern holds for the relative rank-one reducibility, i.e. relative rank-one reducibility only occurs within $P_{Θ_i} = M_{Θ_i}N_{Θ_i}$ for $1 ≤ i ≤ t$. Then we have

**Corollary 2.2** (Product formula).

$$\#(JH(I^G_P(σ))) = \prod_{i=1}^t \#(JH(I^M_{N_{Θ_i}}(σ))).$$

3. **CLOZEL’S FINITENESS CONJECTURE OF SPECIAL EXPONENTS**

In this section, as an application of Theorem 2.1 we will prove Clozel’s finiteness conjecture of special exponents proposed in [Clo85] which plays an essential role in proving Howe’s finiteness conjecture. Note that Clozel’s finiteness conjecture may be checked directly for classical groups from Moeglin–Tadic’s work on the classification of discrete series (cf. [MT02] Moedrit). As the conjecture is much of a quantitative result, it should be proved with little forces, instead of resorting to such a big stick. In what follows, we first recall some notions.

There is no harm to assume that $G$ is of compact center. Recall that for a discrete series representation $π$ of $G$, we write its associated supercuspidal support as $σ$ which is a supercuspidal representation of some Levi subgroup $M$ of $G$. Denote by $ω_σ$ the unramified part of the central character of $σ$, i.e. $ω_σ ∈ a^*_M,C$. Such a character is called a special exponent.

**Clozel’s finiteness conjecture.** The set of special exponents is finite.

Before turning to the proof, let us first talk about the main idea.

Under the **Induction Assumption**, i.e.

Clozel’s finiteness conjecture holds for the relative rank-one case.
One can prove Clozel’s finiteness conjecture for the general case via Theorem 2.1. Roughly speaking, with the help of Muller type theorem of generalized principal series, one knows that the decomposition of $I^G_P(σ)$ is governed by the relative rank-one reducibility and the $R$-group $R_σ$. On the other hand, one knows that a full induced representation can not be a discrete series. In view of Theorem 2.1 thus in order to ensure $ω_σ$ is a special exponent, those roots associated to the relative rank-one reducibility and $R_σ$ must generate the whole space $a^*_M,C$. Then the conjecture follows from an easy fact of linear algebra, i.e. an invertible matrix has only one solution.

To be more precise, let $P_Θ = M_ΘN$ be a standard parabolic subgroup of $G$ with $Θ ⊂ Δ$, and let $σ$ be a supercuspidal representation of $M_Θ$. Decomposing $Θ$ into irreducible pieces
$$Θ = Θ_1 \cup ⋯ \cup Θ_t$$
As $W_{M_\sigma}$ acts on $M_\Theta$, then it preserves the decomposition up to sign, so does $R_\sigma$, i.e. preserving

$$\pm\Theta = \pm\Theta_1 \sqcup \cdots \sqcup \pm\Theta_n.$$

Under the action of $R_\sigma$ on $\pm\Theta$, we have a new decomposition of $\pm\Theta$ into irreducible pairs, i.e.

$$R_\sigma \to S_{\pm n},$$

where $S_{\pm n}$ is the “pseudo”-permutation group, i.e.

$$S_{\pm n} := \{(a_{i_1} \cdots a_{i_k}) : a_{i_j} \in \{\pm 1, \ldots, \pm n\}\}/\pm.$$

Here $\pm$-equivalence means that

$$(a_{i_1} \cdots a_{i_k}) = ((-a_{i_1}) \cdots (-a_{i_k})).$$

For each simple permutation $s = (a_{i_1} \cdots a_{i_k})$, we define the associated roots as, up to scalar,

$$\Phi_s := \{e_{i_j} - e_{i_l} : 1 \leq j < l \leq k\},$$

where $e_{i_j}$ is the component character on $\Theta_{i_j}$.

Proof. It suffices to prove the finiteness of special exponents for only one parabolic subgroup, like $P_\Theta = M_\Theta N$ with $\sigma$ a supercuspidal representation of $M_\Theta$ and $\Theta \subset \Delta$.

Considering the set $\Phi_s$ of roots associated to the relative rank-one reducibility and the $R$-group $R_\sigma$, if

$$\text{Span}_C \Phi_s \neq a^*_M \cdot C,$$

then one knows that, up to associated forms,

$$\text{Span}_C \Phi_s = a^*_L \cdot C,$$

for some Levi $L > M$. Then by Theorem 2.1 we know that

$$I^L_I(\tau) \text{ is irreducible for all } \tau \in JH(I^L_M(\sigma)),$$

which can not be discrete series. Therefore

$$\text{Span}_C \Phi_s = a^*_M \cdot C.$$

Which in turn says that the set of real parts of $\{\omega_\sigma\}_\sigma$ is finite as there are only finitely many rank-one reducibility hyperplanes in $a^*_M$ by the Induction Assumption.

As for the set of imaginary parts of $\{\omega_\sigma\}_\sigma$, the finiteness follows from the facts that

(i) There are finitely many $R_\sigma$, and $R_\sigma$ is finite. This says that there are only finitely many linearly independent subsets.

(ii) Each element in $R_\sigma$ is of finite order. This says that there are only finitely many solutions for each linearly independent subset.

Combining the finiteness of the real parts and imaginary parts, Clozel’s finiteness conjecture holds under the Induction Assumption. □

Corollary 3.1. Let $\tilde{G}$ be a finite central covering of $G$, then Clozel’s finiteness conjecture holds for $\tilde{G}$ under the Induction Assumption.

Proof. This follows from the Induction Assumption that there are only finitely many rank-one reducibility hyperplanes in $a^*_M$ and the R-group theory in [Luo17]. □

Remark 1. The Induction Assumption is a byproduct of the profound Langlands–Shahidi theory for generic $\sigma$s (cf. [Sha90]). But for non-generic $\sigma$s, following Moeglin–Tadic’s classification of discrete series, it is true for classical groups (cf. [Moe03, Moe, MT02, Moe07]).

Remark 2. Clozel’s finiteness conjecture is also known for low rank groups of which their unitary duals are completely known (cf. [Han06, HM10, Kon01, Luo18c, Luo18d, Mat10, Mui98, ST93, Sch14]).
PROOF OF CLOZEL’S FINITENESS CONJECTURE OF SPECIAL EXPONENTS: A REDUCTION STEP

References

[BDKV84] Joseph N. Bernstein, Pierre Deligne, David A. Kazhdan, and Marie France Vignéras, *Représentations des groupes réductifs sur un corps local*, vol. 8, Hermann Paris, 1984.

[BZ77] L.N. Bernstein and Andrey V. Zelevinsky, *Induced representations of reductive $p$-adic groups. I*, Ann. Sci. École Norm. Sup. (4) 10 (1977), no. 4, 441–472.

[Cas95] William Casselman, *Introduction to the theory of admissible representations of reductive $p$-adic groups*, Preprint (1995).

[Clo85] Laurent Clozel, *Sur une conjecture de Howe. I*, Compos. Math. 56 (1985), no. 1, 87–110.

[Gold94] David Goldberg, *Reducibility of induced representations for $Sp(2n)$ and $SO(n)$*, Amer. J. Math. 116 (1994), no. 5, 1101–1151.

[Han06] Marcela Hanzer, *The unitary dual of the Hermitian quaternionic group of split rank 2*, Pacific J. Math. 226 (2006), no. 2, 353–388.

[BDKV84] Joseph N. Bernstein, Pierre Deligne, David A. Kazhdan, and Marie France Vignéras, *Représentations des groupes réductifs sur un corps local*, vol. 8, Hermann Paris, 1984.

[Sch14] Claudia Schoemann, *A proof of Langlands’ conjecture on Plancherel measures: complementary series of Unitary Dual of $p$-adic $U(5)$*, Preprint (2017).

[Sil80] Allan J. Silberger, *Introduction to Harmonic Analysis on Reductive $p$-adic Groups*. (MN-23): Based on lectures by Harish-Chandra at The Institute for Advanced Study, 1971-73, Princeton university press, 1979.

[Moe07] Colette Moeglin, *Classification des séries discrètes pour certains groupes classiques $p$-adiques*, Harmonic Analysis, Group Representations, Automorphic Forms And Invariant Theory: In Honor of Roger E. Howe, World Scientific, 2007, pp. 209–245.

[Moe03] Colette Moeglin, *La formule de Plancherel pour les groupes $p$-adiques*, in *Group Representations, Automorphic Forms And Invariant Theory: In Honor of Roger E. Howe*, World Scientific, 2007, pp. 209–245.

[Win78] Norman Winarsky, *Reducibility of principal series representations of $p$-adic Chevalley groups*, Amer. J. Math. 100 (1978), no. 5, 941–956.

[Ze180] Andrei V. Zelevinsky, *Induced representations of reductive $p$-adic groups. II. On irreducible representations of $GL(n)$*, Ann. Sci. École Norm. Sup. (4), vol. 13, Elsevier, 1980, pp. 165–210.

Department of Mathematical Sciences, Chalmers University of Technology and the University of Gothenburg, Chalmers tvågata 3, 412 96 Göteborg

E-mail address: caihua@chalmers.se