Research Article

On Using the Conventional and Nonconventional Measures of the Auxiliary Variable for Mean Estimation

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In this paper, we propose an improved new class of exponential-ratio-type estimators for estimating the finite population mean using the conventional and the nonconventional measures of the auxiliary variable. Expressions for the bias and MSE are obtained under large sample approximation. Both simulation and numerical studies are conducted to validate the theoretical findings. Use of the conventional and the nonconventional measures of the auxiliary variable is very common in survey research, but we observe that this does not add much value in many of the estimators except for our proposed class of estimators.

1. Introduction

Many research papers have appeared in the literature where authors have used the conventional and the nonconventional measures of the auxiliary variable to enhance the efficiency of estimators. Recently, Gulzar et al. [1] used the nonconventional measures for population variance. Many other researchers have contributed to this area. Singh et al. [2] suggested a class of linear combinations of exponential-ratio- and product-type estimators for mean estimation. Gupta and Shabbir [3] introduced a class of ratio-in-difference-type estimators using the conventional measures for the population mean. Singh et al. [4] presented an improved family of exponential-ratio-type estimator in simple random sampling for population mean. Haq and Shabbir [5] introduced an improved family of ratio-type estimators in simple and stratified sampling. Yadav and Kadilar [6] proposed an exponential family of ratio-type estimators by using conventional measures for estimating the population mean. Shabbir et al. [7] presented a new family of estimators for finite population mean in simple random sampling. Grover and Kaur [8] suggested a generalized class of exponential-ratio-type exponential estimators using the conventional measures for mean estimation. Kadilar [9] discussed a new exponential-type estimator for mean estimation. Irfan et al. [10] suggested a generalized ratio-exponential-type estimator using the conventional measures. Also, Irfan et al. [11] used both conventional and nonconventional measures in their proposed estimator. Ali et al. [12] used the robust-regression-type estimators for mean estimation of sensitive variables, and Shahzad et al. [13] suggested the L-moments based on calibration variance estimators in their recent work.

In our study, we propose a new generalized class of exponential-ratio-type estimators by using the conventional and the nonconventional measures of the auxiliary variable and compare our proposed estimator with several existing estimators.

Consider a finite population \( \Lambda = \{ \Lambda_1, \Lambda_2, \ldots, \Lambda_N \} \) of \( N \) units. A sample of size \( n \) units is drawn from this population using simple random sampling without replacement (SRSWOR). Let \( y_i \) and \( x_i \) be the observed values of the study variable \( (Y) \) and the auxiliary variable \( (X) \), respectively. Let \( \bar{y} = n^{-1} \sum_{i=1}^{n} y_i \) and \( \bar{x} = n^{-1} \sum_{i=1}^{n} x_i \), respectively, be the sample means and \( \bar{Y} = N^{-1} \sum_{i=1}^{N} y_i \) and \( \bar{X} = N^{-1} \sum_{i=1}^{N} x_i \) be the corresponding population means.
Let \((C_y = Y^{-1}S_y, C_x = X^{-1}S_x)\) be the coefficient of variations and \((S_y = \sqrt{(N-1)^{-1}\sum_{i=1}^{N}(y_i - \bar{Y})^2}, \quad S_x = \sqrt{(N-1)^{-1}\sum_{i=1}^{N}(x_i - \bar{X})^2})\) be the standard deviations of \((Y, X)\), respectively. Let \(\rho_{yx} = (S_xS_y)^{-1}S_{yx}\) be the correlation coefficient and \(S_{yx} = (N-1)^{-1}\sum_{i=1}^{N}(y_i - \bar{Y})(x_i - \bar{X})\) be the covariance between the variables indicated by the subscripts. We define the following error terms to obtain the bias and MSE expressions: \(\Theta_0 = (\bar{Y}/\bar{Y}) - 1\) and \(\Theta_1 = (\pi/\bar{X}) - 1\) such that \(E(\Theta_1) = 0, (i = 0, 1)\), \(E(\Theta_0^2) = \gamma C_y^2\), \(E(\Theta_1^2) = \gamma C_x^2\), and \(E(\Theta_0\Theta_1) = \gamma \rho_{yx}C_yC_x = \gamma \rho_{yx}\), where \(\gamma = (n^{-1} - n^{-1})\).

Now we discuss some of the conventional and non-conventional measures of the auxiliary variable which are used in our study. Arthur Lyon Bowley (1869–1957), a British Statistician, introduced a term skewness \((\beta_1(x))\) based on median and the two quartiles, and kurtosis \((\beta_2(x))\) was originated by Karl Pearson (1857–1936). Antonine Augustin Cournot was the first to use the term median \((M_d)\) in 1843. The midrange \((M_g)\), i.e., \(M_g = ((X_{(1)} + X_{(N/2)})/2)\) was introduced by Robert K. Merton. Tukey [14] used the idea of Trimean \((TM)\) \((T_M = ((Q_1 + 2Q_2 + Q_3)/4)\), where \((Q_1, Q_2, Q_3)\) are the first, second, and third quartiles of the auxiliary variable which were discussed by Tukey [15]. The term quartile deviation \((Q_0)\) \(= (Q_0(x) - Q_1(x))/2)\) was used in “Proceeding of the Royal Statistical Society of London” in late 19th century. Hodge and Lehmann [16] used the measure \((H_L)\) \((H_L = \text{median}((X_j + X_k)/2), 1 \leq j < k \leq N)\), for estimation of location based on ranks.

### 2. Some Existing Estimators

We discuss the following mean estimators that exist in the literature:

(i) The sample mean estimator is \(\hat{Y}^{(0)} = \bar{Y}\), and its variance is given by

\[
\text{Var} (\hat{Y}^{(0)}) = Y^2C_y^2.
\]

(ii) The usual ratio estimator proposed by Cochran [17] when the regression line \(Y\) on \(X\) passes through origin is given by

\[
\hat{Y}^{(R)} = \bar{Y}\left(\frac{\bar{X}}{\bar{X}}\right).
\]

where \(\bar{X}\) is the known population mean of the auxiliary variable \(X\). The performance of ratio estimator is better as compared to the usual mean estimator \(\bar{Y}\) when \(\rho_{yx} > (0.25C_x/C_y)\).

The bias and MSE, respectively, of \(\hat{Y}\) are given by

\[
B(\hat{Y}^{(R)}) \approx Y\left[C_y^2 - C_{yx}\right],
\]

\[
\text{MSE}(\hat{Y}^{(R)}) \approx Y^2\left[C_y^2 + C_x^2 - 2C_{yx}\right].
\]

(iii) Bahl and Tuteja [18] suggested the exponential-ratio-type estimator given by

\[
\hat{Y}^{(EP)} = \bar{Y}\exp\left(\frac{\bar{X} - \bar{X}}{\bar{X} + \bar{X}}\right).
\]

The bias and MSE, respectively, of \(\hat{Y}^{(EP)}\) are given by

\[
B(\hat{Y}^{(EP)}) \approx Y\left[C_y^2 - \frac{1}{2}C_{yx}\right],
\]

\[
\text{MSE}(\hat{Y}^{(EP)}) \approx Y^2\left[C_y^2 + \frac{1}{4}C_x^2 - C_{yx}\right].
\]

The exponential ratio estimator \(\hat{Y}^{(EP)}\) is superior to the usual mean estimator \(\hat{Y}^{(0)}\) and ratio estimator \(\hat{Y}^{(R)}\) if \(\rho_{yx} > (0.25C_x/C_y)\) and \(\rho_{yx} < (0.75C_x/C_y)\), respectively.

(iv) The regression or difference estimator [19, 20] is given by

\[
\hat{Y}^{(D)} = \bar{Y} + \omega(\bar{X} - \bar{X}),
\]

where \(\omega\) is a constant.

The minimum MSE of \(\hat{Y}^{(D)}\) at \(\omega_{\text{opt}} = (\bar{Y}\rho_{yx}C_y)/C_x\) is given by

\[
\text{MSE}(\hat{Y}^{(D)})_{\text{min}} = Y^2C_y^2(1 - \rho_{yx}^2).
\]

The difference estimator always performs better than usual sample mean estimator, ratio estimator, and exponential ratio estimator if \(\rho_{yx}^2 > 0, (C_x - C_y\rho_{yx})^2 > 0\), and \((0.5C_x - C_y\rho_{yx})^2 > 0\), respectively.

(v) Singh et al.’s estimator [2] is given by

\[
\hat{Y}^{(S)} = \bar{Y}\left[ s \exp\left(\frac{\bar{X} - \bar{X}}{\bar{X} + \bar{X}}\right) + (1 - s)\exp\left(\frac{\bar{X} - \bar{X}}{\bar{X} + \bar{X}}\right)\right],
\]

where \(s\) is a constant.

The minimum MSE of \(\hat{Y}^{(S)}\) at \(s_{\text{opt}} = 0.5 + ((\rho_{yx}C_y)/C_x)\) is given by

\[
\text{MSE}(\hat{Y}^{(S)})_{\text{min}} = Y^2C_y^2(1 - \rho_{yx}^2) = \text{MSE}(\hat{Y}^{(D)})_{\text{min}}.
\]

(vi) Singh et al. [4] suggested the following estimator:

\[
\hat{Y}^{(S)}_{i} = \bar{Y}\exp\left(\frac{\mu(\bar{X} - \bar{X})}{\mu(\bar{X} - \bar{X}) + 2\nu}\right),
\]

where \(\mu\) and \(\nu\) are the functions of known population parameters of the auxiliary variable. Some members of the family of estimators \(\hat{Y}^{(S)}_{i}\) \((i = 1, 2, \ldots, 12)\) are given in Table 1.
The bias and MSE, respectively, of $\widetilde{P}^{(S)}_i$ are given by
\begin{equation}
B\left(\widetilde{P}^{(S)}_i\right) = y^2\left[\frac{1}{3}C_{xy}^2 - \frac{1}{2}\varphi_i C_{xy}\right],
\end{equation}
\begin{equation}
\text{MSE}\left(\widetilde{P}^{(S)}_i\right) = y^4\left[C_{xy}^2 + \frac{1}{4}\varphi_i^2 C_{xy}^2 - \varphi_i^2 C_{xy}\right],
\end{equation}
where $\varphi_i = (uX/(uX+v)) (i = 1, 2, \ldots, 12)$. We can get different values of $\varphi_i$ (i.e., $u = 1, v = 0$, $u = 1, v = C_{xy}$, $u = C_{xy}, v = \rho_{xy}$), $u = \beta_1(x), v = \beta_2(x)$, $u = 1, v = \rho_{xy}$, $u = \beta_1(x), v = 1, v = \rho_2(x)$, $u = Q_{xy}, v = T_{xy}$, $u = M_R, v = H_L$, $u = Q_{xy}, v = Q_{xy}$, $u = \rho_{xy}, v = \beta_2(x)$) resulting in $\varphi_1 = 1, \varphi_2 = (X/(X + C_{xy}))$, $\varphi_3 = (C_{xy}/(X + \rho_{xy}))$, $\varphi_4 = ((\beta_1(x)X)/X + \rho_{xy})$, $\varphi_5 = ((\beta_2(x)X)/X + \beta_1(x))$, $\varphi_6 = (X/(X + \rho_{xy}))$, $\varphi_7 = (Q_{xy}/(X + T_{xy}))$, $\varphi_8 = (Q_{xy}/(X + M_R))$, $\varphi_9 = (H_L/X + T_{xy})$, $\varphi_{10} = (X/(X + Q_{xy}))$, $\varphi_{11} = (\rho_{xy}/(X + Q_{xy}))$, $\varphi_{12} = (\rho_{xy}/(X + \beta_2(x)))$, respectively.

(vii) Yadav and Kadilar [6] suggested the following estimator:
\begin{equation}
\widehat{Y}^{(Y)} = k\varphi \exp\left(\frac{u(X - \bar{X})}{u(X - \bar{X}) + 2v}\right),
\end{equation}
where $k$ is a constant. Some members of the family of estimators $\widehat{Y}^{(Y)}_i$ ($i = 1, 2, \ldots, 12$) are given in Table 1.

Table 1: Members of Singh et al. [4], Yadav and Kadilar [6], and Grover and Kaur [8].

| $u$ | $v$ | Singh et al. [4] | Yadav and Kadilar [6] | Grover and Kaur [8] |
|-----|-----|------------------|---------------------|---------------------|
| 1   | 0   | $\beta_1(x)$    | $\beta_1(x)$       | $\beta_1(x)$       |
| $C_x$ | $\beta_2(x)$ | $\beta_2(x)$ | $\beta_2(x)$ | $\beta_2(x)$ |
| $C_x$ | $\beta_3(x)$ | $\beta_3(x)$ | $\beta_3(x)$ | $\beta_3(x)$ |
| $\beta_1(x)$ | $\beta_4(x)$ | $\beta_4(x)$ | $\beta_4(x)$ | $\beta_4(x)$ |
| $\beta_2(x)$ | $\beta_5(x)$ | $\beta_5(x)$ | $\beta_5(x)$ | $\beta_5(x)$ |
| $\beta_3(x)$ | $\beta_6(x)$ | $\beta_6(x)$ | $\beta_6(x)$ | $\beta_6(x)$ |
| $\beta_4(x)$ | $\beta_7(x)$ | $\beta_7(x)$ | $\beta_7(x)$ | $\beta_7(x)$ |
| $\beta_5(x)$ | $\beta_8(x)$ | $\beta_8(x)$ | $\beta_8(x)$ | $\beta_8(x)$ |
| $\beta_6(x)$ | $\beta_9(x)$ | $\beta_9(x)$ | $\beta_9(x)$ | $\beta_9(x)$ |
| $\beta_7(x)$ | $\beta_{10}(x)$ | $\beta_{10}(x)$ | $\beta_{10}(x)$ | $\beta_{10}(x)$ |
| $\beta_8(x)$ | $\beta_{11}(x)$ | $\beta_{11}(x)$ | $\beta_{11}(x)$ | $\beta_{11}(x)$ |
| $\beta_9(x)$ | $\beta_{12}(x)$ | $\beta_{12}(x)$ | $\beta_{12}(x)$ | $\beta_{12}(x)$ |

The bias and MSE, respectively, of $\widehat{P}^{(Y)}_i$ are given by
\begin{equation}
B(\widehat{P}^{(Y)}_i) = k\varphi \exp\left(\frac{u(X - \bar{X})}{u(X - \bar{X}) + 2v}\right),
\end{equation}
\begin{equation}
\text{MSE}(\widehat{P}^{(Y)}_i) = k^2\varphi^2 \exp\left(\frac{u(X - \bar{X})}{u(X - \bar{X}) + 2v}\right),
\end{equation}
where $\varphi = (u(X/(uX+v))) (i = 1, 2, \ldots, 12)$. We can get different values of $\varphi_i$ (i.e., $u = 1, v = 0$, $u = 1, v = C_{xy}$, $u = C_{xy}, v = \rho_{xy}$), $u = \beta_1(x), v = \beta_2(x)$, $u = 1, v = \rho_{xy}$, $u = \beta_1(x), v = 1, v = \rho_{xy}$, $u = Q_{xy}, v = T_{xy}$, $u = M_R, v = H_L$, $u = Q_{xy}, v = Q_{xy}$, $u = \rho_{xy}, v = \beta_2(x)$) resulting in $\varphi_1 = 1, \varphi_2 = (X/(X + C_{xy}))$, $\varphi_3 = (C_{xy}/(X + \rho_{xy}))$, $\varphi_4 = ((\beta_1(x)X)/X + \rho_{xy})$, $\varphi_5 = ((\beta_2(x)X)/X + \beta_1(x))$, $\varphi_6 = (X/(X + \rho_{xy}))$, $\varphi_7 = (Q_{xy}/(X + T_{xy}))$, $\varphi_8 = (Q_{xy}/(X + M_R))$, $\varphi_9 = (H_L/X + T_{xy})$, $\varphi_{10} = (X/(X + Q_{xy}))$, $\varphi_{11} = (\rho_{xy}/(X + Q_{xy}))$, $\varphi_{12} = (\rho_{xy}/(X + \beta_2(x)))$, respectively.

(viii) Kadilar [9] suggested the following exponential-type estimator:
\begin{equation}
\widehat{Y}^{(K)} = \gamma Y^{m} \exp\left(\frac{X - X}{X + Y}\right),
\end{equation}
where $m$ is a constant.

The bias and MSE, respectively, of $\widehat{Y}^{(K)}$ at $m_{\text{opt}} = 0.5 - ((\beta_{12}X)/C_{xy})$ are given by
\begin{equation}
B(\widehat{Y}^{(K)}) = Y^{m}_k\left[\frac{(m - 0.5)C_{xy} + C_{xy}^2 m^2}{2 + 3(1 - m)}\right],
\end{equation}
\begin{equation}
\text{MSE}(\widehat{Y}^{(K)}) = Y^{m}_k C_{xy}^2 (1 - \rho_{xy}^2) = \text{MSE}(\widehat{Y}^{(D)})_\text{min},
\end{equation}
\begin{equation}
\text{MSE}(\widehat{Y}^{(K)}) = Y^{m}_k C_{xy}^2 (1 - \rho_{xy}^2) = \text{MSE}(\widehat{Y}^{(D)})_\text{min}.
\end{equation}

(ix) Grover and Kaur’s estimator [8] is given by
\begin{equation}
\widehat{Y}^{(GK)} = [g_1\varphi + g_2(X - \bar{X})] \exp\left(\frac{u(X - \bar{X})}{u(X - \bar{X}) + 2v}\right),
\end{equation}
where $g_1, g_2$ are given by
\begin{equation}
g_1 = \frac{Y^{m}_k}{\gamma}, \quad g_2 = \frac{Y^{m}_k}{\gamma} X.
\end{equation}
where $g_1$ and $g_2$ are constants. Some members of the family of estimators $\hat{Y}_i^{(GK)}$ ($i = 1, 2, \ldots, 12$) are given in Table 1.

\[
B(\hat{Y}^{(GK)}) = (g_1 - 1)\bar{Y} + g_1\bar{Y}C_g + g_2\bar{X}D_g,
\]

\[
\text{MSE}(\hat{Y}^{(GK)})_{\text{min}} = \bar{Y}^2 \left[ 1 - \frac{(A_gD_g^2 + B_gC_g - 2C_gD_gE_g + 2B_gC_g + D_g^2 - 2D_gE_g + B_g)}{(A_gB_g - E_g^2 + B_g)} \right],
\]

where $A_g = \gamma(C_g^2 + \varphi^2C_x^2 - 2\varphi_xC_{xy})$, $B_g = \gamma C_g^2$, $C_g = \gamma((\gamma/3)\varphi_xC_{xy})$, $D_g = ((\gamma^2C_x^2)/2)$, and $E_g = \gamma(\varphi_xC_x - C_{xy})$.

(xi) Recently, Irfan et al. [10] suggested the following exponential-type estimator:

\[
\tilde{Y}^{(Ia)} = I_{1a}\bar{Y} \frac{u(\bar{X} - \bar{x})}{u(\bar{X} + \bar{x})} + I_{2a}\bar{X} - \bar{X}) \exp \left( \frac{u(\bar{X} - \bar{x})}{u(\bar{X} + \bar{x}) + 2v} \right),
\]

where $I_{1a}$ and $I_{2a}$ are constants. Some members of the family of estimators $\tilde{Y}_i^{(Ia)}$ ($i = 1, 2, \ldots, 12$) are given in Table 2.

\[
B(\tilde{Y}^{(Ia)}) = (I_{1a} - 1)\bar{Y} + I_{1a}\bar{Y}C_a + I_{2a}\bar{X}D_a,
\]

\[
\text{MSE}(\tilde{Y}^{(Ia)})_{\text{min}} = \bar{Y}^2 \left[ 1 - \frac{(A_aD_a^2 + B_aC_a - 2C_aD_aE_a + 2B_aC_a + D_a^2 - 2D_aE_a + B_a)}{(A_aB_a - E_a^2 + B_a)} \right],
\]

where $A_a = \gamma(C_a^2 + 3\varphi^2C_x^2 - 4\varphi_xC_{xy})$, $B_a = \gamma C_a^2$, $C_a = \gamma((\gamma^2C_x^2)/2)$, and $E_a = \gamma((\gamma^2C_{xy}^2)/C_{xy})$.

Shabbir et al. [7] suggested the following transformed exponential-ratio-difference-type estimator:

\[
\tilde{Y}^{(SH)} = \left[ s_1\bar{Y} + s_2(\bar{X} - \bar{x}) + \Omega \right] \exp \left( \frac{u(\bar{X} + \bar{x})}{u(\bar{X} + \bar{x}) + 2v} \right),
\]

where $\Omega = (\gamma/2) \left[ \exp(2v) + \frac{u(\bar{X} - \bar{x})}{u(\bar{X} + \bar{x})} + \exp(2v) \right]$ and $s_1$ and $s_2$ are constants. The members of the family of estimators $\tilde{Y}_i^{(SH)}$ ($i = 1, 2, \ldots, 12$) are given in Table 2.

The bias and minimum MSE, respectively, of $\tilde{Y}^{(SH)}$, at $s_{1\text{(opt)}} = \left( (D_sE_s - B_sC_s) / (A_sB_s - E_s^2 + B_s) \right)$ and $s_{2\text{(opt)}} = \left( (D_sE_s - B_sC_s) / (A_sB_s - E_s^2 + B_s) \right)$, are given by

\[
B(\tilde{Y}^{(SH)}) = s_1\bar{Y}D_s + s_2\bar{X}B_s \frac{1}{2},
\]

\[
\text{MSE}(\tilde{Y}^{(SH)})_{\text{min}} = \bar{Y}^2 \left[ \gamma(C_s^2 + 0.25\varphi^2C_x^2 - \varphi_xC_{xy}) - \frac{(A_sD_s^2 + B_sC_s^2 - 2C_sD_sE_s)}{(A_sB_s - E_s^2)} \right],
\]
where \( A_x = 1 + \gamma(C_y^2 + \varphi_1^2C_x^2 - 2\psi_1C_{xy}), \) \( B_x = \gamma C_x^2, \)
\( C_x = \gamma^2 C_x^2 (3/4)\varphi_1^2 C_x^2 - (3/2)\psi_1 C_{xy}, \) \( D_x = \gamma (0.5 \varphi_1 C_x^2 - C_{xy}), \) and \( E_x = \gamma (\varphi_1 C_x^2 - C_{xy}). \)

(xii) On the lines of Shabbir et al. [7], Irfan et al. [11] suggested an estimator, which is given by

\[
\hat{Y}_i^{(1)} = [I_{1b}y + I_{2b}(\overline{X} - \overline{X}) + \Omega] \exp \left( \frac{u(\overline{X} - \overline{X})}{u(\overline{X} + \overline{X}) - 2\nu} \right),
\]

(30)

where \( \Omega = (\pi/2)(\exp(\overline{X} - \overline{X}/(\overline{X} + \overline{X})) + \exp((\overline{X} - \overline{X})/(\overline{X} + \overline{X}))) \) and \( I_{1b} \) and \( I_{2b} \) are constants. The members of the family of estimators \( \hat{Y}_i^{(1)} \) \((i = 1, 2, \ldots, 12)\) are given in Table 3.

The bias and minimum MSE, respectively, of \( \hat{Y}_i^{(1)} \), at

\[
I_{1b(\text{opt})} = ((D_yE_b - B_yC_y)/(A_yB_y - E_y^2)) \quad \text{and} \quad I_{2b(\text{opt})} = ((C_yE_b - A_yD_y)/(A_yB_y - E_y^2)),
\]

are given by

\[
B\left( \hat{Y}_i^{(1)} \right) = \gamma Y\left[ 1 + 3\psi_1 C_x^2 - 1/2\gamma C_{xy} \right] + I_{1b} Y\left[ 3/8 \psi_1^2 C_x^2 - 1/2 \gamma C_{xy} \right] + I_{2b} Y B_x \frac{1}{2}
\]

\[
\text{MSE}\left( \hat{Y}_i^{(1)} \right)_{\min} = \gamma Y\left[ 1 + 0.25\psi_1^2 C_x^2 - \psi_1 C_{xy} \right] - \frac{A_y D_y^2 + B_y C_y^2 - 2C_y D_y E_y}{(A_y B_y - E_y^2)}.
\]

(31)

(32)

where

\[
B_y = \gamma C_x^2,
\]

\[
C_y = \gamma \left( C_y^2 + \left( \frac{5}{8} \psi_1^2 + 1 \right) C_x^2 - \frac{3}{2} \gamma C_{xy} \right),
\]

\[
D_y = \gamma (0.5 \gamma \psi_1 C_x^2 - C_{xy}),
\]

\[
E_y = \gamma (\psi_1 C_x^2 - C_{xy}),
\]

\[
\psi_1 = \frac{u \overline{X}}{u \overline{X} - v C_x},
\]

\[
\psi_2 = \frac{\overline{X}}{\overline{X} - C_x},
\]

\[
\psi_3 = \frac{C_x \overline{X}}{\overline{X} - \rho_{yx}},
\]

\[
\psi_4 = \frac{\beta_1(x) \overline{X}}{\overline{X} - \rho_{yx}},
\]

\[
\psi_5 = \frac{\beta_2(x) \overline{X}}{\overline{X} - \beta_1(x)},
\]

\[
\psi_6 = \frac{\overline{X}}{\overline{X} - \rho_{yx}},
\]

\[
\psi_7 = \frac{Q_D \overline{X}}{\overline{X} - T_M},
\]

\[
\psi_8 = \frac{Q_D Y}{\overline{X} - M_R},
\]

\[
\psi_9 = \frac{H_M Y}{\overline{X} - T_M},
\]

\[
\psi_{10} = \frac{Y}{\overline{X} - Q_D},
\]

\[
\psi_{11} = \frac{\rho_{yx} Y}{\overline{X} - Q_D},
\]

\[
\psi_{12} = \frac{\rho_{yx} Y}{\overline{X} - \beta_2(x)}.
\]

3. Proposed Estimator

We propose a fairly simple class of exponential-ratio-type estimators using the conventional and nonconventional measures as given below:

\[
\tilde{Y} = [T_1 Y + T_2] \exp \left( \frac{u(\overline{X} - \overline{X})}{u(\overline{X} + \overline{X}) + 2\nu} \right).
\]

(34)

where \( T_1 \) and \( T_2 \) are constants and \( u \) and \( \nu \) are the known conventional and nonconventional measures of the auxiliary variable. Various members of the family of estimators \( \tilde{Y}^{(p)} \) \((i = 1, 2, \ldots, 12)\) are given in Table 3. The purpose of constructing this new class of estimators is to see its behavior when using both conventional and nonconventional measures.

Rewriting \( \tilde{Y}^{(p)} \) in terms of errors up to first order of approximation, we have
\[
(\hat{Y}^{(p)} - Y) \approx (T_1 - 1)Y + T_1Y\left[\Theta_0 - \frac{1}{2}\Phi_0\Theta_1 - \frac{1}{2}\Phi_0\Theta_0\Theta_1 + \frac{3}{8}\Phi_0^2\Theta_1^2 + T_2\left[1 - \frac{1}{2}\Phi_0\Theta_1 + \frac{3}{8}\Phi_0^2\Theta_1^2\right]\right].
\] (35)

The bias of \(\tilde{Y}^{(p)}\) to first degree of approximation is given by
\[
B(\tilde{Y}^{(p)}) = (T_1 - 1)Y + T_1Y\Theta_0 + T_2D_i.
\] (36)

\[
\text{MSE}(\tilde{Y}^{(p)}) \approx (T_1 - 1)^2Y^2 + T_1^2Y^2A_i + T_2^2B_i - 2T_1Y^2C_i - 2T_2YD_i + 2T_1T_2YE_i,
\] (37)

where \(A_i = \gamma(C_y^2 + \phi_i^2C_x^2 - 2\phi_iC_{yx}),\ B_i = 1 + \gamma\phi_i^2C_x^2,\ C_i = \gamma((3/8)\phi_i^2C_x^2 - (1/2)\phi_iC_{yx}),\ D_i = 1 + (3/8)\phi_i^2C_x^2,\ \text{and}\ E_i = 1 + \gamma\phi_i^2C_x^2 - \phi_iC_{yx}.

The optimum values of \(T_1\) and \(T_2\) are \(T_{1(\text{opt})} = ((B_iC_i - D_iE_i + B_i)/ (A_iB_i - E_i^2 + B_i))\) and \(T_{2(\text{opt})} = ((Y_k(\text{opt})C_i - C_iE_i + D_i - E_i))/(A_iB_i - E_i^2 + B_i)).\)

\[
\text{MSE}(\tilde{Y}^{(p)})_{\text{min}} = Y^2\left[1 - \frac{(A_iD_i^2 + B_iC_i^2 - 2C_iD_iE_i + 2B_iC_i + D_i^2 - 2D_iE_i + B_i)}{(A_iB_i - E_i^2 + B_i)}\right].
\] (38)

4. Comparison of Estimators
Now we compare the proposed class of estimators with other existing estimators discussed here.

**Condition 1.** By (14) and (38), \(\text{MSE}(\tilde{Y}^{(p)})_{\text{min}} < \text{MSE}(\tilde{Y}^{(0)})\) if \(\gamma(C_y^2 + \phi_i^2C_x^2 - 2\phi_iC_{yx}) - 1 + \frac{\Delta_1}{\Delta_2} > 0.\) (42)

**Condition 2.** By (4) and (37), \(\text{MSE}(\tilde{Y}^{(p)})_{\text{min}} < \text{MSE}(\tilde{Y}^{(R)})\) if \(\gamma(C_y^2 + C_x^2 - 2C_{yx}) - 1 + \frac{\Delta_1}{\Delta_2} > 0.\) (39)

**Condition 3.** By (7) and (38), \(\text{MSE}(\tilde{Y}^{(p)})_{\text{min}} < \text{MSE}(\tilde{Y}^{(EP)})\) if \(\gamma(C_y^2 + \frac{1}{4}C_x^2 - C_{yx}) - 1 + \frac{\Delta_1}{\Delta_2} > 0.\) (40)

**Condition 4.** By (9), (11), (20), and (38), \(\text{MSE}(\tilde{Y}^{(p)})_{\text{min}} < \text{MSE}(\tilde{Y}^{(D)})_{\text{min}}\) if \(\gamma(C_y^2(1 - \phi_i^2C_{yx}) - 1 + \frac{\Delta_1}{\Delta_2} > 0.\) (41)

**Condition 5.** By (14) and (38), \(\text{MSE}(\tilde{Y}^{(p)})_{\text{min}} < \text{MSE}(\tilde{Y}^{(S)})\) if \(\frac{\gamma(C_y^2 + \phi_i^2C_x^2 - 2\phi_iC_{yx}) - 1 + \frac{\Delta_1}{\Delta_2}}{\Delta_1 - \Delta_2} > 0.\) (43)

**Condition 6.** By (17) and (38), \(\text{MSE}(\tilde{Y}^{(p)})_{\text{min}} < \text{MSE}(\tilde{Y}^{(YK)})_{\text{min}}\) if \(\frac{Y_{\text{opt}}^2}{\Delta_1 - \Delta_2} > 0.\) (44)

**Condition 7.** By (23) and (38), \(\text{MSE}(\tilde{Y}^{(p)})_{\text{min}} < \text{MSE}(\tilde{Y}^{(GK)})_{\text{min}}\) if \(\Delta_1 - \Delta_1B_g + B_g^2 - 2B_gC_g + C_g^2 - 2D_gE_g + B_g + \Delta_2 - \Delta_2 \geq 0,\) (45)

where \(\Delta_1B_g = A_gD_g^2 + B_gC_g^2 - 2C_gD_gE_g + 2B_gC_g + D_g^2 - 2D_gE_g + B_g + \Delta_2 - \Delta_2 \geq 0.\) (46)
Table 2: Members of Irfan et al. [10] and Shabbir et al. [7].

| \(u\) | \(v\) | Irfan et al. [10] | Shabbir et al. [7] |
|------|------|------------------|------------------|
| 1    | 0    | \(\hat{Y}_1^{(i_1)} = I_{1a} \mathbf{x} + 1_{2a} (\mathbf{x} - \mathbf{x})\) | \(\hat{Y}_1^{(S)} = [s_1 \mathbf{y} + s_2 (\mathbf{x} - \mathbf{x}) + \Omega(\mathbf{x}) \] |
| 1    | \(C_x\) | \(\hat{Y}_2^{(i_1)} = I_{1a} \mathbf{x} + 1_{2a} (\mathbf{x} - \mathbf{x})\) | \(\hat{Y}_1^{(S)} = [s_1 \mathbf{y} + s_2 (\mathbf{x} - \mathbf{x}) + \Omega(\mathbf{x}) \] |
| \(C_x\) | \(\rho_{xy}\) | \(\hat{Y}_3^{(i_1)} = I_{1a} \mathbf{x} + 1_{2a} (\mathbf{x} - \mathbf{x})\) | \(\hat{Y}_3^{(S)} = [s_1 \mathbf{y} + s_2 (\mathbf{x} - \mathbf{x}) + \Omega(\mathbf{x}) \] |
| \(\beta_1(\mathbf{x})\) | \(\rho_{xy}\) | \(\hat{Y}_4^{(i_1)} = I_{1a} \mathbf{x} + 1_{2a} (\mathbf{x} - \mathbf{x})\) | \(\hat{Y}_4^{(S)} = [s_1 \mathbf{y} + s_2 (\mathbf{x} - \mathbf{x}) + \Omega(\mathbf{x}) \] |
| \(\beta_2(\mathbf{x})\) | \(\beta_2(\mathbf{x})\) | \(\hat{Y}_5^{(i_1)} = I_{1a} \mathbf{x} + 1_{2a} (\mathbf{x} - \mathbf{x})\) | \(\hat{Y}_5^{(S)} = [s_1 \mathbf{y} + s_2 (\mathbf{x} - \mathbf{x}) + \Omega(\mathbf{x}) \] |
| 1    | \(\rho_{xy}\) | \(\hat{Y}_6^{(i_1)} = I_{1a} \mathbf{x} + 1_{2a} (\mathbf{x} - \mathbf{x})\) | \(\hat{Y}_6^{(S)} = [s_1 \mathbf{y} + s_2 (\mathbf{x} - \mathbf{x}) + \Omega(\mathbf{x}) \] |
| \(Q_D\) | \(T_M\) | \(\hat{Y}_7^{(i_1)} = I_{1a} \mathbf{x} + 1_{2a} (\mathbf{x} - \mathbf{x})\) | \(\hat{Y}_7^{(S)} = [s_1 \mathbf{y} + s_2 (\mathbf{x} - \mathbf{x}) + \Omega(\mathbf{x}) \] |
| \(Q_D\) | \(M_R\) | \(\hat{Y}_8^{(i_1)} = I_{1a} \mathbf{x} + 1_{2a} (\mathbf{x} - \mathbf{x})\) | \(\hat{Y}_8^{(S)} = [s_1 \mathbf{y} + s_2 (\mathbf{x} - \mathbf{x}) + \Omega(\mathbf{x}) \] |
| \(\rho_{xy}\) | \(Q_D\) | \(\hat{Y}_9^{(i_1)} = I_{1a} \mathbf{x} + 1_{2a} (\mathbf{x} - \mathbf{x})\) | \(\hat{Y}_9^{(S)} = [s_1 \mathbf{y} + s_2 (\mathbf{x} - \mathbf{x}) + \Omega(\mathbf{x}) \] |
| \(\rho_{xy}\) | \(Q_D\) | \(\hat{Y}_{10}^{(i_1)} = I_{1a} \mathbf{x} + 1_{2a} (\mathbf{x} - \mathbf{x})\) | \(\hat{Y}_{10}^{(S)} = [s_1 \mathbf{y} + s_2 (\mathbf{x} - \mathbf{x}) + \Omega(\mathbf{x}) \] |

where \(\Delta_{12} = A_n D_n^2 + B_n C_n^2 - 2C_n D_n E_n + 2B_n C_n + D_n^2 - 2D_n E_n + B_n\) and \(\Delta_{21} = A_n B_n - E_n^2 + B_n\).

Table 3: Members of Irfan et al. [11] and proposed class of estimators.

| \(u\) | \(v\) | Irfan et al. [11] | Proposed class of estimators |
|------|------|------------------|------------------|
| 1    | 0    | \(\hat{Y}_1^{(i_1)} = I_{1a} \mathbf{x} + 1_{2a} (\mathbf{x} - \mathbf{x}) + \Omega(\mathbf{x}) \] | \(\hat{Y}_1^{(P)} = [T_1 \mathbf{x} + T_2 \exp(1) \) |
| 1    | \(C_x\) | \(\hat{Y}_2^{(i_1)} = I_{1a} \mathbf{x} + 1_{2a} (\mathbf{x} - \mathbf{x}) + \Omega(\mathbf{x}) \] | \(\hat{Y}_2^{(P)} = [T_1 \mathbf{x} + T_2 \exp(2) \) |
| \(C_x\) | \(\rho_{xy}\) | \(\hat{Y}_3^{(i_1)} = I_{1a} \mathbf{x} + 1_{2a} (\mathbf{x} - \mathbf{x}) + \Omega(\mathbf{x}) \] | \(\hat{Y}_3^{(P)} = [T_1 \mathbf{x} + T_2 \exp(3) \) |
| \(\beta_1(\mathbf{x})\) | \(\rho_{xy}\) | \(\hat{Y}_4^{(i_1)} = I_{1a} \mathbf{x} + 1_{2a} (\mathbf{x} - \mathbf{x}) + \Omega(\mathbf{x}) \] | \(\hat{Y}_4^{(P)} = [T_1 \mathbf{x} + T_2 \exp(4) \) |
| \(\beta_2(\mathbf{x})\) | \(\beta_2(\mathbf{x})\) | \(\hat{Y}_5^{(i_1)} = I_{1a} \mathbf{x} + 1_{2a} (\mathbf{x} - \mathbf{x}) + \Omega(\mathbf{x}) \] | \(\hat{Y}_5^{(P)} = [T_1 \mathbf{x} + T_2 \exp(5) \) |
| 1    | \(\rho_{xy}\) | \(\hat{Y}_6^{(i_1)} = I_{1a} \mathbf{x} + 1_{2a} (\mathbf{x} - \mathbf{x}) + \Omega(\mathbf{x}) \] | \(\hat{Y}_6^{(P)} = [T_1 \mathbf{x} + T_2 \exp(6) \) |
| \(Q_D\) | \(T_M\) | \(\hat{Y}_7^{(i_1)} = I_{1a} \mathbf{x} + 1_{2a} (\mathbf{x} - \mathbf{x}) + \Omega(\mathbf{x}) \] | \(\hat{Y}_7^{(P)} = [T_1 \mathbf{x} + T_2 \exp(7) \) |
| \(Q_D\) | \(M_R\) | \(\hat{Y}_8^{(i_1)} = I_{1a} \mathbf{x} + 1_{2a} (\mathbf{x} - \mathbf{x}) + \Omega(\mathbf{x}) \] | \(\hat{Y}_8^{(P)} = [T_1 \mathbf{x} + T_2 \exp(8) \) |
| \(\rho_{xy}\) | \(Q_D\) | \(\hat{Y}_9^{(i_1)} = I_{1a} \mathbf{x} + 1_{2a} (\mathbf{x} - \mathbf{x}) + \Omega(\mathbf{x}) \] | \(\hat{Y}_9^{(P)} = [T_1 \mathbf{x} + T_2 \exp(9) \) |
| \(\rho_{xy}\) | \(Q_D\) | \(\hat{Y}_{10}^{(i_1)} = I_{1a} \mathbf{x} + 1_{2a} (\mathbf{x} - \mathbf{x}) + \Omega(\mathbf{x}) \] | \(\hat{Y}_{10}^{(P)} = [T_1 \mathbf{x} + T_2 \exp(10) \) |

where \(\Delta_{12} = A_n D_n^2 + B_n C_n^2 - 2C_n D_n E_n + 2B_n C_n + D_n^2 - 2D_n E_n + B_n\) and \(\Delta_{21} = A_n B_n - E_n^2 + B_n\).

Condition 9. By (29) and (38), \(\text{MSE}(\hat{Y}^{(P)}) < \text{MSE}(\hat{Y}^{(S)})\) if

\[
\frac{\Delta_1}{\Delta_2} - \frac{\delta_{11} + \delta_{12}}{\delta_{22}} > 0,
\]

where \(\Omega(\mathbf{x}) = (\mathbf{x} - \mathbf{x})\), \(\delta_{11} = A_n D_n^2 + B_n C_n^2 - 2C_n D_n E_n + \) and \(\Delta_{12} = A_n B_n - E_n^2 + B_n\), \(\delta_{22} = C_n^2 + 0.25 \varphi \Omega(\mathbf{x})^2 - \varphi_1 C_n^2\).
Condition 10. By (32) and (38), \( \text{MSE}(\bar{Y}^{(SP)})_{\text{min}} < \text{MSE}(\bar{Y}^{(I)})_{\text{min}} \) if
\[
\left[ \frac{\Delta_{iY}}{\Delta_{2}} - \frac{\Delta_{iY}}{\Delta_{2}} + \Delta_{3iB} \right] > 0,
\] (47)
where \( \Delta_{iY} = A_bD_b^2 + B_bC_b - 2C_bD_bE_b \) and \( \Delta_{3iB} = A_bB_b - E_b^2 + B_b, \Delta_{3iB} = C_b^2 + 0.25\psi_i^2C_b^2 - \psi_iC_b\).

5. Numerical Examples

Both simulation and numerical studies are conducted to observe the performances of different estimators.

5.1. Simulation Study. In this section, a simulation study is conducted to assess the performances of all estimators considered here. We consider two finite populations of size 1000 generated from a bivariate normal distribution with the same theoretical means of \([Y, X]\) as \(\mu = [5, 5]\) but different covariance matrices as given below.

Population 1:
\[
\Sigma = \begin{bmatrix} 9 & 1.9 \\ 1.9 & 4 \end{bmatrix}.
\] (48)

Population 2:
\[
\Sigma = \begin{bmatrix} 6 & 1.1 \\ 1.1 & 2 \end{bmatrix}.
\] (49)

For each population, we consider a sample of sizes 50 and 100. The following steps are performed to carry out the simulation study.

Step 1. Select a SRSWOR of size \(n\) from a population of size \(N\).

Step 2. Use a sample data from Step 1 to find the MSE values of all the estimators.

Step 3. Steps 1 and 2 are repeated 10,000 times.

Step 4. Obtain 10,000 values for MSEs.

Step 5. Average of 10,000 values obtained in Step 4 represents the simulated MSE of each estimator.

The simulated MSEs based on Populations 1 and 2 for sample sizes 50 and 100 are given in Table 4.

| Table 4: MSE values under simulation for Populations 1 and 2 when using \(u = 1, v = 0\). | Population 1 | Population 2 |
|---|---|---|
| | 50 | 100 | 50 | 100 |
| \(\bar{Y}^{(0)}\) | 0.197069 | 0.087232 | 0.12917 | 0.053533 |
| \(\bar{Y}^{(I)}\) | 0.181363 | 0.083289 | 0.11917 | 0.053420 |
| \(\bar{Y}^{(SP)}\) | 0.165596 | 0.076492 | 0.111221 | 0.051525 |
| \(\bar{Y}^{(P)}\) | 0.164650 | 0.076273 | 0.110307 | 0.051128 |
| \(\bar{Y}^{(1)}\) | 0.165481 | 0.076373 | 0.110646 | 0.051170 |
| \(\bar{Y}^{(2)}\) | 0.165596 | 0.076214 | 0.110902 | 0.051525 |
| \(\bar{Y}^{(3)}\) | 0.164704 | 0.076914 | 0.111221 | 0.051409 |
| \(\bar{Y}^{(4)}\) | 0.166345 | 0.076924 | 0.113335 | 0.052623 |

5.2. Real Datasets. We use the following 7 real datasets for a numerical study.

Population 1 (source: Singh and Chaudhary [21]):

- \(Y = \) area under wheat crop in acres during 1974 in 34 villages.
- \(X = \) area under wheat crop in acres during 1971 in 34 villages.

The summary statistics are \(N = 34, n = 20, X = 856.4117, X = 208.8823, C_x = 0.8561, C_x = 0.7205, \rho_{xy} = 0.4491, \beta_1(x) = 0.9782, \beta_2(x) = 0.0978, Q_D = 80.25, T_M = 162.25, M_R = 284.5, \) and \(H_L = 190.0\).

Population 2 (source: Cochran [19]):

- \(Y = \) number of inhabitants (in 1000’s) in 1930.
- \(X = \) number of inhabitants (in 1000’s) in 1920.

The summary statistics are \(N = 49, n = 12, X = 127.7959, X = 103.1429, C_y = 0.9634, C_x = 1.0122, \rho_{xy} = 0.9817, \beta_1(x) = 2.2553, \beta_2(x) = 5.1412, Q_D = 38.50, T_M = 72.75, M_R = 254.5, \) and \(H_L = 77.25\).

Population 3 (source: Singh and Mangat [22]):

- \(Y = \) number of tube wells.
- \(X = \) net irrigated area in hectares for 69 villages.

The summary statistics are \(N = 69, n = 15, X = 135.2608, X = 345.7536, C_y = 0.8422, C_x = 0.8479, \rho_{xy} = 0.9224, \beta_1(x) = 2.3808, \beta_2(x) = 7.2159, Q_D = 138.0, T_M = 274.5, M_R = 900, \) and \(H_L = 277.0\).

Population 4 (source: Singh and Mangat [22]):

- \(Y = \) average duration of sleep in hours.
- \(X = \) age of a person.

The summary statistics are \(N = 30, n = 10, X = 6.3767, X = 66.9333, C_y = 0.1633, C_x = 0.1436, \rho_{xy} = -0.8669, \beta_1(x) = 0.2984, \beta_2(x) = -0.7686, Q_D = 7.25, T_M = 66.5, M_R = 69.0, \) and \(H_L = 67.0\).

Population 5 (source: Gujarati [23, p. 433]):

- \(Y = \) average miles per gallons.

\(\bar{Y} = \) average miles per gallons.
Table 5: PRE of different estimators w.r.t $\bar{Y}^{(0)}$ for all populations using $u = 1, v = 0.$

| Estimator | Pop. 1 | Pop. 2 | Pop. 3 | Pop. 4 | Pop. 5 | Pop. 6 | Pop. 7 |
|-----------|--------|--------|--------|--------|--------|--------|--------|
| $\bar{Y}^{(0)}$ | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 |
| $\bar{Y}^{(R)}$ | 105.00 | 437.85 | 639.81 | 30.32 | 56.74 | 100.97 | 46.80 |
| $\bar{Y}^{(EP)}$ | 125.14 | 408.93 | 307.93 | 51.13 | 74.82 | 124.73 | 70.22 |
| $\bar{Y}^{(D)}, \bar{Y}^{(S)}, \bar{Y}^{(K)}$ | 125.26 | 2757.47 | 670.34 | 402.44 | 191.23 | 124.73 | 115.69 |
| $\bar{Y}^{(GK)}$ | 125.13 | 408.93 | 307.93 | 51.13 | 74.82 | 124.73 | 70.22 |
| $\bar{Y}^{(YK)}$ | 127.13 | 2898.92 | 683.17 | 402.78 | 191.64 | 131.30 | 116.08 |
| $\bar{Y}^{(Ia)}$ | 126.95 | 410.92 | 309.96 | 51.45 | 75.33 | 131.19 | 70.74 |
| $\bar{Y}^{(Sh)}$ | 127.53 | 3269.75 | 698.26 | 403.00 | 191.68 | 132.89 | 116.15 |
| $\bar{Y}^{(L)}$ | 127.53 | 3269.75 | 698.26 | 403.00 | 191.68 | 132.89 | 116.15 |
| $\bar{Y}^{(P)}$ | 717.88 | 11691.20 | 2755.20 | 2084.59 | 4286.18 | 681.20 | 803.92 |

Table 6: PRE of different estimators w.r.t $\bar{Y}^{(0)}$ for all populations using $u = 1, v = C_x.$

| Estimator | Pop. 1 | Pop. 2 | Pop. 3 | Pop. 4 | Pop. 5 | Pop. 6 | Pop. 7 |
|-----------|--------|--------|--------|--------|--------|--------|--------|
| $\bar{Y}^{(0)}$ | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 |
| $\bar{Y}^{(S)}$ | 125.12 | 401.24 | 306.95 | 51.20 | 74.84 | 124.73 | 70.22 |
| $\bar{Y}^{(GK)}$ | 127.13 | 2894.41 | 683.11 | 402.78 | 191.64 | 131.29 | 116.08 |
| $\bar{Y}^{(YK)}$ | 126.93 | 403.21 | 308.97 | 51.52 | 75.36 | 131.17 | 70.80 |
| $\bar{Y}^{(Ia)}$ | 127.46 | 2777.65 | 676.42 | 405.41 | 192.03 | 132.77 | 116.44 |
| $\bar{Y}^{(Sh)}$ | 127.53 | 3249.28 | 698.08 | 403.00 | 191.68 | 132.90 | 116.15 |
| $\bar{Y}^{(L)}$ | 127.53 | 3280.52 | 698.35 | 403.00 | 191.68 | 132.98 | 116.15 |
| $\bar{Y}^{(P)}$ | 722.77 | 11882.80 | 2768.17 | 2093.54 | 4295.75 | 686.07 | 807.59 |

Table 7: PRE of different estimators w.r.t $\bar{Y}^{(0)}$ for all populations using $u = C_x, v = \rho_{yx}.$

| Estimator | Pop. 1 | Pop. 2 | Pop. 3 | Pop. 4 | Pop. 5 | Pop. 6 | Pop. 7 |
|-----------|--------|--------|--------|--------|--------|--------|--------|
| $\bar{Y}^{(0)}$ | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 |
| $\bar{Y}^{(S)}$ | 125.13 | 401.55 | 306.67 | 48.27 | 73.73 | 124.73 | 69.75 |
| $\bar{Y}^{(GK)}$ | 127.13 | 2894.59 | 683.09 | 402.82 | 191.65 | 131.29 | 116.09 |
| $\bar{Y}^{(YK)}$ | 126.94 | 403.52 | 308.70 | 48.61 | 74.25 | 131.17 | 70.27 |
| $\bar{Y}^{(Ia)}$ | 127.46 | 2777.79 | 676.40 | 406.17 | 192.07 | 132.74 | 116.45 |
| $\bar{Y}^{(Sh)}$ | 127.53 | 3250.12 | 698.03 | 403.11 | 191.69 | 132.87 | 116.15 |
| $\bar{Y}^{(L)}$ | 127.53 | 3280.06 | 698.37 | 402.96 | 191.68 | 132.98 | 116.14 |
| $\bar{Y}^{(P)}$ | 722.11 | 11874.74 | 2771.85 | 1726.02 | 3877.55 | 685.02 | 777.66 |

$X =$top speed miles per hour of 81 cars.
The summary statistics are $N = 81, \ n = 18,$ $\overline{Y} = 33.8346, \overline{X} = 112.457, \ C_x = 0.2972, \ C_y = 0.1256, \ \rho_{yx} = 0.50, \ b(x) = 1.9016, \ b(x) = 4.1454, Q_D = 5.0, \ T_M = 109.5, \ M_R = 127.5, \ \text{and} \ H_L = 108.75.$

Population 6 (source: Singh and Chaudhary [21]):

$Y =$area under wheat crop in acres during 1973 in 34 villages.
The summary statistics are $N = 34, \ n = 10, \ \overline{Y} = 856.4117, \overline{X} = 199.4412, \ C_x = 0.8561, \ C_y = 0.7531, \ \rho_{yx} = 0.4453, \ b(x) = 1.1823, \ b(x) = 1.0445, Q_D = 89.375, T_M = 165.562, \ M_R = 320.0, \ \text{and} \ H_L = 184.0.$

Population 7 (source: Gujarati [23, p. 433]):

$Y =$average miles per gallons.
Table 8: PRE of different estimators w.r.t $\tilde{Y}^{(0)}$ for all populations using $u = \beta_1 (x), v = \rho_{yx}$.

| Estimator | Pop. 1 | Pop. 2 | Pop. 3 | Pop. 4 | Pop. 5 | Pop. 6 | Pop. 7 |
|-----------|--------|--------|--------|--------|--------|--------|--------|
| $\tilde{Y}^{(0)}$ | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 |
| $\tilde{Y}^{(S)}$ | 125.13 | 405.58 | 307.47 | 49.80  | 74.75  | 124.73 | 70.39  |
| $\tilde{Y}^{(GK)}$ | 127.13 | 2896.96 | 683.14 | 402.80 | 191.64 | 131.29 | 116.08 |
| $\tilde{Y}^{(YK)}$ | 126.94 | 407.56 | 309.51 | 50.12  | 75.26  | 131.18 | 70.91  |
| $\tilde{Y}^{(Ia)}$ | 127.46 | 2779.64 | 676.47 | 405.75 | 192.03 | 132.75 | 116.43 |
| $\tilde{Y}^{(SH)}$ | 127.53 | 3260.80 | 698.18 | 403.05 | 191.68 | 132.88 | 116.14 |
| $\tilde{Y}^{(P)}$ | 720.99 | 11773.98 | 2761.12 | 1907.81 | 4258.56 | 683.63 | 814.09 |

Table 9: PRE of different estimators w.r.t $\tilde{Y}^{(0)}$ for all populations using $u = \beta_1 (x), v = \rho_{yx}$.

| Estimator | Pop. 1 | Pop. 2 | Pop. 3 | Pop. 4 | Pop. 5 | Pop. 6 | Pop. 7 |
|-----------|--------|--------|--------|--------|--------|--------|--------|
| $\tilde{Y}^{(0)}$ | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 |
| $\tilde{Y}^{(S)}$ | 124.91 | 405.55 | 307.54 | 50.96  | 74.91  | 124.72 | 70.03  |
| $\tilde{Y}^{(GK)}$ | 127.10 | 2896.94 | 683.14 | 402.79 | 191.64 | 131.28 | 116.08 |
| $\tilde{Y}^{(YK)}$ | 126.66 | 407.54 | 309.57 | 51.28  | 75.42  | 131.16 | 70.56  |
| $\tilde{Y}^{(Ia)}$ | 127.36 | 2779.63 | 676.47 | 405.47 | 192.03 | 132.71 | 116.44 |
| $\tilde{Y}^{(SH)}$ | 127.46 | 3260.73 | 698.19 | 403.01 | 191.68 | 132.85 | 116.15 |
| $\tilde{Y}^{(P)}$ | 787.24 | 11773.98 | 2761.24 | 2060.52 | 4321.19 | 688.53 | 793.55 |

Table 10: PRE of different estimators w.r.t $\tilde{Y}^{(0)}$ for all populations using $u = 1, v = \rho_{yx}$.

| Estimator | Pop. 1 | Pop. 2 | Pop. 3 | Pop. 4 | Pop. 5 | Pop. 6 | Pop. 7 |
|-----------|--------|--------|--------|--------|--------|--------|--------|
| $\tilde{Y}^{(0)}$ | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 |
| $\tilde{Y}^{(S)}$ | 125.13 | 401.47 | 306.86 | 50.74  | 74.69  | 124.73 | 70.11  |
| $\tilde{Y}^{(GK)}$ | 127.10 | 2896.94 | 683.14 | 402.79 | 191.64 | 131.28 | 116.08 |
| $\tilde{Y}^{(YK)}$ | 126.66 | 407.54 | 309.57 | 51.28  | 75.42  | 131.16 | 70.56  |
| $\tilde{Y}^{(Ia)}$ | 127.36 | 2779.63 | 676.47 | 405.47 | 192.03 | 132.71 | 116.44 |
| $\tilde{Y}^{(SH)}$ | 127.46 | 3260.73 | 698.19 | 403.01 | 191.68 | 132.85 | 116.15 |
| $\tilde{Y}^{(P)}$ | 787.24 | 11773.98 | 2761.24 | 2060.52 | 4321.19 | 688.53 | 793.55 |

Table 11: PRE of different estimators w.r.t $\tilde{Y}^{(0)}$ for all populations using $u = Q_d, v = T_M$.

| Estimator | Pop. 1 | Pop. 2 | Pop. 3 | Pop. 4 | Pop. 5 | Pop. 6 | Pop. 7 |
|-----------|--------|--------|--------|--------|--------|--------|--------|
| $\tilde{Y}^{(0)}$ | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 |
| $\tilde{Y}^{(S)}$ | 125.10 | 394.85 | 305.65 | 54.94  | 78.39  | 124.73 | 70.39  |
| $\tilde{Y}^{(GK)}$ | 127.10 | 2890.67 | 683.03 | 402.74 | 191.63 | 131.27 | 116.07 |
| $\tilde{Y}^{(YK)}$ | 126.94 | 396.80 | 307.66 | 55.25  | 78.88  | 131.13 | 72.94  |
| $\tilde{Y}^{(Ia)}$ | 127.46 | 2774.88 | 676.41 | 405.52 | 191.93 | 132.68 | 116.38 |
| $\tilde{Y}^{(SH)}$ | 127.50 | 3249.88 | 698.06 | 403.02 | 191.68 | 132.87 | 116.15 |
| $\tilde{Y}^{(P)}$ | 720.99 | 11876.99 | 2769.31 | 2031.02 | 4233.73 | 684.08 | 797.96 |
### Table 12: PRE of different estimators w.r.t $\hat{Y}^{(0)}$ for all populations using $u = Q_D, v = M_R$.

| Estimator | Pop. 1 | Pop. 2 | Pop. 3 | Pop. 4 | Pop. 5 | Pop. 6 | Pop. 7 |
|-----------|--------|--------|--------|--------|--------|--------|--------|
| $\hat{Y}^{(0)}$ | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 |
| $\hat{Y}^{(S)}$ | 125.07 | 364.40 | 300.62 | 55.08 | 78.88 | 124.70 | 72.44 |
| $\hat{Y}^{(GK)}$ | 127.12 | 2872.92 | 682.73 | 402.74 | 191.63 | 131.25 | 116.07 |
| $\hat{Y}^{(YK)}$ | 126.86 | 366.28 | 302.59 | 55.38 | 79.37 | 131.07 | 72.94 |
| $\hat{Y}^{(Ia)}$ | 127.43 | 2764.35 | 675.88 | 404.67 | 191.92 | 132.60 | 116.38 |
| $\hat{Y}^{(SH)}$ | 127.50 | 3154.95 | 696.93 | 402.89 | 191.66 | 132.77 | 116.13 |
| $\hat{Y}^{(P)}$ | 127.55 | 3250.10 | 698.97 | 403.09 | 191.70 | 132.95 | 116.16 |
| | 742.09 | 12980.43 | 2855.80 | 2718.68 | 6448.68 | 704.53 | 952.27 |

### Table 13: PRE of different estimators w.r.t $\hat{Y}^{(0)}$ for all populations using $u = H_L, v = T_M$.

| Estimator | Pop. 1 | Pop. 2 | Pop. 3 | Pop. 4 | Pop. 5 | Pop. 6 | Pop. 7 |
|-----------|--------|--------|--------|--------|--------|--------|--------|
| $\hat{Y}^{(0)}$ | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 |
| $\hat{Y}^{(S)}$ | 125.12 | 401.76 | 306.78 | 51.57 | 75.01 | 124.72 | 70.51 |
| $\hat{Y}^{(GK)}$ | 127.13 | 2894.72 | 683.10 | 402.78 | 191.64 | 131.29 | 116.08 |
| $\hat{Y}^{(YK)}$ | 126.93 | 403.73 | 308.81 | 51.89 | 75.52 | 131.17 | 71.03 |
| $\hat{Y}^{(Ia)}$ | 127.46 | 2777.89 | 676.41 | 405.33 | 191.03 | 132.72 | 116.43 |
| $\hat{Y}^{(SH)}$ | 127.52 | 3250.67 | 698.05 | 403.01 | 191.68 | 132.90 | 116.15 |
| $\hat{Y}^{(P)}$ | 127.53 | 3279.76 | 698.36 | 403.01 | 191.68 | 132.90 | 116.15 |
| | 723.68 | 11869.39 | 2770.36 | 2146.79 | 6463.21 | 687.02 | 820.74 |

### Table 14: PRE of different estimators w.r.t $\hat{Y}^{(0)}$ for all populations using $u = 1, v = Q_D$.

| Estimator | Pop. 1 | Pop. 2 | Pop. 3 | Pop. 4 | Pop. 5 | Pop. 6 | Pop. 7 |
|-----------|--------|--------|--------|--------|--------|--------|--------|
| $\hat{Y}^{(0)}$ | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 |
| $\hat{Y}^{(S)}$ | 122.04 | 252.99 | 306.78 | 54.19 | 75.73 | 121.69 | 73.30 |
| $\hat{Y}^{(GK)}$ | 126.96 | 2811.58 | 683.10 | 402.75 | 191.64 | 130.55 | 116.07 |
| $\hat{Y}^{(YK)}$ | 123.56 | 255.14 | 308.81 | 54.50 | 76.24 | 126.90 | 73.79 |
| $\hat{Y}^{(Ia)}$ | 126.96 | 2767.31 | 676.41 | 404.82 | 192.00 | 130.59 | 116.35 |
| $\hat{Y}^{(SH)}$ | 127.15 | 2913.59 | 683.43 | 402.91 | 191.67 | 131.23 | 116.12 |
| $\hat{Y}^{(P)}$ | 128.27 | 4958.19 | 742.29 | 403.07 | 191.68 | 137.32 | 116.17 |
| | 1365.82 | 20418.93 | 5286.47 | 2559.99 | 4675.49 | 1388.08 | 1010.57 |

### Table 15: PRE of different estimators w.r.t $\hat{Y}^{(0)}$ for all populations using $u = \rho_{ys}, v = Q_D$.

| Estimator | Pop. 1 | Pop. 2 | Pop. 3 | Pop. 4 | Pop. 5 | Pop. 6 | Pop. 7 |
|-----------|--------|--------|--------|--------|--------|--------|--------|
| $\hat{Y}^{(0)}$ | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 |
| $\hat{Y}^{(S)}$ | 117.96 | 251.52 | 210.40 | 47.08 | 73.37 | 117.26 | 57.51 |
| $\hat{Y}^{(GK)}$ | 126.87 | 2810.84 | 677.78 | 402.84 | 191.65 | 130.24 | 116.16 |
| $\hat{Y}^{(YK)}$ | 119.38 | 253.68 | 212.26 | 47.83 | 73.89 | 122.13 | 58.15 |
| $\hat{Y}^{(Ia)}$ | 126.78 | 2767.73 | 671.21 | 406.54 | 192.08 | 129.90 | 117.01 |
| $\hat{Y}^{(SH)}$ | 126.98 | 2910.95 | 682.86 | 403.16 | 191.69 | 131.58 | 116.32 |
| $\hat{Y}^{(P)}$ | 245.87 | 5081.53 | 753.70 | 402.95 | 191.67 | — | 116.12 |
| | 2445.85 | 20609.07 | 5537.93 | 1596.88 | 3752.54 | 2631.19 | 362.22 |
$X=$cubic feet of cab space of 81 cars.

The summary statistics are $N=81$, $n=18$, $\bar{Y}=33.8346$, $\bar{X}=98.7654$, $C_y=0.2972$, $C_x=0.2258$, $\rho_{yx}=-0.3683$, $\beta_1(x)=0.5902$, $\beta_2(x)=0.9202$, $Q_{D}=\bar{Y}^2=12.0$, $\gamma_M=101.0$, $\gamma_R=105.0$, and $H_1=98.0$.

The results based on Populations 1–7 are given in Tables 5–16, where we use the following expression to obtain the percent relative efficiency (PRE):

$$\text{PRE} = \frac{\text{Var}(\hat{\mu}_i^{(0)})}{\text{MSE}(\hat{\mu}_i^{(j)})\text{ or } \text{MSE}(\hat{\mu}_i^{(j)})_{\min}} \times 100, \tag{50}$$

where $j=0, R, EP, D=(S_1, K), S_2, GK, YK, I_a, SH, I_b, P$.

The results based on 7 real datasets are given in Tables 5–16.

In Tables 5–16, PRE values are given based on summary statistics of seven real datasets to observe the performances of all estimators. One can see that the estimators $\hat{\gamma}^{(j)}(\gamma, j=D, S_1, K)$, $\hat{\gamma}^{(j)}(\gamma, j=SH, I_b)$, and $\hat{\gamma}^{(j)}(\gamma, j=EP, S_2)$ are equally efficient. Also, some estimators $\hat{\gamma}^{(j)}(\gamma, j=R, EP, S_2, YK)$ show very poor performances in Populations 4, 5, and 7 because of negative correlations. In Tables 5–16, all of the estimators have very large PRE for Population 2 due to the highest value of $\rho_{yx}=0.9817$. Also, in all cases, the proposed estimator shows the best performance.

### Table 6: PRE of different estimators w.r.t $\hat{\gamma}^{(0)}$ for all populations using $u = \rho_{yx}$, $v = \beta_2(x)$.

| Estimator | Pop. 1 | Pop. 2 | Pop. 3 | Pop. 4 | Pop. 5 | Pop. 6 | Pop. 7 |
|-----------|--------|--------|--------|--------|--------|--------|--------|
| $\hat{\gamma}^{(0)}$ | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 |
| $\hat{\gamma}^{(S_1)}$ | 125.13 | 372.64 | 299.21 | 51.53 | 73.63 | 124.72 | 69.49 |
| $\hat{\gamma}^{(GK)}$ | 127.13 | 2877.70 | 682.65 | 402.78 | 191.65 | 131.27 | 116.09 |
| $\hat{\gamma}^{(YK)}$ | 126.94 | 374.53 | 301.18 | 51.85 | 74.15 | 131.12 | 70.02 |
| $\hat{\gamma}^{(Ia)}$ | 127.47 | 2767.73 | 675.76 | 405.34 | 192.07 | 132.66 | 116.46 |
| $\hat{\gamma}^{(SH)}$ | 127.53 | 3175.50 | 696.68 | 402.99 | 191.68 | 132.81 | 116.15 |
| $\hat{\gamma}^{(Ib)}$ | 127.53 | 3331.29 | 699.12 | 403.01 | 191.68 | 132.93 | 116.14 |
| $\hat{\gamma}^{(P)}$ | 719.36 | 12705.24 | 2876.09 | 2140.11 | 3841.23 | 696.43 | 763.87 |

### Data Availability

The data used to support the findings of this study are included within the article.

### Conflicts of Interest

The authors declare that they have no conflicts of interest.

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