EXISTENCE OF OPTIMAL SOLUTIONS TO LAGRANGE PROBLEM FOR A FRACTIONAL NONLINEAR CONTROL SYSTEM WITH RIEZMANN-LIOUVILLE DERIVATIVE

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ABSTRACT. In the paper, a nonlinear control system containing the Riemann-Liouville derivative of order \( \alpha \in (0, 1) \) with a nonlinear integral performance index is studied. We discuss the existence of optimal solutions to such problem under some convexity assumption. Our study relies on the implicit function theorem for multivalued mappings.

1. Introduction. In the last years, fractional calculus has been proved to be a useful tool in the modelling of many phenomena in various fields of science. It is successfully applied, among other things, in physics (cf. [16], [17]), mechanics (cf. [35]), viscoelasticity and electrochemistry (cf. [5], [10], [11], [28], [38]). Recently, the very popular subjects of research are fractional calculus of variations (cf. [8], [9], [18], [27], [29], [30], [31]) and fractional optimal control (cf. [1], [3], [12], [13], [14], [21], [37]). In [1], [2], [15], [21], [24] numerical methods for solving fractional optimal control problems are presented.

In our paper, we consider the following fractional optimal control problem

\[
(D^\alpha_{a+} x)(t) = f(t, x(t), u(t)), \quad t \in [a, b] \subset \mathbb{R} \ a.e.,
\]

\[
(I^1_{a+} x)(a) = x_0,
\]

\[
u(t) \in M \subset \mathbb{R}^m, \quad t \in [a, b],
\]

\[
J(x, u) = \int_a^b f_0(t, x(t), u(t)) dt \to \min,
\]

where \( f : [a, b] \times \mathbb{R}^n \times M \to \mathbb{R}^n \), \( f_0 : [a, b] \times \mathbb{R}^n \times M \to \mathbb{R}, \ 0 < \alpha < 1 \). Initial condition (2) is called an initial memory value condition (cf. [33]). The first who introduced it in the study of fractional problems of the calculus of variations were Almeida and Torres (cf. [4, Theorem 6.4]). Necessary optimality conditions for the

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above problem have been obtained in [22], in the form of the Pontryagin maximum principle.

We investigate this problem in aspect of the existence of optimal solutions. Our paper is a continuation of research contained in [23] and [34]. In [23], the above problem with a linear control system (1) is studied. The existence of optimal solutions has been obtained there under convexity of the function \(f_0\) with respect to \(u\) as well as compactness and convexity of the set \(M\). In [34], existence of solutions to problem (1)–(4) has been obtained for \(f_0\) convex in \((x,u)\) and \(M = \mathbb{R}^n\). In this note, we get such a result under more general convexity assumption, namely, we assume convexity of the so called extended velocities set. This assumption is less restrictive than conditions used in [23] and [34]. A proof of the main result of the paper is based on the implicit function theorem for multivalued mappings.

The paper is organized as follows. In Section 2, we give the basic notions and facts concerning the fractional calculus. In Section 3, we obtain existence of optimal solutions to problem (1)–(4). A simple illustrative example is presented in Section 4.

2. Preliminaries. In this section, we recall some basic definitions and results concerning the fractional calculus (cf. [18], [25], [36]) as well as prove some additional facts which are used throughout this paper.

Let \(\alpha > 0\) and \(f \in L^1([a,b], \mathbb{R}^n)\). The left sided Riemann-Liouville integral of order \(\alpha\) is defined as

\[
(I_{a+}^\alpha f)(t) := \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau, \quad t \in [a,b] \text{ a.e.}
\]

Additionally, we put

\[
(I_{a+}^0 f)(t) = f(t), \quad t \in [a,b] \text{ a.e.}
\]

Let \(\alpha \in (0,1)\), \(1 \leq p < \infty\). By \(AC_{a+}^{\alpha,p} = AC_{a+}^{\alpha,p}([a,b], \mathbb{R}^n)\) we denote the set of all functions \(f : [a,b] \to \mathbb{R}^n\) of the form

\[
f(t) = \frac{1}{\Gamma(\alpha)} \frac{c}{(t-a)^{1-\alpha}} + (I_{a+}^\alpha \varphi)(t), \quad t \in [a,b] \text{ a.e.,}
\]

with \(c \in \mathbb{R}^n\), \(\varphi \in L^p([a,b], \mathbb{R}^n)\).

Now, let \(\alpha \in (0,1)\) and \(f \in L^1([a,b], \mathbb{R}^n)\). We say that the function \(f\) possesses the left-sided Riemann-Liouville derivative \(D_{a+}^\alpha f\) of order \(\alpha\) if \(I_{a+}^{1-\alpha} f\) has an absolutely continuous representant a.e. on \([a,b]\). In such a case we write \(I_{a+}^{1-\alpha} f \in AC([a,b], \mathbb{R}^n)\) and put

\[
(D_{a+}^\alpha f)(t) := \frac{d}{dt}(I_{a+}^{1-\alpha} f)(t), \quad t \in [a,b] \text{ a.e.}
\]

One can show (cf. [7]) that \(f \in AC_{a+}^{\alpha,p}\) if and only if there exists the derivative \(D_{a+}^\alpha f \in L^p([a,b], \mathbb{R}^n)\). In such a case, \(D_{a+}^\alpha f = \varphi\) a.e. on \([a,b]\) and \((I_{a+}^{1-\alpha} f)(a) = c\) (here \(\varphi\) and \(c\) are taken from (5)).

We have the following two technical results

Lemma 2.1. Let \(\alpha \in (0,1)\) and \(\varphi : [a,b] \to \mathbb{R}^n\) be a measurable function such that

\[
|\varphi(t)| \leq c(t-a)^{\lambda}, \quad t \in [a,b] \text{ a.e.,}
\]

where \(\lambda > 0\).
where \( c \geq 0, \lambda > -1 \). Then \( \varphi \in L^1([a, b], \mathbb{R}^n) \) and

\[
(I_{a+}^\alpha \varphi)(t) \leq c \frac{\Gamma(\lambda + 1)}{\Gamma(\alpha + \lambda + 1)} (t - a)^{\alpha + \lambda}, \quad t \in [a, b] \text{ a.e.}
\]

Proof. Since \( \lambda + 1 > 0 \), therefore

\[
\int_a^b |\varphi(t)| dt \leq \frac{c}{\lambda + 1} (b - a)^{\lambda + 1} < \infty.
\]

So, \( \varphi \in L^1([a, b], \mathbb{R}^n) \). Moreover,

\[
|I_{a+}^\alpha \varphi(t)| \leq \frac{c}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha - 1}(s - a)^\lambda ds = c \frac{\Gamma(\lambda + 1)}{\Gamma(\alpha + \lambda + 1)} (t - a)^{\alpha + \lambda}, \quad t \in [a, b] \text{ a.e.}
\]

The proof is completed. \( \square \)

Lemma 2.2 ([19]). If \( f \in AC([a, b], \mathbb{R}^n) \), then

\[
f \in I_{a+}^{-\alpha}(L^1) := \{ z : [a, b] \to \mathbb{R}^n; \quad z = I_{a+}^{-\alpha} \psi \text{ a.e. on } [a, b], \psi \in L^1(a, b, \mathbb{R}^n) \}
\]

for any \( \alpha \in (0, 1) \).

In [39, Theorem 1] the following fractional Gronwall lemma has been proved.

Lemma 2.3. Assume \( \beta > 0, 0 < T \leq +\infty, a(t) \) is a nonnegative locally integrable function on \([0, T]\) and \( g(t) \) is a nonnegative, nondecreasing, continuous, upper bounded function on \([0, T]\), \( u(t) \) is nonnegative and locally integrable function on \([0, T]\) such that:

\[
u(t) \leq a(t) + \int_0^t (t - s)^{\beta - 1}u(s)ds, \quad t \in [0, T]. \tag{7}\]

Then

\[
u(t) \leq a(t) + \int_0^t \left[ \sum_{n=1}^{\infty} \frac{(g(t)\Gamma(\beta))^n}{\Gamma(n\beta)} (t - s)^{n\beta - 1}a(s) \right] ds, \quad t \in [0, T]. \tag{8}\]

Using the above lemma, we shall prove the following extension of [39, Corollary 2]

Lemma 2.4. Let \( u \) be a function from Lemma 2.3 satisfying condition (7) with functions

\[
a(t) = c \frac{\Gamma(\lambda + 1)}{\Gamma(\beta + \lambda + 1)} t^{\beta + \lambda}, \quad g(t) = R \quad t \in [0, T),
\]

where \( \beta > 0, \lambda > -1, c \geq 0 \) and \( R \geq 0 \). Then,

\[
u(t) \leq c \Gamma(\lambda + 1) t^{\beta + \lambda} E_{\beta, \beta + \lambda + 1} \left( MT(\beta)t^\beta \right), \quad t \in [0, T),
\]

where \( E_{\beta, \eta} \) is the generalized Mittag-Leffler function defined by

\[
E_{\beta, \eta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta \eta + k)}.
\]
Proof. Let us note that functions $a$ and $g$ satisfy assumptions of Lemma 2.3. Consequently, condition (8) is satisfied. It is easy to check that

$$
\int_0^t \left[ \sum_{n=1}^{\infty} \left( \frac{(g(t)\Gamma(\beta))^n}{\Gamma(n\beta)} \right)^n (t-s)^{n\beta-1}a(s) \right] ds =
$$

$$
= \int_0^t \left[ \sum_{n=1}^{\infty} \left( \frac{(R\Gamma(\beta))/\Gamma(n\beta)}{\Gamma(n\beta)} \Gamma(\lambda + 1) \right) (t-s)^{n\beta-1}c \Gamma(\lambda + 1) \right] ds =
$$

$$
c\Gamma(\lambda + 1) \left[ \sum_{n=1}^{\infty} \frac{(R\Gamma(\beta)t)^n}{\Gamma(n\beta + \beta + \lambda + 1)} \right] =
$$

Thus, using condition (8) we obtain

$$
u(t) \leq c \frac{\Gamma(\lambda + 1)}{\Gamma(\beta + \lambda + 1)} t^{\beta+\lambda} + c\Gamma(\lambda + 1)t^{\beta+\lambda} \sum_{n=1}^{\infty} \frac{(R\Gamma(\beta)t^n)}{\Gamma(n\beta + \beta + \lambda + 1)} =
$$

$$
= c\Gamma(\lambda + 1)t^{\beta+\lambda} \left( \frac{1}{\Gamma(\beta + \lambda + 1)} + \sum_{n=1}^{\infty} \frac{(R\Gamma(\beta)t^n)}{\Gamma(n\beta + \beta + \lambda + 1)} \right) =
$$

$$
= c\Gamma(\lambda + 1)t^{\beta+\lambda}E_{\beta,\beta+\lambda+1}(R\Gamma(\beta)t^\beta), \quad t \in [0, T).
$$

The proof is completed. \qed

From the above lemma, we immediately obtain

**Corollary 1.** *Under assumptions of Lemma 2.4, with*

$$
a(t) = \frac{c_1}{\Gamma(\beta + 1)} t^\beta + \frac{c_2}{\Gamma(\beta + \lambda + 1)} t^{\beta+\lambda}, \quad t \in [0, T),
$$

*where $\lambda > -1$, $c_1$, $c_2 \geq 0$, it holds*

$$
u(t) \leq c_1 t^{\beta} E_{\beta,\beta+1}(R\Gamma(\beta)t^\beta) + c_2 \Gamma(\lambda + 1)t^{\beta+\lambda}E_{\beta,\beta+\lambda+1}(R\Gamma(\beta)t^\beta)
$$

*(9)*

*for all $t \in [0, T)$.*

**Remark 1.** From the proofs of lemmas 2.3 and 2.4 it follows that the interval $[0, T]$ can be replaced with an arbitrary interval $[a, b] \subset \mathbb{R}$. In such a case, inequality (9) takes the form

$$
u(t) \leq c_1 (t-a)^\beta E_{\beta,\beta+1}(R\Gamma(\beta)(t-a)^\beta) +
$$

$$
+ c_2 \Gamma(\lambda + 1)(t-a)^{\beta+\lambda}E_{\beta,\beta+\lambda+1}(R\Gamma(\beta)(t-a)^\beta)
$$

*(10)*

*for all $t \in [a, b]$.*

3. **Fractional nonlinear optimal control problem.**

3.1. **Homogeneous problem.** Let us consider the optimal control problem (1)–(4) with $x_0 = 0$. i.e.

$$
(D^\alpha_a x)(t) = f(t, x(t), u(t)), \quad t \in [a, b] \text{ a.e.}
$$

*(11)*

$$
(\Gamma_{a+}^{1-\alpha} x)(a) = 0,
$$

*(12)*

$$
u(t) \in M \subset \mathbb{R}^m, \quad t \in [a, b],
$$

*(13)*

$$
J(x, u) = \int_a^b f_0(t, x(t), u(t)) dt \to \min,
$$

*(14)*
where \( f : [a, b] \times \mathbb{R}^n \times M \to \mathbb{R}^n \), \( f_0 : [a, b] \times \mathbb{R}^n \times M \to \mathbb{R} \), \( 0 < \alpha < 1 \).

Let \( 1 \leq p < \infty \) and

\[
\mathcal{U}_M := \{ u \in L^p([a, b], \mathbb{R}^m) : u(t) \in M, \ t \in [a, b] \}.
\]

In [22], the following theorem on the existence and uniqueness of a solution \( x_u \in \text{AC}^{\alpha, p}_{a+} \) to system (11)–(12), corresponding to a fixed control \( u \in \mathcal{U}_M \), has been proved.

**Theorem 3.1.** Let \( \alpha \in (0, 1) \) and \( 1 \leq p < \infty \). If

(1) \( f(\cdot, x, u) \) is measurable on \([a, b]\) for all \( x \in \mathbb{R}^n \), \( u \in M \), \( f(t, x, \cdot) \) is continuous on \( M \) for \( t \in [a, b] \) a.e. and all \( x \in \mathbb{R}^n \);

(2) there exists \( N > 0 \) such that

\[
|f(t, x_1, u) - f(t, x_2, u)| \leq N|x_1 - x_2|
\]

for \( t \in [a, b] \) a.e. and all \( x_1, x_2 \in \mathbb{R}^n \), \( u \in M \);

(3) there exist \( w \in L^p([a, b], \mathbb{R}) \) and \( \gamma \geq 0 \) such that

\[
|f(t, 0, u)| \leq w(t) + \gamma|u|
\]

for \( t \in [a, b] \) a.e. and all \( u \in M \),

then, for any fixed \( u \in \mathcal{U}_M \), there exists a unique solution \( x_u \in \text{AC}^{\alpha, p}_{a+} \) to (11)–(12).

**Remark 2.** The solution \( x_u \) to system (11)–(12), obtained in [22], is given by

\[
x_u(t) = (I^n_{a+} \varphi_u)(t), \quad t \in [a, b] \text{ a.e.},
\]

where \( \varphi_u \) is a fixed point of the operator \( S : L^p([a, b], \mathbb{R}^n) \to L^p([a, b], \mathbb{R}^n) \) defined by

\[
S(\varphi)(t) = f(t, (I^n_{a+} \varphi)(t), u(t)), \quad t \in [a, b] \text{ a.e.}
\]

Consequently,

\[
x_u(t) = (I^n_{a+} (f(\cdot, x_u(\cdot), u(\cdot)))(t), \quad t \in [a, b] \text{ a.e.} \quad (15)
\]

In the proof of the main result (Theorem 3.5), we shall use the following lemmas.

**Lemma 3.2.** Let \( \alpha \in (0, 1) \), \( 1 \leq p < \infty \) and assume that conditions (1f), (2f) are satisfied. If there exist constants \( c_1 \geq 0 \), \( c_2 \geq 0 \) such that

\[
|f(t, 0, u)| \leq c_1 + c_2(t - a)^\lambda
\]

for \( t \in [a, b] \) a.e. and all \( u \in M \), with \( \lambda > -\frac{1}{p} \), then there exists a function \( h \in L^p([a, b], \mathbb{R}^+) \) (independent on control \( u \)) such that

\[
|x_u(t)| \leq h(t), \quad t \in [a, b] \text{ a.e.}, \quad (17)
\]

for any \( u \in \mathcal{U}_M \).

**Proof.** First, let us note that assumption \( \lambda > -\frac{1}{p} \) implies condition (3f). Consequently, for every control \( u \in \mathcal{U}_M \), there exists a unique solution \( x_u \in \text{AC}^{\alpha, p}_{a+} \) to system (11)–(12), corresponding to \( u \). Moreover, using conditions (15), (2f), (16) and Lemma 2.1, we obtain

\[
|x_u(t)| \leq (I^n_{a+}|f(\cdot, x_u(\cdot), u(\cdot))|(t) \leq \leq (I^n_{a+}|f(\cdot, x_u(\cdot), u(\cdot)) - f(\cdot, 0, u(\cdot))|(t) + (I^n_{a+}|f(\cdot, 0, u(\cdot))|(t) \leq \leq N(I^n_{a+}|x_u(\cdot)|)(t) + (I^n_{a+}(c_1 + c_2(\cdot - a)^\lambda))(t) \leq N(I^n_{a+}|x_u(\cdot)|)(t) + d(t)
\]

for \( t \in [a, b] \) a.e. and \( u \in \mathcal{U}_M \), where \( d(t) = \frac{c_1(\alpha)}{t(\alpha + 1)} + c_2\frac{\Gamma(\lambda + 1)}{\Gamma(\alpha + \lambda + 1)}(t - a)^{\alpha + \lambda} \).
Consequently, we obtain the inequality
\[ |x_u(t)| \leq N |x_u(t)| + d(t), \quad t \in [a, b] \text{ a.e., } u \in \mathcal{U}_M. \]
Finally, using inequality (10) from Remark 1, we obtain
\[ |x_u(t)| \leq h(t), \quad t \in [a, b] \text{ a.e., } u \in \mathcal{U}_M, \]
where
\[
\begin{align*}
    h(t) &= c_1(t-a)^\alpha E_{\alpha,\alpha+1}(N(t-a)^\alpha) + \\
    &+ c_2 \Gamma(\lambda + 1)(t-a)^{\alpha+\lambda} E_{\alpha,\alpha+\lambda+1}(N(t-a)^\alpha), \quad t \in [a, b].
\end{align*}
\]
Since \( \lambda > -\frac{1}{\alpha} \), therefore \( h \in L^p([a, b], \mathbb{R}^+) \).

The proof is completed. \( \square \)

**Remark 3.** If \( \alpha > \frac{1}{p} \) and \( M \) is bounded, then assumption (16) can be replaced with condition (3f).

**Lemma 3.3.** If all assumptions of Lemma 3.2 are satisfied then there exists a function \( v \in L^p([a, b], \mathbb{R}^+) \) such that
\[
|(D_{\alpha}^+ x_u)(t)| \leq v(t), \quad t \in [a, b] \text{ a.e., } u \in \mathcal{U}_M.
\]

**Proof.** Let us put \( v(t) := w(t) + Nh(t) \), where \( w(t) = c_1 + c_2(t-a)^\lambda \in L^p([a, b], \mathbb{R}^+) \)
the function from Lemma 3.2. \( N \) is the Lipschitz constant from assumption (2f) and \( h \) is the function from condition (17). Then, we obtain
\[
|(D_{\alpha}^+ x_u)(t)| = |f(t, x_u(t), u(t))| \leq |f(t, x_u(t), u(t)) - f(t, 0, u(t))| + |f(t, 0, u(t))| \leq N |x_u(t)| + w(t) \leq v(t), \quad t \in [a, b] \text{ a.e., } u \in \mathcal{U}_M.
\]

**3.2. Existence of optimal solutions.**

**Definition 3.4.** We say that a pair \( (x_*, u_*) \in AC_{\alpha+}^p \times \mathcal{U}_M \) is an optimal solution to problem (11)–(14), if \( x_* \) is a solution to system (11)–(12), corresponding to control \( u_* \) and
\[
J(x_*, u_*) \leq J(x, u)
\]
for all pairs \( (x, u) \in AC_{\alpha+}^p \times \mathcal{U}_M \) satisfying (11)–(12).

**Remark 4.** In view of the uniqueness of the solution to control system (11)–(12), we can replace the functional \( J \) with a functional \( \hat{J} : \mathcal{U}_M \to \mathbb{R} \) given by
\[
\hat{J}(u) = \int_a^b f_0(t, x_u(t), u(t)) dt.
\]

We have (cf. [20, Proposition 8.1.2.6]).

**Proposition 1.** Let \( g = g(t, u) : [a, b] \times M \to \mathbb{R}^n \) be measurable in \( t \) and continuous in \( u \). Then, for any measurable function \( H : [a, b] \to \mathbb{R}^n \), the set
\[
\{ t \in [a, b] ; \quad H(t) \in g(t, M) \}
\]
is measurable.

In an elementary way, one can prove:
Proposition 2. Let $W$ be a metric space, $Z$ - a compact metric space and $Y$ - a metric space with a metric $\rho$. If $f : W \times Z \to Y$ is continuous and $w_i \to w_0$ in $W$, then

$$\lim_{i \to \infty} f(w_i, z) = f(w_0, z)$$

uniformly with respect to $z \in Z$, i.e.

$$\forall \varepsilon > 0 \ \exists i_0 \in \mathbb{N} \ \forall i \geq i_0 \ \forall z \in Z \ \rho(f(w_i, z), f(w_0, z)) < \varepsilon.$$

Now, we shall prove the main result of this paper, namely, a theorem on the existence of optimal solutions to problem (11)--(14). We have

**Theorem 3.5.** Let $\alpha \in (0, 1)$ and $1 \leq p < \infty$. Moreover, let us assume that

(a) the set $M$ is compact,
(b) $f$ satisfies assumptions of Lemma 3.2,
(c) $f_0(\cdot, x, u)$ is measurable on $[a, b]$ for all $x \in \mathbb{R}^n$ and $u \in M$,
(d) $f_0(t, \cdot, \cdot)$ is continuous on $\mathbb{R}^n \times M$ for a.e. $t \in [a, b],$
(e) the set of extended velocities $V = \{(v_0, v) \in \mathbb{R}^{1+n} : \exists u \in M \ v_0 = f_0(t, x, u) \text{ and } v = f(t, x, u)\}$ is convex, for $t \in [a, b]$ a.e., $x \in \mathbb{R}^n$,
(f) for any function $\kappa \in L^p([a, b], \mathbb{R}^n)$ there exists a function $\psi \in L^1([a, b], \mathbb{R}^n)$ such that

$$|f_0(t, x, u)| \leq \psi(t)$$

for a.e. $t \in [a, b], |x| \leq \kappa(t)$ and $u \in M$.

Then problem (11) -- (14) possesses an optimal solution $(x_*, u_*) \in AC_{\alpha} \times U_M$.

**Proof.** Let us consider an extended problem of the form:

$$\begin{align*}
(D_{\alpha}^a, \tilde{x}) (t) &= \tilde{f}(t, \tilde{x}(t), u(t)), \quad t \in [a, b] \text{ a.e.,} \\
&= (I_{\alpha}^{1-a} \tilde{x}) (a) = 0, \\
&= u(t) \in M \subset \mathbb{R}^m, \quad t \in [a, b],
\end{align*}$$

where $\tilde{x} = (x^0, x)$ and $\tilde{f} : [a, b] \times \mathbb{R}^{1+n} \times M \to \mathbb{R}^{1+n}$ is given by

$$\tilde{f}(t, \tilde{x}, u) = (f_0(t, x, u), f(t, x, u)).$$

It is clear that in order to find the optimal solution $(x_*, u_*) = (x_{u_*}, u_*)$ to problem (11)--(14), it suffices to find a control $u_*$ such that the extended response $\tilde{x}_* = (x^0_*, x_*)$ satisfies the following equality

$$\tilde{J}(u_*) = (I_{\alpha}^{1-a} x^0_*) (b) = \inf \{\tilde{J}(u) : u \in U_M\}.$$

Let us denote

$$s = \inf \{\tilde{J}(u), \quad u \in U_M\}.$$

From assumption (f) and condition (17) it follows that $-\infty < s < \infty$.

Let $(u_i)_{i \in \mathbb{N}} \in U_M$ be a minimizing sequence of functional $\tilde{J}$, i.e.

$$\lim_{i \to \infty} \tilde{J}(u_i) = s,$$

$(x_i)_{i \in \mathbb{N}}$ - the sequence of corresponding solutions to problem (11) - (12) and $(\tilde{x}_i)_{i \in \mathbb{N}} = (x^0_i, x_i)_{i \in \mathbb{N}}$ - the sequence of corresponding solutions to extended problem (21) - (22).

**Step 1.** We shall show that there exist functions

$$x^0_0 \in I_{\alpha}^a (L^1([a, b], \mathbb{R})) \quad \text{and} \quad x_* \in I_{\alpha}^a (L^1([a, b], \mathbb{R}^n))$$

such that
such that
\[ D_{\alpha+}^{\alpha} \tilde{x}_t = (D_{\alpha+}^{\alpha}x_0, D_{\alpha+}^{\alpha}x_1) \to (D_{\alpha+}^{\alpha}x_0, D_{\alpha+}^{\alpha}x_* ) = D_{\alpha+}^{\alpha} \tilde{x}_* \]
weakly in \( L^1([a,b],\mathbb{R}^{1+n}) \). Let us note that
\[ (I_{\alpha+}^{1-\alpha} \tilde{x}_t)_{t \in \mathbb{N}} = (I_{\alpha+}^{1-\alpha} x_0^0, I_{\alpha+}^{1-\alpha} x_1) \subset AC([a,b],\mathbb{R}^{1+n}) \]
First, we shall show that there exist functions satisfying (24) and a subsequence of the sequence \( (I_{\alpha+}^{1-\alpha} \tilde{x}_t)_{t \in \mathbb{N}} \), convergent uniformly to function \( I_{\alpha+}^{1-\alpha} \tilde{x}_* = (I_{\alpha+}^{1-\alpha} x_0^0, I_{\alpha+}^{1-\alpha} x_*) \) \in AC([a,b],\mathbb{R}^{1+n}) \).
Indeed, Lemma 3.2 and assumptions (2f), (3f) imply the uniform boundedness of the sequence \( (I_{\alpha+}^{1-\alpha} \tilde{x}_t)_{t \in \mathbb{N}} \). The fact that this sequence is uniformly absolutely continuous on \([a,b]\) follows from Lemma 3.3. Moreover, from Lemma 3.2, assumption (f) and absolute continuity of an integral it follows that the sequence of derivatives \( (\frac{d}{dt}(I_{\alpha+}^{1-\alpha} x(t)))_{t \in \mathbb{N}} \) is uniformly absolutely integrable on \([a,b]\). It means that the sequence \( (I_{\alpha+}^{1-\alpha} x(t))_{t \in \mathbb{N}} \) is uniformly absolutely continuous on \([a,b]\). Finally, the estimation
\[ |(I_{\alpha+}^{1-\alpha} x(t))| \leq \int_a^t |f_0(s, x_1(s), u(t))|ds \leq \int_a^t |\psi(s)|ds \leq ||\psi||_{L^1} \]
for a.e. \( t \in [a,b] \) and all \( t \in \mathbb{N} \) guarantees the uniform boundedness of the sequence \( (I_{\alpha+}^{1-\alpha} x(t))_{t \in \mathbb{N}} \) on \([a,b]\).
Consequently, using [23, Theorem 8], we conclude, that there exist a subsequence, still denoted by \( (I_{\alpha+}^{1-\alpha} \tilde{x}_t)_{t \in \mathbb{N}} \), and a function \( \tilde{\nu} = (\nu^0, \nu) \in AC([a,b],\mathbb{R}^{1+n}) \) such that
\[ I_{\alpha+}^{1-\alpha} \tilde{x}_t \Rightarrow \tilde{\nu} \text{ uniformly on } [a,b]. \]
From Lemma 2.2 it follows that \( \tilde{\nu} \in I_{\alpha+}^{1-\alpha}(L^1([a,b],\mathbb{R}^{1+n})) \), so,
\[ \tilde{\nu} = (\nu^0, \nu) = (I_{\alpha+}^{1-\alpha} x_0^0, I_{\alpha+}^{1-\alpha} x_*) \]
with \( x_0^0 \in L^1([a,b],\mathbb{R}) \) and \( x_* \in L^1([a,b],\mathbb{R}^n) \). Let us put \( \tilde{x}_* = (x_0^0, x_*) \).
Then
\[ I_{\alpha+}^{1-\alpha} \tilde{x}_* \in AC([a,b],\mathbb{R}^{1+n}) \]
and
\[ I_{\alpha+}^{1-\alpha} \tilde{x}_t \Rightarrow I_{\alpha+}^{1-\alpha} \tilde{x}_* \text{ uniformly on } [a,b]. \]
Thus,
\[ (I_{\alpha+}^{1-\alpha} \tilde{x}_*)(a) = 0. \]
Consequently, we have
\[ \int_a^t (D_{\alpha+}^{\alpha} \tilde{x}_t)(s)ds = \int_a^t \frac{d}{dt}(I_{\alpha+}^{1-\alpha} \tilde{x}_t)(s)ds = (I_{\alpha+}^{1-\alpha} \tilde{x}_t)(t) - (I_{\alpha+}^{1-\alpha} \tilde{x}_t)(a) =
\]
\[ = (I_{\alpha+}^{1-\alpha} \tilde{x}_t)(t) \Rightarrow (I_{\alpha+}^{1-\alpha} \tilde{x}_*)(t) = (I_{\alpha+}^{1-\alpha} \tilde{x}_*)(t) - (I_{\alpha+}^{1-\alpha} \tilde{x}_*)(a) =
\]
\[ = \int_a^t \frac{d}{dt}(I_{\alpha+}^{1-\alpha} \tilde{x}_*)(s)ds = \int_a^t (D_{\alpha+}^{\alpha} \tilde{x}_*)(s)ds, \quad t \in [a,b] \text{ a.e.} \]
Moreover, Lemma 3.3 and condition (20) imply
\[ \|D_{a+}^\alpha \bar{x}_l\|_{L^1} = \int_a^b |(D_{a+}^\alpha \bar{x}_l)(s)|ds \leq \int_a^b |\psi(s)|ds + \int_a^b |v(s)|ds = \text{const}. \]
Finally, since
\[ |D_{a+}^\alpha \bar{x}_l(t)| \leq \psi(t) + v(t), \quad t \in [a, b] \ a.e., \ l \in \mathbb{N}, \]
and \( \psi + v \in L^1([a, b], \mathbb{R}^n) \), therefore functions \( D_{a+}^\alpha \bar{x}_l \) are uniformly absolutely integrable. It means (cf. [6, Section VII A. §2]) that
\[ D_{a+}^\alpha \bar{x}_l \rightharpoonup D_{a+}^\alpha \bar{x}_* \quad \text{weakly in} \ L^1([a, b], \mathbb{R}^{1+n}). \] (27)
Let us observe that (25) and (26) imply relations (24).

**Step 2.** Now, we shall prove that
\[ (D_{a+}^\alpha \bar{x}_*) (t) = \bar{f}(t, x_*(t), M), \quad t \in [a, b] \ a.e. \] (28)
Indeed, first, let us observe that from the weak convergence (27) and complete continuity of the operator \( I_{a+}^\alpha : L^1 \to L^1 \) (cf. [32, Lemma 1.1]) it follows that
\[ x_l \xrightarrow{l \to \infty} x_* \quad \text{strongly in} \ L^1. \]
Consequently, there exists a subsequence, still denoted by \( (x_l)_{l \in \mathbb{N}} \), convergent to \( x_*(t) \) a.e. on \([a, b]\). In other words,
\[ x_l(t) \xrightarrow{l \to \infty} x_*(t) \quad t \in S_1, \] (29)
where \( S_1 \subset [a, b] \) and \( \mu(S_1) = b - a \) (\( \mu \) denotes one-dimensional Lebesgue measure).

Let us define the set
\[ S := \{ t \in [a, b] : (D_{a+}^\alpha \bar{x}_*) (t) \notin \bar{f}(t, x_*(t), M) \}. \]
From Proposition 1 it follows that \( S \) is measurable. Let us suppose that \( \mu(S) > 0 \), so also \( \mu(S \cap S_1) > 0 \). Let us fix \( t \in S \cap S_1 \). Assumptions (a) and (e) guarantee convexity and compactness of the set \( \bar{f}(t, x_*(t), M) \). Consequently, there exist a constant \( \gamma(t) \in \mathbb{R} \) and a vector \( b(t) \in \mathbb{R}^{1+n} \setminus \{0\} \) such that
\[ b(t) \bar{f}(t, x_*(t), u) < \gamma(t) < b(t)(D_{a+}^\alpha \bar{x}_*)(t) \]
for all \( u \in M \). From the convergence (29) and Proposition 2 it follows that there exists \( l_0 \in \mathbb{N} \) such that
\[ b(t) \bar{f}(t, x_l(t), u) < \gamma(t) \]
for \( l \geq l_0 \) and \( u \in M \). In particular,
\[ b(t) \bar{f}(t, x_l(t), u_l(t)) < \gamma(t) \]
for \( l \geq l_0 \) and consequently,
\[ \limsup_{l \to \infty} \left( b(t) \bar{f}(t, x_l(t), u_l(t)) \right) \leq \gamma(t). \]
Thus,
\[ \limsup_{l \to \infty} \left( b(t) \bar{f}(t, x_l(t), u_l(t)) \right) < b(t)(D_{a+}^\alpha \bar{x}_*)(t). \]
Since the last inequality is strong, therefore
\[ \limsup_{l \to \infty} \left( q \bar{f}(t, x_l(t), u_l(t)) \right) < q(D_{a+}^\alpha \bar{x}_*)(t). \]
for some $q \in \mathbb{Q}$, where $\mathbb{Q}$ is the set of points $q = (q_1, \ldots, q_{n+1})$ such that $q_i$, $i = 1, \ldots, n+1$, are rational numbers. It means that

$$S \cap S_1 \subset \bigcup_{q \in \mathbb{Q}} \{ t \in [a; b]; \lim_{l \to \infty} \left( q \tilde{f}(t, x_l(t), u_l(t)) \right) < q(D_{a+}^{\alpha} \tilde{x}_*)(t) \}.$$ 

From the fact that $\mu(S \cap S_1) > 0$ it follows that there exists $q_0 \in \mathbb{Q}$ such that $\mu(A) > 0$,

$$A = S \cap S_1 \cap \{ t \in [a, b]; \lim_{l \to \infty} \left( q_0 \tilde{f}(t, x_l(t), u_l(t)) \right) < q_0(D_{a+}^{\alpha} \tilde{x}_*)(t) \}.$$ 

Then, from Fatou’s lemma, we obtain

$$\int_a^b \chi_A(t) q_0(D_{a+}^{\alpha} \tilde{x}_*)(t) dt = \int_A q_0(D_{a+}^{\alpha} \tilde{x}_*)(t) dt >$$

$$\geq \limsup_{l \to \infty} \int_A q_0 \tilde{f}(t, x_l(t), u_l(t)) dt \geq$$

$$\geq \limsup_{l \to \infty} \int_A q_0 \tilde{f}(t, x_l(t), u_l(t)) dt =$$

$$= \limsup_{l \to \infty} \int_A q_0(D_{a+}^{\alpha} \tilde{x}_*)(t) dt =$$

$$= \limsup_{l \to \infty} \int_a^b \chi_A(t) q_0(D_{a+}^{\alpha} \tilde{x}_*)(t) dt.$$ 

Since the function $t \mapsto \chi_A(t) q_0$ is essentially bounded, therefore the above strong inequality contradicts the weak convergence of the sequence $(D_{a+}^{\alpha} \tilde{x}_l)_{l \in \mathbb{N}}$ to $D_{a+}^{\alpha} \tilde{x}_*$ in $L^1([a, b], \mathbb{R}^{1+n})$. It means that inclusion (28) is satisfied.

Using the implicit function theorem for multivalued mappings ([26, Sec. II, Theorem 3.12]), we assert that there exists a measurable function $u_* : [a, b] \to M$ such that

$$(D_{a+}^{\alpha} \tilde{x}_*)(t) = f(t, \tilde{x}_*(t), u_*(t)), \quad t \in [a, b] \ a.e.$$ 

Thus,

$$(D_{a+}^{\alpha} x_*)(t) = f(t, x_*(t), u_*(t)), \quad t \in [a, b] \ a.e.$$ 

and

$$(D_{a+}^{\alpha} x^0_*)(t) = f_0(t, x_*(t), u_*(t)), \quad t \in [a, b] \ a.e. \quad (30)$$ 

Since $M$ is bounded, it is clear that $u_* \in \mathcal{U}_M$.

To finish the proof, we shall show that $u_*$ is a minimizer of functional $\hat{J}$. Indeed, from conditions (25) and (26) it follows that

$$I_{a+}^{1-\alpha} x^0_* \in AC([a, b], \mathbb{R}) \quad \text{and} \quad (I_{a+}^{1-\alpha} x^0_*)(a) = 0.$$ 

Consequently, from equality (30), we obtain

$$I_{a+}^{1-\alpha} x^0_*(t) = \frac{1}{t-a} \int_a^t f_0(t, x_*(t), u_*(t)) dt, \quad t \in [a, b].$$
In particular,
\[
\hat{J}(u^*) = \int_a^b f_0(t, x^*(t), u^*(t)) \, dt = (I_{a+}^{1-\alpha} x^0)(b) = \lim_{l \to \infty} (I_{a+}^{1-\alpha} x_l^0)(b) = \]
\[
= \lim_{l \to \infty} \int_a^b f_0(t, x_l(t), u_l(t)) \, dt = \lim_{l \to \infty} \hat{J}(u_l) = s = \inf \{ \hat{J}(u), \quad u \in \mathcal{U}_M \}.
\]
The proof is completed.

\[\hat{J}(u_l) = \]
\[= \lim_{l \to \infty} \int_a^b f_0(t, x_l(t), u_l(t)) \, dt = \lim_{l \to \infty} \hat{J}(u_l) = s = \inf \{ \hat{J}(u), \quad u \in \mathcal{U}_M \}.
\]

Remark 5. Assumption (f) may be formulated only for function \(\kappa(\cdot) = h(\cdot)\), where \(h\) is given by formula (18).

3.3. Nonhomogeneous problem. Now, let us consider the following nonhomogeneous fractional optimal control problem
\[
(D_a^{\alpha} y)(t) = g(t, y(t), u(t)), \quad t \in [a, b] \text{ a.e.} \tag{31}
\]
\[
(I_{a+}^{1-\alpha} y)(a) = y_0, \tag{32}
\]
\[
u(t) \in M \subset \mathbb{R}^n, \quad t \in [a, b] \tag{33}
\]
\[
J_1(y, u) = \int_a^b g_0(t, y(t), u(t)) \, dt \to \min, \tag{34}
\]
where \(g : [a, b] \times \mathbb{R}^n \times M \to \mathbb{R}^n\), \(g_0 : [a, b] \times \mathbb{R}^n \times M \to \mathbb{R}\), \(y_0 \in \mathbb{R}^n\), \(0 < \alpha < 1\).

Definition 3.6. We say, that a pair \((y^*, u^*) \in AC_{a+}^{\alpha,p} \times \mathcal{U}_M\) is an optimal solution to problem (31)–(34), if \(y^*\) is the solution of the control system (31)–(32), corresponding to the control \(u^*\) and
\[
J_1(y^*, u^*) \leq J_1(y, u)
\]
for any pair \((y, u) \in AC_{a+}^{\alpha,p} \times \mathcal{U}_M\) satisfying (31) - (32).

It is easy to show that if a pair \((x^*(\cdot), u^*(\cdot)) \in AC_{a+}^{\alpha,p} \times \mathcal{U}_M\) is an optimal solution to problem (11)–(14) with functions \(f\) and \(f_0\) of the form
\[
f(t, x, u) = g \left( t, x + \frac{y_0}{\Gamma(\alpha)} \frac{1}{(t-a)^{1-\alpha}}, u \right), \tag{35}
\]
\[
f_0(t, x, u) = g_0 \left( t, x + \frac{y_0}{\Gamma(\alpha)} \frac{1}{(t-a)^{1-\alpha}}, u \right), \tag{36}
\]
then the pair
\[
(y^*(\cdot), u^*(\cdot)) = \left( x^*(\cdot) + \frac{y_0}{\Gamma(\alpha)} \frac{1}{(t-a)^{1-\alpha}}, u^*(\cdot) \right) \in AC_{a+}^{\alpha,p} \times \mathcal{U}_M \tag{37}
\]
is an optimal solution to problem (31)–(34).

Using Theorem 3.5, we shall prove the following

Theorem 3.7. Let \(\alpha \in (0, 1)\) and \(1 \leq p < \infty\). If
(A) the set \(M\) is compact,
(B) \(g\) satisfies assumptions (1f) – (2f) of Theorem 3.1,
Remark 6. In the case of $1 - p < \frac{1}{\lambda} < \frac{1}{\alpha - 1}$, the following condition implies (C) with $\lambda \in \left( -\frac{1}{p}, \alpha - 1 \right)$: there exist constants $c_3 > 0$, $c_4 > 0$ such that
\[ |g(t, 0, u)| \leq c_3 + c_4(t - a)^\lambda \]
for a.e. $t \in [a, b]$ and all $u \in M$. Indeed, since $1 - p < \frac{1}{\lambda} < \frac{1}{\alpha - 1}$, therefore the interval $\left( -\frac{1}{p}, \alpha - 1 \right)$ is nonempty. Moreover, since $g$ is lipschitzian in $y$, therefore
\[ \left| g\left(t, \frac{1}{\Gamma(\alpha)} \frac{y_0}{(t - a)^{1-\alpha}}, u\right) \right| \leq |g(t, 0, u)| + \frac{N|y_0|}{\Gamma(\alpha)}(t - a)^{\alpha - 1} \leq c_3 + c_4(t - a)^\lambda + \frac{N|y_0|}{\Gamma(\alpha)}(t - a)^{\alpha - 1} \]
for $t \in [a, b]$ a.e. and all $u \in M$. If $t - a \geq 1$, then $(t - a)^{\alpha - 1} \leq 1$ and, consequently,
\[ \left| g\left(t, \frac{1}{\Gamma(\alpha)} \frac{y_0}{(t - a)^{1-\alpha}}, u\right) \right| \leq c_3 + \frac{N|y_0|}{\Gamma(\alpha)} + c_4(t - a)^\lambda. \]
If $0 < t - a < 1$, then $(t - a)^{\alpha - 1} < (t - a)^\lambda$ for $\lambda < \alpha - 1$ and, consequently,
\[ \left| g\left(t, \frac{1}{\Gamma(\alpha)} \frac{y_0}{(t - a)^{1-\alpha}}, u\right) \right| \leq c_3 + \left( c_4 + \frac{N|y_0|}{\Gamma(\alpha)} \right)(t - a)^\lambda. \]
4. Example. Let us consider the following fractional optimal control problem:

\[
(D_{0+}^\alpha y)(t) = Ay(t) + Bu^3(t), \quad t \in [0, 2] \text{ a.e.}
\]

\[
(I_{0+}^{1-\alpha} y)(0) = y_0,
\]

\[
u(t) \in [-1, 1], \quad t \in [0, 2],
\]

\[
J_1(y, u) = \int_0^2 (y_1(t) - y_2(t) + u^3(t)) dt \to \min,
\]

where \(y = (y_1, y_2) \in \mathbb{R}^2\), \(A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\), \(B = \begin{bmatrix} -1 \\ -1 \end{bmatrix}\), \(y_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\), \(\alpha = \frac{1}{2}\), \(p = \frac{3}{2}\).

In this case

\[
g(t, y, u) = Ay + Bu^3,
\]

\[
g_0(t, y, u) = ((1, -1), y) + u^3.
\]

Of course,

\[
A^k = (A^T)^k = 0, \quad k \geq 2,
\]

\[
g_y(t, y, u) = A, \quad (g_0)_y(t, y, u) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.
\]

It is easy to check that all assumptions of Theorem 3.7 are satisfied. In particular, the set \(\tilde{V}\) from assumption (F) is convex, although both functions \(g\) and \(g_0\) are not convex with respect to variable \(u\). Consequently, there exists a pair \((y^*(t), u^*(t))\) which is an optimal solution to investigated problem. Using the maximum principle (cf. [22, Theorem 9]) we assert that there exists \(\lambda(\cdot) \in I_{2-}^2(L^3)\) such that

\[
(D_{2-}^2 \lambda)(t) = A^T \lambda(t) + \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad t \in [0, 2] \text{ a.e.}
\]

\[
(I_{2-}^1 \lambda)(2) = 0.
\]

Moreover

\[
u^*_3(t) - \lambda(t) Bu^3(t) = \min_{u \in [-1, 1]} \{u^3 - \lambda(t) Bu^3\}
\]

for \(t \in [0, 2] \text{ a.e.}\).

From [22, Theorem 11] it follows that a solution of problem (43) - (44) is given by

\[
\begin{bmatrix} \lambda_1(t) \\ \lambda_2(t) \end{bmatrix} = \begin{bmatrix} \frac{(2 - t)\frac{1}{2}}{\Gamma(\frac{3}{2})} \\ \frac{(2 - t)\frac{1}{2}}{\Gamma(\frac{3}{2})} - \frac{(2 - t)}{\Gamma(2)} \end{bmatrix}, \quad t \in [0, 2].
\]

Consequently, condition (45) is equivalent to the following one

\[
u^*_3(t) - (2 - t)u^3_3(t) = \min_{u \in [-1, 1]} \{u^3 - (2 - t)u^3\} = \begin{cases} t - 1 & t \in [0, 1] \\ 1 - t & t \in (1, 2] \end{cases}
\]

for \(t \in [0, 2] \text{ a.e.}\).
Thus
\[ u_\ast(t) = \begin{cases} 1 & t \in [0, 1] \ a.e. \\ -1 & t \in (1, 2] \ a.e. \end{cases} \]

From [22, Theorem 10] it follows that a solution of system (39) - (40), corresponding to \( u_\ast(\cdot) \) is given by
\[
y_\ast(t) = \Phi(t)y_0 + \int_0^t \Phi(t-s)Bu_\ast(s)ds =
\]
\[
= \Phi(t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{cases} \int_0^t \Phi(t-s)Bds & t \in [0, 1] \ a.e. \\ \int_0^1 \Phi(t-s)Bds - \int_1^t \Phi(t-s)Bds & t \in [1, 2] \ a.e. \end{cases}
\]

where
\[
\Phi(t) = \sum_{k=0}^{\infty} \frac{A^k t^{\alpha(k+1)-1}}{\Gamma(\alpha(k+1))}.
\]

Consequently,
\[
y_\ast(t) = \begin{cases} 1 - t - \frac{t^{\frac{1}{2}}}{\Gamma(\frac{3}{2})} & t \in [0, 1] \ a.e. \\ t^{\frac{1}{2}} - \frac{1}{\Gamma(\frac{1}{2})} - \frac{t^{\frac{1}{2}}}{\Gamma(\frac{1}{2})} + t - 1 + \frac{(t-1)^{\frac{1}{2}}}{\Gamma(\frac{1}{2})} & t \in [1, 2] \ a.e. \\ \end{cases}
\]

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