Buckling of micropolar beams by an improved first order deformation theory

K N Betancourt\(^1\), K Soncco\(^2\) and R Arciniega\(^1,3\)

\(^1\)Department of Civil Engineering, Universidad Peruana de Ciencias Aplicadas, Lima, Peru
Email: u201410150@upc.edu.pe\(^a\); u201412519@upc.edu.pe\(^b\); roman.arciniega@upc.edu.pe\(^c\)

Abstract. In this paper, we present a variational formulation to study the buckling behavior of micropolar beams by using an improved 3D deformation theory. A micropolar continuum applied to beams has been developed using its natural Lagrangian kinematic relations. The Rodriguez rotation measure was used to describe the rotational degrees of freedom. Additionally, a Taylor expansion was performed to linearize the kinematic relations. For the buckling analysis, the Trefftz criterion procedure was applied. A finite element model was derived for the solution of the variational problem using spectral interpolation functions for a higher convergence rate and for avoiding shear locking problems. The results describe the influence of the micropolar parameters and size-dependent behavior. Finally, the model was used to evaluate the buckling loads of simply-supported functionally graded beams considering experimental material parameters.

1. Introduction
Nowadays, the discoveries and applications of new materials that evidence local and non-local mechanical behaviors have increased. Moreover, as new technologies are developed, the use of nanocomponents and nanostructures is preferred. As size decreases, several studies have shown the importance and influence of size-dependent mechanical models [1,2]. A Cosserat continuum [3] is related to a non-classical scheme because it uses independent rotations that allow the evaluation of microstructures and their size-dependent behavior. Applications of micropolar or Cosserat continuum have been developed in civil engineering [4,5], soil mechanics [6,7], and more.

Studies regarding micropolar mechanics have mainly focused on bending and vibration responses. Ansari et al. [8] applied the Finite Element Method for the analysis of microbeams free vibration response. They proposed a nonclassical 3D element for studying the influence of micropolar parameters on vibration frequencies. Penta [9] analyzed the buckling behavior of Euler Bernoulli Vierendeel beams using a homogenized theory. His study has shown the influence of the micropolar continuum on the critical load of Vierendeel girder beams. Carrera and Zozulya [10] developed a unified beam theory for micropolar schemes on analytical bending responses. Ramezani et al. [11] studied the bending response of cantilever micropolar beams. The present authors [12] also investigated the bending response of micropolar beams, comparing their improved shear deformation theory against Hassanpour’s [13] formulation.

Functionally graded materials (FGM) were first discovered in Japan [14], which are the result of the gradual variation of two phases along the desired direction. These materials enhance mechanical and thermal properties and have been applied in aerospace engineering, tribology, nanotechnology, and biology. Even though there are several studies about FGM beams, there are few on micropolar buckling models.
The current research proposes a model to perform buckling analysis using the incremental-fundamental procedure (Treffitz criterion) on micropolar beams through a variational formulation. Several examples are provided and verified with the literature. The authors show numerical results for FGM beams to evaluate the influence of the power-law index in the buckling load, by using the following mixture rule and power-law function through the thickness $x^2$ of the beam

$$P(x^2) = P_b V_b + P_t V_t, \quad V_t = (x^2/h + 1/2)^n, \quad V_b = 1 - V_t,$$

where $P_b$ and $P_t$ are the material properties at the bottom and top layers of the element, $V_b$ and $V_t$ are their respective volume fractions and $n$ is the power-law index.

2. Theoretical Formulation

2.1. Micropolar theory
In the micropolar theory, the kinematics relations of a point are characterized by rotational and transactional degrees of freedom. The nonlinear kinematic relations can be found on Eringen’s work [15], in which he defines a stretch and wryness tensor as:

$$E = F^T Q,$$

$$\Gamma = -\frac{1}{2} \varepsilon : (Q^T \text{Grad} Q),$$

where $F$ is the deformation gradient, $Q$ is the third-order proper orthogonal microrotation tensor, and $\varepsilon$ is the third-order skew Ricci tensor.

2.2. Micropolar material parameters
For a linear micropolar isotropic solid, stress-strain relations for the stress tensor $\sigma$ and the couple stress tensor $\chi$ are stated as [16]

$$\sigma = \lambda \text{tr} \varepsilon + (\mu + \kappa) \varepsilon + \mu \varepsilon^T,$$

$$\chi = \beta_1 \text{tr} \gamma + \beta_2 \gamma^T + \beta_3 \gamma,$$

where $\mu, \lambda, \kappa, \beta_1, \beta_2, \beta_3$ micropolar elastic constants. Parameters $\mu$ and $\kappa$ relates to the Lamé shear modulus $\mu$ according to [17] as

$$\mu = \frac{\mu + 1}{2} \kappa,$$

and $\lambda$ is the first Lamé parameter. Furthermore, Young’s modulus $E$ and Poisson’s ratio $\nu$ can be defined in terms of these micropolar elastic constants by the following equations

$$E = \frac{(2\mu + \kappa)(3\lambda + 2\mu + \kappa)}{2\lambda + 2\mu + \kappa}, \quad \nu = \frac{\lambda}{2\lambda + 2\mu + \kappa}.$$  

2.3. Beam theory
Let $\{x^i\}$ be a set of Cartesian coordinates with orthonormal basis $\{e_i\}, x^1$ is the neutral axis of the beam and $x^2$ is the thickness axis. The displacement and microrotation fields are assumed to be of the following form (see [18])

$$v(x^1, x^2) = u(x^1) + x^2 \phi(x^1) + (x^2)^2 \psi(x^1),$$
\[ \theta(x^1) = \theta(x^1). \]  \tag{7} 

where \( \mathbf{u} = u_ie_i \) represents the displacement vector of the neutral axis, while \( \varphi = \varphi_1e_1 \) and \( \psi = \psi_2e_2 \) are difference vectors with \( i = 1,2 \). The displacement field contains five independent variables and one quadratic term \( \psi \) to avoid poisson locking. The microrotation field is determined as \( \theta = \theta_3e_3 \), which represent the microrotation along the transversal axis \( x^3 \) of the beam. 

For the given displacement and microrotation fields in equations (6) and (7) and by (2), the stretch and wryness measures are work-conjugate to virtual strain measures in the micropolar continuum, as stated in [20]. Nonlinear terms are only considered in the first component of the stretch tensor according to the assumptions proposed by Hasanyan and Waas [19]. The configuration solution of the micropolar beam is defined by the set \( \Phi \equiv (\mathbf{u}, \varphi, \psi, \theta) \). Thus, 

\[ G(\Phi, \delta\Phi) = G_{int}(\Phi, \delta\Phi) - G_{ext}(\Phi, \delta\Phi), \]

\[ G(\Phi, \delta\Phi) = \int_{x^1} (\mathbf{N}^{(0)} \cdot \delta\mathbf{e}^{(0)} + \mathbf{N}^{(1)} \cdot \delta\mathbf{e}^{(1)} + \mathbf{M}^{(0)} \cdot \delta\mathbf{\gamma}^{(0)}) dx^1 + \int_{x^1} (\mathbf{p} \cdot \delta\mathbf{u} + \mathbf{m} \cdot \delta\theta) dx^1, \]  \tag{9} 

where \( \delta\Phi \equiv (\delta\mathbf{u}, \delta\varphi, \delta\psi, \delta\theta) \). \( \mathbf{N}^{(i)} \) are the force stress resultants, \( \mathbf{M}^{(0)} \) is the micropolar stress resultant, \( \mathbf{p} \) are the body forces acting on the beam per unit length, and \( \mathbf{m} \) are the body moment forces, respectively. The force stress resultants \( \mathbf{N}^{(i)} \) are of the form

\[ \mathbf{N}^{(i)} = \sum_{j=1}^{1} \mathbb{B}^{(i+j)} \mathbf{e}^{(j)}, \quad i = 0,1, \]  \tag{10} 

while moment stress resultants \( \mathbf{M}^{(0)} \) have the form

\[ \mathbf{M}^{(i)} = \mathbb{M}^{(0)} \mathbf{\gamma}^{(0)}. \]  \tag{11} 

For the definition of \( \mathbf{N}^{(i)} \) and \( \mathbf{M}^{(0)} \), it is necessary to define the material stiffness coefficients \( \mathbb{B} \) and \( \mathbb{M} \) as

\[ \mathbb{B}(k) = \int_{-h/2}^{h/2} (x^2)^k \mathbb{C} dx^2, \quad k = 0,1,2, \]

\[ \mathbb{M}^{(0)} = h\mathbb{D}, \]  \tag{12} 

where \( h \) represents the thickness of the beam.

The fourth order micropolar tensors \( \mathbb{C} \) and \( \mathbb{D} \) are stated by means of the definitions from [16].
\[ \mathbb{C} = \lambda I \otimes I + (\mu + \kappa) e_a \otimes e_b \otimes e_a \otimes e_b + \mu e_a \otimes I \otimes e_a, \]  
\[ \mathbb{D} = \beta_1 I \otimes I + \beta_2 e_a \otimes I \otimes e_a + \beta_3 e_a \otimes e_a \otimes e_a \]  

where \( a, b = 1, 2, 3 \). From equations (11) to (13), the final expression of the virtual work can be stated as

\[ \mathcal{G}(\Phi, \delta \Phi) = \int_{x^1} \sum_{i=0}^{1} \sum_{j=0}^{1} \delta \epsilon^{(i)} \mathbb{B}^{(i+j)} \epsilon^{(j)} dx^1 + \int_{x^1} \delta \gamma^{(0)} \mathbb{M}^{(0)} \gamma^{(0)} dx^1 
- \int_{x^1} (\mathbf{p} \cdot \delta \mathbf{u} + \mathbf{m} \cdot \delta \mathbf{\theta}) dx^1. \]  

2.5. Stability analysis

The stability analysis considers two systems: the incremental and the fundamental. The former defines the perturbation and the second the membrane state \([21, 22]\). Based on the principle of virtual work, the displacement field is defined as: \( \Phi = \Phi^F + \Phi^I \), where \( F \) stands for fundamental and \( I \), for incremental. For the independent displacement variables, as the paper is focused on beams, the only displacement with incremental perturbation is the axial \( u_1(\chi^1) = u_1^F(\chi^1) + u_1^I(\chi^1) \); other independent variables will only exhibit membrane behavior. With the new definition, the variation of the potential energy is defined as:

\[ \mathcal{G}(\Delta \Phi) = \mathcal{G}(\Phi^F + \Phi^I) - \mathcal{G}(\Phi^F) \geq 0 \]  

This expression is expanded in a Taylor’s expansion, as in:

\[ \mathcal{G}(\Delta \Phi) = \mathcal{G}(\Phi, \delta \Phi) + \frac{1}{2!} \mathcal{G}(\Phi, \delta \Phi, \delta^2 \Phi) + \frac{1}{3!} \mathcal{G}(\Phi, \delta \Phi, \delta^2 \Phi, \delta^3 \Phi) + \cdots \]  

Based on Brush and Almroth \([23]\), the above expansion is reduced into the second variational term by the minimum energy theorem, which its second term is positively defined for stability phenomenon. Therefore, the critical buckling load is obtained when the variation of the potential energy is zero, leading to:

\[ \delta \mathcal{G}(\Phi, \delta \Phi, \delta^2 \Phi) = 0 \]  

Replacing our fundamental-incremental scheme in the above expression, the following is derived:

\[ \delta \mathcal{G}(\Phi^I, \delta \Phi^I, \delta^2 \Phi^I) + \delta \mathcal{G}(\Phi^{F1}, \delta \Phi^{F1}, \delta^2 \Phi^{F1}) = 0 \]  

The expression in equation (18) states that the potential energy can be divided into the incremental displacement and stress resultants, and into the fundamental-incremental terms, which define the buckling load and its eigenvalues.

3. Results

A Finite Element procedure is developed using a 4-element mesh with \( \mathbf{p} = 4 \) interpolation functions for primary variables to avoid shear locking problems. The adopted basis functions are \( C^0 \) interpolant polynomials of Gauss–Lobatto-Legendre (GLL) quadrature points \([24]\), which are suitable for high-order expansions. Explicitly, the one-dimensional basis functions of the order \( \mathbf{p} = m - 1 \) are expressed using the \( p \)-order Legendre polynomial \( P_{m-1} \), as shown:

\[ \phi^{(j)}(\xi) = \frac{(1 - \xi^2)P_{m-1}'(\xi)}{m(m - 1)P_{m-1}(\xi_j)(\xi - \xi_j)} \]
For the validity of the proposal, nondimensional buckling stresses for various length to height ratios of simply-supported micropolar beams are compared against Hasanyan and Waas [25]. Additional results are presented in the last section using experimental material parameters obtained from [26] for micropolar and functionally graded beams.

3.1. Validation of the proposed model
The proposal is compared against the results of Hasanyan and Waas [25] for homogeneous beams. Nondimensional parameters are used according to the following equations

\[
\hat{k} = \kappa / \bar{\mu}, \quad \hat{\lambda} = \lambda / \bar{\mu}, \quad \hat{\beta}_3 = \beta_3 \eta^2 / \bar{\mu}, \quad P_{cr}^{\hat{\kappa}} = P_{cr} / (2A\bar{\mu}), \quad \hat{s} = 2L/h\pi, \quad \eta = \pi / L,
\]

where \( A \) is the transversal area of the beam, \( P_{cr} \) is the buckling load of the beam and \( \hat{s} \) is a slenderness ratio.

In figure 1, the normalized buckling load \( P_{cr}^{\hat{\kappa}} \) is compared for different values of \( \hat{k} \), with \( \hat{\lambda} = 2 \) and \( \hat{\beta}_3 = 0 \). The influence of the micropolar parameter \( \kappa \) in the buckling behavior of beams is shown. As the value of these parameters increases, the critical load is higher, which means that the stiffness of the beam increases. On the other hand, while the length to thickness ratio increases, results show convergence. Hence, as the slenderness of the beam grows, the influence of micropolar couple modulus on the buckling load decreases.

![Figure 1. Normalized buckling load with \( k \neq 0 \)](image)

In figure 2, the normalized buckling load \( P_{cr}^{\hat{\kappa}} \) is analyzed for several values of \( \hat{\beta}_3 \), taking \( \hat{\lambda} = 2 \) and \( \hat{k} = 0 \). The influence of \( \hat{\beta}_3 \) on the buckling load is evaluated as it is related to the curvature strain tensor on the micropolar continuum. The change of \( \beta_3 \) enhances the stiffness of the beam. Higher values of gamma turn into higher influence of these rotations on the overall stiffness.
Figure 2. Normalized buckling load with $\beta_3 \neq 0$

3.2. Applications using experimental material properties in FGM micropolar beams

Additional results are further developed using experimental material parameters from the bibliography [26]. Simply-supported beams with different slenderness ratios are evaluated for various power-law exponents. The beam is composed of aluminum at the top with the following material parameters: $E_t = 70.85 \text{ GPa}$, $\nu_t = 0.33$, $\kappa_t = 1.3155 \times 10^{-5} \text{ GPa}$, $\beta_{3t} = 0.59664 \text{ kN}$, $\beta_{2t} = 0.1585 \text{ kN}$; and by epoxy at the bottom: $E_b = 5.31 \text{ GPa}$, $\nu_b = 0.4$, $\kappa_b = 1.3234 \times 10^{-4} \text{ GPa}$, $\beta_{3b} = 3.3349 \text{ kN}$, $\beta_{2b} = 0.1028 \text{ kN}$. The poison ratios were used to calculate the Lamé coefficients for the two materials and then it was assumed an only varying Young modulus. The results of the analysis are shown in figure 3. The buckling load is normalized using $\bar{\mu}_t$ as in the previous examples for different values of the power-law index.

Figure 3. Normalized buckling load for various power-law exponents.
4. Conclusions

The present research has developed a finite element model for an improved first-order micropolar beam using the Eringen's kinematics tensors linearized by a Taylor expansion. High order terms where neglected. Furthermore, Trefftz’s criterion has been used to evaluate the buckling phenomenon. First, the influence of the micropolar parameters on the buckling load was analyzed for different length to thickness ratios. Results were verified against the literature. It was shown that the couple modulus \( \kappa \) has a directly proportional relationship to the buckling load. However, as the length to thickness relation increases, its impact is reduced leading to a convergence of the results. Another parameter with a similar behavior is the micropolar modulus \( \beta_3 \). Although its influence is like \( \kappa \), it does not vanish as the slenderness ratio of the beam increases. Finally, the effect of the power-law index in FGM micropolar beams was also calculated using material parameters found in the literature. New results for size-dependent micropolar parameters were obtained showing that the power-law index reduces the buckling load in micropolar beams. In the following research, results concerning geometrical non-linear behavior are expected using the non-linear micropolar stretch and wryness tensors for the analysis of the post-buckling phenomena in beams under different boundary conditions.

References

[1] Cuenot S, Frétiligny C, Demoustier-Champagne S and Nysten B 2004 Surface tension effect on the mechanical properties of nanomaterials measured by atomic force microscopy Phys. Rev. B - Condens. Matter Mater. Phys. 69 1–5
[2] Ma Q and Clarke D R 1995 Size dependent hardness of silver single crystals J. Mater. Res. 10 853–63
[3] Cosserat, E., Cosserat F 1909 Théorie des corps déformables (Paris: A. Hermann et fils)
[4] Romanoff J, Jelovica J, Reddy J N and Remes H 2020 Post-buckling of web-core sandwich plates based on classical continuum mechanics: success and needs for non-classical formulations Meccanica 6
[5] Nampally P, Karttunen A T and Reddy J N 2019 Nonlinear finite element analysis of lattice core sandwich beams Eur. J. Mech. A/Solids 74 431–9
[6] Manzari M T 2004 Application of micropolar plasticity to post failure analysis in geomechanics Int. J. Numer. Anal. Methods Geomech. 28 1011–32
[7] Mašín D, Tamagnini C, Viggiani G and Costanzo D 2006 Directional response of a reconstituted fine-grained soil - Part II: Performance of different constitutive models Int. J. Numer. Anal. Methods Geomech. 30 1303–36
[8] Ansari R, Norouzzadeh A, Shakouri A H, Bazdid-Vahdati M and Rouhi H 2018 Finite element analysis of vibrating micro-beams and -plates using a three-dimensional micropolar element Thin-Walled Struct. 124 489–500
[9] Penta F 2020 Buckling analysis of periodic Vierendeel beams by a micro-polar homogenized model Acta Mech. 231 2399–424
[10] Carrera E and Zozulya V V. 2019 Carrera unified formulation (CUF) for the micropolar beams: Analytical solutions Mech. Adv. Mater. Struct. 0 1–25
[11] Ramezani S, Naghdabadi R and Sohrabpour S 2009 Analysis of micropolar elastic beams Eur. J. Mech. A/Solids 28 202–8
[12] Betancourt K N, Soncco K and Arciniega R A 2020 Bending Analysis of Micropolar Beams LACCEI 2020 Proceedings (accepted for publication) pp 29–31
[13] Hassanpour S and Heppler G R 2016 Comprehensive and easy-to-use torsion and bending theories for micropolar beams Int. J. Mech. Sci. 114 71–87
[14] Hirano T, Yamada T, Teraki J, Niino M and Kumakawa A 1988 A study on a functionally gradient material design system for a thrust chamber Proc. 16th Int. Symp. Sp. Technol. Sci.
[15] Eringen A and Wegner J 2003 Nonlocal Continuum Field Theories vol 56
[16] Eremeyev, V.A.,Lebedev, L.P., Altenbach H 2013 Foundations of Micropolar Mechanics (Springer-Verlag Berlin Heidelberg)
[17] Cowin S C 1970 An incorrect inequality in micropolar elasticity theory Zeitschrift für Angew. Math. und Phys. ZAMP 21 494–7
[18] Arciniega R A and Reddy J N 2007 Large deformation analysis of functionally graded shells *Int. J. Solids Struct.* **44** 2036–52

[19] Hasanyan A D and Waas A M 2015 On the buckling of a two-dimensional micropolar strip *J. Appl. Mech. Trans. ASME* **82**

[20] Pietraszkiewicz W and Eremeyev V A 2009 On natural strain measures of the non-linear micropolar continuum *Int. J. Solids Struct.* **46** 774–87

[21] Arciniega R A, Gonçalves P B and Reddy J N 2004 Buckling and postbuckling analysis of laminated cylindrical shells using the third-order shear deformation theory *Int. J. Struct. Stab. Dyn.* **4** 293–312

[22] Ayala S, Vallejos A and Arciniega R A 2020 Buckling Analysis of Functionally Graded Timoshenko Beams *LACCEI 2020 Proceedings* (To be submitted)

[23] Brush D O and Almroth B O 1975 *Buckling of Bars, Plates and Shells* (Mcgraw-hill)

[24] Karniadakis, G. E.; Sherwin S 2005 *Spectral/hp Element Methods for CFD.*

[25] Hasanyan A D and Waas A M 2015 On the Buckling of a Two-Dimensional Micropolar Strip *J. Appl. Mech.* **82**

[26] Abadikhah H and Folkow P D 2017 A systematic approach to derive dynamic equations for homogeneous and functionally graded micropolar plates *Procedia Eng.* **199** 1429–34