On asymptotic formulae in some sum–product questions *

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Annotation.

In this paper we obtain a series of asymptotic formulae in the sum–product phenomena over the prime field $\mathbb{F}_p$. In the proofs we use usual incidence theorems in $\mathbb{F}_p$, as well as the growth result in $\text{SL}_2(\mathbb{F}_p)$ due to Helfgott. Here some of our applications:

• a new bound for the number of the solutions to the equation $(a_1 - a_2)(a_3 - a_4) = (a'_1 - a'_2)(a'_3 - a'_4)$, $a_i, a'_i \in A$, $A$ is an arbitrary subset of $\mathbb{F}_p$,
• a new effective bound for multilinear exponential sums of Bourgain,
• an asymptotic analogue of the Balog–Wooley decomposition theorem,
• growth of $p_1(b) + 1/(a + p_2(b))$, where $a, b$ runs over two subsets of $\mathbb{F}_p$, $p_1, p_2 \in \mathbb{F}_p[x]$ are two non–constant polynomials,
• new bounds for some exponential sums with multiplicative and additive characters.

1 Introduction

Let $p$ be an odd prime number, and $\mathbb{F}_p$ be the finite field. Having two sets $A, B \subset \mathbb{F}_p$, define the sumset, the product set and the quotient set of $A$ and $B$ as

$$A + B := \{a + b : a \in A, b \in B\},$$
$$AB := \{ab : a \in A, b \in B\},$$

and

$$A/B := \{a/b : a \in A, b \in B, b \neq 0\},$$
correspondingly. Our paper is devoted to so–called the sum–product phenomenon in $\mathbb{F}_p$, which was developed in papers [1–17], [27–43], [48–53], [57], [58] and in many others. This is extensively growing area of mathematics with plenty of applications to Number Theory, Additive Combinatorics, Computer Science and Dynamical Systems. It seems like at the moment there is ”the second wave” of results and applications in this field see, e.g., [2], [33–39], [43] and this wave is connected with a fundamental incidence result of Rudnev [42] (see a simple proof of his theorem in [59] and also the famous Guth–Katz [19] solution of the Erdős distinct distance problem which contains the required technique for such incidence results), as well as with more

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applicable energy versions of the sum–product phenomenon [2], [11], [29], [33], [36] and [43]. The sum–product phenomenon asserts that either the sumset or the product set of a set must be large up to some natural algebraic constrains. One of the strongest form of this principle is the Erdős–Szemerédi conjecture [14] which says that for any sufficiently large set $A$ and an arbitrary $\epsilon > 0$ one has

$$\max \{|A + A|, |AA|\} \gg |A|^{2-\epsilon}.$$  

At the moment the best results in the direction can be found in [52], [28], [29], [43] and in [1], [41] for $\mathbb{R}$ and $\mathbb{F}_p$, respectively. For example, let us recall the main results from [1], [41].

**Theorem 1** Let $A \subseteq \mathbb{F}_p$ be an arbitrary set and $|A| < p^{5/8}$. Then

$$\max\{|A + A|, |AA|\} \gg |A|^{1+1/5}. \quad (1)$$  

As one can see the bound above works for small sets only and this is an usual thing for the results in this area. On the other hand, the exact behaviour of the maximum in (1) and other sum–product quantities are known just for very large sets having its sizes comparable to the characteristic $p$, see, e.g., [15], [16], [57], [58]. Even in strong recent paper [53] containing an optimal estimate for the number of point/lines incidences in the case of Cartesian products we have just an upper bound for such incidences but not an asymptotic. The first result in the sum–product theory which gives us an asymptotic formula for a sum–product quantity and which works for sets of any sizes was proved in [36, Theorem 10].

**Theorem 2** Let $A \subseteq \mathbb{F}_p$ be a set and $Q(A)$ be the number of collinear quadruples in $A \times A$. Then

$$Q(A) = \frac{|A|^8}{p^2} + O(|A|^5 \log |A|). \quad (2)$$  

Further for the number $T(A)$ of collinear triples in $A \times A$ one has

$$T(A) = \frac{|A|^6}{p} + O(p^{1/2}|A|^{7/2}). \quad (3)$$

It is known that formula (2) is sharp up to logarithms but (3) is probably not, see [36].

One of the aims of our paper is to prove a series of new asymptotic formulae in the considered area. In the proofs we use usual incidence theorems in $\mathbb{F}_p$, see [42], [53] and other papers, as well as the growth result in $\text{SL}_2(\mathbb{F}_p)$ due to Helfgott [22]. So, our another aim is to obtain some new applications (also, see recent papers [32], [31] where other applications were found) of classical graph (group) expansion phenomena, see [9], [21], [18], [22], [44] and others.

Our first asymptotic formula concerns to the quantity

$$T^+_k(A) := |\{(a_1, \ldots, a_k, a'_1, \ldots, a'_k) \in A^{2k} : a_1 + \cdots + a_k = a'_1 + \cdots + a'_k\}|$$

(similarly one can define its multiplicative analogue $T^*_k(A)$) in the case when $A$ is a multiplicative subgroup of $\mathbb{F}_p^*$ (see Theorem [25] below). In papers [4], [5], [12], [50] just upper bounds for $T^+_k(A)$ can be found but not an asymptotic formula.
Theorem 3 Let $\Gamma \subseteq \mathbb{F}_p^*$ be a multiplicative subgroup. Then for any $k \geq 1$ one has

$$0 \leq T^+_k(\Gamma) - \frac{|\Gamma|^{2k+1}}{p} \leq 2^{3k^2} (C_* \log^4 p)^k - |\Gamma|^{2k+1 - \frac{(k+7)}{2}} T^+_2(\Gamma),$$

(4)

where $C_* > 0$ is an absolute constant.

In Section 5 we obtain new asymptotic formulae and bounds for the quantities

$$|(a_1 - a_2) \cdots (a_{2k-1} - a_{2k}) = (a'_1 - a'_2) \cdots (a'_{2k-1} - a'_{2k}) : a_i, a'_i \in A|$$

as well as for

$$|a_1 a_2 + \cdots + a_{2k-1} a_{2k} = a'_1 a'_2 + \cdots + a'_{2k-1} a'_{2k} : a_i, a'_i \in A|$$

It allows us to improve estimates for exponential sums of Petridis and Shparlinski [39], see Corollary 45 below.

Corollary 4 Given three sets $X, Y, Z \subseteq \mathbb{F}_p$, $|X| \geq |Y| \geq |Z|$ and three complex weights $\rho = (\rho_{x,y})$, $\sigma = (\sigma_{x,z})$, $\tau = (\tau_{y,z})$ all bounded by one, we have

$$\sum_{x \in X, y \in Y, z \in Z} \rho_{x,y} \sigma_{x,z} \tau_{y,z} e(xyz) \ll |X|^{7/8} |Y|^{29/32} |Z|^{29/32} (|Y||Z|)^{-1/3072},$$

provided $|Y|, |Z| < p^{48/97}$.

Moreover we obtain a new effective bound for such sums in an optimal range $|X||Y||Z| \geq p^{1+\delta}$ (and for higher sums). Previously, Bourgain [6, Theorem 1] obtained $(\delta/10)^{10^4}$ instead of $\frac{8 \log(8/\delta)}{\delta} + 4$, see formula [22] below. In general, our saving has the form $p^{-\delta/(C_1 \log(C_2 r/\delta))}$ for $r$ sets instead of $p^{-(\delta/r)C_r}$ from [6, Theorem A]. Here $C, C_1, C_2 > 0$ are some absolute constants.

Theorem 5 Let $X, Y, Z \subseteq \mathbb{F}_p$ be arbitrary sets such that for some $\delta > 0$ the following holds

$$|X||Y||Z| \geq p^{1+\delta}.$$

(5)

Then

$$\sum_{x \in X, y \in Y, z \in Z} e(xyz) \ll |X||Y||Z| \cdot p^{-\frac{\delta}{8 \log(8/\delta) + 4}}.$$

(6)

Our next result is an asymptotic version of the Balog–Wooley [2] decomposition Theorem, also, see [29], [41], [48] (consult Theorem 48 and Corollary 49 below). In particular, it gives us an asymptotic variant of Theorem [4] (signs $\lesssim, \gtrsim$ mean some powers of logarithm of $|A|$).
Theorem 6 Let $A \subseteq \mathbb{F}_p$ be a set and let $1 \leq M \leq p/(2|A|)$ be a parameter. There exist two disjoint subsets $B$ and $C$ of $A$ such that $A = B \sqcup C$ and
\[ 0 \leq T_2^+(B) - \frac{|B|^4}{p} \leq \frac{|A|^{2/3}|B|^{7/3}}{M}, \]
and for any set $X \subseteq \mathbb{F}_p$ one has
\[ T_2^+(C, X) \lesssim \frac{M^2|X^2|A|}{p} + M^{3/2}|A||X|^{3/2}. \]

In particular, for any set $A \subseteq \mathbb{F}_p$ either
\[ |A + A| \geq 5^{-1} \min\{|A|^{6/5}, p/2\} \]
or
\[ |AA| \gtrsim \min\{p|A|^{-2/5}, |A|^{6/5}\}. \]

In the last Section 7 we consider the expansion in $\text{SL}_2(\mathbb{F}_p)$ and obtain some combinatorial applications of the celebrated Helfgott’s growth result. Our first theorem concerns to the intersection of the inverses of additively rich sets $A$, see Corollary 55.

Theorem 7 Let $A, B \subseteq \mathbb{F}_p$, $|B| \geq p^\varepsilon$, $\varepsilon > 0$ and $|A + B| \leq K|A|$. Then for any $\lambda \neq 0$ one has
\[ \left| \left\{ \frac{1}{a_1} - \frac{1}{a_2} = \lambda : a_1, a_2 \in A \right\} \right| \leq \frac{K^2|A|^2}{p} + 2K|A|p^{-1/2k+2}, \]
where $k = k(\varepsilon)$. Also,
\[ T_2^+(1/A, 1/B) - \frac{K^2|A|^2|B|^2}{p} \ll K^{5/4}|A|^{5/4}|B|^{3/2} + K^2|A|^2. \]

Theorem above can be extended to general polynomial maps (and even to rational functions), see Corollary 61.

Theorem 8 Let $p_1, p_2 \in \mathbb{F}_p[x]$ be any non–constant polynomials. Then for any $A, B \subseteq \mathbb{F}_p$, $|B| \geq p^\varepsilon$, $\varepsilon > 0$ one has
\[ 0 \leq \left| \left\{ \frac{p_1(b)}{a + p_2(b)} = p_1(b') + \frac{1}{a' + p_2(b')} : a, a' \in A, b, b' \in B \right\} \right| - \frac{|A|^2|B|^2}{p} \leq \frac{|A|^2|B|^2}{p} \]
\[ \leq 2|A||B|^2p^{-1/2k+2}, \]
where $k = k(\varepsilon, \deg p_1, \deg p_2)$. In particular,
\[ \left| \left\{ \frac{p_1(b)}{a + p_2(b)} : a \in A, b \in B \right\} \right| \gg \min\{p, |A|^{1/2k+2}\}. \]
Also, we break the square–root barrier for exponential sums of the form (and many other exponential sums)
\[
e \left( y \left( \frac{1}{x + b_1} + b_2 \right) \right), \quad e \left( y \left( \frac{p_1(b) + 1}{x + p_2(b)} \right) \right),
\]
\[
\chi \left( y + b_2 + \frac{1}{x + b_1} \right), \quad \chi \left( y + p_1(b) + \frac{1}{x + p_2(b)} \right),
\]
see Corollaries 56, 62 below. Here the variables \( x, y \) belong to some sets \( X, Y \), further \( b_1, b_2 \in B \), \(|B| > p^\varepsilon\), \( e \) and \( \chi \) are any non–principal additive/multiplicative characters and \( p_1, p_2 \in \mathbb{F}_p[x] \) are non–constant polynomials.

Finally, we obtain an expansion result of another sort (see Corollary 67 from Section 8).

**Corollary 9** Let \( A \subseteq \mathbb{F}_p, B_1, B_2, B_3 \subseteq \mathbb{F}_p, B := |B_1| = |B_2| = |B_3| > p^\varepsilon \). Suppose that \(|B_3 - B_1B_2| \leq B^2p^{-\varepsilon} \). Then there is \( \delta = \delta(\varepsilon) > 0 \) such that
\[
\left| \left\{ \frac{a + b_1}{ab_2 + b_3} : a \in A, b_j \in B_j \right\} \right| \gg \min\{p, |A|p^\delta\}. \tag{10}
\]

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## 2 Notation

In this paper \( p \) is an odd prime number, \( \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} \) and \( \mathbb{F}_p^* = \mathbb{F}_p \setminus \{0\} \). We denote the Fourier transform of a function \( f : \mathbb{F}_p \to \mathbb{C} \) by \( \widehat{f} \),
\[
\widehat{f}(\xi) = \sum_{x \in \mathbb{F}_p} f(x)e(-\xi \cdot x), \tag{11}
\]
where \( e(x) = e^{2\pi ix/p} \). We rely on the following basic identities. The first one is called the Plancherel formula and its particular case \( f = g \) is called the Parseval identity
\[
\sum_{x \in \mathbb{F}_p} f(x)\overline{g(x)} = \frac{1}{p} \sum_{\xi \in \mathbb{F}_p} \widehat{f}(\xi)\overline{\widehat{g}(\xi)}. \tag{12}
\]
Another particular case of (12) is
\[
\sum_{y \in \mathbb{F}_p} \left| \sum_{x \in \mathbb{F}_p} f(x)\overline{g(y - x)} \right|^2 = \frac{1}{p} \sum_{\xi \in \mathbb{F}_p} |\widehat{f}(\xi)|^2|\widehat{g}(\xi)|^2, \tag{13}
\]
and the identity
\[
f(x) = \frac{1}{p} \sum_{\xi \in \mathbb{F}_p} \widehat{f}(\xi)e(\xi \cdot x) \tag{14}
\]
is called the inversion formula. Further let \( f, g : \mathbb{F}_p \to \mathbb{C} \) be two functions. Put
\[
(f * g)(x) := \sum_{y \in \mathbb{F}_p} f(y)g(x - y) \quad \text{and} \quad (f \circ g)(x) := \sum_{y \in \mathbb{F}_p} f(y)g(y + x). \tag{15}
\]

Then
\[
\hat{f} \ast \hat{g} = \hat{f \circ g} \quad \text{and} \quad \hat{f} \circ \hat{g} = \hat{f \ast g}. \tag{16}
\]

Put \( E^+(A, B) \) for the common additive energy of two sets \( A, B \subseteq \mathbb{F}_p \) (see, e.g., [56]), that is,
\[
E^+(A, B) = \{|(a_1, a_2, b_1, b_2) \in A \times A \times B \times B : a_1 + b_1 = a_2 + b_2\}|.
\]

If \( A = B \), then we simply write \( E^+(A) \) instead of \( E^+(A, A) \) and the quantity \( E^+(A) \) is called the additive energy in this case. Clearly,
\[
E^+(A, B) = \sum_x (A * B)(x)^2 = \sum_x (A \circ B)(x)^2 = \sum_x (A \circ A)(x)(B \circ B)(x)
\]
and by (13),
\[
E(A, B) = \frac{1}{p} \sum_\xi |\hat{A}(\xi)|^2|\hat{B}(\xi)|^2. \tag{17}
\]

Also, notice that
\[
E^+(A, B) \leq \min\{|A|^2|B|, |B|^2|A|, |A|^{3/2}|B|^{3/2}\}. \tag{18}
\]

Sometimes we write \( E^+(f_1, f_2, f_3, f_4) \) for the additive energy of four real functions, namely,
\[
E^+(f_1, f_2, f_3, f_4) = \sum_{x, y, z} f_1(x)f_2(y)f_3(x + z)f_4(y + z).
\]

It can be shown using the Hölder inequality (see, e.g., [56]) that
\[
E^+(f_1, f_2, f_3, f_4) \leq (E^+(f_1)E^+(f_1)E^+(f_1)E^+(f_1))^{1/4}. \tag{19}
\]

In the same way define the common multiplicative energy of two sets \( A, B \subseteq \mathbb{F}_p \)
\[
E^\times(A, B) = \{|(a_1, a_2, b_1, b_2) \in A \times A \times B \times B : a_1b_1 = a_2b_2\}|.
\]

Certainly, the multiplicative energy \( E^\times(A, B) \) can be expressed in terms of multiplicative convolutions similar to (15). Further the definitions and the formulae above take place in an arbitrary abelian group \( \mathbb{G} \). If there is no difference between \( E^+ \) and \( E^\times \) or there is the only operation on the considered group \( \mathbb{G} \), then we write just \( E \).

Sometimes we use representation function notations like \( r_{AB}(x) \) or \( r_{A+B}(x) \), which counts the number of ways \( x \in \mathbb{F}_p \) can be expressed as a product \( ab \) or a sum \( a + b \) with \( a \in A, b \in B \), respectively. For example, \(|A| = r_{A-A}(0)\) and \( E^+(A) = r_{A+A-A-A}(0) = \sum_x r^2_{A-A}(x) = \sum_x r^2_{A-A}(x) \). In this paper we use the same letter to denote a set \( A \subseteq \mathbb{F}_p \) and its characteristic function \( A : \mathbb{F}_p \to \{0, 1\} \). Thus \( r_{A+B}(x) = (A * B)(x) \), say. Having \( P \subseteq A - A \) we write \( \sigma_P(A) := \sum_{x \in P} r_{A-A}(x) \). Also, we write \( f_A(x) \) for the balanced function of a set \( A \subseteq \mathbb{F}_p \), namely, \( f_A(x) = A(x) - |A|/p \).
Now consider two families of higher energies. Firstly, let
\[
T^+_k(A) := |\{(a_1, \ldots, a_k, a'_1, \ldots, a'_k) \in A^{2k} : a_1 + \cdots + a_k = a'_1 + \cdots + a'_k\}| = \frac{1}{p} \sum_\xi |\hat{A}(\xi)|^{2k}. \tag{20}
\]
It is useful to note that
\[
T^+_{2k}(A) = \left| \{(a_1, \ldots, a_{2k}, a'_1, \ldots, a'_{2k}) \in A^{4k} : (a_1 + \cdots + a_k) + (a_{k+1} + \cdots + a_{2k}) = \right.
\]
\[
= (a'_1 + \cdots + a'_k) + (a'_{k+1} + \cdots + a'_{2k}) \right| = \sum_{x,y,z} r_{kA}(x)r_{kA}(y)r_{kA}(x+z)r_{kA}(y+z), \tag{21}
\]
so one can rewrite \(T^+_{2k}(A)\) via the additive energy of the function \(r_{kA}(x)\). Sometimes we use \(T^+_k(f)\) for an arbitrary function \(f\). It is easy to check that
\[
T^+_k(f) \leq \|f\|^2 \cdot T^+_{k-1}(f), \tag{22}
\]
and hence by the Parseval identity
\[
T^+_k(f) \leq \|f\|^{2k-2} \cdot \|f\|_2^2. \tag{23}
\]
Secondly, for \(k \geq 2\), we put
\[
E^+_k(A) = \sum_{x \in F_p} (A \circ A)(x)^k = \sum_{x \in F_p} r_{A-A}^k(x) = E^+(\Delta_k(A), A^k), \tag{24}
\]
where
\[
\Delta_k(A) := \{(a, a, \ldots, a) \in A^k\}.
\]
Thus \(E^+_2(A) = T^+_2(A) = E^+(A)\). Also, notice that we always have \(|A|^k \leq E^+_k(A) \leq |A|^{k+1}\) and moreover
\[
E^+_k(A) \leq |A|^{k-l} E^+_l(A), \quad \forall l \leq k. \tag{25}
\]
Finally, let us remark that by definition \((24)\) one has \(E^+_1(A) = |A|^2\). Similarly, one can define \(E^+(f)\) for an arbitrary function \(f\). From the inversion formula and the Parseval identity, it follows that
\[
E^+_k(f) = p^{1-k} \sum_{x_1 + \cdots + x_k = 0} |\hat{f}(x_1)|^2 \cdots |\hat{f}(x_k)|^2 \geq 0. \tag{26}
\]
Some further results about the properties of the energies \(E^+_k\) can be found in [45]. Again, sometimes we use \(E^+_k(f)\) for an arbitrary function \(f\) and the first formula from \((24)\) allows to define \(E^+_k(A)\) for any positive \(k\). It was proved in [48, Proposition 16] that \((E^+_k(f))^{1/2k}\) is a norm for even \(k\) and a real function \(f\). The fact that \((T^+_k(f))^{1/2k}\) is a norm is contained in [56] and follows from a generalization of inequality \((10)\).

We write \(\delta\{x\} = 1\) if \(x = 0\) and \(\delta\{x\} = 0\) otherwise.

All logarithms are to base 2. The signs \(\ll\) and \(\gg\) are the usual Vinogradov symbols. If we have a set \(A\), then we will write \(a \ll b\) or \(b \gg a\) if \(a = O(b \cdot \log^c |A|),\), \(c > 0\). When the constants in the signs depend on a parameter \(M\), we write \(\ll_M\) and \(\gg_M\). For a positive integer \(n\), we set \([n] = \{1, \ldots, n\}\). We do not normalize \(L_p\)-norms of functions. So, \(\|f\|_p = (\sum_x |f(x)|^p)^{1/p}\) for any complex function \(f\).
3 Preliminaries

First of all, we need a general design bound for the number of incidences. Let $\mathcal{P} \subseteq \mathbb{F}_q^3$ be a set of points and $\Pi$ be a collection of planes in $\mathbb{F}_q^3$. Having $p \in \mathcal{P}$ and $\pi \in \Pi$ we write

$$I(p, \pi) = \begin{cases} 1 & \text{if } q \in \pi \\ 0 & \text{otherwise} \end{cases}$$

So, $I$ is $|\mathcal{P}| \times |\Pi|$ matrix. If $\mathcal{P} = \mathbb{F}_q^3$ and $\Pi$ is the family of all planes in $\mathbb{F}_q^3$, then we obtain the matrix $I$ and $I$ is a submatrix of $\mathcal{T}$. One can easily calculate $\mathcal{T}^* I$ and $I \mathcal{T}^*$ in the projective plane $\mathbb{P} \mathbb{F}_q^3$ and check that both of this matrices have form $aI + bJ$, where $a, b$ some numbers, $I$ is the identity matrix and $J$ is the matrix having all entries equal 1’s see, e.g., [57, 58]. In particular, vectors $(1, \ldots, 1)$ of the correspondent lengths are the first eigenvectors of $\mathcal{T}^* I$ and $I \mathcal{T}^*$. Moreover, one can check that in our case of points and planes the following holds $a = q^2$ and $b = q + 1$ (see [57, 58]). Having these facts in mind and using the singular decomposition (see, e.g., [23]), we derive that for any functions $\alpha : \mathcal{P} \to \mathbb{C}$, $\beta : \Pi \to \mathbb{C}$ one has

$$|\sum_{p, \pi} I(p, \pi) \alpha(p) \beta(\pi)| = |\sum_{p, \pi} \mathcal{T}(p, \pi) \alpha(p) \beta(\pi)| \leq q \|\alpha\|_2 \|\beta\|_2,$$

provided either $\sum_{p \in \mathcal{P}} \alpha(p) = 0$ or $\sum_{\pi \in \Pi} \beta(\pi) = 0$. Of course, similar arguments work not just for points/planes incidences but, e.g., points/lines incidences and so on.

A much more deep result on incidences is contained in [42] (or see [36, Theorem 8] and the proof of Corollary 2 from [33]). In the proof of formula (29) one should use an incidence bound from [35, Section 3].

**Theorem 10** Let $p$ be an odd prime, $\mathcal{P} \subseteq \mathbb{F}_p^3$ be a set of points and $\Pi$ be a collection of planes in $\mathbb{F}_p^3$. Suppose that $|\mathcal{P}| \leq |\Pi|$ and that $k$ is the maximum number of collinear points in $\mathcal{P}$. Then the number of point–planes incidences satisfies

$$\mathcal{I}(\mathcal{P}, \Pi) \ll \frac{|\mathcal{P}| |\Pi|}{p} + |\mathcal{P}|^{1/2} |\Pi| + k|\mathcal{P}|.$$

More precisely,

$$\mathcal{I}(\mathcal{P}, \Pi) - \frac{|\mathcal{P}| |\Pi|}{p} \ll |\mathcal{P}|^{1/2} |\Pi| + k|\mathcal{P}|.$$

**Corollary 11** Let $\alpha, \beta$ be non–negative functions, $C \subseteq \mathbb{F}_p$ be a set. Suppose that

$$\max\{\|\alpha\|_1 \|\beta\|^{-1}_1 \|\alpha\|_2^{-1} \|\beta\|_2, \|\alpha\|^{-1}_1 \|\beta\|_1 \|\alpha\|_2 \|\beta\|^{-1}_2\} \leq |C|^{1/2} \leq \frac{\|\alpha\|_1 \|\beta\|_1}{\|\alpha\|_2 \|\beta\|_2}$$

and put $L = \log(\|\alpha\|_1 \|\beta\|_1 |C|/(\|\alpha\|_2 \|\beta\|_2))$. Then

$$\sum_x x^2 \alpha \beta + C(x) - \frac{(\|\alpha\|_1 \|\beta\|_1 |C|)^2}{p} \ll L^4 \|\alpha\|_1 \|\beta\|_1 \|\alpha\|_2 \|\beta\|_2 |C|^{3/2},$$
and
\[
\sum_x r_{\alpha(\beta + C)}^2(x) - \frac{\|\alpha\|_1 \|\beta\|_1 |C|^2}{p} \ll L^4 \|\alpha\|_1 \|\beta\|_1 \|\alpha\|_2 \|\beta\|_2 |C|^{3/2}.
\] (32)

**Proof.** We obtain (31) because the proof of (32) is similar. Let \(f(x) = C(x) - |C|/p\). Then
\[
\sum_x r_{\alpha(\beta + f)}^2(x) = \frac{\|\alpha\|_1 \|\beta\|_1 |C|^2}{p} + \sum_x r_{\alpha(\beta + f)}^2(x).
\]
Split the level set of \(\alpha, \beta\) into level sets \(P_j(\alpha), P_j(\beta)\) where the functions \(\alpha, \beta\) differ at most twice, correspondingly. Clearly, there are at most \(L\) such sets because if, say, \(\alpha(x) \leq \varepsilon := 2^{-2} |C|^{-1/2} |\beta|^{-1} \|\alpha\|_2 \|\beta\|_2\), then
\[
\varepsilon \|\alpha\|_1 |C|^{2/2} |\beta|_2^2 \leq 2^{-2} |C|^{3/2} \|\alpha\|_1 \|\beta\|_1 \|\alpha\|_2 \|\beta\|_2,
\]
so it is negligible and hence the inequality \(2\varepsilon \leq \|\alpha\|_1\) gives the required bound. Using the pigeonhole principle and positivity of the operator \(r_{f-f}(x - y)\), we find some \(\Delta_A, \Delta_B\) and \(A \subseteq P_j(\alpha), B \subseteq P_j(\beta)\) such that
\[
\sigma := \sum_x r_{\alpha(\beta + f)}^2(x) \ll L^4 \Delta_A^2 \Delta_B^2 \sum_x r_{\alpha(\beta + f)}^2(x).
\]
On the one hand, in view of (27) the last sum is bounded by
\[
\sigma \leq L^4 \Delta_A^2 \Delta_B^2 |\beta|_2 |A||B| \leq L^4 \Delta_A^2 \Delta_B^2 |A||B||C|.
\] (33)
On the other hand, using Theorem 10 (one can consult paper 11)
\[
\sum_x r_{AB+C}^2(x) = \frac{|A|^2 |B|^2 |C|^2}{p} \leq \sum_x r_{AB+C}^2(x) \ll \frac{|A|^2 |B|^2 |C|^2}{p} + \frac{|A||B||C|^{3/2} + |A||B||C| \max\{|A|, |B|, |C|\}}.
\]
If the second term in the last formula dominates, then we are done. If the first term is the largest one, then \(p \leq (|A||B||C|)^{1/2}\) and (33) gives us
\[
\sigma \leq L^4 \Delta_A^2 \Delta_B |A||B||C| \leq L^4 \Delta_A^2 \Delta_B |A||B||C|^{3/2} \leq L^4 \|\alpha\|_1 \|\beta\|_1 \|\alpha\|_2 \|\beta\|_2 |C|^{3/2}
\]
as required. Finally, condition (30) implies that the third term is negligible. This completes the proof. \(\Box\)

Now we obtain a simple asymptotic formula for the number of points/lines incidences in the case when the set of points form a Cartesian product.

**Lemma 12** Let \(A, B \subseteq \mathbb{F}_p\) be sets, \(|A| \leq |B|\), \(P = A \times B\), and \(L\) be a collection of lines in \(\mathbb{F}_p^2\). Then
\[
\mathcal{I}(P, L) - \frac{|A||B||C|}{p} \ll |A|^{3/4} |B|^{1/2} |C|^{3/4} + |L| + |A||B|.
\] (34)
Proof. Let \( f(x) = B(x) - |B|/p \) be the balanced function of the set \( B \). Then, using the natural notation, we get
\[
\mathcal{I}(\mathcal{P}, \mathcal{L}) = \frac{|A||B|}{p} \mathcal{I}(A) + \mathcal{I}(f \otimes A, \mathcal{L}),
\]
where we count the number of incidences with the weight \( f(x)A(a) \). Using the design bound for points/lines incidences, we obtain
\[
\mathcal{I}(f \otimes A, \mathcal{L}) \leq \|f\|_2 (|A||\mathcal{L}|)^{1/2} \leq (|A||B|\mathcal{L}|)^{1/2}.
\] (35)
By an analogue of the Szemerédi-Trotter theorem in \( \mathbb{F}_p \), see [53] (or [33, Theorem 7], [36, Theorem 9]), we have
\[
\mathcal{I}(f \otimes A, \mathcal{L}) \ll |A|^{3/4} |\mathcal{L}|^{3/4} + |\mathcal{L}| + |A||B|,
\] (36)
provided \( |A||\mathcal{L}| \leq p^2 \). But if \( |A||\mathcal{L}| > p^2 \), then by (35), we see that
\[
\mathcal{I}(f \otimes A, \mathcal{L}) \leq (p|A||B|\mathcal{L}|)^{1/2} \leq |A|^{3/4}|B|^{1/2}|\mathcal{L}|^{3/4}
\]
and the last bound is even better than (36). This completes the proof. \( \square \)

We need a lemma from [50] which is a consequences of the main result from [42] or Theorem 10.

**Lemma 13** Let \( A, Q \subseteq \mathbb{F}_p \) be two sets, \( A, Q \neq \{0\} \), \( M \geq 1 \) be a real number, and \( |QA| \leq M|Q| \). Then
\[
E^+(Q) \leq C_\ast \left( \frac{M^2|Q|^4}{p} + \frac{M^{3/2}|Q|^3}{|A|^{1/2}} \right),
\] (37)
where \( C_\ast \geq 1 \) is an absolute constant.

The second lemma can be obtained in the same vein.

**Lemma 14** Let \( A, B \subseteq \mathbb{F}_p \), and \( |A| + |B| \leq K|A| \). Then
\[
E^+(1/A, 1/B) \leq \frac{K^2|A|^2|B|^2}{p} \ll K^{5/4}|A|^{5/4}|B|^{3/2} + K^2|A|^2.
\] (38)

**Proof.** Indeed, for any \( \alpha, \beta \) the following holds
\[
\left( \frac{1}{\alpha} + \frac{1}{\beta} \right)^{-1} = \frac{\alpha\beta}{\alpha + \beta} = \beta - \beta^2 \cdot \frac{1}{\alpha + \beta}.
\]
Hence
\[
E^+(1/A, 1/B) \leq \left| \{ b_1 - b_1^2x = b_2 - b_2^2y : b_1, b_2 \in B, x, y \in (A+B)^{-1} \} \right| = \mathcal{I}(\mathcal{P}, \mathcal{L}),
\]
where \( \mathcal{P} = (A + B) \times (A + B) \), \( \mathcal{L} = \{l_{b_1,b_2}\} \) and line \( l_{b_1,b_2} \) is defined by the equation \( b_1 - b_1^2x = b_2 - b_2^2y \). Applying Lemma 12, we get
\[
\mathcal{I}(\mathcal{P}, \mathcal{L}) - \frac{|A + B|^2|B|^2}{p} \ll |A + B|^{5/4}|B|^{3/2} + |A + B|^2.
\]

Clearly, \( |B| \leq |A + B| \leq K|A| \) and hence
\[
\mathbb{E}^+(1/A, 1/B) - \frac{K^2|A|^2|B|^2}{p} \ll K^{5/4}|A|^{5/4}|B|^{3/2} + K^2|A|^2
\]
as required. □

The next result is a slight generalization of [50, Lemma 10].

**Lemma 15** Let \( f \) be a real function and \( P \subseteq \mathbb{F}_p^* \) be a set. Then for any \( k \geq 1 \) one has
\[
\left( \sum_{x \in P} r_{f^2}^k(x) \right)^4 \leq \|f\|_{2k}^4 \mathbb{E}_x^{2k}(f) \mathbb{E}^+(P). \tag{39}
\]

**Proof.** We have
\[
\left( \sum_{x \in P} r_{f^2}^k(x) \right)^2 = \left( \sum_{x_1,\ldots,x_k} \prod_{j=1}^k f(x_j) \sum_y P(y) f(y + x_1) \ldots f(y + x_k) \right)^2 \leq \|f\|_{2k}^{2k} \sum_{x_1,\ldots,x_k} \left| \sum_y P(y) f(y + x_1) \ldots f(y + x_k) \right|^2 = \|f\|_{2k}^{2k} \sum_x r_{P-P}(x) r_{f^2}^k(x).
\]
Hence by the Cauchy–Schwarz inequality, we obtain
\[
\left( \sum_{x \in X} r_{f^2}^k(x) \right)^4 \leq \|f\|_{2k}^4 \mathbb{E}_x^{2k}(f) \mathbb{E}^+(P)
\]
as required. □

Let \( A, B, C, D \subseteq \mathbb{F}_p \) be four sets. By \( Q(A, B, C, D) \) we denote the number of *collinear quadruples* in \( A \times A, B \times B, C \times C, D \times D \). If \( A = B = C = D \), then we write \( Q(A) \) for \( Q(A, A, A, A) \). Recent results on the quantity \( Q(A) \) can be found in [38] and [36]. It is easy to see (or consult [36]) that
\[
Q(A, B, C, D) = \left\{ \left. \begin{array}{l}
\frac{b - a'}{b - a} = \frac{c - a'}{c - a} = \frac{d - a'}{d - a} : a, a' \in A, b, b' \in B, c, c' \in C, d, d' \in D \end{array} \right\} \right| \tag{40}
\]
\[
= \sum_{a, a' \in A} \sum_x r_{(B-a)/a'}(x) r_{(C-a)/a'}(x) r_{(D-a)/a'}(x) r_{(D-a)/a'}(x). \tag{41}
\]
Notice that in (40), we mean that the condition, say, \( b = a \) implies \( c = d = b = a \) or, in other words, that all four points \( (a, a'), (b, b'), (c, c'), (d, d') \) have the same abscissa. More rigorously, the summation in (41) should be taken over \( \mathbb{F}_p \cup \{+\infty\} \), where \( x = +\infty \) means that the denominator in any fraction \( x = \frac{b' - a'}{b - a} \) from, say, \( r_{(B-a'}/(B-a)}(x) \) equals zero. Anyway, it is easy to see that the contribution of the point \(+\infty\) is at most \( O(M^5) \), where \( M = \max\{|A|, |B|, |C|, |D|\} \), and hence it is negligible (see, say, Theorem 2 above). Further defining a function \( q_{A,B,C,D}(x, y) \) (see (42)) as
\[
q_{A,B,C,D}(x, y) := \left\{ \frac{b - a}{c - a} = x, \frac{d - a}{c - a} = y : a \in A, b \in B, c \in C, d \in D \right\},
\]
we obtain another formula for the quantity \( Q(A, B, C, D) \), namely,
\[
Q(A, B, C, D) = \sum_{x,y} q_{A,B,C,D}^2(x, y).
\]

An optimal (up to logarithm factors) upper bound for \( Q(A) \) was obtained in [36], [38], see Theorem 2 from the Introduction. We need a simple lemma about the same bound for a generalization of the quantity \( Q(A) \). The proof is analogous to the proof [51, Lemma 6] and [49, Lemma 5].

**Lemma 16** Let \( A, B \subseteq \mathbb{F}_p \) be two sets, \( |B| \leq |A| \leq \sqrt{p} \). Then
\[
Q(B, A, A, A) \ll |A|^{15/4} |B|^{5/4} \log^2 |A| + T(A).
\] (43)

It is known [11, Proposition 2.5] that \( T(A) \ll |A|^{9/2} \), provided \( |A| \leq p^{2/3} \) (also, see Theorem 2 from the Introduction). So, the term \( T(A) \) in (43) is negligible if \( A \) and \( B \) have comparable sizes, say.

Proposition 16 from [43] contains a combinatorial lemma, see Lemma 17 below.

**Lemma 17** Let \((G, +)\) be an abelian group. Also, let \( A \subseteq G \) be a set, \( P \subseteq A - A \), \( P = -P \). Then there is \( A_s \subseteq A \) and a number \( q \), \( q \leq |A_s| \) such that for any \( x \in A_s \) one has \( r_{A+P}(x) \geq q \), and \( \sum_{x \in P} r_{A-A}(x) \sim |A_s|q \).

Another combinatorial result is [43, Theorem 13].

**Theorem 18** Let \((G, +)\) be an abelian group. Also, let \( A \subseteq G \) be a set, \( K \geq 1 \) be a real number, and \( k \geq 2 \) be an integer. Suppose that \( E^+(A) \geq |A|^3/K \). Then there are sets \( A_s \subseteq A \), \( P \subseteq A - A \) such that \( |A_s| \geq |A|/(8kK) \), \( |P| \leq 8kK|A| \) and for any \( a_1, \ldots, a_k \in A_s \) one has
\[
|A \cap (P + a_1) \cap \cdots \cap (P + a_k)| \geq \frac{|A|}{4K}.
\] (44)
We need a result on the energy of a set which is obtained using the eigenvalues method, see [46], [47], [36], [37]. In this form an analogue of the result above was appeared first time in [36, Theorem 28]. One can decrease number of logarithms slightly but it is not our aim.

**Theorem 19** Let $A$ be a finite subset of an abelian group $(G, +)$. Suppose there are parameters $D_1$ and $D_2$ such that

$$E_3^+(A) \leq D_1 |A|^3$$

and for any finite set $B \subset G$

$$E^+(A, B) \leq D_2 |A||B|^{3/2}.$$  

Then

$$E^+(A) \ll D_1^{6/13}D_2^{2/13}|A|^{32/13} \log^{12/13} |A|.$$  

It is implicit in the proof of Theorem 19 that the bound for $E(A, B)$ only needs to hold for $|B| \leq 4|A|^4/E^+(A)$.

Theorem 19 implies the following bound for the multiplicative energy of a subset of $F_p$ with large additive energy.

**Corollary 20** Let $A \subseteq F_p$ and $E^+(A) \geq |A|^3/K$. Then there is $A_* \subseteq A$, $|A_*| \geq |A|/(16K)$ such that for any $B \subseteq F_p$ the following holds

$$E^x(A_*, B) \ll K^{4/3}|A|^2|B|^2 \frac{p}{K^{7/2}|B|^{3/2}|A|}, \quad (45)$$

and if $|A|K \leq \sqrt{p}$, then

$$E^x_3(A_*) \ll K^{23/4}|A|^3 \log^2 |A|. \quad (46)$$

In particular, if $|A|K \leq \sqrt{p}$, then

$$E^x(A_*) \ll K^{83/26}|A|^{32/13}. \quad (47)$$

**Proof.** Applying Theorem 18 with $k = 3$, we find two sets $A, P \subseteq A - A$, $|A_*| \geq |A|/(24K)$, $|P| \leq 24K|A|$ such that for any $a_1, a_2, a_3 \in A_*$ one has

$$|A \cap (P + a_1) \cap (P + a_2) \cap (P + a_3)| \geq \frac{|A|}{4K}. \quad (48)$$

Then

$$E^x(A_*, B) \leq (|A|/4K)^{-2} \left| \{(a - p)b = (a' - p')b' : a, a' \in A, b, b' \in B, p, p' \in P \} \right|. \quad (49)$$

Clearly, the number of the solutions to equation (49) can be interpreted as points/planes incidences. Hence applying Theorem 10 we obtain

$$E^x(A_*, B) \ll (|A|/K)^{-2} \left( \frac{|A|^2|B|^2|P|^2}{p} + (|A||B||P|)^{3/2} + |A||B||P| \max\{|A|, |B|, |P|\} \right). \quad (50)$$
In view of the desired bound (45) one can assume that $|B| \geq K^7$, $|A| \geq |B|^{1/2}K^{7/2}$ (otherwise trivial bounds (15), namely, $E^\delta(A_*,B) \leq \min\{|A||B|^2, |A|^2|B|\}$ work better). Also, (48) implies, trivially, $|P| \geq |A|/(4K)$ and we can assume that $|B| \leq 4|A|^4/E^+(A) \ll K|A|$. Thus it is easy to check that the third term in (50) is negligible and using $|P| \ll K|A|$, we obtain (45).

To prove (46) we notice that in view of (48) and (40) one has
\[
E^\delta_3(A_*) = |\{\alpha/\alpha' = \beta/\beta' = \gamma/\gamma' : \alpha, \alpha', \beta, \beta', \gamma, \gamma' \in A_*\}| \leq
\leq (|A|/4K)^{-2} \left| \left\{ \begin{array}{l} b-a \cr b'-a' \end{array} = \frac{c-a}{c'-a'} = \frac{d-a}{d'-a'} : a, a', b, b', c, c', d, d' \in P \right\} \right| \ll
\ll (|A|/K)^{-2}Q(A,P,P,P).
\]
Suppose that $|A| \leq |P| \leq \sqrt{p}$. One can assume that $K \leq |A|^{4/23}$ because otherwise there is nothing to prove. It remains to estimate $Q(A,P,P,P)$ and we have by Lemma 16 that
\[
Q(A,P,P,P) \ll |P|^{15/4}|A|^{5/4} \log^2 |A| + T(P).
\] (51)

Thus in view of $T(P) \ll |P|^{9/2}$, see [1, Proposition 2.5] the second term in (51) is negligible. Then applying Lemma 16 and the bound $|P| \leq 24K|A|$, we obtain (46). If $|A| > |P|$, then we get even better estimate for $E^\delta_3(A_*)$. Finally, using Theorem 19 we derive from (45), (46) the desired bound (47) (because $|B| \leq 4|A|^4/E^+(A)$ and $|A|K \leq \sqrt{p}$ we see that the second term in (45) dominates). This completes the proof. $\square$

In [37] some better bounds for the energy were obtained but they work in a situation which is opposite to Corollary 20, namely, when the product set (not the sumset) is small.

Now consider the group $\text{SL}_2(\mathbb{F}_p)$ of matrices
\[
g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{F}_p, \quad ad - bc = 1,
\]
which acts on $\mathbb{F}_p$ by
\[
gz := \frac{az + b}{cz + d}, \quad z \in \mathbb{F}_p.
\]
There are two important subgroups in $\text{SL}_2(\mathbb{F}_p)$. Let $B$ be the standard Borel subgroup of upper–triangular matrices, namely, elements of $B$ are
\[
b = b_{r,q} = \begin{pmatrix} r & q \\ 0 & r^{-1} \end{pmatrix}, \quad q \in \mathbb{F}_p, \quad r \in \mathbb{F}_p \setminus \{0\}.
\]
Also, let $U \subseteq B$ be the standard unipotent subgroup. In other words, elements of $U$ are
\[
u = u_q = \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix}, \quad q \in \mathbb{F}_p.
\]

Having a group which is acting of a set, one can define a convolution which is slightly generalizes the ordinary convolution.
Definition 21 Let $F : \text{SL}_2(\mathbb{F}_p) \to \mathbb{C}$ and $f : \mathbb{F}_p \to \mathbb{C}$ be two functions. Define the convolution of $F \ast f : \mathbb{F}_p \to \mathbb{C}$ as

$$(F \ast f)(x) := \sum_{g \in \text{SL}_2(\mathbb{F}_p)} F(g) f(g^{-1}x).$$

Let us mention a well–known lemma (see [9], [21], [44] and other papers) on convolutions in $\text{SL}_2(\mathbb{F}_p)$ which follows from the well–known Frobenius Theorem [20] on representations of $\text{SL}_2(\mathbb{F}_p)$. For the sake of completeness we add the proof of this lemma in the Appendix.

Lemma 22 Let $f : \mathbb{F}_p \to \mathbb{C}$ be a function such that $\sum_x f(x) = 0$. Then for any functions $F : \text{SL}_2(\mathbb{F}_p) \to \mathbb{C}$ and $\varphi : \mathbb{F}_p \to \mathbb{C}$ one has

$$\sum_{x \in \mathbb{F}_p} (F \ast f)(x) \varphi(x) \leq 2p\|F\|_2 \|\varphi\|_2 \|f\|_2.$$ (52)

Finally, we need the classification of subgroups of $\text{SL}_2(\mathbb{F}_p)$, see [54].

Theorem 23 Let $p$ be a prime and $p \geq 5$. Then any subgroup of $\text{SL}_2(\mathbb{F}_p)$ is isomorphic to one of the following subgroups:

1. Finite groups $A_4$, $S_4$, $A_5$.
2. The dihedral groups of order $4 \left( \frac{p+1}{2} \right)$ and their subgroups.
3. A Borel subgroup of order $p(p-1)$ and its subgroups.

We finish this section recalling the celebrated result of Helfgott [22] on the growth in $\text{SL}_2(\mathbb{F}_p)$.

Theorem 24 Let $A \subseteq \text{SL}_2(\mathbb{F}_p)$. Assume that $|A| < p^{3-\delta}$ for $\delta > 0$ and $A$ is not contained in any proper subgroup of $\text{SL}_2(\mathbb{F}_p)$. Then there is a positive function $\kappa(\delta) > 0$ such that

$$|AAA| \gg_{\delta} |A|^{1+\kappa(\delta)}.$$ 

4 First results

Throughout this section $\Gamma$ is a multiplicative subgroup of $\mathbb{F}_p^\times$. Such subgroups were studied by various authors and many deep results about subgroups were obtained, e.g., [5], [3], [12], [30], [50] and others. In this section we find upper bounds for $T_k^+(f)$, $E_k^+(f)$ and for the exponential sums over $f$, where $f$ is an arbitrary $\Gamma$–invariant function, that is, $f(x\gamma) = f(x)$ for all $\gamma \in \Gamma$. The main difference between our new theorems and results from [50] is, firstly, that we consider general functions $f$ and, secondly, the absence of any restrictions on size of support of $f$ (but not on size of $\Gamma$, of course) similar to $\mathbb{R}$ where we have no such restrictions, see our previous paper [50].

We begin with the quantity $T_k^+(f)$ and we use $T_k^+(f)$ in bounds below to make our results sharper. Of course, one can replace this quantity to $\|f\|_1^2 \|f\|_2^2$ (see formula (23)) of by something even smaller using Lemma [13]
Theorem 25 Let $f$ be a $\Gamma$-invariant complex function with $\sum_x f(x) = 0$. Then for any $k \geq 1$ one has
\[ T_{2s}^+(f) \leq 2^{2k^2} (C_* \log^4 p)^{k-1} \cdot \|f\|^{2k+1-4} \|\Gamma\|^{-\frac{(k-1)}{2}} T_2^+(f), \] (53)
where $C_*$ is the absolute constant from Lemma 13.

Proof. For $k = 1$ bound (53) is trivial, so below we will assume that $k \geq 2$. Fix any $s \geq 2$ and put $L = L_s := s \log p$. Our aim is to prove
\[ T_{2s}^+(f) \leq 128C_* L_s \|f\|^{2s} T_s^+(f) |\Gamma|^{-1/2}. \] (54)

After that we use induction and obtain
\[ T_{2s}^+(f) \leq (128C_*)^{k-1} \log^{4(k-1)} p \cdot 2^{4((k-1)+1)+\cdots+2} \|f\|^{2k+4} |\Gamma|^{-\frac{(k-1)}{2}} T_2^+(f) = \\
2^{2k^2+5k-11}(C_* \log^4 p)^{k-1} \cdot \|f\|^{2k+1-4} |\Gamma|^{-\frac{(k-1)}{2}} T_2^+(f) \leq 2^{2k^2}(C_* \log^4 p)^{k-1} \cdot \|f\|^{2k+1-4} |\Gamma|^{-\frac{(k-1)}{2}} T_2^+(f) \] and this coincides with (53).

To prove (54) we notice that by formula (21) one has
\[ T_{2s}^+(f) = \sum_{x,y,z} r_{sf}(x)r_{sf}(y)r_{sf}(x+z)r_{sf}(y+z). \]

Here as usual we have denoted by $r_{sf}(x)$ the function $r_{f,\ldots,f}(x)$, where the number of $f$'s in the sum is $s$. We give two upper bounds for $T_{2s}^+(f)$ and first of all, notice that from the last formula, it follows that $T_{2s}^+(f)$ equals
\[ \sigma := |\Gamma|^{-2} \sum_{\gamma_1,\gamma_2 \in \Gamma} \sum_{a,b,c,d} r_{sf}(a)r_{sf}(b)r_{sf}(c)r_{sf}(d) \cdot \delta\{a + \gamma_1 b = c + \gamma_2 d\} \]
plus the term $E$ which corresponds to $a, b, c, d$ equals zero (see below). Consider the set of points $P \subseteq \mathbb{P}_p$, each point $p$ indexed by $(\gamma_1, c, d)$ and the set of planes $\Pi \subseteq \mathbb{P}_p$ indexed by $(a, b, \gamma_2)$ and each $\pi = \pi_{a,b,\gamma_2} \in \Pi$ has the form $\pi : a + xb = y + xz$. Then in terms of formula (27) one has $\delta\{a + \gamma_1 b = c + \gamma_2 d\} = I(p, \pi)$ for $p = (\gamma_1, c, d)$ and $\pi = \pi_{a,b,\gamma_2}$. By the assumption $\sum_x f(x) = 0$. It follows that $\sum_x r_{sf}(x) = 0$ and hence
\[ \sigma = |\Gamma|^{-2} \sum_{\gamma_1,\gamma_2} f_\Gamma(\gamma_1)f_\Gamma(\gamma_2) \sum_{a,b,c,d} r_{sf}(a)r_{sf}(b)r_{sf}(c)r_{sf}(d) \cdot \delta\{a + \gamma_1 b = c + \gamma_2 d\}, \] (55)

where $f_\Gamma(x) = \Gamma(x) - |\Gamma|/p$ is the balanced function of $\Gamma$. In a similar way, considering for all nonzero $x$ the function $R(x) = |\Gamma|^{-1} \sum_y f_\Gamma(y)r_{sf}(xy^{-1})$, we obtain
\[ \sigma = |\Gamma|^{-4} \sum_{\gamma_1,\gamma_2,\gamma_3,\gamma_4} f_\Gamma(\gamma_1)f_\Gamma(\gamma_2)f_\Gamma(\gamma_3)f_\Gamma(\gamma_4) \sum_{a,b,c,d} r_{sf}(a)r_{sf}(b)r_{sf}(c)r_{sf}(d) \cdot \delta\{a + \gamma_1 b = \gamma_3 c + \gamma_4 d\} \]
\[ = \sum_{x,y,z} R(x)R(y)R(x+z)R(y+z). \] (56)
Clearly, $R(0) = 0$, further $R(x) = r_{sf}(x)$, $x \neq 0$ and $\|R\|_\infty = \|r_{sf}\|_\infty$ if one considers the function $r_{sf}$ as a function on $\mathbb{F}_p^*$ only. Also, notice that $\|f_t\|_1 < 2|\Gamma|$ and hence
\[
\|R\|_1 \leq |\Gamma|^{-1} \|r_{sf}\|_1 |f_t|_1 < 2 \|r_{sf}\|_1 \leq 2 \|f\|_1^4.
\]

Now put $\rho = T_{2s}(f)/(64 \|f\|_1^{3s})$. Since
\[
\sum_{x,y,z : |R(x)| \leq \rho} R(x)R(y)R(x+z)R(y+z) \leq 8\rho \|f\|_1^{3s} = T_{2s}(f)/8,
\]
it follows that
\[
T_{2s}(f) \leq \frac{3}{2} \sum_{x,y,z} R(x)R(y)R(x+z)R(y+z) + \frac{3}{2} \mathcal{E},
\]
where the sum $\sum_{x,y,z}$ above (we denote it as $T_{2s}'(f)$) is taken over nonzero variables $x, y, x+z, y+z$ with $|R(x)|, |R(y)|, |R(x+z)|, |R(y+z)| > \rho$ and by (52)
\[
\mathcal{E} \leq 4 \rho_{sf}(0) \sum_{y,z} r_{sf}(x)r_{sf}(y)r_{sf}(x+z)r_{sf}(y+z) \leq 4 \rho_{sf}(0) \|f\|_1^4 T_{2s}^+(f) \leq 4 \|f\|_2^2 \|f\|_1^{2s-2} T_{2s}^+(f).
\]

Let us compare the obtained estimate for $\mathcal{E}$ with the upper bound in (51). By the assumption $f$ is $\Gamma$–invariant function and hence $\|f\|_1 = |\Gamma| \sum_{\xi \in \mathbb{F}_p^*/\Gamma} |f(\xi)|$, as well as
\[
\|f\|_2^2 = |\Gamma| \sum_{\xi \in \mathbb{F}_p^*/\Gamma} |f(\xi)|^2 \leq |\Gamma|^{-1} \|f\|_1^2.
\]
In particular,
\[
\mathcal{E} \leq 4 \|f\|_1^2 T_{2s}^+(f)|\Gamma|^{-1} \leq 4 \|f\|_1^2 T_{2s}^+(f)|\Gamma|^{-1/2}.
\]
Thus the obtained estimate for $\mathcal{E}$ is much smaller than the upper bound for $T_{2s}^+(f)$ in (51). Hence if (51) holds, then there is nothing to prove and in the opposite case, we get
\[
\mathcal{E} \leq T_{2s}^+(f)/(32C_s L^4),
\]
so it is negligible. Also, we can assume that $T_{2s}'(f) > 0$ because otherwise there is nothing to prove.

Put $P_j = \{x \mid \rho 2^{j-1} < \rho_{sf}(x) \leq \rho 2^j\} \subseteq \mathbb{F}_p^*$, $j \in \mathbb{N}$. By the Hölder inequality or, alternatively, counting trivial solutions $a_j = a'_j$ to equation (20), we have $T_{2s}^+(f) \geq \|f\|_2^{2s}$. If (51) does not hold, then, in particular,
\[
T_{2s}^+(f) \geq 2^7 \|f\|_1^2 T_{2s}^+(f)|\Gamma|^{-1/2} \geq 2^7 \|f\|_1^2 \|f\|_1^{2s} \|f\|_2^{2s} |\Gamma|^{-1/2}
\]
and hence the possible number of the sets $P_j$ does not exceed $L$. Indeed, for any $x$ one has $|r_{sf}(x)| \leq \|f\|_1^{2s-2} \|f\|_2^2$ and hence $\rho 2^{j-1} = 2^{j-7} T_{2s}^+(f) \|f\|_1^{-3s}$ must be less than $\|f\|_1^{2s-2} \|f\|_2^2$. 

otherwise the correspondent set \( P_j \) is empty. In other words, using the H"{o}lder inequality one more time, as well as bound (60), we obtain

\[
2^{-7} \leq \|f\|_1\|f\|_2^2/T_2^+(f) \leq \|f\|_1\|f\|_2^2(2s-2)\|\Gamma\|_1\|f\|_2^2 \leq \rho_{s-1}\|\Gamma\|_1/2^7 \leq \rho_{s}2^{s-1}\|\Gamma\|_1/2^7 < \rho_{s}/2^7
\]
as required. By the Dirichlet principle there is \( \Delta = \rho 2^{j_0-1} \), and a set \( P = P_{j_0} \) such that

\[
T_2^+(f) \leq \frac{3}{2}L^4(2\Delta)^4\mathbb{E}^+(P) + \frac{3}{2}\mathcal{E} = T_2'(f) + \frac{3}{2}\mathcal{E}.
\]

(61)

Indeed, putting \( g_i(x) = P_i(x)r_{sf}(x) \), and using (59), we get

\[
\sum_{x,y,z} r_{sf}(x)r_{sf}(y)r_{sf}(x+z)r_{sf}(y+z) \leq \sum_{i,j,k,l=1}^L \sum_{x,y,z} g_i(x)g_j(y)g_k(x+z)g_l(y+z) \leq \sum_{i,j,k,l=1}^L (\mathbb{E}^+(g_i)\mathbb{E}^+(g_j)\mathbb{E}^+(g_k)\mathbb{E}^+(g_l))^{1/4} = \left(\sum_{i=1}^L (\mathbb{E}^+(g_i))^{1/4}\right)^4 \leq L^3\sum_{i=1}^L \mathbb{E}^+(g_i) \leq L^4 \max_i \mathbb{E}^+(g_i).
\]

Certainly, the sum \( \sum_{x,y,z} R(x)R(y)R(x+z)R(y+z) \) can be estimated in a similar way and one can check that all functions \( R_i(x) = |\Gamma|^{-1} \sum_y f_1(y)r_i(xy^{-1}) \) have zero mean and \( \|R_i\|_\infty \leq \|r_i\|_\infty \). Moreover we always have \( |P|\Delta^2 \leq T_2^+(f) \) and

\[
|P|\Delta \leq \sum_{x \in P} |r_{sf}(x)| \leq \sum_x |r_{sf}(x)| \leq \sum_x r_{sf}(x) = \|f\|_1^2.
\]

(62)

Using Lemma 13 we obtain

\[
\mathbb{E}^+(P) \leq C_\star \left(\frac{|P|^4}{p} + \frac{|P|^3}{|\Gamma|^{1/2}}\right).
\]

Hence

\[
T_2^+(f) \leq 3(16C_\star)L^4 \left(\frac{\Delta^4|P|^4}{p} + \frac{\Delta^4|P|^3}{|\Gamma|^{1/2}}\right).
\]

(63)

Suppose that the second term in (63) dominates. Then in view of \( |P|\Delta^2 \leq T_2^+(\Gamma) \) and \( |P|\Delta \leq \|f\|_1^2 \), we have

\[
|P|^3 \Delta^4 = (P\Delta)^2 P\Delta^2 \leq \|f\|^2_1T_2^+(\Gamma).
\]

In other words, the second term in (63) does not exceed

\[
3(16C_\star)L^4\|f\|_1^2T_2^+(\Gamma)|\Gamma|^{-1/2},
\]

(64)

and inequality (64) (also, recall bound (58)) is proved.
If the first term in (63) dominates, then we notice that $|P||\Gamma|^{1/2} \geq p$ and use another bound. By $\sum_x f(x) = 0$, formulae (53), (61) and (27), as well as the last estimate, we have

$$T_{2s}'(f) \leq 3 \cdot 8L^4 p(4|P|^2)^2|\Gamma|^{-1} = 3 \cdot 2^7|P|^3 |\Delta|^3|\Gamma|^{-1/2} \leq 3 \cdot 16|P|^3 |\Delta|^3|\Gamma|^{-1/2}$$

(65)

for $|\Gamma| \geq 2^6$. Of course quantity (65) is less than the second term in (63). If $|\Gamma| < 2^6$, then it is easy to check that (53) takes place. Combining the obtained bound (65) with (64), we see that in any case

$$T_{2s}'(f) \leq 6(16C_s)L^4\|f\|_1^2 T_s^+(f)|\Gamma|^{-1/2}.$$ 

Finally, using (57), we have

$$T_{2s}'(f) \leq 128C_s L^4\|f\|_1^2 T_s^+(f)|\Gamma|^{-1/2}.$$ 

This completes the proof.  

Now we are ready to obtain an upper bound for the exponential sums over any $\Gamma$–invariant function $f$.

**Corollary 26** Let $\Gamma \subseteq \mathbb{F}_p^*$ be a multiplicative subgroup, $|\Gamma| \geq p^\delta$, $\delta > 0$, and let $f$ be a $\Gamma$–invariant complex function with $\sum_x f(x) = 0$. Then for all sufficiently large $p$ one has

$$\max_{\xi} \hat{f}(\xi) \ll \|f\|_1 \cdot p^{-\frac{\delta}{2^7+2^6-1}}.$$ 

(66)

Further we have a nontrivial upper bound $o(\|f\|_1)$ for the maximum in (66) if

$$\log |\Gamma| \geq \frac{C\log p}{\log \log p},$$

(67)

where $C > 2$ is any constant.

**Proof.** By $p$ denote the maximum in (66). It is attained at some nonzero $\xi$ because $\sum_x f(x) = 0$. Then by Theorem 25, a trivial bound which follows from (23), namely, $T_2^+(f) \leq \|f\|_1^2 \|f\|_2^2$ and formula (20), we obtain

$$|\Gamma| p^{2k+1} \leq pT_{2k}^+(f) \leq p2^{3k^2} (C_s \log^4 p)^{k-1} \cdot \|f\|_1^{2k+1-2}\|f\|_2^2 |\Gamma|^{-\frac{1-k}{2}}.$$ 

(68)

Using formula (57), we get

$$|\Gamma| p^{2k+1} \leq p2^{3k^2} (C_s \log^4 p)^{k-1} \cdot \|f\|_1^{2k+1} |\Gamma|^{-\frac{(k+3)}{2}}.$$ 

(69)

Put $k = \lceil 2\log p/\log |\Gamma| + 4 \rceil \leq 2/\delta + 5$, say. Also, notice that

$$\frac{p\log^{4(k-1)} p}{|\Gamma|^{k/2}} \leq 1$$
because \( k \geq 2 \log p/ \log |\Gamma| + 4 \) and \( p \) is a sufficiently large number depending on \( \delta \) (the choice of \( k \) is slightly larger than \( 2 \log p/ \log |\Gamma| \) to "kill" \( p \) by division by \( |\Gamma|^{k/2} \), as well as logarithms \( \log^4(k-1) \)). Taking a power \( 1/2^{k+1} \) from both parts of (68), we see in view of (69) that
\[
\rho \ll \|f\|_1 \cdot |\Gamma|^{-\frac{7}{2^{k+2}}} \ll \|f\|_1 \cdot p^{-\frac{s}{27+25}}.
\]

To prove the second part of our corollary just notice that the same choice of \( k \) gives something nontrivial if \( 2^k \ll \varepsilon \log |\Gamma| \) for any \( \varepsilon > 0 \). In other words, it is enough to have
\[
k \leq \frac{2 \log p}{\log |\Gamma|} + 5 \leq \log \log |\Gamma| - \log(1/\varepsilon).
\]

It means that the inequality \( \log |\Gamma| \geq C \log p/(\log \log p) \) for any \( C > 2 \) is enough. This completes the proof. \( \square \)

Let us obtain a new general bound for \( E^+_k(f) \).

**Theorem 27** Let \( f \) be a \( \Gamma \)-invariant function real function with \( \sum_x f(x) = 0 \). Then for positive integer \( k \geq 2 \) either
\[
E^+_{2k+1}(f) \leq 32C^1/4(1 + \log(\|f\|_1\|f\|_2^{-1}))\|f\|_2^{2k+1}E^+_k(f)|\Gamma|^{-1/8}
\]

or
\[
E^+_{2k+1}(f) \leq 2\|f\|_2^{2k+2}.
\]

Here \( C^* \) is the absolute constant from Lemma 13. In particular, if \( k \) is chosen as
\[
|\Gamma|^{-\frac{1}{8}} \geq (32C^1/4(1 + \log(\|f\|_1\|f\|_2^{-1}))^{k-1}\|f\|_2^{-1}\|f\|_2^{-2}, \tag{70}
\]

then \( E^+_{2k+1}(f) \leq 2\|f\|_2^{2k+2} \).

**Proof.** Fix an even integer \( l \geq 1 \) and prove that either
\[
E^+_4(f) \leq 32C^1/4(1 + \log(\|f\|_1\|f\|_2^{-1}))\|f\|_2^4E^+_4(f)|\Gamma|^{-1/8} \tag{71}
\]

or
\[
E^+_4(f) \leq 2\|f\|_2^{8l} \tag{72}
\]

After that it requires just to use induction to see
\[
E^+_{2k+1}(f) \leq (32C^1/4(1 + \log(\|f\|_1\|f\|_2^{-1}))^{k-1}\|f\|_2^{2k+1}\cdots E^+_4(f)|\Gamma|^{-(k-1)/8} \leq (32C^1/4(1 + \log(\|f\|_1\|f\|_2^{-1}))^{k-1}\|f\|_2^{2k+2-8}E^+_4(f)|\Gamma|^{-(k-1)/8}.
\]

Trivially, \( E^+_4(f) \leq \|f\|_2^4\|f\|_1^4 \) and hence
\[
\|f\|_2^{2k+2} \leq E^+_{2k+1}(f) \leq (32C^1/4(1 + \log(\|f\|_1\|f\|_2^{-1}))^{k-1}\|f\|_2^{2k+2-2}\|f\|_2^4|\Gamma|^{-(k-1)/8}.
\]
Thus if
\[ |\Gamma|^{k+1/8} \geq (32C_8^{1/4}(1 + \log(\|f\|_1\|f\|_2^{-1}))^{k-1}\|f\|_1^2\|f\|_2^{-2}, \]
then, clearly, \( E_{2k+1}^+(f) \leq 2\|f\|_2^{k+1}. \)

We give two upper bounds for \( E_{4l}^+(f) \). Firstly, let us remark that for any positive integer \( n \) there exists a function \( F \) such that
\[ r_{f-F}^n(x) = r_{F-F}(x). \] (73)

Indeed, from the definition of the required function \( F \) and formula (26) one has
\[ |\hat{F}(x)|^2 = p^{1-n}\sum_{y_1 + \cdots + y_n = x} |\hat{f}(y_1)|^2 \cdots |\hat{f}(y_n)|^2 \geq 0 \]
and we can choose the Fourier transform of \( F \) taking, say, a positive square root of the left–hand side of the previous formula. It defines our function \( F \) (but not uniquely, even in the case \( n = 1 \) one can take \( F(x) = f(x) \) or \( F(x) = f(-x) \), say). In particular, by the Parseval identity, we get
\[ \|F\|_2^2 = p^{-1}\|\hat{F}\|_2^2 = p^{-n}\sum_x \sum_{y_1 + \cdots + y_n = x} |\hat{f}(y_1)|^2 \cdots |\hat{f}(y_n)|^2 = \|f\|_2^{2n}. \] (74)

To obtain another proof of the last equality just substitute \( x = 0 \) into (73) and notice that \( (F \circ F)(0) = \|F\|_2^2 = r_{f-F}^n(0) = (F \circ f)^n(0) = \|f\|_2^{2n} \). Applying these arguments for \( n = 4l - 1 \), we obtain
\[ E_{4l}^+(f) = \sum_x r_{f-f}(x) r_{f-f}^{4l-1}(x) = \sum_x r_{f-f}(x) r_{F-F}(x). \]

By the assumption the function \( f \) is \( \Gamma \)-invariant. Thus
\[ E_{4l}^+(f) = |\Gamma|^{-2} \sum_{\gamma_1, \gamma_2 \in \Gamma} \sum_{a,b,c,d} F(a)f(b)F(c)f(d) \cdot \delta\{a + \gamma_1b = c + \gamma_2d\}. \]

Consider the set of points \( P \subseteq \mathbb{R}_p^3 \), each point \( p \) indexed by \( (\gamma_1, c, d) \) and the set of planes \( \Pi \subseteq \mathbb{R}_p^3 \) indexed by \( (a, b, \gamma_2) \) and each \( \pi \in \Pi \) has the form \( \pi : a + xb = y + \gamma_2z \). Then we have as in Theorem [25] that \( \delta\{a + \gamma_1b = c + \gamma_2d\} = \mathcal{I}(p, \pi) \). By the assumption \( \sum_x f(x) = 0 \). Besides \( \|F\|_2 = \|f\|_2^{8l-2} \). Hence by (27), we have
\[ E_{4l}^+(f) \leq p\|f\|_2^{8l}|\Gamma|^{-1}. \] (75)

Now let us give another bound for \( E_{4l}^+(f) \). Put \( g(x) = r_{f-f}^l(x) \), \( L = 2 + 2\log(\|f\|_1\|f\|_2^{-1}) \) and \( E_{4l}^+(f) = E_{4l}^+(f) - \|f\|_2^{8l} \). We will assume below that \( E_{4l}^+(f) \geq 2^{-1}E_{4l}^+(f) \) because otherwise the required inequality (72) follows immediately. Similarly, we can assume that \( E_{4l}^+(f) \geq \|f\|_2^{8l} \) because otherwise \( E_{4l}^+(f) \leq 2\|f\|_2^{8l} \) and we are done. Further put \( \rho^{4-1/l} = 2^{-1}\|f\|_2^{8l}\|f\|_1^{-2} \) and \( P_j = \{ x : \rho^{2^{-1}} < g(x) \leq \rho^{2^j} \} \). Clearly,
\[ \sum_{x : g(x) < \rho} g^4(x) < \rho^{4-1/l}\|f\|_2^{8l} = 2^{-1}\|f\|_2^{8l}. \]
Thus the number of the sets $P_j$ does not exceed $L$. Indeed, for any $x$ one has $g(x) \leq \| f \|_2^2$ and hence $2^{j-1}p$ must be less than $\| f \|_2^2$ because otherwise $P_j$ is empty. Whence for $j \geq 3$
\[ 2^{j-2}\| f \|_2^2 \| f \|_2^{-2} \leq 2^{(j-1)(4-1)/l} \| f \|_2^2 \| f \|_2^{-2} = 2^{(j-1)(4-1)/l} \rho^{4-1/l} \leq \| f \|_2^{8l-2} \]
and $j \leq 2 + 2 \log(\| f \|_1/\| f \|_2^{-1}) := L$ (if $j < 3$, then the last bound holds trivially). Notice that
\[ \log(\| f \|_1/\| f \|_2^{-1}) \geq 0. \]
Using the Dirichlet principle, we find a set $P = P_{\gamma_0}$ and a positive number $\Delta = \rho^{2^{j-1}}$ such that $P = \{ x : \Delta < g(x) \leq 2\Delta \} \subseteq \mathbb{F}_p$ and

\[ E'_d(f) \leq 2L \sum_{x \in P} r^d_{f-f}(x) \leq 2L\| f \|_2^{3l} \sum_{x \in P} r^{d/2}_{f-f} (x) := \| f \|_2^{3l} \sigma. \quad (76) \]

Applying Lemma 15 combining with Lemma 13 we obtain

\[ \sigma \leq 2L(2\Delta)^{3/2} \sum_{x \in P} r^d_{f-f}(x) \leq 2^{5/2}C_4^l \Delta^{3/2}\| f \|_2 (E_{2l}^+(f))^{1/4} \left( \frac{|P|^4}{p} + \frac{|P|^3}{|\Gamma|} \right)^{1/4} \leq \]

\[ \leq 2^{5/2}C_4^l \Delta^{3/2}\| f \|_2 \frac{\Delta^6}{p} (\sum_{x \in P} r^{d/2}_{f-f} (x) )^{1/4} \left( \frac{\Delta^6}{|\Gamma|} \right)^{1/2} \quad (77) \]

Suppose that the second term in (77) dominates. Since $l$ is an even number, we have $\Delta |P| \leq E^+_d(f)$, $\Delta^2 |P| \leq E^+_d(f)$ and hence $\Delta^3 |P| \leq (E^+_d(f))^3$. It follows that

\[ 2^{-1}E^+_d(f) \leq E'_d(f) \leq 8C_4^l \| f \|_2^4 E^+_d(f)|\Gamma|^{-1/8} \]
and we obtain (77). Now if the first term in (77) dominates, then $|P| |\Gamma|^{1/2} \geq p$ and returning to (76), we have

\[ E'_d(f) \leq 32L \Delta^4 |P|. \]

Multiplying this inequality by $|P|$ and using $\Delta^2 |P| \leq E^+_d(f)$, we get

\[ |P| E'_d(f) \leq 32L (E^+_d(f))^2. \]

Recalling (75), applying the inequality $|P| |\Gamma|^{1/2} \geq p$ and the last bound, we obtain

\[ E^+_d(f) \leq p \| f \|_2^4 |\Gamma|^{-1} \leq |P| \| f \|_2^4 |\Gamma|^{-1/2} \leq 32L \| f \|_2^4 (E^+_d(f))^2 |\Gamma|^{-1/2} (E'_d(f))^{-1}. \]

Whence in view of the inequality $2^{-1}E^+_d(f) \leq E'_d(f)$, we have

\[ E^+_d(f) \leq 8L^{1/2} \| f \|_2^4 E^+_d(f)|\Gamma|^{-1/4}. \]

Thus we see that the required inequality (71) takes place in any case. This completes the proof.

\[ \square \]

**Remark 28** The upper bound in Theorem 27 is optimal. Indeed, let $\chi(x)$ be the Legendre symbol. In other words, if $R$ is the set of quadratic residues and $\chi_0(x)$ is the trivial character, then $\chi(x) = 2R(x) - \chi_0(x)$. Let $\Gamma \subseteq R$ be a multiplicative subgroup. Then $\chi(x)$ is a real $\Gamma$-invariant function and $\sum_x \chi(x) = 0$. By standard formulas for characters see, e.g., [3] one has for any $k \geq 2$ that $E^+_k(\chi) = \sum_x (\chi \circ \chi)^k(x) = (p-1)^k + (p-1)(-1)^k \sim p^k \sim \| \chi \|_2^{2k}$. 

Remark 29 Let $f$ be a real $\Gamma$–invariant function with zero mean and let $T = \|f\|_1/\|f\|_2$. By (57) we have $T \geq |\Gamma|^{1/2}$. Choosing an integer $k = C \log T/\log |\Gamma|$ with sufficiently large constant $C > 0$, we satisfy condition (70). Under this condition, we get

$$|\Gamma| \left( \max_{x \neq 0} |(f \circ f)(x)| \right)^{2k+1} \leq 2\|f\|^{2k+2}.$$ 

It follows that

$$\max_{x \neq 0} |(f \circ f)(x)| \leq \|f\|^2 (2|\Gamma|^{-1})^{1/2k+1}. \quad (78)$$

Thus we have obtained a non–trivial upper bound for the quantity $\max_{x \neq 0} |(f \circ f)(x)|$ if the condition

$$\log |\Gamma| \gg \frac{\log T}{\log \log T}$$

holds. Of course, we last bound implies $\log T \ll |\Gamma|$. Some applications of such sort of bounds can be found in papers [3], [50].

Corollary 30 Let $\Gamma \subseteq \mathbb{F}_p^*$ be a multiplicative subgroup, $|\Gamma| \geq p^\delta$ and $f$ be a $\Gamma$–invariant real function with $\sum_x f(x) = 0$. Also, let $k$ is chosen as

$$|\Gamma|^{\frac{k+1}{2}} \geq (32C^1_\gamma (1 + \log(\|f\|_1 \|f\|_2^{-1})))^{k-1} \|f\|^2 \|f\|_2^{-2}, \quad (79)$$

and $s = \lceil 2\log \|f\|_1/\log(|\Gamma|/2) \rceil$. Then

$$E_{2k+1}^\alpha(f + 1) \leq 3\|f\|^{2k+2}_{2k+1+1}.$$ 

Proof. For any integer $l$, we have

$$E_l^\alpha(f + 1) = \|f\|_2^2 + |f^{2l}(-1)| + \sum_{x \neq 0, 1} r_{(f+1)/(f+1)}(x) \leq \|f\|_2^2 + \sum_{x \neq 0, 1} r_{(f+1)/(f+1)}(x).$$

Here we have used that $\sum_x f(x) = 0$. Further for any $\alpha \neq 0, 1$ put $f_\alpha(x) = f(\alpha^{-1}x)$. Then

$$r_{(f+1)/(f+1)}(x) = r_{f^-f}(x - 1).$$

Take $k$ such that

$$|\Gamma|^{\frac{k-1}{2}} \geq (32C^1_\gamma (1 + \log(\|f\|_1 \|f\|_2^{-1})))^{k-1} \|f\|^2 \|f\|_2^{-2}.$$ 

As in Corollary 29, we have for any $x \neq 0, 1$

$$|\Gamma| \left( \max_{y \neq 0} |(f^x \circ f)(y)| \right)^{2k+1} \leq \sum_y r_{\gamma \cdot f^x}^{2k+1}(y) \leq (E_{2k+1}^\alpha (f^x) E_{2k+1}^\alpha (f))^{1/2} = E_{2k+1}^\alpha(f).$$
Here we have used the Cauchy–Schwarz inequality. Applying Theorem \(27\) we obtain
\[
\max_{y \neq 0} |(f^x \circ f)(y)| \leq \|f\|_2^2 \cdot (2|\Gamma|^{-1})^{1/2k+1}.
\]
Thus for any \(s\), we have
\[
E_{2k+1+s+1}^+ (f + 1) \leq 2\|f\|_2^{2k+2+s+2} + \|f\|_2^{2k+2+s} \|f\|_1^2 (2/|\Gamma|)^s = \|f\|_2^{2k+2+s+2} (2 + \|f\|_1^2 (2/|\Gamma|)^s).
\]
Taking \(s\) such that
\[
(|\Gamma|/2)^s \geq \|f\|_2^2,
\]
or, in other words, \(s \geq 2 \log \|f\|_1 / \log(|\Gamma|/2)\), we obtain the required result. This completes the proof.

For example, let \(f(x) = Q(x) - |Q|/p\), where \(Q\) is any \(\Gamma\)-invariant set, \(\log |Q| \ll |\Gamma|\). Then \(k \sim \log |Q| / \log |\Gamma|\) and \(s \sim \log |Q| / \log |\Gamma|\), so we have the same bound as in Theorem \(27\) for more or less the same order \(l \sim k, s\) of the energy \(E_l(f)\).

The next result shows that smallness of the energy \(E_k^+\) allows to obtain upper bounds for sums of types (\(81\)) and (\(82\)). The arguments of the proof are rather general. Estimate (\(82\)) allows to give an alternative proof of formula (\(75\)). Also, putting \(s = 2^k\) and \(B = P\) in formula (\(82\)) one can derive Lemma \(15\).

**Corollary 31** Let \(\Gamma \subseteq \mathbb{F}_p^*\) be a multiplicative subgroup and \(f\) be a \(\Gamma\)-invariant function real function with \(\sum_x f(x) = 0\). If \(k\) is chosen as
\[
|\Gamma|^{-s} \geq (32C^1 \cdot (1 + \log(|f|_1 \|f\|_2^{-1})))^{k-1} \|f\|_1^2 \|f\|_2^{-2}, \tag{80}
\]
then for any set \(B \subseteq \mathbb{F}_p\) one has
\[
p^{-1} \sum_x (\widehat{f}(x))^2 r_{B-B}(x) \leq \|f\|_2^2 |B| (\frac{2E^+(B)}{p|B|^2})^{1/2k+1} \tag{81}
\]
and for any function \(g : \mathbb{F}_p \rightarrow \mathbb{C}\) and a positive integer \(s \leq 2^k\) the following holds
\[
\left| \sum_{x \in B} (g \circ f)^s (x) \right| \leq |B| \|g\|_2^2 \|f\|_2^2 \left( \frac{2E^+(B)}{|B|^4} \right)^{s/2^{k+2}}. \tag{82}
\]

**Proof.** Denote by \(\sigma\) the sum from (\(81\)) and put \(\mu(x) = p^{-1} |\widehat{f}(x)|^2\). Clearly,
\[
\sigma = \sum_{x \in B} (\mu * B)(x).
\]
Then using the Hölder inequality, we obtain
\[
\sigma^{2k} \leq |B|^{2k-1} \sum_{x \in B} (\mu * 2^k \mu * B)(x) = |B|^{2k-1} \sum_{x} (\mu * 2^k \mu)(x)(B \circ B)(x).
\]
Applying the Hölder inequality one more time, as well as formula (26), we get
\[
\sigma^{2k+1} \leq \|B\|^{2k+1-2}E^+(B)T_{2k}^+(\mu) = p^{-1}\|B\|^{2k+1-2}E^+(B)E_{2k+1}^+(f).
\]
By our choice of the parameter \(k\) and Theorem [27] we have \(E_{2k+1}^+(f) \leq 2\|f\|_2^{2k+2}\) and hence
\[
\sigma \leq \|B\|\|f\|_2^2 \left(\frac{2E^+(B)}{p\|B\|^2}\right)^{1/2k+1}.
\]

It remains to prove (82). Using the Hölder inequality twice, one has
\[
\left(\sum_{x \in B} (g \circ f)^s(x)^{2k+2/s}\right)^{2k+2/s-4} \left(\sum_{x \in B} (g \circ f)^2(x)\right)^4 =
\]
\[
= |B|^{2k+2/s-4} \left(\sum_{y_1, \ldots, y_{2k}} g(y_1) \cdots g(y_{2k}) \sum_{x \in B} f(y_1 + x) \cdots f(y_{2k} + x)\right)^4 \leq
\]
\[
\leq |B|^{2k+2/s-4}\|g\|_2^{2k+2} \left(\sum_{x} (f \circ f)^{2k}(B \circ B)(x)\right)^2 \leq |B|^{2k+2/s-4}\|g\|_2^{2k+2} E_{2k+1}^+(f)E^+(B).
\]
By our choice of the parameter \(k\) and Theorem [27] we have \(E_{2k+1}^+(f) \leq 2\|f\|_2^{2k+2}\). Hence
\[
\left|\sum_{x \in B} (g \circ f)^s(x)\right| \leq \|g\|_2^s\|f\|_2^s|B| \left(\frac{2E^+(B)}{|B|^4}\right)^{s/2k+2}.
\]
as required. \(\square\)

Estimate (82) shows that the smallness of \(E_k(A)\) energy implies that the sums from this inequality are small. It is easy to see that the reverse direction takes place as well. Indeed, suppose in contrary that \(E_n(A) \geq M|A|^n\) for a parameter \(M \geq 1\) and for all positive integers \(n\). Also, let \(l\) be a positive integer and let \(P = P_l\) be a set as in the proof of Theorem [27] such that \(E_{l+1}(A) \sim |P|\Delta^{l+1}\) and \(\Delta < r_{A-A}(x) \leq 2\Delta\) on \(P\). Then \(|P| \gtrsim M\), and using our assumption (let \(s = 1\) for simplicity)
\[
E_{l+1}(A) \lesssim \Delta^l \sigma_P(A) \ll \Delta^l |A||P||P|^{-\varepsilon} \lesssim |A|E_l(A)M^{-\varepsilon},
\]
where \(\varepsilon > 0\) is a constant from our assumption. So, after \(t\) applications of this argument, we get
\[
M|A|^{l+t} \leq E_{l+t}(A) \lesssim |A|^tE_l(A)M^{-\varepsilon t} \leq |A|^{t+l+1}M^{-\varepsilon t}
\]
and hence after \(t\) steps such that \(M^{\varepsilon t+1} \gtrsim |A|\) we obtain a contradiction and it means that, in particular, \(E_{t+1}(A) \leq M|A|^t\). For example, if \(M = |A|^\delta\), then \(E_{t+1}(A) \leq |A|^{t+\delta}\) for \(t \gg \frac{1}{\varepsilon^t}\), say.
5 On some sum–product quantities with six and eight variables

For any set $A \subseteq \mathbb{F}_p$ let
\[
D^x(A) = D_2^x(A) := |\{(a_1 - a_2)(a_3 - a_4) = (a_1' - a_2')(a_3' - a_4') : a_i, a_i' \in A\}|, \tag{83}
\]
and more generally for $k \geq 1$
\[
D_k^x(A) := |\{(a_1 - a_2) \ldots (a_{2k-1} - a_{2k}) = (a_1' - a_2') \ldots (a_{2k-1}' - a_{2k}') : a_i, a_i' \in A\}|. \tag{84}
\]

Clearly, $D_1^x(A) = E^+(A)$. Sometimes, we need $D_k^x(A, B)$ for two sets $A, B$ and even more generally $D_k^x(\alpha, \beta)$ for two functions $\alpha, \beta$.

Our task is to estimate the quantities $D^x(A)$, $D_k^x(A)$. The quantity $D^x(A)$ (and similar $D_k^x(A)$) can be interpreted as the number of incidences between points and planes (see details in [1])
\[
(a_1 - a_2)\lambda = (a_1' - a_2')\mu, \tag{85}
\]

counting with the weights $|\{a_3 - a_4 = \lambda : a_3, a_4 \in A\}|$ and $|\{a_3' - a_4' = \mu : a_3', a_4' \in A\}|$.

**Theorem 32** Let $A \subseteq \mathbb{F}_p$ be a set. Then
\[
D^x(A) - \frac{|A|^8}{p} \ll (\log |A|)^2|A|^5(E^+(A))^{1/2}. \tag{86}
\]

Moreover, for all $k \geq 2$ one has
\[
D_k^x(A) - \frac{|A|^{4k}}{p} \ll (\log |A|)^4|A|^{4k - 2 - 2^{-k + 2}}E^+(A)^{1/2^{k-1}}. \tag{87}
\]

Generally, for any non–negative function $\alpha$ and $\beta(x) = A(x)$, the following holds
\[
D_k^x(\alpha, \beta) - \frac{\|\alpha\|^2_k\|\beta\|^2_k}{p} \ll L^x(\|\alpha\|_1\|\beta\|_1)^{2k-2}(\|\alpha\|_2\|\beta\|_2)^{2-2^{-k+2}}E^+(\alpha, \beta)^{1/2^{k-1}},
\]
where $L := \log(\|\alpha\|_1\|\beta\|_1|A|/(\|\alpha\|_2\|\beta\|_2))$.

**Proof.** We have
\[
D^x(A) = \sum_{\lambda, \mu} r_{A-A}(\lambda)r_{A-A}(\mu)n(\lambda, \mu),
\]
where $n_{A,A}(\lambda, \mu) = \sum_x r_{A-A}(\lambda)(x)r_{A-A}(\mu)(x)$. Consider the balanced function $f(x) = f_A(x) = A(x) - |A|/p$. Then we have
\[
D^x(A) = \frac{|A|^8}{p} + \sum_{\lambda, \mu} r_{A-A}(\lambda)r_{A-A}(\mu)n_{f,f}(\lambda, \mu) = \frac{|A|^8}{p} + \sigma.
\]

Our task is to estimate the error term $\sigma$. Put $L = \log |A|$. Splitting the sum, we get
\[
\sigma \ll \sum_{i,j=1}^L \sum_{\lambda, \mu} n_{f,f}(\lambda, \mu)r_{A-A}^{(i)}(\lambda)r_{A-A}^{(j)}(\mu),
\]
where by \( r_{A-A}^{(j)}(\mu) \) we have denoted the restriction of the function \( r_{A-A} \) on some set \( P_j \) with \( \Delta_j < r_{A-A}^{(j)}(\mu) \leq 2\Delta_j, \mu \in P_j \) and \( \Delta_j > 0 \) is some number. Clearly, the operator \( n_{f,f}(\lambda, \mu) \) is non-negatively defined and hence

\[
\sigma \ll L \sum_{j=1}^{L} \sum_{\lambda, \mu} n_{f,f}(\lambda, \mu) r_{A-A}^{(j)}(\lambda) r_{A-A}^{(j)}(\mu).
\]

By the pigeonhole principle there is some \( \Delta = \Delta_j \) and \( P = P_j \) such that

\[
\sigma \ll L^2 \Delta^2 \sum_{\lambda, \mu} n_{f,f}(\lambda, \mu) P(\lambda) P(\mu). \tag{88}
\]

Since \( \sum_x f(x) = 0 \), we derive from (84) and (27) that

\[
\sigma \ll L^2 |A|^2 |P| \Delta^2. \tag{89}
\]

Now we obtain another bound for \( \sigma \). Using Theorem 10, we get

\[
\sigma \ll L^2 |A|^3 (\Delta |P|) \cdot (\Delta^2 |P|)^{1/2}. \tag{90}
\]

Since \( P \subseteq A-A \) it is easy to see that the term \( |A|^2 |P| \max\{|A|, |P|\} \) is negligible comparable to \( |A|^3 |P|^{3/2} \). Suppose that the second term in the last formula dominates. Then

\[
\sigma \ll L^2 |A|^3 (\Delta |P|). \tag{91}
\]

Clearly,

\[
\Delta |P| \leq \sum_{x} r_{A-A}^{(j)}(x) \leq \sum_{x} r_{A-A}(x) \leq |A|^2,
\]

and

\[
\Delta^2 |P| \leq \sum_{x} (r_{A-A}^{(j)}(x))^2 \leq E^+(A).
\]

Hence

\[
\sigma \ll L^2 |A|^5 (E^+(A))^{1/2}
\]

and we are done. If the first term in formula (90) is the largest one, then \( p \leq |A| |P|^{1/2} \) and inequality (89) gives us

\[
\sigma \leq L^2 p |A|^2 \Delta^2 |P| \ll L^2 |A|^3 \Delta^2 |P|^{3/2}.
\]

We see that it is smaller than the second term in (90) and hence we have proved (85). Another way to bound (88) is just use estimate (29) of Theorem 10.

To obtain (86) we firstly, notice that

\[
D^x(A) = \frac{|A|^8}{p} + \sum_{\lambda, \mu} r_{A-A}(\lambda)r_{A-A}(\mu)n_{f,f}(\lambda, \mu) = \frac{|A|^8}{p} + \sum_{\lambda, \mu} r_{f-f}(\lambda)r_{f-f}(\mu)n_{f,f}(\lambda, \mu) = \frac{|A|^8}{p} + \sigma
\]
and using the Dirichlet principle as above, we can find a set $P$ and a number $\Delta$ such that $\Delta < \|r_{f-f}(\mu)\| \leq 2\Delta$ on $P$ and

$$\sigma \ll L^2\Delta^2 \sum_{\lambda,\mu} n_{f,f}(\lambda,\mu)P(\lambda)P(\mu)$$

(from the Fourier transform, say, it is easy to see that $n_{f,f}(\lambda,\mu) \geq 0$ for any function $f$ but actually one can avoid this step of the proof). Secondly, we have

$$\Delta|P| \leq \sum_{x \in P} |r_{f-f}(x)| \leq \sum_{x} (r_{A-A}(x) + |A|^2/p) \leq 2|A|^2,$$

and

$$\Delta^2|P| \leq \sum_{x \in P} (r_{f-f}(x))^2 \leq E^+(f).$$

Thus one can refine the upper bound for $\sigma$, namely,

$$\sigma = D^x(f) \ll L^2|A|^5(E^+(f))^{1/2}.$$  

Similarly as above, we get for any $k \geq 2$

$$D^x_k(f) \ll L^2|A|^{2k+1}(D^x_{k-1}(f))^{1/2}.$$  

Hence by induction

$$D^x_k(f) \ll L^4|A|^{4k-2-2^{-k+2}}E^+(f)^{1/2k-1} \leq L^4|A|^{4k-2-2^{-k+2}}E^+(A)^{1/2k-1}.$$  

(92)

If the third term in (91) dominates (we are considering the quantity $D^x_k(f)$ now), then the term $|A|^{4k-2}$ appears but it is easy to check that it does not exceed the last estimate in (92) because $E^+(A) \geq |A|^2$. Finally, to obtain (87) use Corollary 11 to estimate the required number of incidences with weights $\alpha, \beta$. Using

$$E^+(\alpha, \beta) \leq \min\{||\alpha||_2||\beta||_2||\alpha||_1||\beta||_1, ||\alpha||_2^2||\beta||_2^2, ||\alpha||^2_1||\beta||^2_2\}$$  

(93)

as well as $E^+(\alpha, \beta) \geq ||\alpha||^2_2||\beta||^2_2$ (at other steps of our iterative procedure similar bounds work) and $||\alpha||_1 \geq ||\alpha||_2$ for $\alpha(x) \geq 0$, one can check that all conditions (30) are satisfied. This completes the proof. □

**Remark 33** Similarly, one can obtain an upper bound for the quantity

$$D'(A) = \{|(a_1 a_2 + a_3 a_4) = a'_1 a'_2 + a'_3 a'_4 : a_i, a'_i \in A\|,$$

as well as for higher energies $D'_k(A) = T^*_k(r_{A^k}), D'_k(A, B)$ and even $D'_k(\alpha, \beta)$ for an arbitrary non–negative function $\alpha$ and $\beta = A(x)$. In this case $E^+$ in (85)–(87) should be changed to $E^x$. Notice that our bound for $D'_k(A)$ is better than the correspondent bound in [6, Theorem 2]. Besides some additional conditions on $A$ and $k$ are required in [6].

Is easy to see that our error term for $D'_k(A)$ cannot be significantly improved for large $k$. Indeed, considering $A$ to be a small interval $[n]$, we get $|AA| \geq |A|^{2-\varepsilon}$ and $|kAA| \ll_k |A|^2$. Hence one cannot obtain something better than a quadratic saving for the error term.
Remark 34 The same method works for convex sets \cite{25} in the real setting where one obtains

$$T_k^+(A) \ll (\log |A|)^4 |A|^{2k-2-2^{-k}+3} \log^2 |A|^{2k-2+2^{-k}+1}$$

for any convex set \(A\), and we have used that \(E^+(A) \ll |A|^{5/2}\). This coincides with the main result of \cite{23} up to logarithms. Applying the best known bound for the additive energy of a convex set (see \cite{47}), namely, \(E^+(A) \lesssim |A|^{32/13}\), we obtain an improvement.

Theorem above immediately implies a consequence on the growth of the products of the differences (here we use a trivial upper bound for the energy \(E^+(A) \leq |A|^3\)).

Corollary 35 Let \(A \subseteq \mathbb{F}_p\) be a set. Then for any \(\varepsilon > 0\) and an arbitrary integer \(k \geq 1\) one has

$$|(A - A)^k| \gg \min\{p, |A|^{2-2^{-k}-\varepsilon}\}.$$ 

Another quick consequence of Theorem 32 is (also, see Section 4 from \cite{36})

Corollary 36 Let \(A \subseteq \mathbb{F}_p\) be a set, \(|A| \leq p^{9/16}\). Then

$$|(A - A)(A - A)| \gg \min\{p, |A|^{3/2+c}\},$$

where \(c > 0\) is an absolute constant.

Proof. By Theorem 32 and the Cauchy–Schwarz inequality, we obtain

$$|A|^8 \ll Q \left( \frac{|A|^8}{p} + |A|^5 (E^+(A))^{1/2} \log^2 |A| \right), \quad (94)$$

where \(Q = |(A - A)(A - A)|\). Thus if the first term in (94) dominates, then we are done. Further if \(E^+(A) \leq |A|^{3-\varepsilon}\), where \(\varepsilon > 0\) is some small constant, then we are done. If not, then put \(M = |A|^\varepsilon\) and apply the Balog–Szemerédi–Gowers Theorem \cite{56}, finding \(A' \subseteq A, |A'| \gg_M |A|\) and \(|A' + A'| \ll_M |A'|\). Using Theorem 4 from \cite{36}, we have for any \(a \in A\) that

$$|(A - A)(A - A)| \geq |(A - a)(A - a)| \gg_M |A|^{14/9},$$

provided \(|A| \leq p^{9/16}\). This completes the proof. \(\square\)

The same argument works for the set \(\frac{A - A}{4}\) but in this situation much better bounds are known, see \cite{40}, \cite{55}. Lower bound for the sets of the form \((A - A)(A - A), (A - A)/(A - A)\) in general fields \(\mathbb{F}_q\) can be found in paper \cite{34}.

Similar arguments allow us to formulate the second part of Theorem 2 as

Theorem 37 Let \(A \subseteq \mathbb{F}_p\) be a set and \(T(A)\) of collinear triples in \(A \times A\). Then

$$0 \leq T(A) - \frac{|A|^6}{p} \ll \min\left\{ p^{1/2} |A|^{7/2}, |A|^{9/2} \right\}. $$
Proof. We use the arguments from [53]. Put \( f(x) = A(x) - |A|/p \). It is easy to see that the quantity \( T(A) \) equals the number of incidences between the planes

\[
\frac{x}{a'' - a'} - \frac{a'}{a'' - a'} - \alpha y + z = 0
\]

and the points \((a, \frac{1}{a'' - a'}, \frac{\alpha'}{a'' - a'})\). Hence as in the proof of Theorem 32 we have

\[
T(A) = \frac{|A|^6}{p} + \sigma,
\]

where the sum \( \sigma \) counts the number of incidences (95) with the weight \( f(a)f(\alpha) \). Hence by (27), we get \( \sigma \leq |A|^3 \leq |A|^{9/2} \). If \(|A|^{3/2} \geq p\), then \( \sigma \leq |A|^3 \leq |A|^{9/2} \). If \(|A|^{3/2} < p\), then \( \sigma \leq |A|^3 + \sigma \leq |A|^{9/2} \). In any case \( \sigma \ll |A|^{9/2} \). Combining this with the bound from Theorem 2, we obtain the required result. This completes the proof.

For any sets \( A, B, C \subseteq \mathbb{F}_p \) put

\[
N(A) = |\{(a - c) = (b - c') : a, a' \in A, b, b' \in B, c, c' \in C\}|.
\]

We write \( N(A) \) if \( A = B = C \). Now we prove an upper bound for the quantity \( N(A) \) which is better than \( O(|A|^{9/2}) \) for sets \( A \) with small energies \( E^\times(A) \) and \( E^+(A) \), namely, when \( (E^\times(A))^2E^+(A) \ll |A|^{8-\varepsilon}, \varepsilon > 0 \).

**Corollary 38** Let \( A, B \subseteq \mathbb{F}_p \) be sets. Then

\[
N(B, A) - \frac{|A|^4|B|^2}{p} \ll (E^\times(B))^{1/2}(E^+(A))^{1/4}|A|^{5/2} \log |A|.
\]

Proof. Put \( f(x) = A(x) - |A|/p \). We have

\[
N(B, A) = \sum_\lambda r_{B/B}(\lambda)r_{(A-A)/(A-A)}(\lambda) = \frac{|A|^4|B|^2}{p} + \sum_\lambda r_{B/B}(\lambda)r_{(f-f)/(A-A)}(\lambda) = \frac{|A|^4|B|^2}{p} + \sigma.
\]

By the Cauchy–Schwarz inequality, we get

\[
\sigma^2 \leq \left( \sum_\lambda r_{B/B}(\lambda)r_{(f-f)/(A-A)}(\lambda) \right)^2 \leq \sum_\lambda r_{B/B}(\lambda) \cdot \sum_\lambda r_{(f-f)/(A-A)}(\lambda) \leq E^\times(B)\left( D^\times(A) - \frac{|A|^8}{p} \right).
\]
Using Theorem 32, we obtain the required result.

Similarly to $N(A)$, put

$N'(A) = |\{a_1a_2 + a_3 = a'_1a'_2 + a'_3 : a_i, a'_i \in A\}|.$ \hfill \(96\)

**Corollary 39** Let $A \subseteq \mathbb{F}_p$ be a set. Then

$$N'(A) - \frac{|A|^6}{p} \ll |A|^{5/2} E^+(A)^{1/2} E^*(A)^{1/4} \log |A|.$$  

**Proof.** Put $f(x) = A(x) - |A|/p$. As in the proof of Theorem 32, we have

$$N'(A) = \frac{|A|^6}{p} + \sigma,$$

where the sum $\sigma$ counts the number of the solutions to equation (96) with the weights $f(a_j), f(a'_j)$. Thus by Theorem 10, the Dirichlet principle and the Hölder inequality, we get

$$\sigma = E^+(f, r_{ff}) \leq (E^+(f))^2 (E^+(r_{ff}))^{1/2} \ll \log |A| \cdot \Delta E^+(A)^{1/2} E^+(r_{ff}, P)^{1/2} \ll$$

$$\ll \log |A| \cdot \Delta E^+(A)^{1/2} \left( \frac{|A|^4 |P|^2}{p} + |A|^3 |P|^{3/2} \right)^{1/2}, \hfill \(97\)$$

where $\Delta < |r_{ff}(x)| \leq 2\Delta$ on the set $P$. Here we have used the definition of $E^+(r_{ff}, P)$, namely,

$$E^+(r_{ff}, P) = |\{p + a_1a_2 = p' + a'_1a'_2 : p, p' \in P\}|,$$

counting with the weights $f(a_1)f(a_2), f(a'_1)f(a'_2)$. Again one can assume that the second term in (97) dominates. Hence using $\Delta |P| \ll |A|^2, \Delta^2 |P| \leq E^*(f) \leq E^*(A)$, we obtain

$$\sigma \ll \log |A| \cdot E^+(A)^{1/2} |A|^{5/2} E^*(A)^{1/4}$$

as required. \hfill \(\square\)

The results above imply an estimate for some average sums of the energies (other results on such quantities can be found in [43]).

**Corollary 40** Let $A \subseteq \mathbb{F}_p$ be a set. Then

$$\sum_{x \in X} E^+(A, xA) - \frac{|X||A|^4}{p} \ll |A|^{5/4} E^+(A)^{3/4} E^*(X)^{1/4} \log^{1/2} |A|.$$
Proof. Indeed, putting \( f(x) = A(x) - |A|/p \), we have
\[
\sigma := \sum_{x \in X} E^+(A, xA) = \sum_{\lambda} r_{A-A}(\lambda)r_{(A-x)A}(\lambda) = \frac{|X||A|^4}{p} + \sum_{\lambda} r_{A-A}(\lambda)r_{(f-f)A}(\lambda).
\]

Hence by the Hölder inequality, we get
\[
\sigma - \frac{|X||A|^4}{p} \ll E^+(A)^{1/2} \left( \sum_{\lambda} r^2_{(f-f)A}(\lambda) \right)^{1/2}.
\]

Using the Hölder inequality one more time, we obtain
\[
\sum_{\lambda} r^2_{(f-f)A}(\lambda) \leq E^+(X)^{1/2} \left( D^+(A) - \frac{|A|^8}{p} \right)^{1/2}.
\]

Applying Theorem 32, we have finally
\[
\sigma \ll |A|^{5/4}E^+(A)^{3/4}E^+(X)^{1/4} \log^{1/2} |A|
\]
as required.

Notice that it was proved in [43, Corollary 8] that for any \( A \subseteq \mathbb{F}_p \), \(|A| \leq p^{3/5}\) there exist two disjoint sets \( B \) and \( C \) of \( A \), each of cardinality \( \geq |A|/3 \), such that \( E^+(B)^3E^+(C)^2 \ll |A|^{14} \).

Unconditional upper bounds for \( D^+(A) \), \( D'(A) \) and multilinear exponential sums

The aim of this section is to prove

**Theorem 41** Let \( A \subseteq \mathbb{F}_p \) be a set, \(|A| \leq p^{2846/4991}\). Then for any \( c < \frac{1}{153} \) one has
\[
D^+(A) \ll |A|^{13/2-c}.
\]

Further if \(|A| \leq p^{48/97}\), then for any \( c_1 < \frac{1}{192} \) the following holds
\[
D^+(A) \ll |A|^{13/2-c_1}.
\]

We need two lemmas from [36, Section 4.5].

**Lemma 42** Let \( A \subseteq \mathbb{F}_p \) be a set and \(|A+A| = M|A|\), \(|A| \leq p^{13/23}M^{25/92}\). Then for any \( \alpha \in \mathbb{F}_p \) one has
\[
E^+(A+\alpha) \lesssim M^{51/26}|A|^{32/13}.
\]
Lemma 43 Let $A \subseteq \mathbb{F}_p$ be a set and $|AA| = M|A|$, $|A| \leq p^{13/23}M^{10/23}$. Then for any $\alpha \in \mathbb{F}_p^*$ one has

$$E^\times(A + \alpha) \lesssim M^{33/13}|A|^{32/13}. $$

We have a connection between the quantities $E^\times(A - \alpha)$ and $D^\times(A)$, namely

$$D^\times(A) \leq |A|^4 \max_{\alpha \in A} E^\times(A - \alpha). \quad (98)$$

Indeed, just fix four variables $a_1, a_2, a_3, a_4$ and

$$D^\times(A) \lesssim |A|^4 |A'|^{5/2-1/26} K^{102/13} M^{51/13},$$

where $M = |A'|/|A|$. The condition of the lemma takes place if

$$|A'| \leq p^{13/23} (K^4 (|A'|/|A|)^2)^{25/92} \quad (99)$$

and we will check (99) later. After that consider $A \setminus A'$ and continue our algorithm with this set. We obtain disjoint sets $A_1 = A'$, $A_2$, , and, clearly, $\sum_j |A_j| \leq |A|$. Finally, in view of (98) and the norm property of $E^\times(\cdot)$, we get an upper bound for $D$, namely,

$$D \lesssim |A|^{13/2} K^{-1/2} + K^{102/13} |A|^4 \left( \sum_j (|A_j|^{5/2-1/26} (|A_j|/|A|)^{51/13})^{1/4} \right)^4 \lesssim$$

$$\lesssim |A|^{13/2} K^{-1/2} + K^{102/13} |A|^4 |A|^{5/2-1/26}. $$

Optimizing over $K$, that is, taking $K = |A|^{1/217}$ we obtain the required bound because condition (99) follows from

$$|A'|^{42} |A|^{50} \leq |A|^{92} \leq p^{52} K^{100}$$

or, in other words, from $|A| \leq p^{(52+100)/217}/92 = p^{2846/4991}$. Also, in view of the condition $|A| \leq p^{2846/4991}$ the term $|A|^8/p$ is negligible.

Similarly, using bound (17) of Corollary 20 and the same calculations, we see that

$$D \lesssim |A|^{13/2} K^{-1/2} + K^{83/26} |A|^4 |A|^{5/2-1/26} + K^{83/26} \left( \sum_j |A_j| (|A_j|^{5/2-1/26})^{1/4} \right)^4 \lesssim$$

$$\lesssim |A|^{13/2} K^{-1/2} + K^{83/26} |A|^4 |A|^{5/2-1/26}. $$

Optimizing over $K$, that is, taking $K = |A|^{1/96}$ we obtain the required bound because the condition $|A|^K \leq \sqrt{p}$ follows from $|A| \leq p^{48/97}$. This completes the proof. □
Remark 44 The same arguments, combining with Lemma 43 (or its refinement from [37]) allow us to prove that either $D'(A) \ll |A|^{13/2-c}$ or $D'(A + \alpha) \ll |A|^{13/2-c}$ for any $\alpha \neq 0$ and all sufficiently small sets $A$ (also, see Remark [43]). Here $c > 0$ is an absolute constant.

Given three sets $X, Y, Z \subseteq \mathbb{F}_p$ and three complex weights $\alpha = (\alpha_x)_{x \in X}$, $\beta = (\beta_y)_{y \in Y}$, $\gamma = (\gamma_z)_{z \in Z}$ all bounded by one, put

$$S(X, Y, Z; \alpha, \beta, \gamma) = \sum_{x \in X, y \in Y, z \in Z} \alpha_x \beta_y \gamma_z e(xyz).$$

Similarly, for some complex weights $\rho = (\rho_{x,y})$, $\sigma = (\sigma_{x,z})$, $\tau = (\tau_{y,z})$ all bounded by one, we define

$$T(X, Y, Z; \rho, \sigma, \tau) = \sum_{x \in X, y \in Y, z \in Z} \rho_{x,y} \sigma_{x,z} \tau_{y,z} e(xyz).$$

Such sums were studied in [39]. Using Corollary 38 and Theorem 11 we improve [39] Theorems 1.3 and refine [39] Theorems 1.1 for sets with small energies (similar bound can be obtained for a correspondent sum with four variables, see [39] Theorem 1.2, 1.4 for large range).

Corollary 45 Let $|X| \geq |Y| \geq |Z|$. Then

$$S(X, Y, Z; \alpha, \beta, \gamma) \ll \log^{1/4} |Y| \cdot p^{1/4} |X|^{3/4} |Y|^{5/8} |Z|^{1/2} (E^X(Z))^{1/8} (E^Y(Y))^{1/16} + |X|^{3/4} |Y||Z|,$$

and if $|Y| < p^{48/97}$, then

$$T(X, Y, Z; \rho, \sigma, \tau) \ll p^{1/8} |X|^{7/8} |Y|^{29/32} |Z|^{29/32} (|Y||Z|)^{-1/3072}.$$

A series of applications of upper bounds for $S(X, Y, Z; \alpha, \beta, \gamma)$, $T(X, Y, Z; \rho, \sigma, \tau)$ can be found in the same paper [39]. Now we obtain a quantitative form of the main result of [5].

Theorem 46 Let $X, Y, Z \subseteq \mathbb{F}_p$ be arbitrary sets. Then for any $k \geq 2$ one has

$$\sum_{x \in X, y \in Y, z \in Z} e(xyz) \ll |X||Y||Z| \left( |Z|^{-2^{-(k+1)}} + \left( \frac{p}{|X||Y||Z|} \right)^{2^{-(k+1)}} (|X||Y|)^{2^{-2k}} \right). \quad (100)$$

More generally, for any non-negative functions $\alpha(x)$, $\beta(y)$, $\gamma(z)$ the following holds

$$\sum_{x,y,z} \alpha(x) \beta(y) \gamma(z) e(xyz) \ll p^{6(1)} \|\alpha\|_1 \|\beta\|_1 \|\gamma\|_1 \times$$

$$\times \left( \frac{\|\gamma\|_2}{\|\gamma\|_1} \right)^{2^{-(k+1)}} + \left( \frac{p \|\alpha\|_2 \|\beta\|_2 \|\gamma\|_2}{\|\alpha\|_2 \|\beta\|_2} \right)^{2^{-(k+1)}} \left( \frac{\|\alpha\|_1 \|\beta\|_1}{\|\alpha\|_2 \|\beta\|_2} \right)^{2^{-2k}}. \quad (101)$$
Proof. Let $S$ be the sum from (100). Using the Cauchy–Schwarz inequality several times, we get for any $k$
\[
|S|^2 \leq |Z|^{2k-1} \sum_{\lambda,z} (r_{XY} \ast_{2k} r_{XY})(\lambda)Z(z)e(\lambda z).
\]

Applying the Cauchy–Schwarz inequality one more time, combining with the Parseval identity, we obtain
\[
|S|^{2k+1} \leq |Z|^{2k+1-2D_2k(X,Y)p}|Z|^{2k+1-1}.
\] (102)

Put $l = 2^k$. By an analogue of Theorem 32 for $D'_j(X,Y)$, see Remark 33 and bound (18), we have
\[
|S|^{2k+1} \leq p|Z|^{2k+1-1} \left( \frac{(|X||Y|)^{2k+1}}{p} + (\log |X||Y|)^8 \left( \frac{E^X(X,Y)}{|X||Y|} \right)^{2-l+1} \right)
\]
\[
\leq p|Z|^{2k+1-1} \left( \frac{(|X||Y|)^{2k+1}}{p} + (\log |X||Y|)^8 (|X||Y|)^{2-l} (|X||Y|)^{2k+1-1} \right)
\]
as required. Similarly, to obtain (101) just use Corollary 11 (actually, in this case we do not need in sharp asymptotic formulae but just in the incidences results as in Theorem 10) to estimate the required number of incidences with weights $\alpha$, namely,
\[
D_2^k(\alpha, \beta) \ll L^8 \cdot \frac{(||\alpha||1||\beta||1)^{2k+1}}{p} + L^8 \cdot (||\alpha||1||\beta||1)^{2k+1-2}(||\alpha||2||\beta||2)^2(2-l+2)(E^X(\alpha, \beta))^{2-l+1} \ll
\]
\[
\ll L^8 \cdot (||\alpha||1||\beta||1)^{2k+1} \left( \frac{1}{p} + \frac{||\alpha||2^2||\beta||2^2}{||\alpha||1||\beta||1} \right)^{2-l+1}
\]
and apply the previous arguments. Here we have put
\[
L = \log(||\alpha||1||\beta||1||\gamma||1(||\alpha||2||\beta||2||\gamma||2)^{-1}) \ll \log p
\]
and also we have used the bound $E^X(\alpha, \beta) \leq ||\alpha||2||\beta||2||\alpha||1||\beta||1$ as in (103). This completes the proof.

One can obtain Theorem 16 for general weights like in [6]. Also, it is known that an analogue of our result (and un upper bound for the multilinear exponential sums as well) for more than three sets follows from the case of three sets, see [6 Section 8]. We demonstrate this just for exponential sums with three and four sets, see explicit bounds (104), (105) below. In general, one can use a simple inequality which takes place for any even $k$ and non–negative functions $\alpha_j$
\[
\left| \sum_{a_1, \ldots, a_r} \alpha_1(a_1) \ldots \alpha_r(a_r)e(a_1 \ldots a_r) \right|_{k_1 \ldots k_r} \leq \left( \prod_{j=1}^{r} ||\alpha_j'||1 \right)^{k_1 \ldots k_r} \sum_{a_1, \ldots, a_r} (\alpha_1 *_{k_1} \alpha_1)(a_1) \ldots (\alpha_r *_{k_r} \alpha_r)(a_r)e(a_1 \ldots a_r)
\]
and insure that the dependence on \( r \) in the saving has the form \( p^{-\delta/(C_1 \log(C_2 r/\delta))} r \), where \( C_1, C_2 > 0 \) are absolute constants. In this case we do not need in sharp asymptotic formulae but just the incidences results as in Theorem 10. The dependence in \([6, \text{Theorem A}]\) was \( p^{-\delta/(\delta r)^c} r \), where \( C > 0 \) is another absolute constant, so our result is better.

Corollary 47 Let \( X, Y, Z \subseteq \mathbb{F}_p \) be arbitrary sets such that for some \( \delta > 0 \) the following holds

\[
|X||Y||Z| \geq p^{1+\delta}.
\]

Then

\[
\sum_{x \in X, y \in Y, z \in Z} e(xyz) \ll |X||Y||Z| \cdot p^{-\delta/4 \log(\delta/\delta/\delta)} + 1.
\]

Finally, let \( r = 4 \), then for any sets \( A_1, \ldots, A_r \subseteq \mathbb{F}_p \) with \( \prod_{j=1}^{r} |A_j| \geq p^{1+\delta} \) one has

\[
\sum_{a_1 \in A_1, \ldots, a_r \in A_r} e(a_1 \ldots a_r) \ll \prod_{j=1}^{r} |A_j| \cdot p^{-\delta/4(\log(\delta/\delta/\delta/\delta)) + 1}.
\]

Proof. To obtain (103) we want to use Theorem 46. Put \( l = 2^k \). Then using crude bounds \( |X|, |Y| \leq p \), we get

\[
\left( \frac{p}{|X||Y||Z|} \right)^{2^{-k+1}} (|X||Y|)^{2^{-2^k}} \leq p^{-\delta/(2l) + 2^{-l+1}} \leq p^{-\delta/(4l)},
\]

provided \( \frac{2l}{\delta} \geq 8 \). It is easy to see that \( l = \lceil 2 \log(8/\delta) \rceil \geq 6 \) is enough. Applying Theorem 46, we see that the second term in (100) is at most \( |X||Y||Z| \cdot p^{-\delta/4(\log(\delta/\delta/\delta/\delta))} \). The first term in this formula equals

\[
|X||Y||Z| \cdot |Z|^{-1/2l} \leq p^{-\delta/(4l)},
\]

provided \( |Z| \geq p^{\delta/2} \) and hence this bound has the same quality. Suppose that \( |Z| < p^{\delta/2} \). Then by (103), we obtain \(|X||Y| \geq p^{1+\delta/2} \) and by a trivial bound for double exponential sums, we get

\[
\sum_{x \in X, y \in Y, z \in Z} e(xyz) \ll |Z| \cdot \sqrt{p|X||Y|} = |X||Y||Z| \cdot \sqrt{p/(|X||Y|)} \leq |X||Y||Z| \cdot p^{-\delta/4}
\]

which is even better.

Now to get (105) we can replace all \( A_i(x) \) to \( A_i(x) - |A_i|/p \) if we want. Let \( S \) be the sum from (105). Also, let \( \eta(x) = r A_1 A_2(x) \) and \( L = \log(|A_1||A_2||A_3|) \). Applying the Hölder inequality, we obtain

\[
|S|^{2n} \leq ([A_3 \ldots A_r])^{2n-1} \sum_{a_3 \in A_3, \ldots, a_r \in A_r} \left| \sum_x \eta(x)e(xa_3 \ldots a_r) \right|^{2n} =
\]
Applying the Hölder inequality as in Theorem 46 (see estimate (102)), combining with an analogue of (57) for $D'$ and using $r = 4$, we obtain for any $l$

$$|A_3|^{-1} \sum_x (\eta *_{2n} \eta)(x) \sum_{a_3 \in A_3, \ldots, a_r \in A_r} e(xa_3 \ldots a_r) \leq \frac{|A_4|}{|A_3|} \left( \frac{pD_2(\eta *_{2n} \eta, A_3)}{|A_4|} \right)^{1/4l} \ll \ll \frac{|A_4|}{|A_3|} \left( \frac{p|A_4|^{-1}L^8(|A_1|^2|A_2|^{2n}|A_3|)^{4l-2}(|\eta *_{2n} \eta|^2|A_3|)^{1-2^{-2l+1}}(E^\times(\eta *_{2n} \eta, A_3))^{1/2}}{|A_4|} \right)^{1/4l} \ll \ll \frac{|A_4|}{|A_3|} \left( \frac{p|A_4|^{-1}L^{16}(|A_1|^2|A_2|^{2n}|A_3|)^{4l-2}(|A_1||A_2|)^{4n-1+2^{-2l+1}}|A_3||A_3|^{1/2}}{|A_4|} \right)^{1/4l} \ll \ll |A_4|(|A_1||A_2|)^{8n} \left( \frac{p}{|A_1||A_2||A_3||A_4|} \cdot L^{16}(|A_1||A_2|)^{2^{-2l+1}}|A_3|^{2^{-2l+1}} \right)^{1/4l}.
$$

Here we have used the fact that $E^\times(\eta *_{2n} \eta, A_3) \leq |\eta *_{2n} \eta|^2|A_3|^{2}$. So, taking $l = n$ such that $n = \lceil 0.5 \log(200/\delta) \rceil$, we obtain the required result. This completes the proof.

\section{An asymptotic variant of the Balog–Wooley decomposition Theorem}

Now we prove a result in the spirit of [2, 29, 41, 43]. The difference between our Theorem 48 and these results is that we have an asymptotic formula for the energy. Of course in formuale (106), (107) below the additive and multiplicative energy can be swapped (for some other sets $B$ and $C$) and moreover can be replaced to other energies (see [43]).

\textbf{Theorem 48} Let $A \subseteq \mathbb{F}_p$ be a set and let $1 \leq M \leq p/(2|A|)$ be a parameter. There exist two disjoint subsets $B$ and $C$ of $A$ such that $A = B \sqcup C$ and

$$E^+(B) - \frac{|B|^4}{p} \leq \frac{|A|^{2/3}|B|^{7/3}}{M}, \quad (106)$$

and for any set $X \subseteq \mathbb{F}_p$ one has

$$E^\times(C, X) \leq \frac{M^2|X|^2|A|^2}{p} + M^{3/2}|A||X|^{3/2}. \quad (107)$$

\textbf{Proof}. Our proof is a sort of an algorithm similar to the arguments of the proof of Theorem 41. At the first step put $B = A$ and $C = \emptyset$. Suppose that we have constructed $B$ at some step of our algorithm. Write $f_B(x) = B(x) - |B|/p$. Then $E^+(B) = \frac{|B|^4}{p} + E^+(f_B, B)$. If

$$E^+(f_B, B) = \sum_x B(x)r_{f_B + B - B}(x) \leq \frac{|A|^{2/3}|B|^{7/3}}{M},$$

then...
then we are done. If not, then \( E^+(f_B, B) \geq \frac{|A|^{2/3}|B|^{7/3}}{M} \) and by the pigeonhole principle we find a set \( P \) such that \( \Delta < |r_{f_B-B}(x)| \leq 2\Delta \) for all \( x \in P \) and
\[
\frac{|A|^{2/3}|B|^{7/3}}{M} \leq E^+(f_B, B) \leq \Delta \sum_x B(x)|r_{f_B+B}(x)| \leq \Delta \sum_x B(x)r_{B+B}(x) + \frac{\Delta|B|^2|P|}{p} \leq \\
\leq \Delta \sum_x B(x)r_{B+B}(x) + \frac{|B|^4}{p} \leq 2\Delta \sum_x B(x)r_{B+B}(x).
\]
Here we have used the assumption \( M \leq p/(2|A|) \). Using Lemma 17 with \( P = P \) and \( A = B \), we find a set \( B_0 \subseteq B \) and a number \( q, q \leq |B_0| \) such that for any \( x \in B_0 \) one has \( r_{B+B}(x) \geq q \), and \( \sum_x B(x)r_{B+B}(x) \sim |B_0|q \). We have
\[
E^+(B_0, X) \leq q^{-2}\left| \{ (b+p)x = (b'+p')x' : x, x' \in X, b, b' \in B, p, p' \in P \} \right|.
\]
Using Theorem 10 and the definition of \( q \), we obtain
\[
E^+(B_0, X) \ll q^{-2}\left( \frac{|X|^2|B|^2|P|^2}{p} + (|X||B||P|)^{3/2} + |X||B||P| \max\{|X|, |B|, |P|\} \right) \ll \\
(E^+(f_B, B))^{-2}|B_0|^2\Delta^2 \left( \frac{|X|^2|B|^2|P|^2}{p} + (|X||B||P|)^{3/2} + |X||B||P| \max\{|X|, |B|, |P|\} \right) .
\]
Now clearly,
\[
\Delta|P| \leq \sum_x r_{B-B}(x) + |B|^2 \leq 2|B|^2 .
\]
and
\[
\Delta^2|P| \leq \sum_x r_{f_B-B}(x) = E^+(f_B, B) .
\]
Then using the last formulae, the fact that \( B \subseteq A \) and returning to (108), we obtain
\[
E^+(B_0, X) \leq (E^+(f_B, B))^{-2}|B_0|^2 \times \\
\times \left( \frac{|X|^2|B|^6}{p} + |X|^{3/2}|B|^{7/2}(E^+(f_B, B))^{1/2} + \Delta^2|X||B||P| \max\{|X|, |B|, |P|\} \right) \leq \\
\frac{M^2|X|^2|B_0|^2}{p} + M^{3/2}|A|^{-1}|B_0|^2|X|^{3/2} + M^2|A|^{-4/3}|B|^{-11/3}\Delta^2|B_0|^2|X||P| \max\{|X|, |B|, |P|\} .
\]
Suppose that the third term in the last estimate is negligible. After that we consider \( B \setminus B_0 \) and continue our algorithm with this set. We obtain disjoint sets \( A_1 = B_0, A_2, \ldots \) and let \( C \) be its union. Finally, in view of the norm property of \( E^+(\cdot, X) \), we get an upper bound for \( E^+(C, X) \), namely,
\[
E^+(C, X) \leq \left( \sum_j (E^+(A_j, X))^{1/2} \right)^2 \leq \left( \frac{M^2|X|^2}{p} + M^{3/2}|A|^{-1}|X|^{3/2} \right) \cdot \left( \sum_j |A_j| \right)^2 .
\]
\[ \leq \frac{M^2|X|^2|A|^2}{p} + M^{3/2}|A||X|^{3/2}. \]  

(112)

It remains to check that the third term in (111) is negligible. From (112), it follows that

\[ M^3 \leq |X| \leq |A|^2/M^3 \]  

(113)
because otherwise there is nothing to prove. Since, \( E^+(f_B, B) \geq \frac{|A|^{2/3}|B|^{7/3}}{M} \) we easily derive

\[ |B| \geq E^+(f_B, B)^{1/3} \gg |A|/M^{3/2}. \]  

(114)

Using these bounds, as well as a trivial upper estimate \( E^+(f_B, B) \leq |B|^3 \) one can quickly check that

\[ E^+(f_B, B) \leq M^{-1/2}|X|^{-1/2}|B|^{11/3}|A|^{1/3}, \quad E^+(f_B, B) \leq M^{-1/2}|X|^{1/2}|B|^{8/3}|A|^{1/3}, \]

and

\[ |B|^{1/3}M^{1/2} \leq |X|^{1/2}|A|^{1/3} \]

and thus indeed the third term in (111) is negligible. This completes the proof. \( \square \)

Notice that one cannot obtain an asymptotic formula as in (106) for both sets \( B \) and \( C \). Indeed, it would imply that \( |B + B|, |CC| \gg p \) but there are sets \( A \) having small sumsets and product sets, just put \( A = P \cap \Gamma \), where \( P \) is a suitable arithmetic progression and \( \Gamma \) is a subgroup.

Now let us obtain a result on the sum–product phenomenon (of course one can replace below + to * and vice versa).

\[ |A + A| \geq 5^{-1}\min\{|A|^{6/5}, p/2\} \]

or

\[ |AA| \geq \min\{p|A|^{-2/5}, |A|^{6/5}\}. \]

**Proof.** Apply Theorem 48 with \( M = |A|^{1/5} \). We find two disjoint subsets \( B \) and \( C \) of \( A \) such that \( A = B \cup C \) and estimates (106), (107) take place. If \( |B| \geq |A|/2 \) and \( M|A| = |A|^{6/5} \leq p/2 \), then by (106) and the Cauchy–Schwarz inequality one has

\[ |B|^4 \leq |B + B| \left( \frac{|B|^4}{p} + \frac{|A|^{2/3}|B|^{7/3}}{M} \right) \leq |A + A| \cdot 3\frac{|A|^{2/3}|B|^{7/3}}{2M} \]

and hence \( |A + A| \geq 5^{-1}|A|^{6/5} \). If \( |B| \geq |A|/2 \), then just consider a maximal set \( A' \subseteq A \) of size \( |A'|^{6/5} \leq p/2 \) and use the previous arguments. Finally, if \( |C| \geq |A|/2 \), then putting \( X = A \) in (107), we obtain

\[ |AA| \geq |AC| \geq \frac{|A|^2|C|^2}{E_X(A, C)} \geq \min\{p|A|^{-2/5}, |A|^{6/5}\} \]

as required. \( \square \)
7 Asymptotic formulae in $\text{SL}_2(\mathbb{F}_p)$

Now we consider the action of $\text{SL}_2(\mathbb{F}_p)$ on $\mathbb{F}_p$ and we begin with our version (see Theorem 50 below) of so-called $L_2$–flattering lemma from [7] (also, see [9], [44]) which is a direct consequence of the celebrated Helfgott’s Theorem 24. The proof of Theorem 50 can be found in the Appendix.

**Theorem 50** Let $\mu$ be a symmetric probability measure on $\text{SL}_2(\mathbb{F}_p)$ such that for a parameter $K \geq 1$ one has

- $\mu(g\Gamma) \leq K^{-1}$ for any proper subgroup $\Gamma \subset \text{SL}_2(\mathbb{F}_p)$, $g \in \text{SL}_2(\mathbb{F}_p)$ and
- $\|\mu\|_\infty \leq K^{-1}$.

Then for any integer $k \leq K^{c_*}$ the following holds

$$0 \leq \|\mu *_{2^k} \mu\|_2^2 - |\text{SL}_2(\mathbb{F}_p)|^{-1} \leq C_* k^{c_*} K^{-c_* k},$$

where $c_* \in (0, 1)$, $C_* > 1$ are absolute constants.

Now we derive some consequences of Theorem 50 to sum–product phenomenon and we begin with some generalizations of arguments from [31]. Transformations

$$y = \frac{-1}{x + a}, \quad y = \frac{-1}{x + a} + b$$

correspond to $\text{SL}_2(\mathbb{F}_p)$ matrices

$$s' = \begin{pmatrix} 0 & -1 \\ 1 & a \end{pmatrix} \in S', \quad s_{a,b} = \begin{pmatrix} b & -1 + ab \\ 1 & a \end{pmatrix} \in S.$$  

The collections $S', S$ of such matrices are clearly connected with continued fractions

$$[a_1, a_2, \ldots] = \frac{1}{a_1 + \frac{1}{a_2 + \ldots}}$$

and correspond to classical continuants (see, e.g., [26]), as well as continuants (entries) of the product of two matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_2 \end{pmatrix}.$$

We need several properties of the set $S$ and the first one can be found in [31] (or see the proof of Lemma 58 and Remark 60 below). It is easy to check that Lemma 51 does not hold for the set $S'$.

**Lemma 51** Suppose that in the definition of the set $S$ one has $a \in B_1$, $b \in B_2$. For any $g_1, g_2 \in \text{SL}_2(\mathbb{F}_p)$ the following holds

$$|g_1 B_1 \cap g_2 \cap S| \leq \max\{|B_1|, |B_2|\}. \quad (116)$$

Moreover, for any dihedral subgroup $\Gamma$ one has

$$|g_1 \Gamma g_2 \cap S| \leq 8 \max\{|B_1|, |B_2|\}. \quad (117)$$
Theorem 50 combining with Lemma 51 gives a consequence for continued fractions (consider the following two-step transformation \( \frac{1}{a + \frac{1}{x + b}} \) with the correspondent map from \( \text{SL}_2(\mathbb{F}_p) \) and a well-known connection of continued fractions with continuants, see, e.g., [26]). Another way to derive Theorem 52 is iteratively apply Corollary 61 below but this way gives worse bounds.

**Theorem 52** Let \( A \subseteq \mathbb{F}_p \) be a set \( |A| > p^{\varepsilon}, \varepsilon > 0 \). Then for any \( k > C \frac{1}{\varepsilon} \), where \( C > 0 \) is an absolute constant and for any \( x \in \mathbb{F}_p \) one has

\[
|\{x = [a_1, a_2, \ldots, a_k] : a_j \in A\}| = \frac{|A|^k}{p^{1+o(1)}}.
\]

Now we can formulate our "counting lemma". Here \( S \) can be any set of matrices satisfying (116), (117). Having a function \( f : \mathbb{F}_p \to \mathbb{C} \) by \( \langle f \rangle \) denote \( \sum_{x \in \mathbb{F}_p} f(x) \).

**Lemma 53** Let \( f_1, f_2 : \mathbb{F}_p \to \mathbb{C} \) be functions and \( |S| > p^{\varepsilon} \). The number of the solutions to the equation

\[
sa_1 = a_2,
\]

counting with weights \( f_1(a_1), f_2(a_2) \), and with the restriction \( s \in S \) is

\[
|S|\langle f_1 \rangle\langle f_2 \rangle p + 2\theta\|f_1\|_2\|f_2\|_2|S|p^{-1/2k+2},
\]

where \( |\theta| \leq 1 \) and \( k = k(\varepsilon) \).

**Proof.** Denote by \( \sigma \) the number of the solutions to equation (118). In terms of the generalized convolution, we have

\[
\sigma = \sum_x f_2(x)(S \ast f_1)(x).
\]

Let \( f(x) = f_1(x) - \langle f_1 \rangle/p \). Then

\[
\sigma = \frac{|S|\langle f_1 \rangle\langle f_2 \rangle p}{|S|\langle f_1 \rangle\langle f_2 \rangle} + \sum_x f_2(x)(S \ast f)(x) = \frac{|S|\langle f_1 \rangle\langle f_2 \rangle p}{|S|\langle f_1 \rangle\langle f_2 \rangle} + \sigma_\ast.
\]

Using the Cauchy–Schwarz inequality, we get

\[
\sigma_\ast^2 \leq \|f_2\|_2^2 \sum_x (S \ast f)^2(x) = \|f_2\|_2^2 \sum_x f(x)(S^{-1} \ast S \ast f)(x).
\]

Here \( \mu(x) := (S^{-1} \ast S)(x) : \text{SL}_2(\mathbb{F}_p) \to \mathbb{R} \) is the usual convolution on the group \( \text{SL}_2(\mathbb{F}_p) \). Notice that \( \mu(x) = \mu(x^{-1}) \). Also, \( \|f\|_2^2 = \|f_1\|_2^2 - (f_1)^2/p \leq \|f_1\|_2^2 \). Thus

\[
\sigma_1 := \sigma_\ast^2 \leq \|f_2\|_2^2 \sum_x f(x)(\mu \ast f)(x)
\]
and we obtain by the iteration of the previous arguments (also, see the proof of Corollary 31) that for any \( k \) one has
\[
\sigma_1^{2k} \leq \|f_2\|_2^{2k+1} \|f_1\|_2^{2k+1-2} \sum f(x)(\mu *_{2k} \mu * f)(x),
\]
(121)
where in \( \mu *_{2k} \mu \) the convolution on \( SL_2(\mathbb{F}_p) \) is taken \( 2^k - 1 \) times (so, we have written the function \( \mu \) exactly \( 2^k \) times). Now applying Lemma 22, we get
\[
\sigma_s \leq \|f_1\|_2 \|f_2\|_2 : (2p|\mu *_{2k} \mu|_2)^{1/2k+1} := \|f_1\|_2 \|f_2\|_2 (2pT_{2k+1}(S))^{1/2k+1}.
\]
(122)
Here
\[
T_{2l}(S) = \|\{s_1^{-1}s_1' \ldots s_i^{-1}s_i' = w_1^{-1}w_1' \ldots w_i^{-1}w_i' : s_j, s_j', w_j, w_j' \in S\}\|.
\]
Trivially, we have \( \|\mu\|_\infty = |S|^{-1} \). Suppose that \( p \) is sufficiently large such that \( |S| > p^\varepsilon > 60 \), say, and hence in view of Lemma 51 we avoid all subgroups (1)–(3) from Theorem 23 in the sense that the conditions of Theorem 50 take place with \( K = |S|^{1/2}/8 \). Thus by Theorem 50 we find some \( k = k(\varepsilon) \) such that
\[
T_{2k}(S) \leq 2|S|^{2k+1} p^{-3}.
\]
Hence in view of (122), we get
\[
\sigma_s \leq 2\|f_1\|_2 \|f_2\|_2 |S| p^{-1/2k+2}
\]
as required. \( \square\)

**Remark 54** From the proof of Theorem 50, it follows that the optimal choice of \( k \) is \( k \sim \log p / \log |S| \). On the other hand, bound (119) is nontrivial if \( k \ll \log \log p \). So, one can check that the assumption \( |S| > p^\varepsilon \) can be relaxed to \( \log |S| \gg \log p / \log \log p \) and under this condition we obtain a nontrivial bound in (119).

Now we obtain an interesting consequence ofLemma 53 to sets with small doubling (another result of the same sort about the products of sets with small doubling is contained in 31). Combining Corollary 55 and Lemma 14 from Section 3 we derive Theorem 7 from the Introduction.

**Corollary 55** Let \( A_1, A_2, B \subseteq \mathbb{F}_p, |B| \geq p^\varepsilon, \varepsilon > 0 \) and \( |A_1 + B| \leq K_1|A_1|, |A_2 + B| \leq K_2|A_2| \). Then the number of the solutions to the equation
\[
r_{A_1^{-1}A_2^{-1}}(1) = \left\{ \frac{1}{a_1} - \frac{1}{a_2} = 1 : a_1 \in A_1, a_2 \in A_2 \right\}
\]
(123)
is at most
\[
\frac{K_1K_2|A_1||A_2|}{p} + 2(K_1K_2|A_1||A_2|)^{1/2} p^{-1/2k+2},
\]
(124)
where \( k = k(\varepsilon) \).
Proof. Clearly, the number of the solutions to the equation \((123)\) does not exceed
\[
|B|^{-2} \left\{ \frac{1}{x-b} - \frac{1}{y-c} = 1 : x \in A_1 + B, y \in A_2 + B, b, c \in B \right\}.
\]
In other words, we have
\[
xy - (b + 1)y - (c - 1)x + (b + 1)(c - 1) + 1 = 0
\]
or, equivalently, in terms of SL_2(\mathbb{F}_p) actions \(s_{-(b+1),c-1}x = y\), where
\[
s_{-(b+1),c-1} = \begin{pmatrix} c-1 & -1 - (b+1)(c-1) \\ 1 & -(b+1) \end{pmatrix} \in S.
\]
Applying Lemma \([53]\) to sets \(- (B + 1), C - 1\), we obtain the required bound. This completes the proof.

Thus when \(K_1, K_2\) are small and sizes of \(A_1, A_2\) are close to \(p^\varepsilon\), our upper bound \((124)\) is close to the right asymptotic formula for \(r_{A_1^{-1} - A_2^{-1}}(1)\). The same can be proved in the case of \(\Gamma\)-invariant sets \(Q_1, Q_2\) for its intersection \(|Q_1 \cap (Q_2 + x)|\), where \(x \neq 0\) is an arbitrary, see Section \([4]\). So, these two phenomena are parallel to each other.

Now let us obtain an application to estimates for some exponential sums.

**Corollary 56** For any functions \(f, g : \mathbb{F}_p \to \mathbb{C}, \langle f \rangle = 0\), and for any set \(B\) with \(|B| \geq p^\varepsilon, \varepsilon > 0\) one has
\[
\sum_{x,y} f(x)g(y) \sum_{b_1, b_2 \in B} e \left( y \left( \frac{1}{x+b_1} + b_2 \right) \right) \ll \|f\|_2 \|g\|_2 \sqrt{p} |B|^{2} p^{-\delta}. \tag{125}
\]
Further for any nontrivial multiplicative character \(\chi\), we get
\[
\sum_{x,y} f(x)g(y) \sum_{b_1, b_2 \in B} \chi \left( y + b_2 + \frac{1}{x+b_1} \right) \ll \|f\|_2 \|g\|_2 \sqrt{p} |B|^{2} p^{-\delta}, \tag{126}
\]
and for \(|Y| \geq p^\varepsilon\), one has
\[
\sum_{x,y} f(x)Y(y) \sum_{b_1, b_2 \in B} \chi \left( y + b_2 + \frac{1}{x+b_1} \right) \ll \varepsilon \|f\|_1 |B|^2 |Y| \cdot \left( \frac{p^{1/2-\delta} \|f\|_2^2}{\|f\|_1^2} \right)^{\varepsilon}. \tag{127}
\]
Here \(\delta = o(\varepsilon) > 0\).

Proof. Using the Cauchy–Schwarz inequality, we obtain
\[
\left| \sum_{x,y} f(x)g(y) \sum_{b_1, b_2 \in B} e \left( y \left( \frac{1}{x+b_1} + b_2 \right) \right) \right|^2 \leq
\]
\[ \leq \|g\|_p^2 \cdot \sum_{x,x'} f(-x) \overline{f(-x')} \cdot |\{s_{b_1,b_2}x = s_{b_1',b_2'}x' : b_1, b_1' \in -B, b_2, b_2' \in B\}| = \|g\|_p^2 \cdot \sigma'. \]

Applying the arguments of the proof of Lemma 53 and the assumption \( \sum_x f(x) = 0 \), we have
\[ \sigma' \leq 2\|f\|_1^2 |B|^4 p^{-1/2k+2} \]
as required.

To obtain (126), we use the usual properties of multiplicative characters (see, e.g., [24]) to derive
\[ \sigma := \left| \sum_{x,y} f(x)g(y) \sum_{b_1, b_2 \in B} \chi \left( y + b_2 + \frac{1}{x + b_1} \right) \right| \ll \|g\|_p^2 \cdot \sum_{x,x'} f(-x) \overline{f(-x')} \cdot |\{s_{b_1,b_2}x = s_{b_1',b_2'}x' : b_1, b_1' \in -B, b_2, b_2' \in B\}| \]
and repeat the arguments.

Now let us use the usual Burgess’ method, see, e.g., [21]. Namely, by the Hölder inequality and Weil’s result (see [24, Theorem 11.23]), we get for any positive integer \( k \)
\[ \sigma^{2k} \leq ((\|f\|_1 |B|^2)^{2k-2} \cdot \sum_{x,x'} f(x) \overline{f(x')} \left| \left\{ b_2 + \frac{1}{x + b_1} = b_2' + \frac{1}{x' + b_1'} : b_1, b_1', b_2, b_2' \in B \right\} \right| \times \]
\[ \times \left( (2k)^k p |Y|^k + 2k \sqrt{p} |Y|^{2k} \right). \]

By Lemma 53, we find \( l = l(\varepsilon) \) such that
\[ \sum_{x,x'} f(x) \overline{f(x')} \left| \left\{ b_2 + \frac{1}{x + b_1} = b_2' + \frac{1}{x' + b_1'} : b_1, b_1', b_2, b_2' \in B \right\} \right| \leq 2\|f\|_1^2 |B|^4 p^{-1/2l+2}. \quad (128) \]

Taking constant \( k = [1/2\varepsilon] \) such that \(|Y|^k \gg \varepsilon (2k)^k \sqrt{p} \), we obtain
\[ \sigma \ll \varepsilon \|f\|_1 |B|^{2p^{-1/2l+\delta}} \cdot \left( \frac{\sqrt{p} \|f\|_1^2}{\|f\|_1^2} \right)^{1/2k} \leq \|f\|_1 |B|^2 |Y| \cdot \left( \frac{p^{1/2-\delta} \|f\|_1^2}{\|f\|_1^2} \right)^{\varepsilon}. \]

Here we have denoted \( 1/2^{l+2} \) as \( \delta \). This completes the proof. \( \square \)

Remark 57 The results above are nontrivial (suppose for simplicity that \( f(x) = X(x) \) for some set \( X \) if \( |X| \gg p^{1/2-\delta} \) and the restriction to the lower bound for size of \( B \) can be extended to \( \log |B| \gg \log p/\log \log p \).

Now we consider some one-parametric families of matrices in \( \text{SL}_2(F_p) \) for which the above methods can be applied.
Lemma 58 Let $B \subseteq \mathbb{F}_p$ and $S_{r_1,r_2} \subseteq SL_2(\mathbb{F}_p)$ be a set of the form

$$s_b = \begin{pmatrix} 1 & r_1(b) \\ r_2(b) & 1 + r_1(b)r_2(b) \end{pmatrix} \in S_{r_1,r_2} \subseteq SL_2(\mathbb{F}_p),$$

where $r_1 = p_1/q_1, r_2 = p_2/q_2$ are non-constant rational functions such that

$$\{p_1p_2, p_1q_2, p_2q_1, q_1q_2\}, \{p_1q_1q_2, p_1p_2q_1, p_1^2p_2, q_1^2q_2, q_1^2p_2\}$$

are linearly independent over $\mathbb{F}_p$. Put

$$M := \max\{\deg(p_1), \deg(p_2), \deg(q_1), \deg(q_2)\}.$$

Then for any $g_1, g_2 \in SL_2(\mathbb{F}_p)$ one has

$$|g_1B_2 \cap S_{r_1,r_2}| \leq 2M. \quad (129)$$

Moreover, for any dihedral subgroup $\Gamma$ one has

$$|g_1\Gamma g_2 \cap S_{r_1,r_2}| \leq 12M. \quad (130)$$

The same holds when $\{1, r_1, r_2\}$ are linearly dependent.

Proof. Take $a \in B_1, b \in B_2$ and consider the equation

$$\begin{pmatrix} x & qx + y/r \\ z & qz + w/r \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} r & q \\ 0 & r^{-1} \end{pmatrix} = \begin{pmatrix} 1 & r_1 \\ r_2 & 1 + r_1r_2 \end{pmatrix} \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} =$$

$$= \begin{pmatrix} X + r_1Z & Y + r_1W \\ r_2X + (1 + r_1r_2)Z & r_2Y + (1 + r_1r_2)W \end{pmatrix}.$$

From $xr = X + r_1Z, zr = r_2X + (1 + r_1r_2)Z$, we have

$$zX + r_1zZ = r_2xX + xZ + r_1r_2xZ. \quad (131)$$

If $Z = 0$, then from $XW - YZ = 1$ one derives $X \neq 0$ and we arrive to $xr_2 = z$. Since $xw - yz = 1$, it follows that $x, z$ cannot be zero simultaneously and hence $r_2$ is a constant. Similarly, we see that $x \neq 0$. By assumption $p_1p_2, p_1q_2, p_2q_1, q_1q_2$ are linearly independent. Hence multiplying (131) by $q_1q_2$, we obtain a non-zero polynomial (with the non-vanishing term $xZp_1p_2$) of degree at most $2M$. Thus equation (131) has at most $2M$ solutions.

Now consider any dihedral subgroup which is just a product of a cyclic group of order four (of order two in $PSL_2(\mathbb{F}_p)$) and a cyclic group of order $(p \pm 1)/2$. It is easy to see that the conjugate class of any element of $SL_2(\mathbb{F}_p)$ is the set of elements having the same trace and that an element with trace $\pm 2$ is conjugated to $\pm \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$, see [54, (6.3)]. We have considered the case of elements with trace $\pm 2$ already. As for the remained case take any matrix of trace $2\alpha$ and of the form $r_\varepsilon(\alpha, \beta) = \begin{pmatrix} \alpha & \varepsilon \beta \\ \beta & \alpha \end{pmatrix}$, where $\alpha^2 - \varepsilon \beta^2 = 1$ and $\alpha \neq \pm 1$ (hence $\varepsilon \neq 0$). One can
check that for any $n$ the element $r^n_\varepsilon(\alpha, \beta)$ has the same form, i.e. $r^n_\varepsilon(\alpha, \beta) = r_\varepsilon(\alpha_n, \beta_n)$ for some $\alpha_n, \beta_n \in \mathbb{F}_p$ and $\alpha^2_n - \varepsilon \beta^2_n = 1$. Then as above

\[
\begin{pmatrix}
\alpha x + \beta y & \varepsilon x + \alpha y \\
\alpha z + \beta w & \varepsilon z + \alpha w
\end{pmatrix} = \begin{pmatrix} x & y \\
z & w \end{pmatrix} \begin{pmatrix} \alpha & \varepsilon \beta \\
\beta & \alpha \end{pmatrix} = \begin{pmatrix} 1 & r_1 \\
r_2 & 1 + r_1 r_2 \end{pmatrix} \begin{pmatrix} X & Y \\
Z & W \end{pmatrix} = \\
\begin{pmatrix} X + r_1 Z & Y + r_1 W \\
r_2 X + (1 + r_1 r_2) Z & r_2 Y + (1 + r_1 r_2) W \end{pmatrix}.
\]

From this we have $X + r_1 Z = \alpha x + \beta y$, $Y + r_1 W = \varepsilon x + \alpha y$. Hence we can find $\alpha, \beta$ via $r_1$, provided $\varepsilon x^2 \neq y^2$. After that, we get

\[Z = \alpha z + \beta w - r_2 (X + r_1 Z).\]  

(132)

Since $\alpha, \beta$ can be linearly expressed via $r_1$, we obtain a contradiction with the linear independence of $p_1 p_2, p_1 q_2, p_2 q_1, q_1 q_2$ provided $Z \neq 0$ (the term $Z p_1 p_2$ does not vanish). If $Z = 0$, then $W \neq 0$ and

\[
\varepsilon \beta z + \alpha w = r_2 Y + (1 + r_1 r_2) W.
\]

(133)

We know that $\alpha, \beta$ can be founded via $r_1$. It gives us a contradiction with linear independence of $p_1 p_2, p_1 q_2, p_2 q_1, q_1 q_2$ (the term $W p_1 p_2$ does not vanish).

It remains to consider the case $\varepsilon x^2 = y^2$. Then we obtain an analogue of (132)

\[W = \varepsilon \beta z + \alpha w - r_2 (Y + r_1 Z).\]  

(134)

If $\varepsilon^2 z^2 \neq w^2$, then from $\alpha z + \beta w = r_2 X + (1 + r_1 r_2) Z$, $\varepsilon \beta z + \alpha w = r_2 Y + (1 + r_1 r_2) W$ we can find $\alpha, \beta$ which depends linearly on $Z + r_2 (X + r_1 Z)$, $W + r_2 (Y + r_1 W)$ and substituting them into $X + r_1 Z = \alpha x + \beta y$, we obtain a contradiction with linear independence of $p_1 p_2, p_1 q_2, p_2 q_1, q_1 q_2$, provided $Z \neq 0$. More precisely, we get a dependence of the form

\[A r_1 r_2 + B r_2 + Z r_1 + C = 0,
\]

where $A, B, C \in \mathbb{F}_p$ are some constants. If $A \neq 0$, then in the linear dependence between $p_1 p_2, p_1 q_2, p_2 q_1, q_1 q_2$ we have the non–vanishing term $A p_1 p_2$. Now if $A = 0$, then $\{1, r_1, r_2\}$ are linearly dependent. Let $\omega = X + r_1 Z$. Then from $\alpha x + \beta y = \omega$, $\alpha z + \beta w = Z + r_2 \omega$, we obtain $\alpha = \omega (w - yr_2) - yZ$, $\beta = \omega (xr_2 - z) + xZ$. It gives us in view of $\varepsilon x^2 = y^2$ that

\[1 = \alpha^2 - \varepsilon \beta^2 = (\omega (w - yr_2) - yZ)^2 - \varepsilon (\omega (xr_2 - z) + xZ)^2 = \\
= \omega^2 ((w - yr_2)^2 - 2 \varepsilon (xr_2 - z)^2) - 2 \omega Z (y (w - yr_2) + \varepsilon (xr_2 - z)) = \\
= \omega^2 (w^2 - \varepsilon z^2 - r_2 (yw - xz \varepsilon)) - 2 \omega Z (yw - xz \varepsilon). \]

(135)

Since $\varepsilon^2 z^2 \neq w^2$, further $yw - xz \varepsilon \neq 0$, $r_2$ depends linearly on $r_1$ and $\omega = X + r_1 Z$, $Z \neq 0$ it follows that (135) gives a nontrivial equation on $r_1$ of degree three. Now if $Z = 0$, then $\omega = X \neq 0$ and from (135) we have a linear equation on $r_2$.

Finally, consider the case $\varepsilon x^2 = y^2$ and $\varepsilon^2 z^2 = w^2$. Put $Q_1 = X + r_1 Z$ and $Q_2 = Z + r_2 (X + r_1 Z) = Z + r_2 Q_1$. Then from $Q_1 = \alpha x + \beta y$, $Q_2 = \alpha z + \beta w$, we obtain $\alpha = wQ_1 - yQ_2$, $\beta = -zQ_1 + xQ_2$. Hence using $\varepsilon x^2 = y^2$ and $\varepsilon^2 z^2 = w^2$, we have

\[1 = \alpha^2 - \varepsilon \beta^2 = (wQ_1 - yQ_2)^2 - \varepsilon (-zQ_1 + xQ_2)^2 = 2Q_1 Q_2 (\varepsilon xz - wy) =
\]
Remark 60

and our independency conditions are satisfied.

Let rational functions $r_1, r_2$ are just non–constant polynomials. Then $q_1 = q_2 = 1$ and our independency conditions are satisfied.

Example 59

If $Z \neq 0$, then the last identity gives us a contradiction with linear independence of $\{p_1q_2, p_1p_2q_1, p_1^2p_2, q_1^2q_2, q_1^2p_2\}$ because the term $2Z^2(\varepsilon xz - wy)p_1^2p_2$ does not vanish. If $Z = 0$, then from the same equation we obtain $1 = 2X^2(\varepsilon xz - wy)r_2$ and hence $r_2$ is a constant.

Lastly, a careful analysis of the proof shows that the same arguments work in the case when $\{1, r_1, r_2\}$ are linearly dependent. This completes the proof. □

Corollary 61

Let $p_1, p_2 \in \mathbb{F}_p[x]$ be any non–constant polynomials. Then for any $A, B \subseteq \mathbb{F}_p$, $|B| \geq p^\varepsilon$, $\varepsilon > 0$ one has

$$\left| \left\{ \frac{1}{a + p_2(b)} : a \in A, b \in B, \deg(a) \leq \deg(p_2) \right\} \right| \leq 2|A||B|^{1/2k+2}$$

(136)

where $k = k(\varepsilon, \deg p_1, \deg p_2)$. In particular,

$$\left| \left\{ \frac{1}{a + p_2(b)} : a \in A, b \in B \right\} \right| \gg \min\{p, |A|p^{1/2k+2}\}.$$  

(137)

Proof. Indeed, in terms of the set $S_{p_1, p_2}$ the number of the solutions to (136) is

$$sa = s'a', \quad a, a' \in A, s, s' \in S_{p_2, p_1}$$

or, equivalently, $(s')^{-1}sa = a'$ (we can assume that $a + p_2$ and $ap_1 + 1 + p_1p_2$ are nonzero). Using Lemma 58, combining with Lemma 57, we obtain the required result. □

Of course Corollary 61 does not hold if either $p_1$ or $p_2$ has zero degree. Using the same arguments as in the proof of Corollary 56, we derive

Corollary 62

Let $B \subseteq \mathbb{F}_p$ and $p_1, p_2, q_1, q_2 \in \mathbb{F}_p[x]$ such that

$$\{p_1, q_1\}, \{p_2, q_2\}, \{p_1p_2, p_1q_2, q_1p_2, q_1q_2\}, \{p_1q_1q_2, p_1p_2q_1, p_1^2p_2, q_1^2q_2, q_1^2p_2\}$$

are linearly independent over $\mathbb{F}_p$. Put

$$M := \max\{\deg(p_1), \deg(p_2), \deg(q_1), \deg(q_2)\}.$$
For any functions \( f, g : \mathbb{F}_p \to \mathbb{C}, \sum_x f(x) = 0 \), and for any set \( B \) with \( |B| \geq p^\varepsilon, \varepsilon > 0 \) one has
\[
\sum_{x,y} f(x)g(y) \sum_{b \in B} e\left( y \left( \frac{q_1(b)q_2(b)x + p_1(b)q_2(b)}{p_2(b)q_1(b)x + q_1(b)q_2(b) + p_1(b)p_2(b)} \right) \right) \ll_M \|f\|_2\|g\|_2 \sqrt{p}|B|^{p-\delta}.
\]

Further for any nontrivial multiplicative character \( \chi \) and \( \lambda \neq 0 \), we get
\[
\sum_{x,y} f(x)g(y) \sum_{b \in B} \chi\left( y + \frac{q_1(b)q_2(b)x + p_1(b)q_2(b)}{p_2(b)q_1(b)x + q_1(b)q_2(b) + p_1(b)p_2(b)} \right) \ll_M \|f\|_2\|g\|_2 \sqrt{p}|B|^{p-\delta}.
\]

Here \( \delta = \delta(\varepsilon, M) > 0 \).

Again, using the usual Burgess method one can obtain a non–trivial bound for sum \((139)\) in the regime when \( \varepsilon \) \( f(x) = X(x), \quad g(y) = Y(y) \) for simplicity) \( |X| \geq p^\varepsilon, |B| \geq p^\varepsilon \) and \( |X| \gg \varepsilon p^{1/2-\delta} \), see Remark 57 and the proof of Corollary 56.

The same method, combining with the results from [32] concerning rich lines (not rich hyperbolas) in \( \mathbb{F}_p \times \mathbb{F}_p \) allows to prove

**Theorem 63** For any functions \( f, g : \mathbb{F}_p \to \mathbb{C} \) and any sets \( A, B \) with \( |B| \geq p^\varepsilon, |A| < p^{1-\varepsilon}, \varepsilon > 0 \) one has
\[
\sum_{x,y} f(x)g(y) \sum_{b_1, b_2 \in B} e(y(a + b_1)b_2) \ll \|f\|_2\|g\|_2 \sqrt{p}|B|^2 p^{-\delta},
\]
where \( \delta = \delta(\varepsilon) > 0 \).

Again the result is nontrivial (suppose for simplicity that \( f(x) = X(x) \) for some set \( X \)) if \( |X| \gg \varepsilon p^{1/2-\delta} \) and the restriction to the lower bound for size of \( B \) can be extended to \( \log |B| \gg \log p/\log \log p \).

## 8 On \( \text{GL}_2(\mathbb{F}_p) \)–actions

In this section we consider the set of all non–degenerate matrices \( \text{GL}_2(\mathbb{F}_p) \) with coefficients from \( \mathbb{F}_p \). Also, let \( G \) be its subset. By \( \det G \) denote the set \( \det G := \{ \det g : g \in G \} \subseteq \mathbb{F}_p^* \) and because \( \text{GL}_2(\mathbb{F}_p) \) is acting on \( \mathbb{F}_p \) we can consider \( G(A) := \{ ga : g \in G, a \in A \} \) for any \( A \subseteq \mathbb{F}_p \). Of course there is no expanding result similar to Theorem 24 in \( \text{GL}_2(\mathbb{F}_p) \) but nevertheless one can easily obtain

**Proposition 64** Let \( G \subseteq \text{GL}_2(\mathbb{F}_p) \) be a set of matrices, \( A \subseteq \mathbb{F}_p \) and \( \varepsilon > 0 \) be a real number. Suppose that
\[
\begin{align*}
&\circ |G| > p^\varepsilon, \\
&\circ |\det G| \leq |G|p^{-\varepsilon}, \\
&\circ \sum_{x \in A} (G^{-1} \ast G)(x) \leq p^{-\varepsilon} \sum_{x \in \text{SL}_2(\mathbb{F}_p)} (G^{-1} \ast G)(x) \text{ for any proper subgroup } \Gamma \subseteq \text{SL}_2(\mathbb{F}_p), s \in \text{SL}_2(\mathbb{F}_p).
\end{align*}
\]
Then there is \( \delta = \delta(\varepsilon) > 0 \) such that
\[
|G(A)| \gg \min\{p, |A|p^\delta\}.
\]
Moreover and notice that \( \subset \) and by the assumption we see that for any proper subgroup \( \Gamma \) and hence \( \Delta \) holds that 

\[
\rho := \sum_{x \in \text{SL}_2(\mathbb{F}_p)} (G^{-1} \ast G)(x) = \sum_{\lambda} |G_{\lambda}|^2 \ll L \sum_{\lambda \in \Lambda} |G_{\lambda}|^2 := L \rho_1 .
\]

Using the assumption \( |\Lambda| \leq |\det G| \leq |G|^{-\varepsilon} \) and the Cauchy–Schwarz inequality, we see that 

\[
\rho \geq |\det (G)|^{-1} \left( \sum_{\lambda} |G_{\lambda}| \right)^2 = \frac{|G|^2}{|\det G|} \geq |G|^{-\varepsilon}
\]

and hence \( \Delta \gg p^\varepsilon/L \). By the definition of the set \( \Lambda \) and the Cauchy–Schwarz inequality, we obtain 

\[
|A|^2 \Delta^2 |\Lambda|^2 \leq \left( \sum_{\lambda \in \Lambda} |G_{\lambda}| |A| \right)^2 = \left( \sum_{\lambda \in \Lambda} \sum_{x \in X} (G_{\lambda} \ast A)(x) \right)^2 \leq |\Lambda| |X| \sum_{\lambda \in \Lambda} \sum_{x \in X} (G_{\lambda} \ast A)^2(x) = |\Lambda| |X| \sum_{\lambda \in \Lambda} \sum_{x \in X} (G_{\lambda}^{-1} \ast G_{\lambda} \ast A)(x)A(x) = |\Lambda| |X| \sigma .
\]

Further consider \( f(x) = \sum_{\lambda \in \Lambda} (G_{\lambda}^{-1} \ast G_{\lambda})(x) \) with \( \|f\|_1 = \rho_1 \) and the measure \( \mu(x) = f(x)/\rho_1 \) and notice that 

\[
\sigma = \sum_{x \in A} (f \ast A)(x) .
\]

Moreover \( \|\mu\|_\infty \ll |\Lambda| \Delta/(|\Lambda| \Delta^2) = \Delta^{-1} \ll L^{-\varepsilon} \), and by the assumption we see that for any proper subgroup \( \Gamma \subset \text{SL}_2(\mathbb{F}_p) \), \( s \in \text{SL}_2(\mathbb{F}_p) \) the following holds 

\[
\mu(s \Gamma) = \rho_1^{-1} \sum_{x \in s \Gamma} \sum_{\lambda \in \Lambda} (G_{\lambda}^{-1} \ast G_{\lambda})(x) \leq \rho_1^{-1} \sum_{x \in s \Gamma} (G^{-1} \ast G)(x) \ll L \rho_1^{-1} \sum_{x \in s \Gamma} (G^{-1} \ast G)(x) \leq L p^{-\varepsilon} .
\]

Using the arguments as in the proof of Lemma \[53\] we see that for some \( \delta = \delta(\varepsilon) > 0 \) one has 

\[
\sum_{\lambda \in \Lambda} \sum_{x} (G_{\lambda}^{-1} \ast G_{\lambda} \ast A)(x)A(x) - \frac{|A|^2 \rho_1}{p} \ll |A| \rho_1 p^{-\delta} .
\]

Hence, returning to \[14\], we obtain 

\[
|A|^2 \Delta^2 |\Lambda|^2 \ll |\Lambda| |X| \left( \frac{|A|^2}{p} + |A| p^{-\delta} \right) \sum_{\lambda \in \Lambda} |G_{\lambda}|^2 \ll |\Lambda|^2 \Delta^2 |X| \left( \frac{|A|^2}{p} + |A| p^{-\delta} \right) .
\]

It follows that 

\[
|X| \gg \min\{p, |A| p^\delta \} .
\]

This completes the proof. \[\square\]
Remark 65 One can see from the proof that the condition $|\det G| \leq |G|p^{-\varepsilon}$ can be refined to $\sum_\lambda |G_\lambda|^2 \geq |G|p^{\varepsilon}$.

Now we give the proof of a simple consequence of the theorem above (the constants 100 in (142), (143) are not really important and can be certainly decreased).

Lemma 66 Let $B_1, B_2, B_3 \subseteq \mathbb{F}_p$ and $G \subseteq \text{GL}_2(\mathbb{F}_p)$ be a set of the form

$$s_{b_1, b_2, b_3} = \begin{pmatrix} 1 & b_1 \\ b_2 & b_3 \end{pmatrix} \in G.$$  

Let $M = \max\{|B_1|, |B_2|, |B_3|\}$. Then for any $g_1, g_2 \in \text{GL}_2(\mathbb{F}_p)$ one has

$$\sigma_{g_1B_2}(S) \leq 100M^4. \quad (142)$$

Moreover, for any dihedral subgroup $\Gamma$ one has

$$\sigma_{g_1\Gamma g_2}(S) \leq 100M^4. \quad (143)$$

Proof. Take any $r_0, r_1, r_2, r_3 \in \mathbb{F}_p$ and consider the equation

$$\begin{pmatrix} x \ y \\ z \ w \end{pmatrix} = \begin{pmatrix} x/r & y/r \\ z/r & w/r \end{pmatrix} = \begin{pmatrix} r_0 & r_1 \\ r_2 & r_3 \end{pmatrix} \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} = \begin{pmatrix} r_0X + r_1Z & r_0Y + r_1W \\ r_2X + r_3Z & r_2Y + r_3W \end{pmatrix}.$$  

From $xr = r_0X + r_1Z$, $zr = r_2X + r_3Z$, we have

$$r_0zr + r_1zr = r_2zr + r_3zr.$$  

(144)

If $Z = 0$, then from $XW - YZ \neq 0$ one obtains $X \neq 0$ and we arrive to $xr_2 = zr_0$. Since $xw - yz \neq 0$, it follows that either $x$ or $z$ does not vanish and hence either $r_0$ or $r_2$ can be found uniquely. Similarly, one can consider the case $x = 0$ (Z is zero or not) and arrive to $r_0X = -r_1Z$. Hence either $r_0$ or $r_1$ can be found uniquely. Finally, if $Z \neq 0$ and $x \neq 0$, then we can find $r_3$ from (144). Now our matrix from $\text{SL}_2(\mathbb{F}_p) \cap G^{-1}G$ has the form

$$\begin{pmatrix} b_3 - b_1b_2 \\ -b_2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} b_3 - b_1b_2 \\ -b_2 \\ 0 \end{pmatrix} = \begin{pmatrix} b_3 - b_1b_2 \\ b_2 - b_1b_3 \\ b_1b_2 - b_1b_3 \end{pmatrix}$$

and such that

$$b_3 - b_1b_2 = b_3 - b_1b_2 \neq 0.$$  

(145)

We need to estimate the number of the solutions to (145) with some restrictions as $xr_2 = zr_0$, $r_0X = -r_1Z$ and so on. The appeared systems of two polynomial equations are rather concrete, so it is not difficult task. Another way is to use the Bézout Theorem.
First of all notice that fixing a variable of equation (145), we obtain the contribution at most $M^4$ into the sum $\sigma_{g_1B_2}(S)$. Secondly, if one expression among $r_0, r_2, r_3$ is fixed, then we substitute one appropriate variable into (145) and obtain the contribution at most $M^4$ into the sum $\sigma_{g_1B_2}(S)$. If $r_1$ is a constant, then we consider two cases: $b_1 = 0$ and $b_1 \neq 0$. The last case allows to substitute $b'_3$ into (145) and obtain a linear equation relatively to $b_3$ with the main coefficient $(b_1 - b'_1)$. Totally it gives the contribution at most $4M^4$. Now if $r_0 = C r_2$ for some $C \neq 0$, then we substitute $b'_2$ into (145), obtain a linear equation relatively to $b_2$ and get the contribution at most $2M^4$. If $r_0 = C r_1$, then either $b_1 = C^{-1}$ (contribution $M^4$) or $b_1 \neq C^{-1}$ and hence the substitution $b'_3$ into (145) gives us an equation of degree at most three and with five variables from $B_j$. Similarly, in the case when we find $r_3$ from (144) one obtains an equation degree at most three and with five variables from $B_j$. It gives at most $3M^4$ solutions and the total contribution into the sum $\sigma_{g_1B_2}(S)$ can be estimated roughly as $100M^4$.

Now let us have deal with the case of a dihedral subgroup which is just a product of a cyclic group of order four (of order two in $\text{PSL}_2(\mathbb{F}_p)$) and a cyclic group of order $(p \pm 1)/2$. Then we use arguments from the proof of Lemma 58 and consider $r_\epsilon(\alpha, \beta) = \begin{pmatrix} \alpha & \epsilon \beta \\ \beta & \alpha \end{pmatrix}$, where $\alpha^2 - \epsilon \beta^2 = 1$ and $\alpha \neq \pm 1$ (hence $\epsilon \neq 0$). Again, one can check that for any $n$ the element $r_\epsilon^n(\alpha, \beta)$ has the same form, i.e. $r_\epsilon^n(\alpha, \beta) = r_\epsilon(\alpha_n, \beta_n)$ for some $\alpha_n, \beta_n \in \mathbb{F}_p$ and $\alpha_n^2 - \epsilon \beta_n^2 = 1$. Then as above

\[
\begin{pmatrix} \alpha x + \beta y & \epsilon \beta x + \alpha y \\ \alpha z + \beta w & \epsilon \beta z + \alpha w \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} \alpha & \epsilon \beta \\ \beta & \alpha \end{pmatrix} = \begin{pmatrix} r_0 & r_1 \\ r_2 & r_3 \end{pmatrix} \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} = 
\end{align*}

\[
= \begin{pmatrix} r_0 X + r_1 Z & r_0 Y + r_1 W \\ r_2 X + r_3 Z & r_2 Y + r_3 W \end{pmatrix}.
\]

From this we have $r_0 X + r_1 Z = \alpha x + \beta y$, $r_0 Y + r_1 W = \epsilon \beta x + \alpha y$. Hence we can find $\alpha, \beta$ via $r_0, r_1$, provided $\epsilon x^2 \neq y^2$. After that, we get

\[
Z = Z(r_0 r_3 - r_1 r_2) = r_0 (\alpha x + \beta w) - r_2 (r_0 X + r_1 Z). \tag{146}
\]

Since $\alpha, \beta$ can be linearly expressed via $r_0, r_1$, we find $r_2 = (b_2 - b'_2)/(b_3 - b_1 b_2)$ from the last expression, provided $r_0 X + r_1 Z \neq 0$. It is easy to check that it gives us a nontrivial linear equation relatively to $b'_3$ because $r_0, r_1$ do not depend on $b'_3$. The possibility $r_0 X + r_1 Z = 0$ was considered above. In any case it gives the contribution at most $3M^4 + 6M^4 = 9M^4$ into the sum $\sigma_{g_1g_2}(S)$.

It remains to consider the case $\epsilon x^2 = y^2$. Then we obtain an analogue of (146)

\[
W = W(r_0 r_3 - r_1 r_2) = r_0 (\epsilon \beta z + \alpha w) - r_2 (r_0 Y + r_1 W). \tag{147}
\]

If $\epsilon x^2 \neq w^2$, then from $\alpha z + \beta w = r_2 X + r_3 Z$, $\epsilon \beta z + \alpha w = r_2 Y + r_3 W$ we can find $\alpha, \beta$ as some linear combinations of $r_2, r_3$ and substituting them into $r_0 X + r_1 Z = \alpha x + \beta y$, we find $r_0$ or $r_1$ uniquely, provided $X \neq 0$ or $Z \neq 0$ (or use the arguments as above applying (146), (147)). In any case it gives the contribution at most $18M^4$ into the sum $\sigma_{g_1g_2}(S)$. Finally, consider the case $\epsilon x^2 = y^2$ and $\epsilon z^2 = w^2$. Put $Q_1 = r_0 X + r_1 Z$ and $Q_2 = r_2 X + r_3 Z$. Also, let $d = \det \begin{pmatrix} x & y \\ z & w \end{pmatrix} \neq 0$. Then from $Q_1 = \alpha x + \beta y$, $Q_2 = \alpha z + \beta w$, we obtain $\alpha = -d^{-1}(w Q_1 - \epsilon x Q_2) = -d^{-1}(\epsilon x Q_1 - \alpha x Q_2)$ and $\beta = -d^{-1}(w Q_2 - \epsilon y Q_1) = -d^{-1}(\epsilon y Q_2 - \beta y Q_1)$.
\begin{align*}
yQ_2, \beta = d^{-1}(-zQ_1 + xQ_2). \text{ Hence } \\
d^2 = d^2(\alpha^2 - \varepsilon \beta^2) = (wQ_1 - yQ_2)^2 - \varepsilon(zQ_1 + xQ_2)^2 = 2Q_1Q_2(\varepsilon xz - wy) \\
= 2(\varepsilon xz - wy)(r_0X + r_1Z)(r_2X + r_3Z).
\end{align*}

Using the definitions of \( r_j \) and formula (145) to exclude \( b_3' \), we have
\[ d^2(b_3 - b_1b_2)^2 = 2(\varepsilon xz - wy)((b_3 - b_1b_2 + b_1b_2') - b_1b_2)X + (b_1b_3 - b_1b_3')Z)((b_3 - b_2')X + (b_3 - b_1b_2')Z). \]

In gives us a quadratic equation on \( b_2 \) and the leading coefficient of this equation is
\[ d^2b_1^2 + 2(\varepsilon xz - wy)X(Zb_1^2 + X(b_1' + b_1)). \]

If this coefficient does not vanished, then it gives us the contribution at most \( 2M^4 \) into the sum \( \sigma_{g_1g_2}(S) \). Suppose that \( b_1 \neq 0 \). In this case because \( d \neq 0 \) our coefficient vanishes iff \( X \neq 0 \) and \( (2(\varepsilon xz - wy)Z + d^2X^{-1})b_1^2 + 2(\varepsilon xz - wy)X(b_1' + b_1) = 0 \). Since \( (\varepsilon xz - wy) \neq 0 \), \( X \neq 0 \), we find \( b_1' \) uniquely. So, it gives us the contribution at most \( 2M^4 \) into the sum \( \sigma_{g_1g_2}(S) \). This completes the proof.

Now we are ready to obtain a consequence of Proposition 64. Of course one can replace an upper bound for \( |B_3 - B_1B_2| \) to a lower bound for \( \sum x r_{r_2B_2B_3}(x) \), see Remark 65 and formula (149) below. Other refinements are possible if one can estimate the number of the solutions to equation (145) with a fixed variable and under other restrictions, say,
\[ C_0r_0 + C_1r_1 + C_2r_2 + C_3r_3 = 0, \quad \delta \neq (C_0, C_1, C_2, C_3) \in \mathbb{F}_p^4, \]
more accurately (we need \( p^{-\varepsilon} \sum x r_{r_2B_2B_3}(x) \) bound).

**Corollary 67** Let \( A \subseteq \mathbb{F}_p, B_1, B_2, B_3 \subseteq \mathbb{F}_p, B := |B_1| = |B_2| = |B_3| > p^{\varepsilon} \). Suppose that \( |B_3 - B_1B_2| \leq B^2p^{-\varepsilon} \). Then there is \( \delta = \delta(\varepsilon) > 0 \) such that
\begin{equation}
\left\lvert \left\{ \frac{a + b_1}{ab_2 + b_3} : a \in A, b_1 \in B_2 \right\} \right\rvert \gg \min\{p, |A|^\delta\}. \tag{148}
\end{equation}

**Proof.** Let \( G = S_{b_1,b_2,b_3}, b_j \in B_2 \) and define \( G_\lambda \) as in Proposition 64. Since \( |B_3 - B_1B_2| \leq B^2p^{-\varepsilon} \), it follows that \( |\det G| \leq |G|B^{-1}p^{-\varepsilon} \). Further by Lemma 66 for any proper subgroup \( \Gamma \subseteq \text{SL}_2(\mathbb{F}_p) \), we have
\[ \sum_{x \in \mathbb{F}_p^2} (G^{-1} * G)(x) \ll B^4, \]
and
\[ \sum_{x \in \text{SL}_2(\mathbb{F}_p)} (G^{-1} * G)(x) = \sum z r_{r_2B_2B_3}(z) \geq \frac{|B_1||B_2||B_3|^2}{|B_3 - B_1B_2|} \geq B^4p^\varepsilon. \tag{149} \]

Thus all conditions of Proposition 64 take place for sufficiently large \( p \). This completes the proof. \( \square \)
9 Appendix

This section contains the proof of Lemma 22 and Theorem 50. Also, we obtain an upper bound for the energy $E_k$, see Theorem 68 and Corollary 69 below.

The proof of Lemma 22. Having a function $f$ with $\sum_x f(x) = 0$, we consider the matrix $M(g, x) := f(g^{-1}x)$ of size $|\SL_2(\mathbb{F}_p)| \times p$ and its singular value decomposition [23]

$$M(g, x) = \sum_{j=1}^p \lambda_j u_j(g) v_j(x).$$ (150)

One can assume that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p \geq 0$. Let $\vec{u} = (1, \ldots, 1)$ be the vector having $p$ ones. Since for any $y \in \mathbb{F}_p$ one has

$$(M^* M \vec{u})(y) = \sum_x \sum_g M(g, x) \overline{M(g, y)} = \sum_x \sum_g f(g^{-1}x) f(g^{-1}y) =$$

$$= \sum_g \sum_x f(g^{-1}y) f(g^{-1}x) = p^{-1} |\SL_2(\mathbb{F}_p)| \cdot \left| \sum_x f(x) \right|^2 = 0,$$

it follows that $\lambda_p = 0$. Here we have used the fact that for any $x \in \mathbb{F}_p$ the following holds $|\text{Stab}(x)| = p^{-1} |\SL_2(\mathbb{F}_p)|$. Further

$$\sum_{j=1}^{p-1} |\lambda_j|^2 = \sum_{j=1}^p |\lambda_j|^2 = \sum_g \sum_x |M(g, x)|^2 = \sum_g \sum_x |f(g^{-1}x)|^2 =$$

$$= |\SL_2(\mathbb{F}_p)| \cdot \sum_x |f(x)|^2 = (p^3 - p) \|f\|_2^2.$$ (151)

It is easy to check that if $\vec{v}(x) \in \mathbb{F}_p^p$ is an eigenvector of $T := M^* M$, then for any $g \in \SL_2(\mathbb{F}_p)$ the vector $\vec{v}(gx)$ is another eigenvector of $T$ and moreover $T g \vec{v} = g T \vec{v}$. Thus the following linear operator $Y_g$ defined by the formula $(Y_g h)(x) := h(gx)$, where $h$ belongs to any eigenspace for $M$, defines a representation because, obviously, $Y_{g_1} Y_{g_2} = Y_{g_1 g_2}$. By the famous Frobenius result [20] the dimension of all nontrivial irreducible representations of $\SL_2(\mathbb{F}_p)$ is at least $(p - 1)/2$. It follows that for any eigenfunction $\vec{v}$, $\vec{v} \neq \vec{u}$ the multiplicity of the correspondent eigenvalue is at least $(p - 1)/2$ (see details in [44], [10], [21]). Hence in view of (151), we obtain $\lambda_1 \leq 2p \|f\|_2$. Finally, by formula (150), the orthogonality of the systems of functions $u_j$ and $v_j$, the Hölder inequality, as well as our upper bound for $\lambda_1$, we have

$$\sum_x (F \ast f)(x) \varphi(x) = \sum_{g,x} M(g,x) F(g) \varphi(x) = \sum_{j=1}^{p-1} \lambda_j \langle F, \pi_j \rangle \langle \varphi, v_j \rangle \leq 2p \|F\|_2 \|\varphi\|_2 \|f\|_2.$$ 

This completes the proof. \(\square\)

The proof of Theorem 50. Clearly, one can assume that $K$ is sufficiently large because otherwise (115) is trivial. Put $s = |\SL_2(\mathbb{F}_p)|$ and write $f(x) = \mu(x) - s^{-1}$. Then $\sum_x f(x) = 0$,
From (153), we have that it follows that $K$ and we obtain a rough upper bound for $||f||_{1} \leq 1 + 1 = 2$ and by induction one can check that $(f \ast_{l} f)(x) = \mu_{l} \mu(x) - s^{-1}$, $l \in \mathbb{N}$. It gives that

$$T_{2l}(\mu) = \|\mu \ast_{2l} \mu\|_{2}^{2} = s^{-1} + \|f \ast_{2l} f\|_{2}^{2} = s^{-1} + T_{2l}(f).$$

So, our task is to estimate $\|f \ast_{2l} f\|_{2}^{2}$. Clearly, it is enough to prove the following bound

$$\|f \ast_{2l} f\|_{2}^{2} \ll \|f \ast_{l} f\|_{2}^{2} \cdot K^{-c_{r}} := \|f \ast_{l} f\|_{2}^{2} / M,$$  \hspace{2cm} (152)

where $c_{r} > 0$ is an absolute constant and $l \in \mathbb{N}$. Since

$$\|f \ast_{2l} f\|_{2}^{2} = \sum_{x}(f \ast_{2l} f)(xy)(f \ast_{2l} f)(x)(f \ast_{2l} f)(y),$$  \hspace{2cm} (153)

it follows that by an analogue of Lemma [22] (applied to the natural action of $\text{SL}_{2}(\mathbb{F}_{p})$ onto $\text{SL}_{2}(\mathbb{F}_{p})$, see details in [18], say) that

$$\|f \ast_{2l} f\|_{2}^{2} \leq 2p\|f \ast_{2l} f\|_{2} + \|f \ast_{l} f\|_{2}^{2} / 2$$

and this implies

$$\|f \ast_{2l} f\|_{2}^{2} \leq 4p^{2}\|f \ast_{l} f\|_{2}^{4}.$$  \hspace{2cm} (154)

Thus we can assume that

$$\|f \ast_{l} f\|_{2}^{2} \geq \frac{1}{4M^{2}}$$

because otherwise there is nothing to prove. Now put $r(x) = (f \ast_{l} f)(x)$. Then, clearly,

$$\|r\|_{\infty} \leq \|r\|_{1} = \sum_{x}\left| (\mu \ast_{l} \mu)(x) - \frac{1}{s} \right| \leq 1 + 1 = 2.$$

From (153), we have

$$T_{2l}(f) := \|f \ast_{2l} f\|_{2}^{2} = \sum_{x,y,z,w} r(x)r(y)r(z)r(w).$$

Put $\rho = T_{2l}(f)/(8\|r\|_{1}^{2})$. Since

$$\sum_{x,y,z,w : |r(x)| < \rho} r(x)r(y)r(z)r(w) \leq \rho\|r\|_{1}^{3} \leq T_{2l}(f)/8,$$

it follows that

$$\rho^{2j-1} \ll |r(x)| \leq \rho 2^{j}$$

Put $P_{j} = \{x \in \text{SL}_{2}(\mathbb{F}_{p}) : \rho 2^{j-1} < |r(x)| \leq \rho 2^{j}\}$ and $L = 8 + \log(2l) \cdot \log K$. By the assumption $k \leq K^{c_{r}}$ and hence $L \leq k \log K \leq K^{-c_{r}} \log K$. Since $\sum_{x} \mu(x) = 1$, it follows that $\|\mu\|_{\infty} \geq s^{-1}$ and we obtain a rough upper bound for $K$, namely, $K \leq s$. So, choosing $c_{r}$ to be sufficiently...
small, we can suppose that for sufficiently large \( p \) one has \( L < p^\varepsilon \) with a given \( \varepsilon > 0 \). Clearly, we can assume that \( T_{2k}(f) \geq K^{-k} \) and hence for any \( x \) such that \( |r(x)| \geq p \) one has

\[
2^j \cdot 2^{-4} K^{-\log 2 t} \|r\|^{-\frac{3}{2}} \leq 2^j \cdot 2^{-4} T_{2k}(f) \|r\|^{-\frac{3}{2}} = \rho 2^{j-1} < |r(x)| \leq \|r\|_\infty \leq 2.
\]

It follows that the possible number of the sets \( P_j \) does not exceed \( L \). Thus as in the proof of Theorem 2.5, we see that there is \( P = P_0, \Delta = 2^{j_0-1} \rho \) such that

\[
T_{2t}(f) \leq 2L^4(2\Delta)^4E(P).
\]

Clearly, \( \Delta^2|P| \leq T_t(f) \) and if \((152)\) does not hold, then it gives us

\[
M^{-1}T_t(f) \leq T_{2t}(f) \leq 2^5 L^4 \Delta^4 |P|^3 \leq 2^5 L^4 T_t(f) |P|.
\]

On the other hand, by the second assumption of the theorem and \( K \leq s \), we get

\[
T_t(f) = T_t(\mu) + s^{-1} \leq \|\mu\|_\triangleleft + s^{-1} \leq \|\mu\|_\infty + s^{-1} \leq 2K^{-1}.
\]

Hence, combining the last two bounds, we obtain

\[
|P| \geq \frac{K}{2^6LM}.
\]

Similarly, we have

\[
M^{-1}T_t(f) \leq T_{2t}(f) \leq 2^5 L^4 \Delta^4 |P|^3 \leq 2^5 L^4 T_t(f) (|P|\Delta)^2,
\]

and hence

\[
|P|\Delta \geq \frac{1}{2^3 L^2 M^{1/2}}.
\]

Also, we have

\[
|P|\Delta \leq \sum_{x \in P} |r(x)| \leq \sum_x |r(x)| = \|r\|_1.
\]

So, we see that if \((152)\) has no place, then

\[
T_t(f)/M \leq T_{2t}(f) \leq 2L^4(2\Delta)^4E(P) = 2^5 L^4 \Delta^4 |P|^3 (E(P)/|P|^3) \leq 2^5 L^4 (\Delta|P|)^2 T_t(f)(E(P)/|P|^3).
\]

In other words

\[
E(P) \geq |P|^3 \cdot 2^{-5} L^{-4} M^{-1} (\Delta|P|)^{-2} \geq 2^{-7} L^{-4} M^{-1} = \zeta|P|^3.
\]

By the Balog–Szemerédi–Gowers Theorem in non–commutative case (see, e.g., [56 Corollary 2.46]), we see that there is \( P_* \subseteq P, |P_*| \gg \zeta |P|, |P_*P_*^{-1}| \ll \zeta^{-C} |P_*|, \) where \( C > 0 \) is an absolute constant. The fact \( |P_*P_*^{-1}| \ll \zeta^{-C} |P_*| \) implies that there is a symmetric set \( H, |H| \ll \zeta^{-C'} |P_*| \), containing the identity, and a set \( |X| \ll \zeta^{-C'} \) such that \( P_* \subseteq XH \) and

\[
HH \subseteq XH \subseteq HXX, \quad \text{and} \quad HH \subseteq HX \subseteq XXH
\]
see [56] Proposition 2.43. Here $C' > 0$ is another absolute constant. Clearly, inclusions (158) combining with $|X| \ll \zeta^{-C'}$, imply

\[ |H^3| = |H \cdot H^2| \leq |H^2 \cdot X| \leq |H \cdot X^2| \leq |X|^2 |H| \ll \zeta^{-2C'} |H|. \tag{159} \]

Further since $P_\ast \subseteq XH$, we see that there is $x \in X$ such that

\[ |H| \geq |P_\ast \cap xH| \geq |P_\ast|/|X| \gg \zeta^{C'} |P_\ast|. \tag{160} \]

By the inclusion $P_\ast \subseteq P$ and the definition of the set $P$, we have

\[ \Delta \zeta^{C'} |P_\ast| \ll |\Delta| P_\ast \cap xH| \leq \sum_{z \in P_\ast \cap xH} |r(x)| \leq \mu(xH) + |P_\ast|/s \leq 2\mu(xH) \tag{161} \]

because otherwise in view of our choice of $\Delta \geq \rho$, we obtain

\[ T_{2L}(f)/(8\|r\|_1^3) = \rho \leq \Delta \ll \zeta^{-C'}/s \]

and hence by (154), we derive $(M$ is sufficiently small comparable to $p$ and $L < p^\ast$ for sufficiently small $\epsilon$, depending on $C'$ only)

\[ T_{2L}(f) \ll L^{4C'} M^{C'} p^{-3} \ll L^{4C'} M^{C' + 1} T_L(f)/p \leq T_L(f)/M \]

as required in (152). Thus we see that (161) takes place and whence by (157)

\[ \mu(xH) \gg \Delta \zeta^{C'} |P_\ast| \gg |\Delta| P_\ast \zeta^{C'} L^{-2} M^{-1/2} \gg L^{-4C''} M^{-C''}. \tag{162} \]

Finally,

\[ |H| \ll \zeta^{-C'} |P_\ast| \leq \zeta^{-C'} T_L(f)/\Delta^2 \leq \zeta^{-C'} T_L(f)/\rho^2 \ll \zeta^{-C'} T_L(f)/T_{2L}(f) \leq p^{5/2}, \tag{163} \]

say, because otherwise (just the last inequality should be explained) in view of (154) and sufficiently small $M$, say, $M \leq p^{1/12}$, we get

\[ T_{2L}(f) \ll \zeta^{-C'}/p \ll \zeta^{-C'/2} p^{-5/4} T_L^{1/2}(f) \ll T_L(f)/M. \]

Using (163), lower bound (156), estimate (160) and recalling (159), we see in view of Theorem 24 that either our set $H$ belongs to some proper subgroup $\Gamma$ or

\[ M^{2C'} L^{8C''} \gg \zeta^{-2C'} \gg |H|^c \gg (\zeta^{C'} |P_\ast|)^c \gg (\zeta^{C'+C} |P|)^c \gg K^c \zeta^{-C''}. \tag{164} \]

In the last inequality we have used lower bound (156). Now suppose that $H$ belongs to a subgroup. By our assumption $\mu(g\Gamma) \leq K^{-1}$ for any proper subgroup $\Gamma \subset \text{SL}_2(\mathbb{F}_p)$, $g \in \text{SL}_2(\mathbb{F}_p)$. Hence (162) gives us

\[ K^{-1} \geq \mu(x\Gamma) \geq \mu(xH) \gg L^{-4C''} M^{-C''}. \tag{165} \]

Finding $M$ satisfying both (164), (165) and using the assumption $k \leq K^{c'}$, we obtain the required dependence $M$ on $K$. This completes the proof. \hfill \Box

Using the same method of the proof (one can check that a non–commutative analogue of Lemma 15 takes place and also, see [50] Theorem 27 and Theorem 27 above), we obtain
Theorem 68 Let $\mu$ be a symmetric probability measure on $\SL_2(\mathbb{F}_p)$ such that for a parameter $K \geq 1$ one has
- $\mu(g\Gamma) \leq K^{-1}$ for any proper subgroup $\Gamma \subset \SL_2(\mathbb{F}_p)$, $g \in \SL_2(\mathbb{F}_p)$ and
- $\|\mu\|_{\infty} \leq K^{-1}$.
Put $f(x) = \mu(x) - |\SL_2(\mathbb{F}_p)|^{-1}$. Then there is $k \ll \frac{\log(\|\mu\|^{-1})}{\log K}$ such that $E_{2k}(f) \leq 2\|f\|_2^{2k+1}$.

Theorem 68 immediately implies (see the proof of bound (82) from Corollary 31).

Corollary 69 Let $\mu$ be a symmetric probability measure on $\SL_2(\mathbb{F}_p)$ such that for a parameter $K \geq 1$ one has
- $\mu(g\Gamma) \leq K^{-1}$ for any proper subgroup $\Gamma \subset \SL_2(\mathbb{F}_p)$, $g \in \SL_2(\mathbb{F}_p)$ and
- $\|\mu\|_{\infty} \leq K^{-1}$.
Put $f(x) = \mu(x) - |\SL_2(\mathbb{F}_p)|^{-1}$. Then there is $k \ll \frac{\log(\|\mu\|^{-1})}{\log K}$ such that for an arbitrary function $h : \SL_2(\mathbb{F}_p) \to \mathbb{C}$ and any set $A \subseteq \SL_2(\mathbb{F}_p)$ and a positive integer $s \leq 2^k$ one has
\[
\left| \sum_{x \in A} (h * f)^s(x) \right| \leq |A|^{\frac{s}{2}} \|h\|_2^s \|f\|_2^s \left( \frac{2E(A)}{|A|^4} \right)^{s/2k+2}. \tag{166}
\]

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