Differential Characters on Orbifolds and String Connections I.

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Abstract. In this paper we introduce the Cheeger-Simons cohomology of a global quotient orbifold. We prove that the Cheeger-Simons cohomology of the orbifold is isomorphic to its Beilinson-Deligne cohomology. Furthermore we construct a string connection (à la Segal) from a global gerbe with connection over the loop orbifold refining the corresponding differential character.

1. Introduction

In our previous papers we have introduced both the concept of \( n\)-gerbe \( L \) with connection over an orbifold \( X \), and the definition of orbifold Beilinson-Deligne cohomology. Our original motivation was the understanding of the concepts of \( B \)-field and discrete torsion in orbifold string theories [11, 13, 14] (cf. [1, 17, 2, 19, 21]). We showed that a \( B \)-field in mathematical terminology is the same as a 1-gerbe with connection over an orbifold (at the low energy limit, of course) and discrete torsion is a particular kind of flat \( B \)-field. String theory thus motivates the consideration of gerbes with connection over orbifolds. But the impact of considering gerbes goes beyond string theory. For example Y. Ruan [18] has recently implemented the usage of gerbes and string connections in order to obtain the twisted version of the Chen-Ruan quantum cohomology of an orbifold.

Given a smooth manifold \( M \) and unitary line bundle with connection \((L, A)\) over \( M \), we can consider consider its holonomy as a map

\[
\text{hol}: Z_1(M) \to U(1).
\]

We define \( \chi \) to be

\[
\chi := -\frac{1}{2\pi} \log \text{hol}.
\]

If we consider the curvature of \( L \) as a 2-form \( \omega \) on \( M \) we have obtained a pair \((\chi, \omega)\) with

\[
\chi: Z_1(M) \to \mathbb{R}/\mathbb{Z}
\]

and

\[
\chi(\partial c) = \int_c \omega \mod \mathbb{Z}
\]

whenever \( c \) is a smooth 2-chain.

Following Cheeger-Simons [7] we will denote by \( \hat{H}_\chi^q(M) \) the \( q^{th} \)-group of differential characters of \( M \). The previous discussion with line bundles refers only to the case \( q = 2 \). In the general case we have to substitute the line bundle by

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a \((q-2)\)-gerbe with connection. The holonomy becomes now a homomorphism
\(Z_{q-1}(M) \to \mathbb{R}\).

Isomorphism classes of gerbes with connection are classified by the so-called
Beilinson-Deligne cohomology \([4, 14]\), denoted by \(H^q(X; \mathbb{Z}(q))\). It is a remarkable
theorem the one that states that
\[ H^q(X; \mathbb{Z}(q)) \cong \check{H}^q_{cs}(M). \]

For this is stating that the holonomy and curvature of a gerbe completely determine
its isomorphism class.

In section 2 of this paper we generalize the previous picture to the case of an
orbifold of the form \(X = [M/G]\) (the previous manifold case being when \(G = 1\)).
One of the main difficulties is that of defining what we mean for a differential
character on an orbifold. This is not completely obvious and there would be several
possibilities for the definition. We offer one that allows a generalization of the
main results to the orbifold setting. Our main construction actually works fine
for orbifolds resulting for a Lie group acting smoothly on a manifold with finite
stabilizers.

In any case our main result is then

**Theorem 1.0.1.** For the orbifold \(X = [M/G]\), the Beilinson-Deligne cohomology
and the Cheeger-Simons cohomology are canonically isomorphic.

In section 3 below we further refine this theorem. A gerbe with a connection
contains more information that its isomorphism class, and therefore more information
that just its differential character. The refinement we put forward in this paper is
a string connection associated to the gerbe with connection over the orbifold. Here
we mean a string connection in the sense of Segal \([20]\) and not in the closely related
sense of Stolz-Teichner. In the manifold case \((G = 1)\) this can be encoded by a
functor from the category \(S^1(M)\), whose objects are maps from \((p-1)\)-manifolds
to \(M\) and whose morphisms are maps of \(p\)-cobordisms to \(M\), to the category of
one-dimensional vector spaces.

For example in the case of the line bundle the objects of \(S^1(M)\) are points in \(M\)
and its morphisms are paths between those points. The holonomy of a line bundle
affords us a functor from \(S^1(M)\) to one-dimensional vector spaces. The vector space
associated to a point is simply the fiber of the line bundle at that point.

Going up one level a gerbe with connection produces via transgression a line
bundle over the loop space of the manifold, and additionally a functor from \(S^2(M)\)
to one-dimensional vector spaces by holonomy \([6, 12]\). In \([12]\) we provided a general-
ization of the loop space for an orbifold called the loop groupoid \(LX\).

The second main result of this paper is the following.

**Theorem 1.0.2.** Let \(\xi\) be a global gerbe with connection over \(X = [M/G]\) and \(E\)
the line bundle with connection induced by it via transgression. Then \(\xi\) permits to
define a string connection \(U\) over the line bundle \(E\) of the loop groupoid \(LX\). This
string connection refines the corresponding differential character.

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birthday.
2. Differential Characters

In [12] we have put forward the definition of an $n$-gerbe with connective structure over an orbifold (i.e. étale, proper foliation groupoid). We define them there as cocycles of a cohomology theory called the Beilinson-Deligne (BD) cohomology. In this section we will define the orbifold version of the Cheeger-Simons (CS) cohomology [7] and then we will show that these two theories are isomorphic.

For smooth manifolds and certain algebraic varieties the isomorphism of this section has been proved by several authors [5, 8, 9]. Here we are concerned with the orbifold case and so, modifying Brylinski’s definition [4] of BD-cohomology and Hopkins-Singer’s [10] of CS-cohomology, we will work out this theorem in the equivariant case, in which we have group actions.

2.1. Cheeger-Simons Cohomology. To make the exposition manageable we will start by considering orbifolds\(^1\) of the type $X = [M/G]$ where $M$ is a finite dimensional, paracompact, smooth manifold with a smooth action of a finite group $G$. In this paragraph we will define the $G$-invariant CS-cohomology and in the next section 2.1.2 we will concentrate on the equivariant CS-cohomology.

2.1.1. The case of a finite group $G$. We want to define a cohomology theory of orbifolds $[M/G]$, which encodes the notion of a $G$-invariant $q$-form with integral periods. If $M_G := M \times_G EG$ and $\Omega^q_G(M)^G$ stands for the $G$-invariant closed $q$-forms over $M$, we want to construct a theory that naturally fits in the upper-left corner of the diagram

\[
\begin{array}{ccc}
\Omega^q_G(M)^G & \rightarrow & H^q(M_G; \mathbb{Z}) \\
\downarrow & & \downarrow \\
H^q(M_G; \mathbb{R}) & \rightarrow & H^q(M_G; \mathbb{R}).
\end{array}
\]

We will do this via the complex of smooth cochains. When $G = \{1\}$ this recovers the classical case due to Cheeger and Simons. We will follow closely Hopkins and Singer’s alternative definition [10 Sect. 3.2].

Let the complex $\hat{C}(q)^n(X)$ be given by

\[
\hat{C}(q)^n(X) = \begin{cases} 
C^n(M \times EG; \mathbb{Z})^G \times C^{n-1}(M \times EG; \mathbb{R})^G \times \Omega^q(M)^G & n \geq q \\
C^{n-1}(M \times EG; \mathbb{Z})^G \times C^{n-2}(M \times EG; \mathbb{R})^G & n < q,
\end{cases}
\]

with differential

\[
d(c, h, \omega) := (\delta c, \omega - c - \delta h, d\omega)
\]

\[
d(c, h) = \begin{cases} 
(\delta c, c - \delta h, 0) & (c, h) \in \hat{C}(q)^{n-1}(X) \\
(\delta c, -c - \delta h) & \text{otherwise}
\end{cases}
\]

An $n$-form $\omega \in \Omega^n(M)$ defines a smooth $n$-cochain in $C^n(M; \mathbb{R})$ through integration; we will use the same symbol for the form and the associated smooth cochain.

\(^1\)We should not confuse the orbifold $X = [M/G]$ with the quotient space $X = M/G$, we will remember all along the orbifold structure, including the stabilizers of the $G$-action on $M$.\]
Pulling the cochain back, via the projection map $M \times EG \to M$, we get the $n$-cochain in $C^n(M \times EG; \mathbb{R})$ that we have called $\tilde{\omega}$; as $G$ is finite $\tilde{\omega}$ defines a cochain in $C^n(M \times EG; \mathbb{R})^G$. By $C^n(M \times EG)^G$ we mean the $G$-invariant cochains\footnote{This construction cannot be generalized to the case when $G$ is a general Lie group. For when $G$ is a finite group $H^*(C^*(M \times EG; \mathbb{Z})^G) \cong H^*(C^*(M \times GEG; \mathbb{Z}))$, and we need the finiteness condition in order to have a pushforward map at the cochain level $C^n(M \times EG)^G \to C^n(M \times GEG)$.}.

**Definition 2.1.1.** The cohomology of the complex $\hat{C}(q)^n(X)$ is the Cheeger-Simons cohomology of $X$, i.e

$$\hat{H}(q)^n(X) := H(\hat{C}(q)^n(X))$$

The homotopy Cartesian square

$$\begin{CD}
\hat{C}(q)^n(X) @>>> \Omega^{*\geq q}(M)^G \\
\downarrow @. \downarrow \\
C^n(M \times EG; \mathbb{Z})^G @>>> C^n(M \times EG; \mathbb{R})^G
\end{CD}$$

yields a Mayer-Vietoris sequence

$$\cdots \to \hat{H}(q)^n(X) \to H^n(M_G; \mathbb{Z}) \times H^n(\Omega^{*\geq q}(M)^G) \to H^n(M_G; \mathbb{R}) \to \hat{H}(q)^{n+1}(X) \to \cdots$$

that leads to natural isomorphisms

$$\hat{H}(q)^n(X) = \begin{cases} H^n(M_G; \mathbb{Z}) & n > q \\ H^{n-1}(M_G; \mathbb{R}/\mathbb{Z}) & n < q, \end{cases}$$

and a short exact sequence

$$(2.1.1) \quad 0 \to H^{q-1}(M_G; \mathbb{R}/\mathbb{Z}) \to \hat{H}(q)^q(X) \to \Omega^q_0(M)^G \to 0.$$
Definition 2.1.3. An equivariant differential q-form on \( M_G \) is a sequence \( \{ \omega_k \}_{k \in \mathbb{N}} \) of differential q-forms \( \omega_k \in \Omega^q(M^k_G) \) such that \( \rho_{k,k'} \omega_{k'} = \omega_k \) with \( k' \geq k \) and \( \rho_{k,k'} \) is the inclusion of manifolds \( M^k_G \hookrightarrow M^{k'}_G \). The vector space of such forms will be called \( \Omega^n_G(M) \).

As the exterior derivative commutes with pullbacks, in our case \( d(\rho_{k,k'} \omega_{k'}) = \rho_{k,k'} (d \omega_{k'}) \), then it makes sense to define the exterior derivative of \( \omega \) as the sequence \( d \omega = \{ d \omega_k \}_{k \in \mathbb{N}} \). Let

\[
\cdots \to \Omega^{q-1}_G(M) \xrightarrow{d} \Omega^n_G(M) \xrightarrow{d} \Omega^{q+1}_G(M) \xrightarrow{d} \cdots
\]

be the equivariant De Rham complex of \( M \). Its cohomology \( \check{H}^*(\Omega^n_G(M)) \) is the equivariant De Rham cohomology of \( M \), \( \check{H}^*_{DR,G}(M) \). It is well known [3] that there is a canonical isomorphism

\[
\check{H}^*_{DR,G}(M) \cong H^*_G(M; \mathbb{R})
\]

where \( H^*_G(M; \mathbb{R}) := H^*(M_G; \mathbb{R}) \) is the equivariant cohomology of \( M \).

We will write \( C^G_k(M; R) \) (\( C^k_G(M; \mathbb{Z}) \)) for the \( k \)-chains (rep. \( k \)-cochains) on \( M_G \) with coefficients in the ring \( R \). Define the equivariant Cheeger-Simons complex \( \check{C}(q)_G^*(M) \) by

\[
\check{C}(q)_G^*(M) = \left\{ \begin{array}{ll}
C^n_G(M; \mathbb{Z}) \times C^{n-1}_G(M; \mathbb{R}) \times \Omega^n_G(M) & n \geq q \\
C^n_G(M; \mathbb{Z}) \times C^{n-2}_G(M; \mathbb{R}) & n < q,
\end{array} \right.
\]

with differential

\[
d(c, h, \omega) := (\delta c, \bar{\omega} - c - \delta h, d \omega)
\]

\[
d(c, h) = \left\{ \begin{array}{ll}
(\delta c, -c - \delta h, 0) & (c, h) \in \check{C}(q)_G^{q-1}(M) \\
(\delta c, -c - \delta h) & \text{otherwise}
\end{array} \right.
\]

An \( n \)-form \( \omega \in \Omega^n_G(M) \) defines a smooth \( n \)-cochain \( \hat{\omega} \) in the following sense. For \( S \in C^n_G(M; \mathbb{R}) \) a \( n \)-chain there exist \( k \in \mathbb{N} \) such that the image of \( S \) in \( M_G \) is included in the subspace \( M^k_G \). Define

\[
\hat{\omega}(S) := \int_S \omega_k.
\]

It is clear from the definition that \( \hat{\omega} \) is independent of \( k \).

Definition 2.1.4. The cohomology of the complex \( \check{C}(q)_G^*(M) \) is the equivariant Cheeger-Simons cohomology of \( M \), i.e

\[
\check{H}^*_G(M) := H^*_G(\check{C}(q)_G^*(M))
\]

The homotopy Cartesian square

\[
\begin{array}{ccc}
\check{C}(q)_G^*(M) & \longrightarrow & \Omega^n_G(M) \\
\downarrow & & \downarrow \\
C^q_G(M; \mathbb{Z}) & \longrightarrow & C^q_G(M; \mathbb{R})
\end{array}
\]

yields a Mayer-Vietoris sequence

\[
\cdots \to \check{H}^n_G(M) \to H^*_G(M; \mathbb{Z}) \times H^n(\Omega^{\geq q}_G(M)) \to \\
H^n_G(M; \mathbb{R}) \to \check{H}^{n+1}_G(M) \to \cdots
\]
that leads to natural isomorphisms
\[ \mathcal{H}(q)_G^\alpha(M) = \begin{cases} H^n_G(M; \mathbb{Z}) & n > q \\ H^{n-1}_G(M; \mathbb{R}/\mathbb{Z}) & n < q, \end{cases} \]
and a short exact sequence
\[ (2.1.2) \quad 0 \to H^{q-1}_G(M; \mathbb{R}/\mathbb{Z}) \to \mathcal{H}(q)_G^\alpha(M) \xrightarrow{\alpha} \Omega^q_{G,0}(M) \to 0. \]
where \( \Omega^q_{G,0}(M) \) stands for the equivariant closed \( j \)-forms with integer periods, (a form \( \omega \) has integer periods if \( \hat{\omega}(S) \in \mathbb{Z} \) whenever \( \partial S = 0 \)).

Note that when the group \( G \) is finite any \( G \)-invariant differential form \( \omega \) over \( M \) pulls back to an equivariant differential form \( \omega \) with
\[ \omega_k := \frac{1}{|G|} \sum_{g \in G} g^*(\pi_k^*\omega) \]
via the projection \( \pi_k: M \times V_{k+n,n} \to M \). It is clear that the map \( \gamma: \Omega^q(M)^G \to \Omega^q_G(M) \) is injective. So if we consider the image of \( \gamma(\Omega^q_0(M)^G) \) in \( \Omega^q_{G,0}(M) \) an then we consider the inverse image of this set via the map \( \alpha \) of the short exact sequence
\[ (2.1.3) \quad 0 \to H^{q-1}_G(M; \mathbb{R}/\mathbb{Z}) \to \alpha^{-1}(\gamma(\Omega^q_0(M)^G)) \xrightarrow{\alpha} \Omega^q_0(M)^G \to 0. \]
So, in view of \([2.1.2]\) and \([2.1.3]\) we get

**Lemma 2.1.5.** The CS-cohomology of the orbifold \([M/G]\) injects in the \( G \)-equivariant CS-cohomology of \( M \). Moreover
\[ \mathcal{H}(q)^\alpha([M/G]) \cong \alpha^{-1}(\gamma(\Omega^q_0(M)^G)). \]
Therefore the CS-cohomology of \([M/G]\) consist of the classes in the \( G \)-equivariant CS-cohomology of \( M \) that are obtained via a \( G \)-invariant closed differential forms over \( M \) with integer periods.

The previous description allows us to go one step further to define the **equivariant differential characters** of \( M \) generalizing the original construction of Cheeger and Simons [7].

**Definition 2.1.6.** A \( G \) equivariant differential character of \( M \) of degree \( q \) consists of a pair \( (\chi, \omega) \) with
\[ \chi: Z_{q-1}(M_G; \mathbb{Z}) \to \mathbb{R}/\mathbb{Z} \]
a character defined over the group of \( q-1 \)-cycles, and \( \omega \in \Omega^q_G(M) \) an equivariant differential \( q \)-form such that for every smooth \( q \)-chain \( S \in C_q(M_G; \mathbb{R}) \)
\[ \chi(\partial S) = \hat{\omega}(S). \]
We will denote by \( \mathcal{H}^q_{cs,G}(M) \) the \( q \)th equivariant group of CS differential characters of \( M \).

As indicated in [10] the map
\[ (2.1.4) \quad \mathcal{H}^q_{G}(M) \xrightarrow{\chi, \omega} \mathcal{H}^q_{cs,G}(M) \]
where \( \chi(z) := h(z) \mod \mathbb{Z} \), is an isomorphism.

Following the spirit of Proposition \([2.1.4]\) we can define the **differential characters** of \([M/G]\) when \( G \) is a finite group, as the equivariant characters \( (\chi, \omega) \) such...
that $\omega$ belongs to the image of some $G$-invariant closed form with integer periods $\omega \in \Omega^q(M)^G$ under the map $\gamma$.

**Definition 2.1.7.** The $q^{th}$ group of differential characters over the orbifold $[M/G]$ will be denoted by $\hat{H}^q([M/G])$.

As before we get an isomorphism

\[
\hat{H}(q)((M/G)) \cong \hat{H}^q_c([M/G])
\]

**2.2. Beilinson-Deligne Cohomology.** BD-cohomology was discovered by Beilinson and Deligne for the purpose of having a cohomology theory for algebraic varieties which includes singular cohomology and the intermediate Jacobians of Griffiths. We will deal with a smooth analog of this theory.

Recall that for a $X$-sheaf, where $X = [M/G]$, we mean a sheaf $F$ over $M$ on which $G$ acts continuously. If $F$ is abelian, the cohomology groups $H^n(X; F)$ are defined as the cohomology groups of the complex

\[
\Gamma(M, T^0) \to \Gamma(M, T^1) \to \cdots
\]

where $F \to T^0 \to T^1 \to \cdots$ is a resolution of $F$ by injective $X$ sheaves and $\Gamma(M, T^j)^G$ are the $G$-invariant sections. When the abelian sheaf $F$ is locally constant (for example $F = \mathbb{Z}$) is a result of Moerdijk [15] that $H^*(X; F) \cong H^*(BX; F)$ where the left hand side is sheaf cohomology and the right hand side is simplicial cohomology of $BX \simeq M_G$ with coefficients in $F$.

Let $A^p_X$ denote the $X$-sheaf of differential $p$-forms and $Z_X$ the constant $\mathbb{Z}$ valued $X$ sheaf with $Z_X \to A^0_X$ the natural inclusion of constant into smooth functions.

**Definition 2.2.1.** The smooth BD complex $Z(q)$ is the complex of $X$ sheaves

\[
Z_X \to A^0_X \xrightarrow{d} A^1_X \xrightarrow{d} \cdots \xrightarrow{d} A^{q-1}_X,
\]

and the hypercohomology groups $H^*(X, Z(q))$ are called the **smooth Beilinson-Deligne cohomology** of $X$.

Now, let $U(1)(q)$ be the complex of sheaves

\[
U(1)(q) \xrightarrow{\sqrt{-1}d\log} A^0_X \xrightarrow{d} A^1_X \xrightarrow{d} \cdots \xrightarrow{d} A^{q-1}_X
\]

where $U(1)_X$ is the sheaf of $U(1)$-valued functions. Because of the quasi-isomorphism between $Z(q)$ and $U(1)(q)[−1]$, i.e.

\[
(2.2.1) \quad \begin{array}{cccccccc}
\mathbb{Z}(p)_X & \xrightarrow{d} & A^0_X & \xrightarrow{d} & A^1_X & \xrightarrow{d} & \cdots & \xrightarrow{d} & A^{q-1}_X \\
\exp(-i\theta) & & & & & & & & \\
U(1)(q) & \xrightarrow{\sqrt{-1}d\log} & A^0_X & \xrightarrow{d} & A^1_X & \xrightarrow{d} & \cdots & \xrightarrow{d} & A^{q-1}_X
\end{array}
\]

there is an isomorphism of hypercohomologies

\[
H^{n-1}(X, U(1)(q)) \cong H^n(X, \mathbb{Z}(p)).
\]

We need to use a more computational approach to this cohomology theory, basically because we will be using 3-cocycles in order to define a string connection, and so we will use a Čech description of the BD-cohomology. In order to make the exposition less lengthy, we are going to make use of some results that can be found
in our previous paper \cite{12}. As $M$ is paracompact, for the orbifold $X = [M/G]$ (or better, the proper étale foliation groupoid with objects $X_0 = M$ and morphisms $X_1 = M \times G$) we can find a smooth étale Leray groupoid $G$ together with a Morita map $G \rightarrow X$, making $G$ and $X$ Morita equivalent. Being Leray means that the spaces $G_n$ of $n$-composable morphisms of $G$ are diffeomorphic to a disjoint union of contractible open sets. In the case when $G = \{1\}$ (i.e. $X = M$) this amounts to finding a contractible open cover of $M$ such that all the finite intersections of this cover are either contractible or empty and then making $G_n$ to be the disjoint union of all intersections of $n$ sets in the cover.

Let’s denote by $\check{C}^\ast(G; U(1)(q))$ the total complex

$$
\check{C}^0(G; U(1)(q)) \xrightarrow{\delta - d} \check{C}^1(G; U(1)(q)) \xrightarrow{\delta + d} \check{C}^2(G; U(1)(q)) \xrightarrow{\delta - d} \cdots
$$

induced by the double complex

$$(2.2.3) : 
\begin{array}{cccc}
\delta & \delta & \cdots & \delta \\
\downarrow & \downarrow & \cdots & \downarrow \\
\Gamma(G_2, U(1)_G) & \Gamma(G_2, A^1_G) & \cdots & \Gamma(G_2, A^{q-1}_G) \\
\downarrow & \downarrow & \cdots & \downarrow \\
\Gamma(G_1, U(1)_G) & \Gamma(G_1, A^1_G) & \cdots & \Gamma(G_1, A^{q-1}_G) \\
\downarrow & \downarrow & \cdots & \downarrow \\
\Gamma(G_0, U(1)_G) & \Gamma(G_0, A^1_G) & \cdots & \Gamma(G_0, A^{q-1}_G)
\end{array}
$$

with $(\delta + (-1)^i d)$ as coboundary operator, where the $\delta$’s are the maps induced simplicial structure of the nerve of the category $G$ and $\Gamma(G_i, A^1_G)$ stands for the global sections of the sheaf that induces $A^1_G$ over $G_i$ (see \cite{12}). Then the Čech hypercohomology of the complex of sheaves $U(1)(q)$ is defined as the cohomology of the Čech complex $\check{C}(G; U(1)(q))$:

$$
\check{H}^\ast(G; U(1)(q)) := H^\ast(\check{C}(G; U(1)(q))).
$$

As the $G_i$’s are diffeomorphic to a disjoint union of contractible sets – Leray – then the previous cohomology actually matches the hypercohomology of the complex $U(1)(q)$, so we get

**Lemma 2.2.2.** The cohomology of the Čech complex $\check{C}^\ast(G, U(1)(q))$ is isomorphic to the hypercohomology of the complex of sheaves $U(1)(q)$ and as $G \rightarrow X$ are isomorphic, then

$$
\check{H}^\ast(G, U(1)(q)) \cong H^\ast(G; U(1)(q)) \cong H^\ast(X; U(1)(q)).
$$

As we are only interested in the case $X = [M/G]$ we can make a more explicit description of the Leray groupoid $G$. Take a contractible open cover $\{U_i\}_{i \in I}$ of $M$ such that all the finite intersections of the cover are either contractible or empty, and with the property that for any $g \in G$ and any $i \in I$ there exists $j \in I$ so that $U_i g = U_j$. Define $G_0$ as the disjoint union of the $U_i$’s with $G_0 \xrightarrow{\rho} M = X_0$ the
natural map. Take $G_1$ as the pullback square

$$
\begin{array}{ccc}
G_1 & \longrightarrow & M \times G \\
\downarrow & & \downarrow \text{sxt} \\
G_0 \times G_0 & \overset{\rho \times \rho}{\longrightarrow} & M \times M
\end{array}
$$

where $s(m, g) = m$ and $t(m, g) = mg$. This defines the proper étale Leray groupoid $G$ and by definition it is Morita equivalent to $X$.

**Lemma 2.2.3.** There is a natural short exact sequence

\[
(2.2.4) \quad 0 \to \tilde{H}^{q-1}(G; \mathbb{R}/\mathbb{Z}) \xrightarrow{\sigma} H^{q-1}(G; U(1)(q)) \xrightarrow{\delta} \Omega^q_0(M)^G \to 0.
\]

**Proof.** The map $\sigma$ is obtained by the inclusion of the locally constant $\mathbb{R}/\mathbb{Z}$-valued $G$-sheaf into $U(1)_G$. It follows that $j$ is injective. Now let’s consider an element $[f] \in \tilde{H}^{q-1}(G; U(1)(q))$. It will consist of the $q$-tuple $(\theta_0, \ldots, \theta_{q-1})$ with $\theta_0 \in \Gamma(G_{q-1}, U(1)_G)$ and $\theta_i \in \Gamma(G_{q-1-i}, A^*_q)$ that satisfies the cocycle condition $d\theta_i + (-1)^{q-1}\delta\theta_{i+1} = 0$.

From the construction of $G$ we see that we can think of $G_1$ as the disjoint union of all the intersections of two sets on the base times the group $G$, i.e.

$$
G_1 = \bigcup_{(i,j) \in I \times I} U_i \cap U_j \times G
$$

where the arrows in $U_i \cap U_j \times \{g\}$ start in $U_i|v_j$ and end in $(U_j|v_i)g$.

By the cocycle condition we know that

$$
g^*\theta_{q-1}|_{U_{ij}g} - \theta_{q-1}|_{U_i} = d\theta_{q-2}|_{U_{ij} \times \{g\}} \quad \text{in} \quad U_{ij}
$$

where $U_{ij} = U_i \cap U_j$. So if we define $q$-forms $\omega_i$ locally by $\omega_i := d\theta_i|_{U_i}$ is easy to see that once all are glued together they will induce a global $q$-form $\omega$ over $M$ which is $G$-invariant. The globality is obtained by taking $g = 1$ and noting that $\omega_i$ and $\omega_j$ agree in the intersection and the invariance is easily seen by taking $i = j$.

As $\omega$ is defined locally by exact forms then it follows that $\omega$ is exact. We define $\kappa([f]) := \omega$; it is well defined because if $f' = (\theta_0', \ldots, \theta_{q-1}')$ is cohomologous to $f$ then $\theta_{q-1}' - \theta_{q-1}$ is exact, so $f$ and $f'$ define the same $q$-form.

We are now left to prove that $\kappa$ is surjective and that $\ker(\kappa) \subset \text{Im}(\sigma)$. We will do so by looking at the double complexes used in the proof of the De Rham theorem and at the one by the Čech description of the complex of sheaves $\mathbb{Z}(q)$. Recall that if $\mathbb{R}_G$ is the $G$-sheaf of locally constant $\mathbb{R}$-valued functions then we know that the complex $22$

$$
\mathcal{A}_G^0 \xrightarrow{\partial} \mathcal{A}_G^1 \xrightarrow{\partial} \cdots
$$

is a resolution of injective sheaves.

If we have a BD class $[\theta_0, \ldots, \theta_{q-1}]$ as before, such that its image under $\kappa$ is zero, i.e $\omega = 0$, then the $q-1$-form given by $\phi_{q-1}$ is closed. As the groupoid is Leray, by a successive application of the Poincaré lemma, we can find a chain $(\alpha_0, \ldots, \alpha_{q-2}) \in \tilde{C}^{q-2}(G; U(1)(q))$ such that

$$
(\theta_0, \ldots, \phi_{q-1}) + (d + (-1)^{q-2}\delta)(\alpha_0, \ldots, \alpha_{q-2}) = (\theta_0', 0, \ldots, 0).
$$
Then $\theta_0^q$ is locally constant (because $d \log \theta_0^q = 0$) and $\delta \theta_0^q = 1$, so it defines a Čech cocycle with values in the $\mathbb{R}/\mathbb{Z} \times \mathbb{G}$ sheaf. This implies that the kernel of $\kappa$ is included in the image of $\sigma$.

Now, a $G$-invariant $q$-form with integer periods $\omega$, via the De Rham theorem, defines forms $\phi_i \in \Gamma(H_1 \mathfrak{a}_G^\circ, \mathfrak{a}_G^\circ)$ and a cocycle in $c \in \Gamma(H_1 \mathfrak{g}, \mathbb{R}_G)$ such that $d\phi_i + (-1)^{q-1} \delta \phi_{i+1} = 0$, $\delta \phi_0 + (-1)^{q-1} c = 0$, and $\delta c = 0$ (here we are making use of the quasi-isomorphism of $\mathfrak{a}_G^\circ$).

As $\omega$ has integer periods then there exist $c' \in \Gamma(H_1 \mathfrak{g}, \mathbb{Z}_G)$ and $h \in \Gamma(H_1 \mathfrak{g}, \mathbb{R}_G)$ such that $c' = \delta h + c$, then $(c, \phi_0 + (-1)^{q-2} h, \phi_1, \ldots, \phi_{q-1})$ is a BD cocycle for the complex of $\mathbb{G}$ sheaves $\mathbb{Z}(\mathfrak{g})$. Its BD-cohomology class under the map $\kappa$ is $\omega$. So $\kappa$ is surjective.

The sequence is short exact.

As the groupoids $\mathfrak{g}$ and $\mathfrak{X}$ are Morita equivalent\(^3\) then we have the short exact sequence

\[(2.2.5) \quad 0 \rightarrow H^{q-1}(\mathfrak{X}; \mathbb{R}/\mathbb{Z}) \rightarrow H^q(\mathfrak{X}; \mathbb{Z}(\mathfrak{g})) \rightarrow \Omega^q_0(\mathfrak{M})^G \rightarrow 0.
\]

**Theorem 2.2.4.** For the orbifold $\mathfrak{X} = [M/G]$, with $G$ a finite group, the Beilinson-Deligne cohomology and the Cheeger-Simons cohomology are canonically isomorphic.

**Proof.** In view of the short exact sequences 2.1.1 and 2.2.6 is just a matter of constructing a map from $H^q([M/G]; \mathbb{Z}(\mathfrak{g}))$ to $H^q([M/G])$. This turns out to be somewhat subtle in the case of orbifolds and it is actually given by the string connection described in the next chapter. For now we can bypass this by a careful use of our definition of equivariant CS-cohomology.

What we will actually do is to define a map from $H^q([M/G]; \mathbb{Z}(\mathfrak{g}))$ to the group of differential characters of $[M/G]$, namely $\check{H}^q_\mathfrak{g}([M/G])$ (see 2.1.5). It consists of pairs $(\chi, \gamma(\omega))$ with $\omega \in \Omega^q_0(\mathfrak{M})^G$ and

\[\chi: Z_{q-1}(M_G; \mathbb{Z}) \rightarrow \mathbb{R}/\mathbb{Z}\]

such that for any smooth $q$ chain $z$ we have

\[\chi(\partial z) = \hat{\gamma}(\omega)(z) \mod \mathbb{Z}.\]

For an element $[\xi] \in H^q([M/G]; \mathbb{Z}(\mathfrak{g}))$ via 2.2.5 we obtain a form $\omega \in \Omega^q_0(\mathfrak{M})^G$. So we only need to define $\chi$ and we will do so by defining its value on the $(q-1)$-dimensional, compact boundaryless submanifolds of $M_G$. Then let $\Sigma$ be a compact $(q-1)$-dimensional smooth manifold without boundary and $\phi: \Sigma \rightarrow M_G$ a smooth map. Then there exist $k \in \mathbb{N}$ such that $\phi: \Sigma \rightarrow M_G^k$. As the group $G$ is finite we can pullback the $G$-bundle $\pi_k: M \times V_{k+n,n} \rightarrow M \times G V_{k+n,n}$ via $\phi$ and we call such $G$-bundle $P$, i.e. $P$ is a $G$-bundle over $\Sigma$ and is the pullback square of the following diagram

\[
\begin{array}{ccc}
P & \rightarrow & M \times V_{k+n,n} \\
\downarrow & & \downarrow \pi_k \\
\Sigma & \phi & M \times G V_{k+n,n}.
\end{array}
\]

---

\(^3\)Two such groupoids are Morita equivalent if and only if they represent the same orbifold (see for example [18]).
Composing $\phi$ with the projection map $pr: M \times V_{k+n,n} \to M$ we obtain a $G$-equivariant map $\hat{\phi} := pr \circ \phi: P \to M$. Note that this map can also be seen as a map of orbifolds $\hat{\phi}: [P/G] \to [M/G]$. Pulling back $[\xi]$ via $\hat{\phi}$ we obtain a class $[\hat{\phi}^* \xi]$ in $H^q([P/G]; \mathbb{Z}(q))$. As $\Omega^q(P)^G = \Omega^q(\Sigma) = \{0\}$ then the class $[\hat{\phi}^* \xi]$ is isomorphic to a class $[\rho] \in H^{q-1}([P/G]; \mathbb{Z}/\mathbb{Z}) = H^q(\Sigma; \mathbb{Z}/\mathbb{Z})$. Define $\chi(\Sigma) := \rho(\Sigma)$.

It is well defined because if $\rho' = \rho + \delta \kappa$ with $\kappa \in C^{q-2}(\Sigma; \mathbb{R})$, as $\partial \Sigma = 0$ then $\rho'(\Sigma) = \rho(\Sigma) + \delta(\kappa)(\Sigma) = \rho(\Sigma) + \kappa(\partial \Sigma) = \rho(\Sigma)$.

Taking $[\xi] \mapsto (\chi, \gamma(\omega))$ we get a map

$$H^q([M/G]; \mathbb{Z}(q)) \to \hat{H}^q_{cs}([M/G])$$

that commutes with the short exact sequences $2.1.1$ and $2.2.5$. By the 5-lemma the result follows.  

3. String Connections

In this section we will focus our attention to gerbes with connection over the orbifold $[M/G]$ although most of the constructions could be generalized to $n$-gerbes with connection.

In order to make the exposition clearer let us introduce this section by explaining the idea of a string connection in the case of a manifold $M$ without any group action.

A gerbe with connection over $M$ induces a line bundle $E$ with connection over the free loop space $LM$ of $M$ via a transgression map. The connection over $LM$ allows one to do parallel transport over a path $\gamma: [0,1] \to LM$ defining an invertible linear operator $A_\gamma$ between the fibers $E_{\gamma(0)}$ and $E_{\gamma(1)}$. This path $\gamma$ in the loop space could be seen also as a 2 dimensional submanifold of $M$ of genus zero with boundary components the loops $\gamma(0)$ and $\gamma(1)$. But we would like to do more than just being able to do parallel transport over a tube, we would like to do a more general transport through an embedded oriented Riemann surface with boundary in $M$. This will give us a transport operator

$$U_\Sigma: E_{\gamma_1} \otimes \cdots \otimes E_{\gamma_p} \to E_{\gamma_{p+1}} \otimes \cdots \otimes E_{\gamma_{p+q}}$$

to each smooth surface $\Sigma$ in $M$ which has $p$ incoming parametrized boundary circles $\gamma_1, \ldots, \gamma_p$ and $q$ outgoing parametrized outgoing circles $\gamma_{p+1}, \ldots, \gamma_{p+q}$. The case $p = q = 1$ and $\Sigma$ a torus is depicted below.

\begin{center}
\begin{tikzpicture}
\draw[thick] (-2,0) to [out=90,in=270] (2,0);
\draw[thick] (-2,0) to [out=90,in=270] (0,2);
\draw[thick] (2,0) to [out=90,in=270] (0,2);
\draw[thick] (0,2) to [out=90,in=270] (0,-2);
\draw[thick] (0,-2) to [out=90,in=270] (-2,0);
\draw[thick] (0,0) circle (2);
\node at (-1,1) {$\gamma_1$};
\node at (1,1) {$\gamma_2$};
\node at (0,-2) {$\Sigma$};
\end{tikzpicture}
\end{center}

As for an ordinary connection $\gamma \mapsto A_\gamma$, the properties of the assignment $\Sigma \mapsto U_\Sigma$ are that is transitive with respect to concatenating Riemann surfaces, and that it is parametrization independent in the sense that it does not change if $\Sigma \to M$ is replaced by the composite $\Sigma' \to \Sigma \to M$ where $\Sigma' \to \Sigma$ is a diffeomorphism.

\footnote{In the case when $G = \{1\}$ and the orbifold is simply a smooth manifold, the theorem was previously obtained in [6].}
Actually this is also true for self-gluing of an incoming and an outgoing components in the same Riemann surface.

A gerbe with connection over $M$ allows one to define such operator $U$. In this case the closed 3-form $\omega$ with integer periods (the field strength, or curvature) must be such that the value of $\omega(v)$ on an element $v$ of 3-volume at $x$ bounded by a surface $\Sigma$ is given by

$$\exp(\sqrt{-1} \omega(v)) = U_{\Sigma},$$

(3.0.7)

where $\Sigma$ is regarded as a path in the loop space from a point loop at $x$ to itself (therefore $U_{\Sigma}$ is an invertible $\mathbb{C}$-linear map from the fiber at the constant loop $x$ to itself, hence we can associate it a complex number of norm one). This operator $U$ is what Segal has called a “String Connection” [20].

If we were to define

$$U_{\Sigma} := \exp(\sqrt{-1} \chi(\Sigma))$$

when $\partial v = \Sigma$ where $(\chi, \omega)$ is the differential character given by the gerbe with connection we can see that formula 3.0.7 follows from the fact that

$$\chi(\Sigma) = \omega(\Sigma) \mod \mathbb{Z}.$$

So the gerbe with connection allows one to define the operator $U$ on surfaces that are boundaries, but in order to do it for any other surface we need to do more.

Coming back to orbifolds, we will explain how the operator $U$ is defined for a specific gerbe with connection over $[M/G]$; we will call this class of gerbes global.

Definition 3.0.5. A global gerbe with connection $\xi$ is a Beilinson-Deligne cocycle over $[M/G]$ whose data is given by global forms\footnote{General gerbes require Leray representatives that are Morita equivalent to the given group action [12].}. Namely $\xi$ will consist of the forms $B \in \Omega^2(M)$, $A_g \in \Omega^1(M)$ and $\rho_{g,h} : M \to \text{U}(1)$ for $g, h \in G$ such that it is a cocycle in the double complex 2.2.3, i.e.

$$g^*B - B = dA_g$$

(3.0.8)

$$A_g + g^*A_h - A_{gh} = \sqrt{-1}d\log \rho_{g,h}.$$  

(3.0.9)

The curvature $\omega \in \Omega^3_0(M)$ of $\xi$ is $G$-invariant and we know that $\omega = dB$.

A gerbe with connection $\xi$ induces a complex line bundle over the loop orbifold of $[M/G]$ and the transport operator will act on its fibers. Here we need to recall the definition of the loop orbifold, and as the gerbe we have in mind is global it suffices to take orbifold maps of principal bundles over the circle to $[M/G]$. The general definition of the loop orbifold can be found in [14] and the definition of the line bundle induced by $\xi$ is in [12].

A loop over the orbifold $[M/G]$ will consist of a map $\phi : Q \to M$ of a $\Gamma$-principal bundle $Q$ over the circle $S^1$ and a homomorphism $\phi_\# : \Gamma \to G$ such that $\phi$ is $\phi_\#$-equivariant. Let’s call this space of loops by $\mathcal{L}[M/G]$. It has a natural action of the group $G$ as follows. For $h \in G$ let $\psi := \phi \cdot h$ where $\psi(x) := \phi(x)h$ and $\psi_\#(\tau) = h^{-1}\phi_\#(\tau)h$, then $\psi : Q \to M$ and is $\psi_\#$-equivariant.

Definition 3.0.6. The groupoid given by $(\mathcal{L}[M/G])/G$ is what we call the loop orbifold.
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Note that we could have just taken only $\Gamma$ principal bundles where $\Gamma$ is a finite cyclic group $\mathbb{Z}_m$. This because the relevant information of such maps lie on the holonomy of the circle on $[M/G]$, and this is characterized by a conjugacy class in $G$. We did not do so because it will be needed in such generality to simplify what follows.

The topology of the loop groupoid is given by the compact-open topology of the space of maps of a given principal bundle. Then loops defined over two different principal bundles are in different connected components of the loop orbifold.

The line bundle $E$ associated to the gerbe $\xi$ is obtained by defining a groupoid map from $(\mathcal{L}[M/G])/G$ to $U(1)$. So it is a trivial line bundle $E = \mathbb{C} \times \mathcal{L}[M/G]$ over $\mathcal{L}[M/G]$ with a $G$ action. Define

$$F : \mathcal{L}[M/G] \times G \to U(1)$$

$$F(\phi, g) \mapsto \exp \left( \frac{\sqrt{-1}}{|\Gamma|} \int_Q \phi^* A_g \right)$$

where $\phi : Q \to M$ is a loop with $Q$ a $\Gamma$-principal bundle. Using equation (3.0.9) and the fact that $\partial Q = 0$ it follows that

$$F(\phi, g)F(\phi \cdot g, h) = F(\phi, gh),$$

meaning that the map $F$ is a map of groupoids, therefore defining an action on the line bundle $E$.

Now we want to define a string connection over this line bundle $E$ given by the gerbe with connection $\xi$. We need to consider the equivalent over an orbifold of a Riemann surface with boundary. This will consist of a map $\Phi : P \to M$ of a $\Gamma$-principal bundle $P$ over an oriented Riemann surface $\Sigma$ ($\Gamma$ finite) and a homomorphism $\Phi_\# : \Gamma \to G$ such that $\Phi$ is $\Phi_\#$-equivariant. Note that there is a natural action of the group $G$ on $\Phi$. It is defined in the same way as for loops.

Let the boundary of $\Sigma$ have $p$ incoming parametrized circles and $q$ outgoing. Then the boundary $\partial P$ of $P$ will consist of $p$ incoming orbifold loops $\gamma_i : Q_i \to M$ $1 \leq i \leq p$ with the induced orientation, and $q$ outgoing ones $\gamma_j : Q_j \to M$, $p + 1 \leq j \leq p + q$ with the opposite orientation so that $\partial P = \bigsqcup_i Q_i \sqcup \bigsqcup_j Q_j$. Here the $Q_i$’s and the $Q_j$’s are $\Gamma$-principal bundles over the circle. As our line bundle is trivial (without the $G$ action) the operator $U_\phi$ is just a complex number. Define

$$U_\phi : E_{\gamma_1} \otimes \cdots \otimes E_{\gamma_p} = \mathbb{C} \to \mathbb{C} = E_{\gamma_{p+1}} \otimes \cdots \otimes E_{\gamma_{p+q}}$$

$$U_\phi := \exp \left( \frac{\sqrt{-1}}{|\Gamma|} \int_P \Phi^* B \right).$$

The only thing left to prove for $U$ to be a string connection over $E$ is that it is compatible with the $G$ action. The concatenation property and the invariance under diffeomorphisms that fix the boundary follow from the fact that the operator is defined through an integral over the surface $P$. So we want the following diagram
to be commutative for any \( g \in G \)

\[
\bigotimes_{i=1}^p E_{\gamma_i} \xrightarrow{\exp\left( \frac{1}{\gamma_i} \int_{Q_i} \gamma_i^* A_g \right)} \bigotimes_{j=p+1}^{p+q} E_{\gamma_j}
\]

\[
\bigotimes_{i=1}^p E_{\gamma_i g} \xrightarrow{\exp\left( \frac{1}{\gamma_i} \int_{Q_i} \gamma_i^* A_g \right)} \bigotimes_{j=p+1}^{p+q} E_{\gamma_j g}
\]

and its commutativity follows from Stokes theorem and the relation 3.0.8

\[
\int_p \Phi^* g^* B - \int_p \Phi^* B = \int_p \Phi^* dA_g = \int_p \Phi^* A_g
\]

\[
= \sum_{i=1}^p \int_{Q_i} \phi^* A_g - \sum_{j=p+1}^{p+q} \int_{Q_j} \phi^* A_g.
\]

The previous string connection is also compatible with the connection that we have associated to the loop orbifold in [12]. Let’s recall the construction. The connection for us will be a linear functional \( \Delta \) on the tangent space of the loop orbifold (a 1-form on the loop orbifold). For \( \phi : Q \to M \) with \( Q \) a \( \Gamma \)-principal bundle and a tangent vector to it, namely a section \( \mu : Q \to \phi^* TM \) such that \( \mu \) is \( \phi_\# \) equivariant we define

\[
\langle \Delta_\phi, \mu \rangle := \frac{1}{|\Gamma|} \int_Q \Phi^* (i_\mu B)
\]

where \( i_\mu \) is contraction on the direction of \( \mu \). In [12], we have proved that \( \Delta \) together with \( F \), the gluing information of the bundle \( E \) (see 3.0.10), form a Beilinson Deligne cocycle over the loop orbifold, hence a line bundle with connection over it.

So we can conclude this paper with the following result

**Theorem 3.0.7.** Let \( \xi \) be a global gerbe with connection over \([M/G]\) and \( E \) the line bundle with connection induced by it via transgression. Then \( \xi \) permits to define a string connection \( U \) over the line bundle \( E \) of the loop groupoid \( (\mathcal{L}[M/G])/G \).

We actually have done more. We claim that we can construct a string connection over the loop orbifold of a general orbifold (smooth Deligne-Mumford stack) from any gerbe with connection. This is the subject of the sequel to this paper.

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