THREE STRONGLY HYPERBOLIC METRICS ON PTOLEMY SPACES

YINGQING XIAO AND ZHANQI ZHANG

Abstract: Recently, strongly hyperbolic space as certain analytic enhancements of Gromov hyperbolic space was introduced by B. Nica and J. Špakula. In this note, we prove that the log-metric \( \log(1 + d) \) on a Ptolemy space \((X, d)\) is a strongly hyperbolic metric. Using our result, we construct three metrics on a Ptolemy metric space and prove they are strongly hyperbolic.

Key Words: Ptolemy space, strongly hyperbolic space, Gromov hyperbolicity.

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1. Introduction

In the field of geometric function theory, the hyperbolic metric plays an important role. In higher dimensional Euclidean spaces, the hyperbolic metric exists only in balls and half-spaces and the lack of hyperbolic metric in general domains has been a primary motivation for introducing the so-called hyperbolic-type metrics in the sense of Gromov. For example, \( \tilde{j} \)-metric, Apollonian metric, Seittenranta’s metric, half apollonian metric, scale-invariant Cassinian metric and Möbius-invariant Cassinian metric (see [4, 6, 7, 8, 9, 10, 11, 13, 14] and the references therein). All these metrics are defined in terms of distance functions and can be classified into one point metrics or two-point metrics based on the number of boundary points used in their definitions. Recently, in the paper [1], the authors proposed an approach to construct a metric from the one-point metrics. More precisely, let \((X, d)\) be an arbitrary metric space. For each \(p \in X\), they defined a distance function \( \tau_p \) on \(X \setminus \{p\}\), by

\[
\tau_p(x, y) = \log(1 + 2 \frac{d(x, y)}{\sqrt{d(p, x)} \sqrt{d(p, y)}})
\]

and proved that for each \(p \in X\), the distance function \( \tau_p \) is Gromov hyperbolic with \( \delta = \log 3 + \log 2 \). In fact, the following more general distance function was first introduced by O. Dovgoshey, P. Hariri and
M. Vuorinen in [5].
\[
    h_{D,c}(x, y) = \log(1 + c \frac{d(x, y)}{\sqrt{d_D(x)d_D(y)}})
\]
where \( D \) is a nonempty open set in a metric space \((X, d)\) and \( d_D(x) = \text{dist}(x, \partial D) \), \( c \geq 2 \). They shown that \( h_{D,c} \) is a metric and 2 is the best possible.

Although hyperbolicity yields a very satisfactory theory, for certain analytic purposes, hyperbolicity by itself is not enough, and one needs certain enhancements. In the paper [14], the authors introduced the notion of strongly hyperbolic space and given certain enhancements. They shown that strongly hyperbolic spaces are Gromov hyperbolic spaces that are metrically well-behaved at infinity, and under weak geodesic assumptions, the strongly space are strongly bolic as well. They shown that CAT(−1) spaces are strongly hyperbolic and also shown that the Green metric defined by a random walk on a hyperbolic group is strongly hyperbolic. Since the strongly hyperbolic space has better properties, it is interesting to determine which hyperbolic metric in geometric function theory is a strongly hyperbolic metric or to construct a strongly hyperbolic metric on a given metric space. We consider this problem in Ptolemy spaces in this paper.

Firstly, we show that the log-metric of a Ptolemy space is a strongly hyperbolic metric. That is, we show that if \((X, d)\) is a Ptolemy space, then \((X, \log(1 + d))\) is a strongly hyperbolic space. Using our result, we can show that the metric space \((X, S_p)\) is also a strongly hyperbolic space. Here
\[
    S_p(x, y) = \log(1 + \frac{d(x, y)}{[1 + d(x, p)][1 + d(y, p)]})
\]
for a fix point \( p \in X \) and \( x, y \in X \).

Secondly, motivated by the recent works of A. G. Aksov, Z. Ibragimov and W. Whiting in [II], we construct a strongly hyperbolic metric on a Ptolemy metric space. To formulate the results of our paper, for each \( p \in X \), we define a distance function \( \chi_p \) on \( X \setminus \{p\} \), by
\[
    \chi_p(x, y) = \log(1 + \frac{d(x, y)}{d(p, x)d(p, y)}).
\]
We prove that if \((X, d)\) is a Ptolemy space, for each \( p \in X \), the distance function \( \chi_p \) is a strongly hyperbolic metric. We also consider the distortion of the above metric \( \chi_p \) under Möbius maps of a punctured ball in \( \mathbb{R}^n \).
2. Strongly hyperbolic metrics on Ptolemy spaces

We begin by recalling some basic notions and facts. Let \( X \) be a metric space, fix a base point \( o \in X \), the Gromov product of \( x, x' \in X \) with respect to \( o \) is defined as

\[
(x|x')_o := \frac{1}{2}(|ox| + |ox'| - |xx'|).
\]

Note that \((x|x')_o \geq 0\) by the triangle inequality.

**Definition 1** (Gromov). A metric space \( X \) is \( \delta \)-hyperbolic, where \( \delta \geq 0 \), if

\[
(x|y)_o \geq \min\{ (x|z)_o, (z|y)_o \} - \delta
\]

for all \( x, y, z, o \in X \).

In the paper [14], the authors given the following enhancements of hyperbolicity.

**Definition 2** ([14], Definition 4.1). We say that a metric space is strongly hyperbolic with parameter \( \epsilon > 0 \) if

\[
\exp(-\epsilon(x|y)_o) \leq \exp(-\epsilon(x|z)_o) + \exp(-\epsilon(z|y)_o)
\]

for all \( x, y, z, o \in X \); equivalently, the four-point condition

\[
\exp\left(\frac{\epsilon}{2}(|xy| + |zt|)\right) \leq \exp\left(\frac{\epsilon}{2}(|xz| + |yt|)\right) + \exp\left(\frac{\epsilon}{2}(|xt| + |zy|)\right)
\]

holds for all \( x, y, z, t \in X \).

The authors mentioned the motivation for considering this notion of strongly hyperbolic is the following theorem in the paper [14].

**Theorem 1** ([14], Theorem 4.2). Let \( X \) be a strongly hyperbolic space with parameter \( \epsilon \). Then \( X \) is an \( \epsilon \)-good, \( \log 2/\epsilon \)-hyperbolic space. Furthermore, \( X \) is strongly bolic provided that \( X \) is roughly geodesic.

Strongly bolic metric spaces was considered by V. Lafforgue in [12] in relation with conjecture of Baum-Connes. Here for hyperbolic spaces \((X, d)\) which are roughly geodesic, strong bolicity in the sense of Lafforgue [12] amounts to the following: for every \( \eta, r > 0 \), there exists \( R > 0 \) such that \( d(x, y) + d(z, t) \leq r \) and \( d(x, z) + d(y, t) \geq R \) imply that \( d(x, t) + d(y, z) \leq d(x, z) + d(y, t) + \eta \).

From the above theorem [14] we know that the strongly hyperbolic space has better properties than general hyperbolic spaces. Thus it is interesting to construct a strongly hyperbolic metric on a metric space.
Definition 3. A metric space \((X,d)\) is called Ptolemy space if the following Ptolemy inequality
\[
d(x_1, x_2)d(x_3, x_4) \leq d(x_1, x_4)d(x_2, x_3) + d(x_1, x_3)d(x_2, x_4)
\]
holds for all quadruples \(x_1, x_2, x_3, x_4 \in X\).

Lemma 1. Suppose \((X,d)\) is a metric space and \(x_i \in X\) for \(i = 1, 2, 3, 4\). Then
\[
d(x_1, x_2) + d(x_3, x_4) \leq d(x_1, x_3) + d(x_1, x_4) + d(x_2, x_3) + d(x_2, x_4).
\]

Proof  By the triangle inequality, we have
\[
\begin{align*}
d(x_1, x_2) & \leq d(x_1, x_3) + d(x_3, x_2), \\
d(x_1, x_2) & \leq d(x_1, x_4) + d(x_4, x_2), \\
d(x_3, x_4) & \leq d(x_3, x_1) + d(x_1, x_4), \\
d(x_3, x_4) & \leq d(x_3, x_2) + d(x_2, x_4).
\end{align*}
\]

We sum the above four inequalities and obtain that
\[
d(x_1, x_2) + d(x_3, x_4) \leq d(x_1, x_3) + d(x_1, x_4) + d(x_2, x_3) + d(x_2, x_4).
\]

\[\square\]

Theorem 2. Suppose that \((X,d)\) is a Ptolemy space, then the metric space \((X, \log(1 + d))\) is a strongly hyperbolic space with parameter \(\epsilon = 2\).

Proof  Let \(x_1, x_2, x_3, x_4 \in X\), we introduce the following notations for convenience. \(\rho_{ij} = \log(1 + d(x_i, x_j))\), \(d_{ij} = d(x_i, x_j)\) for all \(i, j \in \{1, 2, 3, 4\}\). Thus
\[
\rho_{ij} = \log(1 + d_{ij}).
\]

Now, we need to show that
\[
\exp(\rho_{12} + \rho_{34}) \leq \exp(\rho_{13} + \rho_{24}) + \exp(\rho_{14} + \rho_{23}),
\]
which is equivalent to the following inequality,
\[
(1 + d_{12})(1 + d_{34}) \leq (1 + d_{13})(1 + d_{24}) + (1 + d_{14})(1 + d_{23}).
\]

Notice that \((X,d)\) is a Ptolemy space, by Lemma 1 we have
\[
(1 + d_{12})(1 + d_{34}) = 1 + d_{12} + d_{34} + d_{12}d_{34} \\
\leq 2 + d_{13} + d_{24} + d_{14} + d_{23} + d_{14}d_{23} + d_{13}d_{24} \\
= (1 + d_{13})(1 + d_{24}) + (1 + d_{14})(1 + d_{23}).
\]

Thus, we show that the metric space \((X, \log(1 + d))\) is a strongly hyperbolic hyperbolic space with parameter \(\epsilon = 2\). \[\square\]
Let \((X, d)\) be any metric space, fix a base point \(p \in X\), and the following distance function \(s_p\) was considered in the paper [2],

\[
s_p(x, y) = \frac{d(x, y)}{[1 + d(x, p)][1 + d(y, p)]}
\]

for \(x, y \in X\). Sometimes this is a distance function, but in general it may not satisfy the triangle inequality. In this paper, we have the following result.

**Theorem 3.** Suppose \((X, d)\) is a Ptolemy space and \(p \in X\). Then \((X, s_p)\) is also a Ptolemy space.

**Proof.** Firstly, we prove that \(s_p\) is a metric. Obviously, \(s_p(x, y) \geq 0\), \(s_p(x, y) = s_p(y, x)\) and \(s_p(x, y) = 0\) if and only if \(x = y\). So it is enough to show that the triangle inequality holds. That is, for all \(x, y, z \in X \setminus \{p\}\),

\[
s_p(x, y) \leq s_p(x, z) + s_p(z, y),
\]

which is equivalent to

\[
d(x, y)[1 + d(z, p)] \leq d(x, z)[1 + d(y, p)] + d(y, z)[1 + d(x, p)].
\]

Since \((X, d)\) is a Ptolemy space, the above inequality holds naturally, which implies that \(s_p\) is a metric on \(X\).

Now, we show that \((X, s_p)\) also is a Ptolemy space. For any \(x_i \in X\) for \(i = 1, 2, 3, 4\). Set \(p_i = 1 + d(p, x_i)\) and \(d_{ij} = d(x_i, x_j)\), thus \(s_p(x_i, x_j) = d_{ij}/p_ip_j\) for \(i, j \in \{1, 2, 3, 4\}\). Since \((X, d)\) is a Ptolemy space, we have

\[
d_{12}d_{34} \leq d_{13}d_{24} + d_{14}d_{23},
\]

Thus

\[
\frac{d_{12}d_{34}}{p_1p_2p_3p_4} \leq \frac{d_{13}d_{24}}{p_1p_2p_3p_4} + \frac{d_{14}d_{23}}{p_1p_2p_3p_4}.
\]

That is

\[
s_p(x_1, x_2)s_p(x_3, x_4) \leq s_p(x_1, x_3)s_p(x_2, x_4) + s_p(x_1, x_4)s_p(x_2, x_3),
\]

which implies that \((X, s_p)\) also is a Ptolemy space. \(\square\)

Using \(s_p\), we define the following metric \(S_p\) on \(X\) by

\[
S_p(x, y) = \log(1 + s_p(x, y)).
\]

According to Theorem 2, we have the following result.

**Theorem 4.** Suppose \((X, d)\) is a Ptolemy and \(p \in X\). The metric space \((X, S_p)\) is a strongly hyperbolic space with parameter \(\epsilon = 2\). Thus \((X, S_p)\) is a \(\log 2/2\)-hyperbolic space.
Suppose \((X, d)\) is a metric space. For each \(p \in X\), A. G. Aksov, Z. Ibragimov and W. Whiting defined a distance function \(\tau_p\) on \(X \setminus \{p\}\) in [1] by
\[
\tau_p(x, y) = \log(1 + 2 \frac{d(x, y)}{\sqrt{d(p, x)} \sqrt{d(p, y)}}).
\]
They obtained the following result.

**Theorem 5** ([1], Theorem 2.1 and Lemma 4.1). Let \((X, d)\) be a Ptolemy space and let \(p \in X\) be an arbitrary point. Then the distance function \(\tau_p\) is a metric on \(X \setminus \{p\}\). In particular, the space \((X \setminus \{p\}, \tau_p)\) is Gromov hyperbolic with \(\delta = \log 3 + \log 2\).

Motivated by the definition of \(\tau_p\), for each \(p \in X\), we define a distance function \(\chi_p\) on \(X \setminus \{p\}\) by
\[
\chi_p(x, y) = \log(1 + \frac{d(x, y)}{d(p, x)d(p, y)}).
\]
Usually, \(\chi_p\) is not a metric on \(X \setminus \{p\}\). But, when \((X, d)\) is a Ptolemy space, we have the following result.

**Theorem 6.** Let \((X, d)\) be a Ptolemy metric space and let \(p \in X\) be an arbitrary point. Then the distance function \(\chi_p\) is a metric on \(X \setminus \{p\}\).

**Proof.** Obviously, \(\chi_p(x, y) \geq 0\), \(\chi_p(x, y) = \chi_p(y, x)\) and \(\chi_p(x, y) = 0\) if and only if \(x = y\). So it is enough to show that the triangle inequality holds. That is, for all \(x, y, z \in X \setminus \{p\}\),
\[
\chi_p(x, y) \leq \chi_p(x, z) + \chi_p(z, y),
\]
which is equivalent to
\[
\frac{d(x, y)}{d(x, p)d(y, p)} \leq \frac{d(x, z)}{d(x, p)d(z, p)} + \frac{d(y, z)}{d(y, p)d(z, p)} + \frac{d(x, z)d(y, z)}{d(z, p)^2d(x, p)d(y, p)}.
\]
That is
\[
(1) \quad d(x, y)d(z, p) \leq d(x, z)d(y, p) + d(y, z)d(x, p) + \frac{d(x, z)d(y, z)}{d(z, p)}.
\]
Since \((X, d)\) is a Ptolemy space, the above inequality [1] holds naturally, which completes the proof. \(\square\)

**Lemma 2.** Suppose \((X, d)\) is a Ptolemy metric space and \(x_i \in X\) for \(i = 0, 1, 2, 3, 4\). Set \(p_i = d(x_0, x_i)\) and \(d_{ij} = d(x_i, x_j)\) for \(i, j \in \{1, 2, 3, 4\}\). Then
\[
p_{35}p_{412} + p_{15}p_{234} \leq p_{15}p_{34}d_{24} + p_{25}p_{14} + p_{2}p_{3}d_{14} + p_{1}p_{4}d_{23}.
\]
Proof. By the Ptolemy inequality, we have
\[ p_3 p_4 d_{12} \leq p_3 p_1 d_{24} + p_3 p_2 d_{14}, \]
\[ p_3 p_4 d_{12} \leq p_4 p_2 d_{13} + p_1 p_4 d_{23}, \]
\[ p_1 p_2 d_{34} \leq p_1 p_3 d_{24} + p_1 p_4 d_{23}, \]
\[ p_1 p_2 d_{34} \leq p_2 p_4 d_{13} + p_2 p_3 d_{14}. \]
We sum the above four inequalities and obtain that
\[ p_3 p_4 d_{12} + p_1 p_2 d_{34} \leq p_1 p_3 d_{24} + p_2 p_4 d_{13} + p_2 p_3 d_{14} + p_1 p_4 d_{23}. \]
\[ \square \]

Using the above lemma 2, we obtain the following result.

**Theorem 7.** Let \((X, d)\) be a Ptolemy metric space and let \(p \in X\) be an arbitrary point. Then the metric space \((X \setminus \{p\}, \chi_p)\) is strongly hyperbolic space with parameter 2. Thus \((X \setminus \{p\}, \chi_p)\) is \(\log 2/2\)-hyperbolic space.

Proof. Let \(x_1, x_2, x_3, x_4 \in X \setminus \{p\}\), we introduce the following notations for convenience. \(d_{ij} = d(x_i, x_j)\), \(p_i = d(p, x_i)\) and \(\rho_{ij} = \chi_p(x_i, x_j)\) for \(i, j \in \{1, 2, 3, 4\}\). Thus
\[ \rho_{ij} = \log(1 + \frac{d_{ij}}{p_i p_j}) \]
for \(i, j \in \{1, 2, 3, 4\}\). Now, we need to show that
\[ e^{(\rho_{12} + \rho_{34})} \leq e^{(\rho_{13} + \rho_{24})} + e^{(\rho_{14} + \rho_{23})}, \]
which is equivalent to the following inequality
\[
(1 + \frac{d_{12}}{p_1 p_2})(1 + \frac{d_{34}}{p_3 p_4}) \leq (1 + \frac{d_{13}}{p_1 p_3})(1 + \frac{d_{24}}{p_2 p_4})
+ (1 + \frac{d_{14}}{p_1 p_4})(1 + \frac{d_{23}}{p_2 p_3}).
\]
That is
\[
\frac{d_{12}}{p_1 p_2} + \frac{d_{34}}{p_3 p_4} + \frac{d_{12}}{p_1 p_2} \frac{d_{34}}{p_3 p_4} \leq \frac{d_{13}}{p_1 p_3} + \frac{d_{24}}{p_2 p_4} + \frac{d_{13}}{p_1 p_3} \frac{d_{24}}{p_2 p_4}
+ \frac{d_{14}}{p_1 p_4} + \frac{d_{23}}{p_2 p_3} + \frac{d_{14}}{p_1 p_4} \frac{d_{23}}{p_2 p_3} + 1,
\]
which is equivalent to the following inequality
\[
p_3 p_4 d_{12} + p_1 p_2 d_{34} + d_{12} d_{34} \leq p_2 p_4 d_{13} + p_1 p_3 d_{24} + d_{13} d_{24}
+ p_2 p_3 d_{14} + p_1 p_4 d_{23} + d_{14} d_{23}
+ p_1 p_2 p_3 p_4.
\]
Since \((X, d)\) is a Ptolemy space, we have
\[d_{12}d_{34} \leq d_{13}d_{24} + d_{14}d_{23}.\]
From Lemma 2, we have
\[p_3p_4d_{12} + p_1p_2d_{34} \leq p_2p_4d_{13} + p_1p_3d_{24} + p_2p_3d_{14} + p_1p_4d_{23}.\]
Thus, the above inequality holds, which implies that \((X \setminus \{p\}, \chi_p)\) is a strongly space with parameter 2. From Theorem 1, we know that \((X \setminus \{p\}, \chi_p)\) is a log 2/2-hyperbolic space.

3. Distortion property under Möbius transformations

In the following, we use the notation \(\mathbb{R}^n, n \geq 2\) for the Euclidean-dimensional space. The Euclidean distance between \(x, y \in \mathbb{R}^n\) is denoted by \(|x - y|\). Given \(x \in \mathbb{R}^n\) and \(r > 0\), the open ball centered at \(x\) with radius \(r\) is denoted by \(B^n(x, r) := \{y \in \mathbb{R}^n : |x - y| < r\}\). Denote by \(B^n := B^n(0, 1)\), the unit ball in \(\mathbb{R}^n\). One of our objectives in this section is to study the distortion property of our metric under Möbius maps from a punctured ball onto another punctured ball. Distortion properties of the scale-invariant Cassinian metric of the unit ball under Möbius maps has been studied in [10]. Recently, in the [13], M. R. Mohapatra and S. K. Sahoo also considered the distortion of the \(\tilde{\tau}\)-metric under Möbius maps of a punctured ball.

**Theorem 8.** Let \(a \in B^n\) and \(f: B^n \setminus \{0\} \to B^n \setminus \{a\}\) be a Möbius map with \(f(0) = a\). Then for \(x, y \in B^n \setminus \{0\}\), we have
\[
\chi_0(x, y) \leq \chi_a(f(x), f(y)) \leq \chi_0(x, y) - \log(1 - |a|^2).
\]
The equalities hold if and only if \(a = 0\).

**Proof.** If \(a = 0\), the proof is trivial since \(f(x) = Ax\) for some orthogonal matrix \(A\). Now we assume that \(a \neq 0\). Let \(\sigma\) be the inversion in the sphere \(S^{n-1}(a^*, r) = \{x \in \mathbb{R}^n : |x - a^*| = r\}\), where
\[
a^* = \frac{a}{|a|^2}, \quad r = \sqrt{|a^*|^2 - 1} = \frac{\sqrt{1 - |a|^2}}{|a|}.
\]
Note that the sphere \(S^{n-1}(a^*, r)\) is orthogonal to \(S^{n-1}\) and that \(\sigma(a) = 0\). In particular, \(\sigma\) is a Möbius map with \(\sigma(\mathbb{B}^n \setminus \{a\}) = \mathbb{B}^n \setminus \{0\}\). Recall that
\[
\sigma(x) = a^* + \left(\frac{r}{|x - a^*|}\right)^2(x - a^*).
\]
Then \(\sigma \circ f\) is an orthogonal matrix (see, for example, [10, Theorem 3.5.1(i)]). In particular,
\[
|\sigma(f(x)) - \sigma(f(y))| = |x - y|.
\]
By computation, we have

\[ |\sigma(x) - \sigma(y)| = \frac{r^2|x - y|}{|x - a^*||y - a^*|}. \]

Thus

\[ |\sigma(f(x)) - \sigma(f(y))| = \frac{r^2|f(x) - f(y)|}{|f(x) - a^*||f(y) - a^*|} = |x - y|, \]

which implies that

\[ |f(x) - f(y)| = \frac{|x - y|}{r^2} |f(x) - a^*||f(y) - a^*|. \]

Since \( f(0) = a \), we have

\[ |f(x) - a| = \frac{|f(x) - a^*||a - a^*|}{|a^*|^2 - 1}|x| \quad \text{and} \quad |f(y) - a| = \frac{|f(y) - a^*||a - a^*|}{|a^*|^2 - 1}|y|. \]

Notice that

\[ \chi_0(x, y) = \log(1 + \frac{|x - y|}{|x||y|}) \]

and

\[ \chi_a(f(x), f(y)) = \log(1 + \frac{|f(x) - f(y)|}{|f(x) - a||f(y) - a|}). \]

We have

\[
\begin{align*}
\chi_a(f(x), f(y)) &= \log(1 + \frac{|f(x) - f(y)|}{|f(x) - a||f(y) - a|}) \\
&= \log(1 + \frac{|x - y| |a^*|^2 - 1}{|x||y| |a - a^*|^2}) \\
&= \log(1 + \frac{1}{1 - |a|^2} \frac{|x - y|}{|x||y|}).
\end{align*}
\]

Since \( |a| < 1 \), we have \( 1 \leq \frac{1}{1 - |a|^2} \). Thus

\[
1 + \frac{|x - y|}{|x||y|} \leq 1 + \frac{1}{1 - |a|^2} \frac{|x - y|}{|x||y|} \leq \frac{1}{1 - |a|^2} + \frac{1}{1 - |a|^2} \frac{|x - y|}{|x||y|}.
\]

So

\[ \chi_0(x, y) \leq \chi_a(f(x), f(y)) \leq \chi_0(x, y) - \log(1 - |a|^2). \]

Obviously, the equalities hold if and only if \( a = 0 \).

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YINGQING XIAO, COLLEGE OF MATHEMATICS AND ECONOMETRICS, HUNAN UNIVERSITY, CHANGSHA, 410082, CHINA
E-mail address: ouxyq@hnu.edu.cn

ZHANQI ZHANG (Corresponding author), COLLEGE OF MATHEMATICS AND ECONOMETRICS, HUNAN UNIVERSITY, CHANGSHA, 410082, P. R. CHINA
E-mail address: rateriver@sina.com