EXPONENTIAL CONVEXIFYING OF POLYNOMIALS

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Abstract. Let $X \subset \mathbb{R}^n$ be a convex closed and semialgebraic set and let $f$ be a polynomial positive on $X$. We prove that there exists an exponent $N \geq 1$, such that for any $\xi \in \mathbb{R}^n$ the function $\varphi_N(x) = e^{N|x-\xi|^2} f(x)$ is strongly convex on $X$. When $X$ is unbounded we have to assume also that the leading form of $f$ is positive in $\mathbb{R}^n \setminus \{0\}$. We obtain strong convexity of $\Phi_N(x) = e^{e^{N|x|^2}} f(x)$ on possibly unbounded $X$, provided $N$ is sufficiently large, assuming only that $f$ is positive on $X$. We apply these results for searching critical points of polynomials on convex closed semialgebraic sets.

1. Introduction

In [3] we considered several questions concerning convexification of a polynomial $f$ which is positive on a closed convex set $X \subset \mathbb{R}^n$. One of the main results in [3], is the following [Theorem 5.1]: if $X$ is a compact set then there exists a positive integer $N$ such that the function

$$\phi_N(x) = (1 + |x|^2)^N f(x)$$

(1.1)

is strongly convex on $X$. Moreover, explicit estimates for the exponent $N$ were given in [3]. They depend on the diameter of $X$, the size of coefficients of the polynomial $f$ and on the minimum of $f$ on $X$. In fact a stronger version of (1.1) was given in [3]; there exists an integer $N$, which can be explicitly estimated, such that the polynomials

$$\phi_{N,\xi} = (1 + |x-\xi|^2)^N f(x), \quad \xi \in X,$$

are strongly convex on $X$. The fact that $N$ can be chosen independent of $\xi$ was crucial for a construction of an algorithm which for a given polynomial $f$, positive in the convex compact semialgebraic set $X$, produces a sequence $a_\nu \in X$ starting from an arbitrary point $a_0 \in X$, defined by induction: $a_\nu = \text{argmin}_x \phi_{N,a_{\nu-1}}$, i.e., $a_\nu \in X$ is the unique point of $X$ at which $\phi_{N,a_{\nu-1}}$ has a global minimum on $X$. The sequence $a_\nu$ converges to a lower critical point of $f$ on $X$ (see [3, Theorem 7.5]).
In the case of non-compact closed convex set $X$ the results mentioned above require an additional assumption, that the leading form $f_d$ of $f$, satisfy

$$
(1.2) \quad f_d(x) > 0 \quad \text{for} \quad x \in \mathbb{R}^n \setminus \{0\}.
$$

Under this assumption we have that: if a polynomial $f$ is positive on $X$ then for any $R > 0$ there exists $N_0$ such that for each $\xi \in X$, $|\xi| \leq R$, $N > N_0$ the polynomial $\phi_{N,\xi}$ is strongly convex on $X$.

The assumption $(1.2)$ is necessary for local convexity of $\phi_{N,\xi}$ in a neighborhood of infinity, see [3, Proposition 6.3]. However, this assumption is not sufficient to obtain convexity of the polynomial $\phi_{N,\xi}$ for some fixed $N > 0$ independent of $\xi \in X$. For instance the polynomial $f(x) = 1 + x^2$, has this property, cf. [3, Example 4.5].

The main goal of this paper is to study convexification of polynomials functions by exponential factors of the form $e^{N|x-\xi|^2}$ or by double exponential of the form $e^{e^{N|x-\xi|^2}}$. Surprisingly they play distinct roles. We set

$$
\varphi_{N,\xi}(x) := e^{N|x-\xi|^2} f(x).
$$

and prove the following (see Theorem 2.3 and Corollary 2.4): if a polynomial $f$ is positive on a compact and convex set $X \subset \mathbb{R}^n$, than there exists effectively computed number $N_0$ such that for any $N > N_0$ and $\xi \in \mathbb{R}^n$ the function $\varphi_{N,\xi}(x)$ is strongly convex on $X$.

If $X$ is not compact, we obtain the above assertions under the assumption $(1.2)$, see Theorem 3.3. In general the assumption $(1.2)$ can not be ommited as we show in Example 3.6.

Surprisingly convexification in the noncompact case without assumption $(1.2)$ is possible using double exponential factors. Namely in Theorems 4.1 and 4.6 we prove that: if $X \subset \mathbb{R}^n$ is a convex and closed semialgebraic set and $f$ is a polynomial positive on $X$, then for any $R > 0$ there exists effectively computed number $N_0$ such that for any $N > N_0$ and any $\xi \in \mathbb{R}^n$, $|\xi| \leq R$, the function

$$
\Phi_{N,\xi}(x) := e^{e^{N|x-\xi|^2}} f(x)
$$

is strongly convex on $X$.

In the case when $X$ is a convex and closed set, but non necessary semialgebraic, the result still holds (Theorems 4.4 and 4.7) under an additional assumption

$$
(1.3) \quad \inf\{f(x) : x \in X\} \geq m > 0.
$$

In the above theorems one can replace $\Phi_{N,\xi}$ by the function

$$
\Phi_{N,\xi}(x) := e^{Ne^{N|x-\xi|^2}} f(x).
$$
It turns out that convexification of polynomials using exponential function is somehow more natural and powerful than the convexification by the factors of the form \((1 + |x - \xi|^2)^N\) done in [3]. In particular it applies also to the noncompact case and the explicit formulae for the exponent \(N\) are nicer.

We believe that the results mentioned above could be of interest, also to study \(o\)-minimal structures expanded by the exponent function. It fits particularly to the structure \(\mathbb{R}^{\exp}\) semialgebraic sets expanded by the exponent function. The remarkable fact that \(\mathbb{R}^{\exp}\) is indeed an \(o\)-minimal structure was established by A. Wilkie [5]. It would be interesting to explain a different power of exponential and double exponential for convexification.

The main difficulty when determining explicitly the number \(N\) such that the function \(\varphi_N\) is strongly convex on a convex compact set \(X\), comes from an effective estimation of the number \(m\) in (1.3) and the number \(R = \max\{|x| : x \in X\}\). Using results of G. Jeronimo, D. Perrucci, E. Tsigaridas [2] we show in Theorem 2.6) and Theorem 2.3 how it is feasible when \(X\) is a compact semialgebraic set described by polynomial inequalities with integer coefficients and \(f\) is also a polynomial with integer coefficients (see Theorem 2.7).

As an application to optimization we propose an algorithm which produces, starting from an arbitrary point \(a_0 \in X\), a sequence \(a_\nu \in X\) which tends to a lower critical point of a polynomial \(f\) restricted to \(X\) or to infinity. We assume that \(X \subset \mathbb{R}^n\) is a closed convex semialgebraic set and \(f\) a polynomial which is bounded from below on \(X\). Then by adding to \(f\) an appropriate constant we may assume that \(f \geq m > 0\) on \(X\). If \(X\) is unbounded we assume also condition (1.2). Hence by the above mentioned theorems we obtain strong convexity of \(\varphi_\xi(x) = e^{|x-\xi|^2}f(x)\) for \(\xi \in X\). Let us choose any \(a_0 \in X\) and set by induction: \(a_\nu = \arg\min_X \varphi_{N,a_\nu-1}\). Then we prove that the sequence \(a_\nu\) tends to a lower critical point of a polynomial \(f\) restricted to \(X\) or to infinity. Note that computing \(a_\nu\), that is minimizing \(\varphi_{N,a_\nu-1}\) on \(X\), is usually easier since the function is convex. This type of algorithm, based on convexification, is called sometimes \textit{proximal}, see for instance [1]. Observe that computing the critical point of \(\varphi_{N,a_\nu-1}\) involves only algebraic equations.

The paper is organized as follows. In Section 2 we prove that the function \(\varphi_N\) in one variable is strongly convex on a closed interval \(I \subset \mathbb{R}\), provided \(f(x) > m\) for \(x \in I\) and some \(m > 0\) and \(N \in \mathbb{R}\) is sufficiently large. We also estimate from above the number \(N\). In Section 2.2 we consider this problem in the several variables case on a compact convex set \(X\) (see Theorem 2.3). In Sections 3 and 4 we consider the case when the set \(X\) is not compact.
2. Convexifying polynomials

2.1. Convexifying $C^2$-functions in one variable. In this section we prove that if $f$ is a function of class $C^2$ positive on a closed interval $I \subset \mathbb{R}$ (not necessary compact), then for $N$ large enough the function $t \mapsto e^{Nt^2}f(t)$ is strongly convex on $I$.

Let $f : \mathbb{R} \to \mathbb{R}$ be $C^2$ function. For any $N \in \mathbb{R}$ and $p, q \in \mathbb{R}$ we define the following function:

$$\varphi_{N,p,q}(t) := e^{N(t^2+pt+q)}f(t), \ t \in \mathbb{R}.$$ 

For positive numbers $m, D$ we put

$$\mathcal{N}(m, D) := \frac{D}{2m} + \frac{D^2}{2m^2}.$$ 

**Lemma 2.1.** Let $f$ be a function of class $C^2$, which is positive on a closed interval $I \subset \mathbb{R}$. Let $m, D \in \mathbb{R}$ be such that

(2.1) $0 < m \leq \inf\{f(t) : t \in I\}$,

and

(2.2) $|f'(t)| \leq D, \ |f''(t)| \leq D$ for $t \in I$.

Assume that $p^2 \leq 4q$ and

(2.3) $N > \mathcal{N}(m, D)$

then

$$\varphi''_{N,p,q}(t) \geq -\frac{D^2}{m} - D + 2Nm > 0$$

for $t \in I$, thus $\varphi_{N,p,q}$ is strongly convex on $I$.

**Proof.** By definition of $\varphi_{N,p,q}$ we have

$$\varphi''_{N,p,q}(t) = e^{N(t^2+pt+q)}[N^2(2t+p)^2f(t)+2N(2t+p)f'(t)+2f(t)+f''(t)].$$

Hence, from the assumptions we obtain

(2.4) $\varphi''_{N,p,q}(t) \geq e^{N(t^2+pt+q)}[N^2(2t+p)^2m - 2N|2t+p|D + 2Nm - D]$ for $t \in I$.

Note that the function

$$\mathbb{R} \ni \lambda \mapsto N^2m\lambda^2 - 2ND\lambda + 2Nm - D$$

attains its minimum, equal $-\frac{D^2}{m} - D + 2Nm$, at the point $\lambda = \frac{D}{Nm}$. Thus for $N > \mathcal{N}(m, D)$ we have

$$N^2m\lambda^2 - 2ND|\lambda| + 2Nm - D > 0$$

for any $\lambda \in \mathbb{R}$. Therefore

$$\varphi''_{N,p,q}(t) \geq -\frac{D^2}{m} - D + 2Nm > 0 \ \text{for} \ t \in I,$$

which implies that $\varphi_{N,p,q}$ is strongly convex on $I$. \qed
From Lemma 2.1 we immediately obtain

Corollary 2.2. Let $f$ be a function of class $C^2$ on a closed interval $I \subset \mathbb{R}$. Let $m, D \in \mathbb{R}$ be such that
\begin{equation}
(2.5) \quad m < \inf \{ f(t) : t \in I \},
\end{equation}
and
\begin{equation}
(2.6) \quad |f'(t)| \leq D, \quad |f''(t)| \leq D, \quad \text{for} \quad t \in I.
\end{equation}

Than for any $\xi \in \mathbb{R}$ and any $N \geq 1$ the function
\[ \psi_{N,\xi}(t) = e^{N(t-\xi)^2}[f(t) - m + D], \quad t \in I. \]
is strongly convex on $I$. In particular the function
\[ \varphi(t) = e^{(t-\xi)^2}[f(t) - m + D], \quad t \in I, \]
is strongly convex on $I$.

2.2. Convexifying polynomials in several variables. We will show that the function $\varphi_N$ in $n$ variables is strongly convex on a compact convex set $X \subset \mathbb{R}^n$, provided $f$ is a polynomial positive on $X$ and $N$ is sufficient large.

Let $f \in \mathbb{R}[x]$ be a real polynomial in $x = (x_1, \ldots, x_n)$ of the form
\begin{equation}
(2.7) \quad f = \sum_{j=0}^{d} \sum_{|\nu|=j} a_\nu x^\nu,
\end{equation}
where $a_\nu \in \mathbb{R}$, $x^\nu = x_1^{\nu_1} \cdots x_n^{\nu_n}$ and $|\nu| = \nu_1 + \cdots + \nu_n$ for $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{N}^n$ (we assume that $0 \in \mathbb{N}$). For $R > 0$ we denote
\[ D_n(f, R) := \max \left\{ 1, \sum_{j=1}^{d} \sum_{|\nu|=j} j|a_\nu|R^{j-1}, \sum_{j=1}^{d} \sum_{|\nu|=j} j(j-1)|a_\nu|R^{j-2} \right\}. \]

Theorem 2.3. Let $f \in \mathbb{R}[x]$ be a polynomial which is positive on a compact and convex set $X \subset \mathbb{R}^n$. Let $R = \max \{ |x| : x \in X \}$ and
\[ 0 < m \leq \min \{ f(x) : x \in X \}. \]

Than for any $\xi \in \mathbb{R}^n$, any $D \geq D_n(f, R)$ and any real $N > N(m, D)$ the function $\varphi_{N,\xi}(x) := e^{N|x-\xi|^2}f(x)$ is strongly convex on $X$.

Proof. Let
\begin{equation}
(2.8) \quad A := \{(\alpha, \beta) \in \mathbb{R}^n \times \mathbb{R}^n : \langle \alpha, \beta \rangle = 0, \ |\beta| = 1 \},
\end{equation}
where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product on $\mathbb{R}^n$. Set
\[ \gamma_{\alpha,\beta}(t) := \beta t + \alpha, \quad t \in \mathbb{R}. \]
Clearly, the family of all curves $\gamma_{\alpha,\beta}$, where $(\alpha, \beta) \in A$ describes all affine lines in $\mathbb{R}^n$. Denote by $B \subset A$ the set of all $(\alpha, \beta) \in A$ such that
the line parametrized by $\gamma_{\alpha,\beta}$ intersects the set $X$. Then $B$ is a compact set and

$$B \subset \{(\alpha, \beta) \in A : |\alpha| \leq R\}.$$ 

We will prove that for any $(\alpha, \beta) \in B$ and $N > \mathcal{N}(m, D)$ the function $\varphi_{N, \xi} \circ \gamma_{\alpha,\beta}$ is strongly convex on

$$I_{\alpha,\beta} := \{t \in \mathbb{R} : \gamma_{\alpha,\beta}(t) \in X\}.$$ 

Because $X$ is a compact and convex set, so $I_{\alpha,\beta}$ is a closed interval or only one point.

It is obvious that for $(\alpha; \beta) \in B$ the set $\{t \in \mathbb{R} : |\gamma_{\alpha,\beta}(t)| \leq R\}$ is an interval, which contains the point 0 or it is equal to $\{0\}$. Denote this interval by $[-R_{\alpha,\beta}, R_{\alpha,\beta}]$ (under convention $[0, 0] = \{0\}$). Then

$$I_{\alpha,\beta} \subset [-R_{\alpha,\beta}, R_{\alpha,\beta}] \subset [-R; R].$$

Let $f$ be of the form (2.7). Than for $t \in I_{\alpha,\beta}$ we have $|\gamma_{\alpha,\beta}(t)| \leq R$. Let us fix $(\alpha, \beta) \in B$. Then

$$|(f \circ \gamma_{\alpha,\beta})'(t)| \leq \sum_{j=1}^{d} \sum_{|\nu| = j} j |a_{\nu}| R^{j-1}$$

and

$$|(f \circ \gamma_{\alpha,\beta})''(t)| \leq \sum_{j=1}^{d} \sum_{|\nu| \leq j} j(j-1) |a_{\nu}| R^{j-2}.$$ 

Consequently,

$$|(f \circ \gamma_{\alpha,\beta})'(t)| \leq D, \quad |(f \circ \gamma_{\alpha,\beta})''(t)| \leq D \quad \text{for} \quad t \in I_{\alpha,\beta}.$$ 

Take any $\xi \in \mathbb{R}^n$, then

$$|\gamma_{\alpha,\beta}(t) - \xi|^2 = \langle \beta t + \alpha - \xi, \beta t + \alpha - \xi \rangle$$

$$= t^2 - 2\langle \beta, \xi \rangle t + |\alpha|^2 - 2\langle \alpha, \xi \rangle + |\xi|^2,$$

then for $p = -2\langle \beta, \xi \rangle$ and $q = |\alpha|^2 - 2\langle \alpha, \xi \rangle + |\xi|^2$, we have $p^2 \leq 4q$ and

$$\varphi_{N} \circ \gamma_{\alpha,\beta}(t) = e^{N(t^2 + pt + q)} f(\gamma_{\alpha,\beta}(t)).$$

So, by Lemma 2.1 we get that $(\varphi_{N} \circ \gamma_{\alpha,\beta})''(t) \geq \frac{-p^2}{m} - D + 2Nm > 0$ for $t \in I_{\alpha,\beta}$ and $\varphi_{N,\xi}$ is strongly convex on $X$, provided $N > \mathcal{N}(m, D)$. \qed

From Theorem 2.3 we obtain the following corollary.

**Corollary 2.4.** Let $f \in \mathbb{R}[x]$ and let $X \subset \mathbb{R}^n$ be a compact and convex set. Let $R = \max\{|x| : x \in X\}$ and let $m \in \mathbb{R}$ be a constant such that

$$m \leq \min\{f(x) : x \in X\}.$$ 

Than for any $D > D_n(f, R)$ and any $\xi \in \mathbb{R}^n$, the function

$$\varphi_{\xi}(x) := e^{\frac{|x - \xi|^2}{r}} [f(x) - m + D], \quad x \in \mathbb{R}^n$$


is strongly convex on $X$.

By a similar argument as in the proof of Theorem 2.3 we obtain the following fact.

**Remark 2.5.** Let $f: \mathbb{R}^n \to \mathbb{R}$ be a function of class $C^2$ and let $X \subset \mathbb{R}^n$ be a compact and convex set. Assume that $m, D \in \mathbb{R}$ are numbers satisfying

$$m < \min\{f(x) : x \in X\}$$

and the first and second directional derivatives of $f$ in directions of vectors of length 1, are bounded by $D$ on $X$. Then the function

$$\varphi_\xi(x) = e^{\|x-\xi\|^2}[f(x) - m + D], \quad x \in \mathbb{R}^n, \quad \xi \in \mathbb{R}^n$$

is strongly convex on $X$.

### 2.3. Convexifying polynomials with integer coefficients.

For actual applications of Theorem 2.3 it is important to compute the number $N(m, D)$ for a given convex semialgebraic set $X$ and a polynomial $f$ which is positive on $X$. Hence the main difficulty is to compute (or rather estimate) $m = \min\{f(x) : x \in X\}$ and $R = \max\{|x| : x \in X\}$. This actually possible if we suppose that $f$ has integer coefficients and $X$ is described by equations and inequalities with integer coefficients.

More precisely, let $X \subset \mathbb{R}^n$, $n \geq 2$, be a compact semialgebraic set of the form

$$(2.9) \quad X = \{x \in \mathbb{R}^n : g_1(x) = 0, \ldots, g_l(x) = 0, g_{l+1}(x) \geq 0, \ldots, g_k(x) \geq 0\},$$

where $g_1, \ldots, g_k \in \mathbb{Z}[x]$. Under the above notations G. Jeronimo, D. Perrucci, E. Tsigaridas in [2] proved that

**Theorem 2.6.** Let $f, g_1, \ldots, g_k \in \mathbb{Z}[x]$ be polynomials with degrees bound by an even integer $d$ and coefficients of absolute values at most $H$, and let $\hat{H} = \max\{H, 2n + 2k\}$. If $f(x) > 0$ for $x \in X$ and $X$ of the form $\mathbf{(2.9)}$ is compact, then

$$\min\{f(x) : x \in X\} \geq \left(2^{4n-\frac{3}{2}} \hat{H} d^n\right)^{-n^2 d^m}.$$

For a positive real number $H$ and positive integers $d, n, k$ we put

$$b(n, d, H, k) = \left(2^{4n-\frac{3}{2}} \max\{H, 2n + 2k\} d^n\right)^{-n^2 d^m}.$$

From Theorems 2.6 and 2.3 we immediately obtain

**Theorem 2.7.** Let $X \subset \mathbb{R}^n$ be a compact and convex semialgebraic set of the form $\mathbf{(2.9)}$ and let $f, g_1, \ldots, g_k \in \mathbb{Z}[x]$ be polynomials with degrees bound by an even integer $d$ and coefficients of absolute values at most $H$. Set

$$R = \sqrt{[b(n + 1, \max\{d, 4\}, H, k + 2)]^{-1} - 1}, \quad m = b(n, d, H, k).$$
Then
\[ (2.10) \quad \max \{|x| : x \in X\} \leq R. \]
Moreover, if \( f(x) > 0 \) for \( x \in X \), then for any \( D \geq D_n(f, R), \ N > N(m, D) \) and for any \( \xi \in \mathbb{R}^n \) the function
\[ \varphi_{N, \xi}(x) := e^{N|x-\xi|^2}f(x) \]
is strongly convex on \( X \).

Proof. By Theorem 2.6 we have \( 0 < m \leq \min \{f(x) : x \in X\} \). Let
\[ Y = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} : x \in X, \ (1+|x|^2)y^2-1 = 0, \ y \geq 0\}, \]
and let \( h(x, y) = y^2 \). Then \( Y \subset \mathbb{R}^{n+1} \) is a compact semialgebraic set defined by \( k + 2 \) polynomial equations and inequalities of degrees bounded by \( \max \{d, 4\} \). Moreover, the absolute values of coefficients of those polynomials and \( h \) are bounded by \( H \). Then, by Theorem 2.7,
\[ \min \{h(x, y) : (x, y) \in Y\} \geq b(n+1, \max \{d, 4\}, H, k+2) \]
and consequently we obtain (2.10). Summing up, Theorem 2.3 gives the assertion. □

3. Convexifying polynomials on non-compact sets

In this section we will show that the function \( \varphi_N(x) = e^{N|x|^2}f(x) \) in \( n \) variables is strongly convex on a closed and convex set \( X \subset \mathbb{R}^n \) (not necessary compact), provided the polynomial \( f \) takes values larger than a certain number \( m > 0 \), the leading form of a polynomial \( f \) has only positive values and \( N \) is sufficiently large.

3.1. Convexifying polynomials in one variable. For a polynomial \( f \in \mathbb{R}[t] \) of the form \( f(t) = a_0t^d + a_1t^{d-1} + \cdots + a_d, \ a_0, \ldots, a_d \in \mathbb{R}, \ a_0 \neq 0 \), we put
\[ K(f) := 2 \max_{1 \leq i \leq d} \left| \frac{a_i}{a_0} \right|^{1/i}. \]

Lemma 3.1. Let \( f \in \mathbb{R}[t] \) be a polynomial of degree \( d > 0 \) which is positive on a closed interval \( I \subset \mathbb{R} \) (not necessary compact). Let \( m \in \mathbb{R} \) be a positive number such that
\[ \inf \{f(t) : t \in I\} \geq m. \]
Let \( g_N \in \mathbb{R}[t] \) be a polynomial of the form
\[ g_N = 2Nf^2 - (f')^2 + ff'', \]
and let \( \Theta_N \in \mathbb{R}[t, \xi] \) be a polynomial of the form
\[ (3.1) \quad \Theta_N(t, \xi) := 4N^2(t-\xi)^2f(t) + 4N(t-\xi)f'(t) + 2Nf(t) + f''(t) \]
for \( N \in \mathbb{R} \) and \( N \geq 1 \). Then for \( N \geq \mathcal{N}(m, D) \), where \( D \geq D_1(f, R) \) and \( R \geq \max\{K(f), K(g_1)\} \), we have
\[
\Theta_N(t, \xi) > 0 \quad \text{for} \ (t, \xi) \in I \times \mathbb{R}.
\]

**Proof.** Consider the following quadratic function in \( x \)
\[
4N^2x^2f(t) + 4Nx f'(t) + 2N f(t) + f''(t).
\]
Then its discriminant is of the form \( \Delta(t) = -16N^2g_N(t) \). Take \( R \geq \max\{K(f), K(g_1)\}, D \geq D_1(f, R) \) and \( N > \mathcal{N}(m, D) \). Then we have
\[
g_N(t) \geq 2Nf^2(t) - D^2 - f(t)D \geq f^2(t) \left( 2N - \frac{D^2}{m^2} - \frac{D}{m} \right) > 0
\]
for \( t \in I, |t| \leq R \). On the other hand \( g_N(t) \geq g_1(t) > 0 \) for \( t \in I \), \( |t| \geq R \). So \( \Delta(t) < 0 \) for \( t \in I \) and we deduce the assertion. \( \square \)

**Theorem 3.2.** Let \( f \in \mathbb{R}[t] \) be a polynomial of degree \( d > 0 \) and let \( I \subset \mathbb{R} \) be a closed interval (not necessary compact). Assume that there exists \( m \in \mathbb{R} \) such that
\[
0 < m \leq \inf \{f(t) : t \in I\}.
\]
Let \( R > \max\{K(f), K(2f^2 - (f')^2 + ff'')\} \) and \( D \geq D_1(f, R) \). Then for any \( N \in \mathbb{R}, N > \mathcal{N}(m, D) \), and any \( \xi \in \mathbb{R} \) the function
\[
\varphi_{N, \xi}(t) = e^{N(t-\xi)^2}f(t)
\]
is strongly convex on \( I \).

**Proof.** It suffices to observe that \( \varphi''_{N, \xi}(t) = e^{N(t-\xi)^2}\Theta_N(t, \xi) \) and apply Lemma 3.1. \( \square \)

### 3.2. Convexifying polynomials in several variables

**Theorem 3.3.** Let \( X \subset \mathbb{R}^n \) be a convex closed set. Assume that \( f \) is a polynomial of degree \( d > 0 \) which is positive on \( X \),
\[
\text{(3.2)} \quad f_d^{-1}(0) = \{0\}
\]
and there exists \( m > 0 \) such that
\[
\text{(3.3)} \quad \inf \{f(x) : x \in X\} \geq m.
\]
Then there exists \( N_0 \in \mathbb{N} \) such that for any integer \( N \geq N_0 \) and any \( \xi \in \mathbb{R}^n \) the function \( \varphi_{N, \xi}(x) = e^{N|x-\xi|^2}f(x) \) is strongly convex on \( X \).

**Proof.** Take any line of the form \( \gamma_{\alpha, \beta}(t) = \beta t + \alpha \), where \( \alpha, \beta \in \mathbb{R}^n \), \( |\beta| = 1 \) and \( \langle \alpha, \beta \rangle = 0 \). Then
\[
(\varphi_{N, \xi} \circ \gamma_{\alpha, \beta})(t) = e^{N(t^2+|\alpha|^2-2(\beta, \xi)t-2(\alpha, \xi))}f(\gamma_{\alpha, \beta}(t)).
\]
Then
\[(\varphi_{N,\xi} \circ \gamma_{\alpha,\beta})''(t) = e^{N(t^2+a^2-2(\beta,\xi) t-2(\alpha,\xi))}[4N^2((f \circ \gamma_{\alpha,\beta})'(t)) y^2
+ 4N(f \circ \gamma_{\alpha,\beta})'(t) y + 2N(f \circ \gamma_{\alpha,\beta})(t) + (f \circ \gamma_{\alpha,\beta})''(t)],\]
where \(y = t + \langle \beta, \xi \rangle\). Consider the function in the square bracket as a quadratic function in \(y\). Then its discriminant is of the form
\[\Delta = -16N^2[2N(f \circ \gamma_{\alpha,\beta})^2(t) + (f \circ \gamma_{\alpha,\beta})(t)(f \circ \gamma_{\alpha,\beta})'(t) - ((f \circ \gamma_{\alpha,\beta})'(t))^2].\]

Note that \((f \circ \gamma_{\alpha,\beta})'(t)\) and \((f \circ \gamma_{\alpha,\beta})''(t)\) are the first and the second directional derivatives of \(f\) at \(\gamma_{\alpha,\beta}(t)\) in the direction \(\beta\) and \(|\beta| = 1\).

Observe that there exists \(N_0\) such that for any \(N \geq N_0\) we have \(\Delta < 0\). Indeed, it suffices to prove that for any \(x \in X\) and any \(\beta \in \mathbb{R}^n\), \(|\beta| = 1\) we have
\[(3.4) \quad 2N f(x)^2 + f(x) \partial_\beta^2 f(x) - (\partial_\beta f(x))^2 > 0.\]

If \(f_d(x) < 0\) for \(x \in \mathbb{R}^n \setminus \{0\}\) then the set \(X\) is compact and the inequality follows from the assumption that \(f(x) \geq m\) for \(x \in X\). Indeed, let \(D \geq \max\{|\partial_\beta f(x)|, |\partial_\beta^2 f(x)|\}\) for \(x \in X\), \(|\beta| = 1\). Since \(X\) is compact, then
\[2N f(x)^2 + f(x) \partial_\beta^2 f(x) - (\partial_\beta f(x))^2 \geq 2Nm^2 - mD - D^2 > 0\]
for \(N > \mathcal{N}(m, D)\). This gives (3.4).

Consider the case when \(f_d(x) > 0\) for \(x \in \mathbb{R}^n \setminus \{0\}\), and let
\[f_{d*} = \inf\{f_d(x) : x \in S_n\},\]
where \(S_n\) is the unit sphere in \(\mathbb{R}^n\), i.e., \(S_n = \{x \in \mathbb{R}^n : |x| = 1\}\).

Let \(f\) be a polynomial of the form (2.7). We set
\[\|f\| := \sum_{|\nu| \leq d} |a_\nu|.\]
Then \(\|f\| \geq \|f_d\| \geq f_{d*}\). If \(f_{d*} > 0\) then we set
\[\mathcal{K}(f) := \frac{2\|f\|}{f_{d*}}\]
and
\[m(f) := f_{d*} - \sum_{j=0}^{d-1} \mathcal{K}(f)^{j-d} \sum_{|\nu| = j} |a_\nu|.\]

In the further part of the proof we will need the following lemma.

**Lemma 3.4.** If \(d = \deg f > 0\) and \(f_{d*} > 0\), then \(m(f) > 0\) and \(f(x) \geq m(f)|x|^d\) for any \(x \in \mathbb{R}^n\) such that \(|x| \geq \mathcal{K}(f)\).
Proof. Put
\[ h(t) := f_d t^d - d\sum_{j=0}^{d-1} \left| a_{\nu} \right| t^j. \]
Since \( \frac{\|f\|}{f_d} \geq 1 \), then
\[ K(h) = 2 \max_{1 \leq i \leq d} \left| \sum_{|\nu|=d-i} a_{\nu} \right|^{|1/i} f_d \]
and since \( h'(t) > 0 \) for \( t > K(h) \), then \( h(|x|) \geq h(|x|) > 0 \) for \( |x| \geq |x| \). Moreover, \( m(f)K(f)^d = h(|x|) \), so \( m(f) > 0 \). On the other hand
\[ m(f)|x|^d \leq \left( f_d - d\sum_{j=0}^{d-1} |x|^j \sum_{|\nu|=j} |a_{\nu}| \right) |x|^d = h(|x|) \leq f(x) \]
for \( |x| \geq |x| \). This gives the assertion of Lemma 3.4. \( \square \)

Take \( R \geq |x| \), and \( D \geq D_n(f, R) \) then for \( N \geq N(m, D) \) we have
\[ 2Nf(x)^2 + f(x)\partial_\beta f(x) - (\partial_\beta f(x))^2 \geq 2Nm^2 - mD - D^2 > 0 \]
for \( |x| \leq R \).

For \( |x| \geq R \), we have
\[ 2Nf(x)^2 + f(x)\partial_\beta f(x) - (\partial_\beta f(x))^2 \]
\[ \geq 2Nm^2(f)|x|^{2d} - m(f)|x|^dD_n(f, |x|) - D^2_n(f, |x|). \]
Since for \( |x| \geq 1 \),
\[ D_n(f, |x|) \leq D_n(f, 1)|x|^{d-1}, \]
then for \( |x| \geq R \) and \( |\beta| = 1 \) we have
\[ 2Nf(x)^2 + f(x)\partial_\beta f(x) - (\partial_\beta f(x))^2 \]
\[ \geq 2Nm^2(f)|x|^{2d} - m(f)D_n(f, 1)|x|^{2d-1} - D^2_n(f, 1)|x|^{2d-2} \]
\[ \geq |x|^{2d}[2Nm^2(f) - m(f)D_n(f, 1) - D^2_n(f, 1)] > 0. \]
for \( N > N(m(f), D_n(f, 1)) \). This gives (3.4). Moreover, there exists \( \epsilon > 0 \) such that
\[ 2Nf(x)^2 + f(x)\partial_\beta f(x) - (\partial_\beta f(x))^2 > \epsilon. \]
for any \( x \in X \) and \( |\beta| = 1 \).

From (3.4) and the above there follows that \( \Delta < -16N^2\epsilon \) for any \( \alpha, \beta \) and \( t \) such that \( \gamma_{\alpha, \beta}(t) \in X \). Since \( f(x) \geq m \) for \( x \in X \), then \( (\varphi_{N, \xi} \circ 2N^2\epsilon)(t) \geq \frac{\epsilon}{m} \) if \( \gamma_{\alpha, \beta}(t) \in X \). This gives the assertion. \( \square \)

By analogous argument as for Theorem 3.3 and under notations of the proof we obtain the following corollary.
Corollary 3.5. Let \( f \in \mathbb{R}[x] \) be a polynomial of degree \( d \) and let \( X \subset \mathbb{R}^n \) be a convex and closed set. Under assumptions of Theorem 3.3 and notations of the proof, for any \( R \geq \mathbb{R}(f) \), \( D \geq D_n(f, R) \) and \( N > \max\{N(m, D), N_m(f, D_n(f, 1))\} \), and any \( \xi \in \mathbb{R}^n \) the function \( \varphi_{N, \xi}(x) = e^{N|x-\xi|^2}f(x) \) is strongly convex on \( X \). In particular the function
\[
\varphi_{\xi}(x) := e^{\|x-\xi\|^2}[f(x) - m + D]
\]
is strongly convex on \( X \).

The assumption (3.2) that \( f_d(x) \neq 0 \) for \( x \neq 0 \), in Theorem 3.3 cannot be omitted as the following example shows.

Example 3.6. Let \( f \in \mathbb{R}[x, y, z] \) be a polynomial of the form
\[
f(x, y, z) = (y^2 + z^2 + 1) [(x - 1)^2(x + 1)^2 + (yz + 1)^2 + y^2].
\]
Since \( (y^2 + z^2 + 1) [(yz + 1)^2 + y^2] \geq \frac{1}{2} \) for \( y, z \in \mathbb{R}^2 \) then we easily see that
\[
f(x, y, z) \geq \frac{1}{2} \quad \text{for} \quad (x, y, z) \in \mathbb{R}^3.
\]
Note that \( \deg f = 6 \), and the leading form \( f_6(x, y, z) = (y^2 + z^2)(x^4 + y^2z^2) \) has nontrivial zeroes.

Now take any \( N \in \mathbb{R} \) and \( \varphi_N(x, y, z) = e^{N(x^2+y^2+z^2)}f(x, y, z) \). Then for \( \xi \neq 0 \) we have
\[
\varphi_N(0, \xi^{-1}, -\xi) = e^{N(\xi^{-2}+\xi^2)}(\xi^{-2} + \xi^2 + 1)(1 + \xi^{-2})
\]
and
\[
\varphi_N(-1, \xi^{-1}, -\xi) = \varphi_N(1, \xi^{-1}, -\xi) = e^{N(\xi^{-2}+\xi^2)}(\xi^{-2} + \xi^2 + 1)e^{N}\xi^{-2}.
\]
Hence for sufficiently large \( \xi \),
\[
\varphi_N(-1, \xi^{-1}, -\xi) = \varphi_N(1, \xi^{-1}, -\xi) < \varphi_N(0, \xi^{-1}, -\xi),
\]
therefore \( \varphi_N \) can not be a convex function.

The assumption (3.2) in Theorem 3.3 cannot be replaced by a condition \( \lim_{|x| \to \infty} f(x) = \infty \). Indeed, consider a modification of the previous example of the form
\[
f_k(x, y, z) = (y^2 + z^2 + 1)^k [(x - 1)^2(x + 1)^2 + (yz + 1)^2 + y^2],
\]
where \( k \geq 2 \). Then \( \lim_{|(x, y, z)| \to \infty} f(x, y, z) = \infty \) and the function \( \varphi_N(x, y, z) = e^{N(x^2+y^2+z^2)}f_k(x, y, z) \) is not convex for any \( N \in \mathbb{R} \) by the previous argument.

It turns out that the use of a double exponential function leads to a convexity of an appropriate function on \( X \). We show it in the next section.
4. Double exponential convexifying polynomials

In this section, without the assumption that the leading form of a polynomial $f \in \mathbb{R}[x]$ in $n$ variables has only positive values, we will show that the function $\Phi_N(x) = e^{N|x|^2} f(x)$ is strongly convex on a closed and convex semialgebraic set $X \subset \mathbb{R}^n$ (not necessary compact), provided the polynomial $f$ takes positive values on $X$ and $N$ is sufficiently large.

**Theorem 4.1.** Let $X \subset \mathbb{R}^n$ be a closed and convex semialgebraic set, and let $f \in \mathbb{R}[x]$ be a polynomial which has only positive values on $X$. Then there exists $N_0 \in \mathbb{R}$ such that for any $N \geq N_0$ the function $\Phi_N(x) = e^{N|x|^2} f(x)$ is strongly convex on $X$.

**Proof.** Let $f$ be of the form (2.7), $d = \deg f$. Then $f(x)$, , the first and second directional derivatives of $f$ in directions of vectors of length 1 at $x \in X$, are bounded by $D(1 + |x|^d)$, where

$$
\hat{D} := |a_0| + \sum_{|\nu|=1} |a_\nu| + \sum_{j=2}^d \sum_j (j-1)|a_\nu|.
$$

Take an affine line in $\mathbb{R}^n$ of the form

$$
\gamma_{\alpha,\beta}(t) := \beta t + \alpha, \quad t \in \mathbb{R},
$$

where $(\alpha, \beta) \in A$ and the set $A$ is defined in (2.8). Then $|\beta| = 1$, $(\alpha, \beta) = 0$, and $|\gamma_{\alpha,\beta}(t)|^2 = t^2 + |\alpha|^2$. Let write the second derivative of $\Phi_N \circ \gamma_{\alpha,\beta}$ in the form

$$
\left(\Phi_N \circ \gamma_{\alpha,\beta}\right)''(t) = e^{N(t^2 + |\alpha|^2)} (a(t)t^2 + b(t)t + c(t)),
$$

where

$$
a(t) = 4N^2(e^{N(t^2+|\alpha|^2)} + e^{2N(t^2+|\alpha|^2)}) f \circ \gamma_{\alpha,\beta}(t),
$$

$$
b(t) = 4Ne^{N(t^2+|\alpha|^2)} f \circ \gamma_{\alpha,\beta}'(t),
$$

$$
c(t) = 2Ne^{N(t^2+|\alpha|^2)} f \circ \gamma_{\alpha,\beta}(t) + (f \circ \gamma_{\alpha,\beta})''(t)
$$

The discriminant of the polynomial $P_t(\lambda) = a(t)\lambda^2 + b(t)\lambda + c(t)$ is of the form

$$
\Delta = 16N^2 e^{2N(t^2+|\alpha|^2)} \left[ \left( (f \circ \gamma_{\alpha,\beta})'(t) \right)^2 \right.
$$

$$
- f \circ \gamma_{\alpha,\beta}(t) (f \circ \gamma_{\alpha,\beta})''(t) \left( 1 - e^{-N(t^2+|\alpha|^2)} \right)
$$

$$
- 2N (f \circ \gamma_{\alpha,\beta})^2(t) \left( 1 + e^{N(t^2+|\alpha|^2)} \right).
$$


So, by the choice of the number $\tilde{D}$, we have

$$\Delta \leq 32N^2e^{2N(t^2 + |\alpha|^2)} \left[ \tilde{D}^2 \left( 1 + |\gamma_{\alpha,\beta}(t)|^d \right)^2 - N(f \circ \gamma_{\alpha,\beta})^2(t) \left( 1 + e^{N|\gamma_{\alpha,\beta}(t)|^2} \right) \right].$$

Since the set $X$ is semialgebraic and $f^{-1}(0) \cap X = \emptyset$, then by Hörmander-Lojasiewicz inequality, see e.g. [4, Corollary 2.4], there exist $C, K, \mathcal{L} > 0$, where $K, \mathcal{L} \in \mathbb{Z}$, $K \geq d$, depend on $d$ and the complexity of $X$, (i.e., degrees and the number of polynomials describing $X$) such that

$$f(x) \geq C \left( 1 + |x|^K \right)^{-\mathcal{L}} \quad \text{for} \quad x \in X.$$

Moreover, the numbers $K, \mathcal{L}$ are effectively computable. By the above,

$$\Delta \leq 32N^2e^{2N(t^2 + |\alpha|^2)} \left( 1 + |\gamma_{\alpha,\beta}(t)|^K \right)^{-\mathcal{L}} \left[ \tilde{D}^2 \left( 2 + |\gamma_{\alpha,\beta}(t)|^K \right)^{\mathcal{L} + 2} - NC^2 \left( 1 + e^{N|\gamma_{\alpha,\beta}(t)|^2} \right) \right].$$

If $N$ is large enough, then for any $x \in \mathbb{R}^n$ we have

$$\tilde{D}^2(2 + |x|^K)^{\mathcal{L} + 2} < NC^2(1 + e^{N|x|^2}).$$

Therefore $\Delta < 0$, so $P_t(\lambda) > 0$ for any $\lambda \in \mathbb{R}$. Consequently

$$(\Phi_N \circ \gamma_{\alpha,\beta})''(t) = e^{N(t^2 + |\alpha|^2)} P_t(t) > 0,$$

for $t \in \mathbb{R}$. Note that $\lim_{|t| \to \infty}(\Phi_N \circ \gamma_{\alpha,\beta})''(t) = +\infty$, hence there exists $\mu > 0$ such that $(\Phi_N \circ \gamma_{\alpha,\beta})''(t) \geq \mu$ for $t \in \mathbb{R}$. Moreover, the number $\mu$ can be chosen independent of $\gamma_{\alpha,\beta}$. This gives the assertion. \qed

**Remark 4.2.** The number $N_0$ in Theorem 4.1 can be effectively computed, provided we can estimate the constant $C$. More precisely, under notations in the proof, if $k > (\mathcal{L} + 2)K$, then for $|x| \geq 1$ we have

$$NC^2 \left( 1 + e^{N|x|^2} \right) \geq NC^2 + \sum_{j=0}^{k} \frac{C^2N^{j+1}}{j!} |x|^{2j} > \tilde{D}^2 \left( 2 + |x|^K \right)^{\mathcal{L} + 2}$$

for

$$N > k! \max_{i=0, \ldots, \mathcal{L}+2} \left( \tilde{D}^2C^{-2}2^{\mathcal{L}+2-i} \left( \frac{\mathcal{L} + 2}{i} \right) \right).$$

If additionally $N \geq \tilde{D}^2C^{-2}3^{\mathcal{L} + 2}$, then the above inequality holds for any $x \in \mathbb{R}^n$.

**Remark 4.3.** We cannot omit the assumption in Theorem 4.1 that the set $X$ is semialgebraic. For instance if $f(x, y) = -y^2 + y$ and $X = \{(x, y) \in \mathbb{R}^2 : e^{-x} \leq y \leq \frac{1}{2}, \ x \geq 0\}$, then $f(x, y) > 0$ on $X$, but the function $\Phi_N(x, y)$ is not convex on $X$ for any $N \in \mathbb{R}$. 

Assuming that $f(x) \geq m$ on $X$, for some $m > 0$, we can omit the assumption in Theorem 4.1 on semialgebraicity of $X$. More precisely, by a similar argument as in the proof of Theorem 4.1 we obtain

**Theorem 4.4.** Let $X \subset \mathbb{R}^n$ be a closed and convex set, and let $f \in \mathbb{R}[x]$ be a polynomial such that $f(x) \geq m$ for $x \in X$ and some $m > 0$. Then there exists $N_0 \in \mathbb{R}$ such that for any $N \geq N_0$ the function $\Phi_N(x) = e^{N|x|^2}f(x)$ is strongly convex on $X$.

**Remark 4.5.** By a similar argument as for the proof of Theorem 4.4 we obtain that the assertion of this Theorem occurs not only for the function $\Phi_N$ but also for the function $\Phi_N(x) = e^{N|x|^2}f(x)$. More precisely, we have:

Let $X \subset \mathbb{R}^n$ be a closed and convex set, and let $f \in \mathbb{R}[x]$ be a polynomial such that $f(x) \geq m$ for $x \in X$ and some $m > 0$. Then there exists $N_0 \in \mathbb{R}$ such that for any $N \geq N_0$ the function $\Phi_N$ is strongly convex on $X$.

A similar argument as for Theorem 4.1 gives the following theorems.

**Theorem 4.6.** Let $X \subset \mathbb{R}^n$ be a convex closed semialgebraic set and let $r > 0$. If $f$ is a polynomial such that

\[(4.1) \quad f(x) > 0 \quad \text{for} \quad x \in X,\]

then there exists $N_0 \in \mathbb{N}$ such that for any integer $N \geq N_0$ and any $\xi \in \mathbb{R}^n$, $|\xi| \leq r$, the function $\Phi_{N,\xi}(x) = e^{N|x-\xi|^2}f(x)$ is strongly convex on $X$.

Moreover, there exists $\alpha \in \mathbb{R}$ such that the function $\Phi_{\xi}(x) = e^{N|x-\xi|^2}[f(x) + \alpha], \quad x \in \mathbb{R}^n$ is strongly convex on $X$, provided $\xi \in \mathbb{R}^n$, $|\xi| \leq r$.

**Theorem 4.7.** Let $X \subset \mathbb{R}^n$ be a convex closed set and let $r > 0$. Assume that $f$ is a polynomial of degree $d > 0$ such that there exists $m \in \mathbb{R}$ such that

\[(4.2) \quad 0 < m < \inf\{f(x) : x \in X\}.\]

Then there exists $N_0 \in \mathbb{N}$ such that for any integer $N \geq N_0$ and any $\xi \in \mathbb{R}^n$, $|\xi| \leq r$ the function $\Phi_{N,\xi}(x) = e^{N|x-\xi|^2}f(x)$ is strongly convex on $X$.

Moreover, there exists $\alpha \in \mathbb{R}$ such that the function $\Phi_{\xi}(x) = e^{N|x-\xi|^2}[f(x) + \alpha], \quad x \in \mathbb{R}^n$ is strongly convex on $X$, provided $\xi \in \mathbb{R}^n$, $|\xi| \leq r$.

It is impossible to obtain $N$ in the above theorem, such that the function $\Phi_{N,\xi}$ is convex for any $\xi \in X$ as the following example shows.
Example 4.8. Let \( f \in \mathbb{R}[x, y, z] \) be a polynomial of the form
\[
f(x, y, z) = ((yz + 1)^2 + y^2) \left( (xz^2 - 1)^2 \left( xz^2 + 1 \right)^2 + y^2 + z^2 + 1 \right).
\]
Analogously as in Example 3.6 we see that \( f(x, y, z) \geq \frac{1}{2} \) for \((x, y, z) \in \mathbb{R}^3\), and the leading form \( f_{16}(x, y, z) = y^2 z^{10} x^4 \) has nontrivial zeroes.

Now take any \( N \in \mathbb{R} \) and \( \Phi_N(x, y, z) = e^{e^{N(x^2+y^2+z^2)}} f(x, y, z) \). Then for \( \xi = (0, t^{-1}, -t) \), \( t > 0 \) we have
\[
\frac{\partial^2 \Phi_N(\xi)}{\partial x^2}(\xi) = e \left( 2N f(\xi) + \frac{\partial^2 f}{\partial x^2}(\xi) \right).
\]
Since
\[
f(\xi) = 2t^{-2} + t^{-4} + 1
\]
and
\[
\frac{\partial^2 f}{\partial x^2}(\xi) = -4t^2,
\]
then we easily see that \( \frac{\partial^2 \Phi_N(\xi)}{\partial x^2}(\xi) < 0 \) for sufficiently large \( t \). So, \( \Phi_N,\xi \) can not be a convex function.

Remark 4.9. It is worth noting that the use of triple exponential convexifying \( \phi_N(x) = e^{e^{N|x|^2}} f(x) \) of a polynomial \( f \) does not improve convexity of the function \( \phi_N,\xi(x) = e^{e^{N|x-\xi|^2}} f(x) \) regardless of \( \xi \in X \).

5. Algorithm for searching lower critical points

5.1. Searching lower critical points in a compact set. In this part we give an algorithm which produces, starting from an arbitrary point, a sequence of points converging to a lower critical point of a polynomial on a convex compact semialgebraic set. A similar algorithm was proposed in [3].

Let \( X \subset \mathbb{R}^n \) be a closed set and let \( f \) be a function of class \( C^1 \) in a neighborhood \( U \subset \mathbb{R}^n \) of \( X \). We denote the set of lower critical points of the function \( f \) on the set \( X \) by \( \Sigma_X f \). It is obvious that the set of ordinary critical points \( \Sigma f \) of the function \( f \) is contained in the set \( \Sigma_X f \).

Our algorithm for approximation of lower critical points of \( f \) is based on the iteration of computation of the smallest value of the strongly convex function \( \varphi_\xi \) on the convex and compact set \( X \). More precisely, let
\[
R \geq \max\{|x| : x \in X\}.
\]
Take any polynomial $f \in \mathbb{R}[x]$ of the form (2.7). Let

$$m = - \sum_{j=0}^{d} R^j \sum_{|\nu|=j} |a_\nu|,$$

and let

$$D > D_n(f, 2R).$$

Then we have

$$f(x) - m + D \geq D$$

for $x \in X$, and from Corollary 2.4, we have that for any $\xi \in X$, the function

$$\varphi_\xi(x) = e^{||x-\xi||^2} [f(x) - m + D], \quad x \in \mathbb{R}^n$$

is $\mu$-strongly convex on $X$ for some $\mu > 0$. Since we are looking for lower critical points of $f$, so without loss of generality, we may assume that $-m + D = 0$, therefore

$$\varphi_\xi(x) = e^{||x-\xi||^2} f(x), \quad x \in \mathbb{R}^n$$

is $\mu$-strongly convex function in $X$ for any $\xi \in X$.

Any strictly convex function $\varphi$ defined on a compact and convex set $X$ has the unique point, denoted by $\arg\min_X \varphi$, in which the function $\varphi$ has the minimal value on the set $X$. Therefore, choosing an arbitrary point $a_0 \in X$, we can determine by induction a sequence $a_\nu \in X$, $\nu \in \mathbb{N}$, in the following way

$$a_\nu := \arg\min_X \varphi_{a_{\nu-1}} \text{ for } \nu \geq 1.$$  \hspace{1cm} (5.1)

**Theorem 5.1.** Let $X \subset \mathbb{R}^n$ be a compact convex semialgebraic set and $f : \mathbb{R}^n \rightarrow \mathbb{R}$ a positive polynomial on $X$. Let $a_\nu$ be a sequence defined as $a_\nu := \arg\min_X \varphi_{a_{\nu-1}}$ with $a_0 \in X$. Then the limit

$$a_* = \lim_{\nu \rightarrow \infty} a_\nu$$

exist and $a_* \in \Sigma_X f$.

The proof of Theorem 5.1 follows word by word the proof of Theorem 6.5 in [3], where we should use the following three lemmas instead of the corresponding lemmas in [3].

**Lemma 5.2.** For any $\nu \in \mathbb{N}$, we have

$$|a_{\nu+1} - a_\nu| = \text{dist}(a_\nu, f^{-1}(f(a_{\nu+1})) \cap X).$$

**Lemma 5.3.** For any $\nu \in \mathbb{N}$ we have

$$f(a_{\nu+1}) \leq \frac{f(a_\nu) - \frac{\mu}{2}|a_{\nu+1} - a_\nu|^2}{e^{|a_{\nu+1} - a_\nu|^2}}.$$  

In particular the sequence $f(a_\nu)$ is decreasing.
Proof. Since $\varphi$ is strongly convex, the definition of $a_{\nu+1}$ implies that the function

$$[0,1] \ni t \mapsto \varphi_{a_{\nu}}(a_{\nu} + t(a_{\nu+1} - a_{\nu}))$$

decrease, so $\langle a_{\nu+1} - a_{\nu}, \nabla \varphi_{a_{\nu}}(a_{\nu+1}) \rangle \leq 0$. Again by the fact that $\varphi_{a_{\nu}}$ is $\mu$-strictly convex, we get

$$f(a_{\nu}) \geq f(a_{\nu+1})e^{|a_{\nu} - a_{\nu+1}|^2} + \frac{\mu}{2}|a_{\nu} - a_{\nu+1}|.$$ 

This gives the assertion. \qed

We can also adapt the following lemma ([3, Lemma 6.3]).

**Lemma 5.4.** Let $f : [0, \eta] \to \mathbb{R}$ be a $C^1$ function such that $0 < f \leq C$ and $f' \leq -\eta$ on $[0, \eta]$ for some $C \geq \frac{1}{2}$ and $\eta > 0$. Assume that $\varphi(x) = e^{x^2}f(x)$ is strictly convex on $[0, \eta]$. Then $b_1 := \text{argmin}_{[0,\eta]} \varphi \geq \frac{\eta}{2C}$. Hence $f(0) - f(b_1) \geq \frac{\eta^2}{2C}$.

**Remark 5.5.** The function $\varphi_{a_{\nu-1}}$ is defined by using the function exp. However, to determine the minimum value of this function on a compact convex semialgebraic set $X$, it is enough to solve only polynomial equations and inequalities. More precisely, the set $X$ is the union of a finite collection of basic semialgebraic sets, so we may assume that $a_{\nu} \in X = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \ldots, g_k(x) \geq 0\}$, where $g_1, \ldots, g_k \in \mathbb{R}[x]$. Then

$$X = \{x \in \mathbb{R}^n : g_1(x)e^{||x - a_{\nu-1}||^2} \geq 0, \ldots, g_k(x)e^{||x - a_{\nu-1}||^2} \geq 0\}.$$ 

Therefore, when applying Lagrange Multipliers or Karush-Kuhn-Tucker Theorem to compute the point $a_{\nu}$, it is enough to solve a system of polynomial equations and inequalities.

### 5.2. Searching lower critical points in an unbounded set.

Let $X \subset \mathbb{R}^n$ be a convex and closed semialgebraic set. Let $f \in \mathbb{R}[x]$ be a polynomial of degree $d > 0$ of the form (2.7) and let $f_d$ be the leading form of $f$. Assume that $f_{d*} > 0$.

Then by Theorem 3.3, we may effectively compute a real number $N \geq 1$ such that the function $\varphi_{N,\xi}(x) = e^{N||x - \xi||^2}f(x)$ for $\xi \in \mathbb{R}^n$ is strongly convex on $X$. Moreover, $\varphi_{N,\xi}(x) \geq f(x) \geq m(f)||x||^d$ for $x \in X$, $||x|| \geq \mathbb{K}(f)$, so we have

$$\lim_{x \in X, ||x|| \to \infty} \varphi_{N,\xi}(x) = +\infty.$$ 

Then we may uniquely determine the sequence

$$a_{\nu} := \text{argmin}_X \varphi_{a_{\nu-1}} \quad \text{for } \nu \geq 1.$$ 

Analogous argument as for Theorem 5.1 gives the following theorem.
Theorem 5.6. Let $a_\nu$ be a sequence defined by (5.2), Then the limit
\[ a_* = \lim_{\nu \to \infty} a_\nu \]
exist and $a_* \in \Sigma_X f$.

Remark 5.7. If $X \subset \mathbb{R}^n$ is a closed and convex semialgebraic set and a polynomial $f \in \mathbb{R}[x]$ is positive on $X$ and it is proper on $X$ (i.e., $\lim_{x \in X, \|x\| \to \infty} f(x) = +\infty$), then by Theorem 4.6 one can repeat the argument from Theorem 5.1 and obtain a sequence $a_\nu \in X$ such that $\lim_{\nu \to \infty} a_\nu = a_* \in \Sigma_X f$.

If we assume only that $f(x) > 0$ on $X$, then the sequence $a_\nu$ can tend to infinity. Moreover, in the construction of $a_\nu$ we have to change $N$ step by step.

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