Topological Defects Formation after Inflation on Lattice Simulation

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We consider the formation of topological defects after inflation. In order to take into account the effects of the rescattering of fluctuations, we integrate the classical equation that describes the evolution of a complex scalar field on the two-dimensional lattice with a slab symmetry. The growth of fluctuations during preheating is found not to be enough for defect formation, and rather a long stage of the rescattering of fluctuations after preheating is necessary. We conclude that the topological defects are not formed if the breaking scale \( \eta \) is larger than \( \sim (2 - 3) \times 10^{16} \text{GeV} \).

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I. INTRODUCTION

Inflation \( \text{[1,2]} \) was invented in order to overcome several problems in the standard hot big bang universe, such as the flatness and horizon problems. Moreover, harmful topological defects created before inflation were diluted away, and are almost absent in the present universe. If some topological defects such as monopoles or domain walls are produced after inflation, they are disastrous since they will soon dominate the energy density of the universe. Therefore, the reheating temperature cannot be as high as the grand unified theory (GUT) scale so as not to produce the GUT monopoles.

It was recently recognized that topological defects may be formed during preheating \( \text{[3,4]} \). Preheating is the very beginning of the reheating process, and occurs because of the parametric resonant effects \( \text{[3,4]} \). Its essence is an induced effect in the sense that the presence of the produced particles stimulates further decay of the coherently oscillating scalar field into those particles \( \text{[10]} \). This phenomenon is thus peculiar to bosonic particles that obey the Bose-Einstein statistics.

In the preheating stage, very large non-thermal fluctuations are produced, \( \langle \delta \phi^2 \rangle \sim c^2 M_p^2 \) where \( M_p \) is the Planck mass and \( c = 10^{-2} - 10^{-3} \). These fluctuations change the shape of the effective potential of the field \( \phi \) to restore its symmetry if the potential \( V(\phi) \) is of spontaneous symmetry-breaking type. Later, when the amplitude of these fluctuations is redshifted away by the cosmic expansion, the symmetry is spontaneously broken and topological defects may be created. Thus, the mechanism for producing the topological defects is somewhat similar to the Kibble mechanism in high temperature theory. However, we showed in Ref. \( \text{[5]} \) that the amplitude of the fluctuations does not grow larger than that of the homogeneous mode for \( V(\phi) = \lambda (\phi^2 - \eta^2)^2 \), where \( \lambda \) is the self-coupling constant. It implies that the symmetry restoration does not take place and hence the topological defects are not produced. Therefore, we considered that topological defects are formed through another mechanism which is based on the following two facts. The first one is that there are two minima of the effective potential in the radial direction in the simple U(1) theory. The second is that some fluctuations exist initially at the preheating stage. They are produced during inflation and stretched far beyond the horizon to become classical fields. Therefore, the initial condition for the homogeneous field in some region in the universe is different from that in another region in the universe. The dynamics of each region in the universe is thus different and the final value of the field (the minimum into which the field settles down) is different, which leads to the defect formation.

However, in Ref. \( \text{[3]} \), we used the Hartree approximation so that the interactions between particles with different momenta are not fully included. As we will see later, rescatterings \( \text{[11–15]} \) are quite important to determine the dynamics of the defect formation as suggested by Refs. \( \text{[16,17]} \). To deal with the effect of the rescattering, we must go on to the lattice simulation \( \text{[14,15]} \). Therefore, in this paper, we integrated the equations describing the dynamics in a real space on the lattice. There are some difficulties in the simulations on lattice. One of the difficulties is how we can implement the quantum fluctuations of the fields on the lattice. We rely on the idea of Ref. \( \text{[1]} \) and do not discuss in detail (actually, it is not sensitive to the initial conditions for the fluctuations if the order of magnitude is properly taken \( \text{[13]} \)). The other one is the box size and the lattice size. The lattice size must be small enough for identifying the defects. The box size should not be taken too much smaller than the horizon size at the end of the simulation, otherwise we miss the defects whose density is \( O(10) \) per horizon. We also take the proper lattice and box sizes so that we can see the resonance effects. From these requirements, we calculate the evolution of a complex scalar field in a two-dimensional lattice abandoning to calculate it in three dimensions because of the lack of machine memories.

II. MODEL

Let us consider the wine bottle potential

\[ V(\phi) = \lambda (\phi^2 - \eta^2)^2 \]
\[
V(\Phi) = \frac{\lambda}{2} (|\Phi|^2 - \eta^2)^2,
\]
where \(\Phi\) is the complex scalar field, \(\lambda\) is the very small self coupling constant, and \(\eta\) is the breaking scale. This model has a global U(1) symmetry, and cosmic strings are formed when the symmetry is spontaneously broken. The equation of motion becomes
\[
\ddot{\Phi} + 3H \dot{\Phi} - \frac{1}{a^2} \nabla^2 \Phi + \frac{\lambda}{2} (|\Phi|^2 - \eta^2) \Phi = 0,
\]
where \(H\) is the Hubble parameter and the dot denotes differentiation with respect to time \(t\). Rescaling as
\[
a(\tau) = \sqrt{\frac{\lambda}{\eta}} \Phi(0),
\]
\[
\phi = \Phi a(\tau),
\]
\[
\xi = \sqrt{L} \Phi(0),
\]
where \(\Phi(0) \equiv |\Phi(0)|\), then, setting \(a(0) = 1\), Eq.(4) becomes
\[
\ddot{\phi} - \frac{a''}{a} \phi - \nabla^2 \phi + (|\phi|^2 - \tilde{\eta}^2 a^2) \phi = 0,
\]
where \(\tilde{\eta} \equiv \eta / \Phi(0)\) and the prime denotes differentiation with respect to \(\tau\). The second term of RHS can be omitted since the energy density of the universe behaves like radiation at the early time and also the scale factor becomes very large later. Therefore, we assume that the universe is radiation dominated. In this case, the rescaled Hubble parameter, \(h(\tau) \equiv H(\tau) / \sqrt{L} \Phi(0)\), and the scale factor \(a(\tau)\) become
\[
h(\tau) = \frac{\sqrt{2}}{3} a^{-2}(\tau),
\]
and
\[
a(\tau) = \frac{\sqrt{2}}{3} \tau + 1,
\]
respectively, when \(\Phi\) is assumed to be an inflaton (even if \(\Phi\) is not an inflaton, results are the same as in the case of rescaling the breaking scale in an appropriate way, see Ref. [5]) and we put \(a(0) = 1\). Since the physical length and the horizon grow proportional to \(a\) and \(a^2\), respectively, the rescaled horizon grows proportional to \(a\). The initial length of the horizon is \(\ell_{h}(0) = 3 / \sqrt{2} \approx 2.12\). Therefore, the box size should be larger than the horizon size at the end of calculation, \(\ell_{h}(\tau) = \ell_{h}(0) a(\tau)\).

The width of the topological defect is \((\sqrt{\lambda} \eta)^{-1}\) which corresponds to \((\tilde{\eta} a(\tau))^{-1}\) in the rescaled scale. Since it decreases with time, one lattice length should be at least comparable with the defect width at the end of the calculation.

Furthermore, in order to see the resonance effects, both the lattice size should be small enough and the box size should be large enough. Typical resonant momentum is \(k \approx \sqrt{\lambda} \Phi(0)\) at the beginning. Since, in the \(\lambda|\Phi|^4\) theory, the resonance band does not change because of cosmic expansion, the lattice size should be smaller than \(\Delta x \sim k_{res}^{-1} \sim (\sqrt{\lambda} \Phi(0))^{-1}\). It corresponds to \(\sim 1\) in a rescaled unit. In addition, it is necessary to have the box size large enough for a good resolution in the resonance band.

Three requirements lead us to take very large lattice sizes at the expense of one dimension in space. Thus we integrate Eq.(4) on a 4096 \(\times 4096\) lattice with a slab symmetry in a \(z\)-direction. Therefore, the defects that we see on the lattice are the cosmic strings stretched infinitely along the \(z\)-direction.

For the initial conditions we take
During preheating, only those modes in the resonance band grow exponentially through the parametric resonance effect (see Fig. 2). Since the mode equations of the fluctuations in $x$- and $y$-directions can be written as
\[
\delta x''_k + [\tilde{k}^2_x + 3x^2 - \tilde{\eta}^2 a^2]\delta x_k = 0,
\]
\[
\delta y''_k + [\tilde{k}^2_y + x^2 - \tilde{\eta}^2 a^2]\delta y_k = 0,
\]
where back reactions are neglected and $\tilde{k} = k/\sqrt{\Phi_0}$, the resonance mode for each direction is $3/2 < \tilde{k}^2 < \sqrt{3}$ and $\tilde{k}^2_y < 1/2$. As we mentioned above, the fluctuation in $y$-direction grows faster. In the actual calculation, the fastest growing mode is found to be $k \approx 0.47$, which coincides with the initial horizon size, $k^{-1} \approx \ell_h(0)$. We also change the initial amplitude of fluctuations. The difference only appears in the time at the end of preheating $\tau_{ph}$, but its change does not shorten the duration of the rescattering so much [see Eq. (13) below].

After the exponential growth of the fluctuations (the preheating stage), there is a rather long stage of oscillations of the homogeneous mode with small amplitude, which ends when the field settles down into the minimum of the potential at $\tau \approx 700$. During this stage, rescatterings of the fluctuations become important. We can see that the amplitude of the fluctuations slightly grows in Fig. 2, but redistribution of the spectrum is much more efficient.

Let us consider the criterion for the defect formation. The most simple idea is to compare the time when the amplitude of fluctuations grows because of the parametric resonance ($\tau_{ph}$) with that when the field settles down into the minimum of its potential ($\tau_{fall}$). More precisely, $\tau_{ph}$ is the time when the homogeneous field decays through the parametric resonance until the back reactions of the created fluctuations cannot be neglected.
This usually happens when the amplitude of fluctuations becomes as large as that of the homogeneous mode. It is estimated from

\[ \delta \varphi(0) \exp(\mu \tau_{ph}) \sim 1, \]  

where \( \mu \) is the effective growth exponent. In the case of Eq. (12), the maximum value of \( \mu \) is \( \mu_{\text{max}} \simeq 0.147 \) [18, 15]. Since \( \tau_{ph} \simeq 135 \) for \( \delta \varphi(0) = 10^{-7} \) which we take in Fig. 2, \( \tau_{ph} \simeq 150 \) for \( \delta \varphi(0) = 10^{-8} \), and \( \tau_{ph} \simeq 180 \) in our calculations, we get \( \mu_{\text{eff}} \sim 0.12. \)

† \( \mu_{\text{eff}} \) is estimated from Eq. (13). Since the resonance does not occur from the very beginning and becomes effective a little later, the more conventional estimation is \( \mu \simeq 0.141. \) Actually, the square of amplitude of the fluctuation grows from \( \sim 3 \times 10^{-15} \) to \( \sim 0.1 \) during resonance (\( \Delta \tau = 110 \)) for \( \delta \varphi(0) = O(10^{-7}). \)

\[ \langle \delta \varphi^2 \rangle \simeq 0.2, \]  

and they cannot affect the dynamics of the field wholly. This fact was also seen in the previous work using the Hartree approximations which contain the forward scatterings only [14]. Therefore, we conclude that the symmetry cannot be restored only through the parametric resonance. However, after the preheating ends, the mode-mixing (rescattering) of the fluctuations becomes important for determining whether or not the topological defects are produced.

Now we are going further into the detail of the rescattering. At the end of preheating, the occupation numbers of the fluctuations are very large only in the resonance band. The typical time scale for the rescattering is estimated as follows [14]. The cross section for a scattering is naively given by

\[ \sigma \sim \lambda^2 / E^2. \]

Here \( E \sim \sqrt{\lambda} \Phi \) is the typical energy (momentum) of the produced particle. However, since a huge amount of particles are present after preheating, the cross section is much larger because of the Bose enhancement. Therefore the phase space distribution function (the number density in \( k \)-mode)

\[ n_k \sim n_\varphi / E^3 \]  

should be multiplied to \( \sigma \). After all, the scattering rate becomes

\[ \Gamma \sim \sigma n_k n_\varphi \sim \sqrt{\Lambda} \Phi. \]  

Using Eqs. (3) and (8), we get a typical time scale for the rescattering \( \tau_{\text{scat}} \sim O(1) \). Then the occupation number at higher momenta will be significant within \( \Delta \tau \sim 30 \). Notice that the effective rescattering is essentially the same mechanism as the parametric resonance in the sense that these are stimulated effects. The process \( \delta \varphi(k_{\text{res}}) + \delta \varphi(k_{\text{res}}) \rightarrow \delta \varphi(k_{\text{low}}) + \delta \varphi(k_{\text{high}}) \) first occurs. Typically,

\[ \Gamma \sim \sigma n_k n_\varphi \sim \sqrt{\Lambda} \Phi. \]
enhancement in the first place such as: 

\[ \delta \varphi(k_{\text{res}}) + \delta \varphi(k_{\text{res}}) \rightarrow \varphi_0 + \delta \varphi(2k_{\text{res}}), \]
\[ \delta \varphi(2k_{\text{res}}) + \varphi_0 \rightarrow \delta \varphi(k_{\text{res}}) + \delta \varphi(k_{\text{res}}), \]
\[ \delta \varphi(2k_{\text{res}}) + \delta \varphi(2k_{\text{res}}) \rightarrow \delta \varphi(k_{\text{res}}) + \delta \varphi(3k_{\text{res}}), \]
\[ \delta \varphi(3k_{\text{res}}) + \varphi_0 \rightarrow \delta \varphi(k_{\text{res}}) + \delta \varphi(2k_{\text{res}}). \]  

These are efficient because of the Bose enhancement, since the occupation number with \( k = 0 \) (homogeneous mode) is large. In this picture some major processes are as follows:

\[ \delta \varphi(k_{\text{res}}) + \varphi_0 \rightarrow \delta \varphi(k_{\text{res}}) + \delta \varphi(k_{\text{res}}), \]
\[ \delta \varphi(k_{\text{res}}) + \delta \varphi(k_{\text{res}}) \rightarrow \delta \varphi(k_{\text{res}}) + \delta \varphi(k_{\text{res}}), \]
\[ \delta \varphi(k_{\text{res}}) + \delta \varphi(k_{\text{res}}) \rightarrow \varphi_0 + \delta \varphi(k_{\text{res}}) + \delta \varphi(2k_{\text{res}}). \]  

The solid line denotes the average. 

We find that the topological defects when \( \eta < 10^{16}\text{GeV} \) at most and can thus conclude that duration of rescattering needs \( \tau \gtrsim 200. \)

Here we investigate cases for \( \eta = 10^{16}\text{GeV} \) and \( \eta = 2 \times 10^{16}\text{GeV} \) in some extent. In the latter case, there are only a few strings in the horizon size (the whole region at the end of calculation), as seen in Fig. 3. The identification of the defects is done by observing the winding at the space point on the lattice as is seen in Fig. 7. Since half of strings are in fact anti-strings and each pair of the string and anti-string is very close to each other, it will annihilate and disappear very soon, which means that the case of \( \eta = 2 \times 10^{16}\text{GeV} \) may be harmless for cosmological history. Actually, the numbers of defects at the end of calculations are 2, 4, 2, and 16 for four runs. On the other hand, for the case of \( \eta = 10^{16}\text{GeV} \), even though the number of strings in the horizon decreases as time goes on, they still exist in a certain amount (see Fig. 8).
FIG. 9. Evolution of the defects for $\eta = 10^{16}$GeV on the lattice at (a) $\tau = 1300$, (b) $\tau = 1600$, (c) $\tau = 1800$, and (d) $\tau = 2000$. Dots and circles denote defects and anti-defects, respectively.

We expect that it will settle down to some kind of a scaling solution later. The evidence that topological defects will survive annihilations is that the average number of strings at the end time of calculation ($\tau = 2000$) is a little larger and the positions of strings and anti-strings are not always pair-like, which is seen in Fig. 9 (d) as opposed for $\eta = 2 \times 10^{16}$GeV in Fig. 6.

IV. CONCLUSIONS

We have considered the formation of topological defects after inflation. To this end, we have integrated the equation which describes the evolution of a complex scalar field with the potential $V(\Phi) = \lambda(|\Phi|^2 - \eta^2)^2/2$ on the two-dimensional lattice with a slab symmetry in the $z$-direction in order to include both parametric resonance effects and rescatterings of fluctuations. We have found that fluctuations produced during preheating do not lead to the defect formation even if their amplitudes become the same order of magnitude as that of the homogeneous mode at the end of preheating. We also have found that the effects of the rescattering of the fluctuations are essential and a rather long duration of the rescattering is necessary for the topological defects to be formed as suggested by Refs. [16,17]. Fluctuations in the resonance mode scatter off each other to make fluctuations with higher and lower momenta. They also knock the field off from the zero mode to broaden and smooth the spectrum, which in turn stimulates further decays of the homogeneous mode into all the modes. Therefore, the topological defects are formed in a similar way as the Kibble mechanism. These processes take a lot of time so that it is necessary for the field not to fall into the minimum of the potential too soon after preheating. This constrains the value of the breaking scale. From our calculations, we conclude that the topological defects are not formed if $\eta \gtrsim (2 - 3) \times 10^{16}$GeV.

The constraint on the breaking scale is higher than that of our previous result in Ref. [5] where the Hartree approximation is adopted. In both cases, the amplitude of fluctuations at the end of preheating is not larger than that of the homogeneous mode (compare Figs. 2 with 10). There are some differences between the calculations with the Hartree approximation and the lattice simulation during preheating. The fluctuations in the $y$-direction have a similar shape, but the amplitude obtained with the use of the Hartree approximation is a little smaller. The fluctuation in $x$-direction in the calculation with the Hartree approximation does not grow until a very late time, which differs completely from the lattice one. The reason is that most of the rescattering effects are neglected in the Hartree approximation while the rescattering is very efficient for producing fluctuations in both $x$- and $y$-directions in the lattice calculations. Therefore, it is not surprising that the critical breaking scale for the formation of topological defects becomes higher in the lattice simulations.

Note added. After submission of this paper, we noticed a paper, Ref. [19], in which numerical calculations were done in three dimensions, and similar results were reported. We agree with the authors of Ref. [19] that rescatterings are less effective in two dimensions to some extent. However, there is little difference in the resonance effects. Actually, $\mu \simeq 0.141$ in ours [see the footnote below Eq.(13)] while $\simeq 0.147$ in Ref. [19]. Moreover, the
maximum value of fluctuations is almost the same. The advantage of two-dimensional simulations is that the box size can be taken as large as the horizon size at the end of calculations while the lattice size can be taken small enough to identify defects. Therefore, we can estimate the average numbers of defects per horizon size.

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