GEOMETRIC ANALYSIS OF HYPER-STRESSES

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ABSTRACT. A geometric analysis of high order stresses in continuum mechanics is presented. Virtual velocity fields take their values in a vector bundle $W$ over the $n$-dimensional space manifold. A stress field of order $k$ is represented mathematically by an $n$-form valued in the dual of the vector bundle of $k$-jets of $W$. While only limited analysis can be performed on high order stresses as such, they may be represented by non-holonomic hyper-stresses, $n$-forms valued in the duals of iterated jet bundles. For non-holonomic hyper-stresses, the analysis that applies to first order stresses may be iterated. In order to determine a unique value for the tangent surface stress field on the boundary of a body and the corresponding edge interactions, additional geometric structure should be specified, that of a vector field transversal to the boundary.

1. Introduction

The theory of hyper-stresses in continuum mechanics, e.g. Toupin (1962, 1964), Mindlin (1964, 1965), accounts for phenomena not accounted for by the standard theory of stresses, such as edge interactions and surface tension. Although five decades have passed since this pioneering body of work has been published, various aspects of higher-order continuum mechanics are still under current research, e.g. dell’Isola et al. (2012, 2015); Fosdick (2016); Mariano (2007); Münch and Neff (2016); Podio-Guidugli (2015).

This work is concerned with geometric analysis of smooth stresses of order $k$ in continuum mechanics. In Segev (1986), for the setting where both the body $B$ and space $S$ objects of continuum mechanics are modeled as general differentiable manifolds, a hyper-stress theory was proposed in which the fundamental object is the configuration space $Q$ containing all $C^k$-embeddings of the body into space. Using results on manifolds of mappings (e.g. Palais (1968); Michor (1980); Hirsch (1976)), it follows that the configuration space may be given the structure of a Banach manifold. The tangent space $T_\kappa Q$, at a generic configuration of the body $\kappa : B \to S$, is interpreted physically as the space of virtual velocities. It may be identified with the space of $C^k$-sections, vector fields, of some vector bundle $W$, where the space of sections is equipped with the $C^k$-topology. A generalized force $F$ of order $k$ at the configuration $\kappa$ is defined to

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be a continuous linear functional on the tangent space $T_\kappa Q$ and the value of the action of a force on a generalized velocity is interpreted as the corresponding virtual power.

It is shown there that forces may be represented by measures valued in the dual of the $k$-jet bundle, $J^k W$, of $W$. Locally, these measures are represented by a collection of tensors valued measures of orders 1 to $k$. These representing measures are referred to as variational (hyper-) stresses. The relation between a force system containing the forces of order $k$ on all subbodies of $B$ and a $k$-hyper-stress field, the analog of Cauchy’s postulates, is studied in Segev (1986); Segev and DeBotton (1991) for the general case of stress fields that are as irregular as measures.

In the smooth case, the measures of the variational stress are represented by smooth sections $S$ of the fiber bundle $L(J^k W, \wedge^n T^* B) = (J^k W)^* \otimes \wedge^n T^* B$ so that the value of the stress field at a point $x \in B$ is a linear mapping $(J^k W)_x \rightarrow \wedge^n T^* B$. Thus, the power expended by the force $F$ for the generalized velocity $w$ is given by

$$F(w) = \int_B S(j^k w),$$

where $S(j^k w)$ is the $n$-form whose value at $x \in B$ is $S(x)(j^k w(x))$, so that the integration above is well defined.

For the standard continuum mechanics case, $k = 1$, a procedure given in Segev (2002, 2013) and outlined in Section 3 makes it possible to write (1.1) in the form

$$F(w) = \int_B b(w) + \int_{\partial B} t(w),$$

where $b$, the body force, is a section of $L(W, \wedge^n T^* B)$, satisfies

$$\text{div} S + b = 0, \quad \text{in } B,$$

and $t$, the surface force, satisfies a generalization of Cauchy’s formula

$$\rho \circ \sigma = t, \quad \text{on } \partial B.$$ 

Here, $\sigma$, the traction stress, is a section of $L(W, \wedge^{n-1} T^* B)$ that generalizes the Cauchy stress, and $\rho$ is the restriction of forms defined on $TB$ to $T\partial B$. The traction stress is determined by the variational stress. It is emphasized that for the setting of general manifolds, two distinct objects represent the two functions of the classical stress object, namely, acting on derivative of velocities to produce power, and determining the surface force for various subbodies. The divergence operator for manifolds, as mentioned above and defined below, generalizes the standard divergence operator of second order tensors.

In this paper we study the geometric structure required to provide the analogous construction for smooth hyper-stresses of order $k$. In particular, we consider the geometric structure needed to determine the edge interactions induced by hyper-stresses using integral transformations in analogy with the analysis in dell’Isola et al. (2012, 2015). It is shown below that the setting of iterated jet
bundles, $J^1(J^1W)$ for instance, is preferable to that of higher jet bundles, for instance, $J^2W$, respectively. Using iterated jet bundles makes it possible to apply the procedure for standard continuum mechanics, inductively.

Hyper-stresses in bodies induce tangent surface stresses on the corresponding boundaries. However, it is shown that on general differentiable manifolds, the induced surface stress, and hence the edge interactions, are not unique. For the unique determination of the tangent surface stress, one needs at least some specified vector field which is transversal to the boundary or an equivalent structure. The situation is similar to that described in Epstein and Ten (1973); Epstein and de León (1998), where shell theory is considered. Evidently, for the case of a Riemannian manifold, the unit normal vector field provides such a transversal field naturally.

Section 2 introduces the relevant terminology and notation used for jet bundles associated with vector bundles. Section 3 reviews the relevant constructions of Segev (2002) regarding smooth stress distributions on manifolds as outlined above. Section 4 is concerned with hyper-stresses of order $k$, their representations and their invariant components. Some of the difficulties related to the analysis of hyper-stresses are indicated. Section 5 considers iterated jet bundles (see Saunders (1989)). Iterated jet bundles are of interest as their sections may have additional forms of incompatibility in comparison with sections of jet bundles. Forms valued in the duals of iterated jet bundles are referred to here as non-holonomic hyper-stresses. These are considered in Section 6. Due to the inductive nature of iterated jet bundles, it is sufficient to study the properties of the iterated jet bundle $J^1(J^1W)$. The vector bundle $W$ itself may be a jet bundle, or an iterated jet bundle, of some other vector bundle. It is noted that every hyper-stress may be represented by non-holonomic hyper-stresses. The properties of non-holonomic hyper-stresses, in particular, the corresponding integral transformations associated with their action, are analyzed in this section for the case of general manifolds. Section 7 shows how the introduction of a particular vector field which is transversal to the boundary of a body induces a unique stress of a lower order on the boundary. Finally, in Section 8, the edge interactions induced by the non-holonomic hyper-stress are computed.

2. Notation and Preliminaries

All manifolds considered here are viewed as chains or manifolds with corners so that we may use the Stokes theorem for integration of forms.

2.1. Jets in general. We will use the same scheme of notation as in Segev (2013) and we will often use the same notation for a mapping and variables in the co-domains thereof. Let $\pi : W \rightarrow S$ be a vector bundle, a section $w : S \rightarrow W$ of $\pi$ is represented locally in the form

$$(x^1, \ldots, x^n) \mapsto (x^1, \ldots, x^n, w^1(x^i), \ldots, w^d(x^i)), \quad (2.1)$$
where \((x^1, \ldots, x^n)\) is a local coordinate system, and a local basis \(\{g_1, \ldots, g_d\}\) was used for the fibers of \(W\). Let \(I = (i_1, \ldots, i_n)\), for non-negative integers \(i_j\), be a multi-index and let \(|I| = \sum_{j=1}^n i_j\). We use the notation

\[
\frac{\partial^{|I|}}{\partial x^I} = \frac{\partial^{|I|}}{\partial x^{i_1} \cdots \partial x^{i_n}}.
\]  

(2.2)

Two sections \(w\) and \(w'\) have the same \(k\)-jet at \(x_0 \in S\) if

\[
\frac{\partial^{|I|} w^\alpha}{\partial x^I}(x_0) = \frac{\partial^{|I|} w'^\alpha}{\partial x^I}(x_0)
\]  

(2.3)

for all \(I\) such that \(|I| \leq k\) and all \(\alpha = 1, \ldots, d\). Clearly, if this condition holds in one vector bundle chart in a neighborhood of \(x_0\), it will hold in any other chart and it induces an equivalence relation on the vector space \(C^k(W) = C^k(\pi)\) of \(C^k\)-sections of the vector bundle. An equivalence class for this relation is a \(k\)-jet at \(x_0\). Given a section \(w\), the jet it induces at \(x_0\)—the jet of \(w\) at \(x_0\)—will be denoted as \(j^k(w)(x_0)\). Given a chart in a neighborhood of \(x_0\), \(j^k(w)(x_0)\) is represented by

\[
\left\{ w^\alpha_i (x_0) := \frac{\partial^{|I|} w^\alpha}{\partial x^I}(x_0^i) \mid |I| \leq k, \ \alpha = 1, \ldots, d \right\}.
\]  

(2.4)

The collection of all \(k\)-jets at \(x_0 \in S\) is the \(k\)-jet space of the vector bundle at \(x_0\) and is denoted as \(J^k_{x_0} W\). The \(k\)-jet bundle \(J^k W\) is the collection of all \(k\)-jets at the various points in \(S\) so that

\[
J^k W = \bigcup_{x \in S} J^k_x W.
\]  

(2.5)

By convention, \(J^0 W = W\). A natural vector bundle structure

\[
\pi^k : J^k W \rightarrow S,
\]

(2.6)

is available on the jet bundle by which \(\pi^k(A) = x\) if \(A \in J^k_x W\). The linear structure on the fibers is given by \(a_1 A_1 + a_2 A_2 = j^k(a_1 w_1 + a_2 w_2)(x)\), for \(A_1, A_2\) in \(J^k_x W\), \(a_1, a_2 \in \mathbb{R}\), and representing sections \(w_1\) and \(w_2\). Evidently, the result is independent of the choice of representative sections. The fiber \(J^k_x W\) of this vector bundle over \(x \in S\) is isomorphic with

\[
W_x \oplus L(T_x S, W_x) \oplus \cdots \oplus L^p_S(T_x S, W_x) \oplus \cdots \oplus L^k_S(T_x S, W_x),
\]  

(2.7)

where \(L^p_S(T_x S, W_x)\) denotes the vector space of \(p\)-multilinear symmetric mappings from \(T_x S\) to \(W_x\). Thus, an element in \(J^k_x W\) is represented locally in the form

\[
(A_1^{\alpha_0}, A_1^{\alpha_1}, \ldots, A_1^{\alpha_p}, \ldots, A_1^{\alpha_k}) = (A_1^{\alpha}),
\]  

(2.8)

where \(p = 0, \ldots, k, |I_p| = p\), \(\alpha = (\alpha_0, \ldots, \alpha_k)\), \(\alpha_p = 1, \ldots, d\), and evidently, \(A_1^{\alpha_0}\) represents an element of \(W_x\). Each section \(w\) of \(W\) induces a section \(j^k w\) of the \(k\)-th jet bundle and if fact we have a continuous linear injection

\[
j^k : C^k(W) \rightarrow C^0(J^k W),
\]  

(2.9)
where \( CP(U) \) represents the vector space of sections of the vector bundle \( U \) of class \( p \). For additional information on jet bundles, some of which will be used in the following sections, see Saunders (1989).

A jet bundle has also the natural projections

\[
\pi^k_p : J^k W \to J^p W, \quad 0 \leq p \leq k, \tag{2.10}
\]

classified by \( \pi^k_p(A) = J^p (w)(x) \) where \( x = \pi^k(A) \) and \( w \) is any section of \( W \) that represents \( A \). The mapping \( \pi^k_p \) is a vector bundle morphism over \( S \).

Let \( \phi : W \to U \) be a fiber bundle morphism over the base manifold \( S \). For an element \( A \in J^k W \), represented by \( j^k w(x) \), where \( w \) is a section of \( W \), set \( j^k \phi(A) = j^k (\phi \circ w)(x) \in J^k U \). In this way, one defines the \( k \)-lift of \( \phi \), the vector bundle morphism

\[
j^k \phi : J^k W \to J^k U. \tag{2.11}
\]

Evidently,

\[
\pi^k_p \circ j^k \phi = j^p \phi \circ \pi^k_r. \tag{2.12}
\]

2.2. \textbf{Vertical sub-bundles.} For \( 0 \leq r < k \), we say that \( A \in J^k U |_x \) is \( r \)-vertical if for one (and hence any) section \( u \) representing \( A \), \( j^r u(x) = 0 \), or equivalently, if \( \pi^k_r(A) = 0 \). If \( A \) is \( r \)-vertical, then, its local representatives satisfy \( A^p_{II} = 0 \) for all \( p = |I| \leq r \). Thus, locally

\[
V^r J^k U |_x \cong L^{r+1}_S(T_x S, U_x) \oplus \cdots \oplus L^k_S(T_x S, U_x). \tag{2.13}
\]

The collection of \( r \)-vertical elements is a vector sub-bundle of the jet bundle and we denote it by \( V^r J^k U \), \( i.e., V^r J^k U = \text{Kernel } \pi^k_r \). One has the natural vector bundle inclusion

\[
i^r : V^r J^k U \to J^k U. \tag{2.14}
\]

For the particular case of the first jet bundle \( J^1 U \), the only vertical sub-bundle is \( V^0 J^1 U \), and we will often omit the zero superscript and write just \( V J^1 U \).

For the case, \( r = k - 1 \), one has a natural isomorphism

\[
V^{k-1} J^k U |_x \cong L^k_S(T_x S, U_x), \quad V^{k-1} J^k U \cong L^k_S(T S, U). \tag{2.15}
\]

The vertical subbundle \( V^{k-1} J^k U \) will be referred to as the completely vertical sub-bundle of the \( k \)-jet bundle. In the particular case \( k = 1, r = 0 \), it follows that \( V J^1 U \) is naturally isomorphic with \( L(T S, U) \).

\textbf{Remark 2.1.} In view of (2.13), one may be tempted to view elements of \( V^r J^k U \) as elements of \( J^{k-r-1}(L^{r+1}_S(T S, U)) \). However, it may be easily verified that there is no such invariant correspondence.
2.3. Some details on 1-jets. Let \( w : S \rightarrow W \) be a section of \( \pi : W \rightarrow S \), then, \( j^1 w(x), x \in S \), is represented locally in the form \((x^i, w^\alpha(x), w^\beta_j(x))\), and \( \pi^j_0(j^1 w(x)) \) is obviously \( w(x) \) which is represented locally in the form \((x^i, w^\alpha)\). The tangent at \( x \) to the section \( w, T_x w : T_x S \rightarrow T_{w(x)} W \), is represented locally by

\[
(x^i, v^j) \mapsto \left( x^i, w^\alpha(x), v^j, \sum_k w^\beta_k v^k \right).
\]  

(2.16)

Since \( T\pi : TW \rightarrow TS \) is represented locally by \((x^i, w^\alpha, \dot{x}^i, \dot{w}^\beta) \mapsto (x^i, \dot{x}^i)\), any linear mapping \( A : T_x S \rightarrow T_{w(x)} W \) satisfying the condition \( T\pi \circ A = \text{Id} \), induces a unique element \( A \in j^1 W_x \) with \( \pi^j_0(A) = w(x) \).

Let \( \mathcal{V} \) be a submanifold of \( S \) and for the natural embedding \( r^\mathcal{V} : \mathcal{V} \rightarrow S \), let \( T r^\mathcal{V} : T\mathcal{V} \rightarrow TS \) be the tangent mapping. Thus, with \( i, j = 1, \ldots, n \) and \( a, \beta = 1, \ldots, \text{dim } \mathcal{V} \), \( r^\mathcal{V} \) is represented in the form \((y^a), (\dot{y}^b)\) and \( T r^\mathcal{V} \) is represented by \((y^a, \dot{y}^b) \mapsto (i^1_\mathcal{V}(y^a), \Sigma b i^1_{\mathcal{V}, b} \dot{y}^b)\). One has the pullback \( i^*_{\mathcal{V}} : i^*_{\mathcal{V}} W \rightarrow \mathcal{V} \), of the vector bundle \( \pi \) onto \( \mathcal{V} \), the natural inclusion \( \pi^* r^\mathcal{V} : i^*_{\mathcal{V}} W \rightarrow W \), its tangent \( T(\pi^* r^\mathcal{V}) : T(i^*_{\mathcal{V}} W) \rightarrow TW \), the mapping \( i^*_{\mathcal{V}, \pi} : C^1(\pi) \rightarrow C^1(i^*_{\mathcal{V}}(\pi)) \) — which is simply the restriction of sections of \( \pi \) to \( \mathcal{V} \), and the corresponding jet bundle \( \pi^1(i^*_{\mathcal{V}}, \pi) : j^1(i^*_{\mathcal{V}} W) \rightarrow \mathcal{V} \). Thus, we will often use the notation \( W|\mathcal{V} \) for the pullback. Locally, \( \pi^* r^\mathcal{V} \) is represented in the form \((y^a, u^\alpha) \mapsto (i^1(y^a), u^\alpha)\), and \( T(\pi^* r^\mathcal{V}) \) is represented in the form \((y^a, u^\alpha, \dot{y}^b, \dot{u}^\beta) \mapsto (i^1(y^a), u^\alpha, \Sigma b i^1_{\mathcal{V}, b} \dot{y}^b, \dot{u}^\beta)\). Similarly, one may consider the pullback \( \pi^*_{\mathcal{V}}(\pi^1) : i^*_{\mathcal{V}} W \rightarrow \mathcal{V} \) with the natural inclusion \( \pi^*_{\mathcal{V}}(r^\mathcal{V}) : i^*_{\mathcal{V}}(j^1 W) \rightarrow j^1 W \).

There is a natural restriction mapping \( \rho = j^1 \circ i^*_{\mathcal{V}, \pi} : i^*_{\mathcal{V}}(j^1 W) \rightarrow j^1(i^*_{\mathcal{V}} W) \) whereby \( j^1 w(y) \mapsto j^1(i^*_{\mathcal{V}, \pi} w(y)), y \in \mathcal{V} \), and it is noted that \( j^1 \) on the right is the jet extension on the submanifold \( \mathcal{V} \) which we may also write as \( j^1_{\mathcal{V}} \).

3. Simple Stresses

As a primitive mathematical object pertaining to stress theory for continuum mechanics of order 1 we take the variational stress, a smooth section \( S \) of the vector bundle \( L(J^1 W, \wedge^n \pi^* S) \) for some vector bundle \( W \rightarrow S \), where \( \wedge^n \pi^* S \) is the vector bundle of \( n \)-alternating covariant tensors over \( S \). For motivation, see Segel [1986, 2003, 2013]. In particular, for an \( n \)-dimensional submanifold with boundary \( B \subset S \), one is interested in the linear functional, the force,

\[
F_B : w \mapsto \int_B S(j^1 w)
\]  

(3.1)

which is interpreted as the virtual power performed by the variational stress \( S \) for the virtual generalized velocity field \( w \) inside the region \( B \). Here, the jet extension of \( w \) generalizes the traditional gradient to the setting of differentiable manifolds.
Locally, $S$ is represented in the form \((x^i, S_i^0, S_i^1, ..., S_i^n)\), or in detail, denoting the natural base vectors induced by a chart as \(\partial_i = \partial/\partial x^i\), the local representation is

\[
\left( \sum_{\alpha} S_{1...n\alpha}^0 + \sum_{i, \alpha} S_{1...n\alpha}^1 \otimes \partial_i \right) \otimes g^\alpha \otimes (dx^1 \wedge \cdots \wedge dx^n). \tag{3.2}
\]

Consequently, \(S(w)\) is represented locally by

\[
\left( \sum_{\alpha} S_{1...n\alpha} w^\alpha + \sum_{i, \alpha} S_{1...n\alpha} w_i^\alpha \right) dx^1 \wedge \cdots \wedge dx^n. \tag{3.3}
\]

For a vector bundle \(V \to S\), let \(\wedge^p(T^*S, V)\) denote the bundle of \(V\)-valued \(p\)-forms, \(i.e.,\) the vector bundle over \(S\) whose fiber at \(x\) is the vector space of \(p\)-alternating multilinear mappings from \(T_xS\) to \(V_x\). Consider the isomorphism

\[
\text{tr} : \wedge^p(T^*S, V^*) \to L(V, \wedge^p T^*S) \tag{3.4}
\]

defined as follows. For \(T \in \wedge^p(T^*S, V^*)\), \(T^\text{tr} = \text{tr}(T)\) is given by

\[
T^\text{tr}(v)(u_1, \ldots, u_p) = T(u_1, \ldots, u_p)(v). \tag{3.5}
\]

Thus, for a variational stress \(S\) one may consider \(S^\text{tr} = \text{tr}^{-1}(S)\)—an \(n\)-form on \(S\) valued in the dual of the jet bundle.

3.1. **Traction stresses.** Consider the inclusion \(i_V : VJ^1W \to J^1W\). Then, the dual vector bundle morphism \(i_V^* : (J^1W)^* \to (VJ^1W)^* \equiv L(W, TS)\) is a projection represented locally in the form \((x^i, r_p, R_q^i) \mapsto (x^i, R_q^i)\)—the restriction of \(R \in (J^1W)^*\) to vertical elements of the jet bundle. Thus, \(i_V^*(R)(A) \in VJ^1W\), is represented by \(\sum_{i, q} R_q^i A_i^q\). Similarly, for a section \(S\) of \(L(J^1W, \wedge^n T^*S)\), \(i_V^*(S) := i_V^* \circ S\), a section of \(L(VJ^1W, \wedge^d T^*S)\), is given by \(i_V^*(S)(x)(A) = S(x)(i_V(A)) \in \wedge^n T^*S\).

The evaluation \(i_V^*(S)(x)(A)\) is represented by \(\sum_{j, \alpha} S_{1...n\alpha}^j A_j^\alpha dx^1 \wedge \cdots \wedge dx^n\) and so \(i_V^*(S)\) is represented in the form

\[
\sum_{j, \alpha} S_{1...n\alpha}^j \partial_j \otimes g^\alpha \otimes (dx^1 \wedge \cdots \wedge dx^n). \tag{3.6}
\]

The object \(i_V^*(S)\) is the symbol of the linear differential operator \(S\) as defined in Palais (1968).

Using the isomorphism \(VJ^1W \equiv L(TS, W)\), we view \(i_V^*(S)\) as a section of

\[
L(L(TS, W), \wedge^n T^*S) \equiv L(TS, W)^* \otimes \wedge^n T^*S,
\]

\[
\equiv L(W, TS) \otimes \wedge^n T^*S, \tag{3.7}
\]

\[
\equiv W^* \otimes TS \otimes \wedge^n T^*S.
\]

It follows that a section of \(\wedge^n(T^*S, L(W, TS))\) may be represented locally in the form \(\sum_a \varphi^a \otimes v_a \otimes \theta\) for an \(n\)-form \(\theta\) and pairs \(v_a, \varphi^a\) of sections of \(TS\) and \(W^*\).
respectively. We can use the contraction of the second and first factors in the product to obtain \( \sum_a \phi^a \otimes (v_{a, \theta}) \). Thus, we have a natural mapping
\[
C : L(L(TS, W), \wedge^n T^*S) \longrightarrow W^* \otimes \wedge^{n-1} T^*S \\
\equiv L(W, \wedge^{n-1} T^*S). \tag{3.8}
\]
The mapping \( C \) is represented locally by
\[
\sum_{j, a} S^1_{j, a} \partial_j \otimes \sigma^a \otimes (dx^1 \wedge \cdots \wedge dx^n) \longmapsto \sum_{j, a} S^1_{j, a} \sigma^a \otimes \left( \partial_j - (dx^1 \wedge \cdots \wedge dx^n) \right),
\]
where a superimposed “hat” indicates the omission of the specified term.

The mapping
\[
p_\sigma := C \circ i_\nu^* : L(L(TS, W), \wedge^n T^*S) \longrightarrow L(W, \wedge^{n-1} T^*S) \tag{3.10}
\]
associates a section \( \sigma = p_\sigma \circ S \) of \( L(W, \wedge^{n-1} T^*S) \) with a variational stress \( S \).

We refer to a section of \( L(W, \wedge^{n-1} T^*S) \) as a traction stress. Such a section is represented locally by \((x', \sigma_1 \cdots \sigma_\alpha (x'))\), or specifically, by
\[
\sum_{k, r} \sigma_1 \cdots \sigma_\alpha g^a \otimes (dx^1 \wedge \cdots \wedge dx^n). \tag{3.11}
\]
The transposed, \( \sigma^T \), is represented by
\[
\sum_{j, a} \sigma_1 \cdots \sigma_\alpha dx^1 \wedge \cdots \wedge \tilde{dx}^j \wedge \cdots \wedge dx^n \otimes g^a \tag{3.12}
\]
and \( \sigma(w) \) is represented locally by
\[
\sum_{j, a} \sigma_1 \cdots \sigma_\alpha w^a dx^1 \wedge \cdots \wedge \tilde{dx}^j \wedge \cdots \wedge dx^n. \tag{3.13}
\]
We conclude that in case \( \sigma = p_\sigma(S) \), then,
\[
\sigma_1 \cdots \sigma_\alpha = (-1)^{j-1} S^1_{j, a} \tag{3.14}
\]

For each \((n-1)\)-dimensional oriented submanifold \( \mathcal{V}' \subset S \), in particular, the boundary \( \partial B \) of an \( n \)-dimensional submanifold with boundary \( B \subset S \), one may integrate \( \sigma(w) \) over \( \mathcal{V}' \), and evaluate
\[
\int_{\mathcal{V}'} i_\nu^* (\sigma(w)). \tag{3.15}
\]
Here, \( i_\nu : \mathcal{V}' \rightarrow S \) is the natural inclusion so that \( i_\nu^* \) is the restriction of forms.

We conclude that
\[
v_\mathcal{V}' = i_\nu^* \circ \sigma \tag{3.16}
\]
is the surface force induced by \( \sigma \) and the integral above represents the power produced by the traction. The relation \( (3.16) \) is a generalization of the traditional Cauchy formula.
3.2. The divergence of stress and field equations. The divergence, $\text{div} \, S$, of the variational stress field $S$ is a section of $L(W, \wedge^n T^* S)$ which is defined invariantly by (see Segev (2002, 2013))

$$\text{div} \, S(w) = d(p_\sigma(S)(w)) - S(j^1(w)), \quad (3.17)$$

for every differentiable vector field $w$. To present the local expression for $\text{div} \, S$ we first note that if $\sigma = p_\sigma(S)$, then $d(\sigma(w))$ is represented locally by

$$\sum_{j,\alpha} d(\sigma_{1...j...n\alpha} w^\alpha) \wedge dx^1 \wedge \cdots \wedge dx^j \wedge \cdots \wedge dx^n$$

$$= \sum_{i,j,\alpha} (\sigma_{1...j...n\alpha} w^\alpha)_i dx^i \wedge dx^1 \wedge \cdots \wedge dx^j \wedge \cdots \wedge dx^n,$$

$$= \sum_{j,\alpha} (\sigma_{1...j...n\alpha} w^\alpha)_j (-1)^{j-1} dx^1 \wedge \cdots \wedge dx^n,$$

$$= \sum_{j,\alpha} (S^l_{1...n\alpha} w^\alpha)_j dx^1 \wedge \cdots \wedge dx^n. \quad (3.18)$$

Using Equation (3.3), the local expression for $\text{div} \, S(w)$ is therefore

$$\sum_{j,\alpha} \left[ (S^l_{1...n\alpha} w^\alpha)_j - \left( \sum_{\alpha} R_{1...n\alpha} w^\alpha + \sum_{j,\alpha} S^l_{1...n\alpha} w^\alpha \right) \right] dx^1 \wedge \cdots \wedge dx^n$$

$$= \sum_{j,\alpha} (S^l_{1...n\alpha,j} - R_{1...n\alpha}) w^\alpha dx^1 \wedge \cdots \wedge dx^n \quad (3.19)$$

so that $\text{div} \, S$ is represented locally by

$$\sum_{j,\alpha} (S^l_{1...n\alpha,j} - R_{1...n\alpha}) g^\alpha \otimes (dx^1 \wedge \cdots \wedge dx^n). \quad (3.20)$$

It is noted that in the case where $R_{1...n\alpha} = 0$ locally, the expression for the divergence reduces to the traditional expression for the divergence of a tensor field in a Euclidean space.

Given a variational stress $S$, and setting

$$b = - \text{div} \, S, \quad (3.21)$$

for every $n$-dimensional submanifold with boundary $B \subset \mathcal{S}$, a force $F_B$ may be represented in the form

$$F_B(w) = \int_B S(j^1(w)) = \int_B b(w) + \int_{\partial B} \tau_{\partial B}(w) \quad (3.22)$$

which is our generalization of the principle of virtual work.
4. High Order Stresses

For continuum mechanics of order greater than one, the fundamental object we consider is the \( k \)-th order variational stress which is a smooth section of the vector bundle \( L(J^k U, \wedge^n T^* S) \), for some vector bundle \( U \to S \), sections of which are interpreted as virtual generalized velocities. (See Segev [1986] and Segev and DeBotton [1991] for motivation.) Thus, the virtual power performed by a \( k \)-th order variational stress \( S \) for the virtual generalized velocity \( u \) in a body \( B \subset S \) is given by the action of the functional

\[
F_B : u \mapsto \int_B S(J^k u). \tag{4.1}
\]

Observing \((2.7)\), it follows that the fiber, \( L(J^k U, \wedge^n T^* S)_x \) of \( L(J^k U, \wedge^n T^* S) \) at \( x \in S \) is isomorphic with

\[
\left( U_x^* \oplus L(T_x S, U_x)^* \oplus \cdots \oplus L^k_x(T_x S, U_x)^* \right) \otimes \wedge^n T^*_x S, \tag{4.2}
\]

where the isomorphism depends on the charts used. Let \( S^p \) denote the component of the representative of \( S \) in \( L^p_x(TS, W)^* \otimes \wedge^n T^*_x S \). It follows that the stress may be represented locally in the form \((S^0, S^1, \ldots, S^k)\), where \( S^p \) is an array in the form \( S^p_{1 \ldots n^p} \), and \( I \) is a multi-index with \(|I| = p \). The action \( S(A) \) for an element \( A \in J^k U \) is given by

\[
\sum_{p=|I| \leq k, \alpha} S^p_{1 \ldots n^p} A^\alpha_I dx^1 \wedge \cdots \wedge dx^n. \tag{4.3}
\]

For \( A = J^k u \),

\[
S(A) = \sum_{p=|I| \leq k, \alpha} S^p_{1 \ldots n^p} u^\alpha_I dx^1 \wedge \cdots \wedge dx^n. \tag{4.4}
\]

Explicitly, \( S \) is given locally in the form

\[
\sum \left( S^0_{1 \ldots n^0} + \sum_s S^1_{1 \ldots n^s} \partial_{i_1} \cdots \partial_{i_k} \right) \otimes \omega \otimes dx^1 \wedge \cdots \wedge dx^n, \tag{4.5}
\]

and the action is

\[
S(A) = \sum_S S^p_{1 \ldots n^p} u^\alpha_I dx^1 \wedge \cdots \wedge dx^n, \tag{4.6}
\]

where the sums are taken over all \( i_1, \ldots, i_k = 1, \ldots, n \), and \( p = 0, \ldots, k \). Evidently, the arrays \( S^p_{1 \ldots n^p} \) are symmetric in all \( i \) indices.

It is our objective to represent the virtual power for high order stresses \((4.1)\) in a form analogous to \((3.22)\).

4.1. Significant components of hyper-stresses. The inclusion \( \iota^*_V \) \((2.14)\) of the vertical subbundles induces a projection

\[
\iota^*_V : L(J^k U, \wedge^n T^* S) \to L(V^r J^k U, \wedge^n T^* S), \tag{4.7}
\]
by $i_V^r(S) := S \circ i_V^r$. Evidently, the representatives of $i_V^r(S)$ depend only on $S^{pl}_{1...na}$ for $p > r$. In fact, $L(V^r j^k U, \wedge^n T^* S)_x$ is isomorphic with

$$\left( L_S^{r+1}(T_x S, U_x)^* \oplus \cdots \oplus L_S^k(T_x S, U_x)^* \right) \otimes \wedge^n T^*_x S. \quad (4.8)$$

Specifically, for the representation of a hyper-stress as in (4.5), $i_V^r(S)$ is

$$\sum (S^{r+1}_{1...na} \partial_{i_1} \otimes \cdots \otimes \partial_{i_{r+1}} + \cdots + S^{k}_{1...na} \partial_{i_1} \otimes \cdots \otimes \partial_{i_k}) \otimes g^\alpha \otimes dx^1 \wedge \cdots \wedge dx^n. \quad (4.9)$$

In particular, for the case $r = k - 1$, one has a natural isomorphism

$$L(V^{k-1} j^k U, \wedge^n T^* S) \cong L^k(S(TS, U))^* \otimes \wedge^n T^*_x S \quad (4.10)$$

and a natural

$$i_V^{k-1} : L(j^k U, \wedge^n T^* S) \longrightarrow L_x^k(TS, U)^* \otimes \wedge^n T^*_x S. \quad (4.11)$$

which isolates the significant high-order components of $k$-order stresses. When stresses are viewed as linear differential operators, the significant components are the symbols of the differential operators in the terminology of Palais (1968).

For the particular case $r = 0$, one has, locally,

$$V^0 j^k U|_x \cong L_x^1(T_x S, U_x) \oplus \cdots \oplus L_x^k(T_x S, U_x) \quad (4.12)$$

and

$$i_V^0 : L(j^k U, \wedge^n T^* S) \longrightarrow L(V^0 j^k U, \wedge^n T^* S). \quad (4.13)$$

Thus, $i_V^0(S)$ is represented locally by an element of $[L^1(TS, U)^* \oplus \cdots \oplus L^k_x(TS, U)^*] \otimes \wedge^n T^* S$.

**Remark 4.1.** In order to continue the reduction process in analogy with Section 3, one may consider performing a contraction operator in analogy with Equation (3.5) so that $C(i_V^r(S))$ be represented locally by

$$\sum (S^{r+1}_{1...na} \partial_{i_1} \otimes \cdots \otimes \partial_{i_{r+1}} + \cdots + S^{k}_{1...na} \partial_{i_1} \otimes \cdots \otimes \partial_{i_k}) \otimes g^\alpha \otimes (\partial_{i_l} \wedge dx^1 \wedge \cdots \wedge dx^n). \quad (4.14)$$

However, the simplest case of $L(V^0 j^2 U, \wedge^n T^* S)$ may serve as a counter-example. Let $S \in L(V^0 j^2 U, \wedge^n T^* S)$ be represented in two charts $(x', u'^r)$ and $(x^i, u'^r)$ so that $u'^r = A'^{\alpha'} u^\alpha$, in the forms

$$S = \sum_{l,j,\alpha} (S^{1l}_{1...na} \delta_l + S^{2lj}_{1...na} \delta_l \otimes \delta_j) \otimes g^\alpha \otimes (dx^1 \wedge \cdots \wedge dx^n)
= \sum_{l,j,\alpha'} (S^{1l'}_{1...n'\alpha'} \delta_l + S^{2lj'}_{1...n'\alpha'} \delta_l \otimes \delta_j') \otimes g^\alpha' \otimes (dx'^{l'} \wedge \cdots \wedge dx'^{n'}). \quad (4.15)$$
Thus, one might consider defining $C(S)$ locally by

$$\sum_{i', j', \alpha'} \left( S_{i' \ldots n' \alpha'}^{1} + S_{i' \ldots n' \alpha'}^{2} \partial_{j'} \right) \otimes g^{\alpha'} \otimes (\partial_{j'} \cdot (dx^{1'} \wedge \cdots \wedge dx^{n'}))$$

(4.16)

with an analogous expression for the representation in terms of $(x^{i}, u^{\alpha})$.

Comparing the local expressions for $S(j^{2}u(x))$ for a section $u$ with $u(x) = 0$, and using

$$dx^{1} \wedge \cdots \wedge dx^{n} = Jdx^{1} \wedge \cdots \wedge dx^{n}, \quad J = \det(x^{i}_{j}),$$

(4.19)

one obtains

$$S_{1 \ldots n \alpha}^{1} = J \sum_{\alpha', i', j'} \left[ S_{1 \ldots n' \alpha'}^{1} A_{\alpha}^{\alpha'} x_{i'}^{i} + S_{1 \ldots n' \alpha'}^{2} \partial_{j'}(2A_{\alpha}^{\alpha'} x_{i'}^{j} x_{j'}^{j} + A_{\alpha}^{\alpha'} x_{i'}^{j} x_{j'}^{j}) \right].$$

(4.20)

$$S_{1 \ldots n \alpha}^{2} = J \sum_{\alpha', i', j'} S_{1 \ldots n' \alpha'}^{2} A_{\alpha}^{\alpha'} x_{i'}^{i} x_{j'}^{j}.$$  

(4.21)

Substituting these relations into the analog of Equation (4.16) gives

$$\sum_{i, j, \alpha} \left( S_{1 \ldots n \alpha}^{1} + S_{1 \ldots n \alpha}^{2} \partial_{j} \right) \otimes g^{\alpha} \otimes (\partial_{j} \cdot (dx^{1} \wedge \cdots \wedge dx^{n}))$$

$$= J \sum_{\alpha', i', j'} \left[ S_{1 \ldots n' \alpha'}^{1} A_{\alpha}^{\alpha'} x_{i'}^{i} + S_{1 \ldots n' \alpha'}^{2} \partial_{j'}(2A_{\alpha}^{\alpha'} x_{i'}^{j} x_{j'}^{j} + A_{\alpha}^{\alpha'} x_{i'}^{j} x_{j'}^{j}) \right] \otimes A_{\mu}^{\alpha} g^{\beta}$$

$$\otimes (x^{i}_{\mu} \partial_{\nu} \cdot (dx^{1} \wedge \cdots \wedge dx^{n}))/J.$$  

(4.22)

The last expression is not of the form (4.16) because of the second term in the sum. We conclude, therefore, that the proposed definition of the contraction is not invariant. In fact, had the contraction been invariant, this would imply an invariant decomposition $S \mapsto (S^{1}, S^{2})$.

**Remark 4.2.** One could also consider applying the contraction to an element $S$ of $L(V^{k-1} F^{n} U) = L(T^{k}_{S}(T^{*}S, U), \wedge^{n} T^{*}S)$. Thus, for a representation

$$S = \sum_{\alpha, i, \ldots, k} S_{1 \ldots n \alpha}^{1} \partial_{i} \otimes \cdots \otimes \partial_{k} \otimes g^{\alpha} \otimes (dx^{1} \wedge \cdots \wedge dx^{n}),$$

(4.23)
one may set \( C(S) \in L(L^{-1}_S(TS, U), \wedge^{n-1} T^* S) \) by
\[
C(S) = \sum_{\alpha, i_1, \ldots, i_k}^\infty s_{\alpha i_1 \ldots i_k} \partial_{i_1} \otimes \cdots \otimes \partial_{i_k} \otimes g^\alpha \otimes (\partial_{i_1} + (dx^1 \wedge \cdots \wedge dx^n)). \tag{4.24}
\]

While this definition is invariant, it does not lead to any meaningful result because we do not have an object in \( L^{-1}_S(TS, U) \) to apply \( C(S) \) to. In particular, it is observed that the only section of \( U \) whose jet is \( r \)-vertical, for any \( r \), is the zero section. As we will show below, further contractions vanish due to the symmetry of \( s_{\alpha i_1 \ldots i_k} \) relative to the \( i \)-indices.

Another unsuccessful attempt would be to identify elements of \( V^p J^k U \) as elements of \( J^{k-p-1}(L^{p+1}_S(TS, U)) \). Again, a local expression may be shown to be non-invariant.

A way to overcome these difficulties will be suggested below by embedding the space of \( k \)-jets in the space of iterated jets.

5. Iterated Jet Bundles

The method proposed below for manipulating hyper-stresses uses the representation of hyperstresses by non-holonomic hyper-stresses, which are defined on iterated jet bundles. This section describes the basic definitions associated with iterated jet bundles.

5.1. The iterated jet bundle \( J^1(J^1 U) \). Since for any vector bundle \( U, W = J^1 U \rightarrow S \) is also a vector bundle, one may consider the vector bundle \( J^1 W = J^1(J^1 U) \rightarrow S \). In particular, using the subscript \( J^1 U \) to indicate that \( J^1 U \) is the vector bundle to which the jet projections correspond, we have projections
\[
\pi_{0j}^{J^1 U} : J^1(J^1 U) \rightarrow J^1 U, \quad \pi_{ij}^{J^1 U} : J^1(J^1 U) \rightarrow S. \tag{5.1}
\]

As any section \( A \) of \( J^1 U \), or of \( \pi_{ij}^{J^1 U} \) to be specific, is locally of the form
\[
x \mapsto (u^\alpha(x), A_i^\beta(x)) = (A_0^\alpha(x), A_1^{i\alpha}(x)), \tag{5.2}
\]
an element \( B \) of the iterated jet bundle at the point \( x \in S \) is of the form
\[
(x^1, B_0^{0\alpha_0}, B_1^{1\alpha_1}, B_2^{2\alpha_2}, B_3^{3\alpha_3}).
\]

Here, \((B_0^{0\alpha_0}, B_1^{1\alpha_1})\) represent \( B_0 = \pi_{0j}^{J^1 U}(B) \)—the value of a section \( A \) of \( J^1 U \) at \( x \), and \((B_2^{2\alpha_2}, B_3^{3\alpha_3})\) represent the derivative of the section \( A \) at \( x \), i.e.,
\[
(B_2^{2\alpha_2}, B_3^{3\alpha_3}) = (A_2^{i\alpha_2}, A_3^{i\alpha_3}).
\]

It is noted that there is a natural vector bundle inclusion \( \iota : J^2 U_x \rightarrow J^1(J^1 U)_x \) such that \( \iota : J^2 U_x \rightarrow J^1(J^1 U)_x \) is given as follows. Let \( u \) be a section of \( U \) that represents an element \( B \in J^2 U_x \). Then, \( w = j^1 u \) is a section of \( J^1 U \) whose jet is the target element \( \iota(B) = j^1 w(x) = j^1(j^1 u)(x) \) in \( J^1(J^1 U) \). Thus, locally,
\[
(u^{0\alpha_0}, u_1^{i\alpha_1}, u_2^{i\alpha_2}) \mapsto (u^{0\alpha_0}, u_1^{i\alpha_1}, u_2^{i\alpha_2}, u_3^{i\alpha_3}). \tag{5.3}
\]
Evidently, the result is independent of the section chosen and locally the inclusion is in the form

\[ (A^0_{\alpha_0}, A^{1\alpha_1}_{i_1}, A^{2\alpha_2}_{i_1,i_2}, A^{3\alpha_3}_{i_1,i_2,i_3}) \hookrightarrow (A^0_{\alpha_0}, A^{1\alpha_1}_{i_1}, A^{1\alpha_2}_{i_2}, A^{2\alpha_3}_{i_1,i_2,i_3}). \tag{5.4} \]

Thus, the image of \( \iota \) contains elements for which the second and third groups of components are identical and the components in the fourth group are symmetric.

It is observed finally (see Saunders, 1989, p. 169) that there is no natural inverse to \( \iota \), i.e., a projection \( J^1(U) \rightarrow J^2U \).

An additional projection

\[ j^1\pi^1_{0U} : J^1(J^1U) \rightarrow J^1U, \tag{5.5} \]

may be defined as follows. Consider the lift (see (2.11)) \( j^1\pi^1_{0U} : J^1(J^1U) \rightarrow J^1U \). Then, if \( B \in J^1(J^1U)_x \), is represented by a section \( A \) of \( J^1U \), \( j^1\pi^1_{0U}(A) \) is given by \( j^1\pi^1_{0U}(B) = j^1(\pi^1_{0U} \circ A)(x) \). Locally, \( J^1\pi^1_{0U} \) is represented by

\[ (x', B^{0\alpha_0}_{i_1}, B^{1\alpha_1}_{i_2}, B^{2\alpha_2}_{i_3,i_4}, B^{3\alpha_3}_{i_1,i_2,i_3}) \hookrightarrow (x', B^{0\alpha_0}_{i_1}, B^{2\alpha_2}_{i_2}). \tag{5.6} \]

Various levels of “compatibility” or “holonomicity” may be considered for sections of the iterated jet bundle. Let \( B \) be a section of \( \pi^1_{J^1U} \) and \( B_0 = \pi^1_{J^1U} \circ B \). The simplest condition requires that \( B_0 \) is compatible, that is, \( B_0 = j^1u \) for some section \( u \) of \( U \). It follows that \( B^\beta_{\alpha_\gamma} = u^\beta_j \) and that \( \pi^1_{J^1U} \circ B = j^1(\pi^1_{0U} \circ \pi^1_{J^1U} \circ B) \). Another compatibility condition is that \( B = j^1A_0 \), where \( A_0 \) is a section of \( J^1U \). In such a case, \( j^1(\pi^1_{0U} \circ B) = B \), and if \( (B^{0\alpha_0}_{i_1}, B^{1\alpha_1}_{i_2}) \) represent \( A_0 \), then, \( B^{2\alpha_2}_{i_2} = B^{0\alpha_0}_{i_1} \) and \( B^{3\alpha_3}_{i_1,i_2} = B^{1\alpha_1}_{i_2,i_3} \). If these conditions hold, one refers to \( B \) as semi-holonomic. A section \( u \) of \( U \) induces a holonomic section \( B \) of \( J^1(J^1U) \) by setting \( A_0 = j^1u \), \( B = j^1A_0 = j^1(j^1u) \). In other words, \( j^1(\pi^1_{0U} \circ \pi^1_{J^1U} \circ B) = B \) and it is observed that \( j^1(\pi^1_{0U} \circ B) = j^1(\pi^1_{J^1U} \circ B) \). Note that if no holonomicity is imposed, \( B^{3\alpha_3}_{i_1,i_2} \) need not be symmetric in the \( i_3, i_4 \) indices.

Next, we consider vertical subbundles whose sections are non-holonomic. The projection \( \pi^1_{J^1U} \) determines the subbundle

\[ V_{23}J^1(J^1U) := \kernel \pi^1_{0J^1U}, \tag{5.7} \]

whose elements are of the form \( (x', B^{0\alpha_0}_{i_1} = 0, B^{1\alpha_1}_{i_2} = 0, B^{2\alpha_2}_{i_3}, B^{3\alpha_3}_{i_1,i_2,i_3}) \). We will use \( \iota_{23} \) to denote its inclusion in \( J^1(J^1U) \). In addition, we have the vector subbundle \( V_{13}J^1(J^1U) = \kernel j^1\pi^1_{0U} \) whose elements are of the form \( (x', B^{0\alpha_0}_{i_1} = 0, B^{1\alpha_1}_{i_2}, B^{2\alpha_2}_{i_3} = 0, B^{3\alpha_3}_{i_1,i_2}) \). One can easily confirm that

\[ V_{13}J^1(J^1U) = \kernel j^1\pi^1_{0U} \cong J^1(VJ^1U), \tag{5.8} \]

where \( VJ^1U = \kernel \pi^1_{0U} \). We will denote its inclusion in \( J^1(J^1U) \) by \( \iota_{13} \).

As a result, one has the vector subbundle of completely vertical iterated jets

\[ V_JJ^1(J^1U) := \kernel \pi^1_{0J^1U} \cap \kernel j^1\pi^1_{0U}. \tag{5.9} \]
whose elements are represented in the form \( (x^i, B^{0\alpha_0} = 0, B^{1\alpha_1}_i = 0, B^{2\alpha_2}_{i_2} = 0, B^{3\alpha_3}_{i_3}) \). We will use \( \iota_3 \) to denote its inclusion in \( J^1(J^1U) \). Evidently,

\[
V_3J^1(J^1U) \cong L^2(TS, U).
\]

We conclude that there is a natural inclusion

\[
\iota_3 : V^1J^2U \cong L^2_3(TS, U) \hookrightarrow V_3J^1(J^1U) \cong L^2(TS, U)
\]

of completely vertical 2-jets in the vector subbundle of completely vertical iterated jets. In addition, there is a natural projection induced by symmetrization

\[
\pi_S : V_3J^1(J^1U) \cong L^2(TS, U) \longrightarrow V^1J^2U \cong L^2_3(T_0S, U_x), \quad B \longmapsto \frac{1}{4}(B + B^T).
\]

Finally, the vector subbundle \( V_{123}J^1(J^1U) = \text{Kernel}(\pi_0^{1U} \circ \pi_0^{1J^1U}) \) contains elements represented in the form \( (x^i, B^{0\alpha_0} = 0, B^{1\alpha_1}_i, B^{2\alpha_2}_{i_2}, B^{3\alpha_3}_{i_3}) \). The inclusion \( V_{123}J^1(J^1U) \hookrightarrow J^1(J^1U) \) will be denoted in analogy by \( \iota_{123} \).

5.2. The iterated jet bundle \( J^p(J^rU) \). In more general situations, one may consider the iterated jet bundle \( J^p(J^rU) \). Again, one has an inclusion

\[
J^{p+r}_{p,r} : J^{p+r}U \hookrightarrow J^p(J^rU)
\]

given as follows. Let \( u \) be a section that represents an element of \( (J^{p+r}U)_x \). Then, \( J^rU \) is a section of the vector bundle \( J^rU \rightarrow S \), and so, \( J^p(J^rU)(x) \in (J^p(J^rU))_x \).

We also observe that the \( q \)-lift of this inclusion is

\[
J^{q+p+r}_{q,p,r} : J^{q+p+r}U \longrightarrow J^q(J^p(J^rU)).
\]

Thus, using the inclusion \( J^{q+p+r+q}_{q,p,r} : J^{q+p+r+q}U \rightarrow J^q(J^{p+r}U) \), gives the inclusions

\[
J^{q+p+r}_{q,p,r} \circ J^{p+r+q}_{q,p,r} : J^{q+p+r+q}U \hookrightarrow J^q(J^p(J^rU)).
\]

Evidently, one can continue inductively and include any jet bundle in a multiply-iterative jet bundle. For a vector bundle \( U \), we will use the notation

\[
\iota_k : J^kU \longrightarrow (J^1)^kU := \underbrace{J^1(\cdots \cdot J^1(J^1U) \cdots)}_{k\text{-times}}
\]

for the vector bundle injection of the \( k \)-th jet bundle in the \( k \)-times iterated \( (J^1)^kU \) jet bundle of \( U \).

6. Non-Holonomic Hyper-Stresses

Just as elements of \( L(J^2U, \wedge^n T^*S) \) represent second order hyper-stresses, we refer to elements of \( L(J^1(J^1U), \wedge^n T^*S) \) as non-holonomic hyper-stresses. The inclusion \( \iota : J^2U \rightarrow J^1(J^1U) \) induces a projection

\[
i^* : L(J^1(J^1U), \wedge^n T^*S) \longrightarrow L(J^2U, \wedge^n T^*S), \quad X \longmapsto X \circ \iota.
\]

Thus, every 2-hyper-stress may be represented by a non-holonomic hyper-stress.
This argument applies also to the inclusion of the $k$-times iterated jet bundle and so one has a representation

$$t'_{it} : L(J^kU, \wedge^nT^*S) \rightarrow L(J^kU, \wedge^nT^*S). \quad (6.2)$$

For this reason, we will continue the analysis for 2-hyper-stresses only. The general case follows inductively.

6.1. **Basic properties.** A non-holonomic hyper-stress field may be represented in the form

$$(x^i, X_1^{1i}, X_2^{2i}, X_3^{3i}, X_4^{4i}), \quad (6.3)$$

or more explicitly, in the form

$$\left(\sum_{\alpha} x^{0\alpha}_{1...na} g^{\alpha}, \sum_{\alpha,i} x^{1i}_{1...na} \partial_i \otimes g^{\alpha}, \sum_{\alpha,i} x^{2i}_{1...na} \partial_i \otimes g^{\alpha}, \sum_{\alpha,i,j} x^{3ij}_{1...na} \partial_i \otimes \partial_j \otimes g^{\alpha}\right) \otimes (dx^1 \wedge \cdots \wedge dx^n), \quad (6.4)$$

where we have omitted the indication of the dependence of the various fields on $x \in \mathcal{S}$.

With the notation introduced above, the action of a non-holonomic hyper-stress on an element $B$ of $J^1(J^1U)$ is given as

$$\sum_{\alpha,i,j} \left( x^0_{1...na} B^{0\alpha} + x^{1i}_{1...na} B^{1i\alpha} + x^{2i}_{1...na} B^{2\alpha} + x^{3ij}_{1...na} B^{3ij\alpha} \right) dx^1 \wedge \cdots \wedge dx^n. \quad (6.5)$$

For the case where $B = J^1 A$, for a section $A$ of $J^1 U$, the expression is

$$\sum_{\alpha,i,j} \left( x^0_{1...na} A^{0\alpha} + x^{1i}_{1...na} A^{1i\alpha} + x^{2i}_{1...na} A^{2\alpha} + x^{3ij}_{1...na} A^{3ij\alpha} \right) dx^1 \wedge \cdots \wedge dx^n. \quad (6.6)$$

For an element $A \in J^2 U$, one has

$$t^*(X)(A) = X(tA)$$

$$= \left( \sum_{\alpha} x^0_{1...na} A^{0\alpha} + \sum_{\alpha,i} (x^{1i}_{1...na} + x^{2i}_{1...na}) A^{1i\alpha} \right) dx^1 \wedge \cdots \wedge dx^n,$$

$$= \left( \sum_{\alpha} x^0_{1...na} A^{0\alpha} + \sum_{\alpha,i} (x^{1i}_{1...na} + x^{2i}_{1...na}) A^{1i\alpha} \right) dx^1 \wedge \cdots \wedge dx^n,$$

$$= \left( \sum_{\alpha} x^0_{1...na} A^{0\alpha} + \sum_{\alpha,i} (x^{1i}_{1...na} + x^{2i}_{1...na}) A^{1i\alpha} \right) dx^1 \wedge \cdots \wedge dx^n,$$

$$= \left( \sum_{\alpha} x^0_{1...na} A^{0\alpha} + \sum_{\alpha,i} (x^{1i}_{1...na} + x^{2i}_{1...na}) A^{1i\alpha} \right) dx^1 \wedge \cdots \wedge dx^n,$$

$$= \left( \sum_{\alpha} x^0_{1...na} A^{0\alpha} + \sum_{\alpha,i} (x^{1i}_{1...na} + x^{2i}_{1...na}) A^{1i\alpha} \right) dx^1 \wedge \cdots \wedge dx^n,$$

$$= \left( \sum_{\alpha} x^0_{1...na} A^{0\alpha} + \sum_{\alpha,i} (x^{1i}_{1...na} + x^{2i}_{1...na}) A^{1i\alpha} \right) dx^1 \wedge \cdots \wedge dx^n.$$

The mapping $t^*$ is therefore a restriction represented locally by

$$\left( x^i, X_1^{1i}, X_2^{2i}, X_3^{3i}, X_4^{4i} \right) \mapsto \left( x^i, X_1^{1i}, X_2^{2i}, X_3^{3i}, X_4^{4i} \right), \quad (6.8)$$
and being surjective, every second order stress $S$ is of the form

$$S = \iota^* X$$

(6.9)

for some non-unique section $X$ of $L(J^1(U), \wedge^n T^* S)$. Thus, whatever properties we deduce for elements of $J^1(U)$ will hold for their restrictions to Image $\iota$.

We focus our attention in this section to the analysis of the action of a non-holonomic hyper-stress field $X$ on a section $A$ of $J^1 U$ in the form

$$A \mapsto \int_B X(j^1(A)),$$

(6.10)

and in particular, the compatible case where $A = j^1 u$, for a section $u$ of $U$.

Using the operator $p_\sigma$ as in (3.10), one can extract a section $Y = p_\sigma X$ of the vector bundle $L(J^1 U, \wedge^n T^* S)$. The local representation of $Y$ is

$$\sum_{j, i, a} \left( Y_{01\ldots j\ldots na}^i + Y_{11\ldots j\ldots na}^i \right) \otimes g^a \otimes (dx^1 \wedge \cdots \wedge dx^j \wedge \cdots \wedge dx^n)$$

(6.11)

From (6.14) it follows that

$$Y_{01\ldots j\ldots na}^i = (-1)^{j-1} X_{11\ldots j\ldots na}^i, \quad Y_{11\ldots j\ldots na}^i = (-1)^{j-1} X_{11\ldots j\ldots na}^{3ij}.$$  

(6.12)

We may now apply the general definitions of the operators $p_\sigma$ and $\text{div}$ to the case of non-holonomic hyper-stresses. Equation (5.17) assumes the form

$$\text{div} X(A) = d(p_\sigma(X)(A)) - X(j^1 A),$$

(6.13)

in which $\text{div} X$, a section of $L(J^1 U, \wedge^n (T^* S))$ is represented locally by

$$\sum_{a, i, j} \left( X_{11\ldots na}^i - X_{11\ldots na}^i + (X_{11\ldots na}^{1ij} - X_{11\ldots na}^{1ij}) \partial_i \right) \otimes g^i \otimes (dx^1 \wedge \cdots \wedge dx^n).$$

(6.14)

One conclude that for $Y = p_\sigma(X)$,

$$\int_B X(j^1 A) = \int_B d(Y(A)) - \int_B \text{div} X(A) = \int_{\partial B} Y(A) - \int_B \text{div} X(A).$$

(6.15)

Our attention will be focused henceforth on the boundary integral.

6.2. The hyper-surface stress. We will refer to $Y = p_\sigma(X)$ as a hyper-surface stress. The integration of $Y(j^1 w)$ on the boundary $\partial B$ as above induce the special effects associated with hyper-stresses, such as surface tension and edge interaction. Thus, most of the material below is concerned with the analysis of this term.

Let $\mathcal{V}$ be an $(n-1)$-dimensional submanifold of $S$ and let $\rho_\mathcal{V}$ be the restriction of $(n-1)$ forms defined on $S$ to forms on $\mathcal{V}$ which is induced by the inclusion $T^* \mathcal{V} \rightarrow T^* S$. Thus, for a given $Y \in L(J^1 U, \wedge^n T^* S)$ and a submanifold $\mathcal{V}$, $\rho_\mathcal{V} \circ Y \in L((J^1 U)_{|\mathcal{V}}, \wedge^n T^* \mathcal{V})$.

As in the case of stresses, the inclusion $j_0^0 : V^0 J^1 U \rightarrow J^1 U$. induces

$$j_0^0 : L(J^1 U, \wedge^n T^* S) \rightarrow L(L(TS, U), \wedge^n T^* S), \quad Y \mapsto Y \circ j_0^0,$$

(6.16)
represented locally by

\[(x, Y^0_{1...\hat{i}...na}, Y^1_{1...\hat{i}...na}) \mapsto (x, Y^1_{1...\hat{i}...na}), \quad (6.17)\]

where a basis \(\{dx^1 \wedge \cdots \wedge dx^i \wedge \cdots \wedge dx^n\}\) of \(\wedge^{n-1}T^*\mathcal{S}\) has been used.

Let \((y^a), a = 1, \ldots, n-1\) be local coordinates in an open set of \(\mathcal{V}\) so that \((y^1, \ldots, y^{n-1}, x^n)\) are local coordinates in \(\mathcal{S}\). The local representative of \(\rho_Y \circ Y\) is of the form

\[\sum_{a, \alpha} \left( Y^0_{1...n-1a} + \left( Y^1_{1...n-1a} \partial_a + Y^n_{1...n-1a} \partial_n \right) \right) \otimes g^\alpha \otimes (dy^1 \wedge \cdots \wedge dy^{n-1}) \quad (6.18)\]

and that of \(i^0_Y(Y)\) is

\[\sum_{a, \alpha} \left( Y^1_{1...n-1a} \partial_a + Y^n_{1...n-1a} \partial_n \right) \otimes g^\alpha \otimes (dy^1 \wedge \cdots \wedge dy^{n-1}). \quad (6.19)\]

Here, \(\partial_a\) denotes the tangent basis vector \(\partial/\partial y^a\).

**Remark 6.1.** It is observed that since the bundle \(U\) under consideration is a general vector bundle, it may be, in particular, the jet bundle of any other vector bundle, and so, the procedure above may be used in an iterative manner for hyper-stresses of any order.

In particular, consider the case where \(U = J^{k-2}W\), for a vector bundle \(W\). The local representation of a section of \(U\) is of the form

\[(x, w^\alpha, A^1_{i_1}, \ldots, A^{(k-2)\alpha}_{i_{k-2}j_{k-2}})\]

where each \(i\)-index has two identifiers the first of which indicates the order of the corresponding tensor in accordance with the superscript. Thus, an element of \(J^1U\) is represented locally in of the form

\[(x, w^\alpha, A^1_{i_1}, \ldots, A^{(k-2)\alpha}_{i_{k-2}j_{k-2}}; w^\alpha_{,i_1}, A^1_{i_1j_1}, \ldots, A^{(k-2)\alpha}_{i_{k-2}j_{k-2}j_{k-2}})\]

and an element of \(V^0J^1U \equiv L(T\mathcal{S}, J^{k-2}W)\), for which \(w^\alpha = 0, A^1_{i_1} = 0, \ldots, A^{(k-2)\alpha}_{i_{k-2}j_{k-2}} = 0\), is of the form

\[(x, w^\alpha_{,i_1}, A^1_{i_1j_1}, \ldots, A^{(k-2)\alpha}_{i_{k-2}j_{k-2}j_{k-2}}).\]

It follows that, \(i^0_Y(Y)\) is of the form

\[(x, v^0_{,i_1}, \ldots, v^0_{,i_{n-1}j_1}, v^1_{1\cdots\hat{i}_0\cdots n_a}, v^1_{1\cdots\hat{i}_1\cdots n_a}, \ldots, v^{(k-2)\alpha}_{i_{k-2}j_{k-2}j_{k-2}}). \quad (6.20)\]

In the last equation, the arrays \(Y^p\) are symmetric with respect to the indices \(i_p, \ldots, i_{pp}\).
6.3. **Invariant components of non–holonomic hyper-stresses.** The inclusion mappings of the various vertical subbundles of \( J^1(J^1U) \) make it possible to consider the induced projections, yielding invariant components of non-holonomic hyper-stresses. Thus, we have the projections

\[
i^*_1: L(J^1(J^1U), \wedge^nT^*S) \longrightarrow L(V_1J^1(J^1U), \wedge^nT^*S),
\]

\[
i^*_3: L(J^1(J^1U), \wedge^nT^*S) \longrightarrow L(V_3J^1(J^1U), \wedge^nT^*S),
\]

\[
i^*_5: L(J^1(J^1U), \wedge^nT^*S) \longrightarrow L(V_5J^1(J^1U), \wedge^nT^*S),
\]

(6.21)

all given by compositions with the corresponding inclusions. Locally, these projections are given by

\[
(x^1, x_0, x_1, x_2, x_3, x_{1...n_0}, x_{1...n_1}, x_{1...n_2}, x_{1...n_3}) \mapsto (x^1, x_0, x_1, x_2, x_3, x_{1...n_0}, x_{1...n_1}, x_{1...n_2}, x_{1...n_3}),
\]

\[
(x^1, x_0, x_1, x_2, x_3, x_{1...n_0}, x_{1...n_1}, x_{1...n_2}, x_{1...n_3}) \mapsto (x^1, x_0, x_1, x_2, x_3, x_{1...n_0}, x_{1...n_1}, x_{1...n_2}, x_{1...n_3}),
\]

\[
(x^1, x_0, x_1, x_2, x_3, x_{1...n_0}, x_{1...n_1}, x_{1...n_2}, x_{1...n_3}) \mapsto (x^1, x_0, x_1, x_2, x_3, x_{1...n_0}, x_{1...n_1}, x_{1...n_2}, x_{1...n_3}),
\]

(6.22)

respectively.

Of particular interest is the projection

\[
i^*_3: L(J^1(J^1U), \wedge^nT^*S) \longrightarrow L(V_3J^1(J^1U), \wedge^nT^*S) \cong L(L^2(TS, U), \wedge^nT^*S),
\]

(6.23)

represented locally by

\[
(x^1, x_0, x_1, x_2, x_3, x_{1...n_0}, x_{1...n_1}, x_{1...n_2}, x_{1...n_3}) \mapsto (x^1, x_0, x_1, x_2, x_3, x_{1...n_0}, x_{1...n_1}, x_{1...n_2}, x_{1...n_3}).
\]

(6.24)

We will refer to elements in the image of \( i^*_3 \) as the significant components of the corresponding non–holonomic stresses. Evidently, we have a natural inclusion of significant components of second order hyper-stresses (as in Section 4.1) in the collection of significant components of non–holonomic hyper-stresses and a natural projection given by symmetrization. Specifically, the significant components of a non–holonomic stress are of the form

\[
\sum_{a, i,j} x_{1...n a}^{i,j} \partial_i \otimes \partial_j \otimes g^a \otimes (dx^1 \wedge \cdots \wedge dx^n)
\]

and for second order stresses, \( x_{1...n a}^{i,j} \) are symmetric in the \( i, j \) indices.

6.4. **Contraction operations on significant components.** In analogy with the contraction mapping defined in Section 5 as a particular case of Remark 4.2, one can define a contraction mapping

\[
C^1: L(L^2(TS, U), \wedge^nT^*S) \longrightarrow L(L(TS, U), \wedge^{n-1}T^*S),
\]

\[
\cong TS \otimes U^* \otimes \wedge^nT^*S.
\]

(6.26)

To this end, one can simply use the identification \( L^2(TS, U) \equiv L(TS, L(TS, U)) \) and use the definition in (5.38) and (5.39) for \( W = L(TS, U) \). Consider the composition \( C^1 \circ i^*_3 \). For \( X \in L(J^1(J^1U), \wedge^nT^*S) \) represented as in (6.4), \( C^1 \circ i^*_3(X) \) is...
represented by

\[
\sum_{\alpha, i, j} X^3_{i, j, \ldots, n, a} \partial_{x^a} \otimes g^{\alpha} \otimes \left( \partial_{x^1} \wedge \cdots \wedge dx^n \right) = \sum_{\alpha, i, j} (-1)^{i-1} X^3_{i, j, \ldots, n, a} \partial_{x^a} \otimes (dx^1 \wedge \cdots \wedge d\hat{x}^i \wedge \cdots \wedge dx^n). \tag{6.27}
\]

Let \( X : S \to L(f^1(f^1 U), \wedge^n T^* S) \) be a non-holonomic stress field and let \( V \) be an \((n-1)\)-dimensional submanifold of \( S \). Then, the inclusion \( \iota' : V \to S \) induces a restriction \( \iota' \) of \((n-1)\)-forms onto \( V \) and so we have a field

\[
\tau = \iota' \circ C^1 \circ \iota'_3 \circ X \in L(L(T S, U), \wedge^{n-1} T^* V), \tag{6.28}
\]

the hyper-surface stress induced by the hyper-stress. It is noted that the surface stress is not “tangent” to the surface as it acts on elements of \( L(T S, U) \) and not elements of \( L(T V, U) \). In particular, one cannot perform a meaningful second contraction on \( \tau \) because of this reason.

6.5. **Second contraction.** Alluding to the last remark, one can perform a second contraction on \( C^1 \circ \iota'_3(X) \) to obtain an \((n-2)\)-vector valued form \( C \circ C^1 \circ \iota'_3(X) \) in \( S \). Using the representation in (6.27), \( C \circ C^1 \circ \iota'_3(X) \) is represented by

\[
\sum_{\alpha, i, j} (-1)^{i-1} X^3_{i, j, \ldots, n, a} g^{\alpha} \otimes \left( \partial_{x^1} \wedge \cdots \wedge \hat{d}x^i \wedge \cdots \wedge dx^n \right) \tag{6.29}
\]

This may seem promising because such a form, valued in \( U^n \), would induce edge forces, forces concentrated on submanifolds of dimension \((n-2)\). However, as we show below, this form vanishes identically for symmetric (compatible) significant components. Using

\[
\frac{\partial}{\partial x^j} (dx^1 \wedge \cdots \wedge \hat{d}x^i \wedge \cdots \wedge dx^n) =
\begin{cases}
(-1)^{i-1} dx^1 \wedge \cdots \wedge \hat{d}x^i \wedge \cdots \wedge dx^n, & \text{for } j < i, \\
(-1)^{j-1} dx^1 \wedge \cdots \wedge \hat{d}x^i \wedge \cdots \wedge dx^n, & \text{for } j > i,
\end{cases} \tag{6.30}
\]

it follows that the representation of \( C \circ C^1 \circ \iota'_3(X) \) may be rewritten as

\[
\sum_{\alpha, i > j} (-1)^{i+j} (X^3_{i, j, \ldots, n, a} - X^3_{i, j, \ldots, n, a}) g^{\alpha} \otimes (dx^1 \wedge \cdots \wedge \hat{d}x^i \wedge \cdots \wedge \hat{d}x^j \wedge \cdots \wedge dx^n). \tag{6.31}
\]

Since we expect that for the significant components of second order non-holonomic hyper-stresses, \( X^3_{i, j, \ldots, n, a} = X^3_{i, j, \ldots, n, a} \), the form vanishes identically. This is of course expected as we contract twice a completely anti-symmetric tensor with a symmetric 2-tensor.
6.6. The tangent projection of jets and vertical jets for a given submanifold. Consider \((f^1U)_{|\mathcal{Y}} = \iota_{\gamma}^* f^1U\) for an \((n - 1)\)-dimensional submanifold \(\mathcal{Y}\) of \(\mathcal{S}\) and the jet bundle \(f^1(U_{f\mathcal{Y}}) = f^1(a^\mathcal{Y} U)\). One has a natural projection

\[
\text{pr} : (f^1U)_{|\mathcal{Y}} \longrightarrow J^1(U_{f\mathcal{Y}}), \quad \text{given by} \quad A = j^1u(y) \longmapsto j^1(u_{f\mathcal{Y}}(y)),
\]

(6.32)

for any section \(u\) of \(U\) that represents \(A\). Locally, \(\text{pr}\) is represented by

\[
(y^n, u^a, A^b) \longmapsto (y^n, u^a, B^b), \quad i = 1, \ldots, n, \quad b = 1, \ldots, n - 1,
\]

(6.33)

where \(B^b = A^b\) in an adapted coordinate system.

Evidently, \(\text{pr}\) is linear and we may consider

\[
\text{pr}^* : L((f^1U)_{|\mathcal{Y}}), \wedge^{n-1} T^n \mathcal{Y} \longrightarrow L((f^1U)_{|\mathcal{Y}}), \wedge^{n-1} T^n \mathcal{Y}), \quad Z \longmapsto Z \circ \text{pr}. \quad (6.34)
\]

Locally, if \(Z \in L((f^1U)_{|\mathcal{Y}}), \wedge^{n-1} T^n \mathcal{Y})\) is represented by \((y^n, Z^a_0, Z^b_1)\), then, \(\text{pr}^*(Z)\) is represented by \((y^n, Z^a, Z^n_b)\) with \(Z^n_b = 0\).

We will say that \(A \in (f^1U)_{|\mathcal{Y}} = \iota_{\gamma}^* f^1U\) is vertical if \(\text{pr} A = 0\), so that \(A\) is represented by a section which vanishes at \(\pi^1A\) together with its tangential derivatives. The vertical jets form the vertical subbundle \(V_{\mathcal{Y}}((f^1U)_{|\mathcal{Y}}) = \text{Kernel pr}\). Let \(y^n, a = 1, \ldots, n - 1\), be coordinates in a local chart on \(\mathcal{Y}\) so that \(y^n, y^{n-1}, x^n\), is a chart in \(\mathcal{S}\). If \((y^n, u^a, A^b_0, A^b_n)\) represent an element \(A \in (f^1U)_{|\mathcal{Y}}\), then, \(A\) belongs to the vertical subbundle if and only if \(u^a = 0\) and \(A^b_0 = 0, b = 1, \ldots, n - 1\). Evidently, the fiber \(V_{\mathcal{Y}}((f^1U)_{|\mathcal{Y}})|_y\) is isomorphic to \(L(\mathbb{R}, U_y) \cong U_y\).

Remark 6.2. The normal bundle. While the isomorphism of \(V_{\mathcal{Y}}((f^1U)_{|\mathcal{Y}})|_y\) with \(L(\mathbb{R}, U_y) \cong U_y\) depends on the chart, one may construct a natural isomorphism of \(V_{\mathcal{Y}}((f^1U)_{|\mathcal{Y}})\) with \(L(\Xi, U_{|\mathcal{Y}})\). Here, \(\Xi\) is the subbundle of \(T^* S|\mathcal{Y}\) containing the annihilators of \(T^* \mathcal{Y}, \Xi_y = T_y^* \mathcal{Y}^\perp\), i.e., the one-dimensional subspace of forms \(\xi\), with \(\xi(v) = 0\) for all \(v \in T_y \mathcal{Y}\). Let \(\rho : T^* \mathcal{S} \rightarrow T^* \mathcal{Y}\) be the natural restriction. Then, \(\varphi \in \text{Kernel} \rho\) if and only if \(\rho(\varphi)(v) = \varphi(v) = 0\) for all \(v \in T_y \mathcal{Y}\). Hence, we have the identification \(\Xi \cong \text{Kernel} \rho\).

Let \(\iota_{\Xi} : \Xi \rightarrow T^* \mathcal{S}\) be the natural inclusion. Then, we have the surjection \(\iota_{\Xi}^* : T \mathcal{S} \rightarrow \Xi^*\) and \(\iota_{\Xi}^*(v) = \iota_{\Xi}^*(v')\) if and only if \(\varphi(v - v') = 0\) for any \(\varphi \in \Xi\), i.e., \(v - v' \in T \mathcal{V}\). Thus, each element of \(\Xi^*\) determines a unique element of \(T \mathcal{S}/ T \mathcal{V}\), and one makes the identification \(\Xi^* \cong T \mathcal{S}/ T \mathcal{V}\). In fact, \(\iota_{\Xi}^* : T \mathcal{S} \rightarrow \Xi^* \cong T \mathcal{S}/ T \mathcal{V}\) is simply the natural projection on the quotient.

To construct an isomorphism

\[
\iota_{\Xi} : V_{\mathcal{Y}}((f^1U)_{|\mathcal{Y}}) \longrightarrow L(\Xi, U_{|\mathcal{Y}}),
\]

(6.35)

we first note that \(V_{\mathcal{Y}}((f^1U)_{|\mathcal{Y}})\) is a subbundle of \((V f^1U)_{|\mathcal{Y}} \cong L(T \mathcal{S}, U)|_{\mathcal{Y}}\). In fact, while \((L(T \mathcal{S}, U)|_{\mathcal{Y}} = (T^* \mathcal{S} \cup U)|_{\mathcal{Y}}\),

\[
V_{\mathcal{Y}}((f^1U))_y = T_y \mathcal{V}^\perp \otimes U_y = L(\Xi_y, U_y)
\]

(6.36)

as vertical jets are exactly those elements for which the actions on tangent vectors vanish.
One may consider therefore the inclusion
\[ \iota_{V_Y} : V_Y((J^1 U)|_Y) \rightarrow J^1 U|_Y \]  
(6.37)
and the vertical projection of surface stresses
\[ \iota^*_V : L((J^1 U)|_Y, \wedge^{n-1} T^*\mathcal{W}) \rightarrow L(V_Y((J^1 U)|_Y), \wedge^{n-1} T^*\mathcal{W}), \quad Z \mapsto Z \circ \iota_{V_Y}. \]  
(6.38)
If \( Z \in L((J^1 U)|_Y, \wedge^{n-1} T^*\mathcal{W}) \) is represented by \((y^b, Z^0, Z^1, \ldots |_{n-1}^\beta, Z^{1n, n-1}^\beta)\) so that
\[ Z(A) = \left[ \sum_{\alpha} \left( Z^0_{1, \ldots, n-n-1} u^\alpha + Z^1_{1, \ldots, n-n-1} A^\alpha \right) + \sum_{\alpha, \beta} Z^1_{1, \ldots, n-1} A^\alpha B^\beta \right] dy^{1} \wedge \cdots \wedge dy^{n-1}, \]  
(6.39)
then, \( \iota^*_V (Z) \) is of the form \((y^b, Z^1, \ldots |_{n-1}^\beta)\), which in view of (6.35) is a representation of an element in \( L(L(\Xi, U|_Y), \wedge^{n-1} T^*\mathcal{W}) \equiv \Xi \otimes U^*|_Y \otimes \wedge^{n-1} T^*\mathcal{W}. \)

It is observed that one could define the vertical projection somewhat differently (though we keep the same notation) as
\[ \iota^*_V : L((J^1 U)|_Y, (\wedge^{n-1} T^* S)|_Y) \rightarrow L(V_Y((J^1 U)|_Y), (\wedge^{n-1} T^* S)|_Y), \]  
(6.40)
\[ \equiv \Xi \otimes U^*|_Y \otimes (\wedge^{n-1} T^* S)|_Y, \]
which could then be composed on the left with the restriction of forms. These observations can summarized by the sequences
\[ 0 \rightarrow V_Y((J^1 U)|_Y) \xrightarrow{\iota_{V_Y}} (J^1 U)|_Y \xrightarrow{\text{pr}} J^1 U|_Y \rightarrow 0 \]  
(6.41)
\[ 0 \leftarrow V_Y((J^1 U)|_Y)^* \xleftarrow{\iota^*_V} (J^1 U)^*|_Y \xleftarrow{\text{pr}^*} J^1 U|_Y^* \leftarrow 0, \]
where in the second sequence we wrote to dual bundles, rather than the linear maps into \( \wedge^{n-1} T^*\mathcal{W}, \) for short.

We conclude that the vertical projection \( \iota_{V_Y} (Y) \) of a hyper-surface stress \( Y = p_\sigma (X) \) has an invariant meaning. It is somewhat reminiscent of the bending moment in shell theory.

The restriction of the diagram in (6.41) to the vertical subbundle is of the form
\[ 0 \rightarrow V_Y(V(J^1 U)|_Y) \xrightarrow{\iota_{V_Y}} (V J^1 U)|_Y \xrightarrow{\text{pr}} V J^1 U|_Y \rightarrow 0 \]  
(6.42)
\[ 0 \leftarrow V_Y(V(J^1 U)|_Y)^* \xleftarrow{\iota^*_V} (V J^1 U)^*|_Y \xleftarrow{\text{pr}^*} V J^1 U|_Y^* \leftarrow 0, \]
or,
\[ 0 \rightarrow \Xi \otimes U|_Y \xrightarrow{\iota_{V_Y}} (T^* S \otimes U)|_Y \xrightarrow{\text{pr}} T^* \mathcal{W} \otimes U|_Y \rightarrow 0 \]  
(6.43)
\[ 0 \leftarrow \Xi^* \otimes U^*|_Y \xleftarrow{\iota^*_V} (T S \otimes U^*)|_Y \xleftarrow{\text{pr}^*} T^* \mathcal{W} \otimes U^*|_Y \leftarrow 0, \]
where we have not indicated the restrictions for the various mappings.

6.7. **Tangent surface stresses.** We say that a surface stress \( Z \in L(J^1 U|_\mathcal{V}, \wedge^{n-1} T^* \mathcal{V}) \) is tangent to the submanifold \( \mathcal{V} \) if its vertical component vanishes. That is, \( \iota_{\mathcal{V}}(Z) = 0 \), or alternatively, \( Z^{1n}_{\iota \iota} \) for all \( \beta \). From Equation (6.41), \( Z \) is tangent to \( \mathcal{V} \) when it is in the image of \( pr^* \). It is emphasized that there is no natural projection of surface stresses onto tangent surface stresses.

Let \( Z \in L(J^1 U|_\mathcal{V}, \wedge^{n-1} T^* \mathcal{V}) \) be a surface stress tangent to \( \mathcal{V} \) at some point \( y \in \mathcal{V} \). Then, for a section \( w \) of \( U \), and using the notation above,

\[
Z(j^1 w(y)) = \left[ \sum \alpha Z^0_{\alpha \iota \iota \iota} u^\alpha + \sum_{\alpha, \beta} Z^1_{\alpha \beta} u^\alpha \wedge \right] dy^1 \wedge \cdots \wedge dy^{n-1}. \tag{6.44}
\]

It follows that \( Z \) determines linearly a unique element of \( L(J^1 U|_\mathcal{V}, \wedge^{n-1} T^* \mathcal{V}) \) so that there is a natural isomorphism of the subbundle of tangent surface stresses with \( L(J^1 U|_\mathcal{V}, \wedge^{n-1} T^* \mathcal{V}) \).

Consider the term

\[
I_1 = \int_{\partial B} Y(A) \tag{6.45}
\]

appearing in Equation (6.15). Here \( Y \) is a section of \( L(J^1 U, \wedge^{n-1} T^* S) \) and we consider the case where the section \( A \) of \( J^1 U \) is compatible, that is, there is a section \( u \) of \( U \) with \( A = j^1 u \). Let \( Z \) be the restriction of \( Y \) to \( \partial B \) so that

\[
I_1 = \int_{\partial B} Z(j^1 u). \tag{6.46}
\]

Assuming that \( \partial B \) is piecewise smooth, let \( \mathcal{V} \subset \partial B \) be a smooth \( (n-1) \)-dimensional submanifold. If \( Z \) is tangent to \( \mathcal{V} \), one may use (3.17) for \( W = U|_\mathcal{V} \) and obtain

\[
\int_{\mathcal{V}} Z(j^1 u) = \int_{\mathcal{V}} d(p_\sigma(Z)(u)) - \int_{\mathcal{V}} \text{div} Z(u),
\]

\[
= \int_{\partial \mathcal{V}} p_\sigma(Z)(u) - \int_{\mathcal{V}} \text{div} Z(u), \tag{6.47}
\]

where \( p_\sigma(Z) \) is a section of \( L(U|_\mathcal{V}, \wedge^{n-2} T^* \mathcal{V}) \), i.e., a vector valued \( (n-2) \)-form on \( \mathcal{V} \). Thus, the restriction of \( p_\sigma(Z) \) to \( \partial \mathcal{V} \) induces an edge force on \( \partial \mathcal{V} \).

7. **Additional Geometric Structure**

Our objective in this section is to introduce sufficient geometric structure so that a hyper-surface stress may be decomposed into tangent and transverse components relative to \( \mathcal{V} \). Such a decomposition will make it possible to determine unique edge forces which are induced by the non-holonomic hyper-stress.

In analogy with (3.8) one would attempt to use contraction on \( T S|_\mathcal{V} \otimes \wedge^{n-1} T^* \mathcal{V} \). However, this cannot be done because there is no natural extension of forms from \( \wedge^{n-1} T^* \mathcal{V} \) to \( \wedge^{n-1} T^* S \).
Assuming that $\mathcal{V}$ is orientable, let $n$ be a nowhere vanishing vector field in $TS|_{\mathcal{V}}$ which is transversal to $T\mathcal{V}$ and let $N$ be the induced transverse bundle, that is, $N_y = \{an(y) \mid a \in \mathbb{R}\}$. Evidently, if $\mathcal{S}$ is a Riemannian manifold, the metric induces such a transversal, normal, field. Thus, we have a decomposition

$$TS|_{\mathcal{V}} \cong T\mathcal{V} \oplus N. \quad (7.1)$$

Let $\varphi_n$ be the 1-form on $TS$ that annihilates $T\mathcal{V}$ such that $\varphi_n(n) = 1$ and let $pr_1$ and $pr_2$ be the projections giving the tangent and transverse components of the decomposition. Then,

$$pr_1 = \text{Id} - \varphi_n \otimes n, \quad pr_2 = \varphi_n \otimes n \quad (7.2)$$

and we will use the notation $\nu_\mathcal{V} = pr_1(\nu)$, $\nu_N = pr_2(\nu)$. Using adapted coordinates, $(\nu_\mathcal{V})^a = \sum_i (\delta_i^a - \varphi_i n^a)v^i$, $(\nu_N)^i = \sum_j \varphi_j v^j n^i$, where $\varphi_j$ are the components of $\varphi_n$. It is observed that $pr_2$ induces an isomorphism

$$\Xi^* \cong TS/T\mathcal{V} \rightarrow N, \quad [\nu] \mapsto pr_2(\nu) = \varphi_n(\nu)n. \quad (7.3)$$

Next, we introduce some notation. Evidently, $(T\mathcal{V} \oplus N)^* \cong T^*\mathcal{V} \oplus N^*$. In fact, if $i_1 : T\mathcal{V} \rightarrow TS$ and $i_2 : N \rightarrow TS$ are the natural inclusions, then, the projections $i_1^* : T^*\mathcal{S} \rightarrow T^*\mathcal{V}$ and $i_2^* : T^*\mathcal{S} \rightarrow N^*$ are the restrictions of forms and $pr_1^* = \text{Id}^* - n \otimes \varphi_n$, $pr_2^* = n \otimes \varphi_n$. In particular, for every $\psi \in T^*\mathcal{S}$, it is easy to verify that $\psi = \psi_\mathcal{V} + \psi_N$, where $\psi_\mathcal{V} = \psi - \psi(n)\varphi_n = (\text{Id}^* - n \otimes \varphi_n)(\psi)$ and $\psi_N = \psi(n)\varphi_n = (n \otimes \varphi_n)(\psi)$. In addition $\psi_\mathcal{V}(\nu_\mathcal{V}) = 0$, $\psi_N(\nu_N) = 0$ so that $\psi(\nu) = \psi_\mathcal{V}(\nu_\mathcal{V}) + \psi_N(\nu_N)$. In adapted components, $\psi_\mathcal{V} = \sum_a (\psi_\mathcal{V})_a dy^a$, where $(\psi_\mathcal{V})_a = \psi_a - \sum_j \psi_j n^j \varphi_a$.

It follows that

$$L(TS, U)|_{\mathcal{V}} \cong T^*\mathcal{V} \otimes U|_{\mathcal{V}} \cong T^*\mathcal{V} \otimes U|_{\mathcal{V}} \oplus N^* \otimes U|_{\mathcal{V}} \cong L(T\mathcal{V}, U|_{\mathcal{V}}) \oplus L(N, U|_{\mathcal{V}}), \quad (7.4)$$

and

$$L(L(TS, U)|_{\mathcal{V}}, \wedge^{n-1}T^*\mathcal{V}) \cong L(T\mathcal{V}, U|_{\mathcal{V}}), \wedge^{n-1}T^*\mathcal{V} \oplus L(N, U|_{\mathcal{V}}), \wedge^{n-1}T^*\mathcal{V}) \cong T\mathcal{V} \otimes U|_{\mathcal{V}} \otimes \wedge^{n-1}T^*\mathcal{V} \oplus N \otimes U|_{\mathcal{V}} \otimes \wedge^{n-1}T^*\mathcal{V}. \quad (7.5)$$

We will keep the notation $pr_1$ and $pr_2$ for the two projections of the product in (7.5). The first component, $\overline{\nu} := pr_1(i_1^*Y) = (i_1^*Y)|_{\mathcal{V}} \in T\mathcal{V} \otimes U^1|_{\mathcal{V}} \otimes \wedge^{n-1}T^*\mathcal{V}$, is represented locally by

$$\sum_{a, a} Y^1_{1...n-a} \partial_a \otimes g^a \otimes (dy^1 \wedge \cdots \wedge dy^{n-1}), \quad (7.6)$$

with

$$\overline{\nu}^1_{1...n-a} = \sum_i Y^1_{1...n-a} (\delta^a_i - \varphi_i n^a), \quad (7.7)$$
and \( \text{pr}_2(Y) = \iota^*_Y (\iota^*_Y(Y)) \) \( N \in N \otimes U^*|_{\mathcal{V}} \otimes \wedge^{n-1}T^{\ast}\mathcal{V} \) is represented in the form

\[
\sum_{i,j,\alpha} Y_{1i/n-a}^{ij} \varphi_i n_i \partial_j \otimes g^\alpha \otimes (dy^1 \wedge \cdots \wedge dy^{n-1})
\]

\[
\sum_{i,j,\alpha} Y_{1i/n-a}^{ij} \varphi_i n_i \otimes g^\alpha \otimes (dy^1 \wedge \cdots \wedge dy^{n-1}). \tag{7.8}
\]

The situation is illustrated in the following diagram.

\[
\begin{array}{c}
0 \rightarrow N^* \otimes U|_{\mathcal{V}} \xrightarrow{\text{pr}_2 \otimes 1_{\mathcal{V}}} (T^* S \otimes U)|_{\mathcal{V}} \xrightarrow{\text{pr}} T^* \mathcal{V} \otimes U|_{\mathcal{V}} \rightarrow 0, \\
0 \rightarrow N \otimes U^*|_{\mathcal{V}} \xrightarrow{\text{pr}_2 \otimes 1_N} (TS \otimes U^*)|_{\mathcal{V}} \xrightarrow{\text{pr}^*} T^* \mathcal{V} \otimes U^*|_{\mathcal{V}} \rightarrow 0,
\end{array}
\tag{7.9}
\]

where \( \iota_N \) and \( \rho \) are the natural inclusion of \( N \) and its dual, respectively; \( \text{pr}_2 \) is the isomorphism of \( \mathcal{V}_13 \).

**Remark 7.1.** One might try to decompose a jet into two components, one tangent to \( \mathcal{V} \) and the second normal to \( \mathcal{V} \). The two projections are well defined. However, the projections do not endow \( (J^1 U)|_{\mathcal{V}} \) with a structure of a direct sum. This is because none of the spaces \( J^1(U)|_{\mathcal{V}} \) and \( J^1_0(U)|_{\mathcal{V}} \) is a subspace of \( (J^1 U)|_{\mathcal{V}} \), nor can it be made naturally isomorphic to a sub-bundle. For example, if we were able to identify \( J^1(U)|_{\mathcal{V}} \) with a sub-bundle of \( (J^1 U)|_{\mathcal{V}} \), we would be able to extend jets and we could restrict stresses to \( \mathcal{V} \).

It is now possible to perform contraction on the first component \( \iota^*_Y(Y)|_{\mathcal{V}} \in T^* \mathcal{V} \otimes U^*|_{\mathcal{V}} \otimes \wedge^{n-1}T^{\ast}\mathcal{V} \) and obtain

\[
\tau_Y (Y) = C(\iota^*_Y(Y)) \in U^*|_{\mathcal{V}} \otimes \wedge^{n-2}T^{\ast}\mathcal{V} \cong L(U|_{\mathcal{V}}, \wedge^{n-2}T^{\ast}\mathcal{V}) \tag{7.10}
\]

which is represented in analogy with \( \mathcal{V}_13 \) as

\[
\sum_{a,a} (-1)^{a-1} Y_{1a/n-a} \otimes (dy^1 \wedge \cdots \wedge dy^a \wedge \cdots \wedge dy^{n-1}). \tag{7.11}
\]

Substituting \( \mathcal{V}_7 \) and \( \mathcal{V}_12 \), the representation of \( \tau_Y (\rho_\alpha(X)) \) is given in an adapted coordinate system by

\[
\sum_{i,a} (-1)^{a-1} (-1)^{n-1} X_{ia/n-a} \otimes (dy^1 \wedge \cdots \wedge dy^a \wedge \cdots \wedge dy^{n-1}). \tag{7.12}
\]

The field \( \tau_Y (Y) \) represents tangent surface traction stress, as expected.

The construction leading to the determination of the tangent surface traction (hyper-) stress is summarized in the following diagram..
The generalized surface divergence of \( Y \), a section of \( L(J^1 U)|_Y \wedge T^* Y' \) is defined now by

\[
\text{div}_Y (j^1 u) = d(\tau_Y(Y)(u)) - Y(j^1 u).
\]

(7.14)

With the representation of \( d(\tau_Y(Y)(u)) \) by

\[
\left( \sum_{\alpha, a} Y^{i a}_{1 \ldots n - 1 \alpha} u^\alpha + \sum_{\alpha, a} Y^{i a}_{1 \ldots n - 1 \alpha} u^\alpha \right) dy^1 \wedge \cdots \wedge dy^{n-1},
\]

(7.15)

and the representation of \( Y(j^1 u) \) by

\[
\left( \sum_{\alpha} Y^0_{1 \ldots n - 1 \alpha} u^\alpha + \sum_{\alpha, a} Y^{i a}_{1 \ldots n - 1 \alpha} u^\alpha + \sum_{\alpha, i, j} Y^{i j}_{1 \ldots n - 1 \alpha} \varphi_i u^\alpha_j n^l \right) dy^1 \wedge \cdots \wedge dy^{n-1},
\]

(7.16)

div\(_Y \) (j\(^1\) \( u \)) is represented by

\[
\left( \sum_{\alpha, a} \overline{Y}^{i a}_{1 \ldots n - 1 \alpha, a} u^\alpha - \sum_{\alpha} Y^0_{1 \ldots n - 1 \alpha} u^\alpha - \sum_{\alpha, i, j} Y^{i j}_{1 \ldots n - 1 \alpha} \varphi_i u^\alpha_j n^l \right) dy^1 \wedge \cdots \wedge dy^{n-1},
\]

(7.17)

which evidently depends on the derivative of the vector field in the direction of the transverse vector \( n \). This is of course the reason why the divergence of the hyper-surface stress acts on the jet of generalized velocity \( u \) rather than the values of the field itself as in the case of simple stresses.

**Remark 7.2.** One may attempt to view the term \( \sum_{\alpha, a, j} Y^{i a j}_{1 \ldots n - 1 \alpha} \varphi_i u^\alpha_j n^l \) as the action of \( \iota^*_{\psi Y} \) (\( Y \)). However, as opposed to (6.40), \( u^\alpha_j \) cannot be viewed as components of an element in \( \Xi \wedge U^* Y' \). It is only the additional structure induced by the vector field \( n \) that induces the annihilator \( \varphi_i u^\alpha_j n^l \). (It is true, of course,
that any other annihilator of \( T \nu \) may be expressed as a product of \( \varphi_i u^a n^i \) with a scalar valued function.)

Remark 7.3. Evidently, the derivatives
\[
\nabla^{i_1 \ldots i_{n-1} a} = \sum_i [Y^{i_1 \ldots i_{n-1} a} (\delta^a_i - \varphi_i n^a)],
\]
include the derivatives of the transverse vector field \( n \) and the associated form \( \varphi \) which indicate the “curvature” associated with the field \( n \).

Remark 7.4. Clearly, for Riemannian manifolds, the unit normal vector fields provide the necessary structure for all \((n-1)\)-dimensional submanifolds.

8. Boundary Stress and Edge Interactions

Using the definition of the surface divergence in (7.14), one can now write the boundary integral corresponding to \( Y^j u \) in (6.15) as
\[
\int_{\partial B} Y^j u = \int_{\partial B} d(\tau_{\nu}(Y)(u)) - \int_{\partial B} \text{div}_{\partial B} Y^j u.
\]

If \( \partial B \) is smooth and \( n \) is a smooth field, one may use Stokes’s theorem and conclude that
\[
\int_{\partial B} d(\tau_{\nu}(Y)(u)) = \int_{\partial \partial B} \tau_{\nu}(Y)(u) = 0. \tag{8.1}
\]

On the other hand, it will be assumed henceforth that
\[
\partial B = \bigcup_{m=1}^M \nu_m, \tag{8.2}
\]
where each \( \nu_m \) is a smooth \((n-1)\)-dimensional submanifold of \( S \) so that the intersection
\[
E_{lm} = \partial \nu_l \cap \partial \nu_m \tag{8.3}
\]
is either empty or an \((n-2)\)-dimensional submanifold with boundary of \( S \). We will refer to \( E_{lm} \) as the edge between \( \nu_l \) and \( \nu_m \). Naturally, it is assumed that the field \( n \) is smooth in each \( \nu_m \) so that the same applies to both \( \tau_{\nu_m} \) and \( \text{div}_{\nu_m} \).

Thus,
\[
\int_{\partial B} Y^j u = \sum_{m=1}^M \int_{\nu_m} d(\tau_{\nu_m}(Y)(u)) - \sum_{m=1}^M \int_{\nu_m} \text{div}_{\nu_m} Y^j u
\]
\[
= \sum_{m=1}^M \int_{\partial \nu_m} \tau_{\nu_m}(Y)(u) - \sum_{m=1}^M \int_{\nu_m} \text{div}_{\nu_m} Y^j u, \tag{8.4}
\]
\[
= \sum_{m>l} \int_{E_{ml}} (\tau_{\nu_m}(Y) + \tau_{\nu_l}(Y))(u) - \sum_{m=1}^M \int_{\nu_m} \text{div}_{\nu_m} Y^j u.
\]
The integrals over the edges \( E_{ml} \) represent edge interactions, as one would expect.
It follows that Equation (6.15) may be rewritten for \( A = j^1 u \) as

\[
\int_B X(j^1(j^1u)) = \sum_{m \geq 1} \int_{E_m} (\tau\psi_m(Y) + \tau\psi_l(Y))(u) - \sum_{m=1}^M \int_{\gamma_m} \text{div}\psi_m Y(j^1u) \\
- \int_B \text{div} X(j^1u).
\]

One observes that since \( \text{div} X \) is a section of \( L(j^1U, \wedge^n(T^*\mathcal{S})) \), a simple variational stress, we may apply the definition of the generalized divergence (3.17) to it, and so

\[
\text{div} X(j^1u) = d(p_\sigma(\text{div} X)(u)) - \text{div}(\text{div} X)(u). \tag{8.5}
\]

Here, similarly to a traction stress \( p_\sigma(\text{div} X) \) is a section of \( L(U, \wedge^{n-1}T^*\mathcal{S}) \) and \( \text{div}(\text{div} X) \) is a section of \( L(U, \wedge^nT^*\mathcal{S}) \), similarly to a body force. Using (3.20) and (6.14), \( \text{div}(\text{div} X) \), a section of \( L(U, \wedge^{n-1}T^*\mathcal{S}) \), is represented locally by

\[
\sum_{a,i,j}(X_{1...na,ij}^3j - X_{1...na,i}^1i - X_{1...na,i}^2i + X_{1...na}^0)g^\alpha \otimes (dx^1 \wedge \cdots \wedge dx^n). \tag{8.6}
\]

Equation (3.14) implies that \( p_\sigma(\text{div} X) \), a section of \( L(U, \wedge^{n-1}T^*\mathcal{S}) \), is represented by

\[
\sum_{i,j,\alpha}(-1)^{i-1}(X_{1...na,j}^3i - X_{1...na}^1i)g^\alpha \otimes (dx^1 \wedge \cdots \wedge dx^i \wedge \cdots \wedge dx^n) \tag{8.7}
\]

and its restriction to \( \partial B \) is represented in chart adapted to the submanifold in the form (only the term above with \( i = n \) does not vanish in the restriction)

\[
\sum_{j,\alpha}(-1)^{n-1}(X_{1...na,j}^3n - X_{1...na}^1n)g^\alpha \otimes (dy^1 \wedge \cdots \wedge dy^{n-1}) \tag{8.8}
\]

—a “body force”-like object on the boundary.

We conclude that,

\[
\int_B X(j^1(j^1u)) = \sum_{m \geq 1} \int_{E_m} (\tau\psi_m(p_\sigma(X)) + \tau\psi_l(p_\sigma(X)))(u) - \sum_{m=1}^M \int_{\gamma_m} \text{div}\psi_m p_\sigma(X)(j^1u) \\
- \int_{\partial B} p_\sigma(\text{div} X)(u) + \int_B \text{div}(\text{div} X)(u).
\]
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