Pasting and Reversing Operations over some Vector Spaces

Primitivo Acosta-Humánez, Adriana Lorena Chuquen, and Ángela Mariette Rodríguez

To Angie Marcela Acosta, with occasion of her 15th birthday

Abstract. Pasting and Reversing operations have been used successfully over the set of integer numbers, simple permutations, rings and recently over a generalized vector product. In this paper, these operations are defined from a natural way to be applied over vector spaces. In particular we study Pasting and Reversing over vectors, matrices and we rewrite some properties for polynomials as vector space. Finally we study some properties of Reversing through linear transformations as for example an analysis of palindromic and antipalindromic vector subspaces.

Keywords and Phrases. Antipalindromic vector, linear transformation, palindromic vector, Pasting, Reversing, vector space.

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Introduction

Pasting and Reversing are natural processes that the people do day after day, we paste two objects when we put them together as one object, and we reverse one object when we reflect it over a symmetry axis. We can apply these processes over words, thus, Pasting of lumber with jack is lumberjack, while Reversing of lumber is rebmul. A celebrate phrase of Albert Einstein is I prefer pi, in which Reversing of this phrase is itself and for instance this is a palindromic phrase, another palindromic phrases can be found in http://www.palindromelist.net.

Similarly to palindromic phrases, we can think in palindromic poetries, where each line can be palindromic or the hole poetry is palindromic. The following poems can be found at http://www.trauerfreuart.de/palindrome-poems.htm

Deific? A poem
Same deficit sale: doom mood. Elastic if edemas?
Loops secreting in a doom mood. An igniter: cesspool.
Set agony care till in a doom mood. An illiteracy: no gates.
Senile fileting: I am, God, doom mood. Dogma: ignite lifelines.
Straws? Send a snowfield in a doom mood. An idle, if won sadness warts.
Me, opacified.
Put in us - sun it up
Put in rubies, I won’t be demandable.
Balderdash: sure fire bottle fill-in.
Raw, put in urn action, I’m odd.
Local law: put in ruts. Awareness elates pure gnawed limekiln. Us: sunlike, mildew, anger, upset.
A lessen era was: turn it up! Wall, a cold domino: it can run it up.
Warn: ill I felt to be rife. Rush! Sad.
Red label, bad name, debt nowise.
I burn it up

Space caps
Seed net: tabard. No citadel like sun is but spirit. Sense can embargo to get on.
Still amiss: a pyro-memoir, an ecstasy.
A detail, if fades, paler, tall, affined dusk.
Row no risks, asks ironwork, sudden - if fall at relapsed affiliate - days at scenario:
memory, pass! I’m all. It’s not ego to grab menaces. Next. I rip stubs in use, killed at icon. Drab attendees.

One mathematical theory to express these processes as operations was developed by the first author in [1, 2], followed recently by [3, 4, 5].

In [2] is introduced the concept Pasting of positive integers to obtain their squares as well their squares roots. Five years later, in [1], are defined in a general way the concepts Pasting and Reversing to obtain genealogies of simple permutations in the right block of Sarkovskii order which contains the powers of two. Two years later, in [4], were applied Pasting and Reversing, as well palindromicity and antipalindromicity, over the ring of polynomials, differential rings and mathematical games incoming from M. Tahan’s book The man who counted. Another approaches for reversed polynomials, palindromic polynomials and antipalindromic polynomials can be found in [6, 11]. One year later, in [3], is applied Reversing over matrices to study a generalized vector product, in particular were studied relationships between palindromicity and antipalindromicity with such generalized vector product. Finally, in the preprint [5] were applied Pasting and Reversing over simple permutations with mixed order 4n + 2, following [1].

The aim of this paper is to study Pasting and Reversing, as well palindromicity and antipalindromicity, over vector spaces (vectors, matrices, polynomials, etc.). Some properties are analyzed for vectors, matrices and polynomials as vector spaces, in particular we prove that $W_a \subset V$ (set of antipalindromic vectors of $V$) and $W_p \subset V$ (set of palindromic vectors of $V$) are vector subspaces of $V$ ($V$ is a vector space). Moreover, $V = W_a \oplus W_p$ and in consequence $\dim(V) = \dim(W_a) + \dim(W_p)$.

The reader does not need a high mathematical level to understand this paper, is enough with a basic knowledge of linear algebra and matrix theory, see for example the books given in references [7, 10]. Finally, as butterfly effect, we hope that the results and approaches presented here can be used and implemented in the teaching of basic linear algebra for undergraduate level.

1. Pasting and Reversing over Vectors

In this section we study Pasting and Reversing over vectors in two different approaches, the first one corresponds to an analysis without linear transformaions, using basic definitions and properties of vectors. We consider the field $K$ and the vector space $V = K^n$. 
1.1. A first study without linear transformations. Here we study Pasting and Reversing over vectors using the basic definitions and properties of vectors. In this way, any student beginner of linear algebra can understand the results presented. We start giving the definition of Reversing.

**Definition 1.1.** Let be \( v = (v_1, v_2, \ldots, v_n) \in K^n \). Reversing of \( v \), denoted by \( \tilde{v} \), is given by \( \tilde{v} = (v_n, v_{n-1}, \ldots, v_1) \).

Definition 1.1 lead us to the following proposition.

**Proposition 1.2.** Consider \( v \) and \( \tilde{v} \) as in Definition 1.1. The following statements hold:

1. \( \tilde{\tilde{v}} = v \)
2. \( av + bw = a\tilde{v} + b\tilde{w} \), being \( a, b \in K \) and \( v, w \in V \)
3. \( v \cdot w = \tilde{v} \cdot \tilde{w} \)
4. \( v \times w = \tilde{v} \times \tilde{v} \) for all \( v, w \in K^3 \)

**Proof.** The proof is done according to each item:

1. Due to \( v \in K^n \), then by Definition 1.1 we have that \( \tilde{v} = (v_n, v_{n-1}, \ldots, v_1) \) and for instance \( \tilde{v} = (v_1, v_2, \ldots, v_n) = v \).
2. Let \( z = av + bw \), where \( a, b \in K \) and \( v, w \in V \). Therefore

\[
z = (av_1 + bw_1, \ldots, av_n + bw_n)
\]

and in consequence

\[
\tilde{z} = (av_n + bw_n, \ldots, av_1 + bw_1).
\]

By basic theory of linear algebra, particularly by properties of vectors, we have that

\[
z = (av_m, av_{m-1}, \ldots, av_1) + (bw_{m}, bw_{m-1}, \ldots, bw_1),
\]

which implies that

\[
\tilde{z} = a(v_n, v_{n-1}, \ldots, v_1) + b(w_n, w_{n-1}, \ldots, w_1)
\]

and for instance \( av + bw = a\tilde{v} + b\tilde{w} \).
3. Let assume \( v = (v_1, v_2, \ldots, v_n) \) and \( w = (w_1, w_2, \ldots, w_n) \), thus

\[
v \cdot w = v_1 w_1 + \ldots + v_n w_n.
\]

Now, \( \tilde{v} = (v_n, v_{n-1}, \ldots, v_1) \) and \( \tilde{w} = (w_n, w_{n-1}, \ldots, w_1) \), so we obtain

\[
\tilde{v} \cdot \tilde{w} = v_n w_n + v_{n-1} w_{n-1} + \ldots + v_1 w_1 = v \cdot w.
\]
4. Consider \( v, w \in K^3 \), where \( v = (v_1, v_2, v_3), w = (w_1, w_2, w_3) \). The vector product between \( v \) and \( w \) is given by

\[
v \times w = \begin{vmatrix} e_1 & e_2 & e_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix},
\]

by Definition 1.1 we have that

\[
\tilde{v} \times \tilde{w} = \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}.
\]
Now, by properties of determinants (interchanging rows and columns) we obtain
\[
\begin{vmatrix}
| w_2 & w_1 \\
v_2 & v_1 \\
| w_3 & w_1 \\
v_3 & v_1 \\
| w_3 & w_2 \\
v_3 & v_2 \\
\end{vmatrix}
= \begin{vmatrix}
| e_1 & e_2 & e_3 \\
w_3 & w_2 & w_1 \\
v_3 & v_2 & v_1 \\
\end{vmatrix},
\]
for instance \(v \times w = \tilde{w} \times \tilde{v}\).

\[\square\]

Definition 1.3 lead us to the following definition.

**Definition 1.3.** The vectors \(v\) and \(w\) are called palindromic vector and antipalindromic vector respectively whether they satisfy \(\tilde{v} = v\) and \(\tilde{w} = -w\).

Definition 1.3 lead us to the following result.

**Proposition 1.4.** The following statements hold.

1. The sum of two palindromic vectors belonging to \(K^n\) is a palindromic vector belonging to \(K^n\).
2. The sum of two antipalindromic vectors belonging to \(K^n\) is an antipalindromic vector belonging to \(K^n\).
3. The vector product of two palindromic vectors belonging to \(K^3\) is an antipalindromic vector belonging to \(K^3\).
4. The vector product of two antipalindromic vectors belonging to \(K^3\) is the vector \((0, 0, 0)\).
5. The vector product of one palindromic vector belonging to \(K^3\) with one antipalindromic vector belonging to \(K^3\) is a palindromic vector belonging to \(K^3\).

**Proof.** We prove each statement according to its item.

1. Let \(v\) and \(w\) be palindromic vectors. We have that \(\tilde{v} + \tilde{w} = v + w\).
   In consequence, \(v + w\) is a palindromic vector.
2. Let \(v\) and \(w\) be antipalindromic vectors. We have that \(\tilde{v} + \tilde{w} = -v - w = -(v + w)\).
   In consequence, \(v + w\) is an antipalindromic vector.
3. Let assume \(v = (v_1, v_2, v_1)\) and \(w = (w_1, w_2, w_1)\). The vector product between \(v\) and \(w\) is
   \(v \times w = (v_2w_1 - w_2v_1, v_1w_2 - v_2w_1) = z\).
   Now, by Definition 1.1 we obtain
   \(\tilde{z} = (v_1w_2 - v_2w_1, v_1w_2 - v_2w_1)\)
   and
   \(-\tilde{z} = (v_2w_1 - v_1w_2, v_1w_2 - v_2w_1)\).
   Forinstance \(z = -\tilde{z}\) and owing to Definition 1.3 \(v \times w\) is an antipalindromic vector.
4. Let assume \(v = (-v_1, 0, v_1)\) and \(w = (-w_1, 0, w_1)\), then the vector product between \(v\) and \(w\) is given by \(v \times w = (0, 0, 0)\).
(5) Consider \( v = (v_1, v_2, v_1) \) and \( w = (-w_1, 0, w_1) \), the vector product between \( v \) and \( w \) is given by \( v \times w = (v_2w_1, -2v_1w_1, v_2w_1) \). Now, denoting 
\[ z := v \times w, \]
we have by Definition 1.4 that \( z = (v_2w_1, -2v_1w_1, v_2w_1) \).
Thus, by Definition 1.3 we obtain that \( v \times w \) is a palindromic vector.

\[ \square \]

**Remark 1.5.** In [3] were studied, in a more general way, the vector product for palindromic and antipalindromic vectors. There was used a generalized vector product and were obtained some results involving the palindromicity and antipalindromicity of vectors. For completeness we present in section 3 such results with proofs in detail.

Now we proceed to introduce the concept of Pasting over vectors.

**Definition 1.6.** Consider \( v \in K^n \) and \( w \in K^m \), then \( v \circ w \) is given by
\[
(v_1, v_2, \ldots, v_n) \circ (w_1, w_2, \ldots, w_m) = (v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_m)
\]

The following properties are consequences of Definition 1.6.

**Proposition 1.7.** If \( V = K^n \) and \( W = K^m \), then \( V \circ W \cong K^{n+m} \)

**Proof.** Let \( B_n = \{b_1, b_2, \ldots, b_n\} \) and \( B_m = \{c_1, c_2, \ldots, c_m\} \) basis of \( K^n \) and \( K^m \) respectively. Due to \( v \in K^n \) and \( w \in K^m \), we have by Definition 1.6 that \( v \circ w \in K^{n+m} \), then there exists a basis \( B_{n+m} = \{d_1, d_2, \ldots, d_{n+m}\} \) belonging to \( K^{n+m} \), for instance \( K^n \circ K^m \cong K^{n+m} \).

As immediate consequence of Proposition 1.7 we obtain the following result.

**Corollary 1.8.** \( \dim(V \circ W) = \dim V + \dim W \)

**Proposition 1.9.** The following statements holds.

1. \( \bar{v} \circ \bar{w} = \bar{w} \circ \bar{v} \)
2. \( (v \circ w) \circ z = v \circ (w \circ z) \)

**Proof.** We consider each item separately.

1. Consider \( V = K^n \) and \( W = K^m \). Suppose that \( v = (v_1, v_2, \ldots, v_n) \in V \) and \( w = (w_1, w_2, \ldots, w_m) \in W \). Owing to Definition 1.6 we have that
\[
v \circ w = (v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_m) = z.
\]
Now, by Definition 1.4 we obtain
\[
\bar{z} = (w_m, w_{m-1}, \ldots, w_1, v_n, v_{n-1}, \ldots, v_1),
\]
which by Definition 1.6 lead us to
\[
\bar{z} = (w_m, w_{m-1}, \ldots, w_1) \circ (v_n, v_{n-1}, \ldots, v_1)
\]
and therefore
\[
\bar{v} \circ \bar{w} = \bar{w} \circ \bar{v}.
\]
2. Let assume \( V = K^n \), \( W = K^m \), \( Z = K^\ell, v \in V, w \in W \) and \( z \in Z \). By Definition 1.6 we have that
\[
(v \circ w) \circ z = (v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_m) \circ (z_1, z_2, \ldots, z_\ell),
\]
which implies that
\[
(v \circ w) \circ z = (v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_m, z_1, z_2, \ldots, z_\ell).
\]
Again, in virtue of Definition 1.6, we have that
\[(v \circ w) \circ z = (v_1, v_2, \ldots, v_n) \circ ((w_1, w_2, \ldots, w_m) \circ (z_1, z_2, \ldots, z_\ell)),\]
thus we can conclude that
\[(v \circ w) \circ z = v \circ (w \circ z).\]
\[\square\]

**Proposition 1.10.** Let \( V = K^n \) be a vector space. Consider \( W_p \) and \( W_a \) as the sets of palindromic and antipalindromic vectors of \( V \) respectively. The following statements hold.

1. \( W_p \) is a vector subspace of \( V \),
2. \( \dim W_p = \left\lceil \frac{n}{2} \right\rceil \),
3. \( W_a \) is a vector subspace of \( V \),
4. \( \dim W_a = \left\lfloor \frac{n}{2} \right\rfloor \),
5. \( V = W_p \oplus W_a \),
6. \( \forall v \in V, \exists (w_p, w_a) \in W_p \times W_a \) such that \( v = w_p + w_a \).

**Proof.** We consider each statement separately.

(1) Suppose that \( a, b \in K \) and \( v, w \in W_p \). For instance we have that \( v = \hat{v} \) and \( w = \hat{w} \). By Proposition 1.2 we observe that
\[a \hat{v} + b \hat{w} = a \hat{v} + b \hat{w} \in W_p,\]
in consequence, \( W_p \) is a vector subspace of \( V \).

(2) We analyze the cases when \( n \) is even and also when \( n \) is odd.

(a) Consider \( V = K^n \) and we start assuming that \( n = 2k \). If \( v \in W_p \), then
\[(v_1, v_2, \ldots, v_{2k-1}, v_{2k}) = (v_{2k}, v_{2k-1}, \ldots, v_2, v_1),\]
for instance, we have that
\[
\begin{align*}
v_1 &= v_{2k} \\
v_2 &= v_{2k-1} \\
\vdots \\
v_k &= v_{k+1},
\end{align*}
\]
which lead us to \( v = (v_1, v_2, \ldots, v_k, v_k, \ldots, v_2, v_1) \). In this way we write the vector \( v \) as follows:
\[v = v_1(1, 0, \ldots, 0, 0, \ldots, 1) + \ldots + v_k(0, 0, \ldots, 1, 1, \ldots, 0, 0), \quad v_i \in K.\]

The set of vectors of the previous linear combination are palindromic and linearly independent vectors, for instance they are a basis for \( W_p \) and in consequence
\[\dim W_p = k = \left\lceil \frac{2k}{2} \right\rceil = \left\lceil \frac{n}{2} \right\rceil.\]

(b) Consider \( V = K^n \) and now we assume that \( n = 2k - 1 \). If \( v \in W_p \), then
\[(v_1, v_2, \ldots, v_{2k-2}, v_{2k-1}) = (v_{2k-1}, v_{2k-2}, \ldots, v_2, v_1).\]
Thus, we have that

\[
\begin{align*}
v_1 &= v_{2k-1} \\
v_2 &= v_{2k-2} \\
&\vdots \\
v_{k-1} &= v_{k+1},
\end{align*}
\]

that is, \( k - 1 \) pairs plus the fixed component \( v_k \). This lead us to express the vector \( v \) as follows

\[
v = (v_1, v_2, \ldots, v_{k-1}, v_k, v_{k-1}, \ldots, v_2, v_1)
\]

and for instance we have that

\[
v = v_1(1, 0, \ldots, 0, 0, \ldots, 1) + \ldots + v_k(0, 0, \ldots, 0, 1, 0, \ldots, 0), \quad v_i \in K.
\]

The set of vectors of the previous linear combination are palindromic and linearly independent vectors, for instance they are a basis for \( W_p \) and in consequence

\[
dim W_p = k = \left\lceil \frac{2k - 1}{2} \right\rceil = \left\lceil \frac{n}{2} \right\rceil.
\]

In this way, we have proven that for all \( n \in \mathbb{Z}^+ \), \( \dim W_p = \left\lceil \frac{n}{2} \right\rceil \).

(3) Suppose that \( a, b \in K \) and \( v, w \in W_a \). For instance we have that \( \tilde{v} = -v \) and \( \tilde{w} = -w \). By Proposition \[1,2\] we can observe that

\[
av + bw = a\tilde{v} + b\tilde{w} = -(av + bw)
\]

and for instance \( av + bw \in W_a \), which implies that \( W_a \) is a vector subspace of \( V \).

(4) We analyse the cases when \( n \) is even as well when \( n \) is odd.

(a) Consider \( V = K^n \) and we can start assuming that \( n = 2k \). If \( v \in W_a \), then

\[
(v_1, v_2, \ldots, v_{2k-1}, v_{2k}) = -(v_{2k}, v_{2k-1}, \ldots, v_2, v_1),
\]

for instance, we have that

\[
\begin{align*}
v_1 &= -v_{2k} \\
v_2 &= -v_{2k-1} \\
&\vdots \\
v_k &= -v_{k+1},
\end{align*}
\]

which lead us to

\[
v = (v_1, v_2, \ldots, v_k, -v_k, \ldots, -v_2, -v_1).
\]

This implies that

\[
v = v_1(1, 0, \ldots, 0, 0, \ldots, -1) + \ldots + v_k(0, 0, \ldots, 1, -1, \ldots, 0, 0), \quad v_i \in K.
\]

The set of vectors of the previous linear combination are antipalindromic and linearly independent vectors, for instance they are a basis for \( W_a \) and in consequence

\[
dim W_a = k = \left\lfloor \frac{2k}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor.
\]
(b) Consider $V = K^n$ and now we suppose that $n = 2k - 1$. If $v \in W_a$, then

$$(v_1, v_2, \ldots, v_{2k-2}, v_{2k-1}) = -(v_{2k-1}, v_{2k-2}, \ldots, v_2, v_1).$$

Thus, we obtain that

$$v_1 = -v_{2k-1}$$
$$v_2 = -v_{2k-2}$$
$$\vdots$$
$$v_{k-1} = -v_{k+1},$$

that is, $k - 1$ pairs plus the fixed component $v_k = 0$. This leads us to express the vector $v$ as follows:

$$v = (v_1, v_2, \ldots, v_{k-1}, 0, -v_{k-1}, \ldots, -v_2, -v_1)$$

and for instance we have that

$$v = v_1(1, 0, \ldots, 0, 0, \ldots, 0) + \ldots + v_{k-1}(0, 0, \ldots, 1, 0, \ldots, 0), \quad v_i \in K.$$ 

The set of vectors of the previous linear combination are antipalindromic and linearly independent vectors, for instance they are a basis for $W_a$ and in consequence

$$\dim W_a = k - 1 = \left\lfloor \frac{2k - 1}{2} \right \rfloor = \left\lfloor \frac{n}{2} \right \rfloor.$$ 

In this way we have proven that for all $n \in \mathbb{Z}^+$, $\dim W_a = \left\lfloor \frac{n}{2} \right \rfloor$.

(5) Due to $W_p \cap W_a = \{0\}$ and $\dim W_p + \dim W_a = \left\lfloor \frac{n}{2} \right \rfloor + \left\lfloor \frac{n}{2} \right \rfloor = n = \dim V$, therefore $W_p + W_a = V$ and in consequence $W_p \oplus W_a = V$.

(6) Consider $v \in V$, we can observe that

$$w_p = \frac{v + \bar{v}}{2}$$

is a palindromic vector. In the same way we can observe that

$$w_a = \frac{v - \bar{v}}{2}$$

is an antipalindromic vector and for instance $v = w_p + w_a, \forall v \in V$. \hfill \Box

1.2. Reversing as linear transformation. We denote by $\mathcal{R}$ the transformation Reversing, i.e., for all $v \in K^n$, $\mathcal{R}v = \bar{v}$. Thus, we summarize some previous results in the next proposition.

Proposition 1.11. The following statements hold

(1) $\mathcal{R}$ is a linear transformation.

(2) The transformation matrix of $\mathcal{R}$ is given by

$$\bar{I}_n = \begin{pmatrix}
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 1 & 0 \\
\vdots & & & & \\
0 & 1 & \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & 0
\end{pmatrix} \in \mathcal{M}_{n \times n}(K).$$

(3) $\mathcal{R}$ is an automorphism of $K^n$. 
Proof. We proceed according to each item.

(1) For any scalar $\alpha \in K$ and any pair of vectors $v, w \in K^n$ we obtain
$$R(v + w) = v + w = \tilde{v} + \tilde{w} = Rv + Rw \quad \text{and} \quad R(\alpha v) = \alpha \tilde{v} = \alpha Rv.$$

(2) By definition of Reversing we have $Q(\lambda) = \lambda^2 - 1$.

(3) We observe that $\dim \ker(R - id) = \left\lceil \frac{n}{2} \right\rceil$, $\dim \ker(R + id) = \left\lfloor \frac{n}{2} \right\rfloor$.

(4) Minimial and characteristic polynomial of $R$ are given respectively by
$$Q(\lambda) = \lambda^2 - 1, \quad P(\lambda) = \left\{ \begin{array}{ll} (\lambda + 1)^m (\lambda - 1)^m & \text{for } n = 2m \\ (\lambda + 1)^m (\lambda - 1)^{m+1} & \text{for } n = 2m + 1 \end{array} \right.$$

(5) $K^n = \ker(R - id) \oplus \ker(R + id)$.

(6) $\dim \ker(R - id) = \left\lceil \frac{n}{2} \right\rceil$, $\dim \ker(R + id) = \left\lfloor \frac{n}{2} \right\rfloor$.

(7) $\forall v \in K^n$ let $F_p(v)$ be a palindromic vector and let $F_a(v)$ be an antipalindromic vector. Then $F_p$ and $F_a$ (Palindromicing and Antipalindromicing transformations respectively) are isomorphisms from $K^n$ to $\ker(R - id)$ and from $K^n$ to $\ker(R + id)$ respectively. Furthermore, $\forall v \in K^n$, $v = F_p(v) + F_a(v)$.

PASTING AND REVERSING OVER VECTOR SPACES 9

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(8) $\exists n \in \mathbb{N}$ such that $n$ is a power of 2 and $n = \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor$.

(9) Palindromicing and Antipalindromicing transformations, denoted by $F_p$ and $F_a$ respectively, are given by
$$F_p : K^n \rightarrow \ker(R - id), \quad F_a : K^n \rightarrow \ker(R + id)$$

Due to $R$ and $id$ are linear transformations defined over $K^n$, then
$$F_p = \frac{R - id}{2} \quad \text{and} \quad F_a = \frac{R + id}{2}.$$
are linear transformations over $K^n$. Owing to $\ker(F_p) = \ker(F_a) = 0 \in K^n$, then $F_p$ and $F_a$ are monomorphisms. In the same way, due to $\text{Im}(F_p) = \ker(R - id)$ and $\text{Im}(F_a) = \ker(R + id)$, then $F_p$ and $F_a$ are epimorphisms. For instance, $F_p$ and $F_a$ are isomorphisms from $K^n$ to $\ker(R - id)$ and from $K^n$ to $\ker(R + id)$ respectively. Finally, we can see that any vector $v \in K^n$ can be expressed as the sum of a palindromic vector (obtained through $F_p$) with an antipalindromic vector (obtained through $F_a$). That is, $F_p(v)$ is a palindromic vector and $F_a(v)$ is an antipalindromic vector, furthermore, $v = F_p(v) + F_a(v)$.

Remark 1.12. can see $R$ as a particular case of a linear transformation associated to a permutation matrix. Recall that $A_\sigma$ is a permutation matrix, defined over a given $\sigma \in S_n$, whether its associated linear transformation $R_\sigma$ is given by

$$R_\sigma : K^n \rightarrow K^n \quad (v_1, \ldots, v_n) \mapsto (v_{\sigma(1)}, \ldots, v_{\sigma(n)}).$$

This means that Reversing corresponds to $R_\sigma = R$, where the permutation $\sigma$ and the matrix $A_\sigma$ are given respectively by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \ldots & n-1 & n \\ n & n-1 & n-2 & \ldots & 2 & 1 \end{pmatrix}, \quad A_\sigma = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ 1 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}.$$  

2. Pasting and Reversing over Polynomials

In [4] we studied Pasting and Reversing over polynomials from an different approach, we studied these operations focusing on the ring structure for polynomials. In this section we rewrite some properties of Pasting and Reversing over polynomials, but considering to the polynomials as a vector space. Thus, we apply the previous results for vectors, which we gave in Section [1].

Along this section we consider $(K_n[x], +, \cdot)$ as the vector space of the polynomials of degree less than or equal to $n$ over the field $K$. This vector space is isomorphic to $(K^{n+1}, +, \cdot)$. In this context we do not impose conditions over the polynomials just like the conditions given in [4], for example, we do not need that $x \nmid P(x)$. The following result summarizes the properties given in Section [4] for polynomials.

**Proposition 2.1.** Consider $P \in K_n[x]$, $Q \in K_m[x]$ and $R \in K_s[x]$, the following statements hold.

1. $\tilde{P} = P$
2. $\tilde{P} \circ \tilde{Q} = \tilde{Q} \circ \tilde{P}$
3. $(P \circ Q) \circ R = P \circ (Q \circ R)$
4. $a\tilde{P} + b\tilde{Q} = a\tilde{P} + b\tilde{Q}$, being $a, b \in K$ and $P, Q \in K_n[x]$
5. The sum of two palindromic polynomials of degree $n$ is a palindromic polynomial of degree $n$.
6. The sum of two antipalindromic polynomials of degree $n$ is an antipalindromic polynomial of degree $n$. 
we recover some results given in [4, 11].

In this section we consider the vector space $M_{\mathbb{R}}^n$ consisting of polynomials as vectors. The proof is done using Proposition 1.11 and properties of Pasting proven in Section 1.1.

Remark 2.2. As we can see, this section is a rewriting of Section 1 without new results for polynomials as vector spaces, only we suggest the proofs based on the definition and properties of $\mathcal{R}$. Another interesting thing of this section is that we recover some results given in [4, 11].

3. Pasting and Reversing over Matrices

In this section we consider the vector space $\mathcal{M}_{n \times m}$ (matrices of size $n \times m$ with elements belonging to $K$) which is isomorphic to $K^{nm}$. We present here different approaches for Pasting and Reversing.

3.1. Pasting and Reversing by rows or columns. We can see any matrix as a row vector of its column vectors as well as a column vector of its row vectors. Thus, to matrices we can introduce Pasting and Reversing by rows and columns respectively. Let us denote by $\tilde{A}_r$ Reversing of the row vectors $v_i \in K^n$ of $A$ and by $\tilde{A}_c$ Reversing of the column vectors $c_j \in K^m$ of $A$, where $1 \leq i \leq m$ and $1 \leq j \leq n$. For instance,

$$\tilde{A}_r = \begin{pmatrix} \tilde{v}_1 \\ \tilde{v}_2 \\ \vdots \\ \tilde{v}_n \end{pmatrix} = \begin{pmatrix} \mathcal{R}v_1 \\ \mathcal{R}v_2 \\ \vdots \\ \mathcal{R}v_n \end{pmatrix}, \quad \tilde{A}_c = \begin{pmatrix} \tilde{c}_1 \\ \tilde{c}_2 \\ \vdots \\ \tilde{c}_m \end{pmatrix} = \begin{pmatrix} \mathcal{R}c_1 \\ \mathcal{R}c_2 \\ \vdots \\ \mathcal{R}c_m \end{pmatrix}.$$

Let us denote by $\mathcal{R}_r A := \tilde{A}_r$ and $\mathcal{R}_c A := \tilde{A}_c$, where $\mathcal{R}_r$ and $\mathcal{R}_c$ for suitability will be called $r$-Reversing and $c$-Reversing respectively. Owing to $\tilde{v}_i = v_i I_m$ and $\tilde{c}_i = I_n c_j$ for $1 \leq i \leq n$, $1 \leq j \leq m$ we obtain that $\tilde{A}_r = A I_m$ and $\tilde{A}_c = I_n A$. Therefore $\mathcal{R}_r A = A I_m$ and $\mathcal{R}_c A = I_n A$ and for instance we can define palindromicity and antipalindromicity by rows and columns respectively.
Now, we can assume \( A \in \mathcal{M}_{n \times m}(K) \), \( B \in \mathcal{M}_{q \times m}(K) \) and \( C \in \mathcal{M}_{n \times p}(K) \) given as follows:

\[
A = \begin{pmatrix}
  v_1 \\
v_2 \\
  \vdots \\
v_n
\end{pmatrix} = (f_1 \ f_2 \ \cdots \ f_m), \quad v_i \in K^m, \ f_j^T \in K^n, \ 1 \leq i \leq n, \ 1 \leq j \leq m,
\]

\[
B = \begin{pmatrix}
  s_1 \\
s_2 \\
  \vdots \\
s_q
\end{pmatrix} = (g_1 \ g_2 \ \cdots \ g_m), \quad s_i \in K^m, \ g_j^T \in K^q, \ 1 \leq i \leq q, \ 1 \leq j \leq m,
\]

\[
C = \begin{pmatrix}
  w_1 \\
w_2 \\
  \vdots \\
w_n
\end{pmatrix} = (h_1 \ h_2 \ \cdots \ h_p), \quad w_i \in K^p, \ h_j^T \in K^n, \ 1 \leq i \leq n, \ 1 \leq j \leq p.
\]

As we can see, in agreement with Section 1, we transformed the column vectors \( f_i \), \( g_j \) and \( h_i \) in the form of row vectors through the transposition of matrices \( f_j^T, g_j^T \) and \( h_j^T \) are row vectors). Thus, we can define both Pasting by rows (denoted by \( \circ_r \)) over the matrices \( A \) and \( C \) and Pasting by columns (denoted by \( \circ_c \)) over the matrices \( A \) and \( B \) as follows.

\[
A \circ_r C = \begin{pmatrix}
  z_1 \\
z_2 \\
  \vdots \\
z_n
\end{pmatrix}, \quad z_i = v_i \circ w_i, \quad A \circ_c B = \begin{pmatrix}
  y_1 \\
y_2 \\
  \vdots \\
y_n
\end{pmatrix}, \quad y_i^T = f_i^T \circ g_i^T.
\]

From now on we paste column vectors directly without the use of transposition of vectors. Thus, Pasting of column vectors \( f_i \) and \( g_i \) is \( f_i \circ g_i \). For instance, \( f_i^T \circ g_i^T = (f_i \circ g_i)^T \).

**Proposition 3.1.** Consider matrices \( A \), \( B \) and \( C \) under the previous assumptions. The following statements hold.

1. \( \mathcal{R}_r^2 A = A \), \( \mathcal{R}_c^2 A = A \).
2. \( \mathcal{R}_r(A \circ_r B) = (\mathcal{R}_r B) \circ_r (\mathcal{R}_r A) \), \( \mathcal{R}_c(A \circ_c B) = (\mathcal{R}_c B) \circ_c (\mathcal{R}_c A) \).
3. \( (A \circ_r B) \circ_r C = A \circ_r (B \circ_r C) \), \( (A \circ_c B) \circ_c C = A \circ_c (B \circ_c C) \).
4. \( \mathcal{R}_r(\alpha A + \beta B) = \alpha \mathcal{R}_r A + \beta \mathcal{R}_r B \), \( \mathcal{R}_c(\alpha A + \beta B) = \alpha \mathcal{R}_c A + \beta \mathcal{R}_c B \), \( \alpha, \beta \in K \).
5. If \( V = \mathcal{M}_{n \times m}(K) \) and \( W = \mathcal{M}_{n \times p}(K) \), then \( V \circ W = \mathcal{M}_{\mathcal{M}_{n \times m}(K) \times \mathcal{M}_{n \times p}(K)} \).

In the same way, if \( T = \mathcal{M}_{m \times n}(K) \) and \( S = \mathcal{M}_{m \times n}(K) \), then \( T \circ S = \mathcal{M}_{\mathcal{M}_{m \times n}(K) \times \mathcal{M}_{m \times n}(K)} \).
6. Let \( W^*_p \) and \( W^*_p \) be the set of palindromic matrices by rows and columns of \( \mathcal{M}_{n \times m}(K) \) respectively, then the sets \( W^*_p \) and \( W^*_p \) are vector subspaces of \( \mathcal{M}_{n \times m}(K) \).
7. \( \dim W^*_p = n \left[ \frac{m}{2} \right] \), \( \dim W^*_p = m \left[ \frac{n}{2} \right] \).
8. Let \( W^*_a \) and \( W^*_a \) be the set of antipalindromic matrices by rows and columns of \( \mathcal{M}_{n \times m}(K) \) respectively, then \( W^*_a \) and \( W^*_a \) are vector subspaces of \( \mathcal{M}_{n \times m}(K) \).
(9) \( \dim W^r_a = n \left\lceil \frac{m}{2} \right\rceil \), \( \dim W^c_a = m \left\lceil \frac{n}{2} \right\rceil \)

(10) The sum of two palindromic matrices by rows (resp. by columns) of the same vector space is a palindromic matrix by rows (resp. by columns).

(11) The sum of two antipalindromic matrices by rows (resp. by columns) of the same vector space is an antipalindromic matrix by rows (resp. by columns).

(12) \( \mathcal{M}_{n \times m}(K) = W^r_p \oplus W^r_a = W^c_p \oplus W^c_a \).

(13) \( \forall A \in \mathcal{M}_{n \times m}(K), \exists (A^p_r, A^p_c, A^c_r, A^c_c) \in W^r_p \times W^r_a \times W^c_p \times W^c_a \) such that \( A = A^c_r + A^c_c = A^r_p + A^r_a \).

(14) \( A \circ_r B = A((I_n \circ_c 0_{(n-m) \times m}) \circ_r 0_{n \times p}) + 0_{n \times m} \circ_r B, A \in \mathcal{M}_{n \times m}(K), B \in \mathcal{M}_{n \times p}(K) \),

\( A \circ_c B = A((I_n \circ_c 0_{n \times (m-q)}) \circ_c 0_{n \times p}) + 0_{n \times m} \circ_c B, A \in \mathcal{M}_{n \times m}(K), B \in \mathcal{M}_{p \times m}(K) \).

PROOF. From (1) to (13) we proceed as in the proofs of Section 1 using the properties of \( \mathcal{R} \). (14) is consequence of the definition of Pasting by rows and columns.

REMARK 3.2. There are a lot of mathematical software in where Pasting of matrices is very easy, for example, in Matlab Pasting by rows is very easy: \( \{A, B\} \), as well by columns \( \{A; B\} \), however we can build our own program using our approach given in the previous proposition, following the same structure of Pasting of polynomials as in [3]. Thus, we paste matrices by rows and columns using the item (14) in Proposition 3.1. The interested reader can proof the statements of this paper concerning to Pasting using such equations.

The following proposition summarizes some properties derived from Pasting and Reversing by rows and columns with respect to classical matrix operations.

PROPOSITION 3.3. The following statements hold.

(1) \( (\mathcal{R}_c A)^T = R_c(A^T), (\mathcal{R}_r A)^T = R_r(A^T) \).

(2) \( (A \circ_r B)^T = A^T \circ_r B^T, (A \circ_c B)^T = A^T \circ_c B^T \).

(3) \( \mathcal{R}_r(AB) = A(\mathcal{R}_r B), \mathcal{R}_c(AB) = (\mathcal{R}_c A)B \).

(4) \( \det(\mathcal{R}_r A) = \det(\mathcal{R}_c A) = (-1)^{\left\lfloor \frac{m}{2} \right\rfloor} \det A \).

(5) \( (\mathcal{R}_r(A))^{-1} = \mathcal{R}_r(A^{-1}), (\mathcal{R}_c(A))^{-1} = \mathcal{R}_c(A^{-1}) \), \( \det A \neq 0 \).

(6) The product of two palindromic matrices by rows (resp. by columns) is a palindromic matrix by rows (resp. by columns).

(7) The product of two antipalindromic matrices by rows (resp. by columns) is an antipalindromic matrix by rows (resp. by columns).

(8) \( AB \neq 0 \) is a palindromic matrix by rows (resp. \( AB \neq 0 \) is a palindromic matrix by columns) if and only if \( B \) is a palindromic matrix by rows (resp. \( A \) is a palindromic matrix by columns).

(9) \( AB \neq 0 \) is an antipalindromic matrix by rows (resp. \( AB \neq 0 \) is an antipalindromic matrix by columns) if and only if \( B \) is an antipalindromic matrix by rows (resp. \( A \) is an antipalindromic matrix by columns).

PROOF. We proceed according to each item

(1) We see that \( \mathcal{R}_c A = A^T n = (I_n A^T)^T = (R_n A^T)^T \) and for instance \( \mathcal{R}_c A^T = (\mathcal{R}_c A)^T \).

(2) In the same way, we see that \( \mathcal{R}_r A = I_n A = (A^T I_n)^T = (R_r A^T)^T \) and for instance \( \mathcal{R}_c A^T = (\mathcal{R}_r A)^T \).
(2) Assume $A \in \mathcal{M}_{n \times m}(K)$ and $B \in \mathcal{M}_{n \times p}(K)$. Let $v_i$ and $w_i$ be the row vectors of $A$ and $B$ respectively. Thus $v_i \circ w_i$, $i = 1, \ldots, n$, are the row vectors of $A \circ B$, then $(v_i \circ w_i)^T = v_i^T \circ w_i^T$ are the column vectors of $(A \circ B)^T = A^T \circ B^T$. In the same way we can assume $A \in \mathcal{M}_{m \times n}(K)$ and $B \in \mathcal{M}_{p \times m}(K)$. Let $c_j$ and $d_j$ be the column vectors of $A$ and $B$ respectively. Therefore $c_j \circ d_j$, $j = 1, \ldots, m$, are the column vectors of $A \circ B$, then $(c_j \circ d_j)^T = c_j^T \circ d_j^T$ are the row vectors of $(A \circ B)^T = A^T \circ B^T$.

(3) Consider $A \in \mathcal{M}_{n \times m}(K)$ and $B \in \mathcal{M}_{m \times p}(K)$, therefore $R_r(AB) = I_n(AB) = (I_nA)B = (R_rA)B$. In the same way, $R_c(AB) = (AB)I_p = A(BI_p) = A(R_cB)$.

(4) Consider $A \in \mathcal{M}_{n \times n}(K)$, for instance we obtain $\det(R_rA) = \det(I_n) = \det(A) = \det(A) = \det(\tilde{I}_n) = \det(AI_n) = \det(R_cA)$. Now, it is follows by induction that we can transform $\tilde{I}_n$ into $I_n$ through $\left\lfloor \frac{n}{2} \right\rfloor$ elementary operations, therefore, $\det(\tilde{I}_n) = (-1)^{\left\lfloor \frac{n}{2} \right\rfloor}$.

(5) Assume $A \in \mathcal{M}_{n \times n}(K)$, being $A \neq 0$. For instance, $(R_cA)^{-1} = (A^{-1})^{-1} = I_n^{-1}A^{-1} = \tilde{I}_nA^{-1} = R_r(A^{-1})$. In the same way $(R_cA)^{-1} = (I_nA)^{-1} = A^{-1}I_n^{-1} = A^{-1}\tilde{I}_n = R_c(A^{-1})$.

(6) Assume $A \in \mathcal{M}_{m \times n}(K)$ and $B \in \mathcal{M}_{m \times p}(K)$, such that $R_rA = A$ and $R_cB = B$, therefore $R_r(AB) = (R_rA)B = AB$. In the same way, assuming $R_cA = A$ and $R_cB = B$ we obtain $R_c(AB) = AR_cB = AB$.

(7) Assume $A \in \mathcal{M}_{m \times n}(K)$ and $B \in \mathcal{M}_{m \times p}(K)$ such that $R_rA = -A$ and $R_cB = -B$, therefore $R_r(AB) = (R_rA)B = -AB$. In the same way, assuming $R_cA = -A$ and $R_cB = -B$ we obtain $R_c(AB) = AR_cB = -AB$.

(8) Assume $A \in \mathcal{M}_{n \times m}(K)$ and $B \in \mathcal{M}_{m \times p}(K)$, being $AB \neq 0$. By previous item we see that if $A$ is palindromic by rows (resp. if $B$ is palindromic by columns), then $AB$ is palindromic by rows (resp. then $AB$ is palindromic by columns). Now, suppose that $R_r(AB) = AB \neq 0$, for instance $(R_rA)B = (I_nA)B = AB$, which implies that $R_rA = A$. In the same way, if we assume $R_c(AB) = AB \neq 0$, for instance $(R_cA)B = (I_nB)B = AB \neq 0$, which implies that $R_cB = B$.

(9) Assume $A \in \mathcal{M}_{n \times m}(K)$ and $B \in \mathcal{M}_{m \times p}(K)$, being $AB \neq 0$. By previous item we see that if $A$ is antipalindromic by rows (resp. if $B$ is antipalindromic by columns), then $AB$ is antipalindromic by rows (resp. then $AB$ is antipalindromic by columns). Now, suppose that $R_r(AB) = -AB \neq 0$, for instance $(R_rA)B = (I_nA)B = -AB$, which implies that $R_rA = -A$. In the same way, if we assume $R_c(AB) = -AB \neq 0$, for instance $(R_cA)B = A(I_nB) = -AB \neq 0$, which implies that $R_cB = -B$.

Now, in a natural way, we can introduce the sets

$$W_{pp} := W^r_p \cap W^c_p, \quad W_{pa} := W^r_p \cap W^c_a, \quad W_{ap} := W^r_a \cap W^c_p, \quad W_{aa} := W^r_a \cap W^c_a.$$ 

The sets $W_{pp}$ and $W_{aa}$ correspond to the set of double palindromic matrices and the set of double antipalindromic matrices respectively.

**Proposition 3.4.** The following statements hold.

1. $W_{pp}$, $W_{pa}$, $W_{ap}$ and $W_{aa}$ are vector subspaces of $\mathcal{M}_{n \times m}(K)$. 


(2) \( \dim W_{pp} = \left[ \frac{n}{2} \right]\left[ \frac{m}{2} \right], \) \( \dim W_{pa} = \left[ \frac{n}{2} \right]\left[ \frac{m}{2} \right], \) \( \dim W_{ap} = \left[ \frac{n}{2} \right]\left[ \frac{m}{2} \right] \) and \( \dim W_{aa} = \left[ \frac{n}{2} \right]\left[ \frac{m}{2} \right]. \)

(3) \( \mathcal{M}_{n \times m}(K) = W_{pp} \oplus W_{pa} \oplus W_{ap} \oplus W_{aa}. \)

(4) \( \forall A \in \mathcal{M}_{n \times m}(K), \exists (A_{pp}, A_{pa}, A_{ap}, A_{aa}) \in W_{pp} \times W_{pa} \times W_{ap} \times W_{aa} \) such that \( A = A_{pp} + A_{pa} + A_{ap} + A_{aa}. \)

PROOF. The intersection of vector subspaces is a vector subspace. The rest is obtained through elementary properties of vector subspaces and by the nature of \( W_{pp}, W_{pa}, W_{ap} \) and \( W_{aa}. \)

Finally, according to Remark 3.3, we present the results concerning to the relationship between Reversing and the generalized vector product of \( n - 1 \) vectors of \( K^n, \) which are given in [3, §3].

Let \( M_1 = (m_{11}, m_{12}, \ldots, m_{1n}), \ldots, M_{n-1} = (m_{(n-1),1}, a_{(n-1),2}, \ldots, m_{(n-1),n}), \) be \( n - 1 \) vectors belonging to \( K^n. \) It is known that the generalized vector product of these vectors is given by

\[
\times (M_1, M_2, \ldots, M_{n-1}) = \sum_{k=1}^{n} (-1)^{1+k} \det(M{(k)}) e_k,
\]

being \( e_k \) the \( k \)-th element of the canonical basis for \( K^n \) and \( M{(k)} \) is the square matrix obtained after the deleting of the \( k \)-th column of the matrix \( M = (m_{ij}) \in \mathcal{M}_{(n-1) \times n}(K), \) for more information see [3, 10]. Therefore, the matrix \( M{(k)} \) is a square matrix of size \( (n - 1) \times (n - 1) \) and is given by

\[
M{(k)} = (m_{i,j}^{(k)}) = \begin{cases} (m_{i,j}), & \text{whether } j < k \\ (m_{i,j+1}), & \text{whether } j \geq k \end{cases}, \quad M = \begin{pmatrix} M_1 \\ M_2 \\ \vdots \\ M_{n-1} \end{pmatrix}.
\]

PROPOSITION 3.5. Consider the matrix \( M = (m_{ij}) \in \mathcal{M}_{(n-1) \times n}(K), \) then \( \mathcal{R}_r(M{(k)}) = M{(n-k+1)} \tilde{I}_{n-1}, \) for \( 1 \leq k \leq n. \)

PROOF. We know that \( \mathcal{R}_r M = M \tilde{I}_n = (m_{i,n-j+1}), 1 \leq j \leq n. \) Therefore

\[
\mathcal{R}_r M{(k)} = \mathcal{R}_r (m_{i,j}^{(k)}) = \begin{cases} (m_{i,n-j+1}), & \text{whether } j < k, \\ (m_{i,n-j+1}), & \text{whether } j \geq k \end{cases}.
\]

On the other hand,

\[
M{(n-k+1)} \tilde{I}_{n-1} = (m_{i,j}^{(n-k+1)}) \tilde{I}_{n-1} = \begin{cases} (m_{i,j}) \tilde{I}_{n-1}, & \text{whether } j < n-k+1 \\ (m_{i,j+1}) \tilde{I}_{n-1}, & \text{whether } j \geq n-k+1 \end{cases}
\]

\[
= (m_{i,(n-1)-j+1}) = (m_{i,n-j}), \quad \begin{cases} (m_{i,(n-j)}), & \text{whether } n-j < n-k+1 \\ (m_{i,(n-j)+1}), & \text{whether } n-j \geq n-k+1 \end{cases}
\]

\[
= \begin{cases} (m_{i,(n-j)}), & \text{whether } j > k-1 \\ (m_{i,(n-j)+1}), & \text{whether } j \leq k-1 \end{cases}, \quad \begin{cases} (m_{i,n-j}), & \text{whether } j \geq k \\ (m_{i,n-j+1}), & \text{whether } j < k \end{cases}
\]

for instance \( \mathcal{R}_r(M{(k)}) = M{(n-k+1)} \tilde{I}_{n-1}, \) for \( 1 \leq k \leq n. \)

PROPOSITION 3.6. Consider the vectors \( M_i \in K^n, \) where \( 1 \leq i \leq n - 1. \) Then

\[
\times (\mathcal{R}_r M_1, \mathcal{R}_r M_2, \ldots, \mathcal{R}_r M_{n-1}) = (-1)^{\left[ \frac{n}{2} \right]} \mathcal{R}_r (\times (M_1, M_2, \ldots, M_{n-1})).
\]
Proof. For suitability we write

$$
\mathfrak{M} = \times (R_r M_1, R_r M_2, \ldots, R_r M_{n-1}).
$$

Now, applying the generalized vector product we obtain

$$
\begin{align*}
\mathfrak{M} &= \sum_{k=1}^{n} (-1)^{k+1} \det (R_r (M^{(k)})) e_k \\
&= \sum_{k=1}^{n} (-1)^{k+1} \det (M^{(n-k+1)} I_{n-1}) e_k \\
&= \sum_{k=1}^{n} (-1)^{k+1} \det (M^{(n-k+1)}) \det (I_{n-1}) e_k \\
&= \det (I_{n-1}) \sum_{k=1}^{n} (-1)^{n-k} \det (M^{(k)}) e_{n-k+1} \\
&= (-1)^{n+1} \det (I_{n-1}) \sum_{k=1}^{n} (-1)^{k+1} \det (M^{(k)}) e_{n-k+1} \\
&= (-1)^{n+1} \det (I_{n-1}) \left( \sum_{k=1}^{n} (-1)^{k+1} \det (M^{(k)}) e_k \right) I_n \\
&= (-1)^{n+1} \det (I_{n-1}) R_r (\times (M_1, M_2, \ldots, M_{n-1}))
\end{align*}
$$

and for instance,

$$
\begin{align*}
\mathfrak{M} &= (-1)^{n+1} (-1)^{\frac{n-1}{2}} R_r (\times (M_1, M_2, \ldots, M_{n-1})) \\
&= \left\{ \begin{array}{ll}
(-1)^{\frac{n+1}{2}} R_r (\times (M_1, M_2, \ldots, M_{n-1})), & n = 2k \\
(-1)^{\frac{2n+1}{2}} R_r (\times (M_1, M_2, \ldots, M_{n-1})), & n = 2k - 1
\end{array} \right.
\end{align*}
$$

Thus we conclude the proof.

Remark 3.7. If $M$ is a palindromic matrix by rows, then the minors $M^{(k)}$ have at least $\left\lfloor \frac{n}{2} \right\rfloor - 1$ pair of equal columns. This implies that for $n \geq 4$, the minors have at least one pair of equal columns and for instance $\det (M^{(k)}) = 0$ for all $1 \leq k \leq n$, which lead us to

$$
\times (M_1, M_2, \ldots, M_{n-1}) = 0 \in K^n.
$$

This means that the generalized vector product of $(n-1)$ palindromic vectors belonging to $K^n$ is interesting whenever $1 \leq n \leq 3$. The same result is obtained when we assume $M$ as an antipalindromic matrix by rows, so we recover the previous results given in Section 1 with respect to the vector product. Moreover, some rows of $M$ can be palindromic vectors, while the rest can be antipalindromic vectors, in this way, we can obtain similar results.
3.2. Pasting and Reversing simultaneously by rows and columns. As in previous sections, following Section 1, we can consider the matrices

\[ A = (v_{11}, \ldots, v_{1m}, \ldots, v_{nm}), \quad B = (w_{11}, \ldots, w_{1q}, \ldots, w_{pq}) \]

as vectors. To avoid confusion in this section, we use \( \hat{A} \) instead of \( A \) to denote Reversing of \( A \). Thus, we can see, in a natural way, that

\[ \mathcal{R}A = \hat{A} = A\hat{I}_{nm} = (v_{nm}, \ldots, v_{n1}, \ldots, v_{1m}, \ldots, v_{11}) \]

and also for \( n = p \) or \( m = q \) (exclusively) that

\[ A \odot B = (v_{11}, \ldots, v_{1m}, \ldots, v_{nm}, w_{11}, \ldots, w_{1q}, \ldots, w_{pq}) \]

We come back to express \( \mathcal{R}A \) and \( A \odot B \) in term of matrices instead of vectors, i.e.,

\[
\mathcal{R}A = \begin{pmatrix} v_{nm} & \ldots & v_{n1} \\ \vdots & & \vdots \\ v_{1m} & \ldots & v_{11} \end{pmatrix}, \quad A \odot B = \begin{cases} \\
\begin{pmatrix}
  v_{11} & \ldots & v_{1m} & w_{11} & \ldots & w_{1q} \\
  \vdots & & \vdots & \vdots & & \vdots \\
  v_{n1} & \ldots & v_{nm} & w_{p1} & \ldots & w_{pq}
\end{pmatrix} & n = p, m \neq q \\
\begin{pmatrix}
  v_{11} & \ldots & v_{1m} \\
  \vdots & & \vdots \\
  v_{n1} & \ldots & v_{nm} \\
  w_{11} & \ldots & w_{1p} \\
  \vdots & & \vdots \\
  w_{p1} & \ldots & w_{pq}
\end{pmatrix} & n \neq p, m = q
\end{cases}
\]

We say that any matrix \( P \), with the conditions established above, is a palindromic matrix whether \( \hat{P} = \mathcal{R}P = P \). In the same way, we say that any matrix \( A \), with the conditions established above, is an antipalindromic matrix whether \( \hat{A} = \mathcal{R}A = -A \). Note that \( \mathcal{R}(I_n) = \hat{I}_n \) where \( I_n \) is written in the matrix form, but it should keep in mind that \( \mathcal{R}(I_n) = I_n \hat{I}_n \), where \( I_n \) is written as vector. Thus, we arrive to the following elementary result.

**Lemma 3.8.** Consider \( M \in \mathcal{M}_{n \times m}(K) \). Then

\[ \mathcal{R}A = \begin{pmatrix} \mathcal{R}v_n \\ \vdots \\ \mathcal{R}v_1 \end{pmatrix}, \quad \text{where} \quad A = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \]

**Proof.** It is followed by definition of Reversing in matrices seen as vectors. \( \square \)

The following proposition summarizes the previous results for matrices as vectors.

**Proposition 3.9.** Consider \( A \in \mathcal{M}_{n \times m}(K) \), \( B \in \mathcal{M}_{p \times q}(K) \) and \( C \in \mathcal{M}_{r \times s}(K) \) satisfying the conditions established above, the following statements hold.

\begin{enumerate}
  \item \( \mathcal{R}^2 A = A \)
  \item \( \mathcal{R}(A \odot B) = \mathcal{R}(B) \odot \mathcal{R}(A) \)
  \item \( (A \odot B) \odot C = A \odot (B \odot C) \)
  \item \( \mathcal{R}(bA + cB) = b\mathcal{R}(A) + c\mathcal{R}(B) \) where \( b, c \in K \) and \( A, B \in \mathcal{M}_{n \times m}(K) \).
  \item If \( V = \mathcal{M}_{n \times m}(K) \) and \( W = \mathcal{M}_{p \times q}(K) \), then \( V \odot W = \mathcal{M}_{r \times s}(K) \), where either \( n = p = r, m \neq q \) and \( s = m + q \), or \( n \neq p, m = q \) and \( r = n + p \).
\end{enumerate}
(6) Let $W_p$ be the set of palindromic matrices of $\mathcal{M}_{n \times m}(K)$, then $W_p$ is a vector subspace of $\mathcal{M}_{n \times m}(K)$.
(7) $\dim W_p = \left\lfloor \frac{nm}{2} \right\rfloor$
(8) Let $W_a$ be the set of antipalindromic matrices of $\mathcal{M}_{n \times m}(K)$, then $W_a$ is a vector subspace of $\mathcal{M}_{n \times m}(K)$.
(9) $\dim W_a = \left\lfloor \frac{nm}{2} \right\rfloor$
(10) The sum of two palindromic matrices of the same vector space is a palindromic matrix.
(11) The sum of two antipalindromic matrices of the same vector space is an antipalindromic matrix.
(12) $\mathcal{M}_{n \times m}(K) = W_p \oplus W_a$.
(13) $\forall A \in \mathcal{M}_{n \times m}(K), \exists (A_p, A_a) \in W_p \times W_a$ such that $A = A_p + A_a$.

PROOF. Proceed as in the proofs of Section 1 using Lemma 3.8. □

The following result shows the relationship of Reversing with matrices classical operations.

PROPOSITION 3.10. The following statements hold.
(1) $\mathcal{R}(I_n) = I_n$
(2) $\mathcal{R} = \mathcal{R}_c \mathcal{R}_r = \mathcal{R}_r \mathcal{R}_c$.
(3) $\mathcal{R}(AB) = \mathcal{R}(A) \mathcal{R}(B)$.
(4) $(\mathcal{R}(A))^{-1} = \mathcal{R}(A^{-1})$, det $A \neq 0$.
(5) det$(\mathcal{R}(A)) = $ det $A$.
(6) Tr$(\mathcal{R}(A)) = $ Tr $A$.
(7) $\mathcal{R}(A^T) = (\mathcal{R}(A))^T$.
(8) The product of two palindromic matrices is a palindromic matrix.
(9) The product of two antipalindromic matrices is a palindromic matrix.
(10) The product of one palindromic matrix with one antipalindromic matrix is an antipalindromic matrix.

PROOF. We prove each item.
(1) Due to $I_n$ can be seen as the vector
$$(1,0,\ldots,0,0,1,\ldots,0,\ldots,0,1) \in K^{n^2},$$
then
$$\mathcal{R}(I_n) = I_n R_n I_{n^2} = (1,0,\ldots,0,0,1,\ldots,0,\ldots,0,1) = I_n.$$
(2) Assume $A \in \mathcal{M}_{n \times m}(K)$ and $B \in \mathcal{M}_{m \times r}(K)$. Thus, $AB \in \mathcal{M}_{n \times r}(K)$,$$
\mathcal{R}A = \hat{A} \in \mathcal{M}_{n \times m}, \mathcal{R}B = \hat{B} \in \mathcal{M}_{m \times r} \text{ and } \mathcal{R}(AB) = \hat{A} \hat{B} \in \mathcal{M}_{n \times r}.$$
Suppose that $A = [a_{ij}]_{n \times m}, B = [b_{ij}]_{m \times r}, AB = C = [c_{ij}]_{n \times r}$ and
$$\hat{C} = [\hat{d}_{ij}]_{n \times r}, \text{ for instance}$$
$$c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}, \quad \hat{d}_{ij} = c_{(n+1-i)(r+1-j)} = \sum_{k=1}^{m} a_{(n+1-i)k} b_{k(r+1-j)},$$
which implies that $\hat{C} = \hat{A} \hat{B}$ and then $\mathcal{R}(AB) = \mathcal{R}(A) \mathcal{R}(B)$.
(3) Assume $A \in \mathcal{M}_{n \times n}(K)$, being det $A \neq 0$. Therefore
$$\mathcal{R}(AA^{-1}) = \mathcal{R}(A) \mathcal{R}(A^{-1}) = \mathcal{R}(I_n) = I_n.$$ For instance, $(\mathcal{R}(A))^{-1} = \mathcal{R}(A^{-1})$. 

(4) Assume $A \in \mathcal{M}_{n \times n}(K)$. Due to $\mathcal{R}(A)$ is obtained throughout $2k$ elementary operations of $A$, interchanging $k$ rows and interchanging $k$ columns, then $\det(\mathcal{R}(A)) = (-1)^{2k} \det A = \det A$.

(5) Assume $A = [a_{ij}]_{n \times n} \in \mathcal{M}_{n \times n}(K)$. Thus

$$\text{Tr} \hat{A} = \sum_{i=1}^{n} a_{(n+1-i)(n+1-i)} = a_{nn} + a_{(n-1)(n-1)} + \cdots + a_{22} + a_{11} = \sum_{k=1}^{n} a_{kk} = \text{Tr} A.$$

(6) Assume $A = [a_{ij}]_{n \times m} \in \mathcal{M}_{n \times m}(K)$ and $\hat{A}^T = [d_{ij}]_{m \times n} \in \mathcal{M}_{m \times n}(K)$.

We see that $d_{ij} = a_{(m+1-j)(n+1-i)} = c_{ji}$, where $(A)^T = [c_{ji}]_{m \times n}$ and for instance $\hat{A}^T = (\hat{A})^T$, which implies that $\mathcal{R}(A^T) = (\mathcal{R}(A))^T$.

(7) Assume $\hat{A} = A$ and $\hat{B} = B$. Therefore, $\hat{A}B = \hat{A}B = AB$.

(8) Assume $\hat{A} = -A$ and $\hat{B} = -B$. Therefore, $\hat{A}B = \hat{A}B = AB$.

(9) Assume $\hat{A} = -A$ and $\hat{B} = B$. Therefore, $\hat{A}B = \hat{A}B = -AB$.

At this point, we have considered Pasting over an special case of matrices. As we can see, it can be confused when both matrices have the same size, how can we paste them? Another natural question is: how can we paste to matrices whenever $n \neq p$ and $m \neq q$? To avoid this difficulty we introduce Pasting by blocks, which will be denoted by $\circ \circ$. Consider matrices $A \in \mathcal{M}_{n \times m}(K)$ and $B \in \mathcal{M}_{r \times s}(K)$, Pasting by blocks of $A$ with $B$ is given by

$$A \circ \circ B := \begin{pmatrix} A & 0_{n \times s} \\ 0_{r \times m} & B \end{pmatrix} \in \mathcal{M}_{(n+r) \times (m+s)}(K).$$

It is well known that Pasting by blocks corresponds to a particular case of block matrices, also called partitioned matrices, see [8, 9]. The following result, although is known from block matrices point of view, is consequence of Proposition [8, 9] and Proposition 5.10 considering $\mathcal{R}$ as above and $\circ \circ$ instead of $\circ$.

**Proposition 3.11.** Consider the matrices $A \in \mathcal{M}_{n \times m}(K)$, $B \in \mathcal{M}_{p \times q}(K)$ and $C \in \mathcal{M}_{r \times s}(K)$, the following statements hold.

1. $\mathcal{R}(A \circ \circ B) = \mathcal{R}(B) \circ \circ \mathcal{R}(A)$
2. $(A \circ \circ B) \circ \circ C = A \circ \circ (B \circ \circ C)$
3. If $V = \mathcal{M}_{n \times m}(K)$ and $W = \mathcal{M}_{r \times s}(K)$, then $V \circ \circ W = \mathcal{M}_{(n+r) \times (m+s)}(K)$, where $r = n + p$ and $s = m + q$.
4. $(A \circ \circ B)^T = A^T \circ \circ B^T$.
5. $\det(A \circ \circ B) = \det A \det B$.
6. $\text{Tr}(A \circ \circ B) = \text{Tr} A + \text{Tr} B$.
7. $(A \circ \circ B)^{-1} = A^{-1} \circ \circ B^{-1}$, $\det(AB) \neq 0$.

**Proof.** According to each item:

1. follow directly from the definition of Pasting by blocks, Reversing and Pasting of vectors.
2. is due to $A \circ \circ B = A \oplus B$.
3. to (7) are known properties of block matrices.

**Final Remarks**

This paper is an invitation to undergraduate students and teachers of linear algebra to explore Pasting and Reversing in advanced topics of linear algebra, as
well in other branches of mathematics and physics. Here we solved one question proposed in [4], which relates Pasting and Reversing with vector spaces and matrix theory. However, there are a lot of questions about the applications of Pasting and Reversing, as well generalizations of these operations. The interested reader can try to solve the following natural questions.

- What properties for Pasting and Reversing hold when we use $A_{\sigma}$ as the companion matrix of the linear transformation $R_{\sigma}$, where $\sigma \in S_n$ and $\sigma^2 \neq e$? In particular, what happens when $R_{\sigma}^n = e$, being $R_{\sigma}^{n-1} \neq e$?
- How can we apply Pasting and Reversing in mathematical physics and dynamical systems? It is easy to see that Pauli spin matrices can be written in term of Reversing and the symplectic matrix can be written in term of Pasting. Furthermore, for reversible systems this approach has been applied successfully, see [11].

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(P. Acosta-Humánez) Universidad del Norte, Barranquilla - Colombia
E-mail address: pacostahumanez@unicino.edu.co

(A. Chuquen) Universidad Sergio Arboleda, Bogotá - Colombia

(A. Rodríguez) Universidad Nacional de Colombia, Bogotá - Colombia