We reproduce the result on $\pi_P$ from [5] below and we want to highlight the concavity of the infection probability as a function of the vaccination coverage.

**Proposition 1.** [5] Given any $P \in [0, 1]$, there exists a unique $\pi_P$ that is strictly decreasing and concave in $P$ until $P$ reaches the elimination threshold $P_{crit}$. Furthermore, $\pi_P = 1 - \frac{1}{R_0(1-P)}$ for any $P < P_{crit}$, and $\pi_P = 0$ for any $P \geq P_{crit}$.

We now proceed to analyzing the game. Let $\sigma_i \in [0,1]$ denote the probability that player $i$ chooses vaccination. $\sigma = (\sigma_1, \ldots, \sigma_n)$ denotes a mixed-strategy profile. The expected payoff for player $i$ from randomization with $\sigma_i$ can be expressed as follows:

$$EU_i(\sigma_i, \sigma_{-i}) = \frac{u(R)}{\mu} - \sigma_i C - (1 - \sigma_i)d_{R_0}LE[\pi_{P(\sigma)}],$$

where $E[\pi_{P(\sigma)}]$ denotes the expected infection probability given the mixed-strategy profile $\sigma$.

**Definition 1.** A strategy profile $\sigma^* = (\sigma^*_1, \ldots, \sigma^*_n) \in [0,1]^n$ is a totally mixed-strategy Nash equilibrium for the game $G$ if we have for any $i \in N$

$$\sigma_i \in (0,1),$$

and for all $\sigma^*_i \in [0,1],$

$$EU_i(\sigma^*_i, \sigma_{-i}^*) \geq EU_i(\sigma_i, \sigma_{-i}^*).$$

In what follows, we shall focus on the case in which $C \leq \pi_0d_{R_0}L$. If $C > \pi_0d_{R_0}L$, the only equilibrium outcome is zero vaccination coverage.

**Characterization of all Nash equilibria**

The following proposition characterizes the set of pure-strategy equilibrium outcomes.

**Proposition 2.** For $k = 0, 1, \ldots, \nu_{crit} - 1$, $\nu = k + 1$ is the pure strategy equilibrium outcome for $\frac{C}{d_{R_0}L} \in (\pi_{k+1}, \pi_k]$.

**Proof of Proposition 2.** For any $i \in V(b)$, she has no incentive to deviate because $C \leq d_{R_0}L\pi_k$; for any $i \notin V(b)$, she has no incentive to deviate because $C > d_{R_0}L\pi_{k+1}$. $\nu < k + 1$ cannot arise in pure strategy equilibrium because some $i \notin V(b)$ can always be better off by not taking the vaccination. $\nu > k + 1$ cannot arise in pure strategy equilibrium because some $i \in V(b)$ can always be better off by taking the vaccination. Lastly, note that $\nu > \nu_{crit}$ cannot arise in equilibrium because the infection probability vanishes.
We now show a general characterization of all mixed-strategy equilibria. Let \( \mathcal{M} \) be the set of players using mixed strategies, and \( |\mathcal{M}| = m \). The next proposition characterizes all the mixed-strategy equilibria for \( m > 1 \).

**Proposition 3.** Given \( n > R_0 \) for \( m > 1 \) and \( v \leq \min\{v_{\text{crit}} - 1, n - m\} \), \( \langle v, m \rangle \) arises as a mixed-strategy equilibrium outcome for \( \frac{C}{d_{R_0}} \in (\pi_{v+m-1}, \pi_v) \) with \( \sigma = \sigma^* \) and is uniquely determined by

\[
\frac{C}{d_{R_0}}L = \sum_{k=0}^{m-1} \pi_{v+k} \left( \frac{m-1}{k} \right) \sigma^k (1 - \sigma^*)^{m-1-k}. \tag{5}
\]

**Proof of Proposition 3.** A mixed-strategy Nash equilibrium requires that every player in \( \mathcal{M} \) is indifferent between vaccination and non-vaccination, i.e.,

\[
EU_i(vc, \sigma^*_i) = EU_i(nv, \sigma^*_i) \quad \text{for any } i \in \mathcal{M}. \tag{6}
\]

It follows that

\[
\frac{1}{\mu}u(R) - C = \frac{1}{\mu}u(S) - d_{R_0}L \mathbb{E}[\pi_{P(v)}].
\]

where \( \mathbb{E}[\pi_{P(v)}] \) denote the expected infection probability given \( v = k \). Note that the additional vaccination arising from mixed-strategy follows the Poisson binomial distribution with success probabilities \( \sigma^*\), we obtain

\[
\frac{C}{d_{R_0}}L = \sum_{k=0}^{m-1} \pi_{v+k} \sum_{\nu \in \mathcal{P}(M_{-i}, k)} \prod_{j \in \nu} \prod_{l \in M_{-i}} (1 - \sigma_l) \tag{7}
\]

for any \( i \in \mathcal{M} \). Consider this system of equations (characterizing the indifference conditions for the players in set \( \mathcal{M} \) where \( v \leq \min\{v_{\text{crit}} - 1, n - m\} \), we claim that for any mixed strategy equilibria with \( m > 1 \), the mixed-strategy profile \( \sigma \) is unique as shown by the following two lemmas.

**Lemma 1.** There exists a solution to (7) for \( \frac{C}{d_{R_0}} \in (\pi_{v+m-1}, \pi_v) \) s.t. \( \sigma_i = \sigma^* \) for any \( i \in \mathcal{M} \).

**Proof of Lemma 1.** The system (7) reduces to

\[
\frac{C}{d_{R_0}}L = \sum_{k=0}^{m-1} \pi_{v+k} \left( \frac{m-1}{k} \right) \sigma^k (1 - \sigma^*)^{m-1-k}. \tag{8}
\]

By intermediate value theorem, there exists \( \sigma^* \in (0, 1) \) such that the above equation holds.

**Lemma 2.** (7) has at most one solution.

**Proof of Lemma 2.** Define vector-valued function \( H : [0, 1]^n \to \mathbb{R}^n \) where every
component function

\[ H_i := dR_0 r \sum_{k=0}^{m-1} \pi_{v+k} \sum_{V \in \mathcal{P}(M_{-i})} \prod_{j \in V \setminus M_{-i}} \sigma_j \prod_{l \in M_{-i} \cap V} (1 - \sigma_l). \]

It is easy to check that \( H \) is continuously differentiable on \((0, 1)^n\). The system of equations (7) is equivalent to \( \sigma_i = H_i(\sigma) \) for all \( i \in N \). Suppose there exists two solutions \( \sigma^* \) and \( \sigma' \) such that \( |\sigma^* - \sigma'| > 0 \). By mean value inequality (Rudin, 1976), we have

\[ \|\sigma^* - \sigma'\| = \|H(\sigma^*) - H(\sigma')\| \leq \|DH(\xi)\| \cdot \|\sigma^* - \sigma'\| \quad (9) \]

where \( \xi \in (0, 1)^n \) and \( DH(\xi) \) is the Jacobian matrix evaluated at \( \xi \). Since the row vectors of \( DH(\xi) \) are linearly dependent, \( DH(\xi) \) is not invertible and thus \( \|DH(\xi)\| = 0 \). It follows that \( \|\sigma^* - \sigma'\| \leq 0 \). The requires a contradiction.

Combining the two lemmas, we reach the conclusion that in any mixed strategy equilibrium with \( m > 1 \), mixing probabilities must be unique and identical across players.

Now (5) implies the best response of any \( i \in M \). Any \( i \in V(b) \) has no incentive to deviate since her incentive constraint \( C < dR_0 L E[\pi_{k-1}] \) can be simplified using (5) as

\[ \frac{C}{dR_0 L} < \sigma^{m-1} \pi_{v+m-2} + (1 - \sigma)^{m-1} \pi_{v-1}, \]

which holds under \( \frac{C}{dR_0 L} \in (\pi_{v+m-1}, \pi_v) \). Any \( i \notin V(b) \) has no incentive to deviate since her incentive constraint \( C > dR_0 L E[\pi_{P(i)}] \) can be simplified using (5) as

\[ \frac{C}{dR_0 L} > \sigma^{m-1} \pi_{v+m} + (1 - \sigma)^{m-1} \pi_{v+1}, \]

which again holds under \( \frac{C}{dR_0 L} \in (\pi_{v+m-1}, \pi_v) \). This completes the proof.

It follows from the general proposition that there exists a unique totally mixed strategy equilibrium where \( m = n \).

**Corollary 1.** There exists a unique totally mixed strategy equilibrium, where \( \sigma_i^* = \sigma^* \) and is implicitly defined by

\[ \frac{1}{dR_0 L} = \sum_{k=0}^{n} \binom{v}{k} \left( 1 - \frac{1}{R_0} - \frac{k}{n} \right) \binom{n}{k} \sigma^*^k (1 - \sigma^*)^{n-1-k}. \quad (10) \]
Equilibrium Vaccination Coverage

Let $\mathcal{P}^*(r, R_0)$ denote the probability distribution over the vaccination coverage induced in the mixed-strategy equilibrium. The following proposition shows that an increase in the relative benefit $r$ leads to an equilibrium distribution yielding an unambiguously higher vaccination coverage.

**Proposition 4.** For any $R_0 > 1$, $\mathcal{P}^*(r, R_0)$ first-order stochastically dominates (FOSD) $\mathcal{P}^*(r', R_0)$ if and only if $r > r'$.

**Proof of Proposition 4.** Let $\nu$ and $\nu'$ be the equilibrium number of vaccinated people given the relative benefit $r$ and $r'$ respectively. For future use, denote $F_\nu(x)$ as the CDF of a random variable $\nu$. We start by working directly with $\sigma$ as shown in the following lemma.

**Lemma 3.** $\nu$ FOSD $\nu'$ if and only if $\sigma > \sigma'$.

**Proof of Lemma 3.** $\nu$ FOSD $\nu'$ if and only if $F_\nu(x) \leq F_{\nu'}(x)$ for any $x \in \{1, \ldots, n\}$. Since both $\nu$ and $\nu'$ follow binomial distribution with $n$ trials, we remain to show $\frac{dF_\nu(x)}{d\sigma} \leq 0$.

Now consider for any $x$, the derivative

$$
\frac{dF_\nu(x)}{d\sigma} = \sum_{k=1}^{x} \binom{n}{k} \sigma^{k-1} (1 - \sigma)^{n-k} - \sum_{k=0}^{x} \binom{n-k}{k} \sigma^{k} (1 - \sigma)^{n-k-1}
$$

$$
= n \left( \sum_{k=1}^{x} \binom{n-1}{k-1} \sigma^{k-1} (1 - \sigma)^{n-k} - \sum_{k=0}^{x} \binom{n-1}{k} \sigma^{k} (1 - \sigma)^{n-k-1} \right)
$$

$$
= n \left( F_\nu(x-1) - F_\nu(x) \right) \leq 0
$$

where $\tilde{\nu} \sim Bin(n-1, \sigma)$. $lacksquare$

It remains to show that $\sigma^*$ is monotonically increasing in $r$. By implicit function theorem, taking partial derivative of the vaccine uptake likelihood $\sigma^*(r, R_0)$ with respect to $r$ gives us

$$
\frac{\partial \sigma^*}{\partial r} = \frac{1}{d_{\infty} r^2} \sum_{k=0}^{\nu_{crit}} ((n-1)\sigma^* - k) \left( 1 - \frac{1}{R_0} \frac{k}{n} \right) \binom{n}{k} \sigma^{k-1} (1 - \sigma)^{n-2-k},
$$

which is positive for any $r > \frac{R_0}{(K_0 - 1)d_{\infty}}$.

Therefore, $\nu$ FOSD $\nu'$ and equivalently, $P(r, R_0)$ FOSD $P(r', R_0)$ if and only if $r > r'$. $lacksquare$

The stochastic dominance presented in Proposition 4 implies that $Pr^*(P \geq P_{crit})$, the equilibrium probability for the society to achieve the vaccination coverage needed to obtain herd immunity, is monotonically increasing in $r$ and converges to 1 as $r$ goes to infinity. By nature of mixed-strategy equilibria, it is impossible to obtain $Pr^*(P \geq P_{crit}) = 1$ and obliterate an epidemic. However, the society can still
approximate the complete immunity via voluntary vaccination. The following corollary summarizes this discussion.

**Corollary 2.** $\Pr^*(P \geq P_{\text{crit}}) \to 1$ as $r \to \infty$.

**Proof of Corollary 2.** By Proposition 4, $\Pr^*(P > P_{\text{crit}}) = 1 - M_r(v_{\text{crit}})$ is monotonically increasing in $r$. Furthermore, as $r \to \infty$, $\sigma^* \to 1$, and $M_r(v_{\text{crit}}) \to 0$. □

We now investigate the role of concavity of the long-run infection probability in facilitating the elimination of an epidemic. Let $P^L(r, R_0)$ be the equilibrium vaccination coverage in the linearized environment with $\pi^L = 1 - \frac{1}{R_0} - P$ for any $P < P_{\text{crit}}$, and $\pi^L = 0$ for any $P \geq P_{\text{crit}}$. Then we have the following result.

**Proposition 5.** For any $r > \frac{R_0}{(R_0 - 1)d_{R_0}}$ and $R_0 > 1$, $P^*(r, R_0)$ FOSD $P^L(r, R_0)$.

**Proof of Proposition 5.** By Proposition 3, the totally mixed strategy equilibrium in the linearized environment $\sigma^L = \sigma^L$ is implicitly defined by

$$\frac{1}{d_{R_0}} = \sum_{k=0}^{v_{\text{crit}}} \binom{n}{k} \left( 1 - \frac{1}{R_0} \right) \left( \frac{n - 1}{k} \right) \sigma^L \left( 1 - \sigma^L \right)^{n-1-k}.$$  \hspace{1cm} (11)

By Pascal’s rule, $\binom{n-1}{k} \leq \binom{n}{k}$ for any $k < n$ and thus $\sigma^L \leq \sigma^*$. By lemma 3, $P^*(r, R_0)$ FOSD $P^L(r, R_0)$. □

We now address a comparative static question regarding $R_0$. That is, are the players more likely to reach immunity if they are faced with a more threatening epidemic? An exogenous increase in the reproduction ratio $R_0$ has two competing effects on how easily the society can achieve herd immunity via voluntary vaccination. On the one hand, it raises the long-run probability of infection $\pi_P$, meaning that individuals in the mixed-strategy Nash equilibrium are more likely to vaccinate. On the other hand, a higher $R_0$ also increases the critical level needed for herd immunity $P_{\text{crit}}$. Figure 4 shows that the first effect dominates the second effect so that it is easier to achieve herd immunity when $R_0$ is higher. This result is summarized in the following proposition.

**Proposition 6.** For any $r \in \left(0, \frac{R_0}{(R_0 - 1)d_{R_0}}\right]$, $P^*(r, R_0)$ FOSD $P^*(r, R_0')$ if and only if $R_0 > R_0'$.

**Proof of Proposition 6.** By implicit function theorem, taking partial derivative of the vaccine uptake likelihood $\sigma^*(r, R_0)$ with respect to $R_0$ gives us

$$\frac{\partial \sigma^*}{\partial R_0} = \frac{\sum_{k=0}^{v_{\text{crit}}} R_0^{-2} \binom{n}{k} \sigma^* \left( 1 - \sigma^* \right)^{n-1-k} + \frac{d_{R_0}}{d_{R_0}}}{\sum_{k=0}^{v_{\text{crit}}} \left( n - 1 \right) \sigma^* - k \left( 1 - \frac{1}{R_0} \right) \left( \binom{n}{k} \right) \sigma^* \left( 1 - \sigma^* \right)^{n-2-k'}} \hspace{1cm} (12)$$

which is positive for $r > \frac{R_0}{(R_0 - 1)d_{R_0}}$. By lemma 3, $P^*(r, R_0)$ FOSD $P^*(r, R_0')$ if and only if $R_0 > R_0'$. □
This result implies that a more contagious disease is unambiguously easier to deal with. A higher $R_0$ encourages people to get vaccinated voluntarily. Hence, epidemics like Ebola, with substantially low $R_0$, are particularly difficult to control based on voluntary vaccination.