1. **Introduction**

In this article we shall study some geometric properties of a non-trivial square tile (a non-trivial square tile is a non-constant function on a square).

Consider infinitely many copies of this single square tile and cover the plane with them, without gaps and without overlaps (a tiling of the plane), with the vertices making a square point lattice. The question we ask ourselves in this article is the following: if there is a rotation center of order 4 what kind of geometric properties has the drawing in the tile?

1.1. **Trivial patterns.** In this article we represent rotation centers of order 4 by small black squares; those of order 2 by small circles; reflections are represented by red lines. We do not represent glide reflections. The border of fundamental regions are represented by yellow lines.

![Figure 1](image)

Take a square tile like the one that is shown in Figure 1a. If one translates it (Figure 1b), rotates it around a vertex (as in Figure 2b) or around the center of...
one edge (as in Figure 2a), one can obtain different patterns. These are the trivial ones and these operations (translations, rotations around a vertex or, of order 2, around the center of one edge) are also called trivial.

In the trivial patterns the rotation centers of order 4 can only be in the vertices, and the rotation centers of order 2 can only be in the vertices or in the center of the edges; these are the trivial centers.

All compositions with a 1968 tile by Querubim Lapa (see page 31, in [1]) have rotation centers of order 2 and 4 that are obtained in this way.

However, a 1966 Eduardo Nery tile (Figures 3 and 28–30) can help us to see that there are tilings of the plane with infinitely many copies of a single and “simple” tile that are not trivial.

1.2. **A Eduardo Nery tile.** In 1966, the Portuguese artist Eduardo Nery designed a pattern tile, with remarkable properties (see Figures 3 and 28–30). It can be seen in several locations in Portugal (see References [1], [3] and [4]) such as:
a) the agency of the former Nacional Ultramarino Bank, Torres Vedras (1971–1972);
b) the Mértola Health Centre (1981);
c) the Contumil Railway Station, Oporto (65 panels in three passenger platforms, 1992–1994).

If one assembles four of these tiles around a corner making a square and then translates it parallelly to the edges, covering the plane, one obtains eleven (11) different tilings with symmetry by reflexion and twelve \((12 = 6 \times 2)\) without it. In the eleven (11) tilings are represented eight (8) wallpaper groups: \(pm\) (1),

- \(p2mm\)
- \(c2mm\)
- \(p4gm\)
p2mm (1), p2mg (1), p2gg (1), cm (3), c2mm (1), p4mm (1) and p4gm (2). In the
twelve (12) tilings without symmetry by reflexion are represented two wallpaper
groups: p1 (10) and p2 (2). The meaning of these notations on wallpaper groups
can be seen in References [2], [5], [6].

Three of these tilings are represented in Figures 4 and 5. They show rotations
centers of order 2 in the interior of the tiles and rotation centers of order 4 in the
middle of some edges. This means that they are not trivial. In the Appendix,
Figures 28-30 show the same patterns in coloured tilings.

Figure 6 represents all the symmetries of this Eduardo Nery tile.

These interesting three examples of non trivial tilings of the plane (in particular,
the one of Figures 5 and 30) encouraged this search for an answer to the question
that we asked ourselves in the beginning: if there is a rotation center of order 4
what kind of geometric properties has the drawing in the tile?

2. THE GENERAL RULES AND THEIR EXCEPTIONS

In the following, remember that we represent rotation centers of order 4 by
small black squares; those of order 2 by small circles; reflections are represented
by red lines. We do not represent glide reflections. The border of fundamental
regions are represented by yellow lines.

Consider a p4 pattern on the plane (R², the referential of the pattern) with
rotation centers of order 4 located in the points (r, s) ∈ Z² (r and s are integer
numbers).

Let (p, q), (r, s) ∈ Z². Consider the coordinate system defined by the origin
(0, 0) and the vectors u = (p, q), v = (−q, p). Then

\[ (1, 0) = \frac{1}{p^2 + q^2} (pu - qv), \]
\[(0, 1) = \frac{1}{p^2 + q^2} (qu + pv).\]

Hence, in the referential with \((0, 0)\) as origin and \(u\) and \(v\) as vectors (the referential of the tile; see Figure 7), the point \((r, s)\) \(\in\mathbb{Z}^2\), in the referential of the pattern, has coordinates
\[
(r, s) \rightarrow \frac{1}{p^2 + q^2} (rp + sq, sp - rq).
\]

In the following we shall always represent the domain of a tile as the square \([0, 1]^2\) (see Figure 7). In a tiling of the plane, with infinitely many copies of this square tile, each tile is located in a region \([p, p + 1] \times [q, q + 1]\), with \(p, q \in \mathbb{Z}\). If no confusion is possible we use the domain to denote each tile.

The majority of the tiles that one can see are of one of four types. However there are eight exceptions to these general rules.

2.1. The first type of general tiles. Take \(p\) and \(q\) both even or both odd numbers (\(p + q\) is even) and the square with vertices in \((0, 0), (p, q), (-q, p), (p - q, p + q)\), in the referential of the pattern, or any translation of this square. A tile of this type is obtained by intersecting this square with the pattern in the plane.

The rotation centers of order 4 are located, in the referential of the tile, at
\[
(a_0, b_0) + \frac{1}{p^2 + q^2} (rp + sq, sp - rq),
\]
for some \((a_0, b_0) \in [0, 1]^2\), and with \(r, s \in \mathbb{Z}\).
Define \( p_1 = \frac{p-q}{2}, q_1 = \frac{p+q}{2} \); then \( p = p_1 + q_1 \) and \( q = q_1 - p_1 \). Notice that for any \( p_1, q_1 \in \mathbb{Z} \), \( p_1 + q_1 \) and \( q_1 - p_1 \) are both even or both odd numbers. Then, the rotation centers of order 4 are located, in the referential of the tile, at

\[
(a_0, b_0) + \frac{1}{2} (p_1^2 + q_1^2) (r (p_1 + q_1) + s (q_1 - p_1), s (p_1 + q_1) + r (p_1 - q_1)),
\]

for some \((a_0, b_0) \in [0, 1]^2\), and with \( r, s \in \mathbb{Z} \).

The number of such rotation centers is an even number, \( p^2 + q^2 = 2 (p_1^2 + q_1^2) \).

\[\text{Figure 8.}\]

In Figure 8 we consider \( p = q = 1 \) and represent two tiles. In the left one there is a center at \((1/2, 1/2)\) and the other centers are trivial, located in the vertices. In the right one there is a center at \((a_0, b_0)\) and the other at \((a_0 + 1/2, b_0 + 1/2)\). We only represent \( p4 \) versions of the tiles; there are also \( p4mm \) versions and \( p4gm \) versions.

\[\text{Figure 9.}\]
In Figure 9 we consider \( p = 2 \) and \( q = 0 \) and represent two tiles. In the left one there are centers at \((1/2, 0), (0, 1/2), (1/2, 1/2), (1, 1/2), (1/2, 1)\) and the others are located in the vertices. In the right one there is a center at \((a_0, b_0)\) and the others at \((a_0 + 1/2, b_0), (a_0 + 1/2, b_0 + 1/2)\) and \((a_0, b_0 + 1/2)\). We only represent \( p4 \) versions of the tiles; there are also \( p4mm \) versions and \( p4gm \) versions.

2.2. **The second type of general tiles.** Take \( p \) and \( q \) such that if one of them is even the other is odd \((p + q \text{ is odd})\) and take the square with vertices in \((0, 0), (p, q), (-q, p), (p - q, p + q)\), in the referential of the pattern. A tile of this second type is obtained by intersecting this square with the pattern in the plane.

The rotation centers of order 4 are located, in the referential of the tile, at

\[
\frac{1}{p^2 + q^2} (rp + sq, sp - rq) ,
\]

with \( r, s \in \mathbb{Z} \).

The number of such rotation centers is an odd number, \( p^2 + q^2 \).

In Figure 10 we represent two tiles. In the left we consider \( p = 1 \) and \( q = 0 \); there are only centers located in the vertices. In the right we consider \( p = 1 \) and \( q = 2 \); there are centers at \((2/5, 1/5), (4/5, 2/5), (3/5, 4/5), (1/5, 3/5)\), and the others are located in the vertices. We only represent \( p4 \) versions of the tiles; there are also \( p4mm \) versions and \( p4gm \) versions. We can also consider interchanging \( p \) and \( q \); the figures are the same but seen as in a mirror.

2.3. **The third type of general tiles.** Take \( p \) and \( q \) such that if one of them is even the other is odd \((p + q \text{ is odd})\) and take the square with vertices in \((-p/2, -q/2), (p/2, q/2), (p/2 - q, p + q/2), (-p/2 - q, p - q/2)\), in the referential of the pattern. A tile of this third type is obtained by intersecting this square with the pattern in the plane.
The rotation centers of order 4 are located, in the referential of the tile, at

$$(a, b) + \frac{1}{p^2 + q^2} (rp + sq, sp - rq),$$

with $r, s \in \mathbb{Z}$, $(a, b) = \left(\frac{1}{2}, 0\right)$ or $(a, b) = \left(0, \frac{1}{2}\right)$.

The number of such rotation centers is a odd number, $p^2 + q^2$.

In Figure 11, we represent two tiles. In the left we consider $p = 1$ and $q = 0$; there are centers located at $(1/2, 0)$ and $(1/2, 1)$. In the right we consider $p = 1$ and $q = 2$; there are centers at $(1/2, 0), (1/2, 1), (3/10, 2/5), (9/10, 1/5), (7/10, 3/5)$ and $(3/10, 2/5)$. We only represent $p4$ versions of the tiles; there are also $p4mm$ versions and $p4gm$ versions. We can also consider interchanging $p$ and $q$; the figures are the same but seen as in a mirror.

2.4. The fourth type of general tiles. Take $p$ and $q$ both odd numbers and the square with vertices in $(0, 0), \frac{1}{2} (p, q), \frac{1}{2} (p - q, p + q), \frac{1}{2} (-q, p)$, in the referential of the pattern. A tile of this type is obtained by intersecting this square with the pattern in the plane.

The rotation centers of order 4 are located, in the referential of the tile, at

$$(a, b) + \frac{2}{p^2 + q^2} (rp + sq, sp - rq),$$

with $r, s \in \mathbb{Z}$, $(a, b) = (0, 0)$ or $(a, b) = (1, 0)$, indifferently. Notice that if there is a rotation center at $(0, 0)$, then there is another one at $(1, 1)$; the same happens with $(1, 0)$ and $(0, 1)$.

The number of such rotation centers is not an integer; it is $(p^2 + q^2)/4$, where $p^2 + q^2$ is even.
In Figure 12 we represent two tiles. In the left we consider \( p = q = 1 \); it is a tile without symmetries. In the right we consider \( p = 1 \) and \( q = 3 \); there are centers at \( (3/5, 1/5) \), \( (2/5, 4/5) \), and the others are located in the vertices \( (0, 0) \) and \( (1, 1) \). We only represent \( p4 \) versions of the tiles; there are also \( p4mm \) versions and \( p4gm \) versions. We can also consider interchanging \( p \) and \( q \); the figures are the same but seen as in a mirror.

2.5. **The exceptions.** There are eight exceptions to these four general rules. All of them have two rotation centers of order 4 in the middle of two edges.

In the first type of these tiles these two edges have a common vertex. Assume that the rotation centers of order 4 are at \( (1/2, 0) \) and \( (0, 1/2) \). Then, there are also rotation centers of order 2 at \( (1/4, 1/4) \), \( (3/4, 1/4) \), \( (3/4, 3/4) \) and \( (1/4, 3/4) \). This tile with reflections, has the diagonal that contains \( (0, 0) \) a reflection line (\( p4gm \)). This is exactly the case of Eduardo Nery tile. See Figure 13.

In the second and third types of these tiles these two edges have no common vertex. Assume that the rotation centers of order 4 are at \( (1/2, 0) \) and \( (1/2, 1) \).
In the second type there is translation invariance by the vectors \((1/2, 1/2)\) and \((1/2, -1/2)\). This tile with reflections, has a reflection line containing the points \((1/2, 0)\) and \((1/2, 1)\) \((p4mm)\), or a reflection line containing the points \((0, 1/2)\) and \((1, 1/2)\) \((p4gm)\). See Figure 14.

![Figure 14](image1.png)

In the third type there is no translation invariance and one has rotation centers of order 2 at \((1/4, 1/4)\), \((3/4, 1/4)\), \((3/4, 3/4)\) and \((1/4, 3/4)\). This tile with reflections, has a reflection line containing the points \((1/2, 0)\) and \((1/2, 1)\) \((p4mm)\), or a reflection line containing the points \((0, 1/2)\) and \((1, 1/2)\) \((p4gm)\). See Figure 15.

![Figure 15](image2.png)

3. Proofs

3.1. The general rules. Consider a tiling of the plane with infinitely many copies of a square tile. Each tile is located in a region \([p, p + 1] \times [q, q + 1]\), with \(p, q \in \mathbb{Z}\). This tiling has a non-trivial rotation center of order 4 in the point \((a, b) \in [0, 1]^2\). Non-trivial means that it is not one of the four vertices of the tile.

We shall assume w.l.g. that \(0 \leq b \leq a \leq 1/2, \ a > 0\).
3.1.1. The action of a rotation on the tiles. In this section’s figures we denote the vertices in the different tiles in the following way (see Figures 16 and 17):

a) For the tile $[0, 1]^2$:
   (1, 1) is “1”, (0, 1) is “2”, (0, 0) is “3” and (1, 0) is “4”.

b) For the tile $[-1, 0] \times [0, 1]$:
   (0, 1) is “a1”, (−1, 1) is “a2”, (−1, 0) is “a3” and (0, 0) is “a4”.

c) For the tile $[0, 1] \times [-1, 0]$:
   (1, 0) is “b1”, (0, 0) is “b2”, (0, −1) is “b3” and (1, −1) is “b4”.

d) For the tile $[-1, 0] \times [-1, 0]$:
   (0, 0) is “c1”, (−1, 0) is “c2”, (−1, −1) is “c3” and (0, −1) is “c4”.

e) For the tile $[1, 2] \times [0, 1]$:
   (2, 1) is “d1”, (1, 1) is “d2”, (1, 0) is “d3” and (2, 0) is “d4”.

f) For the tile $[1, 2] \times [-1, 0]$:
   (2, 0) is “e1”, (1, 0) is “e2”, (1, −1) is “e3” and (2, −1) is “e4”.

![Figure 16](image-url)

A single square tile can be located in, at most, four different ways in every $[p, p + 1] \times [q, q + 1]$. We denote A, B, C and D the different ways of locating the tile in $[-1, 0] \times [0, 1]$. We denote E, F, G and H the different ways of locating the tile in $[0, 1] \times [-1, 0]$: 
The rotation of order 4 around $(a, b)$ generates the following transformations on the tile (see figures 16–24). In these tables “4” means rotation center of order 4, “2” means rotation center of order 2 and “tr” means translation:

|       | A       | B       | C       | D       |
|-------|---------|---------|---------|---------|
| A / E | a₁ / b₁ | 1       | 2       | 3       | 4       |
| B / F | a₂ / b₂ | 2       | 3       | 4       | 1       |
| C / G | a₃ / b₃ | 3       | 4       | 1       | 2       |
| D / H | a₄ / b₄ | 4       | 1       | 2       | 3       |

|       | A       | B       | C       | D       |
|-------|---------|---------|---------|---------|
| 4     | (a, b)  | (a, b)  | (a, b)  | (a, b)  |
| 4     | $\left(\frac{1}{2} + a, \frac{1}{2} + b\right)$ | $(1 + a - b, a + b)$ | $\left(\frac{1}{2} - b, \frac{1}{2} + a\right)$ | $(a + b, b - a)$ |
| 2     | $\left(\frac{1}{2} + a, b\right)$ | $\left(\frac{1+a-b}{2}, \frac{1+a+b}{2}\right)$ | - | - |
| tr    | -       | -       | $(-2a, 1 - 2b)$ | $(a - b, a + b)$ |
Each set of transformations, in these cases (A, B, C, D, E, F, G and H), generates a $p_4$ pattern in the plane with $pu_i + qv_i$ translation invariance, where $u_i$ and $v_i$ are

$$
\begin{array}{cccc}
  i & \rightarrow & A & B \\
  u_i & \left(\frac{1}{2}, \frac{1}{2}\right) & (1-b, a) & \left(\frac{1}{2} - a - b, \frac{1}{2} + a - b\right) \\
  v_i & \left(-\frac{1}{2}, \frac{1}{2}\right) & (-a, 1-b) & \left(-\frac{1}{2} - a + b, \frac{1}{2} - a - b\right)
\end{array}
$$

$$
\begin{array}{cccc}
  i & \rightarrow & E & F \\
  u_i & \left(\frac{1}{2}, \frac{1}{2}\right) & (a, b) & \left(\frac{1}{2} - a + b, \frac{1}{2} - a - b\right) \\
  v_i & \left(-\frac{1}{2}, \frac{1}{2}\right) & (-b, a) & \left(a + b - \frac{1}{2} - a + b\right)
\end{array}
$$

For the tile located at $[-1,0] \times [-1,0]$, the results are

$$
\begin{array}{cccc}
  c_1 = 1 & c_1 = 2 & c_1 = 3 & c_1 = 4 \\
  4 & (a, b) & (a, b) & (a, b) \\
  4 & (a, 1+b) & \left(\frac{1}{2} - a - b, \frac{1}{2} + a + b\right) & (-b, a) & \left(\frac{1}{2} + a + b, \frac{1}{2} - a + b\right) \\
  2 & \left(\frac{1}{2} + a, \frac{1}{2} + b\right) & \left(\frac{a-b}{2}, \frac{1+a+b}{2}\right) & - & - \\
  \text{tr} & - & - & (2a, 2b) & (a - b - 1, a + b)
\end{array}
$$

$$
\begin{array}{cccc}
  i & \rightarrow & (c_1 = 1) & (c_1 = 2) & (c_1 = 3) & (c_1 = 4) \\
  u_i & (1, 0) & \left(\frac{1}{2} - b, \frac{1}{2} + a\right) & (a - b, a + b) & \left(\frac{1}{2} + b, \frac{1}{2} - a\right) \\
  v_i & (0, 1) & \left(-\frac{1}{2} - a, \frac{1}{2} - b\right) & (-a - b, a - b) & \left(a - \frac{1}{2}, \frac{1}{2} + b\right)
\end{array}
$$

Notice that there are more transformations involving, for example, the interaction of the tiles $[-1,0] \times [0,1]$ and $[0,1] \times [-1,0]$:
### Figure 18.

### Figure 19.

|   | AE    | AF    | AG    | AH    |
|---|-------|-------|-------|-------|
| 4 | \((a, b)\) | \(\left\{ \frac{1}{2} + a - b, \frac{1}{2} + a + b \right\}\) | \((1 - b, a)\) | \(\left\{ \frac{1}{2} + a + b, \frac{1}{2} - a + b \right\}\) |
| 4 | \((a, 1 + b)\) | \((-b, a)\) | \((1 - a, -b)\) | \((1 + b, 1 - a)\) |
| 2 | \(\left\{ \frac{1}{2} + a, \frac{1}{2} + b \right\}\) | \(\left\{ \frac{1+a-b}{2}, \frac{a+b}{2} \right\}\) | - | - |
| tr | - | - | \((2a, 2b)\) | \((-a + b, 1 - a - b)\) |

|   | BE    | BF    | BG    | BH    |
|---|-------|-------|-------|-------|
| 4 | \(\left\{ \frac{1}{2} + a - b, \frac{1}{2} + a + b \right\}\) | \((1 - b, a)\) | \(\left\{ \frac{1}{2} - a - b, \frac{1}{2} + a - b \right\}\) | \((1 - a, 1 - b)\) |
| 4 | \((a, 1 + b)\) | \((-b, a)\) | \((1 - a, -b)\) | \((1 + b, 1 - a)\) |
| 2 | - | \(\left\{ \frac{1}{2} - b, \frac{1}{2} + a \right\}\) | \(\left\{ 1 - \frac{a+b}{2}, \frac{1+a-b}{2} \right\}\) | - |
| tr | \((1 - a - b, a - b)\) | - | - | \((-2b, 2a)\) |
3.1.2. Case E. In this case the transformations are the three rotations of order 4 ($\sigma_0$, $\sigma_1$ and $\sigma_2$) with centers located at $(a, b)$, $(1/2 + a, 1/2 + b)$ and $(a - 1/2, 1/2 + b)$, and the rotation of order 2 ($\sigma_3$) with centers located at $(a, 1/2 + b)$. These transformations are compatible in the sense that they generate a pattern $p4$ in the plane.

\[
\begin{array}{cccc}
\text{CE} & \text{CF} & \text{CG} & \text{CH} \\
4 & (b, 1-a) & \left(\frac{1}{2} - a - b, \frac{1}{2} + a - b\right) & (1-a, 1-b) & \left(\frac{1}{2} - a + b, \frac{1}{2} - a - b\right) \\
4 & (a, 1+b) & (-b, a) & (1-a, -b) & (1+b, 1-a) \\
2 & - & - & \left(\frac{1}{2} - a, \frac{1}{2} - b\right) & \left(\frac{1}{2} + a, \frac{1}{2} - a\right) \\
\text{tr} & (2a, 2b) & (-a + b, 1 - a - b) & - & - \\
\end{array}
\]

\[
\begin{array}{cccc}
\text{DE} & \text{DF} & \text{DG} & \text{DH} \\
4 & \left(\frac{1}{2} + a + b, \frac{1}{2} - a + b\right) & (a, b) & \left(\frac{1}{2} - a + b, \frac{1}{2} - a - b\right) & (b, 1-a) \\
4 & (a, 1+b) & (-b, a) & (1-a, -b) & (1+b, 1-a) \\
2 & \left(\frac{a+b}{2}, \frac{1-a+b}{2}\right) & - & - & \left(\frac{1}{2} + b, \frac{1}{2} - a\right) \\
\text{tr} & - & - & (-2b, 2a) & (1-a - b, a - b) \\
\end{array}
\]

$i \to \text{AE/BF/CG/DH} \quad \text{AF/BG/CH/DE} \quad \text{AG/BH/CE/DF} \quad \text{AH/BE/CF/DG}$

\[
\begin{array}{cccc}
u_i & (1, 0) & \left(\frac{1}{2} + a, \frac{1}{2} + b\right) & (a-b, a+b) & \left(\frac{1}{2} - a, \frac{1}{2} - b\right) \\
v_i & (0, 1) & \left(-\frac{1}{2} - b, \frac{1}{2} + a\right) & (-a-b, a-b) & \left(b - \frac{1}{2}, \frac{1}{2} - a\right) \\
\end{array}
\]

**Figure 20.**
**Proposition 3.1.** The tile is such that it has the pattern \(p4\) generated by the four transformations in the plane \((\mathbb{R}^2)\).

**Proof.** Consider the dashed square \(gr\) contained in the tile (Figure 20). \(gr\) is a generating region of the pattern \(p4\) in the plane. Its rotation by \(\sigma_3\) covers \([0,a] \times [b+1/2,1]\). Its rotation by \(\sigma_0\), \(\sigma_0^2\) and \(\sigma_0^3\) covers \([0,a] \times [0,b+1/2]\) and \([a,a+1/2] \times [0,b]\). Its rotation by \(\sigma_1\), \(\sigma_1^2\) and \(\sigma_1^3\) covers \([a+1/2,1] \times [b,1]\) and \([a,a+1/2] \times [b+1/2,1]\). The rotation of \([a,a+b] \times [b+1/2,1-a+b]\) by \(\sigma_1\) covers \([a+1/2,1] \times [0,b]\). 

3.1.3. Case F. In this case the transformations are a translation by \(u = (a+b, b-a)\), two rotations of order 4 (\(\sigma_0\) and \(\sigma_1\)) with centers located at \((a,b)\) and \((a-b,a+b)\), and a rotation of order 2 (\(\sigma_2\)) with center located at \((a+b, a+b)\). These transformations are compatible in the sense that they generate a pattern \(p4\) in the plane.

![Figure 21](image-url)

**Proposition 3.2.** The covered region of the tile, by the pattern \(p4\) in the plane, is the union of \([2a,1] \times [0,a+b]\) and the translations by \(nu\) of \([0,2a] \times [0,1]\) and (whenever \(a+b < \frac{1}{2}\) and \(b < a\)) by \(-nu\) of \([2a,1-2b] \times [a+b,2a], n = 0,1,2,\ldots\)

**Proof.** Consider the dashed triangle \(gr\) contained in the tile (Figure 21); \(gr\) is a generating region of the pattern \(p4\) in the plane. Its rotations by \(\sigma_0\) covers the quadrilateral with vertices at \((0,0)\), \(P\), \((2a,2b)\) and \((a-b,a+b)\); \(P = (2(a^2+b^2)/(a+b),0)\). The triangle with vertices at \(P\), \((2a,0)\) and \((2a,2b)\) is the translation by \(u\) of the triangle with vertices \(((a-b)^2/(a+b),a-b)\), \((a-b,a-b)\) and \((a-b,a+b)\). Hence, we have covered a quadrilateral with vertices at \((0,0)\), \((2a,0)\), \((2a,2b)\) and \((a-b,a+b)\). Its rotation by \(\sigma_1\), \(\sigma_1^2\) and \(\sigma_1^3\) covers the rectangle \([0,2a] \times [0,2a+2b]\).
If one rotates this last rectangle by $\sigma_0^{-1}$, one covers $[2a, 3a + b] \times [0, a + b]$. Let us use the notations
\[
\alpha_n = (n + 3) a + (n + 1) b, \quad \text{for } 0 \leq n \leq \frac{1 - 3a - b}{a + b},
\]
\[
= 1, \quad \text{for } n \geq \frac{1 - 3a - b}{a + b};
\]
\[
\gamma_n = (n + 2) a + (n + 2) b, \quad \text{for } 0 \leq n \leq \frac{1 - 2a - 2b}{a + b},
\]
\[
= 1, \quad \text{for } n \geq \frac{1 - 2a - 2b}{a + b},
\]
and call $R_n$ the region union of $[0, 2a] \times [0, \gamma_n]$ and $[2a, \alpha_n] \times [0, a + b]$. This region of the tile has the pattern $p4$ of the plane for $n = 0$.

Assume that, for some $n \geq 0$, the region $R_n$ is covered.

Rotating $R_n$ by $\sigma_1$, one covers also the rectangle $[a - b, 2a] \times [\gamma_n, \gamma_{n+1}]$. The rectangle $[0, a - b] \times [\gamma_n, \gamma_{n+1}]$ is the translation by $(-u)$ of $[a + b, 2a] \times [\gamma_n - a + b, \gamma_{n+1} - a + b]$. Hence, the region $[0, 2a] \times [\gamma_n, \gamma_{n+1}]$ is covered, and so is the region $[0, 2a] \times [0, \gamma_{n+1}]$.

If one rotates this last rectangle by $\sigma_0^{-1}$, one covers $[2a, \alpha_{n+1}] \times [0, a + b]$. Hence, $R_{n+1}$ is covered.

Notice that if $\alpha_n = 1$, then $\gamma_{n+1} = 1$, and if $\gamma_{n+1} = 1$, then $\alpha_n = 1$. Therefore, for some $n \geq 0$, $R_n$ is the union of $[0, 2a] \times [0, 1]$ and $[2a, 1] \times [0, a + b]$.

If $a + b < 1/2$ and $b < a$, then rotating $[0, 2a] \times [0, 1]$ by $\sigma_1^{-1}$, one covers $[2a, 1 - 2b] \times [a + b, 2a]$.

Finally, the covered region of the tile, by the pattern $p4$ in the plane, is the union of $[2a, 1] \times [0, a + b]$ and the translations by $nu$ of $[0, 2a] \times [0, 1]$ and (whenever $a + b < 1/2$ and $b < a$) by $-nu$ of $[2a, 1 - 2b] \times [a + b, 2a]$, $n = 0, 1, 2, \ldots$ \(\square\)

Notice that the covered region contains
\[
\left\{ (x, y) \in [0, 1]^2 : y \leq 1 + (a + b)^{-1} (b - a) (x - a + b) \right\}.
\]

3.1.4. Case G. Assume that $(a, b) \neq (1/2, 0)$. In this case the transformations are a translation by $u = (1 - 2a, -2b)$, and three rotations of order 4 ($\sigma_0$, $\sigma_1$, and $\sigma_2$) with centers located at $(a, b)$, $(1/2 + b, 1/2 - a)$ and $(1/2 - b, a - 1/2)$ (see Figure 22). These transformations are compatible in the sense that they generate a pattern $p4$ in the plane.

Notice that if $(a, b) = (1/2, 0)$, the transformations collapse: $u = 0$ and all the centers are located at $(1/2, 0)$.  


In Figure 23 the black rectangle, $R_n$, has vertices at $(\alpha_n, 0)$, $(\beta_n, 0)$, $(\alpha_n, \gamma_n)$ and $(\beta_n, \gamma_n)$, with

$$0 \leq \alpha_n \leq a < \frac{1}{2} + b \leq \beta_n \leq 1,$$

and

$$0 < \frac{1}{2} - a + b \leq \gamma_n \leq 1.$$

The red and the green rectangles are rotations of the black one by $\sigma_0$, $\sigma_0^{-1}$, $\sigma_1^{-1}$ and $\sigma_1$, in the left hand side (of Figure 23), and by $\sigma_0$, $\sigma_0^{-1}$, $\sigma_1^{-1}$ and $\sigma_1^2$, in the
right hand side (of Figure 23). The points \( P_1, \ldots, P_4, Q_1, \ldots, Q_6 \) have coordinates:

\[
\begin{align*}
P_1 &= (a + b - \gamma_n, -a + b + \beta_n) \\
P_2 &= (a + b, -a + b + \beta_n) \\
P_3 &= (a + b, 1 - a + b - \alpha_n) \\
P_4 &= (a + b + \gamma_n, 1 - a + b - \alpha_n) \\
Q_1 &= (2a - \beta_n, 2b - \gamma_n) \\
Q_2 &= (1 - a + b, -a - b + \alpha_n) \\
Q_3 &= (a + b + \gamma_n, 1 - a + b - \beta_n) \\
Q_4 &= (a + b - \gamma_n, -a + b + \alpha_n) \\
Q_5 &= (a - b, a + b - \beta_n) \\
Q_6 &= (1 + 2b - \alpha_n, 1 - 2a - \gamma_n).
\end{align*}
\]

The coordinates \( \alpha_n, \beta_n \) and \( \gamma_n \) are defined as follows:

\[
\begin{align*}
\alpha_0 &= a, \quad \beta_0 = \frac{1}{2} + b, \quad \gamma_0 = \frac{1}{2} - a + b; \\
\alpha_{n+1} &= a + b - \gamma_n, \quad \text{if } a + b - \gamma_n \geq 0, \\
&= 0, \quad \text{if } a + b - \gamma_n \leq 0; \\
\beta_{n+1} &= a + b + \gamma_n, \quad \text{if } a + b + \gamma_n \leq 1, \\
&= 1, \quad \text{if } a + b + \gamma_n \geq 1; \\
\gamma_{n+1} &= -a + b + \beta_n, \quad \text{if } a + b \leq \frac{1}{2}, \\
&= 1 - a + b - \alpha_n, \quad \text{if } a + b \geq \frac{1}{2}.
\end{align*}
\]

Notice that \( \gamma_n \leq 1 - a + b \), for every \( n = 0, 1, 2, \ldots \)

**Proposition 3.3.** The tile is such that the region

\[[0, 2a] \times [0, 1 - a + 3b] \cup [2a, 1] \times [0, 1 - a + b]\]

has the pattern \( p_4 \) generated by the four transformations in the plane \((\mathbb{R}^2)\).

**Proof.** 1. Assume that \( a + b \leq \frac{1}{2} \) and \( b > 0 \). Then

\[
\begin{align*}
\alpha_n &= 2a - (n - 1) b - \frac{1}{2}, \quad \text{for } 1 \leq n \leq \frac{4a + 2b - 1}{2b}, \\
&= 0, \quad \text{for } n \geq \frac{4a + 2b - 1}{2b};
\end{align*}
\]
\[ \beta_n = (n + 1) b + \frac{1}{2}, \quad \text{for } n \leq \frac{1 - 2b}{2b} \]
\[ = 1, \quad \text{for } n \geq \frac{1 - 2b}{2b} \]

\[ \gamma_n = (n + 1) b - a + \frac{1}{2}, \quad \text{for } n \leq \frac{1}{2b} \]
\[ = 1 - a + b, \quad \text{for } n \geq \frac{1}{2b}. \]

Notice that

a) \[ 4a + 2b - 1 \leq 1 - 2b < 1; \]

b) for every \( n \geq 0, \)
\[ (a + b - \gamma_n) - (2a - \beta_n) = b > 0; \]

c) \[ 2b - \gamma_n = b - bn + a - \frac{1}{2} \leq -bn \leq 0; \]

d) \[ \alpha_n - a - b = a - bn - \frac{1}{2} \leq -b (n + 1) < 0; \]

e) however,
\[ (a + b + \gamma_n) - (1 - a + b) = a + b (n + 1) - \frac{1}{2}, \]
\[ 1 - a + b - \beta_n = \frac{1}{2} - a - bn; \]

hence, if
\[ \frac{1 - 2a - 2b}{2b} < n < \frac{1 - 2a}{2b}, \]

there is a a small rectangle with vertices at \((1 - a + b, 0), (1/2 + (n + 2) b, 0), (1/2 + (n + 2) b, 1/2 - a - nb)\) and \((1 - a + b, 1/2 - a - nb)\).

This rectangle is the translated by \( u \) of the rectangle with vertices \((a + b, 2b), (-\frac{1}{2} + (n + 2) b + 2a, 2b), (-\frac{1}{2} + (n + 2) b + 2a, 1/2 - a - nb + 2b)\) and \((a + b, 1/2 - a - nb + 2b)\).

f) As
\[ \beta_n - \left(-\frac{1}{2} + (n + 2) b + 2a\right) = 1 - 2a - b \geq 0, \]

g) and
\[ \gamma_n - \left(\frac{1}{2} - a - nb + 2\right) = (2n - 1) b \geq 0, \]

this last rectangle is contained in \( R_n \) for \( n \geq 1. \)
2. Assume that \( a + b \geq \frac{1}{2} \) and \( a < \frac{1}{2} \). Then

\[
\alpha_n = (n + 1) a - \frac{n}{2}, \quad \text{for } n \leq \frac{2a}{1 - 2a}
\]
\[
= 0, \quad \text{for } n \geq \frac{2a}{1 - 2a}
\]

\[
\beta_n = 2b - (n - 1) a + \frac{n}{2}, \quad \text{for } 1 \leq n \leq \frac{2 - 2a - 4b}{1 - 2a}
\]
\[
= 1, \quad \text{for } n \geq \frac{2 - 2a - 4b}{1 - 2a}
\]

\[
\gamma_n = -(n + 1) a + b + \frac{n + 1}{2}, \quad \text{for } n \leq \frac{1}{1 - 2a}
\]
\[
= 1 - a + b, \quad \text{for } n \geq \frac{1}{1 - 2a}.
\]

Notice that

a) \[
2 - 2a - 4b \leq 2a < 1;
\]

b) for every \( n \leq \frac{2a}{1 - 2a} \)

\[
(1 + 2b - \alpha_n) - (a + b + \gamma_n) = \frac{1}{2} - a > 0;
\]

c) for every \( n \leq \frac{1}{1 - 2a} \)

\[
1 - 2a - \gamma_n = \frac{1}{2} (2a - 1) (n - 1) - b \leq 0;
\]

d) for every \( 1 \leq n \leq \frac{2 - 2a - 4b}{1 - 2a} \)

\[
a + b - \beta_n = \frac{1}{2} n (2a - 1) - b < 0;
\]

e) however,

\[
(a + b - \gamma_n) - (a - b) = b + \frac{1}{2} (2a - 1) (n + 1),
\]

\[
-a + b + \alpha_n = b + \frac{1}{2} n (2a - 1);
\]

hence, if

\[
\frac{2a + 2b - 1}{1 - 2a} < n < \frac{2b}{1 - 2a};
\]

there is a small rectangle with vertices at \((a - \frac{1}{2}(1 - 2a)(n + 1), 0)\), \((a - b, 0)\), \((a - b, b + (1/2)n(2a - 1))\) and \((a - \frac{1}{2}(1 - 2a)(n + 1), b + (1/2)n(2a - 1))\).
This rectangle is the rotation by $\sigma_2$ of the rectangle with vertices $(1 - a - b, 0)$, $(1 - a - (1/2)n(1 - 2a), 0)$, $(1 - a - (1/2)n(1 - 2a), -b + 1/2(1 - 2a)(n + 1))$ and $(1 - a - b, -b + 1/2(1 - 2a)(n + 1))$.

As

\[ (1 - a + b - \alpha_n) - \left(-b + \frac{1}{2} (1 - 2a) (n + 1)\right) = \frac{1}{2} - a + 2b \geq 0, \]

\[ (a + b + \gamma_n) - \left(1 - a - \frac{1}{2} n (1 - 2a)\right) = a + 2b - \frac{1}{2} + n (1 - 2a) \]

\[ > 3a + 4b - \frac{3}{2} \geq b; \]

this last rectangle is contained in the red rectangle with vertex $P_4$.

3. Assume that $a < 1/2$ and $b = 0$.

In this case $u = (1 - 2a, 0)$. The generating region, $gr$, of Figure 22 is the square with vertices at $(a, 0)$, $(1/2, 0)$, $(1/2, 1/2 - a)$ and $(a, 1/2 - a)$. Rotating it around the center $(1/2, 1/2 - a)$, one obtains another square with vertices at $(a, 0)$, $(1 - a, 0)$, $(1 - a, 1 - 2a)$ and $(a, 1 - 2a)$. Translating this last square by $pu$, with $p \in \mathbb{Z}$, one obtains $[0, 1] \times [0, 1 - 2a]$. Rotating this region around the center $(1/2, 1/2 - a)$, one obtains the union of $[0, 1] \times [0, 1 - 2a]$ with $[a, 1 - a] \times [0, 1 - a]$. Translating it by $pu$, with $p \in \mathbb{Z}$, one obtains $[0, 1] \times [0, 1 - a]$.

4. Assume that $b > 0$ and $a = 1/2$.

In this case $u = (0, -2b)$. The generating region, $gr$, of Figure 22 is the square with vertices at $(1/2, 0)$, $(1/2 + b, 0)$, $(1/2 + b, b)$ and $(1/2, b)$. Rotating it around the center $(1/2, b)$, one obtains another square with vertices at $(1/2 - b, 0)$, $(1/2 + b, 0)$, $(1/2 + b, 2b)$ and $(1/2 - b, 2b)$. Translating this last square by $pu$, with $p \in \mathbb{Z}$, one obtains $[1/2 - b, 1/2 + b] \times [0, 1]$. Rotating this region around the center $(1/2, b)$, one obtains the union of $[1/2 - b, 1/2 + b] \times [0, 1]$ with $[0, 1] \times [0, 2b]$. Translating it by $pu$, with $p \in \mathbb{Z}$, one obtains $[0, 1] \times [0, 1]$.

3.1.5. Case H. In this case the transformations are a translation by $u = (1 - a + b, 1 - a - b)$, two rotations of order 4 ($\sigma_0$ and $\sigma_7$) with centers located at $(a, b)$ and $(a + b, 1 - a + b)$, and a rotation of order 2 ($\sigma_2$) with center located at $(1 + a + b, 1 - a + b)$. These transformations are compatible in the sense that they generate a pattern $p4$ in the plane.

**Proposition 3.4.** The tile is such that it has the pattern $p4$ generated by the four transformations in the plane ($\mathbb{R}^2$).

**Proof.** Consider the dashed triangle $gr$ contained in the tile (Figure 24); $gr$ is a generating region of the pattern $p4$ in the plane. Its rotation by $\sigma_0$, $\sigma_0^2$
and $\sigma_0^3$ covers the union of $[a, 1] \times [0, b]$, the quadrilateral with vertices $(0, b)$, $(a, b)$, $(a + b, 1 - a + b)$ and $P_2$; $P_2 = (0, 2b + (1 - 2a) (1 - a - b)/(1 - a + b))$. Its rotation by $\sigma_2$ covers the triangle with vertices $(a + b, 1 - a + b)$, $(1, 0)$ and $P_1$; $P_1 = (1, 1 - a + b^2/(1 - a))$. The triangle with vertices $(a + b, 1 - a + b)$, $P_1$ and $(1, 1 - a + b)$, is the rotation by $\sigma_1$ of a triangle in $\mathsf{gr}$. The triangle with vertices $(0, 1 - a + b)$, $P_2$ and $(a + b, 1 - a + b)$, is the rotation by $\sigma_1^{-1}$ of a triangle in $\mathsf{gr}$.

Hence, the region union of $[a, 1] \times [0, b]$ and $[0, 1] \times [b, 1 - a + b]$ is covered. Its rotation by $\sigma_0^2$ covers $[0, a] \times [0, b]$. The rotation of $[0, 1] \times [0, 1 - a + b]$ by $\sigma_1$ and $\sigma_1^{-1}$ covers $[0, 1] \times [1 - a + b, 1]$. Therefore the proposition is proved.  

3.1.6. **Periodicity of the tile pattern.** Let us briefly point out that all the transformations involving cases A, B, C, D, E, F, G and H, transform the generating regions ($\mathsf{gr}$, see Figures 20–24), covering the tile.

Remember now that

| $i$ | A              | B              | C              | D              |
|-----|----------------|----------------|----------------|----------------|
| $u_i$ | $\left(\frac{1}{2}, \frac{1}{2}\right)$ | $(1 - b, a)$ | $\left(\frac{1}{2} - a - b, \frac{1}{2} + a - b\right)$ | $(a, b)$ |
| $v_i$ | $\left(-\frac{1}{2}, \frac{1}{2}\right)$ | $(-a, 1 - b)$ | $\left(-\frac{1}{2} - a + b, \frac{1}{2} - a - b\right)$ | $(-b, a)$ |

| $i$ | E              | F              | G              | H              |
|-----|----------------|----------------|----------------|----------------|
| $u_i$ | $\left(\frac{1}{2}, \frac{1}{2}\right)$ | $(a, b)$ | $\left(\frac{1}{2} - a + b, \frac{1}{2} - a - b\right)$ | $(b, 1 - a)$ |
| $v_i$ | $\left(-\frac{1}{2}, \frac{1}{2}\right)$ | $(-b, a)$ | $\left(a + b - \frac{1}{2}, \frac{1}{2} - a + b\right)$ | $(a - 1, b)$ |
1. For all the cases involving the three tiles $[0,1]^2$, $[-1,0] \times [0,1]$ and $[0,1] \times [-1,0]$ one has the following:

a) For the seven cases AE, AF, AG, AH, BE, CE and DE, one has

$$pu_A + qv_A = \frac{1}{2} (p - q, p + q) = p \left( \frac{1}{2}, \frac{1}{2} \right) + q \left( -\frac{1}{2}, \frac{1}{2} \right)$$

or

$$pu_E + qv_E = \frac{1}{2} (p - q, p + q) = p \left( \frac{1}{2}, \frac{1}{2} \right) + q \left( -\frac{1}{2}, \frac{1}{2} \right);$$

b) BF

$$pu_B + qv_B + qu_F - pv_F = (p, q) = p (1, 0) + q (0, 1);$$

c) BG

$$(p + q) u_B + (q - p) v_B + pu_G + qv_G = q \left( \frac{3}{2}, \frac{1}{2} \right) - p \left( -\frac{3}{2}, \frac{1}{2} \right);$$

d) BH

$$pu_B + qv_B + pu_H + qv_H = p (1, 1) + q (-1, 1);$$

e) CF

$$pu_C + qv_C + (p + q) u_F + (q - p) v_F = p \left( \frac{1}{2}, \frac{1}{2} \right) + q \left( -\frac{1}{2}, \frac{1}{2} \right);$$

f) CG

$$pu_C - qv_C + qu_G + pv_G = q (1, 0) + p (0, 1);$$

g) CH

$$pu_C - qv_C + (p + q) u_H + (p - q) v_H = q \left( \frac{3}{2}, \frac{1}{2} \right) + p \left( -\frac{1}{2}, \frac{3}{2} \right);$$

h) DF

$$pu_D + qv_D - qv_F - pu_F = (0, 0);$$

i) DG

$$(p + q) u_D + (p - q) v_D + pu_G - qv_G = p \left( \frac{1}{2}, \frac{1}{2} \right) - q \left( -\frac{1}{2}, \frac{1}{2} \right);$$

j) DH

$$pu_D + qv_D + qu_H - pv_H = p (1, 0) + q (0, 1).$$

Hence, for the nine cases AE, AF, AG, AH, BE, CE, DE, CF and DG the tile has a pattern with rotation centers of order 4 at:

$$(a, b) + p \left( \frac{1}{2}, \frac{1}{2} \right) + q \left( -\frac{1}{2}, \frac{1}{2} \right);$$

for the cases BF, CG, and DH:

$$(a, b) + p (1, 0) + q (0, 1);$$
for the case BH: 

\[(a, b) + p (1, 1) + q (-1, 1); \]

for the case BG: 

\[(a, b) + p \left( \frac{1}{2}, \frac{3}{2} \right) + q \left( -\frac{1}{2}, \frac{3}{2} \right); \]

for the case CH: 

\[(a, b) + p \left( \frac{3}{2}, \frac{1}{2} \right) + q \left( -\frac{1}{2}, \frac{3}{2} \right); \]

with \(p, q \in \mathbb{Z};\) for the case DF there is no conclusion.

2. Remember that 

\[
i \rightarrow (c_1 =) 1 \quad (c_1 =) 2 \quad (c_1 =) 3 \quad (c_1 =) 4
\]

\[
\begin{array}{llll}
  i \quad & u_i & (1, 0) & \left( \frac{1}{2} - b, \frac{1}{2} + a \right) \\
  & v_i & (0, 1) & \left( -\frac{1}{2} - a, \frac{1}{2} - b \right)
\end{array}
\]

\[
\begin{array}{llll}
  (c_1 =) & (a - b, a + b) & \left( \frac{1}{2} + b, \frac{1}{2} - a \right) \\
  (c_1 =) & (a - b, a - b) & \left( a - \frac{1}{2}, \frac{1}{2} + b \right)
\end{array}
\]

In the following the notations \(BG_i, CH_i, DF_i,\) mean that \(c_1 = i;\) also \(u = u_D = u_F\) and \(v = v_D = v_F\) (see Figures 16 and 17).

For all the cases involving the three tiles \([0, 1]^2, [-1, 0] \times [0, 1], [0, 1] \times [-1, 0]\) and \([-1, 0] \times [-1, 0]\) one has the following:

a) For the three cases BG1, CH1 and DF1, one has 

\[pu_1 + qv_1 = (p, q) = p (1, 0) + q (0, 1);\]

b) BG2 

\[pu_B + qv_B - pu_2 - qv_2 = q \left( \frac{1}{2}, \frac{1}{2} \right) + p \left( \frac{1}{2}, -\frac{1}{2} \right);\]

\[pu_G + qv_G + (p + q) u_2 + (q - p) v_2 = p \left( \frac{3}{2}, \frac{1}{2} \right) + q \left( -\frac{1}{2}, \frac{3}{2} \right);\]

c) BG3 

\[(q - p) u_B + (p + q) v_B + pu_3 - qv_3 = q (1, 1) + p (-1, 1);\]

\[pu_G + qv_G + pu_3 + qv_3 = p \left( \frac{1}{2}, \frac{1}{2} \right) + q \left( -\frac{1}{2}, \frac{1}{2} \right);\]

d) BG4 

\[pu_B + qv_B + pu_4 + qv_4 = p \left( \frac{3}{2}, \frac{1}{2} \right) + q \left( -\frac{1}{2}, \frac{3}{2} \right);\]

\[pu_G + qv_G - (p + q) u_4 + (p - q) v_4 = p \left( -\frac{1}{2}, \frac{1}{2} \right) - q \left( \frac{1}{2}, \frac{1}{2} \right);\]
e) CH2

\[ pu_H + qv_H + pu_2 + qv_2 = p \left( \frac{1}{2}, \frac{3}{2} \right) + q \left( -\frac{3}{2}, \frac{1}{2} \right); \]
\[ -pu_C + qv_C + (p + q) u_2 + (p - q) v_2 = q \left( \frac{1}{2}, \frac{1}{2} \right) + p \left( -\frac{1}{2}, \frac{1}{2} \right); \]

f) CH3

\[ (p + q) u_H + (-p + q) v_H + pu_3 + qv_3 = p (1, 1) + q (-1, 1); \]
\[ pu_C + qv_C + qu_3 - pv_3 = p \left( \frac{1}{2}, \frac{1}{2} \right) + q \left( -\frac{1}{2}, \frac{1}{2} \right); \]

g) CH4

\[ pu_H - qv_H - pu_4 + qv_4 = q \left( \frac{1}{2}, \frac{1}{2} \right) + p \left( -\frac{1}{2}, \frac{1}{2} \right); \]
\[ pu_C + qv_C + (p - q) u_4 + (p + q) v_4 = p \left( \frac{1}{2}, \frac{3}{2} \right) + q \left( -\frac{3}{2}, \frac{1}{2} \right); \]

h) DF2

\[ pu - qv + qu_2 + pv_2 = q \left( \frac{1}{2}, \frac{1}{2} \right) + p \left( -\frac{1}{2}, \frac{1}{2} \right); \]

i) DF3

\[ (q - p) u - (p + q) v + pu_3 + qv_3 = (0, 0); \]

j) DF4

\[ -pu + qv + qu_4 + pv_4 = q \left( \frac{1}{2}, \frac{1}{2} \right) + p \left( -\frac{1}{2}, \frac{1}{2} \right). \]

Hence, for the eleven cases BG1, CH1, DF1, BG2, BG3, BG4, CH2, CH3, CH4, DF2 and DF4 the tile has a pattern with rotation centers of order 4 at:

\[ (a, b) + p \left( \frac{1}{2}, \frac{1}{2} \right) + q \left( -\frac{1}{2}, \frac{1}{2} \right); \]

with \( p, q \in \mathbb{Z}; \) for the case DF3 there is no conclusion.

3. In the following the notation DF3i indicates that \( c_1 = 3 \) and \( d_1 = i; \) also, \( u = u_D = u_F, v = v_D = v_F, u_i = u_{DF3i} \) and \( v_i = v_{DF3i} \) (see Figure [17]).
|     | DF31              | DF32              | DF33              | DF34              |
|-----|-------------------|-------------------|-------------------|-------------------|
| 4   | $a + b - \frac{1}{2}$, 1/2 - a + b | $1 - b, a$        | $\frac{1}{2} - a + b$, 3/2 - a - b | $1 - a, 1 - b$    |
| 4   | $a - \frac{1}{2}, 1/2 + b$ | $1 - a, -b$       | $\frac{1}{2} + b, 3/2 - a$ | $1 + b, 1 - a$    |
| 4   | -                 | $1 - a - b$, a - b | -                 | $1 - a + b$, 1 - a - b |
| 2   | -                 | -                 | $1 - a, \frac{1}{2} - b$ | -                 |
|     | tr ($1 - a - b$, a - b) | (2b - 1, 1 - 2a) | -                 | -                 |
|     | tr ($a - b - 1$, a + b) | (a + b - 1, $1 - a + b$ ) | -                 | -                 |
|     | tr (2a - 1, 2b) | -                 | -                 | -                 |

| $i \rightarrow$ | $(d_1 = 1)$ | $(d_1 = 2)$ | $(d_1 = 3)$ | $(d_1 = 4)$ |
|------------------|-------------|-------------|-------------|-------------|
| $u_i$ | $\frac{1}{2} - a + b$, 1/2 - a - b | $1 - 2a - b$, a - 2b | $\frac{1}{2} - 2a + 2b$, 3/2 - 2a - 2b | $1 - 2a + b$, 1 - a - 2b |
| $v_i$ | $a + b - \frac{1}{2}$, 1/2 - a + b | $2b - a$, 2 - a | $2a + 2b - \frac{3}{2}$, 1/2 - 2a + 2b | $a + 2b - 1$, 1 - 2a + b |

Notice that

$$v_1 - u_1 = (2a - 1, 2b), \quad u_2 = (1 - a - b, a - b) - (a, b),$$

$$\left(1 - a, \frac{1}{2} - b\right) - (a, b) = \left(1 - 2a, \frac{1}{2} - 2b\right),$$

$$u_3 = \left(1 - 2a, \frac{1}{2} - 2b\right) + \left(2b - \frac{1}{2}, 1 - 2a\right),$$

$$u_4 = (1 - a + b, 1 - a - b) - (a, b).$$

a) DF31

$$(p - q) u + (p + q) v + pu_1 + qv_1 = p \left(\frac{1}{2}, \frac{1}{2}\right) + q \left(-\frac{1}{2}, \frac{1}{2}\right);$$

b) DF32

$$(2p + q) u + (2q - p) v + pu_2 + qv_2 = p (1, 0) + q (0, 1);$$
c) DF33

\[(2p - 2q)u + (2p + 2q)v + pu_3 + qv_3 = p \left( \frac{1}{2}, \frac{3}{2} \right) + q \left( -\frac{3}{2}, \frac{1}{2} \right); \]

d) DF34

\[(2p - 2q)u + (2p + 2q)v + pu_4 + qv_4 = p (1, 1) + q (-1, 1).\]

Hence, for the case DF31, the tile has a pattern with rotation centers of order 4 at:

\[(a, b) + p \left( \frac{1}{2}, \frac{1}{2} \right) + q \left( \frac{1}{2}, \frac{1}{2} \right);\]

for the case DF32, the tile has a pattern with rotation centers of order 4 at:

\[(a, b) + p (1, 0) + q (0, 1);\]

for the case DF34, the tile has a pattern with rotation centers of order 4 at:

\[(a, b) + p (1, 1) + q (-1, 1);\]

with \(p, q \in \mathbb{Z};\) for the case DF33 there is no conclusion.

4. In the following the notation DF33i, means that \(c_1 = 3, d_1 = 3\) and \(e_1 = i;\) also, \(u = u_D = u_F, v = v_D = v_F, u_i = u_{DF33i} \) and \(v_i = v_{DF33i}\) (see Figure 17).
\[ i \rightarrow (e_1 =) 1 \quad (e_1 =) 2 \quad (e_1 =) 3 \quad (e_1 =) 4 \]

| \( u_i \) | \( (b, 1 - a) \) | \( (1/2 - a + b, 1/2 - a - b) \) | \( (1 - 2a + b, 1 - a - 2b) \) | \( (3/2 - 2a, 1/2 - 2b) \) |
|\( v_i \) | \( (a - 1, b) \) | \( (a + b - 1/2, 1/2 - a + b) \) | \( (a + 2b - 1, 2b - 1/2, 3/2 - 2a) \) | \( (1/2 - a + b, 1 - 2a + b) \) |

Notice that
\[ u_1 = (a + b, 1 - a + b) - (a, b), \quad u_2 + v_2 = (2b, 1 - 2a), \]
\[ u_3 = (1 - a + b, 1 - a - b) - (a, b), \]
\[ \left( \frac{1}{2} + b, 1 - a \right) - (a, b) = \left( \frac{1}{2} - a + b, 1 - a - b \right), \]
\[ u_4 = \left( \frac{1}{2} - a + b, 1 - a - b \right) + \left( 1 - a - b, a - b - \frac{1}{2} \right). \]
a) DF331
\[ pu + qv + qu_1 - pv_1 = p (1, 0) + q (0, 1); \]
b) DF332
\[ (p - q) u + (p + q) v + pu_2 + qv_2 = p \left( \frac{1}{2}, \frac{1}{2} \right) + q \left( -\frac{1}{2}, \frac{1}{2} \right); \]
c) DF333
\[ (2p - q) u + (p + 2q) v + pu_3 + qv_3 = p (1, 1) + (-1, 1); \]
d) DF334
\[ 2pu + 2qv + pu_4 + qv_4 = p \left( \frac{3}{2}, \frac{1}{2} \right) + q \left( -\frac{1}{2}, \frac{3}{2} \right); \]
\[ -pu_{DF33} - qv_{DF33} + (p - q) u_4 + (p + q) v_4 = p \left( \frac{1}{2}, \frac{1}{2} \right) + q \left( -\frac{1}{2}, \frac{1}{2} \right). \]

Hence, for the cases DF332 and DF334, the tile has a pattern with rotation centers of order 4 at:
\[ (a, b) + p \left( \frac{1}{2}, \frac{1}{2} \right) + q \left( -\frac{1}{2}, \frac{1}{2} \right); \]
for the case DF331, the tile has a pattern with rotation centers of order 4 at:
\[ (a, b) + p (1, 0) + q (0, 1); \]
for the case DF333, the tile has a pattern with rotation centers of order 4 at:
\[ (a, b) + p (1, 1) + q (-1, 1); \]
with \( p, q \in \mathbb{Z}. \)

Hence, we have proved
Proposition 3.5. Let \((a, b)\) the rotation center of order 4 with \(0 \leq b \leq a \leq 1/2\), \(a > 0\), \((a, b) \neq (1/2, 0)\). Consider the tiles \([0, 1]^2\) and \([0, 1] \times [-1, 0]\), and their four possibilities: E, F, G and H. The tile has a pattern \(p_4\) with the following possible translation invariances:

1) F, G and H: \(p (1, 0) + q (0, 1), \ p, q \in \mathbb{Z}\).
2) F and H: \(p (1, 1) + q (-1, 1), \ p, q \in \mathbb{Z}\).
3) E, F, G and H: \(p \left( \frac{1}{2}, \frac{1}{2} \right) + q \left( -\frac{1}{2}, \frac{1}{2} \right), \ p, q \in \mathbb{Z}\).

3.1.7. Mathematical classification of tiles in the general case. Consider a tile in \(\mathbb{R}^2\) like it is shown in Figure 7. Let \(n\) the number of rotation centers of order 4 in the tile. Assume that there are rotation centers of order 4 located at \((a, b)\) and \((a + \alpha, b + \beta)\), with \(\alpha > 0\) and \(\beta \geq 0\), and that, in these conditions, \((a + \alpha, b + \beta)\) is the nearest from \((a, b)\). Then, the rotation centers of order 4 are located at

\[
(a, b) + r (\alpha, \beta) + s (-\beta, \alpha),
\]

with \(r, s \in \mathbb{Z}\). The number of rotation centers in the tile is

\[
n = \frac{p^2 + q^2}{\alpha^2 + \beta^2}.
\]

We consider three possibilities.

1. There is a rotation center at \((a + 1, b)\) (as already seen this can be the situation in the cases F, G and H; we call them F1, G1 and H1). Then, there are \(p, q \in \mathbb{Z}\) such that

\[
p (\alpha, \beta) + q (-\beta, \alpha) = (1, 0);
\]

hence

\[
\alpha = \frac{p}{p^2 + q^2}, \ \beta = \frac{-q}{p^2 + q^2},
\]

\[
n = p^2 + q^2 \text{ and the centers are located at the points.}
\]

\[
(a, b) + \frac{1}{p^2 + q^2} (rp + sq, sp - rq).
\]

2. There is a rotation center at \((a + 1, b + 1)\) (as already seen this can be the situation in the cases F and H; we call them F2 and H2). Then, there are \(p, q \in \mathbb{Z}\) such that

\[
p (\alpha, \beta) + q (-\beta, \alpha) = (1, 1);
\]

hence

\[
\alpha = \frac{p + q}{p^2 + q^2}, \ \beta = \frac{p - q}{p^2 + q^2},
\]

\[
n = \frac{p^2 + q^2}{2} \text{ and the centers are located at the points}
\]

\[
(a, b) + \frac{1}{p^2 + q^2} (r (p + q) + s (q - p), s (p + q) + r (p - q)).
\]
3. There is a rotation center at \((a + \frac{1}{2}, b + \frac{1}{2})\) (as already seen this can be the situation in the cases E, F, G and H; we call them E3, F3, G3 and H3). Then, there are \(p, q \in \mathbb{Z}\) such that

\[
p (\alpha, \beta) + q (-\beta, \alpha) = \left( \frac{1}{2}, \frac{1}{2} \right);
\]

hence

\[
\alpha = \frac{p + q}{2(p^2 + q^2)}, \quad \beta = \frac{p - q}{2(p^2 + q^2)},
\]

\(n = 2(p^2 + q^2)\) and the centers are located at the points

\[
(a, b) + \frac{1}{2(p^2 + q^2)} (r (p + q) + s (q - p), s (p + q) + r (p - q)).
\]

**These are always in the first type of general tiles.** The cases E3, F3, G3 and H3 are in the first type of general tiles.

Hence, let us now consider the other five cases: F1, F2, G1, H1, H2.

Remember that

\[
\begin{array}{cccc}
  i & E & F & G \\
  u_i & (a, b) & \left( \frac{1}{2} - a + b, \frac{1}{2} - a - b \right) & (b, 1 - a) \\
  v_i & (-b, a) & \left( a + b - \frac{1}{2}, \frac{1}{2} - a + b \right) & (a - 1, b)
\end{array}
\]

and let \((x, y)\) the coordinates of the rotation centers of order 4. In the following we use the notations \(p, q, p_i, q_i, r, s, r_i, s_i \in \mathbb{Z}, i = 0, 1, \ldots\)

**F1.** In this case

\[
(x, y) = (a, b) + \frac{1}{p^2 + q^2} (r p + s q, s p - r q),
\]

\[
(a, b) = \left( \frac{r_0 p + s_0 q}{p^2 + q^2}, \frac{s_0 p - r_0 q}{p^2 + q^2} \right)
\]

\[
(x, y) = \frac{1}{p^2 + q^2} ((r + r_0) p + (s + s_0) q, (s + s_0) p - (r + r_0) q),
\]

\[
(x, y) = \frac{1}{p^2 + q^2} (r_1 p + s_1 q, s_1 p - r_1 q).
\]

These are included in the first and second types of general tiles.
F2. In this case

\[(x, y) = (a, b) + \frac{1}{p^2 + q^2} (r(p + q) + s(q - p), s(p + q) + r(p - q)),\]

\[(x, y) = \frac{1}{p^2 + q^2} ((r + r_0)(p + q) + (s + s_0)(q - p)),\]

\[(s + s_0)(p + q) + (r + r_0)(p - q)),\]

\[(x, y) = \frac{1}{p^2 + q^2} (r_1(p + q) + s_1(q - p), s_1(p + q) + r_1(p - q)).\]

Let \(p_1 = p + q\) and \(q_1 = q - p\). Then \(p = \frac{p_1 - q_1}{2}\) and \(q = \frac{p_1 + q_1}{2}\) and

\[(x, y) = \frac{2}{p_1^2 + q_1^2} (r_1p_1 + s_1q_1, s_1p_1 - r_1q_1).\]

a) If \(p + q\) is odd, then \(p_1\) and \(q_1\) are odd numbers and these are the fourth type of general tiles.

b) If \(p + q\) is even, then \(p_1\) and \(q_1\) are even numbers. Let \(p_1 = 2p_2\) and \(q_1 = 2q_2\); then

\[(x, y) = \frac{1}{p_2^2 + q_2^2} (r_1p_2 + s_1q_2, s_1p_2 - r_1q_2).\]

These are included in the first and second types of general tiles.

G1. In this case

\[(x, y) = (a, b) + \frac{1}{p^2 + q^2} (r(p + q), s(p - r)),\]

\[(a, b) = \left(\frac{1}{2} + \frac{1}{2} - \frac{(r_0 + s_0)p + q(r_0 - s_0)}{p^2 + q^2}, \frac{1}{2} - \frac{(r_0 - s_0)p + q(r_0 + s_0)}{p^2 + q^2}\right).\]

a) If \(p + q\) is even, these are included in the first type of general tiles.

b) If \(p + q\) is odd,
   i) if \(r_0 + s_0\) is even, let
   \[r_1 = r - \frac{r_0 + s_0}{2},\quad s_1 = s + \frac{r_0 - s_0}{2},\]
   \[(x, y) = \left(\frac{1}{2}, 0\right) + \frac{1}{p^2 + q^2} (r_1p + s_1q, s_1p - r_1q).\]
   
   ii) if \(r_0 + s_0\) is odd, then \(r_0 + s_0 - p - q\) and \(r_0 - s_0 - p + q\) are even. Let
   \[r_1 = \frac{r_0 + s_0 - p - q}{2},\quad s_1 = \frac{r_0 - s_0 - p + q}{2};\]
   then \(r_0 + s_0 = 2r_1 + p + q\) and \(r_0 - s_0 = 2s_1 + p - q\) and
   \[(a, b) = \left(-\frac{p(r_1 + q_1)}{p^2 + q^2}, \frac{1}{2} + \frac{ps_1 + qr_1}{p^2 + q^2}\right).\]
Let $r_2 = r - r_1$ and $s_2 = s + s_1$;

$$(x, y) = \left(0, \frac{1}{2}\right) + \frac{1}{p^2 + q^2} \left(r_2p + s_2q, s_2p - r_2q\right).$$

This case b) represents the third type of general tiles.

**H1.** In this case

$$(x, y) = (a, b) + \frac{1}{p^2 + q^2} \left(rp + sq, sp - rq\right),$$

$$(a, b) = \left(1 - \frac{s_0p - r_0q}{p^2 + q^2}, \frac{r_0p + s_0q}{p^2 + q^2}\right).$$

a) If $p + q$ is odd, these are included in the first type of general tiles.

b) If $p + q$ is even, let

$$r_1 = r + p - s_0, \quad s_1 = s + q + r_0;$$

$$(x, y) = \frac{1}{p^2 + q^2} \left(r_1p + s_1q, s_1p - r_1q\right).$$

These represent the second type of general tiles.

**H2.** In this case

$$(x, y) = (a, b) + \frac{1}{p^2 + q^2} \left(r(p + q) + s(q - p), s(p + q) + r(p - q)\right),$$

$$(a, b) = \left(1 - \frac{s_0(p + q) + r_0(p - q)}{p^2 + q^2}, \frac{r_0(p + q) + s_0(q - p)}{p^2 + q^2}\right).$$

Let

$$r_1 = r - s_0, \quad s_1 = s + r_0.$$

$$(x, y) = (1, 0) + \frac{1}{p^2 + q^2} \left(r_1(p + q) + s_1(q - p), s_1(p + q) + r_1(p - q)\right).$$

Let $p_1 = p + q$ and $q_1 = q - p$. Then $p = \frac{p_1 - q_1}{2}$ and $q = \frac{p_1 + q_1}{2}$

$$(x, y) = (1, 0) + \frac{2}{p_1^2 + q_1^2} \left(r_1p_1 + s_1q_1, s_1p_1 - r_1q_1\right).$$

Notice that $p_1$ and $q_1$ are both even or both odd.

a) If $p_1$ and $q_1$ are both odd, these are the fourth type of general tiles.

b) If $p_1$ and $q_1$ are both even, let $p_1 = 2p_2$ and $q_1 = 2q_2$. Then

$$(x, y) = (1, 0) + \frac{1}{p_2^2 + q_2^2} \left(r_1p_2 + s_1q_2, s_1p_2 - r_1q_2\right).$$

These are included in the first and second types of general tiles.
3.2. **The exceptions.** Assume that the rotation center of order 4 is in the middle of the edge of one tile. Divide the tile in four identical squares \(a_1-a_2-a_3-a_4, a_2-a_3-a_4-a_1, b_1-b_2-b_3-b_4\) and \(c_1-c_2-c_3-c_4\), like it is shown in Figure 25.

![Figure 25](image)

**Figure 25.**

![Figure 26](image)

**Figure 26.**

Figure 26 shows the rotation center of order 4 in the middle of the common edge of two tiles and two neighbor tiles. There are sixteen ways of choosing these last tiles:

a) In six of them

\[
\begin{align*}
a_1 &\equiv a_3 \equiv b_2 \equiv b_4 \equiv c_1 \equiv c_3, \\
a_2 &\equiv a_4 \equiv b_1 \equiv b_3 \equiv c_2 \equiv c_4;
\end{align*}
\]

the tile is like it is shown in Figure 27a;
b) in two of them

\[ a_1 \equiv a_3 \equiv c_1 \equiv c_3, \]
\[ a_2 \equiv a_4 \equiv c_2 \equiv c_4, \]
\[ b_1 \equiv b_3, \quad b_2 \equiv b_4; \]

the tile is like it is shown in Figure 27b;

c) in one of them

\[ a_1 \equiv b_4 \equiv c_1, \quad a_2 \equiv b_1 \equiv c_2, \]
\[ a_3 \equiv b_2 \equiv c_3, \quad a_4 \equiv b_3 \equiv c_4; \]

the tile is like it is shown in Figure 27c;

d) in three of them

\[ a_1 \equiv b_2 \equiv c_3, \quad a_2 \equiv b_3 \equiv c_4, \]
\[ a_3 \equiv b_4 \equiv c_1, \quad a_4 \equiv b_1 \equiv c_2; \]

the tile is like it is shown in Figure 27d;

e) in one of them

\[ b_1 \equiv c_2, \quad b_2 \equiv c_3, \quad b_3 \equiv c_4, \quad b_4 \equiv c_1; \]

the tile is like it is shown in Figure 27e;

f) in two of them

\[ a_1 \equiv a_3 \equiv b_2 \equiv b_4, \]
\[ a_2 \equiv a_4 \equiv b_1 \equiv b_3, \]
\[ c_1 \equiv c_3, \quad c_2 \equiv c_4; \]

the tile is like it is shown in Figure 27f;
g) in one of them

\[ b_1 \equiv b_3 \equiv c_2 \equiv c_4, \quad b_2 \equiv b_4 \equiv c_1 \equiv c_3, \]

\[ y_1 \equiv a_3, \quad y_2 \equiv y_3 \equiv a_4, \quad y_4 \equiv a_1; \]

there are four possibilities for the y’s, but only one of them \( (a_1 \equiv a_3, a_2 \equiv a_4) \) gives a different tile; it is like is shown in Figure 27g.

The tiles in Figure 27a and 27d can be obtained by the general rule, making \( p = 2 \) and \( q = 0 \) in the first one, and \( p = 1 \) and \( q = 1 \) in the second one. The tile in Figure 27e can be obtained by the general rule, making \( p = 1 \) and \( q = 0 \); it is of the third type.

The tiles in Figure 27b and 27f are identical.

Hence, if we do not consider reflections, there are three exceptions to the general rule, that are shown in Figures 13–15.

Considering reflections, it is not difficult to see that there are the eight possibilities represented in Figures 13–15.
Appendix: Three patterns with Eduardo Nery tile

Figure 28.
Figure 29.
Figure 30.
REFERENCES

[1] A. Almeida, Azulejaria Modernista, Moderna e Contemporânea. Lisboa, Museu Nacional do Azulejo, 1968.
http://mnazulejo.imc-ip.pt/Data/Documents/Cursos/azulejar2009/AA_01.pdf

[2] M. A. Armstrong, Groups and Symmetry. Berlin: Springer-Verlag, 1988.

[3] http://www.eduardonery.pt/
(see work in architecture / urban space / public art / pattern tiles).

[4] P. Henriques, Rocha de Sousa, S. Vieira: Eduardo Nery. Exposição Retrospectiva: Tapeçaria, Azulejo, Mosaico, Vitral (1961–2003). Museu Nacional do Azulejo, Lisboa, IPM, 2003.
http://mnazulejo.imc-ip.pt/pt-PT/museu/publicacoes/ContentList.aspx

[5] J. Rezende, Plane periodical puzzles with numbers. http://arxiv.org/abs/1106.0953

[6] D. Schattschneider, The Plane Symmetry Groups: Their Recognition and Notation, American Mathematical Monthly, Volume 85, Issue 6, 439–450.
http://www.math.fsu.edu/~quine/MB_10/schattschneider.pdf

Grupo de Física-Matemática da Universidade de Lisboa, Av. Prof. Gama Pinto
2, 1649-003 Lisboa, Portugal, and Departamento de Matemática, Faculdade de Ciências da Universidade de Lisboa
E-mail address: rezende@ciic.fc.ul.pt