POLYNOMIAL EXTENSIONS OF SEMISTAR OPERATIONS

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Abstract. We provide a complete solution to the problem of extending arbitrary semistar operations of an integral domain $D$ to semistar operations of the polynomial ring $D[X]$. As an application, we show that one can reobtain the main results of the papers [1] and [2] concerning the problem in the special cases of stable semistar operations of finite type or semistar operations defined by families of overrings. Finally, we investigate the behavior of the polynomial extensions of the most important and classical operations such as $d_D$, $v_D$, $t_D$, $w_D$ and $b_D$ operations.

1. Preliminaries

Let $D$ be an integral domain with quotient field $K$. Let $\mathcal{F}(D)$ denote the set of all nonzero $D$-submodules of $K$ and let $\mathcal{F}(D)$ be the set of all nonzero fractional ideals of $D$, i.e., $E \in \mathcal{F}(D)$ if $E \in \mathcal{F}(D)$ and there exists a nonzero $d \in D$ with $dE \subseteq D$. Let $f(D)$ be the set of all nonzero finitely generated $D$-submodules of $K$. Then, obviously, $f(D) \subseteq \mathcal{F}(D) \subseteq \mathcal{F}(D)$.

Following Okabe-Matsuda [14], a semistar operation of $D$ is a map $\star : \mathcal{F}(D) \to \mathcal{F}(D)$, $E \mapsto E^\star$, such that, for all $x \in K \setminus \{0\}$ and for all $E, F \in \mathcal{F}(D)$, the following properties hold:

$(\star_1)$ $(xE)^\star = xE^\star$;
$(\star_2)$ $E \subseteq F$ implies $E^\star \subseteq F^\star$;
$(\star_3)$ $E \subseteq E^\star$ and $E^{**} := (E^\star)^\star = E^\star$.

A (semi)star operation is a semistar operation that, restricted to $\mathcal{F}(D)$, is a star operation (in the sense of [8, Section 32]). It is easy to see that a semistar operation $\star$ of $D$ is a (semi)star operation if and only if $D^\star = D$.

If $\star$ is a semistar operation of $D$, then we can consider a map $\star_f : \mathcal{F}(D) \to \mathcal{F}(D)$ defined as follows:

$E^\star_f := \bigcup \{F^\star \mid F \in f(D) \text{ and } F \subseteq E\}$ for each $E \in \mathcal{F}(D)$.

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It is easy to see that $\#_j$ is a semistar operation of $D$, which is called the semistar operation of finite type associated to $\#$. Note that, for each $F \in f(D)$, $F^* = F^*\#_j$.

A semistar operation $\#$ is called a semistar operation of finite type if $\# = \#_j$.

If $\#_1$ and $\#_2$ are two semistar operations of $D$ such that $E^{*\#_1} \subseteq E^{*\#_2}$ for all $E \in \mathcal{F}(D)$, then we say that $\#_1 \leq \#_2$. This is equivalent to say that $(E^{*\#_1})^{*\#_2} = E^{*\#_2} = (E^{*\#_2})^{*\#_1}$ for each $E \in \mathcal{F}(D)$. Obviously, for each semistar operation $\#$ of $D$, we have $\#_j \leq \#$. Let $d_D$ (or, simply, $d$) be the identity semistar operation of $D$; clearly $d \leq \#$ for all semistar operations $\#$ of $D$. Let $e_D$ (or, simply, $e$) be the trivial semistar operation of $D$, defined by $E^e = K$ for each $E \in \mathcal{F}(D)$; clearly $\# \leq e$ for all semistar operations $\#$ of $D$.

Let $\#$ be a semistar operation of $D$. We say that a nonzero integral ideal $I$ of $D$ is a quasi-$\#$-ideal if $I^* \cap D = I$, a quasi-$\#$-prime ideal if it is a prime quasi-$\#$-ideal, and a quasi-$\#$-maximal ideal if it is maximal in the set of all proper quasi-$\#$-ideals.

A quasi-$\#$-maximal ideal is a prime ideal. It is easy to prove that each proper quasi-$\#$-ideal is contained in a quasi-$\#$-maximal ideal. More details can be found in [6, page 4781]. We will denote by $\text{QMax}^\#$ $(D)$ (respectively, $\text{QSpec}^\#$ $(D)$) the set of all quasi-$\#$-maximal ideals (respectively, quasi-$\#$-prime ideals) of $D$. When $\#$ is a (semi)star operation, the notion of quasi-$\#$-ideal coincides with the “classical” notion of integral $\#$-ideal (i.e., a nonzero integral ideal $I$ such that $I^* = I$).

If $\Delta$ is a set of prime ideals of $D$, then the semistar operation $\#_\Delta$ defined by

$$E^{*\#_\Delta} := \bigcap \{ E^D_P \mid P \in \Delta \} \text{ for each } E \in \mathcal{F}(D)$$

is called the spectral semistar operation of $D$ associated to $\Delta$. A semistar operation $\#$ of $D$ is called a spectral semistar operation if there exists a subset $\Delta$ of the prime spectrum of $D$, $\text{Spec}(D)$, such that $\# = \#_\Delta$.

A semistar operation $\#$ is stable if $(E \cap F)^* = E^* \cap F^*$ for each $E, F \in \mathcal{F}(D)$. Clearly, spectral semistar operations are stable [4, Lemma 4.1(3)].

When $\#$ is a semistar operation of $D$ and $\Delta := \text{QMax}^\#(D)$, we set $\# := \#_\Delta$, i.e.,

$$E^{\#} := \bigcap \{ E^D_P \mid P \in \text{QMax}^\#(D) \} \text{ for each } E \in \mathcal{F}(D),$$

or equivalently,

$$E^{\#} = \bigcup \{(E : J) \mid J \in f(D), J \subseteq D, \text{ and } J^* = D^* \} \text{ for each } E \in \mathcal{F}(D).$$

Then $\#$ is a stable semistar operation of finite type, which is called the stable semistar operation of finite type associated to $\#$. It is known that if a semistar operation $\#$ is stable and of finite type, then $\# = \#$ [4, Corollary 3.9(2)].

By $v_D$ (or, simply, by $v$) we denote the $v$-semistar operation of $D$ defined as usual, that is, $E^v := (D : (D : E))$ for each $E \in \mathcal{F}(D)$ and $E^v := K$ for each $E \in \mathcal{F}(D) \setminus \mathcal{F}(D)$. By $t_D$ (or, simply, by $t$) we denote $(v_D)$, the semistar operation of finite type associated to $v_D$ and by $w_D$ (or just by $w$) the stable semistar operation of finite type associated to $v_D$ (or, equivalently, to $t_D$), considered by F.G. Wang and R.L. McCasland in [14] (cf. also [5]); i.e., $w_D := \#_D = \#_D$. Clearly $w_D \leq t_D \leq v_D$.

Moreover, it is easy to see that for each (semi)star operation $\#$ of $D$, we have $\# \leq v_D$, $\#_j \leq t_D$, and $\# \leq w_D$ (cf. also [8, Theorem 34.1(4)]).
Let $\ast$ be a semistar operation of $D$ and let $F \in f(D)$. We say that $F$ is $\ast$-\textit{eab} (respectively, $\ast$-\textit{ab}) if, for all $G, H \in f(D)$ (respectively, for all $G, H \in \overline{F}(D)$), $(FG)^\ast \subseteq (FH)^\ast$ implies $G^\ast \subseteq H^\ast$. The operation $\ast$ is said to be \textit{eab} (respectively, \textit{ab}) if each $F \in f(D)$ is $\ast$-\textit{eab} (respectively, $\ast$-\textit{ab}). An \textit{ab} operation is obviously an \textit{eab} operation.

Using the fact that, given $F \in f(D)$, $F$ is $\ast$-\textit{eab} if and only if $((FH)^\ast : F^\ast) = H^\ast$ for each $H \in f(D)$ \cite{7, Lemma 8}, we can associate to each semistar operation $\ast$ of $D$ an \textit{eab} semistar operation $\ast_a$ of finite type, which is called the \textit{eab semistar operation associated to} $\ast$ and defined as follows: for each $F \in f(D)$ and for each $E \in \overline{F}(D)$,

$$F^{\ast_a} := \bigcup \{((FH)^\ast : H^\ast) \mid H \in f(D)\},$$

$$E^{\ast_a} := \bigcup \{F^{\ast_a} \mid F \subseteq E, F \in f(D)\}$$

\cite[Definition 4.4 and Proposition 4.5]{5}. The previous construction, in the ideal systems setting, is essentially due to P. Jaffard \cite{13} and F. Halter-Koch \cite{10}, \cite{11}. Obviously, $(\ast_a)_a = \ast_a$. Moreover, when $\ast = \ast_f$, then $\ast$ is \textit{ab} if and only if $\ast = \ast_a$ \cite[Proposition 4.5(5)]{5}. We call the semistar operation $b_D := (d_D)_a$ the $b$-operation of $D$. It is easy to see that $b_D$ is a (semi)star operation of $D$ if and only if $D$ is integrally closed.

Given a family of semistar operations $\{\ast_\lambda \mid \lambda \in \Lambda\}$ of $D$, the semistar operation $\land \ast_\lambda$ of $D$ is defined by

$$E^{\land \ast_\lambda} := \bigcap \{E^{\ast_\lambda} \mid \lambda \in \Lambda\} \text{ for each } E \in \overline{F}(D).$$

Let $T := \{T_\lambda \mid \lambda \in \Lambda\}$ be a set of overrings of $D$. We denote by $\ast_{\{T_\lambda\}}$ the semistar operation of $D$ defined by $E^{\ast_{\{T_\lambda\}}} := ET_\lambda$ for each $E \in \overline{F}(D)$ and by $\land_T$ the semistar operation $\land \{\ast_{\{T_\lambda\}} \mid \lambda \in \Lambda\}$.

For a semistar operation $\ast$ of $D$, we say that a valuation overring $V$ of $D$ is a $\ast$-valuation overring of $D$ provided $F^\ast \subseteq FV$ (or, equivalently, $F^*V = FV$) for each $F \in f(D)$. Let $\mathcal{V}(\ast)$ be the family of all $\ast$-valuation overrings of $D$. Then the semistar operation $\land_{\mathcal{V}(\ast)}$ of $D$ is an ab semistar operation \cite[page 2098]{7}; clearly,

$$\land_{\mathcal{V}(\ast)} = \land_{\mathcal{V}(\ast_f)}.$$ Note that

$$\ast_a = \land_{\mathcal{V}(\ast)}, \text{ in particular, } b_D = \land_{\mathcal{V}(d_D)}.$$ This result follows from \cite[Proposition 4.1(5)]{7}.

We now consider the polynomial ring $D[X]$ over $D$. A semistar operation $\ast$ of $D[X]$ is called an \textit{extension} (respectively, a \textit{strict extension}) of a semistar operation $\ast$ of $D$ if $E^\ast = E[X]^\ast \cap K$ (respectively, $E^\ast[X] = E[X]^\ast$) for all $E \in \overline{F}(D)$.

Given a semistar operation $\ast$ of $D[X]$, set $E^\ast_0 := (E[X])^\ast \cap K$ for each $E \in \overline{F}(D)$. Then $\ast_0$ is a semistar operation of $D$ and $\ast$ is an extension of $\ast_0$. By \cite[Lemma 5]{2}, $(\ast_0)_0 = (\ast_0)_f$ and $(\ast_0)_0 = (\ast_0)$. It is easy to see that $(d_{D[X]}^\ast)_0 = d_D$ and $(w_{D[X]}^\ast)_0 = w_D$, and therefore, $(t_{D[X]}^\ast)_0 = t_D$ and $(w_{D[X]}^\ast)_0 = w_D$. In fact, it is known that:

$$(E[X])^{w_{D[X]}} = E^{w_D}[X] \text{ for all } E \in F(D),$$

$$(E[X])^{d_{D[X]}} = E^{d_D}[X] \text{ and } (E[X])^{w_{D[X]}} = E^{w_D}[X] \text{ for all } E \in \overline{F}(D).$$

Thus $t_{D[X]}$ and $w_{D[X]}$ are strict extensions of $t_D$ and $w_D$, respectively.
The present work is devoted to the following problem: how to extend “in a canonical way” an arbitrary semistar operation of $D$ to the polynomial ring $D[X]$. The first attempts to extend a semistar operation of $D$ to a semistar operation of $D[X]$ were done by G. Picozza [13] and then by the first two authors of this paper [1, 2]. Their study is focused on the stable semistar operations of finite type. In this paper, we provide a complete solution to this problem in the most general setting. As an application, we show that, in the particular cases of stable semistar operations of finite type or semistar operations defined by families of overrings, we reobtain the main results given in [1] and [2]. Finally, we investigate the behavior of the polynomial extensions of some operations among the most important and classical ones such as the $d_D, v_D, t_D, w_D$ and $b_D$ operations.

To be more precise, in Section 2, we show that there always exists the maximum in the set of all strict extensions to the polynomial ring $D[X]$ of a given semistar operation $\star$ on $D$. Let $\mathbf{\star}^*$ denote this semistar operation of $D[X]$. After giving an explicit description of $\mathbf{\star}^*$, we show that such a semistar operation is never of finite type and we investigate the properties of $\mathbf{\star}^*$; in particular, $(\mathbf{\star}^*)_f = (\mathbf{\star}^*)_d$ is the largest finite-type strict extension of $\star_f$ and $\mathbf{\star}^* = \mathbf{\star}^*$ is the largest stable finite-type strict extension of $\mathbf{\star}^*$. As an application, we consider some of the classical operations and we prove that $(\mathbf{\star}^{wD})_f = (\mathbf{\star}^{dD})_f = t_{D[X]}$ and $\mathbf{\star}^{wD} = \mathbf{\star}^{dD} = w_{D[X]}$. Moreover, for the trivial operations, we have $\mathbf{\star}^{wD} \leq e_{D[X]}$ and $d_{D[X]} \leq (\mathbf{\star}^{dD})_f$, with $d_{D[X]} = (\mathbf{\star}^{dD})_f$ if and only if $D$ is a field.

After having observed that each semistar operation $\star$ of $D$ admits infinitely many strict extensions to $D[X]$ and, among them, the largest one is $\mathbf{\star}^*$; in Section 3 we show the existence of the smallest strict extension to $D[X]$. Unlike the largest strict extension, the smallest strict extension, denoted by $\mathbf{\star}^*$, is not in general described in an explicit form. However, in case of stable semistar operations of finite type, we prove that $\mathbf{\star}^* = \widetilde{\mathbf{\star}}$, where $A^{[\mathbf{\star}]} := \bigcap\{AD_Q[X] \mid Q \in QMax^*(D)\}$ for each $A \in \mathbf{F}(D[X])$.

In the last section, we generalize some results concerning the polynomial extensions of a stable finite-type semistar operation to the polynomial extensions of a semistar operation defined by a given family of overrings of $D$. As an application of the main result of the section, we obtain $\mathbf{\star}_a^{bD} = (\mathbf{\star}_a^{bD})_a \leq (\mathbf{\star}_a^{bD})_a \leq \mathbf{\star}_a^{bD}$, with $b_{D[X]} = (\mathbf{\star}_a^{bD})_a$ if and only if $D$ is a field.

2. Polynomial Strict Extensions of General Semistar Operations

The goal of the present section is to define in a canonical way an extension to the polynomial ring $D[X]$ of a given semistar operation $\star$ of $D$.

For $A \in \mathbf{F}(D[X])$ with $A \subseteq K[X]$, we denote by $c_D(A)$ the $D$-submodule of $K$ generated by the contents $c_D(f)$ for all $f \in A$, i.e., $c_D(A) := \sum_{f \in A} c_D(f)$. Then, obviously, $A \subseteq c_D(A)[X]$.

**Theorem 2.1.** Let $D$ be an integral domain with quotient field $K$, let $\star$ be a semistar operation of $D$, and let $T$ be an overring of $D$ such that $T^* = T$. For each $A \in \mathbf{F}(D[X])$, set
Therefore, the opposite inclusion, let \( (\star) \) assume that \( A \subseteq (T[X] : A) \), we have \( (E[X])^\star \subseteq A^\star \). For the opposite inclusion, let \( z \in (T[X] : E[X]) \). Then, obviously, \( z \in K[X] \). Write \( z = z_0 + z_1 X + \cdots + z_n X^n \), with \( z_i \in K \). Then \( c_D(zE[X]) = z_0 E + z_1 E + \cdots + z_n E \). Therefore, \( z^{-1}(c_D(zE[X]))^\star \supseteq z^{-1}(z_0 E + z_1 E + \cdots + z_n E)^\star \supseteq z^{-1}(z_0 E + z_1 E + \cdots + z_n E)^\star \). Thus, we have \( (E[X])^\star \supseteq A^\star \). Since it is the largest strict extension of \( \star \) to \( D[X] \), in fact, it is the largest strict extension of \( \star \) to \( D[X] \).

Proof. From the definition, it follows immediately that \( A^\star \in \bigl(D[X]\bigr)^\star \).

**Claim 1.** Let \( E \subseteq T \), then \( (E[X])^\star = E^\star \). Let \( (E[X])^\star = E^\star \).

Since \( c_D(E[X]) = E + 1 \in (T[X] : E[X]) \), we have \( (E[X])^\star \subseteq A^\star \). For the opposite inclusion, let \( z \in (T[X] : E[X]) \). Then, obviously, \( z \in X[K] \). Write \( z = z_0 + z_1 X + \cdots + z_n X^n \), with \( z_i \in K \). Then \( c_D(zE[X]) = z_0 E + z_1 E + \cdots + z_n E \). Therefore, \( z^{-1}(c_D(zE[X]))^\star \supseteq z_0 E + z_1 E + \cdots + z_n E \). Thus, we have \( (E[X])^\star \subseteq A^\star \) and hence \( (E[X])^\star = E^\star \). Also, since \( E^\star \subseteq \bigl(D[X]\bigr)^\star \) and \( E^\star \subseteq T^\star = T \), we have \( (E[X])^\star = (E^\star)^\star = E^\star \).

**Claim 2.** For each \( \alpha \in K(X) \setminus \{0\} \) and \( A \subseteq \bigl(D[X]\bigr)^\star \), \( (\alpha A)^\star = \alpha A^\star \).

It follows from the fact that \( (T[X] : A) = \alpha^{-1}(T[X] : \alpha A) \).

**Claim 3.** If \( A_1, A_2 \subseteq \bigl(D[X]\bigr)^\star \) and \( A_1 \subseteq A_2 \), then \( A_1^\star \subseteq A_2^\star \).

This is a straightforward consequence of the definition.

**Claim 4.** For each \( A \subseteq \bigl(D[X]\bigr)^\star \), \( A \subseteq A^\star \).

Let \( z \in (T[X] : A) \setminus \{0\} \). Then \( zA \subseteq c_D(zA)[X] \subseteq (c_D(zA))^\star \), and hence \( A \subseteq z^{-1}(c_D(zA))^\star \). Therefore, \( A \subseteq A^\star \).

**Claim 5.** For each \( A \subseteq \bigl(D[X]\bigr)^\star \), \( (A^\star)^\star = A^\star \).

From Claims 3 and 4, \( A^\star \subseteq (A^\star)^\star \). For the opposite inclusion, we may assume that \( A^\star \neq K(X) \). Let \( z \in (T[X] : A) \setminus \{0\} \). By Claims 2, 3, and 3, and 3, we have \( z(A^\star)^\star = z(A^\star)^\star = (zA)^\star \subseteq ((c_D(zA))^\star \subseteq ((c_D(zA))^\star = ((c_D(zA))^\star = ((c_D(zA))^\star = ((c_D(zA))^\star = \cdots \subseteq (c_D(zA))^\star \). Since \( z \) is an arbitrary nonzero element of \( (T[X] : A) \), we have \( (A^\star)^\star \subseteq A^\star \).

(1) Claims 2–5 show that \( A^\star \) is a semistar operation of \( D[X] \).

(2) Note that, by Claim 1, \( D[X]^\star = D^\star \).

(3) is a direct consequence of the definition.

(4) By (1) and Claim 1, \( A^\star \) is a strict extension of \( \star \) to \( D[X] \). In order to show that \( A^\star \) is the largest strict extension of \( \star \) to \( D[X] \) and \( A \subseteq \bigl(D[X]\bigr)^\star \), if \( (K[X] : A) = 0 \), then clearly \( A^\star \subseteq K(X) = A^\star \). Assume that \( (K[X] : A) \neq 0 \) and let \( z \in (K[X] : A) \setminus \{0\} \). Then \( zA \subseteq c_D(zA)[X] \) and so \( zA^\star = zA^\star \subseteq (c_D(zA)[X])^\star = (c_D(zA))^\star \), i.e., \( A^\star \subseteq z^{-1}(c_D(zA))^\star \). Hence \( A^\star \subseteq A^\star \).
Remark 2.2. For each $E \in \mathcal{F}(D)$, set

$$E^{\ast*} := \begin{cases} E^*, & \text{if } (D^*: E) \neq (0), \\ K, & \text{otherwise}. \end{cases}$$

Then $\ast_e$ is a semistar operation of $D$ with $\ast \leq \ast_e$ and $(D^*)^{\ast_e} = D^*$. Using Claims 1 and 2, we can easily show that $\hat{\Delta}^\ast_D$, is an extension of $\ast_e$ to $D[X]$. In particular, $\hat{\Delta}^\ast \leq \hat{\Delta}^{\ast_e} \leq \hat{\Delta}^\ast$.

Theorem 2.3. Let $\ast$ be a semistar operation of $D$ and let $\Delta^\ast$ denote the strict extension $\Delta^\ast_K$ of $\ast$ to $D[X]$ introduced in Theorem 2.1. Then:

1. $(\Delta^\ast)_j = (\Delta^\ast')_j$ is the largest finite-type strict extension of $\ast_j$.

2. $\tilde{\Delta} = \tilde{\Delta}^\ast$ is the largest stable finite-type strict extension of $\tilde{\ast}$.

Proof. (1) Since $\Delta^\ast$ is a strict extension of $\ast$, it follows immediately that $(\Delta^\ast)_j$ is a strict extension of $\ast_j$. Then, by Theorem 2.1(4), $(\Delta^\ast)_j \leq \Delta^\ast$ and hence $(\Delta^\ast)_j \leq (\Delta^\ast')_j$. Since the opposite inequality is obvious, we have $(\Delta^\ast)_j = (\Delta^\ast')_j$. Now, let $\hat{\ast}^\ast$ be a finite-type strict extension of $\ast_j$, then $\hat{\ast}^\ast \leq \Delta^\ast$ by Theorem 2.1(4), and hence $\hat{\ast}^\ast = \flat \leq (\Delta^\ast')_j = (\Delta^\ast)_j$.

(2) We will show first that $\Delta^\ast$ is a strict extension of $\tilde{\ast}$. Let $E \in \mathcal{F}(D)$. Since $\Delta^\ast$ is an extension of $\ast$, $\tilde{\Delta}^\ast$ is an extension of $\hat{\ast}$ [Lemma 5], and hence we have $E^*[X] \subseteq (E[X])^{\tilde{\Delta}^\ast}$. Let $0 \neq f \in (E[X])^{\tilde{\Delta}^\ast} \subseteq (E[X])^{\Delta^\ast} = E^*[X] \subseteq K[X]$. Then $f J \in E[X]$ for some $J \in f(D[X])$ such that $J \subseteq D[X]$ and $J^{\prime} = (D[X])^{\ast}$. Since $J \subseteq c_D(J[X]) \subseteq D[X]$ and $J^{\prime} = (D[X])^{\ast}$, we have $c_D(f J) = (c_D(J[X]))^{\ast} = (D[X])^{\ast} = D^*[X]$, i.e., $(c_D(J))^* = D^*$. Write $J = (g_1, g_2, \ldots, g_n)$. Since $f g_i \in f J \subseteq E[X]$, $c_D(f g_i) \subseteq E$ for all $i = 1, 2, \ldots, n$. Let $m := \deg f$. Then, by Dedekind-Mertens Lemma [Theorem 28.1], $c_D(f c_D(g))^{m+1} = c_D(f g)c_D(g)^m \subseteq E$ for all $i = 1, 2, \ldots, n$, and so

$$c_D(f)((c_D(g_1))^{m+1} + (c_D(g_2))^{m+1} + \cdots + (c_D(g_n))^{m+1}) \subseteq E.$$

Note that $(c_D(g_1))^{m+1} + (c_D(g_2))^{m+1} + \cdots + (c_D(g_n))^{m+1}$ is a finitely generated integral ideal of $D$. Also, from the equation $(c_D(g_1) + c_D(g_2) + \cdots + c_D(g_n))^* = (c_D(J))^* = D^*$, it easily follows that

$$(c_D(g_1))^{m+1} + (c_D(g_2))^{m+1} + \cdots + (c_D(g_n))^{m+1} = D^*.$$

Therefore, $c_D(f)((c_D(g_1))^{m+1} + (c_D(g_2))^{m+1} + \cdots + (c_D(g_n))^{m+1}) \subseteq E$ implies that $c_D(f) \in E^*[X]$, i.e., $f \in E^*[X]$. Thus, we have $(E[X])^{\tilde{\Delta}^\ast} \subseteq E^*[X]$ and hence $E^*[X] = (E[X])^{\tilde{\Delta}^\ast}$. Therefore, $\tilde{\Delta}^\ast$ is a strict extension of $\tilde{\ast}$ that is stable and of finite type.

By Theorem 2.1(4), $\tilde{\Delta}^\ast \leq \tilde{\Delta}^\ast$ and hence $\hat{\Delta}^\ast \leq \tilde{\Delta}^\ast$. Since the opposite inequality is obvious, we have $\hat{\Delta}^\ast = \tilde{\Delta}^\ast$. Let $\tilde{\ast}$ be a stable finite-type strict extension of $\tilde{\ast}$. Then $\tilde{\ast} \leq \hat{\Delta}^\ast$ and hence $\tilde{\ast} = \tilde{\ast} \leq \hat{\Delta}^\ast$.

Corollary 2.4. Let $t_D[X]$ and $w_D[X]$ be the $t$-semistar operation and the $w$-semistar operation of $D[X]$, respectively. Then:
(1) \((\triangledown^D)_{\triangledown} = (\triangledown^D)_{\triangledown} = t_{D[X]}\).
(2) \(\triangledown^D = \triangledown^D = w_{D[X]}\).

**Proof.** (1) Since \(t_{D[X]}\) is the largest finite-type (semi)star operation of \(D[X]\) and it is a strict extension of \(t_D\) (as observed in Section 1), it is the largest finite-type strict extension of \(t_D\) and hence, by Theorem 2.3(1), \(t_{D[X]} = (\triangledown^D)_{\triangledown} = (\triangledown^D)_{\triangledown}\).

(2) It follows from Theorem 2.3(2) and the fact that \(t_{D[X]} = w_{D[X]}\). Indeed, \(w_{D[X]} = \tilde{t}_{D[X]} = (\triangledown^D)_{\triangledown} = \triangledown = \triangledown = \triangledown^D\). □

It is natural to ask whether \((\triangledown^\triangledown)_{\triangledown} = \triangledown^\triangledown\) and \(\tilde{\triangledown} = \tilde{\triangledown}\). The next proposition provides the negative answer to that.

**Proposition 2.5.** With the notation of Theorem 2.3, the semistar operation \(\triangledown^\star\) of \(D[X]\) is not of finite type for any semistar operation \(\star\) of \(D\).

**Proof.** Let \(A := \bigcup_{n=1}^{\infty} D[X]\), then \(A \in \overline{F}(D[X])\) and \((K[X] : A) = (0)\). Hence \(A^\triangledown^\star = K(X)\) by definition. Next, if \(B \in F(D[X])\) and \(B \subseteq A\), then \(B \subseteq \frac{1}{X}D[X]\) for some \(m \geq 1\). So \(B^\triangledown^\star \subseteq \frac{1}{X}D[X]\) holds by Theorem 2.1(3). Also, since \(A^\triangledown^\star\) is not a strict extension of \(A^\star\), it follows from Theorem 2.3(2) and the fact that \(w_{D[X]} = \tilde{t}_{D[X]} = (\triangledown^D)_{\triangledown} = \triangledown = \triangledown = \triangledown\). □

In the following, we compare \(\triangledown^D, \triangledown^B, \) and \(v_{D[X]}\).

**Corollary 2.6.** Let \(v_{D[X]}\) be the \(v\)-semistar operation of \(D[X]\). Then:

1. \(\triangledown^D \leq \triangledown^B = v_{D[X]}\). Moreover, \(\triangledown^D = v_{D[X]}\) if and only if \(D = K\).
2. \((v_{D[X]})_{\triangledown} = t_{D[X]} = (\triangledown^D)_{\triangledown} = (\triangledown^B)_{\triangledown} \leq \triangledown^D\).

**Proof.** (1) The inequality \(\triangledown^D = \triangledown^B \leq \triangledown^D\) holds by Theorem 2.1(3). Also, since \(\triangledown^B\) is a (semi)star operation of \(D[X]\), \(\triangledown^B \leq v_{D[X]}\). Now, for \(A \in F(D[X])\), we have

\[
A^\triangledown^D = \bigcap \{z^{-1}(c_D(zA))^v[X] \mid 0 \neq z \in (D[X] : A)\} \\
= \bigcap \{z^{-1}(c_D(zA)[X])^v[D[X]] \mid 0 \neq z \in (D[X] : A)\} \\
\supseteq \bigcap \{z^{-1}(zA)^v[D[X]] \mid 0 \neq z \in (D[X] : A)\} \\
= \bigcap \{z^{-1}(zA)^v[D[X]] \mid 0 \neq z \in (D[X] : A)\} \\
= A^\triangledown^D
\]

and for \(A \in \overline{F}(D[X]) \setminus F(D[X])\), we have \(A^\triangledown^D = K(X) = A^\triangledown^D\). Thus we obtain the equality \(\triangledown^D = v_{D[X]}\). If \(D = K\), then obviously \(\triangledown^D = \triangledown^D = \triangledown^D = v_{D[X]}\). Assume that \(D \neq K\). Then \(K \in \overline{F}(D) \setminus F(D)\) for each \(E \in \overline{F}(D) \setminus F(D)\), \(E[X]^v[D[X]] = K(X)\) but \(E^\triangledown^D \neq K[X]\). Thus \(\triangledown^D = v_{D[X]}\) is not a strict extension of \(v_D\) and hence \(\triangledown^D \leq \triangledown^B = v_{D[X]}\).

(2) is an easy consequence of Corollary 2.3(1) and Proposition 2.7. □

**Remark 2.7.** (a) The statement (1) of the previous corollary can be stated more precisely as follows: for each \(A \in \overline{F}(D[X])\), we have

\[
A^{\triangledown[D]} = \begin{cases} 
K(X) = A^{\triangledown[D]} = A^{\triangledown[D]} & \text{if } (K[X] : A) = (0), \\
K(X) = A^{\triangledown[D]} \supseteq A^{\triangledown[D]} & \text{if } (K[X] : A) \neq (0) \text{ but } (D[X] : A) = (0), \\
A^{\triangledown[D]} = A^{\triangledown[D]} & \text{if } (D[X] : A) \neq (0).
\end{cases}
\]
(b) If $D$ is a Krull domain, then $D[X]$ is also a Krull domain, and hence $t_{D[X]}$, $\star^{PD}$, $\star^D$, and $v_{D[X]}$ coincide when they are restricted to $F(D[X])$. Therefore, the star operation $\star^{PD}_{F(D[X])}$ can be of finite type, in contrast with the semistar operation $\star^D$ (Proposition 2.9). On the other hand, if $D$ is a TV-domain such that $D[X]$ is not a TV-domain (see [3]), then by (a), $\star^{PD}_{F(D[X])} = \star^{VD}_{F(D[X])} = v_{D[X]}|F(D[X]) \neq t_{D[X]}|F(D[X])$. Thus $\star^{PD}_{F(D[X])}$ (and $\star^D$) may not be of finite type.

(c) Note that $\star^{PD} \leq e_{D[X]}$ and $d_{D[X]} \leq (\star^{PD})_f$; moreover, $d_{D[X]} = (\star^{PD})_f$ if and only if $D = K$. The first proper inequality is obvious, because $\star^{PD} \leq e_{D[X]}$ and $(D[X])^{\star^D} = D^{\star^D}[X] = K[X] \subseteq K(X) = (D[X])^{\star^{PD}}$. The second inequality is also obvious, because $d_{D[X]}$ is the smallest semistar operation of $D[X]$. If $D = K$, then $(\star^{PD})_f = (\star^{PD})_f = t_{D[X]} = d_{D[X]}$ by Corollary 2.8. Assume that $D \neq K$.

Let $\alpha$ be a nonzero nonunit element of $D$ and let $A = (\alpha, X)D[X]$. Since $(K[X]: A) = K[X]$ and $c_D(z\alpha) = c_D(z)$ for all $z \in K[X]\setminus(0)$, $A = A_D^D \subseteq (A^{\star^D})_f = A^{\star_D} = D[X]$. In fact, since $1 \in (K[X]: A)$, $A^{\star_D} \subseteq D[X]$; on the other hand, $A^{\star^D} = \left\{z^{-1}c_D(z\alpha)[X] \mid 0 \neq z \in K[X]\right\} = \left\{z^{-1}c_D(z)[X] \mid 0 \neq z \in K[X]\right\} \supseteq \left\{z^{-1}z[D[X]] \mid 0 \neq z \in K[X]\right\} = D[X]$.

According to Theorem 2.1, any semistar operation $\star$ of $D$ admits a strict extension (in fact, the largest strict extension) to $D[X]$. In the following, we show that, in fact, there exist infinitely many strict extensions to $D[X]$ of $\star$.

Lemma 2.8. Let $\star$ be a semistar operation of $D$ and let $\star'$ and $\star''$ be two extensions of $\star$ to $D[X]$. If $\star' \leq \star''$ and $\star''$ is a strict extension of $\star$, then $\star'$ is also a strict extension of $\star$.

Proof. For each $E \in F(D)$, we have $E^\star[X] \subseteq (E[X])^\star' \subseteq (E[X])^\star'' = E^\star[X]$, and hence $(E[X])^\star' = E^\star[X]$. □

Proposition 2.9. For each semistar operation $\star$ of $D$, there exists a strictly increasing infinite sequence of semistar operations of $D[X]$ which are all strict extensions of $\star$.

Proof. Let $\{f_i\}_{i=1}^\infty$ be a set of countably infinite nonassociate irreducible polynomials in $K[X]$. For each $n \geq 1$, let $\star^*_n := (\bigwedge \{\star(K[X]_{(f_i)}) \mid i \geq n\}) \vee \star^*$, i.e., for each $A \in F(D[X])$, $A^\star := (\bigcap_{i \geq n} A K[X]_{(f_i)}) \cap A^\star^*$. Then each $\star^*_n$ is a strict extension to $D[X]$ of $\star$ such that $\star^*_1 \leq \star^*_2 \leq \cdots \leq \star^*$ (Lemma 2.8 and Theorem 2.1(i)). Let $n < m$ and let $B := K[X]_{(f_m)}$. Then $B \in F(D[X])$ and $BK[X]_{(f_m)} = K[X]$ for all $i \neq n$. Since $(K[X]: B) = (0)$, $B^\star = K(X)$. Therefore, $B^\star^*_n = K(X)$, while $B^\star^*_m = B$. Thus $\star^*_n \neq \star^*_m$. □

Remark 2.10. We do not know whether, for a given arbitrary star operation $\star$ of $D$, there exists a strictly increasing sequence of (strict) star operation extensions to $D[X]$ of $\star$. However, we can show that it does hold if $\star$ is of finite type and $D$ admits a $\star$-valuation overring which is not equal to the quotient field $K$.

Let $V$ be a nontrivial $\star$-valuation overring of $D$ with maximal ideal $M$ and let $P := M \cap D$. Then $P$ is a nonzero prime ideal of $D$.

Case 1. $D/P$ is infinite.
Let \( \{a_i\}_{i=1}^{\infty} \) be a set of elements of \( D \) such that \( \bar{a}_i \neq \bar{a}_j \) in \( D/P \) for all \( i \neq j \). Then the ideals \( N_i := M + (X - a_i)V[X] \) are distinct maximal ideals of \( V[X] \). We denote by \( \triangleleft \) the strict star operation extension to \( D[X] \) of \( * \) (defined as in the semistar operation case, but obviously only on nonzero fractional ideals of \( D[X] \)). For each \( n \geq 1 \), define \( A^{\triangleleft n} := (\bigcap_{i \geq n} AV[X]_{N_i}) \cap A^{\triangleleft} \) for all \( A \in F(D[X]) \). Since \( V \) is a \( * \)-valuation overring of \( D \) and \( \triangleleft \) is a strict star operation extension to \( D[X] \) of \( * \), each \( \star_n \) is a strict star operation extension to \( D[X] \) of \( * \) such that \( \star_1 \leq \star_2 \leq \cdots \). Choose a nonzero element \( c \in P \) and let \( A_i := (c, X - a_i)D[X] \) for each \( i \geq 1 \). Then, since \( cD(X - a_i) = D \), we have \( A_1^{\triangleleft} = D[X] \). Let \( n < m \). Then \( A_n^{\triangleleft} \subseteq N_nV[X]_{N_n} \cap D[X] \subseteq N_n[X] \cap D[X] \subseteq D[X] \), while \( A_m^{\triangleleft} = (\bigcap_{i \geq m} V[X]_{N_i}) \cap D[X] = D[X] \). Thus \( \star_n \neq \star_m \).

**Case 2.** \( D/P \) is finite.

Since \( D/P \) is a finite field, for each \( n \geq 1 \), there exists a monic irreducible polynomial \( f_n \in (D/P)[X] \) of degree \( n \). Then, for \( n \neq m \), \( f_n \neq f_m \). \( (D/P)[X] = (D/P)[X] \), and so, \( \forall n \geq 1 \), define \( A_n^{\triangleleft} := (\bigcap_{i \geq n} AV[X]_{N_i}) \cap A^{\triangleleft} \) for all \( A \in F(D[X]) \). Then, as above, each \( \star_n \) is a strict star operation extension to \( D[X] \) of \( * \) such that \( \star_1 \leq \star_2 \leq \cdots \). Choose a nonzero element \( c \in P \) and set \( A_n := (c, f_n)D[X] \). Then, since \( cD(f_n) = D \), we have \( A_n^{\triangleleft} = D[X] \). Let \( n < m \). Then \( A_n^{\triangleleft} \subseteq N_nV[X]_{N_n} \cap D[X] \subseteq N_n[X] \cap D[X] \subseteq D[X] \), while \( A_m^{\triangleleft} = (\bigcap_{i \geq m} V[X]_{N_i}) \cap D[X] = D[X] \). Thus \( \star_n \neq \star_m \).

## 3. Relationship among strict extensions

In Section 2, we have shown that each semistar operation of \( D \) admits the largest strict extension to \( D[X] \), by defining its precise form. The next proposition provides the existence of the smallest strict extension. Unlike the largest strict extension, the smallest strict extension is not described in an explicit form in general. However, for a stable semistar operation of finite type, we are able to provide a complete description of its smallest strict extension.

**Proposition 3.1.** Let \( D \) be an integral domain and let \( * \) be a semistar operation of \( D \). Set \( \{ \star_\lambda \mid \lambda \in \Lambda \} \) the set of all the semistar operations of \( D[X] \) extending \( * \). Then \( \lambda^* := \wedge \{ \star_\lambda \mid \lambda \in \Lambda \} \) is the smallest semistar operation of \( D[X] \) extending \( * \). Moreover, it is a strict extension of \( * \).

**Proof.** Note that, by definition, for all \( E \in \mathcal{F}(D) \), \( (E[X])^{\lambda^*} = \bigcap \{(E[X])^{\star_\lambda} \mid \lambda \in \Lambda \} \). Since \( (E[X])^{\lambda^*} \cap K = E^{\star_\lambda} \) for each \( \lambda \in \Lambda \), we deduce immediately that \( (E[X])^{\lambda^*} \cap K = E^{\star_\lambda} \). Also, \( \lambda^* \) is a strict extension of \( * \) to \( D[X] \) by Theorem 2.1(4), and so \( \lambda^* \leq \lambda^* \). Therefore, \( \lambda^* \) is a strict extension of \( * \) by Lemma 2.3. \( \square \)

We know that for any semistar operation \( * \) of \( D \), \( (\lambda^*)_i = (\lambda^*)_j \leq \lambda^* \) and \( \lambda^* = (\lambda^*) \leq \lambda^* \). We will see now what happens for the semistar operation \( \lambda^* \).
Lemma 3.2. Let $\star$, $\star'$ and $\star''$ be semistar operations of $D$ and let $\star^*$ be the smallest strict extension of $\star$ to $D[X]$ introduced in Proposition 3.1. Then:

1. $\star^* = \wedge\{ \star^* | \star^*$ is a semistar operation of $D[X]$ such that $\star_0^* \geq \star \}$.
2. If $\star' \leq \star''$, then $\star^* \leq \star^*$.
3. Every semistar operation $\star$ of $D[X]$ such that $\star^* \leq \star \leq \star^*$ is a strict extension of $\star$.

Proof. (1) Set $\hat{\star} := \wedge\{ \star^* | \star^*$ is a semistar operation of $D[X]$ such that $\star_0^* \geq \star \}$. It is obvious that $\hat{\star} \leq \star^*$. From this inequality, we obtain that $\star \leq \hat{\star} = (\star^*)_0 = \star$ and therefore $\hat{\star}$ is an extension of $\star$ to $D[X]$. By the minimality of $\star^*$, we deduce that $\hat{\star}$ coincides with $\star^*$.

(2) is a straightforward consequence of (1).

(3) is an easy consequence of Lemma 3.2.

Proposition 3.3. Let $\star$ be a semistar operation of $D$ and let $\star^*$ be the semistar operation of $D[X]$ introduced in Proposition 3.1. Then:

1. $\star^*$ is a semistar operation of finite type (and hence $\star^* = (\star^*)_0 \leq (\star^*)_0 \leq (\star^*)_0 \leq (\star^*)_0$).
2. $\star^*$ is a stable semistar operation of finite type (and hence $\star^* = \tilde{\star} \leq \star^* \leq \tilde{\star} = \tilde{\star} \leq \star^*$).

Proof. (1) By [2, Lemma 5], we have $\star_j = ((\star^*)_0) = ((\star^*)_0) \leq (\star^*)_0 = \star_j$, and hence $(\star^*)_0 \leq \star^*$ are both extensions of $\star_j$. By the minimality of $\star^*$, they must be equal. Thus $\star^*$ is a strict extension of $\star_j$ of finite type. The parenthetical statement is a straightforward consequence of Theorem 2.3(1) and Proposition 3.1.

(2) By [2, Lemma 5], $(\tilde{\star})_0 = (\tilde{\star})_0 = \tilde{\star} = \star^*$, and hence $\tilde{\star} \leq \star^*$ are both extensions of $\tilde{\star}$. Then, by the minimality of $\tilde{\star}$, we have $\tilde{\star} = \star^*$. Thus $\star^*$ is stable and of finite type. The parenthetical statement is a straightforward consequence of Theorem 2.3(2) and Proposition 3.1.

Remark 3.4. (a) It can happen that $(\star^*)_0 \leq (\star^*)_0$. For instance, if $D$ is not a field, then $\lambda^{dD} = d_D(X) \leq (\lambda^{dD})_0$ (see Remark 2.7(c)). But, at the moment, we do not know if it is possible that $\lambda^{dD} \leq (\lambda^{dD})_0$.

(b) It can happen that $\lambda^\sim \leq \lambda^\sim$. For instance, let $D$ be an integral domain, not a field, with $d_D = w_D$. Then $\lambda^{wD} = d_D(X)$, $(\lambda^{wD}) = w_D(X)$ by Corollary 2.4 but $d_D(X) \neq w_D(X)$. So $\lambda^{wD} = \lambda^{wD} \leq \lambda^{wD}$. On the other hand, we do not know whether it is possible that $\lambda^\sim \leq \lambda^\sim$.

Let $\star$ be a semistar operation of $D$. In the paper [2], the authors introduced the following semistar operations $[\hat{\star}]$ and $\langle \hat{\star} \rangle$ of $D[X]$ for each $A \in \mathcal{F}(D[X])$,

$$
A^{[\hat{\star}]} := \cap\{ ADQ[X] | Q \in \text{QMax}^\star(D) \},
A^{\langle \hat{\star} \rangle} := \cap\{ ADQ(X) | Q \in \text{QMax}^\sim \} \cap AK[X].
$$

They showed that both are stable finite-type strict extensions of $\hat{\star}$ [2, Corollary 18]. We will compare $\hat{\star}$, $\lambda^\sim$, $[\hat{\star}]$, and $\langle \hat{\star} \rangle$. 
Theorem 3.5. Let $\star$ be a semistar operation of $D$. Then $\lambda^{\tilde{\star}} = \tilde{\lambda}$, i.e.,

$$A^{\lambda^{\tilde{\star}}} = \bigcap \{ AD_Q[X] \mid Q \in \text{QMax}^\star(D) \}$$

for each $A \in \overline{\mathcal{F}}(D[X])$.

Proof. Since $\tilde{\lambda}$ is an extension of $\lambda$ to $D[X]$, it suffices to show that $\tilde{\lambda} \leq \lambda^{\tilde{\star}}$. Note that $\lambda^{\tilde{\star}}$ and $\tilde{\lambda}$ are stable semistar operations of finite type, so $\lambda^{\tilde{\star}} = \lambda^{\tilde{\star}}$ and $\tilde{\lambda} = \tilde{\lambda}$. Therefore, for each $A \in \overline{\mathcal{F}}(D[X])$, we have:

$$A^{\lambda^{\tilde{\star}}} = \bigcup \{(A : J) \mid J \subseteq D[X], J \subseteq D[X], \text{ and } J^{\lambda^{\tilde{\star}}} = (D[X])^{\lambda^{\tilde{\star}}} = D^*[X]\}$$

$$\supseteq \bigcup \{(A : H[X]) \mid H \subseteq D, \text{ and } (H[X])^{\lambda^{\tilde{\star}}} = D^*[X]\}$$

$$= \bigcup \{(A : H[X]) \mid H \subseteq D, \text{ and } H^*[X] = D^*[X]\}$$

$$= \bigcup \{(A : H) \mid H \subseteq f(D), H \subseteq D, \text{ and } H^*[X] = D^*[X]\}$$

$$= A^{[\tilde{\lambda}]}.$$ 

Thus the conclusion $\lambda^{\tilde{\star}} = \tilde{\lambda}$ follows. 

Finally we will show that $\lambda^{\star} = \tilde{\lambda}$. For this purpose, we need to extend a couple of results which are known for the $t$-operation case to a more general semistar operation setting.

Given a semistar operation $\star$ of $D$, let $v_D(D^*)$ be the semistar operation of $D$ defined by $E^{v_D(D^*)} := (ED^*)^{\star D} = (D^* : (D^* : E))$ for each $E \in \overline{\mathcal{F}}(D)$ and set $t_D(D^*) := v_D(D^*)$. Then $E^{t_D(D^*)} = (ED^*)^{\star D}$ for each $E \in \overline{\mathcal{F}}(D)$. It is also obvious that $\star \leq v_D(D^*)$ and $\star \leq t_D(D^*)$.

Lemma 3.6. Let $\star$ be a semistar operation of $D$ and let $t(D^*[X]) := t_D[X](D^*[X])$ be the semistar operation of $D[X]$ introduced above. If $M$ is a quasi-$t(D^*[X])$-maximal ideal of $D[X]$ with $M \cap D \neq (0)$, then $M = (M \cap D)[X]$.

Proof. Let $M$ be a quasi-$t(D^*[X])$-maximal ideal of $D[X]$ with $M \cap D \neq (0)$. Since $M^{t(D^*[X])} = (MD^*[X])^{t(D^*[X])}$ is a proper ideal of $D^*[X]$, there exists a $t_D-[X]$-maximal ideal $N$ of $D^*[X]$ containing $(MD^*[X])^{t(D^*[X])}$. Since $N \cap D^* \neq (0)$, $N = (N \cap D^*)[X]$ [12, Proposition 1.1]. Therefore, it follows that $M = N \cap D[X] = (N \cap D^*)[X] \cap D[X] = (M \cap D)[X]$.

Lemma 3.7. Let $\star$ be a semistar operation of $D$ and let $t(D^*) := t_D(D^*)$, $t(D^*[X]) := t_D[X](D^*[X])$ as above. If $Q$ is a nonzero prime ideal of $D[X]$ such that $Q \cap D = (0)$ and $c_D(Q)^{t(D^*)} = D^*$, then $Q$ is a quasi-$t(D^*[X])$-maximal ideal of $D[X]$.

Proof. It is clear that $Q$ is a quasi-$t(D^*[X])$-prime ideal of $D[X]$. Suppose $Q$ is not a quasi-$t(D^*[X])$-maximal ideal of $D[X]$ and let $M$ be a quasi-$t(D^*[X])$-maximal ideal of $D[X]$ with $Q \subseteq M$. Since the containment is proper, $M \cap D \neq (0)$. By Lemma 3.6, $M = (M \cap D)[X]$. Note that $M = M^{t(D^*[X])} = (M \cap D)^{t(D^*[X])}$ and hence that $(M \cap D)^{t(D^*)} = M \cap D$. Since $Q \subseteq M$, we have $c_D(Q) \subseteq c_D(M) = M \cap D$ and so $c_D(Q)^{t(D^*)} \subseteq (M \cap D)^{t(D^*)} \subseteq D^*$, which is a contradiction. 

\[\square\]
Proposition 3.8. Let \( \star \) be a semistar operation of \( D \). Then

\[
\text{QMax}(\star^r)(D[X]) = \{ Q \in \text{Spec}(D[X]) \mid Q \cap D = (0) \text{ and } \mathfrak{c}_D(Q)^{\star^r} = D^{\star^r} \} \cup \{ P[X] \mid P \in \text{QMax}^{\star^r}(D) \}.
\]

Proof. Let \( t(D^\star) \) and \( t(D^\star[X]) \) be as in Lemma 3.7. Then \( \star^r \leq t(D^\star) \) and \( (\star^r)_f \leq t(D[X])^{\star^r} = t(D^\star[X]) \). Let \( Q \) be a prime ideal of \( D[X] \) with \( Q \cap D = (0) \) and \( \mathfrak{c}_D(Q)^{\star^r} = D^{\star^r} \). Then, obviously, \( \mathfrak{c}_D(Q)^{t(D^\star)} = D^{\star^r} \), and thus, by Lemma 3.7, \( Q \) is a quasi-\( t(D^\star[X]) \)-maximal ideal of \( D[X] \). This implies that \( Q \) is a quasi-\( (\star^r)_f \)-prime ideal of \( D[X] \). Let \( P \in \text{QMax}^{\star^r}(D) \). Since \( (\star^r)_f \) is a strict extension of \( \star^r \), \( (P[X])^{(\star^r)_f} = P^{\star^r}[X] \) and hence \( (P[X])^{(\star^r)_f} \cap D[X] = P^{\star^r}[X] \cap D[X] = P[X] \). This implies that \( P[X] \) is a quasi-\( (\star^r)_f \)-prime ideal of \( D[X] \).

Therefore, for the equality of the statement, it suffices to show that if \( M \) is a prime ideal of \( D[X] \) such that \( M \cap D \neq (0) \) and \( \mathfrak{c}_D(M)^{\star^r} = D^{\star^r} \), then \( M^{(\star^r)_f} = (D[X])^{(\star^r)_f} \). Choose a nonzero \( a \in M \cap D \) and a nonzero \( g \in M \) with \( \mathfrak{c}_D(g)^{\star^r} = D^{\star^r} \). Then, for each nonzero element \( z \in (K[X] : (a,g)) \subseteq K[X] \), \( \mathfrak{c}_D(zg)^{\star^r} = \mathfrak{c}_D(z)^{\star^r} \) by Dedekind-Mertens Lemma, and so \( \mathfrak{c}_D(z(a,g))^{\star^r} = (\mathfrak{c}_D(z) + \mathfrak{c}_D(g))^{\star^r} = (\mathfrak{c}_D(z) + \mathfrak{c}_D(zg))^{\star^r} = \mathfrak{c}_D(z)^{\star^r} \). Therefore, we have

\[
(D[X])^{(\star^r)_f} \supseteq (a,g)^{\star^r} = \bigcap \{ z^{-1}(\mathfrak{c}_D(z(a,g))^{\star^r})[X] \mid 0 \neq z \in (K[X] : (a,g)) \}
= \bigcap \{ z^{-1}(\mathfrak{c}_D(z))^{\star^r}[X] \mid 0 \neq z \in (K[X] : (a,g)) \}
= \bigcap \{ z^{-1}(\mathfrak{c}_D(z))^{(\star^r)_f} \mid 0 \neq z \in (K[X] : (a,g)) \}
\subseteq \bigcap \{ z^{-1}(zD[X])^{(\star^r)_f} \mid 0 \neq z \in (K[X] : (a,g)) \}
= (D[X])^{(\star^r)_f}.
\]

Thus \( (a,g)^{\star^r} = (D[X])^{(\star^r)_f} \), and hence \( M^{(\star^r)_f} = (D[X])^{(\star^r)_f} \).

Theorem 3.9. Let \( D \) be an integral domain with quotient field \( K \) and let \( \star \) be a semistar operation of \( D \). Then \( \star^r = \langle \bar{\star} \rangle \), i.e.,

\[
A^{\bar{\star}} = \bigcap \{ AD_Q(X) \mid Q \in \text{QMax}^{\star^r} \} \cap AK[X]
\]

for each \( A \in \mathcal{F}(D[X]) \).

Proof. By Proposition 3.8 and [2] Remark 20(2), we have \( \text{QMax}(\star^r)(D[X]) = \text{QMax}^{\bar{\star}}(D[X]) \) and hence \( (\star^r)_f = \langle \bar{\star} \rangle \). By Theorem 2.3 \( (\star^r)_f = (\star^r)_f = \bar{\star} \), and by [2] Proposition 16, \( (\bar{\star}) = (\bar{\star}) \). Thus the conclusion \( \bar{\star}^r = \langle \bar{\star} \rangle \) follows.

4. Semistar Operations Defined by Families of Overrings

In the present section, we generalize some known results concerning the polynomial extensions of a stable finite-type semistar operation, to the case where the semistar operation is defined by a given family of overrings of \( D \).

Lemma 4.1. Let \( T \) be an overring of \( D \) and let \( \star_T \) (respectively, \( \star_{\{T[X] \}} \)) be the semistar operation of \( D \) (respectively, of \( D[X] \)) defined by \( E^{\star_T} := ET \) for each \( E \in \mathcal{F}(D) \) (respectively, \( A^{\star_{\{T[X] \}}} := AT[X] \) for each \( A \in \mathcal{F}(D[X]) \)). Then \( \mathfrak{A}^{\star_T} = \star_{\{T[X] \}} \leq \mathfrak{A}^{\star_{\{T[X] \}}} \).
Proof. It is clear that $\lambda^\ast(T)$, $\bigstar(T[X])$, and $\triangle^\ast(T)$ are all strict extensions of $\ast(T)$. Since $\bigstar(T[X])$ is of finite type but $\triangle^\ast(T)$ is not of finite type (Proposition 2.10), we have $\lambda^\ast(T) \leq \bigstar(T[X]) \leq \triangle^\ast(T)$. For the equality $\lambda^\ast(T) = \bigstar(T[X])$, let $A \in \overline{F}(D[X])$. Then $A\lambda^\ast(T) = (AD[X])\lambda^\ast(T) \supseteq A(D[X])\lambda^\ast(T) = AD^\ast(T)[X] = AT[X] = A\bigstar(T[X])$. This implies $\lambda^\ast(T) \geq \bigstar(T[X])$. Therefore, the equality $\lambda^\ast(T) = \bigstar(T[X])$ holds.

Now we consider a semistar operation given by an arbitrary family of overrings. Let $\mathcal{T} := \{T_\lambda \mid \lambda \in \Lambda\}$ be a set of overrings of $D$, and let $\wedge_{\mathcal{T}} := \bigwedge\{\bigstar(T_\lambda) \mid \lambda \in \Lambda\}$, i.e., $E\wedge_{\mathcal{T}} := \bigcap_\lambda ET_\lambda$ for each $E \in \overline{F}(D)$. Let $\mathcal{T}[X] := \{T_\lambda[X] \mid \lambda \in \Lambda\}$ and let $\wedge_{\mathcal{T}[X]} := \bigwedge\{\bigstar(T_\lambda[X]) \mid \lambda \in \Lambda\}$, i.e., $A\wedge_{\mathcal{T}[X]} := \bigcap_\lambda AT_\lambda[X]$ for each $A \in \overline{F}(D[X])$.

**Proposition 4.2.** With the notation recalled above, we have

$$\lambda^\wedge_{\mathcal{T}} \leq \wedge_{\mathcal{T}[X]} \leq \lambda^\wedge_{\mathcal{T}[X]}.$$

Proof. We easily deduce from the definitions that, for each $E \in \overline{F}(D)$,

$$(E^\wedge_{\mathcal{T}[X]}[X])\wedge_{\mathcal{T}[X]} = (E[X])\wedge_{\mathcal{T}[X]} = E^\wedge_{\mathcal{T}[X]}[X],$$

and hence $\wedge_{\mathcal{T}[X]}$ is a strict extension of $\wedge_{\mathcal{T}}$. By the minimality of $\lambda^\wedge_{\mathcal{T}}$ and the maximality of $\lambda^\wedge_{\mathcal{T}[X]}$, we immediately obtain that $\lambda^\wedge_{\mathcal{T}} \leq \wedge_{\mathcal{T}[X]} \leq \lambda^\wedge_{\mathcal{T}[X]}$. □

**Remark 4.3.** Given a semistar operation $\ast$ of an integral domain $D$ which is not a field, let $\mathcal{T} := \{D_Q \mid Q \in \text{QMax}^\ast(D)\}$. Then $\wedge_{\mathcal{T}} = \bigwedge^\ast$ and $\wedge_{\mathcal{T}[X]} = [\bigwedge^\ast]$, and hence by Theorem 3.3 we have $\lambda^\wedge_{\mathcal{T}} = \lambda^\ast = [\bigwedge^\ast] = \wedge_{\mathcal{T}[X]} \leq \lambda^\wedge_{\mathcal{T}[X]} = \lambda^\wedge_{\mathcal{T}[X]}$. For the general case, i.e., for an arbitrary family of overrings $\mathcal{T}$, it would be interesting to know under which conditions $\lambda^\wedge_{\mathcal{T}}$ coincides with $\wedge_{\mathcal{T}[X]}$.

For a given semistar operation $\ast$, we investigate the relationship among the following semistar operations:

$$\lambda^\wedge_{\mathcal{V}}$, $\lambda^\wedge_{\mathcal{V}^\ast}$, $(\lambda^\ast)_a$, $(\lambda^\ast)_a$, $\wedge_{\mathcal{V}(\ast)[X]}$$

where $\mathcal{V}(\ast)$ is the family of all $\ast$–valuation overrings of $D$. Recall that, when $\ast = d_D$ and $\mathcal{V} := \mathcal{V}(d_D)$ is the family of all valuation overrings of $D$, then $(d_D)_a = b_D = \wedge_{\mathcal{V}}$. Set $[b_D] := \wedge_{\mathcal{V}[X]}$. Then, from Proposition 4.2, we immediately deduce that:

**Corollary 4.4.** For any integral domain $D$,

$$\lambda^{b_D} \leq [b_D] \leq \lambda^{b_D}.$$

For tackling the general question, we need the following lemma.

**Lemma 4.5.** Let $\ast$ be a semistar operation of $D$ and let $\bigstar$ be a strict extension of $\ast$ to $D[X]$. Then $\bigstar_a$ is a strict extension of $\ast_a$.

Proof. We start by proving some results of independent interest.

**Claim 1.** If $H$ is a nonzero finitely generated integral ideal of $D[X]$, then

$$\sum_{g \in H} (c_D(g))^r = \left(\sum_{g \in H} c_D(g)\right)^r$$

for all $r \geq 1$. 
The inclusion \((\subseteq)\) is obvious. The opposite inclusion \((\supseteq)\) follows from the observation that for an arbitrary choice of \(g_1, g_2, \ldots, g_m \in H\), \(c_D(g_1) + c_D(g_2) + \cdots + c_D(g_m) = c_D(g)\) for some \(g \in H\) (we can put \(g := g_1 + X^{\deg(g_1)-1}g_2 + X^{\deg(g_2)+\deg(g_3)+\cdots+\deg(g_{m-1})+m-1}g_m\).

**Claim 2.** If \(E\) and \(H\) are nonzero finitely generated integral ideals of \(D\) and \(D[X]\), respectively, then
\[
((E[X]H)^\star : H^\star) \subseteq E^*\![X].
\]

Note first that \(((E[X]H)^\star : H^\star) \subseteq K[X]\). Indeed, since \(HK[X] = hK[X]\) for some \(h \in K[X]\), we have
\[
((E[X]H)^\star : H^\star) \subseteq ((E[X]HK[X])^\star : (HK[X])^\star) \subseteq ((HK[X])^\star : (K[X])^\star)
\]
\[
= (K^*[X] : K^*[X])
\]
\[
= (K[X] : K[X]) = K[X].
\]

Let \(f \in ((E[X]H)^\star : H^\star) \subseteq K[X]\). Then,
\[
fH \subseteq (E[X]H)^\star \subseteq (E[X]c_D(H)[X])^\star = (Ec_D(H))^\star[X].
\]

Let \(m := \deg(f)\) and let \(g \in H\). Then, by the previous observation,
\[
c_D(f)c_D(g)^{m+1} = c_D(fg)c_D(g)^m \subseteq (Ec_D(H))^\star c_D(H)^m \subseteq (Ec_D(H)^{m+1})^\star.
\]
and so
\[
Ec_D(f)(\sum_{g \in H} c_D(g)^{m+1}) \subseteq (Ec_D(H)^{m+1})^\star.
\]

By Claim 1, we deduce that
\[
c_D(f)c_D(H)^{m+1} \subseteq (Ec_D(H)^{m+1})^\star.
\]
Therefore, \(c_D(f) \subseteq ((Ec_D(H)^{m+1})^\star : (c_D(H)^{m+1})^\star)\). Since \(c_D(H)^{m+1}\) is a finitely generated ideal of \(D\), \(((Ec_D(H)^{m+1})^\star : (c_D(H)^{m+1})^\star) \subseteq E^*\![X]\). Thus we deduce that \(f \in E^*\![X]\).

From Claim 2, it easily follows that for each \(E \in \mathcal{F}(D)\), \((E[X])^\star \subseteq E^*\![X]\). Since the opposite inclusion is obvious, the proof is completed.

**Proposition 4.6.** Let \(\star\) be a semistar operation of \(D\). Then
\[
\lambda^{\star}_{a} \leq (\lambda^{\star}_{a})_{a} \leq (\lambda^{\star}_{a})_{a} \leq (\lambda^{\star}_{a})_{a} \leq \lambda^{\star}_{a}.
\]

**Proof.** By Lemma 4.3, we have \(\lambda^{\star}_{a} \leq (\lambda^{\star}_{a})_{a}\), and since \(\lambda^{\star}_{a}\) is of finite type, \(\lambda^{\star}_{a} \leq (\lambda^{\star}_{a})_{a}\). Also, from the first inequality, we have \((\lambda^{\star}_{a})_{a} \leq ((\lambda^{\star}_{a})_{a} = (\lambda^{\star}_{a})_{a}\). Thus, we get \(\lambda^{\star}_{a} \leq (\lambda^{\star}_{a})_{a} \leq (\lambda^{\star}_{a})_{a}\). Similarly, \((\lambda^{\star}_{a})_{a} \leq (\lambda^{\star}_{a})_{a} \leq (\lambda^{\star}_{a})_{a}\), where the strict inequalities follow from Proposition 2.5. Moreover, from the first inequality, we also have \((\lambda^{\star}_{a})_{a} = ((\lambda^{\star}_{a})_{a} \leq (\lambda^{\star}_{a})_{a}\). Thus, we get \((\lambda^{\star}_{a})_{a} \leq (\lambda^{\star}_{a})_{a} \leq \lambda^{\star}_{a}\). Finally, since \(\lambda^{\star}_{a} \leq \lambda^{\star}_{a}\), obviously we have \((\lambda^{\star}_{a})_{a} \leq (\lambda^{\star}_{a})_{a}\).

**Remark 4.7.** (a) It can happen that \(\lambda^{\star}_{a} \leq (\lambda^{\star}_{a})_{a}\), i.e., in general, \(\lambda^{\star}_{a}\) is not an eab semistar operation. For instance, let \(D\) be a Prüfer domain, not a field, and let \(\star = d_D = b_D\). Then \(\lambda^{\star}_{a} = \lambda b_D = \lambda d_D = d_D[X] \neq b_D[X] = (d_D[X])_{a} = (\lambda^{\star}_{a})_{a}\).

(b) It can happen that \((\lambda^{\star}_{a})_{a} \leq (\lambda^{\star}_{a})_{a}\). For instance, if \(D\) is not a field, then \(b_D[X] = (d_D[X])_{a} = (\lambda^{\star}_{a})_{a} \leq (\lambda d_D)_{a}\). Indeed, let \(a\) be a nonzero nonunit element of \(D\) and let \(A := (a, X)D[X]\). Since \(A\) is a finitely generated integral ideal of \(D[X]\), \(A^{\lambda d_D} \supseteq A^{\lambda d_D} = D[X]\) (see Remark 2.7(c)). On the other hand, recall
that $A^{b_D[X]} = \bigcap AW$, where $W$ ranges over the valuation overrings of $D[X]$, and hence that $A^{b_D[X]} \subseteq (D[X])^b_{D[X]} = D^{bd}[X] = \overline{D}[X]$, where $\overline{D}$ is the integral closure of $D$. Let $N$ be a maximal ideal of $\overline{D}$ containing $\alpha$. Then, $N + (X)$ is a prime ideal of $\overline{D}[X]$. By [8, Theorem 19.6], there exists a valuation overring $W$ of $\overline{D}[X]$ such that $N + (X)$ is the center of $W$ on $\overline{D}[X]$. Hence $A^{b_D[X]} \subseteq AW \cap \overline{D}[X] \subseteq (N + (X))W \cap \overline{D}[X] = N + (X)$. Therefore, $b_{D[X]} \neq (\Delta^{bd})_a$.

(c) If $\star = *f$, then $(\lambda^{*a})_a = (\lambda^*)_a$. Because, if $\star = *f$, then $\star \leq *a$ and hence $\lambda^* \leq \lambda^{*a}$. Consequently, $(\lambda^{*a})_a \leq (\lambda^*)_a$ and hence the equality holds. However, we do not know whether it is possible in general that $(\lambda^{*a})_a \leq (\lambda^*)_a$. This problem is related with the inequality $\lambda^* \leq (\lambda^*)_f$. If $\lambda^* = (\lambda^*)_f$, then we have $(\lambda^{*a})_a = (\lambda(*a))_a = (\lambda(*a))^* = (\lambda^*)_a$. As mentioned in Remark 4.9, we do not know whether the equality $\lambda^* = (\lambda^*)_f$ holds or not.

(d) Without much difficulty, we can show that the set of $\Delta^*$-valuation overrings of $D[X]$ is the set $\{K[X]_{(f)} \mid f \text{ is an irreducible polynomial of } K[X]\} \cup \{V(X) \mid V \text{ is a } *\text{-valuation overring of } D\}$. However, we do not have any information about the $\Delta^*$-valuation overrings of $D[X]$, and thus we do not know whether it is possible that $(\Delta^*)_a \leq (\lambda^*)_a$.

**Corollary 4.8.** Let $D$ be an integral domain with quotient field $K$. Then,

$$\lambda^{bd} \leq [b_D] \leq b_{D[X]} = (\lambda^{bd})_a = [b_D]_a \leq (\Delta^{bd})_a \leq \Delta^{bd}.$$  

Moreover, $b_{D[X]} = (\Delta^{bd})_a$ if and only if $D = K$.

**Proof.** By Remark 4.7(c), $b_{D[X]} = (d_{D[X]})_a = (\lambda^{d_D})_a = (\lambda^{bd})_a$, and by [2] Proposition 15, $b_{D[X]} = [b_D]_a$. In order to show that $[b_D] \leq b_{D[X]}$, let $E \in \mathcal{F}(D[X])$ and let $W$ be a valuation overring of $D[X]$. Then $V := W \cap K$ is a valuation overring of $D$, and hence $EW \supseteq EV[X] \supseteq E^{bd}[X]$. Therefore, $E^{bd}[X] \supseteq E^{bd}[X]$. If $D = K$, then by Remark 4.7(c), $d_{D[X]} = (\Delta^{bd})_a$, and hence $b_{D[X]} = (d_{D[X]})_a = ((\Delta^{bd})_a)^* = (\Delta^{bd})_a_{*a}$. If $D \neq K$, then $b_{D[X]} \leq (\Delta^{bd})_a$ (see Remark 4.7(b)). Since $(\Delta^{bd})_a \leq (\Delta^{bd})_a_{*a}$, it immediately follows that $b_{D[X]} \leq (\Delta^{bd})_a$. □

**Remark 4.9.** It can happen that $[b_D] \leq b_{D[X]}$. For instance, let $D$ be a Prüfer domain which is not a field. Then $[b_D] = [d_D] = d_{D[X]} \leq b_{D[X]}$.

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