The Density functional method for nuclear systems. 
History and perspectives

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Abstract. The earlier attempts to construct a density functional for nuclear systems are shortly reviewed. Some results obtained with the functional derived by Beiner, Mas and myself are then recalled, as well as the extension to excited states. The universality of the functional is discussed. A way to improve the efficiency of the functional is proposed, in which the parameters of the functional are allowed to vary according to the underlying shell structure of each nucleus.

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1. Foreword.

I should apologise to colleagues actually working in this field : I haven’t followed the recent developments, and thus references to relevant papers published in the last 10-15 years will be missing.

What will be presented here is merely the history of the earlier attempts to put the density functional method for nuclear systems on a valuable track, without any attempt to deliver a semi-exhaustive review on the subject. This paper is mostly based on personal experience, thus many references to similar or related works are missing. The basic motivation is to suggest a way to go beyond the present status of the density functional.

This paper is dedicated to my friend and colleague Marcel Beiner, who passed away some years ago after a long illness.

My first meetings with physicists from the Eastern countries goes back to the Summer school in the Tatra mountains in 1962 “Selected Topics in Nuclear Theory”. A. Sandulescu, who is honoured during this present meeting, was among the participants.

2. Introduction

Since the earlier days of nuclear physics, the problem of modelising the atomic nucleus as an ensemble of bound particles has retained the attention of theorists. The first attempt to calculate the binding energy per particle in the homogeneous medium (infinite nuclear matter) was achieved by Euler [1]. He used plane wave, Gaussian forces, and worked
up to second order in the perturbation theory. Details of his calculations are quoted in the book of G. Brown on the nuclear many-body problem [2].

Short after second world war, descriptions of finite systems started, first using the Thomas-Fermi approximation for the kinetic energy and the concept of local density. In this respect, we shall quote the work of P. Gombas, with two main papers entitled "Statistische Theorie der Atomkern" [3]. The basic idea is to count the number of available states in a box, or in a sphere, assuming plane wave as eigenfunctions. This leads immediately to an estimate of the density in function of the Fermi momentum $k_f$, taking into account the degeneracy due to the spin and isospin degrees of freedom:

$$\rho \propto k_f^3.$$  \hspace{1cm} (1)

By using this relation, the total energy of a nucleus can be calculated by using plane wave in first order, as a function of $k_f$, then transformed into a function of $\rho(r)$ and integrated over the nuclear volume. Various techniques can be applied to get the ground state energy by minimisation. Gombas used Gaussians to parametrised $\rho(r)$ and looked for the minimum in the parameter space.

Several authors have followed this method to either to directly estimate binding energies of finite nuclei, or to derive the proton and neutron numbers dependence of a mass formula. Let us quote here the earlier papers of Skyrme [4], Seyler and Blanchard [5], Bodmer [6], Hara [7] and the work by Kumar, Lecouteur and Roy [8].

However, the first person to formulate the total binding energy of a nucleus under the form of a density functional was Wilets. His model appeared in an article with Berg studying the nuclear surface [9]. Their functional reads

$$E[\rho] = \int \left[ k_{\text{eff}} \rho^{5/3} + \zeta(\rho) \frac{(\nabla \rho(r))^2}{\rho(r)} + \epsilon(\rho) \right] dv$$ \hspace{1cm} (2)

The first term is the kinetic energy at the Thomas-Fermi approximation. The second contribution contains the Weizsäcker correction to the Thomas-Fermi expression and surface effect, and the third is the potential energy in nuclear matter. The Coulomb energy is neglected; proton and neutron densities are assumed to be proportional. The variational principle was applied together with the constraint on the particle number $A$

$$\int \rho dv = A.$$ \hspace{1cm} (3)

Few cases were solved in the one-dimensional space, which corresponds to semi infinite nuclear matter. The effect of Coulomb energy and non proportional proton and neutron densities were considered later by Wilets.

3. The La Jolla time.

I became involved in the density functional industry while working as a post-doc by Keith Brueckner at La Jolla. He had recently achieved nuclear matter calculations for various ratio of proton to neutron numbers with Coon and Dabrowsky [10]. At the same time, Walter Kohn was also professor at La Jolla. He had established his theorem with
Hohenberg proving that the ground state of any fermion systems could be described by a functional of the one-body density \[11\]. The theorem proves the existence of the functional but does not say how to derive it. Brueckner gave the nuclear matter results to S. Jorna, J.-R. Buchler and myself with the aim of constructing a density functional based on the Thomas-Fermi approximation to describe finite nuclei.

Our first goal was to obtain the nuclear matter potential energy as a function of \( \rho \) from the numbers provided us by Brueckner. On very simple and natural arguments, based on dimensionality, we found that the most appropriate expression was an expansion in the Fermi momentum \( k_f \). It means an expansion in \( \rho^{1/3}(r) \), which has the dimension of a length and corresponds to the local density approximation. The potential energy per particle took the following form

\[
\frac{\epsilon}{A} = a_1 [1 + s_1 \alpha^2(r)] \rho(r) + a_2 [1 + s_2 \alpha^2(r)] \rho^{4/3} + a_3 [1 + s_3 \alpha^2(r)] \rho^{5/3}(r).
\]

The neutron-proton asymmetry was efficiently taken into account by the factor \( \alpha^2(r) \) defined by

\[
\alpha(r) = \frac{\rho_n(r) - \rho_p(r)}{\rho(r)}; \quad \rho(r) = \rho_n(r) + \rho_p(r),
\]

where \( \rho_n(r) \) and \( \rho_p(r) \) are the neutron and proton densities, respectively. The fit to the Brueckner, Coon and Dabrowsky results was quite good, as it can be judged from the original drawing displayed in fig. 1.

Surface effects were included very much in the same way as proposed by Wilets. They originate from the first order corrections to the Thomas-Fermi approximation due to Weizsäcker \[12\], as well as the finite range of nuclear forces. In both cases, next order would involve the density Laplacians. Such contributions are more delicate to handle numerically, giving rise to fourth order differential equations and generating instabilities. Moreover the convergence of the series expansion is not guaranteed. Consequently, it is safer to take the first order terms as effective components of the functional.

Finally, the constructed functional took the following form

\[
E[\rho(r)] = a_k \rho^{5/3}(r) + \eta_p \left| \nabla \rho(r) \right|^2 + \eta_n |\nabla \rho(r)|^2 + \epsilon(\rho(r)) + E_{\text{coul}}.
\]

The last component is the Coulomb energy, which contains the direct term and an approximate exchange term. The field variables can be \( \rho(r) \) and \( \alpha(r) \) or \( \rho_n(r) \) and \( \rho_p(r) \). The total energy is obtained by integration over the whole space

\[
E_{\text{tot}} = \int E[\rho_n(r), \rho_p(r)] d^3r
\]

with the constraints on the particle numbers

\[
\int \rho_n(r) d^3r = N; \quad \int \rho_p(r) d^3r = Z.
\]

Applying the Euler-Lagrange minimisation procedure with respect to \( \rho_n(r) \) and \( \rho_p(r) \) generates a system of two coupled second order differential equations. Simultaneously, the Poisson equation for the charge density is solved to get the electric field seen by the protons.
As a very first attempt, these equations were solved for proportional proton and neutron densities, taking $\alpha = 0$ and assuming spherical symmetry. The functional is left with the single free parameter $\eta$ [13]. The binding energies of the $N = Z$ nuclei were reproduced correctly up to $A \propto 100$, with the main characteristics: the increase of the binding energies with $A$ up to a saturation point followed by a slow decrease. The original drawing is displayed in fig 2.

Separated proton and neutron densities were considered in a subsequent work [14]. Departure from spherical symmetry was achieved with R. Clark [15]. Ellipsoidal shapes were found to lower the ground state energy with respect to a spherical shape for $Z \geq 106$. The mean field experienced by a single neutron or a single proton was obtained by taking the first derivative of the functional with respect to $\rho_n(r)$ or $\rho_p(r)$, respectively. Solving the Schrödinger equation with this mean field as potential yields the single particle energies. This was the subject of the Lin thesis, and applied to the isotope shift of nuclear charge distributions, in particular to the $^{40}\text{Ca}$ and $^{48}\text{Ca}$ shift [16].

Finally, the density functional were also used to calculate the interaction potential between to colliding nuclei in the sudden approximation:

$$V(\vec{r}_1 - \vec{r}_2) = E[\rho_1 + \rho_2] - E[\rho_1] - E[\rho_2].$$

(9)

When confronted to experiments, the calculated binding energies and charge radii were reproduced at a semi-quantitative level. The discrepancies arise from three main points: the properties of nuclear matter entering through $\epsilon(\rho)$, the shell effects and the fact that, at very low density, the usual model of nuclear matter has to be corrected for clustering of nucleons. The easiest part is to play with the parameters of $\epsilon(\rho)$. They can be fixed by imposing the energy per particle $E_0(\rho_0)/A$ at the equilibrium density $\rho_0$ for $\alpha = 0$ and the “compressibility”, namely the value of the $\epsilon$ second derivative at $\rho_0$. The asymmetry coefficient has also to be considered. Such adjustments result in a satisfactory global description of ground state properties [17]. Further improvements require the introduction of shell effects.

4. Shell structure.

The shell structure observed in nuclei arises from the Pauli principle. It has important implication at the level of the one-body density. In fact, the Thomas-Fermi approach yields averaged nuclear densities and cannot reproduce the density fluctuations due to the Pauli correlations. The simplest way to introduce the shell structure is to expand the one-body density on a basis of orthonormal functions:

$$\rho_q(r) = \sum_j |\psi_{q,j}(r)|^2.$$  

(10)

Here $q$ is the isospin state and $j$ the set of quantum numbers attached to each single particle state. This procedure is known as the Kohn-Sham approximation [18]. The total energy is thus a functional of the single particle wave functions. Applying the
Euler-Lagrange principle leads to a set of second order differential equation similar to the Hartree-Fock system.

Two supplementary contributions have to be added in order to build a functional able to come close to the observed binding energies and charge densities. The first one is the pairing effect, the second is a spin-orbit component. In our work, the pairing was treated at the BCS approximation, matrix elements being calculated by using the Hamada-Johnson potential [19]. We greatly benefited from the careful study that Marcel Beiner had achieved in this topics for his PhD thesis. The pairing correlations have the important property of giving the probability occupation \( v_j^2 \leq 1 \) of the \( \{ j \} \) state.

The origin of the spin-orbit component is still debated, though its presence is well demonstrated experimentally. Our choice was more or less dictated by various models. We verified that the precise radial shape of this term was not decisive. The adopted expression was simple but sufficient to get the correct energy splitting among spin-orbit partners.

Finally the total energy was given by

\[
E_{\text{tot}} = \int \left[ \frac{\hbar^2}{2m} \sum_{q,j} |\vec{\nabla}\psi_{q,j}(\vec{r})|^2 + \epsilon(\rho(\vec{r})) + \eta_0|\vec{\nabla}\rho(\vec{r})|^2 + \eta_1|\vec{\nabla}(\rho_n(\vec{r}) - \rho_p(\vec{r}))|^2 \\
+ \epsilon_{\text{coul}} - \frac{\lambda}{2}(\rho_n^+ + \rho_p^+)|\vec{\nabla} \cdot \vec{J}| d^3r + E_{\text{pairing}} \right] .
\]

(11)

Here, the current \( \vec{J} \) is defined by

\[
\vec{J} = -i \sum_{q,j} \psi_{q,j}^* (\vec{\sigma} \wedge \vec{\nabla}) \psi_{q,j} ,
\]

(12)

with \( \vec{\sigma} \) being the Pauli matrices. This functional contains 12 free parameters. They were adjusted on the properties of 4 nuclei, \( ^{40}\text{Ca}, \; ^{48}\text{Ca}, \; ^{56}\text{Ni} \) and \( ^{208}\text{Pb} \), reproducing the binding energies, charge radii and separation energies. As far as the symmetry energy is concerned, the functional was fitting the infinite neutron matter of Buchler and Ingber [20].

The functional has been applied to a large ensemble of nuclei [21]. A first set of about 250 nuclei situated along the line of stability has been extended to proton and neutron rich nuclei, up to the drip line for light element [22]. Binding energies have been published in the 1975 Nuclear Data table [23].

The differences between calculated and experimental binding energies show a structure. The agreement is excellent for magic nuclei. The discrepancies reach about 1-3 Mev for other spherical nuclei. Being maximum in the middle of the major shells they climb up to 20 Mev for deformed nuclei.

The situation is similar with the charge radii, the discrepancies exhibiting the same behaviour with respect to the shell structure. Besides the charge radii, the charge densities were also well reproduced. The sample displayed in fig 3 underlines the remarkable agreement in the nuclear surface [21]. However, the shell effects affecting the interior part is only qualitatively reproduced. This is an important point to stress: according to the Kohn theorem, the correct functional must produce the correct binding...
energy for the correct ground state density. Thus, the fit of the charge distributions is of key importance. The discrepancies are a clear indication that the way the shell effects have been incorporated is not sufficient. We conjecture that other correlations have be included besides the pairing contribution.

5. Predictability

Reproducing measured quantities is a first step in establishing the validity of a model. However, its predictive power is more interesting. The high quality of the fits to magic and specially doubly magic nuclei was a confidence criterion to predict properties of similar nuclei. This was the case with $^{132}\text{Sn}$, for which a binding energy of 1102.5 Mev and a charge radius of 4.71 fm, were predicted, and later confirmed experimentally: 1102.8 Mev and 4.709 fm, respectively [24, 25].

The position of the neutron drip line was well reproduced for light nuclei. It allows us to predict $^{26}\text{O}$ to be unbound, which was confirmed. Furthermore, because the model solved Hatree-Fock BCS equations, it gives a natural criterion to fix magic numbers. It relies on the fact that for magic numbers the occupation probability $v_{q,j}$ of a $\{q,j\}$ state is 0 or 1. Consequently, a magic N or Z corresponds to

$$\sigma_q = \sum_j v_{q,j} u_{q,j} = 0$$

(13)

with $u_{q,j} = 1 - v_{q,j}$ and $v_{q,j}^2 + u_{q,j}^2 = 1$. In other words, the magic character of N or Z is defined as the stability of the energy gap between occupied and empty shells with respect to the pairing correlations. Conversely, the maximum of $\sigma_q$ corresponds to the middle of major shells, a place were deformed nuclei occur. Two examples are displayed in fig. 4.

This criterion was applied to neutron rich nuclei. $N = 16$ was predicted to be magic, which was confirmed later on. The next neutron magic number, close to the neutron drip line is $N = 34$. On the side of the superheavy elements, $N = 182$ and $N = 228$ are found to be magic. Less attention has been paid to proton rich nuclei, for which the magic character of proton number is lost. For stable or almost stable nuclei, we could not find any magic proton number beyond $Z = 82$. This situation is due to the Coulomb force, which increases with $Z^2$ while the strong interaction increases with $A$. The Coulomb force being repulsive, there is a necessity for the proton orbitals to be confined over the nuclear volume to exist as bound or discrete states. In other words the proton orbitals are sensitive to the Coulomb and centrifugal barriers. This phenomenon increases the density of proton states close to the Fermi surface and thus diminishes the gap between orbitals.

Note that our calculations assume spherical symmetry, which is not necessarily a safe ansatz beyond approximately $Z = 106$. However, as long as the neutrons are spherically distributed, this hypothesis is a sound one. This is the case for $N= 182$ and 228. A point which merits to be further investigated.
To the extend that the symmetry energy is well taken into account, proton and neutron properties are treated at the same level. This is reflected in the binding energies as well as in the one- and two-particle separation energies. Both quantities are actually reproduced in a similar way. In such a case, the neutron densities are predicted with the same degree of accuracy as the charge densities.

Another firm prediction: bubble nuclei, namely nuclei with a large particle depression in the centre, do not exist as ground states.

6. Improving the functional.

For years, the density functional method was considered less reliable than the Hartree-Fock-Bogoliubov approach using Skyrme or Gogny interactions. The pioneering work in this field is due Brink and Vautherin [26]. They were followed by many others. Attempts to use forces as realistic as possible were achieved by Negele [27], as well as Campi and Sprung [28].

The two approaches have some part in common, since Skyrme of Gogny forces are effective two-body density dependent interactions based on nuclear matter properties [29]. The great advantage of the functional is that it does not postulate a specific 2-body effective force. The different components can be derived independently and thus the functional is submitted to less constraints. This aspect is particularly efficient for the spin-orbit contribution and the pairing matrix elements.

The functional is expected to be universal, in other words it must describe the ensemble of fermion systems belonging to the same class. In particular, this universality is expressed by the form of $\epsilon(\rho)$, that is its expansion in powers of $k_f$. The dimensionality argument that was advocated from the nuclear matter calculations, get recently support from effective field theory [30].

Consequently, the universality of the functional is to be found in the potential energy form $\epsilon(\rho)$ and the gradient terms besides contributions which have a conventional character.

The parameters are determined from the properties of a specific class of nuclei. Here use was made of double magic nuclei, and their one-body densities were expanded on the self-consistent basis obtained from the Hartree-Fock BCS calculations. This appeals to the two following remarks.

1. If indeed the charge densities of the doubly magic nuclei are reproduced accurately, it means we have touched the “true functional”. As we have seen from the $^{208}$Pb result, this goal is not totally achieved.

2. The parameters determined from a given ensemble of nuclei are not necessarily valid for another ensemble. The parameter values reflect the underlying complicated many-body structure. This last is sensitive to various types of correlations, which vary with the density of states near the Fermi surface as exemplified by the pairing correlations.

Consequently, the parameters should be allowed to vary with the nucleus according
to its substructures. Obviously, the variation should be smooth and under control. An ad hoc adjustment case by case would be unacceptable. The behaviour of the discrepancies recorded for the binding energies suggests to take advantage of the amount of pairing correlations through the $\sigma_q$ to drive these changes.

To be explicit, the parameters we used were adjusted to magic nuclei. A similar procedure for deformed nuclei would lead to a different set of parameters. We conjecture that both sets could be linked by simple functions of the $\sigma_q$. To examples bring support for this conjecture.

6.1. A perturbative approach to the binding energies.

A perturbative estimate of the four quasi-particle contribution was achieved with Behrman [31]. Since the variational principle leads to Hartree-Fock-BCS equations, the ground state is stable against one-particle one-hole excitations. We therefore assume the dominant correction to come from two-phonon configurations. We take each phonon to have spin and parity $2^+$, and couple them together to give zero angular momentum. For the two-body residual interaction, we use a simple ad hoc parametrisation of the two-body matrix elements. Considering $A, B, a, b$ single-particle state in the spherical representation and denoting $\bar{a}, \bar{b}$ the time reversal states of $a, b$, we ended up with a correction of the form

$$\Delta E = \sum_{A,B,a,b} \frac{|<BCS|w|BCS >_{A,B,a,b}|^2}{E_A + E_B + E_a + E_b} \cdot \frac{(1 - \delta_{Aa})(1 - \delta_{Ab})(1 - \delta_{Ba})(1 - \delta_{Bb})}{(1 - \delta_{Aa})(1 - \delta_{Ab})(1 - \delta_{Ba})(1 - \delta_{Bb})}.$$ (14)

The matrix element was given by

$$<BCS|W|BCS >_{A,B,a,b} = -G\sqrt{2J + 1}[u_A u_B v_a v_b][u_a u_b v_A v_B] \{2\delta_{t_zA} + t_zB, t_zA + t_zB \}
- \eta[1 + (-)^{j_A + j_B + jA + j_B} \delta_{t_zA + t_zB, t_zA + t_zB}]
+ \eta[(-)^{j_A + j_B - J} + (-)^{j_a + j_b - J} \delta_{t_zA + t_zB, t_zA + t_zB}]$$ (15)

Here,

$$\eta = [(1 + \delta_{AB})(1 + \delta_{ab})]^{-1},$$ (16)

and $G$ is an adjustable constant. The Kronecker symbol $\delta$ of $t_z$ ensures charge conservation with $t_z = \pm 1/2$ for protons and neutrons, respectively. The $v_j$ are the occupation probability and the $E_j$ the single particle energies with respect to the BCS ground state. The summation runs over all single particle levels which conform to the constraints on angular momentum, parity and charge conservation which we have imposed. In practice the calculation was restricted to single particle state with $v^2_j$ or $u^2_j$ between 0.99 and 0.01. We have verified that more than 90 % of the contribution comes from configurations within this truncated space.

Results are displayed in fig 5 for $G = 5.144$ MeV. It shows the clear improvement brought by taking into account the two-phonon correlations on the binding energies.
6.2. A mass formula with coefficients depending on valence nucleons.

Let us consider the Weizsäcker-Bethe mass formula and allow its parameters to depend on the number of valence nucleons [32]. Assume the magic numbers $N_{m}$ to be known, we choose the simplest probabilistic way of counting the valence nucleons. It is nothing but the product of the number of particle pairs by the number of hole pairs between two magic numbers $N_{m1,q} < N_{m2,q}$ [33]. With

$$n_{q} = N_{q} - N_{m1,q} \quad (17)$$

the number of particles above $N_{m1,q}$, we define

$$f_{q} = n_{q} \frac{\Delta_{q} - n_{q}}{\Delta_{q}^{2}} \quad ; \quad \Delta_{q} = N_{m2,q} - N_{m1,q} \quad (18)$$

The function $f_{q}$ is zero at each end of a major shell and maximum in the middle. It more or less simulates the $\sigma_{q}$ of the Hartree-Fock-BCS calculations, ignoring subshell structures. The mass formula reads

$$E(N, Z) = -a_{v}(f_{p}, f_{n})A + a_{r}\frac{(N-Z)^{2}}{A} + a_{s}R^{2}(f_{p}, f_{n}) + a_{c}\frac{Z^{2}}{R(f_{p}, f_{n})} \quad (19)$$

For the radius, we found the following form to yield a very good fit to experimental charge radii (to 1 % or better)

$$R(f_{p}, f_{n}) = r_{0}(1 + 0.0069[f_{p} + f_{n} + f_{p}f_{n}]) \quad (20)$$

with

$$r_{0} = 1.291[0.9428 - 0.3382\frac{(N - z)^{2}}{A^{2}} + 33.55A^{-2}]a^{1/3} \quad (21)$$

The four parameters of $r_{0}$ have been fitted on the charge radii of $^{16}O$, $^{40}Ca$, $^{48}Ca$, $^{132}Sn$ and $^{208}Pb$. The binding energies of this set of magic nuclei has been used to fix the four parameters of the mass formula. Proton and neutron radii were supposed to be proportional, which is a crude approximation valid only for stable nuclei.

The following parameters have been determined :

$$a_{v} = 15.5935 \text{ MeV}; a_{s} = 11.29 \text{ MeVfm}^{-2} \quad (22)$$

$$a_{r} = 23.59 \text{ MeV} \quad ; \quad a_{c} = 0.9428 \text{ MeVfm}.$$  

The mass formula has been applied to stable nuclei with $50 \leq Z \leq 82$. In this case, it fits the binding energies of $^{132}Sn$ and $^{208}Pb$. If the $f_{q}$ are set to zero, the binding energies of other nuclei are overestimated, up to 15 MeV in the middle of the major shell. The discrepancies are reduced by a factor 2 or more if the $f_{q}$ are introduced in the radii $R$. Another factor of 2 is gained if a correction of the form

$$E_{corr} = [1. + \frac{1.1}{1. + f_{p} + f_{n} + f_{p}f_{n}}](f_{p} + f_{n}) \text{ MeV} \quad (23)$$

is added to the volume term. This correction is independent on the number of particles except through the $f_{q}$ factors. It thus becomes negligible as $A \rightarrow \infty$. The difference between experimental and theoretical binding energies are displayed in fig 6. Most of
the results are situated within a band 3.0 MeV wide, which is remarkable, considering
the crudeness of the occupation number approximation which has been used, and the
simplicity of the model. It gives us a hint that a more sophisticated method based on
the $\sigma_q$ determined self-consistently should be quite efficient.

The exercise shows also that one of the major improvement came from a better
description of the radii. It suggests that, at the level of $\epsilon(\rho)$, the most important change
with $\sigma_q$ would be to lower the equilibrium density with increasing $\sigma_q$.

7. Excited states.

The density functional describes essentially the ground states. This is the gist of the
Kohn theorem. Thus the question of the excited states generated by a functional sounds
irrelevant. However, to the extent that the functional represent a semi-classical field
theory of a many-particle ground state, the Landau theory of quasi-particle excitation
should apply, at least in first approximation.

In this framework, the quasi-particle interaction is defined as the second derivative
of the functional with respect to the basic field variables, i.e. the proton and neutron
densities in each spin state. For the particle-hole interaction, it reads [34]

$$W_{ph}(\vec{r}_1 - \vec{r}_2) = \delta(\vec{r}_1 - \vec{r}_2) \sum_{s,t,s',t'} \frac{1}{6} \left( 1 + (-1)^{s-s'} \vec{\sigma}_1 \cdot \vec{\sigma}_2 \right) \left( 1 + (-1)^{t-t'} \vec{\tau}_1 \cdot \vec{\tau}_2 \right) \left[ \frac{\partial^2 \epsilon(\rho,t,\rho_{s',t'})}{\partial \rho_{st} \partial \rho_{s't'}}, \right].$$

Attempts to calculate selected excited levels have shown qualitative agreements with
experimental results, specially for the collective states. Use was made of an approximate
RPA technique. The 2$^+$ and 3$^-$ levels of spherical nuclei were found between 0.5 and
1.0 MeV too high. Beyond excitation energies, transition densities were also calculated,
and transition probabilities were in good agreement with experiments [35].

For instance, for the lowest 3$^-$ state of $^{208}$Pb the excitation energy was found to be
3.2 MeV compared to the experimental value of 2.6 MeV. The corresponding transition
density was used to calculate the proton-$^{208}$Pb differential cross section with excitation
of this 3$^-$. The calculations were made at 1 Gev incident energy by means of the Glauber
model and found in good agreement with experimental data [36].

Although the description of excited states with the density functional method
cannot compete with more specialised approaches designed to fit nuclear spectra, it
should give a way to test functionals beyond ground state properties. For instance, we
have verified that the functionals used with Beiner and Mas yield ground state stable
against spin excitations. This is not automatically the case, and some Skyrme forces
were known to have this deficiency.

On the other hand, if a quasi particle description of low lying quadrupole and
octupole states is not able to reproduce the bulk experimental behaviours, it suggests
that some kind of correlations should be added explicitly in the functional. This
point has not retained sufficient attention till now, as far as I know. It merits more investigations.

8. Conclusions.

The density functional method has proven to be a sound approach to the nuclear many-body problem. The universality of the functional is contained in the functional expression through the $\rho^{1/3}$ expansion of the potential energy and the gradient terms. The effective field theory has brought support to the potential energy expansion in $\rho^{1/3}$, obtained first from nuclear matter calculations. The coefficients of the successive terms are not accurately predicted by a fundamental theory. They are fitted on a sample of selected nuclei. Thus they are supposed to vary with nuclei according to criterions based on the concept of magic nucleons numbers, valence nucleons, the density of single particle levels near the Fermi surface and the lowest excitation energies. We conjecture that such criterions are well represented by the particle number dispersions of the BCS states.

Finally, the universality of the functional should incite its application to other systems of particles in strong interaction, like the hypernuclei.

Playing with the common means of access to information, asking for papers concerning the nuclear applications of the density functional method produces an enormous amount of references. There is no point here to propose a list of relevant articles. However, I would like to mention the works of the Bulgarian group with Antonov, Petkov and Stoitsov. They proposed a original approach to the problem in the momentum space [37].

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