The quintessence field as a perfect cosmic fluid of constant pressure

Wenzhong Liu, Jun Ouyang and Huan-Xiong Yang

Interdisciplinary Center for Theoretical Study, University of Science and Technology of China, Hefei, 200026, P. R. China
Kavli Institute for Theoretical Physics China, CAS, Beijing 100190, China

E-mail: wzhliu@mail.ustc.edu.cn, yangjun@mail.ustc.edu.cn, hyang@ustc.edu.cn

Abstract. We study the cosmology of a quintessence scalar field which is equivalent to a non-barotropic perfect fluid of constant pressure. The coincidence problem is alleviated by such a quintessence equation-of-state that interpolates between plateau of zero at large redshifts and plateau of minus one as the redshift approaches to zero. The quintessence field is neither a unified dark matter nor a mixture of cosmological constant and cold dark matter, relying on the facts that the quintessence density contrasts of sub-horizon modes would undergo a period of late-time decline and the squared sound speeds of quintessence perturbations do not vanish. What a role does the quintessence play is dynamic dark energy, its clustering could remarkably reduce the growth rate of the density perturbations of non-relativistic matters.

PACS numbers: 98.80.Jk, 98.80.Cq
1. Introduction

Recent cosmic observations, including Type Ia Supernovae [11, 12], Large Scale Structure (LSS) [3, 4, 5, 6, 7, 8] and Cosmic Microwave Background (CMB) [9, 10, 11, 12] have independently indicated that the evolution of the universe is currently dominated by a homogeneously distributed cosmic fluid with negative pressure, the so-called dark energy. The dark energy fills the universe making up of order 74% of its energy budget. The remaining energy fraction in the present universe is about 22% occupied by the pressureless cold dark matter and 4% occupied by baryons. Despite many years of research and much progress, the nature and the origin of dark energy does still remain as an open issue.

Phenomenologically, the best candidate of dark energy that fits perfectly the observation data is the so-called cosmological constant (CC) Λ, introduced at first by Einstein in his gravitational field equations as a Lagrange multiplier ensuring the constancy of the 4-volume of the universe [13, 14]. However, CC explanation of dark energy encounters some fundamental obstacles in physics, in particular, the fine-tuning problem and the coincidence problem [15]. From the particle physics perspective, CC should be interpreted as the density of vacuum energy. The physical CC should contain quantum corrections from the zero-point energies of matter fields, which is close to Planck density $M_P^4$ ($M_P = 1/\sqrt{8\pi G}$ is the reduced Planck mass) in magnitude. What the fine-tuning problem has to face is the great discrepancy between the observed value of CC and its quantum correction, the former is only $10^{-123}$ times in magnitude as the latter [16, 17, 18, 19, 20, 21]. Even if this fine-tuning problem could be evaded, the coincidence problem as to why both energy densities of the observed dark energy and the dark matter are of the same order at present epoch remains, due to the fact that CC is time independent and non-dynamical.

The resolution of fine-tuning problem has probably to wait for the advent of a satisfied quantum theory of gravity in 4-dimensional spacetime. As a temporary expedient in cosmology community, the physical CC is assumed to vanish exactly [22] † and it is conjectured that the dark energy which drives the late time cosmic accelerated expansion is dynamical. The observations actually say little about the evolution of the equation of state (EoS) of dark energy. Researchers have proposed lots of alternative models (see Reviews [23, 15, 24] and references therein) to explain the late time cosmic acceleration and alleviate the corresponding coincidence problem. The well-known Chaplygin gas model [25] and its generalizations, e.g., the models proposed in Refs. [26, 27, 28] are among these approaches.

The Chaplygin gas and its generalizations are of the so-called barotropic fluids whose pressure depends only upon the energy density [15]. What distinguishes them from other dynamic dark energy models is that, at background level, they get rid of

† This assumption does actually prohibit the interpretation of CC as the energy density of vacuum fields. CC does not fluctuate, but the vacuum fields fluctuate and their fluctuations couple universally to gravitation. We thank B. Deiss for pointing it to us.
the coincidence problem by unifying the dark energy and dark matter into a single dark substance. The gas behaves as a pressureless matter for very large redshift, however, it becomes a CC as the redshift is small. Due to the non-vanishing squared speed of sound, unfortunately, the Chaplygin gas and its generalizations would have fatal flaws for explaining tiny but stable CMB anisotropies [29, 30]. It follows from Sandvik’s analysis in Ref. [30] that the unique reliable Chaplygin gas like model is the perfect fluid of constant pressure. That the pressure of a fluid is kept invariant makes the squared sound speed vanish identically, resulting in the disappearance of the embarrassed oscillation-instability issue in density perturbations. The density contrasts of constant-pressure barotropic fluid decay monotonously during evolution, implying that the fluid does not unify the dark energy and cold dark matter into a single dark substance. It is merely a mixture of CC and a non-relativistic matter, and is equivalent to the standard Λ-CDM model to all orders in perturbation theory. Consequently, the coincidence problem remains.

The fluid models provide at most some effective descriptions of the dynamic dark energy. In view of the superstring/M-theory, a developing but promising candidate of quantum gravity, the realization of dark energy with some scalar fields is a more fundamental description than the perfect fluid models. The scalar fields are ubiquitous in superstring/M-theory, they arise naturally as dilaton or the compactification moduli. A scalar field with a canonical kinetic energy is dubbed quintessence [31], which is among the most studied candidates for dynamic dark energy. The quintessence typically involves a single scalar field with a particular self-interaction potential, allowing the vacuum energy to become dominant only recently. The quintessence potential has generally to be fine-tuned ad hoc to solve the coincidence problem. At background level, a quintessence field is generally equivalent to a perfect fluid whose EoS evolves from plus one at early times to approximately minus one at present epoch, with some unphysical ingredients introduced into the description of the evolution of Universe.

In this paper, we intend to study the cosmology of the quintessence counterpart of a perfect fluid of constant pressure. Different from Chaplygin gas like models, a quintessence field can not be viewed as a barotropic fluid in general [32]. Whether the quintessence field of constant pressure is equivalent to Λ-CDM model or provides a unified description to dark energy and dark matter are open issues. Our main motivation in this paper is to establish a dynamic dark energy model with a quintessence scalar field whose EoS evolves from zero at early times to nearly minus one at present and potential is not required to be fine-tuned severely, so that the model is brought back to life in search of solutions of the coincidence problem. We also try to examine the possibility to build a unified description of dark energy and dark matter in such a scenario. It is shown that the constant pressure requirement enables the quintessence EoS interpolates between two plateaus, one corresponds to cold dark matter in the large redshift epoch, another to CC as the redshift comes near zero. The quintessence potential comes completely from the constant pressure assumption. It is factorized into the product of the squared Hubble rate and a dimensionless quantity, dubbed the
The quintessence field as a perfect cosmic fluid of constant pressure

reduced quintessence potential, which has also two plateaus so that the fine-tuning of the potential is unnecessary. Our investigation in the linear perturbation theory shows that the squared sound speeds of quintessence perturbations oscillate around unity, and the corresponding density contrasts decay monotonously at late times during their evolution. We conclude that the quintessence field of constant pressure could only play the role of a dynamic dark energy, which is neither a unified dark matter nor a mixture of CC and some non-relativistic matter.

Throughout the paper we adopt the Planck units \( c = \hbar = M_P = 1 \).

2. Model

We begin with the assumption that at the background level our universe is described by a flat Robertson-Walker metric

\[
d s^2 = -dt^2 + a^2(t)d\vec{x}^2
\]

in which fills a mixture of a quintessence scalar \( \phi(t) \) and a perfect fluid. The quintessence field is described by Lagrangian

\[
\mathcal{L} = -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi)
\]

but the extra fluid by its pressure \( P_f \) and energy density \( \rho_f \). There are several well-known candidates for such a cosmic fluid, e.g., radiation, baryonic dust, the cold dark matter (CDM), the cosmological constant \( \Lambda \), or mixture of them. As usual, we use \( \omega_f \) to denote the EoS of this extra fluid, defined by \( \omega_f = P_f/\rho_f \). The quintessence EoS is similarly defined as \( \omega_\phi = P_\phi/\rho_\phi \), where \( P_\phi = \frac{1}{2} \dot{\phi}^2 - V(\phi) \) and \( \rho_\phi = \frac{1}{2} \dot{\phi}^2 + V(\phi) \) are respectively the pressure and energy density when the quintessence field is also viewed as a perfect fluid. \( \omega_\phi \approx 1 \) if kinetic term dominates while \( \omega_\phi \approx -1 \) if potential term dominates. If both kinetic and potential terms are almost equivalently important, \( \omega_\phi \approx 0 \). It is also remarkable for the EoS of a quintessence field not to cross over the cosmological constant boundary \( \omega_{CC} = -1 \) [33, 13, 24].

In the flat Robertson-Walker background, Einstein’s gravitational field equations become the so-called Friedmann equations:

\[
3H^2 = \frac{1}{2} \dot{\phi}^2 + V(\phi) + \rho_f
\]

\[
\dot{H} = -\frac{1}{2} \dot{\phi}^2 - \frac{1}{2} (1 + \omega_f) \rho_f
\]

where \( H = \dot{a}/a \) is the Hubble’s expansion rate of the universe for which we denote its present value as \( H_0 \). The extra fluid is assumed not to interact with the quintessence scalar field, \( \dot{\rho}_f + 3H(1 + \omega_f)\rho_f = 0 \). Consequently, the evolution of the scalar field is subject to the following Klein-Gordon equation,

\[
\ddot{\phi} + 3H \dot{\phi} + V_{,\phi} = 0
\]

which is very the energy conservation equation of the scalar field at the background level. To be convenient, we will use the so-called e-folding number \( N = \ln a \) as the time
variable from now on, $-\infty < N < +\infty$, and let $N = 0$ represent the present epoch. A prime will represent a derivative with respect to $N$, unless otherwise specified. Because $t \to N = \ln a$ is merely a time transformation, both coordinates $(t, \vec{x})$ and $(N, \vec{x})$ are comoving ones [34]. In terms of $N$, Eqs. (3), (4) and (5) are recast as:

$$3H^2 = V(\phi) + \frac{1}{2}H^2\phi'^2 + \rho_f$$

(6)

$$H^2\phi'' = \left[ V(\phi) + \frac{1}{2}(1 - \omega_f)\rho_f \right] \phi' + V_{,\phi}(\phi) = 0$$

(8)

respectively.

We want to study a quintessence model in this paper whose EoS interpolates between the pressureless matter dominant epoch at large red shifts ($z \gtrsim 1$§ and the dark energy dominant epoch at $z \approx 0$. The merits of such a model are as follows. Firstly, there are few unphysical ingredients of EoS different from $\omega = 1/3$, 0 and $\omega = -1$ involved in the alleviation of the coincidence problem by a quintessence field. Besides, the model provides the possibility to unify the dark energy and dark matter into a single dark substance, the so-called unified dark matter or quartessence [35]. Here we introduce a dimensionless quantity $U(\phi)$, referred to as the reduced quintessence potential from now on, so that $V(\phi)$ is factorized,

$$V(\phi) = 3H^2U(\phi)$$

(9)

Then we can translate Klein-Gordon equation (8) into:

$$H^2\left[ \frac{1}{2} \frac{d}{dN} + 3U + \frac{3}{2}(1 - \omega_f)\Omega_f \right] \phi'^2 - 6(1 - U - \Omega_f) = 0$$

(10)

where $\Omega_f = \rho_f/3H^2$ is the reduced energy density of the extra fluid. The first Friedman equation (6) can equivalently be expressed as,

$$\frac{1}{2}\phi'^2 = 3(1 - U - \Omega_f)$$

(11)

In terms of Eq.(11), Klein-Gordon equation Eq.(10) turns out to be an identity. This is indeed the case. As we have explained, among the two Friedmann equations (6), (7) and the Klein-Gordon equation (8), only two of them are independent.

What looks fascinating here is that Eq.(11) is similar to the characteristic equation of an instanton in $(1 + 1)$ Euclidean space if the potential $U(\phi)$ has more than one vacua [36, 37]. We want to know if the cosmological observations could put some restrictions on the possible forms of $U(\phi)$. With $U(\phi)$, the energy density and pressure of the quintessence field are rewritten as,

$$P_\phi = H^2 \left( \frac{1}{2}\phi'^2 - 3U \right)$$

(12)

$$\rho_\phi = H^2 \left( \frac{1}{2}\phi'^2 + 3U \right)$$

(13)

§ The red-shift parameter $z$ is defined as $z = e^{-N} - 1$. Therefore, $z \gtrsim 1$ corresponds to $N \lesssim -0.693$. 

The quintessence field as a perfect cosmic fluid of constant pressure

5
We can further simplify Eqs. (12) and (13) into:

\[ \rho_\phi = 3H^2(1 - \Omega_f) \]  
\[ P_\phi = 3H^2(1 - 2U - \Omega_f) \]  

by exploiting Eq. (11). The cosmology of the model at the background level is determined completely by quintessence EoS \( \omega_\phi \) and the effective EoS parameter \( \omega_{eff} = (P_\phi + P_f)/(\rho_\phi + \rho_f) \) of the mixture fluid. Obviously,

\[ \omega_\phi = 1 - \frac{2U}{1 - \Omega_f} \]  

and

\[ \omega_{eff} = 1 - 2U - (1 - \omega_f)\Omega_f \]  

Provided \( \omega_{eff}|_{N=0} < -1/3 \), the late-time accelerated expansion of our universe occurs. Therefore, the magnitude of the reduced quintessence potential at the present epoch must obey the inequality: \[ \frac{2}{3} - \frac{1}{2}(1 - \omega_f)\Omega_f^{(0)} < U|_{N=0} \]. For example, if the extra cosmic fluid is the non-relativistic matter, \( \omega_f \approx 0, \Omega_f^{(0)} \approx 0.26 \), we have \( U|_{N=0} \gtrsim 0.54 \).

The interpolation of \( \omega_\phi \) between zero and minus one can be achieved if the reduced quintessence potential \( U(\phi) \) has at least two plateaus, one is at \( U \approx 1 - \frac{1}{2}(1 - \omega_f)\Omega_f \) for the large red-shift epoch and another is at \( U \approx 1 - \frac{1}{2}(1 - \omega_f)\Omega_f \) as the red-shift parameter \( z \) approaches to zero. To this end, the quintessence field under consideration is defined to have a vanishing adiabatic sound speed \( c^2_{\phi, ad} = \partial P_\phi/\partial \rho_\phi \). In other words, the pressure \( P_\phi \) is supposed not to change during the evolution of the universe. Keeping \( P_\phi \) invariant implies that,

\[ \omega'_\phi = 3\omega_\phi(1 + \omega_\phi) \]  

Obviously, the expected plateaus in the \( \omega_\phi \sim N \) curve are guaranteed by this condition, one is \( \omega_\phi = 0 \) and another is \( \omega_\phi = -1 \). Eq. (18) can be translated into the definition equation for the reduced quintessence potential,

\[ U' + 3[(1 - U)(1 - 2U) - (1 - U + \omega_f U)\Omega_f] = 0 \]  

The quintessence field is expected to play the role of dark energy at low red shifts to excite the late time cosmic acceleration. The magnitude of the reduced potential \( U \) must be larger than \( \frac{1}{6}(1 - \Omega_f) \) so that \( P_\phi < 0 \). For this scalar behaves as the pressureless non-relativistic matter at large red shifts, the Hubble rate \( H \) have to ascend sharply with respect to the increase of the red-shift parameter.

In the absence of the extra fluid, \( \Omega_f = 0 \), Eq. (19) can be solved analytically, with a closed form solution given below:

\[ U(N) = \frac{1 - \alpha + (2\alpha - 1)e^{3N}}{2 - 2\alpha + (2\alpha - 1)e^{3N}} \]  

where \( \alpha := U|_{N=0} \) is an integration parameter which stands for the value of the reduced quintessence potential \( U(\phi) \) at present. As pointed out previously, \( \alpha > 2/3 \). Because \( \alpha \neq 0 \), we see that: \( 1/2 \leq U(N) \leq 1 \).
The quintessence field as a perfect cosmic fluid of constant pressure

Substitution of Eq. (20) into Eqs. (11) and (7) leads to,

\[
\phi(N) = \frac{2}{\sqrt{3}} \left[ \tanh^{-1} \frac{1}{\sqrt{2(1-\alpha)}} - \tanh^{-1} \frac{\sqrt{2(1-\alpha) + (2\alpha - 1)e^{3N}}}{2(1-\alpha)} \right]
\]

(21)

and

\[
H^2 = H_0^2 \left[ (2\alpha - 1) + 2(1 - \alpha)e^{-3N} \right]
\]

(22)

To guarantee the positivity of \( H^2 \) during its evolution, \( 1/2 \leq \alpha \leq 1 \). Therefore, the parameter \( \alpha \) takes its value in the region \( 2/3 < \alpha \leq 1 \). Combination of Eqs. (21) and (22) with Eq. (9) enables us to express the quintessence potential as an explicit function of the scalar field itself. The result is,

\[
V(\phi) = \frac{3H_0^2}{4} \left[ 3(2\alpha - 1) + (3 - 2\alpha) \cosh(\sqrt{3}\phi) - 2\sqrt{2(1-\alpha)} \sinh(\sqrt{3}\phi) \right]
\]

(23)

There exist potentially three different interpretations of the quintessence field under consideration. The first possibility is simply to interpret it as a dynamic dark energy. Alternatively it could be interpreted as a unified dark matter (quartessence). With these two interpretations, the quintessence field is characterized by its energy density and pressure given below,

\[
\rho_\phi = 3H_0^2 \left[ (2\alpha - 1) + 2(1 - \alpha)e^{-3N} \right]
\]

(24)

\[
P_\phi = -3H_0^2(2\alpha - 1)
\]

(25)

Though the pressure \( P_\phi \) is a negative constant, the energy density \( \rho_\phi \) interpolates between that of a non-relativistic matter, \( \rho_\phi \approx 6H_0^2(1-\alpha)e^{-3N} \), for \( N \ll 1 \) and that of a cosmological constant, \( \rho_\phi \approx 3H_0^2(2\alpha - 1) \), for \( N \gg 0 \). The evolution of quintessence EoS is found to be,

\[
\omega_\phi = -\frac{(2\alpha - 1)e^{3N}}{2(1 - \alpha) + (2\alpha - 1)e^{3N}}
\]

(26)

In Figure 1, we plot the evolution of \( \omega_\phi \) by choosing \( \alpha = 0.87 \). As expected, \( \omega_\phi \) interpolates between minus one for \( N \gg 1 \) and zero for \( N \lesssim -2 \).

The third possibility is to interpret the quintessence field under consideration as a mixture of CC and some non-relativistic matter, \( \rho_\phi = \rho_{\text{CC}} + \rho_M \). In this scenario, the dark energy is understood as a cosmological constant, \( \rho_{\text{CC}} = -P_{\text{CC}} = 3H_0^2(2\alpha - 1) \), but the non-relativistic matter is characterized by \( \rho_M = 6H_0^2(1 - \alpha)e^{-3N} \) and \( P_M = 0 \). The EoS parameters of two components are \( \omega_{\text{CC}} = -1 \) and \( \omega_M = 0 \), respectively. Because of \( \omega_{\text{CC}} = -1 \), there is no interaction between the components. The quintessence EoS \( \omega_\phi \), as given in Eq. (26), should be understood as the effective EoS of such a mixture, \( \omega_\phi = \omega_{\text{CC}}\Omega_{\text{CC}} + \omega_M\Omega_M = -\Omega_{\text{CC}} \), where the dimensionless energy densities of the components are as follows,

\[
\Omega_{\text{CC}}(N) = \frac{(2\alpha - 1)}{(2\alpha - 1) + 2(1 - \alpha)e^{-3N}}
\]

(27)

|| When \( \alpha = 1 \), the quintessence scalar field under consideration degenerates to the cosmological constant \( \Lambda = 3H_0^2 \).
The quintessence field as a perfect cosmic fluid of constant pressure

\[ \frac{8}{-4} \frac{-3}{-2} \frac{-1}{2} \frac{1}{2} \]

\[ N = \frac{\ln a}{-0.8} \]

\[ \Omega \Phi \]

Figure 1. Evolution of EoS \( \omega_\phi \) given in Eqs.\,(26) for parameters \( \alpha = 0.87 \). As expected, two plateaus exist. One plateau corresponds to \( \omega_\phi \approx -1 \) for \( N \gtrsim 1 \), another plateau corresponds to \( \omega_\phi \approx 0 \) for \( N \lesssim -2 \). \( \omega_\phi \) crosses over the acceleration boundary \(-1/3\) at \( N \approx -0.37 \).

\[ \Omega_M(N) = \frac{2(1 - \alpha) e^{-3N}}{(2\alpha - 1) + 2(1 - \alpha) e^{-3N}} \]  

Notice that \( \Omega_M(0) = 2(1 - \alpha) \). Relying on the observational constraint \( \Omega_M(0) \approx 0.26 \), the best-fit value of the parameter \( \alpha \) might be \( \alpha \approx 0.87 \). With respect to such an interpretation, the quintessence model under consideration is merely a reformulation of the standard \( \Lambda \)-CDM model.

All of three interpretations of the quintessence field of a constant pressure appear to be plausible at the background level. They can, however, be distinguished from each other if we study the evolution of the matter perturbations. This will be addressed in the next section.

When the contribution of the extra cosmic fluid is taken into account,

\[ \Omega_f = \left( \frac{H_0^2}{H^2} \right) \Omega_{f}^{(0)} e^{-3N(1+\omega_f)} \]  

Eqs. \,(27) and \,(31) become,

\[ (H^2)' + 6H^2(1 - U) - 3H_0^2\Omega_{f}^{(0)}(1 - \omega_f)e^{-3N(1+\omega_f)} = 0 \]

\[ \phi'' = 6(1 - U) - 6\Omega_{f}^{(0)} \frac{H_0^2}{H^2} e^{-3N(1+\omega_f)} \]

respectively, and the condition \,(19) for preserving \( P_\phi \) to be a constant during the evolution of the Universe is recast as:

\[ U' + 3 \left[ (1 - U)(1 - 2U) - \Omega_{f}^{(0)} \frac{H_0^2}{H^2} (1 - U + \omega_f U)e^{-3N(1+\omega_f)} \right] = 0 \]

The closed-form solutions to Eqs.\,(30), \,(31) and \,(32) are not available in general. However, if the extra fluid is of extremely non-relativistic, \( \omega_f = 0 \), these equations
The quintessence field as a perfect cosmic fluid of constant pressure

Figure 2. Parameter space of $\alpha$ and $\Omega_f^{(0)}$. The solid black lines stand for the critical cases, from bottom to the top corresponding to $\alpha = (1 - \Omega_f^{(0)})/2$, $\alpha = 1 - \Omega_f^{(0)}$ and $\alpha = 1 - \Omega_f^{(0)}/2$ respectively. The dashed line corresponds to the acceleration boundary $6\alpha - 4 + 3\Omega_f^{(0)} = 0$. All of the parameters $(\Omega_f^{(0)}, \alpha)$ within the region of gray shadow above this dashed line allow the existence of late time cosmic acceleration.

can also be solved exactly. The corresponding solution is as follows:

$$\phi(N) = \frac{2}{\sqrt{3}} \sqrt{\frac{2(1 - \alpha - \Omega_f^{(0)})}{2 - 2\alpha - \Omega_f^{(0)}}} \left[ \tanh^{-1} \frac{1}{\sqrt{2 - 2\alpha - \Omega_f^{(0)}}} \right]$$

$$- \tanh^{-1} \left[ \frac{2 - 2\alpha - \Omega_f^{(0)} + (2\alpha - 1 + \Omega_f^{(0)})e^{3N}}{2 - 2\alpha - \Omega_f^{(0)}} \right]$$

$$H^2 = H_0^2 \left[ (2\alpha - 1 + \Omega_f^{(0)}) + (2 - 2\alpha - \Omega_f^{(0)})e^{-3N} \right]$$

$$U = \frac{1 - \alpha - \Omega_f^{(0)} + (2\alpha - 1 + \Omega_f^{(0)})e^{3N}}{2 - 2\alpha - \Omega_f^{(0)} + (2\alpha - 1 + \Omega_f^{(0)})e^{3N}}$$

where as before, $\alpha = U|_{N=0}$. Eq. (35) that gives the closed form of the reduced quintessence potential in the presence of extra fluid does differ from Eq. (20) but recovers it for $\Omega_f^{(0)} = 0$. Substitution of Eqs. (35) and (29) into (16) and (17) leads to the following expressions of the quintessence EoS,

$$\omega_{\phi} = -\frac{(2\alpha - 1 + \Omega_f^{(0)})e^{3N}}{2(1 - \alpha - \Omega_f^{(0)}) + (2\alpha - 1 + \Omega_f^{(0)})e^{3N}}$$

and the effective EoS of the mixture,

$$\omega_{eff} = -\frac{(2\alpha - 1 + \Omega_f^{(0)})e^{3N}}{2 - 2\alpha - \Omega_f^{(0)} + (2\alpha - 1 + \Omega_f^{(0)})e^{3N}}$$

Manifestly, $\omega_{\phi} \lesssim \omega_{eff}$. Eqs. (34), (36) and (37) implies that the behavior of the mixed quintessence field and the extra fluid is determined by two parameters, i.e., the initial
The quintessence field as a perfect cosmic fluid of constant pressure

The quintessence field as a perfect cosmic fluid of constant pressure value of the reduced quintessence potential $\alpha$ and the initial dimensionless density $\Omega_f^{(0)}$ of the extra fluid. The late time accelerated expansion occurs if these two parameters are subject to the inequality $6\alpha - 4 + 3\Omega_f^{(0)} > 0$. Except for $\alpha = 1 - \Omega_f^{(0)}/2$, these two parameters have also to satisfy the constraint $\alpha \leq 1 - \Omega_f^{(0)}$ to guarantee both $\omega_\phi$ and $\omega_{eff}$ nonsingular. In principle, $\alpha$ and $\Omega_f^{(0)}$ are two free parameters, however, there exist three degenerate but important cases where they are dependent upon one another.

The first case corresponds to $\alpha = (1 - \Omega_f^{(0)})/2$, for which $\omega_\phi = \omega_{eff} = 0$, both of the quintessence field and the extra fluid act as non-relativistic matters. In the second case, $\alpha = 1 - \Omega_f^{(0)}/2$, $\omega_{eff} = -1$, the mixture of the quintessence field and the extra fluid behaves as a CC. The third degenerate case is characterized by equations $\alpha = 1 - \Omega_f^{(0)}$ and $\omega_\phi = -1$, in which the quintessence field itself is nothing but a CC. It is conspicuous that the standard Λ-CDM is recovered in the third case if we take the values of the two parameters as $\alpha \approx 0.74$ and $\Omega_f^{(0)} \approx 0.26$.

![Figure 3. Evolution of EoS $\omega_\phi$ and $\omega_{eff}$ versus $N = \ln a$ for parameters $\alpha = 0.7$, $\omega_f = 0$ and $\Omega_f^{(0)} \approx 0.26$. The black curve stands for $\omega_{eff}$ while the gray curve describes $\omega_\phi$. Both curves interpolate between two plateaus of zero and of minus one.](image)

Because $\omega_\phi \leq \omega_{eff}$, it is earlier for $\omega_\phi$ to cross the acceleration boundary of $\omega = -1/3$. In particular, if $\alpha = 1 - \Omega_f^{(0)}$, $\omega_\phi = -1$ identically but $\omega_{eff}$ approaches to the onset of acceleration only at $N = -\frac{1}{3} \ln [2(1 - \Omega_f^{(0)})/\Omega_f^{(0)}]$. Even if the quintessence field provides a unified description for both dark energy and dark matter, it plays merely the role of dark energy once $\omega_\phi$ crosses the acceleration boundary. Consequently, the budget of cold dark matter at present epoch have to be balanced by some other cosmic ingredients. In this paper, we use simply an extra fluid of $\omega_f = 0$ to mimic the mixture of baryonic dust and (at least partial) cold dark matter so that the observational constraint on the abundance of non-relativistic matter at present epoch is saturated, $\Omega_f^{(0)} \approx 0.26$. For comparison we will consider four situations, corresponding to $\alpha = 0.62$, 0.66, 0.70 and 0.74, respectively. As have pointed out, the last case is a paraphrase of the standard Λ-CDM model where the quintessence field behaves exactly as a cosmological constant,
The quintessence field as a perfect cosmic fluid of constant pressure

$\omega_\phi = -1$. The first three cases put forward models of dynamic dark energy in which the EoS of the quintessence scalar field interpolate between two plateaus of zero and of minus one, so that the notorious coincidence problem with which the prototypic $\Lambda$-CDM model is embarrassed is greatly alleviated. The existence of plateaus in the $\omega_\sim N$ curve implies that the severe fine-tuning of the quintessence potential is not required.

|     | $\alpha$ | $\Omega_f^{(0)}$ | $\omega_\phi^{(0)}$ | $\omega_{\text{eff}}^{(0)}$ | $N_c$ |
|-----|---------|-----------------|---------------------|---------------------|-------|
| Case 1 | 0.62   | 0.26            | -0.68              | -0.50              | -0.23 |
| Case 2 | 0.66   | 0.26            | -0.78              | -0.58              | -0.34 |
| Case 3 | 0.70   | 0.26            | -0.89              | -0.66              | -0.45 |
| $\Lambda$-CDM | 0.74 | 0.26            | -1.00              | -0.74              | -0.58 |

Table 1. Parameter choice for the quintessence model under consideration. We use $\omega_\phi^{(0)}$ and $\omega_{\text{eff}}^{(0)}$ to denote the values of quintessence field EoS and the effective EoS of the mixture of quintessence field and the extra non-relativistic fluid at present epoch. $N_c$ stands for the onset for $\omega_{\text{eff}}$ crosses the acceleration boundary $-1/3$.

3. perturbations

Dark energy influences not only the expansion rate, it influences also the growth rate of matter perturbations. To clarify what the role does the quintessence field of constant pressure play during the evolution of the universe, in this section we study the matter perturbations at the linear perturbation level. We start with the perturbed Robertson-Walker metric,

$$ds^2 = -(1 + 2\Psi)H^{-2}dN^2 + e^{2N}(1 + 2\Phi)\delta_{ij}dx^idx^j$$

in Newtonian gauge and for convenience let us work out from beginning a single Fourier mode $k$ so that the perturbed quantities $\Phi = \Phi(N) e^{i\vec{k} \cdot \vec{x}}$, $\Psi = \Psi(N) e^{i\vec{k} \cdot \vec{x}}$ and so on. The perturbed energy-momentum tensor of the extra fluid is described by the perturbed energy density $\delta \rho_f = -T_0^0$, the perturbed pressure $\delta P_f = \frac{1}{3} \delta T_i^i$, and the velocity divergence $\theta_f$ defined by $ik_j \delta T_{0j}^j = -(1 + \omega_f)\rho_f \theta_f$. For the quintessence field, these perturbed quantities read:

$$\delta \rho_\phi = H^2(\phi' \phi' - \Psi \phi'^2) + V_\phi \phi$$

$$\delta P_\phi = H^2(\phi' \phi' - \Psi \phi'^2) - V_\phi \phi$$

$$\theta_\phi = \lambda^{-2} \frac{\varphi}{\phi'}$$

where $\lambda = He^N/k$, and $\varphi := \delta \phi$ denotes the field fluctuation. The perturbation equations for a perfect fluid with density contrast $\delta_X$ and velocity divergence $\theta_X$ are:

$$\delta_X' = 3(\omega_X - c_X^2)\delta_X - (1 + \omega_X)(\theta_X + 3\phi')$$

$$\theta_X' = -\frac{\theta_X}{2} \left[1 - 6\omega_X - 3\omega_{\text{eff}} + \frac{2\omega_X'}{1 + \omega_X}\right] + \lambda^{-2} \left[\frac{\delta_X^2}{1 + \omega_X} + \Psi\right]$$
The quintessence field as a perfect cosmic fluid of constant pressure

where $X$ is either $\phi$ or $f$, and $c_X^2 := \delta P_X/\delta \rho_X$ the squared sound speed of fluid $X$. The evolution of Bardeen potentials $\Phi$ and $\Psi$ is subject to Einstein’s gravitational equations. The result is,

$$\Phi = 3\hat{\lambda}^2 \left[ \frac{1}{2} \Omega_f \delta_f + \frac{1}{2}(1 - \Omega_f) \delta_\phi + \Psi - \Phi' \right]$$

$$\Phi' = \Psi - \frac{3}{2} \hat{\lambda}^2 \left[ (1 + \omega_f) \Omega_f \theta_f + (1 + \omega_\phi)(1 - \Omega_f) \theta_\phi \right]$$

$$\Psi = -\Phi$$

We assume that the extra fluid is of barotropic so that $\omega_f = \omega_f(\rho_f)$ and $c_f^2 = dP_f/d\rho_f$. The quintessence field, on the other hand, cannot be regarded as a barotropic fluid in general. When $X$ in Eqs.(42) and (43) stands for the quintessence field $\phi$ of constant pressure, $\omega_\phi$ is given in Eq.(16), but the squared sound speed $c_\phi^2$ does not coincide with its adiabatic counterpart $c_{\phi,ad}^2$.

$$c_\phi^2 = \frac{H^2(\phi' \phi' - \Psi \phi^2) - V_{\phi \phi}}{H^2(\phi' \phi' - \Psi \phi^2) + V_{\phi \phi}}$$

In other words, $c_\phi^2$ depends on the detailed behavior of both perturbation and background quantities. There is no reason for $c_\phi^2$ vanishes identically. In fact, if we put ourselves in the quintessence rest frames, we have $\phi = 0$ and hence $c_\phi^2 = 1$. Without loss of generality, we assume $c_\phi^2 \neq 0$ in the following. Eq.(43) for $\theta_\phi$ turns out to become an identity, but Eq.(42) for $\delta_\phi$ reduces to:

$$\varphi'' + \left( 3 + \frac{H'}{H} \right) \varphi' + \left( \hat{\lambda}^{-2} + \frac{V_{\phi \phi}}{H^2} \right) \varphi + 4\phi' \Phi' - 2\frac{V_{\phi \phi}}{H^2} \Phi = 0$$

where,

$$V_{\phi} = -\frac{3}{2} H^2 \phi'$$

$$V_{\phi \phi} = \frac{9}{4} H^2 (3 - 2U - \Omega_f)$$

At late times ($z \lesssim 10^4$) all modes of interest have entered the comoving horizon. In view of observations, the typical scales relevant to the galaxy matter power spectrum correspond to the wavenumbers $k \lesssim 0.1h$ Mpc$^{-1}$ or equivalently $k \lesssim 300 H_0$. Consequently, Eqs.(42), (43) for the extra fluid perturbations and Eq.(48) for the quintessence fluctuations should be solved for $30 H_0 \lesssim k \lesssim 100 H_0$, for which $\hat{\lambda} \ll 1$. Notice that $\Phi \sim \mathcal{O}(\hat{\lambda}^2)$ and $U \sim \mathcal{O}(1)$. The coefficient of field fluctuation $\varphi$ in Eq.(48) is dominated by $\hat{\lambda}^{-2}$ when $\hat{\lambda} \ll 1$, which is the case on sub-horizon scales $k \gg H_0$ if $N$ is not very negative. Let us consider temporarily the case of $\Omega_f = 0$ and solve numerically Eq.(48) with initial conditions $\varphi|_{N=0} = 0$ and $\varphi'|_{N=0} = 1$. The solution tells that the field fluctuations oscillate around zero on sub-horizon scales, with nearly vanishing average values over many oscillations. As illustrated in Figure 4, the field fluctuations of sub-horizon modes ($30 H_0 \lesssim k \lesssim 300 H_0$) could approximately be regarded as zero for $N \gtrsim -3$. This conclusion can even be extrapolated to some earlier time, say i.e., $N \gtrsim -2.4$. 

\[\text{i.e., } N \gtrsim -2.4.\]
The quintessence field as a perfect cosmic fluid of constant pressure

$N \gtrsim -5$, with slightly poor accuracy. It holds also for parameters $\alpha$ and $\Omega_f^{(0)}$ assuming other values as in Table 1. Consequently, the velocity divergence $\theta_{\phi}$ of sub-horizon quintessence modes oscillates around zero and its sound speed oscillates around one, and for $N \gtrsim -5$, we have $\theta_{\phi} \approx 0$ and $c_{\phi}^2 \approx 1$. The approximate equality $\theta_{\phi} \approx 0$ for $N \gtrsim -5$ further implies the decoupling of Eq. (42) from Eq. (43) for $X = \phi$. The former turns out to become a first-order differential equation of $\delta_{\phi}$ for $\dot{\lambda} \ll 1$,

$$\delta_{\phi}^\prime + 3(1 - \omega_{\phi})\delta_{\phi} - \frac{9H_0^2}{2k^2}e^{2N}(1 + \omega_{\phi})\delta_{\phi} = 0$$

(51)

The solution of Eq. (51) takes the following closed form as $k \rightarrow \infty$,

$$\delta_{\phi} = \frac{\delta_{\phi}(0)e^{-3N}}{2(1 - \alpha) + (2\alpha - 1)e^{3N}}$$

(52)

Unfortunately, such a density contrast falls sharply before the universe enters the late-time accelerated expansion. It contributes little to the galaxy clustering. This is the same as the sub-horizon density contrasts of the barotropic fluid of constant pressure [30]. In fact, $\delta_{\phi}$ vanishes identically for $\alpha = 0.74$ and $\Omega_f^{(0)} = 0.26$ because in this case the quintessence field plays the role of a pure CC [4]. The quintessence field of constant pressure could not take charge of structure formation even if it unifies the dark energy and some (fuzzy) dark matter [4] into a single dark substance. Notice that the sound speed of the mixture of CC and some non-relativistic matter vanishes identically [15]. That the physical sound speed of quintessence field does not vanish obviously mismatches this fact, which expels definitely the possibility to interpret the quintessence field of constant pressure as a mixture of CC and non-relativistic matters. It is better to interpret the quintessence field of constant pressure as a dynamic dark energy, it unifies at most the dark energy and a fraction of (fuzzy) dark matter. The quintessence field clusters, but such a clustering decays monotonously during the evolution that it takes no direct responsibility for the structure formation on large scales. The structure formation in the proposed scenario must have other impellers.

The cold dark matter which is believed to answer for the structure formation can effectively be described as an extremely non-relativistic perfect fluid. We should take $\Omega_f \neq 0$ and $\omega_f = 0$ in our model building. In such a scenario, the extra fluid is a mixture of cold dark matter and baryons. Though Eq. (48) describes still damped oscillations around zero so that $\varphi \approx 0$ for $N$ not to be very negative ($N \gtrsim -5$) and the fluctuations of quintessence field decay on sub-horizon scales, the sub-horizon fluctuations of the quintessence field are strongly damped by the potential $V_{\phi\phi}$.

+ When $\omega_{\phi} = -1$, Eq. (51) is solved by either $\delta_{\phi} \sim e^{-6N}$ or $\delta_{\phi} = 0$. However, the former is a specious solution because it conflicts with Eq. (43) for $X = \phi$ under the assumptions of $\theta_{\phi} \approx 0$ and $c_{\phi}^2 \approx 1$.

* According to Eq. (50), the mass of quintessence field reads,

$$m_{\phi} = \sqrt{V_{\phi\phi}} = \frac{3}{2}H_0\sqrt{2\alpha + \Omega_f^{(0)} - 1 + 2(2 - 2\alpha - \Omega_f^{(0)})e^{-3N}}$$

For the parameters given in Table 1, say $\alpha = 0.62$ and $\Omega_f^{(0)} = 0.26$, we have $m_{\phi}|_{N\approx -5} \approx 2700H_0 \approx 4 \times 10^{-30}$ eV, which is much less than the possible mass limit ($m_\nu < 1.43$ eV) of the massive neutrinos [39].
The quintessence field as a perfect cosmic fluid of constant pressure

The quintessence field as a perfect cosmic fluid of constant pressure

non-relativistic fluid grow steadily. It follows from Eqs.\([42]\), \([43]\), \([44]\), \([45]\) and \([46]\) that, for \(\dot{\lambda} \ll 1\),

\[
\delta'_{\phi} + a_0 \delta_{\phi} = b_0 \delta_f + b_1 \delta'_f \\
\delta''_f + c_1 \delta'_f + c_0 \delta_f = d_0 \delta_{\phi} + d_1 \delta'_\phi
\]

where,

\[
a_0 = 3(1 - \omega_{\phi}) + \frac{9}{2} \dot{\lambda}^2 (1 + \omega_{\phi})(1 - \Omega_f) \\
b_0 = \frac{9}{2} \dot{\lambda}^2 (1 + \omega_{\phi})\Omega_f \\
b_1 = -\frac{9}{2} \dot{\lambda}^2 (1 + \omega_{\phi})\Omega_f \\
c_0 = -\frac{3}{2} \Omega_f - \frac{9}{4} \dot{\lambda}^2 \left[2\Omega'_f - 3\Omega^2_f - (1 + 9 \omega_{eff})\Omega_f\right] \\
c_1 = \frac{1}{2} (1 - 3 \omega_{eff}) + \frac{9}{2} \dot{\lambda}^2 \left[\Omega'_f - (1 + 3 \omega_{eff})\Omega_f\right] \\
d_0 = \frac{3}{2} (1 - \Omega_f) - \frac{9}{4} \dot{\lambda}^2 \left[2\Omega'_f + 3\Omega_f (1 - \Omega_f) + (1 + 9 \omega_{eff})(1 - \Omega_f)\right] \\
d_1 = \frac{9}{2} \dot{\lambda}^2 (1 - \Omega_f)
\]

In obtaining these coefficients the assumption \(\theta_{\phi} \approx 0\) on sub-horizon scales has been made use of, which could be justified only when \(N\) is not very negative. If we take the extreme sub-horizon limit, \(k \to \infty\), we can further simplify Eqs.\([53]\) and \([54]\) as,

\[
\delta'_{\phi} + 3(1 - \omega_{\phi})\delta_{\phi} = 0 \\
\delta''_f + \frac{1}{2} (1 - 3 \omega_{eff})\delta'_f - \frac{3}{2} \Omega_f \delta_f = \frac{3}{2} (1 - \Omega_f) \delta_{\phi}
\]
The quintessence field as a perfect cosmic fluid of constant pressure

Eq. (62), as expected, describes a decaying field density contrast \( \delta_\phi \). Eq. (63), on the other hand, allows a growing density contrast \( \delta_f \) for matter perturbations. It follows from Eq. (63) that the quintessence density contrast plays a role of external source to the evolution of the density contrast of the non-relativistic fluid. Provided this external source is ignored, Eq. (63) can approximately be solved by a closed-form solution,

\[
\delta_f = C_1 e^{3\gamma_- N} {}_2F_1(\gamma_-, \gamma_+ + 1/2, 1 + \gamma_+ + \gamma_-; \xi) + C_2 e^{-3\gamma_+ N} {}_2F_1(-\gamma_- + 1/2, -\gamma_+, 1 - \gamma_+ - \gamma_-; \xi) \tag{64}
\]

where \( C_1 \) and \( C_2 \) are two integration constants,

\[
\gamma_\pm = \frac{1}{12} \left[ \sqrt{2\alpha - 23\Omega_f^{(0)} - 2} \pm 1 \right], \quad \xi = \frac{2\alpha + \Omega_f^{(0)} - 1}{2\alpha + \Omega_f^{(0)} - 1} e^{3N} \tag{65}
\]

and \( {}_2F_1(a, b, c; \xi) \) stands for the hypergeometric function. In case \( \alpha = 1 - \Omega_f^{(0)} \), the quintessence field degenerates to a CC (i.e., \( \delta_\phi = 0 \)) and the solution given in Eq. (64) becomes of exact,

\[
\delta_m = C_1 e^{3N} {}_2F_1\left(1/3, 1, 1; \frac{\alpha e^{3N}}{\alpha - 1}\right) + C_2 e^{-3N/2} {}_2F_1\left(1/6, -1/2, 1/6; \frac{\alpha e^{3N}}{\alpha - 1}\right) \tag{66}
\]

Figure 5. Evolution of density contrasts of matter perturbations of sub-horizon mode \( k \approx 40H_0 \). The curves in the bottom-up order describe the matter density contrast \( \delta_f \) as the numerical solution of Eqs. (53) and (54) for \( \alpha = 0.62, 0.66, 0.70 \) and 0.74, respectively. In all cases the parameter \( \Omega_f^{(0)} \) is assumed to be 0.26. Among these curves the gray one corresponds to the sub-horizon matter density contrast in the standard \( \Lambda \)-CDM model. The initial conditions are assigned as \( \delta_f|_{N=-4.62} \approx \delta_\phi|_{N=-4.62} \approx 0.006 \) and \( \delta_\phi|_{N=-4.62} \approx 10^{-6} \). The larger the parameter \( \alpha \) is, the faster the matter density contrast grows in the proposed scenario.

If the external source term is taken into account, a closed-form solution to Eq. (63) seems to be impossible. Furthermore, the fact that the typical smallest scale relevant to the galaxy matter power spectrum corresponds to \( k \approx 300H_0 \) requires us to solve Eqs. (53) and (54) instead for sub-horizon modes satisfying observational constraint.
The quintessence field as a perfect cosmic fluid of constant pressure

$k \lesssim 30H_0$ [30]. The key scale in the matter power spectrum is the matter-radiation equality scale $k_{eq} \sim 100$ Mpc (i.e., $k_{eq} \approx 40H_0$), which defines the turn-around in the spectrum. These equations can be solved numerically if some appropriate initial conditions are assigned. We use Eqs. (53) and (54) to evolve the perturbations from $N \approx -4.62$ until today. The numerical solution of $\delta_f$ with initial conditions $\delta_f|_{N=-4.62} \approx 0.006$ and $\delta_\phi|_{N=-4.62} \approx 10^{-5}$ is plotted in Figure 5 and Figure 6 for the typical mode $k \approx 40H_0$, where we take the relevant parameters $\alpha$ and $\Omega_{f}^{(0)}$ as in Table 1. The hierarchy between the initial conditions $\delta_f|_{N=-4.62}$ and $\delta_\phi|_{N=-4.62}$ has been fine-tuned so that $\delta_\phi$ vanishes effectively as the quintessence field mimics only a CC in $\Lambda$-CDM model. As expected, in all cases the matter density contrasts grow steadily while the quintessence density contrasts decline for $N \gtrsim -3$. Except for the $\Lambda$-CDM model where $\delta_\phi$ falls sharply to zero, the quintessence density contrasts undergo a period of increase preceded their final-stage decline. This probably is a clue that there could not be any observable Sachs-Wolfe effect [38] [15] related to the quintessence fluctuations provided we fine-tune appropriately the initial condition of $\delta_\phi$. From Figure 5 we see that the sub-horizon matter density contrasts in the proposed scenario of dynamic energy models grow remarkably slower than that in $\Lambda$-CDM model. Due to the feedback of non-vanishing quintessence density contrasts, see Figure 6, the matter perturbations in these models enter the accelerated expansion epoch later than those in $\Lambda$-CDM. The onset of nonlinearity occurs when the density perturbations obey $\delta_f \gtrsim 0.3$. For a comoving scale of 100 Mpc, i.e., $k \approx 40H_0$, this occurs at redshift $z^* \sim 1$ [34] which corresponds to $N^* \approx -0.693$. It follows from Figure 5 that $\delta_f|_{N^*} \approx 0.09, 0.12, 0.17$ in the proposed scenarios of $\alpha \approx 0.62, 0.66, 0.70$ and $\delta_f|_{N^*} \approx 0.28$ in the standard $\Lambda$-CDM model, respectively. The matter density contrast as a function of comoving wavenumber at $N^*$ is also shown in Figure 7.

The growth of matter perturbations can be measured by the so-called growth rate, which is defined as the ratio of the derivative of the matter density contrast with respect to enfolding time to matter density contrast itself, i.e., $f = \delta_f'/\delta_f$ in the present scenario. In Figure 8, we plot the evolution of the growth rates of matter perturbations of mode $k \approx 40H_0$ for $\alpha = 0.62, 0.66$ and 0.70 as well as their analogue in $\Lambda$-CDM. Because the cosmological constant does not cluster, it has no influence on the evolution of matter perturbations, in the standard $\Lambda$-CDM model the growth rate of matter perturbation remains almost invariant before it enters acceleration. If the role of dark energy is played by the quintessence scalar field of constant pressure, however, the clustering of quintessence perturbations does certainly affect the growth of matter perturbations. In the proposed scenario, although the quintessence density contrast does not take charge of the large scale structures, it acts as an external driving force in the evolution equation (54) of matter density contrast. Such an influence results in remarkable decline in $f \sim N$ curve prior to the epoch of acceleration. As shown in Figure 8, the smaller the parameter $\alpha$ is, the greater the quintessence field contributes to (fuzzy) dark matter and the faster

\[ i.e., \ z \approx 100. \] This is a typical redshift in the cosmological epoch dominated by non-relativistic matter.
The quintessence field as a perfect cosmic fluid of constant pressure

Figure 6. Evolution of quintessence density contrasts of sub-horizon mode $k \approx 40H_0$. The parameter choice is the same as Figure 5, so are the initial conditions. The curves correspond to $\alpha = 0.62$, 0.66, 0.70 and 0.74 from the top to the bottom. In the first cases the quintessence density contrasts increase during the period $-4.4 \lesssim N \lesssim -3$ and then decrease gradually. In the last case which corresponds to $\Lambda$-CDM model, the quintessence field is almost equivalent to a cosmological constant and its density contrast drops sharply to zero. The decrease of $\delta \phi$ as $N \gtrsim -3$ excludes the possibility for the considered quintessence field to unify dark energy and the cold dark matter into a single dark substance. On the other hand, the increase of $\delta \phi$ prior to $N \approx -3$ implies that there could not be any observable Sachs-Wolfe effect related to the quintessence fluctuations if the initial condition for $\delta \phi$ is fine-tuned appropriately.

Figure 7. The matter density contrasts on sub-horizon scales at the onset of nonlinearity $N^* \approx -0.693$. The parameter choice is as Table 1. The curves correspond to $\alpha = 0.62$, 0.66, 0.70 and 0.74 from the bottom upward.

the sub-horizon perturbations of non-relativistic matters decline. When the cosmological acceleration begins, $N \gtrsim -0.693$, the magnitude of $\delta \phi$ drops effectively to zero and no longer provides robust driving force, so the growth rates of matter perturbations ebb steadily in all cases.
4. Conclusion

In this paper we have studied the cosmology of a quintessence scalar field which can be viewed as a non-barotropic perfect fluid of constant pressure in linear perturbation theory. The assumption of constant pressure ensures the interpolation of the quintessence EoS between the plateau of zero at large red shifts and that of minus one at small red shifts. Not this characteristic alleviates greatly the coincidence problem only, it brings also few unphysical ingredients into the description of evolution of universe. The potential of the quintessence field is determined completely from the requirement of constant pressure, which is unnecessary to be severely fine-tuned in the alleviation of coincidence problem. The adiabatic sound speed of the quintessence field vanishes, however, its physical sound speed depends upon the details of perturbations and equals effectively to unity for modes within the deep comoving horizon. The quintessence field does not unify the dark energy and the whole cold dark matter into a single dark substance, it is also not a mixture of these two dark substances. What the role it really play during evolution is plausibly a dynamic dark energy that could cluster. This clustering is measured by a decaying density contrast. Though it is not directly responsible to the formation of large scale structure, the clustering of the quintessence perturbations could effectively alleviate the growth rates of matter perturbations as an external driving force.

Acknowledgments

We would like to thank Y. F. Cai, M. Z. Li, J. X. Lu and M. L. Yan for valuable discussions. The work was supported in part by NSFC under No. 11235010.
The quintessence field as a perfect cosmic fluid of constant pressure

References

[1] A. G. Riess, A. V. Filippenko, P. Challis, A. Clocchiatti, A. Diercks, M. Peter, R. L. Gilliland, C. J. Hogan, S. Jha, R. P. Kirshner, and B. Leibundgut, “Observational Evidence from Supernovae for an Accelerating Universe and a Cosmological Constant,” Astron. J. 116 (1998) 1009, 9805201v1

[2] S. Perlmutter, G. Aldering, G. Goldhaber, R. A. Knop, P. Nugent, P. G. Castro, S. Deustua, S. Fabbro, A. Goobar, D. E. Groom, I. M. Hook, A. G. Kim, M. Y. Kim, J. C. Lee, N. J. Nunes, R. Pain, C. R. Pennypacker, R. Quimby, C. Lidman, R. S. Ellis, M. Irwin, R. G. McMahon, P. Ruizapuente, N. Walton, B. Schaefer, B. J. Boyle, A. V. Filipenko, T. Matheson, A. S. Fruchter, N. Panagia, H. J. M. Newberg, W. J. Couch, and T. S. C. Project, “Measurements of ω and Λ from 42 High redshift Supernovae,” The Astrophysical Journal 517 (June, 1999) 565–586, 9812133

[3] M. Tegmark, M. Strauss, M. Blanton, K. Abazajian, S. Dodelson, H. Sandvik, X. Wang, D. Weinberg, I. Zehavi, N. Bahcall, F. Hoyle, D. Schlegel, R. Scocccimarro, M. Vogeley, A. Berlind, T. Budavari, A. Connolly, D. Eisenstein, D. Finkbeiner, J. Frieman, J. Gunn, L. Hui, B. Jain, D. Johnston, S. Kent, H. Lin, R. Nakajima, R. Nichol, J. Ostriker, A. Pope, R. Scranton, U. Seljak, R. Sheth, A. Stebbins, A. Szalay, I. Szapudi, Y. Xu, J. Annis, J. Brinkmann, S. Burles, F. Castander, I. Csabai, J. Loveday, M. Doi, M. Fukugita, B. Gillespie, G. Hennessy, D. Hogg, v. Ivezic, G. Knapp, D. Lamb, B. Lee, R. Lupton, T. McKay, P. Kunszt, J. Munn, L. Oonnel, J. Peoples, J. Pier, M. Richmond, C. Rockosi, D. Schneider, C. Stoughton, D. Tucker, D. Vanden Berk, B. Yanny, and D. York, “Cosmological parameters from SDSS and WMAP,” Physical Review D 69 (May, 2004) 103501.

[4] M. Tegmark, M. R. Blanton, M. A. Strauss, F. Hoyle, D. Schlegel, R. Scocccimarro, M. S. Vogeley, D. H. Weinberg, I. Zehavi, A. Berlind, T. Budavari, A. Connolly, D. J. Eisenstein, D. Finkbeiner, J. A. Frieman, J. E. Gunn, A. J. S. Hamilton, L. Hui, B. Jain, D. Johnston, S. Kent, H. Lin, R. Nakajima, R. C. Nichol, J. P. Ostriker, A. Pope, R. Scranton, U. Seljak, R. K. Sheth, A. Stebbins, A. S. Szalay, I. Szapudi, L. Verde, Y. Xu, J. Annis, N. A. Bahcall, J. Brinkmann, S. Burles, F. J. Castander, I. Csabai, J. Loveday, M. Doi, M. Fukugita, J. R. Gott III, G. Hennessy, D. W. Hogg, v. Ivezic, G. R. Knapp, D. Q. Lamb, B. C. Lee, R. H. Lupton, T. A. McKay, P. Kunszt, J. A. Munn, L. Oonnel, J. Peoples, J. R. Pier, M. Richmond, C. Rockosi, D. P. Schneider, C. Stoughton, D. L. Tucker, D. E. Vanden Berk, B. Yanny, and D. G. York, “The Three dimensional Power Spectrum of Galaxies from the Sloan Digital Sky Survey,” The Astrophysical Journal 606 (May, 2004) 702–740, 0310725

[5] M. Tegmark, D. J. Eisenstein, M. a. Strauss, D. H. Weinberg, M. R. Blanton, J. a. Frieman, M. Fukugita, J. E. Gunn, A. J. S. Hamilton, G. R. Knapp, R. C. Nichol, J. P. Ostriker, N. Padmanabhan, W. J. Percival, D. J. Schlegel, D. P. Schneider, R. Scocccimarro, U. Seljak, H.-J. Seo, M. Swanson, A. S. Szalay, M. S. Vogeley, J. Yoo, I. Zehavi, K. Abazajian, S. F. Anderson, J. Annis, N. a. Bahcall, B. Bassett, A. Berlind, J. Brinkmann, T. Budavari, F. Castander, A. Connolly, I. Csabai, M. Doi, D. P. Finkbeiner, B. Gillespie, K. Glazebrook, G. S. Hennessy, D. W. Hogg, v. Ivezic, B. Jain, D. Johnston, S. Kent, D. Q. Lamb, B. C. Lee, H. Lin, J. Loveday, R. H. Lupton, J. a. Munn, K. Pan, C. Park, J. Peoples, J. R. Pier, A. Pope, M. Richmond, C. Rockosi, R. Scranton, R. K. Sheth, A. Stebbins, C. Stoughton, I. Szapudi, D. L. Tucker, D. E. V. Berk, B. Yanny, and D. G. York, “Cosmological constraints from the SDSS luminous red galaxies,” Physical Review D 74 (Dec., 2006) 123507.

[6] U. Seljak, A. Makarov, P. McDonald, S. Anderson, N. Bahcall, J. Brinkmann, S. Burles, R. Cen, M. Doi, J. Gunn, v. Ivezic, S. Kent, J. Loveday, R. Lupton, J. Munn, R. Nichol, J. Ostriker, D. Schlegel, D. Schneider, M. Tegmark, D. Berk, D. Weinberg, and D. York, “Cosmological parameter analysis including SDSS Lyα forest and galaxy bias: Constraints on the primordial spectrum of fluctuations, neutrino mass, and dark energy,” Physical Review D 71 (May, 2005) 103515.
The quintessence field as a perfect cosmic fluid of constant pressure

[7] J. K. Adelman-McCarthy, J. K. McCarthy, M. A. Agueros, S. S. Allam, K. S. J. Anderson, S. F. Anderson, J. Annis, N. A. Bahcall, I. K. Baldry, J. C. Barentine, A. Berlind, M. Bernardi, M. R. Blanton, W. N. Boroski, H. J. Brewington, J. Brinchmann, J. Brinkmann, R. J. Brumner, T. Budavari, L. N. Carey, M. A. Carr, F. J. Castander, A. J. Connolly, I. Csabai, P. C. Czarapata, J. J. Dalcanton, M. Doi, F. Dong, D. J. Eisenstein, M. L. Evans, X. Fan, D. P. Finkbeiner, S. D. Friedman, J. A. Frieman, M. Fukugita, B. Gillespie, K. Glazebrook, J. Gray, E. K. Grebel, J. E. Gunn, V. K. Gurbani, E. de Haas, P. B. Hall, F. H. Harris, M. Harvanek, S. L. Hawley, J. Hayes, J. S. Hendry, G. S. Hennessy, R. B. Hindsley, C. M. Hirata, C. J. Hogan, D. W. Hogg, D. J. Holmgren, J. A. Holtzman, S. Ichikawa, V. Ivezic, S. Jester, D. E. Johnston, A. M. Jorgensen, M. Jurić, S. M. Kent, S. J. Kleinman, G. R. Knapp, A. Y. Kniazev, R. G. Kron, J. Krzesinski, N. Kuropatkin, D. Q. Lamb, H. Lampeitl, B. C. Lee, R. F. Leger, H. Lin, D. C. Long, J. Loveday, R. H. Lupton, B. Margon, D. Martinez-Galarlo, R. Mandelbaum, T. Matsubara, P. M. McGeehe, T. A. McKay, A. Meiksin, J. A. Munn, R. Nakajima, T. Nash, E. H. Neilson, Jr., H. J. Newberg, P. R. Newman, R. C. Nichol, T. Nicinski, M. Nieto-antisteban, A. Nitta, W. Ollulane, S. Okamura, R. Owen, N. Padmanabhan, G. Pauls, J. Peoples, Jr., J. R. R. Piers, A. C. Pope, D. Pourbaix, T. R. Quinn, G. T. Richards, M. W. Richmond, C. M. Rockosi, D. J. Schlegel, D. P. Schneider, J. Schroeder, R. Scranon, U. Seljak, E. Sheldon, K. Shimassoku, J. A. Smith, V. Smolčić, S. A. Sneden, C. Stoughton, M. A. Strauss, M. SubbaRao, A. S. Szalay, I. Szapudi, P. Szkody, M. Tegmark, A. R. Thakar, D. L. Tucker, A. Uomoto, D. E. Vanden Berk, J. Vanden Berk, M. S. Vogey, W. Voges, N. P. Vogt, L. M. Walkowicz, D. H. Weinberg, A. A. West, S. D. M. White, Y. Xu, B. Yanny, D. R. Yocum, D. G. York, I. Zehavi, S. Zibetti, and D. B. Zucker, “The Fourth Data Release of the Sloan Digital Sky Survey,” The Astrophysical Journal Supplement Series 162 (Jan., 2006) 38–48, 0507711.

[8] K. Abazajian, J. K. Adelman-McCarthy, M. A. Agueros, S. S. Allam, K. S. J. Anderson, S. F. Anderson, J. Annis, N. A. Bahcall, I. K. Baldry, S. Bastian, A. Berlind, M. Bernardi, M. R. Blanton, J. Bochanski, W. N. Boroski, H. J. Brewington, J. W. Briggs, J. Brinkmann, R. J. Brumner, T. Budavári, L. N. Carey, F. J. Castander, A. J. Connolly, K. R. Covey, I. Csabai, J. J. Dalcanton, M. Doi, F. Dong, D. J. Eisenstein, M. L. Evans, X. Fan, D. P. Finkbeiner, S. D. Friedman, J. A. Frieman, M. Fukugita, B. Gillespie, K. Glazebrook, J. Gray, E. K. Grebel, J. E. Gunn, V. K. Gurbani, P. B. Hall, M. Hamabe, D. Harbeck, F. H. Harris, H. C. Harris, M. Harvanek, S. L. Hawley, J. Hayes, T. M. Heckman, J. S. Hendry, G. S. Hennessy, R. B. Hindsley, C. J. Hogan, D. W. Hogg, D. J. Holmgren, J. A. Holtzman, S.-i. Ichikawa, T. Ichikawa, v. Ivezic, S. Jester, D. E. Johnston, A. M. Jorgensen, M. Jurić, S. M. Kent, S. J. Kleinman, G. R. Knapp, A. Y. Kniazev, R. G. Kron, J. Krzesinski, N. Kuropatkin, D. Q. Lamb, H. Lampeitl, B. C. Lee, H. Lin, D. C. Long, J. Loveday, R. H. Lupton, E. Mannery, B. Margon, D. Martínez-Delgado, T. Matsubara, P. M. McGeehe, T. A. McKay, A. Meiksin, B. Menard, J. A. Munn, T. Nash, H. Lin, H. Newberg, J. Nichol, T. Nicinski, M. N. Santos-Tebanan, A. Nitta, S. Okamura, W. O’Mullane, R. Owen, N. Padmanabhan, G. Pauls, J. Peoples, J. R. Pier, A. C. Pope, D. Pourbaix, T. R. Quinn, G. T. Richards, M. W. Richmond, H.-W. Rix, C. M. Rockosi, D. J. Schlegel, D. P. Schneider, J. Schroeder, R. Scranon, U. Seljak, E. Sheldon, K. Shimassoku, J. A. Smith, V. Smolčić, S. A. Sneden, C. Stoughton, M. A. Strauss, M. SubbaRao, A. S. Szalay, I. Szapudi, P. Szkody, M. Tegmark, A. R. Thakar, D. L. Tucker, A. Uomoto, D. E. Vanden Berk, J. Vanden Berk, M. S. Vogey, W. Voges, N. P. Vogt, L. M. Walkowicz, D. H. Weinberg, A. A. West, S. D. M. White, Y. Xu, B. Yanny, D. R. Yocum, D. G. York, I. Zehavi, S. Zibetti, and D. B. Zucker, “The Fourth Data Release of the Sloan Digital Sky Survey,” The Astronomical Journal 129 (Mar., 2005) 1755–1759, 0410239.

[9] D. N. Spergel, R. Bean, O. Dore, M. R. Nolta, C. L. Bennett, J. Dunkley, G. Hinshaw, N. Jarosik, E. Komatsu, L. Page, H. V. Peiris, L. Verde, M. Halpern, R. S. Hill, A. Kogut, M. Limon, S. S.
The quintessence field as a perfect cosmic fluid of constant pressure

Meyer, N. Odegard, G. S. Tucker, J. L. Weiland, E. Wollack, and E. L. Wright, “Wilkinson Microwave Anisotropy Probe (WMAP) Three Year Results: Implications for Cosmology,” *The Astrophysical Journal Supplement Series* 170 (June, 2007) 377–408, [10].

[10] L. Page, G. Hinshaw, E. Komatsu, M. R. Nolta, D. N. Spergel, C. L. Bennett, C. Barnes, R. Bean, O. Dore, J. Dunkley, M. Halpern, R. S. Hill, N. Jarosik, A. Kogut, M. Limon, S. S. Meyer, N. Odegard, H. V. Peiris, G. S. Tucker, L. Verde, J. L. Weiland, E. Wollack, and E. L. Wright, “Three year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Polarization Analysis,” *The Astrophysical Journal Supplement Series* 170 (June, 2007) 335–376, [11].

[11] G. Hinshaw, M. R. Nolta, C. L. Bennett, R. Bean, O. Dore, M. R. Greason, M. Halpern, R. S. Hill, N. Jarosik, A. Kogut, E. Komatsu, M. Limon, N. Odegard, S. S. Meyer, L. Page, H. V. Peiris, D. N. Spergel, G. S. Tucker, L. Verde, J. L. Weiland, E. Wollack, and E. L. Wright, “Three year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Temperature Analysis,” *The Astrophysical Journal Supplement Series* 170 (June, 2007) 288–334, [12].

[12] N. Jarosik, C. Barnes, M. R. Greason, R. S. Hill, M. R. Nolta, N. Odegard, J. L. Weiland, R. Bean, C. L. Bennett, O. Dore, M. Halpern, G. Hinshaw, A. Kogut, E. Komatsu, M. Limon, S. S. Meyer, L. Page, D. N. Spergel, G. S. Tucker, E. Wollack, and E. L. Wright, “Three year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Beam Profiles, Data Processing, Radiometer Characterization, and Systematic Error Limits,” *The Astrophysical Journal Supplement Series* 170 (June, 2007) 263–287, [13].

[13] M. Kunz and D. Sapone, “Crossing the phantom divide,” *Physical Review D* 74 (2006) 123503, [14].

[14] S. Nobbenhuis, “The cosmological constant problem, an inspiration for new physics,” *Arxiv preprint gr-qc/0609011* (2006).

[15] L. Amendola and S. Tsujikawa, *Dark Energy, Theory and Observations*. CUP, 2010.

[16] S. Kachru, M. Schulz, and E. Silverstein, “Self-tuning flat domain walls in 5D gravity and string theory,” *Physical Review D* 62 (2000) 045021, [17].

[17] S.-H. Henry Tye and I. Wasserman, “Brane World Solution to the Cosmological Constant Problem,” *Physical Review Letters* 86 (Feb., 2001) 1682–1685, [18].

[18] J. Yokoyama, “Cosmological Constant From Degenerate Vacua,” *Physical Review Letters* 88 (Mar., 2002) 8, [19].

[19] G. L. Kane, M. J. Perry, and A. N. Zytkow, “A Possible Mechanism for Generating a Small Positive Cosmological Constant,” *arXiv:hep-th/0311152* (2003) [20].

[20] S. Mukohyama and L. Randall, “Dynamical Approach to the Cosmological Constant,” *Physical Review Letters* 92 (2004) 4, [21].

[21] A. Dolgov and F. Urban, “Dynamical vacuum energy via adjustment mechanism,” *Physical Review D* 77 (2008) 083503, [22].

[22] B. M. Deiss, “Cosmic Dark Energy Emerging from Gravitationally Effective Vacuum Fluctuations,” *arXiv:1209.5386* (2012) [23].

[23] E. J. Copeland, M. Sami, and S. Tsujikawa, “Dynamics of dark energy,” *International Journal of Modern Physics D* 15 (2006) 1753–1936, [24].

[24] Y.-F. Cai, E. N. Saridakis, M. R. Setare, and J.-Q. Xia, “Quintom cosmology: Theoretical implications and observations,” *Physics Reports* 493 (2010) 1–60, [25].

[25] A. Kamenshchik, U. Moschella, and V. Pasquier, “An alternative to quintessence,” *Physics Letters B* 511 (July, 2001) 265–268, [26].

[26] M. Bento, O. Bertolami, and a. Sen, “Generalized Chaplygin gas, accelerated expansion, and dark energy-matter unification,” *Physical Review D* 66 (2002) 043507, [27].

[27] N. Bilić, G. Tupper, and R. Viollier, “Unification of dark matter and dark energy: the inhomogeneous Chaplygin gas,” *Physics Letters B* 535 (2002) 17, [28].

[28] N. Bilić, G. Tupper, and R. Viollier, “Cosmological tachyon condensation,” *Physical Review D* 80 (2009) 023515, [29].
The quintessence field as a perfect cosmic fluid of constant pressure

[29] L. Amendola, F. Finelli, C. Burigana, and D. Carturan, “WMAP and the generalized Chaplygin gas,” *Journal of Cosmology and Astroparticle Physics* **2003** (2003) 005–005.

[30] H. Sandvik, M. Tegmark, M. Zaldarriaga, and I. Waga, “The end of unified dark matter?,” *Physical Review D* **69** (2004) 123524, [0311544](https://dx.doi.org/10.1103/PhysRevD.69.123524).

[31] B. Ratra and P. J. Peebles, “Cosmological Consequences of a Rolling Homogeneous Scalar Field,” *Phys. Rev. D* **37** (1988) 3406.

[32] A. J. Christopherson and K. a. Malik, “The non-adiabatic pressure in general scalar field systems,” *Physics Letters B* **675** (2009) 159–163.

[33] A. Vikman, “Can dark energy evolve to the phantom?,” *Physical Review D* **71** (2005) 023515, [0407107](https://dx.doi.org/10.1103/PhysRevD.71.023515).

[34] G. F. R. Ellis, R. Maartens, and M. A. H. MacCallum, *Relativistic Cosmology*. CUP, 2012.

[35] M. Makler, S. Q. de Oliveira, and I. Waga, “Constraints on the generalized Chaplygin gas from supernovae observations,” *Physics Letters B* **555** (2003) 1.

[36] R. Rajaraman, *Solitons and instantons*. Amsterdam: North-Holland, 1982.

[37] T. Vachaspati, *Kinks and domain walls*. CUP, 2006.

[38] S. Dodelson, *Modern Cosmology*. Elsevier(Singapore) Pte Ltd, 2003.

[39] J. Kristiansen, O. y. Elgarøy, and H. k. Dahle, “Using the cluster mass function from weak lensing to constrain neutrino masses,” *Physical Review D* **75** (2007) 083510, [0611761](https://dx.doi.org/10.1103/PhysRevD.75.083510).