The Schrödinger system $H = -\frac{1}{2} e^{\Upsilon(t-t_0)} \partial_{xx} + \frac{1}{2} \omega^2 e^{-\Upsilon(t-t_0)} x^2$

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Abstract. In this paper, we attack the specific time-dependent Hamiltonian problem $H = -\frac{1}{2} e^{\Upsilon(t-t_0)} \partial_{xx} + \frac{1}{2} \omega^2 e^{-\Upsilon(t-t_0)} x^2$. This corresponds to a time-dependent mass (TM) Schrödinger equation. We give the specific transformations to i) the more general quadratic (TQ) Schrödinger equation and to ii) a different time-dependent oscillator (TO) equation. For each Schrödinger system, we give the Lie algebra of space-time symmetries, the number states, the coherent states, the squeezed-states and the time-dependent $\langle x \rangle$, $\langle p \rangle$, $(\Delta x)^2$, $(\Delta p)^2$, and uncertainty product.
1 Introduction

In recent work [1, 2], we discussed general time-dependent quadratic (TQ) Schrödinger equations of the form

\begin{equation}
S_1 \Phi(x, t) = \{- [1 + k(t)] P^2 + 2T + h(t)D + g(t)P \\
-2h^{(2)}(t)X^2 - 2h^{(1)}(t)X - 2h^{(0)}(t)I \} \Phi(x, t) = 0.
\end{equation}

(1)

It was shown how to solve them by i) first performing a unitary transformation to a time-dependent mass (TM) equation, and then ii) making a change of time variable to yield a time-dependent oscillator (TO) equation which in principle can be solved. One can then work backwards to find the TM and TQ solutions.

Elsewhere [3], we went into detail on how to solve a specific subclass of cases, TM equations with only time-dependent \( P^2 \) and \( X^2 \) terms. [We will refer to equations from this paper as, e.g., Eq. (3)-22.] These Hamiltonians are parametrized as

\begin{equation}
\hat{H}_2 = \frac{1}{2} e^{-2\nu(t)} P^2 + h^{(2)}(t)e^{2\nu(t)} X^2.
\end{equation}

(2)

In this paper, we will demonstrate this procedure by examining the TM system

\begin{equation}
\hat{H}_2 = \frac{1}{2} \Upsilon(t-t_o) P^2 + \frac{\omega^2}{2} e^{-\Upsilon(t-t_o)} X^2.
\end{equation}

(3)

Starting with foresight with what we know to be the associated TQ equation [4], this problem is:

\begin{equation}
TQ : \quad S_1 \Phi(x, t) = \{- P^2 + 2T + \Upsilon D - \omega^2 X^2 \} \Phi(x, t) = 0,
\end{equation}

(4)

\begin{equation}
H_1 = \frac{1}{2} P^2 - \frac{\Upsilon}{2} D + \frac{\omega^2}{2} X^2.
\end{equation}

(5)

Note that this TQ equation is actually time-independent.

This TQ equation yields a TM equation by the unitary transformation

\begin{equation}
R(0, \nu, 0) = \exp \{-i \nu D \} = \exp \left\{-i \frac{\Upsilon}{2} (t-t_o) D \right\},
\end{equation}

(6)
where $\nu$ is a real function of $t$. This yields:

$$TM : \hat{S}_2\hat{\Theta}(x, t) = \left\{ -e^{\Upsilon(t-t_0)} P^2 + 2T - \omega^2 e^{-\Upsilon(t-t_0)} X^2 \right\} \hat{\Theta}(x, t) = 0,$$

(7)

from which the $TM$ Hamiltonian of Eq. (3) follows. This $TM$ equation is the defining equation of our problem. On its own, it has been the object of a number of investigations [1], [5]-[18].

The $TM$ equation yields a $TO$ equation by the change of time variable:

$$t' - t'_o = \frac{1}{\Upsilon} \left[ e^{\Upsilon(t-t_o)} - 1 \right].$$

(8)

One finds:

$$TO : \quad S_3\Psi(x, t') = \left\{ -P^2 + 2T' - \frac{\omega^2}{[1 + \Upsilon(t-t_o)]^2} X^2 \right\} \Psi(x, t') = 0,$$

(9)

$$H_3 = \frac{1}{2} P^2 + \frac{\omega^2}{2[1 + \Upsilon(t-t_o)]^2} X^2.$$

(10)

Depending upon the sign of $\Upsilon$, certain restrictions upon $t'$ apply. If $\Upsilon > 0$, then $t' - t'_o$ will lie in the interval $[0, +\infty)$. If $\Upsilon < 0$, $t' - t'_o$ must lie in the interval $[0, 1/|\Upsilon|)$. Because i) three equations ($TQ$, $TM$, $TO$) are being considered, ii) there is a condition on the relative sizes of $|\Upsilon|^2$ and $\omega^2$, and iii) the sign of $\Upsilon$ is important, there are 18 separate cases that can be discussed (6 for each Schrödinger equation). In what now follows we will look at specific cases for each equation as illustrative examples.

In Section 2, we derive time-dependent functions from which one can compute the time-dependent Lie symmetries that form the basis of the oscillator algebras. In Sections 4, 5, and 6, number states, coherent states, and squeezed states are obtained. In Section 7, we obtain coherent-state and squeezed-state expectation values, uncertainties, and uncertainty products for the three systems, and discuss the classical motion.

2 Solutions for the Time-Dependent Functions

2.1 The $TO$ Functions

To find the symmetries associated with the $TO$ equation, $S_3\Psi = 0$, we must obtain the complex solutions $\xi$ and $\bar{\xi}$ to the differential Eq. (3)-19. For our problem, this equation has the form:

$$\ddot{\gamma} + \frac{\omega^2}{[1 + \Upsilon(t' - t'_o)]^2} \gamma = 0.$$

(11)
The solutions $\xi$ and their complex conjugates $\bar{\xi}$ satisfy the Wronskian (12)

$$W_{t'}(\xi, \bar{\xi}) = -i,$$

where subscript indicates that $\xi$ and $\bar{\xi}$ are to be differentiated with respect to $t'$.

Make the following change of variables in Eq. (11):

$$\tau = 1 + \Upsilon(t' - t'_o), \quad w(\tau) = \gamma(t).$$

With this change of variables, Eq. (11) becomes

$$\frac{d^2 w}{d\tau^2} + \frac{\omega^2}{\Upsilon^2 \tau^2}w = 0.$$  (14)

When $\Upsilon > 0$, then $\tau \in [1, \infty)$. When $\Upsilon < 0$, then $\tau \in (0, 1]$. In either case, $\tau$ is always positive.

The general solutions to Eq. (14) are derived in Appendix A. There are three different classes of solutions, depending upon the relative magnitudes of $\Upsilon^2$ and $\omega^2$. The three classes of real solutions to Eq. (14) are:

1. $\Upsilon^2 > 4\omega^2$  
   $$w_1(\tau) = \sqrt{\tau} \exp\left(-\frac{\Delta^2}{2} \ln \tau\right), \quad w_2(\tau) = \sqrt{\tau} \exp\left(-\frac{\Delta^2}{2} \ln \tau\right).$$  (15)

2. $\Upsilon^2 = 4\omega^2$  
   $$w_1(\tau) = \sqrt{\tau}, \quad w_2(\tau) = \sqrt{\tau} \ln \tau.$$  (16)

3. $\Upsilon^2 < 4\omega^2$  
   $$w_1(\tau) = \sqrt{\tau} \cos\left(-\frac{\Delta^2}{2} \ln \tau\right), \quad w_2(\tau) = \sqrt{\tau} \sin\left(-\frac{\Delta^2}{2} \ln \tau\right),$$
   $$\Delta^2 \equiv \left|1 - 4\omega^2/\Upsilon^2\right|.$$  (17)

Real solutions to Eq. (14) are obtained from these solutions by combining Eqs. (13) and (15) to (17). We write the solutions as

$$\gamma_1(t') = C_1 w_1(\tau), \quad \gamma_2(t') = C_2 w_2(\tau).$$  (19)

The constants $C_1$ and $C_2$ are chosen so that the Wronskian of the real solutions is

$$W_{t'}(\gamma_1, \gamma_2) = \gamma_1 \dot{\gamma}_2 - \dot{\gamma}_1 \gamma_2 = 1.$$  (20)

Both $C_1$ and $C_2$ will depend upon the magnitude and sign of $\Upsilon$ and this gives rise to two subclasses of solutions. For example, for class-1 solutions, when $\Upsilon > 0$, we find that

$$C_1 = \sqrt{\frac{1}{\Upsilon \Delta}} = C_2.$$  (21)
For class-2 solutions, when $\Upsilon < 0$ ($\Upsilon = -|\Upsilon|$), then

$$C_1 = \sqrt{\frac{1}{|\Upsilon|\Delta}} = -C_2.$$  \hfill (22)

The complex solutions satisfying the Wronskian (12) are

$$\xi(t') = \sqrt{\frac{1}{2\Upsilon}} (\gamma_1(t') + i\gamma_2(t')),$$  \hfill (23)

and their complex conjugates, $\bar{\xi}(t')$. Citing the class-1 and class-2 examples again, we obtain solutions depending upon the sign of $\Upsilon$. For class-1 ($\Upsilon > 0$) we have

$$\xi(t') = \sqrt{\frac{1}{2\Upsilon\Delta}} \left( e^{-\Delta\frac{2}{\hbar} \ln \tau} + i e^{\Delta\frac{2}{\hbar} \ln \tau} \right),$$  \hfill (24)

and for class-2 ($\Upsilon < 0$)

$$\xi(t') = \sqrt{\frac{1}{2|\Upsilon|\Delta}} \left( e^{-\Delta\frac{2}{\hbar} \ln \tau} - i e^{\Delta\frac{2}{\hbar} \ln \tau} \right).$$  \hfill (25)

In order to compute the $os(1)$ generators, expectation values, uncertainties, and uncertainty products associated with the $TO$ Schrödinger equation, several other functions are required. These functions, $\phi_j$, $j = 1, 2, 3$, [see Eq. (3)-20], and their first and second derivatives can all be calculated from $\xi$ and $\bar{\xi}$ and their derivatives. They are given in Appendix B, Table B-1.

### 2.2 The TM Functions

The $TM$ functions, $\dot{\xi}$, $\ddot{\xi}$, $\dddot{\xi}$, $\hat{\phi}_j$, $\dot{\hat{\phi}}_j$, $\ddot{\phi}_j$, $j = 1, 2, 3$, can be obtained from the composition of the corresponding $TO$ functions and the mapping of Eq. (3). [See Eq. (3)-32.] These $TM$ functions are compiled in Appendix B, Table B-2.

### 2.3 The TQ Functions

The $TQ$-functions, $\Xi_P$ and $\Xi_X$, their complex conjugates, and $C_{3,T}$, $C_{3,D}$, and $C_{3,X^2}$ can be computed from the $TM$ functions in Table B-2 using Eqs. (3)-41 to (3)-44. In particular,

$$\Xi_P(t) = \dot{\xi}(t)e^{\nu} = \xi(t)e^{-\chi/2}, \quad \Xi_X(t) = \dot{\xi}(t)e^{-\nu} = \xi(t)e^{\chi/2}$$  \hfill (26)

since $\kappa = 0$ and where $\chi = \Upsilon(t - t_o)$. The $TQ$-functions are listed in Appendix B, Table B-3.
2.4 Initial Values of Functions

Initial values for some of these functions are also needed. For $TO$, the initial values are obtained by setting $t' = t'_o$ in the functions in Table B-1. Similarly, for $TM$ and $TQ$, the initial values result by setting $t = t_o$ in the functions in Tables B-2 and B-3. When $t' = t'_o$ for $TO$, the variable $\tau = 1$, regardless of the sign of $\Upsilon$. When $t = t_o$ for $TM$ and $TQ$, the variable $\chi = 0$.

3 The Algebras

Although each class of solutions will produce a Schrödinger algebra, ($SA^2_1$), we are only interested in its oscillator subalgebra os(1). So, the three Schrödinger equations will have six isomorphic os(1) algebras: three corresponding to $\Upsilon > 0$ and three to $\Upsilon < 0$, for a total of 18 isomorphic Lie algebras. We use the following notation to distinguish between the eighteen different cases, i.e., systems:

Notation: \{TX, R(\Upsilon, \omega), \pm\}, where $TX = \{TO, TM, TQ\}$, $R(\Upsilon, \omega)$ specifies one of the three relationships between $\Upsilon^2$ and $4\omega^2$, and $\pm$ indicates the sign of $\Upsilon$.

The basis operators for the $os(1)$ algebras, $\{M_j, J_j\pm\}$, can be constructed from the appropriate functions in Tables B-1, B-2, and B-3, using the third operator in Eq. (3)-17 and Eq. (3)-18 for $TO$, Eqs.(3)-30 and (3)-31 for $TM$, and Eq. (3)-40 for the $TQ$ system. There are too many algebras to list the operators of each. So, as examples, we give the operators for a few of the systems $\{TX, R(\Upsilon, \omega), \pm\}$:

A Case of a $TO$ os(1) Algebra:

\{TO, \Upsilon^2 < 4\omega^2 +\} : 

\begin{align*}
J_{3-} &= i\sqrt{\frac{1}{15}} \exp\left[i\frac{\Delta}{2} \ln \tau\right] \left\{\sqrt{\tau}P - \frac{\Upsilon}{2} (1 + i\Delta) \frac{1}{\sqrt{\tau}} X \right\}, \\
J_{3+} &= i\sqrt{\frac{1}{15}} \exp\left[-i\frac{\Delta}{2} \ln \tau\right] \left\{-\sqrt{\tau}P + \frac{\Upsilon}{2} (1 - i\Delta) \frac{1}{\sqrt{\tau}} X \right\}, \\
M_3 &= \frac{2}{\tau_3} \tau T' - \frac{1}{\Delta} D, \\
\end{align*}

(27)

where $\tau$ is given by Eq. (13).
Cases of $os(1)$ TM Algebras: Recalling that $\chi = \Upsilon(t - t_0)$:

\[
\{TM, \Upsilon^2 < 4\omega^2, +\} : \quad \hat{J}_2^- = i\sqrt{1/\Upsilon} e^{\chi/2} e^{i\hat{\Delta} \chi} \left\{ P - \frac{\Upsilon}{2} (1 + i\Delta) e^{-\chi} X \right\}, \\
\hat{J}_2^+ = i\sqrt{1/\Upsilon} e^{\chi/2} e^{-i\hat{\Delta} \chi} \left\{ -P + \frac{\Upsilon}{2} (1 - i\Delta) e^{-\chi} X \right\}, \\
\hat{M}_2 = \frac{2}{1+\Delta} T - \frac{1}{\Delta} D.  \tag{28}
\]

\[
\{TM, \Upsilon^2 < 4\omega^2, -\} : \quad \hat{J}_2^- = i\sqrt{1/\Upsilon} e^{\chi/2} e^{-i\hat{\Delta} \chi} \left\{ P + \frac{\Upsilon}{2} (1 - i\Delta) e^{-\chi} X \right\}, \\
\hat{J}_2^+ = i\sqrt{1/\Upsilon} e^{\chi/2} e^{i\hat{\Delta} \chi} \left\{ -P - \frac{\Upsilon}{2} (1 + i\Delta) e^{-\chi} X \right\}, \\
\hat{M}_2 = \frac{2}{1+\Delta} T + \frac{1}{\Delta} D.  \tag{29}
\]

\[
\{TM, \Upsilon^2 = 4\omega^2, +\} : \quad \hat{J}_2^- = i\sqrt{1/\Upsilon} e^{\chi/2} \left\{ (1 + i\chi) P - \Upsilon e^{-\chi} \left[ \frac{1}{2} + i \left( 1 + \frac{1}{2}\chi \right) \right] \right\}, \\
\hat{J}_2^+ = i\sqrt{1/\Upsilon} e^{\chi/2} \left\{ -(1 - i\chi) P + \Upsilon e^{-\chi} \left[ \frac{1}{2} - i \left( 1 + \frac{1}{2}\chi \right) \right] \right\}, \\
\hat{M}_2 = \frac{1}{\Upsilon} (1 + \chi^2) T - \frac{1}{\Upsilon} (1 + \chi)^2 D + \frac{1}{\Upsilon} e^{-\chi}(1 + \chi)X^2.  \tag{30}
\]

\[
\{TM, \Upsilon^2 > 4\omega^2, +\} \\
\hat{J}_2^- = i\sqrt{1/\Upsilon} e^{\chi/2} \left\{ \left( e^{-\hat{\Delta} \chi} + i e^{\hat{\Delta} \chi} \right) P - \frac{\Upsilon}{2} e^{-\chi} \left[ (1 - \Delta) e^{-\hat{\Delta} \chi} + i(1 + \Delta) e^{\hat{\Delta} \chi} \right] X \right\}, \\
\hat{J}_2^+ = i\sqrt{1/\Upsilon} e^{\chi/2} \left\{ -\left( e^{-\hat{\Delta} \chi} - i e^{\hat{\Delta} \chi} \right) P + \frac{\Upsilon}{2} e^{-\chi} \left[ (1 - \Delta) e^{-\hat{\Delta} \chi} - i(1 + \Delta) e^{\hat{\Delta} \chi} \right] X \right\}, \\
\hat{M}_2 = \frac{1}{\Upsilon \Delta} \left( e^{-\Delta \chi} + e^{\Delta \chi} \right) T - \frac{1}{\Upsilon \Delta} \left[ (1 - \Delta) e^{-\Delta \chi} + (1 + \Delta) e^{\Delta \chi} \right] D \\
-\frac{\Upsilon}{\Delta} e^{-\chi} \left[ -(1 - \Delta) e^{-\Delta \chi} + (1 + \Delta) e^{\Delta \chi} \right] X^2.  \tag{31}
\]

A Case of an $os(1)$ TQ Algebra:

\[
\{TQ, \ Upsilon^2 < 4\omega^2, + \} : \quad J_{1-} = i\sqrt{1/\Upsilon} \exp \left( i\frac{\Delta}{2} \chi \right) \left[ P - \frac{\Upsilon}{2} (1 + i\Delta) X \right],  \\
J_{1+} = i\sqrt{1/\Upsilon} \exp \left( -i\frac{\Delta}{2} \chi \right) \left[ -P + \frac{\Upsilon}{2} (1 - i\Delta) X \right],  \\
M_1 = \frac{2}{1+\Delta} T.  \tag{32}
\]

4 Number States

Combining Section 4 of Ref. [3] with the time-dependent functions for this problem given in Tables B-1 to B-3, one can exploit the $os(1)$ algebraic structure to obtain the number states
for the 18 systems characterized by \{TX, R(\Upsilon, \omega), \pm\}. We give examples below:

**A Case of TO Number States:** For the system \{TO, \Upsilon^2 < 4\omega^2, +\}, for which the Lie symmetry operators are given in Eq. (27), the wave functions are

\[
\Psi_n(x, t) = \frac{1}{\sqrt{2\pi n!}} \exp \left( i \frac{\Upsilon}{2} x^2 \right) H_n \left( \frac{\Upsilon}{2} x \right) \left( \frac{\Upsilon}{2\pi} \right)^{\frac{1}{4}} \exp \left[ -\frac{i}{2} \left( n + \frac{1}{2} \right) \Delta \ln \tau \right].
\]  

[See Eq. (3)-54.] Here and below, the \(H_n\) are Hermite polynomials.

**A Case of TM Number States:** For the system \{TM, \Upsilon^2 < 4\omega^2, +\}, for which the Lie symmetry operators are given in Eq. (31), the wave functions are [see Eq. (3)-58]

\[
\Theta_n(x, t) = \frac{1}{\sqrt{2\pi n!}} \exp \left( i \frac{\Upsilon}{2} x^2 \right) H_n \left( \frac{\Upsilon}{2} x \right) \left( \frac{\Upsilon}{2\pi} \right)^{\frac{1}{4}} \exp \left[ -\frac{i}{2} \left( n + \frac{1}{2} \right) \Delta \chi \right].
\]  

**A Case of TQ Number States:** For the system \{TQ, \Upsilon^2 < 4\omega^2, +\}, for which the Lie symmetry operators are given in Eq. (32), the wave functions are [see Eq. (3)-63]

\[
\Phi_n(x, t) = \frac{1}{\sqrt{2\pi n!}} \exp \left( i \frac{\Upsilon}{2} x^2 \right) H_n \left( \frac{\Upsilon}{2} x \right) \exp \left( -\frac{\Upsilon}{2} x^2 \right) \left( \frac{\Upsilon}{2\pi} \right)^{\frac{1}{4}} \exp \left[ -\frac{i}{2} \left( n + \frac{1}{2} \right) \Delta \chi \right].
\]  

5 Coherent States

Combining the time-dependent functions for this problem given in Tables B-1 to B-3 (this time) with Section 5 of Ref. [3], one can exploit the os(1) algebraic structure to obtain the coherent states for the 18 systems characterized by \{TX, R(\Upsilon, \omega), \pm\}. We give examples below:

**A Case of TO Coherent States:** For specific cases, we use the appropriate \(t'\)-dependent functions in Table B-1. For example, for the case \{TO, \Upsilon^2 < 4\omega^2, +\}, the coherent states are [see Eq. (3)-66]

\[
\Psi_\alpha(x, t') = \left( \frac{\Upsilon}{2\pi} \right)^{\frac{1}{4}} e^{-\frac{i}{4} (1 + i \Delta) \ln \tau} \exp \left\{ -\frac{1}{2} \left( \frac{\Upsilon}{2\pi} \right) \left[ x - X_3^+(\alpha) \right]^2 \right\} \times \exp \left\{ i \left[ (\frac{\Upsilon}{\pi}) x^2 + \left( \frac{\Upsilon}{2\pi} \right) \left( x - \frac{1}{2} X_3^+(\alpha) \right) X_3^- (\alpha) \right] \right\},
\]  

\[
X_3^+(\alpha) = p_o \left( \frac{2\pi}{\sqrt{\Upsilon^2}} \right) \sin \left( \frac{\Delta}{2} \ln \tau \right) + x_o \left( \frac{\sqrt{\Upsilon^2}}{\Delta} \right) \left[ \Delta \cos \left( \frac{\Delta}{2} \ln \tau \right) - \sin \left( \frac{\Delta}{2} \ln \tau \right) \right],
\]  

\[
X_3^- (\alpha) = p_o \left( \frac{2\pi}{\sqrt{\Upsilon^2}} \right) \cos \left( \frac{\Delta}{2} \ln \tau \right) - x_o \left( \frac{\sqrt{\Upsilon^2}}{\Delta} \right) \left[ \cos \left( \frac{\Delta}{2} \ln \tau \right) + \Delta \sin \left( \frac{\Delta}{2} \ln \tau \right) \right].
\]
Recall that \( \tau = 1 + \Upsilon(t' - t_0) \).

**A Case of TM Coherent States:** For \( \{TM, \Upsilon^2 < 4\omega^2, +\} \), we find from Eq. (3)-68 that

\[
\hat{\Theta}_\alpha(x, t) = \left( \frac{\Upsilon \Delta}{2\pi} \right)^{\frac{1}{4}} e^{-\frac{i}{4}(1+i\chi)} \exp \left\{ -\frac{1}{2} \left( \frac{\Upsilon \Delta}{2\omega^2} \right) \left[ x - \hat{X}_2^\pm(\alpha) \right]^2 \right\} \times \exp \left\{ i \left[ \left( \frac{\Upsilon}{4\omega^2} \right) x^2 + \left( \frac{\Upsilon \Delta}{2\omega^2} \right) \left( x - \frac{i}{2} \hat{X}_2^\pm(\alpha) \right) \hat{X}_2^\pm(\alpha) \right] \right\},
\]

(39)

Here, \( \hat{X}_2^\pm(\alpha) = p_o \left( \frac{2\chi/\Upsilon}{\Delta} \right) \sin \left( \frac{\chi}{2} \right) + x_o \left( \frac{\chi/\Upsilon}{\Delta} \right) \left[ \Delta \cos \left( \frac{\chi}{2} \right) - \sin \left( \frac{\chi}{2} \right) \right] \),

(40)

and

\[ X_2^\pm(\alpha) = p_o \left( \frac{2\chi/\Upsilon}{\Delta} \right) \cos \left( \frac{\chi}{2} \right) - x_o \left( \frac{\chi/\Upsilon}{\Delta} \right) \left[ \cos \left( \frac{\chi}{2} \right) + \Delta \sin \left( \frac{\chi}{2} \right) \right]. \]

(41)

A Case of TQ Coherent States: Coherent states for each TQ-system [Eq. (3)-74] of this problem can be computed from the functions in Table B-3. For example, the DOCS wave function for \( \{TQ, \Upsilon^2 < 4\omega^2, +\} \) is

\[
\Phi_\alpha(x, t) = \left( \frac{\Upsilon \Delta}{2\pi} \right)^{\frac{1}{4}} e^{-i\chi/4} \exp \left\{ -\frac{1}{2} \left( \frac{\Upsilon \Delta}{2\omega^2} \right) \left[ x - X_1^+(\alpha) \right]^2 \right\} \times \exp \left\{ i \left[ \left( \frac{\Upsilon}{4\omega^2} \right) x^2 + \left( \frac{\Upsilon \Delta}{2\omega^2} \right) \left( x - \frac{i}{2} X_1^+(\alpha) \right) X_1^-(\alpha) \right] \right\},
\]

(42)

Here, \( X_1^+(\alpha) = p_o \left( \frac{2\chi/\Upsilon}{\Delta} \right) \sin \left( \frac{\chi}{2} \right) + x_o \left( \frac{\chi/\Upsilon}{\Delta} \right) \left[ \Delta \cos \left( \frac{\chi}{2} \right) - \sin \left( \frac{\chi}{2} \right) \right] \),

(43)

and

\[ X_1^-(\alpha) = p_o \left( \frac{2\chi/\Upsilon}{\Delta} \right) \cos \left( \frac{\chi}{2} \right) - x_o \left( \frac{\chi/\Upsilon}{\Delta} \right) \left[ \cos \left( \frac{\chi}{2} \right) + \Delta \sin \left( \frac{\chi}{2} \right) \right]. \]

(44)

6 Squeezed States

Finally, combining the time-dependent functions for this problem given in Tables B-1 to B-3 (this time) with Section 6 of Ref. [3], one can exploit the os(1) algebraic structure to obtain the coherent states for the 18 systems characterized by \( \{TX, R(\Upsilon, \omega), \pm\} \). We give examples below:

**A Case of TO Squeezed States:** The squeezed-state wave functions for the \( \{TO, \Upsilon^2 < 4\omega^2, +\} \) system are obtained from Eq. (3)-91 and the functions in Table B-1. They have the
form

\[ \Psi_{\alpha, z}(x, t') = \left( \frac{1}{2\pi Q_3} \right)^{\frac{1}{4}} \left(\frac{e^{-\frac{\chi}{2} \ln \tau} + e^{\frac{\chi}{2} \ln \tau - \theta}}{e^{\frac{3}{2} \ln \tau} + e^{\frac{3}{2} \ln \tau - \phi}} \tanh r \right)^{\frac{1}{4}} \exp \left\{ -\frac{1}{4} \frac{1}{Q_3} \left[ x - X^+_{3}(\alpha, z) \right]^2 \right\} \]

\times \exp \left\{ i \left[ \frac{1}{4} \frac{R_3}{Q_3} x^2 + \frac{1}{2} \frac{1}{Q_3} \left( x - \frac{1}{2} X^+_{3}(\alpha, z) \right) X^-_{3}(\alpha, z) \right]\right\}, \quad (45)

\[ Q_3 = \frac{\tau}{ \Upsilon \Delta} \left(\cosh 2r + \cos (\Delta \ln \tau - \theta) \sinh 2r \right) \quad (46) \]

\[ R_3 = \frac{\tau}{ \Upsilon \Delta} \cosh 2r + \frac{\cos (\Delta \ln \tau - \theta) - \Delta \sin (\Delta \ln \tau - \theta) \sinh 2r \cosh 2r + \cos (\Delta \ln \tau - \theta) \sinh 2r, \quad (47)\]

where \( X^+_{3}(\alpha) \) and \( X^-_{3}(\alpha) \) are given in Eqs. (B7) and Eq. (B8). Also, from Eq. (3-94), one has

\[ X^-_{3}(\alpha, z) = X^+_{3}(\alpha) \cosh 2r + Y^+_{3}(\alpha, \theta) \sinh 2r, \quad (48) \]

\[ Y^+_{3}(\alpha, \theta) = p_o \left( \frac{2\chi}{\Upsilon \Delta} \right) \cos \left( \frac{\chi}{2} \chi - \theta \right) - x_o \left( \frac{\chi}{2} \chi - \theta \right) \cos \left( \frac{\chi}{2} \chi - \theta \right) - \Delta \sin \left( \frac{\chi}{2} \chi - \theta \right) \right]. \quad (49) \]

**A Case of TM Squeezed States:** The squeezed-state wave functions for the \( \{TM, \Upsilon^2 < 4\omega^2, +\} \) system are obtained from Eq. (3-96) and the functions in Table B-2. They have the form

\[ \Theta_{\alpha, z}(x, t) = \left( \frac{1}{2\pi Q_2} \right)^{\frac{1}{4}} \left(\frac{e^{-i\frac{\chi}{2} \ln \tau} + e^{i\frac{\chi}{2} \ln \tau - \theta}}{e^{i\frac{\chi}{2} \ln \tau} + e^{-i\frac{\chi}{2} \ln \tau + \theta}} \tanh r \right)^{\frac{1}{4}} \exp \left\{ -\frac{1}{4} \frac{1}{Q_2} \left[ x - \hat{X}^+_{2}(\alpha, z) \right]^2 \right\} \]

\times \exp \left\{ i \left[ \frac{1}{4} \frac{\hat{R}_2}{Q_2} x^2 + \frac{1}{2} \frac{1}{Q_2} \left( x - \frac{1}{2} \hat{X}^+_{2}(\alpha, z) \right) \hat{X}^-_{2}(\alpha, z) \right]\right\}, \quad (50)

\[ \hat{Q}_2 = \frac{\chi}{ \Upsilon \Delta} \left(\cosh 2r + \cos (\Delta \ln \tau - \theta) \sinh 2r \right) \quad (51) \]

\[ \hat{R}_2 = \frac{\tau}{ \Upsilon \Delta} \cosh 2r + \frac{\cos (\Delta \chi - \theta) - \Delta \sin (\Delta \chi - \theta) \sinh 2r \cosh 2r + \cos (\Delta \chi - \theta) \sinh 2r, \quad (52)\]

where \( \hat{X}^+_{2}(\alpha) \) and where \( \hat{X}^-_{2}(\alpha) \) are given in Eqs. (40) and (41). Also, from Eq. (3-101),

\[ \hat{X}^-_{2}(\alpha, z) = \hat{X}^+_{2}(\alpha) \cosh 2r + \hat{Y}^{-}_{2}(\alpha, \theta) \sinh 2r, \quad (53) \]

\[ \hat{Y}^{-}_{2}(\alpha, \theta) = p_o \left( \frac{2\chi}{\Upsilon \Delta} \right) \cos \left( \frac{\chi}{2} \chi - \theta \right) - x_o \left( \frac{\chi}{2} \chi - \theta \right) \cos \left( \frac{\chi}{2} \chi - \theta \right) - \Delta \sin \left( \frac{\chi}{2} \chi - \theta \right) \right]. \quad (54) \]

**A case of TQ Squeezed States:** Finally, for the specific system \( \{TQ, \Upsilon^2 < 4\omega^2, +\} \), the squeezed-state wave functions are obtained from Eq. (3-103) and the functions in Table B-3.
They are

\[ \Phi_{\alpha,z}(x,t) = \left( \frac{1}{2\pi Q_1} \right)^{\frac{1}{4}} \left( \frac{e^{-i\frac{Q_1}{4}x} + e^{i\frac{Q_1}{4}x}}{e^{i\frac{Q_1}{4}x} + e^{-i\frac{Q_1}{4}x}} \tanh r \right)^{\frac{1}{4}} \exp \left\{ -\frac{1}{4} \frac{1}{Q_1} \left[ x - X_1^+(\alpha, z) \right]^2 \right\}, \]

\[ \times \exp \left\{ i \left[ \frac{1}{4} R_1 \frac{1}{Q_1} x^2 + \frac{1}{2Q_1} \left( x - \frac{1}{2} X_1^+(\alpha, z) \right) X_1^-(\alpha, z) \right] \right\}, \quad (55) \]

\[ Q_1 = \frac{1}{\Upsilon \Delta} (\cosh 2r + \cos(\Delta \ln \tau - \theta) \sinh 2r), \quad (56) \]

\[ \frac{R_1}{Q_1} = \frac{\Upsilon (\cosh 2r + [\cos(\Delta \chi - \theta) - \Delta \sin(\Delta \chi - \theta)] \sinh 2r)}{\cosh 2r + \cos(\Delta \chi - \theta) \sinh 2r}, \quad (57) \]

where \( X_1^+(\alpha) \) and \( X_1^-(\alpha) \) are given in Eqs. (43) and (44). Also, from Eqs. (3)-107 and (3)-108, one has

\[ X_1^-(\alpha, z) = X_1^-(\alpha) \cosh 2r + Y_1^-(\alpha, \theta) \sinh 2r, \quad (58) \]

\[ Y_1^-(\alpha, \theta) = p_o \left( \frac{2}{\Upsilon \Delta} \right) \cos \left( \frac{\Delta \chi - \theta}{4} \right) - x_o \left( \frac{1}{4} \right) \cos \left( \frac{\Delta \chi - \theta}{2} \right) - \Delta \sin \left( \frac{\Delta \chi - \theta}{2} \right). \quad (59) \]

7 Expectation Values

7.1 The Dynamical Variables \( \langle x \rangle \) and \( \langle p \rangle \)

Expectation values of \( \langle x \rangle \) and \( \langle p \rangle \) were obtained for general time-dependent quadratic Hamiltonians in Ref. [2] using algebraic methods. Applying these results to our specific Hamiltonians (see Eqs. (88), (87) and (86) of [2], respectively), we now give the expectation values \( \langle x \rangle \) and \( \langle p \rangle \) as a function of time, \( \Upsilon \), and \( \omega \), for the TO, TM, and TQ systems, respectively. These are identical for both the coherent-state and squeezed-state expectation values, as they should be.

First, in Table 1, we give \( \langle x \rangle \).

In Figures 1-3, we show examples of the quantum motion, \( \langle x \rangle \), for the three cases: \( \Upsilon^2 > 4\omega^2 \), \( =, < \) \( 4\omega^2 \). Each figure contains the TO, TM, and TQ solutions.

Beginning with Figure 1, we consider the case \( \Upsilon^2 > 4\omega^2 \). The TO system, starting at time \( (t' - t_o) = 0 \), begins at \( \langle x \rangle = 1 \) but then goes negative after \( (t' - t_o) = 3 \). The TM system has \( \langle x \rangle \) positive but exponentially small for large negative time. It reaches a maximum near \( (t - t_o) = 0 \). Then it becomes negative, growing exponentially, for positive time. In the TQ
system, $\langle x \rangle$ is exponentially large and positive for large negative time, goes through unity, and is exponentially large but negative for large positive time.

Table 1. $\langle x \rangle$, as a function of $\Upsilon$ and $\omega$, for the $TO$, $TM$, and $TQ$ systems. $\Delta^2 = |1 - 4\omega^2/\Upsilon^2|$, $\tau = [1 + \Upsilon(t' - t'_o)]$, $\chi = \Upsilon(t - t_o)$.  

| Class | $\langle x \rangle$ |
|-------|---------------------|
| $\Upsilon^2 > 4\omega^2$ | |
| $TO$ | $\frac{\delta}{\Delta} \sqrt{\tau} \left[ \Delta \cos \left( \frac{\Delta}{2} \ln \tau \right) - \sin \left( \frac{\Delta}{2} \ln \tau \right) \right] + \frac{2\omega}{\Delta} \sqrt{\tau} \sin \left( \frac{\Delta}{2} \ln \tau \right)$ |
| $TM$ | $\frac{\delta}{\Delta} e^{\chi/2} \left[ \Delta \cos \left( \frac{\Delta}{2} \chi \right) - \sin \left( \frac{\Delta}{2} \chi \right) \right] + \frac{2\omega}{\Delta} e^{\chi/2} \sin \left( \frac{\Delta}{2} \chi \right)$ |
| $TQ$ | $\frac{\delta}{\Delta} \left[ \Delta \cos \left( \frac{\Delta}{2} \chi \right) - \sin \left( \frac{\Delta}{2} \chi \right) \right] + \frac{2\omega}{\Delta} \sin \left( \frac{\Delta}{2} \chi \right)$ |
| $\Upsilon^2 = 4\omega^2$ | |
| $TO$ | $x_o \sqrt{\tau} \left( 1 - \frac{1}{2} \ln \tau \right) + \frac{2\omega}{\Delta} \sqrt{\tau} \ln \tau$ |
| $TM$ | $x_o e^{\chi/2} \left( 1 - \frac{1}{2} \chi \right) + \frac{2\omega}{\Delta} e^{\chi/2} \chi$ |
| $TQ$ | $x_o \left( 1 - \frac{1}{2} \chi \right) + \frac{2\omega}{\Delta} \chi$ |
| $\Upsilon^2 < 4\omega^2$ | |
| $TO$ | $\frac{\delta}{\Delta} \sqrt{\tau} \left[ \Delta \cos \left( \frac{\Delta}{2} \ln \tau \right) - \sin \left( \frac{\Delta}{2} \ln \tau \right) \right] + \frac{2\omega}{\Delta} \sqrt{\tau} \sin \left( \frac{\Delta}{2} \ln \tau \right)$ |
| $TM$ | $\frac{\delta}{\Delta} e^{\chi/2} \left[ \Delta \cos \left( \frac{\Delta}{2} \chi \right) - \sin \left( \frac{\Delta}{2} \chi \right) \right] + \frac{2\omega}{\Delta} e^{\chi/2} \sin \left( \frac{\Delta}{2} \chi \right)$ |
| $TQ$ | $\frac{\delta}{\Delta} \left[ \Delta \cos \left( \frac{\Delta}{2} \chi \right) - \sin \left( \frac{\Delta}{2} \chi \right) \right] + \frac{2\omega}{\Delta} \sin \left( \frac{\Delta}{2} \chi \right)$ |

Continuing to Figure 2, we have $\Upsilon^2 = 4\omega^2$. We find that the $TO$ system, starting at time $(t - t_o) = 0$, begins at $\langle x \rangle = 1$, rises slightly, and then decreases. Eventually $\langle x \rangle = 1$ goes negative, near $(t' - t'_o) = 15$, as $-\sqrt{\tau} \ln t'$. The $TM$ system is exponentially small and positive for large negative time, reaches a maximum near $(t - t_o) = 0$, and then becomes exponentially large and negative for positive time. In the $TQ$ system, $\langle x \rangle$ varies as a straight line having negative slope with respect to time.

Lastly, in Figure 3, we have $\Upsilon^2 < 4\omega^2$. The $TO$ system, starting at time $(t' - t'_o) = 0$, begins at $\langle x \rangle = 1$, rises slightly, and then decreases, going negative after $(t' - t'_o) = 3$. Observe that all three $TO$ curves, up to this point, have been similar in nature. But this time there is a difference. This curve has $\cos - \sin$ oscillations with respect to $\ln t'$. The curve reaches a minimum near $(\langle x \rangle, (t' - t'_o)) \approx (-40, 1000)$, then reaches a maximum near $(1400, 10^6), \approx (1400, 10^6)$,
Figure 1: For the case $\Upsilon^2 > 4\omega^2$, we show the expectation values $\langle x \rangle$ as a function of time. Time is $(t' - t'_o)$ for the $TO$ system (dashes) and $(t - t_o)$ for the $TM$ (line) and the $TQ$ (thick line) systems. Here, $x_o = p_o = 1$, $\Upsilon = 5$, and $\omega = 2$. The $TO$ system is restricted to $(t' - t'_o) > 0$. But for $(t' - t'_o) < 0$ this curve matches on to the $TO$ curve for $\Upsilon < 0$; i.e., $\Upsilon = -5$ and $\omega = 2$.

Figure 2: For the case $\Upsilon^2 = 4\omega^2$, the expectation values $\langle x \rangle$ as a function of time. Time is $(t' - t'_o)$ for the $TO$ system (dashes) and $(t - t_o)$ for the $TM$ (line) and the $TQ$ (thick line) systems. Here, $x_o = p_o = 1$, $\Upsilon = 4$, and $\omega = 2$. The $TO$ system is restricted to $(t' - t'_o) > 0$. But for $(t' - t'_o) < 0$ this curve matches on to the $TO$ curve for $\Upsilon < 0$; i.e., $\Upsilon = -5$ and $\omega = 2$. 
Figure 3: For the case $\Upsilon^2 < 4\omega^2$, the expectation values $\langle x \rangle$ as a function of time. Time is $(t' - t'_o)$ for the $TO$ system (dashes) and $(t - t_o)$ for the $TM$ (line) and the $TQ$ (thick line) systems. Here, $x_o = p_o = 1$, $\Upsilon = 3$, and $\omega = 2$. The $TO$ system is restricted to $(t' - t'_o) > 0$. But for $(t' - t'_o) < 0$ this curve matches on to the $TO$ curve for $\Upsilon < 0$; i.e., $\Upsilon = -5$ and $\omega = 2$.

and so on. The $TM$ system has $\langle x \rangle$ being exponentially small for large negative time, going through unity, and then turning rapidly negative. Once again, up to this point, all three $TM$ curves have apparently been similar in nature. But this curve has $\cos - \sin$ oscillations about it. Therefore, for large positive time, the exponential growth is oscillatory. For example, the first minimum past zero is at $(\langle x \rangle, (t - t_o)) \approx (-40, 2)$ The $TQ$ system shows normal-looking oscillatory motion for $\langle x \rangle$.

In Table 2 we give $\langle p \rangle$ for this problem.

7.2 The Classical Motion

For coherent states and squeezed states, $\langle x \rangle$ and $\langle p \rangle$ should obey the classical Hamiltonian equations of motion:

$$\dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x}. \quad (60)$$
Consider the classical Hamiltonians associated with the Schrödinger equations:

\[ T_O : \quad H = \frac{p^2}{2} + \frac{\omega^2 x^2}{2}, \quad \text{(61)} \]

\[ T_M : \quad \dot{H} = e^{\Upsilon(t-t_0)} \frac{p^2}{2} + \frac{\omega^2 e^{-\Upsilon(t-t_0)} x^2}{2}, \quad \text{(62)} \]

\[ T_Q : \quad H = \frac{p^2}{2} - \frac{\Upsilon}{2} xp + \frac{\omega^2 x^2}{2}, \quad \text{(63)} \]

Applying these to the classical equations of motion (60) one obtains ("dot" is \( d/dt \) or \( d/dt' \) as appropriate)

\[ T_O : \quad \dot{x} = p, \quad \dot{p} = -\frac{\omega^2}{\Upsilon} x, \quad \text{(64)} \]

\[ T_M : \quad \dot{x} = e^{\chi} p, \quad \dot{p} = -\omega^2 e^{\chi} x, \quad \text{(65)} \]

\[ T_Q : \quad \dot{x} = p - \frac{\Upsilon}{2} x, \quad \dot{p} = -\omega^2 x + \frac{\Upsilon}{2} p. \quad \text{(66)} \]

Table 2. \( \langle p \rangle \), as a function of \( \Upsilon \) and \( \omega \), for the \( T_O \), \( T_M \), and \( T_Q \) systems. \( \Delta^2 = |1 - 4\omega^2/\Upsilon^2|, \tau = [1 + \Upsilon(t' - t'_0)], \chi = \Upsilon(t - t_0). \)

| Class  | \( \langle p \rangle \) |
|--------|---------------------|
| \( \Upsilon^2 > 4\omega^2 \) | \( -2x_o \frac{\Delta}{\frac{\Upsilon^2}{4\Delta}} \sinh \left( \frac{\Delta}{2} \ln \tau \right) + \frac{p_o}{\tau^2} \left[ \Delta \cosh \left( \frac{\Delta}{2} \ln \tau \right) + \sinh \left( \frac{\Delta}{2} \ln \tau \right) \right] \) |
| \( \Upsilon^2 = 4\omega^2 \) | \( -x_o \frac{1}{\sqrt{\tau}} \ln \tau + p_o \frac{1}{\sqrt{\tau}} \left( 1 + \frac{1}{2} \ln \tau \right) \) |
| \( \Upsilon^2 < 4\omega^2 \) | \( -2x_o \frac{\Delta}{\frac{\Upsilon^2}{4\Delta}} \sin \left( \frac{\Delta}{2} \ln \tau \right) + \frac{p_o}{\tau^2} \left[ \Delta \cos \left( \frac{\Delta}{2} \ln \tau \right) + \sin \left( \frac{\Delta}{2} \ln \tau \right) \right] \) |
The reader can verify that all the expectation values in Tables 1 and 2 satisfy these equations of motion. This specifically demonstrates the general results for quadratic time-dependent Hamiltonians derived in \[2\].

### 7.3 Uncertainties

Similarly, one can calculate \(\langle x^2 \rangle\) and \(\langle p^2 \rangle\) and hence the uncertainties \((\Delta x)^2\) and \((\Delta p)^2\). The uncertainty products \((\Delta x)^2(\Delta p)^2\) for all three types of Schrödinger equations then follow. These are shown in Tables 3, 4, and 5.

Table 3. \((\Delta x)^2\) for the \(TO\), \(TM\), and \(TQ\) systems when \(\Upsilon > 0\). For \(\Upsilon < 0\), change \(\Upsilon\) to \(|\Upsilon|\) and \(\theta\) to \(-\theta\) in the following.

| Class | \((\Delta x)^2\) |
|-------|------------------|
| \(\Upsilon^2 > 4\omega^2\) |
| \(TO\) | \(\frac{7}{24\Delta} \left\{ s^2 \left[ \cosh (\Delta \ln \tau) - \sinh (\Delta \ln \tau) \cos \theta + \sin \theta \right] + \frac{1}{4s^2r} \left[ \cosh (\Delta \ln \tau) + \sinh (\Delta \ln \tau) \cos \theta - \sin \theta \right] \right\} \) |
| \(TM\) | \(\frac{5e^\chi}{24\Delta} \left\{ s^2 \left[ \cosh (\Delta \chi) - \sinh (\Delta \chi) \cos \theta + \sin \theta \right] + \frac{1}{4s^2} \left[ \cosh (\Delta \chi) + \sinh (\Delta \chi) \cos \theta - \sin \theta \right] \right\} \) |
| \(TQ\) | \(\frac{1}{24\Delta} \left\{ s^2 \left[ \cosh (\Delta \chi) - \sinh (\Delta \chi) \cos \theta + \sin \theta \right] + \frac{1}{4s^2} \left[ \cosh (\Delta \chi) + \sinh (\Delta \chi) \cos \theta - \sin \theta \right] \right\} \) |
| \(\Upsilon^2 = 4\omega^2\) |
| \(TO\) | \(\frac{7}{44\Delta} \left\{ s^2 \left[ (1 + \ln^2 \tau) + (1 - \ln^2 \tau) \cos \theta + 2 \ln \tau \sin \theta \right] + \frac{1}{4s^2r} \left[ (1 + \ln^2 \tau) - (1 - \ln^2 \tau) \cos \theta - 2 \ln \tau \sin \theta \right] \right\} \) |
| \(TM\) | \(\frac{5e^\chi}{44\Delta} \left\{ s^2 \left[ (1 + \chi^2) + (1 - \chi^2) \cos \theta + 2 \chi \sin \theta \right] + \frac{1}{4s^2} \left[ (1 + \chi^2) - (1 - \chi^2) \cos \theta - 2 \chi \sin \theta \right] \right\} \) |
| \(TQ\) | \(\frac{1}{44\Delta} \left\{ s^2 \left[ (1 + \chi^2) + (1 - \chi^2) \cos \theta + 2 \chi \sin \theta \right] + \frac{1}{4s^2} \left[ (1 + \chi^2) - (1 - \chi^2) \cos \theta - 2 \chi \sin \theta \right] \right\} \) |
| \(\Upsilon^2 < 4\omega^2\) |
| \(TO\) | \(\frac{7}{24\Delta} \left\{ s^2 \left[ 1 + \cos (\Delta \ln \tau - \theta) \right] + \frac{1}{4s^2r} \left[ 1 - \cos (\Delta \ln \tau - \theta) \right] \right\} \) |
| \(TM\) | \(\frac{5e^\chi}{24\Delta} \left\{ s^2 \left[ 1 + \cos (\Delta \chi - \theta) \right] + \frac{1}{4s^2} \left[ 1 - \cos (\Delta \chi - \theta) \right] \right\} \) |
| \(TQ\) | \(\frac{1}{24\Delta} \left\{ s^2 \left[ 1 + \cos (\Delta \chi - \theta) \right] + \frac{1}{4s^2} \left[ 1 - \cos (\Delta \chi - \theta) \right] \right\} \) |
Table 4. \((\Delta p)^2\) for the TO, TM, and TQ systems when \(\Upsilon > 0\).
For \(\Upsilon < 0\), change \(\Upsilon\) to \(|\Upsilon|\) and \(\theta\) to \(-\theta\) in the following.
\(\Delta^2 = |1 - 4\omega^2/\Upsilon^2|\), \(\tau = [1 + \Upsilon(t' - t_0)]\), \(\chi = \Upsilon(t - t_0)\), \(s = \exp r\).

| Class | \((\Delta p)^2\) |
|-------|------------------|
| \(\Upsilon^2 > 4\omega^2\) | \(\frac{\Upsilon}{2\pi} \left\{ s^2 \left\{ (1 + \Delta^2) \left[ \cosh (\Delta \ln \tau - \sinh (\Delta \ln \tau) \cos \theta \right] + (1 - \Delta^2) \sin \theta + 2\Delta \left[ \sinh (\Delta \ln \tau) - \cosh (\Delta \ln \tau) \cos \theta \right] \right\} + \frac{1}{\Upsilon} \left\{ (1 + \Delta^2) \left[ \cosh (\Delta \ln \tau) + \sinh (\Delta \ln \tau) \cos \theta \right] - (1 - \Delta^2) \sin \theta + 2\Delta \left[ \sinh (\Delta \ln \tau) + \cosh (\Delta \ln \tau) \cos \theta \right] \right\} \right\} \right\} |
| \(\Upsilon^2 = 4\omega^2\) | \(\frac{\Upsilon}{2\pi} \left\{ s^2 \left\{ \left[ \frac{1}{2} + (1 + \frac{1}{2} \ln \tau)^2 \right] + \left[ \frac{1}{2} - (1 + \frac{1}{2} \ln \tau)^2 \right] \cos \theta + (1 + \frac{1}{2} \ln \tau) \sin \theta \right\} + \frac{1}{\Upsilon} \left\{ \left[ \frac{1}{2} + (1 + \frac{1}{2} \ln \tau)^2 \right] - \left[ \frac{1}{2} - (1 + \frac{1}{2} \ln \tau)^2 \right] \cos \theta - (1 + \frac{1}{2} \ln \tau) \sin \theta \right\} \right\} \right\} |
| \(\Upsilon^2 < 4\omega^2\) | \(\frac{\Upsilon}{2\pi} \left\{ s^2 \left\{ (1 + \Delta^2) + (1 - \Delta^2) \cos (\Delta \ln \tau - \theta) - 2\Delta \sin (\Delta \ln \tau - \theta) \right\} + \frac{1}{\Upsilon} \left\{ (1 + \Delta^2) - (1 - \Delta^2) \cos (\Delta \ln \tau - \theta) + 2\Delta \sin (\Delta \ln \tau - \theta) \right\} \right\} \right\} |

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Table 5. \((\Delta x)^2(\Delta p)^2\) for the TO, TM, and TQ systems when \(\Upsilon > 0\). For \(\Upsilon < 0\), replace \(\theta\) by \(-\theta\) in the following.
\[
\Delta^2 = |1 - 4\omega^2/\Upsilon^2|, \quad \tau = |1 + \Upsilon(t' - t_o)|, \quad \chi = \Upsilon(t - t_o), \quad s = \exp r.
\]

| Class | \((\Delta p)^2\) |
|-------|------------------|
| \(\Upsilon^2 > 4\omega^2\) | \[
\begin{aligned}
&\text{TO} & 1 + \frac{\pi}{12} \left\{ s^2 \left[ \sin^2 (\theta) - \sin^2 (\chi) \right] + \frac{\pi}{6} \left[ - \sin (\theta) + (1 + \Delta \cos \theta) \sinh (\Delta \ln \tau) + (\Delta - \cos \theta) \sinh (\Delta \ln \tau) \right] \right\} \\
&\text{TM} & 1 + \frac{\pi}{12} \left\{ s^2 \left[ \sin^2 (\theta) - \sin^2 (\chi) \right] + \frac{\pi}{6} \left[ - \sin (\theta) + (1 + \Delta \cos \theta) \sinh (\Delta \ln \tau) + (\Delta - \cos \theta) \sinh (\Delta \ln \tau) \right] \right\} \\
&\text{TQ} & 1 + \frac{\pi}{12} \left\{ s^2 \left[ \sin^2 (\theta) - \sin^2 (\chi) \right] + \frac{\pi}{6} \left[ - \sin (\theta) + (1 + \Delta \cos \theta) \sinh (\Delta \ln \tau) + (\Delta - \cos \theta) \sinh (\Delta \ln \tau) \right] \right\}
\end{aligned}
\]
| \(\Upsilon^2 = 4\omega^2\) | \[
\begin{aligned}
&\text{TO} & 1 + \frac{\pi}{12} \left\{ s^2 \left[ \frac{1}{2} (1 + \ln \tau)^2 + (1 + \ln \tau) \right] + \frac{\pi}{12} \left[ -2 \ln (1 + \ln \tau) + \frac{\pi}{6} (1 - 2\ln \tau - \ln^2 \tau) \cos \theta \right] \right\} \\
&\text{TM} & 1 + \frac{\pi}{12} \left\{ s^2 \left[ \frac{1}{2} (1 + \chi)^2 + (1 + \chi) \sin \theta + \frac{\pi}{6} (1 - 2\chi - \chi^2) \cos \theta \right] + \frac{\pi}{12} \left[ -2 \chi - \chi^2 \right] \right\} \\
&\text{TQ} & 1 + \frac{\pi}{12} \left\{ s^2 \left[ \frac{1}{2} (1 + \chi)^2 + (1 + \chi) \sin \theta + \frac{\pi}{6} (1 - 2\chi - \chi^2) \cos \theta \right] + \frac{\pi}{12} \left[ -2 \chi - \chi^2 \right] \right\}
\end{aligned}
\]
| \(\Upsilon^2 < 4\omega^2\) | \[
\begin{aligned}
&\text{TO} & 1 + \frac{\pi}{12} \left\{ s^2 \left[ \sin (\Delta \ln \tau - \theta) - \sin (\Delta \ln \tau - \theta) \right] + \frac{\pi}{12} \left[ - \sin (\Delta \ln \tau - \theta) + \Delta \sin (\Delta \ln \tau - \theta) \right] \right\} \\
&\text{TM} & 1 + \frac{\pi}{12} \left\{ s^2 \left[ \sin (\Delta \chi - \theta) - \sin (\Delta \chi - \theta) \right] + \frac{\pi}{12} \left[ - \sin (\Delta \chi - \theta) + \Delta \sin (\Delta \chi - \theta) \right] \right\} \\
&\text{TQ} & 1 + \frac{\pi}{12} \left\{ s^2 \left[ \sin (\Delta \chi - \theta) - \sin (\Delta \chi - \theta) \right] + \frac{\pi}{12} \left[ - \sin (\Delta \chi - \theta) + \Delta \sin (\Delta \chi - \theta) \right] \right\}
\end{aligned}
\]

8 Conclusion

Our results extend and complement the seminal work of Dodonov and Man’ko [10] as well as the work of Cheng and Fung [17]. These authors focused on solving the Schrödinger equation with Hamiltonian (8). In both papers the authors use a symmetry approach. Dodonov and Man’ko obtained a set of number states similar to ours for the weakly damped case (\(\Upsilon^2 < 4\omega^2\)), the strongly damped case (\(\Upsilon^2 > 4\omega^2\)), and the critically damped case (\(\Upsilon^2 = 4\omega^2\)). For each damping case Cheng and Fung used evolution operator and Lie methods to compute the expectation.
values of $x$ and $p$ in coherent and squeezed states.

In our work, we have shown how Lie symmetry methods can be used to obtain number states, coherent states, and squeezed states not only for the $TM$-type Schrödinger equations with Hamiltonian (3), but also for systems of related $TQ$ and $TO$ equations. The general transformation method developed in [1, 2, 3] demonstrates the intimate connection between these classes of Schrödinger equations, their solutions, and the expectation values. Furthermore, for each class of Schrödinger equation, we have proven that the coherent- and squeezed-state expectation values of position and momentum satisfy Hamilton’s equations.

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Appendix A: Solutions to Eq. (14)

The second-order ordinary differential equation

$$\frac{d^2w}{ds^2} + As^{-2}w = 0, \quad (67)$$

is an Euler-type equation with a regular singular point at $s = 0$. [The Wronskian of the solutions to Eq. (67) is a constant [21].] Assuming that $s > 0$, we look for solutions of the form

$$w(s) = s^k. \quad (68)$$

Substituting (68) into Eq. (67) yields a quadratic for $k$ and solutions

$$k(k - 1) + A = 0, \quad k = \frac{1}{2} \left(1 \pm \sqrt{1 - 4A}\right). \quad (69)$$

The solutions are characterized by value and sign of the discriminant $1 - 4A$.

$$k = \frac{1}{2} (1 \pm \sigma\Delta), \quad \sigma = \sqrt{\text{sign}(1 - 4A)}, \quad \Delta^2 = |1 - 4A|. \quad (70)$$

Now, let us examine the various real solutions of Eq. (67).
(i) $1 - 4A > 0$: In this case, $\sigma = 1$. The real solutions and their Wronskian are

$$w_1(s) = s^{(1-\Delta)/2} = \sqrt{s} \exp \left(-\frac{\Delta}{2} \ln s \right), \quad (71)$$

$$w_2(s) = s^{(1+\Delta)/2} = \sqrt{s} \exp \left(\frac{\Delta}{2} \ln s \right), \quad (72)$$

$$W_s(w_1, w_2) = \Delta. \quad (73)$$

(ii) $1 - 4A = 0$: In this case, the two roots are equal. The two solutions are

$$w_1 = \sqrt{s}, \quad w_2 = \sqrt{s} \ln s. \quad (74)$$

The second is obtained by standard methods [21]. The Wronskian of these two solutions is

$$W_s(w_1, w_2) = 1. \quad (75)$$

(iii) $1 - 4A < 0$: In this case, $\sigma = i$ and we obtain two solutions that are complex conjugates of one another. By taking appropriate linear combinations of the two complex solutions, we obtain real solutions $w_1$ and $w_2$

$$w_1(s) = \sqrt{s} \cos \left(\frac{\Delta}{2} \ln s \right), \quad w_2(s) = \sqrt{s} \sin \left(\frac{\Delta}{2} \ln s \right). \quad (76)$$

Their Wronskian is given by

$$W_s(w_1, w_2) = \Delta/2. \quad (77)$$

The nonzero Wronskians demonstrate that in all cases the two real solutions are linearly independent of each other [21].
Appendix B: Time-dependent $TO$, $TM$, and $TQ$ functions

Table B-1. Time dependent functions for the $TO$ system. The functions $\xi, \xi, \phi_2,$ and $\dot{\phi}_2$ can be obtained by taking the complex conjugate of $\xi, \dot{\xi}, \phi_1,$ and $\dot{\phi}_1,$ respectively. $\Delta^2 = |1 - 4\omega^2/\Upsilon^2|, \tau = [1 + \Upsilon(t' - t')]$.

| $\Upsilon > 0$ | $\Upsilon < 0$ |
|----------------|----------------|
| $T^2 > 4\omega^2$ | $T^2 > 4\omega^2$ |
| $\xi(t')$ | $\xi(t')$ |
| $\sqrt{\frac{1}{2\Delta^2}} \sqrt{\tau} \left( e^{-\frac{\Delta}{2} \ln \tau} + ie^{\frac{\Delta}{2} \ln \tau} \right)$ | $\sqrt{\frac{1}{2\Delta^2}} \sqrt{\tau} \left( e^{-\frac{\Delta}{2} \ln \tau} - ie^{\frac{\Delta}{2} \ln \tau} \right)$ |
| $\dot{\xi}(t')$ | $-\frac{i}{\Delta^2} \sqrt{\tau} \left( (1 + \Delta)e^{-\frac{\Delta}{2} \ln \tau} - i(1 + \Delta)e^{\frac{\Delta}{2} \ln \tau} \right)$ |
| $\phi_3(t')$ | $\phi_3(t')$ |
| $\frac{1}{\Delta^2} \left( e^{-\Delta \ln \tau} + e^{\Delta \ln \tau} \right)$ | $\frac{1}{\Delta^2} \left( e^{-\Delta \ln \tau} + e^{\Delta \ln \tau} \right)$ |
| $\dot{\phi}_3(t')$ | $\dot{\phi}_3(t')$ |
| $\frac{1}{\Delta} \left( (1 - \Delta)e^{-\Delta \ln \tau} + (1 + \Delta)e^{\Delta \ln \tau} \right)$ | $\frac{1}{\Delta} \left( (1 - \Delta)e^{-\Delta \ln \tau} + (1 + \Delta)e^{\Delta \ln \tau} \right)$ |
| $\phi_1(t')$ | $\phi_1(t')$ |
| $\frac{1}{\Delta^2} \left( e^{-\Delta \ln \tau} - e^{\Delta \ln \tau} + 2i \right)$ | $\frac{1}{\Delta^2} \left( e^{-\Delta \ln \tau} - e^{\Delta \ln \tau} - 2i \right)$ |
| $\dot{\phi}_1(t')$ | $\dot{\phi}_1(t')$ |
| $\frac{i}{\Delta^2} \left( (1 - \Delta)e^{-\Delta \ln \tau} - (1 + \Delta)e^{\Delta \ln \tau} + 2i \right)$ | $\frac{i}{\Delta^2} \left( (1 - \Delta)e^{-\Delta \ln \tau} - (1 + \Delta)e^{\Delta \ln \tau} - 2i \right)$ |

| $\Upsilon > 0$ | $\Upsilon < 0$ |
|----------------|----------------|
| $T^2 = 4\omega^2$ | $T^2 = 4\omega^2$ |
| $\xi(t')$ | $\xi(t')$ |
| $\sqrt{\frac{1}{\Upsilon}} \sqrt{\tau} (1 + i \ln \tau)$ | $\sqrt{\frac{1}{\Upsilon}} \sqrt{\tau} (1 - i \ln \tau)$ |
| $\dot{\xi}(t')$ | $\dot{\xi}(t')$ |
| $\sqrt{\frac{1}{\Upsilon}} \sqrt{\tau} \left[ \frac{1}{2} + i \left( 1 + \frac{1}{2} \ln \tau \right) \right]$ | $\sqrt{\frac{1}{\Upsilon}} \sqrt{\tau} \left[ \frac{1}{2} - i \left( 1 + \frac{1}{2} \ln \tau \right) \right]$ |
| $\phi_3(t')$ | $\phi_3(t')$ |
| $\frac{1}{\Upsilon} \left( 1 + \ln^2 \tau \right)$ | $\frac{1}{\Upsilon} \left( 1 + \ln^2 \tau \right)$ |
| $\dot{\phi}_3(t')$ | $\dot{\phi}_3(t')$ |
| $\frac{1}{\Upsilon} \left( 1 + \ln \tau \right)^2$ | $- \left( 1 + \ln \tau \right)^2$ |
| $\phi_1(t')$ | $\phi_1(t')$ |
| $\frac{1}{2\Upsilon} \sqrt{\tau} \left( 1 - \ln^2 \tau + 2i \ln \tau \right)$ | $\frac{1}{2\Upsilon} \sqrt{\tau} \left( 1 - \ln^2 \tau - 2i \ln \tau \right)$ |
| $\dot{\phi}_1(t')$ | $\dot{\phi}_1(t')$ |
| $\frac{i}{2\Upsilon} \left[ 1 - \ln^2 \tau - 2i \ln \tau + 2(1 + \ln \tau) \right]$ | $- \frac{i}{2} \left[ 1 - \ln^2 \tau - 2i \ln \tau - 2i(1 + \ln \tau) \right]$ |

| $\Upsilon > 0$ | $\Upsilon < 0$ |
|----------------|----------------|
| $T^2 < 4\omega^2$ | $T^2 < 4\omega^2$ |
| $\xi(t')$ | $\xi(t')$ |
| $\sqrt{\frac{1}{\Upsilon}} \sqrt{\tau} e^{\frac{\Delta}{2} \ln \tau}$ | $\sqrt{\frac{1}{\Upsilon}} \sqrt{\tau} e^{-\frac{\Delta}{2} \ln \tau}$ |
| $\dot{\xi}(t')$ | $\dot{\xi}(t')$ |
| $\sqrt{\frac{1}{\Upsilon}} \sqrt{\tau} \left( 1 + i\Delta \right) e^{\frac{\Delta}{2} \ln \tau}$ | $\sqrt{\frac{1}{\Upsilon}} \sqrt{\tau} \left( 1 - i\Delta \right) e^{-\frac{\Delta}{2} \ln \tau}$ |
| $\phi_3(t')$ | $\phi_3(t')$ |
| $\frac{\Delta}{\Upsilon} \tau$ | $\frac{\Delta}{\Upsilon} \tau$ |
| $\dot{\phi}_3(t')$ | $\dot{\phi}_3(t')$ |
| $\frac{\Delta}{\Upsilon} \tau$ | $\frac{\Delta}{\Upsilon} \tau$ |
| $\phi_1(t')$ | $\phi_1(t')$ |
| $\frac{1}{\Upsilon} \tau e^{i\Delta \ln \tau}$ | $\frac{1}{\Upsilon} \tau e^{-i\Delta \ln \tau}$ |
| $\dot{\phi}_1(t')$ | $\dot{\phi}_1(t')$ |
| $\frac{1}{\Upsilon} \tau e^{i\Delta \ln \tau}$ | $\frac{1}{\Upsilon} \tau e^{-i\Delta \ln \tau}$ |
Table B-2. Time dependent functions for the \( T M \) system. The functions \( \xi, \dot{\xi}, \phi_2, \) and \( \dot{\phi}_2 \) can be obtained by taking the complex conjugate of \( \xi, \dot{\xi}, \phi_1, \) and \( \dot{\phi}_1, \) respectively. \( \Delta^2 = |1 - 4\omega^2 / \Upsilon^2|, \ \chi = \Upsilon(t - t_0). \)

| \( \Upsilon > 0 \) | \( \Upsilon < 0 \) |
|---|---|
| \( \dot{\xi}(t) \) | \( \sqrt{\frac{1}{2\Upsilon e^{x/2}}} \left( e^{-\frac{\Delta x}{2} + i e \frac{x}{2}} \right) - \sqrt{\frac{1}{2|\Upsilon| e^{x/2}}} \left( e^{-\frac{\Delta x}{2} - i e \frac{x}{2}} \right) \) |
| \( \ddot{\xi}(t) \) | \( -\frac{i}{4\Upsilon} \left[ (1 - \Delta) e^{-\Delta x} + (1 + \Delta) e^{\Delta x} \right] \) |
| \( \dot{\phi}_2(t) \) | \( \frac{1}{\Upsilon} e^{x} \left( e^{-\Delta x} + e^{\Delta x} \right) \) |
| \( \ddot{\phi}_2(t) \) | \( \frac{1}{\Upsilon} e^{x} \left( e^{-\Delta x} - e^{\Delta x} - 2i \right) \) |
| \( \dot{\phi}_1(t) \) | \( \frac{1}{\Upsilon} \left[ (1 - \Delta) e^{-\Delta x} + (1 + \Delta) e^{\Delta x} \right] \) |
| \( \ddot{\phi}_1(t) \) | \( \frac{1}{\Upsilon} \left[ (1 - \Delta) e^{-\Delta x} - e^{\Delta x} \right] \) |

| \( \Upsilon = 4\omega^2 \) |
|---|---|
| \( \dot{\xi}(t) \) | \( \sqrt{\frac{1}{2\Upsilon e^{x/2}}} \left[ \frac{1}{2} + i \left( 1 + \frac{1}{2} \chi \right) \right] - \sqrt{\frac{1}{2|\Upsilon| e^{x/2}}} \left[ \frac{1}{2} - i \left( 1 + \frac{1}{2} \chi \right) \right] \) |
| \( \ddot{\xi}(t) \) | \( -\frac{i}{4\Upsilon} e^{x} \left( 1 + \chi^2 \right) \) |
| \( \dot{\phi}_2(t) \) | \( \frac{1}{\Upsilon} e^{x} \left( 1 + \chi^2 \right) \) |
| \( \ddot{\phi}_2(t) \) | \( 2\Upsilon e^{-\chi} \left( 1 + \chi \right) \) |
| \( \dot{\phi}_1(t) \) | \( \frac{1}{\Upsilon} e^{x} \left( 1 - \chi^2 + 2i \chi \right) \) |
| \( \ddot{\phi}_1(t) \) | \( -\frac{i}{4\Upsilon} e^{x} \left[ 1 - \chi^2 - 2i \chi \right] \) |

| \( \Upsilon < 4\omega^2 \) |
|---|---|
| \( \dot{\xi}(t) \) | \( \sqrt{\frac{1}{2\Upsilon e^{x/2}}} \left[ \frac{1}{2} + i \left( 1 + \frac{1}{2} \chi \right) \right] \) |
| \( \ddot{\xi}(t) \) | \( -\frac{i}{4\Upsilon} e^{x} \left( 1 + \chi^2 \right) \) |
| \( \dot{\phi}_2(t) \) | \( \frac{1}{\Upsilon} e^{x} \left( 1 + \chi^2 \right) \) |
| \( \ddot{\phi}_2(t) \) | \( 2\Upsilon e^{-\chi} \left( 1 + \chi \right) \) |
| \( \dot{\phi}_1(t) \) | \( \frac{1}{\Upsilon} e^{x} \left( 1 - \chi^2 + 2i \chi \right) \) |
| \( \ddot{\phi}_1(t) \) | \( -\frac{i}{4\Upsilon} e^{x} \left[ 1 - \chi^2 - 2i \chi \right] \) |
Table B-3. Time dependent functions for the $TQ$ system. The functions $\Xi_P$ and $\Xi_X$ can be obtained by taking the complex conjugate of $\Xi_P$ and $\Xi_X$, respectively. $\Delta^2 = |1 - 4\omega^2/Y^2|$, $\chi = \gamma(t - t_0)$.

|               | $\gamma > 0$ | $\gamma < 0$ |
|---------------|-------------|-------------|
| $\Xi_P(t)$    | $\sqrt{\frac{1}{2\Delta}} \left( e^{-\Delta \chi} + ie^{\Delta \chi} \right)$ | $\sqrt{\frac{1}{2\Delta}} \left( e^{-\Delta \chi} - ie^{\Delta \chi} \right)$ |
| $\Xi_X(t)$    | $\sqrt{\frac{1}{\Delta}} \left[ (1 - \Delta)e^{-\Delta \chi} + (1 + \Delta)e^{\Delta \chi} \right]$ | $\sqrt{\frac{1}{\Delta}} \left[ (1 - \Delta)e^{-\Delta \chi} - (1 + \Delta)e^{\Delta \chi} \right]$ |
| $C_{3,T}(t)$  | $\frac{1}{\Delta} \left( e^{-\Delta \chi} + e^{\Delta \chi} \right)$ | $\frac{1}{\Delta} \left( e^{-\Delta \chi} - e^{\Delta \chi} \right)$ |
| $C_{3,D}(t)$  | $\frac{1}{\Delta} \left[ -e^{-\Delta \chi} + e^{\Delta \chi} \right]$ | $\frac{1}{\Delta} \left[ -e^{-\Delta \chi} - e^{\Delta \chi} \right]$ |
| $C_{3,X2}(t)$ | $\frac{1}{\gamma} \left[ -(1 - \Delta)e^{-\Delta \chi} + (1 + \Delta)e^{\Delta \chi} \right]$ | $\frac{1}{\gamma} \left[ -(1 - \Delta)e^{-\Delta \chi} - (1 + \Delta)e^{\Delta \chi} \right]$ |

|               | $\gamma = 4\omega^2$ |
|---------------|--------------------------|
| $\Xi_P(t)$    | $\sqrt{\frac{1}{\gamma}} (1 + i\chi)$ |
| $\Xi_X(t)$    | $\sqrt{\frac{1}{\gamma}} \left[ \frac{1}{2} + i \left( 1 + \frac{1}{2} \chi \right) \right]$ |
| $C_{3,T}(t)$  | $\frac{1}{\gamma} \left( 1 + \chi^2 \right)$ |
| $C_{3,D}(t)$  | $\chi$ |
| $C_{3,X2}(t)$ | $-\frac{1}{\gamma} \left( 1 + \chi \right)$ |

|               | $\gamma < 4\omega^2$ |
|---------------|--------------------------|
| $\Xi_P(t)$    | $\sqrt{\frac{1}{\gamma}} \left( e^{-\Delta \chi} + e^{\Delta \chi} \right)$ |
| $\Xi_X(t)$    | $\sqrt{\frac{1}{\gamma}} \left( 1 + i\Delta \right) e^{-\frac{1}{2} \Delta \chi}$ |
| $C_{3,T}(t)$  | $\frac{1}{\gamma} \left( 1 + e^{\Delta \chi} \right)$ |
| $C_{3,D}(t)$  | $\frac{1}{\gamma} \left( 1 + e^{-\Delta \chi} \right)$ |
| $C_{3,X2}(t)$ | $\frac{1}{\gamma} \left( 1 + e^{-\Delta \chi} \right)$ |

$C_3, T(t)$, $T^2 = \frac{1}{Y} \left( 1 + \chi \right)$

$C_3, D(t)$, $T^2 = \frac{1}{Y} \left( 1 - \chi \right)$

$C_3, X2(t)$, $T^2 = \frac{1}{Y} \left( 1 - \chi \right)$
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