SOME CLUSTER TILTING MODULES FOR WEIGHTED SURFACE ALGEBRAS

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Abstract. Non-singular weighted surface algebras satisfy the necessary condition found in [6] for existence of cluster tilting modules. We show that any such algebra whose Gabriel quiver is bipartite, has a module satisfying the necessary exterior vanishing condition. We show that it is 3-cluster tilting precisely for non-singular triangular or spherical algebras, but not for any other weighted surface algebra with bipartite Gabriel quiver.

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1. Introduction

A module $M$ of a finite-dimensional algebra $A$ is an $n$-cluster tilting module (or maximal $(n-1)$-orthogonal) provided

$$\text{add}(M) = \{ N \mid \text{Ext}^i(M, N) = 0 \text{ for } 1 \leq i \leq n-1 \}$$

$$= \{ N \mid \text{Ext}^i(N, M) = 0 \text{ for } 1 \leq i \leq n-1 \},$$

(see [12], [13]). We would like to know whether non-singular weighted surface algebras have cluster tilting modules. Weighted surface algebras are a class of tame symmetric algebras, periodic as bimodules, of period 4 (see [7] and [8], [9]). This means that they satisfy the necessary condition found in [6], requiring that all non-projective modules should have bounded periodic resolutions. As observed in [6] if such an algebra has an $n$-cluster tilting module then the only option is $n = 3$.

Here we study weighted surface algebras which have a bipartite Gabriel quiver, which means that in the presentation as in [8] (see also [9]) it has many virtual arrows. We introduce a module $M$, defined in the same way for each of the algebras, which satisfies $\text{Ext}^1(M, M) = 0$ and $\text{Ext}^2(M, M) = 0$. We show that it is 3-cluster tilting when $\Lambda$ is either a triangle algebra $T(\lambda)$, or a spherical algebra $S(\lambda)$ (see §3 for the definition) with $\lambda \in K$ and $\lambda \neq 0, 1$. We also show that for any other weighted surface algebra whose Gabriel quiver is bipartite, $M$ cannot be a direct summand of a 3-cluster tilting module.

The algebra $T(\lambda)$ occurs in various places in the literature. It is an algebra with $k = 1$ in the family $Q(3A)_1$ of algebras of quaternion type, in [2]. Furthermore, it occurs with the name $B_{1,1}(\lambda)$ in [3]. As well, it occurs in [11] with the name $A_1(\lambda)$. In [8] it is called the triangle algebra $T(\lambda)$ (in Example 3.4). Similarly the spherical algebra $S(\lambda)$ was introduced in [8] (Example 3.6).

Spherical algebras are a special case of the family of algebras which come from the triangulation $T(n)$ of the sphere as defined in Example 7.5 of [10]. We call these algebras $n$-spherical; when $n = 2$ they are the same as the spherical algebras. One may also observe that the spherical algebra with $n = 1$ with the multiplicities 2, 2, 1 is the same as the triangle algebra (see [8], Example 3.3).

Algebras whose Gabriel quiver is the same as that of $T(\lambda)$, allowing a multiplicity $k > 1$, coincide (up to a scalar parameter) with the algebras $Q(3A)_2^k$ in the labelling of [11]. When the
characteristic of $K$ is 2 these occur as the basic algebras of blocks of finite groups. Very recently B. Böhmler and R. Marcinczik proved using computer calculations that for $k = 2$, it has a 3-cluster tilting module (see [2]).

Much of this note was written five years ago, when talking to Idun Reiten about [3], and it was extended first when spherical algebras had been discovered, and then again, inspired by an email from R. Marcinczik (for which I am grateful).

2. Preliminaries

Throughout $K$ is an algebraically closed field, of arbitrary characteristic. Assume $\Lambda$ is a finite-dimensional symmetric $K$-algebra. We recall some identities for the stable category $\text{mod} \Lambda$.

1. $D\text{Ext}^1(M, N) \cong \text{Hom}(\tau^{-1}N, M)$, and in this case $\tau \cong \Omega^2$.

2. $\text{Ext}^i(U, V) \cong \text{Hom}(\Omega^i U, V) \cong \text{Hom}(U, \Omega^{-i} V)$.

This implies that $\dim \text{Ext}^i(M, N) = \dim \text{Ext}^1(N, \Omega^{i+2} M)$. The algebras we consider have the property that all non-projective indecomposable right $\Lambda$-modules are $\Omega$ periodic of periods dividing 4. This gives us the following, we refer to this as ext symmetry.

**Corollary 2.1.** Assume $\Lambda$ is symmetric and all modules have $\Omega$-period dividing 4. Then for all $M, N$ we have $\dim \text{Ext}^2(M, N) = \dim \text{Ext}^1(N, M)$ as vector spaces.

This simplifies the search for 3-cluster tilting modules. If we know that $\text{Ext}^1(N, X) = 0$ and $\text{Ext}^1(X, N) = 0$ then automatically $\text{Ext}^2(X, N) = 0$ and $\text{Ext}^2(N, X) = 0$.

3. The algebras

3.1. Weighted surface algebras. We review the definition from [9], for details see [7], [8], [9].

Assume $Q$ is a finite quiver. Denote by $KQ$ the path algebra of $Q$ over $K$. We will consider algebras of the form $A = KQ/I$ where $I$ is an ideal of $KQ$ which contains all paths of length $\geq m$ for some $m > 0$, so that the algebra is finite-dimensional and basic. The Gabriel quiver $Q_A$ of $A$ is then the full subquiver of $Q$ obtained from $Q$ by removing all arrows $\alpha$ with $\alpha + I \in R^2 Q + I$.

A quiver $Q$ is 2-regular if for each vertex $i \in Q_0$ there are precisely two arrows starting at $i$ and two arrows ending at $i$. Such a quiver has an involution on the arrows, $\alpha \mapsto \bar{\alpha}$, such that for each arrow $\alpha$, the arrow $\bar{\alpha}$ is the arrow $\neq \alpha$ such that $s(\alpha) = s(\bar{\alpha})$.

A triangulation quiver is a pair $(Q, f)$ where $Q$ is a (finite) connected 2-regular quiver, with at least two vertices, and where $f$ is a fixed permutation of the arrows such that $t(\alpha) = s(f(\alpha))$ for each arrow $\alpha$, and such that $f^3$ is the identity. The permutation $f$ uniquely determines a permutation $g$ of the arrows, defined by $g(\alpha) := f(\alpha)$ for any arrow $\alpha$. We assume throughout that $(Q, f)$ is a triangulation quiver. To give the presentations of the algebras in question, we use the following notation. For each arrow $\alpha$, we fix

- $m_{\alpha} \in \mathbb{N}^*$ a weight, constant on $g$-cycles, and
- $c_{\alpha} \in K^*$ a parameter, constant on $g$-cycles, and define
- $n_{\alpha} :=$ the length of the $g$-cycle of $\alpha$, $B_{\alpha} := \alpha g_1(\alpha) \cdots g_{m_{\alpha} n_{\alpha} - 1}(\alpha)$ the path along the $g$-cycle of $\alpha$ of length $m_{\alpha} n_{\alpha}$, $A_{\alpha} := \alpha g_1(\alpha) \cdots g_{m_{\alpha} n_{\alpha} - 2}(\alpha)$ the path along the $g$-cycle of $\alpha$ of length $m_{\alpha} n_{\alpha} - 1$. 
Definition 3.1. We say that an arrow $\alpha$ of $Q$ is virtual if $m_{\alpha}n_{\alpha} = 2$, that is $A_{\alpha}$ has length 1. Note that this condition is preserved under the permutation $g$, and that virtual arrows form $g$-orbits of sizes 1 or 2.

We assume that the following conditions hold.
(1) $m_{\alpha}n_{\alpha} \geq 2$ for all arrows $\alpha$, and
(2) $m_{\alpha}n_{\alpha} \geq 3$ for all arrows $\alpha$ such that $\bar{\alpha}$ is virtual and $\bar{\alpha}$ is not a loop, and $m_{\alpha}n_{\alpha} \geq 4$ for all arrows $\alpha$ such that $\bar{\alpha}$ is virtual and $\bar{\alpha}$ is a loop.

Condition (1) is a general assumption, and (2) is needed to eliminate two small algebras (see [8]).

We also assume that $Q$ has at least three vertices. With this, the definition of a weighted surface algebra (as revised in [9]) is as follows.

Definition 3.2. The algebra $\Lambda = \Lambda(Q,f,\bullet,c) = KQ/I$ is a weighted surface algebra if $(Q,f)$ is a triangulation quiver, with $|Q_0| \geq 2$, and $I = I(Q,f,\bullet,c)$ is the ideal of $KQ$ generated by:
(1) $\alpha f(\alpha) - c_{\alpha} A_{\bar{\alpha}}$ for all arrows $\alpha$ of $Q$,
(2) $\alpha f(\alpha) g(f(\alpha))$ for all arrows $\alpha$ of $Q$ unless $f^2(\alpha)$ is virtual, or unless $f(\bar{\alpha})$ is virtual and $m_{\bar{\alpha}} = 1$, $n_{\bar{\alpha}} = 3$.
(3) $ag(\alpha) f(\alpha)$ for all arrows $\alpha$ of $Q$ unless $f(\alpha)$ is virtual, or unless $f^2(\alpha)$ is virtual and $m_{f(\alpha)} = 1$, $n_{f(\alpha)} = 3$.

The Gabriel quiver $Q_\Lambda$ is the subquiver of $Q$ obtained by removing all virtual arrows.

We recall a few properties.
(1) Any such algebra is symmetric and tame.
(2) The dimension of $e_i \Lambda$ is equal to $m_{\alpha}n_{\alpha} + m_{\bar{\alpha}}n_{\bar{\alpha}}$ where $\alpha, \bar{\alpha}$ are the arrows starting at $i$.

The relations also imply that $c_{\alpha} B_{\alpha} = c_{\bar{\alpha}} B_{\bar{\alpha}}$ in $\Lambda$. One can show that this spans the socle of $e_i \Lambda$.

We wish to define a module $M$ such that $\text{Ext}^1(M,M) = 0$ and $\text{Ext}^2(M,M) = 0$, as a candidate to be 3-cluster tilting. This can be done for a weighted surface algebra whose quiver is bipartite; this requires that each triangle of $f$ must contain a virtual arrow. Such a quiver can be thought of made up of three building blocks, first a quiver of the form

\[
\begin{array}{ccc}
\gamma_1 & \sigma_1 & \gamma_2 \\
\cdots \\
& \xi_n & \sigma_n \\
\delta_n & \xi_{n+1} & \\
\end{array}
\]

where the shaded triangles define the $f$-orbits.

Next, quivers of the form

\[
\begin{array}{ccc}
\varepsilon & 1 & \alpha \\
& \beta & \\
\end{array}
\]

or

\[
\begin{array}{ccc}
\gamma & 2' \\
\delta & 3 \\
\end{array}
\]

\[
\begin{array}{ccc}
\varepsilon' & 2' & \\
\end{array}
\]

are also allowed. The quivers above are used to construct the module $M$.
We describe the quivers and algebras we consider. We always take the multiplicities at 2-cycles of $g$ equal to 1, and at loops we take multiplicity 2. That is, all arrows in 2-cycles and loops are virtual and not part of the Gabriel quiver.

3.2. Algebras with Gabriel quiver

3.2.1. We take the quiver $Q$ obtained by gluing the second and the third type above, identifying vertex 2 with vertex $2'$. The permutation $g$ is of the form $(\alpha \gamma \delta \beta)(\varepsilon)(\varepsilon')$. Let $m_\alpha = k \geq 2$. The case $k = 1$ is special, this gives the triangular algebra, called $T(\lambda)$ in [8], here $\lambda \neq 1$. With suitable choice of $c_\bullet$, the presentation of the weighted surface algebra induces the (Gabriel) presentation of $T(\lambda)$:

\[
\begin{align*}
\alpha \beta \alpha &= \alpha \gamma \delta, \\
\delta \beta \alpha &= \lambda (\delta \gamma \delta), \\
\beta \alpha \beta &= \gamma \delta \beta, \\
\beta \alpha \gamma &= \lambda (\gamma \delta \gamma), \\
\alpha \beta \alpha \gamma &= 0, \\
\beta \alpha \beta \alpha &= 0, \\
\delta \gamma \beta &= 0, \\
\gamma \delta \gamma \delta &= 0, \\
\alpha \beta \alpha \beta &= 0, \\
\delta \gamma \delta \gamma &= 0, \\
\delta \beta \alpha &= 0.
\end{align*}
\]

One can show that $T(\lambda)$ and $T(\mu)$ are not isomorphic for $\lambda \neq \mu$. The weighted surface algebras with the same quiver and $\varepsilon, \varepsilon'$ virtual loops have Gabriel quiver denoted by $Q(3A)^{2k}$ in [11] (which is the algebra with parameter $k$ in the the family $Q(3A)^{2k}$ of [5]).

3.3. Spherical algebras. We have the algebras whose quiver is given by the first building block where we identify $a_1 = a_{n+1}$, for $n \geq 2$. The case $n = 2$ gives the algebra $S(\lambda)$, called spherical algebra, introduced in [8], Example 3.6, as follows.

where the four shaded triangles denote the $f$-orbits. We take all multiplicities equal to 1, the presentation induced by the weighted surface algebra presentation is, with suitable choice of $c_\bullet$,

\[
\begin{align*}
\alpha \beta \nu &= \omega \nu, \\
\beta \nu \delta &= \lambda \beta \gamma \sigma, \\
\nu \delta \alpha &= \lambda \gamma \sigma \alpha, \\
\delta \alpha \beta &= \delta \omega, \\
\gamma \sigma \varphi &= \nu \delta \varphi, \\
\sigma \varphi \omega &= \lambda \sigma \alpha \beta, \\
\varphi \omega &= \lambda \alpha \beta \gamma, \\
\omega \gamma \sigma &= \omega \nu \delta, \\
\alpha \beta \nu \delta &= 0, \\
\beta \nu \delta \varphi &= 0, \\
\nu \delta \alpha \beta \nu &= 0, \\
\delta \alpha \beta \gamma &= 0, \\
\gamma \sigma \omega \gamma &= 0, \\
\sigma \omega \nu &= 0, \\
\omega \gamma \sigma &= 0, \\
\beta \gamma \sigma &= 0, \\
\sigma \alpha \nu &= 0, \\
\delta \omega &= 0, \\
\omega \nu \delta &= 0, \\
\beta \nu \delta &= 0, \\
\delta \alpha \beta &= 0, \\
\sigma \varphi &= 0, \\
\sigma \omega &= 0, \\
\omega \gamma &= 0.
\end{align*}
\]
3.4. The $n$-spherical algebra. When $n \geq 2$, the permutation $g$ is of the form
\[ \prod_{i=1}^{n} (\xi_i \eta_i) \cdot (\gamma_1 \sigma_1 \gamma_2 \sigma_2 \ldots \gamma_n \sigma_n) \cdot (q_n \delta_n q_{n-1} \delta_{n-1} \ldots q_1 \delta_1). \]
We take the multiplicities for the $2n$-cycles to be $m, m' \geq 1$, and write $c, c'$ for the parameters at these cycles.

3.5. A mixed algebra. We can glue together the three building blocks by identifying $2 = a_1$, and $2' = a_{n+1}$. In this case, the permutation $g$ is the product of one large cycle with $n$ cycles of length 2, and two loops:
\[ \prod_{i=1}^{n} (\xi_i \eta_i) \cdot (\gamma_1 \sigma_1 \gamma_2 \sigma_2 \ldots \gamma_n \sigma_n \gamma \delta \gamma_n \delta_n \gamma_n-1 \delta_{n-1} \ldots q_1 \delta_1 \beta \alpha)(\varepsilon)(\varepsilon'). \]
We take again the multiplicities equal to 1 on 2-cycles of $g$, or $m_{\gamma_i} = m$, and the parameter function with value 1 on each virtual arrow. These algebras were not studied in previous papers but they fit into the same scheme.

4. Construction of the module $M$

Let $\Lambda$ be one of the algebras as described above. Let $\Gamma \subset Q_0$ be the set of vertices which are not adjacent to a virtual arrow.

**Definition 4.1.** Let $M$ be the (right) $\Lambda$-module
\[ M := \Lambda \oplus \left( \bigoplus_{i \in \Gamma} S_i \right) \oplus \left( \bigoplus_{\nu \in \Gamma'} \Omega^2(S_\nu) \right) \]

In the following we write down the details for the case of the $n$-spherical algebra, for the other algebras they are essentially the same. In this case $\Gamma = \{ a_i \mid 1 \leq i \leq n \}$.

4.1. The $\Omega$-translates of the simple modules. For the algebra in question, the dimensions of the indecomposable projectives are:
\[ \dim P_{a_i} = 2n(m + m'), \quad \dim P_{b_i} = 2nm + 2, \quad \dim P_{\delta_i} = 2nm' + 2. \]
Let $a_i \in \Gamma$. The structure of $\Omega^{\pm 1}(S_{a_i})$ can be seen from the presentation of the algebra. The module $\Omega^2(S_{a_i})$ has dimension 5, the Loewy structure is
\[ b_{i-1} \quad d_i \quad a_i \quad b_i \quad d_{i-1}. \]
That is, the module has a 'simple waist'. Now let $\nu \in Q \setminus \Gamma$ we set $U_\nu := \Omega^2(S_\nu)$. Then $\Omega(U_\nu) = \Omega^{-1}(S_\nu)$ and $\Omega^{-1}(U_\nu) = \Omega(S_\nu)$, their structure can also be seen from the presentation. We describe $U_\nu$.

**Lemma 4.2.** The module $U_{b_i}$ is uniserial of length $2nm' - 1$, with composition series
\[ U_{b_i} = U(a_i, d_{i-1}, a_{i-1}, d_{i-2}, a_{i-2}, \ldots, a_1) \]
The module $U_{d_i}$ is uniserial of length $2nm - 1$, with composition series
\[ U_{d_i} = U(a_{i+1}, b_{i+1}, a_{i+2}, b_{i+2}, \ldots, a_i) \]
(taking indices modulo $n$ and writing $a_i, d_i, b_i$ meaning the corresponding simple module).
Proof We compute $U_{b_i}$, that is $\Omega^2(S_{b_i}) = \Omega(S_{b_i}) = \{ x \in P_{a_2} \mid \sigma_1 x = 0 \} \subset P_{a_2}$. From the relations for the algebra, we have

$$\sigma_1 \vartheta_1 \delta_1 = c A_{\sigma_1} = c \sigma_1 A'_{\sigma_1}$$

Hence $\sigma_1 \psi = 0$ if we set

$$\psi = \psi_{\vartheta_1 \delta_1} := \vartheta_1 \delta_1 - c A'_{\sigma_1}.$$ 

One exhibits a basis for $\psi \Lambda$, showing that it has the same dimension as $\Omega^2(S_{b_1})$, hence we have equality. The submodule structure follows directly. The case $U_{d_i}$ is similar. □

**Proposition 4.3.** We have $\text{Ext}^1(M, M) = 0$ and $\text{Ext}^2(M, M) = 0$.

Proof By ext symmetry, it suffices to show that for any non-projective indecomposable summand $X$ of $M$ we have $\text{Ext}^1(M, X) = 0$ and $\text{Ext}^1(X, M) = 0$. For this, we use the following short exact sequences:

Let $a_i \in \Gamma$,

(1) \hspace{1cm} $0 \to \Omega(S_{a_i}) \to P_{a_i} \to S_{a_i} \to 0$, \hspace{1cm} $0 \to \Omega^2(S_{a_i}) \to P_{a_i} \oplus P_{d_i} \to \Omega(S_{a_i}) \to 0$

Consider a vertex $\nu$ not in $\Gamma$, let $\nu = c_i$

(2) \hspace{1cm} $0 \to \Omega^{-1} S_{b_i} \to P_{a_i} \to U_{b_i} \to 0$, \hspace{1cm} $0 \to S_{b_i} \to P_{b_i} \to \Omega^{-1}(S_{b_i}) \to 0$

Let $\nu = d_i$, then

(3) \hspace{1cm} $0 \to \Omega^{-1}(S_{d_i}) \to P_{a_i+1} \to U_{d_i} \to 0$, \hspace{1cm} $0 \to S_{d_i} \to P_{d_i} \to \Omega^{-1}(S_{d_i}) \to 0$

We apply the functor $\text{Hom}_A(-, X)$ to the above exact sequences.

(I) Assume $X = S_{a_j}$ for some $j$. We know from the quiver that $\text{Ext}^1(S_{a_j}, S_{a_j}) = 0$ already. To show that $\text{Ext}^2(S_{a_j}, S_{a_j}) = 0$ we apply the functor to the second sequence in (1). From the structure of $\Omega^2(S_{a_j})$ we see directly that $\text{Hom}(\Omega^2(S_{a_j}), S_{a_j}) = 0$ and hence $\text{Ext}^2(S_{a_j}, S_{a_j}) = 0$.

We have $\text{Ext}^1(U_{b_j}, X) = 0$ since $\text{Hom}(\Omega^{-1}(S_{b_j}), S_{a_j}) = 0$ furthermore $\text{Ext}^2(U_{b_j}, X) = 0$ since $\text{Hom}(S_{b_j}, S_{a_j}) = 0$. Similarly one shows $\text{Ext}^1(U_{d_i}, X) = 0$ and $\text{Ext}^2(U_{d_i}, X) = 0$.

(II) Now assume $X = U_{b_j}$ for some $j$. First, by dimension shift $\text{Ext}^1(U_{\nu}, U_{b_j}) \cong \text{Ext}^1(S_{\nu}, S_{b_j}) = 0$ for any $\nu$ of valency 1, from the quiver. Next, consider $\text{Ext}^2(U_{\nu}, X)$, by applying the functor $\text{Hom}(-, X)$ to the second exact sequence in (2). We have $\text{Hom}(S_{\nu}, U_{b_j}) = 0$ (the socle of $U_{b_j}$ is always some $S_a$), and hence $\text{Ext}^2(U_{\nu}, X) = 0$.

Now consider $\text{Ext}^t(S_{a_j}, X)$ for $t = 1, 2$. By the ext symmetry, it is isomorphic to $\text{Ext}^t(X, S_{a_j})$ for $t = 1, 2$. By part (I) we know that it is zero.

The proof for $X = U_{d_i}$ is analogous. □

**Remark 4.4.** For possible later use, we write down sequences which may be used to show $\text{Ext}^1(X, M) = 0$ and $\text{Ext}^2(X, M) = 0$: Let $a_i \in \Gamma$,

(1*) \hspace{1cm} $0 \to S_{a_i} \to P_{a_i} \to \Omega^{-1}(S_{a_i}) \to 0$, \hspace{1cm} $0 \to \Omega^{-1}(S_{a_i}) \to P_{b_i} \oplus P_{d_i} \to \Omega^{-2}(S_{a_i}) \to 0$

Consider a vertex $\nu$ not in $\Gamma$, let $\nu = c_i$

(2*) \hspace{1cm} $0 \to U_{b_i} \to P_{a_i+1} \to \Omega(S_{b_i}) \to 0$, \hspace{1cm} $0 \to \Omega(S_{b_i}) \to P_{b_i} \to S_{b_i} \to 0$

Let $\nu = d_i$, then

(3*) \hspace{1cm} $0 \to U_{d_i} \to P_{a_i} \to \Omega(S_{d_i}) \to 0$, \hspace{1cm} $0 \to \Omega(S_{d_i}) \to P_{d_i} \to S_{d_i} \to 0$. 
5. Ext vanishing and 3-cluster tilting

We would like to determine when \( M \) is 3-cluster tilting. Hence take \( X \) indecomposable and not projective, and assume

\[
\text{Ext}^1(M, X) = 0 = \text{Ext}^2(M, X).
\]

By ext symmetry, we get for free that \( \text{Ext}^1(X, M) = 0 = \text{Ext}^2(X, M) \). The aim is to show that \( X \) is in \( \text{add}(M) \), or if not, to identify \( X \).

**Lemma 5.1.** The socle and the top of \( X \) belong to \( \text{add}(\oplus_i S_{a_i}) \).

**Proof** Let \( \nu \) be a vertex \( \neq a_i \) for any \( i \). Apply the functor \( \text{Hom}(-, X) \) to the second sequence of (2), this gives the exact sequence

\[
0 \to \text{Hom}(\Omega^{-1}(S_{b_i}), X) \to \text{Hom}(P_{b_i}, X) \to \text{Hom}(S_{b_i}, X) \to 0
\]

Any homomorphism \( P_{b_i} \to X \) must map the socle to zero, otherwise it would be split. Hence it lies in \( \text{Hom}(\Omega^{-1}(S_{b_i}), X) \) and therefore the first two terms are isomorphic. Hence the last term is zero, as required. To show that also \( \text{Hom}(X, S_{a_i}) = 0 \) we use a sequence from (2*). □

**Lemma 5.2.** We have \( \text{Hom}(\Omega(X), S_{a_i}) = 0 \) and \( \text{Hom}(S_{a_i}, \Omega^{-1}(X)) = 0. \)

**Proof** Since \( \text{Ext}^1(X, S_{a_i}) = 0 \), from a minimal projective cover of \( X \) we obtain the exact sequence

\[
0 \to \text{Hom}(X, S_{a_i}) \to \text{Hom}(P_X, S_{a_i}) \to \text{Hom}(\Omega(X), S_{a_i}) \to 0
\]

The first two terms are isomorphic since we start with a projective cover. Hence the last is zero. Similarly by using an injective hull we get \( \text{Hom}(\Omega^{-1}(X), S_{a_i}) = 0. \) □

Let \( \mathcal{X} \) be the category of \( A \)-modules which have socle and top in \( \text{add}(S_{a_i}) \). This category is equivalent to \( \text{mod} - e\Lambda e \). where \( e \) is the idempotent \( e := \sum_i e_{a_i} \). An equivalence is given by the functor \( V \mapsto V e \), with inverse the composite of \((-) \otimes_{e\Lambda e} (e\Lambda) \) follows by factoring out the largest \( A \)-submodule \( V' \) with \( V' e = 0 \) (see for example [4]).

We may write down quiver and presentation of the algebra \( e\Lambda e \). The arrows are \( x_i := \gamma_i \sigma_i \) and \( y_i := \rho_i \delta_i \), for \( 1 \leq i \leq n \) where \( x_i : a_i \to a_{i+1} \) and \( y_i : a_{i+1} \to a_i \). From the relations for \( \Lambda \) we see \( x_i y_i = 0 \) and \( y_i x_{i-1} = 0 \). That is, \( e\Lambda e \) is special biserial. Moreover, for any \( i \), the longest non-zero monomial \( x_i x_{i+1} \ldots \) is up to a scalar equal to the longest non-zero monomial \( y_{i-1} y_{i-2} \ldots \), and this gives the socle relations.

**Lemma 5.3.** The module \( X \) has simple socle and top.

**Proof** The module \( X \), and as well, all projectives (injectives) \( P_{a_i} \) belong to the category \( \mathcal{X} \), and hence we may fix an injective hull, or projective cover, of \( X \) by identifying with the image of a suitable injective hull, or projective cover, of \( X e \), in \( \text{mod} \ e\Lambda e \). The indecomposable \( e\Lambda e \)-modules are 'strings' or 'bands', and their injective hulls or projective covers may be written down explicitly.

Assume the socle of \( X \) is not simple, then consider the injective hull \( I_X \), it has at least two indecomposable summands, say it is \( \oplus_{i \in R} P_{a_i} \). We may assume, with the above convention, and taking \( X \to I_X \) as inclusion, that \( X \) has a generator \( \omega = (\omega_1, \omega_2, 0, \ldots) \) such that \( e\Lambda \omega \) has socle of length two, and moreover, that \( \omega x_j = (\omega_1 x_j, 0, \ldots) \) and \( \omega y_j = (0, \omega_2 y_{j-1}, 0, \ldots) \) and \( \omega x_r = 0 \), \( \omega y_s = 0 \) for all other generators \( x_r, y_s \) of \( e\Lambda e \). This implies then that \( \omega J \subseteq X \) where \( J \) is the radical of \( \Lambda \). Now consider \( \pi : I_X \to \Omega(X) \). The element \( \pi(\omega_1, 0, \ldots) \) is non-zero (since
\[ \omega \text{ is a generator for } X. \] Furthermore, \([\pi(\omega_1, \ldots)]J = \pi[(\omega_1, 0)J = 0 \text{ since } (\omega_1, 0, \ldots)J \text{ is contained in } X. \] Now \( \pi(\omega_1, 0) = \pi(\omega_1, 0)e, \) (since topX is in \( \text{add}(\oplus S_{a_i}) \). Hence for some \( i \) we have \( \text{Hom}(S_{a_i}, \Omega^{-1}(X)) \neq 0. \) This contradicts the previous Lemma.

Similarly by exploiting a projective cover, one shows that the top of \( X \) must be simple. □

**Proposition 5.4.** The module \( X \) is uniserial.

*Proof* If \( X \) is not uniserial then \( Xe \) is not uniserial (using the structure of the projectives in this case). Then \( Xe \) is a 'band module'. This means that \( X \) contains a submodule isomorphic to the second socle of some \( P_{a_j} \). That is \( \text{Hom}(\Omega^2(S_{a_j}), X) \neq 0. \)

Applying \( \text{Hom}(-, X) \) to the exact sequence
\[
0 \rightarrow \Omega^2(S_{a_j}) \xrightarrow{-} P := P_{b_j} \oplus P_{d_{j-1}} \rightarrow \Omega(S_{a_j}) \rightarrow 0
\]
gives an exact sequence, that is a non-zero homomorphism \( \theta : \Omega^2(S_{a_j}) \rightarrow X \) factors through \( \iota, \) say \( \theta = \psi \circ \iota. \) The kernel of \( \theta \) is the socle of \( \Omega^2(S_{a_j}) \) which also is the socle of \( P. \) We factor out these socles, then for the induced maps we have
\[
\tilde{\theta} = \tilde{\psi} \circ \tilde{\iota}.
\]
Now, the map \( \tilde{\psi} \) on the socle of \( \tilde{P} \) is non-zero on each component. It follows that the image of \( \tilde{\psi} \) has Loewy length equal to the Loewy length of \( P/\text{soc}P. \)

Note that all modules \( P_{a_i} \) have the same Loewy length \( \ell. \) As well \( P_{b_j} \oplus P_{d_{j-1}} \) has Loewy length \( \ell. \) Hence the Loewy length of \( P/\text{soc}P \) is \( \ell - 1. \) The image of \( \tilde{\psi} \) is contained in the radical of \( X, \) which is the unique maximal submodule. It follows that the Loewy length of \( X \) is \( \ell. \) But this means that \( X \) must be projective, a contradiction. This shows that \( Xe \) is uniserial, and then from the structure of the projectives, also \( X \) is uniserial. □

We summarize. We have shown that if \( X \) is indecomposable and not projective such that \( \text{Ext}^1(M, X) = 0 = \text{Ext}^2(M, X) \) then
\[ (\ast) \ X \text{ is uniserial, and } \text{soc}X \text{ and top}X \text{ are in } \text{add}(\oplus_i S_{a_i}). \] That is, \( X \) is a subquotient of some \( U_{b_i} \) or \( U_{d_j}. \)

We show now that if \( X \) is any module satifying \((\ast)\) then \( \text{Ext}^1(M, X) = 0 \) and \( \text{Ext}^2(M, X) = 0. \)

**Lemma 5.5.** Let \( X = U(a_j, b_j, a_{j+1}, b_{j+1}, \ldots, a_l) \), a subquotient of some \( U_i. \) Then
\[
\text{Ext}^1(M, X) = 0 \text{ and } \text{Ext}^2(M, X) = 0.
\]

*Proof* We use the sequences in the proof of Proposition 4.3. We apply the functor \( (-, X) := \text{Hom}(-, X) \) to the exact sequences in (1). We start with the second, this gives
\[
0 \rightarrow (\Omega(S_{a_j}), X) \xrightarrow{-} (P_{b_j} \oplus P_{d_{j-1}}, X) \rightarrow (\Omega^2(S_{a_j}), X) \rightarrow \text{Ext}^1(\Omega(S_{a_j}), X) \rightarrow 0
\]
We see that \( \text{Hom}(\Omega^2(S_{a_j}), X) = 0 \) (\( X \) is uniserial). Hence the ext space is zero. Moreover, it follows that the first two terms are isomorphic, which we can use for the first sequence:
\[
0 \rightarrow (S_{a_i}, X) \rightarrow Xe_{a_i} \rightarrow Xe_{b_i} \oplus Xe_{d_i} \rightarrow \text{Ext}^1(S_{a_i}, X) \rightarrow 0
\]
In our case, \( Xe_{d_i} = 0. \) Note that in the composition series we have length two subquotients \( a_r, b_r, \) except that for \( l = r \) we have an extra copy of \( a_l. \) Hence if \( i = l \) then the first term is \( K, \) and \( |Xe_{a_i}| = 1 + |Xe_{b_i}| \) and ext is zero. Suppose \( i \neq l, \) then the first term is zero and the second and third are isomorphic. Again ext is zero.
Next, we apply \((-,X)\) to the sequences in (3). Since \(S_d\) does not occur in \(X\), the functor takes the second sequence to zero. From the first sequence we get
\[
0 \rightarrow \text{Hom}(U_{d_i}, X) \rightarrow Xe_{a_i+1} \rightarrow 0 \rightarrow \text{Ext}^1(U_{d_i}, X) \rightarrow 0
\]
and the ext space is zero.

Now consider \((-,X)\) applied to sequences in (2). The second sequence gives \(\text{Hom}(\Omega^{-1}(S_{b_i}), X) \cong Xe_{b_i}\) and \(\text{Ext}^1(\Omega^{-1}(S_{b_i}), X) = 0\). Consider the first sequence, this gives
\[
0 \rightarrow \text{Hom}(U_{b_i}, X) \rightarrow Xe_{a_i} \rightarrow Xe_{b_i} \rightarrow \text{Ext}^1(U_{b_i}, X) \rightarrow 0
\]
If the top (ie \(S_{a_i}\)) of \(U_{b_i}\) is not the same as the socle of \(X\) then the hom space is zero and the second and third term are isomorphic, and ext is zero. Suppose \(i = l\), then the first term is \(K\), and \(|Xe_{a_i}| = 1 + |Xe_{b_i}|\) and again the ext space is zero. □

**Corollary 5.6.** Assume \(\Lambda\) is the triangle algebra, or the spherical algebra. Then \(M\) is 3-cluster tilting.

**Proof** For these algebras, all indecomposables satisfying (*) are in \(\text{add}(M)\). □

Consider an \(n\)-spherical algebra for \(n \geq 3\) and \(m = m' = 1\). Then the (finite) set of modules \(X\) satisfying (*) contains all modules of the form
\[
\mathcal{U}(a_i, b_i, a_{i+1}), \quad \mathcal{U}(a_i, d_{i-1}, a_{i-1}).
\]
To have a 3-cluster tilting module with \(M\) as a summand, we would need to take \(\widetilde{M} = M \oplus \mathcal{V}\) where \(\mathcal{V}\) is the direct sum of all modules satisfying (*). However, \(\widetilde{M}\) has self-extensions. For example there is a non-split exact sequence
\[
0 \rightarrow \mathcal{U}(a_2, b_2, a_3) \rightarrow S_{a_2} \oplus \mathcal{U}(a_1, b_1, a_2, b_2, a_3) \rightarrow \mathcal{U}(a_1, b_1, a_2) \rightarrow 0
\]
Hence \(M\) cannot be extended to a 3-cluster tilting module for the \(n\)-spherical algebra when \(n \geq 3\).

We also consider the algebra with triangular quiver and \(k \geq 2\). In this case the list of uniserial modules \(X\) which are subquotients of \(U_1\) and \(U_3\) contains the modules
\[
\mathcal{U}(2, 3, 2), \quad \mathcal{U}(2, 1, 2)
\]
Let \(\widetilde{M} = M \oplus \mathcal{V}\) where \(\mathcal{V}\) is the direct sum of all indecomposable modules satisfying (*). This is not a 3-cluster tilting module since it has self-extensions: we have the non-split exact sequence
\[
0 \rightarrow \mathcal{U}(2, 3, 2) \rightarrow S_2 \oplus \mathcal{U}(2, 1, 2, 3, 2) \rightarrow \mathcal{U}(2, 1, 2) \rightarrow 0
\]

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