Functionals of spatial point processes having a density with respect to the Poisson process

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Abstract

$U$-statistics of spatial point processes given by a density with respect to a Poisson process are investigated. In the first half of the paper general relations are derived for the moments of the functionals using kernels from the Wiener-Itô chaos expansion. In the second half we obtain more explicit results for a system of $U$-statistics of some parametric models in stochastic geometry. In the logarithmic form functionals are connected to Gibbs models. There is an inequality between moments of Poisson and non-Poisson functionals in this case, and we have a version of the central limit theorem in the Poisson case.

Keywords: difference of a functional, limit theorem, moments, $U$-statistics

Classification: 60G55, 60D05

1 Introduction

Recently the investigation of functionals of Poisson point processes using differences and Wiener-Itô chaos expansion has been developed, cf. [3]. In [9] central limit theorems for $U$-statistics of Poisson processes were derived based on Malliavin calculus and the Stein method. In the present paper we study functionals of non-Poisson point processes given by a density w.r.t. a Poisson process. Instead of modification of a general approach to moments for Poisson processes [4] using the diagram formula [7] we apply the $L_2$ expansion of a product of functionals [3]. Specially $U$-statistics are of interest and formulas for their moments, more explicitly for the mean and covariance, are derived which involve conditional intensities. The product of a functional and a density is further studied in a
logarithmic form using the characterization theorem for Gibbs processes from [1].

In the second part of the paper parametric models for point processes of interacting particles [5] are investigated as a special case of the general theory. We concentrate on lower-dimensional particles, namely interacting segments in the plane and interacting plates in the three-dimensional space. Their natural $U$-statistics correspond to the total length (area) and characteristics of intersections. First and second moments are presented in a closed form using explicit formulas for conditional intensities. Limitations on the parameter space are indicated which allow for repulsive interactions. Finally in the Poisson case using results from [4] the central limit theorem for a vector of $U$-statistics of the model is discussed.

2 Moments of functionals of point processes having a density

Consider a bounded Borel set $B \subset \mathbb{R}^d$ with Lebesgue measure $|B| > 0$ and a measurable space $(\mathbb{N}, \mathcal{N})$ of integer-valued finite measures on $B$. $\mathcal{N}$ is the smallest $\sigma$-algebra which makes the mappings $x \mapsto x(A)$ measurable for all Borel sets $A \subset B$ and all $x \in \mathbb{N}$. A random element having a.s. values in $(\mathbb{N}, \mathcal{N})$ is called a finite point process. Let a Poisson point process $\eta$ on $B$ have finite intensity measure $\lambda$ with no atoms and distribution $P_{\eta}$ on $\mathcal{N}$. We consider a finite point process $\mu$ on $B$ given by a density $p$ w.r.t. $\eta$, i.e. with distribution $P_{\mu}$

$$dP_{\mu}(x) = p(x)dP_{\eta}(x), \quad x \in \mathbb{N},$$

where $p : \mathbb{N} \to \mathbb{R}_+$ is measurable satisfying

$$\int_{\mathbb{N}} p(x)dP_{\eta}(x) = 1.$$

For a measurable functional $F : \mathbb{N} \to \mathbb{R}$, $F(\mu)$ is a random variable. As described in [1], p.61, integer-valued finite measures can be represented in this context by $n$-tuples of points corresponding to their support ($n$ is variable). Throughout the paper we will apply this representation without using its explicit notation from [1]. We deal with $L_p$ spaces, $1 \leq p < +\infty$, of functions on various measure spaces. For $F \in L_1(P_{\mu})$ it holds

$$\mathbb{E}F(\mu) = \mathbb{E}[F(\eta)p(\eta)].$$}

We formulate this assertion in terms of moments of a functional.

Lemma 1. Let $F \in L_m(P_{\mu})$, $G_m(x) = F^m(x)p(x)$, $m = 1, 2, \ldots$. Then the $m$-th moment

$$\mathbb{E}F^m(\mu) = \mathbb{E}G_m(\eta), \quad m = 1, 2, \ldots,$$

specially for the mean and variance we have

$$\mathbb{E}F(\mu) = \mathbb{E}G_1(\eta), \quad \text{var} F(\mu) = \mathbb{E}G_2(\eta) - [\mathbb{E}G_1(\eta)]^2.$$
Proof: It holds $\mathbb{E}F^m(\mu) = \int F^m(x)dP_\mu(x) = \int F^m(x)p(x)dP_\eta(x) = \mathbb{E}G_m(\eta)$, then $\mathbb{E}F(\mu) = \mathbb{E}G_1(\eta)$.

$$\text{var} F(\mu) = \int (F(x) - \mathbb{E}F(\mu))^2dP_\mu(x) = \int (F(x) - \mathbb{E}G_1(\eta))^2p(x)dP_\eta(x) =$$

$$= \int F^2(x)p(x)dP_\eta(x) - 2\mathbb{E}G_1(\eta) \int F(x)p(x)dP_\eta(x) + [\mathbb{E}G_1(\eta)]^2 =$$

$$= \int F^2(x)p(x)dP_\eta(x) - [\mathbb{E}G_1(\eta)]^2.$$

□

For a functional $F$, $y \in B$, one defines the difference operator $D_yF$ for a point process $\mu$ as a random variable

$$D_yF(\mu) = F(\mu + \delta_y) - F(\mu),$$

where $\delta_y$ is a Dirac measure at the point $y$. Inductively for $n \geq 2$ and $(y_1, \ldots, y_n) \in B^n$ we define a function

$$D^n_{y_1, \ldots, y_n}F = D^1_{y_1}D^{n-1}_{y_2, \ldots, y_n}F,$$

where $D^1_y = D_y$, $D^0F = F$. Operator $D^n_{y_1, \ldots, y_n}$ is symmetric in $y_1, \ldots, y_n$ and symmetric functions $T^\mu_n F$ on $B^n$ are defined as

$$T^\mu_n F(y_1, \ldots, y_n) = \mathbb{E}D^n_{y_1, \ldots, y_n}F(\mu),$$

$n \in \mathbb{N}$, $T^\mu_0 F = \mathbb{E}F(\mu)$, whenever the expectations exist. We write $T^n_\mu F$ for $T^n_{\eta}F$.

For the functionals of a Poisson process Theorem 1.1 in [3] says that given $F$, $\tilde{F} \in L^2(P_\eta)$ it holds

$$\mathbb{E}[F(\eta)\tilde{F}(\eta)] = \mathbb{E}F(\eta)\mathbb{E}\tilde{F}(\eta) + \sum_{n=1}^{\infty} \frac{1}{n!} \langle T^n_\mu F, T^n_\mu \tilde{F} \rangle_n, \quad (5)$$

where $\langle ., . \rangle_n$ is the scalar product in $L^2(\lambda^n)$.

Using the Wiener-Itô chaos expansion and diagram formula, [4] in Corollary 3.5 derive a general formula for $m$-th moment of a functional of the Poisson process. Instead of trying to modify this formula to $Fp^m$ in order to obtain $\mathbb{E}F^m(\mu)$, we use (5) to get explicit formulas for the first two moments in the case of $U$-statistics.

### 2.1 Explicit formulas for $U$-statistics

A $U$-statistic $F$ of order $k \in \mathbb{N}$ driven by a function $f$ is a functional of a finite point process $\mu$ defined as

$$F(\mu) = \sum_{(x_1, \ldots, x_k) \in \mu^k} f(x_1, \ldots, x_k), \quad (6)$$
where \( f : B \to \mathbb{R} \) is a function symmetric w.r.t. to the permutations of its variables, \( f \in L_1(\lambda^k) \). Here \( \mu^k_x \) is the set of \( k \)-tuples of different points of \( \mu \). By the Campbell theorem [10] we have

\[
\mathbb{E} F(\eta) = \int_B \ldots \int_B f(x_1, \ldots, x_k) \lambda(dx_1 \ldots dx_k)
\]

(we write \( \lambda(dx_1 \ldots dx_k) \) instead of \( \lambda(dx_1) \ldots \lambda(dx_k) \)) and for \( F \in L_2(P_\eta) \) using (5) we have

\[
\text{var} F(\eta) = \sum_{i=1}^k i! \binom{k}{i}^2 \times \int_{B^k} \left( \int_{B^{k-i}} f(y_1, \ldots, y_i, x_1, \ldots, x_{k-i}) \lambda(dx_1 \ldots dx_{k-i}) \right)^2 \lambda(dy_1, \ldots, dy_i).
\]

It is easily derived [9] that for \( U \)-statistic of order \( k \) it holds

\[
D_n F(\eta_1, \ldots, \eta_n) = \frac{k!}{(k-n)!} \sum_{(x_1, \ldots, x_{k-n}) \in \mu^{k-n}} f(y_1, \ldots, y_n, x_1, \ldots, x_{k-n}) \lambda(dx_1 \ldots dx_{k-n})
\]

for \( n \leq k \), \( D_n F(\eta_1, \ldots, \eta_n) = 0 \) for \( n > k \). Then from the Campbell theorem

\[
T_n F(\eta_1, \ldots, \eta_n) = \frac{k!}{(k-n)!} \int_{B^{k-n}} f(y_1, \ldots, y_n, x_1, \ldots, x_{k-n}) \lambda(dx_1 \ldots dx_{k-n}),
\]

\( n \leq k \), \( T_n F(\eta_1, \ldots, \eta_n) = 0, n > k \).

Let \( \mu \) be a finite point process with density \( p \) satisfying

\[
p(x) > 0 \Rightarrow p(\tilde{x}) > 0
\]

for all \( \tilde{x} \subset x \). For the (Papangelou) conditional intensity of \( \mu \), see [1], it holds

\[
\lambda^*(u, x) = \frac{p(x \cup \{u\})}{p(x)}, \quad x \in \mathbb{N}, \ u \in B, \ u \notin x,
\]

here probability \( P(u \in \mu) = 0 \). For \( n \geq 1 \) we use analogously a.s.

\[
\lambda^*_n(u_1, \ldots, u_n, x) = \frac{p(x \cup \{u_1, \ldots, u_n\})}{p(x)}, \quad u_1, \ldots, u_n \in B,
\]

the conditional intensity of \( n \)-th order of \( \mu \), \( \lambda^*_0 \equiv 1 \). We observe that \( \lambda^*_n \) is symmetric in the variables \( u_1, \ldots, u_n \). A point process \( \mu \) with conditional intensity \( \lambda^* \) has intensity function

\[
\rho(u) = \mathbb{E} \lambda^*(u, \mu).
\]

**Lemma 2.** It holds

\[
T_n \rho(y_1, \ldots, y_n) = \sum_{J \subset \{1, \ldots, n\}} (-1)^{n-|J|} \mathbb{E} \lambda^*_n(\{y_j, j \in J\}, \mu),
\]

where \( |J| \) is the cardinality of \( J \), assuming that the expectations exist.
Proof: From \[ D_{y_1,\ldots,y_n}^n(\eta) = \sum_{J \subset \{1,\ldots,n\}} (-1)^{n-|J|} p(\eta \cup \{y_j, j \in J\}). \] Then
\[
T_n p(y_1, \ldots, y_n) = \mathbb{E} D_{y_1,\ldots,y_n}^n(\eta) =
\int \sum_{J \subset \{1,\ldots,n\}} (-1)^{n-|J|} p(x \cup \{y_j, j \in J\}) \frac{dP_y(x)}{p(x)}
\]
and (12) follows. □

Theorem 1. Let \( F_j \) be \( U \)-statistics of order \( k_j \), \( j = 1, \ldots, m \), such that
\[
\prod_{j=1}^m F_j \in L_2(P_\eta)
\]
and the density \( p \in L_2(P_\eta) \). Then it holds
\[
\mathbb{E} \left[ \prod_{j=1}^m F_j(\mu) \right] = \mathbb{E} \left[ \prod_{j=1}^m F_j(\eta) \right] + \sum_{n=1}^q \frac{1}{n!} (T_n \prod_{j=1}^m F_j, T_n p)_n,
\]
where \( q = \sum_{i=1}^m k_i \).

Proof: Using formulas (2) and (5) with \( \mathbb{E} p(\eta) = 1 \) we claim that
\[
T_n \prod_{j=1}^m F_j = 0, \quad n > q.
\]

For two \( U \)-statistics \( F, G \) of order \( k, l \) driven by \( f, g \), respectively, we have
\[
D_y FG(\eta) = \sum_{(x_1,\ldots,x_k) \in (\eta,y)^k_\eta} f(x_1,\ldots,x_k) \sum_{(z_1,\ldots,z_l) \in (\eta,y)^l_{\eta_\neq}} g(z_1,\ldots,z_l) -
\sum_{(x_1,\ldots,x_k) \in \eta_\neq^k} f(x_1,\ldots,x_k) \sum_{(z_1,\ldots,z_l) \in \eta_\neq^l} g(z_1,\ldots,z_l).
\]

Only terms where \( y \) is among variables (either in one or both sums) in the first product on the right side do not cancel with any term in the second product. Thus for the second difference there is one place less for variables (since \( y \) is fixed). After \( k + l \) differences all places are occupied and \( D_{y_1,\ldots,y_{k+l}}^{k+l}(\eta) \) is independent of the Poisson process. Therefore the \((k + l + 1)\)-st difference is zero and (14) holds for a product of two functionals. From the same reasoning with more than two \( U \)-statistics (14) follows. □

Next we evaluate the first and second moments more explicitly.

Theorem 2. For a \( U \)-statistic \( F \in L_2(P_\eta) \) of order \( k \) and density \( p \in L_2(P_\eta) \) it holds
\[
\mathbb{E} F(\mu) = \int_{B_k} f(x_1,\ldots,x_k) \mathbb{E} [\lambda_k^*(x_1,\ldots,x_k, \mu)] \lambda(d(x_1,\ldots,x_k)).
\]
Proof: Denote $C_n^j$ the set of all combinations $c = \{ c_1, \ldots, c_j \}$ from $\{1, \ldots, n\}$. We put (9) and (12) into (13) with $m = 1$ and obtain

$$E F(\mu) = \sum_{n=0}^{k} \frac{1}{n!} \int_{B^n} \frac{k!}{(k-n)!} \times$$

$$\int_{B^{k-n}} f(x_1, \ldots, x_n, x_1, \ldots, x_{k-n}) \lambda(d(x_1, \ldots, x_{k-n})) \times$$

$$\sum_{j=0}^{n} (-1)^{n-j} \sum_{c \in C_n^j} \mathbb{E} \lambda^*_j(y_{c_1}, \ldots, y_{c_j}, \mu) \lambda(d(y_1, \ldots, y_n)) =$$

$$= \sum_{j=0}^{k} \sum_{n=j}^{k} (-1)^{n-j} \binom{k}{n} \times$$

$$\int_{B^k} \sum_{c \in C_n^j} \mathbb{E} \lambda^*_j(y_{c_1}, \ldots, y_{c_j}, \mu) f(x_1, \ldots, y_k) \lambda(d(y_1, \ldots, y_k)).$$

The cardinality of $C_n^j$ is $\binom{n}{j}$ and the identity

$$\sum_{n=j}^{k} (-1)^{n-j} \binom{k}{n} \binom{n}{j} = 0, \ j < k$$

holds, see [2], p.39, identity 11. Thus for each fixed $j < k$ it follows that the inner sum over $n$ in (16) vanishes, while the remaining value $j = k$ yields the result. \hfill $\square$

**Lemma 3.** Let $F, G$ be $U$-statistics of order $k, l$ driven by functions $f, g$, respectively, $k \leq l$. Then

$$F(\mu)G(\mu) = \sum_{j=0}^{k} \binom{k}{j} \frac{l!}{(l-k+j)!} \sum_{(x_1, \ldots, x_{l+j}) \in \mu_{x+j}^+} f(x_1, \ldots, x_k) g(x_1, \ldots, x_{k-j}, x_{k+1}, \ldots, x_{l+j}).$$

specially

$$F^2(\mu) = \sum_{j=0}^{k} \binom{k}{j} \frac{1}{j!} \sum_{(x_1, \ldots, x_{l+j}) \in \mu_{x+j}^+} f(x_1, \ldots, x_k) f(x_1, \ldots, x_{k-j}, x_{k+1}, \ldots, x_{k+j}).$$

**Proof:** The product $FG$ of $U$-statistics is a sum of $k+1$ terms, which are sums (over $k+j$ distinct points from $\mu$) of products $f(x_1, \ldots, x_k)g(y_1, \ldots, y_l)$, where $k-j$ variables appear simultaneously in both lists of variables of the product, $j = 0, 1, \ldots, k$. Their first occurrence is independent of the order (since all orders
are present in the inner sum) while their second occurrence is dependent on the order. Therefore coefficients at the inner sums are equal to
\[ \binom{k}{j} \left( \frac{l - k + j}{l - k + j} \right) (k - j)! \]
which leads to the result. □

Let in Lemma 3
\[ h(x_1, \ldots, x_{l+j}) = f(x_1, \ldots, x_k)g(x_{k-j}, x_{k+1}, \ldots, x_{l+j}) \]
We have that \( h \) is not necessarily symmetric in all \( l + j \) variables, so we use its symmetrization
\[ \bar{h}(x_1, \ldots, x_{l+j}) = \frac{1}{(l+j)!} \sum_{\pi_{l+j}} h(x_{m_1}, \ldots, x_{m_{l+j}}), \]
where \( \pi_{l+j} \) is the set of all permutations of indices \( 1, \ldots, l + j \).

**Theorem 3.** For \( U \)-statistics \( F,G \) from Lemma 3, \( f,g \) nonnegative, \( FG \in L_2(P_\eta) \), \( p \in L_2(P_\eta) \), it holds
\[ \mathbb{E}[F(\mu)G(\mu)] = \sum_{j=0}^{k} \binom{k}{j} \frac{l!}{(l - k + j)!} \left( \int_{B^{l+j}} h(x_1, \ldots, x_{l+j}) \mathbb{E}[^{\ast}_{l+j}(x_1, \ldots, x_{l+j}, \mu)] \lambda(d(x_1, \ldots, x_{l+j})). \right. \]
Specially for \( F = G \),
\[ \mathbb{E}F^2(\mu) = \sum_{j=0}^{k} \binom{k}{j} \frac{k!}{j!} \int_{B^{k+j}} h(x_1, \ldots, x_{k+j}) \mathbb{E}[^{\ast}_{k+j}(x_1, \ldots, x_{k+j}, \mu)] \lambda(d(x_1, \ldots, x_{k+j})). \]

**Proof:** Expectation of (17) is a linear combination of expectations of
\[ H_j(\mu) = \sum_{(x_1, \ldots, x_{l+j}) \in \mu_{l+j}^{i+j}} h(x_1, \ldots, x_{l+j}) = \sum_{(x_1, \ldots, x_{l+j}) \in \mu_{l+j}^{i+j}} \bar{h}(x_1, \ldots, x_{l+j}), \]
which are \( U \)-statistics of order \( l + j \). \( FG \in L_2(P_\eta) \) implies \( H_j \in L_2(P_\eta) \), \( j = 0, \ldots, k \), so we can apply Theorem 2 to \( H_j(\mu) \), \( j = 0, \ldots, k \), and from Lemma 3 the result follows. □

The assumptions of the above Theorems can be verified using formula for the expectation of a nonnegative functional of a Poisson process, see [6], p.15:
\[ \mathbb{E}[F(\eta)] = e^{-\lambda(B)} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{B} \ldots \int_{B} F(u_1, \ldots, u_n) \lambda(d(u_1, \ldots, u_n)). \]
Example 1. Consider $k = 1$, $C \subset B$ measurable and $U$-statistic

$$F(\mu) = \sum_{y \in \mu} f(y) = \mu(C), \quad f(y) = 1_{\{y \in C\}}.$$  

Let $\beta > 0$, $0 \leq \gamma \leq 1$, $r > 0$ be parameters, $\mu$ a Strauss point process [11] on $B \subset \mathbb{R}^d$ bounded with density

$$p(x) = \alpha \beta^n(x) \gamma(s(x)), \quad s(x) = \sum_{y,z \in x} 1_{||z-y|| \leq r},$$  

w.r.t. the Poisson point process with Lebesgue intensity measure $\lambda$, $\alpha$ is the normalizing constant, $n(x)$ the number of points in $x \in \mathbb{N}$. Here conditional intensities

$$\lambda^*(u,x) = \beta \gamma \gamma(t(u,x), u \notin x),$$

$$\lambda^*_2(y_1,y_2,x) = \beta^2 \gamma^2 \gamma(\sum_{y \in x} 1_{||y-y_2|| \leq r}),$$  

where $t(u,x) = \sum_{y \in x} 1_{||u-y|| \leq r}$. The assumptions of Theorems 2 and 3 are verified using (19), since e.g. $p^2(x) \leq \alpha^2 \beta^2 n(x)$ and

$$\sum_{n=0}^{\infty} \frac{\beta^2 n!}{n!} \approx \infty,$$

analogously for $F^2$, $F^4$. Thus we obtain

$$\mathbb{E}\mu(C) = \beta \int_C \mathbb{E}[\gamma(t(\mu))] \lambda(dy),$$

$$\mathbb{E}[\mu(C)^2] = \beta \int_C \mathbb{E}[\gamma(t(\mu))] \lambda(dy) +$$

$$+ \beta^2 \int_C \mathbb{E}[\gamma(t(\mu))] \lambda(dy).$$

Example 2. The special case of Strauss process with $\gamma = 1$ in (20) is Poisson process $\eta_{\beta}$ with deterministic constant conditional intensities $\lambda^*_n(u,\eta_{\beta}) = \beta^n$, $n = 1,2,\ldots$ and constant intensity function $\beta$, cf. [11]. An easy exercise is to verify that formula [7] for $\eta_{\beta}$ is a special case of [18].

2.2 Functionals in logarithmic form

In Lemma 1 we used the relation

$$\mathbb{E}F^m(\mu) = \mathbb{E}[F^m(\eta)p(\eta)], \quad m = 1,2,\ldots,$$

where $\eta$ is a Poisson process and $\mu$ a point process with probability density $p$ w.r.t. $\eta$. Consider a functional on $\mathbb{N}$

$$H_m = \log(F^m p) = m \log F + \log p, \quad m = 1,2,\ldots$$  

under the assumption $H_m \in L_1(P_\eta)$. From Jensen inequality we have

$$\log \mathbb{E}F^m(\mu) \geq \mathbb{E}H_m(\eta).$$
According to Theorem 4.3 in \cite{1} \( \lambda^*(u, x) \), \( x \in \mathbb{N}, u \in B \), is a conditional intensity of a point process \( \mu \) satisfying \( \lambda \) if and only if it can be expressed in the form

\[
\lambda^*(u, x) = \exp \left[ V_1(u) + \sum_{y \in x} V_2(u, y) + \sum_{\{y_1, y_2\} \subset x} V_3(u, y_1, y_2) + \ldots \right],
\]

(23)

where \( V_k : B^k \rightarrow \mathbb{R} \cup \{-\infty\} \) is called the potential of order \( k \). Then the density is that of a Gibbs process

\[
p(x) = \exp \left[ V_0 + \sum_{y \in x} V_1(y) + \sum_{\{y_1, y_2\} \subset x} V_2(y_1, y_2) + \ldots \right].
\]

(24)

Consequently

\[
\log p(x) = V_0 + \sum_{y \in x} V_1(y) + \sum_{\{y_1, y_2\} \subset x} V_2(y_1, y_2) + \ldots
\]

is a sum of a constant and \( U \)-statistics.

Assume that there is only a finite number \( l \) of sums on the right side of (23) and further that

\[
F(\eta) = \exp \left[ \sum_{(x_1, \ldots, x_k) \in \eta_{\mu}^k} f(x_1, \ldots, x_k) \right].
\]

(25)

Then \( \log F \) is a \( U \)-statistics of order \( k \) and \( H_m \) is a finite sum of \( U \)-statistics.

### 3 Stochastic geometry functionals

Let \( B \subset \mathbb{R}^l \), \( l \in \mathbb{N} \) be a bounded Borel set with positive Lebesgue measure, \( X \) a germ-grain process \( \text{\cite{10}} \) of germs \( z \in B \) and compact grains \( K_z \subset \mathbb{R}^l \), typically \( z \in K_z \). For a realization \( x \) of the germ-grain process denote \( U_x \) the union of all grains. Consider a probability density \( \text{\cite{5}} \)

\[
p(x) = c_\nu^{-1} \exp(\nu G(U_x)),
\]

(26)

of \( X \) w.r.t. a given reference Poisson point process \( \eta \). Here \( \nu = (\nu_1, \ldots, \nu_d) \) is a vector of real parameters, \( c_\nu \) a normalizing constant, \( G(U_x) \in \mathbb{R}^d \) is a vector of geometrical characteristics of \( U_x \). In the exponent of \( \text{\cite{20}} \) there is the scalar product in \( \mathbb{R}^d \).

The intensity of the reference process depends on a specific model, we consider process of interacting segments in \( \mathbb{R}^2 \) or interacting plates in \( \mathbb{R}^3 \) where we study natural \( U \)-statistics. Consider first \( B \subset \mathbb{R}^2 \),

\[
Y = B \times (0, b] \times [0, \pi),
\]

(27)
where \( b > 0 \) is an upper bound for the segment length. \((\mathbb{N}, \mathcal{N})\) is here the measurable space of integer-valued finite measures on \( Y \) or equivalently of \( n \)-tuples of points of \( Y \) with variable \( n \). The Poisson process \( \eta \) on \( Y \) has intensity measure \( \lambda \),

\[
\lambda(d(z, r, \phi)) = \rho(z)dzQ(dr)V(d\phi),
\]

where \( z \) denotes the location of the segment centre, \( r \) the segment length and \( \phi \) its axial orientation, \( Q, V \) are probability measures, \( V \) nondegenerate, \( \rho \) a bounded intensity function of germs on \( B \).

The segment process \( \mu \) has the density (26) with \( \nu = (\nu_1, \nu_2) \), we assume \( \nu_2 \leq 0 \) to guarantee that \( p \) is a probability density. Further

\[
G(U_x) = (L(U_x), N(U_x)),
\]

where \( L \) is the total length of all segments and \( N \) the total number of intersections between segments. Thus if \( l \) is the length of an individual segment

\[
L(U_\mu) = \sum_{s \in \mu} l(s)
\]

is \( U \)-statistics of the first order and

\[
N(U_\mu) = \frac{1}{2} \sum_{s, t \in \mu_2^*} 1_{[s \cap t \neq \emptyset]}
\]

is \( U \)-statistics of the second order.

Similarly we consider \( B \subset \mathbb{R}^3 \) and a Poisson process \( \eta \) in

\[
Y = B \times (0, b] \times S^2,
\]

where \( S^2 \) is the unit hemisphere in \( \mathbb{R}^3 \), with intensity measure \( \lambda \) on \( Y \)

\[
\lambda(d(z, r, \phi)) = \rho(z)dzQ(dr)V(d\phi)
\]

where \( z \) denotes the location of circular plate centre, \( r \) the radius of the plate and \( \phi \) its normal orientation. The point process \( \mu \) of circular plates has the density (26) w.r.t. \( \eta \) with \( \nu = (\nu_1, \nu_2, \nu_3) \), we assume \( \nu_2 \leq 0, \nu_3 \leq 0 \). Further

\[
G(U_x) = (S(U_x), L(U_x), N(U_x)),
\]

where \( S \) is the total area of plates, \( L \) the total length of intersection lines and \( N \) the total number of intersection points of triplets of plates. Let \( A \) be the area of a single plate, \( l \) the length of a single intersection segment, we define

\[
S(U_\mu) = \sum_{s \in \mu} A(s)
\]

which is \( U \)-statistics of the first order,

\[
L(U_\mu) = \frac{1}{2} \sum_{(s,t) \in \mu_2^*} l(s \cap t)
\]
Proof: It is $P(y \in \mu) = 0$ for $y \in Y$ and
\[
\lambda^*(y, \mu) = \frac{p(\mu \cup \{ y \})}{p(\mu)} = e^{pG(U_{\mu \cup \{ y \}}) - pG(U_{\mu})} = e^{pD_yG(U_{\mu})} = \mathcal{E}_\mu(y) \ a.s.
\]
For $n = 2$, $D_{y_1y_2}^2 G(U_{\mu}) = G(U_{\mu \cup y_1 \cup y_2}) - G(U_{\mu \cup y_1}) - G(U_{\mu \cup y_2}) + G(U_{\mu})$, $G(U_{\mu \cup y_1 \cup y_2}) - G(U_{\mu}) = D_{y_1y_2}^2 G(U_{\mu}) + D_{y_2} G(U_{\mu}) + D_{y_y} G(U_{\mu})$ and
\[
\lambda^*_2(y_1, y_2, \mu) = e^{pG(U_{\mu \cup y_1 \cup y_2}) - pG(U_{\mu})} = \\
eq \exp(\nu(D_{y_1} G(U_{\mu}) + D_{y_2} G(U_{\mu}) + D_{y_1y_2}^2 G(U_{\mu}))) = \mathcal{E}_\mu(y_1, y_2) \ a.s.
\]
Analogously we obtain the result for \( n = 3, \ldots, 6 \), e.g. it holds

\[
G(U_{\mu \cup Y_1 \cup Y_2 \cup Y_3}) - G(U_{\mu}) = \\
= \sum_{i=1}^{3} D_{y_i}G(U_{\mu}) + \sum_{i<j} D_{y_i y_j}G(U_{\mu}) + D_{y_1 y_2 y_3}G(U_{\mu}) = Q_3.
\]

\[\square\]

**Theorem 5.** Let \( \mu \) be the process of circular plates on \( Y \) \((32)\) with density \((26)\), \( \nu_2 \leq 0, \nu_3 \leq 0 \). Then

\[
\mathbb{E}S(U_{\mu}) = \int_Y \mathbb{E}[\mathcal{E}_\mu(y)]A(y)\lambda(dy),
\]

\[
\mathbb{E}L(U_{\mu}) = \frac{1}{2} \int_{Y^2} \mathbb{E}[\mathcal{E}_\mu(y_1, y_2)]l(y_1 \cap y_2)\lambda(d(y_1, y_2)),
\]

\[
\mathbb{E}N(U_{\mu}) = \frac{1}{6} \int_{Y^3} \mathbb{E}[\mathcal{E}_\mu(y_1, y_2, y_3)]1_{[y_1 \cap y_2 \cap y_3 \neq \emptyset]}\lambda(d(y_1, y_2, y_3)),
\]

\[
\mathbb{E}[S(U_{\mu})^2] = \int_Y \mathbb{E}[\mathcal{E}_\mu(y)]A(y)^2\lambda(dy) + \\
+ \int_{Y^2} \mathbb{E}[\mathcal{E}_\mu(y_1, y_2)]A(y_1)A(y_2)\lambda(d(y_1, y_2)),
\]

\[
\mathbb{E}[L(U_{\mu})^2] = \frac{1}{2} \int_{Y^2} \mathbb{E}[\mathcal{E}_\mu(y_1, y_2)]l(y_1 \cap y_2)^2\lambda(d(y_1, y_2)) + \\
+ \int_{Y^3} \mathbb{E}[\mathcal{E}_\mu(y_1, y_2, y_3)]l(y_1 \cap y_2)l(y_3 \cap y_1)\lambda(d(y_1, y_2, y_3)) + \\
+ \frac{1}{4} \int_{Y^4} \mathbb{E}[\mathcal{E}_\mu(y_1, y_2, y_3, y_4)]l(y_1 \cap y_2)l(y_3 \cap y_4)\lambda(d(y_1, \ldots, y_4)),
\]

\[
\mathbb{E}[N(U_{\mu})^2] = \frac{1}{6} \int_{Y^3} \mathbb{E}[\mathcal{E}_\mu(y_1, y_2, y_3)]1_{[y_1 \cap y_2 \cap y_3 \neq \emptyset]}\lambda(d(y_1, y_2, y_3)) + \\
+ \frac{1}{2} \int_{Y^4} \mathbb{E}[\mathcal{E}_\mu(y_1, \ldots, y_4)]1_{[y_1 \cap y_2 \cap y_3 \neq \emptyset]}1_{[y_4 \cap y_2 \cap y_3 \neq \emptyset]}\lambda(d(y_1, \ldots, y_4)) + \\
+ \frac{1}{4} \int_{Y^5} \mathbb{E}[\mathcal{E}_\mu(y_1, \ldots, y_5)]1_{[y_1 \cap y_2 \cap y_3 \neq \emptyset]}1_{[y_4 \cap y_3 \cap y_5 \neq \emptyset]}\lambda(d(y_1, \ldots, y_5)) + \\
+ \frac{1}{36} \int_{Y^6} \mathbb{E}[\mathcal{E}_\mu(y_1, \ldots, y_6)]1_{[y_1 \cap y_2 \cap y_3 \neq \emptyset]}1_{[y_4 \cap y_3 \cap y_6 \neq \emptyset]}\lambda(d(y_1, \ldots, y_6)).
\]
Let us verify the assumptions of Theorems 2 and 3 from which the formulas follow. For \( x \in \mathbb{N} \) with \( n(x) = n \) we have estimates
\[
S(U_x) \leq \pi b^2 n, \quad L(U_x) \leq 2b \left( \frac{n}{2} \right), \quad N(U_x) \leq \left( \frac{n}{3} \right).
\]
Since \( \nu_2 \leq 0, \nu_3 \leq 0 \) we have
\[
p^2(x) \leq \text{const.} \exp(2\nu_1 \pi b^2 n(x)),
\]
and from (19)
\[
\sum_{n=0} \frac{\lambda(Y)^n}{n!} \exp(2\nu_1 \pi b^2 n) < +\infty.
\]
Concerning the powers of \( U \)-statistics \( S(U_x), L(U_x), N(U_x) \) an analogous estimate of (19) is finite.

From Theorem 3 one can also obtain explicit formulas for mixed moments of \( U \)-statistics, e.g.
\[
\mathbb{E}[L(U_\mu)N(U_\mu)] = \frac{1}{2} \int Y \mathbb{E}[\mathcal{E}_\mu(y_1, y_2)] l(y_1 \cap y_2) 1_{[y_1 \cap y_2 \cap y_3 \neq \emptyset]} \lambda(d(y_1, y_2, y_3)) +
\]
\[
+ \frac{1}{2} \int Y \mathbb{E}[\mathcal{E}_\mu(y)] l(y_1 \cap y_2) 1_{[y_1 \cap y_3 \cap y_4 \neq \emptyset]} \lambda(d(y_1, y_2)) +
\]
\[
+ \frac{1}{12} \int Y^3 \mathbb{E}[\mathcal{E}_\mu(y_1, y_2)] l(y_1 \cap y_2) 1_{[y_1 \cap y_4 \cap y_5 \neq \emptyset]} \lambda(d(y_1, y_2, y_3)).
\]

We obtain similar results for the segment process \( \mu \) in \( \mathbb{R}^2 \) with \( U \)-statistics \( G(U_x) \) in (29). Here we have for \( y, y_i \in Y \) \( y, y_i \notin x, x \in \mathbb{N} \)
\[
D_y G(U_x) = \left( \sum_{s \in x} \frac{l(y)}{1_{[s \cap y \neq \emptyset]}} \right), \quad D_{y_1 y_2}^2 G(U_x) = \left( 0_{[y_1 \cap y_2 \neq \emptyset]} \right).
\]
Define analogously \( \mathcal{E}_\mu(y) = \exp(\nu D_y G(U_\mu)), \mathcal{E}_\mu(y_1, y_2) = \exp(\nu Q_m), m = 2, 3, 4, \)
\[
Q_m = \sum_{i=1}^m D_{y_i} G(U_\mu) + \sum_{1 \leq i < j \leq m} D_{y_i y_j}^2 G(U_\mu).
\]

Observe as in Theorem 4 that a.s.
\[
\mathcal{E}_\mu(y_1, \ldots, y_m) = \lambda^*_m(y_1, \ldots, y_m), \quad m = 1, 2, 3, 4.
\]

**Corollary 1.** Let \( \mu \) be the segment process on \( Y \) (27) with density (26), \( \nu_2 \leq 0, \) then for \( U \)-statistics \( (30) \) and (31) we have
\[
\mathbb{E}L(U_\mu) = \int Y \mathbb{E}[\mathcal{E}_\mu(y)] l(y) \lambda(dy),
\]
\[
\mathbb{E}N(U_\mu) = \frac{1}{2} \int Y^2 \mathbb{E}[\mathcal{E}_\mu(y_1, y_2)] 1_{[y_1 \cap y_2 \neq \emptyset]} \lambda(d(y_1, y_2)),
\]
we write \( H \) of a vector of \( U \) functions, \( l \)

In order to study the statistics \( H \) consider one of the three choices: \( E \) means of log \( l \) which is a finite Gibbsian form, cf. (24) with \( \nu \) in (21) having in mind that the process \( \mu \)

Here we deal with 3.1 Geometric functionals in logarithmic form

Here we deal with

\[ H_m(\eta) = m \log F(\eta) + \log p(\eta), \ m = 1, 2, \ldots \]

in (21) having in mind that the process \( \mu \) with density \( p \) w.r.t. \( \eta \) is related by means of \( \log \mathbb{E}F^m(\mu) \geq \mathbb{E}H_m(\eta) \). Now consider the density (20) where

\[ \log p(x) = -\log c_\nu + \nu_1 S(U_x) + \nu_2 L(U_x) + \nu_3 N(U_x) \]

which is a finite Gibbsian form, cf. (24) with \( l = 3 \) non-constant terms. For \( F(x) \) consider one of the three choices: \( F(x) = e^{S(U_x)}, e^{L(U_x)}, e^{N(U_x)} \), accordingly we write \( H_{m1}, H_{m2}, H_{m3} \), respectively:

\[
\begin{align*}
H_{m1}(\eta) &= -\log c_\nu + (m + \nu_1)S(U_\eta) + \nu_2 L(U_\eta) + \nu_3 N(U_\eta) \\
H_{m2}(\eta) &= -\log c_\nu + \nu_1 S(U_\eta) + (m + \nu_2) L(U_\eta) + \nu_3 N(U_\eta) \\
H_{m3}(\eta) &= -\log c_\nu + \nu_1 S(U_\eta) + \nu_2 L(U_\eta) + (m + \nu_3) N(U_\eta)
\end{align*}
\]

In order to study the statistics \( H_m^p \) we need to investigate multivariate behavior of a vector of \( U \)-statistics, e.g. for the process of plates in \( \mathbb{R}^3 \)

\[ (S(U_\eta), L(U_\eta), N(U_\eta)) \]

Generally for \( l \geq 1 \) and \( i = 1, \ldots, l \) let \( k_i \in \mathbb{N}, f^{(i)} \in L_1(\lambda^{k_i}) \) be symmetric functions,

\[ F^{(i)}(\eta) = \sum_{(x_1, \ldots, x_{k_i}) \in \eta^{k_i}} f^{(i)}(x_1, \ldots, x_{k_i}) \]
Consider Poisson processes \( \eta_a \) with intensity measures \( \lambda_a = a \lambda \), \( a > 0 \) and \( g_i : (0, \infty) \to \mathbb{R} \) with \( g_i(a) \neq 0 \) for all \( a > 0 \). Following [4] U-statistics

\[
F_a^{(i)}(\eta_a) = g_i(a) \sum_{(x_1, \ldots, x_k) \in \eta_a^{k_i+1}} f^{(i)}(x_1, \ldots, x_k)
\]

are transformed to

\[
\hat{F}_a^{(i)} = g_i(a)^{-1} a^{-(k_i+1)} (F_a^{(i)} - \mathbb{E} F_a^{(i)}).
\]

The asymptotic covariances are

\[
C_{ij} = \lim_{a \to \infty} \text{cov}(\hat{F}_a^{(i)}, \hat{F}_a^{(j)}) = \int T_1 F_a^{(i)}(x) T_1 F_a^{(j)}(x) \lambda(dx), \quad i, j \in \{1, \ldots, l\}.
\]

Define \( k = k_1 + \cdots + k_l \) and

\[
J_i = \{ j : k_1 + \cdots + k_{i-1} < j \leq k_1 + \cdots + k_i \}, \quad i = 1, \ldots, l.
\]

Let \( \pi = \{ J_i, \ 1 \leq i \leq l \} \) and let \( \Pi_{k_1, \ldots, k_l} \subset \Pi_k \) (\( \Pi_k \) is the system of all partitions of \( \{1, \ldots, k\} \)) be the set of all \( \sigma \in \Pi_k \) such that \( |J \cap J'| \leq 1 \) for all \( J, J' \in \pi \) and all \( J' \in \sigma \). For a partition \( \sigma \in \Pi_{k_1, \ldots, k_l} \) we define the function \( (\otimes^l_{j=1} f_j) : Y^{[\sigma]} \to \mathbb{R} \) by replacing all variables of the tensor product \( \otimes^l_{j=1} f_j \) that belong to the same block of \( \sigma \) by a new common variable, \( |\sigma| \) is the number of blocks in \( \sigma \). For an \( l \)-dimensional centered Gaussian random vector \( X \) with covariance matrix \( C = (C_{ij})_{i,j=1,\ldots,l} \), assuming that

\[
\int |(\otimes^l_{j=1} f_j)_{[\sigma]}| d\lambda^{[\sigma]} < \infty, \quad \sigma \in \Pi_{k_1, \ldots, k_l},
\]

[4] show that \( (\hat{F}_a^{(1)}, \ldots, \hat{F}_a^{(l)}) \) converges in distribution to \( X \) when \( a \to \infty \).

Moreover, the convergence under the distance between \( l \)-dimensional random vectors \( X, Y \)

\[
d_3(X, Y) = \sup_{g \in \mathcal{H}} |\mathbb{E} g(X) - \mathbb{E} g(Y)|,
\]

where \( \mathcal{H} \) is the system of functions \( h \in C^3(\mathbb{R}^l) \) with

\[
\max_{1 \leq i_1 \leq l} \sup_{x \in \mathbb{R}^l} \left| \frac{\partial^2 h(x)}{\partial x_{i_1} \partial x_{i_2}} \right| \leq 1, \quad \max_{1 \leq i_1 \leq l} \sup_{x \in \mathbb{R}^l} \left| \frac{\partial^3 h(x)}{\partial x_{i_1} \partial x_{i_2} \partial x_{i_3}} \right| \leq 1
\]

implies convergence in distribution. Based on the multi-dimensional Malliavin-Stein inequality derived in [8] for the distance \( d_3 \) of a random vector from centered Gaussian random vector with covariance matrix \( C \), [4] show that under the assumption

\[
\int |T_1 F_a^{(i)}|^3 d\lambda < \infty, \quad i = 1, \ldots, l,
\]

there exists a constant \( c \) such that

\[
d_3((\hat{F}_a^{(1)}, \ldots, \hat{F}_a^{(l)}), X) \leq ca^{-\frac{1}{2}}, \quad a \geq 1.
\]
Example 3. Consider the Poisson segment process on $Y$ with intensity measure $\lambda$ and the $U$-statistics $L(U_{\eta})$ and $N(U_{\eta})$ of order $k_1 = 1, k_2 = 2$, respectively. We have $g_1(a) \equiv g_2(a) \equiv 1$ and in (34)

$$C_{11} = \int_Y l(s)^2 \lambda(ds), \quad C_{22} = \int_Y \lambda(\{s : s \cap t \neq \emptyset\})^2 \lambda(dt),$$

$$C_{12} = 2 \int_Y l(y)\lambda(\{s : s \cap y \neq \emptyset\})\lambda(dy).$$

The assumption transforms to conditions:

$$\int_Y l(s)1_{[s \cap t \neq \emptyset]}\lambda(d(s,t,u)) = \lambda^2(\{(t,u) \in Y^2; t \cap u \neq \emptyset\}) \int_Y l(s)d\lambda(s) < \infty,$$

$$\int_Y l(s)1_{[s \cap t \neq \emptyset]}\lambda(d(s,t)) < \infty$$

since the partitions of interest are

$$\{(\{1\}, \{2\}, \{3\}), (\{1\}, \{2\}, \{3\}), (\{1\}, \{3\}, \{2\})$$

and the last two of them are equivalent. The assumption transforms to conditions:

$$\int_Y l(s)^3\lambda(ds) < \infty, \quad \int_Y \lambda(\{s : s \cap y \neq \emptyset\})^3\lambda(dy) < \infty.$$
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