E-INFINITY STRUCTURES OVER $\mathcal{L}$-ALGEBRAS

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ABSTRACT. In this paper we introduce the concept of $\mathcal{L}$-algebras, which can be seen as a generalization of the structure determined by the Eilenberg-Mac lane transformation and Alexander-Whitney diagonal in chain complexes. In this sense, our main result states that $\mathcal{L}$-algebras are endowed with an $E_\infty$-coalgebra structure, like the one determined by the Barrat-Eccles operad in chain complexes. This results implies that the canonical $\mathcal{L}$-algebra of spaces contains as much homotopy information as its usually associated $E_\infty$-coalgebras, suggesting $\mathcal{L}$-algebras as a tool for the study of homotopy types.

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1. INTRODUCTION

The central notion of this paper is the algebraic structure called $\mathcal{L}$-algebra. Introduced by Alain Prouté in several talks since the eighties and never published (Max Planck Institut-Bonn 1986, Louvain-la Neuve 1987, Freie Universität-Berlin 1988, Seminar Keller-Maltsiniotis-Paris 2010), $\mathcal{L}$-algebras have been thought to be highly related to the homotopy type of spaces by using an internal structure that models the diagonals which determines invariants like Steenrod operations. $\mathcal{L}$-algebras are also inspired in Segal’s $\Gamma$ structures (see [13]), and can be thinking as a kind of co-version of $\Gamma$ spaces.

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Eilenberg-Mac Lane transformation plays a central role in \( \mathcal{L} \)-algebras, where it is the prototype (motivation) for the product of \( \mathcal{L} \)-algebras. It is interesting to notice the existence of a preprint of Tom Leinster (see [2]), which present a similar object.

Using a homotopy inverse of the structural quasi-isomorphism \( \mu \) of an \( \mathcal{L} \)-algebra \( A \) we can define a coproduct on its main element \( A[1] \). Indeed, we only have to take the composition of an homotopy inverse of \( \mu : A[1] \otimes A[1] \to A[2] \) with the morphism \( s_1 : A[1] \to A[2] \). Observe that this coproduct in general is not associative. But, the structure of \( \mathcal{L} \)-algebra makes this coproduct associative and commutative up to homotopy. Moreover, the homotopies also satisfy to be associative and commutative up to homotopy, and this property is maintained on the next levels of homotopies, generating a system of higher homotopies. The classical case where this happens is in the context of chain complexes associated to a simplicial set.

Higher homotopies can be organized into an \( E_\infty \)-coalgebra. Such a structure was exhibited in particular by Smith (see [14]) and alternatively by Sánchez-Guevara using a simplification of Smith’s operad (see [12]). We will generalize these descriptions in the context of \( \mathcal{L} \)-algebras with values in the category DGA-Mod, in other words, we will prove that the main element of an \( \mathcal{L} \)-algebra \( A \) is equipped with an \( E_8 \)-coalgebra structure describing the system of higher homotopies associated to the coproducts induced by the structural quasi-isomorphism of \( A \).

The main difference with the case of chain complexes associated to a simplicial set, where the process begins with the Alexander-Whitney diagonal, which is an associative coproduct, is that in general we don’t have the associativity. Then, in order to model the higher homotopies we have to consider an \( E_\infty \)-operad that will have several generators in degree 0, and not only one like the operads \( \mathcal{S} \) and \( \mathcal{R} \). In the section ??, we will construct an \( E_\infty \)-operad that we denote \( \mathcal{K} \). The construction is made by infinitely many steps, in the sense that we construct a sequence of operads \( \{ \mathcal{K}_n \}_{n \geq 2} \), in such a way that \( \mathcal{K}_i \) is a suboperad of \( \mathcal{K}_{i+1} \). The operads \( \mathcal{K}_i \) are not \( E_\infty \)-operads, but they will be almost \( E_\infty \)-operads, in the sense that until arity \( i \) they will satisfy the \( E_\infty \)-conditions. Finally, the \( E_\infty \)-operad \( \mathcal{K} \) is obtained by taking the inductive limit of this sequence of operads.

One of the characteristics of this construction is the use of a technique that we call polynomial operads. It will create a new operad from an \( \mathcal{S} \)-module containing an \( \mathcal{S} \)-submodule with an operadic structure, in such a way that this operadic structure is preserved in the resulting operad. This done by using amalgamated sums in the category of operads. The section 3.2 is completely dedicated to the description of this technique.

In the final part we exhibit the main element \( A[1] \) of an \( \mathcal{L} \)-algebra as an \( E_\infty \)-coalgebra. Again, this will be possible due to the sequence of operads that define \( \mathcal{K}_i \), in the sense that it will be sufficient to exhibit \( A[1] \) as a \( \mathcal{K}_i \)-coalgebra for each \( i \), because the universal property of colimits will induce the \( \mathcal{K} \)-coalgebra structure on \( A[1] \). Moreover, our construction that \( A[1] \) is a \( E_\infty \)-coalgebra is functorial. This proves that an \( \mathcal{L} \)-algebra quasi-isomorphic to \( A(X) \) contains at least as many homotopy information as a \( E_\infty \)-coalgebra structure on \( C_*(X) \), such as the one described by Smith.

In [4], Mandell describes an \( E_\infty \)-algebra structure on the normalized cochain complex associated to a simplicial space, which under some finiteness hypothesis gives an invariant for the weak homotopy type of the space. Our results suggest that \( \mathcal{L} \)-algebras are also pertinent in order to describe the weak homotopy type of spaces.
2. Preliminaries

Let \( \Lambda \) be a commutative ring with unity, a differential graded module over \( \Lambda \) or simply DG module is a graded \( \Lambda \)-module \( M \) together with a homogeneous morphism \( \partial : M \to M \) of degree \(-1\), called differential, such that \( \partial^2 = 0 \). When provided with homogeneous morphisms \( \epsilon : M \to k \) and \( \eta : k \to M \), called augmentation and coaugmentation, respectively, such that \( \epsilon \circ \eta = \text{id} \), \( M \) is said to be a differential graded module with augmentation or DGA module. The category of DG modules and DGA modules, both with homogeneous morphisms, is denoted DG-Mod and DGA-Mod, respectively. Signs in this graded context follows Koszul convention (cf. [3]).

\( M \) is acyclic if its augmentation \( \epsilon : M \to \Lambda \) induces an isomorphism in homology, and \( M \) is said to be contractible if \( \epsilon \) is a homotopy equivalence (cf. [10]). A contracting chain homotopy of \( M \), is a degree 1 morphism \( h : M \to M \) homotopy from 0 to \( \text{Id}_M \). In other words, \( h \) satisfies, \( \partial h + h \partial = 1 \). If \( H_*(M) = 0 \), \( M \) is said to be null-homotopic. A morphism \( f : M \to N \) of degree \( k \) is said null-homotopic if it is homotopic to 0, that is, there is a morphism \( h : M \to N \) of degree \( k+1 \) such that \( \partial h = \partial - (-1)^k h \partial - f \).

When \( M \) has a contracting chain homotopy then \( M \) is null-homotopic, and as a converse, if \( M \) projective, null-homotopic and bounded below, then it has a contracting chain homotopy, which is constructed inductively. A similar proof lead to the following result.

**Lemma 2.1.** Let \( f : L \to N \) be a homogeneous morphism of DG modules of degree \( k \in \mathbb{Z} \). Let \( L' \) be a DG submodule of \( L \) and \( P \) graded submodule of \( L \) such that \( P \) is projective and bounded below, and in each degree we have the decomposition \( L_i = L'_i \oplus P_i \). Write \( f' \) for the restriction of \( f \) to \( L' \). Suppose that the homology of \( N \) is zero and that \( f' \) is null-homotopic by a homotopy \( h : L' \to N \). Then there exists a homotopy \( H \) on \( L \) extending \( h \), which makes \( f \) null-homotopic.

**Theorem 2.2** (Relative Lifting Theorem). Let \( f : M \to N \) and \( \varphi : L \to N \) morphisms of DG modules of degree \( l \) and \( k \), respectively. Suppose that \( f \) is a quasi-isomorphism, and let \( L' \) be a DG submodule of \( L \) and \( P \) graded submodule of \( L \) such that \( P \) is projective and bounded below, and in each degree we have the decomposition \( L_i = L'_i \oplus P_i \). Write \( \varphi' \) for the restriction of \( \varphi \) to \( L' \). Suppose there is a morphism \( \alpha' : L' \to M \) of DG modules of degree \( k-l \) that lifts \( \varphi' \) up to homotopy along \( f \). Then there exists an extension \( \alpha : L \to M \) of \( \alpha' \), that lifts \( \varphi \) up to homotopy along \( f \). Moreover, the homotopy can be choose to be an extension of the homotopy associated to \( \alpha' \).

\[
\begin{array}{ccc}
M & \xrightarrow{f} & N \\
\downarrow \alpha & & \downarrow \varphi \\
L & \xrightarrow{\varphi'} & L'
\end{array}
\]

(2.1)

**Proof.** Let \( C(f) \) be the mapping cone of \( f \). Let \( u : N \to C(f) \) the inclusion \( x \mapsto \begin{pmatrix} 0 \\ x \end{pmatrix} \) and \( h' : L' \to N \) the homotopy from \( \varphi' \) to \( f \circ \alpha' \). Then we can easily check that
\( (\alpha', h') : L' \to C(f) \) is a homotopy to 0 of \( u \circ \varphi' = \begin{pmatrix} 0 \\ \varphi' \end{pmatrix} \). Lemma 2.1 says there exists a homotopy to zero \( (H_1, H_2) : L \to C(f) \) of \( u \circ \varphi \) extending \( \begin{pmatrix} \alpha' \\ h' \end{pmatrix} \). So we have,

\[
\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \partial_{C(f)} \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} + (-1)^k \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} \partial_L
\]

\[
= \begin{pmatrix} -(-1)^l \partial_M H_1 + (-1)^k H_1 \partial_L \\ fH_1 + \partial_N H_2 + (-1)^k H_2 \partial_L \end{pmatrix}
\]

(2.2)

This gives the following equations.

\[
\partial_M H_1 = (-1)^{l+k} H_1 \partial_L
\]

\[
\varphi - fH_1 = \partial_N H_2 + (-1)^k H_2 \partial_L
\]

The first says that \( H_1 \) is a morphism of DG-modules and the second that \( H_2 \) is a homotopy from \( fH_1 \) to \( \varphi \). Finally, we take \( \alpha = H_1 \) as the lift of \( \varphi \) along \( f \). \( \square \)

Let \( k \) be a field. For \( n \) positive integer, \( \Sigma_n \) denote the symmetric group of \( n \) elements and \( k[\Sigma_n] \) its group ring over \( k \) (cf. [6], [15]).

**Proposition 2.3** (Acyclic extension). Let \( M \) be a \( k[\Sigma_n] \)-free finitely generated DGA module over \( k \). Then there exists a \( k[\Sigma_n] \)-free finitely generated acyclic DGA module \( N \) over \( k \), with \( M_0 = N_0 \) and \( M \) as DGA submodule.

**Proof.** For the modules on \( k[\Sigma_n] \), we consider the adjunction \( L \dashv U : \text{Set} \to \text{Mod}_k[\Sigma_n] \), where \( U \) is the forgetful functor. For every module \( N \), the counit gives the surjection \( \epsilon_N : LU(N) \to N \), which will be denoted \( p : PN \to N \). Given a DGA module \( M \) we denote \( ZM \) its submodule of cycles. On \( ZM \) the differential is 0, then we extend the meaning of \( P \) to graded modules, we keep the same notation for the extended morphism \( p : PZM \to ZM \). Consider the composition \( d = i \circ p : PZM \to M \), where \( i \) is the canonical inclusion of \( ZM \) in \( M \). \( ZM \) can seen as a submodule of \( PZM \), then \( p : PZM \to ZM \) is a retraction for this inclusion and if \( m \in ZM \), then \( d(m) = m \).

Observe that in the mapping cone of \( d : i \circ p : PZM \to M, C(d) \), all the cycles of \( M \) are now boundaries and also on \( C(d) \) will appear new cycles. Indeed, let \( m \in M \) cycle, recall that the differential of \( C(d) \) is given by \( \begin{pmatrix} 0 & 0 \\ d & \partial_M \end{pmatrix} \). Then in \( C(d) \) we have \( \partial \begin{pmatrix} m \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ m \end{pmatrix} \) which means that \( m \) is a boundary. If it happens that \( m \) is already a boundary in \( M \), that is there exists \( n \) such that \( \partial_M(n) = m \), then \( \begin{pmatrix} m \\ -n \end{pmatrix} \) is a cycle in \( C(d) \). From this also notice that if all the cycles of \( M \) have degree at least \( k \), then all the cycles in \( C(d) \) will have at least degree \( k + 1 \).

Let \( M \) \( k[\Sigma_n] \)-free finitely generated DGA module, and denote \( W \) the kernel of the augmentation \( \epsilon : M \to k \) and consider the \( k[\Sigma_n] \)-linear morphisms for \( n \geq 1 \),

\[
PZC(d_n) \xrightarrow{d_{n+1}} C(d_n)
\]

where \( d_1 \) is \( d : PZW \to W \), and \( d_{n+1} \) is \( d : PZC(d_n) \to C(d_n) \). Then we have that \( W \) is included in \( C(d_1) \) and \( C(d_n) \) is included in \( C(d_{n+1}) \). With this we can define a
DGA-module $N$ that satisfies the conditions of the theorem by taking the colimit of the following diagram,

$$\begin{array}{c}
M & \xleftarrow{W} & C(d_1) & \rightarrow & C(d_2) & \rightarrow \cdots
\end{array}$$

where all the arrows are the respective canonical inclusions. Observe that we can reduce the size of this acyclic extension by considering in the first step only the cycles of degree 0 of $W$, and for the construction of $d_{n+1}$, considering only the cycles of degree $n$ of the last mapping cone. \(\square\)

3. Operads

In the following we consider differential graded modules over a field $k$. An operad $P$ in the monoidal category $\text{DGA-Mod}$ is a collection of DGA-modules $\{P(n)\}_{n \geq 1}$ together with right actions of the symmetric group $\Sigma_n$ on each component $P(n)$, and morphisms of the form $\gamma : P(r) \otimes P(i_1) \otimes \cdots \otimes P(i_r) \rightarrow P(i_1 + \cdots + i_r)$, which satisfy the usual conditions of existence of an unit, associativity and equivariance. Morphisms $\gamma$ will be called composition morphisms of the operad. A morphism between operads $f : P \rightarrow Q$ is a collection of DGA-morphisms $f_n : P(n) \rightarrow Q(n)$ of degree 0, respecting units, composition and equivariance. The category of operads is denoted $\mathcal{OP}$ (cf. [7], [5], [3]).

Two fundamental examples of operads are the endomorphism operad and the co-endomorphism operad, because their behavior inspired the definition of operad. For $M \in \text{DGA-Mod}$, the operad $\text{End}(M)$ of endomorphisms of $M$ is defined by $\text{End}(M)(n) = \text{Hom}(M^\otimes n, M)$, with unit $\eta : k \rightarrow \text{End}(M)(1)$ given by $\eta(1) = \text{Id}_M$, right action of $\Sigma_n$ over $\text{End}(M)$ induced by the left action of $\Sigma_n$ over $M^\otimes n$, and obvious composition applications. For the coendomorphism operad $\text{Coend}(M)$ we take $\text{Coend}(N)(n) = \text{Hom}(N, N^\otimes n)$ with unit $\eta(1) = \text{Id}_N$, right action of $\Sigma_n$ over $\text{Coend}(N)$ is the induced by the right action of $\Sigma_n$ over $N^\otimes n$ and obvious compositions.

An important feature in the theory of operads are its representations. That is, when the abstract operations of the operads are interpreted as concrete application over an object in the ground category. This passage from the abstract to the concrete is made through morphisms of type $P(n) \rightarrow \text{Hom}(A^\otimes n, A)$. In this sens, an element of $\mathcal{P}$ is realized as an $n$-ary operation over $A$. This association must be coherent with composition operations and symmetric groups actions.

An algebra over the operad $\mathcal{P}$, or $\mathcal{P}$-algebra, is a DGA module $A$, together with a morphism of operads from $\mathcal{P}$ to $\text{End}(A)$. Similarly, a coalgebra is a DGA module $C$, together with a morphism of operad from $\mathcal{P}$ to $\text{Coend}(C)$.

In the symmetric monoidal category $\text{DGA-Mod}$, for every $Y$ DGA module the functor $- \otimes Y$ is left adjunct of the functor $\text{Hom}(Y, -)$. Denote $\theta$ the natural bijection $\theta_{X,Z} : \text{Hom}(X \otimes Y, Z) \rightarrow \text{Hom}(X, \text{Hom}(Y, Z))$ given by this adjunction. Then, for a morphism of operads $f : \mathcal{P} \rightarrow \text{End}(A)$, each component $f_n : P(n) \rightarrow \text{Hom}(A^\otimes n, A)$ determines a morphism of DGA modules $\varphi_n : P(n) \otimes A \rightarrow A^\otimes n$, given by $\varphi_n = \theta^{-1}(f_n)$. This allows defining $P$-algebras and $P$-coalgebras equivalently by a collection $\{\varphi_n\}_{n \geq 1}$ of morphisms of DGA modules $\varphi_n : P(n) \otimes C^\otimes n \rightarrow C$, and $\varphi_n : P(n) \otimes C \rightarrow C^\otimes n$, respectively, which satisfying the usual conditions of associativity, unit and equivariance.
If we forget composition morphisms of an operad \( P \), the collection of DGA modules with right actions that remains is called an \( S \)-module. They form a category denoted \( S \)-Mod, which has all colimits and limits because it is a category of diagrams over DGA-Mod.

The forgetful functor \( U : \mathcal{OP} \to S \)-Mod has a right adjoint denoted \( F : S \)-Mod \( \to \mathcal{OP} \), called the free operad functor (cf. [11],[5]). This adjunction defines for every operad \( Q \) and \( S \)-module \( M \), the natural bijection,

\[
\theta : \mathcal{OP}(F(M), Q) \to S \text{-Mod}(M, U(Q))
\]

With unit and counit denoted by \( \eta \) and \( \epsilon \), respectively. Recall that \( \eta_M : M \to UF(M) \) and \( \epsilon_P : FU(P) \to P \).

The category of operads has all small colimits (cf. [1]), which will be used to construct the \( E_8 \)-operad \( K \) in section 5.

3.1. \( E_8 \)-Operads.

**Definition 3.1** (\( E_8 \)-Operad). An operad \( P \) on the category DGA-Mod is called an \( E_8 \)-operad if each component \( P(n) \) is a \( k[\Sigma_n] \)-free resolution of \( k \).

**Definition 3.2** (\( E_8 \)-algebra and \( E_8 \)-coalgebra). We call a \( E_8 \)-algebra any \( P \)-algebra with \( P \) an \( E_8 \)-operad. And in the same way, an \( E_8 \)-coalgebra is an \( P \)-coalgebra where the operad \( P \) is an \( E_8 \)-operad.

We introduce a notion of morphism between \( E_8 \)-coalgebras which is well suited for our purpose.

**Definition 3.3.** Let \( P \) be an \( E_8 \)-operad on the category DGA-Mod, and let \( A, B \) \( P \)-coalgebras. A morphism \( f : A \to B \) of \( P \)-coalgebras is a morphism of DGA-Mod which preserves the \( P \)-coalgebra structure up to homotopy, that is, the following diagram

\[
\begin{array}{ccc}
P(n) \otimes A & \xrightarrow{\varphi^A_n} & A^\otimes n \\
\downarrow 1 \otimes f & & \downarrow f^\otimes n \\
P(n) \otimes B & \xrightarrow{\varphi^B_n} & B^\otimes n
\end{array}
\]

is commutative up to homotopy for every \( n > 0 \), where \( \varphi^A_n \) and \( \varphi^B_n \) are the associated morphisms of the \( P \)-coalgebra structure of \( A \) and \( B \), respectively. The category of coalgebras on the operad \( P \) is denoted \( P \)-CoAlg.

3.2. Polynomial Operads. The polynomial operads construction is a technique used to create an operad from an \( S \)-module with an \( S \)-submodule having an operadic structure, in such a way that this operadic structure is preserved. Recall that we denote by \( U \) the forgetful functor from operads to \( S \)-modules.

**Definition 3.4.** \( \mathcal{C} \) is the category such that,

1. The objects are pairs of the form \( (\mathcal{E}, M) \), where \( M \) is a \( S \)-module and \( \mathcal{E} \) is an operad such that \( U(\mathcal{E}) \) is a \( S \)-submodule of \( M \). The canonical inclusion is denoted by \( i_\mathcal{E} : U(\mathcal{E}) \to M \).
(2) A morphism from \((E,M)\) to \((F,N)\), is a pair \((f,\overline{f})\) with \(f : E \to F\) morphism of operads, and \(\overline{f} : M \to N\) morphism of \(S\)-modules, such that the following diagram commutes.

\[
\begin{array}{ccc}
M & \xrightarrow{\overline{f}} & N \\
i_E & & \uparrow i_F \\
U(E) & \xrightarrow{U(f)} & U(F)
\end{array}
\]

Essentially, a morphism from \((E,M)\) to \((F,N)\) in \(\mathcal{C}\) is morphism of \(S\)-modules from \(M\) to \(N\) that sends \(U(E)\) to \(U(F)\) and respects the operadic structure of \(E\).

**Definition 3.5.** We define \(\mathcal{U} : \mathcal{OP} \to \mathcal{C}\) to be the functor forgetful which sends every operad \(E\) to the pair \((E, U(E))\). That is, every operad is sent to the pair formed by itself and its underlying \(S\)-module.

**Theorem 3.6.** The functor \(\mathcal{U} : \mathcal{OP} \to \mathcal{C}\) has a left adjoint. We denote this adjoint by \(\mathcal{J}\), and the image of \((E,M)\) under \(\mathcal{J}\) by \(E[r]_M\) and called the polynomial operad on \(M\) with coefficients in \(E\).

**Proof.** We can associate to every \((E,M) \in \mathcal{C}\) the following diagram in \(\mathcal{OP}\),

\[
\begin{array}{ccc}
FU(E) & \xrightarrow{\epsilon} & \mathcal{E} \\
F(i_E) & & \downarrow \alpha \\
F(M) & \xrightarrow{\beta} & \mathcal{E}[M]
\end{array}
\]

where \(\epsilon : FU \to 1_{\mathcal{OP}}\) is the counit of the adjunction \(F \dashv U : \mathcal{S} \to \mathcal{OP}\). This association is functorial by the naturality of the counit \(\epsilon\) and the definition of \((f,\overline{f})\) as a morphism in \(\mathcal{C}\). Thus we have a functor \(Cm\) from \(\mathcal{C}\) to the category of diagrams in \(\mathcal{OP}\) of the form \(\cdots \leftarrow \bullet \rightarrow \bullet \rightarrow \cdots\). Then, we define the functor \(\mathcal{J} : \mathcal{C} \to \mathcal{OP}\) to be the composition of \(Cm\) with the functor of colimits on \(\mathcal{OP}\).

In order to prove that we have the adjunction \(\mathcal{J} \dashv \mathcal{U} : \mathcal{C} \to \mathcal{OP}\), we only have to construct for every object \((E,M) \in \mathcal{C}\) an universal arrow \(\Psi\) from \((E,M)\) to \(\mathcal{U}\mathcal{J}(E,M) = (\mathcal{E}[M], U(\mathcal{E}[M]))\).

Let \((E,M)\) be an object in \(\mathcal{C}\) and consider the following diagram given by the colimit \(\mathcal{J}(E,M)\).

\[
\begin{array}{ccc}
FU(E) & \xrightarrow{\epsilon} & \mathcal{E} \\
F(i_E) & & \downarrow \alpha \\
F(M) & \xrightarrow{\beta} & \mathcal{E}[M]
\end{array}
\]

Now consider the couple of arrows \((\alpha, \theta(\beta))\), where \(\theta\) is the isomorphism,

\[
\mathcal{OP}(F(M), P) \xrightarrow{\theta} \mathcal{S}(M, U(P))
\]
Proposition 3.9. For every pair of θ. This couple will be our universal arrow Ψ. Checking
that Ψ is a morphism in C and that it satisfies the universal property, is routine. □

The universal arrow in the proof of theorem 3.6 extends to the unit of the adjunction
\( J \dashv \Omega : C \to \mathcal{OP} \). We keep the notation Ψ for this unit. The universal property for the
unit Ψ : 1_\varepsilon \to \Omega J\), gives the following result.

**Proposition 3.7.** Let \((E, M) \in C\) and \(A \in \mathcal{OP}\). For every morphism \((f, \bar{f}) : (E, M) \to \Omega(A) = (A, U(A))\), there exists an unique morphism of operads \(\varphi : E[M] \to A\), such that
\(U(\varphi)\Psi = \bar{f}\). So we have the following commutative diagram.

\[
\begin{array}{ccc}
(E, M) & \xrightarrow{\Psi} & (\Omega[\varphi]) \\
(f, \bar{f}) \downarrow & & \downarrow \varphi, U(\varphi) \\
(A, U(A)) & & \\
\end{array}
\]

(3.7)

In the following we proof some properties of polynomial operads construction which
will be used in defining operad K in section 5.

**Definition 3.8.** Let \(\theta_n : S\text{-Mod} \to S\text{-Mod}\) be the functor which sends each S-module \(M\) to the S-module \(\theta_n(M)\) such that \(\theta_n(M)(k) = M(k)\) if \(k \leq n\) and \(\theta_n(M)(k) = 0\) if \(k > n\). The S-modules in the image of the functor \(\theta_n\) are called \(n\)-S-modules.

**Proposition 3.9.** For every pair of S-modules \(M, N\) we have

\[
\theta_n(M \circ N) = \theta_n(\theta_n(M) \circ \theta_n(N))
\]

(3.8)

**Proof.** Observe that in a composition the arities of the operations composed have lower
arities than the resulting operations, because we only consider operations with arities \(\geq 1\). □

**Definition 3.10.** Let \(\mathcal{OP}_n\) be the subcategory of the category of operads which are null
for arities \(> n\). We call the objects of \(\mathcal{OP}_n\), \(n\)-operads. \(T_n : \mathcal{OP} \to \mathcal{OP}_n\) is the functor
which assigns to each operad the quotient \(n\)-operad determined by the operad ideal of
operations of arities \(> n\). In other words, the resulting operad has null compositions
when the resulting operation has arity \(> n\). \(I : \mathcal{OP}_n \to \mathcal{OP}\) is the inclusion functor and
satisfies \(T_n \circ I = 1_{\mathcal{OP}_n}\), for every \(n > 1\).

Next proposition says that in order to construct the part of \(F(M)\) with arity at most
\(n\), is enough to consider operations of \(M\) with arity at most \(n\).

**Proposition 3.11.** For \(n > 1\), \(T_n \circ F = T_n \circ F \circ \theta_n\)

**Proof.** By induction over the construction of \(F\). \(\theta_n F_{k+1}(M) = \theta_n(I \oplus (M \circ F_k(M)))\), which
is equal to \(\theta_n(I \oplus \theta_n(M \circ F_k(M)))\), by proposition 3.9 this is \(\theta_n(I \oplus (\theta_n(M) \circ \theta_n(F_k(M))))\)
and by induction hypothesis we obtain \(\theta_n F_{k+1}(M) = \theta_n(I \oplus (\theta_n(M) \circ F_k(\theta_n(M))))\), and
then \(\theta_n\) commute with the colimit defining \(F\) because it is filtering. Finally \(\theta_n\) becomes \(T_n\) when the operad structure is added. □

**Proposition 3.12.** For every \(n > 1\) we have the adjunction \(I \dashv T_n : \mathcal{OP} \to \mathcal{OP}_n\).
Proof. We only have to check for every $n$-$\mathcal{S}$-module $Q$ that the identity $1_Q : T_n I(Q) \to Q$ in $\mathcal{OP}_n$ is a universal arrow from the functor $T_n$ to the object $Q$. Observe that every morphism of the form $g : T_n(P) \to Q$ determines a morphism $\varphi : P \to I(Q)$ by taking $\varphi(k) = 0$ if $k > n$ and equal to $g$ in lower degrees.

\[ T_n I(Q) \xrightarrow{1} Q \xleftarrow{g} T_n(P) \]

(3.9)

\[ T_n FU(\mathcal{E}) \xrightarrow{\epsilon} \mathcal{E} \]
\[ F(i) \downarrow \quad \alpha \downarrow \]
\[ F(M) \xrightarrow{\beta} \mathcal{E}[M] \]

(3.10)

Definition 3.13. An $n$-isomorphism $f : P \to Q$ of operads is a morphism of operads such that $T_n(f)$ is an isomorphism of $n$-operads.

Lemma 3.14. In the following diagram from the definition of polynomial operads,

\[ T_n FU(\mathcal{E}) \xrightarrow{T_n(\epsilon)} T_n \mathcal{E} \]
\[ T_n F(i) \downarrow \quad T_n(\alpha) \downarrow \]
\[ T_n F(M) \xrightarrow{T_n(\beta)} T_n \mathcal{E}[M] \]

(3.11)

if $i$ is an $n$-isomorphism (of $\mathcal{S}$-modules) then the operad morphism $\alpha : \mathcal{E} \to \mathcal{E}[M]$ is an $n$-isomorphism.

Proof. By $T_n F(i) = (T_n F \circ \theta_n)(i)$, but $\theta_n(i)$ is an isomorphism, so that $T_n F(i)$ is also an isomorphism, which means that $F(i)$ is an $n$-isomorphism. By 3.12 the following diagram is cocartesian,

so that $T_n(\alpha)$ is an isomorphism, which means that $\alpha$ is an $n$-isomorphism. \qed

4. $\mathcal{L}$-Algebras

Associated to chain complexes of DGA modules, there are applications like the Eilenberg-Maclane transformation and the Alexander-Whitney diagonal, both gives important information about the homotopy type of spaces (cf. [7]). Under certain conditions the Alexander-Whitney diagonal can be described by unicity (cf. [8], [9]). Following these ideas, with $\mathcal{L}$-algebras Alain Prouté associated to each space a structure with abstract properties inspired in such of chain complexes that allow unicity of Alexander-Whitney diagonal.
4.1. The Category $\mathcal{L}$. With $\mathcal{L}$-algebras our principal interest is to model the relations describing the behavior of diagonals in chain complexes. This can be done by using an approach similar to simplicial objects, that is, defining $\mathcal{L}$-algebras as contravariant functors from a suitable category. This category will be denoted $\mathcal{L}$.

The objects of the category used by G. Segal, $\Gamma$ (see [13]) are the finite sets, and a morphism from $x$ to $y$ is an application $f : x \to \mathcal{P}(y)$ such that $z_1 \neq z_2$ implies $f(z_1) \cap f(z_2) = \emptyset$. Then we have the isomorphisms of categories $\Gamma^{\text{op}} \cong \mathcal{L}$ and $\mathcal{L}^{\text{op}} \cong \Gamma$.

**Definition 4.1.** We define $\mathcal{L}$ to be the category with objects totally ordered sets $[n] = \{1, \ldots, n\}$ for $n > 0$ and $[0] = \emptyset$, the empty set. Morphisms in $\mathcal{L}$ are taken to be all the partial maps between these sets.

An morphism $\alpha : [n] \to [m]$ of $\mathcal{L}$ could be described by a pair $(\text{Dom}(f), f)$, where $\text{Dom}(f) \subseteq [n]$ is the domain of $f$.

When a cocartesian category has a zero object is called pointed cocartesian category. Furthermore, if the zero and the sum are explicitly given, the category is called strict pointed cocartesian category. $\mathcal{L}$ is a strict pointed cocartesian category, with the strictly associative sum $[n] + [m] := [n + m]$, and with zero object $[0]$.

**Definition 4.2.** In $\mathcal{L}$ we identify the following arrows.

1. The face operator $d_i : [n] \to [n + 1]$ is defined for $1 \leq i \leq n + 1$ by:

   $$d_i(x) = \begin{cases} x & \text{if } x < i \\ x + 1 & \text{if } x \geq i \end{cases}$$

   When $n = 0$, the only face operator $d_1 : [0] \to [1]$ is the universal morphism from $[0]$.

2. The degeneracy operator $s_i : [n] \to [n - 1]$ is defined for $1 \leq i \leq n$ by:

   $$s_i(x) = \begin{cases} x & \text{if } x \leq i \\ x - 1 & \text{if } x > i \end{cases}$$

   In the case $n = 1$, the only degeneracy operator $s_1 : [1] \to [0]$ is the universal morphism to $[0]$.

3. In $\mathcal{L}$, any injective map $i : [n] \to [m]$ of the form $([n], i)$ has a unique minimal retraction, denoted by $\tilde{i} : [m] \to [n]$, in other words, $\tilde{i}$ is the only morphism with domain given by the image of $i$ and which satisfies the relation $\tilde{i} \circ i = 1_{[n]}$. In particular, the minimal retraction associated to the face operator $d_i$ will be denoted $\zeta_i$. For $d_1 : [0] \to [1]$, its minimal retraction $\zeta_1 : [1] \to [0]$ coincide with $s_1 : [1] \to [0]$.

All morphisms of $\mathcal{L}$ can be generated by sum and compositions of the following five arrows.

$$\begin{array}{c}
[0] & \xrightarrow{d_1} & [1] & \xrightarrow{1} & [2] & \xleftarrow{s_1} & \end{array}$$

$\mathcal{P}(y)$ is the set of subsets of $y$. 

\text{\footnote{2}}
Category \( \mathcal{L} \) can be characterized as the free strictly associative pointed cocartesian category on one object, as the following proposition shows.

**Proposition 4.3.** Let \( \mathcal{C} \) be a strictly associative pointed cocartesian category, and \( X \) an object of \( \mathcal{C} \). Then there is a unique functor \( F : \mathcal{L} \rightarrow \mathcal{C} \) preserving zero and coproducts and such that \( F([1]) = X \).

**Proof.** Indeed, \( F([n]) \) must be the \( n \)-fold sum \( X + \cdots + X \) and \( F([0]) \) must be the zero object of \( \mathcal{C} \). The five morphisms above have mandatory images by \( F \), this means that \( F(1) = 1_X \), \( d_1 \) and \( \zeta_1 \) are send to the unique morphisms \( 0 \rightarrow X \) and \( X \rightarrow 0 \), where \( 0 \) is the zero object of \( \mathcal{C} \), the image of \( s_1 \) is the codiagonal of \( [1] \), that is the morphisms \( [1] + [1] \rightarrow [1] \) obtained by the universal property of coproduct, so its image by \( F \) must be \( X + X \rightarrow X \), the codiagonal of \( X \), which is well defined because \( \mathcal{C} \) is cocartesian, that is, the sum is well defined. And \( \sigma \) which is the canonical twisting arrow of the sum \( [1] + [1] \), should be send to the canonical twisting arrow of \( X + X \). \( \square \)

Similarly, the opposite category \( \mathcal{L}^{\text{op}} \) of \( \mathcal{L} \) is characterized as the free strictly associative pointed cartesian category generated by \([1]\).

### 4.2. \( \mathcal{L} \)-Algebras

An \( \mathcal{L} \)-algebra is a contravariant functor from \( \mathcal{L} \) to a category with a notion of homology, together with a natural transformation \( \mu \), which is called the product of the \( \mathcal{L} \)-algebra. Homotopy coherence is reflected in requiring product \( \mu \) induces isomorphisms in homology. Then, \( \mathcal{L} \)-algebra are defined in categories equipped with quasi-isomorphisms, that is, with a distinguished class of arrows, called quasi-isomorphisms, which forms a subcategory. The only categories of this kind we will use are DGA-Mod and DGA-Alg, both over a field \( k \), where being an quasi-isomorphisms means inducing an isomorphism in homology.

**Definition 4.4 (\( \mathcal{L} \)-algebra).** Let \((\mathcal{C}, \otimes, k, T)\) be a strict symmetric monoidal category with quasi-isomorphisms. An \( \mathcal{L} \)-algebra \( A \) with values in the category \( \mathcal{C} \) consists of a functor,

\[
A : \mathcal{L}^{\text{op}} \rightarrow \mathcal{C}
\]

with a natural transformation \( \mu : \otimes \circ (A \times A) \rightarrow A \circ + \).

The morphism in \( \mathcal{C} \) that \( \mu \) associates to each pair \( [(n), [m]] \in \mathcal{L} \times \mathcal{L} \), goes from \( A[n] \otimes A[m] \rightarrow A[n + m] \) and is written \( \mu_{[n],[m]} \). The image of any arrow \( \alpha \) is simply written again as \( \alpha \), but this image goes in the opposite direction. Then, for every pair of arrows \( \alpha : [p] \rightarrow [n] \) and \( \beta : [q] \rightarrow [m] \) in \( \mathcal{L} \) we have the following commutative diagram.

\[
\begin{array}{ccc}
A[n] \otimes A[m] & \xrightarrow{\mu_{[n],[m]}} & A[n + m] \\
\downarrow{\alpha \otimes \beta} & & \downarrow{\alpha + \beta} \\
A[p] \otimes A[q] & \xrightarrow{\mu_{[p],[q]}} & A[p + q]
\end{array}
\]

The functor \( A \) and the natural transformation \( \mu \) are required to satisfy the following conditions.

1. **Associativity:** \( \mu \circ (\mu \otimes 1) = \mu \circ (1 \otimes \mu) \).
(2) **Commutativity:** Let \([n], [m] \in \mathcal{L}\), and \(\tau : [m + n] \to [n + m]\) be the twisting morphism of \([m] + [n]\), then the following diagram commutes.

\[
\begin{array}{ccc}
A[n] \otimes A[m] & \xrightarrow{\mu} & A[n + m] \\
\downarrow T & & \downarrow \tau \\
A[m] \otimes A[n] & \xrightarrow{\mu} & A[m + n]
\end{array}
\]

(4.6)

(3) **Unit:** The image of \([0]\) by \(\mathcal{A}\) is \(k\) and \(\mu_{[0],[n]} = \mu_{[n],[0]} = 1\).

(4) **Coherence:** For every pair \(([n],[m])\), the morphism \(\mu_{[n],[m]}\) is a quasi-isomorphism. The natural transformation \(\mu\) is called the product or the structural quasi-isomorphism of \(\mathcal{A}\). Also, in order to simplify the expressions we drop the indexes of \(\mu_{[n],[m]}\) and simply write \(\mu\) when necessary.

There is an degenerated case of \(\mathcal{L}\)-algebra when \(A[n] = A[1]^{\otimes n}\) for every \(n \leq 1\) and \(\mu\) is taken to be the identity, which implies that \(s_1 : A[1] \to A[1] \otimes A[1]\) is a commutative coproduct. Indeed, in \(\mathcal{L}\) we have the following commutative diagram.

\[
\begin{array}{ccc}
[1] & \xleftarrow{s_1} & [2] \\
\downarrow s_1 & & \downarrow \tau \\
[2] & \underset{s_1}{\Rightarrow} & A[2] \xleftarrow{\mu=1} A[1] \otimes A[1]
\end{array}
\]

(4.7)

Taking \(T = \tau\) and \(T \circ s_1 = s_1 : A[1] \to A[1] \otimes A[1]\). The fact that the coproduct is commutative implies that higher homotopies of diagonals are zero, making this \(\mathcal{L}\)-algebras not very interesting for us. The reason is, \(\mathcal{L}\)-algebras are supposed to model the behavior of systems of diagonals like the one found in the chain complex associated to a simplicial set. In that case the diagonals obtained from homotopy inverses of the Eilenberg-Mac Lane transformation are not commutative, because of the existence of Steenrod operations.

**Proposition 4.5** (The category of \(\mathcal{L}\)-algebras). \(\mathcal{L}\)-algebras with values in a category \(\mathcal{C}\) together with natural transformations \(f : \mathcal{A} \to \mathcal{B}\) such that

1. \(f\) preserves the product of \(\mathcal{L}\)-algebras, that is, for every pair \(([n],[m])\) the following diagram is commutative.

\[
\begin{array}{ccc}
A[n] \otimes A[m] & \xrightarrow{\mu_\mathcal{A}} & A[n + m] \\
\downarrow f_{[n] \otimes f_{[m]}} & & \downarrow f_{[n + m]} \\
B[n] \otimes B[m] & \xrightarrow{\mu_\mathcal{B}} & B[n + m]
\end{array}
\]

(4.8)

2. \(f_{[0]} : A[0] \to B[0]\) is the identity of \(k\).

form a category. This category will be denoted \(\mathcal{L}(\mathcal{C})\).

**Proof.** Let \(\mathcal{A}, \mathcal{B}, \mathcal{C}\) be three \(\mathcal{L}\)-algebras, \(f : \mathcal{A} \to \mathcal{B}\) and \(g : \mathcal{B} \to \mathcal{C}\), be two morphisms of \(\mathcal{L}\)-algebras. It suffice to check that the composition of natural transformation \(g \circ f\) is a morphism of \(\mathcal{L}\)-algebras. Second condition is trivial and first condition is consequence of the following commutative diagrams for \(f\) and \(g\).
For an \( \mathcal{L} \)-algebra \( A \) with values in the category \( \mathcal{C} \), if \( \mathcal{C} \) is the category of DGA algebras, \( A \) is called a multiplicative \( \mathcal{L} \)-algebra. \( \mathcal{L} \)-algebras are designed to represent the 0-reduced simplicial sets and multiplicative \( \mathcal{L} \)-algebras will represent the 0-reduced simplicial groups.

4.3. The Monoidal Category of \( \mathcal{L}(\mathcal{C}) \). Let \( T : \mathcal{L}^{\text{op}} \to \mathcal{C} \) be the functor defined by \( T[n] = k \) for every \( n \geq 0 \) and \( T(\alpha) = 1_k \) for every morphism in \( \mathcal{L} \). Together with the natural transformation \( \mu : \otimes \circ (T \times T) \to T \circ + \) defined by \( \mu_{[n],[m]} = 1_k \) for all \( [n],[m] \in \mathcal{L} \), the functor \( T \) is an \( \mathcal{L} \)-algebra, it is called the trivial \( \mathcal{L} \)-algebra with values in \( \mathcal{C} \) and is denoted \( k \).

**Proposition 4.6.** Let \( \mathcal{C} \) be the category DGA-modules. Then the trivial \( \mathcal{L} \)-algebra \( k \) is a zero object in \( \mathcal{L}(\mathcal{C}) \).

**Proof.** \( k \) is the zero object of DGA-Mod, then, for any \( \mathcal{L} \)-algebra \( A \), this defines unique DGA-morphisms \( i_n : k \to A[n] \) and \( p_n : A[n] \to k \) \( (n \geq 0) \), which coincide with the coaugmentation and augmentation of \( A[n] \), respectively. The associated natural transformations \( i : k \to A \) and \( p : A \to k \) are morphisms of \( \mathcal{L} \)-algebras by commutativity of the following diagram.

\[
\begin{array}{c}
\eta_n \otimes \eta_m \\
A[n] \otimes A[m] \xrightarrow{i} A[n] \xrightarrow{\mu} A[n + m]
\end{array}
\]

4.3. The Monoidal Category of \( \mathcal{L}(\mathcal{C}) \). Let \( T : \mathcal{L}^{\text{op}} \to \mathcal{C} \) be the functor defined by \( T[n] = k \) for every \( n \geq 0 \) and \( T(\alpha) = 1_k \) for every morphism in \( \mathcal{L} \). Together with the natural transformation \( \mu : \otimes \circ (T \times T) \to T \circ + \) defined by \( \mu_{[n],[m]} = 1_k \) for all \( [n],[m] \in \mathcal{L} \), the functor \( T \) is an \( \mathcal{L} \)-algebra, it is called the trivial \( \mathcal{L} \)-algebra with values in \( \mathcal{C} \) and is denoted \( k \).

**Proposition 4.6.** Let \( \mathcal{C} \) be the category DGA-modules. Then the trivial \( \mathcal{L} \)-algebra \( k \) is a zero object in \( \mathcal{L}(\mathcal{C}) \).

**Proof.** \( k \) is the zero object of DGA-Mod, then, for any \( \mathcal{L} \)-algebra \( A \), this defines unique DGA-morphisms \( i_n : k \to A[n] \) and \( p_n : A[n] \to k \) \( (n \geq 0) \), which coincide with the coaugmentation and augmentation of \( A[n] \), respectively. The associated natural transformations \( i : k \to A \) and \( p : A \to k \) are morphisms of \( \mathcal{L} \)-algebras by commutativity of the following diagram.

\[
\begin{array}{c}
\eta_n \otimes \eta_m \\
A[n] \otimes A[m] \xrightarrow{i} A[n] \xrightarrow{\mu} A[n + m]
\end{array}
\]

**Proposition 4.7.** Let \( A \) and \( B \) be two \( \mathcal{L} \)-algebras. Let \( P \) be the functor \( P : \mathcal{L}^{\text{op}} \to \mathcal{C} \) defined by,

1. \( P[n] = A[n] \otimes B[n] \) for all \( [n] \in \mathcal{L} \).
2. \( P(\alpha) = A(\alpha) \otimes B(\alpha) \) for all \( \alpha \) morphism in \( \mathcal{L} \).

Let \( \mu_{P} : \otimes \circ (P \times P) \to P \circ + \) be the natural transformation given by the following composition.
We need to check that for every pair of morphisms of \( \mathcal{L} \)-algebras, \( f : \mathcal{A} \to \mathcal{B}, \quad g : \mathcal{C} \to \mathcal{D} \), there is a morphism of \( \mathcal{L} \)-algebras \( f \otimes g : \mathcal{A} \otimes \mathcal{C} \to \mathcal{B} \otimes \mathcal{D} \).

Define \( f \otimes g \) as \( (f \otimes g)_n = f_n \otimes g_n : A[n] \otimes C[n] \to B[n] \otimes D[n] \). Let \( \alpha : [m] \to [n] \) morphism of \( \mathcal{L} \), and consider the following diagram.

\[
\begin{align*}
\begin{array}{ccc}
A[n] \otimes C[n] & \xrightarrow{(f \otimes g)_n} & B[n] \otimes D[n] \\
\downarrow \alpha \otimes \alpha & & \downarrow \beta \otimes \beta \\
A[m] \otimes C[m] & \xrightarrow{(f \otimes g)_n} & B[m] \otimes D[m]
\end{array}
\end{align*}
\]
This diagram is commutative because,

\[(4.14)\]
\[(f \otimes g)_m \circ (\alpha \otimes \alpha) = (f_m \otimes g_m) \circ (\alpha \otimes \alpha) = (f_m \circ \alpha) \otimes (g_m \otimes \alpha) = (\beta \circ f_n) \otimes (\beta \circ g_n) \quad (f \text{ and } g \text{ are morphism of } \mathcal{L}\text{-algebras}) = (\beta \otimes \beta) \circ (f \otimes g)_n\]

\(f \otimes g\) preserves the quasi-isomorphism \(\mu\) by the following diagram.

\[(4.15)\]
\[
\begin{array}{ccc}
(A \otimes C)(n) \otimes (A \otimes C)(m) & \xrightarrow{(f \otimes g)_n \otimes (f \otimes g)_m} & (B \otimes D)(n) \otimes (B \otimes D)(m) \\
(A \otimes C)(n + m) & \xrightarrow{(f \otimes g)_{n+m}} & (B \otimes D)(n + m)
\end{array}
\]

Commutativity follows because,

\[(f \otimes g)_{n+m} \circ \mu_{A \otimes C} = (f_{n+m} \otimes g_{n+m})(\mu_A \otimes \mu_C)(1 \otimes T \otimes 1) = (f_{n+m} \mu_A \otimes g_{n+m} \mu_C)(1 \otimes T \otimes 1) = (\mu_B(f_n \otimes f_m) \otimes \mu_D(g_n \otimes g_m))(1 \otimes T \otimes 1) = (\mu_B \otimes \mu_D)(1 \otimes T \otimes 1)(f_n \otimes g_n \otimes f_m \otimes g_m) = \mu_{B \otimes D} \circ ((f \otimes g)_n \otimes (f \otimes g)_m)\]

Next proposition proof is straightforward.

**Proposition 4.10.** The category \(\mathcal{L}(\mathcal{C})\) is a strict symmetric monoidal category with unit, where product is given by the tensor product of \(\mathcal{L}\) and unit is the trivial \(\mathcal{L}\)-algebra \(k\).

**Definition 4.11.** Let \(A\) be an \(\mathcal{L}\)-algebra. The module \(A[1]\) is called the main element of \(A\). The associated forgetful functor from \(\mathcal{L}(\mathcal{C})\) to \(\mathcal{C}\) is denoted by \(U\). In fact we have a collection indexed by \(n \geq 0\) of forgetful functors \(U_n : \mathcal{L}(\mathcal{C}) \to \mathcal{C}\), with \(U_n(A) = A[n]\).

**Definition 4.12.** Let \(A\) be an \(\mathcal{L}\)-algebra. The homology of \(A\) is defined to be the homology of its main element.

**Remark 4.13.** The homology of \(\mathcal{L}\)-algebras is equal to the composition of functors \(H_* \circ U\), where \(H_*\) is the homology functor in \(\mathcal{C}\).

**Definition 4.14.** Let \(f : A \to B\) be a morphism of \(\mathcal{L}\)-algebras with values in \(\mathcal{C}\). The morphism \(f\) is called quasi-isomorphism if the induced morphism \(U(f)\) in \(\mathcal{C}\) by the forgetful functor, is a quasi-isomorphism.

**Proposition 4.15.** Let \(f : A \to B\) be a morphism of \(\mathcal{L}\)-algebras with values in \(\mathcal{C}\). If \(U_k(f)\) is a quasi-isomorphism in \(\mathcal{C}\) for some \(k\), then \(U_{kn}(f)\) is a quasi-isomorphism for every \(n \geq 0\). In particular if \(f\) is a quasi-isomorphism then \(U_n(f)\) is a quasi-isomorphism for every \(n \geq 1\).
Proof. We proceed by induction. The hypothesis says that $f_k : A[k] \to B[k]$ is a quasi-isomorphism. Now, the following diagram is commutative because $f$ is a morphism of $L$-algebras

\[
\begin{array}{ccc}
A[k] \otimes A[k(n-1)] & \xrightarrow{f_k \otimes f_{k(n-1)}} & B[k] \otimes B[k(n-1)] \\
\mu_A & & \mu_B \\
A[kn] & \xrightarrow{f_{kn}} & B[kn]
\end{array}
\]

(4.17)

The tensor product $f_k \otimes f_{k(n-1)}$ is a quasi-isomorphism since $k$ is a field. Then $f_{kn}$ is a quasi-isomorphism. □

Definition 4.16. The equivalence relation on $L$-algebras spanned by quasi-isomorphisms will be called again quasi-isomorphism.

The concept of $L$ is inspired by the fact that any natural diagonal of chain complexes $C_*(X) \to C_*(X) \otimes C_*(X)$, is determined by a zig-zag of natural morphisms $C_*(X) \to C_*(X \times X) \leftarrow C_*(X) \otimes C_*(X)$, where the first arrow is the morphism induced by the simplicial diagonal $X \to X \times X$ and the second arrow is the Eilenberg-Mac Lane transformation. In this section we will proceed to describe the $L$-algebra structure on the chain complexes that have as product the Eilenberg-Mac Lane transformation, using a canonical way to associate to each simplicial set (not necessarily 0-reduced) an $L$-algebra whose main element is its chain complex.

Proposition 4.17. Let $sSet_\ast$ be the category of pointed simplicial set. Then for every simplicial set $X$, there is an unique functor $S_X : L^{op} \to sSet_\ast$ preserving zeros, mapping sums to products, and $[1]$ to $X$.

Proof. $sSet_\ast$ is a strict pointed cartesian category, then by proposition 4.3 the result follows. □

Proposition 4.18. Let $X$ be a pointed simplicial set. Then composition,

\[
C_\ast \circ S_X : L^{op} \to \mathcal{C}
\]

together with the Eilenberg-Mac Lane transformation is an $L$-algebra.

Proof. It follows easily from Eilenberg-Mac Lane transformation properties. □

Definition 4.19. The $L$-algebra associated to the pointed simplicial set $X$ will be called the canonical $L$-algebra of $X$, and denoted by $A_X$.

$L$-algebra $A_X$ for $n \geq 1$ take the values $A_X[n] = C_\ast(X^n)$ with product $\mu_{A_X}$the Eilenberg-Mac Lane transformation $\nabla_{n,m} : C_\ast(X^n) \otimes C_\ast(X^m) \to C_\ast(X^{n+m})$ Let $\ast$ be the base point of $X$, then the image by $A_X$ for a morphism $\alpha : [m] \to [n]$ in $\mathcal{L}$ is given by the following formula.

\[
\begin{array}{ccc}
A_X[n] & \xrightarrow{A_X(\alpha)} & A_X[m] = C_\ast(X^m) \\
(x_1, \ldots, x_n) & \xrightarrow{(x_\alpha(1), \ldots, x_\alpha(m))} & (x_1, \ldots, x_n)
\end{array}
\]

(4.19)
Where \( x_{\alpha(j)} = * \) for each \( j \) not in \( \text{Dom}(\alpha) \). Note that taking \( X = * \) the simplicial point, \( A_\ast \) is the trivial \( L \)-algebra \( k \).

In the case of a simplicial group (who will be pointed by its unit), we have an extra structure.

**Proposition 4.20.** Let \( H \) be a simplicial group, then \( A_H \) is a multiplicative \( L \)-algebra.

**Proof.** For every \( n \geq 0 \), \( H^n \) and \( A_H[n] = C_*(H^n) \) is a differential graded algebra (with the Pontrjagin product). Since the Eilenberg-Mac Lane is a morphism of algebras, the functor \( A \) maps simplicial groups to multiplicative \( L \)-algebras. \( \square \)

5. **Principal results**

5.1. **The Operad \( K \).** In this section is constructed an \( E_\infty \)-operad \( K \) by using the inductive colimit of a suitable collection of operads \( \{K_n\}_{n \geq 2} \).

This construction starts with a \( S \)-module \( M \) concentrated in arity 2 and such that \( M(2) \) is suppose to be a DGA-module \( k[\Sigma_2] \)-free resolution of \( k \). Operad \( K_2 \) is taken as the free operad on \( M \), which makes that \( K_2 \) will have like \( M \) a \( k[\Sigma_2] \)-free resolution of \( k \) with all abstract binary operations coded by \( K_2 \).

However, in higher arities \( K_2 \) fails to have free resolutions of \( k \). For example, there is no guarantee we can found in \( K_2(3) \) appropriate homotopies linking trees of degree 0 with nodes any generating element \( x \) of degree 0 in \( K_2(2) \), like the ones in the following picture.

\[
(5.1)
\]

Next step extend operad \( K_2 \) into an operad \( K_3 \) having on component \( K_2(3) \) all missing homotopies by using the polynomial operads technique discussed in subsection 3.2.

This process is repeated with each component in order to include all possible homotopies, creating a nested collection of operad which inductive limit will have all required homotopies to form an \( E_\infty \)-operad, the operad \( K \).

**Definition 5.1.** Let \( M \) be a \( k[\Sigma_n] \)-free finitely generated DGA-module. The acyclic extension of \( M \) is the associated acyclic DGA-module given by proposition 2.3. It will be denoted by \( X(M) \).

**Definition 5.2.** Let \( M \) be a DGA-module \( k[\Sigma_2] \)-free resolution of \( k \). For \( n \geq 2 \), \( K_n \) is the operad defined by induction as follows.

1. \( K_2 = F(M) \), the free operad on \( M \), with \( M \) seen as \( S \)-module concentrated in arity 2.
2. \( K_{n+1} = K_n[M_n] \), the polynomial operad on \( M_n \) with coefficients in \( K_n \), defined in proposition 3.6 and \( M_n \) is the \( S \)-module given by:

\[
(5.2)
M_n(i) = \begin{cases} 
K_n(i) & i \neq n + 1 \\
X(K_n(n + 1)) & i = n + 1 
\end{cases}
\]
\( K_2 = F(M) \) is only formed from compositions of arity 2 operations in \( M \), and is not acyclic for arities \( \geq 3 \). \( K_2(3) \) is extended to an acyclic DGA-module \( X(K_2(3)) \), Inclusion \( K_2 \hookrightarrow M_2 \), which is strict only in arity 3, increase \( K_2 \) with new operations in arity 3 not decomposable in terms of operations of arity 2. Now, operad \( K_3 = K_2[M_2] \) is formed by compositions all operations in \( K_2 \) and the new arity 3 operations, with \( K_3 \) matching \( K_2 \) in arity 2, and in general, the extension \( K_n \to K_{n+1} \) is the identity for arities \( \leq n \).

Formally, canonical inclusions \( K_n \hookrightarrow K_{n+1} \) are given by arrows \( \alpha_n \) from the construction of each \( K_{n+1} \).

\[
\begin{align*}
    FU(K_n) & \xrightarrow{\epsilon_{K_n}} K_n \\
    F(i_{K_n}) & \downarrow \quad \quad \downarrow \alpha_n \\
    F(M_n) & \xrightarrow{\beta_n} K_n[M_n] = K_{n+1}
\end{align*}
\]

But \( \alpha_n : K_n \to K_{n+1} \) behaves like the identity for arities \( \leq n \) in the sense of lemma 3.14, allowing the following definition.

**Definition 5.3.** The operad \( K \) is defined to be the inductive limit of the collection of operads,

\[
K_2 \xrightarrow{\alpha_2} K_3 \xrightarrow{\alpha_3} \cdots \xrightarrow{\alpha_{n-1}} K_n \xrightarrow{\alpha_n} \cdots
\]

**Proposition 5.4.** The operad \( K \) is an \( E_\infty \)-operad.

**Proof.** We have \( K(2) = K_2(2) = M_2 \), a \( k[\Sigma_2] \)-free resolution of \( k \), and by construction \( K(n+1) = K_{n+1}(n+1) = M_n = X(K_n(n+1)) \), which is acyclic and \( k[\Sigma_{n+1}] \)-free.

\[\square\]

5.2. \( E_\infty \)-Structures in \( \mathcal{L} \)-Algebras. Following theorem is the principal objective of this paper, it exhibits the nature of main elements of \( \mathcal{L} \)-algebras as \( E_\infty \)-coalgebras. The outline of the proof is, we start by showing that main element \( A[1] \) of an \( \mathcal{L} \)-algebra \( A \) is a \( K_n \)-coalgebra for all \( n > 1 \). Then, using the fact that operad \( K \) is the inductive limit of these operads, \( A[1] \) can have an induced \( K \)-coalgebra structure, in other words, \( A[1] \) is an \( E_\infty \)-coalgebra.

**Theorem 5.5** (Main Theorem). Let \( K \) be the \( E_\infty \)-operad defined in 5.3. Then, there exists a functor \( \mathcal{F} \)

\[
\mathcal{L} \text{-Alg} \xrightarrow{\mathcal{F}} K \text{-CoAlg}
\]

which associates a \( K \)-coalgebra \( \mathcal{F}(A) \) to each \( \mathcal{L} \)-algebra \( A \) and satisfies the following conditions.

1. The underlying DGA module of \( \mathcal{F}(A) \) is \( A[1] \).
2. For every \( n \geq 1 \), the morphism \( \varphi_n : K(n) \otimes A[1] \to A[1]^{\otimes n} \), given by the \( K \)-coalgebra structure defined on \( A[1] \) by \( \mathcal{F} \), makes the following diagram commutative up to homotopy.

\[
\begin{align*}
    A[1]^{\otimes n} & \xrightarrow{\mu} A[n] \\
    K(n) \otimes A[1] & \xrightarrow{\varphi_n} A[n] \\
    & \xrightarrow{n_1 (\otimes 1)} A[n]
\end{align*}
\]

(5.6)
an operad morphism \( K \) of \((5.8)\) and the action of \( \Sigma_2 \) makes the following diagram commutative up to homotopy,

\[
\begin{array}{ccc}
A[1]^{\otimes i} & \overset{\mu}{\longrightarrow} & A[i] \\
\downarrow{\varphi_i} & & \downarrow{s_1(\epsilon \otimes 1)} \\
K_n(i) \otimes A[1] & & \\
\end{array}
\]

where \( \mu \) is given by the structural quasi-isomorphism of \( A \) and \( s_1 \) is the image by \( A \) of the only morphism in \( L \) of the form \( ([n], \alpha) : [n] \to [1] \).

**Proof of theorem [5.5].** We use the fact that the operad \( E_\infty \)-operad \( K \) is the inductive limit of the sequence of operads,

\[
K_2 \subset \cdots \subset K_n \subset \cdots \subset K
\]

in order to proceed by induction. We first show for all \( n \geq 2 \) that \( A[1] \) has an structure of \( K_n \)-coalgebra which satisfies the second condition of the theorem. That is, there exists an operad morphism \( \overline{F}_n : K_n \to \text{Coend}(A[1]) \), such that the associated morphisms \( \varphi_i : K_n(i) \otimes A[1] \to A[1]^{\otimes i} \), makes the following diagram commutative up to homotopy,

\[
\begin{array}{ccc}
A[1]^{\otimes i} & \overset{\mu}{\longrightarrow} & A[i] \\
\downarrow{\varphi_i} & & \downarrow{s_1(\epsilon \otimes 1)} \\
K_n(i) \otimes A[1] & & \\
\end{array}
\]

- **Case \( K_2 \):** Recall that \( K_2 \) is the free operad on the \( S \)-module \( M_2 \) concentrated in arity 2. To show that \( A[1] \) is a \( K_2 \)-coalgebra, we define an \( \Sigma_2 \)-equivariant morphism from \( M_2(2) \) to \( \text{Coend}(A[1])(2) \) using the relative lifting theorem \( \text{[2.2]} \). In order to satisfy the condition on \( K_2 \) and then, the \( K_2 \)-coalgebra structure is obtained as a consequence of the universal property of free operads.

Defining a \( \Sigma_2 \) morphism from \( M_2(2) \) to \( \text{Coend}(A[1])(2) \) is equivalent to define a morphism of DGA-\( k[\Sigma_2] \)-modules,

\[
(5.9) \quad \overline{\varphi}_2 : K_2(2) \otimes A[1] \to A[1] \otimes A[1]
\]

Recall that \( M_2(2) = K_2(2) \), the action of \( \Sigma_2 \) on \( A[1] \otimes A[1] \) is the permutation of factors and the action of \( \Sigma_2 \) on \( K_2(2) \otimes A[1] \) maps \( x \otimes a \) to \( x\sigma \otimes a \). Now consider the following diagram,

\[
\begin{array}{ccc}
A[1] \otimes A[1] & \overset{\mu}{\longrightarrow} & A[2] \\
\downarrow{\varphi_2} & & \downarrow{s_0(\epsilon \otimes 1)} \\
K_2(2) \otimes A[1] & & \\
\end{array}
\]

where \( \epsilon \) is the augmentation of \( K_2(2) \), \( s_0 : A[1] \to A[2] \) is the image by \( A \) of the only arrow in \( L \) of the form \( ([2], \alpha) : [2] \to [1] \) and \( \mu \) is the structural quasi-isomorphism of \( A \).

The DGA-\( k[\Sigma_2] \)-morphism \( \varphi_2 \) that makes the diagram commutative up to homotopy is obtained with the theorem \( \text{[2.2]} \) by taking \( L' = 0 \). This complete the existence of a \( \Sigma_2 \)-equivariant morphism from \( M_2(2) \) to \( \text{Coend}(A[1])(2) \) and therefore, we have a morphism \( F_2 \) of \( S \)-modules from \( M_2 \) to \( \text{Coend}(A[1]) \), which behaves on \( M_2(2) \) as \( \varphi_2 \) and as 0 on \( M_2(i), i \neq 2 \).

Now, consider the following diagram,
\[
M_2 \xrightarrow{F_2} \mathcal{K}_2 = F(M_2)
\]

(5.11)

where the upper arrow is given by the inclusion of \(S\)-modules. The universal property of the free operad \(\mathcal{K}_2\) says there is an unique morphism of operads \(F_2\) making the diagram commutative. This morphism \(F_2\) gives the \(\mathcal{K}_2\)-coalgebra structure on \(A[1]\) that we wanted.

**Case \(K_n\):** Suppose we have a sequence of operad morphisms \(F_2, \ldots, F_{n-1}\), such that, for \(i < n\), \(F_i : K_i \to \text{Coend}(A[1])\) and \(F_i\) satisfies the second condition of the theorem. As we have seen in the construction of \(K_i\)'s, the operad \(K_{n-1}\) as \(S\)-module, can be embedded as a direct factor in a \(S\)-module \(M_{n-1}\) with component \(M_{n-1}(n)\) acyclic and \(k[\Sigma_n]\)-free. Then we have \(M_{n-1} = K_{n-1} \oplus P\), where the component \(P(j)\) is \(k[\Sigma_j]\)-free for \(j > 0\).

Observe that this defines an object \((K_{n-1}, M_{n-1})\) and morphism in \(\mathcal{C}\),

\[
F_{n-1} : (K_{n-1}, M_{n-1}) \to (\text{Coend}(A[1]), U(\text{Coend}(A[1])))
\]

which behaves as \(F_{n-1}\) on \(K_{n-1}\) and as 0 on \(P\). In order to satisfy the second condition of the theorem we focus our attention in the \(\Sigma_j\)-equivariant morphism given by \(F_{n-1}\) on the component \(K_{n-1}(j), j > 0\). We will extend this morphism on the components \(M_{n-1}(j)\) or equivalently, define a DGA\(-k[\Sigma_j]\) morphism \(\phi_j\) from \(M_{n-1}(j) \otimes A[1]\) to \(A[1]^\otimes j\). In order to do that, consider the diagram,

\[
A[1]^\otimes j \xrightarrow{\mu} A[j] \xrightarrow{s_0 \circ (\varepsilon \otimes 1)} A[j]
\]

\[
\xymatrix{ A[1]^\otimes j \ar[rr]^\mu \ar[dr]_{\phi_j} & & A[j] \ar[dl]^ {s_0 \circ (\varepsilon \otimes 1)} \\
& M_{n-1}(j) \otimes A[1] & \ar[ul] \ar[dl] \ar[ull] \\
& K_{n-1}(j) \otimes A[1] &}
\]

(5.13)

where \(\phi_j\) is the \(k[\Sigma_n]\)-morphism induced by \(F_{n-1}\). By hypothesis, the morphism \(\phi_j\) makes commutative up to homotopy the outer triangle of the diagram. Then the existence of \(\phi_j\) follows after applying the relative lifting theorem \([2,2]\) with \(L = M_{n-1}(j) \otimes A[1]\) and \(L' = K_{n-1}(j) \otimes A[1]\).

Observe that the \(\Sigma_j\)-equivariant morphism from \(M_{n-1}(j)\) to \(\text{Coend}(A[1])(j)\) induced by \(\phi_j\) behaves like \(F_{n-1}\) over \(K_{n-1}(j)\). Denote by \(F_{n-1}\) the morphism of \(S\)-modules given by this data. Then \((F_{n-1}, F_{n-1}) : (K_{n-1}, M_{n-1}) \to (\text{Coend}(A[1]), U(\text{Coend}(A[1])))\) is a morphism of \(\mathcal{C}\), and consider the following diagram.

\[
(K_{n-1}, M_{n-1}) \xrightarrow{\psi} (K_{n-1}[M_{n-1}], U(K_{n-1}[M_{n-1}]))
\]

\[
\xymatrix{(K_{n-1}[M_{n-1}], U(K_{n-1}[M_{n-1}])) \ar[dr]^{(F_{n-1}, U(F_{n-1}))} & \\
(K_{n-1}, M_{n-1}) \ar[ur]_{(F_{n-1}, U(F_{n-1}))} &}
\]

(5.14)
The operad morphism $\overline{F}_n$ making the diagram commutative follows by proposition 3.7. Then, $\overline{F}_n$ gives the $K_n$-coalgebra structure on $A[1]$ needed to complete the inductive step.

$A[1]$ is a $K$-coalgebra: We now proceed with the final part of the proof and exhibit $A[1]$ as a $K$-coalgebra. Consider the following cocone of operads on the diagram given by

\[
\begin{array}{ccc}
\mathcal{K} & \xrightarrow{F} & \operatorname{Coend}(A[1]) \\
\vdots & \vdots & \vdots \\
K_2 & \xleftarrow{\ldots} & K_n & \xleftarrow{\ldots}
\end{array}
\]

The inductive part of the proof exhibited the operad $\operatorname{Coend}(A[1])$, together with the morphisms $\overline{F}_n$’s, as a cocone on the $K_i$’s. By definition $\mathcal{K}$ is also a cocone on the $K_i$’s. Then the universal property of colimits says that there exists an unique morphism of operads $\overline{F}$ from $\mathcal{K}$ to $\operatorname{Coend}(A[1])$ commutative on these two cocones. The morphism $\overline{F} : \mathcal{K} \to \operatorname{Coend}(A[1])$ exhibit $A[1]$ as an $\mathcal{K}$-coalgebra with the conditions stated by the theorem.

**Functoriality:** Let $f : A \to B$ be a morphism of $\mathcal{L}$ and consider the following diagram.

\[
\begin{array}{ccc}
A[n] \otimes^n & \xrightarrow{f_n \otimes^n} & B[n] \otimes^n \\
\varphi_n^A & \xleftarrow{\mu^A} & A[n] & \xrightarrow{f_n} & B[n] & \xleftarrow{\mu^B} \\
K[n] \otimes A[1] & \xleftarrow{1 \otimes f_1} & K[n] \otimes B[1] \\
\end{array}
\]

The two triangles are commutative up to homotopy by the second condition of the theorem and the inner diagrams are commutative because $f$ is a morphism of $\mathcal{L}$-algebras. The commutative up to homotopy of the outer diagram follows from this and the fact that $\mu$ is a quasi-isomorphism. This shows that our construction is functorial and completes the proof. $\square$

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