BLOWING UP SEQUENCES OF CONSTANT MEAN CURVATURE TORI WITH FIXED SPECTRAL GENUS IN THE EUCLIDEAN 3-SPACE TO MINIMAL SURFACES

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Abstract. We investigate conditions under which a blowup, i.e. a rescaling in the parameter domain and in the ambient space, of a sequence of constant mean curvature torus immersions into the Euclidean 3-space produces a sequence of which a subsequence converges to a non-trivial minimal surface immersion.

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1. Introduction

We consider sequences \((f_n)\) of smooth constant mean curvature (cmc) torus immersions into the Euclidean 3-space \(\mathbb{R}^3\) and investigate under which circumstances there exists a “blowup”, i.e. a rescaling of both the parameter of the \(f_n\) and of the ambient space \(\mathbb{R}^3\), such that with these rescalings, a subsequence of the \(f_n\) converges to a non-trivial surface immersion \(\tilde{f}\) into \(\mathbb{R}^3\). Note that by a well-known result due to Pinkall/Sterling, and independently to Hitchin, any such \(f_n\) has finite spectral genus, and in this paper we will always assume that all \(f_n\) have the same spectral genus \(g < \infty\). It will turn out that in the non-trivial case where the sequence \((f_n)\) does not itself have a subsequence that converges to a non-trivial surface immersion, the limiting surface described by \(\tilde{f}\) is a minimal surface.

The most important method we use for this investigation is the spectral theory for the integrable systems that correspond to the relevant finite type surfaces in \(\mathbb{R}^3\). They are the integrable system of the sinh-Gordon equation, which corresponds to cmc surfaces in \(\mathbb{R}^3\), and the integrable system of the Korteweg-de Vries equation (KdV equation), which corresponds to minimal surfaces in \(\mathbb{R}^3\). Our approach is based on the idea that any cmc surface or minimal surface of finite spectral genus is uniquely determined by a polynomial Killing field, which is a matrix-valued polynomial in the spectral parameter \(\lambda\) that depends only on the infinitesimal geometry of the surface at a single base point \(z_0\). We will find that to the blowup of the cmc tori that was described above, there corresponds a blowup of the spectral parameter \(\lambda\) and of the polynomial Killing fields such that the convergence of the blown up sequence of immersions is equivalent to the convergence of the blown up polynomial Killing fields. Because we can estimate the growth of the coefficients of the polynomial Killing fields corresponding to the \(f_n\) as \(n \to \infty\), we can use this equivalence to find situations in which the blowup of the \(f_n\) we describe has a convergent subsequence.

In Section \(\S\) we describe the spectral theory for cmc tori in \(\mathbb{R}^3\). The treatment we give here, especially of polynomial Killing fields, closely follows the description of the spectral theory for cmc surfaces in \(S^2 \times \mathbb{R}\) in \([\text{H-K-S-1}]\). In Section \(\S\) we describe the spectral theory for minimal surfaces in \(\mathbb{R}^3\), which occur as the limit of our blown-up sequence of cmc surfaces. While most of the spectral theory is of course well-known, the adaption of polynomial Killing fields and of the Pinkall-Sterling iteration to this case appears to be new. In Section \(\S\) we finally find conditions under which a
suitable blowup of the $f_n$ converges to a minimal surface in the sense described above, describe this blowup, and prove its convergence.

2. The integrable system for constant mean curvature surfaces

This paper is concerned with surfaces and curves in $\mathbb{R}^3$. We denote the standard inner product of $\mathbb{R}^3$ by $\langle \cdot, \cdot \rangle$. In the context of Wirtinger derivatives $f_z, f_{\bar{z}}$ of functions $f$ mapping into $\mathbb{R}^3$, we denote by $\langle \cdot, \cdot \rangle$ also the $\mathbb{C}$-bilinear continuation of that inner product of $\mathbb{R}^3$ to $\mathbb{C}^3$.

Surface immersions into $\mathbb{R}^3$. We begin by considering a smooth immersion $f : X \rightarrow \mathbb{R}^3$ of a 2-dimensional manifold $X$ into $\mathbb{R}^3$. There exists the structure of a Riemann surface on $X$ such that the immersion $f$ becomes conformal, and we always regard $X$ as a Riemann surface in this way. We choose a holomorphic coordinate $z = x + iy$ of $X$ and express the fundamental geometric quantities of $f$ locally with respect to $z$. Due to $f$ being a conformal immersion with respect to $z$ we have

$$\|f_x\|^2 = \|f_y\|^2 = 2\langle f_z, f_{\bar{z}} \rangle > 0 \text{ and } \langle f_z, f_y \rangle = 0,$$

hence the Riemannian metric on $X$ induced by $f$ is locally given by

$$g = e^\omega (dx^2 + dy^2) = e^\omega \, dz \, d\bar{z} \quad \text{with} \quad \omega = \ln(2\langle f_z, f_{\bar{z}} \rangle).$$

The smooth real-valued function $\omega$ is called the conformal factor of $f$ (with respect to the coordinate $z$). Let $N$ be a unit normal field for $f$ (at least locally on the domain of the coordinate $z$). The mean curvature $H$ of $f$ is one-half the trace of the shape operator $S = g^{-1}h$ and therefore given by

$$H = e^{-\omega} \cdot \frac{1}{2} \langle f_{xx} + f_{yy}, N \rangle = \langle f_z, f_{\bar{z}} \rangle^{-1} \cdot \langle f_{zz}, N \rangle.$$

Moreover the Hopf differential $Q \, dz^2$ is the $dz^2$-component of the second fundamental form and therefore given by

$$Q \, dz^2 \quad \text{with} \quad Q = \langle f_{zz}, N \rangle.$$

The zeros of $Q$, i.e. the points where the second fundamental form of $f$ is diagonal, are called umbilical points of $f$. The integrability condition $f_{z\bar{z}} = f_{\bar{z}z}$ for a surface immersion into $\mathbb{R}^3$ is expressed by the equations of Gauss and Codazzi, which with respect to the coordinate $z$ are given by

$$2\omega_{z\bar{z}} + H^2 e^{\omega} - 4 |Q|^2 e^{-\omega} = 0 \quad \text{(2.1)}$$

$$Q = e^{\omega} H_z. \quad \text{(2.2)}$$

The connection form and the extended frame for cmc immersions. We now turn our attention to constant mean curvature (cmc) surface immersions $f$ into $\mathbb{R}^3$, i.e. to the case where the mean curvature function $H$ is constant and non-zero. Then the Codazzi equation (2.2) shows that the Hopf differential $Q \, dz^2$ is holomorphic. Thus $f$ is either totally umbilical (this happens only if $f$ parameterises part of a round sphere in $\mathbb{R}^3$), or else the umbilical points of $f$ are discrete. In the latter case, around a non-umbilical point the coordinate $z$ can be chosen such that the function $Q$ describing the Hopf differential $Q \, dz^2$ is constant and non-zero; we will always choose $z$ in such a way in the sequel. In this setting the Codazzi equation (2.2) reduces to $0 = 0$, so the Gauss equation (2.1) is the sole condition of integrability for cmc immersions. Note that if we consider $H = \frac{1}{2}$ and $|Q| = \frac{1}{4}$, then Equation (2.1) reduces to the sinh-Gordon equation

$$\Delta \omega + \sinh(\omega) = 0.$$

We now consider the family of connection forms

$$\alpha = \omega = \frac{1}{4} \left( -4Q e^{-\omega/2} \begin{pmatrix} \omega_z & 2H e^{\omega/2} \lambda^{-1} \\ -\omega_z & -\omega_{\bar{z}} \end{pmatrix} \right) \, dz + \frac{1}{4} \left( -2H e^{\omega/2} \lambda \begin{pmatrix} -\omega_{\bar{z}} & 4Q e^{-\omega/2} \\ \omega_{\bar{z}} & \omega_{z} \end{pmatrix} \right) \, d\bar{z}. \quad \text{(2.3)}$$
which depends on a new variable, the spectral parameter $\lambda \in \mathbb{C}^*$. Note that these local 1-forms with respect to different coordinates $z$ piece together to global $\mathfrak{sl}(2, \mathbb{C})$-valued smooth 1-forms on $X$. An explicit calculation shows that the Maurer-Cartan equation $d\alpha_{\lambda} + \frac{1}{2} [\alpha_{\lambda}, \alpha_{\lambda}] = 0$ for $\alpha_{\lambda}$ does not depend on the value of $\lambda$, and is equivalent to the Gauss equation (2.1). It follows that if the data $(\omega, H, Q)$ correspond to a cmc immersion and hence satisfy the Gauss equation (2.1), then the initial value problem of the partial differential equation

$$dF_{\lambda} = F_{\lambda} \alpha_{\lambda} \quad \text{with} \quad F_{\lambda}(z_0) = 1$$

has a unique solution $F_{\lambda} : X \to \text{SL}(2, \mathbb{C})$. $F_{\lambda}$ is called the extended frame of $f$. Because $\alpha_{\lambda}$ depends holomorphically on the spectral parameter $\lambda \in \mathbb{C}^*$, also $F_{\lambda}$ depends holomorphically on $\lambda$. Note that because $\omega$ is real-valued and $H$ is real, we have

$$\alpha_{\lambda^{-1}} = -\overline{\alpha_{\lambda}} \quad \text{and therefore} \quad F_{\lambda^{-1}} = F_{\lambda}^{-1}.$$  

(2.4)

In particular for $|\lambda| = 1$ we have $\alpha_{\lambda} \in \mathfrak{su}(2)$ and $F_{\lambda} \in \text{SU}(2)$.

**The Sym-Bobenko formula.** The following Sym-Bobenko formula, see [3, Section 5], shows how the immersion $f$ can be reconstructed from the extended frame $F_{\lambda}$. For the purpose of describing this formula we identify $\mathbb{R}^3$ as an oriented Euclidean space with $\mathfrak{su}(2)$ via

$$(x_1, x_2, x_3) \in \mathbb{R}^3 \mapsto \frac{1}{2} \begin{pmatrix} x_3 & x_1 + ix_2 \\ x_1 - ix_2 & -x_3 \end{pmatrix} \in \mathfrak{su}(2).$$

The inner product on $\mathfrak{su}(2)$ that corresponds to the usual inner product on $\mathbb{R}^3$ is

$$(X, Y) := -2 \text{tr}(XY) \quad \text{for} \quad X, Y \in \mathfrak{su}(2).$$

The cross product $\times$ of $\mathbb{R}^3$ corresponds under this identification to the Lie bracket (commutator) of elements of $\mathfrak{su}(2)$.

**Proposition 2.1.** Let a real-valued, smooth function $\omega$ and constants $H \in \mathbb{R}^+$, $Q \in \mathbb{C}^*$ be given so that these data satisfy the Gauss equation (2.1). Further choose a Sym point $\lambda_s \in S^1$. Then

$$f = -\frac{1}{H} i \lambda_s \frac{\partial F}{\partial \lambda} F^{-1} \bigg|_{\lambda = \lambda_s}$$

(2.5)

is an immersion $X \to \mathfrak{su}(2) \cong \mathbb{R}^3$ with induced metric $e^{\omega} d\zeta d\bar{\zeta}$, constant mean curvature $H$ and Hopf differential $\lambda_s^{-1}Q d\zeta^2$. The tangential directions of $f$ are given by

$$f_x = e^{\omega/2} F_{\frac{1}{2}} \begin{pmatrix} 0 & \lambda_s^{-1} \\ \lambda_s & 0 \end{pmatrix} F^{-1} \quad \text{and} \quad f_y = e^{\omega/2} F_{\frac{1}{2}} \begin{pmatrix} 0 & i\lambda_s^{-1} \\ -i\lambda_s & 0 \end{pmatrix} F^{-1},$$

(2.6)

and a unit normal field for $f$ is given by

$$N = F_{\frac{1}{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} F^{-1}.$$  

(2.7)

In particular with the choice $\lambda_s = 1$ for the Sym point, the Sym-Bobenko formula (2.5) recovers the original immersion $f$ from which the data $(\omega, H, Q)$ are derived (up to a rigid motion of $\mathbb{R}^3$). By varying $\lambda_s$ over $S^1$, we obtain an entire family of cmc immersions which all have the same induced metric $e^{\omega} d\zeta d\bar{\zeta}$ and the same value $H$ of the constant mean curvature, but whose Hopf differential varies as $\lambda_s^{-1}Q d\zeta^2$. This family is called the associated family of the original immersion $f$.

**Proof.** We first note that because $\alpha_{\lambda}$ is $\mathfrak{su}(2)$-valued on the circle $S^1 \ni \lambda$, $F_{\lambda}$ takes values in $\text{SU}(2)$ there. Thus $G = \frac{\partial F}{\partial \lambda}$ is tangential to $\text{SU}(2)$ at $\lambda = \lambda_s$, and therefore $f$ indeed maps into $T_1 \text{SU}(2) \cong \mathfrak{su}(2)$. By differentiating the equation $dF = F\alpha$ with respect to $\lambda$, one sees that $G$ solves the partial differential equation $dG = G\alpha + F\beta$ with

$$\beta = \frac{\partial \alpha}{\partial \lambda} = -\frac{1}{2} H e^{\omega/2} \begin{pmatrix} 0 & \lambda^{-2} d\zeta \\ d\bar{\zeta} & 0 \end{pmatrix}.$$  

(2.8)
We now calculate
\[ df = -\frac{i\lambda}{H} (dGF^{-1} - G F^{-1} dF) = -\frac{i\lambda}{H} (\lambda F F^{-1} - G F^{-1} F F^{-1}) = -\frac{i\lambda}{H} (\lambda F F^{-1} F F^{-1} F F^{-1}) = -\frac{i\lambda}{H} \lambda F F^{-1} F F^{-1} F F^{-1}. \]

By inserting Equation (2.8) we obtain
\[ f_z = \lambda^{-1} e^{\omega/2} F \frac{\partial}{\partial z} \left( \begin{array}{c} 0 \\ 1 \\ 0 \\ 0 \end{array} \right) F^{-1} \quad \text{and} \quad f_{\bar{z}} = \lambda e^{\omega/2} F \frac{\partial}{\partial \bar{z}} \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right) F^{-1}. \]

Via the equations \( f_x = f_z + f_{\bar{z}} \) and \( f_y = \alpha (f_z - f_{\bar{z}}) \), Equations (2.9) follow, and then Equation (2.7) follows from \( N = \frac{f_x \times f_y}{|f_x \times f_y|} \). We also calculate
\[ 2\langle f_z, f_{\bar{z}} \rangle = 2e^{\omega} \cdot \left\langle \frac{1}{2} \left( \begin{array}{c} 0 \\ 1 \\ 0 \\ 0 \end{array} \right), \frac{1}{2} \left( \begin{array}{c} 0 \\ 0 \\ 1 \\ 0 \end{array} \right) \right\rangle = e^{\omega}, \]
whence it follows that \( f \) is a conformal immersion with the induced metric \( e^{\omega} \, dz \, d\bar{z} \).

We moreover obtain from Equations (2.9) and (2.3):
\[
\begin{align*}
 f_{zz} &= \frac{1}{2} \omega_z f_z + [FU F^{-1}, f_z] = \omega_z f_z + \lambda^{-1} Q N \\
 f_{\bar{z}z} &= \frac{1}{2} \omega_{\bar{z}} f_z + [FV F^{-1}, f_z] = \frac{1}{2} e^{\omega} H N \\
 f_{z\bar{z}} &= \frac{1}{2} \omega_z f_{\bar{z}} + [FV F^{-1}, f_{\bar{z}}] = \omega_z f_{\bar{z}} + \lambda Q N .
\end{align*}
\]

Thus the Hopf differential of \( f \) is given by \( \langle f_z, f_{\bar{z}} \rangle^{-1} \cdot \langle f_{z\bar{z}}, N \rangle = 2e^{-\omega} \cdot \frac{1}{2} e^{\omega} H = H \).

\[ \square \]

The placement of the factors \( H \) and \( Q \), as well as of \( e^{\pm \omega/2} \) in the connection forms given by (2.3) is somewhat unusual. The reason for our choice is that it facilitates the blow-up of cmc immersions described in Section 4 where the limit \( H \to 0 \) is counteracted by a blowup of the spectral parameter \( \lambda \). The relationship to the more common form of \( \alpha \) is described in the following remark.

**Remark 2.2.** It is a classically well-known fact that for any constant mean curvature \( (H \neq 0) \)-surface, one of the two parallel surfaces in the distance \( 1/H \) is again a constant mean curvature surface. In the situation described above, this fact is reflected in the following way:

For any solution \( (e^{\omega}, H, Q) \) of the equations of Gauss and Codazzi (2.1), (2.2) with \( H, Q \neq 0 \) constants, by the substitution \( (e^{\omega}, H, Q) \mapsto (e^{\omega}, \bar{H}, \bar{Q}) = (e^{-\omega}, 2|Q|, \bar{H}) \) one obtains another solution of (2.1), (2.2). The connection form \( \hat{\alpha}_\lambda \) that corresponds to the latter solution is
\[
\hat{\alpha}_\lambda = \frac{1}{4} \left( \begin{array}{cc}
-\omega_z & 4|Q| e^{-\omega/2} \lambda^{-1} \\
-2H Q |Q| & e^{\omega/2}
\end{array} \right) \, dz + \frac{1}{4} \left( \begin{array}{cc}
\omega_{\bar{z}} & 2H Q |Q| e^{\omega/2} \\
4|Q| & e^{-\omega/2} \lambda^{-1}
\end{array} \right) \, d\bar{z} = -(g.\alpha_\lambda)^t ,
\]
where \( g.\alpha_\lambda \) denotes the gauge transformation of \( \alpha_\lambda \) with
\[
g = \left( \begin{array}{cc}
\frac{Q}{|Q|} & \lambda^{-1} \\
0 & \lambda^{1/2}
\end{array} \right) .
\]

For general \( g \) that can vary in both \( z \) and \( \lambda \), this gauge transformation is defined by
\[
g.\alpha = g^{-1} \alpha g + g^{-1} dg \quad \text{and} \quad g.F = F g ,
\]
so that the partial differential equation \( d(g.F) = (g.F)(g.\alpha) \) is maintained. In the specific situation here, \( g \) does not depend on \( z \) however, so simply \( g.\alpha = g^{-1} \alpha g \). Thus we see that the solution \( \hat{F} \) of \( d\hat{F} = \hat{F} \hat{\alpha} \), \( \hat{F}(z = 0) = 1 \) is \( \hat{F} = (g.F)^{1-g^{-1}} = F^{1-g^{-1}} \), where we additionally used that \( g^t = g \) holds. By inserting this equation into the Sym-Bobenko formula (2.5) and evaluating, we
see that the immersion \( \tilde{f} \) corresponding to the data \((e^\omega, \bar{H}, \bar{Q})\) is given by “the other variant” of the Sym-Bobenko formula

\[
\tilde{f} = -\frac{i\lambda}{\bar{H}} \frac{\partial \bar{F}}{\partial \lambda} F^{-1} = \frac{1}{\bar{H}} \left( F \frac{i}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} F^{-1} + i\lambda \frac{\partial F}{\partial \lambda} F^{-1} \right) .
\]

By comparison with Proposition 2.7 we see that \(-\frac{\bar{H}}{H} \tilde{f}^t = f - \frac{1}{4} \pi N\) holds, and hence up to the rotation-reflection \(X \mapsto -X^t\) of \(\mathbb{R}^3\) and the scaling factor \(\frac{\bar{H}}{H} = 2\frac{\omega(\bar{H})}{\omega(H)}\), \(\tilde{f}\) is the parallel surface of the cmc surface described by \(f\) that is also a cmc surface.

**Spectral data.** We now describe spectral data for this integrable system. For this purpose we suppose that the solution \(\omega\) of the Gauss equation (2.1) and hence the corresponding connection form \(\alpha\) is (at least) simply periodic, i.e. there exists a (minimal) period \(T \in \mathbb{C}^*\) such that \(\omega(z + T) = \omega(z)\) holds for all \(z\). Of course, in this situation, the extended frame \(F\) is not generally periodic, however its departure from being periodic is measured by the monodromy (with base point \(z_0\)) \(M_{\omega_0}(\lambda) = F_\lambda(\lambda_0)^{-1} \cdot F_\lambda(\lambda_0 + T)\). The dependence of the monodromy on the base point \(z_0\) is described by the differential equation \(dM = [M, \alpha]\), and consequently we have

\[
M_{\omega_1}(\lambda) = F_\lambda(z_1)^{-1} \cdot M_{\omega_0}(\lambda) \cdot F_\lambda(z_1) \quad \text{for any other base point } z_1 .
\]

It follows that the eigenvalues of \(M_{\omega_0}(\lambda)\) and the holomorphic function \(\Delta(\lambda) = \text{tr} M_{\omega_0}(\lambda)\) do not depend on the choice of the base point \(z_0\). We collect the eigenvalues \(\mu\) of \(M_{\omega_0}(\lambda)\) in the multiplier curve

\[
\Sigma = \{ (\lambda, \mu) \in \mathbb{C}^* \times \mathbb{C}^* \mid \mu^2 - \Delta(\lambda) \mu + 1 = 0 \} .
\]

The multiplier curve is a typically singular complex curve embedded in \(\mathbb{C}^* \times \mathbb{C}^*\) with infinite arithmetic genus. It is hyperelliptic over \(\mathbb{C}^*\) in the sense that the holomorphic map \(\Sigma \to \mathbb{C}^*, (\lambda, \mu) \mapsto \lambda\) is a branched, two-fold covering map, and the holomorphic involution \(\sigma: \Sigma \to \Sigma, (\lambda, \mu) \mapsto (\lambda, \mu^{-1})\) interchanges the two sheets of this covering map. The reality condition (2.4) implies

\[
M_{\omega_0}(\bar{\lambda}^{-1}) = \overline{M_{\omega_0}(\lambda)^t}^{-1} \quad \text{for } \lambda \in \mathbb{C}^*
\]

and therefore \(\Sigma\) also has an anti-holomorphic involution \(\rho: \Sigma \to \Sigma, (\lambda, \mu) \mapsto (\bar{\lambda}^{-1}, \bar{\mu}^{-1})\) which commutes with \(\sigma\).

For a fixed base point \(z_0\), the eigenvectors of \(M_{\omega_0}(\lambda)\) define a holomorphic line bundle \(\Lambda_{\omega_0}\) on the complex curve \(\Sigma\). If we write \(M_{\omega_0}(\lambda) = \begin{pmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda) \end{pmatrix}\) with the holomorphic functions \(a, b, c, d: \mathbb{C}^* \to \mathbb{C}\), then eigenvectors \((v_1, v_2)^t \in \mathbb{C}^2\) of \(M(\lambda)\) corresponding to the eigenvalue \(\mu\) are characterised by either of the two equivalent equations

\[
\begin{align*}
(a(\lambda) - \mu)v_1 + b(\lambda)v_2 &= 0 , \\
c(\lambda)v_1 + (d(\lambda) - \mu)v_2 &= 0 .
\end{align*}
\]

It follows that \((1, \frac{a(\lambda) - \mu}{b(\lambda)}, \frac{c(\lambda)}{d(\lambda)} - \mu, 1)\) are meromorphic sections of \(\Lambda_{\omega_0}\).

We now consider the case that the surface described by \(f\) resp. the solution \(\omega\) has finite type. This means that the \(\Lambda_{\omega_0}\)-halfway desingularisation \([K-L-S-S, \text{Section 4}]\) \(\Sigma_0\) of \(\Sigma\), i.e. the maximal one-sheeted, branched covering of \(\Sigma\) to which the generalised divisor corresponding to the line bundle \(\Lambda_{\omega_0}\) can be lifted, has finite arithmetic genus, meaning \(\dim H^1(\Sigma_0, \mathcal{O}) < \infty\). It was shown by Hitchin \([H]\) and independently by Pinkall/Sterling \([P-S]\) that any solution \(\omega\) of the sinh-Gordon equation (or of (2.1)) that is doubly periodic (i.e. has two \(\mathbb{R}\)-linear independent periods \(T_1, T_2\)) has finite type. This is the case in particular if \(f\) parameterises a cmc torus in \(\mathbb{R}^3\). The complex curve \(\Sigma_0\), which can still have singularities, is called the open spectral curve. By definition there exists a holomorphic line bundle \(\Lambda_{\omega_0}\) on \(\Sigma_0\) that projects to the line bundle \(\Lambda_{\omega_0}\) on \(\Sigma\). The objects on \(\Sigma\) induce a two-fold branched holomorphic covering \(\lambda: \Sigma_0 \to \mathbb{C}^*\), a holomorphic involution \(\sigma: \Sigma_0 \to \Sigma_0\) that interchanges the two sheets of the covering \(\lambda\) and an
anti-holomorphic involution $\rho : \Sigma^o \rightarrow \Sigma^o$ with $\sigma \circ \rho = \rho \circ \sigma$ and $\lambda \circ \rho = \lambda^{-1}$. Because $\Sigma^o$ has finite genus, it can be compactified at $\lambda = 0$ and $\lambda = \infty$ to give a compact complex curve $\Sigma$ that is hyperelliptic above $\mathbb{P}^1$ in the usual sense by the natural extension of the holomorphic involution $\sigma$. We call $\Sigma$ the spectral curve of $f$ or of $\omega$. An investigation of the asymptotic behaviour of the monodromy $M_{z_0}(\lambda)$ near $\lambda = 0$ and $\lambda = \infty$, see [Hi] Section 3 or [Kl] Sections 4, 5, shows that this compactification adds only a single point each at $\lambda = 0$ and $\lambda = \infty$ to $\Sigma^o$ and that these two added points are regular branch points of $\Sigma$. It follows that $\Sigma$ can be realised as a sub-variety in $\mathbb{C}^2$ as

$$\Sigma = \{(\lambda, \nu) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid \nu^2 = \lambda a(\lambda)\},$$

where $a(\lambda)$ is a polynomial in $\lambda$ of even degree $2g$ and $g = g(\Sigma)$ is the arithmetic genus of $\Sigma$. We have $a(0) \neq 0$ and we normalise $a(\lambda)$ such that $a(0) = -\frac{1}{2} HQ$ holds. Moreover, the reality condition on $\Sigma$ implies

$$a(\lambda) = \lambda^{2g} \cdot \overline{a(\lambda^{-1})}.$$ 

With respect to this realisation of $\Sigma$, the involutions $\sigma$ and $\rho$ are given by

$$\sigma : \Sigma \rightarrow \Sigma, \ (\lambda, \nu) \mapsto (\lambda, -\nu) \quad \text{and} \quad \rho : \Sigma \rightarrow \Sigma, \ (\lambda, \nu) \mapsto (\lambda^{-1}, \lambda^{-1} \nu).$$

**Polynomial Killing fields.** It was shown in [K-L-S-S] Section 4 that for every holomorphic function $\varphi$ on $\Sigma$ there exists one and only one holomorphic $(2 \times 2)$-matrix valued function $N(\lambda)$ in $\lambda$ such that $\Lambda_{z_0}$ is the eigenline bundle of $N$ and $\varphi$ is the corresponding eigenfunction, meaning that $N(\lambda)s = \varphi(\lambda)s$ holds for every holomorphic section $s$ of $\Lambda_{z_0}$. If we apply this statement to the original eigenfunction $\mu$ on the multiplier curve $\Sigma$, we recover the original monodromy $M_{z_0}$ which gave rise to $\Sigma$. But we now apply the statement to the anti-symmetric holomorphic function $\varphi = \frac{\xi}{\lambda}$ on $\Sigma$. The resulting $(2 \times 2)$-matrix-valued holomorphic function $\xi_{z_0} = \xi_{z_0}(\lambda)$, which satisfies

$$\xi_{z_0}(\lambda)s = \frac{\nu}{\lambda}s \quad \text{for every holomorphic section } s \text{ of } \Lambda_{z_0},$$

is called the polynomial Killing field for $\omega$ (at the base point $z_0$). Because the eigenfunction $\xi$ is anti-symmetric with respect to the hyperelliptic involution of $\Sigma$, $\xi_{z_0}(\lambda)$ is trace-free, i.e. $\xi_{z_0}$ maps into $\mathfrak{sl}(2, \mathbb{C})$. Because $\Sigma$ is a compact complex curve (after the compactification described above), $\xi_{z_0}$ is a polynomial in $\lambda$ and $\lambda^{-1}$, and the equation $\det(\xi_{z_0}) = \frac{\nu}{\lambda} \cdot \frac{u}{-\lambda} = -\frac{1}{\lambda} a(\lambda)$ which considered near $\lambda = 0$ shows that the lowest power of $\lambda$ that occurs in $\xi_{z_0}$ is $\lambda^{-1}$. A more precise investigation of the asymptotic behaviour of $M_{z_0}(\lambda)$ near $\lambda = 0$ in fact shows $\text{Res}_{\lambda = 0}(\xi_{z_0}) = \frac{1}{2} HQ e^{\omega(z_0)/2}$. Additionally the reality conditions imply the following reality condition for $\xi_{z_0}$:

$$\xi_{z_0}(\lambda) = -\lambda^{g-1} \xi_{z_0}(\lambda^{-1}).$$

Therefore the highest power of $\lambda$ that occurs in $\xi_{z_0}$ is $\lambda^g$. To summarise, $\xi_{z_0}$ is of the form

$$\xi_{z_0}(\lambda) = \sum_{k=-1}^{g} \xi_{z_0,k} \lambda^k \quad \text{with} \quad \xi_{z_0,k} = \begin{pmatrix} u_k & v_k \\ w_k & -u_k \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{C}) \quad \text{for} \quad k \in \{-1, \ldots, g\},$$

where

$$\xi_{z_0,-1} = \begin{pmatrix} 0 & \frac{1}{2} HQ e^{\omega(z_0)/2} \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \xi_{z_0,g-(k+1)} = -\bar{\xi}_{z_0,k} \quad \text{for} \quad k \in \{-1, \ldots, g\}.$$ 

Note that $\lambda \xi_{z_0}(\lambda)$ is a polynomial in $\lambda$ and $\lambda \xi_{z_0} s = \nu s$ holds for any holomorphic section of $\Lambda_{z_0}$, hence we have

$$-\lambda \det(\xi_{z_0}) = -\lambda^{-1} \det(\lambda \xi_{z_0}) = -\lambda^{-1} \nu (-\nu) = a(\lambda)$$

and thus for $\lambda = 0$: $v_{-1} \cdot w_0 = a(0) = -\frac{1}{2} HQ$. We thus obtain

$$v_{-1} = \frac{1}{2} HQ e^{\omega(z_0)/2} \quad \text{and} \quad w_0 = -Q e^{-\omega(z_0)/2}.$$
Concerning the dependence of the polynomial Killing field $\xi_{z_0}$ on the base point $z_0$, we note that for a different base point $z_1$ we have $\Lambda_{z_1} = F(z_1)^{-1} \Lambda_{z_0}$ due to Equation (2.10), and therefore

$$\xi_{z_1} = F(z_1)^{-1} \xi_{z_0} F(z_1).$$

(2.18)

Hence concerning differentiation with respect to the base point $z$, the family $\xi = \xi_z$ of polynomial Killing fields fulfills the differential equation

$$d \xi + [\alpha_\lambda, \xi] = 0.$$  

(2.19)

By decomposing this differential equation with respect to powers of $\lambda$ and entries of the $(2 \times 2)$-matrices, one sees that the polynomial Killing field $\xi$ can be reconstructed from the “initial condition” $v_{-1} = \frac{1}{2} H e^{\omega/2}$ by an iterative process. This process was introduced by Pinkall/Sterling in [P-S] in the course of their proof that cmc tori are of finite type, and is now called the Pinkall-Sterling iteration.

Proposition 2.3. Let a solution $(\omega, H, Q)$ of the Gauss equation (2.1) with a smooth real-valued function $\omega$ and constants $H > 0$, $Q \in \mathbb{C}^*$ be given. We suppose that this solution is of finite type $g$ and write the corresponding polynomial Killing field $\xi = \xi_z$ in the form

$$\xi = \sum_{k=-1}^{g} \xi_k \lambda^k$$

with

$$\xi_k = \begin{pmatrix} u_k & \tau_k e^{\omega/2} \\ \sigma_k e^{-\omega/2} & -u_k \end{pmatrix},$$

where $u_k, \tau_k, \sigma_k$ are smooth, complex-valued functions in $z$. Then we have $\tau_{-1} = \frac{1}{2} H$, $u_{-1} = \sigma_{-1} = 0$ and for every $k \geq 0$:

$$\tau_{k,z} = -\frac{1}{2Q} (u_{k,zz} - \omega_z u_{k,z})$$

(2.20)

$$\tau_{k,\bar{z}} = 2\bar{Q} e^{-\omega} u_k$$

(2.21)

$$u_{k+1} = \frac{1}{H} (\tau_{k,z} + \omega_z \tau_k)$$

(2.22)

$$\sigma_{k+1} = -\frac{1}{Q} (u_{k+1,\bar{z}} + \frac{1}{2} H e^{\omega} \tau_k).$$

Moreover every $u_k$ solves the linearisation of the Gauss equation (2.1):

$$2u_{k,\bar{z}} + (H^2 e^{\omega} - 4|Q|^2 e^{-\omega}) u_k = 0.$$  

(2.23)

Proof. We write

$$\alpha_\lambda = (U_{-1} \lambda^{-1} + U_0) dz + (V_0 + V_1 \lambda) d\bar{z}$$

with

$$U_{-1} = \frac{1}{2} \begin{pmatrix} 0 & He^{\omega/2} \\ 0 & 0 \end{pmatrix}, \quad U_0 = \frac{1}{4} \begin{pmatrix} \omega_z & 0 \\ -4Q e^{-\omega/2} & -\omega_z \end{pmatrix},$$

$$V_0 = \frac{1}{4} \begin{pmatrix} -\omega_{\bar{z}} & 4\bar{Q} e^{-\omega/2} \\ 0 & \omega_{\bar{z}} \end{pmatrix}$$

and

$$V_1 = \frac{1}{2} \begin{pmatrix} 0 & 0 \\ -He^{\omega/2} & 0 \end{pmatrix}. $$

We separate the differential equation (2.19) into its $dz$-part and its $d\bar{z}$-part, and also into the individual powers of $\lambda$ that occur. In this way we obtain the equations

$$\xi_{k,z} + [U_0, \xi_k] + [U_{-1}, \xi_{k+1}] = 0$$

$$\xi_{k,\bar{z}} + [V_1, \xi_{k-1}] + [V_0, \xi_k] = 0.$$
for all $k \in \{-1, \ldots, g\}$. By evaluating the brackets and separating the entries of the matrices, we obtain the following equations:

\begin{align}
(2.24) & \quad u_{k,z} + Q \tau_k + \frac{1}{2} H e^\omega \sigma_{k+1} = 0 \\
(2.25) & \quad u_{k,\bar{z}} + \frac{1}{2} H e^\omega \bar{\tau}_k + \bar{Q} \sigma_k = 0 \\
(2.26) & \quad \tau_{k,z} + \omega_z \tau_k - H u_{k+1} = 0 \\
(2.27) & \quad e^{\omega/2} \tau_{k,\bar{z}} - 2Q e^{-\omega/2} u_k = 0 \\
(2.28) & \quad e^{\omega/2} \sigma_{k,z} - 2Q e^{-\omega/2} u_k = 0 \\
(2.29) & \quad \sigma_{k,\bar{z}} + \omega_z \sigma_k - H u_{k-1} = 0.
\end{align}

The right-hand equation of (2.20) follows from Equation (2.27). Equation (2.21) follows from Equation (2.24), and Equation (2.22) follows from Equation (2.25).

From Equation (2.24) for $k - 1$ and Equation (2.25) for $k$ we obtain

\begin{equation}
-\frac{3}{2} H e^{-\omega}(u_{k-1,z} + Q \tau_{k-1}) = \sigma_k = -\frac{1}{Q} (u_{k,\bar{z}} + \frac{1}{2} H e^\omega \tau_{k-1})
\end{equation}

and therefore

\begin{equation}
\left( \frac{H}{Q} e^\omega - \frac{3Q}{2} e^{-\omega} \right) \tau_{k-1} - \frac{3}{2} H e^{-\omega} u_{k-1,z} + \frac{1}{Q} u_{k,\bar{z}} = 0.
\end{equation}

By substituting $u_{k-1}$ using Equation (2.27) and $u_k$ using Equation (2.26) we get

\begin{equation}
\left( \frac{H}{2Q} e^\omega - \frac{3Q}{4} e^{-\omega} \right) \tau_{k-1} - \frac{1}{Q} H e^{\omega} (e^{\omega} \tau_{k-1}, \bar{z})_z + \frac{1}{Q} H e^{\omega} (\tau_{k-1}, z + \omega z \tau_{k-1})_z = 0,
\end{equation}

thus

\begin{equation}
\left( \frac{1}{Q} H e^{\omega} - \frac{3Q}{4} e^{-\omega} \right) \tau_{k-1} = 0
\end{equation}

and hence again the Maurer-Cartan equation (2.1).

By differentiating the right hand half of (2.30) by $z$ and applying Equation (2.26) we moreover obtain

\begin{equation}
\sigma_{k,z} = -\frac{1}{Q} (u_{k,\bar{z}} + \frac{1}{2} H e^\omega (\tau_{k-1,z} + \omega z \tau_{k-1})) = -\frac{1}{Q} u_{k,\bar{z}} - \frac{H}{2Q} e^\omega u_k.
\end{equation}

On the other hand we have $\sigma_{k,z} = 2Q e^{-\omega} u_k$ by Equation (2.25). By equating these two presentations of $\sigma_{k,z}$ we obtain Equation (2.23) for $u_k$.

Separately, by differentiating Equation (2.24) with respect to $z$ we get

\begin{equation}
u_{k,z} + Q \tau_k z + \frac{1}{2} H e^\omega (\sigma_{k+1,z} + \omega z \sigma_{k+1}) = 0.
\end{equation}

Equation (2.24) implies

\begin{equation}
\frac{1}{2} H e^\omega \sigma_{k+1} = -u_{k,z} - Q \tau_k
\end{equation}

and Equations (2.28) and (2.26) imply

\begin{equation}
\sigma_{k+1,z} = 2Q e^{-\omega} u_{k+1} = \frac{3Q}{2} e^{-\omega} (\tau_{k,z} + \omega z \tau_k).
\end{equation}

By inserting Equations (2.32) and (2.33) into Equation (2.31), and solving for $\tau_{k,z}$, we finally obtain the equation on the left-hand side of (2.20).

According to Equation (2.23), the Taylor coefficients of the diagonal entries of the polynomial Killing field are solutions of the linearisation of the Gauss equation (2.1). Thus they can be interpreted as infinitesimal deformations in the space of conformal metrics at the metric given by $e^{\omega} \text{d} z \text{d} \bar{z}$. This is the reason why $\xi$ is called a polynomial “Killing field”.

The Pinkall-Sterling iteration permits to reconstruct the entries of the polynomial Killing field corresponding to a given conformal factor $\omega$ corresponding to a cmc immersion of finite type $g$ from the “initial condition” $\omega_{-1} = \frac{1}{2} H e^{\omega/2}$. Indeed, the Gauss equation (2.1) and its linearisation (2.24) imply that in each step, the integrability condition $(\tau_{k,z})_z = (\tau_{k,\bar{z}})_z$ for $\tau_k$ is satisfied, thus the system of partial differential equations (2.20) has a solution $\tau_k$.

This solution is only unique up
to an additive constant $C_k$. The condition that $\omega$ has finite type $g$ is equivalent to the property that the $C_k$ can be chosen in such a way that $u_0 = 0$ holds; then also $\tau_0 = 0$ holds.

As a corollary to the equations of the Pinkall-Sterling iteration, the lowest entries of the polynomial Killing field

$$\xi = \frac{1}{2} \begin{pmatrix} 0 & H e^{\omega/2} \\ 0 & 0 \end{pmatrix} \lambda^{-1} + \sum_{k=0}^g \begin{pmatrix} u_k & v_k \\ w_k & -u_k \end{pmatrix} \lambda^k$$

can be written down explicitly: We have $\tau_{-1} = \frac{1}{2} H \omega$. From Equation (2.21) we obtain $u_0 = \frac{1}{H} (\tau_{-1} + \omega_2 \tau_{-1}) = \frac{1}{2} \omega_2$, and then we obtain from Equation (2.22) and the Gauss equation (2.1) $\sigma_0 = -\frac{1}{Q} \left( \frac{1}{2} \omega_2 + \frac{1}{2} H^2 e^\omega \right) = -Q e^{-\omega} \left( \text{thereby recovering the right-hand half of (2.17)} \right)$. Equations (2.20) now give the following system of partial differential equations for $\tau_0$:

$$\tau_{0,z} = -\frac{1}{4Q} (\omega_{zzz} - \omega_z \omega_{zz} \omega_z) \quad \tau_{0,z} = -\bar{Q} e^{-\omega} \omega_z,$$

which is due to the Gauss equation (2.1) solved by $\tau_0 = -\frac{1}{4Q} (\omega_{zz} - \frac{1}{2} \omega^2_z) + C_0$ with a constant $C_0$.

Repeating one more step of the iteration gives

$$u_1 = -\frac{1}{4Q} \omega_{zzz} - \frac{1}{2} \omega^3_z - 4QC_0 \omega_z$$

$$\sigma_1 = -\frac{1}{4Q} \omega^2_z + 2 \omega_{zz} + 8QC_0 \omega_z$$

$$\tau_1 = \frac{1}{64Q} \omega_{zzzz} - \omega_{zzz} \omega_z + \frac{1}{2} \omega^2_{zz} - \frac{3}{2} \omega_{zz} \omega_z^2 + \frac{3}{8} \omega^4_z - 4C_0 (\omega_{zz} - \frac{1}{2} \omega^2_z) + C_1.$$

From these calculations and the reality condition (2.15) we see that the polynomial Killing field $\xi$ corresponding to the conformal factor $\omega$ has the form

$$(2.34) \quad \xi = \begin{pmatrix} 0 & \frac{1}{2} H e^{\omega/2} \\ 0 & 0 \end{pmatrix} \lambda^{-1} + \begin{pmatrix} \frac{1}{2} \omega_z & -e^{\omega/2} (\frac{1}{4Q} (\omega_{zz} - \frac{1}{2} \omega^2_z) - C_0) \\ -\frac{1}{2} \omega_z & -\frac{1}{2} \omega_z \end{pmatrix} \lambda^{-1} + \begin{pmatrix} -\frac{1}{2} \omega_z & -\frac{1}{2} \omega_z \\ 0 & 0 \end{pmatrix} \lambda^{-1} + \ldots$$

In the Pinkall-Sterling iteration, specific choices of integration constants for $\tau_0$ yield the polynomial Killing field $\xi$ of the conformal factor $\omega$. We may more broadly consider all solutions to (2.19) of the form

$$\xi = \sum_{k=-1}^d \xi_k \lambda^k \quad \text{with} \quad \xi_k = \begin{pmatrix} u_k \\ \sigma_k \end{pmatrix} \begin{pmatrix} e^{\omega/2} \\ -u_k \end{pmatrix},$$

where $u_k, \tau_k, \sigma_k$ are smooth, complex-valued functions of $z$ and $\tau_{-1} = \frac{1}{2} H, \sigma_{-1} = \sigma_{-1} = 0$. Each such expression is termed a polynomial Killing field of $\omega$. Not all choices of integration constants in the Pinkall-Sterling iteration process will yield solutions of finite degree but we restrict our attention to those which do.

**Proposition 2.4.** Assume that we are given as above a solution $(\omega, H, Q)$ of the Gauss equation (2.1) whose spectral curve has finite arithmetic genus $g$. Then the polynomial Killing field $\xi(\lambda)$ defined by (2.14) is the unique polynomial Killing field of $\omega$ of minimal degree.

**Proof.** We begin by showing that there is only one polynomial Killing field of minimal degree $d_0$. Suppose that $\xi^i, i = 1, 2$ are both polynomial Killing fields of $\omega$

$$\xi^i = \sum_{k=-1}^{d_0} \xi^i_k \lambda^k \quad \text{with} \quad \xi_k = \begin{pmatrix} u_k^i \\ \sigma_k^i \end{pmatrix} \begin{pmatrix} e^{\omega/2} \\ -u_k^i \end{pmatrix}.$$
By definition,
\[ \xi^1 - \xi^2 = \left( 0, \frac{1}{2} H e^{\omega/2}, 0 \right). \]
Consider the difference \( \xi^D = \xi^1 - \xi^2. \) If it is nonzero then there is a smallest \( l, \) \( 0 \leq l \leq d_0 \) for which \( \xi^D_l \neq 0. \) From Equations (2.20), (2.21) and (2.22) we see that \( u_l = \sigma_l = 0 \) whilst \( \tau_l \) is a nonzero constant. Then \( \lambda^{l-1} \frac{H}{2\pi} \xi^D \) is a polynomial Killing field \( \lambda^{l-1} \xi^D \) for \( \omega \) of degree strictly lower than \( d_0. \) Hence the difference \( \xi^D \) must vanish identically.

We proceed now to show that the minimal degree of a polynomial Killing field for \( \omega \) is indeed the spectral genus \( g. \) 

\[ \square \]

To sum up this discussion, we consider the space of cmc potentials of spectral genus \( g \)

\[ (2.35) \quad \mathcal{P}_g = \left\{ \zeta_\lambda = \sum_{k=1}^{g} \zeta_k \lambda_k \bigg| \zeta_k = -\frac{\zeta_{g-(k+1)}}{} \in \mathfrak{sl}(2, \mathbb{C}), \zeta_{-1} \in \{0 \in \mathbb{R}^+\}, \text{tr}(\zeta_{-1} \zeta_0) \neq 0 \right\}. \]

Then the polynomial Killing field \( \xi = \xi_z \) corresponding to the conformal factor \( \omega \) of a cmc immersion, seen as a function depending on the base point \( z, \) has values in \( \mathcal{P}_g \) and is a solution of the differential equation (2.19). On \( \mathcal{P}_g \) we moreover consider the 1-form-valued linear map (compare [H-K-S-1, Equation (3.3)])

\[ (2.36) \quad \mathcal{P}_g \to \Omega^1(\mathbb{C}) \otimes \mathfrak{sl}(2, \mathbb{C}), \zeta \mapsto \alpha_\lambda(\zeta) := \begin{pmatrix} \frac{1}{2} u_0 & v_1 \lambda^{-1} \\ w_0 & -\frac{1}{2} u_0 \end{pmatrix} \text{d}z - \begin{pmatrix} \frac{1}{2} u_0 & w_0 \\ v_0 & w_0 \end{pmatrix} \text{d}\zeta. \]

By comparison of Equation (2.34) with Equation (2.3) we see that \( \alpha_\lambda(\zeta) = \alpha_\lambda, \) where \( \alpha_\lambda(\zeta) \) is defined by Equation (2.36) and \( \alpha_\lambda \) is defined by Equation (2.3) with respect to the conformal factor \( \omega \) to which the polynomial Killing field \( \xi \) conforms. In particular, all cmc polynomial Killing fields \( \xi \) of spectral genus \( g \) are solutions of the differential equation

\[ (2.37) \quad d\xi + [\alpha_\lambda(\xi), \xi] = 0 \quad \text{on} \ \mathcal{P}_g, \]

and from such a solution, say \( \xi = \begin{pmatrix} 0 & v_1 \\ 0 & 0 \end{pmatrix} \lambda^{-1} + \ldots, \) the corresponding conformal factor \( \omega \) can be recovered by means of the equation \( \frac{1}{2} H e^{\omega/2} = v_1. \) In this sense the definition of the space of potentials (2.35) together with the linear map (2.35) characterises the integrable system of the Gauss equation for constant mean curvature immersions into \( \mathbb{R}^3 \) on the level of polynomial Killing fields.

**Symes’ method.** The polynomial Killing field encodes all information of the spectral curve \( \Sigma \) and the eigenline bundle \( \Lambda_{z_0} \) via the the characterisation that \( \Sigma \) is the complex curve defined by the characteristic equation of \( \lambda \xi_{z_0} \) and \( \Lambda_{z_0} \) is the eigenline bundle of \( \lambda \xi_{z_0} \) seen as a holomorphic line bundle on \( \Sigma. \) Therefore it is expected that the extended frame \( F_\lambda \) and hence (via the Sym-Bobenko formula (2.3)) the cmc immersion \( f_\lambda \) can be recovered from the polynomial Killing field \( \xi_{z_0} \) (at a fixed, arbitrarily chosen base point \( z_0 \)). The process by which this is accomplished is known as **Symes’ method** [5]. We give a description of this method below, which closely follows [H-K-S-1, Section 3]. For this purpose we need the Iwasawa decomposition of the loop group \( \Lambda(2, \mathbb{C}) : \)

**Proposition 2.5.** For \( 0 < r < \infty \) we denote the circle \( S^1_r = \partial B(0, r) = \{ \lambda \in \mathbb{C} \mid |z| = r \} \) and the annulus \( A_r = \{ \lambda \in \mathbb{C} \mid \min(r, r^{-1}) < |\lambda| < \max(r, r^{-1}) \}. \) We consider the loop groups

\[ \Lambda_r \mathbb{SL}(2, \mathbb{C}) = \{ \Phi : S^1_r \to \mathbb{SL}(2, \mathbb{C}) \text{ analytic} \} \]

\[ \Lambda_r \mathbb{SU}(2) = \{ F \in \Lambda_r \mathbb{SL}(2, \mathbb{C}) \mid F \text{ extends analytically to } A_r \text{ and } F[S^1_r] \subset \mathbb{SU}(2) \} \]

\[ \Lambda_r^+ \mathbb{SL}(2, \mathbb{C}) = \{ \Phi \in \Lambda_r \mathbb{SL}(2, \mathbb{C}) \mid \Phi \text{ extends analytically to } B(0, r) \} \]

\[ \Lambda_r^+ \mathbb{SL}(2, \mathbb{C}) = \{ \Phi \in \Lambda_r^+ \mathbb{SL}(2, \mathbb{C}) \mid |\Phi|_{\lambda=0} = \begin{pmatrix} \rho & c \\ 0 & \rho^{-1} \end{pmatrix} \text{ with some } \rho > 0 \text{ and } c \in \mathbb{C} \} \]
In the case \( r = 1 \) we omit the subscript \( r \) from the loop groups.

The map

\[
\text{iwasawa}_r : \Lambda_r \text{SU}(2) \times \Lambda_r^+ \mathbb{R}\text{SL}(2, \mathbb{C}) \rightarrow \Lambda_r \text{SL}(2, \mathbb{C}), \quad (F, B) \mapsto F \cdot B
\]

is a real analytic diffeomorphism onto \( \Lambda_r \text{SL}(2, \mathbb{C}) \). For given \( \Phi \in \Lambda_r \text{SL}(2, \mathbb{C}), \) \( (F, B) = \text{iwasawa}_r^{-1}(\Phi) \) is called the \( r \)-Iwasawa decomposition of \( \Phi \).

**Proposition 2.6.** Let a polynomial Killing field \( \zeta \in \mathcal{P}_g \) be given. We fix a base point \( z_0 \) and let \( \Phi_\lambda(z) = F_\lambda(z) \cdot B_\lambda(z) \) be the Iwasawa decomposition (Proposition 2.3) of \( \Phi_\lambda(z) = \exp((z - z_0)\zeta) \).

In this situation \( F_\lambda(z) \) extends holomorphically to \( \mathbb{C}^* \ni \lambda \), we have \( dF_\lambda = F_\lambda \alpha_\lambda \), and for \( \lambda_s \in S^1 \), the Sym-Bobenko formula (2.3) with this \( F_\lambda \) gives a conformal cmc immersion \( f \) into \( \mathbb{R}^3 \) with conformal metric \( e^{2\zeta} dz \, d\bar{z} \). Here \( \omega \) is the real-valued function characterised by \( \xi = (0, \frac{1}{2} \text{Re} e^{\zeta/2}) \lambda^{-1} + \ldots \) for the unique solution \( \xi : \mathbb{C} \rightarrow \mathcal{P}_g \) of Equation (2.37) with \( \xi_{z_0} = 0 \), and \( \alpha_\lambda \) is defined by Equation (2.3) with respect to this \( \omega \).

**Proof.** (Compare [H-K-S-1, Proposition 3.2].) It suffices to show \( F_\lambda^{-1} dF_\lambda = \alpha_\lambda(\xi) \). We have \( F_\lambda = \Phi_\lambda \cdot B_\lambda^{-1} \), so \( F_\lambda \) is obtained from \( \Phi_\lambda \) by a gauge transformation with the gauge \( B_\lambda^{-1} \). It follows that the corresponding connection form \( F_\lambda^{-1} dF_\lambda \) is obtained from \( \Phi_\lambda^{-1} d\Phi_\lambda = \zeta \, dz \) by the formula

\[
F_\lambda^{-1} dF_\lambda = B_\lambda \zeta \, B_\lambda^{-1} \, dz - dB_\lambda \cdot B_\lambda^{-1} = \xi_\lambda \, dz - dB_\lambda \cdot B_\lambda^{-1}.
\]

Here the second equals sign follows from Equation (2.18) and the fact that \( \Phi_\lambda = F_\lambda \cdot B_\lambda \) commutes with \( \zeta \). By the properties of the Iwasawa decomposition, \( B_\lambda \) extends holomorphically to \( \lambda = 0 \), thus the series expansion of the right hand side of Equation (2.38) in \( \lambda \) can only contain powers \( \lambda^k \) with \( k \geq -1 \). On the other hand, the reality condition (2.15) for \( \zeta \) implies that

\[
F_{\lambda^{-1}} = F_{\lambda^{-1}} \imath
\]

holds also in the present situation. Therefore also \( F_{\lambda^{-1}} \) only contains powers \( \lambda^k \) of \( \lambda \) with \( k \geq -1 \). Combining these two statements, we see that \( F_\lambda^{-1} dF_\lambda \) contains only the powers \( \lambda^k \) of \( \lambda \) with \( k \in \{-1, 0, 1\} \). We thus write

\[
F_\lambda^{-1} dF_\lambda = (A'_{-1} \lambda^{-1} + A'_{1} \lambda) \, dz + (A''_{-1} \lambda^{-1} + A''_{0} + A''_{1} \lambda) \, d\bar{z}
\]

with \( A'_{-1}, A'_{1} \in \mathfrak{sl}(2, \mathbb{C}) \).

Here the reality condition (2.39) implies

\[
A''_{-1} = -A'_{1}, \quad A''_{0} = -A'_{1}^c \text{ and } A''_{1} = -A'_{-1}^c.
\]

We now write

\[
B_{\lambda=0} = \begin{pmatrix} \rho & c \\ 0 & \rho^{-1} \end{pmatrix} \quad \text{and} \quad \xi_\lambda = \sum_{k=-1}^{g} \xi_k \lambda^k \quad \text{with} \quad \xi_{-1} = (0, v_{-1}) \quad \text{and} \quad \xi_k = (u_k, v_k, w_k, v_k) \quad \text{with smooth functions} \quad \rho : \mathbb{C} \rightarrow \mathbb{R}_+ \quad \text{and} \quad c, u_k, v_k, w_k : \mathbb{C} \rightarrow \mathbb{C}.
\]

It then follows from Equation (2.38) that

\[
A'_{-1} = \xi_{-1},
\]

\[
A'_{1} = 0,
\]

\[
A''_{0} = (\frac{4}{\rho^2} B_{\lambda=0}^{-1} B_{\lambda=0}^{-1} = \begin{pmatrix} u_0 - \rho^{-1} \rho_2 & v_0 - \rho c_2 + c \rho_2 \\ u_0 + \rho^{-1} \rho_2 & -u_0 + \rho^{-1} \rho_2 \end{pmatrix},
\]

\[
A''_{1} = -(\frac{4}{\rho^2} B_{\lambda=0}^{-1} B_{\lambda=0}^{-1} = \begin{pmatrix} -\rho - \rho c_2 + c \rho_2 \\ 0 & -\rho^{-1} \rho_2 \end{pmatrix}.
\]

By comparing the two representations of \( A''_{0} = -A''_{1}^c \) that are obtained from Equations (2.43) and (2.44) and using the fact that \( \rho \) is real-valued, we see that \( -(u_0 - \rho^{-1} \rho_2) = -\rho^{-1} \rho_2 \) and hence \( u_0 = 2 \rho^{-1} \rho_2 \) holds, that \( -u_0 = -\rho c_2 + c \rho_2 \) and hence \( w_0 = \rho \bar{c}_z - \bar{c} \rho_z \) holds, and that
Proposition 2.7. The polynomial \( a(\lambda) \) has exactly \( g \) zeros \( \lambda_1, \ldots, \lambda_g \in B(0,1) \setminus \{0\} \) (counted with multiplicity) inside the unit disk, which we number such that \( |\lambda_1| \leq \ldots \leq |\lambda_g| \). Then \( \lambda_1, \ldots, \lambda_g, \bar{\lambda}_1^{-1}, \ldots, \bar{\lambda}_g^{-1} \) are all zeros of \( a(\lambda) \), and thus we have

\[
a(\lambda) = -\frac{1}{2} HQ \prod_{k=1}^{g} (1 - \lambda_k^{-1}\lambda) \cdot (1 - \bar{\lambda}_k\lambda) .
\]

Proof. We have \( a(0) = -\frac{1}{2} HQ \neq 0 \), so \( \lambda = 0 \) is not a zero of \( a(\lambda) \). If some \( \lambda \in \mathbb{C}^* \) is a zero of \( a(\lambda) \), then \( \bar{\lambda}^{-1} \) also is a zero of \( a(\lambda) \) due to the reality condition (2.13). There are no zeros of \( a(\lambda) \) on \( S^1 \). It follows that out of the \( 2g \) many zeros of \( a(\lambda) \), exactly \( g \) are inside the unit disk, which we denote by \( \lambda_1, \ldots, \lambda_g \) as in the proposition, and then the other \( g \) zeros are \( \bar{\lambda}_1^{-1}, \ldots, \bar{\lambda}_g^{-1} \in \mathbb{C} \setminus \overline{B}(0,1) \). Now we note that the expression on the right hand side of Equation (2.46) is a polynomial of degree \( 2g \) in \( \lambda \) which has the same zeros as \( a(\lambda) \) and whose value at \( \lambda = 0 \) equals \( -\frac{1}{2} HQ = a(0) \). This proves Equation (2.46).

Similarly, the polynomial Killing field \( \xi_{z_0} \) and the values of \( \omega \) and \( \omega_z \) at \( z = z_0 \) can be reconstructed from the spectral divisor, i.e. the divisor on \( \Sigma \) that describes the spectral line bundle \( \Lambda_{z_0} \). This stems from the fact that if we write

\[
\xi_{z_0} = \begin{pmatrix} u(\lambda) & v(\lambda) \\ w(\lambda) & -u(\lambda) \end{pmatrix},
\]

where \( u(\lambda), \lambda v(\lambda) \) and \( w(\lambda) \) are polynomials of degree \( g \), then \( \psi = \frac{u - \lambda u(\lambda)}{\lambda v(\lambda)} \) is a meromorphic function on \( \Sigma \) such that \( \xi_{z_0} \left( \begin{pmatrix} 1 \\ \psi \end{pmatrix} \right) = \xi \left( \begin{pmatrix} 1 \\ \psi \end{pmatrix} \right) \) holds, thus \( (1, \psi) \) is a section of \( \Lambda_{z_0} \) without zeros. Hence a representative divisor for the line bundle \( \Lambda_{z_0} \) is given by the polar divisor of \( \psi \). A pole of \( \psi \) can only occur at points where \( \lambda v(\lambda) = 0 \). At such points, we have \( \lambda u(\lambda) = \pm \nu \) due to Equation (2.16), and hence a pole of \( \psi \) occurs at some \( (\lambda, \nu) \in \Sigma \) if and only if \( \lambda u(\lambda) = -\nu \) and \( \lambda v(\lambda) = 0 \) holds. It follows that the support of the spectral divisor consists of those points \( (\beta_k, \nu_k) \in \Sigma \) where \( (\lambda v(\lambda) = 0 \) and \( \nu_k = -\beta_k u(\beta_k) \) holds. Because \( \lambda v \) is a polynomial of degree \( g \), there are exactly \( g \) such points. Again it is useful to make the reconstruction process explicit.
Proposition 2.8. Let \((\beta_k, \nu_k) \in \Sigma\) be the points in the support of the spectral divisor, where \(k = 1, \ldots, g\). Then we have:

\begin{align}
(2.48) & \quad e^{\omega(z_0)} = 2(-1)^g \frac{\bar{Q}}{H} \prod_{k=1}^g \beta_k = \frac{2|Q|}{H} \prod_{k=1}^g |\beta_k| \quad \text{where} \quad (-1)^g \frac{\bar{Q}}{H} \prod_{k=1}^g \beta_k \in \mathbb{R}_+
\end{align}

\begin{align}
(2.49) & \quad \omega_z(z_0)^2 = 8HQ \sum_{k=1}^g (\lambda_k^{-1} - \beta_k^{-1} + \bar{\lambda}_k - \bar{\beta}_k)
\end{align}

\begin{align}
(2.50) & \quad u(\lambda) = -\sum_{k=1}^g \beta_k^{-1} \nu_k \chi_k(\lambda)
\end{align}

\begin{align}
(2.51) & \quad \chi_k(\lambda) = \prod_{k' \neq k} \frac{\lambda - \beta_{k'}}{\beta_k - \beta_{k'}} = -\frac{2}{H} e^{-\omega(z_0)/2} \frac{\lambda v(\lambda)}{\lambda - \beta_k}
\end{align}

\begin{align}
(2.52) & \quad v(\lambda) = \frac{1}{2} He^{\omega(z_0)/2} \lambda^{-1} \left( \prod_{k=1}^g (1 - \beta_k^{-1} \lambda) \right)
\end{align}

\begin{align}
(2.53) & \quad w(\lambda) = -Qe^{-\omega(z_0)/2} \prod_{k=1}^g (1 - \bar{\beta}_k \lambda).
\end{align}

Proof. It was discussed above the proposition that the \(\beta_1, \ldots, \beta_g\) are the zeros of the polynomial \(\lambda v(\lambda)\) of degree \(g\). Because of \(\lambda v(\lambda)|_{\lambda=0} = v_{-1} \neq 0\) they are all in \(\mathbb{C}^*\). Moreover, the zeros of the polynomial \(w(\lambda)\) with \(w(0) = w_0\) are \(\bar{\beta}_1, \ldots, \bar{\beta}_g\) due to the reality condition (2.15). Thus we have

\begin{align}
(2.54) & \quad \lambda v(\lambda) = v_{-1} \prod_{k=1}^g (1 - \beta_k^{-1} \lambda) \quad \text{and} \quad w(\lambda) = w_0 \prod_{k=1}^g (1 - \bar{\beta}_k \lambda).
\end{align}

By inserting (2.17) into these equations, Equations (2.52) and (2.53) follow. On the other hand, it also follows from the reality condition (2.15) together with the product formula for \(v(\lambda)\) in (2.51) that

\begin{align*}
w(\lambda) & = -\lambda^g (\lambda v(\lambda))^{-1} = -\lambda^g \bar{v}_{-1} \prod_{k=1}^g (1 - \bar{\beta}_k^{-1} \lambda^{-1})
\end{align*}

\begin{align*}
& = (-1)^{g+1} \bar{v}_{-1} \prod_{k=1}^g \bar{\beta}_k^{-1} \cdot \prod_{k=1}^g (1 - \bar{\beta}_k \lambda) .
\end{align*}

By comparison with the product formula for \(w(\lambda)\) in (2.54) we see that

\begin{align*}
w_0 & = (-1)^{g+1} \bar{v}_{-1} \prod_{k=1}^g \bar{\beta}_k^{-1} \quad \text{and thus} \quad \frac{\bar{v}_{-1}}{w_0} = (-1)^{g+1} \prod_{k=1}^g \bar{\beta}_k
\end{align*}

holds. Due to Equation (2.17) we have \(\frac{\bar{v}_{-1}}{w_0} = -\frac{H}{2Q} e^{\omega(z_0)}\) and hence

\begin{align*}
e^{\omega(z_0)} & = (-1)^g \frac{2Q}{H} \prod_{k=1}^g \bar{\beta}_k.
\end{align*}

Because of \(e^{\omega(z_0)} > 0\) we deduce \((-1)^g Q \prod_{k=1}^g \bar{\beta}_k \in \mathbb{R}_+\), in particular \(Q \prod_{k=1}^g \bar{\beta}_k = \bar{Q} \prod_{k=1}^g \beta_k\). This proves Equation (2.48).

The polynomial \(u(\lambda)\) has degree \(g - 1\) (indeed \(u_g = -\bar{u}_{-1} = 0\) by the reality condition (2.15)), and is therefore uniquely determined by the \(g\)-many equations \(u(\beta_k) = -\beta_k^{-1} \nu_k\). It is therefore given by Equation (2.50), where we define \(\chi_k(\lambda)\) as the unique polynomial of degree \(g - 1\) with
\( \chi_k(\beta_{k'}) = 0 \) for \( k' \neq k \) and \( \chi_k(\beta_k) = 1 \). Then the first equals sign in (2.51) is obvious. Due to the fact that \( \lambda v(\lambda) \) is a polynomial of degree \( g \) whose zeros are exactly \( \beta_1, \ldots, \beta_g \), we also have

\[
\chi_k(\lambda) = \frac{\lambda v(\lambda)}{v(\beta_k) \cdot (\lambda - \beta_k)}.
\]

By Equation (2.52) we have

\[
(\lambda v)'(\beta_k) = -\frac{1}{2} He^{\omega(z_0)/2} \prod_{k' \neq k} (1 - \beta^{-1}_{k'} \beta_k),
\]

and this implies the second equals sign in (2.51).

Finally, by Equation (2.16) we have

\[
a(\lambda) = -\lambda \det(\xi_{z_0}) = \lambda u(\lambda)^2 + (\lambda v(\lambda)) \cdot w(\lambda),
\]

therefore

\[
a'(\lambda) = u(\lambda)^2 + 2\lambda u(\lambda) u'(\lambda) + (\lambda v(\lambda)) \cdot w(\lambda) + (\lambda v(\lambda)) \cdot w'(\lambda)
\]

and hence

(2.55)

\[
a'(0) = u_0^2 + v_0 w_0 + v_1 w_1.
\]

On the other hand, from Equations (2.46), (2.52) and (2.53) we obtain

\[
a'(0) = \frac{1}{2} HQ \sum_{k=1}^{g} (\lambda_{-1}^{-1} + \bar{\lambda}_k)
\]

\[
v_{-1} = \frac{1}{2} He^{\omega(z_0)/2}
\]

\[
v_0 = -\frac{1}{2} He^{\omega(z_0)/2} \sum_{k=1}^{g} \beta_{-1}^{-1}
\]

\[
w_0 = -Qe^{-\omega(z_0)/2}
\]

\[
w_1 = Qe^{-\omega(z_0)/2} \sum_{k=1}^{g} \bar{\beta}_k.
\]

Thus we obtain from Equation (2.55)

\[
u_0^2 = a'(0) - v_0 w_0 - v_1 w_1 = \frac{1}{2} HQ \sum_{k=1}^{g} (\lambda_{-1}^{-1} - \beta_{-1}^{-1} + \bar{\lambda}_k - \bar{\beta}_k).
\]

Due to \( u_0 = \frac{1}{4} \omega_z(z_0) \), Equation (2.49) follows.

**The isospectral set.** Let a polynomial \( a(\lambda) \) of degree \( 2g \) that satisfies \( a(0) = -\frac{1}{2} HQ \) and the reality condition (2.13) be given. Then \( a(\lambda) \) defines a spectral curve \( \Sigma \) by means of Equation (2.12).

We can ask about the set of polynomial Killing fields \( \zeta \in \mathcal{P}_g \) that correspond to the spectral curve \( \Sigma \); because of Equation (2.16) this condition is equivalent to the equation \(-\lambda \det(\zeta) = a(\lambda)\). The set of \( \zeta \in \mathcal{P}_g \) that solve this equation is called the isospectral set

\[
I(a) = \{ \zeta \in \mathcal{P}_g \mid -\lambda \det(\zeta) = a(\lambda) \}.
\]

It is known that \( I(a) \) is a compact and connected, real-\( g \)-dimensional variety. There exists an action of \( \mathbb{R}^g \) on \( I(a) \) which is called the isospectral flow. It is infinitesimally given by the vector fields on \( I(a) \) described by

(2.56)

\[
\frac{\partial \zeta}{\partial t} = \left[(\zeta \lambda^n)_+, \zeta\right] = -\left[(\zeta \lambda^n)_-, \zeta\right],
\]

see [H-K-S-1, Section 4].
Replacing $\xi_{x_0}$ by $\zeta \in \mathcal{P}_g$, we define $u(\lambda), v(\lambda), w(\lambda)$ by Equation (2.37), the number $\omega(\zeta) \in \mathbb{R}$ by $v_{-1} = \frac{1}{2}He^{\omega(\zeta)}$, and the $\lambda_k, \beta_k, \nu_k$ characterising the associated spectral curve and spectral divisor as before. Then Propositions 2.7 and 2.8 (except for Equation (2.49)) hold in the present situation.

The following proposition describes estimates for $e^{+\omega(\zeta)}$ and for the $\beta_k$. Via Proposition 2.8 these estimates can be used to estimate the coefficients of polynomial Killing fields $\zeta \in I(a)$. This will be very important for the construction of our blow-ups. For the following proposition we assume that the zeros $\lambda_k$ of $a(\lambda)$ are ordered such that

$$0 < |\lambda_1| \leq \ldots \leq |\lambda_g| < 1$$

holds. Moreover we define $\rho(\lambda) = \min\{|\lambda|, |\lambda|^{-1}\} \in (0, 1)$ for any $\lambda \in \mathbb{C}^*$ and assume that the zeros $\beta_1, \ldots, \beta_g$ of $\lambda v(\lambda)$ are ordered such that

$$0 < \rho(\beta_1) \leq \ldots \leq \rho(\beta_g) < 1$$

holds.

**Proposition 2.9.** In the situation described above, we have for any $\zeta \in I(a)$:

1. $|\lambda_k| \leq |\beta_k| \leq |\lambda_k|^{-1}$ for all $k \in \{1, \ldots, g\}$.
2. $\frac{|Q|}{|\lambda_k|} \prod_{k=1}^g |\lambda_k| \leq e^{\omega(\zeta)} \leq \frac{|Q|}{\omega(\zeta)} \prod_{k=1}^g |\lambda_k|^{-1}$.

In this estimate, the lower resp. the upper bound for $e^{-\omega(\zeta)}$ is attained when $u(\zeta) = 0$ and $\beta_k = \lambda_k$ resp. $\beta_k = \lambda_k^{-1}$ holds for all $k \in \{1, \ldots, g\}$.

**Proof.** Let $\beta$ be a continuous function on $I(a)$ such that $(\lambda v)(\beta) = 0$ holds. The continuous, real-valued function $|\beta|$ is defined on the compact variety $I(a)$ and therefore attains its maximum and its minimum. These extremal points are critical points of the infinitesimal isospectral flow (2.56). Now assume that for a critical point $\zeta \in I(a)$ of the isospectral flow, $u(\lambda) \neq 0$ would hold. Then there would exist some $n > 0$ such that the upper triangular part of $(\zeta \lambda^n)_+$ does not vanish. Therefore the action of the infinitesimal isospectral group on $\zeta$ can change the value of $\lambda v$ at $\beta$ in arbitrary direction. This is a contradiction to $|\beta|$ being extremal at $\zeta$. Thus we have $u(\lambda) = 0$, i.e. $\zeta$ is off-diagonal. For such $\zeta \in I(a)$ we have

$$a(\lambda) = -\Lambda \det(\zeta) = \lambda v(\lambda) \cdot w(\lambda).$$

This shows that for extremal $\zeta \in I(a)$, the set $\{\lambda_k, \lambda_k^{-1} \mid 1 \leq k \leq g\}$ of zeros of $a(\lambda)$ is equal to the set $\{\beta_k, \beta_k^{-1} \mid 1 \leq k \leq g\}$ of zeros of $\lambda v(\lambda) \cdot w(\lambda)$. Because of our ordering (2.57) of the $\lambda_k$ and (2.58) of the $\beta_k$, we in fact have $\{\lambda_k, \lambda_k^{-1}\} = \{\beta_k, \beta_k^{-1}\}$ for all $k$ (after possibly rearranging roots of the same absolute value). In particular we have either $|\beta_k| = |\lambda_k|$ (then $|\beta_k|$ attains a minimum here) or $|\beta_k| = |\lambda_k|^{-1}$ (then $|\beta_k|$ attains a maximum). This proves (1). The estimate (2) follows by simply applying (1) to Equation (2.48). \qed

**3. Polynomial Killing fields and Symes’ method for minimal surface immersions**

We now develop the analogue of Polynomial Killing fields and Symes’ method for a different integrable system, which is associated to minimal surface immersions into $\mathbb{R}^3$. As we will see in Section 4, it will occur as the blow-up of a sequence of solutions of the sinh-Gordon equation under certain circumstances.

**Minimal immersions into the 3-space.** Again let $f : X \to \mathbb{R}^3$ with $X \subset \mathbb{C}$ be a conformal surface immersion into $\mathbb{R}^3$. In relation to $f$ we again use the notations introduced at the beginning of Section 2 but now assume that $f$ is minimal, meaning that its mean curvature $H$ vanishes. We again consider $f$ near a non-umbilical point and assume that the coordinate $z$ on $X$ is chosen such that the function $Q$ describing the Hopf differential $Q \, dz^2$ is constant and non-zero. In this
setting the Codazzi equation (2.2) again reduces to \(0 = 0\), whereas the Gauss equation (2.1) means that the negative \(-\omega\) of the conformal factor of \(f\) is a solution of Liouville’s equation
\[
-\omega_{z\bar{z}} = -2|Q|^2 e^{-\omega}.
\]

To define an extended frame for \(f\), we again consider a family \(\alpha_\lambda\) of connection forms with respect to a spectral parameter \(\lambda \in \mathbb{C}\) (unlike in the cmc case, \(\lambda = 0\) is now permitted)
\[
\alpha = \alpha_\lambda = \frac{1}{4} \begin{pmatrix}
\omega_z \\
4Q e^{-\omega/2}
\end{pmatrix} dz + \frac{1}{4} \begin{pmatrix}
-\omega_z \\
\omega_z
\end{pmatrix} d\bar{z}.
\]

Note that in comparison to the cmc connection form (2.3), \(\lambda^{-1}\) has been replaced by \(\lambda\). This has been done so that \(\alpha_\lambda\) can be defined for \(\lambda \in \mathbb{C}\) (rather than \(\lambda \in \mathbb{C}^* \cup \{\infty\}\)) and also so that the Sym point (see below) is at \(\lambda_s = 0\) (rather than \(\lambda_s = \infty\), which would complicate the Sym-Bobenko formula). Again one checks that Liouville’s equation (3.1) is equivalent to the Maurer-Cartan equation for \(\alpha_\lambda\). Thus the initial value problem
\[
dF_\lambda = F_\lambda \alpha_\lambda \quad \text{with} \quad F_\lambda(z_0) = 1
\]
has for every \(\lambda \in \mathbb{C}\) a unique solution \(F_\lambda : X \to \text{SL}(2, \mathbb{C})\), called the extended frame of \(f\). Again, \(F_\lambda\) depends holomorphically on \(\lambda\), and due to \(\alpha_{\lambda=0} = \text{su}(2)\), \(F_{\lambda=0}\) maps into \(\text{SU}(2)\). Because this reality condition holds only for a discrete subset of \(\lambda \in \mathbb{C}\), there is no formula that is analogous to the right-hand side equation of (2.4) in the present situation.

However there is a variant of the Sym-Bobenko formula in this setting, which is analogous to the Weierstrass representation for minimal surfaces in \(\mathbb{R}^3\), compare [B, Section 3]. When one compares this variant of the Sym-Bobenko formula to the version for cmc immersions (Equation (2.5)), one notes that the factor \(\lambda \frac{\partial F}{\partial \alpha}\) in the expression for the immersion \(f\) is replaced by an \(\text{SL}(2, \mathbb{C})\)-valued function \(G = G(z, \lambda)\) that solves the inhomogeneous linear differential equation \(dG = G\alpha + F\beta\) with a \(\beta\) that is different from \(\frac{\partial \alpha}{\partial \alpha}\).

**Proposition 3.1.** Let a real-valued, smooth function \(\omega\) and a constant \(Q \in \mathbb{C}^*\) be given so that these data satisfy Liouville’s equation (3.1). Moreover let \(\varphi \in \mathbb{R}\) and consider the \(\text{sl}(2, \mathbb{C})\)-valued connection form
\[
\beta_\lambda = e^{\omega/2} \frac{1}{2} \begin{pmatrix}
0 & e^{-i\varphi} \, dz \\
\omega/2 & 0
\end{pmatrix}
\]
and a solution \(G = G(z, \lambda)\) of the differential equation \(dG = G\alpha + F\beta\) with \(G(z_0, \lambda = 0) \in \text{su}(2)\). Then
\[
f = G F^{-1} \bigg|_{\lambda = 0}
\]
is a minimal immersion \(X \to \text{su}(2) \cong \mathbb{R}^3\) with induced metric \(e^\omega \, dz \, d\bar{z}\) and Hopf differential \(e^{-i\varphi} Q \, dz^2\). The tangential directions \(f_x, f_y\) of \(f\) and the unit normal field \(N\) of \(f\) are again given by Equation (2.6) with \(\lambda_s = e^{i\varphi}\) and Equation (2.7) respectively.

**Proof.** The proof is similar to the proof of Proposition 2.1. We again have
\[
df = dG \cdot F^{-1} - G F^{-1} dF F^{-1} = G \alpha F^{-1} + F \beta F^{-1} - G F^{-1} F \alpha F^{-1} = F \beta F^{-1}.
\]

For \(\lambda = 0\) we have \(F(z, \lambda = 0) \in \text{SU}(2)\) and \(\beta_{\lambda=0} \in \text{su}(2)\), hence \(df\) maps into \(\text{su}(2)\). Because we also have \(f(z_0) = G(z_0, \lambda = 0) \in \text{su}(2)\), it follows that \(f\) indeed maps into \(\text{su}(2) \cong \mathbb{R}^3\). The statements on the tangential and normal directions of \(f\) also follow from the formula \(df = F \beta F^{-1}\). Moreover we have
\[
f_x = e^{\omega/2} e^{-i\varphi} F_{1/2} \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) F^{-1} \quad \text{and} \quad f_z = e^{\omega/2} e^{i\varphi} F_{1/2} \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) F^{-1},
\]
and therefore
\[
2(f_x, f_z) = 2 e^\omega : \left( \frac{i}{2} \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right), \frac{i}{2} \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \right) = e^\omega,
\]
whence it follows that $f$ is a conformal immersion with the induced metric $e^\omega \, dz \, d\bar{z}$. We moreover obtain from Equations (3.4) and (3.2):

$$f_{zz} = \frac{1}{2} \omega_z f_z + [FUF^{-1}, f_z] = \omega_z f_z + e^{-4\varphi} Q N$$
$$f_{z\bar{z}} = \frac{1}{2} \omega_{\bar{z}} f_z + [FVF^{-1}, f_z] = 0$$
$$f_{\bar{z}\bar{z}} = \frac{1}{2} \omega_{\bar{z}} f_{\bar{z}} + [FVF^{-1}, f_{\bar{z}}] = \omega_{\bar{z}} f_{\bar{z}} + e^{4\varphi} \bar{Q} N.$$ 

Thus the Hopf differential of $f$ is given by $(f_{zz}, N) dz^2 = e^{-4\varphi} Q \, dz^2$ and the mean curvature of $f$ vanishes.

The proof is analogous to the one of Proposition 2.3. We write

$$\alpha = (U_0 + U_1 \lambda) \, dz + V_0 \, d\bar{z}$$

where $U_0 = \frac{1}{4} \begin{pmatrix} \omega_z & 0 \\ -4Q e^{-\omega/2} & -\omega_z \end{pmatrix}$, $U_1 = \frac{1}{4} \begin{pmatrix} 0 & e^{\omega/2} \\ 0 & 0 \end{pmatrix}$ and $V_0 = \frac{1}{4} \begin{pmatrix} -\omega_z & 4\bar{Q} e^{-\omega/2} \\ 0 & \omega_{\bar{z}} \end{pmatrix}$.

**Polynomial Killing fields for minimal immersions.** Analogously as for the cmc case, we consider polynomial Killing fields for the integrable system of minimal surface immersions of finite type $g \in \mathbb{N}$. For this integrable system, the space of potentials is

$$\mathcal{P}_{g}^{KdV} = \left\{ \zeta_\lambda = \sum_{k=1}^{-g} \zeta_k \lambda^k \middle| \zeta_k \in \mathfrak{sl}(2, \mathbb{C}), \zeta_1 \in \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}, \text{tr}(\zeta_1 \zeta_0) \neq 0 \right\}.$$ 

By analogy with the situation for the sinh-Gordon integrable system, we call the members of $\mathcal{P}_{g}^{KdV}$ (KdV-)polynomial Killing fields. However note that in contrast to that case, these polynomial Killing fields minus their “initial term” $\zeta_1 \lambda$ are polynomials in $\lambda^{-1}$, not in $\lambda$. This corresponds to the substitution of $\lambda$ with $\lambda^{-1}$ in the expression for $\alpha_\lambda$ in Equation (3.2). Like in the cmc case however, the dependence of the polynomial Killing field $\xi = \xi_{z_0}$ for a solution $\omega$ of Liouville’s equation (3.1) on the base point $z_0$ is described by the differential equation

$$d\xi + [\alpha_\lambda, \xi] = 0,$$

where $\alpha_\lambda$ is now the connection form given by Equation (3.2), and we impose the “initial condition” $v_1 = \frac{1}{2} e^{\omega/2}$ for $\xi = \left( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) \lambda + \ldots$. The differential equation (3.5) can be expressed as an iterative algorithm for the computation of the coefficients of the polynomial Killing field $\xi$ which is analogous to the Pinkall-Sterling iteration in the cmc case:

**Proposition 3.2.** Let a solution $(\omega, Q)$ of Liouville’s equation (3.1) with a smooth real-valued function $\omega$ and a constant $Q \in \mathbb{C}^*$ be given. We suppose that this solution is of finite type $g$ and write the corresponding polynomial Killing field $\xi = \xi_z$ in the form

$$\xi = \sum_{k=1}^{-g} \xi_k \lambda^k \quad \text{with} \quad \xi_k = \begin{pmatrix} u_k \\ \sigma_k \frac{e^{\omega/2}}{\tau_k} \end{pmatrix},$$

where $u_k, \tau_k, \sigma_k$ are smooth, complex-valued functions in $z$. Then we have $\tau_1 = \frac{1}{4}$, $u_1 = \sigma_1 = 0$ and for every $k = 0, \ldots, -g$:

$$\tau_{k,z} = -\frac{1}{2} \frac{1}{Q}(u_{k,zz} - \omega_z u_{k,z}),$$
$$\tau_{k,\bar{z}} = 2 \bar{Q} e^{-\omega} u_k,$$
$$u_{k-1} = 2(\tau_{k,z} + \omega_z \tau_k),$$
$$\sigma_{k-1} = -\frac{1}{Q} u_{k-1,\bar{z}}.$$ 

Moreover every $u_k$ solves the linearisation of Liouville’s equation (3.1):

$$u_{k,zz} + 2|Q|^2 e^{-\omega} u_k = 0.$$ 

**Proof.** The proof is analogous to the one of Proposition 2.3. We write $\alpha_\lambda = (U_0 + U_1 \lambda)dz + V_0 \, d\bar{z}$ with

$$U_0 = \frac{1}{4} \begin{pmatrix} \omega_z & 0 \\ -4Q e^{-\omega/2} & -\omega_z \end{pmatrix}, \quad U_1 = \frac{1}{4} \begin{pmatrix} 0 & e^{\omega/2} \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad V_0 = \frac{1}{4} \begin{pmatrix} -\omega_z & 4\bar{Q} e^{-\omega/2} \\ 0 & \omega_{\bar{z}} \end{pmatrix}.$$
We separate the differential equation \( (2.19) \) into its \( dz \)-part and its \( d\bar{z} \)-part, and also into the individual powers of \( \lambda \) that occur. In this way we obtain the equations
\[
\xi_{k,z} + [U_0, \xi_k] + [U_1, \xi_{k-1}] = 0 \\
\xi_{k,\bar{z}} + [V_0, \xi_k] = 0
\]
for all \( k \in \{1, \ldots, -g\} \). By evaluating the brackets and separating the entries of the matrices, we obtain the following equations:
\[
\begin{align*}
&\xi_{k,z} + Q\tau_k + \frac{1}{4} e^{\omega} \sigma_{k-1} = 0 \\
&\xi_{k,\bar{z}} + Q\sigma_k = 0 \\
&\tau_{k,z} + \omega_z \tau_k - \frac{1}{2} u_{k-1} = 0 \\
&e^{\omega/2} \tau_{k,\bar{z}} - 2Qe^{-\omega/2} u_k = 0 \\
&e^{\omega/2} \sigma_{k,z} - 2Qe^{-\omega/2} u_k = 0 \\
&\sigma_{k,\bar{z}} + \omega_z \sigma_k = 0.
\end{align*}
\]

The right-hand equation of (3.6) follows from Equation (3.13), Equation (3.7) follows from Equation (3.12), and Equation (3.8) follows from Equation (3.11).

By differentiating (3.11) by \( z \), and by (3.14), we get two different expressions for \( \sigma_{k,z} \):
\[
\frac{1}{Q} u_{k,\bar{z}} = \sigma_{k,z} = 2Q e^{-\omega} u_k
\]
and this equation implies (3.9).

By differentiating Equation (3.10) with respect to \( z \) we get
\[
\xi_{k,zz} + Q\tau_{k,z} + \frac{1}{4} e^{\omega}(\sigma_{k-1,z} + \omega_z \sigma_{k-1}) = 0.
\]
Equation (3.10) implies
\[
\frac{1}{4} e^{\omega} \sigma_{k-1} = -u_{k,z} - Q\tau_k
\]
and Equations (3.14) (for \( k - 1 \)) and (3.12) imply
\[
\sigma_{k-1,z} = 2Q e^{-\omega} u_{k-1} = 4Q e^{-\omega}(\tau_{k,z} + \omega_z \tau_k).
\]

By inserting Equations (3.17) and (3.18) into Equation (3.16), and solving for \( \tau_{k,z} \), we obtain the equation on the left-hand side of (3.6).

The iteration of Proposition 3.2 again permits to write down the lowest terms of the polynomial Killing field \( \xi \) corresponding to some solution \( \omega \) of Liouville’s equation explicitly in terms of \( \omega \) and its derivatives. Again there is an integration constant \( C_k \) associated to every \( \tau_k \), and one starts with the largest value for \( k \) and then uses Equations (3.6) - (3.8) to work downwards. In this way, one finds
\[
\xi = \left( \begin{array}{cc}
0 & \frac{1}{4} e^{\omega/2} \\
\frac{1}{2} \omega_z & -Qe^{-\omega/2}
\end{array} \right) \lambda + \left( \begin{array}{cc}
-\frac{1}{2} \omega_z & -e^{\omega/2}(\frac{1}{4}(\omega_{zz} - \frac{1}{2} \omega_z^2) - C_0) \\
-Qe^{-\omega/2} & -\frac{1}{2} \omega_z
\end{array} \right) + \ldots.
\]

By comparison with (3.2) we see that for the polynomial Killing field \( \xi \) corresponding to some solution \( \omega \) of Liouville’s equation (3.1) we have \( \alpha_\lambda = \alpha_\lambda^{KdV}(\xi) \), where \( \alpha_\lambda \) is defined by Equation (5.2) with this \( \omega \), and \( \alpha_\lambda^{KdV}(\xi) \) is defined by the linear map
\[
\mathcal{P}_g^{KdV} \rightarrow \Omega^1(\mathbb{C}) \otimes \mathfrak{sl}(2, \mathbb{C}), \; \zeta \mapsto \alpha_\lambda^{KdV}(\zeta) := \left( \begin{array}{cc}
\frac{1}{2} u_0 & v_1 \\
\frac{1}{2} v_0 & -\frac{1}{2} u_0
\end{array} \right) d\bar{z} - \left( \begin{array}{cc}
\frac{1}{2} u_0 & v_1 \\
\frac{1}{2} v_0 & -\frac{1}{2} u_0
\end{array} \right) \frac{1}{2} \omega_z d\bar{z}.
\]

Thus we see that similar to the cmc situation, any polynomial Killing field \( \xi = \xi_z \) of degree \( g \) solves with regard to the dependence on the base point \( z \) the differential equation
\[
d\xi + [\alpha_\lambda^{KdV}(\xi), \xi] = 0.
\]
For such a solution, say $\xi = \left(\begin{array}{c} 0 \\ v_1 \\ 0 \end{array}\right) \lambda + \ldots$, the corresponding solution of Liouville’s equation (5.1) is the real-valued function $\omega$ with $\frac{1}{4} e^{\omega/2} = v_1$.

**Reconstruction of the minimal immersion from the polynomial Killing field.** It will turn out that by a variation of Symes’ method, this kind of polynomial Killing field gives rise to extended frames of minimal surface immersions into $\mathbb{R}^3$, from which the immersions themselves can be obtained by the variant of the Sym-Bobenko formula given in Proposition 3.3.

In this process the Iwasawa decomposition is replaced by the following variant of the Birkhoff decomposition. We consider the following loop subgroups of $\text{ASL}(2, \mathbb{C})$

$\Lambda^+ \text{SL}(2, \mathbb{C}) = \{ \Phi \in \text{ASL}(2, \mathbb{C}) \mid \Phi \text{ extends analytically to } \mathbb{C} \}$

$\lambda^+ \text{SU}(2) \text{SL}(2, \mathbb{C}) = \{ \Phi \in \text{ASL}(2, \mathbb{C}) \mid \Phi|_{\lambda=0} \in \text{SU}(2) \}$

$\Lambda^- \text{SL}(2, \mathbb{C}) = \{ \Phi \in \text{ASL}(2, \mathbb{C}) \mid \Phi \text{ extends analytically to } \mathbb{P}^1 \setminus \{0\} \}$

$\Lambda^\text{−,R} \text{SL}(2, \mathbb{C}) = \{ \Phi \in \Lambda^- \text{SL}(2, \mathbb{C}) \mid \Phi|_{\lambda=\infty} = 1 \}$

$\Lambda^\text{−,R} \text{SL}(2, \mathbb{C}) = \{ \Phi \in \Lambda^- \text{SL}(2, \mathbb{C}) \mid \Phi|_{\lambda=\infty} = \left(\begin{array}{cc} \rho & c \\ 0 & \rho^{-1} \end{array}\right) \}$ with some $\rho > 0$ and $c \in \mathbb{C}$.

**Proposition 3.3** (Modified Birkhoff decomposition). The map

$$
\Lambda^+ \text{SU}(2) \text{SL}(2, \mathbb{C}) \times \Lambda^- \text{R} \text{SL}(2, \mathbb{C}) \rightarrow \text{ASL}(2, \mathbb{C}), (F, B) \mapsto F \cdot B
$$

is a real analytic diffeomorphism onto an open and dense subset $U$ of $\text{ASL}(2, \mathbb{C})$, called the big cell of $\text{ASL}(2, \mathbb{C})$.

**Proof.** The usual loop group Birkhoff decomposition shows the existence of the big cell $U$ of $\text{ASL}(2, \mathbb{C})$ along with the complex analytic diffeomorphism

$$
\Lambda^+ \text{SL}(2, \mathbb{C}) \times \Lambda^- \text{SL}(2, \mathbb{C}) \rightarrow \text{ASL}(2, \mathbb{C}), (g_+, g_-) \mapsto g_+ \cdot g_-
$$
ono onto $U$. Now let $\Phi \in U \subset \text{ASL}(2, \mathbb{C})$ be given, and let $\Phi = g_+ \cdot g_-$ with $g_+ \in \Lambda^+ \text{SL}(2, \mathbb{C})$ and $g_- \in \Lambda^- \text{SL}(2, \mathbb{C})$ be the usual Birkhoff decomposition of $\Phi$. By the classical Iwasawa decomposition of the complex, semi-simple Lie group $\text{SL}(2, \mathbb{C})$, $g_+(0) \in \text{SL}(2, \mathbb{C})$ can be decomposed as $g_+(0) = ub$ with $u \in \text{SU}(2)$ and $b = \left(\begin{array}{cc} \rho & c \\ 0 & \rho^{-1} \end{array}\right)$ where $\rho > 0$ and $c \in \mathbb{C}$. Here $u$ and $b$ depend real analytically on $g_+(0)$. We now define $F = g_+ \cdot b$ and $B = b \cdot g_-$. We then have

$$
F \in \Lambda^+ \text{SL}(2, \mathbb{C}) \quad \text{with} \quad F(0) = g_+(0) \cdot b^{-1} = u \in \text{SU}(2), \quad \text{so} \quad F \in \Lambda^+ \text{SU}(2) \text{SL}(2, \mathbb{C})
$$

and

$$
B \in \Lambda^- \text{SL}(2, \mathbb{C}) \quad \text{with} \quad B(\infty) = b \cdot g_-(\infty) = b, \quad \text{so} \quad B \in \Lambda^- \text{R} \text{SL}(2, \mathbb{C})
$$

and

$$
F \cdot B = g_+ b^{-1} b g_- = g_+ g_- = \Phi.
$$

This proves the proposition. □

**Proposition 3.4.** Let a polynomial Killing field $\zeta \in \mathcal{P}_g^{KdV}$ be given. We fix a base point $z_0$ and let $\Phi_\lambda(z) = F_\lambda(z) \cdot B_\lambda(z)$ be the modified Birkhoff decomposition (Proposition 3.3) of $\Phi_\lambda(z)$ = $\exp((z - z_0)\zeta)$. In this situation we have $\frac{dF_\lambda}{d\lambda} = F_\lambda \alpha_\lambda$, and thus the formula (3.3) with this $F_\lambda$ gives a conformal minimal immersion $f$ into $\mathbb{R}^3$ with conformal metric $e^\omega \frac{dz \overline{dz}}{4}$. Here $\omega$ is the real-valued function characterised by $\xi = \left(\begin{array}{c} 0 \\ \frac{1}{4} e^{\omega/2} \end{array}\right) \lambda + \ldots$ for the unique solution $\xi : \mathbb{C} \rightarrow \mathcal{P}_g^{KdV}$ of Equation (3.21) with $\xi_{z_0} = \zeta$, and $\alpha_\lambda$ is defined by Equation (3.2) with respect to this $\omega$.

**Proof.** Like in the proof of Proposition 2.6 it suffices to show $F_\lambda^{-1} \frac{dF_\lambda}{d\lambda} = \alpha_\lambda^{KdV}(\xi)$. Due to the properties of the modified Birkhoff decomposition, $F_\lambda$ extends holomorphically to $\lambda = 0$, and
therefore the Laurent series expansion of $F^{-1}_\lambda \, dF_\lambda$ with respect to $\lambda$ can only contain powers $\lambda^k$ with $k \geq 0$. On the other hand, we again have the equation

$$F^{-1}_\lambda \, dF_\lambda = B_\lambda \zeta B^{-1}_\lambda \, dz - dB_\lambda \cdot B^{-1}_\lambda = \xi_\lambda \, dz - dB_\lambda \cdot B^{-1}_\lambda.$$  

Because $B_\lambda$ extends holomorphically to $\lambda = \infty$, the series expansion of the right hand side of Equation (3.22) can only contain powers $\lambda^k$ with $k \leq 1$. In summary this shows that the series expansion of $F^{-1}_\lambda \, dF_\lambda$ contains only the powers $\lambda^1$ and $\lambda^0$. We thus write

$$F^{-1}_\lambda \, dF_\lambda = (A_0' + A_1' \lambda) \, dz + (A_0'' + A_1'' \lambda) \, d\bar{z} \quad \text{with} \quad A_0', A_1' \in \text{su}(2),$$

Because $F_{\lambda=0}$ maps into SU(2) we have $F^{-1}_{\lambda=0} \, dF_\lambda|_{\lambda=0} \in \Omega^1(\mathbb{C}) \otimes \text{su}(2)$, and therefore we have

$$A_0' = -A_0''.$$

We now write

$$B_{\lambda=\infty} = \left( \begin{array}{cc} \rho & c \\ 0 & \rho^{-1} \end{array} \right) \quad \text{and} \quad \zeta_\lambda = \sum_{k=1}^{\infty} \zeta_k \lambda^k$$

with smooth functions $\rho : \mathbb{C} \rightarrow \mathbb{R}_+$ and $c, u_k, v_k, w_k : \mathbb{C} \rightarrow \mathbb{C}$. It then follows from Equation (3.22) that

$$A_0' = \zeta_1 = \left( \begin{array}{c} 0 \\ v_1 \end{array} \right),$$

$$A_0'' = 0,$$

$$A_0' = \zeta_0 - (\frac{d}{dz} B_{\lambda=\infty}) B^{-1}_{\lambda=\infty} = \left( \begin{array}{ccc} u_0 - \rho^{-1} \rho_x & v_0 - \rho c_x + c \rho_x \\ w_0 & 0 \end{array} \right),$$

$$A_0'' = -(\frac{d}{\bar{z}} B_{\lambda=\infty}) B^{-1}_{\lambda=\infty} = \left( \begin{array}{ccc} \rho^{-1} \rho_x & -\rho c_x + c \rho_x \\ 0 & \rho^{-1} \rho_x \end{array} \right).$$

From Equations (3.23) and (3.27) we see (using the fact that $\rho$ is real-valued)

$$A_0' = \left( \begin{array}{ccc} \rho^{-1} \rho_x & 0 \\ \rho c_x - c \rho_x & -\rho^{-1} \rho_x \end{array} \right)$$

and by comparing this representation of $A_0'$ with Equation (3.26) we obtain

$$u_0 = 2\rho^{-1} \rho_x, \quad v_0 = \rho c_x - c \rho_x \quad \text{and} \quad w_0 = \rho \bar{c}_x - \bar{c} \rho_x.$$

By inserting these equations into Equations (3.26) and (3.27) we obtain

$$A_0' = \left( \begin{array}{ccc} \frac{1}{2} u_0 & 0 \\ 0 & -\frac{1}{2} u_0 \end{array} \right) \quad \text{and} \quad A_0'' = \left( \begin{array}{ccc} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{array} \right).$$

These equations, together with Equations (3.24) and (3.25) show the claimed statement $F^{-1}_\lambda \, dF_\lambda = \sigma^K_\lambda (\phi)$. 

\begin{example}

See Section 3.5, Example 6. Aside from the flat plane, the helicoid is the only ruled minimal surface in $\mathbb{R}^3$. It has the conformal parameterisation

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}^3, (x, y) \rightarrow (\sinh(x) \cos(y), \sinh(x) \sin(y), y).$$

Indeed, it is easy to check that

$$\langle f_x, f_x \rangle = \langle f_y, f_y \rangle = \cosh(x)^2 \quad \text{and} \quad \langle f_x, f_y \rangle = 0$$

holds. Therefore the metric of the helicoid is with respect to the conformal coordinate $z = x + iy$ determined by the above parameterisation given by $e^{2z} \, dz \, d\bar{z}$ with $e^{2\omega} = 2 \langle f_z, f_{\bar{z}} \rangle = \cosh(x)^2$ and hence $\omega = 2 \ln(\cosh(x))$. A further explicit computation shows that the Hopf differential of $f$ is given by $iQ \, dz \, d\bar{z}$ with $Q = \frac{1}{2}$. The function $\omega$ satisfies

$$\omega_z = \omega_{\bar{z}} = \frac{1}{\cosh(x)} \quad \text{and} \quad \omega_{zz} = \frac{1}{2 \cosh(x)^2} = \frac{1}{2} e^{-\omega} = \omega_{z\bar{z}}.$$ 

\end{example}
The latter equation proves explicitly that the function $\omega$ is a solution of Liouville’s equation (3.1). This implies in particular that the conformal metric $\omega$ varies with $x$ and therefore the ruling lines are not parameterised by constant velocity in the parameterisation $f$. These ruling lines also are one of the two families of asymptotic lines of $f$ (i.e. of null lines of the second fundamental form of $f$). Because the Hopf differential of $f$ is a purely imaginary multiple of $dz^2$, the other family of asymptotic lines is parameterised by the parallels of the $y$-axis in $\mathbb{R}^2$; they correspond to the helixes on the helicoid $f$.

Via the iteration of Proposition 3.2 we can compute a polynomial Killing field for the helicoid parameterised by $f$. From Equation (3.19) we obtain:

$$
\begin{align*}
    v_1 &= \frac{1}{2}e^{\omega/2} = \frac{1}{2} \cosh(x) \\
    u_0 &= \frac{1}{2} \omega_z = \frac{1}{2} \tanh(x) \\
    w_0 &= -Qe^{-\omega/2} = -\frac{1}{2} \cosh(x)^{-1} \\
    v_0 &= -e^{\omega/2}(\frac{1}{16}((\omega_{zz} - \frac{1}{2} \omega_z^2) - C_0) = -e^{\omega/2}(2e^{-\omega} - 1 - C_0).
\end{align*}
$$

By choosing the integration constant for $v_0$ as $C_0 = -1$, we obtain $v_0 = -2e^{-\omega/2} = -2\cosh(x)^{-1}$. In Proposition 3.2 we have $\tau_0 = e^{-\omega/2}v_0 = -2\cosh(x)^{-2} = -2e^{-\omega}$ and therefore by Equation (3.7)

$$
u_{-1} = 2(\tau_{0,z} + \omega_z \tau_0) = 2(2\omega_z e^{-\omega} + \omega_z \cdot (-2)e^{-\omega}) = 0.$$

By Equations (3.6) – (3.8) it follows that also all the lower terms of the polynomial Killing field vanish (if one chooses $C_k = 0$ for the integration constants for $\tau_k$ where $k \leq -1$). This shows that the helicoid has spectral genus 0, and the corresponding polynomial Killing field is given by

$$
\xi = \begin{pmatrix} 0 & \frac{1}{2} \cosh(x) \\ 0 & 0 \end{pmatrix} \lambda + \begin{pmatrix} \frac{1}{2} \tanh(x) & -2 \cosh(x)^{-1} \\ -\frac{1}{2} \cosh(x)^{-1} & -\frac{1}{2} \tanh(x) \end{pmatrix} .
$$

To use Symes’ method and the Sym-Bobenko formula to recover the immersion $f$ from this polynomial Killing field, we need to choose the proper value for the constant $\varphi$ from Proposition 3.4. This constant determines the phase of the Hopf differential of the constructed immersion, which equals $e^{-i\varphi} Q dz^2$. Because the Hopf differential of $f$ is $iQ dz^2$, one should choose $\varphi = -\pi$ in Proposition 3.4 to recover $f$.

4. **Blowing up cmc tori to minimal surfaces**

We now let a sequence $(f_n)_{n \in \mathbb{N}}$ of smooth cmc torus immersions $f_n : \mathbb{C} \to \mathbb{R}^3$ of fixed spectral genus $g$ be given. Without loss of generality we may suppose that the parameterisations are chosen in such a way that the natural coordinate $z \in \mathbb{C}$ is a conformal coordinate for all of them and that the Hopf differential of $f_n$ is given by $Q dz^2$ with a fixed $Q \in \mathbb{C}^*$ and moreover we suppose that the $f_n$ are scaled in the destination space $\mathbb{R}^3$ such that the value of the mean curvature of the $f_n$ is some fixed $H > 0$. We write the Riemannian metric on $\mathbb{C}$ that is induced by $f_n$ as $e^{\omega_n} dz \, d\bar{z}$ with the smooth function $\omega_n : \mathbb{C} \to \mathbb{R}_+$. For each $n \in \mathbb{N}$ we choose base points $z_{n,0} \in \mathbb{C}$ at first arbitrarily (below we will make restrictions on this choice) and introduce the objects defined in Section 2 for $f_n$ with respect to this base point (where we attach the subscript $n$ to the corresponding symbols). In particular we consider the polynomial Killing field $\zeta_n \in \mathbb{P}_g$ with the base point $z_{n,0}$, the polynomials $a_n(\lambda) = -\lambda \det(\zeta_n)$ defining the corresponding spectral curve $\Sigma_n$, the zeros $\lambda_{n,1}, \ldots, \lambda_{n,g}$ of $a_n(\lambda)$ with $|\lambda_{n,k}| < 1$, and the points $(\beta_{n,1}, \nu_{n,1}), \ldots, (\beta_{n,g}, \nu_{n,g})$ in the support of the the spectral divisor defined by $\zeta_n$. We will assume that the $\lambda_{n,k}$ and the $\beta_{n,k}$ are ordered as in (2.57) and (2.58) for every $n \in \mathbb{N}$.
If $|\lambda_{n,1}|$ is bounded away from zero as $n \to \infty$, then all $|\lambda_{n,k}|$ are bounded away from zero by (2.57) and all $|\beta_{n,k}|$ are bounded and bounded away from zero by Proposition 2.9(1). It then follows from Proposition 2.8 that all of the finitely many coefficients of $\zeta_n \in P_g$ are bounded as $n \to \infty$. Because the linear space $P_g$ is finite-dimensional, after we pass to a subsequence, $\zeta_n$ converges to a $\zeta_\infty \in P_g$ as $n \to \infty$. This limiting polynomial Killing field is non-zero because $-\lambda \det(\zeta_n) = \lim_{n \to \infty} a_n(\lambda)$ is a polynomial of degree $2g$ which is non-zero at $\lambda = 0$ and has exactly the zeros $\lim_{n \to \infty} \lambda_{n,k}$ where $1 \leq k \leq g$ inside the unit disk. Moreover, the associated connection form $\alpha(\zeta_n)$ which characterises the integrable system associated to $\zeta_n$ converges non-trivially to an $\alpha(\zeta_\infty)$ that is of the same form. Therefore Symes’ method applied to $\zeta_\infty$ yields a cmc immersion $f_\infty$ into $\mathbb{R}^3$ which is the limit of the corresponding subsequence of the $f_n$ by the Sym-Bobenko formula (2.5). This case is not relevant to our interests because we are interested in the case where we obtain a solution of a different integrable system by taking the (blow-up) limit of the $\zeta_n$. So we will assume that $\lambda_{n,1} \to 0$ as $n \to \infty$ in the sequel.

To obtain convergence in this setting we construct a blow-up of the coordinates $z$ and $\lambda$, and of the immersions $f_n$ and their polynomial Killing fields $\zeta_n$. More explicitly, we choose sequences $(\ell_n), (r_n), (s_n), (h_n)$ of positive real numbers, introduce a new coordinate $\tilde{z}$ on the domain $\mathbb{C}$ of $f_n$ and spectral parameter $\tilde{\lambda}$ by

$$\lambda = \ell_n \tilde{\lambda}, \quad z = z_{n,0} + r_n \tilde{z},$$

(4.1)

and then rescale the immersion $f_n$ and the polynomial Killing field $\zeta_n$ in the following way:

$$f_n(\tilde{z}) = h_n^{-1}(f_n(z_{n,0} + r_n \tilde{z}) - f_n(z_{n,0})) \quad \text{and} \quad \tilde{\zeta}_n(\tilde{\lambda}) = s_n \zeta_n(\ell_n \tilde{\lambda}).$$

(4.2)

Note that the relationship between the new coordinates $(\tilde{\lambda}, \tilde{z})$ and the original coordinates $(\lambda, z)$ depends on $n$. For the construction of the blow-up we will consider the new coordinates $(\tilde{\lambda}, \tilde{z})$ as being independent of $n$ and consequently let the corresponding values of $(\lambda, z)$ vary with $n$ (although we do not indicate this dependency with a subscript $n$ for the sake of notational sanity).

Also note that $\tilde{z}$ is a conformal coordinate for $f_n$, that the induced Riemannian metric of $f_n$ is given by $e^{\omega_n(z)} d\tilde{z} d\tilde{z}$ with the conformal factor $e^{\omega_n(z)} = h_n^{-2} r_n^2 e^{\omega_n(z_{n,0} + r_n \tilde{z})}$, and that $f_n$ has the constant mean curvature $h_n H$ and the Hopf differential $h_n^{-1} r_n^2 Q d\tilde{z}^2$. We also express the $\lambda_{n,k}$ and the $\beta_{n,k}$ in the $\tilde{\lambda}$-coordinates by

$$\tilde{\lambda}_{n,k} = \ell_n^{-1} \lambda_{n,k} \quad \text{and} \quad \tilde{\beta}_{n,k} = \ell_n^{-1} \beta_{n,k}.$$  

Our objective in the blow-up construction is to choose the $\ell_n, r_n, s_n$ as sequences converging to zero in such a way that both the blow-up polynomial Killing fields $\tilde{\zeta}_n$ and the associated connection forms $\alpha(\tilde{\zeta}_n)$ converge in the blow-up coordinates to non-trivial quantities. For this first blow-up construction, we want to make our choice in such a way that none of the branch points of the spectral curve $\Sigma_n$, corresponding to the zeros $\lambda_{n,k}$ of the polynomial $a_n(\lambda)$, moves into $\tilde{\lambda} = 0$ in the blow-up coordinates. This will give us a KdV spectral curve and the situation described in Section 3 in the blow-up. This is effected by making the choice $\ell_n = |\lambda_{n,1}|$. We will then see that with the proper choice of $h_n$ also the $f_n$ converge to a surface immersion into $\mathbb{R}^3$ and that the limiting immersion is minimal.

**Theorem 4.1.** In the situation described above, suppose that the following conditions are satisfied:

(a) $\lim_{n \to \infty} \lambda_{n,1} = 0$.

(b) For all $k \neq k'$, the sequences $\ell_n^{-1} \cdot (\beta_{n,k} - \beta_{n,k'})$ and $\ell_n^{-1} \cdot (\beta_{n,k}^{-1} - \beta_{n,k'}^{-1})$ are bounded away from zero.

(c) $|\lambda_{n,1}|^{-1} \cdot e^{\omega_n(z_{n,0})}$ is bounded and bounded away from zero.

We then choose

$$\ell_n = |\lambda_{n,1}|, \quad r_n = s_n = \ell_n^{1/2} \quad \text{and} \quad h_n = \ell_n.$$

(4.3)
There then exists a subsequence of the \((f_n)\), again denoted by \((f_n)\), so that the rescaled polynomial Killing fields defined in (12) converge to some non-zero \(\tilde{\zeta}_\infty(\tilde{\lambda})\). With respect to this subsequence the following also holds:

1. There exists \(1 \leq \tilde{d} \leq g\) so that \(\tilde{\lambda}_{n,k}\) converges to some \(\tilde{\lambda}_{\infty,k} \in \mathbb{C} \setminus \mathbb{D}\) for \(k \leq \tilde{d}\) and \(\tilde{\lambda}_{n,k} \to \infty\) for \(k > \tilde{d}\). The spectral curve \(\tilde{\Sigma}_\infty\) corresponding to \(\tilde{\zeta}_\infty\) has genus \(\tilde{g} = \frac{1}{2}\tilde{d}\).

2. Let \(\lambda^{KdV} = \tilde{\lambda}^{-1}\) and \(\zeta^{KdV}(\lambda^{KdV}) = \tilde{\zeta}_\infty((\lambda^{KdV})^{-1})\), then \(\zeta^{KdV} \in \mathcal{P}_{\tilde{d}}^{KdV}\) holds.

Moreover write

\[
\alpha_{\tilde{n},\tilde{\lambda}}(\tilde{\zeta}_n) = \tilde{U}_n d\tilde{z} + \tilde{V}_n d\tilde{\bar{z}} \quad \text{and} \quad \alpha_{\lambda^{KdV}}^{KdV}(\zeta^{KdV}) = U_\infty^{KdV} d\tilde{z} + V_\infty^{KdV} d\tilde{\bar{z}}.
\]

Then \(\tilde{U}_n\) and \(\tilde{V}_n\) converge for \(n \to \infty\) to \(U_\infty^{KdV}\) and \(V_\infty^{KdV}\), respectively.

3. Let \(F_n(z,\lambda)\) be the extended frame corresponding to \(f_n\) and let \(\tilde{F}_n(\tilde{z},\tilde{\lambda}) = F_n(z_{n,0} + r_n \tilde{z},\ell_n \tilde{\lambda})\). Then \(\tilde{F}_n\) converges for \(n \to \infty\) locally uniformly in \(z \in \mathbb{C}\) and \(\tilde{\lambda} \in \mathbb{C}^* \cup \{\infty\}\) to some \(\tilde{F}_\infty(\tilde{z},\tilde{\lambda})\). \(\tilde{F}_\infty\) depends holomorphically on \(\tilde{\lambda}\) and \(\tilde{F}_\infty(\tilde{z},\tilde{\lambda} = \infty) \in SU(2)\) holds.

Let \(\tilde{F}_n^{KdV}(\tilde{z},\lambda^{KdV}) = \tilde{F}_n(\tilde{z},(\lambda^{KdV})^{-1})\) and let \(\tilde{\zeta}^{KdV}_\infty : \tilde{z} \ni C \to \mathbb{P}_{\tilde{d}}^{KdV}\) be the solution of (3.21) with \(\tilde{\zeta}^{KdV}_\infty(0) = \zeta^{KdV}\). Then \(d\tilde{F}_\infty^{KdV} = F_\infty^{KdV} \alpha^{KdV}(\zeta^{KdV})\) holds.

4. The functions \(\tilde{\omega}_n\) converge locally uniformly to a real-valued, smooth function \(\tilde{\omega}_\infty\).

5. The sequence of blown-up cmc immersions \(\tilde{f}_n\) converges locally uniformly to the conformal minimal surface immersion \(\tilde{f}_\infty : C \to \mathbb{R}^3\) defined by Equation (3.1) with \(F = F_\infty^{KdV}\). The immersion \(\tilde{f}_\infty\) induces the metric \(e^{2\tilde{\omega}_\infty} d\tilde{z} d\tilde{\bar{z}}\), and has the Hopf differential \(Q d\tilde{z} d\tilde{\bar{z}}\).

**Remark 4.2.** The condition (b) of Theorem 4.1 ensure in particular that the divisors corresponding to the spectral line bundle \(\Lambda_n\) and to its dual bundle converge to non-special divisors on the limiting spectral curve in the blow-up.

Due to Equation (2.18) condition (c) can be interpreted as a condition on the rate of convergence of the \(\beta_{n,k}\) as \(n \to \infty\).

**Proof.** Due to our choice of \(\ell_n\), \(r_n\) and \(h_n\) in (4.3) we have \(e^{\tilde{\omega}_n(0) = h_n^{-2} r_n^2 e^{\omega_n(z_{n,0})} = |\lambda_{n,1}|^{-1} e^{\omega_n(z_{n,0})}\), thus it follows from condition (c) that we can choose a subsequence of \((f_n)\) so that \(\tilde{\omega}_n(0)\) converges to some real number \(\tilde{\omega}_\infty(0)\). We will see below that \(\tilde{\omega}_n(\tilde{z})\) also converges for \(\tilde{z} \neq 0\) so that (3) is satisfied.

We now at first investigate how the spectral data, namely the polynomial \(a_n(\lambda)\) which define the spectral curve, the anti-symmetric function \(\nu\) on the spectral curve \(\Sigma_n\), and the spectral line bundle \(\Lambda_n\) on the spectral curve behave under the blow-up. The roots \(\lambda_{n,1},\ldots,\lambda_{n,g}\) of \(a_n(\lambda)\) inside the unit disk correspond to \(\tilde{\lambda}_{n,k} = \ell_n^{-1} \lambda_{n,k}\). Due to our choice (4.3) and the ordering (2.57), we have

\[
1 = |\tilde{\lambda}_{n,1}| \leq |\tilde{\lambda}_{n,2}| \leq \ldots \leq |\tilde{\lambda}_{n,g}|.
\]

Note that for \(k \geq 2\) it is possible for the sequence \((|\tilde{\lambda}_{n,k}|)_{n \in \mathbb{N}}\) to be unbounded. By passing to a subsequence, we can arrange that for every \(k \geq 1\) the sequence \((\tilde{\lambda}_{n,k})_{n \in \mathbb{N}}\) converges to some \(\tilde{\lambda}_{\infty,k} \in \mathbb{P}^1 \setminus \mathbb{D}\). There exists \(1 \leq d \leq g\) so that \(\tilde{\lambda}_{\infty,k} \in \mathbb{C}\) for \(k \leq d\) and \(\tilde{\lambda}_{\infty,k} = \infty\) for \(k > d\). By Proposition 2.7 we then have

\[
a_n(\ell_n \tilde{\lambda}) = -\frac{1}{2} HQ \prod_{k=1}^g (1 - \tilde{\lambda}_{n,k}^{-1} \tilde{\lambda}) \cdot (1 - \ell_n \tilde{\lambda}_{n,k} \tilde{\lambda}).
\]
Due to $|\lambda_{n,k}| < 1$ for all $n, k$, we have $\ell_n \lambda_{n,k} \to 0$ for any $k$; and moreover $\lambda_{n,k}^{-1} \to 0$ for $k > \tilde{d}$. This shows that the polynomials $\tilde{a}_n(\tilde{\lambda}) := a_n(\ell_n \lambda)$ converge for $n \to \infty$ to the polynomial
\[
\tilde{a}_\infty(\tilde{\lambda}) = -\frac{1}{2}HQ \prod_{k=1}^{\tilde{d}} (1 - \lambda_{\infty,k}^{-1} \tilde{\lambda}) .
\]
Clearly, the polynomial $\tilde{a}_\infty(\tilde{\lambda})$ has degree $\tilde{d}$ and its zeros are exactly $\lambda_{\infty,1}, \ldots, \lambda_{\infty,\tilde{d}}$. The spectral curve corresponding to $a_n(\lambda)$ is given by
\[
\Sigma_n = \{(\lambda, \nu) \mid \nu^2 = \lambda a_n(\lambda)\} .
\]
On $\Sigma_n$ we thus have
\[
\nu^2 = \lambda a_n(\lambda) = \ell_n \lambda a_n(\ell_n \lambda) = \ell_n \lambda \tilde{a}_n(\lambda) .
\]
Thus by blowing up the holomorphic function $\nu$ on $\Sigma_n$ by
\[
\nu = \ell_n^{1/2} \tilde{\nu}, \quad \text{we have } \tilde{\nu}^2 = \tilde{\lambda} \tilde{a}_n(\tilde{\lambda}) .
\]
The function $\tilde{\nu}$ defines the rescaled (blown-up) spectral curve
\[
\tilde{\Sigma}_n = \{(\tilde{\lambda}, \tilde{\nu}) \mid \tilde{\nu}^2 = \tilde{\lambda} \tilde{a}_n(\tilde{\lambda})\} ,
\]
which is naturally biholomorphic to $\Sigma_n$ via the biholomorphic map
\[
\Theta_n : \tilde{\Sigma}_n \to \Sigma_n, (\tilde{\lambda}, \tilde{\nu}) \mapsto (\ell_n \tilde{\lambda}, \ell_n^{1/2} \tilde{\nu}) .
\]
It follows that if we view $\tilde{\nu} = \tilde{\nu}(\tilde{\lambda})$ as a two-valued, holomorphic function on $C \ni \tilde{\lambda}$, then $\tilde{\nu}$ converges to a holomorphic function on the hyperelliptic complex curve
\[
\tilde{\Sigma}_\infty = \{(\tilde{\lambda}, \tilde{\nu}) \mid \tilde{\nu}^2 = \tilde{\lambda} \tilde{a}_\infty(\tilde{\lambda})\} ,
\]
which we may regard as the limit of the curves $\tilde{\Sigma}_n$. Because $\tilde{\lambda} \tilde{a}_\infty(\tilde{\lambda})$ is a polynomial of degree $\tilde{d} + 1$, the complex curve $\tilde{\Sigma}_\infty$ has genus $\tilde{g} = \lfloor \frac{1}{2} \tilde{d} \rfloor$.

Further we let $(\beta_{n,1}, \nu_{n,1}), \ldots, (\beta_{n,g}, \nu_{n,g}) \in \Sigma_n$ be the divisor points of the section $(1, \psi_n)$ defining the line bundle $\Lambda_n$. Then $(\tilde{\beta}_{n,k}, \tilde{\nu}_n, k) = (\ell_n^{-1} \beta_{n,k}, \ell_n^{-1/2} \nu_{n,k}) \in \tilde{\Sigma}_n$ are the corresponding points for the line bundle $\tilde{\Lambda}_n$ on $\Sigma_n$. We again assume the ordering of the $\beta_{n,k}$ given by (2.58) for every $n$, then we have by Proposition [2.9](1)
\[
1 \leq |\tilde{\lambda}_{n,k}| \leq |\tilde{\beta}_{n,k}| \leq \ell_n^{-2} |\lambda_{n,k}|^{-1} ,
\]
By again passing to a subsequence, we may assume that $\tilde{\beta}_{n,k}$ converges for $n \to \infty$ to some $\tilde{\beta}_{\infty,k} \in \mathbb{P}^1$. Note that $|\tilde{\beta}_{\infty,k}| \geq 1$, but $\tilde{\beta}_{\infty,k} = \infty$ is possible for all $k$. Because of $\tilde{\nu}_{n,k}^2 = \beta_{n,k} a_n(\beta_{n,k})$ we can also achieve that $\tilde{\nu}_{n,k}$ converges to some $\tilde{\nu}_{\infty,k}$ so that $(\tilde{\beta}_{\infty,k}, \tilde{\nu}_{\infty,k}) \in \tilde{\Sigma}_\infty$. Due to condition (b), the finite $\tilde{\beta}_{\infty,k}$ are pairwise unequal and in particular the divisor $\sum_k (\tilde{\beta}_{\infty,k}, \tilde{\nu}_{\infty,k})$ on $\tilde{\Sigma}_\infty$ is non-special.

We will show that for the subsequence of the $(\zeta_n)$ we have now chosen, $\tilde{\zeta}_n$ converges to some $\tilde{\zeta}_\infty$ as $n \to \infty$, and that the additional convergence statements in (2)–(4) hold. For this purpose we write
\[
\zeta_n(\lambda) = \begin{pmatrix} u_n(\lambda) & v_n(\lambda) \\ w_n(\lambda) & -u_n(\lambda) \end{pmatrix} \quad \text{and} \quad \tilde{\zeta}_n(\tilde{\lambda}) = \begin{pmatrix} \tilde{u}_n(\tilde{\lambda}) & \tilde{v}_n(\tilde{\lambda}) \\ \tilde{w}_n(\tilde{\lambda}) & -\tilde{u}_n(\tilde{\lambda}) \end{pmatrix} .
\]
We have by Equation (2.52)
\[
\tilde{\lambda} \tilde{u}_n(\tilde{\lambda}) = s_n (1 - \beta_{n,k}^{-1} \tilde{\lambda}) = \frac{1}{2} H \ell_n^{-1/2} e^{\omega_n(\zeta_n, n) / 2} \prod_{k=1}^{g} (1 - \beta_{n,k}^{-1} \ell_n^{-1} \tilde{\lambda}) = \frac{1}{2} H e^{\tilde{\omega}_n(0) / 2} \prod_{k=1}^{g} (1 - \beta_{n,k}^{-1} \tilde{\lambda}) .
\]
Hence \( \tilde{\lambda} \tilde{v}_n(\tilde{\lambda}) \) converges for \( n \to \infty \) to the polynomial

\[
\tilde{\lambda} \tilde{v}_\infty(\tilde{\lambda}) = \frac{1}{2} H e^{\tilde{\omega}_\infty(0)/2} \prod_k (1 - \tilde{\beta}_{\infty,k}^{-1} \tilde{\lambda}) .
\]

We define \( \chi_{n,k}(\lambda) = \prod_{k' \neq k} \frac{\lambda - \beta_{n,k'}}{\beta_{n,k} - \beta_{n,k'}} \), compare Equation \((2.51)\), then we have

\[
\chi_{n,k}(\ell_n \tilde{\lambda}) = \prod_{k' \neq k} \frac{\ell_n \tilde{\lambda} - \beta_{n,k'}}{\tilde{\beta}_{n,k} - \beta_{n,k'}} = \prod_{k' \neq k} \frac{\tilde{\lambda} - \tilde{\beta}_{n,k'}}{\tilde{\beta}_{n,k} - \beta_{n,k'}} =: \tilde{\chi}_{n,k}(\tilde{\lambda}) .
\]

If \( \tilde{\beta}_{\infty,k} \neq \infty \), then \( \tilde{\chi}_{n,k}(\tilde{\lambda}) \) converges for \( n \to \infty \) to

\[
\tilde{\chi}_{\infty,k}(\tilde{\lambda}) = \prod_{k' \neq k} \frac{\tilde{\lambda} - \tilde{\beta}_{\infty,k'}}{\tilde{\beta}_{\infty,k} - \beta_{n,k'}} ;
\]

note that the denominator cannot become zero because of \((b)\). If \( \tilde{\beta}_{\infty,k} = \infty \), then

\[
\tilde{\chi}_{n,k}(\tilde{\lambda}) = \tilde{\beta}_{n,k}^{-1} \prod_{k' \neq k} \frac{\tilde{\beta}_{\infty,k'}^{-1} \tilde{\lambda}^{-1}}{\tilde{\beta}_{\infty,k} - \beta_{n,k'}}
\]

converges to zero; note that condition \((b)\) again implies that the denominator is bounded away from zero. It follows by Equation \((2.50)\) that

\[
\tilde{u}_n(\tilde{\lambda}) = s_n u_n(\ell_n \tilde{\lambda}) = -\ell_n^{1/2} \sum_{k=1}^g \beta_{n,k}^{-1} \nu_n,k \chi_{n,k}(\lambda) = -\sum_{k=1}^g \tilde{\beta}_{n,k}^{-1} \nu_n,k \tilde{\chi}_{n,k}(\tilde{\lambda})
\]

converges for \( n \to \infty \) to the polynomial

\[
\tilde{u}_\infty(\tilde{\lambda}) = \sum_{k=1, \ldots, g, \tilde{\beta}_{\infty,k} \neq 0} \tilde{\beta}_{\infty,k}^{-1} \tilde{\nu}_{\infty,k} \tilde{\chi}_{\infty,k}(\tilde{\lambda}) .
\]

Before we complete the proof of the convergence of \( \tilde{\zeta}_n \), we investigate the behavior of the spectral line bundle \( \Lambda_n \) corresponding to the polynomial Killing field \( \zeta_n(\lambda) \). We saw in Section \(2\) that \((1, \psi_n)\) with \( \psi_n = \frac{\nu - \lambda u_n(\lambda)}{\lambda v_n(\lambda)} \) is a holomorphic section of \( \Lambda_n \). Therefore the rescaled version \( \tilde{\Lambda}_n = \Theta_n^* \Lambda_n \) of the line bundle \( \Lambda_n \) is determined by the holomorphic section \((1, \tilde{\psi}_n)\), where the meromorphic function \( \tilde{\psi}_n \) on \( \tilde{\Sigma}_n \) is defined by \( \tilde{\psi}_n = \psi_n \circ \Theta_n \), i.e., \( \tilde{\psi}_n(\tilde{\lambda}, \tilde{\nu}) = \psi_n(\ell_n \tilde{\lambda}, \ell_n^{1/2} \tilde{\nu}) \). Now we have

\[
\tilde{\psi}_n(\tilde{\lambda}, \tilde{\nu}) = \psi_n(\ell_n \tilde{\lambda}, \ell_n^{1/2} \tilde{\nu}) = \frac{\ell_n^{1/2} \tilde{\nu} - \ell_n \tilde{\lambda} u_n(\ell_n \tilde{\lambda})}{\ell_n \tilde{\lambda} v_n(\ell_n \tilde{\lambda})} = \frac{\ell_n^{1/2} \tilde{\nu} - \ell_n s_n^{-1} \tilde{\lambda} u_n(\tilde{\lambda})}{\ell_n s_n^{-1} \tilde{\lambda} v_n(\tilde{\lambda})} = \frac{\tilde{\nu} - \tilde{\lambda} u_n(\tilde{\lambda})}{\tilde{\lambda} v_n(\tilde{\lambda})} .
\]

Therefore \( \tilde{\psi}_n \) converges for \( n \to \infty \) to the non-constant, meromorphic function \( \tilde{\psi}_\infty = \frac{\tilde{\nu} - \tilde{\lambda} u_\infty(\tilde{\lambda})}{\tilde{\lambda} v_\infty(\tilde{\lambda})} \) on \( \tilde{\Sigma}_\infty \). In this sense, the line bundles \( \tilde{\Lambda}_n \) on \( \tilde{\Sigma}_n \) converge for \( n \to \infty \) to the holomorphic line bundle \( \tilde{\Lambda}_\infty \) on \( \tilde{\Sigma}_\infty \) determined by the holomorphic section \((1, \tilde{\psi}_\infty)\).

There exists one and only one \((2 \times 2)\)-matrix-valued holomorphic function \( \tilde{\zeta}_\infty(\tilde{\lambda}) \) which has the line bundle \( \tilde{\Lambda}_\infty \) on \( \tilde{\Sigma}_\infty \) as eigenline bundle and the function \( \frac{\tilde{\nu}}{\tilde{\lambda}} \) on \( \tilde{\Sigma}_\infty \) as corresponding eigenvalue function, i.e. so that \( \tilde{\zeta}_\infty(\tilde{\lambda}) \cdot \left( \begin{smallmatrix} 1 \\ \frac{1}{\psi_\infty} \end{smallmatrix} \right) = \frac{\tilde{\nu}}{\tilde{\lambda}} \left( \begin{smallmatrix} 1 \\ \frac{1}{\psi_\infty} \end{smallmatrix} \right) \) holds, see \[KLSS\] Section 4. Because we have

\[
\tilde{\zeta}_n(\tilde{\lambda}) \cdot \left( \begin{smallmatrix} \frac{1}{\psi_n(\tilde{\lambda}, \tilde{\nu})} \\ \psi_n(\ell_n \tilde{\lambda}, \ell_n^{1/2} \tilde{\nu}) \end{smallmatrix} \right) = s_n \tilde{\zeta}_n(\ell_n \tilde{\lambda}) \cdot \left( \begin{smallmatrix} \frac{1}{\psi_n(\ell_n \tilde{\lambda}, \ell_n^{1/2} \tilde{\nu})} \\ \psi_n(\ell_n \tilde{\lambda}, \ell_n^{1/2} \tilde{\nu}) \end{smallmatrix} \right) = \frac{\ell_n^{1/2} \tilde{\nu} - \ell_n \tilde{\lambda} u_n(\ell_n \tilde{\lambda})}{\ell_n \tilde{\lambda} v_n(\ell_n \tilde{\lambda})} = \frac{\tilde{\nu} - \tilde{\lambda} u_n(\tilde{\lambda})}{\tilde{\lambda} v_n(\tilde{\lambda})}
\]

for every \( n \), it follows that \( \tilde{\zeta}_\infty(\tilde{\lambda}) \) is the limit of \( \tilde{\zeta}_n(\tilde{\lambda}) \) for \( n \to \infty \). Because \( \tilde{a}_\infty(\tilde{\lambda}) = -\tilde{\lambda} \det(\tilde{\zeta}_\infty) \) is of degree \( \tilde{d} \), the highest power of \( \tilde{\lambda} \) that occurs in \( \tilde{\zeta}_\infty \) is \( \tilde{d} \). This shows that \( \tilde{\zeta}_\infty^{\text{KdV}} \), defined in \((2)\), is in \( \mathcal{P}_d^{\text{KdV}} \).
We have $dz = r_n \, d\tilde{z}$ and $\tilde{z}_{n,k} = s_n \cdot \tilde{k} \cdot \zeta_{n,k}$ and therefore due to our choice $r_n = s_n$

$$\alpha_{\ell_n, \lambda}(\zeta_n) = \begin{pmatrix} u_{n,0} & v_{n,0} \cdot \lambda^{-1} \cdot v_{n,0} \\ w_{n,0} & -u_{n,0} \end{pmatrix} r_n \, d\tilde{z} = \begin{pmatrix} \bar{u}_{n,0} & \bar{v}_{n,0} \cdot \lambda^{-1} \cdot \bar{v}_{n,0} \\ \bar{w}_{n,0} & -\bar{u}_{n,0} \end{pmatrix} \, \bar{d}\tilde{z}.$$

By our preceding results on the convergence of $\tilde{z}_n$ and $\ell_n \to 0$, the second claim of (2) follows.

To prove the remaining statements of (3)–(5), we use the fact that by Symes’ method for the cmc integrable system (Proposition 2.4) the extended frame $F_n(z, \lambda)$ corresponding to $f_n$ occurs in the Iwasawa decomposition (Proposition 2.5) $\Phi_n(z, \lambda) = F_n(z, \lambda) \cdot B_n(z, \lambda)$ of $\Phi_n(z, \lambda) = \exp((z - z_{n,0}) \zeta_n)$. Similarly, by Symes’ method for the integrable system of minimal surfaces (Proposition 3.1), the factor $F_{\text{KdV}}$ in the modified Birkhoff decomposition (Proposition 3.3) $\Phi_{\text{KdV}}(\tilde{z}, \lambda^{\text{KdV}}) = F_{\text{KdV}}(\tilde{z}, \lambda^{\text{KdV}}) \cdot B_{\text{KdV}}(\tilde{z}, \lambda^{\text{KdV}})$ of $\Phi_{\text{KdV}}(\tilde{z}, \lambda^{\text{KdV}}) = \exp(\tilde{z} \cdot \zeta^{\text{KdV}})$ is the extended frame of a minimal surface immersion $\tilde{f}_\infty$, which is obtained by the Sym-Bobenko formula of Proposition 3.1.

We now also consider $\hat{\Phi}_n(\tilde{z}, \hat{\lambda}) = \exp(\tilde{z} \cdot \hat{\zeta}_n)$. For $n \to \infty$, $\hat{\zeta}_n(\hat{\lambda})$ converges to $\zeta^{\text{KdV}}(\lambda^{-1})$ and therefore $\hat{\Phi}_n(\tilde{z}, \hat{\lambda})$ converges to $\Phi_{\text{KdV}}(\tilde{z}, \lambda^{-1})$. On the other hand we have

$$\hat{\Phi}_n(\tilde{z}, \hat{\lambda}) = \exp(\tilde{z} \cdot \hat{\zeta}_n(\hat{\lambda})) = \exp(\tilde{z} \cdot s_n \cdot \zeta_n(\ell_n \hat{\lambda})) = \exp(r_n \cdot \tilde{z} \cdot \zeta_n(\ell_n \hat{\lambda})) = \Phi_n(z_{n,0} + r_n \tilde{z}, \ell_n \hat{\lambda}), \quad \hat{\Phi}_n(\tilde{z}, \hat{\lambda}) = B_n(z_{n,0} + r_n \tilde{z}, \ell_n \hat{\lambda}).$$

By the properties of the extended frame $F_n$, $\hat{F}_n(\tilde{z}, \hat{\lambda})$ is holomorphic for $\hat{\lambda} \in \mathbb{C}^*$ and we have $\hat{F}_n(\tilde{z}, \hat{\lambda}) \in SU(2)$ for $|\hat{\lambda}| = \ell_n^{-1}$, i.e. $|\lambda^{\text{KdV}}| = \ell_n$. Moreover, by the properties of the Iwasawa decomposition, $\hat{B}_n(\tilde{z}, \hat{\lambda})$ is holomorphic for $\tilde{z} \in B(0, \ell_n^{-1})$, i.e. for $\lambda^{\text{KdV}} \in B(\infty, \ell_n^{-1})$, and is equal to an upper triangular matrix with real diagonal entries for $\hat{\lambda} = 0$, i.e. for $\lambda^{\text{KdV}} = \infty$. This shows that the $(\ell_n^{-1})$-Iwasawa decomposition (3.3) of $\hat{\Phi}_n(\tilde{z}, \hat{\lambda})$ converges to the modified Birkhoff decomposition of $\Phi_{\text{KdV}}(\tilde{z}, \lambda^{\text{KdV}} = \lambda^{-1})$. This implies (3).

We now consider also the basepoint-dependent polynomial Killing fields $\tilde{\xi}_n(\tilde{z}, \hat{\lambda})$ with $\tilde{\xi}_n(0, \hat{\lambda}) = \tilde{\xi}_n(\hat{\lambda})$. By Equation (2.18) we have

$$\tilde{\xi}_n(\tilde{z}, \hat{\lambda}) = \tilde{F}_n(\tilde{z}, \hat{\lambda})^{-1} \cdot \tilde{\xi}_n(\hat{\lambda}) = \hat{F}_n(\tilde{z}, \hat{\lambda})^{-1} \cdot \hat{\xi}_n(\hat{\lambda}) = \hat{F}_n(\tilde{z}, \hat{\lambda})^{-1} \cdot \hat{\xi}_n(\hat{\lambda}).$$

Because of $\hat{F}_n(\tilde{z}, \hat{\lambda}) \to \hat{F}_\infty(\tilde{z}, \hat{\lambda})$ and $\hat{\xi}_n(\hat{\lambda}) \to \xi_\infty(\hat{\lambda})$, we see that $\tilde{\xi}_n(\tilde{z}, \hat{\lambda})$ converges locally uniformly in $\tilde{z} \in \mathbb{C}$ to a function $\xi_\infty(\tilde{z}, \hat{\lambda})$. $\Phi_{\text{KdV}}(\tilde{z}, \lambda^{\text{KdV}}) = \xi_\infty(\tilde{z}, (\lambda^{\text{KdV}})^{-1})$ maps into $\mathcal{P}_{(d)}$, and is a solution of $d\xi_{\text{KdV}} + [\alpha_{\lambda^{\text{KdV}}}^{\text{KdV}}(\xi_{\text{KdV}}), \xi_{\text{KdV}}] = 0$ with $\xi_{\text{KdV}}(0, \lambda^{\text{KdV}}) = \xi_{\infty}(\lambda^{\text{KdV}})$. The lower left entry of the $(\lambda^{\text{KdV}})^0$-term of $\tilde{\xi}_n$ is equal to $-Q \cdot e^{-\hat{\omega}_n}/2$ by Equation (2.17). Therefore $\tilde{\omega}_n$ converges locally uniformly to the smooth, real-valued function $\hat{\omega}_\infty$ such that the lower left entry of the $(\lambda^{\text{KdV}})^0$-term of $\xi_{\text{KdV}}$ is equal to $-Q \cdot e^{-\hat{\omega}_\infty}/2$, giving (4). Due to Equation (3.19) this function $\hat{\omega}_\infty$ indeed corresponds to the induced metric of $\hat{f}_\infty$. We also note that the sequence $(\alpha_{\lambda}(\xi_n))_{n \in \mathbb{N}}$ converges to $\alpha_{\lambda_{\text{KdV}}}^{\text{KdV}}(\xi_{\text{KdV}}) = \xi_{\infty}(\lambda_{\text{KdV}}^{\text{KdV}})$.

Finally we show the convergence of the sequence $(\tilde{f}_n)$ of blown up cmc immersions. For this purpose we express the cmc immersions $f_n$ by the Sym-Bobenko formula (2.5) with respect to some Sym point $\lambda_0 = e^{i\varphi} \in S^1$. Indeed after applying suitable translations in $\mathbb{R}^3$ so that $f_n(z_{n,0}) = 0$, we have

$$f_n(z) = G_n \cdot F_n^{-1}_{\lambda=e^{i\varphi}}$$

with $G_n(z, \lambda) := -\frac{1}{P} \cdot \hat{\lambda} \cdot \frac{\partial F_n}{\partial \hat{\lambda}}.$
and therefore
\[ \hat{f}_n(z) = h_n^{-1} f_n(z,0+r_n z) = h_n^{-1} G_n(z,0+r_n z,\lambda) F_n^{-1}(z,0+r_n z,\lambda) \bigg|_{\lambda = e^{i\varphi}} = \tilde{G}_n(z,\lambda = e^{i\varphi}) \cdot \tilde{F}_n(z,\lambda = \ell_n^{-1} e^{i\varphi})^{-1}, \]
where we recall \( \tilde{F}_n(z,\lambda) = F_n(z,0+r_n z,\ell_n \lambda) \) and define \( \tilde{G}_n(z,\lambda) = h_n^{-1} \cdot G_n(z,0+r_n z,\lambda) \)
(note that we do not blow up the parameter \( \lambda \) of \( \tilde{G}_n \)). We thus have
\[ \tilde{G}_n(z,\lambda) = -h_n^{-1} i \frac{\partial F_n(z,0+r_n z,\lambda)}{\partial \lambda}. \]

We have \( dF_n = F_n \alpha_n \), where \( \alpha_n(\lambda) = \alpha_\lambda(\xi_n) \) is given by Equation (2.3) with respect to the solution \( \omega = \omega_n \), and therefore \( dF_n = \partial \omega_n = \partial h_n \alpha_n + F \partial \alpha_n \), whence \( d\tilde{G}_n = \tilde{G}_n \alpha_n + \tilde{F}_n \beta_n \)
follows with
\[ \beta_n = -h_n^{-1} i \frac{\partial \alpha_n}{\partial \lambda} \equiv h_n^{-1} i \frac{\partial \omega_n}{\partial \lambda} \cdot \frac{e^{i\varphi}}{2} \begin{pmatrix} \lambda^{-1} d\lambda \\ 0 \end{pmatrix} = e^{i\varphi} \begin{pmatrix} 0 \\ 2 \end{pmatrix} \begin{pmatrix} \lambda d\lambda \\ 0 \end{pmatrix} \cdot \frac{e^{i\varphi}}{2} \cdot d\lambda. \]

Here we again used the fact that due to our choices of blow-up factors \( \xi_n \) we have \( h_n^{-1} e^{i\varphi} d\lambda = e^{i\varphi} d\lambda \). We now consider these objects at \( \lambda = e^{i\varphi} \) which equation is equivalent to \( \lambda = \ell_n^{-1} e^{i\varphi} \) and take the limit as \( n \to \infty \). Then \( \tilde{F}_n(z,\lambda) \) converges to \( \tilde{F}_\infty(z,\infty) = F_{KdV}(z,0) \) and the factors of \( d\lambda \) and of \( d\tilde{z} \) in \( \alpha_n, \lambda \) converge to the corresponding factors in \( \alpha_{KdV,0}(e^{i\varphi}) \). Moreover the said factors of \( \beta_n \) converge to the corresponding factors of
\[ \beta_\infty = e^{i\varphi} \begin{pmatrix} 0 \\ 2 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} \begin{pmatrix} e^{i\varphi} d\lambda \\ d\lambda \end{pmatrix} \cdot \frac{e^{i\varphi}}{2} \cdot d\lambda. \]

This shows that \( \tilde{G}_n(z,\lambda) \) converges at least for \( \lambda = e^{i\varphi} \) to the solution of \( d\tilde{G}_\infty = \tilde{G}_\infty \alpha_{\infty, KdV,0} + \tilde{F}_\infty \beta_\infty \). And then \( \tilde{f}_n = \tilde{G}_n \cdot \tilde{F}_n^{-1} \) converges to \( \tilde{G}_\infty \cdot \tilde{F}_\infty^{-1} =: \tilde{f}_\infty \), which is by Proposition 3.1 a minimal conformal immersion with induced metric \( e^{i\varphi} d\lambda d\tilde{z} \) and Hopf differential \( Q e^{i\varphi} d\tilde{z}^2 \). Thus (5) is shown when we choose the Sym point \( \lambda_n = e^{i\varphi} = 1 \). \( \square \)

**Example 4.3.** In the situation investigated in this section, one special case is of particular interest to us. We choose \( H = 1 \) and \( Q = \frac{1}{2} \), and the Sym point \( \lambda_s = -i \) in Proposition 2.1 resp. \( \varphi = -\frac{\pi}{2} \) in Proposition 3.7. Suppose that the spectral genus \( g = 2d + 1 \) is odd, that all the \( \lambda_{n,k} \) go to zero as \( n \to \infty \), but that for \( k \geq 2 \) the \( \lambda_{n,k} \) go to zero at a slower rate than \( \lambda_{n,1} \) does, meaning that
\[ \frac{|\lambda_{n,1}|}{|\lambda_{n,1}|} \to \infty. \]
Moreover we suppose that \( \beta_{n,1} \) goes to zero at the same rate as \( \lambda_{n,1} \) does, i.e. \( \beta_{n,1} = |\lambda_{n,1}|^{-1} \beta_{n,1} \) has a limit \( \beta_{\infty,1} \in \mathbb{C} \setminus D \). Due to the inequality of Proposition 2.9(1), we have \( |\beta_{n,k}| \to \infty \) for \( k \geq 2 \), and we suppose that these divisor points are pairwise conjugate to each other in the sense that
\[ |\beta_{n,2j}| < 1 \quad \text{and} \quad |\beta_{n,2j+1}| = |\beta_{n,2j}^{-1}| \quad \text{holds for} \ j = 1, \ldots, d. \]

Equation (2.48) shows that the hypotheses (a)–(c) of Theorem 4.7 are satisfied in the present situation. In Theorem 4.7 we have \( \check{d} = 1 \) and therefore the blow up minimal immersion \( \tilde{f}_\infty \) has spectral genus \( \frac{1}{2} = 0 \). The limiting polynomial Killing field \( \zeta_{KdV}^\infty \) at the base point \( \tilde{z} = 0 \) is therefore of the form
\[ \zeta_{KdV}^\infty = \begin{pmatrix} 0 \\ v_1 \\ 0 \end{pmatrix} \lambda_{KdV} + \begin{pmatrix} u_0 \\ w_0 \\ -u_0 \end{pmatrix} \in P_{0,KdV}. \]

Here we have by Equations (3.19) and (2.48)
\[ v_1 = \frac{1}{4} e^{i\varphi}(0) = \frac{1}{4} \lim_{n \to \infty} (\ell_n e^{i\varphi}(z_n,0)) = \frac{1}{4} \lim_{n \to \infty} \left( \frac{2Q}{H} \ell_n^{-1} |\beta_{n,1}| \right)^{1/2} \geq \frac{1}{4} \beta_{\infty,1}. \]
Therefore there exists $x \in \mathbb{R}$ with $v_1 = \frac{1}{4} \cosh(x)$. We then have by Equation (3.19)
$$w_0 = -\frac{1}{4} Q v_1^{-1} = -\frac{1}{2} \cosh(x)^{-1}.$$ 
By a finite rescaling of the coordinate $\tilde{z}$ we can obtain that $u_0 = \frac{1}{2} (\tilde{\omega}_\infty) \tilde{z} = \frac{1}{2} \tanh(x)$ holds, and by choosing the integration constant for $v_0$ in the Pinkall-Sterling iteration of Proposition 3.2 appropriately, we can moreover obtain $v_0 = -2 \cosh(x)^{-1}$. To sum up, after these transformations we have
$$\zeta^{\text{KdV}}_\infty = \begin{pmatrix} 0 & \frac{1}{4} \cosh(x) \\ 0 & 0 \end{pmatrix} \lambda^{\text{KdV}} + \begin{pmatrix} \frac{1}{2} \tanh(x) & -2 \cosh(x)^{-1} \\ -\frac{1}{2} \cosh(x)^{-1} & -\frac{1}{2} \tanh(x) \end{pmatrix} \in \mathcal{P}_0^{\text{KdV}}.$$ 
This is the polynomial Killing field of a helicoid in $\mathbb{R}^3$, as we saw in Example 3.5. Under the stated circumstances, the blowup of a sequence of cmc tori at the fastest possible rate ($\ell_n = |\lambda_n, 1|$) therefore produces a helicoid in $\mathbb{R}^3$.

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