ALMOST CONFORMAL TRANSFORMATION IN A CLASS OF RIEMANNIAN MANIFOLDS

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Abstract. We consider a 3-dimensional Riemannian manifold \( V \) with a metric \( g \) and an affinor structure \( q \). The local coordinates of these tensors are circulant matrices. In \( V \) we define an almost conformal transformation. Using that definition we construct an infinite series of circulant metrics which are successively almost conformally related. In this case we get some properties.

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1. Preliminaries

We consider a 3-dimensional Riemannian manifold \( M \) with a metric tensor \( g \) and two affine tensors \( q \) and \( S \) such that: their local coordinates form circulant matrices. So these matrices are as follows:

\[
\begin{align*}
g_{ij} &= \begin{pmatrix} A & B & B \\ B & A & B \\ B & B & A \end{pmatrix}, & A > B > 0,
\end{align*}
\]

where \( A \) and \( B \) are smooth functions of a point \( p(x^1, x^2, x^3) \) in some \( F \subset \mathbb{R}^3 \),

\[
\begin{align*}
q^j_i &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, & S^j_i &= \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.
\end{align*}
\]

We note by \( V \) the class of manifolds like \( M \).

Let \( M \) be in \( V \) and \( \nabla \) be the connection of \( g \). Let us give some results for \( M \) in \( V \), obtained in [1].

\[
\begin{align*}
q^1 &= E; & g(qu, qv) &= g(u, v), & u, v \in \chi M.
\end{align*}
\]

\[
\begin{align*}
\nabla q &= 0 \quad \Leftrightarrow \quad \text{grad} A = \text{grad} B. S.
\end{align*}
\]

\[
\begin{align*}
0 < B < A \quad \Rightarrow \quad g \text{ is positively defined.}
\end{align*}
\]

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2. Almost conformal transformation

Let $M$ be in $V$. We note $f_{ij} = g_{ik}q_k^j + g_{kj}q_k^i$, i.e.

$$f_{ij} = \begin{pmatrix} 2B & A + B & A + B \\ A + B & 2B & A + B \\ A + B & A + B & 2B \end{pmatrix}.$$  \hfill (6)

We calculate $\det f_{ij} = 2(A - B)^2(A + 2B) \neq 0$, so we accept $f_{ij}$ for local coordinates of another metric $f$. Further, we suppose $\alpha$ and $\beta$ are two smooth functions in $F \subset \mathbb{R}^3$ and we construct the metric $g_1$, as follows:

$$g_1 = \alpha g + \beta f.$$  \hfill (7)

We say that equation (7) define an almost conformal transformation, noting that if $\beta = 0$ then (7) implies the case of the classical conformal transformation in $M$ [2].

From (1), (6) and (7) we get the local coordinates of $g_1$:

$$g_{1,ij} = \begin{pmatrix} \alpha A + 2\beta B & \beta A + (\alpha + \beta)B & \beta A + (\alpha + \beta)B \\ \beta A + (\alpha + \beta)B & \alpha A + 2\beta B & \beta A + (\alpha + \beta)B \\ \beta A + (\alpha + \beta)B & \beta A + (\alpha + \beta)B & \alpha A + 2\beta B \end{pmatrix}.$$  \hfill (8)

We see that $f_{ij}$ and $g_{1,ij}$ are both circulant matrices.

**Theorem 2.1.** Let $M$ be a manifold in $V$, also $g$ and $g_1$ be two metrics of $M$, related by (4). Let $\nabla$ and $\nabla$ be the corresponding connections of $g$ and $g_1$, and $\nabla q = 0$. Then $\nabla q = 0$ if and only if, when

$$\text{grad} \alpha = \text{grad} \beta S.$$  \hfill (9)

**Proof.** At first we suppose (9) is valid. Using (9) and (4) we can verify that the following identity is true:

$$\text{grad} \alpha A + 2\beta B = \text{grad} \beta A + (\alpha + \beta)B S.$$  \hfill (10)

The identity (10) is analogue to (1), and consequently we conclude $\nabla q = 0$.

Inversely, if $\nabla q = 0$ then analogously to (4) we have (10). Now (1) and (10) imply (9). So the theorem is proved.  \hfill $\square$

Note. We see that (10) is a system of partial differential equations. In this case we know that this system has a solution [5].

Let $w = w(x(p), y(p), z(p))$ be an arbitrary vector in $T_p M$, $p \in M$, $M \subset V$, such that $qw \neq w$. For the metric $g$ of $M$ we suppose $0 < B < A$, i.e. $g$ is positively defined (see [5]).

Let $\varphi$ be the angle between $w$ and $qw$ with respect to $g$. Then thank’s to (1), (2) and (5) we get $\cos \varphi = \frac{g(w, qw)}{g(w, w)}$, and we note that $\varphi \in (0, \frac{2\pi}{3})$ [1].

**Lemma 2.2.** Let $g_1$ be the metric given by (7). If $0 < \beta < \alpha$ and $g$ is positively defined, then $g_1$ is also positively defined.

**Proof.** For $g_1$ we have that $\alpha A + 2\beta B - (\beta A + (\alpha + \beta)B) = (\alpha - \beta)(A - B) > 0$. Analogously to (6) we state that $g_1$ is positively defined.  \hfill $\square$

**Lemma 2.3.** Let $w = w(x(p), y(p), z(p))$ be in $T_p M$, $p \in M$, $M \subset V$, $qw \neq w$. Let $g$ and $g_1$ be the metrics of $M$, related by (4). Then we have

$$g_1(w, w) = \alpha g(w, w) + 2\beta g(w, qw)$$

and

$$g_1(w, qw) = \beta g(w, w) + (\alpha + \beta)g(w, qw).$$  \hfill (11)
Proof. Using (1) and (2) we find
\[
g(w, w) = A(x^2 + y^2 + z^2) + 2B(xy + yz + zx) \\
g(w, qw) = B(x^2 + y^2 + z^2) + (A + B)(xy + yz + zx).
\]

Now, we use (8) and (12) after some computations we get (11). □

**Theorem 2.4.** Let \( w = w(x(p), y(p), z(p)) \) be a vector in \( T_p M, \ p \in M, \ M \subset V, \ qw \neq w \). Let \( g \) and \( g_1 \) be two positively defined metrics of \( M \), related by (7). If \( \varphi \) and \( \varphi_1 \) are the angles between \( w \) and \( qw \), with respect to \( g \) and \( g_1 \) respectively, then the following equation is true
\[
\cos \varphi_1 = \frac{\beta + (\alpha + \beta)\cos \varphi}{\alpha + 2\beta \cos \varphi}.
\]

Proof. Since \( g \) and \( g_1 \) are both positively defined metrics we can calculate \( \cos \varphi \) and \( \cos \varphi_1 \), respectively [2]. Then by using (11) from Lemma 2.2 and Lemma 2.3 we get (13). □

We note \( \varphi \in (0, \frac{2\pi}{3}) \). Theorem 2.4 implies immediately the assertions:

**Corollary 2.5.** If \( \varphi_1 \) is the angle between \( w \) and \( qw \) with respect to \( g_1 \) then \( \varphi_1 \in (0, \frac{2\pi}{3}) \).

**Corollary 2.6.** Let \( \varphi \) and \( \varphi_1 \) be the angles between \( w \) and \( qw \) with respect to \( g \) and \( g_1 \). Then
1) \( \varphi = \frac{\pi}{2} \) if and only if when \( \varphi_1 = \arccos \frac{\beta}{\alpha} \);
2) \( \varphi_1 = \frac{\pi}{2} \) if and only if when \( \varphi = \arccos \left( -\frac{\beta}{\alpha + \beta} \right) \).

Further, we consider an infinite series of the metrics of \( M \) in \( V \) as follows:
\[
g_0, \ g_1, \ g_2, \ldots, \ g_n, \ldots
\]
where
\[
g_0 = g, \ g_n = \alpha g_{n-1} + \beta f_{n-1}, \ f_{n-1,10} = g_{n-1,10} a_0^q + g_{n-1,10} a_0^q, \ 0 < \beta < \alpha.
\]

By the method of the mathematical induction we can see that the matrix of every \( g_n \) is circulant one and every \( g_n \) is positively defined.

**Theorem 2.7.** Let \( w = w(x(p), y(p), z(p)) \) be in \( T_p M, \ p \in M, \ M \subset V, \ qw \neq w \). Let \( \varphi_n \) be the angle between \( w \) and \( qw \) with respect to metric \( g_n \) from (17). Then the infinite series:
\[
\varphi_0, \ \varphi_1, \ \varphi_2, \ldots, \ \varphi_n, \ldots
\]
is converge and \( \lim \varphi_n = 0 \).

Proof. Using the method of the mathematical induction and Theorem 2.4 we obtain
\[
\cos \varphi_n = \frac{\beta + (\alpha + \beta)\cos \varphi_{n-1}}{\alpha + 2\beta \cos \varphi_{n-1}}
\]
as well as \( \varphi_n \in (0, \frac{2\pi}{3}) \). From (15) we get
\[
\cos \varphi_n - \cos \varphi_{n-1} = \frac{\beta(1 - \cos \varphi_{n-1})(1 + 2 \cos \varphi_{n-1})}{\alpha + 2\beta \cos \varphi_{n-1}}.
\]
The equation (16) implies \( \cos \varphi_n > \cos \varphi_{n-1} \), so the series \( \{\cos \varphi_n\} \) is increasing one and since \( \cos \varphi_n < 1 \) then it is converge. From (15) we have \( \lim \cos \varphi_n = 1 \), so \( \lim \varphi_n = 0 \). □
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