On solutions for the $b$-family of peakon equations

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Abstract: We investigate a family of peakon equations, labelled by two parameters $b$ and $\kappa$, all of which admit one-peakon solutions in a unified form. The well known Camassa-Holm equation and Degasperis-Procesi equation are derived from the $b$-family peakon equations by choosing $b = 2$ and 3, respectively. For all values of $b$, their two peakon-type solutions are shown in difference form and in weak sense. At the same time, the dynamic behaviors are shown in difference phenomenons, such as the peaked waves collide elastically in the case $b = 2$, while the peaked waves collide inelastically in other cases.

Keywords: peakon solution, dynamic behavior, $b$-family of peakon equations

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1 Introduction

In recent years, there has been considerable interest in the following a family of partial differential equations \cite{1,2}

\[ u_t + b\kappa u_x - u_{xxx} + (b+1)u u_x = bu_x u_{xx} + uu_{xxx}, \]  

where $b$ and $\kappa$ are constants. Eq. (1) was called a $b$-family of peakon equations.

The first equation to be distinguished within this class is the dispersionless version of the integrable Camassa-Holm (CH) equation

\[ u_t + 2\kappa u_x - u_{xx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \]  

which is the $b = 2$ case of Eq. (1). This equation was derived physically as a shallow water wave equation by Camassa and Holm \cite{3}. Indeed, Eq. (2) was earlier reported by Fuchssteiner and Fokas \cite{4} as a bi-Hamiltonian generalization of the KdV equation. Eq. (2) shares most of the important properties of an integrable system of KdV type, for example, the existence of Lax pair formalism \cite{5}, the bi-Hamiltonian structure \cite{6} and can be solved by the inverse scattering method \cite{5,6}, Darboux transformation method \cite{7,8} and Hirota bilinear method \cite{9} and so on \cite{10,11,12}.

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When $\kappa > 0$, the CH equation (2) has smooth solitary waves. It has a peculiar property that when $\kappa \to 0$ the solutions become piecewise smooth and gas corners at their crests, such solutions are weak solutions of the CH equation with $\kappa = 0$ and are called peakons [3, 10]. The CH equation (2) has attracted considerable attention due to its complete integrability for all values of $\kappa$ and it has been proven that the peakons with $\kappa = 0$ and all smooth solitary waves for this equation (2) are orbitally stable [13, 14]. However, to the best of our knowledge, there are no reports on weak solutions related to the multi-peakon wave solution of the CH equation (2) with $\kappa \neq 0$. One of the aim for this paper is to derive some peakon type solutions of the CH equation (2) with $\kappa \neq 0$. We show that it is possible to construct the multi-peakon solution of the CH equation (2) by simply superimposing the single peakon solutions and solving for the evolution of their amplitudes and the positions of their peaks. At the same time, we further discuss their peakon-type solutions and analyze particularly dynamic behavior when the two peakons collide elastically.

For $b = 3$, Eq. (1) reduces to the Degasperis-Procesi (DP) equation [15, 16]

$$u_t + 3\kappa u_x - uu_x + 4u u_{xx} = 3u_x u_{xx} + uu_{xxx}. \tag{3}$$

This equation can be also be considered as a model for shallow water dynamics and be found to be completely integrable. Similar to the CH equation, the DP equation possesses the Lax pair, bi-Hamiltonian structure and also admits peakon dynamics.

Motivated by the Ref. [17, 18, 19], our main purpose in this paper is to investigate multi-peakon solutions of the $b$-family equations (1) with arbitrary $b$. This paper is organized as follows. In section 2, we look for multi-peakon type solutions to the $b$-family of peakon equations (1) arise form the solutions of the systems of ODEs. In particular, the dynamic behaviors of the two-peakon type solutions of the two cases $b = 1$ and $b \neq 1$ are discussed in detail. Moreover, the dynamical systems for $N$ peakon solutions of the $b$-family of peakon equations are given. Thus, some conclusions can be concluded in section 3.

2 Multi-peakon solutions to the $b$-family of peakon equations

The $b$-family of peakon equations (1) can be rewritten as

$$m_t + bm u_x + um_x = 0, \quad m = u - u_{xx} + \kappa. \tag{4}$$

In the next, we shall derive multi-peakon type solutions to the $b$-family of peakon equations (1).
2.1 One-peakon solution to the b-family of peakon equations

Let us suppose that one-peakon solution of the \( b \)-family of peakon equations is of the following form

\[
 u(x, t) = p_1(t) e^{-|x-q_1(t)|} + r_1(t),
\]

where \( p_1(t), q_1(t) \) and \( r_1(t) \) are functions of \( t \) needed to be determined. In this case, it is obvious to see that function \( u(x, t) \) do not has the first order derivative at the point \( x = q_1(t) \), but we can obtain their derivatives \( u_x, m, m_x \) and \( m_t \) in the weak sense as follows

\[
 u_x = -p_1 \text{sgn}(x - q_1) e^{-|x-q_1|}, \quad m = 2p_1 \delta(x - q_1),
\]

\[
 m_x = 2p_1 \delta'(x - q_1), \quad m_t = 2p_1 \delta(x - q_1) - 2p_1q_1 \delta'(x - q_1),
\]

where \( \delta(x - q_1) \) denotes delta distribution function.

Substituting (5)-(7) into Eq. (4) and integrating in the distribution sense, we can readily get

\[
 p_1 t = 0, \quad q_1 t = p_1 - \kappa, \quad r_1 t = 0, \quad r_1 + \kappa = 0.
\]

From (8), it is easy to see that we may have

\[
 p_1 = c, \quad q_1 = (c - \kappa) t + c_0, \quad r_1 = -\kappa,
\]

where \( c \) and \( c_0 \) are two arbitrary integration constants. Substituting (9) into Eq. (5), we obtain a one-peakon solution of the \( b \)-family of peakon equations (1) as follows

\[
 u = c e^{-|x-(c-\kappa)t-c_0|} - \kappa.
\]

For arbitrary value of \( \kappa \), the unified one-peakon solution (10) is similar to the result of the Ref. [20, 21].

2.2 Two-peakon dynamical system to the b-family of peakon equations

We assume that the \( b \)-family of peakon equations (1) admits two-peakon soliton as follows

\[
 u = p_1 e^{-|x-q_1|} + p_2 e^{-|x-q_2|} - \kappa,
\]

where \( p_1, p_2, q_1 \) and \( q_2 \) are functions of \( t \) needed to be determined. By a direct calculation in the distribution sense, we have

\[
 u_x = -p_1 \text{sgn}(x - q_1) e^{-|x-q_1|} - p_2 \text{sgn}(x - q_2) e^{-|x-q_2|},
\]

\[
 m = 2p_1 \delta(x - q_1) + 2p_2 \delta(x - q_2).
\]
\[ m_x = 2p_1 \delta' (x - q_1) + 2p_2 \delta' (x - q_2), \quad (14) \]

\[ m_t = 2p_{1t} \delta (x - q_1) - 2p_1 q_{1t} \delta' (x - q_1) + 2p_{2t} \delta (x - q_2) - 2p_2 q_{2t} \delta' (x - q_2). \quad (15) \]

Substituting (11)-(15) into (4) and integrating through test functions yield the following ODE dynamical system

\[ p_{1t} = (b - 1)p_1 p_2 \text{sgn} (q_1 - q_2) e^{-|q_1 - q_2|}, \quad (16) \]

\[ p_{2t} = (b - 1)p_1 p_2 \text{sgn} (q_2 - q_1) e^{-|q_2 - q_1|}, \quad (17) \]

\[ q_{1t} = p_1 + p_2 e^{-|q_1 - q_2|} - \kappa, \quad (18) \]

\[ q_{2t} = p_1 e^{-|q_2 - q_1|} + p_2 - \kappa. \quad (19) \]

We shall discuss all possible values of \( b \) in the following cases.

**Case 1** \( b = 1. \)

Take \( b = 1 \), Eqs. (16)-(19) become the following equations

\[ p_{1t} = p_{2t} = 0, \quad (20) \]

\[ q_{1t} = p_1 + p_2 e^{-|q_1 - q_2|} - \kappa, \quad (21) \]

\[ q_{2t} = p_1 e^{-|q_2 - q_1|} + p_2 - \kappa. \quad (22) \]

Solving Eqs. (20)-(22), we have

\[ p_1 = c_1, \quad p_2 = c_2, \quad (23) \]

\[ q_1 = (c_1 - \kappa)t - \frac{c_2}{c_1 - c_2} \ln(1 + e^{-(c_1 - c_2)t - c_0}) + C_1, \quad (24) \]

\[ q_2 = (c_2 - \kappa)t - \frac{c_1}{c_1 - c_2} \ln(1 + e^{-(c_1 - c_2)t - c_0}) + C_2, \quad (25) \]

where \( c_0, c_1, c_2, C_1 \) and \( C_2 \) are integration constants. Substituting (23)-(25) into (11), we get a two-peakon solution

\[ u = c_1 e^{-|x - q_1|} + c_2 e^{-|x - q_2|} - \kappa, \quad (26) \]

where \( q_1 \) and \( q_2 \) are given in (24)-(25). The dynamic behaviors of the two-peakon wave (20) are shown in Figure 1. Figure 1 is shown two-peakon waves collide inelastically.

**Case 2** \( b \neq 1. \)

According to Eqs. (16)-(17), we have

\[ (p_1 + p_2)_t = 0. \quad (27) \]
By the above equation, it is easy to see that there are the following relations

\[ p_1 = p, \quad p_2 = -p + \Gamma, \]  

(28)

where \( p = p(t) \) is a function of \( t \), and \( \Gamma \) is an integration constant.

Without loss of generality, we take \( Q = q_1 - q_2 > 0 \) and combine Eqs. (18)-(19), we are able to get

\[ Q_t = (2p - \Gamma)(1 - e^{-Q}). \]  

(29)

From Eqs. (16)-(17) and (28), we have

\[ Q = -\ln \left( -\frac{p_t}{(b-1)p(p-\Gamma)} \right). \]  

(30)

Combining (29) and (30) leads to

\[-(b-1)p(p-\Gamma)p_{tt} = (b-1)p(p-\Gamma)(2p-\Gamma)p_t + (b-2)(2p-\Gamma)p_t^2. \]  

(31)

For Eq. (31), we shall give the following subcases \((b = 2)\) and \((b \neq 1, \Gamma \neq 0)\).

**Subcase 2.1** \( b = 2 \).

Let choosing \( b = 2 \), we have

\[ p_t + p^2 - \Gamma p = d, \]  

(32)

where \( d \) is a integrate constant. For get the solutions of the Eq. (32), we shall discuss the following three cases.

(i) For \( d > -\frac{\Gamma^2}{4} \), Eq. (32) leads to

\[ p = \frac{\Gamma}{2} + \Delta \coth \Delta(t - c_0), \quad \Delta = \sqrt{d + \frac{1}{4}\Gamma^2}. \]  

(33)

Substituting (30) and (33) into (18)-(19), we obtain

\[ q_1 = \left( \frac{\Gamma}{2} - \kappa \right) t + \ln \left| \frac{\Gamma}{2} \sinh \Delta(t - c_0) + \Delta \cosh \Delta(t - c_0) \right| + C_1, \]  

(34)

\[ q_2 = \left( \frac{\Gamma}{2} - \kappa \right) t - \ln \left| -\frac{\Gamma}{2} \sinh \Delta(t - c_0) + \Delta \cosh \Delta(t - c_0) \right| + C_2, \]  

(35)

where \( \Delta = \sqrt{d + \frac{1}{4}\Gamma^2} \) and \( c_0, C_1, C_2 \) are integrable constants.

Therefore, we have the first type two-peakon solutions of the CH equation (2) as

\[ u = \left[ \frac{\Gamma}{2} + \Delta \coth \Delta(t - c_0) \right] e^{-|x-q_1|} + \left[ \frac{\Gamma}{2} - \Delta \coth \Delta(t - c_0) \right] e^{-|x-q_2|}. \]  

(36)
where \(q_1\) and \(q_2\) are given in (34)-(35). In Figure 2, we have described the interaction processes of the first type two-peakon solutions (36). The head on elastic collision of two-peakon waves (36) is illustrated in Figure 2. This interaction is very similar to that of two solitons of the KdV equation.

(ii) For \(d < -\frac{\Gamma^2}{4}\), Eq. (32) leads to

\[
p = \Gamma + \Omega \tan(\Omega(t - c_0)), \quad \Omega = \sqrt{-d - \frac{1}{4} \Gamma^2}.
\] (37)

Substituting (30) and (37) into (18)-(19), we obtain

\[
q_1 = \left(\frac{\Gamma - \kappa}{2}\right) t - 2 \ln |\cos(\Omega(t - c_0))| + \ln \left|\frac{\Gamma}{2} \cos(\Omega(t - c_0)) + \Omega \sin(\Omega(t - c_0))\right| + C_1,
\] (38)

\[
q_2 = \left(\frac{\Gamma - \kappa}{2}\right) t + 2 \ln |\cos(\Omega(t - c_0))| - \ln \left|\frac{\Gamma}{2} \cos(\Omega(t - c_0)) + \Omega \sin(\Omega(t - c_0))\right| + C_2,
\] (39)

where \(\Omega = \sqrt{-d - \frac{1}{4} \Gamma^2}\) and \(c_0, C_1, C_2\) are integrable constants.

So we have second type two-peakon solutions for the CH equation (2)

\[
u = \left[\frac{\Gamma}{2} + \Omega \tan(\Omega(t - c_0))\right] e^{-|x-q_1|} + \left[\frac{\Gamma}{2} - \Omega \tan(\Omega(t - c_0))\right] e^{-|x-q_2|},
\] (40)

where \(q_1\) and \(q_2\) are given in (38)-(39). In Figure 3, the period structure of the second type two-peakon solutions (40) is described. It can be seen that this solution (40) has singularities.

(iii) For \(d = -\frac{\Gamma^2}{4}\), Eq. (32) leads to

\[
p = \Gamma + \frac{1}{t - c_0},
\] (41)

where \(c_0\) is an integrable constant. Substituting (30) and (41) into (18)-(19), we obtain

\[
q_1 = \left(\frac{\Gamma - \kappa}{2}\right) t + \ln \left|\frac{\Gamma}{2} (t - c_0) + 1\right| + C_1,
\] (42)

\[
q_2 = \left(\frac{\Gamma - \kappa}{2}\right) t - \ln \left|\frac{\Gamma}{2} (t - c_0) + 1\right| + C_2,
\] (43)

where \(c_0, C_1\) and \(C_2\) are integrable constants.

So we have third type two-peakon solutions for the CH equation (2)

\[
u = \left[\frac{\Gamma}{2} + \frac{1}{t - c_0}\right] e^{-|x-q_1|} + \left[\frac{\Gamma}{2} - \frac{1}{t - c_0}\right] e^{-|x-q_2|},
\] (44)

where \(q_1\) and \(q_2\) are given in (42)-(43). Figure 4 shows the profiles of the third type two-peakon solutions (44).

Subcase 2.2. \(b \neq 1, \Gamma \neq 0\).
In this case, we obtain a particular solutions of Eq. (31) as

$$p = \frac{A\Gamma}{A + B e^{-(b-1)\Gamma t}},$$  \hspace{1cm} (45)

where $A$ and $B$ are integrable constants. Substituting (45) into (28) and (18)-(19), we get

$$p_1 = \frac{A\Gamma}{A + Be^{-(b-1)\Gamma t}}, \hspace{1cm} p_2 = -\frac{A\Gamma}{A + Be^{-(b-1)\Gamma t}} + \Gamma,$$

$$q_1 = (\Gamma - \kappa)t + C_1, \hspace{1cm} q_2 = (\Gamma - \kappa)t + C_2,$$  \hspace{1cm} (46)

where $C_1$ and $C_2$ are integrable constants. Substituting (46)-(47) into (11), we have a solution of the $b$-family of peakon equations

$$u = \frac{A\Gamma}{A + Be^{-(b-1)\Gamma t}} e^{-|x-q_1|} + \left[-\frac{A\Gamma}{A + Be^{-(b-1)\Gamma t}} + \Gamma\right] e^{-|x-q_2|},$$  \hspace{1cm} (48)

where $q_1$ and $q_2$ are given in (47). Figure 5 plot the solitoff structure of the $b$-family of peakon equations for six different values of $b$.

### 2.3 $N$-peakon solution to the $b$-family of peakon equations

In general, we suppose an $N$-peakon solution of the $b$-family of peakon equations (1) has the following form

$$u = \sum_{j=1}^{N} p_j(t) e^{-|x-q_j(t)|} - \kappa.$$  \hspace{1cm} (49)

Substituting (49) into the $b$-family of peakon equations (11) and integrating through test functions, we obtain the $N$-peakon dynamical system as follows

$$p_{jt} = (b-1)p_j \sum_{k=1}^{N} p_k \text{sgn}(q_j - q_k) e^{-|q_j - q_k|},$$  \hspace{1cm} (50)

$$q_{jt} = \sum_{k=1}^{N} p_k e^{-|q_j - q_k|} - \kappa, \hspace{1cm} (j = 1, 2, \ldots, N).$$  \hspace{1cm} (51)

Thus, $N$-peakon solutions of the $b$-family of peakon equations with $\kappa \neq 0$ are obtained by simply superimposing the single peakon solutions and solving for the evolving of their amplitudes $p_j$ and the positions of their peakons $q_j$.

### 3 Conclusions

The $b$-family peakon equations (11) are generalizing nonlinear model. Remarkably, every member (free $b$ and $\kappa$) of the $b$-family equations (11) admits unified one-peakon solutions as form of (10).
In particular, the $b$-family peakon equations (1) include the two special integrable models, namely the Camassa-Holm and Degasperis-Procesi equations ($b = 2$ and 3).

In this paper, the multi-peakon solutions of the $b$-family equations (1) with $b = 1$ and $b \neq 1$ are discussed in detail. The results in this paper are wider than those already known [3]. At the same time, the dynamic behaviors of obtained solutions are illustrated through some figures. Taking into account the obtained results, we believe that Eqs. (50)-(51) relating to the $b$-family equations (1) deserves further investigation, such as in the cases of $N \geq 3$.

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Figure 1. (a) 3D graphs of the two-peakon solutions defined by (26) with $c_0 = 0$, $c_1 = 2$, $c_2 = 1$, $C_1 = C_2 = 0$, $\kappa = \frac{1}{10}$. (b) Contour plot of the two-peakon solutions defined by (26).

Figure 2. (a) 3D graphs of the first type two-peakon solutions defined by (36) with $c = \frac{1}{100}$, $c_0 = C_1 = C_2 = 0$, $\kappa = \frac{1}{100}$, $d = \frac{1}{4}$. (b) Contour plot of the first type two-peakon solutions defined by (36).
Figure 3. (a) 3D graphs of the second type two-peakon solutions defined by (40) with $c = c_0 = C_1 = C_2 = 0, \kappa = \frac{1}{100}, d = -1$. (b) Contour plot of the second type two-peakon solutions defined by (40).

Figure 4. (a) 3D graphs of the third type two-peakon solutions defined by (44) with $c = 2, c_0 = C_1 = C_2 = 0, \kappa = 1$. (b) Contour plot of the third type two-peakon solutions defined by (44).
Figure 5. 3D graphs of the soliton structure of the solutions defined by (48) with $C_1 = 6, C_2 = 1, \Gamma = 2, \kappa = A = B = 1, \ (a) \ b = 2, \ (b) \ b = 3, \ (c) \ b = 4.$