Simultaneous Integer Relation Detection and Its an Application

Chen Jing-wei \ §, Feng Yong \ § \ §, Qin Xiao-lin \ § \ § and Zhang Jing-zhong \ § 

Laboratory of Computer Reasoning and Trustworthy Computation, University of Electronic Science and Technology of China, Chengdu 610054, China

¶ Laboratory for Automated Reasoning and Programming, Chengdu Institute of Computer Applications, Chinese Academy of Sciences, Chengdu 610041, China

velen.chan@163.com {yongfeng, qinxl}@casit.ac.cn zjz101@yahoo.com.cn

ABSTRACT

Let \( x_1, \ldots, x_t \in \mathbb{R}^n \). A simultaneous integer relation (SIR) for \( x_1, \ldots, x_t \) is a vector \( m \in \mathbb{Z}^n \setminus \{0\} \) such that \( X^T m = 0 \) for \( i = 1, \ldots, t \). In this paper, we propose an algorithm SIRD to detect an SIR for real vectors, which constructs an SIR within \( O(n^2 + n^3 \log \lambda(X)) \) arithmetic operations, where \( \lambda(X) \) is the least Euclidean norm of SIRs for \( x_1, \ldots, x_t \). One can easily generalize SIRD to complex number field. Experimental results show that SIRD is practical and better than another detecting algorithm in the literature.

1. INTRODUCTION

Let \( x_1, \ldots, x_t \) be vectors in \( \mathbb{R}^n \), and denote \((x_1, \ldots, x_t)\) by \( X \). A simultaneous integer relation (SIR) for \( x_1, \ldots, x_t \) is a vector \( m \in \mathbb{Z}^n \setminus \{0\} \) such that \( X^T m = 0 \), i.e. \( x_i^T m = 0 \) for \( i = 1, \ldots, t \). For short, we also call \( m \) an SIR for \( X \).

When \( t = 1 \), we say that \( m \) is an integer relation for \( x_1 \). The problem of detecting integer relations for a rational or real vector is quite old. Historical surveys can be found in [5, 14, 11, 17, 13]. Among these integer relation detecting algorithms, the HJLS algorithm [16, 17] and the PSLQ algorithm [12, 13] have been used frequently.

In the present paper, using the technique to construct the hyperplane matrix in HJLS and a generalized method of the matrix reduction from PSLQ, we propose an algorithm SIRD, which can be used to detect an SIR for \( t \) real vectors. The cost of our algorithm is at most \( O(n^2 + n^3 \log \lambda(X)) \) exact arithmetic operations for detecting an SIR for \( X \), where \( \lambda(X) \) is the least Euclidean norm of SIRs for \( X \). Furthermore, our detecting algorithm SIRD either always finds an SIR for \( X \) if one exists or proves that there are no SIRs for \( X \) of norm less than a given size. Experimental results show that SIRD is practical.

In application, we successfully apply SIRD to find the minimal polynomial of an algebraic number \( \alpha \in \mathbb{C} \) with degree and height at most \( n \) and \( H \) respectively from its an approximation \( \bar{\alpha} \) satisfying \( \max_{1 \leq i \leq n} |\alpha_i - \bar{\alpha}_i| < \epsilon \), and propose the corresponding algorithm MPF, where the minimal polynomial of an algebraic number \( \alpha \) is the unique primitive polynomial \( p(x) \in \mathbb{Z}[x] \) of least degree such that \( p(\alpha) = 0 \). In fact, for \( i \) from 1 to \( n \) we run SIRD with \( v_1 = (1, \text{Re}(\bar{\alpha}), \ldots, \text{Re}(\bar{\alpha}))^T, v_2 = (0, \text{Im}(\bar{\alpha}), \ldots, \text{Im}(\bar{\alpha}))^T \) as its input and then an exact SIR for \( v_1, v_2 \) has been detected. We provide a sufficient controlling on \( \epsilon \) and prove that such an \( \epsilon \) is sufficient to enable an exact SIR for \( v_1 \) and \( v_2 \) to be also an SIR for \((1, \text{Re}(\bar{\alpha}), \ldots, \text{Re}(\bar{\alpha}))^T \) and \((0, \text{Im}(\bar{\alpha}), \ldots, \text{Im}(\bar{\alpha}))^T \), where \( \epsilon \) depends only on \( n \) and \( H \), as in [13]. It implies the correctness of MPF and is better than already existing results in [18, 25].

1.1 Related Works

In [16, 17], J. Hastad, B. Just, J. C. Lagarias, and C. P. Schnorr not only presented the HJLS algorithm and the first rigorous proof of a ‘polynomial time’ bound for a relation finding algorithm but also proposed a simultaneous relations algorithm (see [17] section 5), whereas HJLS is numerically unstable. The unstable examples can be found in [12, 13]. In their draft [26], C. Rössner and C. P. Schnorr studied the case of \( t = 2 \) by using a modified HJLS algorithm. But for the moment, [26] is still in a preliminary state with some open problems. The PSLQ algorithm, together with related lattice reduction schemes such as LLL [21], was named one of ten ‘algorithms of the twentieth century’ by the publication Computing in Science and Engineering (see [11, 9]), and is now extensively used in Experimental Mathematics, with applications such as identification of multiple zeta constants, a new formula for \( \pi \), finding algebraic relations and so on (see [4, 3, 2]). Moreover, PSLQ is numerically stable and can be easily generalized to complex number field and Hamiltonian quaternion number field (see [13]), but it is not suitable to detect an SIR for several real vectors.

Permission to make digital or hard copies of all or part of this work for personal or classroom use is granted without fee provided that copies are not made or distributed for profit or commercial advantage and that copies bear this notice and the full citation on the first page. To copy otherwise, to republish, to post on servers or to redistribute to lists, requires prior specific permission and/or a fee.

Copyright 200X ACM X-XXXXXX-XX/XX/XX ...$10.00.
The SIRD algorithm in this paper is to detect an SIR for \( t \) real vectors and can be applied to detect an integer relation in \( \mathbb{Z}^n \) for a complex vector or a Hamilton quaternion number vector. A significant body of experimental data shows that SIRD is practical and better than the HJLs simultaneous relations algorithm.

In fact, the MPF algorithm in this paper is a positive answer to the following interesting question: Suppose we are given an approximation to an algebraic number \( \alpha \), and two bounds on the degree and the size of the coefficients of its minimal polynomial respectively. Is it possible to infer the minimal polynomial? The question was raised, independently, by Manuel Blum in theoretical cryptography (see [19, 20]) and the last author of this paper in automated reasoning (see [20]). The first complete answer to this question, KLL algorithm, was presented by R. Kannan, A.K. Lenstra and L. Lovász in [19, 20] by using the celebrated lattice reduction algorithm LLL [21]. In the computer algebra system Maple, the built-in function \texttt{PolynomialTools:-MinimalPolynomial()} is a function to find a polynomial of degree \( n \) (or less) with small integer coefficients which has the given approximation \( \tau \) of an algebraic number as one of its roots and is based on KLL algorithm. The correctness of the polynomial returned by the built-in function depends on the accuracy of the approximation (see Maple’s Help).

From another aspect, the minimal polynomial of an algebraic number \( \alpha \) with exact degree \( n \) can be found by detecting an integer relation for the vector \( v = (1, \alpha, \ldots, \alpha^n)^T \). Besides HJLs, B. Just also presented an algorithm to detect integer relations for a given vector consists of algebraic numbers in [13]. We can apply Just’s algorithm or HJLs to the vector \( v \) for finding the minimal polynomial of \( \alpha \). However, both Just’s algorithm and HJLs are not numerically stable, as mentioned previously. All these algorithms are based on LLL. Two authors of this paper presented a method to reconstruct a rational number from its an approximation by using continued fraction in [30]. It may be viewed as an answer to a special case of the question. Based on PSLQ, one can find algebraic relations, such as [3, 7, 9, 11], whereas these articles did not involve the minimal polynomial finding. The authors of this paper also presented an algorithm in [24] for finding the minimal polynomial of a real algebraic number from its an approximation. However, these PSLQ based algorithms can not deal with complex algebraic numbers since PSLQ only outputs a relation in Gaussian integer ring for a complex vector.

Fortunately, our simultaneous integer relation detection algorithm SIRD in present paper can be used to overcome these pitfalls. Applying SIRD to one or two real vectors, we present another affirmative answer, the MPF algorithm, to the question above. We show that MPF is a more efficient minimal polynomial finding algorithm comparing with the algorithms in [13, 25] and provide a sufficient condition on the error controlling, from which we can claim that the polynomial returned by MPF is the exact minimal polynomial of the algebraic number that we only know an approximate value and two bounds on its degree and height. Although a similar even better complexity can be obtained by KLL, MPF has its own meaning since it is a new method without using LLL reduction.

**Road-map.** In section 2 and 3 we first give some preliminaries, and then present the SIRD algorithm and analyze it. We report on some experimental results about the performance of SIRD in section 4 apply SIRD to find the minimal polynomial of an algebraic number from its an approximation and propose the MPF algorithm in section 5 in which we also analyze MPF and present the result of error controlling. We conclude this paper with section 6.

**Notations.** Throughout this paper, \( \mathbb{Z}, \mathbb{R}, \) and \( \mathbb{C} \) are the sets of integers, real numbers, and complex numbers respectively. The real and imaginary parts of \( z \in \mathbb{C} \) will be denoted \( \operatorname{Re}(z) \) and \( \operatorname{Im}(z) \) respectively. For \( c \in \mathbb{R}, |c| = |c + i0| \). All vectors in this paper are column vectors, and will be denoted in bold. If \( x \in \mathbb{R}^n \), then \( |x| \) represents its Euclidean norm, i.e. \( |x| = \sqrt{x \cdot x} \), where \( <, > \) is the inner product of two vectors. We denote \( n \times n \) identity matrix by \( I_n \). Given a matrix \( A = (a_{ij}) \), we denote its transpose by \( A^T \), its trace by \( \operatorname{tr}(A) \), its determinant by \( |A| \), and its Frobenius norm by \( \|A\|_F = (\sum_{i} a_{ii}^2)^{1/2} \), i.e. \( \|A\|_F = (\sum_{i} a_{ii}^2)^{1/2} \). We say that a matrix \( A \) is lower trapezoidal if \( a_{ij} = 0 \) for \( i < j \). \( GL(n, \mathbb{Z}) \) is the group of \( n \times n \) unimodular matrix with entries in \( \mathbb{Z} \). The height of a vector is defined by the maximum of all the absolute values of its entries. For a polynomial \( f(x) = \sum_{n=0} a_n x^n \), we denote by \( \deg(f) \) its degree with respect to \( x \), \( |f| = \max_{n=0} |a_n| \) its one norm, \( \|f\| = (\sum_{n=0} |a_n|^2)^{1/2} \) its Euclidean length, and \( \operatorname{height}(f) = \max_{0 \le i \le n} |a_i| \) its height.

2. PRELIMINARIES

In what follows we always suppose that \( x_1, \ldots, x_t \) are linearly independent vectors in \( \mathbb{R}^n \), where \( x_i = (x_{i,1}, \ldots, x_{i,n})^T \).

Obviously, we have \( t < n \). We denote by \( X \) the matrix \((x_1, \ldots, x_t)\), and suppose that \( X \in \mathbb{R}^{n \times t} \) satisfies

\[
\begin{vmatrix}
  x_{1,n-t+1} & x_{2,n-t+1} & \cdots & x_{t,n-t+1} \\
x_{1,n-t+2} & x_{2,n-t+2} & \cdots & x_{t,n-t+2} \\
\vdots & \vdots & \ddots & \vdots \\
x_{1,n} & x_{2,n} & \cdots & x_{t,n}
\end{vmatrix} \neq 0, \quad (2.1)
\]

unless otherwise specified. For \( X \in \mathbb{R}^{n \times t} \) not satisfying [24], exchanging some rows of \( X \) produces \( X' = CX \), where \( C \) is an appropriate matrix in \( GL(n, \mathbb{Z}) \). And then we detect an SIR for \( X' \). If \( m \) is an SIR for \( X' \), then \( C^T m \) is an SIR for \( X \).

2.1 Hyperplane Matrix

**Definition 2.1 (Hyperplane Matrix).** Let \( X = (x_1, 
\ldots, x_t) \in \mathbb{R}^{n \times t} \). A hyperplane matrix with respect to \( X \) is any matrix \( W \in \mathbb{R}^{n \times (n-t)} \) such that \( X^T W = 0 \) and the columns of \( W \) span \( X^\perp = \{ y \in \mathbb{R}^n : x_i^T y = 0, i = 1, \ldots, t \} \).

Now we introduce a method to construct a hyperplane matrix for \( X \).

Let \( b_1, \ldots, b_n \) form a standard basis of \( \mathbb{R}^n \), i.e. the \( i \)-th entry of \( b_i \) is 1 and others are 0. By performing the process of standard Gram-Schmidt orthogonalization to \( x_1, \ldots, x_t, b_1, \ldots, b_n \) in turn we have

\[
x_1^* = \frac{x_1}{\|x_1\|_2}, \quad x_k^* = \frac{x_k}{\|x_k\|_2}, \quad k = 2, \ldots, t,
\]

\[
b_1^* = \frac{\langle b_1^*, x_1^* \rangle}{\|x_1^*\|_2}, \quad b_1 = \frac{b_1}{\|b_1\|_2},
\]

\[
b_2^* = \frac{\langle b_2^*, x_1^* \rangle}{\|x_1^*\|_2}, \quad b_2 = \frac{b_2}{\|b_2\|_2},
\]

\[
\vdots
\]

\[
b_t^* = \frac{\langle b_t^*, x_1^* \rangle}{\|x_1^*\|_2}, \quad b_t = \frac{b_t}{\|b_t\|_2},
\]

where \( \langle \cdot, \cdot \rangle \) is the inner product of two vectors.
\[ b_i^* = b_i - \sum_{j=1}^{t} \frac{b_i \cdot x_j^*}{\| b_i \|_2^2} x_j^* - \sum_{j=1}^{i-1} \mu_{ij} b_j^*, \]
\[ b_i^* = \frac{b_i - b_{i-1} \mu_{ij} b_j^*}{\| b_i \|_2}, \quad i = 2, \ldots, n, \]
where \( \mu_{ij} = \left\{ \begin{array}{ll}
\frac{b_i \cdot b_j^*}{\| b_j \|_2^2} & \text{if } \| b_i \|_2 \neq 0, \\
0 & \text{if } \| b_i \|_2 = 0. \end{array} \right. \]

**Lemma 2.2.** Let \( x_k, x_k^* \), \( b_j \) and \( b_j^* \) be as above. Then

1. there exist \( t \) elements in \( \{1, \ldots, n\} \) denoted by \( j_1, \ldots, j_t \) such that \( b_{j_1}^* = \cdots = b_{j_t}^* = 0. \)
2. \( b_{n-t+1}^* = \cdots = b_{n}^* = 0. \)

**Proof.** Part 1 easily follows from the process of standard Gram-Schmidt orthogonalization. We next prove \( b_{n-t+1}^* = \cdots = b_{n}^* = 0 \) when \( t \geq 1 \) holds. Set
\[ a_1 x_1 + \cdots + a_t x_t + b_1 b_1 + \cdots + b_{n-t} b_{n-t} = 0. \]
Taking each side as a column vector and observing the last \( t \) components of two sides, we have \( a_1 = \cdots = a_t = 0. \) And since \( b_1, \ldots, b_{n-t} \) are linearly independent, we have \( l_t = \cdots = l_{n-t} = 0. \) Thus the \( n \) vectors \( x_1, \ldots, x_t, b_1, \ldots, b_{n-t} \) are linearly independent. This implies \( b_{n-t+1} = \cdots = b_{n} = 0. \)

**Definition 2.3.** \((H_X)\). For \( X \in \mathbb{R}^{n \times t} \) satisfying \((\mathcal{L})\), define \( H_X \) to be the \( n \times (n-t) \) matrix \((b_1^*, \ldots, b_{n-t}^*)^T \).

**Lemma 2.4.** Let \( X \in \mathbb{R}^{n \times t} \) and \( H_X \) be as above. Then
1. \( H_X^T H_X = I_{n-t} \).
2. \( \|H_X\|_F = \sqrt{n-t}. \)
3. \((x_1^*, \ldots, x_t^*, H_X)\) is an orthogonal matrix.
4. \( X^T H_X = 0 \), i.e., \( H_X \) is a hyperplane matrix of \( X \).
5. \( H_X \) is a lower trapezoidal matrix and every diagonal element of \( H_X \) is nonzero.

**Proof.** Since every two columns of \( H_X \) are orthogonal, part 1 follows. And part 2 follows from part 1. Let \( X^* = (x_1^*, \ldots, x_t^*)^T \). Obviously, \((x_1^*, \ldots, x_t^*, H_X)\) is an orthogonal matrix. From part 3 and standard Gram-Schmidt orthogonalization we have \( X^T H_X = 0 \) and \( X = X^* Q \) respectively, where \( Q \) is an appropriate \( t \times t \) invertible matrix. Thus \( X^T H_X = Q^T X^T H_X = 0 \) and hence that part 4 follows. We now prove part 5. Denote the \( k \)-th element of \( b_1^* \) by \( b_{1,k}^* \). The diagonal elements of \( H_X \) are \( b_{i,i}^* \) for \( i = 1, \ldots, n \). Before normalizing \( b_1^* \), we have \( b_{i,i}^* = 1 - \sum_{k=1}^{i-1} x_{i,k}^* - \sum_{j=1}^{i-1} b_{j,j}^* \), and at the same time, \( 0 \leq \| b_i^* \|_2^2 \leq \| b_{i,i}^* \|_2^2 \). Thus all the diagonal elements of \( H_X \) are nonzero. Now we only need to show that \( H_X \) is lower trapezoidal. From standard Gram-Schmidt orthogonalization, we can check that \( b_{1,k}^* < b_{1,i}^*, b_{k,k}^* \geq 0 \) holds for \( i > k \). This completes the proof.

So far, we have had a method to produce a hyperplane matrix \( H_X \) for \( X \in \mathbb{R}^{n \times t} \). The basic idea is from HJLS (see \([18, 17]\)). The same strategy was also used in PSLQ, however, in which partial sum was adopted instead of Gram-Schmidt orthogonalization.

**Lemma 2.5.** For \( X = (x_1, \ldots, x_t) \in \mathbb{R}^{n \times t} \) define \( P_X = H_X H_X^T \). Then
1. \( P_X = P_X \).
2. \( P_X = I_n - \sum_{i=1}^{t} x_i^* x_i^T. \)
3. \( P_X^2 = P_X \).
4. \( \|P_X\|_F = \sqrt{n-t}. \)
5. \( P_X z = z \) for any \( z \in X^+. \) Particularly, \( P_X m = m \) for any \( m \in \mathbb{R}^{n \times t} \).

**Proof.** The proof of the first part is easy. Let \( U = (x_1^*, \ldots, x_t^*, H_X) \). From Lemma 2.2 we have \( I_n = U U^T = X^T H_X + \sum_{i=1}^{t} x_i^* x_i^T \). Thus part 2 follows. Part 3 and part 4 follow from \( P_X^2 = H_X (H_X^T H_X)^T H_X = H_X H_X^T \). As this theorem easily follows from the proof of Theorem 1 in \([13]\) for the case of \( X \in \mathbb{R}^{n \times t} \).

**Theorem 2.6.** Let \( X \in \mathbb{R}^{n \times t} \) and \( H_X \) be as above. Suppose that for any matrix \( A \in GL(n, \mathbb{Z}) \) there exists an orthogonal matrix \( Q \in \mathbb{R}^{(n-t) \times (n-t)} \) such that \( (h_{i,j}) = AH_X Q \) is lower trapezoidal and all of the diagonal elements of \((h_{i,j})\) satisfy \( h_{i,j} \neq 0 \). Then for any \( m \in \mathbb{R}^{n \times t} \) we have
\[
\frac{1}{\max_{1 \leq j \leq n-t} |h_{j,j}|} = \min_{1 \leq j \leq n-t} \frac{1}{|h_{j,j}|} \leq \|m\|_2. \tag{2.2}
\]
As this theorem easily follows from the proof of Theorem 1 of \([13]\) with little modifications, the detail has been omitted here.

The lower bound given in \((2.2)\) when \( t = 1 \) is consistent with a similar lower bound in \([14, 15]\). Moreover, if a method to reduce the norm of \( H_X \) by multiplication by some unimodular \( A \in GL(n, \mathbb{Z}) \) on the left has been developed, then it will produce an increasing lower bound on \( \lambda(X) \), where \( \lambda(X) \) is the least Euclidean norm of SIRs for \( X \). In fact this theorem suggests a strategy to detect an SIR for \( X \).

### 2.2 Matrix Reduction

We now study how to reduce the hyperplane matrix \( H_X \). First we recall (modified) Hermite reduction in \([13]\).

**Algorithm 1 (Modified Hermite Reduction).**

**Input:** a lower trapezoidal matrix \( H = (h_{i,j}) \in \mathbb{R}^{n \times (n-1)} \) with \( h_{i,j} \neq 0 \).

**Output:** a reducing matrix \( D \) of \( H \).

1. \( D := I_n \).
2. for \( i \) from 2 to \( n \) do
3. for \( j \) from \( i-1 \) by \(-1\) to 1 do
4. \( q := [h_{i,j} / h_{i,j}] \), where \( |c| = |c + 1/2| \) for a real number \( c \).
5. for \( k \) from 1 to \( n \) do
6. \( d_{i,k} := d_{i,k} - q d_{i,k} \).
7. return the \( n \times n \) matrix \( D \).

If Algorithm 1 outputs \( D \) for an \( n \times (n-1) \) matrix \( H \), we say that \( DH \) is the modified Hermite reduction of \( H \) and that \( D \) is the reducing matrix of \( H \). This reduction develops the left multiplying modified Hermite reducing matrix \( D \).

Hermite reduction is also presented in \([13]\), and is equivalent to modified Hermite reduction for a lower triangular matrix \( H \) with \( h_{i,j} \neq 0 \) (see \([13]\) Lemma 3). Both the two equivalent reductions have the following properties:

1. \( P_X = I_n - \sum_{i=1}^{t} x_i^* x_i^T. \)
2. \( P_X^2 = P_X \).
3. \( P_X \| f \| = \sqrt{n-t}. \)
4. \( P_X z = z \) for any \( z \in X^+. \) Particularly, \( P_X m = m \) for any \( m \in \mathbb{R}^{n \times t} \).
1. The reducing matrix $D \in GL(n, \mathbb{Z})$.
2. For all $k > i$, the (modified) Hermite reduced matrix $H' = (h_{i,j}') = DH$ satisfies $|h_{i,i}'| \leq |h_{i,i}|/2$.

In order that the reduced and reducing matrices of $H_X \in \mathbb{R}^{n \times (n-t)}$ satisfy the two properties above, we need the following generalized Hermite reduction.

Algorithm 2 (Generalized Hermite Reduction).

Input: a lower trapezoidal matrix $H = (h_{i,j}) \in \mathbb{R}^{n \times (n-t)}$ with $h_{j,j} \neq 0$.

Output: a reducing matrix $D$ of $H$.

1: $D := I_n$

2: for $i$ from $2$ to $n$ do

3: if $i \leq n - t + 1$ then $temp := i - 1$

4: for $j$ from $temp$ by $-1$ to $1$ do

5: if $q := |h_{i,j}/h_{j,j}| \neq 0$

6: for $k$ from $1$ to $n$ do

7: $d_{i,k} := d_{i,k} - qd_{j,k}$

8: for every two integers $s_1, s_2 \in \{n-t+1, \ldots, n\}$ satisfying $s_1 < s_2$, $h_{s_1,n-t} = 0$ and $h_{s_2,n-t} \neq 0$ do

9: exchange the $s_1$-th row and the $s_2$-th row of $D$.

10: return the $n \times n$ matrix $D$.

If Algorithm 2 output $D$ for an $n \times (n-t)$ matrix $H$, we call $DH$ the generalized Hermite reduction of $H$ and $D$ the reducing matrix of $H$. Obviously, generalized Hermite reduction is equivalent to modified Hermite reduction when $t = 1$. In addition, we can easily check that generalized Hermite reduction remains the two properties mentioned above.

Remark 1. There are two main differences between (modified) Hermite reduction and generalized Hermite reduction. Firstly, the last $t-1$ rows of $H$ will also be reduced by the first $n-t$ rows of $H$ in generalized Hermite reduction, while (modified) Hermite reduction cannot do so. Secondly, generalized Hermite reduction exchanges the $s_1$-th row and the $s_2$-th row of $D$ if $s_1 < s_2$, $h_{s_1,n-t} = 0$ and $h_{s_2,n-t} \neq 0$ (from Step 2 to Step 9). This implies that if $h_{n-t+1,n-t} = 0$ after generalized Hermite reduction then $h_{n-t+2,n-t} = \cdots = h_{n,n-t} = 0$. This property plays an important role in the proof of Lemma 3.1.

3. THE SIRD ALGORITHM

3.1 The Description of SIRD

Using the hyperplane matrix constructing method and generalize Hermite reduction in the previous section we can get a simultaneous integer relation detecting algorithm SIRD.

3.2 Analysis of SIRD

Let $H(k)$ be the result after $k$ iterations of SIRD. Why do we set the parameter $\gamma > 2/\sqrt{3}$ at Step 5? Suppose the $r$ chosen in Step 5 is not $n - t$. In this case we let $\alpha, \beta, \lambda, \delta$ be as in (3.1). Then

\[
\begin{pmatrix}
\alpha \\
\beta \\
\lambda \\
\delta
\end{pmatrix}
\]

is the submatrix of $H(k-1)$ consisting of the $r$ and $r+1$ rows of columns $r$ and $r+1$, where $r < n - t$. After Step 4 has been performed $\lambda$ may not be zero, which makes that

Algorithm 3 (The SIRD Algorithm).

Input: $(x_1, \ldots, x_t) = X \in \mathbb{R}^{n \times t}$ satisfying (2.1).

Output: either output an SIR for $X$ or give a lower bound on $\lambda(X)$.

1: Initialization. Compute the hyperplane matrix $H_X$, set $H := H_X$, $B := I_n$.

2: Reduction. Call Algorithm 2 to reduce $H_X$ producing the reducing matrix $D \in GL(n, \mathbb{Z})$. Set $X := XD^{-1}$, $H := DH_B$, $B := BD^{-1}$.

3: loop

4: Exchange. Let $H = (h_{i,j})$. Choose an integer $r$ such that $\gamma |h_{r,r}| \geq \gamma |h_{i,i}|$ for $1 \leq i \leq n - t$, where $\gamma > 2/\sqrt{3}$. Define the permutation matrix $R$ to be the identity matrix with the $r$ and $(r+1)$-rows exchanged. Update $X := XR$, $H := RH_B$, $B := BR$.

5: Corner. Let

\[
\alpha := h_{r,r}, \quad \beta := h_{r+1,r}, \quad \lambda := h_{r+1,r+1}, \quad \delta := \sqrt{\beta^2 + \lambda^2}.
\] (3.1)

6: Reduction. Call Algorithm 2 to reduce $H_X$ producing $D$. Update $X := XD^{-1}$, $H := DH_B$, $B := BD^{-1}$.

7: Compute $G := 1/\max_{1 \leq j \leq n-t} |h_{j,j}|$. Then there exists no SIR whose Euclidean norm is less than $G$.

8: if $x_j = 0$ for some $1 \leq j \leq n$, or $h_{n-t,n-t} = 0$ then

9: return the corresponding SIR for $X$.

10: end loop

$H$ is not lower trapezoidal. After Step 4 the result is

\[
\begin{pmatrix}
\beta \\
\alpha \\
\lambda \\
\delta
\end{pmatrix}
\begin{pmatrix}
\beta/\delta \\
\alpha/\delta \\
-\lambda/\delta \\
-\alpha/\delta
\end{pmatrix}
= \begin{pmatrix}
\delta/\alpha \\
-\lambda/\alpha \\
-\beta/\alpha \\
-\gamma/\alpha
\end{pmatrix}.
\] (3.2)

Since $r$ is chosen such that $\gamma |h_{r,r}| \geq \gamma |h_{i,i}|$ is as large as possible, and $r < n - t$ we have $|h_{r+1,r+1}|(k-1)| \leq \frac{\gamma}{3} |h_{r,r}|(k-1)|$, hence $|\lambda| \leq \frac{\lambda}{3} |\alpha|$. From the property of generalized Hermite reduction we have that $|\lambda| \leq \frac{\lambda}{4} |\alpha|$, which then gives

\[
|h_{r,r}(k)| = \frac{|\lambda|}{|\alpha|} = \sqrt{\frac{\beta^2 + \lambda^2}{\alpha^2}} \leq \sqrt{\frac{1}{4} + \frac{1}{4}}.
\] (3.3)

Thus $|h_{r,r}|$ is reduced as long as $\sqrt{\frac{1}{4} + \frac{1}{4}} < 1$, i.e. $\gamma > 2/\sqrt{3}$. As was pointed out by Borwein (see [5]), although this increases $h_{r+1,r+1}$, this is not a significant problem. At each step we force the larger diagonal elements of $H$ toward $h_{n-t,n-t}$, where their size can be reduced by at least a factor of 2 when $r = n - t$.

As a matter of fact, the parameter $\gamma$ can be freely chosen in the open interval $(2/\sqrt{3}, +\infty)$.

Lemma 3.1. If $h_{j,j}(k) = 0$ for some $1 \leq j \leq n - t$ and no smaller $k$, then $j = n - t$ and an SIR for $X$ must appear as a column of the matrix $B$.

Proof. By the hypothesis on $k$ we know that all diagonal elements of $H(k-1)$ are not zero. Now, suppose the $r$ chosen in Step 4 is not $n - t$. Since generalized Hermite reduction does not introduce any new zeros on the diagonal, and from the analysis of Step 4 and Step 5 above, we have
that no diagonal element of \( H(k) \) is zero. This contradicts the hypothesis on \( k \) and our assumption that \( r < n - t \) was false. Thus we have \( r = n - t \) after the \((k-1)\)-th iteration has been completed.

Next we show that there must be an SIR for \( X \) appeared as a column of the matrix \( B \). We have \( X^T H X = 0 \) from Lemma 2.4 and hence that \( 0 = X^T B B^{-1} H X = X^T B B^{-1} H X Q = X^T B H(k-1) \), where \( Q \) is an appropriate orthogonal \((n-t) \times (n-t)\) matrix. Let \((z_1, \ldots, z_t)^T = X^T B \), where \( z_i = (z_{i1}, \ldots, z_{in})^T \). Then

\[
\begin{pmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0
\end{pmatrix} = X^T B H(k-1) = \begin{pmatrix}
z_1^T \\
\vdots \\
z_t^T
\end{pmatrix} H(k-1)
\]

\[
= \begin{pmatrix}
\sum_{k=\min-t}^{\min-t} z_{1k} h_{n-k,n-t} \gamma^{k-1} \\
\vdots \\
\sum_{k=\min-t}^{\min-t} z_{tk} h_{n-k,n-t} \gamma^{k-1}
\end{pmatrix} = \begin{pmatrix}
\cdots \\
\sum_{k=\min-t}^{\min-t} z_{1k} h_{n-k,n-t} \gamma^{k-1} \\
\vdots \\
\sum_{k=\min-t}^{\min-t} z_{tk} h_{n-k,n-t} \gamma^{k-1}
\end{pmatrix}.
\]

We know \( h_{n-t+1,n-t}(k-1) = 0 \) and \( h_{n-t,n-t}(k-1) \neq 0 \) from \( h_{n-t,n-t}(k) = 0 \). From Remark 1 and \( h_{n-t+1,n-t}(k-1) = 0 \) we have \( h_{n-t+2,n-t}(k-1) = \cdots = h_{n-t,\min-t}(k-1) = 0 \) which implies the last equality. Since \( h_{n-t,n-t}(k-1) \neq 0 \), it follows that \( z_{1n-t} = \cdots = z_{tn-t} = 0 \). Thus the \((n-t)\)-th column of \( B \) is an SIR for \( X \).

From Theorem 2.6 and Lemma 3.1, the correctness of SIRD has been proved. Moreover, we have

**Theorem 3.2.** Let \( \lambda(X) \) be the least Euclidean norm of any SIR for \( X \). Let \( \mathbf{m} \) be an SIR detected by SIRD. Then \( \|\mathbf{m}\|_2 \leq \gamma^{n-t-1} \lambda(X) \) for all \( \gamma > 2/\sqrt{3} \).

**Proof.** Assume \( r = n - t \) with \( h_{n-t,n-t}(k) \neq 0 \) and \( h_{n-t,n-t}(k+1) = 0 \) at the \(k\)-th iteration of SIRD. Then from Theorem 2.6 and the exchange rule of SIRD we have

\[
\lambda(X) \geq 1/\max_{1 \leq i \leq n-t} |h_{i,i}(k)| \geq \gamma^{t+1-n}/|h_{n-t,n-t}(k)|.
\]

At this time, \( \|\mathbf{m}\|_2 = 1/|h_{n-t,n-t}(k)| \) holds from the same strategy in the proof of Lemma 10 in [13].

**Definition 3.3.** (The \( \Pi \) function). For the \(k\)-th iteration in SIRD, define

\[
\Pi(k) = \prod_{1 \leq i \leq n-t} \min \left\{ \left( \frac{n-t}{2} \lambda(X) \right)^{\left( \frac{t}{2} \right)} \frac{1}{|h_{j,i}(k)|} \right\}^{n-t}.
\]

The routine of analyzing the number of iterations in [13] can be carried over here with redefining the \( \Pi \) function as above. So we state the following lemma directly without proof.

**Lemma 3.4.** For \( k > 1 \) we have

1. \( \left( \frac{n-t}{2} \lambda(X) \right)^{\left( \frac{t}{2} \right)} \geq \Pi(k) \geq 1 \), where \( \lambda(X) \) is the least norm of SIRs for \( X \).
2. \( \Pi(k) \geq \sqrt{\frac{2e}{\gamma^t}} \Pi(k-1) \).

From this lemma, it follows that the \( \Pi \) function is increasing with respect to \( k \) and has an upper bound for a fixed \( \gamma \in (2/\sqrt{3}, +\infty) \). Thus we have

**Theorem 3.5.** If \( X \in R^{n \times t} \) has SIRs, then the number of iterations such that SIRD finds an SIR for \( X \) will be more than

\[
\left( \frac{n}{2} \right)^{\left( \frac{t}{2} \right)} \log(\gamma^{n-t} \lambda(X)).
\]

**Proof.** From Definition 3.3 we can infer \( \Pi(0) \geq 1 \). And by Lemma 3.4 we know that

\[
\Pi(k) \geq \left( \sqrt{\frac{4e^2}{\gamma^t}} \right)^k.
\]

Solving \( k \) from this inequality gives the conclusion, as was to be shown.

**Corollary 3.6.** If \( X \in R^{n \times t} \) has SIRs, then there exists a \( \gamma \) such that SIRD will find an SIR for \( X \) in polynomial time \( O(n^2 + n^3 \log \lambda(X)) \).

**Proof.** Let \( \gamma = 2 \). Then SIRD will construct an SIR for \( X \) in no more than

\[
(n-t)^2(n + t - 1) + (n-t)(n + t - 1) \log \lambda(X)
\]

iterations. SIRD takes \( O(n-t) \) exact arithmetic operations per iteration, and hence that \( O((n-t)^3 + (n-t)^3 \log \lambda(X)) \) exact arithmetic operations is enough to produce an SIR for \( X \). Since \( t < n \), the proof is complete.

**Remark 2.** From this corollary, we can claim that our detecting algorithm always return an SIR for \( X \) if one exists. Additionally, SIRD will produce lower bound on the Euclidean norm of any possible SIRs for \( X \) (Theorem 2.6). Thus SIRD can be used to prove that there are no SIRs for \( X \) of norm less than a given size.

**Remark 3.** PSLQ may be viewed as a particular case of SIRD when \( t = 1 \). Similarly with PSLQ, SIRD can be easily generalized to complex field with \( \gamma > \sqrt{2} \) such that the outputs are in Gaussian integer ring and all conclusions mentioned above hold with corresponding modifications.

**Remark 4.** Moreover, SIRD can also be applied to detect an integer relation in \( Z^n \) for a given complex vector. For example, suppose \( z = x + yj \) in \( C^n \) with vector components \( x, y \in R^n \) where \( I = \sqrt{-1} \). Then SIRD can give an SIR \( \mathbf{m} \) for \( (x, y) \), and hence that \( \mathbf{m} \in Z^n \) is an integer relation for \( z \), but PSLQ only can give a Gaussian integer relation in \( Z[I]^n \). This is one of the biggest differences between SIRD and PSLQ. Furthermore, the matrix reducing method in SIRD is generalized Hermite reduction, which avoids LLL-type reduction. This is a difference not only between SIRD and HJLS, but also between SIRD and PSLQ because that (modified) Hermite reduction is not suitable to detect SIRs any more. And just the generalized Hermite reduction guarantees the correctness of SIRD.

### 4. PERFORMANCE RESULTS

In theory, the costs of SIRD and the HJLS simultaneous relations algorithm (see [17] section 5) are the same as in Corollary 3.6 in the worst case, whereas in practice SIRD usually needs fewer iterations. For \( v_1 = (11, 27, 31)^T \) and \( v_2 = (1, 2, 3)^T \), HJLS outputs \((19, -2, -5)^T \) after 5 iterations while SIRD outputs \((-19, 2, 5)^T \) after only 2 iterations.
Table 1: Comparison of performance results for HJLS and SIRD

| No. | n   | itrHJLS | itrsIRD | tHJLS  | tsIRD  |
|-----|-----|---------|---------|--------|--------|
| 1   | 4   | 15      | 12      | 0.047  | 0.0    |
| 2   | 4   | 13      | 9       | 0.171  | 0.016  |
| 3   | 4   | 21      | 19      | 0.062  | 0.015  |
| 4   | 5   | 25      | 20      | 0.110  | 0.016  |
| 5   | 5   | 27      | 43      | 0.125  | 0.016  |
| 6   | 5   | 21      | 14      | 0.110  | 0.032  |
| 7   | 30  | 51      | 21      | 1.703  | 0.422  |
| 8   | 54  | 34      | 9       | 5.625  | 1.265  |
| 9   | 79  | 34      | 40      | 14.157 | 4.422  |
| 10  | 97  | 37      | 5       | 23.860 | 5.625  |
| 11  | 128 | 45      | 6       | 49.657 | 11.141 |
| 12  | 149 | 29      | 14      | 76.797 | 18.603 |
| 13  | 173 | 26      | 2       | 114.140| 25.000 |
| 14  | 192 | 29      | 2       | 153.078| 33.641 |
| 15  | 278 | 28      | 8       | 440.781| 102.860|
| 16  | 290 | 35      | 6       | 500.562| 118.578|
| 17  | 293 | 23      | 7       | 512.796| 123.265|
| 18  | 305 | 22      | 4       | 581.844| 137.672|
| 19  | 316 | 19      | 3       | 649.032| 147.796|
| 20  | 325 | 18      | 2       | 716.094| 159.813|

Both the SIRD algorithm and the HJLS simultaneous relations algorithm when \( t = 2 \), i.e. detecting an SIR for two vectors, were implemented in Maple. The tests were run on AMD Athalon 7750 processor (2.70 GHz) with 2GB main memory.

The purpose of the trials in Table 1 is to compare the performances of HJLS and SIRD. \( n \) in Table 1 gives the dimension of the relation vector. \( itr_{HJLS} \) and \( itrs_{SIRD} \) are the numbers of iterations of HJLS and SIRD respectively. The columns headed \( t_{HJLS} \) and \( t_{SIRD} \) give the CPU run time respectively of the two algorithms in seconds.

The 20 trials in Table 1 were constructed by Maple’s pseudo random number generator. The first 6 trials are for low dimension, and others for higher dimension. The results show that SIRD appears to be more effective than HJLS. In 18 out of 20 trials, the number of iterations of SIRD is less than that of HJLS. It is still true that SIRD usually needs fewer iterations than HJLS for more tests. This leads that the running time of SIRD is much less than HJLS. With \( n \) increasing, the difference between the efficiency of SIRD and HJLS is increasingly notable. On average, the SIRD running time is about 26.7% of the running time of HJLS. All these results are obtained under the condition that \( \gamma = 2/\sqrt{3} = 10^{-14} \).

The Maple implementation and more tests are available from http://cid-5dbb3d2c11c63a9b.skydrive.live.com/self.aspx/Public/sird.rar.

5. AN APPLICATION

Any SIR detecting algorithm intervenes in many fields of application, such as Diophantine approximating, numerical constants relations finding, etc. In this section, we discuss how to find the minimal polynomial of a complex algebraic number from its an approximation by using SIRD.

5.1 The MPF Algorithm

We say that a complex number \( \alpha \) is an algebraic number if \( \alpha \) is a root of a non-zero polynomial in one variable with integer coefficients. The minimal polynomial of \( \alpha \) is the unique primitive polynomial \( p(x) \in \mathbb{Z}[x] \) of least degree such that \( p(\alpha) = 0 \). The degree and height of \( \alpha \) are the degree and height of its minimal polynomial \( p(x) \) respectively.

In this section, let \( \alpha = a + bI \in \mathbb{C} \) be an algebraic number with degree at most \( n \), height at most \( H \), where \( I = \sqrt{-1} \). Suppose we are given an approximation \( \tilde{\alpha} \) to \( \alpha \) such that

\[
\max_{1 \leq i \leq n} |\alpha^i - \tilde{\alpha}^i| < \epsilon.
\]

Is it possible to infer the minimal polynomial from the approximation? Computer algebra system Maple has an LLL-based procedure, PolynomialTools:-MinimalPolynomial(), for finding the minimal polynomial of an algebraic number from its an approximation, whose basic idea is from [27, 19, 20]. Applying SIRD, we shall give another affirmative answer, the following MPF algorithm, to the question above.

Algorithm 4 (The MPF Algorithm).

Input: an approximation \( \tilde{\alpha} \) to \( \alpha \) satisfying (5.1), a degree bound \( n \), and a height bound \( H \), \( \epsilon \) satisfying (5.3).

Output: the minimal polynomial of \( \alpha \).

1: while \( 2 \leq i \leq n \) do
2: \( \nu := (1, \tilde{\alpha}, \ldots, \tilde{\alpha}^i)^T \)
3: Call SIRD with \( \gamma = 2 \) producing an integer relation \( \mathbf{p}_i = (p_0, p_1, \ldots, p_i)^T \) for \( \nu \)
4: \( p_i := \) the primitive part of \( \sum_{j=0}^{i} p_j x^j \)
5: if \( \text{height}(p_i) > \epsilon^{n-2}/\sqrt{n+H} \) then
6: \( i := i + 1; \) goto Step 1
7: end while

Remark 5. At Step 3 of MPF, \( \mathbf{p}_i \) is an SIR for \( \mathbf{v}_1 = (1, \text{Re} (\tilde{\alpha}), \ldots, \text{Re} (\tilde{\alpha}^i))^T \) and \( \mathbf{v}_2 = (0, \text{Im}(\tilde{\alpha}), \ldots, \text{Im}(\tilde{\alpha}^i))^T \) when \( \text{Im}(\tilde{\alpha}) \neq 0 \).

5.2 Error Controlling

The main idea of our minimal polynomial finding (MPF) algorithm to determine the minimal polynomial of an algebraic number from its an approximation is as follows: We try the value of \( i = 2, \ldots, n \) in order. With \( i \) fixed, we call SIRD for detecting an exact integer relation \( \mathbf{p}_i = (p_0, p_1, \ldots, p_i)^T \) for \( \nu = (1, \tilde{\alpha}, \ldots, \tilde{\alpha}^i)^T \). Then \( p_i(x) \equiv \sum_{j=0}^{i} p_j x^j \) satisfies \( p_i(\tilde{\alpha}) = 0 \), however, from which we can not decide whether \( p_i(\alpha) = 0 \) or not. Hence the most important problem is how to choose an appropriate \( \epsilon \) in (5.1) such that \( p_i(\tilde{\alpha}) = 0 \) implies \( p_i(\alpha) = 0 \). Before describing it in detail, we consider the following example.

Example 1. Let \( \alpha = 2 + \sqrt{3}I \). We know that the minimal polynomial of \( \alpha \) in \( \mathbb{Z}[x] \) is \( 7 - 4x + x^2 \). Let \( \tilde{\alpha} = 2.000 + 1.732I \) be the approximation to \( \alpha \) with four significant digits. Hence \( \mathbf{v}_1 = (1, 2, \ldots, 1)^T \) and \( \mathbf{v}_2 = (0, 1.732, 6.928)^T \). Feeding SIRD \( \mathbf{v}_1, \mathbf{v}_2 \) as its input vectors gives an SIR for \( \mathbf{v}_1, \mathbf{v}_2 \) after 2 iterations. The corresponding matrices \( B \) are

\[
\begin{pmatrix}
2 & 1 & 0 \\
-1 & -1 & 0
\end{pmatrix}
\begin{pmatrix}
7 & 0 & 2 \\
-4 & 0 & -1
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 1 \\
1 & 1 & 0
\end{pmatrix}
\]

It is obvious that the first column of the latter one is an SIR for \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \), and corresponds to the coefficients of the minimal polynomial of \( \alpha \). However, if we take only 3
significant digits for the same data, after 3 iterations SIRD outputs $(1213, -693, 173)^T$, which is an SIR for $(1, 2, 1)^T$ and $(0, 1.73, 6.93)^T$, but does not correspond to the coefficients of the minimal polynomial of \( \alpha \). For this reason, we have to appropriately control the error such that the output of MPF is correct.

**Lemma 5.1.** Let \( f \) be a polynomial in \( \mathbb{Z}[x] \) of degree \( n \). If \( \max_{1 \leq i \leq n} |\alpha^i - \bar{\alpha}^i| < \varepsilon \), then \( |f(\alpha) - f(\bar{\alpha})| \leq \varepsilon \cdot n \cdot \text{height}(f) \).

**Definition 5.2.** (Mahler measure). For any polynomial \( g = \sum_{i=0}^{m} g_i x^i \in \mathbb{Z}[x] \) of degree \( m \) with the complex roots \( z_1, z_2, \ldots, z_m \), we define the Mahler measure \( M(g) \) by

\[
M(g) = \prod_{j=1}^{m} \max\{|1, |z_j|\}.
\]

The Mahler measure of an algebraic number \( \alpha \) is defined to be the measure of its minimal polynomial.

**Lemma 5.3.** (see [23, Lemma 3]) Let \( \alpha_1, \ldots, \alpha_q \) be algebraic numbers of exact degree of \( d_1, \ldots, d_q \) respectively. Define \( D = \left[\mathbb{Q}(\alpha_1, \ldots, \alpha_q) : \mathbb{Q}\right] \). Let \( P \in \mathbb{Z}[x_1, \ldots, x_q] \) have degree at most \( N_h \) in \( x_h \) (\( 1 \leq h \leq q \)). If \( P(\alpha_1, \ldots, \alpha_q) \neq 0 \), then

\[
|P(\alpha_1, \ldots, \alpha_q)| \geq ||P||_1^{-D} \prod_{h=1}^{q} M(\alpha_h)^{-DN_h/d_h},
\]

where \( M(\alpha) \) is the Mahler measure of \( \alpha \).

This lemma gives a lower bound on \( |P(\alpha_1, \ldots, \alpha_q)| \) if \( P(\alpha_1, \ldots, \alpha_q) \neq 0 \) for an arbitrary multivariate polynomial \( P \in \mathbb{Z}[x_1, \ldots, x_q] \). If we apply it to \( g(x) = \sum_{i=0}^{m} g_i x^i \) in \( \mathbb{Z}[x] \), then we have

**Corollary 5.4.** Let \( \alpha \) be an algebraic number with exact degree \( n_0 \) and \( g(x) = \sum_{i=0}^{m} g_i x^i \in \mathbb{Z}[x] \). Suppose both height(\( g \)) and height(\( \alpha \)) are \( \leq H \). If \( g(\alpha) \neq 0 \), then

\[
|g(\alpha)| \geq (m + 1)^{-\left(\frac{n_0}{2}\right)} \cdot (n_0 + 1)^{-\frac{1}{2}} \cdot H^{-(m + q + 1)},
\]

where \( \text{height}(\alpha) \) is the height of \( \alpha \)'s minimal polynomial.

**Proof.** For \( f(x) \in \mathbb{Z}[x] \) with degree \( n \), we have Landau’s inequality: \( M(f) \leq ||f||_2 \) (e.g. see [23, p. 154]).

\[
\text{height}(f) \leq ||f||_1 \leq (n + 1)\text{height}(f), \quad \text{height}(f) \leq ||f||_2 \leq \sqrt{n + \text{height}(f)}.
\]

This corollary easily follows from Lemma 5.3 and the facts above.

Next we investigate how to choose \( \varepsilon \) to enable MPF to correctly return the minimal polynomial of \( \alpha \) from \( \bar{\alpha} \). We denote the exact degree of \( \alpha \) by \( n_0(\leq n) \). For \( 2 \leq i \leq n \), Step 3 in MPF gives a polynomial \( p_i \in \mathbb{Z}[x] \) with degree \( i \) such that \( p_i(\bar{\alpha}) = 0 \). From Corollary 5.4 we know that if \( p_i(\alpha) \neq 0 \), then

\[
|p_i(\alpha)| \geq (i + 1)^{1-\left(\frac{n_0}{2}\right)} \cdot (n_0 + 1)^{-\frac{1}{2}} \cdot H^{1-\left(\frac{n_0}{2}\right)} \geq M, \quad (5.2)
\]

where \( M = (n + 1)^{1-\left(\frac{n_0}{2}\right)} \cdot H^{(n - 2n_0 - 1)} \).

**Theorem 5.5.** Let \( \alpha, \bar{\alpha} \) and \( M \) be as above, and \( p \) a polynomial in \( \mathbb{Z}[x] \) with degree \( \leq n \) and height \( \leq H \). Then there exist some \( \varepsilon > 0 \) such that \( |p(\bar{\alpha})| = 0 \) implies \( p(\alpha) = 0 \).

**Proof.** Set \( \deg(p) = i(\leq n) \). From Lemma 5.1 we have

\[
|p(\alpha)| = |p(\alpha) - p(\bar{\alpha})| \leq \varepsilon \cdot i \cdot H \leq \varepsilon \cdot n \cdot H.
\]

Thus if \( \varepsilon < \frac{M}{H} \), then \( |p(\alpha)| < M \). From 5.2 it follows that \( p(\alpha) = 0 \).

If we substitute \( 2^n - \sqrt{n + 1} \cdot H \) for \( H \), we have

**Corollary 5.6.** Let \( \alpha \) and \( \bar{\alpha} \) be as above and

\[
\varepsilon < 2^{2n^2+4n(n+1)} \cdot \frac{1}{2^n} \cdot H^{-2n}.
\]

Then for \( i \) from 1 to \( n \), an integer relation for \((1, \alpha, \ldots, \alpha^i)^T\) with height \( \leq 2^{-n} \sqrt{n + 1} \cdot H \) is also for \((1, \alpha, \ldots, \alpha^i)^T\).

### 5.3 Correctness and Cost of MPF

Assume that the degree of \( \alpha \) is \( n_0 \) and that \( \varepsilon \) satisfies 5.6.

When \( 2 \leq i \leq n_0 \), there exists no relation for \((1, \alpha, \ldots, \alpha^i)\), which, combined with Corollary 5.6.0, means that \( p_i(x) \) must satisfy the condition in Step 3 of MPF and then go into next iteration. When \( i = n_0 \), we know that the coefficients of the minimal polynomial of \( \alpha \) form an integer relation for \((1, \alpha, \ldots, \alpha^{n_0})\), whose height \( \leq H \), hence Euclidean norm \( \leq n_0 + 1 \cdot H \). This implies that \((1, \alpha, \ldots, \alpha^{n_0})\) has also an integer relation with Euclidean norm \( \leq n_0 + 1 \cdot H \). From Theorem 5.2 we know that the height of the relation SIRD detected will \( \leq 2^{-n} \sqrt{n + 1} \cdot H \). Thus the relation detected by SIRD when \( i = n_0 \) will never satisfy the condition in Step 3 and corresponds an integral multiple of the minimal polynomial of \( \alpha \). Hence the correctness of MPF follows.

From 5.3 we have \( \log \varepsilon \in O(n^2 + n \log H) \). Thus we can give another answer to Blum’s and Zhang’s question without using LLL lattice reduction algorithm.

**Theorem 5.7.** Let \( \alpha \) be an algebraic number and let \( n \) and \( H \) be upper bounds of the degree and height of \( \alpha \) respectively. Suppose we are given an approximation \( \bar{\alpha} \) to \( \alpha \) such that \( \max_{1 \leq i \leq n} |\alpha^i - \bar{\alpha}^i| < \varepsilon \). Then the minimal polynomial of \( \alpha \) can be determined in \( O(n^2 + n^4 \log H) \) arithmetic operations on floating-point numbers having \( O(n^2 + n \log H) \) bit-complexity.

| Digits | Complexity |
|--------|------------|
| KLL[20] | \( O(n^2 + n \log H) \) |
| Just [15] | \( O(n^2 + n^4 \log H) \) |
| QFCZ[23] | \( O(n^2 + n \log H) \) |
| MPF | \( O(n^2 + n^4 \log H) \) |

Table 2: Comparison of different minimal polynomial finding algorithms

Table 2 gives a comparison of the digits and complexity of 4 different minimal polynomial finding algorithms in the worst case. Since the algorithm in [23] can only find the minimal polynomial of a real algebraic number, we don’t compare the complexity with it. It seems that a lower complexity can be achieved by using some new type LLL algorithms, such as L² [23] and H-LLL [24], but when we apply these new algorithms to find the minimal polynomial we have to choose \( \varepsilon \) as in a similar formula with 5.3. Thus multiple precision arithmetic is inevitable.

**Example 3 (con.).** For \( \alpha = 2 + \sqrt{3} \), its minimal polynomial \( 7 - 4x + x^2 \). Set \( n = 2 \) and \( H = 7 \). Computing the error tolerance as in equation 5.3 gives \( \varepsilon < 583443^{-1} \). Corollary 5.6 implies that \( |\log_{10} 583443^{-1}| = 5 \) correct decimal digits are sufficient to guarantee the output is correct. This example also illustrates that \( \varepsilon = 5 \) is only a sufficient condition on error controlling, but not a necessary one.
6. CONCLUSION

The number of iterations and the cost of SIRD algorithm are related to the parameter $\gamma$. For $v_1 = (86, 6, 8, 673)^T$ and $v_2 = (83, 5, 87, 91)^T$, if we choose $\gamma = 1.16$ then SIRD outputs $(-215, 402, 159, 22)^T$ after 12 iterations, however, if we choose $\gamma = 5$, SIRD outputs $(93, 364, 93, -14)^T$ after only 6 iterations. In future work we expect to find the best choice for $\gamma$. Additionally, how to choose the digits such that SIRD under floating-point arithmetic finds an exact SIRd is also in our interests. Finally, we see that the MPF algorithm can be used to factor $f$ in $\mathbb{Z}[x]$ like this: Solve an approximation root with accuracy satisfying equation (5.3), and call MPF for finding its minimal polynomial which corresponds an irreducible factor of $f$, and then repeat the two steps until $f$ has been factored completely. It is symbolic-numeric and different from traditional algorithms based on Hensel lifting.

Acknowledgements. This research was partially supported by the Knowledge Innovation Program of CAS (KJCX 2-YW-S02) and the NSFC (10771205).

7. REFERENCES

[1] Bailey, D., Borwein, J., Kapoor, V., and Weisstein, E. Ten problems in experimental mathematics. American Mathematical Monthly 113, 6 (2006), 481–509.

[2] Bailey, D. H., and Borwein, J. PSLQ: An algorithm to discover integer relations. LBNL Paper LBNL-2144E, (2000). available from http://escholarship.org/uc/item/95p4255b

[3] Bailey, D. H., Borwein, J. M., Calkin, N. J., Girgensohn, R., Luke, D. R., and Moll, V. H. Experimental Mathematics in Action. AK Peters, 2007.

[4] Bailey, D. H., and Broadhurst, D. J. Parallel integer relation detection: techniques and applications. Math. Comput. 70, 236 (2001), 1719 –1736.

[5] Bernstein, L. The Jacobi-Perron algorithm, its theory and application. Lecture Notes in Mathematics 207. Springer, 1971.

[6] Borwein, J., and Corless, R. Emerging tools for experimental mathematics. American Mathematical Monthly 106, 10 (1999), 889–909.

[7] Borwein, J. M., and Lisoněk, P. Applications of integer relation algorithms. Discrete Mathematics (Special issue for FPSAC 1997) 217 (2000), 65–82.

[8] Borwein, P. Computational Excursions in Analysis and Number Theory. Springer, New York, 2002.

[9] Borwein, P., Hare, K. G., and Meichsner, A. Reverse symbolic computations, the identify function. In Proceedings from the Maple Summer Workshop (Maple Software, Waterloo, 2002).

[10] Brentjes, A. J. Multi-dimensional continued fraction algorithms. Mathematisch Centrum Computational Methods in Number Theory, Pt. 2 p 287-319 (see N 84-17999 08-67) (1982).

[11] Dongarra, J., and Sullivan, F. Guest editors’ introduction: the top 10 algorithms. Comput. Sci. Eng. 2, 1 (2000), 22–23.

[12] Ferguson, H. R. P., and Bailey, D. H. Polynomial time, numerically stable integer relation algorithm. Tech. Rep. RNR-91-032, NAS Applied Research Branch, NASA Ames Research Center, Mar. 1992.

[13] Ferguson, H. R. P., Bailey, D. H., and Arno, S. Analysis of PSLQ, an integer relation finding algorithm. Math. Comput. 68, 225 (1999), 351–369.

[14] Ferguson, H. R. P., and Forcade, R. W. Generalization of the Euclidean algorithm for real numbers to all dimensions higher than two. Bull. Amer. Math. Soc. 1, 6 (1979), 912–914.

[15] Ferguson, H. R. P., and Forcade, R. W. Multidimensional Euclidean algorithms. (Crelle’s) Journal für die reine und angewandte Mathematik 334 (1982), 171–181.

[16] Hastad, J., Helfrich, B., Lagarias, J. C., and Schnorr, C. P. Polynomial time algorithms for finding integer relations among real numbers. In STACS ‘86. 1986, pp. 105–118.

[17] Hastad, J., Just, B., Lagarias, J. C., and Schnorr, C. P. Polynomial time algorithms for finding integer relations among real numbers. SIAM Journal on Computing 18, 5 (1989), 859–881.

[18] Just, B. Integer relations among algebraic numbers. In Mathematical Foundations of Computer Science 1989, pp. 314–320.

[19] Kannan, R., Lenstra, A. K., and Lovász, L. Polynomial factorization and nonrandomness of bits of algebraic and some transcendental numbers. In STOC ’84 (1984), pp. 191–200.

[20] Kannan, R., Lenstra, A. K., and Lovász, L. Polynomial factorization and nonrandomness of bits of algebraic and some transcendental numbers. Math. Comput. 50, 181 (1988), 235–250.

[21] Lenstra, A. K., Lenstra, H. W., and Lovász, L. Factoring polynomials with rational coefficients. Math. Ann. 261, 4 (1982), 515–534.

[22] Mignotte, M., and Waldschmidt, M. Linear forms in two logarithms and Schneider’s method. Math. Ann. 231 (1978), 241–267.

[23] Morel, I., Stehlé, D., and Villard, G. H-LLL: using Householder inside LLL. In ISSAC ’09 (2009), pp. 271–278.

[24] Nguyê, P. Q., and Stehlé, D. Floating-point LLL revisited. In EUROCRYPT 2005 (2005), pp. 215–233.

[25] Qin, X.-l., Feng, Y., Chen, J.-w., and Zhang, J.-z. Finding exact minimal polynomial by approximations. In SNC’09 (2009), pp. 125–131.

[26] Rössner, C., and Schnorr, C. P. Diophantine approximation of a plane. (1997). available from http://citeseer.ist.psu.edu/193822.html

[27] Schönhage, A. Factorization of univariate integer polynomials by Diophantine approximation and an improved basis reduction algorithm. In LNCS, vol. 172, 1984, pp. 436–447.

[28] von zur Gathen, J., and Gerhard, J. Modern Computer Algebra. Cambridge University Press, London, 1999.

[29] Yang, L., Zhang, J.-z., and Hou, X.-r. Obtaining exact value by approximate computations. Science in China Series A: Mathematics 50, 9 (2007), 1361–1368.