Formulation for nonaxisymmetric uniformly rotating equilibrium configurations in the second post-Newtonian approximation of general relativity

Hideki Asada and Masaru Shibata

Department of Earth and Space Science, Graduate School of Science
Osaka University, Toyonaka, Osaka 560, Japan

ABSTRACT

We present a formalism to obtain equilibrium configurations of uniformly rotating fluid in the second post-Newtonian approximation of general relativity. In our formalism, we need to solve 29 Poisson equations, but their source terms decrease rapidly enough at the external region of the matter (i.e., at worst $O(r^{-4})$). Hence these Poisson equations can be solved accurately as the boundary value problem using standard numerical methods. This formalism will be useful to obtain nonaxisymmetric uniformly rotating equilibrium configurations such as synchronized binary neutron stars just before merging and the Jacobi ellipsoid.

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*Electronic address: asada@vega.ess.sci.osaka-u.ac.jp
†Electronic address: shibata@vega.ess.sci.osaka-u.ac.jp
I. INTRODUCTION

The last stage of coalescing binary neutron stars (BNS’s) is one of the most promising sources for kilometer size interferometric gravitational wave detectors, LIGO [1] and VIRGO [2]. When the orbital separation of BNS’s becomes \( \sim 700 \text{km} \) as a result of the emission of gravitational waves, it is observed that the frequency of gravitational waves from them becomes \( \sim 10 \text{Hz} \). After then, the orbit of BNS’s shrinks owing to the radiation reaction toward merging in a few minutes [3]. In such a phase, BNS’s are the strongly self-gravitating bound systems, and gravitational waves from them will have various general relativistic (GR) informations. In particular, in the last few milliseconds before merging, BNS’s are in a very strong GR gravitational field because the orbital separation is less than ten times of the Schwarzschild radius of the system. Thus, if we could detect the signal of gravitational waves radiated in the last few milliseconds, we would be able to observe directly the phenomena in the GR gravitational field.

To interpret the implication of the signal of gravitational waves, we need to understand the theoretical mechanism of merging in detail. The little knowledge we have about the very last phase of BNS’s is as follows: When the orbital separation of BNS’s is \( \lesssim 10GM/c^2 \), where \( M \) is the total mass of BNS’s, they move approximately in circular orbits because the timescale of the energy loss due to gravitational radiation \( t_{GW} \) is much longer than the orbital period \( P \) as

\[
\frac{t_{GW}}{P} \sim 15 \left( \frac{dc^2}{10GM} \right)^{5/2} \left( \frac{M}{4\mu} \right),
\]

(1.1)

where \( \mu \) and \( d \) are the reduced mass and the separation of BNS’s. Thus, BNS’s adiabatically evolve radiating gravitational waves. However, when the orbital separation becomes \( 6 - 10GM/c^2 \), they cannot maintain the circular orbit because of instabilities due to the GR gravity [4] or the tidal field [5]. As a result of such instabilities, the circular orbit of BNS’s changes into the plunging orbit to merge. This means that the nature of the signal of gravitational waves changes around the transition between the circular orbit and plunging one. Gravitational waves emitted at this transition region may bring us an
important information about the structure of NS’s because the location where the instability occurs will depend on the equation of state (EOS) of NS sensitively \[5,6\]. Thus, it is very important to investigate the location of the innermost stable circular orbit (ISCO) of BNS’s.

As mentioned above, the ISCO is determined not only by the GR effects, but also by the hydrodynamic one. We emphasize that the tidal effects depend strongly on the structure of NS. Here, NS is a GR object because of its compactness, \(Gm/c^2 R \sim 0.2\), where \(m\) and \(R\) are the mass and radius of NS. Thus, in order to know the location of the ISCO accurately, we need to solve the GR hydrodynamic equations in general. A strategy to search the ISCO in GR manner is as follows; since the timescale of the energy loss is much longer than the orbital period according to Eq. (1), we may suppose that the motion of BNS’s is composed of the stationary part and the small radiation reaction part. From this physical point of view, we may consider that BNS’s evolve quasi-stationally, and we can take the following procedure; first, neglecting the evolution due to gravitational radiation, equilibrium configurations are constructed, and then the radiation reaction is taken into account as a correction to the equilibrium configurations. The ISCO is determined from the point, where the dynamical instability for the equilibrium configurations occurs. It may be a grand challenge, however, to distinguish the stationary part from the nonstationary one in general relativity. As Detweiler has pointed out \[7\], a stationary solution of the Einstein equation with standing gravitational waves, which will be constructed by adding the incoming waves from infinity, may be a valuable approximation to physically realistic solutions. However, these solutions are not asymptotically flat \[7\] because GWs contribute to the total energy of the system and the total energy of GWs inside a radius \(r\) grows linearly with \(r\). The lack of asymptotic flatness forces us to consider only a bounded space and impose boundary conditions in the near zone. Careful consideration will be necessary to find out an appropriate boundary condition for describing the physically realistic system in the near zone.

Recently, Wilson and his collaborators \[8\] proposed a simirelativistic approximation method in order to calculate the equilibrium configuration of BNS’s just before merging.
In their method, they assume the line element as

\[
ds^2 = -(\alpha^2 - \beta_i\beta^i)c^2dt^2 + 2\beta_i cdtdx^i + \psi^4dx^3,
\]

i.e., three metric \(\gamma_{ij}\) is chosen as the conformal flat (i.e., \(\gamma_{ij} = \psi^4\delta_{ij}\)), and solve only the constraint equations in the Einstein equation. In their approach, they claim that they ignore only the contribution of gravitational waves, but it is not correct at all; as shown in previous post-Newtonian (PN) analyses [9–11], the tensor potential term exists in the three metric even if we ignore the radiation reaction of gravitational waves (i.e., \(\psi^{-4}\gamma_{ij} \neq \delta_{ij}\)). Since such a term appears from the second PN order in the PN approximation, the accuracy of their results is less than the 2PN order: In reality, from results by Cook et al. [12] in which they obtain equilibrium configurations of the axisymmetric NS using both the Einstein equation and Wilson’s method, we can see that some quantities obtained from Wilson’s scheme, such as the lapse function, the three metric, the angular velocity, and so on, deviate from the exact solution by about \(O((Gm/Rc^2)^2)\). This seems to indicate that their approach for the system of BNS’s is valid only at the 1PN level from the PN point of view. Furthermore, the meaning of their approximation is obscure: It is not clear at all how to estimate errors due to such an approximation scheme and in which situation but the spherical symmetric system, the scheme based on the assumption of the conformal flatness is justified.

In contrast with Wilson’s method, the meaning of the PN approximation is fairly clear: In the PN approximation, the metric is formally expanded with respect to \(c^{-1}\) assuming the slow motion and weak self-gravity of matter. If we will take into account the next PN order, the accuracy of approximate solutions will be improved. This means that we can estimate the order of magnitude of the error due to the ignorance of higher PN terms. Also, in the PN approximation, we can distinguish the radiation reaction terms, which begin at the 2.5PN order [13], from other terms in the metric. Thus, it is possible to construct the equilibrium configuration of BNS’s without the radiation reaction terms in the 2PN approximation.

We schematically describe two approaches in Tables 1(a) and 1(b). As mentioned
above, in close binary of NS’s, it is important to take into account GR effects to orbital motion as well as to the internal structure of each NS. As for the orbital motion, there exist two parameters; one is the PN parameter $v/c$ and the other is the mass ratio $\eta$ of the reduced mass $\mu$ to the total mass $M$, and both parameters are less than unity. Thus, the physical quantities such as the orbital frequency are expanded with respect to them.

In Table 1(a), we show schematically various levels of approximations in terms of $v/c$ and $\eta$. If all terms in a level are taken into account in the 2PN approximation, we mark $P^2N$, while $W$ means that all terms in the marked level are taken into account in Wilson’s approach. From Table 1(a), we see that the 2PN approximation can include all corrections in $\eta$ up to the 2PN order in contrast with Wilson’s approach. On the other hand, Wilson’s approach will hold completely in the test particle limit, i.e., at $O(\eta^0)$, whereas even in this limit the 2PN approximation is not valid at higher PN orders. As for the internal structure of each NS, there also exist two small parameters; one is the compactness $Gm/c^2R$ and the other is the deformation parameter from its spherical shape, such as an ellipticity $e$.

In this case, the PN approximation becomes an expansion in terms of $Gm/c^2R$. In Table 1(b), we also show various levels of approximation in terms of these parameters. Although Wilson’s approach is exact for spherical NS’s, it is not valid in nonspherical cases even at the 2PN order. On the other hand, in the 2PN approximation, the spherical compact star cannot be obtained correctly in contrast with Wilson’s approach. In this way, the 2PN approximation has a week point: Although it can take into account all effects up to the 2PN order, it is inferior to Wilson’s approach when we take a test-particle limit, $\eta \to 0$, or we describe an exactly spherical NS. However, as shown below, the error due to the ignorance of higher PN terms in those cases is not so large.

To estimate the error due to the ignorance of the higher PN terms, let us compare the GR exact solutions with their PN approximations. First, we consider a small star of mass $\mu$ orbiting a Schwarzschild black hole of mass $m_{bh} \gg \mu$. In this case, we may consider that the small star moves on the geodesic around the Schwarzschild black hole, and the orbital angular velocity becomes $\ldots$
\[ \Omega = \sqrt{\frac{Gm_{bh}}{\bar{r} + Gm_{bh}c^{-2}r^3}}, \quad (1.3) \]

where \( \bar{r} \) is the coordinate radius of the orbit in the harmonic coordinate. In the PN approximation, Eq. (1.3) becomes

\[ \Omega = \sqrt{\frac{Gm_{bh}}{\bar{r}^3}} \left\{ 1 - \frac{3Gm_{bh}}{2\bar{r}c^2} + \frac{15}{8} \left( \frac{Gm_{bh}}{\bar{r}c^2} \right)^2 + O(c^{-6}) \right\}. \quad (1.4) \]

Comparing Eq. (1.3) with Eq. (1.4), it is found that the error size of the 2PN angular velocity is \( \sim 0.3\% \) at \( \bar{r} = 9Gm_{bh}c^{-2} \), and \( \sim 1\% \) at \( \bar{r} = 6Gm_{bh}c^{-2} \). Thus, the 2PN approximation seems fairly good to describe the motion of relativistic binary stars just before coalescence. Next, we consider a spherical NS of a uniform density in order to investigate the applicability of the PN approximation for determination of the internal structure of NS’s. In this model, the pressure, \( P \), and the density, \( \rho = \text{const.} \), are related with each other [14]:

\[
\frac{P}{\rho c^2} = \frac{(1 - 2Gm_{s}^2/c^2R^3)^{1/2} - (1 - 2Gm/c^2R)^{1/2}}{3(1 - 2Gm/c^2R)^{1/2} - (1 - 2Gm_{s}^2/c^2R^3)^{1/2}} \\
= \frac{1}{2} \frac{Gm}{c^2R} \left( 1 - \frac{r_s^2}{R^2} \right) + \frac{G^2m^2}{c^4R^2} \left( 1 - \frac{r_s^2}{R^2} \right) + \frac{G^3m^3}{c^6R^3} \left( \frac{17}{8} - \frac{19r_s^2}{8R^2} + \frac{3r_s^4}{8R^4} - \frac{r_s^6}{8R^6} \right) + O(c^{-8}), \quad (1.5)
\]

where \( r_s \) is the coordinate radius in the Schwarzschild coordinate and terms of order \( c^{-2}, c^{-4} \) and \( c^{-6} \) denote Newtonian, 1PN and 2PN terms respectively. In the second line in Eq. (1.3), we expand the equation in power of \( Gm/c^2R \) regarding it as a small quantity. In fig.1, we shows the error, \( 1 - \tilde{P}/P \), in Newtonian, 1PN and 2PN cases as a function of \( r_s \) for \( R = 5Gm/c^2 \)(solid lines) and \( 8Gm/c^2 \)(dotted lines), where \( \tilde{P} \) denotes the PN approximate pressure. It is found that the discrepancy in the Newtonian treatment is very large, while in the 2PN approximation the error is less than 10%. In this way, we can estimate rigidly the typical error size in the 2PN approximation. Furthermore, the accuracy is fairly good if the NS is not extremely compact; the 2PN approximation will be fairly accurate if the radius of NS is larger than \( \sim 10\text{km} \).

Thus, in the present paper, we develop a formalism to obtain equilibrium configurations of uniformly rotating fluid in the 2PN order as a first step. In section 2, we review the basic equations up to the 2PN order. In section 3, we rewrite the Poisson equation.
for potential functions, which are described in section 2, into useful forms in which the source terms of the Poisson equations decrease rapidly enough ($O(r^{-4})$). In section 4, we show a formulation to obtain numerically equilibrium solutions of uniformly rotating fluid in the 2PN approximation: Taking into account the formulation in the first PN approximation [15], we further rewrite potentials defined in section 3 into a polynomial form in the angular velocity, $\Omega$. Then, we transform the integrated Euler equation into the polynomial form in $\Omega^2$ so that the convergence property in iteration procedures may be much improved. For the sake of analysis for numerical results, we describe the 2PN expression of the conserved quantities, such as the conserved mass, the ADM mass, the total energy and the total angular momentum in section 5. Section 6 is devoted to summary. Throughout this paper, $G$ and $c$ denote the gravitational constant and the speed of light. Hereafter, we use units of $G = 1$.

II. FORMULATION

We write the line element in the following form;

$$ds^2 = -(\alpha^2 - \beta_i \beta^i)c^2 dt^2 + 2\beta_i c dt dx^i + \psi^4 \tilde{\gamma}_{ij} dx^i dx^j,$$  \hspace{1cm} (2.1)

where we define $\det(\tilde{\gamma}_{ij}) = 1$. To fix the gauge condition in the time coordinate, we use the maximal slice condition $K^i_i = 0$, where $K^i_i$ is the trace part of the extrinsic curvature, $K_{ij}$. As the spatial gauge condition, we adopt the transverse gauge $\tilde{\gamma}_{ij,j} = 0$ in order to remove the gauge modes from $\tilde{\gamma}_{ij}$. In this case, up to the 2 PN approximation, each metric variable is expanded as [10]

$$\psi = 1 + \frac{1}{c^2} U + \frac{1}{c^4} (4) \psi + O(c^{-6}),$$  \hspace{1cm} (2.2)

$$\alpha = 1 - \frac{1}{c^2} U + \frac{1}{c^4} \left( \frac{U^2}{2} + X \right) + \frac{1}{c^6} (6) \alpha + O(c^{-7}),$$  \hspace{1cm} (2.3)

$$\beta^i = \frac{1}{c^4} (3) \beta_i + \frac{1}{c^6} (5) \beta_i + O(c^{-7}),$$  \hspace{1cm} (2.4)

$$\tilde{\gamma}_{ij} = \delta_{ij} + \frac{1}{c^4} h_{ij} + O(c^{-5}).$$  \hspace{1cm} (2.5)

As for the energy-momentum tensor of the Einstein equation, we consider the perfect fluid as
\[ T_{\mu\nu} = (\rho c^2 + \rho \varepsilon + P) u_\mu u_\nu + Pg_{\mu\nu}. \]  

(2.6)

For simplicity, we assume that the matter obeys the polytropic equation of state (EOS);

\[ P = (\Gamma - 1)\rho \varepsilon = K \rho^\Gamma, \]  

(2.7)

where \( \Gamma \) and \( K \) are the polytropic exponent and polytropic constant, respectively. Up to the 2PN order, the four velocity is expanded as [16,10]

\[
\begin{align*}
  u^0 &= 1 + \frac{1}{c^2} \left( \frac{1}{2} v^2 + U \right) + \frac{1}{c^4} \left( \frac{3}{8} v^4 + \frac{5}{2} v^2 U + \frac{1}{2} U^2 + (3) \beta_i v^i - X \right) + O(c^{-6}), \\
  u^0 &= -\left[ 1 + \frac{1}{c^2} \left( \frac{1}{2} v^2 - U \right) + \frac{1}{c^4} \left( \frac{3}{8} v^4 + \frac{3}{2} v^2 U + \frac{1}{2} U^2 + X \right) \right] + O(c^{-6}), \\
  u^i &= \frac{v^i}{c} \left[ 1 + \frac{1}{c^2} \left( \frac{1}{2} v^2 + U \right) + \frac{1}{c^4} \left( \frac{3}{8} v^4 + \frac{5}{2} v^2 U + \frac{1}{2} U^2 + (3) \beta_i v^i - X \right) \right] + O(c^{-7}), \\
  u_i &= \frac{v^i}{c} + \frac{1}{c^3} \left( (3) \beta_i + v^i \left( \frac{1}{2} v^2 + 3U \right) \right) + \frac{1}{c^5} \left( (5) \beta_i + (3) \beta_j v^j + h_{ij} v^j \right) \\
  &\quad + v^i \left( \frac{3}{8} v^4 + \frac{7}{2} v^2 U + 4U^2 - X + 4(4) \psi + (3) \beta_j v^j \right) \right) + O(c^{-6}),
\end{align*}
\]

(2.8)

where \( v^i = u^i/u^0 \) and \( v^2 = v^i v^i \). Since we need \( u^0 \) up to 3PN order to obtain the 2PN equations of motion, we derive it here. Using Eq. (2.8), we can calculate \((\alpha u^0)^2\) up to 3PN order as

\[
(\alpha u^0)^2 = 1 + \psi^{-4} \tilde{\gamma}^{ij} u_i u_j \\
= 1 + \frac{v^2}{c^2} + \frac{1}{c^4} \left( 2(3) \beta_j v^j + 4U v^2 + v^4 \right) + \frac{1}{c^6} \left( (3) \beta_j (3) \beta_j + 8(3) \beta_j v^j U + h_{ij} v^i v^j \right) \\
+ 2(5) \beta_i v^i + \left( 4(3) \beta_j v^j + 4(4) \psi + \frac{15}{2} U^2 - 2X \right) v^2 + 8U v^4 + v^6 \right) + O(c^{-7}),
\]

(2.9)

where we use \( \tilde{\gamma}^{ij} = \delta_{ij} - c^{-4} h_{ij} + O(c^{-5}) \). Thus, we obtain \( u^0 \) up to the 3PN order as

\[
\begin{align*}
  u^0 &= 1 + \frac{1}{c^2} \left( \frac{1}{2} v^2 + U \right) + \frac{1}{c^4} \left( \frac{3}{8} v^4 + \frac{5}{2} v^2 U + \frac{1}{2} U^2 + (3) \beta_i v^i - X \right) \\
  &\quad + \frac{1}{c^6} \left( (6) \alpha + \frac{1}{2} \left( (3) \beta_j (3) \beta_j + h_{ij} v^i v^j \right) + (5) \beta_j v^j + 5(3) \beta_j v^j U - 2UX \right) \\
  &\quad + \left( \frac{3}{2} (3) \beta_j v^j + 2(4) \psi + 6U^2 - \frac{3}{2} X \right) v^2 + \frac{27}{8} U v^4 + \frac{5}{16} v^6 \right) + O(c^{-7}).
\end{align*}
\]

(2.10)

Substituting PN expansions of metric and matter variables into the Einstein equation, and using the polytropic EOS, we find that the metric variables obey the following Poisson equations [10];
\[ \Delta U = -4\pi \rho, \]  
\[ \Delta X = 4\pi \rho \left( 2v^2 + 2U + (3\Gamma - 2)\varepsilon \right), \]  
\[ \Delta(4)\psi = -2\pi \rho \left( v^2 + \varepsilon + \frac{5}{2}U \right), \]  
\[ \Delta(3)\beta_i = 16\pi \rho v^i - \dot{U}_i, \]  
\[ \Delta(5)\beta_i = 16\pi \rho \left[ v^i \left( v^2 + 2U + \Gamma \varepsilon \right) + (3)\beta_i \right] - 4U_\beta \left( (3)\beta_i + (3)\beta_j - \frac{2}{3}\delta_{ij}(3)\beta_k \right) - 2(4)\psi_i + \frac{1}{2}(U\dot{U})_i + (3)\beta(\dot{U}_i), \]  
\[ \Delta h_{ij} = \left( UU_{ij} - \frac{1}{3}\delta_{ij}U\Delta_{\text{flat}}U - 3U_iU_j + \delta_{ij}U_kU_k \right) - 16\pi \left( \rho v^i v^j - \frac{1}{3}\delta_{ij}\rho v^2 \right) - \left( (3)\beta_{i,j} + (3)\beta_{j,i} - \frac{2}{3}\delta_{ij}(3)\beta_{k,k} \right) - 2 \left( (X + 2(4)\psi)_i, v_j - \frac{1}{3}\delta_{ij}\Delta(X + 2(4)\psi) \right), \]  
\[ \Delta(6)\alpha = 4\pi \rho \left[ 2v^4 + 2v^2 \left( 5U + \Gamma \varepsilon \right) + (3)\beta - X + (3)\beta v^i \right] - h_{ij}U_{ij} - \frac{3}{2}UU_iU_j + U_i(2(4)\psi - X)_j + \frac{1}{2}(3)\beta_{i,j} \left( (3)\beta_{i,j} + (3)\beta_{j,i} - \frac{2}{3}\delta_{ij}(3)\beta_{k,k} \right), \]

where \( \Delta \) is the flat Laplacian, and \( \cdot \) denotes \( \partial \partial t \).

Equations of motion for fluid are derived from

\[ \nabla_\mu T^\mu = 0. \]  

In this paper, we consider the uniformly rotating fluid around \( z \)-axis with the angular velocity \( \Omega \), i.e.,

\[ v^i = \epsilon_{ijk} \Omega^j x^k = (-y\Omega, x\Omega, 0), \]

where we choose \( \Omega^j = (0, 0, \Omega) \) and \( \epsilon_{ijk} \) is the completely anti-symmetric unit tensor. In this case, the following relations hold;

\[ \left( \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \varphi} \right) Q = \left( \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \varphi} \right) Q_i = \left( \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \varphi} \right) Q_{ij} = 0, \]

where \( Q, Q_i \) and \( Q_{ij} \) are arbitrary scalars, vectors, and tensors, respectively. Then, Eq.(2.18) can be integrated as \[ 17 \]

\[ \int \frac{dP}{\rho c^2 + \rho \varepsilon + P} = \ln u^0 + C, \]
where \( C \) is a constant. For the polytropic EOS, Eq. (2.21) becomes
\[
\ln \left[ 1 + \frac{\Gamma K}{c^2(\Gamma - 1)} \rho^{\Gamma - 1} \right] = \ln u^0 + C, \tag{2.22}
\]
or
\[
1 + \frac{\Gamma K}{c^2(\Gamma - 1)} \rho^{\Gamma - 1} = u^0 \exp(C). \tag{2.23}
\]
Using Eq. (2.10), the 2PN approximation of Eq. (2.22) is written as
\[
H - \frac{H^2}{2c^2} + \frac{H^3}{3c^4} = \frac{v^2}{2} + U + \frac{1}{c^2} \left( 2Uv^2 + \frac{v^4}{4} - X + (3)\beta_i v^i \right)
\]
\[+ \frac{1}{c^4} \left( -(6)\alpha + \frac{1}{2}(3)\beta_i (3)\beta_i + 4(3)\beta_i v^i U - \frac{U^3}{6} + (3)\beta_i v^i v^2 + 2(4)\psi v^2 \right.
\]
\[+ \frac{15}{4} U^2 v^2 + 2Uv^4 + \frac{1}{6} v^6 - UX - v^2 X + (5)\beta_i v^i + \frac{1}{2} h_{ij} v^i v^j \right) + C, \tag{2.24}
\]
where \( H = \Gamma K \rho^{\Gamma - 1}/(\Gamma - 1) \), \( v^2 = R^2 \Omega^2 \) and \( R^2 = x^2 + y^2 \). Note that Eq. (2.24) can be also obtained from the 2PN Euler equation like in the first PN case \[\text{[13].} \] If we solve the coupled equations (2.11-17) and (2.24), we can obtain equilibrium configurations of the non-axisymmetric uniformly rotating body.

### III. DERIVATION OF THE POISSON EQUATION OF COMPACT SOURCES

#### FOR \( h_{ij}, (3)\beta_i \) and \( (5)\beta_i \)

In section 2, we derive the Poisson equations for metric variables. However, the source terms in the Poisson equations for \( (3)\beta_i \), \( (5)\beta_i \), and \( h_{ij} \) fall off slowly as \( r \to \infty \) because these terms behave as \( O(r^{-3}) \) at \( r \to \infty \). These Poisson equations do not take convenient forms when we try to solve them as the boundary value problem in numerical calculation. Hence in the following, we rewrite them into other convenient forms in numerical calculation.

As for \( h_{ij} \), first of all, we split the equation into three parts as \[\text{[14]}\]
\[
\Delta h_{ij}^{(U)} = U (U_{,ij} - \frac{1}{3} \delta_{ij} \Delta U) - 3U_{,i} U_{,j} + \delta_{ij} U_{,k} U_{,k} \equiv -4\pi S_{ij}^{(U)}, \tag{3.1}
\]
\[
\Delta h_{ij}^{(S)} = -16\pi (\rho v^i v^j - \frac{1}{3} \delta_{ij} \rho v^2), \tag{3.2}
\]
\[
\Delta h_{ij}^{(G)} = -((3)\dot{\beta}_{i,j} + (3)\dot{\beta}_{j,i} - \frac{2}{3} \delta_{ij} (3)\dot{\beta}_{k,k})
\]
\[- 2 \left( (X + 2(4)\psi)_{,ij} - \frac{1}{3} \delta_{ij} \Delta (X + 2(4)\psi) \right). \tag{3.3}
\]
The equation for $h_{ij}^{(S)}$ has a compact source, and also the source term of $h_{ij}^{(U)}$ behaves as $O(r^{-6})$ at $r \to \infty$, so that Poisson equations for them are solved easily as the boundary value problem. On the other hand, the source term of $h_{ij}^{(G)}$ behaves as $O(r^{-3})$ at $r \to \infty$, so that it seems troublesome to solve the equation for it as the boundary value problem. In order to solve the equation for $h_{ij}^{(G)}$ as the boundary value problem, we had better rewrite the equation into useful forms. As shown in a previous paper [10], Eq.(3.3) is integrated to give

$$h_{ij}^{(G)} = 2 \frac{\partial}{\partial x^i} \int (\rho v^i) |x - y| d^3y + 2 \frac{\partial}{\partial x^j} \int (\rho v^j) |x - y| d^3y + \delta_{ij} \int \hat{\rho} |x - y| d^3y$$

$$+ \frac{1}{12} \frac{\partial^2}{\partial x^i \partial x^j} \int \hat{\rho} |x - y|^3 d^3y + \frac{\partial^2}{\partial x^i \partial x^j} \int \left( \rho v^2 + 3P - \frac{\rho U}{2} \right) |x - y| d^3y$$

$$- \frac{2}{3} \delta_{ij} \int \frac{\left( \rho v^2 + 3P - \frac{\rho U}{2} \right)}{|x - y|} d^3y. \quad (3.4)$$

Using the relations

$$\hat{\rho} = -(\rho v^j)_{,j} + O(c^{-2}),$$

$$\hat{v}^i = 0,$$

$$v^i x^i = 0, \quad (3.5)$$

Eq.(3.4) is rewritten as

$$h_{ij}^{(G)} = \frac{7}{4} \int (\rho v^i) \frac{x^i - y^i}{|x - y|} d^3y + \frac{7}{4} \int (\rho v^j) \frac{x^j - y^j}{|x - y|} d^3y - \delta_{ij} x^k \int \frac{(\rho v^k)}{|x - y|} d^3y$$

$$- \frac{1}{8} x^k \left[ \frac{\partial}{\partial x^i} \int (\rho v^k) \frac{x^j - y^j}{|x - y|} d^3y + \frac{\partial}{\partial x^j} \int (\rho v^k) \frac{x^i - y^i}{|x - y|} d^3y \right]$$

$$+ \frac{1}{2} \left[ \frac{\partial}{\partial x^i} \int \left( \rho v^2 + 3P - \frac{\rho U}{2} \right) \frac{x^j - y^j}{|x - y|} d^3y + \frac{\partial}{\partial x^j} \int \left( \rho v^2 + 3P - \frac{\rho U}{2} \right) \frac{x^i - y^i}{|x - y|} d^3y \right]$$

$$- \frac{2}{3} \delta_{ij} \int \frac{\left( \rho v^2 + 3P - \frac{\rho U}{2} \right)}{|x - y|} d^3y. \quad (3.6)$$

From Eq.(3.6), it is found that $h_{ij}^{(G)}$ is written as

$$h_{ij}^{(G)} = \frac{7}{4} \left( x^i (3) \hat{P}_j + x^j (3) \hat{P}_i - \hat{Q}_{ij}^{(T)} - \hat{Q}_{ji}^{(T)} \right) - \delta_{ij} x^k (3) \hat{P}_k$$

$$- \frac{1}{8} x^k \left[ \frac{\partial}{\partial x^i} \left( x^j (3) \hat{P}_k - \hat{Q}_{jk}^{(T)} \right) + \frac{\partial}{\partial x^j} \left( x^i (3) \hat{P}_k - \hat{Q}_{ki}^{(T)} \right) \right]$$

$$+ \frac{1}{2} \left[ \frac{\partial}{\partial x^i} \left( x^j Q^{(T)} - Q_j^{(T)} \right) + \frac{\partial}{\partial x^j} \left( x^i Q^{(T)} - Q_i^{(T)} \right) \right] - \frac{2}{3} \delta_{ij} Q^{(T)}, \quad (3.7)$$
where
\[ \Delta_{(3)}P_i = -4\pi \rho v^i, \]  
\[ \Delta Q_{ij}^{(T)} = -4\pi \rho v^i x^j, \]  
\[ \Delta Q^{(I)} = -4\pi \left( \rho v^2 + 3P - \frac{1}{2}\rho U \right), \]  
\[ \Delta Q_i^{(I)} = -4\pi \left( \rho v^2 + 3P - \frac{1}{2}\rho U \right) x^i. \]  

Therefore, \( h_{ij}^{(G)} \) can be deduced from variables which satisfy the Poisson equations with compact sources.

The source terms in the Poisson equations for \( (3)\beta_i \) and \( (5)\beta_i \) also fall off slowly. However, if we rewrite them as
\[ (3)\beta_i = -4(3)P_i - \frac{1}{2} \left( x^i \dot{U} - \dot{q}_i \right), \]  
\[ (5)\beta_i = -4(5)P_i - \frac{1}{2} \left( 2x^i(4)\psi - \eta_i \right), \]

where
\[ \Delta q_i = -4\pi \rho x^i, \]  
\[ \Delta (5)\hat{P}_i = -4\pi \rho \left[ v^i \left( v^2 + 2U + \Gamma \varepsilon \right) + (3)\hat{\beta}_i \right] + U_j \left( (3)\hat{\beta}_{i,j} + (3)\hat{\beta}_{j,i} - \frac{2}{3}\delta_{ij}(3)\hat{\beta}_{k,k} \right) \]  
\[ - \frac{1}{8}(UU)_i - \frac{1}{4}(3)\hat{\beta}_l U_{i,l}, \]  
\[ \Delta \eta_i = -4\pi \rho \left( v^2 + \varepsilon + \frac{5}{2}U \right) x^i, \]  

then \( (3)\beta_i \) and \( (5)\beta_i \) can be obtained by solving the Poisson equations in which the fall-off of the source terms is fast enough, \( O(r^{-5}) \), for numerical calculation. Note that, using the relation \( (3)P_i = \epsilon_{i\ell k}q_k\Omega \) and Eqs. (2.20), \( (3)\beta_i \) and \( (5)\beta_i \) may be written as
\[ (3)\beta_i = \Omega \left\{ -4\epsilon_{i\ell k}q_k + \frac{1}{2} \left( x^i U_{,\psi} - q_{i,\psi} \right) \right\} \equiv \Omega (3)\hat{\beta}_i, \]  
\[ (5)\beta_i = \Omega \left\{ -4(5)\hat{P}_i + \frac{1}{2} \left( 2x^i(4)\psi_{,\psi} - \eta_{i,\psi} \right) \right\}, \]  

where
\[ \Delta_{(5)}\hat{P}_i = -4\pi \rho \left[ \epsilon_{i\ell k}x^k \left( v^2 + 2U + \Gamma \varepsilon \right) + (3)\hat{\beta}_i \right] + U_j \left( (3)\hat{\beta}_{i,j} + (3)\hat{\beta}_{j,i} - \frac{2}{3}\delta_{ij}(3)\hat{\beta}_{k,k} \right) \]  
\[ + \frac{1}{8}(UU_{,\psi})_i - \frac{1}{4}(3)\hat{\beta}_l U_{i,l}. \]
IV. DERIVATION OF BASIC EQUATIONS

In this section, we derive the basic equation which has a suitable form to construct equilibrium configurations of uniformly rotating body in numerical calculation. Although equilibrium configurations can be formally obtained by solving Eq. (2.24) as well as metric potentials, $U, X, (4) \psi, (6) \alpha, (3) \beta_i, (5) \beta_i$ and $h_{ij}$, they do not take convenient forms for numerical calculation. Thus, we here change Eq. (2.24) into other forms appropriate to obtain numerically equilibrium configurations.

In numerical calculation, the standard method to obtain equilibrium configurations is as follows [19,20,15]:

(1) We give a trial density configuration for $\rho$.

(2) We solve the Poisson equations.

(3) Using Eq. (2.24), we give a new density configuration.

These procedures are repeated until a sufficient convergence is achieved. Here, at (3), we need to specify unknown constants, $\Omega$ and $C$. In standard numerical methods [19,20], these are calculated during iteration fixing densities at two points; i.e., if we put $\rho_1$ and $\rho_2$ at $x_1$ and $x_2$ into Eq. (2.24), they become two simultaneous equations for $\Omega$ and $C$. Hence, we can calculate them. However, the procedure is not so simple in the PN case: $\Omega$ is included in the source of the Poisson equations for the variables such as $X, (4) \psi, (6) \alpha, \eta_i, (5) \bar{P}_i, \bar{h}_{ij}^{(S)}$, $Q_{ij}^{(T)}$, $Q_i^{(U)}$ and $Q_i^{(I)}$. Thus, if we use Eq. (2.24) as it is, equations for $\Omega$ and $C$ become implicit equations for $\Omega$. As found in a previous paper [15], in such a situation, the convergence to a solution is very slow. Therefore, we transform those equations into other forms in which the potentials as well as Eq. (2.24) become explicit polynomial equations in $\Omega$.

First of all, we define $q_2, q_{2i}, q_4, q_u, q_e$ and $q_{ij}$ which satisfy

$$\Delta q_2 = -4\pi \rho R^2,$$  \hspace{1cm} (4.1)

$$\Delta q_{2i} = -4\pi \rho R^2 x^i,$$  \hspace{1cm} (4.2)

$$\Delta q_4 = -4\pi \rho R^4,$$  \hspace{1cm} (4.3)

$$\Delta q_u = -4\pi \rho U,$$  \hspace{1cm} (4.4)
\[ \Delta q_e = -4\pi \rho \varepsilon, \quad (4.5) \]
\[ \Delta q_{ij} = -4\pi \rho x^i x^j. \quad (4.6) \]

Then, \( X, (4) \psi, Q^{(l)}, Q_i^{(l)}, \eta_i, (5) \dot{P}_i, Q_{ij}^{(T)}, \) and \( h_{ij}^{(S)} \) are written as

\[ X = -2q_2 \Omega^2 - 2q_u - (3\Gamma - 2)q_e, \quad (4.7) \]
\[ (4) \psi = \frac{1}{2} \left( q_2 \Omega^2 + q_e + \frac{5}{2} q_u \right), \quad (4.8) \]
\[ Q^{(l)} = q_2 \Omega^2 + 3(\Gamma - 1)q_e - \frac{1}{2} q_u \equiv q_2 \Omega^2 + Q_0^{(l)}, \quad (4.9) \]
\[ Q_i^{(l)} = q_{2i} \Omega^2 + Q_0^{(l)}, \quad (4.10) \]
\[ \eta_i = q_{2i} \Omega^2 + \eta_0i, \quad (4.11) \]
\[ (5) \dot{P}_i = \epsilon_{izk} q_{2k} \Omega^2 + (5) P_0i, \quad (4.12) \]
\[ Q_{ij}^{(T)} = \epsilon_{izl} q_{jl} \Omega, \quad (4.13) \]
\[ h_{ij}^{(S)} = 4\Omega^2 \left( \epsilon_{izk} \epsilon_{jkl} q_{kl} - \frac{1}{3} \delta_{ij} q_2 \right), \quad (4.14) \]

where \( Q_0^{(l)}, \eta_0i \) and \( (5) P_0i \) satisfy

\[ \Delta Q_0^{(l)} = -4\pi \left( 3P - \frac{1}{2} \rho U \right) x^i = -4\pi \rho \left( 3(\Gamma - 1)\varepsilon - \frac{1}{2} U \right) x^i, \quad (4.15) \]
\[ \Delta \eta_0i = -4\pi \rho \left( \varepsilon + \frac{5}{2} U \right) x^i, \quad (4.16) \]
\[ \Delta (5) P_0i = -4\pi \rho \left[ \epsilon_{izk} x^k \left( 2U + \Gamma \varepsilon \right) + (3) \dot{\beta}_i \right] + U_j \left( (3) \dot{\beta}_{i,j} + (3) \dot{\beta}_{j,i} - \frac{2}{3} \delta_{ij} (3) \dot{\beta}_{k,k} \right) + \frac{1}{8} (U U_{\varphi}, i) - \frac{1}{4} (3) \dot{\beta}_k U_{k,i} \right) \equiv -4\pi S_2^{(P)} + (4.17) \]

Note that \( (5) \beta_i \) and \( h_{ij}^{(G)} \) are the cubic and quadratic equations in \( \Omega \), respectively, as

\[ (5) \beta_i = \Omega \left[ -4(5) P_{0i} + \frac{1}{2} \left\{ x^i \left( q_e + \frac{5}{2} q_u \right) \right. - \eta_{0i, \varphi} \right\} \right] + \Omega^3 \left[ -4 \epsilon_{izk} q_{2k} + \frac{1}{2} (x^i q_{2i, \varphi} - q_{2i, \varphi}) \right]
\[ \equiv (5) \beta_i^{(A)} \Omega + (5) \beta_i^{(B)} \Omega^3, \quad (4.18) \]
\[ h_{ij}^{(G)} = \frac{1}{2} \left\{ \frac{\partial}{\partial x^j} \left( x^i Q_0^{(l)} - Q_0^{(l)} \right) + \frac{\partial}{\partial x^j} \left( x^j Q_0^{(l)} - Q_0^{(l)} \right) - \frac{4}{3} \delta_{ij} Q_0^{(l)} \right\}
\[ + \Omega^2 \left\{ \frac{1}{2} \left\{ \frac{\partial}{\partial x^j} \left( x^i q_{2j} - q_{2i} \right) + \frac{\partial}{\partial x^j} \left( x^j q_{2j} - q_{2j} \right) - \frac{4}{3} \delta_{ij} q_2 \right\}
\[ - \frac{7}{4} \left( x^i \epsilon_{izk} q_{k,i, \varphi} + x^j \epsilon_{izk} q_{k,j, \varphi} - \epsilon_{izk} q_{k, i, \varphi} - \epsilon_{izk} q_{k, j, \varphi} \right) + \delta_{ij} x^k \epsilon_{kzl} q_{l}
\[ + \frac{1}{8} x^k \left\{ \frac{\partial}{\partial x^i} \left( x^i \epsilon_{kzl} q_{l, \varphi} - \epsilon_{kzl} q_{l, i} \right) + \frac{\partial}{\partial x^j} \left( x^i \epsilon_{kzl} q_{l, \varphi} - \epsilon_{kzl} q_{l, i} \right) \right\} \right\}
\[ \equiv h_{ij}^{(A)} + h_{ij}^{(B)} \Omega^2, \quad (4.19) \]
Finally, we write \( \alpha \) as

\[
(6)\alpha = (6)\alpha_0 + (6)\alpha_2 \Omega^2 - 2q_4 \Omega^4, \tag{4.20}
\]

where \( (6)\alpha_0 \) and \( (6)\alpha_2 \) satisfy

\[
\Delta (6)\alpha_0 = 4\pi \rho \left[ (3\Gamma - 2)\varepsilon U - (3\Gamma - 4)q_e + 3q_u \right] \nonumber
\]

\[
- \left( h_{ij}^{(U)} + h_{ij}^{(A)} \right) U_{,ij} - \frac{3}{2} UU_{,l}U_{,l} + U_{,l} \frac{\partial}{\partial x^l} \left( \frac{9}{2} q_u + (3\Gamma + 1)q_e \right) \nonumber
\]

\[
\equiv -4\pi S^{(\alpha_0)}, \tag{4.21}
\]

\[
\Delta (6)\alpha_2 = 8\pi \rho R^2 \left( 5U + \Gamma \varepsilon + 2(3)\hat{\beta}_{\varphi} \right) - \left( 4\epsilon_{izk}\epsilon_{jzl}q_{kl} - \frac{4}{3} \delta_{ij}q_2 + h_{ij}^{(B)} \right) U_{,ij} + 3q_2 U_{,l} \nonumber
\]

\[
+ \frac{1}{2} (3)\hat{\beta}_{i,j} \left( (3)\hat{\beta}_{i,j} + (3)\hat{\beta}_{j,i} - \frac{2}{3} \delta_{ij}(3)\hat{\beta}_{k,k} \right) \nonumber
\]

\[
\equiv -4\pi S^{(\alpha_2)}. \tag{4.22}
\]

Using the above quantities, Eq.(2.24) is rewritten as

\[
H - \frac{H^2}{2c^2} + \frac{H^3}{3c^4} = A + B\Omega^2 + D\Omega^4 + \frac{R^6}{6c^4}\Omega^6 + C, \tag{4.23}
\]

where

\[
A = U + \frac{1}{c^2} \left( 2q_u + (3\Gamma - 2)q_e \right) + \frac{1}{c^4} \left\{ -(6)\alpha_0 - \frac{U^3}{6} + U \left( 2q_u + (3\Gamma - 2)q_e \right) \right\}, \nonumber
\]

\[
B = \frac{R^2}{2} + \frac{1}{c^2} \left( 2R^2 U + 2q_2 + (3)\hat{\beta}_{\varphi} \right) + \frac{1}{c^4} \left\{ -(6)\alpha_2 + \frac{1}{2} (3)\hat{\beta}_{i(3)}\hat{\beta}_i + 4(3)\hat{\beta}_{\varphi} U \right. \nonumber
\]

\[
+ (3\Gamma - 1)q_e R^2 + \frac{9}{2} q_u R^2 + \frac{15}{4} U^2 R^2 + 2q_2 U + (5)\hat{\beta}_{\varphi}^{(A)} + \frac{1}{2} \left( h_{\varphi\varphi}^{(U)} + h_{\varphi\varphi}^{(A)} \right) \right\}, \nonumber
\]

\[
D = \frac{R^4}{4c^2} + \frac{1}{c^4} \left\{ 2q_4 + (3)\hat{\beta}_{\varphi} R^2 + \frac{7}{3} q_2 R^2 + 2UR^2 + (5)\hat{\beta}_{\varphi}^{(B)} + \frac{1}{2} \left( h_{\varphi\varphi}^{(B)} + 4R^2 q_{RR} \right) \right\}. \tag{4.24}
\]

Note that in the above, we use the following relations which hold for arbitrary vector \( Q_i \)

and symmetric tensor \( Q_{ij} \),

\[
Q_{\varphi} = -yQ_x + xQ_y, \nonumber
\]

\[
Q_{\varphi\varphi} = y^2 Q_{xx} - 2xy Q_{xy} + x^2 Q_{yy}, \nonumber
\]

\[
R^2 Q_{RR} = x^2 Q_{xx} + 2xy Q_{xy} + y^2 Q_{yy}. \tag{4.25}
\]

We also note that source terms of Poisson equations for variables which appear in \( A, B \)

and \( D \) do not depend on \( \Omega \) explicitly. Thus, Eq.(4.23) takes the desired form for numerical calculation.
In this formalism, we need to solve 29 Poisson equations for $U$, $q_x$, $q_y$, $q_z$, $(5) P_{0x}$, $(5) P_{0y}$, $\eta_{0x}$, $\eta_{0y}$, $Q_{0x}^{(I)}$, $Q_{0y}^{(I)}$, $Q_{0z}^{(I)}$, $q_2$, $q_{2x}$, $q_{2y}$, $q_{2z}$, $q_u$, $q_v$, $h_{xx}^{(U)}$, $h_{xy}^{(U)}$, $h_{xz}^{(U)}$, $h_{yy}^{(U)}$, $h_{yz}^{(U)}$, $q_{xx}$, $q_{xy}$, $q_{xz}$, $q_{yz}$, $(6) \alpha_0$, $(6) \alpha_2$ and $q_4$. In Table 2, we show the list of the Poisson equations to be solved. In Table 3, we also summarize what variables are needed to calculate the metric variables $U$, $X$, $(4) \psi$, $(6) \alpha$, $(3) \beta_i$, $(5) \beta_i$, $h_{ij}^{(U)}$, $h_{ij}^{(S)}$, $h_{ij}^{(A)}$ and $h_{ij}^{(B)}$. Note that we do not need $(5) P_{0z}$, $\eta_{0z}$, and $q_{zz}$ because they do not appear in any equation. Also, we do not have to solve the Poisson equations for $h_{zz}^{(U)}$ and $q_{yy}$ because they can be calculated from $h_{zz}^{(U)} = -h_{xx}^{(U)} - h_{yy}^{(U)}$ and $q_{yy} = q_2 - q_{xx}$.

In order to derive $U$, $q_i$, $q_2$, $q_{2i}$, $q_4$, $q_e$ and $q_{ij}$, we do not need any other potential because only matter variables appear in the source terms of their Poisson equations. On the other hand, for $q_u$, $Q_{0i}^{(I)}$, $\eta_{0i}$ and $h_{ij}^{(U)}$, we need the Newtonian potential $U$, and for $(5) P_{0i}$, $(6) \alpha_0$ and $(6) \alpha_2$, we need the Newtonian as well as PN potentials. Thus, $U$, $q_i$, $q_2$, $q_{2i}$, $q_4$, $q_e$ and $q_{ij}$ must be solved first, and then $q_u$, $Q_{0i}^{(I)}$, $\eta_{0i}$, $h_{ij}^{(U)}$, $(5) P_{0i}$ and $(6) \alpha_2$ should be solved. $(6) \alpha_0$ must be solved after we obtain $q_u$ because its Poisson equation involves $q_u$ in the source term. In Table 2, we also list potentials which are included in the source terms of the Poisson equations for other potentials.

The configuration which we are most interested in and would like to obtain is the equilibrium state for BNS’s of equal mass. Hence, we show the boundary condition at $r \to \infty$ for this problem. When we consider equilibrium configurations for BNS’s where the center of mass for each NS is on the x-axis, boundary conditions for potentials at $r \to \infty$ become

\[
U = \frac{1}{r} \int \rho dV + O(r^{-3}), \\
q_x = \frac{n^x}{r^2} \int \rho x^2 dV + O(r^{-4}), \\
q_2 = \frac{1}{r} \int \rho R^2 dV + O(r^{-3}), \\
q_y = \frac{n^y}{r^2} \int \rho y^2 dV + O(r^{-4}), \\
q_e = \frac{1}{r} \int \rho e dV + O(r^{-3}), \\
q_z = \frac{n^z}{r^2} \int \rho z^2 dV + O(r^{-4}), \\
q_u = \frac{1}{r} \int \rho UdV + O(r^{-3}), \\
q_4 = \frac{1}{r} \int \rho R^4 dV + O(r^{-3}),
\]

\[
(5) P_{0x} = \frac{n^x}{r^2} \int S_x^{(P)} x dV + \frac{n^y}{r^2} \int S_y^{(P)} y dV + O(r^{-3}),
\]

\[
(5) P_{0y} = \frac{n^x}{r^2} \int S_x^{(P)} x dV + \frac{n^y}{r^2} \int S_y^{(P)} y dV + O(r^{-3}),
\]
\[ \eta_{0x} = \frac{n^x}{r^2} \int \rho x^2 (\varepsilon + \frac{5}{2} U) dV + O(r^{-4}), \]
\[ \eta_{0y} = \frac{n^y}{r^2} \int \rho y^2 (\varepsilon + \frac{5}{2} U) dV + O(r^{-4}), \]
\[ Q_{0x}^{(I)} = \frac{n^x}{r^2} \int \rho x^2 (3(\Gamma-1)\varepsilon - \frac{1}{2} U) dV + O(r^{-4}), \quad q_{2x} = \frac{n^x}{r^2} \int \rho R^2 x^2 dV + O(r^{-4}), \]
\[ Q_{0y}^{(I)} = \frac{n^y}{r^2} \int \rho y^2 (3(\Gamma-1)\varepsilon - \frac{1}{2} U) dV + O(r^{-4}), \quad q_{2y} = \frac{n^y}{r^2} \int \rho R^2 y^2 dV + O(r^{-4}), \]
\[ Q_{0z}^{(I)} = \frac{n^z}{r^2} \int \rho z^2 (3(\Gamma-1)\varepsilon - \frac{1}{2} U) dV + O(r^{-4}), \quad q_{2z} = \frac{n^z}{r^2} \int \rho R^2 z^2 dV + O(r^{-4}), \quad (4.29) \]
\[ h_{xx}^{(U)} = \frac{1}{r} \int S_{xx}^{(U)} dV + O(r^{-3}), \]
\[ h_{yy}^{(U)} = \frac{1}{r} \int S_{yy}^{(U)} dV + O(r^{-3}), \]
\[ h_{zz}^{(U)} = \frac{3n^y n^z}{r^3} \int S_{yz}^{(U)} yzdV + O(r^{-5}), \]
\[ q_{xx} = \frac{1}{r} \int \rho x^2 dV + O(r^{-3}), \quad q_{xy} = \frac{3n^x n^y}{r^3} \int \rho x^2 y^2 dV + O(r^{-5}), \]
\[ q_{xz} = \frac{3n^x n^z}{r^3} \int \rho x^2 z^2 dV + O(r^{-5}), \quad q_{yz} = \frac{3n^y n^z}{r^3} \int \rho y^2 z^2 dV + O(r^{-5}), \quad (4.31) \]
\[ (6) \alpha_0 = \frac{1}{r} \int S^{(\alpha_0)} dV + O(r^{-3}), \quad (6) \alpha_2 = \frac{1}{r} \int S^{(\alpha_2)} dV + O(r^{-3}), \quad (4.32) \]

where \( dV = d^3x \), and
\[ n^i = \frac{x^i}{r}. \quad (4.33) \]

We note that at \( r \to \infty \), \( S_i^{(P)} \to O(r^{-5}) \), \( S_{ij}^{(U)} \to O(r^{-6}) \), \( S^{(\alpha_0)} \to O(r^{-4}) \) and \( S^{(\alpha_2)} \to O(r^{-4}) \), so that all the above integrals are well defined.

V. CONSERVED QUANTITIES

In this section, we show the conserved quantities in the 2PN approximation because they will be useful to investigate the stability property of equilibrium solutions obtained in numerical calculations.

(1) Conserved mass \([\Box]\);
\[ M_* \equiv \int \rho_* d^3 x, \] (5.1)

where

\[ \rho_* = \rho \alpha u^0 \psi^6 \]
\[ = \rho \left[ 1 + \frac{1}{c^2} \left( \frac{1}{2} v^2 + 3U \right) + \frac{1}{c^4} \left( \frac{3}{8} v^4 + \frac{7}{2} v^2 U + \frac{15}{4} U^2 + 6 \psi + (3) \beta_i v^i \right) + O(c^{-6}) \right]. \] (5.2)

Equation (5.2) may be written as

\[ \rho_* = \rho \left[ 1 + \frac{1}{c^2} \left( \frac{1}{2} v^2 + 3U \right) + \frac{1}{c^4} \left( \frac{3}{8} v^4 + \frac{13}{2} v^2 U + \frac{45}{4} U^2 + 3U \psi + (3) \beta_i v^i \right) + O(c^{-6}) \right]. \] (5.3)

(2) ADM mass [21,10]:

\[ M_{ADM} = \frac{-1}{2\pi} \int \Delta \psi d^3 x \equiv \int \rho_{ADM} d^3 x, \] (5.4)

where

\[ \rho_{ADM} = \rho \left[ 1 + \frac{1}{c^2} \left( v^2 + \varepsilon + \frac{5}{2} U \right) + \frac{1}{c^4} \left( v^4 + \frac{13}{2} v^2 U + \Gamma \varepsilon v^2 + \frac{5}{2} U \psi + \frac{5}{2} U^2 + 5 \psi \right) + 2(3) \beta_i v^i + \frac{1}{32 \pi \rho} (3) \beta_{i,j} \left( (3) \beta_{i,j} + (3) \beta_{j,i} - \frac{2}{3} \delta_{ij} (3) \beta_{k,k} \right) + O(c^{-6}) \right], \] (5.5)

or

\[ \rho_{ADM} = \rho \left[ 1 + \frac{1}{c^2} \left( v^2 + \varepsilon + \frac{5}{2} U \right) + \frac{1}{c^4} \left( v^4 + 9 v^2 U + \Gamma \varepsilon v^2 + 5 \psi + \frac{35}{4} U^2 + \frac{3}{2} (3) \beta_i v^i \right) + O(c^{-6}) \right]. \] (5.6)

(3) Total energy, which is calculated from \( M_{ADM} - M_* \) in the third PN order [10]:

\[ E \equiv \int \rho_E d^3 x, \] (5.7)

where

\[ \rho_E = \rho \left[ \left( \frac{1}{2} v^2 + \varepsilon - \frac{1}{2} U \right) + \frac{1}{c^2} \left( \frac{5}{8} v^4 + \frac{5}{2} v^2 U + \Gamma \varepsilon v^2 + 2 \psi \varepsilon - \frac{5}{2} U^2 + \frac{1}{2} (3) \beta_i v^i \right) + \frac{1}{c^4} \left( \frac{11}{16} v^6 + v^4 (\Gamma \varepsilon + \frac{47}{8} U) + v^2 (4 \psi + 6 \Gamma \varepsilon U + \frac{41}{8} U^2 + \frac{5}{2} (3) \beta_i v^i - X) \right) - \frac{5}{2} U^3 + 2 \Gamma (3) \beta_i v^i \varepsilon + 5 \psi_4 + 5 U (3) \beta_i v^i - \frac{15}{2} U (4) \psi + \frac{5}{4} U^2 \varepsilon \right. \]
\[ \left. + \frac{1}{2} h_{ij} v^i v^j + \frac{1}{2} (3) \beta_i (3) \beta_i \right. \]
\[ \left. + \frac{U}{16 \pi \rho} \left( 2 h_{ij} U_{,ij} + (3) \beta_{i,j} \left( (3) \beta_{i,j} + (3) \beta_{j,i} - \frac{2}{3} \delta_{ij} (3) \beta_{k,k} \right) \right) + O(c^{-6}) \right]. \] (5.8)
Total linear and angular momenta: In the case $K_i^i = 0$, these are calculated from

$$P_i = \lim_{r \rightarrow \infty} \frac{1}{8\pi} \oint K_{ij} n^j dS$$

$$= \lim_{r \rightarrow \infty} \frac{1}{8\pi} \oint \psi^6 K_{ij} n^j dS$$

$$= \frac{1}{8\pi} \int (\psi^6 K^j_j) d^3x$$

$$= \int (J_i + \frac{1}{16\pi} \psi^4 \gamma_{jk,i} K^{jk}) \psi^6 d^3x,$$  

(5.9)

where $J_i = (\rho \epsilon^2 + \rho \epsilon + P) \alpha u^0 u_i$. Up to the 2PN order, the second term in the last line of Eq.(5.9) becomes

$$\frac{1}{16\pi} \int h_{jk,i(3)} \beta_j k d^3x,$$

$$= \frac{1}{16\pi} \int \left[ (h_{jk,i(3)} \beta_j)_k - h_{jk,ik(3)} \beta_j \right] d^3x,$$

$$= \frac{1}{16\pi} \lim_{r \rightarrow \infty} \oint h_{jk,i(3)} \beta_j n^k dS = 0,$$  

(5.10)

where we use $h_{jk} \rightarrow O(r^{-1})$ and $(3) \beta_j \rightarrow O(r^{-2})$ at $r \rightarrow \infty$, and the gauge condition $h_{jk,k} = 0$. Thus, in the 2PN approximation, $P_i$ becomes

$$P_i \equiv \int p_i d^3x,$$  

(5.11)

where

$$p_i = \rho \left[ v^i + \frac{1}{c^2} \left\{ v^i \left( v^2 + \Gamma \epsilon + 6U \right) + (3) \beta_i \right\} + \frac{1}{c^4} \left\{ h_{ij} v^j + (5) \beta_i + (3) \beta_i \left( v^2 + 6U + \Gamma \epsilon \right) 

+ v^i \left\{ 2(3) \beta_i v^i + 10(4) \psi + 6\Gamma \epsilon U + \frac{67}{4} U^2 + \Gamma \epsilon v^2 + 10U v^2 + v^4 - X \right\} \right\} + O(c^{-5}) \right].$$  

(5.12)

The total angular momentum $J$ becomes

$$J = \int p_\varphi d^3x,$$  

(5.13)

where $p_\varphi = -y p_x + x p_y$.

VI. SUMMARY

It is generally expected that there exists no Killing vector in the spacetime of coalescing BNS’s because such a spacetime is filled with gravitational radiation which propagates to
null infinity. However, we may consider coalescing BNS’s as the almost stationary object from physical point of view as described in section 1. Motivated by this idea, in this paper, we have developed a formalism to obtain equilibrium configurations of uniformly rotating fluid up to the 2PN order using the PN approximation. The concept of being “almost” stationary becomes clear in the framework of the PN approximation and, in particular, the stationary rotating objects can exist exactly at the 2PN order, since the energy loss due to the gravitational radiation does occur from the 2.5PN order. There appear, at the 2PN order, tensor potentials $h_{ij}$ which were completely ignored in Wilson’s approach [8]. It should be noted that these tensor potentials play an important role at the 2PN order: This is because they appear in the equations to determine equilibrium configurations as shown in previous sections and they also contribute to the total energy and angular momentum of systems. This means that if we performed the stability analysis ignoring the tensor potentials, we might reach an incorrect conclusion.

In our formalism, we extract terms depending on the angular velocity $\Omega$ from the integrated Euler equation and Poisson equations for potentials, and rewrite the integrated Euler equation as an explicit equation in $\Omega$. This reduction will improve the convergence in numerical iteration procedure. As a result, the number of Poisson equations we need to solve in each step of iteration reaches 29. However, source terms of the Poisson equations decrease rapidly enough, at worst $O(r^{-4})$, in the region far from the source, so that we can solve accurately these equations as the boundary value problem like in the case of the first PN calculations [8]. Thus, the present formalism will be useful to obtain equilibrium configurations for synchronized BNS’s or the Jacobi ellipsoid. These configurations will be obtained in future work.

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**Figure Captions**

Fig. 1 Error of the pressure in the post-Newtonian approximation for the GR compact star of uniform density as a function of the normalized areal radius ($r/R$). Solid and dotted lines show the case $R = 5Gm/c^2$ and $8Gm/c^2$, where $R$ and $m$ are the circumference radius and the mass of star, respectively.
Table 1 (a)

Various levels of approximation in terms of PN expansions($v^2/c^2$) and mass ratio($\eta = \mu/M; \mu = \text{reduced mass}, M = \text{total mass}$). We mark $P^2N$ if all terms in that level are taken into account in the 2PN approximation, while $W$ is marked if Wilson’s approach takes into account all terms in that level. The mark $-$ means that the relevant term does not exist and the levels taken into account by neither approaches are blank. We neglect secular effects due to gravitational radiation reaction in Tables 1(a) and (b). It should be noted that, at $O(\eta^0)$, Wilson’s approach produces exact GR solutions, but it is not justified at the 2PN order even at $O(\eta^1)$.

| PN \ $\eta$ | $\eta^0$ | $\eta^1$ | $\eta^2$ | $O(\eta^3)$ |
|-------------|----------|----------|----------|-------------|
| N           | $P^2N, W$| $-$      | $-$      | $-$         |
| 1PN         | $P^2N, W$| $P^2N, W$| $-$      | $-$         |
| 2PN         | $P^2N, W$| $P^2N$   | $P^2N$   | $-$         |
| $\geq$3PN   | $W$      |          |          |             |

Table 1 (b)

Various levels of approximation in terms of PN expansions($Gm/c^2R$) and ellipticity of a NS($e$). The meanings of $P^2N$ and $W$ are the same as those in Table 1(a). Wilson’s approach produces exact GR solutions in the case of the completely spherical star.

| PN \ $e$ | $e = 0$ | $e \neq 0$ |
|----------|---------|------------|
| N        | $P^2N, W$| $P^2N, W$  |
| 1PN      | $P^2N, W$| $P^2N, W$  |
| 2PN      | $P^2N, W$| $P^2N$     |
| $\geq$3PN| $W$      |             |
Table 2

List of potentials to be solved (column 1), Poisson equations for them (column 2), and other potential variables which appear in the source term of the Poisson equation (column 3). Note that $i$ and $j$ run $x, y, z$. Also, note that we do not have to solve $\eta_{0z}$, $(5)P_{0z}$, $q_{yy}$, $q_{zz}$ and $h_{ij}^{(U)}$.

| Pot. | Eq. | Needed pots. | Pot. | Eq. | Needed pots. |
|------|-----|--------------|------|-----|--------------|
| $U$  | (2.11) | None | $q_{ij}$ | (4.6) | None |
| $q_i$ | (3.14) | None | $Q_{0i}^{(I)}$ | (4.15) | $U$ |
| $q_2$ | (4.1) | None | $\eta_{0i}$ | (4.16) | $U$ |
| $q_{2i}$ | (4.2) | None | $(5)P_{0i}$ | (4.17) | $U$, $q_{i}$ |
| $q_4$ | (4.3) | None | $(6)\alpha_0$ | (4.21) | $U$, $q_e$, $q_u$, $h_{ij}^{(U)}$, $Q_{0i}^{(I)}$ |
| $q_u$ | (4.4) | $U$ | $(6)\alpha_2$ | (4.22) | $U$, $q_2$, $q_i$, $q_{2i}$, $q_{ij}$ |
| $q_e$ | (4.5) | None | $h_{ij}^{(U)}$ | (3.1) | $U$ |
Table 3

Variables to be solved in order to obtain the original metric variables.

| Metric | Variables to be solved | see Eq. |
|--------|------------------------|---------|
| $U$    | $U$                    | (2.11)  |
| $(3)\beta_i$ | $q_i, U$          | (3.17)  |
| $X$    | $q_2, q_u, q_e$       | (4.7)   |
| $(4)\psi$ | $q_2, q_u, q_e$     | (4.8)   |
| $(5)\beta_i^{(A)}$ | $(5)P_{0i}, \eta_{0i}, q_u, q_e$ | (4.18)  |
| $(5)\beta_i^{(B)}$ | $q_{2i}, q_2$      | (4.18)  |
| $(6)\alpha$ | $(6)\alpha_0, (6)\alpha_2, q_4$ | (4.20)  |
| $h_{ij}^{(U)}$ | $h_{ij}^{(U)}$ | (3.1)   |
| $h_{ij}^{(S)}$ | $q_{ij}, q_2$    | (4.14)  |
| $h_{ij}^{(A)}$ | $Q_{0i}^{(I)}, q_u, q_e$ | (4.19)  |
| $h_{ij}^{(B)}$ | $q_{ij}, q_2, q_{2i}, q_i$ | (4.19)  |
