Abstract

Information geometry is an important tool to study statistical models. There are some important examples in statistical models which are regarded as warped products. In this paper, we study information geometry of warped products. We consider the case where the warped product and its fiber space are equipped with dually flat connections and, in the particular case of a cone, characterize the connections on the base space \( \mathbb{R}^+ \). The resulting connections turn out to be the \( \alpha \)-connections with \( \alpha = \pm 1 \).

Keywords

Information geometry · Warped product · \( \alpha \)-connection

1 Introduction

Recently, the study of spaces consisting of probability measures is getting more attention. As tools to investigate such spaces, there are two famous theories in geometry: information geometry and Wasserstein geometry. Information geometry is mainly concerned with finite dimensional statistical models and Wasserstein geometry is concerned with infinite dimensional spaces of probability measures. We can compare these two geometries, for example, on Gaussian distributions.

This paper concerns information geometry of a warped product and, in particular, on a cone, which is a kind of warped product of the line \( \mathbb{R}^+ \) and a manifold. Under some natural assumptions, we characterize connections on the line, with which warped products are constructed. The assumption we set is different from that of [18] and matches examples of statistical models.

Examples of warped product metrics include the denormalizations of the Fisher metric, the Bogoliubov–Kubo–Mori metric and the Fisher metric on the Takano Gaussian space, which is a set of multivariate Gaussian distributions with restricted parameters.
Besides these examples, there are some more statistical models represented as warped products. In [17], it was shown that the Wasserstein Gaussian space, which is the set of multivariate Gaussian distributions on $\mathbb{R}^n$ with mean zero equipped with the $L^2$-Wasserstein metric, has a cone structure and in [14] the relations between Fisher metrics of location scale models and warped product metrics are studied. It seems that warped products get more attention in the field of statistical models than before.

Although information geometry is studied on real manifolds, the theory of statistical manifolds is studied in the field of affine geometry and statistical structures on complex manifolds get more attention as in [7]. Also in this field, warped products are important since they play an important role in the theory of submanifolds in complex manifolds, for example, CR submanifold theory as in [3]. There are many researches extending the theories of CR submanifolds in Kähler manifolds to submanifolds in holomorphic statistical manifolds as in [1]. Statistical structures in [8] and the structures cultivated in this paper are slightly different because we do not care the compatibility of statistical structures and complex structures. This compatibility is expressed in the definition of holomorphic statistical structures in [7].

This paper is organized as follows. In Sect. 2, we briefly review information geometry. Section 3 is devoted to some formulas in warped products. Then in Sect. 4, we study cones and consider necessary conditions for making both the cone and the fiber space to be dually flat. This necessary condition states that there are only two possible connections on the line. The following theorem is one of our main results.

**Theorem** (Theorem 1) Under Assumption 2, we have

$$D_\alpha \tilde{\partial}_t = \frac{1}{t} \tilde{\partial}_t \quad \text{or} \quad -\frac{1}{t} \tilde{\partial}_t,$$

where $t$ is the natural coordinate on the line $\mathbb{R}_{>0}$, which is the base space of the warped product.

By observing examples, these two connections turn out to be the $\alpha$-connections with $\alpha = \pm 1$. An analogous characterization for the Takano Gaussian space is also considered in Sect. 5. In Sect. 6, we discuss dually flat connections on the Wasserstein Gaussian space. We remark that, although it is known in [13] that there is no dually flat proper doubly warped Finsler manifold, what they actually proved is that some coordinates cannot be dual affine coordinates. Thus our claims do not contradict their claim. We also discuss this point in Sect. 6. In the Appendix, we study elliptic distributions.

### 2 Preliminaries

#### 2.1 Information geometry

We briefly review the basics of information geometry, we refer to [2] for further reading. Let $(M, g)$ be a Riemannian manifold and $\nabla$ be an affine connection of $M$. $\mathfrak{X}(M)$ denotes the set of $C^\infty$ vector fields on $M$. We define another affine connection
\[ X g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X^* Z) \]

for \( X, Y, Z \in \mathfrak{X}(M) \). We call \( \nabla^* \) the dual connection of \( \nabla \). We define the torsion and the curvature of \( \nabla \) by

\[ T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y], \quad R(X, Y) Z := [\nabla_X, \nabla_Y] Z - \nabla_{[X, Y]} Z, \]

respectively. If \( R \) satisfies

\[ R(X, Y) Z = k \{ g(Y, Z) X - g(X, Z) Y \} \]

for some \( k \in \mathbb{R} \) and all \( X, Y, Z \in \mathfrak{X}(M) \), \((M, g, \nabla)\) is called a space of constant curvature \( k \). We summarize some important facts on \( \nabla \) and \( \nabla^* \) in the following.

**Proposition 1** Let \( \nabla \) and \( \nabla^* \) be dual affine connections of \( M \). If two of the following conditions hold true, then the other two of them also hold true:

- \( \nabla \) is torsion free,
- \( \nabla^* \) is torsion free,
- \( \nabla g \) is a symmetric tensor,
- \( \frac{\nabla + \nabla^*}{2} \) is the Levi-Civita connection of \( g \).

**Proposition 2** Let \((M, g, \nabla, \nabla^*)\) be a Riemannian manifold with dual affine connections. The curvature with respect to \( \nabla \) vanishes if and only if the curvature with respect to \( \nabla^* \) vanishes.

Let \((M, g, \nabla, \nabla^*)\) be a Riemannian manifold with dual affine connections. If the torsion and the curvature with respect to \( \nabla \) and those of \( \nabla^* \) all vanish, then we say that \((M, g, \nabla, \nabla^*)\) is dually flat. For a local coordinate system \((U; x_1, \ldots, x_n)\), if the Christoffel symbols \( \{\Gamma^k_{ij}\} \) of \( \nabla \) vanish, we call it \( \nabla \)-affine coordinates.

**Proposition 3** Let \((M, g, \nabla, \nabla^*)\) be a Riemannian manifold with dual affine connections. If it is dually flat, then there exist \( \nabla \)-affine coordinates \((x_i)\) and \( \nabla^* \)-affine coordinates \((y_j)\) such that

\[ g \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_j} \right) = \delta_{ij}. \]

The coordinates \{\((x_i), (y_j)\)\} above are called dual affine coordinates. Using dual affine coordinates, we can construct the canonical divergence [2, §3.4].

Next, we introduce the Fisher metric and \( \alpha \)-connections. Consider a family \( S \) of probability distributions on a finite set \( \mathcal{X} \). Suppose that \( S \) is parameterized by \( n \) real-valued variables \([\xi^1, \ldots, \xi^n] \) so that

\[ S := \{ p_{\xi} = p(x; \xi) \mid \xi = [\xi^1, \ldots, \xi^n] \in \Xi \}, \]
where $\Xi$ is an open subset of $\mathbb{R}^n$.

For $\alpha \in \mathbb{R}, u > 0, x \in X$ and $\xi \in \Xi$, we put

$$L^{(\alpha)}(u) := \begin{cases} \frac{2}{1-\alpha} u^{1-\alpha} & (\alpha \neq 1), \\ \log u & (\alpha = 1), \end{cases} \quad l^{(\alpha)}(x; \xi) := L^{(\alpha)}(p(x; \xi)).$$

Then, we define the Fisher metric $g$ as

$$g_{ij}(\xi) := \int \partial_i l^{(\alpha)}(x; \xi) \partial_j l^{(-\alpha)}(x; \xi) \, dx,$$

and $\alpha$-connections $\Gamma^{(\alpha)}_{ij,k}$ as

$$\Gamma^{(\alpha)}_{ij,k}(\xi) := \int \partial_i \partial_j l^{(\alpha)}(x; \xi) \partial_k l^{(-\alpha)}(x; \xi) \, dx,$$

where $g(\nabla_{\partial_i} \partial_j, \partial_k) = \Gamma^{(\alpha)}_{ij,k}$. Note that the Fisher metric does not depend on $\alpha$. We set

$$\mathcal{S} := \{ \tau p_\xi \mid \xi \in \Xi, \tau > 0 \},$$

and call it the denormalization of $S$.

In [2], the Fisher metric and connections on $\mathcal{S}$ are defined as follows. An extension $\tilde{l}$ of $l$ is defined as

$$\tilde{l}^{(\alpha)} = l^{(\alpha)}(x; \xi, \tau) := L^{(\alpha)}(\tau p(x; \xi)).$$

Using this $\tilde{l}$, we define the metric and connections on $\mathcal{S}$ by

$$\tilde{g}_{ij}(\xi) := \int \partial_i \tilde{l}^{(\alpha)} \partial_j \tilde{l}^{(-\alpha)} \, dx, \quad \tilde{\Gamma}^{(\alpha)}_{ij,k} = \int \partial_i \partial_j \tilde{l}^{(\alpha)} \partial_k \tilde{l}^{(-\alpha)} \, dx. \quad (1)$$

### 2.2 Quantum information geometry

Information geometry of density matrices is called quantum information geometry. The set of density matrices $D$ is defined as

$$D := \{ \rho \in \mathbb{P}(n) \mid \text{Tr}(\rho) = 1 \},$$

where $\mathbb{P}(n)$ is the set of $n \times n$ positive definite Hermitian matrices. Parameterizing elements of $D$ as $\rho_\xi$ by $\xi \in \Xi$, the $m$-representation of the natural basis is written as

$$(\partial_i)^{(m)} = \partial_i \rho.$$
The mixture connection $\nabla^{(m)}$ is a connection such that
\[
(\nabla_{\partial_i}^{(m)} \partial_j)^{(m)} = \partial_i \partial_j \rho.
\]

We set
\[
\mathcal{MON} := \left\{ f : \mathbb{R}_{>0} \to \mathbb{R}_{>0} \mid f \text{ is operator monotone, } f(1) = 1, f(t) = tf\left(\frac{1}{t}\right) \right\}.
\]

The monotone metric for $f \in \mathcal{MON}$ is expressed as
\[
g_f^\rho(X, Y) = \text{Tr} \left\{ X^* \frac{1}{(2\pi i)^2} \oint \oint c(\xi, \eta) \frac{1}{\xi - \rho} \frac{1}{\eta - \rho} Y d\xi d\eta \right\},
\]
where $c(x, y) = 1/(yf(x/y))$ and $\xi(t), \eta(t)$ are paths surrounding the positive spectrum of $\rho$. The monotone metric for $f(x) = (x - 1)/\log x$ is called the Bogoliubov–Kubo–Mori (BKM) metric. We refer to [2] and [4] for further reading. It is known that the BKM metric enjoys the following remarkable property.

**Proposition 4** ($\mathcal{D}, \text{BKM}$) equipped with the mixture connection is dually flat.

We can find a proof of this proposition in [2, Theorem 7.1], and the proof does not use the condition that the matrices considered have trace 1. Thus we can prove the proposition below in completely the same way.

**Proposition 5** ($\mathcal{P}(n), \text{BKM}$) equipped with the mixture connection is dually flat.

### 2.3 Takano Gaussian space

In this subsection, we explain some results from [15]. We consider multivariate Gaussian distributions
\[
p(x; \xi) = \frac{1}{(\sqrt{2\pi} \sigma)^n} \prod_{i=1}^{n} \exp \left\{ -\frac{(x_i - m_i)^2}{2\sigma^2} \right\},
\]
where $\xi = (\sigma, m_1, \ldots, m_n) \in L^{(n+1)}$, $L^{(n+1)} := \mathbb{R}_{>0} \times \mathbb{R}^n$.

By a straightforward calculation, we obtain the Fisher metric $G_T$ as
\[
(G_T)_{\sigma \sigma} = \frac{2n}{\sigma^2}, \quad (G_T)_{\sigma i} = (G_T)_{i \sigma} = 0, \quad (G_T)_{ij} = \frac{1}{\sigma^2} \delta_{ij},
\]
where $\partial_{\sigma} = \partial / \partial \sigma$ and $\partial_i = \partial / \partial m_i$, i.e.,
\[
G_T = \frac{1}{\sigma^2} (2n \sigma^2 + dm_1^2 + \cdots + dm_n^2).
\]
Its $\alpha$-connections are

\[
\Gamma_{ij,k}^{(\alpha)} = 0, \quad \Gamma_{ij,\sigma}^{(\alpha)} = \frac{1 - \alpha}{\sigma^3} \delta_{ij}, \quad \Gamma_{i\sigma,k}^{(\alpha)} = -\frac{1 + \alpha}{\sigma^3} \delta_{ik},
\]

\[
\Gamma_{i\sigma,\sigma}^{(\alpha)} = 0, \quad \Gamma_{\sigma\sigma,i}^{(\alpha)} = 0, \quad \Gamma_{\sigma\sigma,\sigma}^{(\alpha)} = -(1 + 2\alpha) \frac{2n}{\sigma^3},
\]

and

\[
\nabla^{(\alpha)}_{\partial_i} \partial_j = \frac{1 - \alpha}{2n\sigma} \delta_{ij} \partial_\sigma, \quad \nabla^{(\alpha)}_{\partial_i} \partial_\sigma = \nabla^{(\alpha)}_{\partial_\sigma} \partial_i = -\frac{1 + \alpha}{\sigma} \partial_i, \quad \nabla^{(\alpha)}_{\partial_\sigma} \partial_\sigma = -\frac{1 + 2\alpha}{\sigma} \partial_\sigma.
\]

In [15], they call $\alpha$-flat if the curvature tensor with respect to the $\alpha$-connection vanishes identically and the following fact is proved.

**Proposition 6** $(L^{(n+1)}, G_T, \nabla^{(\alpha)})$ is a space of constant curvature $-\frac{(1-\alpha)(1+\alpha)}{2n}$. In particular, $(L^{(n+1)}, G_T)$ is $(\pm 1)$-flat.

For simplicity, we call $(L^{(n+1)}, G_T)$ the Takano Gaussian space in this paper.

### 3 Warped products

In this section, we calculate dual affine connections on warped products.

#### 3.1 Koszul formula

Let $\nabla, \nabla^*$ be torsion free dual affine connections on $(M, g)$. For $X, Y, Z, W \in \mathfrak{X}(M)$, let us first see a kind of Koszul formula for $\nabla$. Summing up

\[
Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X^* Z),
\]

\[
Yg(X, Z) = g(\nabla_Y X, Z) + g(X, \nabla_Y^* Z),
\]

\[
-Zg(X, Y) = -g(\nabla_Z X, Y) - g(X, \nabla_Z^* Y),
\]

we get

\[
Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) = g(\nabla_X Y, Z) + g(\nabla_Y X, Z) + g(Y, \nabla_X^* Z - \nabla_Z X) + g(X, \nabla_Y^* Z - \nabla_Z^* Y).
\]

Recalling that we consider torsion free affine connections, we have

\[
2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) + g([X, Y], Z) - g(Y, \nabla_X Z - \nabla_Z X) - g(X, [Y, Z]).
\]

In order to have a further look on $(\nabla_X^* Z - \nabla_Z X)$, we put

\[
a(X, W) := \nabla_X^* W - \nabla_W X,
\]
and calculate
\[
\begin{align*}
a(X, W) - a(W, X) &= (\nabla_X^W W - \nabla_W X) - (\nabla_W^X X - \nabla_X W) = 2[X, W], \\
a(X, W) + a(W, X) &= (\nabla_X^W W - \nabla_W X) + (\nabla_W^X X - \nabla_X W) = -2(P_X W + P_W X),
\end{align*}
\]
where we put
\[
P := \frac{\nabla - \nabla^*}{2}.
\]

Let us collect some properties of \(P\).

**Lemma 1** Let \(f\) be an arbitrary \(C^\infty\) function on \(M\). For any \(X, Y, Z \in \mathfrak{X}(M)\), we have the following equations:

\[
\begin{align*}
P_X Y &= P_Y X, \\
g(P_X Y, Z) &= g(Y, P_X Z), \\
P_{fX} Y &= f P_X Y, \quad P_X fY = f P_X Y.
\end{align*}
\]

**Proof** For (3),
\[
P_X Y - P_Y X = \frac{\nabla_X Y - \nabla_Y X}{2} - \frac{\nabla_X^* Y - \nabla_Y^* X}{2} = \frac{1}{2}(X - Y) = 0.
\]

For (4),
\[
g\left(\frac{\nabla_X - \nabla_X^*}{2}, Z\right) = \frac{1}{2} \left(g(\nabla_X Y, Z) - g(\nabla_X^* Y, Z)\right)
\]
\[
= \frac{1}{2} \left(X g(Y, Z) - g(Y, \nabla_X Z)\right) - \frac{1}{2} \left(Y g(Y, Z) - g(Y, \nabla_X Z)\right)
\]
\[
= \frac{1}{2} \left(g(Y, \nabla_X Z) - g(Y, \nabla_X^* Z)\right)
\]
\[
= g(Y, P_X Z).
\]

For (5), the first equation is clear and we also observe
\[
P_X (f Y) = \frac{\nabla_X (f Y) - \nabla_X^* (f Y)}{2} = X f \frac{Y - Y}{2} + f \frac{\nabla_X Y - \nabla_X^* Y}{2} = f P_X Y.
\]

By the above lemma, we can express \(a\) using \(P\) as
\[
\begin{align*}a(X, W) + a(W, X) &= -2(P_X W + P_W X) = -4P_X W, \\
a(X, W) &= [X, W] - 2P_X W.
\end{align*}
\]

Substituting this into (2), we obtain the following Koszul formula:
\[
2g(\nabla_X Y, Z) = X g(Y, Z) + Y g(X, Z) - Z g(X, Y)
\]
\[ g([X, Y], Z) - g(Y, [X, Z] - 2P_X Z) \]
\[ -g(X, [Y, Z]). \]  

(6)

3.2 O’Neill formulas for affine connections on warped products

Let \((B, g_B), (F, g_F)\) be Riemannian manifolds, \(f\) be a positive \(C^\infty\)-function on \(B\), \(M := B \times_f F\) be the warped product of them equipped with the metric \(G := g_B + f^2 g_F\), and \(D, D^*\) be torsion free dual affine connections on \(M\). Denote by \(L(F), L(B)\) the sets of lifts of vector fields on \(F\) to \(M\), \(B\) to \(M\), respectively. Let \(X, Y, Z \in L(B)\), \(U, V, W \in L(F)\) in the sequel. We will assume the following.

**Assumption 1** \(DX Y \in L(B)\) for all \(X, Y \in L(B)\).

**Lemma 2** Under Assumption 1, we have \(G(D^*_X Y, V) = 0\), \(G(P_X V, Y) = 0\).

**Proof** The first equation follows from Assumption 1 and the fact that \((D + D^*)/2\) is the Levi-Civita connection (recall Proposition 1). To see the second equation, since \(G(Y, V) = G(X, V) = 0\) and \([X, V] = [Y, V] = 0\), we have

\[
2G(D_X Y, V) = XG(Y, V) + YG(X, V) - VG(X, Y) + G([X, Y], V) - G(Y, [X, V] - 2P_X V) - G(X, [Y, V])
\]
\[
= -VG(X, Y) + G([X, Y], V) + G(Y, 2P_X V)
\]
\[
= G(Y, 2P_X V).
\]

By combining this with \(2G(D_X Y, V) = 0\) by Assumption 1, the second equation holds. \(\square\)

Let us modify some formulas on warped products in \([11]\) for the Levi-Civita connections to those for affine connections.

First we express \(D_V X\). On the one hand, \(G(D_X V, Y) = 0\) by the Koszul formula (6) and Lemma 2. On the other hand, it follows from (4) and (6) that

\[
2G(D_X V, W) = XG(V, W) + V G(X, W) - W G(X, V) + G([X, V], W) - G(V, [X, W] - 2P_X W) - G(X, [V, W])
\]
\[
= XG(V, W) + 2G(V, P_X W)
\]
\[
= XG(V, W) + 2G(P_X V, W).
\]

Since

\[
XG(V, W) = 2\frac{Xf}{f} G(V, W)
\]
by the definition of $G$ and $D$ is torsion free, we obtain

$$D_V X = D_X V = \frac{Xf}{f} V + P_X V. \quad (7)$$

Next we consider $D_W V$. Observe that

$$G(D_V W, X) = -G(W, D^*_V X) = -G(W, \frac{Xf}{f} V - P_X V).$$

Using $Xf = G(\nabla f, X)$, (3) and (4), we have

$$G(D_V W, X) = G\left(-\frac{G(V, W)}{f} \nabla f + P_V W, X\right).$$

Thus we obtain

$$\text{Hor} D_V W = -\frac{G(V, W)}{f} \nabla f + \text{Hor} P_V W, \quad (8)$$

where Hor denotes the projection to $TB$.

**Remark 1** As another way to reach these formulas, we can use the fact that $(D + D^*)/2$ is the Levi-Civita connection and formulas in [11]. For example,

$$\left(\frac{D + D^*}{2}\right)_X V = \frac{Xf}{f} V$$

implies

$$D_X V = \frac{Xf}{f} V + \frac{D_X V - D^*_X V}{2} = \frac{Xf}{f} V + P_X V.$$

## 4 Cones

In this section, we specialize our study of warped products to cones. We fix our framework and assumptions (including Assumption 1).

**Assumption 2** Let $B = \mathbb{R}_{>0}$ with the Euclidean metric $g_B$ such that $g_B\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = 1$, $f(t) = t$ and $(\tilde{\nabla}, \tilde{\nabla}^*)$ be dually flat affine connections on $(F, g_F)$. Let $D, D^*$ be dually flat affine connections on $B \times_f F$ and $G$ be its warped product metric. We assume that $D$ satisfies

- $D_X Y$ is horizontal, i.e., $D_X Y \in \mathcal{L}(B)$ for all $X, Y \in \mathcal{L}(B)$,
- $\text{Ver} (D_V W) = \text{Lift} (\tilde{\nabla}_V W)$ for all $V, W \in \mathcal{L}(F)$,

where Ver is the projection to $TF$. 
Remark 2 Denote the curvature with respect to $D$ by $R$ and the curvature with respect to $\tilde{\nabla}$ by $F R$. Denote their duals by $R^*$ and $F R^*$. Note that by Proposition 2, we have $R^* = F R^* = 0$ when $R = F R = 0$.

Remark 3 We only consider the warping functions which give concrete examples.

In the following arguments in this section, we assume this assumption without mentioning. We shall study what $R = F R = 0$ means and characterize admissible connections on $B$ (Theorem 1).

4.1 Calculations of $G(R(U, V)V, U)$

We first consider in vertical directions. We are going to calculate the Gauss equation (the relation between $R$ and $F R$) for affine connections on $F$ and $M$ in a similar way to [11]. For $U, V, W, Q \in \mathcal{L}(F)$, it follows from Assumption 2 that

$$G(D_U D_V W, Q) = G(D_U (\text{Ver} D_V W), Q) + G(D_U (\text{Hor} D_V W), Q) = G(\tilde{\nabla}_U \tilde{\nabla}_V W, Q) + \{UG(II(V, W), Q) - G(II(V, W), D^*_U Q)\}$$

$$= G(\tilde{\nabla}_U \tilde{\nabla}_V W, Q) - G(II(V, W), \text{Hor} D^*_U Q)$$

$$= G(\tilde{\nabla}_U \tilde{\nabla}_V W, Q) - G(II(V, W), II^*(U, Q)),$$

where $II(W, X) := \text{Hor} D_W X$, which is an affine version of the second fundamental form on $F$. Thus we have

$$G(R(U, V)V, U) = G(F R(U, V)V, U) - G(II(V, W), II^*(U, Q)) + G(II(U, W), II^*(V, Q)).$$

(9)

Recall that $F R = 0$ by Assumption 2. Observe from (8) that

$$G(II(V, W), II^*(U, Q))$$

$$= G\left( -\frac{G(V, W)}{f} \text{grad} f + \text{Hor}(P_V W), -\frac{G(U, Q)}{f} \text{grad} f - \text{Hor}(P_U Q) \right)$$

$$= \frac{\|\text{grad} f\|^2}{f^2} G(V, W)G(U, Q) - G(\text{Hor}(P_V W), \text{Hor}(P_U Q))$$

$$+ \frac{G(V, W)G(\text{grad} f, P_U Q)}{f} - \frac{G(U, Q)G(\text{grad} f, P_V W)}{f},$$

and similarly

$$G(II(U, W), II^*(V, Q)) = \frac{\|\text{grad} f\|^2}{f^2} G(U, W)G(V, Q) - G(\text{Hor}(P_U W), \text{Hor}(P_V Q))$$

$$+ \frac{G(U, W)G(\text{grad} f, P_V Q)}{f} - \frac{G(V, Q)G(\text{grad} f, P_U W)}{f}.$$
Substituting these and letting $W = V$ and $Q = U$,

\[
G(R(U, V) V, U) = -\frac{\|\text{grad} f\|^2}{f^2} \left\{ G(V, V) G(U, U) - G(U, V)^2 \right\} \\
+ G(\text{Hor}(P_V V), \text{Hor}(P_U U)) - G(\text{Hor}(P_V V), \text{Hor}(P_U U)) \\
- G(V, V) G(\text{grad} f, P_U U) \\
+ \frac{G(U, U) G(\text{grad} f, P_V V)}{f} + \frac{G(U, V) G(\text{grad} f, P_V U)}{f} \\
- \frac{G(V, U) G(\text{grad} f, P_U V)}{f} ,
\]

where we used (3). Recalling $R = 0$ by Assumption 2, for any $U, V \in \mathcal{L}(F)$, we find

\[
\frac{\|\text{grad} f\|^2}{f^2} \left\{ G(V, V) G(U, U) - G(U, V)^2 \right\} \\
+ G(\text{Hor}(P_V V), \text{Hor}(P_U U)) - G(\text{Hor}(P_V U), \text{Hor}(P_V U)) \\
- \frac{G(V, V) G(\text{grad} f, P_U U)}{f} + \frac{G(U, U) G(\text{grad} f, P_V V)}{f} - \frac{G(U, U) G(\text{grad} f, P_V U)}{f} = 0. 
\]

Focusing on the symmetric and anti-symmetric parts in $U$ and $V$, and recalling $f(t) = t$, we obtain the following two equations:

\[
G(V, V) G(\text{grad} f, P_U U) = G(U, U) G(\text{grad} f, P_V V), \tag{10}
\]

\[
\frac{1}{t^2} \left\{ G(V, V) G(U, U) - G(U, V)^2 \right\} - G(\text{Hor}(P_V V), \text{Hor}(P_U U)) \\
+ G(\text{Hor}(P_V U), \text{Hor}(P_U U)) = 0. \tag{11}
\]

To characterize admissible connections on $B$, we prepare some lemmas.

**Lemma 3** We have

\[
\text{Hor}\left( P_{\frac{V}{\|V\|}} \frac{U}{\|U\|} \right) = \text{Hor}\left( P_{\frac{V}{\|V\|}} \frac{V}{\|V\|} \right)
\]

for any $U, V \in \mathcal{L}(F)$. 
Proof It follows from (10) that
\[ G \left( \text{grad} \, f, \text{Hor} \, P \frac{U}{\|U\|} \right) = G \left( \text{grad} \, f, \text{Hor} \, P \frac{V}{\|V\|} \right) \]
and we get
\[ \text{Hor} \left( P \frac{U}{\|U\|} \right) = \text{Hor} \left( P \frac{V}{\|V\|} \right). \]
\[ \square \]

Hereafter, fix an arbitrary \( x \in F \). Let \( (\xi_i) \) be normal coordinates around \( x \) on \( F \) and denote \( \widetilde{\partial}_i = \text{Lift} \left( \frac{\partial}{\partial \xi_i} \right) \). We also denote \( \widetilde{\partial}_t = \text{Lift} \left( \frac{\partial}{\partial t} \right) \).

Lemma 4 We have
\[ (\text{Hor} P \frac{U}{\|U\|})_{(x,t)} = (t \widetilde{\partial}_t) \text{ or } (-t \widetilde{\partial}_t). \]

Proof Put \( U = \widetilde{\partial}_i \) and \( V = \widetilde{\partial}_j \) (\( i \neq j \)). Then Lemma 3 implies
\[ \frac{\text{Hor} \, P_{(U+V)}(U + V)}{\|U + V\|^2} = \frac{\text{Hor} \, P_U U}{\|U\|^2} = \frac{\text{Hor} \, P_V V}{\|V\|^2}. \]
Note that, for all \( t > 0 \),
\[ \|U + V\|^2_{(x,t)} = \|U\|^2_{(x,t)} + \|V\|^2_{(x,t)} = 2t^2. \]
This yields
\[ \left\{ \text{Hor} P_{(U+V)}(U + V) \right\}_{(x,t)} = 2 (\text{Hor} P_U U)_{(x,t)} = 2 (\text{Hor} P_V V)_{(x,t)}, \]
and hence
\[ (\text{Hor} P_U V)_{(x,t)} = 0. \] (12)
Combining this with (11), we have for all \( t \),
\[ t^2 = G_{(x,t)} (\text{Hor} P_V V, \text{Hor} P_U U) = G_{(x,t)} (\text{Hor} P_U U, \text{Hor} P_U U). \]
This proves the claim. \[ \square \]
4.2 Calculations of $G(R(V, X)X, V)$

Now, we put $X = \tilde{\partial}_t$ and define $k$ by $D_X X = k(t)\tilde{\partial}_t$.

**Lemma 5** We have

$$(P_X \tilde{\partial}_t)_{(x,t)} = \left(\frac{1}{t} \tilde{\partial}_t\right)_{(x,t)} \text{ or } \left(-\frac{1}{t} \tilde{\partial}_t\right)_{(x,t)}$$

for all $t > 0$.

**Proof** Put $V = \tilde{\partial}_t$. Recall that $(\text{Hor } P_V V)_{(x,t)} = t\tilde{\partial}_t$ or $(-t\tilde{\partial}_t)$ by Lemma 4. First, we consider the case $(\text{Hor } P_V V)_{(x,t)} = t\tilde{\partial}_t$. We deduce from (7) that

$$G_{(x,t)}(D^*_V D^*_X X, V) = G \left(-k(t) \left(\frac{1}{t} V - P_X V\right), V\right)$$

$$= -k(t)t + k(t)t$$

$$= 0$$

for all $t > 0$, where the second equality follows since $G(P_X V, V) = G(P_V X, V) = G(X, P_V V) = t$ by (3) and (4). We similarly find from (7) that

$$G_{(x,t)}(D^*_X D^*_V X, V) = XG \left(\frac{1}{t} V - P_X V, V\right) - G \left(\frac{V}{t} - P_X V, D_X V\right)$$

$$= \partial_t \left(\frac{1}{t^2} - t\right) - G \left(\frac{V}{t} - P_X V, \frac{V}{t} + P_X V\right)$$

$$= -G \left(\frac{V}{t}, \frac{V}{t}\right) + G(P_X V, P_X V)$$

$$= -1 + G(P_X V, P_X V).$$

Therefore we obtain for all $t > 0$, since $R^* = 0$,

$$G(P_X V, P_X V) = 1.$$

Next we consider the case $(\text{Hor } P_V V)_{(x,t)} = -t\tilde{\partial}_t$. We have

$$G_{(x,t)}(D_V (D_X X), V) = G \left(k(t) \left(\frac{1}{t} V + P_X V\right), V\right)$$

$$= tk(t) - tk(t)$$

$$= 0$$

and

$$G_{(x,t)}(D_X (D_V X), V) = G \left(D_X \left(\frac{1}{t} V + P_X V\right), V\right).$$
= XG \left( \frac{1}{t} V + P_X V, V \right) - G \left( \frac{V}{t} + P_X V, \frac{V}{t} - P_X V \right) \\
= -1 + G(P_X V, P_X V).

Since R = 0, we have \( G(x, t)(P_X V, P_X V) = 1 \).

For \( U = \tilde{\partial}_j(i \neq j) \), using \((\text{Hor } P_V U)(x, t) = 0\) in (12), (3) and (4), we have

\[
G(x, t)(P_X V, U) = G(x, t)(P_V X, U) = G(x, t)(X, P_V U) = 0.
\]

Moreover \( G(P_X V, X) = G(V, P_X X) = 0 \). Therefore \((P_X V)(x, t)\) and \(V(x, t)\) are linearly dependent for all \( t > 0 \), which proves the claim. \( \square \)

The next result is the aim of this section, which is a characterization of connections on the line \( B \).

**Theorem 1** Under Assumption 2, we have

\[
k(t) = \frac{1}{t} \text{ or } \left( -\frac{1}{t} \right).
\]

**Proof** Put \( V = \tilde{\partial}_t \). When \((\text{Hor } P_V V)(x, t) = -t\tilde{\partial}_t \), we have

\[
G(x, t)(P_X V, V) = G(x, t)(P_V X, V) = G(x, t)(X, P_V V) = -t.
\]

Combining this with Lemma 5, we find

\[
(P_V X)(x, t) = -\frac{1}{t} V.
\]

We similarly find that \( P_V X = \frac{1}{t} V \) if \( \text{Hor } P_V V = t\tilde{\partial}_t \). Hence, we need to consider only the following two cases.

First, we consider the case \((\text{Hor } P_V V)(x, t) = t\tilde{\partial}_t \) and \((P_V X)(x, t) = \frac{1}{t} V \). We have

\[
G(D_V (D_X X), V) = G \left( k(t) \left( \frac{1}{t} V + P_V \tilde{\partial}_t \right), V \right) = 2tk(t),
\]

and

\[
G(D_X (D_V X), V) = XG(D_V X, V) - G(D_V X, D_X^* V) \\
= \partial_t \left\{ G \left( \frac{1}{t} V + P_X V, V \right) \right\} - G \left( \frac{V}{t} + P_X V, \frac{V}{t} - P_X V \right) \\
= \partial_t(t + t) - 1 + 1 = 2.
\]

Hence by \( R = 0 \), we obtain \( k(t) = \frac{1}{t} \). \( \square \)
Next, we consider the case \((\text{Hor} P_V V)(x, t) = -t \tilde{\partial}_t\) and \((P_V X)(x, t) = -\frac{1}{t} V\). We similarly have

\[
G(D_V^v D_X^v X, V) = -k(t)G\left(\frac{1}{t} V - P_v \tilde{\partial}_t, V\right) = -2tk(t),
\]

and

\[
G(D_X^v D_V^v X, V) = \partial_t(t + t) - G\left(\frac{V}{t} - P_X V, \frac{V}{t} + P_X V\right) = 2 - 1 + 1 = 2.
\]

Therefore \(k(t) = -\frac{1}{t}\).

From the above proof, we obtain that \(\text{Hor} (P_\tilde{\partial}_i \tilde{\partial}_i) = t \tilde{\partial}_t\) and \(P_\tilde{\partial}_i X = \frac{1}{t} \tilde{\partial}_i\) if \(k(t) = \frac{1}{t}\), and that \(\text{Hor} (P_\tilde{\partial}_i \tilde{\partial}_i) = -t \tilde{\partial}_t\) and \(P_\tilde{\partial}_i X = -\frac{1}{t} \tilde{\partial}_i\) if \(k(t) = -\frac{1}{t}\).

### 4.3 Example 1: Denormalization

Here we consider the denormalization (recall Sect. 2.1) as an example of warped products with affine connections. Since we can prove the isometry to a warped product in the same way as in Sect. 4.4, we omit a detailed proof and see an explicit expression of an isometry between the warped product and the denormalization.

Let \(\tilde{S}\) be the set of positive finite measures on a finite set \(X\). We define a map \(h\) as

\[
h : \mathbb{R}_{>0} \times S \rightarrow \tilde{S},
\]

\[
(t, p) \mapsto t^2 p/4.
\]

We pull back \(\tilde{g}\) on \(\tilde{S}\) in (1) by \(h\) and define the induced metric \(G\) on \(\mathbb{R}_{>0} \times S\). This \((\mathbb{R}_{>0} \times S, G)\) is a warped product and \(c(t) := t^2 p/4\) is a line of constant speed 1.

Let \(\{\xi_1, \ldots, \xi_n\}\) be a coordinate system of \(S\). We adopt \(\{\tau, \xi_1, \ldots, \xi_n\}\) as a coordinate system of \(\tilde{S}\) and denote its natural basis by \(\tilde{\partial}_i = \frac{\partial}{\partial \xi_i}\) and \(\tilde{\partial}_\tau = \frac{\partial}{\partial \tau}\). For a vector field \(X = X^i \xi_i \in \mathcal{X}(S)\), we define \(\tilde{X} := X^i \tilde{\partial}_i \in \mathcal{X}(\tilde{S})\).

We can set affine connections on the denormalization as in Sect. 2.1. In [2], it is expressed as follows. For \(X, Y \in \mathcal{X}(S)\),

\[
\nabla^{(\alpha)}_X Y = (\nabla^{(\alpha)}_X Y) - \frac{1 + \alpha}{2} (\tilde{X}, \tilde{Y}) \tilde{\partial}_\tau,
\]

\[
\nabla^{(\alpha)}_{\tilde{\partial}_\tau} \tilde{X} = \nabla^{(\alpha)}_X \tilde{\partial}_\tau = -\frac{1 + \alpha}{2} \frac{1}{\tau} \tilde{X},
\]

\[
\nabla^{(\alpha)}_{\tilde{\partial}_i} \tilde{\partial}_\tau = -\frac{1 + \alpha}{2} \frac{1}{\tau} \tilde{\partial}_i.
\]

Also, the metric \(\tilde{g}\) is expressed as

\[
\tilde{g}_{ij} = \tau g_{ij}, \quad \tilde{g}_{i\tau} = 0, \quad \tilde{g}_{\tau\tau} = \frac{1}{\tau}.
\]
We can check that this connection satisfies Assumption 2 by direct calculations, that is to say, the $\alpha$-connection on the denormalization is compatible with the warped product structure and their curvatures vanish at $\alpha = \pm 1$. Let us see that the results we obtained in Sects. 4.1 and 4.2 are also obtained in this situation. We set $\tau = t^2/4$.

Note that $\|\tilde{\partial}_\tau\| := \sqrt{\tilde{g}(\tilde{\partial}_\tau, \tilde{\partial}_\tau)} = 1/\sqrt{\tau}$. We have, by omitting the tilde for simplicity,

$$D \frac{\tilde{\partial}_\tau}{\|\tilde{\partial}_\tau\|} X = \frac{1}{2} \frac{1 - \alpha}{\tau} X = \frac{1 - \alpha}{t} X,$$

$$P_X \frac{\tilde{\partial}_\tau}{\|\tilde{\partial}_\tau\|} \frac{\tilde{\partial}_\tau}{\|\tilde{\partial}_\tau\|} = \frac{1}{2} \left( \frac{1 - \alpha}{t} - \frac{1 + \alpha}{t} \right) X = -\frac{\alpha}{t} X,$$

$$D \frac{\tilde{\partial}_\tau}{\|\tilde{\partial}_\tau\|} \frac{\tilde{\partial}_\tau}{\|\tilde{\partial}_\tau\|} = \frac{1}{2} \sqrt{\tau} \left( (\partial_\tau \sqrt{\tau}) \frac{\tilde{\partial}_\tau}{\|\tilde{\partial}_\tau\|} + \frac{1}{\|\tilde{\partial}_\tau\|} \left( -\frac{1 + \alpha}{2} \frac{\tilde{\partial}_\tau}{\|\tilde{\partial}_\tau\|} \right) \right) = -\frac{\alpha}{2} \frac{\tilde{\partial}_\tau}{\|\tilde{\partial}_\tau\|} = -\frac{\alpha}{t} \frac{\tilde{\partial}_\tau}{\|\tilde{\partial}_\tau\|},$$

where $D = \tilde{\nabla}^{(\alpha)}$. When $\alpha = \pm 1$, these equations are compatible with the connections in Sect. 4.2.

### 4.4 Example 2: BKM cone

Next, we consider $\mathbb{P}(n)$ equipped with the extended BKM metric (recall Sect. 2.2), which we call the BKM cone. We first show that $\mathbb{P}(n)$ with the extended monotone metric (not only the BKM metric) has a warped product structure.

**Proposition 7** $\mathbb{P}(n)$ equipped with the extended monotone metric is a warped product. Precisely, there exists an isometry as follows:

$$\mathbb{R}_{>0} \times t(t) = t \to (\mathbb{P}(n), g^f) \quad (t, \rho) \mapsto \frac{t^2 \rho}{4},$$

where $g^f$ is an arbitrary monotone metric.

**Proof** For simplicity, we calculate $2 \times 2$ matrices as in [4]. The following argument can be easily extended to the $n \times n$ case. For an arbitrary $\rho \in \mathcal{D}$, there exists a unitary matrix $U$ such that $U \rho U^* = \rho_0$, where $\rho_0 = \text{diag}[x, y]$ for some $x, y \in \mathbb{R}$. We set

$$X_1 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad X_4 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

These $X_1, X_2, X_3, X_4$ form an orthogonal basis of every tangent space of $(\mathbb{P}(2), g^f)$. Let us calculate the length of these vectors at $\rho_0$:

$$g^f_{\rho_0}(X_1, X_1) = \frac{1}{(2\pi)^2} \text{Tr} \int c(\xi, \eta) \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\xi-x} & 0 \\ 0 & \frac{1}{\eta-y} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\eta-x} & 0 \\ 0 & \frac{1}{\eta-y} \end{pmatrix} d\xi d\eta$$

$$= \frac{1}{(2\pi)^2} \text{Tr} \int c(\xi, \eta) \begin{pmatrix} \frac{4}{(\xi-x)(\eta-x)} & 0 \\ 0 & 0 \end{pmatrix} d\xi d\eta$$
These calculations show that, for any $k > 0$ and any tangent vectors $X$ and $Y$, we have
\[
g^f_{k\rho}(X, Y) = g^f_{k\rho_0}(UXU^*, UYU^*) = \frac{1}{k} g^f_{\rho_0}(UXU^*, UYU^*) = \frac{1}{k} g^f_{\rho}(X, Y),
\]
where we used the fact that $g^f_{UXU^*}(UXU^*, UYU^*) = g^f_{\rho}(X, Y)$. Thus, we obtain
\[
g^f_{k\rho}(kX, kX) = k g^f_{\rho}(X, X). \tag{13}
\]
We define $h$ as
\[
h : \mathbb{R}_{>0} \times D \to \mathbb{P}(2),
\]
\[
(t, \rho) \mapsto \frac{t^2 \rho}{4}.
\]
We pull back $g^f$ on $\mathbb{P}(n)$ by $h$ and define $G$ on $\mathbb{R}_{>0} \times D$. We show that $G$ is a warped product metric on $\mathbb{R}_{>0} \times D$. We consider the lines
\[
\gamma(t) := \frac{t^2 \rho}{4}, \quad \gamma_0(t) := \frac{t^2 \rho_0}{4}.
\]
Then we find
\[
G_{(t, \rho)}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = g^f_{\gamma(t)}(\gamma'(t), \gamma'(t))
\]
\[
= g^f_{U\gamma(t)U^*}(U\gamma'(t)U^*, U\gamma'(t)U^*)
\]
\[
= g^f_{\gamma_0(t)}(\gamma'_0(t), \gamma'_0(t))
\]
\[
= g^f_{\gamma_0(t)}\left(\begin{pmatrix} tx/2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} tx/2 & 0 \\ 0 & 0 \end{pmatrix}\right)
\]
\[
+ g^f_{\gamma_0(t)}\left(\begin{pmatrix} 0 & 0 \\ 0 & ty/2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & ty/2 \end{pmatrix}\right)
\]
\[
= \left(\frac{tx}{2}\right)^2 \frac{4}{t^2 x} + \left(\frac{ty}{2}\right)^2 \frac{4}{t^2 y}.
\]
Hereafter, we assume \( x + y = 1 \). We now get

\[
G(t, \rho) \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right) = 1. \tag{14}
\]

Let us calculate \( dh \). Since \( h \) is expressed by the natural coordinates of \( \mathbb{R}_{>0} \times \mathcal{D} \) and \( \mathbb{P}(2) \) as

\[
\left( t, \begin{pmatrix} x & z + iw \\ z - iw & 1 - x \end{pmatrix} \right) \mapsto \frac{t^2}{4} \begin{pmatrix} x & z + iw \\ z - iw & 1 - x \end{pmatrix} = \begin{pmatrix} a & b + ic \\ b - ic & d \end{pmatrix},
\]

we have

\[
(Jh)_\rho = \begin{pmatrix}
\frac{tx}{2} & \frac{t^2}{4} & 0 & 0 \\
\frac{tz}{2} & 0 & \frac{t^2}{4} & 0 \\
\frac{tw}{2} & 0 & 0 & \frac{t^2}{4} \\
\frac{t(1-x)}{2} & -\frac{t^2}{4} & 0 & 0
\end{pmatrix}.
\]

The pull-back metric \( G \) satisfies

\[
G\left( \left( \frac{\partial}{\partial z} \right)_{(t, \rho)}, \left( \frac{\partial}{\partial z} \right)_{(t, \rho)} \right) = g^f\left( \frac{t^2}{4} \left( \frac{\partial}{\partial b} \right)_{\mathbb{P}(2)}, \frac{t^2}{4} \left( \frac{\partial}{\partial b} \right)_{\mathbb{P}(2)} \right),
\]

where

\[
\left( \frac{\partial}{\partial b} \right)_{\mathbb{P}(2)} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

Using (13), we obtain

\[
g^f\left( \frac{t^2}{4} \left( \frac{\partial}{\partial b} \right)_{\mathbb{P}(2)}, \frac{t^2}{4} \left( \frac{\partial}{\partial b} \right)_{\mathbb{P}(2)} \right) = t^2 g^f \left( \frac{1}{4} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \frac{1}{4} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)
\]

\[
= t^2 G\left( \left( \frac{\partial}{\partial z} \right)_{(t, \rho)}, \left( \frac{\partial}{\partial z} \right)_{(t, \rho)} \right).
\]

Hence,

\[
G(t, \rho) \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right) = t^2 G(1, \rho) \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right). \tag{15}
\]

The same equation holds for \( \frac{\partial}{\partial x} \) and \( \frac{\partial}{\partial w} \). Combining this with (14), we see that \( G \) is a warped product metric with the warping function \( l(t) = t \). \( \square \)
For $n = 2$, if we take trivial coordinates of $\mathcal{D}$ such as
\[
\rho(x, y, z) = \begin{pmatrix} x & y + iz \\ y - iz & 1 - x \end{pmatrix},
\]
the mixture connection is an affine connection, for which $\{x, y, z\}$ is affine coordinates. For example, we have the following calculation for the mixture connection $\nabla^{(m)}$. Set $X = \frac{\partial}{\partial x}, Y = \phi \frac{\partial}{\partial y}$, for an arbitrary function $\phi$ on $\mathcal{D}$. Then we have
\[
(\nabla^{(m)}_X Y)_\rho = \left\{ \nabla^{(m)}_X \left( \phi \frac{\partial}{\partial y} \right) \right\}_\rho = \left( \frac{\partial \phi}{\partial x} \frac{\partial}{\partial y} \right)_\rho = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)
\]
\[
= \frac{\partial}{\partial x} \left( \phi \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = X(Y\rho).
\]
Similarly, if we take the coordinates of $\mathbb{P}(2)$ such as
\[
\tilde{\rho} = \begin{pmatrix} \alpha & \beta + i\gamma \\ \beta - i\gamma & \zeta \end{pmatrix},
\]
then the connection $D$ whose affine coordinate system is $\{\alpha, \beta, \gamma, \zeta\}$ satisfies $(DXY)_{\tilde{\rho}} = X(Y\tilde{\rho})$.

For another coordinates such as
\[
\tilde{\rho} = \tau \begin{pmatrix} x & y + iz \\ y - iz & 1 - x \end{pmatrix},
\]
we have $D_{\partial_x} \partial_x = 0$ and $D_{\partial_x} \partial_x = D_{\partial_y} \partial_y = D_{\partial_z} \partial_z = 0$. Combining this with $\tilde{\nabla}_{\partial_x} \partial_x = \tilde{\nabla}_{\partial_y} \partial_y = \tilde{\nabla}_{\partial_z} \partial_z = 0$, we see that the connection $D$ defined above satisfies Assumption 2.

**Remark 4** In [9], quantum $\alpha$-connections on the set of positive definite matrices and their dually flatness are studied. Also for the quantum $\alpha$-connections, we can check the same compatibility between connections and the warped product structure as that of the classical denormalization, which we studied in Sect. 4.3.

### 5 Connections on the Takano Gaussian space

In this section, we consider the Takano Gaussian space (recall Sect. 2.3) and show an analogue to Theorem 4.1.

Let $(B, g_B), (F, g_F)$ be Riemannian manifolds and $\nabla^F$ denotes the Levi-Civita connection of $F$. We furnish $M := B \times F$ with a metric $G$ such that
\[
G := b^2 g_B + f^2 g_F.
\]
where $f, b$ are positive functions on $B$. This is the same situation as in the Takano Gaussian space. Denote by $\nabla$ the Levi-Civita connection on $(M, G)$, and by $\mathcal{L}(F)$, $\mathcal{L}(B)$ the sets of lifts of tangent vector fields of $F$ to $M$, $B$ to $M$, respectively. Simple calculations show that $\nabla_X Y$ is horizontal for any $X, Y \in \mathcal{L}(B)$ and $\text{Ver} \nabla V W = \text{Lift}(\nabla^F V W)$ for any $V, W \in \mathcal{L}(F)$.

From now on, we set $M := L^{n+1}$, $F := \{(m_1, \ldots, m_n) | m_i \in \mathbb{R}\}$ and $B := \{\sigma \in \mathbb{R}_{>0}\}$. Let $G$ be the Fisher metric on $M$. Let $D$ be an arbitrary affine connection on $M$. We define the affine connection $\tilde{\nabla}^F$ on the fiber space $F$ by the natural projection of $D$.

We fix our framework.

Assumption 3 We assume that $D$ satisfies

- $D_X Y$ is horizontal, i.e., $D_X Y \in \mathcal{L}(B)$ for any $X, Y \in \mathcal{L}(B)$,
- $\text{Ver}(D_V W) = \text{Lift}(\tilde{\nabla}^F V W)$ for any $V, W \in \mathcal{L}(F)$.

Let $R$ be the curvature with respect to $D$, $F R$ be the curvature with respect to $\tilde{\nabla}^F$, and $R^*$ and $F R^*$ be their duals. We also assume $R = F R = 0$.

Remark 5 If we take an $\alpha$-connection of the Takano Gaussian space as $D$, we see that $(F, \tilde{\nabla}^F)$ is dually flat from the expression of the Christoffel symbols of the Takano Gaussian space in Sect. 2.3.

In the following arguments in this section, we assume this assumption without mentioning. As we saw in Sect. 4.1, the following equations hold:

\begin{align}
G(V, V)G(\text{grad} f, P_U U) &= G(U, U)G(\text{grad} f, P_V V), \\
\frac{\|\text{grad} f\|^2}{f^2} \left\{ G(V, V)G(U, U) - G(U, V)^2 \right\} \\
&- G(\text{Hor} (P_V V), \text{Hor} (P_U U)) + G(\text{Hor} (P_U U), \text{Hor} (P_V U)) = 0.
\end{align}

where $f(\sigma) = \sqrt{2n/\sigma}$. In the following arguments, we set $X = \tilde{\partial}_\sigma, U = \tilde{\partial}_i, V = \tilde{\partial}_j (i \neq j)$, where $\tilde{\partial}_\sigma = \text{Lift} \left( \frac{\partial}{\partial \sigma} \right)$ and $\tilde{\partial}_i = \text{Lift} \left( \frac{\partial}{\partial m_i} \right)$. To characterize the connections on the line, we prepare some lemmas.

Lemma 6 We have

\[
(\text{Hor} P_U U)_{(x, \sigma)} = \frac{1}{2n\sigma} \tilde{\partial}_\sigma \quad \text{or} \quad \left( -\frac{1}{2n\sigma} \tilde{\partial}_\sigma \right).
\]

Proof In the same way as Lemma 3, we have

\[
\text{Hor} P_U U = \text{Hor} P_V V.
\]

Applying (16) to $U + V$ and $V$, we obtain

\[
G(U + V, U + V)G(\text{grad} f, P_V V) = G(V, V)G(\text{grad} f, P_{(U+V)}(U + V)).
\]
Substituting $G(U + V, U + V) = 2/\sigma^2$ and $G(V, V) = 1/\sigma^2$ to the equation above, we get

$$2\text{Hor } P_V V = \text{Hor } P_{(U+V)}(U + V).$$

Hence,

$$\text{Hor } (P_U U + P_V V + 2P_U V) = 2\text{Hor } P_V V.$$

Together with (18), we get

$$\text{(Hor } P_U V))_{(x, \sigma)} = 0.$$

Combining this with (17) and

$$\frac{\| \text{grad } f \|^2}{f^2} = \frac{G(\text{grad } f, \text{grad } f)}{f^2} = \frac{G\left(G_{\sigma\sigma} \partial_{\sigma}, \left(\frac{1}{\sigma}\right) \partial_{\sigma}\right)}{\left(\frac{1}{\sigma^2}\right)} = \left(\frac{\sigma}{2n}\right)^2 G\left(\partial_{\sigma}, \partial_{\sigma}\right) = \frac{1}{2n},$$

we have, for all $t > 0$,

$$\frac{1}{2n} \frac{1}{\sigma^2} \frac{1}{\sigma^2} = G_{(x, \sigma)}(\text{Hor } P_V V, \text{Hor } P_U U) = G_{(x, \sigma)}(\text{Hor } P_U U, \text{Hor } P_U U),$$

which proves the claim. $\square$

We define $k, l$ by $D_X X = k(\sigma) \partial_{\tilde{\sigma}}$ and $D_X X = l(\sigma) \partial_{\tilde{\sigma}}$. Combining $G(\partial_{\sigma}, \partial_{\tilde{\sigma}}) = \frac{2n}{\sigma^2}$ with $\partial_{\sigma} G(\partial_{\tilde{\sigma}}, \partial_{\tilde{\sigma}}) = G(D_{\partial_{\sigma}} \partial_{\tilde{\sigma}}, \partial_{\tilde{\sigma}}) + G(\partial_{\sigma}, D_{\partial_{\tilde{\sigma}}} \partial_{\tilde{\sigma}})$, we obtain

$$-\frac{4n}{\sigma^3} = \frac{2n}{\sigma^2} (k(\sigma) + l(\sigma)).$$

Hence,

$$-\frac{2}{\sigma} = k(\sigma) + l(\sigma)$$

holds.

**Theorem 2** Under Assumption 3, the connection on the line is

$$D_{\partial_{\sigma}} \partial_{\tilde{\sigma}} = \frac{1}{\sigma} \partial_{\tilde{\sigma}} \quad \text{or} \quad D_{\partial_{\sigma}} \partial_{\tilde{\sigma}} = -\frac{3}{\sigma} \partial_{\tilde{\sigma}}.$$

**Proof** We only check the case of $\text{Hor } P_V V = \frac{1}{2n\sigma} \partial_{\tilde{\sigma}}$, because the other case of $\text{Hor } P_V V = -\frac{1}{2n\sigma} \partial_{\tilde{\sigma}}$ follows from the completely same argument. According to the O’Neill formula (7) for affine connections, we have

$$D_V X = D_X V = \frac{\partial_{\sigma} f}{f} V + P_V X = -\frac{1}{\sigma} V + P_X V.$$
Using this, we calculate $G(R^*(V, X)X, V)$. We have

\[
G(D_Y^* D_X^* X, V) = G \left( l(\sigma) \left\{ -\frac{1}{\sigma} V - P_X V \right\}, V \right)
\]

\[
= l(\sigma) \left\{ -\frac{1}{\sigma} G(V, V) - G(X, P_V V) \right\}
\]

\[
= l(\sigma) \left( -\frac{1}{\sigma^2} G_1, \frac{1}{2n\sigma^2} \partial_\sigma \right)
\]

\[
= l(\sigma) \left( -\frac{1}{\sigma^3} \right),
\]

and

\[
G(D_X^* D_Y^* X, V) = XG(D_Y^* X, V) - G(D_Y^* X, D_X^* V)
\]

\[
= \partial_\sigma G \left( \frac{1}{\sigma} V - P_X V, V \right) - G \left( \frac{V}{\sigma}, \frac{V}{\sigma} \right) + G(P_X V, P_X V)
\]

\[
= \partial_\sigma \left( \frac{1}{\sigma} G(V, V) - G(X, P_V V) \right) - \frac{1}{\sigma^2} G(V, V) + G(P_X V, P_X V)
\]

\[
= \frac{5}{\sigma^4} + G(P_X V, P_X V).
\]

Since $R^* = 0$, we obtain

\[
l(\sigma) \left( -\frac{2}{\sigma^3} \right) = \frac{5}{\sigma^4} + G(P_X V, P_X V). \tag{20}
\]

Next, let us calculate $G(R(V, X)X, V)$. We have

\[
G(D_Y D_X X, V) = G \left( k(\sigma)(D_Y X), V \right)
\]

\[
= k(\sigma) G \left( \frac{1}{\sigma} V + P_X V, V \right)
\]

\[
= k(\sigma) \left( -\frac{1}{\sigma^3} + G(P_X V, V) \right)
\]

\[
= k(\sigma) \left( -\frac{1}{\sigma^3} + \frac{2n}{2n\sigma^2} \right) = 0,
\]

and

\[
G(D_X D_Y X, V) = XG(D_Y X, V) - G(D_Y X, D_X^* V)
\]

\[
= \partial_\sigma G \left( \frac{1}{\sigma} V + P_X V, V \right) - G \left( \frac{V}{\sigma}, \frac{V}{\sigma} \right) + G(P_X V, P_X V)
\]
\[ \begin{aligned}
\frac{\partial}{\partial \sigma} \left( -\frac{1}{\sigma^2} + G(X, PV) \right) - \frac{1}{\sigma^2} G(P_X V, P_X V) \\
\frac{\partial}{\partial \sigma} \left( -\frac{1}{\sigma^3} + \frac{1}{\sigma^3} \right) - \frac{1}{\sigma^4} G(P_X V, P_X V) = -\frac{1}{\sigma^4} + G(P_X V, P_X V). 
\end{aligned} \]

Since \( R = 0 \), we have
\[ 0 = -\frac{1}{\sigma^4} + G(P_X V, P_X V). \]

Combining this with (20) and (19), we obtain
\[ l(\sigma) = -\frac{3}{\sigma}, \quad k(\sigma) = \frac{1}{\sigma}. \]

The other case is shown in the same way. \( \square \)

Note that these connections coincide with the \( \alpha \)-connections at \( \alpha = \pm 1 \) in the Takano Gaussian space (recall Sect. 2.3).

6 Discussion: Wasserstein Gaussian space

By Wasserstein Gaussian space, we mean the set of multivariate Gaussian distributions on \( \mathbb{R}^n \) with mean zero equipped with the \( L^2 \)-Wasserstein metric. When we started investigating warped products in information geometry, we thought that we would be able to find dually flat connections on the Wasserstein Gaussian space and calculate its canonical divergence. The scenario we thought was the following. In [16], it is proved that the Wasserstein Gaussian space has a cone structure. Recently in [5], it is proved that we can find dually flat connections on the space of density matrices equipped with the monotone metric. We can apply this result because the SLD metric and the Wasserstein metric on Gaussian distributions are essentially the same on \( D \). [12]. We thought that once we study dually flat affine connections on warped products, we would be able to extend the dually flat connections on the fiber space to the warped product in a natural way. However, it turned out that it is difficult to draw dual affine coordinates of the dually flat connections on warped products we made. Thus, we do not know how to calculate the canonical divergence. Let us explain the difficulty in this section.

In the previous sections, we discussed necessary conditions for warped products and fiber spaces to be dually flat. First, we show that it is also a sufficient condition for the Wasserstein Gaussian space. The question is, when we extend connections on the fiber space to the warped product, whether the warped product with those connections becomes dually flat or not.

According to the arguments in Sect. 4, we now define the connection \( D \) on a cone \( M = \mathbb{R}_{>0} \times_f F \), where \( f(t) = t \) and \( (F, g_F) \) is a Riemannian manifold. We consider the situation that the fiber space is equipped with a dually flat affine connection \( \tilde{\nabla} \). Let \( G := g_B + f^2 g_F \) be the warped product metric on \( M \). Denote the base space by
with a coordinate \( t \in \mathbb{R}_{>0} \) such that \( g_B(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = 1 \). Let \( X := \frac{\partial}{\partial t} \) and \( \{U_i\}_{i=1}^n \subset \mathcal{L}(F) \) be a basis of \( \mathcal{L}(F) \) such that \([U_i, U_j] = 0\) for any \( i, j \in \{1, \ldots, n\} \).

Following Theorem 1 and Lemma 5, define the connection \( D \) by

\[
D_X X = \frac{1}{t} \frac{\partial}{\partial t}, \quad D_X V = D_V X := \frac{2}{t} V, \quad \begin{cases} 
\text{Hor } D_V W := 0, \\
\text{Ver } D_V W := \text{Lift}(\tilde{\nabla}_V W), 
\end{cases}
\]

where \( V, W \) are arbitrary vectors in \( \{U_i\}_{i=1}^n \).

**Proposition 8** \((M, D, D^*)\) is a dually flat space.

**Proof** We only have to check that the curvature vanishes with respect to \( D \). Let \( U, V, W, Q \) be arbitrary vectors in \( \{U_i\}_{i=1}^n \). Note that we have

\[
\text{Hor} D_U V = 0, \quad (21)
\]

and

\[
D_X^* U = D_U^* X = \frac{1}{t} U - \frac{1}{t} U = 0. \quad (22)
\]

We first check that \( R(U, V) W \) vanishes. From (21) and (22), we have

\[
G(R(U, V) W, X) = G(D_U D_V W, X) - G(D_V D_U W, X) = \{ U G(D_V W, X) - G(D_V W, D_U X) \} \\
- \{ V G(D_U W, X) - G(D_U W, D_V X) \} = U G(\text{Hor} D_V W, X) - V G(\text{Hor} D_U W, X) = 0.
\]

Recall from Sect. 4.1 and (8) that

\[
G(II(V, W), II^*(U, Q)) = \frac{\|\text{grad } f\|^2}{f^2} G(V, W) G(U, Q) - G(\text{Hor} P_V W, \text{Hor} P_U Q) \\
+ \frac{G(V, W) G(\text{grad } f, P_U Q) - G(U, Q) G(\text{grad } f, P_V W)}{f} \\
= \frac{1}{t^2} G(V, W) G(U, Q) - G \left( \frac{G(V, W)}{t} \frac{\partial}{\partial t}, G(U, Q) \frac{\partial}{\partial t} \right) \\
+ \frac{1}{t} G(V, W) G \left( \frac{\partial}{\partial t}, \frac{G(U, Q)}{t} \frac{\partial}{\partial t} \right) \\
- \frac{1}{t} G(U, Q) G \left( \frac{\partial}{\partial t}, \frac{G(V, W)}{t} \frac{\partial}{\partial t} \right) = 0,
\]

thus we have \( G(R(U, V) W, Q) = G^F R(U, V) W, Q = 0 \) by (9). Hence, we have \( R(U, V) W = 0 \).

Next, we check that \( R(X, U) V \) vanishes. From (21) and (22), we have

\[
G(R(X, U) V, X) = G(D_X D_U V - D_U D_X V, X)
\]
\[
\begin{align*}
\{XG(D_U V, X) - G(D_U V, D_X^* X)\} - UG(D_X V, X) & \quad \text{=} -UG\left(\frac{2}{t} V, X\right) = 0.
\end{align*}
\]

We also have
\[
\begin{align*}
G(R(X, U)V, Q) = & \{XG(D_U V, Q) - G(D_U V, D_X^* Q)\} \\
& -\{UG(D_X V, Q) - G(D_X V, D_U^* Q)\} \\
& = 2t\{g_F(\tilde{\nabla} U V, Q) - Ug_F(V, Q) + g_F(V, \tilde{\nabla}_U^* Q)\} \\
& = 0.
\end{align*}
\]

Hence, \(R(X, U)V = 0\). In a similar way, we can check \(R(U, X)X = 0\).

\[\square\]

**Remark 6** We denote the Wasserstein Gaussian space over \(\mathbb{R}^n\) by the \(n \times n\) Wasserstein Gaussian space since its elements are represented by \(n \times n\) covariance matrices. For the \(2 \times 2\) Wasserstein Gaussian space, we remark that the existence of a dually flat affine connection \(\tilde{\nabla}\) is guaranteed by \([5]\).

Thus the above proposition implies that we can furnish the \(2 \times 2\) Wasserstein Gaussian space with dually flat affine connections. Though we think it necessary to draw dual affine coordinates to calculate the canonical divergence, it turned out to be difficult. This is because, for example, \(D_X U\) does not vanish, which means that the trivial extension of affine coordinates on the fiber space does not give affine coordinates of the warped product. Here, trivial extension means \(\{t, \xi_1, \ldots, \xi_n\}\) for affine coordinates \(\{\xi_1, \ldots, \xi_n\}\) of the fiber space and the coordinate \(\{t\}\) of the line.

**Remark 7** In \([13]\), it is claimed that there is no dually flat proper doubly warped Finsler manifolds. Let us restrict their argument to Riemannian manifolds. For two manifolds \(M_1\) and \(M_2\) and their doubly warped product \((M_1 \times M_2, G)\), let \((x_i)\) and \((u_\alpha)\) be coordinates of \(M_1\) and \(M_2\), respectively. Then their claim asserts that the coordinates \(((x_i), (u_\alpha))\) on \((M_1 \times M_2, G)\) cannot be affine coordinates for any dually flat connections on \(M_1 \times M_2\) unless \(G\) is the product metric.

For further understanding the relation between affine coordinates and their connections, let us observe the \(2 \times 2\) BKM cone (recall Sect. 4.4). Let \(\tilde{\nabla}\) be an affine connection whose affine coordinate is \(\{a, b, c, d\}\), with which \(2 \times 2\) matrices are expressed as
\[
\begin{pmatrix}
a & c + ib \\
c - ib & d
\end{pmatrix}.
\]

This \(\tilde{\nabla}\) is a dually flat affine connection on the BKM cone. Let \(\tilde{D}\) be an affine connection whose affine coordinate is \(\{t, \alpha, \beta, \gamma\}\), with which \(2 \times 2\) matrices are expressed as
\[
t\begin{pmatrix}
\alpha & \beta + iy \\
\beta - iy & 1 - \alpha
\end{pmatrix}.
\]
Relations of these coordinates are

\[ t = a + d, \quad \alpha = \frac{a}{a + d}, \quad 1 - \alpha = \frac{d}{a + d}, \]

\[ \frac{\partial}{\partial \alpha} = \begin{pmatrix} t \\ 0 \end{pmatrix}, \quad \frac{\partial}{\partial \alpha} = \begin{pmatrix} 0 \\ -t \end{pmatrix}, \quad (a + d) \left( \frac{\partial}{\partial \alpha} - \frac{\partial}{\partial \alpha} \right), \]

\[ \frac{\partial}{\partial \alpha} = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}, \quad \frac{\partial}{\partial \alpha} = \begin{pmatrix} 0 \\ 1 - \alpha \end{pmatrix}, \quad \frac{\partial}{\partial \alpha} + \frac{d}{a + d} \frac{\partial}{\partial d}. \]

Using these relations, we calculate

\[ \tilde{\nabla} \frac{\partial}{\partial \alpha} \frac{\partial}{\partial \alpha} = \tilde{\nabla} (a + d) \left( \frac{a}{a + d} \frac{\partial}{\partial \alpha} + \frac{d}{a + d} \frac{\partial}{\partial \alpha} \right) \]

\[ = (a + d) \left( \frac{\partial}{\partial \alpha} - \frac{\partial}{\partial \alpha} \right) \left( \frac{a}{a + d} \frac{\partial}{\partial \alpha} + (a + d) \left( \frac{\partial}{\partial \alpha} - \frac{\partial}{\partial \alpha} \right) \left( \frac{d}{a + d} \right) \frac{\partial}{\partial \alpha} \right) \]

\[ = (a + d) \left( \frac{d}{(a + d)^2} + \frac{a}{(a + d)^2} \right) \frac{\partial}{\partial \alpha} + (a + d) \left( - \frac{d}{(a + d)^2} - \frac{a}{(a + d)^2} \right) \frac{\partial}{\partial \alpha} \]

\[ = \frac{\partial}{\partial \alpha} - \frac{\partial}{\partial \alpha} \neq 0. \]

On the other hand

\[ \tilde{D} \frac{\partial}{\partial \alpha} \frac{\partial}{\partial \alpha} = 0. \]

Hence, \( \tilde{\nabla} \) and \( \tilde{D} \) are different.

**Appendix**

In this appendix, we observe that elliptic distributions have structures of warped products with warping functions different from those in Sect. 4.

As described in [10], a 1-dimensional random variable \( X \) is said to have an elliptic distribution \( EL_{1}^{h}(\mu, \sigma^2) \) with parameters \( \mu > 0 \) and \( \sigma > 0 \), if its density is expressed as

\[ p_{h}(x|\mu, \sigma) = \frac{1}{\sigma} h \left( \frac{(x - \mu)^2}{\sigma^2} \right) \]

for some function \( h \).

Let \( Z \) be an random variable which has \( EL_{1}^{h}(0, 1) \) distribution and \( W = \{d \log h(Z^2)/d(Z^2)\} \). We set

\[ a = E(Z^2W^2), \quad b = E(Z^4W^2). \]
The Fisher metric $G_F$ of elliptic distributions is expressed in [10, Section 3] as

$$G_F = \frac{4ad\mu^2 + (4b-1)d\sigma^2}{\sigma^2}.$$  \hspace{1cm} (23)

Let $M$ be a parameter space of $EL^h_1(\mu, \sigma^2)$, i.e., $M = \{\mu, \sigma\}$. Setting the parameter

$$t := \sqrt{4b-1} \log \sigma,$$

we have

$$G_F (\tilde{\delta}_t, \tilde{\delta}_t) = G_F \left( \frac{\partial \sigma}{\partial t} \tilde{\delta}_\sigma, \frac{\partial \sigma}{\partial t} \tilde{\delta}_\sigma \right) = \frac{\sigma^2}{4b-1} G_F \left( \frac{\partial}{\partial \sigma}, \frac{\partial}{\partial \sigma} \right) = 1,$$

where $\tilde{\delta}_t = \text{Lift} \left( \frac{\partial}{\partial t} \right)$ and $\tilde{\delta}_\sigma = \text{Lift} \left( \frac{\partial}{\partial \sigma} \right)$. Therefore, $G_F$ is expressed as

$$G_F = dt^2 + f_F(t)^2 d\mu^2,$$

where

$$f_F(t) := \sqrt{4a} \exp \left( - \frac{t}{\sqrt{4b-1}} \right).$$

Since $(M, G_F)$ is two-dimensional, the arguments in Sect. 4.1 do not work and it is difficult to characterize affine connections on $(M, G_F)$ as we demonstrated in previous sections.

**Remark 8** We recall that the Fisher metric $G_T$ of Takano Gaussian space is

$$G_T = \frac{dm_1^2 + \cdots + dm_n^2 + 2n d\sigma^2}{\sigma^2}.$$

In Sect. 5, we investigated Takano Gaussian space with functions $f$ and $b$. We can also express $G_T$ as a warped metric using a single warping function. Setting the parameter

$$t = \sqrt{2n} \log \sigma,$$

we have

$$G_T (\tilde{\delta}_t, \tilde{\delta}_t) = G_T \left( \frac{\partial \sigma}{\partial t} \tilde{\delta}_\sigma, \frac{\partial \sigma}{\partial t} \tilde{\delta}_\sigma \right) = \frac{\sigma^2}{2n} 2n \sigma^2 = 1,$$

where $\tilde{\delta}_t = \text{Lift} \left( \frac{\partial}{\partial t} \right)$ and $\tilde{\delta}_\sigma = \text{Lift} \left( \frac{\partial}{\partial \sigma} \right)$. Therefore, $G_T$ is expressed as

$$G_T = dt^2 + f_T(t)^2 (dm_1^2 + \cdots + dm_n^2),$$

where

$$f_T(t) = \exp \left( - \frac{t}{\sqrt{2n}} \right).$$
**Remark 9** We remark that the same kind of metric is considered in [6], where they consider the set

\[ G = \left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \mid x, y \in \mathbb{R}, \ y > 0 \right\} \]

and the metric

\[ g = \frac{dx^2 + \lambda^2 dy^2}{y^2} \]

with a parameter \( \lambda > 0 \). They define the connections \( \nabla^{(\alpha)} \) on \( G \) as

\[ \nabla^{(\alpha)}_{\partial_x} \partial_x = \frac{1 - \alpha}{\lambda^2 y} \partial_y, \quad \nabla^{(\alpha)}_{\partial_x} \partial_y = \frac{1 + \alpha}{y} \partial_x, \quad \nabla^{(\alpha)}_{\partial_y} \partial_x = -\frac{1 + 2\alpha}{y} \partial_y \]

in [6, Section 4]. By direct calculations, we see that \( (G, g, \nabla^{(\alpha)}) \) are dually flat when \( \alpha = \pm 1 \).

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**Conflict of interest** On behalf of all authors, the corresponding author states that there is no conflict of interest.

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