Gevrey stability of hydrostatic approximate for the Navier–Stokes equations in a thin domain

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Abstract

In this paper, we justify the limit from the Navier–Stokes system in a thin domain to the hydrostatic Navier–Stokes/Prandtl system for the convex initial data with Gevrey 9/8 regularity in $x$.

Keywords: Navier–Stokes equations, hydrostatic Prandtl equation, Gevrey stability, a thin domain

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1. Introduction

In this paper, we consider 2D incompressible Navier–Stokes equations in a thin domain when the depth of the domain and the viscosity coefficient converge to zero simultaneously in a related way:

\[
\begin{aligned}
\frac{\partial U}{\partial t} + U \cdot \nabla U - \varepsilon^2 (\partial_x^2 + \eta \partial_y^2) U + \nabla P &= 0 \quad \text{in } S^\varepsilon \times (0, \infty), \\
\text{div } U &= 0, \\
U|_{y=0} &= U|_{y=\varepsilon} = 0,
\end{aligned}
\]

where $S^\varepsilon = \{(x, y) \in \mathbb{T} \times \mathbb{R} : 0 < y < \varepsilon\}$, and $U(t, x, y), P(t, x, y)$ stand for the velocity and pressure function respectively and $\eta$ is a positive constant independent of $\varepsilon$. The system is
prescribed with the initial data of the form
\[ U|_{t=0} = \left( u_0 \left( x, \frac{y}{\varepsilon} \right), \varepsilon v_0 \left( x, \frac{y}{\varepsilon} \right) \right) = U_0^\varepsilon \quad \text{in } S^\varepsilon. \]

We rescale \((U, P)\) as follows
\[ U(t, x, y) = \left( u^\varepsilon \left( t, x, \frac{y}{\varepsilon} \right), \varepsilon v^\varepsilon \left( t, x, \frac{y}{\varepsilon} \right) \right) \quad \text{and} \quad P(t, x, y) = p^\varepsilon \left( t, x, \frac{y}{\varepsilon} \right). \]

Then the system (1.1) is reduced to the following scaled anisotropic Navier–Stokes system:
\[
\begin{aligned}
\partial_t u^\varepsilon + u^\varepsilon \partial_x u^\varepsilon + v^\varepsilon \partial_y u^\varepsilon - \varepsilon^2 \partial_x^2 u^\varepsilon - \eta \partial_y^2 u^\varepsilon + \partial_x p^\varepsilon &= 0 \quad \text{in } S \times (0, \infty), \\
\varepsilon^2 (\partial_t v^\varepsilon + u^\varepsilon \partial_x v^\varepsilon + v^\varepsilon \partial_y v^\varepsilon - \varepsilon^2 \partial_x^2 v^\varepsilon - \eta \partial_y^2 v^\varepsilon) + \partial_y p^\varepsilon &= 0 \quad \text{in } S \times (0, \infty), \\
\partial_x u^\varepsilon + \partial_y v^\varepsilon &= 0 \quad \text{in } S \times (0, \infty), \\
(u^\varepsilon, v^\varepsilon)|_{y=0,1} &= 0, \\
(u^\varepsilon, v^\varepsilon)|_{t=0} &= (u_0, v_0) \quad \text{in } S,
\end{aligned}
\]

where \(S = \{(x, y) \in \mathbb{T} \times (0,1)\}. \) This is a classical model in geophysical fluid, where the vertical dimension of the domain is very small compared with the horizontal dimension of the domain. For simplicity, we take \(\eta = 1\) in the sequel and denote \(\Delta = \varepsilon^2 \partial_x^2 + \partial_y^2\).

Formally, taking \(\varepsilon \to 0\) in (1.2), we derive the hydrostatic Navier–Stokes/Prandtl system (see \([8, 14]\)):
\[
\begin{aligned}
\partial_t u^p + u^p \partial_x u^p + v^p \partial_y u^p - \partial_x^2 u^p + \partial_x p^p &= 0 \quad \text{in } S \times (0, \infty), \\
\partial_t p^p &= 0 \quad \text{in } S \times (0, \infty), \\
\partial_x u^p + \partial_y v^p &= 0 \quad \text{in } S \times (0, \infty), \\
(u^p, v^p)|_{y=0,1} &= 0, \\
(u^p)|_{t=0} &= u_0 \quad \text{in } S.
\end{aligned}
\]

The goal of this paper is to justify the limit from the scaled anisotropic Navier–Stokes system (1.2) to the hydrostatic Navier–Stokes system (1.3). The first step is to deal with the well-posedness of the system (1.3). Similar to the classical Prandtl equation, nonlinear term \(v^p \partial_x u^p\) will lead to one derivative loss in \(x\) via the direct energy estimate to (1.3). Indeed, the system (1.3) may not be well-posed for general data in the Sobolev space \([14]\). However, the system is well-posed for analytic data \([13]\). A classical result for the Prandtl equation is the well-posedness in the Sobolev space for monotonic data in \(y\) direction \([1, 11, 12]\). This kind of data is forbidden for the system (1.3) due to the boundary condition. Recently, Gerard-Varet et al \([6]\) proved the well-posedness of the system (1.3) for class of convex data in the Gevrey class \(\mathcal{G}^4\).

A natural question is whether the limit could be justified in the Gevrey class \(\mathcal{G}^4\). In a recent work \([13]\), Paicu, Zhang and the third author justified the global (in time) limit for small analytic data. For the data in Gevrey class or Sobolev space, this question is highly nontrivial. In fact, although the Prandtl equation is well-posed in the Sobolev space or Gevrey class (see \([3–5, 9]\)), the question of the inviscid limit of the Navier–Stokes equations in the same spaces remains a challenging problem.
Motivated by the methods introduced in [6, 15], we justify the limit from the system (1.2) to (1.3) for a class of convex data in the Gevrey class $\mathcal{G}$. More precisely, we consider the initial data of the form

$$
u^\epsilon(0, x, y) = u_0(x, y), \quad v^\epsilon(0, x, y) = v_0(x, y),$$

(1.4)

which satisfy the compatibility conditions

$$\partial_x u_0 + \partial_y v_0 = 0, \quad u_0(x, 0) = u_0(x, 1) = v_0(x, 0) = v_0(x, 1) = 0,$$

(1.5)

$$\int_0^1 \partial_x u_0 \, dy = 0, \quad \partial^2_x u_0|_{y=0,1} = \int_0^1 (-\partial_y u_0^2 + \partial_y^2 u_0) \, dy - \int_S \partial^2_x u_0,$$

(1.6)

and the convex condition

$$\inf_S \partial^2_x u_0 = 2\delta_0 > 0.$$  

(1.7)

We further assume that initial data falls into the Gevrey class with the bound

$$\|\partial_y u_0\|_{\mathcal{G}^{N_0-\sigma}} + \|\partial^2_y u_0\|_{\mathcal{G}^{N_0-5\sigma}} = M < +\infty.$$  

(1.8)

Here the Gevrey class normal $\|\cdot\|_{\mathcal{G}^{\sigma}}$ is defined by

$$\|f\|_{\mathcal{G}^{\sigma}} = \|e^{\tau(D_x)^\sigma} f\|_{H^{\beta}},$$

with $\|f\|_{H^{\beta}} = \|f\|_{H^{\beta}(0,1)}$. For simplicity, we drop subscript $\sigma, \tau$ in the notations $\|f\|_{\mathcal{G}^{\sigma}}$ and denote $\|f\|_{\mathcal{G}^{\sigma}}$ for short. Readers can refer to the precise definition in the section 2.

For the data satisfying (1.5)–(1.8), following the proof in [6], one can prove the following local well-posedness result for the system (1.3).

**Theorem 1.1.** Let the initial data $u_0$ satisfy (1.5)–(1.8) with $\sigma \in [\frac{8}{9}, 1]$, $\tau_0 > 0$ and $N_0 \geq 10$. Then there exist $T > 0$ and a unique solution $u^\epsilon$ of (1.3), which satisfies

$$\sup_{t \in [0, T]} \left( \|\partial_y u^\epsilon(t)\|_{\mathcal{G}^{N_0-1}} + \|\partial^2_y u^\epsilon(t)\|_{\mathcal{G}^{N_0-4}} \right) < +\infty,$$

$$\sup_{t \in [0, T]} \partial^2_x u^\epsilon > \delta_0.$$

Now we state the main result of this paper.

**Theorem 1.2.** Let initial data $u_0$ satisfies (1.5)–(1.8) with $\sigma \in [\frac{8}{9}, 1]$, $\tau_0 > 0$ and $N_0 \geq 10$. Then there exists a unique solution of the Navier–Stokes equation (1.2) in $[0, T]$, which satisfies

$$\|(u^\epsilon - u^p, \nu^\epsilon - \nu^p)\|_{L_2(t)^\infty L_\infty} \leq C \epsilon^2,$$

where $(u^p, \nu^p)$ is given by theorem 1.1 and $C$ is a constant independent of $\epsilon$.

**Remark 1.3.** The range $\sigma \in [\frac{8}{9}, 1]$ should not be optimal. According to [7], the optimal range may be $[\frac{7}{8}, 1]$.

Let us sketch main ingredients of our proof and structure of this paper.
• **Introduce the error equation.** In section 3, we introduce the error

\[ u^R = u^r - u^p, \quad v^R = v^r - v^p, \quad p^R = p^r - p^p, \]

which satisfy

\[
\begin{cases}
\partial_t u^R - \Delta u^R + \partial_t p^R + u^r \partial_x u^R + u^p \partial_x u^p + v^r \partial_y u^R + v^p \partial_y u^p - \varepsilon^2 \partial_x^2 u^p = 0, \\
\varepsilon^2 (\partial_y v^R - \Delta v^R) + \partial_y p^R + \varepsilon^2 (\partial_y v^p - \varepsilon^2 \partial_x^2 v^p - \partial_y^2 v^p + u^r \partial_x v^r + v^r \partial_y v^r) = 0.
\end{cases}
\]

The key point is to prove that

\[
\|(u^R, \varepsilon v^R)\|_{L^2 \cap L^\infty} \leq C \varepsilon.
\]

The main difficulty comes from the term \( v^R \partial_y u^R \), since \( v^R \) is controlled via the relation \( v^R = - \int y_0 \partial_x u^R \, dy \), which will lead to one derivative loss in \( x \) variable. In [13], the authors used the analyticity to overcome this difficulty. For the data in the Gevrey class, we have to introduce new ideas.

• **Introduce the vorticity formulation.** In [6], the authors introduced the vorticity formulation of (1.3):

\[
\partial_t \omega - \partial_x^2 \omega + u \partial_x \omega + v \partial_y \omega = 0, \quad \omega = \partial_y u.
\]

If we test \( \omega \) to this equation, then the term \( v \partial_x \omega \) still lose one derivative. In [6], the first key idea is to use the so called hydrostatic trick, i.e., test the vorticity equation by \( \omega \partial_x \omega \), which makes sense under the convex assumption. Indeed, the trouble term vanishes due to

\[
\int_S v \partial_x \omega \frac{\omega}{\partial_x \omega} \, dx \, dy = \int_S v \omega \, dx \, dy = 0.
\]

However, the viscosity term \( \partial_x^2 \omega \) will give rise to new difficulty due to \( \omega \rvert_{y=0,1} \neq 0 \). The second key idea introduced in [6] is to introduce the boundary corrector \( \omega^b \) defined by

\[
\partial_t \omega^b - \partial_x^2 \omega^b = 0, \quad \partial_x \omega^b \rvert_{y=0,1} \approx - \partial_x \int_0^1 u^2 \, dy,
\]

and then use the hydrostatic trick for the equation of \( \omega^m = \omega - \omega^b \).

Motivated by [6, 10, 15], we introduce the vorticity formulation of the error equations in section 3:

\[
\begin{cases}
\partial_t \omega^R - \Delta \omega^R + f = N(\omega^R, \omega^R), \\
(\partial_t + \varepsilon|D|)\omega^R \rvert_{y=0} = \partial_x h^0 + \cdots, \\
(\partial_t - \varepsilon|D|)\omega^R \rvert_{y=1} = \partial_x h^1 + \cdots.
\end{cases}
\] (1.9)

Compared with the vorticity equation in [6], there is an extra term \( \varepsilon|D|\omega^R \rvert_{y=0,1} \) on the boundary. If we test \( \omega^R \) to the first equation in (1.9), we have
Introduce the energy functional
\[ E(t) = E_{\omega^R}(t) + AE_{\omega^L}(t) + E_{\omega^H}(t), \]
where \( A \) is a large constant and
\[
E_{\omega^R}(t) = \sup_{s \in [0, t]} \| \omega^{in}(s) \|_{X^0}^2 + \int_0^t \left\| (\partial_x, \varepsilon \partial_x) \omega^{in} \right\|_{X^0}^2 \, ds + \beta \int_0^t \| \omega^{in} \|_{X^{r+\frac{2}{3}}}^2 \, ds,
\]
and
\[
E_{\omega^L}^1(t) = \sup_{s \in [0, t]} \varepsilon^2 \left\| P_{\geq N(c)}(u^R, \varepsilon v^R)(s) \right\|_{X^{r+1}}^2 + \int_0^t \varepsilon^2 \left\| P_{\geq N(c)}(\partial_x, \varepsilon \partial_x)(u^R, \varepsilon v^R) \right\|_{X^{r+1}}^2 \, ds
+ \beta \int_0^t \varepsilon^2 \left\| P_{\geq N(c)}(u^R, \varepsilon v^R) \right\|_{X^{r+1+\frac{2}{3}}}^2 \, ds,
\]
and
\[
E_{\omega^H}^2(t) = \sup_{s \in [0, t]} \left\| P_{\geq N(c)}(u^R, \varepsilon v^R)(s) \right\|_{X^{r+1-\sigma}}^2
+ \int_0^t \left\| P_{\geq N(c)}(\partial_x, \varepsilon \partial_x)(u^R, \varepsilon v^R) \right\|_{X^{r+1-\sigma}}^2 \, ds + \beta \int_0^t \left\| P_{\geq N(c)}(u^R, \varepsilon v^R) \right\|_{X^{r+1-\frac{2}{3}}}^2 \, ds.
\]

In section 6, using the hydrostatic trick, we can derive
\[
E_{\omega^R}(t) \leq C t \varepsilon^4 + 2 \delta \int_0^t \left\| \mathcal{N} \right\|_{X^{r+\frac{2}{3}}}^2 \, ds + C \epsilon^2 \int_0^t \left\| P_{\geq N(c)}(\partial_x, \varepsilon \partial_x)(u^R, \varepsilon v^R) \right\|_{X^{r+1}}^2 \, ds
+ C \int_0^t \left\| P_{\geq N(c)}(\partial_x, \varepsilon \partial_x)(u^R, \varepsilon v^R) \right\|_{X^{r+1-\sigma}}^2 \, ds.
\]
where the last two terms on the right-hand side come from the following boundary term
in the energy estimate
\[
\left| \int_t^t \left[ \int_T |D| (D)^t \omega^R_0 (D) \omega^N_0 \right]_{t=0}^{t=1} \, dx \, ds \right|,
\]
which is bounded by
\[
\int_0^t \left( \| |D| \omega^m \|_{X^r} + \| |D| \omega^{N\varepsilon} \|_{X^r} \right) \left( \| \partial_x \omega^m \|_{X^r} + \| \omega^m \|_{X^r} \right)
\]
\[
+ \left( \| |D| \omega^{N\varepsilon} \|_{X^{r-\frac{1}{2}}} + \| |D| \partial_x \omega^{N\varepsilon} \|_{X^{r-\frac{1}{2}}} \right) \| \omega^m \|_{X^{r+\frac{1}{2}}} \, ds.
\]

Here \( f_\Phi = \mathcal{F}^{-1}(\omega^{\Phi(t)} \tilde{f}(k)) = \omega^{\Phi(t)} D \) and \( \Phi(t, k) \overset{\text{def}}{=} \tau(t)(k)^\sigma \). Main trouble is to control
the term \( \int_0^t \| |D| \omega^m \|_{X^r}^2 \, ds \). For this, we introduce a key high-low frequency decomposition
for \( \omega^m \). For low frequency part, we have
\[
\int_0^t \| P_{\leq 2N(\varepsilon)} |D| \omega^m \|_{X^r}^2 \, ds \leq CE_{\omega}(t)
\]
and for high frequency part,
\[
\int_0^t \| P_{> N(\varepsilon)} |D| \omega^m \|_{X^r}^2 \, ds \leq C\varepsilon^2 \int_0^t \| P_{> N(\varepsilon)} (\partial_x \varepsilon \partial_x)(u^R, \varepsilon v^R) \|_{X^{r+1}}^2 \, ds
\]
\[
+ \int_0^t \left( \| P_{> N(\varepsilon)} (\partial_x \varepsilon \partial_x)(u^R, \varepsilon v^R) \|_{X^{r+1-\sigma}}^2 + \| \omega^m \|_{X^{r+\frac{1}{2}}} \right) \, ds,
\]
where we take \( N(\varepsilon) = [e^{-\frac{1}{2\sigma}}] \) an integer such that \( \varepsilon[k] \sim (k)^{\frac{1}{2}} \).

In section 7, we will derive the following energy estimate
\[
E_{u^R}(t) + E_{v^R}(t) \leq C \int_0^t \| \omega^m \|_{X^{r+\frac{1}{2}}}^2 \, ds + \delta \int_0^t \| P_{> N(\varepsilon)} (\mathcal{N}, \varepsilon \mathcal{N}) \|_{X^{r+1-\frac{1}{2}}}^2 \, ds.
\]

The most trouble term is \( v^R \partial_x u^R \) in the energy estimate. For it, we use the commutator
estimate to obtain
\[
\| P_{> N(\varepsilon)} (v^R \partial_x u^R) \|_{X^{r+1-\frac{1}{2}}} \leq \| P_{> N(\varepsilon)} v^R \partial_x u^R \|_{X^{r+1-\frac{1}{2}}} + \| P_{> N(\varepsilon)} (v^R \partial_x u^R) - P_{> N(\varepsilon)} v^R \partial_x u^R \|_{X^{r+1-\frac{1}{2}}}
\]
\[
\leq C \left( \| P_{> N(\varepsilon)} v^R \|_{X^{r+\frac{1}{2}}} + \| P_{> N(\varepsilon)} \|_{X^{r+\frac{1}{2}}} + \| v^R \|_{X^{r+\frac{1}{2}}}, \right)
\]
\[
\leq C \left( \| P_{> N(\varepsilon)} v^R \|_{X^{r+\frac{1}{2}}} + \| P_{> N(\varepsilon)} v^R \|_{X^{r+\frac{1}{2}}} + \| \omega^R \|_{X^{r+\frac{1}{2}}} \right)
\]
\[
\leq C \left( \| P_{> N(\varepsilon)} v^R \|_{X^{r+\frac{1}{2}}} + \| P_{> N(\varepsilon)} \omega^R \|_{X^{r+1-\frac{1}{2}}} + \| \omega^R \|_{X^{r+\frac{1}{2}}} \right).
\]
Here we used \( 2 - \sigma \leq 1 + \frac{1}{2} \) and the fact that
\[
\| P_{> N(\varepsilon)} f \|_{X^r} \leq C \| P_{> N(\varepsilon)} f \|_{X^{r+1-\frac{1}{2}}}.
\]
which is the key for the control of $v^R$ instead of the usual control $\|u^R\|_{X^r} \leq \|u^R\|_{X^{r+1}}$. For $(k) \leq N(\varepsilon)$ (i.e., $\varepsilon \leq \langle k \rangle^{-1}$), we have

$$\varepsilon \|P_{\leq N(\varepsilon)} \omega^R\|_{X^{r+1}} \leq \|\omega^R\|_{X^r}.$$ 

Thus, we obtain

$$\varepsilon^2 \int_0^t \|P_{\geq N(\varepsilon)} (v^R \partial_t u^R)\|_{H^{r+1}}^2 \, ds \leq CE(t).$$

**Nonlinear estimates and bootstrap argument.** In section 8, we make the nonlinear estimates for $(\omega^R, u^R, v^R)$. Based on the energy estimates for $(\omega^R, u^R, v^R)$ and nonlinear estimates, we close our energy estimates by using a standard bootstrap argument in section 9.

Throughout this paper, we denote by $C$ a constant independent of $\varepsilon, \beta$. We denote by $N(\varepsilon) = \lceil \varepsilon^{-2/\gamma} \rceil$ an integer.

### 2. Gevrey class and elliptic equation in a strip

#### 2.1. Some estimates in Gevrey class

Let us define

$$f_\Phi = \mathcal{F}^{-1}(e^{\Phi(x)} \hat{f}(k)) = e^{\Phi(x)} f, \quad \Phi(t, k) \overset{\text{def}}{=} \tau(t)(k)$$

Obviously, for $\sigma \in [0, 1]$ and $\tau(t) \geq 0$, $\Phi(t, k)$ satisfies the subadditive inequality

$$\Phi(t, k) \leq \Phi(t, k - \ell) + \Phi(t, \ell).$$

Then we have

$$\|f\|_{X^r_{\sigma, \tau}} = \|f_\Phi\|_{H^0}.$$ 

For functions which only depend on variable $x$, we denote

$$|f|_{X^r_{\sigma, \tau}} = \|f_\Phi\|_{H^0}.$$

It is easy to see that if $r' \geq r$, then $\|f\|_{X^r_{\sigma, \tau}} \geq \|f\|_{X^{r'}_{\sigma, \tau}}$. For simplicity, we drop subscript $\sigma, \tau$ in the notations $\|f\|_{X^r_{\sigma, \tau}}, |f|_{X^r_{\sigma, \tau}}, \phi$ etc. We say that a function $f$ belongs to Gevrey class $1/\beta$ if $\|f\|_{X^r_{\sigma, \tau}} < +\infty$. When $\sigma = 1$, the function is analytic. In the sequel, we always take

$$\tau(t) = \tau_0 e^{-\beta t}, \quad \tau_0 > 0, \quad \beta \geq 1 \quad (\text{to be determined later}).$$

We introduce the frequency cut-off operators $P_{\geq N}$ and $P_{\leq N}$, which are defined by

$$P_{\geq N} f(x) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \chi \left( \frac{k}{N} \right) \hat{f}(k) e^{ikx}, \quad P_{\leq N} f = (1 - P_{\geq N-1}) f.$$ 

Here function $\chi$ is a smooth even function with supp $\chi \in \left[ \frac{1}{2}, +\infty \right) \cup \left( -\infty, -\frac{1}{2} \right]$ and $\chi(x) = 1$ for $|x| \geq 1$.

**Lemma 2.1.** Let $r \geq 0, s_1 > \frac{1}{2}, s > \frac{1}{2}$ and $0 \leq \delta \leq 1$. Then it holds that

$$\|[(D)^{s_1}, f] \partial_t g\|_{L^2} \leq C \|f\|_{H^{s_1}} \|g\|_{H^s} + C \|f\|_{H^{s+1}} \|g\|_{H^{s+\delta}},$$

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\[
\|P_{\geq N} f \partial_s g\|_{H^s} \leq C \|f\|_{H^{s'}} \|P_{\geq N} s\|_{H^s} + C \|f\|_{H^{s+\delta}} \|g\|_{H^{s+\delta}}.
\]

**Proof.** The first inequality is classical (see [2] for example). Here we only present the proof for the second one.

Thanks to Plancherel formula, we have
\[
\|P_{\geq N} f \partial_s g\|_{H^s} = \left\| \langle k \rangle^{\prime} \left( \chi \left( \frac{k}{N} \right) \hat{f} \ast \partial_s \hat{g} - \hat{f} \ast \left( \chi \left( \frac{\ell}{N} \right) \partial_s \hat{g} \right) \right) \right\|_{l^2_k},
\]
\[
\leq \left\| \langle k \rangle^{\prime} \sum_{\ell \in \mathbb{Z}} \left( \chi \left( \frac{k}{N} \right) - \chi \left( \frac{\ell}{N} \right) \right) |\ell| \|\hat{f}(k - \ell)|\hat{g}(\ell)| \right\|_{l^2_k}.
\]
We consider two cases. For $|\ell| \leq 2|k - \ell|$, we have
\[
\langle k \rangle^{\prime} |\ell| \leq C |\ell|^\delta (k - \ell)^{\epsilon + 1 - \delta},
\]
which implies that
\[
\left\| \langle k \rangle^{\prime} \sum_{|\ell| \leq 2|k - \ell|} \left( \chi \left( \frac{k}{N} \right) - \chi \left( \frac{\ell}{N} \right) \right) |\ell| \|\hat{f}(k - \ell)|\hat{g}(\ell)| \right\|_{l^2_k}
\leq C \|f\|_{H^{s+\delta}} \|g\|_{H^{s+\delta}}.
\]
Here we used $s > \frac{1}{2}$. If $|\ell| \geq 2|k - \ell|$, then we have
\[
\frac{|\ell|}{2} \leq |k| \leq \frac{3|\ell|}{2},
\]
and $k, \ell$ must have the same sign. Using the mean value theorem, we get
\[
\chi \left( \frac{k}{N} \right) - \chi \left( \frac{\ell}{N} \right) = \frac{1}{N} \chi^{\prime} \left( \frac{\xi}{N} \right) (k - \ell),
\]
where $\xi$ is some point between $k$ and $\ell$. In this case, we have
\[
|\ell| \left| \chi \left( \frac{k}{N} \right) - \chi \left( \frac{\ell}{N} \right) \right| \leq \chi \left( \frac{\ell}{N^2/2} \right) \left| \frac{\xi}{N} \chi^{\prime} \left( \frac{\xi}{N} \right) \right| \left| k - \ell \right| \left| \frac{|\ell|}{|\xi|} \right|
\leq C \chi \left( \frac{\ell}{N^2/2} \right) |k - \ell|.
\]
Therefore, we obtain
\[
\left\| \langle k \rangle^{\prime} \sum_{|\ell| \geq 2|k - \ell|} \left( \chi \left( \frac{k}{N} \right) - \chi \left( \frac{\ell}{N} \right) \right) |\ell| \|\hat{f}(k - \ell)|\hat{g}(\ell)| \right\|_{l^2_k}
\leq \left\| \sum_{\ell \in \mathbb{Z}} |k - \ell| \|\hat{f}(k - \ell)(\ell)| \chi \left( \frac{\ell}{N^2/2} \right) |\hat{g}(\ell)| \right\|_{l^2_k}
\leq C \|f\|_{H^s} \|P_{\geq N} s\|_{H^s}.
\]
Here we used \( s_1 > \frac{3}{2} \). This shows the second inequality. \( \square \)

**Lemma 2.2.** Let \( r \geq 0 \) and \( s > \frac{1}{2} \). Then for any \( N \in \mathbb{Z}_+ \),

\[
|fg|_{X^r} \lesssim C|f|_{X^r}|g|_{X^r} + C|f|_{X^r}|g|_{X^r},
\]

\[
|P_{>N}(fg)|_{X^r} \lesssim C|f|_{X^r}|P_{>N}g|_{X^r} + C|P_{>N}f|_{X^r}|g|_{X^r}.
\]

**Proof.** By Plancherel formula, we have

\[
|fg|_{X^r} = \left\| \langle k \rangle^r e^{\Phi(k)} \langle \hat{f} \ast \hat{g} \rangle(k) \right\|_{L^2}.
\]

We get by (2.2) that

\[
\left| e^{\Phi(k)} \langle \hat{f} \ast \hat{g} \rangle(k) \right| \leq \left| \sum_{\ell \in \mathbb{Z}} e^{\Phi(k-\ell)} \langle \hat{f} \rangle(k-\ell) e^{\Phi(\ell)} \langle \hat{g} \rangle(\ell) \right|
\]

\[
= \sum_{\ell \in \mathbb{Z}} \langle \hat{f} \rangle(k-\ell) \langle \hat{g} \rangle(\ell) = \langle \hat{f} \rangle \ast \langle \hat{g} \rangle = \mathcal{F}(fg),
\]

where we denote \( f^+ \) by the Fourier transformation inverse of \( \langle \hat{f} \rangle \) and so does \( g^+ \). Then by the classical product estimate in Sobolev space, we obtain

\[
|fg|_{X^r} \leq \|f^+ g^+\|_{H^r} \leq C \left( \|f^+\|_r \|g^+\|_r + \|f^+\|_r \|f^+\|_r \right)
\]

\[
= C \left( \|f|_{X^r}|g|_{X^r} + |f|_{X^r}|g|_{X^r} \right),
\]

where we use the fact that map \( f_\Phi \mapsto f^+ \) preserves the \( L^2 \) norm and \( s > \frac{1}{2} \).

For the second one, we use the fact that if \( 2|k-\ell| \geq \ell \), then

\[
\langle k \rangle' \leq C\langle k-\ell \rangle', \quad \chi \left( \frac{k}{N} \right) \leq \chi \left( \frac{k-\ell}{N/2} \right),
\]

and if \( \|\ell\| \geq 2|k-\ell| \), then

\[
\langle k \rangle' \leq C\langle \ell \rangle', \quad \chi \left( \frac{k}{N} \right) \leq \chi \left( \frac{\ell}{N/2} \right).
\]

Then as in the argument for the first inequality, we have

\[
|P_{>N}(fg)|_{X^r} \leq C\|g^+\|_r \|P_{>N}f\|_{L^2} + C\|f^+\|_r \|P_{>N}g\|_{L^2}
\]

\[
\leq C \left( \|f|_{X^r}|P_{>N}g|_{X^r} + |P_{>N}f|_{X^r}|g|_{X^r} \right).
\]

This shows the second inequality. \( \square \)

**Lemma 2.3.** Let \( r \geq 0 \), \( s_1 > \frac{3}{2} \), \( s > \frac{1}{2} \) and \( 0 \leq \delta \leq 1 \). Then it holds that

\[
\|f \partial_i g\|_{H^s} - f \partial_i g\|_{H^s} \leq C|f|_{X^{s_1}}|g|_{X^{s_1}} + C|f|_{X^{s_1}}|g|_{X^{s_1}},
\]

\[
|P_{>N}(f \partial_i g)|_r - f(P_{>N} \partial_i g)|_{H^s} \leq C|f|_{X^{s_1}}|P_{>N}g|_{X^{s_1}}
\]

\[
+ C|f|_{X^{s_1}}|g|_{X^{s_1}},
\]

for any \( N \in \mathbb{Z}_+ \).
2.2. Elliptic equation in a strip

By Plancherel formula and the argument in lemma 2.2, we have

\[
\| (f \partial_x g) - f \partial_x g \|_{H^s} \leq \left( \sum_{\ell \in \mathbb{Z}} |m(k, \ell)| |f_{\Phi}^+(k - \ell) g_{\Phi}^-(\ell)| \right)^{\frac{1}{2}},
\]

where \( m(k, \ell) = (e^{\Phi(x, \ell)} - e^{\Phi(y, \ell)}) e^{-\Phi(k, -\ell) - \Phi(x, \ell)} \). For \(|\ell| \leq 2|k - \ell|\), we have

\[
\langle k \rangle^r |\ell| \leq C (k - \ell)^{r+1-\delta} |\ell|^\delta, \quad m(k, \ell) \leq C,
\]

which imply that

\[
\left\| \left( \sum_{|\ell| \leq 2|k - \ell|} m(k, \ell) f_{\Phi}^+(k - \ell) |\ell| g_{\Phi}^-(\ell) \right) \right\|_{L^2} \leq C \| f_{\Phi}^+ \|_{H^{r+\delta}} \| g_{\Phi}^- \|_{H^{r+\delta}}^2
\]

\[
\leq C \| f \|_{H^{r+\delta}} |g|_{H^{r+\delta}}.
\]

For \(|\ell| \geq 2|k - \ell|\), we have

\[
\langle k \rangle^r \leq C (|\ell|)^{r-1} \delta, \quad m(k, \ell) \leq C |\ell|^\delta |k - \ell|,
\]

which imply that

\[
\left\| \left( \sum_{|\ell| \geq 2|k - \ell|} m(k, \ell) f_{\Phi}^+(k - \ell) |\ell| g_{\Phi}^-(\ell) \right) \right\|_{L^2} \leq C \| \partial_y f_{\Phi}^+ \|_{H^r} \| g_{\Phi}^- \|_{H^{r+\delta}}
\]

\[
\leq C \| f \|_{H^r} |g|_{H^{r+\delta}}.
\]

This shows the first inequality.

For the second one, we have

\[
\| P_{\geq N}(f \partial_x g) - f(P_{\geq N} \partial_x g) \|_{H^s} \leq \| P_{>N} f \partial_x g \|_{H^s} + \| P_{\geq N} e^{\Phi} f \partial_x g \|_{H^s}
\]

\[
= I_1 + I_2.
\]

It follows from lemma 2.1 that

\[
I_1 \leq C \| f \|_{H^r} |g|_{H^{r+\delta}} + C \| f \|_{H^{r+\delta}} |g|_{H^{r+\delta}}.
\]

Note that \( \chi(\frac{1}{N}) \leq \chi(\frac{1}{2N}) + C(\frac{1}{2N}) \). We infer from the first inequality of this lemma that

\[
I_2 \leq C \| f \|_{H^r} |g|_{H^{r+\delta}} + C \| f \|_{H^{r+\delta}} |g|_{H^{r+\delta}}.
\]

Putting \( I_1 - I_2 \) together, we arrive at the second inequality. \( \square \)

2.2. Elliptic equation in a strip

We denote by \((\Delta_{x,y})^{-1} h\) the solution of the following elliptic equation:

\[
\begin{align*}
\Delta F &= (\partial_x^2 + c^2 \partial_y^2) F = h, \quad (x, y) \in S, \\
F|_{y=0,1} &= 0.
\end{align*}
\]
Let us introduce some notations
\[ K_1(k, y) = \frac{e^{\xi|k|} - e^{-\xi|k|}}{e^{\xi|k|} - e^{-\xi|k|}}, \quad K_2(k, y) = e^{-\xi|k|}, \quad (2.4) \]
and
\[ G_0(k, y) = e^{-\xi|k|}K_1(k, y) - K_1(k, 1 - y) - K_2(k, y) \]
\[ = \frac{2e^{-\xi|k|(1-y)} - 2e^{\xi|k|(1-y)}}{e^{\xi|k|} - e^{-\xi|k|}}. \quad (2.5) \]
\[ G_1(k, y) = K_1(k, y) - e^{-\xi|k|}K_1(k, 1 - y) + K_2(k, 1 - y) \]
\[ = \frac{2e^{\xi|k|y} - 2e^{-\xi|k|y}}{e^{\xi|k|} - e^{-\xi|k|}}. \quad (2.6) \]

**Lemma 2.4.** Let \( F = (\triangle_{x,D})^{-1} h \). It holds that
\[
\hat{F}(k, y) = \frac{e^{-\xi|k|(1-y)}}{2|k|\varepsilon} \int_0^1 K_1(k, y)y\hat{h}(k, y')dy' + \frac{e^{-\xi|k|y}}{2|k|\varepsilon} \int_0^1 K_1(k, 1-y)'\hat{h}(k, y')dy' \]
\[ + \frac{1}{2|k|\varepsilon} \int_0^y K_2(k, y-y')\hat{h}(k, y')dy' - \frac{1}{2|k|\varepsilon} \int_0^y K_2(k, y-y')\hat{h}(k, y')dy'. \]

In particular, we have
\[
\mathcal{F}(\partial_y(\triangle_{x,D})^{-1} h) = \frac{e^{-\xi|k|(1-y)}}{2} \int_0^1 K_1(k, y)y\hat{h}(k, y')dy' - \frac{e^{-\xi|k|y}}{2} \int_0^1 K_1(k, 1-y)'\hat{h}(k, y')dy' \]
\[ + \frac{1}{2} \int_1^y K_2(k, y-y')\hat{h}(k, y')dy' + \frac{1}{2} \int_0^y K_2(k, y-y')\hat{h}(k, y')dy', \]
and
\[
\mathcal{F}(\partial_y(\triangle_{x,D})^{-1} h)|_{y=0} = \frac{1}{2} \int_0^1 (G_0\hat{h})(k, y')dy', \]
\[ \mathcal{F}(\partial_y(\triangle_{x,D})^{-1} h)|_{y=1} = \frac{1}{2} \int_0^1 (G_1\hat{h})(k, y')dy'. \]

**Proof.** Taking the Fourier transformation in \( x \) on \( F \), we get
\[
(\partial^2_y - \xi^2|k|^2)\hat{F} = \hat{h}, \quad \hat{F}|_{y=0,1} = 0.
\]
Then we have
\[
\hat{F}(k, y) = C_1(k)e^{\xi|k|y} + C_2(k)e^{-\xi|k|y} + F_s(k, y),
\]
where
\[
F_s(k, y) = \frac{1}{2|k|\varepsilon} \int_1^y e^{-\xi|k|(y-y')}\hat{h}(k, y')dy' - \frac{1}{2|k|\varepsilon} \int_0^y e^{-\xi|k|(y-y')}\hat{h}(k, y')dy',
\]
and
\[
C_1(k) = \frac{1}{2|k|\varepsilon \left( e^{\xi|k|} - e^{-\xi|k|} \right)} \int_0^1 \left( e^{-\xi|k|(1+y)} - e^{-\xi|k|(1-y)} \right)\hat{h}(k, y)dy.
\]
\[ C_2(k) = \frac{1}{2|k| (e^{-|k|} - e^{|k|})} \int_0^1 (e^{-|k| \epsilon} - e^{|k| \epsilon}) \hat{F}(k, y) dy. \]

Recalling the definitions of \( K_1 \) and \( K_2 \), we obtain the solution formula of \( \hat{F}(k, y) \). The other formulas can be directly obtained from \( \hat{F}(k, y) \). \( \square \)

We also introduce
\[ G_2(k, y) = \partial_y G_0(k, y), \quad G_3(k, y) = \partial_y G_1(k, y). \] (2.7)

It is easy to find that for any \( s \in [1, \infty] \),
\[ \| (K_1, K_2, G_0, G_1) \|_{L^s} \leq C \min \left\{ 1, \frac{1}{\varepsilon^s (1 + |k|)^s} \right\}, \] (2.8)
\[ \| (\partial_y K_1, \partial_y K_2, G_2, G_3) \|_{L^s} \leq C \varepsilon^{1-s} (1 + |k|)^{1-s}. \] (2.9)

Here the constant \( C \) is independent of \( k, \varepsilon \).

3. The vorticity formulation of the error equations

We denote
\[ u^R = u^\varepsilon - u^p, \quad v^R = v^\varepsilon - v^p, \quad p^R = p^\varepsilon - p^p. \]

It is easy to find that
\[
\begin{cases}
\partial_t u^R - \Delta_x u^R + \partial_y p^R + u^\varepsilon \partial_y u^R + u^R \partial_y u^\varepsilon + v^\varepsilon \partial_y u^p + v^R \partial_y u^\varepsilon - \varepsilon^2 \partial_y^2 u^\varepsilon = 0, \\
\varepsilon^2(\partial_y v^R - \Delta_x v^R) + \partial_y p^R + \varepsilon^2(\partial_y v^\varepsilon - \varepsilon^2 \partial_y^2 v^\varepsilon - \partial_y^2 v^p + u^\varepsilon \partial_y v^\varepsilon + v^\varepsilon \partial_y v^p) = 0, \\
\partial_x u^R + \partial_y v^R = 0, \\
(u^R, v^R)|_{y=0} = (u^R, v^R)|_{y=1} = 0, \\
(u^R, v^R)|_{t=0} = 0.
\end{cases}
\] (3.1)

We introduce the vorticity \( \omega^R = \partial_y u^R - \varepsilon^2 \partial_x v^R \), which satisfies
\[ \partial_t \omega^R - \Delta_x \omega^R + f = N(\omega^R, \omega^p), \]
where \( f \) and \( N(\omega^R, \omega^p) \) are defined by
\[ f = f_3 - \varepsilon^2(f_1 + f_2), \] (3.2)
\[ N(\omega^R, \omega^p) = -u^p \partial_y \omega^R - v^p \partial_y \omega^R, \] (3.3)
with
\[ f_1 = -(u^R \partial_y^2 v^p + v^R \partial_y \partial_x v^p), \] (3.4)
\[ f_2 = -(\partial_y \partial_x v^p - \varepsilon^2 \partial_y^2 v^p - \partial_y^2 \partial_x v^p + \partial_y^2 \partial_y \omega^p + u^p \partial_y^2 v^p + v^p \partial_y \partial_x v^p). \] (3.5)
Lemma 3.1. I hold that
\[ f_3 = u^p \partial_t \omega + u^R \partial_t \omega^p + v^p \partial_y \omega^R + v^R \partial_y \omega^p, \quad \omega^p = \partial_y u^p. \] (3.6)

Next let us derive the boundary condition of the vorticity. Thanks to \( \partial_t u^R + \partial_y v^R = 0 \) and \( v^R|_{y=0} = 0 \), there exists \( \phi \) so that
\[ -\partial_t \phi = v^R, \quad \partial_y \phi = u^R - \frac{1}{2\pi} \int_S u^R \, dx \, dy. \]

Since \( \int_S v^R \, dx = 0 \), the function \( \phi \) is periodic in \( x \). Thanks to \( \partial_t \phi|_{y=0} = 0 \) and \( \phi(1, x) - \phi(0, x) = 0 \), we may assume that \( \phi|_{y=0} = 0 \). Thus, there holds that
\[ \Delta \phi = \omega^R \quad \text{in} \quad S, \quad \phi|_{y=0} = 0. \]

This shows that
\[ u^R = \partial_y (\Delta \phi)^{-1} \omega^R + \frac{1}{2\pi} \int_S u^R \, dx \, dy, \] (3.7)
\[ v^R = -\partial_y (\Delta \phi)^{-1} \omega^R. \] (3.8)

Motivated by \([10, 15]\), we have

**Lemma 3.1.** It holds that
\[
(\partial_t + \varepsilon |D|) \omega^R |_{y=0} = \partial_y (\Delta \phi)^{-1} (f - N(\omega^R, \omega^0)) |_{y=0} + \frac{1}{2\pi} \int_S \partial_t u^R \, dx \, dy,
\]
\[
(\partial_t - \varepsilon |D|) \omega^R |_{y=1} = \partial_y (\Delta \phi)^{-1} (f - N(\omega^R, \omega^0)) |_{y=1} + \frac{1}{2\pi} \int_S \partial_t u^R \, dx \, dy.
\]

**Proof.** We only prove the first equality and the second one is similar. We introduce \( \omega^R_{h, 0} \) which is the harmonic extension of \( \omega^R |_{y=0} \), i.e.,
\[
\begin{cases}
\Delta_x \omega^R_{h, 0} = 0, & x \in \mathbb{T}, \ y \in \mathbb{R}_+,
\omega^R_{h, 0}|_{y=0} = \omega^R|_{y=0}.
\end{cases}
\]

We know that
\[ \partial_y \omega^R_{h, 0}|_{y=0} = -\varepsilon |D| \omega^R |_{y=0}. \]

Taking \( \partial_t \) to (3.7) and using \( u^R |_{y=0} = 0 \) and the equation of \( \omega^R \), we obtain
\[
0 = \partial_t u^R |_{y=0} = \partial_y (\Delta \phi)^{-1} \omega^R |_{y=0} + \frac{1}{2\pi} \int_S \partial_t u^R \, dy
= \partial_y (\Delta \phi)^{-1} \left( \Delta_x (\omega^R - \omega^R_{h, 0}) - f + N(\omega^R, \omega^0) \right) |_{y=0} + \frac{1}{2\pi} \int_S \partial_t u^R \, dy
= \partial_y \omega^R |_{y=0} - \partial_y \omega^R_{h, 0}|_{y=0} - \partial_y (\Delta \phi)^{-1} (f - N(\omega^R, \omega^0)) |_{y=0} + \frac{1}{2\pi} \int_S \partial_t u^R \, dy,
\]
where we used \( (\omega^R - \omega^R_{h, 0}) |_{y=0} = 0 \) and \( \Delta_x \omega^R_{h, 0} = 0 \). This shows the first equality. 
\[ \square \]
Based on lemma 2.4, we give more precise formulation of the boundary condition of $\omega^R$. Firstly, by the definition of $f_3$ and using the divergence free condition, we obtain
\[
f_3 = \partial_y(u^p \omega^R + \partial_x^{-1} v^R \partial_1 \omega^p) + u^R \partial_2 \omega^p - \partial_x^{-1} v^R \partial_1 \partial_2 \omega^p + \partial_y(v^p \omega^R),
\]
where $\partial_x^{-1} v^R$ is defined by
\[
\partial_x^{-1} v^R = \begin{cases} 
- \int_0^y u^R \, dz & 0 \leq y \leq \frac{1}{2}, \\
- \int_1^y u^R \, dz & \frac{1}{2} < y \leq 1.
\end{cases}
\] (3.9)

Then by lemma 2.4 and integration by parts (note $v^p \omega^R|_{y=0,1}=0$), we get
\[
\mathcal{F}(\partial_1 (\Delta_{\epsilon,D})^{-1} f)|_{y=0} = \frac{ik}{2} \int_0^1 G_0(k,y) \mathcal{F}(u^p \omega^R + \partial_x^{-1} v^R \partial_1 \omega^p) (k,y) dy \\
- \frac{1}{2} \int_0^1 G_2(k,y) \mathcal{F}(v^p \omega^R) (k,y) dy \\
+ \frac{1}{2} \int_0^1 G_0(k,y) \mathcal{F}(u^R \partial_2 \omega^p - \partial_x^{-1} v^R \partial_x \partial_2 \omega^p \\
- \varepsilon^2 (f_1 + f_2)) (k,y) dy.
\]

Thus, we obtain
\[
\mathcal{F} \left( (\partial_1 + \varepsilon |D|) \omega^R \right) |_{y=0} = ik \mathcal{F} h^0(k) + \mathcal{F} h^0_1 (k) \\
- \mathcal{F} \left( \partial_1 (\Delta_{\epsilon,D})^{-1} (N(\omega^R, \omega^R)) |_{y=0} \right) \\
+ \frac{1}{2 \pi} \int_S \mathcal{F}(\partial_2 u^R) dx \, dy,
\]
where
\[
\mathcal{F} h^0(k) = \frac{1}{2} \int_0^1 (G_0 \mathcal{F}(u^p \omega^R + \partial_x^{-1} v^R \partial_1 \omega^p)) (k,y) dy, \\
\mathcal{F} h^0_1 (k) = - \frac{1}{2} \int_0^1 (G_2 \mathcal{F}(v^p \omega^R)) (k,y) dy \\
+ \frac{1}{2} \int_0^1 (G_0 \mathcal{F}(u^R \partial_2 \omega^p - \partial_x^{-1} v^R \partial_x \partial_2 \omega^p - \varepsilon^2 (f_1 + f_2)) (k,y) dy.
\] (3.10) (3.11)

Similarly, we have
where

\[ Fh_1^1 (k) = \frac{1}{2} \int_0^1 \left( G_1 F (u^R, \omega^R + \partial_x^{-1} v^R \partial_t \omega^R) \right) (k, y) dy, \]  
(3.12)

\[ Fh_1^1 (k) = -\frac{1}{2} \int_0^1 \left( G_1 F (v^R, \omega^R) \right) (k, y) dy \]
\[ + \frac{1}{2} \int_0^1 \left( G_1 F \left( u^R \partial_t \omega^R - \partial_x^{-1} v^R \partial_t \partial_x \omega^R - \epsilon^2 (f_1 + f_2) \right) \right) (k, y) dy. \]
(3.13)

Finally, we conclude that

\[
\begin{cases}
\partial_t \omega^R - \triangle \omega^R + f = N(\omega^R, \omega^R), \\
(\partial_x + \epsilon |D| \omega^R)_{|y=0} = \partial_x h^0 + h^0_1 - \partial_x \left( \triangle \omega^R \right)_{|y=0} + \frac{1}{2\pi} \int_S F(\partial_t u^R) dx \ dy, \\
(\partial_x - \epsilon |D| \omega^R)_{|y=1} = \partial_x h^1 + h^1_1 - \partial_x \left( \triangle \omega^R \right)_{|y=1} + \frac{1}{2\pi} \int_S F(\partial_t u^R) dx \ dy, \\
\omega^R_{|x=0} = 0. 
\end{cases}
\]
(3.14)

Note that \( \partial_x h^0 \) and \( \partial_x h^1 \) are the worst terms.

Let us compute \( \int_S \partial_t u^R \ dx \ dy \). Using the equation of \( u^R \) in (3.1), we find that

\[
\int_S \partial_t u^R \ dx \ dy = \int_S \partial_x^2 u^R \ dx \ dy = \int_S \partial_y \omega^R \ dx \ dy,
\]

which gives

\[
\left| \int_S \partial_t u^R \ dx \ dy \right| \leq \| \partial_x \omega^R \|_{L^1}.
\]
(3.15)

Since \((u^R, v^R)\) satisfies the following elliptic equations

\[
\begin{cases}
\triangle_x u^R = \partial_x \omega^R, \\
\left. u^R \right|_{y=0,1} = 0,
\end{cases}
\]
\[ \begin{cases}
\triangle_x v^R = -\partial_x \omega^R, \\
\left. v^R \right|_{y=0,1} = 0.
\end{cases} \]

we arrive at

\[
\| (u^R, \epsilon v^R, \partial_x u^R, \epsilon \partial_x u^R, \partial_y v^R, \epsilon^2 \partial_x v^R) \|_{X^0} \leq C \| \omega^R \|_{X^0}. \]
(3.16)
4. Boundary layer lift

To handle the worst terms \( \partial_t h^0 \) and \( \partial_t h^1 \), motivated by [6], we introduce a boundary layer lift for the vorticity. More precisely, we consider the heat equation

\[
\begin{aligned}
\begin{cases}
(\partial_t - \triangle_x)\omega^{h,1} = 0, \\
\partial_y \omega^{h,1} |_{y = i} = \partial_t h^i, \\
\omega^{h,1} |_{t = 0} = 0,
\end{cases}
\end{aligned}
\] (4.1)
\]

where \( t \in [0, T] \), \( x \in \mathbb{T} \) and \( y > 0 \) for \( i = 0 \) and \( y < 1 \) for \( i = 1 \). Here \( (h^0, h^1) \) is given by (3.10) and (3.12). We also introduce the boundary layer velocity \((u^{h,1}, v^{h,1})\), which are given by

\[
\begin{aligned}
u^{h,1}(x, y) = \int_{-\infty}^y \omega^{h,1}(x, z)dz, \quad &v^{h,1} = \int_{-\infty}^y \partial_z u^{h,1}(x, z)dz \quad \text{for} \quad y < 1. 
\end{aligned}
\] (4.3)

Motivated by lemma 3.1 in [6], we have the following uniform estimates for \((\omega^{h,i}, u^{h,i}, v^{h,i})\) in Gevrey class \( X' = X'_{\sigma,x} \).

**Lemma 4.1.** Let \( T > 0 \) and \( r \in \mathbb{R} \). The boundary layer vorticity \( \omega^{h,i} \) obeys that

\[
\begin{aligned}
\int_0^T \|\omega^{h,i}\|_{X'}^2 + \| (y - i)\partial_y \omega^{h,i} \|_{X'}^2 \, dt \leq & \frac{C}{\beta^2} \int_0^T |h^i|^2_{X'^{\epsilon + 1, -\frac{2}{3}}} \, dt, \\
\int_0^T \| (y - i)\omega^{h,i} \|_{X'}^2 + \| (y - i)^{\epsilon+1} \partial_t \omega^{h,i} \|_{X'}^2 \, dt \leq & \frac{C}{\beta^2} \int_0^T |h^i|^2_{X'^{\epsilon + 1, -\frac{2}{3}}} \, dt, \\
\int_0^T |v^{h,i}|_{y = 1 - |z|^{\frac{1}{\epsilon}}} \, dt + \| \partial_x \omega^{h,i} |_{y = 1 - |z|^{\frac{1}{\epsilon}}} \|_{X'} \, dt \leq & \frac{C}{\beta^2 M} \int_0^T |h^i|^2_{X'^{\epsilon + 1, -\frac{2}{3}}} \, dt, \\
\int_0^T \| (\partial_x, \epsilon \partial_x) \omega^{h,i} \|_{X'}^2 \, dt \leq & \frac{C}{\beta^2} \int_0^T |h^i|^2_{X'^{\epsilon + 1, -\frac{2}{3}}} \, dt, \\
\sup_{s \in [0,T]} \| \omega^{h,i}(s) \|_{X'}^2 \leq & \frac{C}{\beta^2} \int_0^T |h^i|^2_{X'^{\epsilon + 1, -\frac{2}{3}}} \, dt,
\end{aligned}
\]

and the boundary layer velocity \( u^{h,i} \) obeys that

\[
\begin{aligned}
\int_0^T \|u^{h,i}\|_{X'}^2 \, dt \leq & \frac{C}{\beta^2} \int_0^T |h^i|^2_{X'^{\epsilon + 1, -\frac{2}{3}}} \, dt, \\
\int_0^T \int_0^\infty u^{h,0} \, dy \, dt \leq & \frac{C}{\beta^2} \int_0^T |h^i|^2_{X'^{\epsilon + 1, -\frac{2}{3}}} \, dt, \\
\int_0^T \| (y - i) u^{h,i} \|_{X'}^2 \, dt \leq & \frac{C}{\beta^2} \int_0^T |h^i|^2_{X'^{\epsilon + 1, -\frac{2}{3}}} \, dt, \\
\int_0^T \| \epsilon |D| u^{h,i} \|_{X'}^2 \, dt \leq & \frac{C}{\beta^2} \int_0^T |h^i|^2_{X'^{\epsilon + 1, -\frac{2}{3}}} \, dt, \\
\int_0^T |u^{h,i}|_{y = 1 - |z|^{\frac{1}{\epsilon}}} \, dt \leq & \frac{C}{\beta^2} \int_0^T |h^i|^2_{X'^{\epsilon + 1, -\frac{2}{3}}} \, dt,
\end{aligned}
\]

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and the boundary layer velocity \( v^{b,i} \) obeys that

\[
\int_0^t \| v^{b,i} \|_{X^1}^2 \, ds \leq \frac{C}{\beta T} \int_0^t \| h^{1,2}_i \|_{X^1}^2 \, ds,
\]

\[
\int_0^t \| \varepsilon D v^{b,i} \|_{X^1}^2 \, ds \leq \frac{C}{\beta T} \int_0^t \| h^{1,2}_i \|_{X^1}^2 \, ds,
\]

\[
\int_0^t \| v^{b,i} \|_{X^{1-\beta}}^2 \, ds \leq \frac{C}{\beta T} \int_0^t \| h^{1,2}_i \|_{X^{1-\beta}}^2 \, ds,
\]

\[
\int_0^t \| v^{b,i} \|_{X^{1-\beta}}^2 \, ds \leq \frac{C}{\beta T} \int_0^t \| h^{1,2}_i \|_{X^{1-\beta}}^2 \, ds,
\]

for all \( t \in [0, T] \), \( i = 0, 1 \) and any \( M \geq 0 \).

**Proof.** The proof is almost the same as lemma 3.1 in [6]. Here we just show main idea by proving an inequality.

Thanks to the definition of \( \tau (t) \), we find that \( \omega^{b,0}_\Phi \) satisfies

\[
\begin{aligned}
(\partial_t + \tau_0 \beta(D)^{\sigma} - \Delta_x) \omega^{b,0}_\Phi &= 0, \quad t \in [0, T], \quad (x, y) \in \mathbb{T} \times [0, \infty), \\
\partial_y \omega^{b,0}_\Phi |_{y=0} &= \partial_y h^{b,0}_\Phi, \\
\omega^{b,0}_\Phi |_{t=0} &= 0.
\end{aligned}
\]

(4.4)

For fixed \( x \in \mathbb{T} \), we define that \( h^{b,0}_\Phi(t, x) = 0 \) for \( t \in \mathbb{R} \setminus [0, T] \), and then we consider the extended system of (4.4):

\[
\begin{aligned}
(\partial_t + \tau_0 \beta(D)^{\sigma} - \Delta_x) \bar{\omega}^{b,0}_\Phi &= 0, \quad t \in \mathbb{R}, \quad (x, y) \in \mathbb{T} \times (0, \infty), \\
\partial_y \bar{\omega}^{b,0}_\Phi |_{y=0} &= \partial_y h^{b,0}_\Phi,
\end{aligned}
\]

(4.5)

which satisfies

\[
\bar{\omega}^{b,0}_\Phi = \omega^{b,0}_\Phi, \quad t \in [0, T], \quad \text{and} \quad \bar{\omega}^{b,0}_\Phi = 0, \quad t < 0.
\]

which comes from lemma 3.2 in [6]. Taking Fourier transform in \( t, x \) to obtain

\[
\begin{aligned}
(i \zeta + \tau_0 \beta(k)^{\sigma} + \varepsilon^2 |k|^2) \mathcal{F}_{\mathcal{L}_x} \bar{\omega}^{b,0}_\Phi - \partial_x^2 \mathcal{F}_{\mathcal{L}_x} \bar{\omega}^{b,0}_\Phi &= 0, \\
\partial_y \mathcal{F}_{\mathcal{L}_x} \bar{\omega}^{b,0}_\Phi |_{y=0} &= ik \mathcal{F}_{\mathcal{L}_x} h^{b,0}_\Phi.
\end{aligned}
\]

Then the solution is given by

\[
\mathcal{F}_{\mathcal{L}_x} \bar{\omega}^{b,0}_\Phi(\zeta, k, y) = \frac{-ik \mathcal{F}_{\mathcal{L}_x} h^{b,0}_\Phi(\zeta, k)}{i \zeta + \tau_0 \beta(k)^{\sigma} + \varepsilon^2 |k|^2} e^{-|\zeta| \sqrt{\tau_0 \beta(k)^{\sigma} + \varepsilon^2 |k|^2}}.
\]

(4.6)

For all \( \zeta \) with \( \text{Im} \zeta \leq 0 \), there holds

\[
\left| \sqrt{\tau_0 \beta(k)^{\sigma} + \varepsilon^2 |k|^2} \right| \geq \sqrt{\tau_0 \beta(k)^{\sigma} + \varepsilon^2 |k|^2} - \text{Im} \zeta \geq \sqrt{\tau_0 \beta(k)^{\sigma} + \varepsilon^2 |k|^2},
\]

which along with (4.6) implies

\[
\| \mathcal{F}_{\mathcal{L}_x} \bar{\omega}^{b,0}_\Phi(\zeta, k, y) \|_{L^2(\mathbb{L}^2)}^2 \leq \left\| \frac{C_1 |k|}{(\beta(k)^{\sigma})^{\gamma/4}} \mathcal{F}_{\mathcal{L}_x} h^{b,0}_\Phi \right\|_{L^2(\mathbb{L}^2)}^2,
\]

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This shows by Plancherel’s formula that
\[ \int_0^t \| \omega^{h0} \|_{L^2}^2 \, ds \leq C_1 \frac{1}{\beta^2} \int_0^t |h^{00 \perp}|^2 \, dx. \]

The proof of the other inequalities is similar. But the proof of the fifth inequality is similar to lemma 3.3 in [6].

5. Control the boundary layer lift via the interior vorticity

We introduce the boundary layer profiles
\begin{align}
\omega^{bl}(t, x, y) &= \omega^{h0}(t, x, y) + \omega^{h1}(t, x, y), \\
\bar{u}^{bl}(t, x, y) &= \bar{u}^{h0}(t, x, y) + \bar{u}^{h1}(t, x, y), \\
\bar{v}^{bl}(t, x, y) &= \bar{v}^{h0}(t, x, y) + \bar{v}^{h1}(t, x, y) \\
&= \partial_x \left( \int_y^{+\infty} \bar{u}^{h0}(t, x, z) \, dz + \int_y^{-\infty} \bar{u}^{h1}(t, x, z) \, dz \right),
\end{align}

and the interior vorticity and velocity as follows
\begin{align}
\omega^{in}(t, x, y) &= \omega^{h}(t, x, y) - \omega^{bl}(t, x, y), \\
\bar{u}^{in}(t, x, y) &= \bar{u}^{h}(t, x, y) - \bar{u}^{bl}(t, x, y), \\
\bar{v}^{in}(t, x, y) &= \bar{v}^{h}(t, x, y) - \bar{v}^{bl}(t, x, y).
\end{align}

The following lemma gives the relation between \((\bar{u}^{in}, \bar{v}^{in})\) and \(\omega^{in}\).

**Lemma 5.1.** Let \(\Psi\) solve the elliptic equation
\[ \begin{cases}
\Delta \Psi = 0, \\
\Psi|_{y=0} = \left( \int_0^{+\infty} \bar{u}^{h0}(t, x, z) \, dz + \int_0^{-\infty} \bar{u}^{h1}(t, x, z) \, dz \right), \\
\Psi|_{y=1} = \left( \int_1^{+\infty} \bar{u}^{h0}(t, x, z) \, dz + \int_1^{-\infty} \bar{u}^{h1}(t, x, z) \, dz \right).
\end{cases} \]

Then it holds that
\[ \begin{align*}
\bar{u}^{in} &= \frac{1}{2\pi} \int_S \bar{u}^h \, dx \, dy + \bar{\partial}_y \Psi = \bar{\partial}_y (\Delta \omega) \bar{\partial}_y (\omega^{in} + \varepsilon^2 \bar{\partial}_x \bar{v}^{bl}), \\
\bar{v}^{in} - \bar{\partial}_x \Psi &= -\bar{\partial}_x (\Delta \omega \bar{\partial}_y (\omega^{in} + \varepsilon^2 \bar{\partial}_x \bar{v}^{bl}).
\end{align*} \]

**Proof.** By the construction, we have
\[ \partial_x \bar{u}^{in} + \partial_y \bar{v}^{in} = 0, \quad \omega^{in} = \bar{\partial}_y \bar{u}^{in} - \varepsilon^2 \bar{\partial}_x \bar{v}^{in} - \varepsilon^2 \bar{\partial}_x \bar{v}^{bl}. \]

Thanks to \(\partial_x \bar{u}^{in} + \partial_y \bar{v}^{in} = 0\), there exists a stream function \(\phi\) so that
\[ \begin{align*}
-\bar{\partial}_x \phi &= \bar{v}^{in}, \\
\bar{\partial}_y \phi &= \bar{u}^{in} - \frac{1}{2\pi} \int_S \bar{u}^h \, dx \, dy.
\end{align*} \]
Here $\phi$ is a periodic function in $x$ due to $\int_{x} v^{in} \, dx = 0$. Thanks to $\int_{x} \phi(x, 1) \, dx = \int_{x} \phi(x, 0) \, dx$, we may assume that $\int_{x} \phi(x, 1) \, dx = \int_{x} \phi(x, 0) \, dx = 0$. Thus, we find that

$$\Delta_{t}(\phi + \Psi) = \omega^{in} + \varepsilon^{2} \partial_{x} v^{hi}, \quad (\phi + \Psi)|_{y=0.1} = 0.$$ 

This implies our result. \qed

**Remark 5.2.** Here, we explain the condition $\int_{x} v^{in} \, dx = 0$ which is used in the proof of lemma 5.1. By the divergence free condition $\partial_{t} u^{in} + \partial_{x} v^{in} = 0$, we write

$$v^{in} = \int_{0}^{t} \partial_{t} u^{in}(t, x, y') \, dy' + v^{in}(t, x, 0),$$

and it is easy to derive $\int_{x} \partial_{t} (\int_{0}^{t} u^{in}(t, x, y') \, dy') \, dx = 0$. On the other hand, since $v^{in}(t, x, 0) = v^{R}(t, x, 0) - v^{N}(t, x, 0) = -\partial_{t} \left( \int_{0}^{+\infty} u^{h\phi}(t, x, z) \, dz + \int_{-\infty}^{-\infty} u^{h\phi}(t, x, z) \, dz \right)$, it is easy to see

$$\int_{T} v^{in}(t, x, 0) \, dx = 0.$$ 

Hence, we obtain $\int_{x} v^{in} \, dx = 0$.

**Lemma 5.3.** Let $0 < \alpha_{1} < \alpha_{2}$ and $r \geq 0$. Then it holds that

$$\int_{0}^{t} \| (u^{in}, \varepsilon v^{in}) \|_{X^{r+\frac{1}{2}}}^{2} \, ds \leq C \int_{0}^{t} \| v^{in} \|_{X^{r+\frac{1}{2}}}^{2} \, ds,$$

$$+ \frac{C}{\beta^{2}} \int_{0}^{t} \left( \| h^{2}, h^{1} \|_{X^{r+\frac{3}{2}}}^{2} \right) \, ds,$$

$$\int_{0}^{t} \| (P_{\geq \alpha_{2}N(c)} - P_{\geq \alpha_{1}N(c)}(u^{R}, \varepsilon v^{R})) \|_{X^{r \frac{3}{2}}}^{2} \, ds \leq C \int_{0}^{t} \| (P_{\geq \alpha_{2}N(c)} - P_{\geq \alpha_{1}N(c)}) \|_{X^{r \frac{3}{2}}}^{2} \, ds,$$

and the weighted estimate

$$\int_{0}^{t} \| (P_{\geq \alpha_{2}N(c)} - P_{\geq \alpha_{1}N(c)}(\varepsilon \partial_{x}(\Delta_{t} \cdot) v^{hi} - \varepsilon \phi v^{R}) \|_{X^{r \frac{3}{2}}}^{2} \, ds$$

$$\leq C \int_{0}^{t} \left( \| (P_{\geq \alpha_{2}N(c)} - P_{\geq \alpha_{1}N(c)}) \|_{X^{r \frac{3}{2}}}^{2} \right) \, ds,$$

where the weight function $\phi$ is defined by $\phi(y) = y(1 - y)$.

**Proof.** Using the fact that

$$\| F( (\partial_{x}, \varepsilon \partial_{x})(\Delta_{t} \cdot))^{-1} f \|_{L^{2}} \leq \frac{C}{(1 + \varepsilon |k|)} \| F \|_{L^{2}},$$

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and Lemma 5.1, we infer that
\[
\|\tilde{u}_\Phi^n, \tilde{v}_\Phi^n\|_{L^2_t} \leq \|\tilde{\partial}_x \tilde{\Psi}_\Phi, \tilde{\varepsilon} \tilde{\partial}_x \tilde{\Psi}_\Phi\|_{L^2_t} + \frac{C}{(1 + \varepsilon|k|)} \|\tilde{\omega}_\Phi^h + \varepsilon^2 \tilde{\partial}_x \tilde{\nu}_\Phi^h\|_{L^2_t}
\]
+ \|\tilde{u}_\Phi^h\|_{L^2_t}.

Thanks to the definition of \(\Psi\), we have
\[
\tilde{\Psi}(t, k, y) = \tilde{\Psi}(t, k, y) = K_1(k, y)\tilde{\Psi}|_{y=1} + K_1(k, 1 - y)\tilde{\Psi}|_{y=0}.
\tag{5.8}
\]
By (2.8), (2.9) and following the proof of Lemma 4.1, we can deduce that
\[
\|\tilde{F}_{t,x}(\tilde{\partial}_x \tilde{\Psi}_\Phi, \varepsilon \tilde{\partial}_x \tilde{\Psi}_\Phi)\|_{L^2_t} \leq \|\tilde{\partial}_x \tilde{\varepsilon}|k|\tilde{K}_1\|_{L^2_t}|\tilde{F}_{t,x}\tilde{\Psi}_\Phi|_{y=0.1}|
\leq C(\varepsilon|k|)^\frac{2}{5} \left( \int_0^{+\infty} |\tilde{F}_{t,x}\tilde{u}_\Phi^0(\zeta, k, y)dy| + \int_1^{-\infty} |\tilde{F}_{t,x}\tilde{u}_\Phi^0(\zeta, k, y)dy| \right)
\leq C(\varepsilon|k|)^\frac{2}{5} \left( |\tilde{K}_1| (|\tilde{F}_{t,x}\tilde{h}_\Phi^0| + |\tilde{F}_{t,x}\tilde{h}_\Phi^1|) \right)
\leq C(\varepsilon|k|)^{\frac{2}{5}} \frac{1}{(\beta|k|^\sigma + \varepsilon^2|k|^2)^{\frac{2}{5}}},
\]
and
\[
\frac{C}{(1 + \varepsilon|k|)} \|\tilde{e}^2 \tilde{F}_{t,x}(\tilde{\partial}_x \tilde{v}_\Phi^h)\|_{L^2_t} \leq \frac{\varepsilon^2|k|^3 (|\tilde{F}_{t,x}\tilde{h}_\Phi^0| + |\tilde{F}_{t,x}\tilde{h}_\Phi^1|)}{(1 + \varepsilon|k|)(\beta|k|^\sigma + \varepsilon^2|k|^2)^{\frac{2}{5}}}
\leq C\frac{|\tilde{K}_1| (|\tilde{F}_{t,x}\tilde{h}_\Phi^0| + |\tilde{F}_{t,x}\tilde{h}_\Phi^1|)}{(\beta|k|^\sigma + \varepsilon^2|k|^2)^{\frac{2}{5}}}.
\]
This implies that
\[
\int_0^t \|u^{m, n}, v^{m, n}\|_{X^{r+\frac{3}{5}}}^2 \, ds \leq C \int_0^t \|\omega^m\|_{X^{r+\frac{3}{5}}}^2 \, ds
+ \frac{C}{\beta^\frac{2}{5}} \int_0^t (|h^0, h^1|)^2_{X^{r+1 - \frac{3}{5}}} \, ds + \int_0^t \|u^{h, h}\|_{L^2_t}^2 \, ds
\leq C \int_0^t \|\omega^m\|_{X^{r+\frac{3}{5}}}^2 \, ds + \frac{C}{\beta^\frac{2}{5}} \int_0^t (|h^0, h^1|)^2_{X^{r+1 - \frac{3}{5}}} \, ds.
\]
Here we used the fact that
\[
\int_0^t \|u^{h, h}\|_{L^2_t}^2 \, ds \leq C \int_0^t \|\omega^h\|_{L^2_t}^2 \, ds \leq C \int_0^t \|\omega^m\|_{X^{r+\frac{3}{5}}}^2 \, ds
+ \frac{C}{\beta^\frac{2}{5}} \int_0^t (|h^0, h^1|)^2_{X^{r+1 - \frac{3}{5}}} \, ds.
\]
By (3.7) and (3.8), we infer that for \(\alpha_1 \tilde{N}(\varepsilon) \leq |k| \leq \alpha_2 \tilde{N}(\varepsilon),
\[
\|u^{h, h, 0}(t, k, \cdot)\|_{L^2_t} \leq \frac{C}{(1 + \varepsilon|k|)} \|\tilde{\omega}_\Phi^h(t, k, \cdot)\|_{L^2_t} \leq \frac{C}{(k)^{\frac{2}{5}}} \|\tilde{\omega}_\Phi^h(t, k, \cdot)\|_{L^2_t},
\]
which implies the second inequality.
By (3.7) and lemma 2.4, we find that for $\alpha_1 N(\varepsilon) \leq |k| \leq \alpha_2 N(\varepsilon)$,
\[
\varphi(y) \tilde{u}_0^k = \frac{\varphi(y) e^{-\varepsilon |k|(1-\gamma)}}{2} \int_0^1 K_1(k, y) \tilde{w}_0^k(t, k, y) dy - \frac{\varphi(y) e^{-\varepsilon |k|}}{2} \int_0^1 K_1(k, 1 - y) \tilde{w}_0^k(t, k, y) dy
+ \frac{\varphi(y)}{2} \int_0^{\gamma} K_2(k, y') \tilde{w}_0^k(t, k, y') dy' + \frac{\varphi(y)}{2} \int_0^{\gamma} K_2(k, y - y') \tilde{w}_0^k(t, k, y') dy' = B^1 + B^2 + B^3 + B^4.
\]

By (2.8) and $\|\varphi(y) e^{-\varepsilon |k|y}\|_L^2 \leq \frac{C}{\varepsilon^2 (1 + |k|)^2}$, we get
\[
\|B^1\|_{L^2}^2 + \|B^2\|_{L^2}^2 \leq C \|\varphi(y) e^{-\varepsilon |k|y}\|_L^2 \|K_1\|_{L^2} \|\tilde{w}_0^k\|_{L^2}^2 \\
\leq \frac{C}{\varepsilon^2 (1 + |k|)^2} \|\tilde{w}_0^k\|_{L^2}^2.
\]

Thanks to $\varphi(y) = \varphi(y') + (y - y')(1 - y - y')$, we get by Young's inequality that
\[
\|B^3\|_{L^2} \leq \left\| \int_0^{\gamma} (y - y') K_2(k, y - y') \tilde{w}_0^k(k, y') dy' \right\|_{L^2} + \left\| \int_0^{\gamma} K_2(k, y - y') \varphi \tilde{w}_0^k(k, y') dy' \right\|_{L^2}
\leq \|y K_2(k, y)\|_{L^2} \|\tilde{w}_0^k\|_{L^2} + \|K_2(k, y)\|_{L^2} \|\varphi \tilde{w}_0^k\|_{L^2}
\leq \frac{C}{\varepsilon^2 (1 + |k|)^2} \|\tilde{w}_0^k\|_{L^2}^2 + \frac{C}{\varepsilon (1 + |k|)} \|\varphi \tilde{w}_0^k\|_{L^2}.
\]

Similarly, we have
\[
\|B^4\|_{L^2} \leq \frac{C}{\varepsilon^2 (1 + |k|)^2} \|\tilde{w}_0^k\|_{L^2}^2 + \frac{C}{\varepsilon (1 + |k|)} \|\varphi \tilde{w}_0^k\|_{L^2}.
\]

Summing up, we infer that for $\alpha_1 N(\varepsilon) \leq |k| \leq \alpha_2 N(\varepsilon),$
\[
\|\varphi u_0^k\|_{L^2} \leq \frac{C}{\varepsilon^2 (1 + |k|)^2} \|\tilde{w}_0^k\|_{L^2}^2 + \frac{C}{\varepsilon (1 + |k|)} \|\varphi \tilde{w}_0^k\|_{L^2}.
\]

Thus, we obtain
\[
\|(P_{> \alpha_2 N(\varepsilon)} - P_{\geq \alpha_1 N(\varepsilon)}) \varphi u^k\|_{X^r} \leq C \left( \|(P_{> \alpha_2 N(\varepsilon)} - P_{\geq \alpha_1 N(\varepsilon)}) \varphi \omega^k\|_{X^r} \right) ^{\frac{2}{\alpha}} \\
+ \|(P_{> \alpha_2 N(\varepsilon)} - P_{\geq \alpha_1 N(\varepsilon)}) \omega^R\|_{X^{r-\frac{3}{4}}}.
\]

The estimate of $\varepsilon \varphi \omega^R$ is similar. \qed

**Lemma 5.4.** Let $u^\sigma$ be given in theorem 1.1 and $r \in [1, N_0 - 5]$. Then there exists $\beta_\varepsilon > 1$ such that for $\beta \geq \beta_\varepsilon$ and $\sigma \in [\frac{8}{3}, 1]$, there holds that
\[
\int_0^1 \|h^1 h^1\|_{X^{r+\frac{3}{4}}} ds \leq C \int_0^1 \|\omega^R\|_{X^{r+\frac{3}{4}}} ds, \quad (5.9)
\]
and for $\sigma \in [\frac{8}{3}, 1],$

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\[ \int_0^t |\varepsilon| D(h^0, h^1)^2_{x + \frac{\sigma}{2}} \, ds \leq C \int_0^t \left( \|P \varepsilon \sigma \varepsilon^2 \partial_x v^a\|_{x + \frac{\sigma}{2}}^2 + \|\omega^m\|_{x + \frac{\sigma}{2}}^2 \right) \, ds. \quad (5.10) \]

**Proof.** Let us first prove (5.9). Recalling the definition of \( h^0 \) in (3.10), we have

\[ |h^0|^2_{x + \frac{\sigma}{2}} = \|\langle k \rangle^{r + \frac{\sigma}{2}} h_0 \|^2_{x + \frac{\sigma}{2}} \leq C \|\langle k \rangle^{r + \frac{\sigma}{2}} \|^2_{x + \frac{\sigma}{2}} \int_0^1 (G_0 F(u^R \omega^R) dy \|^2_{x + \frac{\sigma}{2}} + C \|\langle k \rangle^{r + \frac{\sigma}{2}} \|^2_{x + \frac{\sigma}{2}} \int_0^1 (G_0 F(\partial_x v^a \cdot \partial_y v^a) dy \|^2_{x + \frac{\sigma}{2}} \leq C \|u^R \omega^R\|_{x + \frac{\sigma}{2}} + C \|\partial_x v^a \cdot \partial_y v^a\|_{x + \frac{\sigma}{2}}. \]

By lemmas 2.2, 4.1 and 5.3, we get

\[ \int_0^t \|u^R \omega^R\|^2_{x + \frac{\sigma}{2}} \, ds \leq C \int_0^t \|\omega^m\|^2_{x + \frac{\sigma}{2}} \, ds + C \int_0^t \|\omega^m\|^2_{x + \frac{\sigma}{2}} \, ds \leq C \int_0^t \|\omega^m\|^2_{x + \frac{\sigma}{2}} \, ds + \frac{C}{\beta^2} \int_0^t \|(h^0, h^1)^2_{x + \frac{\sigma}{2}} \, ds, \]

and

\[ \int_0^t \|\partial_x^{-1} v^a \cdot \partial_y v^a\|^2_{x + \frac{\sigma}{2}} \, ds \leq C \int_0^t \|u^R\|^2_{x + \frac{\sigma}{2}} \, ds \leq C \int_0^t \|u^R\|^2_{x + \frac{\sigma}{2}} \, ds \leq C \int_0^t \|\omega^m\|^2_{x + \frac{\sigma}{2}} \, ds + \frac{C}{\beta^2} \int_0^t \|(h^0, h^1)^2_{x + \frac{\sigma}{2}} \, ds. \]

This shows that

\[ \int_0^t |h^0|^2_{x + \frac{\sigma}{2}} \, ds \leq C \int_0^t \|\omega^m\|^2_{x + \frac{\sigma}{2}} \, ds + \frac{C}{\beta^2} \int_0^t \|(h^0, h^1)^2_{x + \frac{\sigma}{2}} \, ds. \]

Similarly, we have

\[ \int_0^t |h^1|^2_{x + \frac{\sigma}{2}} \, ds \leq C \int_0^t \|\omega^m\|^2_{x + \frac{\sigma}{2}} \, ds + \frac{C}{\beta^2} \int_0^t \|(h^0, h^1)^2_{x + \frac{\sigma}{2}} \, ds. \]

Choosing \( \beta \geq \beta^* \) with \( \beta^* \) suitably large, we deduce that

\[ \int_0^t \|(h^0, h^1)^2_{x + \frac{\sigma}{2}} \, ds \leq C \int_0^t \|\omega^m\|^2_{x + \frac{\sigma}{2}} \, ds, \quad (5.12) \]

if \( 1 - \frac{\beta^*}{\beta} < \frac{\sigma}{2} \) which is equivalent to \( \sigma \geq \frac{4}{7} \).
Next we prove (5.10). We have
\[ \| | \epsilon | D | h^0 | x_r+1, \varphi | = \| | k |^{r+1} \frac{1}{T} \epsilon | k | \hat{h}_k \| L^2 \]
\[ \leq \left\| \left( 1_{|k| \geq N(\epsilon)} + 1_{|k| < N(\epsilon) - 1} \right) | k |^{r+1} \frac{1}{T} \epsilon | k | \int_0^1 G_0 \mathcal{F}(u^R \omega^R) \, dy \right\| L^2 \]
\[ + \int_0^1 \left( 1_{|k| \geq N(\epsilon)} + 1_{|k| < N(\epsilon) - 1} \right) | k |^{r+1} \frac{1}{T} \epsilon | k | \int_0^1 G_0 \mathcal{F}(\partial_x^{-1} u^R \cdot \partial_x \omega^R) \, dy \right\| L^2 \]
\[ = I_1 + I_2 + I_3 + I_4. \]

For $I_1$, due to $|k| \geq N(\epsilon)$, we have $\epsilon (1 + |k|) \geq c(k)^\frac{2}{3}$ and
\[ \| \epsilon | k | \| G_0 \mathcal{F} \| L^2 \leq C \epsilon | k | \min \left\{ 1, \frac{1}{\epsilon (1 + |k|)^{\frac{2}{3}}} \right\} \leq C(k)^{-\frac{2}{3}}, \quad (5.13) \]
which along with lemma 2.3 gives
\[ \int_0^1 I_1^2 \, ds \leq \int_0^1 \| P_{\geq N(\epsilon)} \left( \frac{u^R}{\varphi} \right) \|_{L^2}^2 \, ds \]
\[ \leq C \int_0^1 \| P_{\geq N(\epsilon)} \omega^R \|_{L^2}^2 \, ds + \int_0^1 \| P_{\geq N(\epsilon)} \left( \frac{u^R}{\varphi} \right) \|_{L^2}^2 \, ds \]
\[ \leq C \int_0^1 \| P_{\geq N(\epsilon)} \left( \partial_x u^R, \epsilon^2 \partial_x v^R \right) \|_{L^2}^2 \, ds + \int_0^1 \| P_{\geq N(\epsilon)} \omega^R \|_{L^2}^2 \, ds \]
\[ \leq C \int_0^1 \| P_{\geq N(\epsilon)} \partial_x u^R, \epsilon^2 \partial_x v^R \|_{L^2}^2 \, ds + C \int_0^1 \| \omega^R \|_{L^2}^2 \, ds. \]

For $I_3$, by (5.13) again and Hardy’s inequality
\[ \| \partial_x^{-1} v^R \|_{L^2} \leq C \| \partial_x \partial_x^{-1} v^R \|_{L^2} \leq C \| u^R \|_{L^2}, \]
we obtain
\[ \int_0^1 I_3^2 \, ds \leq C \int_0^1 \| P_{\geq N(\epsilon)} \left( \partial_x^{-1} u^R \cdot \partial_x \omega^R \right) \|_{L^2}^2 \, ds \]
\[ \leq C \int_0^1 \| P_{\geq N(\epsilon)} u^R \|_{L^2}^2 \, ds \leq C \int_0^1 \| u^R \|_{L^2}^2 \, ds. \]

For $I_2$, using the facts that $\epsilon | k | \| G_0 \mathcal{F} \| L^2 \leq C$ and $\epsilon | k | \| G_0 \| L^2 \leq C(\epsilon | k |)^{\frac{1}{2}} \leq C(k)^{\frac{2}{3}}$ due to $|k| \leq 2N(\epsilon)$, we get by lemma 4.1 and (5.9) that
\[ \int_0^1 I_2^2 \, ds \leq \int_0^1 \| P_{\leq 2N(\epsilon)} \left( \frac{u^R}{\varphi} \mathcal{F}(\omega^R) \right) \|_{L^2}^2 \, ds \]
\[ + \int_0^1 \| P_{\leq 2N(\epsilon)} \left( G_0 \left( \frac{u^R}{\varphi} \mathcal{F}(\omega^R) \right) \right) \|_{L^2}^2 \, ds \]
\[ \leq C \int_0^1 \| u^R \|_{L^2}^2 \, ds. \]
Proposition 5.5. Under the assumptions of lemma 5.4, there holds that

\[
\sup_{s \in [0,1]} \| \omega^{h^4}(s) \|_{X^{r-1+\frac{1}{2}}} + \int_0^1 \| (\partial_s, \varepsilon \partial_s) \omega^{h^4} \|_{X^{r-1+\frac{1}{2}}} \, ds + \beta \int_0^1 \left( \| \omega^h \|_{X^{r-1+\frac{1}{2}}} + \| \varphi \omega^{h^4} \|_{X^{r-1+\frac{1}{2}}} \right) \, ds \leq C \frac{1}{\beta^2} \int_0^1 \| \omega^{h^4} \|_{X^{r+\frac{1}{2}}} \, ds.
\]
Let us conclude this section by the estimates of $h^0_1, h^1_1$.

**Lemma 5.6.** Under the assumptions of lemma 5.4, there holds that

$$
\int_0^t (\dot{h}^0_1, h^1_1)^2_{L^2} \, ds \leq C \varepsilon^2 + C \int_0^t \|\omega^0\|^2_{L^2} \, ds + C \int_0^t \|\partial_t \omega^0\|^2_{L^2} \, ds.
$$

**Proof.** Recalling the definition of (3.11), we have

$$
\int_0^t |h^0_1|_{L^2} \, ds \leq \|k\| \int_0^1 (G_2 \mathcal{F}(v^p \omega^0) \omega^0) dy_{L^2} \, ds + \|k\| \int_0^1 (G_0 \mathcal{F}(u^R \partial_x \omega^0) \omega^0) dy_{L^2} \, ds
$$

$$
+ \|k\| \int_0^1 (\mathcal{G}_0 \mathcal{F}(\dot{u}^R - \partial_x \alpha \omega^0) \partial_t \omega^0) \partial_t \omega^0) dy_{L^2} \, ds
$$

$$
+ \varepsilon^4 \|k\| \int_0^1 (\dot{G}_0 \mathcal{F}(\dot{f}_1) \omega^0) dy_{L^2} \, ds + \varepsilon^4 \|k\| \int_0^1 (\dot{G}_0 \mathcal{F}(\dot{f}_2) \omega^0) dy_{L^2} \, ds
$$

$$
= I_1 + I_2 + I_3 + I_4 + I_5.
$$

Using the facts that $\|G_2\|_{L^1} \leq C$ and

$$
\|v^p \omega^0\|^2_{L^2} \leq C \|\partial_t \omega^0\|^2_{L^2} \leq C,
$$

we get by lemmas 4.1 and 5.4 that

$$
I_1 \leq C \int_0^t \|\varphi^2 \omega^0\|^2_{L^2(U^\infty)} \, ds \leq C \int_0^t \|\omega^0\|^2_{L^2(U^\infty)} \, ds + C \int_0^t \|\varphi^2 \omega^0\|^2_{L^2(U^\infty)} \, ds
$$

$$
\leq C \int_0^t (\|\omega^0\|^2_{L^2} + \|\partial_t \omega^0\|^2_{L^2} + \|\partial_x \omega^0\|^2_{L^2}) \, ds + C \int_0^t \|\varphi^2 \omega^0\|^2_{L^2(U^\infty)} \, ds
$$

$$
\leq C \int_0^t (\|\omega^0\|^2_{L^2} + \|\partial_t \omega^0\|^2_{L^2} + \|\partial_x \omega^0\|^2_{L^2}) \, ds + C \|\partial_x \omega^0\|^2_{L^2(U^\infty)} \, ds.
$$

Here we use the Gagliardo–Nirenberg inequality

$$
\|f\|_{L^3(U^\infty)} \leq C \|f\|_{L^1(U^\infty)}^{\frac{1}{2}} \left( \|f\|_{L^2(U^\infty)}^{\frac{1}{2}} + \|\partial_x f\|_{L^2(U^\infty)}^{\frac{1}{2}} \right).
$$

(5.14)

Similar to (5.11), we have

$$
|I_2| + |I_3| \leq C \int_0^t \|\omega^0\|^2_{L^2} \, ds + \frac{C}{\beta^2} \int_0^t (h^0, h^1)^2_{L^2(U^\infty)} \, ds
$$

$$
\leq C \int_0^t \|\omega^0\|^2_{L^2} \, ds.
$$

For $I_4$ and $I_5$, by lemma 2.2 and (3.16), we have

$$
|I_4| \leq C \varepsilon^2 \int_0^t (\varepsilon u^R, \varepsilon v^R) \|\dot{\omega}^0\|_{L^2} \, ds \leq C \varepsilon^2 \int_0^t \|\omega^0\|^2_{L^2} \, ds.
$$
\[ \leq C \int_0^t \left( \| \omega^{in} \|_{X^{\frac{4}{5}}}^2 + \frac{1}{\beta^2} (h^{0}, h^{1})_{X^{\frac{1}{5}+\frac{1}{5}}}^2 \right) ds \leq C \int_0^t \| \omega^{in} \|_{X^{\frac{4}{5}}}^2 ds \]

and

\[ |I_5| \leq C t e^4. \]

Collecting the estimates \( I_1 - I_5 \), we arrive at

\[ \int_0^t \| h^0 \|_{X^{\frac{4}{5}}} ds \leq C t e^4 + C \int_0^t \| \omega^{in} \|_{X^{\frac{4}{5}}} ds + C \int_0^t \| \partial_t \omega^{in} \|_{X^{\frac{4}{5}}} ds. \]

The estimate of \( h^1 \) is similar. \( \square \)

### 6. Energy estimate via the vorticity equation

By the construction of \( \omega^{in} \), we find that

\[
\begin{align*}
\partial_t \omega^{in} - \Delta \omega^{in} &+ u^R \partial_t \omega^{in} + v^R \partial_t \omega^{in} + v^{in} \partial_t \omega^p \\
+ \epsilon^2 (f_1 + f_2) &= N(\omega^R, \omega^R) - u^R \partial_t \omega^bl - u^R \partial_t \omega^pl - v^R \partial_t \omega^bl - v^R \partial_t \omega^pl,
\end{align*}
\]

\[
\partial_t \omega^{in} |_{r=0} = h^0_1 - \epsilon |D| \omega^{y,1} |_{r=0} - \partial_t (\Delta \tau, D)^{-1} (N(\omega^R, \omega^R)) |_{r=0}
\]

\[
- (\partial_x - \epsilon |D|) \omega^{y,0} |_{r=1} + \frac{1}{2 \pi} \int_S \partial_t u^R dx dy.
\]

\[
\partial_t \omega^{in} |_{r=1} = h^1 + \epsilon |D| \omega^{y,1} |_{r=1} - \partial_t (\Delta \tau, D)^{-1} (N(\omega^R, \omega^R)) |_{r=1}
\]

\[
- (\partial_x - \epsilon |D|) \omega^{y,0} |_{r=1} + \frac{1}{2 \pi} \int_S \partial_t u^R dx dy.
\]

\[
\omega^{in} |_{r=0} = 0,
\]

(6.1)

where \((h^0_1, h^1)\) is given by (3.11) and (3.13), \((u^R, v^R)\) is the solution of (1.3) and \( f_1, f_2 \) are given by (3.4) and (3.5). For simplicity, we denote \( \mathcal{N} = N(\omega^R, \omega^R) \).

**Proposition 6.1.** Let \( \sigma \in [\frac{2}{5}, 1] \) and \( r = N_0 - 7 \). Then there exists \( \beta_0 > 1 \) and \( \delta > 0 \) so that for all \( t \in [0, T] \), \( \beta \geq \beta_0 \) and \( \delta \in (0, \delta) \), there holds that

\[
\sup_{s \in [0, t]} \| \omega^{in}(s) \|_{X^0}^2 + \int_0^t \left( \| \partial_t \omega \|_{X^{\frac{4}{5}}}^2 + \beta \int_0^t \| \omega^{in} \|_{X^{\frac{4}{5}+\delta}}^2 ds \right)
\]

\[
\leq C t e^4 + 2 \delta \int_0^t \| \mathcal{N} \|_{X^{\frac{4}{5}+\delta}}^2 ds + C \epsilon^2 \int_0^t \| P_{\geq N_0} (\partial_x \epsilon \partial_x (u^R, v^R)) \|_{X^{\frac{1}{5}+\delta}}^2 ds
\]

\[
+ C \int_0^t \| P_{\geq N_0} (\partial_x \epsilon \partial_x (u^R, v^R)) \|_{X^{\frac{1}{5}+\delta}}^2 ds.
\]
Proof. Firstly, we derive the equation of $\omega_{\Phi}^\infty$:

$$\begin{align*}
\frac{1}{2} \frac{d}{dt} \left\| \frac{D'}{\sqrt{\partial_\omega^2}} \omega_{\Phi}^\infty \right\|_{L^2}^2 + \beta \left\| \frac{D'}{\sqrt{\partial_\omega^2}} \omega_{\Phi}^\infty \right\|_{L^2}^2 + \left\| \frac{\partial_\omega \mathcal{N}}{\sqrt{\partial_\omega^2}} \omega_{\Phi}^\infty \right\|_{L^2}^2 &= 0.
\end{align*}$$

The worst term in the system is $v^{\infty}_0 \partial_\omega^p$. To handle it, we use the hydrostatic trick. Taking $\langle D' \rangle$ on the both sides of (6.2) and taking $L^2$ inner product with $\frac{\partial_\omega \mathcal{N}}{\sqrt{\partial_\omega^2}} (\partial_\omega^p \geq \delta_0)$, we arrive at

$$\begin{align*}
\frac{1}{2} \frac{d}{dt} & \left\| \frac{D'}{\sqrt{\partial_\omega^2}} \omega_{\Phi}^\infty \right\|_{L^2}^2 + \beta \left\| \frac{D'}{\sqrt{\partial_\omega^2}} \omega_{\Phi}^\infty \right\|_{L^2}^2 + \left\| \frac{\partial_\omega \mathcal{N}}{\sqrt{\partial_\omega^2}} \omega_{\Phi}^\infty \right\|_{L^2}^2 \\
&= \int_S \langle D' \rangle \omega_{\Phi}^\infty \cdot (\partial_\omega \mathcal{N}) \frac{\partial_\omega \mathcal{N}}{\sqrt{\partial_\omega^2}} \omega_{\Phi}^\infty dxdy + \int_S \left( \nabla \partial_\omega \mathcal{N} \right) \omega_{\Phi}^\infty (\partial_\omega^p) dxdy.
\end{align*}$$

Integrating on $[0, t]$ with $t \leq T$ and using $\partial_\omega^p \geq \delta_0$, we obtain
\[ \|\omega^{\text{in}}(t)\|_{X^{\epsilon, \frac{1}{2}}}^2 + 2\beta \int_0^t \|\omega^{\text{in}}\|^2_{X^{\epsilon, \frac{1}{2}}} \, ds + 2 \int_0^t \| (\epsilon \partial_1 \omega^{\text{in}}, \partial_3 \omega^{\text{in}}) \|_{X^{\epsilon, \frac{1}{2}}}^2 \, ds \leq C \int_0^t |T^0| + \cdots + |T^{14}| \, ds. \]

Now we estimate \( T^i, i = 0, \ldots, 14 \).

Estimate of \( T^0 \) and \( T^2 \). It is easy to get
\[
\int_0^t |T^0| \, ds \leq C \int_0^t \|\omega^{\text{in}}\|^2_{X^{\epsilon, \frac{1}{2}}} \, ds + \delta \int_0^t \| (\epsilon \partial_1 \omega^{\text{in}}, \partial_3 \omega^{\text{in}}) \|_{X^{\epsilon, \frac{1}{2}}}^2 \, ds,
\]
\[ \int_0^t |T^2| \, ds \leq C \int_0^t \|\omega^{\text{in}}\|^2_{X^{\epsilon, \frac{1}{2}}} \, ds. \]

Estimate of \( T^4 \). By lemmas 4.1, 5.4, 5.6 and (3.15), we have
\[
\int_0^t |T^4| \, ds \leq C \int_0^t \left( \| (h^0, h^1, \partial_3 + \epsilon |D| \omega^{\text{in}}) \|_{X^{\epsilon, \frac{1}{2}}} + \| (\partial_1, \partial_3) \omega^{\text{in}} \|_{X^{\epsilon, \frac{1}{2}}} \right) \, ds
\]
\[ + \| \partial_1 \omega^{\text{in}} \|_{X^{\epsilon, \frac{1}{2}}} \, ds \leq C \int_0^t \left( \| (h^0, h^1, h^1, h^1) \|_{X^{\epsilon, \frac{1}{2}}} + \| \omega^{\text{in}} \|_{X^{\epsilon, \frac{1}{2}}} \, ds \right)
\]
\[ + \int_0^t \int_T \| |D| (D)^\prime \omega_R \frac{(D)^\prime \omega_R}{\partial_1 \omega^{\text{in}}} \|_{L^1} \, dx \, ds \,
\]
\[ \leq \delta \int_0^t \| (\partial_1, \partial_3) \omega^{\text{in}} \|_{X^{\epsilon, \frac{1}{2}}} \, ds \leq C \epsilon^4 + \delta \int_0^t \| \omega^{\text{in}} \|_{X^{\epsilon, \frac{1}{2}}} \, ds \]
\[ + C \int_0^t \| \omega^{\text{in}} \|_{X^{\epsilon, \frac{1}{2}}} \, ds \leq C \int_0^t \| (h^0, h^1) \|_{X^{\epsilon, \frac{1}{2}}} \, ds. \]

where we used
\[
\int_0^t \| \partial_1 \omega^{\text{in}} \|^2_{L^2} \, ds \leq C \int_0^t \| \partial_3 \omega^{\text{in}} \|^2_{L^2} \, ds + \frac{C}{\beta \tau} \int_0^t \| (h^0, h^1) \|_{X^{\epsilon, \frac{1}{2}}} \, ds.
\]

Let \( y_0, y_1 \in [0, 1] \) so that
\[ \| |D| \omega^{\text{in}}(y_0) \|_{X^{\epsilon, \frac{1}{2}}} \leq \| |D| \omega^{\text{in}} \|_{X^{\epsilon, \frac{1}{2}}}, \quad \| \omega^{\text{in}}(y_1) \|_{X^{\epsilon, \frac{1}{2}}} \leq \| \omega^{\text{in}} \|_{X^{\epsilon, \frac{1}{2}}}. \]

Then we infer that
\[
\left| \int_0^t \int_T \| |D| (D)^\prime \omega_R \frac{(D)^\prime \omega_R}{\partial_1 \omega^{\text{in}}} \|_{L^1} \, dx \, ds \right|
\]
\[ \begin{aligned}
&\leq \left| \int_{0}^{\tau} \int_{y_{0}}^{y_{1}} \left( \frac{\partial}{\partial x} |D| \omega_{bl}^{m} \right) \left| \frac{\partial}{\partial y_{n}} |D| \omega_{bl}^{m} \right| \right| \ dx \ dy
+ \left| \int_{0}^{\tau} \int_{y_{0}}^{y_{1}} \left( \frac{\partial}{\partial x} |D| \omega_{bl}^{m} \right) \left| \frac{\partial}{\partial y_{n}} |D| \omega_{bl}^{m} \right| \right| \ dx \ dy
\end{aligned} \]

By lemmas 4.1 and 5.4, we get

\[ \int_{0}^{\tau} \left( \|\varepsilon|D|\omega_{bl}^{m}\|_{X^{r}} + \|\varepsilon|D|\partial_{x}\omega_{bl}^{m}\|_{X^{r}} \right) \ ds \]

\[ \leq C \int_{0}^{\tau} \left( \|\varepsilon|D|(h^{0}, h^{1})\|_{X^{r+1}} \right) \ ds \]

\[ \leq C \int_{0}^{\tau} \left( \|P_{\geq N(e)}(\partial_{x}, \varepsilon|D)(u^{R}, \varepsilon v^{R})\|_{X^{r+1}} + \|\omega_{bl}^{m}\|_{X^{r+1}} \right) \ ds. \]

For \( \|\varepsilon|D|\omega^{m}\|_{X^{r}} \), we divide the frequency into two parts: \( |k| \geq N(e) \) and \( |k| \leq N(e) \). When \( |k| \leq N(e) \), it holds that \( \varepsilon|k| \leq C(k)^{2} \), which gives

\[ \int_{0}^{\tau} \left( \|P_{\leq 2N(e)}|D|\omega^{m}\|_{X^{r}} \right) \ ds \leq C \int_{0}^{\tau} \|\omega^{m}\|_{X^{r+1}} \ ds. \]

For the high frequency part, by lemmas 4.1 and 5.4, we get

\[ \int_{0}^{\tau} \left( \|P_{\geq N(e)}|D|\omega^{m}\|_{X^{r}} \right) \ ds \leq C \varepsilon^{2} \int_{0}^{\tau} \left( \|P_{\geq N(e)}(\partial_{x}, \varepsilon|D)(u^{R}, \varepsilon v^{R})\|_{X^{r+1}} \ ds + \int_{0}^{\tau} \|\varepsilon|D|\omega^{bl}\|_{X^{r}} \ ds \]

\[ \leq C \varepsilon^{2} \int_{0}^{\tau} \left( \|P_{\geq N(e)}(\partial_{x}, \varepsilon|D)(u^{R}, \varepsilon v^{R})\|_{X^{r+1}} \ ds + \int_{0}^{\tau} \|\varepsilon|D|(h^{0}, h^{1})\|_{X^{r+1}} \ ds \]

\[ \leq C \varepsilon^{2} \int_{0}^{\tau} \left( \|P_{\geq N(e)}(\partial_{x}, \varepsilon|D)(u^{R}, \varepsilon v^{R})\|_{X^{r+1}} \ ds 
+ \int_{0}^{\tau} \left( \|P_{\leq 2N(e)}(\partial_{x}, \varepsilon|D)(u^{R}, \varepsilon v^{R})\|_{X^{r+1}} + \|\omega^{m}\|_{X^{r+1}} \ ds \right) \right). \]
Summing up, we arrive at
\[ \int_0^t |T^1| ds \lesssim \delta \int_0^t \| (\partial_{y_x}, \varepsilon \partial_{y_z}) \omega^m \|_{X^r}^2 \, ds + C t e^4 + C \int_0^t \| \omega^m \|_{X^r+\frac{7}{2}}^2 \, dt' \]
\[ + \delta \int_0^t \| \mathcal{N} \|_{X^r-\frac{7}{4}}^2 \, ds \]
\[ + C \varepsilon^2 \int_0^t \| P_{\geq N(\varepsilon)} (\partial_{y_x}, \varepsilon \partial_{y_z}) (u^R, \varepsilon v^R) \|_{X^r+1}^2 \, ds \]
\[ + C \int_0^t \| P_{\geq N(\varepsilon)} (\partial_{y_x}, \varepsilon \partial_{y_z}) (u^R, \varepsilon v^R) \|_{X^r+1}^2 \, ds. \]

**Estimate of \( I^3 \).** By lemma 2.1, we have
\[ \int_0^t |T^3| ds \leq C \int_0^t (\| \omega^m \|_{X^r} + \| \partial_{y_x} \omega^m \|_{X^r}) \| \omega^m \|_{X^r} \, ds \]
\[ \leq \delta \int_0^t \| \partial_{y_x} \omega^m \|_{X^r}^2 \, ds + C \int_0^t \| \omega^m \|_{X^r}^2 \, ds. \]

**Estimate of \( I^4 \).** By lemmas 2.1 and 5.3, we have
\[ \int_0^t |T^4| ds \leq C \int_0^t \| v^m \|_{X^r-1} \| \omega^m \|_{X^r} \, ds \leq C \int_0^t \| \omega^m \|_{X^r}^2 \, ds \]
where we used the fact that \( v^m = \int_0^t \partial_{y_x} u^m \, dz - v^b |_{y=0} \) and
\[ \int_0^t \| v^b |_{y=0} \|_{X^r-1}^2 \, ds \leq C \int_0^t \| (h^0, h^1) \|_{X^r+1}^2 \, ds \leq C \int_0^t \| \omega^m \|_{X^r}^2 \, ds \]
due to \( \sigma \geq \frac{8}{9} \).

**Estimate of \( I^5 \).** For this term, we need to use the hydrostatic trick. Integration by parts gives
\[ T^5 = \int_S \langle D \rangle^T v^m \langle D \rangle^T (\partial_{y_x} u^m \varepsilon + \varepsilon^2 \partial_{y_x} v^m \varepsilon + \varepsilon^2 \partial_{y_x} v^b \rangle \, dx dy \]
\[ = -\int_S \langle D \rangle^T \partial_{y_x} u^m \langle D \rangle^T u^m \, dx dy - \int_S \varepsilon^2 \langle D \rangle^T \partial_{y_x} v^m \langle D \rangle^T v^m \, dx dy \]
\[ - \varepsilon^2 \int_S \langle D \rangle^T v^m \langle D \rangle^T \partial_{y_x} v^m \, dx dy + \int_T \langle D \rangle^T v^m \langle D \rangle^T u^m \rangle_{y=0} \, dx \]
\[ = -\int_T \langle D \rangle^T v^m \langle D \rangle^T \partial_{y_x} v^m \, dx dy + \int_T \langle D \rangle^T v^m \langle D \rangle^T u^m \rangle \, dx \]
\[ = \int_T \langle D \rangle^T v^m (0) \langle D \rangle^T u^m (0) \, dx \]
\[ = T^{51} + T^{52} + T^{53}. \]

We first consider the boundary term \( T^{52} \). Recalling the boundary condition
\[\begin{align*}
\mathbf{u}^{in}(1) &= -\mathbf{u}^{hl}(1) = -(\mathbf{u}^{h0}(1) + \mathbf{u}^{h1}(1)), \\
\mathbf{v}^{in}(1) &= -\mathbf{v}^{hl}(1) = -(\mathbf{v}^{h0}(1) + \mathbf{v}^{h1}(1)),
\end{align*}\]

it follows from lemmas 4.1 and 5.4 that

\[
\int_0^T \left| \mathbf{T}^{i2} \right| \text{d}s \leq C \left( \int_0^T \left| \mathbf{v}^{i2} \right|_{X^{s+1} + \frac{3}{2}} \text{d}s \right)^{\frac{1}{2}} \left( \int_0^T \left| \mathbf{u}^{i1} \right|_{X^{s+1} + \frac{3}{2}} \text{d}s \right)^{\frac{1}{2}}
\]

\[
\leq \frac{C}{\beta^2} \left( \int_0^T \left| (h^0, h^1) \right|^2_{X^{s+3} + \frac{3}{2}} \text{d}s \right)^{\frac{1}{2}} \left( \int_0^T \left| (h^0, h^1) \right|^2_{X^{s+1} + \frac{3}{2}} \text{d}s \right)^{\frac{1}{2}}
\]

\[
\leq \frac{C}{\beta^2} \int_0^T \left\| \omega^{in} \right\|^2_{X^{s+\frac{3}{2}}} \text{d}s
\]

here we used \(3 - 3\sigma \leq \frac{\sigma}{2} \) due to \(\sigma \geq \frac{\sigma}{2} \). Similarly, we have

\[
\int_0^T \left| \mathbf{T}^{i3} \right| \text{d}s \leq \frac{C}{\beta^2} \int_0^T \left\| \omega^{in} \right\|^2_{X^{s+\frac{3}{2}}} \text{d}s.
\]

By lemmas 5.3 and 4.1, we get

\[
\begin{align*}
\int_0^T \left| \mathbf{T}^{i1} \right| \text{d}s &\leq C \int_0^T \left( \left\| \varepsilon \mathbf{v}^{in} \right\|^2_{X^{s+\frac{3}{2}}} + \left\| \varepsilon \partial_x \mathbf{u}^{hl} \right\|^2_{X^{s+\frac{3}{2}}} \right) \text{d}s \\
&\leq C \int_0^T \left\| \omega^{in} \right\|^2_{X^{s+\frac{3}{2}}} \text{d}s + \frac{C}{\beta^2} \int_0^T \left| (h^0, h^1) \right|^2_{X^{s+1} + \frac{3}{2}} \text{d}s
\end{align*}
\]

\[
+ \frac{C}{\beta^2} \int_0^T \left| (h^0, h^1) \right|^2_{X^{s+2} + \frac{3}{2}} \text{d}s
\]

\[
\leq C \int_0^T \left\| \omega^{in} \right\|^2_{X^{s+\frac{3}{2}}} \text{d}s,
\]

here we used \(-\frac{\sigma}{2} + 2 - \frac{3\sigma}{2} \leq \frac{3}{2} \) due to \(\sigma \geq \frac{\sigma}{2} \). This shows that

\[
\int_0^T \left| \mathbf{T}^{i} \right| \text{d}s \leq C \int_0^T \left\| \omega^{in} \right\|^2_{X^{s+\frac{3}{2}}} \text{d}s.
\]

Estimates of \(T^i_{x=6, 7, 8} \). By lemmas 4.1 and 5.4, we get

\[
\begin{align*}
\int_0^T \left| \mathbf{T}^{i6} \right| \text{d}s &\leq C \int_0^T \frac{||\mathbf{u}^h||}{\phi} \partial_x (\varphi \omega^{hl}) \text{d}s + C \int_0^T \left\| \omega^{in} \right\|^2_{X^{s+\frac{3}{2}}} \text{d}s \\
&\leq C \int_0^T \left\| \varphi \omega^{hl} \right\|^2_{X^{s+1} + \frac{3}{2}} \text{d}s + C \int_0^T \left\| \omega^{in} \right\|^2_{X^{s+\frac{3}{2}}} \text{d}s
\end{align*}
\]

\[
\leq C \int_0^T \left| (h^0, h^1) \right|^2_{X^{s+2} + \frac{3}{2}} + C \int_0^T \left\| \omega^{in} \right\|^2_{X^{s+\frac{3}{2}}} \text{d}s \leq C \int_0^T \left\| \omega^{in} \right\|^2_{X^{s+\frac{3}{2}}} \text{d}s,
\]

where we used \(2 - \frac{7\sigma}{4} \leq \frac{3}{2} \) due to \(\sigma \geq \frac{\sigma}{2} \). Similarly, we have
\[
\int_0^t |T^j| \leq C \int_0^t \left| \frac{\partial}{\partial \bar{z}} \mathbf{\omega} \right|_{H^\infty} \left| \frac{\partial}{\partial \bar{z}} \mathbf{\omega} \right|_{H^\infty} ds
\]
\[
\leq C \int_0^t \left| \frac{1}{\beta^2} |(h^0, h^1)|_{X^{\infty}} \left| \frac{\partial}{\partial \bar{z}} \mathbf{\omega} \right|_{H^\infty} ds
\]
\[
\leq C \int_0^t \left| \frac{\partial}{\partial \bar{z}} \mathbf{\omega} \right|_{H^\infty}^2 ds,
\]
and
\[
\int_0^t |T^j| ds \leq C \int_0^t \left| \frac{\partial}{\partial \bar{z}} \mathbf{\omega} \right|_{H^\infty}^2 ds
\]
\[
\leq C \int_0^t \left| \frac{1}{\beta^2} |(h^0, h^1)|_{X^{\infty}} \left| \frac{\partial}{\partial \bar{z}} \mathbf{\omega} \right|_{H^\infty}^2 ds
\]
\[
\leq C \int_0^t \left| \frac{\partial}{\partial \bar{z}} \mathbf{\omega} \right|_{H^\infty}^2 ds.
\]

Estimates of \( T^j \), \( j = 9, 10, 11 \). By lemmas 2.3, 4.1 and 5.4, we have
\[
\int_0^t |T^j| ds \leq C \int_0^t \left| \frac{\partial}{\partial \bar{z}} \mathbf{\omega} \right|_{H^\infty} ds
\]
\[
\leq C \int_0^t \left| \frac{1}{\beta^2} |(h^0, h^1)|_{X^{\infty}} \left| \frac{\partial}{\partial \bar{z}} \mathbf{\omega} \right|_{H^\infty}^2 ds
\]
\[
\leq C \int_0^t \left| \frac{\partial}{\partial \bar{z}} \mathbf{\omega} \right|_{H^\infty}^2 ds.
\]
Due to \( u^R = -\int_0^t \partial_t u^R ds \), we similarly have
\[
\int_0^t |T^{11}| ds \leq C \int_0^t \left| \left( \frac{\partial}{\partial \bar{z}} \mathbf{\omega} \right) \right|_{H^\infty} \left| \frac{\partial}{\partial \bar{z}} \mathbf{\omega} \right|_{H^\infty} ds
\]
\[
\leq C \int_0^t \left| \frac{1}{\beta^2} \left| \frac{\partial}{\partial \bar{z}} \mathbf{\omega} \right|_{H^\infty} ds
\]
\[
\leq C \int_0^t \left| \frac{\partial}{\partial \bar{z}} \mathbf{\omega} \right|_{H^\infty}^2 ds.
\]
and
\[
\int_0^t |T^{10}| ds \leq C \int_0^t \left| \frac{\partial}{\partial \bar{z}} \mathbf{\omega} \right|_{H^\infty} ds
\]
\[
\leq \delta \int_0^t \left| \frac{\partial}{\partial \bar{z}} \mathbf{\omega} \right|_{H^\infty}^2 ds + C \int_0^t \left| \frac{\partial}{\partial \bar{z}} \mathbf{\omega} \right|_{H^\infty}^2 ds.
\]

Estimates of \( T^j \), \( j = 12, 13, 14 \). By lemmas 2.2, 4.1 and 5.3, it is easy to see that
\[
\int_0^t |T^{12}| ds \leq C \int_0^t \left| \frac{\partial}{\partial \bar{z}} \mathbf{\omega} \right|_{H^\infty} ds + C \int_0^t \left| (\mathbf{u}^R, \varepsilon \mathbf{u}^R) \right|_{H^\infty}^2 ds
\]
\[
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\]
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\[
\begin{array}{c}
\int_0^t \| \omega^{in} \|_{X^{r+\frac{\beta}{2}}}^2 \, ds,
\end{array}
\]
and
\[
\int_0^t |T^{13}| \, ds \leq C \epsilon^4 + C \int_0^t \| \omega^{in} \|_{X^{r+\frac{\beta}{2}}}^2 \, ds,
\]
\[
\int_0^t |T^{14}| \leq C \int_0^t \| \omega^{in} \|_{X^{r+\frac{\beta}{2}}}^2 \, ds + \delta \int_0^t \| N \|_{X^{r-\frac{\beta}}^2}^2 \, ds.
\]

Summing up the estimates of \( T^0 \) – \( T^{14} \), and taking \( \beta \) large enough and \( \delta \) small enough, we deduce the desired result. \( \square \)

We directly deduce from propositions 5.5 and 6.1 that

**Corollary 6.2.** Under the assumption of proposition 6.1, there holds that

\[
\begin{align*}
\sup_{s \in [0,1]} \| \omega^R(s) \|_{X^{r+\frac{\beta}{2}}}^2 &+ \int_0^t \| (\partial_s, \epsilon \partial_s) \omega^R \|_{X^{r+\frac{\beta}{2}}}^2 \, ds \\
+ \beta \int_0^t \left( \| \omega^R \|_{X^{r+\frac{\beta}{2}}}^2 + \| \varphi \omega^R \|_{X^{r+\frac{\beta}{2}}}^2 \right) \, ds &\leq C \epsilon^4 + 2 \delta \int_0^t \| N \|_{X^{r-\frac{\beta}}^2}^2 \, ds + C \epsilon^2 \int_0^t \| P_{\geq \kappa} (\partial_s, \epsilon \partial_s) (u^R, \epsilon v^R) \|_{X^{r+1}}^2 \, ds \\
&+ C \int_0^t \| P_{\geq \kappa} (\partial_s, \epsilon \partial_s) (u^R, \epsilon v^R) \|_{X^{r+1}}^2 \, ds.
\end{align*}
\]

**7. Energy estimate via the velocity equation**

In this section, we are devoted to the estimates for the high frequency part of \((u^R, v^R)\). In this case, we can directly use the velocity equation. Recall that \((u^R, v^R)\) satisfies

\[
\begin{align*}
\partial_t u^R - \Delta_x u^R + \partial_x p^R + u^p \partial_x u^R + u^R \partial_x u^p + v^p \partial_x v^R + v^R \partial_x v^p &+ N_u \quad = 0, \\
\epsilon^2 (\partial_t v^R - \Delta_x v^R + u^p \partial_x v^R + u^R \partial_x v^p + v^p \partial_x v^R + v^R \partial_x v^p + N_v + N_u) &+ \partial_x p^R + \epsilon^2 (\partial_t v^p - \epsilon^2 \partial_x^2 v^R - \partial_x^2 v^p + u^p \partial_x v^p + v^p \partial_x v^p) = 0, \\
\partial_t v^R + \partial_x v^R &+ N_v \quad = 0, \\
(u^R, v^R)_{|t=0} &\quad = (u^R, v^R)_{|\gamma=1} = 0, \\
(u^R, v^R)_{|t=0} &\quad = 0.
\end{align*}
\]

(7.1)

Here \((N_u, N_v)\) is nonlinear term given by

\[
\begin{align*}
N_u = u^R \partial_x u^R + v^R \partial_x v^R, \\
N_v = u^R \partial_x v^R + v^R \partial_x v^R.
\end{align*}
\]
Proposition 7.1. Let \( \sigma \in [\frac{1}{4}, 1] \) and \( r = N_0 - 7 \). Then there exist \( \beta_1 \) and \( T_0 \), such that for any \( \delta \in (0, \delta_0) \), \( \beta \geq \beta_1 \) and \( t \in [0, T_0] \), there holds that

\[
\varepsilon^2 \| P_{\geq N_0}(u^0, v^0)(t) \|_{X_{1+2\delta}}^2 + \beta \varepsilon^2 \int_0^t \| P_{\geq N_0}(u^0, v^0) \|_{X_{1+2\delta}}^2 \, ds + \int_0^t \| P_{\geq N_0}(\partial_{r_1} u^0, \partial_{r_1} v^0) \|_{X_{1+2\delta}}^2 \, ds \\
\leq C \int_0^t \| \omega_{\infty} \|_{X_{1+2\delta}}^2 \, ds + \delta \int_0^t \| P_{\geq N_0}(N_0, \varepsilon N_{\infty}) \|_{X_{1+2\delta}}^2 \, ds,
\]

and

\[
\| P_{\geq N_0}(u^0, v^0)(t) \|_{X_{1+2\delta}}^2 + \beta \int_0^t \| P_{\geq N_0}(u^0, v^0) \|_{X_{1+2\delta}}^2 \, ds + \int_0^t \| P_{\geq N_0}(\partial_{r_1} u^0, \partial_{r_1} v^0) \|_{X_{1+2\delta}}^2 \, ds \\
\leq C \int_0^t \| \omega_{\infty} \|_{X_{1+2\delta}}^2 \, ds + \delta \int_0^t \| P_{\geq N_0}(N_0, \varepsilon N_{\infty}) \|_{X_{1+2\delta}}^2 \, ds.
\]

Proof. Acting operator \( e^{\beta(t)} P_{\geq N_0} \) on the first two equations of (7.1) and taking \( H^{r+1,0} \) inner product with \( P_{\geq N_0}(u^0_0, v^0_0) \), we get by integration by parts that

\[
\frac{1}{2} \frac{d}{dt} \| P_{\geq N_0}(u^0, v^0)(t) \|_{X_{1+2\delta}}^2 + \beta \| P_{\geq N_0}(u^0, v^0) \|_{X_{1+2\delta}}^2 + \| P_{\geq N_0}(\partial_{r_1} u^0, \partial_{r_1} v^0) \|_{X_{1+2\delta}}^2 \\
= \langle P_{\geq N_0}(u^0_0, v^0_0), P_{\geq N_0}(u^0_0, v^0_0) \rangle_{H^{r+1,0}} + \frac{1}{2} \langle P_{\geq N_0}(u^0_0, v^0_0), \partial_{r_1} u^0 + \partial_{r_1} v^0 \rangle_{H^{r+1,0}}
\]

This gives

\[
\| P_{\geq N_0}(u^0, v^0)(t) \|_{X_{1+2\delta}}^2 + 2 \beta \int_0^t \| P_{\geq N_0}(u^0, v^0) \|_{X_{1+2\delta}}^2 \, ds
\]

\[
+ 2 \int_0^t \| P_{\geq N_0}(\partial_{r_1} u^0, \partial_{r_1} v^0) \|_{X_{1+2\delta}}^2 \, ds \leq C \int_0^t |S^1| + \cdots + |S^{14}| \, ds.
\]
Thanks to \( \partial_t u^p + \partial_t u^p = 0, \partial_t u^p + \partial_t u^R = 0 \), we have \( S^i = S^2 = 0 \).

Estimate of \( S^i \). We get by lemmas 2.3, 5.3 and (3.16) that

\[
\int_0^t |S^i| \, ds \leq \int_0^t \left( \|P_{N(\varepsilon)} u^R \|_{X_{r+1}^{\frac{1}{2}}} + \|P_{N(\varepsilon)} (u^R \partial_t u^p\|_{X_{r+1}^{\frac{1}{2}}} \right) \, ds \\
- \int_0^t \left( \|P_{N(\varepsilon)} u^R \|_{X_{r+1}^{\frac{1}{2}}} + \|P_{N(\varepsilon)} (u^R \partial_t u^p\|_{X_{r+1}^{\frac{1}{2}}} \right) \, ds \\
\leq \int_0^t \|P_{N(\varepsilon)} u^R \|_{X_{r+1}^{\frac{1}{2}}} + \|P_{N(\varepsilon)} (u^R \partial_t u^p\|_{X_{r+1}^{\frac{1}{2}}} \right) \, ds \\
+ \int_0^t \|P_{N(\varepsilon)} (u^R \partial_t u^p\|_{X_{r+1}^{\frac{1}{2}}} + \|P_{N(\varepsilon)} (u^R \partial_t u^p\|_{X_{r+1}^{\frac{1}{2}}} \right) \, ds \\
\leq \int_0^t \|P_{N(\varepsilon)} u^R \|_{X_{r+1}^{\frac{1}{2}}} + \|P_{N(\varepsilon)} (u^R \partial_t u^p\|_{X_{r+1}^{\frac{1}{2}}} \right) \, ds \\
+ \|\omega^R\|_{X_{r+1}^{\frac{1}{2}}} \right) \, ds.
\]

Here we used \( 2 - \sigma \leq 1 + \frac{1}{2} \) and \( \varepsilon |k| \geq (k)^{\frac{1}{2}} \) for \( |k| \geq N(\varepsilon) \). Similarly, we have

\[
\int_0^t |S^i| \, ds \leq C \int_0^t \left( \|P_{N(\varepsilon)} (u^R \|_{X_{r+1}^{\frac{1}{2}}} + \|P_{N(\varepsilon)} u^R \|_{X_{r+1}^{\frac{1}{2}}} + \|\omega^R\|_{X_{r+1}^{\frac{1}{2}}} \right) \, ds,
\]

and

\[
\int_0^t |S^i| + |S^i| \, ds \leq C \int_0^t \left( \|P_{N(\varepsilon)} u^R \|_{X_{r+1}^{\frac{1}{2}}} + \|P_{N(\varepsilon)} u^R \|_{X_{r+1}^{\frac{1}{2}}} + \|\omega^R\|_{X_{r+1}^{\frac{1}{2}}} \right) \, ds,
\]

Estimate of \( S^i \). By lemmas 2.3, 5.3 and (3.16), we have

\[
\int_0^t |S^i| \, ds \leq C \int_0^t \left( \|P_{N(\varepsilon)} u^R \|_{X_{r+1}^{\frac{1}{2}}} + \|P_{N(\varepsilon)} u^R \|_{X_{r+1}^{\frac{1}{2}}} + \|\omega^R\|_{X_{r+1}^{\frac{1}{2}}} \right) \, ds,
\]

\[
\int_0^t |S^i| + |S^i| \, ds \leq C \int_0^t \left( \|P_{N(\varepsilon)} u^R \|_{X_{r+1}^{\frac{1}{2}}} + \|P_{N(\varepsilon)} u^R \|_{X_{r+1}^{\frac{1}{2}}} + \|\omega^R\|_{X_{r+1}^{\frac{1}{2}}} \right) \, ds.
\]

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Similarly, we have

\[
\int_0^t |S^0| \, ds \leq C \int_0^t \left( \|P_{\geq N(c)}(\varepsilon v^R)\|_{X_{r+1}^{-1}}^2 + \|P_{\leq N(c)}(\varphi \omega^R)\|_{X_{\frac{4}{3}+}}^2 \right) \, ds,
\]

and

\[
\int_0^t |S^0| \, ds \leq C \int_0^t \|P_{\geq N(c)}u^R\|_{X_{r+1}^{-1}}^2 \, ds
\]

\[
+ \delta \int_0^t \left( \|P_{\geq N(c)}(\partial_x u^R)\|_{X_{r+1}^{-1}}^2 + \|P_{\leq N(c)}(\partial_x u^R)\|_{X_{\frac{4}{3}+}}^2 + \|\partial_x u^R\|_{X_{\frac{4}{3}+}}^2 \right) \, ds
\]

\[
\leq C \int_0^t \left( \|P_{\geq N(c)}u^R\|_{X_{r+1}^{-1}}^2 + \|P_{\leq N(c)}(\partial_x u^R)\|_{X_{\frac{4}{3}+}}^2 + \|\partial_x u^R\|_{X_{\frac{4}{3}+}}^2 \right) \, ds
\]

\[
+ \delta \int_0^t \|P_{\geq N(c)}(\partial_x u^R)\|_{X_{r+1}}^2 \, ds,
\]

and

\[
\int_0^t |S^{10}| \, ds \leq C \int_0^t \left( \|P_{\geq N(c)}(\varepsilon v^R)\|_{X_{r+1}^{-1}}^2 + \|P_{\leq N(c)}(\varphi \omega^R)\|_{X_{\frac{4}{3}+}}^2 + \|\omega^R\|_{X_{\frac{4}{3}+}}^2 \right) \, ds
\]

\[
+ \delta \int_0^t \|P_{\geq N(c)}(\varepsilon \partial_x u^R)\|_{X_{r+1}}^2 \, ds.
\]

Estimate of \(S^{11} - S^{12}\). It is easy to see that

\[
\int_0^t |S^{11}| + |S^{12}| \, ds \leq Ct \varepsilon^4 + C \int_0^t \|P_{\geq N(c)}u^R\|_{X_{r+1}^{-1}}^2 \, ds,
\]

where we used \(v^R = -\int_0^t \partial_x u^R \, dy\) and integration by parts for \(S^{12}\).

Estimate of \(S^{13} - S^{14}\). It is easy to see that

\[
\int_0^t |S^{13}| + |S^{14}| \, ds \leq C \int_0^t \|P_{\geq N(c)}(u^R, \varepsilon v^R)\|_{X_{r+1}^{-1}}^2 \, ds
\]

\[
+ \delta \int_0^t \|P_{\geq N(c)}(\mathcal{N}_a, \varepsilon \mathcal{N}_e)\|_{X_{r+1}^{-1}} \, ds.
\]

Summing up the estimates of \(S^1 - S^{14}\), and then taking \(\beta\) large enough and \(\delta\) small enough, we arrive at

\[
\|P_{\geq N(c)}(u^R, \varepsilon v^R)(t)\|_{X_{r+1}}^2 + \beta \int_0^t \|P_{\geq N(c)}(u^R, \varepsilon v^R)\|_{X_{r+1}^{-1}}^2 \, ds
\]

\[
+ \int_0^t \|P_{\geq N(c)}(\partial_x, \varepsilon \partial_x)(u^R, \varepsilon v^R)\|_{X_{r+1}}^2 \, ds
\]

\[
\leq C \int_0^t \left( \|P_{\leq N(c)}(\varphi \omega^R)\|_{X_{r+1}}^2 + \|P_{\leq N(c)}(\varphi \omega^R)\|_{X_{\frac{4}{3}+}}^2 + \|\omega^R\|_{X_{\frac{4}{3}+}}^2 \right) \, ds
\]

\[
(7.2)
\]

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By lemmas 4.1 and 5.4, we have
\[
\int_0^t \| \varphi \omega^R \|^2_{X^{r+\frac{3}{2}}} \, ds \leq C \int_0^t \| \omega^m \|^2_{X^{r+\frac{3}{2}}} + \| \varphi \omega^R \|^2_{X^{r+\frac{3}{2}}} \, ds
\]
\[
\leq C \int_0^t \| \omega^m \|^2_{X^{r+\frac{3}{2}}} \, ds, \quad \int_0^t \| \omega^R \|^2_{X^{r+\frac{3}{2}}} \, ds
\]
\[
\leq \int_0^t \| \omega^R \|^2_{X^3} \, ds \leq \int_0^t \| \omega^m \|^2_{X^3} + \| \omega^R \|^2_{X^3} \, ds
\]
\[
\leq C \int_0^t \| \omega^m \|^2_{X^{r+\frac{3}{2}}} \, ds,
\]
which along with \( \varepsilon |k| \leq \langle k \rangle^\frac{3}{2} \) for \( |k| \leq N(\varepsilon) \) give
\[
\int_0^t \varepsilon^2 \| P_{\leq N(\varepsilon)}(\varphi \omega^R) \|^2_{X^{r+1}} + \varepsilon^2 \| P_{\leq N(\varepsilon)} \omega^R \|^2_{X^{r+1+\frac{3}{2}}} \, ds \leq C \int_0^t \| \omega^m \|^2_{X^{r+\frac{3}{2}}} + \| \omega^R \|^2_{X^3} \, ds \leq C \int_0^t \| \omega^m \|^2_{X^{r+\frac{3}{2}}} \, ds.
\]
Therefore, it holds that
\[
\varepsilon^2 \| P_{\geq N(\varepsilon)}(u^R, \varepsilon v^R)(t) \|^2_{X^{r+1}} + \beta \varepsilon^2 \int_0^t \| P_{\geq N(\varepsilon)}(u^R, \varepsilon v^R) \|^2_{X^{r+1+\frac{3}{2}}} \, ds
\]
\[
+ \int_0^t \varepsilon^2 \| P_{\geq N(\varepsilon)}(\partial_y, \varepsilon \partial_x)(u^R, \varepsilon v^R) \|^2_{X^{r+1}} \, ds
\]
\[
\leq C \int_0^t \| \omega^m \|^2_{X^{r+\frac{3}{2}}} \, ds + \delta \int_0^t \| P_{\geq N(\varepsilon)}(N_x, \varepsilon N_v) \|^2_{X^{r+1+\frac{3}{2}}} \, ds,
\]
which gives the first result.

The second result follows by taking \( r - \sigma \) instead of \( r \) in (7.2) and noticing that
\[
\int_0^t \| P_{\leq N(\varepsilon)}(\varphi \omega^R) \|^2_{X^{r+1-\sigma}} + \| P_{\leq N(\varepsilon)} \omega^R \|^2_{X^{r+1+\frac{3}{2}}} \, ds
\]
\[
\leq C \int_0^t \| \varphi \omega^R \|^2_{X^{r-\frac{3}{2}}} + \| \omega^R \|^2_{X^3} \, ds
\]
\[
\leq C \int_0^t \| \omega^m \|^2_{X^{r+\frac{3}{2}}} \, ds.
\]
This completes the proof of the proposition. \( \square \)

8. Nonlinear estimates

In this section, we estimate nonlinear terms \( (N, N_x, N_v) \), which are defined by
\[
N = -u^R \partial_y \omega^R - v^R \partial_x \omega^R.
\]
\[ N_u = u^R \partial_t u^R + v^R \partial_t v^R, \quad N_v = u^R \partial_t v^R + v^R \partial_t v^R. \]

For this, let us first assume the following energy bounds:

\[ \sup_{s \in [0,t]} \| \omega^R(s) \|_{X^r}^2 + \int_0^t \| (\partial_x \omega^R, \varepsilon \partial_y \omega^R) \|_{X^r}^2 \leq C \varepsilon^4 \tag{8.1} \]

for any \( t \in [0, T] \).

**Proposition 8.1.** Under the assumption (8.1), there holds that

\[
\int_0^t \| \mathcal{N} \|_{X^r}^2 \, ds \leq C \sup_{s \in [0,t]} \left( \| P_{\geq N(c)} u^R \|_{X^r}^2 + \| \omega^m \|_{X^r}^2 \right) + C \int_0^t \varepsilon^2 \| P_{\geq N(c)} (\partial_x, \varepsilon \partial_y) u^R \|_{X^r}^2 + \| \partial_x \omega^m \|_{X^r}^2 + \| \omega^m \|_{X^r}^2 \, ds,
\]

\[
\int_0^t \| P_{\geq N(c)} (N_u, \varepsilon N_v) \|_{X^r}^2 \, ds \leq C \int_0^t \varepsilon^2 \| P_{\geq N(c)} (\partial_x, \varepsilon \partial_y) (u^R, \varepsilon v^R) \|_{X^r}^2 + \| \omega^m \|_{X^r}^2 \, ds,
\]

for any \( t \in [0, T] \).

**Proof.** By the definition of \( \mathcal{N} \), we have

\[
\int_0^t \| \mathcal{N} \|_{X^r}^2 \, ds \leq \int_0^t \| u^R \partial_t u^R \|_{X^r}^2 \, ds + \int_0^t \| v^R \partial_t v^R \|_{X^r}^2 \, ds = I_1 + I_2.
\]

It follows from lemma 2.2 and (3.16) that

\[
I_1 \leq \int_0^t \frac{u^R}{\varepsilon} \left( \frac{\| \partial_x \omega^R \|_{X^r}^2}{\varepsilon} + \frac{\| \partial_y \omega^R \|_{X^r}^2}{\varepsilon} \right) + \frac{\| \partial_x \omega^R \|_{X^r}^2}{\varepsilon} + \frac{\| \partial_y \omega^R \|_{X^r}^2}{\varepsilon} + \frac{\| \omega^R \|_{X^r}^2}{\varepsilon} \, ds
\]

\[
\leq C \int_0^t \varepsilon^2 \| \partial_x \omega^R \|_{X^r}^2 + \| \omega^R \|_{X^r}^2 \, ds
\]

from which and the fact that

\[
\| \varepsilon^2 \partial_x \omega^R \|_{X^r}^2 \leq \| \varepsilon^2 P_{\geq N(c)} \partial_x \omega^R \|_{X^r}^2 + \| \varepsilon^2 P_{\leq N(c)} \partial_x \omega^R \|_{X^r}^2
\]

\[
\leq C \varepsilon \| P_{\geq N(c)} (\partial_x, \varepsilon \partial_y) (u^R, \varepsilon v^R) \|_{X^r}^2 + C \| \omega^m \|_{X^r}^2 \, ds,
\]

we infer that

\[
I_1 \leq C \int_0^t \varepsilon^2 \| P_{\geq N(c)} (\partial_x, \varepsilon \partial_y) (u^R, \varepsilon v^R) \|_{X^r}^2 + \| \omega^m \|_{X^r}^2 \, ds.
\]
By lemma 2.2 again, we get
\[
I_2 \leq \int_0^t \int \| \frac{\partial \omega^R}{\partial t} \|^2_{L^2(X)} + \| \varphi \partial_t \omega^R \|^2_{X^{-\frac{d}{2}}} + \| \varphi v^R \|^2_{L^2(U^R)} \frac{\| \partial_t \omega^R \|}{\varepsilon} \| \partial_t \omega^R \|^2_{X^{-\frac{d}{2}}} ds
\leq \int_0^t \| \omega^R \|^2_{X^{\frac{d}{2}}} + \| \varphi \partial_t \omega^R \|^2_{X^{-\frac{d}{2}}} + \left( \| \varphi v^R \|^2_{X^{-\frac{d}{2}}} + \| \varphi \partial_t u^R \|^2_{X^{-\frac{d}{2}}} \right) \frac{\| \partial_t \omega^R \|}{\varepsilon} \| \partial_t \omega^R \|^2_{X^{-\frac{d}{2}}} ds
\leq C \int_0^t \varepsilon^4 \| \varphi \partial_t \omega^R \|^2_{X^{-\frac{d}{2}}} ds + \sup_{s \in [0,t]} \varepsilon^2 \| \varepsilon \partial_t u^R \|^2_{X^{-\frac{d}{2}}}.
\]

On the other hand, by lemmas 4.1, 5.4 and (3.16), we have
\[
\int_0^t \| \varphi \partial_t \omega^R \|^2_{X^{-\frac{d}{2}}} \leq C \int_0^t \| \partial_t \omega^R \|^2_{X^{-\frac{d}{2}}} \leq C \int_0^t \| \partial_t \omega^R \|^2_{X^{-\frac{d}{2}}} + 2 \| \omega^R \|^2_{X^{-\frac{d}{2}}} ds
\]
\[
\leq C \int_0^t \| \partial_t \omega^R \|^2_{X^{-\frac{d}{2}}} + \| \varepsilon \partial_t u^R \|^2_{X^{-\frac{d}{2}}} ds,
\]
and
\[
\| \varepsilon^2 \partial_t u^R \|^2_{X^{-\frac{d}{2}}} \leq \| \varepsilon^2 P_{\geq N(\varepsilon)} \partial_t u^R \|^2_{X^{-\frac{d}{2}}} + \| \varepsilon^2 P_{< N(\varepsilon)} \partial_t u^R \|^2_{X^{-\frac{d}{2}}}
\leq C \| \varepsilon P_{\geq N(\varepsilon)} \partial_t u^R \|^2_{X^{-\frac{d}{2}}} + \| \varepsilon \partial_t u^R \|^2_{X^{-\frac{d}{2}}}
\leq C \| P_{\geq N(\varepsilon)} \partial_t u^R \|^2_{X^{\frac{d}{2}}} + \| \varepsilon \partial_t u^R \|^2_{X^{\frac{d}{2}}}.
\]

This shows that
\[
I_2 \leq C \varepsilon^4 \int_0^t \| \partial_t \omega^R \|^2_{X^{-\frac{d}{2}}} + \| \omega^R \|^2_{X^{-\frac{d}{2}}} ds + C \sup_{s \in [0,t]} \left( \| P_{\geq N(\varepsilon)} \partial_t u^R \|^2_{X^{\frac{d}{2}}} + \| \varepsilon \partial_t u^R \|^2_{X^{\frac{d}{2}}} \right).
\]

Now the first inequality follows from the estimates of $I_1$ and $I_2$. Next we estimate $N_d$. Recalling that $N_d = u^s \partial_t u^R + \varepsilon^R \partial_t u^R$, we have
\[
\int_0^t \| P_{\geq N(\varepsilon)} N_d \|^2_{X^{\frac{d}{2}}} ds \leq \int_0^t \| P_{\geq N(\varepsilon)} (\partial_t u^R) \|^2_{X^{\frac{d}{2}}} ds + \int_0^t \| P_{\geq N(\varepsilon)} u^R \|^2_{X^{\frac{d}{2}}} ds
= I_3 + I_4.
\]

By lemma 2.2 and (3.16), we have
\[
I_3 \leq \int_0^t \int \| u^R \|^2_{L^2(U^R)} \frac{1}{\varepsilon} + \| P_{\geq N(\varepsilon)} u^R \|^2_{X^{\frac{d}{2}}} \frac{1}{\varepsilon} \| \partial_t \omega^R \|^2_{X^{-\frac{d}{2}}} \| \partial_t \omega^R \|^2_{X^{-\frac{d}{2}}} ds
\leq C \int_0^t \| \omega^R \|^2_{X^{\frac{d}{2}}} + \| \partial_t u^R \|^2_{X^{\frac{d}{2}}} ds
\leq C \int_0^t \varepsilon^4 \| P_{\geq N(\varepsilon)} \partial_t u^R \|^2_{X^{\frac{d}{2}}} ds.
\]
Proof of theorem 1.2

This section is devoted to proving theorem 1.2.

- **Local well-posedness.** The local well-posedness of the anisotropic Navier–Stokes equations in the Gevrey class can be proved by a standard energy method. Here we omit the details. Let \( T_1 \) be the maximal existence time of the solution.

- **Bootstrap assumption:**

\[
\sup_{s \in [0, t]} \left\| \omega^R(s) \right\|_{X^r-1} + \int_0^t \left\| (\partial_s \omega^R, \varepsilon \partial_s \omega^R) \right\|_{X^r-1}^2 \leq C \varepsilon^4
\]

for any \( t \in [0, T_1] \). Here \( C \) is determined later.

- **Energy functional:**

\[
E(t) = \left\| \omega^R(t) \right\|_{X^r}^2 + A \varepsilon^2 \left\| P_{\geq N(\varepsilon)}(u^R, \varepsilon v^R)(t) \right\|_{X^r+1}^2
\]

\[
+ \left\| P_{\geq N(\varepsilon)}(u^R, \varepsilon v^R)(t) \right\|_{X^{r+1}}^2,
\]

\[
G(t) = \left\| \omega^{in}(t) \right\|_{X^{r+\frac{5}{2}}}^2 + A \varepsilon^2 \left\| P_{\geq N(\varepsilon)}(u^R, \varepsilon v^R)(t) \right\|_{X^{r+1}}^2.
\]
\[ D(t) = \| (\partial_y, \varepsilon \partial_x) \omega_R(t) \|_{L^2}^2 + A \varepsilon^2 \| P_{\geq N(\varepsilon)} (\partial_y, \varepsilon \partial_x) (u^R, \varepsilon v^R) \|_{X^{r+1}}^2 \]

where \( A \) is a large constant determined later.

**Energy estimates.** It follows from propositions 6.1, 7.1 and 8.1 that

\[
\sup_{s \in [0,t]} E(s) + \beta \int_0^t G(s) ds + \int_0^t D(s) ds \\
\leq C \int_0^t G(s) ds + \left( C_A + C_1 \delta \right) \int_0^t D(s) ds + Ct \varepsilon^4 + C_1 \delta \sup_{s \in [0,t]} E(s).
\]

Here \( C_1 \) is independent of \( \delta \). Taking \( \beta \) and \( A \) large enough and \( \delta \) small enough, we obtain

\[
\sup_{s \in [0,t]} E(s) + \beta \int_0^t G(s) ds + \int_0^t D(s) ds \leq Ct \varepsilon^4 \tag{9.1}
\]

for any \( t \in [0, T_1] \).

**Improving the bootstrap assumption.** It follows from corollary 6.2 and (9.1) that

\[
\sup_{s \in [0,t]} \| \omega_R(s) \|_{X^{r-1}}^2 + \int_0^t \| (\partial_y \omega_R, \varepsilon \partial_x \omega_R) \|_{X^{r-1}}^2 \leq Ct \varepsilon^4 \leq \frac{\mathcal{C}}{2} \varepsilon^4.
\]

by choosing \( \mathcal{C} \) so that \( \mathcal{C} \geq 2CT \). This in particular implies that \( T_1 \geq T \).

**Stability in \( L^2 \cap L^\infty \).** By the Sobolev embedding, we get

\[
\| (u^R, \varepsilon v^R) \|_{L^2_x L^\infty_y} \leq C \varepsilon^2.
\]

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**References**

[1] Alexandre R, Wang Y, Xu C-J and Yang T 2015 Well-posedness of the Prandtl equation in Sobolev spaces *J. Am. Math. Soc.* 28 745–84

[2] Bahouri H, Chemin J Y and Danchin R 2011 *Fourier Analysis and Nonlinear Partial Differential Equations* (Grundlehren der mathematischen Wissenschaften 343) (Berlin: Springer)

[3] Chen D, Wang Y and Zhang Z 2018 Well-posedness of the linearized Prandtl equation around a non-monotonic shear flow *Ann. Inst. Henri Poincare* C 35 1119–42

[4] Dietert H and Gérard-Varet D 2019 Well-posedness of the Prandtl equation without any structural assumption *Ann. PDE* 5 8

[5] Gérard-Varet D and Masmoudi N 2015 Well-posedness for the Prandtl system without analyticity or monotonicity *Ann. Sci. École Norm. Sup.* 48 1273–325

[6] Gérard-Varet D, Masmoudi N and Vicol V 2018 Well-posedness of the hydrostatic Navier–Stokes equations (arXiv:1804.04489)
[7] Grenier E, Guo Y and Nguyen T T 2016 Spectral instability of general symmetric shear flows in a two-dimensional channel Adv. Math. 292 52–110

[8] Lagrée P-Y and Lorthois S 2005 The RNS/Prandtl equations and their link with other asymptotic descriptions: application to the wall shear stress scaling in a constricted pipe Int. J. Eng. Sci. 43 352–78

[9] Li W and Yang T 2016 Well-posedness in Gevrey space for the Prandtl equations with non-degenerate critical points (arXiv:1609.08430)

[10] Maekawa Y 2014 On the inviscid limit problem of the vorticity equations for viscous incompressible flows in the half-plane Commun. Pure Appl. Math. 67 1045–128

[11] Masmoudi N and Wong T K 2015 Local-in-time existence and uniqueness of solutions to the Prandtl equations by energy methods Commun. Pure Appl. Math. 68 1683–741

[12] Oleinik O 1967 On the mathematical theory of boundary layer for an unsteady flow of incompressible fluid J. Appl. Math. Mech. 30 951–74

[13] Paicu M, Zhang P and Zhang Z 2019 On the hydrostatic approximate of the Navier–Stokes equations in a thin strip (arXiv:1904.04438)

[14] Renardy M 2009 Ill-posedness of the hydrostatic Euler and Navier–Stokes equations Arch. Ration. Mech. Anal. 194 877–86

[15] Wang C, Wang Y and Zhang Z 2017 Zero-viscosity limit of the Navier–Stokes equations in the analytic setting Arch. Ration. Mech. Anal. 224 555–95