Weak field limit of Reissner-Nordström black hole lensing

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Abstract

We study gravitational lensing by a Reissner-Nordström (RN) black hole in the weak field limit. We obtain the basic equations for the deflection angle and time delay and find analytical expressions for the positions and amplifications of the primary and secondary images. Due to a net positive charge, the separation between images increases, but no change in the total magnification occurs.

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I. INTRODUCTION

Within the next decade, the accuracy of space-based missions dedicated to measuring astrometric positions, parallaxes, and proper motions of stars is expected to attain 1 microarcsecond (µas). Thus progress in observational techniques has made it necessary to take into account many subtle relativistic effects in light propagation. Lensing by massive black holes requires the full relativistic corrections, demanding a treatment of lensing theory to the proper order of approximation at which interesting effects appear for any particular phenomenon.

The gravitational lensing can reveal an unique tool to detect exotic objects in the universe, that, though not yet observed, are not forbidden on a theoretical ground. In addition to the primary and secondary images, a spherically symmetric black hole produces two infinite series of faint relativistic images, formed by light rays winding around the black hole at distances comparable to the gravitational radius. The weak field theory of gravitational lensing suffices to describe the primary and secondary images, but strong field lensing is demanded to consider phenomena close to the horizon. Since the relativistic images are strongly demagnified and the astrometric separations among all the images are really small, a proper consideration of the full system of images requires a treatment to higher orders of approximation of the images formed in the weak field regime.

In the present work, we wish to study the lensing situation then the deflector is a charged or Reissner-Nordström (RN) black hole. Charged rotating black holes are plausible endpoints of the catastrophic gravitational collapse of the most massive magnetized rotating stars. When surrounded by a co-rotating magnetosphere of equal and opposite charge, the system attains a minimum energy configuration. The magnetosphere preserves the black hole from a neutralization due to a selective accretion of charge from the environment and the black hole can be quite stable in a typical astrophysical environment of low density [6].

In this paper, we propose a comprehensive analysis of the weak field limit of gravitational lensing due to a charged black hole. Gravitational lensing in the strong field scenario when the lens is a RN black hole has been discussed in Eiroa et al. [3], who argued that rotation does not lead to qualitatively different results. They found analytical expressions for the positions and amplifications of the two sets of relativistic images, but they did not discuss the effect of a charge on the primary and secondary images.
Bhadra discussed gravitational lensing due to a charged black hole of heterotic string theory with the aim of examining the possible string effects in a strong-field observation, but no significant signatures emerged in lensing observables.

In Section II we perform the weak field limit of the RN metric. In Section III following Fermat’s principle, we calculate analytical expressions for both time delay and deflection angle. In Section IV using a perturbative method, we calculate the positions and magnification of the images. Section V is devoted to some final considerations. In this paper, we will use geometrized units (speed of light in vacuum \( c = 1 \) and gravitational constant \( G = 1 \)).

II. THE RN METRIC

The RN metric is a spherically symmetric solution of the coupled equations of Einstein and of Maxwell. It represents a black hole with a mass \( M \) and a charge \( Q \). The RN metric, in its standard form, reads

\[
ds^2 = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 - r^2 \left(\sin^2 \theta d\phi^2 + d\theta^2\right).
\]

(1)

The metric in Eq. (1) can be expressed in an equivalent isotropic form by introducing a new radius variable, \( \rho \),

\[
r = \rho \left(1 + \frac{M}{\rho} + \frac{M^2 - Q^2}{4\rho^2}\right),
\]

(2)

Substituting Eq. (2) in Eq. (1) gives

\[
ds^2 = \left[\frac{M^2 - 4\rho^2 - Q^2}{(M + 2\rho)^2 - Q^2}\right]^2 dt^2 - \left[1 + 2M \frac{1}{\rho} + \frac{M^2 - Q^2}{4\rho^2}\right]^2 \left\{d\rho^2 + \rho^2 \left(\sin^2 \theta d\phi^2 + d\theta^2\right)\right\}.
\]

(3)

Finally, in quasi-Minkovskian coordinates, the weak field limit of Eq. (3) reads

\[
ds^2 \simeq \left[1 - \frac{2M}{x} + \left(1 + \frac{a^2}{2}\right) \frac{2M^2}{x^2}\right] dx^2 - \left[1 + \frac{2M}{x} + \left(1 - \frac{a^2}{3}\right) \frac{3M^2}{2 x^2}\right] dx^2,
\]

(4)

where \( a \equiv Q/M \leq 1 \).


III. LENSING QUANTITIES

The time delay of the deflected path $p$ relative to the unlensed ray $p_0$ is

$$\Delta T \equiv \int_p n \, dl_p - \int_{p_0} dl_p,$$

being $n$ the effective index of refraction, defined as

$$n \equiv -\frac{g_{i0}}{g_{00}} e^i + \frac{1}{\sqrt{g_{00}}},$$

where $e^i \equiv \frac{dx^i}{dl_p}$ is the unit tangent vector of a ray; $dl_p^2 \equiv \left(-g_{ij} + \frac{g_{00}}{g_{00}}\right) dx^i dx^j$ defines the spatial metric.

For the RN metric, we get

$$dl_p \simeq \left\{1 + M \frac{x}{x} + \frac{1}{4} \frac{a^2 M^2}{x^2}\right\} dl_{eucl},$$

where $dl_{eucl} \equiv \sqrt{\delta_{ij} dx^i dx^j}$ is the Euclidean arc length, and

$$n \simeq 1 + \frac{M}{x} + \frac{1}{2} \frac{a^2 M^2}{x^2}.$$  

Equation (5) can be expressed as a sum of geometrical and potential time delays

$$\Delta T = \Delta T_{\text{geom}} + \Delta T_{\text{pot}}.$$  

The geometrical time delay,

$$\Delta T_{\text{geom}} \equiv \int_p dl_p - \int_{p_0} dl_p,$$

is due to the extra path length relative to the unperturbed ray $p_0$.

It is useful to employ the spatial orthogonal coordinates $(\xi_1, \xi_2, l)$, centred on the lens and such that the $l$-axis is along the incoming light ray direction $e_\text{in}$. The three-dimensional position vector $x$ can be expressed as $x = \xi + l e_\text{in}$. The lens plane, $(\xi_1, \xi_2)$, corresponds to $l = 0$. With these assumptions, the geometrical time delay is

$$\Delta T_{\text{geom}} \simeq \frac{1}{2} \frac{D_d D_s}{D_{ds}} \left| \frac{\xi}{D_d} - \frac{\eta}{D_s} \right|^2,$$

where $D_s$ is the distance from the observer to the source, $D_d$ is the distance from the observer to the deflector and $D_{ds}$ is the distance from the deflector to the source; $\eta$ is the two-dimensional vector position of the source in the source plane.
The potential time delay $\Delta T_{\text{pot}}$ accounts for the retardation of the deflected ray caused by the gravitational field of the lens; it is defined as the difference between the total travel time and the integral of the line element along the deflected path,

$$\Delta T_{\text{pot}} \equiv \int_p n \, dl_p - \int_p dl_p.$$  \hspace{1cm} (11)

To calculate properly the potential time delay, we have to consider the time delay to the post-post-Newtonian (ppN) order $8$. The first contribution derives from the integration of the Newtonian potential on the deflected path, calculated at order $G$. Non-linear interaction of matter with space-time, represented in the metric element by terms which contain $1/x^2$, also contributes. Finally, higher order corrections to the geometrical time delay enter the ppN time delay, since the difference in the distances of closest approach of the deflected and undeflected light rays represents a post-Newtonian quantity and first-order corrections in the calculation of this difference induce a ppN contribution to the time delay.

After lengthy but straightforward calculations, we get

$$\Delta T \simeq -4M \ln \left( \frac{\xi}{\xi_0} \right) + \frac{3\pi}{4} (5 - a^2) M^2 \frac{1}{\xi},$$  \hspace{1cm} (12)

where $\xi_0$ is a scale-length in the lens plane and $\xi$ is the impact parameter.

From the Fermat’s principle $5, 7$, we can derive the deflection angle, i.e. the difference of the initial and final ray direction,

$$\hat{\alpha} \equiv -\nabla_\xi \Delta T_{\text{pot}}.$$  \hspace{1cm} (13)

We get

$$\hat{\alpha}(\xi) \simeq 4M \frac{\xi}{\xi^2} + \frac{3\pi}{4} (5 - a^2) M^2 \frac{\xi}{\xi^3}.$$  \hspace{1cm} (14)

The curvature of the space-time generated by the charge induces a correction to the ppN order. Again, using the Fermat’s principle, the lens equation reads

$$\eta = \frac{D_s}{D_d} \xi - D_{ds} \hat{\alpha}(\xi).$$  \hspace{1cm} (15)

IV. THE LENS MAPPING

Owing to the spherical symmetry of the system, the lens equation reduces to a one-dimensional equation: source, lens and images lie on a straight line on the observer’s sky.
As a natural length scale, we introduce the Einstein radius $R_E$,

$$
\xi_0 = R_E \equiv \sqrt{4M \frac{D_d D_{ds}}{D_s}}.
$$  \hfill (16)

Let us change to a dimensionless variable $x \equiv \xi/\xi_0$. The lens equation reads

$$
y = x - \alpha(x),
$$  \hfill (17)

where $y$ is the position of the source, in units of $D_d D_s/\xi_0$, and $\alpha$ is the scaled deflection angle, $\alpha(x) \equiv \frac{D_d D_{ds}}{\xi_0 D_s} \hat{\alpha}(\xi)$. Owing to symmetry, we will consider source positions $y \geq 0$, whereas the range of $x$ will be taken to be the whole real axis.

The scaled deflection angle becomes

$$
\alpha(x) = \frac{1}{x} + \frac{3\pi}{32} \frac{R_{\text{Sch}}}{R_E} \frac{1}{x|x|}.
$$  \hfill (18)

In the next, we will use the notation $d_{RN} \equiv \frac{3\pi}{32} \frac{R_{\text{Sch}}}{R_E}$.

Under the condition $d_{RN} \ll 1$, that holds in usual astrophysical systems, we can perform a perturbative analysis to obtain approximate solutions of the lens equation. The image position can be expressed as,

$$
x \simeq x(0) + d_{RN} x(1),
$$  \hfill (19)

where $x(0)$ and $x(1)$ denote the zeroth-order solution and the correction to the first-order, respectively. The unperturbed images are solutions of the lens equation for $d_{RN} = 0$. Two unperturbed images form at

$$
x_{(0)}^\pm = \frac{1}{2} \left( 1 \pm \sqrt{1 + 4 \frac{y^2}{y'^2}} \right) y.
$$  \hfill (20)

The primary image $x_+$ lies outside the Einstein ring (on the same side of the source), while the secondary image $x_-$ is inside (on the side opposite the source).

By substituting the expression in Eq. (19) for the perturbed images in the full lens equation, Eq. (17), we obtain the first-order perturbations,

$$
x_{(1)}^\pm = x_{(0)}^\pm + \frac{d_{RN}}{x_{(0)}^\pm \sqrt{y'^2 + 4}}.
$$  \hfill (21)

Critical curves form where the Jacobian of the lens mapping vanishes. It is

$$
A = \frac{y}{x} \frac{dy}{dx} \simeq 1 - \frac{1}{x^3} - \frac{3 - x^2}{|x|^5} d_{RN}.
$$  \hfill (22)
A is singular at the tangential critical curve, a circle centred on the origin and mapped onto the point \( y = 0 \). The effect of a charge is to increase the radius of the tangential circle from 1 to \( 1 + \frac{d_{\text{RN}}}{2} \). No critical radial curve forms.

The amplification \( \mu = 1/|A| \) for each image becomes

\[
\mu^\pm = \frac{y^2 + 2}{2y\sqrt{y^2 + 4}} \pm \left( \frac{1}{2} - \frac{d_{\text{RN}}}{(y^2 + 4)^{3/2}} \right).
\]  

(23)

As noted in [2], a correction \( \propto 1/x^2 \) to the deflection angle does not change the total magnification of a source at first order. Only the amplification factor for each image is corrected. Furthermore, no effects in astrometric shift during microlensing events are produced [4].

V. CONCLUSIONS

We have discussed gravitational lensing due to a RN black hole in the weak field limit. A charge in the lens induces a correction of ppN order in the time delay of images and deflection angle. We found the position and magnification of the images. Effects of the charge occur only for the quantities relative to each image whereas the total amplification curve and the astrometric shift in microlensing events are not affected.

Despite we have developed our lensing formalism for a RN black hole, our procedure is quite general and can be extended to any astrophysical system wherein a correction of ppN order to the lensing quantities arises in the weak field limit. Only the form of the parameter \( d_{\text{RN}} \) changes as a function of the parameters that characterizes the system.

In view of using gravitational lensing either to discover exotic objects in the universe or to test alternative theories of gravity, a methodical study of higher-order effects is demanded. Virbhadra et al. [11] investigated a static and circularly symmetric lens with mass and scalar charge. Such an object can be also treated as a naked singularity. Due to the scalar field, an extra term \( 1/\xi^2 \) appears.

Gravitational lensing in metric theories of gravity can also reproduce some feature of the RN lensing. Such theories can be characterized by an approximate metric element, with dimensionless parameters, derived in the slow-motion and weak field limit, inclusive of both ppN corrections and gravito-magnetic field [8]. When the standard coefficients of the post-Newtonian parametrized expansion reduce to \( \beta = 1 + \frac{a^2}{2} \) and \( \gamma = 1 \), the ppN coefficient \( \epsilon \) to \( 1 - \frac{a^2}{3} \) and the gravito-magnetic one, \( \mu \), to zero, then the metric tensor reduces to the weak
field limit of the RN metric. Using these particular values for the coefficients in the formulae derived for time delay and deflection angle in metric theories of gravity [8], we retrieve the expression in Eqs. (12, 14).

Future astrometric space missions with a resolution of $\mu$as, such as the ARISE project [10], demands a full knowledge of the phenomenology of the primary and secondary images produced in the weak field limit. This is a fundamental step in the detection of relativistic images of charged black hole, that are closed to the nonrelativistic ones but are much less intense [8].

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