Radiation by a relativistic charged particle in self-wakefield in periodic structure

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Abstract

A new elementary mechanism of radiation due to the oscillatory character of a radiation reaction force appearing when a relativistic charged particle moves along a periodic structure without external fields is investigated. It is shown that the non-synchronous spatial harmonics of Cherenkov-type radiation (CR) can give rise to the oscillation of a particle which consequently generates undulator-type radiation (UR). In the spectral region, where the diffraction of generated waves is essential, the radiation manifests itself in the coherent interference of CR and UR. A pure undulator-type radiation takes place only in the wavelength range where the wave diffraction can be neglected. In the case of coherent UR emitted by a bunch of N electrons, the UR power is proportional to $N^4$.

Key words: electrons, periodic structure, wakefield, undulator radiation

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1 Introduction

In this paper we offer a new mechanism of radiation due to the oscillatory character of a reaction force arising as a relativistic charged particle moves through a periodic structure. The impossibility of oscillatory motion of a free charged particle in the self-field has been proved in [1]. However, it is also well known that a charged particle moving at a constant velocity along the periodic structure emits the Cherenkov-type radiation (or the diffraction radiation) [2]. The fields of this radiation, called as wakefields, can be expressed as a spatial-harmonics series expansion according to Floquet’s theorem. The
action of synchronous spatial harmonics of the self-wakefields on the particle results in energy losses associated with the Cherenkov-type radiation. Under certain conditions, the non-synchronous spatial harmonics give rise to the oscillatory motion of the particle that consequently generates the undulator-type radiation. This radiation is the subject of discussion in this article.

2 Methodology

As a periodic structure, we will consider the vacuum corrugated waveguide with a metallic surface. Such structures are commonly used in rf linacs. Let a particle having the ultrarelativistic velocity \( v \), the charge \( e \) and the mass \( m \) moves along the structure with the period \( D \). The longitudinal component of the velocity \( v_z \), parallel to the structure axis, is very close to the velocity of light \( c \). The radiation reaction force and the radiation power have to be found.

For calculating the radiation reaction force we will use the approach developed in [3]. At first, we will suppose that the charged particle is not a point charge, but is distributed with density \( \rho = e f(r - r(t)) \). Here \( r(t) \) is the radius vector of the center of mass of the particle and \( \int f(r) \, d^3r = 1 \). The equation of motion for the center of mass is written as

\[
\frac{d(m \gamma v)}{dt} = e \int \left[ E(r, t) + \frac{v \times H(r, t)}{c} \right] f[r - r(t)] \, d^3r, \tag{1}
\]

where \( \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \) is the Lorentz factor, \( E(r, t) \) and \( H(r, t) \) are, respectively, the electrical and magnetic self-fields of the charge moving in the periodic structure. These fields can be expressed in terms of vector potential \( A(r, t) \) and scalar potential \( \Phi(r, t) \) in the Coulomb calibration \( \text{div} A = 0 \)

\[
E = -\frac{1}{c} \frac{\partial A}{\partial t} - \nabla \Phi, \quad H = \text{rot} A. \tag{2}
\]

The potential electric self-field \( -\nabla \Phi \) does not influence the motion of the center of mass for the distributed charge [3]. So, it will suffice to find \( A(r, t) \) which satisfies the wave equation

\[
\Delta A - \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} = -\frac{4\pi}{c} \rho v + \frac{1}{c} \frac{\partial (\nabla \Phi)}{\partial t}. \tag{3}
\]
We will seek for the solution of Eq. (3) in the form of the Fourier series

\[ A(r) = \text{Re} \left[ \sum_{\lambda} q_{\lambda}(t) A_{\lambda}(r) \right], \tag{4} \]

where \( q_{\lambda}(t) \) are certain unknown functions of time \( t \), \( A_{\lambda}(r) \) is a set of orthogonal eigenfunctions of homogeneous Eq. (3). Inserting Eq. (4) into Eq. (3) yields the equations for \( q_{\lambda}(t) \)

\[ \frac{d^2 q_{\lambda}}{dt^2} + \omega_{\lambda}^2 q_{\lambda} = \frac{e v(t)}{c V_{\text{tot}}} \int A_{\lambda}^*(r) f[r - r(t)] \, d^3r, \tag{5} \]

where \( \omega_{\lambda} \) is a set of eigenfrequencies. We solved these equations with the following initial conditions: \( q_{\lambda}(0) = 0, \frac{dq_{\lambda}(t)}{dt} \big|_{t=0} = 0 \). \( V_{\text{tot}} = MV_{\text{cell}} \) is the volume of the periodic structure. In order to deal with a discrete set of waves we assume that the structure is enclosed in a "periodicity box" containing \( M \to \infty \) cells of volume \( V_{\text{cell}} \) \[2\]. Solving Eq. (5) and using the definitions (2), we rewrite Eq. (1) as

\[ \frac{d}{dt} (m \gamma v) = F(v(t), r(t), t) = -\frac{e^2}{4c^2 V_{\text{tot}}} \times \sum_{\lambda} \left\{ \left[ A_{\lambda}(r(t)) - \frac{v(t) \times \text{rot} A_{\lambda}(r(t))}{i \omega_{\lambda}} \right] e^{i \omega_{\lambda} t} \int_0^t e^{-i \omega_{\lambda} t'} v(t') A_{\lambda}^*(r) \, dt' \right. \\
\left. + \left[ A_{\lambda}(r(t)) + \frac{v(t) \times \text{rot} A_{\lambda}(r(t))}{i \omega_{\lambda}} \right] e^{-i \omega_{\lambda} t} \int_0^t e^{i \omega_{\lambda} t'} v(t') A_{\lambda}^*(r) \, dt' \right\} + c.c. \tag{6} \]

As the radiation reaction force \( F(t) \), unlike the electromagnetic mass, does not depend on the particle size \( r_0 \) (where \( r_0 \) is meant in the laboratory frame of reference), the distribution function \( f[r - r(t)] \) can be replaced by the Dirac \( \delta \)-function, and in the sum of Eq. (6) only the frequencies \( \omega_{\lambda} < c/r_0 \) are taken account \[3\]. The eigenfunctions of the vector potential for the periodic structures with perfectly conducting walls are usually given in the Floquet form \[2\]

\[ A_{\lambda}(r) = \sum_{n=-\infty}^{\infty} g_{\lambda}^{(n)}(r_\perp) e^{ihnz} \tag{7} \]

where \( A_{\lambda}(r) \) is the periodic function of \( z \) with the period \( D \), \( g_{\lambda}^{(n)}(r_\perp) \) is the amplitude of the \( n \)th spatial harmonic dependent on the transverse coordinates \( r_\perp \), \( h \) is a discrete parameter multiple of \( 2\pi/(MD) \) in the interval
\[ (-\pi/D : \pi/D), \quad h_n = h + 2\pi n/D \] is the propagation constant of the \( n \)th spatial harmonic.

The set of eigenfunctions (7) for infinitely long periodic waveguide is physically limited in frequency by the value of electron plasma frequency \( \omega_e \) in the metal. As is known, if \( \omega_\lambda \sim \omega_e \), the conduction of metal walls strongly falls off, and the diffraction conditions in the periodic structure are disrupted. So, in the spectral region \( \omega_e < \omega_\lambda \), where the wave diffraction can be neglected, the periodic waveguide can be considered as a free space. In this part of frequency spectrum, the vector potential is sought as expansion in terms of the plane waves

\[
A_{\lambda l}(r) = c\sqrt{4\pi} a_{\lambda l} e^{ik_\lambda r},
\]

where \( k_\lambda \) is the wave propagation vector; \( a_{\lambda l} \) are the real unit vectors of polarization \((l = 1, 2)\), perpendicular to \( k_\lambda \).

3 The zeroth-order wake force

In the ultrarelativistic limit, the equation of motion (6) can be solved by the method of successive approximations. We will find non-relativistic corrections for the particle velocity \( v_0 \approx c \). As a zeroth order approximation, we consider the uniform motion of the charged particle parallel to the waveguide axis:

\[
v = v_0 = v_0 e_z, \quad r(t) = r_0 + v_0 t
\]

Inserting Eqs. (7) and (9) (for the frequency region \( \omega_\lambda \ll \omega_e \)) into the right part of Eq. (6) we obtain the radiation reaction force in the zeroth-order approximation in the form of a wake force

\[
F(t) = -e^2 \sum_{p=-\infty}^{\infty} w^{(p)} e^{ip\Omega t} + c.c.
\]

where \( \Omega \equiv 2\pi v_0 / D \) and the amplitudes of spatial harmonics of the wake function are defined as

\[
w^{(p)} = \frac{v_0 D}{4e^2 V_{\text{tot}}} \sum_{n=-\infty}^{\infty} \sum_{\lambda_j} g_{z\lambda_j}^{(n)*} \left| v_0 - \frac{d\omega_\lambda}{dh} \right|_{\lambda=\lambda_j} \times \left[ g_{z\lambda_j}^{(n+p)} - i\frac{v_0}{\omega_\lambda} \nabla_{\perp} g_{z\lambda_j}^{(n+p)} - \frac{\Omega p}{\omega_\lambda} g_{\perp\lambda_j}^{(n+p)} \right]
\]
Hereinafter, the amplitudes of spatial harmonics are taken at $r = r_0\perp$ as $g^{(n)}_{\lambda}(r_0\perp) = g^{(n)}_{\lambda}(n \Omega)$, $\omega_{\lambda}$, satisfies the resonance conditions $h v_0 - \omega_{\lambda} = n \Omega$. The wake force (10) is the periodic function of time with the period $D/v_0$. The synchronous harmonic of the force $-2e^2 w_z^{(0)}$ defines the energy losses associated with Cherenkov-type radiation. The power of this radiation $2v_0 e^2 w_z^{(0)}$ agrees with the one given in Ref. [2]. As it is easily seen, the transverse component of the synchronous harmonic of the wake force equals zero, as $w_x^{(0)} = 0$.

In the range of $\omega_e < \omega_{\lambda}$, where the structure is supposed as a free space, there is no radiation in the zeroth order approximation (at $v_0 = const$).

4 The first-order approximation

If the charged particle moves off-axis, it experiences the action of the transverse component of nonsynchronous harmonics of the wake force ($w^{(p)}_\perp \neq 0$). So, we will find non-relativistic corrections for both the velocity $v_0$ and the radius vector $r_\perp$ of the off-axis particle that are caused by the periodic transverse wake force. We assume that the change in the longitudinal velocity is negligible. Putting Eq. (10) into the equation of motion (6) we correct the law of motion

$$v(t) = v_0 + v_\perp(t) = v_0 + ic \sum_{p \neq 0} \frac{b^{(p)}_p}{p} e^{ip\Omega t},$$

$$r(t) = r_{0\perp} + v_0 t + \delta r_\perp(t) = r_{0\perp} + v_0 t + \frac{c}{\Omega} \sum_{p \neq 0} \frac{b^{(p)}_p}{p^2} e^{ip\Omega t},$$

where $b^{(p)}_p$ is the dimensionless vector

$$b^{(p)}_p = \frac{2e^2}{mc\gamma \Omega} \left( w^{(p)}_\perp + w^{(-p)*}_\perp \right).$$

The absolute value $|b^{(p)}|$ is the small parameter.

Substituting Eqs. (12), (13) and (7) into Eq. (6), and multiplying it by $v$, we obtain the power radiation within the accuracy $|b^{(p)}|^2$ in the range $\omega_{\lambda} \ll \omega_e$

$$P \equiv - \lim_{t \to \infty} \frac{1}{t} \int_0^t \langle \dot{v} \rangle \dot{v} \cdot \dot{r} \cdot \dot{t} dt$$

$$= \frac{e^2 \pi v_0}{2c V_{\text{tot}}} \sum_{\lambda} \sum_{n=-\infty}^{\infty} \left[ \delta(h_n v_0 - \omega_{\lambda}) + \delta(h_n v_0 + \omega_{\lambda}) \right]$$
\[ \sum_{p \neq 0} \frac{b^{(p)}}{2p} \left( \frac{c}{p\Omega} \nabla \perp \mathbf{g}_{z\lambda}^{(n+p)} - \frac{ic}{v_0} \mathbf{g}_{n \perp \lambda}^{(n+p)} \right) \right|^2. \quad (15) \]

Replacing the summation over discrete \( h \) by integration at \( M \to \infty \) in Eq. (15) we find that

\[ P = \frac{e^2 v_0 D}{2c V_{cell}} \sum_{n=0}^{\infty} \sum_{\lambda_j} \frac{1}{|v_0 - \frac{d\omega}{dh}|_{\lambda = \lambda_j}} \times \left| \mathbf{g}_{z\lambda_j}^{(n)} + \sum_{p \neq 0} \frac{b^{(p)}}{2p} \left( \frac{c}{p\Omega} \nabla \perp \mathbf{g}_{z\lambda_j}^{(n+p)} - \frac{ic}{v_0} \mathbf{g}_{n \perp \lambda_j}^{(n+p)} \right) \right|^2, \quad (16) \]

where \( \omega_{\lambda_j} \) satisfies the resonance conditions \( h\nu_0 - \omega_\lambda = n\Omega \).

Eq. (16) shows that in the region \( \omega_\lambda \ll \omega_e \) there is coherent interference between the Cherenkov-type radiation and the undulator-type radiation, that is caused by the oscillation of the particle in the nonsynchronous harmonics self-wakefield. As is evident from Eqs. (16) and (14) in this frequency range the total radiation power tends to the CR power with increasing \( \gamma \).

Let us next consider the radiation of the charge particle in the range \( \omega_e < \omega_\lambda \). In analogy with Eq. (15), substituting Eqs. (12), (13) and (8) into Eq. (6), we can obtain the power of the pure undulator-type radiation

\[ P_U \equiv -\lim_{t \to \infty} \frac{1}{t} \int_0^t v(t) \mathbf{F} (v(t), \mathbf{r}(t), t) \, dt \]

\[ = \frac{e^2 c^2 \pi^2}{2V_{tot}} \sum_{\lambda} \sum_{p \neq 0=1} \frac{a_{\lambda p}}{p} - \frac{Da_{z\lambda_k} k_{\perp \lambda} b^{(p)}}{2\pi p^2} \times \left[ \delta (k_{z\lambda} v_0 - p\Omega - \omega_\lambda) + \delta (k_{z\lambda} v_0 - p\Omega + \omega_\lambda) \right]. \quad (17) \]

Here to simplify the calculations, we consider the oscillation of the particle in the dipole limit

\[ \mathbf{k}_\lambda \delta \mathbf{r}_\perp (t) \ll 2\pi. \quad (18) \]

Considering Eq. (18) and the wave dispersion in a free space \( (\omega_\lambda = c k_\lambda) \), we go from the summation over \( \lambda \) in Eq. (17) to integration over \( \omega \) at \( V_{tot} \to \infty \)

\[ P_U = \frac{e^2}{16\pi c} \int_0^{2\pi} d\varphi \int_0^\pi \sin \theta d\theta \int_{\omega_e}^{\omega_{max} \leq c/r_0} \omega^2 d\omega \]
\[
\times \sum_{p \neq 0} \left\{ \frac{|b_x^{(p)}|^2}{p^2} \left[ 1 - R(\omega, \theta, p) \sin^2 \theta \cos^2 \varphi \right] \\
+ \frac{|b_y^{(p)}|^2}{p^2} \left[ 1 - R(\omega, \theta, p) \sin^2 \theta \sin^2 \varphi \right] \\
- \text{Re} \left( \frac{b_x^{(p)} b_y^{(p)}}{p^2} R(\omega, \theta, p) \sin^2(2\varphi) \sin^2 \theta \right) \right\} \\
\times \left\{ \delta[\omega(\beta_0 \cos \theta - 1) - p\Omega] + \delta[\omega(\beta_0 \cos \theta + 1) - p\Omega] \right\},
\]

(19)

where \( R(\omega, \theta, p) \equiv \left( 1 - \frac{\omega}{\beta_0 \gamma} \beta_0 \cos \theta \right)^2 - \left( \beta_0 \frac{\omega}{\gamma} \right)^2 \), \( \theta \) is the angle between the wave vector \( \mathbf{k} \) and the \( \mathbf{Oz} \) axis, \( \varphi \) is the angle between the \( \mathbf{x} \) axis and the \( \mathbf{xOy} \)-plane projection of \( \mathbf{k} \), \( \beta_0 = v_0/c \).

For \( \omega_e < \omega_\lambda \) it is of interest to consider the radiation of the high energy charged particle satisfying the condition \( \omega_e \ll \Omega \gamma^2 \). In this case, integrating over \( \omega, \theta, \varphi \) in Eq. (19) and replacing \( b^{(p)} \) from Eq. (14) we find the total power of pure undulator-type radiation

\[
P_U = \frac{4e^6}{3m^2c^3\gamma^2} \sum_{p=1}^{p_{\text{lim}}} \left| w_{\perp}^{(p)} + w_{\perp}^{(-p)\ast} \right|^2,
\]

(20)

where the number of harmonics in the sum is limited by the condition (18) resulting in \( p \ll p_{\text{lim}} = 2\pi\gamma/\text{max} |b^{(p)}| \). As it follows from Eq. (20), the power grows as \( \sim \gamma^2 \), so in the region \( \omega_e \ll \omega_\lambda \) the UR power can exceed the CR power emitted in the spectral region \( \omega_\lambda \ll \omega_e \).

It should also be stated that, if instead of the above considered point particle there is a bunch of \( N \) electrons with longitudinal and transverse dimensions (\( \sigma_z \) and \( \sigma_\perp \)) which satisfy the both conditions \( \sigma_z \ll D/(2q\gamma^2) \) and \( \sigma_\perp \ll D/(2q\gamma) \), then the radiation is coherent in the frequency region \( \omega \ll 2q\Omega \gamma^2 \). Moreover, as it follows from Eq. (20), the UR power is proportional to \( N^4 \) in the range \( \omega_e \ll \omega \ll 2q\Omega \gamma^2 \)

\[
P_U = \frac{4e^6 N^4}{3m^2c^3\gamma^2} \sum_{p=1}^{q} \left| w_{\perp}^{(p)} + w_{\perp}^{(-p)\ast} \right|^2.
\]

(21)

5 Conclusions

The new radiation mechanism considered above may be of use in undulators based on periodic structures without external fields, where the non-synchronous wake-harmonics of an electron bunch act as pump waves. These
wakefield undulators require no magnetic fields or rf sources needed in present-day FEL. Note also that the undulator-type radiation power is proportional to $\gamma^2$. So, in the future high energy electron rf linacs, in view of deviation of a beam from the linac axis, because of the coherent betatron oscillation of the beam in a focusing system, the interaction of electrons with the spatial non-synchronous harmonics of both an accelerating mode [4] and a wakefield may result in the electron energy loss associated with the spontaneous undulator-type radiation.

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