FROBENIUS-PERRON THEORY FOR PROJECTIVE SCHEMES

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Abstract. The Frobenius-Perron theory of an endofunctor of a \(k\)-linear category (recently introduced in [CG]) provides new invariants for abelian and triangulated categories. Here we study Frobenius-Perron type invariants for derived categories of commutative and noncommutative projective schemes. In particular, we calculate the Frobenius-Perron dimension for domestic and tubular weighted projective lines, define Frobenius-Perron generalizations of Calabi-Yau and Kodaira dimensions, and provide examples. We apply this theory to the derived categories associated to certain Artin-Schelter regular and finite-dimensional algebras.

0. Introduction

The Frobenius-Perron dimension of an endofunctor of a category was introduced by the authors in [CG]. It can be viewed as a generalization of the Frobenius-Perron dimension of an object in a fusion category introduced by Etingof-Nikshych-Ostrik [ENO2005] in early 2000 (also see [EGNO2015, EGO2004, Nik2004]). It is shown in [CG] that the Frobenius-Perron dimension of either \(\text{Ext}^1\) or the suspension of a triangulated category is a useful invariant in several different topics such as embedding problem, Tame and wild dichotomy, complexity of categories. In particular, the Frobenius-Perron invariants have strong connections with the representation type of a category [CG, ZZ].

The definition of the Frobenius-Perron dimension of a category will be recalled in Section 2. In the present paper we continue to develop Frobenius-Perron theory, but we restrict our attention to the bounded derived category of coherent sheaves over a projective scheme. A projective scheme could be a classical (or commutative) one, or a noncommutative one in the sense of [AZ], or a weighted projective line in the sense of [GL]. We refer to Section 3 for some basics concerning weighted projective lines.

Our first goal is to understand the Frobenius-Perron dimension, denoted by \(\text{fpd}\), of the bounded derived category of coherent sheaves over a weighted projective line, which is also helpful for understanding the Frobenius-Perron dimension of the path algebra of an acyclic quiver of \(ADE\) type via the derived equivalence given in Lemma 2.1(2). Let \(X\) be a weighted projective line (respectively, a commutative or noncommutative projective scheme). We use \(\text{coh}(X)\) to denote the abelian category of coherent sheaves over \(X\) and \(D^b(\text{coh}(X))\) to denote the bounded derived category of \(\text{coh}(X)\). Here is the main result in this topic.

\[\text{usp} \quad \text{Mathematics Subject Classification.} \quad \text{Primary 16E35, 16E65, 16E10, Secondary 16B50.} \]

\[\text{Key words and phrases.} \quad \text{Frobenius-Perron dimension, derived category, projective scheme, weighted projective line, noncommutative projective scheme.}\]
Theorem 0.1 (Theorem 2.13). Let $X$ be a weighted projective line that is either domestic or tubular. Then $\text{fpd}(D^b(\text{coh}(X))) = 1$.

Note that Theorem 0.1 is a “weighted” version of [CG, Proposition 6.5(1,2)]. By Lemma 2.1(2), we obtain the Frobenius-Perron dimension of the bounded derived category of finite dimensional representations of acyclic quivers of $ADE$ type.

Our second goal is to introduce Frobenius-Perron (“fp” for short) version of some classical invariants. We will focus on fp-analogues of two important invariants in projective algebraic geometry, namely, Calabi-Yau dimension, and Kodaira dimension.

Let $\text{fp} \kappa$ (respectively, $\text{fp} \text{cy}$) denote the fp version of the Kodaira dimension [Definition 3.5(1)] (respectively, the Calabi-Yau dimension [Definition 3.1(3)]). Both are defined for bounded derived categories of smooth projective schemes or more generally triangulated categories with Serre functor. In algebraic geometry, a Calabi-Yau variety has the trivial canonical bundle. In noncommutative algebraic geometry, a “skew Calabi-Yau” scheme may not have a trivial canonical bundle. Our fp version of the Calabi-Yau dimension covers the case even when the canonical bundle is not trivial. Below is one of the main results in this direction. Note that the definition of $\text{fp} \kappa$ is dependent on a chosen structure sheaf.

Theorem 0.2 (Propositions 3.3 and 3.6). Let $X$ be a smooth projective scheme and $T$ be the triangulated category $D^b(\text{coh}(X))$ with structure sheaf $\mathcal{O}_X$. Then the following hold.

1. $\text{fp} \kappa(T, \mathcal{O}_X) = \kappa(X)$.
2. $\text{fp} \text{cy}(T) = \dim X$. As a consequence, if $X$ is Calabi-Yau, then $\text{fp} \text{cy} T$ equals the Calabi-Yau dimension of $X$.

In the noncommutative case we have

Theorem 0.3 (Theorem 4.5). Let $A$ be a noetherian connected graded Artin-Schelter Gorenstein algebra of injective dimension $d \geq 2$ and AS index $\ell$ that is generated in degree 1. Suppose that $X := \text{Proj} A$ has finite homological dimension and that the Hilbert series of $A$ is rational. Let $T$ be the bounded derived category of $\text{coh}(X)$ and $\mathcal{A}$ be the image of $A$ in $\text{Proj} A$.

1. $\text{fp} \text{cy}(T) = d - 1$.
2. If $\ell > 0$, then $\text{fp} \kappa(T, \mathcal{A}) = -\infty$ and $\text{fp} \kappa^{-1}(T, \mathcal{A}) = \text{GKdim} A - 1$.
3. If $\ell < 0$, then $\text{fp} \kappa(T, \mathcal{A}) = \text{GKdim} A - 1$ and $\text{fp} \kappa^{-1}(T, \mathcal{A}) = -\infty$.
4. If $\ell = 0$, then $\text{fp} \kappa(T, \mathcal{A}) = \text{fp} \kappa^{-1}(T, \mathcal{A}) = 0$.

Similar results are proved for Piontkovski projective lines, see Section 4. Note that the definitions of fp Calabi-Yau dimension and fp Kodaira dimension make sense for bounded derived category of left modules over a finite dimension algebra of finite global dimension. However, we don’t have a complete result for that case. Some examples are given in Section 5.

When we are working with finite dimensional algebras, it is well-known that the global dimension is not a derived invariant. By definition, fp Calabi-Yau dimension is a derived invariant. This suggests that the fp Calabi-Yau dimension is nicer than the global dimension in some aspects. So it is important to study this new invariant. To start we ask the following questions.
Question 0.4. Let $A$ be a finite dimensional algebra of finite global dimension and let $\mathcal{T}$ be the derived category $\mathcal{D}^b(\text{Mod}_{f.d.-A})$.

(1) Is $\text{fpcy}(\mathcal{T})$ always finite?
(2) What is the set of possible values of $\text{fpcy}(\mathcal{T})$ when $A$ varies?
(3) What is the set of possible values of $\text{fpK}(\mathcal{T}, A)$ when $A$ varies?

More questions and some partial answers are given in Section 5. Some other examples are given in [Wi, ZZ].

We have outlined several important applications of Frobenius-Perron invariants in [CG]. Next we would like to mention one surprising application of the Frobenius-Perron curvature defined in [CG, Definition 2.3(4)].

Theorem 0.5. [ZZ, Corollary 0.6] Suppose that the bounded derived categories of representations of two finite acyclic quivers are equivalent as tensor triangulated categories. Then the quivers are isomorphic.

This result is striking because it is well-known that, for two Dynkin quivers with the same underlying Dynkin diagram, their derived categories are triangulated equivalent [BGP, Ha], even if the quivers are non-isomorphic. We hope that different Frobenius-Perron invariants will become effective tools in the study of triangulated categories and monoidal (or tensor) triangulated categories.

This paper is organized as follows. Some background material are provided in Section 1. In Section 2, we review some facts about weighted projective lines and prove Theorem 0.1. In Section 3, we introduce fp-version of Calabi-Yau dimension and Kodaira dimension for Ext-finite triangulated categories with Serre functor and prove Theorem 0.2. In Section 4, fp Calabi-Yau dimension and fp Kodaira dimension are studied for noncommutative projective schemes and Theorem 0.3 is proved there. In Section 5, some partial results, comments and examples are given concerning finite dimensional algebras. Sections 6 and 7 are appendices. The proof of Theorem 0.1 is dependent on some linear algebra computation given in Section 6. This paper can be viewed as a sequel of [CG].

1. Preliminaries and definitions

Throughout let $k$ be a base field that is algebraically closed. Let everything be over $k$.

We are mainly interested in the derived category $\mathcal{D}^b(\text{coh}(X))$ where $X$ is a smooth commutative or noncommutative projective scheme, but most definitions work for more general pre-triangulated (or abelian) categories.

Part of this section is copied from [CG].

1.1. Spectral radius of a square matrix. Let $A$ be an $n \times n$-matrix over complex numbers $\mathbb{C}$. The spectral radius of $A$ is defined to be

$$\rho(A) = \max\{|r_1|, |r_2|, \cdots, |r_n|\}$$

where $\{r_1, r_2, \cdots, r_n\}$ is the complete set of eigenvalues of $A$. When each entry of $A$ is a positive real number, $\rho(A)$ is also called the Perron root or the Perron-Frobenius eigenvalue of $A$.

In order to include the “infinite-dimensional” setting, we extend the definition of the spectral radius in the following way.
Let $A = (a_{ij})_{n \times n}$ be an $n \times n$-matrix with entries $a_{ij}$ in $\mathbb{R}^+ := \mathbb{R} \cup \{\pm \infty\}$. Define $A' = (a'_{ij})_{n \times n}$ where

$$a'_{ij} = \begin{cases} a_{ij} & a_{ij} \neq \pm \infty, \\ x_{ij} & a_{ij} = \infty, \\ -x_{ij} & a_{ij} = -\infty. \end{cases}$$

In other words, we are replacing $\infty$ in the $(i, j)$-entry by a finite real number, called $x_{ij}$, in the $(i, j)$-entry. Or $x_{ij}$ are considered as function or a variable mapping $\mathbb{R} \to \mathbb{R}$.

**Definition 1.1.** Let $A$ be an $n \times n$-matrix with entries in $\mathbb{R}^+$. The spectral radius of $A$ is defined to be

$$\rho(A) := \lim \inf_{\text{all } x_{ij} \to \infty} \rho(A').$$

See [CG, Remark 1.3 and Example 1.4].

1.2. **Frobenius-Perron dimension of a quiver.**

**Definition 1.2.** [CG, Definition 1.6] Let $Q$ be a quiver.

1. If $Q$ has finitely many vertices, then the Frobenius-Perron dimension of $Q$ is defined to be

$$\text{fpd } Q := \rho(A(Q))$$

where $A(Q)$ is the adjacency matrix of $Q$.

2. Let $Q$ be any quiver. The Frobenius-Perron dimension of $Q$ is defined to be

$$\text{fpd } Q := \sup \{\text{fpd } Q'\}$$

where $Q'$ runs over all finite subquivers of $Q$.

1.3. **Frobenius-Perron dimension of an endofunctor.** Let $C$ denote a $k$-linear category. For simplicity, we use $\dim(A, B)$ for $\dim \text{Hom}_C(A, B)$ for any two objects $A$ and $B$ in $C$. Here the second dim is $\dim_k$.

The set of finite subsets of nonzero objects in $C$ is denoted by $\Phi$ and the set of subsets of $n$ nonzero objects in $C$ is denoted by $\Phi_n$ for each $n \geq 1$. It is clear that $\Phi = \bigcup_{n \geq 1} \Phi_n$. We do not consider the empty set as an element of $\Phi$.

**Definition 1.3.** [CG, Definition 2.1] Let $\phi := \{X_1, X_2, \ldots, X_n\}$ be a finite subset of nonzero objects in $C$, namely, $\phi \in \Phi_n$. Let $\sigma$ be an endofunctor of $C$.

1. The adjacency matrix of $(\phi, \sigma)$ is defined to be

$$A(\phi, \sigma) := (a_{ij})_{n \times n}, \quad \text{where } a_{ij} := \dim(X_i, \sigma(X_j)) \ \forall i, j.$$ (1)

2. An object $M$ in $C$ is called a brick [ASS, Definition 2.4, Ch. VII] if

$$\text{Hom}_C(M, M) = k.$$ (2)

If $C$ is a pre-triangulated category [Ne, Definition 1.1.2] with suspension $\Sigma$, an object $M$ in $C$ is called an atomic object if it is a brick and satisfies

$$\text{Hom}_C(M, \Sigma^{-i}(M)) = 0, \ \forall i > 0.$$ (3)

3. $\phi \in \Phi$ is called a brick set (respectively, an atomic set) if each $X_i$ is a brick (respectively, atomic) and

$$\dim(X_i, X_j) = \delta_{ij}.$$
for all \( 1 \leq i, j \leq n \). The set of brick (respectively, atomic) \( n \)-object subsets is denoted by \( \Phi_{n,b} \) (respectively, \( \Phi_{n,a} \)). We write \( \Phi_n = \bigcup_{n \geq 1} \Phi_{n,b} \) (respectively, \( \Phi_n = \bigcup_{n \geq 1} \Phi_{n,a} \)).

**Definition 1.4.** [CG, Definition 2.3] Retain the notation as in Definition 1.3, and we use \( \Phi_b \) as the testing objects. When \( \mathcal{C} \) is a pre-triangulated category, \( \Phi_b \) is automatically replaced by \( \Phi_a \) unless otherwise stated.

1. The \( n \)-th Frobenius-Perron dimension of \( \sigma \) is defined to be
   \[
   \text{fpd}^n(\sigma) := \sup_{\phi \in \Phi_{n,b}} \{ \rho(A(\phi, \sigma)) \}.
   \]
   If \( \Phi_{n,b} \) is empty, then, by convention, \( \text{fpd}^n(\sigma) = 0 \).

2. The Frobenius-Perron dimension of \( \sigma \) is defined to be
   \[
   \text{fpd}(\sigma) := \sup_n \{ \text{fpd}^n(\sigma) \} = \sup_{\phi \in \Phi_b} \{ \rho(A(\phi, \sigma)) \}.
   \]

3. The Frobenius-Perron growth of \( \sigma \) is defined to be
   \[
   \text{fpg}(\sigma) := \sup_{\phi \in \Phi_b} \{ \limsup_{n \to \infty} \log_n(\rho(A(\phi, \sigma^n))) \}.
   \]
   By convention, \( \log_0 0 = -\infty \).

4. The Frobenius-Perron curvature of \( \sigma \) is defined to be
   \[
   \text{fpv}(\sigma) := \sup_{\phi \in \Phi_b} \{ \limsup_{n \to \infty} (\rho(A(\phi, \sigma^n)))^{1/n} \}.
   \]

In this paper, we only use \( \Phi_b \) and \( \Phi_a \) as the testing objects. But in principal one can use other testing objects, see Section 7. We continue to review definitions from [CG].

**Definition 1.5.** [CG, Definition 2.7]

1. Let \( \mathfrak{A} \) be an abelian category. The Frobenius-Perron dimension of \( \mathfrak{A} \) is defined to be
   \[
   \text{fpd} \mathfrak{A} := \text{fpd}(E^1)
   \]
   where \( E^1 = \text{Ext}_\mathfrak{A}^1(-,-) \) is defined as in [CG, Example 2.6(1)]. The Frobenius-Perron theory of \( \mathfrak{A} \) is the collection
   \[
   \{ \text{fpd}^m(E^n) \}_{m \geq 1, n \geq 0}
   \]
   where \( E^n = \text{Ext}_\mathfrak{A}^n(-,-) \) is defined as in [CG, Example 2.6(1)].

2. Let \( \mathcal{T} \) be a pre-triangulated category with suspension \( \Sigma \). The Frobenius-Perron dimension of \( \mathcal{T} \) is defined to be
   \[
   \text{fpd} \mathcal{T} := \text{fpd}(\Sigma).
   \]

3. The fp-global dimension of \( \mathcal{T} \) is defined to be
   \[
   \text{fpgd} \mathcal{T} := \sup \{ n \mid \text{fpd}(\Sigma^n) \neq 0 \}.
   \]

Fix an endofunctor \( \sigma \) of a category \( \mathcal{C} \). For a set of bricks \( B \) in \( \mathcal{C} \) (or a set of atomic objects when \( \mathcal{C} \) is triangulated), we define
   \[
   \text{fpd}^n |_B (\sigma) = \sup \{ \rho(\sigma, \sigma) \mid \sigma := \{ X_1, \cdots, X_n \} \in \Phi_{n,b}, \text{ and } X_i \in B \ \forall i \}.
   \]
Let \( \Lambda := \{ \lambda \} \) be a totally ordered set. We say a set of bricks \( B \) in \( \mathcal{C} \) has a \( \sigma \)-decomposition \( \{ B^\lambda \}_{\lambda \in \Lambda} \) (based on \( \Lambda \)) if the following holds.

1. \( B \) is a disjoint union \( \bigcup_{\lambda \in \Lambda} B^\lambda \).
(2) If $X \in B^\lambda$ and $Y \in B^\delta$ with $\lambda < \delta$, $\text{Hom}_C(X, \sigma(Y)) = 0$.

The following is [CG, Lemma 6.1].

**Lemma 1.6.** [CG, Lemma 6.1] Let $n$ be a positive integer. Suppose that $B$ has a $\sigma$-decomposition $\{B^\lambda\}_{\lambda \in \Lambda}$. Then

$$\text{fpd}^n |_{B(\sigma)} \leq \sup_{\lambda \in \Lambda, m \leq n} \{\text{fpd}^m |_{B^\lambda(\sigma)}\}.$$  

2. Frobenius-Perron theory of weighted projective lines

The main goal of this section is to recall some facts about weighted projective lines and then to prove Theorem 0.1.

2.1. Weighted projective lines. First we recall the definition and some basics about weighted projective lines. Details can be found in [GL, Section 1].

For $t \geq 1$, let $p := (p_0, p_1, \cdots, p_t)$ be a $(t + 1)$-tuple of positive integers, called the weight sequence. Let $D := (\lambda_0, \lambda_1, \cdots, \lambda_t)$ be a sequence of distinct points of the projective line $\mathbb{P}^1$ over $k$. We normalize $D$ so that $\lambda_0 = \infty$, $\lambda_1 = 0$ and $\lambda_2 = 1$ (if $t \geq 2$). Let

$$S := k[X_0, X_1, \cdots, X_t]/(X_i^{p_i} - X_i^{p_0} + \lambda_i X_0^{p_i}, i = 2, \cdots, t).$$

The image of $X_i$ in $S$ is denoted by $x_i$ for all $i$. Let $\mathbb{L}$ be the abelian group of rank $1$ generated by $x_i$ for $i = 0, 1, \cdots, t$ and subject to the relations

$$p_0 x_0^t = \cdots = p_t x_t^t = \cdots = p_t x_t^t =: c^t.$$

The algebra $S$ is $\mathbb{L}$-graded by setting $\deg x_i = x_i$. The corresponding weighted projective line, denoted by $X(p, D)$ or simply $X$, is a noncommutative space whose category of coherent sheaves is given by the quotient category

$$\text{coh}(X) := \frac{\text{gr}^1 S}{\text{gr}_{f.d.} S}.$$  

The weighted projective lines are classified into the following three classes:

$\lambda$ is

- **domestic** if $p$ is $(p, q, (2, 2, n), (2, 3, 3), (2, 3, 4), (2, 3, 5));$
- **tubular** if $p$ is $(2, 3, 6), (3, 3, 3), (2, 4, 4), (2, 2, 2, 2);$  
- **wild** otherwise.

In [Sc, Section 4.4], domestic (respectively, tubular, wild) weighted projective lines are called parabolic (respectively, elliptic, hyperbolic). Let $X$ be a weighted projective line. A sheaf $F \in \text{coh}(X)$ is called torsion if it is of finite length in $\text{coh}(X)$. Let $\text{Tor}(X)$ denote the full subcategory of $\text{coh}(X)$ consisting of all torsion objects. By [Sc, Lemma 4.16], the category $\text{Tor}(X)$ decomposes as a direct product of orthogonal blocks

$$(\text{E2.0.1}) \quad \text{Tor}(X) = \prod_{x \in \mathbb{P}^1 \{\lambda_0, \lambda_1, \cdots, \lambda_t\}} \text{Tor}_x \times \prod_{i=0}^t \text{Tor}_{\lambda_i},$$

where $\text{Tor}_x$ is equivalent to the category of nilpotent representations of the Jordan quiver (with one vertex and one arrow) over the residue field $k_x$ and where $\text{Tor}_{\lambda_i}$ is equivalent to the category of nilpotent representations over $k$ of the cyclic quiver of length $p_i$, see Example 2.3. A simple object in $\text{coh}(X)$ is called ordinary simple (see [GL]) if it is the skyscraper sheaf $O_x$ of a closed point $x \in \mathbb{P}^1 \{\lambda_0, \lambda_1, \cdots, \lambda_t\}.$
Let $\text{Vect}(X)$ be the full subcategory of $\text{coh}(X)$ consisting of all vector bundles. Similar to the elliptic curve case [BB, Section 4], one can define the concepts of degree, rank and slope of a vector bundle on a weighted projective line $X$; details are given in [Sc, Section 4.7] and [LM, Section 2]. For each $\mu \in \mathbb{Q}$, let $\text{Vect}_\mu(X)$ be the full subcategory of $\text{Vect}(X)$ consisting of all semistable vector bundles of slope $\mu$. By convention, $\text{Vect}_\infty(X)$ denotes $\text{Tor}(X)$. By [Sc, Comments after Corollary 4.34], when $X$ is a domestic or tubular weighted projective line, every indecomposable object in $\text{coh}(X)$ is in

$$\bigcup_{\mu \in \mathbb{Q} \cup \{\infty\}} \text{Vect}_\mu(X).$$

Below we collect some nice properties of weighted projective lines. The definition of a stable tube (or simply tube) was introduced in [Ri].

**Lemma 2.1.** [CG, Lemma 7.9] Let $X = X(p,D)$ be a weighted projective line.

1. $\text{coh}(X)$ is noetherian and hereditary.
2. 
   $$D^b(\text{coh}(X)) \cong \begin{cases} 
   D^b(\text{Mod}_{f.d.} -k\mathbb{A}_{p,q}) & \text{if } p = (p,q), \\
   D^b(\text{Mod}_{f.d.} -kD_n) & \text{if } p = (2,2,n), \\
   D^b(\text{Mod}_{f.d.} -kE_6) & \text{if } p = (2,3,3), \\
   D^b(\text{Mod}_{f.d.} -kE_7) & \text{if } p = (2,3,4), \\
   D^b(\text{Mod}_{f.d.} -kE_8) & \text{if } p = (2,3,5).
   \end{cases}$$
3. Let $S$ be an ordinary simple object in $\text{coh}(X)$. Then $\text{Ext}^1_X(S,S) = k$.
4. $\text{fpd}^1(\text{coh}(X)) \geq 1$.
5. If $X$ is tubular or domestic, then $\text{Ext}^1_X(X,Y) = 0$ for all $X \in \text{Vect}_{\mu'}(X)$ and $Y \in \text{Vect}_\mu(X)$ with $\mu' < \mu$.
6. If $X$ is domestic, then $\text{Ext}^1_X(X,Y) = 0$ for all $X \in \text{Vect}_{\mu'}(X)$ and $Y \in \text{Vect}_\mu(X)$ with $\mu' \leq \mu < \infty$. As a consequence, $\text{fpd}(\Sigma|_{\text{Vect}_\mu(X)}) = 0$ for all $\mu < \infty$.
7. Suppose $X$ is tubular or domestic. Then every indecomposable vector bundle $X$ is semi-stable.
8. Suppose $X$ is tubular and let $\mu \in \mathbb{Q}$. Then each $\text{Vect}_\mu(X)$ is a uniserial category. Accordingly indecomposables in $\text{Vect}_\mu(X)$ decomposes into Auslander-Reiten components, which all are stable tubes of finite rank. In fact, for every $\mu \in \mathbb{Q}$,

   $$\text{Vect}_\mu(X) \cong \text{Vect}_\infty(X) = \text{Tor}(X).$$

**Proof.**
1. This is well-known, see [Le, Theorem 2.2]
2. See [KLM, Proposition 5.1(i)] and [GL, 5.4.1].
3. Let $S$ be an ordinary simple object which is of the form $O_x$ for some $x \in \mathbb{P}^1 \setminus \{\lambda_0, \cdots, \lambda_t\}$. Then $S$ is a brick and $\text{Ext}^1_X(S,S) = \text{Ext}^1_X(O_x,O_x) = k$.
4. Follows from (3) by taking $\phi := \{S\}$.
5. This is [Sc, Corollary 4.34(i)].
6. This is [Sc, Comments after Corollary 4.34]. The consequence is clear.
7. [GL, Theorem 5.6(i)].
8. See [Sc, Theorem 4.42] and [GL, Theorem 5.6(iii)].

Our main goal in this section is to prove Theorem 0.1. Eventually one should ask the following question.
Question 2.2. [CG, Question 7.11] Let \( X \) be a weighted projective line. What is the exact value of \( \text{fpd}^n D^b(\text{coh}(X)) \), for \( n \geq 1 \), in terms of other invariants of \( X \)?

2.2. Standard stable tubes [LS, Ri]. In this subsection we would like to understand the (standard) stable tubes in \( \text{Tor}_{\lambda} \), in the decomposition (E2.0.1), which is the Auslander-Reiten quiver of the \( x \)-nilpotent (or \( x \)-torsion) representations of the algebra in the following example.

Example 2.3. Let \( \xi \) be a primitive \( n \)th root of unity. Let \( T_n \) be the algebra

\[
T_n := \frac{k\langle g, x \rangle}{(g^n - 1, gx - \xi xg)}.
\]

This algebra can be expressed by using a group action. Let \( G \) be the group

\[
\{ g \mid g^n = 1 \} \cong \mathbb{Z}/(n)
\]

acting on the polynomial ring \( k[x] \) by \( g \cdot x = \xi x \). Then \( T_n \) is naturally isomorphic to the skew group ring \( k[x] \ast G \). Let \( A_{n-1} \rightarrow \) denote the cycle quiver with \( n \) vertices, namely, the quiver with one oriented cycle connecting \( n \) vertices. It is also known that \( T_n \) is isomorphic to the path algebra of the quiver \( A_{n-1} \).

Let \( \mathfrak{A} \) be the category of finite dimensional left \( T_n \)-modules that are \( x \)-torsion. In this subsection we will show that \( \text{fpd}(\mathfrak{A}) = 1 \) [Corollary 2.12]. We start with somewhat more general setting.

Let \( A \) be any algebra and let \( \text{Mod}_{f.d.} - A \) be the category of finite dimensional left \( A \)-modules. Let \( \Gamma(\text{Mod}_{f.d.} - A) \) denote the Auslander-Reiten quiver with Auslander-Reiten translation \( \tau \).

Let \( \mathcal{C} \) be a component of \( \Gamma(\text{Mod}_{f.d.} - A) \). We say \( \mathcal{C} \) is a self-hereditary component of \( \Gamma(\text{Mod}_{f.d.} - A) \) if for each pair of indecomposable \( A \)-modules \( X \) and \( Y \) in \( \mathcal{C} \), we have \( \text{Ext}^2_A(X, Y) = 0 \).

We now recall some results from the book [SS]. The definitions can be found in [SS]. Let \( \phi := \{ E_1, \cdots, E_r \} \) be a brick set in \( \mathcal{C} \). (In [SS], \( \phi \) is called a finite family of pairwise orthogonal bricks.) The extension category [SS, p.13] of \( \phi \), denoted by \( \mathcal{E}, \mathcal{E}_A, \) or \( \mathcal{E}X^r\mathcal{T}_A(E_1, \cdots, E_r) \), is defined to be the full subcategory of \( \text{Mod}_{f.d.} - A \) whose nonzero objects are all the objects \( M \) such that there exists a chain of submodules

\[
M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_l = 0,
\]

for some \( l \geq 1 \), with \( M_i/M_{i+1} \) isomorphic to one of the bricks \( E_1, \cdots, E_r \) for all \( 0 \leq i < l \). We say \( \{ E_1, \cdots, E_r \} \) is a \( \tau \)-cycle if \( \tau(E_i) = E_{i-1} \) for all \( i \in \mathbb{Z}/(r) \).

We will be using the notation introduced in [SS]. For example, \( E_i[j] \) represents some uniserial object, which is nothing to do with the \( j \)th suspension of \( E_i \).

Theorem 2.4. [SS, Lemma 2.1 and Theorem 2.2 in Ch. X] Let \( \phi := \{ E_1, \cdots, E_r \} \), with \( r \geq 1 \), be a brick set on \( \text{Mod}_{f.d.} - A \). Suppose that \( \phi \) is a \( \tau \)-cycle and a self-hereditary family [SS, p.14]. Then the extension category \( \mathcal{E} \) is an abelian category with the following properties.

1. For each pair \( (i, j) \), with \( 1 \leq i, j \leq r \), there exist a uniserial object \( E_i[j] \) of \( \mathcal{E} \)-length \( l_E(E_i[j]) = j \) in the category \( \mathcal{E} \), and homomorphisms \( u_{ij}: E_i[j - 1] \rightarrow E_i[j] \) and \( p_{ij}: E_i[j] \rightarrow E_{i+1}[j - 1] \),
for \( j \geq 2 \), such that we have two short exact sequences in \( \text{Mod}_{f.d.} - A \)
\[
0 \to E_i[j - 1] \xrightarrow{u_{ij}} E_i[j] \xrightarrow{p'_{ij}} E_{i+1}[j-1] \to 0,
\]
\[
0 \to E_i[1] \xrightarrow{u'_{ij}} E_i[j] \xrightarrow{p_{ij}} E_{i+1}[j-1] \to 0,
\]
where \( p'_{ij} = p_{i+1, j-2} \circ \cdots \circ p_{ij} \) and \( u'_{ij} = u_{ij} \circ \cdots \circ u_{i2} \). Moreover, for each \( j \geq 2 \), there exists an almost split sequence
\[
0 \to E_{i}[j - 1] \xrightarrow{u_{ij}} E_{i+1}[j - 2] \oplus E_{i}[j] \xrightarrow{(u_{i+1,j-1} \circ p_{ij})} E_{i+1}[j - 1] \to 0,
\]
for \( i \in \{1, \ldots, r\} \) and all \( k \in \mathbb{Z} \).

(2) The indecomposable uniserial objects \( E_i[j] \), with \( i \in \{1, \ldots, r\} \) and \( j \geq 1 \), of the category \( \mathcal{E} \), connected by the homomorphisms \( u_{ij} : E_i[j - 1] \to E_i[j] \) and \( p_{ij} : E_i[j] \to E_{i+1}[j - 1] \), form the infinite diagram presented below.

(3) \( \text{Ext}_A^2(X, Y) = 0 \), for each pair of objects \( X \) and \( Y \) of \( \mathcal{E} \).

**Theorem 2.5.** [SS, Theorem 2.6 in Ch. X] Retain the hypothesis as in Theorem 2.4. Then the abelian category \( \mathcal{E} \) has the following properties.

1. Every indecomposable object \( M \) of the category \( \mathcal{E} \) is uniserial and is of the form \( M \cong E_i[j] \), where \( i \in \{1, \ldots, r\} \) and \( j \geq 1 \).
2. The collection of indecomposable objects forms a self-hereditary component, denoted by \( \Xi_\mathcal{E} \), of \( \Gamma(\text{Mod}_{f.d.} - A) \).
3. The component \( \Xi_\mathcal{E} \) is a standard stable tube of rank \( r \) [SS, Definition 1.1 in Ch. X].
4. The objects \( E_1, \ldots, E_r \) form the complete set of objects lying on the mouth [SS, Definition 1.2 in Ch. X] of the tube \( \Xi_\mathcal{E} \).

Let \( \Xi \) be the \( \Xi_\mathcal{E} \) defined as in Theorem 2.5. Let \( D := \text{Hom}_k(-, k) \) be the usual \( k \)-linear dual.

**Corollary 2.6.** [SS, Corollary 2.7 in Ch. X] Retain the hypothesis as in Theorem 2.4.
(1) The only homomorphisms between two indecomposable modules in \( \mathfrak{X} \) are \( \mathbb{k} \)-linear combinations of compositions of the homomorphisms \( u_{ij}, p_{ij} \), and the identity homomorphisms, and they are only subject to the relations arising from the almost split sequences in Theorem 2.4(1).

(2) Given \( i \in \{1, \cdots, r\} \) and \( j \geq 1 \), we have

- (2a) \( \text{End}_A(E_i[j]) \cong \mathbb{k}[t]/(t^m) \), for some \( m \geq 1 \),
- (2b) \( \text{End}_A(E_i[j]) \cong \mathbb{k} \) if and only if \( j \leq r \), and
- (2c) \( \text{Ext}^1_A(E_i[j], E_i[j]) \cong D \text{Hom}_A(E_i[j], \tau E_i[j]) = 0 \) if and only if \( j \leq r-1 \).

(3) If the tube \( \mathfrak{X} \) is homogeneous (namely, \( r = 1 \)), then \( \text{Ext}^3(M, M) \neq 0 \), for any indecomposable \( M \) in \( \mathcal{C} \).

Brick objects in \( \mathfrak{X} \) are determined by Corollary 2.6(2b). To work out all brick sets in \( \mathfrak{X} \), we need to understand the \( \text{Hom} \) between brick objects. Part (2) of the following theorem describes these \( \text{Homs} \).

**Theorem 2.7.** Let \( \mathfrak{X} \) be a standard stable tube of rank \( r \) as used in Theorem 2.5 and Corollary 2.6. Keep the notation as above and assume \( 1 \leq i, j, i', j' \leq r \). Then the following hold.

1. \( \text{End}_\mathfrak{X}(E_i[j]) \cong \mathbb{k} \).
2. \( \text{Hom}_\mathfrak{X}(E_i[j], E_{i'}[j']) \neq 0 \) if and only if \((i', j')\) satisfies one of the following conditions:
   - (2a) \( i' \leq i + j - 1 \) and \( i + j \leq i' + j' \),
   - (2b) \( i' \leq i + j - 1 - r \) and \( i + j \leq i' + j' + r \). Here, if \( i + j - 1 - r < 1 \), then \( \{1 \leq i' \leq i + j - 1 - r \} = \emptyset \).
   - Moreover, if \( \text{Hom}_\mathfrak{X}(E_i[j], E_{i'}[j']) = 0 \), then \( \text{Hom}_\mathfrak{X}(E_i[j], E_{i'}[j']) \cong \mathbb{k} \).

**Proof.** (1) See Corollary 2.6(2).

(2) Since \( \mathfrak{X} \) is a standard stable tube, it is a mesh category [SS, Definition 2.4 in Ch. X]. Moreover, by Corollary 2.6(a), the only homomorphisms between two indecomposable modules in \( \mathfrak{X} \) are \( \mathbb{k} \)-linear combinations of compositions of the homomorphisms \( u_{ij}, p_{ij} \), and the identity homomorphisms, which subject to the relations arising from the almost split sequences in Theorem 2.4(a). By mesh relationship [SS, Definition 2.4 in Ch. X], we obtain the description (2a) and (2b) for all objects \( E_{i'}[j'] \) that satisfy \( \text{Hom}_\mathfrak{X}(E_i[j], E_{i'}[j']) \neq 0 \).

Moreover, if \( j' < j \), \( \text{Hom}_\mathfrak{X}(E_i[j], E_{i'}[j']) \) is generated by composition morphisms

\[
\begin{align*}
&u_{i', j'} \cdots u_{i', i + j - i' + l + 1} \cdots u_{i', i + j - i' + l + 1} p_{i', i + j - i' + l + 1} u_{i', i + j - i' + l + 1} p_{i', i + j - i' + l + 1} \cdots u_{i', j + 1 - p_{i', i + j - i' + l + 1} u_{i', j + 1 - p_{i', i + j - i' + l + 1} u_{i', i + j - i' + l + 1} u_{i, j + 1 - p_{i, i + j - i' + l + 1} u_{i, j + 1 - p_{i, i + j - i' + l + 1} u_{i, j + 1} \cdots u_{i, j + 1} u_{i, j + 1},
\end{align*}
\]

where \( 0 \leq k \leq i' - i, 0 \leq l \leq (i' + j') - (i + j) - 1 \). Here, if \( i + k > r \), then the index \( i + k \) means \( i + k - r \).

If \( i' \geq j \), \( \text{Hom}_\mathfrak{X}(E_i[j], E_{i'}[j']) \) is generated by

\[
\begin{align*}
p_{i, i + j - i' + l + 1} \cdots p_{i, i + j - i' + l + 1} p_{i, i + j - i' + l + 1} u_{i, i + j - i' + l + 1} u_{i, i + j - i' + l + 1} p_{i, i + j - i' + l + 1} u_{i, i + j - i' + l + 1} \cdots u_{i, j + 1} u_{i, j + 1},
\end{align*}
\]

where \( 0 \leq k \leq (i' + j') - (i + j), 0 \leq l \leq i' - i - 1 \). Here, if \( i + l > r \), then the index \( i + l \) means \( i + l - r \). Therefore \( \text{Hom}_\mathfrak{X}(E_i[j], E_{i'}[j']) \cong \mathbb{k} \). \( \square \)

Part (1) of Corollary 2.8 next is just a re-interpretation of Theorem 2.7(2).

**Corollary 2.8.** Let \( \mathfrak{X} \) be a standard stable tube of rank \( r \). Keep the notation as above and put \( E_i[j], 1 \leq i, j \leq r \) in order

\[
E_1[1], E_2[1], \cdots, E_r[1]; E_1[2], E_2[2], \cdots, E_r[2]; \cdots; E_1[r], E_2[r], \cdots, E_r[r],
\]

and denote them by \( X_1, \cdots, X_n \) where \( n = r^2 \).
(1) The $n \times n$ matrix

\[
(\dim \text{Hom}_\mathcal{T}(X_j, X_i))_{n \times n}
\]

has the following form

\[
\begin{pmatrix}
\begin{array}{cccccc}
p^0 & p^1 & p^2 & p^3 & \ldots & p^{r-1} \\
p^0 & \sum_{i=0}^{1} P^i & \sum_{i=1}^{2} P^i & \sum_{i=2}^{3} P^i & \ldots & \sum_{i=r-2}^{r-1} P^i \\
p^0 & \sum_{i=0}^{1} P^i & \sum_{i=1}^{2} P^i & \sum_{i=2}^{3} P^i & \ldots & \sum_{i=r-3}^{r-1} P^i \\
p^0 & \sum_{i=0}^{1} P^i & \sum_{i=1}^{2} P^i & \sum_{i=2}^{3} P^i & \ldots & \sum_{i=r-4}^{r-1} P^i \\
p^0 & \sum_{i=0}^{1} P^i & \sum_{i=1}^{2} P^i & \sum_{i=2}^{3} P^i & \ldots & \sum_{i=r-5}^{r-1} P^i \\
p^0 & \sum_{i=0}^{1} P^i & \sum_{i=1}^{2} P^i & \sum_{i=2}^{3} P^i & \ldots & \sum_{i=r-6}^{r-1} P^i \\
\end{array}
\end{pmatrix}
\]

\[(E2.8.1)\]

where

\[
P = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & 1 & 0 \\
\end{pmatrix}
\]

and where $P^0$ is the identity matrix $I_{r \times r}$ of order $r$.

(2) The $n \times n$-matrix

\[
(\dim \text{Ext}^1_\mathcal{T}(X_j, X_i))_{n \times n}
\]

has the following form

\[
\begin{pmatrix}
\begin{array}{cccccc}
p^{r-1} & p^{r-1} & p^{r-1} & p^{r-1} & \ldots & p^{r-1} \\
p^{r-2} & \sum_{i=r-2}^{r-1} P^i & \sum_{i=r-2}^{r-1} P^i & \sum_{i=r-2}^{r-1} P^i & \ldots & \sum_{i=r-2}^{r-1} P^i \\
p^{r-3} & \sum_{i=r-3}^{r-2} P^i & \sum_{i=r-3}^{r-2} P^i & \sum_{i=r-3}^{r-2} P^i & \ldots & \sum_{i=r-3}^{r-2} P^i \\
p^{r-4} & \sum_{i=r-4}^{r-3} P^i & \sum_{i=r-4}^{r-3} P^i & \sum_{i=r-4}^{r-3} P^i & \ldots & \sum_{i=r-4}^{r-3} P^i \\
p^0 & \sum_{i=0}^{1} P^i & \sum_{i=1}^{2} P^i & \sum_{i=2}^{3} P^i & \ldots & \sum_{i=r-5}^{r-1} P^i \\
\end{array}
\end{pmatrix}
\]

\[(E2.8.2)\]

**Proof.**

(1) This follows from Theorem 2.7(2).

(2) The assertion follows from part (1) and the Serre duality

\[
\text{Ext}^1_\mathcal{T}(E_i[j], E_i[j']) \cong D \text{Hom}_\mathcal{T}(E_i[j'], \tau E_i[j]) = D \text{Hom}_\mathcal{T}(E_i[j'], E_{i-1}[j]).
\]

Some detailed matching of entries is omitted.

We use the following example to illustrate the results in Corollary 2.8.

**Example 2.9.** Let $\mathcal{T}$ be a standard stable tube of rank 3:
Put $E_{ij}, 1 \leq i, j \leq 3$ in order

$$E_1[1], E_2[1], E_3[1]; E_1[2], E_2[2], E_3[2]; E_1[3], E_2[3], E_3[3].$$

Denote this list as $X_1, \ldots, X_9$. Then we have

**Table 1. Hom$_3^j(X_j, X_i)$**

| $X_j = E_1[1]$ | $X_j = E_2[1]$ | $X_j = E_3[1]$ | $X_j = E_1[2]$ | $X_j = E_2[2]$ | $X_j = E_3[2]$ | $X_j = E_1[3]$ | $X_j = E_2[3]$ | $X_j = E_3[3]$ |
|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|
| $E_2[1]$      | 0             | 0             | 0             | 0             | 0             | 0             | 0             | 0             |
| $E_3[1]$      | 0             | 0             | 0             | 0             | 0             | 0             | 0             | 0             |
| $E_2[2]$      | 0             | 0             | 0             | 0             | 0             | 0             | 0             | 0             |
| $E_3[2]$      | 0             | 0             | 0             | 0             | 0             | 0             | 0             | 0             |
| $E_3[3]$      | 0             | 0             | 0             | 0             | 0             | 0             | 0             | 0             |
| $E_3[1]$      | 0             | 0             | 0             | 0             | 0             | 0             | 0             | 0             |
| $E_3[2]$      | 0             | 0             | 0             | 0             | 0             | 0             | 0             | 0             |
| $E_3[3]$      | 0             | 0             | 0             | 0             | 0             | 0             | 0             | 0             |

**Table 2. Ext$_1^j(X_j, X_i)$**

| $X_j = E_1[1]$ | $X_j = E_2[1]$ | $X_j = E_3[1]$ | $X_j = E_1[2]$ | $X_j = E_2[2]$ | $X_j = E_3[2]$ | $X_j = E_1[3]$ | $X_j = E_2[3]$ | $X_j = E_3[3]$ |
|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|
| $E_2[1]$      | 0             | 0             | 0             | 0             | 0             | 0             | 0             | 0             |
| $E_3[1]$      | 0             | 0             | 0             | 0             | 0             | 0             | 0             | 0             |
| $E_2[2]$      | 0             | 0             | 0             | 0             | 0             | 0             | 0             | 0             |
| $E_3[2]$      | 0             | 0             | 0             | 0             | 0             | 0             | 0             | 0             |
| $E_3[3]$      | 0             | 0             | 0             | 0             | 0             | 0             | 0             | 0             |
| $E_2[1]$      | 0             | 0             | 0             | 0             | 0             | 0             | 0             | 0             |
| $E_2[2]$      | 0             | 0             | 0             | 0             | 0             | 0             | 0             | 0             |
| $E_2[3]$      | 0             | 0             | 0             | 0             | 0             | 0             | 0             | 0             |

The corresponding Hom-dimension and Ext$^1$-dimension matrices are

$$\left( \dim \text{Hom}_3^j(X_j, X_i) \right)_{9 \times 9} = \begin{pmatrix} P^0 & P^1 & P^2 \\ P^0 & \sum_{i=0}^1 P^i & \sum_{i=1}^2 P^i \\ P^0 & \sum_{i=0}^1 P^i & \sum_{i=0}^2 P^i \end{pmatrix}$$
and

\[
\dim \text{Ext}^2_\mathcal{X}(X_j, X_i)_{9 \times 9} = \begin{pmatrix}
P^2 & P^2 & P^2 \\
P^1 & \sum_{i=1}^2 P^i & \sum_{i=1}^2 P^i \\
P^0 & \sum_{i=0} P^i & \sum_{i=0} P^i
\end{pmatrix}
\]

where

\[
P = \begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}.
\]

The next lemma shows that if \( H(i_1, \ldots, i_s) \) is a principal submatrix of (E2.8.1) such that \( H(i_1, \ldots, i_s) = I_{s \times s} \), then the corresponding principal submatrix of (E2.8.2) (with the same rows and columns) has spectral radius at most 1.

**Lemma 2.10** (Lemma 6.4). Retain the above notation. Suppose the Hom-matrix is given as in (E2.8.1) and Ext\(^1\)-matrix is given as in (E2.8.2), then \( \rho(A(\phi)) \leq 1 \) for every brick set \( \phi \).

The proof of Lemma 2.10 is given in Appendix A. Note that (E2.8.1)-(E2.8.2) are in fact the transpose of the usual Hom and Ext-matrices. By [CG, Lemma 3.7] (by considering the opposite category), Lemma 2.10 holds for the transpose matrices of (E2.8.1)-(E2.8.2) too.

**Theorem 2.11.** Let \( \mathcal{E} \) and \( \mathcal{X} \) be as in Theorem 2.5. Then \( \text{fpd} \mathcal{E} = 1 \).

**Proof.** By Corollary 2.6(2b), all brick objects in \( \mathcal{E} \) are \( E_i[j] \) for all \( i \in \mathbb{Z}/(r) \) and \( 1 \leq j \leq r \). We can determine all brick sets by using the matrix in Corollary 2.8(1). For each brick set, its Ext\(^1\)-matrix \( M \) was determined by using (E2.8.2). By Lemma 2.10, \( \rho(M) \leq 1 \). On the other hand, let \( \phi = \{ E_i[1], \ldots, E_r[1] \} \), then \( M = P^{r-1} \) and hence \( \text{fpd} \mathcal{E} = \rho(A(\phi, \text{Ext}^1)) = 1 \). Therefore the assertion follows. \( \square \)

We have an immediate consequence. Let \( T_r \) be the algebra in Example 2.3.

**Corollary 2.12.** Let \( \mathfrak{A} \) be the category of finite dimensional left \( x \)-nilpotent \( T_r \)-modules. Then \( \text{fpd} \mathfrak{A} = 1 \).

**Proof.** It is well-known that \( \mathfrak{A} \) is equivalent to the category \( \mathcal{E} \) of rank \( r \). To see this we set \( \text{deg} x = 1 \) and \( \text{deg} g = 0 \). Then the degree zero component of \( T_r \) is isomorphic to \( k^{\oplus r} \) with primitive idempotents \( \{ e_1, \ldots, e_r \} \). Under this setting, \( E_i[j] \) is identify with \( (T_r/(x^i))e_j \) for all \( i, j \). Now the result follows from Theorem 2.11. \( \square \)

### 2.3. Proof of Theorem 0.1

Now we are ready to show Theorem 0.1.

**Theorem 2.13.** Let \( \mathcal{X} \) be a domestic or tubular weighted projective line. Then \( \text{fpd} D^b(\text{coh}(\mathcal{X})) = 1 \).

**Proof.** By Lemma 2.1(4), it suffices to show that \( \text{fpd}(D^b(\text{coh}(\mathcal{X}))) \leq 1 \). By [CG, Theorem 3.5(4)], it is enough to show that

\[
\text{fpd}(\sigma) = \text{fpd}(\text{coh}(\mathcal{X})) \leq 1
\]

where \( \sigma \) is \( \text{Ext}^2_\mathcal{X}(\cdot, \cdot) \).

By [Sc, Corollary 4.34(iii)], every brick (or indecomposable) object is semistable. By Lemma 2.1(5), the class \( \{ \text{Vect}_\mu(\mathcal{X}) \}_{\mu \in \mathbb{Q} \cup \{ \infty \}} \) is a \( \sigma \)-decomposition (see the
definition before Lemma 1.6). By Lemma 1.6, it is enough to show the claim that 
\[ \text{fpd} |_{\text{Vect}_\mu(X)}(\sigma) \leq 1 \] for every \( \mu \).

Case 1: \( X \) is domestic. If \( \mu \) is finite, then \( \text{fpd} |_{\text{Vect}_\mu(X)}(\sigma) = 0 \) by Lemma 2.1(6). If \( \mu = \infty \), then, by (E2.0.1), \( \text{Vect}_\infty(X) := \text{Tor}(X) \) has a decomposition into Auslander-Reiten components, which all are tubes of finite rank. By Theorem 2.11, \( \text{fpd} |_{\text{Tor}(X)}(\sigma) = 1 \). The claim follows.

Case 2: \( X \) is tubular. By Lemma 2.1(8),
\[ \text{Vect}_\mu(X) \cong \text{Vect}_\infty(X) = \text{Tor}(X) \]
for all \( \mu \). Then the proof of Case 1 applies. Therefore the claim follows. \( \square \)

As usual we use \( \mathcal{O}_X \) for the structure sheaf of \( X \) [GL, Sect. 1.5].

**Proposition 2.14.** Let \( X \) be a weighted projective line of wild type. Then
\[ \text{fpd} \ D^b(\text{coh}(X)) \geq \dim \text{Hom}_X(\mathcal{O}_X, \mathcal{O}_X(\omega)) \]
where \( \omega \) is the dualizing element [GL, Sec. 1.2].

**Proof.** Let \( \phi = \{\mathcal{O}_X\} \) which is a brick and atomic object. Then, by definition and Serre duality [GL, Theorem 2.2],
\[ \text{fpd} \ D^b(\text{coh}(X)) \geq \rho(A(\phi, \text{Ext}^1)) = \dim \text{Ext}^1_X(\mathcal{O}_X, \mathcal{O}_X) = \dim \text{Hom}_X(\mathcal{O}_X, \mathcal{O}_X(\omega)). \]
\( \square \)

3. Dimension theory for classical projective schemes

The aim of this section is to introduce Frobenius-Perron versions of two important and related invariants – Calabi-Yau dimension and Kodaira dimension of \( D^b(\text{coh}(X)) \) where \( X \) is a smooth projective scheme.

3.1. A result from [CG]. Let \( X \) be a smooth (irreducible) projective scheme over \( \mathbb{C} \) of positive dimension. By [CG, Proposition 6.5 and 6.7],
\[ \text{fpd}(D^b(\text{coh}(X))) = \begin{cases} 1 & \text{if } X \text{ is } \mathbb{P}^1 \text{ or an elliptic curve,} \\ \infty & \text{otherwise.} \end{cases} \]

3.2. Calabi-Yau dimension. Recall from [Ke1, Section 8.1] that if a Hom-finite category \( C \) has a Serre functor \( S \), then there is a natural isomorphism
\[ \text{Hom}_C(X,Y)^* \cong \text{Hom}_C(Y,S(X)) \]
for all \( X, Y \in C \). A (pre-)triangulated Hom-finite category \( C \) with Serre functor \( S \) is called \( n \)-Calabi-Yau if there is a natural isomorphism
\[ S \cong \Sigma^n =: [n] \]
where \( \Sigma \) is the suspension of \( C \). In this case \( n \) is called the Calabi-Yau dimension of \( C \). (In [Ke2, Section 2.6] it is called weakly \( n \)-Calabi-Yau.) More generally, \( C \) is called a fractional Calabi-Yau category if there is an \( m > 0 \) and there is a natural isomorphism
\[ S^m \cong \Sigma^n =: [n] \]
for some \( n \), see [vR, p.2708] and [Ku, Definition 1.2]. In this case we say \( C \) has Calabi-Yau dimension \( \frac{n}{m} \). Abelian hereditary fractionally Calabi-Yau categories
are classified in [vR]. One key property of Calabi-Yau varieties is that the canonical bundle of these varieties are trivial. However, our definition of fp Calabi-Yau dimension (see Definition 3.1 below) applies to projective schemes that do not have the trivial canonical bundle.

If a Serre functor exists, then it is unique up to isomorphism. In this case we usually use $S$ for the Serre functor. Throughout the rest of this section, let $\mathcal{T}$ be a Hom-finite (pre-)triangulated category with Serre functor $S$. We will define a version of (fractional) Calabi-Yau dimension for $\mathcal{T}$ which is not necessarily a (fractional) Calabi-Yau category.

Recall from Definition 1.4(3) that the Frobenius-Perron growth of a functor $\sigma$ is defined to be

$$fpg(\sigma) := \sup_{\phi \in \Phi_a} \{ \limsup_{n \to \infty} \log_n(\rho(A(\phi, \sigma^n))) \}.$$  

By convention, $\log_0 0 = -\infty$. Similarly, we define a slightly modified version of $fpg$ as follows, which is used to define the fp Calabi-Yau dimension.

**Definition 3.1.** Let $\sigma$ be an endofunctor of $\mathcal{T}$ with Serre functor $S$.

1. The lower Frobenius-Perron growth of $\sigma$ is defined to be

$$fpg(\sigma) := \sup_{\phi \in \Phi_a} \{ \liminf_{n \to \infty} \log_n(\rho(A(\phi, \sigma^n))) \}.$$  

By convention, $\log_0 0 = -\infty$.

2. The spectrum of $\mathcal{T}$ is defined to be

$$Sp(\mathcal{T}) := \{(m, n) \in \mathbb{Z} \times \mathbb{Z}^2 | fpg(S^m \circ \Sigma^{-n}) > -\infty \}.$$  

3. The fp Calabi-Yau dimension of $\mathcal{T}$ is defined to be

$$fpcy(\mathcal{T}) := \lim_{M \to \infty} \left\{ \sup_{|m| \geq M} \left\{ \frac{n}{m} | (m, n) \in Sp(\mathcal{T}) \right\} \right\}.$$  

Next we show that the fp Calabi-Yau dimension exists for various cases.

**Lemma 3.2.** Suppose $\mathcal{T}$ is a pre-triangulated category satisfying the following conditions:

1. $\mathcal{T}$ is Ext-finite [BV, Definition 2.1], namely, for all objects $X, Y \in \mathcal{T}$,

$$\sum_{s \in \mathbb{Z}} \dim \text{Hom}_\mathcal{T}(X, \Sigma^s(Y)) < \infty.$$  

2. $\mathcal{T}$ has a Serre functor $S$.

3. $\mathcal{T}$ is fractional Calabi-Yau of dimension $d = a/b \in \mathbb{Q}$.

Then the following holds.

1. $Sp(\mathcal{T}) \subseteq (b, a)\mathbb{Q}$.

2. If $\mathcal{T}$ contains at least one atomic object, there exists $w \in \mathbb{N}$ such that $(bw, aut) \in Sp(\mathcal{T})$ for all $t \in \mathbb{Z}$.

3. Under the hypothesis of part (2), we have $fpcy(\mathcal{T}) = d$.

**Proof.** (1) Let $(m, n)$ be a pair of integers that is not in $(b, a)\mathbb{Q}$, and let $\sigma = S^m \circ \Sigma^{-n}$. Since $\mathcal{T}$ is fractional Calabi-Yau of dimension $a/b$, we can assume that $S^b \circ \Sigma^{-a}$ is the identity functor. Then, for each $t$, $\sigma^t = \Sigma_t^{(ma-nb)}$ where $ma - nb \neq 0$. By hypothesis (a), for any two objects $X$ and $Y$ in $\mathcal{T}$, $\text{Hom}_\mathcal{T}(X, \Sigma_t^{(ma-nb)}Y) = 0$ for $|t| \gg 0$. Then, for every atomic set $\phi$, it implies that $A(\phi, \sigma^t) = 0$ for $|t| \gg 0$. 


Thus \( \log_{\Theta}(\rho(A, \sigma^m)) = -\infty \) for all \( |t| \gg 0 \). This implies, by definition, that 
\[ \text{fpg}(\sigma) = -\infty, \text{ or } (m, n) \not\in S_p(T). \]
The assertion follows.

(2) Let \( \phi \) be the set of a single atomic object \( X \) in \( T \). Replacing \((b, a)\) by \((bw, aw)\) for some positive integer \( w \) if necessary, we can assume that \( \sigma := S^b \circ \Sigma^{-a} \) is the identity functor. Then \( A(\phi, \sigma^n) \) is the \( 1 \times 1 \)-identity matrix \( I_1 \) for all \( n \). Then \( \log_n(\rho(A, \phi, \sigma^n)) = 0 \) for all \( n \). This implies that fpg(\( \sigma \)) = 0 > -\infty or \( (b, a) \in S(T) \). Similarly, one sees that \((bt, at) \in S(T)\) for all integer \( t \). The assertion follows.

(3) This follows from the definition and parts (1,2).

The next proposition is part (2) of Theorem 0.2, which shows that fpcy is indeed a generalization of Calabi-Yau dimension.

**Proposition 3.3.** Let \( X \) be a smooth irreducible projective variety of dimension \( d \in \mathbb{N} \) and let \( T \) be \( D^b(\text{coh}(X)) \). Then fpcy(\( T \)) = \( d \).

**Proof.** When \( d = 0 \), then \( T = D^b_{\mathbb{F},coh}(\text{Vect}_k) \). It is easy to see that the Serre functor \( S \) is the identity. So the assertion is easily shown. Now we assume that \( d > 0 \).

It suffices to show that \( S_p(T) = \{(t, td) \mid t \in \mathbb{Z}\} \).

By [BO, (7)], the Serre functor \( S \) is equal to \(-\otimes_X \omega_X[d]\) where \( d \) is the dimension of \( X \) and \( \omega_X \) is the canonical bundle of \( X \). Let \( \sigma = S \circ \Sigma^{-d} \). Then \( \sigma \) is the functor \(-\otimes_X \omega_X \). Let \( O_a \) be the skyscraper sheaf of a closed point \( a \in X \). Then \( \sigma \) is an atomic object and \( \sigma(O_a) \cong O_a \). Let \( \phi = \{O_a\} \). Then \( A(\phi, \sigma^n) \) is the \( 1 \times 1 \)-identity matrix \( I_1 \). This implies that \( \log_n(\rho(A, \phi, \sigma^n)) = 0 \) and that fpg(\( \sigma \)) = 0 > -\infty. Therefore \((1, d) \in S(T) \). Similarly, one sees that \((t, td) \in S(T)\) for all \( t \geq 0 \).

For the other implication, let \((m, n) \in \mathbb{Z}^{\geq 2} \setminus \{(t, td) \mid t \in \mathbb{Z}\} \). Let \( \sigma = S^m \circ \Sigma^{-n} \).

We need to show that that fpg\( (\sigma) = -\infty \). Note that \( \sigma = -\otimes_X \omega_X^m \circ \Sigma^{-n+dm} \) where \(-n + dm \not= 0 \). Since \( \text{coh}(X) \) has global dimension \( d \), for all objects \( A \) and \( B \) in \( T \),

\[ \text{Hom}_T(A, \sigma^t(B)) = \text{Hom}_T(A, (B \otimes_X \omega_X^m)[t(-n + dm)]) = 0 \]

for all \( t \gg 0 \). This implies that fpcy\( (\sigma) = -\infty \) as required.

\[ \square \]

### 3.3. Kodaira dimension

First we review the classical definition of the Kodaira dimension.

**Definition 3.4.** [La, Definition 2.1.3 and Example 2.1.5] Let \( X \) be a smooth projective variety and let \( \omega_X \) be the canonical bundle of \( X \).

(1) The **Kodaira dimension** of \( X \) is defined to be

\[ \kappa(X) := \lim_{n \to \infty} \log_n \left( \dim H^0(X, \omega_X^\otimes n) \right). \]

(2) More generally, for a line bundle \( M \), the **Kodaira-Iitaka dimension** of \( M \) is defined to be

\[ \kappa(X, M) := \lim_{n \to \infty} \log_n \left( \dim H^0(X, M^\otimes n) \right). \]

(3) The **anti-Kodaira dimension** of \( X \) is defined to be

\[ \kappa^{-1}(X) := \kappa(X, \omega_X^{-1}). \]

The anti-Kodaira dimension of a scheme was defined in [Sa]. It is classical and well-known that \( \kappa(X), \kappa(X, M) \in \{-\infty, 0, 1, \cdots, \dim X\} \) and that there are \( 0 < c_1 < c_2 \) such that,

\[ c_1 n^{\kappa(X, M)} \leq \dim \text{Hom}_X(O_X, M^\otimes n) \leq c_2 n^{\kappa(X, M)} \quad \forall \ n \gg 0. \]

(E3.4.1)
See [La, Corollary 2.1.37]. By Proposition 3.3, \(\dim X = \text{fpcy}(\mathcal{T})\), which suggests the following definition.

In the following we use the order \(t_1 < t_2\) if \(t_1\) divides \(t_2\). Since we are mainly interested in commutative and noncommutative projective schemes, \(\mathcal{T}\) is equipped with the structure sheaf, denoted by \(\mathcal{O}\).

**Definition 3.5.** Let \(\mathcal{T}\) be a pre-triangulated category and \(\mathcal{T}_*\) denote the pair \((\mathcal{T}, \mathcal{O})\) where \(\mathcal{O}\) is a given special object in \(\mathcal{T}\). Suppose that \(\mathcal{T}\) is Hom-finite with Serre functor \(S\) and that \(\text{fpcy}(\mathcal{T}) = \frac{d}{a/b}\) for some integers \(a, b\).

1. The **fp Kodaira dimension** of \(\mathcal{T}_*\) is defined to be
   \[
   \text{fp} \kappa(\mathcal{T}_*) := \lim_t \left\{ \limsup_{n \to \infty} \dim \text{Hom}_{\mathcal{T}}(\mathcal{O}, (S^b \circ \Sigma^{-an})(\mathcal{O})) \right\}
   \]
   where the first limit ranges over all positive integers \(t\) with order \(<\) as defined before the definition.

2. The **fp anti-Kodaira dimension** of \(\mathcal{T}_*\) is defined to be
   \[
   \text{fp} \kappa^{-1}(\mathcal{T}_*) := \lim_t \left\{ \limsup_{n \to \infty} \dim \text{Hom}_{\mathcal{T}}(\mathcal{O}, (S^{-b} \circ \Sigma^{an})(\mathcal{O})) \right\}
   \]
   where the first limit ranges over all positive integers \(t\) with order \(<\) as defined before the definition.

The following proposition justifies the above definition.

**Proposition 3.6.** Let \(X\) be a smooth irreducible projective variety of dimension \(d \in \mathbb{N}\) and let \(\mathcal{T}\) be \(D^b(\text{coh}(X))\) with structure sheaf \(\mathcal{O} := \mathcal{O}_X\). Then

\[
\text{fp} \kappa(\mathcal{T}_*) = \kappa(X) \quad \text{and} \quad \text{fp} \kappa^{-1}(\mathcal{T}_*) = \kappa^{-1}(X).
\]

**Proof.** By Proposition 3.3, \(\text{fpcy}(\mathcal{T}) = d = \dim X\). So we take \(b = 1\) and \(a = d\) in Definition 3.5. By (E3.4.1), for each \(t \geq 1\),

\[
\lim_{n \to \infty} \dim \text{Hom}_{\mathcal{T}}(\mathcal{O}, (S^b \circ \Sigma^{-an})(\mathcal{O})) = \kappa(X).
\]

The first assertion follows by the definition.

The proof for anti-Kodaira dimension is similar. \(\square\)

Theorem 0.2 follows from Propositions 3.3 and 3.6.

**Remark 3.7.** Assume the hypothesis of Lemma 3.2. Then one can check easily that

\[
\text{fp} \kappa(\mathcal{T}_*) = \text{fp} \kappa^{-1}(\mathcal{T}_*) = 0.
\]

So abstractly (E3.7.1) should be part of the definition of a fractional Calabi-Yau variety (even in the noncommutative setting).

**Proposition 3.8.** If \(\mathcal{T}\) is a triangulated category such that either \(\text{fp} \kappa(\mathcal{T}_*) = \infty\) or \(\text{fp} \kappa^{-1}(\mathcal{T}_*) = \infty\) or \(\text{fpcy}(\mathcal{T}) = \infty\) or \(-\infty\), then \(\mathcal{T}_*\) is not triangulated equivalent to the bounded derived category of a smooth projective scheme.

**Proof.** By definition, \(\text{fpcy}\) is an invariant of a triangulated category, and \(\text{fp} \kappa\) is an invariant of a triangulated category with \(\mathcal{O}\). The assertion follows from Propositions 3.3 and 3.6. \(\square\)
4. INVARIANTS OF NONCOMMUTATIVE PROJECTIVE SCHEMES

In this section we study fp Calabi-Yau dimension and fp Kodaira dimension of noncommutative projective schemes in the sense of [AZ]. An algebra \( A \) is said to be **connected graded** over \( k \) if \( A = k \oplus A_1 \oplus A_2 \oplus \cdots \) with \( A_i A_j \subseteq A_{i+j} \) for all \( i, j, \in \mathbb{N} \). Let \( A \) be a noetherian connected graded algebra. The **noncommutative projective scheme** associated to \( A \) is denoted by \( X := \text{Proj} A \), see [AZ] for the detailed definition of a noncommutative projective scheme. Let \( \text{coh}(X) \) be the category of noetherian objects in \( \text{Proj} A \) and let \( T \) be the triangulated category \( \text{D}^b(\text{coh}(X)) \). Here is a restatement of a nice result of Bondal-Van den Bergh [BV, Theorem 4.2.13].

**Theorem 4.1.** [BV, Theorem 4.2.13] Suppose \( A \) is noetherian and has a balanced dualizing complex and that \( \text{Proj} A \) has finite homological dimension. Then \( T \) has a Serre functor.

One special class of connected graded algebras are the Artin-Schelter regular algebras (see the next definition).

**Definition 4.2.** [AS] Let \( A \) be a connected graded algebra over the base field \( k \). We say \( A \) is **Artin-Schelter Gorenstein** (or **AS Gorenstein**) if the following conditions hold:

(a) \( A \) has finite injective dimension \( d \) on both sides,
(b) \( \text{Ext}^i_A(k, A) = \text{Ext}^i_A(k, A^{op}) = 0 \) for all \( i \neq d \) where \( k = A/A_{\geq 1} \), and
(c) \( \text{Ext}^d_A(k, A) \cong k(\ell) \) and \( \text{Ext}^d_A(k, A^{op}) \cong k(\ell) \) for some integer \( \ell \). This integer \( \ell \) is called the **AS index** of \( A \).

If moreover

(d) \( A \) has finite global dimension,

then \( A \) is called **Artin-Schelter regular** (or **AS regular**).

We collect some well-known facts below. Let \( \pi : \text{Gr} A \rightarrow \text{Proj} A \) be the canonical quotient functor. By abuse of notation, we also apply \( \pi \) to some graded \( A \)-bimodules such as \( \mu A_1 \) in Lemma 4.3 below.

**Lemma 4.3.** Let \( A \) be a noetherian connected graded algebra, let \( X \) be \( \text{Proj} A \), and let \( T \) be \( \text{D}^b(\text{coh}(X)) \).

(1) [Ye, Corollary 4.14] If \( A \) is Artin-Schelter Gorenstein, then \( A \) has a balanced dualizing complex.

(2) [YZ, Corollary 4.3] Suppose \( A \) is Artin-Schelter Gorenstein such that \( X \) has finite homological dimension. Let \( d = \text{injdim} A \), \( \ell \) be the AS index of \( A \) and \( \mu \) be the Nakayama automorphism of \( A \). Then the Serre functor of \( T \) is \( - \otimes_{O} \pi(\mu A^{1})(-\ell)[d - 1] \).

Let \( M \) be a locally finite \( \mathbb{Z} \)-graded module or vector space. The Hilbert series of \( M \) is defined to be

\[
H_M(t) := \sum_{n \in \mathbb{Z}} \dim M_n t^n.
\]

Let \( A \) be a graded algebra and \( s \) be a positive integer. The **\( s \)th Veronese subalgebra** of \( A \) is defined to be

\[
A^{(s)} := \oplus_{n \in \mathbb{Z}} A_{sn}.
\]

The following lemma is well-known.
Lemma 4.4. Let $A$ be a noetherian connected graded algebra generated in degree 1. Let $s$ be a positive integer.

1. $A$ is a finitely generated module over $A^{(s)}$ on both sides.
2. $\text{GKdim } A = \text{GKdim } A^{(s)}$.
3. If the Hilbert series of $A$ is a rational function, then so is the Hilbert series of $A^{(s)}$.
4. If the Hilbert series of $A$ is a rational function, then
   \[
   \limsup_{n \to \infty} \log n (\dim A_n) = \text{GKdim } A - 1.
   \]
5. If $A$ is an Artin-Schelter regular algebra, then the Hilbert series of $A$ is a rational function.

Proof. (1) Connected graded noetherian algebras are finitely generated. So $A$ is generated by $\bigoplus_{i=0}^{s-1} A_i$ over $A^{(s)}$ on both sides.
(2) It follows from part (1) and [MR, Proposition 8.2.9(i)].
(3) This follows from the fact that
   \[
   H_A(t) = \frac{1}{s} \sum_{i=0}^{s-1} H_A(\xi^i t)
   \]
where $\xi$ is an $s$th primitive root of unity.
(4) When $H_A(t)$ is a rational function, $\dim A_n$ is a multi-polynomial function of $n$ in the sense of [Zh, p.399]. Say $d$ is the degree of the multi-polynomial function $\dim A_n$ of $n$. Then $\text{GKdim } A = d + 1$ by using [Zh, (E7)]. This implies that
   \[
   \limsup_{n \to \infty} \log n (\dim A_n) = d = \text{GKdim } A - 1.
   \]
(5) This is [StZ, Proposition 3.1].

Theorem 4.5. Let $A$ be a noetherian connected graded Artin-Schelter Gorenstein algebra of injective dimension $d \geq 2$ that is generated in degree 1. Suppose that $X := \text{Proj } A$ has finite homological dimension. Let $T$ be the bounded derived category of $\text{coh}(X)$. In parts (2, 3, 4, 5, 6), we further assume that the Hilbert series of $A$ is rational. Let $\ell$ be the AS index of $A$.

1. $\text{fpcy}(T) = d - 1$.
2. If $\ell > 0$, then $\text{fpK}(T_s) = -\infty$ and $\text{fpK}^{-1}(T_s) = \text{GKdim } A - 1$.
3. If $\ell < 0$, then $\text{fpK}(T_s) = \text{GKdim } A - 1$ and $\text{fpK}^{-1}(T_s) = -\infty$.
4. If $\ell = 0$, then $\text{fpK}(T_s) = \text{fpK}^{-1}(T_s) = 0$.
5. For all objects $C$ and $D$ in $T$,
   \[
   \limsup_{n \to \infty} \log n (\dim \text{Hom}_T(C, (S \circ \Sigma^{-d})^n(D))) \leq \text{GKdim } A - 1.
   \]
6. For all objects $C$ and $D$ in $T$,
   \[
   \limsup_{n \to \infty} \log n (\dim \text{Hom}_T(C, (S \circ \Sigma^{-d})^{-n}(D))) \leq \text{GKdim } A - 1.
   \]

Proof. (1) Let $\ell$ be the AS index of $A$. There are two different cases: $\ell \leq 0$ and $\ell \geq 0$. The proofs are similar, so we only prove the assertion for the first case.
First we claim that $Sp(T) \subseteq (1, d - 1)\mathbb{Z}$. Suppose $(m, n) \notin (1, d - 1)\mathbb{Z}$. Let $\sigma = S^m \circ \Sigma^{-n} = -\otimes \mathcal{M}^\otimes m[(d - 1)m - n]$ where $\mathcal{M} = \pi(\mu A^1)(-\ell)$ and where
\((d - 1)m - n \neq 0\). Since \(\text{Proj } A\) has finite global dimension, we have that, for all objects \(A\) and \(B\) in \(\mathcal{T}\),

\[
\text{Hom}_T(A, \sigma^t(B)) = \text{Hom}_T(A, (B \otimes \mathcal{M}^\otimes m)[t((d - 1)m - n)]) = 0
\]

for all \(t \gg 0\). This implies that \(\text{fp}_c \sigma = -\infty\). Hence \((m, n) \notin \text{Sp}(\mathcal{T})\) and hence we have proven the claim.

Second we claim that \((1, d - 1)\mathbb{N} \subseteq \text{Sp}(\mathcal{T})\). Let \(\mathcal{O} = \pi(A)\). It is a brick object as \(\text{Hom}_T(\mathcal{O}, \mathcal{O}) = A_0 = \mathbb{k}\). For every \((m, n) = (s, s(d - 1)) \in (1, d - 1)\mathbb{N}\), let \(\sigma = S^m \circ \Sigma^{-n} = - \otimes \mathcal{M}^\otimes s\). Then

\[
\sigma(\mathcal{O}) \cong \mathcal{O}(-\ell s)
\]

and

\[
\text{Hom}_T(\mathcal{O}, \sigma(\mathcal{O})) \cong A_{-\ell s} \neq 0
\]

when \(\ell \leq 0\). This implies that \(\rho(A(\phi, \sigma)) \geq 1\) where \(\phi = \{\mathcal{O}\}\). Similarly, \(\rho(A(\phi, \sigma^s)) \geq 1\) for all \(n\). Consequently, \(\text{fp}_c \sigma = 0 > -\infty\). Therefore \((m, n) \in \text{Sp}(\mathcal{T})\) as desired. Now we have

\[
(1, d - 1)\mathbb{N} \subseteq \text{Sp}(\mathcal{T}) \subseteq (1, d - 1)\mathbb{Z}
\]

which implies that \(\text{fp}_c(\mathcal{T}) = d - 1\).

(2) Let \(w = \text{GKdim } A\) which is at least 2. By Lemma 4.4(2,3,4), for every integer \(s \geq 1\),

\[
\text{(E4.5.1)} \quad \limsup_{n \to \infty} \log_n (\text{dim } A_{sn}) = \text{GKdim } A - 1 = w - 1.
\]

Now assume that \(\ell\) is positive. By Lemma 4.3(2), \(\sigma := S \circ \Sigma^{-(d - 1)}\) is equivalent to \(- \otimes \mathcal{O}(\ell)\) when applied to \(\mathcal{O}\). Thus

\[
\text{(E4.5.2)} \quad \text{Hom}_T(\mathcal{O}, \sigma^n(\mathcal{O})) = A_{-\ell n} = 0
\]

when \(n \geq 0\) and

\[
\text{(E4.5.3)} \quad \text{Hom}_T(\mathcal{O}, \sigma^{-n}(\mathcal{O})) = A_{\ell n}
\]

for \(n \geq 0\). Now (E4.5.2) implies that \(\kappa(\mathcal{T}, \mathcal{O}) = -\infty\), and (E4.5.3) together with (E4.5.1) implies that \(\kappa^{-1}(\mathcal{T}, \mathcal{O}) = w - 1 = \text{GKdim } A - 1\).

(3) Similar to the proof of (2).

(4) Assume that \(\ell = 0\). Then \(\sigma := S \circ \Sigma^{-(d - 1)}\) is equivalent to \(- \otimes \mathcal{O}(\ell)\) when applied to \(\mathcal{O}\). Thus

\[
\text{Hom}_T(\mathcal{O}, \sigma^n(\mathcal{O})) = A_{-\ell n} = A_0 = \mathbb{k}
\]

for all \(n \in \mathbb{Z}\). This implies that \(\text{fp}_c(\mathcal{T}_s) = \text{fp}_c^{-1}(\mathcal{T}_s) = 0\).

(5,6) The proofs are similar. We only consider (5). Note that \(S \circ \Sigma^{-(d - 1)} = - \otimes \pi^\nu A^1(-\ell)\). By [BV, Lemmas 4.2.3 and 4.3.2], \(\mathcal{T}\) is generated by \(\{\mathcal{O}(n)\}_{n \in \mathbb{Z}}\). Hence we can assume that \(\mathcal{C} = \mathcal{O}\) and \(\mathcal{D} = \mathcal{O}(a)[b]\) for some \(a\) and \(b\). Note that [AZ, Theorem 8.1(3)] holds for Artin-Schelter Gorenstein algebras. Then [AZ, Theorem 8.1(3)] implies that

\[
\dim \text{Ext}_A^b(\mathcal{O}, (S \circ \Sigma^{-(d - 1)}) \otimes n(\mathcal{O}(a))) \leq cn^{w - 1}
\]

for some constant \(c\) only dependent on \(a, b\). Therefore the assertion follows. \(\square\)
The noncommutative projective scheme in the sense of [AZ] can be defined for connected graded coherent algebras that are not necessarily noetherian. Here we consider a family of noncommutative projective schemes of non-noetherian Artin-Schelter regular algebras of global dimension two.

Let $W_n$ be the Artin-Schelter regular algebra $k\langle x_1, \ldots, x_n \rangle / (\sum_{i=1}^n x_i^2)$ of global dimension 2. When $n \geq 3$, this algebra is non-noetherian [Zh, Theorem 0.2(1)], but coherent [Pi, Theorem 1.2]. Let $\mathbb{P}^1_n$ denote the noncommutative projective scheme associated to $W_n$ defined in [Pi], which is also denoted by Proj$W_n$. We call $\mathbb{P}^1_n$ a Piontkovski projective line of rank $n$. See [Pi, SSmi] for basic properties of $\mathbb{P}^1_n$. The main result concerning $\mathbb{P}^1_n$ is the following. See [CG] for the definition of fp$c_x$ and fp$v$.

**Theorem 4.6.** Let $\mathbb{P}^1_n$ be a Piontkovski projective line of rank $n \geq 2$. Let $\mathcal{T}_n$ be the derived category $D^b(\text{coh}(\mathbb{P}^1_n))$. Then

1. $\text{fpd}(\mathcal{T}_n) = 1$ for all $n \geq 2$.
2. $\text{fp}\text{gldim}(\mathcal{T}_n) = 1$ for all $n \geq 2$.
3. $\text{fp}c_x(\mathcal{T}_n) = 0$ for all $n \geq 2$.
4. $\text{fp}v(\mathcal{T}_n) = 0$ for all $n \geq 2$.
5. $\text{fp}c_y(\mathcal{T}_n) = 1$ for all $n \geq 2$.
6. $\text{fp}c^{-1}(\mathcal{T}_n)_a = \begin{cases} 1 & n = 2, \\ \infty & n \geq 3. \end{cases}$
7. $\text{fp}c(\mathcal{T}_n)_a = -\infty$ for all $n \geq 2$.

**Proof.** Let $\mathcal{O}$ denote the object $\pi(W_n)$ in $\mathcal{X} := \text{Proj} W_n$. Then $\text{coh}(\mathcal{X})$ is hereditary, and $\mathcal{T}_n$ is Ext-finite with Serre functor $S := - \otimes \mathcal{O}(-2)[1]$ [Pi, Proposition 1.5].

1. By [CG, Theorem 3.5(4)], $\text{fpd}(\mathcal{T}_n) = \text{fpd}(\text{coh}(\mathcal{X}))$. Since $\text{coh}(\mathcal{X})$ is hereditary, $\text{fpd}(\text{coh}(\mathcal{X})) \leq 1$. It remains to show that $\text{fpd}(\text{coh}(\mathcal{X})) \geq 1$. Let $A$ be the simple object in $\text{coh}(\mathcal{X})$ of the form $\pi(W/I)$ where $I$ is the right ideal of $W$ generated by $W(x_3, \ldots, x_n) + x_2$. Then it is routine to verify that $\text{Ext}^1_{\text{coh}(\mathcal{X})}(A, A) = k$. This implies that $\text{fpd}(\text{coh}(\mathcal{X})) \geq 1$ as required.

2. By [CG, Theorem 3.5(1)], $\text{fp}\text{gldim}(\mathcal{T}_n) \leq \text{gldim} \text{coh}(\mathcal{X}) = 1$. By part (1), $\text{fp}\text{gldim}(\mathcal{T}_n) \geq 1$. The assertion follows.

3,4 These follow from the fact that $\text{coh}(\mathcal{X})$ is hereditary.

5. Since the Serre functor $S$ is of the form $- \otimes \mathcal{O}(-2)[1]$, we can almost copy the proof of Proposition 3.3.

6,7 Using the special form of the Serre functor $S$, we can adapt the proofs of Proposition 3.6 and Theorem 4.5. \hfill $\square$

5. **Comments on fp-invariants of finite dimensional algebras**

In this section we give some remarks, comments and examples concerning finite dimensional algebras. Let $\mathcal{T}(A)$ be the derived category $D^b(\text{Mod}_f,d - A)$. The fp Calabi-Yau dimension $\text{fp}c_y(\mathcal{T})$ is defined as in Definition 3.1.

**Example 5.1.** Let $Q$ be a finite acyclic quiver and $A$ be the path algebra $kQ$. Let $\mathcal{T}$ be the bounded derived category $D^b(\text{Mod}_f,d - A)$.

1. If $Q$ is of ADE type, then $\mathcal{T}$ is fractional Calabi-Yau and by Lemma 3.2(3) and [Ke1, Example 8.3(2)],

$$\text{fp}c_y(\mathcal{T}) = \frac{h - 2}{h}$$
where \( h \) is the Coxeter number of \( Q \). In this case, \( \text{fpcy}(T) \) is strictly between 0 and 1.

(2) If \( Q \) is of \( \tilde{A}\tilde{D}\tilde{E} \) type, then, by using Lemma 2.1(2), \( T \) is equivalent to \( D^b(\text{coh}(\mathbb{X})) \) for a weighted projective line \( \mathbb{X} \). Similar to the proof of Theorem 4.5(1), one can show that
\[
\text{fpcy}(D^b(\text{coh}(\mathbb{X}))) = 1
\]
for every weighted projective line \( \mathbb{X} \). (The proof is slightly more complicated and details are omitted). Thus we obtain that
\[
\text{fpcy}(T) = 1.
\]

If \( Q \) is of wild representation type, in some cases, one can show that
\[
\text{fpcy}(T) = 1.
\]

Such an example is the Kronecker quiver with two vertices and \( n \) arrows from the first vertex to the second, see Example 5.7. It is not clear to us if (E5.1.1) holds for all acyclic quivers of wild representation type.

**Lemma 5.2.** Let \( A \) and \( B \) be finite dimensional algebras of finite global dimension. Suppose that both \( T(A) \) and \( T(B) \) are fractional Calabi-Yau of dimension \( d_1 \) and \( d_2 \) respectively. Then \( T(A \otimes B) \) is fractional Calabi-Yau of dimension \( d_1 + d_2 \).

The proof of Lemma 5.2 is straightforward and hence omitted. One immediate question is

**Question 5.3.** Let \( A \) and \( B \) be finite dimensional algebras of finite global dimension. Is then
\[
\text{fpcy}(T(A \otimes B)) = \text{fpcy}(T(A)) + \text{fpcy}(T(B))?
\]

To define fp (anti-)Kodaira dimension of \( T \) as in Definition 3.5, we need to specify an object \( O \) which plays the role of the structure sheaf in algebraic geometry. One choice for \( O \) is the left \( A \)-module \( A \). So we let \( T(A)_* \) be \((T(A), A)\).

**Definition 5.4.** Let \( A \) be a finite dimensional algebra of finite global dimension. Suppose that \( T(A) \) has a fractional fp Calabi-Yau dimension \( \frac{d}{2} \in \mathbb{Q} \).

(1) The \textit{fp Kodaira dimension} of \( A \) is defined to be
\[
\text{fp} \kappa(A) := \text{fp}(T(A)_*) = \lim_{n \to \infty} \left\{ \limsup_{n \to \infty} \text{dim}_{T} \text{Hom}(A, (S^{\text{bnt}} \circ \Sigma^{-\text{ant}})(A)) \right\}
\]
where the first limit ranges over all positive integers \( t \) with order \( < \) as defined before Definition 3.5.

(2) The \textit{fp anti-Kodaira dimension} of \( A \) is defined to be
\[
\text{fp} \kappa^{-1}(A) := \text{fp}^{-1}(T(A)_*) = \lim_{n \to \infty} \left\{ \limsup_{n \to \infty} \text{dim}_{T} \text{Hom}(A, (S^{-\text{bnt}} \circ \Sigma^{\text{ant}})(A)) \right\}
\]
where the first limit ranges over all positive integers \( t \) with order \( < \) as defined before Definition 3.5.

**Remark 5.5.** Suppose \( T(A) \) is fractional Calabi-Yau. Then one can check easily that
\[
\text{(E5.5.1)} \quad \text{fp}(A) = \text{fp}^{-1}(A) = 0.
\]
So if there is a notion of a fractional Calabi-Yau algebra, (E5.5.1) should be a part of the definition.
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Let $A$ be a finite dimensional algebra of finite global dimension. Then the Serre functor is given by $- \otimes A^*$. It is unknown if $\text{fpcy}(T(A))$ always exists. If $\text{fpcy}(T(A))$ exists and is a rational number, then we can define and calculate Kodaira dimension (respectively, anti-Kodaira dimension) of $A$. Here is a list of questions that are related to the $\text{fpcy}$ Kodaira dimension.

**Question 5.6.** Let $A$ and $B$ be two finite dimensional algebra of finite global dimension. Suppose that $\text{fpcy}(T(A))$ and $\text{fpcy}(T(B))$ are rational numbers.

1. Are $\text{fpcy}(A)$ and $\text{fpcy}^{-1}(A)$ less than $\infty$?
2. If $T(A)$ is triangulated equivalent to $T(B)$, is then $\text{fpcy}(A) = \text{fpcy}(B)$ and $\text{fpcy}^{-1}(A) = \text{fpcy}^{-1}(B)$?
3. Suppose both $\text{fpcy}(A)$ and $\text{fpcy}(B)$ are finite, is then $\text{fpcy}(A \otimes B) = \text{fpcy}(A) + \text{fpcy}(B)$?

The first question has a negative answer.

**Example 5.7.** Let $Q_n$ be the Kronecker quiver with two vertices and $n$ arrows from the first vertex to the second. Let $U_n$ be the path algebra of $Q_n$. By [Min, Theorem 0.1], if $n \geq 2$,

$$D^b(\text{Mod}_f.d. - U_n) \cong D^b(\text{coh}(P_{1,n})) =: T_n$$

where $P_{1,n}$ is given in Theorem 4.6. By Theorem 4.6(3), $\text{fpcy}(D^b(\text{Mod}_f.d. - U_n)) = 1$ for all $n \geq 2$. On the other hand, $\text{fpcy}(D^b(\text{Mod}_f.d. - U_1)) = \frac{h - 2}{h} = \frac{1}{3}$ where $h = 3$ is the Coxeter number of the quiver $A_2$ (which is $Q_1$), see [Ke1, Example 8.3(2)].

We claim that $\text{fpcy}(U_n) = -\infty$ and that $\text{fpcy}^{-1}(U_n) = \infty$ when $n \geq 3$. We only prove the second assertion. By the noncommutative Beilinson’s theorem given in [Min, Theorem 0.1] (also see (E5.9.1)), the equivalent (E5.7.1) sends $U_n$ to $O \oplus O(1)$ where $O$ is the structure sheaf of $\mathbb{P}^n_1$.

Note that the Serre functor is $S := - \otimes O(-2)[1]$. Let $A = U_n$. Then

$$\text{fpcy}^{-1}(A) = \lim_{t} \left\{ \limsup_{m \to \infty} \dim \text{Hom}_{T_n}(A, (S^{-tm} \circ \Sigma^{tm})(A)) \right\}$$

$$= \lim_{t} \left\{ \limsup_{m \to \infty} \dim \text{Hom}_{T_n}(O \oplus O(1), (S^{-tm} \circ \Sigma^{tm})(O \oplus O(1))) \right\}$$

$$\geq \lim_{t} \left\{ \limsup_{m \to \infty} \dim \text{Hom}_{T_n}(O, (S^{-tm} \circ \Sigma^{tm})(O)) \right\}$$

$$= \text{fpcy}^{-1}((T_n)_+)$$

$$= \infty$$

where the last equation is Theorem 4.6(6).

We add a few more questions to Question 0.4.

**Question 5.8.** Let $A$ be a finite dimensional algebra of finite global dimension and let $T$ be the derived category $D^b(\text{Mod}_{f.d.} - A)$.

1. By Example 5.7, if $A$ is the path algebra of the quiver $Q_1$, then $\text{fpcy}(T) = \frac{1}{3}$. Is the minimum value of $\text{fpcy}(T)$ equal to $\frac{1}{3}$ for an arbitrary $A$?
2. Is there a value of $\text{fpcy}(T)$ outside of the set

$$R := \left( \sum_{h \geq 3} \frac{h - 2}{h} \right) \cap \mathbb{Q}_{> 0}?$$
By Lemma 5.2 and [Ke1, Example 8.3(2)], every number in $R$ can be realized as $\text{fpcy}(T)$ for some finite dimensional algebra. But we don’t have examples of $\text{fpcy}$ that are outside this range.

Recall that

**Definition 5.9.** [HP, p. 1230] Let $X$ be a smooth projective scheme.

1. A coherent sheaf $E$ on $X$ is called *exceptional* if $\text{Hom}_X(E, E) \cong k$ and $\text{Ext}^i_X(E, E) = 0$ for every $i \geq 0$.
2. A sequence $E_1, \ldots, E_n$ of exceptional sheaves is called an *exceptional sequence* if $\text{Ext}^k_X(E_i, E_j) = 0$ for all $k$ and for all $i > j$.
3. If an exceptional sequence generates $D^b(\text{coh}(X))$, then it is called *full*.
4. If an exceptional sequence satisfies $\text{Ext}^k_X(E_i, E_j) = 0$ for all $k > 0$ and all $i, j$, then it is called a *strongly exceptional sequence*.

The existence of a full exceptional sequence has been proved for many projective schemes. However, on Calabi-Yau varieties there are no exceptional sheaves. When $X$ has a full exceptional sequence $E_1, \ldots, E_n$, then there is a triangulated equivalence

(E5.9.1) \[ D^b(\text{coh}(X)) \cong D^b(\text{Mod}_{f.d} - A) \]

where $A$ is the finite dimensional algebra $\text{End}_X(\bigoplus_{i=1}^n E_i)$. In this setting, the fp Calabi-Yau dimension of $D^b(\text{Mod}_{f.d} - A)$ is equal to $\dim X$, which exists and is finite. In many examples in algebraic geometry, a full exceptional sequence consists of line bundles. Assume this is true. Via (E5.9.1), one sees easily that $\text{fpcy}^{\pm 1}(A) \geq \text{fpcy}^{\pm 1}(X)$. In fact, in many examples, we have $\text{fpcy}^{\pm 1}(A) = \text{fpcy}^{\pm 1}(X)$.

6. **Appendix A: Proof of Lemma 2.10**

As in Section 2, let $r$ be a positive integer. Suppose that a hereditary abelian category $\mathfrak{T}$ has $r^2$ brick objects as given in Corollary 2.8, now labeled as

\[ 1, 2, 3, \ldots, r^2 - 1, r^2, \]

where the matrices

\[ H = (H_{ij})_{r^2 \times r^2} = (\dim \text{Hom}(i, j))_{r^2 \times r^2}, \]

see (E2.8.1), and

\[ E = (E_{ij})_{r^2 \times r^2} = (\dim \text{Hom}(i, \Sigma j))_{r^2 \times r^2}, \]

see (E2.8.2), are given by the following block matrices:
$H = \begin{pmatrix}
(1) & (2) & (3) & (4) & \ldots & (r) \\
(p^0) & p^1 & p^2 & p^3 & \ldots & p^{r-1} \\
(2) & \sum_{i=0}^{r-2} p^i & \sum_{i=1}^{r-1} p^i & \sum_{i=2}^{r-2} p^i & \ldots & \sum_{i=r-2} p^i \\
(3) & \sum_{i=0}^{r-3} p^i & \sum_{i=1}^{r-2} p^i & \sum_{i=2}^{r-3} p^i & \ldots & \sum_{i=r-3} p^i \\
(4) & \sum_{i=0}^{r-4} p^i & \sum_{i=1}^{r-3} p^i & \sum_{i=2}^{r-4} p^i & \ldots & \sum_{i=r-4} p^i \\
\vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\
(r) & \sum_{i=0}^{r-1} p^i & \sum_{i=1}^{r-1} p^i & \sum_{i=2}^{r-1} p^i & \ldots & \sum_{i=r-1} p^i \\
\end{pmatrix}$

$E = \begin{pmatrix}
(1) & (2) & (3) & (4) & \ldots & (r) \\
(p^{r-1}) & p^{r-1} & p^{r-1} & p^{r-1} & \ldots & p^{r-1} \\
(p^{r-2}) & \sum_{i=2}^{r-1} p^i & \sum_{i=1}^{r-2} p^i & \sum_{i=2}^{r-3} p^i & \ldots & \sum_{i=r-2} p^i \\
(p^{r-3}) & \sum_{i=3}^{r-2} p^i & \sum_{i=2}^{r-3} p^i & \sum_{i=3}^{r-4} p^i & \ldots & \sum_{i=r-3} p^i \\
(p^{r-4}) & \sum_{i=4}^{r-3} p^i & \sum_{i=3}^{r-4} p^i & \sum_{i=4}^{r-5} p^i & \ldots & \sum_{i=r-4} p^i \\
\vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\
(p^0) & \sum_{i=1}^{r} p^i & \sum_{i=1}^{r} p^i & \sum_{i=1}^{r} p^i & \ldots & \sum_{i=1}^{r} p^i \\
\end{pmatrix}$

where $P$ is the $r \times r$ permutation matrix

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 1 & 0 \\
\end{pmatrix}
$$
and $P^0$ denotes the $r \times r$ identity matrix. Note that $P^r = P^0$. We will show that $ho(A(\phi)) \leq 1$ for all brick sets $\phi$.

The Hom and Ext matrices given in (E2.8.1)-(E2.8.2) are actually the transpose of the usual Hom and Ext matrices since they follow the convention of Corollary 2.8. See the remark before Theorem 2.11.

First, notice that

$$H^T = \begin{pmatrix}
(1) & (2) & (3) & (4) & \cdots & (r) \\
(1) & P^r & P^r & P^r & \cdots & P^r \\
(2) & P^{r-1} & \sum_{i=0}^{1} P^{r-i} & \sum_{i=0}^{1} P^{r-i} & \cdots & \sum_{i=0}^{1} P^{r-i} \\
(3) & P^{r-2} & \sum_{i=1}^{2} P^{r-i} & \sum_{i=1}^{2} P^{r-i} & \cdots & \sum_{i=1}^{2} P^{r-i} \\
(4) & P^{r-3} & \sum_{i=1}^{3} P^{r-i} & \sum_{i=1}^{3} P^{r-i} & \cdots & \sum_{i=1}^{3} P^{r-i} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
(r) & \sum_{i=r-2}^{r-1} P^{r-i} & \sum_{i=r-3}^{r-1} P^{r-i} & \sum_{i=r-4}^{r-1} P^{r-i} & \cdots & \sum_{i=0}^{r-1} P^{r-i}
\end{pmatrix}$$

Define the following non-negative $r^2 \times r^2$ matrices:

$$F := \begin{pmatrix}
(1) & (2) & (3) & (4) & \cdots & (r) \\
(1) & P^{r-1} & P^{r-1} & P^{r-1} & \cdots & P^{r-1} \\
(2) & P^{r-2} & P^{r-2} & P^{r-2} & \cdots & P^{r-2} \\
(3) & P^{r-3} & P^{r-3} & P^{r-3} & \cdots & P^{r-3} \\
(4) & P^{r-4} & P^{r-4} & P^{r-4} & \cdots & P^{r-4} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
(r) & P^0 & P^0 & P^0 & \cdots & P^0
\end{pmatrix}$$

$$G := \begin{pmatrix}
(1) & (2) & (3) & (4) & \cdots & (r) \\
(1) & P^r & P^r & P^r & \cdots & P^r \\
(2) & P^{r-1} & P^r & P^r & \cdots & P^r \\
(3) & P^{r-2} & P^{r-1} & P^r & \cdots & P^r \\
(4) & P^{r-3} & P^{r-2} & P^{r-1} & P^r & \cdots & P^r \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
(r) & P^1 & P^2 & P^3 & \cdots & P^r
\end{pmatrix}$$

We can see that
Lemma 6.1. Let \( I, J \in \mathbb{R}^d \). Since

Case 1: consider two different cases.

Lemma 6.2. If \( (F_i) \) which implies that \( I \). Case 2: Since \( J \) such that \( \{I, J, J'\} \) \( \phi \).

Proof. Assume without loss of generality that \( m \).

Therefore, \( J, J' \) cannot be in the same brick set, a contradiction.
By Lemma 6.1,
\[ 1 = F_{IJ} = F_{mr+i, mr+j} = (P^{r-1-n})_{ij} = (P^{n+1-r})_{ji}. \]
\[ 1 = F_{I,J'} = F_{mr+i, m'r+j'} = (P^{r-1-n})_{ij'}. \]
Then
\[ 1 = (P^{n+1-r})_{ij}(P^{r-1-n})_{ij'} = \sum_{k=1}^{r} (P^{n+1-r})_{jk}(P^{r-1-n})_{kj'} \]
which implies that \( j = j' \).

By examination of the \( H \) matrix (E6.0.1), we can see that for \( m \geq m' \),
\[ H_{jj'} = H_{mr+j, m'r+j'} = (P^0)_{jj'} + A_{jj'} \]
for some non-negative \( r \times r \) matrix \( A \). Therefore,
\[ H_{jj'} = H_{mr+j, m'r+j'} = (P^0)_{jj'} + A_{jj} = \delta_{jj} + A_{jj} \geq 1 \]
as required. \( \square \)

**Lemma 6.3.** If \( \phi \) is a brick set, for each column \( J \) there exists at most one row \( I \) such that \( A(\phi)_{IJ} \neq 0 \). If there exists a row \( I \) such that \( A(\phi)_{IJ} \neq 0 \), then \( A(\phi)_{IJ} = 1 \).

**Proof.** The proof is similar to the proof of Lemma 6.2 and omitted. \( \square \)

**Lemma 6.4.** If \( \phi \) is a brick set, \( \rho(A(\phi)) \leq 1 \).

**Proof.** By Lemmas 6.2 and 6.3, we have shown that if \( \phi \) is a brick set, \( A(\phi) \) is a matrix with at most one non-zero entry in each row and at most one non-zero entry in each column, such that if any entry is non-zero, it is 1. Then \( A(\phi) \) is almost a permutation matrix. In this case the quiver corresponding to \( A(\phi) \) has cycle number at most 1. By [CG, Theorem 1.8], \( \rho(A(\phi)) \leq 1 \). \( \square \)

7. **Appendix B: Some variants**

Recall that the set of subsets of \( n \) nonzero objects in \( C \) is denoted by \( \Phi_n \) for each \( n \geq 1 \). And let \( \Phi = \bigcup_{n \geq 1} \Phi_n \). In this paper, we use either \( \Phi_b \) or \( \Phi_a \) as testing objects in the definition of fp-invariants [Definition 1.4]. Depending on the situation, we might want to choose a testing set different from \( \Phi_b \) or \( \Phi_a \). Here is a list of possible alternative testing sets.

**Example 7.1.**

1. \( \Phi = \bigcup_{n \geq 1} \Phi_n \).

2. If the category \( C \) is abelian, we can consider “simple sets” as follows. Let \( \Phi_{n,s} \) be the set of \( n \)-object subsets of \( C \), say \( \phi := \{X_1, X_2, \ldots, X_n\} \), where the \( X_i \) are non-isomorphic simple objects in \( C \). Let \( \Phi_s = \bigcup_{n \geq 1} \Phi_{n,s} \).

3. A subset \( \phi = \{X_1, X_2, \ldots, X_n\} \) is called a triangular brick set if each \( X_i \) is a brick object and, up to a permutation, \( \text{Hom}_C(X_i, X_j) = 0 \) for all \( i < j \). Let \( \Phi_{n, tb} \) be the set of all triangular brick \( n \)-sets, and let \( \Phi_{tb} = \bigcup_{n \geq 1} \Phi_{n, tb} \).

4. Now assume that \( C \) is a triangulated category with suspension functor \( \Sigma \). A subset \( \phi = \{X_1, X_2, \ldots, X_n\} \) is called a triangular atomic set if each \( X_i \) is an atomic object and, up to a permutation, \( \text{Hom}_C(X_i, X_j) = 0 \) for all \( i < j \). Let \( \Phi_{n, ta} \) be the set of all triangular atomic \( n \)-sets, and let \( \Phi_{ta} = \bigcup_{n \geq 1} \Phi_{n, ta} \).
Basically, for any property $P$, we can define $\Phi_{n,P}^b$ (respectively, $\Phi_{n,P}^a$) and let $\Phi_P^b$ (respectively, $\Phi_P^a$) be $\bigcup_{n \geq 1} \Phi_{n,P}^b$ (respectively, $\bigcup_{n \geq 1} \Phi_{n,P}^a$). All of the definitions in this paper and in [CG] can be modified after we redefine $\rho(A(\phi, \sigma))$ as follows.

**Definition 7.2.** Let $C$ be a $k$-linear category and let $\phi$ be a set of $n$ nonzero objects, say $\{X_1, \cdots, X_n\}$, in $C$. Let $\sigma$ be a $k$-linear endofunctor of $C$. We define

$$\rho(A(\phi, \sigma)) := \frac{\rho\left((\dim \text{Hom}_C(X_i, \sigma(X_j)))_{n \times n}\right)}{\rho\left((\dim \text{Hom}_C(X_i, X_j))_{n \times n}\right)}.$$

Note that $\rho(A(\phi, \sigma))$ agrees with the original definition when $\phi$ is a brick set. One reason to introduce $P$-versions of fp-invariants is to extend these invariants even if the category contains no brick objects.

**Acknowledgments.** The authors thank the referee for his/her very careful reading and valuable comments and thank Professor Jarod Alper for many useful conversations on the subject. J. Chen was partially supported by the National Natural Science Foundation of China (Grant Nos. 11971398 and 12131018) and the Fundamental Research Funds for Central Universities of China (Grant No. 20720180002). Z. Gao was partially supported by the National Natural Science Foundation of China (Grant No. 61971365). E. Wicks and J.J. Zhang were partially supported by the US National Science Foundation (Grant Nos. DMS-1402863, DMS-1700825 and DMS-2001015). X.-H. Zhang was partially supported by the National Natural Science Foundation of China (Grant No. 11401328). H. Zhu was partially supported by a grant from Jiangsu overseas Research and Training Program for university prominent young and middle-aged Teachers and Presidents, China.

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