ON THE CONSTRUCTION OF MONOPOLES WITH ARBITRARY SYMMETRY BREAKING

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ABSTRACT. We produce finite energy BPS monopoles with prescribed arbitrary symmetry breaking from a new class of solutions to Nahm's equation.

1. INTRODUCTION

Gauge theory is for many of us the study of vector bundles equipped with exceptional objects, such as connections and other fields satisfying certain equations. Amongst the most classical of these objects are monopoles on 3-manifolds, where the data given by a connection $\nabla$ with curvature $F_\nabla$ on a vector bundle $E$ and an endomorphism $\Phi$ of that bundle, called the Higgs field, together being a solution to the Bogomolny equation $F_\nabla = *\nabla\Phi$.

The study of monopoles has been going on for quite some time, and yet, the most basic space on which to study monopoles, that is when the 3-manifold is the Euclidean space $\mathbb{R}^3$, still offers many mysteries. Even when the structure group is the simplest, SU(2), there is still a lot to uncover. We certainly know a lot about the moduli space thanks to [1, 15, 20, 26, 27, 30, 31, 35, 47], but Sen’s conjectures [51] that propose an understanding of the $L^2$-cohomology of the moduli space, have yet to be completely proven, although important steps in this direction have been made recently; see [22, 40, 50]. Understanding monopoles themselves, for instance the locations where Higgs fields are small, is the subject of the magnetic bag conjecture originally formulated by Bolognesi [5] in 2006. While good progress has been made ([4, 21, 24, 41, 42, 53]), it has not been resolved either.

When one cranks up the rank and is willing to consider the bigger structure group SU$(N)$, there is a more vast field on which to play, even while we still stick to $\mathbb{R}^3$ as our base manifold. This playing field was explored using multiple tools: the Nahm transform ([29, 32, 46]), spectral curves ([34, 43]), rational maps ([37–39, 45]).

It is generally assumed, although not universally proven yet, that (under reasonable assumptions, such as finite energy), the Higgs field splits the bundle into eigenbundles at infinity. This splitting is called symmetry breaking. To be more precise (and even more precision is shown...
in Section 2), we have in some gauge and in one direction an asymptotic behavior

$$\Phi = \mu - \frac{1}{2r} \kappa + O\left(\frac{1}{r^2}\right),$$  

(1.1)

with $\mu, \kappa \in su(N)$ satisfying $[\mu, \kappa] = 0$. Thus we can simultaneously diagonalize $\mu$ and $\kappa$. In the case of maximal symmetry breaking, all eigenvalues of $\mu$ are different and the symmetry breaks from $SU(N)$ to $S(U(1) \times \cdots \times U(1))$ at infinity, that is the bundle splits to a sum of complex line bundles, each of which are eigenbundles for both $\mu$ and $\kappa$.

**Minimal symmetry breaking** occurs when there are only two eigenvalues and one of them has multiplicity 1. In that case, the symmetry breaks from $SU(N)$ to $S(U(N - 1) \times U(1))$. The simplest minimal symmetry breaking case is when the structure group is $SU(3)$ and was studied in [16–19]; Dancer’s work has been an inspiration for our desire to understand the Nahm transform for arbitrary symmetry breaking.

In [44], Murray and Singer made four conjectures that are key to understanding monopoles with nonmaximal symmetry breaking and their moduli spaces. This work and its companion paper [10] are partially motivated by these conjectures.

The heuristic of the Nahm transform is clear and well-known, originating in [46]; see also exposition work to it include [36], [7, Chapter 2], and [3, Chapter 5]. In principle, one should find a direct connection between the poles of the Nahm data and the asymptotic behavior of the monopole. This analysis was circumvented in the work of Hurtubise–Murray [33] in the case of maximal symmetry breaking for the classical groups. Their proof uses algebraic geometry and is summarized in the following diagram:

In this paper we introduce a new class of Nahm data and use them to construct monopoles with arbitrary symmetry breaking, without passing through the spectral curve. Our methods and proofs are completely analytic. The full story is recounted in the forthcoming companion paper [10], where we study the reverse direction of the transform for such monopoles.

Without further ado, let us dive into a cartoon picture of our result, which is expanded in full in the remainder of this paper. In the maximal symmetry breaking case, all the eigenvalues of $\mu$ are different, but in the general case they need not be so. To account for multiplicities we must adapt our notation and have distinct eigenvalues $i\lambda_1, \ldots, i\lambda_n$ of $\mu$, with $i\lambda_a$ of
multiplicity $r_a$, and ordered so that $\lambda_1 < \cdots < \lambda_n$. To each eigenvalue $i\lambda_a$ correspond an eigendecomposition of $\kappa$ restricted to that eigenspace, with a certain number of positive eigenvalues $k^+_{a,1}, \ldots, k^+_{a,r_a}$, a certain number of negative eigenvalues $-k^-_{a,1}, \ldots, -k^-_{a,r_a}$, and with $r^0_a$ many zero eigenvalues. This data determines nonnegative integers $m_a$. We then look for solutions to Nahm’s equation

$$T_1(t) = [T_2(t), T_3(t)], \quad T_2(t) = [T_3(t), T_1(t)], \quad \text{and} \quad T_3(t) = [T_1(t), T_2(t)],$$
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on the intervals $(\lambda_a, \lambda_{a+1})$, with $T_a$ taking values in $u(m_a)$. As $t$ approaches either end of $(\lambda_a, \lambda_{a+1})$, the Nahm matrices $T_a$ develop poles whose residue form representations of SU(2) that decomposes into irreducible components of dimensions given by the eigenvalues of $\kappa$. The precise behavior is explained in detail in Section 3.

We then use such solutions to Nahm’s equation to construct compatible Sobolev spaces together with a family of nondegenerate, Dirac-type, Fredholm operators labelled by vectors $x \in \mathbb{R}^3$, called the Dirac–Nahm operators. These operators are direct generalizations of ones used in [27, 46, 47]. Using this family we construct a monopole that break symmetry exactly as in equation (1.1).

In the version of Hurtubise and Murray only one of $r^-_a, r^+_a, r^0_a$ is nonzero at each $\lambda_a$, as they study only the case of maximal symmetry breaking. Hence in that setup one either has a residue on the left or on the right (but not on both) and the corresponding representation is always irreducible. Furthermore, $r^0_a$ can only be zero or one, and in the case where $r^0_a = 1$, the Nahm data must have a discontinuity of rank 1. In the current scenario of arbitrary symmetry breaking, residues can occur both on left and right of the $\lambda_a$, and discontinuities of the data arises in rank $r^0_a$ which can be bigger than 1: all the behaviors can occur at once at any of the eigenvalues.

The minimal symmetry breaking case with structure group SU(3) was studied by Dancer and Leese in [16–19], and in the work of Houghton–Weinberg [28] in the case of symmetry breaking from SU($N$) to $S(U(1) \times U(N-2) \times U(1))$ where the middle eigenvalues of $\mu$ are equal, the eigenvalues of $\kappa$ corresponding to the repeated eigenvalues of $\mu$ are all $i$. This simplifies considerably the analysis and hides the actual general picture we have uncovered. Indeed, the irreducible representation of dimension 1 of SU(2) is trivial, and thus the Nahm data has no pole at the corresponding eigenvalue.

Nahm data occur in the description of other gauge theoretic objects as well. Heuristically, the Nahm transform maps instantons on $S^1 \times \mathbb{R}^3$ to some sort of Nahm data on $S^1$. This correspondence was first studied in full in [9] for structure group SU(2), following groundwork of [48, 49]. This work was extended in [52] to arbitrary rank.
More generally, Nahm data occur in bow diagrams, a blend of intervals (on which the Nahm data live) and quivers (on which ADHM-like data live); this framework was introduced by Cherkis in [11]. That bow data correspond to instantons on multi-Taub–NUT spaces (which are in the next more complicated ALF 4-manifolds after $S^1 \times \mathbb{R}^3$) through the Nahm transform is the aim of the recent work [13, 14].

In all of those cases, the representations occurring as residues at the end of the Nahm intervals are irreducible. Our work opens the possibilities of exploring those cases with reducible representations, producing instantons whose behavior at infinity is more general.

Further on this topic, Cherkis predicts in [12] a potential octonionic version of the Nahm transform. As a start, He in [25] studies at length the moduli space of solutions to the octonionic Nahm's equation, a version obtained from dimensional reduction of the Spin(7) instanton equation instead of the anti-self-dual equation that yields the standard Nahm's equation studied in this paper. The residues of the octonionic Nahm data studied by He at the end of intervals are irreducible. Our work suggests a natural extension, and should be considered in Cherkis’ program.

Throughout the many decades of study of monopoles, there has been a plethora of symmetric monopoles produced and studied: for structure group SU(2), for higher rank with minimal symmetry breaking, on the hyperbolic space. The ideas of this paper have already been used in [8] to produce novel examples of spherically and axially symmetric monopoles and, in fact, characterize all such solutions.

**Organization of the paper.** In Section 2, we briefly outline the gauge theoretic background for BPS monopoles on $\mathbb{R}^3$. In Section 3, we introduce the Nahm data needed for our construction. In Section 4, we study the Fredholm theory of the Dirac–Nahm operator and use it to define the monopoles associated to Nahm data. In Section 5 that the monopoles constructed in Section 4 have the desired symmetry breaking type. Finally, in Appendix A, we show how to use our results to construct monopoles with real orthogonal or compact symplectic structure groups.

2. Monopoles

Let $E$ be a smooth, rank $N$, Hermitian vector bundle over $\mathbb{R}^3$. A pair, $(\nabla, \Phi)$, of an SU($N$) connection, $\nabla$, on $E$ and a traceless, skew-adjoint bundle maps, $\Phi$, of $E$, is a monopole if it satisfies the (Bogomolny) monopole equation

$$F_\nabla = * \nabla \Phi,$$

(2.1)
and has finite Yang–Mills–Higgs energy

$$\mathcal{E}_{\text{YMH}}(\nabla, \Phi) := \frac{1}{4\pi} \int_{\mathbb{R}^3} (|F_\nabla|^2 + |\nabla \Phi|^2) \text{vol.}$$ (2.2)

Note that if equation (2.1) holds then the finiteness of $\mathcal{E}_{\text{YMH}}(\nabla, \Phi)$ is equivalent to

$$|\nabla \Phi| \in L^2(\mathbb{R}^3).$$ (2.3)

We remark, that the monopole equation (2.1) implies

$$\nabla^* \nabla \Phi = 0.$$ (2.4)

Remark 2.1. One can, of course, consider $U(N)$-monopoles as well. In that case, if $(\nabla, \Phi)$ is a solution to equations (2.3) and (2.4), then the pair $(\nabla^{\text{det}(E)}, \text{tr}(\Phi))$ is a finite energy $U(1)$-monopole on $\det(E) := \wedge^N E$. Thus $\text{tr}(\Phi)$ is constant, $\nabla^{\text{det}(E)}$ is gauge-equivalent to the product connection, and $\nabla$ induces a reduction of the structure group to $SU(N)$. In particular, the pair $\left(\nabla, \Phi - \frac{\text{tr}(\Phi)}{N} \mathbb{1}\right)$ is gauge equivalent to a finite energy $SU(N)$-monopole.

We impose asymptotic conditions on $(\nabla, \Phi)$, following [44]: Let $r$ be the euclidean distance (radial coordinate) from the origin on $\mathbb{R}^3$. Let $S^2_\infty$ the “sphere at infinity,” that is the space of oriented lines in $\mathbb{R}^3$ that pass through the origin, equip $S^2_\infty$ with the round metric of radius one, and let $\pi: \left(\mathbb{R}^3 - \{0\}\right) \to S^2_\infty$ be the obvious map. Let $E_\infty \to S^2_\infty$ be a (necessarily topologically trivial) Hermitian vector bundle with structure group $SU(N)$. We then assume that there exist

1. sections $\mu, \kappa \in \Gamma(\mathfrak{su}(E_\infty))$,
2. an $SU(N)$ connection $\nabla^\infty$ on $E_\infty$,
3. and an isomorphism of $E|_{\mathbb{R}^3 - \{0\}}$ and $\pi^*(E_\infty)$,

such that, under the above isomorphism, we have on $\mathbb{R}^3 - \{0\}$ that

$$\nabla = \nabla^\infty + \partial_r \otimes dr + \frac{1}{r} \mathcal{R}_1,$$ (2.5a)

$$\Phi = \mu - \frac{1}{2r} \kappa \otimes dr + \frac{1}{r^2} \mathcal{R}_2,$$ (2.5b)

$$\nabla \Phi = \frac{1}{2r^2} \kappa \otimes dr + \frac{1}{r^3} \mathcal{R}_3,$$ (2.5c)

for some $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_3$ each satisfying

$$\lim_{R \to \infty} \sup \left( \| \mathcal{R}_i \|_{L^\infty(S^2_\infty)} + R \| \nabla \mathcal{R}_i \|_{L^\infty(S^2_\infty)} \right) < \infty.$$
Note that equation (2.1) and conditions (2.5a) to (2.5c) imply

\[
\nabla_\infty \mu = 0, \\
\nabla_\infty \kappa = 0, \\
F_{\nabla_\infty} = \frac{i}{2} \kappa \otimes \text{vol}_{S^2_\infty}.
\]

Thus

\[
\left[ \mu, \kappa \right] = \left[ \mu, 2 * S^2_\infty \left( \frac{i}{2} \kappa \otimes \text{vol}_{S^2_\infty} \right) \right] = -2 * S^2_\infty \left[ F_{\nabla_\infty}, \mu \right] = -2 * S^2_\infty d^2_{\nabla_\infty} \mu = 0.
\]

By equations (2.6b) and (2.6c) \( \nabla_\infty \) is a Yang–Mills connection. Thus, by [23, Theorem 2.1], \( (E_\infty, \nabla_\infty) \) is reducible to a direct sum of Hermitian line bundles in a way that on each line bundle the induced connection is also a Yang–Mills connection, that is, its curvature is \( i \frac{k}{2} \kappa \otimes \text{vol}_{S^2_\infty} \) for some integer \( k \in \mathbb{Z} \). The sum of these integers is necessarily zero.

There are further necessary conditions that the spectra of \( \mu \) and \( \kappa \) ought to satisfy (see [44, Section 4]), but we leave the discussion of those to the next section.

**Remark 2.2.** Given two bundles, \( E_1 \) and \( E_2 \), and monopoles \( (\nabla_1, \Phi_1) \) and \( (\nabla_2, \Phi_2) \) on them, one can construct a new monopole on \( E = E_1 \oplus E_2 \) via

\[
(\nabla, \Phi) = (\nabla_1 \oplus \nabla_2, \Phi_1 \oplus \Phi_2).
\]

More generally, if \( (\nabla_1, \Phi_1) \) and \( (\nabla_2, \Phi_2) \) have unitary (but not necessarily special unitary) structure groups, say \( U(N_1) \) and \( U(N_2) \), then for any \( \lambda_1, \lambda_2 \in \mathbb{R} \), the field configuration

\[
(\nabla, \Phi) = \left( \nabla_1 \oplus \nabla_2, \left( \Phi_1 - i \lambda_1 \mathbb{1}_{N_1 \times N_1} \right) \oplus \left( \Phi_2 - i \lambda_2 \mathbb{1}_{N_2 \times N_2} \right) \right),
\]

is also a \( U(N_1 + N_2) \)-monopole. Moreover, as in Remark 2.1, \( \text{tr}(\Phi_1) \) and \( \text{tr}(\Phi_2) \) are constants, so setting \( \lambda_1 = -\frac{i}{N_1} \text{tr}(\Phi_1) \) and \( \lambda_2 = -\frac{i}{N_2} \text{tr}(\Phi_2) \) yields that

\[
(\nabla, \Phi) = \left( \nabla_1 \oplus \nabla_2, \left( \Phi_1 - \frac{i}{N_1} \text{tr}(\Phi_1) \mathbb{1}_{N_1 \times N_1} \right) \oplus \left( \Phi_2 - \frac{i}{N_2} \text{tr}(\Phi_2) \mathbb{1}_{N_2 \times N_2} \right) \right),
\]

is an \( SU(N_1 + N_2) \)-monopole. We call such a monopole reducible.

Note that if, say, \( N_2 = 1 \), then, as in Remark 2.1, we get that \( \nabla_2 \) is (gauge equivalent to) the product connection and \( \Phi_2 - \frac{i}{N_2} \text{tr}(\Phi_2) \mathbb{1}_{N_2 \times N_2} = 0 \). In this case we call \( (\nabla_2, \Phi_2) \) a flat factor. Conversely, one can always add flat factors to a monopole the above way. This process creates an extra \( k = 0 \) eigenvalue in \( \kappa \).

In this paper we only consider monopoles without flat factors. Note that such monopoles can still be reducible.
3. NAHM DATA WITH NONMAXIMAL SYMMETRY BREAKING

In this section we construct SU(\(N\)) monopoles with prescribed symmetry breaking. By condition (2.5b), the asymptotic behavior of the Higgs field is determined by a pair of elements in \(\Gamma(\text{su}(E_\infty)), \mu\) and \(\kappa\). By equations (2.6a) and (2.6b), these elements have constant spectrum and by equation (2.7), they can be simultaneously diagonalized. Finally, by equation (2.6c), \(i\kappa\) has only integer eigenvalues. For each eigenvalue \(i\lambda \in \text{Spec}\(\mu\)) we then have in some gauge and for some integers \(k_a^{\pm} \geq k_{a+1}^{\pm} > 0\) that

\[ \kappa|_{\ker(\mu - i\lambda)} = \text{diag}\{ik_1^+, ik_2^+, \ldots, 0, \ldots, -ik_2^-, -ik_1^-\}. \]

(The number of positive/zero/negative elements can be zero.)

With that in mind, we define the type of symmetry breaking as follows.

**Definition 3.1.** Let \(n, N \in \mathbb{N}_+\) and \(n \leq N\). A symmetry breaking type of rank \(N\) and size \(n\), \(\mathbb{T} = (\lambda, r, k)\), consists of a triple of \(n\)-tuples

- **eigenvalues:** \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n\),
- **ranks:** \(r = (r_1, r_2, \ldots, r_n) \in \mathbb{N}_+^n\),
- **Chern numbers:** \(k = (k_1, k_2, \ldots, k_n) \in \mathbb{Z}^{r_1} \times \mathbb{Z}^{r_2} \times \cdots \times \mathbb{Z}^{r_n}\),

satisfying conditions enumerated below. These conditions are best formulated using additional notation. We label the entries in the \(k_a\) as

\[ k_a^{\pm} = \sum_{b=1}^{r_a} k_{a,b}^{\pm}, \]

with \(0 < k_{a,1}^+ \leq \cdots \leq k_{a,r_a}^+\), and let

\[ r_a^0 := r_a - r_a^+ - r_a^- \]

be the number of zeros in \(k_a\). For each \(a \in \{1, 2, \ldots, n\}\), we define

\[ k_a^+ := \sum_{b=1}^{r_a} k_{a,b}^+, \]
\[ k_a^- := k_a^+ - k_a^-, \]
\[ m_a := \sum_{b=1}^{r_a} k_{a,b}. \]

Finally, let \(m_0 := 0\) and \(r_0 := 0\). The conditions \(\lambda, r, k\) satisfy are

1. \(\sum_{a=1}^{n} r_a = N\),
2. \(\sum_{a=1}^{n} \lambda_a r_a = 0\),
3. \(\lambda_1 < \lambda_2 < \cdots < \lambda_n\),
(4) \( r_1 = r_1^+ \) and \( r_n = r_n^- \), and 
(5) for each \( a \in \{1, 2, \ldots, n-1\} \),

\[
m_a \geq \max \{ r_a^0 + k_a^+, r_{a+1}^0 + k_{a+1}^- \}.
\]

and \( m_n = 0 \).

**Remark 3.2.** The interpretation of \( \mathbb{T} = (\lambda, r, k) \) is that for all \( a \in \{1, 2, \ldots, n\} \):

1. The imaginary numbers \( i\lambda_a \) are the distinct eigenvalues of the Higgs field at infinity.
2. The nonnegative integers \( r_a^+, r_a^- \), and \( r_a^0 \) are the number of line bundles with positive, negative, and zero Chern number in the holomorphic decomposition of the bundle corresponding to the eigenvalue \( i\lambda_a \).
3. For all \( b \in \{1, 2, \ldots, r_a^\pm\} \) the integer \( \pm k_{a,b}^\pm \) is the nonzero Chern number of a line bundle in the holomorphic decomposition of the bundle \( E_{\infty} \).

Next we define the framing of a symmetry breaking type which can be viewed as an analogue of framed gauge equivalence classes for monopoles.

**Definition 3.3.** A framing of a symmetry breaking type \( \mathbb{T} \), is a pair, \( \mathcal{F} = \left( \left(V_a^+, V_a^-\right)_{a=1}^{n-1}, (C_a)_{a=2}^n\right) \), such that, for all \( a \in \{1, 2, \ldots, n-1\} \), we have

\[
V_a^\pm \subseteq \mathbb{C}^{m_a},
\]

with \( \text{dim}_\mathbb{C}(V_a^+) = k_a^+ \) and \( \text{dim}_\mathbb{C}(V_a^-) = k_{a+1}^- \). Furthermore, for all \( a \in \{2, \ldots, n-1\} \), the maps \( C_a : (V_{a-1}^-)^\perp \to (V_a^+)^\perp \) are unitary isomorphisms.

**Remark 3.4.** Note that \( V_1^+ = \mathbb{C}^{m_1} \) and \( V_{n-1}^- = \mathbb{C}^{m_{n-1}} \)

Let \( \mathbb{H} \) be the field of quaternions and \( \sigma_1, \sigma_2, \sigma_3 \) be the usual basis of \( \text{Im}(\mathbb{H}) \). We also identify \( \text{SU}(2) \) with \( \text{Sp}(1) \) and thus \( \text{su}(2) \) with \( \text{sp}(1) \equiv \text{Im}(\mathbb{H}) \). We consider \( \mathbb{H} \) as a complex right module when tensored with complex vector spaces. Finally, let \( R_k \) denote the irreducible representation of \( \text{su}(2) \) of dimension \( k \).

**Definition 3.5.** A Nahm data with symmetry breaking type \( \mathbb{T} \) is a pair \( \mathcal{N} = (\mathcal{F}, \mathcal{T}) \) consisting of a framing \( \mathcal{F} \) and an \( (n-1) \)-tuple of pairs, \( \mathcal{T} = (d^{(a)}, T^{(a)})_{a=1}^{n-1} \), such that:

1. For all \( a \in \{1, 2, \ldots, n-1\} \), \( d^{(a)} \) is a unitary connection of the bundle \( (\lambda_a, \lambda_{a+1}) \times \mathbb{C}^{m_a} \), \( \text{dom}(T^{(a)}) = (\lambda_a, \lambda_{a+1}) \), and 

\[
T^{(a)} : (\lambda_a, \lambda_{a+1}) \to \mathfrak{u}(m_a)^\otimes 3; \ t \mapsto \left( T_1^{(a)}(t), T_2^{(a)}(t), T_3^{(a)}(t) \right)
\]

is an analytic function. Furthermore, \( \mathcal{T} \) solves Nahm’s equation:

\[
d_{\partial_t} T_1^{(a)} = \left[ T_2^{(a)}, T_3^{(a)} \right], \quad d_{\partial_t} T_2^{(a)} = \left[ T_3^{(a)}, T_1^{(a)} \right], \quad \text{and} \quad d_{\partial_t} T_3^{(a)} = \left[ T_1^{(a)}, T_2^{(a)} \right]. \quad (3.1)
\]
(2) For all \( a \in \{1,2,\ldots,n-1\} \), let \( d^{(a),0} \) be the product connection on \((\lambda_a, \lambda_{a+1}) \times \mathbb{C}^{m_2}\). Then
\[
T_0^{(a)} : (\lambda_a, \lambda_{a+1}) \rightarrow u(m_a); \quad t \mapsto d^{(a),0}_{\partial_t} - d^{(a),0}_{\partial_t}
\]
extends smoothly over \([\lambda_a, \lambda_{a+1}]\).

(3) The decompositions in Definition 3.3 induce embeddings
\[
\begin{align*}
\mathfrak{u}(k^+_a) \oplus \mathfrak{u}(m_a - k^+_a) & \hookrightarrow \mathfrak{u}(m_a), \\
\mathfrak{u}(k^-_{a+1}) \oplus \mathfrak{u}(m_a - k^-_{a+1}) & \hookrightarrow \mathfrak{u}(m_a),
\end{align*}
\]
under which we have for \( 1 \gg \epsilon > 0 \)
\[
T_a^{(\lambda_a + \epsilon)} = \begin{pmatrix}
-\frac{1}{\epsilon} \rho^+_a + O(1) & \mathcal{R}^+_a(e) \\
-(\mathcal{R}^+_a(e))^* & \tau^+_a + O(\epsilon)
\end{pmatrix},
\]
\[
T_a^{(\lambda_{a+1} - \epsilon)} = \begin{pmatrix}
-\frac{1}{\epsilon} \rho^-_{a,a+1} + O(1) & \mathcal{R}^-_{a,a+1}(e) \\
-(\mathcal{R}^-_{a,a+1}(e))^* & \tau^-_{a,a+1} + O(\epsilon)
\end{pmatrix}.
\]
Furthermore, the maps
\[
\hat{\rho}^+_a : \mathfrak{su}(2) \rightarrow \mathfrak{u}(k^+_a); \quad x_1 \sigma_1 + x_2 \sigma_2 + x_3 \sigma_3 \mapsto x_1 \rho^+_1 + x_2 \rho^+_2 + x_3 \rho^+_3,
\]
are Lie algebra homomorphisms that decompose to a direct sum of of irreducible \(\mathfrak{su}(2)\)-representations as
\[
\hat{\rho}^+_a \cong \bigoplus_{b=1}^{r_a^+} R_{k^+_{a,b}},
\]
\[
\hat{\rho}^-_{a+1} \cong \bigoplus_{b=1}^{r^+_{a+1}} R_{k^-_{a+1,b}}.
\]
Finally, the rows corresponding to \(\hat{\rho}^\pm_a\) in \(\mathcal{R}^\pm_{a,a}(\epsilon)\) are \(O(\epsilon^{(k^\pm_{a,b} - 1)/2})\).

(4) For all \( a \in \{2,\ldots,n\} \), let us define the operators
\[
\mathcal{C}_a := \sum_{a=1}^{3} \left( \tau^+_{a,a} \circ C_a - C_a \circ \tau^-_{a,a} \right) \otimes \sigma_a : (V_{a-1}^-)^\perp \otimes \mathbb{H} \rightarrow (V_a^+)^\perp \otimes \mathbb{H}.
\]
Then there are linearly independent elements \(\{x_{1,a}, x_{2,a}, \ldots, x_{r_a^0,a}\} \subset (V_a^+)^\perp\) and quaternions \(\{q_{1,a}, q_{2,a}, \ldots, q_{r_a^0,a}\}\) such that
\[
\mathcal{C}_a = \sum_{\rho=1}^{r_a^0} \left( x_{\rho,a} \otimes (C_a^{-1}(x_{\rho,a}))^* \right) \otimes \left( q_{\rho,a} \otimes q_{\rho,a}^* - \frac{1}{2} |q_{\rho,a}|^2 1_\mathbb{H} \right).
\]
Let
\[
X^+_1 := \mathbb{C}^{m_1} \otimes \mathbb{H},
\]
Definition 3.6. Let the $T$ according to Definition 3.5, with $G$ above discussion, $G^+$.

Next, let us define the appropriate gauge group and moduli space.

Definition 3.6. Let the gauge group be

$$G^T := \bigoplus_{a=1}^{n-1} C^\infty([\lambda_a, \lambda_{a+1}], U(m_a)),$$

with the natural group structure. If $g = (g_a)_{a=1}^{n-1} \in G^T$ and $N = (F, T)$ is a Nahm data according to Definition 3.5, with $F = \left( (V_a^+, V_a^-)_{a=1}^{n-1}, (C_a)_{a=2}^{n-1} \right)$, $T = (d(a), T(a))_{a=1}^{n-1}$, then let

$$g(F) := \left( g_a(\lambda_a)(V_a^+), g_a(\lambda_{a+1})(V_{a+1}^-) \right)_{a=1}^{n-1}, \left( (g_a(\lambda_a)|_{V_a^+}) \circ C_a \circ \left( g_{a-1}(\lambda_a)|_{V_{a-1}^+} \right) \right)_{a=2}^{n-1},$$

$$g(T) := (g_a \circ d(a) \circ g_{a-1}^{-1}, \text{Ad}(g_a) \circ T(a))_{a=1}^{n-1},$$

$$g(N) := (g(F), g(T)).$$

Then $g(N)$ is again a Nahm data with symmetry breaking type $\mathbb{T}$ with framing $g(F)$.

Definition 3.7. Let $B^T$ be the space of all Nahm data with symmetry breaking type $\mathbb{T}$. By the above discussion, $G^T$ acts on $B^T$. We define the moduli space of Nahm data with symmetry breaking type $\mathbb{T}$ as

$$M^T := B^T / G^T.$$
Lemma 3.8. Let $\mathbb{T} = (\lambda, r, k)$ and $\mathbb{T}' = (\lambda', r', k')$ be such that $k = k'$. Then there exist canonical isomorphisms

\[ G^T \cong G^{T'}, \quad (3.5a) \]
\[ B^T \cong B^{T'}, \quad (3.5b) \]
\[ M^T \cong M^{T'}, \quad (3.5c) \]

such that the diagram

\[
\begin{array}{ccc}
G^T & \cong & G^{T'} \\
\downarrow & & \downarrow \\
B^T & \cong & B^{T'} \\
\downarrow & & \downarrow \\
M^T & \cong & M^{T'}
\end{array}
\]

commutes.

Proof. Note that $k$ already determines $r$, the rank $N$, and the size $n$ of a symmetry breaking type. Thus $r = r'$, $N = N'$, and $n = n'$. Thus we can use the ideas of [8, Remark 4.10] as follows: Since the spaces in equations (3.5a) and (3.5b) are both spaces of functions, we construct isomorphisms of their domains and prove that the corresponding pullbacks are the desired isomorphisms. Note that

\[
\bigcup_{a=1}^{n-1} [\lambda_a, \lambda_{a+1}] = [\lambda_1, \lambda_n] \quad \text{and} \quad \bigcup_{a=1}^{n-1} [\lambda'_a, \lambda'_{a+1}] = [\lambda'_1, \lambda'_n].
\]

For all $a \in \{1, 2, \ldots, n-1\}$, let

\[ f_a : [\lambda'_a, \lambda'_{a+1}] \rightarrow [\lambda_a, \lambda_{a+1}] ; \quad t \mapsto \frac{\lambda_{a+1} - \lambda_a}{\lambda'_{a+1} - \lambda'_a} t + \frac{\lambda'_{a+1} - \lambda_{a+1} \lambda'_a}{\lambda'_{a+1} - \lambda'_a}, \]

and let $f^*_{\mathbb{T}, \mathbb{T}'} : [\lambda'_1, \lambda'_n] \rightarrow [\lambda_1, \lambda_n]$ be defined so that if $t \in [\lambda'_a, \lambda'_{a+1}]$, then $f^*_{\mathbb{T}, \mathbb{T}'}(t) = f_a(t)$. Note that $f^*_{\mathbb{T}, \mathbb{T}'}$ is now well-defined and continuous. For all $g = (g_a)_{a=1}^{n-1} \in G^T$, let

\[ f^*_{\mathbb{T}, \mathbb{T}'}(g)(a) := (g_a \circ f_a)_{a=1}^{n-1} \in G^{T'}. \]

This map is the desired isomorphism in equation (3.5b). If $\mathcal{N} = (\mathcal{F}, \mathcal{F}) \in B^T$, such that $\mathcal{F} = (T^{(a)})_{a=1}^{n-1}$, then let

\[ f^*_{\mathbb{T}, \mathbb{T}'}(\mathcal{F}) := \left( \frac{\lambda_{a+1} - \lambda_a}{\lambda'_{a+1} - \lambda'_a} T^{(a)} \circ \left( f_a \mid [\lambda'_a, \lambda'_{a+1}] \right) \right)_{a=1}^{n-1}, \]
\[ f^*_{\mathbb{T}, \mathbb{T}'}(\mathcal{N}) := \left( f^*_{\mathbb{T}, \mathbb{T}'}(\mathcal{F}) , \mathcal{F} \right), \]
then it is easy to check the commutativity of the diagram

\[
\begin{array}{ccc}
\mathcal{G}^T & \xrightarrow{f_{T,T'}} & \mathcal{G}^{T'} \\
\downarrow & & \downarrow \\
\mathcal{B}^T & \xrightarrow{f_{T,T'}} & \mathcal{B}^{T'}
\end{array}
\]

Finally, one can verify, with a little more work, that the map

\[\mathcal{M}^T \ni [\mathcal{N}] \mapsto [f^*_{T,T'}(\mathcal{N})] \in \mathcal{M}^{T'},\]

is well-defined and is an isomorphism, which proves equation (3.5c). \qed

**Remark 3.9.** In Lemma 3.8 "isomorphism" means merely a bijection, as we omit the problem of topologizing \(\mathcal{B}^T\), and hence \(\mathcal{M}^T\). Note however that \(\mathcal{B}^T\) is a finite dimensional Lie-group, thus it has a canonical smooth structure and, of course, the identity map is a diffeomorphism. Furthermore, the map \(f^*_{T,T'}\) constructed in Lemma 3.8 is conjecturally also a diffeomorphism for the appropriate smooth structures.

4. Nahm's construction and Dirac–Nahm operator

For the rest of the paper, we fix a Nahm data, \(\mathcal{N}\), as above, with symmetry breaking type \(\mathbb{T}\).

**Definition 4.1** (Nahm's construction). Let

\[\mathcal{H}_0^\mathcal{N} := \bigoplus_{a=1}^{n-1} L^2(\mathbb{C}^{m_b} \otimes \mathbb{H}) \bigoplus (\mathbb{C}^{m_b} \otimes \mathbb{H})\]

We say that an element

\[\Psi = (\psi_a)_{a=1}^{n-1} \in \bigoplus_{a=1}^{n-1} L^2(\mathbb{C}^{m_b} \otimes \mathbb{H}),\]

satisfies the Nahm boundary conditions, if for all \(a \in \{1, 2, \ldots, n-1\}\), we have

\[0 = \lim_{\epsilon \to 0^+} \Pi_{X_a^+ \oplus Y_a^+} (\psi_a(\lambda_a + \epsilon)),\]  

\[0 = \lim_{\epsilon \to 0^+} \Pi_{X_a^- \oplus Y_a^-} (\psi_a(\lambda_{a+1} - \epsilon)),\]  

\[0 = \lim_{\epsilon \to 0^+} \left( \Pi_{Z_a^+} (\psi_{a+1}(\lambda_{a+1} + \epsilon)) - (C_{a+1} \otimes \mathbb{1}_{\mathbb{H}})(\Pi_{Z_a^-} (\psi_a(\lambda_{a+1} - \epsilon))) \right).\]

Now let

\[\mathcal{H}_1^\mathcal{N} := \left\{ \Psi \in \bigoplus_{a=1}^{n-1} L^2((\mathbb{C}^{m_b} \otimes \mathbb{H}) \big| \Psi \text{ satisfies conditions (4.1a) to (4.1c)} \right\}.

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The Hilbert space structure of $\mathcal{H}_1^N$ is defined by

$$\|\Psi\|^2_{\mathcal{H}_1^N} := \sum_{a=1}^{n-1} \|d(a)\psi_a\|^2_{L^2(\lambda_a, \lambda_{a+1})}.$$  

For each $x \in \mathbb{R}^3$, the Dirac–Nahm operator, $D_x^N: \mathcal{H}_1^N \to \mathcal{H}_0^N$, is defined by

$$D_x^N((\psi_a)_{a=1}^{n-1}) := \left( i d(a) \psi_a + \sum_{a=1}^{n-1} \left( (i T_a(a) + x_a \mathbb{1} \psi_n) \otimes \sigma_a \right) \psi_a \right)_{a=1}^{n-1}.$$  

By Theorem 4.6 below, $(D_x^N)_{x \in \mathbb{R}^3}$ is a smooth family of Fredholm operators of index $-N$ and trivial kernel. Hence $E^N := (E_x^N := \text{coker}(D_x^N))_{x \in \mathbb{R}^3}$ defines a smooth, Hermitian vector bundle of rank $N$ over $\mathbb{R}^3$. Let $\nabla^N$ be the unitary connection on $E^N$ that is induced by the product connection on $\mathbb{R}^3 \times \mathcal{H}_0^N$. Let $M: \mathcal{H}_0^N \to \mathcal{H}_0^N$ be the operator of multiplication by $t$, for all $x \in \mathbb{R}^3$ let $\Pi_{E_x^N}$ be the orthogonal projection onto $E_x^N$, and let $\Phi^N_x$ be the operator

$$\Phi^N_x: E_x^N \to E_x^N; \Psi \mapsto -i (\Pi_{E_x^N} \circ M)(\Psi).$$

**Remark 4.2.** Note that the projections onto $Z_{a+1}$ and $Z_a$ in condition (4.1c) are redundant, as the components of $\Psi$ perpendicular to these spaces are already assumed to vanish in the limit by conditions (4.1a) and (4.1b).

**Remark 4.3.** If $g := (g_a)_{a=1}^{n-1} \in \mathcal{G}$ and $N := g(N)$, then the assignment

$$\Psi = (\psi_a)_{a=1}^{n-1} \mapsto \tilde{g}(\Psi) := (g_a(\psi_a))_{a=1}^{n-1},$$

defines a unitary isomorphisms of Hilbert spaces $\mathcal{H}_i^N \xrightarrow{\tilde{g}} \mathcal{H}_i^{N'}$ ($i \in \{1, 2\}$). Furthermore, for all $x \in \mathbb{R}^3$, we have that $D_x^{N'} = \tilde{g} \circ D_x^N \circ \tilde{g}^{-1}$. Hence $\tilde{g}$ restricts to an isomorphism of Hermitian vector bundles, $\tilde{g}|_{E^N} := g_*: E^N \to E^{N'}$, such that $g_*\left(\nabla^N, \Phi^N\right) = \left(\nabla^{N'}, \Phi^{N'}\right)$.

Using Remark 4.3, we choose the following gauge for the purposes of this paper:

**Definition 4.4.** For all $a \in \{1, 2, \ldots, n-1\}$, let $T_{0}^{a}$ be as defined in equation (3.2), let us pick $t_a \in (\lambda_a, \lambda_{a+1})$, and define

$$g_a: [\lambda_a, \lambda_{a+1}] \to \text{U}(m_a); \ t \mapsto \exp \left( \int_{t_a}^{t} T_{0} \, dt \right),$$

and $g := (g_a)_{a=1}^{n-1} \in \mathcal{G}$. Then note that if $N := g(N)$, then the connections in $N'$ are all product connections. We say that such a Nahm data is in temporal gauge.

**Remark 4.5.** When $N = n$, and thus for all $a \in \{1, 2, \ldots, n\}$, $r_a = 1$, and $N$ is in temporal gauge, then the above construction recovers the one by Hurtubise and Murray in [29, Section 4].
Note that in Definition 4.1 we claimed, without proof, that \((\mathcal{D}_x^N)_{x \in \mathbb{R}^3}\) is a smooth family of Fredholm operators of index \(-N\) and trivial cokernel. Since this still needs verification, let us for all \(x \in \mathbb{R}^3\) and smooth \(\Psi = (\psi_a)_{a=1}^{n-1} \in \mathcal{H}_1^N\) define

\[
\tilde{\mathcal{D}}_x^N(\Psi) := \left( i \frac{d}{dt} \psi_a + \sum_{a=1}^{3} \left( (i T_a^{(a)} + x_a \mathds{1}_{\mathbb{C}^m}) \otimes \sigma_a \right) \psi_a \right)_{a=1}^{n-1}, \tag{4.3}
\]

which is a smooth element of \(\mathcal{H}_0^N\). In order to prove Theorem 5.1, namely that the pair \((\nabla^N, \Phi^N)\) is an SU(\(N\))-monopole that satisfies equation (2.1) and conditions (2.5a) and (2.5b) with the desired asymptotics, we first need to build up the necessary functional analytic framework for the Dirac–Nahm operator (4.2), which we do next in Theorem 4.6.

**Theorem 4.6.** For all \(x \in \mathbb{R}^3\) the following statements hold:

1. The operator \(\tilde{\mathcal{D}}_x^N\) has a unique extension to continuous linear maps from \(\mathcal{H}^N_1\) to \(\mathcal{H}^N_0\), denoted by \(\mathcal{D}_x^N\).
2. The operator \(\mathcal{D}_x^N\) is Fredholm.
3. The kernel of \(\mathcal{D}_x^N\) is trivial and \(\text{index}(\mathcal{D}_x^N) = -N\).

Furthermore, the assignment \(x \mapsto \mathcal{D}_x^N\) is analytic in the operator norm.

**Proof.** Combining Remark 4.3 and Definition 4.4, we can assume, without any loss of generality, that \(\mathcal{N}\) is in temporal gauge. Thus for all \(a \in \{1, 2, \ldots, n-1\}\), \(d^{(a)}\) is the production connection, and hence, for any section \(\chi_a\) over \((\lambda_a, \lambda_{a+1})\), we write \(\hat{\chi}_a := d_t^{(a)} \chi_a\).

In order to prove the first point, we show that the map in equation (4.3) has finite operator norm. Let \(\Psi = (\psi_a)_{a=1}^{n-1} \in \mathcal{H}_1^N\) be smooth. Then

\[
\|\tilde{\mathcal{D}}_x^N(\Psi)\|_{\mathcal{H}_0^N}^2 = \sum_{a=1}^{n-1} \|\tilde{\mathcal{D}}_x^N(\psi_a)\|_{L^2(\lambda_a, \lambda_{a+1})}^2
\]

\[
= \sum_{a=1}^{n-1} \| i \hat{\psi}_a + \sum_{a=1}^{3} \left( (i T_a^{(a)} + x_a \mathds{1}_{\mathbb{C}^m}) \otimes \sigma_a \right) \psi_a \|_{L^2(\lambda_a, \lambda_{a+1})}^2
\]

\[
= \sum_{a=1}^{n-1} \| i \hat{\psi}_a \|_{L^2(\lambda_a, \lambda_{a+1})}^2 \sum_{a=1}^{3} \left( \left( (i T_a^{(a)} + x_a \mathds{1}_{\mathbb{C}^m}) \otimes \sigma_a \right) \psi_a \right)_{L^2(\lambda_a, \lambda_{a+1})}^2
\]

\[
+ \sum_{a=1}^{n-1} \sum_{a=1}^{3} 2 \text{Re} \left( \left( i \hat{\psi}_a \right) \left( \left( (i T_a^{(a)} + x_a \mathds{1}_{\mathbb{C}^m}) \otimes \sigma_a \right) \psi_a \right) \right)_{L^2(\lambda_a, \lambda_{a+1})}
\]

\[
+ \sum_{a=1}^{n-1} \sum_{a=1}^{3} \left( \left( (i T_a^{(a)} + x_a \mathds{1}_{\mathbb{C}^m}) \otimes \sigma_a \right) \psi_a \right)_{L^2(\lambda_a, \lambda_{a+1})}^2,
\]

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The first term can be bounded as
\[ \mathcal{F}_1 = \sum_{a=1}^{n-1} \| i \hat{\psi}_a \|^2_{L^2(\lambda_a, \lambda_{a+1})} = \sum_{a=1}^{n-1} \| \hat{\psi}_a \|^2_{L^2(\lambda_a, \lambda_{a+1})} \leq \| \Psi \|^2_{\mathcal{A}^\gamma}. \]

Let \( h^a \) be the Hermitian structure on \( (\lambda_a, \lambda_{a+1}) \times \mathbb{C}^{m_a} \otimes \mathbb{H} \). Using integration by parts and the skew-adjointness of \( \sigma_a \), for each \( a \in \{1, 2, \ldots, n-1\} \) and \( \alpha \in \{1, 2, 3\} \), the corresponding summand in the second term can be rewritten as
\[
\mathcal{F}_{2,a,a} := 2 \text{Re} \left( \langle i \hat{\psi}_a | [(i T_a^{(a)} + x_a \mathbb{1}_{\mathbb{C}^{m_a}}) \otimes \sigma_a] \psi_a \rangle_{L^2(\lambda_a, \lambda_{a+1})} \right) \\
= \frac{1}{2} \mathcal{F}_{2,a,a} + \text{Re} \left( \langle i \hat{\psi}_a | [(i T_a^{(a)} + x_a \mathbb{1}_{\mathbb{C}^{m_a}}) \otimes \sigma_a] \psi_a \rangle_{L^2(\lambda_a, \lambda_{a+1})} \right) \\
= \frac{1}{2} \mathcal{F}_{2,a,a} + \text{Re} \left( \langle \psi_a | [(i T_a^{(a)} + x_a \mathbb{1}_{\mathbb{C}^{m_a}}) \otimes \sigma_a] i \hat{\psi}_a \rangle_{L^2(\lambda_a, \lambda_{a+1})} \right) \\
- \text{Re} \left( \langle \psi_a | (i T_a^{(a)} \otimes \sigma_a) \psi_a \rangle_{L^2(\lambda_a, \lambda_{a+1})} \right) \\
+ \lim_{t \to \lambda_{a+1}^-} \text{Re} \left( h^a(i \psi_a, (i T_a^{(a)} + x_a \mathbb{1}_{\mathbb{C}^{m_a}}) \otimes \sigma_a) \psi_a \right) \\
= \frac{1}{2} \mathcal{F}_{2,a,a} + \text{Re} \left( \langle \psi_a | [(i T_a^{(a)} + x_a \mathbb{1}_{\mathbb{C}^{m_a}}) \otimes \sigma_a] i \hat{\psi}_a \rangle_{L^2(\lambda_a, \lambda_{a+1})} \right) \\
- \text{Re} \left( \langle \psi_a | (i T_a^{(a)} \otimes \sigma_a) \psi_a \rangle_{L^2(\lambda_a, \lambda_{a+1})} \right) \\
- \lim_{t \to \lambda_a^-} \text{Re} \left( h^a(i \psi_a, (i T_a^{(a)} + x_a \mathbb{1}_{\mathbb{C}^{m_a}}) \otimes \sigma_a) \psi_a \right).
\]

Thus
\[
\mathcal{F}_2 = - \sum_{a=1}^{n-1} \sum_{\alpha=1}^{3} \text{Re} \left( \langle \psi_a | (i T_a^{(a)} \otimes \sigma_a) \psi_a \rangle_{L^2(\lambda_a, \lambda_{a+1})} \right) \\
+ \sum_{a=1}^{n-1} \text{boundary terms from the right at } \lambda_a.
\]
we get that the boundary terms corresponding to $X_a^\pm$ and $Y_a^\pm$ all vanish. Using condition (4.1c), we get that the boundary terms corresponding to $Z_a^+$ and $Z_{a+1}$ cancel each other out. Thus

$$\mathcal{J}_2 = -\sum_{a=1}^{n-1} \sum_{\alpha=1}^3 \text{Re} \left( \langle \psi_a | (T^{(a)}_\alpha \otimes \sigma_a) \psi_a \rangle \right)_{L^2(\lambda_a, \lambda_{a+1})}.$$

For each $a \in \{1, 2, \ldots, n-1\}$, using $\sigma_a \sigma_\beta = -\delta_{a, \beta} \mathbb{1}_\mathbb{H} + \sum_{\gamma=1}^3 \epsilon_{a, \beta, \gamma} \sigma_\gamma$ and equation (3.1), the corresponding summand in the third term can be rewritten as

$$\mathcal{J}_{3,a} := \| (i T^{(a)}_\alpha + x_a \mathbb{1}_{\mathbb{C}^{ma}}) \otimes \sigma_a \|_{L^2(\lambda_a, \lambda_{a+1})}^2$$

$$= -\sum_{a=1}^3 \sum_{\alpha=1}^3 \text{Re} \left( \langle \psi_a | (i T^{(a)}_\alpha + x_a \mathbb{1}_{\mathbb{C}^{ma}}) \otimes \sigma_\alpha \right) \left( (i T^{(a)}_\beta + x_\beta \mathbb{1}_{\mathbb{C}^{ma}}) \otimes \sigma_\beta \right) \langle \psi_a \rangle_{L^2(\lambda_a, \lambda_{a+1})}$$

$$= -\sum_{a=1}^3 \sum_{\alpha=1}^3 \text{Re} \left( \langle \psi_a | (i T^{(a)}_\alpha + x_a \mathbb{1}_{\mathbb{C}^{ma}}) \right) \left( i T^{(a)}_\beta + x_\beta \mathbb{1}_{\mathbb{C}^{ma}} \right) \otimes (\sigma_a \sigma_\beta) \langle \psi_a \rangle_{L^2(\lambda_a, \lambda_{a+1})}$$

$$= -\sum_{a=1}^3 \sum_{\alpha=1}^3 (-\delta_{a, \beta}) \text{Re} \left( \langle \psi_a | (i T^{(a)}_\alpha + x_a \mathbb{1}_{\mathbb{C}^{ma}}) \right) \left( i T^{(a)}_\beta + x_\beta \mathbb{1}_{\mathbb{C}^{ma}} \right) \otimes 1_{\mathbb{H}} \langle \psi_a \rangle_{L^2(\lambda_a, \lambda_{a+1})}$$

$$- \sum_{a=1}^3 \sum_{\beta=1}^3 \sum_{\gamma=1}^3 \epsilon_{a, \beta, \gamma} \text{Re} \left( \langle \psi_a | (i T^{(a)}_\alpha + x_a \mathbb{1}_{\mathbb{C}^{ma}}) \right) \left( i T^{(a)}_\beta + x_\beta \mathbb{1}_{\mathbb{C}^{ma}} \right) \otimes \sigma_\gamma \langle \psi_a \rangle_{L^2(\lambda_a, \lambda_{a+1})}$$

$$= \sum_{a=1}^3 \| (i T^{(a)}_\alpha + x_a \mathbb{1}_{\mathbb{C}^{ma}}) \psi_a \|_{L^2(\lambda_a, \lambda_{a+1})}^2$$

$$- \sum_{a=1}^3 \sum_{\beta=1}^3 \sum_{\gamma=1}^3 \epsilon_{a, \beta, \gamma} \text{Re} \left( \langle \psi_a | (i T^{(a)}_\alpha + x_a \mathbb{1}_{\mathbb{C}^{ma}}) \right) \left( i T^{(a)}_\beta + x_\beta \mathbb{1}_{\mathbb{C}^{ma}} \right) \otimes \sigma_\gamma \langle \psi_a \rangle_{L^2(\lambda_a, \lambda_{a+1})}$$

$$= \sum_{a=1}^3 \| (i T^{(a)}_\alpha + x_a \mathbb{1}_{\mathbb{C}^{ma}}) \psi_a \|_{L^2(\lambda_a, \lambda_{a+1})}^2$$

$$+ \sum_{a=1}^3 \sum_{\beta=1}^3 \sum_{\gamma=1}^3 \epsilon_{a, \beta, \gamma} \text{Re} \left( \langle \psi_a | (i T^{(a)}_\alpha + x_a \mathbb{1}_{\mathbb{C}^{ma}}) \right) \left( i T^{(a)}_\beta + x_\beta \mathbb{1}_{\mathbb{C}^{ma}} \right) \otimes \sigma_\gamma \langle \psi_a \rangle_{L^2(\lambda_a, \lambda_{a+1})}$$

Using the antisymmetry of $\epsilon_{a, \beta, \gamma}$, the sum in the last line vanishes. Furthermore, using Nahm's equation (3.1),

$$\mathcal{J}_3 = \sum_{a=1}^{n-1} \sum_{\alpha=1}^3 \| (i T^{(a)}_\alpha + x_a \mathbb{1}_{\mathbb{C}^{ma}}) \otimes \mathbb{1}_{\mathbb{H}} \psi_a \|_{L^2(\lambda_a, \lambda_{a+1})}^2 - \mathcal{J}_2.$$
Let us recall the classical Hardy’s inequality: if \( u \in L_1^2(0, e) \), such that \( u(0) = 0 \), then
\[
\int_0^1 \frac{|u(t)|^2}{t^2} \, dt \leq 4 \int_0^1 |\dot{u}(t)|^2 \, dt. \tag{4.4}
\]

Using inequality (4.4), we get that there is a positive number \( C^\mathcal{T} \), such that
\[
\left| \sum_{a=1}^{n-1} \sum_{\alpha=1}^3 \| ((i T_a^{(a)} + x_\alpha 1_{\mathbb{C}^m}) \otimes 1_H) \Psi_a \|_{L^2(\lambda_a, \lambda_{a+1})}^2 \right| \leq \left( C^\mathcal{T} + |x| \right)^2 \| \Psi \|_{\mathcal{H}^N_1}^2.
\]

Hence we get that
\[
\| \tilde{D}_x^N (\Psi) \|_{\mathcal{H}^N_0}^2 = \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3
\]
\[
= \sum_{a=1}^{n-1} \| \Psi_a \|_{L^2(\lambda_a, \lambda_{a+1})}^2 + \sum_{a=1}^{n-1} \sum_{\alpha=1}^3 \| ((i T_a^{(a)} + x_\alpha 1_{\mathbb{C}^m}) \otimes 1_H) \Psi_a \|_{L^2(\lambda_a, \lambda_{a+1})}^2
\]
\[
\leq \| \Psi \|_{\mathcal{H}^N_1}^2 + \left( C^\mathcal{T} + |x| \right)^2 \| \Psi \|_{\mathcal{H}^N_1}^2
\]
\[
\leq \left( 1 + \left( C^\mathcal{T} + |x| \right)^2 \right) \| \Psi \|_{\mathcal{H}^N_1}^2.
\]

Since the above formula holds on a dense subset of \( \mathcal{H}^N_1 \), the claim about the unique and continuous extension, \( D_x^N \), is true. Moreover we have \( \| D_x^N \| \leq \sqrt{1 + (C^\mathcal{T} + |x|)^2} \). The first claim is now proved.

Now equation (4.5) still holds for \( D_x^N \) thus every element in ker(\( D_x^N \)) is constant and thus by the boundary condition of equation (4.1) is zero. Hence ker(\( D_x^N \)) is trivial. In fact
\[
\| D_x^N (\Psi) \|_{\mathcal{H}^N_0} \geq \| \Psi \|_{\mathcal{H}^N_1}
\]
holds. Combined with [6, Lemma 4.3.9], this inequality yields that the image of \( D_x^N \) is closed in \( \mathcal{H}^N_0 \).

In order to complete the proof of the second and third claims, it is enough to prove that dim(coker(\( D_x^N \))) = N. Since \( \mathcal{H}^N_0 \) and \( \mathcal{H}^N_1 \) are Hilbert spaces and image(\( D_x^N \)) is closed in \( \mathcal{H}^N_0 \), we can canonically identify the cokernel of \( D_x^N \) with the kernel of \( (D_x^N)^* \). If \( \Psi = (\psi_a)_{a=1}^{n-1} \in \ker \left( (D_x^N)^* \right) \), then for all \( \Psi' = (\psi'_a)_{a=1}^{n-1} \in \mathcal{H}^N_1 \) we have that
\[
0 = \sum_{a=1}^{n-1} \left( i \dot{\psi}'_a + \sum_{\alpha=1}^3 \left( (i T_a^{(a)} + x_\alpha 1_{\mathbb{C}^m}) \otimes \sigma_\alpha \right) \Psi'_a \bigg|_{L^2(\lambda_a, \lambda_{a+1})} \right).
\]  \tag{4.6}

Using functions \( \Psi' \) that satisfy that for all \( a \in \{1, 2, \ldots, n-1\} \)
\[
\psi'_a \in C_c^\infty((\lambda_a, \lambda_{a+1}); \mathbb{C}^m \otimes \mathbb{H}),
\]

we get that
\[ 0 = i\dot{\psi}_a - \sum_{a=1}^{3} \left( (iT_a^{(a)} + x_a \mathbb{1}_c^{y_0}) \otimes \sigma_a \right) \psi_a \]  
(4.7)
on (\lambda_a, \lambda_{a+1}), and in particular that the weak derivatives \( \dot{\psi}_a \) exist.

Now using integration by parts and \( \Psi' \) supported in a neighborhood of \( \lambda_a \), equation (4.6) becomes
\[ 0 = \lim_{\epsilon \to 0^+} h^1(\psi'_1, \psi_1)_{\lambda_1 + \epsilon}, \]  
(4.8a)
on \forall a \in \{2, \ldots, n-1\} : 0 = \lim_{\epsilon \to 0^+} \left( h^a(\psi'_a, \psi_a)_{\lambda_a + \epsilon} - h^{a-1}(\psi'_{a-1}, \psi_{a-1})_{\lambda_a - \epsilon} \right), \]  
(4.8b)
on \[ 0 = \lim_{\epsilon \to 0^+} h^{a-1}(\psi'_{n-1}, \psi_{n-1})_{\lambda_n - \epsilon}. \]  
(4.8c)

By the Picard–Lindelöf Theorem, we have a \( 2m_a \)-dimensional space of solutions to equation (4.7). The theory of singular, regular, ordinary differential equations, as used in [27], tells us furthermore that this \( 2m_a \)-dimensional space decomposes as follows: there is a \( (2m_a - 2k_{a+1}^+) \)-dimensional space of solutions that stay bounded near \( \lambda_a \) from the right and have limits in \( Y_a^+ \oplus Z_a^+ \). Furthermore, for each summand, \( R_{a+1}^{k_a^+} \), in equation (3.4a) there is a \( (k_{a,b}^+ - 1) \)-dimensional space of solutions that are \( O((t - \lambda_a)^{-1}(k_{a,b}^+ - 1)/2) \) (not \( L^2 \)) near \( \lambda_a \) from the right and another \( (k_{a,b}^+ + 1) \)-dimensional space of solutions that are \( O((t - \lambda_a)^{1/2}) \) (thus converges to zero) near \( \lambda_a \) from the right. Similarly, there is a \( (2m_a - 2k_{a+1}^-) \)-dimensional space of solutions that stay bounded near \( \lambda_{a+1} \) from the left. These solutions have limits in \( Y_{a+1}^- \oplus Z_{a+1}^- \).

Furthermore, for each summand \( R_{a+1}^{k_{a+1}^-} \) in equation (3.4b) there is a \( (k_{a+1}^+ - 1) \)-dimensional space of solutions that are \( O((\lambda_{a+1} - t)^{-1}(k_{a,b}^+ - 1)/2) \) (not \( L^2 \)) near \( \lambda_{a+1} \) from the left and another \( (k_{a+1}^+ + 1) \)-dimensional space of solutions that are \( O((\lambda_{a+1} - t)^{1/2}) \) (thus converges to zero) near \( \lambda_{a+1} \) from the left. Thus there is a \( (2m_a - k_{a+1}^+ + r_{a+1}^+) \)-dimensional space of solutions that are in \( L^2 \) near \( \lambda_a \) from the right and all of these have limits in \( Y_a^+ \oplus Z_a^+ \). Similarly, there is a \( (2m_a - k_{a+1}^- + r_{a+1}^-) \)-dimensional space of solutions that are in \( L^2 \) near \( \lambda_{a+1} \) from the left and all of these have limits in \( Y_{a+1}^- \oplus Z_{a+1}^- \).

The above shows that restricting to \( L^2 \) solutions already satisfy equations (4.8a) and (4.8c), while equation (4.8b) yields that for all \( a \in \{2, \ldots, n-1\} \) we have
\[ \lim_{\epsilon \to 0^+} (\psi_a(\lambda_a + \epsilon) - C_a(\psi_{a-1}(\lambda_a - \epsilon))) \in Y_a^+, \]  
(4.9)

For each \( a \in \{1, 2, \ldots, n-1\} \), let \( V_a \) be the \( 2m_a \)-dimensional space of solutions to equation (4.7). Let \( U_0 \subseteq V_1 \) be the subspace of solutions that are \( L^2 \) near \( \lambda_1 \). For each \( a \in \{1, 2, \ldots, n-2\} \), let \( U_a \subseteq V_a \times V_{a+1} \) be the subspace of pairs of solutions that are both \( L^2 \) near \( \lambda_a \) and satisfy condition (4.9), and let \( U_{n-1} \subseteq V_{n-1} \) be the subspace of solutions that are \( L^2 \) near \( \lambda_n \).
above discussion we have for all \( a \in \{1, 2, \ldots, n - 2\} \) that
\[
\dim(U_a) = \underbrace{k_{a+1}^- + r_{a+1}^-}_{\text{solutions terminating at } \lambda_{a+1}} + \underbrace{2m_a - 2k_{a+1}^-}_{\text{solutions continuing through } \lambda_{a+1}} \\
\quad + \underbrace{r_{a+1}^-}_{\text{solutions that jump at } \lambda_{a+1}} + \underbrace{k_{a+1}^+ + r_{a+1}^+}_{\text{solutions starting at } \lambda_{a+1}}
\]
\[
= m_a + m_{a+1} + r_a.
\]
Recall from Definition 3.1 that \( m_0 = r_0 = m_n = 0 \), and thus the above formula is also valid for \( \dim(U_0) \) and \( \dim(U_{n-1}) \). Finally, let us define the map
\[
\mathcal{D} : \bigoplus_{a=0}^{n-1} U_a \to \bigoplus_{a=1}^{n-1} V_a; (\psi_1^+,(\psi_1^-),\psi_2^+),\ldots,(\psi_{n-2}^+,(\psi_{n-2}^-)),(\psi_{n-1}^+,(\psi_{n-1}^-)) \mapsto (\psi_a^+ - (\psi_a^-))_{a=1}^{n-1}.
\]
Since \( \mathcal{D} \) is a map between finite dimensional vector spaces, its index is
\[
\text{index}(\mathcal{D}) = \dim\left(\bigoplus_{a=0}^{n-1} U_a\right) - \dim\left(\bigoplus_{a=1}^{n-1} V_a\right) \\
= \sum_{a=0}^{n-1} (m_a + m_{a+1} + r_a) - \sum_{a=1}^{n-1} 2m_a \\
= \sum_{a=1}^{n} r_a \\
= N.
\]
Furthermore, \( \ker(\mathcal{D}) \cong \ker\left(\left(\mathcal{D}_x^V\right)^*\right) \) via the map
\[
\ker(\mathcal{D}) \to \ker\left(\left(\mathcal{D}_x^V\right)^*\right); (\psi_1^+,(\psi_1^-),\psi_2^+),\ldots,(\psi_{n-2}^+,(\psi_{n-2}^-)),(\psi_{n-1}^+,(\psi_{n-1}^-)) \mapsto (\psi_a^+)_{a=1}^{n-1}.
\]
Hence it is enough to show that \( \text{coker}(\mathcal{D}) \) is trivial. For each \( a \in \{1, 2, \ldots, n - 1\} \), let \( \tilde{V}_a \) be the \( 2m_a \)-dimensional space of solutions to
\[
i\tilde{\psi}_a + \sum_{a=1}^{3} \left( (i T^{(a)}(\tilde{\psi}_a) \| C^{m_a}) \otimes \sigma_a \right) \tilde{\psi}_a = 0
\]
on \( (\lambda_a, \lambda_{a+1}) \). If \( \psi'_a \in \tilde{V}_a \) and \( \psi_a \in V_a \), then \( h^a(\psi'_a, \psi_a) \) is constant. Furthermore (using existence part of the Picard–Lindelöf Theorem again) we can see an isomorphism
\[
\bigoplus_{a=1}^{n-1} \tilde{V}_a \cong \left(\bigoplus_{a=1}^{n-1} V_a\right)^*.
\] (4.10)
Furthermore, we have that
\[
(\text{coker}(\mathcal{D}))^* \cong \left\{ f \in \left(\bigoplus_{a=1}^{n-1} V_a\right)^* \mid f|_{\text{image}(\mathcal{D})} = 0 \right\}
\] (4.11)
through the isomorphism \( F \overset{\sim}{\longrightarrow} F \circ \pi \) defined using the canonical projection
\[
\pi: \bigoplus_{a=1}^{n-1} V_a \rightarrow \left( \bigoplus_{a=1}^{n-1} V_a \right) / \text{image}(\mathcal{D}).
\]
The argument used in [29, middle of page 79] shows that under the isomorphisms given in equations (4.10) and (4.11), \( (\text{coker}(\mathcal{D}))^* \) is identified with \( \text{ker}(\mathcal{D}_X^N) \), which we have already shown to be trivial. The second and third claims are now proved.

Finally, for all \( x, y \in \mathbb{R}^3 \), the operator \( \mathcal{D}_X^N - \mathcal{D}_Y^N \) is just the left-multiplication by the imaginary quaternion \( \sum_{a=1}^{3}(x_a - y_a)\sigma_a \). In particular, we have of the operator norm that \( \| \mathcal{D}_X^N - \mathcal{D}_Y^N \| = |x - y| \), and thus the assignment \( x \mapsto \mathcal{D}_X^N \) is continuous and linear, and thus analytic. The proof is now complete. □

5. The Nahm Transform

We are now ready to state and prove our main theorem.

**Theorem 5.1.** The pair \((\nabla^N, \Phi^N)\) is an \( \text{SU}(N)\)-monopole, that is, it satisfies equations (2.1) and (2.3). Furthermore, conditions (2.5a) to (2.5c) hold and, in some gauge, \( \mu \) and \( \kappa \) satisfy
\[
\mu = \text{diag}(\mu_1, \mu_2, \cdots, \mu_n),
\]
\[
\kappa = \text{diag}(\kappa_1, \kappa_2, \cdots, \kappa_n),
\]
and for all \( a \in \{1, 2, \ldots, n\} \), \( \mu_a, \kappa_a \in \mathfrak{u}(r_a) \) satisfy
\[
\mu_a = i \lambda_a \mathbb{1}_{\mathbb{C}^r_a},
\]
\[
\kappa_a = \text{diag}(ik_{a,1}^+, ik_{a,2}^+, \cdots, 0, \cdots, -ik_{a,2}^-, -ik_{a,1}^-).
\]
Finally, the Yang–Mills–Higgs energy (2.2) of \((\nabla^N, \Phi^N)\) is given by
\[
E_{\text{YMH}}(\nabla^N, \Phi^N) = \sum_{a=1}^{n} \lambda_a k_a.
\]

**Remark 5.2.** In [10], we also show that the above monopole is without flat factors.

**Proof of Theorem 5.1.** First we prove that the pair \((\nabla^N, \Phi^N)\) satisfies equation (2.1). We use the notations of Definition 4.1. Furthermore, for any \( \Psi = (\psi_a)_{a=1}^{n-1} \in \mathcal{H}^F_i \) (with \( i \in \{0, 1\} \)), let us define
\[
\Sigma_a \Psi := \left( (\mathbb{1}_{\mathbb{C}^m_a} \otimes \sigma_a)\psi_a \right)_{a=1}^{n-1}.
\]
Since \( \Sigma_a \) is unitary and skew-adjoint, the assignment
\[
\mathbb{H} \ni \mapsto \text{End}(\mathcal{H}^F_i); \quad x_0 \mathbb{1}_\mathbb{H} + x_1 \sigma_1 + x_2 \sigma_2 + x_3 \sigma_3 \mapsto x_0 \mathbb{1}_{\mathcal{H}^F_i} + x_1 \Sigma_1 + x_2 \Sigma_2 + x_3 \Sigma_3,
\]
makes $\mathcal{H}^N_x$ a quaternionic Hilbert space.

The proof of equation (4.5) can easily be generalized to prove that for a smooth element $\Psi = (\psi_\alpha)_{\alpha=1}^{n-1} \in \mathcal{H}^N_1$ we have

$$H^N_x(\Psi) := \left( D^N_x \right)^* D^N_x (\Psi) = \left( -\ddot{\psi}_\alpha + \sum_{\alpha=1}^{3} \left( i T^{(\alpha)} + x_\alpha \mathbb{I}_{\mathbb{C}^m} \right) \otimes 1_{\mathbb{H}} \right)_{\alpha=1}^{n-1} \psi_\alpha.$$

Note that equation (4.5) implies that there is a positive number, $c^N_{\text{Sobolev}}$, such that

$$\|D^N_x(\Psi)\|_{\mathcal{H}_0^N} \leq c^N_{\text{Sobolev}} (1 + |x|^2) \|\Psi\|_{\mathcal{H}_0^N},$$

and thus $H^N_x$ has a continuous inverse (when regarded as a map to its topological dual) whose restriction to $\mathcal{H}_0^N$, denoted by $G^N_x$, satisfies

$$\|G^N_x\| \leq \frac{1}{c^N_{\text{Sobolev}} (1 + |x|^2)},$$

and $G^N_x$ depends on $x \in \mathbb{R}^3$ analytically. Furthermore, since $H^N_x$ commutes with action of quaternions, for all $\alpha \in \{1, 2, 3\}$ we have that

$$G^N_x \circ \Sigma_\alpha = \Sigma_\alpha \circ G^N_x. \quad (5.3)$$

Next we show that

$$\Pi_{E^N_x} = 1_{\mathcal{H}^N_0} - D^N_x \circ G^N_x \circ \left( D^N_x \right)^*.$$

By construction, $\Pi_{E^N_x}$ is an orthogonal projection, that is $\Pi_{E^N_x}^2 = \Pi_{E^N_x}$ and $\left( \Pi_{E^N_x} \right)^* = \Pi_{E^N_x}$. If $\Psi \in E^N_x = \ker(D^N_x)^*$, then

$$\Pi_{E^N_x}(\Psi) = \Psi - \left( D^N_x \circ G^N_x \circ \left( D^N_x \right)^* \right)(\Psi) = \Psi,$$

and if $\Psi$ is $L^2$-orthogonal to $E^N_x$, then, via a standard usage of the Lax–Milgram theorem, one can show that there exists a (unique) $Y \in \mathcal{H}_{1}^N$, such that $\Psi = D^N_x(Y)$, and thus

$$\Pi_{E^N_x}(\Psi) = \Psi - \left( D^N_x \circ G^N_x \circ \left( D^N_x \right)^* \right)(\Psi)$$

$$= D^N_x(Y) - \left( D^N_x \circ G^N_x \circ \left( D^N_x \right)^* \right)(Y)$$

$$= D^N_x(Y) - D^N_x(Y)$$

$$= 0.$$

Hence, $\Pi_{E^N_x}$ is indeed the $L^2$-orthogonal projection onto $\Pi_{E^N_x}$.
Note that the connection $\nabla^N$ is a Berry connection, in the sense of [2]. Thus by [2, Equation (1.9)], we get that

$$F_{\nabla^N} = \Pi_{E_x^N} \circ \left( d\Pi_{E_x^N} \wedge d\Pi_{E_x^N} \right) \circ \Pi_{E_x^N}. $$

Equivalently, if we write

$$F_{\nabla^N} = \frac{1}{2} \sum_{\alpha, \beta=1}^3 F_{\alpha, \beta} \, dx^\alpha \wedge dx^\beta,$$

then for all $\alpha, \beta \in \{1, 2, 3\}$:

$$F_{\alpha, \beta} = \Pi_{E_x^N} \circ \left( \partial_\alpha \Pi_{E_x^N} \right) \circ \left( \partial_\beta \Pi_{E_x^N} \right) \circ \Pi_{E_x^N} - \Pi_{E_x^N} \circ \left( \partial_\beta \Pi_{E_x^N} \right) \circ \left( \partial_\alpha \Pi_{E_x^N} \right) \circ \Pi_{E_x^N}. $$

We first compute $\partial_\alpha \Pi_{E_x^N}$:

$$\partial_\alpha \Pi_{E_x^N} = - \partial_\alpha \left( D_x^N \circ G_x^N \circ \left( D_x^N \right)^* \right)$$

$$= - \left( \partial_\alpha D_x^N \circ G_x^N \circ \left( D_x^N \right)^* \right) - D_x^N \circ \left( \partial_\alpha G_x^N \circ \left( D_x^N \right)^* \right) - D_x^N \circ \left( \partial_\alpha \left( \left( D_x^N \right)^* \right) \right)$$

$$= - \left( \partial_\alpha D_x^N \circ G_x^N \circ \left( D_x^N \right)^* \right) - D_x^N \circ \left( \partial_\alpha G_x^N \circ \left( D_x^N \right)^* \right) - D_x^N \circ \left( \partial_\alpha \left( \left( D_x^N \right)^* \right) \right).$$

By equation (4.2), we get that $\partial_\alpha D_x^N = \Sigma_\alpha$. Using the above equation and that

$$\Pi_{E_x^N} \circ D_x^N = \left( D_x^N \right)^* \circ \Pi_{E_x^N} = 0,$$

we get

$$\Pi_{E_x^N} \circ \left( \partial_\alpha \Pi_{E_x^N} \right) = - \Pi_{E_x^N} \circ \Sigma_\alpha \circ \left( D_x^N \right)^*,$$

(5.4a)

$$\left( \partial_\beta \Pi_{E_x^N} \right) \circ \Pi_{E_x^N} = D_x^N \circ G_x^N \circ \Sigma_\beta \circ \Pi_{E_x^N}. $$

(5.4b)

Combining equations (5.4a) and (5.4b) yields

$$\Pi_{E_x^N} \circ \left( \partial_\alpha \Pi_{E_x^N} \right) \circ \left( \partial_\beta \Pi_{E_x^N} \right) \circ \Pi_{E_x^N} = - \Pi_{E_x^N} \circ \Sigma_\alpha \circ \left( D_x^N \right)^* \circ D_x^N \circ G_x^N \circ \Sigma_\beta \circ \Pi_{E_x^N}$$

$$= - \Pi_{E_x^N} \circ \Sigma_\alpha \circ G_x^N \circ H_x^N \circ G_x^N \circ \Sigma_\beta \circ \Pi_{E_x^N}$$

$$= - \Pi_{E_x^N} \circ \Sigma_\alpha \circ G_x^N \circ \Sigma_\beta \circ \Pi_{E_x^N}$$

$$= - \Pi_{E_x^N} \circ \Sigma_\alpha \circ \Sigma_\beta \circ G_x^N \circ \Pi_{E_x^N}$$

$$= \delta_{\alpha, \beta} \Pi_{E_x^N} \circ G_x^N \circ \Pi_{E_x^N} - \sum_{\gamma=1}^3 \epsilon_{a, \beta, \gamma} \Pi_{E_x^N} \circ \Sigma_\gamma \circ G_x^N \circ \Pi_{E_x^N},$$

and hence

$$F_{\alpha, \beta} = - 2 \sum_{\gamma=1}^3 \epsilon_{a, \beta, \gamma} \Pi_{E_x^N} \circ \Sigma_\gamma \circ G_x^N \circ \Pi_{E_x^N}. $$

(5.5)

Similarly, we can compute the components of $\nabla^N \Phi^N$: Let $M$ be as in Definition 4.1, and thus $\Phi^N = -i \Pi_{E_x^N} \circ M$. Note that $M$ is independent of $x$. Fix $\alpha \in \{1, 2, 3\}$. Then, using equation (5.4a),

$$\text{...}$$
we get
\[ \nabla^N_a \Phi^N = \left( \Pi_{E^N_x} \circ \partial_a \right) \circ \left( -i \Pi_{E^N_x} \circ M \right) \big|_{E^N_x} \]
\[ = -i \Pi_{E^N_x} \circ \left( \partial_a \Pi_{E^N_x} \right) \circ M \circ \Pi_{E^N_x} + i \Pi_{E^N_x} \circ M \circ \left( \partial_a \Pi_{E^N_x} \right) \circ \Pi_{E^N_x}. \]

Using equations (5.4a) and (5.4b) again we get
\[ \nabla^N_a \Phi^N = i \Pi_{E^N_x} \circ \Sigma_a \circ G^N_x \circ \left( D^N_x \right)^* \circ M \circ \Pi_{E^N_x} - i \Pi_{E^N_x} \circ M \circ D^N_x \circ G^N_x \circ \Sigma_a \circ \Pi_{E^N_x}. \]

Using the definitions of \( D^N_x \) and \( M \), we get
\[ M \circ D^N_x \big|_{E^N_x} = \left( D^N_x \right)^* \circ M \big|_{E^N_x} = i 1_{E^N_x}. \]

Putting these together, and using equation (5.3), we get
\[ \nabla^N_a \Phi^N = -2 \Pi_{E^N_x} \circ \Sigma_a \circ G^N_x \circ \Pi_{E^N_x}. \]  

(5.6)

Now equations (5.5) and (5.6) imply that the pair \( (\nabla^N, \Phi^N) \) satisfies equation (2.1). Since \( \Phi^N \) is skew-adjoint, the structure group can be reduced to \( U(N) \). By Remark 2.1, the trace of \( \Phi^N \) is constant and if that constant is zero. Then \( (\nabla^N, \Phi^N) \) is, in fact, an SU(\( N \))-monopole.

Next we show that
\[ \nabla^N \Phi^N - \nabla^N_\partial \Phi^N \otimes dr = O\left( \frac{1}{|x|^r} \right). \]

(5.7)

Using an appropriate change of basis from SO(3), we can assume, without any loss of generality, that \( x_1 = |x|, x_2 = x_3 = 0 \), and thus equation (5.7) is equivalent to
\[ \forall \alpha \in \{2, 3\} : \nabla^N_\alpha \Phi^N = O\left( \frac{1}{|x|^r} \right). \]

Using another change of basis, we can assume that \( \alpha = 2 \). Let \( \Psi \in E^N_x \) be unit norm. It is enough to show that
\[ \langle \Psi | \nabla^N_2 \Phi^N | \Psi \rangle_{E^N_x} = 2 \langle \Sigma_2 \Psi | G^N_3(\Psi) \rangle_{H^N_0} = O\left( \frac{1}{|x|^r} \right), \]
for all such \( \Psi \). Let \( Y := G^N_3(\Psi) \). Then \( (D^N_3)^* D^N_3(\Psi) = \Psi \) and thus, for \( |x| \) large, using equation (4.5), we get
\[ \frac{1}{2} |x|^2 \| Y \|^2_{H^N_0} \leq \| D^N_3 Y \|^2_{H^N_0} = \langle \Psi | Y \rangle_{H^N_0} \leq \| Y \|_{H^N_0}, \]
yielding (for \( |x| \gg 1 \))
\[ \| Y \|_{H^N_0} \leq \frac{2}{|x|^2}, \]
\[ \| D^N_3 Y \|_{H^N_0} \leq \frac{\sqrt{2}}{|x|}. \]

Thus we have
\[ -\langle \Sigma_1 \Psi | \Sigma_3 Y \rangle_{H^N_0} = -\frac{1}{|x|} \langle \langle x | \Sigma_1 \Psi | \Sigma_3 \rangle Y \rangle_{H^N_0} \]
Using that we get
\[
\frac{1}{2} \langle \Psi | \nabla^N_2 \Phi^N \Psi \rangle_{E_x^N} = \langle \Sigma_2 \Psi | \nabla G_x(\Psi) \rangle_{H^0_{\mathcal{N}}}
= \langle \Sigma_2 \Psi | \nabla \rangle_{H^0_{\mathcal{N}}}
= \langle \Sigma_2 \circ \Sigma_1 \Psi | \nabla \rangle_{H^0_{\mathcal{N}}}
= - \langle \Sigma_1 \Psi | \Sigma_2 \nabla \rangle_{H^0_{\mathcal{N}}}
= O\left( \frac{1}{|x|^3} \right).
\]

Hence equation (5.7) holds. Similar argument shows that if \( \alpha \in \{2, 3\} \), then
\[
\nabla^N_{\alpha} \nabla^N_1 \Phi^N = O\left( \frac{1}{|x|^3} \right),
\]
which, together with equation (5.7), shows that \( R^2 \nabla^N_1 \Phi^N \) converges to a section of \( su(E_\infty) \).

Furthermore, we can also prove analogously that
\[
\nabla^N_{\alpha} \nabla^N_1 \Phi^N = O\left( \frac{1}{|x|^3} \right), \quad (5.8)
\]
which gives that for any \( k \in \mathbb{N} \), this convergence holds in the \( C^k \)-topology. Setting
\[
\kappa := \lim_{R \to \infty} 2R^2 \nabla^N_1 \Phi^N,
\]
yields condition (2.5c). Using equation (5.8) again, we also get equation (2.6b).

For each \( R \in \mathbb{R}_+ \), let \( (\nabla^R, \Phi^R) \) be the pullbacks to \( S^2_\infty \), via the canonical maps \( S^2_\infty \to S^2_R \), of the connection induced by \( \nabla^N \) on \( E|_{S^2_R} \) and \( \Psi^N|_{S^2_R} \). Let us fix an orthonormal basis of \( E_0 \), call \( \{ \Psi_1, \Psi_2, \ldots, \Psi_N \} \). Now for all \( x \in \mathbb{R}^3 \) and \( a \in \{1, 2, \ldots, N\} \), let \( \Psi_a(x) \in E_x \) be the parallel transport of \( \Psi_a \) via \( \nabla^N \) along the oriented straight line from the origin to \( x \). The sections obtained give us a global trivialization of \( E \). We show that in this trivialization, the connection matrices of \( \nabla^R \) and the matrix of \( \Phi^R \) both converge to smooth sections on \( S^2_\infty \). Let, for all \( a, b \in \{1, 2, \ldots, N\} \), \( \alpha \in \{1, 2, 3\} \), and \( R, u \in \mathbb{R}_+ \)
\[
\Phi^R_{a,b} := \langle \Psi_a | \Phi^R \Psi_b \rangle_{H^0_{\mathcal{N}}},
A^R_{a,a,b} := \langle \Psi_a | \nabla^R \Psi_b \rangle_{H^0_{\mathcal{N}}}.
\]
Now let us compute
\[
\partial_R \Phi^R_{a,b} = \partial_R \left( h(\Psi_a, \Phi^R) \right) = \langle \partial_R \Psi_a | \Phi^R \Psi_b \rangle_{H^0_{\mathcal{N}}} + \langle \Psi_a | (\partial_R \Phi^R) \Psi_b \rangle_{H^0_{\mathcal{N}}} + \langle \Psi_a | \Phi^R \partial_R \Psi_b \rangle_{H^0_{\mathcal{N}}}.
\]
The first and last terms on the right are zero, since $\Psi_a$ and $\Psi_b$ are parallel in the radial direction. The middle term can be rewritten as

$$\partial_R \Phi^R_{a,b} = \langle \Psi_a | (\nabla^N_r \Phi^N) \Psi_b \rangle_{\mathcal{H}^N_0}.$$ 

Thus we get

$$\left| \partial_R \Phi^R_{a,b} \right| = \left| \langle \Psi_a | (\nabla^N_r \Phi^N) \Psi_b \rangle_{\mathcal{H}^N_0} \right| = O\left(\frac{1}{R^2}\right),$$

which means that $\Phi^R$ is pointwise convergent, to a section $\mu$, on $E_\infty$, as $R \to \infty$. As before, without any loss of generality, we can assume that $x_1 = R$ and $x_2 = x_3 = 0$. Let $\alpha \in \{2, 3\}$. Then,

$$\partial_R A^R_{\alpha,a,b} = \partial_R \langle \Psi_a | \nabla^R_{\alpha} \Psi_b \rangle_{\mathcal{H}^N_0} = \langle \Psi_a | \nabla^N_1 \nabla^R_{\alpha} \Psi_b \rangle_{\mathcal{H}^N_0} = \langle \Psi_a | F_{1,\alpha} \Psi_b \rangle_{\mathcal{H}^N_0} = O\left(\frac{1}{R}\right).$$

Thus $\nabla^R$ is pointwise convergent to a connection, $\nabla^\infty$, on $E_\infty$, as $R \to \infty$, proving condition (2.5a).

Using equation (5.7) we can similarly prove that for any $k \in \mathbb{N}$, the convergence, $(\nabla^R, \Phi^R) \to (\nabla^\infty, \mu)$ also holds in the $C^k$-topology. Furthermore, bound in equation (5.7) is strong enough to prove that $2R^2 \nabla^R \Phi^R$ converges to a section $\kappa$ for which we can show, using equation (2.1), that it satisfies equation (2.6c). Thus we also have condition (2.5b)

Let now $\alpha \in \{2, 3\}$, and compute using equation (5.7)

$$\nabla^\infty_a \mu = \lim_{R \to \infty} \left( R \nabla^N_1 \Phi^N \right) = 0,$$

which proves equation (2.6a). Hence we also get $\nabla^\infty$ is a Yang–Mills connection.

Finally, we prove that the spectra of $\mu$ and $\kappa$ satisfy equations (5.1a) and (5.1b), which also immediately yield equation (5.2), via the usual Bogomolny trick for monopoles. This can be done analogously to the proof of [27, Theorem 2.8], where Hitchin constructs approximate (unit length) eigenvectors of $\Phi$, say $\Psi$, satisfying

$$\Phi \Psi = i \left( \lambda - \frac{k}{2|x|} \right) \Psi + O\left(\frac{1}{|x|}\right).$$

The same can now be done for each $a \in \{1, 2, \ldots, n\}$, sign $\pm$, and $b \in \{1, 2, \ldots, k^\pm_a\}$, with $\lambda = \lambda_a$ and $k = k^\pm_{a,b}$, which completes the proof.

\[\Box\]

**APPENDIX A. MONOPOLES WITH REAL ORTHOGONAL AND COMPACT SYMPLECTIC STRUCTURE GROUPS**

So far we have only viewed $(\nabla^N, \Phi^N)$ as an SU($N$)-monopole. However, given an arbitrary subgroup $G \subseteq$ SU($N$), one can produce G-monopole through introducing further restrictions on the Nahm data. In the maximal symmetry breaking case this was proven in [29, Section 5].
for real orthogonal and compact symplectic groups, and in [34] for arbitrary compact Lie

groups. The methods can easily be adapted to the case of nonmaximal symmetry breaking.

Here we only outline the cases when $G$ is either a real orthogonal group, $SO(k)$, or a

compact symplectic groups, $Sp(k)$. The reduction to real orthogonal groups is equivalent to the

existence of a $(\nabla^N, \Phi^N)$-compatible real structure, $\hat{C}$, on $E^N_x$, that is, a covariantly constant,

real linear bundle map that commutes with $\Phi^N$, anti-commutes with the multiplication by

$i$, and is an involution, that is $\hat{C}^2 = 1_{E^N_x}$. Similarly, the reduction to compact symplectic

groups is equivalent to the existence of a $(\nabla^N, \Phi^N)$-compatible quaternionic structure, $\hat{J}$, on

$E^N_x$, that is, a covariantly constant, real linear bundle map that anti-commutes with $\Phi^N$, anti-

commutes with the multiplication by $i = \sigma_1$, and is a complex structure, that is $\hat{J}^2 = -1_{E^N_x}$.

Let $\mathcal{N}$ be a Nahm data in temporal gauge and that for all $a \in \{1, 2, \ldots, n-1\}$ satisfies

$$\lambda_a = -\lambda_{n-a},$$

$$r^+_a = r^-_{n-a},$$

$$r^0_a = r^0_{n-a},$$

and for all $b \in \{1, 2, \ldots, r^+_a\}$

$$k^\pm_{a,b} = -k^\mp_{n-a,b},$$

and thus

$$k^\pm_a = k^\mp_{n-a},$$

$$k_a = -k_{n-a},$$

$$m_a = m_{n-a}.$$  

Let $\iota: \mathbb{R} \to \mathbb{R}; t \mapsto -t$. Then $\iota(\lambda_a, \lambda_{a+1}) = (\lambda_{n-a-1}, \lambda_{n-a})$.

**Orthogonal groups:** Let us further assume that for all $t \in (\lambda_a, \lambda_{a+1})$ and $\alpha \in \{1, 2, 3\}$:

$$T^{(a)}(t) = \overline{T^{(a)}(-t)}.$$  

Let $c_a: \mathbb{C}^{m_a} \otimes \mathbb{H} \to \mathbb{C}^{m_{n-a}} \otimes \mathbb{H} \cong \mathbb{C}^{m_a} \otimes \mathbb{H}$ given via

$$c_a(v \otimes q) = \overline{v} \otimes \overline{q},$$

and then, for all $\Psi = (\psi_a)_{a=1}^{n-1} \in \mathcal{H}^N_0$, let us define

$$C(\Psi) = (c_{n-a}(\psi_{n-a}) \circ \iota)_{a=1}^{n-1}.$$  

Then $C$ is a real structure on $\mathcal{H}^N_0$ and for all $x \in \mathbb{R}^3$, $C$ preserves $E^N_x$. Let $\hat{C}_x := C|_{E^N_x}$

and consider $\hat{C}$ as a bundle map. Then we have

$$\nabla^N \hat{C} = 0, \quad \hat{C} \circ \Phi^N = \Phi^N \circ \hat{C}.$$  

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Thus \( (\nabla^N, \Phi^N) \) is an SO(\(N\))-monopole.

**Compact symplectic groups:** Let us further assume that for all \( t \in (\lambda_a, \lambda_{a+1}) \) and \( \alpha \in \{1, 2, 3\} \):

\[
T^{(a)}(t) = -\overline{T^{(a)}(-t)}.
\]

Let \( j_a : \mathbb{C}^m \otimes \mathbb{H} \to \mathbb{C}^{m_n-a} \otimes \mathbb{H} \cong \mathbb{C}^{m_n} \otimes \mathbb{H} \) given via

\[
j_a(\nu \otimes q) = \overline{\nu} \otimes (q \sigma_2),
\]

and then, for all \( \Psi = (\psi_a)_{a=1}^{n-1} \in \mathcal{H}_{0}^N \), let us define

\[
J(\Psi) = (j_{n-a}(\psi_{n-a}) \circ i)_{a=1}^{n-1}
\]

Then \( J \) is a real structure on \( \mathcal{H}^N_0 \) and for all \( x \in \mathbb{R}^3 \), \( J \) preserves \( E^N_x \). Let \( J_x := J|_{E^N_x} \) and consider \( J \) as a bundle map. Then we have

\[
\nabla^N J = 0, \quad \text{and} \quad \hat{J} \circ \Phi^N = -\Phi^N \circ \hat{J}.
\]

Thus \( N \) is necessarily even and \( (\nabla^N, \Phi^N) \) is an Sp(\(\frac{N}{2}\))-monopole.

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**Acknowledgment.** The first author acknowledges the support of the Natural Sciences and Engineering Research Council of Canada (NSERC), RGPIN-2019-04375.

The authors are thankful to Mark Stern for the many illuminating discussions about gauge theory, in general, and this project, in particular, that have greatly contributed to the completion of this paper. They also wish to thank Sergey Cherkis, Jacques Hurtubise, Gonçalo Oliveira, and Thomas Walpuski for their feedback and insight.

While working on various parts of this project, the second author also enjoyed the hospitality of the University of Waterloo, the Perimeter Institute, the Fields Institute, and Duke University.

Both authors enjoyed opportunities to advance the research of this paper at the conference “Geometric and analytic aspects of moduli spaces” in Hannover in 2019, and at the BIRS online workshop “Geometry, Analysis, and Quantum Physics of Monopoles” in 2021.

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