ON INTERSECTION FORMS OF DEFINITE 4-MANIFOLDS BOUNDED BY A RATIONAL HOMOLOGY 3-SPHERE

DONG HEON CHOE AND KYUNGBAE PARK

Abstract. We show that, if a rational homology 3-sphere $Y$ bounds a positive definite smooth 4-manifold, then there are finitely many negative definite lattices, up to the stable-equivalence, which can be realized as the intersection form of a smooth 4-manifold bounded by $Y$. To this end, we make use of constraints on definite forms bounded by $Y$ induced from Donaldson’s diagonalization theorem, and correction term invariants due to Frøyshov, and Ozsváth and Szabó. In particular, we prove that all spherical 3-manifolds satisfy such finiteness property.

1. Introduction

Throughout this paper we assume that all manifolds are compact and oriented. We say a 4-manifold $X$ is bounded by a 3-manifold $Y$ if $Y$ is homeomorphic to the boundary of $X$ and the orientation of $Y$ inherits the orientation of $X$ in the standard way.

The intersection pairing $Q_X$ on $H_2(X; \mathbb{Z})/\text{Tors}$ of a 4-manifold $X$ is an integer-valued, symmetric, bilinear form over $\mathbb{Z}^{b_2(X)}$. If the boundary of $X$ is a rational homology 3-sphere or empty, $Q_X$ is nondegenerate. In this case, the intersection form $Q_X$ algebraically forms a lattice. In particular, if $X$ is closed, then $Q_X$ is unimodular, i.e. $\det(Q_X) = \pm 1$.

The remarkable works of Donaldson and Freedman in the early 1980’s portray a big difference between topological and smooth categories in dimension 4 for the answer to the following question:

Question. Which negative definite, unimodular lattices are realized as the intersection form of a closed 4-manifold?

Freedman showed that any unimodular definite lattice can be realized as the intersection form of a closed, topological 4-manifold [Fre82]. On the other hand, Donaldson’s diagonalization theorem asserts that the only standard diagonal one can be realized as the intersection form of a closed, smooth 4-manifold [Don83, Don87].

In this paper, we would like to study this phenomenon for 4-manifolds with a fixed boundary. We say a lattice $\Lambda$ is smoothly (resp. topologically) bounded by a 3-manifold $Y$ if $\Lambda$ can be realized as the intersection form of a smooth (resp. topological) 4-manifold with the boundary $Y$. It follows easily by connected summing $\mathbb{CP}^2$ to a 4-manifold realizing $\Lambda$ that if $Y$ bounds a lattice $\Lambda$, then it also does $\Lambda \oplus \langle -1 \rangle$. We call this procedure the stabilization of $\Lambda$.

Definition 1.1. Two negative definite lattices $\Lambda_1$ and $\Lambda_2$ are stable-equivalent if $\Lambda_1 \oplus \langle -1 \rangle^m \cong \Lambda_2 \oplus \langle -1 \rangle^n$ for some non-negative integers $m$ and $n$. 

2010 Mathematics Subject Classification. 57M27, 57N13, 57R58.

Key words and phrases. Smooth 4-manifolds; intersection forms; spherical 3-manifolds; integral lattices.
Let $\mathcal{I}(Y)$ (resp. $\mathcal{I}^{\text{TOP}}(Y)$) denote the set of all negative definite lattices that can be smoothly (resp. topologically) bounded by $Y$, up to the stable-equivalence. In terms of these notations, aforementioned Freedman and Donaldson’s results can be interpreted as

$$\mathcal{I}^{\text{TOP}}(S^3) = \{[\Lambda] \mid \Lambda: \text{any unimodular negative definite lattice}\}$$

and

$$\mathcal{I}(S^3) = \{[-1]\}.$$ 

Following the Freedman’s results, Boyer studied a realization problem for topological 4-manifolds with a fixed boundary $Y$ [Boy86]. Roughly speaking, any forms presenting the linking pairing of $Y$ can be realized. Hence one can easily observe the following:

**Theorem 1.2.** Let $Y$ be a rational homology 3-sphere. Then

$$|\mathcal{I}^{\text{TOP}}(Y)| = \infty.$$

**Proof.** For the linking pairing of a 3-manifold $Y$, it is known by Edmonds [Edm05, Section 6] that there is a definite form presenting it. This form is realized by a topological 4-manifold $W$ by Boyer’s result. Now, we have infinitely many definite lattices bounded by $Y$, up to the stabilization, by connected summing $W$ and closed topological 4-manifolds with non-standard definite intersection forms. □

The main interest of this article is whether the finiteness of $\mathcal{I}(Y)$ holds for any rational homology 3-sphere $Y$, similarly to the case of $S^3$. Although we conjecture that $\mathcal{I}(Y)$ is finite for any rational homology 3-sphere $Y$, it has been known only for manifolds with small correction term invariant. Recall that the correction term is a rational-valued invariant for rational homology 3-spheres, due to Frøyshov in Seiberg-Witten theory [Frø96] and Ozsváth and Szabó in Heegaard Floer theory [OS03]. The correction term is known to give constraints on definite lattices smoothly bounded by $Y$. Let $\Lambda$ be a negative definite lattice bounded by $Y$. Then the following inequality is satisfied

$$\delta(\Lambda) \leq d(Y),$$

where $\delta$ is a rational-valued invariant for definite lattices defined in [Elk95a]: see also Section 2. On the other hand, the finiteness of the number of stable classes of definite unimodular lattices are known only for small $\delta \leq 6$, purely algebraically [Elk95a, Elk95b, Gau07, NV03]. Therefore, we can conclude that $\mathcal{I}(Y)$ is finite for any integral homology 3-sphere $Y$ with $d(Y) \leq 6$. For rational homology spheres with sufficiently small $\delta$, we deduce a similar finiteness of $\mathcal{I}(Y)$ from the result of Owens and Strle in [OS12a] about non-unimodular lattices. Our main results are that $\mathcal{I}(Y)$ is finite under some additional conditions.

**Theorem 1.3.** Let $Y_1$ and $Y_2$ be rational homology 3-spheres. If there is a negative definite smooth cobordism from $Y_1$ to $Y_2$ and $|\mathcal{I}(Y_2)| < \infty$, then $|\mathcal{I}(Y_1)| < \infty$.

Since we know $|\mathcal{I}(S^3)| < \infty$ by Donaldson’s diagonalization theorem, we have the following corollary.

**Corollary 1.4.** Let $Y$ be a rational homology 3-sphere. If $Y$ bounds a positive (resp. negative) definite smooth 4-manifold, then there are only finitely many negative (resp. positive) definite lattices, up to the stable-equivalence, which can be realized as the intersection form of a smooth 4-manifold bounded by $Y$.

In other words, if $\mathcal{I}(-Y) \neq \emptyset$, then $|\mathcal{I}(Y)| < \infty$. 

Proof. Let $W$ be a positive definite smooth 4-manifold with the boundary $Y$. Then we construct a negative definite cobordism from $Y$ to $S^3$ by reversing the orientation of the punctured $W$.  

Remark. We say a 4-manifold $X$ is negative (resp. positive) definite if $b_2(X) = b_2^+(X)$ (resp. $b_2(X) = b_2^-(X)$). In particular, a 4-manifold with $b_2 = 0$ is considered to be both positive and negative definite.

In [Boy86, Corollary 0.4], Boyer showed that there are only finitely many homeomorphism types of simply-connected 4-manifolds which have given intersection form and boundary. Our results give a bit of answer to the geography problem of simply-connected, smooth 4-manifolds with a fixed boundary.

Corollary 1.5. If $Y$ bounds a positive (resp. negative) definite smooth 4-manifold, then there are finitely many homeomorphism types of simply-connected, negative (resp. positive) definite smooth 4-manifolds bounded by $Y$, up to the stabilization.

To prove our main theorem, we consider a set of lattices $L$ defined purely algebraically in terms of invariants of a given 3-manifold $Y$. This set $L$ contains negative definite lattices, up to the stable-equivalence, which satisfy the conditions for a definite lattice to be smoothly bounded by $Y$, induced from the correction term invariants and fundamental obstructions from the algebraic topology. See Section 2.2 for details of these conditions. Then Theorem 1.3 readily follows after we show the finiteness of $L$.

Theorem 1.6. Let $\Gamma_1$ and $\Gamma_2$ be fixed negative definite lattices, and $C > 0$ and $D \in \mathbb{Z}$ be constants. Define $L(\Gamma_1, \Gamma_2; C, D)$ to be the set of negative definite lattices $\Lambda$, up to the stable-equivalence, satisfying the following conditions:

- $\det(\Lambda) = D$,
- $\delta(\Lambda) \leq C$, and
- $\Gamma_1 \oplus \Lambda$ embeds into $\Gamma_2 \oplus \langle -1 \rangle^N$, $N = rk(\Gamma_1) + rk(\Lambda) - rk(\Gamma_2)$.

Then $L(\Gamma_1, \Gamma_2; C, D)$ is finite.

Our proof of Theorem 1.6 is highly inspired by the work of Owens and Strle in [OS12a], where they studied non-unimodular lattices in terms of the lengths of characteristic covectors. The key idea of our proof is to improve one of their inequalities on the length of a characteristic covector, enough to give an upper bound on the rank of the lattices in $L(\Gamma_1, \Gamma_2; C, D)$.

Remark. We remark that not all lattices which represent a stable class in $\mathcal{I}(Y)$ can be realized as the intersection form of a smooth 4-manifold bound by $Y$. It is well known that the Poincaré homology 3-sphere $\Sigma$, oriented as the boundary of the $-E_8$-plumbed 4-manifold, can be obtained by $(-1)$-framed surgery along the left-handed trefoil knot. Hence $[(-1)] = [0] \in \mathcal{I}(\Sigma)$. Whereas one can prove, using a constraint from the Donaldson’s diagonalization theorem, $\Sigma$ cannot bound any 4-manifold with $b_2 = 0$, i.e. the empty lattice cannot be realized. See Example 2.2.

Seifert fibered spaces such that $\mathcal{I}(Y) \subset \mathcal{I}(Y)$. It is interesting to ask which 3-manifolds satisfy the condition in Corollary 1.3, i.e. which 3-manifolds bound a positive definite smooth 4-manifold. Since many of 3-manifolds, including all Seifert fibered rational homology 3-spheres, bound a definite smooth 4-manifold of both signs.

It is well known that any lens space satisfy such property. Note that lens spaces can be obtained by double covering of $S^3$ branched along 2-bridge knots. As generalizing
lens spaces to this direction, the double branched covers of $S^3$ along quasi-alternating links are known to bound definite smooth 4-manifolds of both signs (with trivial first homology) [OS05b, Proof of Lemma 3.6]. Also notice that lens spaces can be obtained by Dehn-surgery along the unknot. In [OS12b], Owens and Strle classified 3-manifolds obtained by Dehn-surgery on torus knots which can bound definite smooth 4-manifolds of both signs.

In Section 4, we consider this question for another class of 3-manifolds, Seifert-fibered rational homology 3-spheres, which also contains lens spaces. In particular, we completely determine which spherical 3-manifolds bound definite smooth 4-manifolds of both signs, and finally show that any spherical 3-manifold $Y$ has the property that $|I(Y)| < \infty$.

**Theorem 1.7.** Let $Y$ be a spherical 3-manifold. Then, there are finitely many stable classes of negative definite lattices which can be realized as the intersection form of a smooth 4-manifold bounded by $Y$, i.e. $|I(Y)| < \infty$.

**Further questions.**

*Generalizing our results to 3-manifolds with $b_1 > 0$.* Note that a 3-manifold with $b_1 > 0$ might bound a 4-manifold with degenerate intersection form. Hence, we generalize $I(Y)$ to be the set of stable classes of lattices which can be realized as the maximal nondegenerate subspace of the intersection form of a negative semi-definite 4-manifolds bounded by a 3-manifold $Y$.

On the other hand, the correction term invariants for rational homology 3-spheres has been generalized for some 3-manifolds with $b_1 > 0$ by Levine and Ruberman in [LR14], and by Behrens and Golla in [BG18]. These generalized correction term invariants also give restrictions on semi-definite intersection forms of 4-manifolds bounded by a 3-manifold. Hence we expect the analogous finiteness results of $I(Y)$ for any closed oriented 3-manifold $Y$ under suitable conditions, but we leave this for the future study.

*Determining the order of $I$. Although we know $|I(S^3)| = 1$ from Donaldson’s diagonalization theorem, determining the exact order of $I(\cdot)$ for other than $S^3$ seems a difficult problem. For instance, the correction term of the Poincaré homology sphere $\Sigma$ is known as 2. On the other hand, in [Elk95b] Elkies showed that there are only 15 stable classes of negative definite, unimodular lattices with $\delta \leq 2$. Since the trivial lattice and $-E_8$ lattice are in fact realized to be bounded by $\Sigma$, we have $2 \leq |I(\Sigma)| \leq 15$. Then what is the exact value of $|I(\Sigma)|$?

### 2. Preliminary

In this section we collect some background materials that will be used to prove our main theorems.

#### 2.1. Lattices.

A lattice of rank $n$ is a free abelian group $\mathbb{Z}^n$ equipped with an integer-valued, nondegenerate, symmetric, bilinear form, $Q: \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$.

Let $\Lambda = (\mathbb{Z}^n, Q)$ be a lattice. By tensoring $\Lambda$ with $\mathbb{R}$, one can extend $Q$ to a symmetric bilinear form over the vector space $\mathbb{R}^n$. We define the signature of $\Lambda$ to be the signature of $Q$. We say $\Lambda$ is positive (resp. negative ) definite if the signature of $\Lambda$ equals to the (resp. negative) rank of $\Lambda$.

By fixing a basis $\{v_1, \ldots, v_n\}$ for $\Lambda$, we can represent $\Lambda$ by an $n \times n$ matrix, $[Q(v_i, v_j)]$. The determinant of a lattice $\Lambda$, $\det(\Lambda)$, is the determinant of a matrix.
representation of $\Lambda$. In particular, if the determinant of a lattice is $\pm 1$, or equivalently a corresponding matrix is invertible over $\mathbb{Z}$, we say the lattice is unimodular.

The dual lattice $\Lambda^* := \text{Hom}(\Lambda, \mathbb{Z})$ can be identified with the set of elements in $\xi \in \Lambda \otimes \mathbb{R}$ such that $\xi \cdot w \in \mathbb{Z}$ for any $w \in \Lambda$. We call $\xi$ in $\Lambda^*$ a characteristic covector if $\xi \cdot w \equiv w \cdot w$ modulo 2 for any $w \in \Lambda$. We say a lattice $\Lambda_1$ embeds into $\Lambda_2$ if there is a monomorphism from $\Lambda_1$ to $\Lambda_2$ preserving bilinear forms.

The lattice $\langle p \rangle$ denotes the lattice of rank 1 represented by the matrix $[p]$. The standard negative definite lattice of rank $n$ is the lattice $\langle -1 \rangle^n$, the direct sum of $n$ copies of $\langle -1 \rangle$. For a negative definite lattice $\Lambda$ with rank $n$, we define $\delta$-invariant of $\Lambda$ as

$$\delta(\Lambda) := \frac{n - \min_{\xi \in \text{Char}(\Lambda)} |\xi \cdot \xi|}{4},$$

where $\text{Char}(\Lambda)$ is the set of characteristic covectors of $\Lambda$. Note that a negative definite lattice $\Lambda$ can be uniquely decomposed as $\Lambda' \oplus \langle -1 \rangle^m$ so that $\Lambda'$ does not contain any vector with square $-1$, and $\delta(\Lambda) = \delta(\Lambda')$.

### 2.2 Restrictions on lattices bounded by a rational homology 3-sphere.

We recall some constraints on lattices bounded by a given rational homology 3-sphere. We also refer the readers to Owens and Strle’s survey paper [OS05a] for more detail.

**Topological obstruction.** Suppose $Y$ bounds a lattice $\Lambda = (\mathbb{Z}^n, Q)$ and $X$ is a 4-manifold realizing $\Lambda$. From the homology long exact sequence of the pair $(X, Y)$, we have

$$|H^2(Y; \mathbb{Z})| = |\det(\Lambda)|t^2$$

for some integer $t$. See [OS06, Lemma 2.1] for example. In particular, $\det(\Lambda)$ divides $|H^2(Y; \mathbb{Z})|$. We remark that there are more detailed conditions regarding the linking form of $\Lambda$ and the linking pairing of $Y$, but we only recall the above simple property for our purpose.

Donaldson’s Theorem. The celebrated Donaldson’s diagonalization theorem can be used to give a constraint on definite lattices smoothly bounded by a 3-manifold. Recall the Donaldson’s theorem.

**Theorem 2.1** ([Don87, Theorem 1]). Suppose $X$ is a closed, smooth 4-manifold. If the intersection form of $X$ is negative definite, then it is isometric to the standard negative definite lattice $(\mathbb{Z}^n, \langle -1 \rangle^n)$.

Suppose a rational homology 3-sphere $Y$ bounds a positive definite, smooth 4-manifold $W$. If $Y$ bounds a negative definite lattice $\Lambda$, then we can construct a negative definite, closed, smooth 4-manifold by gluing a 4-manifold $X$ realizing $\Lambda$ with $W$ along $Y$. By Donaldson theorem, the intersection form of the closed smooth 4-manifold $X \cup_Y -W$ is the standard negative definite of the rank, $\text{rk}(\Lambda) + \text{rk}(Q_W)$. Then, from the Mayer-Vietoris sequence for the pair $(X \cup_Y -W, Y)$,

$$\cdots \to H_2(Y) \to H_2(X) \oplus H_2(-W) \to H_2(X \cup_Y -W) \to \cdots,$$

we have an embedding of $\Lambda \oplus -Q_W$ into $\langle -1 \rangle^{\text{rk}(\Lambda) + \text{rk}(Q_W)}$.

**Example 2.2.** The Poincaré homology sphere $-\Sigma$, oriented as the boundary of $E_8$-plumbed 4-manifold, naturally bounds the $E_8$ lattice which is positive definite. It is well known that $-E_8$ lattice cannot be embedded into the standard negative definite lattice: see [LL11, Lemma 3.3] for example. Therefore, $-\Sigma$ cannot bound any negative definite smooth 4-manifold (including a 4-manifold with trivial intersection form), i.e. $\mathcal{I}(-\Sigma) = \emptyset$. 

Correction terms. Let \( Y \) be a rational homology 3-sphere and \( t \) be a spin\(^c\) structure over \( Y \). In [OS03], Ozsváth and Szabó defined a rational valued invariant for \((Y, t)\) called the correction term or \(d\)-invariant, denoted by \( d(Y, t) \). It is an analogous invariant to Frøyshov’s in Seiberg-Witten theory [Frø96]. Among many important properties of the correction term, it gives a constraint on a definite 4-manifold bounded by \( Y \).

**Theorem 2.3** ([OS03, Theorem 9.6]). If \( X \) is a negative definite, smooth 4-manifold bounded by \( Y \), then for each spin\(^c\)-structure \( s \) over \( X \)

\[
c_1(s)^2 + n \leq 4d(Y, s|Y),
\]

where \( c_1(\cdot) \) denotes the first Chern class, \( n \) is the rank of \( H_2(X; \mathbb{Z}) \), and \( s|Y \) is the restriction of \( s \) over \( Y \).

Since any characteristic covector in \( H^2(X; \mathbb{Z})/Tors \) is identified with \( c_1(s) \) for a spin\(^c\)-structure \( s \) on \( X \), we have the following.

**Proposition 2.4.** If a negative definite lattice \( \Lambda \) is smoothly bounded by a rational homology 3-sphere \( Y \), then

\[
\delta(\Lambda) \leq d(Y),
\]

where

\[
d(Y) := \max_{t \in \text{Spin}^c(Y)} d(Y, t).
\]

3. Finiteness of the number of definite lattices bounded by a rational homology 3-sphere

The purpose of this section is to prove Theorem 1.6 and consequently Theorem 1.3.

First, recall the following well-known fact: see [MH73, p. 18] for example.

**Lemma 3.1.** There are finitely many isomorphism classes of definite lattices which have a given rank and determinant.

We first show a special case of Theorem 1.6 in which \( \Gamma_1 \) and \( \Gamma_2 \) are the trivial empty lattices.

**Proposition 3.2.** Let \( C > 0 \) and \( D \in \mathbb{Z} \) be constants. There are finitely many negative definite lattices \( \Lambda \), up to the stable-equivalence, which satisfy the following conditions:

- \( \det \Lambda = D \)
- \( \delta(\Lambda) \leq C \), and
- \( \Lambda \) embeds into \( (-1)^{rk(\Lambda)} \) with prime index.

**Proof.** Let \( \Lambda \) be a negative definite lattice of rank \( n \) that satisfies the conditions above. Without loss of generality, we may assume that there is no vector of square \(-1\) in \( \Lambda \). By Lemma 3.1 the theorem follows if we find an upper bound for the rank of \( \Lambda \), only depending on \( D \) and \( C \). Let \( \{e, e_1, \ldots, e_{n-1}\} \) be the standard basis of the standard negative definite lattice \( (-1)^n \), i.e. \( e^2 = -1 \), \( e \cdot e_i = 0 \), \( e_i \cdot e_j = -\delta_{ij} \) for \( i, j = 1, \ldots, n-1 \).

Let \( p \) be the index of the embedding \( \iota \) of \( \Lambda \) into \( (-1)^n \). If \( p = 1 \), then the embedding is an isomorphism and so \( \Lambda \) should be the empty lattice, up to the stable-equivalence. Now suppose \( p \) is an odd prime. Then the cokernel of \( \iota \) is a cyclic group of order \( p \), and it is generated by \( e + \Lambda \) since \( e \not\in \Lambda \). Observe that \( -e_i \in s_i e + \Lambda \) for some odd integer \( s_i \in [-p + 1, p - 1] \). Consider a set of elements of \( \Lambda \),

\[
\mathcal{B} := \{pe, e_1 + s_1 e, \ldots, e_{n-1} + s_{n-1} e\}.
\]
Since the determinant of the coordinates matrix corresponding to the above set equals to \( p \), it is in fact a basis for \( \Lambda \). The matrix representation of \( \Lambda \) with respect to the basis is given as

\[
Q = -\begin{pmatrix}
p^2 & ps_1 & ps_2 & \ldots & ps_{n-1} 
ps_1 & 1 + s_1^2 & s_1s_2 & \ldots & s_1s_{n-1} 
ps_2 & s_1s_2 & 1 + s_2^2 & \ddots & \vdots 
\vdots & \vdots & \ddots & \ddots & s_{n-2}s_{n-1} 
ps_{n-1} & s_1s_{n-1} & \ldots & s_{n-2}s_{n-1} & 1 + s_{n-1}^2
\end{pmatrix}.
\]

We also compute the inverse of \( Q \) as follows

\[
Q^{-1} = \begin{pmatrix}
\frac{1}{p} & \frac{s_1}{p} & \frac{s_2}{p} & \ldots & \frac{s_{n-1}}{p} 
\frac{s_1}{p^2} & \frac{s_1^2}{p} & \frac{s_1s_2}{p} & \ldots & \frac{s_{n-1}s_1}{p} 
\frac{s_2}{p^2} & \frac{s_1s_2}{p^2} & \frac{s_2^2}{p} & \ddots & \vdots 
\vdots & \vdots & \ddots & \ddots & \frac{s_{n-1}s_2}{p} 
\frac{s_{n-1}}{p^2} & \frac{s_1s_{n-1}}{p^2} & \frac{s_2s_{n-1}}{p^2} & \ldots & -I_{(n-1) \times (n-1)}
\end{pmatrix}.
\]

Note that \( Q^{-1} \) represents the dual lattice of \( \Lambda \) with respect to the dual basis of \( B \). Hence a characteristic covector \( \xi \) of \( \Lambda \) can be written as a vector

\[
\xi = (k, k_1, \ldots, k_{n-1}),
\]

where \( k \) is an odd integer and \( k_i \)'s are even integers, in terms of the dual basis of \( Q \) since \( p^2 \) is odd and \( 1 + s_i^2 \) is even for each \( i \). From the matrix \( Q^{-1} \) we compute

\[
|\xi \cdot \xi| = \frac{1}{p^2}(k^2 + \sum_{i=1}^{n-1}(ks_i - pk_i)^2).
\]

Applying Lemma 3.3 below,

\[
\min\{|\xi \cdot \xi| : \xi \text{ characteristic covector of } \Lambda\} \leq \frac{n + 2}{3}.
\]

Therefore, by Proposition 2.4,

\[
\delta(\Lambda) = \frac{n - \min_{\xi \in \text{Char}(\Lambda)} |\xi \cdot \xi|}{4} \leq C,
\]

and we conclude that

\[
n \leq 6C + 1.
\]

In the case of \( p = 2 \), the lattice \( \Lambda \) admits a basis

\[
\{2e, e_1 + e, \ldots, e_{n-1} + e\},
\]

and hence 0 vector is characteristic. Therefore,

\[
\delta(\Lambda) = \frac{n}{4} \leq C.
\]

By Lemma 3.1, there are only finitely many negative definite lattices satisfying the given conditions.

Now, we prove the following algebraic lemma used in the proof above.
Lemma 3.3. For an odd prime \( p \) and odd integers \( s_1, s_2, \ldots, s_{n-1} \) in \([-p+1, p-1]\), there exist an odd integer \( k \) and even integers \( k_1, k_2, \ldots, k_{n-1} \) such that
\[
k^2 + \sum_{i=1}^{n-1} (ks_i - pk_i)^2 < \frac{n+2}{3}p^2.
\]

Proof. For an odd prime \( p \), consider the set \( K := \{-p+2, -p+4, \ldots, p-2\} \) of odd integers in the interval \([-p+1, p-1]\). Note, for each \( k \) and \( s_i \) in \( K \), there is a unique even integer \( k_i \) such that \( ks_i - pk_i \in K \). Denote this \( k_i \) by \( k_i(k, s_i) \). Since \( ks_i \equiv k's_i \pmod{2p} \) implies \( k \equiv k' \pmod{2p} \) for \( k \) and \( k' \) in \( K \), we obtain \( \{ks_i - p \cdot k_i(k, s_i)\mid k \in K\} = K \) for each \( s_i \in K \). Therefore,
\[
\sum_{k \in K} \sum_{i=1}^{n-1} (ks_i - p \cdot k_i(k, s_i))^2 = (n-1) \cdot 2(1^2 + 3^2 + \cdots + (p-2)^2)
\]
\[
= \frac{n-1}{3}p(p-1)(p-2).
\]
Since \( |K| = p-1 \), there exists \( k \in K \) such that
\[
\sum_{i=1}^{n-1} (ks_i - p \cdot k_i(k, s_i))^2 \leq \frac{n-1}{3}p(p-2).
\]
Since \( |k| < p \), we obtain the desired inequality. \( \square \)

Now, Theorem 1.6 is proved by applying a similar argument of Proposition 3.2.

First, observe the following.

Lemma 3.4. Let \( \Gamma_1 \) and \( \Gamma_2 \) be negative definite lattices. Then the set of stable classes of lattices,
\[
\mathcal{C}(\Gamma_1, \Gamma_2) := \{(\text{Im}u)^\perp \mid \iota: \Gamma_1 \hookrightarrow \Gamma_2 \oplus \langle -1 \rangle^N, \text{an embedding for some } N \in \mathbb{N}\}/\sim,
\]
is finite.

Proof. First, we claim that the set
\[
\{\text{Im}u\}^\perp \mid \iota: \Gamma_1 \hookrightarrow \Gamma_2 \oplus \langle -1 \rangle^N, \text{an embedding}\}/\sim
\]
is stabilized for some large enough \( N \). Let \( \{v_1, \ldots, v_{\text{rk}(\Gamma_1)}\} \) be a basis for \( \Gamma_1 \). By considering the representations of \( v_i \) in terms of a basis for \( \Gamma_2 \oplus \langle -1 \rangle^N \), if
\[
N > \sum_{i=1}^{\text{rk}(\Gamma_1)} |v_i \cdot v_i|,
\]
then there is a vector \( e \in \langle -1 \rangle^N \) such that \( e \cdot e = -1 \) and \( e \cdot v_i = 0 \) for all \( i = 1, \ldots, \text{rk}(\Gamma_1) \). Hence in order to prove the finiteness of \( \mathcal{C}(\Gamma_1, \Gamma_2) \), it is enough to consider some fixed large \( N \).

Let \( \{w_j\}_{j=1}^m \) be a basis for \( \Gamma_2 \oplus \langle -1 \rangle^N \). Note that an embedding of \( \Gamma_1 \) into \( \Gamma_2 \oplus \langle -1 \rangle^N \) can be presented by the system of equations
\[
v_i = \sum_{j=1}^m a_{i,j}w_j,
\]
where \( a_{i,j} \) are integers. Since \( \Gamma_1 \) and \( \Gamma_2 \oplus \langle -1 \rangle^N \) are definite, the possible choices of \( a_{i,j} \) are finite for each \( i, j \), and hence the number of possible embedding maps is also finite. \( \square \)
Proof of Theorem 1.6. Fix negative definite lattices $\Gamma_1$ and $\Gamma_2$, and constants $C > 0$ and $D \in \mathbb{Z}$. Let $\Lambda$ be a negative definite lattice which satisfies the conditions in the theorem. Without loss of generality, we may assume that there is no square $-1$ vector in $\Lambda$. By Lemma 3.1, the theorem follows if we show that the rank of $\Lambda$ is bounded by some constant only depending on $\Gamma_1$, $\Gamma_2$, $C$ and $D$.

From the third condition of $\Lambda$, there is an embedding 

$$i|_{\Gamma_1} : \Gamma_1 \hookrightarrow \Gamma_2 \oplus \langle -1 \rangle^N,$$

where $N = \text{rk}(\Lambda) + \text{rk}(\Gamma_1) - \text{rk}(\Gamma_2)$. Let $(\text{Im}(i|_{\Gamma_1}))^\perp \cong \langle -1 \rangle^n \oplus E$, where $E$ is a lattice without square $-1$ vectors and $n = \text{rk}(\Lambda) - \text{rk}(E)$. Note that $[E]$ is one of the elements in the finite set $C(\Gamma_1, \Gamma_2)$ in Lemma 3.4.

Now we need to find a bound of the rank of $\Lambda$ embedded in $\langle -1 \rangle^n \oplus E$. This will be obtained by the similar argument in the proof of Theorem 3.2. The main difference is that we have an extra summand $E$.

Let $i'$ be an embedding $\Lambda$ into $\langle -1 \rangle^n \oplus E$, and $p$ be the index of $i'$. If $n = 0$, i.e. the rank of $\Lambda$ is same as the rank of $E$, then we have a bound of the rank of $\Lambda$ by Lemma 3.4. Similarly, if $p = 1$, then $\Lambda \cong \langle -1 \rangle^n \oplus E$ and we have the same rank bound of $\Lambda$.

Now suppose $n \neq 0$ and that $p$ is an odd prime. Let

$$\{e, e_1, \ldots, e_{n-1}, f_1, \ldots, f_r\}$$

be a basis for $\langle -1 \rangle^n \oplus E$ so that $e^2 = -1$, $e \cdot e_i = 0$, $e_i \cdot e_j = -\delta_{ij}$ and $e \cdot f_j = e_i \cdot f_j = 0$ for any $i, j$ and $E$ is generated by $\{f_1, \ldots, f_r\}$. By the same argument in Proposition 3.2 we can choose a basis for $\Lambda$,

$$\{pe, e_1 + s_1e, \ldots, e_{n-1} + s_{n-1}e, f_1 + t_1e, \ldots, f_r + t_re\}$$

where $s_i$’s are odd integers in $[-p+1, p-1]$ and $t_j$’s are integers in $[-p+1, p-1]$. Now with respect to the dual coordinates for this basis, write a characteristic covector $\xi$ as

$$\xi = (k, k_1, \ldots, k_{n-1}, l_1, \ldots, l_r)$$

where $k$ is odd, $k_i$’s are even and $l_j \equiv (f_j + t_je) \cdot (f_j + t_je) \pmod{2}$ for each $i, j$. To find the matrices of $\Lambda$ and $\Lambda^{-1}$, introduce an $(n + r) \times (n + r)$ matrix $M$ and a $r \times r$ matrix $A$ as

$$M_{ij} := \begin{cases} p & \text{if } i = 1, j = 1 \\ 1 & \text{if } i = j, 2 \leq j \leq n + r \\ s_{j-1} & \text{if } i = 1, 2 \leq j \leq n \\ t_{j-n} & \text{if } i = 1, n + 1 \leq j \leq n + r \\ 0 & \text{otherwise,} \end{cases}$$

and

$$A_{ij} := f_i \cdot f_j.$$ 

Note that $M$ represents the embedding of $\Lambda$ into $\langle -1 \rangle^n \oplus E$ and $A$ represents $E$. By the basis in (1), $\Lambda$ and the dual of $\Lambda$ are represented as follows:

$$Q_\Lambda = M^t \left( \begin{array}{cc} -I_{n \times n} & 0 \\ 0 & A \end{array} \right) M$$

and

$$Q_\Lambda^{-1} = M^{-1} \left( \begin{array}{cc} -I_{n \times n} & 0 \\ 0 & A^{-1} \end{array} \right) (M^t)^{-1} = -M^{-1}(M^t)^{-1} + M^{-1} \left( \begin{array}{cc} 0 & 0 \\ 0 & A^{-1} + I_{r \times r} \end{array} \right) (M^t)^{-1}.$$
We obtain that
\[ |\xi \cdot \xi| \leq \frac{1}{p^2} (k^2 + \sum_{i=1}^{n-1} (ks_i - pk_i)^2) + |S(k, l_1, \ldots, l_r, t_1, \ldots, t_r)|, \]
for some function $S$. We emphasize that the function $S$ is independent to $n$, since it is obtained from the last term of Equation (2). Then by Lemma 3.3,
\[ \min \{|\xi \cdot \xi| : \xi \text{ characteristic covector of } L\} \leq \frac{n}{3} + |S| \]
Therefore, we obtain
\[ n \leq \frac{3}{2} (4C + |S|) \]
from $\delta(\Lambda) \leq C$. The case that $p = 2$ is easier to find a similar bound for $n$ by applying the same argument.

For an arbitrary index $p$, we use an idea in [OS12a] to have a sequence of embeddings
\[ \Lambda = E_0 \hookrightarrow E_1 \hookrightarrow E_2 \hookrightarrow \cdots \hookrightarrow E_s = (-1)^n \oplus E \]
such that each embedding $E_i \hookrightarrow E_{i+1}$ has a prime index. The length of this steps is also bounded by some constant related to $D$. Moreover, $\delta(E_i) \leq \delta(\Lambda) \leq C$ for any $i$ since $E_i^* \hookrightarrow \Lambda^*$ and so $\text{Char}(E_i) \subset \text{Char}(\Lambda)$. Thus we complete the proof of theorem by an induction along each prime index embedding. □

**Proof of Theorem 1.3** Let $W$ be a negative definite, smooth cobordism from $Y_1$ and $Y_2$. If $X$ is a negative definite 4-manifold with the boundary $Y_1$, then $X \cup Y_1 W$ is a negative definite 4-manifold bounded by $Y_2$. Moreover, the intersection form of $X$ embeds into the intersection form of $X \cup Y_1 W$. By the necessary conditions for a negative definite lattice to be bounded by a rational homology sphere discussed in Section 2.2, $\mathcal{I}(Y_1)$ is a subset of the union of
\[ \mathcal{L}(Q_W, \Gamma; \max_{t \in \text{Spin}^c(Y_1)} d(Y_1, t), D) \]
over all integers $D$ dividing $|H_1(Y_1; \mathbb{Z})|$ and $[\Gamma] \in \mathcal{I}(Y_2)$. Then Theorem 1.3 follows from Theorem 1.6. □

4. **Definite lattices bounded by Seifert fibered rational homology 3-spheres**

In this section, we discuss which Seifert fibered 3-manifolds satisfy the condition in Corollary 1.3. In particular, we completely classify spherical 3-manifolds $Y$ such that both $\mathcal{I}(Y)$ and $\mathcal{I}(-Y)$ are nonempty, and show that $|\mathcal{I}(Y)| < \infty$ for any spherical 3-manifold $Y$.

4.1. **Seifert fibered spaces.** Seifert fibered 3-manifolds are a large class of 3-manifolds that contains 6 geometries among Thurston’s 8 geometries of 3-manifolds. A Seifert fibered rational homology 3-sphere can be represented by a Seifert form
\[ (e_0; (a_1, b_1), \ldots, (a_k, b_k)), \]
where $e_0$, $a_i$’s are integers, $b_i$’s are positive integers and $\gcd(a_i, b_i) = 1$, and Dehn-surgery diagram of the corresponding 3-manifold is depicted in Figure 1 for the case that $k = 3$. Let $M(e_0; (a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k))$ denote the corresponding 3-manifold. A Seifert fibered 3-manifold $M(e_0; (a_1, b_1), \ldots, (a_k, b_k))$ naturally bounds a 4-manifold
constructed by the plumbing diagram in Figure 1 in which $\alpha_{ij} \in \mathbb{Z}$ are determined by the following Hirzebruch-Jung continued fraction:

$$\frac{a_i}{b_i} = [\alpha_1^i, \alpha_2^i, \ldots, \alpha_l_i^i] = \alpha_1^i - \frac{1}{\alpha_2^i - \frac{1}{\ldots - \frac{1}{\alpha_l_i^i}}}$$

where $\alpha_j^i \geq 2$ for $1 \leq i \leq k$ and $2 \leq j \leq l_i$.

It follows easily from the blow-up procedure and Rolfsen’s twist that $M(e_0; (a_1, b_1), \ldots, (a_k, b_k))$ and $M(e_0 \pm 1; (a_1, b_1), \ldots, (\pm 1, 1), \ldots, (a_k, b_k))$ and $M(e_0; (a_1, b_1), \ldots, (a_j, b_j), \ldots, (a_k, b_k))$ and $M(e_0 + n; (a_1, b_1), \ldots, (a_j, b_j - na_j), \ldots, (a_k, b_k))$ represent the same homeomorphic 3-manifold respectively: see [GS99, Chapter 5.3]. Hence any Seifert fibered rational homology 3-sphere admits a canonical Seifert form, $(e_0; (a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k))$ such that $a_i > b_i > 0$ for all $1 \leq i \leq k$. We refer the form by the normal form of a Seifert fibered rational homology 3-sphere. We call the 4-manifold, obtained from the normal form of a Seifert fibered 3-manifold, the canonical plumbed 4-manifold of the Seifert fibered 3-manifold.

**Lemma 4.1.** The intersection form of the corresponding plumbed 4-manifold of a normal form $M(e_0; (a_1, b_1), \ldots, (a_k, b_k))$ is negative definite if and only if $e(M) := e_0 + \frac{b_1}{a_1} + \cdots + \frac{b_k}{a_k} < 0$.

**Proof.** This is directly obtained by diagonalizing the corresponding intersection form of the plumbed 4-manifold after extending over $\mathbb{Q}$. □

We refer $e(M)$ by the Euler number of a Seifert form $M$. Notice that the Euler number is in fact an invariant for the Seifert fibered 3-manifold $Y$, and $e(-Y) = -e(Y)$.

**Proposition 4.2.** Any Seifert fibered rational homology 3-sphere can bound positive or negative definite smooth 4-manifolds.

**Proof.** Let $Y$ be a Seifert fibered rational homology 3-sphere. By the linking pairing of the surgery diagram in Figure 1 a rational homology 3-sphere $Y$ has a nonzero Euler number. If $e(Y) < 0$, then $Y$ bounds a negative definite 4-manifold by the above lemma. If $e(Y) > 0$, then $-Y$ bounds a negative definite 4-manifold since $e(-Y) = -e(Y) < 0$. Thus $Y$ bounds a positive definite one. □

Now we introduce a condition for a Seifert fibered rational homology 3-sphere to bound definite 4-manifolds of both signs.
$e_0 - a_1 / b_1 - a_2 / b_2 - a_3 / b_3$

**Figure 1.** The surgery diagram and plumbing diagram of Seifert manifold $M(e_0; (a_1, b_1), (a_2, b_2), (a_3, b_3))$.

**Proposition 4.3.** Let $Y$ be a Seifert fibered rational homology 3-sphere of the normal form

$$(e_0; (a_1, b_2), \ldots, (a_k, b_k)).$$

If $e_0 + k \leq 0$, then $Y$ bounds both positive and negative definite smooth 4-manifolds, i.e. both $I(Y)$ and $I(-Y)$ are not empty.

**Proof.** Let $Y$ be a Seifert 3-manifold with the normal form $(e_0; (a_1, b_2), \ldots, (a_k, b_k))$ such that $e_0 + k \leq 0$. By the previous proposition, $Y$ bounds a negative definite 4-manifold. To find a positive definite bounding of $Y$, consider the plumbed 4-manifold $X$ corresponding to $-Y \approx M(-e_0 - k; (a_1, a_1 - b_1), \ldots, (a_k, a_k - b_k))$. Note that $b_2^+(X) = 1$. By blowing up $(e_0 + k)$ points on the sphere in $X$ corresponding to the central vertex, we get a sphere with self intersection 0 in $X \# (e_0 + k)\mathbb{CP}^2$. By doing a surgery on this sphere, we obtain a desired negative definite 4-manifold. More precisely, we remove the interior of the tubular neighborhood of the sphere, $S^2 \times D^2 \subset X \# (e_0 + k)\mathbb{CP}^2$, and glue $D^3 \times S^1$ along the boundary, and it reduces $b_2^+(X \# (e_0 + k)\mathbb{CP}^2)$ by 1. □

**Remark.** This proposition can be alternatively proved by the fact that these Seifert fibered spaces can be obtained by the branched double covers of $S^3$ along alternating Montesinos links. See [MO07, Section 4].

Note that the condition in Proposition 4.3 is not a necessary condition. For example, the Brieskorn manifold,

$$\Sigma(2, 3, 6n + 1) \cong M(-1, (2, 1), (3, 1), (6n + 1, 1))$$

bounds both negative and positive definite 4-manifolds since $e(M) < 0$ and it can be obtained by (+1)-surgery of $S^3$ on the $n$-twist knot.

On the other hand, the inequality $e_0 + k \leq 0$ is sharp since the Brieskorn manifold $\Sigma(2, 3, 5) \cong M(-2, (2, 1), (3, 2), (5, 4))$, of which $e_0 + k = 1$, cannot bound any positive definite 4-manifold by the constraint from Donaldson’s diagonalization theorem. Note that $\Sigma(2, 3, 5)$ is the Poincaré homology sphere $\Sigma$ in Example 2.2.

### 4.2. Spherical 3-manifolds

A 3-manifold is spherical if it admits a metric of constant curvature +1. It is well known that a spherical 3-manifold has a finite fundamental group, and conversely a closed 3-manifold with a finite fundamental group is spherical by the elliptization theorem due to Perelman.
Spherical 3-manifolds are divided into 5-types: C (cyclic), D (dihedral), O (octahedral) and I (icosahedral) type, in terms of their fundamental groups. Note that spherical 3-manifolds are Seifert fibered, and their normalized Seifert forms are given as follows, up to the orientation of the manifolds [Sei33]:

- Type C: $M(e_0; (a_1, b_1))$
- Type D: $M(e_0; (2, 1), (2, 1), (a_3, b_3))$
- Type T: $M(e_0; (2, 1), (3, b_2), (3, b_3))$
- Type O: $M(e_0; (2, 1), (3, b_2), (4, b_3))$
- Type I: $M(e_0; (2, 1), (3, b_2), (5, b_3))$

where $e_0 \leq -2$, $a_i > b_i > 0$ and $gcd(a_i, b_i) = 1$. Notice that the manifolds are oriented so that their canonical plumbed 4-manifolds are negative definite.

We claim that most of the spherical 3-manifolds can bound smooth definite 4-manifolds of both signs, except the following cases:

\[
\begin{align*}
T_1 &= M(-2; (2, 1), (3, 2), (3, 2)), \\
O_1 &= M(-2; (2, 1), (3, 2), (4, 3)), \\
I_1 &= M(-2; (2, 1), (3, 2), (5, 4)), \\
I_7 &= M(-2; (2, 1), (3, 2), (5, 3)).
\end{align*}
\]

Remark that we follow the notations of Bhupal and Ono in [BOT12] for this class of 3-manifolds.

**Proposition 4.4.** The manifolds $T_1$, $O_1$, $I_1$ and $I_7$ cannot bound a positive definite smooth 4-manifolds.

**Proof.** In [LL11] Lemma 3.3, Lecuona and Lisca showed that if $1 < \frac{a_1}{b_1} \leq \frac{a_2}{b_2} \leq \frac{a_3}{b_3}$ and $1 < \frac{b_2}{a_2} + \frac{b_3}{a_3}$, then the intersection lattice of the plumbing associated to $M(-2; (a_1, b_1), (a_2, b_2), (a_3, b_3))$ cannot be embedded into a negative definite standard lattice.

Observe that the manifolds, $T_1$, $O_1$, $I_1$ and $I_7$ satisfy the conditions of the lemma. Hence these manifolds cannot bound any positive definite 4-manifolds by the standard argument using Donaldson’s theorem. \(\square\)

**Proposition 4.5.** Any spherical 3-manifolds except $T_1$, $O_1$, $I_1$ and $I_7$ can bound both positive and negative definite smooth 4-manifolds.

**Proof.** Since we orient a spherical 3-manifold to bound a natural negative definite 4-manifold, it is enough to construct a positive definite one with the given boundary $Y$. Type-C manifolds are lens spaces, and it is well known that they bound positive definite 4-manifolds either.

Let $(n, q)$ be a pair of integers such that $1 < q < n$ and $gcd(n, q) = 1$, and $\frac{n}{q} = [b, b_1, \ldots, b_{r-1}, b_r]$. We denote by $D_{n,q}$ the dihedral manifold $M(-b; (2, 1), (2, 1), (q, bq - n))$.

The canonical plumbed 4-manifold of $D_{n,q}$ is given in Figure 2. If $b > 2$, there is a positive definite bounding by Proposition 4.3. In the case $b = 2$, we can check that $-D_{n,q} \cong M(-1; (2, 1), (2, 1), (q, n - q))$, and the corresponding plumbed 4-manifold $X$ satisfies $b_X^+(X) = 1$. As seen in Figure 4, we get a 2-sphere with self-intersection 0 after blowing down twice from $X$, and the sphere intersects algebraically twice with the sphere of self-intersection $-c + 3$. Hence the sphere with self-intersection 0 is
homologically essential, and we obtain a desired negative definite 4-manifold by a surgery along the sphere.

In the other cases (tetrahedral, octahedral and icosahedral cases), we can apply similar argument except the 4-cases. □

By Proposition 4.5 and Theorem 1.3, we know that the most of the spherical manifolds have finitely many stable classes of definite lattices to bound them. Finally, we show that the exceptional cases of spherical 3-manifolds also satisfy such finiteness property.

Proof of Theorem 1.7. Note that $I_1 \cong \Sigma$, and in this case we have $|I(\Sigma)| \leq 15$ from the lattice theoretic result in [Elk95b] as mentioned in the introduction. We utilize this for the other cases.

Observe that the canonical plumbed 4-manifold $X_{T_1}$ of $T_1$ can be embedded in the canonical plumbed 4-manifold $X_{\Sigma}$ of $\Sigma$. See Figure 4. Let $W$ be a 4-manifold

Figure 2. The plumbing graph of the canonical plumbed 4-manifold of $D_{n,q}$, where $\frac{n}{q} = [b, b_1, \ldots, b_{r-1}, b_r]$.

Figure 3. The canonical plumbed 4-manifold of $-D_{n,q}$ and the configuration after blow-down twice, where $\frac{n}{n-q} = [c, c_1, \ldots, c_k]$. 
constructed by removing $X_{T_1}$ from $X_{\Sigma}$. Then $W$ is a negative definite cobordism from $T_1$ to $\Sigma$. The finiteness of $I(T_1)$ follows from Theorem 1.3.

Since the canonical plumbed 4-manifold $O_1$ is also embedded in $X_{\Sigma}$, we have $|I(O_1)| < \infty$. For the manifold $I_7$, we blow up on the sphere in $X_{\Sigma}$ to contain the canonical plumbed 4-manifold of $I_7$. Then the same argument works to show that $|I(I_7)| < \infty$. □

As we mentioned in the introduction, it is known that the 3-manifolds obtained by the double branched cover of quasi-alternating links in $S^3$ bound both positive and negative definite 4-manifolds with trivial $H_1$. Indeed, there are some family of Seifert fibered 3-manifolds that are not obtained by double branched cover on a quasi-alternating link but can be shown to bound definite 4-manifolds of both signs by our result. For example, the dihedral manifolds $D_{n,n-1}$ are such 3-manifolds.

**Proposition 4.6.** The dihedral manifold $D_{n,n-1}$ cannot bound a positive definite smooth 4-manifold with trivial $H_1$, and consequently cannot be obtained by the double branched cover of $S^3$ along a quasi-alternating link in $S^3$.

**Proof.** Let $Y$ be $D_{n,n-1}$ manifold, and $X$ be the canonical plumbed 4-manifold of $Y$. If $W$ is a positive definite 4-manifold bounded by $Y$, then, as usual, $Q_X \oplus -Q_W$ embeds into $(-1)^{\text{rk}(Q_X)} + \text{rk}(Q_W)$. First observe that an embedding $\iota$ of $Q_X$ to a standard definite lattice is unique, up to the automorphism of the standard definite lattice, as depicted in Figure 5 in terms of the standard basis $\{e_1, e_2, \ldots, e_{n+1}\}$. For this unique embedding, we have $(\text{Im} \iota)^\perp \cong (-1)^{\text{rk}(Q_W)}$.

Since $H_1(X; \mathbb{Z})$ is trivial, $-Q_W$ is isomorphic to $(\text{Im} \iota)^\perp \cong (-1)^{\text{rk}(Q_W)}$. Suppose $H_1(W)$ is trivial. Then $H^2(W, \partial W)$ and $H^2(W)$ are torsion free, and $H^2(Y)$ have to be trivial from the following long exact sequence:

$$
\cdots \to H^2(W, \partial W) \xrightarrow{\cup} H^2(W) \to H^2(Y) \to H^3(W, \partial W) = 0
$$
However, we know that $H^2(Y)$ is non-trivial.

Remark. Recently, all quasi-alternating Montesinos links are completely classified due to Issa in \cite{Iss17}. Since any Seifert fibered rational homology 3-sphere is the double branched cover of $S^3$ along a Montesinos link, the above proposition might be followed from his result.

Acknowledgments

The authors would like to thank Jongil Park and Ki-Heon Yun for interests on this project and helpful discussions, and Matt Hedden and Brendan Owens for valuable comments on an earlier version of this paper.

REFERENCES

[BG18] S. Behrens and M. Golla. Heegaard Floer correction terms, with a twist. *Quantum Topol.*, 9(1):1–37, 2018.

[BO12] M. Bhupal and K. Ono. Symplectic fillings of links of quotient surface singularities. *Nagoya Math. J.*, 207:1–45, 2012.

[Boy86] S. Boyer. Simply-connected 4-manifolds with a given boundary. *Trans. Amer. Math. Soc.*, 298(1):331–357, 1986.

[Don83] S. K. Donaldson. An application of gauge theory to four-dimensional topology. *J. Differential Geom.*, 18(2):279–315, 1983.

[Don87] S. K. Donaldson. The orientation of Yang-Mills moduli spaces and 4-manifold topology. *J. Differential Geom.*, 26(3):397–428, 1987.

[Edm05] A. L. Edmonds. Homology lens spaces in topological 4-manifolds. *Illinois J. Math.*, 49(3):827–837 (electronic), 2005.

[Elk95a] N. D. Elkies. A characterization of the $\mathbb{Z}^n$ lattice. *Math. Res. Lett.*, 2(3):321–326, 1995.

[Elk95b] N. D. Elkies. Lattices and codes with long shadows. *Math. Res. Lett.*, 2(5):643–651, 1995.

[Fre82] M. H. Freedman. The topology of four-dimensional manifolds. *J. Differential Geom.*, 17(3):357–453, 1982.

[Fra96] K. A. Frøyshov. The Seiberg-Witten equations and four-manifolds with boundary. *Math. Res. Lett.*, 3(3):373–390, 1996.

[Gau07] M. Gaulter. Characteristic vectors of unimodular lattices which represent two. *J. Théor. Nombres Bordeaux*, 19(2):405–414, 2007.

[GS99] R. E. Gompf and A. I. Stipsicz. *4-manifolds and Kirby calculus*, volume 20 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 1999.

[Iss17] A. Issa. The classification of quasi-alternating montesinos links, 2017. to appear in Proc. Amer. Math. Soc., available at arXiv:1701.08425.

[LL11] A. G. Lecuona and P. Lisca. Stein fillable Seifert fibered 3-manifolds. *Algebr. Geom. Topol.*, 11(2):625–642, 2011.

[LR14] A. S. Levine and D. Ruberman. Generalized Heegaard Floer correction terms. In *Proceedings of the Gökova Geometry-Topology Conference 2013*, pages 76–96. Gökova Geometry/Topology Conference (GGT), Gökova, 2014.

[MH73] J. Milnor and D. Husemoller. *Symmetric bilinear forms*. Springer-Verlag, New York-Heidelberg, 1973. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 73.
[MO07] C. Manolescu and B. Owens. A concordance invariant from the Floer homology of double branched covers. *Int. Math. Res. Not. IMRN*, (20):Art. ID rnm077, 21, 2007.

[NV03] G. Nebe and B. Venkov. Unimodular lattices with long shadow. *J. Number Theory*, 99(2):307–317, 2003.

[OS03] P. Ozsváth and Z. Szabó. Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary. *Adv. Math.*, 173(2):179–261, 2003.

[OS05a] B. Owens and S. Strle. Definite manifolds bounded by rational homology three spheres. In *Geometry and topology of manifolds*, volume 47 of *Fields Inst. Commun.*, pages 243–252. Amer. Math. Soc., Providence, RI, 2005.

[OS05b] P. Ozsváth and Z. Szabó. On the Heegaard Floer homology of branched double-covers. *Adv. Math.*, 194(1):1–33, 2005.

[OS06] B. Owens and S. Strle. Rational homology spheres and the four-ball genus of knots. *Adv. Math.*, 200(1):196–216, 2006.

[OS12a] B. Owens and S. Strle. A characterization of the $\mathbb{Z}^n \oplus \mathbb{Z}^d$ lattice and definite nonunimodular intersection forms. *Amer. J. Math.*, 134(4):891–913, 2012.

[OS12b] B. Owens and S. Strle. Dehn surgeries and negative-definite four-manifolds. *Selecta Math.* (N.S.), 18(4):839–854, 2012.

[Sei33] H. Seifert. Topologie Dreidimensionaler Gefaserter Räume. *Acta Math.*, 60(1):147–238, 1933.

DEPARTMENT OF MATHEMATICAL SCIENCE, SEOUL NATIONAL UNIVERSITY, SEOUL 08826, REPUBLIC OF KOREA

E-mail address: honey8276@snu.ac.kr

SCHOOL OF MATHEMATICS, KOREA INSTITUTE FOR ADVANCED STUDY, SEOUL 02455, REPUBLIC OF KOREA

E-mail address: kbpark@kias.re.kr

URL: newton.kias.re.kr/~kbpark