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Combinatorial expressions of Hopf polynomial invariants

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Abstract

In 2017 Aguiar and Ardila provided a generic way to construct polynomial invariants of combinatorial objects using the notions of Hopf monoids and characters of Hopf monoids. The polynomials constructed this way are often subject to reciprocity theorems depending on the antipode of the associated Hopf monoid, i.e. while they are defined over positive integers, it is possible to find them a combinatorial interpretation over negative integers. In the same article Aguiar and Ardila then give a cancellation-free grouping-free formula for the antipode on generalized permutahedra and apply their constructions over some examples. In this work, we give a combinatorial interpretation of these polynomials over both positive integers and negative integers for the Hopf monoids of generalized permutahedra and hypergraphs and for every character on these two Hopf monoids. In the case of hypergraphs, we present two different proofs for the interpretation on negative integers, one using Aguiar and Ardila’s antipode formula and one similar to the way Aval et al. defined a chromatic polynomial for hypergraphs in arXiv:1806.08546. We then deduce similar results on other combinatorial objects including graphs, simplicial complexes and building sets.

Keywords: Hopf monoids, polynomial invariants, Generalized permutahedra, Hypergraphs, Colorings.
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1 Introduction

The notion of Hopf monoid first appeared, albeit not under this name, in André Joyal’s introductory work to the theory of species [14], and was built in the continuity of works from Joni and Rota [18] [13]. A full theory of Hopf monoids, as well as the denomination of Hopf monoid, was later developed by Aguiar and Mahajan in [2], [3]. As with Hopf algebras, Hopf monoids give an algebraic framework to deal with operations of merging (product) and splitting (co-product) combinatorial objects. Using the theory of Hopf monoids, Aguiar and Ardila showed in [1] how to construct polynomial invariants, recovering the chromatic polynomial of graphs, the Billera-Jia-Reiner polynomial of matroids and the strict order polynomial of posets. Their construction of such a polynomial invariant requires a character, that is a multiplicative linear map, so that on a fixed Hopf monoid, we can construct a polynomial invariant $\chi^\zeta$ for every character $\zeta$ on this Hopf monoid. Furthermore, Aguiar and Ardila give a way to compute these polynomial invariants on negative integers thereby recovering reciprocity theorems associated to these combinatorial objects. This computation is based on the notion of antipode of a Hopf monoid, which plays the role of inversion in Hopf theory. They complete this result by providing a cancellation-free and grouping-free formula for the antipode of all the Hopf monoids they define: generalized permutahedra, matroids, posets, hypergraphs, simple hypergraphs, simplicial complexes, building sets, graphs (with two different Hopf structure), partitions and set of paths. Their results were used in [4] to define a chromatic polynomial over hypergraphs subject to a reciprocity theorem and in the author’s PhD thesis [15], from which this work is in great part an extract.

In this work we apply Aguiar and Ardila’s construction of polynomial invariants to the Hopf monoids of generalized permutahedra and all its sub-monoids defined in [1]: hypergraphs, simple hypergraphs, simplicial complexes, building sets, graphs, partitions and set of paths. Our results are a direct extension to Aval et al. results [4] in the sense that we give a combinatorial description of $\chi^\zeta$, for all these Hopf monoids and all characters over them while they only do so for one character per Hopf monoid and do not consider generalized permutahedra. The descriptions we provide come in the form of counting combinatorial objects in link with objects of the considered Hopf monoid. For example, as we will show in Theorem 2.5 and Theorem 3.12 for $P$ a generalized permutahedron and $h$ a hypergraph we have the following general formulas for $\chi^\zeta$:

$$\chi^\zeta(P)(n) = \sum_{Q \leq P} \zeta(Q)|N^P_Q(n)| \quad \text{and} \quad \chi^\zeta(h)(n) = \sum_{f \in A_h} \zeta(f(h))|C_{h,f,n}|,$$

where $N^P_Q$ is the closed normal fan of $P$, $A_h$ is the set of acyclic orientations of $h$ and $C_{h,f,n}$ is the set of colorings of $h$ with $n$ colors strictly compatible with the orientation $f$. To obtain these kind of formulas for all the Hopf monoids we consider, we follow a similar process to the one Aguiar and Ardila used to obtain their antipode formulas: we begin by the case of the Hopf monoid of generalized permutahedra and deduce the results for other Hopf monoid from their injection in this first Hopf monoid. Our results generalize many existing results in the literature e.g. [4], [6] and [11].

This work is organized as follows. In Section 1 we give general context on Hopf monoids and some definitions on decompositions which are standard objects when working with Hopf structures. In Section 2 we present the Hopf monoid structure of generalized permutahedra defined in [1] and we provide a combinatorial interpretation to the polynomial invariants of the Hopf monoid of generalized permutahedra. In Section 3 we do the same with the Hopf monoid of hypergraphs but providing two proofs for the combinatorial interpretation of the polynomial invariants over negative integers. Finally in Section 4 we obtain similar results for the Hopf monoids defined in [1] over simple hypergraphs, simplicial complexes, building sets, graphs, partitions and set of paths.

Context

Definitions and background on Hopf monoids

In all this paper, otherwise stated, $V$ will always denote a finite set and $V_1$ and $V_2$ two disjoint sets such that $V = V_1 \sqcup V_2$. The letter $n$ always denotes a non negative integer and we denote
by \([n]\) the set \(\{1, \ldots, n\}\). All vector spaces appearing in this paper are defined over a field of characteristic 0 denoted by \(K\).

We recall here basic definitions on Hopf monoids. We refer the reader to [7] for a more information on the theory of species and to [3] for more information on Hopf monoids.

**Definition 1.1.** A **linear species** \(S\) consists of the following data:

- For each finite set \(V\), a vector space \(S[V]\) of finite dimension,
- For each bijection of finite sets \(\sigma : V \to V'\), a linear map \(S[\sigma] : S[V] \to S[V']\). These maps should be such that \(S[\sigma_1 \circ \sigma_2] = S[\sigma_1] \circ S[\sigma_2]\) and \(S[Id] = Id\).

Let \(R\) and \(S\) be two linear species. The linear species \(R\) is a **linear sub-species** of \(S\) if \(R[V]\) is a sub-space of \(S[V]\) for every finite set \(V\) and \(R[\sigma] = S[\sigma]\) for every bijection of finite sets \(\sigma\). A **morphism of linear species** from \(R\) to \(S\) is a collection of linear maps \(f_V : R[V] \to S[V]\) such that for each bijection \(\sigma : V \to V'\), we have \(f_{V'} \circ R[\sigma] = S[\sigma] \circ f_V\). For easier reading, we will often forget the index \(V\).

We will use the term **species** to refer to linear species.

**Definition 1.2.** A **connected Hopf monoid** in linear species is a linear species \(M\) where \(M[\emptyset] = K\) and which is equipped with a **product** and a **co-product** linear maps:

\[
\mu_{V_1, V_2} : M[V_1] \otimes M[V_2] \to M[V_1 \sqcup V_2], \quad \Delta_{V_1, V_2} : M[V_1 \sqcup V_2] \to M[V_1] \otimes M[V_2],
\]

which are subject to the following axioms, where we use the classical infix notation \(-\cdot-\) for the product.

- **Naturality.** The maps \(\mu : M \cdot M \to M\) and \(\Delta : M \to M \cdot M\) are morphisms of linear species.
- **Unitality.** For \(x \in M[V]\), \(x \cdot 1_K = x = 1_K \cdot x\), i.e. merging with the unit does not change our objects.
- **Co-unity.** \(\Delta_{V, \emptyset} = Id \otimes 1_K\) and \(\Delta_{\emptyset, V} = 1_K \otimes Id\), i.e. splitting on the empty set does not change our objects.
- **Associativity.** For \(x \in M[V_1]\), \(y \in M[V_2]\) and \(z \in M[V_3]\), \((x \cdot y) \cdot z = (x \cdot y) \cdot z\), i.e. the results of merging three objects does not depend on the order in which we merge them.
- **Co-associativity.** \(\Delta_{V_1, V_2} \otimes Id \circ \Delta_{V_1 \sqcup V_2, V_3} = Id \otimes \Delta_{V_2, V_3} \circ \Delta_{V_1, V_2 \sqcup V_3}\), i.e. the result of splitting an object in three does not depend on the order in which we split it.
- **Compatibility.** Let \(V = V_1 \sqcup V_2 = V_3 \sqcup V_4\) and for \(i \in \{1, 2\}\) and \(j \in \{3, 4\}\) denote by \(V_{ij}\) the set \(V_i \cap V_j\). Then we have the following commutative diagram:

\[
\begin{array}{ccc}
M[V_1] \otimes M[V_2] & \xrightarrow{\mu_{V_1, V_2}} & M[V] \\
\Delta_{V_1, V_4} \otimes \Delta_{V_2, V_4} & \downarrow & \Delta_{V_3, V_4} \\
M[V_1] \otimes M[V_3] \otimes M[V_2] \otimes M[V_4] & \xrightarrow{\mu_{V_1, V_2} \otimes \mu_{V_3, V_4}} & M[V_1] \otimes M[V_3] \otimes M[V_2] \otimes M[V_4]
\end{array}
\]

\(i.e.\) merging and splitting is the same than splitting and merging.

We use the term **Hopf monoid** to refer to a connected Hopf monoid in linear species.

A Hopf **sub-monoid** of a Hopf monoid \(M\) is a sub-species of \(M\) stable under the product and co-product maps.

A **morphism of Hopf monoids** is a morphism of linear species which preserves the products, co-products and unit \(1_K\).

The **co-opposite Hopf monoid** of a Hopf monoid \(M\) is the Hopf monoid \(M^{\text{cop}}\) over the same species than \(M\) and with the product and co-product defined by: \(\mu_{V_1, V_2}^{M^{\text{cop}}} = \mu_{V_1, V_2}^M\) and \(\Delta_{V_1, V_2}^{M^{\text{cop}}} = \Delta_{V_2, V_1}^M\).
Remark 1. The axioms of associativity \[^1\] and co-associativity \[^2\] make it so that we can naturally extend the definitions of the structure maps over any decomposition of \(V\). For \(V_1, \ldots, V_n\) a decomposition of \(V\):

\[
\mu_{V_1,\ldots,V_n} : M[V_1] \otimes \cdots \otimes M[V_n] \to M[V] \quad \Delta_{V_1,\ldots,V_n} : M[V] \to M[V_1] \otimes \cdots \otimes M[V_n],
\]

are respectively defined by iterating any kind of maps of the form \(Id^{\otimes k} \otimes \mu_{V_i,V_{i+1}} \otimes Id^{\otimes l}\) and of the form \(Id^{\otimes k} \otimes \Delta_{V_i,V_{i+1}} \otimes Id^{\otimes l}\), as long as the domains and co-domains coincide.

A \textit{decomposition} of a finite set \(V\) is a sequence of pairwise disjoint subsets \(S = (V_1, \ldots, V_l)\) such that \(V = \bigsqcup_{i=1}^l V_i\). A \textit{composition} of a finite set \(V\) is a decomposition of \(V\) without empty parts. We will write \(S \vdash V\) for \(S\) a decomposition of \(V\), \(S \vdash V\) if \(S\) is a composition, \(|S| = l\) the \textit{length} of a decomposition and \(|V|\) the number of elements in the decomposition.

\textbf{Definition 1.3.} The \textit{antipode} of a Hopf monoid \(M\) is the species morphism \(S : M \to M\) defined by \(S_\emptyset = \text{Id}\) and for \(V \neq \emptyset\),

\[
S_V(x) = \sum_{V_1,\ldots,V_n \vdash V} (-1)^n \mu_{V_1,\ldots,V_n} \circ \Delta_{V_1,\ldots,V_n}(x).
\]

This formula is known as \textit{Takeuchi’s formula}.

A \textit{character} \(\zeta\) of a Hopf monoid \(M\) is a collection of linear maps \(\{\zeta_V : M[V] \to \mathbb{k}\}_V\) compatible with the product and sending the unit on the unit.

We say that a character is a \textit{characteristic function} if it takes values in the set \(\{0, 1\}\). A \textit{discrete} element of a Hopf monoid \(M\) is an element which can be obtained as a product of elements of size 1, \textit{i.e.} \(x \in M[V]\) is discrete if \(V = \{v_1, \ldots, v_n\}\) and there exists \(x_1 \in M[\{v_1\}], \ldots, x_n \in M[\{v_n\}]\) such that \(x = \mu_{\{v_1\},\ldots,\{v_n\}}(x_1 \otimes \cdots \otimes x_n)\). The \textit{basic character} of any Hopf monoid \(M\) is then the characteristic function of discrete elements of \(M\). We will denote it by \(\zeta_1\).

The main object of study of this paper are the following maps.

\textbf{Definition 1.4.} Let \(M\) be a Hopf monoid and \(\zeta\) a character on \(M\). For \(x \in M[V]\) define the map

\[
\chi^{(M,\zeta)}_V (x)(n) = \sum_{V = V_1 \sqcup \cdots \sqcup V_n} \zeta_V \circ \mu_{V_1,\ldots,V_n} \circ \Delta_{V_1,\ldots,V_n}(x).
\]

Depending on how clear it is from the context, we will not specify \(\zeta\) and/or \(M\) and use the notations \(\chi^M\), \(\chi^\zeta\) or \(\chi\) to designate the map thus defined. These maps have very interesting properties as shown in the following theorem and proposition.

\textbf{Theorem 1.5} (Proposition 16.1 and Proposition 16.2 in \[13]\). Let \(M\) be a Hopf monoid, \(\zeta\) a character on \(M\) and \(\chi^{(M,\zeta)}\) be the collection of maps of Definition 1.4. Then \(\chi^{(M,\zeta)}_V (x)(n)\) is a polynomial invariant in \(n\) such that:

1. \(\chi^{(M,\zeta)}_V (x)(1) = \zeta(x)\),
2. \(\chi^{(M,\zeta)}_\emptyset = 1\) and \(\chi^{(M,\zeta)}_{V_1 \sqcup V_2} (x \cdot y) = \chi^{(M,\zeta)}_{V_1} (x) \chi^{(M,\zeta)}_{V_2} (y)\),
3. \(\chi^{(M,\zeta)}_V (x)(-n) = \chi^{(M,\zeta)}_V (S_V(x))(n)\).

\textbf{Proposition 1.6} (Proposition 16.3 in \[13]\). Let \(M\) and \(N\) be two Hopf monoids, \(\zeta^M\) and \(\zeta^N\) be two character on these Hopf monoids and \(\phi : M \to N\) be a morphism of Hopf monoids compatible with the characters: \(\zeta^N \circ \phi = \zeta^M\). Then \(\chi^N \circ \phi = \chi^M\).
Decomposition, compositions and colorings

We give here more details on these objects which will often use throughout this paper. We briefly presented the species of decompositions and compositions in the previous subsection. We want to seamlessly pass from one notion to the other, so we give a few explanations on this bijection. Given an integer \( n \), the canonical bijection between decompositions of \( V \) of size \( n \) and colorings of \( V \) with \([n]\) is given by:

\[
b_{V,n} : \{ f : V \to [n] \} \to \{ P \in D\text{comp}[V] \mid |l(P)| = n \}
\]

\[
f \mapsto (f^{-1}(1), \ldots, f^{-1}(n)).
\]

If it is clear from the context what are \( V \) and \( n \), we will write \( b \) instead of \( b_{V,n} \). If \( P \) is a partition we will also refer to \( b^{-1}(P) \) by \( P \) so that instead of writing “i such that \( v \in P_i \)” and “i and j such that \( v \in P_i, v' \in P_j \) and \( i < j \)” we can just write \( P(v) \) and \( P(v) < P(v') \). Similarly, if \( P \) is a function we will refer to \( b(P) \) by \( P \) so that \( P_i = P^{-1}(i) \). Remark that \( b_{V,n} \) induces a bijection between compositions and part orderings.

Let us now give some usual operations and definitions over decompositions and compositions. Let \( V \) and \( W \) be two disjoint sets and \( P = (P_1, \ldots, P_l) \succeq V \) and \( Q = (Q_1, \ldots, Q_k) \succeq W \) be two compositions. We call product of \( P \) and \( Q \) and we denote by \( P \cdot Q \) the composition \((P_1, \ldots, P_l, Q_1, \ldots, Q_k)\). We call shuffle product of \( P \) and \( Q \) the set \( sh(P, Q) \) defined by \( sh(P, Q) = \{ R \mid (b^{-1}(R))_V = b^{-1}(P) \text{ and } (b^{-1}(R))_W = b^{-1}(P') \} \), where for \( f \) a map with domain \( V \) and \( W \subseteq V \), the map \( f_{|W} \) is defined by \( f_{|W}(v) = f(v) \) for all \( v \in W \).

Let \( P' = (P_{1,1}, \ldots, P_{1,k_1}, P_{2,1}, \ldots, P_{2,k_2}, \ldots, P_{l,k_l}) \) be another composition of \( V \). We say that \( P' \) refines \( P \) and write \( P' \prec P \) if \( P_i = \bigcup_{j=1}^{k_i} P_{i,j} \) for \( 1 \leq i \leq l \).

2 Generalized permutahedra

Using Aguiar and Ardila’s results in [1], we give here an explicit combinatorial interpretation of the polynomial invariant in Definition 1.4 and its reciprocity theorem over the Hopf monoid of generalized permutahedra. The two theorems of this section are direct generalizations of Proposition 17.3 and 17.4 [1].

2.1 Hopf monoid structure

Let us first give the Hopf monoid structure of generalized permutahedra defined in [1] while introducing some classical definitions over generalized permutahedra. We refer the reader interested in generalized permutahedra to the vast literature on the subject (see for example [17], [10], [16]).

A generalized permutahedron in \( R^V \) is a polytope in \( R^V \) whose edges are parallel to the vectors \( v - v' \) for \( v, v' \in V \). We denote by \( GP \) the species of generalized permutahedra. We call the elements of the dual \((R^V)^* = R^V \) directions. For \( P \in GP[V] \) and \( y \) a direction in \( R^V \), we call maximum face of \( P \) in the direction \( y \) or \( y \)-maximum face of \( P \) the generalized permutahedron \( P_y = \{ p \in P \mid y(p) \geq y(q) \text{ for all } q \in P \} \). In particular, the faces of dimension 1 are the edges of \( P \).

We denote by \( Q \leq P \) for \( Q \) a face of \( P \) and \( Q < P \) if additionally \( Q \neq P \). We denote by \( L(P) \) the face lattice of \( P \), which is the poset of faces of \( P \) ordered with the previously defined order. We denote by \( |Q; P| \) the interval \( \{ P' \mid Q \leq P' \leq P \} \) and by \( |Q; P| \) the same interval but with strict inequality on the right side.

For each face \( Q \) of \( P \) define the normal cones as:

\[
N^+_y(Q) = \{ y \in R^V \mid P_y = Q \}
\]

\[
N_y(Q) = \{ y \in R^V \mid Q \text{ is a face of } P_y \}.
\]
Proposition 2.1 (Theorem 3.15 in [10]). Let $V$ be a finite set, $W \subseteq V$ and $P$ be a generalized permutahedron in $\mathbb{R}V$. Denote by $1_{W}$ the direction defined by $1_{W}(v) = 1$ if $v \in W$ and $1_{W}(v) = 0$ if $v \in V \setminus W$. Then there exist two generalized permutahedra $P_{|W}$ in $\mathbb{R}W$ and $P_{|W}$ in $\mathbb{R}V \setminus W$ such that $P_{|W} = P_{|W} + P_{|W}$.

Theorem 2.2 (Theorem 5.3 in [1]). The species $GP$ as a Hopf monoid structure given by

\[ \mu_{V_{1}, V_{2}} : GP[V_{1}] \otimes GP[V_{2}] \to GP[V] \]

\[ \Delta_{V_{1}, V_{2}} : GP[V] \to GP[V_{1}] \otimes GP[V_{2}] \]

\[ P \otimes Q \mapsto P + Q \]

where $P + Q = \{ p + q | p \in P, q \in Q \}$ is the Minkowski sum of $P$ and $Q$.

The Hopf monoid $GP$ has the following formula for the antipode:

Theorem 2.3 (Theorem 7.1 in [1]). The antipode of $GP$ is given by the cancellation-free and grouping-free formula:

\[ S_{V}(P) = \sum_{Q \leq P} (-1)^{|V| - \dim Q} Q. \]

We end this subsection by mentioning a Hopf sub-monoid of $GP$ which will use later on. Denote by $\Delta_{V} = \text{conv}(v | v \in V)$ the standard simplex of $\mathbb{R}V$. A hypergraphic polytope is a Minkowski sum of standard simplices. We denote by $HGP$ the species of hypergraphic polytopes.

Proposition 2.4 (Proposition 19.5 in [1]). The species $HGP$ is a Hopf sub-monoid of $GP$.

2.2 Polynomial invariant and reciprocity theorem

We begin by giving an interpretation of $\chi$ over non negative integers. Most definitions necessary to state our theorem were given in previous subsection, but we still need to introduce two simple notions. Let $P \in GP[V]$ be a generalized permutahedron and remark that as maps from $V$ to $[n]$, colorings are elements of $\mathbb{R}V$ i.e. directions.

- For $\zeta$ a character of $GP$, $P$ is a $\zeta$-face if $\zeta(P) \neq 0$.
- For $Q$ a face of $P$ and $c$ a coloring of $V$, $Q$ and $c$ are said to be strictly compatible if $P_{c} = Q$. They are said to be compatible if $Q \leq P_{c}$. We respectively denote by $N_{P}(Q)_{n} = [n]^{V} \cap N_{P}(Q)$ and $N_{P}(Q)_{n} = [n]^{V} \cap N_{P}(Q)$ the set of colorings with $[n]$ strictly compatible with $Q$ and compatible with $Q$.

Theorem 2.5. Let $\zeta$ be a character of $GP$, $V$ be a finite set and $P \in GP[V]$ a generalized permutahedra. Then

\[ \chi_{\zeta}(P)(n) = \sum_{Q \leq P} \zeta(Q) |N_{P}(Q)_{n}|. \]

In particular, if $\zeta$ is a characteristic function, then $\chi_{\zeta}(P)(n)$ is the number of strictly compatible pairs of $\zeta$-faces of $P$ and colorings with $[n]$.

Proof. We have:

\[ \chi_{\zeta}(P)(n) = \sum_{V = V_{1} \cup \cdots \cup V_{n}} \zeta_{V} \circ \mu_{V_{1}, \ldots, V_{n}} \circ \Delta_{V_{1}, \ldots, V_{n}}(P) \]

\[ = \sum_{D \vdash V, l(D) = n} \zeta_{V} \circ \mu_{D} \circ \Delta_{D}(P) \]

\[ = \sum_{D \text{ coloring of } V \text{ with } [n]} \zeta_{V}(P_{D}) \]

\[ = \sum_{Q \leq P} \sum_{D \text{ coloring of } V \text{ with } [n]} \zeta_{V}(Q) \]

\[ = \sum_{Q \leq P} \zeta_{V}(Q) |N_{P}(Q)_{n}|. \]
where the third equality comes from Proposition 5.4 which states that for $D = (V_1, \ldots, V_n)$ a decomposition/coloring of $V$, the $D$-maximum face of $P$ is given by $P_D = \mu_D \circ \Delta_D(P)$. \hfill $\square$

**Example 2.6.** Let $\zeta$ be the characteristic function of graphic polytopes i.e. polytopes which can be written as a Minkowski sum of standard simplices of dimension 1. Let $P = \Delta_{e_1,e_2,e_3} + \Delta_{e_1,e_4}$:

\[
P = \begin{array}{ccc}
(2,0,0,0) & & (0,0,1,1) \\
(1,1,0,0) & & (1,0,1,0) \\
(1,0,0,0) & & 
\end{array}
\]

where the hidden point is of coordinate $(0,1,0,1)$. From Theorem 2.5 we know that $\chi^\zeta(P)(3)$ is the number of strictly compatible pairs of $\zeta$-faces of $P$ and colorings with $[2]$. The $\zeta$-faces of $P$ are its three rectangular faces which correspond to the sums $\Delta_{e_1,e_2} + \Delta_{e_1,e_4}$, $\Delta_{e_1,e_3} + \Delta_{e_1,e_4}$ and $\Delta_{e_2,e_3} + \Delta_{e_1,e_4}$. Each of these faces is strictly compatible with exactly one coloring with $[2]$. These colorings are respectively $2e_1^1 + 2e_2^1 + e_3^1 + 2e_4^2$, $2e_1^1 + e_2^2 + e_3^2 + 2e_4^4$ and $e_1^1 + 2e_2^2 + 2e_3^2 + e_4^4$. Hence we have $\chi^\zeta(P)(2) = 3$.

The basic character $\zeta_1$ of $GP$ is equal to 1 on points and 0 elsewhere. There is a particular interpretation for this character which is presented in [1]. For $P \in GP[V]$ a generalized permutahedra and $y$ a direction in $RV$, $y$ is said to be $P$-generic if $P_y$ is a point.

**Corollary 2.7** (Theorem 9.2 (v) in [12] and Proposition 17.3 in [1]). Let $V$ be a finite set and $P \in GP[V]$ a generalized permutahedra. Then $\chi^{\zeta_i}(P)(n)$ is the number of $P$-generic colorings of $V$ with $[n]$.

**Proof.** By definition the $\zeta_1$-faces are the points and a coloring $c$ is strictly compatible with a face $Q$ if $P_c = Q$. $\chi^{\zeta_i}(P)$ is then counting the number of colorings $c$ such that $P_c$ is a point i.e. the number of $P$-generic colorings. \hfill $\square$

We now give the reciprocity theorem associated with these polynomials. A character $\zeta$ of $GP$ is said to be even if the $\zeta$-faces are of even dimension.

**Theorem 2.8.** Let $\zeta$ be a character of $GP$, $V$ be a finite set and $P \in GP[V]$ a generalized permutahedra then

\[
\chi^\zeta_V(h)(-n) = \sum_{Q \leq P} (-1)^{|V|- \dim Q} \zeta(Q)|\mathcal{N}_P(Q)_n|.
\]

Furthermore if $\zeta$ is an even characteristic function then $(-1)^{|V|} \chi^\zeta(P)(-n)$ is the number of compatible pairs of $\zeta$-faces of $P$ and colorings with $[n]$. In particular, $(-1)^{|V|} \chi^\zeta(P)(-1)$ is the number of $\zeta$-faces.

**Proof.** Lemma 17.2.1 in [1] state that

\[
\sum_{Q \leq P} (-1)^{\dim Q} Q_y = \sum_{Q \leq P-y} (-1)^{\dim Q} Q.
\]

Hence, for any direction $y$ we have:

\[
\sum_{Q \leq P} (-1)^{\dim Q} \zeta(Q)_y = \sum_{Q \leq P-y} (-1)^{\dim Q} \zeta(Q).
\]
Beginning with Theorem 2.3 and Theorem 1.5.3 we have:

\[ \chi(P)(-n) = \chi(S(P))(n) = \chi \left( \sum_{Q \leq P} (-1)^{|V(Q)| - \dim Q} \chi(Q)(n) \right) \]

\[ = \sum_{Q \leq P} (-1)^{|V(Q)| - \dim Q} \chi(Q)(n) \]

\[ = \sum_{Q \leq P} (-1)^{|V(Q)| - \dim Q} \sum_{R \leq Q} \zeta(R) |N^o_Q(R)| \]

\[ = \sum_{Q \leq P} (-1)^{|V(Q)| - \dim Q} \sum_{R \leq Q} \zeta(R) \sum_{y : V \to [n]} 1 \]

\[ = \sum_{Q \leq P} (-1)^{|V(Q)| - \dim Q} \sum_{y : V \to [n]} \zeta(Q_y) \]

\[ = \sum_{y : V \to [n]} (-1)^{|V|} \sum_{Q \leq P} (-1)^{\dim Q} \zeta(Q_y) \]

\[ = \sum_{y : V \to [n]} (-1)^{|V|} \sum_{Q \leq P_{-y}} (-1)^{\dim Q} \zeta(Q) \]

\[ = \sum_{Q \leq P} (-1)^{|V(Q)| - \dim Q} \zeta(Q) \sum_{y : V \to [n]} 1 \]

\[ = \sum_{Q \leq P} (-1)^{|V(Q)| - \dim Q} \zeta(Q)|N_P(Q)|, \]

where the last equality comes from the fact that \( P_{-y} = P_{n+1-y} \) and \( n + 1 - y \) as co-domain \([n]\) is and only if \( y \) has co-domain \([n]\). □

**Example 2.9.** The character \( \zeta \) of Example 2.6 is not even but the \( \zeta \)-faces of the polytope \( P \) of Example 2.6 are of even dimension hence \((-1)^{|V(P)|} \chi^\zeta(P)(-2)\) is equal to the number of compatible pairs of \( \zeta \)-faces of \( P \) and colorings with \([2]\). For each rectangular face of \( P \), its compatible colorings with \([2]\) are the one given in Example 2.6 along with the colorings \( e_1^4 + e_3^4 + e_5^4 + e_4^3 \) and \( 2e_1^3 + 2e_2^2 + 2e_3^3 + 2e_4^3 \). Hence \( \chi^\zeta(P)(-2) = 9 \).

**Corollary 2.10** (Theorem 6.3 and Theorem 9.2 (v) in [12] and Proposition 17.4 in [1]). Let \( V \) be a finite set and \( P \in GP[V] \) a generalized permutahedra. Then, we have:

\[ (-1)^{|V|} \chi^\zeta(P)(-n) = \sum_{c : V \to [n]} \text{vertices of } P_c. \]

**Proof.** From Theorem 2.8 we have that \((-1)^{|V|} \chi^\zeta(P)(-n)\) is the number of compatible pairs of points and coloring with \([n]\). Since the points compatible with a coloring \( c \) are by definition the points in \( P_c \), formula 2.10 follows. □

### 3 Polynomial invariants and reciprocity theorems on the Hopf monoid of hypergraphs

As in the previous section, we give here an explicit combinatorial interpretation of the polynomial invariant and its reciprocity theorem over the Hopf monoid of hypergraphs defined in [1]. We expand more on this Hopf monoid and give two proofs of both the combinatorial interpretation of \( \chi \) over positive and non negative integers. One of these proofs is self contained and the other one uses the results of the previous section.

A hypergraph over \( V \) is a multiset \( h \) of non empty parts of \( V \) called edges. In this context the elements of \( V \) are called vertices of \( h \). The species of hypergraphs is denoted by \( HG \). Note that
two hypergraphs over different sets can never be equal, e.g \( \{\{1, 2, 3\}, \{2, 3, 4\}\} \in HG[[4]] \) is not the same as \( \{\{1, 2, 3\}, \{2, 3, 4\}\} \in HG[[4] \cup \{a, b\}] \). This is illustrated in the following figure.

Two hypergraphs with same edges but over different sets.

Remark 2. When working only with hypergraphs, we will prefer two different ways of drawing the edges depending on the context. The same edge \( \{a, b, c\} \) can thus be drawn in the two following ways:

The species \( HG \) has a Hopf monoid structure with the product the disjoint union and the co-product given by \( \Delta_{V_1, V_2}(h) = h|_{V_1} \otimes h/_{V_1} \) where
\[
\begin{align*}
h|_V &= \{e \in h \mid e \subseteq V\} \quad \text{and} \quad h/_{V} = \{e \cap V^c \mid e \not\subseteq V\}.
\end{align*}
\]

Example 3.1. For \( V = [5] \), \( V_1 = \{1, 2, 5\} \) and \( V_2 = \{3, 4\} \), we have the following co-product:

The Hopf structure defined here is different from the one defined and studied in [5] but there are similar results in both this paper and [4]. This is due to the fact that the notion of acyclic orientations plays a huge role when working on hypergraphs. Up to some minor subtleties, our notion of acyclic orientations is the same as the one in [5]: we trade the notion of flats for acyclic orientations where all the vertices of an edge can be a target. We preferred to define them as maps over hypergraphs with certain properties while they are defined as compositions of edges which induce directed acyclic graphs. We will point out the common results when they appear.

While the proofs using results of the previous section are rather short, the self contained proofs are more involved. Subsection 3.1 presents some preliminary results. The combinatorial interpretations of \( \chi(n) \) and \( \chi(-n) \) and their self contained proofs are in subsection 3.2 and subsection 3.3. The proof using the previous section is presented in subsection 3.4.

### 3.1 Generalized Faulhaber’s polynomials

As stated in Theorem 1.5, for any character \( \zeta \) and any hypergraph \( h \), \( \chi^\zeta(h)(n) \) is a polynomial in \( n \). The objects defined here are useful tools to show and exploit this polynomial dependency and were introduced in [4].

Let \( p = (p_1, p_2, \ldots, p_t) \) be a finite sequence of positive integers. We define the generalized Faulhaber polynomial over \( p \), \( F_p \), the function over the integers given by, for \( n \in \mathbb{N} \):

\[
F_p(n) = \sum_{0 \leq k_1 < \cdots < k_t \leq n-1} k_1^{p_1} \cdots k_t^{p_t}.
\]
Note that if \( t > n \), then \( F_p(n) = \sum_{j=0}^{d_t-1} = 0 \).

As their names suggest, these functions are polynomials.

**Proposition 3.2** (Proposition 13 in [4]). Let \( p_1, p_2, \ldots, p_t \) be integers and define \( d_k = \sum_{i=1}^{k} p_i + k \) for \( 1 \leq k \leq t \). Then \( F_{p_1, \ldots, p_t} \) is a polynomial of degree \( d_t \) whose constant coefficient is null and whose \((d_t - i)\)-th coefficient is given by

\[
\min_{j_{t-1}=0} \cdots \min_{j_1=0} \prod_{k=1}^{t} \left( \frac{d_k - j_k - 1}{j_k - j_{k-1}} \right) B_{j_k-j_{k-1}} \]

where \( j_t = i \) and \( j_0 = 0 \), and the \( B_j \) numbers are the Bernoulli numbers with the convention \( B_1 = -1/2 \).

**Remark 3.** These polynomials also generalize Stirling numbers of the first kind: denote by \( F_{t,k}(n) \) the generalized Faulhaber polynomial associated to the sequence of size \( k \) with all elements equal to 1. \( F_{t,k}(n) = \sum_{0 \leq k_1 < \cdots < k_{t-1} \leq n-1} \prod_{i=1}^{t-1} k_i \) is indeed the absolute value of the coefficient of \( x^{n-k} \) in \( x(x-1) \cdots (x-n+1) \) and hence \( F_{t,k}(n) = s(n,k) \).

**Lemma 3.3** (Lemma 21 in [4]). Let \( p \) be a sequence of positive integers of length \( t \). We then have

\[
F_p(-n) = (-1)^{d_t} \sum_{p < q} F_q(n+1),
\]

where \( d_t \) is defined in the same way as in Proposition 3.2.

### 3.2 Chromatic polynomials of hypergraphs

Before stating our results on \( \chi \) in Theorem 3.12, recall that a coloring of \( V \) with \([n]\) is a map from \( V \) to \([n]\) and that there is a canonical bijection between decompositions and colorings.

**Definition 3.4.** Let \( h \) be a hypergraph over \( V \) and \( c \) be a coloring. For \( v \in e \in h \), we say that \( v \) is a maximal vertex of \( e \) (for \( c \)) if \( v \) is of maximal color in \( e \) and we call the maximal color of \( e \) (for \( c \)) the color of a maximal vertex of \( e \). We say that a vertex \( v \) is a maximal vertex (for \( c \)) if it is a maximal vertex of an edge and that a color is a maximal color (for \( c \)) if it is the maximal color of an edge.

If \( W \subseteq V \) is a subset of vertices, the order of appearance of \( W \) (for \( V \)) is the composition forget\(_W\) \( c \cap W \) where \( c \cap W = (c_1 \cap W, \ldots, c_{\chi(c)} \cap W) \) and the map forget\(_W\) sends any decomposition to the composition obtained by dropping the empty parts.

When working with colorings, by a slight abuse of notation, for \( W \) a set of vertices of the same color for \( c \), we will denote by \( c(W) \) their color. This extends \( c \) to a map from monochromatic sets of vertices to \([n]\).

**Example 3.5.** We represent a hypergraph along a coloring on \( V = \{a, b, c, d, e, f\} \) with \([1, 2, 3, 4]\):

```
\begin{tikzpicture}
  \node (a) at (0,0) [circle, fill=black] {a}; \node (b) at (1,0) [circle, fill=black] {b}; \node (c) at (2,0) [circle, fill=black] {c}; \node (d) at (3,0) [circle, fill=black] {d}; \node (e) at (0,1) [circle, fill=black] {e}; \node (f) at (2,1) [circle, fill=black] {f};
  \draw (a) -- (b) -- (c) -- (d) -- (a);
  \draw (c) -- (e) -- (f) -- (c);
  \draw (a) -- (e);
\end{tikzpicture}
```

The maximal vertex of \( e_1 \) is \( a \) and the maximal vertices of \( e_3 \) are \( c \) and \( d \). The maximal color of \( e_2 \) is \( 3 \). The order of appearance of \( \{a, c, d, e\} \) is \( (\{e\}, \{c,d\}, \{a\}) \).

**Definition 3.6.** Let \( h \) be a hypergraph over \( V \). An admissible orientation \( f \) of \( h \) is a map from \( h \) to \( \mathcal{P}(V) \setminus \{\emptyset\} \) such that \( f(e) \subseteq e \) for every edge \( e \). It is an acyclic orientation if there is no sequence of distinct edges \( e_1, \ldots, e_k \) such that \( f(e_i) \cap f(e_{i+1}) \neq \emptyset \) or \( \emptyset \subset f(e_i) \cap e_{i+1} \subset f(e_{i+1}) \) for \( 1 \leq i < k \) and \( f(e_k) \cap f(e_1) \neq \emptyset \), where \( f_{x}(e) = f(e) \setminus f(e) \).
Example 3.7. We represent here a hypergraph with two different orientations. One is cyclic and the other is discrete and acyclic.

![cyclic orientation](image1)

![acyclic discrete orientation](image2)

Given a hypergraph $h$ and $f$ an admissible orientation of $h$, the image of $h$ by $f$, $f(h)$ is also a hypergraph: $f(h) = \{f(e) | e \in h\}$.

Definition 3.8. Let $h$ be a hypergraph over $V$ and $f$ an admissible orientation of $h$.

- For $c$ coloring of $V$, $c$ and $f$ are said to be compatible if for every edge $e$ the elements of $f(e)$ are maximal in $e$ for $c$. They are said to be strictly compatible if $f(e)$ is exactly the set of maximal elements of $e$ for $c$. We denote by $\mathcal{C}_{h,f,n}$ the set of colorings of $V$ with $[n]$ compatible with $f$ and by $C_{h,f,n}$ the set of those with strict compatibility.

- For $\zeta$ a character of $HG$, $f$ is said to be a $\zeta$-orientation of $h$ if $\zeta(f(h)) \neq 0$.

Example 3.9. The coloring given in Example 3.5 has three compatible acyclic orientations: both send $e_1$ on $a$, $e_2$ on $c$ and $e_4$ on $b$, but $e_3$ can be either sent over $c$, $d$ or $\{c,d\}$. Among these three only the last one is strictly compatible.

Here is an example of an acyclic orientation of a hypergraph with a compatible coloring and a strictly compatible coloring with $\{1,2,3\}$:

![compatibility](image3)

![strict compatibility](image4)

- Let $\zeta$ be the characteristic function of hypergraphs which connected components are either of size 3 or an isolated vertex. The preceding orientation is not a $\zeta$-orientation since the image of the hypergraph by this orientation is the hypergraph

![hypergraph](image5)
Here is an example of a \( \zeta \)-orientation and the image of the hypergraph by it:

Remark that with these definitions, given a coloring \( c \) and a hypergraph \( h \), there is a unique orientation of \( h \) strictly compatible with \( c \), which is defined by \( f(e) = \{ v \in e \mid c(v) = \max(c(e)) \} \). Furthermore, this orientation is necessarily acyclic. Indeed, suppose \( e_1, \ldots, e_k \) is a directed cycle in \( f \), then \( f(e_k) \cap f_s(e_1) \neq \emptyset \) implies that \( c(f(e_k)) < c(f(e_1)) \) and for \( 1 \leq i < k \) either \( f(e_i) \cap f_s(e_{i+1}) \), which would imply \( c(f(e_i)) < c(f(e_{i+1})) \) or \( f(e_i) \cap f(e_{i+1}) \neq \emptyset \) which would imply \( c(f(e_i)) = c(f(e_{i+1})) \). This then gives us \( c(f(e_i)) < c(f(e_1)) \) which is absurd. We will denote by \( \text{max}_e \) this orientation.

With the same kind of reasoning, any coloring compatible with a cyclic orientation must be monochromatic on directed cycles. The study of \( C_{\text{h},f,n} \) and \( \overline{C}_{\text{h},f,n} \) is hence more interesting when \( f \) is acyclic and we have an expression of \( C_{\text{f},h,n} \) in terms of generalized Faulhaber polynomials. Recall that for every hypergraph \( h \) over \( V \), we have a decomposition of \( V \) in the set of isolated vertices and vertices in an edge: \( V = I(h) \sqcup NI(h) \).

**Proposition 3.10.** Let \( h \) be a hypergraph over \( V \), \( f \) be an acyclic orientation of \( h \). Define \( P_{\text{h},f} \) and \( P'_{\text{h},f} \) as the set of compositions \( P \models f(h) \) such that for \( e \) and \( e' \) two edges of \( h \),

- if \( f(e) \cap f(e') \neq \emptyset \) then \( P(f(e)) = P(f(e')) \),
- else if \( f(e) \cap f_s(e') \neq \emptyset \) then \( P(f(e)) < P(f(e')) \) for \( P \in P_{\text{h},f} \) and \( P(f(e)) \leq P(f(e')) \) for \( P \in P'_{\text{h},f} \).

We then have

\[
|C_{\text{h},f,n}| = n^{\|I(h)\|} \sum_{P \in P_{\text{h},f}} F_{p_1,\ldots,p_l(P)}(n) \quad \text{and} \quad |\overline{C}_{\text{h},f,n}| = n^{\|I(h)\|} \sum_{P \in P'_{\text{h},f}} F_{p_1,\ldots,p_l(P)}(n + 1),
\]

where for every composition \( P \), \( p_i = |\overline{P}_i| \) and \( \overline{P}_i = NI(f(h))^c \cap \left( \bigcup_{e \in f^{-1}(p_i)} e \right) \cap \bigcup_{j < i} \overline{P}_j^e \).

**Example 3.11.** To prove the formula for \( |C_{\text{h},f,n}| \) we will show that there is a bijection between the set of strictly compatible colorings and the set

\[
\bigsqcup_{P \in P_{\text{h},f}} \bigsqcup_{0 \leq k_1 < k_2 < \ldots < k_l(P) \leq n - 1} \prod_{i \leq l(P)} [k_i] \overline{P}_i.
\]

We give here an example of how this bijection will work. Let \( H, F \) and \( C \) be the hypergraph, the
acyclic orientation and the strictly compatible coloring with \(\{1, 2, 3, 4, 5, 6\}\) represented here:

The image of \(C\) is obtained in the following way:

- \(P = (d, ij, ab) \in P_{H,F}\) is the relative order of the maximal vertices for \(C\),
- \(k_1 = 1, k_2 = 4\) and \(k_3 = 5\) are the colors of the vertices in \(P\) shifted by \(-1\), the vertices in \(P_i\) being of color \(k_i + 1\),
- we then have \(\tilde{P}_1 = \{f\}, \tilde{P}_2 = \{c, g, h\}\) and \(\tilde{P}_3 = \{e\}\) and the image of \(C\) is the triplet \((f_1, f_2, f_3)\) defined by

\[
\begin{align*}
f_1 : \{f\} &\rightarrow \{1\} \\
& f \mapsto 1 \\
f_2 : \{c, g, h\} &\rightarrow \{1, 2, 3, 4\} \\
& c \mapsto 3 \\
& g \mapsto 3 \\
& h \mapsto 4 \\
f_3 : \{e\} &\rightarrow \{1, 2, 3, 4, 5\} \\
& e \mapsto 2.
\end{align*}
\]

**Proof.** We first prove the formula for \(|C_{h,f,n}|\). The term \(n^{[l(h)]}\) in the formula is trivially obtained and we hence consider that \(h\) has no isolated vertices.

Informally, the formula can be obtained by the following reasoning. To choose a coloring strictly compatible with \(f\), one can proceed in the following way:

1. choose a part ordering of the sets of maximal vertices: \(P \in P_{h,f}\),
2. choose the color of these vertices: \(k_1 + 1, \ldots, k_{l(P)} + 1\),
3. choose the colors of the yet uncolored vertices which are in the same edge than vertices of minimal color in \(f(h): k_1^{\tilde{P}_1}\); then those in the same edge than a vertices of second minimal color in \(f(h): k_2^{\tilde{P}_2}\), etc.

More formally, we show that there exists a bijection between the set of strictly compatible colorings and the set

\[
\bigsqcup_{P \in P_{h,f}} \bigsqcup_{0 \leq k_1 < \cdots < k_l(P) \leq n-1} \prod_{1 \leq i \leq l(P)} [k_i]^{\tilde{P}_i},
\]
where \([k_i]_{\tilde{P}_i}\) is the set of maps from \(\tilde{P}_i\) to \([k_i]\).

For any subset \(A\) of \(\mathbb{N}\), we denote by \(\text{bij}_A\) the unique increasing bijection from \(A\) to \([|A|]\). Let \(c\) be a strictly compatible coloring, that is to say, \(f = \max_c\). We begin by constructing its image by the announced bijection. Recall that \(c\) extends to a map with domain monochromatic set of vertices and hence \(c(f(h))\) is the set of maximal colors of \(c\). Define:

- the part ordering of the set \(f(h)\) of maximal vertices: \(P = \text{bij}_{c(f(h))} \circ c\). Here again we consider \(c\) as a map from monochromatic set of vertices to \([n]\).
- The colors of the maximal vertices: \(k_i = c(P_i) - 1\) for \(1 \leq i \leq l(P)\).
- The remaining vertices: \(\tilde{P}_i = NI(f(h))^c \cap \left( \bigcup_{e \in f^{-1}(P_i)} e \right) \bigcap_{j < i} \tilde{P}_j^c\) for \(1 \leq i \leq l(P)\).

The part ordering \(P\) is in \(P_{h,f}\) because the strict compatibility of \(c\) with \(f\) implies that if \(f(e) \cap f(e') \neq \emptyset\) then \(c(f(e)) = c(f(e) \cap f(e')) = c(f(e'))\) and else if \(f(e) \cap f(e') \neq \emptyset\) then \(c(f(e)) < c(f(e'))\) and \(\text{bij}_{c(f(h))}\) is increasing by definition. The sequence \(k_1, \ldots, k_{l(P)}\) is increasing since \(c(P_i) = c(\text{bij}_{c(f(h))} \circ c)^{-1}(i) = \text{bij}_{c(f(h))}^{-1}(i)\) and \(\text{bij}_{c(f(h))}\) is increasing by definition. The vertices in \(\tilde{P}_i\) are the vertices which share an edge with a set of vertices in \(P_i\left(\bigcup_{e \in f^{-1}(P_i)} e\right)\) but which are not maximal \((NI(f(h))^c)\). Since the set of vertices in \(P_i\) are of color \(c(P_i) = k_i + 1\), the colors of the vertices in \(\tilde{P}_i\) are necessarily in \([k_i]\). Hence the map \(c_{\tilde{P}_i}\), that is to say \(c\) restricted to \(\tilde{P}_i\) is indeed of co-domain \([k_i]\). We then define the image of \(c\) as the tuple \((c_{\tilde{P}_1}, \ldots, c_{\tilde{P}_{l(P)}})\).

Let us now consider the other direction of the bijection. Let be a partition \(P \in P_{h,f}\), a sequence of integers \(1 \leq k_1 < \cdots < k_{l(P)}\) and \((c_1, \ldots, c_{l(P)}) \in \prod_{1 \leq l \leq l(P)} [k_l]_{\tilde{P}_i}\). Define \(c : V \to [n]\) by \(c|_{\tilde{P}_i} = c_i\) and \(c(f(h))|_{v} = c|_{\tilde{P}_i}^{-1}(k_{l+1}, \ldots, k_{l(P)+1})\). This is well defined since if \(v \in f(e) \cap f(e')\) then \(f(e) \cap f(e') \neq \emptyset\) then \(P(f(e)) = P(f(e'))\). This map \(c\) has indeed domain \(V\) since \((\tilde{P}_1, \ldots, \tilde{P}_{l(P)}, NI(f(h)))\) is a partition of \(V\). Let us show that \(c\) is a coloring strictly compatible with \(f\). If \(v, v'\) are two vertices in \(f(e)\) then by definition \(c(v) = c(v')\). Let now be \(v \in e \setminus f(e)\). If \(v \in f(e')\) then necessarily \(f(e) \cap f(e') = \emptyset\) because otherwise \(e', e, e'\) would be a directed cycle. Indeed, \(v\) is an exit of \(e'\) and an entry of \(e\) and \(f(e) \cap f(e') \neq \emptyset\) would imply that \(e'\) share an exit with \(e\), but not all \((v)\). Hence, by definition of \(P_{h,f}\), \(P(f(e')) < P(f(e))\) and so \(c(v) = c(f(e'))\).

- If \(v \notin f(h)\) then \(v \in \tilde{P}_i\) with \(i \leq P(f(e))\) and we do have the desired inequality: \(c(v) = c_i(v) \leq k_i < k_i + 1 \leq k_{l(P(f(e)))} + 1 = c(f(e))\).

We conclude this first part of the proof by remarking that the two previous constructions are inverse functions.

The proof for the formula of \(|c_{h,f,n}|\) is the same except that we now show a bijection with

\[
\bigcup_{P \in P_{h,f}} \bigcup_{0 \leq k_1 \leq k_2 \leq \cdots \leq k_{l(P)} \leq n} \prod_{1 \leq l \leq l(P)} [k_i]_{\tilde{P}_i},
\]

and that to do so the only change with what precedes is choosing \(k_i = c(P_i)\) instead of \(k_i = c(P_i) - 1\).

We now state the first theorem of this section:

**Theorem 3.12.** Let \(\zeta\) be a character of \(HG\), \(V\) be a finite set and \(h \in HG[V]\) a hypergraph. Then

\[
\chi^\zeta(h)(n) = \sum_{f \in A_h} \zeta(f(h)) |c_{h,f,n}|.
\]

In particular, if \(\zeta\) is a characteristic function, then \(\chi^\zeta(h)(n)\) is the number of strictly compatible pairs of acyclic \(\zeta\)-orientations of \(h\) and colorings with \([n]\). In this case we call \(\chi^\zeta\) the \(\zeta\)-chromatic polynomial of \(h\).
Proof. Recall the definition of $\chi^C$ from [14]:

$$\chi^C_V(n) = \sum_{D \in \text{comp}[V], |V_D| = n} \zeta_V \circ \mu_D \circ \Delta_D(h).$$

The bijection $b$ defined at [1] provides a bijection between the decomposition of size $n$ of $V$ and the colorings of $V$ with $[n]$. To prove our theorem, we just need to show that for $D$ a coloring of $V$, $\max_D(h) = \mu_D \circ \Delta_D$ since we would then have

$$\chi^C_V(n) = \sum_{D \in \text{comp}[V], |V_D| = n} \zeta_V \circ \mu_D \circ \Delta_D(h)$$

$$= \sum_{D \text{ coloring of } f \text{ with } [n]} \zeta_V(\max_D(h))$$

$$= \sum_{f \in A_n} \zeta_V(f(h))[C_{h,f,n}].$$

We prove this by induction over $n$. Let $D = (D_1, D_2)$ be a decomposition of $V$ of size 2. Then $\mu_D \circ \Delta_D(h) = h|D_1 \cup h/D_2 = \{ e \in h | e \subseteq D_1 \} \cup \{ e \cap D_2 | e \subseteq D_1 \}$. We then need to show that $\max_D(e) = e$ when $e \subseteq D_1$ and $\max_D(e) = e \cap D_2$ else. If $e \subseteq D_1$, then that means that all the vertices in $e$ are maximal in $e$ of color 1, and hence $\max_D(e) = e$. If $e \nsubseteq D_1$, then the maximal vertices of $e$ are of the one of color 2 i.e. $\max_D(e) = e \cap D_2$. This concludes the case $n = 2$.

Suppose now that this statement is true for $n - 1$ and let $D$ be a decomposition of $V$ of size $n$. Denote by $W$ the set $V \setminus D_n$ and by $D|W = (D_1, \ldots, D_{n-1})$ the restriction of $D$ to $W$ (as a map). By induction we have that

$$\mu_D \circ \Delta_D(h) = \mu_W \circ \mu_D \circ \Delta_D(h)$$

$$= \mu_W \circ \mu_D \circ \Delta_D(h)$$

and hence we must show that $\max_D(e) = \max_{D|W}(e)$ when $e \subseteq W$ and $\max_D(e) = e \cap D_n$ otherwise. The first assertion is straightforward by definition of the restriction and the second assertion is proven in an analogous way that the case $n = 2$. This concludes this proof.

It is clear that connected hypergraphs play the role of prime elements in $HG$. This means that every hypergraph can be uniquely written as a product of connected elements, up to the order. When defining a character, we hence only need to define it on the connected hypergraphs. In particular, we only define characteristic functions by specifying the connected hypergraphs with value 1.

Example 3.13. Let $\zeta_3$ be the characteristic function of edges of size three and isolated vertices. Then $\chi^{C_3}$ counts the colorings such that there is exactly three maximal vertices by edge and edges do not share maximal vertices.

Let now $\zeta_1$ be the characteristic function of connected hypergraphs over three vertices and isolated vertices. Then $\chi^C$ counts the number of colorings such that reducing each edge to its maximal vertices gives us a hypergraph where connected components are either either of size 3 or an isolated vertex.

Of the three following colorings with $\{1, 2, 3\}$, the first two are counted by $\chi^{C_3}$ but not the third. None of them are counted by $\chi^{C_1}$.
Recall that the basic character $ζ_1$ is the characteristic function of discrete hypergraphs. We have a particular interpretation of $χ^{ζ_1}$.

**Corollary 3.14** (Theorem 18 in [4]). Let $V$ be a finite set and $h ∈ HG[V]$ a hypergraph. Then $χ^{ζ_1}(h)(n)$ is the number of colorings of $V$ such that every edge of $h$ has only one maximal vertex.

**Proof.** This proof is straightforward: the $ζ_1$-orientations are exactly the discrete orientations and the colorings compatible with such orientations are colorings where each edge has exactly one maximal vertex.

**Example 3.15.** The coloring given in Example 3.5 is not counted in $χ^{ζ_1}(h)(4)$ since $e_3$ has two maximal vertices. However by changing the color of $d$ to 2 we do obtain a coloring where every edge has only one maximal vertex.

Let $g$ be the hypergraph $\{\{1, 2, 3\}, \{2, 3, 4\}\} ∈ HG[[4]]$ represented in Figure 1. We then have $χ^{ζ_1}_4(h)(n) = n^4 - \frac{8}{3}n^3 + \frac{5}{2}n^2 - \frac{5}{6}n$ and we verify that, for example, $χ^{ζ_1}_4(h)(2) = 3$.

### 3.3 Reciprocity theorem

We now give the reciprocity theorem which gives us an expression of $χ^ζ$ over negative integers as well as a combinatorial interpretation when possible. A character $ζ$ of $HG$ is odd if $ζ(h) = 0$ for every $h$ with a connected component with an even number of vertices. This can also be expressed by stating that the only connected hypergraphs on which $ζ$ is not null have odd number of vertices.

Denote by $cc(h)$ the number of connected components of $h$.

**Theorem 3.16.** Let $ζ$ be a character of $HG$, $V$ be a finite set and $h ∈ HG[V]$ a hypergraph then

$$χ^ζ(h)(−n) = \sum_{f \in A_h} (-1)^{cc(f(h))}ζ(f(h))|C_{h, f, n}|,$$

Furthermore, if $ζ$ is an odd characteristic function then $(-1)^{|V|}χ^ζ(h)(−n)$ is the number of compatible pairs of acyclic $ζ$-orientations of $h$ and colorings with $[n]$. In particular we have in this case that $(-1)^{|V|}χ^ζ(h)(−1)$ is the number of acyclic $ζ$-orientations of $h$.

**Corollary 3.17** (Theorem 24 in [4]). Let $V$ be a finite set and $h ∈ HG[V]$ a hypergraph. Then $(-1)^{|V|}χ^{ζ_1}(h)(−n)$ is the number of compatible discrete acyclic orientation of $h$ and colorings with $[n]$. In particular, we have now that $(-1)^{|V|}χ^{ζ_1}(h)(−1)$ is the number of discrete acyclic orientations of $h$.

**Example 3.18.** For any any hypergraph $h$ over $V$ and any odd character $ζ$ of $HG$, we have $χ_V(h)(n) ≤ (-1)^{|V|}χ_V(h)(−n)$. This comes from the fact that any strictly compatible pair is a compatible pair. This is observed for $ζ = ζ_1$ and $h = \{\{1, 2, 3\}, \{2, 3, 4\}\} ∈ HG[[4]]$:

$$χ_{[4]}(h)(n) = n^4 - \frac{8}{3}n^3 + \frac{5}{2}n^2 - \frac{5}{6}n < n^4 + \frac{8}{3}n^3 + \frac{5}{2}n^2 + \frac{5}{6}n = (-1)^4χ_{[4]}(h)(−n).$$

We also verify that $h$ does have $χ_{[4]}(h)(−1) = 7$ acyclic discrete orientations ($3 \times 3$ orientations minus the two cyclic orientations).

As announced at the beginning of this section, we give here a self-contained proof which uses the previous results on generalized Faulhaber’s polynomials and on compatible colorings. Except for the definition of some objects, this proof is essentially the same than the one given in [4] for the case of the basic character and we will also need the following lemma.

Recall from subsection 1 the classical definitions over compositions: product, shuffle product and refinement.

**Lemma 3.19** (Lemma 23 in [4]). Let $V$ be a set and $P ∈ V$ a composition of $V$. We have the identity:

$$\sum_{Q ≤ P} (-1)^{|Q|} = (-1)^{|V|}.$$

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Furthermore, let \( q \) be a directed acyclic graph on \( V \) and consider the constrained set 
\[ C(g, P) = \{ Q \prec P \mid \forall (v, v') \in g, Q(v) < Q(v') \}. \]
We have the more general identity:
\[
\sum_{Q \in C(g, P)} (-1)^{|Q|} = \begin{cases} 
0 & \text{if there exists } (v, v') \in g \text{ such that } P(v') < P(v), \\
(-1)^{|V|} & \text{if not.} 
\end{cases}
\]

While we interpreted Lemma 3.19 as a result on graphs and partitions, it can also be seen as
a result on posets and linear extensions.

We can now give our first proof to Theorem 3.16

**Proof of Theorem 3.16.** From Theorem 3.12, Proposition 3.10 and Lemma 3.3, we have that:
\[
\chi^c_V(h)(-n) = (-n)^{|I(h)|} \sum_{f \in A_h} \zeta(f(h)) \sum_{P \in P_{h,f}} (-1)^{(P)} \sum_{\sum_{i=1}^{|P|} p_i = |NI(f(h))| - |NI(I(h))|} F_q(n + 1).
\]

Let \( P \) be a composition. We then have:

- \( \sum_{i=1}^{|P|} p_i = |NI(h)| - |NI(f(h))|, \) since \( (\overline{P}_1, \ldots, \overline{P}_{(f)}), NI(f(h)) \) is a partition of \( NI(h) \).
- The map 
  \[
  \phi : \{ Q \models f(h) \mid P \prec Q \} \rightarrow \{ q \models (|NI(f(h))| - |NI(I(h))|) \mid (p_1, \ldots, p_{(f)}) \prec q \}
  \]
  \[
  Q \mapsto ([\overline{Q}_1], \ldots, [\overline{Q}_{(f)}])
  \]
  is a bijection (\( \overline{Q}_i \) is defined in the same way that \( \overline{P}_i \) in Proposition 3.10). Indeed, the two sets have same cardinality \( \sum_{k=0}^{(f)} \binom{(P)}{k} \): in both cases we choose the number \( k + 1 \) of elements of the composition and then which consecutive elements of the composition to merge: \( \binom{(P)}{k} \).

Furthermore the map \( \phi \) is a surjection, since the composition \( (q_1, \ldots, q_k) \) with \( q_i = \sum_{j=j_i} p_j \) is the image of the composition \( Q_1, \ldots, Q_k \) with \( Q_i = \bigcup_{j=j_i} p_j \), \( P \). This comes from the fact that for any two disjoint sets of sets \( A, B \) we have \( \bigcup_{e \in A, e \in B} e = \bigcup_{e \in A} e \bigcup_{e \in B} e = \bigcup_{e \in A} e \bigcup_{e \in B} e \) and that the sets \( f^{-1}(P_i) \) are pairwise disjoint.

These two remarks lead to:
\[
\chi^c_V(h)(-n) = n^{|I(h)|} \sum_{f \in A_h} (-1)^{|V| - |NI(f(h))|} \zeta(f(h)) \sum_{P \in P_{h,f}} (-1)^{(P)} \sum_{P \prec Q} F_{\phi(Q)}(n + 1)
\]
\[
= n^{|I(h)|} \sum_{f \in A_h} (-1)^{|V| - |NI(f(h))|} \zeta(f(h)) \sum_{Q \models f(h)} \left( \sum_{P \prec Q} (-1)^{(P)} \right) F_{\phi(Q)}(n + 1).
\]

Let \( g \) be the graph with vertices the connected components of \( f(h) \) and with an oriented edge from a connected component \( h_1 \) to another connected component \( h_2 \) if there is \( e_1 \in h_1 \) and \( e_2 \in h_2 \) such that \( f(e_1) \cap f(e_2) = \emptyset \). Since \( f \) is acyclic, \( g \) is a directed acyclic graph. We can see the compositions in \( P_{h,f} \) and \( P_{h,f} \) (Proposition 3.10) as compositions over connected components of \( f(h) \) since for such compositions, two edges of \( f(h) \) with a non empty intersection must be in the same part by definition. Remarking then that with this point of view \( P \prec Q \mid P \in P_{h,f} \) = \( C(g, Q) \), Lemma 3.19 leads to:
\[
\chi_V(h)(-n) = n^{|I(h)|} \sum_{f \in A_h} (-1)^{|V| - |NI(f(h))|} \zeta(f(h)) \sum_{P \models f(h)} (-1)^{|cc(f(h)) - |I(f(h))| + |cc(f(h))|} \zeta(f(h)) \sum_{P \prec Q} F_{P_{1,...,P_{(f)}}}(n + 1)
\]
\[
= n^{|I(h)|} \sum_{f \in A_h} (-1)^{|V| - |NI(f(h))| - |I(f(h))| + |cc(f(h))|} \zeta(f(h)) \sum_{P \models f(h)} F_{P_{1,...,P_{(f)}}}(n + 1)
\]
\[
= n^{|I(h)|} \sum_{f \in A_h} (-1)^{|cc(f(h))|} |C_{f,h,n}|,
\]
..
where the last equality is Proposition 3.10.

To complete this proof, note that when $\zeta$ is odd, $\zeta(f(h)) \neq 0$ implies that $|V| - cc(f(h))$ is even since each connected component $h'$ of $f(h)$ participate for $V(h') - 1$ which is even. \hfill $\Box$

### 3.4 Alternative proof

Let us now give the second proof. As announced this proof is shorter and we will prove both Theorem 3.12 and Theorem 3.16 at the same time.

This proof rests on the existence of a bijection between hypergraphs and hypergraphic polytopes. Indeed for $h \in HG[V]$ a hypergraph, denote by $\Delta_h$ the Minkowski sum $\sum_{e \in h} \Delta_e$. Then the map $\Delta : h \mapsto \Delta_h$ clearly is a species isomorphism from $HG$ to $HGP$. We further extend the similarities of hypergraphs and hypergraphic polytopes with the following lemma.

**Lemma 3.20.** Let $V$ be a finite set and $h$ a hypergraph over $V$. Then the faces of $\Delta_h$ are exactly the hypergraphic polytopes $\Delta_{f(h)}$ for $f \in \mathcal{A}_h$. Furthermore, for $f$ an acyclic orientation of $h$, $C_{h,f,n} = \mathcal{N}_h^\Delta(\Delta_{f(h)})_n$ and $\overline{C}_{h,f,n} = \mathcal{N}_h(\Delta_{f(h)})_n$.

**Remark.** This lemma is equivalent to Theorem 2.18 of [5]. Our approach and notations being different than in [5], we preferred to give an alternative proof of this lemma.

**Proof.** Let be $f \in \mathcal{A}_h$. We first show that $\Delta_{f(h)}$ is indeed a face of $\Delta_h$. Let $c$ be a coloring strictly compatible with $f$, i.e. $f = \text{max}_c$. We show that $\Delta_{f(h)}$ is the $c$-maximum face of $\Delta_h$. Let $p$ be a point in $\Delta_h$. Then by definition of $\Delta_h$ $p$ can be written as $\sum_{e \in h} \sum_{v \in e} a_{e,v}v$ where for each edge $e$ the $a_{e,v}$ are positive real numbers summing to one: $\sum_{v \in e} a_{e,v} = 1$. We then have that,

$$c(p) = \sum_{e \in h} \sum_{v \in e} a_{e,v}c(v) = \sum_{e \in h} \left( c(f(e)) \sum_{v \in f(e)} a_{e,v} + \sum_{v \in f_s(e)} a_{e,v}c(v) \right),$$

which is maximum when $a_{e,v} = 0$ for every edge $e$ and every $v \in f_s(e)$. This implies that the $c$-maximum face of $\Delta_h$ is the set of points of the form $\sum_{e \in h} \sum_{v \in f(e)} a_{e,v}v$ with $\sum_{v \in f(e)} a_{e,v} = 1$. This is exactly $\Delta_{f(h)}$.

Let now $Q$ be a face of $\Delta_h$. For $P$ a polytope and $y$ a direction, the $y$-maximum face $P_y$ does not depend on the exact values of $y$ but only on the induced order $v < v'$ if $y(v) < y(v')$. Let hence be $c$ a coloring with value in $[n]$ such that $Q = P_c$. Then by what precedes, $Q = \Delta_{\text{max}_c(h)}$.

The equalities between the sets of compatible colorings and the cones directly follow from the preceding. \hfill $\Box$

This lemma together with Proposition 1.6 is enough to give the desired proof.

**Proof of Theorem 3.16.** Let $\zeta$ be a character of $HG$ and define $\zeta'$ the character of $GP$ defined by $\zeta'(P) = \zeta(h)$ if $P = \Delta_h$ is a hypergraphic polytope and $\zeta'(P) = 0$ else. Let $h$ be a hypergraph over $V$. Then, applying Proposition 1.6, Theorem 2.5 and finally Lemma 3.20 we have:

$$\chi_V^{HG,\zeta}(h)(n) = \chi_V^{GP,\zeta'}(\Delta_h)(n) = \sum_{Q \leq \Delta_h} \zeta'(Q)|\mathcal{N}_h^\Delta(Q)_n|$$

$$= \sum_{f \in \mathcal{A}_h} \zeta(f(h))|\mathcal{N}_h^\Delta(\Delta_{f(h)})_n|$$

$$= \sum_{f \in \mathcal{A}_h} \zeta(f(h))|C_{h,f,n}|.$$  

The formula over non positive integers is obtained analogously. \hfill $\Box$
Remark 5. As a corollary from Lemma 3.20 and Theorem 2.3 we have that the antipode of $HG$ is given by the cancellation-free and grouping-free expression

$$S_V(h) = \sum_{f \in A_h} (-1)^{cc(f(h))} f(h).$$

While this expression express the antipode only in term of acyclic orientations, faces of polytopes have much more apparent structure than acyclic orientations and are easier and more intuitive to work with.

4 Other Hopf monoids

In this section we use Theorem 3.12 and Theorem 3.16 to obtain similar results on other Hopf monoids, more precisely the Hopf monoids from sections 19 to 24 of [1]. The general method used here is to use the fact that these Hopf monoids can be seen as sub-monoids of (most of the times) the Hopf monoid of simple hypergraphs, and then present an interpretation of what is an acyclic orientation on these particular Hopf monoids. More precisely, Proposition 1.6 tells us that if we have a morphism $\phi : M \to SHG$ then $\chi^{M,\zeta} \circ \phi = \chi^{SHG,\zeta} \circ \phi$ for $\zeta$ any character of $SHG$. Since $\phi$ will be injective in our cases, restricting its co-domain to is image makes it an isomorphism. We then have $\chi^{M,\zeta} = \chi^{SHG,\zeta} \circ \phi^{-1} \circ \phi$ for $\zeta$ any character of $M$. What remains is to find a combinatorial interpretation of acyclic orientations on the objects of $M$.

Remark 6. Note that this is exactly how we obtained our second proof of the combinatorial interpretation of $\chi^{HG}$ Here, since we are only working on purely combinatorial objects, the morphism $\phi$ is simpler than in the case of $\Delta : HG \to HGP$.

Not only the results of this section generalize a lot of other results, but they are also obtained with a uniform approach. We provide details at the beginning of each subsection on the links between our results and already existing ones.

4.1 Simple hypergraphs

A simple hypergraph over $V$ is a set $h$ of non empty parts of $V$. $SHG$ admits a similar Hopf monoid structure to $HG$. In fact its structure maps can be defined in the same way

$$\mu_{V_1,V_2} : SHG[V_1] \otimes SHG[V_2] \to SHG[V], \quad \Delta_{V_1,V_2} : SHG[V] \to SHG[V_1] \otimes SHG[V_2]$$

where we also have $h|_V = \{ e \in h \mid e \subseteq V \}$ and $h/V = \{ e \cap V^c \mid e \not\subseteq V \}$. The difference with $HG$ is that here we are working with sets instead of multisets. So even if two edges $e_1$ and $e_1$ are such that $e_1 \cap V = e_2 \cap V = e$, there will only be one edge $e$ in $h/V$.

As this structure is very similar to the one over hypergraphs, it is of no surprise that the polynomial invariants also have similar expression.

Proposition 4.1. Let $\zeta$ be a character of $SHG$, $V$ be a finite set and $h \in SHG[V]$ a simple hypergraph. We then have:

$$\chi^\zeta_V(h)(n) = \sum_{f \in A_h} \zeta(f(h))|C_{h,f,n}|,$$

$$\chi^\zeta_V(h)(-n) = \sum_{f \in A_h} (-1)^{cc(f(h))}\zeta(f(h))|\overline{C}_{h,f,n}|.$$

If $\zeta$ is a characteristic function, then $\chi^\zeta_V(h)(n)$ is the number of strictly compatible pairs of acyclic $\zeta$-orientations of $h$ and colorings with $[n]$. Furthermore, if $\zeta$ is odd $(-1)^{|V|}\chi^\zeta_V(h)(-n)$ is the number of compatible ones. In particular, $(-1)^{|V|}\chi^\zeta_V(h)(-1)$ is the number of acyclic $\zeta$-orientations of $h$. 

Proof. Let mult be the species morphism from \( SHG \) to \( HG \) which sends a simple hypergraph on the same hypergraph. It is the right inverse of \( D \) which sends a hypergraph to its domain. Let \( \zeta \) be a character of \( SHG \), and remark that \( D \) is a morphism of Hopf monoids so \( \zeta \circ D \) is a character of \( HG \). Then Proposition \([1,6]\) with \( \phi = D \) gives us, for \( h \) a simple hypergraph

\[
\chi^{HG,\zeta \circ D}(\text{mult}(h))(n) = \chi^{SHG,\zeta}(D(\text{mult}(h)))(n) = \chi^{SHG,\zeta}(h)(n).
\]

The result follows, since \( A_{\text{mult}(h)} = A_h \). Moreover, the same goes with (strictly) compatible colorings. \( \square \)

Corollary 4.2. Let \( V \) be a finite set and \( h \in SHG[V] \) a hypergraph. Then \( \chi^{\zeta_1}(h)(n) \) is the number of colorings of \( V \) such that every edge of \( h \) has only one maximal vertex and \( (-1)^{|V|}\chi^{\zeta_1}(h)(-n) \) is the number of compatible discrete acyclic orientation of \( h \) and colorings with \( [n] \). In particular, \( (-1)^{|V|}\chi^{\zeta_1}(h)(-1) \) is the number of discrete acyclic orientations of \( h \).

4.2 Graphs

Recall that a graph over \( V \) is a hypergraph whose edges are all of cardinality 2. The species \( G \) is not stable under the restriction in \( HG \) but it still admits a close Hopf monoid structure. The structure maps are given by

\[
\begin{align*}
\mu_{V_1, V_2} : G[V_1] \otimes G[V_2] &\rightarrow G[V] \\
g_1 \otimes g_2 &\mapsto g_1 \sqcup g_2 \\
\Delta_{V_1, V_2} : G[V] &\rightarrow G[V_1] \otimes G[V_2] \\
g &\mapsto g|_{V_1} \otimes g|_{V_2},
\end{align*}
\]

where \( g|_{V_2} \) is defined in the same way as in \( HG \) and \( g|_{V_1} = \{e \in g \mid e \subseteq V_1\} \) is the contraction of \( V_2 \) to \( g \) as a hypergraph i.e. \( g|_{V_1} = g|_{V_2} \).

Usually an orientation of a graph is what we call here a discrete orientation. We hence introduce the notion of partial orientation of a graph which we think is more intuitive than the notion of admissible orientation in the case of graphs. A partial orientation of a graph \( g \) is a discrete orientation of a sub-graph \( h \) of \( g \). The partial orientations of \( g \) are in bijection with the admissible orientations of \( g \) by the map \( \kappa \) which sends a partial orientation \( f \) on the admissible orientation \( \kappa_f \) defined by \( \kappa_f(e) = f(e) \), if \( f(e) \) is defined, and \( \kappa_f(e) = e \) else. For \( f \) a partial orientation of \( g \), we denote by \( f(g) \) the sub-graph of \( g \) formed of the non-oriented edges. One can think of it as if we followed the oriented edges while erasing them behind us. This is the same than the graph obtained by leaving aside the edges of size 1 in \( \kappa_f(e) \). A partial orientation is acyclic if it is trivial (no edge is oriented) or it is not possible to complete it in order to obtain a cycle. It is equivalent to say that its image by \( \kappa \) is acyclic and we will consider \( A_g \) as the set of acyclic partial orientations on \( g \) in the rest of this subsection.

A coloring \( c \) of \( V \) is strictly compatible (resp compatible) with a partial orientation \( f \) of \( g \) if \( f(e) = \max_c(e) \) (resp \( f(e) \subseteq \max_c(e) \)) when \( f(e) \) is defined and the rest of the edges are monochromatic, \( \max_c(e) = e \).

Example 4.3. We represent here a cyclic partial orientation and a coloring with \( \{1, 2\} \) along its strictly compatible partial orientation.

For \( \zeta \) a character of \( G \), a \( \zeta \)-partial orientation of \( g \) is a partial orientation \( f \) such that \( \zeta(f(g)) \neq 0 \). A character \( \zeta \) is odd if the connected graphs on which it is not null have an odd number of vertices.
Remark 7. In the literature, the preferred notion is that of pairs of flats $F$ and discrete acyclic orientations of the quotient graph $g/F$. A flat $F$ of a graph $g$ is a sub-graph of $g$ of the form $\mu_D \circ \Delta_D(g) = g|D_1 \sqcup \ldots \sqcup g|D_n$ for $D \vdash g$. The quotient $g/F$ is then the graph obtained by deleting the edges in $F$ and merging all the vertices which shared a connected component in $F$. The bijection with acyclic orientations of $g$ is again easy to see: send a pair $(F, \alpha)$ of a flat and a discrete acyclic orientation of $g/F$ on the acyclic orientation $f$ defined by $f(e) = \alpha(e)$ if $e \notin F$ and $f(e) = e$ else.

We preferred the notion of acyclic partial orientation which is more coherent in our context. All our results over hypergraphs were expressed in terms of acyclic orientations and not pairs of flat and acyclic orientation, as is done in [5]. Still, we also give our result in term of flats for the sake of completeness. For $\xi$ a character of $G$, a $\xi$-flat of $g$ is a flat on which $\xi$ is not null.

Proposition 4.4. Let $\xi$ be a character of $G$, $V$ be a finite set and $g \in G[V]$ a graph. We then have:

$$\chi_V^\xi(g)(n) = \sum_{f \in A_g} \sum_{F \in \text{Flats}(g)} \sum_{a \in A_{g/F}} \xi(F)|C_{g,f,n}|,$$

$$\chi_V^\xi(g)(-n) = \sum_{f \in A_g} \sum_{F \in \text{Flats}(g)} \sum_{a \in A_{g/F}} \xi(F)|C_{g,F,a,n}|.$$

If $\xi$ is a characteristic function, then $\chi^\xi(g)(n)$ is the number of strictly compatible pairs of acyclic $\xi$-partial orientations of $g$ and colorings with $[n]$. Furthermore, if $\xi$ is odd $(-1)^{|V|}\chi_V^\xi(g)(-1)$ is the number of compatible ones. In particular, $(-1)^{|V|}\chi_V^\xi(g)(-1)$ is the number of acyclic $\xi$-partial orientations of $g$.

Proof. Let $HG_{\leq 2}$ be the sub-species of $HG$ of hypergraphs with edges of size at most 2. The species $HG_{\leq 2}$ is stable under product and co-product and is hence a Hopf sub-monoid. Let $s : HG_{\leq 2} \to G$ be the species morphism which forget edges of size 1. Then $s : HG_{\leq 2} \to G$ is a morphism of Hopf monoid. Let $\xi$ be a character of $G$. Proposition 1.6 gives us, for $g$ a graph,

$$\chi^{HG_{\leq 2};\xi\circ s}(g)(n) = \chi^{G;\xi}(s(g))(n) = \chi^{G;\xi}(g).$$

This concludes the proof.

A proper coloring of a graph is a coloring such that no edge has its two vertices of the same color. The chromatic polynomial of a graph $g$ is the polynomial $T_g$ such that $T_g(n)$ is the number of proper colorings with $n$ colors.

Corollary 4.5 (Proposition 18.1 in [1]). Let $g$ be a graph. Then $\chi^{G;\xi_1}(g)(n) = T_g(n)$.

Proof. From Corollary 3.14 we know that $\chi^{G;\xi_1}(g)(n)$ is the number of colorings with $[n]$ such that each edge has a unique maximal vertex. In the case of a graph, this is equivalent to saying that no edge has its two vertices of the same color, i.e. it is a proper coloring.

In particular, by evaluating $\chi^{G;\xi_1}$ on non positive integers, we recover the classical reciprocity theorem of Stanley [20].

Remark 8. As was the case for simple hypergraphs and hypergraphs, the polynomial invariants of the Hopf sub-monoid of simple graphs $SG$ of $SHG$ admit the same formulas than the polynomial invariants defined there.

4.3 Simplicial complexes

In [6] Benedetti, Hallam, and Machacek constructed a combinatorial Hopf algebra of simplicial complexes. In particular they obtained results over some polynomial invariant which we generalize in this subsection.
An abstract simplicial complex, or simplicial complex, on $V$ is a collection $C$ of subsets of $V$, called faces, such that any non-empty subset of a face is a face i.e. $I \in C$ and $\emptyset \subseteq J \subset I$ implies $J \in C$. We denote by $SC$ the species of simplicial complexes. Proposition 21.1 of [1], states that the species $SC$ of simplicial complexes is a sub-monoid of $SHG$.

Let us now give a simple lemma which will be useful in this subsection and the next one, see Figure 2 for an example of what this lemma is about.

**Lemma 4.6.** Let $V$ be a finite set, $h \in SHG[V]$ be a simple hypergraph and $f$ an acyclic orientation of $h$. Let and $e$ and $e'$ two edges of $h$ of size at least 2 such that $e' \subset e$. Then if $f(e) \cap e' \neq \emptyset$, necessarily $f(e) \cap e' = f(e')$.

![Two counter examples of Lemma 4.6](image)

We see that we have a cycle in both cases.

**Proof.** Let $e$ and $e'$ be two such edges. Suppose there exists $v \in f(e')$ such that $v \notin f(e) \cap e'$. Then $f(e') \cap f_s(e') \neq \emptyset$ and since $f(e) \cap e' \neq \emptyset$, we have either $f(e) \cap f_s(e') \neq \emptyset$ or the strict inclusions $\emptyset \subsetneq f(e) \cap f(e') \subsetneq f(e')$. This makes the sequence $e', e$ a cycle. Hence $f(e') \subsetneq f(e) \cap e'$.

Suppose now there exists $v \in f(e) \cap e'$ such that $v \notin f(e')$. Then similarly to the previous case, $f(e) \cap f_s(e') \neq \emptyset$ and either $f(e') \cap f_s(e')$ or $\emptyset \subsetneq f(e') \cap f(e) \subsetneq f(e')$. Hence $f(e) \cap e' \subsetneq f(e')$ and so $f(e') = f(e) \cap e'$.

The 1-skeleton of a simplicial complex is the simple graph formed by its faces of cardinality 2.

**Lemma 4.7.** Let be $V$ a finite set, $C \in SC[V]$ and $g$ the 1-skeleton of $C$. Then $A_C \cong A_g$.

**Proof.** The fact that every acyclic orientation of $C$ gives rise to an acyclic orientation of $g$ is clear: if $f \in A_C$, then a cycle in $f_g$ is also a cycle in $f$ and hence it is not possible. This is a bijection because from Lemma 4.6 an orientation of a simplicial complex only needs to be defined on its faces of size 2.

For $C$ a simplicial complex and $f$ an acyclic orientation of its 1-skeleton, we will also denote by $f$ the image of $f$ by this bijection.

**Proposition 4.8.** Let $\zeta$ be a character of $SC$, $V$ be a finite set, $C \in SC[V]$ be a simplicial complex and $g$ be the 1-skeleton of $C$. We then have:

$$\chi^C_V(C)(n) = \sum_{f \in A_g} \zeta(f(C))|C_{g,f,n}|;$$

$$\chi^C_V(C)(-n) = \sum_{f \in A_g} (-1)^{cc(f(g))}\zeta(f(C))|\overline{C}_{g,f,n}|.$$ 

If $\zeta$ is a characteristic function, then $\chi^C_V(C)(n)$ is the number of strictly compatible pairs of acyclic $\zeta$-orientations of $C$ and colorings with $[n]$. Furthermore, if $\zeta$ is odd $(-1)^{|V|}\chi^C_V(C)(-n)$ is the number of compatible ones. In particular, $(-1)^{|V|}\chi^C_V(C)(-1)$ is the number of acyclic $\zeta$-orientations of $C$.

**Proof.** This is just a direct application of Lemma 4.7. We just observe that while we do have $C_{C,f,n} = C_{g,f,n}$, $\overline{C}_{C,f,n} = \overline{C}_{g,f,n}$ and $cc(f(C)) = cc(f(g))$, we do not have $f(C) = f(g)$, $f(g)$ being the 1-skeleton of $f(C)$. Hence we can not replace $\zeta(f(C))$ and $\zeta$-orientation of $C$ with $\zeta(f(g))$ and $\zeta$-orientation of $g$. □
We say that a character $\zeta$ of $SC$ is downward compatible if $\zeta(C) = \zeta(g)$ for any simplicial complex $C$ and $g$ its 1-skeleton. The map which add to a graph all the edges of size 1 is an injective map from $SG$ to $SHG$. We consider $SG$ as a sub-species of $SHG$ under this map.

**Corollary 4.9.** Let $\zeta$ be a downward compatible character of $SC$, $C \in SC[V]$ be a simplicial complex and $g$ be the 1-skeleton of $C$. Then $\chi_{V}^{SHG, \zeta}(C)(n) = \chi_{V}^{SG, \zeta}(g)(n)$.

**Corollary 4.10** (Corollary 28 in [14]). Let $V$ be a finite set, $C \in SC[V]$ be a simplicial complex and $g$ be the 1-skeleton of $C$. Then $\chi_{V}^{\zeta_1}(C)$ is the chromatic polynomial of $g$.

### 4.4 Building sets

Building sets and graphical building sets have been studied in a Hopf algebraic context by Grujić in [11] where he gave results in link with the one obtained here.

Building sets were independently introduced by De Concini and Procesi in [8] and by Schmitt in [19]. A building set on $V$ is collection $B$ of subsets of $V$, called connected sets, such that if $I, J \in B$ and $I \cap J \neq \emptyset$ then $I \cup J \in B$ and for all $v \in V$, $\{v\} \in B$. We denote by $BS$ the species of building sets. By Proposition 22.3 of [1] the species $BS$ of building sets is a sub-monoid of $SHG$.

**Definition 4.11** ([9, 17]). Let $V$ be a finite set and $B$ a building set on $V$. Let $F$ be a forest of rooted trees whose vertices are labelled by the elements of a partition $\pi$ of $V$. We denote by $\leq$ the relation “is a descendent of” over $\pi$ implied by $F$ and we denote by $F_{\leq p} = \bigcup_{q \leq p} q$ for $p \in \pi$.

The forest $F$ is then a $B$-forest if it satisfies the three following conditions.

1. If $r$ is a root of $F$, then $F_{\leq r}$ is the set of vertices of the connected component containing $r$.
2. For any node $p$, $F_{\leq p} \in B$.
3. For any $k \geq 2$ and pairwise incomparable nodes $p_1, \ldots, p_k$, $\bigcup F_{p_i} \notin B$.

We denote by $B-F$ the set of $B$-forest of $B$ and call $B$-trees the $B$-forests of $B$ when $B$ is a connected.

**Lemma 4.12.** Let $V$ be a finite set and $B$ a connected building set on $V$. Then the $B$-trees also admit the following inductive definition.

- The unique $B$-tree of the building set on a singleton $\{v\}$ is the rooted tree with only its root $\{v\}$.
- If $V$ is not a singleton, let $r$ be a subset of $V$ and denote by $V_1, \ldots, V_k$ the maximal connected component of $B$ which does not intersect with $r$. For $1 \leq i \leq k$ let $B_i$ be the connected building set associated to $V_i$, which is defined by $B_i = \{e \in B \mid e \subset V_i\}$ and let $T_i$ be a $B_i$-tree. Then the tree rooted on $r$ obtained by doing the disjoint union of the $T_i$ and adding an edge between $r$ and the roots of the $T_i$ is a $B$-tree.

**Remark 9.** This lemma is in a way the same as saying that there is a bijection between $B$-forest and nested sets of $B$ ([9, 17]). The formulation given here is more adapted to what we need in the sequel.

**Proof.** We begin by showing that the tree defined in such a way are indeed $B$-trees. We do this by induction over $|V|$. If $V$ is a singleton then it is obvious. Suppose now $V$ is not a singleton and let $r, V_1, \ldots, V_k$ and $B_1, \ldots, B_k$ be as defined in the lemma.

1. To show the first item, we prove that $V_1, \ldots, V_k$ is a partition of $V \setminus r$ because then, by denoting by $r_i$ the root of $T_i$,

$$T_{\leq r} = \cup_{p \leq r} p = r \cup_{p < r} p = r \cup_{i} \cup_{p \leq r_i} p = r \cup_{i} T_{i, \leq r_i} = r \cup V_i = V,$$

where the fourth equality is obtained by induction. Suppose $V_i \cap V_j \neq \emptyset$. Then since $B$ is a building set, $V_i \cup V_j \in B$, and since neither $V_i$ nor $V_j$ intersect with $r$, their union does not
either. This contradicts the maximality of \( V_i \) and \( V_j \) and it is hence not possible. Let now \( v \in V \setminus r \) then since \( \{v\} \in V \), there exists a maximal edge connected component not intersecting \( V \) which contains \( v \).

2. We already showed the second case in the case of \( r \). Let \( p \neq r \) be a node of \( T \). Then there exists \( i \) such that \( p \) is a node of \( T_i \) and hence by induction \( T_{\leq p} \in B_i \subset B \).

3. Let be \( l \geq 2 \) and \( p_1, \ldots, p_l \) be pairwise incomparable nodes of \( T \). Suppose that \( \bigcup p_i \in B \). Then if there exists \( j \) such that all the \( p_i \) are in \( T_j \), all the \( p_i \) would be subset of \( V_j \) and we would have \( \bigcup p_i \in B_j \) which is not possible by induction. So there is at least two indices \( i, j \) such that \( p_i \in T_k \) and \( p_j \in T_k \) (the \( p_i \)'s can not be equal to \( r \) since \( r \) is comparable to every node). Then \( \bigcup p_i \) is in \( B \) but in no \( B_i \) and hence it intersects with \( r \) by definition. This is not possible since no \( p_i \) intersects with \( r \). This shows that \( \bigcup p_i \notin B \).

Let now \( T \) be a \( B \)-tree, \( r \) be its root, \( T_1, \ldots, T_k \) be its direct sub-trees and \( r_1, \ldots, r_k \) be their respective roots. For \( 1 \leq i \leq k \), let \( V_i = T_{\leq r_i} = T_{i \leq r_i} \) and \( B_i \) be the connected building set associated to \( V_i \). Then by definition of \( B \)-trees, for \( 1 \leq i \leq k \), \( T_i \) is a \( B_i \)-tree. We only need to show that the \( V_i \) are the maximal connected sets which do not intersect \( r \) to conclude. Clearly they do not intersect \( r \) since they are union of nodes and all nodes are pairwise disjoint. Suppose \( W \) is a connected set not intersecting \( r \) and such that \( V_i \subseteq W \) for one of the \( V_i \)'s. Since \( V_1, \ldots, V_k \) forms a partition of \( V \setminus r \), \( W \) must intersect with some \( V_{j_1}, \ldots, V_{j_l} \), \( j_m \neq i \) for \( 1 \leq l \leq l \). Suppose without loss of generality that \( j_m = m \) and \( i = l+1 \). Then \( W \cup_{1 \leq j \leq l} V_j \in B \) since \( B \) is a building set. But this is not possible since this is also equal to \( \bigcup_{1 \leq j \leq l+1} V_j = \bigcup_{1 \leq j \leq l+1} T_{\leq r_j} \) and the \( r_j \)'s are pairwise incomparable. Hence the \( V_i \) are maximal.

**Example 4.13.** We represented here an induction step of the construction of a \( B \)-tree. The set \( W \) is in red and the connected sets in blue are the maximal connected sets not intersecting \( W \).

![Diagram](image.png)

We also represent a \( B \)-tree obtained by beginning with the above choice:

![Diagram](image.png)

Let \( V \) be a set, \( F \) a rooted forest on \( V \) and \( c \) a coloring of \( V \). We say that \( F \) and \( c \) are (strictly) compatible if \( c \) is a (strictly) increasing map on \( F \). Respectively denote by \( C_{F,n} \) and \( \overline{C}_{F,n} \) the set of colorings strictly compatible and compatible with \( F \).

**Lemma 4.14.** Let \( V \) be a finite set and \( B \) a building set on \( V \). Then there is a bijection \( A_B \cong B \cdot F \) which preserves (strict) compatibility with colorings.

**Proof.** First remark that for \( B \) and \( B' \) two building sets over two disjoint sets we have \( A_{B \cup B'} = A_B \times A_{B'} \) and \( B \cup B' \cdot F = B \cdot F \times B' \cdot F \), so it is sufficient to prove this lemma on connected building sets and \( B \)-trees.

We construct a bijection \( b \) between \( A_B \) and \( B \cdot F \) by induction on \( |V| \). If \( V \) is a singleton then there is a unique \( B \)-tree possible and a unique acyclic orientation possible and the bijection is trivial. Suppose now \( V \) is not a singleton. Let \( f \) be an acyclic orientation of \( B \). Let \( V_1, \ldots, V_k \) be the maximal connected sets not intersecting \( f(V) \) and \( B_1, \ldots, B_k \) their associated connected building sets. Then for \( i \in [k] \), \( f|_{V_i} \) is an acyclic orientation of \( B_i \) and \( T_i = b(f|_{V_i}) \) is a \( B_i \)-tree for
1 ≤ i ≤ k. Define b(f) as the rooted tree in f(V) obtained by doing the disjoint union of the T_i and adding an edge between f(V) and the roots of the T_i.s. Then b(f) is a B-tree by Lemma 4.12.

Let now T be a B-tree and r, V_1, ..., V_k and B_1, ..., B_k as defined in Lemma 4.12. Let B_0 be the set of edges intersecting with r and let b^{-1}(T) be the orientation defined by b^{-1}(T)(e) = b^{-1}(T_i)(e) if e ∈ B_i and b^{-1}(T)(e) = r if e ∈ B_0. Suppose there exists e_1, ..., e_k a cycle in b^{-1}(T). Since r, V_1, ..., V_k is partition of V, this cycle must be entirely contained in one of the B_i. Since by induction this cycle can not be entirely in a B_i, it must be contained in B_0. But for every edge in e ∈ B_0 b^{-1}(f)(e) = r. Hence b^{-1}(f) is acyclic.

The fact that in the two preceding constructions the root of the B-tree is the image of the connected component along with the induction hypothesis enable us to conclude that the b and b^{-1} thus defined are indeed inverse functions. It also gives us that b preserves (strict) compatibility with colorings, since for a (strictly) compatible coloring c of f the color c(f(V)) is necessarily the maximal color: f(V) = max_c(V) = \{v ∈ V | c(v) is maximal in V\}.

Let B be a building set and F be a B-forest. The F-induced building set is the building set whose connected sets are obtained by taking the non empty intersection of each connected set with the maximum node possible in F. We denote it by B ∩ F. For ζ a character of BS we say that F is a ζ-B-forest if ζ(B ∩ F) ≠ 0.

**Proposition 4.15.** Let ζ be a character of BS, V be a finite set and B ∈ BS[V] be a building set. We then have:

\[
\begin{align*}
\chi^ζ_V(B)(n) &= \sum_{F ∈ B.F} \zeta(B ∩ F)|C_{F,n}|, \\
\chi^ζ_V(B)(-n) &= \sum_{F ∈ B.F} (-1)^{cc(B ∩ F)}\zeta(B ∩ F)|\overline{C}_{F,n}|.
\end{align*}
\]

If ζ is a characteristic function, then \(χ^ζ_V(B)(n)\) is the number of strictly compatible pairs of ζ-B-forests and colorings with [n]. Furthermore, if ζ is odd \((-1)^{|V|}\chi^ζ_V(B)(-n)\) is the number of compatible ones. In particular, \((-1)^{|V|}\chi^ζ_V(B)(-1)\) is the number of ζ-B-forest.

**Proof.** Again, this is a direct application of the previous lemma. We just need to remark from the construction of the bijection of Lemma 4.14 that with our definition of B ∩ F we have B ∩ F = b^{-1}(F)(B).

### 4.5 Graphs, ripping and sewing

The species SG of simple graphs admits another Hopf monoid structure than the one given in subsection 4.2. This Hopf monoid is defined in Definition 23.2 of [1] and its structure maps are given by

\[
\begin{align*}
\mu_{V_1,V_2} & : SG[V_1] \otimes SG[V_2] → SG[V] & \Delta_{V_1,V_2} & : SG[V] → SG[V_1] \otimes SG[V_2] \\
g_1 \otimes g_2 & → g_1 \sqcup g_2 & g & → g|_{V_1} \otimes g/_{V_1},
\end{align*}
\]

where g|_{V_1} is the sub-graph of g induced by V_1 and g/_{V_1} is the simple graph on V_2 with an edge between v and v' if there is a path from v to v' in which all the vertices which are not ends are in V_1. These two operations are respectively called **ripping out V_1** and **sewing through V_1**.

Here is an example of co-product with the set \(V = \{a,b,c,d,e,f\}\) and \(V_1 = \{b,c,e\}\) and \(V_2 = \{a,d,f\}\):

![Diagram](image-url)
Definition 4.16 (Definition 23.1 in [1]). Let be \( g \in SG[V] \). A tube is a subset \( W \subseteq V \) such that \( g|_W \) is connected. The set of tubes of \( g \) is a building set called \textit{graphical building set} of \( g \) and which we denote \( \text{tubes}(g) \).

By Proposition 23.3 of [1] we know that \( g \mapsto \text{tubes}(g) \) is a Hopf monoid morphism from \( SG \) to \( BS \).

Definition 4.17. Let be \( V \) be a finite set and \( g \in SG,V[V] \) a connected simple graph. A partitioning tree of \( g \) is a rooted tree whose vertices are labelled by the elements of a partition of \( V \) and which is inductively defined by the following.

- If \( V \) is a singleton, then the unique partitioning tree is the trivial tree with \( V \) as sole vertex.
- Else choose \( W \subset V \) and a partitioning tree for each connected component of \( g_{V \setminus W} \). The tree with root \( W \) and direct sub-trees these partitioning trees is then a partitioning tree of \( g \).

If \( g \) is not connected anymore, a partitioning forest of \( g \) is the disjoint union of partitioning trees of each connected component of \( g \). We denote by \( PF(g) \) the set of partitioning forest of \( g \).

Let \( g \) be a simple graph and \( F \) be a partitioning forest of \( g \). The graph \textit{ripped and sewed through} \( F \) is the graph \( g_T \) obtained by the following process. Begin with \( g_T = \emptyset \) and iteratively repeat the following: for each leaf \( V \) of \( F \), add \( g_V \) to \( g_F \) and sew \( g \) through \( V \). Delete all the leaves of \( F \) and repeat the previous operation. The process terminates when \( F \) is empty. For \( \zeta \) a character of \( SG \), we say that \( F \) is a \( \zeta \)-partitioning forest if \( \zeta(g_F) \neq 0 \).

Proposition 4.18. Let \( \zeta \) be a character of \( SG \), \( V \) be a finite set and \( g \in SG,V[V] \) be a simple graph. We then have:

\[
\chi^\zeta_V(g)(n) = \sum_{F \in PF(g)} \zeta(g_F)|C_{F,n}|,
\]
\[
\chi^\zeta_V(g)(-n) = \sum_{F \in PF(g)} (-1)^{|cc(g_F)|}\zeta(g_F)|\overline{C}_{F,n}|.
\]

If \( \zeta \) is a characteristic function, then \( \chi^\zeta(g)(n) \) is the number of strictly compatible pairs of \( \zeta \)-partitioning forests of \( g \) and colorings with \( [n] \). Furthermore, if \( \zeta \) is odd \((-1)^{|V|}\chi^\zeta(g)(-1) \) is the number of compatible ones. In particular, \((-1)^{|V|}\chi^\zeta(g)(-1) \) is the number of \( \zeta \)-partitioning forests of \( g \).

Proof. Since \( \chi^{SG,\zeta}(g)(n) = \chi^{BS,\zeta \circ \text{tubes}^{-1} \circ \text{tubes}(g)}(n) \), we only must verify that the partitioning forest of \( g \) are the \( \text{tubes}(g) \)-forest and that \( \text{tubes}(g_F) = \text{tubes}(g) \cap F \). We do this in the case where \( g \) is connected, the other case being a direct consequence from this one. This is quite straightforward: in the definition of a partitioning tree, the set \( W \) is a subset of \( V = \text{tubes}(g) \) and if \( V_1, \ldots, V_k \) be the set of vertices of the connected components of \( g_{V \setminus W} \), then there are the maximal connected sets of \( \text{tubes}(g) \) which does not intersect with \( g \). This and Lemma 4.12 show that the definition of partitioning trees of \( g \) is the same than that of \( \text{tubes}(g) \)-trees and hence they are the same objects.

Let us now show that \( \text{tubes}(g_T) = \text{tubes}(g) \cap T \). Let \( V_1, \ldots, V_k \) be the nodes of \( T \) starting with the leaves and going the way up until the root. Denote by \( D \) the partition \( V_1, \ldots, V_k \). Then by definition of \( g_T \), \( g_T = \mu_D \circ \Delta_D(g) \). Since \( g \) is a Hopf monoid morphism, we have that \( \mu_D \circ \Delta_D(g)(\mu_D) = \mu_D \circ \Delta_D(\text{tubes}(g)) = \max_{\emptyset}(\text{tubes}(g)) = b(T)(\text{tubes}(g)) = \text{tubes}(g) \cap T \), where \( \max_{\emptyset} \) is the unique strictly compatible orientation of \( D \) and \( b \) is the bijection defined in Lemma 4.14.

We have a particular interpretation of this proposition for the basic character.
Corollary 4.19 (Corollary 34 in [3]). Let $V$ be a finite set and $g \in SG[V]$ a simple graph. Then $\chi^\zeta_V(g)(n)$ is the number of colorings of $V$ with $[n]$ such that every path in $g$ with ends of the same color has a vertex of color strictly greater than the colors of the ends.

Proof. Again we only show this in the case of $g$ connected. Since $\zeta_1$ is the characteristic function of discrete elements, the $\zeta_1$-partitioning trees are the trees where every node is a singleton. Let $T$ be a partitioning tree of $g$ and $c$ a coloring strictly compatible with $T$. Let $\{v\}$ and $\{v'\}$ be two nodes of $T$ of the same color. Then they are incomparable by definition of strict compatibility. Hence we have a unique vertex of maximal color. Then each connected component of $g_{V \setminus v}$ must also have one vertex of maximal color. Hence we have an inductive way to construct a tree strictly compatible with $c$. Since this way coincides with the definition of partitioning tree whose vertices are singletons, this concludes the proof. \qed

4.6 Partitions

Recall that a partition of $V$ is a set $\{P_1, \ldots, P_k\}$ of disjoint non empty sets, called parts, such that $\sqcup_i P_i = V$. The linear species $\Pi$ admits a Hopf monoid structure with structure maps:

$$
\mu_{V_1, V_2} : \Pi[V_1] \otimes \Pi[V_2] \to \Pi[V] \\
\Delta_{V_1, V_2} : \Pi[V] \to \Pi[V_1] \otimes \Pi[V_2]
$$

where for $\pi = \{P_1, \ldots, P_k\}$, $\pi|_{P_i}$ is the partition of $V_i$ obtained by taking the intersection with $V_i$ of each part $P_i$ and forgetting the empty parts.

A cliquey graph is a simple graph which is the disjoint union of cliques. The species morphism which sends a partition $\{P_1, \ldots, P_k\}$ on the cliquey graph composed of the cliques on $P_1, \ldots, P_k$. By Proposition 24.2 of [4], this is a Hopf monoid morphism $\Pi \to SG^{\text{cop}}$ with the ripping and sewing Hopf structure on $SG$.

We say that a partition $\tau$ refines a partition $\pi$, and denote $\tau \prec \pi$ is the parts of $\pi$ are the unions of the parts of $\tau$. For $D$ a decomposition, denote by $\pi(D)$ the partition obtained by forgetting the order and the empty parts of $D$. We say that a decomposition $D$ is induced by $\pi$ is $\pi(D) \prec \pi$. For $\zeta$ a character of $\Pi$, we say that a decomposition $D$ is a $\zeta$-decomposition if $\zeta(\pi(D)) \neq 0$.

Proposition 4.20. Let $\zeta$ be a character of $\Pi$, $V$ be a finite set and $\pi \in \Pi[V]$ be a partition. We then have:

$$
\chi^\zeta_V(\pi)(n) = \sum_{\tau \prec \pi} \zeta(\tau) \ell(\tau)! \binom{n}{\ell(\tau)}
$$

If $\zeta$ is a characteristic function, then $\chi^\zeta(\pi)(n)$ is the number of $\zeta$-decomposition of size $n$ induced by $\pi$.

Proof. As done in the previous section, since $\zeta$ is multiplicative we only need to look at the trivial partition $\pi = \{V\}$. Let $g$ be the image of $\pi$ by the previously defined morphism $i.e.$ $g$ is the cliquey graph over $V$. For $W \subset V$, it is clear that $g_{V \setminus W}$ is the cliquey graph over $V \setminus W$ and the partitioning trees of $g$ are then chain trees. These are in bijection with compositions of $V$. By definition of $g_T$, for $T = T_1, \ldots, T_k$ such a tree/decomposition, $g_T$ is the disjoint union of the cliquey graphs over $V_1, \ldots, V_k$. A coloring strictly compatible with a line tree with $k$ vertices is an increasing map from $[k]$ to $[n]$. This is equal to $\binom{n}{k}$ and is the number of decomposition of size $n$ which reduce to the line tree when forgetting the empty parts. Hence we have $\chi^\zeta_V(\{V\})(n) = \sum_{P \sqcup V} \zeta(\pi(P)) \binom{n}{\ell(P)}$ which is equal to the desired formula grouping by partitions $\tau$ such that $\pi(P) = \tau$. \qed
We do not give the formula for the negative integers here since the formula for the non-negative ones is sufficiently explicit and the objects are simple enough that the notion of compatible colorings is not particularly revealing.

4.7 Set of paths

A word on $V$ is a total ordering of $V$. The paths on $V$ are the words on $V$ quotiented by the relation $w_1 \cdots w_{|V|} \sim w_{|V|} \cdots w_1$. A set of paths $\alpha$ of $V$ is a partition $\{V_1, \ldots, V_k\}$ of $V$ with a path $s_i$ on each part $V_i$ and we will write $\alpha = s_1 \ldots s_l$. We denote by $\text{Path}$ the species of set of paths. The species $\text{Path}$ of sets of paths admits a Hopf monoid structure with structure maps

$$
\mu_{V_1, V_2} : \text{Path}[V_1] \otimes \text{Path}[V_2] \to \text{Path}[V], \quad \Delta_{V_1, V_2} : \text{Path}[V] \to \text{Path}[V_1] \otimes \text{Path}[V_2]
$$

where $\mu_{V_1, V_2}$ is the set of paths obtained by replacing each occurrence of an element of $V_1$ in $\alpha$ by the separation symbol | and removing the multiplicity of |.

Example 4.21. For $V = \{a, b, c, d, e, f, g\}$ and $V_1 = \{b, c, e\}$ and $V_2 = \{a, d, f, g\}$, we have:

$$
\Delta_{V_1, V_2}(bfc|aed) = be \otimes f|a|d.
$$

For $\alpha = s_1 \ldots s_l$ a set of path, denote by $l(\alpha)$ the simple graph whose connected components are the paths induced by $s_1, \ldots, s_l$. By Proposition 25.1 of [1] we know that $\alpha \mapsto l(\alpha)$ is a morphism of Hopf monoids from $\text{Path}$ to $\text{SG}^{\text{cop}}$.

Example 4.22. For $V = \{a, b, c, d, e, f, g\}$ and $\alpha = bfc|aed$, $l(\alpha)$ is the following graph:

```
  --
 b | e | c
  --
  a | d | f
  --
```

We only give the interpretation for $\chi^{\xi_1}$ here.

Proposition 4.23 (Corollary 38 in [3]). Let $V$ be a finite set and $\alpha \in \text{Path}[V]$ be a path on $V$. Then $\chi^{\xi_1}(\alpha)(n)$ is the number of strictly compatible pairs of binary trees with $|V|$ vertices and colorings with $[n]$ and $\chi^{\xi_1}(\alpha)(-n)$ is the number of compatible pairs of binary trees with $|V|$ vertices and colorings with $[n]$. In particular $\chi^{\xi_1}(\alpha)(-1) = C_{|V|}$ where $C_n = \frac{1}{n+1} \binom{2n}{n}$ is the $n$-th Catalan number.

Proof. First remark that by definition of $\chi$, we have $\chi^{\text{SG}^{\text{cop}}} = \chi^{\text{SG}}$ and so

$$
\chi^{\text{Path}, \xi_1}(\alpha)(n) = \chi^{\text{SG}, \xi_1}(l(\alpha))(n).
$$

Fix one of the two total orderings of $V$ induced by $\alpha$ so that we can consider the left and the right of a vertex $v$ of $l(\alpha)$. Then each vertex of $l(\alpha)$ is totally characterised by the number of vertices on its left (and on its right) and hence the partitioning trees of $l(\alpha)$ with singletons for vertices are exactly the binary trees with $|V|$ vertices.

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