SOME RESULTS ON $\varphi$-CONVEX FUNCTIONS AND GEODESIC $\varphi$-CONVEX FUNCTIONS

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Abstract. As a generalization of geodesic function, in the present paper, we introduce the notion of geodesic $\varphi$-convex function and deduce some basic properties of $\varphi$-convex function and geodesic $\varphi$-convex function. We also introduce the concept of geodesic $\varphi$-convex set and $\varphi$-epigraph and investigate a characterization of geodesic $\varphi$-convex functions in terms of their $\varphi$-epigraphs.

1. Introduction

Convex sets and convex functions play an important role in the study of the theory of non-linear programming and optimization. But in many situations only convexity is not enough to provide a satisfactory solution of a problem. Hence it is necessary to generalize the concept of convexity notion. Again, due to the curvature and torsion of a Riemannian manifold highly nonlinearity appears in the study of convexity on such a manifold. Geodesics are length minimizing curves, and the notion of geodesic convex function arises naturally on a complete Riemannian manifold and such a concept is investigated recently in [4, 12].

In 2016 Eshaghi Gordji et. al. [3] defined the notion of the $\varphi$-convex function and deduced Jensen and Hadamard type inequalities for such functions. In the present paper, we have deduced some other properties of $\varphi$-convex functions. Again, generalizing the concept of $\varphi$-convex function, we have introduced the notion of geodesic $\varphi$-convex function on a complete Riemannian manifold and showed its existence by a proper example (see, Ex: 2.1). Convex sets on a Riemannian manifold have been generalized in different ways such as geodesic $E$-convex function [6], geodesic semi-$E$-convex function [7], geodesic semi $E$-$b$-convex functions [8] etc. We have also introduced a new class of sets, called, geodesic $\varphi$-convex sets on a complete Riemannian

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manifold.

The paper is organized as follows. Section 2 deals with the rudimentary facts of convex functions and convex sets. Section 3 is devoted to the study of some properties of mean-value like inequalities of \( \varphi \)-convex functions. In section 4, we study some properties of geodesic \( \varphi \)-convex functions on a complete Riemannian manifold and prove that such notion is invariant under a diffeomorphism. We also study sequentially upper bounded functions on a complete Riemannian manifold and prove that supremum of such sequence of functions is a geodesic \( \varphi \)-convex function. We also obtain a condition for which a geodesic \( \varphi \)-convex function has a local minimum. The last section is concerned with \( \varphi \)-epigraphs on a complete Riemannian manifold and obtain a characterization of geodesic \( \varphi \)-convex functions in terms of their \( \varphi \)-epigraphs
(see, Theorem. 5.1).

2. Preliminaries

In this section, we recall some definition and known results of convex and \( \varphi \)-convex functions and also some results about Riemannian manifolds which will be used throughout the paper. For the detailed discussion of Riemannian manifold we refer \[9\]. Let \( I = [a, b] \) be an interval in \( \mathbb{R} \) and \( \varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) be a bifunction \[3\].

**Definition 2.1.** A function \( f : I \to \mathbb{R} \) is called convex if for any two points \( x, y \in I \) and \( t \in [0, 1] \)

\[
f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y).
\]

**Definition 2.2.** \[3\] A function \( f : I \to \mathbb{R} \) is called \( \varphi \)-convex if

\[
f(tx + (1 - t)y) \leq f(y) + t\varphi(f(x), f(y)),
\]

for all \( x, y \in I \) and \( t \in [0, 1] \).

Especially, if \( \varphi(x, y) = x - y \) then \( \varphi \)-convex function reduces to a convex function.

**Definition 2.3.** \[3\] The function \( \varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is called nonnegatively homogeneous if \( \varphi(\lambda x, \lambda y) = \lambda \varphi(x, y) \) for all \( x, y \in \mathbb{R} \) and for all \( \lambda \geq 0 \), and called additive if \( \varphi(x_1 + x_2, y_1 + y_2) = \varphi(x_1, y_1) + \varphi(x_2, y_2) \) for all \( x_1, x_2, y_1, y_2 \in \mathbb{R} \). If \( \varphi \) is both nonnegatively homogeneous and additive then \( \varphi \) is called nonnegatively linear.
Recently, Hanson’s [5] generalized convex sets and introduced the concept of invex sets, defined follows.

**Definition 2.4.** [5] A set $K \subseteq \mathbb{R}$ is said to be invex if there exists a function $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n$ such that

$$x, y \in K, \lambda \in [0, 1] \Rightarrow y + \lambda \eta(x, y) \in K.$$  

**Definition 2.5.** [3] Let $K \subseteq \mathbb{R}$ be an invex set with respect to $\eta$. A function $f : K \to \mathbb{R}$ is said to be $G$-preinvex with respect to $\eta$ and $\psi$ if

$$f(y + t\eta(x, y)) \leq f(y) + t\psi(f(x), f(y)),$$

for all $x, y \in K, t \in [0, 1]$.

Let $(M, g)$ be a complete Riemannian manifold with Riemannian metric $g$ and Levi-Civita connection $\nabla$. We recall that a geodesic is a smooth curve $\alpha$ whose tangent is parallel along the curve $\alpha$, that is, $\alpha$ satisfies the equation $\nabla_{\dot{\alpha}(t)} \frac{d\alpha(t)}{dt} = 0$. We shall denote the geodesic arc connecting $x \in M$ and $y \in M$ by $\alpha_{xy} : [0, 1] \to M$ such that $\alpha_{xy}(0) = x \in M, \alpha_{xy}(1) = y \in M$.

**Definition 2.6.** [12] A non-empty subset $A$ of $M$ is called totally convex, if it contains every geodesic $\alpha_{xy}$ of $M$, whose end points $x$ and $y$ belong to $A$.

**Definition 2.7.** [12] If $A$ is a totally convex set of $M$, then $f : A \to \mathbb{R}$ is called geodesically convex if

$$f(\alpha_{xy}(t)) \leq (1 - t)f(x) + tf(y)$$

holds for every $x, y \in A$ and $t \in [0, 1]$. If the inequality is strict then $f$ is called strictly geodesically convex.

Generalizing the notion of $\varphi$-convex function in Riemannian manifold, we introduce the concept of geodesic $\varphi$-convex function, defined as follows:

**Definition 2.8.** If $A$ is a totally convex set in $M$, then $f : A \to \mathbb{R}$ is called geodesic $\varphi$-convex if

$$f(\alpha_{xy}(t)) \leq f(x) + t\varphi(f(y), f(x)),$$

holds for every $x, y \in A$ and $t \in [0, 1]$. 

Remark. We note that if $f$ is differentiable, then $f$ is called geodesic $\varphi$-convex if and only if

$$df_x \dot{\alpha}_{xy} \leq \varphi(f(y), f(x)),$$

where $df_x$ is the differential of $f$ at the point $x \in A \subset M$ and dot denotes the differentiation with respect to $t$.

If the above inequality is strict then $f$ is called strictly geodesic $\varphi$-convex function.

Geodesic convex functions are obviously geodesic $\varphi$-convex, where $\varphi(x, y) = x - y$. In the following example we show that geodesic $\varphi$-convex function on $M$ need not be geodesic convex.

Example 2.1. Let $M = \mathbb{R} \times S^1$ and define $f : M \to \mathbb{R}$ by $f(x, s) = x^3$. Then $f$ is not geodesic convex in $M$. Now define $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $\varphi(x, y) = x^3 - y^3$. We note that for any two points $(x, s_1)$ and $(y, s_2)$ the geodesic joining them is a portion of a helix of the form $\alpha_{xy}(t) = (tx + (1 - t)y, e^{i(\theta_1 + (1-t)\theta_2)})$, for $0 \leq t \leq 1$, where $e^{i\theta_1} = s_1$ and $e^{i\theta_2} = s_2$ for some $\theta_1, \theta_2 \in [0, 2\pi]$. Hence

$$f(\alpha(t)) = (tx + (1-t)y)^3$$

$$= t^3(x^3 - 3x^2y + 3xy^2 - y^3) + t^2(3x^2y - 6xy^2 + 3y^3) + t(3xy^2 - 3y^3) + y^3$$

$$\leq y^3 + t(x^3 - y^3)$$

$$= f(y, s_2) + t\varphi(f(x, s_1), f(y, s_2)).$$

This proves that $f$ is geodesic $\varphi$-convex in $M$.

Barani et. al. [1] extended the work of [5] to Riemannian manifold and defined geodesic invex sets and geodesic $\eta$-preinvex functions.

Definition 2.9. [1] Let $M$ be an $n$-dimensional Riemannian manifold and $\eta : M \times M \to TM$ be a function such that for every $x, y \in M$, $\eta(x, y) \in T_y M$. A non-empty subset $S$ of $M$ is said to be geodesic invex set with respect to $\eta$ if for every $x, y \in S$, there exists a unique geodesic $\alpha_{xy} : [0, 1] \to M$ such that

$$\alpha_{xy}(0) = y, \; \dot{\alpha}_{xy}(0) = \eta(x, y), \; \alpha_{xy}(t) \in S, \; \forall \; t \in [0, 1].$$

Definition 2.10. [1] Let $S$ be an open subset of $M$ which is geodesic invex with respect to $\eta : M \times M \to TM$. A function $f : S \to \mathbb{R}$ is said to be geodesic $\eta$-preinvex if

$$f(\alpha_{xy}(t)) \leq tf(x) + (1 - t)f(y).$$
for every $x, y \in S$, $t \in [0, 1]$.

We generalize above definition as follows.

**Definition 2.11.** Let $S \subseteq M$ be geodesic invex with respect to $\eta$. A function $f : S \to \mathbb{R}$ is said to be geodesic $\varphi$-preinvex with respect to $\eta$ and $\varphi$, if

$$f(\alpha_{xy}(t)) \leq f(y) + t\varphi(f(x), f(y)),$$

for every $x, y \in S$, $t \in [0, 1]$.

### 3. Some properties of $\varphi$-convex functions

The notion of $\varphi$-convex functions have been studied in [3], where Jensen type inequality and Hermite-Hadamard type inequality have been deduced for $\varphi$-convex function. This section deals with some properties of $\varphi$-convex functions. Let $f : I \to \mathbb{R}$ be a $\varphi$-convex function. For any two points $x_1$ and $x_2$ in $I$ with $x_1 < x_2$, each point $x$ in $(x_1, x_2)$ can be expressed as

$$x = tx_1 + (1-t)x_2,$$

where $t = \frac{x_2 - x}{x_2 - x_1}$.

Hence a function $f$ is $\varphi$-convex if

$$f(x) \leq f(x_2) + \frac{x_2 - x}{x_2 - x_1}\varphi(f(x_1), f(x_2)),$$

for $x_1 < x < x_2$ in $I$.

Rearranging the above terms we get

$$\frac{f(x_2) - f(x)}{x_2 - x} \geq \frac{\varphi(f(x_1), f(x_2))}{x_1 - x_2} \text{ for } x_1 < x < x_2 \text{ in } I. \quad (3.1)$$

So, a function $f$ is $\varphi$-convex if it satisfies the inequality (3.1).

**Theorem 3.1.** If $f : I \to \mathbb{R}$ is differentiable and $\varphi$-convex in $I$ and $f(x_1) \neq f(x_2)$ then there exist $\xi, \eta \in (x_1, x_2) \subset I$ such that $f'(\xi) \geq \frac{\varphi(f(x_1), f(x_2))}{f(x_1) - f(x_2)} f'(\eta) \geq f'(\eta)$.

**Proof.** Since $f$ is $\varphi$-convex, from (2.1) we get

$$\frac{f(x_2) - f(x)}{x_2 - x} \geq \frac{\varphi(f(x_1), f(x_2))}{x_1 - x_2} \text{ for } x \in (x_1, x_2) \subset I,$$

$$= \frac{\varphi(f(x_1), f(x_2)) f(x_2) - f(x_1)}{f(x_1) - f(x_2)} \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Now applying mean-value theorem, the above inequality yields
\[ f'(\xi) \geq \frac{\varphi(f(x_1), f(x_2))}{f(x_1) - f(x_2)} f'(\eta) \tag{3.2} \]

for some \( \xi \in (x_1, x) \subset (x_1, x_2) \) and \( \eta \in (x_1, x_2) \). Again by setting \( t = 1 \) in (1.1) we get \( \varphi(f(x_1), f(x_2)) \geq f(x_1) - f(x_2) \). Hence (3.2) implies that

\[ f'(\xi) \geq \frac{\varphi(f(x_1), f(x_2))}{f(x_1) - f(x_2)} f'(\eta) \geq f'(\eta). \tag{3.3} \]

\[ \square \]

**Theorem 3.2.** Let \( f : I \rightarrow \mathbb{R} \) be a differentiable \( \varphi \)-convex function. Then for each \( x, y, z \in I \) such that \( x < y < z \) the following inequality holds:

\[ f'(y) + f'(z) \leq \frac{\varphi(f(x), f(y)) + \varphi(f(y), f(z))}{x - z}. \]

**Proof.** Since \( f \) is \( \varphi \)-convex in each interval \([x, y]\) and \([y, z]\), hence

\[ f(tx + (1 - t)y) \leq f(y) + t\varphi(f(x), f(y)), \]

and

\[ f(ty + (1 - t)z) \leq f(z) + t\varphi(f(y), f(z)). \]

From the above two inequalities, we obtain

\[ \frac{f(tx + (1 - t)y) - f(y) + f(ty + (1 - t)z) - f(z)}{t} \leq \varphi(f(x), f(y)) + \varphi(f(y), f(z)). \]

Now setting \( t \rightarrow 0 \), we have

\[ f'(y)(x - y) + f'(z)(y - z) \leq \varphi(f(x), f(y)) + \varphi(f(y), f(z)). \]

Again \( z > y \) implies that \( x - z < x - y \) and \( x < y \) implies that \( x - z < y - z \). Thus we get \( (x - z)(f'(y) + f'(z)) \leq f'(y)(x - y) + f'(z)(y - z). \)

Hence we obtain

\[ f'(y) + f'(z) \leq \frac{\varphi(f(x), f(y)) + \varphi(f(y), f(z))}{x - z}. \]

\[ \square \]
4. Properties of geodesic $\varphi$-convex functions

Theorem 4.1. Let $A$ be a totally convex set in $M$. Then a function $f : A \to \mathbb{R}$ is geodesic $\varphi$-convex if and only if for each $x, y \in A$, the function $g_{xy} = f \circ \alpha_{xy}$ is $\varphi$-convex on $[0, 1]$.

Proof. Let us suppose that $g_{xy}$ is $\varphi$-convex on $[0, 1]$. Then for each $t_1, t_2 \in [0, 1]$,

$$g_{xy}(st_2 + (1 - s)t_1) \leq g_{xy}(t_1) + s\varphi(g_{xy}(t_2), g_{xy}(t_1)), \forall s \in [0, 1].$$

Now taking $t_1 = 0$ and $t_2 = 1$ we get

$$g_{xy}(s) \leq g_{xy}(0) + s\varphi(g_{xy}(1), g_{xy}(0)),$$

i.e.,

$$f(\alpha_{xy}(s)) \leq f(x) + s\varphi(f(y), f(x)), \forall x, y \in A \text{ and } \forall s \in [0, 1].$$

Conversely, let $f$ be a $\varphi$-convex function. Now restricting the domain of $\alpha_{xy}$ to $[t_1, t_2]$, we get a geodesic joining $\alpha_{xy}(t_1)$ and $\alpha_{xy}(t_2)$. Now reparametrize this restriction,

$$\gamma(s) = \alpha_{xy}(st_2 + (1 - s)t_1), \ s \in [0, 1].$$

Since $f(\gamma(s)) \leq f(\gamma(0)) + s\varphi(f(\gamma(1)), f(\gamma(0)))$,

i.e., $f(\alpha_{xy}(st_2 + (1 - s)t_1)) \leq f(\alpha_{xy}(t_1)) + s\varphi(f(\alpha_{xy}(t_2)), f(\alpha_{xy}(t_0)))$,

hence

$$g_{xy}(st_2 + (1 - s)t_1) \leq g_{xy}(t_1) + s\varphi(g_{xy}(t_2), g_{xy}(t_1)).$$

So, $g_{xy}$ is $\varphi$-convex in $[0, 1]$. $\blacksquare$

Theorem 4.2. Let $A \subset M$ be a totally convex set, $f : A \to \mathbb{R}$ geodesic convex function and $g : I \to \mathbb{R}$ be non-decreasing $\varphi$-convex function with $\text{Range}(f) \subseteq I$. Then $g \circ f : A \to \mathbb{R}$ is a geodesic $\varphi$-convex function.

Proof. Since $f$ is geodesic convex in $A$, hence for any $x, y \in A$

$$f(\alpha_{xy}(t)) \leq (1 - t)f(x) + tf(y),$$

where $\alpha_{xy} : [0, 1] \to M$ is a geodesic arc connecting $x$ and $y$. Now as $g$ is non-decreasing and $\varphi$-convex, we have

$$g \circ f(\alpha_{xy}(t)) \leq g((1 - t)f(x) + tf(y)) \leq g \circ f(x) + t\varphi(g \circ f(y), g \circ f(x)).$$

Hence $g \circ f$ is geodesic $\varphi$-convex in $A$. $\blacksquare$
Remark. In Theorem 4.2, if $g$ is strictly $\varphi$-convex, then $g \circ f$ is also strictly geodesic $\varphi$-convex.

**Theorem 4.3.** Let $f_i : A \to \mathbb{R}$ be geodesic $\varphi$-convex functions for $i = 1, 2, \cdots, n$ and $\varphi$ be nonnegatively linear. Then for $\lambda_i \geq 0$, $i = 1, 2, \cdots, n$, the function $f = \sum_{i=1}^{n} \lambda_i f_i : A \to \mathbb{R}$ is geodesic $\varphi$-convex.

**Proof.** Let $x, y \in A$ then

$$f(\alpha_{xy}) = \sum_{i=1}^{n} \lambda_i f_i(\alpha_{xy})$$

$$\leq \sum_{i=1}^{n} \lambda_i [f_i(x) + t\varphi(f(y), f(x))]$$

$$= \sum_{i=1}^{n} [\lambda_i f_i(x) + t\varphi(\lambda_i f(y), \lambda_i f(x))]$$

$$= f(x) + t\varphi(f(y), f(x)).$$

Hence $f$ is geodesic $\varphi$-convex in $A$. $\square$

Let $M$ and $N$ be two complete Riemannian manifolds and $\nabla$ be the Levi-Civita connection of $M$. If $F : M \to N$ be a diffeomorphism, then $F_*\nabla = \nabla_1$ is an affine connection of $N$. Hence if $\gamma$ is a geodesic in $(M, \nabla)$, then $F \circ \gamma$ is also a geodesic in $(N, \nabla_1)$ [12, pp. 66].

**Theorem 4.4.** Let $f : A \to \mathbb{R}$ be a geodesic $\varphi$-convex function. If $F : (M, \nabla) \to (N, \nabla_1)$ is a diffeomorphism then $f \circ F^{-1} : F(A) \to \mathbb{R}$ is a geodesic $\varphi$-convex function, where $\nabla_1 = F_*\nabla$.

**Proof.** Let $x, y \in A$ and $\alpha_{xy}$ be a geodesic arc joining $x$ and $y$. Since $F$ is a diffeomorphism, hence $F(A)$ is totally geodesic [12] and $F \circ \alpha_{xy}$ is a geodesic joining $F(x)$ and $F(y)$. Now we have

$$(f \circ F^{-1})(F(\alpha_{xy})) = f(\alpha_{xy})$$

$$\leq f(x) + t\varphi(f(y), f(x))$$

$$= (f \circ F^{-1})(F(x)) + t\varphi((f \circ F^{-1})(F(y)), (f \circ F^{-1})(F(x)))$$

i.e., $f \circ F^{-1}$ is geodesic $\varphi$-convex on $F(A)$. $\square$
Theorem 4.5. Let \( f : B \to \mathbb{R} \) be a geodesic \( \varphi \)-convex function and \( \varphi \) be bounded from above on \( f(B) \times f(B) \) with an upper bound \( M_\varphi \), where \( B \) is a convex set in \( \mathbb{R}^n \) with \( \text{Int}(B) \neq \emptyset \). Then \( f \) is continuous on \( \text{Int}(B) \).

Proof. Let \( a \in \text{Int}(B) \). Then there exists an open ball \( B(a, h) \subset \text{Int}(B) \) for some \( h > 0 \). Now choose \( r \) \((0 < r < h)\) such that the closed ball \( \tilde{B}(a, r + \epsilon) \subset B(a, h) \) for some arbitrary small \( \epsilon > 0 \). Choose any \( x, y \in \tilde{B}(a, r) \). Set \( z = y + \frac{\epsilon}{\|y - x\|}(y - x) \) and \( t = \frac{\|y - x\|}{\epsilon + \|y - x\|} \). Then it is obvious that \( z \in \tilde{B}(a, r + \epsilon) \) and \( y = tz + (1 - t)x \). Thus \( f(y) \leq f(x) + t\varphi(f(z), f(x)) \leq f(x) + tM_\varphi \).

Hence we get
\[
f(y) - f(x) \leq tM_\varphi \leq \frac{\|y - x\|}{\epsilon}M_\varphi = K\|y - x\|
\]
where \( K = M_\varphi/\epsilon \). And if we interchange the position of \( x \) and \( y \), then we also get \( f(x) - f(y) \leq K\|y - x\| \). Hence \( |f(x) - f(y)| \leq K\|y - x\| \). Since \( \tilde{B}(a, r) \) is arbitrary hence \( f \) is continuous on \( \text{Int}(B) \).

Definition 4.1. A bifunction \( \varphi : \mathbb{R}^2 \to \mathbb{R} \) is called sequentially upper bounded if
\[
\sup_n \varphi(x_n, y_n) \leq \varphi(\sup_n x_n, \sup_n y_n)
\]
for any two bounded real sequences \( \{x_n\} \) and \( \{y_n\} \).

Example 4.1. The functions \( \varphi(x, y) = x + y \) and \( \psi(x, y) = xy, \ \forall (x, y) \in \mathbb{R}^2 \) are sequentially upper bounded functions.

Proposition 4.6. Let \( A \subseteq M \) be totally convex set and \( \{f_n\}_{n \in \mathbb{N}} \) be a non-empty family of geodesic \( \varphi \)-convex functions on \( A \), where \( \varphi \) is sequentially upper bounded. If \( \sup_n f_n(x) \) exists for each \( x \in A \) then \( f(x) = \sup_n f_n(x) \) is also a geodesic \( \varphi \)-convex function.

Proof. Let \( x, y \in A \) and \( \alpha_{xy} : [0, 1] \to M \) be a geodesic connecting \( x \) and \( y \). Then
\[
f(\alpha_{xy}(t)) = \sup_n f_n(\alpha_{xy}(t))
\]
\[
\leq \sup_n (f_n(x) + t\varphi(f_n(y), f_n(x)))
\]
\[
\leq \sup_n f_n(x) + t\sup_n \varphi(f_n(y), f_n(x))
\]
\[
\leq \sup_n f_n(x) + t\varphi(\sup_n f_n(y), \sup_n f_n(x))
\]
\[
= f(x) + t\varphi(f(y), f(x))
\]
Hence \( f \) is a geodesic \( \varphi \)-convex function on \( A \). \( \square \)
Theorem 4.7. Let $f : A \to \mathbb{R}$ be a geodesic $\varphi$-convex function. If $f$ has local minimum at $x_0 \in \text{Int}(A)$, then $\varphi(f(x), f(x_0)) \geq 0$ for all $x \in A$.

Proof. Since $x_0 \in \text{Int}(A)$, $B(x_0, r) \subset A$ for some $r > 0$. Take any point $x \in A$. Then there exists a geodesic $\alpha_{x_0x} : [0, 1] \to M$ belonging to $A$ and $f(\alpha_{x_0x}(t)) \leq f(x_0) + t\varphi(f(x), f(x_0))$. Now there exists $\xi$ such that $0 < \xi \leq 1$ and $\alpha_{x_0x}(t) \in B(x_0, r)$, $\forall t \in [0, \xi]$.

As $f$ has local minimum at $x_0$, we obtain

$$f(x_0) \leq f(\alpha_{x_0x}(\xi)) \leq f(x_0) + \xi\varphi(f(x), f(x_0)).$$

Thus,

$$\varphi(f(x), f(x_0)) \geq 0, \forall x \in A.$$  \[\square\]

Theorem 4.8. Let $f : A \to \mathbb{R}$ be geodesic $\varphi$-convex and $\varphi$ be bounded from above on $f(A) \times f(A)$ with an upper bound $M_\varphi$. Then $f$ is continuous on $\text{Int}(A)$.

Proof. Let $a \in \text{Int}(A)$ and $(U, \psi)$ be a chart containing $a$. Since $\psi$ is a differmorphism so using Theorem 4.5, $f \circ \psi^{-1} : \psi(U \cap \text{Int}(A)) \to \mathbb{R}$ is also $\varphi$-convex and hence continuous. Therefore, we get $f = f \circ \psi^{-1} \circ \psi : U \cap \text{Int}(A) \to \mathbb{R}$ is continuous. Since $a$ is arbitrary, $f$ is continuous on $\text{Int}(A)$. \[\square\]

Proposition 4.9. Let $\{\varphi_n : n \in \mathbb{N}\}$ be a collection of bifunctions such that $f : A \to \mathbb{R}$ is geodesic $\varphi_n$-convex functions for each $n$. If $\varphi_n \to \varphi$ as $n \to \infty$ then $f$ is also a geodesic $\varphi$-convex function.

Proposition 4.10. Let $\{\varphi_n : n \in \mathbb{N}\}$ be a collection of bifunctions such that $f : A \to \mathbb{R}$ is geodesic $\sum_{i=1}^{n} \varphi_i$-convex function for each $n$. If $\sum_{n \in \mathbb{N}} \varphi_n$ converges to $\varphi$ then $f$ is also a geodesic $\varphi$-convex function.

Theorem 4.11. If $f : A \to \mathbb{R}$ is strictly geodesic $\varphi$-convex with $\varphi$ as antisymmetric function, then $df_x \hat{\alpha}_{xy} \neq df_y \hat{\alpha}_{xy}$ for any $x, y \in A, x \neq y$.

Proof. Let $\alpha_{xy} : [0, 1] \to M$ be a geodesic with starting point $x$ and ending point $y$. Now define $\alpha_{yx}(t) = \alpha_{xy}(1-t), \ t \in [0, 1]$. Then $\alpha_{yx}$ is a geodesic with starting point $y$ and
ending point $x$ and $df_y \dot{\alpha}_{yx} = -df_y \dot{\alpha}_{xy}$.

On contrary, suppose that $df_x \dot{\alpha}_{xy} = df_y \dot{\alpha}_{xy}$. Now since $f$ is geodesic $\varphi$-convex, we get

$$df_x \dot{\alpha}_{xy} < \varphi(f(y), f(x))$$

and

$$df_y \dot{\alpha}_{xy} < \varphi(f(x), f(y)).$$

Since $df_y \dot{\alpha}_{yx} = -df_y \dot{\alpha}_{xy}$, we get

$$-df_y \dot{\alpha}_{xy} < \varphi(f(x), f(y)).$$

Using antisymmetry property of $\varphi$, we obtain

$$df_y \dot{\alpha}_{xy} > \varphi(f(y), f(x)).$$

Since $df_x \dot{\alpha}_{xy} = df_y \dot{\alpha}_{xy}$, we get

$$df_x \dot{\alpha}_{xy} > \varphi(f(y), f(x)).$$

Hence $\varphi(f(y), f(x)) < \varphi(f(y), f(x))$, which is a contradiction. Hence $df_x \dot{\alpha}_{xy} \neq df_y \dot{\alpha}_{xy}$. $\square$

**Theorem 4.12.** Let $S \subseteq M$ be a geodesic invex set with respect to $\eta$. Suppose that $f : S \to \mathbb{R}$ is geodesic $\varphi$-convex and $g : I \to \mathbb{R}$ is a non-decreasing $G$-preinvex function with respect to $\phi$ and $\psi$ such that range $(f) \subseteq I$. Then $gof : I \to \mathbb{R}$ is a geodesic $\psi$-convex function.

**Proof.** Let $x, y \in S$. Since $f$ is a geodesic $\varphi$-convex function, we have

$$f(\alpha_{xy}(t)) \leq f(y) + t\varphi(f(x), f(y)).$$

Since $g$ is non-decreasing $G$-preinvex function, we have

$$g(f(\alpha_{xy}(t))) \leq g(f(y) + t\varphi(f(x), f(y)))$$

$$\leq g(f(y)) + t\psi(g(f(x)), g(f(y))).$$

$\square$
5. \( \varphi \)-Epigraphs

In this section we have introduced the notion of \( \varphi \)-epigraphs on complete Riemannian manifolds and obtained a characterization of geodesic \( \varphi \)-convex function in terms of their \( \varphi \)-epigraphs.

**Definition 5.1.** A set \( B \subseteq M \times \mathbb{R} \) is said to be geodesic \( \varphi \)-convex if and only if for any two points \( (x, \alpha), (y, \beta) \in B \) imply

\[
(\alpha_{xy}(t), \alpha + t\varphi(\beta, \alpha)) \in B, \quad 0 \leq t \leq 1.
\]

If \( A \) is a totally geodesic convex subset of \( M \) and \( K \) is an invex subset of \( \mathbb{R} \) with respect to \( \varphi \), then \( A \times K \) is geodesic \( \varphi \)-convex.

Now the \( \varphi \)-epigraph of \( f \) is defined by

\[
E_{\varphi}(f) = \{(x, \alpha) \in M \times \mathbb{R} : f(x) \leq \alpha\}.
\]

**Definition 5.2.** A function \( \varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is non-decreasing if \( x_1 \leq x_2 \) and \( y_1 \leq y_2 \) implies \( \varphi(x_1, y_1) \leq \varphi(x_2, y_2) \) for \( x_1, x_2, y_1, y_2 \in \mathbb{R} \).

**Theorem 5.1.** Let \( A \) be a totally geodesic convex subset of \( M \) and \( \varphi \) be non-decreasing. Then \( f : A \to \mathbb{R} \) is geodesic \( \varphi \)-convex if and only if \( E_{\varphi}(f) \) is a geodesic \( \varphi \)-convex set.

**Proof.** Suppose that \( f : A \to \mathbb{R} \) is a geodesic \( \varphi \)-convex. Let \( (x, \alpha), (y, \beta) \in E_{\varphi}(f) \). Then \( f(x) \leq \alpha \) and \( f(y) \leq \beta \). Since \( f \) is geodesic \( \varphi \)-convex, hence

\[
f(\alpha_{xy}(t)) \leq f(x) + t\varphi(f(y), f(x)),
\]

for any geodesic \( \alpha_{xy} : [0, 1] \to M \) connecting \( x \) and \( y \). Since \( \varphi \) is non-decreasing, so

\[
f(\alpha_{xy}(t)) \leq \alpha + t\varphi(\beta, \alpha), \quad \forall t \in [0, 1].
\]

Hence

\[
(\alpha_{xy}(t), \alpha + t\varphi(\beta, \alpha)) \in E_{\varphi}(f) \quad \forall t \in [0, 1].
\]

That is, \( E_{\varphi}(f) \) is a geodesic \( \varphi \)-convex set.

Conversely, assume that \( E_{\varphi}(f) \) is a geodesic \( \varphi \)-convex set. Let \( x, y \in A \). Then \( (x, f(x)), (y, f(y)) \in E_{\varphi}(f) \). Hence for any \( t \in [0, 1] \),

\[
(\alpha_{xy}(t), f(x) + t\varphi(f(y), f(x))) \in E_{\varphi}(f),
\]
which implies that
\[ f(\alpha_{xy}(t)) \leq f(x) + t\varphi(f(y), f(x)), \]
for any \( x, y \in A \) and for any geodesic \( \alpha_{xy} : [0, 1] \to M \). So, \( f \) is a geodesic \( \varphi \)-convex function. \( \square \)

**Theorem 5.2.** Let \( A_i, i \in \Lambda \), be a family of geodesic \( \varphi \)-convex sets. Then their intersection \( A = \bigcap_{i \in \Lambda} A_i \) is also a geodesic \( \varphi \)-convex set.

**Proof.** Let \( (x, \alpha), (y, \beta) \in A \). Then for each \( i \in \Lambda \), \((x, \alpha), (y, \beta) \in A_i \). Since each \( A_i \) is geodesic \( \varphi \)-convex for each \( i \in \Lambda \), hence
\[
(\alpha_{xy}(t), \alpha + t\varphi(\beta, \alpha)) \in A_i, \quad \forall t \in [0, 1].
\]
This implies
\[
(\alpha_{xy}(t), \alpha + t\varphi(\beta, \alpha)) \in \bigcap_{i \in \Lambda} A_i, \quad \forall t \in [0, 1].
\]
Hence \( A \) is a geodesic \( \varphi \)-convex set. \( \square \)

As a direct consequence of Theorem 5.1 and Theorem 5.2, we get the following corollary.

**Corollary.** Let \( \{f_i\}_{i \in \Lambda} \) be a family of geodesic \( \varphi \)-convex functions defined on a totally convex set \( A \subseteq M \) which are bounded above and \( \varphi \) be non-decreasing. If the \( \varphi \)-epigraphs \( E_{\varphi}(f_i) \) are geodesic \( \varphi \)-convex sets, then \( f = \sup_{i \in \Lambda} f_i \) is also a geodesic \( \varphi \)-convex function on \( A \).

**References**

[1] Barani, A. and Pouryayevali, M.R., *Invex sets and preinvex functions on Riemannian manifolds*, J. Math. Anal. Appl., 328, (2007), 767-779.

[2] Delavar, M. R. and Dargomir, S. S., *On \( \eta \)-convexity*, Math. Inequal. Appl., 20(1), (2017), 203-216.

[3] Eshaghi Gordji, M., Rostamian Delavar, M. and De La Sen, M., *On \( \varphi \)-convex functions*, J. Math. Inequal., 10(1), (2016), 173-183.

[4] Greene, R. E. and Shiohama, G., *Convex functions on complete noncompact manifolds: Topological structure*, Invent. Math., 63, (1981), 129-157.

[5] Hanson, M.A., *On sufficiency of the Kuhn-Tucker conditions*, J. Math. Anal. Appl., 80 (1981) 545-550.

[6] Iqbal, A., Ali, S. and Ahmad, I., *On Geodesic E-Convex sets, Geodesic E-Convex Functions and E-Epigraphs*, J. Optim. Theory. Appl., 155(1), (2012), 239-251.

[7] Iqbal, A., Ahmad, I. and Ali, S., *Some properties of geodesic semi-E-convex functions*, Nonlinear Anal., 74, (2011), 6805-6813.

[8] Kilicman, A. and Saleh,W., *Some properties of geodesic semi E-b-vex functions*, Open Math., 13, (2015), 795-804.
[9] Lang, S., *Fundamentals of Differential Geometry*, Grad. Texts in Math., Vol. 191, Springer, New York, 1994.

[10] Lindner, W., *Some properties of the set of discontinuity points of a monotonic function of several variables*, Folia. Math., 11(1), (200), 53-57.

[11] Mohan, S. R. and Neogy, S. K., *On Invex Sets and Preinvex Functions*, J. Math. Anal. Appl., 189(1), (1995), 901-908.

[12] Udrişte, C., *Convex Functions and Optimization Methods on Riemannian Manifolds*, Kluwer Academic Publisher, 1994.

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