Pre-Asymptotic Analysis of the Scattering Problem

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The pre-asymptotic analysis of the multichannel scattering problem for particles with an arbitrary spin and short-range interactions has been presented. The complete operator-valued dependence of the scattered differential flux on the distance to the target exactly consistent with the unitarity condition has been obtained.

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1. INTRODUCTION

As known, because of the local conservation of the current density, the radial flux of particles emitted into a given solid angle is independent of the distance \( R \) even from an anisotropic point stationary source of classical particles, light rays, or an incompressible liquid.

In the wave picture, such an independence is valid only for the flux of a spherical divergent (convergent) wave \( e^{\pm ikR}/R \) and gives the same inverse-square law \( \propto R^{-2} \) for the event rate and the independence of the differential scattering cross section \( f^2 \) from \( R \) [1–3]. We analyze the possible violation of this law. Being a purely wave effect, it occurs because the exact scattered wave is nonspherical, i.e., because of the next, pre-asymptotic, terms of its asymptotic expansion in \( R^{-S} \). This expansion is obtained in this work in the explicit operator form for all orders \( S \geq 1 \) taking into account the conservation of the corresponding current.

A growing interest in the pre-asymptotic analysis of the scattering problem is due both to the results of new experiments and their analysis [4–6] and to the development of the theory itself [7–10]. In contrast to the long-range Coulomb potential, which allows only the nonasymptotic analysis [7, 8] based on the exact Coulomb solutions, an arbitrary short-range local or nonlocal interaction allows the pre-asymptotic analysis based on exact free solutions [9, 10]. For example, such an analysis is possible for the scattering of neutrons on nuclei with the inclusion of all neutral channels of such a reaction [1–3]. Further, the results of the pre-asymptotic analysis of single-channel scattering performed in [9, 10] are generalized and refined to this case. We use the system of units where \( \hbar = 1 \).

2. ASYMPTOTIC EXPANSION OF THE WAVEFUNCTION

The asymptotic expansion of the wavefunction follows from the operator expansion of the free Green’s function

\[
\frac{e^{\pm ik(R-x)}}{4\pi R-x} = \frac{\chi_{L_n}(\mp ikR)}{4\pi R} e^{\pm ik(n, x)}
\]

\[
\frac{e^{\pm ikR}}{4\pi R} \left[ 1 + \sum_{S=1}^{\infty} \sum_{\mu=1}^{S} \frac{\prod_{l=1}^{S} \left[ \mathcal{L}_n - \mu(\mu - 1) \right]}{S!(\mp 2ikR)^S} \frac{e^{\pm ik(n, x)}}{4\pi R} \right].
\]

The expansion is written for \( |x| = r < R \); vectors \( R = \mathbf{n} \), where \( \mathbf{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \phi) \) and \( x = \mathbf{r} \); and operators \( \mathbf{\nabla}_R = \mathbf{n}\partial_R + R^{-1} \mathbf{\nabla}_n \), where \( \mathbf{\nabla}_n = (0, \partial_\theta, (\sin \theta)^{-1}\partial_\phi) \) in the spherical basis; \( \mathcal{L}_n = \mathbf{L}_n^2 = -\mathbf{\nabla}_n^2 \), where \( \mathbf{L}_n = -i(\mathbf{n} \times \mathbf{\nabla}_n) \) is the orbital angular momentum operator; and \( \Lambda_n = \sqrt{\mathcal{L}_n^2 + \frac{1}{4} - \frac{1}{2}} \). The equality in Eq. (1) is a brief operator representation [6, 9] of the multipole expansion of the Green’s function [1] involving the following plane wave expansion [2]:

\[
e^{\pm ik(n, x)} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \psi_{l0}(kr) \sum_{m=-l}^{l} Y_{lm}^m(\mathbf{n}) Y_{lm}^m(v). \]
Here and below, $Y^m_l(n) = \langle n| l, m \rangle$ is a spherical function, $\psi_{\alpha}^{1/2}(kr) = (2i)^{l+1} [i \chi_r(-ikr) - i \chi_r(ikr)]$ is the regular solution of the free radial equation, where

$$\chi_{\ell}(z) = \frac{2z^{1/2}}{\pi} J_{\ell+1/2}(z), \quad z = ikr, \quad (3)$$

is the irregular solution of this equation, where $K_{\ell}(z)$ is the modified Bessel function of the third kind \cite{11}. For integer $l$ values, the irregular solution given by (3) is expressed in terms of elementary functions \cite{2,3,11}:

$$\chi_{\ell}(z) \rightarrow e^{-2z} \sum_{S=0}^{l} \frac{(l+S)!}{S!(l-S)(2z)^S}, \quad z = \mp ikR, \quad (4)$$

for which $\chi_{\ell}(izkR) Y^m_l(n) = \chi_{\ell}(z) Y^m_l(n)$. (5)

The series in the parameter $S$ in Eq. (1) for $2\ell_n \gg l(l+1)$ is the known asymptotic expansion \cite{11} of function (3), which becomes the finite sum given by Eq. (4) at integer $l \ll \Lambda_n$.

Following \cite{9}, we substitute Eqs. (1) into the Lippmann–Schwinger equation for the scattering wavefunction without rearrangement in the center-of-mass system \cite{1}:

$$\Psi_{\alpha}^{+}(k_{\alpha}\alpha, R; a) = \Psi_{\beta}^{0}(k_{\alpha}\alpha, R; a) + \int d^3 x d^3 y \langle a| G_{\beta}^+(E; R, x) V^+(x, y) \Psi_{\alpha}^{+}(k_{\alpha}\alpha, y)| a \rangle, \quad (6)$$

for the free Green’s function $G_{\beta}^+$ and interaction $V^+$ as operators in the subspace of states of the target $1\alpha(a)$ channels with the reduced mass $m_a$ and target energies $E_{\alpha(a)}$ at the momentum of the $a$th incident particle $k_{\alpha} = k_{\alpha} \mathbf{x}$, where $\kappa_{\alpha(a)} = [2m_a(E - \varepsilon_{\alpha(a)})]^{1/2}$:

$$\langle a| G_{\beta}^+(E; R, x) | a \rangle = \frac{-m_a}{2\pi} \int_{|R-x|} \Phi_{\beta a}(a) \Phi_{\beta a}^+(a'), \quad (8)$$

and similarly for $V^+$. The vectors $\mathbf{R} = \mathbf{R_a}, \mathbf{x}, \mathbf{y}$ specify the coordinates of the $\alpha$th external particle and $a\rightarrow a', a' \rightarrow a$ are sets of vectors of the relative coordinates of particles of the target $(bc)$ in its different states related to the channels $\alpha(a)$, $\beta(a)$ with the $\alpha$th external particle corresponding to the separation of the total Hamiltonian into the free Hamiltonian $H^a$ of the channels $\alpha(a)$ with the Green’s function $G_{\alpha}^z(E) = (E \pm i0 - H^a)^{-1}$ (8) and the interaction $V^a$ in them between the $\alpha$th particle and target:

$$H = H^a + V^a, \quad H^a = P^2_a/(2m_a) + \hat{H}_{bc}^{(a)}, \quad (9)$$

Integrals \cite{1} over the internal variables $a'$ of the target states (7), implicitly entering Eq. (6), are given explicitly in Eq. (14). The separation of the total Hamiltonian in Eq. (9) with the free Hamiltonian $H^a$ and the Hamiltonian of the target $\hat{H}_{bc}^{(a)}$ distinguishes in Eq. (12) the subgroup of channels $\beta(a)$ with excitations of different bound states of the target $(bc)$, which does not contain continuous spectrum states corresponding to its decay \cite{2}. Nevertheless, the sum \cite{8} over $\beta(a) \rightarrow \beta$ \cite{1,2,12} includes it in this complete and orthonormalized system of eigenfunctions (12) and, thereby, includes it in the system of functions (13).

The channel indices $\alpha$ and $\beta$ and the variables $a$ in Eqs. (6)–(13) can include discrete indices of spin degrees of freedom of the system \cite{1}. Choosing a common fixed quantization axis for all spins, we can expand below the scattering amplitude $f_{\alpha(a)}(k_{\alpha}, k_{\alpha'})$ only in the spherical functions $Y^m_l(n)$ with the quantum numbers and variables of the final states (see Eq. (30)) \cite{1}.

In Eq. (6), the matrix elements $\Psi_{\alpha-\beta}^{\pm}(x)$ of the interaction $V^a$ appearing in Eq. (9) are given by the expression

$$\Psi_{\alpha-\beta}^{\pm}(x) = \int d^3 y \int d^3 a' \Phi_{\alpha}^+(a') \int d^3 a'' \times \langle a'| V^a(x, y) a'' \rangle \Psi_{\beta}^{\pm}(k_{\alpha}, a, y; a'')(2\pi)^{3/2}. \quad (14)$$

Let them be either finite functions of $r = |x|$ vanishing at $r > r_0$ or decreasing at $r \rightarrow \infty$ faster than $C_n r^{-N}$ with any power $N$. Then note that two presumed additions similar to \cite{9},

$$\Delta_{R_{\beta(a)}} = \frac{\chi_{\Lambda_n}(\mp ikR)}{2\pi R} \int_{r=0}^{R} d^3 x e^{\mp ikR} w_{\beta(a)}^{(z)}(x), \quad (15)$$

either vanish at $r > r_0$ because of the finiteness of the norm $||\Psi|| = \sup_{y, a, \alpha} |\Psi_{\alpha}^{z}(k_{\alpha}, a, y; a)|$ or also decrease at
\[ r \to \infty \] faster than \( r^{-N} \) with any power \( N \gg 1 \). As a result, according to estimates similar to [9] at

\[ \sup_{β} \int d^{3} y \int d^{3} x \left| \int d^{3} a^{′} Φ_{β}^{a}(a^{′})|Ψ_{β}(x, y)|a^{′} \rangle \right| \leq \frac{C_{N}}{r}, \]

\[ \bar{C}_{N} = C_{N}||Ψ_{β}(2π)^{3/2}, \quad |Δ_{β}f_{β}^{α}| < \frac{2m_{α}C_{N}}{(N-2)R^{N-2}}, \quad (16) \]

\[ |Δ_{β}f_{β}^{α}| < \frac{2m_{α}C_{N}}{(N-3)R^{N-2}}\left[ 1 + O(R^{-1}) \right], \]

the indicated substitution gives the explicit operator form of the asymptotic expansion of the scattering wavefunction without rearrangement:

\[ Ψ_{α}^{β}(k_{β}α R; a) = (2π)^{-3/2} \sum_{β(α)} Φ_{β}(a) \]

\[ \times \left\{ δ_{βα}(e^{ik_{β}R}) + \frac{X_{α}(ik_{β}R)}{R} f_{βα}^{α}(k_{β}n; k_{α}) \right\}, \quad (17) \]

which is expressed here in terms of only the physical scattering amplitude on the energy shell given by the formula

\[ f_{βα}^{α}(k_{β}n; k_{α}) = -\frac{m_{α}}{2π} \int d^{3} x e^{ik_{β}Rn} W^{α(2)}(x). \quad (18) \]

This amplitude determines the differential and total scattering cross section from the \( α(α) \) channel to the \( β(α) \) channel [1, 2, 12]:

\[ \frac{dσ_{βα}}{dΩ(n)} = k_{β}k_{α} \left| f_{βα}^{α}(k_{β}n; k_{α}) \right|^{2}, \quad (19) \]

\[ \sigma_{βα} = \frac{k_{β}}{k_{α}} \int dΩ(n) \left| f_{βα}^{α}(k_{β}n; k_{α}) \right|^{2}. \quad (20) \]

The contribution from the incident plane wave (13) is absent in Eq. (17) for all inelastic channels with \( β ≠ α \). The same consideration and estimates for collisions with rearrangement lead to expressions similar to Eqs. (17) and (18), where, however, the “incident” wave more complex than Eq. (13) and the scattered wave in Eq. (17) correspond to different targets and the contribution from the “incident” wave in the \( α(α) \) channel is orthogonal to the bound states of the final target in the \( β(β) \) channel [1, 2]. Expansion (1) is now performed in Eq. (17) in powers of the distance \( R \to \bar{R} = \bar{R}_{β} \) between another final target \( α(α) \) and another final external particle \( β(β) \) scattered in a different direction \( R = \bar{R}v \) in the \( β(β) \) final channel. That is, \( α = α(α) \) and \( Ψ_{α}^{β}(k_{β}α R; a)≡ Ψ_{β}^{α}(k_{β}α R; b) \), we now have \( \bar{V}^{α} \to \bar{V}^{β} \), \( m_{α} \to m_{β} \), \( \bar{β} \to β(β) \), and \( n \to v \) in Eqs. (8), (14), and (18), and the substitution \( k_{β}/k_{α} \to \bar{v}_{β}/\bar{v}_{α} \) for the corresponding velocities \( \bar{v}_{β} = k_{β}/m_{β} \) and \( \bar{v}_{α} = k_{α}/m_{α} \) should be made in Eqs. (19) and (20) [1] (cf. [10]).

In any case, the exact asymptotic expansion of the scattering wavefunction for a short-range potential is obtained by the simple replacement \( e^{±ik_{β}R} \to X_{α}(±ik_{β}R) \) of the exponential in the expression for a spherical wave by the operator-valued function (1) acting only on the angular variables \( n \) (or \( v \)) of the corresponding scattering amplitude (18).

Considering the expression in the curly brackets in Eq. (17) as the asymptotic expansion of the function \( η_{βα}^{α}(R) \), we write

\[ Ψ_{α}^{β}(k_{β}α R; a) = (2π)^{-3/2} \sum_{β(α)} Φ_{β}(a) η_{βα}^{α}(R), \quad (21) \]

\[ e^{±ik_{β}R} R \left[ f_{βα}^{α}(k_{β}n; k_{α}) + \sum_{S=1}^{∞} h_{S}^{α}(k_{β}n; k_{α}) \right], \quad (22) \]

\[ h_{S}^{α}(k_{β}n; k_{α}) = \frac{1}{S!} \sum_{μ=1}^{S} [δ_{βα} - μ(μ - 1)] f_{βα}^{α}(k_{β}n; k_{α}). \]

This means that the coefficients \( h_{S}^{α} \) of the asymptotic expansion (22) are observable together with \( f_{βα}^{α} \).

3. SCATTERED DIFFERENTIAL FLUX, UNITARITY CONDITION, AND OPTICAL THEOREM

Following [2, 9], we consider the radial flux element through a small element of the spherical surface \( R^{2}dΩ(n) \) for the off-diagonal current density \( J_{αβ}(qγ, k_{α}, R; a, a') \) constructed on the wavefunctions from Eqs. (6) and (10), taken in the form of Eqs. (21) and (17) at \( R = Rn, \quad qγ = kγy, \quad k_{α} = k_{α}x, \) and \( δ_{R} = (n · \bar{V}_{R}) = \bar{δ}_{R} - \bar{δ}_{R}. \) Because of the current conservation \( [\bar{V}_{R} · J_{αβ}(qγ, k_{α}, R; a, a')] = 0 \) on the equations of motion (9)–(12), the total flux through any closed surface should be zero, whereas the radial flux element is

\[ R^{2}dΩ(n) (n · J_{αβ}(qγ, k_{α}, R; a, a')) \]

\[ = R^{2}dΩ(n) - \frac{1}{2i} \left[ Ψ_{α}^{β} (qγ, R; a, a') \bar{δ}_{R} Ψ_{β}^{α} (k_{β}α, R; a) \right] \]

\[ = R^{2}dΩ(n) \frac{1}{(2π)^{3}} \sum_{β} \sum_{α} Φ_{α}^{β} (a) Φ_{β}^{α} (a) \frac{1}{2i} \left[ Ψ_{α}^{β} (R) \bar{δ}_{R} Ψ_{β}^{α} (R) \right]. \]

Here and below, arrows over symbols indicate the direction of action of operators, and \( z_{β} = -ik_{β}R, \) etc.
Substitution (17) gives the flux element (23) as the sum of three terms having a clear physical meaning:

\[
\frac{R^2}{2i}\left[\eta_{\beta\alpha}^{*}(R)\delta_{R}\eta_{\alpha\beta}(R)\right]^{(1)} - \frac{R^2}{2i}(n \cdot (k_\alpha + q_\gamma))e^{iR(n \cdot (k_\alpha - q_\gamma))} + \frac{1}{2i}(2)
\]

\[
\times \frac{f_{\beta\alpha}^{*}(k_\beta n; q_\gamma)}{i} \left[\chi_{\zeta}(\gamma_\alpha)(\gamma_\zeta)\right]f_{\beta\alpha}(k_\beta n; k_\alpha, \kappa, \gamma_\alpha, \alpha_\beta, \beta_\alpha, \kappa_\beta, \alpha_\gamma, \beta_\gamma, \kappa_\gamma)
\]

\[
+ \frac{i}{2}\left[\delta_{\gamma_\beta, \alpha_\beta}e^{i(n \cdot \gamma)}\left[z_\beta(n \cdot \gamma) + 1 - z_\beta \frac{\partial}{\partial z_\beta}\right] - \frac{\gamma_\alpha, \beta_\alpha, \kappa_\beta, \alpha_\gamma, \beta_\gamma, \kappa_\gamma}{\gamma_\alpha, \beta_\alpha, \kappa_\beta, \alpha_\gamma, \beta_\gamma, \kappa_\gamma}ight].
\]

\[
(24)
\]

As usual [2, 3], the first term \((1)\) corresponds to the incident fluxes, the second one \((2)\) now describes the scattered flux, and the third term \((3)\) corresponds to their interference. Here and below, we use the following elementary identities for derivatives and Wronskians [2] of arbitrary functions \(\psi, \phi, w, g\) and functions \(\chi\) of \(z\):

\[
[w(z)\psi(z)\delta_{\gamma_\beta, \alpha_\beta}g(z)\phi(z)] = w(z)\psi(z)\phi(z) + [w(z)\psi(z)\delta_{\gamma_\beta, \alpha_\beta}g(z)] = 0,
\]

\[
+w(z)g(z)[\psi(z)\delta_{\gamma_\beta, \alpha_\beta}\phi(z)], \quad [\phi(z)\delta_{\gamma_\beta, \alpha_\beta}g(z)] = 0,
\]

\[
\frac{\partial}{\partial z}[w(z)\phi(z)] = \psi(z)\delta_{\gamma_\beta, \alpha_\beta}g(z), \quad \frac{\partial}{\partial z}[g(z)\psi(z)] = -\psi(z)\delta_{\gamma_\beta, \alpha_\beta}g(z),
\]

\[
\frac{\partial}{\partial z}[w(z)\phi(z)] = \psi(z)\delta_{\gamma_\beta, \alpha_\beta}g(z), \quad \frac{\partial}{\partial z}[g(z)\psi(z)] = -\psi(z)\delta_{\gamma_\beta, \alpha_\beta}g(z).
\]

\[
(25)
\]

\[
(26)
\]

\[
(27)
\]

Taking into account the completeness and orthonormality of the system of target functions given by Eqs. (7) and (12),

\[
\int d^3x \Phi_{\alpha}^{*}(x)\Phi_{\beta}(x) = \delta_{\alpha\beta},
\]

\[
(28)
\]

these identities determine the result of calculation of the total flux from Eqs. (23) and (24) integrating over these functions at \(a^* = a\). The multiplication by \(\delta_{\gamma_\beta, \alpha_\beta}\) and summation over all channels in Eq. (23) reduce the contribution from the integral \(\int d\Omega(n)\) at \(\gamma = \alpha\) to integral (42) from [9], which is zero. Since the operator \(\Lambda_n\) is self-adjoint on the unit sphere and because of identity (27), the same operations reduce the contribution to the result of integration (23) from the term \((2)\) to the form of the right-hand side of the unitarity condition [1–3]:

\[
\int \sum_{(\beta_{\gamma\alpha})} k_\alpha \left[\frac{d\Omega(n)}{d\Omega(n)} f_{\beta\alpha}^{*}(k_\beta n, q_\gamma) f_{\beta\alpha}(k_\beta n, k_\alpha, \kappa, \gamma_\alpha, \alpha_\beta, \beta_\alpha, \kappa_\beta, \alpha_\gamma, \beta_\gamma, \kappa_\gamma\right].
\]

\[
(29)
\]

Here, as above, the asterisk \(*\) stands for complex conjugation and/or for Hermitian conjugation in the case of spin indices. Therefore, the same operations should transform the contribution from the term \((3)\) to the left-hand side of the unitarity condition. Indeed, the integral \(\int d\Omega(n)\) at \(\beta = \beta = \gamma\) (or \(\alpha\)) can be determined using the following always possible expansion of the scattering amplitude mentioned in Section 2:

\[
f_{\beta\alpha}^{+}(k_\beta n; k_\alpha, \kappa, \gamma_\alpha, \alpha_\beta, \beta_\alpha, \kappa_\beta, \alpha_\gamma, \beta_\gamma, \kappa_\gamma) = \frac{1}{\sum_{(\beta_{\gamma\alpha})}} k_\alpha \left[\frac{d\Omega(n)}{d\Omega(n)} f_{\beta\alpha}^{*}(k_\beta n, q_\gamma) f_{\beta\alpha}(k_\beta n, k_\alpha, \kappa, \gamma_\alpha, \alpha_\beta, \beta_\alpha, \kappa_\beta, \alpha_\gamma, \beta_\gamma, \kappa_\gamma\right].
\]

\[
(30)
\]

which effectively changes \(f_{\beta\alpha}(n)\) to \(f_{\beta\alpha}^{+}(n)\) in the last line of Eq. (24). Taking into account Eq. (5), this expansion provides the partial contribution to \(\int d\Omega(n)\) in the form (omitting \(B_{\beta\alpha}^{im}\))

\[
\int \frac{d\Omega(n)}{d\Omega(n)} e^{i(n \cdot \gamma)} Y_{l_\alpha}^{m}(n).
\]

\[
(31)
\]

According to expansion (2) and orthonormality of the spherical functions,

\[
\frac{A(z_{\gamma})}{z_{\gamma}} = 2\pi Y_{l_\alpha}^{m}(v) \left[\chi_{\zeta}(z_{\gamma}) - 1\right] \frac{\chi_{\zeta}(z_{\gamma})}{z_{\gamma}},
\]

\[
(32)
\]

A different method of calculation of the contribution from the term \((3)\) in the spinless case [9] shows that \(\chi_{\zeta}(z_{\gamma})\) and \(\chi_{\zeta}(z_{\gamma})\) describe here interference in the forward and backward directions with \((n \cdot v) = 1\) and \((n \cdot v) = -1\), respectively. Then, according to Eqs. (25)–(27), interference in the backward direction is absent in all orders of \(R^s\) and the contribution from interference in the forward direction in all orders of \(R^s\) is again reduced to a spherical function, which thus results in the amplitude given by Eq. (30) depending already on \(v\) and, finally, in the left-hand side of the unitarity condition [1, 2] (with the minus sign). That is, according to Eq. (32),

\[
\int d\Omega(n)\left[\frac{d\Omega(n)}{d\Omega(n)} f_{\beta\alpha}^{*}(k_\beta n; k_\alpha, \kappa, \gamma_\alpha, \alpha_\beta, \beta_\alpha, \kappa_\beta, \alpha_\gamma, \beta_\gamma, \kappa_\gamma\right] = -\frac{4\pi}{2i} \left[\frac{d\Omega(n)}{d\Omega(n)} f_{\beta\alpha}(k_\beta n; k_\alpha, \kappa, \gamma_\alpha, \alpha_\beta, \beta_\alpha, \kappa_\beta, \alpha_\gamma, \beta_\gamma, \kappa_\gamma\right] - \frac{4\pi}{2i} \left[\frac{d\Omega(n)}{d\Omega(n)} f_{\beta\alpha}^{*}(k_\beta n; k_\alpha, \kappa, \gamma_\alpha, \alpha_\beta, \beta_\alpha, \kappa_\beta, \alpha_\gamma, \beta_\gamma, \kappa_\gamma\right].
\]

\[
(33)
\]

The vanishing of the total flux as the sum of Eqs. (29) and (33) gives the unitarity condition [1–3] and, taking into account Eqs. (19) and (20), leads at \(\gamma = \alpha\) and \(v = \chi\) to the optical theorem. Therefore, the unitarity condition and optical theorem are applicable not only in the far-field region but also in the near-field one at
finite distances from the scattering target. The differential flux scattered without rearrangement from the \( \alpha(\alpha) \) channel, which explicitly depends on this distance, is determined by the term (2) in Eq. (24) as

\[
\frac{dS_\alpha(R)}{d\Omega(n)} = \frac{1}{2ik_\alpha} \left| f_{\beta\alpha}(k_\beta n; k_\alpha) \right|^2 \sum_{n=0}^{\infty} \frac{1}{2i k_\alpha} \left| f_{\beta\alpha}(k_\beta n; k_\alpha) \right|^2 \tag{34}
\]

This quantity is experimentally measured at finite distances \( R \) instead of the sum of differential cross sections (19), and, at sufficiently large values \( k_\beta R > 1 \), it really has the form

\[
\frac{dS_\alpha(R)}{d\Omega(n)} = \frac{k_\beta}{k_\alpha} \left| f_{\beta\alpha}(k_\beta n; k_\alpha) \right|^2 - \frac{1}{k_\alpha} \text{Im} \left[ f_{\beta\alpha}(k_\beta n; k_\alpha) \right] + \frac{1}{(2k_\beta R)^2} \left| f_{\beta\alpha}(k_\beta n; k_\alpha) \right|^2 - \left| f_{\beta\alpha}(k_\beta n; k_\alpha) \right|^2 \tag{35}
\]

According to [10], Eq. (35) with \( k_\beta/k_\alpha \rightarrow \nu_\beta/\nu_\alpha \) remains valid, including the relativistic correction \( \mathcal{O}(P_\alpha^2) \) to the kinetic energy operator in Eq. (9).

For the same reasons [2], discussed in Section 2, only the contribution of differential fluxes (2) of the form of Eq. (34) holds in Eq. (24) applied to collisions with rearrangement, where the scattering amplitude \( f_{\beta\alpha}(k_\beta n; k_\alpha) \) is defined as the coefficient of the divergent spherical wave in the wavefunction \( \Psi(R) = \Psi(\rho) \) from Eqs. (10) and (6) at \( R \rightarrow \infty \) [1, 2, 12]. We recall that the determination of the inelastic scattering cross section requires the rearrangement of the asymptotic expression for \( \Psi(R) \) even in the single-channel case [3].

The substitution of Eq. (30) into Eq. (34) gives the corresponding partial expansion for the scattered differential flux at \( z_0 = -ik_\beta R \):

\[
\frac{dS_\alpha(R)}{d\Omega(n)} = \sum_{l=0}^{\infty} k_\beta \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sum_{j=0}^{l} \sum_{l=0}^{l} a_l(n, j) \sum_{l=0}^{l} a_l(\alpha, \beta) \left| f_{\beta\alpha}(k_\beta n; k_\alpha) \right|^2 \tag{38}
\]

which is certainly consistent with Eq. (37) taking into account Eq. (27) and the orthonormality of the spherical functions \( Y_{lm}^m(n) \). The corresponding analog of Eq. (35) is obtained by the substitution of the expansion for the Wronskian similar to [9]:

\[
\frac{1}{2} \left| \chi_j(z_\beta) \right|^2 \chi_j(-z_\beta) = 1 + \Delta_{j\ell} \int_{-\infty}^{\infty} \frac{d\zeta}{\zeta} \chi_j(\zeta) \chi_j(-\zeta) + O((k_\beta R)^2), \tag{39}
\]

taken with the necessary accuracy in the powers of \( R^{-5} \) at

\[
\Delta_{j\ell} = j(j + 1) - l(l + 1), \quad \Delta_{j\ell} = j(j + 1) + l(l + 1),
\]

as

\[
\frac{1}{2} \left| \chi_j(z_\beta) \right|^2 \chi_j(-z_\beta) = 1 + \Delta_{j\ell} \int_{-\infty}^{\infty} \frac{d\zeta}{\zeta} \chi_j(\zeta) \chi_j(-\zeta) + \Delta_{j\ell} \int_{-\infty}^{\infty} \frac{d\zeta}{\zeta} \chi_j(\zeta) \chi_j(-\zeta) + \frac{3}{8} \Delta_{j\ell} \left[ \chi_j^2 - \chi_j^2(\zeta) \chi_j(\zeta) \right]^2 \tag{40}
\]

The calculation of the coefficients \( A_l(l, j) \) shows their very strong dependence on any of their integer parameters \( n, l, j > 0 \). Note that \( A_l(l, j) = 1 \) for any \( l \) and \( j \) values and the several lowest coefficients \( A_l(l, j) \) are given in Table 1.
4. CONCLUSIONS

To summarize, using the operator expansion (1) of the free Green’s function, we have derived the exact asymptotic expansion given by Eqs. (21) and (22) for wavefunctions of multichannel scattering of particles with an arbitrary spin for a wide class of short-range interactions. The coefficients of this expansion are determined only by the physical scattering amplitude (18). The derived expansion has confirmed the applicability of the unitarity condition given by Eqs. (29) and (33) and of the optical theorem not only in the far-field region but also in the near-field one, where the scattered differential flux given by Eqs. (34) and (38), which explicitly depends on the distance $R$, assumes at finite distances $R$ the role of the differential scattering cross section given by Eqs. (19) and (36). At quite large values $k_R > 1$, this flux can be represented by the first terms of the expansions given by Eq. (35) or Eqs. (38) and (40).

This circumstance makes it possible to expect that the application of these relations to processing of the results of scattering experiments with a sufficiently short variable base $R$ will not only give additional information on the corresponding interactions [6] but also allow estimating finer quantum effects [10].

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