Robust unknown input observer design for uncertain interval type-2 T–S fuzzy systems subject to time-varying delays

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ABSTRACT
This paper investigates the problem of robust unknown input observers design for fault detection of Takagi–Sugeno fuzzy systems. In order to handle uncertainties related to membership functions and rule-base, in this study interval type-2 fuzzy sets are employed as activation functions. The system is supposed to be affected by parameter uncertainties and time-varying delays, which makes the design procedure more challenging. Furthermore, to achieve better results in the detection of faults, a multi-objective optimization index is considered so as to get a residual signal with the most possible sensitivity to the fault and least one to other signals. This issue will lead to some design constraints in the terms of linear matrix inequalities. Two case studies are provided to show the validity of the proposed method. In addition, the superiority of interval type-2 fuzzy sets compared to type-1 sets is investigated in the simulation part.

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1. Introduction

Control problem and observer design for nonlinear systems have been attracted more attentions in recent years. Nonlinear models, which are able to model real systems in the large domain of operation, have been used in control problems extensively. However, obtained models may be too complicated to design a control law or an observer. Takagi–Sugeno (T–S) models introduce a solution to this problem (Takagi & Sugeno, 1985). This approach can be used in the modelling of large kinds of systems. The main idea of this approach is the blending of several local linear model or affine models in different zones that describes the dynamic of nonlinear model in a small region of the operating space. The main advantages of this approach is decreasing of computational efforts compared to nonlinear methods.

Type–1 T–S fuzzy models (Sugeno & Kang, 1988) have been used for the purpose of analysis and design of control systems extensively. For instance, in Tong, Li, Feng, and Li (2011), a combination of adaptive-backstepping and dynamic surface control technique is proposed for the control of multi-input-multi-output nonlinear systems. In this case, fuzzy logic is used for the approximation of nonlinear functions. Moreover, a fuzzy observer is designed to approximate immeasurable states of the system. Therefore, this method should be refereed to the observer-based techniques in control of nonlinear systems. In the case of stochastic nonlinear switched systems with modelled dynamics, a control design is investigated in Li, Sui, and Tong (2016). Type–1 T–S fuzzy systems, which are presented in Li, Sun, Wu, and Lam (2015), Zhang, Han, and Jia (2015), Ben Hamouda, Ayadi, and Langlois (2016) and Li and Tong (2016), use type–1 fuzzy sets with a certain membership degree. If the parameter uncertainties of the system were considered, membership degrees would have not be certain. On the other hand, Lyapunov method in stability proofs will lead to more conservative conditions.

Type–2 fuzzy sets was introduced to deal with the uncertainties in a better way (Coupland & John, 2007; Mendel, John, & Liu, 2006; Mendel & Liu, 2007). Type–2 fuzzy logic systems are theoretically considered as the summation of infinite number type–1 fuzzy logic systems. Therefore, more information such as parameter uncertainties are considered in such systems. Type–2 fuzzy logic systems are known for their performance in encountering uncertainties related to membership functions compared to type–1 fuzzy sets. This issue has been studied well in the literature (Hassani & Zarei, 2015; Li, Sun, Wu, & Lam, 2015; Lu, Shi, Lam, & Zhao, 2015).

Moreover, the problem of observer design for T–S fuzzy systems is still challenging. Different kind of
observers have been investigated in the literature for different purposes. For example, in Tong, Huo, and Li (2014), a fuzzy adaptive observer is designed so as to estimate immeasurable states and actuator faults in a large-scaled nonlinear system. An unknown input fault detection observer for the sensor fault detections of T–S fuzzy systems is addressed in Chadli, Abdo, and Ding (2013). The authors have considered sensor fault as an auxiliary state variable and based on this concept, have proceed their H∞/H∞ problem.

However, the problem of unknown input observer design for interval type-2 T–S fuzzy systems has not been addressed well in the literature. Motivated by this consideration, the main subject of current work is observer design for fault detection of interval type-2 T–S fuzzy systems. In addition, considering time-varying delays and parameter uncertainties, have made the problem more challenging. In observer design, to generate a residual signal with the most possible sensitivity to the fault and the least possible sensitivity to other signals, an optimization index of H∞/H∞ is employed. This issue will lead to introduction of two new theorems with constraints in terms of linear matrix inequalities (LMIs). These constraints should be met simultaneously; to this end, an iterative optimization algorithm is used.

In order to show the superiority of the proposed method for observer design, two examples are provided in the simulation section. First, a numerical example is used to introduction of two new theorems with constraints under them the residual signal shows the least possible sensitivity to other signals.

2. Main results

This section presents how the proposed observer gains will be obtained. In this regards, as mentioned earlier, the proposed observer is going to be designed in a way that satisfies two properties:

1. Residual signal shows the most possible sensitivity to the fault signal.
2. Residual signal shows the least possible sensitivity to other signals.

In order to design such observer, two theorems are given. The first one, provides constraints under them the residual signal shows the most possible sensitivity to the fault and the second one will show constraints under them residual signal shows the least possible sensitivity to other signals.

2.1. T–S fuzzy system

Consider an interval type-2 T–S fuzzy system with p rules which the ith one is:

\[ \dot{x}(t) = (A_i + \Delta A_i)x(t) + (A_{di} + \Delta A_{di})x(t - \tau_i(t)) + (B_i + \Delta B_i)u(t) + (B_{di} + \Delta B_{di})u(t - \tau_2(t)) + E_i d(t) + H_i w(t) + F_i f(t) \]

where \( \bar{M}_i^j, j = 1, 2, \ldots, \psi \) are interval type-2 fuzzy sets correspond to the ith rule in which \( \psi = 1, 2, \ldots, p \). \( x(t) \in \mathbb{R}^n \) is state vector, \( u(t) \in \mathbb{R}^q \) is known input, \( y(t) \in \mathbb{R}^p \) is output vector, \( d(t) \in \mathbb{R}^m \) is disturbance, \( w(t) \in \mathbb{R}^r \) unknown input and \( f(t) \in \mathbb{R}^f \) is fault signal. Furthermore, \( \tau_1(t) \) and \( \tau_2(t) \) are time-varying delays with the properties \( 0 < \tau_1(t) < \bar{\tau}_1 \) and \( 0 < \tau_2(t) < \bar{\tau}_2 \) (Ahmadizadeh, Zarei, & Karimi, 2014). \( A_i^{nxn} \) and \( A_{di}^{nxn} \) are system matrices, \( B_i^{nxq} \) and \( B_{di}^{nxq} \) are input matrices, \( E_i^{nxm}, H_i^{nxr} \) and \( F_i^{nxf} \) are distribution matrices of disturbance, unknown input and fault, respectively. Furthermore, \( C_i^{np} \), \( D_i^{pxm} \), \( F_i^{nxr} \) and \( F_i^{nxf} \) are used to show the distribution of disturbance, unknown input and fault signals in system output. In addition, matrices \( \Delta A_i, \Delta A_{di}, \Delta B_i, \Delta B_{di} \) are supposed to be parameter uncertainties with bounded norm.

Based on fuzzy systems theory, firing interval related to the ith rule can be computed as follows:

\[ \bar{w}_i(x(t)) = [w^{ll}_i(x(t)), w^{ur}_i(x(t))], \quad i = 1, 2, \ldots, p \]

where

\[ w^{ll}_i(x(t)) = \mu_{\bar{M}_i^j}(f_1(x(t))) \times \cdots \times \mu_{\bar{M}_i^j}(f_\psi(x(t))), \]

\[ w^{ur}_i(x(t)) = \bar{\mu}_{\bar{M}_i^j}(f_1(x(t))) \times \cdots \times \bar{\mu}_{\bar{M}_i^j}(f_\psi(x(t))). \]

In Equations (3) and (4), \( \mu_{\bar{M}_i^j}(f_\alpha(x(t))) \) and \( \bar{\mu}_{\bar{M}_i^j}(f_\alpha(x(t))) \) demonstrate lower and upper membership functions with the property of \( \mu_{\bar{M}_i^j}(f_\alpha(x(t))) \leq \mu_{\bar{M}_i^j}(f_\alpha(x(t))) \leq 1 \) and \( \mu_{\bar{M}_i^j}(f_\alpha(x(t))) \leq \bar{\mu}_{\bar{M}_i^j}(f_\alpha(x(t))) \). In this case, the inferred
fuzzy system will be obtained as following:

\[
\dot{x}(t) = \sum_{i=1}^{p} w_i(x(t))[A_i + \Delta A_i]x(t) \\
+ (A_{di} + \Delta A_{di})x(t - \tau_1(t)) + (B_{i} + \Delta B_{i})u(t) \\
+ (B_{di} + \Delta B_{di})u(t - \tau_2(t)) + E_i(t) \\
+ H_i w(t) + F_i f(t)
\]

\[
y(t) = Cx(t) + Dd(t) + F_w w(t) + F_y f(t),
\]

where

\[
w_i(x(t)) = w_i^T(x(t))υ_i(x(t)) + w_i^U(x(t))\bar{υ}_i(x(t)) \tag{6}
\]

are weighting coefficients of interval-type-2 T–S fuzzy system, which are satisfying the following constraints:

\[
w_i(x(t)) \in [0, 1], \quad \sum_{i=1}^{p} w_i(x(t)) = 1.
\]

It is evident that system described in Equation (5) contains parameter uncertainties, time-varying delays in states and known inputs, and is affected by disturbance and unknown input. To investigate the problem of observer design for a system, which is close to real applications, such uncertainties are required to be considered in the system model. Based on this concept, the results can be extended to real world applications as the aim of research is so.

In Equation (6), variables \(υ_i(x(t))\) and \(\bar{υ}_i(x(t))\) are in general nonlinear functions with the property of \(υ_i(x(t)) + \bar{υ}_i(x(t)) = 1\). Due to uncertainties system is involved with, these functions are unknown and in many cases they are chosen as constants.

### 2.2. Unknown input T–S fuzzy observer

In this paper, following structure is chosen for the proposed observer:

\[
\dot{z}(t) = \sum_{i=1}^{p} w_i(x(t))[A_{ri} z(t) + A_{rui} z(t - \tau_1(t)) + B_{rui} u(t) \\
+ B_{rui} u(t - \tau_2(t)) + N_{rif} y(t) + N_{rui} y(t - \tau_1(t))] \\
\dot{x}(t) = z(t) + L_1 y(t) \\
\dot{y}(t) = C x(t) \\
r(t) = V(y(t) - \bar{y}(t)), \tag{7}
\]

where matrices \(A_{ri}, A_{rui}, B_{rui}, N_{rif}, N_{rui}\) are observer gains which should be computed. By considering such an structure for the observer, in order to proceed designing procedure, one needs to define estimation error as \(e(t) = x(t) - \hat{x}(t)\). Defining following matrices:

\[
T = I - L_1 C \\
L_2 = A_{ri} L_1 - N_{rif} \\
L_3 = A_{rui} L_1 - N_{rui}
\]

and setting these constraints:

\[
L_3 = 0 \\
T A_i + L_2 C - A_{ri} = 0 \\
T A_{di} + L_3 C - A_{rui} = 0 \\
TB_i - B_{ri} = 0 \\
TB_{di} - B_{rui} = 0 \\
TE_i + L_2 D = 0 \\
L_1 D = 0
\]

will result in error dynamics as follows:

\[
\dot{e}(t) = \sum_{i=1}^{p} w_i(x(t))[A_{ri} e(t) + A_{rui} e(t - \tau_1(t)) + K_w \bar{w}(t) \\
+ K_{ri} \bar{f}(t) + T \Delta A_{di}(t)x(t - \tau_2(t)) \\
+ T \Delta B_i(t) u(t) + T \Delta B_{di}(t) u(t - \tau_2(t))],
\]

where

\[
K_{wi} = [TH_i + L_2 F_w - L_1 F_w], \quad K_{ri} = [TF_i + L_2 F_y - L_1 F_y], \\
\bar{w}(t) = [w(t) \bar{w}(t)], \quad \bar{f}(t) = [f(t) \bar{f}(t)]. \tag{11}
\]

Furthermore, it is straightforward that:

\[
r(t) = V(y(t) - \bar{y}(t)) \\
= V(C e(t) + D d(t) + F_w w(t) + F_y f(t)) \\
= V C e(t) + V P_1 \bar{w}(t) + V P_2 \bar{f}(t), \tag{12}
\]

where \(P_1 = [F_w 0]\) and \(P_2 = [F_y 0]\).

It is clear from Equation (12) that residual signal is affected directly by fault signal. This is the main goal in Unknown Input Observer (UIO) designing.

### 3. Observer design

In this section, two theorems are derived such that the residual signal is generated with the most possible sensitivity to fault, and the least possible sensitivity to other signals. According to Equation (7), to achieve a residual with the least possible sensitivity to \(\bar{w}(t)\), following criteria for the least amount of \(\alpha\) should be met:

\[
\|G_{ref,\bar{w}(t)}\| \leq \alpha, \tag{13}
\]

where \(G_{ref,\bar{w}(t)}\) stands for the transformation function between \(r_{ref}\) and \(\bar{w}(t)\). First theorem provides constraints that satisfy criteria (13).
Theorem 3.1: If there were symmetric positive definite matrices $Q,S,P$ and $Z^*$, matrices $X, Y, \Psi^*_1, \Psi^*_2$ and $W$, positive scalars $\alpha$ and $\delta$, such that the following LMI constraint holds for $i = 1, 2, \ldots, p$

$$
\begin{bmatrix}
\dot{\theta}_i & PA_{hi}^* - X + Y^T PK_{hi}^* + W^T + C^TZ^*P1 \\
* & -(1 - h_{di})Y - Y^T - W^T \\
* & * - \alpha^2 I + P^T Z^*P1 \\
\end{bmatrix} < 0
$$

and also following equations hold:

$$
\Psi^*_i D = 0, \quad (15)
$$

$$
P E_i - \Psi^*_i C E_i + \Psi^*_2 D = 0, \quad (16)
$$

$$
Z^* D = 0. \quad (17)
$$

Then, following system is asymptotically stable and $\|G_{ref, \hat{w}(t)}\| \leq \alpha$.

$$
\begin{align*}
\dot{e}_{ref, \hat{w}(t)}(t) &= \sum_{i=1}^{p} w_i(x(t)) [A_{hi} e_{ref, \hat{w}(t)}(t) \\
&+ A_{idi} e_{ref, \hat{w}(t)}(t - \tau_i(t)) + K_w \hat{w}(t)] \\
r(t) &= V^* C e_{ref, \hat{w}(t)}(t) + V^* P_1 \hat{w}(t).
\end{align*}
\quad (18)
$$

Proof: The objective is to achieve $\|G_{ref, \hat{w}(t)}\| \leq \alpha$ index, which is equivalent to:

$$
\|G_{ref, \hat{w}(t)}\| = \frac{\|\dot{r}\|}{\|w\|} = \frac{\int_{t-\tau_1(t)}^{t} r_{ref, \hat{w}(t)}^T(\tau) r_{ref, \hat{w}(t)}(\tau) d\tau}{\int_{t-\tau_1(t)}^{t} \hat{w}(t) \hat{w}(t) d\tau} \leq \alpha^2
$$

$$
J_{ref, \hat{w}(t)} = \int_{t-\tau_1(t)}^{t} r_{ref, \hat{w}(t)}^T(\tau) r_{ref, \hat{w}(t)}(\tau) d\tau - \alpha^2 \hat{w}^T(t) \hat{w}(t) d\tau \leq 0. \quad (19)
$$

The rest of the proof is based on Lyapunov indirect method. To this end, Lyapunov–Krasovskii functional (LKF) candidate is considered as follows:

$$
\begin{align*}
V(t) &= e_{ref, \hat{w}(t)}^T(t) e_{ref, \hat{w}(t)}(t) \\
&+ \int_{t-\tau_1(t)}^{t} e_{ref, \hat{w}(t)}^T(\tau) S e_{ref, \hat{w}(t)}(\tau) d\tau \\
&+ \int_{h_{di}}^{\theta} \int_{t-\theta}^{t} \dot{e}_{ref, \hat{w}(t)}^T(\tau) Q \dot{e}_{ref, \hat{w}(t)}(\tau) d\tau d\theta.
\end{align*}
\quad (20)
$$

In order to calculate first derivative of proposed LKF, Leibniz integral formula is needed (Ahmadizadeh et al., 2014):

$$
\frac{d}{dt} \left( \int_{a(x)}^{b(x)} f(x, t) \, dt \right) = f(x, b(x)) b'(x) - f(x, a(x)) a'(x) + \int_{a(x)}^{b(x)} f_x(x, t) \, dt.
$$

Furthermore, the following equation is hold for all $X,Y$ and $W$ with appropriate dimensions:

$$
2[e_{ref, \hat{w}(t)}^T(t) X + e_{ref, \hat{w}(t)}^T(t - \tau_1(t)) Y + \hat{w}(t) W] \\
\times \left[ e_{ref, \hat{w}(t)}(t) - e_{ref, \hat{w}(t)}(t - \tau_1(t)) \\
- \int_{t-\tau_1(t)}^{t} \dot{e}_{ref, \hat{w}(t)}(\tau) d\tau \right] = 0. \quad (22)
$$

Therefore, the first derivative of proposed LKF in Equation (20) can be computed as follows:

$$
\dot{V}(t) = 2e_{ref, \hat{w}(t)}^T(t) P \dot{e}_{ref, \hat{w}(t)}(t) + e_{ref, \hat{w}(t)}^T(t) S e_{ref, \hat{w}(t)}(t) \\
- [1 - \tau_1(t)] e_{ref, \hat{w}(t)}^T(t - \tau_1(t)) S e_{ref, \hat{w}(t)}(t - \tau_1(t)) \\
+ \tilde{h}_{di} e_{ref, \hat{w}(t)}^T(\tau) Q \dot{e}_{ref, \hat{w}(t)}(\tau) \\
+ \int_{t+\tilde{h}_{di}}^{t} \dot{e}_{ref, \hat{w}(t)}^T(\tau) Q \dot{e}_{ref, \hat{w}(t)}(\tau) d\tau \\
+ 2[e_{ref, \hat{w}(t)}^T(t) X + e_{ref, \hat{w}(t)}^T(t - \tau_1(t)) Y + \hat{w}(t) W] \\
\times \left[ e_{ref, \hat{w}(t)}(t) - e_{ref, \hat{w}(t)}(t - \tau_1(t)) \\
- \int_{t-\tau_1(t)}^{t} \dot{e}_{ref, \hat{w}(t)}(\tau) d\tau \right].
$$

Based on constraints expressed about the main system (1), it could be concluded that:

$$
\dot{V}(t) \leq 2e_{ref, \hat{w}(t)}^T(t) P \dot{e}_{ref, \hat{w}(t)}(t) + e_{ref, \hat{w}(t)}^T(t) S e_{ref, \hat{w}(t)}(t) \\
- [1 - \tau_1(t)] e_{ref, \hat{w}(t)}^T(t - \tau_1(t)) S e_{ref, \hat{w}(t)}(t - \tau_1(t)) \\
+ \tilde{h}_{di} e_{ref, \hat{w}(t)}^T(\tau) Q \dot{e}_{ref, \hat{w}(t)}(\tau) \\
+ \int_{t+\tilde{h}_{di}}^{t} \dot{e}_{ref, \hat{w}(t)}^T(\tau) Q \dot{e}_{ref, \hat{w}(t)}(\tau) d\tau \\
+ 2[e_{ref, \hat{w}(t)}^T(t) X + e_{ref, \hat{w}(t)}^T(t - \tau_1(t)) Y + \hat{w}(t) W] \\
\times \left[ e_{ref, \hat{w}(t)}(t) - e_{ref, \hat{w}(t)}(t - \tau_1(t)) \\
- \int_{t-\tau_1(t)}^{t} \dot{e}_{ref, \hat{w}(t)}(\tau) d\tau \right].
$$

\quad (23)}
Considering property of $V(\infty) > 0$, index (19) can be written as:

$$J_{ref,\tilde{w}(t)} = \int_{t_{\tau_1}}^{t} \left( J_{ref,\tilde{w}(t)} \right)^{T} \left( J_{ref,\tilde{w}(t)} \right) + \alpha^{2} \tilde{w}(t)^{T} \tilde{w}(t) + V(t) \, d\tau \leq 0. \quad (25)$$

In this regards, it is straightforward that:

$$J_{ref,\tilde{w}(t)} = \int_{0}^{\infty} \left[ VCE_{ref,\tilde{w}(t)} + VP_{1}\tilde{w}(t) \right]^{T}$$

$$\times \left[ VCE_{ref,\tilde{w}(t)} + VP_{1}\tilde{w}(t) \right] - \alpha^{2} \tilde{w}(t)^{T} \tilde{w}(t)$$

$$+ 2e_{ref,\tilde{w}(t)}^{T} e_{ref,\tilde{w}(t)} + A_{kd}^{T} e_{ref,\tilde{w}(t)} (t - \tau_{1}(t)) + K_{w} \tilde{w}(t)$$

$$+ A_{kd}^{T} e_{ref,\tilde{w}(t)} (t - \tau_{1}(t)) + e_{ref,\tilde{w}(t)}^{T} \psi_{ref,\tilde{w}(t)} (t)$$

$$- (1 - \hat{\lambda}_{d}) e_{ref,\tilde{w}(t)}^{T} \psi_{ref,\tilde{w}(t)} (t - \tau_{1}(t)) \psi_{ref,\tilde{w}(t)} (t - \tau_{1}(t))$$

$$+ \hat{\lambda}_{d} \left[ \sum_{i=1}^{p} q(t) A_{rf} e_{ref,\tilde{w}(t)} (t) + K_{w} \tilde{w}(t) \right]$$

$$\times \left[ \sum_{i=1}^{p} w_{i}(t) (A_{rf} e_{ref,\tilde{w}(t)} (t) + K_{w} \tilde{w}(t)) \right] + A_{kd}^{T} e_{ref,\tilde{w}(t)} (t - \tau_{1}(t))$$

$$\times \int_{t_{\tau_1}}^{t} e_{ref,\tilde{w}(t)}^{T} \psi_{ref,\tilde{w}(t)} (\tau) \, d\tau$$

$$+ 2[e_{ref,\tilde{w}(t)}(t) + e_{ref,\tilde{w}(t)}(t - \tau_{1}(t))] Y + \tilde{w}(t) W$$

$$\times \left[ \right.$$

$$\left. e_{ref,\tilde{w}(t)}^{T}(t) - e_{ref,\tilde{w}(t)}^{T}(t - \tau_{1}(t)) \right]$$

$$\left. - \int_{t_{\tau_1}}^{t} e_{ref,\tilde{w}(t)}^{T}(\tau) \, d\tau \right]. \quad (26)$$

With some computational efforts, it will be concluded that:

$$J_{ref,\tilde{w}(t)} \leq \int_{0}^{\infty} \sum_{i=1}^{p} w_{i}(t) \left[ e_{ref,\tilde{w}(t)}^{T}(t) \right]$$

$$\times \left[ e_{ref,\tilde{w}(t)}^{T}(t - \tau_{1}(t)) \right]^{T}$$

$$\times \left[ C^{T} V^{T} C + 2 P A_{rf} + S + 2X \quad PA_{fdi} - X + Y^{T} \quad S(1 - \hat{\lambda}_{d}) - 2Y \right]$$

$$\times \left[ \right.$$

$$\left. C^{T} V^{T} P_{1} + W^{T} + PK_{w} \quad -W^{T} \quad -\alpha^{2} \right]$$

$$\left. \right]$$

$$\times \left[ \right.$$

$$\left. e_{ref,\tilde{w}(t)}^{T}(t) \right]$$

$$\times \left[ e_{ref,\tilde{w}(t)}^{T}(t - \tau_{1}(t)) \right]^{T}$$

$$\times \left[ A_{rf}^{T} A_{fdi}^{T} \quad A_{fdi}^{T} Q K_{wi} \right]$$

$$\times \left[ \right.$$

$$\left. \right.$$
Therefore, it is straightforward that:

\[
J_{\text{ref}, \tilde{w}(t)} \leq \int_0^\infty \left\{ \sum_{i=1}^p w_i(t) \Theta_i - \int_{t-h_n}^t \Gamma^T(t, \tau) Q^{-1} \Gamma(t, \tau) \, d\tau \right\} \, dt,
\]

where

\[
v(t) = \begin{bmatrix} e_{\text{ref}, \tilde{w}(t)}(t) \\ e_{\text{ref}, \tilde{w}(t)}(t - \tau_1(t)) \\ \tilde{w}(t) \end{bmatrix}
= \begin{bmatrix} E_{11} & PA_{\bar{h}} - X + Y^T \\ * & -S(1 - \bar{h}_d) - Y - Y^T \\ * & * \\ C^T V^T V P_1 + W^T + PK_{wi} \\ -W^T \\ -\alpha^2 I + P_{11}^T V^T P_1 \end{bmatrix}
\]

and

\[
E_{11} = PA_{\bar{h}} + A_{\bar{h}}^T P + S + X + X^T + C^T V^T V C
\]

\[
\Theta_i = v^T(t) \Theta_i v(t) + \hat{h}_d v(t) \begin{bmatrix} A_{\bar{h}}^T \\ A_{\bar{h}}^T_{\bar{f}z} \\ A_{\bar{f}z}^T \\ K_{wi} \\ K_{wi}^T \end{bmatrix}^T \begin{bmatrix} X & 0 & X \\ 0 & Y & 0 \\ 0 & 0 & W \end{bmatrix} \begin{bmatrix} X \\ Y \\ W \end{bmatrix} v(t)
\]

\[
\Gamma(t, \tau) = \dot{e}_{\text{ref}, \tilde{w}(t)}(\tau) + v(t)
= \begin{bmatrix} \dot{X} \\ Y \\ W \end{bmatrix},
\]

Therefore, it is straightforward that:

\[
J_{\text{ref}, \tilde{w}(t)} \leq \int_0^\infty \sum_{i=1}^p w_i(t) \Theta_i \, dt \leq 0.
\]

In order to satisfy Equation (91), it is necessary that \( \Theta_i \leq 0 \) for \( i = 1, 2, \ldots, p \). Applying Schur complement to this equation results in LMI constraint of Theorem 3.1. Proof is completed.

In the next step, to generate a residual with the most possible sensitivity to fault signal, criteria \( \|G_{\text{ref}, \tilde{f}(t)}\| \geq \beta \) should be met. Theorem 3.2 provides conditions to satisfy this criteria.

**Theorem 3.2:** If there were symmetric positive definite matrices \( Q, S, P \) and \( Z^* \), matrices \( X, Y, \Psi_{\bar{y}}^1, \Psi_{\bar{y}}^2 \) and \( W \), positive scalars \( \beta, v \) and \( \delta \), such that following LMI constraint holds for \( i = 1, 2, \ldots, p \):

\[
\begin{bmatrix} \Theta_i & \Psi_{\bar{y}}^1 D = 0, \\ PA_{\bar{f}z}^* - X + Y^T \\ * -S(1 - \bar{h}_d) - Y - Y^T + vl + I \\ * * \\ * * \\ PK_{wi}^* + W^T + C^T Z^* P_2 \\ -F^T \\ -\beta^2 I + P_{11}^* Z^* P_2 - \epsilon vl \\ * \\ \sqrt{\hat{h}_d} X \sqrt{\hat{h}_d} (PA_{\bar{f}z}^*)^T \\ \sqrt{\hat{h}_d} Y \sqrt{\hat{h}_d} (PA_{\bar{f}z}^*)^T \\ \sqrt{\hat{h}_d} F \sqrt{\hat{h}_d} (PK_{wi}^*)^T \\ -Q \\ * -2\delta P + \delta^2 Q \end{bmatrix} < 0
\]

and also the following equations hold:

\[
PE_j - \Psi_{\bar{y}}^1 CE_i + \Psi_{\bar{y}}^2 D = 0, \quad Z^* D = 0.
\]

Then, the following system is asymptotically stable and \( \|G_{\text{ref}, \tilde{f}(t)}\| \geq \beta \).

\[
\bar{e}(t) = \sum_{i=1}^p w_i(x(t))[A_{\bar{h}} e(t) + A_{\bar{f}z} e(t - \tau_1(t)) + K_{wi} \tilde{w}(t)]
\]

\[
r(t) = V^* C e(t) + V^* P_1 \tilde{w}(t).
\]

**Proof:** The proof of Theorem 3.2 is just the same as what was derived for Theorem 3.1, with a different performance index as follows:

\[
J_{\text{ref}, \tilde{f}(t)} \geq \int_0^\infty [r_{\text{ref}, \tilde{f}(t)}^T(t) r_{\text{ref}, \tilde{f}(t)}(t) - \beta^2 \dot{f}(t) \dot{f}(t)] \, dt \geq 0.
\]

The rest of proof is such as what was applied to Theorem 3.1.

**Remark 3.1:** To design a residual signal with the most possible sensitivity to the fault and least sensitivity to other signals, Theorems 3.1 and 3.2 should be met simultaneously. This issue is needed to be considered in simulations. This issue can be solved with an iterative algorithm (Ahmadizadeh et al., 2014). Based on this algorithm, to solve these LMIs simultaneously, at first, a large value for \( \alpha \) and an small one for \( \beta \) should be considered. Using
these parameters, all LMIs should be solved. If the results was feasible, designer should decrease $\alpha$ and increase $\beta$ and thereafter, feasibility of LMIs should be checked. This method should be continued until the LMI constraints were found are infeasible. The last values of $\alpha$ and $\beta$ are chosen as the result of the multi-objective optimization problem.

4. Simulation results

4.1. Numerical example

To show the effectiveness of proposed method in UIO designing, here a numerical example is provided. As mentioned before, in order to get a residual signal with the most possible sensitivity to the fault and the least sensitivity to other signals, both theorems presented in this paper should be met simultaneously. System parameters are supposed to be as follows:

$A_1 = \begin{bmatrix} -3.8 & 1.5 & -0.5 \\ 0.5 & -3 & 1 \\ -0.3 & 0.7 & -0.2 \end{bmatrix}$,

$A_2 = \begin{bmatrix} -1.8 & 0.5 & -0.5 \\ 5 & -3 & 0.9 \\ -0.3 & 0.7 & -2.4 \end{bmatrix}$,

$A_{d1} = \begin{bmatrix} 0.4 & 0.1 & -0.2 \\ 0.1 & -0.8 & 0.2 \\ 0.7 & -0.1 & 0.5 \end{bmatrix}$,

$A_{d2} = \begin{bmatrix} 0.4 & -0.1 & -0.1 \\ 0.1 & -0.3 & 0.2 \\ 0.1 & -0.1 & 0.5 \end{bmatrix}$,

$B_1 = \begin{bmatrix} 0.1 \\ 0.2 \\ -0.4 \end{bmatrix}$, $B_2 = \begin{bmatrix} 0 \\ 0.4 \\ -0.1 \end{bmatrix}$,

$B_{d1} = \begin{bmatrix} -0.3 \\ 0 \\ 0.1 \end{bmatrix}$, $B_{d2} = \begin{bmatrix} 0.5 \\ -0.2 \\ 0 \end{bmatrix}$,

$E_1 = \begin{bmatrix} -0.4 \\ 0.1 \\ -0.3 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 \\ 0.2 \\ -0.3 \end{bmatrix}$,

$H_1 = \begin{bmatrix} 0.1 \\ 0.2 \\ -0.4 \end{bmatrix}$, $H_2 = \begin{bmatrix} 0.6 \\ -0.3 \\ -0.1 \end{bmatrix}$,

$F_1 = \begin{bmatrix} 0.6 \\ -0.5 \\ 0.4 \end{bmatrix}$, $F_1 = \begin{bmatrix} 0.1 \\ 0.2 \\ -0.3 \end{bmatrix}$,

$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $F_y = \begin{bmatrix} 0.2 \\ 0.8 \\ -1.2 \end{bmatrix}$.

$F_w = \begin{bmatrix} 0.2 \\ 0.8 \\ -1.2 \end{bmatrix}$, $D = \begin{bmatrix} 0.2 \\ 0.5 \\ -0.1 \end{bmatrix}$.

Solving LMI constraints in Theorems 3.1 and 3.2, will lead to following gain matrices of proposed observer. To solve these LMIs, $\alpha = 1.4$ and $\beta = 0.37$ will result in feasible LMIs. Consequently, the design parameters are obtained as follows:

$N_{1f1} = \begin{bmatrix} -0.36 & -1.33 & -0.006 \\ -0.19 & -0.48 & 0.06 \end{bmatrix}$,

$N_{1f2} = \begin{bmatrix} -0.07 & 0.5 & 0.54 \\ -0.17 & -0.41 & 0.74 \end{bmatrix}$,

$N_{2f1} = \begin{bmatrix} 0.28 & -0.13 & -0.1 \\ 0.09 & -0.41 & -0.03 \\ 0.55 & -0.26 & -0.21 \end{bmatrix}$,

$N_{2f2} = \begin{bmatrix} 0.13 & -0.06 & -0.04 \\ 0.09 & -0.04 & -0.03 \\ 0.08 & -0.04 & -0.02 \end{bmatrix}$,

$V = \begin{bmatrix} 1.35 & -0.87 & -1.64 \\ -0.84 & 0.54 & 1.02 \\ -1.48 & 0.96 & 1.83 \end{bmatrix}$.

In UIO design for interval type-2 T–S fuzzy systems, upper and lower membership functions are supposed to be as follows:

$w_I(w(t)) = 0.95 - \frac{0.925}{1 + e^{-\frac{\tau_1(t)+15}{15}}}$,

$w_{II}(w(t)) = 0.95 - \frac{0.925}{1 + e^{-\frac{\tau_1(t)+4.5}{15}}}$,

$w_I(x(t)) = 0.05 + \frac{0.925}{1 + e^{-\frac{\tau_1(t)+15}{15}}}$,

$w_{II}(x(t)) = 0.05 + \frac{0.925}{1 + e^{-\frac{\tau_1(t)+4.5}{15}}}$.

Furthermore, parameters $\tau_1$ and $\tau_2$ are considered to be constant and equal to 0.5. Step signal is chosen as known input and time-varying delays are as $\tau_1(t) = 0.5(1 + \sin(t))$ and $\tau_2(t) = 0.2(1 - \sin(t))$. Unknown input and disturbance are supposed to be white Gaussian noise with power of 0.005 and $t/(1 + t^2)$, respectively.

To check the validity of proposed observer in detection of faults, for the first case, a sinusoidal fault with amplitude of 0.5 and frequency of $\pi$ is supposed to affect the system. Such a fault is shown in Figure 1. As a simulation, this fault is applied to the system from $t = 4$ s to $t = 6$ s. An optimal residual signal, should have the value of 0 before and after the presence of fault and model...
the fault accurately during its presence. But, as you know due to uncertainties about the system and observer gains, such an optimal residual signal cannot be obtained. However, the more close residual signal to such a signal, the better decision about the presence of fault. The effects of this sinusoidal fault on residual signals are demonstrated in Figure 2. It is clear that residual signals are deformed after the fault occurrence. Additional information about the type of fault (sinusoidal) and even its amplitude and frequency, can be obtained from residual signals.

To make a decision about fault occurrence in the system (fault detection), a performance index should be considered. Here in this paper, root mean square norm of residual signal (just 1st residual signal has been simulated) is as performance index. It is defined as follows:

\[
\|r\|_{2,d} > J_{th} \\
\|r\|_{2,d} = \int_{t_1}^{t_2} r^T(t)r(t) \, dt, \quad d = t_2 - t_1 \tag{38}
\]

\[
J_{th} = (\sup \|r\|_{2,d})_{\bar{f}(t)=0}.
\]

Performance index (38) is computed for both conditions, when the system is fault free, and when the system is affected by sinusoidal fault. Results are shown for type-1 and interval type-2 observers in Figure 3. It is evident that both observers are able to detect fault presence at \(t = 4\) s. However, interval type-2 observer has generated residual signals with bigger amplitude during the period

![Figure 1](image1.png)  
**Figure 1.** Sinusoidal fault applied to the system.

![Figure 2](image2.png)  
**Figure 2.** Residuals correspond to sinusoidal fault; comparison between type-1 and interval type-2 UIOs. (a) 1st residual signal, (b) 2nd residual signal and (c) 3rd residual signal.

![Figure 3](image3.png)  
**Figure 3.** Residual evaluation regards to sinusoidal fault. (a) Type-1 observer and (b) interval type-2 observer.
As another case, effects of an Abrupt fault is going to be investigated. This fault with amplitude of 0.5 is demonstrated in Figure 4. Same as what proposed for sinusoidal fault, here residuals variations due to existence of this fault are shown in Figure 5. Furthermore, performance index (38), for the case of abrupt fault is as Figure 6.

As it was shown through simulations, both observers were able to detect faults in the system. However, when fault is in the incipient stages, the performance of observers is more important. Because, when the fault has low amplitude, it shows that fault is in its initial stages. In this case, fault should be detected and the system should be repaired, otherwise, the system will face with a big damage. Therefore, it is important to detect fault in the first stages.

To compare the results of this paper with the method proposed in Ahmadizadeh et al. (2014), in the case of sinusoidal fault, the amplitude of fault is reduced to 0.1 to simulate an incipient fault. Performance index (38) for this fault is shown in Figure 7. It is obvious that when the amplitude of the fault is small, type-1 observer proposed in Ahmadizadeh et al. (2014) can detect the fault presence; however, with a considerable delay, while proposed interval type-2 observer make a more appropriate decision about fault occurrence.

![Figure 4. Abrupt fault applied to the system.](image)

![Figure 5. Residuals correspond to abrupt fault; comparison between type-1 and interval type-2 UIOs. (a) 1st residual signal, (b) 2nd residual signal and (c) 3rd residual signal.](image)

![Figure 6. Residual evaluation regards to abrupt fault. (a) Type-1 observer and (b) interval type-2 observer.](image)
4.2. A single link manipulator

In this section, to show the applicability of the proposed method, fault detection of a single link manipulator with revolute joints, which is actuated by a DC motor, is studied. This system is depicted in Figure 8 and its model can be described as (Ichalal, Marx, Ragot, & Maquin, 2010):

\[
\begin{align*}
\dot{\theta}(t) &= \omega_m(t) \\
\dot{\omega}_m(t) &= \frac{k}{J_m}(\theta_l(t) - \theta_m(t)) - \frac{b}{J_m}\omega_m(t) + \frac{K_r}{J_m}u(t) \\
\dot{\theta}_l(t) &= \omega_l(t) \\
\dot{\omega}_l(t) &= -\frac{k}{J_l}(\theta_l(t) - \theta_m(t)) - \frac{mg}{J_l}\sin(\theta_l(t)),
\end{align*}
\]

(39)

where \(\theta_m(t)\) is the angular position of the motor, \(\omega_m(t)\) shows the angular velocity of the motor, \(\dot{\theta}_l\) demonstrates the angular position of the link and \(\dot{\omega}_l(t)\) stands for angular speed of the link. State representation of this system is as follows:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) + Ew(t) + f(x(t)) \\
y(t) &= Cx(t) + Gw(t),
\end{align*}
\]

(40)

Furthermore, T–S fuzzy model of the system can be obtained based on a nonlinear sector transformation approach (Tanaka & Wang, 2004) such as follows:

\[
\dot{x}(t) = \sum_{i=1}^{p}[A_ix(t) + B_iu(t)]
\]

(41)

However, to investigate the problem of fault detection for this system, distribution of disturbance and fault are added to the system (41). Based on this concept, this system can be represented by T–S fuzzy model with following components:

\[
A_1 = 
\begin{bmatrix}
0 & 1 & 0 & 0 \\
-48.6 & -1.25 & 48.6 & 0 \\
0 & 0 & 0 & 1 \\
19.5 & 0 & -22.83 & 0
\end{bmatrix}
\]

\[
A_2 = 
\begin{bmatrix}
0 & 1 & 0 & 0 \\
-48.6 & -1.25 & 48.6 & 0 \\
0 & 0 & 0 & 1 \\
19.5 & 0 & -18.77 & 0
\end{bmatrix}
\]
Iterative algorithm proposed in Remark 1 will lead to parameters $\alpha = 1.1$ and $\beta = 0.55$. These choices of $\alpha$ and $\beta$ will result in observer gains such as follows:

$$B_1 = B_2 = \begin{bmatrix} 0 & 21.6 & 0 & 0 \end{bmatrix}^T, \quad D = \begin{bmatrix} 0.5 \\ 0.4 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} -0.7 \\ 0.68 \\ -0.43 \\ 0.1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0.9 \\ 0.4 \\ -0.9 \\ 0.2 \end{bmatrix}, \quad H_1 = \begin{bmatrix} 0.1 \\ 0.3 \\ -0.8 \\ 0.1 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 0.4 \\ -0.2 \\ -0.5 \\ 0.1 \end{bmatrix}, \quad F_{x_1} = \begin{bmatrix} -0.45 \\ 1.04 \\ 0.2 \end{bmatrix}, \quad F_{x_2} = \begin{bmatrix} -0.2 \\ 0.45 \\ -0.6 \\ 0.5 \end{bmatrix}, \quad F_y = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}, \quad F_w = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$B_{f_1} = B_{f_2} = \begin{bmatrix} -30.833 & -5.2642 & 33.4345 & -1.418 \\ 0.8574 & -5.0082 & 4.1487 & 0.599 \\ 9.7089 & -2.8605 & -5.2731 & 1.775 \\ 64.1292 & 10.4989 & -69.6251 & 4.112 \end{bmatrix}$$

$$A_{f_1} = \begin{bmatrix} -34.2372 & -7.1871 & 31.7336 & -5.572 \\ 2.5688 & -4.1081 & 4.0554 & 2.270 \\ 10.8849 & -2.5676 & -4.5552 & 2.951 \\ 64.4534 & 9.0961 & -64.5876 & 4.437 \end{bmatrix}$$

$$A_{f_2} = \begin{bmatrix} -30.833 & -5.2642 & 33.4345 & -1.418 \\ 0.8574 & -5.0082 & 4.1487 & 0.599 \\ 9.7089 & -2.8605 & -5.2731 & 1.775 \\ 64.1292 & 10.4989 & -69.6251 & 4.112 \end{bmatrix}$$

$$B_{f_1} = B_{f_2} = \begin{bmatrix} 14.1038 \\ -2.4592 \\ -4.3953 \\ -19.9351 \end{bmatrix}$$

$$N_{f_1} = \begin{bmatrix} -14.9827 & 23.1658 \\ -2.7415 & 0.6954 \\ 1.9655 & -5.3886 \\ 37.2289 & -49.1280 \end{bmatrix}$$

The same as previous, an abrupt fault with the domain of 0.5, which starts at $t = 3$ s and ends at $t = 5$ s of the simulation affects the system. Residual signals and evaluation of the first residual are demonstrated in Figure 9. The results clearly show how the proposed method is effective in fault detection of practical systems.

5. Conclusion

In this paper, a new method in UIO design for fault detection of interval type-2 T–S fuzzy systems was proposed. The considered system was supposed to be affected by time-varying delays and parameter uncertainties that made the design procedure more challenging. In this regards, two new theorems were introduced to show how the observer gains can be calculated in a way that residual signal shows the most possible sensitivity to the fault and least possible sensitivity to other signals. Necessary conditions for observer design were expressed in the terms of LMIs. To show validness of proposed method, the detection of two different kinds of faults were demonstrated. In this regard, type-1 and interval type-2 observers were compared. Both observers are able to detect the faults as soon as occurrence in the system. Moreover, the detection of a fault with small amplitude was addressed to show the superiority of the proposed method. The results illustrate how the proposed observer is able to detect fault in an appropriate time, while type-1 fuzzy observers can detect the fault with a considerable delay. Furthermore, to show how the proposed method can be used in real systems, fault detection of a single link manipulator was studied in simulation section. It was shown how this
observer can deal with a fault in such a system and detect it in an appropriate time.

As further work, it is well known that in some applications, premise variables of T–S fuzzy systems are a function of system states which may not be partially or completely measurable. In these cases, the premise variables should be estimated to use in the system. Such a problem will lead to new aspects in UIO design, therefore, the results can be extended to the case of immeasurable premise variables. Furthermore, this UIO is used for normal T–S fuzzy systems, while one can use the proposed method in switched T–S fuzzy systems or networked control systems. Moreover, in the case of observer-based control problems, the proposed method can be useful in better estimation.

**Disclosure statement**

No potential conflict of interest was reported by the authors.

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