Non-commutative integrability, exact solvability and the Hamilton–Jacobi theory

Sergio Grillo

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Abstract
The non-commutative integrability (NCI) is a property fulfilled by some Hamiltonian systems that ensures, among other things, the exact solvability of their corresponding equations of motion. The latter means that an “explicit formula” for the trajectories of these systems can be constructed. Such a construction rests mainly on the so-called Lie theorem on integrability by quadratures. It is worth mentioning that, in the context of Hamiltonian systems, the NCI has been for around 40 years, essentially, the unique criterium for exact solvability expressed in the terms of first integrals (containing the usual Liouville–Arnold integrability criterium as a particular case). Concretely, a Hamiltonian system with \( n \) degrees of freedom is said to be non-commutative integrable if we know a set of independent first integrals \( F_1, \ldots, F_l \) such that: the kernel of the \( l \times l \) matrix with coefficients \( \{ F_i, F_j \} \), where \( \{ \cdot, \cdot \} \) denotes the canonical Poisson bracket, has dimension \( 2n - l \) (isotropy); and each bracket \( \{ F_i, F_j \} \) is functionally dependent on \( F_1, \ldots, F_l \) (closure). In this paper, we develop two procedures for constructing the trajectories of a Hamiltonian system which only require isotropic first integrals (closure condition is not needed). One of them is based on an extended version of the geometric Hamilton–Jacobi theory, and does not rely on the above mentioned Lie’s theorem. We do all that in the language of functions of several variables.

1 Introduction

Consider an ordinary differential equation (ODE) of the form

\[ \dot{y}(t) = f(t, y(t)) . \]
It is well-known that the continuity of $f$ on an open subset $U \subseteq \mathbb{R}^2$ is enough to ensure the existence of a solution passing through each point of $U$ (see for instance [2,10]). But, which is the expression of such a solution? Can we find an explicit formula for it? In general, we do not know how to do that. In the particular case in which the continuous function $f$ is of the form

$$f(t, y) = g(t)/h(y)$$

(and consequently $h(y) \neq 0$ for all $y$), the unique solution passing through $(t_0, y_0) \in U$ is given by

$$\int_{y_0}^{y(t)} h(s) \, ds = \int_{t_0}^{t} g(s) \, ds,$$

or equivalently, in terms of a primitive or quadrature $H$ (resp. $G$) of $h$ (resp. $g$), the solution is the curve satisfying

$$H(y(t)) = H(y_0) + G(t) - G(t_0).$$

Since, $H'(y) = h(y) \neq 0$ for all $y$, it is clear that $H$ is injective, so above equation can be univocally solved for $y(t)$. In any case, for the given function $f$, we could transform the original ODE into an algebraic equation (with the same solutions). In general, this is the best we can do in order to solve an ODE (or a system of them). When this happens, one uses to say that the given ODE is exactly solvable. And when the data of the algebraic equation, as in the present case, is given by primitives of the original data, one says that the solutions can be constructed, or integrated, up to quadratures. Since this is the usual situation, we shall use the phrases “exactly solvable” and “integrable up to quadratures” as synonyms.

In the context of Hamiltonian systems, the exact solvability of their equations of motion is ensured by the non-commutative integrability property. A Hamiltonian system defined by $H : \mathbb{R}^{2n} \to \mathbb{R}$ is non-commutative integrable (NCI), super-integrable or Mischenko-Fomenko integrable [17] (see also [11] and references therein), if functions $F_1, \ldots, F_l : \mathbb{R}^{2n} \to \mathbb{R}$ such that:

1. (independence) the rank of the Jacobian matrix of $F = (F_1, \ldots, F_l)$ is $l$;
2. (first integrals) $\{F_i, H\} = 0$ for all $i$;
3. (isotropy) the matrix with coefficients $\{F_i, F_j\}$ has a kernel of dimension $2n - l$;
4. (closure) for each $i, j$ there exists a function $P_{ij} : \text{Im}F \subseteq \mathbb{R}^l \to \mathbb{R}$ such that $\{F_i, F_j\} = P_{ij} \circ (F_1, \ldots, F_l)$;

are known.¹ Here, $\{\cdot, \cdot\}$ denotes the canonical Poisson bracket. The usual way of proving that the trajectories of such a system can be explicitly constructed relays on the Lie theorem on integrability by quadratures [4,14] (for recent extensions of the theorem, see [6,7]).

¹ What is important here is not the existence of the functions $F_1, \ldots, F_l$, but the fact that we know them. In order to emphasize that, sometimes a NCI system is defined as a pair $(H, F)$, with $F = (F_1, \ldots, F_l)$ satisfying conditions above.
We are omitting another conditions that sometimes appear in the definition of a NCI system, as the compactness and connectedness of the common level sets of the functions \( F_1, \ldots, F_l \). Such additional conditions ensure certain qualitative behavior of the system, and also some geometric properties (see [11] for a review), in which we are not interested right now.

Note that the isotropy condition implies that \( l \geq n \). In the particular case in which \( l = n \), we have the usual notion of integrability: Liouville–Arnold or commutative integrability (CI) [3,15]. In such a case, the last two conditions collapse into the isotropy condition only, which says that \( \{ F_i, F_j \} = 0 \) for all \( i, j \).

If the functions \( F_1, \ldots, F_l \) are just defined (or satisfy above conditions) on an open subset \( U \subseteq \mathbb{R}^{2n} \), we shall say that the system is NCI along \( U \) (and CI along \( U \) when \( l = n \)). And, if for each point of \( \mathbb{R}^{2n} \) we know an open neighborhood \( U \) such that the system is NCI along \( U \), we shall say that the system is locally NCI (and locally CI when \( l = n \)).

It is worth mentioning that, around every point of \( \mathbb{R}^{2n} \), excluding the critical points of the system, there always exist functions \( F_i \)'s satisfying the conditions (1) to (4). This is a direct consequence of the Carathéodory-Jacobi-Lie theorem (see for instance [13]). However, in the definition of NCI, we are not asking the existence of such functions, but the knowledge of them. In fact, it is this knowledge (and not just the existence) what enable us to construct up to quadratures the trajectories of the system (via the above mentioned Lie’s theorem).

At this point, we can formulate the following theoretical question: are all the properties defining a NCI system essential in order to ensure the exact solvability of a Hamiltonian system? In this work, we show that conditions (1), (2) and (3) are enough for such a purpose. We do that by following two different ways.

- Firstly we prove that, from an independent set of isotropic first integrals, i.e. functions satisfying (1), (2) and (3), we can construct, around almost every point of the phase space, a set of functions also satisfying (4). Here, by “almost” we mean that the construction works on an open dense subset of \( \mathbb{R}^{2n} \). This implies that, in such a subset, the system is locally NCI and, consequently, using the Lie’s theorem, is also exactly solvable there. It is worth mentioning that, in order to construct the trajectories of the system, we need to use all the first integrals, not just the original isotropic ones.
- Secondly, we use a generalized version of the Hamilton–Jacobi theory [9]. In this case, we develop an alternative procedure for constructing the trajectories of the system that only uses the given isotropic first integrals (no need for constructing additional first integrals). Such a procedure is an extension of the usual construction of canonical transformations via the Hamilton’s characteristic functions (which is the main aim of the standard Hamilton–Jacobi theory [3,8]). In particular, the Lie’s theorem is not needed this time. Also, the integration is ensured in the whole of the phase space (not only along a dense subset).

Part of the content of this paper already appeared in [9], but in that reference such a content was presented in the language of the symplectic geometry. Here, we make a rather different presentation (of the results and their proofs), using (when possible) only elementary concepts of the calculus of several variables. Our aim is to reach a more
general public, with no background in differential geometry. We can do that simply because the exact solvability is, essentially, a local aspect of a dynamical system.

Summarizing, the main goal of the paper is two-fold:

a. To show the following theoretical result: “for a given Hamiltonian system, the knowledge of a set of isotropic first integrals is enough to integrate its equations of motion up to quadratures.”

b. To state and prove above result by using the language of functions of several variables.

The paper is organized as follows. In Sect. 2 we prove the well-known fact that local NCI implies exact solvability. Although the proof is rather standard, we do it with some detail because we want to highlight the kind of procedures which are involved in the construction of the trajectories. Then, at the end of the section, we give our first proof of the result described in the point a above. In Sect. 3 we make a brief review of the standard Hamilton–Jacobi theory, emphasizing its relationship with the local CI. In Sect. 4 we present the extension of the Hamilton–Jacobi theory that appears in Ref. [9]. Instead of working in the context of the symplectic geometry, as in the mentioned paper, we shall work in the simpler framework of functions of several variables. In Sect. 5 we show the relationship between the extended theory and the NCI. Finally, in Sect. 6, we elaborate a new procedure for constructing (up to quadratures) the trajectories of a Hamiltonian system, based on the above mentioned extension of the Hamilton–Jacobi theory. This constitutes a second proof of our main result (see point a again).

We shall assume that the reader is familiar with the basic ideas related to Hamiltonian systems [3,8] and to the calculus of several variables. Nevertheless, below, we introduce some notation and recall some useful concepts and results associated to those subjects.

Notation, conventions and some basic concepts.

- Throughout this paper, all the functions will be of class $C^\infty$ on an open subset $A$ of some $\mathbb{R}^m$. For instance, if we say that a function is left (resp. right) invertible, we shall be assuming that it has a left (resp. right) inverse of class $C^\infty$. So, if we have a left and right invertible function, then such a function is a **diffeomorphism**: class $C^\infty$, bijective and with inverse of class $C^\infty$.

- Given $F : A \subseteq \mathbb{R}^m \to \mathbb{R}^k$, we shall denote by $DF (\mathbf{x}) \in \text{Mat}(k \times m, \mathbb{R})$ the Jacobian matrix or differential of $F$ at the point $\mathbf{x} \in A$, i.e. the $k \times m$ real matrix with coefficients

$$
[DF (\mathbf{x})]_{ij} = \frac{\partial F_i}{\partial x_j} (\mathbf{x}), \quad i = 1, \ldots, k, \quad j = 1, \ldots, m,
$$

where each $F_i$ (resp. $x_j$) is a component of $F$ (resp. $\mathbf{x}$). We shall also see $DF (\mathbf{x})$ as a linear transformation from $DF (\mathbf{x}) : \mathbb{R}^m \to \mathbb{R}^k$. If $k = 1$ (i.e. $F = F_1$), $DF (\mathbf{x})$ is a row vector that we shall sometimes denote

$$
DF (\mathbf{x}) = \frac{\partial F}{\partial \mathbf{x}} (\mathbf{x}).
$$
• By the **rank** of \( F \) at \( x \) we shall mean the number \( \text{rank} F (x) := \dim \text{Im} \{ DF (x) \} \). A set of \( k \) functions \( F_1, \ldots, F_k : A \subseteq \mathbb{R}^m \to \mathbb{R} \) is **independent** if the rank of \( F := (F_1, \ldots, F_k) : A \subseteq \mathbb{R}^m \to \mathbb{R}^k \) is \( k \) for all \( x \in A \). In other words, the linear transformation \( DF (x) : \mathbb{R}^m \to \mathbb{R}^k \) is surjective for all \( x \). It can be shown that, in such a case, \( \text{Im} F \subseteq \mathbb{R}^k \) is an open subset. Given a set of independent functions \( F_1, \ldots, F_k \), for every \( \lambda \in \text{Im} F \) we shall say that each pre-image \( F^{-1} (\lambda) \) is a **manifold of dimension** \( m - k \). On the other hand, given another function \( G : A \to \mathbb{R} \), we shall say that \( G \) is **dependent on** \( F_1, \ldots, F_k \) if there exists \( P : \text{Im} F \to \mathbb{R} \) such that \( \lambda = P \circ F \); and \( G \) is **locally dependent on** \( F_1, \ldots, F_k \) if for each \( x \in A \) there exists an open neighborhood \( U \) of \( x \) and a function \( P : F \circ (U) \subseteq \mathbb{R}^k \to \mathbb{R} \) such that \( G|_U = P \circ (F_1, \ldots, F_k)|_U \).

• In this paper, we shall restrict ourself to Hamiltonian systems whose phase space is contained in \( \mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n \). This is because we are only interested in local aspects of these systems. So, the Hamiltonians will be functions \( H : A \subseteq \mathbb{R}^{2n} \to \mathbb{R} \). Nevertheless, for simplicity, we shall usually assume that \( A = \mathbb{R}^{2n} \). Denoting the points of \( \mathbb{R}^n \) by \( (q, p) = (q^1, \ldots, q^n, p_1, \ldots, p_n) \), the canonical equations for a Hamiltonian \( H \) are

\[
\dot{q}(t) = \frac{\partial H}{\partial p} (q(t), p(t)) \quad \text{and} \quad \dot{p}(t) = -\frac{\partial H}{\partial q} (q(t), p(t)). \quad (1.1)
\]

• Given two functions \( F, G : A \subseteq \mathbb{R}^{2n} \to \mathbb{R} \), its canonical Poisson bracket \( \{ F, G \} : A \subseteq \mathbb{R}^{2n} \to \mathbb{R} \) is given by

\[
\{ F, G \} (x) = DF (x) \cdot J \cdot (DG (x))^t, \quad (1.2)
\]

where

\[
J = \begin{bmatrix} 0_n & I_n \\ -I_n & 0_n \end{bmatrix} \in \text{Mat} (2n \times 2n, \mathbb{R}) \quad (1.3)
\]

and \( 0_n \) (resp. \( I_n \)) denotes the \( n \times n \) null (resp. identity) matrix. Note that \( J^{-1} = -J = J^t \).

• By a **vector field** on \( A \subseteq \mathbb{R}^m \) we shall mean a function \( X : A \to \mathbb{R}^m \). A set of vector fields \( X_1, \ldots, X_r \) on \( A \) is **linearly independent** if so is the set of vectors \( X_1 (x), \ldots, X_r (x) \in \mathbb{R}^m \) for all \( x \in A \). Given two vector fields \( X \) and \( Y \), its **Lie bracket** \([ X, Y ]\) is the vector field given by

\[
[X, Y] (x) = X (x) \cdot (DY (x))^t - Y (x) \cdot (DX (x))^t.
\]

Every vector field \( X \) on \( A \) defines a dynamical system whose trajectories, also called the **integral curves** of \( X \), are the functions \( \gamma : I \subseteq \mathbb{R} \to A \) such that

\[
\frac{d}{dt} \gamma (t) = X (\gamma (t)). \quad (1.4)
\]
Given a manifold $N \subseteq A$ defined by a function $F$, we shall say that $X$ is **tangent to** $N$, or that $N$ is an **invariant manifold** for $X$, if

$$X(x) \cdot (DF(x))^T = 0, \quad \forall x \in N.$$  \hspace{1cm} (1.5)

It can be shown that this is the same as saying that all the integral curves of $X$ passing through $N$ are entirely contained in $N$ (for a proof, see Ref. [5]).

- Given a function $H : A \subseteq \mathbb{R}^{2n} \rightarrow \mathbb{R}$, the vector field $X_H$ given by

$$X_H(x) = -DH(x) \cdot J = J \cdot (DH(x))^T$$  \hspace{1cm} (1.6)

is called **Hamiltonian vector field** associated to $H$. It is easy to see [combining (1.1), (1.4) and (1.6)] that the integral curves of $X_H$ are exactly the trajectories of $H$. Also, given functions $F$ and $G$ on $A \subseteq \mathbb{R}^{2n}$, it can be shown that

$$[X_F, X_G] = -X_{\{F,G\}}.$$  \hspace{1cm} (1.7)

### 2 NCI, isotropy and Lie integrability

In this section, in the first place, we give a proof of the well-known fact that local NCI implies exact solvability. We do that for later convenience, in order to highlight the kind of procedures involved in the construction of the trajectories of a Hamiltonian system. This will enable us to compare the different integration procedures that appear along the paper.

Secondly, we present the first proof of our main result: one of the conditions appearing in the definition of NCI, the closure condition, is no needed for ensuring exact solvability.

#### 2.1 From NCI to exact solvability

Let us show that a locally NCI system is exactly solvable. The proof will be based on the theorem below. Before stating and proving it, let us introduce some terminology.

We shall say that “a function $F : A \subseteq \mathbb{R}^m \rightarrow \mathbb{R}^k$ **can be constructed**” if its domain $A$ and their values $F(x)$ (for all $x \in A$)

- are simply known;
- they can be determined by making a finite number of arithmetic operations (as the calculation of a determinant) and/or solving a finite set of linear equations (which actually can be reduced to arithmetic operations);
- or they can be expressed in terms of the derivatives and/or lateral inverses of another (known) functions.

If the expression of $F$ also involves primitives of another functions, we shall say that “$F$ **can be constructed up to quadratures.**”
Theorem 1 (Lie integrability by quadratures) Given a vector field $X$ on $\mathbb{R}^m$ tangent to an $r$-dimensional manifold $N \subseteq \mathbb{R}^m$, if we know $r$ linearly independent vector fields $X_1, \ldots, X_r$ tangent to $N$ such that $[X_i, X_j](x) = [X_i, X](x) = 0$ for all $i, j$ and all $x \in N$, then the integral curves of $X$ contained inside $N$ can be constructed up to quadratures.

Proof Given $p \in N$, if $X(p) = 0$, then the integral curve through $p$ is the constant function. So, let us assume that $X(p) \neq 0$. We are going to construct (up to quadratures), around $p$, a set of local coordinates $(y_1, \ldots, y_r)$ for the manifold $N$ where the equations of motion adopt the form

$$\dot{y}_1(t) = 1, \quad \dot{y}_2(t) = \cdots = \dot{y}_r(t) = 0. \quad (2.1)$$

Since the vector fields $X, X_1, \ldots, X_r$ are tangent to $N$, $N$ is $r$-dimensional and the $X_i$’s are independent, then we can write $X(p)$ as a linear combination of the vectors $X_i(p)$’s. Since $X(p) \neq 0$, such a linear combination must have some non-null coefficient. Let us assume that the first coefficient is non-null (otherwise, we can reorder the vector fields). This means that the vectors $X(p), X_2(p), \ldots, X_r(p)$ are independent. By continuity, there exists an open neighborhood $U \subseteq N$ of $p$ where the vector fields $X, X_2, \ldots, X_r$ are independent. From now on, let us write $X = X_1$.

Let $(x_1, \ldots, x_r)$ be local coordinates for $N$ defined on $U$ (shrinking $U$ if needed). Since the vector fields $X_1, \ldots, X_r$ are tangent to $N$, the related directional derivatives of a function $f : U \to \mathbb{R}$ can be written

$$X_i \cdot (Df)_t = \sum_{j=1}^r b_{ij} \frac{\partial f}{\partial x_j},$$

for certain functions $b_{ij} : U \subseteq N \to \mathbb{R}$. Now, for each $k = 1, \ldots, r$, consider the equations

$$\sum_{j=1}^r b_{ij} \frac{\partial f}{\partial x_j} = \delta_{ik}, \quad i = 1, \ldots, r,$$

being $\delta_{ik}$ the Kronecker delta. Since the vector fields $X_i$’s are independent along $U$, then the matrix with coefficients $b_{ij}$’s must be invertible. So, last equations are equivalent to

$$\frac{\partial f}{\partial x_i} = (b^{-1})_{ik}, \quad i = 1, \ldots, r. \quad (2.2)$$

On the other hand, it is easy to show that condition $[X_i, X_j] = 0$ is equivalent to

$$\frac{\partial}{\partial x_j} (b^{-1})_{ik} = \frac{\partial}{\partial x_i} (b^{-1})_{jk}, \quad i, j, k = 1, \ldots, r,$$

It is well-known that this kind of coordinates always exist around non-critical points of any vector field. What is important here it is not their existence, but the fact that they can be constructed.
what implies that Eq. (2.2) can be solved by quadratures. In fact, for each $k$, the general solution $y_k$ is given by the formula

$$y_k(x_1, \ldots, x_r) = \sum_{i=1}^{r} \int_{x_{0,i}}^{x_i} \left( b^{-1} \right)_{ik} \left( x_{0,1}, \ldots, x_{0,i-1}, t, x_{i+1}, \ldots, x_r \right).$$

We can choose the numbers $x_{0,i}$’s as the coordinates of $p$. In such a case, it is clear that the functions $y_1, \ldots, y_r$ define a new coordinate system of $N$ around $p$. In particular, they are independent. Moreover, they satisfy

$$X_1 \cdot (Dy_k)^t = \delta_{1k}, \quad k = 1, \ldots, r,$$

what implies precisely Eq. (2.1). Then, the integral curves $(x_1(t), \ldots, x_r(t))$ of the field $X = X_1$ around $p$ are given by the algebraic equations

$$y_1(x_1(t), \ldots, x_r(t)) = t + y_{0,1}, \quad y_j(x_1(t), \ldots, x_r(t)) = y_{0,j},$$

for $j = 2, \ldots, r$, which can be univocally solved for the $x_i(t)$’s because the functions $y_i$ are independent. Since all that can be done around any point of $N$, the theorem is proved.

**Remark** As we said in the Introduction, along all of this paper, the phrases “the system is exactly solvable” and “the trajectories of the system can be constructed up to quadratures” will be used as synonyms.

Now, suppose that we have a NCI system (as defined in the Introduction) with Hamiltonian function $H : \mathbb{R}^{2n} \to \mathbb{R}$ and independent first integrals $F_1, \ldots, F_l$.

**Example 1** As example of a NCI system, we can consider the isotropic harmonic oscillator with 3 degrees of freedom. Its Hamiltonian function $H : \mathbb{R}^6 \to \mathbb{R}$ is given by

$$H(q, p) = p_1^2 + p_2^2 + p_3^2 + (q_1)^2 + (q_2)^2 + (q_3)^2.$$

Beside $H$, the functions $H_i, P_{ij} : \mathbb{R}^6 \to \mathbb{R}, 1 \leq i, j \leq 3$, with

$$H_i(q, p) = p_i^2 + (q_i)^2 \quad \text{and} \quad P_{ij}(q, p) = q_j p_i - q_i p_j,$$

are also first integrals for the system. It can be shown that $H, H_1, H_2, P_{12}$ and $P_{13}$ are independent inside an open dense subset $A \subseteq \mathbb{R}^6$, and for dimensional reasons they must be isotropic (see point (3)) and must satisfy the closure condition (see point (4)). So, the system is NCI along $A$. In this case, since the number of independent first integrals is equal to $2n - 1$, it says that the system is maximally superintegrable.
Define $F := (F_1, \ldots, F_l)$ and for each $\lambda \in \text{Im} F$ (the range of $F$) consider the level set

$$F^{-1}(\lambda) = \left\{ \mathbf{x} \in \mathbb{R}^{2n} : F(\mathbf{x}) = \lambda \right\}.$$  

Note that:

- $\mathbb{R}^{2n}$ is a (disjoint) union of the subsets $F^{-1}(\lambda)$;
- (independence) each subset $F^{-1}(\lambda)$ is a manifold of dimension $r := 2n - l$;
- (first integrals) each trajectory of the system is contained inside some level set $F^{-1}(\lambda)$, i.e. the Hamiltonian vector field $X_H$ is tangent to each manifold $F^{-1}(\lambda)$.

As a consequence, in order to find all the trajectories of our system, it is enough to look for them on each $r$-dimensional manifold $F^{-1}(\lambda)$.

**Proposition 1** Under above conditions and notation, for each $\lambda_0 \in \text{Im} F$, a set of vector fields $X_1, \ldots, X_r$ on $\mathbb{R}^{2n}$ tangent to $F^{-1}(\lambda_0)$ and such that

$$\left[ X_i, X_j \right](\mathbf{x}) = \left[ X_i, X_H \right](\mathbf{x}) = 0, \quad \forall \mathbf{x} \in F^{-1}(\lambda_0),$$

can be constructed.

**Proof** The closure condition [see point (4) above] says that

$$\left\{ F_i, F_j \right\}(\mathbf{x}) = P_{ij}(\lambda), \quad \forall \mathbf{x} \in F^{-1}(\lambda),$$

for some functions $P_{ij}$. Note that the last equation determines completely each function $P_{ij}$. So, the functions $P_{ij}$’s are known. On the other hand, the isotropy condition [see point (3)] ensures, for each $\lambda$, the existence of $r = 2n - l$ linearly independent vectors $v^1_\lambda, \ldots, v^r_\lambda \in \mathbb{R}^l$ such that

$$\sum_{j=1}^l P_{ij}(\lambda) \left( v^k_\lambda \right)_j = 0, \quad i = 1, \ldots, l, \quad k = 1, \ldots, r.$$  

Moreover, it can be shown that, given $\lambda_0 \in \text{Im} F$, we can find a neighborhood $V$ of $\lambda_0$ and functions (of class $C^\infty$)

$$\lambda \in V \mapsto \left( v^k_\lambda \right)_j \in \mathbb{R}, \quad j = 1, \ldots, l,$$

satisfying above equation. (We just have to make standard linear manipulations). Consider the related vector fields [see Eq. (1.6)]

$$X^\lambda_i := \sum_{j=1}^l \left( v^k_\lambda \right)_j X_{F_j}, \quad i = 1, \ldots, r.$$  

It is easy to see that they are linearly independent. In addition, using the point (2) and Eq. (1.7), we have

$$- [X_{F_j}, X_H] = X_{\{F_j, H\}} = 0, \quad j = 1, \ldots, l,$$

what implies that

$$[X_i^\lambda, X_H] = 0, \quad i = 1, \ldots, r. \quad (2.5)$$

And, since

$$[X_{Fa}, X_{Fb}] (x) = - \sum_{k=1}^l {\partial P_{ab} \over \partial \lambda_k} (F (x)) X_{F_k} (x)$$

[using again (1.6) and (1.7)], then

$$[X_i^\lambda, X_j^\lambda] (x) = - \sum_{k=1}^l (v_i^\lambda)_a {\partial P_{ab} \over \partial \lambda_k} (F (x)) (v_j^\lambda)_{b} X_{F_k} (x).$$

But, for all $\lambda \in U$, taking into account that $P_{ij} = - P_{ji}$,

$$(v_i^\lambda)_{a} {\partial P_{ab} \over \partial \lambda_k} (\lambda) (v_j^\lambda)_{b} = {\partial \over \partial \lambda_k} \left( (v_i^\lambda)_{a} P_{ab} (\lambda) (v_j^\lambda)_{b} \right) - {\partial (v_i^\lambda)_{a} \over \partial \lambda_k} P_{ab} (\lambda) (v_j^\lambda)_{b} - (v_i^\lambda)_{a} P_{ab} (\lambda) {\partial (v_j^\lambda)_{b} \over \partial \lambda_k} = 0 - 0 - 0 = 0,$$

so,

$$[X_i^\lambda, X_j^\lambda] (x) = 0, \quad \forall x \in F^{-1} (\lambda). \quad (2.6)$$

Finally, since

$$X_j^\lambda (x) \cdot (DF_i (x))' = \sum_{k=1}^l P_{ik} (F (x)) (v_j^\lambda)_{k} = 0$$

for all $i, j$ and $x \in F^{-1} (\lambda)$, then the vector fields $X_i^\lambda$ are tangent to $F^{-1} (\lambda)$, for all $\lambda \in V$ [see Eq. (1.5)]. Accordingly, it is enough to define $X_i := X_i^{\lambda_0}$. \qed

Combining above proposition and the Lie’s theorem, it is clear that the trajectories of a NCI system can be constructed up to quadratures. For such a construction, we need to follow the steps below. Given the functions $F_1, \ldots, F_l$:

1. construct, around each $\lambda_0 \in \text{Im} F$, the functions given by (2.3);
2. construct the vector fields $X^1_{\lambda_0}, \ldots, X^r_{\lambda_0}$ by using Eq. (2.4);
3. apply the (proof of the) Lie’s theorem to each manifold $F^{-1}(\lambda_0)$.

If the system is just locally NCI, above construction can be made on each open subset $U \subseteq \mathbb{R}^{2n}$ along which the system is NCI. Since those subsets cover the whole of $\mathbb{R}^{2n}$, again we can construct up to quadratures all the trajectories. Concluding,

**Theorem 2** Every (locally) NCI system is exactly solvable.

### 2.2 From isotropy to NCI

Now, let us see that, from a set of isotropic first integrals, we can construct another local first integrals that make the system a locally NCI system (unless on an open dense subset) and, consequently, exactly solvable. First, we need several auxiliary results.

**Lemma 1** Consider a function $G : A \subseteq \mathbb{R}^m \to \mathbb{R}$ and a set of independent functions $F_1, \ldots, F_k : A \subseteq \mathbb{R}^m \to \mathbb{R}$. If there exists $P : \text{Im}F \subseteq \mathbb{R}^k \to \mathbb{R}$ such that $G = P \circ (F_1, \ldots, F_k)$, i.e. $G$ is dependent on $F_1, \ldots, F_k$, then

$$\text{rank} \hat{F}(x) = k, \quad \forall x \in A,$$

where $\hat{F} := (F_1, \ldots, F_k, G)$. Reciprocally, if above condition holds, then the function $G$ is locally dependent on $F_1, \ldots, F_k$.

**Proof** For the first statement, note that $k \leq \text{rank} \hat{F}(x) \leq k + 1$. But if $\text{rank} \hat{F}(x) = k + 1$, then the functions $F_1, \ldots, F_k, G$ are independent and the equality $G = P \circ (F_1, \ldots, F_k)$ does not hold for any $P$. For the converse, use the constant rank theorem (see for instance Ref. [5], Theorem 7.1).

**Lemma 2** Given a function $G : A \subseteq \mathbb{R}^m \to \mathbb{R}$ and a set of independent functions $F_1, \ldots, F_k : A \subseteq \mathbb{R}^m \to \mathbb{R}$, the subset $A$ can be written as a disjoint union $A = D \cup I \cup B$ where $D$ and $I$ are open subsets, $B$ is a closed set (relative to $A$) with empty interior, the function $G$ is locally dependent on $F_1, \ldots, F_k$ along $D$ and $F_1, \ldots, F_k, G$ are independent along $I$.

**Proof** Define

$$R_j := \left\{ x \in A : \text{rank} \hat{F}(x) = j \right\},$$

with $\hat{F}$ as in the previous lemma. It is clear that $R_j = \emptyset$ if $j \neq k, k + 1$. Accordingly, $A = R_k \cup R_{k+1}$. Of course, $R_k = A - R_{k+1}$ (i.e. $R_k \cap R_{k+1} = \emptyset$) and, since $R_{k+1}$ is open (because $k + 1$ is the maximal rank), then $R_k$ is closed inside $A$ and we can write $R_k = \text{int}R_k \cup \partial R_k$ (here “int” and “$\partial$” are the interior and the border relative to $A$). Thus, the lemma follows by taking

$$D := \text{int}R_k, \quad I := R_{k+1} \quad \text{and} \quad B := \partial R_k,$$

3 Perhaps the result is quite expected, but, as far as we know, its proof is not published anywhere.
Lemma 3  Given a set of independent functions $F_1, \ldots, F_l$, and defining $F := (F_1, \ldots, F_l)$, the following statements are equivalent:

1. $F_1, \ldots, F_l$ are isotropic;
2. the function $F$ satisfies
   \[
   \dim \left[ \text{Ker} \left[ DF \cdot J \cdot (DF)^t \right] \right] = 2n - l; \quad (2.7)
   \]
3. the function $F$ satisfies
   \[
   \text{Ker} [DF] \subseteq \text{Im} \left[ J \cdot (DF)^t \right] = J \cdot \text{Im} \left[ (DF)^t \right]. \quad (2.8)
   \]

Proof  It is easy to show that
\[
\{ F_i, F_j \} = [DF \cdot J \cdot (DF)^t]_{ij},
\]
so the equivalence between 1 and 2 is immediate. Now, let us show the equivalence between 2 and 3. Note that, since $DF$ is surjective (because the functions $F_i$’s are independent), its transpose $(DF)^t$ is injective and, consequently, since $J$ is injective too, we have that
\[
\dim \left[ \text{Ker} \left[ DF \cdot J \cdot (DF)^t \right] \right] = \dim \left[ (J \cdot (DF)^t)^{-1} \cdot \text{Ker} [DF] \right] = \dim \left[ \text{Im} \left[ J \cdot (DF)^t \right] \cap \text{Ker} [DF] \right]. \quad (2.9)
\]
On the other hand, the surjectivity of $DF$ also says that $\dim [\text{Ker} [DF]] = 2n - l$. So, using (2.9), it follows that (2.7) holds if and only if (2.8) holds. □

Remark  Given a linear space $V$ and a subspace $U \subseteq V$, the annihilator of $U$, i.e. the linear forms vanishing on $U$, will be denoted $U^0$. Recall that, if $W \subseteq U$, then $U^0 \subseteq W^0$.

Lemma 4  Given a set of independent isotropic functions $F_1, \ldots, F_k : A \subseteq \mathbb{R}^m \to \mathbb{R}$, if we add a new function $F_{k+1} : A \subseteq \mathbb{R}^m \to \mathbb{R}$ such that $F_1, \ldots, F_k, F_{k+1}$ is an independent set, then such a bigger set is also isotropic.

Proof  Define $F := (F_1, \ldots, F_k)$ and $\hat{F} := (F_1, \ldots, F_k, F_{k+1})$. It is clear that
\[
\text{Ker} \left[ D\hat{F} \right] \subseteq \text{Ker} [DF], \quad (2.10)
\]
and consequently (see the last remark)
\[
(\text{Ker} [DF])^0 \subseteq \left( \text{Ker} \left[ D\hat{F} \right] \right)^0. \quad (2.11)
\]
On the other hand, a well-known fact of linear algebra tells us that
\[(\text{Ker}[DF])^0 = \text{Im}[DF]'.\] (2.12)

So, according to Lemma 3, the isotropy condition on \(F\) (see Eq. (2.8)) is equivalent to
\[
\text{Ker}[DF] \subseteq J \cdot (\text{Ker}[DF])^0.
\]

Finally, combining the last equation with (2.10) and (2.11) we have that \(\hat{F}\) is isotropic too. \(\square\)

Now, the announced construction.

**Proposition 2** Suppose that \(F_1, \ldots, F_k : A \subseteq \mathbb{R}^m \rightarrow \mathbb{R}\) are functions satisfying (1), (2) and (3). Then, around every point of an open dense subset of \(A\), a set of functions \(F_{k+1}, \ldots, F_l\) can be constructed such that \(F_1, \ldots, F_l\) satisfy the conditions (1), (2), (3) and (4).

**Proof** Let us consider, for each \(r \in \mathbb{N}\), the set of pairs
\[
S_r := \{(i, j) : i, j \in \{1, \ldots, r\}, i < j\}
\]
and a bijection \(\phi_r : S_r \rightarrow \{1, \ldots, r (r - 1) / 2\}\). (This is just to simplify the notation).

Given \((i, j) \in S_k\) with \(\phi_k(i, j) = a\), consider the set of first integrals \(F_1, \ldots, F_k, G^a\), where \(G^a := \{F_i, F_j\}\) (recall that the Poisson bracket of two first integrals is also a first integral). According to Lemma 2, we can decompose \(A\) as a disjoint union \(A = D^a \cup I^a \cup B^a\) with the properties mentioned in such a lemma. It is clear that, in the open subset \(\hat{D} := \bigcap_{a=1}^{k(k-1)/2} D^a\), the condition (4) is locally fulfilled for the functions \(F_1, \ldots, F_k\). Then, around every point of \(\hat{D}\), conditions (1), (2), (3) and (4) are true for \(F_1, \ldots, F_k\). Note that the complement of \(\hat{D}\) (inside \(A\)) is given by the union
\[
\bigcup_{a=1}^{k(k-1)/2} (I^a \cup B^a).
\]

Then, we can write
\[
A = \hat{D} \cup (\bigcup_a I^a) \cup \hat{B},
\]
where \(\hat{B} := \bigcup_{a=1}^{k(k-1)/2} B^a\) is a closed set (relative to \(A\)) with empty interior.

Let us focus on each open subset \(I^a\). There, the functions \(F_1, \ldots, F_k, F_{k+1} := G^a\) satisfy (1) by definition of \(I^a\), (2) because they are first integrals, and, according to Lemma 4, they also satisfy (3). Now, given \((i', j') \in S_{k+1}\) with \(\phi_{k+1}(i', j') = b\), consider the set of first integrals \(F_1, \ldots, F_k, F_{k+1}, G^b\), with \(G^b := \{F_{i'}, F_{j'}\}\), and the decomposition \(I^a = D^{a,b} \cup I^{a,b} \cup B^{a,b}\) of Lemma 2. Again, around every point of
each open subset \( \hat{D}^a = \bigcap_{b=1}^{k(k+1)/2} D^{a,b} \), the conditions (1), (2), (3) and (4) are true for the function \( F_1, \ldots, F_k, F_{k+1} \). And we can write

\[
I^a = \hat{D}^a \cup \left( \bigcup_b I^{a,b} \right) \cup \hat{B}^a,
\]

where each \( \hat{B}^a := \bigcup_{b=1}^{k(k+1)/2} B^{a,b} \) is a closed set (relative to \( I^a \)) with empty interior. Applying this procedure \( s \) times, we shall arrive at:

- open subsets \( I^{a_1, \ldots, a_s} \), contained in \( I^{a_1, \ldots, a_s-1} \), where the functions

\[
F_1, \ldots, F_k, F_{k+1} := G^{a_1,}, F_{k+2} := G^{a_2,}, \ldots, F_{k+s} := G^{a_s,}
\]

satisfy (1), (2) and (3);

- open subsets \( \hat{D}^{a_1, \ldots, a_{s-1}} \) where

\[
F_1, \ldots, F_k, F_{k+1} = G^{a_1,}, F_{k+2} = G^{a_2,}, \ldots, F_{k+s-1} = G^{a_{s-1,}}
\]

satisfy (1), (2), (3) and (4) around each point of it;

- and closed sets (relative to \( I^{a_1, \ldots, a_{s-1}} \)) with empty interior \( \hat{B}^{a_1, \ldots, a_{s-1}} \) such that

\[
I^{a_1, \ldots, a_{s-1}} = \hat{D}^{a_1, \ldots, a_{s-1}} \cup \left( \bigcup_{a_1} I^{a_1, \ldots, a_{s-1}} \right) \cup \hat{B}^{a_1, \ldots, a_{s-1}}.
\]

When \( s = m - k \), we shall have open subsets \( I^{a_1, a_2, \ldots, a_{m-k}} \) where

\[
F_1, \ldots, F_k, F_{k+1} = G^{a_1,}, F_{k+2} = G^{a_2,}, \ldots, F_m = G^{a_{m-k}}
\]

are independent, i.e. \( F = (F_1, \ldots, F_m) \) has rank \( m \) there, which is the maximal one. Then, in the next step,

\[
I^{a_1, \ldots, a_{m-k}, b} = B^{a_1, \ldots, a_{m-k}, b} = \emptyset
\]

for all \( b \), so \( I^{a_1, \ldots, a_{m-k}} = \hat{D}^{a_1, \ldots, a_{m-k}} \). As a consequence, we can write \( A = D \cup B \) where

\[
D = \hat{D} \cup \left( \bigcup_a \hat{D}^a \right) \cup \left( \bigcup_{a_1, a_2} \hat{D}^{a_1, a_2} \right) \cup \ldots \cup \left( \bigcup_{a_1, \ldots, a_{m-k}} \hat{D}^{a_1, \ldots, a_{m-k}} \right)
\]

is an open subset such that, around every point of it, functions \( F_{k+1}, \ldots, F_l \) can be constructed for which \( F_1, \ldots, F_l \) satisfy conditions (1), (2), (3) and (4), and

\[
B = \hat{B} \cup \left( \bigcup_a \hat{B}^a \right) \cup \left( \bigcup_{a_1, a_2} \hat{B}^{a_1, a_2} \right) \cup \ldots \cup \left( \bigcup_{a_1, \ldots, a_{m-k-1}} \hat{B}^{a_1, \ldots, a_{m-k-1}} \right)
\]

is a set with empty interior. The last fact says that \( D \) is dense inside \( A \), what ends our proof.

\[\square\]

Combining Theorem 2 and the last proposition, we easily have that:
Theorem 3 Given a Hamiltonian system, if we know a set of isotropic and independent first integrals, then the system is exactly solvable along an open dense subset of the phase space.

Remark 1 Given a Hamiltonian system, the existence of a set of local isotropic first integrals can be easily shown as a consequence of the Carathéodory-Jacobi-Lie theorem [13]. But in the theorem above (as in the definition of a NCI system) it is not the existence of a set of isotropic first integrals what is required, but the knowledge of such a set. Otherwise, the construction of the trajectories can not be done.

Example 2 Fix three symmetric linear transformations $T_i : \mathbb{R}^3 \to \mathbb{R}^3$, $i = 1, 2, 3$. Suppose that, for some non-null vector $v_0 \in \mathbb{R}^3$, the set $\{T_1(v_0), T_2(v_0), T_3(v_0)\}$ is a basis for $\mathbb{R}^3$. Then, the same is true inside an open neighborhood $U \subseteq \mathbb{R}^3$ of $v_0$. Denote by $\{e_1, e_2, e_3\}$ the canonical basis of $\mathbb{R}^3$. Now, for each $v \in U$, consider the linear isomorphism $\gamma(v) : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\gamma(v)(e_i) = T_i(v),$$

and denote by $\hat{\gamma}(v)$ its inverse. Finally, let $\Gamma_i^k(v)$ and $\hat{\Gamma}_k^i(v)$ be the matrix coefficients (in the canonical basis) of above linear transformations, i.e.

$$\gamma(v)(e_i) = \Gamma_i^k(v)e_k \quad \text{and} \quad \hat{\gamma}(v)(e_i) = \hat{\Gamma}_k^i(v)e_k.$$ 

Sum over repeated indices convention is assumed. It is clear that each number $\Gamma_i^k(v)$ depends linearly on $v$, so, if $v = v_l e_l$, we can write

$$\Gamma_i^k(v) = \Gamma_i^{kl} v_l$$

for some numbers $\Gamma_i^{kl}$. But the same is not true, in general, for the numbers $\hat{\Gamma}_k^i(v)$, which are rational functions of the variables $v_l$. Also, since each $T_i$ is symmetric and $\hat{\gamma}(v) = (\gamma(v))^{-1}$, it is clear that

$$\Gamma_i^{kl} = \Gamma_i^{lk} \quad \text{and} \quad \left(\Gamma_i^{kl} v_l\right) \left(\hat{\Gamma}_k^i(v)\right) = \delta_i^j.$$ 

With all that, consider a Hamiltonian function $H : \mathbb{R}^3 \times U \to \mathbb{R}$ defined by the formula

$$H(q, p) = \Gamma_1^{kl} p_k p_l + \hat{\Gamma}_k^1(p) q^k.$$ 

It can be shown that the functions $F_1 = H$,

$$F_2(q, p) = \Gamma_2^{kl} p_k p_l + \hat{\Gamma}_k^3(p) q^k, \quad F_3(q, p) = \Gamma_3^{kl} p_k p_l - \hat{\Gamma}_k^2(p) q^k$$

and

$$F_4(q, p) = \frac{1}{2} \left[\left(\hat{\Gamma}_k^2(p) q^k\right)^2 + \left(\hat{\Gamma}_k^3(p) q^k\right)^2\right]$$
define a set of independent isotropic first integrals for $H$ which does not satisfy the closure condition. Thus, according to theorem above, each one of these systems is integrable by quadratures. Analogous examples can be given for higher dimensional phase spaces. 

**Remark 2** Theorem 3 can be seen as a new criterium (i.e. a sufficient condition) for exact solvability of Hamiltonian systems, which is weaker than the NCI. Its usefulness is very clear: if for a given Hamiltonian system we know a set of isotropic first integrals (which do not necessarily satisfy the closure condition, as in the above example), then we can be sure that the system is exactly solvable. But such a new criterium does not give rise to new exactly solvable systems, beyond the NCI ones, as it is clear from Proposition 2. In other words, it is not possible to give examples of Hamiltonian systems satisfying this new criterium which are not (in essence) NCI systems. In this sense, the examples given in Example 2 have nothing special, since they define NCI systems too. Moreover, it can be shown that they are also CI ones (with the aid of another set of first integrals).\(^4\) We presented such examples just for illustrative purposes.

Concluding, we have given our first proof to the main result of the paper. It says that the closure condition is not essential, a priori, for ensuring exact solvability. Nevertheless, if we go over the above results, we can see that, in order to build up the trajectories, we previously need to construct more first integrals, in such a way that the resulting entire set of first integrals gives rise to a NCI system. Thus, at the end of the day, closure condition is involved in the integration process. This does not contradict our result, it simply says that in its proof such a condition still plays an important role. In the last section of the paper we shall present an alternative procedure for integrating the Hamilton equations that only uses the isotropic first integrals. In other words, we give a second proof of our result in which the closure condition is not used at all.

### 3 The standard Hamilton–Jacobi theory

The main idea behind the Hamilton–Jacobi theory is to find coordinates where the equations of motion of a Hamiltonian system adopt a very simple form \([3,8]\). Let us review such an idea.

Consider a Hamiltonian system with $n$ degrees of freedom defined by a Hamiltonian function $H$. The (time independent) **Hamilton–Jacobi equation (HJE)** is

$$
\frac{\partial}{\partial q} \left[ H \left( q, \frac{\partial W}{\partial q}(q) \right) \right] = 0, \quad (3.1)
$$

whose unknown is a function $W : \mathbb{R}^n \to \mathbb{R}$. The solutions $W$ of such an equation are called Hamilton's characteristic functions.

\(^4\) Given a NCI system, it can be shown that there exists a set of local first integrals for the system satisfying the properties of CI (see Ref. [11]). But in general, at the best of our knowledge, such first integrals can not be constructed up to quadratures. Thus, we can not ensure that the system is (locally) CI.
In practice, $W$ is usually defined only along an open subset of $\mathbb{R}^n$. In such a case, one says that $W$ is a **local solution**. Nevertheless, to simplify the notation, we shall assume that the domain is always the entire space.

One is actually interested in finding a “big enough” family of such solutions or, more precisely, a function $W : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ such that each

$$W_\lambda := W (\cdot, \lambda) : \mathbb{R}^n \rightarrow \mathbb{R} : q \mapsto W (q, \lambda), \quad \lambda \in \mathbb{R}^n,$$

is a solution of the HJE and

$$\det \left[ \frac{\partial^2 W}{\partial \lambda \partial q} (q, \lambda) \right] \neq 0, \quad \forall (q, \lambda) \in \mathbb{R}^{2n}. \quad (3.2)$$

Each function $W_\lambda$ is called a **partial solution** of the HJE.

**Remark** Note that the HJE implies that $H (q, \frac{\partial W}{\partial q} (q, \lambda))$ only depends on $\lambda$, i.e.

$$H \left( q, \frac{\partial W}{\partial q} (q, \lambda) \right) = h (\lambda) \quad (3.3)$$

for some function $h : \mathbb{R}^n \rightarrow \mathbb{R}$.

The condition (3.2) is the same as asking that the function $\Sigma : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{2n}$, given by

$$\Sigma (q, \lambda) = \left( q, \frac{\partial W}{\partial q} (q, \lambda) \right), \quad (3.4)$$

is a **local** diffeomorphism. This means that, for every couple of points $(q, \lambda), (q, p) \in \mathbb{R}^{2n}$ such that $\Sigma (q, \lambda) = (q, p)$, there exist open neighborhoods $V$ and $U$ of $(q, \lambda)$ and $(q, p)$, respectively, such that $\Sigma (V) = U$ and the restriction of $\Sigma$ to $V$ is a diffeomorphism with its image $U$.

**Remark** Unless a confusion may arise, every local inverse of $\Sigma$ will be indicated simply as $\Sigma^{-1}$ (no mention to the domain or codomain).

We shall also ask $\Sigma$ to be surjective. A function $\Sigma$ [given by (3.4)] with all these properties is called a **complete solution** of the HJE. If $\Sigma$ is not defined along all of $\mathbb{R}^n \times \mathbb{R}^n$, but along an open subset of it, we shall say that $\Sigma$ is a **local complete solution**.

Given a complete solution $\Sigma$, it can be shown [3,8] that the equations

$$Q = \frac{\partial W}{\partial \lambda} (q, \lambda), \quad p = \frac{\partial W}{\partial q} (q, \lambda), \quad (3.5)$$

define a new set of (local) canonical coordinates $Q = Q (q, p)$ and $\lambda = \lambda (q, p)$ around every point of the phase space, in terms of which the canonical Hamilton equations
[recall (1.1)] read

\[ \dot{Q}(t) = \frac{\partial K}{\partial \lambda}(Q(t), \lambda(t)), \quad \dot{\lambda}(t) = -\frac{\partial K}{\partial Q}(Q(t), \lambda(t)), \]

where \( K(Q, \lambda) = H(q(Q, \lambda), p(Q, \lambda)) \). The crucial point here is that [see Eq. (3.3)]

\[ K(Q, \lambda) = H(q(Q, \lambda), \frac{\partial W}{\partial q}(q(Q, \lambda), \lambda)) = h(\lambda), \]

and consequently the equations of motion translate to

\[ \dot{Q}(t) = \frac{\partial h}{\partial \lambda}(\lambda(t)), \quad \dot{\lambda}(t) = 0, \]

which can be easily solved. In fact, the general solution is given by

\[ Q(t) = Q_0 + t \frac{\partial h}{\partial \lambda}(\lambda_0), \quad \lambda(t) = \lambda_0. \]

Moreover, the trajectories \((q(t), p(t))\) of the system can be obtained through the algebraic equations [see (3.5)]

\[ \frac{\partial W}{\partial \lambda}(q(t), \lambda_0) = Q_0 + t \frac{\partial h}{\partial \lambda}(\lambda_0) \quad \text{and} \quad p(t) = \frac{\partial W}{\partial q}(q(t), \lambda_0). \]

We just must solve the first equation for \( q(t) \), which can be done because of condition (3.2).

On the other hand, the functions \( F_i(q, p) := \lambda_i(q, p) \) are local first integrals of the system and, since they are conjugate momenta, they are in involution, i.e. \( \{ F_i, F_j \} = 0 \) for all \( i, j = 1, \ldots, n \). Summing up,

**Theorem 4** Consider a Hamiltonian system with \( n \) degrees of freedom. If we know a complete solution \( \Sigma \) of the HJE for such a system, then the latter can be exactly solved. Moreover, the system is locally commutative integrable by means of the local first integrals

\[ F_i(q, p) = \lambda_i(q, p) = \left[ \Sigma^{-1}(q, p) \right]_{n+i}, \quad i = 1, \ldots, n. \]

The second affirmation in the last theorem establishes a deep connection between commutative integrability and the Hamilton–Jacobi theory: given a complete solution of the HJE, we have, around every point of \( \mathbb{R}^{2n} \), \( n \) local independent first integrals in involution. A reciprocal result is also true, under an additional assumption. Suppose that we have a set of \( n \) functions \( F_1, \ldots, F_n \) such that the set of vectors

\[ \left\{ \frac{\partial F_1}{\partial p}(q, p), \ldots, \frac{\partial F_n}{\partial p}(q, p) \right\} \]
is l.i. for all \((q, p)\). One says that the functions \(F_1, \ldots, F_n\) are \textit{vertically independent}. (This implies, in particular, that the involved functions are independent). Now, denote \(\pi : \mathbb{R}^{2n} \to \mathbb{R}^n\) the projection \(\pi (q, p) = q\) and define \(F := (F_1, \ldots, F_n) : \mathbb{R}^{2n} \to \mathbb{R}^n\). It can be shown that \((\pi, F) : \mathbb{R}^{2n} \to \mathbb{R}^n \times \mathbb{R}^n\) is a local diffeomorphism. Moreover, if the functions \(F_i\) are first integrals for a Hamiltonian \(H\) and they are in involution, then each inverse \(\Sigma := (\pi, F)^{-1}\) is a (local) complete solution of the HJE for \(H\). Thus: \textit{given \(n\) vertically independent first integrals in involution, we have, around every point of \(\mathbb{R}^{2n}\), a local complete solution of the HJE.}

All above results will be shown in the next sections, in a more general context.

4 An extended Hamilton–Jacobi theory

We have said in the last section that there is a deep connection between the (standard) Hamilton–Jacobi theory and the commutative integrability. Based on Ref. [9], we shall present below a slightly extension of such a theory which is intimately related to the non-commutative integrability. (Another extension of the Hamilton–Jacobi theory related to NCI has been developed in [12]).

4.1 Re-writing the HJE

Fix a Hamiltonian function \(H : \mathbb{R}^{2n} \to \mathbb{R}\). From now on, given a function \(F : A \subseteq \mathbb{R}^m \to \mathbb{R}\), by \(\nabla F (x)\) we shall denote the column vector \((DF (x))^t\). Given a solution \(W\) of \((3.1)\), let us define \(\sigma : \mathbb{R}^n \to \mathbb{R}^{2n}\) as

\[
\sigma (q) := (q, \hat{\sigma} (q)) := (q, (\nabla W (q))^t).
\]

It is clear that \(\sigma\) satisfies

\[
\nabla (H \circ \sigma) = 0 \quad \text{and} \quad \nabla \times \hat{\sigma}^t = 0. \tag{4.1}
\]

Reciprocally, given \(\sigma : \mathbb{R}^n \to \mathbb{R}^{2n}\) of the form

\[
\sigma (q) := (q, \hat{\sigma} (q)) \tag{4.2}
\]

and fulfilling \((4.1)\), then \((\hat{\sigma} (q))^t = \nabla W (q)\) for some function \(W\) satisfying the HJE. So, we can think of \((4.1)\) as the HJE and take the functions \(\sigma\) of the form \((4.2)\) as their unknowns. In these terms, the complete solutions will be given by a family of solutions \(\sigma_\lambda\) such that \(\Sigma (q, \lambda) = \sigma_\lambda (q)\) is a surjective local diffeomorphism. But we shall consider a further modification of \((4.1)\).

\textbf{Remark 3} Note that a function of the form \((4.2)\) has a left inverse (of class \(C^\infty\)), and consequently the same is true for each differential \(D\sigma (q)\). One of its left inverses is the projection \(\pi : \mathbb{R}^{2n} \to \mathbb{R}^n : (q, p) \mapsto q\). In particular, \(D\sigma (q) : \mathbb{R}^n \to \mathbb{R}^{2n}\) is
injective for all \( q \), what means that

\[
\dim \left[ \text{Im} \left[ D\sigma (q) \right] \right] = n, \quad \forall q \in \mathbb{R}^n. \quad (4.3)
\]

Fix a solution \( \sigma \) of (4.1). On the one hand, since

\[
\nabla (H \circ \sigma) (q) = (D\sigma (q))^t \cdot (\nabla H \circ \sigma (q)),
\]

we have that

\[
\nabla H \circ \sigma (q) \in \text{Ker} \left[ (D\sigma (q))^t \right], \quad \forall q \in \mathbb{R}^n. \quad (4.4)
\]

On the other hand, in terms of the matrix \( J \) [see Eq. (1.3)], the condition \( \nabla \times \hat{\sigma}^t = 0 \) is equivalent to

\[
(D\sigma (q))^t \cdot J \cdot D\sigma (q) = 0, \quad (4.5)
\]

what says that

\[
\text{Im} \left[ J \cdot D\sigma (q) \right] \subseteq \text{Ker} \left[ (D\sigma (q))^t \right].
\]

Since [recall Eq. (2.12)]

\[
\text{Ker} \left[ (D\sigma (q))^t \right] = (\text{Im} \left[ D\sigma (q) \right])^0,
\]

then [see (4.3)]

\[
\dim \left[ \text{Ker} \left[ (D\sigma (q))^t \right] \right] = 2n - \dim \left[ \text{Im} \left[ D\sigma (q) \right] \right] = 2n - n = n. \quad (4.6)
\]

Accordingly, using that \( \dim \left[ \text{Im} \left[ J \cdot D\sigma (q) \right] \right] = \dim \left[ \text{Im} \left[ D\sigma (q) \right] \right] \) (since \( J \) is invertible), we have the equality

\[
\text{Ker} \left[ (D\sigma (q))^t \right] = \text{Im} \left[ J \cdot D\sigma (q) \right]. \quad (4.7)
\]

So, combining Eqs. (4.4) and (4.7),

\[
\nabla H \circ \sigma (q) \in \text{Im} \left[ J \cdot D\sigma (q) \right], \quad \forall q \in \mathbb{R}^n. \quad (4.8)
\]

Concluding,

**Proposition 3** A function of the form (4.2) satisfies (4.1) if and only if satisfies [see (4.5) and (4.8)]

\[
\nabla H \circ \sigma (q) \in J \cdot \text{Im} \left[ D\sigma (q) \right] \quad \text{and} \quad (D\sigma (q))^t \cdot J \cdot (D\sigma (q)) = 0, \quad (4.9)
\]

for all \( q \in \mathbb{R}^n \).
Remark The first part of (4.9) is important from the geometric point of view, because it says that the image of $\sigma$ defines an invariant manifold for the Hamiltonian system. The second part says that such a manifold is Lagrangian [1,16].

4.2 Generalized solutions of the HJE

Now, the announced extension.

**Definition 1** A **generalized (partial) solution** of the HJE for $H$ is a left invertible function $\sigma : \mathbb{R}^r \rightarrow \mathbb{R}^{2n}$ (see Remark 3), for some natural $r$, satisfying [see Eq. (4.9)]

$$\nabla H \circ \sigma (x) \in J \cdot \text{Im} \left( D\sigma (x) \right) \quad \text{and} \quad (D\sigma (x))^t \cdot J \cdot (D\sigma (x)) = 0, \quad (4.10)$$

for all $x \in \mathbb{R}^r$. And a **generalized complete solution** of the HJE for $H$ is a family of partial solutions $\sigma_\lambda : \mathbb{R}^r \rightarrow \mathbb{R}^{2n}$ with the same left inverse, and with $\lambda \in \mathbb{R}^l$ for some natural $l$, such that

$$\Sigma : \mathbb{R}^r \times \mathbb{R}^l \rightarrow \mathbb{R}^{2n} : (x, \lambda) \mapsto \sigma_\lambda (x)$$

is a surjective local diffeomorphism.

When the function $\sigma$ (resp. $\Sigma$) is defined along a proper open subset of $\mathbb{R}^r$ (resp. $\mathbb{R}^r \times \mathbb{R}^l$), we shall say that $\sigma$ (resp. $\Sigma$) is a **generalized local solution** (resp. **generalized local complete solution**) of the HJE.

**Remark** In Ref. [9], we have called “solutions” to the functions $\sigma$ satisfying just the first part of (4.10), and “isotropic solutions” to those $\sigma$ that also satisfy the second part. This is because the image of $\sigma$ defines an isotropic [1,16] invariant manifold.

The values of $r$ and $l$ are not arbitrary.

**Proposition 4** If $\sigma : \mathbb{R}^r \rightarrow \mathbb{R}^{2n}$ is a generalized solution of the HJE for $H$, then $r \leq n$. And if $\Sigma : \mathbb{R}^r \times \mathbb{R}^l \rightarrow \mathbb{R}^{2n}$ is a generalized complete solution, then $r + l = 2n$.

**Proof** The second part of (4.10) is equivalent to

$$\text{Im} \left[ J \cdot D\sigma (x) \right] \subseteq \text{Ker} \left[ (D\sigma (x))^t \right]. \quad (4.11)$$

Since $\sigma$ is left invertible, then

$$\dim \left[ \text{Im} \left[ D\sigma (x) \right] \right] = r, \quad (4.12)$$

and consequently

$$\dim \left[ \text{Ker} \left[ (D\sigma (x))^t \right] \right] = 2n - \dim \left[ \text{Im} \left[ D\sigma (x) \right] \right] = 2n - r. \quad (4.13)$$

Finally, since $J$ is invertible, $\dim \left[ \text{Im} \left[ J \cdot D\sigma (x) \right] \right] = \dim \left[ \text{Im} \left[ D\sigma (x) \right] \right]$. So, from the last three equations we have that $r \leq 2n - r$, or equivalently, $r \leq n$.

Regarding the second affirmation of the proposition, the fact that $\Sigma$ must be a local diffeomorphism implies that $l + r = 2n$ (see Ref. [5]).
Remark Consider a function $\sigma$ satisfying the second part of (4.10). If $\sigma$ also satisfies the first part, it is clear that

$$\nabla (H \circ \sigma) (x) = 0, \quad \forall x \in \mathbb{R}^r. \quad (4.14)$$

But the converse is not true [i.e. if $\sigma$ satisfies (4.14), it is not true, in general, that it satisfies the first part of (4.10)], as happens for the $r = n$ case. This is precisely because the inclusion (4.11) is strict for $r < n$ [compare (4.6) and (4.13)].

The next result gives an alternative way of describing the generalized partial and complete solutions of the HJE which will be useful later.

**Proposition 5** A left invertible function $\sigma$ satisfying the second part of (4.10) is a generalized solution of the HJE for $H$ if and only if

$$\nabla H \circ \sigma (x) = J \cdot D\sigma (x) \cdot v_\sigma^H (x),$$

with

$$v_\sigma^H (x) := - [D \Pi (\sigma (x))] \cdot J \cdot (\nabla H \circ \sigma (x)), \quad (4.15)$$

being $\Pi : \mathbb{R}^{2n} \to \mathbb{R}^r$ some left inverse of $\sigma$. And a family of left invertible functions $\sigma_\lambda$, with common left inverse $\Pi$, defines a generalized complete solution $\Sigma$ if and only if $\Sigma$ is a surjective local diffeomorphism,

$$\nabla (H \circ \Sigma) (x, \lambda) = (D\Sigma (x, \lambda))^t \cdot J \cdot D\Sigma (x, \lambda) \cdot \left( \begin{array}{c} v_{\sigma_\lambda}^H (x) \\ 0 \end{array} \right) \quad (4.16)$$

[where $v_{\sigma_\lambda}^H (x)$ is given by (4.15)] and

$$\left( \begin{array}{c} w \\ 0 \end{array} \right)^t \cdot (D\Sigma (x, \lambda))^t \cdot J \cdot D\Sigma (x, \lambda) \cdot \left( \begin{array}{c} v \\ 0 \end{array} \right) = 0, \quad (4.17)$$

for all column vectors $v, w \in \mathbb{R}^r$.

**Proof** The condition $\nabla H \circ \sigma (x) \in J \cdot \text{Im} (D\sigma (x))$ is equivalent to the existence of a column vector $v \in \mathbb{R}^r$ such that

$$\nabla H \circ \sigma (x) = J \cdot D\sigma (x) \cdot v.$$

Let $\Pi$ be a left inverse of $\sigma$, i.e. $\Pi \circ \sigma = \text{id}_{\mathbb{R}^r}$. Note that $[D \Pi (\sigma (x))] \cdot D\sigma (x) = I_r$. Multiplying by $[D \Pi (\sigma (x))] \cdot J$ to the left both members of above equation, and using that $J^{-1} = -J$, we have

$$[D \Pi (\sigma (x))] \cdot J \cdot (\nabla H \circ \sigma (x)) = -v,$$
which proves the first part of the proposition. Now, consider a complete solution $\Sigma$ given by functions $\sigma_\lambda$ (all of them with the same left inverse $\Pi$). Then, according to the last result, they must satisfy

$$\nabla H \circ \sigma_\lambda (x) = J \cdot D\sigma_\lambda (x) \cdot v_H^{\sigma_\lambda} (x).$$

Since

$$D\sigma_\lambda (x) \cdot v_H^{\sigma_\lambda} (x) = D\Sigma (x, \lambda) \cdot \left( \begin{array}{c} v_H^{\sigma_\lambda} (x) \\ 0 \end{array} \right),$$
multiplying (4.18) by $(D\Sigma (x, \lambda))^t$ to the left, we have precisely the Eq. (4.16). To prove (4.17), it is enough to check that

$$(D\Sigma (x, \lambda))^t \cdot J \cdot D\Sigma (x, \lambda) \cdot \left( \begin{array}{c} v \\ 0 \end{array} \right) = \left( \begin{array}{c} 0 \\ v' \end{array} \right)$$

for some column vector $v'$, which is true thanks to the second part of (4.10) (for each $\sigma_\lambda$). The reciprocal follows reversing the previous steps. $\square$

5 Complete solutions and isotropic first integrals

Now, let us see that, related to any generalized complete solution $\Sigma$, we have a set of $l$ independent (local) first integrals. Let

$$p : \mathbb{R}^r \times \mathbb{R}^l \to \mathbb{R}^l : (x, \lambda) \mapsto \lambda$$

be the projection onto the second factor. Around a given point of $\mathbb{R}^{2n}$, fix a local inverse $\Sigma^{-1}$ of $\Sigma$ and define $F = (F_1, \ldots, F_l) = p \circ \Sigma^{-1}$. (Note that $F$ is only defined on the open subset where $\Sigma^{-1}$ is defined). Since

$$DF = \left[ Dp \circ \Sigma^{-1} \right] \cdot (D\Sigma)^{-1},$$

$D\Sigma$ is non-singular and $Dp$ is surjective, then $DF$ is also surjective, which means that the functions $F_i$’s are independent. Let us show that they are first integrals for $H$. The Poisson bracket between each $F_i$ and $H$ is [see (1.2)]

$$\{F_i, H\} = DF_i \cdot J \cdot \nabla H.$$}

So, the bracket vanishes for all $i = 1, \ldots, l$ if and only if $DF \cdot J \cdot \nabla H = 0$, or equivalently,

$$J \cdot \nabla H (q, p) \in \text{Ker} \left[ DF (q, p) \right].$$
In terms of points \((x, \lambda) = \Sigma^{-1}(q, p)\), this means that

\[
 J \cdot \nabla H (\Sigma (x, \lambda)) \in \text{Ker} \left[ DF (\Sigma (x, \lambda)) \right].
\]  

(5.4)

It is easy to see that

\[
 \text{Ker} [Dp (x, \lambda)] = \mathbb{R}^r \times \{0\},
\]

and accordingly [recall (5.2)]

\[
 \text{Ker} [DF (\Sigma (x, \lambda))] = (D \Sigma (x, \lambda)) \cdot \text{Ker} [Dp (x, \lambda)]
\]

\[
 = \text{Im} \left[ \frac{\partial \Sigma}{\partial x} (x, \lambda) \right] = \text{Im} [D\sigma_\lambda (x)].
\]  

(5.5)

Thus, Eq. (5.4) is equivalent to

\[
 \nabla H (\Sigma (x, \lambda)) = \nabla H \circ \sigma_\lambda (x) \in J \cdot \text{Im} [D\sigma_\lambda (x)],
\]

which is precisely the first part of (4.10). As a consequence, the functions \(F_i\)’s define a set of \(l\) independent first integrals for \(H\). Now, let us prove that they are isotropic. To do that, we just need the Lemma 3. In fact, combining (4.11) and (5.5) we have exactly the inclusion (2.8). Finally, according to Lemma 3, the functions \(F_1, \ldots, F_l\) are isotropic. So, given a complete solution, we have constructed a set of local independent isotropic first integrals around every point of the phase space. Moreover, the “inverse construction” can also be made, as we show below.

**Theorem 5** Given a complete solution \(\Sigma : \mathbb{R}^r \times \mathbb{R}^l \rightarrow \mathbb{R}^{2n}\) and a point of \(\mathbb{R}^{2n}\), a set of \(l\) independent and isotropic first integrals is defined by the formula [see (5.1)]

\[
 F = (F_1, \ldots, F_l) := p \circ \Sigma^{-1},
\]

being \(\Sigma^{-1}\) a local inverse of \(\Sigma\) around the given point. Reciprocally, given a set of \(l\) independent and isotropic first integrals \(F_1, \ldots, F_l\) and a point of \(\mathbb{R}^{2n}\), a generalized local complete solution with image around such a point can be constructed.

**Proof** The first implication have been proved above. Let us show the second one. Fix a point \((q^0, p^0)\) of \(\mathbb{R}^{2n}\). The independence of the functions \(F_i\)’s ensures that the \(l \times 2n\) matrix \(DF (q^0, p^0)\) has \(l\) columns linearly independent. For simplicity, let us write \(q_i = y_i\) and \(p_i = y_{n+i}\). Suppose that the mentioned columns are

\[
 \begin{bmatrix}
 \frac{\partial F_1}{\partial y_i} \\
 \frac{\partial F_2}{\partial y_i} \\
 \vdots \\
 \frac{\partial F_l}{\partial y_i}
\end{bmatrix}, \quad \begin{bmatrix}
 \frac{\partial F_1}{\partial y_i} \\
 \frac{\partial F_2}{\partial y_i} \\
 \vdots \\
 \frac{\partial F_l}{\partial y_i}
\end{bmatrix}, \ldots, \begin{bmatrix}
 \frac{\partial F_1}{\partial y_i} \\
 \frac{\partial F_2}{\partial y_i} \\
 \vdots \\
 \frac{\partial F_l}{\partial y_i}
\end{bmatrix}.
\]
Now, call $z_{r+k} = y_k$, $k = 1, \ldots, l$, and call $z_1, \ldots, z_r$ to the rest of $y$’s (in some order). Finally, define $\Pi = (\Pi_1, \ldots, \Pi_r) : \mathbb{R}^{2n} \to \mathbb{R}^r$ such that $\Pi_i (q, p) = z_i$, $i = 1, \ldots, r$. It is easy to see that the function $(\Pi, F) : \mathbb{R}^{2n} \to \mathbb{R}^r \times \mathbb{R}^l$ is locally invertible around $(q^0, p^0)$. It is enough to check that the differential $D (\Pi, F) (q^0, p^0)$ is a full rank $2n \times 2n$ matrix. Let us prove that any local inverse $\Sigma := (\Pi, F)^{-1}$ is a complete solution. To simplify the notation, we shall assume that $\Sigma$ is globally defined. Note first that each function $\sigma_\lambda : \mathbb{R}^r \to \mathbb{R}^{2n}$ is left inverted by $\Pi$. In fact, $$(\Pi, F) (\sigma_\lambda (x)) = (x, \lambda),$$ so $\Pi \circ \sigma_\lambda (x) = x$. It rests to show that each $\sigma_\lambda$ satisfies (4.10). To do that, it is enough to imitate the steps we made above. Since the functions $F_i$’s are first integrals, we know that Eqs. (5.3) and (5.4) hold. From the very definition of $\Sigma$, it is clear that $F = p \circ \Sigma^{-1}$, and consequently Eq. (5.5) also holds. Combining (5.4) and (5.5), we have the first part of (4.10) for each $\sigma_\lambda$. Finally, let us prove the second part. The isotropy of $F$ says that (2.8) holds, which combined with (5.5) gives rise to (4.11). But the last equation says precisely that the second part of (4.10) is fulfilled, as we wanted to show. \hfill $\Box$

**Example** Consider a Hamiltonian $H$ and a related set of isotropic first integrals as those given in Example 2. Define $F := (F_1, \ldots, F_4) : \mathbb{R}^3 \times U \to \mathbb{R}^4$. It is easy to see that $$(\Pi, F) : \mathbb{R}^3 \times U \to \mathbb{R}^2 : (q, p) \mapsto (p_1, p_2)$$ is transverse to $F$, i.e. $(\Pi, F)$ is locally invertible. According to the proof of above theorem, each local inverse $\Sigma$ is a (local) complete solution of the generalized HJE for $F_1 = H$. Let us construct one of them. It is clear that its domain must be included in $\mathbb{R}^2 \times \mathbb{R}^4$, whose points we shall denote $(x, \lambda) = (x_1, x_2, \lambda_1, \ldots, \lambda_4)$, and its codomain is $\mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3$, so we can write $\Sigma$ as a 2-component function $\Sigma = (\Sigma^1, \Sigma^2)$. Denote by $x_3$ a function of $(x_1, x_2, \lambda_1, \ldots, \lambda_4)$ satisfying the algebraic equation $$2 \lambda_4 - \left( \lambda_2 - \Gamma^{ij}_{1} x_i x_j \right)^2 - \left( \lambda_3 - \Gamma^{ij}_{3} x_i x_j \right)^2 = 0.$$ Then, an inverse of $(\Pi, F)$ is given by the formulae

$$\Sigma^{1,i} (x, \lambda) = \Gamma^{jk}_{i} x_k \alpha^i \quad \text{and} \quad \Sigma^{2,i} (x, \lambda) = x_i,$$  

(5.6)

where

$$\alpha^1 = \lambda_1 - \Gamma^{ij}_{1} x_i x_j, \quad \alpha^2 = -\lambda_3 - \Gamma^{ij}_{3} x_i x_j \quad \text{and} \quad \alpha^3 = \lambda_2 - \Gamma^{ij}_{2} x_i x_j.$$  

$\diamondsuit$
Summarizing, at a local level, having a generalized complete solution is the same as having a set of isotropic first integrals. As a consequence, using the results of Sect. 2.2, from a generalized complete solution we can construct a set of local first integrals that make our Hamiltonian system, along an open dense subset of the phase space, a locally NCI system. This gives rise to an extension of the Theorem 4 (and its “converse,” commented below it) to the present generalized context. Also, this tells us that the knowledge of a generalized complete solution ensures exact solvability. Note however that, according to the procedures presented so far, in order to construct the trajectories of the system, such a generalized complete solution is not enough (we also need the rest of first integrals constructed in Proposition 2).

In the next section we shall develop an alternative procedure which only uses a generalized (local) complete solution. This is because such a procedure does not rest on the Lie theorem on integrability by quadratures. Also, the new procedure enable us to find all the trajectories of the system, not only those contained in a dense subset of the phase space.

6 An alternative procedure for integration

Let us see that a generalized complete solution defines (up to quadratures) a transformation of the canonical equations into a set of algebraic equations.

**Proposition 6** Given a generalized complete solution \( \Sigma : \mathbb{R}^r \times \mathbb{R}^l \rightarrow \mathbb{R}^{2n} \) for \( H \), we can construct up to quadratures a function \( h : \mathbb{R}^l \rightarrow \mathbb{R} \) and a function \( W : \mathbb{R}^r \times \mathbb{R}^l \rightarrow \mathbb{R} \) such that

\[
H \circ \Sigma (x, \lambda) = h(\lambda), \quad \forall x \in \mathbb{R}^r,
\]

and

\[
\theta (\Sigma (x, \lambda)) \cdot \frac{\partial \Sigma}{\partial x} (x, \lambda) = \frac{\partial W}{\partial x} (x, \lambda), \quad \forall x \in \mathbb{R}^r,
\]

where \( \theta : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n} : (q, p) \mapsto (p, 0) \).

**Proof** Suppose that \( \Sigma \) is given by a family of partial solutions \( \sigma_\lambda \). Since \( \nabla (H \circ \sigma_\lambda) = 0 \) for each \( \lambda \) [recall (4.14)], then \( H \circ \sigma_\lambda (x) \) do not depends on \( x \), but only on \( \lambda \). This defines a function \( h : \mathbb{R}^l \rightarrow \mathbb{R} \) by the formula

\[
h(\lambda) := H \circ \sigma_\lambda (x) = H \circ \Sigma (x, \lambda),
\]

as we claim above [see (6.1)]. Now, let us construct \( W \). The condition \((D\sigma_\lambda) J \cdot D\sigma_\lambda = 0\), if we write \( \sigma_\lambda = (\sigma_\lambda^1, \sigma_\lambda^2) \), says exactly that

\[
\frac{\partial}{\partial x_k} \left[ \sum_{j=1}^n (\sigma_\lambda^2) \frac{\partial}{\partial x_i} (\sigma_\lambda^1) \right] - \frac{\partial}{\partial x_i} \left[ \sum_{j=1}^n (\sigma_\lambda^2) \frac{\partial}{\partial x_k} (\sigma_\lambda^1) \right] = 0,
\]
what implies that
\[ \sum_{j=1}^{n} \left( \sigma_{\lambda}^j \right) \frac{\partial}{\partial x_i} \left( \sigma_{\lambda}^1 \right) = \frac{\partial}{\partial x_i} W_{\lambda} \]

for some function $W_{\lambda}$. Moreover, as it is well-known, Eq. (6.3) ensures that each function $W_{\lambda}$ can be obtained up to quadratures. Finally, using the function $\theta$ defined above, it is easy to show that
\[
\sum_{j=1}^{n} \left( \sigma_{\lambda}^j \right) \frac{\partial}{\partial x_i} \left( \sigma_{\lambda}^1 \right) \left( x \right) = \left[ \theta \left( \sigma_{\lambda} \left( x \right) \right) \right] \frac{\partial}{\partial x} \sigma_{\lambda} \left( x \right),
\]

so, defining $W \left( x, \lambda \right) := W_{\lambda} \left( x \right)$, the Eq. (6.2) easily follows. \(\square\)

**Remark** If the partial solutions $\sigma_{\lambda}$ are of the form (4.2), then
\[
\theta \left( \Sigma \left( q, \lambda \right) \right) \cdot \frac{\partial \Sigma}{\partial q} \left( q, \lambda \right) = \left( \hat{\sigma}_{\lambda} \left( q \right), 0 \right) \cdot \left( \frac{\partial I_n}{\partial q} \left( q \right) \right) = \hat{\sigma}_{\lambda} \left( q \right).
\]

Thus, the function $W$ in the Eq. (6.2) is an extension of the idea of Hamilton’s characteristic function.

**Remark** In terms of the projection $p$ [see Eq. (5.1)], the function $h$ can be characterized by the equality
\[
H \circ \Sigma = h \circ p.
\]

**Example** Coming back to Example 2, for the local complete solution given by the Eq. (5.6), since $F \circ \Sigma \left( x, \lambda \right) = (\lambda_1, \ldots, \lambda_4)$, and consequently $H \circ \Sigma \left( x, \lambda \right) = \lambda_1$, it follows that
\[
h \left( \lambda \right) = \lambda_1.
\]

On the other hand, $W$ is given by
\[
W \left( x, \lambda \right) = \int_{x_{0,1}}^{x_1} \left( \Sigma_{i}^2 \frac{\partial \Sigma_{1,i}^{1}}{\partial x_1} \right) \left( t, x_2, \lambda \right) dt + \int_{x_{0,2}}^{x_2} \left( \Sigma_{i}^2 \frac{\partial \Sigma_{1,i}^{1}}{\partial x_2} \right) \left( x_{0,1}, t, \lambda \right) dt.
\]

Now, let us define $\varphi : \mathbb{R}^r \times \mathbb{R}^l \rightarrow \mathbb{R}^l$ as
\[
\varphi \left( x, \lambda \right) := \frac{\partial W}{\partial \lambda} \left( x, \lambda \right) - \theta \left( \Sigma \left( x, \lambda \right) \right) \cdot \frac{\partial \Sigma}{\partial \lambda} \left( x, \lambda \right).
\]

\(\diamond\)
**Proposition 7**  The linear map $\frac{\partial \varphi}{\partial x}(x, \lambda): \mathbb{R}^r \to \mathbb{R}^l$ is injective for all $(x, \lambda)$.

**Proof**  It is easy to see that, for every column vector $v \in \mathbb{R}^r$,

$$
\frac{\partial \varphi}{\partial x}(x, \lambda) \cdot v = \left( \frac{\partial}{\partial \lambda} \left[ \theta(\Sigma(x, \lambda)) \right] \cdot \frac{\partial \Sigma}{\partial x}(x, \lambda) - \frac{\partial}{\partial x} \left[ \theta(\Sigma(x, \lambda)) \right] \cdot \frac{\partial \Sigma}{\partial \lambda}(x, \lambda) \right) \cdot v
$$

(6.7)

where we are using that $D_p(x, \lambda) = (0_{l \times r} I_l)$ [see Eq. (5.1)]. We know from (4.17) that

$$
(D \Sigma(x, \lambda))' \cdot J \cdot D \Sigma(x, \lambda) \cdot \begin{pmatrix} v \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ v' \end{pmatrix}
$$

(6.8)

for some column vector $v' \in \mathbb{R}^l$, and consequently

$$
\frac{\partial \varphi}{\partial x}(x, \lambda) \cdot v = v'.
$$

If $v' = 0$, since $D \Sigma(x, \lambda)$ and $J$ are bijective linear maps, then $v$ must be zero, what ends our proof.

**Proposition 8**  A curve $(q(t), p(t))$ is a trajectory for $H$ with initial condition $(q_0, p_0)$ if and only if, given a local inverse of $\Sigma$ around $(q_0, p_0)$, the curve $(x(t), \lambda(t)) := \Sigma^{-1}(q(t), p(t))$ satisfies

$$
\varphi(x(t), \lambda_0) = \varphi(x_0, \lambda_0) - t D h(\lambda_0) \quad \text{and} \quad \lambda(t) = \lambda_0,
$$

(6.9)

with $(x_0, \lambda_0) := \Sigma^{-1}(q_0, p_0)$.

**Proof**  Consider a trajectory $(q(t), p(t))$ with initial condition $(q_0, p_0)$ and a local inverse $\Sigma^{-1}$ around $(q_0, p_0)$, and define $(x(t), \lambda(t)) := \Sigma^{-1}(q(t), p(t))$ and $(x_0, \lambda_0) := \Sigma^{-1}(q_0, p_0)$. (Of course, $x(0) = x_0$ and $\lambda(0) = \lambda_0$). We known from the previous section that

$$
\lambda(t) = p \circ \Sigma^{-1}(q(t), p(t))
$$

is constant. Then, $\lambda(t) = \lambda_0$. So, it is enough to see that

$$
\frac{d}{dt} [\varphi(x(t), \lambda_0)]' = -\nabla h(\lambda_0).
$$

(6.10)

On the one hand, Eq. (6.7) tells us that
\[
\frac{d}{dt} [\varphi (x(t), \lambda_0)]^t = Dp (x(t), \lambda_0) \cdot (D \Sigma (x(t), \lambda_0))^t \cdot J \cdot D \Sigma (x(t), \lambda_0) \cdot \begin{pmatrix} v \\ 0 \end{pmatrix}, \quad (6.11)
\]

with \( v^t = \dot{x}(t) \). On the other hand, using (1.1),

\[
\begin{pmatrix} v \\ 0 \end{pmatrix} = \frac{d}{dt} (x(t), \lambda(t))^t = D \Sigma^{-1} (q(t), p(t)) \cdot \begin{pmatrix} (\dot{q}(t))^t \\ (\dot{p}(t))^t \end{pmatrix} \]

\[
= D \Sigma^{-1} (q(t), p(t)) \cdot \begin{pmatrix} \frac{\partial H}{\partial p} (q(t), p(t))^t \\ -\frac{\partial H}{\partial q} (q(t), p(t))^t \end{pmatrix} \]

\[
= (D \Sigma (x(t), \lambda_0))^{-1} \cdot J \cdot (\nabla H (\Sigma (x(t), \lambda_0))). \quad (6.12)
\]

Thus,

\[
Dp (x(t), \lambda_0) \cdot (D \Sigma (x(t), \lambda_0))^t \cdot J \cdot D \Sigma (x(t), \lambda_0) \cdot \begin{pmatrix} v \\ 0 \end{pmatrix} = -Dp (x(t), \lambda_0) \cdot (D \Sigma (x(t), \lambda_0))^t \cdot \nabla H (\Sigma (x(t), \lambda_0)) \]

\[
= -Dp (x(t), \lambda_0) \cdot \nabla (H \circ \Sigma) (x(t), \lambda_0) = -\nabla h (\lambda_0), \quad (6.13)
\]

where we have used that

\[
\nabla (H \circ \Sigma) (x, \lambda) = (Dp (x, \lambda))^t \cdot \nabla h (\lambda) \quad (6.14)
\]

[see Eq. (6.4)] and

\[
Dp (x, \lambda) \cdot (Dp (x, \lambda))^t = (0_{l \times r} I_l) \cdot \begin{pmatrix} 0_{r \times l} \\ I_l \end{pmatrix} = I_l.
\]

This ends the proof of the first affirmation. For the converse, we must show the equality [see the first row of Eq. (6.12)]

\[
J \cdot D \Sigma (x(t), \lambda_0) \cdot \begin{pmatrix} v \\ 0 \end{pmatrix} = -\nabla H (\Sigma (x(t), \lambda_0)). \quad (6.15)
\]

Combining (6.10) and (6.11) we have that

\[
Dp (x(t), \lambda_0) \cdot (D \Sigma (x(t), \lambda_0))^t \cdot J \cdot D \Sigma (x(t), \lambda_0) \cdot \begin{pmatrix} v \\ 0 \end{pmatrix} = -\nabla h (\lambda_0),
\]
and multiplying by \((Dp(x(t), \lambda_0))^T\) [see Eq. (6.14)]

\[
(Dp(x(t), \lambda_0))^T \cdot Dp(x(t), \lambda_0) \cdot (D \Sigma(x(t), \lambda_0))^T \cdot J \cdot D \Sigma(x(t), \lambda_0) \cdot \begin{pmatrix} v \\ 0 \end{pmatrix}
\]

\[
= - \nabla (H \circ \Sigma)(x(t), \lambda_0) = -(D \Sigma(x(t), \lambda_0))^T \cdot \nabla H(\Sigma(x(t), \lambda_0))
\]

But, from Eq. (6.8) and the fact that

\[
(Dp(x, \lambda))^T \cdot Dp(x, \lambda) \cdot \begin{pmatrix} 0 \\ v' \end{pmatrix} = \begin{pmatrix} 0_{r \times l} \\ I_l \end{pmatrix} \cdot \begin{pmatrix} 0 \\ v' \end{pmatrix} = \begin{pmatrix} 0 \\ v' \end{pmatrix}
\]

for all column vectors \(v' \in \mathbb{R}^l\), the equality

\[
(D \Sigma(x(t), \lambda_0))^T \cdot J \cdot D \Sigma(x(t), \lambda_0) \cdot \begin{pmatrix} v \\ 0 \end{pmatrix}
\]

\[
= -(D \Sigma(x(t), \lambda_0))^T \cdot \nabla H(\Sigma(x(t), \lambda_0))
\]

follows, which clearly implies Eq. (6.15).

\[\square\]

**Example** In the case of the Example 2, because of (6.5), the algebraic equations are

\[
\phi^1(x(t), \lambda_0) = \phi^1(x_0, \lambda_0) - t \quad \text{and} \quad \phi^i(x(t), \lambda_0) = \phi^i(x_0, \lambda_0),
\]

for \(i \geq 2\), with

\[
\phi(x, \lambda) = \int_{x_{0,1}}^{x_1} \frac{\partial}{\partial \lambda} \left( \Sigma_2^2 \frac{\partial \Sigma^{1,i}}{\partial x_1} \right) (t, x_2, \lambda) \, dt + \int_{x_{0,2}}^{x_2} \frac{\partial}{\partial \lambda} \left( \Sigma_2^2 \frac{\partial \Sigma^{1,i}}{\partial x_2} \right) (x_{0,1}, t, \lambda) \, dt
\]

\[
- \left( \Sigma_2^2 \frac{\partial \Sigma^{1,i}}{\partial \lambda} \right) (x, \lambda).
\]

Summing up, giving a generalized complete solution, last proposition tells us that we can transform the canonical Hamilton equations into a set of algebraic equations [see (6.9)] whose data can be obtained up to quadratures. Moreover, Proposition 7 combined with the Implicit Function Theorem ensures that Eq. (6.9) can be solved for \(x(t)\), i.e. the solutions of the algebraic equations can be constructed. In other words,

**Theorem 6** Given a Hamiltonian system, if we know\(^{5}\) a generalized complete solution solution for it, then such a system is exactly solvable (along the whole of the phase space).

Moreover, the trajectories of the system can be constructed up to quadratures by following a procedure different to that related to the NCI, which can be described as follows. Given \(\Sigma\):

\(^{5}\) As we emphasized in Remark 1, we are asking the knowledge of a complete solution, not just its existence.
1. construct (up to quadratures) the functions \( h \) and \( W \) from Eqs. (6.1) and (6.2), respectively;
2. construct from Eq. (6.6) the function \( \varphi \);
3. fix a pair \((x_0, \lambda_0)\) and solve the Eq. (6.9) for \( x(t) \) by inverting \( \varphi \);
4. define \((q(t), p(t)) = \Sigma(x(t), \lambda_0)\).

Theorem above combined with Theorem 5 constitutes our second proof of the main assertion of this paper: “for a given Hamiltonian system, the knowledge of a set of isotropic first integrals is enough to integrate its equations of motion up to quadratures.” Unlike our first proof, the closure condition do not appear in any step of the present one.

In parallel with Remark 2, we can see Theorem 6 as another criterium for exact solvability, weaker than NCI. It is also true in this case (because of Proposition 2 and Theorem 5) that such a criterium does not give rise to new exactly solvable systems (different from the NCI ones). Nevertheless, the theoretical contribution of the theorem is undeniable: we are substantially weakening the hypothesis of a classical result. On the other hand, we think that its practical contribution lies on its proof, which constitutes an integration procedure completely different from the usual ones (where, in particular, the Lie’s theorem is not involved at all).

### 7 Conclusions

In order to summarize the results of this work, given a Hamiltonian system defined by \( H \), we can say that, if we have a set of independent isotropic first integrals for \( H \), we can construct its trajectories by following two ways:

**based on the Lie integrability theorem:**
- following the steps of the proof of Proposition 2, construct the subset \( D \) and, around each point of \( D \), construct the functions that makes the system locally NCI;
- using such functions, apply the points 1, 2 and 3 described at the end of Sect. 2.1.

**based on the generalized Hamilton–Jacobi theory:**
- following the steps of the proof of Theorem 5, construct around each point of the phase space a generalized local complete solution of the HJE;
- using such solutions, apply the points 1, 2, 3 and 4 described below Theorem 6.

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**Compliance with ethical standards**

**Conflict of interest** The author declares that he has no conflict of interest.

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