Multipliers and embedding operators with application to abstract differential equations
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ABSTRACT

In this paper, Mikhlin and Marcinkiewicz–Lizorkin type operator-valued multiplier theorems in weighted Lebesgue-Bochner spaces are studied. Using these results one derives embedding theorems in $E_0$-valued weighted Sobolev-Lions type spaces $W^{l}_{p,\gamma}(\Omega;E_0,E)$, where $E_0$, $E$ are two Banach spaces, $E_0$ is continuously and densely embedded into $E$. One proves that, there exists a smoothest interpolation space $E_\alpha$, between $E_0$ and $E$, such that the differential operator $D^\alpha$ acts as a bounded linear operator from $W^{l}_{p,\gamma}(\Omega;E_0,E)$ to $L^p_{\gamma}(\Omega;E_\alpha)$. By using these results the $L^p_{\gamma}$-separability properties of elliptic operators and regularity properties of appropriate degenerate differential operators are studied. In particular, we prove that the associated differential operator is positive and also is a generator of an analytic semigroup. Moreover, the maximal $L^p_{\gamma}$-regularity properties of Cauchy problem for abstract parabolic equation and system of infinity many parabolic equations is obtained.

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1. Introduction

Fourier multipliers in vector-valued function spaces have been studied e.g. in [4], [18], [31]. Operator-valued Fourier multipliers have been investigated in [5], [8 – 11] and [29]. Mikhlin type Fourier multipliers in scalar weighted spaces have been studied e.g. in [13], [28]. Moreover, operator-valued Fourier multipliers in weighted abstract $L_p$ spaces were investigated e.g. in [2] and [16]. In [6, 12, 13] singular integral operators with operator-valued kernel were studied in weighted $L_p$-spaces. Embedding theorems in vector-valued function spaces are studied e.g. in [14, 15], [20-26]. Regularity properties of differential-operator equations (DOEs) have been studied e.g. in [1], [2], [7, 8], [22 – 25],
operator equation

$$\|D^\alpha u\|_{L_{p,\gamma}(\Omega; E(A))} \leq C_{\mu} \left[ h^\mu \|u\|_{W_{p,\gamma}^l(\Omega; E(A), E)} + h^{- (1-\mu)} \|u\|_{L_{p,\gamma}(\Omega; E)} \right]$$

for $u \in W_{p,\gamma}^l(\Omega; E(A), E)$, where $A$ is a positive operator in $E$ and

$$l = (l_1, l_2, ..., l_n), \quad \alpha = (\alpha_1, \alpha_2, ..., \alpha_n), \quad |\alpha : l| = \sum_{k=1}^{n} \frac{\alpha_k}{l_k},$$

$$0 < h \leq h_0 < \infty, \quad 0 < \mu < 1 - |\alpha : l|.$$  

This fact generalizes and improves the results [3, § 9, 27, § 1.7] for scalar Sobolev space, the result [14] for one dimensional Sobolev-Lions spaces and the results [15], [22] for Hilbert-space valued class. Finally, we consider the differential-equation equation

$$Lu = \sum_{|\alpha| = 2l} a_\alpha D^\alpha u + Au + \sum_{|\alpha| < 2l} A_\alpha(x) D^\alpha u + \lambda u = f, \quad (1.1)$$

where $a_\alpha$ are complex numbers, $A$, $A_\alpha(x)$ are linear operators in a Banach space $E$ and $\lambda$ is a complex parameter.

We say that the problem (1.1) is $L_{p,\gamma}(R^n; E)$-separable if there exists a unique solution $u \in W_{p,\gamma}^2(R^n; E(A), E)$ of (1.1) for all $f \in L_{p,\gamma}(R^n; E)$ and there exists a positive constant $C$ depend only on $p$ and $\gamma$ such that the following coercive uniform estimate holds

$$\sum_{|\alpha| \leq 2l} |\lambda|^{1-\frac{|\alpha|}{2l}} \|D^\alpha u\|_{L_{p,\gamma}(R^n; E)} + \|Au\|_{L_{p,\gamma}(R^n; E)} \leq C \|f\|_{L_{p,\gamma}(R^n; E)}. \quad (1.2)$$

Estimate (1.2) implies that if $f \in L_{p,\gamma}(R^n; E)$ and $u$ is a solution of (1.1) then all terms of equation (1.1) belong to $L_{p,\gamma}(R^n; E)$ (i.e. all terms are separable in $L_{p,\gamma}(R^n; E)$). The above estimate implies that the inverse of the differential operator generated by (1.1) is bounded from $L_{p,\gamma}(R^n; E)$ to $W_{p,\gamma}^2(R^n; E(A), E)$.

By using the separability properties of (1.1) we show that the Cauchy problem for the parabolic equation

$$\partial_t u + \sum_{|\alpha| = 2l} a_\alpha D^\alpha u + Au = f(t, x), \quad t \in (0, \infty), \quad x \in R^n, \quad (1.3)$$

[29 – 30]. A comprehensive introduction to DOEs and historical references may be found in [1] and [30].
\[ u(0, x) = 0, \quad x \in \mathbb{R}^n \]

is well-posed in weighted spaces \( L_{p, \gamma}(\mathbb{R}^n; E) \) with mixed norm, where \( p = (p, p_1) \).

The paper is organized as follows. In Section 2, the necessary tools from Banach space theory and some background materials are given. In sections 3, the multiplier theorems in vector-valued weighted Lebesgue spaces are proved. In Section 4, by using these multiplier theorems, embedding theorems in \( E \)-valued weighted Sobolev type spaces are shown. Finally, in sections 5-8 the separability properties of (1.1), (1.3) and also regularity properties of appropriate degenerate differential operators are established.

2. Notations and background

Let \( E \) be a Banach space and let \( \gamma = \gamma(x), x = (x_1, x_2, ..., x_n) \) be a positive measurable function on the measurable subset \( \Omega \subset \mathbb{R}^n \). Let \( L_{p, \gamma}(\Omega; E) \) denote the weighted Lebesgue-Bochner space, i.e. the space of all strongly measurable \( E \)-valued functions that are defined on \( \Omega \) with the norm

\[
\| f \|_{L_{p, \gamma}} = \left( \int \| f(x) \|^p_E \gamma(x) \, dx \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty,
\]

\[
\| f \|_{L_{\infty, \gamma}} = \sup_{x \in \Omega} \| f(x) \|_E \gamma(x) \quad \text{for} \quad p = \infty.
\]

For \( p(x) \equiv 1 \), the space \( L_{p, \gamma}(\Omega; E) \) will be denoted by \( L_p = L_p(\Omega; E) \).

The weight \( \gamma \) is said to be satisfy an \( A_p \) condition, i.e. \( \gamma \in A_p, 1 < p < \infty \), if there is a positive constant \( C \) such that

\[
\sup_Q \left( \frac{1}{|Q|} \int_Q \gamma(x) \, dx \right) \left( \frac{1}{|Q|} \int_Q \gamma^{-\frac{1}{p-1}}(x) \, dx \right)^{p-1} \leq C
\]

for all for all cubes \( Q \subset \mathbb{R}^n \).

The Banach space \( E \) is called a UMD-space and written as \( E \in \text{UMD} \) if only if the Hilbert operator

\[
(Hf)(x) = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y} \, dy
\]

is bounded in the space \( L_p(R, E), p \in (1, \infty) \) (see e.g. [8]). UMD spaces include \( L_p, l_p \) spaces, Lorentz spaces \( L_{pq}, p, q \in (1, \infty) \) and Morrey spaces (see e.g. [20]).

A Banach space \( E \) has a property \( (\alpha) \) (see e.g. [19]) if there exists a constant \( \alpha \) such that

\[
\left\| \sum_{i,j=1}^N \alpha_{ij} \varepsilon_i \varepsilon_j x_{ij} \right\|_{L_2(\Omega \times \Omega; E)} \| dx \leq \alpha \left\| \sum_{i,j=1}^N \varepsilon_i \varepsilon_j x_{ij} \right\|_{L_2(\Omega \times \Omega; E)}
\]
for all \( N \in \mathbb{N} \), \( x_{i,j} \in E \), \( \alpha_{ij} \in \{0, 1\} \), \( i, j = 1, 2, ..., N \), and all choices of independent, symmetric, \( \{-1, 1\} \)-valued random variables \( \varepsilon_1, \varepsilon_2, ..., \varepsilon_N, \varepsilon'_1, \varepsilon'_2, ..., \varepsilon'_N \) on probability spaces \( \Omega, \Omega' \). For example the spaces \( L_p(\Omega), 1 \leq p < \infty \) has the property \((\alpha)\).

Let \( C \) be the set of complex numbers and
\[
S_\varphi = \{ \xi; \; \xi \in C, \; |\arg \xi| \leq \varphi \} \cup \{0\}, \; 0 \leq \varphi < \pi.
\]
Let \( E_1 \) and \( E_2 \) be two Banach spaces. \( B(E_1, E_2) \) denotes the space of bounded linear operators from \( E_1 \) to \( E_2 \). For \( E_1 = E_2 = E \) it will be denote by \( B(E) \).

A linear operator \( A \) is said to be positive in a Banach space \( E \), with bound \( M \), if \( D(A) \) is dense in \( E \) and
\[
\left\| (A + \xi I)^{-1} \right\|_{B(E)} \leq M (1 + |\xi|)^{-1}
\]
with \( \xi \in S_\varphi, \varphi \in [0, \pi] \), where \( M \) is a positive constant and \( I \) is an identity operator in \( E \). Sometimes instead of \( A + \xi I \), we will write \( A + \xi \) or \( A\xi \). It is known \([27, \S 1.15.1]\) there exist fractional powers \( A^\theta \) of the positive operator \( A \).

**Definition 2.1.** A positive operator \( A \) is said to be \( R \)-positive in the Banach space \( E \) if there exists \( \varphi \in [0, \pi) \) such that the set
\[
\left\{ \xi (A + \xi I)^{-1} : \xi \in S_\varphi \right\}
\]
is \( R \)-bounded (see e.g. \([29]\]).

Let \( E(A^\theta) \) denote the space \( D(A^\theta) \) with graphical norm defined as
\[
\left\| u \right\|_{E(A^\theta)} = \left( \left\| u \right\|^p + \left\| A^\theta u \right\|^p \right)^{\frac{1}{p}}, \; 1 \leq p < \infty, \; -\infty < \theta < \infty.
\]
Let \( (E_1, E_2)_{\theta, \varphi} \) denote the interpolation space obtained from \( \{E_1, E_2\} \) by the \( K \)-method \([27, \S 1.3.1]\), where \( \theta \in (0, 1), \; p \in [1, \infty) \).

We denote by \( D(R^n; E) \) the space of \( E \)-valued \( C_\infty \) function with compact support, equipped with the usual inductive limit topology and \( S(E) = S(R^n; E) \) denote the \( E \)-valued Schwartz space of rapidly decreasing smooth functions. For \( E = \mathbb{C} \) we simply write \( D(R^n) \) and \( S = S(R^n) \), respectively. Let \( D'(R^n; E) = B(D(R^n), E) \) denote the space of \( E \)-valued distributions and let \( S'(E) = S'(R^n; E) \) denote a space of linear continuous mapping from \( S(R^n) \) into \( E \). The Fourier transform for \( u \in S'(R^n; E) \) is defined by
\[
F(u)(\varphi) = u(F(\varphi)), \; \varphi \in S(R^n).
\]
Let \( \gamma \) be such that \( S(R^n; E_1) \) is dense in \( L_{p,\gamma}(R^n; E_1) \). A function
\[
\Psi \in C(1)(R^n; B(E_1, E_2))
\]
is called a multiplier from \( L_{p,\gamma}(R^n; E_1) \) to \( L_{q,\gamma}(R^n; E_2) \) if there exists a positive constant \( C \) such that
\[
\left\| F^{-1} \Psi(\xi) Fu \right\|_{L_{q,\gamma}(R^n; E_2)} \leq C \left\| u \right\|_{L_{p,\gamma}(R^n; E_1)}
\]
for all $u \in S(R^n; E_1)$.

We denote the set of all multipliers from $L_{p,\gamma}(R^n; E_1)$ to $L_{q,\gamma}(R^n; E_2)$ by $M^{q,\gamma}_{p,\gamma}(E_1, E_2)$.

A set $K \subset B(E_1, E_2)$ is called $R$-bounded (see e.g. [9, § 3.1]) if there is a constant $C > 0$ such that for all $T_1, T_2, ..., T_m \in K$ and $u_1, u_2, ..., u_m \in E_1$, $m \in \mathbb{N}$

$$\int_0^1 \left\| \sum_{j=1}^m r_j(y)T_j u_j \right\|_{E_2} dy \leq C \int_0^1 \left\| \sum_{j=1}^m r_j(y)u_j \right\|_{E_1} dy,$$

where \{r_j\} is a sequence of independent symmetric $\{-1;1\}$-valued random variables on $[0,1]$. The smallest $C$ for which the above estimate holds is called the $R$-bound of $K$ and denoted by $R(K)$.

**Definition 2.2.** The Banach space $E$ satisfies the multiplier condition with respect to $p \in (1, \infty)$ and to the weighted function $\gamma$ if for all $\Psi \in C^{(\alpha)}(R^n; B(E))$ the inequality

$$R \left\{ \left\| D_x^\alpha \Psi (\xi) : \xi \in R^n \setminus \{0\}, \right\| : \alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \right\} \leq K_\alpha < \infty \quad (2.1)$$

for $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n), \alpha_i \in (0,1)$ implies that $\Psi \in M_{p,\gamma}^{q,\gamma}(E)$.

Note that, if $E_1$ and $E_2$ are UMD spaces and $\gamma(x) \equiv 1$, then by virtue of operator valued multiplier theorems (see e.g. [9 - 12], [30]) we obtain that $\Psi$ is a Fourier multiplier in $L_p(R^n; E)$.

Let $\Omega$ be a domain on $R^n$ and let $l = (l_1, l_2, ..., l_n) \in \mathbb{N}^n$. Assume $E_0$ is continuously and densely belongs to $E$. Here, $W_{p,\gamma}^l(\Omega; E_0, E)$ denotes the anisotropic weighted Sobolev-Lions type space of functions $u \in L_{p,\gamma}(\Omega; E_0)$ which have generalized derivatives $\partial^k_{x_k} \in L_{p,\gamma}(\Omega; E)$ with norm

$$\left\| u \right\|_{W_{p,\gamma}^m(\Omega; E_0, E)} = \left\| u \right\|_{L_{p,\gamma}(\Omega; E_0)} + \sum_{k=1}^n \left\| \partial^k_{x_k} u \right\|_{L_{p,\gamma}(\Omega; E)} < \infty.$$

For $l_1 = l_2 = ... = l_n = m$ we denote $W_{p,\gamma}^l(\Omega; E_0, E)$ by $W_{p,\gamma}^m(\Omega; E_0, E)$ as an isotropic weighted Sobolev-Lions space.

### 3. Operator-valued multiplier results in weighted Lebesgue spaces

Let $E_1$, $E_2$ be Banach spaces. We put

$$X = L_{p,\gamma}(R^n; E_1) \text{ and } Y = L_{p,\gamma}(R^n; E_2).$$

By following Theorems 3. 6 and 3.7 of [9] we will show the following multiplier theorems:
Theorem 3.1. Let $\gamma \in A_p$, $p \in (1, \infty)$. Assume $E_1, E_2$ are UMD spaces with property $(\alpha)$ and let

$$M \in C^{(n)}(R^n \setminus \{0\} ; B(E_1, E_2)).$$

If

$$R \left\{ \xi^\beta D^\beta_M(\xi) : \xi \in R^n \setminus \{0\} \right\} \leq C_\beta < \infty$$

for all $\beta = (\beta_1, \beta_2, ..., \beta_n), \beta_i \in \{0, 1\}$, then $M$ is a multiplier from $X$ to $Y$ with

$$\|M\|_{B(X,Y)} \leq \sum_{\beta_i \in \{0, 1\}} C_\beta.$$

If $n = 1$ the result remains true without $E_1$ having property $(\alpha)$.

Theorem 3.2. Let $\gamma \in A_p$, $p \in (1, \infty)$. Let $E_1, E_2$ be UMD spaces and let $M \in C^{(n)}(R^n \setminus \{0\} ; B(E_1, E_2))$.

If

$$R \left\{ |\xi|^{\beta} D^\beta_M(\xi) : \xi \in R^n \setminus \{0\} \right\} \leq C_\beta < \infty$$

for all $\beta = (\beta_1, \beta_2, ..., \beta_n), \beta_i \in \{0, 1\}$, then $M$ is a multiplier from $X$ to $Y$ with

$$\|M\|_{B(X,Y)} \leq \sum_{\beta_i \in \{0, 1\}} C_\beta.$$

To prove Theorem 3.1 we need the following result:

The following Propositions A$_1$ and A$_2$ are due to Clément, de Pagter, Sukochev and Witvliet, see [5].

Proposition A$_1$. Let $\Delta_{E_1}^{E_1}$ and $\Delta_{E_2}^{E_2}$ be unconditional Schauder decompositions of the Banach spaces $E_1$ and $E_2$ respectively, with unconditional constants $C_{E_1}$ and $C_{E_2}$. Further let $\{T_j : j \in \mathbb{Z}^n\}$ be an $R$-bounded family in $B(E_1, E_2)$ with $T_j \Delta_{E_1}^{E_1} = \Delta_{E_2}^{E_2} T_j \Delta_{E_1}^{E_1}$ for all $j \in \mathbb{N}$. Then the series

$$Tu = \sum_{j=1}^{\infty} T_j \Delta_{E_1}^{E_1} u$$

converges for every $u \in E_1$ and defines a bounded operator $T : E_1 \to E_2$ with

$$\|T\| \leq C_{E_1} C_{E_2} R(\{T_j : j \in \mathbb{Z}^n\}).$$

Proposition A$_2$. Assume $E$ is a Banach space that has property$(\alpha)$, $\Delta = \{\Delta_k\}_{k=1}^{\infty}$ is an unconditional Schauder decomposition and $Q \subset B(E)$ is an $R$-bounded collection of operators. Then the set

$$S := \left\{ \sum_{k=0}^{\infty} T_k \Delta_k : T_k \in Q \text{ such that } T_k \Delta_k = \Delta_k T_k \text{ for all } k \in \mathbb{N} \right\}$$

is $R$-bounded in $E$. 

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Let $Ω \subset R^n$. By using the same reasoning as used in [5, Lemma 3.17] we have:

**Lemma 3.1.** Let $γ \in A_p, p \in (1, \infty)$. Assume $E$ is a Banach spaces. For $φ \in L_∞(Ω)$ we denote by $M_φ = M_φ^X$ the associated multiplication operator in $X = L_{p,γ}(Ω; E)$. Then the collection

$$\{M_φ : φ \in L_∞(Ω), \|φ\|_∞ ≤ 1\}$$

is $R$–bounded in $X$.

From Lemma 3.1 we obtain

**Corollary 3.1.** Let $γ \in A_p, p \in (1, \infty)$. Assume $E_1$ and $E_2$ are Banach spaces. For $φ \in L_∞(Ω)$ we denote by $M_φ^X$ and $M_φ^Y$ the associated multiplication operators in $X = L_{p,γ}(Ω; E_1)$ and $Y = L_{p,γ}(Ω; E_2)$ respectively. If the set $K \subset B(X, Y)$ is $R$–bounded, then the family

$$\{M_φ^X TM_φ^Y : φ, ψ \in L_∞(R^n), \|φ\|_∞, \|ψ\|_∞ ≤ 1, T \in K\}$$

is $R$–bounded as well.

For $k = nr + j, r \in Z, j \in \{1, 2, ..., n\}$ let

$$\mathbb{D}_k = \{ξ = (ξ_1, ξ_2, ..., ξ_n) \in R^n, |ξ_i| < 2^{r+1} \text{ for } i \in \{1, 2, ..., j-1\}, 2^r ≤ |ξ_j| < 2^{r+1}, |ξ_i| < 2^r \text{ for } i \in \{j+1, ..., n\}\}.$$

For $ν = (ν_1, ν_2, ..., ν_n) \in Z^n$ let

$$\mathbb{D}_ν = \{ξ \in R^n \setminus \{0\}, 2^{ν_i-1} ≤ |ξ_j| < 2^{ν_i} \text{ for } i \in \{1, 2, ..., n\}\}.$$

From [2, Proposition A_4] we have:

**Lemma 3.2.** Let $γ \in A_p, p \in (1, \infty)$ and let $E$ be a UMD space (respectively, UMD space with property $(α)$). Then for any choice of signs $ε_k, k \in Z$ (respectively, $ε_k, k \in Z^n$) the function $ψ : R^n \to C$ with $ψ(ξ) = ε_k$ for $ξ \in \mathbb{D}_k$ (respectively, $ψ(ξ) = ε_ν$ for $ξ \in \mathbb{D}_ν, ν \in Ω$) is a $M_{p,γ}^∞(E)$ multiplier.

Let $E$ be a Banach space. The $(n–\text{dimensional})$ Riesz projection operator $R$ is defined by

$$RF = F^{-1}_\chi(0,∞)N Ff, \ f \in S(R^n; E),$$

where $\chi(Ω)$ denotes the characteristic function of $Ω \subset R^n$.

Let

$$R_jf = F^{-1}_j(0,∞)F_jf \text{ for } f \in S(R^n; E), j = 1, 2, ..., n,$$

where $F_j$ denote the one-dimensional Fourier transform with respect to variable $x_j$ and $\chi_j$ denotes the characteristic function of the halfspace

$$R^n_j = \{x = (x_1, x_2, ..., x_n) \in R^n, x_j > 0\}.$$

**Lemma 3.3.** Assume $γ \in A_p$ for $p \in (1, \infty)$ and $E$ is a UMD space. Then $R$ defines a bounded operator in $L_{p,γ}(R^n; E)$.  

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Proof. Since $\gamma \in A_p$, then by [12, Corollary 2.10] (or [6, Theorem 4]) the Hilbert operator is bounded in $L_{p,\gamma}(R; E)$. It is known that $R_1 = \frac{1}{2\pi i} (i\pi I - H)$, where $I$ is the identity operator. By using this relation we obtain that Riesz projection operator $R_1$ is bounded in $L_{p,\gamma}(R; E)$. Hence, one-dimensional Riesz projection $R_j$ also are defined bounded operators in $L_{p,\gamma}(R; E)$. It is not hard to see that

$$R = \prod_{j=1}^{n} R_j,$$

i.e. $R$ is bounded operator in $L_{p,\gamma}(R; E)$.

For $j = (j_1, j_2, \ldots, j_n) \in \mathbb{Z}^n$ let $D_j$ be the dyadic interval associated with $j$, i.e.

$$D_j = \prod_{k=1}^{n} [2^{j_k}, 2^{j_k+1})$$

and

$$Q = Q_{a,b} = \prod_{k=1}^{n} (a_k, b_k),$$

where $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n) \in R^n$.

Consider the operator

$$\Phi_{a,b} f = F^{-1} \chi \left( Q_{a,b} \right) F f \quad \text{for} \quad f \in S(R^n; E).$$

**Lemma 3.4.** Assume $\gamma \in A_p$ for $p \in (1, \infty)$ and $E$ is a UMD space. Then for each $a, b \in R^n$ the operator $f \rightarrow \Phi_{a,b} f$ is bounded in $L_{p,\gamma}(R^n; E)$. Moreover, the set $\{ \Phi_{a,b} : a, b \in R^n \}$ is an $R$–bounded family in $B(L_{p,\gamma}(R^n; E))$.

**Proof.** We first look at characteristic functions of sets of the form

$$C_a = \prod_{k=1}^{n} [a_k, \infty).$$

We can $F^{-1} \chi_{C_a} F f$ expressed as:

$$\Phi_{a} f = F^{-1} \chi_{C_a} F f = e^{i\alpha_1 \tau_1} R_1 e^{-i\alpha_1 \tau_1} \cdots e^{i\alpha_n \tau_n} R_1 e^{-i\alpha_n \tau_n} f(\tau_1, \ldots, \tau_n)$$

for

$$\tau_1, \ldots, \tau_n \in R^n.$$

We see that the set $\{ \Phi_{a} : a \in R^n \}$ is $R$–bounded in view of Proposition 3.1.

Setting $C_b = \prod_{k=1}^{n} [-\infty, b_k]$ we analogously get that the set $\{ \Phi_{b} : b \in R^n \}$ is $R$–bounded as well, where

$$\Phi_{b} f = F^{-1} \chi_{C_b} F f \quad \text{for} \quad f \in S(R^n; E).$$

Since $\Phi_{a,b} = \Phi_{a} \Phi_{b}$, the result follows because the pointwise product of $R$–bounded sets is again $R$–bounded.
Assume $E_1$ and $E_2$ are UMD spaces. We put
\[ X_1 = R \left( L_{p,\gamma}(R^n; E_1) \right), \quad Y_1 = R \left( L_{p,\gamma}(R^n; E_2) \right). \]

Let $\{A_j : j \in \mathbb{Z}^n\}$ be a decomposition of $(0, \infty)^n$ in intervals such that for each compact $K \subset (0,\infty)^n$ the set $\{A_j \cap K : j \in \mathbb{Z}^n\}$ is finite. Assume further that the families $\{\Delta_{j,1}^X : j \in \mathbb{Z}^n\}$ and $\{\Delta_{j,1}^Y : j \in \mathbb{Z}^n\}$ of the corresponding Fourier multipliers, i.e.
\[ \Delta_{j,1}^X = F_{E_1}\chi_{A_j}F_{E_1}^{-1}, \quad \Delta_{j,1}^Y = F_{E_2}\chi_{A_j}F_{E_2}^{-1} \]
are unconditional Schauder decompositions of $X$ and $Y$ respectively, where $F_E$ and $F_E^{-1}$ denote the Fourier and inverse Fourier transforms. For $k \in \mathbb{N}$ we now cut each interval $A_j$ in $2^{kn}$ smaller ones by decomposing it in each coordinate direction into $2^k$ pieces. These new smaller intervals are denoted by $A^k_{j,l}$, where $j \in \mathbb{Z}^n$ and $l \in \{0,1,\ldots,2^k-1\}^n$.

Let $M$ be a function on $R^n$ with values in a Banach space $B(E_1, E_2)$. Assume that $M$ is constant operator on the intervals $A^k_{j,l}$, and denote by $M^k_{j,l}$, the corresponding value of $M$. Next we show that an operator-valued function which is constant on the $A^k_{j,l}$’s is a Fourier multiplier from $X$ to $Y$ if it satisfies a certain inequality involving $R$–bounds.

**Proposition 3.1.** Assume $\gamma \in \Lambda_p$ for $p \in (1, \infty)$ and $E_1$, $E_2$ are UMD spaces. Further let $M : R^n \rightarrow B(E_1, E_2)$ be a function which is constant on each $A^k_{j,l}$ and zero on $R^n \setminus (0, \infty)^n$. Assume that
\[ \sum_{r=\alpha}^{\beta(2^k-1)} R \left( \left\{ \sum_{\nu \in (0,1)^n, \nu \leq \beta} (-1)^{|\nu|} M^k_{j,\beta(-\nu)} : j \in \mathbb{Z}^n \right\} \right) = C_{\beta,k} < \infty \]
for every multiindex $\beta \in (0,1)^n$ and $k \in \mathbb{Z}$. Then $M$ is a Fourier multiplier from $X$ into $Y$. The norm of $T = F_{E_2}^{-1}MF_{E_1}$ may be estimated by
\[ ||T|| \leq C_XC_YC_Q \sum_{\beta \in (0,1)^n} C_{\beta,k} \]
where $C_X$ and $C_Y$ are the unconditional constants and $C_Q$ is the $R$–bound found in Lemma 3.4.

**Proof.** By Lemma 3.4, each $\chi_{A^k_{j,l}}$ is a Fourier multiplier in $X$. We denote the operators $F_{E_1}\chi_{A^k_{j,l}}F_{E_1}^{-1}$ by $\Delta_{j,l}$. For $f \in S(R^n; E_1)$ we get
\[ Tf = F_{E_2}^{-1}MF_{E_1}f = F_{E_2}^{-1} \sum_{j=-\infty}^{\infty} M_{\chi_{A_j}}F_{E_1}f. \]

Then by using the same reasoning as used in [9, Theorem 3.3] we obtain
\[ Tf = \sum_{j=-\infty}^{\infty} T_j \Delta_{j}^X f, \]
where $T_j$ are operators defined by

$$T_j = \sum_{\beta \in (0,1)^n} \sum_{\nu = 0}^{\beta \cdot (2^k - 1)} \sum_{\nu \in (0,1)^n, \nu \leq \beta} (-1)^{[\nu]} \sum_{l=\nu}^{2^k - 1} M_{j,\beta(\nu-\nu)}^k \Delta_{j,l}^k.$$

Since $M_{j,l}^k \Delta_{j,l} = M_{j,l}^k \Delta_{j,l}$ and $\Delta_{j,l} \Delta_{j,l}^k = \Delta_{j,l}^k \Delta_{j,l}^k$, we have $\Delta_{j,l}^k \Delta_{j,l}^k = T_j \Delta_{j,l}^k$. Moreover, since $\{\Delta_{j,l}^k : j \in \mathbb{Z}^n\}$ and $\{\Delta_{j,l}^k : j \in \mathbb{Z}^n\}$ are unconditional Schauder decompositions of the spaces $X$, $Y$ respectively and $S (R^n; E_1)$ is dense in $X$, it remains to prove that the family $\{T_j : j \in \mathbb{Z}^n\}$ is $R$-bounded. This step is derived as in [9, Theorem 3.3], i.e. we show that

$$R((T_j : j \in \mathbb{Z}^n)) \leq C Q \sum_{\beta \in (0,1)^n} C_{\beta,k}.$$

Then in view of Proposition A1 we have $T \in B (X; Y)$ with

$$\|T\| \leq C X C E R((\{T_j : j \in \mathbb{Z}^n\})) \leq C X C E Q \sum_{\beta \in (0,1)^n} C_{\beta,k}.$$

In a similar way as [9, Proposition 3.4] it can be shown the following proposition. It will be used to prove the Mikhlin theorem by approximating the given function $\Psi : R^n \rightarrow B(X, Y)$ by piecewise constant multipliers and is a generalization of the same result from [7] for unweighted spaces $L_p (R^n; E)$.

**Proposition 3.2.** Assume $\gamma \in A_p$ for $p \in (1, \infty)$ and $E_1$, $E_2$ are Banach spaces. Let $M, M_N \in L^{loc}_1 (R^n, B (E_1, E_2))$ be Fourier multipliers from $X$ to $Y$ such that $M_N \rightarrow M$ in $L^{loc}_1 (R^n, B (E_1, E_2))$. If $E_2$ reflexive and the sequence

$$\{T_N\} = \{F^{-1} M_N F, N \in \mathbb{N}\}$$

is uniformly bounded in $B (X, Y)$, then the operator $T := F_{E_2}^{-1} M F_{E_1}$ is a bounded operator from $X$ to $Y$ with

$$\|T\| \leq \lim_{N \rightarrow \infty} \inf \|T_N\|.$$

The next lemma states that the family of dyadic intervals in $R^n$ can be used to build up an unconditional Schauder decomposition of $R (X)$ provided $E$ is a UMD space with property $(\alpha)$.

**Lemma 3.5.** Assume $\gamma \in A_p$ for $p \in (1, \infty)$ and $E$ is a UMD space. For $j = (j_1, j_2, \ldots, j_n) \in \mathbb{Z}^n$ let $D_j$ be the dyadic interval defined by (3.1) and $\Delta_j := F^{-1} \chi_{D_j} F$. Then:

(a) If $n = 1$, then the family $\{\Delta_j : j \in \mathbb{Z}^n\}$ is an unconditional Schauder decomposition of $X_1 = R (L_p, (R^n; E))$;

(b) If $E$ has property $(\alpha)$, then the assertion of part (a) is true for arbitrary $n$. 

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Proof. (a) It is clear that the \( \Delta_j \)'s are projections in \( L_{p,\gamma}(R^n; E) \) and that \( \Delta_j \Delta_{j'} = \delta_{j,j'} \Delta_j \). Let \( 1, 2, ... \) be any enumeration of \( \mathbb{Z} \). We have to prove that

\[
T_N f := \sum_{k=1}^{N} \Delta_k f \rightarrow f \text{ in } X_1 \text{ as } N \rightarrow \infty.
\]

This convergence is clear for \( f \in S(0, \infty; E) \). In view of a \( 3\varepsilon \)-argument it remains to show that the set \( \{T_N : N \in \mathbb{N}\} \) is uniformly bounded. To this aim we define the function \( m_N : \mathbb{R} \rightarrow \mathbb{R} \) by

\[
m_N (x) = \begin{cases} 
1 & \text{when } x \in \bigcup_{k=1}^{N} D_k, \\
-1 & \text{when } x \text{ is elsewhere and } N \in \mathbb{N}.
\end{cases}
\]

By Proposition A$_{4}$ of [2] we get that each \( m_N (x) \) is a Fourier multiplier in \( L_{p,\gamma}(R^n; E) \). Moreover, the proof the Proposition A$_{4}$ in [2] shows that the family \( \{F^{-1}m_N F\} \) is uniformly bounded. Hence, we get

\[
\|T_N\| = \left\| \sum_{k=1}^{N} F^{-1}\chi_{D_k} F \right\| = \left\| \sum_{k=1}^{N} F^{-1}\chi_{\bigcup_{k=1}^{N} D_k} F \right\| = \\
\frac{1}{2} \left\| \sum_{k=1}^{N} F^{-1} \left( \chi_{(0,\infty)} + m_N \right) F \right\| \leq \frac{1}{2} \left( \|R\| + \sup_{N \in \mathbb{N}} \|F^{-1}m_N F\| \right) < \infty.
\]

This gives the assertion (a). By Proposition A$_{2}$ we get that the collection \( \left\{ \sum_{k \in G} \Delta_k : G \subset \mathbb{Z} \right\} \) is \( R \)-bounded which in view of Proposition A$_{1}$ yields that the product of two unconditional Schauder decompositions is again an unconditional Schauder decomposition. The general case now follows by induction.

Proof of Theorem 3.1. Without loss of generality we assume \( M (\xi) = 0 \) for \( \xi \notin (0, \infty)^n \). To apply Propositions 3.1 and 3.2 we use the decomposition of Lemma 3.5 to approximate \( M \). Now, we cut each \( D_j \) into \( 2^{nk} \) pieces and define

\[
M_{j,r}^k := M \left( 2^{n} r^{i} + r^{i-k}, ..., 2^{n} r^{j} + r^{j-k} \right), \quad k \in \mathbb{Z}, \ r, \ j \in \mathbb{Z}^n,
\]

where

\[
0 \leq r_i \leq 2^k - 1.
\]

In view of Proposition 3.1 we have to estimate the \( R \)-bounds

\[
\beta, (2^k - 1) \sum_{r=\alpha} R \left( \left\{ \sum_{\nu \in (0,1)^n, \nu \leq \beta} (-1)^{[\nu]} M_{j,\beta(r-\nu)} : j \in \mathbb{Z}^n \right\} \right)
\]

for all \( \beta \in (0,1)^n \) independently of \( k \). For \( \beta = (0,0,...,0) \) this expression is trivially bounded by \( R(\{M(\xi), \ \xi \neq 0\}) \). For \( \beta \neq 0 \) let \( i \) be the smallest index with \( \beta_i = 1 \). Every \( \nu \) with \( \nu_i = 0 \) and \( \nu \leq \beta \) has a term \( \nu \) with \( \nu_m = \nu_m \) for
m \neq i \text{ and } \tilde{\nu}_i = 1. \text{ Now, by using the same reasoning as used in the proof of Theorem 3.6 of [9] by Corollary 3.1 we get the desired estimate}

\[ \sum_{r=\alpha}^{\beta} R \left( \left\{ \sum_{\nu \in (0,1)^n, \nu \leq \beta} (-1)^{|\nu|} M_{j,\beta(r-\nu)}^{k,j} : j \in \mathbb{Z}_n \right\} \right) \leq CR \left( \left\{ \xi^\beta D^\beta M : \xi \in (0,\infty)^n \right\} \right) \leq C.C \beta \]

which completes the proof.

**Remark 3.1.** If $E_1$ does not have property ($\alpha$), we can use another decomposition of $R^n$ to get an unconditional Schauder decomposition of $L_{p,\gamma}(R^n; E_1)$. But without property ($\alpha$) we have to impose stronger conditions on $M$ to get $L_{p,\gamma}$ boundedness of the corresponding multiplier operator.

**Proof of Theorem 3.2.** For $j \in \mathbb{Z}$, let $s(j) \in \mathbb{Z}$ and $t(t) \in \{1, 2, ... n\}$ be the unique numbers satisfying $j = ns + t$. Set

\[ D_j = (0, 2^s - 1) \times (2^s, 2^{s+1}) \times (0, 2^s)^{n-s} \]

and define $\Delta_j = F^{-1} \chi_{D_j} F$. Let $j = ns + t$ be the unique representation of $j$. For $k \in \mathbb{Z}$, $r \in \mathbb{Z}^n$ with $0 \leq r_i \leq 2^k - 1$ define the operator $M_{j,r}^k$ by

\[ M_{j,r}^k = M(y_1, y_2, ..., y_n), \]

where

\[ y_i = r_i 2^{s+1-k} \text{ for } i \in \{1, 2, ..., t-1\} \]
\[ y_t = 2^s + r_t 2^{s-k}, \]
\[ y_i = r_i 2^{s-k}, \text{ for } i \in \{t+1, t+2, ..., n\}. \]

Then, by reasoning as the proof of Theorem 3.7 in [9] we get the assertion.

### 4. Embeding theorems in Sobolev-Lions type spaces

The embedding of Sobolev-Lions spaces play important role in the regularity theory of PDE with operator coefficients. In this section, we show continuity of embedding operators in anisotropic Sobolev-Lions spaces.

Let

\[ X = L_{p,\gamma}(R^n; E), \quad Y = W^{l,\gamma}_{p,\gamma}(R^n; E(A), E), \]

\[ l = (l_1, l_2, ..., l_n), \quad \alpha = (\alpha_1, \alpha_2, ..., \alpha_n), \quad \kappa = |\alpha| : l = \sum_{k=1}^{n} \frac{\alpha_k}{l_k}, \]

\[ \xi = (\xi_1, \xi_2, ..., \xi_n) \in R^n, \quad |\xi|^\alpha = \prod_{k=1}^{n} |\xi_k|^\alpha_k. \]

From [22, Lemma 3.1] we have
Lemma 4.1. Assume $A$ is a $\varphi$– positive linear operator on a Banach space $E$. Then for any $h > 0$ and $0 \leq \mu \leq 1 - \varsigma$ the operator-function

$$
\Psi (\xi) = \Psi_h (\xi) = |\xi|^\alpha A^{1-\varsigma-\mu}h^{-\mu} \left[ A + \sum_{k=1}^{n} |\xi_k|^{|k| + h^{-1}} \right]^{-1}
$$

is bounded in $E$ uniformly with respect to $\xi \in \mathbb{R}^n$ and $h > 0$ i.e. there exists a constant $C_\mu$ such that

$$
\|\Psi_h (\xi)\|_{B(E)} \leq C_\mu
$$

for all $\xi \in \mathbb{R}^n$ and $h > 0$.

One of main result of this section is the following:

Theorem 4.1. Let $\gamma \in A_\mu$ for $p \in (1, \infty)$. Assume $E$ is an UMD space and $A$ is a $\varphi$– positive operator in $E$. Then for $0 \leq \mu \leq 1 - \varsigma$ the embedding

$$
D^\alpha Y \subset L_{p, \gamma} (\mathbb{R}^n; E (A^{1-\varsigma-\mu}))
$$

is a continuous and there exists a constant $C_\mu > 0$ depending only on $\mu, p, \gamma$ such that

$$
\|D^\alpha u\|_{L_{p, \gamma}(\mathbb{R}^n; E(A^{1-\varsigma-\mu}))} \leq C_\mu \left[ h^\mu \|u\|_Y + h^{-(1-\mu)} \|u\|_X \right]
$$

for $u \in Y$ and $0 < h \leq h_0 < \infty$.

Proof. It is clear to see that

$$
A^{1-\alpha-\mu}D^\alpha u = F^{-1}F A^{1-\varsigma-\mu} D^\alpha u = F^{-1}A^{1-\varsigma-\mu} F D^\alpha u
$$

$$
F^{-1}A^{1-\varsigma-\mu} (i\xi)^\alpha F u = F^{-1} (i\xi)^\alpha A^{1-\varsigma-\mu} F u.
$$

Hence, denoting $Fu$ by $\hat{u}$, we get from (4.3) the following estimate

$$
C_2 \|F^{-1} (i\xi)^\alpha A^{1-\varsigma-\mu} \hat{u}\|_X \leq \|D^\alpha u\|_{L_{p, \gamma}(\mathbb{R}^n; E(A^{1-\varsigma-\mu}))} \leq C_1 \|F^{-1} (i\xi)^\alpha A^{1-\varsigma-\mu} \hat{u}\|_X,
$$

where $C_1, C_2$ are positive constants depending only of $p$ and $\gamma$. Similarly, there exist positive constants $M_1$ and $M_2$ such that for $u \in Y$ we have

$$
M_1 \|u\|_Y \leq \|F^{-1} \hat{u}\|_X + \sum_{k=1}^{n} \|F^{-1} [(i\xi_k)^{l_k} \hat{u}]\|_X \leq M_2 \|u\|_Y.
$$

Therefore, for proving the inequality (4.2) it suffices to show

$$
\|F^{-1} (i\xi)^\alpha A^{1-\varsigma-\mu} \hat{u}\|_X \leq
$$

$$
C_\mu (h^\mu \|F^{-1} \hat{u}\|_X + \sum_{k=1}^{n} \|F^{-1} [(i\xi_k)^{l_k} \hat{u}]\|_X + h^{-(1-\mu)} \|F^{-1} \hat{u}\|_X).
$$

(4.4)
Therefore, the inequality (4.2) will follow if we prove the following estimate
\[
\| F^{-1} \left[ \xi^n A^{1-\alpha} \hat{u} \right] \|_X \leq C_\mu \| F^{-1} G(\xi) \hat{u} \|_X .
\] (4.5)
for \( u \in Y \), where
\[
G(\xi) = h^\mu \left[ A + \sum_{k=1}^n |\xi_k|^{l_k} + h^{-(1-\mu)} \right].
\]
Due to positivity of \( A \), the operator function \( G(\xi) \) has a bounded inverse in \( E \) for all \( \xi \in \mathbb{R}^n \) and \( h > 0 \). So, we can set
\[
F^{-1} \xi^n A^{1-\alpha} \hat{u} = F^{-1} \xi^n A^{1-\alpha} G^{-1}(\xi) \left[ h^\mu \left( A + \sum_{k=1}^n |\xi_k|^{l_k} \right) + h^{-(1-\mu)} \right] \hat{u}. 
\] (4.6)
The inequality (4.5) will follow immediately from (4.6) if we can prove that the operator-function \( \Psi_h = \xi^n A^{1-\alpha} G^{-1}(\xi) \) is a multiplier in \( M_{\mu,\gamma}^* (E) \) uniformly with respect to \( h \). So, by Theorem 3.1 it suffices to show that the set
\[
B(\xi, h) = \left\{ \xi^j D^\beta \Psi_h(\xi) ; \ \xi \in \mathbb{R}^n \setminus \{0\}, \ \beta_j \in \{0,1\} \right\}
\]
is \( R \)-bounded uniformly in \( h \), i.e.
\[
\sup_h R \{ B(\xi, h) \} \leq M. \tag{4.7}
\]
By Lemma 4.1 there exists a constant \( C_\mu > 0 \) such that the following uniform estimate holds
\[
\| \Psi_h(\xi) \|_{B(E)} \leq C_\mu. \tag{4.8}
\]
Let first, \( \beta = (\beta_1, \ldots, \beta_n) \) where \( \beta_k = 1 \) and \( \beta = 0 \) for \( j \neq k \). Then, by using the resolvent properties of \( A \) we obtain
\[
\left| \frac{\partial}{\partial \xi_k} \Psi_h(\xi) \right| \leq \prod_{k=1}^n (i)_{|\alpha_k|} \alpha_k |\xi_{\alpha_1} \cdots \xi_{\alpha_{k-1}} \xi_{\alpha_k}^{-1} \cdots \xi_{\alpha_n|^|}|
\]
\[
\left\| A^{1-\alpha-\mu} \left[ h^\mu \left( A + \sum_{k=1}^n |\xi_k|^{l_k} \right) + h^{-(1-\mu)} \right] \right\| +
\]
\[
|\xi|^\alpha \left\| A^{1-\alpha-\mu} \left[ h^\mu \left( A + \sum_{k=1}^n |\xi_k|^{l_k} \right) + h^{-(1-\mu)} \right] \right\| h \left| \xi_k \right|^{l_k-1} \leq C_\mu \left| \xi_k \right|^{-1}, \ k = 1, 2 \ldots n.
\]
Repeating the above process, we obtain that there exists a constant \( C_\mu > 0 \) depending only \( \mu \) such that
\[
\left| \xi^\beta \right| \left\| D^\beta \Psi_h(\xi) \right\|_{B(E)} \leq C_\mu
\]
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for $\beta = (\beta_1, ..., \beta_n)$, $\beta_k \in \{0, 1\}$ and for all $\xi \in \mathbb{R}^n$, $\xi \neq 0$. Due to $R$-positivity of $A$ and by (4.9) we obtain that the set

$$B_0(\xi) = \{AG^{-1}(\xi, h); \xi \in \mathbb{R}^n \setminus \{0\}, \beta_j \in \{0, 1\}\}$$

is $R$ bounded uniformly in $h$. Then, by virtue of Kahane’s contraction principle [8, Lemma 3.5] and by (4.9) we obtain that the set

$$B_0(\xi, h) = \{AD^{-2}(\xi, h); \xi \in \mathbb{R}^n \setminus \{0\}, \beta_j \in \{0, 1\}\}$$

is uniformly $R$-bounded. Moreover, by using the inequalities of moment for positive operators and Young’s we get that

$$\|\Psi_h(\xi)u\| \leq C\mu\left(\|Au\| + \sum_{k=1}^{n} |\xi_k|^{-1} \|u\|\right),$$

(4.10)

where

$$u = G^{-1}(\xi, h)f, f \in E.$$

Then thanks to $R$-boundedness of $B_i(\xi, \lambda)$ we have

$$\int_0^1 \left\| \sum_{j=1}^{m} r_j(y) B_i(\eta_j, h) u_j \right\|_E dy \leq C \int_0^1 \left\| \sum_{j=1}^{m} r_j(y) u_j \right\|_E dy,$$

(4.11)

for all $\xi_1, \xi_2, ..., \xi_m \in \mathbb{R}^n$, $\eta_j = (\xi_{j1}, \xi_{j2}, ..., \xi_{jn}) \in \mathbb{R}^n$, $u_1, u_2, ..., u_m \in E$, $m \in \mathbb{N}$, where \(\{r_j\}\) is a sequence of independent symmetric $\{-1, 1\}$-valued random variables on $[0, 1]$. Thus, in view of Kahane’s contraction principle, additional and product properties of $R$-bounded operators and (4.10), (4.11) we obtain

$$\int_0^1 \left\| \sum_{j=1}^{m} r_j(y) \Psi(\eta_j, h) u_j \right\|_E dy \leq C \int_0^1 \left\| \sum_{j=1}^{m} B_i(\eta_j, h) r_j(y) u_j \right\|_E dy \leq$$

(4.12)

$$C \int_0^1 \left\| \sum_{j=1}^{m} r_j(y) u_j \right\|_E dy.$$

The estimate (4.12) implies $R$-boundedness of the set $B(\xi, h)$, which implies the assertion.

It is possible to state Theorem 4.1 in a more general setting. For this aim, we use the concept of extension operator.

**Condition 4.1.** Let $\gamma \in A_p$ for $p \in (1, \infty)$. Let $A$ be a positive operator in UMD space $E$. Assume a region $\Omega \subset \mathbb{R}^n$ such that there exists bounded linear extension operator $B$ from $W^{1, \gamma}_p(\Omega, E; A)$ to $Y$ for $1 < p < \infty$.

**Remark 4.1.** If $\Omega \subset \mathbb{R}^n$ is a region satisfying the strong $l$-horn condition (see [3], p.117 for $E = C$, $A = I$ and $\gamma(x) \equiv 1$) then for $1 < p < \infty$ there
exists a bounded linear extension operator from $W^1_p(\Omega) = W^1_p(\Omega; \mathbb{C}, \mathbb{C})$ to $W^1_p(R^n) = W^1_p(R^n; \mathbb{C}, \mathbb{C})$.

**Theorem 4.2.** Assume conditions of Theorem 4.1 and Condition 4.1 are satisfied. Then for $0 \leq \mu \leq 1 - \kappa$ the embedding

$$D^\alpha W^1_{p,\gamma}(\Omega; E(A), E) \subset L_{p,\gamma}(\Omega; E(A^{1-\kappa-\mu}))$$

is continuous and there exists a constant $C_\mu$ depending only of $\mu$, $p$, $\gamma$ such that

$$\|D^\alpha u\|_{L_{p,\gamma}(\Omega; E(A^{1-\kappa-\mu}))} \leq C_\mu \left[ h^\mu \|u\|_{W^1_{p,\gamma}(\Omega; E(A), E)} + h^{-(1-\mu)} \|u\|_{L_{p,\gamma}(\Omega; E)} \right].$$

**Proof.** It is suffices to prove the estimate (4.10). Let $B$ be a bounded linear extension operator from $W^1_{p,\gamma}(\Omega; E(A), E)$ to $W^1_{p,\gamma}(R^n; E(A), E)$ and let $B_\Omega$ be the restriction operator from $R^n$ to $\Omega$. Then for any $u \in W^1_{p,\gamma}(\Omega; E(A), E)$ we have

$$\|D^\alpha u\|_{L_{p,\gamma}(\Omega; E(A^{1-\kappa-\mu}))} = \|D^\alpha B_\Omega Bu\|_{L_{p,\gamma}(\Omega; E(A^{1-\kappa-\mu}))} \leq C_\mu \left[ h^\mu \|Bu\|_{W^1_{p,\gamma}(R^n; E(A), E)} + h^{-(1-\mu)} \|Bu\|_{L_{p,\gamma}(R^n; E)} \right].$$

**Result 4.1.** Assume the conditions of Theorem 4.2 are satisfied. Then for $u \in W^1_{p,\gamma}(\Omega; E(A), E)$ we have the following multiplicative estimate

$$\|D^\alpha u\|_{L_{p,\gamma}(\Omega; E(A^{1-\kappa-\mu}))} \leq C_\mu \|u\|_{W^1_{p,\gamma}(\Omega; E(A), E)}^{1-\mu} \|u\|_{L_{p,\gamma}(\Omega; E)}^\mu. \quad (4.11)$$

Indeed, setting

$$h = \|u\|_{L_{p,\gamma}(\Omega; E)} \cdot \|u\|_{W^1_{p,\gamma}(\Omega; E(A), E)}^{-1}$$

in (4.10) we obtain (4.11).

**Theorem 4.3.** Suppose conditions of Theorem 4.1 are hold. Then for $0 < \mu < 1 - \kappa$ the embedding

$$D^\alpha Y \subset L_{p,\gamma}(R^n; (E(A), E))_{\kappa+\mu,p}$$

is continuous and there exists a constant $C_\mu$ depending only of $\mu$, $p$, $\gamma$ such that

$$\|D^\alpha u\|_{L_{p,\gamma}(R^n; (E(A), E))_{\kappa+\mu,p}} \leq h^\mu \|u\|_{Y} + h^{-(1-\mu)} \|u\|_{X} \quad (4.12)$$

for $u \in Y$ and $0 < h \leq h_0 < \infty$.

**Proof.** It is sufficient to prove the estimate (4.12) for $u \in Y$. By definition of interpolation spaces $(E(A), E)_{\kappa+\mu,p}$ (see [27, §1.14.5]) the estimate (4.12) is equivalent to the inequality
\[
\left\| F^{-1} y^{1-\kappa-\mu} \left[ A^{\kappa+\mu} (A + y)^{-1} \right] \xi \tilde{u} \right\|_{L_{p,\gamma}(R^{n+1};E)} \leq C_{\mu} \left\| F^{-1} \left[ h^{\mu} \left( A + \sum_{k=1}^{n} \xi k^{l_k} + h^{-(1-\mu)} \right) \right] \tilde{u} \right\|_{L_{p,\gamma}(R^n;E)}.
\]

By multiplier properties, the inequality (4.13) will follow immediately if we will prove that the operator-function

\[
\Psi = (i\xi)^{\alpha} y^{1-\kappa-\mu} \frac{1}{A^{\kappa+\mu}} (A + y)^{-1} \left[ h^{\mu} \left( A + \sum_{k=1}^{n} \xi k^{l_k} \right) + h^{-(1-\mu)} \right]^{-1}
\]
is a multiplier from \( X \) to \( L_{p,\gamma}(R^n;L_p(\cdot;E)) \). This fact is proved by the same manner as Theorem 4.1. Therefore, we get the estimate (4.12).

In a similar way, as the Theorem 4.2 we obtain

**Theorem 4.4.** Suppose conditions of Theorem 4.2 are hold. Then for \( 0 < \mu < 1 - \kappa \) the embedding

\[
D^{\alpha} W^{l}_{p,\gamma}(\Omega; E(A), E) \subset L_{p,\gamma}\left(\Omega; (E(A), E)_{\kappa+\mu,p}\right)
\]
is continuous and there exists a constant \( C_{\mu} \) depending only of \( \mu, p, \gamma \) such that

\[
\| D^{\alpha} u \|_{L_{p,\gamma}(\Omega,(E(A), E)_{\kappa+\mu,p})} \leq C_{\mu} \left( h^{\mu} \| u \|_{W^{l}_{p,\gamma}(\Omega; E(A), E)} + h^{-(1-\mu)} \| u \|_{L_{p,\gamma}(\Omega; E)} \right)
\]

for \( u \in W^{l}_{p,\gamma}(\Omega; E(A), E) \) and \( 0 < h \leq h_0 < \infty \).

**Result 4.2.** Suppose the conditions of Theorem 4.2 are hold. Then for \( u \in W^{l}_{p,\gamma}(\Omega; E(A), E) \) we have the following multiplicative estimate

\[
\| D^{\alpha} u \|_{L_{p,\gamma}(\Omega; (E(A), E)_{\kappa+\mu,p})} \leq C_{\mu} \left( \| u \|_{W^{l}_{p,\gamma}(\Omega; (E(A), E))}^{1-\mu} + \| u \|_{L_{p,\gamma}(\Omega; E)}^{\mu} \right).
\]

Indeed setting \( h = \| u \|_{L_{p,\gamma}(\Omega; E)} \cdot \| u \|_{W^{l}_{p,\gamma}(\Omega; (E(A), E))}^{-1} \) in (4.14) we obtain (4.15).

From Theorem 4.2 we obtain

**Result 4.3.** Assume the conditions of Theorem 4.2 are satisfied for \( l_1 = l_2 = \ldots = l_n = m \). Then for \( 0 \leq \mu \leq 1 - \kappa \) the embedding

\[
D^{\alpha} W^{m}_{p,\gamma}(\Omega; E(A), E) \subset L_{p,\gamma}\left(\Omega; (E(A^{1-\kappa-\mu}))\right)
\]
is continuous and there exists a constant \( C_{\mu} \) depending only of \( \mu, p, \gamma \) such that

\[
\| D^{\alpha} u \|_{L_{p,\gamma}(\Omega; (E(A^{1-\kappa-\mu}))} \leq C_{\mu} \left( h^{\mu} \| u \|_{W^{m}_{p,\gamma}(\Omega; E(A), E)} + h^{-(1-\mu)} \| u \|_{L_{p,\gamma}(\Omega; E)} \right)
\]

for \( u \in W^{m}_{p,\gamma}(\Omega; E(A), E) \) and \( 0 < h \leq h_0 < \infty \) where

\[
\kappa = \frac{\alpha}{m}.
\]
Result 4.3. If $E = H$, where $H$ is a Hilbert space and $p_k = q_k = 2$, $\Omega = (0, T), \lambda = l_2 = \ldots = l_m = m$, $A = A^\times \geq c^2 I$, $\gamma(x) \equiv 1$ then we obtain the well known Lions-Peetre [14] result. Moreover, the result of Lions-Peetre is improving even in the one dimensional case and for non selfadjoint positive operators $A$.

From Theorems 4.2 we obtain

Result 4.4. Suppose the conditions of Theorem 4.2 are satisfied for $\gamma(x) \equiv 1$. Then for $0 \leq \mu \leq 1 - \infty$ the embedding

$$D^\alpha W^I_p (\Omega; E(A), E) \subset L_p (\Omega; E(A^{1-\infty-\mu}))$$

is continuous and there exists a constant $C_\mu > 0$ depending only of $\mu, p, \gamma$ such that

$$\|D^\alpha u\|_{L_p (\Omega; E(A^{1-\infty-\mu}))} \leq C_\mu \left[ h^\mu \|u\|_{W^I_p (\Omega; E(A), E)} + h^{-1-\mu} \|u\|_{L_p (\Omega; E(A))} \right]$$

for $u \in W^I_p (\Omega; E(A), E)$ and $0 < h \leq h_0 < \infty$.

Moreover, if $\Omega$ is a bounded domain in $R^n$ and $A^{-1}$ is a compact operator in $E$, then for $0 < \mu \leq 1 - \infty$ the embedding

$$D^\alpha W^I_p (\Omega; E(A), E) \subset L_p (\Omega; E(A^{1-\infty-\mu}))$$

is compact.

If $E = C$, $A = I$, $\gamma(x) \equiv 1$ we get the embedding $D^\alpha W^I_p (\Omega) \subset L_p (\Omega)$ proved in [3] for Sobolev spaces $W^I_p (\Omega)$.

Let $s > 0$. Consider the following sequence space (see e.g. [27, § 1.18])

$$l^s_q = \{u = \{u_i\}; i = 1, 2, \ldots, \infty, u_i \in \mathbb{C}\}$$

with the norm

$$\|u\|_{l^s_q} = \left( \sum_{i=1}^{\infty} 2^{i\nu} |u_i|^p \right)^{\frac{1}{p}} < \infty, \quad \nu \in (1, \infty).$$

Note that, $l^0_q = l_q$. Let $A$ be infinite matrix defined in $l_q$ such that $D(A) = l^s_q$, $A = [\delta_{ij} 2^n]$, where $\delta_{ij} = 0$, when $i \neq j$, $\delta_{ij} = 1$, when $i = j = 1, 2, \ldots, \infty$.

It is clear to see that the operator $A$ is positive in $l_q$. From Theorem 4.2 we obtain the following results:

Result 4.5. Suppose the conditions of Theorem 4.2 are satisfied for $E = C$. Then for $0 \leq \mu \leq 1 - \infty, 1 < p < \infty$ the embedding

$$D^\alpha W^I_p (\Omega, l^s_q; l_q) \subset L_p (\Omega, l^s_q; l_q)$$

is continuous and there exists a constant $C_\mu > 0$ depending only of $\mu, p, q, \gamma$ such that

$$\|D^\alpha u\|_{L_p (\Omega, l^s_q; l_q)} \leq C_\mu \left[ h^\mu \|u\|_{W^I_p (\Omega; l^s_q; l_q)} + h^{-1-\mu} \|u\|_{L_p (\Omega; l^s_q; l_q)} \right]$$
for \( u \in W^{l}_{p,\gamma}(\Omega, t_{q}^{s}, t_{q}^{l}) \) and \( 0 < h \leq h_{0} < \infty \).

**Result 4.6.** Suppose the conditions of Theorem 4.2 are hold for \( E = \mathbb{C} \). Then for \( 0 < \mu \leq 1 - \varkappa \), \( 1 < p < \infty \) the embedding

\[
D^{\alpha}W^{l}_{p,\gamma}(\Omega, t_{q}^{s}, t_{q}^{l}) \subset L^{p,\gamma}(\Omega, t_{q}^{(1-\varkappa-\mu)})
\]

is compact.

**Result 4.7.** For \( 0 \leq \mu \leq 1 - \varkappa \), \( 1 < p < \infty \) the embedding

\[
D^{\alpha}W^{l}_{p}(\Omega, t_{q}^{s}, t_{q}^{l}) \subset L^{p}(\Omega, t_{q}^{(1-\varkappa-\mu)})
\]

is a continuous and there exists a constant \( C_{\mu} > 0 \), depending only of \( \mu, p, q, \gamma \) such that

\[
\|D^{\alpha}u\|_{L^{p}(\Omega, t_{q}^{(1-\varkappa-\mu)})} \leq C_{\mu} \left[ h^{\mu}\|u\|_{W^{l}_{p}(\Omega, t_{q}^{s})} + h^{-1-\mu}\|u\|_{L^{p}(\Omega, t_{q})} \right]
\]

for \( u \in W^{l}_{p}(\Omega, t_{q}^{s}, t_{q}^{l}) \) and \( 0 < h \leq h_{0} < \infty \).

Note that, these results haven’t been obtained with classical method until now.

### 5. Separable differential operators in weighted Lebesque spaces

Firstly, consider the leading part of the equation (1.1), i.e. consider the following equation

\[
L_{0}u = \sum_{|\alpha|=2l} a_{\alpha}D^{\alpha}u + Au + \lambda u = f,
\]

where \( a_{\alpha} \) are complex numbers, \( l \in \mathbb{N} \), \( A \) is a linear operator in a Banach space \( E \) and \( \lambda \) is a complex parameter.

Let

\[
X = L^{p,\gamma}(R^{n}; E), \quad Y = W^{2l}_{p,\gamma}(R^{n}; E(A), E).
\]

**Condition 5.1.** Let

(a) \( K(\xi) = \sum_{|\alpha|=2l} a_{\alpha} (i\xi_{1})^{\alpha_{1}} (i\xi_{2})^{\alpha_{2}} \cdots (i\xi_{n})^{\alpha_{n}} \in S(\varphi_{1}) \)

for \( 0 \leq \varphi_{1} < \pi \);

(b) There exists the positive constat \( M_{0} \) so that

\[
|K(\xi)| \geq M_{0} \sum_{k=1}^{n} \xi_{k}^{2l} \quad \text{for all } \xi \in R^{n}, \xi \neq 0.
\]

In this section we prove the following result
Theorem 5.1. Suppose the following conditions hold:
(1) Condition 5.1 is hold;
(2) $\gamma \in A_p$ for $p \in [1, \infty]$;
(3) $A$ is a $R$–positive operator in UMD space $E$ for $0 \leq \varphi < \pi - \varphi_1$.

Then for all $f \in X$ and $\lambda \in S(\varphi_1)$ equation (6.1) has an unique solution $u$ that belongs to space $Y$ and the coercive uniform estimate holds
\[
\sum_{|\alpha| \leq 2l} |\lambda|^{1 - \frac{|\alpha|}{2l}} ||D^\alpha u||_X + ||Au||_X \leq C ||f||_X.
\] (5.2)

Proof. By applying the Fourier transform to the equation (5.1) we get
\[
[K (\xi) + A + \lambda] \hat{u} (\xi) = f \hat{\cdot} (\xi),
\] (5.3)
where
\[
K (\xi) = \sum_{|\alpha| = 2l} a_\alpha (i\xi_1)^{\alpha_1} (i\xi_2)^{\alpha_2} ... (i\xi_n)^{\alpha_n}.
\]
Since $K (\xi) \in S(\varphi_1)$ for all $\xi \in R^n$, the operator $A + K (\xi)$ is invertible in $E$. So, we obtain that the solution of the equation (5.3) can be represented in the form
\[
u (x) = F^{-1} [A + K (\xi) + \lambda]^{-1} f \hat{\cdot}.
\] (5.4)

By using (5.4) we have
\[
||Au||_X = \left\| F^{-1} A [A + K (\xi) + \lambda]^{-1} f \hat{\cdot} \right\|_X,
\]
\[
||D^\alpha u||_X = \left\| F^{-1} (i\xi_1)^{\alpha_1} (i\xi_2)^{\alpha_2} ... (i\xi_n)^{\alpha_n} [A + K (\xi) + \lambda]^{-1} f \hat{\cdot} \right\|_X.
\]

Hence, it is suffices to show that the operator-functions
\[
\sigma_{1, \lambda} (\xi) = A [A + K (\xi) + \lambda]^{-1},
\]
\[
\sigma_{2, \lambda} (\xi) = \sum_{|\alpha| \leq 2l} \xi_1^{\alpha_1} \xi_2^{\alpha_2} ... \xi_n^{\alpha_n} |\lambda|^{1 - \frac{|\alpha|}{2l}} [A + K (\xi) + \lambda]^{-1}
\]
are multipliers in $X$. To see this, it is suffices to show that the following collections
\[
\left\{ \xi^\beta D^\beta \sigma_{1, \lambda} (\xi) : \xi \in R^n \setminus \{0\}, \beta \in U_n \right\}, \left\{ \xi^\beta D^\beta \sigma_{2, \lambda} (\xi) : \xi \in R^n \setminus \{0\}, \beta \in U_n \right\}
\]
are $R$–bounded in $E$ uniformly in $\lambda$, where
\[
U = \left\{ \beta = (\beta_1, ..., \beta_n), \beta_i \in \{0, 1\} \right\}.
\]

Due to $R$–positivity of $A$, the set
\[
\left\{ \sigma_{1, \lambda} (\xi) : \xi \in R^n \setminus \{0\}, \beta \in U_n \right\}
\]
is $R$-bounded. Moreover, by using the same reasoning as used in the proof of Theorem 4.1 and in view of (3) condition we obtain that the set

$$\{\sigma_{2, \lambda} (\xi) : \xi \in \mathbb{R}^n \setminus \{0\}, \beta \in U_n\}$$

is $R$-bounded uniformly in $\lambda \in S(\varphi_1)$. Then by virtue of Kahane’s contraction principle, by product properties of the collection of $R$-bounded operators (see e.g. Lemma 3.5., Proposition 3.4. in [8]) and due to $R$-positivity of operator $A$ we obtain

$$\sup_{\lambda \in S(\varphi_1)} R \left\{ \xi^\beta D^\beta \sigma_{1, \lambda} (\xi) : \xi \in \mathbb{R}^n \setminus \{0\}, \beta \in U_n \right\} \leq C, \quad (5.4)$$

$$\sup_{\lambda \in S(\varphi_1)} R \left\{ \xi^\beta D^\beta \sigma_{2, \lambda} (\xi) : \xi \in \mathbb{R}^n \setminus \{0\}, \beta \in U_n \right\} \leq C.$$

The estimates (5.4) by Theorem 3.1 imply that the operator functions $\sigma_{1, \lambda} (\xi)$ and $\sigma_{2, \lambda} (\xi)$ are $L_{p, \gamma}$($\mathbb{R}^n; E$) multipliers.

Let $L_0$ denote the differential operator in $X$ that generated by problem (5.1) for $\lambda = 0$, that is

$$D (L_0) = Y, \quad L_0 u = \sum_{|\alpha| = 2l} a_\alpha D^\alpha u + Au.$$

The estimate (5.2) implies that the operator $L_0$ has a bounded inverse from $X$ into $Y$. We denote by $L$ differential operator in $X$ that generated by problem (1.1), i.e.

$$D (L) = Y, \quad L u = L_0 u + L_1 u, \quad L_1 u = \sum_{|\alpha| \leq 2l} A_\alpha (x) D^\alpha u.$$

**Theorem 5.2.** Suppose all conditions of Theorem 5.1 are hold and

$$A_\alpha (x) A^{-\left(1 - \frac{|\alpha|}{2l} - \mu \right)} \in L_\infty (\mathbb{R}^n; B (E)) \quad \text{for } 0 < \mu < 1 - \frac{|\alpha|}{2l}.$$

Then for all $f \in X$ and $\lambda \in S(\varphi_1)$ with sufficiently large $|\lambda|$ equation (1.1) has an unique solution $u$ that belongs to space $Y$ and the uniform coercive estimate holds

$$\sum_{|\alpha| \leq 2l} |\lambda|^{-1 - \frac{|\alpha|}{2l}} \|D^\alpha u\|_X + \|Au\|_X \leq C \|f\|_X. \quad (5.5)$$

**Proof.** In view of condition on $A_\alpha (x)$ and by virtue of Theorem 4.1 there is $h > 0$ such that

$$\|L_1 u\|_X \leq \sum_{|\alpha| < 2l} \|A_\alpha (x) D^\alpha u\|_X \leq C \sum_{|\alpha| < 2l} \left\| A^{1 - \frac{|\alpha|}{2l} - \mu} D^\alpha u \right\|_X \leq \quad (5.6)$$
\[ h^\mu \left( \sum_{|\alpha| = 2l} \| D^\alpha u \|_X + \|(A + \lambda) u\|_X \right) + h^{-(1-\mu)} \| u \|_X \]

for \( u \in Y \). Then from estimates (5.2) and (5.6) for \( u \in Y \) we have
\[
\| L_1 u \|_X \leq C \left[ h^\mu \|(L_0 + \lambda) u\|_X + h^{-(1-\mu)} \| u \|_X \right]. \tag{5.7}
\]

Since \( \| u \|_X = \frac{1}{h} \|(L_0 + \lambda) u - L_0u\|_X \) for \( u \in Y \) we get
\[
\| u \|_X \leq \frac{1}{h} \|(L_0 + \lambda) u\|_X + \| L_0 u \|_X \leq 1 \tag{5.8}
\]
\[
\frac{1}{\lambda} \|(L_0 + \lambda) u\|_X + \frac{M}{\lambda} \left( \sum_{|\alpha| = 2l} \| D^\alpha u \|_X + \| A u \|_X \right).
\]

From estimates (5.7) and (5.8) for \( u \in Y \) we obtain
\[
\| L_1 u \|_X \leq Ch^\mu \|(L_0 + \lambda) u\|_X + CM\lambda^{-1}h^{-(1-\mu)} \|(L_0 + \lambda) u\|_X. \tag{5.9}
\]

Then choosing \( h \) and \( \lambda \) such that \( Ch^\mu < 1, CMh^{-(1-\mu)} < \lambda \), from (5.9) for sufficiently large \( \lambda \) we have
\[
\left\| L_1 (L_0 + \lambda)^{-1} \right\|_{B(X)} < 1. \tag{5.10}
\]

Since we have the relation
\[
(L + \lambda)^{-1} = (L_0 + \lambda)^{-1} \left[ I + L_1 (L_0 + \lambda)^{-1} \right]^{-1}
\]
so by using the estimates (5.5), (5.10) and the perturbation theory of linear operators we obtain the assertion.

From Theorem 5.2 we obtain the following results:

**Result 5.1.** Assume the conditions of Theorem 5.2 are satisfied. Then there exists a constant \( C_1 \) and \( C_2 \) depending only on \( p, \gamma \) such that
\[
C_1 \| u \|_Y \leq \|(L + d) u\|_X \leq C_2 \| u \|_Y
\]
for all \( u \in Y \) and for sufficiently large \( d > 0 \).

**Result 5.2.** Assume the conditions of Theorem 5.2 are satisfied. Then the resolvent operator \( (L + \lambda)^{-1} \) satisfies the following coercive sharp estimate holds
\[
\sum_{|\alpha| \leq 2l} |\lambda|^{1-|\alpha|} \left\| D^\alpha (L + \lambda)^{-1} \right\|_{B(X)} + \left\| A (L + \lambda)^{-1} \right\|_{B(X)} \leq C
\]
for \( \lambda \in S(\varphi_1) \).

The Result 5.2 implies that operator \( L \) is positive operator in \( X \). Then by virtue of [27, §1.14.5] the operator \( L \) is a generator of an analytic semigroup in \( X \) for \( \varphi \in \left( \frac{2}{2}, \pi \right) \).

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6. The Cauchy problem for abstract parabolic equation

Consider now, the Cauchy problem (1.3). In this section we obtain the existence and uniqueness of the maximal regular solution of problem (1.3). First of all we show

**Theorem 6.1.** Assume the conditions of Theorem 5.1 are satisfied. Then the operator $L_0$ is $R$-positive in $X$.

**Proof.** Theorem 5.1 implies that the operator $L_0$ is positive in $X$. We have to prove the $R$-boundedness of the set

$$\sigma(\lambda) = \{\lambda (L_0 + \lambda)^{-1} : \lambda \in S_\varphi\}.$$ 

From Theorem 5.1 we have

$$\lambda (L_0 + \lambda)^{-1} f = F^{-1} \Phi (\xi, \lambda) f,$$

for $f \in X$, where

$$\Phi (\xi, \lambda) = \lambda (A + L_0 (\xi) + \lambda)^{-1}, \quad L_0 (\xi) = \sum_{|\alpha|=2l} a_\alpha \xi^\alpha.$$ 

By definition of $R$-boundedness, it is sufficient to show that the operator function $\Phi (\xi, \lambda)$ (depended on variable $\lambda$ and parameters $\xi, \varepsilon$) is uniformly bounded multiplier in $X$. In a similar manner one can easily show that $\Phi (\xi, \lambda)$ is multiplier in $X$. Then, by definition of $R$-boundedness we have

$$\int_0^1 \left\| \sum_{j=1}^m r_j (y) \lambda_j (L_0 + \lambda_j)^{-1} f_j \right\|_X \, dy = \int_0^1 \left\| \sum_{j=1}^m r_j (y) F^{-1} \Phi (\xi, \lambda_j) f_j \right\|_X \, dy =

\int_0^1 \left\| F^{-1} \sum_{j=1}^m r_j (y) \Phi (\xi, \lambda_j) f_j \right\|_X \, dy \leq C \int_0^1 \left\| \sum_{j=1}^m r_j (y) f_j \right\|_X \, dy$$

for all $\xi_1, \xi_2, \ldots, \xi_m \in R^n, \lambda_1, \lambda_2, \ldots, \lambda_m \in S_\varphi, f_1, f_2, \ldots, f_m \in X, m \in \mathbb{N}$, where $\{r_j\}$ is a sequence of independent symmetric $\{-1, 1\}$-valued random variables on $[0, 1]$. Hence, the set $\sigma(\lambda)$ is $R$-bounded.

For $p = (p, p_1), R^{n+1}_+ = R_+ \times R^n, F = L_{p, \gamma} (R^{n+1}_+; E)$ will be denoted the space of all $E$-valued $p$-summable weighted functions with mixed norm, i.e. the space of all measurable functions $f$ defined on $R^{n+1}_+$ for which

$$\|f\|_{L_{p, \gamma}(R^{n+1}_+; E)} = \left( \int_{R_+} \left( \int_{R^n} \|f(x)\|^p \gamma(x) \, dx \right)^{\frac{p_1}{p}} \, dt \right)^{\frac{1}{p}} < \infty.$$
Analogously, $F_0 = W^{1,2l}_{p,\gamma} (R^{n+1}_+, E(A), E)$ denotes the Sobolev-Lions space with corresponding mixed norm, i.e.

$$F_0 = \{ u: u \in F, \frac{\partial u}{\partial t} \in F, D^\alpha u \in F, |\alpha| = 2l, \}$$

$$\|u\|_Y = \| \frac{\partial u}{\partial t} \|_F + \sum_{|\alpha| = 2l} \| D^\alpha u \|_F + \| Au \|_F < \infty.$$  

The main result of this section is the following:

**Theorem 6.2.** Assume all conditions of Theorem 5.1 hold for $\varphi \in (\frac{\pi}{2}, \pi)$ and $p_1 \in (1, \infty)$. Then for $f \in F$ problem (1.3) has a unique solution $u \in F_0$ satisfying

$$\|\partial_t u\|_F + \sum_{|\alpha| = 2l} \| D^\alpha u \|_F + \| Au \|_F \leq C \| f \|_F.$$  

**Proof.** So, the problem (1.3) can be expressed as

$$\frac{du}{dt} + L_0 u(t) = f(t), \quad u(0) = 0, \quad t \in (0, \infty).$$  

By the Result 5.2 the operator $L_0$ is positive in $X$. The Theorem 6.1 implies that $L_0$ is $R$-positivity in $X$ for $\varphi \in (\frac{\pi}{2}, \pi)$. Then by virtue of [29, Th. 4.10] we obtain that, for $f \in L_{p_1}(R_+; X)$ the Cauchy problem (6.2) has a unique solution $u \in F_0$ satisfying

$$\|D_t u\|_{L_{p_1}(R_+; X)} + \|L_0 u\|_{L_{p_1}(R_+; X)} \leq C \| f \|_{L_{p_1}(R_+; X)}.$$  

In view of Result 5.1 the operator $L_0$ is separable in $X$, i.e, the estimate (6.3) implies (6.1).

**7. Degenerate abstract differential equations**

Let us consider the problem

$$Lu = \sum_{|\alpha| = 2l} a_\alpha D^{[\alpha]} u + Au + \sum_{|\alpha| < 2l} A_\alpha (x) D^{[\alpha]} u + \lambda u = f,$$  

(7.1)

where $A, A_\alpha$ are linear operators in a Banach space $E$ and $\lambda$ is a complex parameter, where

$$D^{[\alpha]}_k = \left( \gamma_k (x_k) \frac{\partial}{\partial x_k} \right)^{\alpha_k}, \quad D^{[\alpha]} = D^{[\alpha_1]}_1 D^{[\alpha_2]}_2 ... D^{[\alpha_n]}_n,$$

here $\gamma_k (x)$ are positive measurable functions on $R^n$.

Let

$$W^{[l]}_{p,\gamma} (\Omega, E_0, E) = \left\{ u \in L_p (\Omega; E_0), \ D^{[l_k]}_k u \in L_p (\Omega; E) \right\},$$

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∥u∥_{W^{[2]}_{p,γ}(Ω;E_0,E)} = ∥u∥_{L^p(Ω;E_0)} + \sum_{k=1}^{n} ∥D_k^{[1]} u∥_{L^p(Ω;E)} < ∞.

Here,

\begin{align*}
X &= L^p(R^n;E), \quad Y = W^{[2]}_{p,γ}(R^n;E(A),E).
\end{align*}

Let

\begin{align*}
\int_0^{x_k} \gamma_k^{-1} (y) \, dy < \infty, \quad k = 1, 2, ..., n. \tag{7.2}
\end{align*}

**Remark 7.1.** Under the substitution

\begin{align*}
\tau_k &= \int_0^{x_k} \gamma_k^{-1} (y) \, dy \tag{7.3}
\end{align*}

the spaces \(X\) and \(Y\) are mapped isomorphically onto the weighted spaces \(L^p,\tilde{γ}(R^n;E)\), \(W^{[2]}_{p,\tilde{γ}}(R^n;E(A),E)\) where

\begin{align*}
\tilde{γ} = \tilde{γ} (\tau) = \prod_{k=1}^{n} \gamma_k (x_k (τ_k)), \quad τ = (τ_1, τ_2, ..., τ_n).
\end{align*}

Moreover, under the transformation (7.3) the problem (7.1) is mapped to

the undegenerate problem (1.1) considered in the weighted space \(L^p,\tilde{γ}(R^n;E)\).

**Condition 7.1.** Assume (7.1) holds and \(γ_k (x_k (τ_k)) \in A_p\) for \(k = 1, 2, ..., n\) and \(p \in (1, ∞)\).

From Theorem 5.2 and Remark 7.1 we obtain the following results:

**Result 7.1.** Assume the conditions of Theorem 5.2 are satisfied. Then for all \(f \in X\) and \(λ \in S(φ_1)\) with sufficiently large \(|λ|\) equation (1.1) has an unique solution \(u\) that belongs to \(Y\) and the uniform coercive estimate holds

\begin{align*}
\sum_{|α| \leq 2l} |λ|^{-\frac{|α|}{2p}} \left\| D^{[α]} u \right\|_X + ∥Au∥_X \leq C ∥f∥_X .
\end{align*}

Let \(G\) denote the operator in \(X\) generated by the problem (7.1).

**Result 7.2.** Assume the conditions of Theorem 5.2 and the Condition 7.1 are satisfied. Then the resolvent operator \((L + λ)^{-1}\) satisfies the following sharp estimate

\begin{align*}
\sum_{|α| \leq 2l} |λ|^{-\frac{|α|}{2p}} \left\| D^{[α]} (G + λ)^{-1} \right\|_{B(X)} + ∥A(G + λ)^{-1}∥_{B(X)} \leq C
\end{align*}

for \(λ \in S(φ_1)\).

The Result 5.2 implies that operator \(G\) is positive operator in \(X\). Then by virtue of [27, §1.14.5] the operator \(G\) is a generator of an analytic semigroup in \(X\) for \(φ \in \left(\frac{π}{2}, π\right)\).
Consider the Cauchy problem for degenerate parabolic equation

\[ \partial_t u + \sum_{|\alpha|=2l} a_\alpha D^{[\alpha]}u + Au = f(t,x), \quad t \in (0, \infty), \quad x \in \mathbb{R}^n, \]  

\[ u(0,x) = 0, \quad x \in \mathbb{R}^n, \]

where \( a_\alpha \) are complex numbers and \( A \) is a linear operator in a Banach space \( E \).

For \( p = (p,p_1) \), let \( \Phi = L_p^p(R_+^{n+1} ; E) \) denotes \( L_p^p(R_+^{n+1} ; E) \) for \( \gamma(x) \equiv 1 \).

Analogously, \( \Phi_0 = W^{1,2l}_p(R_+^{n+1}, E(A), E) \) denotes the Sobolev-Lions space with corresponding mixed norm, i.e.

\[ \Phi_0 = \{ u : u \in \Phi, \ \frac{\partial u}{\partial t} \in \Phi, \ D^{[\alpha]}u \in \Phi, \ |\alpha| = 2l, \] 

\[ \|u\|_{\Phi_0} = \left\| \frac{\partial u}{\partial t} \right\|_\Phi + \sum_{|\alpha|=2l} \left\| D^{[\alpha]}u \right\|_\Phi + \|Au\|_\Phi < \infty. \]

From Theorem 6.2 and Remark 7.1 we obtain the following results:

**Result 7.3.** Assume all conditions of Theorem 5.1 and the Condition 7.1 are satisfied for \( \varphi \in \left( \pi/2, \pi \right) \) and \( p_1 \in (1, \infty) \). Then for all \( f \in \Phi \) problem (7.4) has a unique solution \( u \in \Phi_0 \) satisfying

\[ \left\| \frac{\partial u}{\partial t} \right\|_\Phi + \sum_{|\alpha|=2l} \left\| D^{[\alpha]}u \right\|_\Phi + \|Au\|_\Phi \leq C \|f\|_\Phi. \]

8. Maximal regularity properties of infinite many system of parabolic equations

Consider the Cauchy problem for infinite many system of parabolic equations

\[ \partial_t u_i(t,x) \sum_{|\alpha|=2l} a_\alpha D^{[\alpha]}u_i(t,x) + \sum_{j=1}^{\infty} a_{ij} u_j(t,x) = f_i(t,x), \quad x \in \mathbb{R}^n, \quad t \in (0, \infty), \]

\[ u(0,x) = 0, \quad \text{for a.e.} \ x \in \mathbb{R}^n, \quad i = 1, 2, \ldots, N, \quad N \in \mathbb{N}, \]

where \( a_\alpha \) and \( a_{ij} \) are complex numbers.

**Condition 8.1.** Let

\[ a_{ij} = a_{ji}, \quad \sum_{i,j=1}^{N} a_{ij} \xi_i \xi_j \geq C_0 |\xi|^2, \quad \text{for} \ \xi \neq 0. \]

Let

\[ u = \{ u_j \}, \quad Au = \left\{ \sum_{j=1}^{N} a_{ij} u_j \right\}, \quad i, \ j = 1, 2, \ldots N, \]
Since
\[ A \]
where \( \lambda \) is easy to see that \( A \) generates a positive operator in \( l_q \).\] 

**Proof.** Let \( E = l_q \), \( A \) be a matrix such that \( A = [a_{ij}] \), \( i, j = 1, 2, ...N \). It is easy to see that

\[ B (\lambda) = \lambda (A + \lambda)^{-1} = \frac{\lambda}{D(\lambda)} [A_{ji} (\lambda)], i, j = 1, 2, ...N, \]

where \( D(\lambda) = \det (A - \lambda I) \), \( A_{ji} (\lambda) \) are entries of the corresponding adjoint matrix of \( A - \lambda I \). Since the matrix \( A \) is symmetric and positive definite, it generates a positive operator in \( l_q \) for \( q \in (1, \infty) \). For all \( u_1, u_2, ..., u_\mu \in l_q \), \( \lambda_1, \lambda_2, ..., \lambda_\mu \in \mathbb{C} \) and independent symmetric \( \{-1, 1\} \)-valued random variables \( r_k(y), k = 1, 2, ..., \mu, \mu \in \mathbb{N} \) we have

\[
\int_{\Omega} \left| \sum_{k=1}^{\mu} r_k(y) B(\lambda_k) u_k \right|^q dy \leq C \sum_{j=1}^{N} \left| \sum_{k=1}^{\mu} \frac{\lambda_k}{D(\lambda_k)} A_{ji} (\lambda_k) \right|^q \int_{\Omega} \left| \sum_{k=1}^{\mu} r_k(y) u_{kj} \right|^q dy. \quad (8.4)
\]

Since \( A \) is symmetric and positive definite, we have

\[
sup_{k,i} \sum_{j=1}^{N} \left| \frac{\lambda_k}{D(\lambda_k)} A_{ji} (\lambda_k) \right|^q \leq C. \quad (8.5)
\]

From (8.4) and (8.5) we get

\[
\int_{\Omega} \left| \sum_{k=1}^{\mu} r_k(y) B(\lambda_k) u_k \right|^q dy \leq C \int_{\Omega} \left| \sum_{k=1}^{\mu} r_k(y) u_k \right|^q dy.
\]
i.e., the operator $A$ is $R$-positive in $l_q$. Hence, by Theorem 6.2 we obtain the assertion.

**Remark 8.1.** There are a lot of $R-$positive operators in different concrete Banach spaces. Therefore, putting concrete Banach spaces instead of $E$, and concrete differential, pseudo differential operators, or finite, infinite matrices instead of $A$, by virtue of Theorems 5.2 and 6.2 we can obtained the different class of maximal regular partial differential equations or system of equations.

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