STABILITY OF BRANCHED PULL-BACK PROJECTIVE FOLIATIONS

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Abstract. We prove that, if \( n \geq 3 \), a singular foliation \( \mathcal{F} \) on \( \mathbb{P}^n \) which can be written as pull-back, where \( \mathcal{G} \) is a foliation in \( \mathbb{P}^2 \) of degree \( d \geq 2 \) with one or three invariant lines in general position and \( f : \mathbb{P}^n \to \mathbb{P}^2 \), \( \deg(f) = \nu \geq 2 \), is an appropriated rational map, is stable under holomorphic deformations. As a consequence we conclude that the closure of the sets \( \{ \mathcal{F} = f^*(\mathcal{G}) \} \) are new irreducible components of the space of holomorphic foliations of certain degrees.

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1. Introduction

Let \( \mathcal{F} \) be a holomorphic singular foliation on \( \mathbb{P}^n \) of codimension 1, \( \Pi_n : \mathbb{C}^n+1 \setminus \{0\} \to \mathbb{P}^n \) be the natural projection and \( \mathcal{F}^* = \Pi_n^*(\mathcal{F}) \). It is known that \( \mathcal{F}^* \) can be defined by an integrable 1-form \( \Omega = \sum_{j=0}^{n} A_j dz_j \) where the \( A_j \)'s are homogeneous polynomials of the same degree \( k + 1 \) satisfying the Euler condition:

\[
\sum_{j=0}^{n} z_j A_j \equiv 0.
\]

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The singular set \( S(\mathcal{F}) \) is given by \( S(\mathcal{F}) = \{ A_0 = \ldots = A_n = 0 \} \) and is such that \( \text{codim}(S(\mathcal{F})) \geq 2 \). The integrability condition is given by
\[
\Omega \wedge d\Omega = 0.
\]

The form \( \Omega \) will be called a homogeneous expression of \( \mathcal{F} \). The degree of \( \mathcal{F} \) is, by definition, the number of tangencies (counted with multiplicities) of a generic linearly embedded \( \mathbb{P}^1 \) with \( \mathcal{F} \). If we denote it by \( \text{deg}(\mathcal{F}) \) then \( \text{deg}(\mathcal{F}) = k \). The set of homogeneous 1-forms which satisfy (1.1) and (1.2) will be denoted by \( \Omega^1(n, k + 1) \).

We denote the space of foliations of a fixed degree \( k \) in \( \mathbb{P}^n \) by \( \text{Fol}(k, n) \). Due to the integrability condition and the fact that \( S(\mathcal{F}) \) has codimension \( \geq 2 \), we see that \( \text{Fol}(k, n) \) can be identified with a Zariski’s open set in the variety obtained by projectivizing the space of forms \( \Omega \) which satisfy (1.1) and (1.2), i.e \( \mathbb{P}\Omega^1(n, k + 1) \).

It is in fact an intersection of quadrics. To obtain a satisfactory description of \( \text{Fol}(k; n) \) (for example, to talk about deformations) it would be reasonable to know the decomposition of \( \text{Fol}(k; n) \) in irreducible components. This leads us to the following:

**Problem:** Describe and classify the irreducible components of \( \text{Fol}(k; n) \) \( k \geq 3 \) on \( \mathbb{P}^n \), \( n \geq 3 \).

One can exhibit some kind of list of components in every degree, but this list is incomplete. In the paper \([CLN1]\), the authors proved that the space of holomorphic codimension one foliations of degree 2 on \( \mathbb{P}^n \), \( n \geq 3 \), has six irreducible components, which can be described by geometric and dynamic properties of a generic element.

We refer the curious reader to \([CLN1]\) and \([LN0]\) for a detailed description of them. There are known families of irreducible components in which the typical element is a pull-back of a foliation on \( \mathbb{P}^2 \) by a rational map. Given a generic rational map \( f: \mathbb{P}^n \dashrightarrow \mathbb{P}^2 \) of degree \( \nu \geq 1 \), it can be written in homogeneous coordinates as \( f = (F_0, F_1, F_2) \) where \( F_0, F_1 \) and \( F_2 \) are homogeneous polynomials of degree \( \nu \). Now consider a foliation \( \mathcal{G} \) on \( \mathbb{P}^2 \) of degree \( d \geq 2 \). We can associate to the pair \( (f, \mathcal{G}) \) the pull-back foliation \( \mathcal{F} = f^* \mathcal{G} \). The degree of the foliation \( \mathcal{F} \) is \( \nu(d + 2) - 2 \) as proved in \([CLN1]\). Denote by \( PB(d, \nu; n) \) the closure in \( \text{Fol}(\nu(d + 2) - 2, n) \), \( n \geq 3 \) of the set of foliations \( \mathcal{F} \) of the form \( f^* \mathcal{G} \). Since \( (f, \mathcal{G}) \to f^* \mathcal{G} \) is an algebraic parametrization of \( PB(d, \nu; n) \) it follows that \( PB(d, \nu; n) \) is an unirational irreducible algebraic subset of \( \text{Fol}(\nu(d + 2) - 2, n) \), \( n \geq 3 \). We have the following result:

**Theorem 1.1.** \( PB(d, \nu; n) \) is a unirational irreducible component of \( \text{Fol}(\nu(d + 2) - 2, n) \); \( n \geq 3 \), \( \nu \geq 1 \) and \( d \geq 2 \).

The case \( \nu = 1 \), of linear pull-backs, was proven in \([CaLN]\), whereas the case \( \nu > 1 \), of nonlinear pull-backs, was proved in \([CLNE]\). The search for new components of pull-back type was started in the Ph.D thesis of the author \([CS]\). There we began to consider branched rational maps and foliations with algebraic invariant sets of positive dimensions.

Let \( \mathcal{F} \) be a holomorphic foliation on \( \mathbb{P}^n \) which can be written as \( \mathcal{F} = f^* (\mathcal{G}) \), where \( \mathcal{G} \) is a foliation in \( \mathbb{P}^2 \) of degree \( d \geq 2 \) with three invariant lines in general position, say \( (XYZ) = 0 \), and \( f: \mathbb{P}^n \dashrightarrow \mathbb{P}^2 \), \( \text{deg}(f) = \nu \geq 2 \), \( f = \left(F_0^a : F_1^b : F_2^c \right) \). Denote by \( PB(k, \nu, \alpha, \beta, \gamma) \) the closure in \( \text{Fol}(k, n) \), \( n \geq 3 \) of the set of foliations \( \mathcal{F} \) of the form \( f^* \mathcal{G} \). The degree of the foliation \( \mathcal{F} \) is \( k = \nu \left[ (d - 1) + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right] - 2 \), as proved in \([CS]\). Since \( (f, \mathcal{G}) \to f^* \mathcal{G} \) is an algebraic parametrization of
where each coefficient of \( \tilde{\eta} \) that are homogeneous polynomials without common factors satisfying

\[
\text{Definition 2.1.}
\]

\[
\begin{cases}
\text{foliations with one invariant line. Let us describe this last case: Let } XA \\
\text{on such that } \ell \\
\text{Theorem 1.2. } PB(k, \nu, \alpha, \beta, \gamma) \text{ is a unirational irreducible component of } \text{Fol}(k, n) \text{ for all } n \geq 3, \text{deg}(F_0).\alpha = \text{deg}(F_1).\beta = \text{deg}(F_2).\gamma = \nu \geq 2, (\alpha, \beta, \gamma) \in \mathbb{N}^3 \text{ such that } 1 < \alpha < \beta < \gamma \text{ and } d \geq 2.
\end{cases}
\]

In this paper we continue looking for new components of branched pull-back-type. In this direction we extend the previous result to case where \( \alpha = \beta \geq 1 \). We observe that in the case \( \alpha = \beta > 1 \) we continue dealing with foliations in \( \mathbb{P}^2 \) with three invariant lines in general position. On the other hand, in the situation \( \alpha = \beta = 1 \) we need to consider another set of foliations in \( \mathbb{P}^2 \). That is, we need foliations with one invariant line. Let us describe this last case: Let \( \mathcal{G} \) be a foliation on \( \mathbb{P}^2 \) with one invariant straight line, say \( \ell \). Consider coordinates \((X, Y, Z) \in \mathbb{C}^3\) such that \( \ell = \Pi_2(Z = 0) \), where \( \Pi_2 : \mathbb{C}^3 \setminus \{0\} \to \mathbb{P}^2 \) is the natural projection. The foliation \( \mathcal{G} \) can be represented in these coordinates by a polynomial 1-form of the type \( \Omega = ZA(X, Y, Z)\,dX + ZB(X, Y, Z)\,dY + C(X, Y, Z)\,dZ \) where by (1) \( XA + YB + C = 0 \). Let \( f : \mathbb{P}^n \to \mathbb{P}^2 \) be a rational map represented in the coordinates \((X, Y, Z) \in \mathbb{C}^3 \) and \( W \in \mathbb{C}^{n+1} \) by \( \tilde{f} = (F_0, F_1, F_2) \) where \( F_0, F_1 \) and \( F_2 \in \mathbb{C}[W] \) are homogeneous polynomials without common factors satisfying

\[
\text{deg}(F_0) = \text{deg}(F_1) = \gamma . \text{deg}(F_2) = \nu.
\]

The pull back foliation \( f^*(\mathcal{G}) \) is then defined by

\[
\tilde{\eta}_{[f, \mathcal{G}]}(W) = [F_2 (A \circ F)\,dF_0 + F_2 (B \circ F)\,dF_1 + \gamma (C \circ F)\,dF_2],
\]

where each coefficient of \( \tilde{\eta}_{[f, \mathcal{G}]}(W) \) has degree \( \Gamma = \nu \left[d + 1 + \frac{1}{\Gamma}\right] - 1 \). The crucial point here is that the mapping \( f \) sends the hypersurface \( (F_2 = 0) \) contained in its critical set over the line invariant by \( \mathcal{G} \).

Let \( PB(\Gamma - 1, \nu, \alpha, \gamma) \) be the closure in \( \text{Fol}(\Gamma - 1, n) \) of the set \( \{ [\tilde{\eta}_{[f, \mathcal{G}]}] \} \). It is an unirational irreducible algebraic subset of \( \text{Fol}(\Gamma - 1, n) \). We will return to this point in Section 4. We observe that the arguments for the cases \( \alpha = \beta = 1 \) and \( \alpha = \beta > 1 \) are similar. Hence we can unify the two situations in a unique statement. The main result of this work is:

**Theorem A.** \( PB(\Gamma - 1, \nu, \alpha, \gamma) \) is an unirational irreducible component of \( \text{Fol}(\Gamma - 1, n) \) for all \( n \geq 3 \), \( \text{deg}(F_0).\alpha = \text{deg}(F_1).\beta = \text{deg}(F_2).\gamma = \nu \geq 2 \), such that \( \alpha \geq 1 \), \( \gamma \geq 2 \), \( \nu \geq 2 \) and \( d \geq 2 \) are integers.

2. BRANCHED RATIONAL MAPS

Let \( f : \mathbb{P}^n \to \mathbb{P}^2 \) be a rational map and \( \tilde{f} : \mathbb{C}^{n+1} \to \mathbb{C}^3 \) is its natural lifting in homogeneous coordinates. The indeterminacy locus of \( f \) is, by definition, the set \( I(\tilde{f}) = \Pi_n \left( \tilde{f}^{-1}(0) \right) \). We characterize the set of rational maps used throughout this text as follows:

**Definition 2.1.** We denote by \( BRM(n, \nu, \alpha, \gamma) \) the set of maps \( \{ f : \mathbb{P}^n \to \mathbb{P}^2 \} \) of degree \( \nu \) given by \( f = (F_0 : F_1 : F_2) \) where \( F_0, F_1 \) and \( F_2 \) are homogeneous polynomials without common factors, with \( \text{deg}(F_0).\alpha = \text{deg}(F_1).\alpha = \text{deg}(F_2).\gamma = \nu \), where \( \nu \geq 2 \), \( \alpha \geq 1 \) and \( \gamma \geq 2 \) are integers.
Let us fix some coordinates \((z_0, ..., z_n)\) on \(\mathbb{C}^{n+1}\) and \((X, Y, Z)\) on \(\mathbb{C}^3\) and denote by \((F_0^\alpha, F_1^\alpha, F_2^\alpha)\) the components of \(f\) relative to these coordinates. Let us note that the indeterminacy locus \(I(f)\) is the intersection of the three hypersurfaces \((F_0 = 0)\), \((F_1 = 0)\) and \((F_2 = 0)\).

**Definition 2.2.** We say that \(f \in BRM(n, \nu, \alpha, \gamma)\) is generic if for all \(p \in \hat{f}^{-1}(0) \setminus \{0\}\) we have \(dF_0(p) \wedge dF_1(p) \wedge dF_2(p) \neq 0\).

This is equivalent to saying that \(f \in BRM(n, \nu, \alpha, \gamma)\) is generic if \(I(f)\) is the transverse intersection of the 3 hypersurfaces \((F_0 = 0)\), \((F_1 = 0)\) and \((F_2 = 0)\). As a consequence we have that the set \(I(f)\) is smooth. For instance, if \(n = 3\), \(f\) is generic and \(deg(f) = \nu\), then by Bezout’s theorem \(I(f)\) consists of \(\frac{\nu^3}{\alpha^2\gamma}\) distinct points with multiplicity \(\alpha^2\gamma\). If \(n = 4\), then \(I(f)\) is a smooth connected algebraic curve in \(\mathbb{P}^4\) of degree \(\frac{\nu^3}{\alpha^2\gamma}\). In general, for \(n \geq 4\), \(I(f)\) is a smooth connected algebraic submanifold of \(\mathbb{P}^n\) of degree \(\frac{\nu^3}{\alpha^2\gamma}\) and codimension three.

Denote \(\nabla F_k = (\frac{\partial F_k}{\partial z_0}, ..., \frac{\partial F_k}{\partial z_n})\). Consider the derivative matrix

\[
M = \begin{bmatrix}
\frac{\partial F_0}{\partial z_0} & \frac{\partial F_0}{\partial z_1} \\
\frac{\partial F_1}{\partial z_0} & \frac{\partial F_1}{\partial z_1} \\
\frac{\partial F_2}{\partial z_0} & \frac{\partial F_2}{\partial z_1}
\end{bmatrix}.
\]

The critical set of \(\hat{f}\) is given by the points of \(\mathbb{C}^{n+1} \setminus 0\) where \(\text{rank}(M) \leq 3\); it is the union of two sets. The first is given by the set of \(P \in \mathbb{C}^{n+1} \setminus 0\) = \(X_1\) such that the rank of the following matrix

\[
N = \begin{bmatrix}
\nabla F_0 \\
\nabla F_1 \\
\nabla F_2
\end{bmatrix}
\]

is smaller than 3. The second is the subset

\[
X_2 = \{P \in \mathbb{C}^{n+1} \setminus 0 \mid (F_0^{\alpha-1}) (F_1^{\alpha-1}) (F_2^{\gamma-1})(P) = 0\}.
\]

Denote \(P(f) = \Pi_{\nu}(X_1 \cup X_2)\). The set of generic maps will be denoted by \(Gen(n, \nu, \alpha, \gamma)\). We state the following result whose proof is standard in algebraic geometry:

**Proposition 2.3.** \(Gen(n, \nu, \alpha, \gamma)\) is a Zariski dense subset of \(BRM(n, \nu, \alpha, \gamma)\).

Once the case of foliations which are pull-backs of three invariant straight have been already discussed in \([CS]\). We will concentrate only on the case where \(\alpha = 1\). The case \(\alpha > 1\) is obtained following the same ideas.

### 3. Foliations with One Invariant Line

#### 3.1. Basic facts

Denote by \(I_1(d, 2)\) the set of the holomorphic foliations on \(\mathbb{P}^2\) of degree \(d \geq 2\) that leaves the line \(Z = 0\) invariant. We observe that any foliation which has 1 invariant straight line can be carried to one of these by a linear automorphism of \(\mathbb{P}^2\). The relation \(XA + YB + C = 0\) enables to parametrize \(I_1(d, 2)\) as follows

\[
H^0(\mathbb{P}^2, O_{\mathbb{P}^2}(d-1))^\times 2 \rightarrow H^0(\mathbb{P}^2, O_{\mathbb{P}^2}(d-1))^\times 3
\]

\[
(A, B) \mapsto (A, B, -XA - YB).
\]

We let the group of linear automorphisms of \(\mathbb{P}^2\) act on \(I_1(d, 2)\). After this procedure we obtain a set of foliations of degree \(d\) that we denote by \(I_1(d, 2)\).
We are interested in making deformations of foliations and for our purposes we need a subset of $I_1(d,2)$ with good properties (foliations having few algebraic invariant curves and only hyperbolic singularities). We explain this properties in detail. Let $q \in U$ be an isolated singularity of a foliation $\mathcal{G}$ defined on an open subset of $U \subset \mathbb{C}^2$. We say that $q$ is nondegenerate if there exists a holomorphic vector field $X$ tangent to $\mathcal{G}$ in a neighborhood of $q$ such that $DX(q)$ is nonsingular. In particular, $q$ is an isolated singularity of $\mathcal{G}$. The characteristic numbers of $q$ are the quotients $\lambda$ and $\lambda^{-1}$ of the eigenvalues of $DX(q)$, which do not depend on the vector field $X$ chosen. If $\lambda \notin \mathbb{Q}_+$, then $\mathcal{G}$ exhibits exactly two (smooth and transverse) local separatrices at $q$, $S^+_q$ and $S^-_q$ with eigenvalues $\lambda_q^+$ and $\lambda_q^-$ and which are tangent to the characteristic directions of a vector field $X$. The characteristic numbers (also called Camacho-Sad index) of these local separatrices are given by

$$I(\mathcal{G}, S^+_q) = \frac{\lambda_q^-}{\lambda_q^+} \text{ and } I(\mathcal{G}, S^-_q) = \frac{\lambda_q^+}{\lambda_q^-}.$$ 

The singularity is hyperbolic if the characteristic numbers are nonreal. We introduce the following spaces of foliations:

1. $ND(d,2) = \{ \mathcal{G} \in \text{Fol}(d,2); \text{the singularities of } \mathcal{G} \text{ are nondegenerate}\}$,
2. $\mathcal{H}(d,2) = \{ \mathcal{G} \in ND(d,2); \text{any characteristic number } \lambda \text{ of } \mathcal{G} \text{ satisfies } \lambda \in \mathbb{C}\setminus\mathbb{R} \}$.

It is a well-known fact [LN2] that $\mathcal{H}(d,2)$ contains an open and dense subset of $\text{Fol}(d,2)$. Denote by $A(d) = I_1(d,2) \cap \mathcal{H}(d,2)$. Observe that $A(d)$ is a Zariski dense subset of $I_1(d,2)$. Concerning the set $ND(d,2)$, we have the following result, proved in [LN2].

**Proposition 3.1.** Let $\mathcal{G}_0 \in ND(d,2)$. Then $\# \text{Sing}(\mathcal{G}_0) = d^2 + d + 1 = N(d)$. Moreover if $\text{Sing}(\mathcal{G}_0) = \{p^0_1, ..., p^0_N\}$ where $p^0_i \neq p^0_j$ if $i \neq j$, then there are connected neighborhoods $U_j \ni p^0_j$, pairwise disjoint, and holomorphic maps $\phi_j : U \subset ND(d,2) \rightarrow U_j$, where $U \ni \mathcal{G}_0$ is an open neighborhood, such that for $\mathcal{G} \in U$, $(\text{Sing}(\mathcal{G}) \cap U_j) = \phi_j(\mathcal{G})$ is a nondegenerate singularity. In particular, $ND(d,2)$ is open in $\text{Fol}(2,d)$. Moreover, if $\mathcal{G}_0 \in \mathcal{H}(d,2)$ then the two local separatrices as well as their associated eigenvalues depend analytically on $\mathcal{G}$.

In the paper [LNSSSg] which is related to the topological rigidity of foliations on $\mathbb{P}^2$ in the spirit of Ilyashenko’s works. The authors have proved the following useful result see [LNSSSg] Theorem 3, p.385].

**Theorem 3.2.** Let $d \geq 2$. There exists an non empty open and dense subset $M(d) \subset A(d)$, such that if $\mathcal{G} \in M(d)$ then the only algebraic invariant curve of $\mathcal{G}$ is the line.

4. **RAMIFIED PULL-BACK COMPONENTS - GENERIC CONDITIONS**

Let us fix a coordinate system $(X,Y,Z)$ on $\mathbb{P}^2$ and denote by $\ell$ the straight line that corresponds to the plane $Z = 0$ in $\mathbb{C}^3$, respectively. Let us denote by $\bar{M}(d)$ the subset $M(d) \cap I_1(d,2)$. 

Definition 4.1. Let $f \in Gen(n, \nu, 1, \gamma)$. We say that $G \in M(d)$ is in generic position with respect to $f$ if $|Sing(G) \cap Y_2| = \emptyset$, where

$$Y_2(f) = Y_2 := \Pi_2 \left[ \tilde{f} \{ w \in \mathbb{C}^{n+1} \mid dF_0(w) \wedge dF_1(w) \wedge dF_2(w) = 0 \} \right]$$

and $\ell$ is $G$-invariant.

In this case we say that $(f, G)$ is a generic pair. In particular, when we fix a map $f \in Gen(n, \nu, 1, \gamma)$ the set $A = \{ G \in M(d) \mid |Sing(G) \cap Y_2| = 0 \}$ is an open dense subset in $M(d)$ [LN, Sc], Since $VC(f)$ is an algebraic curve in $\mathbb{P}^2$. The set $U_1 := \{ (f, G) \in Gen(n, \nu, 1, \gamma) \times M(d) \mid |Sing(G) \cap Y_2| = 0 \}$ is an open dense subset of $Gen(n, \nu, 1, \gamma) \times M(d)$. Hence the set $W := \{ \tilde{\eta}_{f, G} \mid (f, G) \in U_1 \}$ is an open and dense subset of $PB(\Gamma - 1, 1, \nu, 1, \gamma)$.

Proposition 4.2. If $F$ comes from a generic pair, then the degree of $F$ is

$$\nu \left[ d + 1 + \frac{1}{\gamma} \right] - 2.$$

The proof of this fact can be obtained as in the case treated in [CS].

Consider the set of foliations $\mathcal{I}_1 (d, 2)$, $d \geq 2$, and the following map:

$$\Phi : BRM(n, \nu, 1, \gamma) \times \mathcal{I}_1 (d, 2) \rightarrow \mathcal{F}_0 (\Gamma - 1, n)$$

$$(f, G) \rightarrow f^*(\overline{G}) = \Phi (f, G).$$

The image of $\Phi$ can be written as:

$$\Phi (f, G) = [F_2 (A \circ F) dF_0 + F_2 (B \circ F) dF_1 + \gamma (C \circ F) dF_2].$$

Recall that $\Phi (f, G) = \tilde{\eta}_{f, G}$. More precisely, let $PB(\Gamma - 1, 1, \nu, 1, \gamma)$ be the closure in $\mathcal{F}_0 (\Gamma - 1, n)$ of the set of foliations $F$ of the form $f^*(\overline{G})$, where $f \in BRM(n, \nu, 1, \gamma)$ and $G \in \mathcal{I}_1 (2, d)$. Since $BRM(n, \nu, 1, \gamma)$ and $\mathcal{I}_1 (2, d)$ are irreducible algebraic sets and the map $(f, G) \rightarrow f^*(\overline{G}) \in \mathcal{F}_0 (\Gamma - 1, n)$ is an algebraic parametrization of $PB(\Gamma - 1, 1, \nu, 1, \gamma)$, we have that $PB(\Gamma - 1, 1, \nu, 1, \gamma)$ is an irreducible algebraic subset of $\mathcal{F}_0 (\Gamma - 1, n)$. Moreover, the set of generic pull-back foliations $\{ F \mid F = f^*(\overline{G}) \}$, where $(f, G)$ is a generic pair, is an open (not Zariski) and dense subset of $PB(\Gamma - 1, 1, \nu, 1, \gamma)$ for $\gamma \geq 2 \in \mathbb{N}$, $\nu \geq 2 \in \mathbb{N}$ and $d \geq 2 \in \mathbb{N}$.

5. Description of generic ramified pull-back foliations on $\mathbb{P}^n$

5.1. The Kupka set. Let $\tau$ be a singularity of $G$ and $V_\tau = \overline{f^{-1}(\tau)}$. If $(f, G)$ is a generic pair then $V_\tau \setminus I(f)$ is contained in the Kupka set of $F$. As an example we detail the case where $\tau$ is a singularity over the invariant line, say $\tau = [1 : 0 : 0]$. Fix $p \in V_\tau \setminus I(f)$. There exist local analytic coordinate systems such that $f(x, y, z) = (x, y') = (u, v)$. Suppose that $G$ is represented by the 1-form $\omega; the hypothesis of $G$ being of Hyperbolic-type implies that we can suppose $\omega(u, v) = \lambda_1 u(1 + R(u, v)) dv - \lambda_2 u du$, where $\lambda_2 \in \mathbb{C} \setminus \mathbb{R}$. We obtain $\tilde{\omega}(x, y) = f^*(\omega) = (y^{-1})(\lambda_1 \alpha x(1 + R(x, y')) dy - \lambda_2 y dx) = (y^{-1}) \tilde{\omega}(x, y)$ and so $d \tilde{\omega}(p) \neq 0$. Therefore if $p$ is as before it belongs to the Kupka-set of $F$. For the other points the argumentation is analogous. This is the well known Kupka-Reeb phenomenon, and we say that $p$ is contained in the Kupka-set of $F$. It is known that this local product structure is stable under small perturbations of $F$ for instance, see [K], [G.LN].
5.2. Generalized Kupka and quasi-homogeneous singularities. In this section we will recall the quasi-homogeneous singularities of an integrable holomorphic 1-form. They appear in the indeterminacy set of \( f \) and play a central role in great part of the proof of Theorem B.

**Definition 5.1.** Let \( \omega \) be an holomorphic integrable 1-form defined in a neighborhood of \( p \in \mathbb{C}^3 \). We say that \( p \) is a Generalized Kupka (GK) singularity of \( \omega \) if \( \omega(p) = 0 \) and either \( d\omega(p) \neq 0 \) or \( p \) is an isolated zero of \( d\omega \).

Let \( \omega \) be an integrable 1-form in a neighborhood of \( p \in \mathbb{C}^3 \) and \( \mu \) be a holomorphic 3-form such that \( \mu(p) \neq 0 \). Then \( d\omega = i_Z(\mu) \) where \( Z \) is a holomorphic vector field.

**Definition 5.2.** We say that \( p \) is a quasi-homogeneous singularity of \( \omega \) if \( p \) is an isolated singularity of \( Z \) and the germ of \( Z \) at \( p \) is nilpotent, that is, if \( L = DZ(p) \) then all eigenvalues of \( L \) are equals to zero.

This definition is justified by the following result that can be found in [LN2] or [C.CA.G.LN]:

**Theorem 5.3.** Let \( p \) be a quasi-homogeneous singularity of an holomorphic integrable 1-form \( \omega \). Then there exists two holomorphic vector fields \( S \) and \( Z \) and a local chart \( U := (x_0, x_1, x_2) \) around \( p \) such that \( x_0(p) = x_1(p) = x_2(p) = 0 \) and:

(a) \( \omega = \lambda x_0 \frac{\partial}{\partial x_0} + p_1 x_1 \frac{\partial}{\partial x_1} + p_2 x_2 \frac{\partial}{\partial x_2} \), \( \lambda \in \mathbb{Q}^+ \) \( d\omega = i_Z(dx_0 + dx_1 + dx_2) \) and \( Z = (\text{rot}(\omega)) \);

(b) \( S = p_0 x_0 \frac{\partial}{\partial x_0} + p_1 x_1 \frac{\partial}{\partial x_1} + p_2 x_2 \frac{\partial}{\partial x_2} \), where, \( p_0, p_1, p_2 \) are positive integers with \( \text{g.c.d}(p_0, p_1, p_2) = 1 \);

(c) \( p \) is an isolated singularity for \( Z \), \( Z \) is polynomial in the chart \( U := (x_0, x_1, x_2) \) and \( [S, Z] = \ell Z \), where \( \ell \geq 1 \).

**Definition 5.4.** Let \( p \) be a quasi-homogeneous singularity of \( \omega \). We say that it is of the type \( (p_0 : p_1 : p_2; \ell) \), if for some local chart and vector fields \( S \) and \( Z \) the properties \( (a), (b) \) and \( (c) \) of the Theorem 5.3 are satisfied.

We can now state the stability result, whose proof can be found in [C.CA.G.LN]:

**Proposition 5.5.** Let \( (\omega_s)_{s \in \Sigma} \) be a holomorphic family of integrable 1-forms defined in a neighborhood of a compact ball \( B = \{ z \in \mathbb{C}^3 : |z| \leq \rho \} \), where \( \Sigma \) is a neighborhood of \( 0 \in \mathbb{C}^k \). Suppose that all singularities of \( \omega_0 \) in \( B \) are GK and that \( \text{sing}(d\omega_0) \subset \text{int}(B) \). Then there exists \( \epsilon > 0 \) such that if \( s \in B(0, \epsilon) \subset \Sigma \), then all singularities of \( \omega_s \) in \( B \) are GK. Moreover, if \( 0 \in B \) is a quasi-homogeneous singularity of type \( (p_0 : p_1 : p_2; \ell) \) then there exists a holomorphic map \( B(0, \epsilon) \ni s \mapsto z(s) \), such that \( z(0) = 0 \) and \( z(s) \) is a GK singularity of \( \omega_s \) of the same type (quasi-homogeneous of the type \( (p_0 : p_1 : p_2; \ell) \), according to the case).

Let us describe \( F = f^*(\mathcal{G}) \) in a neighborhood of a point \( p \in I(f) \). It is easy to show that there exists a local chart \( (U, (x_0, x_1, x_2, y) \in \mathbb{C}^3 \times \mathbb{C}^{n-2}) \) around \( p \) such that the lifting \( \tilde{f} \) of \( f \) is of the form \( \tilde{f}|_U = (x_0, x_1, x_2) : U \to \mathbb{C}^3 \). In particular \( \tilde{F}|_{U(p)} \) is represented by the 1-form

\[
\eta(x_0, x_1, x_2, y) = x_2A(x_0, x_1, x_2^2)dx_0 + x_2B(x_0, x_1, x_2^2)dx_1 + \gamma C(x_0, x_1, x_2^2)dx_2.
\]
Let us now obtain the vector field $S$ as in Theorem 5.3. Consider the radial vector field $R = X \frac{\partial}{\partial x} + Y \frac{\partial}{\partial y} + Z \frac{\partial}{\partial z}$. Note that in the coordinate system above it transforms into

$$x_0 \frac{\partial}{\partial x_0} + x_1 \frac{\partial}{\partial x_1} + \frac{1}{\gamma} x_2 \frac{\partial}{\partial x_2}.$$ 

Since the eigenvalues of $S$ have to be integers, after a multiplication by $\gamma$ we obtain

$$S = \gamma x_0 \frac{\partial}{\partial x_0} + \gamma x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}.$$ 

Let us concentrate in the case $n = 3$.

**Lemma 5.6.** If $\eta$ and $S$ are as above then we have $L_S \eta = [1 + \gamma (1 + d)] \eta$.

**Proof.** We just have to use Cartan’s formula for the Lie’s derivative, $L_S \eta = i_S d\eta + d(i_S \eta)$. The details are left for the reader.

**Lemma 5.7.** If $p \in I(f)$ then $p$ is a quasi-homogeneous singularity of $\eta$.

**Proof.** First of all note that $i_S \eta = 0$. From the computations obtained in lemma 5.6 we have that $L_S \eta = m \eta$, where $m = [1 + \gamma (1 + d)]$. This implies that the singular set of $\eta$ is invariant under the flow of $S$. The vector field $Z$ such that $\eta = i_S i_Z (dx_0 \wedge dx_1 \wedge dx_2)$ is given by

$$Z = Z_0(x_0, x_1, x_2) \frac{\partial}{\partial x_0} + Z_1(x_0, x_1, x_2) \frac{\partial}{\partial x_1} + Z_2(x_0, x_1, x_2) \frac{\partial}{\partial x_2}$$

where for $i = 0, 1$ we have $Z_i(x_0, x_1, x_2) = \hat{A}_i(x_0, x_1, x_2) \gamma \eta$ and $Z_2(x_0, x_1, x_2) = x_2 \hat{A}_2(x_0, x_1, x_2)$ moreover for $i = 0, 1$ the polynomials $\hat{A}_i(0, 0, 0) = 0$ and $\hat{A}_2(0, 0, 0) = 0$. We observe that these polynomials are not unique. On the other hand, they have to satisfy the following relations:

$$A(x_0, x_1, x_2) = \gamma x_1 \hat{A}_2(x_0, x_1, x_2) - \hat{A}_1(x_0, x_1, x_2)$$

$$B(x_0, x_1, x_2) = \hat{A}_0(x_0, x_1, x_2) - \gamma x_0 \hat{A}_2(x_0, x_1, x_2)$$

$$C(x_0, x_1, x_2) = x_0 \hat{A}_1(x_0, x_1, x_2) - x_1 \hat{A}_1(x_0, x_1, x_2)$$

We must show that the origin is an isolated singularity of $\eta$ and all eigenvalues of $DZ(0)$ are 0. By straightforward computation we find that the Jacobian matrix $DZ(0)$ is the null matrix, hence all its eigenvalues are null. Since all singular curves of $\mathcal{F}$ in a neighborhood $(U, (x_0, x_1, x_2))$ of 0 are of Kupka type, as proved in Section 5.1 it follows that the origin is an isolated singularity of $\mathcal{Z}$. Note that the unique singularities of $\eta$ in the neighborhood $(U, (x_0, x_1, x_2))$ of 0 come from $f^* \text{Sing}(\mathcal{G})$; this follows from the fact that $\text{Sing}(\mathcal{G}) \cap (VC(f) \setminus \ell) = \emptyset$. On the other hand we have seen that $(f)^{-1}(\text{sing}(\mathcal{G})) \cap I(f)$ is contained in the Kupka set of $\mathcal{F}$. Hence the point $p$ is an isolated singularity of $d\eta$ and thus an isolated singularity of $\mathcal{Z}$. \hfill \Box

As a consequence, in the case $n = 3$ any $p \in I(f)$ is a quasi-homogeneous singularity of type $[\gamma : \gamma : 1]$. In the case $n \geq 4$ the argument is analogous. Moreover, in this case there will be a local structure product near any point $p \in I(f)$. In fact in the case $n \geq 4$ we have:

**Corollary 5.8.** Let $(f, \mathcal{G})$ be a generic pair. Let $p \in I(f)$ and $\eta$ an 1-form defining $\mathcal{F}$ in a neighborhood of $p$. Then there exists a 3-plane $\Pi \subset \mathbb{C}^n$ such that $d(\eta)|\Pi$ has an isolated singularity at $0 \in \Pi$.

**Proof.** Immediate from the local product structure. \hfill \Box
5.3. **Deformations of the singular set.** In this section we give some auxiliary lemmas which assist in the proof of Theorem A. We have constructed an open and dense subset $W$ inside $PB(\Gamma - 1, \nu, 1, 1, \gamma)$ containing the generic pull-back foliations. We will show that for any foliation $\mathcal{F} \in W$ and any germ of a holomorphic family of foliations $(\mathcal{F}_t)_{t \in (\mathbb{C}, 0)}$ such that $\mathcal{F}_0 = \mathcal{F}$ we have $\mathcal{F}_t \in PB(\Gamma - 1, \nu, 1, 1, \gamma)$ for all $t \in (\mathbb{C}, 0)$.

**Lemma 5.9.** There exists a germ of isotopy of class $C^\infty$, $(I(t))_{t \in (\mathbb{C}, 0)}$ having the following properties:

(i) $I(0) = I(f_0)$ and $I(t)$ is algebraic and smooth of codimension 3 for all $t \in (\mathbb{C}, 0)$.

(ii) For all $p \in I(t)$, there exists a neighborhood $U(p, t) = U$ of $p$ such that $\mathcal{F}_t$ is equivalent to the product of a regular foliation of codimension 3 and a singular foliation $\mathcal{F}_{p,t}$ of codimension one given by the 1-form $\eta_{p,t}$.

**Remark 5.10.** The family of 1-forms $\eta_{p,t}$, represents the quasi-homogeneous foliation given by the Proposition 5.3.

Proof. See [LN0] lemma 2.3.2, p.81.

**Remark 5.11.** In the case $n > 3$, the variety $I(t)$ is connected since $I(f_0)$ is connected. The local product structure in $I(t)$ implies that the transversal type of $\mathcal{F}_t$ is constant. In particular, $\mathcal{F}_{p,t}$ does not depend on $p \in I(t)$. In the case $n = 3$, $I(t) = p_1(t), ..., p_k(t), ..., p_3(t)$ and we can not guarantee a priori that $\mathcal{F}_{p,t} = \mathcal{F}_{p,j}$, if $i \neq j$.

The singular set of $\mathcal{G}_0$ can be divided in two subsets $\mathcal{S}_W(\mathcal{G}_0)$, $\mathcal{S}_t(\mathcal{G}_0)$. We know that $\#\mathcal{S}_W(\mathcal{G}_0) = d^2$, $\#\mathcal{S}_t(\mathcal{G}_0) = (d + 1)$. Let $\tau \in Sing(\mathcal{G}_0)$ and $K(\mathcal{F}_0) = \cup_{\tau \in Sing(\mathcal{G}_0)} V_\tau \setminus I_0$ where $V_\tau = f_0^{-1}(\tau)$. As in Lemma 5.9 let us consider a representative of the germ $(\mathcal{F}_t)_t$, defined on a disc $D_\delta := (|t| < \delta)$.

**Lemma 5.12.** There exist $\epsilon > 0$ and smooth isotopies $\phi_\tau : D_\epsilon \times V_\tau \rightarrow \mathbb{P}^n$, $\tau \in Sing(\mathcal{G}_0)$, such that $V_\tau(t) = \phi_\tau(\{t\} \times V_\tau)$ satisfies:

(a) $V_\tau(t)$ is an algebraic subvariety of codimension two of $\mathbb{P}^n$ and $V_\tau(0) = V_\tau$

(b) $I(t) \subset V_\tau(t)$ for all $\tau \in Sing(\mathcal{G}_0)$ and for all $t \in D_\epsilon$. Moreover, if $\tau \neq \tau'$, and $\tau, \tau' \in Sing(\mathcal{G}_0)$, we have $V_\tau(t) \cap V_{\tau'}(t) = I(t)$ for all $t \in D_\epsilon$ and the intersection is transversal.

(c) $V_\tau(t) \setminus I(t)$ is contained in the Kupka-set of $F_\tau$ for all $\tau \in Sing(\mathcal{G}_0)$ and for all $t \in D_\epsilon$. In particular, the transversal type of $\mathcal{F}_t$ is constant along $V_\tau(t) \setminus I(t)$.

Proof. See [LN0] lemma 2.3.3, p.83.

6. **Proof of Theorem A**

6.1. **End of the proof of Theorem A**. We divide the end of the proof of Theorem A in two parts. In the first part we construct a family of rational maps $f_t : \mathbb{P}^n \dashrightarrow \mathbb{P}^2$, $f_t \in Gen(n, \nu, 1, \gamma)$, such that $(f_t)_{t \in D_\epsilon}$ is a deformation of $f_0$ and the subvarieties $V_\tau(t) \subset Sing(\mathcal{G}_t)$, are fibers of $f_t$ for all $t$. In the second part we show that there exists a family of foliations $(\mathcal{G}_t)_{t \in D_\epsilon}, \mathcal{G}_t \in A$ (see Section 3) such that $\mathcal{F}_t = f_t^*(\mathcal{G}_t)$ for all $t \in D_\epsilon$. 


6.1.1. Part 1. Let us define the family of candidates that will be a deformation of the mapping $f_0$. Set $V_a = f_0^{-1}(a)$, $V_b = f_0^{-1}(b)$, $V_c = f_0^{-1}(c)$, where $a = [0 : 0 : 1]$, $b = [0 : 1 : 0]$ and $c = [1 : 0 : 0]$ and denote by $V_{\tau^*} = f_0^{-1}(\tau^*)$, where $\tau^* \in \text{Sing}(G_0) \setminus \{a, b, c\}$. In this coordinate system the points $b$ and $c$ belong to $\ell$.

**Proposition 6.1.** Let $(f_t)_{t \in D_{\ell}}$ be a deformation of $f_0 = f_0^*(G_0)$, where $(f_0, G_0)$ is a generic pair, with $G_0 \in A$, $f_0 \in \text{Gen}(n, \nu, 1, \gamma)$ and $\deg(f_0) = \nu \geq 2$. Then there exists a deformation $(f_t)_{t \in D_{\ell}}$ of $f_0$ in $\text{Gen}(n, \nu, 1, \gamma)$ such that:

(i) $V_a(t), V_b(t)$ and $V_c(t)$ are fibers of $(f_t)_{t \in D_{\ell}}$.

(ii) $I(t) = I(f_t), \forall t \in D_{\ell}$.

**Proof.** Let $\tilde{f}_0 = (F_0, F_1, F_2) : \mathbb{C}^{n+1} \to \mathbb{C}^3$ be the homogeneous expression of $f_0$. Then $V_a, V_b$ and $V_c$ appear as the complete intersections $(F_1 = F_2 = 0)$, $(F_0 = F_2 = 0)$, and $(F_0 = F_1 = 0)$ respectively. Hence $I(f_0) = V_a \cap V_b = V_a \cap V_c = V_b \cap V_c$. It follows from [Ser, section 4.6, p.235-236] that $V_a(t)$ is a complete intersection, say $V_a(t) = (F_0(t) = F_1(t) = 0)$, where $(F_0(t))_{t \in D_{\ell}}$ and $(F_1(t))_{t \in D_{\ell}}$ are deformations of $F_0$ and $F_1$ and $D_{\ell}$ is a possibly smaller neighborhood of $0$. Moreover, $F_0(t) = 0$ and $F_1(t) = 0$ meet transversely along $V_a(t)$. In the same way, it is possible to define $V_b(t)$ and $V_b(t)$ as complete intersections, say $(\tilde{F}_1(t) = F_2(t) = 0)$ and $(\tilde{F}_2(t) = F_1(t) = 0)$ respectively, where $(F_j(t))_{t \in D_{\ell}}$ and $(\tilde{F}_j(t))_{t \in D_{\ell}}$ are deformations of $F_j$, $0 \leq j \leq 2$.

We will prove that we can find polynomials $P_0(t), P_1(t)$ and $P_2(t)$ such that $V_a(t) = (P_1(t) = P_2(t) = 0), V_b(t) = (P_0(t) = P_2(t) = 0)$ and $V_c(t) = (P_0(t) = P_1(t) = 0)$. Observe first that since $F_0(t)$, $F_1(t)$ and $F_2(t)$ are near $F_0, F_1$ and $F_2$ respectively, they meet as a regular complete intersection at:

$$J(t) = (F_0(t) = F_1(t) = F_2(t) = 0) = V_a(t) \cap (F_2(t) = 0).$$

Hence $J(t) \cap (\tilde{F}_1(t) = 0) = V_c(t) \cap V_a(t) = I(t)$, which implies that $I(t) \subset J(t)$. Since $I(t)$ and $J(t)$ have $\frac{n^2}{3}$ points, we have that $I(t) = J(t)$ for all $t \in D_{\ell}$.

**Remark 6.2.** In the case $n \geq 4$, both sets are codimension-three smooth and connected submanifolds of $\mathbb{P}^n$, implying again that $I(t) = J(t)$. In particular, we obtain that

$$I(t) = (F_0(t) = F_1(t) = F_2(t) = 0) \subset (\tilde{F}_j(t) = 0), 0 \leq j \leq 2.$$
In an analogous way we have that \( \hat{F}_0(t) = F_0(t) + m(t)F_1(t) + n(t)F_2(t) \) for the polygonal \( \hat{F}_0(t) \). Now observe that \( V(\hat{F}_0(t), \hat{F}_2(t)) = V(F_0(t) + m(t)F_1(t), F_3(t)) \) where \( m(t) \in \mathbb{C} \) satisfying \( m(0) = 0 \). Hence we can define the family of polynomials as being \( P_0(t) = F_0(t) + m(t)F_1(t), P_1(t) = F_1(t) \) and \( P_2(t) = F_2(t) \). This defines a family of mappings \((f_t)_{t \in D',} : \mathbb{P}^3 \rightarrow \mathbb{P}^2 \), and \( V_{\delta}(t), V_{\delta}(t) \) and \( V_{\gamma}(t) \) are fibers of \( f_t \) for fixed \( t \). Observe that, for \( \epsilon' \) sufficiently small, \((f_t)_{t \in D'} \) is generic in the sense of definition 3.2, and its indeterminacy locus \( I(f_t) \) is precisely \( I(t) \). Moreover, since \( Gen(3, \nu, 1, \gamma) \) is open, we can suppose that this family \((f_t)_{t \in D'} \) is in \( Gen(3, \nu, 1, \gamma) \). This concludes the proof of proposition 5.10. \( \square \)

We observe that this family can be considered also as a family of mappings \((\mathcal{F}_t)_{t \in D'} : \mathbb{P}^3 \rightarrow \mathbb{P}^2_{[\gamma, \gamma, 1]} \), where \( \mathcal{F}_t = (F_0(t), F_1(t), F_2(t)) \) where \( \mathbb{P}^2_{[\gamma, \gamma, 1]} \) denotes the weighted projective plane with weights \((\gamma, \gamma, 1) \). Moreover, using the map

\[
\begin{align*}
f'_{\mathbf{w}} : \mathbb{P}^2_{[\gamma, \gamma, 1]} & \rightarrow \mathbb{P}^2 \\
(x_0 : x_1 : x_2) & \rightarrow (x_0 : x_1 : x_2^{\gamma})
\end{align*}
\]

we can factorize \( f_t \) as being \( f_t = f'_{\mathbf{w}} \circ \mathcal{F}_t \) as shown in the diagram below:

\[
\begin{array}{ccc}
\mathbb{P}^3 & \xrightarrow{f_t} & \mathbb{P}^2 \\
\downarrow \mathcal{F}_t & & \downarrow f'_{\mathbf{w}} \\
\mathbb{P}^2_{[\gamma, \gamma, 1]} & & \mathbb{P}^2
\end{array}
\]

Now we will prove that the remaining curves \( V_{\gamma}(t) \) are also fibers of \( f_t \). In the local coordinates \( X(t) = (x_0(t), x_1(t), x_2(t)) \) near some point of \( I(t) \) we have that the vector field \( S \) is diagonal and the components of the map \( f_t \) are written as follows:

\[
\begin{align*}
P_0(t) &= u_0x_0(t) + x_1(t)x_2(t)h_{0u} \\
P_1(t) &= u_1x_1(t) + x_0(t)x_2(t)h_{1u} \\
P_2(t) &= u_2x_2(t) + x_0(t)x_1(t)h_{2u}
\end{align*}
\]

where the functions \( u_{it} \in \mathcal{O}^*(\mathbb{C}^3, 0) \) and \( h_{it} \in \mathcal{O}^*(\mathbb{C}^3, 0), 0 \leq i \leq 2 \). Note that when the parameter \( t \) goes to 0 the functions \( h_{it}(t), 0 \leq i \leq 2 \) also goes to 0. We want to show that an orbit of the vector field \( S \) in the coordinate system \( X(t) \) that extends globally like a singular curve of the foliation \( \mathcal{F}_t \) is a fiber of \( f_t \).

**Lemma 6.4.** Any generic orbit of the vector field \( S \) that extends globally as singular curve of the foliations \( \mathcal{F}_t \) is also a fiber of \( f_t \) for fixed \( t \).

**Proof.** To simplify the notation we will omit the index \( t \). Let \( \delta(s) \) be a generic orbit of the vector field \( S \) (here by a generic orbit we mean an orbit that is not a coordinate axis). We can parametrize \( \delta(s) \) as \( s \rightarrow (as^{\gamma}, bs^{\gamma}, cs) \), \( a \neq 0, b \neq 0, c \neq 0 \). Without loss of generality we can suppose that \( a = b = c = 1 \). We have

\[
f_s(\delta(s)) = [(s^{\gamma}u_0 + s^{(1+\gamma)}h_{0u}) : (s^{\gamma}u_1 + s^{(1+\gamma)}h_{1u}) : (su_2 + s^{2\gamma}h_{2u})^{\gamma}].
\]

Hence we can extract the factor \( s^{\gamma} \) from \( f_s(\delta(s)) \) and we obtain

\[
f_s(\delta(s)) = [(u_0 + sh_0) : (u_1 + sh_1) : (u_2 + s^{2\gamma}h_{2u})^{\gamma}].
\]

Since \( V_{\gamma} \) is a fiber, \( f_0(V_{\gamma}) = [d : e : f] \in \mathbb{P}^2 \) with \( d \neq 0, e \neq 0, f \neq 0 \). If we take a covering of \( I(f) = \{p_1, ..., p_{\nu} \} \) by small open balls \( B_j(p_j) \), \( 1 \leq j \leq \frac{n}{\nu} \).
the set $V_{\gamma} \setminus \cup_j B_j(p_j)$ is compact. For a small deformation $f_t$ of $f_0$ we have that $f_t[V_{\gamma}(t) \setminus \cup_j B_j(p_j)(t)]$ stays near $f[V_{\gamma} \setminus \cup_j B_j(p_j)]$. Hence for $t$ sufficiently small the components of expression \[G\] do not vanish both inside as well as outside of the neighborhood $\cup_j B_j(p_j)(t)$.

This implies that the components of $f_t$ do not vanish along each generic fiber that extends locally as a singular curve of the foliation $F_t$. This is possible only if $f_t$ is constant along these curves. In fact, $f_t(V_{\gamma}(t))$ is either a curve or a point. If it is a curve then it cuts all lines of $\mathbb{P}^2$ and therefore the components should be zero somewhere. Hence $f_t(V_{\gamma}(t))$ is constant and we conclude that $V_{\gamma}(t)$ is a fiber.

Observe also that when we make a blow-up with weights $(\gamma, \gamma, 1)$ at the points of $I(f_t)$ we solve completely the indeterminacy points of the mappings $f_t$ for each $t$. □

6.1.2. Part 2. Let us now define a family of foliations $(\mathcal{G}_t)_{t \in D}$, $\mathcal{G}_t \in \mathcal{A}$ (see Section \[I\]) such that $F_t = \mathcal{G}_t^* (\mathcal{F}_t)$ for all $t \in D$. Firstly we consider the case $n = 3$. Instead of utilize the foliation $\mathcal{F}$ obtained as the foliation $\mathcal{F}^* \mathcal{G}$, the idea that we will utilize in this part of the proof is to consider $\mathcal{F}$ on $\mathbb{P}^n$ defined as the foliation pull-back foliation from $\mathbb{P}^n$ to $\mathbb{P}^2_{[\gamma, \gamma, 1]}$:

$$\mathcal{F} : \mathbb{P}^n \to \mathbb{P}^2_{[\gamma, \gamma, 1]}$$

$$\mathcal{F} \eta \to \eta.$$

once they define the same foliation. Let $M_{[\gamma, \gamma, 1]}(t)$ be the family of “complex algebraic threefolds” obtained from $\mathbb{P}^3$ by blowing-up with weights $(\gamma, \gamma, 1)$ at the $\frac{n^3}{3}$ points $p_1(t), ..., p_j(t), ..., p_{n^2}(t)$ corresponding to $I(t)$ of $\mathcal{F}_t$; and denote by

$$\pi_w(t) : M_{[\gamma, \gamma, 1]}(t) \to \mathbb{P}^3$$

the blowing-up map. The exceptional divisor of $\pi_w(t)$ consists of $\frac{n^3}{3}$ orbifolds $E_j(t) = \pi_w(t)^{-1}(p_j(t))$, $1 \leq j \leq \frac{n^3}{3}$, which are weighted projective planes of the type $\mathbb{P}^2_{[\gamma, \gamma, 1]}$. More precisely, if we blow-up $\mathcal{F}_t$ at the point $p_j(t)$, then the restriction of the strict transform $\pi_w^* \mathcal{F}_t$ to the exceptional divisor $E_j(t) = \mathbb{P}^2_{[\gamma, \gamma, 1]}$ is the same quasi-homogeneous 1-form that defines $\mathcal{F}_t$ at the point $p_j(t)$. Using the map

$$f_w : \mathbb{P}^2_{[\gamma, \gamma, 1]} \to \mathbb{P}^2$$

$$(x_0 : x_1 : x_2) \to (x_0 : x_1 : x_2^3)$$

it follows that we can push-forward the foliation to $\mathbb{P}^2$. Let us denote by $\text{Fol}_{[\gamma, \gamma, 1]}^3$ the set of $\{\hat{\mathcal{G}}\}$ saturated foliations of degree $d' = \gamma(d + 1) + 1$ on $\mathbb{P}^2_{[\gamma, \gamma, 1]}$ with one invariant line in general position and $H_1(d, 2)$ the subsets of saturated foliations with an invariant line in $\mathbb{P}^2$ respectively. The mapping $f_w : \mathbb{P}^2_{[\gamma, \gamma, 1]} \to \mathbb{P}^2$ induces a natural isomorphism $(f_w)_* : H_1(d, 2) \to \text{Fol}_{[\gamma, \gamma, 1]}^3$. With this process in mind we produce a family of holomorphic foliations in $\mathcal{A} \subset H_1(d, 2)$. This family is the “holomorphic path” of candidates to be a deformation of $\hat{\mathcal{G}}_0$. In fact, since $(\mathcal{A}' = f_w)_* (\mathcal{A})$ is an open set inside $\text{Fol}_{[\gamma, \gamma, 1]}^3$ we can suppose that this family is inside $\mathcal{A}$. Hence using the mapping $f_w$, we can transport holomorphic from $\mathcal{A}$ to $\mathcal{A}'$ and vice-versa.

We fix the exceptional divisor $E_1(t)$ to work with and we denote by $\hat{\mathcal{G}}_t \in \mathcal{A}'$ the restriction of $\pi^*_w \mathcal{F}_t$ to $E_1(t)$. As we have seen, this process produces foliations in $\mathcal{A}'$ up to a linear automorphism of $\mathbb{P}^2_{[\gamma, \gamma, 1]}$. Consider the family of mappings
We will consider the family \((\tilde{f}_t)_{t \in D_{\epsilon}}\) as a family of rational maps \(\tilde{f}_t : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2_{[\gamma,1]}\) defined in Proposition 6.1. We decrease \(\epsilon\) if necessary. Note that the map

\[
\tilde{f}_t \circ \pi_w(t) : M_{[\gamma,1]}(t) \cup_j E_j(t) \to E_1(t) \simeq \mathbb{P}^2_{[\gamma,1]}
\]

extends holomorphically, that is, as an orbifold mapping, to

\[
\hat{f}_t : M_{[\gamma,1]}(t) \to E_1(t) \simeq \mathbb{P}^2_{[\gamma,1]}.
\]

This is due to the fact that each orbit of the vector field \(\mathcal{S}_t\) determines an equivalence class in \(\mathbb{P}^2_{[\gamma,1]}\) and is a fiber of the map

\[
(x_0(t), x_1(t), x_2(t)) \to (x_0(t), x_1(t), x_2(t)).
\]

The mapping \(\tilde{f}_t\) can be interpreted as follows. Each fiber of \(\tilde{f}_t\) meets \(p_j(t)\) once, which implies that each fiber of \(\hat{f}_t\) cuts \(E_1(t)\) once outside of the singular line in \([M_{[\gamma,1]}(t) \cap E_1(t)]\). Since \(M_{[\gamma,1]}(t) \cup_j E_j(t)\) is biholomorphic to \(\mathbb{P}^3 I(t)\), after identifying \(E_1(t)\) with \(\mathbb{P}^2_{[\gamma,1]}\) we can imagine that if \(q \in M_{[\gamma,1]}(t) \cup_j E_j(t)\) then \(\hat{f}_t(q)\) is the intersection point of the fiber \(\hat{f}_t^{-1}(\hat{f}_t(q))\) with \(E_1(t)\). We obtain a mapping

\[
\hat{f}_t : M_{[\gamma,1]}(t) \to \mathbb{P}^2_{[\gamma,1]}.
\]

It can be extended over the singular set of \(M_{[\gamma,1]}(t)\) using Riemann’s Extension Theorem. This is due to the fact that the orbifold \(M_{[\gamma,1]}(t)\) has singular set of codimension 2 and these singularities are of the quotient type; therefore it is a normal complex space. We shall also denote this extension by \(\hat{f}_t\) to simplify the notation. We remark that the blowing-up with weights \((\gamma, \gamma, 1)\) can completely solve the indeterminacy set of \(\tilde{f}_t\) or \(\hat{f}_t\) for each \(t\) as the reader can check. With all these ingredients we can define the foliation \(\tilde{F}_t = \tilde{F}_t(\tilde{G}_t) = \tilde{f}_t^* (\tilde{G}_t) \in PB(\Gamma - 1, \nu, 1, 1, \gamma)\). This foliation is a deformation of \(\tilde{F}_0\). Based on the previous discussion let us denote \(\mathcal{F}_1(t) = \pi_w(t)^*(\mathcal{F}_1)\) and \(\tilde{F}_1(t) = \pi_w(t)^*(\tilde{F}_1)\).

**Lemma 6.5.** If \(\mathcal{F}_1(t)\) and \(\tilde{F}_1(t)\) are the foliations defined previously, we have that

\[
\mathcal{F}_1(t)|_{E_1(t) \simeq \mathbb{P}^2_{[\gamma,1]}} = \tilde{G}_t = \tilde{F}_1(t)|_{E_1(t) \simeq \mathbb{P}^2_{[\gamma,1]}}
\]

where \(\tilde{G}_t\) is the foliation induced on \(E_1(t) \simeq \mathbb{P}^2_{[\gamma,1]}\) by the quasi-homogeneous 1-form \(\eta_{p_1(t)}\).

**Proof.** In a neighborhood of \(p_1(t) \in I(t)\), \(\mathcal{F}_1\) is represented by the quasi-homogeneous 1-form \(\eta_{p_1(t)}\). This 1-form satisfies \(i_{S_t} \eta_{p_1(t)} = 0\) and therefore naturally defines a foliation on the weighted projective space \(E_1(t) \simeq \mathbb{P}^2_{[\gamma,1]}\). This proves the first equality. The second equality follows from the geometrical interpretation of the mapping \(\hat{f}_t : M_{[\gamma,1]}(t) \to \mathbb{P}^2_{[\gamma,1]}\), since \(\tilde{F}_1(t) = \hat{f}_t^* (\tilde{G}_1)\).

Now we choose an affine chart of the space \(\mathbb{P}^2_{[\gamma,1]}\). This affine chart is biholomorphic to \(\mathbb{C}^2\). In this affine chart for each \(t\) the foliation \(\tilde{G}_t\) has \(d^2\) singular points.

Let \(\tau_1(t)\) be a singularity of \(\tilde{G}_t\) outside of the line at infinity. Since the map \(t \to \tau_1(t) \in \mathbb{P}^2_{[\gamma,1]}\) is holomorphic, there exists a holomorphic family of automorphisms of \(\mathbb{P}^2_{[\gamma,1]}\) such that \(\tau_1(t) = [0 : 0 : 1] \in E_1(t) \simeq \mathbb{P}^2_{[\gamma,1]}\) is kept fixed. Observe that such a singularity has non algebraic separatrices at this point. Fix
a local analytic coordinate system \((x_t, y_t)\) at \(\tau_1(t)\) such that the local separatrices are \((x_t = 0)\) and \((y_t = 0)\), respectively. Here we are considering the affine chart of \(\mathbb{P}^2_{[\gamma, \gamma, 1]}\) which is biholomorphic to \(\mathbb{C}^2\). This is useful because the foliations \(G_t\) and \(\tilde{G}_t\) in this local coordinates are at least biholomorphic equivalents. Observe that the local smooth hypersurfaces along \(\tilde{V}_{\tau_1(t)} = \tilde{f}_t^{-1}(\tau_1(t))\) defined by \(\tilde{X}_t := (x_t \circ \tilde{f}_t = 0)\) and \(\tilde{Y}_t := (y_t \circ \tilde{f}_t = 0)\) are invariant for \(\tilde{F}_1(t)\). Furthermore, they meet transversely along \(\tilde{V}_{\tau_1(t)}\). On the other hand, \(\tilde{V}_{\tau_1(t)}\) is also contained in the Kupka set of \(F_1(t)\). Therefore there are two local smooth hypersurfaces \(X_t := (x_t \circ f_t = 0)\) and \(Y_t := (y_t \circ f_t = 0)\) invariant for \(F_1(t)\) such that:

1. \(X_t\) and \(Y_t\) meet transversely along \(\tilde{V}_{\tau_1(t)}\).
2. \(X_t \cap \pi_w(t)^{-1}(p_1(t)) = (x_t = 0) = \tilde{X}_t \cap \pi_w(t)^{-1}(p_1(t))\) and \(Y_t \cap \pi_w(t)^{-1}(p_1(t)) = (y_t = 0) = \tilde{Y}_t \cap \pi_w(t)^{-1}(p_1(t))\) (because \(F_1(t)\) and \(\tilde{F}_1(t)\) coincide on \(E_1(t) \simeq \mathbb{P}^2\)).
3. \(X_t\) and \(Y_t\) are deformations of \(X_0 = \tilde{X}_0\) and \(Y_0 = \tilde{Y}_0\), respectively.

**Lemma 6.6.** \(X_t = \tilde{X}_t\) for small \(t\).

**Proof.** Let us consider the projection \(\hat{f}_t : M_{[\gamma, \gamma, 1]}(t) \to \mathbb{P}^2_{[\gamma, \gamma, 1]}\) on a neighborhood of the regular fibre \(\hat{V}_{\tau_1(t)}\), and fix local coordinates \(x_t, y_t\) on \(\mathbb{P}^2_{[\gamma, \gamma, 1]}\) such that \(X_t := (x_t \circ \hat{f}_t = 0)\). For small \(\epsilon\), let \(H_\epsilon = (y_t \circ \hat{f}_t = \epsilon)\). Thus \(\hat{\Sigma}_\epsilon = \hat{X}_t \cap H_\epsilon\) are (vertical) compact curves, deformations of \(\hat{\Sigma}_0 = \hat{V}_{\tau_1(t)}\). Set \(\Sigma_\epsilon = X_t \cap \tilde{H}_\epsilon\). The \(\Sigma_\epsilon\)s, as the \(\tilde{\Sigma}_\epsilon\)s, are compact curves (for \(t\) and \(\epsilon\) small), since \(X_t\) and \(\tilde{X}_t\) are both deformations of the same \(X_0\). Thus for small \(t\), \(X_t\) is close to \(\tilde{X}_t\). It follows that \(\hat{f}_t(\Sigma_\epsilon)\) is an analytic curve contained in a small neighborhood of \(\tau_1(t)\), for small \(\epsilon\). By the maximum principle, we must have that \(\hat{f}_t(\Sigma_\epsilon)\) is a point, so that \(\hat{f}_t(X_t) = \hat{f}_t(\cup_\epsilon \Sigma_\epsilon)\) is a curve \(C\), that is, \(X_t = \hat{f}_t^{-1}(C)\). But \(X_t\) and \(\tilde{X}_t\) intersect the exceptional divisor \(E_1(t) = \mathbb{P}^2_{[\gamma, \gamma, 1]}\) along the separatrix \((x_t = 0)\) of \(G_t\) through \(\tau_1(t)\). This implies that \(X_t = \hat{f}_t^{-1}(C) = \hat{f}_t^{-1}(x_t = 0) = \tilde{X}_t\). \(\square\)

We have proved that the foliations \(F_t\) and \(\tilde{F}_t\) have a common local leaf: the leaf that contains \(\pi_w(t)\left(\tilde{X}_t \setminus \tilde{V}_{\tau_1(t)}\right)\) which is not algebraic. Let \(D(t) := \text{Tang}(F(t), \tilde{F}(t))\) be the set of tangencies between \(F(t)\) and \(\tilde{F}(t)\). This set can be defined by \(D(t) = \{Z \in \mathbb{C}^4; \Omega(t) \land \tilde{\Omega}(t) = 0\}\), where \(\Omega(t)\) and \(\tilde{\Omega}(t)\) define \(F(t)\) and \(\tilde{F}(t)\), respectively. Hence it is an algebraic set. Since this set contains an immersed non-algebraic surface \(X_t\), we necessarily have that \(D(t) = \mathbb{P}^3\). This proves Theorem B in the case \(n = 3\).

Suppose now that \(n \geq 4\). The previous argument implies that if \(Y\) is a generic 3–plane in \(\mathbb{P}^n\), we have \(F(t) \cap Y = \tilde{F}(t) \cap Y\). In fact, such planes cut transversely every strata of the singular set, and \(I(t)\) consists of \(\mathbb{P}^3\) points. This implies that \(f_t\) is generic for \(|t|\) sufficiently small. We can then repeat the previous argument, finishing the proof of Theorem A.

Recall from Definition 22 the concept of a generic map. Let \(f \in BRM(n, \nu, 1, 1, \gamma)\), \(I(f)\) its indeterminacy locus and \(F\) a foliation on \(\mathbb{P}^n\), \(n \geq 3\). Consider the following properties:
If \( n = 3 \), at any point \( p_j \in I(f) \) \( \mathcal{F} \) has the following local structure: there exists an analytic coordinate system \((U_{p_j}, Z_{p_j})\) around \( p_j \) such that \( Z_{p_j}(p_j) = 0 \in (C^3, 0) \) and \( \mathcal{F}|_{(U_{p_j}, Z_{p_j})} \) can be represented by a quasi-homogeneous 1-form \( \eta_{p_j} \) (as described in the Lemma 5.7) such that

(a) \( \text{Sing}(d\eta_{p_j}) = 0 \),

(b) 0 is a quasi-homogeneous singularity of the type \([\gamma : \gamma : 1]\).

If \( n \geq 4 \), \( \mathcal{F} \) has a local structure: the situation for \( n = 3 \) “times” a regular foliation in \( C^{n-3} \).

\[ P_2: \] There exists a fibre \( f^{-1}(q) = V(q) \) such that \( V(q) = f^{-1}(q) \setminus I(f) \) is contained in the Kupka-Set of \( \mathcal{F} \) and \( V(q) \) is not contained in \((F_2 = 0)\).

\[ P_3: \] \( V(q) \) has transversal type \( X \), where \( X \) is a germ of vector field on \((C^2, 0)\) with a non algebraic separatrix and such that \( 0 \in C^2 \) is a non-degenerate singularity with eigenvalues \( \lambda_1 \) and \( \lambda_2 \), \( \lambda_2 / \lambda_1 \notin R \).

Lemma 6.6 allows us to prove the following results:

**Theorem B.** In the conditions above, if properties \( P_1, P_2 \) and \( P_3 \) hold then \( \mathcal{F} \) is a pull back foliation, \( \mathcal{F} = f^* (\hat{\mathcal{G}}) \), where \( \hat{\mathcal{G}} \) is of degree \( d \geq 2 \) on \( \mathbb{P}^2 \) with one invariant line.

Let us denote by \( \text{Fol}'_{d'}[2, (\gamma, \gamma, 1)] \) the set of \{\( \hat{\mathcal{G}} \)\} saturated foliations of degree \( d' \) on \( \mathbb{P}^2_{[\gamma, \gamma, 1]} \) with one invariant line. According to this notation the previous Theorem can be re-written as:

**Theorem C.** In the conditions above, if properties \( P_1, P_2 \) and \( P_3 \) hold then \( \mathcal{F} \) is a pull back foliation, \( \mathcal{F} = \mathcal{F}|_{f^* (\hat{\mathcal{G}})} \), where \( \hat{\mathcal{G}} \in \text{Fol}'_{d'}[2, (\gamma, \gamma, 1)] \)

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