Intersection curve of two parametric surfaces in Euclidean $n$-space

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Abstract. The aim of this paper is to study the differential geometric properties of the intersection curve of two parametric surfaces in Euclidean $n$-space. For this aim, we first present the $m$th order derivative formula of a curve lying on a parametric surface. Then, we obtain curvatures and Frenet vectors of the transversal intersection curve of two parametric surfaces in Euclidean $n$-space. We also provide computer code produced in MATLAB to simplify determining the coefficients relative to Frenet frame of higher order derivatives of a curve.

1. Introduction

Recently, the surface-surface intersection problem has attracted much attention due to its importance in computer aided geometric design. This attention is due to the fact that determining the intersection curve, i.e. its parametric representation, of two surfaces is not easy in most cases. However, even if the intersection curve cannot be represented parametrically, it has been shown in recent studies that the differential geometric properties of the intersection curve can be still computed. If the tangent planes of two intersecting surfaces overlap at an intersection point, the intersection is called a tangential intersection, otherwise it is called a transversal intersection in Euclidean 3-space $\mathbb{E}^3$. The majority of the recent studies focus on transversal intersection rather than tangential intersection. The reason for this is the easy computation (vector product of normal vectors of two surfaces) of the tangential direction in transversal intersection.

Differential geometry of intersection curves of two parametric surfaces in Euclidean 3-space has been studied by some authors using different methods.
[10, 21]. These studies have been extended to $\mathbb{E}^4$ for the intersection of three parametric hypersurfaces [6, 9, 10, 14] and to $\mathbb{E}^5$ for the intersection of four parametric hypersurfaces [15]. Since surfaces can also be defined by their implicit equations, differential geometry of the intersection curve of two implicit surfaces has been studied in $\mathbb{E}^3$ by [2, 12, 20, 21] and of three implicit hypersurfaces has been studied in $\mathbb{E}^4$ by [3, 4, 7, 14, 19]. There also exist some studies for the intersection curves of different type surfaces in $\mathbb{E}^3$ [10, 18, 21] and in $\mathbb{E}^4$ [1, 8, 10, 14]. On the other hand, by using the wedge product of two vectors in $(n+1)$-dimensions, Goldman [12] derived a closed formula for the first curvature of the transversal intersection of $n$-implicit hypersurfaces in $(n+1)$-dimensions. Aléssio [5] derived the second and the third curvatures of intersection curves of $(n-1)$-implicit hypersurfaces in $\mathbb{E}^n$ by generalizing the method of Goldman. Recently, the intersection curve of $(n-1)$ transversally intersecting hypersurfaces has been studied in $\mathbb{E}^n$ by [17].

If we take into account all of these recent studies, it is seen that all intersection problems have been considered for the intersection of $(n-1)$-hypersurfaces in $\mathbb{E}^n$. However, to the best of our knowledge, differential geometry of the intersection curve of two surfaces has not been studied in higher dimensions. Since the normal space of a surface is $(n-2)$-dimensional and the vector product is defined for $(n-1)$-vectors in $n$-space, computing the tangential direction even for the transversal intersection of two surfaces in $\mathbb{E}^n$ is not as evident as in $\mathbb{E}^3$.

The purpose of this paper is to consider the intersection problem of two parametric surfaces in $\mathbb{E}^n$, $n \geq 4$, and study the differential geometry of their intersection curve. First we obtain the $m$th order derivative formula for a curve lying on a parametric surface. We also generate orthogonal bases for the normal spaces of the intersecting surfaces. By using the $m$th order derivative formula and orthogonal bases of normal spaces of the intersecting surfaces, we present a method which enables us to compute all curvatures and Frenet vectors of the intersection curve of two parametric surfaces in $\mathbb{E}^n$. We also provide a computer code produced in MATLAB to simplify determining the coefficients relative to the Frenet frame of higher order derivatives of a curve.

This paper is organized as follows. Section 2 includes the definition of the vector product of $(n-1)$-vectors in $\mathbb{R}^n$, Frenet formulas and derivatives up to order 4 of a curve lying on a surface in $\mathbb{E}^n$. We present the $m$th order derivative formula of a curve lying on a parametric surface in Section 3. Section 4 involves a method which enables us to compute all curvatures and Frenet vectors of the intersection curve of two parametric surfaces in $\mathbb{E}^n$. We provide two examples as an application of the method in Section 5.
2. Preliminaries

**Definition 1.** Let \( \{e_1, e_2, \ldots, e_n\} \) denote the standard basis in \( \mathbb{R}^n \). The vector product of the vectors \( a_1 = \sum_{j=1}^{n} a_{1j}e_j \), \( a_2 = \sum_{j=1}^{n} a_{2j}e_j \), \ldots, \( a_{n-1} = \sum_{j=1}^{n} a_{n-1j}e_j \) is defined by (see [16])

\[
H = a_1 \times a_2 \times \cdots \times a_{n-1} = \begin{vmatrix}
e_1 & e_2 & \cdots & e_n \\
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n}
\end{vmatrix}
\]

The product \( H \) in \( \mathbb{R}^n \) is a vector perpendicular simultaneously to all the \( e_i \) \( (1 \leq i \leq n-1) \) and its norm is given by the formula (see [16])

\[
||H|| = ||a_1|| \cdot ||a_2|| \cdots ||a_{n-1}|| \cdot K,
\]

where

\[
K = \begin{vmatrix}
1 & \cos \alpha_{12} & \cdots & \cos \alpha_{1,n-1} \\
\cos \alpha_{21} & 1 & \cdots & \cos \alpha_{2,n-1} \\
\vdots & \vdots & \ddots & \vdots \\
\cos \alpha_{n-1,1} & \cos \alpha_{n-1,2} & \cdots & 1
\end{vmatrix}^{1/2}
\]

with \( \cos \alpha_{ij} = \frac{\langle e_i, e_j \rangle}{||e_i|| \cdot ||e_j||} \).

Let \( S \subset \mathbb{E}^n \) be a regular surface given by \( X(u_1, u_2) \) and \( \alpha \) be a unit-speed curve with the arc-length \( s \) lying on \( S \). If we denote the Frenet frame of \( \alpha \) by \( \{V_1(s), V_2(s), \ldots, V_n(s)\} \), then the Frenet formulas are given by (see [11])

\[
\begin{align*}
V_1'(s) &= k_1(s)V_2(s), \\
V_i'(s) &= -k_{i-1}(s)V_{i-1}(s) + k_i(s)V_{i+1}(s), \quad 2 \leq i \leq n-1, \\
V_n'(s) &= -k_{n-1}(s)V_{n-1}(s),
\end{align*}
\]

where \( k_1(s), k_2(s), \ldots, k_{n-1}(s) \) are the curvatures of \( \alpha(s) \). Using the Frenet formulas, we can write the derivatives of the curve \( \alpha \) as

\[
\begin{align*}
\alpha' &= V_1, \\
\alpha'' &= k_1V_2, \\
\alpha''' &= -k_1^2V_1 + k_1'V_2 + k_1k_2V_3, \\
\alpha^{(4)} &= -3k_1k_1'V_1 + (-k_1^2 + k_1'' - k_1k_2')V_2 + (2k_1'k_2 + k_1k_2')V_3 + k_1k_2k_3V_4, \\
&\vdots \\
\alpha^{(n)} &= \{\ldots\}V_1 + \{\ldots\}V_2 + \ldots + \{\ldots\}V_{n-1} + k_1k_2k_3\ldots k_{n-1}V_n.
\end{align*}
\]
In addition, since the curve \( \alpha(s) \) lies on the surface \( S \), we can write \( \alpha(s) = X(u_1(s), u_2(s)) \). Then we have

\[
\alpha'(s) = \sum_{i_1=1}^{2} X_{i_1} u_{i_1}', \quad (2.1)
\]

\[
\alpha''(s) = \sum_{i_1=1}^{2} X_{i_1} u_{i_1}'' + \sum_{i_1,i_2=1}^{2} X_{i_1,i_2} u_{i_1}' u_{i_2}', \quad (2.2)
\]

\[
\alpha'''(s) = \sum_{i_1=1}^{2} X_{i_1} u_{i_1}'' + 3 \sum_{i_1,i_2=1}^{2} X_{i_1,i_2} u_{i_1}' u_{i_2}' + \sum_{i_1,i_2,i_3=1}^{2} X_{i_1,i_2,i_3} u_{i_1}' u_{i_2}' u_{i_3}', \quad (2.3)
\]

\[
\alpha^{(4)}(s) = \sum_{i_1=1}^{2} X_{i_1} u_{i_1}'' + 4 \sum_{i_1,i_2=1}^{2} X_{i_1,i_2} u_{i_1}' u_{i_2}' + 3 \sum_{i_1,i_2=1}^{2} X_{i_1,i_2} u_{i_1}'' u_{i_2}'' + 6 \sum_{i_1,i_2,i_3=1}^{2} X_{i_1,i_2,i_3} u_{i_1}' u_{i_2}' u_{i_3}' + \sum_{i_1,i_2,i_3,i_4=1}^{2} X_{i_1,i_2,i_3,i_4} u_{i_1}' u_{i_2}' u_{i_3}' u_{i_4}', \quad (2.4)
\]

where \( X_i = \frac{\partial X}{\partial u_i} \), \( X_{ij} = \frac{\partial^2 X}{\partial u_i \partial u_j} \), \ldots

3. Higher order derivatives of a curve lying on a parametric surface

We will need higher order derivatives of a curve lying on a surface. For that reason, we start by presenting the following theorem which enables to calculate higher order derivatives of a surface curve in \( \mathbb{E}^n \).

**Theorem 1.** Let \( S \subset \mathbb{E}^n \), \( n \geq 4 \), be a regular surface given by its parametric equation \( X(u_1, u_2) \) and \( \alpha \) be a unit-speed curve with arc-length \( s \) lying on \( S \) given by \( \alpha(s) = X(u_1(s), u_2(s)) \). Then, the \( m \)th order derivative of \( \alpha \) is obtained by

\[
\alpha^{(m)} = C_1 \sum_{i_1=1}^{2} k_{i_1} X_{i_1} u_{i_1}^{(m)} + \sum_{r_1,r_2=1}^{\left\lfloor \frac{m}{2} \right\rfloor} \left( C_2 \sum_{i_1,i_2=1}^{2} k_{i_1,i_2} X_{i_1,i_2} u_{i_1}^{(m-r_1)} u_{i_2}^{(r_1)} \right) + \sum_{r_1,r_2,r_3=1}^{\left\lfloor \frac{m}{3} \right\rfloor} \left( C_3 \sum_{i_1,i_2,i_3=1}^{2} k_{i_1,i_2,i_3} X_{i_1,i_2,i_3} u_{i_1}^{(m-r_1-1)} u_{i_2}^{(r_2)} u_{i_3}^{(r_1-r_2+1)} \right) + \sum_{r_1,r_2,r_3,r_4=1}^{\left\lfloor \frac{m}{4} \right\rfloor} \left( C_4 \sum_{i_1,i_2,i_3,i_4=1}^{2} k_{i_1,i_2,i_3,i_4} X_{i_1,i_2,i_3,i_4} u_{i_1}^{(m-r_1-2)} u_{i_2}^{(r_2)} u_{i_3}^{(r_1-r_2-2+2)} \right) 
\]

\vdots
where

\[
C_1 = \binom{m}{i}, \quad C_2 = \binom{m}{r_1, r_2, \ldots, r_m}, \quad C_3 = \binom{m}{r_1 + 1, r_2 + 1, \ldots, r_m}, \\
C_4 = \binom{m}{r_1 + 2, r_2 + 2, \ldots, r_m}, \quad \ldots, \quad C_m = \binom{m}{r_1 + (m-1), r_2 + (m-1), \ldots, r_m}, \quad \text{(each } C_\ell \text{ includes } \ell \text{ combinations)}
\]

the factors \( C_1, 1 < i < m, \) are determined by

\[
\binom{m}{r_1 + (i-2), r_2 + (i-2), \ldots, r_m + (i-2)},
\]

the terms \( b \) of each \( \binom{a}{b} \) represents the exponent of \( u_i \) in which the terms \( u_{i_1}^{(t_1)}, u_{i_2}^{(t_2)}, \ldots, u_{i_m}^{(t_m)} \) are written in the order such that \( t_1 \geq t_2 \geq \ldots \geq t_m \), and if \( t_i \leq 0, 1 \leq i \leq m, \) the related sum term is canceled (see appendix A for explanation). The coefficients \( k_{i_1}, k_{i_1i_2}, \ldots, k_{i_1i_2\ldots i_{m-1}i_m} \) are calculated as shown in Table 1, where the integer \( q \) in each \( q! \) represents the number of repeated exponents.

| \( u_{i_1}^{(t_1)} \) & \( u_{i_2}^{(t_2)} \) & \( u_{i_3}^{(t_3)} \) & \( u_{i_4}^{(t_4)} \) & \( u_{i_5}^{(t_5)} \) & \( u_{i_m}^{(t_m)} \) | \( k_{i_1}, k_{i_1i_2}, \ldots, k_{i_1i_2\ldots i_{m-1}i_m} \) |
|---|---|---|---|---|---|
| \( u_{i_1}^{(t_1)} \) & \( u_{i_2}^{(t_2)} \) & \( u_{i_3}^{(t_3)} \) & \( u_{i_4}^{(t_4)} \) & \( u_{i_5}^{(t_5)} \) & \( u_{i_m}^{(t_m)} \) | \( 1 \) |
| \( u_{i_1}^{(t_1)} \) & \( u_{i_2}^{(t_2)} \) & \( u_{i_3}^{(t_3)} \) & \( u_{i_4}^{(t_4)} \) & \( u_{i_5}^{(t_5)} \) & \( u_{i_m}^{(t_m)} \) | \( 2 \) |
| \( u_{i_1}^{(t_1)} \) & \( u_{i_2}^{(t_2)} \) & \( u_{i_3}^{(t_3)} \) & \( u_{i_4}^{(t_4)} \) & \( u_{i_5}^{(t_5)} \) & \( u_{i_m}^{(t_m)} \) | \( 2! \) |
| \( u_{i_1}^{(t_1)} \) & \( u_{i_2}^{(t_2)} \) & \( u_{i_3}^{(t_3)} \) & \( u_{i_4}^{(t_4)} \) & \( u_{i_5}^{(t_5)} \) & \( u_{i_m}^{(t_m)} \) | \( 3! \) |
| \( u_{i_1}^{(t_1)} \) & \( u_{i_2}^{(t_2)} \) & \( u_{i_3}^{(t_3)} \) & \( u_{i_4}^{(t_4)} \) & \( u_{i_5}^{(t_5)} \) & \( u_{i_m}^{(t_m)} \) | \( 3! \) |

**Proof.** Let us prove it by induction.
Step I: Let us show that it is true for \( m = 2 \). In this case, \( r_1 = 1 \). Then, we may write

\[
\alpha''(s) = \left(\frac{2}{2}\right) \sum_{i_1=1}^{2} \frac{1}{k_{i_1}} X_{i_1} u''_{i_1} + \left(\frac{1}{1}\right) \sum_{i_1,i_2=1}^{2} \frac{1}{k_{i_1} k_{i_2}} X_{i_1 i_2} u'_{i_1} u'_{i_2}
\]

\[
= \frac{1}{k_1} X_1 u''_1 + \frac{1}{k_2} X_2 u''_2 + 2 \frac{1}{k_{11}} X_{11} u'_1 u'_1 + 2 \frac{1}{k_{12}} X_{12} u'_1 u'_2 + 2 \frac{1}{k_{21}} X_{21} u'_1 u'_1 + 2 \frac{1}{k_{22}} X_{22} u'_2 u'_2.
\]

From Table 1 given above, we have \( k_1 = k_2 = 1, k_{11} = k_{12} = k_{21} = k_{22} = 2 \), which yields

\[
\alpha''(s) = X_1 u''_1 + X_2 u''_2 + X_{11} u'_1 u'_1 + X_{12} u'_1 u'_2 + X_{21} u'_2 u'_1 + X_{22} u'_2 u'_2
\]

as given in (2.2). Thus, the statement holds for \( m = 2 \).

Step II: We assume that the formula (3.1) is true for \( m \).

Step III: Let us show that the formula is true for \( m + 1 \). By taking the derivative of both sides of \( \alpha^{(m)} \), we obtain

\[
\alpha^{(m+1)} = C_1 \left[ \sum_{i_1,i_2=1}^{2} \frac{1}{k_{i_1}} X_{i_1 i_2} u^{(m)}_{i_1} u''_{i_2} + \sum_{i_1=1}^{2} \frac{1}{k_{i_1}} X_{i_1} u^{(m+1)}_{i_1} \right]
\]

\[
+ \sum_{r_1=1}^{\lfloor \frac{m}{2} \rfloor} C_2 \left[ \sum_{i_1,i_2,i_3=1}^{2} \frac{1}{k_{i_1} k_{i_2} k_{i_3}} X_{i_1 i_2 i_3} u^{(m-r_1)}_{i_1} u^{(r_1)}_{i_2} u''_{i_3} \right]
\]

\[
+ \sum_{r_1,r_2=1}^{\lfloor \frac{m}{2} \rfloor} C_3 \left[ \sum_{i_1,i_2,i_3,i_4=1}^{2} \frac{1}{k_{i_1} k_{i_2} k_{i_3} k_{i_4}} X_{i_1 i_2 i_3 i_4} u^{(m-r_1-1)}_{i_1} u^{(r_2)}_{i_2} u^{(r_1-r_2+1)}_{i_3} u''_{i_4} \right]
\]

\[
+ \sum_{r_1,r_2,r_3=1}^{\lfloor \frac{m}{2} \rfloor} \left[ \sum_{i_1,i_2,i_3,i_4,i_5=1}^{2} \frac{1}{k_{i_1} k_{i_2} k_{i_3} k_{i_4} k_{i_5}} X_{i_1 i_2 i_3 i_4 i_5} u^{(m-r_1-1)}_{i_1} u^{(r_2)}_{i_2} u^{(r_1-r_2+1)}_{i_3} u^{(r_1-r_2+1)}_{i_4} \right]
\]

\[
\vdots
\]
or

\[
\alpha^{(m+1)} = C_1 \sum_{i_1=1}^{m} \frac{1}{k_{i_1}} X_{i_1} u_i^{(m+1)} + C_1 \sum_{i_1,i_2=1}^{m} \frac{1}{k_{i_1} k_{i_2}} X_{i_1 i_2} u_{i_1}^{(m)} u_{i_2}' + \sum_{r_1=1}^{\lfloor m/2 \rfloor} \left( C_2 \sum_{i_1,i_2,i_3=1}^{m} \frac{1}{k_{i_1} k_{i_2} k_{i_3}} X_{i_1 i_2 i_3} u_{i_1}^{(m-r_1)} u_{i_2}^{(r_1)} u_{i_3}' \right) + \sum_{r_1,r_2=1}^{\lfloor m/2 \rfloor} \left( C_3 \sum_{i_1,i_2,i_3=1}^{m} \frac{1}{k_{i_1} k_{i_2} k_{i_3}} X_{i_1 i_2 i_3} u_{i_1}^{(m-r_1)} u_{i_2}^{(r_1)} u_{i_3}^{(r_2+2)} \right) + \ldots + C_m \sum_{i_1,i_2,\ldots,i_{m+1}=1}^{m} \frac{1}{m!} X_{i_1 i_2 \ldots i_{m+1}} u_{i_1}' u_{i_2}' \ldots u_{i_{m+1}}'.
\]

If we consider the values of \( r_1, r_2 \in \left[1, \left\lfloor \frac{m}{2} \right\rfloor \right] \), the sums in 1, 2, 3, ..., \( m+1 \) given above can be expressed as a single sum as follows:

1: \( \binom{m+1}{m+1} \sum_{i_1=1}^{m} \frac{1}{k_{i_1}} X_{i_1} u_i^{(m+1)}. \)

2: \( \sum_{r_1=1}^{\lfloor m/2 \rfloor} \left( \binom{m+1}{m+1-r_1} \sum_{i_1,i_2=1}^{m} \frac{1}{k_{i_1} k_{i_2}} X_{i_1 i_2} u_{i_1}^{(m-r_1)} u_{i_2}^{(r_1)} \right). \)
\[ g: \sum_{r_1, r_2=1}^{m+1} \binom{m+1}{m-r_1} \binom{r_1+1}{r_2} \sum_{i_1, i_2, i_3=1}^{2} \frac{1}{K_{i_1 i_2 i_3}} X_{i_1 i_2 i_3} u_{i_1}^{(m-r_1)} u_{i_2}^{(r_2)} u_{i_3}^{(r_1-r_2+1)} \]

\[ m+1: \binom{m+1}{1} \binom{(m-1)}{1} \binom{(1)}{1} \sum_{i_1, i_2, ..., i_{m+1}=1}^{2} \frac{1}{(m+1)!} X_{i_1 i_2, ..., i_{m+1}} u_{i_1}' u_{i_2}' ... u_{i_{m+1}}'. \]

Then, the sum \( 1 + 2 + 3 + ... + m+1 \) gives us the result. \( \square \)

4. Transversal intersection curve of two parametric surfaces in \( \mathbb{E}^n \)

Let \( S_1 \) and \( S_2 \) be regular intersecting surfaces given by their parametric equations

\[ X(u_1, u_2) = (f_1(u_1, u_2), f_2(u_1, u_2), ..., f_n(u_1, u_2)) \]

and

\[ Y(w_1, w_2) = (g_1(w_1, w_2), g_2(w_1, w_2), ..., g_n(w_1, w_2)) \]

in \( \mathbb{E}^n, n \geq 4 \), respectively. Then, the tangent spaces of \( S_1 \) and \( S_2 \) are spanned by \( \{X_{u_1}, X_{u_2}\} \) and \( \{Y_{w_1}, Y_{w_2}\} \), respectively. Let \( \alpha(s) \) be their intersection curve with arc-length parameter \( s \).

Let us consider the set

\[ W = \{X_{u_1}, X_{u_2}, Y_{w_1}, Y_{w_2}\}. \]

For the tangent vector fields in \( W \) we have the following cases at an intersection point \( \alpha(s) \).

**Case 1.** If \( W \) is linearly independent, then \( \alpha(s) \) is an isolated point of such intersection.

**Case 2.** If \( W \) has only two linearly independent vectors at an intersection point, then the tangent spaces of \( S_1 \) and \( S_2 \) overlap at that point. This kind of intersection is called a tangential intersection. We exclude such intersections.

**Case 3.** If \( W \) has three linearly independent vectors at an intersection point, then the tangent spaces of \( S_1 \) and \( S_2 \) differ at that point. This kind of intersection is called a transversal intersection. We focus on such intersections.

Let us consider the case 3 and denote the Frenet frame of \( \alpha \) by \( \{V_1, V_2, ..., V_n\} \) at the intersection point \( P = X(u_1(0), u_2(0)) = Y(w_1(0), w_2(0)) \).
4.1. Tangent vector \((\alpha')\). We need bases of the normal spaces of \(S_1\) and \(S_2\) to obtain the tangent vector \(V_1\). To obtain a basis of the normal space of \(S_1\), we choose \(n - 3\) vectors from the standard basis \(\{e_1, e_2, ..., e_n\}\) of \(\mathbb{R}^n\) to be linearly independent with \(\{X_{u_1}, X_{u_2}\}\). Without loss of generality, we assume that \(\{X_{u_1}, X_{u_2}, e_1, e_2, ..., e_{n-3}\}\) is linearly independent at \(P\). Let us define

\[
\begin{align*}
N_1^X &= X_{u_1} \times X_{u_2} \times e_1 \times e_2 \times \ldots \times e_{n-3}, \\
N_2^X &= X_{u_1} \times X_{u_2} \times N_1^X \times e_1 \times \ldots \times e_{n-4}, \\
N_3^X &= X_{u_1} \times X_{u_2} \times N_1^X \times N_2^X \times e_1 \times \ldots \times e_{n-5}, \\
&\vdots \\
N_{n-2}^X &= X_{u_1} \times X_{u_2} \times N_1^X \times N_2^X \times \ldots \times N_{n-3}^X.
\end{align*}
\]  

(4.1)

It is clear from the above equations that \(\{N_1^X, N_2^X, ..., N_{n-2}^X\}\) constitutes an orthogonal basis for the normal space of \(S_1\) at \(P\). Similarly, assuming that \(\{Y_{w_1}, Y_{w_2}, e_1, e_2, ..., e_{n-3}\}\) is linearly independent at \(P\) and defining

\[
\begin{align*}
N_1^Y &= Y_{w_1} \times Y_{w_2} \times e_1 \times e_2 \times \ldots \times e_{n-3}, \\
N_2^Y &= Y_{w_1} \times Y_{w_2} \times N_1^Y \times e_1 \times \ldots \times e_{n-4}, \\
N_3^Y &= Y_{w_1} \times Y_{w_2} \times N_1^Y \times N_2^Y \times e_1 \times \ldots \times e_{n-5}, \\
&\vdots \\
N_{n-2}^Y &= Y_{w_1} \times Y_{w_2} \times N_1^Y \times N_2^Y \times \ldots \times N_{n-3}^Y.
\end{align*}
\]  

(4.2)

yields the orthogonal basis \(\{N_1^Y, N_2^Y, ..., N_{n-2}^Y\}\) for the normal space of \(S_2\) at \(P\). Let

\[\mathcal{N} = \{N_1^X, N_2^X, ..., N_{n-2}^X, N_1^Y, N_2^Y, ..., N_{n-2}^Y\}\]

and \(d\) be the dimension of the subspace spanned by \(\mathcal{N}\). It is clear that the value of \(d\) can be \(n - 2, n - 1\) or \(n\). These values of \(d\) correspond to the following intersection types:

a) If \(d = n\), then the intersection point \(P\) is an isolated point.

b) If \(d = n - 2\), then the normal spaces of the surfaces \(S_1\) and \(S_2\) overlap at \(P\), i.e., we have a tangential intersection.

c) If \(d = n - 1\), then we have a transversal intersection at \(P\).

Since we consider the transversal intersection, let us assume that \(d = n - 1\) and \(\{N_1^X, N_2^X, ..., N_{n-2}^X, N_1^Y, \ldots, N_{n-2}^Y\}\), \(\ell \in \{1, 2, \ldots, n - 2\}\) is linearly independent at \(P\). Since \(V_1 \perp N_j^X\), \(1 \leq j \leq n - 2\) and \(V_1 \perp N_1^Y\), the tangent vector of the intersection curve can be obtained by

\[
V_1 = \frac{N_1^X \times N_2^X \times \ldots \times N_{n-2}^X \times N_1^Y}{\|N_1^X \times N_2^X \times \ldots \times N_{n-2}^X \times N_1^Y\|}.
\]  

(4.3)

Thus, if we take the dot product of both sides of

\[V_1 = X_{u_1}u_1' + X_{u_2}u_2'
\]
with $X_{u_1}, X_{u_2}$ and of
\[ V_1 = Y_{w_1} w'_1 + Y_{w_2} w'_2 \]
with $Y_{w_1}, Y_{w_2}$, the coefficients $u'_1, u'_2$ and $w'_1, w'_2$ can be found by
\[ u'_1 = \frac{G\langle V_1, X_{u_1} \rangle - F\langle V_1, X_{u_2} \rangle}{EG - F^2}, \quad u'_2 = \frac{E\langle V_1, X_{u_2} \rangle - F\langle V_1, X_{u_1} \rangle}{EG - F^2} \]
and
\[ w'_1 = g\langle V_1, Y_{w_1} \rangle - f\langle V_1, Y_{w_2} \rangle, \quad w'_2 = e\langle V_1, Y_{w_2} \rangle - f\langle V_1, Y_{w_1} \rangle, \]
where $E, F, G$ and $e, f, g$ are the first fundamental form coefficients of $S_1$ and $S_2$, respectively.

4.2. Second derivative ($\alpha''$). Since $\alpha''$ is orthogonal to $V_1$, we may write
\[ \alpha'' = \lambda_1 N^X_1 + \lambda_2 N^X_2 + \ldots + \lambda_{n-2} N^X_{n-2} + \lambda_{n-1} N^X_\ell. \] (4.4)
We must determine the coefficients $\lambda_i$, $1 \leq i \leq n - 1$. If we take the dot product of both sides of (4.4) with $N^X_1, N^X_2, \ldots, N^X_{n-2}, N^X_\ell$, respectively, we obtain the system of linear equations depending on $\lambda_i$, $1 \leq i \leq n - 1$, as
\[ \langle N^X_1, N^X_1 \rangle \lambda_1 + \langle N^Y_1, N^X_1 \rangle \lambda_{n-1} = \langle \alpha'', N^X_1 \rangle \]
\[ \langle N^X_2, N^X_2 \rangle \lambda_2 + \langle N^Y_2, N^X_2 \rangle \lambda_{n-1} = \langle \alpha'', N^X_2 \rangle \]
\[ \vdots \]
\[ \langle N^X_{n-2}, N^X_{n-2} \rangle \lambda_{n-2} + \langle N^Y_{n-2}, N^X_{n-2} \rangle \lambda_{n-1} = \langle \alpha'', N^X_{n-2} \rangle \]
\[ \langle N^X_\ell, N^X_\ell \rangle \lambda_1 + \langle N^Y_\ell, N^X_\ell \rangle \lambda_2 + \ldots + \langle N^Y_{n-2}, N^X_{n-2} \rangle \lambda_{n-1} = \langle \alpha'', N^X_\ell \rangle. \]
For the coefficient determinant
\[ \det A = \begin{vmatrix} \langle N^X_1, N^X_1 \rangle & 0 & \ldots & 0 & \langle N^Y_1, N^X_1 \rangle \\ 0 & \langle N^X_2, N^X_2 \rangle & \ldots & 0 & \langle N^Y_2, N^X_2 \rangle \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \langle N^X_{n-2}, N^X_{n-2} \rangle & \langle N^Y_{n-2}, N^X_{n-2} \rangle \\ \langle N^X_\ell, N^X_\ell \rangle & \langle N^Y_\ell, N^X_\ell \rangle & \ldots & \langle N^X_{n-2}, N^X_\ell \rangle & \langle N^Y_{n-2}, N^X_\ell \rangle \end{vmatrix} \]
of this system, by using Definition 1 we have
\[ \det A = \|N^X_1 \times N^X_2 \times \ldots \times N^X_{n-2} \times N^Y_\ell \|^2 \neq 0. \]
Using the Cramer's method, the solution of this system is then obtained as
\[ \lambda_i = \frac{1}{\det A} \left\{ \langle \alpha'', N^X_\ell \rangle A_{(n-1)i} + \sum_{j=1}^{n-2} \langle \alpha'', N^X_j \rangle A_{ji} \right\}, \quad 1 \leq i \leq n - 1, \] (4.5)
where $A_{ji}$ denotes the $(j, i)$ cofactor of the matrix $A$. Besides, if we use (2.2), we have

$$
\langle \alpha'', \mathbf{N}^X_j \rangle = \sum_{i_1, i_2=1}^{2} \langle \mathbf{X}_{i_1i_2}^X, \mathbf{N}^X_j \rangle u'_{i_1} u'_{i_2},
$$

$$
\langle \alpha'', \mathbf{N}^Y_j \rangle = \sum_{i_1, i_2=1}^{2} \langle \mathbf{Y}_{i_1i_2}^Y, \mathbf{N}^Y_j \rangle w'_{i_1} w'_{i_2}.
$$

(4.6)

Since the right sides of the equations in (4.6) are known, substitution of (4.6) into (4.5) yields $\lambda_i$, which allows us to compute $\alpha''$ by using (4.4). Hence, $u''_1, u''_2$ and $w''_1, w''_2$ can be obtained by using (2.2). Also, using the Gram-Schmidt orthogonalization method, we find

$$
\mathbf{E}_2 = \alpha'' - \langle \alpha'', \mathbf{V}_1 \rangle \mathbf{V}_1, \quad \mathbf{V}_2 = \frac{\mathbf{E}_2}{\| \mathbf{E}_2 \|}.
$$

(4.7)

Then, since $\alpha'' = k_1 \mathbf{V}_2$, the first curvature of the intersection curve is obtained by $k_1 = \langle \alpha'', \mathbf{V}_2 \rangle$.

**4.3. Third derivative ($\alpha'''$).** For the third derivative of the intersection curve, we may write

$$
\alpha''' = -k_1^2 \mathbf{V}_1 + k'_2 \mathbf{V}_2 + k_1 k_2 \mathbf{V}_3
$$

$$
= -k_1^2 \mathbf{V}_1 + \mu_1 \mathbf{N}^X_1 + \mu_2 \mathbf{N}^X_2 + \ldots + \mu_{n-2} \mathbf{N}^X_{n-2} + \mu_{n-1} \mathbf{N}^Y_\ell.
$$

(4.8)

Similarly, by taking the dot product of both sides of (4.8) with $\mathbf{N}^X_1, \mathbf{N}^X_2, \ldots, \mathbf{N}^X_{n-2}, \mathbf{N}^Y_\ell$, respectively, we obtain a system of linear equations depending on $\mu_i$ which has the solution

$$
\mu_i = \frac{1}{\det A} \left\{ \left\langle \alpha''', \mathbf{N}^Y_\ell \right\rangle A_{(n-1)i} + \sum_{j=1}^{n-2} \left\langle \alpha''', \mathbf{N}^X_j \right\rangle A_{ji} \right\}, 1 \leq i \leq n-1.
$$

(4.9)

If we use (2.3), we have

$$
\langle \alpha''', \mathbf{N}^X_j \rangle = 3 \sum_{i_1, i_2=1}^{2} \langle \mathbf{X}_{i_1i_2}^X, \mathbf{N}^X_j \rangle u'''_{i_1} u'_{i_2},
$$

$$
\sum_{i_1, i_2, i_3=1}^{2} \langle \mathbf{X}_{i_1i_2i_3}^X, \mathbf{N}^X_j \rangle u'_{i_1} u'_{i_2} u'_{i_3}.
$$

(4.10)

$$
\langle \alpha''', \mathbf{N}^Y_\ell \rangle = 3 \sum_{i_1, i_2=1}^{2} \langle \mathbf{Y}_{i_1i_2}^Y, \mathbf{N}^Y_\ell \rangle w'''_1 w'_2 w'_3.
$$

(4.11)

Since the right sides of the equations in (4.10) and (4.11) are known, substitutions of (4.10) and (4.11) into (4.9) yield $\mu_i, 1 \leq i \leq n-1$, which enables us to compute $\alpha'''$ via (4.8). Hence, $u'''_1, u'''_2$ and $w'''_1, w'''_2$ can be found by using (2.3). Using the Gram-Schmidt orthogonalization method, we obtain

$$
\mathbf{E}_3 = \alpha''' - \langle \alpha''', \mathbf{V}_1 \rangle \mathbf{V}_1 - \langle \alpha''', \mathbf{V}_2 \rangle \mathbf{V}_2, \quad \mathbf{V}_3 = \frac{\mathbf{E}_3}{\| \mathbf{E}_3 \|}.
$$

(4.12)
Then, if \( k_1 \neq 0 \), the second curvature of the intersection curve can be obtained by

\[
k_2 = \frac{\langle \alpha'', V_3 \rangle}{k_1}.
\]

(4.13)

We also have

\[
k'_1 = \langle \alpha''', V_2 \rangle.
\]

4.4. Fourth derivative \((\alpha'^{(4)})\). Similarly to the second and third derivatives, we may write

\[
\alpha'^{(4)} = -3k_1k'_1V_1 + (-k^3_1 + k''_1 - k_1k'_2) V_2 + (2k'_1k_2 + k_1k''_2) V_3 + k_1k_2k_3 V_4,
\]

\[
= -3k_1k'_1V_1 + \eta_1N_1^X + \eta_2N_2^X + \ldots + \eta_{n-2}N_{n-2}^X + \eta_{n-1}N_{n-1}^Y,
\]

(4.14)

where

\[
\eta_h = \frac{1}{\det A} \left\{ \langle \alpha'^{(4)}, N_\ell^Y \rangle A_{(n-1)i} + \sum_{j=1}^{n-2} \langle \alpha'^{(4)}, N_j^X \rangle A_{ji} \right\},
\]

(4.15)

1 \leq i \leq n - 1, and \( \langle \alpha'^{(4)}, N_j^X \rangle \) and \( \langle \alpha'^{(4)}, N_\ell^Y \rangle \) can be obtained by using (2.4). Then, we find

\[
E_4 = \alpha'^{(4)} - \langle \alpha'^{(4)}, V_1 \rangle V_1 - \langle \alpha'^{(4)}, V_2 \rangle V_2 - \langle \alpha'^{(4)}, V_3 \rangle V_3,
\]

\[
V_4 = \frac{E_4}{\|E_4\|}.
\]

(4.16)

If \( k_2 \neq 0 \), then the third curvature of the intersection curve is obtained by

\[
k_3 = \frac{\langle \alpha'^{(4)}, V_4 \rangle}{k_1k_2}.
\]

(4.17)

Also, we obtain \( k''_1 \) and \( k'_2 \) by using \( \langle \alpha'^{(4)}, V_2 \rangle \) and \( \langle \alpha'^{(4)}, V_3 \rangle \), respectively.

4.5. Higher order derivatives \((\alpha'^{(m)}, m \geq 5)\). Similarly, for the \( m \)th order derivative of the intersection curve, we can write

\[
\alpha'^{(m)} = d_1V_1 + d_2V_2 + \ldots + d_{m-1}V_{m-1} + k_1k_2k_3\ldots k_{m-1}V_m
\]

\[
= d_1V_1 + \xi_1N_1^X + \xi_2N_2^X + \ldots + \xi_{n-2}N_{n-2}^X + \xi_{n-1}N_{n-1}^Y.
\]

(4.18)

We need to determine \( \xi_i, 1 \leq i \leq n - 1 \), and only the coefficient \( d_1 \) to find \( \alpha'^{(m)} \). For \( \xi_i \), we have

\[
\xi_i = \frac{1}{\det A} \left\{ \langle \alpha'^{(m)}, N_\ell^Y \rangle A_{(n-1)i} + \sum_{j=1}^{n-2} \langle \alpha'^{(m)}, N_j^X \rangle A_{ji} \right\}, \quad 1 \leq i \leq n - 1,
\]

(4.19)

where \( \langle \alpha'^{(m)}, N_j^X \rangle \) and \( \langle \alpha'^{(m)}, N_\ell^Y \rangle \) can be obtained by using (3.1).

On the other hand, as seen in (4.14), for \( m \geq 4 \) the coefficients \( d_1, d_2, \ldots, d_{m-1} \) consist of not only the curvatures \( k_i \) but also the derivatives
of these curvatures. However, the curvatures and their derivatives which are included in $d_1$ can be obtained by using the previously found Frenet vectors and actual derivatives. For example, $d_1$ includes $k'_1, k''_1, k'''_1$ and $k'_2$ for $\alpha^{(6)}$ in which all are known except $k'''_2$. But, $k'''_1$ can be obtained by using $\langle \alpha^{(5)}, V_2 \rangle$. We should note that computing the coefficients $d_1, d_2, \ldots, d_{m-1}$ manually is difficult and time-consuming for higher order derivatives. To overcome this difficulty, we also provide a code (see appendix B) in MATLAB R2018a which produces all $d_i$ for the desired $m$. Thus, we can find $\alpha^{(m)}, m \geq 5$, by using (4.18). If we again use (3.1), we can find $u_1^{(m)}, u_2^{(m)}, w_1^{(m)}, w_2^{(m)}$. Also, using the Gram-Schmidt orthogonalization method, we obtain

$$E_m = \alpha^{(m)} - \sum_{i=1}^{m-1} \langle \alpha^{(m)}, V_i \rangle V_i, \quad V_m = \frac{E_m}{\|E_m\|}, \quad 5 \leq m \leq n-1. \quad (4.20)$$

Finally, the last Frenet vector $V_n$ can be obtained by $V_n = V_1 \times V_2 \times \cdots \times V_{n-1}$. Hence, if we assume $k_i \neq 0, i \geq 3$, the $m$th curvature of the intersection curve is obtained from

$$k_m = \frac{\langle \alpha^{(m)}, V_m \rangle}{k_1k_2\ldots k_{m-1}}, \quad 5 \leq m \leq n-1. \quad (4.21)$$

**Remark 1.** We assume that the curvatures of the intersection curve do not vanish at the intersection point. If any curvature vanishes at an intersection point, we can follow the algorithm given by [21] to obtain the Frenet vectors.

### 5. Examples

**5.1. Example in $E^4$.** Let us consider the intersection of the Whitney sphere $S_1$ given by the parametrization

$$X(u_1, u_2) = \frac{1}{1 + (u_1^2 + u_2^2)^2} \left( u_1(1 + u_1^2 + u_2^2), u_1(u_1^2 + u_2^2 - 1), u_2(1 + u_1^2 + u_2^2), u_2(u_1^2 + u_2^2 - 1) \right),$$

and the surface $S_2$ given by the parametrization

$$Y(w_1, w_2) = (w_1, w_2, w_1 w_2, w_2^2)$$

in $E^4$. These surfaces are projected perspective[13] from the point $(0, 0, 0, 2)$ into the hyperplane $x_4 = 0$ and their projections are displayed in Figure 1.

Let us find the Frenet vectors and curvatures of their intersection curve $\alpha$ at the intersection point $P = X(1, 0) = Y(1, 0) = (1, 0, 0, 0)$. The tangent vectors of these surfaces at $P$ are

| $X_{u_1}$ | $(0, 1, 0, 0)$ |
| $X_{u_2}$ | $(0, 0, 1, 0)$ |
| $Y_{w_1}$ | $(1, 0, 0, 0)$ |
| $Y_{w_2}$ | $(0, 1, 0, 0)$ |
FIGURE 1. Projections of the intersecting surfaces $S_1$ and $S_2$.

It is easy to see that we have a transversal intersection at $P$ since $\{X_{u_1}, X_{u_2}, Y_{w_1}\}$ is linearly independent. Then, since $\{X_{u_1}, X_{u_2}, e_1\}$ is linearly independent, by using (4.1) we obtain the basis vectors of the normal space of $S_1$ as

$$N^X_1 = X_{u_1} \times X_{u_2} \times e_1 = -e_4,$$
$$N^X_2 = X_{u_1} \times X_{u_2} \times N^X_1 = -e_1.$$  

Similarly, since $\{Y_{w_1}, Y_{w_2}, e_4\}$ is linearly independent at $P$, by using (4.2), we obtain the basis vectors of the normal space of $S_2$ as

$$N^Y_1 = Y_{w_1} \times Y_{w_2} \times e_4 = e_3,$$
$$N^Y_2 = Y_{w_1} \times Y_{w_2} \times N^Y_1 = -e_4.$$  

Hence, $\{N^X_1, N^X_2, N^Y_1\}$ is linearly independent, i.e. $d = 3$. Thus, by using (4.3) we find the tangent vector of the intersection curve at $P$ as

$$V_1 = \frac{N^X_1 \times N^X_2 \times N^Y_1}{\|N^X_1 \times N^X_2 \times N^Y_1\|} = (0, -1, 0, 0),$$

which yields $u_1' = -1$, $u_2' = 0$, $w_1' = 0$, $w_2' = -1$.

For the second order derivatives of the given surfaces at $P$ we have

$$X_{u_1u_1} = (-3, -1, 0, 0), \quad Y_{w_1w_1} = (0, 0, 0, 0),$$
$$X_{u_1u_2} = (0, 0, -1, 1), \quad Y_{w_1w_2} = (0, 0, 0, 0),$$
$$X_{u_2u_2} = (-1, 1, 0, 0), \quad Y_{w_2w_2} = (0, 0, 2, 0).$$

Since we may write

$$\alpha'' = \lambda_1 N^X_1 + \lambda_2 N^X_2 + \lambda_3 N^Y_1,$$
by using (4.5) and (4.6) we obtain $\alpha_1 = 0$, $\alpha_2 = 3$, $\alpha_3 = 2$. Thus, we have $\alpha'' = (-3, 0, 2, 0)$ and $u_1'' = 1$, $u_2'' = 2$, $u_3'' = -3$, $u_4'' = 0$ at $P$. Also, we obtain the second Frenet vector of the intersection curve as $V_2 = \left( -\frac{3}{\sqrt{13}}, 0, \frac{2}{\sqrt{13}}, 0 \right)$, and the first curvature as $k_1 = \sqrt{13}$ at $P$.

For the third order derivatives of the given surfaces at $P$ we have

\[
X_{u_1u_1u_1} = (9, -9, 0, 0), \quad Y_{w_1w_1w_1} = (0, 0, 0, 0),
\]
\[
X_{u_1u_2u_1} = (0, 0, -1, -3), \quad Y_{w_1w_1w_1} = (0, 0, 0, 0),
\]
\[
X_{u_1u_2u_2} = (-1, -3, 0, 0), \quad Y_{w_1w_2w_2} = (0, 0, 2, 0),
\]
\[
X_{u_2u_2u_2} = (0, 0, -3, 3), \quad Y_{w_2w_2w_2} = (0, 0, 0, 6).
\]

If we use (4.8), we may write

\[
\alpha''' = -k_1^2 V_1 + \mu_1 N_1^X + \mu_2 N_2^X + \mu_3 N_1^Y,
\]

where $\mu_1 = 6$, $\mu_2 = \mu_3 = 0$. Thus, we have $\alpha''' = (0, 13, 0, -6)$ and $u_1''' = 1$, $u_2''' = -6$, $u_3''' = 0$, $u_4''' = 13$ at $P$. The third Frenet vector is found as $V_3 = (0, 0, 0, -1)$, and the second curvature is obtained as $k_2 = \frac{6}{\sqrt{13}}$ at $P$.

If we continue in a similar manner, we find

\[
V_4 = \left( -\frac{2}{\sqrt{13}}, 0, -\frac{3}{\sqrt{13}}, 0 \right), \quad k_3 = \frac{47}{\sqrt{13}}.
\]

5.2. Example in $\mathbb{E}^5$. Let $S_1$ and $S_2$ be the surfaces given by

\[
X(u_1, u_2) = (u_2^2, u_2, u_1^3, u_1, u_1^4),
\]
\[
Y(w_1, w_2) = (w_1, w_2, w_2^3, w_2, w_2^3),
\]

respectively. Let us find the Frenet vectors and curvatures of their intersection curve $\alpha$ at the intersection point $P = X(0, 0) = Y(0, 0) = (0, 0, 0, 0, 0)$. The tangent vector fields of these surfaces are

\[
X_{u_1} = (0, 0, 3u_1^2, 1, 4u_1^3), \quad X_{u_2} = (2u_2, 1, 0, 0, 0),
\]
\[
Y_{w_1} = (1, 0, 0, 0, 2w_1), \quad Y_{w_2} = (0, 1, 3w_2^2, 1, 0).
\]

Then, since $\{X_{u_1}, X_{u_2}, e_1, e_3\}$ is linearly independent at $P$, by using (4.1) we obtain the basis vectors of the normal space of $S_1$ as

\[
N_1^X = X_{u_1} \times X_{u_2} \times e_1 \times e_3 = e_5,
\]
\[
N_2^X = X_{u_1} \times X_{u_2} \times N_1^X \times e_1 = e_3,
\]
\[
N_3^X = X_{u_1} \times X_{u_2} \times N_1^X \times N_2^X = -e_1.
\]

Similarly, since $\{Y_{w_1}, Y_{w_2}, e_2, e_3\}$ is linearly independent at $P$, by using (4.2) we obtain the basis vectors of the normal space of $S_2$ as

\[
N_1^Y = Y_{w_1} \times Y_{w_2} \times e_2 \times e_3 = e_5,
\]
\[
N_2^Y = Y_{w_1} \times Y_{w_2} \times N_1^Y \times e_2 = e_3,
\]
\[
N_3^Y = Y_{w_1} \times Y_{w_2} \times N_1^Y \times N_2^Y = -e_1.
\]
In this case, \( \{N_1^X, N_2^X, N_3^X, N_3^Y\} \) is linearly independent, i.e. \( d = 4 \) at \( P \). Thus, by using (4.3) we find the tangent vector of the intersection curve at \( P \) as

\[
V_1 = \frac{N_1^X \times N_2^X \times N_3^X \times N_3^Y}{\|N_1^X \times N_2^X \times N_3^X \times N_3^Y\|} = \left(0, \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)
\]

which yields \( u'_1 = \frac{1}{\sqrt{2}}, u'_2 = \frac{1}{\sqrt{2}}, u'_3 = 0, w'_2 = \frac{1}{\sqrt{2}} \).

For the nonzero second order derivatives of the given surfaces at \( P \) we have \( X_{u_2u_2} = 2e_1, Y_{u_1u_1} = 2e_5 \). Since we may write

\[
\alpha'' = \lambda_1 N_1^X + \lambda_2 N_2^X + \lambda_3 N_3^X + \lambda_4 N_3^Y,
\]

by using (4.5) and (4.6) we obtain \( \lambda_1 = \lambda_2 = \lambda_4 = 0, \lambda_3 = 1 \). Thus, we have \( \alpha'' = N_3^X = e_1 \) and \( u''_1 = u''_2 = w''_2 = 0, w''_1 = 1 \) at \( P \). Also, we obtain the second Frenet vector as \( V_2 = (1, 0, 0, 0, 0) \), and the first curvature as \( k_1 = 1 \) at \( P \).

For the nonzero third order derivatives of the given surfaces at \( P \) we have \( X_{u_1u_1u_1} = 6e_3, Y_{u_2u_2u_2} = 6e_3 \). If we use (4.8), we may write

\[
\alpha''' = -V_1 + \mu_1 N_1^X + \mu_2 N_2^X + \mu_3 N_3^X + \mu_4 N_3^Y,
\]

where \( \mu_1 = \mu_3 = \mu_4 = 0, \mu_2 = -\frac{3}{\sqrt{2}} \). Thus, we have

\[
\alpha''' = \left(0, -\frac{1}{\sqrt{2}}, \frac{3}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \quad \text{and} \quad u'''_1 = u'''_2 = w'''_2 = -\frac{1}{\sqrt{2}}, w'''_1 = 0
\]

at \( P \). The third Frenet vector is found as \( V_3 = (0, 0, 1, 0, 0) \), and the second curvature is obtained as \( k_2 = \frac{3\sqrt{2}}{2} \) at \( P \).

If we continue in a similar manner, we find

\[
V_4 = (0, 0, 0, 0, 1), \quad V_5 = \left(0, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \quad k_3 = 2\sqrt{2}, \quad k_4 = 0.
\]

6. Conclusion

Differential geometric properties of the intersection curve of two parametric surfaces are studied in Euclidean \( n \)-space. We first present the \( m \)th order derivative formula of a curve lying on a parametric surface, and then, we obtain curvatures and Frenet vectors of the transversal intersection curve of two parametric surfaces in Euclidean \( n \)-space. We also provide a computer code produced in MATLAB to simplify determining the coefficients according to the Frenet frame of higher order derivatives of a curve.
Appendix A

Let us find the fifth order derivative of $\alpha$ by using (3.1). In this case, since $m = 5$, we may write

$$\alpha^{(5)} = C_1 \sum_{i_1=1}^{5} \frac{1}{k_{i_1}} X_{i_1} u_{i_1}^{(5)} + \sum_{r_1=1}^{5} \left( C_2 \sum_{i_1,i_2=1}^{5} \frac{1}{k_{i_1} k_{i_2}} X_{i_1,i_2} u_{i_1}^{(5-r_1)} u_{i_2}^{(r_1)} \right)$$

$$+ \sum_{r_1,r_2=1}^{5} \left( C_3 \sum_{i_1,i_2,i_3=1}^{5} \frac{1}{k_{i_1} k_{i_2} k_{i_3}} X_{i_1,i_2,i_3} u_{i_1}^{(5-r_1-1)} u_{i_2}^{(r_2)} u_{i_3}^{(r_1-r_2+1)} \right)$$

$$+ \sum_{r_1,r_2,r_3=1}^{5} \left( C_4 \sum_{i_1,i_2,i_3,i_4=1}^{5} \frac{1}{k_{i_1} k_{i_2} k_{i_3} k_{i_4}} X_{i_1,i_2,i_3,i_4} u_{i_1}^{(5-r_1-2)} u_{i_2}^{(r_2)} u_{i_3}^{(r_3)} u_{i_4}^{(r_1-r_2-r_3+2)} \right)$$

$$+ C_5 \sum_{i_1,i_2,i_3,i_4,i_5=1}^{5} \frac{1}{5!} X_{i_1,i_2,i_3,i_4,i_5} u_{i_1}' u_{i_2}' u_{i_3}' u_{i_4}' u_{i_5}'. \tag{A.1}$$

If we consider the definitions of $C_i$, $1 \leq i \leq 5$, we get

$$\alpha^{(5)} = \binom{5}{5} \sum_{i_1=1}^{5} \frac{1}{k_{i_1}} X_{i_1} u_{i_1}^{(5)} + \binom{5}{4} \sum_{i_1,i_2=1}^{5} \frac{1}{k_{i_1} k_{i_2}} X_{i_1,i_2} u_{i_1}^{(4)} u_{i_2}'$$

$$+ \binom{5}{3} \sum_{i_1,i_2,i_3=1}^{5} \frac{1}{k_{i_1} k_{i_2} k_{i_3}} X_{i_1,i_2,i_3} u_{i_1}''' u_{i_2}' u_{i_3}'$$

$$+ \binom{5}{2} \binom{5}{2} \sum_{i_1,i_2,i_3=1}^{5} \frac{1}{k_{i_1} k_{i_2} k_{i_3}} X_{i_1,i_2,i_3} u_{i_1}'' u_{i_2}''$$

$$+ \binom{5}{2} \binom{5}{2} \sum_{i_1,i_2,i_3=1}^{5} \frac{1}{k_{i_1} k_{i_2} k_{i_3}} X_{i_1,i_2,i_3} u_{i_1}'' u_{i_2}''$$

$$+ \binom{5}{2} \binom{5}{2} \sum_{i_1,i_2,i_3=1}^{5} \frac{1}{k_{i_1} k_{i_2} k_{i_3}} X_{i_1,i_2,i_3} u_{i_1}'' u_{i_2}''$$

$$+ \binom{5}{2} \binom{5}{4} \sum_{i_1,i_2,i_3,i_4=1}^{5} \frac{1}{k_{i_1} k_{i_2} k_{i_3} k_{i_4}} X_{i_1,i_2,i_3,i_4} u_{i_1}^{(5-r_1-2)} u_{i_2}^{(r_2)} u_{i_3}^{(r_3)} u_{i_4}^{(r_1-r_2-r_3+2)}$$

$$+ \binom{5}{4} \binom{5}{4} \sum_{i_1,i_2,i_3,i_4,i_5=1}^{5} \frac{1}{5!} X_{i_1,i_2,i_3,i_4,i_5} u_{i_1}' u_{i_2}' u_{i_3}' u_{i_4}' u_{i_5}'. \tag{A.2}$$
However, since the orders of derivatives in the sums
\[
\sum_{i_1,i_2,i_3=1}^{2} \frac{1}{k_{i_1 i_2 i_3}} X_{i_1 i_2 i_3} u_{i_1} u_{i_2} u_{i_3}^{(0)} \quad \text{and} \quad \sum_{i_1,i_2,i_3=1}^{2} \frac{1}{k_{i_1 i_2 i_3}} X_{i_1 i_2 i_3} u_{i_1} u_{i_2} u_{i_3}^{''}
\]
do not satisfy the conditions in Theorem 1, these sum terms must be canceled, i.e. we have
\[
\alpha^{(5)} = 2 \sum_{i_1=1}^{2} \frac{1}{k_{i_1}} X_{i_1} u_{i_1}^{(5)} + 5 \sum_{i_1,i_2=1}^{2} \frac{1}{k_{i_1 i_2}} X_{i_1 i_2} u_{i_1}^{(4)} u_{i_2}'
\]
\[
+ 10 \sum_{i_1,i_2=1}^{2} \frac{1}{k_{i_1 i_2}} X_{i_1 i_2} u_{i_1}^{''} u_{i_2}'' + 20 \sum_{i_1,i_2,i_3=1}^{2} \frac{1}{k_{i_1 i_2 i_3}} X_{i_1 i_2 i_3} u_{i_1}^{''} u_{i_2}' u_{i_3}''
\]
\[
+ 30 \sum_{i_1,i_2,i_3=1}^{2} \frac{1}{k_{i_1 i_2 i_3}} X_{i_1 i_2 i_3} u_{i_1}^{''} u_{i_2}' u_{i_3}'
\]
\[
+ \sum_{r_1,r_2,r_3=1}^{2} \left( C_4 \sum_{i_1,i_2,i_3,i_4=1}^{2} \frac{1}{k_{i_1 i_2 i_3 i_4}} X_{i_1 i_2 i_3 i_4} u_{i_1}^{(5-r_1-2)} u_{i_2}^{(r_2)} u_{i_3}^{(r_3)} u_{i_4}^{(r_1-r_2-r_3+2)} \right)
\]
\[
+ 120 \sum_{i_1,i_2,i_3,i_4,i_5=1}^{2} \frac{1}{5!} X_{i_1 i_2 i_3 i_4 i_5} u_{i_1} u_{i_2}' u_{i_3}' u_{i_4}' u_{i_5}'.
\]

Similarly, the orders of the derivatives in the underlined sum I are given below:

| Cases | $r_1$ | $r_2$ | $r_3$ | $t_1$ | $t_2$ | $t_3$ | $t_4$ |
|-------|-------|-------|-------|-------|-------|-------|-------|
| i     | 1     | 1     | 1     | 2     | 1     | 1     | 1     |
| ii    | 1     | 1     | 2     | 2     | 1     | 1     | 0     |
| iii   | 1     | 2     | 1     | 2     | 2     | 1     | 0     |
| iv    | 1     | 2     | 2     | 2     | 2     | 2     | -1    |
| v     | 2     | 1     | 1     | 1     | 1     | 1     | 2     |
| vi    | 2     | 1     | 2     | 1     | 1     | 2     | 1     |
| vii   | 2     | 2     | 1     | 1     | 2     | 1     | 1     |
| viii  | 2     | 2     | 2     | 1     | 2     | 2     | 0     |

It is clear here that only the orders in Case i satisfy the condition of Theorem
1. Therefore, for the sum $I$ we only have

$$\left( \begin{array}{cccc} 5 & 3 & 2 & 1 \\ 2 & 1 & 1 & 1 \end{array} \right) \sum_{i_1,i_2,i_3,i_4=1}^2 \frac{1}{k_{i_1i_2i_3i_4}} u_{i_1}' u_{i_2}' u_{i_3}' u_{i_4}' .$$  \hspace{1cm} (A.4)

If we substitute (A.4) into (A.3), since $k_{i_1} = 1!$, $k_{i_1i_2} = 1!$, $k_{i_1i_2i_3} = 2!$, $k_{i_1i_2i_3i_4} = 3!$ according to Table 1, we obtain

$$\alpha^{(5)} = \sum_{i_1=1}^2 X_{i_1} u_{i_1}^{(5)} + 5 \sum_{i_1,i_2=1}^2 X_{i_1i_2} u_{i_1}' u_{i_2}' + 10 \sum_{i_1,i_2,i_3=1}^2 X_{i_1i_2i_3} u_{i_1}'' u_{i_2}'' u_{i_3}'' +$$

$$+ 10 \sum_{i_1,i_2,i_3,i_4=1}^2 X_{i_1i_2i_3i_4} u_{i_1}' u_{i_2}' u_{i_3}' u_{i_4}' +$$

$$+ \sum_{i_1,i_2,i_3,i_4,i_5=1}^2 X_{i_1i_2i_3i_4i_5} u_{i_1}' u_{i_2}' u_{i_3}' u_{i_4}' u_{i_5}' .$$

**Appendix B**

```matlab
n=input('enter the size of the space: n=');
syms s
T1{:}= sprintfc('k%d(s)',1:n-1);
syms(T1{:})
A1=[T1{:}];
B1=str2sym(A1);
T2{:}= sprintfc('V%d(s)',1:n);
syms(T2{:})
A2=[T2{:}];
B2=str2sym(A2);
C(1)=k1(s)*V2(s);
for i=2:n-1
  C(i)=-B1(i-1)*B2(i-1)+ B1(i)*B2(i+1);
end
C(n)=-B1(n-1)*B2(n-1);
D=diff(B2,s);
E=k1(s)*V2(s);
F(1)=V1(s);
F(2)=k1(s)*V2(s);
alpha(1)=V1(s);
alpha(2)=k1(s)*V2(s);
F(3)=diff(E,s);
alpha(3)=subs(F(3),D(2),C(2));
```
for i=4:n
    alpha(i)=diff(alpha(i-1),s);
    G(1)=subs(alpha(i),D(1),C(1));
    for j=2:n
        G(j)=subs(G(j-1),D(j),C(j));
    end
    alpha(i)=G(n);
end
for i=1:n
    fprintf(’alpha(%d)=%s
’,i,alpha(i))
end

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Conflict of interest

The authors declare that they have no conflict of interest.

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