Constructing balanced equations of motion for particles in general relativity: the harmonic gauge case

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Abstract

Based in the framework of article [1], where we have presented the general problems one encounters in the construction of balanced equations of motions for particles in relativistic theories of gravity, we present in this work the explicit balanced equations of motion for a compact object in general relativity in the harmonic gauge.

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1 Introduction

The excitement of witnessing this new era of gravitational wave observations [2, 3, 4, 5, 6, 7, 8] coming from binary black holes and binary neutron stars, poses also the challenge to have at hand the most convenient models of the physical system, at every stage of their dynamics; in order to cope with the description of the data acquisition, which is ever increasing. In particular we would like to contribute with models that are useful for the description of the dynamics of coalescence binary systems and their specific relation with the gravitational emitted radiation. Our work concentrates in the construction of equations of motion of compact objects, subjected to the back reaction due to their emission of gravitational radiation.

In reference [1], we have presented the general necessary framework to construct balanced equations of motion for particle in general relativistic theories. In this work we present this framework applied to the case of general relativity in the harmonic gauge.

The target system we have in mind is a binary gravitationally bound isolated interactive system. When considering a compact object as isolated, in the framework of isolated spacetimes, one realizes that in general one can ascribe a flat background to the spacetime. More concretely, in the asymptotic region one can always write the metric as

\[ g = \eta + h_{\text{asy}}; \]  

where \( \eta \) is a flat metric associated to inertial frames in the asymptotic region and \( h_{\text{asy}} \) the tensor where all the physical information is encoded. But, there are as many flat metrics \( \eta_{\text{asy}} \) as there are BMS proper supertranslation generators. And also one knows that the difficulties in finding appropriate rest frames comes from the existence of gravitational radiation [11, 12, 13]. At this point it is important to remark that we have shown in the past that the way to circumvent this problem is to recur to the notion of center of mass at future null infinity. The notion of center of mass is in turn related to the definition of angular momentum. We have solved this problem in [14], where a supertranslation free definition of intrinsic angular momentum was introduced along with its related center of mass frame. In other words, for each point at future null infinity, we have a way to single out a unique decomposition of the metric in the form [1], with an appropriately selected flat background.

It is for this reason that gravitational radiation should be the first quantity to be taken into account for the discussion of back reaction on the motion of compact objects. Therefore in calculating the appropriate equations of motion for particles we take this as our starting point; so that the root of the difficulties is taken care at the beginning of our approach.

We adopt here the viewpoint explained in [1] in which we assume there exists an exact metric \( g \) that corresponds to an isolated binary system of compact objects; which it can be decomposed in a form:

\[ g = \eta + h_A + h_B + h_{AB}; \]  

where \( \eta \) is a flat metric, \( h_A \) is proportional to a parameter \( M_A \), that one can think is some kind of measure of the mass of system \( A \), similarly \( h_B \) is proportional to a parameter \( M_B \), and \( h_{AB} \) is proportional to both parameters. To study the gravitational radiation emitted
by the motion of particle $A$, we model the asymptotic structure of a sub-metric
\[ g_A = \eta + h_A; \]
and to describe the rest of the system, we use a sub-metric
\[ g_B = \eta + h_B. \]
For more details see article [1], where it is also explained that the appropriate choice of the flat metric should be related to a local notion of center of mass frame.

Although we will be studying the dynamics of system $A$, to simplify the notation we will avoid using a subindex $A$, whenever possible.

We present in the next section the problematic of the interior region. Then, in section 3 we present the ingredients of the exterior asymptotic problem. A discussion of the relation between these two topics is presented in section 4. The balanced equations of motion calculated in the harmonic gauge are introduced in section 5. In the last section we make some final remarks on the subject of this work. Some auxiliary relations have been included in an Appendix.

2 The interior problem

The subject of the interior problem is treated through the approximation based on the relaxed covariant form of the field equations, as studied in [1].

When solving the relaxed field equations we use as background a flat background with metric $\eta$. The relaxed solutions is further restricted to satisfy a dynamical equation on an auxiliary metric $g(B)$, that takes into account the existence of the rest of the system, that we call $B$.

We start by presenting a summary of its elements.

2.1 The decomposition of the metric

The study of the particle paradigm can be decomposed in an interior problem and an asymptotic problem. In the interior problem one studies the determination of the space-time geometry in terms of the dynamics of the binary system. The basic assumption takes the total metric as consisting of a flat background metric plus a correcting terms that has the information of the system. In what follows we present the basic equations that describe the interior problem in terms of a general background metric $\eta$; not necessarily flat.

Let us consider four independent auxiliary functions $x^\mu$, with $\mu = 0, 1, 2, 3$. Then let us observe that
\[ g^{ab}\nabla_a x^a = g^{ab} \partial_a x^b = g^{ab} \partial_a x^b - g^{ab}(\Gamma^e_{ab} \partial_e x^b). \]

Then, if $I^e_\mu$ denotes the inverse of $\partial_e x^b$, which exists by assumption of the independence of the set $x^\mu$, one has
\[ g^{ab} \Gamma^e_{ab} = (g^{ab} \nabla_a x^b - g^{ab} \partial_a x^b) I^e_\mu = H^\mu I^e_\mu; \]

where we are using
\[ H^\mu = -g^{ab} \nabla_a \nabla_b x^\mu + g^{ab} \partial_a \partial_b x^\mu. \]

Alternatively, let us define the gauge vector
\[ \mathcal{H}^e = H^\mu I^e_\mu; \]
which implies
\[ H^\mu = \partial_e x^e \mathcal{H}^e = \mathcal{H}(x^\mu); \]
then one has
\[ g^{ab} \Gamma^e_{ab} = \mathcal{H}^e. \]

Then the Ricci tensor can be expressed by
\[ R_{ac} = \Theta_{ac} + \frac{1}{2} g^{bd} (\Theta_{bad} h_{ec} + \Theta_{bcd} e_{hc} + 2 \Theta_{bac} h_{ed}) \]
\[ + \frac{1}{2} g^{bd} \partial_b h_{ac} - \partial_a (g_{ec} \mathcal{H}^e) + g_{ed} \Gamma^e_{ac} \mathcal{H}^d \]
\[ - g^{bf} g_{ed} \Gamma^d_{fb} - \frac{1}{2} (\Gamma^d_{ab} g_{cd} + \Gamma^d_{bc} g_{bd}). \]

The field equations are
\[ R_{ac} = -8\pi \kappa \left( T_{ac} - \frac{1}{2} g_{ac} g_{bd} T_{bd} \right). \]
In writing equation \(16\) in a coordinate frame, without any reference to \(\eta\), one would obtain the analogous expression without the \(\Theta\) terms, and where all the appearance of \(\partial\) derivatives are replace by coordinate derivatives.

Alternatively one can use the form of the field equations in terms of a slightly different logic.

If we use the expression of the Ricci tensor as given by \(16\) in \(17\), namely

\[
\frac{1}{2} g^{bd} \partial_b \partial_d h_{ac} - \partial_a (g_{ce} \partial^e h^{ce}) + g_{cd} \Gamma^d_{ac} \partial^e h_{ce} + 2 \Theta_{bca} \epsilon^{e} h_{ed}
\]

\[
- \frac{1}{2} g^{bf} \partial_d \Gamma^d_{bf} + \frac{1}{2} \left( \Gamma^d_{bcd} + \Gamma^d_{bdc} \right) - \frac{1}{2} \left( \Gamma^d_{bad} + \Gamma^d_{adb} \right)
\]

\[= -8\pi \kappa \left( T_{ac} - \frac{1}{2} g_{ac} g^{bd} T_{bd} \right); \tag{18} \]

we will refer to this as the relaxed field equations\(16\), where one has not assumed that \(\mathcal{R}^e = g^{ab} \Gamma_{a,b}^e\).

Using the standard harmonic gauge technique, one would say: solve the relaxed field equation in the coordinate frame, with \(H^\mu = 0\), and then require the equation

\[g^{bd} \nabla_b \nabla_d x^\mu = 0. \tag{19} \]

In the standard approach one makes use of coordinate basis; therefore the previous statement would be the complete story. However in our case, \(H^\mu\) has a second term where two covariant derivatives of \(x^\mu\) with respect to the metric \(\eta\) appears. At this point it is important to notice that if we have the solutions \(x^\mu\) from \(19\) then, on constructing \(\eta\) with these as a Cartesian coordinate system, one would obtain \(H^\mu = 0\).

In some occasions it is preferable to work with a different set of equations. In this respect, several authors have indicated that actually to request equation \(19\) is equivalent\(17 \ 18 \ 16\) to demand

\[g^{ab} \nabla_a T_{bc} = 0. \tag{20} \]

When dealing with Einstein equations in the relaxed form, and treating the vacuum case, equation \(20\) is understood as the condition that the divergence of the Einstein tensor must be zero (which of course is identically zero in the non-relaxed form).

In reference \(15\) we have related this approach to the results of Friedric\(19\). We have also indicated in this reference how to set up an iterative approximation scheme to solve the field equations.

The first order correction for the metric is calculated from

\[
\frac{1}{2} \eta^{bd} \partial_b \partial_d \eta^{(1)}_{ac} = -8\pi \left( T^{(0)}_{ac} - \frac{1}{2} \eta_{ab} \eta^{cd} T^{(0)}_{cd} \right); \tag{21} \]

where we are using geometric units.

### 2.3 The gravitational field from one particle in first order

Let us consider a massive point particle with mass \(M\) describing, in a flat space-time \((\mathcal{M}^0, \eta_{ab})\), a timelike world line curve \(C\) which in a Cartesian coordinate system \(x^a\) reads

\[x^\mu = z^\mu(\tau_0), \tag{22} \]

with \(\tau_0\) meaning the proper time of the particle along \(C\).

The unit tangent vector to \(C\), with respect to the flat background metric is

\[v^\mu = \frac{dz^\mu}{d\tau_0}. \tag{23} \]

that is,

\[\eta(v, v) = 1. \tag{24} \]

Now, for each point \(Q = z(\tau_0)\) of \(C\), we draw a future null cone \(\mathcal{C}_Q\) with vertex in \(Q\). If we call \(x^\mu_P\) the Minkowskian coordinates of an arbitrary point on the cone \(\mathcal{C}_Q\), then we can define the retarded radial distance on the null cone by

\[r = v_\mu (x^\mu_P - z^\mu(\tau_0)). \tag{25} \]

The energy momentum tensor \(T^{(0)}_{ab}\) of a point particle is proportional to \(mv_a v_b\); where \(m\) is the mass and \(v^a\) its four velocity. We are distinguishing between the unit tangent vector \(v^a\), with respect to the metric \(\eta\), and the four velocity vector \(v^a\), because we would like to consider the possibility to normalize the vector \(v\) with respect to a different metric. In order to represent a point particle \(T^{(0)}_{ab}\) must also be proportional to a three dimensional delta function that has support on the world line of the particle.

We will assume that the particle does not have multipolar structure. Then, given an arbitrary Minkowskian frame \((x^0, x^1, x^2, x^3)\), we will express the energy momentum by

\[T_{ab}(x^0 = z^0(\tau), x^1, x^2, x^3) = M v_a(\tau) v_b(\tau)
\]

\[
\frac{\delta(x^1 - z^1(\tau)) \delta(x^2 - z^2(\tau)) \delta(x^3 - z^3(\tau))}{\sqrt{- \det g(3)}} ; \tag{26} \]

where \(\tau\) is the chosen parametrization.

Then, for a source of the form \(26\), independently of the parametrization, the solution to the linear field equation is

\[k^{(1)}_{ab} = -4M v_a v_b - \frac{2}{r}\eta_{ab}; \tag{27} \]

so that in general

\[g^{(1)}_{(A)ab} = \left( 1 + \frac{2M_A}{r} \right) \eta_{ab} - \frac{4M_A}{r} v_a v_b. \tag{28} \]

In these equations we have considered the definition

\[v_a \equiv \eta_{ab} v^b; \tag{29} \]

however it should be emphasized that the vector \(v^b\) is not normalized with the flat metric \(\eta\) as we will see below.
It is curious that the inverse of this first order metric can be calculated exactly; namely

\[ g^{(1)ab} = \frac{1}{1 + 2\bar{M}_\eta} \eta^{ab} + \frac{4M}{r} \eta^{ab} v^a v^b; \]

(30)

if the vector field \( v^a \) is normalized with \( \eta \), and

\[ g^{(1)ab} = \frac{1}{1 + 2\bar{M}_\eta} \eta^{ab} \]

\[ + \frac{4M}{r} \frac{1}{[1 - 2\bar{M}_\eta (2\eta(v, v) - 1)]} v^a v^b; \]

(31)

if \( \eta(v, v) \neq 1 \). In any case, at first order these two versions of the inverse coincide.

### 2.4 Lowering and raising indices for the velocity vectors and normalization conditions

We require

\[ 1 = g(B)_{ab} v^a(\sigma_A) v^b(\sigma_A) \]

\[ = (\eta_{ab} + \bar{h}(B)_{ab}) v^a(\sigma_A) v^b(\sigma_A), \]

(32)

and therefore

\[ v^a(\sigma_A) v^b(\sigma_A) = \eta_{ab} - \bar{h}(B)_{ab} v^a(\sigma_A) v^b(\sigma_A). \]

(33)

Let us note that the two velocity vectors are proportional

\[ v^b = T v^b. \]

(34)

So that one has

\[ 1 = g_B(v, v) = T^2 g_B(v, v) = T^2 (1 + h_B(v, v)); \]

(35)

which gives \( T \) in terms of \( v \) and \( g_B \).

For later reference, we will use \( \tau \) to denote the proper time with respect to the metric \( g_B \) and \( \tau_0 \) for the symbol of the proper time with respect to the metric \( \eta \).

### 2.5 Local dynamics at zero order in back reaction

The expression for the momentum and the local dynamics at zero order has been discussed in [1]. We here just mentioned the fundamental expressions.

Let \( e^a_i \) be a frame basis (an orthonormal frame with respect to metric \( g_B(B) \)); where, as before, we are distinguishing between the abstract indices \( (a, b, c, ...) \) and the numeric frame indices \( (\tilde{a}, \tilde{b}, \tilde{c}, \ldots) \) and let \( \theta^\sigma_\sigma \) be its coframe. Then the local notion of momentum for particle \( \sigma_A \) is given by

\[ P(\sigma_A) = M_A g_B(B)(v_A, e^a_i). \]

(36)

The local dynamics, at zero order in back reaction due to gravitational radiation, is determined by

\[ M_A g_B(B)_{ab} \left( v^a(\sigma_A) \right\} e_c^b \right\} = 0. \]

(37)

When considering below the effects of back reaction, the right hand side will turn different from zero.

### 3 The exterior asymptotic problem

#### 3.1 Total momentum

Given any section \( S \) at future null infinity, the total momentum of a generic spacetime, in terms of an inertial (Bondi) frame [10], is normally given by

\[ \mathcal{P}^{\mu} = -\frac{1}{4\pi} \int_{S_0} \bar{l}_0^\mu (\Psi_2^2 + \sigma_0 \sigma_0) dS^2, \]

(38)

where the auxiliary null vector \( \bar{l}_0(x^2, x^3) \), is defined in terms of the angular coordinates \( (x^2, x^3) \), by

\[ \bar{l}_0^\mu (x^2, x^3) \equiv \left( 1, \sin(\theta) \cos(\phi), \sin(\theta) \sin(\phi), \cos(\theta) \right) \]

\[ = \left( 1, \zeta + \zeta, \zeta - \zeta, \zeta - 1 \right); \]

(39)

where \( \mu, \nu, \ldots = 0, 1, 2, 3 \), and we are using either the standard sphere angular coordinates \( (\theta, \phi) \) or the complex angular coordinates \( \zeta = e^{i\theta} \cot(\frac{\phi}{2}) \); and here a dot means \( \frac{\partial}{\partial t} \); i.e., the partial derivative with respect to the inertial asymptotic time \( \bar{t} \). \( \Psi_2^2 \) is a component of the Weyl tensor in the GHP [20] notation and \( \sigma_0 \) is the leading order behavior of the spin coefficient \( \sigma \) in terms of the asymptotic coordinate \( \bar{r} \). So the set of intrinsic inertial coordinates are \( (\bar{u}, \theta, \phi) \) or \( (\bar{u}, \zeta, \bar{\zeta}) \).

#### 3.2 The total momentum and flux from charge integrals

The total momentum for the monopole particle can be calculated using the charge integral \( Q_S \) of the Riemann tensor technique as explained in [11]. In this subsection we will use the notation of reference [14] to reproduce the essential equations of our first work.

Using the equations of reference [20] one can obtain expressions for the asymptotic fields with respect to an inertial reference frame: the radiation field

\[ \Psi_3 = \frac{\partial \sigma_0}{\partial \bar{r}} + \dot{\sigma} \left( \frac{1}{\bar{r}} \right), \]

(40)

the relation between the leading order behavior of the shear and primed shear

\[ \dot{\sigma}_0 = -\dot{\sigma}_0', \]

(41)

the time derivative of the leading order behavior of the \( \Psi_2 \)

\[ \dot{\Psi}_2 = \frac{\partial \sigma_0}{\partial \bar{r}} + \sigma_\sigma \Psi_4, \]

(42)

and the leading order behavior of the radiation component

\[ \Psi_4 = \dot{\sigma}_0'. \]

(43)
In the calculation for the flux of the momentum one takes $w_1 = 0$, and obtains [1]:

$$\dot{Q}_S(w) = 4 \int [2w_1(\vec{a}_0^\prime \sigma_0')] dS^2 + c.c.$$  \hspace{1cm} (44)

Therefore we can encode the radiation flux of momentum in terms of the $(\sigma_0' \sigma_0')$ factor; which is gauge invariant as discussed in [1].

It follows that the time variation of the Bondi momentum, of a generic spacetime, can be expressed by

$$\dot{\mathcal{F}}^\mu = -\frac{1}{4\pi} \int_S \tilde{V} \sigma_0' \sigma_0' dS^2 \equiv -\mathcal{F}^\mu;$$  \hspace{1cm} (45)

that is, $\mathcal{F}^\mu$ is the total momentum flux.

Normally one expresses the dynamics in terms of the proper time of the particle. As explained in reference [1] the relation between the asymptotic inertial time $\tilde{u}$ and the dynamical time $u$ can be expressed as

$$\frac{\partial \tilde{u}}{\partial u} = \tilde{V}(u, \zeta, \bar{\zeta}) > 0,$$  \hspace{1cm} (46)

that is, in terms of the time derivative of the inertial time $\tilde{u}$ with respect to the non-inertial time $u$. This in turn introduces the new information in the scalar $\tilde{V}(u, \zeta, \bar{\zeta})$; which will appear in the new expressions.

Then the time derivative of the total momentum with respect to the new dynamical time is given by

$$\frac{d\mathcal{F}^\mu}{du} = -\frac{1}{4\pi} \int_S \tilde{V} \sigma_0' \sigma_0' dS^2 = -\mathcal{F}^\mu;$$  \hspace{1cm} (47)

where now $\mathcal{F}^\mu$ is the instantaneous momentum flux with respect to the time $u$. Therefore, when considering non-inertial times, one only needs to calculate the radiation scalar $\sigma_0'$ to evaluate the flux of gravitational radiation.

4 The link between the interior and asymptotic problem

4.1 Preliminary considerations

It is appropriate at this point to recall the experience gained in the case of charged particles in Minkowski spacetime. Forcing the equation of motion to balance the electromagnetic radiation we have derived [21] a new general equation of motion that in a particular case gives well known Lorentz-Dirac equations of motion.

We will now apply the notion of balance equations of motion to the case of a binary system in general relativity.

So the technique is as follows. We know that the asymptotic equations imply the balance of Bondi total momentum with the total flux of momentum at future null infinity. We use this as a tool to find the correct correction to the zero order equations of motion that would satisfy this balance equation.

Also, let us note that the equations must be local, in the sense that the corrections must be expressed in terms of local quantities describing the geometry and the motion. These restrictions lead us to a natural first order correction to the equations of motion that we describe below.

4.2 The Asymptotic structure

We have indicated above that in these kind of approximations one should not focus on the concept of spin coefficient $\sigma$, but instead should consider as more representative of the radiation content the spin coefficient $\sigma'$; since, independent of the gauges one always have

$$\sigma_0' = \frac{1}{2} \frac{\partial h_{0mn}}{\partial u};$$  \hspace{1cm} (48)

and

$$\Psi_4' = \frac{\partial \sigma_0'}{\partial u};$$  \hspace{1cm} (49)

where $\tilde{u}$ is an asymptotic inertial null retarded coordinate.

Let us review here the set of dynamical times one could use. Several of the asymptotic equations are presented in terms of an inertial frame, which uses the inertial asymptotic time $\tilde{u}$. However, since the interior dynamics is either presented in terms of the proper times $\tau$ or $\tau_0$, other null coordinates naturally appear that we mention next.

The asymptotic null function $u$ is defined so that at zero order in the strength of gravitational radiation $\lambda\Pi$ it agrees with the proper time, i.e. $u = \tau$. However, when gravitational radiation effects are considered then $\frac{du}{d\tau}$ becomes a new degree of freedom of the problem.

Among the two natural dynamical times we have the relation $\frac{d\tilde{u}}{d\tau_0} = \frac{4}{\Phi}$ (see equation (55) above.).

The tensor $h$ for a one particle system is:

$$h^{(1)}_{ab} = -4M \left( \begin{array}{c} V^a V_b - \frac{1}{2} \eta_{ab} \end{array} \right).$$  \hspace{1cm} (49)

So that the components with respect to the inertial null tetrad vectors are:

$$h_{ll} = h_{ab} \bar{a}^b \bar{a}^a = -\frac{4MA}{r} \bar{V}^2 V^2;$$  \hspace{1cm} (50)

$$h_{mm} = \frac{4MA}{r} \bar{V}^2 V^2 \bar{V}_\eta V_\eta;$$  \hspace{1cm} (51)

$$h_{ln} = -\frac{2MA}{r} \left( \bar{V}^2 V^2 V_\eta - \frac{1}{2} \right);$$  \hspace{1cm} (52)

$$h_{mm} = \frac{4MA}{r} \left( \bar{V}^2 V^2 \bar{V}_\eta V_\eta \cdot 1 \right);$$  \hspace{1cm} (53)

$$h_{nn} = \frac{4MA}{r} \left( \bar{V}^2 V^2 V_\eta - \frac{1}{2} \right);$$  \hspace{1cm} (54)

$$h_{nm} = \frac{2MA}{r} \left( \bar{V}^2 V^2 \bar{V}_\eta V_\eta \cdot 1 \right);$$  \hspace{1cm} (55)

and the complex conjugate deduced from them. The details of the notation are described in the appendix. However we should also remark that the asymptotic radial coordinate $r$ is defined with respect to the intrinsic
system; and that one has the relation \( r = V_0 \dot{r} + \mathcal{O}(\dot{r}^0) \); so that
\[
\frac{1}{2} h_{0mm} = -2M \frac{\gamma^2}{V_0} (\partial_0 V_0)^2;
\] (57)
and therefore
\[
\sigma_0' = -4M_A \left( \frac{\gamma^2}{V_0} \partial_0 \tilde{V}_0 \tilde{\partial}_0 V_0 + \frac{\gamma}{V_0} \tilde{\gamma} (\partial_0 V_0)^2 \right)
- \frac{1}{2} \frac{\gamma^2}{V_0} \left( \partial_0 V_0 \right)^2; \tag{58}
\]
where a tilde dot means \( \partial_0 \); i.e. derivative with respect to the asymptotic inertial time \( \tilde{u} \).

Relating this calculation of the radiation field with the general discussion of balanced equations of motion presented in our previous article, it is clear that this expression should be understood as being evaluated at the retarded time \( u \) with respect to the first order metric \( g_{\lambda} \); although actually it has been evaluated at the flat background retarded time. Then, taking into account the general relation (60) one can also express
\[
\tilde{V}_0 = V_0 \frac{d\sigma}{d\tau} = \frac{d\gamma}{d\tilde{V}_0}; \tag{59}
\]
where now a dot means \( \partial_\tau \); i.e. derivative with respect to the asymptotic intrinsic time \( \sigma \), and with the new notation
\[
\sigma \equiv \frac{\partial \tilde{u}}{\partial \tau}. \tag{60}
\]
Employing the proper time \( \sigma_0 \), one can also express
\[
\tilde{V}_0 = \frac{d\gamma}{d\sigma_0}; \tag{61}
\]
where we are using the notation
\[
\sigma_0 = \frac{\partial \tilde{u}}{\partial \sigma_0}. \tag{62}
\]
Similarly, we can express
\[
\tilde{\gamma} = \frac{d\sigma}{d\sigma_0}. \tag{63}
\]
Then, one can write the radiation field in terms of \( \sigma_0 \) as
\[
\sigma_0' = -4M_A \left( \frac{\gamma^2}{V_0} \partial_0 \tilde{V}_0 \tilde{\partial}_0 V_0 + \frac{\gamma}{V_0} \tilde{\gamma} (\partial_0 V_0)^2 \right)
- \frac{1}{2} \frac{\gamma^2}{V_0} \left( \partial_0 V_0 \right)^2. \tag{64}
\]
In this expression it should be observed that \( V_0 = V_\eta + \mathcal{O}(\dot{\lambda}) \); since when no gravitational radiation effects are present, the structure of the asymptotic spacetime agrees with that of an stationary spacetime, and so there is one preferred asymptotic flat background metric, and in this situation one chooses the interior flat background metric to agree with the asymptotic one; which coincides with the center of mass reference frame in both regimes. In this way translations in the interior would agree with translations in the asymptotic region. Then since, the radiation field \( \sigma_0' \) is by definition order \( \lambda \); we can use at first order in gravitational radiation the expression
\[
\sigma_0' = -4M_A \left( \frac{\gamma^2}{V_0} \partial_0 \tilde{V}_0 \tilde{\partial}_0 V_0 + \frac{\gamma}{V_0} \tilde{\gamma} (\partial_0 V_0)^2 \right)
- \frac{1}{2} \frac{\gamma^2}{V_0} \left( \partial_0 V_0 \right)^2. \tag{65}
\]
This is the expression that is useful for further calculations of the equation of motion, since it is already in terms of the flat proper time, and Minkowskian quantities.

5 Balanced particle dynamics

Following our previous work, we define the flux vector with respect to the proper time \( \tau_0 \) as:
\[
\mathbf{F}_0^\mu = -\frac{1}{4\pi} \int_S l_0^{\mu} V_0 \sigma_0' \sigma_0' dS^2; \tag{66}
\]
where we use the definition (62). So that at lowest order we can just express it as:
\[
\mathbf{F}_0^\mu = -\frac{1}{4\pi} \int_S l_0^{\mu} V_0 \sigma_0' \sigma_0' dS^2. \tag{67}
\]
Then, as explained in our previous work, the equation of motion in the final form is:
\[
M_A \left( v^a \partial_a v^b + \gamma_a^b c c^a v^c + \frac{1}{\mu} \frac{d\gamma}{d\tau_0} + \frac{w}{\mu} \right) v^b = \frac{1}{\chi \gamma} \mathbf{F}_0^\mu. \tag{68}
\]
when expressed in terms of the \( \gamma_0 \) dynamical time.

Contracting this equation with \( \eta^a d^a \) gives
\[
M_A \left( \gamma_a^b c c^a v^c \eta_0 d^0 d^c + \frac{1}{\mu} \frac{d\gamma}{d\tau_0} + \frac{w}{\mu} \right) = \frac{1}{\chi \gamma} \mathbf{F}_0^\mu \eta_0 d^0 d^c; \tag{69}
\]
and it remains the equation of motion
\[
M_A a^b = M_A f^b_\perp + \frac{1}{\chi \gamma} \mathbf{F}_0^\mu (\eta_0^b - c d^b); \tag{70}
\]
where we are using the notation:
\[
a^b \equiv v^a d_b v^a, \tag{71}
\]
and
\[
f^b_\perp \equiv -\gamma_a^b c c^a v^c (\eta_0^b - c d^b); \tag{72}
\]
which, it should be remarked, only depends on the background geometry \( g_B \) and \( v \).

Equation (69) is understood as an equation for \( w \). The main dynamical equation is then (70); where the possible degree of freedom \( \chi \) depends on the detail nature of gauge being used. In our case, if we call \( u' \) the asymptotic coordinate associated to the interior time \( \tau_0 \); then,
in the isolated case, in the harmonic gauge, at the first order, a null function \( u^\ast \) will be asymptotically defined by a relation of the form \( u' = u^\ast + r^\epsilon(r) \), as is known from the Schwarzschild case. But what is important in our case is that asymptotically one has \( du' = du^\ast \), at fixed \( r \). It is reasonable to assume this behavior in this setting, so that we take \( \chi = 1 \) in our model.

Let us note that these balanced equations of motion indicate that the back reaction due to gravitational radiation is described by the force \( F_0^\mu \), which in turn is determined by \( a_0 \), that is given in terms of the mechanical information of the particle. So, although the equations of motion are completely specified, they are complicated enough that we plan to calculate them numerically; when applying the model to specific systems.

6 Final comments

In this work we have presented the equations of motion for particles that take into account the first order back reaction due to gravitational radiation, using the framework of general relativity, with Hilbert-Einstein field equations, in the harmonic gauge. We have used in this work for the submetric \( g_A \) the structure that comes from the first order iteration of the field equations; but our work can be extended to higher orders. This is the natural first calculation of the balanced equations of motion approach, we have presented in [1]; since most of the literature dealing with equations of motion use the harmonic gauge, as for example in classical post-Newtonian works. In order to build a model with greater predictive power, we need to develop our equations of motion to higher orders. The complicated extension to higher order calculations of our will be presented in the future.

One of the points to be remarked is the identification between the interior center of mass flat background metric, with the asymptotic center of mass flat metric in the asymptotic region. This plays a central conceptual role in our approach; but this point is completely missing in other works on the subject of equations of motion.

We plan to apply these equations of motion to a variety of physical systems, including the observed gravitational wave data.

It is a priori difficult to advance in what regime our approach should or should not compare with previous ones as for example post-Newtonian approaches; since the starting points are completely different, and our model is designed to deal at first order with back reaction of gravitational radiation. Also, because our equations are still relativistic, in terms of individual proper times; instead of the universal time approach for the post-Newtonian works. In [22] we have presented a technique to calculate retarded effects on flat background to arbitrary precision; that we intend to apply to this model. This is another difference with the classical post-Newtonian approach, in which the calculation of the retarded effects are related to the particular order of the approximation.

Before applying our model to specific cases, it is difficult to forecast a qualitative comparison with the classical post-Newtonian approach. But in any case, we would like to study the comparison in detail.

In a future work we will present the corresponding balanced equations of motion in general relativity in the null gauge; where it will be clear that the approach changes completely, but that all the general considerations presented in [1] also apply.

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A Interior reference frames at our disposal

In what follows we mention some reference frames and coordinates we have at our disposal to carry out the calculation of the back reaction to the particle due to the gravitational radiation emitted.

A.1 Relations among coordinates and null vectors in the flat background

The inertial system

Let us denote with \( y^\mu \) the standard Cartesian coordinates and with \((\bar{x}^0 = \bar{u}, \bar{x}^1 = \bar{r}, \bar{x}^2, \bar{x}^3)\), where \((\bar{x}^2, \bar{x}^3) = (\theta, \phi)\) or \(\zeta = \frac{\bar{x}^2+i\bar{x}^3}{2}\), the corresponding null polar coordinates; then, the relation between them is given by

\[
y^\mu = \bar{u} \delta^\mu_0 + \bar{r} \bar{l}^\mu(\zeta, \bar{\zeta});
\]

with

\[
\bar{l}^\mu(\zeta, \bar{\zeta}) = \bar{l}^\mu_0(\zeta, \bar{\zeta});
\]

defined in (78).

The intrinsic non-inertial system

Let \( z^\mu(\tau_0) \) be the evolution of the particle with proper time \( \tau_0 \). We define a null function \( u' \) as the future null cones emanating from \( z^\mu(\tau_0) \), such that \( u' = \tau_0 \) at the world line of the particle.

If \((x^0 = u', x^1 = r, x^2, x^3)\), where \((x^2, x^3) = \Theta^\prime, \varphi^\prime\) or \(\zeta' = \frac{x^2+i x^3}{2}\), are the null polar coordinates adapted to an arbitrary timelike curve, determined by \( z(u')^\mu \), then one has

\[
y^\mu = z^\mu(u') + r \, l^\mu(u', \zeta', \zeta').
\]

Note that

\[
(y^\mu - z^\mu(u')) l_\mu = r \, l^\mu l_\mu = 0,
\]

and that

\[
(y^\mu - z^\mu(u')) v_\mu = r \, l^\mu v_\mu = r;
\]

so that

\[
l^\mu = \frac{y^\mu - z^\mu(u')}{(y^\nu - z^\nu(u')) v_\nu},
\]
Note also that
\[
\frac{\partial r}{\partial y^\nu} = \delta^\nu_{\mu} v_\mu - v_\nu \frac{\partial u^\nu}{\partial y^\mu} v_\mu + (y^\mu - z^\mu(u')) a_\mu \frac{\partial u^\nu}{\partial y^\mu},
\]
(79)

Also we have
\[
\delta^\nu_{\mu} = v^\nu \frac{\partial u^\nu}{\partial y^\mu} + \frac{\partial r}{\partial y^\nu} l^\mu + \frac{\partial \mu}{\partial y^\nu};
\]
which implies
\[
\Omega^\mu_{\nu} = \frac{\partial u^\nu}{\partial y^\mu} + r l^\nu.
\]
(80)

Therefore we have
\[
\frac{\partial u^\nu}{\partial y^\mu} = l^\nu,
\]
and
\[
\frac{\partial r}{\partial y^\mu} = v_\nu + ((y^\mu - z^\mu(u')) a_\mu - 1) l^\nu;
\]
which are needed for the derivatives of \( h_{ab} \) with respect to intrinsic coordinates.

Given a fixed point \( y^\mu \) one has to take different space like directions, and therefore different angular coordinates, for the two null vectors to reach the fixed point. But, if given a particular future null cone determined by the apex \( z(u') \), one also chooses an inertial frame with origin at this apex, then, from equations (73) and (74) one deduces that at this cone the two null vectors \( l^\mu(u, \zeta', \bar{\zeta}') \) and \( \bar{l}^\mu_0(\zeta', \bar{\zeta}') \) must be proportional; so that
\[
l^\mu(u, \zeta', \bar{\zeta}') = \alpha(u, \zeta', \bar{\zeta}') \bar{l}^\mu_0(\zeta', \bar{\zeta}');
\]
(81)

with \( \alpha > 0 \). But then we have
\[
1 = l^\mu v_\mu = \alpha \bar{l}^\mu_0 v_\mu;
\]
(82)

which implies that
\[
\frac{1}{\alpha} = V_\eta,
\]
(83)

and also
\[
l^\mu(u, \zeta', \bar{\zeta}') = \frac{1}{V_\eta(u, \zeta', \bar{\zeta}')} \bar{l}^\mu_0(\zeta', \bar{\zeta}');
\]
(84)

A.2 The other null tetrad vectors in the flat background

The inertial frame

Let us define the complex vector
\[
\hat{m}(y^\nu) = m(\hat{r}^\nu) = \hat{r} m(\hat{r}) = \hat{r} \sqrt{\frac{2P_0}{\hat{r}}} \frac{\partial \mu}{\partial \zeta} = \partial_0(\hat{l}^\nu);
\]
(85)

and let us act on the Cartesian coordinates; so that
\[
\hat{m}(y^\nu) = \hat{m}^\mu = \hat{m}(\hat{r}^\nu) = \hat{r} \hat{m}(\hat{r}) = \hat{r} \sqrt{\frac{2P_0}{\hat{r}}} \frac{\partial \mu}{\partial \zeta} = \partial_0(\hat{l}^\nu);
\]
(86)

where we recall that \( \partial_0 \) is the edth operator in the GHP\[20\] notation for the unit sphere. Therefore, we see that the definition (85) agrees with the simple recipe
\[
\hat{m}(\zeta, \bar{\zeta}) = \partial_0(\hat{l}^\nu(\zeta, \bar{\zeta})).
\]
(87)

Let us also note that
\[
0 = \partial_0(\hat{l}^\nu \hat{r}_\mu) = 2 \partial_0(\hat{l}^\nu) \hat{r}_\mu;
\]
(88)

so that it agrees with the orthogonality condition between \( \hat{l} \) and \( \hat{m} \).

It remains to see if one could express the other null tetrad vector \( \hat{n} \) also in terms of the initial expression for \( \hat{l} \). Let us consider an expression of the form
\[
\hat{n}^\mu(\zeta, \bar{\zeta}) = \alpha \hat{l}^\mu(\zeta, \bar{\zeta}) + \beta \partial_0 \partial_0 \hat{l}^\mu(\zeta, \bar{\zeta});
\]
(89)

for local coefficients \( \alpha \) and \( \beta \) that we set next.

Let us recall that the operator edth satisfies the property
\[
\partial_0 \partial_0 Y_{lm} = - \frac{l(l+1)}{2} Y_{lm};
\]
(90)

so that
\[
\partial_0 \partial_0 \hat{l}_i = - i \hat{l}_i;
\]
(91)

for \( i = 1, 2, 3 \). Then since,
\[
\sum_i (\hat{l}^i)^2 = 1;
\]
(92)

one has
\[
\hat{l}_i \partial_0 \partial_0 \hat{l}^i = 1;
\]
(93)

which indicates that one must take \( \beta = 1 \). Then, the positive \( \alpha \) is set by the condition that \( \hat{n} \) must be null; so that
\[
0 = 2 \alpha \beta \hat{l}_i \partial_0 \partial_0 \hat{l}^i + \beta^2 (\sum_i (\hat{l}^i)^2) = 2 \alpha - 1;
\]
(94)

so that
\[
\hat{n}^\mu(\zeta, \bar{\zeta}) = \frac{1}{2} \hat{l}^\mu(\zeta, \bar{\zeta}) + \partial_0 \partial_0 \hat{l}^\mu(\zeta, \bar{\zeta}).
\]
(95)

It is convenient at this point to note the difference between the scalar
\[
V_\eta = \hat{l}^\mu v_\mu;
\]
with \( \tau \) the proper time with respect to the first order background metric \( g_B \), for the case of the binary system \( (A, B) \). Let us emphasize that the velocity vectors are proportional, as indicated in (34), so that
\[
V_M = \frac{\partial_0}{\tau}.
\]
(96)
We see then that
\[ v_\mu \hat{n}^\mu = \partial_0 V_M; \] (101)
while
\[ v_\mu \hat{n}^\mu = \frac{1}{2} V_-; \] (102)
where \( V_- = V_M (-v^\nu); \) that is, it is the corresponding \( V_M \) for which the spacelike components \( v^\mu \) of the velocity \( v^\mu \) have been replaced by their negative values. We also define the corresponding \( V_{\eta -} = V_{\eta} (-v^\nu) \), so that one can also write
\[ v_\mu \hat{n}^\mu = \frac{1}{2} V_{\eta -}. \] (103)

Let \( A^\nu \) and \( B^\nu \) be any two vector fields that do not depend on the angular variables, and let us define \( A = \hat{l}^\nu A_\mu \) and \( B = \hat{l}^\nu B_\mu \). Then, calculating the scalar product of the two vectors one obtains
\[
\eta_{ab} A^a B^b = \left( \hat{l}^{\nu} \hat{l}^\nu - \hat{n}^{\mu} \hat{n}^\nu - \hat{n}^{\nu} \hat{n}^\nu \right) A_\mu B_\nu \\
= A B + A \partial_0 \partial_0 B - B \partial_0 \partial_0 A - \partial_0 A \partial_0 B; \]

which is a very useful equation that is used frequently; as for example in the components of \( h_{ab} \) in terms of the null tetrad base. In particular we have
\[ 1 = V_\eta^2 + 2 V_\eta \partial_0 V_\eta - 2 \partial_0 V_\eta \partial_0 V_\eta, \] (105)
and
\[ \Upsilon^2 = V_\eta^2 + 2 V_M \partial_0 \partial_0 V_M - 2 \partial_0 V_M \partial_0 V_M \\
= V_M V_- - 2 \partial_0 V_M \partial_0 V_M; \] (106)
from which
\[ \partial_0 \partial_0 V_M = \frac{1}{2} (V_- - V_M); \] (107)
without forgetting \( \hat{l}_M \).

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