Poincaré-Hopf Theorem for Isolated Determinantal Singularities

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Abstract

Let $X \subset \mathbb{C}^r$ be compact $d$-variety with isolated determinantal singularities and $\omega$ be a 1-form on $X$ with a finite number of singularities (in the stratified sense). Under some technical conditions on $r$ we use two generalization of Poincaré-Hopf index with the goal of proving a Poincaré-Hopf Type Theorem for $X$.

Introduction

The Poincaré-Hopf Theorem can be seen as a bridge between combinatorial algebraic topology and differential topology and the Euler characteristic is the main stone in this connection. The Euler characteristic is a very important and well known invariant which appears in mathematics since the first years in primary school and goes up to highlight applications in theoretical physics.

To compute the Euler characteristic on the differentiable side of a smooth variety it is necessary to consider the Poincaré Hopf index. However, to adapt this concept on singular varieties, we need to generalize the Poincaré-Hopf index to the singular case. In this context, many generalizations can be considered, such as the different approaches presented in [15, 14, 6, 16].

In [5], the authors present a proof of this type of result in the case where these isolated singularities are complete intersections. In this context, we have the existence and unicity of smoothing, which makes possible to define a generalization for the Poincaré-Hopf index.

The next step to continue the research is to use these new indices to find a proof of Poincaré-Hopf Theorem for compact varieties with isolated singularities of determinantal type. In this work, we consider compact varieties with isolated determinantal singularities. To obtain a version of Poincaré-Hopf
Theorem in this case, we use techniques similar to the ones used in [5], and some interesting new results about determinantal singularities.

Let $X$ be a compact variety with isolated codimension 2 determinantal singularities. In [20], using the unicity of the smoothing, the authors define the Milnor number of $X$ as the middle Betti number of a generic fiber of the smoothing. In a more general setting of determinantal varieties, the results depend on the Euler Characteristic of the stabilization given by the essential smoothing. In that paper, the authors also connect this invariant with the Ebeling and Gusein-Zade index of the 1-form given by the differential of a generic linear projection defined on the variety.

The cases that we consider in this paper is not covered by the ICIS setting. The non-standard behaviour of our setting can be seen because, for instance, we have non-smoothable and smoothable singularities and even in the smoothable case we split in two cases: unicity or not unicity of the smoothing. We consider two different Poincaré-Hopf index generalizations: one, denoted by $\text{Ind}_{PH}$, was considered by Ebeling and Gusein-Zade in [8] and can be seen as a generalization of the GSV-index [14] and, the other, by $\text{Ind}_{PHN}$ defined by Ebeling and Gusein-Zade in [8].

In Section 1, we present the basic results about determinantal varieties and indices of 1-forms and in Section 2, we prove our main result.

Acknowledgements

The authors are grateful to professors Brasselet, Seade and Ruas for their important suggestions about the theme of this paper. We also thank professors Ebeling and Zach for the fruitful conversations about their work, which is essential in this paper, during the Thematic Program on Singularity Theory, at IMPA, Rio de Janeiro, Brazil.

The first author was supported by FAPESP, under grant 2019/21181-02, and by CNPq, under grant 303046/2016-3. The second author was partially supported by Proex ICMC/USP in a visit to So Carlos, where part of this work was developed. The third author was supported by FAPESP, grant 2015/25191-9. The authors also thank PROBAL (CAPES-DAAD), grant 88881.198862/2018-01.
1 Basic Definitions

Let \( M_{n,p} \) be the set of all \( n \times p \) matrices with complex entries, \( M^t_{n,p} \subset M_{n,p} \) the subset of matrices that have rank less than \( t \), with \( 1 \leq t \leq \min(n,p) \). It is possible to show that \( M^t_{n,p} \) is a singular algebraic variety of codimension \( (n-t+1)(p-t+1) \) with singular locus \( M^{t-1}_{n,p} \) (see [7]). The set \( M^t_{n,p} \) is called generic determinantal variety.

**Definition 1.1.** Let \( F = (F_{ij}(x)) \) be an \( n \times p \) matrix whose entries are complex analytic functions on \( U \subset \mathbb{C}^r \), \( 0 \in U \) and \( f \) the function defined by the \( t \times t \) minors of \( F \). We say that \( X \) is a determinantal variety if \( X \) is defined by the equation \( f = 0 \) and the codimension of \( X \) is \( (n-t+1)(p-t+1) \).

Using [8] and [23], we present formulas of the Poincaré-Hopf type for compact varieties with isolated determinantal singularities. In order to apply [23] we need to consider a more general case of essentially isolated determinantal singularities (EIDS) defined by Ebeling e Gusein-Zade. For that, we recall the definition of essentially nonsingular point.

**Definition 1.2.** A point \( x \in X = F^{-1}(M^t_{n,p}) \) is called essentially nonsingular if, at this point, the map \( F \) is transversal to the corresponding stratum \( M^i_{n,p} \setminus M^{i-1}_{n,p} \) of the variety \( M^t_{n,p} \), where \( i = \text{rk}F(x) + 1 \).

Now we present the definition of essentially singular point at the origin.

**Definition 1.3.** A germ \( (X,0) \subset (\mathbb{C}^r,0) \) of a determinantal variety has an isolated essentially singular point at the origin if it has only essentially non-singular points in a punctured neighbourhood of the origin in \( X \).

Let \( (X,0) \subset (\mathbb{C}^r,0) \) be the germ of an analytic equidimensional variety. It is well known that complete intersections are smoothable and for a determinantal singularity, the existence and uniqueness of the smoothing do not occur in general. Because of that Ebeling and Guzein-Zade introduced the following definition.

**Definition 1.4.** An essential smoothing \( \tilde{X} \) of the EIDS \( (X,0) \) is a subvariety lying in a neighbourhood \( U \) of the origin in \( \mathbb{C}^r \) and defined by a perturbation \( \tilde{F} : U \to M_{n,p} \) of the germ \( F \) such that \( \tilde{F} \) is transversal to all the strata \( M^i_{n,p} \setminus M^{i-1}_{n,p} \) with \( i \leq t \).
An essential smoothing is not smooth in general, its singular locus is 
\( \tilde{F}^{-1}(M_{n,p}^{t-1}) \) and 
\[
\tilde{X} = \bigcup_{1 \leq i \leq t} \tilde{F}^{-1}(M_{n,p}^i \setminus M_{n,p}^{i-1}).
\]

If \( X = F^{-1}(M_{n,p}^t) \) is an EIDS, \( 1 \leq t \leq \min\{n, p\} \), an essential smoothing of \( X \) is a genuine smoothing if and only if \( r < (n - t + 2)(p - t + 2) \) (see \[8\] for more details).

In \[20\], the authors obtain the following results that can be seen as a Lé-Greuel type formula for germs of Cohen-Macaulay determinantal varieties of codimension 2 with isolated singularity at the origin.

**Theorem 1.1.** (\[20\]) Let \( (X, 0) \subset (\mathbb{C}^4, 0) \) be the germ of a determinantal surface with isolated singularity at the origin. Then,
\[
m_2(X) = \mu(p^{-1}(0) \cap X) + \mu(X),
\]
where \( m_2(X) \) is the second polar multiplicity of \( X \).

The \( m_2(X) \) multiplicity here is a generalization presented by Pereira and Ruas, in the determinantal context, to Gaffney’s multiplicity \( m_d(X) \) defined in \[11\] for isolated complete intersection singularities. When \( \dim(X) = 3 \), we obtain an expression which reduces to the Lé-Greuel formula when \( b_2(X_t) = 0 \).

**Proposition 1.1.** (\[20\]) Let \( (X, 0) \subset (\mathbb{C}^5, 0) \) be the germ of a determinantal variety of codimension 2 with isolated singularity at the origin. Then,
\[
m_3(X) = \mu(p^{-1}(0) \cap X) + \mu(X) + b_2(X_t),
\]
where \( b_2(X_t) \) is the 2-th Betti number of the generic fiber of \( X_t \) and \( m_3(X) \) is the polar multiplicity of \( X \).

In \[23\], the author studies the topology of essentially isolated determinantal singularities and obtains a result that describes the homotopy type of the Milnor fiber of an EIDS. Let \( L_{n,p}^k = M_{n,p}^t \cap H_k \) be the \( k \)-th complex link of \( M_{n,p}^t \), where \( H_k \) is a plane of codimension \( k \) in general position with the generic determinantal variety \( M_{n,p}^t \) out of the origin (see \[22\]).

**Proposition 1.2.** (\[23\], Corollary 3.5) Let \( (X, 0) \) be an EIDS given by a holomorphic map germ \( F : (\mathbb{C}^r, 0) \to (M_{n,p}, 0) \) such that \( X = F^{-1}(M_{n,p}^t) \)
is smoothable. If $F_u$ is a stabilization of $F$ and $\overline{X}_u = F_u^{-1}(M_{n,p})$ is the determinantal Milnor fiber, then

$$\overline{X}_u \simeq_s L_{n,p}^{r,np} \vee \bigvee_{i=1}^{s} S^d,$$

where $d = r - (n - t + 1)(p - t + 1) = \dim(X)$, $L_{n,p}^{t,k} = H_k \cap M_{n,p}^t$ and $H_k$ is a codimensional $k$ hyperplane in general position out of the origin.

If $(X,0) \subset (\mathbb{C}^r,0)$ is a smoothable determinantal singularity of codimension 2, we have $r < (n - n + 2)((n + 1) - n + 2) = 6$. Moreover using the previous result and Example 3.6 of [23], we have

$$\overline{X}_u \simeq_s L_{2,3}^{2(n+1)-r} \vee \bigvee_{i=1}^{s} S^d,$$

and the generic determinantal complex link associated can be calculated by

$$L_{2,3}^{2,k} \simeq \begin{cases} S^2 & \text{if } k \in \{1, 2\} \\ \bigvee_{i=1}^{e-1} S^1 & \text{if } k = 3 \\ e \text{ points} & \text{if } k = 4 \\ \emptyset & \text{otherwise} \end{cases}$$

with $k = n(n + 1) - r$.

a) If $X$ is a determinantal surface in $\mathbb{C}^4$, then

$$X_u \simeq_s L_{2,3}^{2,2} \vee \bigvee_{i=1}^{s} S^2 \simeq \bigvee_{i=1}^{s+1} S^2.$$  

We note that in this case the Milnor number defined in [20] is the number of spheres appearing on the previous bouquet.

b) If $X$ is a determinantal 3-variety in $\mathbb{C}^5$, then

$$X_u \simeq_s L_{2,3}^{2,1} \vee \bigvee_{i=1}^{s} S^3 \simeq S^2 \vee \bigvee_{i=1}^{s} S^3.$$  

Using Propositions [1,1] and [1,2], since $b_2(X_t) = 1$ ([23]), we obtain the following consequence.

**Corollary 1.1.** Let $(X,0) \subset (\mathbb{C}^5,0)$ be the germ of a determinantal variety of codimension 2 with isolated singularity at the origin. Then,

$$m_3(X) = \mu(p^{-1}(0) \cap X) + \mu(X) + 1.$$
2 Index of 1-Forms on Determinantal Varieties

Let \((X, 0)\) be an EIDS represented by a matrix \(F = (F_{ij}(x)), x \in \mathbb{C}^r\) and \(\tilde{X}\) an essential smoothing of it. In [8], Ebeling and Gusein-Zade define indices of 1-forms on EIDS. If \(X\) is a smoothable singularity then the following definition coincides with the definition presented in [8], Section 3.4.

**Definition 2.1.** Let \((X, 0) \subset (\mathbb{C}^r, 0)\) be a EIDS and \(\omega\) a 1-form on \((X, 0)\). The Poincaré-Hopf index (PH-index), \(\text{Ind}_{\text{PH}}\omega = \text{Ind}_{\text{PH}}(\omega, X, 0)\), of \(\omega\) on \((X, 0)\) is the sum of the indices of the zeros of a generic perturbation \(\tilde{\omega}\) of the 1-form \(\omega\) on the essential smoothing \(\tilde{X}\) appearing in the preimage of a neighbourhood of the origin in \((\mathbb{C}^r, 0)\).

In the case where \(\tilde{X}\) is singular we can consider the Poincaré-Hopf-Nash index (PHN-index) that can be defined as follows. Let \(\mathcal{X}\) be the total space of the Nash transform of the variety \(\tilde{X}\), \(\mathbb{T}\) the Nash bundle over \(\tilde{X}\), and \(\Pi : \mathcal{X} \to \tilde{X}\) the associate projection. The 1-form \(\omega\) defines a nonvanishing section \(\hat{\omega}\) of the dual bundle \(\hat{\mathbb{T}}^*\) over the preimage of the intersection \(\tilde{X} \cap S_\epsilon\) of the variety \(\tilde{X}\) with the sphere \(S_\epsilon\) centered at the origin. Notice that \(\mathcal{X}\) is a smooth manifold as it was proved in [8], p. 06.

**Definition 2.2.** [8] The Poincaré-Hopf-Nash index of the 1-form \(\omega\) on the EIDS \((X, 0)\), \(\text{Ind}_{\text{PHN}}(\omega, X, 0) = \text{Ind}_{\text{PHN}}\omega\), is the obstruction to extending the nonzero section \(\hat{\omega}\) of the dual Nash bundle \(\hat{\mathbb{T}}^*\) from the preimage of the boundary \(S_\epsilon = \partial B_\epsilon\) of the ball \(B_\epsilon\) to the preimage of its interior, i.e., to the manifold \(\mathcal{X}\) or, more precisely, its value (as an element of the cohomology group \(H^2d(\Pi(\tilde{X} \cap B_\epsilon), \Pi(\tilde{X} \cap S_\epsilon))\) on the fundamental class of the pair \((\Pi(X \cap B_\epsilon), \Pi(\tilde{X} \cap S_\epsilon))\).

The next proposition is a key ingredient to prove the formulas we present in the following.

**Proposition 2.1.** [8] Let \(l : M_{n,p} \to \mathbb{C}\) be a generic linear form, and let \(L^t_{n,p} = M^t_{n,p} \cap l^{-1}(1)\). Then, for \(t \leq n \leq p\), one has

\[
\chi(L^t_{n,p}) = (-1)^t \binom{n-1}{t-1}^t.
\]

As an immediate consequence, we obtain the following formula.
Corollary 2.1. If \( l : M_{n,p} \to \mathbb{C}^k \) is a linear projection, then
\[
\chi(L_{l_{n,p}}^{t,k}) = (-1)^t \binom{n-k}{t-1},
\]
where \( L_{l_{n,p}}^{t,k} = M_{l_{n,p}}^t \cap l^{-1}(\delta), \delta \neq 0 \).

**Proof.** Let \( l : M_{n,p} \to \mathbb{C}^k \) be a linear projection and \( l^{-1}(\delta) \) is a plane of codimension \( k \) in \( M_{l_{n,p}}^t \), with \( \delta \neq 0 \). Then \( l^{-1}(\delta) \) is isomorphic to \( M_{n-k,p-1} \times \{ \xi \} \), where \( \xi \) is a complex number. Then the result follows by induction applying the previous result on the right space of matrices. \( \square \)

In [21, 5] the authors proved a Poncaré-Hopf type theorem for compact varieties with isolated complete intersection singularities. Using [20] we can extend this result in the case of codimension 2 determinantal varieties with isolated singularities.

We start considering \( r < (n - t + 2)(p - t + 2), i.e., smoothable determinantal varieties. In this case, the relation between the PHN-index and the radial index (present in [9]) is given by
\[
\text{Ind}_{PHN}(\omega, X, 0) = \text{Ind}_{rad}(\omega, X, 0) + (-1)^{\dim(X)}\chi(X_u),
\]
with \( \chi(X_u) = \chi(X_u) - 1 \).

Let \( X \subset \mathbb{C}^r \) be a compact surface with isolated singularities \( p_1, \ldots, p_l \) such that the germ \( X_i = (X, p_i), i = 1, \ldots, l \), is a germ of isolated singularity and \( \omega \) is a 1-form on \( X \) with isolated singularities. Then we use the following formula that is a consequence of the definition of the radial index:
\[
\text{Ind}_{rad}(\omega, X_i, p_i) = 1 + \sum_{j=1}^{s_i} \text{Ind}_{PHN}(\omega, X_i, q_j^i),
\]
where \( \omega \) is a 1-form on \( X_i \setminus B(p_i, \epsilon_i^j) \) which coincides with \( \omega \) on \( S \cap \partial B(p_i, \epsilon_i^j) \) and with a radial form on \( X \cap \partial B(p_i, \epsilon_i^j) \) and \( q_j^i \) are the singularities of \( \omega \) on \( X_i \) (see [5]).

**Theorem 2.1.** Let \( S \subset \mathbb{C}^4 \) be a compact surface with isolated singularities \( p_1, \ldots, p_l \) such that the germ \( (S, p_i), i = 1, \ldots, l \), is a determinantal surface with isolated singularity and \( \omega \) a 1-form on \( S \) with isolated singularities. Then
\[
\sum \text{Ind}_{PHN}(\omega, S, p_i) = \chi(S) + \sum \mu(S_i),
\]
where $S_i = S \cap B(p_i, \epsilon)$ with $1 \leq i \leq l$.

**Proof.** Let $0 < \epsilon << 1$ such that the representative of the germ of $S$ on $p_i$, $S_i = S \cap B(p_i, \epsilon)$, is a determinantal surface with isolated singularity with $1 \leq i \leq l$ (of the same type, given by $2 \times 2$ minors of a $2 \times 3$ matrix). Then $\omega|_{S_i}$ has isolated singularities at $p_i$ where $l + 1 \leq i \leq l + k$.

If the value of the 1-form $\omega$ is not positive on a fixed outward looking normal vector field on the boundary of a tubular neighbourhood of $S$, we can choose $\epsilon'_i < \epsilon_i$ such that $B(p_i, \epsilon'_i)$ are the balls coming from the construction of the radial index. If $S' = S \setminus \bigcup B(p_i, \epsilon'_i)$, with $i \in \{1, \ldots, l\}$, then $S'$ is a surface with boundary and

$$
\sum_{j=1}^{k} Ind_{PHN}(\omega, S', p_{l+j}) = \chi(S').
$$

In fact, the Euler characteristic of $S$ is equal the sum of Euler characteristic of $S'$ and the number of singularities of $S$ since each $S_i$ is contractible. On the other hand,

$$
\sum Ind_{PHN}(\omega, S, p_i) = \sum_{i=1}^{l} Ind_{PHN}(\omega, S_i, p_i) + \sum_{j=1}^{k} Ind_{PHN}(\omega, S', p_{l+j}) \tag{7}
$$

Using Equations (4) and (5), we have

$$
\sum Ind_{PHN}(\omega, S, p_i) = 
\sum_{i=1}^{l} \left( Ind_{rad}(\omega, S_i, p_i) + (-1)^2 \overline{\chi}(S_i) \right) + \sum_{j=1}^{k} Ind_{PHN}(\omega, S', p_{l+j}) =
\sum_{i=1}^{l} \left( 1 + \sum_{j=1}^{s_i} Ind_{PHN}(\overline{\omega}, S_i, q_j^i) + \overline{\chi}(S_i) \right) + \sum_{j=1}^{k} Ind_{PHN}(\omega, S', p_{l+j}),
$$

where $\overline{\omega}$ is a 1-form on $S_i \setminus B(p_i, \epsilon'_i)$ which coincides with $\omega$ on $S \cap \partial B(p_i, \epsilon_i)$ and with a radial form on $S \cap \partial B(p_i, \epsilon'_i)$ and $q_j^i$ are the singularities of $\overline{\omega}$ on $S_i$.

Hence,

$$
\sum Ind_{PHN}(\omega, S, p_i) = k + \chi(S') + \sum \overline{\chi}(S_i) = \chi(S) + \sum_{i=1}^{l} \mu(S_i).
$$

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Remark 2.1. For simple space curves investigated by Frühbis-Krüger in [1] we can prove an analogous result using the Milnor number of an arbitrary reduced curve singularity defined by Buchweitz and Greuel in [4].

Theorem 2.2. Let \( X \subset \mathbb{C}^5 \) be a 3-variety with isolated singularities \( p_1, \ldots, p_l \) such that the germ of \( X \) at \( p_i, i = 1, \ldots, l \), is a determinantal 3-variety with isolated singularities and \( \omega \) a 1-form on \( X \). Then

\[
\sum \text{Ind}_{PHN}(\omega, X, p_i) = \chi(X) - l - \sum_{i=1}^{l} \mu(X_i)
\]

(8)

where \( X_i = X \cap B(p_i, \epsilon) \), for \( 0 < \epsilon << 1 \).

Proof. Let \( 0 < \epsilon << 1 \) such that the representative of the germ \( X_i = X \cap B(p_i, \epsilon) \) of \( X \) on \( p_i \) is a 3-determinantal variety with isolated singularity and let \( q_j \) be the isolated singularities of \( \omega \) on \( X_i \). Using similar arguments as in the proof of Theorem 2.1 we have

\[
\sum \text{Ind}_{PHN}(\omega, X, p_i) = \chi(X') - \sum_{i=1}^{l} \chi(X_i) + \#\{\text{singular points of } X\} = \\
= \chi(X) - \sum_{i=1}^{l} \mu(X_i) - \sum_{i=1}^{l} b_2(X_u) = \\
= \chi(X) - \left( \sum_{i=1}^{l} (\mu(X_i) + 1) \right).
\]

Corollary 2.2. In the setting of Theorem 2.2, if \( X_i \) are simple singularities, then

\[
\sum \text{Ind}_{PHN}(\omega, X, p_i) = \chi(X) - \sum_{i=1}^{l} \tau(X_i),
\]

(9)

where \( \tau(X_i) \) is the Tjurina number of \( X_i \) (see [20]).
Theorem 2.3. Let \( X \subset \mathbb{C}^r \) be a compact \( d \)-variety with isolated determinantal singularities \( \{ p_1, \ldots, p_l \} \), with \( r < (n - t + 2)(p - t + 2) \) and \( \omega \) be a 1-form on \( X \) with a finite number of singularities (in the stratified sense) of the same type. Then

\[
\sum \text{Ind}_{PHN}(\omega, X, p_i) = \chi(X) + (-1)^d l \left( 1 + (-1)^d + (-1)^t \left( \frac{n-1}{t-1} \right) \right)
\]

Proof. Let \( \{ U_i \}_{i \in I} \) be a finite covering of \( X \) such that each \( X_i = X \cap U_i \) can be described as \( F_i^{-1}(M_{n,p}^t) \), where \( X_i \) has isolated singularity at \( p_i \) and \( r < (n - t + 2)(p - t + 2) \). Then \( \omega|_{X_i} \) has isolated singularities at \( p_i \) where \( l + 1 \leq i \leq k \).

If the value of the 1-form \( \omega \) is not positive on a fixed outward looking normal vector field on the boundary of a tubular neighbourhood of \( X \), we can choose \( \epsilon'_i < \epsilon_i \) such that \( B(p_i, \epsilon'_i) \) are the balls coming from the construction of the radial index. If \( X' = X \setminus \bigcup B(p_i, \epsilon_i) \), then \( X' \) is a determinantal variety with boundary and \( \sum_{j=1}^{k} \text{Ind}_{PHN}(\omega, X', p_{l+j}) = \chi(X') \). In fact, \( \chi(X) = \chi(X') + k \) since each \( X_i \) is contractible with \( i \in \{ 1, \ldots, l \} \). Nevertheless,

\[
\sum \text{Ind}_{PHN}(\omega, X, p_i) = \sum_{i=1}^{l} \text{Ind}_{PHN}(\omega, X_i, p_i) + \sum_{j=1}^{k} \text{Ind}_{PHN}(\omega, X', p_{l+j})
\]

Using Equation (4), we have

\[
\sum \text{Ind}_{PHN}(\omega, X, p_i) = \sum_{i=1}^{l} (\text{Ind}_{rad}(\omega, X_i, p_i) + (-1)^d \chi(X_u) + \sum_{j=1}^{k} \text{Ind}_{PHN}(\omega, X', p_{l+j})
\]

\[
= \sum_{i=1}^{l} \left( 1 + \sum_{j=1}^{s_i} \text{Ind}_{PHN}(\omega, X_i, q'_j) + (-1)^d \chi(X_u) \right) + \sum_{i=1}^{k} \text{Ind}_{PHN}(\omega, X', p_{l+j})
\]

\[
= l + \chi(X') + \sum_{i=1}^{l} (-1)^d \chi(X_u) = \chi(X) + \sum_{i=1}^{l} (-1)^d \chi(X_u).
\]

Moreover, we know that \( X_u \cong_{ht} L_{n,p}^{t,n-p-r} \vee \bigvee_{v=1}^{r_t} S^d \) and

\[
\chi(L_{n,p}^t) = (-1)^t \left( \frac{n-1}{t-1} \right).
\]
where \( t \leq n \leq p \). Then
\[
\bar{\chi}(X_u) = (-1)^t \binom{n-1}{t-1} + \sum_{v=1}^{r_i} ((-1)^d + 1)
\]
Therefore,
\[
\sum Ind_{PHN}(\omega, X, p_i) = \chi(X) + \sum_{i=1}^{l} (-1)^d \bar{\chi}(X_u)
\]
\[
= \chi(X) + \sum_{i=1}^{l} \left( (-1)^{d+t} \binom{n-1}{t-1} + 1 + (-1)^d \right)
\]
\[
= \chi(X) + (-1)^d l \left( 1 + (-1)^d + (-1)^t \binom{n-1}{t-1} \right)
\]

\[\blacksquare\]

**Remark 2.2.** Let us denote by \( Eu_{M_{n,p}}(0) \) the Euler obstruction of \( M_{n,p}^t \) at the origin. Using the previous result and the formula
\[
Eu_{M_{n-1,p}}(0) = \binom{n-1}{t-1}
\]
presented in [13], we obtain

a) If \( d \) is odd, then
\[
\chi(X) = \sum Ind_{PHN}(\omega, X, p_i) + l(-1)^{d+t-1} Eu_{M_{n-1,p}}(0).
\]
b) If \( d \) is even, then
\[
\chi(X) = \sum Ind_{PHN}(\omega, X, p_i) + l(-1)^{d+t-1} Eu_{M_{n-1,p}}(0) - 2l.
\]
That means that the Euler characteristic of \( X \) measure, in some sense, the difference between the \( PHN \)-index on \( X \) and the Euler obstruction of \( M_{n-1,p}^t \) at the origin. For more details about the Euler obstruction see [16] and [6].
In the next result, we consider non-smoothable determinantal singularity with isolated singularity, i.e., \( r = (n - t + 2)(p - t + 2) \). In this case, the relation between the PHN- and the radial index present in [9] reduces to

\[
\text{Ind}_{\text{PHN}}(\omega, X, 0) = \\
\text{Ind}_{\text{rad}}(\omega, X, 0) + (-1)^{\dim(X)} \chi(X_u) + (-1)^{n+p+1}(n - t + 1)\chi((X_u)_{t-1}).
\]

(11)

**Theorem 2.4.** Let \( X \subset \mathbb{C}^r \) be a compact \( d \)-variety with rigid isolated determinantal singularities \( \{p_1, \ldots, p_l\} \) and \((X^i_u, 0)\) the germ of \( X_u \) on \( p_i \). Let \( \omega = dp \) be a 1-form on \( X \) with a finite number of singularities (in the stratified sense) and \( p_i \) the singularities of \( \omega|_{X_u^i} \), where \( l + 1 \leq i \leq k^* \). Then

\[
\sum \text{Ind}_{\text{PHN}}(\omega, X, p_i) = \chi(X) + l(-1)^d\chi(X^i_u) + (-1)^{n+p+1}(n - t + 1)\sum_{i=1}^{l} \chi((X^i_u)_{t-1}).
\]

(12)

**Proof.** Let \( \{U_i\}_{i \in I} \) be a finite covering of \( X \) such that each \( X^i_u = X \cap U_i \) can be describe as

\[
F_i : U_i \to M_{n,p}
\]

where \( F_i^{-1}(X^i_{n,p}) \) has isolated singularity at \( p_i \) and \( r = (n - t + 2)(p - t + 2) \), with \( \dim(X^i_u) = d \). Using arguments similar to previous ones, we have

\[
\sum \text{Ind}_{\text{PHN}}(\omega, X, p_i) = \sum_{i=1}^{l} \text{Ind}_{\text{PHN}}(\omega, X \cap U_i, p_i) + \sum_{j=1}^{k} \text{Ind}_{\text{PHN}}(\omega, X', p_{l+j})
\]
Using relation (11), we have

\[ \sum \text{Ind}_{PHN}(\omega, X, p_i) = \]

\[ = \sum_{i=1}^{l} (\text{Ind}_{rad}(\omega, X^l_i, p_i) + (-1)^d X^l_i + (-1)^{n+p+1}(n-t+1)\chi((X^l_i)_t^{-1}) + \]

\[ + \sum_{j=1}^{k} \text{Ind}_{PHN}(\omega, X', p_{l+j}) = \]

\[ = \sum_{i=1}^{l} \left( 1 + \sum_{j=1}^{s_i} \text{Ind}_{PHN}(\omega, X^l_i, q_j^i) + (-1)^d X^l_i + (-1)^{n+p+1}(n-t+1)\chi((X^l_i)_t^{-1}) \right) \]

\[ + \sum_{i=1}^{k} \text{Ind}_{PHN}(\omega, X', p_{l+j}) = \]

\[ = l + \chi(X') + l(-1)^d X^l + \sum_{i=1}^{l} (-1)^{n+p+1}(n-t+1)\chi((X^l_i)_t^{-1}) = \]

\[ = \chi(X) + l(-1)^d X^l_i + l(-1)^{n+p+1}(n-t+1)\chi((X^l_i)_t^{-1}). \]

\[ \square \]

**Remark 2.3.**

a) In the setting of Corollary 2.1, we can explicitly calculate the sum of indices in terms of Newton binomial.

b) Notice that if the singularity $X_i$ is rigid for any $i \in \{1, \ldots, l\}$, then

\[ \sum \text{Ind}_{PHN}(\omega, X, p_i) = \chi(X) + l(-1)^{n+p}(n-t+1). \]

Here we conclude this work where it was delivered a generalization of the Poincaré-Hopf theorem for compact determinantal varieties in the both smoothable and non-smoothable cases.
References

[1] A. Frühbis-Krüger, Classification of Simple Space Curves
Singularities, Comm. in Alg., 27 (8), pp. 3993-4013, (1999).

[2] A. Frühbis-Krüger, A. Neumer, Communications in Algebra, 1532-4125, 38, Issue 2, pp. 454-495, (2010).

[3] B. Tessier, Variétés Polaires 2: Multiplicités Polaires, Sections Planes, et Conditions de Whitney, Actes de la conférence de géométrie algébrique à la Rábida, Springer Lecture Notes, 961, pp. 314-491, (1981).

[4] R. O. G. Buchweitz, G. M. Greuel, The Milnor Number and Deformations of Complex Curve Singularities, Inventiones Mathematicae, 58, pp. 241-281, (1980).

[5] J.-P. Brasselet, J. Seade, and T. Suwa, Vector Fields on
Singular Varieties, 1987, Lecture Notes in Mathematics (2009), Springer.

[6] J.-P. Brasselet M.-H. Schwartz, Sur les classes de Chern
d’un ensemble analytique complexe, Astérisque, 82-83, 93-147, 1981.

[7] W. Bruns, U. Vetter, Determinantal Rings, Springer-Verlang, New York, (1998).

[8] W. Ebeling, S. M. Gusein-Zade, On indices of 1-forms on
determinantal singularities, Tr. Mat. Inst. Steklova, 267, pp. 119-131, (2009).

[9] W. Ebeling, S. M. Gusein-Zade, Radial Index and Euler
Obstruction of a 1-form on a singular varieties, Geometriae
Dedicata, 113, pp. 231-241, (2005).

[10] T. Gaffney, Polar Multiplicities and Equisingularity of Map
Germs, Topology, 32, pp. 185-223, (1993).

[11] T. Gaffney Multiplicities and equisingularity of ICIS germs,
Invent math 123, 209-220, (1996).
[12] T. Gaffney, N. Grulha Jr., The multiplicity polar theorem, collections of 1-forms and Chern numbers, Journal of Singularities, v. 7, pp. 36-29, (2013).

[13] T. Gaffney, N. Grulha Jr., M. A. S. Ruas, The local Euler obstruction and topology of the stabilization of associated determinantal varieties, Mathematische Zeitschrift, v. 291, pp. 905-930, (2019).

[14] X. Gmez-Mont, J. Seade and A. Verjovsky, The index of a holomorphic flow with an isolated singularity, Math. Ann., 291, 737-751, 1991.

[15] H. C. King, D. Trotman, Poincaré-Hopf theorems on singular spaces, Proceedings of the London Mathematical Society, Vol. 108, 682-703, 2014.

[16] R. MacPherson, Chern classes for singular algebraic varieties, Ann. of Math., 100, 423-432, 1974.

[17] W. J. Milnor, Morse Theory / Based on lecture notes by M. Spivak and R. Wells, Annals of Mathematics Studies, 51, New Jersey, (1963).

[18] Lê D. T., Calcul du Nombre de Cycles Évanouissants d’une Hypersurface Complexe, Ann. Inst. Fourier 23, (1973), no. 4, 261-270.

[19] J. J. Nuñó-Ballesteros, B. Oréfice, J. N. Tomazella, The vanishing Euler Characteristic of an isolated determinantal singularity, Israel Journal of Mathematics, v. X, pp. 1-21, (2013).

[20] M. S. Pereira, M. A. S. Ruas, Codimension Two Determinantal Varieties with Isolated Singularities, Mathematica Scandinavica (Papirform), v. 115, p. 161-172, (2014).

[21] J. Seade, T. Suwa, An adjunction formula for local complete intersections, International Journal of Mathematics, Vol. 09, No. 06, pp. 759-768 (1998).
[22] M. Tibar, *Bouquet decomposition of the Milnor fibre*, Topology (1995), 227–241.

[23] M. Zach, *Bouquet Decomposition For Determinantal Milnor Fibers*, arXiv e-prints, arXiv:1804.02220, https://ui.adsabs.harvard.edu/abs/2018arXiv180402220Z.

[24] J. Wahl, *Smoothings of normal surface singularities*, Topology, 20, pp. 219–246, (1981).