Robustness of operator quantum error correction with respect to initialization errors

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In the theory of operator quantum error correction (OQEC), the notion of correctability is defined under the assumption that states are perfectly initialized inside a particular subspace, a factor of which (a subsystem) contains the protected information. If the initial state of the system does not belong entirely to the subspace in question, the restriction of the state to the otherwise correctable subsystem may not remain invariant after the application of noise and error correction. It is known that in the case of decoherence-free subspaces and subsystems (DFSs) the condition for perfect unitary evolution inside the code imposes more restrictive conditions on the noise process if one allows imperfect initialization. It was believed that these conditions are necessary if DFSs are to be able to protect imperfectly encoded states from subsequent errors. By a similar argument, general OQEC codes would also require more restrictive error-correction conditions for the case of imperfect initialization. In this study, we examine this requirement by looking at the errors on the encoded state. In order to quantitatively analyze the errors in an OQEC code, we introduce a measure of the fidelity between the encoded information in two states for the case of subsystem encoding. A major part of the paper concerns the definition of the measure and the derivation of its properties. In contrast to what was previously believed, we obtain that more restrictive conditions are not necessary neither for DFSs nor for general OQEC codes. This is because the effective noise that can arise inside the code as a result of imperfect initialization is such that it can only increase the fidelity of an imperfectly encoded state with a perfectly encoded one.

I. INTRODUCTION

Operator quantum error correction (OQEC) is a generalized approach to protecting quantum information from noise, which unifies in a common framework previously proposed error correction schemes, including the standard method of active error correction as well as the passive method of decoherence-free subspaces and subsystems (for a recent generalization including entanglement-assisted error correction, see ). This approach employs the most general encoding for the protection of information—encoding in subsystems of the Hilbert space of a system (see also Ref. ). The concept of noiseless subsystem is a cornerstone in this theory, as it serves as a basis for the definition of correctable subsystem and error correction in general. This concept is defined through the assumption of perfect initialization of the state of the system inside a particular subspace. In practice, however, perfect initialization of the state may not be easy to achieve. Hence, it is important to understand to what extent the preparation requirement can be relaxed.

As shown in Ref. , in order to ensure perfect noiselessness of a subsystem in the case of imperfect initialization, the noise process has to satisfy more restrictive conditions than those required in the case of perfect initialization. It was believed that these conditions are necessary if a noiseless (or more generally decoherence-free) subsystem is to be robust against arbitrarily large initialization errors. The fundamental relation between a noiseless subsystem and a correctable subsystem implies that in the case of imperfect initialization, more restrictive conditions would be needed for OQEC codes as well.

In this paper we show that with respect to the ability of a code to protect from errors, more restrictive conditions are not necessary. For this purpose, we define a measure of the fidelity between the encoded information in two states for the case of subsystem encoding. We first give an intuitive motivation for the definition, and then study the properties of the measure. We then show that the effective noise that can arise inside the code due to imperfect initialization under the standard conditions, is such that it can only increase the fidelity of the encoded information with the information encoded in a perfectly prepared state. This robustness against initialization errors is shown to hold also when the state is subject to encoded operations.

II. REVIEW OF CONDITIONS FOR NOISELESS SUBSYSTEMS AND OQEC CODES

For simplicity, we consider the case where information is stored in only one subsystem, i.e., we consider a decomposition of the system’s Hilbert space of the form

\[ \mathcal{H}^S = \mathcal{H}^A \otimes \mathcal{H}^B \otimes \mathcal{K}, \]

where the sector \( \mathcal{H}^A \) (also called a subsystem) is used for encoding of the protected information. Let \( \mathcal{B}(\mathcal{H}) \) denote the set of linear operators on a finite-dimensional Hilbert space \( \mathcal{H} \). In the OQEC formalism, noise is represented by a completely positive trace-preserving (CPTP) linear map or a noise channel \( \mathcal{E} : \mathcal{B}(\mathcal{H}^S) \to \mathcal{B}(\mathcal{H}^S) \). Every such
map can be written in the Kraus form [21] 
\[ \mathcal{E}(\sigma) = \sum_i E_i \sigma E_i^\dagger, \text{ for all } \sigma \in \mathcal{B}(\mathcal{H}^S), \]  
(2)
where the Kraus operators \{E_i\} \subseteq \mathcal{B}(\mathcal{H}^S) satisfy 
\[ \sum_i E_i^\dagger E_i = I^S. \]  
(3)

The subsystem \( \mathcal{H}^A \) in the decomposition (1) is called noiseless with respect to the channel \( \mathcal{E} \), if 
\[ \text{Tr}_B\{(\mathcal{P}^{AB} \circ \mathcal{E})(\sigma)\} = \text{Tr}_B\{\sigma\}, \]  
for all \( \sigma \in \mathcal{B}(\mathcal{H}^S) \) such that \( \sigma = \mathcal{P}^{AB}(\sigma) \),
where \( \mathcal{P}^{AB}(\cdot) = \mathcal{P}^{AB}(\cdot) \mathcal{P}^{AB} \) and \( \mathcal{P}^{AB} \) is the projector of \( \mathcal{H}^S \) onto \( \mathcal{H}^A \otimes \mathcal{H}^B \). Similarly, a correctable subsystem is one for which there exists a correcting CPTP map \( \mathcal{R} : \mathcal{B}(\mathcal{H}^S) \to \mathcal{B}(\mathcal{H}^S) \), such that the subsystem is noiseless with respect to the map \( \mathcal{R} \circ \mathcal{E} \):
\[ \text{Tr}_B\{(\mathcal{P}^{AB} \circ \mathcal{R} \circ \mathcal{E})(\sigma)\} = \text{Tr}_B\{\sigma\}, \]  
for all \( \sigma \in \mathcal{B}(\mathcal{H}^S) \) such that \( \sigma = \mathcal{P}^{AB}(\sigma) \).

The definition of noiseless subsystem (4) implies that the information encoded in \( \mathcal{B}(\mathcal{H}^A) \) remains invariant after the process \( \mathcal{E} \), if the initial density operator of the system \( \rho(0) \) belongs to \( \mathcal{B}(\mathcal{H}^A \otimes \mathcal{H}^B) \). If, however, one allows imperfect initialization, \( \rho(0) \neq \mathcal{P}^{AB}(\rho(0)) \), this need not be the case. Consider the “initialization-free” analogue of the definition (4):
\[ \text{Tr}_B\{(\mathcal{P}^{AB} \circ \mathcal{R} \circ \mathcal{E})(\sigma)\} = \text{Tr}_B\{\mathcal{P}^{AB}(\sigma)\}, \]  
(6)
for all \( \sigma \in \mathcal{B}(\mathcal{H}^S) \).

Obviously Eq. (6) implies Eq. (4), but the reverse is not true. As shown in [20], the definition (6) imposes more restrictive conditions on the channel \( \mathcal{E} \) than those imposed by (4). To see this, consider the form of the Kraus operators \( E_i \) in the block basis corresponding to the decomposition (1). From a result derived in [20] it follows that the subsystem \( \mathcal{H}^A \) is noiseless in the sense of Eq. (4), if and only if the Kraus operators have the form
\[ E_i = \begin{bmatrix} I^A \otimes C_i^B & D_i \\ 0 & G_i \end{bmatrix}, \]  
(7)
where the upper left block corresponds to the subspace \( \mathcal{H}^A \otimes \mathcal{H}^B \), and the lower right block corresponds to \( \mathcal{K} \).

The completeness relation (3) implies the following conditions on the operators \( C_i^B, D_i, \) and \( G_i \):
\[ \sum_i C_i^B C_i^B = I^B, \]  
(8)
\[ \sum_i I^A \otimes C_i^B D_i = 0, \]  
(9)
\[ \sum_i (D_i^\dagger D_i + G_i^\dagger G_i) = I_K. \]  
(10)

In the same block basis, a perfectly initialized state \( \rho \) and its image under the map (7) have the form
\[ \rho = \begin{bmatrix} \rho_1 & 0 \\ 0 & 0 \end{bmatrix}, \]  
\[ \mathcal{E}(\rho) = \begin{bmatrix} \rho_1 & 0 \\ 0 & 0 \end{bmatrix}, \]  
(11)
where \( \rho'_1 = \sum_i I^A \otimes C_i^B \rho_1 I^A \otimes C_i^B \). Using the linearity and cyclic invariance of the trace together with Eq. (8), we obtain
\[ \text{Tr}_B\{(\mathcal{P}^{AB} \circ \mathcal{E})(\rho)\} = \text{Tr}_B\{\sum_i I^A \otimes C_i^B \rho_1 I^A \otimes C_i^B\} \]  
\[ = \text{Tr}_B\{\rho_1 \sum_i I^A \otimes C_i^B C_i^B\} = \text{Tr}_B\{\mathcal{P}^{AB}(\rho)\}, \]  
(12)
i.e., the reduced operator on \( \mathcal{H}^A \) remains invariant.

On the other hand, an imperfectly initialized state \( \tilde{\rho} \) and its image have the form
\[ \tilde{\rho} = \begin{bmatrix} \tilde{\rho}_1 & \tilde{\rho}_2 \\ \tilde{\rho}_2^\dagger & \tilde{\rho}_3 \end{bmatrix}, \]  
\[ \mathcal{E}(\tilde{\rho}) = \begin{bmatrix} \tilde{\rho}_1 & \tilde{\rho}_2 \\ \tilde{\rho}_2^\dagger & \tilde{\rho}_3 \end{bmatrix}. \]  
(13)
Here \( \tilde{\rho}_2 \) and/or \( \tilde{\rho}_3 \) are non-vanishing, and
\[ \tilde{\rho}_1 = \sum_i (I^A \otimes C_i^B \tilde{\rho}_1 I^A \otimes C_i^B + D_i \tilde{\rho}_2 I^A \otimes C_i^B \]  
\[ + I^A \otimes C_i^B \tilde{\rho}_2 D_i^\dagger + D_i \tilde{\rho}_3 D_i^\dagger), \]  
\[ \tilde{\rho}_2 = \sum_i (I^A \otimes C_i^B \tilde{\rho}_2 G_i^\dagger + D_i \tilde{\rho}_3 G_i^\dagger), \]  
(15)
\[ \tilde{\rho}_3 = \sum_i G_i \tilde{\rho}_3 G_i^\dagger. \]  
(16)
In this case, using the linearity and cyclic invariance of the trace together with Eq. (8) and Eq. (9), we obtain
\[ \text{Tr}_B \{ (p^{AB} \circ \mathcal{E})(\hat{\rho}) \} = \text{Tr}_B \{ \sum_i (I^A \otimes C_i^B \hat{\rho}_1 I^A \otimes C_i^B + D_i \hat{\rho}_2 I^A \otimes C_i^B + I^A \otimes C_i^B \hat{\rho}_2 D_i^\dagger + D_i \hat{\rho}_3 D_i^\dagger) \} \]
\[ = \text{Tr}_B \{ \hat{\rho}_1 \sum_i \underbrace{I^A \otimes C_i^B C_i^B}_{I^A \otimes I^B} \} + \text{Tr}_B \{ \sum_i \underbrace{I^A \otimes C_i^B D_i}_{I^A \otimes 0} \hat{\rho}_2 \} + \text{Tr}_B \{ \sum_i \underbrace{I^A \otimes C_i^B D_i^\dagger}_{I^A \otimes 0} \hat{\rho}_2 \} + \text{Tr}_B \{ \sum_i D_i \hat{\rho}_3 D_i^\dagger \} \]
\[ = \text{Tr}_B \hat{\rho}_1 + \text{Tr}_B \{ \sum_i D_i \hat{\rho}_3 D_i^\dagger \} \neq \text{Tr}_B \hat{\rho}_\| \equiv \text{Tr}_B \{ p^{AB}(\hat{\rho}) \}, \quad (17) \]

i.e., the reduced operator on \( \mathcal{H}^A \) is not preserved. It is easy to see that the reduced operator would be preserved for every imperfectly initialized state if and only if we impose the additional condition
\[ D_i = 0, \quad \text{for all } i. \quad (18) \]

This further restriction to the form of the Kraus operators is equivalent to the requirement that there are no transitions from the subspace \( \mathcal{K} \) to the subspace \( \mathcal{H}^A \otimes \mathcal{H}^B \) under the process \( \mathcal{E} \). This is in addition to the requirement that no states leave \( \mathcal{H}^A \otimes \mathcal{H}^B \), which is ensured by the vanishing lower left blocks of the Kraus operators \( \tilde{R} \). Condition \( (18) \) automatically imposes an additional restriction on the error-correction conditions, since if \( \mathcal{R} \) is an error-correcting map in this “initialization-free” sense, the map \( \mathcal{R} \circ \mathcal{E} \) would have to satisfy Eq. \( (13) \). But is this constraint necessary from the point of view of the ability of the code to correct further errors?

Notice that since \( \hat{\rho} \) is a positive operator, \( \hat{\rho}_3 \) is positive, and hence \( \text{Tr}_B \{ \sum_i D_i \hat{\rho}_3 D_i^\dagger \} \) is positive. The reduced operator on subsystem \( \mathcal{H}^A \), although unnormalized, can be regarded as a (partial) probability mixture of states on \( \mathcal{H}^A \). The noise process modifies the original mixture (\( \text{Tr}_B \hat{\rho}_1 \)) by adding to it another partial mixture (the positive operator \( \text{Tr}_B \{ \sum_i D_i \hat{\rho}_3 D_i^\dagger \} \)). Since the weight of any state already present in the mixture can only increase by this process, this should not worsen the faithfulness with which information is encoded in \( \hat{\rho} \). In order to make this argument rigorous, however, we need a measure that quantifies the faithfulness of the encoding.

\section{III. FIDELITY BETWEEN THE ENCODED INFORMATION IN TWO STATES}

\subsection{A. Motivating the definition}

If we consider two states with density operators \( \tau \) and \( \nu \), a good measure of the faithfulness with which one state represents the other is given by the fidelity between the states:
\[ F(\tau, \nu) = \text{Tr} \sqrt{\tau \nu} \sqrt{\tau}. \quad (19) \]

This quantity can be thought of as a square root of a generalized “transition probability” between the two states \( \tau \) and \( \nu \) as defined by Uhlmann [23]. Another interpretation due to Fuchs [22] gives an operational meaning of the fidelity as the minimal overlap between the probability distributions generated by all possible generalized measurements on the states:
\[ F(\tau, \nu) = \min_{\{M_i\}} \sum_i \text{Tr} \{ M_i \tau \} \sqrt{\text{Tr} \{ M_i \nu \}}. \quad (20) \]

Here, minimum is taken over all positive operators \( \{M_i\} \) that form a positive operator-valued measure (POVM) \( (21) \). \( \sum_i M_i = I^S \).

In our case, we need a quantity that compares the encoded information in two states. Clearly, the standard fidelity between the states will not do since it measures the similarity between the states on the entire Hilbert space. The encoded information, however, concerns only the reduced operators on subsystem \( \mathcal{H}^A \). In view of this, we propose the following

\textbf{Definition 1.} Let \( \tau \) and \( \nu \) be two density operators on a Hilbert space \( \mathcal{H}^S \) with decomposition \( (1) \). The fidelity between the information encoded in subsystem \( \mathcal{H}^A \) in the two states is given by:
\[ F^A(\tau, \nu) = \max_{\tau', \nu'} F(\tau', \nu'), \quad (21) \]

where maximum is taken over all density operators \( \tau' \) and \( \nu' \) that have the same reduced operators on \( \mathcal{H}^A \) as \( \tau \) and \( \nu \): \( \text{Tr}_B \{ p^{AB}(\tau') \} = \text{Tr}_B \{ p^{AB}(\tau) \}, \quad \text{Tr}_B \{ p^{AB}(\nu') \} = \text{Tr}_B \{ p^{AB}(\nu) \} \).

The intuition behind this definition is that by maximizing over all states that have the same reduced operators on \( \mathcal{H}^A \) as the states being compared, we ensure that the measure does not penalize for differences between the states that are not due specifically to differences between the reduced operators.

\subsection{B. Properties of the measure}

\textbf{Property 1 (Symmetry).} Since the fidelity is symmetric with respect to its inputs, it is obvious from Eq. \( (21) \) that \( F^A \) is also symmetric:
\[ F^A(\tau, \nu) = F^A(\nu, \tau). \quad (22) \]
Although intuitive, the definition \((21)\) does not allow for a simple calculation of \(F^A\). We now derive an equivalent form for \(F^A\), which is simple and easy to compute. Let \(\mathcal{P}_K(\cdot) = P_K(\cdot)P_K\) denote the superoperator projector on \(B(K)\), and let
\[
\rho^A \equiv \text{Tr}_B\{\mathcal{P}_{AB}(\rho)\}/\text{Tr}\{\mathcal{P}_{AB}(\rho)\}
\] (23)
denote the normalized reduced operator of \(\rho\) on \(H^A\).

**Theorem 1.** The definition \((21)\) is equivalent to
\[
F^A(\tau, \nu) = F(\tau^*, \nu^*) \leq F(\Pi(\tau^*), \Pi(\nu^*)),
\] (26)
where \(\Pi(\cdot) = \mathcal{P}_{AB}(\cdot) + \mathcal{P}_K(\cdot)\). But the states \(\Pi(\tau^*)\) and \(\Pi(\nu^*)\) satisfy
\[
\text{Tr}_B\{\mathcal{P}_{AB}(\Pi(\tau^*))\} = \text{Tr}_B\{\mathcal{P}_{AB}(\tau)\},
\] (27)
\[
\text{Tr}_B\{\mathcal{P}_{AB}(\Pi(\nu^*))\} = \text{Tr}_B\{\mathcal{P}_{AB}(\nu)\},
\] (28)
i.e., they are among those states over which the maximum in Eq. \((21)\) is taken. Therefore,
\[
F^A(\tau, \nu) = F(\Pi(\tau^*), \Pi(\nu^*)).
\] (29)

Using Eq. \((19)\) and the fact that in the block basis corresponding to the decomposition \((11)\) the states \(\Pi(\tau^*)\) and \(\Pi(\nu^*)\) have block-diagonal forms, it is easy to see that
\[
F(\Pi(\tau^*), \Pi(\nu^*)) = \tilde{F}(\mathcal{P}_{AB}(\tau^*), \mathcal{P}_{AB}(\nu^*))
\] + \[
\frac{\mathcal{P}_{AB}(\tau^*)}{\text{Tr}\{\mathcal{P}_{AB}(\tau)\}} \times \frac{\mathcal{P}_{AB}(\nu^*)}{\text{Tr}\{\mathcal{P}_{AB}(\nu)\}},
\] (30)
\[
\tilde{F}(\mathcal{P}_K(\tau^*), \mathcal{P}_K(\nu^*)) = \text{Tr}\{\mathcal{P}_K(\tau)\} \times \text{Tr}\{\mathcal{P}_K(\nu)\},
\] (31)
\[
F(\mathcal{P}_K(\tau^*), \mathcal{P}_K(\nu^*)) = \text{Tr}\{\mathcal{P}_K(\tau)\} \times \text{Tr}\{\mathcal{P}_K(\nu)\},
\] (32)
\[
\tilde{F}(\mathcal{P}_K(\tau^*), \mathcal{P}_K(\nu^*)) = \text{Tr}\{\mathcal{P}_K(\tau)\} \times \text{Tr}\{\mathcal{P}_K(\nu)\}.
\] (33)
Since \(\tau^*\) and \(\nu^*\) should maximize the right-hand side of Eq. \((30)\), and the only restriction on \(\mathcal{P}_K(\tau^*)\) and \(\mathcal{P}_K(\nu^*)\) is \(\text{Tr}\{\mathcal{P}_K(\tau^*)\} = \text{Tr}\{\mathcal{P}_K(\tau)\}, \text{Tr}\{\mathcal{P}_K(\nu^*)\} = \text{Tr}\{\mathcal{P}_K(\nu)\}\), we must have
\[
F\left(\frac{\mathcal{P}_K(\tau^*)}{\text{Tr}\{\mathcal{P}_K(\tau)\}}, \frac{\mathcal{P}_K(\nu^*)}{\text{Tr}\{\mathcal{P}_K(\nu)\}}\right) = 1,
\] (34)
i.e.,
\[
\frac{\mathcal{P}_K(\tau^*)}{\text{Tr}\{\mathcal{P}_K(\tau)\}} = \frac{\mathcal{P}_K(\nu^*)}{\text{Tr}\{\mathcal{P}_K(\nu)\}}.
\] (35)
Thus we obtain
\[
\tilde{F}(\mathcal{P}_K(\tau^*), \mathcal{P}_K(\nu^*)) = \text{Tr}\{\mathcal{P}_K(\tau)\} \times \text{Tr}\{\mathcal{P}_K(\nu)\}.
\] (36)
This completes the proof.

We next provide an operational interpretation of the measure \(F^A\). For this we need the following

**Lemma.** The function \(F^A(\tau, \nu)\) defined in Eq. \((25)\) equals the minimum overlap between the statistical distributions generated by all local measurements on subsystem \(H^A\):
\[
F^A(\tau, \nu) = \min_{\{M_i\}} \sum_i \sqrt{\text{Tr}\{M_i\} \times \text{Tr}\{M_i\}}
\] (37)
where \(M_i = M_i^A \otimes I^B\), \(\sum_i M_i = I^A \otimes I^B\), \(M_i^A > 0\), for all \(i\).

Note that since the operators \(M_i\) do not form a complete PVM on the entire Hilbert space, the probability distributions \(p_i(\cdot) = \text{Tr}\{M_i\} \times \text{Tr}\{M_i\}\) generated by such measurements generally do not sum up to 1. This reflects the fact that a measurement on subsystem \(H^A\) requires a projection onto the subspace \(H^A \otimes H^B\), i.e., it is realized through post-selection.

**Proof.** Using that
\[
\text{Tr}\{M_i\} = \text{Tr}\{M_i^A \otimes I^B\}\mathcal{P}_{AB}(\tau)\}
\] = \[
\text{Tr}\{\mathcal{P}_{AB}(\tau)\} \times \text{Tr}\{M_i^A \otimes I^B\}\mathcal{P}_{AB}(\tau)\}
\] = \[
\text{Tr}\{\mathcal{P}_{AB}(\tau)\} \times \text{Tr}\{M_i^A \otimes I^B\}\mathcal{P}_{AB}(\tau)\}
\] (38)
we can write Eq. \((37)\) in the form
\[
F^A(\tau, \nu) = \sqrt{\text{Tr}\{\mathcal{P}_{AB}(\tau)\} \times \text{Tr}\{\mathcal{P}_{AB}(\tau)\}}\times \min_{\{M_i\}} \sum_i \sqrt{\text{Tr}\{M_i^A \otimes I^B\}\mathcal{P}_{AB}(\tau)\} \times \text{Tr}\{M_i^A \otimes I^B\}\mathcal{P}_{AB}(\tau)\}.
\] (39)
From Eq. (21), we see that (39) is equivalent to (25).

**Theorem 2.** \( F^A(\tau, v) \) equals the minimum overlap

\[
F^A(\tau, v) = \min_{\{M_i\}_{i \geq 0}} \sqrt{\text{Tr}\{M_i \tau\}} \sqrt{\text{Tr}\{M_i \nu\}}
\]

between the statistical distributions generated by all possible measurements of the form \( M_0 = P_K, M_i = M_i^A \otimes I^B \) for \( i \geq 1 \), \( \sum_{i \geq 0} M_i = I^S \).

**Proof.** The proof follows from Eq. (21) and Eq. (37).

Note that the measure \( F^A \) compares the information stored in subsystem \( \mathcal{H}^A \), which is the information extractable through local measurements on \( \mathcal{H} \). The last result reflects the intuition that extracting information encoded in \( \mathcal{H}^A \) involves a measurement that projects on the subspaces \( \mathcal{H}^A \otimes \mathcal{H}^B \) or \( K \).

**Property 2 (Normalization).** From the definition (21) it is obvious that

\[
F^A(\tau, v) \leq F^A(\tau, \tau) = 1, \quad \tau \neq v.
\]

From Eq. (21) we can now see that

\[
F^A(\tau, v) = 1, \quad \text{iff} \quad \text{Tr}_B\{P^{AB}(\tau)\} = \text{Tr}_B\{P^{AB}(v)\},
\]

as one would expect from a measure that compares only the encoded information in \( \mathcal{H}^A \).

**Proposition.** Using that the maximum in Eq. (21) is attained for states of the form \( \Pi(\tau^*) \) and \( \Pi(v^*) \) (Eq. (29)) where \( \tau^* \) and \( v^* \) satisfy Eq. (31) and Eq. (39), without loss of generality we can assume that for all \( \tau \) and \( v \),

\[
F^A(\tau, v) = F(\tau^*, v^*),
\]

where

\[
\tau^* = \text{Tr}_B\{P^{AB}(\tau)\} \otimes |0^B\rangle \langle 0^B| + \text{Tr}\{P_K(\tau)|0_K\rangle \langle 0_K|, \quad (44)
\]

\[
v^* = \text{Tr}_B\{P^{AB}(v)\} \otimes |0^B\rangle \langle 0^B| + \text{Tr}\{P_K(v)|0_K\rangle \langle 0_K|, \quad (45)
\]

with \( |0^B\rangle \) and \( |0_K\rangle \) being some fixed states in \( \mathcal{H}^B \) and \( K \), respectively.

**Property 3 (Strong concavity and concavity of the square of \( F^A \)).** The form of \( F^A \) given by Eqs. (43)–(45) can be used for deriving various useful properties of \( F^A \) from the properties of the standard fidelity. For example, it implies that for all mixtures \( \sum_i p_i \tau_i \) and \( \sum_i q_i v_i \) we have

\[
F^A(\sum_i p_i \tau_i, \sum_i q_i v_i) = F(\sum_i p_i \tau_i^*, \sum_i q_i v_i^*). \quad (46)
\]

This means that the property of strong concavity of the fidelity (25) (and all weaker concavity properties that follow from it) as well as the concavity of the square of the fidelity (29), are automatically satisfied by the measure \( F^A \).

**Definition 2.** Similarly to the concept of angle between two states (25) which can be defined from the standard fidelity, we can define an angle between the encoded information in two states:

\[
\Lambda^A(\tau, v) = \arccos F^A(\tau, v). \quad (47)
\]

**Property 4 (Triangle inequality).** From Eqs. (43)–(45) it follows that just as the angle between states satisfies the triangle inequality, so does the angle between the encoded information:

\[
\Lambda^A(\tau, v) \leq \Lambda^A(\tau, \phi) + \Lambda^A(\phi, v). \quad (48)
\]

**Property 5 (Monotonicity of \( F^A \) under local CPTP maps).** We point out that the monotonicity under CPTP maps of the standard fidelity does not translate directly to the measure \( F^A \). Rather, as can be seen from Eq. (21), \( F^A \) satisfies monotonicity under local CPTP maps on \( \mathcal{H}^A \):

\[
F^A(\mathcal{E}(\tau), \mathcal{E}(v)) \geq F^A(\tau, v) \quad (49)
\]

for

\[
\mathcal{E} = \mathcal{E}^A \otimes \mathcal{E}^B \otimes \mathcal{E}_K, \quad (50)
\]

where \( \mathcal{E}^A, \mathcal{E}^B \) and \( \mathcal{E}_K \) are CPTP maps on operators over \( \mathcal{H}^A, \mathcal{H}^B \) and \( K \), respectively.

**Comment.** There exist other maps under which \( F^A \) is also non-decreasing. Such are the maps which take states from \( \mathcal{H}^A \otimes \mathcal{H}^B \) to \( K \) without transfer in the opposite direction. But in general, maps which couple states in \( \mathcal{H}^A \otimes \mathcal{H}^B \) with states in \( K \), or states in \( \mathcal{H}^A \) with states in \( \mathcal{H}^B \), do not obey this property. For example, a unitary map which swaps the states in \( \mathcal{H}^A \) and \( \mathcal{H}^B \) (assuming both subsystems are of the same dimension) could both increase or decrease the measure depending on the states in \( \mathcal{H}^B \). Similarly, a unitary map exchanging states between \( \mathcal{H}^A \otimes \mathcal{H}^B \) and \( K \) could give rise to both increase or decrease of the measure depending on the states in \( K \).

Finally, the monotonicity of \( F^A \) under local CPTP maps implies

**Property 6 (Contractivity of the angle under local CPTP maps).** For CPTP maps of the form (50), \( \Lambda^A \) satisfies

\[
\Lambda^A(\mathcal{E}(\tau), \mathcal{E}(v)) \leq \Lambda^A(\tau, v). \quad (51)
\]

**IV. ROBUSTNESS OF OQEC WITH RESPECT TO INITIALIZATION ERRORS**

Let us now consider the fidelity between the encoded information in an ideally prepared state (11) and in a state which is not perfectly initialized (13):

\[
F^A(\rho, \tilde{\rho}) = \sqrt{\text{Tr}_B \rho_1 \text{Tr}_B \tilde{\rho}_1} F^A(\rho^A, \tilde{\rho}^A) + 0 \quad (52)
\]

\[
= \text{Tr} \sqrt{\text{Tr}_B \rho_1 \text{Tr}_B \tilde{\rho}_1} \text{Tr}_B \rho_1 \equiv \tilde{F}(\text{Tr}_B \rho_1, \text{Tr}_B \tilde{\rho}_1).
\]
After the noise process $\mathcal{E}$ with Kraus operators $\{\mathcal{E}\}$, the imperfectly encoded state transforms to $\mathcal{E}(\hat{\rho})$. Its fidelity with the perfectly encoded state becomes

$$F^A(\rho, \mathcal{E}(\hat{\rho})) = \tilde{F}(\text{Tr}_B \rho_1, \text{Tr}_B \hat{\rho}_1)$$

$$= \tilde{F}(\text{Tr}_B \rho_1, \text{Tr}_B \hat{\rho}_1 + \text{Tr}_B \{\sum_i D_i \hat{\rho}_3 D_i^\dagger\}), \quad (53)$$

where we have used the expressions for $\text{Tr}_B \rho_1$ and $\text{Tr}_B \hat{\rho}_1$ obtained in Eq. (12) and Eq. (17). As we pointed out earlier, the operator $\text{Tr}_B \{\sum_i D_i \hat{\rho}_3 D_i^\dagger\}$ is positive. Then from the concavity of the square of the fidelity $\tilde{F}$, it follows that

$$F^A(\rho, \mathcal{E}(\hat{\rho})) \geq F^A(\rho, \hat{\rho}). \quad (55)$$

We see that even if the “initialization-free” constraint [13] is not satisfied, no further decrease in the fidelity occurs as a result of the process. The effective noise (the term $\text{Tr}_B \{\sum_i D_i \hat{\rho}_3 D_i^\dagger\}$) that arises due to violation of that constraint, can only decrease the initialization error.

The above result can be generalized to include the possibility for information processing on the subsystem.

Imagine that we want to perform a computational task which ideally corresponds to applying the CPTP map $\mathcal{C}^A$ on the encoded state. In general, the subsystem $\mathcal{H}^A$ may consist of many subsystems encoding separate information units (e.g. qubits), and the computational process may involve many applications of error correction. The noise process itself generally acts continuously during the computation. Let us assume that all operations following the initialization are performed fault-tolerantly [20, 27, 28, 29, 30] so that the overall transformation $\mathcal{C}$ on a perfectly initialized state succeeds with an arbitrarily high probability (for a model of fault-tolerant quantum computation on subsystems, see e.g. [31]). This means that the effect of $\mathcal{C}$ on the reduced operator of a perfectly initialized state is

$$\text{tr}_B \rho_1 \rightarrow \mathcal{C}^A(\text{tr}_B \rho_1) \quad (56)$$

up to an arbitrarily small error.

**Theorem 4.** Let $\mathcal{C}$ be a CPTP map whose effect on reduced operator of every perfectly initialized state $\{\mathcal{E}\}$ is given by Eq. (53) with $\mathcal{C}^A$ being a CPTP map on $\mathcal{B}(\mathcal{H}^A)$. Then the fidelity between the encoded information in a perfectly initialized state $\{\mathcal{E}\}$ and an imperfectly initialized state [13] does not decrease under $\mathcal{C}$:

$$F^A(\mathcal{C}(\rho), \mathcal{C}(\hat{\rho})) \geq F^A(\rho, \hat{\rho}). \quad (57)$$

**Proof.** From Eq. (56) it follows that the map $\mathcal{C}$ has Kraus operators with vanishing lower left blocks, similarly to [7]. If the state is not perfectly initialized, an argument similar to the one performed earlier shows that the reduced operator on the subsystem transforms as $\text{Tr}_B \hat{\rho}_1 \rightarrow \mathcal{C}^A(\text{Tr}_B \hat{\rho}_1) + \hat{\rho}_{\text{err}}$, where $\hat{\rho}_{\text{err}}$ is a positive operator which appears as a result of the possibly non-vanishing upper right blocks of the Kraus operators. Using an argument analogous to [54] and the monotonicity
of the fidelity under CPTP maps $F_{\rho}$, we obtain

$$F^A(\rho, \rho) = F(\rho^A(BB\rho_1), \rho^A(BB\rho_1) + \rho^A_{\text{err}}) + 0$$

$$\geq F(\rho^A(BB\rho_1), \rho^A(BB\rho_1))$$

$$\geq \sqrt{\text{Tr} \rho_1 \sqrt{\text{Tr} \rho_1} F(\rho^A(\rho), \rho^A(\rho_{\text{err}}))}$$

$$\geq \sqrt{\text{Tr} \rho_1 \sqrt{\text{Tr} \rho_1} F(\rho^A(\rho), \rho^A(\rho_{\text{err}}))}$$

$$= F(\rho^A(BB\rho_1), \rho^A(BB\rho_1))$$

Again, the preparation error is not amplified by the process. The problem of how to deal with preparation errors has been discussed in the context of fault-tolerant computation on standard error-correction codes, e.g., in [32]. The situation for general OQEC is similar—if the initial state is known, the error can be eliminated by repeating the encoding. If the state to be encoded is unknown, the preparation error generally cannot be corrected. Nevertheless, encoding would still be worthwhile as long as the initialization error is smaller than the error which would result from leaving the state unprotected.

V. CONCLUSION

In summary, we have shown that a noiseless subsystem is robust against initialization errors without the need for modification of the noiseless subsystem conditions. Similarly, we have argued that general OQEC codes are robust with respect to imperfect preparation in their standard form. This property is compatible with fault-tolerant methods of computation, which is essential for reliable quantum information processing. In order to rigorously prove our result, we introduced a measure of the fidelity $F^A(\tau, \nu)$ between the encoded information in two states. The measure is defined as the maximum of the fidelity between all possible states which have the same reduced operators on the subsystem code as the states being compared. We derived a simple form of the measure and discussed many of its properties. We also gave an operational interpretation of the quantity.

Since the concept of encoded information is central to quantum information science, the fidelity measure introduced in this paper may find various applications. It provides a natural means for extending key concepts such as the fidelity of a quantum channel [10] or the entanglement fidelity [25] to the case of subsystem codes.

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