A Holographic Interpretation
of
Asymptotically de Sitter Spacetimes

by

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ABSTRACT

One of the remarkable outcomes of the AdS/CFT correspondence has been the generalization of Cardy’s entropy formula for arbitrary dimensionality, as well as a variety of anti-de Sitter scenarios. More recently, related work has been done in the realm of asymptotically de Sitter backgrounds. Such studies presume a well-defined dS/CFT duality, which has not yet attained the credibility of its AdS analogue. In this paper, we derive and interpret generalized forms of the Cardy entropy for a selection of asymptotically de Sitter spacetimes. These include the Schwarzschild-de Sitter black hole (as a review of [hep-th/0112093]), the Reissner-Nordstrom-de Sitter black hole and a special class of topological de Sitter solutions. Each of these cases is found to have interesting implications in the context of the proposed correspondence.
1 Introduction

It is commonly believed that the holographic principle will be an essential constituent of any valid theory of quantum gravity [1, 2]. This principle (which follows from Bekenstein’s entropy bound [3]) implies that, if given a physical system, the relevant degrees of freedom must suitably live on a surface enclosing that system. To take it a step further, given a physical theory in \(d+1\)-dimensional spacetime, one would expect the existence of a \(d\)-dimensional theory that can capably describe the same physics.

In spite of its esoteric origins, the holographic principle has explicitly been realized by way of the AdS/CFT correspondence [4, 5]. That is, a duality has been demonstrated between \(d+1\)-dimensional anti-de Sitter (AdS) spacetimes and appropriately defined, \(d\)-dimensional conformal field theories (CFTs). More specifically, the thermodynamics at the horizon of an AdS black hole can be identified with the thermodynamics of a CFT that lives on an asymptotic boundary of the AdS background [6].

Recently, Verlinde [7] has utilized this AdS/CFT duality to demonstrate a result of considerable significance. This author was able to generalize Cardy’s well-known entropy formula (for a 2-dimensional CFT) [8] to a theory of arbitrary dimensionality. The essence of Verlinde’s work was the identification of Cardy’s central charge (the number of massless particle species) with the Casimir (or sub-extensive) contribution to the CFT energy. Of further significance, this Casimir energy can be fixed, precisely, by way of the AdS/CFT duality (provided that the conformal symmetry at the boundary has properly been exploited).

This generalized Cardy entropy, or Cardy-Verlinde formula, has since been extended to a variety of asymptotically AdS spacetimes (see, for instance, Ref.[9]), as well as dynamical-boundary scenarios (see, for instance, Ref.[10]). The range of validity is quite remarkable given the “modest” scope of the original Cardy formula.

In analogy to the AdS/CFT duality, a correspondence between de Sitter (dS) spacetimes and CFTs has similarly been proposed [12]. (For earlier work along this line, see Refs.[13]-[15].) At the simplest level, any dS spacetime is related to an AdS space by a trivial reversal in the sign of the cosmological

\[\text{There has been an abundance of work in these areas. One might consult Ref.[11] for a listing of most of the relevant citations.}\]
constant. However, this sign change has significant repercussions that can hinder a quantum (or semi-classical) interpretation of dS geometries \cite{20}. For instance, dS spacetimes lack a globally timelike Killing vector, a spatial infinity, an objective observer and a string-theoretical description.

In spite of these complications, considerable progress has still been made towards establishing a dS/CFT correspondence \cite{12}-\cite{48}. The crucial points are as follows: (i) the CFT is a Euclidean one that lives on a spacelike boundary at temporal infinity and (ii) the dS cosmological horizon adopts the role played by the black hole horizon in the AdS/CFT duality. Some of the more recent studies have considered generalizing the Cardy-Verlinde formalism for a dS (bulk) scenario \cite{18,39,42,43,45,47,48}. Although such treatments have achieved only qualified success (see discussion below), there does indeed appear to be a Cardy-Verlinde-like description of the boundary entropy in a dS context.

Of particular interest to the current paper is a very recent study by Halyo \cite{47}. (For earlier, closely related works, see Refs.\cite{39,42}.) This author considered a Schwarzschild-de Sitter (SdS) spacetime and implemented a somewhat novel approach in deriving the generalized Cardy-Verlinde formula of interest.\footnote{Such studies have, in large part, been motivated by recent evidence that our universe has a positive cosmological constant \cite{49}. Thus, by investigating the dS/CFT duality, one might hope to understand “realistic” quantum gravity in a holographic way.} The premise of this approach is that a Cardy-Verlinde-like form (for the entropy of a suitable boundary theory) can be obtained with only three pieces of input: the bulk metric, the conformal symmetry on the asymptotic boundary and the Hawking temperature at the cosmological horizon. Along with this derivation, it was shown that the boundary theory can be interpreted in terms of Strominger’s realization: Euclidean time evolution in a dS bulk is dual to a renormalization group flow \cite{34}. (Also see Ref.\cite{35}.)

In the current paper, our analysis begins with a review of Haylo’s procedure in the context of an SdS-black hole bulk. We then “break new ground” by extending this treatment to a pair of interesting cases: the Reissner-Nordstrom-de Sitter (RNdS) black hole and a certain class of asymptotically de Sitter solutions lacking a black hole horizon. This latter class of “topo-
logical” de Sitter (TdS) spacetimes was first considered by Cai, Myung and Zhang [38] and has since been investigated in Refs. [42, 45, 48].

Early indicators from the cited studies suggest that TdS solutions can be used to circumvent many of the difficulties associated with the “conventional” dS/CFT correspondence. These problematic points include a negative energy on the boundary (which implies a non-unitary CFT), a failure to incorporate the thermodynamics of the (usual) black hole horizon into the CFT framework, and formal breakdowns that occur in a dynamical-boundary scenario. On the other hand, TdS solutions have the undesirable feature of a naked singularity. It is just possible, however, that there is some CFT that remains well defined while effectively describing this singular bulk. (Noting that a dual boundary theory is not necessarily sensitive to the intricate details of the corresponding bulk geometry [51].) On this basis, we propose that such solutions do indeed merit further investigation.

The rest of the paper is organized as follows. In the next section, by way of review [47], we derive a generalized form of the Cardy-Verlinde entropy for a SdS background. Also, a brief account is given on how the asymptotic boundary theory can be interpreted in terms of renormalization group flows. In Section 3, we generalize the prior formalism to accommodate a RNdS background. That is, we consider the implications (at the conformal boundary) when charge is incorporated into an asymptotically dS background. In Section 4, the analytic program is further extended to the case of a TdS bulk spacetime. We also elaborate on the distinctions that exist between these topological solutions and “conventional” asymptotically dS spacetimes. Section 5 closes with a brief summary and discussion.

2 Schwarzschild-dS/CFT Correspondence

We begin the formal analysis by considering a Euclidean CFT on the asymptotic boundary of a Schwarzschild-de Sitter (SdS) background. The primary

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4In fact, the original motivation for the study of TdS spacetimes [38] was to test an earlier conjecture on cosmological singularities [35].

5Since conformal symmetry in the bulk is broken by the presence of a black hole, a prospectively dual boundary theory is, strictly speaking, not necessarily a conformal one [34]. Nonetheless, for convenience sake, we will continue to refer to the relevant boundary theories as CFTs.
focus of this section is to derive a generalized form of the Cardy-Verlinde formula for this scenario. We again note that this is essentially a review of a derivation found in Ref. [47].

Let us first consider an \( n+2 \)-dimensional SdS metric in manifestly static coordinates [20]:

\[
ds^2_{n+2} = -h(r)dt^2 + \frac{1}{h(r)}dr^2 + r^2d\Omega^2_n, \tag{1}
\]

\[
h(r) = 1 - \frac{r^2}{L^2} - \frac{\omega_n M}{r^{n-1}}, \tag{2}
\]

\[
\omega_n = \frac{16\pi G_{n+2}}{n V_n}. \tag{3}
\]

Here, \( L \) is the curvature radius of the asymptotically dS background (i.e., \( L^{-2} = 2\Lambda/n(n+1) \), where \( \Lambda \) is the positive cosmological constant), \( d\Omega^2_n \) denotes the line element of an \( n \)-dimensional constant-curvature hypersurface of positive curvature, \( V_n \) is the volume of this hypersurface, \( G_{n+2} \) is the \( n+2 \)-dimensional Newton constant, and \( M \) is a constant of integration. \( M \) can be identified with the conserved mass of a SdS black hole and is always non-negative.

Note that a positive mass and a positively curved hypersurface are necessary conditions for the existence of an apparent black hole horizon. In fact, the allowed range of mass values is given by \( 0 \leq M \leq M_N \), where:

\[
M_N = \frac{2l^{n-1}}{(n+1)\omega_n} \left( \frac{n-1}{n+1} \right)^{\frac{n-1}{n+1}} \tag{4}
\]

is the mass of an \( n+2 \)-dimensional Nariai black hole [12]. In this Nariai case, the black hole horizon coincides with the cosmological horizon, whereas the cosmological horizon typically lies on the outside (i.e., further from the singularity at \( r = 0 \)). On the other hand, \( M = 0 \) describes a “pure” dS space having no black hole horizon; only a cosmological horizon at \( r = L \).

In general, the cosmological horizon, \( r = R \leq L \), corresponds to the largest root of \( h(r) \). This identification leads to a defining relation for the black hole mass:

\[
M = \frac{R^n}{\omega_n} - \frac{R^{n+1}}{\omega_n L^2}. \tag{5}
\]
From a holographic perspective, the quantity of immediate interest, rather than the mass of the black hole, is the corresponding excitation of gravitational energy as measured at $I^+$ (i.e., future spacelike infinity).\footnote{Asymptotically dS spacetimes actually have a pair of spacelike asymptotic boundaries: $I_+$ at future infinity and $I_-$ at past infinity. We will typically be considering the CFT at future infinity.} As discussed in Section 1, there are inherent difficulties in defining conserved charges in an asymptotically dS spacetime. Nevertheless, Balasubramanian \textit{et al.} \cite{35} have recently calculated the gravitational energy of interest by suitably adapting the Brown-York tensor \cite{53}.

Applying the results of Ref.\cite{35} and subtracting off the vacuum-energy contribution from the anomalous Casimir effect (which is non-zero when $n+2$ is odd), we obtain the following excitation energy:

$$E_{SdS} = -M.$$ \hfill (6)

Significantly, the negative sign implies that the presence of a black hole actually lowers the total bulk energy with respect to the total energy for a pure dS spacetime (for which the gravitational excitation has been calibrated to zero).

Let us now consider the asymptotic form of the SdS metric at future spacelike infinity. First, though, we note the following. For a given choice of coordinates covering an asymptotically dS spacetime, $I_+$ can be defined by the limit $\tau \to +\infty$; where $\tau$ is an appropriately defined timelike coordinate \cite{20}. In the case of our static coordinate system \cite{1}, $r$ becomes the relevant timelike coordinate (close to this asymptotic boundary), as $t$ and $r$ switch their usual roles when an observer passes through the cosmological horizon.\footnote{Since the exterior region (with respect to the cosmological horizon) is inaccessible to an internally located observer, we are referring to a hypothetical, omnipotent spectator.} Hence, the SdS metric \cite{1} at $I_+$ should be determined by way of the following limit:

$$\lim_{r \to \infty} \left[ \frac{L^2}{r^2} ds_{n+2}^2 \right] = dt^2 + L^2 d\Omega_n^2,$$ \hfill (7)

Evidently, this asymptotic limit yields a spacelike metric describing a relatively simple geometry: $\mathcal{R} \times S^n$. Moreover, the resulting metric can also be identified with that of an $n+1$-dimensional Euclidean CFT (on $I_+$). It is appropriate, however, to rescale the boundary coordinates so that the radius
of the spatial surfaces coincides with the radial distance of the cosmological horizon \([7]\). (In doing so, we are exploiting the conformal symmetry of the asymptotic boundary \([4, 5, 6]\).) That is, \(t \rightarrow tL/R\). As a consequence of this rescaling, it follows that the CFT energy should be “red shifted” from \(E_{SdS}\) by the same factor of \(L/R\).

On the basis of the above arguments and Eqs.(5,6), the CFT energy is given as follows:

\[
E_{\text{CFT}} = \frac{nC R^n}{24 L^{n+1}} \left[1 - \frac{L^2}{R^2}\right],
\]

where we have defined:

\[
C \equiv \frac{3L^nV_n}{2\pi G_{n+2}}.
\]

\(C\) can readily be identified with the Cardy-like “central charge” \([8]\) of the CFT corresponding to a pure dS spacetime \([12]\). We will follow Ref.\([47]\) and presume that \(C\) remains a fixed quantity for an asymptotically dS theory.

As it turns out, the CFT energy can be expressed in terms of a pair of separable contributions: \(E_{\text{CFT}} = E_E + E_C\). The first term, \(E_E = nC R^n/24L^{n+1}\), is strictly positive and can be identified as the extensive energy of an \(n\)-dimensional CFT gas \([7]\). It follows that the second term, \(E_C = -E_EL^2/R^2\), must be a sub-extensive contribution. Notably, \(E_C\) is always negative and can be identified with the Casimir energy of the CFT \([7]\). Keep in mind that \(E_{\text{CFT}}\) must be negative for a non-vanishing black hole mass, since \(L > R\) if \(M > 0\).

Next, let us consider the Hawking temperature associated with the cosmological horizon. Applying the well-known procedure of Gibbons and Hawking \([50, 54]\) along with Eqs.(2,5), we have:

\[
T_{SdS} = -\frac{1}{4\pi} \frac{dh}{dr}_{r=R} = \frac{R}{4\pi L^2} \left[(n+1) - (n-1)\frac{L^2}{R^2}\right].
\]

It follows that the temperature of the CFT is given as the red-shifted value of the above. That is:

\[
T_{\text{CFT}} = \frac{L}{R} T_{SdS} = \frac{1}{4\pi L} \left[(n+1) - (n-1)\frac{L^2}{R^2}\right].
\]

We will now use a standard thermodynamic relation:

\[
T_{\text{CFT}} = \left(\frac{\partial E_{\text{CFT}}}{\partial S_{\text{CFT}}}\right)_V
\]
to obtain the CFT entropy. More precisely, we begin by fixing the boundary volume $V = V_n R^n$ (i.e., fixing $R$), while treating $L$ (but not $C$) as a variable quantity. We then obtain an expression for $\delta S_{CFT}$ by dividing $\delta E_{CFT}$ with $T_{CFT}$. As it so happens, the resulting variation in entropy can be trivially integrated to yield $S_{CFT}$. Note that there is an arbitrary constant of integration, which will always be fixed to vanish. This choice can be justified with an appeal to the black hole area law (see below).

Applying the above methodology and Eqs.(8,11), we find that Eq.(12) is satisfied by the following:

$$S_{CFT} = \frac{\pi CR^n}{6L^n} = \frac{4\pi}{n} R \sqrt{E_E|E_C|}.$$  

Remarkably, this CFT entropy agrees with the Bekenstein-Hawking area law [55, 56], since it can easily be verified that $S_{CFT} = V/4G_{n+2}$. (Note that $V$ is clearly the $n+2$-dimensional generalization of a horizon surface area.) However, this should not be interpreted as a derivation of the Bekenstein-Hawking formula, but rather as a check on consistency and a demonstration that the CFT entropy is unambiguously defined.

The resulting CFT entropy can also be identified as a modified version of the Cardy formula [8]. In fact, the only explicit difference between Eq.(13) (for SdS) and the Cardy-Verlinde formula (for Schwarzschild-AdS) [7] is the need for absolute value bars. Note that the above formalism also holds for a pure dS space. In this case, one sets $R = L$ to obtain $E_{CFT} = 0$, $E_C = -E_E$, $T_{CFT} = 1/2\pi L$ and $S_{CFT} = \frac{\sqrt{c}}{24}$. (Note that $V_l$ is clearly the $n+2$-dimensional generalization of a horizon surface area.) However, this should not be interpreted as a derivation of the Bekenstein-Hawking formula, but rather as a check on consistency and a demonstration that the CFT entropy is unambiguously defined.

It is useful to rewrite this generalized Cardy-Verlinde expression (13) so that it more closely resembles the original Cardy formulation. Let us first define the following:

$$c \equiv \frac{24}{n} R|E_C|,$$

$$L_o \equiv \frac{1}{n} R E_{CFT}.$$  

The modified Cardy-Verlinde formula (13) then takes on the form:

$$S_{CFT} = 2\pi \sqrt{\frac{c}{6} \left[ L_o + \frac{c}{24} \right]}.$$  

As for the dual CFT of an (asymptotically) AdS spacetime, $L_o$ represents the product of the total energy and the radius, while $c/24$ is a shift caused by
the Casimir effect. What is of interest here are the relative signs: $L_o$ is now a negative quantity (or vanishing for pure dS), while the “Casimir shift” is now positive (since $c \geq 0$). Conversely, $L_o$ is positive and the Casimir shift is negative for the Schwarzschild-AdS case. Note that $c$ is analogous to the Cardy central charge and, as anticipated, goes to $C$ as $R \to L$ (i.e., for pure dS).

We have observed that, from a holographic perspective, the SdS “picture” is quite similar to that of its Schwarzschild-AdS counterpart. However, there are some notable issues that are specific to the SdS scenario. For instance, $E_{CFT} < 0$ is a precarious outcome, as it implies that the boundary theory fails to be unitary. (Note that this negative energy is especially problematic in a dynamical boundary scenario.) Given this non-unitarity and that the SdS spacetime lacks conformal symmetry (it is broken by the black hole), there is no reason to expect, a priori, that the bulk theory can be described by a dual CFT. Furthermore, the boundary entropy, $S_{CFT}$, is bounded from above by its value for pure dS space (given that a negative $L_o$ is vanishing in this limit). Such an upper limit on the accessible degrees of freedom would appear to be a counter-intuitive result. Finally, the modified Cardy-Verlinde formula fails to incorporate the thermodynamics of the black hole; only the cosmological horizon has been considered.

Some of these complications are effectively negated when viewed from a perspective that has been argued for by Halyo. The premise of these arguments is based, in large part, on Strominger’s identification of a dS/CFT-inspired duality between Euclidean time evolution and an appropriate renormalization group (RG) flow. We briefly summarize this point of view in the following discussion.

Let us first point out that the SdS black hole is known to be unstable; this is by virtue of the Hawking temperature being higher at the black hole horizon than at the cosmological horizon. Given this instability, it follows that the black hole mass, $M$, gradually decreases as time increases. (Accordingly, the negative boundary energy, $E_{CFT}$, gradually increases towards zero.) After the black hole completely evaporates, what remains behind is a pure dS spacetime, which is, of course, perfectly stable.

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8However, we note that, from a bulk perspective, such an upper bound on the entropy is a well-accepted quirk of asymptotically dS spacetimes.

9Also see Ref. [35]. For earlier work on RG flows in a holographic setting, consult, for instance, Refs. [31, 37, 58].
Let us now consider this evaporation process in the context of a RG flow. Strominger argues [34] that the time evolution (in the dS bulk) is expected to be dual with the reverse of a RG flow on the boundary. That is, a flow from an infrared fixed point (a conformally invariant point with a relatively low number of degrees of freedom) to an ultraviolet fixed point (comparatively high number of degrees of freedom). We can identify the CFT for pure dS as an ultraviolet fixed point, given the obvious conformal symmetry and that it corresponds to an upper bound on the entropy. It then follows that the boundary theory for a SdS black hole (a point of relatively low entropy) should evolve with time towards this pure-dS fixed point of maximal entropy. Notably, this viewpoint is in perfect agreement with the thermodynamic behavior that we discussed above.

Moreover, since the boundary theory must inevitably reach the ultraviolet fixed point, it can always be interpreted as an asymptotically conformal theory. Of further note, as time evolves (i.e., RG flows from infrared to ultraviolet) the energy on the boundary monotonically increases from a negative value towards zero. That is, the boundary theory effectively behaves in a quasi-unitary manner.

Also of interest, the Nariai black hole assumes the role of the (unstable) infrared fixed point in this RG-flow picture. This follows from the Nariai black hole having the largest allowable mass (cf. Eq.(4)) and, hence, the smallest (or most negative) allowed values for the CFT entropy and energy.

Finally, we point out that one issue remains conspicuously unresolved; namely, the failure of the generalized Cardy-Verlinde formula to account for the thermodynamics of the black hole horizon. We will have more to say on this matter in Section 4.

3 Reissner-Nordstrom-dS/CFT Correspondence

In this section, we will apply the prior techniques to the case of a Reissner-Nordstrom-de Sitter (RNdS) background. Such a spacetime represents the black hole solutions of an Einstein-Maxwell action having a positive cosmological constant.

\[^{10}\text{In this context, reverse implies that degrees of freedom are being integrated out as time devolves.}\]
We begin here with the RN\(\text{d}S\) metric for an \(n+2\)-dimensional spacetime in static coordinates \([59]\):

\[
\begin{align*}
    ds^2_{n+2} &= -u(r)dt^2 + \frac{1}{u(r)}dr^2 + r^2d\Omega_n^2,
    \quad (17)\\
    u(r) &= 1 - \frac{r^2}{L^2} - \frac{\omega_n M}{r^{n-1}} + \frac{Q^2}{r^{2n-2}},
    \quad (18)
\end{align*}
\]

where \(Q\) describes the conserved charge of the associated black hole and all other parameters are as previously defined.

Here, there are two possibilities. Either the RN\(\text{d}S\) black holes are magnetically charged or electrically charged \([60]\). Since the electrically charged black holes can lose their charge through the emission of particles (and thereby decay into SdS black holes), we will focus on the former case; thus implying that the charge is a fixed quantity.

Typically for \(n \geq 2\), there will be three positive (real) roots of \(u(r)\); with the outermost root describing a cosmological horizon, and the remaining pair describing inner and outer black hole horizons.

The allowed range of mass values (assuming that naked singularities are forbidden) can now be expressed as \(M_{\text{min}} \leq M \leq M_{\text{max}}\). Here, \(M_{\text{max}}\) corresponds to the so-called charged Nariai solution, in which case the outermost black hole horizon coincides with the cosmological horizon. Meanwhile, \(M_{\text{min}}\) corresponds to an extremal black hole (i.e., the inner and outer black hole horizons coincide). In general, these bounds are difficult to solve analytically. However, it is only necessary to know that they exist and are well defined. Keep in mind, though, that \(M_{\text{min}} > 0\) and \(M_{\text{max}} > M_N\) if \(Q^2 > 0\).

Similar to before, we can determine the location of the cosmological horizon, \(r = R\), by solving for the largest root of \(u(r)\). This identification yields the following defining relation for the black hole mass:

\[
M = \frac{R^{n-1}}{\omega_n} - \frac{R^{n+1}}{\omega_n L^2} + \frac{Q^2}{\omega_n R^{n-1}}.
\quad (19)
\]

As discussed in the prior section, one obtains the CFT energy by taking the negative of the red-shifted mass. That is, \(E_{\text{CFT}} = -ML/R\) or:

\[
E_{\text{CFT}} = \frac{nC}{24} \frac{R^n}{L^{n+1}} \left[ 1 - \frac{L^2}{R^2} + \frac{L^2 Q^2}{R^{2n}} \right].
\quad (20)
\]
Note that we have also reversed the relative sign on the charge term; this step follows from the CFT at $I_+$ being a Euclidean one \[12\] (which necessitates a complexification of charge along with time \[54\]). It is convenient to separate the CFT energy into three distinct portions: $E_{CFT} = E_E + E_C + E_Q$. The positive extensive contribution ($E_E$) and the negative Casimir (or sub-extensive contribution, $E_C$) are defined exactly as in the uncharged case. However, there is now an additional “electrostatic” contribution ($E_Q$), which depends on the charge and is clearly non-negative. This charge-induced portion, $E_Q$, can possibly be interpreted as a zero-temperature background energy; which implies that it should make no contribution to the CFT entropy \[9\]. We will demonstrate below that this is indeed the case.

As for the prior model, the CFT temperature corresponds to the redshifted value of the Hawking temperature at the cosmological horizon. So it follows that:

$$T_{CFT} = -\frac{L}{4\pi R} \frac{du}{dr} \bigg|_{r=R} = \frac{1}{4\pi L} \left[ (n + 1) - (n - 1) \frac{L^2}{R^2} + (n - 1) \frac{L^2 Q^2}{R^{2n}} \right]. \quad (21)$$

By applying the appropriate thermodynamic relation:

$$T_{CFT} = \left( \frac{\partial E_{CFT}}{\partial S_{CFT}} \right)_{V,Q} \quad (22)$$

along with Eqs.(20,21), we should be able to deduce the form of the CFT entropy. Utilizing the same approach as in the prior section (see the discussion leading up to Eq.(13)), but now fixing both $V$ and $Q$, we ultimately find the following:

$$S_{CFT} = \frac{\pi C R^n}{6L^n} = \frac{4\pi}{n} R \sqrt{E_E |E_C|}. \quad (23)$$

This CFT entropy is formally identical to that of the uncharged scenario and is, of course, in agreement with the Bekenstein-Hawking area law: $S_{CFT} = V_n R^n/4G_{n+2}$. As anticipated, the CFT entropy has no explicit knowledge about the electrostatic contribution.\[11\] This verifies the conjectured status of $E_Q$ as a zero-temperature background energy (as opposed to

\[11\]This entropy does, however, have an implicit dependence on the charge. This occurs via the location of the cosmological horizon; cf. Eq.(19).
a thermodynamic excitation). That is, the electrostatic contribution should be subtracted from the total energy before the degrees of freedom are totaled up.

As an aside, one can define a chemical potential in the usual way: \( \Phi_{CFT} \equiv \frac{\partial E_{CFT}}{\partial Q} \) (with all other parameters being fixed). This leads to the first law of thermodynamics for a constant-volume system:

\[
dE_{CFT} = T_{CFT} dS_{CFT} + \Phi_{CFT} dQ,
\]

where \( \Phi_{CFT} = nCQ/12L^{n-1}R^n \).

Finally, we point out that, because of the duplicity of the entropy expression, the arguments regarding RG flows (at the end of Section 2) will retain their validity in the case of this charged model. The only significant difference is the identity of the fixed points. For the current analysis, the ultraviolet fixed point corresponds to the (stable) extremal RNdS black hole and the infrared fixed point is described by the (unstable) charged Nariai black hole. Recall that these are the solutions of minimal and maximal allowed mass, respectively.

4 Topological-dS/CFT Correspondence

In this section, we consider a certain brand of asymptotically dS solutions that were first proposed by Cai, Myung and Zhang [38] and received further consideration in Refs. [42, 45, 48]. These topological-de Sitter (TdS) solutions can be obtained with a sign reversal of the mass term in the SdS metric. If the mass is kept positive, this reversal eliminates the black hole horizon and, hence, gives rise to an undesirable naked singularity. However, it is within the realm of possibility that a suitably dual CFT remains well defined even in the presence of the bulk singularity. In this regard, we point out that such a boundary theory is not necessarily sensitive to the fine-grained details of its corresponding background. (See, for instance, Ref.[51].)

Although the above argument is rather speculative, TdS solutions provide a natural means for resolving some of the difficulties associated with the dS/CFT correspondence. For instance, the issue of incorporating the thermodynamics of a dS-black hole horizon (into a CFT framework) would trivially be negated. Moreover, the “mass reversal” has been shown to induce a positive CFT energy on the asymptotic boundary. In view of these
desirable features, we argue that such solutions deserve further investigation and proceed on this basis.

Let us now apply the prior treatment to the case of a TdS background geometry. The TdS metric, for an \( n+2 \)-dimensional spacetime, can be expressed in the following static form \[38\]:

\[
\begin{align*}
    ds^2_{n+2} &= -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2d\Omega^2_n, \\
    f(r) &= k - \frac{r^2}{L^2} + \frac{\omega_n M}{r^{n-1}}.
\end{align*}
\]  

(25)

(26)

Here, \( M \) is a mass parameter that is to be regarded as a non-negative quantity (in spite of the sign reversal from Eq.(2)), \( k \) is a constant of integration that describes the geometry of the cosmological horizon, and all other parameters are as previously defined. Without loss of generality, \( k \) can be set to +1,0 or -1; describing a spatial slicing that is respectively elliptic, flat and hyperbolic.\[12\] It is interesting to note that the mass parameter can now increase without bound.

There is only one positive (real) root of \( f(r) \), and this locates the position of the cosmological horizon, \( R \). On this basis, the following relation for the mass parameter can be obtained:

\[
M = \frac{R^{n+1}}{\omega_n L^2} - k \frac{R^{n-1}}{\omega_n}.
\]  

(27)

As usual, we are particularly interested in the gravitational excitation energy as measured at \( I_+ \). By applying the formalism of Refs.\[35\], Cai et al.\[38\] obtained the following intuitive result for this energy:

\[ E_{T_{dS}} = M. \]  

(28)

Note that the anomalous Casimir contribution has, as usual, been subtracted off. Significantly, a non-vanishing mass now induces an excitation of positive gravitational energy.

\[\text{\[12\]By contrast, in the SdS model, the presumed existence of a black hole horizon necessitates the choice of } k = +1.\]
As priorly discussed, we obtain the CFT energy by red shifting the gravitational excitation with a factor of $L/R$. That is:

$$E_{CFT} = \frac{nC}{24} \frac{R^n}{L^{n+1}} \left[ 1 - k \frac{L^2}{R^2} \right].$$

(29)

Note that the initial conditions require this energy to be a non-negative quantity.

Once again, we are able to express this CFT (total) energy as a sum of identifiable contributions: $E_{CFT} = E_E + E_C$. The extensive term, $E_E = nCR^n/24L^{n+1}$, is always positive (as was also found for the SdS theory). Meanwhile, the Casimir (sub-extensive) contribution is given by:

$$E_C = -k \frac{L^2}{R^2} E_E$$

(30)

and can be positive, negative or vanishing; depending on the choice of $k$. (By contrast, the SdS Casimir energy is always negative.)

As demonstrated before, the CFT temperature is obtainable as the red-shifted value of the Hawking temperature at the cosmological horizon. More explicitly:

$$T_{CFT} = \frac{L}{4\pi R} \left. \frac{df}{dr} \right|_{r=R} = \frac{1}{4\pi L} \left[ (n+1) - (n-1)k \frac{L^2}{R^2} \right].$$

(31)

For the purpose of calculating the CFT entropy, let us now reconsider the following relation:

$$T_{CFT} = \left( \frac{\partial E_{CFT}}{\partial S_{CFT}} \right)_V.$$ 

(32)

Adopting the approach of the prior sections, we obtain:

$$S_{CFT} = \frac{\pi CR^n}{6L^n} = \frac{4\pi}{n} R \sqrt{E_E \frac{E_C}{k}}.$$ 

(33)

Note that the quantity $|k^{-1}E_C|$ translates to $nCR^{n-2}/24L^{n-1} > 0$, regardless of the choice for $k$. That is, the CFT entropy contains no explicit information about this TdS geometrical parameter. (Although $k$ does have implicit influence via $R$.)
As found for the previous models, $S_{CFT}$ agrees with the anticipated Bekenstein-Hawking form: $V/4G_{n+2}$. Moreover, the entropy can be identified as a (slightly) modified version of the Cardy-Verlinde formula [8, 7]. Note that the above formalism also holds for a pure dS space. In this case, one sets $R = L$ and $k = 1$ to obtain $E_{CFT} = 0$, $E_C = -E_E$, $T_{CFT} = 1/2\pi L$ and $S_{CFT} = V/nL^n/4G_{n+2}$.

As for the SdS model, it is informative to re-express $S_{CFT}$ into a form that resembles the original Cardy relation. With this in mind, let us define:

$$c \equiv \frac{24}{n} R \left| \frac{E_C}{k} \right|, \quad (34)$$

$$L_o \equiv \frac{1}{n} R E_{CFT}. \quad (35)$$

The modified Cardy-Verlinde formula (33) then becomes:

$$S_{CFT} = 2\pi \sqrt{\frac{c}{6}} \left[ L_o + k \frac{c}{24} \right]. \quad (36)$$

Notably, this relation is closer to the original Cardy-Verlinde formulation [8, 7] than was found for the SdS analysis [14]. First of all, $L_o$ is a positive quantity for the TdS-bulk case. (Conversely, it is negative for the SdS scenario.) Secondly, consider the relative shift as induced by the Casimir effect. Given a dS context, this can only be negative for a TdS-bulk spacetime. This occurs when $k = -1$, which happens to be the condition for a positive Casimir energy (cf. Eq. (30)). In this sense, it is the case of a hyperbolic horizon geometry which most closely resembles the Schwarzschild-AdS “template”.

Let us now recall the discussion on RG flows at the end of Section 2. For the current TdS scenario, such an interpretation is hindered by difficulties in identifying the relevant fixed points. One notable exception being the infrared fixed point for the $k = 1$ case, as the associated pure-dS spacetime (i.e., the $M = 0$ solution) conveniently imposes a lower bound on $S_{CFT}$. Alternatively, the massless limit is not so well defined if $k = 0$ or $-1$. For these topologies, the cosmological horizon disappears when (or before) $M$ goes to zero. However, the existence of such exotic bulk geometries is not necessarily an issue, as we argue below.

In interpreting the boundary theory of a TdS background, we first take note of the key finding: the CFT (total) energy is a strictly non-negative
quantity. This implies that the boundary theory is a unitary one that is capable of describing bulk spacetime events; for instance, the emission of Hawking radiation from the cosmological horizon.

Let us now consider the following picture. In the distant past, the TdS bulk is assumed to be in a relatively massive state (i.e., large $M$ and, thus, large $S_{CFT}$). It follows that the cosmological horizon will necessarily radiate until it achieves a state of thermal equilibrium with the emitted radiation. (Such an equilibrium state follows by virtue of there being no black hole horizon in this model.) When this occurs, there will be zero net radiation and the bulk will have settled into a state of comparatively low mass (i.e., small $M$ and, thus, small $S_{CFT}$). However, we conjecture that this “final” value of mass is not so low that the cosmological horizon (in the case of $k = 0$ or $-1$) will be jeopardized (as discussed above). To put it another way, time evolution in the bulk is dual with a RG flow from ultraviolet to infrared boundary points that are at least effectively fixed.

Along with the apparent unitarity of the boundary theory, the above picture is supported by an intuitively satisfying correspondence: the boundary degrees of freedom are directly correlated with the gravitational mass in the bulk. This is at least the case for $k = -1$ and $k = 0$, where it is clear that an increasing (decreasing) $M$ always corresponds to an increasing (decreasing) $S_{CFT}$.

On the other hand, this monotonic mass-entropy trend is not so evident for the case of $k = +1$. With this topological choice, a numerical analysis will likely be required to establish the true mass-entropy relationship. However, the negative Casimir energy for $k = +1$ suggests that this may be the “least physical” case of the three. Interestingly, peculiar behavior linked to a negative Casimir energy has been detected for another background: a “topological” AdS black hole with $k = -1$. 

In some sense, TdS solutions are “mirror images” of their AdS analogues, with the cosmological horizon replacing the AdS black hole horizon. Furthermore, in view of the relative sign of the Casimir energy, a TdS hyperbolic horizon ($k = -1$) should be regarded as the “reflection” of an AdS spherical horizon ($k = +1$) and vice versa.

\footnote{From an AdS perspective, topological refers to asymptotically AdS-black hole solutions having either a hyperbolic ($k = -1$) or flat ($k = 0$) hypersurface \cite{61}. Note that $k = +1$ describes the “usual” Schwarzschild-AdS black hole.}
5 Conclusion

In summary, we have considered generalized forms of the Cardy-Verlinde entropy formula \[8, 7\] in the context of asymptotically de Sitter spacetimes. Towards this end, we have applied a program of study that is based on a prior work by Halyo \[47\]. The premise of this methodology is that one can derive the entropy of an appropriately dual CFT by using only the bulk metric, the asymptotic conformal symmetry and the Hawking radiation of the cosmological horizon \[50\]. The validity of this approach was substantiated by the reproduction (in all examined cases) of the Bekenstein-Hawking area law, which is expected to be a fundamental feature of any quantum theory of gravity \[55, 56\].

Our analysis considered three distinct cases having dS asymptotics. First of all, we examined the Schwarzschild-de Sitter black hole, which was essentially a review of the originating work \[47\]. In particular, we derived the entropy of the dual CFT, and demonstrated that it adopts a Cardy-Verlinde-like form. We also argued that, in spite of apparent conceptual difficulties (such as a non-unitary CFT), the proposed SdS/CFT duality can still fit into a self-consistent framework. These arguments were based on an observation by Strominger \[34\]: Euclidean time evolution in a de Sitter space is dual to a renormalization group flow from an infrared to an ultraviolet fixed point.

Secondly, we considered the Reissner-Nordstrom-de Sitter black hole. We found that the addition of charge into the model results in an “electrostatic” contribution to the CFT energy. However, the charge made no explicit contribution to the boundary entropy, which implies that the electrostatic energy can be interpreted as a zero-temperature background (rather than a thermodynamic excitation). This outcome was anticipated \[9\], but not necessarily obvious.

Finally, we applied the analytic program to a special class of “topological” de Sitter solutions. Such solutions correspond to a sign reversal in the mass term of the corresponding Schwarzschild-dS scenario. For this case, the CFT entropy most closely resembled the original Cardy-Verlinde formulation (for AdS spacetimes) \[8, 7\]. Furthermore, the boundary energy was found to be positive, in stark contrast to the priorly studied cases. This is a desirable outcome, as it suggests a unitary theory at the conformal boundary. With this property of unitarity, we were able to conceptualize a duality between an apparent flow in the boundary theory and time evolution in the TdS bulk.
Let us further point out two notable failures of SdS-type models (with regard to the proposed holographic duality) that TdS solutions can seemingly resolve. These are as follows: (i) the issue of how to incorporate the properties of the black hole horizon into the CFT thermodynamic relations (there is no such horizon for TdS spacetimes) and (ii) a negative CFT energy can become especially problematic in the context of a dynamic-boundary scenario.

In view of the above considerations, TdS spacetimes appear to be promising candidates for the realization of a dS/CFT correspondence. However, we again stress that TdS solutions have a naked singularity as an inevitable consequence. It is possible that a well-defined dual CFT exists in spite of this singular behavior; however, until this existence can be established, the outcomes of Section 4 should be regarded as speculative.

In conclusion, a holographic interpretation of asymptotically de Sitter spacetimes continues to have a few unresolved issues; whether it is viewed from the perspective of a black hole (SdS, RNdS) or a topological (TdS) bulk. Nonetheless, in weighing all the evidence, we find that the results of this analysis come out in support of the proposed dS/CFT correspondence.

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