On Feynman’s Approach to the Foundations of Gauge Theory

M. C. Land‡, N. Shnerb§, and L. P. Horwitz‡§

‡School of Physics and Astronomy
Raymond and Beverly Sackler Faculty of Exact Sciences
Tel Aviv University, Ramat Aviv, Israel

§Department of Physics
Bar-Ilan University
Ramat Gan, Israel

Abstract

In 1948, Feynman showed Dyson how the Lorentz force law and homogeneous Maxwell equations could be derived from commutation relations among Euclidean coordinates and velocities, without reference to an action or variational principle. When Dyson published the work in 1990, several authors noted that the derived equations have only Galilean symmetry and so are not actually the Maxwell theory. In particular, Hojman and Shepley proved that the existence of commutation relations is a strong assumption, sufficient to determine the corresponding action, which for Feynman’s derivation is of Newtonian form. In a recent paper, Tanimura generalized Feynman’s derivation to a Lorentz covariant form with scalar evolution parameter, and obtained an expression for the Lorentz force which appears to be consistent with relativistic kinematics and relates the force to the Maxwell field in the usual manner. However, Tanimura’s derivation does not lead to the usual Maxwell theory either, because the force equation depends on a fifth (scalar) electromagnetic potential, and the invariant evolution parameter cannot be consistently identified with the proper time of the particle motion. Moreover, the derivation cannot be made reparameterization invariant; the scalar potential causes violations of the mass-shell constraint which this invariance should guarantee.

In this paper, we examine Tanimura’s derivation in the framework of the proper time method in relativistic mechanics, and use the technique of Hojman and Shepley to study the unconstrained commutation relations. We show that Tanimura’s result then corresponds to the five-dimensional electromagnetic theory previously derived from a
Stueckelberg-type quantum theory in which one gauges the invariant parameter in the proper time method. This theory provides the final step in Feynman’s program of deriving the Maxwell theory from commutation relations; the Maxwell theory emerges as the “correlation limit” of a more general gauge theory, in which it is properly contained.

1 Introduction

In 1990, Dyson [1] published a derivation, originally due to Feynman, of the Lorentz force law in the form

\[ m \frac{d^2 x^i}{dt^2} = E^i(t, x) + \epsilon^{ijk} \frac{dx_j}{dt} H_k(t, x), \]  

and of the homogeneous Maxwell equations

\[ \nabla \cdot H = 0 \quad \nabla \times E + \frac{\partial}{\partial t} H = 0, \]  

which assumes only the canonical commutation relations among Euclidean position and velocity,

\[ \begin{bmatrix} x^i \, x^j \end{bmatrix} = 0, \]  

\[ m \begin{bmatrix} x^i \, \dot{x}^j \end{bmatrix} = i \hbar \delta^{ij}, \]  

and Newton’s second law

\[ m \ddot{x}^i = F^i(t, x, \dot{x}), \]  

where \( \dot{x}^i = dx^i/dt \) and \( i, j = 1, 2, 3 \). Feynman’s program of deriving the Maxwell theory without reference to either canonical momenta or a Lagrangian was based on the hypothesis that commutation relations between \( x \) and \( \dot{x} \) form a fundamental basis for mechanics, but constitute a weaker set of initial assumptions, and might therefore lead to a more general theory. Dyson explained that Feynman had shown him this derivation in October 1948 but never published it because, “He was exploring possible alternatives to the standard theory;” which this proof, by producing the usual equations, failed to provide.

Dyson’s publication of Feynman’s derivation provoked a debate in the literature [2, 3, 4, 5], with several papers challenging the identification of the derived equations as Maxwell’s theory. These authors argue that although the Lorentz covariance of (1.2) may be assumed \textit{a posteriori}, Lorentz covariance of the inhomogeneous Maxwell equations conflicts with the
Euclidean assumptions of the “proof.” In fact, as pointed out by [3] the homogeneous equations may be regarded as Galilean or Lorentz covariant; however, the inhomogeneous Maxwell equations are not Galilean covariant and the Lorentz force equations are not Lorentz covariant.

This conflict of symmetries was demonstrated in a stronger fashion by Hojman and Shepley [4] and by Hughes [5] who place Feynman’s program in the context of the inverse problem of the calculus of variations and demonstrate that conditions (1.3) and (1.4) are sufficient to establish the self-adjointness of the differential equations (1.3). Given self-adjointness, it is well known [6] that the right hand side of (1.1) is the most general admissible form for \( F^i(t, x, \dot{x}) \), and that this system of differential equations is equivalent to the Lagrangian formulation

\[
L = \frac{1}{2} m \delta_{ij} \dot{x}^i \dot{x}^j + \delta_{ij} \dot{x}^i A^j(t, x) + A_0(t, x)
\]

where

\[
H = \nabla \times A \quad \mathbf{E} = - \left( \frac{\partial}{\partial t} A + \nabla A_0 \right).
\]

The potentials \( A_0 \) and \( A \) must exist by virtue of (1.2). Hence, a Lagrangian and well-defined canonical momenta exist, and these are of Galilean form. Furthermore, the inhomogeneous equations of Maxwell form, which can be obtained by adding kinetic terms for the fields, would not be Lorentz covariant since the source current is not Lorentz covariant. Therefore, Feynman’s argument essentially re-derives the conditions on velocity dependent forces in nonrelativistic mechanics. These conditions are essential to the self-adjointness of the differential equations (a consequence of the commutation relations), but do not reveal the full dynamical structure implied by the commutation relations.

In an attempt to achieve a properly relativistic version of Feynman’s derivation, Tanimura [7] has presented a modified argument which maintains manifest Lorentz covariance (in \( d \)-dimensional spacetime) throughout. Employing the approach which is the basis of the “proper time formalism” [8, 9, 10, 11, 12], Tanimura assumes the commutation relations

\[
\left[ x^\mu, x^\nu \right] = 0, \quad m \left[ x^\mu, \dot{x}^\nu \right] = -i\hbar g^\mu\nu(x),
\]

and the force law

\[
m \ddot{x}^\mu = F^\mu(x, \dot{x}),
\]
where $\mu, \nu = 0, \ldots, d - 1$, and $\dot{x} = \frac{dx}{d\tau}$, where $\tau$ is a Lorentz invariant parameter. In analogy to Feynman’s derivation, he obtains expressions which are formally similar to the usual covariant form of the Lorentz force in curved space

$$m\ddot{x}^\mu = F^\mu(x, \dot{x}) = G^\mu(x) + F^\mu_\nu(x) \dot{x}^\nu - m\Gamma^\mu_\nu_\rho \dot{x}^\nu \dot{x}^\rho,$$

(1.11)

where $\Gamma^\mu_\nu_\rho$ is the Levi-Civita connection

$$\Gamma^\mu_\nu_\rho = \frac{1}{2} (\partial_\rho g^\mu_\nu + \partial_\nu g^\mu_\rho - \partial_\mu g^\nu_\rho)$$

(1.12)

(so that the terms proportional to $m$ are the covariant derivative of the velocity), and the homogeneous Maxwell equations,

$$\partial_\mu F^\nu_\rho + \partial_\nu F^\rho_\mu + \partial_\rho F^\mu_\nu = 0.$$  

(1.13)

The vector field $G^\mu(x)$ satisfies

$$\partial_\mu G^\nu - \partial_\nu G^\mu = 0.$$  

(1.14)

From (1.13) and (1.14) one sees that $F^\mu_\nu$ and $G^\mu$ may be derived from a $d$-vector and a scalar potential, respectively.

However, the expressions which Tanimura derives are not the Maxwell theory, either. In addition to the usual antisymmetric second rank tensor $F^\mu_\nu$, the electromagnetic field includes the “new” $d$-vector field $G^\mu$, and both of these fields contribute to the Lorentz force. But, $F^\mu_\nu$ and $G^\mu$ are completely decoupled in the field equations, so that while $G^\mu(x)$ could act non-trivially on particle motions, no interaction with the usual electromagnetic field is available to control the scalar potential. Moreover, Tanimura finds that the variable $\tau$ introduced to parameterize the world lines cannot be taken to be the proper time of the motion (the proper time constraint $\dot{x}^\mu \dot{x}_\mu = 1$ and (1.3) imply $\dot{x}^\mu = 0$) and must be treated as a “new” Lorentz scalar, independent of the phase space coordinates. Similarly, the derivation is not invariant under general reparameterization — only affine transformations of $\tau$ preserve the structure of the equations. Since these differences from standard Maxwell electrodynamics originate in the framework of the proper time method, they merit further study and explanation.

In this paper, we shall show that the appearance of a “new” scalar potential and a “new” scalar time are necessary and related consequences of the unconstrained commutation relations assumed in the formulation of the problem. By assuming, in (1.8) and (1.3), that the
2\(d\) components of position and velocity are independent, Tanimura defines an unconstrained canonical problem in \(d + 1\)-dimensions with an implicit symplectic structure analogous to that of the Newtonian problem posed by Feynman. Applying the technique of Hojman and Shepley to the relativistic case, we shall construct the equivalent Lagrangian and Hamiltonian formulations of this canonical problem, which turn out to be naturally Lorentz and gauge invariant. Since the form of the Lagrangian is determined by the commutation relations, any constraints imposed on the system subsequently must be consistent with (1.8) and (1.9). Thus, if we require that the parameter \(\tau\) be \textit{a priori} proportional to the proper time, then the relation

\[
\dot{x}^\mu \dot{x}_\mu = (ds/d\tau)^2 = \text{constant}
\]  

would be constitute a constraint on the velocities, contradicting the assumptions of the problem. Similarly, the validity of operations of the form

\[
\frac{d}{d\tau} [x^\mu(\tau), \dot{x}^\mu(\tau)] = [\dot{x}^\mu(\tau), \dot{x}^\mu(\tau)] + [x^\mu(\tau), \ddot{x}^\mu(\tau)]
\]

(a central step in the Feynman and Tanimura derivations) becomes questionable when \(\tau\) is not completely independent of the coordinates and velocities. For the same reasons, general reparameterization invariance is inconsistent with equations (1.8) and (1.9), because such invariance corresponds to a constraint on the phase space \([13]\). Following Hojman and Shepley, we shall be led to an action in which reparameterization invariance is clearly absent. We will further demonstrate that the “new” gauge degree of freedom, which appears through the field \(G^\mu\), acts as a compensation field for the dependence of local gauge transformations on the parameter \(\tau\), the “new” dimension independent of the spacetime variables. So, just as in the Newtonian problem, it is the dimension of the dynamical problem which determines the number of gauge degrees of freedom. Loosely speaking, one may think of the proper time constraint in the conventional Maxwell theory as removing the additional gauge degree of freedom (we will sharpen this point below).

The problem of implementing the canonical commutation relations

\[
[x^\mu, p_\nu] = -i\hbar \delta^\mu_\nu
\]

on the 8 dimensional phase space coordinates is an old one, whose principal difficulties are clearly seen in the course of Tanimura’s derivation. Since the observed time is a coordinate in Minkowski space, it may not be used as a parameter of system evolution (in light of the
Feynman-Stueckelberg interpretation of negative energy states as propagating backward in time, the motion need not even be monotonic in $t$). But if one parameterizes the coordinates with a Lorentz invariant time (which by the arguments of the previous paragraph, may not be identified with the proper time of the phase space) and maintains the notion of a definite particle mass, then one must adequately handle the proper time (or mass-shell) constraint. In the original formulation of the so-called proper time method, the constraint was applied $a$ posteriori to the solutions of the unconstrained problem, which is permissible as long as the interactions preserve the mass-shell dynamically (we shall examine this point below). In more recent formulations, the constrained theory is rewritten in a form in which a Lagrange multiplier enforces the constraint dynamically. In both of these methods, one associates with the system a manifestly covariant canonical mechanics with invariant evolution parameter, permitting the application of techniques from nonrelativistic mechanics. In addition, as pointed out by Schwinger, since the physical interactions are independent of the evolution parameter, the proper time method preserves the symmetries of the system.

In a different approach to the proper time formalism, introduced by Horwitz and Piron, the invariant evolution parameter $\tau$ is regarded as a true physical time, playing the role of the Newtonian time in nonrelativistic mechanics. One is then led to a symplectic mechanics in which manifest Poincaré covariance plays the role of Galilean covariance in Newtonian mechanics (see also [21]). In this theory, the proper time relation is not a constraint at all. The value of $\dot{x}^\mu\dot{x}_\mu$ is a dynamical quantity; it may be constant only for appropriate (for example, electromagnetic) interactions. The relaxation of the constraint permits the consideration of interactions of a more general type, and in particular allows the construction of consistent relativistic potential models. The Hamiltonian form of this mechanics leads naturally to a Schrödinger type equation, which has been solved for the relativistic bound state and scattering problems. Arguing that local gauge invariance of the Schrödinger equation should include gauge transformations dependent on the evolution parameter, Sa’ad, Horwitz, and Arshansky introduced a compensation field associated with the invariant time $\tau$. This new potential leads to an off-shell electromagnetic theory (so called because the interactions explicitly take the electromagnetic fields off mass shell), which nevertheless has the Maxwell theory as its ($\tau$-static) equilibrium limit. The quantum field theory of off-shell electromagnetism has been developed and provides a basis for the
empirical evaluation of this structure [25, 26, 27].

We examine here Tanimura’s work in the context of the proper time method. We will show that the force and field equations obtained by Tanimura are just those of the off-shell theory. By adapting the techniques of Hojman and Shepley, we will demonstrate that Tanimura’s assumptions lead to the Lagrangian of off-shell electromagnetism. Thus, while Tanimura’s derivation does not lead directly to the Maxwell theory, it does lead to a proper generalization of Maxwell’s electromagnetism, which goes over to the Maxwell theory in a systematic manner, and therefore fulfills Feynman’s program of providing a path from the canonical commutation relations to Maxwell’s theory, as well as a more general theory.

That Tanimura’s derivation leads to a set of equations associated with an off-shell dynamical theory leads to an interesting connection between commutation relations and gauge degrees of freedom. This connection is of particular importance as the proper time method is increasingly used with interactions depending explicitly on the proper time parameter [28]. As we shall demonstrate below, such interactions require that the “new” gauge field associated with the proper time dimension, possess a non-trivial relationship to the usual gauge fields and are inconsistent with the requirements of on-shell dynamics.

In Section 2, we review Tanimura’s covariant derivation in curved spacetime, and obtain equations (1.11), (1.13) and (1.14) from equations (1.8) and (1.9). In Section 3, we provide a brief review of the inverse problem in the calculus of variations and the work of Hojman and Shepley. We use these results to obtain Tanimura’s equations from self-adjointness. In Section 4, we present the theory of off-shell electromagnetism as the local gauge theory of covariant quantum mechanics with invariant evolution parameter, and shall show that it is precisely the theory derived in Sections 2 and 3. We then show how the theory differs from Maxwell electrodynamics as a dynamic theory, but reduces to it in the \( \tau \)-static limit. In the context of the off-shell theory, we will discuss the related issues of constraints and gauge freedom. In Section 5, the presentation of Section 3 is generalized to the case of non-Abelian gauge fields, and the results are compared with the equations obtained by Tanimura.
2 The Lorentz Force in Curved Spacetime

We now review the derivation given by Tanimura [7] in a pseudo-Riemannian manifold. In this theory, we treat the event as a point in the \(d\)-dimensional curved spacetime whose coordinates \(x^\mu(\tau)\), \((\mu = 0, 1, \ldots, d - 1)\) evolve according to the invariant parameter \(\tau\). This approach has the advantage that the conjugate velocities are derivatives of the event coordinates with respect to \(\tau\), a well defined procedure even in curved spacetime [29].

We consider the metric \(g_{\mu\nu}(x)\) (which becomes \(\eta_{\mu\nu} = \text{diag}(+1, -1, \ldots, -1)\) in flat space), so that the coordinates and velocities \(\dot{x}^\mu(\tau)\) are assumed to satisfy the commutation relations

\[
[x^\mu, x^\nu] = 0 \tag{2.1}
\]

\[
m[x^\mu, \dot{x}^\nu] = -i\hbar g^{\mu\nu}(x) \tag{2.2}
\]

and the equations of motion are

\[
m\ddot{x}^\mu = F^\mu(\tau, x, \dot{x}). \tag{2.3}
\]

We regard \(x^\mu(\tau)\), its \(\tau\)-derivatives and functions of these quantities as quantum operators in a Heisenberg picture. The field equations and the Lorentz force may then be interpreted, in the Ehrenfest sense, as relations among the expectation values which correspond to relations among classical quantities. Generalizing Tanimura, we allow the force \(F^\mu(\tau, x, \dot{x})\) to be a function of \(\tau\). Equation (2.2) implies that for any function \(q(x)\)

\[
[\dot{x}^\mu, q(x)] = \frac{i\hbar}{m} \frac{\partial q}{\partial x^\mu}. \tag{2.4}
\]

We differentiate (2.2) with respect to \(\tau\)

\[
m[\ddot{x}^\mu, \dot{x}^\nu] + m[x^\mu, \ddot{x}^\nu] = -i\hbar \partial_\rho g^{\mu\nu}(x)\dot{x}^\rho \tag{2.5}
\]

and define \(W^{\mu\nu} = -W^{\nu\mu}\) by

\[
W^{\mu\nu} = -\frac{m^2}{i\hbar} [\dot{x}^\mu, \dot{x}^\nu]. \tag{2.6}
\]

From the Jacobi identity,

\[
[x^\lambda, [\dot{x}^\mu, \dot{x}^\nu]] + [\dot{x}^\mu, [x^\nu, x^\lambda]] + [\dot{x}^\nu, [x^\lambda, \dot{x}^\mu]] = 0 \tag{2.7}
\]
and (2.2), we show that

\[
[x^\lambda, W^{\mu \nu}] = -\frac{m^2}{i\hbar} [x^\lambda, [\dot{x}^\mu, \dot{x}^\nu]] \\
= -\frac{m^2}{i\hbar} \left( [[x^\lambda, \dot{x}^\mu], \dot{x}^\nu] + [\dot{x}^\mu, [x^\lambda, \dot{x}^\nu]] \right) \\
= m\left( [g^\lambda{}^\mu, \dot{x}^\nu] + [\dot{x}^\mu, g^\lambda{}^\nu] \right) \\
= -i\hbar(\partial^\nu g^\lambda{}^\mu - \partial^\mu g^\lambda{}^\nu).
\]

(2.8)

So, defining \( F^{\mu \nu} = -F^{\nu \mu} \) by

\[
F^{\mu \nu} = W^{\mu \nu} - m\langle (\partial^\nu g^{\lambda}{}^\mu - \partial^\mu g^{\lambda}{}^\nu)\dot{x}^\lambda \rangle
\]

(2.9)

where the brackets \( \langle \ldots \rangle \) represent Weyl ordering of the non-commuting operators, we find from (2.8) and (2.9)

\[
[x^\sigma, F^{\mu \nu}] = 0,
\]

(2.10)

which shows that \( F^{\mu \nu} \) is independent of \( \dot{x} \). When lowering indices, we define

\[
\dot{x}_\mu = \langle g_{\mu \nu}(x)\dot{x}^\nu \rangle.
\]

(2.11)

With the identity

\[
-\frac{m^2}{i\hbar} [\dot{x}_\mu, \dot{x}_\nu] = -\frac{m^2}{i\hbar} \langle g_{\mu \alpha}\dot{x}^\alpha, \langle g_{\nu \beta}\dot{x}^\beta \rangle \rangle = \langle g_{\mu \alpha}g_{\nu \beta}W^{\alpha \beta} \rangle - m\langle \partial_\mu \langle g_{\nu \alpha}\dot{x}^\alpha \rangle - \partial_\nu \langle g_{\mu \beta}\dot{x}^\beta \rangle \rangle
\]

(2.12)

one sees that

\[
F_{\mu \nu} = g_{\mu \alpha}g_{\nu \beta}F^{\alpha \beta} = -\frac{m^2}{i\hbar} [\dot{x}_{\mu}, \dot{x}_{\nu}]
\]

(2.13)

and the Jacobi identity then gives

\[
\partial_\mu F_{\nu \rho} + \partial_\nu F_{\rho \mu} + \partial_\rho F_{\mu \nu} = 0.
\]

(2.14)

Rearranging equation (2.5) and using (2.6),

we see that

\[
m[x^\mu, \ddot{x}^\nu] = \frac{i\hbar}{m} F^{\mu \nu} + 2i\hbar\langle \Gamma^{\mu \nu \lambda} \dot{x}_\lambda \rangle
\]

(2.15)

where

\[
\Gamma^{\nu \lambda \mu} = -\frac{1}{2}\left( \partial^\mu g^{\lambda \nu} + \partial^\lambda g^{\mu \nu} - \partial^\nu g^{\lambda \mu} \right)
\]

(2.16)
is the Levi-Civita connection. Let us define $G^\mu$ through the equation

$$F^\mu = m\ddot{x}^\mu = G^\mu(x, \dot{x}, \tau) + \langle F^{\mu\nu} \dot{x}_\nu \rangle - m\langle \Gamma^{\mu\lambda\nu} \dot{x}_\lambda \dot{x}_\nu \rangle.$$ \hspace{1cm} (2.17)

Then,

$$[x^\lambda, G^\mu] = [x^\lambda, F^\mu] - \Gamma^{\mu\nu\rho} [x^\lambda, \dot{x}_\nu] + m \Gamma^{\mu\rho} [x^\lambda, \dot{x}_\rho] = \frac{i\hbar}{m} F^{\lambda\mu} + 2 i\hbar \langle \Gamma^{\mu\rho} \dot{x}_\rho \rangle + \frac{i\hbar}{m} F^{\mu\nu} \delta^\lambda_\nu - i\hbar \langle \Gamma^{\mu\rho} \delta^\lambda_\rho \dot{x}_\rho + \Gamma^{\mu\rho} \dot{x}_\nu \delta^\lambda_\rho \rangle = 0,$$ \hspace{1cm} (2.18)

so that $G^\mu$ is independent of $\dot{x}$. We may then define the force as

$$G^\mu + \langle F^{\mu\nu} \dot{x}_\nu \rangle = m[\ddot{x}^\mu + \langle \Gamma^{\mu\lambda\nu} \dot{x}_\lambda \dot{x}_\nu \rangle] = m\frac{D\dot{x}^\mu}{D\tau}.$$ \hspace{1cm} (2.19)

Since

$$m\ddot{x}^\mu = m\frac{d}{d\tau} \langle g^{\mu\nu} \dot{x}_\nu \rangle,$$ \hspace{1cm} (2.20)

when we lower the index of $G^\mu$ (by (2.18), a tensor) we find that

$$G_\nu = g_{\nu\alpha} F^\alpha - \langle g_{\nu\alpha} F^{\alpha\beta} \dot{x}_\beta \rangle + m\langle g_{\nu\alpha} \Gamma^{\alpha\beta\gamma} \dot{x}_\beta \dot{x}_\gamma \rangle.$$ \hspace{1cm} (2.21)

We write the first term on the right hand side of (2.21) in the form

$$g_{\nu\alpha} F^\alpha = m \langle g_{\nu\alpha} \ddot{x}^\alpha \rangle = m \frac{d}{d\tau} \langle g^{\alpha\beta} \dot{x}_\beta \rangle = m \langle g_{\nu\alpha} g^{\alpha\beta} \dot{x}_\beta + g_{\nu\alpha} \partial^\gamma g^{\alpha\beta} \dot{x}_\beta \dot{x}_\gamma \rangle = m \dot{x}_\nu + m \langle g_{\nu\alpha} \partial^\gamma g^{\alpha\beta} \dot{x}_\beta \dot{x}_\gamma \rangle.$$ \hspace{1cm} (2.22)

Since the indices $\beta$ and $\gamma$ of $\partial^\gamma g^{\alpha\beta}$ occur in (2.22) in symmetric combination, we may write

$$\frac{1}{2} (\partial^\gamma g^{\alpha\beta} + \partial^\beta g^{\alpha\gamma}) = -\Gamma^{\alpha\beta\gamma} + \frac{1}{2} \partial^\alpha g^{\beta\gamma}.$$ \hspace{1cm} (2.23)

Equations (2.21), (2.22), and (2.23) imply

$$G_\nu = m \ddot{x}_\nu + \frac{1}{2} m \langle g_{\nu\alpha} \partial^\alpha g^{\beta\gamma} \dot{x}_\beta \dot{x}_\gamma \rangle - \langle g_{\nu\alpha} F^{\alpha\beta} \dot{x}_\beta \rangle = m \ddot{x}_\nu + \frac{1}{2} m \langle \partial_\nu g^{\alpha\beta} \dot{x}_\alpha \dot{x}_\beta \rangle - \langle F_{\nu\alpha} g^{\alpha\beta} \dot{x}_\beta \rangle.$$ \hspace{1cm} (2.24)
Using Equations (2.4), (2.13), and (2.24) we obtain

\[
[\dot{x}_\mu, G_\nu] = m [\dot{x}_\mu, \ddot{x}_\nu] + \left( \frac{i}{2} \hbar \partial_\mu \partial_\nu g^{\alpha\beta} \dot{x}_\alpha \dot{x}_\beta - \frac{i\hbar}{2m} \partial_\nu g^{\alpha\beta} (F_{\mu\alpha} \dot{x}_\beta + \dot{x}_\alpha F_{\mu\beta}) - \frac{\hbar}{m} \partial_\mu (F_{\nu\alpha} g^{\alpha\beta} \dot{x}_\beta) + \frac{i\hbar}{m} \partial_\nu g^{\alpha\beta} (F_{\mu\alpha} \dot{x}_\beta) \right. \\
\left. - \frac{i\hbar}{m} \partial_\nu g^{\alpha\beta} (F_{\nu\alpha} g^{\alpha\beta} \dot{x}_\beta) + \frac{i\hbar}{m^2} F_{\nu\alpha} g^{\alpha\beta} F_{\mu\beta} \right].
\]

(2.25)

Finally antisymmetrization with respect to the indices \(\mu\) and \(\nu\) gives

\[
[\dot{x}_\mu, G_\nu] - [\dot{x}_\nu, G_\mu] = m [\dot{x}_\mu, \ddot{x}_\nu] - \left( \frac{i\hbar}{m} \partial_\mu (F_{\nu\alpha} g^{\alpha\beta} \dot{x}_\beta) - \frac{i\hbar}{m} \partial_\nu g^{\alpha\beta} (F_{\mu\alpha} \dot{x}_\beta) \right. \\
\left. - \frac{i\hbar}{m} \partial_\nu g^{\alpha\beta} (F_{\nu\alpha} g^{\alpha\beta} \dot{x}_\beta) + \frac{i\hbar}{m^2} F_{\nu\alpha} g^{\alpha\beta} F_{\mu\beta} \right].
\]

(2.26)

Therefore, using (2.14),

\[
\partial_\mu G_\nu - \partial_\nu G_\mu + \frac{\partial F_{\mu\nu}}{\partial \tau} = 0.
\]

(2.27)

Regarding equations (2.19), (2.14), and (2.27) in the Ehrenfest sense, we may summarize the classical theory as

\[
m \frac{D\dot{x}_\mu}{D\tau} = m [\ddot{x}_\mu + \Gamma^{\mu\nu\rho} \dot{x}_\nu \dot{x}_\rho] = G_\mu + F_{\mu\nu} \dot{x}_\nu
\]

(2.28)

\[
\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0
\]

(2.29)

\[
\partial_\mu G_\nu - \partial_\nu G_\mu + \frac{\partial F_{\mu\nu}}{\partial \tau} = 0
\]

(2.30)

We see that the expressions for the Lorentz force and the conditions on the fields reduce to equations (1.11), (1.14), and (1.13) when the metric and the fields are taken to be \(\tau\)-independent.

Let us introduce the definitions

\[
x^d = \tau \quad \partial_d = \partial_\tau \quad F_{\mu d} = -F_{d\mu} = G_\mu.
\]

(2.31)

We may then combine (2.14) and (2.27) as

\[
\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0
\]

(2.32)
(for $\alpha, \beta, \gamma = 0, \cdots, d$), which shows that the two form $F$ is closed on the $(d+1)$-dimensional manifold $(\tau, x)$. Hence, if for example, this manifold is contractable, then $F$ is an exact form which can be obtained as the derivative of some potential with the form $F = dA$. The Lorentz force equation becomes

$$m \frac{D\dot{x}^\mu}{D\tau} = F^\mu_{\nu}(\tau, x) \dot{x}^\nu + G^\mu(\tau, x)$$

$$= F^\mu_{\nu}(\tau, x) \dot{x}^\nu + F^\mu_\beta(\tau, x) \dot{x}^\beta.$$

(2.33)

Following Dyson and Feynman, we may observe that given equation (2.32), the two-form $F^{\alpha\beta}$ is determined if we know functions $\rho$ and $j^\mu$ such that

$$D_\alpha F^{\mu\alpha} = j^\mu \quad (2.34)$$

$$D_\alpha F^{d\alpha} = \rho. \quad (2.35)$$

where $D_\alpha$ is the covariant derivative.

Unlike the Newtonian case, the Lorentz covariance of these expressions is manifest, and we expect that $j^\mu$ transforms as a $d$-vector and $\rho$ transforms as a scalar. By denoting $\rho = j^d$, equations (2.34) and (2.35) can be written compactly as

$$D_\alpha F^{\beta\alpha} = j^\beta \quad (2.36)$$

where, due to the antisymmetry of $F^{\beta\alpha}$, $j^\beta$ is conserved

$$D_\alpha j^\alpha = 0. \quad (2.37)$$

Notice that (2.37) admits a formal $d + 1$-dimensional symmetry (as does the homogeneous field equation (2.32)), owing to the sum on $\alpha = 0, \cdots, d$. However, the physical Lorentz covariance of the matter currents breaks this formal symmetry.

In Section 4, we will return to the equations derived here formally, and examine their meaning as a covariant canonical mechanics.
3 Aspects of the Inverse Problem in the Calculus of Variations

In [4], Hojman and Shepley set out to prove that Feynman’s program of finding the Maxwell theory from equations (1.3) — (1.5) without a Lagrangian, was in principle impossible, because these equations are sufficient to establish the existence of a unique Lagrangian of electromagnetic form. To accomplish this goal, they demonstrate a new connection between the commutation relations and well-established results from the inverse problem in the calculus of variations, a theory which concerns the conditions under which a system of differential equations may be derived from a variational principle. We briefly review elements of this theory and use these results to derive Tanimura’s equations from self-adjointness.

For the situation relevant to Lagrangian mechanics, we consider a set of ordinary second order differential equations of the form

\[ F_k(\tau, q, \dot{q}, \ddot{q}) = 0 \quad \dot{q}^j = \frac{dq^j}{d\tau} \quad \ddot{q}^j = \frac{d^2q^j}{d\tau^2} \quad j, k = 1, \ldots, n . \]  

(3.1)

Under variations of the path

\[
\begin{align*}
q(\tau) &\rightarrow q(\tau) + dq(\tau) \\
\dot{q}(\tau) &\rightarrow \dot{q}(\tau) + d\dot{q}(\tau) = \dot{q}(\tau) + \frac{d}{d\tau}dq(\tau) \\
\ddot{q}(\tau) &\rightarrow \ddot{q}(\tau) + d\dddot{q}(\tau) = \dddot{q}(\tau) + \frac{d^2}{d\tau^2}dq(\tau) \ , 
\end{align*}
\]  

(3.2)

the function \( F_k(\tau, q, \dot{q}, \ddot{q}) \) admits the variational one-form defined by

\[
dF_k = \frac{\partial F_k}{\partial q^j} dq^j + \frac{\partial F_k}{\partial \dot{q}^j} d\dot{q}^j + \frac{\partial F_k}{\partial \ddot{q}^j} d\ddot{q}^j ,
\]  

(3.3)

and the variational two-form,

\[
dq^k dF_k = \frac{\partial F_k}{\partial q^j} dq^k \wedge dq^j + \frac{\partial F_k}{\partial \dot{q}^j} dq^k \wedge d\dot{q}^j + \frac{\partial F_k}{\partial \ddot{q}^j} dq^k \wedge d\ddot{q}^j \]

(3.4)

where the 3n path variations \( (dq^k, d\dot{q}^k, d\ddot{q}^k) \) for \( k = 1, \ldots, n \) are understood to be linearly independent. The system of differential equations \( F_k(\tau, q, \dot{q}, \ddot{q}) \) is called self-adjoint if there exists a two-form \( \Omega_2(dq, d\dot{q}) \) such that for all admissible variations of the path,

\[
dq^k dF_k(dq) = \frac{d}{d\tau} \Omega_2(dq, d\dot{q}) .
\]  

(3.5)
Through integration by parts, one may show \[6\] that such a two-form exists and is unique up to an additive constant, if and only if
\[
\frac{\partial F_i}{\partial \ddot{q}^k} = \frac{\partial F_k}{\partial \ddot{q}^i} \quad (3.6)
\]
\[
\frac{\partial F_i}{\partial q^k} + \frac{\partial F_k}{\partial q^i} = \frac{d}{d\tau} \left[ \frac{\partial F_i}{\partial \dot{q}^k} + \frac{\partial F_k}{\partial \dot{q}^i} \right] \quad (3.7)
\]
\[
\frac{\partial F_i}{\partial q^k} - \frac{\partial F_k}{\partial q^i} = \frac{1}{2} \frac{d}{d\tau} \left[ \frac{\partial F_i}{\partial \dot{q}^k} - \frac{\partial F_k}{\partial \dot{q}^i} \right]. \quad (3.8)
\]

Equations (3.6) – (3.8) are known as the Helmholtz conditions \[31, 32\]. Introducing the notation
\[
\delta = dq^k_{\beta} \frac{\partial}{\partial q^k_{\beta}} \quad q^k_{\beta} = \left( \frac{d}{d\tau} \right)^{\beta} q^k \quad \beta = 0, 1, 2, \quad (3.9)
\]
it follows that
\[
\delta^2 = dq^k_{\beta} \wedge dq^l_{\alpha} \frac{\partial^2}{\partial q^k_{\beta} \partial q^l_{\alpha}} = 0, \quad (3.10)
\]
which permits the equivalence of a set of self-adjointness differential equations to a Lagrangian formulation to be easily demonstrated \[33\]. Given the Lagrangian \(L\),
\[
\delta L = \frac{\partial L}{\partial q^k} dq^k + \frac{\partial L}{\partial \dot{q}^k} d\dot{q}^k = \left[ -\frac{d}{d\tau} \frac{\partial L}{\partial \ddot{q}^k} + \frac{\partial L}{\partial q^k} \right] dq^k + \frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{q}^k} dq^k \right) = F_k dq^k + \frac{d}{d\tau} \Omega_1 \quad (3.11)
\]
Therefore,
\[
\delta^2 = 0 \implies -dq^k \delta F_k + \frac{d}{d\tau} \delta \Omega_1 = dq^k \delta F_k + \frac{d}{d\tau} \Omega_2 = 0 \quad (3.12)
\]
which demonstrates self-adjointness. Conversely, if \(F_k\) is self-adjoint, then \(dq^k \delta F_k - \frac{d}{d\tau} \Omega_2 = 0\) and since \(\delta^2 = 0\),
\[
\frac{d}{d\tau} \Omega_2 = \delta \frac{d}{d\tau} \Omega_1. \quad (3.13)
\]
Therefore,
\[
0 = dq^k \delta F_k - \frac{d}{d\tau} \Omega_2 = \delta (dq^k \delta F_k - \frac{d}{d\tau} \Omega_1) = \delta L \quad (3.14)
\]
and one obtains the differential equations \(F_k = 0\) by variation of \(L\) under \(\tau\)-integration.

For the second order equations considered here, it also follows \[3\] from self-adjointness that the most general form of \(F_k\) is
\[
F_k(\tau, q, \dot{q}, \ddot{q}) = A_{kj}(\tau, q, \dot{q}) \ddot{q}^j + B_k(\tau, q, \dot{q}). \quad (3.15)
\]
To see this, notice that $F_k$ is independent of $d^3q^i/dt^3$, so that the right hand side of (3.1) must be independent of $\ddot{q}^i$. Inserting (3.15) into (3.6) – (3.8), one finds the Helmholtz conditions on $A_{kj}$ and $B_k$

$$A_{ij} = A_{ji},$$
$$\frac{\partial A_{ij}}{\partial q^k} = \frac{\partial A_{kj}}{\partial q^i},$$

(3.16)

$$\frac{\partial B_i}{\partial q^i} + \frac{\partial B_j}{\partial q^i} = 2 \left[ \frac{\partial}{\partial \tau} + \dot{q}^k \frac{\partial}{\partial q^k} \right] A_{ij},$$

(3.17)

$$\frac{\partial B_i}{\partial q^i} - \frac{\partial B_j}{\partial q^i} = \frac{1}{2} \left[ \frac{\partial}{\partial \tau} + \dot{q}^k \frac{\partial}{\partial q^k} \right] \left( \frac{\partial B_i}{\partial \dot{q}^i} - \frac{\partial B_i}{\partial \dot{q}^j} \right),$$

(3.18)

along with the useful identity

$$\frac{\partial A_{ij}}{\partial q^k} - \frac{\partial A_{kj}}{\partial q^i} = \frac{1}{2} \frac{\partial}{\partial \dot{q}^j} \left( \frac{\partial B_i}{\partial \dot{q}^k} - \frac{\partial B_i}{\partial \dot{q}^i} \right).$$

(3.19)

In the domain of invertibilty of the $A_{jk}$, one can write (3.15) as

$$F_k(\tau, q, \dot{q}, \ddot{q}) = A_{kj}(\tau, q, \dot{q})[\ddot{q}^j - f^j],$$

(3.20)

where

$$f^j(\tau, q, \dot{q}) = -(A^{-1})^{jk} B_k.$$

(3.21)

Inserting $B_k = -A_{kj} f^j$ into (3.17) and (3.18), the Helmholtz conditions for the form (3.20) become

$$A_{ij} = A_{ji},$$
$$\frac{\partial A_{ij}}{\partial q^k} = \frac{\partial A_{kj}}{\partial q^i},$$

(3.22)

$$\frac{D}{D\tau} A_{ij} = -\frac{1}{2} \left[ A_{ik} \frac{\partial f^k}{\partial q^i} + A_{jk} \frac{\partial f^k}{\partial q^i} \right],$$

(3.23)

$$\frac{1}{2} \frac{D}{D\tau} \left[ A_{ik} \frac{\partial f^k}{\partial \dot{q}^i} - A_{jk} \frac{\partial f^k}{\partial \dot{q}^i} \right] = A_{ik} \frac{\partial f^k}{\partial q^i} - A_{jk} \frac{\partial f^k}{\partial q^i},$$

(3.24)

where

$$\frac{D}{D\tau} = \frac{\partial}{\partial \tau} + \dot{q}^k \frac{\partial}{\partial q^b} + f^k \frac{\partial}{\partial \dot{q}^k}$$

(3.25)

is the total time derivative subject to the constraint

$$\ddot{q}^k - f^k(\tau, q, \dot{q}) = 0.$$  

(3.26)

The identity (3.19) becomes

$$\frac{\partial A_{ij}}{\partial q^k} - \frac{\partial A_{kj}}{\partial q^i} = -\frac{1}{2} \frac{\partial}{\partial \dot{q}^j} \left( A_{in} f^n - \frac{\partial}{\partial q^i} (A_{kn} f^n) \right).$$

(3.27)
Within the domain of applicability of the inverse function theorem, (3.26) is equivalent to (3.13), and the Helmholtz conditions become the necessary and sufficient conditions for the existence of an integrating factor $A_{jk}$ such that

$$F_k = A_{kj}(\tau, q, \dot{q})[\ddot{q}^j - f^j] = \frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{q}^k} \right) - \frac{\partial L}{\partial q^k}. \quad (3.28)$$

The Helmholtz conditions for this form have been rederived in an elegant way in terms of the Lie derivative along $f^k$ [4].

In [4], Hojman and Shepley prove that given the quantum mechanical commutation relations

$$[X^i(\tau), \dot{X}^j(\tau)] = i\hbar G^{ij}, \quad (3.29)$$

the classical function

$$g^{ij} = \lim_{\hbar \rightarrow 0} G^{ij} \quad (3.30)$$

has an inverse

$$\omega_{ij} = (g^{-1})_{ij} \quad (3.31)$$

which satisfies the Helmholtz conditions (3.22) – (3.24). Therefore, the assumption of commutation relations, (1.3) and (1.4) [or (1.8) and (1.9)], is sufficiently strong to establish the existence of an equivalent Lagrangian for the classical problem associated with the quantum commutators. For the Newtonian case, in which $g^{ij} = \delta^{ij}$ and $\tau \rightarrow t$, it is shown in [4, 5] that the Helmholtz conditions lead to the Lagrangian (1.6), with field equations (1.7). In [6], Santilli discusses the classical case, applying (3.6) – (3.8) to (3.20) for the case $A_{ij} = \delta_{ij}$, and arrives at (1.1) and (1.2) without explicitly writing the Lagrangian.

We now adapt Santilli’s argument to the type of curved space discussed in Section 2. Starting with the Helmholtz conditions and the metric $g_{\mu\nu}(x)$, we will be led to the equations derived by Tanimura. We take the function $A_{\mu\nu} = g_{\mu\nu}(x)$ in equation (3.30) to be a Riemannian metric independent of $\dot{x}$, so that equations (3.22) are satisfied automatically. Since $g_{\mu\nu}$ does not depend on $\dot{x}^\mu$, equation (3.23) becomes

$$\frac{D}{D\tau} g_{\mu\nu} = \dot{x}^\sigma \frac{\partial}{\partial x^\sigma} g_{\mu\nu} = -\frac{1}{2} \left[ \frac{\partial f_\mu}{\partial \dot{x}^\nu} - \frac{\partial f_\nu}{\partial \dot{x}^\mu} \right] \quad (3.32)$$

and equation (3.27) becomes

$$-\frac{1}{2} \frac{\partial}{\partial \dot{x}^\nu} \left[ \frac{\partial f_\mu}{\partial \dot{x}^\sigma} - \frac{\partial f_\sigma}{\partial \dot{x}^\mu} \right] = \frac{\partial g_{\mu\nu}}{\partial x^\sigma} - \frac{\partial g_{\sigma\nu}}{\partial x^\mu}. \quad (3.33)$$

16
Acting on (3.32) with $\partial/\partial \dot{x}^\sigma$ and exchanging $(\nu \leftrightarrow \sigma)$, we obtain
\[ g_{\mu\sigma,\nu} = -\frac{1}{2} \left[ \frac{\partial^2 f_\mu}{\partial \dot{x}^\sigma \partial \dot{x}^\nu} + \frac{\partial^2 f_\sigma}{\partial \dot{x}^\mu \partial \dot{x}^\nu} \right] \] (3.34)
where $g_{\mu\sigma,\nu} = \partial g_{\mu\sigma}/\partial x^\nu$. Combining (3.33) and (3.34), we find
\[ \frac{1}{2} \frac{\partial^2 f_\mu}{\partial \dot{x}^\sigma \partial \dot{x}^\nu} = -\frac{1}{2} (g_{\mu\nu,\sigma} + g_{\mu\sigma,\nu} - g_{\sigma\nu,\mu}) = -\Gamma_{\mu\sigma\nu} \] (3.35)
where $\Gamma_{\mu\sigma\nu}$ is defined as in (2.16). From (3.35) we see that the most general expression for $f_\mu(\tau, x, \dot{x})$ is
\[ f_\mu = -\Gamma_{\mu\nu\sigma} \dot{x}^\nu \dot{x}^\sigma - \rho_{\mu\nu}(\tau, x) \dot{x}^\nu - \sigma_\mu(\tau, x). \] (3.36)
Now from (3.32) we find
\[ \dot{x}^\sigma \frac{\partial g_{\mu\nu}}{\partial x^\sigma} = \frac{1}{2} \left[ 2\Gamma_{\mu\sigma\nu} \dot{x}^\nu + 2\Gamma_{\nu\sigma\nu} \dot{x}^\nu + \rho_{\mu\nu} + \rho_{\nu\mu} \right]. \] (3.37)
Using
\[ (\Gamma_{\mu\sigma\nu} + \Gamma_{\nu\mu\sigma}) \dot{x}^\sigma = g_{\mu\nu,\sigma} \dot{x}^\sigma \] (3.38)
we find that all terms except for those in $\rho_{\mu\nu}$ cancel, so that
\[ 0 = \rho_{\mu\nu} + \rho_{\nu\mu}. \] (3.39)
We now apply equation (3.24) which becomes
\[ \frac{1}{2} \frac{D}{D\tau} \left[ g_{\mu\nu} \frac{\partial f_\sigma}{\partial \dot{x}^\nu} - g_{\nu\sigma} \frac{\partial f_\nu}{\partial \dot{x}^\mu} \right] = g_{\mu\sigma} \frac{\partial f_\sigma}{\partial \dot{x}^\nu} - g_{\nu\sigma} \frac{\partial f_\nu}{\partial \dot{x}^\mu} \] (3.40)
Using (3.36) to expand the left hand side,
\[ \frac{1}{2} \frac{D}{D\tau} \left[ \frac{\partial f_\mu}{\partial \dot{x}^\nu} - \frac{\partial f_\nu}{\partial \dot{x}^\mu} \right] = -\frac{1}{2} \frac{D}{D\tau} \left[ \frac{\partial}{\partial \dot{x}^\nu} \left( \Gamma_{\mu\lambda\sigma} \dot{x}^\lambda \dot{x}^\sigma + \rho_{\mu\lambda}(\tau, x) \dot{x}^\lambda \right. \right. \] \[ \left. \left. + \sigma_\mu(\tau, x) \right) - (\mu \leftrightarrow \nu) \right] \] \[ = -\frac{1}{2} \frac{D}{D\tau} \left[ 2(\Gamma_{\mu\lambda\nu} - \Gamma_{\nu\mu\lambda}) \dot{x}^\lambda + \rho_{\mu\nu} - \rho_{\nu\mu} \right] \] \[ = -\left( \frac{\partial}{\partial \tau} + \dot{x}^\sigma \frac{\partial}{\partial x^\sigma} + f_\sigma \frac{\partial}{\partial \dot{x}^\sigma} \right) \left[ (g_{\mu\lambda,\nu} - g_{\nu\lambda,\mu}) \dot{x}^\lambda + \rho_{\mu\nu} \right] \] \[ = -(g_{\mu\sigma,\nu} - g_{\nu\sigma,\mu}) f_\sigma - \rho_{\mu\nu,\tau} \] \[ - \dot{x}^\lambda \dot{x}^\sigma (g_{\mu\lambda,\nu\sigma} - g_{\nu\lambda,\mu\sigma}) + \dot{x}^\lambda \rho_{\mu\nu,\lambda}. \] (3.41)
where $\rho_{\mu \nu, \tau} = \partial \rho_{\mu \nu} / \partial \tau$, and we have used
\[
2(\Gamma_{\mu \nu \lambda} - \Gamma_{\nu \mu \lambda}) \dot{x}^\lambda = \dot{x}^\lambda (-g_{\nu \lambda \mu} + g_{\mu \lambda \nu} + g_{\nu \mu \lambda} - g_{\nu \lambda \nu} - g_{\mu \nu \lambda}) \\
= 2\dot{x}^\lambda (g_{\mu \lambda \nu} - g_{\nu \lambda \mu}). 
\quad (3.42)
\]
From (3.36), we have
\[
f_{\mu, \nu} = -\left[ \Gamma_{\mu \lambda \sigma} \dot{x}^\lambda \dot{x}^\sigma + \rho_{\mu \lambda} (\tau, x) \dot{x}^\lambda + \sigma_{\mu} (\tau, x) \right]_{, \nu} \\
= -\left[ \Gamma_{\mu \lambda \sigma} \dot{x}^\lambda \dot{x}^\sigma + \rho_{\mu \lambda, \nu} \dot{x}^\lambda + \sigma_{\mu, \nu} \right] 
\quad (3.43)
\]
so that the right hand side of (3.40) is
\[
f_{\mu, \nu} - f_{\nu, \mu} - g_{\mu \sigma, \nu} f^\sigma + g_{\nu \sigma, \mu} f^\sigma = \\
= -\left[ (\Gamma_{\mu \lambda \sigma, \nu} - \Gamma_{\nu \lambda \sigma, \mu}) \dot{x}^\lambda \dot{x}^\sigma + (\rho_{\mu \lambda, \nu} - \rho_{\nu \lambda, \mu}) \dot{x}^\lambda + \sigma_{\mu, \nu} - \sigma_{\nu, \mu} \right] \\
- (g_{\mu \sigma, \nu} - g_{\nu \sigma, \mu}) f^\sigma. 
\quad (3.44)
\]
Now, equating (3.41) and (3.44) and canceling common terms, we are left with
\[
\frac{\partial \rho_{\mu \nu}}{\partial \tau} + \dot{x}^\lambda \rho_{\mu \nu, \lambda} = \dot{x}^\lambda (\rho_{\mu \lambda, \nu} - \rho_{\nu \lambda, \mu}) + \sigma_{\mu, \nu} - \sigma_{\nu, \mu}. 
\quad (3.45)
\]
Since the $\dot{x}^\lambda$ are arbitrary, we have the two expressions
\[
\frac{\partial \rho_{\mu \nu}}{\partial \tau} = \frac{\partial \sigma_{\mu}}{\partial x^\nu} - \frac{\partial \sigma_{\nu}}{\partial x^\mu} 
\quad (3.46) \\
\partial_\lambda \rho_{\mu \nu} + \partial_\mu \rho_{\nu \lambda} + \partial_\nu \rho_{\lambda \mu} = 0 
\quad (3.47)
\]
Comparing (3.47) with (2.14), (3.46) with (2.27), and (3.36) with (2.17), we see that we may identify
\[
F_{\mu \nu} = -\rho_{\mu \nu} \quad \text{and} \quad G_{\mu} = -\sigma_{\mu}. 
\quad (3.48)
\]
This identification demonstrates explicitly that Tanimura’s equations (generalized to permit explicit $\tau$ dependence of the fields and metric) are simply the conditions on the most general velocity dependent forces which may be obtained from a variational principle.

To make the connection with the off-shell electromagnetic theory, we now derive the Lagrangian in flat Minkowski space. For this case, where
\[
A_{\mu \nu} = g_{\mu \nu} \rightarrow \eta_{\mu \nu} = \text{diag}(1, -1, \cdots, -1) 
\quad (3.49)
\]
Tanimura’s equations reduce to

\[ \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0 \]
\[ \partial_\mu G_\nu - \partial_\nu G_\mu + \frac{\partial F_{\mu\nu}}{\partial \tau} = 0 \]
\[ m \ddot{x}^\mu = G_\mu(\tau, x) + F^{\mu\nu}(\tau, x) \dot{x}_\nu \] (3.50)

Following [8], we observe that (3.28) implies

\[ \eta_{\mu\nu}[m \ddot{x}^\nu - f^\nu] = \frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{x}^\mu} \right) - \frac{\partial L}{\partial x^\mu} \]
\[ = \frac{\partial^2 L}{\partial \dot{x}_\mu \partial \ddot{x}^\nu} \ddot{x}^\nu + \frac{\partial^2 L}{\partial \dot{x}_\mu \partial \dot{x}_\nu} \dot{x}_\nu \]
\[ + \frac{\partial^2 L}{\partial \dot{x}_\mu \partial \tau} - \frac{\partial L}{\partial x^\mu} \] (3.51)

so that

\[ m \eta_{\mu\nu} = \frac{\partial^2 L}{\partial \dot{x}_\mu \partial \ddot{x}^\nu} \] (3.52)
\[ -\eta_{\mu\nu} f^\nu = \frac{\partial^2 L}{\partial \dot{x}_\mu \partial \dot{x}_\nu} \dot{x}_\nu + \frac{\partial^2 L}{\partial \dot{x}_\mu \partial \tau} \] (3.53)

The solution to the equation \( \partial^2 L_{\text{kinetic}} / \partial \dot{x}_\mu \partial \ddot{x}^\nu = m \eta_{\mu\nu} \) is unique up to terms of the type which may be absorbed in (3.53). Therefore, we see that \( L \) may consist of the quadratic term integrated from (3.52), plus terms at most linear in \( \dot{x}^\mu \). So we may write

\[ L = \frac{1}{2} m \dot{x}^\mu \dot{x}_\mu + A_\mu(\tau, x) \dot{x}^\mu + \phi(\tau, x) . \] (3.54)

Applying (3.53) to \( L \) yields the Lorentz force if we identify the fields \( A_\mu \) and \( \phi \) as the potentials for the field strengths

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \] (3.55)
\[ G_\mu = \partial_\mu \phi - \partial_\tau A_\mu . \] (3.56)

We may recover the inhomogeneous field equations (2.34) and (2.35) by introducing to the action the kinetic terms

\[ \int d^4x d\tau \left[ \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \pm \frac{1}{2} G_\mu G^\mu \right] . \] (3.57)

Applying the Euler-Lagrange equations to the action which includes these terms leads to the source equations, with the identification of the classical currents as

\[ j^\mu(\tau, y) = \dot{x}^\mu(\tau) \delta^4(y - x(\tau)) \] (3.58)
\[ \rho(\tau, y) = \delta^4(y - x(\tau)) . \] (3.59)
If the fields are considered $\tau$-independent \textit{a priori}, as by Tanimura, the currents are

\begin{equation}
\begin{aligned}
j^\mu(y) &= \int d\tau \dot{x}^\mu(\tau) \delta^4(y - x(\tau)) \\
\rho(y) &= \int d\tau \delta^4(y - x(\tau)).
\end{aligned}
\end{equation}

Equations (2.35) and (3.61) indicate that the $\tau$-independent $G^\mu$ field in Tanimura’s formulation must have a non-trivial form related to the path of the particle event, and also, influence the particle motions through the Lorentz force, with no coupling to $F^\mu\nu$ which could control the interaction. In the next section, we arrive at (3.54) by a different argument and examine the dynamical system which it describes.

### 4 Implications for Gauge Theory in Covariant Quantum Mechanics

In this section, we show the consequences of the analysis discussed above for a Lorentz and gauge covariant quantum mechanics. We first review the structure of such a theory.

In 1951, Schwinger \cite{12} represented the Green’s functions of the Dirac field as a parametric integral and formally transformed the Dirac problem into a dynamical one, in which the integration parameter acts as a proper time according to which a “Hamiltonian” operator generates the evolution of the system through spacetime. Applying Schwinger’s method to the Klein-Gordon equation, one obtains an equation for the Green’s function (we take $\hbar = 1$ in the following)

\begin{equation}
G = \frac{1}{(p - eA)^2 + m^2}
\end{equation}

given by

\begin{equation}
G(x, x') = \langle x | G | x' \rangle = i \int_0^\infty ds e^{-im^2s} \langle x | e^{-i(p - eA)^2s} | x' \rangle.
\end{equation}

The function

\begin{equation}
G(x, x'; s) = \langle x(s) | x'(0) \rangle = \langle x | e^{-i(p - eA)^2s} | x' \rangle
\end{equation}

satisfies

\begin{equation}
\frac{i}{\partial s} \langle x(s) | x'(0) \rangle = (p - eA)^2 \langle x(s) | x'(0) \rangle = K \langle x(s) | x'(0) \rangle
\end{equation}

with the boundary condition

\begin{equation}
\lim_{s \to 0} \langle x(s) | x'(0) \rangle = \delta^4(x - x').
\end{equation}
DeWitt regarded (4.4) as defining the Green’s function for the Schrödinger equation
\[ i \frac{\partial}{\partial s} \psi_s(x) = K \psi_s(x) = (p - eA)^2 \psi_s(x), \tag{4.6} \]
an equation which Stueckelberg had also written as the basis for a covariant quantum mechanical formalism which includes the description of pair annihilation. Feynman wrote equation (4.6) in order to obtain the path integral for the Klein-Gordon equation and regarded the integration of the Green’s function with the weight \( e^{-im^2s} \), as the requirement that asymptotic solutions be on mass-shell. The usual Feynman propagator \( \Delta_F(x - x') \) emerges naturally from the assumption that \( G(x, x'; s) = 0 \) whenever \( s < 0 \). Thus \( \Delta_F \) corresponds to a \( G(x, x'; s) \) which is causal in the classical sense of no response before stimulus (before, in the sense of \( s \)). Schwinger employed the proper time method as a way to exploit the techniques of nonrelativistic mechanics in relativistic quantum theory, and found that it provided a natural approach to perturbation theory and regularization.

In 1973, Horwitz and Piron constructed a canonical formalism for the relativistic classical and quantum mechanics of many particles. In order to formulate a generalized Hamilton’s principle, they introduce a Lorentz invariant evolution parameter \( \tau \), which they call the historical time and regard as corresponding to the ordering relation of successive events in spacetime. For a one-particle system, the equations of motion are
\[ \frac{dx^\mu}{d\tau} = \frac{\partial K}{\partial p_\mu}, \quad \frac{dp^\mu}{d\tau} = -\frac{\partial K}{\partial x_\mu}, \tag{4.7} \]
where \( \mu, \nu = 0, 1, 2, 3 \) and \( K(x^\mu, p^\mu) \) a Lorentz scalar. Taking \( K = p^2/2M \) leads to the usual description of the relativistic motion of a free particle:
\[ \dot{x}^0 = \frac{p^0}{M}, \quad \dot{x}^i = \frac{p^i}{M}, \quad p^\mu = \text{constant} \tag{4.8} \]
and so
\[ \frac{dx^i}{dt} = \frac{p^i}{p^0}, \quad \dot{x}^2 = -\frac{m^2}{M^2} = \text{constant} \tag{4.9} \]
with \( m/M \) scaling \( \tau \) to the proper time. This formalism leads naturally to a Schrödinger equation for the quantum theory, which is identical to the Stueckelberg equation (4.6) for the Klein-Gordon problem (for \( 2M = 1 \)).

Saad, Horwitz, and Arshansky later argued that the local gauge covariance of the Schrödinger equation should include transformations which depend on \( \tau \), as well as on the
spacetime coordinates. This requirement of full gauge covariance leads to a theory of five
gauge compensation fields, since gauge transformations are functions on the five dimensional
space \((\tau,x)\). Under local gauge transformations of the form
\[
\psi(x,\tau) \rightarrow e^{ie\Lambda(x,\tau)}\psi(x,\tau)
\] (4.10)
the equation
\[
-(i\partial_\tau - e_0 a_4)\psi(x,\tau) = \frac{1}{2M}(p^\mu - e_0 a^\mu)(p_\mu - e_0 a_\mu)\psi(x,\tau)
\] (4.11)
is covariant, when the compensation fields transform as
\[
a_\mu(x,\tau) \rightarrow a_\mu(x,\tau) + \partial_\mu \Lambda(x,\tau) \quad a_4(x,\tau) \rightarrow a_4(x,\tau) + \partial_\tau \Lambda(x,\tau).
\] (4.12)
This Schrödinger equation (4.11) leads to the five dimensional conserved current [compare
with equation (2.37)]
\[
\partial_\mu j^\mu + \partial_\tau j^4 = 0
\] (4.13)
where
\[
j^\mu = |\psi(x,\tau)|^2 \quad j^\mu = \frac{-i}{2M}(\psi^*(\partial^\mu - ie_0 a^\mu)\psi - \psi(\partial^\mu - ie_0 a^\mu)\psi^*).
\] (4.14)
In analogy to nonrelativistic quantum mechanics the squared amplitude of the wave function
may be interpreted as the probability of finding an event at \((\tau,x)\). Equation (4.13) may be
written as \(\partial_\alpha j^\alpha = 0\), with \(\alpha = 0, 1, 2, 3, 4\).

From (4.11) we can write the classical Hamiltonian as
\[
K = \frac{1}{2M}(p^\mu - e_0 a^\mu)(p_\mu - e_0 a_\mu) - e_0 a_4
\] (4.15)
and using (4.17) we find
\[
M \dot{x}^\mu = (p^\mu - e_0 a^\mu)
\] (4.16)
which enables us to write the classical Lagrangian,
\[
L = \dot{x}^\mu p_\mu - K
= \frac{1}{2}M\dot{x}^\mu \dot{x}_\mu + e_0 \dot{x}^\mu a_\mu + e_0 a_4.
\] (4.17)
Comparing (4.17) with (3.54), we see that identifying the scalar potential \(\phi(\tau,x)\) with \(e_0 a_4\)
and the vector potential \(A_\mu(\tau,x)\) with \(e_0 a_\mu\), this Lagrangian describes the same theory which
we derived from the Helmholtz conditions and from Tanimura’s commutation relations (1.8) and (1.9). We may find the Lorentz force [36] by applying the Euler-Lagrange equations to (4.17), which in the notation of (2.33), i.e., $\alpha, \beta = 0, 1, 2, 3, 4$, is

$$M \ddot{x}^\mu = f_{\mu 4} + f_{\mu \nu} \dot{x}^\nu,$$  \hspace{1cm} (4.18)

where

$$f_{\mu \nu} = \partial_\mu a_\nu - \partial_\nu a_\mu \quad f_{\mu 4} = \partial_\mu a_4 - \partial_4 a_\mu$$  \hspace{1cm} (4.19)

Since $f_{\mu \nu}$ and $f_{\mu 4}$ are defined in terms of potentials, the field equations, (2.14) and (2.27) [or (3.47) and (3.46)] follow by definition.

Comparing (4.18) with (1.11) and (3.36), we identify $f_{\mu 4}$ with the “new” field $G_{\mu}$ found by Tanimura (and the field $\sigma_\mu$ which emerges from the derivation based on the Helmholtz conditions). This justifies our claim in Section 1 that the appearance of the “new” scalar degree of gauge freedom in Tanimura’s derivation corresponds to invariance of the theory under gauge transformations which depend upon the evolution parameter $\tau$.

We observe that the four equations (4.18) imply [36] that

$$\frac{d}{d\tau} \left( \frac{1}{2} M \dot{x}^2 \right) = M \ddot{x}^\mu \dot{x}_\mu = \dot{x}^\mu (f_{\mu 4} + f_{\mu \nu} \dot{x}^\nu) = \dot{x}^\mu f_{\mu 4}$$ \hspace{1cm} (4.20)

By replacing $f_{\mu 4}$ with $-f_{4\mu}$ and using $f_{44} \equiv 0$, we can write equation (4.21) as

$$\frac{d}{d\tau} \left( \frac{1}{2} M \dot{x}^2 \right) = f_{4\alpha} \dot{x}^\alpha$$ \hspace{1cm} (4.21)

which formally becomes the “fifth” component of the Lorentz force law. If we define the $4 \oplus 1$ vector

$$U^\alpha = (-\frac{1}{2} \dot{x}^2, \dot{x}^\mu)$$ \hspace{1cm} (4.22)

then (4.20) and (4.21) can be written as

$$M \frac{d}{d\tau} U^\alpha = f_{\alpha \beta} \dot{x}^\beta.$$ \hspace{1cm} (4.23)

We are now in a position to address the mass-shell condition $\dot{x}^2 = \text{constant}$ discussed in Section 1. Since the right hand side of (4.18) is the most general expression which may appear as a Lorentz force, we may conclude from (4.21) that the conditions for the dynamical (as opposed to asymptotic) conservation of $\dot{x}^2 = \text{constant}$, are

$$G_{\mu} = f_{4\mu} = 0 \quad \text{and} \quad \partial_\tau f^{\mu\nu} = 0$$ \hspace{1cm} (4.24)
The second condition follows from (2.27) for $G^\mu = 0$; that is $f^{\mu\nu}$ must be a $\tau$-static field. Thus, we see that the most general interaction which preserves the proper time constraint is of conventional Maxwell type, as employed by Schwinger [12] in his original use of the proper time method. In this way, we may make precise the claim that the proper time constraint suppresses the “new” gauge degree of freedom. Notice that the fields in these expressions are explicitly $\tau$-dependent.

The action, given by

$$S = \int d\tau \frac{1}{2} M \dot{x}^\mu \dot{x}_\mu + e_0 \dot{x}^\mu a_\mu(\tau, x) + e_0 a_4(\tau, x)$$

(4.25)

is clearly not reparameterization invariant, corresponding to the absence of the mass-shell constraint. But, under the conditions (4.24), the Lagrangian has no explicit $\tau$-dependence (see (4.19)), and so the Hamiltonian is conserved. From (4.15) and (4.16), we see that under those circumstances, $K \rightarrow M \dot{x}^2/2$. For those interactions which respect the proper time relation and keep particles on mass-shell dynamically, the $\tau$-derivative of the proper time is essentially the constant of motion conserved by Noether’s theorem for the $\tau$-translation symmetry. Thus, the mass-shell relation has the status, classically, of a conservation law rather than a constraint.

When we add as the dynamical term for the gauge field, $(\lambda/4) f_\alpha f^{\alpha\beta} f^{\alpha\beta}$ where $\lambda$ is a dimensional constant, the equations for the field are found to be

$$\partial_\beta f^{\alpha\beta} = e_0 f^{\alpha} = e j^\alpha$$

(4.26)

$$\epsilon^{\alpha\beta\gamma\delta} \partial_\alpha f_{\beta\gamma} = 0$$

(4.27)

where $f_{\alpha\beta} = \partial_\alpha a_\beta - \partial_\beta a_\alpha$, and

$$j^\mu(\tau, y) = \dot{x}^\mu(\tau) \delta^4(y - x(\tau))$$

(4.28)

$$j^4(\tau, y) = \rho(\tau, y) = \delta^4(y - x(\tau)).$$

(4.29)

We identify $e_0/\lambda$ as the dimensionless Maxwell charge (it follows from (4.32) below that $e_0$ has dimension of length). The currents (4.28) and (4.29) have the form of (3.58) and (3.59), so that (4.26) is the flat space equivalent of (2.36). Similarly, (4.27) is the homogeneous equation (2.32) and (1.46 1.47).
The Lagrangian for the free electromagnetic field has a formal five dimensional symmetry. Analysis of the resulting wave equation \[24\] for the sourceless case shows that the symmetry group of the equations can be either \(O(3,2)\) or \(O(4,1)\), depending on the signature of the \(\tau\) index. Since \(U^\alpha\) (see \[4.22\]) transforms as a Lorentz scalar plus Lorentz 4-vector, rather than as a vector of \(O(3,2)\) or \(O(4,1)\), the presence of sources breaks this formal symmetry to \(O(3,1)\). This situation is analogous to the nonrelativistic case discussed in Section 1, in which the homogeneous field equations \[4.2\] may be regarded as \(O(3,1)\) covariant, while the source dynamics \[4.4\], may be seen as having a Galilean symmetry; in a consistent theory of sources and fields, only the common \(O(3)\) symmetry survives.

Since the 4-vector part of the current in \[4.14\] is not conserved by itself, it may not be the source for the Maxwell field. However, integration of \[4.14\] over \(\tau\), with appropriate boundary conditions, leads to \(\partial_\mu J^\mu = 0\), where

\[
J^\mu(x) = \int_{-\infty}^{\infty} d\tau j^\mu(x, \tau) \tag{4.30}
\]

so that we may identify \(J^\mu\) as the source of the Maxwell field. Under appropriate boundary conditions, integration of \[4.26\] over \(\tau\) implies

\[
\partial_\mu F^{\mu\nu} = e J^\mu \tag{4.31}
\]

\[
e^{\mu\rho\lambda\sigma} \partial_\mu F_{\rho\sigma} = 0
\]

where

\[
F^{\mu\nu}(x) = \int_{-\infty}^{\infty} d\tau f^{\mu\nu}(x, \tau) \tag{4.32}
\]

\[
A^\mu(x) = \int_{-\infty}^{\infty} d\tau a^\mu(x, \tau)
\]

so that \(a^\alpha(x, \tau)\) has been called the pre-Maxwell field.

In the pre-Maxwell theory, interactions take place between events in spacetime rather than between worldlines. Each event, occurring at \(\tau\), induces a current density in spacetime which disperses for large \(\tau\), and the continuity equation \[4.13\] states that these current densities evolve as the event density \(j^4\) progresses through spacetime as a function of \(\tau\). As noted above, if \(j^4 \to 0\) as \(|\tau| \to \infty\) (pointwise in spacetime), then the integral of \(j^\mu\) over \(\tau\) may be identified with the Maxwell current. This integration has been called concatenation \[38\] and provides the link between the event along a worldline and the notion of a particle, whose support is the entire worldline. Concatenation is evidently related to the integration performed
in the Stueckelberg theory, and following Feynman’s interpretation, places the electromagnetic field on the zero mass-shell. The Maxwell theory has the character of an equilibrium limit of the microscopic pre-Maxwell theory. For further discussion and applications see [22, 25, 26, 27, 39, 40, 41] and references contained therein.

5 Non-Abelian Gauge Theory

It was shown in Section 4 that while the mass-shell condition may not be maintained as a constraint on the phase space, the quantity \( \dot{x}^\mu \dot{x}_\mu \) is a constant of the motion when the “new” gauge field \( G_\mu = F_{4\mu} \) vanishes and \( F^{\mu\nu} \) is \( \tau \)-independent. Under these conditions, the Lagrangian has no explicit \( \tau \)-dependence and the conserved Hamiltonian is precisely \( \dot{x}^\mu \dot{x}_\mu \).

We remark briefly on the case of a non-Abelian gauge field, in which a \( \tau \)-dependent quantity appears in the Lagrangian, without changing these conditions on the fields.

In [42], C. R. Lee employed Feynman’s method to derive equations of motion for a particle interacting with a classical non-Abelian gauge field, in the form originally given by Wong [43]. By studying the Heisenberg equations of motion for the Hamiltonian of the Dirac equation in the presence of an SU(2) gauge field, Wong formulated the following structure:

\[
\begin{align*}
m \dddot{\xi}_\mu & = g f_{\mu\nu} \cdot I(\tau) \dot{\xi}^\nu \quad \text{(5.1)} \\
\ddot{I} & = -g b_\mu \times I \dot{\xi}^\mu \quad \text{(5.2)} \\
f_{\mu\nu} & = \partial_\mu b_\nu - \partial_\nu b_\mu + g b_\mu \times b_\nu \quad \text{(5.3)} \\
\dot{f}_{\mu\nu} + g b^\mu \times f_{\mu\nu} & = -\dot{I}_\nu \quad \text{(5.4)} \\
b^\mu & = b_{a\mu} I^a \quad f_{\mu\nu} = f_{a\mu\nu} I^a \quad [I^a, I^b] = i\hbar \varepsilon^{abc} I_c. \quad \text{(5.5)}
\end{align*}
\]

where \( \xi^\mu(\tau) \) is the particle world line operator as parameterized by the Lorentz invariant scalar \( \tau \) and where the \( I^a(\tau) \) are an operator representation of the generators of a non-Abelian gauge group. By virtue of (5.3), one has the inhomogeneous equation

\[
\mathcal{D}_\mu f_{\nu\mu} + \mathcal{D}_\nu f_{\rho\mu} + \mathcal{D}_\rho f_{\nu\mu} = 0, \quad \text{(5.6)}
\]

where the covariant derivative is

\[
(\mathcal{D}_\mu f_{\nu\mu})_a = \partial_\mu f_{a\mu\nu} - \varepsilon_{abc} b_{b\mu} f_{c\mu\nu}. \quad \text{(5.7)}
\]
Lee [42] followed Feynman’s method, supplementing assumptions (1.3) — (1.5) with the relations (5.5) and
\[
[x_i, I^a(t)] = 0 \quad \hat{I} + g b_i \times I \hat{x}^i = 0 \tag{5.8}
\]
for \(i = 1, 2, 3\), and in analogy to Feynman’s derivation, arrived at the Newtonian equivalent of Wong’s equations.

Tanimura [7] generalized Lee’s derivation to \(d\)-dimensional flat Minkowski space and a general gauge group by supplementing equations (2.1) and (2.2) with
\[
[I^a, I^b] = i \hbar f^{abc} I^c \tag{5.9}
\]
\[
[x^\mu, I^a] = 0 \tag{5.10}
\]
\[
m \ddot{x}^\mu = F^\mu(x, \dot{x}, I) = F^\mu_a(x, \dot{x}) I^a \tag{5.11}
\]
and
\[
\dot{I}^a = f^{abc} A_{b\mu}(x) \dot{x}^\mu I^c \tag{5.12}
\]
The results of his derivation are
\[
m \ddot{x}^\mu = G^\mu_a(x) I^a + F^\mu_{a\nu}(x) I^a \dot{x}^\nu \tag{5.13}
\]
where the fields satisfy
\[
(D_\mu G_\nu - D_\nu G_\mu)_a = 0 \tag{5.14}
\]
\[
(D_\mu F_{\nu\rho} + D_\nu F_{\rho\mu} + D_\rho F_{\mu\nu})_a = 0, \tag{5.15}
\]
and where the form of the covariant derivative is
\[
(D_\mu F_{\nu\rho})_a = \partial_\mu F_{a\nu\rho} - f^{bc} A_{b\mu} F_{c\nu\rho}. \tag{5.16}
\]
The field strength \(F_{\nu\rho}\) is related to \(A_{b\mu}\) through
\[
f^{abc} \left( F_{a\mu\nu} - (\partial_\mu A_{a\nu} - \partial_\nu A_{a\mu} - f^{de} A_{d\mu} A_{e\nu}) \right) = 0. \tag{5.17}
\]
The non-Abelian theory may be examined from the point of view of Section 3, by generalizing the Helmholtz conditions to take account of classical non-Abelian gauge fields according to Wong’s formulation. To achieve this, we associate with variations \(dq\) of the path \(q(\tau)\), a
variation $dI^a$ of the generators $I^a$, which may be understood as the variation of the orientation of the tangent space under $q(\tau) \rightarrow q(\tau) + dq(\tau)$. The explicit form of this variation follows from (5.12): for small $d\tau$,

$$dI^a = f_c^{ab}[A_{b\mu}(\tau, x) \, dx^\mu + \phi_b(\tau, x) d\tau]I^c$$

(5.18)

where we have allowed an explicit $\tau$-dependence for the gauge field, and have included a Lorentz scalar gauge field $\phi_a$, in analogy with the Abelian case. Now, the quantity $M = M_a I^a$ undergoes the variation of the path

$$(\tau, x) \rightarrow (\tau + d\tau, x + dx)$$

(5.19)

according to

$$dM = (dM_a) I^a + M_a (dI^a)$$

$$= \left( \frac{\partial M_a}{\partial \tau} d\tau + \frac{\partial M_a}{\partial x^\mu} dx^\mu + \frac{\partial M_a}{\partial \dot{x}^\mu} d\dot{x}^\mu + \frac{\partial M_a}{\partial \ddot{x}^\mu} d\ddot{x}^\mu \right) I^a + M_a [f_c^{ab} A_{b\mu} \, dx^\mu + \phi_b d\tau]I^c$$

$$= \left[ \frac{\partial M_a}{\partial \tau} - f_a^{bc} \phi_b M_c \right] I^a d\tau + \left[ \frac{\partial M_a}{\partial x^\mu} - f_a^{bc} A_{b\mu} M_c \right] I^a dx^\mu + \frac{\partial M_a}{\partial \dot{x}^\mu} I^a d\dot{x}^\mu + \frac{\partial M_a}{\partial \ddot{x}^\mu} I^a d\ddot{x}^\mu$$

$$= D_\tau M d\tau + D_\mu M dx^\mu + \frac{\partial M}{\partial \dot{x}^\mu} d\dot{x}^\mu + \frac{\partial M}{\partial \ddot{x}^\mu} d\ddot{x}^\mu$$

(5.20)

in which the spacetime part of the covariant derivative $D_\mu$ has the form of (5.16), and a similar covariant derivative for the $\tau$ component appears which contains $\phi_a$. Now, the entire structure presented in Section 3 follows with the replacements

$$\frac{\partial}{\partial x^\mu} \rightarrow D_\mu \quad \frac{\partial}{\partial \tau} \rightarrow D_\tau ,$$

(5.21)

so that the Helmholtz conditions become

$$A_{\mu\nu} = A_{\nu\mu} \quad \frac{\partial A_{\mu\nu}}{\partial \dot{x}^\sigma} = \frac{\partial A_{\nu\mu}}{\partial \dot{x}^\sigma}$$

(5.22)

$$\frac{D}{D\tau} A_{\mu\nu} = -\frac{1}{2} \left[ A_{\mu\sigma} \frac{\partial f^\sigma}{\partial \dot{x}^\nu} + A_{\nu\sigma} \frac{\partial f^\sigma}{\partial \dot{x}^\mu} \right]$$

(5.23)

$$\frac{1}{2} \frac{D}{D\tau} \left[ A_{\mu\sigma} \frac{\partial f^\sigma}{\partial \dot{x}^\mu} - A_{\nu\sigma} \frac{\partial f^\sigma}{\partial \dot{x}^\nu} \right] = A_{\mu\sigma} D_\nu f^\sigma - A_{\nu\sigma} D_\mu f^\sigma$$

(5.24)

where

$$\frac{D}{D\tau} = D_\tau + \dot{x}^\sigma D_\sigma + f^\sigma \frac{\partial}{\partial \dot{x}^\sigma}$$

(5.25)
is the total $\tau$ derivative subject to

$$\ddot{x}_\mu - f_{\alpha\mu}(\tau, x, \dot{x})I^\alpha = 0 \quad (5.26)$$

Since Hojman and Shepley’s argument relates only to the commutation relations among the coordinates, not to the structure of the forces, their result carries over unchanged.

We now apply equations (5.22) — (5.26) to the case of flat spacetime. Since $A_{\mu\nu} = g_{\mu\nu} = \eta_{\mu\nu}$ is constant, (5.22) is trivially satisfied and (5.23) becomes

$$\partial f_\mu \partial \dot{x}_\nu + \partial f_\nu \partial \dot{x}_\mu = 0 = \implies \frac{\partial^2 f_\mu}{\partial \dot{x}_\nu \partial \dot{x}_\lambda} + \frac{\partial^2 f_\nu}{\partial \dot{x}_\mu \partial \dot{x}_\lambda} = 0 \quad (5.27)$$

Recalling the identity (3.27), we may also write (since the metric carries no group indices)

$$\frac{\partial^2 f_\mu}{\partial \dot{x}_\nu \partial \dot{x}_\lambda} - \frac{\partial^2 f_\nu}{\partial \dot{x}_\mu \partial \dot{x}_\lambda} = 0 \quad (5.28)$$

so that

$$\frac{\partial^2 f_\mu}{\partial \dot{x}_\nu \partial \dot{x}_\lambda} = 0 \quad (5.29)$$

Therefore, the most general form of $f_\mu$ is

$$f_\mu = f_{\mu\nu}(\tau, x)\dot{x}_\nu + g_\mu(\tau, x) \quad (5.30)$$

where (5.27) requires that $f_{\mu\nu} + f_{\nu\mu} = 0$. Finally, applying (5.24) we find

$$\frac{1}{2} \frac{D}{D\tau} \left[ \partial f_\mu \partial \dot{x}_\nu - \partial f_\nu \partial \dot{x}_\mu \right] = \mathcal{D}_\nu f_\mu - \mathcal{D}_\mu f_\nu$$

$$\frac{1}{2} \frac{D}{D\tau}[f_{\mu\nu} - f_{\nu\mu}] = \mathcal{D}_\nu f_{\mu\lambda} \dot{x}_\lambda + \mathcal{D}_\nu g_\mu - \mathcal{D}_\mu f_{\nu\lambda} \dot{x}_\lambda + \mathcal{D}_\mu g_\nu$$

$$(\mathcal{D}_\tau + \dot{x}_\lambda \mathcal{D}_\lambda)f_{\mu\nu} = \dot{x}_\lambda(\mathcal{D}_\nu f_{\mu\lambda} - \mathcal{D}_\mu f_{\nu\lambda}) + \mathcal{D}_\nu g_\mu - \mathcal{D}_\mu g_\nu \quad (5.31)$$

Since $\dot{x}_\mu$ is arbitrary, we find that

$$\mathcal{D}_\lambda f_{\mu\nu} + \mathcal{D}_\mu f_{\nu\lambda} + \mathcal{D}_\nu f_{\lambda\mu} = 0 \quad (5.32)$$

$$\mathcal{D}_\tau f_{\mu\nu} + \mathcal{D}_\mu g_\nu - \mathcal{D}_\nu g_\mu = 0 \quad (5.33)$$

Now, in analogy to the Abelian case, we may write

$$L = \frac{1}{2} m \ddot{x}_\mu \dot{x}_\mu + A_{\alpha\mu}(\tau, x)I^\alpha(\tau)\dot{x}_\mu + \phi_\alpha(\tau, x)I^\alpha(\tau) \quad (5.34)$$
Applying the Euler-Lagrange equations to (5.34), we obtain
\[ \frac{d}{d\tau} [m\ddot{x}^\mu + A_{a\mu} I^a] = \frac{\partial}{\partial x^\mu} [A_{a\nu} I^a \dot{x}^\nu + \phi_a I^a] \]

\[ m\ddot{x}^\mu + \frac{\partial A_{a\mu}}{\partial \tau} I^a + \frac{\partial A_{a\mu}}{\partial x^\nu} \dot{x}^\nu I^a + A_{a\mu} \dot{I}^a = \frac{\partial A_{a\nu}}{\partial x^\mu} \dot{x}^\nu I^a + \frac{\partial \phi_a}{\partial x^\mu} I^a \]  

(5.35)

Rearranging terms and using (5.18) to express \( \dot{I}^a \), we find
\[ m\ddot{x}^\mu = \left[ \frac{\partial A_{a\nu}}{\partial x^\mu} \dot{x}^\nu - \frac{\partial A_{a\mu}}{\partial x^\nu} \dot{x}^\nu \right] I^a - A_{a\mu} \dot{I}^a \]

\[ = \left[ \frac{\partial A_{a\nu}}{\partial x^\mu} \dot{x}^\nu - \frac{\partial A_{a\mu}}{\partial x^\nu} \dot{x}^\nu \right] I^a - A_{a\mu} \frac{\partial A_{a\nu}}{\partial \tau} I^a + \frac{\partial \phi_a}{\partial x^\mu} I^a - \frac{\partial A_{a\mu}}{\partial \tau} I^a \]

\[ = \left[ \frac{\partial A_{a\nu}}{\partial x^\mu} \dot{x}^\nu - \frac{\partial A_{a\mu}}{\partial x^\nu} \dot{x}^\nu + f_{a}^{\;bc} A_{a\mu} A_{b\nu} \right] \dot{x}^\nu I^a + \left[ \frac{\partial \phi_a}{\partial x^\mu} - \frac{\partial A_{a\mu}}{\partial \tau} + f_{a}^{\;bc} A_{a\mu} \phi_b \right] I^a. \]  

(5.36)

Comparing (5.36) with (5.30), we may express the field strengths in terms of the potentials as
\[ f_{\mu\nu} = \left[ \frac{\partial A_{a\nu}}{\partial x^\mu} - \frac{\partial A_{a\mu}}{\partial x^\nu} + f_{a}^{\;bc} A_{a\mu} A_{b\nu} \right] \dot{x}^\nu I^a \]

\[ g_{\mu} = \left[ \frac{\partial \phi_a}{\partial x^\mu} - \frac{\partial A_{a\mu}}{\partial \tau} + f_{a}^{\;bc} A_{a\mu} \phi_b \right] I^a, \]  

(5.37)

from which it follows that (5.32) and (5.33) are satisfied. We remark that since Tanimura did not include a Lorentz scalar potential and his fields were assumed to be \( \tau \)-independent, there is no potential in his formulation from which the field \( G_{\mu} \) in (5.13) and (5.14) could be derived. On the other hand, by the argument of Hojman and Shepley, equations (5.13) are equivalent to the Lagrangian given in (5.34), so that the scalar potential must be present to obtain a non-zero \( G_{\mu} \). Unlike the Abelian case, the potentials appear in the covariant derivative, and so the “new” gauge potential will mix with all quantities whose covariant derivatives are calculated.

As in (2.31), we may introduce the definitions
\[ x^d = \tau \quad \partial_\tau = \partial_d \quad f_{\mu d} = -f_{d \mu} = g_{\mu}. \]  

(5.38)

We may then combine (5.32) and (5.33) as
\[ \partial_\alpha f_{\beta \gamma} + \partial_\beta f_{\gamma \alpha} + \partial_\gamma f_{\alpha \beta} = 0 \]  

(5.39)
(for \(\alpha, \beta, \gamma = 0, \cdots, d\)). The Lorentz force equation becomes

\[
m\ddot{x}^\mu = f_a^{\mu\nu} \dot{x}_\nu I^a + g_a^\mu I^a = f_a^{\mu\nu} \dot{x}_\nu + f_a^{\mu}_d I^a \dot{x}^d = f_a^{\mu}_\beta \dot{x}^\beta. \quad (5.40)
\]

where

\[
f_{\alpha\beta} = \left[ \frac{\partial A_{\alpha\beta}}{\partial x^\alpha} - \frac{\partial A_{\alpha\alpha}}{\partial x^\beta} + f_{\alpha\beta} A_{\alpha\alpha} A_{\beta\beta} \right] I^a \quad (5.41)
\]

recovers the usual relationship of the field strength tensor to the non-Abelian potential.

We finally examine the conservation of \(\dot{x}^\mu \dot{x}_\mu\). In the non-Abelian case, the mass-shell condition becomes (compare with (4.20))

\[
\frac{d}{d\tau} \left( \frac{1}{2} m \dot{x}^2 \right) = m \dot{x}^\mu \dot{x}_\mu = \dot{x}^\mu I^a (g_{a\mu} + f_{a\mu\nu} \dot{x}^\nu) = \dot{x}^\mu g_{a\mu} I^a = 0 \quad (5.42)
\]

which implies

\[
g_{a\mu} = 0 \quad \text{and} \quad \partial_\tau f^{a\mu\nu} = 0 \quad (5.43)
\]

where the second expression follows from (5.33). Notice that \(I^a(\tau)\) introduces a \(\tau\)-dependence which is present in the Lagrangian even when the fields are \(\tau\)-independent. But one may easily compute the Hamiltonian from (5.34) as

\[
K = \dot{x}^\mu \frac{\partial L}{\partial \dot{x}^\mu} - L = \frac{1}{2} m \dot{x}^\mu \dot{x}_\mu - \phi_a I^a \quad (5.44)
\]

so that (5.42) tells us that the Hamiltonian is conserved under the conditions (5.43), in apparent contradiction to the explicit \(\tau\)-dependence of the Lagrangian. We may check, however, by explicit calculation that

\[
\frac{\partial L}{\partial \tau} = A_{a\mu} \dot{x}^\mu \dot{I}^a + \phi_a \dot{I}^a = f^{abc} A_{a\mu} A_{b\nu} \dot{x}^\mu \dot{x}^\nu I^c + \phi_a \dot{I}^a = \phi_a \dot{I}^a. \quad (5.45)
\]

Thus, despite the explicit appearance of \(\tau\) in the Lagrangian, the structure of the non-Abelian field guarantees that \(\partial L/\partial \tau = 0\) in the absence of the “new” gauge potential \(\phi_a\), leading to the preservation of the mass-shell as conservation of the Hamiltonian, as in the Abelian case.

### 6 Conclusion

In Feynman’s 1948 derivation and in the powerful technique of Hojman and Shepley, one sees that the form of the commutation relations between position and velocity (defined as
a derivative with respect to an independent time parameter) determines the most general form of the forces which may act on those phase space variables, and that these forces are of a gauge type. Since the commutation relations determine the form of the Lagrangian to be a scalar with respect to the group which preserves the metric, and since the evolution parameter $\tau$ is an independent variable, the Hamiltonian will also be a scalar and will generate translations of this parameter. The gauge group may include $\tau$-dependent gauge transformations, requiring a conjugate compensation field. The source-free equations for the gauge fields then admit a formal spacetime symmetry which is larger than the symmetry group of original phase space, in which the parameter $\tau$ plays the role of a coordinate. In the case of Feynman’s assumption of Newtonian mechanics, the appearance of the fourth gauge field $A_0$ compensates for $t$-dependent gauge transformations, and the homogeneous field equations are formally consistent with a 4-dimensional symmetry, which could be $O(3,1)$. Similarly, in the case of Tanimura’s assumption of $2d$ independent phase space variables with an $O(d-1,1)$ symmetry, the appearance of the $d+1$st gauge field compensates for $\tau$-dependent gauge transformations, and the homogeneous field equations are formally consistent with a $d+1$-dimensional symmetry, possibly $O(d,1)$ or $O(d-1,2)$. From the Lorentz force law for the $O(d-1,1)$ theory, one may find the $d+1$st expression which explicitly relates the “new” field with the exchange of mass between the field and the sources. Similarly, in the $O(3)$ case derived by Feynman, one may derive, from the three independent components of the Lorentz force law, a fourth expression which relates the $t$-dependent $E$-field to the exchange of scalar energy between the field and the sources.

For the case of the $O(3,1)$ theory, one has a means of arriving at the Maxwell theory of electrodynamics from commutation relations. Deriving the Maxwell theory in this manner provides a clear picture of the way in which the usual on-shell dynamics is a proper restriction of the general off-shell theory.

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