\textbf{$A_4$ Flavour Model for Dirac Neutrinos: Type I and Inverse Seesaw}

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Abstract

We propose two different seesaw models namely, type I and inverse seesaw to realise light Dirac neutrinos within the framework of $A_4$ discrete flavour symmetry. The additional fields and their transformations under the flavour symmetries are chosen in such a way that naturally predicts the hierarchies of different elements of the seesaw mass matrices in these two types of seesaw mechanisms. For generic choices of flavon alignments, both the models predict normal hierarchical light neutrino masses with the atmospheric mixing angle in the lower octant. Apart from predicting interesting correlations between different neutrino parameters as well as between neutrino and model parameters, the model also predicts the leptonic Dirac CP phase to lie in a specific range $-\pi/3 \text{ to } \pi/3$. While the type I seesaw model predicts smaller values of absolute neutrino mass, the inverse seesaw predictions for the absolute neutrino masses can saturate the cosmological upper bound on sum of absolute neutrino masses for certain choices of model parameters.

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1 Introduction

Although the observations of non-zero neutrino mass and large leptonic mixing have been confirmed by several neutrino experiments in the last two decades [1–7], three important issues related to neutrino physics are yet not settled. They are namely, (a) nature of neutrinos: Dirac or Majorana, (b) mass hierarchy of neutrinos: normal ($m_3 > m_2 > m_1$) or inverted ($m_2 > m_1 > m_3$) and (c) leptonic CP violation. The present status of different neutrino parameters can be found in the latest global fit analysis [8, 9]. While neutrino oscillation experiments are insensitive to the nature of neutrinos, experiments looking for lepton number violating signatures can probe the Majorana nature of neutrinos. Neutrinoless double beta decay ($0\nu\beta\beta$) is one such lepton number violating process which has been searched for at several experiments without any positive result so far but giving stricter bounds on the effective neutrino mass. Cosmology experiments are also giving tight constraints on the lightest neutrino mass from the measurement of the sum of absolute neutrino masses $\sum m_i \leq 0.17$ eV [10], disfavouring the quasi-degenerate regime of light neutrino masses.

Although negative results at $0\nu\beta\beta$ experiments do not prove that the light neutrinos are of Dirac nature, it is nevertheless suggestive enough to come up with scenarios predicting Dirac neutrinos with correct mass and mixing. There have been several proposals already that can generate tiny Dirac neutrino masses [11–24]. While most of these scenarios explain the origin of tiny Dirac mass through some type of seesaw mechanisms at tree or loop level, there are some scenarios [13,24] which consider an additional scalar doublet apart from the standard model (SM) one which acquire a tiny vacuum expectation value (vev) naturally due to the presence of a softly broken global symmetry. These Dirac neutrino mass models also incorporate additional symmetries like $U(1)_{B-L}, Z_N, A_4$ in order to generate a tiny neutrino mass of purely Dirac type with specific mixing patterns. These symmetries play a crucial role either in forbidding a tree level Dirac mass term between left handed lepton doublet and right handed neutrino singlet or a Majorana mass term of right handed neutrino singlet. In this work, we particularly look at the possibility of a flavour symmetric scenario for Dirac neutrinos within the well motivated $A_4$ flavour symmetry group. The details of this non-abelian discrete group is given in Appendix A and can also be found in several review articles [25]. Although there are many $A_4$ realisations of seesaw mechanisms for Majorana neutrinos (see [26] and references there in), there are not many studies done in the context of Dirac neutrinos. Recently there have been some attempts in this direction, specially for type I seesaw [22], type II seesaw [23] and neutrinoophilic two Higgs doublet model [24] for Dirac neutrinos.

In the present work, we propose two different seesaw scenarios for Dirac neutrinos namely, type I and inverse seesaw within the framework of $A_4$ flavour symmetry. Type I seesaw for Dirac neutrinos with $A_4$ flavour symmetry was also proposed recently by the authors of [22] along with its correlation to dark matter stability. In this work, we propose a more minimal version of type I seesaw as we do not incorporate dark matter into account. We also incorporate additional $Z_N$ discrete symmetries in such a way that naturally explains the hierarchy of different terms in the neutrino mass matrix. Here we note that type I seesaw for Majorana neutrinos were proposed long back [27]. We then propose an inverse seesaw realisation of Dirac neutrinos within the $A_4$ flavour symmetric framework. For earlier works on this seesaw mechanism for Majorana neutrinos, one
may refer to \cite{28}. Unlike canonical seesaw models, the inverse seesaw can be a low scale framework where the singlet heavy neutrinos can be at or below the TeV scale without any fine tuning of Yukawa couplings. In the Majorana neutrino scenario, this is possible due to softly broken global lepton number symmetry by the singlet mass term. In the present case, we however, have a conserved lepton number global symmetry due to the purely Dirac nature of light neutrinos. Therefore, it is no longer possible to use soft $U(1)_L$ global symmetry breaking argument to generate a tiny singlet mass term. In spite of that, we generate a tiny singlet neutrino mass term at next to leading order by appropriately choosing $Z_N$ discrete symmetries. Such discrete symmetries make sure that such a term do not arise at leading order so that its smallness can be naturally explained from higher order terms. Similar to the type I seesaw case, here also we can naturally explain the hierarchy of different terms present in the inverse seesaw mass matrix. In both of these models, the antisymmetric term arising out of the products of two $A_4$ triplets plays a non-trivial role in generating the correct neutrino mixing. We can obtain the tribimaximal (TBM) mixing from the symmetric contribution of the product of the two triplet flavons while nonzero $\theta_{13}$ is generated from the anti-symmetric contribution \cite{24}. Such anti-symmetric contribution from $A_4$ triplet products can play a non-trivial role in generating nonzero $\theta_{13}$ in Majorana neutrino scenarios (through Dirac Yukawa coupling appearing in type I seesaw) as well \cite{29}. The Dirac neutrino mass matrix can completely dictate the observed neutrino mixing in this construction in case where the charged lepton mass matrix is diagonal. However, in some cases, the charged lepton mass matrix can be non-trivial and has an important contribution to lepton mixing.

Both of the discrete flavour symmetric constructions for type I and inverse seesaw mechanisms show highly predictive nature of the models for generic choices of flavon alignments. The anti-symmetric contribution arising from the Dirac nature of neutrinos not only generate nonzero $\theta_{13}$ but also shows deviations from maximal value of atmospheric mixing angle, favoured by the latest global fit data \cite{8,9}. Interestingly, $\theta_{23}$ is found to be in the lower octant in our models. Now, due to the particular flavour structures of the models, only normal hierarchy for neutrino mass spectrum is allowed, another interesting prediction of the model. In addition to this, we also constrain the absolute neutrino masses and Dirac CP phase, that can be probed at ongoing and future experiments. The model can also be falsified by any future observation of $0\nu\beta\beta$.

This letter is organised as follows. In Section 2 we present complete $A_4$ flavour symmetric models for type-I and inverse seesaw scenario respectively. Complete phenomenology of the associated models and their predictions are also presented in this section. Then we conclude in Section 3 and included a short note on $A_4$ multiplication rules involved in our analysis in the Appendix A.

\section{$A_4$ Flavour Model with Dirac Neutrinos}

\subsection{Dirac Type I seesaw}

Unlike in the canonical seesaw mechanism for Majorana neutrinos \cite{27} where we incorporate the presence of three (at least two) Majorana heavy neutrinos, here we introduce two copies of Weyl fermions $N_L$ and $N_R$ per generation, which are charged under discrete $Z_4 \times Z_3$ symmetry as given in Table 1. Here $N_{L,R}$ can also be considered to be part of
a heavy Dirac fermion whose mass can arise either as a bare mass term or from flavons
depending upon their transformations under the flavour symmetries. In Table 1, we also
show the relevant SM fields, required flavon fields as well as their transformations under
the flavour symmetry. It can be seen from the symmetry transformations that a Dirac
mass term for light neutrinos can not be written at tree level. However, we can write
down mass term for heavy neutrinos as well as coupling between light and heavy neu-
trinos, so that the effective light neutrino Dirac mass can be generated from a seesaw
mechanism.

| Fields | L  | e_R, µ_R, τ_R | H  | ν_R | N_L | N_R | φ_S | φ_T | ξ | χ |
|--------|----|----------------|----|-----|-----|-----|-----|-----|----|----|
| A_4    | 3  | 1, 1', 1''     | 1  | 3   | 3   | 3   | 3   | 3   | 1  |
| Z_4    | i  | -i             | 1  | 1   | -1  | -1  | 1   | 1   | -i |
| Z_3    | ω  | ω              | 1  | ω²  | ω   | ω   | 1   | ω   | 1  |

Table 1: Field content and transformation properties under $A_4 \times Z_4 \times Z_3$ symmetry.

The relevant Lagrangian for charged lepton sector can be written as

$$\mathcal{L}_l = \frac{y_e}{\Lambda} (\bar{L}_e \phi_T) H e_R + \frac{y_\mu}{\Lambda} (\bar{L}_\mu \phi_T) H \mu_R + \frac{y_\tau}{\Lambda} (\bar{L}_\tau \phi_T) H \tau_R.$$  (1)

For generic flavon vev alignment $\langle \phi_T \rangle = (v_T, v_T, v_T)$ the corresponding mass matrix is
given by

$$m_l = \frac{v v_T}{\Lambda} \begin{pmatrix} y_e & y_\mu & y_\tau \\ y_e & \omega y_\mu & \omega^2 y_\tau \\ y_e & \omega^2 y_\mu & \omega y_\tau \end{pmatrix},$$  (2)

where $\Lambda$ is the cut-off scale of the theory and $y_e, y_\mu, y_\tau$ are respective coupling constants.
This matrix can be diagonalised by using the magic matrix $U_\omega$, given by

$$U_\omega = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}.$$  (3)

Now, the Lagrangian for neutrino sector can be written as

$$\mathcal{L} = y_D \chi \bar{\nu}_R \nu_R / \Lambda + y_{d'} \chi^2 \bar{N}_L \nu_R / \Lambda + y_\xi \bar{N}_L N_R + y_s \phi_S \bar{N}_L N_R + y_a \phi_S \bar{N}_L N_R + h.c.$$  (4)

where the subscripts 3s, 3a correspond to symmetric and anti-symmetric parts of triplet
products in the $S$ diagonal $A_4$ basis, given in Appendix A. From these contributions, we
obtain the mass matrices in $(\nu_L, N_R), (N_L, \nu_R), (N_L, N_R)$ basis as

$$M_D = \frac{y_D v v_\chi}{\Lambda} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, M'_D = \frac{y_{d'} v^2 \chi}{\Lambda} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$  and

$$M = \begin{pmatrix} x & 0 & s + a \\ 0 & x & 0 \\ s - a & 0 & x \end{pmatrix}$$ with $\langle \chi \rangle = v_\chi, \langle \xi \rangle = v_\xi, \langle \phi_S \rangle = (0, v_S, 0)$  (5)
respectively. Such vev alignment for one of the $A_4$ triplet, $\phi_S$ in this case, is widely used in the $S$ diagonal basis of $A_4$ and can be realised in a natural way by minimisation the scalar potential \[24,30–33\]. In the $T$ diagonal basis other possible vev alignment (e.g. where the first component of the triplet gets vev) is adopted \[34\]. Here it is worth mentioning that in the present set-up other possible vev alignments (e.g. where the first component of the triplet gets vev) is adopted \[34\]. Here it is worth mentioning that in the present set-up other possible vev alignments like $\langle \phi_S \rangle = (v_S, 0, 0)$ or $\langle \phi_S \rangle = (0, 0, v_S)$ are unable to reproduce correct neutrino mixing as observed by the experiments. The vev of the SM Higgs is denoted by $v$. Here we have defined $x = y_\xi v_\xi$, $s = y_s v_s$, $a = y_a v_S$ and $y_D, y_D', y_t, y_s, y_a$ are respective coupling constants involved in the neutrino Lagrangian. Note that $s$ and $a$ are the symmetric and antisymmetric contributions originated from $A_4$ multiplication, mentioned earlier. This antisymmetric part only contribute in the mass matrix if neutrinos are Dirac particles \[24\] or in the Dirac neutrino mass matrix used in canonical seesaw mechanism for Majorana light neutrinos \[29\]. On the other hand, only the symmetric part contributes in a Majorana neutrino mass matrix as the anti-symmetric part identically vanishes. Here we will find that this antisymmetric part, originated due to the Dirac nature of neutrinos, plays an instrumental role in the rest of the analysis and crucially dictates the neutrino masses and mixing. Now, the light Dirac neutrino mass matrix in this type I seesaw like scenario can be written as

$$m_\nu = -M_D' M^{-1}_D$$

$$= -\frac{y_D y_D' vv_3}{\Lambda^2} M^{-1}$$

$$= -\lambda \begin{pmatrix}
  x & 0 & -(a + s) \\
  0 & x^2 - s^2 + x^2 & 0 \\
  a - s & 0 & x
\end{pmatrix},$$

where $\lambda = \frac{y_D y_D' v v_3}{\Lambda^2 (a^2 - s^2 + x^2)}$ is a dimensionless quantity. It should be noted that the simple type I seesaw formula written above for light Dirac neutrinos is obtained under the assumption $M_D, M_D' \ll M$ which is justified as the latter is generated at leading order whereas $M_D, M_D'$ arise at dimension five level only due to the chosen particle content and their symmetry transformations. Now we define a Hermitian matrix as

$$\mathcal{M} = m_\nu m_\nu^\dagger$$

$$= |\lambda|^2 \begin{pmatrix}
  |x|^2 + |a + s|^2 & 0 & x(a - s)^* - x^*(a + s) \\
  0 & \frac{a^2 - s^2 + x^2}{x} & 0 \\
  x^*(a - s) - x(a + s)^* & 0 & |x|^2 + |a - s|^2
\end{pmatrix}.$$  \hspace{1cm} (11)

This matrix can be diagonalised by a unitary matrix $U_{13}$, given by

$$U_{13} = \begin{pmatrix}
  \cos \theta & 0 & \sin \theta e^{-i\psi} \\
  0 & 1 & 0 \\
  -\sin \theta e^{i\psi} & 0 & \cos \theta
\end{pmatrix},$$

through the relation $U_{13}^\dagger \mathcal{M} U_{13} = \text{diag}(m_1^2, m_2^2, m_3^2)$. Here we find the mass eigenvalues
the lepton mixing matrix $U$ is given by

$$
m_1^2 = \kappa^2 \left[ 1 + \alpha^2 + \beta^2 - \sqrt{2 \alpha \beta \cos(\phi_{ax} - \phi_{sx})} \right] + 4 \left( \alpha^2 \sin^2 \phi_{ax} + \beta^2 \cos^2 \phi_{sx} \right)$$

$$
m_2^2 = \kappa^2 \left[ 1 + \alpha^4 + \beta^4 + 2 \alpha^2 \cos 2\phi_{ax} - 2 \beta^2 \cos 2\phi_{sx} - 2 \alpha^2 \beta^2 \cos 2(\phi_{sx} - \phi_{ax}) \right],$$

$$
m_3^2 = \kappa^2 \left[ 1 + \alpha^2 + \beta^2 + \sqrt{2 \alpha \beta \cos(\phi_{ax} - \phi_{sx})} \right] + 4 \left( \alpha^2 \sin^2 \phi_{ax} + \beta^2 \cos^2 \phi_{sx} \right)$$

where we have defined $\kappa^2 = |x|^2|x|^2$, $\alpha = |a|/|x|$, $\beta = |s|/|x|$, $\phi_{sx} = \phi_s - \phi_x$, $\phi_{sx} = \phi_a - \phi_x$ with $s = |s|e^{i\phi_s}$, $a = |a|e^{i\phi_a}$ and $x = |x|e^{i\phi_x}$ respectively. From these definitions it is clear that $\alpha$ is associated with the antisymmetric contribution whereas $\beta$ is related to the symmetric contribution in the Dirac neutrino mass matrix. Now, we obtain the rotation angle and phase involved in $U_{13}$ as

$$
\tan 2\theta = \frac{\beta \cos \phi_{sx} \cos \psi - \alpha \sin \phi_{ax} \sin \psi}{\alpha \beta \cos(\phi_{sx} - \phi_{ax})}
$$

and

$$
\tan \psi = -\frac{\alpha \sin \phi_{ax}}{\beta \cos \phi_{sx}}
$$

respectively. Now the final lepton mixing matrix is given by

$$
U = U_\alpha U_{13},
$$

and the $U_{e3}$ element of the Pontecorvo Maki Nakagawa Sakata (PMNS) leptonic mixing matrix is given by $\frac{1}{\sqrt{3}}(\cos \theta + \sin \theta e^{-i\psi})$. The PMNS mixing matrix is parametrised as

$$
U_{PMNS} = \begin{pmatrix}
c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\
-s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & \frac{s_{13}}{\sqrt{3}} \\
-s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & \frac{s_{23}}{\sqrt{3}}c_{13}
\end{pmatrix}.
$$

Comparing $U_{e3}$ from the model with the one in the standard PMNS leptonic mixing matrix $U_{PMNS}$, we obtain

$$
\sin \theta_{13}e^{-i\delta} = \frac{1}{\sqrt{3}}(\cos \theta + \sin \theta e^{-i\psi}).
$$

Now, $\sin \theta_{13}$ and $\delta$ can be parametrised in terms of $\theta$ and $\psi$ as

$$
\sin^2 \theta_{13} = \frac{1}{3}(1 + \sin 2\theta \cos \psi) \quad \text{and} \quad \tan \delta = \frac{\sin \theta \sin \psi}{\cos \theta + \sin \theta \cos \psi}.
$$

Such correlation between $\sin \theta_{13}$ (i.e., $U_{e3}$) and the model parameters can easily be obtained and can also be found in [24, 35, 38] for other scenarios. From equations (16)-(21) it is clear that, all the mixing angles ($\theta_{13}, \theta_{12}, \theta_{23}$) and Dirac CP phase ($\delta$) involved in the lepton mixing matrix $U_{PMNS}$ are functions of four parameters namely, $\alpha$, $\beta$, $\phi_{ax}$ and $\phi_{sx}$. Now, using $3\sigma$ allowed range [9] of the three mixing angles ($\theta_{13}, \theta_{12}, \theta_{23}$), in figure [1] we have shown the constrained range of $\alpha$, $\beta$, $\phi_{ax}$ and $\phi_{sx}$. In figure [1] the blue dots
Figure 1: Allowed regions of $\beta$-$\alpha$ (left panel) and $\phi_{ax}$-$\phi_{sx}$ (right panel) planes from the $3\sigma$ global fit values of $\theta_{13}$, $\theta_{12}$ and $\theta_{23}$ \cite{9} represented by the blue dots. Red dots in each plot also satisfy $3\sigma$ allowed range for the ratio ($r$) of solar to atmospheric mass squared differences \cite{9}.

represent the allowed points in the $\alpha$-$\beta$ plane (left panel) and $\phi_{ax}$-$\phi_{sx}$ (right panel) plane respectively. In addition to the bounds obtained from the mixing angles, the parameter space can be further constrained in order to satisfy the ratio of solar to atmospheric mass squared differences, defined as

$$r = \frac{\Delta m^2_\odot}{|\Delta m^2_A|} = \frac{\Delta m^2_{21}}{|\Delta m^2_{31}|}. \quad (22)$$

From equation (13-15) and equation (22) it is evident that this ratio $r$ is also function of $\alpha$, $\beta$, $\phi_{sx}$ and $\phi_{ax}$ (which are appearing in the expression for the mixing angles). Once again using the $3\sigma$ range of the neutrino mass squared differences we find the allowed ranges

Figure 2: Left panel: Estimation for $\kappa$ (in eV) as a function of $\alpha$. Right panel: Prediction for absolute neutrino masses (orange, blue, brown and red for $m_1$, $m_2$, $m_3$ and $\sum m_i$, respectively) as a function of $\alpha$. In both cases the parameter space simultaneously satisfies $3\sigma$ allowed range of $\theta_{13}$, $\theta_{12}$, $\theta_{23}$ and $r$ \cite{9} as shown in figure 1.

for $\alpha$, $\beta$, $\phi_{sx}$ and $\phi_{ax}$ given by the red dots in the both panels of figure 1. Therefore, these red dots represents the regions of model parameters that satisfy the complete neutrino
This reveals that the allowed range of \( \alpha \approx 0.6-1.6 \) corresponds to \( \beta \approx 0.4-2.0 \) as evident for the left panel of figure 1. On the other hand, the right panel plot of figure 1 shows that few disconnected regions are allowed in the \( \phi_{sx}-\phi_{ax} \) parameter space.

Now, using these allowed values (obtained from figure 1) for the parameters (\( \alpha, \beta, \phi_{sx} \) and \( \phi_{ax} \)) and the best fit value for the solar mass squared difference \( \Delta m_{\odot}^2 = 7.5 \times 10^{-5} \text{ eV}^2 \), we can find the common factor \( \kappa \) appearing in the absolute neutrino mass eigenvalues using the relation

\[
\kappa = \sqrt{\frac{\Delta m_{\odot}^2 / \{1 + \alpha^4 + \beta^4 + 2\alpha^2 \cos 2\phi_{ax} - 2\beta^2 \cos 2\phi_{sx} - 2\alpha^2 \beta^2 \cos 2(\phi_{sx} - \phi_{ax})\}}{-[1 + \alpha^2 + \beta^2 - (2\alpha \beta \cos(\phi_{ax} - \phi_{sx}))^2 + 4(\alpha^2 \sin^2 \phi_{ax} + \beta^2 \cos^2 \phi_{sx})]^2}}.
\]

Here we have used equations (13) and (14) to deduce the above correlation. In left panel of figure 2 we have plotted the allowed values for \( \kappa \) (in eV) as a function of \( \alpha \) for the solar mass squared difference \( \Delta m_{\odot}^2 = 7.5 \times 10^{-5} \text{ eV}^2 \), we can find the common factor \( \kappa \) appearing in the absolute neutrino mass eigenvalues using the relation

\[
\kappa = \sqrt{\frac{\Delta m_{\odot}^2 / \{1 + \alpha^4 + \beta^4 + 2\alpha^2 \cos 2\phi_{ax} - 2\beta^2 \cos 2\phi_{sx} - 2\alpha^2 \beta^2 \cos 2(\phi_{sx} - \phi_{ax})\}}{-[1 + \alpha^2 + \beta^2 - (2\alpha \beta \cos(\phi_{ax} - \phi_{sx}))^2 + 4(\alpha^2 \sin^2 \phi_{ax} + \beta^2 \cos^2 \phi_{sx})]^2}}.
\]

Here we have used equations (13) and (14) to deduce the above correlation. In left panel of figure 2 we have plotted the allowed values for \( \kappa \) (in eV) as a function of \( \alpha \), where we also find that the allowed range \( \alpha \approx 0.6-1.6 \) restricts \( \kappa \) to fall in the range 0.012-0.03 eV. Now using the estimation for \( \kappa \) as in left-panel of figure 2 we can find the absolute neutrino masses using equations (13)-(15). In the right panel of figure 2 we have plotted the individual absolute neutrino masses (where orange, blue, brown dots stand for \( m_1, m_2 \) and \( m_3 \) respectively) as well as their sum \( \sum m_i \) denoted by the red dots) as a function of \( \alpha \). Here we find that the allowed ranges for the absolute neutrino masses (obeying
normal hierarchy) are given by $m_1 \approx 0.0060 - 0.0023$ eV, $m_2 \approx 0.0105 - 0.0090$ eV, $m_3 \approx 0.0547 - 0.0481$ eV and $\sum m_i \approx 0.0707 - 0.0596$ eV when $\alpha$ is in the range 0.6-1.6. In the present setup, an inverted hierarchy of light neutrino mass spectrum however cannot be accommodated, an interesting prediction that will undergo tests in several ongoing and near future experiments.

![Figure 4: Correlations between different light neutrino parameters for $3\sigma$ allowed range of $\theta_{13}$, $\theta_{12}$, $\theta_{23}$ and $r$.](image)

Now, from equations (17) and (21), we find that the Dirac CP phase can be evaluated once we find allowed parameter space in the present model. Therefore using the allowed regions for $\alpha$, $\beta$, $\phi_{sx}$ and $\phi_{ax}$ as obtained in figure 1, we can find the predictions for the Dirac CP phase $\delta$ within this framework. In figure 3 we have shown the prediction for $\delta$ as a function of $\phi_{ax}$, $\phi_{sx}$, $\alpha$ and it is clear that the model predicts $\delta$ to be in the range $-\pi/3 \lesssim \delta \lesssim \pi/3$.

Next, to understand the correlation between associated observables in the present type I seesaw framework we present few schematics in figure 4. Here the upper left panel represents the correlation between $\delta$ and $\theta_{23}$. This figure shows that in our setup $\theta_{23}$ falls in the lower octant when $\delta$ lies between $-\pi/3$ to $\pi/3$. This is in good agreement with all three global fit analysis [8,9,39] where the best fit value for $\theta_{23}$ prefers to be in the lower octant (although for $3\sigma$ range both octants are possible) for the normal hierarchy of light neutrino masses. And in our case, only normal hierarchy of neutrino mass spectrum is allowed. Now, in the other two panels of figure 4 we have plotted our allowed parameter space in $\sin^2 \theta_{23}$-$m_1$ and $\delta$-$\sum m_i$ plane respectively. From $\sin^2 \theta_{23}$ versus $m_1$ plot it is
clear that as $m_1$ approaches towards its maximum, $\theta_{23}$ also tends towards the maximal value. Now in the left panel of figure 4 we plot sum of the absolute neutrino masses $\sum m_i$ as a function of lightest light neutrino mass $m_1$ and it falls well below the Planck upper limit \cite{10} (as shown by the shaded region). In this figure the splitting in the sum of absolute mass is due to $3\sigma$ uncertainties in the solar and atmospheric mass squared differences \cite{9}. Now, on the other hand, $\delta$ versus $\sum m_i$ plot in the lower right panel of figure 4 shows that for $\delta$ within the range $-\pi/3$ to $\pi/3$, $\sum m_i$ ranges between 0.0707 eV to 0.0596 eV indicating higher value of $\sum m_i$ is allowed only when $\delta \neq 0$. It is interesting to note that, the predicted values of $\sum m_i$ lie well below the cosmological upper bound $\sum m_i \leq 0.17$ eV \cite{10}.

2.2 Dirac Inverse seesaw

In usual inverse seesaw model, the complete neutral fermion mass matrix is $9 \times 9$ whose structure in the $(\nu_L, N_R, S_R)$ basis

\[
M_\nu = \begin{pmatrix}
0 & m_D^T & 0 \\
m_D & 0 & M^T \\
0 & M & \mu
\end{pmatrix}
\]  

(23)

where $m_D$ is the usual Dirac neutrino mass. The lepton number violation occurs only through the $3 \times 3$ block denoted by $\mu$ so that this term can be naturally small. Block diagonalisation of the above mass matrix results in the effective light neutrino mass matrix as,

\[
m_\nu = m_D^T (M^T)^{-1} \mu M^{-1} m_D
\]  

(24)

Unlike canonical seesaw where the light neutrino mass is inversely proportional to the lepton number violating Majorana mass term of singlet neutrinos, here the light neutrino mass is directly proportional to the singlet mass term $\mu$. The heavy neutrino masses are proportional to $M$. Here, even if $M \sim 1$ TeV, correct neutrino masses can be generated for $m_D \sim 10$ GeV, say if $\mu \sim 1$ keV. Such small $\mu$ term is natural as $\mu \rightarrow 0$ helps in recovering the global lepton number symmetry $U(1)_L$ of the model. Thus, inverse seesaw is a natural TeV scale seesaw model where the heavy neutrinos can remain as light as a TeV and Dirac mass can be as large as the charged lepton masses and can still be consistent with sub-eV light neutrino masses.

In this section, we wish to construct a similar mass matrix for Dirac neutrinos so that the smallness of light Dirac neutrino mass can be generated naturally by a TeV scale seesaw. Since lepton number is conserved for Dirac neutrinos, we consider it as a conserved global symmetry of the model, similar to the type I seesaw discussed above. The field content of the proposed model is given in table 2. The $A_4$ symmetry is augmented by $Z_4 \times Z_3 \times Z_2$ discrete symmetries in order to make sure that the desired strengths of different elements of the inverse seesaw mass matrix are naturally obtained.

The Lagrangian for the above field content can be written as

\[
\mathcal{L}_Y = \frac{y_e}{\Lambda} (\bar{L} \phi_T) H e_R + \frac{y_\mu}{\Lambda} (\bar{L} \phi_T) H \mu_R + \frac{y_\tau}{\Lambda} (\bar{L} \phi_T) H \tau_R + \frac{\bar{L} H N_R}{\Lambda} (y_e \xi + y_\mu \phi_S + y_\tau \phi_S) + \frac{\bar{\nu}_R N_L \zeta}{\Lambda} + Y_{NS} \bar{S}_R N_L \zeta + Y'_{NS} \bar{S}_L N_R \zeta + \frac{Y_S}{\Lambda^2} \bar{S}_L S_R \phi^R.
\]  

(25)
We consider the vev alignment (similar to the one present in the previous subsection) of the flavons as

\[
\langle \phi_T \rangle = (v_T, v_T, v_T), \quad \langle \phi_S \rangle = (0, v_S, 0), \quad \langle \xi \rangle = v_{\xi}, \quad \langle \zeta \rangle = v_{\zeta}, \quad \langle \eta \rangle = v_{\eta}, \quad \langle \phi' \rangle = v_{\phi'}. 
\]  

Effective light neutrino mass in this scenario can be written as,

\[
m_\nu = M_{RN} (M'_{NS})^{-1} M_{S} M^{-1}_{NS} M_{\nu N}. 
\]

From the Lagrangian presented in equation (25), we can find the mass matrices involved in the neutrino sector after symmetry breaking ($A_4$ as well as electroweak) as

\[
M_{RN} = \frac{Y_{RN}}{\Lambda} v_{\xi} v_{S} I, \quad M_{NS} = Y_{NS} v_{\xi} I, \quad M'_{NS} = Y'_{NS} v_{\xi} I, \quad M_{S} = \frac{Y_{S}}{\Lambda^2} v_{\phi'} I
\]

\[
M_{\nu N} = v_{v S} \left( \begin{array}{ccc}
  x & 0 & s + a \\
  0 & x & 0 \\
  s - a & 0 & x 
\end{array} \right). 
\]

Here, $x = y_{\xi} v_{\xi}$, $s = y_{a} v_{s}$ and $a = y_{a} v_{s}$ respectively where $s$ and $a$ stands for symmetric and antisyemtric contributions originated from $A_4$ multiplication similar to the type I seesaw case discussed before. The couplings $Y_{RN}, Y_{NS}, Y'_{NS}, Y_{S}, y_{\xi}, y_{a}, y_{a}$ are the Yukawa couplings given in the above Lagrangian and $\Lambda$ is the cut-off scale. Again, here we emphasise that the antisymmetric part of $A_4$ triplet products particularly contribute to any Dirac type mass matrix involved in the neutrino seesaw formula and the associated phenomenology crucially depends on this contribution. Since the construction of the charged lepton sector is exactly identical with type I seesaw scenario, it can again be diagonalised by the magic matrix $U_\omega$ given in equation (3). To diagonalise the neutrino mass matrix let us define the Hermitian mass matrix as before.

\[
M = m_\nu m_{\nu}^\dagger = |\lambda|^2 \left( \begin{array}{ccc}
  |x|^2 + |s + a|^2 & 0 & x(s - a)^* + x^*(s + a) \\
  0 & |x|^2 & 0 \\
  x(s - a) + x(s + a)^* & 0 & |x|^2 + |s - a|^2 
\end{array} \right), 
\]

where $\lambda = \frac{Y_{RN} Y_{S} v_{\phi'}}{Y_{\xi S} Y_{NS} \Lambda^2}$. Here we have assumed the vev of all the scalar flavons (except the SM Higgs) to be same and denoted by $f$, i.e., $v_S = v_\xi = v_\zeta = v_\eta = v_{\phi'} = f$. The Hermitian matrix $M$ can be diagonalised by a unitary matrix $U_{13}$ as given in equation (12), obeying
\[ U_{13}^\dagger MU_{13} = \text{diag}(m_1^2, m_2^2, m_3^2), \]

where the two parameters \( \theta \) and \( \psi \) appearing in \( U_{13} \) are found to be

\[
\tan 2\theta = \frac{\alpha \sin \phi_{ax} \sin \psi - \beta \cos \phi_{ax} \cos \psi}{\alpha \beta \cos(\phi_{ax} - \phi_{ax})} \quad \text{and} \quad \tan \psi = -\frac{\alpha \sin \phi_{ax}}{\beta \cos \phi_{ax}}. \tag{30}
\]

Here, \( \alpha = |a|/|x|, \beta = |s|/|x|, \phi_{ax} = \phi_x - \phi_s, \phi_{ax} = \phi_a - \phi_s \) with \( s = |s|e^{i\phi_s}, a = |a|e^{i\phi_a} \) and \( x = |x|e^{i\phi_x} \) respectively. Hence \( \alpha \) is basically associated with the antisymmetric contribution whereas \( \beta \) is related to the symmetric contribution in the Dirac neutrino mass matrix. The final lepton mixing matrix in this case is also governed by the mixing matrix, \( U = U_{13}^\dagger U_{13} \) involving contributions from both charged lepton and neutrino sector. Therefore, the correlation of \( \theta_{13} \) (and \( \delta \)) with \( \theta \) (and \( \psi \)) in this case is similar to the one presented in the type I seesaw case as given by equation (21).

![Figure 5: Allowed regions of \( \beta \) vs \( \alpha \) (left panel) and \( \phi_{ax} \) vs \( \phi_{sx} \) (right panel) for 3\( \sigma \) allowed range of \( \theta_{13}, \theta_{12} \) and \( \theta_{23} \) represented by the blue dots. Red dots in each plot also satisfies 3\( \sigma \) allowed range for the the solar to atmospheric mass-squared ratio \( r \) along with upper limit on sum of the thee light neutrinos \( \sum m_i \leq 0.17 \text{ eV}^{10} \), representing the actual allowed parameter space.](image)

After diagonalisation of the Hermitian matrix as given in equation (29), the real, positive squared mass eigenvalues are obtained as

\[
m_1^2 = \kappa^2 \left[ 1 + \alpha^2 + \beta^2 - \sqrt{\left(2\alpha\beta \cos(\phi_{ax} - \phi_{ax})\right)^2 + 4\left(\alpha^2 \sin^2 \phi_{ax} + \beta^2 \cos^2 \phi_{ax}\right)} \right] \tag{31}
\]

\[
m_2^2 = \kappa^2, \tag{32}
\]

\[
m_3^2 = \kappa^2 \left[ 1 + \alpha^2 + \beta^2 + \sqrt{\left(2\alpha\beta \cos(\phi_{ax} - \phi_{ax})\right)^2 + 4\left(\alpha^2 \sin^2 \phi_{ax} + \beta^2 \cos^2 \phi_{ax}\right)} \right] \tag{33}
\]

where we have defined \( \kappa^2 = |\lambda|^2|x|^2 \). Here we find that, both neutrino mixing angles and masses are functions of parameters like \( \alpha, \beta, \phi_{ax} \) and \( \phi_{ax} \) as evident from equations (30) and (31-33) respectively. Using similar strategy, we again try to constrain the involved parameter space (\( \alpha, \beta, \phi_{ax} \) and \( \phi_{ax} \)) as illustrated in figure 5. The blue dots in both left (in \( \alpha-\beta \) plane) and right (in \( \phi_{sx}-\phi_{sx} \) plane) panel satisfies 3\( \sigma \) allowed range for the neutrino mixing angles \( [9] \). Then we impose the constraints (varying within 3\( \sigma \) range) coming from the ratio of the two mass squared differences as defined in equation (22). The red dots in both the panels of figure 5 shows allowed ranges of the parameter space,
after taking both these constraints (mixing angles and mass squared difference ratios) into account. In the left panel of figure 5 we find that, corresponding to $\alpha$ in the range 0.2 to 1.7, $\beta$ is restricted within 2.5. The right panel of the same plot reveals that a few disconnected regions in the $\phi_{sx}$-$\phi_{ax}$ plane are allowed. Note that here we have also used the recent upper bound on sum of the thee light neutrinos $\sum m_i \leq 0.17$ eV \[10\] to constrain the parameter space and afterwards we analyse only those regions which satisfy this limit. Now, the common factor ($\kappa$) appearing in the neutrino mass eigenvalues shown in equations (31-33) can be evaluated using

$$\kappa = \sqrt{\Delta m^2_\odot \{1 - [1 + \alpha^2 + \beta^2 - \sqrt{(2\alpha\beta \cos(\phi_{ax} - \phi_{sx}))^2 + 4(\alpha^2 \sin^2 \phi_{ax} + \beta^2 \cos^2 \phi_{sx})]}\}}. \quad (34)$$

In figure 6 left panel we show the estimates of $\kappa$ (in eV) as a function of $\alpha$. Also, it is worth mentioning that due to particular flavour structure of the this inverse seesaw scenario $m_2$ coincides with $\kappa$ as given in equation 32. Now, our prediction for absolute neutrino masses (with orange, blue, brown dots representing $m_1$, $m_2$, $m_3$ and $\sum m_i$ respectively) and Dirac CP phase $\delta$ (right panel) for $3\sigma$ allowed range of $\theta_{13}, \theta_{12}, \theta_{23}, r$ \[9\] along with with upper limit on sum of the thee light neutrinos $\sum m_i \leq 0.17$ eV \[10\].

In figure 6 right panel we predict the absolute neutrino masses (orange, blue, brown, red for $m_1$, $m_2$, $m_3$ and $\sum m_i$ respectively) and Dirac CP phase $\delta$ (right panel) for $3\sigma$ allowed range of $\theta_{13}, \theta_{12}, \theta_{23}$, $r$ \[9\] along with upper limit on sum of the thee light neutrinos $\sum m_i \leq 0.17$ eV \[10\].

Now, to illustrate the prediction for Dirac CP phase and its dependence on the parameters of the model, in figure 7 we present the allowed regions for $\delta$ as a function of $\phi_{ax}$ (upper left panel), $\phi_{sx}$ (upper right panel) and $\alpha$ (bottom panel) respectively. From these plots it turns out that in this inverse seesaw scenario, the allowed value for $\delta$ lies in appropriately range $-\pi/3$ to $+\pi/3$, similar to what we saw for type I seesaw case before.

Finally, to understand the correlation between observables associated with neutrino masses and mixings in this inverse framework, we refer to figure 8. In the upper left
Figure 7: Predictions for Dirac CP phase $\delta$ (in radian) as a function of $\phi_{ax}, \phi_{sx}$ and $\alpha$ for $3\sigma$ allowed range of $\theta_{13}, \theta_{12}, \theta_{23}, r$ along with the limit on sum of the thee light neutrinos $\Sigma m_i \leq 0.17$ eV [10].

panel of this figure a correlation between $\delta$ and $\theta_{23}$ is presented and we find that for $\delta$ in the range $-\pi/3$ to $\pi/3$, $\theta_{23}$ always falls in the lower octant. As mentioned earlier, this is in good agreement with all three global analysis [8,9,39] where the best fit value for $\theta_{23}$ prefers to be in the lower octant (although for $3\sigma$ range both the octants are possible) for the normal hierarchy of light neutrino masses. Here we remind ourself that only normal hierarchy of neutrino mass is allowed in the present scenario. In the upper right panel of figure 8 we have plotted the allowed parameter space in $\sin^2 \theta_{23}-m_1$ plane whereas the bottom left panel represents the allowed region in the $\delta-\Sigma m_i$ plane. From $\sin^2 \theta_{23}$ versus $m_1$ plot it is clear that smaller the lightest neutrino mass $m_1$, more likely is the deviation of $\theta_{23}$ from its maximal value. In the bottom left panel of figure 8 the purple dots show the model predictions for $\Sigma m_i$ corresponding to the lightest light neutrino mass $m_1$, representing a high mass regime for the light neutrinos. Here, the region bounded by the solid lines represent $3\sigma$ uncertainty in the mass squared differences and the shaded region stands for the disallowed region by the Planck upper limit [10]. Finally, $\delta$ vs $\Sigma m_i$ plot in the bottom right panel shows that all regions of $\delta$ (between $-\pi/3$ to $\pi/3$) allowed with $\Sigma m_i$ ranging in between 0.067 eV to 0.17 eV indicating higher value of $\Sigma m_i$ is only possible when $\delta \neq 0$. Such high values of $\Sigma m_i$ can saturate the cosmological upper bound $\Sigma m_i \leq 0.17$ eV [10] which can indirectly constrain the Dirac CP phase as well.

It is observed that the allowed range of the lightest neutrino mass is different in inverse seesaw case compared to what is obtained for type I seesaw. This is evident
from the right panels of figure 2 and figure 6 for type I and inverse seesaw respectively. This can be explained from the difference in light neutrino mass eigenvalue expressions given in equations (13), (14), (15) for type I seesaw and equations (31), (32), (33) for inverse seesaw. As can be seen from these expressions, the second mass eigenvalue \( m_2 \) expression is very different in the two cases due to the \( A_4 \) flavour symmetric construction and the governing seesaw mechanism. Due to this difference, constraint coming from the ratio of solar to atmospheric mass squared differences \( r \) in these two scenarios are such that the inverse seesaw scenario permits a relatively larger allowed parameter space (for \( \alpha \) and \( \beta \)) satisfying neutrino oscillation data. This is evident from the left panel of figure 1 (for type I seesaw) and figure 5 (for inverse seesaw) respectively, where red dots represents allowed parameter space and one can find that relatively smaller values for \( \alpha \) and \( \beta \) are allowed for inverse seesaw compared to the type-I seesaw scenario. These smaller values of \( \alpha \) and \( \beta \) for inverse seesaw case actually yields larger value for the common factor \( \kappa \) (evaluated using equation (34)) appearing in the absolute light neutrino masses and hence generates larger value for neutrino mass compared to type I case.

### 3 Conclusion

We have studied two different seesaw scenarios for light Dirac neutrinos namely, type I and inverse seesaw within the framework of \( A_4 \) flavour symmetry to explain lepton...
masses and mixing. In both the cases, the $A_4$ symmetry is augmented by additional discrete symmetries in order to make sure that the correct hierarchy between different terms appearing in the complete neutral fermion mass matrix is naturally obtained without making any ad hoc assumptions. This is done by generating relatively smaller terms at next to leading order compared to the large terms in the seesaw matrix. Since lepton number is a global conserved symmetry in both the cases, all the mass matrices involved are of Dirac type and hence the $A_4$ triple products contain the anti-symmetric component. This anti-symmetric part plays a crucial role in generating the correct neutrino phenomenology by explicitly breaking $\mu - \tau$ symmetries which give rise to vanishing reactor mixing angle. Since we use the $S$ diagonal basis of $A_4$ for Dirac neutrino case, the charged lepton mass matrix is also non trivial in our scenarios and hence can contribute to the leptonic mixing matrix.

For generic choices of $A_4$ flavon alignments, we find that both the models are very predictive in terms of predicting the light neutrino mass spectrum and hierarchy, leptonic CP phase as well as the octant of atmospheric mixing angle. While both of them predicts normal hierarchical pattern of light neutrino masses with the atmospheric mixing angle lying in the lower octant, in agreement with the latest global fit neutrino oscillation data, they also predict the lepton Dirac CP phase to lie in specific range $-\pi/3$ to $\pi/3$. While the type I seesaw predicts the sum of light neutrino masses to be small, the inverse seesaw scenario predicts it to be high and can saturate the cosmological upper bound $\sum m_i \leq 0.17$ eV. Apart from this, the models also predict interesting correlation between neutrino observables like Dirac CP phase, atmospheric mixing angle, light neutrino masses so that measuring one can shed light on the other. Both the models can also predict the absence of lepton number violation and hence can not be tested in ongoing and future neutrinoless double beta decay experiments. Also, the inverse seesaw model can naturally predict lighter heavy neutrino spectrum compared to type I seesaw and hence can have other phenomenological consequences. Such a detailed analysis is left for future investigations.

Apart from different predictions for light neutrino parameters, the two seesaw scenarios discussed here can also be distinguished by observing different phenomena they give rise to. Since the light neutrino mass in inverse seesaw mechanism is primarily governed by the smallness of the $\mu$ term in (24), the right handed neutrinos can have masses near the TeV scale and at the same time can have sizeable Yukawa couplings with the light neutrinos, giving rise to interesting possibilities at collider experiments \cite{40}. This interesting feature makes it different from ordinary type I seesaw, where TeV scale right handed neutrino mass has to be compensated by tiny Yukawa couplings. We may also get distinguishable features in terms of predictions of these two models, if we also incorporate the quark sector mixing \cite{30}. For simplicity, we have considered the quark sector particles to be singlet under the $A_4$ symmetry and leave a more general study including quarks and leptons to future studies.

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A4 Multiplication Rules

A4, the symmetry group of a tetrahedron, is a discrete non-abelian group of even permutations of four objects. It has four irreducible representations: three one-dimensional and one three dimensional which are denoted by 1, 1', 1'' and 3 respectively, being consistent with the sum of square of the dimensions $\sum n_i^2 = 12$. We denote a generic permutation $(1, 2, 3, 4) \rightarrow (n_1, n_2, n_3, n_4)$ simply by $(n_1n_2n_3n_4)$. The group $A_4$ can be generated by two basic permutations $S$ and $T$ given by $S = (4321), T = (2314)$. This satisfies $S^2 = T^3 = (ST)^3 = 1$

which is called a presentation of the group. Their product rules of the irreducible representations are given as

\[
\begin{align*}
1 \otimes 1 &= 1 \\
1' \otimes 1' &= 1'' \\
1' \otimes 1'' &= 1 \\
1'' \otimes 1'' &= 1'
\end{align*}
\]

\[
\begin{align*}
3 \otimes 3 &= 1 \otimes 1' \otimes 1'' \otimes 3_a \otimes 3_s
\end{align*}
\]

where $a$ and $s$ in the subscript corresponds to anti-symmetric and symmetric parts respectively. Denoting two triplets as $(a_1, b_1, c_1)$ and $(a_2, b_2, c_2)$ respectively, their direct product can be decomposed into the direct sum mentioned above. In the $S$ diagonal basis, the products are given as

\[
\begin{align*}
1 &\sim a_1a_2 + b_1b_2 + c_1c_2 \\
1' &\sim a_1a_2 + \omega^2b_1b_2 + \omega c_1c_2 \\
1'' &\sim a_1a_2 + \omega b_1b_2 + \omega^2 c_1c_2 \\
3_s &\sim (b_1c_2 + c_1b_2, c_1a_2 + a_1c_2, a_1b_2 + b_1a_2) \\
3_a &\sim (b_1c_2 - c_1b_2, c_1a_2 - a_1c_2, a_1b_2 - b_1a_2)
\end{align*}
\]

In the $T$ diagonal basis on the other hand, they can be written as

\[
\begin{align*}
1 &\sim a_1a_2 + b_1c_2 + c_1b_2 \\
1' &\sim c_1c_2 + a_1b_2 + b_1a_2 \\
1'' &\sim b_1b_2 + c_1a_2 + a_1c_2 \\
3_s &\sim \frac{1}{3}(2a_1a_2 - b_1c_2 - c_1b_2, 2c_1c_2 - a_1b_2 - b_1a_2, 2b_1b_2 - a_1c_2 - c_1a_2) \\
3_a &\sim \frac{1}{2}(b_1c_2 - c_1b_2, a_1b_2 - b_1a_2, c_1a_2 - a_1c_2)
\end{align*}
\]
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