Variational properties of a pumped dynamical system

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We have earlier constructed a generalized entropy concept to show the direction of time in an evolution following from a Markov generator. In such a dynamical system, the entity found changes in a monotonic way starting from any initial state of the system. In this paper, we generalize the treatment to the case when population is pumped into the system from levels not explicitly considered. These populations then pass through the coupled levels and exit by decay to levels outside the system. We derive the form of the equation of motion and relate it to our earlier treatments. It turns out that the formalism can be generalized to the new situation. Its physically relevant features are demonstrated, and the behaviour obtained is illustrated by numerical treatment of the standard two-level system with pumping and relaxation included.

PACS numbers: 03.65.Yz, 03.65.Ta, 05.70.-a

I. INTRODUCTION

All irreversible evolution models should carry some signature of the direction of time. In the theory of dynamical systems [1] such an entity is called a Lyapunov function. This defines uniquely the forward direction of time. In a physical context, the function may be considered to be a generalization of the entropy concept. In the case of approach to thermal equilibrium, the description should consistently reproduce what we take to be known from thermodynamics. The procedure for finding this description will, of course, depend on the model we offer for the time evolution. The general concept of the direction of physical time is discussed rather comprehensively in [2].

The modern theory of kinetic equations is compiled in the text [3]. The quantum counterparts are presented in [4]. Recent progress in quantum dynamics has created a separate research field named open systems dynamics [5]; its more mathematical aspects are presented in [6].

The contemporary focus on information measures has led to the introduction of a multitude of generalized entropies [7] and [8]. The various expressions have different properties and different uses. In our earlier work, we have attempted to describe the evolution of a general dynamical system by constructing a state functional displaying the direction of time by monotonically changing. This may then be regarded as a generalized entropy or, alternatively, as a Lyapunov function.

In general, the time evolution equation of a physical system is of the first order and generated by a linear operator on the object defining the state of the system. For convenience, the generator is here taken to be time independent, which restricts the applicability of our approach. The time step is taken to depend only on the state at the present time. This is usually called a Markovian evolution in physics. Within these restrictions we have been able to develop a rather detailed theory for the general case.

In [9] the solution was presented for a restricted class of evolution equations. In the case of a driven two-level system with decay, the solution was presented in [10]. As the cases discussed that far assumed no degeneracy, this was treated in [11]. The case of a thermal reservoir was introduced in [12].

The method introduced by us displays a monotonically evolving quantity. This serves to define a forward direction of time. In our paper [13], we derive the proper time-inversed behavior of irreversible time evolution. The related minimum property of entropy production is discussed in [14].

In this paper, we want to apply the theory to a case not usually addressed by the formal evolution theory. We consider the case where the physics takes place in a state space pumped by externally induced means. The population in the space considered is thus increasing with time, but this is counteracted by decay channels out of it. This system is not of the canonical Lindblad form [15], but it retains the physical interpretation by letting the density matrix elements denote the density of active particles instead of the customary probabilities. In laser physics and nonlinear spectroscopy this system has frequently been taken to model the physical situation investigated [16]. As most experimental laser physics is still described by equations of this class, we regard it to be important to relate such equations to the more recent approach developed by us.

In Sec. 2, we introduce the concept of a pumped system and its quantum mechanical description by a linear master equation. In Sec. 3 we discuss the physical situation as it differs from a genuine Lindblad case. Finally we apply the
theory introduced earlier to this case, and extract its characterization of the direction of time. The theory is then, in Sec. 4, illustrated by applying it to the case of a pumped decaying two-level system. This model has played a central role in both laser spectroscopy and quantum optics [17]. The formalism, however, also applies to Markovian rate equations [18]. Finally the conclusions from the treatment are discussed in Sec. 5.

II. THE PUMPED SYSTEM

A. The time evolution equation

The terminology of quantal state evolution assigns the term open system to any quantum system in contact with an environment. Ordinarily the system is probabilistically closed, i.e. we assume that the density matrix of the system retains the probability interpretation and is normalized. Consequently all probability exchange between the states, reversible as well as dissipative, occurs between levels in the system. The general memory-less expression for such evolution is given by the Lindblad form of the generating equation.

There is, however, a class of time evolution equations that do not conform to this model. There is a flow of probability through the system; the states are pumped by some technical method and the influx of probability is compensated by states decaying to unobserved lower levels. Such models are termed pumped systems here. Naturally the state space could be enlarged so that all pumping and decay mechanisms are included in the dynamics of the evolution. It is, of course, always possible to consider a space large enough to eliminate the need for a phenomenological master equation, but it is not always an expedient or even a practical possibility.

The time evolution equation will now hold several different pieces:

1. The decays inside and out of the system are described by a relaxation operator $\mathcal{R}$, which we assume to be linear and memoryless. The time derivative of the density matrix in the absence of pumping and with $H = 0$, where $H$ is the Hamilton operator, would be

$$\partial_t \rho = \mathcal{R} \rho.$$  

2. The pumping is described by a random process which at time $t_0$ introduces a system in the state

$$| \psi \rangle = \sum_n c_n (0) | \varphi_n \rangle,$$

where $\{ | \varphi_n \rangle \}$ is an arbitrary basis on the state space of the system under consideration. At a later time this will, in the absence of relaxation, evolve to the state

$$| \psi (t) \rangle = \sum_n c_n (t - t_0) | \varphi_n \rangle.$$

We now form an ensemble accumulated by all systems introduced at times $t_0 \leq t$. This is described by the state

$$\tilde{\rho}_{nm}(t) = \int_{-\infty}^{t} c_n (t - t_0) c^*_m (t - t_0) P_{\psi}(t_0) dt_0,$$

where $P_{\psi}(t_0)$ is the probability of introducing a system in state $\psi$ into the ensemble at time $t_0$. This is assumed to be normalized over the states introduced:

$$\sum_{\psi} P_{\psi}(t) = 1.$$  

The contribution from (4) to the time evolution is given by two terms,

$$\partial_t \tilde{\rho}_{nm}(t) = c_n (0) c^*_m (0) P_{\psi}(t) + \int_{-\infty}^{t} \partial_t [c_n (t - t_0) c^*_m (t - t_0)] P_{\psi}(t_0) dt_0.$$  

We now introduce the density operator averaged over the pumped states in (2) by setting

$$\sum_{\psi} \tilde{\rho}_{nm}(t) \rightarrow \rho_{nm}(t).$$
We may then write the averaged time evolution as
\[ \partial_t \rho_{nm}(t) = \Lambda_{nm}(t) + C_{nm}(t), \]  
(8)
where the inhomogeneous term is given by
\[ \Lambda_{nm}(t) = \sum_{\psi} c_n(0) c_m^*(0) P_{\psi}(t). \]  
(9)

Because the Hamiltonian is taken to be the same for all systems pumped into the ensemble, we let the state coefficients satisfy the ordinary Schrödinger equation in the form
\[ \partial_t c_n(t-t_0) = -i \sum_k H_{nk} c_k(t-t_0), \]  
(10)
where \( H \) is the generator of the unitary time evolution inside the system. Using this in (6) we obtain
\[ C_{nm}(t) = -i \langle [H, \rho(t)] \rangle_{nm}, \]  
(11)
where we have introduced the ensemble density matrix from (7).

Now, collecting terms from (1) and (8), and using (11), we find the time evolution equation for the pumped ensemble to be
\[ \partial_t \rho = \Lambda(t) - i [H, \rho(t)] + R \rho(t). \]  
(12)
This is the most general linear and memoryless evolution equation we encounter. The Lindblad form falls under this more general class if we set \( \Lambda(t) = 0 \). If this is not the case, then the relaxation term \( R \rho \) cannot be of the Lindblad form, which on its own would ensure conservation of the probability. As the term \( \Lambda \) does not do this, its influence must be compensated by \( R \rho \).

Not being of the Lindblad form, Eq. (12) does not manifestly generate completely positive time evolution. However, properly used, it does not violate the physical interpretation of the density matrix. The pumping in Eq. (9) is of the form of a density matrix and thus cannot cause any troubles. The damping operator (1) has to be introduced such that no abnormal results emerge. This implies inequalities between the decay constants, but these are no different from the corresponding relations encountered in Markovian rate processes, Ref. [18]. The Hamiltonian evolution, naturally, induces only physically acceptable changes.

B. Properties of pumped evolution

The equation (12) may be written in a notation generalizing the Lindblad case as
\[ \partial_t \rho = \Lambda + L \rho. \]  
(13)
In agreement with the concept of pumping a state, it follows that
\[ \Lambda_{nn} = \sum_{\psi} c_n(0) c_n^*(0) P_{\psi}(t) \geq 0; \]  
(14)
this term can only add to the population of the state. The pumping mechanism is usually incoherent, and \( \Lambda \) is not expected to have any non-diagonal contributions, because of the ensemble average.

Conservation of probability requires that the steady state, if it exists, satisfies
\[ Tr (R \rho_0) = -Tr \Lambda. \]  
(15)
The operator \( R \) is thus essentially a negative operator.

If no pumping occurs, \( \Lambda = 0 \), there are two possibilities:

- After an infinite time, all population decays out of the system and the only steady state is the one with all populations vanishing, \( \rho_0(\infty) = 0 \). In this situation, the relation
  \[ L \rho = 0 \]  
(16)
has got no nonvanishing solutions; all eigenvalues of \( L \) are nonzero.
• There is a part of $\mathcal{L}$ which allows for a steady state with nonvanishing density matrix

$$\mathcal{L}_0 \rho^0 = 0; \rho^0 \neq 0. \quad (17)$$

This concerns a part of the state space which has no decay channel out of it, and population ending up here will remain trapped. Thus no part of the pumped population can go into this subspace, because then probability would accumulate here without limit. It thus describes a part of the state space which is totally decoupled from the rest of the system: No coherent transfer nor any irreversible decay can couple this to the rest. It is consequently a system which can be treated by the ordinary theory; all its population derives from its initial state and no pumping will affect it. In the subsequent discussion we omit this possibility.

The steady state solution of (13) becomes

$$\rho_0 = -\mathcal{L}^{-1} \Lambda. \quad (18)$$

According to what is said above, the operator $\mathcal{L}$ has got no zero eigenvalues, and the expression (18) will exist in all physically acceptable situations.

The ensemble density matrix elements $\rho_{nn} \geq 0$ all denote the population on level $n$ but they no longer necessarily relate to any probability; in particular it need not hold that

$$Tr \rho_0 = Tr \rho(t). \quad (19)$$

Note that the special initial condition, $\rho(t_0) = 0$, is perfectly acceptable and may still imply $\rho_0 = \rho(\infty) \neq 0$. The off-diagonal elements of $\rho$ retain their role as determining the multipole moments of the system.

With this notation we may write (13) as

$$\frac{\partial}{\partial t} \rho = \mathcal{L} (\rho - \rho_0); \quad (20)$$

with the notation

$$\delta \rho \equiv \rho - \rho_0. \quad (21)$$

This allows us to write

$$\frac{\partial}{\partial t} \delta \rho = \mathcal{L} \delta \rho, \quad (22)$$

which relates the present case to our earlier theory.

### III. PHYSICS OF TIME EVOLUTION

#### A. Formal properties

The objects $\rho$ representing the state of the system obey the general dynamic time evolution equation of the form

$$\frac{\partial}{\partial t} \rho = \frac{\partial}{\partial t} \delta \rho = \Lambda + \mathcal{L} \rho = \mathcal{L} \delta \rho. \quad (23)$$

They can be taken to belong to a linear manifold of elements denoted by $| \rho \rangle \rangle$. This space can be equipped with a natural inner product by writing

$$\langle \langle \rho_1 | \rho_2 \rangle \rangle \equiv Tr (\rho_1^\dagger \rho_2). \quad (24)$$

The time evolution operator is not assumed to be Hermitian with respect to the inner product defined. Consequently we need to introduce right eigenvectors

$$\mathcal{L} | x_\nu \rangle \rangle = \lambda_\nu | x_\nu \rangle \rangle \quad (25)$$

and left eigenvectors

$$\mathcal{L}^\dagger | y_\nu \rangle \rangle = \lambda_\nu^* | y_\nu \rangle \rangle. \quad (26)$$
From these relations follows that
\[ \langle \langle y_\mu \mid L \mid x_\nu \rangle \rangle = \lambda_\nu \langle \langle y_\mu \mid x_\nu \rangle \rangle \]
(27)
\[ \langle \langle L^\dagger y_\mu \mid x_\nu \rangle \rangle = \lambda_\mu \langle \langle y_\mu \mid x_\nu \rangle \rangle. \]

Thus if \( \lambda_\nu \neq \lambda_\mu \) the states are mutually orthogonal and may be normalized against each other
\[ \langle \langle y_\nu \mid x_\mu \rangle \rangle = \delta_{\nu\mu}. \]
(28)
We assume in the following that both the right and left eigenstates form complete basis sets. This is, in particular, true if their corresponding eigenvalues are nondegenerate, but in general they are complex.

With these definitions, we can present spectral representations for the operators
\[ L = \sum_\nu \mid x_\nu \rangle \langle y_\nu \mid \]
\[ L^\dagger = \sum_\nu \mid y_\nu \rangle \langle x_\nu \mid. \]
(29)
When both sets of eigenstates are complete, the identity operator can be written as
\[ I = \sum_\nu \mid x_\nu \rangle \langle y_\nu \mid \]
\[ \mid x_\nu \rangle \langle y_\nu \mid = \sum_\nu \mid y_\nu \rangle \langle x_\nu \mid. \]
(30)

The inner product between states belonging to the right eigenvalues have no simple relations. In order to obtain a simple situation, we introduce the mapping
\[ \Omega : \{ \mid x_\nu \rangle \} \Rightarrow \{ \mid y_\nu \rangle \} \]
by setting
\[ \Omega = \sum_\nu \mid y_\nu \rangle \langle y_\nu \mid \]
(31)

The inverse mapping is seen to be
\[ \Omega^{-1} = \sum_\nu \mid x_\nu \rangle \langle x_\nu \mid. \]
(33)
These are positive Hermitian operators, and it is possible to define a new metric based on the bilinear form defined as
\[ M_{\Omega}[\rho_1, \rho_2] \equiv \langle \langle \rho_1 \mid \Omega \mid \rho_2 \rangle \rangle \equiv Tr(\rho_1^\dagger \Omega \rho_2). \]
(34)
This has got all the properties of an inner product, and it thus defines a topology in the space of all quantum states. Within this product we have
\[ M_\Omega(\mid x_\nu \rangle, \mid x_\mu \rangle) = \langle \langle x_\nu \mid y_\mu \rangle \rangle = \delta_{\nu\mu}. \]
(35)
A similar construction is possible on the states \( \{ \mid y_\nu \rangle \} \) by the use of \( \Omega^{-1} \). The operator \( \Omega \) may be considered as a metric operator on the manifold of physical states.

By direct calculation we find that
\[ \Omega L \Omega^{-1} = \sum_\nu \mid y_\nu \rangle \lambda_\nu \langle \langle x_\nu \rangle \rangle = L^\dagger. \]
(36)
Here as in the following, we assume that complex conjugation denoted by * affects only c-numbers, not states.

From (36), we derive the relations
\[ \Omega L = L^\dagger \Omega ; \quad L \Omega^{-1} = \Omega^{-1} L^\dagger \]
\[ \Omega L^* = L^\dagger \Omega, \quad L^* \Omega^{-1} = \Omega^{-1} L^{\dagger}, \]
(37)
forming the basis for the considerations below.
If $L$ has real eigenvalues, then $L^* = L$ and the operator is Hermitian with respect to the new inner product $M$ defined above. If the eigenvalues occur in complex conjugate pairs, i.e. for each $|x_\mu\rangle\rangle$ there exists an $|x_\mu^*\rangle\rangle$ such that

$$L| x_\mu^* \rangle\rangle = \lambda_\mu^* | x_\mu^* \rangle\rangle,$$

we find from

$$\lambda_\mu \langle\langle y_\mu | x_\mu^* \rangle\rangle = \langle\langle y_\mu | L | x_\mu^* \rangle\rangle = \lambda_\mu^* \langle\langle y_\mu | x_\mu^* \rangle\rangle,$$

that

$$\langle\langle y_\mu | x_\mu^* \rangle\rangle \equiv \langle\langle x_\mu | \Omega | x_\mu^* \rangle\rangle = 0;$$

these states are thus orthogonal in the $M$ metric. We expect the eigenvalues of physical evolution operators to correspond to damped oscillations at frequencies given by the energy differences in the system. We call this case physical evolution.

The time dependent state that is a solution of the evolution equation (23) is given by

$$| \delta \rho(t) \rangle\rangle = \sum_{\nu} r_\nu \exp (\lambda_\nu t) | x_\nu \rangle\rangle,$$

where

$$r_\nu = \langle\langle y_\nu | \delta \rho(0) \rangle\rangle.$$

From the physical meaning of the time evolution operator in a pumped system, we derive the following conclusions:

- There exists no steady state $| x_0 \rangle\rangle$ with eigenvalue $\lambda_0 = 0$. Asymptotically, the time evolution will lead to $\rho(\infty) = \rho_0$.
- Any solution must approach this in a smooth manner, thus we require $\text{Re} \ \lambda_\nu < 0$ for all $\nu$.

### B. The direction of time

As the system undergoes irreversible time evolution, it singles out one direction of time defining the forward progress. In classical dynamics, we can achieve this by introducing a monotonically changing variational function; in system theory this is called a Lyapunov function.

We find easily that the definition

$$M_\Omega(\delta \rho, \delta \rho) = \text{Tr} (\delta \rho^\dagger \Omega \delta \rho) \equiv \langle\langle \delta \rho | \Omega | \delta \rho \rangle\rangle$$

provides a function changing monotonically with time. We have namely

$$\partial_t M_\Omega [(\rho - \rho_0), (\rho - \rho_0)] = \text{Tr} (\partial_t \delta \rho^\dagger \Omega \delta \rho) + \text{Tr} (\delta \rho^\dagger \Omega \partial_t \delta \rho)$$

$$= \text{Tr} (\delta \rho^\dagger L^* \Omega \delta \rho) + \text{Tr} (\delta \rho^\dagger \Omega L \delta \rho)$$

$$= \text{Tr} (\delta \rho^\dagger \Omega (L^* + L) \delta \rho)$$

$$= 2 \text{Re} \ \text{Tr} (\delta \rho^\dagger \Omega L \delta \rho).$$

where we have used (37). Using the expansion

$$| \delta \rho(t) \rangle\rangle = \sum_{\nu} r_\nu \exp (\lambda_\nu t) | x_\nu \rangle\rangle$$

we obtain

$$\partial_t M_\Omega(\delta \rho, \delta \rho) = \sum_{\nu} | r_\nu |^2 (\lambda_\nu^* + \lambda_\nu) \exp [(\lambda_\nu^* + \lambda_\nu) t] \leq 0.$$
because all eigenvalues have negative real parts. The rate \( \frac{\Delta_2}{\Gamma_2} \) decreases with time and when steady state is reached, the development ceases. This is thus a proper variational operator which determines the direction of time towards the eventual steady state. In particular we note that the pumped system, in general, lacks a concept of thermal equilibrium.

The expression \( \frac{\Delta_2}{\Gamma_2} \) represents the natural extension of the purity to the case of irreversible time evolution. If we want an entity extensive in the combination of uncoupled subsystems, we may introduce the entropy-like expression

\[
S_{\Omega} = \pm \log M_{\Omega};
\]

the choice of sign depends on the interpretation of the quantity.

**IV. APPLICATION: PUMPED TWO-LEVEL SYSTEM**

We apply the considerations above to the generic driven two-level system with pumping and decay out of the system. The time evolution equation is now

\[
\frac{\partial}{\partial t} \begin{bmatrix} \rho_{22} \\ \rho_{21} \\ \rho_{12} \\ \rho_{11} \end{bmatrix} = \begin{bmatrix} \Lambda_2 & -\Gamma_2 -iV & -iV & 0 \\ 0 & -iV (i\omega - \gamma) & 0 & iV \\ iV & 0 & -iV (i\omega + \gamma) & -iV \\ 0 & iV & -iV & -\Gamma_1 \end{bmatrix} \begin{bmatrix} \rho_{22} \\ \rho_{21} \\ \rho_{12} \\ \rho_{11} \end{bmatrix}.
\]

(47)

Here \( \omega \) is the detuning of the driving field, \( V \) is the Rabi type coupling between the levels, and \( \Gamma \) and \( \gamma \) are the population and coherence decay rates. These define the time constants \( T_1 \) and \( T_2 \) respectively. The coherence decay satisfies the constraint

\[
\gamma \geq \frac{1}{2} (\Gamma_1 + \Gamma_2).
\]

(48)

This follows if we set \( V = \Lambda = 0 \) and impose the condition

\[
\rho_{11}(t)\rho_{22}(t) \geq \rho_{12}(t)\rho_{21}(t).
\]

(49)

**A. Analytic results**

It is straightforward to obtain the complex eigenvalues of the matrix in (47) and verify that all real parts correspond to decay. The steady state of (47) is also easily computed, but the ensuing expression is too unwieldy to allow an intuitive interpretation. One quantity of interest is the asymptotic population difference

\[
\rho_{22}(\infty) - \rho_{11}(\infty) = \left( \frac{\Lambda_2}{\Gamma_2} - \frac{\Lambda_1}{\Gamma_1} \right) \left( 1 - \frac{\eta^2 V^2}{\omega^2 + \gamma^2 + \eta^2 V^2} \right).
\]

(50)

The deviations from the pumping difference without coupling is in the form of a power-broadened Lorentzian, where the rate of incoherence is measured by the dimensionless parameter

\[
\eta^2 = \frac{2\gamma (\Gamma_1 + \Gamma_2)}{\Gamma_1 \Gamma_2} \geq \frac{(\Gamma_1 + \Gamma_2)^2}{\Gamma_1 \Gamma_2} \geq 4.
\]

(51)

From the result (50) we obtain some general observations:

The population difference goes to zero:

- If \( \frac{\Lambda_2}{\Gamma_2} = \frac{\Lambda_1}{\Gamma_1} \), then each level loses on the average as much as it gains.
- If \( V \to \infty \). In this case the flopping between the levels is so fast that the population spends roughly half its time on each level, and consequently their differences are smeared out.

On the other hand, the difference goes to the value \( \frac{\Lambda_2}{\Gamma_2} - \frac{\Lambda_1}{\Gamma_1} \) if the coupling becomes inefficient. This happens when

- The coupling is weak, \( V \to 0 \).
- The decoherence rate is strong, \( \gamma \to \infty \).
FIG. 1: The figures show the total population $\rho_{22} + \rho_{11}$ (solid line), $\rho_{22}$ (dashed line), $\rho_{11}$ (dotted line) and the absolute value of $\rho_{21}$ (dash-dotted line) as a function of time. In Case 1 (top left), the pumping parameters are $\Lambda_1 = 0, \Lambda_2 = 1.0$, the decay rates are $\Gamma_2 = 0, \Gamma_1 = 1.0, \gamma = 0.5$, the detuning is $\omega = 0$ and the coupling $V = 2.0$. There is zero population in the initial state. In Case 2 (top right) the parameters and initial state are the same as in Case 1 but with $\omega = -1.0$. In Case 3 (bottom left) the parameters are $\Lambda_1 = 1.0, \Lambda_2 = 1.0, \Gamma_2 = 1.0, \Gamma_1 = 1.0, \gamma = 1.0, \omega = -1.0$ and $V = 5.0$, and the initial state has $\rho_{22}(0) = 1$, with other density matrix elements equal to zero. In Case 4 (bottom right) parameters and initial state are as in Case 3, but with $\omega = -5.0$. All numerical values are given in terms of scaled dimensionless physical variables, as explained in the text.

B. Numerical results

The time evolution equation \( E \) contains 7 parameters with dimensions \([t^{-1}]\). One may be eliminated by scaling the time rate, but the parameter space is still too large to allow a full systematic mapping. Some conclusions are, however, readily obtained.

When we pose initial conditions $\rho(0)$ and numerically integrate the equations \( E \) we can see the manifestations of the remarks above. The resulting time evolution is exemplified in Fig. 1. After scaling the physical parameters in Eq. (47) by a suitably selected time parameter, we have 6 remaining dimensionless variables. The numerical values used in the computations are these dimensionless parameters, which are given in the figures.

Fast flopping rates $V$ tend to make the final populations equal and the coherences zero. The decay times are determined by the relaxation rates, and during the time evolution the population difference tends to change sign $V/\gamma$ times; increasing the detuning, we find that $\omega$ tends to decrease the amplitude of these oscillations. The effect of $\omega$ is, however, less than that of the coupling $V$.

In Fig. 2, the functional $M_\Omega$ is plotted for the cases in Fig. 1. We see that in each case, the expectation value of $\Omega$ decays smoothly and monotonically.

Some details of these four cases chosen as illustration are given in the Appendix. As seen there, all eigenvalues imply damping as we expect from the physical interpretation. In all four cases, there are two purely damped modes. These are the ones describable as pure rate variables. We have, in fact, integrated a large number of parameter combinations, and we have found no anomalous behavior; all cases show smooth and regular behavior. The results are very similar to the ones in the cases presented here. We expect our analysis to apply to all cases.

V. CONCLUSION

The idea in our line of research has been to find expressions uniquely indicating the direction of time at each instant. This is particularly urgent in the field of quantum optics where the time evolution is conventionally described by
FIG. 2: The expectation value of $\Omega$ as a function of time for the same four cases as in Fig. 1, from bottom to top Case 1 (solid line), Case 2 (dashed line), Case 3 (dotted line) and Case 4 (dash-dotted line). In all cases, normalisation is chosen as $\langle \rho(0)|\Omega|\rho(0) \rangle = 1$. As the curves are very similar, they have been shifted vertically by 0.5 with respect to each other. The asymptotic value for the unshifted $\langle \rho(t)|\Omega|\rho(t) \rangle$ when $t \to \infty$ is zero in all four cases, and the behaviour is monotonic as a function of time. All numerical values are given in terms of scaled dimensionless physical variables, as explained in the text.

Phenomenological master equations. In particular, the state space of interest comprises only a part of all possible states of the physical system. Thus an interpretation in terms of quantum concepts is natural, but also Markovian rate models fall under our concepts [9]. In fact, the rate equation situation is introducing an approach [18], which constitutes a precursor to our method.

In mathematics, the related formalism has been known for some time under the name of “symmetrization” [19, 20]. The Prigogine school has introduced a similar description of irreversible time evolution [21], but their formalism is based on a different physical argument and less concisely defined than our present approach. We base all our expressions on precisely defined mathematical concepts. When applicable our formulation leads to uniquely determined numerical evaluations.

The approach is limited to linear evolution equations with time-independent generators. The linearity is natural if we wish time evolution in independent subsystems to remain uncoupled. Violating this condition takes us into a totally different category of models. To introduce generators depending on time is possible, but the ensuing theory then has to deal with time dependent eigenvalues and eigenelements. This is possible but leads to a theory too complex to give transparent physical interpretations.

We feel that even with its restrictions, our method is general enough to illuminate the properties of irreversible system dynamics.

Appendix A: Details of the numerical simulations

Below we give the steady states and the eigenvalues for $\mathcal{L}$ relating to cases 1-4 plotted in Figs. 1 and 2. The steady states $\rho_0$ agree with Eq. (50), and all real parts of eigenvalues are negative as required. The parameters are as follows:

- Case 1: $\Lambda_1 = 0, \Lambda_2 = 1.0, \Lambda_{21} = 0, \Gamma_2 = 0, \Gamma_1 = 1.0, \gamma = 0.5, \omega = 0, V = 2.0$
  $\rho_0 = (1.0625, -0.25i, 0.25i, 1)^T$
  $\lambda_1 = -0.5 - 3.9686i, \lambda_2 = -0.5 + 3.9686i, \lambda_3 = \lambda_4 = -0.5$

- Case 2: $\Lambda_1 = 0, \Lambda_2 = 1.0, \Lambda_{21} = 0, \Gamma_2 = 0, \Gamma_1 = 1.0, \gamma = 0.5, \omega = -1.0, V = 2.0$
  $\rho_0 = (1.3125, -0.5 - 0.25i, -0.5 + 0.25i, 1)^T$
  $\lambda_1 = -0.5000 - 4.0945i, \lambda_2 = -0.5000 + 4.0945i, \lambda_3 = -0.6221, \lambda_4 = -0.3779$

- Case 3: $\Lambda_1 = 1.0, \Lambda_2 = 1.0, \Lambda_{21} = 0, \Gamma_2 = 1.0, \Gamma_1 = 1.0, \gamma = 1.0, \omega = 1.0, V = 5.0$
  $\rho_0 = (1, 0, 0, 1)^T$
  $\lambda_1 = -1.0 - 10.0499i, \lambda_2 = -1.0 + 10.0499i, \lambda_3 = \lambda_4 = -1.0$
• Case 4: $\Lambda_1 = 1.0, \Lambda_2 = 1.0, \Lambda_{21} = 0, \Gamma_2 = 1.0, \Gamma_1 = 1.0, \gamma = 1.0, \omega = 5.0, V = 5.0$

$\rho_0 = (1, 0, 0, 1)^T$

$\lambda_1 = -1.0 - 11.1803 i, \lambda_2 = -1.0 + 11.1803 i, \lambda_3 = \lambda_4 = -1.0.$

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