Homogenization of Poisson and Stokes equations in the whole space

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Abstract

We consider the homogenization of the Poisson and the Stokes equations in the whole space perforated with periodically distributed small holes. The periodic homogenization in bounded domains is well understood, following the classical results in [25, 4, 1, 2]. In this paper, we show that these classical homogenization results in a bounded domain can be extended to the whole space $\mathbb{R}^d$. Our results cover all three cases corresponding to different sizes of holes and cover all $d \geq 2$.

Keywords: Poisson and Stokes equations; homogenization; perforated domains; whole space.

1 Introduction

The homogenization of the Poisson and the Stokes equations in a bounded domain perforated with a large number of small holes has been systematically studied in many literatures following the classical papers [4] for the Poisson equation and [25, 1, 2] for the Stokes equations.

The perforated domain under consideration is described as following. Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$ be an open domain and let $\varepsilon, a_\varepsilon$ be small parameters satisfying $0 < a_\varepsilon \leq \varepsilon \leq 1$. The holes are denoted by $T_{\varepsilon,k}$ which are assumed to be closed, bounded, and simply connected, with $C^1$ boundary; $\delta_i, i = 1, 2$ are fixed positive numbers. The perforation parameters $\varepsilon$ and $a_\varepsilon$ are used to measure the mutual distance of the holes and the size of the holes, respectively, and $\varepsilon x_0 = \varepsilon x_0 + \varepsilon k$ determine the locations of the holes. The perforated domain $\Omega_\varepsilon$ is then defined as

$$
\Omega_\varepsilon := \Omega \setminus \bigcup_{k \in \mathbb{Z}^d} T_{\varepsilon,k}.
$$

In this paper, we consider the following Dirichlet problems of the Poisson and the Stokes equations in $\Omega_\varepsilon$:

$$
\begin{cases}
-\Delta u_\varepsilon = f, & \text{in } \Omega_\varepsilon, \\
u_\varepsilon = 0, & \text{on } \partial \Omega_\varepsilon,
\end{cases}
$$

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\[
\begin{aligned}
-\Delta v_\varepsilon + \nabla p_\varepsilon &= g, \quad \text{in } \Omega_\varepsilon, \\
\text{div } v_\varepsilon &= 0, \quad \text{in } \Omega_\varepsilon, \\
v_\varepsilon &= 0, \quad \text{on } \partial \Omega_\varepsilon.
\end{aligned}
\]

Cioranescu and Murat \cite{4} considered (1.2) and Allaire \cite{1, 2} considered (1.3), where \( \Omega \) is assumed to be a bounded domain with smooth boundary (for example a bounded \( C^1 \) domain), and the external forces \( f \in L^2(\Omega), \ g \in L^2(\Omega; \mathbb{R}^d) \). In their studies, instead of giving specific assumptions on the holes configurations as in (1.1), some abstract framework of hypotheses is imposed. It was shown that when the number of holes goes to infinity and the size of the holes goes to zero simultaneously, the solution approaches an effective state governed by certain homogenized equations which are defined in homogeneous domains — domains without holes. The homogenized equations are crucially determined by the ratio between the size of the holes and the mutual distance between the holes. Precisely, Allaire \cite{1, 2} showed that the homogenized equations for (1.3) are determined by the ratio \( \sigma_\varepsilon \) defined as following:

\[(1.4) \quad \sigma_\varepsilon := \left( \frac{\varepsilon^d}{d^{d-2}} \right)^{\frac{1}{2}} \quad \text{if } d \geq 3; \quad \sigma_\varepsilon := \varepsilon \left| \log \frac{d\varepsilon}{\varepsilon^d} \right|^{\frac{1}{2}} \quad \text{if } d = 2.\]

If \( \lim_{\varepsilon \to 0} \sigma_\varepsilon = 0 \) corresponding to the supercritical case of large holes, the homogenized system is the Darcy’s law; if \( \lim_{\varepsilon \to 0} \sigma_\varepsilon = \infty \) corresponding to the subcritical case of small holes, the limit system remains to be the same Stokes equations; if \( \lim_{\varepsilon \to 0} \sigma_\varepsilon = \sigma_* \in (0, +\infty) \) corresponding to the case of critical size of holes, the homogenized equations are governed by the Stokes-Brinkman equations — a combination of the Darcy’s law and the original Stokes equations.

The homogenization studies in \cite{4} and \cite{1, 2} are extended in different perspectives. More complicated models in fluid mechanics are considered, see \cite{20}, \cite{9}, \cite{8, 7, 19}, and the references therein. By employing the idea of cell problem introduced by Tartar \cite{25}, a new unified proof covering different sizes of holes is given in \cite{16} and \cite{18} for the homogenization of the Poisson equation and the Stokes system, respectively. Another direction of research is to consider more general holes configurations without periodicity, see \cite{6, 10, 14, 5, 12, 13}. Without periodicity, Hillairet \cite{14} considered the Stokes problem with nonzero boundary values on the holes, where the modelling goes back to \cite{6}. In \cite{6} and \cite{14}, the minimal distance between the holes is assumed to be much larger than the size of the holes. Very recently in \cite{5, 12, 13}, the random homogenization of the Poisson equation and the Stokes equations is studied. Particularly in \cite{12, 13}, randomly distributed spherical holes are considered, where the centers of the holes are distributed according to a Poisson point process. They imposed very weak assumptions on the holes configurations, where the holes are allowed to be very close or even overlap. For the random homogenization study in \cite{5}, the overlap of holes is negligible in probability. In \cite{10, 14, 5, 12, 13}, the critical size of holes is considered in a bounded domain in \( \mathbb{R}^d \) with \( d \geq 3 \) and the Brinkman type equations are derived.

So far, most of the results are obtained for bounded domains. Recently in \cite{15}, along with others, H"ofler and Velazquez studied the homogenization of the Poisson and the Stokes problems in the whole space in three and higher dimensional setting. They employed the reflection method and derived the Brinkman type equations in the limit. In \cite{15}, a new abstract framework in functional analysis was built to describe the homogenization problems, and show the connection between the method of reflections and such abstract framework. Unlike the holes configurations (1.1) considered in this paper, no periodicity is assumed in \cite{15}. Instead, some general assumptions are imposed on the holes, such as the size, the minimal distance, the upper bound of the total capacity, the convergence and the lower bound of the average capacity. See Conditions 1.1 and 1.2, Assumption
1.7 in [15]. The analysis in [15] relies on the notation of *screening length*, which goes back to [22, 24]. In [23, 24], using the *screening estimate*, the homogenization in unbounded domains for the Poisson equation is also studied.

In [15], only the critical case $\lim_{\varepsilon \to 0} \sigma_\varepsilon = \sigma_* \in (0, +\infty)$ is included, and it seems their method is not compatible to the other two noncritical cases. Indeed, Condition 1.1 in [15] which ensures the boundedness of the total capacity of the holes is not satisfied for the supercritical case of large holes; Assumption 1.7 in [15] which ensures the positivity of the lower bound of the average capacity is satisfied only if the ratio $\sigma_\varepsilon$ is bounded as $\varepsilon \to 0$. This is not satisfied for the subcritical case of small holes. The analysis in [15] does not work for two dimensional case. One main issue is that in two dimensions, the decay of the Green function for the Laplace or the Stokes operator is weaker and this makes it more difficult to bound the interaction between holes. Such bounds are needed for the method of reflections used in [15]. Another issue is on the characterization of the homogeneous Sobolev spaces $D^{1,2}_0(\mathbb{R}^2)$ when $d = 2$. See Remarks 2.2. In this paper, we use different method and we will cover all the three cases (critical, supercritical, and subcritical) and all $d \geq 2$.

## 2 Main results

In this section, we state our main homogenization results in the whole space $\Omega = \mathbb{R}^d$. We first introduce some function spaces and related properties.

### 2.1 Some function spaces

Let $E$ be a locally Lipschitz domain in $\mathbb{R}^d$. For any $1 \leq q \leq \infty$ and $m \in \mathbb{Z}_+$, $W^{m,q}(E)$ denotes the classical Sobolev space, and $W^{m,\infty}_0(E)$ denotes the completion of $C_c^\infty(E)$ in $W^{m,q}(E)$. Here $C_c^\infty(E)$ is the space of smooth functions with compact support. We use $W^{-1,q}(E)$ to denote the dual space of $W^{1,q}_0(E)$. For $1 \leq q < \infty$, $W^{1,q}(\mathbb{R}^d) = W^{1,q}_0(\mathbb{R}^d)$.

If the functions are vector valued in $\mathbb{R}^n$, we use the notations $W^{m,q}(E; \mathbb{R}^n)$, $W^{m,\infty}_0(E; \mathbb{R}^n)$, $C_c^\infty(E; \mathbb{R}^n)$, and so on. Let $C_c^\infty(E; \mathbb{R}^d)$ be the space of divergence free functions in $C_c^\infty(E; \mathbb{R}^d)$. We use $\langle \cdot, \cdot \rangle_{X',X}$ to denote the dual pair between a Banach space $X$ and its dual space $X'$. We often omit the subscript and simply write $\langle \cdot, \cdot \rangle$ if it is clear from the context.

We now recall some concepts of the homogeneous Sobolev spaces. The materials are mainly taken from Chapter II.6 and II.7 of Galdi’s book [11]. Let $1 \leq q < \infty$. We define the linear space

\[(2.1) \quad D^{1,q}(E) = \{ u \in L^1_{\text{loc}}(E) : \| \nabla u \|_{L^q(E)} < \infty \}, \quad |u|_{D^{1,q}(E)} := \| \nabla u \|_{L^q(E)}.
\]

The space $D^{1,q}$ is generally not a Banach space. After introducing the equivalent classes

\[ [u] = \{ u + c, \ c \in \mathbb{R} \text{ is a constant} \}, \quad \text{for any } u \in D^{1,q}(E), \]

the space $D^{1,q}(E)$ of all equivalence classes $[u]$ equipped with the norm

\[ \| [u] \|_{D^{1,q}(E)} := |u|_{D^{1,q}(E)} = \| \nabla u \|_{L^q(E)} \]

is a Banach space.

The semi-norm $| \cdot |_{D^{1,q}(E)}$ introduced in (2.1) defines a norm in $C_c^\infty(E)$. We introduce the Banach space $D^{1,q}_0(E)$ which is the completion of $C_c^\infty(E)$ with respect to the norm $| \cdot |_{D^{1,q}(E)}$. We denote by $D^{-1,q}(E)$ the dual space of $D^{1,q}_0(E)$. 

3
For any open set \( E \) in \( \mathbb{R}^d \), there holds the following Gagliardo-Nirenberg-Sobolev inequality: for each \( 1 \leq q < d \), there exists a constant \( C \) depending only on \( q \) and \( d \) such that for all \( u \in C_c^\infty(E) \), there holds

\[
\|u\|_{L^q(E)} \leq C(q,d)\|\nabla u\|_{L^q(E)}, \quad \frac{1}{q^*} = \frac{1}{p} - \frac{1}{d}.
\]

By density argument, the same inequality (2.2) holds for all \( u \in D^{1,q}_0(E) \) with \( 1 \leq q < d \). This means \( D^{1,q}_0(E) \) is continuously embedded into \( L^q(E) \) if \( 1 \leq q < d \). Moreover, if \( 1 \leq q < d \), Galdi [11, equation (II.7.14)] gave an equivalent characterization of \( D^{1,q}_0(E) \):

\[
D^{1,q}_0(E) = \{ u \in D^{1,q}(E) : u \in L^q(E) \text{ such that } \psi u \in W^{1,q}_0(E) \text{ for any } \psi \in C_c^\infty(\mathbb{R}^d) \},
\]

with the equivalent norm

\[
\| \cdot \|_{D^{1,q}_0(E)} := \| \cdot \|_{D^{1,q}(E)} + \| \cdot \|_{L^q(E)}.
\]

It becomes more tricky if \( q \geq d \). Particularly if \( q \geq d \) and \( E = \mathbb{R}^d \), there holds (see (II.7.16) of [11])

\[
D^{1,q}_0(\mathbb{R}^d) = D^{1,q}(\mathbb{R}^d) = \{ [u] = u + c : \nabla u \in L^q(\mathbb{R}^d), \ c \text{ is a constant} \}.
\]

### 2.2 Homogenization results

We first give our assumptions on the source functions.

**Assumption 2.1.** Let \( d \geq 2 \).

(i) For the critical case \( \lim_{\varepsilon \to 0} \sigma_\varepsilon = \sigma_* \in (0, +\infty) \), we assume \( f \in W^{-1,2}(\mathbb{R}^d) \) and \( g \in W^{-1,2}(\mathbb{R}^d; \mathbb{R}^d) \).

(ii) For the supercritical case \( \lim_{\varepsilon \to 0} \sigma_\varepsilon = 0 \), we assume \( f \in L^2(\mathbb{R}^d) \) and \( g \in L^2(\mathbb{R}^d; \mathbb{R}^d) \).

(iii) For the subcritical case \( \lim_{\varepsilon \to 0} \sigma_\varepsilon = \infty \), we assume \( f \in D^{-1,2}(\mathbb{R}^d) \) and \( g \in D^{-1,2}(\mathbb{R}^d; \mathbb{R}^d) \).

In [4], [1, 2], and many other literatures, the source functions are assumed to be in \( L^2(\Omega) \), which is a subspace of \( W^{-1,2}(\Omega) \) or \( D^{-1,2}(\mathbb{R}^d) \) when \( \Omega \) is bounded. Actually this choice can be relaxed for the critical and subcritical cases, where \( W^{-1,2}(\Omega) \) source functions will be good. In bounded domains, the classical Poincaré inequality can always be applied. But we loose the uniformness of the Poincaré type inequality for the subcritical case in the whole space. We need better source functions in \( D^{-1,2}(\mathbb{R}^d) \) for this case. We do not need more restrictions for the other two cases compared to the study in a bounded domain. We give a remark on Assumption 2.1 (iii):

**Remark 2.2.** A sufficient condition for Assumption 2.1 (iii) is the following:

\[
f \in L^{\frac{2d}{d+2}}(\mathbb{R}^d) \text{ if } d \geq 3; \quad f \in L^2(\mathbb{R}^d), \ f \text{ is compactly supported and } \int f = 0 \text{ if } d = 2.
\]

If \( d \geq 3 \), the number \( \frac{2d}{d+2} \) is actually the Lebesgue conjugate number of the component \( 2^* \) from the Gagliardo-Nirenberg-Sobolev inequality (2.2). This ensures \( L^{\frac{2d}{d+2}}(\mathbb{R}^d) \) is continuously embedded into \( D^{-1,2}(\mathbb{R}^d) \).

The \( 2d \) case is more tricky. In this case \( D^{1,2}_0(\mathbb{R}^2) = D^{1,2}(\mathbb{R}^2) \) the functions in which can only be defined up to an addition of some constant. To ensure \( f \in D^{-1,2}(\mathbb{R}^2) \), necessarily \( \langle f, 1 \rangle = 0 \) which
is equivalent to \( \int f = 0 \) if \( f \) is integrable. If \( f \in L^2_0(\mathbb{R}^d) \) (the subscript 0 means zero average) and \( f \) is compactly supported, then applying the Poincaré’s inequality

\[
\|u - \langle u \rangle_{\text{supp} f}\|_{L^2(\text{supp} f)} \leq C(\text{supp} f)\|\nabla u\|_{L^2(\text{supp} f)}
\]

implies

\[
\int_{\mathbb{R}^d} f u \, dx = \int_{\text{supp} f} (u - \langle u \rangle_{\text{supp} f}) \, dx \leq C(\text{supp} f)\|f\|_{L^2}\|\nabla u\|_{L^2(\text{supp} f)}.
\]

This means \( f \in D^{-1,2}(\mathbb{R}^d) \). Here \( \langle u \rangle_{\text{supp} f} \) denotes the average of \( u \) in \( \text{supp} f \). We remark that the constant \( C(\text{supp} f) \) depends on the size of \( \text{supp} f \).

Remark 2.3. To ensure the well-posedness of the Poisson problem (1.2) and the Stokes problem (1.3), weaker assumptions \( f \in W^{-1,2}(\mathbb{R}^d) \), \( g \in W^{-1,2}(\mathbb{R}^d; \mathbb{R}^d) \) will be sufficient for all three cases and for all \( d \geq 2 \). See Proposition 3.1.

Before stating the theorems, we introduce the following convention: for each \( u \in W^{1,2}_0(\Omega_\varepsilon) \), we will naturally treat \( u \) as a function in \( W^{1,2}_0(\mathbb{R}^d) \) by imposing

\[
u = 0 \quad \text{on} \quad \mathbb{R}^d \setminus \Omega_\varepsilon = \bigcup_{k \in \mathbb{Z}^d} T_{\varepsilon,k}.
\]

For the Poisson problem (1.2), we have the following result where the limits are taken up to possible extractions of subsequences.

**Theorem 2.4.** Let \( \Omega = \mathbb{R}^d \), \( d \geq 2 \). Let \( f \) satisfy Assumption 2.1. Then for each fixed \( \varepsilon \in (0, 1) \), the Poisson problem (1.2) admits a unique weak solution \( u_\varepsilon \in W^{1,2}_0(\Omega_\varepsilon) \). Moreover, we have the following description of the limit system related to different sizes of holes:

(i) If \( \lim_{\varepsilon \to 0} \sigma_\varepsilon = 0 \) corresponding to the case of large holes, we have

\[
\sigma_\varepsilon^{-2} u_\varepsilon \rightharpoonup u \quad \text{weakly in} \quad L^2(\mathbb{R}^d),
\]

where \( u \) satisfies

\[
u = \bar{w} f.
\]

(ii) If \( \lim_{\varepsilon \to 0} \sigma_\varepsilon = \sigma_* \in (0, +\infty) \) corresponding to the case of critical size of holes, we have

\[
u_{\varepsilon} \rightharpoonup u \quad \text{weakly in} \quad W^{1,2}_0(\mathbb{R}^d),
\]

where \( u \) solves the Laplace-Brinkman equation:

\[
-\Delta u + \sigma_*^{-2} \bar{w}^{-1} u = f, \quad \text{in} \quad \mathbb{R}^d.
\]

(iii) If \( \lim_{\varepsilon \to 0} \sigma_\varepsilon = \infty \) corresponding to the case of small holes, we have

\[
u_{\varepsilon} \to u \quad \text{strongly in} \quad D^{1,2}_0(\mathbb{R}^d),
\]

where \( u \) satisfies the Poisson equation

\[
-\Delta u = f, \quad \text{in} \quad \mathbb{R}^d.
\]
Here in cases (i) and (ii), \( \bar{w} \) is a positive constant solely determined by the model hole \( T \) and is given in (6.1).

For the Stokes problem, we have the following theorem. Even with the presence of the pressure term, no additional assumption is needed. Again, the limits are taken up to possible extractions of subsequences.

**Theorem 2.5.** Let \( \Omega = \mathbb{R}^d, \ d \geq 2. \) Let \( g \) satisfy Assumption 2.1. Then the Stokes problem (1.3) admits a unique weak solution \( (v_\varepsilon, p_\varepsilon) \in W^{1,2}_0(\Omega; \mathbb{R}^d) \times L^2_{\text{loc}}(\Omega) \) where the uniqueness of \( p_\varepsilon \) is defined up to modulating constants. And there exists an extension \( \tilde{p}_\varepsilon \in L^2_{\text{loc}}(\mathbb{R}^d) \) of the pressure such that \( \tilde{p}_\varepsilon = p_\varepsilon \) in \( \Omega_\varepsilon \). Moreover, we have the following homogenization results:

(i) If \( \lim_{\varepsilon \to 0} \sigma_\varepsilon = 0 \), there holds

\[ \sigma_\varepsilon^{-2} v_\varepsilon \to v \text{ weakly in } L^2(\mathbb{R}^d; \mathbb{R}^d) \]

and \( \tilde{p}_\varepsilon = p_\varepsilon^{(1)} + p_\varepsilon^{(2)} \) with

\[ \nabla p_\varepsilon^{(1)} \to \nabla p \text{ weakly in } L^2(\mathbb{R}^d; \mathbb{R}^d), \quad p_\varepsilon^{(2)} \to 0 \text{ strongly in } L^2(\mathbb{R}^d). \]

Moreover, the limit \( (v, p) \in L^2(\mathbb{R}^d; \mathbb{R}^d) \times D^{1,2}(\mathbb{R}^d) \) satisfies the Darcy’s law:

\[
\begin{aligned}
\begin{cases}
v = A(g - \nabla p), & \text{in } \mathbb{R}^d, \\
\text{div } v = 0, & \text{in } \mathbb{R}^d.
\end{cases}
\end{aligned}
\]

(ii) If \( \lim_{\varepsilon \to 0} \sigma_\varepsilon = \sigma_* \in (0, +\infty) \), then

\[ v_\varepsilon \to v \text{ weakly in } W^{1,2}_0(\mathbb{R}^d; \mathbb{R}^d) \]

and \( \tilde{p}_\varepsilon = p_\varepsilon^{(1)} + p_\varepsilon^{(2)} \) with

\[ \nabla p_\varepsilon^{(1)} \to \nabla p^{(1)} \text{ weakly in } W^{m,2}(\mathbb{R}^d; \mathbb{R}^d) \text{ for all } m \in \mathbb{N}, \quad p_\varepsilon^{(2)} \to p^{(2)} \text{ weakly in } L^2(\mathbb{R}^d). \]

Let \( p = p^{(1)} + p^{(2)}. \) Then \( (v, p) \in W^{1,2}_0(\mathbb{R}^d; \mathbb{R}^d) \times (C^\infty(\mathbb{R}^d) + L^2(\mathbb{R}^d)) \) satisfies the Stokes-Brinkman equations:

\[
\begin{aligned}
\begin{cases}
-\Delta v + \nabla p + \sigma_*^{-2} A^{-1} v = g, & \text{in } \mathbb{R}^d, \\
\text{div } v = 0, & \text{in } \mathbb{R}^d.
\end{cases}
\end{aligned}
\]

(iii) If \( \lim_{\varepsilon \to 0} \sigma_\varepsilon = \infty \), we have

\[ v_\varepsilon \to v \text{ strongly in } D^{1,2}_0(\mathbb{R}^d; \mathbb{R}^d), \]

and \( \tilde{p}_\varepsilon = p_\varepsilon^{(1)} + p_\varepsilon^{(2)} \) with

\[ \nabla p_\varepsilon^{(1)} \to 0 \text{ strongly in } L^2(\mathbb{R}^d; \mathbb{R}^d), \quad p_\varepsilon^{(2)} \to p \text{ weakly in } L^2(\mathbb{R}^d). \]

Moreover, the limit \( (v, p) \in D^{1,2}_0(\mathbb{R}^d; \mathbb{R}^d) \times L^2(\mathbb{R}^d) \) solves the Stokes equations:

\[
\begin{aligned}
\begin{cases}
-\Delta v + \nabla p = g, & \text{in } \mathbb{R}^d, \\
\text{div } v = 0, & \text{in } \mathbb{R}^d.
\end{cases}
\end{aligned}
\]
Here in cases (i) and (ii), $A$ is a constant positive definite matrix determined by the model hole $T$ and given later in (5.4).

Our theorems extend the pioneering results in [4, 1, 2] to the whole space. Unlike in a bounded domain, a lot nice properties in the perforated whole space are missing, such as Poincaré inequality, Bogovskii operator, and higher integrability implying lower ones. These properties are often used in the study of homogenization. Still, we derived the same uniform estimates and the same limit systems in the whole space. This is the main novelty of this paper. We remark that our results also extend the homogenization results in [15] to two dimensional case in periodic holes configurations: for any $d \geq 2$ and any source term in $W^{-1,2}(\mathbb{R}^d)$, we derived the Brinkman type equations for the critical case.

Another novelty of this paper is the method of proof. The perforated whole space is a bad domain in the sense that it is not bounded, nor is its complementary due to the infinite holes distributed all over the space; this means it is not an exterior domain either. However, the same reason makes the domain a good one: one can benefit from the zero boundary conditions on the holes which are everywhere in $\mathbb{R}^d$ and obtain a Poincaré type inequality (see Lemma 3.2 given later). This is observed for bounded domains by Tartar [25] for the special case where the mutual distance is comparable to the size of the holes, and is generalized by Allaire [2]. The same idea applies to unbounded domains as shown in Lemma 4.5 in [15]. This Poincaré type inequality can be used to close the energy estimates for (1.2) and (1.3), and to deduce the uniform estimates for the velocity field. The only issue for this Poincaré type inequality is that one has an unbounded estimate constant as $\varepsilon \to 0$ for the subcritical case of small holes. This is the reason that we assume the source term in $D^{-1,2}(\mathbb{R}^d)$ in Assumption 2.1 for the subcritical case.

For the Stokes problem, additional difficulties arise due to the pressure term. An observation is that the restriction operator constructed in [1] only relies on local properties. It turns out that it can be applied to the whole space. Then following [25] and [2], the extension of the pressure can be defined by using the restriction operator through a dual pair. We introduce suitable frequency cut-off functions for different cases and deduce uniform estimates for the pressure extension. Given the desired uniform estimates, we employ a modified cell problem and use a unified approach to prove the homogenization results, as in [16] and [18].

The rest of the paper is devoted to the proof of our theorems. We will show the proof details only for the Stokes problem. The Poisson case can be done similarly and we only give a sketch in Section 6; actually the proof is easier without the extra troubles caused by the pressure. The paper is organized as following. In Section 3, we prove a preliminary result concerning the well-posedness of the Stokes problem (1.3) for each fixed $\varepsilon > 0$. Then in Section 4 we deduce our desired uniform estimates. We finally derive the limit system in Section 5.

In the sequel, we use $C$ to denote a universal positive constant independent of $\varepsilon$.

### 3 Solvability of the Stokes problem

We shall prove the following result:

**Proposition 3.1.** Let $\Omega = \mathbb{R}^d$, $d \geq 2$ and let $g \in W^{-1,2}(\Omega_\varepsilon;\mathbb{R}^d)$. For each fixed $\varepsilon \in (0, 1)$, the Stokes problem (1.3) admits a unique weak solution $(v_\varepsilon, p_\varepsilon) \in W^{1,2}_0(\Omega_\varepsilon;\mathbb{R}^d) \times L^2_{\mathrm{loc}}(\Omega_\varepsilon)$ such that $\operatorname{div} v_\varepsilon = 0$ and

$$
\int_{\Omega_\varepsilon} \nabla v_\varepsilon : \nabla \varphi \, dx - \int_{\Omega_\varepsilon} p_\varepsilon \operatorname{div} \varphi \, dx = \langle g, \varphi \rangle, \quad \text{for all } \varphi \in C_c^\infty(\Omega_\varepsilon;\mathbb{R}^d).
$$


The uniqueness of \( p_\varepsilon \) is defined up to adding constants. Moreover, there hold the estimates

\[
\|\nabla v_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C(1 + \sigma_\varepsilon), \quad \|v_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C\sigma_\varepsilon(1 + \sigma_\varepsilon),
\]

where \( \sigma_\varepsilon \) is the ratio given in (1.4).

Here we provide a simple proof by approximating the unbounded domain \( \Omega_\varepsilon \) by bounded ones. The key point is the following Poincaré type inequality in perforated domains. Note that Proposition 3.1 can also be proved by using the classical variational method together with the following Poincaré type inequality.

**Lemma 3.2.** Let \( R > 1 \) and the holes \( T_{\varepsilon,k} \) be given in (1.1). Define

\[
\Omega_{\varepsilon,R} := B(0, R) \setminus \bigcup_{k \in K_{\varepsilon,R}} T_{\varepsilon,k}, \quad K_{\varepsilon,R} := \{ k \in \mathbb{Z}^d : \varepsilon Q_k \subset B(0, R) \}.
\]

Then there holds

\[
\|u\|_{L^2(\Omega_{\varepsilon,R})} \leq C\sigma_\varepsilon\|\nabla u\|_{L^2(\Omega_{\varepsilon,R})}, \quad \text{for all } u \in W^{1,2}_0(\Omega_{\varepsilon,R}),
\]

where \( \sigma_\varepsilon \) is given in (1.4) and \( C \) is independent of \( R \). The above result holds for \( R = \infty \):

\[
\|u\|_{L^2(\Omega_\varepsilon)} \leq C\sigma_\varepsilon\|\nabla u\|_{L^2(\Omega_\varepsilon)}, \quad \text{for all } u \in W^{1,2}_0(\Omega_\varepsilon).
\]

Similar results have been shown in [2] (see Lemma 3.4.1 therein) for bounded domains. In [15, Lemma 4.5], the critical case in \( \mathbb{R}^3 \) was considered. Lemma 3.2 can be proved similarly. For the convenience of the readers, we briefly reproduce it below.

**Proof of Lemma 3.2.** We will only prove (3.4). The proof of (3.3) can be done similarly. We will assume \( u \in C^\infty_c(\Omega_\varepsilon) \); for general \( u \in W^{1,2}_0(\Omega_\varepsilon) \), the result follows by density argument. By (1.1), we observe that for each \( k \in \mathbb{Z}^d \),

\[
B(\varepsilon x_k, \delta_1 a_\varepsilon) \subset \subset T_{\varepsilon,k} \subset \subset B(\varepsilon x_k, \delta_2 \varepsilon) \subset \subset \varepsilon Q_k \subset \subset B(\varepsilon x_k, 2\varepsilon) \subset \bigcup_{|k' - k| \leq 3} \varepsilon Q_{k'},
\]

where

\[
|k - k'| = |(k_1, \ldots, k_d) - (k'_1, \ldots, k'_d)| := \sum_{i=1}^d |k_i - k'_i|.
\]

For each \( x \in \Omega_\varepsilon \), there exists \( k \in \mathbb{Z}^d \) such that \( x \in \varepsilon Q_k \). Denote

\[
r_x = |x - \varepsilon x_k|, \quad \omega_x = \frac{x - \varepsilon x_k}{|x - \varepsilon x_k|}.
\]

By the fact \( u = 0 \) on \( T_{\varepsilon,k} \supset B(\varepsilon x_k, \delta_1 a_\varepsilon) \), we have

\[
u(x) = u(\varepsilon x_k + r_x \omega_x) - u(\varepsilon x_k + \delta_1 a_\varepsilon \omega_x)
= \int_{\delta_1 a_\varepsilon}^{r_x} \frac{d}{ds} u(\varepsilon x_k + s \omega_x) \, ds
= \int_{\delta_1 a_\varepsilon}^{r_x} (\nabla u)(\varepsilon x_k + s \omega_x) \cdot \omega_x \, ds.
\]
By H"{o}lder's inequality, direct calculation gives

\[ \|u\|_{L^2(\overline{\Omega}_{\varepsilon R})}^2 \leq \int_{B(\varepsilon x_k, 2\varepsilon) \setminus B(\varepsilon x_k, \varepsilon a_{\varepsilon})} |u(x)|^2 \, dx = \int_{\delta_{1a_{\varepsilon}}}^{2\varepsilon} \int_{\delta_{1a_{\varepsilon}}}^{2\varepsilon} |u(\varepsilon x_k + r\omega)|^2 r^{d-1} \, d\omega \, dr \]
\[ \leq \int_{\delta_{1a_{\varepsilon}}}^{2\varepsilon} \int_{\delta_{1a_{\varepsilon}}}^{2\varepsilon} \left( \int_{\delta_{1a_{\varepsilon}}}^{2\varepsilon} (\nabla u)(\varepsilon x_k + s\omega) \cdot \omega \, ds \right)^2 r^{d-1} \, d\omega \, dr \]
\[ \leq \int_{\delta_{1a_{\varepsilon}}}^{2\varepsilon} \int_{\delta_{1a_{\varepsilon}}}^{2\varepsilon} \left( \int_{\delta_{1a_{\varepsilon}}}^{2\varepsilon} s^{-d+1} \, ds \right) \left( \int_{\delta_{1a_{\varepsilon}}}^{2\varepsilon} s^{d-1} |\nabla u(\varepsilon x_k + s\omega)|^2 \, ds \right) \, dr \, d\omega \]
\[ \leq \left( \int_{\delta_{1a_{\varepsilon}}}^{2\varepsilon} r^{d-1} \left( \int_{\delta_{1a_{\varepsilon}}}^{2\varepsilon} s^{-d+1} \, ds \right) \right) \left( \int_{\delta_{1a_{\varepsilon}}}^{2\varepsilon} \int_{\delta_{1a_{\varepsilon}}}^{2\varepsilon} s^{d-1} |\nabla u(\varepsilon x_k + s\omega)|^2 \, ds \, d\omega \right) \]
\[ \leq C \int_{\delta_{1a_{\varepsilon}}}^{2\varepsilon} r^{d-1} \, dr \int_{\delta_{1a_{\varepsilon}}}^{2\varepsilon} s^{-d+1} \, ds \int_{B(\varepsilon x_k, 2\varepsilon) \setminus B(\varepsilon x_k, \varepsilon a_{\varepsilon})} |\nabla u(x)|^2 \, dx. \]

We then deduce from (3.6) that

\[ \|u\|_{L^2(\overline{\Omega}_{\varepsilon R})}^2 \leq C \varepsilon^2 \log \left( \frac{\varepsilon}{a_{\varepsilon}} \right) \|u\|_{L^2(B(\varepsilon x_k, 2\varepsilon))}^2, \quad \text{if } d = 2, \]
\[ \|u\|_{L^2(\overline{\Omega}_{\varepsilon R})}^2 \leq C \varepsilon^d a_{\varepsilon}^{-d+2} \|u\|_{L^2(B(\varepsilon x_k, 2\varepsilon))}^2, \quad \text{if } d \geq 3. \]

Recall the definition of \( \sigma_{\varepsilon} \) in (1.4). By (3.5) and (3.7), we obtain

\[ \|u\|_{L^2(\overline{\Omega}_{\varepsilon R})}^2 \leq C \sigma_{\varepsilon}^2 \|\nabla u\|_{L^2(B(\varepsilon x_k, 2\varepsilon))}^2 \leq C \sigma_{\varepsilon}^2 \sum_{|k' - k| \leq 3} \|\nabla u\|_{L^2(B(\varepsilon \overline{\Omega}_{k'}))}^2. \]

Thus,

\[ \|u\|_{L^2(\overline{\Omega}_{\varepsilon R})}^2 = \sum_{k \in \mathbb{Z}^d} \|u\|_{L^2(\overline{\Omega}_{\varepsilon R})}^2 \leq C \sigma_{\varepsilon}^2 \sum_{k \in \mathbb{Z}^d} \sum_{|k' - k| \leq 3} \|\nabla u\|_{L^2(B(\varepsilon \overline{\Omega}_{k'}))}^2 \]
\[ \leq C \sigma_{\varepsilon}^2 \sum_{k' \in \mathbb{Z}^d} \|\nabla u\|_{L^2(B(\varepsilon \overline{\Omega}_{k'}))}^2 \sum_{|k'| \leq 3} 1 \]
\[ \leq C \sigma_{\varepsilon}^2 \sum_{k' \in \mathbb{Z}^d} \|\nabla u\|_{L^2(\overline{\Omega}_{\varepsilon R})}^2. \]

This is our desired inequality (3.4).

\[ \square \]

**Proof of Proposition 3.1.** Now we consider the following Stokes equations in bounded domain \( \Omega_{\varepsilon, R} \) with \( R > 1 \):

\[ \begin{cases} -\Delta \mathbf{v}_{\varepsilon, R} + \nabla p_{\varepsilon, R} = \mathbf{g}, & \text{in } \Omega_{\varepsilon, R}, \\ \text{div} \mathbf{v}_{\varepsilon, R} = 0, & \text{in } \Omega_{\varepsilon, R}, \\ \mathbf{v}_{\varepsilon, R} = 0, & \text{on } \partial \Omega_{\varepsilon, R}. \end{cases} \]

For the bounded \( C^1 \) domain \( \Omega_{\varepsilon, R} \), the classical theory (see [11] for instance) ensures the existence and uniqueness of weak solution \( (\mathbf{v}_{\varepsilon, R}, p_{\varepsilon, R}) \in W^{1,2}_0(\Omega_{\varepsilon, R}; \mathbb{R}^d) \times L^2(\Omega_{\varepsilon, R}) \) in the classical weak sense: \( \text{div} \mathbf{v}_{\varepsilon, R} = 0 \) and

\[ \int_{\Omega_{\varepsilon, R}} \nabla \mathbf{v}_{\varepsilon, R} : \nabla \varphi \, dx = (\mathbf{g}, \varphi), \quad \text{for all } \varphi \in C^\infty_{c, \text{div}}(\Omega_{\varepsilon, R}; \mathbb{R}^d). \]
For the pressure, by (3.13), we may apply Lemma IV.1.1 in [11] to ensure that there exists clearly \( \text{div} \mathbf{v} \) (3.13) and we shall assume \( g \) the pressure so far. The goal of this section is to prove the uniform estimates given in Theorem 2.5. In particular, we do not have any uniform estimates for homogenized models in Theorem 2.5. Choosing \( u \) We know from Proposition 3.1 that the Stokes problem (1.3) admits a unique weak solution \( (\mathbf{v}, p) \) Here \( L^2_0(\Omega_{\varepsilon,R}) \) denotes the space of \( L^2(\Omega_{\varepsilon,R}) \) functions which are of zero average. By density argument, the test functions in (3.9) can be taken as arbitrary divergence free functions in \( W^{1,2}_0(\Omega_{\varepsilon,R}; \mathbb{R}^d) \). Choosing the solution \( \mathbf{v}_{\varepsilon,R} \) itself as a test function in (3.9) implies

\[
\|\nabla \mathbf{v}_{\varepsilon,R}\|_{L^2(\Omega_{\varepsilon,R})} \leq C\|g\|_{W^{-1,2}(\Omega_{\varepsilon,R})}\|\mathbf{v}_{\varepsilon,R}\|_{W^{1,2}(\Omega_{\varepsilon,R})} \\
\leq C\|g\|_{W^{-1,2}(\Omega_{\varepsilon,R})}(\|\mathbf{v}_{\varepsilon,R}\|_{L^2(\Omega_{\varepsilon,R})} + \|\nabla \mathbf{v}_{\varepsilon,R}\|_{L^2(\Omega_{\varepsilon,R})}).
\]

Applying Lemma 3.2 in (3.10) gives

\[
\|\nabla \mathbf{v}_{\varepsilon,R}\|_{L^2(\Omega_{\varepsilon,R})} \leq C\|g\|_{W^{-1,2}(\Omega_{\varepsilon})}(1 + \sigma_{\varepsilon})\|\nabla \mathbf{v}_{\varepsilon,R}\|_{L^2(\Omega_{\varepsilon,R})}.
\]

Thus,

\[
\|\nabla \mathbf{v}_{\varepsilon,R}\|_{L^2(\Omega_{\varepsilon,R})} \leq C\|g\|_{W^{-1,2}(\Omega_{\varepsilon})}(1 + \sigma_{\varepsilon}), \quad \|\mathbf{v}_{\varepsilon,R}\|_{L^2(\Omega_{\varepsilon,R})} \leq C\|g\|_{W^{-1,2}(\Omega_{\varepsilon})}\sigma_{\varepsilon}(1 + \sigma_{\varepsilon}).
\]

We can extend \( \mathbf{v}_{\varepsilon,R} \) by zero on \( B(0, R)^c \) and obtain a bounded family \( \{\mathbf{v}_{\varepsilon,R}\}_{R>1} \) in \( W^{1,2}(\Omega_{\varepsilon}; \mathbb{R}^d) \). Then up to a subsequence,

\[
\mathbf{v}_{\varepsilon,R} \to \mathbf{v}_{\varepsilon} \text{ weakly in } W^{1,2}_0(\Omega_{\varepsilon}; \mathbb{R}^d) \text{ as } R \to \infty.
\]

Clearly \( \text{div} \mathbf{v}_{\varepsilon} = 0 \) and the estimates in (3.2) follow from (3.11) and (3.12). We shall show that \( \mathbf{v}_{\varepsilon} \) is a weak solution to the original Stokes problem (1.3). Indeed, for any \( \varphi \in C^\infty_c(\Omega_{\varepsilon}; \mathbb{R}^d) \), there exists \( R_{\varphi} > 1 \) such that \( \text{supp} \varphi \subset \Omega_{\varepsilon,R_{\varphi}} \). This means that \( \varphi \) is a proper test function in (3.9) for all \( R \geq R_{\varphi} \). Testing (3.9) by \( \varphi \), passing \( R \to \infty \), and applying (3.12) gives

\[
\int_{\Omega_{\varepsilon}} \nabla \mathbf{v}_{\varepsilon} : \nabla \varphi \, dx = \langle \mathbf{g}, \varphi \rangle, \quad \text{for any } \varphi \in C^\infty_c(\Omega_{\varepsilon}; \mathbb{R}^d).
\]

For the pressure, by (3.13), we may apply Lemma IV.1.1 in [11] to ensure that there exists \( p_{\varepsilon} \in L^2_{\text{loc}}(\Omega_{\varepsilon}) \) such that (3.1) is satisfied.

The uniqueness can be derived in a classical way. Let \( (\mathbf{u}_{\varepsilon}, q_{\varepsilon}) \in W^{1,2}_0(\Omega_{\varepsilon}; \mathbb{R}^d) \times L^2_{\text{loc}}(\Omega_{\varepsilon}) \) be a solution to (1.3) with \( \mathbf{g} = 0 \), that means \( \text{div} \mathbf{u}_{\varepsilon} = 0 \) and

\[
\int_{\Omega_{\varepsilon}} \nabla \mathbf{u}_{\varepsilon} : \nabla \varphi \, dx - \int_{\Omega_{\varepsilon}} q_{\varepsilon} \text{div} \varphi \, dx = 0, \quad \text{for all } \varphi \in C^\infty_c(\Omega_{\varepsilon}; \mathbb{R}^d).
\]

Choosing \( \mathbf{u}_{\varepsilon} \) as a test function in (3.14) implies \( \|\nabla \mathbf{u}_{\varepsilon}\|_{L^2(\Omega_{\varepsilon})} = 0 \) and then \( \mathbf{u}_{\varepsilon} = 0 \), due to \( \mathbf{u}_{\varepsilon} \in W^{1,2}_0(\Omega_{\varepsilon}; \mathbb{R}^d) \). Thus,

\[
\int_{\Omega_{\varepsilon}} q_{\varepsilon} \text{div} \varphi \, dx = 0, \quad \text{for all } \varphi \in C^\infty_c(\Omega_{\varepsilon}; \mathbb{R}^d).
\]

This implies \( \nabla q_{\varepsilon} = 0 \) and \( q_{\varepsilon} \) is a constant in \( \Omega_{\varepsilon} \). We completed the proof of Proposition 3.1.

\[\Box\]

4 Uniform estimates

We know from Proposition 3.1 that the Stokes problem (1.3) admits a unique weak solution \( (\mathbf{v}_{\varepsilon}, p_{\varepsilon}) \) for merely \( \mathbf{g} \in W^{-1,2}(\Omega_{\varepsilon}; \mathbb{R}^d) \). However the estimates in (3.2) are not enough to derive the homogenized models in Theorem 2.5. In particular, we do not have any uniform estimates for the pressure so far. The goal of this section is to prove the uniform estimates given in Theorem 2.5 and we shall assume \( \mathbf{g} \) satisfy Assumption 2.1.
4.1 Estimates of $v_{\varepsilon}$

We will estimate $v_{\varepsilon}$ case by case. We first consider the critical case $\lim_{\varepsilon \to 0} \sigma_{\varepsilon} = \sigma_* \in (0, \infty)$. In this case $\{\sigma_{\varepsilon}\}_{0 < \varepsilon < 1}$ is bounded, so we can directly apply Proposition 3.1 to obtain

\begin{equation}
\|v_{\varepsilon}\|_{W^{1,2}(\Omega_{\varepsilon})} \leq C\|g\|_{W^{-1,2}(\mathbb{R}^d)}.
\end{equation}

We then consider the subcritical case $\lim_{\varepsilon \to 0} \sigma_{\varepsilon} = \infty$. In this case, we assume $g \in D^{-1,2}(\mathbb{R}^d; \mathbb{R}^d)$. By Theorem 3.1, taking $v_{\varepsilon}$ as a test function in (3.1) gives

\begin{equation}
\|\nabla v_{\varepsilon}\|_{L^2(\Omega_{\varepsilon})} \leq C\|g\|_{D^{-1,2}(\Omega_{\varepsilon})}\|v_{\varepsilon}\|_{D_0^{1,2}(\Omega_{\varepsilon})} \leq C\|g\|_{D^{-1,2}(\mathbb{R}^d)}\|\nabla v_{\varepsilon}\|_{L^2(\Omega_{\varepsilon})}.
\end{equation}

This implies

\begin{equation}
\|\nabla v_{\varepsilon}\|_{L^2(\Omega_{\varepsilon})} \leq C\|g\|_{D^{-1,2}(\mathbb{R}^d)}.
\end{equation}

Unfortunately, we merely have an unbounded estimate for the $L^2$ norm of $v_{\varepsilon}$ by using the Poincaré inequality in Lemma 3.2:

\begin{equation}
\|v_{\varepsilon}\|_{L^2(\Omega_{\varepsilon})} \leq C\sigma_{\varepsilon}\|\nabla v_{\varepsilon}\|_{L^2(\Omega_{\varepsilon})} \leq C\sigma_{\varepsilon}\|g\|_{D^{-1,2}(\mathbb{R}^d)}.
\end{equation}

However, if $d \geq 3$, since $v_{\varepsilon} \in W_0^{1,2}(\Omega_{\varepsilon}; \mathbb{R}^d)$, the Gagliardo-Nirenberg-Sobolev inequality gives

\begin{equation}
\|v_{\varepsilon}\|_{L^2(\Omega_{\varepsilon})} \leq C\|\nabla v_{\varepsilon}\|_{L^2(\Omega_{\varepsilon})} \leq C\|g\|_{D^{-1,2}(\mathbb{R}^d)}.
\end{equation}

We finally consider the supercritical case $\lim_{\varepsilon \to 0} \sigma_{\varepsilon} = 0$ where we assume $g \in L^2(\mathbb{R}^d; \mathbb{R}^d)$. Then taking $v_{\varepsilon}$ as a test function in (3.1) and using the Poincaré inequality in Lemma 3.2 gives

\begin{equation}
\|\nabla v_{\varepsilon}\|_{L^2(\Omega_{\varepsilon})} \leq C\|g\|_{L^2(\Omega_{\varepsilon})}\|v_{\varepsilon}\|_{L^2(\Omega_{\varepsilon})} \leq C\sigma_{\varepsilon}\|g\|_{L^2(\mathbb{R}^d)}\|\nabla v_{\varepsilon}\|_{L^2(\Omega_{\varepsilon})}.
\end{equation}

Hence,

\begin{equation}
\|\nabla v_{\varepsilon}\|_{L^2(\Omega_{\varepsilon})} \leq C\sigma_{\varepsilon}\|g\|_{L^2(\mathbb{R}^d)}, \quad \|v_{\varepsilon}\|_{L^2(\Omega_{\varepsilon})} \leq C\sigma_{\varepsilon}^2\|g\|_{L^2(\mathbb{R}^d)}.
\end{equation}

We summarize the above estimates in (4.1)–(4.5) into the following proposition where the weak limits are taken up to extracting subsequences.

**Proposition 4.1.** Let $\Omega = \mathbb{R}^d$, $d \geq 2$ and $g$ satisfy Assumption 2.1. Then we have the following estimates for the solution $v_{\varepsilon}$ obtained in Proposition 3.1:

(i) For the critical case $\lim_{\varepsilon \to 0} \sigma_{\varepsilon} = \sigma_* \in (0, +\infty)$,

\begin{equation}
\|v_{\varepsilon}\|_{W^{1,2}(\Omega_{\varepsilon})} \leq C\|g\|_{W^{-1,2}(\mathbb{R}^d)}.
\end{equation}

Hence, $v_{\varepsilon} \rightharpoonup v$ weakly in $W_0^{1,2}(\mathbb{R}^d; \mathbb{R}^d)$ with $\text{div} v = 0$.

(ii) For the subcritical case $\lim_{\varepsilon \to 0} \sigma_{\varepsilon} = \infty$,

\begin{equation}
\|\nabla v_{\varepsilon}\|_{L^2(\Omega_{\varepsilon})} + \sigma_{\varepsilon}^{-1}\|v_{\varepsilon}\|_{L^2(\Omega_{\varepsilon})} \leq C\|g\|_{D^{-1,2}(\mathbb{R}^d)}.
\end{equation}

If $d \geq 3$, $\|v_{\varepsilon}\|_{L^{\frac{2d}{d-2}}(\Omega_{\varepsilon})} \leq C\|g\|_{D^{-1,2}(\mathbb{R}^d)}$. Hence, $v_{\varepsilon} \rightharpoonup v$ weakly in $D_0^{1,2}(\mathbb{R}^d; \mathbb{R}^d)$ with $\text{div} v = 0$.

(iii) For the supercritical case $\lim_{\varepsilon \to 0} \sigma_{\varepsilon} = 0$,

\begin{equation}
\|\nabla v_{\varepsilon}\|_{L^2(\Omega_{\varepsilon})} \leq C\sigma_{\varepsilon}\|g\|_{L^2(\mathbb{R}^d)}, \quad \|v_{\varepsilon}\|_{L^2(\Omega_{\varepsilon})} \leq C\sigma_{\varepsilon}^2\|g\|_{L^2(\mathbb{R}^d)}.
\end{equation}

Hence, $\sigma_{\varepsilon}^{-2}v_{\varepsilon} \rightharpoonup v$ weakly in $L^2(\mathbb{R}^d; \mathbb{R}^d)$ with $\text{div} v = 0$.

In above proposition, $\text{div} v = 0$ follows directly from $\text{div} v_{\varepsilon} = 0$. 

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4.2 Estimates of the pressure

It is more tricky to estimate the pressure. To do this, we will employ the restriction operator constructed by Allaire in [1, 2] (see also earlier in [25] for a specific case). Firstly we observe that Allaire’s construction relies essentially on analysis in the neighbourhood of each single hole. After checking the argument in Section 2.2 [1], his construction also works for unbounded domains:

**Proposition 4.2.** Let $\Omega = \mathbb{R}^d$, $d \geq 2$. For any $u \in W^{1,2}_0(\mathbb{R}^d; \mathbb{R}^d)$, define $R_\varepsilon(u)$ as following:

$$
R_\varepsilon(u)(x) := u(x), \quad \text{if } x \in \mathbb{R}^d \setminus \bigcup_{k \in \mathbb{Z}^d} B(\varepsilon x_k, \delta_2 \varepsilon),
$$

$$
R_\varepsilon(u)(x) := u_{\varepsilon,k}(x), \quad \text{if } x \in B(\varepsilon x_k, \delta_2 \varepsilon) \setminus T_{\varepsilon,k}, \quad \text{for each } k \in \mathbb{Z}^d,
$$

where $u_{\varepsilon,k}$ solves

$$
\begin{cases}
- \Delta u_{\varepsilon,k} + \nabla p_{\varepsilon,k} = -\Delta u, & \text{in } B(\varepsilon x_k, \delta_2 \varepsilon) \setminus T_{\varepsilon,k}, \\
\text{div } u_{\varepsilon,k} = \text{div } u + \frac{1}{|B(\varepsilon x_k, \delta_2 \varepsilon) \setminus T_{\varepsilon,k}|} \int_{T_{\varepsilon,k}} \text{div } u \, dx, & \text{in } B(\varepsilon x_k, \delta_2 \varepsilon) \setminus T_{\varepsilon,k}, \\
u_{\varepsilon,k} = u, & \text{on } \partial B(\varepsilon x_k, \delta_2 \varepsilon), \\
u_{\varepsilon,k} = 0, & \text{on } \partial T_{\varepsilon,k}.
\end{cases}
$$

Then $R_\varepsilon$ defines a linear operator (named restriction operator) from $W^{1,2}_0(\mathbb{R}^d; \mathbb{R}^d)$ to $W^{1,2}_0(\Omega_\varepsilon; \mathbb{R}^d)$ such that

(i) $u \in W^{1,2}_0(\Omega_\varepsilon; \mathbb{R}^d) \implies R_\varepsilon(\widetilde{u}) = u$ in $\Omega_\varepsilon$, where $\widetilde{u}$ is the zero extension of $u$ in $\mathbb{R}^d$.

(ii) $u \in W^{1,2}_0(\mathbb{R}^d; \mathbb{R}^d)$, $\text{div } u = 0$ in $\mathbb{R}^d \implies \text{div } R_\varepsilon(u) = 0$ in $\Omega_\varepsilon$.

(iii) For each $u \in W^{1,2}_0(\mathbb{R}^d; \mathbb{R}^d)$, $\|\nabla R_\varepsilon(u)\|_{L^2(\Omega_\varepsilon)} \leq C (\|\nabla u\|_{L^2(\mathbb{R}^d)} + \sigma^{-1}_\varepsilon \|u\|_{L^2(\mathbb{R}^d)})$, and by the Poincaré inequality in Lemma 3.2, there holds $\|R_\varepsilon(u)\|_{L^2(\Omega_\varepsilon)} \leq C (\sigma_\varepsilon \|\nabla u\|_{L^2(\mathbb{R}^d)} + \|u\|_{L^2(\mathbb{R}^d)})$.

The proof of Proposition 4.2 can be done exactly as in [1] and we omit it. We remark that a $W^{1,q}$ version of Allaire’s restriction operator is shown by the author in [17] for $3/2 < q < 3$ in a bounded domain in $\mathbb{R}^3$.

Applying the restriction operator $R_\varepsilon$, the extension of the pressure, denoted by $\tilde{p}_\varepsilon$, is defined by the following formula (see the original idea of Tartar in [25] in bounded domains):

$$
\langle \nabla \tilde{p}_\varepsilon, \varphi \rangle_{W^{-1,2}(\mathbb{R}^d),W^{1,2}_0(\mathbb{R}^d)} = \langle \nabla \tilde{p}_\varepsilon, R_\varepsilon(\varphi) \rangle_{W^{-1,2}(\Omega_\varepsilon),W^{1,2}_0(\Omega_\varepsilon)}, \quad \text{for all } \varphi \in W^{1,2}_0(\mathbb{R}^d; \mathbb{R}^d),
$$

where $p_\varepsilon$ is the pressure of the Stokes problem (1.3). Note that the above formulation (4.6) is well defined due to the three properties of $R_\varepsilon$ in Proposition 4.2:

- Property (iii) and the estimates of $v_\varepsilon$ in Proposition 4.1 ensure $\nabla \tilde{p}_\varepsilon \in W^{-1,2}(\mathbb{R}^d; \mathbb{R}^d)$.
- Property (ii) ensures the compatibility: for each $\varphi \in W^{1,2}_0(\mathbb{R}^d; \mathbb{R}^d)$ with $\text{div } \varphi = 0$, then one deduces naturally from (4.6) that $\langle \nabla \tilde{p}_\varepsilon, \varphi \rangle = \langle \nabla \tilde{p}_\varepsilon, R_\varepsilon(\varphi) \rangle = 0$.
- Property (i) ensures $\nabla \tilde{p}_\varepsilon = \nabla p_\varepsilon$ in $\Omega_\varepsilon$. 

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Now we are in the position to deduce the uniform estimates of $\tilde{p}_\varepsilon$, case by case. We will repeatedly use the estimates of $v_\varepsilon$ in Proposition 4.1.

We first consider the critical case $\lim_{\varepsilon \to 0} \sigma_\varepsilon = \sigma_* \in (0, \infty)$ where we have $\|v_\varepsilon\|_{W^{1,2}(\Omega_\varepsilon)} \leq C$ from Proposition 4.1. Then, by Property (iii) in Proposition 4.2, using the Stokes equations (1.3), we obtain for all $\varphi \in W_0^{1,2}(\mathbb{R}^d; \mathbb{R}^d)$ that

$$\left| \langle \nabla \tilde{p}_\varepsilon, \varphi \rangle_{\mathbb{R}^d} \right| = \left| \langle \nabla p_\varepsilon, R_\varepsilon(\varphi) \rangle_{\Omega_\varepsilon} \right| = \left| \langle \Delta v_\varepsilon + g, R_\varepsilon(\varphi) \rangle_{\Omega_\varepsilon} \right| \leq C \left( \|\nabla v_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|g\|_{W^{-1,2}(\mathbb{R}^d)} \right) \|R_\varepsilon(\varphi)\|_{W^{1,2}(\Omega_\varepsilon)} \leq C \left( \|\nabla \varphi\|_{L^2(\mathbb{R}^d)} + \|\varphi\|_{L^2(\mathbb{R}^d)} \right).$$

This means the family $\{\nabla \tilde{p}_\varepsilon\}_{0<\varepsilon<1}$ is bounded in $W^{-1,2}(\mathbb{R}^d; \mathbb{R}^d)$. Thus, by the characterization of Sobolev space $W^{1,2}(\mathbb{R}^d)$ using Fourier transforms, we have

$$\|\nabla \tilde{p}_\varepsilon\|_{W^{-1,2}(\mathbb{R}^d)}^2 = C_d \int_{\mathbb{R}^d} |\xi|^2 (1 + |\xi|^2)^{-1} |\mathcal{F}[\tilde{p}_\varepsilon](\xi)|^2 \, d\xi \leq C,$$

where $\mathcal{F}[\cdot]$ denotes the Fourier transform and $C_d$ is a constant depending only on the dimension $d$. Let $\chi \in C_\infty^\infty(B(0,2))$ be a cutoff function such that $0 \leq \chi \leq 1$ and $\chi = 1$ on $B(0,1)$. We can decompose $\tilde{p}_\varepsilon = p^{(1)}_\varepsilon + p^{(2)}_\varepsilon$ with

$$p^{(1)}_\varepsilon := \chi(D)\tilde{p}_\varepsilon, \quad p^{(2)}_\varepsilon := (1 - \chi(D))\tilde{p}_\varepsilon,$$

where $\chi(D)$ is the Fourier multiplier with symbol $\chi(\xi)$. Then by (4.7), direct calculation implies

$$\|\nabla p^{(1)}_\varepsilon\|_{W^{1,2}(\mathbb{R}^d)}^2 = C_d \int_{|\xi| \leq 2} |\xi|^2 (1 + |\xi|^2)^m |\chi(\xi)|^2 |\mathcal{F}[\tilde{p}_\varepsilon](\xi)|^2 \, d\xi \leq C_d 5^{m+1} \int_{|\xi| \leq 2} |\xi|^2 (1 + |\xi|^2)^{-1} |\mathcal{F}[\tilde{p}_\varepsilon](\xi)|^2 \, d\xi \leq C 5^m, \text{ for all } m \in \mathbb{N},$$

and

$$\|p^{(2)}_\varepsilon\|_{L^2(\mathbb{R}^d)}^2 = C_d \int_{|\xi| \geq 1} |1 - \chi(\xi)|^2 |\mathcal{F}[\tilde{p}_\varepsilon](\xi)|^2 \, d\xi \leq 2 C_d \int_{|\xi| \geq 1} |\xi|^2 (1 + |\xi|^2)^{-1} |\mathcal{F}[\tilde{p}_\varepsilon](\xi)|^2 \, d\xi \leq C.$$

This implies $\nabla p^{(1)}_\varepsilon \in \cap_{m=1}^\infty W^{m,2}(\mathbb{R}^d; \mathbb{R}^d) = \mathcal{S}(\mathbb{R}^d; \mathbb{R}^d)$ the Schwartz class.

Now we deal with the supercritical case $\lim_{\varepsilon \to 0} \sigma_\varepsilon = 0$ where we have $\|\nabla v_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C \sigma_\varepsilon$ and $\|v_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C \sigma_\varepsilon^2$ from Proposition 4.1. Then, by Property (iii) in Proposition 4.2, using (1.3) and Lemma 3.2, we obtain for all $\varphi \in W_0^{1,2}(\mathbb{R}^d; \mathbb{R}^d)$ that

$$\left| \langle \nabla \tilde{p}_\varepsilon, \varphi \rangle_{\mathbb{R}^d} \right| = \left| \langle \Delta v_\varepsilon + g, R_\varepsilon(\varphi) \rangle_{\Omega_\varepsilon} \right| \leq C \|\nabla v_\varepsilon\|_{L^2(\Omega_\varepsilon)} \|\nabla R_\varepsilon(\varphi)\|_{L^2(\Omega_\varepsilon)} + C \|g\|_{L^2(\mathbb{R}^d)} \|R_\varepsilon(\varphi)\|_{L^2(\Omega_\varepsilon)} \leq C \sigma_\varepsilon \|\nabla R_\varepsilon(\varphi)\|_{L^2(\mathbb{R}^d)} + C \sigma_\varepsilon \sigma_\varepsilon \|\nabla R_\varepsilon(\varphi)\|_{L^2(\mathbb{R}^d)} \leq C (\sigma_\varepsilon \|\nabla \varphi\|_{L^2(\mathbb{R}^d)} + \|\varphi\|_{L^2(\mathbb{R}^d)}).$$

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For fixed $\varepsilon > 0$, consider the semiclassical Sobolev space $W^{1,2}_{\sigma_\varepsilon}(\mathbb{R}^d)$ armed with the norm
\[
\|u\|_{W^{1,2}_{\sigma_\varepsilon}(\mathbb{R}^d)} := \left(\sigma_\varepsilon^2 \|\nabla \varphi\|_{L^2(\mathbb{R}^d)}^2 + \|\varphi\|_{L^2(\mathbb{R}^d)}^2\right)^{\frac{1}{2}} = C_d \left(\int_{\mathbb{R}^d} (1 + \sigma_\varepsilon^2 |\xi|^2)|\mathcal{F}[u](\xi)|^2 \, d\xi\right)^{\frac{1}{2}}.
\]

Then by (4.10), the family $\{\nabla \tilde{p}_\varepsilon\}_{0 < \varepsilon < 1}$ is bounded in $W^{-1,2}_{\sigma_\varepsilon}(\mathbb{R}^d; \mathbb{R}^d) = (W^{1,2}_{\sigma_\varepsilon}(\mathbb{R}^d; \mathbb{R}^d))'.$ This means
\[
\|\nabla \tilde{p}_\varepsilon\|_{W^{-1,2}_{\sigma_\varepsilon}(\mathbb{R}^d; \mathbb{R}^d)} = C_d \int_{\mathbb{R}^d} |\xi|^2 (1 + \sigma_\varepsilon^2 |\xi|^2)^{-1} |\mathcal{F}[\tilde{p}_\varepsilon](\xi)|^2 \, d\xi \leq C.
\]

Let $\chi \in C_c^\infty(B(0, 2))$ be the same cut-off function. We decompose $\tilde{p}_\varepsilon = p^{(1)}_\varepsilon + p^{(2)}_\varepsilon$ with
\[
p^{(1)}_\varepsilon := \chi(\sigma_\varepsilon D) \tilde{p}_\varepsilon, \quad p^{(2)}_\varepsilon := (1 - \chi(\sigma_\varepsilon D)) \tilde{p}_\varepsilon.
\]

Observing
\[
(1 + \sigma^2_\varepsilon |\xi|^2)^{-1} \geq 1/5 \quad \text{for all } |\xi| \leq 2\sigma_\varepsilon^{-1}, \quad |\xi|^2 (1 + \sigma^2_\varepsilon |\xi|^2)^{-1} \geq \sigma_\varepsilon^{-2}/2 \quad \text{for all } |\xi| \geq \sigma_\varepsilon^{-1}.
\]

Then by (4.11) we have
\[
\|\nabla p^{(1)}_\varepsilon\|_{L^2(\mathbb{R}^d)}^2 = C_d \int_{|\xi| \leq 2\sigma_\varepsilon^{-1}} |\xi|^2 |\chi(\sigma_\varepsilon \xi)|^2 |\mathcal{F}[\tilde{p}_\varepsilon](\xi)|^2 \, d\xi
\]
\[
\leq 5 C_d \int_{|\xi| \leq 2\sigma_\varepsilon^{-1}} |\xi|^2 (1 + \sigma_\varepsilon^2 |\xi|^2)^{-1} |\mathcal{F}[\tilde{p}_\varepsilon](\xi)|^2 \, d\xi
\]
\[
\leq C,
\]
and
\[
\|p^{(2)}_\varepsilon\|_{L^2(\mathbb{R}^d)}^2 = C_d \int_{|\xi| \geq \sigma_\varepsilon^{-1}} |1 - \chi(\sigma_\varepsilon \xi)|^2 |\mathcal{F}[\tilde{p}_\varepsilon](\xi)|^2 \, d\xi
\]
\[
\leq 2\sigma_\varepsilon^2 C_d \int_{|\xi| \geq \sigma_\varepsilon^{-1}} |\xi|^2 (1 + \sigma_\varepsilon^2 |\xi|^2)^{-1} |\mathcal{F}[\tilde{p}_\varepsilon](\xi)|^2 \, d\xi
\]
\[
\leq C \sigma_\varepsilon^2.
\]

For the subcritical case $\lim_{\varepsilon \to 0} \sigma_\varepsilon = \infty$ where we have $\|\nabla \varphi\|_{L^2(\Omega_\varepsilon)} \leq C$ from Proposition 4.1. Then for all $\varphi \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^d)$,
\[
|\langle \nabla \tilde{p}_\varepsilon, \varphi \rangle_{\mathbb{R}^d}| = |\langle \nabla \varphi \rangle_{\mathbb{R}^d}|
\]
\[
\leq C(\|\nabla \varphi\|_{L^2(\Omega_\varepsilon)} + \|g\|_{D^{-1,2}(\mathbb{R}^d)} \|\nabla R_\varepsilon(\varphi)\|_{L^2(\Omega_\varepsilon)}
\]
\[
\leq C(\|\nabla \varphi\|_{L^2(\mathbb{R}^d)} + \sigma_\varepsilon^{-1} \|\varphi\|_{L^2(\mathbb{R}^d)}).
\]

We may employ the analysis in the supercritical case and decompose $\tilde{p}_\varepsilon = p^{(1)}_\varepsilon + p^{(2)}_\varepsilon$ with
\[
p^{(1)}_\varepsilon := \chi(\sigma_\varepsilon D) \tilde{p}_\varepsilon, \quad p^{(2)}_\varepsilon := (1 - \chi(\sigma_\varepsilon D)) \tilde{p}_\varepsilon
\]
and deduce from (4.14) that
\[
\|\nabla p^{(1)}_\varepsilon\|_{L^2(\mathbb{R}^d)} \leq C \sigma^{-1}_\varepsilon, \quad \|p^{(2)}_\varepsilon\|_{L^2(\mathbb{R}^d)} \leq C.
\]

We summarize the above estimates (see (4.8), (4.9), (4.12), (4.13), (4.15)) into the following proposition where the limits are taken up to possible extractions of subsequences.
Proposition 4.3. Let $\Omega = \mathbb{R}^d$, $d \geq 2$ and $g$ satisfy Assumption 2.1. Then we have the following estimates for the pressure extension $\tilde{p}_\varepsilon$ defined by (4.6):

(i) For the critical case $\lim_{\varepsilon \to 0} \sigma_\varepsilon = \sigma_* \in (0, +\infty)$, there exists a decomposition $\tilde{p}_\varepsilon = p_\varepsilon^{(1)} + p_\varepsilon^{(2)}$ with

$$\|\nabla p_\varepsilon^{(1)}\|_{W^{m,2}(\mathbb{R}^d)} \leq C(m) \text{ for all } m \in \mathbb{N}, \quad \|p_\varepsilon^{(2)}\|_{L^2(\mathbb{R}^d)} \leq C.$$

Hence, $\nabla p_\varepsilon^{(1)} \rightharpoonup \nabla p^{(1)}$ weakly in $W^{m,2}(\mathbb{R}^d; \mathbb{R}^d)$ for all $m \in \mathbb{N}$, $p_\varepsilon^{(2)} \to p^{(2)}$ weakly in $L^2(\mathbb{R}^d)$.

(ii) For the supercritical case $\lim_{\varepsilon \to 0} \sigma_\varepsilon = 0$, there exists a decomposition $\tilde{p}_\varepsilon = p_\varepsilon^{(1)} + p_\varepsilon^{(2)}$ with

$$\|\nabla p_\varepsilon^{(1)}\|_{L^2(\mathbb{R}^d)} \leq C, \quad \|p_\varepsilon^{(2)}\|_{L^2(\mathbb{R}^d)} \leq C\sigma_\varepsilon.$$

Then $\nabla p_\varepsilon^{(1)} \to \nabla p$ weakly in $L^2(\mathbb{R}^d; \mathbb{R}^d)$ and $p_\varepsilon^{(2)} \to 0$ strongly in $L^2(\mathbb{R}^d)$.

(iii) For the subcritical case $\lim_{\varepsilon \to 0} \sigma_\varepsilon = \infty$, there exists a decomposition $\tilde{p}_\varepsilon = p_\varepsilon^{(1)} + p_\varepsilon^{(2)}$ with

$$\|\nabla p_\varepsilon^{(1)}\|_{L^2(\mathbb{R}^d)} \leq C\sigma_\varepsilon^{-1}, \quad \|p_\varepsilon^{(2)}\|_{L^2(\mathbb{R}^d)} \leq C.$$

Then $\nabla p_\varepsilon^{(1)} \to 0$ strongly in $L^2(\mathbb{R}^d; \mathbb{R}^d)$ and $p_\varepsilon^{(2)} \to p$ weakly in $L^2(\mathbb{R}^d)$.

For all the above three cases, $\{\tilde{p}_\varepsilon\}_{0 < \varepsilon < 1}$ is bounded in $L^2_{loc}(\mathbb{R}^d)$. Since $p_\varepsilon$ coincides with $\tilde{p}_\varepsilon$ in $\Omega_\varepsilon$, so $\{p_\varepsilon\}_{0 < \varepsilon < 1}$ is bounded in $L^2_{loc}(\Omega_\varepsilon)$.

The estimates and weak convergence of $\{v_\varepsilon\}_{0 < \varepsilon < 1}$ and $\{\tilde{p}_\varepsilon\}_{0 < \varepsilon < 1}$ in Theorem 2.5 have been shown in Propositions 4.1 and 4.3. To complete the proof of Theorem 2.5, it is left to show the limit equations and the strong convergence of $\nabla v_\varepsilon$ in $L^2(\mathbb{R}^d)$ for the subcritical case. This will be done in the next section.

5 Limit equations

The goal of this section is to show the limit equations and finish the proof of Theorem 2.5. To achieve such a goal, a natural way is to pass $\varepsilon \to 0$ in the weak formulation of (1.3). In this limit passage, there is an issue on the choice of test functions. Since the homogenized system is defined in $\mathbb{R}^d$, so one needs to choose test functions in $C_c^{\infty}(\mathbb{R}^d)$. However $C_c^{\infty}(\mathbb{R}^d)$ functions are not proper test functions for the Stokes problem (1.3) in $\Omega_\varepsilon$, for which the test functions should be chosen in $C_c^{\infty}(\Omega_\varepsilon)$. Hence, a proper surgery on the test functions need to be done and this surgery plays a crucial role in the study of the homogenization problems. Tartar [25] and Allaire [1, 2] used different ideas to solve this issue. As in [18], we use Tartar’s idea of cell problem.

5.1 Cell problem

We generalize Tartar’s idea and consider the following modified cell problem introduced in [18]:

\[
\begin{align*}
-\Delta w_\eta^i &+ \nabla q_\eta^i = cT^2 \varepsilon_i, & &\text{in } Q_\eta := Q_0 \setminus (\eta T), \\
\text{div } w_\eta^i & = 0, & &\text{in } Q_\eta, \\
w_\eta^i & = 0, & &\text{on } \eta T, \\
(w_\eta^i, q_\eta^i) & \text{ is } Q_0\text{-periodic}.
\end{align*}
\]
Here \( \{e^i\}_{i=1,...,d} \) is the standard Euclidean coordinate of \( \mathbb{R}^d \); \( \eta = a_\varepsilon / \varepsilon \in (0,1] \), and \( c_\eta \) is defined as
\[
(5.1) \quad c_\eta := \left| \log \eta \right|^{-\frac{1}{2}} \text{ if } d = 2; \quad c_\eta := \eta^{\frac{d-2}{2}} \text{ if } d \geq 3.
\]

We collect some basic facts from Section 2 in [18]:
\[
(5.2) \quad \| \nabla w_\eta \|^2_{L^2(\Omega_\varepsilon)} \leq C c_\eta, \quad \| w_\eta \|^2_{L^2(\Omega_\varepsilon)} \leq C, \quad \| q_\eta \|^2_{L^2(\Omega_\varepsilon)} \leq C c_\eta.
\]

Then as \( \varepsilon \to 0 \), up to possible extractions of subsequences,
\[
(5.3) \quad w_\eta \to w \text{ weakly in } W^{1,2}(Q_0), \quad w_\eta \to w \text{ strongly in } L^2(Q_0), \quad c_\eta^{-1} q_\eta \to q \text{ weakly in } L^2(Q_0).
\]

Thus,
\[
(5.4) \quad A(\eta)_{i,j} := c_\eta^{-2} \int_{Q_\eta} \nabla w_\eta^i : \nabla w_\eta^j \, dx = \int_{Q_\eta} (w_\eta^i)_{j} \, dx \to \bar{w}_{j}^i := \int_{Q_0} w_j^i \, dx,
\]
with \( A := (\bar{w}_{j}^i)_{1 \leq i,j \leq d} \) a symmetric positive definite matrix. Moreover, the main theorem in [3, Section 0] says that \( A = M^{-1} \), where \( M \) is the permeability tensor introduced in [1] or [3].

Then define
\[
\begin{align*}
& w_{\eta,\varepsilon}^i(\cdot) := w_\eta^i\left(\frac{\cdot}{\varepsilon}\right), \quad q_{\eta,\varepsilon}^i(\cdot) := q_\eta^i\left(\frac{\cdot}{\varepsilon}\right),
\end{align*}
\]
which solve
\[
\begin{cases}
-\Delta w_{\eta,\varepsilon}^i + \varepsilon^{-1} \nabla q_{\eta,\varepsilon}^i = \varepsilon^{-2} c_\eta^2 e^i = \sigma_{\varepsilon}^{-2} e^i, & \text{in } \varepsilon Q_0 \setminus (a_\varepsilon T), \\
\text{div } w_{\eta,\varepsilon}^i = 0, & \text{in } \varepsilon Q_0 \setminus (a_\varepsilon T), \\
w_{\eta,\varepsilon}^i = 0, & \text{on } a_\varepsilon T,
\end{cases}
\]
(5.5)

Here we used the fact \( c_\eta \varepsilon^{-1} = \sigma_{\varepsilon}^{-1} \). For each \( R > 1 \), by (5.2) and the periodicity of \( (w_\eta^i, q_\eta^i) \), direct calculation gives
\[
(5.6) \quad \| w_{\eta,\varepsilon}^i \|^2_{L^2(B(0,R))} \leq C(R), \quad \| q_{\eta,\varepsilon}^i \|^2_{L^2(B(0,R))} \leq C(R) c_\eta, \quad \| \nabla w_{\eta,\varepsilon}^i \|^2_{L^2(B(0,R))} \leq C(R) \sigma_{\varepsilon}^{-1},
\]
where the constant \( C(R) \) depends only on \( R \). By (5.3), again using the periodicity of \( (w_\eta^i, q_\eta^i) \) gives
\[
(5.7) \quad w_{\eta,\varepsilon}^i \to \bar{w}^i \text{ weakly in } L^2_{\text{loc}}(\mathbb{R}^d), \quad c_\eta^{-1} q_{\eta,\varepsilon}^i \to \bar{q}^i \text{ weakly in } L^2_{\text{loc}}(\mathbb{R}^d),
\]
as \( \varepsilon \to 0 \), up to extracting subsequences. Here \( \bar{q}^i := \int_{Q_0} q^i \, dx \).

### 5.2 Limit Passages

Clearly \( w_{\eta,\varepsilon}^i \) vanishes on the holes \( T_{\varepsilon,k} \) for all \( k \in \mathbb{Z}^d \). Thus, given a scalar function \( \phi \in C^\infty_c(\mathbb{R}^d) \), there holds \( w_{\eta,\varepsilon}^i \phi \in W_0^{1,2}(\Omega_\varepsilon; \mathbb{R}^d) \). We can take \( w_{\eta,\varepsilon}^i \phi \) as a test function in the weak formulation of (1.3) and deduce
\[
(5.8) \quad \int_{\mathbb{R}^d} \nabla \phi : \nabla (w_{\eta,\varepsilon}^i \phi) \, dx - \int_{\mathbb{R}^d} \bar{p}_\varepsilon \text{div} (w_{\eta,\varepsilon}^i \phi) \, dx = \langle \mathbf{g}, (w_{\eta,\varepsilon}^i \phi) \rangle.
\]
Since $v_\varepsilon$ and $w_{\eta,\varepsilon}^i$ both vanish on the holes and $\tilde p_\varepsilon$ coincides with $p_\varepsilon$ in $\Omega_\varepsilon$, the integrals in (5.8) are the same if we replace $\mathbb{R}^d$ by $\Omega_\varepsilon$ which should be the correct one in the weak formulation of (1.3). By (5.5), direct calculation gives

$$
\int_{\mathbb{R}^d} \nabla v_\varepsilon : (w_{\eta,\varepsilon}^i \phi) \, dx = \int_{\mathbb{R}^d} \nabla v_\varepsilon : (w_{\eta,\varepsilon}^i \phi) \, dx - \int_{\mathbb{R}^d} (\nabla \phi \otimes v_\varepsilon) : \nabla w_{\eta,\varepsilon}^i \, dx \\
+ \int_{\mathbb{R}^d} \nabla (\phi v_\varepsilon) : \nabla w_{\eta,\varepsilon}^i \, dx \\
= \int_{\mathbb{R}^d} \nabla v_\varepsilon : (w_{\eta,\varepsilon}^i \phi) \, dx - \int_{\mathbb{R}^d} (\nabla \phi \otimes v_\varepsilon) : \nabla w_{\eta,\varepsilon}^i \, dx \\
+ \varepsilon^{-1} \int_{\mathbb{R}^d} \operatorname{div} (\phi v_\varepsilon) q_{\eta,\varepsilon}^i \, dx + \sigma_{\varepsilon}^{-2} \int_{\mathbb{R}^d} (\phi v_\varepsilon) \cdot e^i \, dx.
$$

By Proposition 4.3 and $\operatorname{div} w_{\eta,\varepsilon}^i = 0$, we have

$$
\int_{\mathbb{R}^d} \tilde p_\varepsilon \operatorname{div} (w_{\eta,\varepsilon}^i \phi) \, dx = - \int_{\mathbb{R}^d} \nabla p_{\varepsilon}^{(1)} : (w_{\eta,\varepsilon}^i \phi) \, dx + \int_{\mathbb{R}^d} p_{\varepsilon}^{(2)} \operatorname{div} (w_{\eta,\varepsilon}^i \phi) \, dx \\
= - \int_{\mathbb{R}^d} \nabla p_{\varepsilon}^{(1)} : (w_{\eta,\varepsilon}^i \phi) \, dx + \int_{\mathbb{R}^d} p_{\varepsilon}^{(2)} \nabla \phi \cdot w_{\eta,\varepsilon}^i \, dx.
$$

We will pass $\varepsilon \to 0$ case by case in the following subsections. Propositions 4.1 and 4.3 will be used multiple times. The limits are often taken up to extractions of subsequences and we will not repeat this point.

The constant $C$ in the following argument will often depend on the size of $\operatorname{supp} \phi$ due to the local integrability of $w_{\eta,\varepsilon}^i$ and $q_{\eta,\varepsilon}^i$, see (5.6). We may not emphasize this dependency if it is clear from the context. Anyway, once $\phi$ is given, the constant $C(\operatorname{supp} \phi)$ is fixed.

### 5.3 Super-critical case with large holes

For the super-critical case $\lim_{\varepsilon \to 0} \sigma_{\varepsilon} \to 0$, we assume $g \in L^2(\mathbb{R}^d; \mathbb{R}^d)$. In this case, we recall some estimates that we are going to use right away (see Propositions 4.1 and 4.3):

$$
\|\nabla v_\varepsilon\|_{L^2(\mathbb{R}^d)} \leq C \sigma_{\varepsilon}, \quad \|v_\varepsilon\|_{L^2(\mathbb{R}^d)} \leq C \sigma_{\varepsilon}^2,
$$

$$
\tilde p_\varepsilon = p_{\varepsilon}^{(1)} + p_{\varepsilon}^{(2)} \text{ with } \|p_{\varepsilon}^{(1)}\|_{L^2(\mathbb{R}^d)} \leq C, \quad \|p_{\varepsilon}^{(2)}\|_{L^2(\mathbb{R}^d)} \leq C \sigma_{\varepsilon}.
$$

We first estimate the right-hand side of (5.9). By (5.6), (5.7), (5.11), we have

$$
\int_{\mathbb{R}^d} \nabla v_\varepsilon : (w_{\eta,\varepsilon}^i \phi) \, dx \leq C \|\nabla v_\varepsilon\|_{L^2(\mathbb{R}^d)} \|w_{\eta,\varepsilon}^i\|_{L^2(\operatorname{supp} \phi)} \leq C(\operatorname{supp} \phi) \sigma_{\varepsilon} \to 0,
$$

$$
\int_{\mathbb{R}^d} (\nabla \phi \otimes v_\varepsilon) : \nabla w_{\eta,\varepsilon}^i \, dx \leq C \|v_\varepsilon\|_{L^2(\mathbb{R}^d)} \|\nabla w_{\eta,\varepsilon}^i\|_{L^2(\operatorname{supp} \phi)} \leq C(\operatorname{supp} \phi) \sigma_{\varepsilon}^2 \sigma_{\varepsilon}^{-1} \to 0.
$$

Here $C(\operatorname{supp} \phi)$ depends on the size of the compact set $\operatorname{supp} \phi$. Using the divergence free condition $\operatorname{div} v_\varepsilon = 0$ and observing $\varepsilon^{-1} c_{\eta} = \sigma_{\varepsilon}^{-1}$ implies

$$
\varepsilon^{-1} \int_{\mathbb{R}^d} \operatorname{div} (\phi v_\varepsilon) q_{\eta,\varepsilon}^i \, dx \leq \varepsilon^{-1} \int_{\mathbb{R}^d} \nabla \phi \cdot v^i_\varepsilon q_{\eta,\varepsilon}^i \, dx \\
\leq C \varepsilon^{-1} \|v_\varepsilon\|_{L^2(\mathbb{R}^d)} \|q_{\eta,\varepsilon}^i\|_{L^2(\operatorname{supp} \phi)} \\
\leq C \varepsilon^{-1} \sigma_{\varepsilon}^2 c_{\eta} = C \sigma_{\varepsilon} \to 0.
$$
Since $\sigma_\varepsilon^{-2} \mathbf{v}_\varepsilon \to \mathbf{v}$ weakly in $L^2(\mathbb{R}^d; \mathbb{R}^d)$, then

\[(5.14) \quad \sigma_\varepsilon^{-2} \int_{\mathbb{R}^d} \phi \mathbf{v}_\varepsilon \cdot \mathbf{e}^i \, dx \to \int_{\mathbb{R}^d} \phi \mathbf{v} \cdot \mathbf{e}^i \, dx.\]

For the terms related to the pressure in (5.10), by (5.11) and (5.7), we have

\[(5.15) \quad \int_{\mathbb{R}^d} \tilde{p}_\varepsilon \text{div} (w_{\eta, \varepsilon}^i \phi) \, dx = - \int_{\mathbb{R}^d} \nabla p_\varepsilon^{(1)} \cdot (w_{\eta, \varepsilon}^i \phi) \, dx + \int_{\mathbb{R}^d} p_\varepsilon^{(2)} \nabla \phi \cdot w_{\eta, \varepsilon}^i \, dx \to - \int_{\mathbb{R}^d} \nabla p \cdot (\bar{w}^i \phi) \, dx.\]

For the source term,

\[(5.16) \quad \langle \mathbf{g}, (w_{\eta, \varepsilon}^i \phi) \rangle = \int_{\mathbb{R}^d} \mathbf{g} \cdot (w_{\eta, \varepsilon}^i \phi) \, dx \to \int_{\mathbb{R}^d} \mathbf{g} \cdot (\bar{w}^i \phi) \, dx.\]

Thus, by (5.12)–(5.16), passing $\varepsilon \to 0$ in (5.8) implies

\[
\int_{\mathbb{R}^d} \phi \mathbf{v} \cdot \mathbf{e}^i \, dx + \int_{\mathbb{R}^d} \nabla p \cdot (\bar{w}^i \phi) \, dx = \int_{\mathbb{R}^d} \mathbf{g} \cdot \bar{w}^i \phi \, dx.
\]

This is the Darcy’s law in $\mathbb{R}^d$:

\[
\mathbf{v} = A(\mathbf{g} - \nabla p), \quad \text{in } \mathbb{R}^d,
\]

where $A = (\bar{w}_j^i)_{1 \leq i, j \leq d}$, which is the constant positive definite matrix defined in (5.4).

### 5.4 Critical case

For the critical case $\lim_{\varepsilon \to 0} \sigma_\varepsilon = \sigma_* \in (0, +\infty)$, we have (see Propositions 4.1 and 4.3, (5.6) and (5.7)):

\[(5.17) \quad \|\mathbf{v}_\varepsilon\|_{W^{1,2}(\mathbb{R}^d)} \leq C,
\]

\[
\tilde{p}_\varepsilon = p_\varepsilon^{(1)} + p_\varepsilon^{(2)} \quad \text{with } \|\nabla p_\varepsilon^{(1)}\|_{W^{2,m}(\mathbb{R}^d)} \leq C \quad \text{for all } m \in \mathbb{N}, \|p_\varepsilon^{(2)}\|_{L^2(\mathbb{R}^d)} \leq C,
\]

\[
\|w_{\eta, \varepsilon}^i\|_{W^{1,2}(B(0,R))} \leq C(R) \quad \text{for all } R > 1.
\]

Then by Rellich-Kondrachov compact embedding theorem, $w_{\eta, \varepsilon}^i \to \bar{w}^i$ weakly in $W^{1,2}(B(0,R); \mathbb{R}^d)$, $w_{\eta, \varepsilon}^i \to \bar{w}^i$ strongly in $L^2(B(0,R); \mathbb{R}^d)$, and $\mathbf{v}_\varepsilon \to \mathbf{v}$ strongly in $L^2(B(0,R); \mathbb{R}^d)$, for all $R > 1$. Choose $R$ large such that supp$\phi \subset B(0,R)$. We then have for the right-hand side of (5.9):

\[(5.18) \quad \varepsilon^{-1} \int_{\mathbb{R}^d} \text{div} (\phi \mathbf{v}_\varepsilon) \, q_i^2 \, dx = \varepsilon^{-1} c_\eta \int_{\mathbb{R}^d} \text{div} (\phi \mathbf{v}_\varepsilon) (c_\eta^{-1} q_i^2) \, dx = \sigma_*^{-1} \int_{\mathbb{R}^d} \nabla \phi \cdot \mathbf{v}_\varepsilon (c_\eta^{-1} q_i) \, dx \to \sigma_*^{-1} \int_{\mathbb{R}^d} \nabla \phi \cdot \mathbf{v} \, q_i \, dx = 0,
\]

\[
\sigma_*^{-2} \int_{\mathbb{R}^d} \phi \mathbf{v}_\varepsilon \cdot \mathbf{e}^i \, dx \to \sigma_*^{-2} \int_{\mathbb{R}^d} \phi \mathbf{v} \cdot \mathbf{e}^i \, dx,
\]
where we used the fact that \( \bar{w}^i \) and \( \bar{q}^i \) are constant.

Similarly, for the terms related to the pressure in (5.10), we have

\[
\int_{\mathbb{R}^d} p_\varepsilon \, \text{div} (w^i_{\eta, \varepsilon} \phi) \, dx = - \int_{\mathbb{R}^d} \nabla p^{(1)}_\varepsilon \cdot (w^i_{\eta, \varepsilon} \phi) \, dx + \int_{\mathbb{R}^d} p^{(2)}_\varepsilon \nabla \phi \cdot w^i_{\eta, \varepsilon} \, dx \\
- \int_{\mathbb{R}^d} \nabla p^{(1)}_\varepsilon \cdot (\bar{w}^i \phi) \, dx + \int_{\mathbb{R}^d} p^{(2)}_\varepsilon \nabla \phi \cdot \bar{w}^i \, dx = \int_{\mathbb{R}^d} p \, \text{div} (\bar{w}^i \phi) \, dx,
\]

where \( p = p^{(1)} + p^{(2)} \).

For the source term, since \( w^i_{\eta, \varepsilon} \phi \to \bar{w}^i \phi \) weakly in \( W^{1,2}(\mathbb{R}^d; \mathbb{R}^d) \), there holds

\[
\langle g, w^i_{\eta, \varepsilon} \phi \rangle \to \langle g, \bar{w}^i \phi \rangle.
\]

Finally, by (5.18), (5.19) and (5.20), passing \( \varepsilon \to 0 \) in (5.8) implies

\[
\int_{\mathbb{R}^d} \nabla v : \nabla (\bar{w}^i \phi) \, dx + \sigma^2_\varepsilon \int_{\mathbb{R}^d} \phi v \cdot e^i \, dx - \int_{\mathbb{R}^d} p \, \text{div} (\bar{w}^i \phi) \, dx = \langle g, \bar{w}^i \phi \rangle.
\]

This is the Brinkman type equations in the sense of distribution in \( \mathbb{R}^d \):

\[
\sigma^2_\varepsilon v = A(g - \nabla p + \Delta v) \iff -\Delta v + \nabla p + \sigma^{-1}_\varepsilon A^{-1} v = g.
\]

### 5.5 Subcritical case with small holes

For the subcritical case \( \lim_{\varepsilon \to 0} \sigma_\varepsilon \to \infty \), we recall the estimates (see Propositions 4.1 and 4.3):

\[
\|\nabla v_\varepsilon\|_{L^2(\mathbb{R}^d)} \leq C, \quad \|v_\varepsilon\|_{L^2(\mathbb{R}^d)} \leq C \sigma_\varepsilon, \\
\tilde{p}_\varepsilon = p^{(1)}_\varepsilon + p^{(2)}_\varepsilon \text{ with } \|\nabla p^{(1)}_\varepsilon\|_{L^2(\mathbb{R}^d)} \leq C \sigma^{-1}_\varepsilon, \quad \|p^{(2)}_\varepsilon\|_{L^2(\mathbb{R}^d)} \leq C.
\]

By (5.6) and (5.7), for each \( R > 1 \) we have \( \|\nabla w^i_{\eta, \varepsilon}\|_{L^2(B(0,R))} \leq C(R) \sigma_\varepsilon^{-1} \to 0 \) and \( w^i_{\eta, \varepsilon} \to \bar{w}^i \) strongly in \( L^2(B(0,R); \mathbb{R}^d) \). Thus, as \( \varepsilon \to 0 \),

\[
\int_{\mathbb{R}^d} \nabla v_\varepsilon : \nabla (w^i_{\eta, \varepsilon} \phi) \, dx = \int_{\mathbb{R}^d} \nabla v_\varepsilon : (w^i_{\eta, \varepsilon} \otimes \nabla \phi) \, dx + \int_{\mathbb{R}^d} \nabla v_\varepsilon : \nabla w^i_{\eta, \varepsilon} \phi \, dx \\
\to \int_{\mathbb{R}^d} \nabla v : (\bar{w}^i \otimes \nabla \phi) \, dx = \int_{\mathbb{R}^d} \nabla v : \nabla (\bar{w}^i \phi) \, dx,
\]

\[
\int_{\mathbb{R}^d} \tilde{p}_\varepsilon \, \text{div} (w^i_{\eta, \varepsilon} \phi) \, dx = - \int_{\mathbb{R}^d} \nabla p^{(1)}_\varepsilon \cdot (w^i_{\eta, \varepsilon} \phi) \, dx + \int_{\mathbb{R}^d} p^{(2)}_\varepsilon \nabla \phi \cdot w^i_{\eta, \varepsilon} \, dx \to \int_{\mathbb{R}^d} p \, \text{div} (\bar{w}^i \phi) \, dx,
\]

and

\[
\langle g, (w^i_{\eta, \varepsilon} \phi) \rangle \to \langle g, (\bar{w}^i \phi) \rangle.
\]

By (5.22)–(5.24), passing \( \varepsilon \to 0 \) in (5.8) implies

\[
\int_{\mathbb{R}^d} \nabla v : \nabla (\bar{w}^i \phi) \, dx - \int_{\mathbb{R}^d} p \, \text{div} (\bar{w}^i \phi) \, dx = \langle g, \bar{w}^i \phi \rangle.
\]

This gives

\[
\int_{\mathbb{R}^d} \nabla v : \nabla (A \varphi) - p \, \text{div} (A \varphi) \, dx = \langle g, A \varphi \rangle, \quad \forall \varphi \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^d),
\]

on
which means
\[ (5.25) \quad -\Delta v + \nabla p = g, \]
in the sense of distribution in \( \mathbb{R}^d \), due to the positivity of \( A \).

At the end, we show the strong convergence \( v_\varepsilon \to v \) in \( D_0^{1,2}(\mathbb{R}^d; \mathbb{R}^d) \). Taking \( v_\varepsilon \) as a test function in the weak formulation of (1.3), using the weak convergence \( \nabla v_\varepsilon \to \nabla v \) in \( L^2(\mathbb{R}^d; \mathbb{R}^{d \times d}) \), and passing \( \varepsilon \to 0 \) implies
\[ \lim_{\varepsilon \to 0} \| \nabla v_\varepsilon \|^2_{L^2(\mathbb{R}^d)} = (g, v). \]

Taking \( v \) as a test function to (5.25) gives
\[ \| \nabla v \|^2_{L^2(\mathbb{R}^d)} = (g, v). \]
This gives \( \lim_{\varepsilon \to 0} \| \nabla v_\varepsilon \|_{L^2(\mathbb{R}^d)} = \| \nabla v \|_{L^2(\mathbb{R}^d)} \) resulting in \( \nabla v_\varepsilon \to \nabla v \) strong in \( L^2(\mathbb{R}^d; \mathbb{R}^{d \times d}) \). We thus complete the proof of Theorem 2.5.

6 The Poisson problem

The proof of the homogenization results of the Poisson problem is similar to the Stokes case but only easier. We briefly show some steps. The following result corresponds to Propositions 3.1 and 4.1. The limits are taken up to possible extractions of subsequences.

**Proposition 6.1.** Let \( \Omega = \mathbb{R}^d \), \( d \geq 2 \) and \( f \) satisfy Assumption 2.1. Then the Poisson problem (1.2) admits a unique solution \( u_\varepsilon \in W^{1,2}_0(\Omega_\varepsilon) \) with the following estimates:

(i) For the critical case \( \lim_{\varepsilon \to 0} \sigma_\varepsilon = \sigma_* \in (0, +\infty) \),
\[ \| u_\varepsilon \|_{W^{1,2}(\Omega_\varepsilon)} \leq C \| f \|_{W^{-1,2}(\mathbb{R}^d)}. \]
Hence, \( u_\varepsilon \to u \) weakly in \( W^{1,2}_0(\mathbb{R}^d) \).

(ii) For the subcritical case \( \lim_{\varepsilon \to 0} \sigma_\varepsilon = \infty \),
\[ \| \nabla u_\varepsilon \|_{L^2(\Omega_\varepsilon)} + \sigma_\varepsilon^{-1} \| u_\varepsilon \|_{L^2(\Omega_\varepsilon)} \leq C \| f \|_{D^{-1,2}(\mathbb{R}^d)}. \]
If \( d \geq 3 \), \( \| u_\varepsilon \|_{L^{\frac{2d}{d-2}}(\Omega_\varepsilon)} \leq C \| f \|_{D^{-1,2}(\mathbb{R}^d)}. \) Hence, \( u_\varepsilon \to u \) weakly in \( D_0^{1,2}(\mathbb{R}^d) \).

(iii) For the supercritical case \( \lim_{\varepsilon \to 0} \sigma_\varepsilon = 0 \),
\[ \| \nabla u_\varepsilon \|_{L^2(\Omega_\varepsilon)} \leq C \sigma_\varepsilon \| f \|_{L^2(\mathbb{R}^d)}, \quad \| u_\varepsilon \|_{L^2(\Omega_\varepsilon)} \leq C \sigma_\varepsilon^2 \| f \|_{L^2(\mathbb{R}^d)}. \]
Hence, \( \sigma_\varepsilon^{-2} u_\varepsilon \to u \) weakly in \( L^2(\mathbb{R}^d) \).

The corresponding cell problem is
\[
\begin{aligned}
-\Delta w_\eta &= c_\eta^2, & &\text{in } Q_\eta := Q_0 \setminus (\eta T), \\
w_\eta &= 0, & &\text{on } \eta T, \\
w_\eta &\text{is } Q_0 \text{-periodic},
\end{aligned}
\]
where $c_\eta$ is the same as before, see (5.1). The solution satisfies
\[
\|\nabla w_\eta\|_{L^2(Q_0)} \leq Cc_\eta, \quad \|w_\eta\|_{L^2(Q_0)} \leq C.
\]

Then define
\[
w_{\eta,\varepsilon}(\cdot) := w_\eta(\frac{\cdot}{\varepsilon}),
\]
which solves
\[
\begin{cases}
-\Delta w_{\eta,\varepsilon} = \varepsilon^{-2}c_\eta^2 = \sigma_\varepsilon^{-2}, & \text{in } \varepsilon Q_0 \setminus (a_\varepsilon T), \\
w_{\eta,\varepsilon} = 0, & \text{on } a_\varepsilon T, \\
w_{\eta,\varepsilon} & \text{is } \varepsilon Q_0\text{-periodic.}
\end{cases}
\]
Clearly $w_{\eta,\varepsilon}$ vanishes on the holes. For each $R > 1$, by (5.2) and the periodicity of $w_\eta$, direct calculation gives
\[
\|w_{\eta,\varepsilon}\|_{L^2(B(0,R))} \leq C(R), \quad \|\nabla w_{\eta,\varepsilon}\|_{L^2(B(0,R))} \leq C(R)\sigma_\varepsilon^{-1},
\]
where the constant $C(R)$ depends only on $R$. Using one more time the periodicity of $w_\eta$ implies
\begin{equation}
(6.1) \quad w_{\eta,\varepsilon} \rightharpoonup \bar{w} \text{ weakly in } L^2_{\text{loc}}(\mathbb{R}^d),
\end{equation}
as $\varepsilon \to 0$, up to possible extraction of subsequences. Here $\bar{w} := \int_{Q_0} w \, dx$ where $w$ is the weak limit of $w_\eta$ in $L^2(Q_0)$.

For each $\phi \in C_c^\infty(\mathbb{R}^d)$, testing (1.2) by $\phi w_{\eta,\varepsilon}$ gives
\begin{equation}
(6.2) \quad \int_{\mathbb{R}^d} \nabla u_\varepsilon : \nabla (\phi w_{\eta,\varepsilon}) \, dx = \langle f, (w_{\eta,\varepsilon}\phi) \rangle.
\end{equation}
It is left to pass $\varepsilon \to 0$ in (6.2). This can be done case by case similarly as the Stokes problem and we will not repeat the details.

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