A New Proof of the Channel Coding Theorem via Hypothesis Testing in Quantum Information Theory

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Abstract

A new proof of the direct part of the quantum channel coding theorem is shown based on a standpoint of quantum hypothesis testing. A packing procedure of mutually noncommutative operators is carried out to derive an upper bound on the error probability, which is similar to Feinstein’s lemma in classical channel coding. The upper bound is used to show the proof of the direct part along with a variant of Hiai-Petz’s theorem in quantum hypothesis testing.

Keywords

Channel coding theorem, hypothesis testing, Hiai-Petz’s theorem, quantum relative entropy

1 Introduction

Let \( H \) be a Hilbert space which represents a physical system of information carrier. We suppose \( \dim H < \infty \) for mathematical simplicity. Let \( \mathcal{L}(H) \) be the set of linear operators on \( H \) and define the totality of density operators on \( H \) by
\[
S(H) \overset{\text{def}}{=} \{ \rho \in \mathcal{L}(H) \mid \rho = \rho^*, \text{Tr}[\rho] = 1 \}.
\]

We will treat a quantum channel defined by a mapping \( x \in X \mapsto \rho_x \in S(H) \), where \( X \) is a finite set of input alphabets and each \( \rho_x \) represents the quantum state of the output signal.

An encoding-decoding system of the message over the \( n \)-th extension of the channel is described as follows. Each message \( k \in \{1, \ldots, M_n\} \) is encoded to a codeword \( u^k = x^k_1 \otimes \cdots \otimes x^k_n \) in a codebook \( C_n = \{ u^1, \ldots, u^{M_n} \} \subseteq X^n \) by an encoder, where \( X^n \) is the \( n \)-th direct product of \( X \), and the codeword \( u^k \) is mapped to \( \rho_{u^k} = x^k_1 \otimes \cdots \otimes x^k_n \in S(H^\otimes n) \) through the channel. The decoding process, which is called a decoder, is represented by a set \( X^n = \{ X_1, X_2, \ldots, X_{M_n} \} \) of nonnegative operators on \( H^\otimes n \) satisfying \( \sum_{k=1}^{M_n} X_k \leq I_n \), implying that \( X^n \) with \( X_0 \overset{\text{def}}{=} I_n - \sum_{k=1}^{M_n} X_k \) becomes a quantum measurement on \( H^\otimes n \) taking its value in \( \{0, 1, \ldots, M_n\} \). A pair of encoding and decoding processes \((C^n, X^n)\) is called a code with cardinality \( M_n \) or with transmission rate \( R_n = \log M_n/n \).

The probability that the decoder outputs a message \( k \) when a message \( l \) is sent is given by \( \text{Tr}[\rho_{u^l} X_k] \). Thus, assuming that all messages arise with the uniform probability, the average error probability of the code \((C^n, X^n)\) is given by
\[
\text{Pe}(C^n, X^n) \overset{\text{def}}{=} \frac{1}{M_n} \sum_{k=1}^{M_n} (1 - \text{Tr}[\rho_{u^l} X_k]) \text{.}
\]
Our interest lies in asymptotically achievable transmission rates with arbitrarily small error. The channel capacity is defined as the supremum of those values:

\[
C \overset{\text{def}}{=} \sup \left\{ R \mid \exists \{(C^n, X^n)\}_{n=1}^\infty \text{ such that } \lim_{n \to \infty} \Pr(C^n, X^n) = 0 \text{ and } \liminf_{n \to \infty} \frac{1}{n} \log M_n \geq R \right\}. \tag{3}
\]

In order to describe the quantum channel coding theorem, let us introduce the quantum mutual information $I$ as follows. Let $\mathcal{P}(\mathcal{X})$ be the totality of the probability distributions on $\mathcal{X}$, and let $\sigma_p \overset{\text{def}}{=} \sum_{x \in \mathcal{X}} p(x) \rho_x$ be the mixture state by some $p \in \mathcal{P}(\mathcal{X})$. Then the quantum mutual information is defined by

\[
I(p) \overset{\text{def}}{=} H(\sigma_p) - \sum_{x \in \mathcal{X}} p(x) H(\rho_x)
= \sum_{x \in \mathcal{X}} p(x) D(\rho_x \| \sigma_p), \tag{4}
\]

where $H(\rho) \overset{\text{def}}{=} -\operatorname{Tr}[\rho \log \rho]$ is the von Neumann entropy and $D(\rho \| \sigma) \overset{\text{def}}{=} \operatorname{Tr}[\rho (\log \rho - \log \sigma)]$ is the quantum relative entropy. One of the most significant theorems in quantum information theory is the quantum channel coding theorem:

\[
C = \max_{p \in \mathcal{P}(\mathcal{X})} I(p). \tag{5}
\]

The theorem is composed of two inequalities: the direct part ($\geq$) and the converse part ($\leq$). The direct part is given by showing the existence of a good code, and was established in the middle of the 1990’s by Holevo and independently by Schumacher and Westmoreland after the breakthrough by Hausladen et al. The classical counterpart of the method used there is thought as a variant of the joint typical decoding along with the random coding technique. On the other hand, the converse part, concerned with nonexistence of too good code, goes back to 1970’s works by Holevo and Winter. In 1999, another proof of the direct part was also given by Winter, in which he developed the method of type in the quantum setting followed by a greedy construction of a good code.

In classical channel coding, the most transparent proof of the direct part is thought to be Feinstein’s proof (see also) (see also [3]), the essence of which is described below. Let $w^n(x^n|x^n)$ be a channel matrix transmitting an input sequence $x^n = x_1 \ldots x_n \in \mathcal{X}^n$ to an output sequence $y^n = y_1 \ldots y_n \in \mathcal{Y}^n$, where $\mathcal{X}^n$ and $\mathcal{Y}^n$ are the $n$-th direct products of the sets $\mathcal{X}$ and $\mathcal{Y}$ of input alphabets and output alphabets, respectively. The channel is called stationary memoryless if $w^n(x^n|x^n) = w(y_1|x_1) \ldots w(y_n|x_n)$ holds with a one-shot channel $w(y|x) \in \mathcal{X} \times \mathcal{Y}$. If the input $X^n = X_1 \ldots X_n$ is a random variable taking its value in $\mathcal{X}^n$ subject to a probability distribution $p^n(x^n) \in \mathcal{X}^n$, then the output $Y^n = Y_1 \ldots Y_n$ and the pair $(X^n, Y^n)$ also become random variables subject to $q^n(y^n) = \sum_{x^n} p^n(x^n) w(y^n|x^n)$ and $p^n(x^n) w^n(y^n|x^n)$, respectively. In the first part of Feinstein’s proof, it is shown that if we have

\[
\Pr \left\{ \frac{1}{n} \log \frac{w^n(Y^n|X^n)}{q^n(Y^n)} \leq a \right\} \longrightarrow 0 \quad (n \to \infty) \tag{6}
\]

for a real number $a$, then we can construct a reliable code with transmission rate arbitrarily near $a$. In the second part, assuming that $p^n(x^n)$ is the independently and identically distributed (i.i.d.) extension of a probability distribution $p(x) \in \mathcal{X}$ and $w^n(y^n|x^n)$ is a stationary memoryless channel, (6) is shown for $a < I(X; Y)$ by the law of large numbers, where $I(X; Y)$ is the classical mutual information for the random variables $(X, Y)$ subject to $p(x) w(y|x)$. As for the first part, no assumption such as the stationary memoryless property for the channel is needed, which led to a general formula for the classical channel capacity by Verdú-Han as one of the landmarks in the information-spectrum method. As ways of providing the first part, two different methods are known so far; the random coding technique and the packing algorithm, both of which give us an important insight into the construction of good codes.
In this paper a new proof of the direct part for the stationary memoryless quantum channel is shown based on a limiting theorem in quantum hypothesis testing, which is regarded as a variant of Hiai-Petz’s theorem, combined with a packing procedure for operators following Winter to obtain a good code. The limiting theorem in quantum hypothesis testing is thought to be a substitute for the law of large numbers used to show in classical information theory. The approach used here is regarded as an attempt to develop the information-spectrum method in quantum channel coding, which was followed by Hayashi-Nagaoka with further developments using the random coding technique to obtain a general formula for the channel capacity in the quantum setting. It should be noted here, however, any packing algorithm to derive the general formula is not established yet.

2 Relation with Hypothesis Testing

In the sequel, \( \sigma_p = \sum_{x} p(x) \rho_x \) is written as \( \sigma \) omitting the subscript \( p \) when no confusion is likely to arise. Let us define block diagonal matrices

\[
\hat{\rho} \overset{\text{def}}{=} \begin{pmatrix}
\vdots & 0 \\
p(x) \rho_x & \ddots \\
0 & \ddots & 0
\end{pmatrix}, \quad \hat{\sigma} \overset{\text{def}}{=} \begin{pmatrix}
\vdots & 0 \\
p(x) \sigma & \ddots \\
0 & \ddots & \ddots
\end{pmatrix},
\]

which are density operators in \( S(\bigoplus_{x \in X} \mathcal{H}) \) and denoted as

\[
\hat{\rho} = \bigoplus_{x \in X} p(x) \rho_x, \quad \hat{\sigma} = \bigoplus_{x \in X} p(x) \sigma,
\]

respectively. It is important to note that the quantum mutual information is nothing but the quantum relative entropy between \( \hat{\rho} \) and \( \hat{\sigma} \), i.e., \( D(\hat{\rho}\|\hat{\sigma}) = I(p) \). In the same way the \( n \)-th tensor powers of \( \hat{\rho} \) and \( \hat{\sigma} \) are given by the following block diagonal matrices

\[
\hat{\rho} \otimes^n = \bigoplus_{x^n \in X^n} p^n(x^n) \rho_{x^n}, \quad \hat{\sigma} \otimes^n = \bigoplus_{x^n \in X^n} p^n(x^n) \sigma \otimes^n
\]

in \( S(\bigoplus_{x^n \in X^n} \mathcal{H} \otimes^n) \), respectively, where we used the following notations for each \( x^n = x_1 x_2 \ldots x_n \in X^n \)

\[
p^n(x^n) \overset{\text{def}}{=} p(x_1)p(x_2)\ldots p(x_n), \quad \rho_{x^n} \overset{\text{def}}{=} \rho_{x_1} \otimes \rho_{x_2} \otimes \ldots \otimes \rho_{x_n}.
\]

Here, let \( \mathcal{E}_{\hat{\rho} \otimes^n}(\hat{\rho} \otimes^n) \) be the pinching defined in Appendix A. Applying Lemma 3 in Appendix A inductively, we can show that

\[
\mathcal{E}_{\hat{\rho} \otimes^n}(\hat{\rho} \otimes^n) = \bigoplus_{x^n \in X^n} p^n(x^n) \mathcal{E}_{\hat{\rho} \otimes^n}(\rho_{x^n}).
\]

In order to relate the above observation to the channel coding theorem, let us introduce the quantum hypothesis testing problem here, which is explained concisely in Appendix B, and examine the error probability of a test as follows. Given a Hermitian operator \( X = \sum_i x_i E_i \), define the projection \( \{ X > 0 \} \) by

\[
\{ X > 0 \} \overset{\text{def}}{=} \sum_{i : x_i > 0} E_i.
\]
With the above notation, we define a test for the hypotheses $\hat{\rho} \otimes^n$ and $\hat{\sigma} \otimes^n$ as
\[
\hat{S}_n(a) \overset{\text{def}}{=} \left\{ \mathcal{E}_{\hat{\sigma} \otimes^n}(\hat{\rho} \otimes^n) - e^{na} \hat{\sigma} \otimes^n > 0 \right\} = \bigoplus_{x^n \in \mathcal{X}^n} \left\{ \mathcal{E}_{\hat{\sigma} \otimes^n}(\hat{\rho}_{x^n}) - e^{na} \hat{\sigma} \otimes^n > 0 \right\},
\] (14)
where $a$ is a real parameter and the last equality follows from (12). The error probability of the first kind for the test is written as follows by using the notation $\rho_{x^n} \overset{\text{def}}{=} \mathcal{E}_{\sigma \otimes^n}(\hat{\rho}_{x^n})$
\[
\hat{\alpha}_n \left( \hat{S}_n(a) \right) \overset{\text{def}}{=} \text{Tr} \left[ \hat{\rho} \otimes^n \left( I_n - \hat{S}_n(a) \right) \right] = \sum_{x^n \in \mathcal{X}^n} p^n(x^n) \text{Tr} \left[ \rho_{x^n} \left\{ \rho_{x^n} - e^{na} \sigma \otimes^n \leq 0 \right\} \right],
\] (15)
which tends to zero exponentially if $a < D(\hat{\rho} \parallel \hat{\sigma})$ by Lemma 4 in Appendix B [16]. Thus we have shown the following lemma as an analogue of (6) in quantum channel coding.

**Lemma 1** For $\forall a < I(\rho)$, we have
\[
\lim_{n \to \infty} \sum_{x^n \in \mathcal{X}^n} p^n(x^n) \text{Tr} \left[ \rho_{x^n} \left\{ \rho_{x^n} - e^{na} \sigma \otimes^n \leq 0 \right\} \right] = 0.
\] (16)

### 3 Greedy Construction of a Code

In the previous section, we have shown the achievability of the quantum mutual information in the sense of Lemma 1. Needless to say, this does not directly mean the achievability concerned with the error probability nor the transmission rate, since the tests $\{ \rho_{x^n} - e^{na} \sigma \otimes^n > 0 \}$ for different $x^n$ do not commute each other and can not be realized simultaneously on $H \otimes^n$. In this section we will give a packing procedure of the tests to meet the condition of the quantum measurement on $H \otimes^n$, which leads to the following lemma.

**Lemma 2** Let $x \in \mathcal{X} \mapsto \rho_x \in \mathcal{S}(H)$ be a quantum channel, and let $\sigma \overset{\text{def}}{=} \sum_{x \in \mathcal{X}} p(x) \rho_x$ be the mixture state by a probability distribution $p \in \mathcal{P}(\mathcal{X})$. Suppose that the following condition holds for the pinchin $\rho_{x^n} \overset{\text{def}}{=} \mathcal{E}_\sigma(\rho_x)$ with some real numbers $\delta \geq 0$ and $c > 0$:
\[
\sum_{x \in \mathcal{X}} p(x) \text{Tr} \left[ \rho_{x^n} \left\{ \rho_{x^n} - c \sigma > 0 \right\} \right] \geq 1 - \delta.
\] (17)

Then, for any real numbers $\gamma > 0$ and $\eta > 0$, there exist a codebook $C = \{ u_k \}_{k=1}^M \subseteq \mathcal{X}$ with cardinality $M$ and a decoder $X = \{ X_k \}_{k=1}^M$ on $H$ such that
\[
\frac{\gamma}{\gamma + \delta} \min \{ \eta, 1 - \delta - \gamma - 2\sqrt{\eta} \} \leq \frac{M}{c},
\] (18)
\[
\frac{1}{M} \sum_{k=1}^M (1 - \text{Tr} \left[ \rho_{u_k} X_k \right]) \leq \delta + \gamma + 2\sqrt{\eta}.
\] (19)

Before proceeding to the proof of Lemma 2, let us consider the $n$-th extension of the lemma replacing the symbols above as follows:
\[
\mathcal{X} \leftarrow \mathcal{X}^n, \quad \rho_x \leftarrow \rho_{x^n}, \quad p \leftarrow p^n, \quad \sigma \leftarrow \sigma \otimes^n, \quad e \leftarrow e^{na}, \quad \gamma \leftarrow \gamma_n, \quad \eta \leftarrow e^{-n\lambda}, \quad (20)
\]
where $a$ is a real number and $\lambda > 0$ is an arbitrarily small number. Letting $\overline{p}_{x^n} = \mathcal{E}\otimes^n (\rho_{x^n})$, we can easily see that (17) is satisfied by setting the following $\delta_n(a)$ in place of $\delta$,

$$
\delta_n(a) \overset{\text{def}}{=} \sum_{x^n \in \mathcal{X}^n} p^n(x^n) \text{Tr} \left[ \overline{p}_{x^n} \{ \overline{p}_{x^n} - e^{na} \sigma^{\otimes n} \leq 0 \} \right].
$$

(21)

Therefore we obtain the following theorem using Lemma 2.

**Theorem 1** With the above notation, there exist a codebook $C^n = \{ u^k \}_{k=1}^{M_n} \subseteq \mathcal{X}^n$ with cardinality $M_n$ and a decoder $X^n = \{ X_k \}_{k=1}^{M}$ on $\mathcal{H}^{\otimes n}$ such that

$$
\frac{1}{\gamma_n} \min \left\{ e^{-n\lambda}, 1 - \delta_n(a) - \gamma_n - 2\sqrt{e^{-n\lambda}} \right\} \leq e^{-na} M_n,
$$

(22)

and

$$
\text{Pe}(C^n, X^n) \leq \delta_n(a) + \gamma_n + 2\sqrt{e^{-n\lambda}}.
$$

(23)

For any $a < I(p)$, we have $\lim_{n \to \infty} \delta_n(a) = 0$ by Lemma 3, and hence we can choose $\gamma_n$ such that $\lim_{n \to \infty} \gamma_n = 0$ and $\lim_{n \to \infty} \frac{\gamma_n}{\gamma_n + \delta_n(a)} = 1$. Therefore the above theorem yields

$$
\liminf_{n \to \infty} \frac{1}{n} \log M_n \geq a,
$$

(24)

and

$$
\lim_{n \to \infty} \text{Pe}(C^n, X^n) = 0,
$$

(25)

since $\lambda > 0$ can be arbitrarily small. Thus we have given a new proof of the direct part of the quantum channel coding theorem except that the proof of Lemma 2 remains to be shown.

**Proof of Lemma 3** To begin with, let us define a set of candidates for codewords by

$$
\mathcal{X}' \overset{\text{def}}{=} \{ x \in \mathcal{X} \mid \text{Tr}[\overline{p}_x \{ \overline{p}_x - c \sigma > 0 \}] \geq 1 - \delta - \gamma \}. \quad (26)
$$

Then the probability of $\mathcal{X}'$ is bounded below as

$$
p(\mathcal{X}') \overset{\text{def}}{=} \sum_{x \in \mathcal{X}'} p(x) \geq \frac{\gamma}{\gamma + \delta},
$$

(27)

which is verified as follows:

$$
1 - \delta \leq \sum_{x \in \mathcal{X}} p(x) \text{Tr}[\overline{p}_x \{ \overline{p}_x - c \sigma > 0 \}]
$$

$$
= \sum_{x \in \mathcal{X}'} p(x) \text{Tr}[\overline{p}_x \{ \overline{p}_x - c \sigma > 0 \}] + \sum_{x \in \mathcal{X} \setminus \mathcal{X}'} p(x) \text{Tr}[\overline{p}_x \{ \overline{p}_x - c \sigma > 0 \}]
$$

$$
\leq p(\mathcal{X}') + (1 - p(\mathcal{X}'))(1 - \delta - \gamma).
$$

(28)

Utilizing the normalization technique for operators developed by Winter (see the proof of Theorem 10 in [4]), a codebook $C = \{ u_k \}_{k=1}^{M} \subseteq \mathcal{X}^j$ and a decoder $X = \{ X_k \}_{k=1}^{M}$ are constructed by the following greedy algorithm, along with the operators $S_k \in \mathcal{L}(\mathcal{H}) (k = 0, 1, \ldots, M)$ for normalization.

(a) Let $S_0 = 0$.

(b) Repeat the following procedures for $k = 1, 2, \ldots$

\(\text{(b-1)}\) If there exists an $x \in \mathcal{X} \setminus \{ u_1, \ldots, u_{k-1} \}$ such that $\text{Tr}[\overline{p}_x S_{k-1}] \leq \eta$, then choose such an $x$ arbitrarily and define

$$
u_k \overset{\text{def}}{=} x,
$$

$$
X_k \overset{\text{def}}{=} \sqrt{I - S_{k-1} \{ \overline{p}_{u_k} - c \sigma > 0 \}} \sqrt{I - S_{k-1}},
$$

$$
S_k \overset{\text{def}}{=} S_{k-1} + X_k,
$$

else go to (c).
(b-2) Let $k \leftarrow k + 1$ and go back to (b-1).

(c) Letting $M \leftarrow k$, $\mathcal{C} \triangleq \{u_k\}_{k=1}^M$ and $X \triangleq \{X_k\}_{k=1}^M$, end the algorithm.

It should be noted here that each of $X_k$ and $S_k$ commutes with $\sigma$, and each $S_k$ satisfies the following inequalities

$$0 \leq S_k \leq I,$$  \hspace{1cm} (29)

which follows from

$$0 \leq \{p_x - c\sigma > 0\} \leq I$$  \hspace{1cm} (30)

and induction. Moreover it holds that for $\forall x \in \mathcal{A}' \setminus \mathcal{C}$

$$\text{Tr}[\rho_{ux} S_{k-1}] \leq \eta,$$  \hspace{1cm} (31)

while for any codeword $u_k \in \mathcal{C}$ we have $\text{Tr}[\rho_{ux} S_{k-1}] \leq \eta$. Thus, applying Winter’s gentle measurement lemma (Lemma 9 in [7], see Appendix C), we obtain

$$\left\|\rho_{ux} - \sqrt{I - S_{k-1}} \rho_{ux} \sqrt{I - S_{k-1}}\right\|_1 \leq 2\sqrt{\eta}.$$  \hspace{1cm} (32)

Therefore, using the property of the pinching (41), we have

$$\text{Tr}[\rho_{ux} X_k] = \text{Tr}[\rho_{ux} X_k]$$
$$= \text{Tr}\left[\sqrt{I - S_{k-1}} \rho_{ux} \sqrt{I - S_{k-1}} \{\rho_{ux} - c\sigma > 0\}\right]$$
$$= \text{Tr}\left[\rho_{ux} \{\rho_{ux} - c\sigma > 0\}\right] - \text{Tr}\left[\left(\rho_{ux} - \sqrt{I - S_{k-1}} \rho_{ux} \sqrt{I - S_{k-1}}\right) \{\rho_{ux} - c\sigma > 0\}\right]$$
$$\geq \text{Tr}[\rho_{ux} \{\rho_{ux} - c\sigma > 0\}] - 2\sqrt{\eta}$$
$$\geq 1 - \delta - \gamma - 2\sqrt{\eta},$$  \hspace{1cm} (33)

where the last inequality follows from $u_k \in \mathcal{A}'$, and hence

$$\text{Tr}[\rho_{ux} S_M] = \text{Tr}\left[\rho_{ux} \left(\sum_{k=1}^M X_k\right)\right]$$
$$\geq \text{Tr}[\rho_{ux} X_k]$$
$$\geq 1 - \delta - \gamma - 2\sqrt{\eta}.$$  \hspace{1cm} (34)

Now, in order to evaluate the cardinality of the code $M$, we will estimate the lower and upper bound of $\text{Tr}[\sigma S_M]$. Using (31) and (34), the lower bound is obtained as

$$\text{Tr}[\sigma S_M] = \sum_{x \in \mathcal{X}} p(x) \text{Tr}[\rho_{ux} S_M]$$
$$\geq \sum_{x \in \mathcal{X}'} p(x) \text{Tr}[\rho_{ux} S_M]$$
$$\geq \sum_{x \in \mathcal{X}'} p(x) \min \{\eta, 1 - \delta - \gamma - 2\sqrt{\eta}\}$$
$$\geq \frac{\gamma}{\gamma + \delta} \min \{\eta, 1 - \delta - \gamma - 2\sqrt{\eta}\},$$  \hspace{1cm} (35)
where the last inequality follows from (27). On the other hand, \( \text{Tr}[\sigma S] \) is bounded above as

\[
\text{Tr}[\sigma S] = \sum_{k=1}^{M} \text{Tr}[\sigma X_k] \\
= \sum_{k=1}^{M} \text{Tr} \left[ \sqrt{I - S_{k-1}} \sigma \sqrt{I - S_{k-1}} \{ \sigma_u - c \sigma > 0 \} \right] \\
\leq \sum_{k=1}^{M} \text{Tr} \left[ \sigma \{ \sigma_u - c \sigma > 0 \} \right],
\]

(36)
since we have \( \sqrt{I - S_{k-1}} \sigma \sqrt{I - S_{k-1}} \leq \sigma \), which follows from (29) and the fact that each \( \sqrt{I - S_{k-1}} \) commutes with \( \sigma \). Moreover, (36) is bounded above further as

\[
\text{Tr}[\sigma S] \leq \sum_{k=1}^{M} \frac{1}{c} \text{Tr} \left[ \sigma_u \{ \sigma_u - c \sigma > 0 \} \right] \\
\leq \frac{M}{c},
\]

(37)
by the definition of \( \{ \sigma_u - c \sigma > 0 \} \). Now the assertion (18) is shown by combining the lower bound (35) with the upper bound (37), and (19) follows from (33).

### 4 Concluding Remarks

We have revisited the direct part of the quantum channel coding theorem [2] [3] considering the relation with the hypothesis testing problem, and an upper bound on the error probability similar to Feinstein’s lemma [10] has been obtained. The approach used here is regarded as an attempt to develop the information-spectrum method [13] in quantum channel coding.

### Appendices

#### A A property of the pinching

We summarize a property of the pinching related to a \( \ast \)-subalgebra in \( \mathcal{L}(\mathcal{H}) \) for readers’ convenience. In the sequel, we denote the \( \ast \)-algebra generated by operators \( \{ I, A_1, A_2, \ldots \} \subseteq \mathcal{L}(\mathcal{H}) \) as \( \| A_1, A_2, \ldots \| \). Given a Hermitian operator \( A \in \mathcal{L}(\mathcal{H}) \), let

\[
A = \sum_{i=1}^{v} a_i E_i
\]

(38)
be its spectral decomposition, where each \( E_i \) is the projection corresponding to an eigenvalue \( a_i \) different from others and \( v \) is the number of the eigenvalues. Then we can easily see that \( [A] = [E_1, \ldots, E_v] \) holds. Actually, it is clear that we have the inclusion \( [A] \subseteq [E_1, \ldots, E_v] \) from (38). On the other hand, considering the following function for each \( i \),

\[
f_i(x) \overset{\text{def}}{=} \prod_{j:j \neq i} \frac{x - a_j}{a_i - a_j},
\]

(39)
we have \( E_i = f_i(A) \), which yields the converse inclusion \( [A] \supseteq [E_1, \ldots, E_v] \).
Let us denote the commutant of \([A]\) as

\[
[A]^\prime \overset{\text{def}}{=} \{ B \in L(H) \mid \forall C \in [A], BC = CB \}
= \{ B \in L(H) \mid BA = AB \}
= \{ B \in L(H) \mid BE_i = E_i B (i = 1, \ldots, v) \}. \tag{40}
\]

Then the pinching \(E_A : L(H) \to [A]^\prime \subseteq L(H)\) is defined as the projection of an operator to the *-subalgebra \([A]^\prime\) so that

\[
\forall C \in [A]^\prime, \quad \langle\langle B, C \rangle\rangle = \langle\langle E_A(B), C \rangle\rangle \tag{41}
\]

holds for \(\forall B \in L(H)\), where

\[
\langle\langle A, B \rangle\rangle \overset{\text{def}}{=} \text{Tr}[A^* B] \tag{42}
\]

is the Hilbert-Schmidt inner product. Note that the pinching can be written explicitly as

\[
E_A(B) = \sum_{i=1}^{v} E_i B E_i. \tag{43}
\]

We used the following property of the pinching to show \([12]\) in the section \([3]\).

**Lemma 3** Let \(A, B, C, D \in L(H)\), and define block diagonal matrices on \(H \oplus H\) by \(X \overset{\text{def}}{=} A \oplus B\) and \(Y \overset{\text{def}}{=} C \oplus D\) with the same notation as used in the section \([3]\). Then we have

\[
E_Y(X) = E_C(A) \oplus E_D(B). \tag{44}
\]

**Proof:** For any operator \(Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \in [Y]^\prime\), \(YZ = YZ\) yields \(Z_{11} \in [C]^\prime\) and \(Z_{22} \in [D]^\prime\) by a direct calculation. Thus, we have

\[
\text{Tr}[XZ] = \text{Tr}[AZ_{11}] + \text{Tr}[BZ_{22}]
= \text{Tr}[E_C(A)Z_{11}] + \text{Tr}[E_D(B)Z_{22}]
= \text{Tr}[(E_C(A) \oplus E_D(B)) Z]. \tag{45}
\]

Now the assertion has been proved, since \(E_C(A) \oplus E_D(B) \in [Y]^\prime\) is obvious and \(Z \in [Y]^\prime\) is arbitrary. \(\blacksquare\)

**B Hiai-Petz’s Theorem and its variants**

In this appendix we summarize Hiai-Petz’s theorem \([17]\) and its variants \([16]\) in quantum hypothesis testing, which is used to show Lemma \([3]\). Given \(\rho\) and \(\sigma\) in \(S(H)\), let us consider the hypothesis testing problem about hypotheses \(H_0 : \rho \overset{\otimes n}{\in} S(H^{\otimes n})\) and \(H_1 : \sigma \overset{\otimes n}{\in} S(H^{\otimes n})\). The problem is to decide which hypothesis is true based on a two-valued quantum measurement \(\{X_0, X_1\}\) on \(H^{\otimes n}\), where the subscripts 0 and 1 indicate the acceptance of \(H_0\) and \(H_1\), respectively. In the sequel, an operator \(A_n \in L(H^{\otimes n})\) satisfying inequalities \(0 \leq A_n \leq I_n\) is called a test, since \(A_n\) is identified with the measurement \(\{A_n, I_n - A_n\}\). For a test \(A_n\), the error probabilities of the first kind and the second kind are, respectively, defined by

\[
\alpha_n(A_n) \overset{\text{def}}{=} \text{Tr}[\rho \overset{\otimes n}{\in} (I_n - A_n)], \quad \beta_n(A_n) \overset{\text{def}}{=} \text{Tr}[\sigma \overset{\otimes n}{\in} A_n]. \tag{46}
\]

For \(0 < \forall \varepsilon < 1\), let us define

\[
\beta_n^*(\varepsilon) \overset{\text{def}}{=} \min\{\beta_n(A_n) \mid A_n : \text{test}, \alpha_n(A_n) \leq \varepsilon\}, \tag{47}
\]
and recall the quantum relative entropy:

\[ D(\rho\|\sigma) \overset{\text{def}}{=} \text{Tr}[\rho(\log \rho - \log \sigma)]. \]  

(48)

Then we have the following theorem, which is called the quantum Stein’s lemma:

\[ \lim_{n \to \infty} \frac{1}{n} \log \beta_n^*(\varepsilon) = -D(\rho\|\sigma). \]  

(49)

The first proof of (49) was shown by two inequalities. One is the direct part given by Hiai-Petz [17]:

\[ \limsup_{n \to \infty} \frac{1}{n} \log \beta_n^*(\varepsilon) \leq -D(\rho\|\sigma), \]  

(50)

and the other is the converse part given by Ogawa-Nagaoka [20].

Preceding the direct part (50), Hiai-Petz [17] proved the following theorem

\[ D(\rho\|\sigma) = \lim_{n \to \infty} \frac{1}{n} \sup_{M_n} D_{M_n}(\rho^\otimes_n\|\sigma^\otimes_n), \]  

(51)

where the supremum is taken over the set of quantum measurements on \( \mathcal{H}^\otimes_n \), and \( D_{M_n}(\rho^\otimes_n\|\sigma^\otimes_n) \) is the classical relative entropy (Kullback divergence) between the probability distributions \( \{\text{Tr}[\rho^\otimes_n M_{n,i}]\} \) and \( \{\text{Tr}[\sigma^\otimes_n M_{n,i}]\} \). Note that the monotonicity of the quantum relative entropy \[22] \[23] yields

\[ D(\rho\|\sigma) \geq \frac{1}{n} D_{M_n}(\rho^\otimes_n\|\sigma^\otimes_n) \]  

for any measurement \( M_n \), and in addition there exists a measurement that attains the equality if and only if \( \rho \) and \( \sigma \) mutually commute. Hiai-Petz combined (51) with the classical hypothesis testing problem to show the direct part (50).

In recent developments [21] [16], we found direct proofs of (50) without using the achievability of the information quantity (51). Here, we will make use of a simple test defined below with the notation (13) and the pinching \( E_{\sigma^\otimes_n}(\rho^\otimes_n) \) (see Appendix A):

\[ \overline{S}_n(a) \overset{\text{def}}{=} \left\{ E_{\sigma^\otimes_n}(\rho^\otimes_n) - e^{na} \sigma^\otimes_n > 0 \right\}, \]  

(52)

which satisfies the following lemma (see Theorem 2 in [16]).

**Lemma 4 (Ogawa-Hayashi)** For \( 0 \leq s \leq 1 \), we have

\[ \alpha_n(\overline{S}_n(a)) \leq (n + 1)^{\dim \mathcal{H}} e^{nas - \psi(s)}, \]  

(53)

\[ \beta_n(\overline{S}_n(a)) \leq e^{-na}, \]  

(54)

where

\[ \psi(s) \overset{\text{def}}{=} -\log \text{Tr} [\rho^\otimes_n \sigma^\otimes_n e^{-s(\rho^\otimes_n + \sigma^\otimes_n)}]. \]  

(55)

Observing that \( \psi(0) = 0 \) and \( \psi'(0) = D(\rho\|\sigma) \), we can show that for \( \forall a < D(\rho\|\sigma) \)

\[ \lim_{n \to \infty} \alpha_n(\overline{S}_n(a)) = 0, \quad \beta_n(\overline{S}_n(a)) \leq e^{-na}. \]  

(56)

### C Winter’s Lemma

In this appendix Winter’s gentle measurement lemma (Lemma 9 in [7]) is explained. The original proof of the lemma by Winter is based on a study of the relation between the trace norm distance and the fidelity. Here, a direct proof of the lemma is given for readers’ convenience accompanied by a little improvement of the constant in the upper bound.
Lemma 5 (Winter) For $\forall \rho \in \mathcal{S}(\mathcal{H})$ and $\forall X \in \mathcal{L}(\mathcal{H})$ satisfying inequalities $0 \leq X \leq I$, we have

$$\left\| \rho - \sqrt{X} \rho \sqrt{X} \right\|_1 \leq 2 \sqrt{\text{Tr}[\rho(I-X)]}. \quad (57)$$

Proof: A direct calculation yields

$$\left\| \rho - \sqrt{X} \rho \sqrt{X} \right\|_1 = \left\| (I - \sqrt{X}^{\perp}) \rho - \sqrt{X} \rho \sqrt{X} \right\|_1$$

$$\leq \left\| (I - \sqrt{X}) \rho \right\|_1 + \left\| \sqrt{X} \rho (I - \sqrt{X}) \right\|_1 \quad (58)$$

$$= \text{Tr} \left[ (I - \sqrt{X}) \sqrt{\rho} \cdot \sqrt{\rho} \right] + \text{Tr} \left[ \sqrt{X} \sqrt{\rho} \cdot \sqrt{\rho} (I - \sqrt{X}) \right]$$

$$\leq \sqrt{\text{Tr} \left[ \rho (I - \sqrt{X})^2 \right]} \cdot \text{Tr}[\rho] + \sqrt{\text{Tr}[\rho X] \cdot \text{Tr} \left[ \rho (I - \sqrt{X})^2 \right]} \quad (59)$$

$$\leq 2 \sqrt{\text{Tr}[\rho(I-X)]}, \quad (60)$$

where (58) follows from the triangle inequality, (59) follows from the Cauchy-Schwartz inequality for operators, and we used $(1 - \sqrt{x})^2 \leq (1 - x)$ for $0 \leq x \leq 1$ and $\text{Tr}[\rho X] \leq 1$ in (60).

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