A Note on Chromatic Sum

Meysam Alishahi and Ali Taherkhani

Department of Mathematical Sciences
Shahid Beheshti University, G.C.,
P.O. Box 19839-63113, Tehran, Iran
malishahi@sbu.ac.ir
a_taherkhani@sbu.ac.ir

Abstract

The chromatic sum $\Sigma(G)$ of a graph $G$ is the smallest sum of colors among proper coloring with the natural number. In this paper, we introduce a necessary condition for the existence of graph homomorphisms. Also, we present $\Sigma(G) < \chi_f(G) |G|$ for every graph $G$.

Key words: chromatic sum, graph homomorphism, Fractional chromatic number.

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1 Introduction and Preliminaries

We consider finite undirected graphs with no loops and multiple edges and use [4] for the notions and notations not defined here. Let $G$ be a graph and $c$ be a proper coloring of it, define $\Sigma_c(G) = \sum_{v \in V(G)} c(v)$. The vertex-chromatic sum of $G$, denoted by $\Sigma(G)$, is defined as $\min\{\Sigma_c(G) | c$ is a proper coloring of $G\}$. The vertex-strength of $G$ denoted by $s(G)$, or briefly by $s$, is the smallest number $s$ such that there is a proper coloring $c$ with $s$ colors where $\Sigma_c(G) = \Sigma(G)$. Clearly, $s(G) \geq \chi(G)$ and equality does not always hold. In fact, for every positive integer $k$, almost all trees satisfy $s > k$; see [7]. Chromatic sum has been investigated in literature [1, 2, 3, 5, 6, 7, 10].

In [10], Thomassen et al. obtained several bounds for chromatic sum for general graphs. The first is a rather natural result of an application of a greedy algorithm: $\Sigma(G) \leq n + e$, where $n$ and $e$ are the number of vertices and edges of $G$, respectively. Also, they presented an upper and lower limit for the chromatic sum in terms of $e$. They showed that $\sqrt{8e} \leq \Sigma(G) \leq \frac{3}{2}(e + 1)$ and these bounds are sharp.

Let $G$ and $H$ be two graphs. A homomorphism $\sigma$ from a graph $G$ to a graph $H$ is a map $\sigma : V(G) \rightarrow V(H)$ such that $uv \in E(G)$ implies $\sigma(u)\sigma(v) \in E(H)$. The set of all homomorphisms from $G$ to $H$ is denoted by $\text{Hom}(G, H)$. An isomorphism of $G$ to $H$ is a homomorphism $f : G \rightarrow H$ which is a vertex and edge bijective homomorphism. An isomorphism $f : G \rightarrow G$ is called an automorphism of $G$, and the set of all automorphism of $G$ is denoted by $\text{Aut}(G)$.

Suppose $m \geq 2n$ are positive integers. We denote by $[m]$ the set $\{1, 2, \cdots, m\}$, and denote by $\binom{[m]}{n}$ the collection of all $n$-subsets of $[m]$. The Kneser graph $KG(m, n)$

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1 Corresponding author. Tel.: +98 2129902917.
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has vertex set \( \binom{m}{n} \), in which \( A \sim B \) if and only if \( A \cap B = \emptyset \). The graph \( KG(5, 2) \) is named Petersen graph that is denoted by \( P \). It was conjectured by Kneser in 1955 and proved by Lovász [8] in 1978 that \( \chi(KG(m, n)) = m - 2n + 2 \).

The fractional chromatic number of a graph \( G \), denoted by \( \chi_f(G) \), is the infimum of the ratios \( \frac{m}{n} \) such that there is a homomorphism from \( G \) to \( KG(m, n) \). It is known [9] that the infimum in the definition can be attained, and hence can be replaced by the minimum. It is easy to see \( \chi_f(G) \leq \chi(G) \). On the other hand, the ratio \( \frac{\chi(G)}{\chi_f(G)} \) can be arbitrary large, see [9].

In next section we present a necessary condition for existence of graph homomorphisms in terms of chromatic sum. Next, we introduce an upper bound for chromatic sum based on fractional chromatic number.

## 2 Graph Homomorphism and Chromatic Sum

Graph homomorphism is a fundamental concept in graph theory, where it is related to many important concepts and problems in the field. It is well-known that in general it is a hard problem to decide whether there exists a homomorphism from a given graph \( G \) to a given graph \( H \), and consequently, it is interesting to obtain necessary conditions for the existence of such mappings. In this regard, we have the following theorem.

**Theorem 1.** Let \( G \) and \( H \) be two graphs such that \( H \) is a vertex transitive graph. If \( \sigma : G \rightarrow H \) is a homomorphism, then

\[
\frac{\Sigma(G)}{|G|} \leq \frac{\Sigma(H)}{|H|}.
\]

**Proof.** Let \( \text{Aut}(H) = \{ f_1, f_2, \ldots, f_t \} \) and \( \tilde{G} = \bigcup_{i=1}^{t} G_i \) that \( G_i \) is an isomorphic copy of \( G \). Define \( \tilde{\sigma} : \tilde{G} \rightarrow H \) such that its restriction to \( G_i \) is \( f_i \circ \sigma \). Since \( H \) is a vertex transitive graph, one can easily show that for every \( v \in V(H) \), \( |\tilde{\sigma}^{-1}(v)| = \frac{|G|}{|H|} \) and it is independent of \( v \). Now, suppose \( c \) is a proper coloring of \( H \) such that \( \Sigma_c(H) = \Sigma(H) \). For any vertex \( v \in V(\tilde{G}) \), set \( \tilde{c}(v) = c(\tilde{\sigma}(v)) \). Obviously, \( \tilde{c} \) is a proper coloring of \( \tilde{G} \) and also \( \Sigma_{\tilde{c}}(\tilde{G}) = \frac{|G|}{|H|} \times \Sigma(H) \). Therefore, there is an \( i \) such that \( \Sigma_{\tilde{c}|_{G_i}}(G_i) \leq \frac{|G|}{|H|} \times \Sigma(H) \) and since \( G = G_i \), \( \Sigma(G) \leq \frac{|G|}{|H|} \times \Sigma(H) \) which is the desired conclusion. \( \blacksquare \)

Theorem 1 provides a necessary condition for the existence of graph homomorphisms. Here we show that The Petersen graph \( P \) has the same chromatic number and circular chromatic number. One can check that \( \Sigma(P) = 19 \) and \( \Sigma(K_8) = 15 \). Therefore, as an application of the previous theorem, there is no homomorphism from \( P \) to \( K_8 \).

It is well-known that the chromatic sum is an NP-complete problem[7]. In this regard, finding upper and lower bounds for chromatic sum is useful. It was shown in [8] that \( \Sigma(G) \leq \binom{\chi(G)+1}{2} |G| \). Since \( \Sigma(K_n) = \frac{n(n+1)}{2} \), if we set \( H = K_{\chi_f(G)} \), then Theorem 1 implies this bound. Here we obtain an upper bound for the chromatic sum in terms of fractional chromatic number.
For an independent set $S$ in a graph $G$ the following inequality is an immediate consequence of the definition of the chromatic sum \((2)\),

$$
\Sigma(G) \leq |G| + \Sigma(G \setminus S).
$$

(1)

**Theorem 2.** For every graph $G$, we have

$$
\Sigma(G) < \chi_f(G)|G|.
$$

**Proof.** Assume that $\chi_f(G) = \frac{m}{n}$ and $\text{Hom}(G, KG(m, n)) \neq \emptyset$. In view of equation \([1]\) we have $\Sigma(KG(m, n)) \leq \left(\binom{m}{n}\right) + \Sigma(KG(m - 1, n))$. Hence $\Sigma(KG(m, n)) \leq \sum_{i=0}^{m-2n-1} \left(\binom{m-i}{n}\right) + \Sigma(KG(2n, n))$. On the other hand, $\sum_{i=0}^{m-2n-1} \left(\binom{m-i}{n}\right) = \left(\frac{m+1}{n+1}\right) - \left(\frac{2n+1}{n+1}\right)$ and $\Sigma(KG(2n, n)) = \frac{3}{2} \binom{2n}{n}$. Therefore, $\Sigma(KG(m, n)) \leq \left(\frac{m+1}{n+1}\right) - \left(\frac{n-1}{2n+2}\right) \binom{2n}{n}$.

Now, since $\text{Hom}(G, KG(m, n)) \neq \emptyset$, Theorem 2 implies that

$$
\Sigma(G) \leq \left(\frac{m+1}{n+1} - \left(\frac{n-1}{2n+2}\right) \binom{2n}{n}\right) |G|.
$$

Furthermore, $\frac{m+1}{n+1} - \left(\frac{n-1}{2n+2}\right) \binom{2n}{n} \leq m/n = \chi_f(G)$, as desired. \(\blacksquare\)

In particular, if $G$ is a vertex transitive graph, $\chi_f(G) = \frac{|G|}{\alpha(G)}$ and hence $\Sigma(G) < \frac{|G|^2}{\alpha(G)}$. Furthermore, $e(G) = \frac{\Delta(G)|G|}{2}$. If $\chi_f(G) \leq \frac{\Delta(G)}{2}$, then $\chi_f(G)|G| < \frac{3}{2}(e(G) + 1)$. Therefore, the bound in Theorem 2 is better than the upper bound $\frac{3}{2}(e(G) + 1)$ (see [10]).

On the other hand, in view of Theorem 1 we have

$$
\Sigma(G) \geq \frac{\omega(G) + 1}{2} |G|
$$

where $G$ is a vertex transitive graph and $\omega(G)$ is the size of the largest clique in it.

Also, it is a known result that the ratio $\frac{\chi_f(G)}{\chi_f(G_n)}$ can be arbitrary large (see [9]). Let $G = \{G_i\}_{i \in \mathbb{N}}$ such that $\frac{\chi_f(G_n)}{\chi_f(G_n)} \to \infty$. We can assume that $G_n$ is critical for all $n$ ($G$ is critical if $\chi(G \setminus v) < \chi(G)$ for every $v \in V(G)$). Thus, $e(G_n) \geq \frac{|G_n||\chi(G_n)|}{2}$ and we also have $\frac{\Delta(G_n)}{\chi_f(G_n)} \to \infty$. It means the bound in Theorem 2 is better than the upper bound $\frac{3}{2}(e(G) + 1)$ for the graphs in $G$.

In Theorem 2 we used an upper bound of $\Sigma(KG(m, n))$, but we do not know the exact value of $\Sigma(KG(m, n))$. The improvement of this upper bound yields an improvement in Theorem 2.

**Problem 1** What is the exact value of $\Sigma(KG(m, n))$? Is it true that $\Sigma(KG(m, n)) = \left(\binom{m}{n}\right) \left(\frac{m+1}{n+1} - \left(\frac{n-1}{2n+2}\right) \binom{2n}{n}\right)$?

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References

[1] Paul Erdős, Ewa Kubicka, and Allen J. Schwenk. Graphs that require many colors to achieve their chromatic sum. In *Proceedings of the Twentieth Southeastern Conference on Combinatorics, Graph Theory, and Computing (Boca Raton, FL, 1989)*, volume 71, pages 17–28, 1990.

[2] H. Hajiabolhassan, M. L. Mehrabadi, and R. Tusserkani. Minimal coloring and strength of graphs. *Discrete Math.*, 215(1-3):265–270, 2000.

[3] H. Hajiabolhassan, M. L. Mehrabadi, and R. Tusserkani. Tabular graphs and chromatic sum. *Discrete Math.*, 304(1-3):11–22, 2005.

[4] Pavol Hell and Jaroslav Nešetřil. *Graphs and homomorphisms*, volume 28 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2004.

[5] Tao Jiang and Douglas B. West. Coloring of trees with minimum sum of colors. *J. Graph Theory*, 32(4):354–358, 1999.

[6] Ewa Kubicka. The chromatic sum of a graph: history and recent developments. *Int. J. Math. Math. Sci.*, (29-32):1563–1573, 2004.

[7] Ewa Kubicka and Allen J. Schwenk. An introduction to chromatic sums. *Proc. ACM Computer Science Conference*, Louisville(Kentucky):39–45, 1989.

[8] L. Lovász. Kneser's conjecture, chromatic number, and homotopy. *J. Combin. Theory Ser. A*, 25(3):319–324, 1978.

[9] Edward R. Scheinerman and Daniel H. Ullman. *Fractional graph theory*. Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons Inc., New York, 1997. A rational approach to the theory of graphs, With a foreword by Claude Berge, A Wiley-Interscience Publication.

[10] Carsten Thomassen, Paul Erdős, Yousef Alavi, Paresh J. Malde, and Allen J. Schwenk. Tight bounds on the chromatic sum of a connected graph. *J. Graph Theory*, 13(3):353–357, 1989.