NON-VANISHINGNESS OF BETTI NUMBERS OF EDGE IDEALS

KYOUKO KIMURA

Abstract. Given finite simple graph one can associate the edge ideal. In this paper we discuss the non-vanishingness of the graded Betti numbers of edge ideals in terms of the original graph. In particular, we give a necessary and sufficient condition for a chordal graph on which the graded Betti number does not vanish and characterize the graded Betti number for a forest. Moreover we characterize the projective dimension for a chordal graph.

1. Introduction

Let $G$ be a finite simple graph, that is, finite graph with no loop and no multiple edge. We denote its vertex set by $V = V(G)$ and its edge set by $E(G)$. Let $K$ be a field and $S = K[V] := K[x : x \in V]$ a polynomial ring with $\deg x = 1$. The edge ideal of $G$ is a squarefree monomial ideal $I(G) \subset S$ generated by all products $x_ix_j$ with $\{x_i, x_j\} \in E(G)$. We are interested in describing invariants of $I(G)$ in terms of $G$.

Let us consider a minimal graded free resolution of $S/I(G)$ over $S$:

$$0 \longrightarrow \bigoplus_j S(-j)^{\beta_{p,j}} \longrightarrow \cdots \longrightarrow \bigoplus_j S(-j)^{\beta_{1,j}} \longrightarrow S \longrightarrow S/I(G) \longrightarrow 0.$$ 

The integers $\beta_{i,j}(S/I(G)) := \beta_{i,j}$ is called the $i$th graded Betti number of $S/I(G)$ in degree $j$. The length $p$ of the resolution is called the projective dimension of $S/I(G)$ over $S$, denoted by $\text{pd} S/I(G)$, that is,

$$\text{pd} S/I(G) = \max\{i : \beta_{i,j}(S/I(G)) \neq 0 \text{ for some } j\}.$$ 

Also, the (Castelnuovo–Mumford) regularity of $S/I(G)$ is defined by

$$\text{reg} S/I(G) := \max\{j - i : \beta_{i,j}(S/I(G)) \neq 0\}.$$ 

In this paper, we focus on these invariants, which have studied by many authors, e.g., [2, 7, 9, 10, 11, 12, 14, 16]. In particular, Zheng [16] characterized the projective dimension and the regularity for a forest, which is a graph with no cycle. Later Hà and Van Tuyl [7] extended this characterization of the regularity to that for a chordal graph. Here a finite graph is called chordal if each cycle of $G$ whose length is more than 3 has a chord. On the other hand, Katzman [11] proved some results on the non-vanishingness of the graded Betti numbers. For other problems and results on this area we refer to [6].

In this paper, we give a sufficient condition on which the graded Betti number does not vanish (Theorem 3.1). This is a generalization of Katzman’s result. Moreover
we proved that the condition is also a necessary one for chordal graphs (Theorem 4.1). As a result, we characterize the non-vanishingness of the graded Betti numbers and thus, the projective dimension for chordal graphs. Furthermore we give a characterization of the graded Betti numbers for forests.

The organization of the paper is as follows. In Section 2, we recall some definitions on graphs and results by Zheng [10], Hà and Van Tuyl [7], and Katzman [11] mentioned above. Also in this section we introduce the notion of the strongly disjoint set of bouquets on a graph which plays an important role in our characterization. Next in Section 3, we discuss the non-vanishingness of the graded Betti numbers. In particular, we prove Theorem 3.1. In Section 4, we provide refined results of Theorem 3.1 for chordal graphs or forests: Theorem 4.1. Then we also have a characterization of the projective dimension for a chordal graph. However, in Section 5, we give another characterization of it which has a rather weaker condition. As an application of this, we recover the result that the projective dimension of the edge ideal of a chordal graph coincides with its big height (Corollary 5.6), which was proved by Morey and Villarreal [13, Corollary 3.33] for more general graphs.

2. Strongly disjoint set of bouquets

In this section, we prepare some definitions on graphs and recall some known results on our problem.

Let $G$ be a finite simple graph with the vertex set $V$ and the edge set $E(G)$. Let $e, e'$ be two distinct edges of $G$. The distance between $e$ and $e'$ in $G$, denoted by $\text{dist}_G(e, e')$, is defined by the minimum length $\ell$ among sequences $e_0 = e, e_1, \ldots, e_\ell = e'$ with $e_{i-1} \cap e_i \neq \emptyset$, where $e_i \in E(G)$. If there is no such a sequence, we define $\text{dist}_G(e, e') = \infty$. We say that $e$ and $e'$ are $3$-disjoint in $G$ if $\text{dist}_G(e, e') \geq 3$. A subset $\mathcal{E} \subset E(G)$ is said to be pairwise $3$-disjoint if every pairs of distinct edges $e, e' \in \mathcal{E}$ are $3$-disjoint in $G$; see [7, Definitions 2.2 and 6.3].

The graph $B$ with $V(B) = \{w, z_1, \ldots, z_d\}$ and $E(B) = \{\{w, z_i\} : i = 1, \ldots, d\}$ ($d \geq 1$) is called a bouquet.

$$B: \quad \begin{array}{c}
  \bullet \\
  z_1 \\
  z_2 \\
  \ldots \\
  z_d \\
  w
\end{array}$$

Then the vertex $w$ is called the root of $B$, the vertices $z_i$ flowers of $B$, and the edges $\{w, z_i\}$ stems of $B$; see [10, Definition 1.7]. We call a bouquet which is a subgraph of a graph $G$ a bouquet of $G$. Let $\mathcal{B} = \{B_1, B_2, \ldots, B_j\}$ be a set of bouquets of $G$. We set

- $F(\mathcal{B}) := \{z \in V : z$ is a flower of some bouquet in $\mathcal{B}\}$,
- $R(\mathcal{B}) := \{w \in V : w$ is a root of some bouquet in $\mathcal{B}\}$,
- $S(\mathcal{B}) := \{s \in E(G) : s$ is a stem of some bouquet in $\mathcal{B}\}$.

A type of $\mathcal{B}$ is defined by $(\# F(\mathcal{B}), \# R(\mathcal{B}))$. We define a disjointness on the set of bouquets of $G$. 

**Definition 2.1.** A set $\mathcal{B} = \{B_1, B_2, \ldots, B_j\}$ of bouquets of $G$ is said to be **strongly disjoint** in $G$ if the following 2 conditions are satisfied:

1. $V(B_k) \cap V(B_\ell) = \emptyset$ for all $k \neq \ell$.
2. We can choose a stem $s_k$ from each bouquet $B_k \in \mathcal{B}$ so that $\{s_1, s_2, \ldots, s_j\}$ is pairwise 3-disjoint in $G$.

**Remark 2.2.** If $\mathcal{B} = \{B_1, B_2, \ldots, B_j\}$ is a strongly disjoint set of bouquets in $G$, then any two vertices those belong to $R(\mathcal{B})$ is not adjacent in $G$. Indeed if not, say the roots of $B_1$ and $B_2$ are adjacent in $G$, then the distance between any stem of $B_1$ and any stem of $B_2$ is 2.

Moreover we give the following definition.

**Definition 2.3.** Let $G$ be a finite simple graph.

1. We say that $G$ **coincides with a strongly disjoint set of bouquets of type $(i, j)$** if there exists a strongly disjoint set $\mathcal{B}$ of bouquets of type $(i, j)$ with $V(G) = F(\mathcal{B}) \cup R(\mathcal{B})$, $E(G) = S(\mathcal{B})$.

2. We say that $G$ **contains a strongly disjoint set of bouquets of type $(i, j)$** if there exists a strongly disjoint set $\mathcal{B}$ of bouquets of type $(i, j)$ with $V(G) = F(\mathcal{B}) \cup R(\mathcal{B})$.

Let $G$ be a finite simple graph on $V$ and $W$ a subset of $V$. The subgraph of $G$ whose vertex set is $W$ and whose edge set is $\{e \in E(G) : e \subseteq W\}$ is called the induced subgraph of $G$ on $W$ and denoted by $G_W$. In the forward sections we consider an induced subgraph of $G$ which contains a strongly disjoint set of bouquets of type $(i, j)$.

Now we recall the results due to Zheng [16] and Hà and Van Tuyl [7], mentioned in Introduction. For a graph $G$, we set

$$d_G := \max\{\#F(\mathcal{B}) : \mathcal{B} \text{ is a strongly disjoint set of bouquets of } G\},$$

$$c_G := \max\{\#\mathcal{E} : \mathcal{E} \subseteq E(G) \text{ is a pairwise 3-disjoint in } G\}.$$}

**Theorem 2.4** (Hà and Van Tuyl [7], Zheng [16]). Let $G$ be a finite simple graph on $V$ and $S = K[V]$. 

1. ([16], Theorem 6.5]). When $G$ is a forest, we have $\text{pd } S/I(G) = d_G$.
2. ([7, Theorem 6.5]). $\text{reg } S/I(G) \geq c_G$.
3. ([7, Theorem 6.8], [16, Theorem 2.18]). When $G$ is a chordal graph, we have $\text{reg } S/I(G) = c_G$.

Also Katzman’s results on graded Betti numbers are as follows:

**Theorem 2.5** (Katzman [11, Lemma 2.2, Proposition 2.5]). Let $G$ be a finite simple graph on $V$ and $S = K[V]$.

1. If there exists a subset $W \subset V$ such that the induced subgraph $G_W$ coincides with a strongly disjoint set of bouquets of type $(i, j)$, then $\beta_{i,i+j}(S/I(G)) \neq 0$.
2. $\beta_{i,k}(S/I(G)) = 0$ when $k > 2i$.
3. The graded Betti number $\beta_{i,2i}(S/I(G))$ coincides with the number of subsets $W$ of $V$ for which the induced subgraph $G_W$ coincides with a strongly disjoint set of bouquets of type $(i, 2i)$. 


Remark 2.6. Theorem 2.4 (2) follows from Theorem 2.5 (1) or (3). Actually the result of Hà and Van Tuyl [7] is in more general situation of hypergraphs.

3. Non-vanishingness of the graded Betti numbers

In this section we provide a sufficient condition for which a graded Betti number of an edge ideal does not vanish. The main result in this section is the following theorem, which is a generalization of Katzman’s result (Theorem 2.5 (1)).

Theorem 3.1. Let $G$ be a finite simple graph on $V$ and $S = K[V]$. Assume that there exists a subset $W$ of $V$ such that the induced subgraph $G_W$ contains a strongly disjoint set of bouquets of type $(i, j)$. Then $\beta_{i,i+j}(S/I(G)) \neq 0$.

In particular, we have $\text{pd} S/I(G) \geq d_G$.

Remark 3.2. Precisely we have that $\beta_{i,i+j}(S/I(G))$ is greater than or equal to the number of subsets $W \subset V$ those satisfy the condition in Theorem 3.1.

Lemma 3.3. Let $G$ be a finite simple graph on $V$. Then for all $i \geq 0$, we have

$$\beta_{i,j}(I(G)) = \sum_{W \subset V, \#W = j} \beta_{i,j}(I(G_W)).$$

Proof. Since $I(G)$ is a squarefree monomial ideal, it is the Stanley–Reisner ideal $I_{\Delta}$ for some simplicial complex $\Delta$. By Hochster’s formula for Betti numbers (see e.g., [3, Theorem 5.5.1]), we have

$$\beta_{i,j}(I(G)) = \beta_{i+1,j}(K[\Delta]) = \sum_{W \subset V, \#W = j} \dim_K \tilde{H}_{j-(i+1)-1}(\Delta_W; K),$$

where $\tilde{H}_i(\Delta; K)$ stands for the $i$th reduced homology group of $\Delta$ and where $\Delta_W$ denotes the restriction of $\Delta$ on $W$: $\Delta_W = \{F \in \Delta : F \subset W\}$. Note that $I_{\Delta_W} = I(G_W)$ for a subset $W \subset V$. Hence again by Hochster’s formula, we have

$$\beta_{i,j}(I(G_W)) = \beta_{i+1,j}(K[\Delta_W]) = \dim_K \tilde{H}_{j-(i+1)-1}(\Delta_W; K)$$

for a subset $W \subset V$ with $\#W = j$. This completes the proof.

In the proof of Theorem 2.5 (1), Katzman used the Taylor resolution. In the proof of Theorem 3.1, we use a Lyubeznik resolution in stead of the Taylor resolution. A Lyubeznik resolution ([13]) is a subcomplex of the Taylor resolution generated by $L$-admissible symbols. It gives a (not necessarily minimal) free resolution for a monomial ideal.

Let $I$ be a monomial ideal of a polynomial ring over $K$ and $m_1, m_2, \ldots, m_\mu$ the minimal system of monomial generators of $I$. Let $e_{\ell_1, \ell_2, \ldots, \ell_i}$ ($\ell_1 < \ell_2 < \cdots < \ell_i$) denotes the free basis of the Taylor resolution of $I$. Recall that the degree of $e_{\ell_1, \ell_2, \ldots, \ell_i}$ is given by the degree of lcm($m_{\ell_1}, m_{\ell_2}, \ldots, m_{\ell_i}$). We say that a symbol $[\ell_1, \ell_2, \ldots, \ell_i] := e_{\ell_1, \ell_2, \ldots, \ell_i}$ is $L$-admissible if for all $t < i$ and for all $q < \ell_t$, the monomial generator $m_q$ does not divide lcm($m_{\ell_t}, m_{\ell_{t+1}}, \ldots, m_{\ell_i}$). Note that the $L$-admissibility as well
as a Lyubeznik resolution of $I$ depends on an order of monomial generators of $I$. An $L$-admissible symbol $[\ell_1, \ell_2, \ldots, \ell_i]$ is said to be maximal if there is no $L$-admissible symbol $[k_1, k_2, \ldots, k_i]$ with $\{\ell_1, \ell_2, \ldots, \ell_i\} \subsetneq \{k_1, k_2, \ldots, k_i\}$. If there exists a maximal $L$-admissible symbol $[\ell_1, \ell_2, \ldots, \ell_i]$ which satisfies the condition
\begin{equation}
\text{lcm}(m_{\ell_1}, \ldots, m_{\ell_i}) \neq \text{lcm}(m_{\ell_1}, \ldots, m_{\ell_i}), \text{ for all } k = 1, 2, \ldots, i,
\end{equation}
then $\beta_{i,j}(S/I(G)) \neq 0$, where $j = \deg[\ell_1, \ell_2, \ldots, \ell_i]$ (see also Barile [1, Remark 1]).

Now we prove Theorem 3.1.

**Proof of Theorem 3.1.** By virtue of Lemma 3.3, it is sufficient to prove the theorem when $\# V = i + j$ and $G$ contains a strongly disjoint set of bouquets of type $(i, j)$. Let $\mathcal{B} = \{B_1, B_2, \ldots, B_j\}$ be such a set of bouquets. Since $\mathcal{B}$ is strongly disjoint, we can choose a stem $s_k$ from each bouquet $B_k \in \mathcal{B}$ so that $S_0 := \{s_1, s_2, \ldots, s_j\}$ is pairwise 3-disjoint in $G$. We set $S' = S(\mathcal{B}) \setminus S_0$ and $E = E(G) \setminus S(\mathcal{B})$. We define an ordering of the edges of $G$ as $S', E, S_0$ and consider the associated ordering on the minimal system of monomial generators of $I(G)$. We consider the symbol $\sigma$ corresponding to $S', S_0$. We claim that $\sigma$ is a maximal $L$-admissible symbol.

The $L$-admissibility of $\sigma$ follows from the assumption that $S_0$ is pairwise 3-disjoint. To prove that $\sigma$ is maximal, we consider the symbol $\tau$ which corresponds to $S', e, S_0$, where $e = \{u, v\} \in E$. Since $V = F(\mathcal{B}) \cup R(\mathcal{B})$ and $\mathcal{B}$ is strongly disjoint, it follows that $\{u, v\} \cap F(\mathcal{B}) \neq \emptyset$. Moreover at least one of $u, v$ belongs to $F(\mathcal{B})$ which is not a vertex of the stems belonging to $S_0$ because $S_0$ is pairwise 3-disjoint. Let $u$ be such a vertex and assume that $u \in V(B_k)$. Then the product of the monomial $uv$ and the monomial which corresponds to $s_k$ is divisible by the monomial corresponding to the stem of $B_k$ whose flower is $u$. Hence $\tau$ is not $L$-admissible. Therefore $\sigma$ is a maximal $L$-admissible symbol.

It is clear that $\sigma$ satisfies the condition (3.1). Since $\deg \sigma = i + j$, we conclude that $\beta_{i,i+j}(S/I(G)) \neq 0$. \hfill $\Box$

4. THE CASE OF CHORDAL GRAPHS

In this section, we prove that the converse of Theorem 3.1 is true when $G$ is a chordal graph. Moreover we give a characterization of the graded Betti numbers of edge ideals of forests. Precisely, the main result of this section is the following theorem.

**Theorem 4.1.** Let $G$ be a finite simple graph on $V$ and $S = K[V]$.

1. Suppose that $G$ is chordal. Then $\beta_{i,i+j}(S/I(G)) \neq 0$ if and only if there exists a subset $W$ of $V$ such that the induced subgraph $G_W$ contains a strongly disjoint set of bouquets of type $(i, j)$.

   In particular, we have $pd S/I(G) = d_G$.

2. When $G$ is a forest, the graded Betti number $\beta_{i,i+j}(S/I(G))$ coincides with the number of subsets $W$ of $V$ with the same condition as in (1).

**Remark 4.2.** We obtain a characterization of the projective dimension for chordal graphs by Theorem 4.1 though we give another characterization in the next section.
Let $G$ be a finite simple graph on $V$. Take a vertex $v \in V$. We say that $u \in V$ is a neighbour of $v$ in $G$ if \( \{u, v\} \) is an edge of $G$. We denote by $N(v)$, the neighbourhood of $v$. For an edge $e \in E(G)$, we denote by $G \setminus e$ the subgraph of $G$ obtained from $G$ by removing the edge $e$.

A graph $G$ is said to be chordal if each cycle in $G$ whose length is more than 3 has a chord. Dirac [4] proved that when $G$ is chordal, there exists a perfect elimination ordering on $E(G)$. This means that any induced subgraph $G'$ of $G$ has a vertex $v$ such that the induced subgraph of $G'$ on the neighbourhood of $v$ in $G'$ is a complete graph.

The next result about the graded Betti numbers of edge ideals of chordal graphs due to Hà and Van Tuyl is a key in our proof of Theorem 4.1. For simplicity, we set $\beta_{-1,0}(I(G)) = 1$ and $\beta_{-1,j}(I(G)) = 0$ if $j \neq 0$. When $G$ is a graph with no edge, we set $\beta_{i,j}(I(G))$ as 1 when $(i, j) = (-1, 0)$ and as 0 otherwise.

**Lemma 4.3** (Hà and Van Tuyl [7, Theorem 5.8]). Let $G$ be a chordal graph on the vertex set $V$. Suppose that $e = \{u, v\}$ is an edge of $G$ such that $G_{N(v)}$ is a complete graph. Set $N(u) = \{v, x_1, \ldots, x_t\}$ and $G' = G_{V \setminus \{u, v, x_1, \ldots, x_t\}}$. Then both $G \setminus e$ and $G'$ are chordal, and

\[
\beta_{i,j}(I(G)) = \beta_{i,j}(I(G \setminus e)) + \sum_{\ell=0}^{i} \binom{t}{\ell} \beta_{i-1-\ell,j-2-\ell}(I(G')).
\]

**Remark 4.4.** The edge set of $G'$ in the above theorem is

\[
E(G') = \{e' \in E(G) : \text{dist}_G(e, e') \geq 3\}.
\]

In the proof of Theorem 4.1 we use (4.1) as the following form:

\[
\beta_{i-1,i+j}(I(G)) = \beta_{i-1,i+j}(I(G \setminus e)) + \sum_{\ell=0}^{i-1} \binom{t}{\ell} \beta_{i-2-\ell,(i-1-\ell)+(j-1)}(I(G')).
\]

First we investigate the relation between a strongly disjoint set of bouquets of $G \setminus e$, $G'$ and that of $G$.

**Lemma 4.5.** Let $G$ be a chordal graph. We use the same notations as in Lemma 4.3

1. Let $\mathcal{B}$ be a strongly disjoint set of bouquets of $G \setminus e$. Then $\mathcal{B}$ is also strongly disjoint in $G$. In particular, $d_G \geq d_{G \setminus e}$.

2. Let $\mathcal{B}'$ be a strongly disjoint set of bouquets of $G'$ and $B$ the bouquet of $G$ whose root is $u$ and whose flowers are $v, x_1, \ldots, x_t$. Then $\mathcal{B}' \cup \{B\}$ is a strongly disjoint set of bouquets of $G$. In particular, $d_G \geq d_{G'} + (t + 1)$.

**Proof.** (1) Let $\mathcal{B} = \{B_1, B_2, \ldots, B_t\}$ be a strongly disjoint set of bouquets in $G \setminus e$, where $e = \{u, v\}$. If one of $u, v$ is not in $R(\mathcal{B}) \cup F(\mathcal{B})$, then it is clear that $\mathcal{B}$ is also strongly disjoint in $G$. Also we easily see that $\mathcal{B}$ is strongly disjoint in $G$ when $u, v$ are the vertices of the same bouquet $B_k$. (In this case, both of $u, v$ are flowers of $B_k$.) Thus we may assume that $u \in V(B_1)$ and $v \in V(B_2)$.

We consider 4 cases.
Case 1: $u, v \in R(B)$. In this case, all flowers of $B_2$ are neighbours of $u$. Then there are no stems $s_1, s_2$ ($s_1 \in E(B_1), s_2 \in E(B_2)$) with $\text{dist}_{G(e)}(s_1, s_2) \geq 3$. This contradicts the assumption that $B$ is strongly disjoint in $G \setminus e$.

Case 2: $u \in R(B)$ and $v \in F(B)$. Since $G_{N(v)}$ is a complete graph, the root of $B_2$ is a neighbour of $u$. This leads a contradiction as in Case 1.

Case 3: $u \in F(B)$ and $v \in R(B)$. Let $w_1$ be the root of $B_1$. The completeness of $G_{N(v)}$ implies that the distance between \{$u, w_1$\} and any stem $s_2$ of $B_2$ in $G \setminus e$ is equal to 2. It follows that $B$ is also strongly disjoint in $G$.

Case 4: $u, v \in F(B)$. Let $w_1, w_2$ be roots of $B_1, B_2$ respectively. Then \{$w_2, u\} \in E(G \setminus e)$ and $\text{dist}_{G(e)}(\{u, w_1\}, \{v, w_2\}) = 2$. Therefore $B$ is also strongly disjoint in $G$.

(2) We observed that $e$ is 3-disjoint with any stem of $B'$ in Remark 4.4. Let $S$ be a set of stems of $B'$ which guarantees the strongly disjointness of $B'$ in $G'$. Then $S \cup \{e\}$ is pairwise 3-disjoint. This guarantees strongly disjointness of $B' \cup \{B\}$.

Now we prove Theorem 4.1.

Proof of Theorem 4.1. (1) By Theorem 3.1, it is sufficient to prove that when $\beta_{i,i+j}(S/I(G)) \neq 0$, there exists a subset $W$ of $V$ such that $G_W$ contains a strongly disjoint set of bouquets of type $(i, j)$. We use induction on $\#E(G)$. If $\#E(G) = 1$, then $\beta_{i,j}(S/I(G)) = 0$ except for $(i, j) = (1, 1)$ and $\beta_{1,1+1}(S/I(G)) = 1$. In this case, $G$ coincides with a strongly disjoint set of bouquets of type $(1, 1)$. Hence the claim is true.

Assume that $\#E(G) \geq 2$. By Lemma 3.3 we may assume that $\#V = i + j$ and may prove that if $\beta_{i,i+j}(S/I(G)) \neq 0$, i.e., $\beta_{i-1,i+j}(I(G)) \neq 0$, then $G$ contains a strongly disjoint set of bouquets of type $(i, j)$. We use the same notation as in Lemma 4.3. Note that $G' = G_{W_0}$ where $W_0 = V \setminus \{u, v, x_1, \ldots, x_t\}$. Since $\#W_0 = i + j - (t + 2)$, the summand of righthand side of the second term of (1.1.2) is 0 except for $\ell = t$. Hence we can rewrite (1.1.2) as

$$\beta_{i-1,i+j}(I(G)) = \beta_{i-1,i+j}(I(G \setminus e)) + \beta_{i-2-t,(i-1-t)+(j-1)}(I(G')) \quad (4.3)$$

By the assumption that $\beta_{i-1,i+j}(I(G)) \neq 0$, one of the summands of righthand side of (4.3) does not vanish.

If $\beta_{i-1,i+j}(I(G \setminus e)) \neq 0$, then by inductive hypothesis, $G \setminus e$ contains a strongly disjoint set of bouquets of type $(i, j)$. By Lemma 4.5 (1), this set of bouquets is also strongly disjoint in $G$.

If $\beta_{i-2-t,(i-1-t)+(j-1)}(I(G')) \neq 0$, then by inductive hypothesis, $G'$ contains a strongly disjoint set $B'$ of bouquets of type $(i - 1 - t, j - 1)$. Let $B$ be the bouquet of $G$ whose root is $u$ and whose flowers are $v, x_1, \ldots, x_t$. By Lemma 4.5 (2), the set of bouquets $B := B' \cup \{B\}$ is strongly disjoint in $G$ and the type is $(i, j)$. Therefore the assertion follows.

(2) By Lemma 3.3 we may prove that if $\#V = i + j$ and $\beta_{i-1,i+j}(I(G)) \neq 0$, then $\beta_{i-1,i+j}(I(G)) = 1$. We proceed the proof by induction on $\#E(G)$. When $\#E(G) = 1$, the claim is trivial. We assume that $\#E(G) \geq 2$. Note that the equality (4.3) holds also in this case. Since $G$ is a forest, the neighbourhood of $v$ in $G$ consists of only $u$. Thus $v$ is an isolated vertices of $G \setminus e$. This implies that $\beta_{i-1,i+j}(I(G \setminus e)) = 0$. Thus $\beta_{i-2-t,(i-1-t)+(j-1)}(I(G')) \neq 0$ by (4.3).
Then we have $\beta_{i-2-t(i-1-t)+(j-1)}(I(G')) = 1$ by inductive hypothesis and we obtain $\beta_{i-1,t+j}(I(G)) = 1$ again by (4.3), as desired.

The next example shows that Theorem 4.1 (1) is false for a general graph.

**Example 4.6.** Let us consider the following non-chordal graph $G_1$ on the vertex set $V = \{1, 2, \ldots, 6\}$:

$$G_1:$$

```
  2 1 6
  3 4 5
```

Actually, $G_1$ is the complete bipartite graph $K_{3,3}$. In particular it is unmixed. The Betti diagram of $S/I(G_1)$ is

$$
j \backslash i \quad | \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5
\hline
0 \quad | \quad 1 \quad 9 \quad 18 \quad 15 \quad 6 \quad 1
1 \quad | \quad 2
$$

Here the entry of $j$th row and $i$th column stands for $\beta_{i,j}(S/I(G_1))$. There is no bouquet with 4 flowers in $G_1$ though $\beta_{4,4+1}(S/I(G_1)) \neq 0$. Also $\text{pd } S/I(G_1) = 5$ while $d_{G_1} = 3$.

The next example shows that Theorem 4.1 (2) is false for chordal graphs.

**Example 4.7.** Let $G_2$ be the chordal graph with the vertex set $V = \{1, 2, 3\}$ and the edge set $\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$:

$$G_2:$$

```
  1
/|
/ |
2 3
```

The Betti diagram of $S/I(G_2)$ is

$$
j \backslash i \quad | \quad 0 \quad 1 \quad 2
\hline
0 \quad | \quad 1 \quad 3 \quad 2
1 \quad | \quad 2
$$

Although $\beta_{2,2+1}(S/I(G_2)) = 2$, a subset of $V$ with cardinality $2 + 1$ is only $V$.

5. **Another characterization of the projective dimension**

We gave a characterization of the projective dimension for chordal graphs in Theorem 4.1. However it seems that it is not easy to read it from the graph. In this section, we introduce another notion of disjointness of a set of bouquets, which is rather weaker than strongly disjointness and give another characterization of the projective dimension for chordal graphs using this notion.
Let $G$ be a finite simple graph and $\mathcal{B} = \{B_1, B_2, \ldots, B_j\}$ a set of bouquets of $G$.

**Definition 5.1.** We say that a set $\mathcal{B} = \{B_1, B_2, \ldots, B_j\}$ of bouquets of $G$ is semi-strongly disjoint in $G$ if the following 2 conditions are satisfied:

1. $V(B_k) \cap V(B_\ell) = \emptyset$ for all $k \neq \ell$.
2. Any two vertices belonging to $R(\mathcal{B})$ are not adjacent in $G$.

As noted in Remark 2.2, if $\mathcal{B}$ is strongly disjoint, then $\mathcal{B}$ is also semi-strongly disjoint. We set

\[ d'_G := \max\{\#F(\mathcal{B}) : \mathcal{B} \text{ is a semi-strongly disjoint set of bouquets of } G\}. \]

In general, the inequality $d_G \leq d'_G$ holds. There exists a graph $G$ with $d_G < d'_G$ as the following example shows.

**Example 5.2.** Let $G_3$ be the following graph:

![Graph G3](image)

Note that $G_3$ is a bipartite graph which is not unmixed. It is easy to see that the distance of any two edge of $G_3$ is at most 2. That is there exists no 3-disjoint edges in $G_3$. Thus the strongly disjoint set of bouquets of $G_3$ consists of only one bouquet. In particular $d_{G_3} = 3$. On the other hand let $B_1$ (resp. $B_2$) be the bouquet whose root is 1 (resp. 6) and whose flowers are 2, 3 (resp. 4, 5). Since $\{1, 6\}$ is not an edge of $G_3$, the set of bouquets $B_1, B_2$ is semi-strongly disjoint and $d'_{G_3} = 4 > 3 = d_{G_3}$.

The following result is the main theorem of this section.

**Theorem 5.3.** Let $G$ be a chordal graph. Then

\[ \text{pd } S/I(G) = d_G = d'_G. \]

**Proof.** By Theorem 4.1 we may prove that $d_G \geq d'_G$. We use induction on $\#E(G)$.

When $\#E(G) = 1$, clearly $d_G = d'_G = 1$ hold.

We consider the case of $\#E(G) \geq 2$. Let $\mathcal{B} = \{B_1, B_2, \ldots, B_j\}$ be a semi-strongly disjoint set of bouquets of $G$ with $\#F(\mathcal{B}) = d'_G$. When $j = 1$, the assertion is clear because in this case, $\mathcal{B}$ is also strongly disjoint. We assume that $j \geq 2$. Let $e = \{u, v\}$ be an edge of $G$ such that $G_{N(v)}$ is a complete graph as in Lemma 4.3. If $e \notin S(\mathcal{B})$, then $\mathcal{B}$ is a set of bouquets of $G \setminus e$ and semi-strongly disjoint also in $G \setminus e$. Hence by inductive hypothesis and Lemma 4.5 (1), we have

\[ d'_G \leq d'_{G \setminus e} = d_{G \setminus e} \leq d_G, \]

as desired.

Next we consider the case where $e \in S(\mathcal{B})$, say $e \in E(B_j)$. First we consider the case where $u$ is a neighbour of the root of $B_1$. In this case the root of $B_j$ must be $v$. Let $B'_j$ be the bouquet obtained by adding $u$ to $B_1$ as a flower and $B'_j$ the one obtained by removing $u$ from $B_j$. Then $\mathcal{B}' := \{B'_1, B_2, \ldots, B_{j-1}, B'_j\}$ is a semi-strongly disjoint set of bouquets of $G$ with $e \notin S(\mathcal{B})$. Thus we have $d_G \geq d'_G$ as shown.
in the above. Therefore we may assume that \( u \) is not a neighbour of the root \( w_k \) of \( B_k \) for any \( k = 1, 2, \ldots, j - 1 \). Then since \( G_{N(u)} \) is a complete graph, we may assume that \( u \) is a root of \( B_j \). Let \( B' \) be the set of bouquets which is obtained by removing all flowers those are neighbours of \( u \) from the set of bouquets \( B_1, B_2, \ldots, B_{j-1} \). Then \( B' \) is a semi-strongly disjoint set of bouquets of \( G' = G_{V \setminus (N(u) \cup \{u\})} \) and \( d'_G - (t+1) \leq \#F(B') \leq d'_G \). Combining this with the inductive hypothesis and Lemma 4.5 (2), we have
\[
d'_G \leq d'_{G'} + (t + 1) = d_{G'} + (t + 1) \leq d_G.
\]
This completes the proof. \( \square \)

We cannot replace the strongly disjointness by the semi-strongly one on a characterization of the graded Betti numbers for a forest in Theorem 4.1 (2).

**Example 5.4.** Let \( G_4 \) be the line graph with 7 vertices and \( B \) the set of bouquets of \( G_4 \) as the following picture:

\[
G_4: \quad \begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\end{array} \quad B: \quad \begin{array}{ccccccc}
1 & 3 & 5 & 7 \\
2 & 4 & 6 \\
\end{array}
\]

It is easy to see that \( B \) is semi-strongly disjoint. But \( B \) is not strongly-disjoint. To see this, let \( B \) be the bouquet of \( B \) whose root is 4. Then the stem of \( B \) which is 3-disjoint with \( \{1, 2\} \) in \( G_4 \) is only \( \{4, 5\} \) but it is not 3-disjoint with \( \{6, 7\} \) in \( G_4 \).

In fact, the type of \( B \) is \((4, 3)\) while the Betti diagram of \( S/I(G_4) \) is given by
\[
\begin{array}{c|cccc}
0 & 0 & 1 & 2 & 3 & 4 \\
\hline
1 & 6 & 5 & \\
2 & 6 & 9 & 3 & \\
\end{array}
\]

and \( \beta_{4,4+3}(S/I(G_4)) = 0 \).

**Remark 5.5.** In general, inequalities \( \text{pd} S/I(G) \geq d_G \) (Theorem 3.1) and \( d'_G \geq d_G \) holds. Then are there any relations between \( \text{pd} S/I(G) \) and \( d'_G \)? The inequality \( \text{pd} S/I(G) \geq d'_G \) holds for many graphs \( G \). But we do not know whether this is always true or not.

Finally, we give an application of Theorem 5.3. Let \( G \) be a finite simple graph on \( V \). A subset \( C \) is said to be a vertex cover of \( G \) if it intersects all edges of \( G \). We say that a vertex cover of \( G \) is minimal if it has no proper subset which is also a vertex cover of \( G \). There is one-to-one correspondence between the vertex covers of \( G \) and the minimal prime divisors of \( I(G) \):
\[
I(G) = \bigcap_{C: \text{a minimal vertex cover of } G} (x_i : i \in C).
\]
The big height of \( I(G) \), denoted by \( \text{bight} I(G) \), is defined as the maximum heights among the minimal prime divisors of \( I(G) \). In general, the inequality \( \text{pd} S/I(G) \geq \text{bight} I(G) \) holds. As a corollary of Theorem 5.3 we recover the following result
which follows from Morey and Villarreal [15, Corollary 3.33] together with Francisco and Van Tuyl [5, Theorem 3.2]; see also Herzog, Hibi, and Zheng [8].

**Corollary 5.6** ([15], [5], [8]). Let $G$ be a chordal graph on $V$. We set $S = K[V]$. Then

$$\text{bight } I(G) = \text{pd } S/I(G).$$

In particular, if $I(G)$ is unmixed, then $S/I(G)$ is Cohen–Macaulay.

**Proof.** We may prove that $\text{pd } S/I(G) \leq \text{bight } I(G)$. Let $\mathcal{B} = \{B_1, B_2, \ldots, B_j\}$ be a semi-strongly disjoint set of bouquets of $G$ with $\#F(\mathcal{B}) = d_G'$. Then $\#F(\mathcal{B}) = \text{pd } S/I(G)$ by Theorem 5.3. We claim that $F(\mathcal{B})$ is a minimal vertex cover of $G$. If $F(\mathcal{B})$ is a vertex cover of $G$, then the minimality of it is clear. Hence we only show that $F(\mathcal{B})$ is a vertex cover of $G$.

Suppose that, on the contrary, there exists an edge $e = \{u,v\} \in E(G)$ with $e \cap F(\mathcal{B}) = \emptyset$. When $u$ is a neighbour of the root of $B_k \in \mathcal{B}$ for some $k$, the set $\mathcal{B}'$ of bouquets obtained by adding $u$ to $B_k$ as a flower is semi-strongly disjoint and satisfies $\#F(\mathcal{B}') = d_G' + 1$. This is a contradiction. Therefore $u, v \notin F(\mathcal{B}) \cup R(\mathcal{B})$. Let $B_e$ denotes the bouquet of $G$ with only one stem $e$. Then $\mathcal{B}'' := \mathcal{B} \cup \{B_e\}$ is a semi-strongly disjoint set of bouquets of $G$ and $\#F(\mathcal{B}'') = d_G' + 1$. This is also a contradiction. 

**Acknowledgments.** The author thanks Professors Jürgen Herzog and Takayuki Hibi for the suggestion to consider the characterization of the graded Betti numbers. She also thanks Professor Akimichi Takemura for giving her attention to chordal graphs. The Author is grateful to Professor Naoki Terai for telling her Corollary 5.6. She is also grateful to Professor Rafael H. Villarreal for pointing out that Corollary 5.6 was proved in a more general setting in [15]. The author thanks Professor Russ Woodroofe for giving her useful suggestions and comments.

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Department of Mathematics, Faculty of Science, Shizuoka University, 836 Ohya, Suruga-ku, Shizuoka 422-8529, Japan

E-mail address: skkimur@ipc.shizuoka.ac.jp