MIRROR DUALITY BETWEEN CALABI–YAU FRACTIONAL COMPLETE INTERSECTIONS

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ABSTRACT. This is an expanded version of the author’s talk at the third annual meeting of International Consortium of Chinese Mathematicians held at USTC in December 2020. In this expository article, we give a survey on joint works with Hosono, Lian, and Yau [HLTY20, LLY22]. We also carry out explicit examples to illustrate the results in enumerative geometry which will appear in our forthcoming papers.

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0. Introduction

Mirror symmetry for singular Calabi–Yau varieties was discovered by Hosono, Lian, Takagi, and Yau in their recent work on the study of the family of $K3$ surfaces arising from double covers over $\mathbb{P}^2$ branched along six lines in general position [HLTY20, HLTY19], which were investigated by Matsumoto, Sasaki, and Yoshida as a higher dimensional analogue of the Legendre family [MSY88, MSY92]. Such a $K3$ surface is singular and admits 15 ordinary double points (ODPs); blowing up at these points gives a crepant resolution. Denote by $[x:y:z]$ the homogeneous coordinates on $\mathbb{P}^2$. The branch locus can be parameterized by linear functions $a_{1j}x + a_{2j}y + a_{3j}z$ with $j = 1, \ldots, 6$. Regarding $(a_{ij})$ as a matrix, we see that the configurations of six lines are parameterized by a GIT quotient

$$P(3, 6) := \text{GL}_3(\mathbb{C}) \backslash \text{M}(3, 6)/\langle (\mathbb{C}^*)^6 \rangle.$$
Here $M(3,6)$ is an open subset in $\text{Mat}_{3\times 6}(\mathbb{C})$ consisting of matrices whose any $3 \times 3$ minors are invertible. The group $\text{GL}_3(\mathbb{C})$ acts on $M(3,6)$ via the usual multiplication on the left and $((\mathbb{C}^*)^6$ acts on $M(3,6)$ via scaling the columns of elements in $M(3,6)$.

The parameter space $P(3,6)$ admits two compactifications – (a) a GIT compactification (a.k.a. the Baily–Borel–Satake compactification) \cite{DO88, Mat93} and (b) a toroidal compactification constructed by Reuvers \cite{Reu06}. However, as Hosono, Lian, Takagi, and Yau pointed out, it is not clear whether or not these compactifications admit a priori the so-called large complex structure limit points (LCSL points). In order to study mirror symmetry, they constructed a new compactification of $P(3,6)$ instead and found LCSL points on it. We briefly explain their idea. The $\text{GL}_3(\mathbb{C})$-action on $P^2$ allows us to rearrange three out of the six lines to the coordinate axes so that the $K3$ family is in fact parameterized by three lines in $P^2$. This procedure is called the partial gauge fixing in \cite{HLTY20}. The $\text{GL}_3(\mathbb{C}) \times (\mathbb{C}^*)^6$ action is reduced to a $(\mathbb{C}^*)^5$ action. It then follows that the period integrals of the $K3$ family satisfy certain GKZ $A$-hypergeometric system with an integral matrix $A \in \text{Mat}_{5 \times 9}(\mathbb{Z})$ and a fractional exponent $\beta \in \mathbb{Q}^5$. The matrix $A$ can be recognized as the integral matrix associated to certain nef-partition on the base $P^2$ and the torus $(\mathbb{C}^*)^5$ can be identified with $L \otimes \mathbb{C}^*$, where $L$ is the lattice relation of $A$. Consequently, $P(3,6)$ admits a toroidal compactification via the secondary fan. It turns out that the standard techniques for Calabi–Yau hypersurfaces and complete intersections in toric varieties are still applicable and results in \cite{HKT95, HLY96, HLY97} can be adapted into the present situation. Because of this striking similarity with the classical complete intersections, we shall call such a double cover a fractional complete intersection. Based on numerical evidences, it is conjectured that the mirror of the $K3$ family is given by certain family of double covers over a del Pezzo surface of degree 6, which is a blow-up of $P^2$ at three torus invariant points (\cite{HLY19} Conjecture 6.3). Note that such a del Pezzo surface can be obtained from Batyrev–Borisov’s duality construction for the nef-partition on $P^2$ associated with the integral matrix $A$ and the conjectured mirror is constructed by taking an appropriate double cover over it.

The purpose of this paper is to introduce the results obtained by Hosono, Lian, Yau and the author in \cite{HLLY20, LLY22}. To summarize, we generalized the construction to higher dimensional bases to produce a pair of singular Calabi–Yau varieties which is conjectured to be a mirror pair. To support the conjecture, the first step is the topological test: one computes their Euler characteristics as well as their Hodge numbers; these have been done in \cite{HLLY20}. The second step is the quantum test: in the 3-fold case, one could carry out the $A$ and $B$ model correlation functions and show that they are related under the mirror map. Here the $B$ model is taken to be the variation of Hodge structures for the equisingular family whereas the $A$
model is taken to be the untwisted part of the genus zero orbifold Gromov–Witten theory since our Calabi–Yau double covers are orbifolds. In the present case, the period integrals are governed by a GKZ $A$-hypergeometric system with a fractional exponent. Mimicking the classical case, we found a close relationship between the principal parts of the operators in the GKZ $A$-hypergeometric system and the cohomology ring of the base of the conjectured mirror Calabi–Yau variety; this leads to a cohomology-valued series first introduced in 1994 by Hosono, Lian, and Yau (a.k.a. Givental’s $I$-function, up to an overall $\Gamma$-factor) which plays a crucial role in mirror symmetry [LLY22].

In this expository paper, we will explain our idea and illustrate our results by carrying out explicit examples.

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1. A SINGULAR DOUBLE COVER CONSTRUCTION

1.1. Batyrev–Borisov’s duality construction. We review the construction of classical mirror pairs of Calabi–Yau complete intersections in toric varieties given by Batyrev and Borisov [BB96]. The dual pairs will be served as bases of singular Calabi–Yau double covers. Let us fix the following notation and terminologies.

- Let $N = \mathbb{Z}^n$ be a lattice of rank $n$ and $M := \text{Hom}_\mathbb{Z}(N, \mathbb{Z})$ be the dual lattice. We denote by $N_\mathbb{R}$ and $M_\mathbb{R}$ the tensor products $N \otimes \mathbb{Z} \mathbb{R}$ and $M \otimes \mathbb{Z} \mathbb{R}$.
- For a complete fan $\Sigma$ in $N_\mathbb{R}$, we denote by $\Sigma(k)$ the set of all $k$-dimensional cones in $\Sigma$. For $\rho \in \Sigma(1)$, the same notation $\rho$ is also used to denote the primitive generator of the corresponding 1-cone.
- The toric variety defined by $\Sigma$ is denoted by $X_{\Sigma,N}$, $X_{\Sigma}$ or simply by $X$ if the context is clear. Let $T = (\mathbb{C}^*)^n$ be its maximal torus. Denote by $D_\rho$ the Weil divisor on $X_{\Sigma}$ determined by $\rho \in \Sigma(1)$.
- Let $D = \sum_\rho a_\rho D_\rho$ be a torus invariant divisor. The polytope of $D$ is defined to be the set
  \[ \Delta_D := \{ m \in M_\mathbb{R} \mid \langle m, \rho \rangle \geq -a_\rho, \forall \rho \in \Sigma(1) \} \]
The integral points in $\Delta_D$ give a canonical basis of $H^0(X, D)$.
- A polytope in $M_\mathbb{R}$ is called lattice polytope if its vertices belong to $M$. For a lattice polytope $\Delta$ in $M_\mathbb{R}$, we denote by $\Sigma_\Delta$ the normal fan of $\Delta$. The toric variety determined by $\Delta$ is denoted by $P_\Delta$. We have $P_\Delta = X_{\Sigma_\Delta}$.
A reflexive polytope $\Delta \subset M_\mathbb{R}$ is a lattice polytope containing the origin $0 \in M_\mathbb{R}$ in its interior and such that the polar dual $\Delta^\vee$ is again a lattice polytope. If $\Delta$ is a reflexive polytope, then $\Delta^\vee$ is also a lattice polytope and satisfies $(\Delta^\vee)^\vee = \Delta$. The normal fan of $\Delta$ is the face fan of $\Delta^\vee$ and vice versa.

For a reflexive polytope $\Delta$, a nef-partition on $\mathbb{P}_\Delta$ is a decomposition

$$\Sigma_\Delta(1) = I_1 \cup \cdots \cup I_r$$

such that $E_s := \sum_{\rho \in I_s} D_\rho$ is numerical effective for each $s$. This gives rise to a Minkowski sum decomposition $\Delta = \Delta_1 + \cdots + \Delta_r$, where $\Delta_i = \Delta_{E_i}$ is the polytope of $E_i$.

We recall Batyrev–Borisov’s duality construction. Let $\Delta$ be a reflexive polytope and $\Sigma_\Delta(1) = I_1 \cup \cdots \cup I_r$ be a nef-partition. We define $\nabla_k := \text{Conv}(\{0\} \cup I_k)$ and $\nabla := \nabla_1 + \cdots + \nabla_r$. It turns out that $\nabla$ is a reflexive polytope in $N_\mathbb{R}$ whose polar polytope is $\nabla^\vee = \text{Conv}(\Delta_1, \ldots, \Delta_r)$ and $\nabla + \cdots + \nabla_r$ corresponds to a nef-partition on $\mathbb{P}_\nabla$, called the dual nef-partition. We denote by $F_1, \ldots, F_r$ the corresponding nef toric divisors on $\mathbb{P}_\nabla$. Then the polytope of $F_j$ is $\nabla_j$.

1.2. A construction of singular double covers. In this subsection, we recall the construction of Calabi–Yau double covers introduced in [HLLY20]. Let $\Delta = \Delta_1 + \cdots + \Delta_r$ be a nef-partition and $\nabla = \nabla_1 + \cdots + \nabla_r$ be the dual nef-partition. Let $X \to \mathbb{P}_\Delta$ and $X^\vee \to \mathbb{P}_\nabla$ be maximal projective crepant partial desingularizations (MPCP desingularizations for short hereafter) for $\mathbb{P}_\Delta$ and $\mathbb{P}_\nabla$ [Bat94]. Note that MPCP desingularizations are smooth and not unique in general. According to the construction, the polytopes $\Delta_i$ and $\nabla_j$ correspond to $E_i$ on $\mathbb{P}_\Delta$ and $F_j$ on $\mathbb{P}_\nabla$, respectively. The nef-partitions on $\mathbb{P}_\Delta$ and $\mathbb{P}_\nabla$ give rise to nef-partitions on $X$ and $X^\vee$ via pullback. To save the notation, the corresponding toric divisors on $X$ and $X^\vee$ and their polytopes will be still denoted by $E_i$, $F_j$ and $\Delta_i$, $\nabla_j$.

**Hypothesis.** Throughout this paper, unless otherwise stated, we shall assume that $X$ and $X^\vee$ are both smooth. Equivalently, we assume that both $\Delta$ and $\nabla$ admit a uni-modular triangulation.

Using these data, we can construct a pair of families of Calabi–Yau varieties.

**Definition 1.1.** Let $\Delta = \Delta_1 + \cdots + \Delta_r$ be a decomposition representing a nef-partition $E_1 + \cdots + E_r$ on $X$. A gauge fixed double cover branched along the nef-partition $\Delta_1 + \cdots + \Delta_r$ over $X$ is the double cover $Y \to X$ constructed from the section

$$s = \prod_{i=1}^r \prod_{j=1}^2 s_{i,j}$$

where $s_{i,1}$ is the global section of $E_i$ corresponding to the lattice point $0 \in \Delta_i$ and $s_{i,2}$ is a smooth global section of $E_i$ such that $\text{div}(s)$ is a simple normal crossing (SNC) divisor.
In the present situation, one can prove that $Y$ is Cohen–Macaulay. Moreover, since the branch locus is a SNC divisor, $Y$ is a normal variety with at worst quotient singularities. Also, the fact that the branch divisor is linearly equivalent to $-2K_X$ implies that $Y$ is Calabi–Yau; namely $\omega_Y \cong \mathcal{O}_Y$.

Deforming the sections $s_i, s_2$ yields a singular Calabi–Yau family $\mathcal{Y} \to V$. Here $V \subset H^0(X, E_1) \times \cdots \times H^0(X, E_r)$ is an open subset where the branch divisor $\text{div}(s)$ is SNC. Likewise, we can apply the construction to the dual nef-partition $\nabla = \nabla_1 + \cdots + \nabla_r$ and obtain another singular Calabi–Yau family $\mathcal{Y}^\vee \to U$, where $U \subset H^0(X^\vee, F_1) \times \cdots \times H^0(X^\vee, F_r)$ is an open subset such that the branch divisor is SNC.

Example 1.2. Let $\Delta = \text{Conv}\{(2, -1), (-1, 2), (-1, -1)\}$. We then have $X =\mathbb{P}_\Delta = \mathbb{P}^2$ since $\mathbb{P}_\Delta$ is already smooth. Consider the trivial nef-partition $\Delta = \Delta_1$. The dual nef-partition is given by $\nabla = \nabla_1 = \Delta^\vee = \text{Conv}\{(1, 0), (0, 1), (-1, -1)\}$.

The toric variety $\mathbb{P}_\nabla$ is the mirror $\mathbb{P}^2$ whose MPCP desingularization $X^\vee \to \mathbb{P}_\nabla$ is smooth. Applying the construction to the present case, we obtain two singular Calabi–Yau families $\mathcal{Y} \to V$ and $\mathcal{Y}^\vee \to U$. The fiber of $\mathcal{Y} \to V$ is a double cover over $X$ branched along three lines and one cubic whereas the fiber of $\mathcal{Y}^\vee \to U$ is a double cover over $X^\vee$ branched along a wheel consisting of nine $\mathbb{P}^1$’s and one smooth elliptic curve.

Example 1.3. Let $\Delta = \text{Conv}\{(-3, 1, 1), (1, -3, 1), (1, 1, -3), (-1, -1, -1)\}$. In the present case, we have $X =\mathbb{P}_\Delta = \mathbb{P}^3$. Consider the trivial nef-partition $\Delta = \Delta_1$. The dual nef-partition is given by $\nabla = \nabla_1 = \Delta^\vee = \text{Conv}\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (-1, -1, -1)\}$.

Note that in this case, $\mathbb{P}_\nabla$ admits a smooth MPCP desingularization $X^\vee \to \mathbb{P}_\nabla$. (See also [LZ21, Appendix] for details.)

Conjecture. Let $Y$ and $Y^\vee$ be the fiber of $\mathcal{Y} \to V$ and $\mathcal{Y}^\vee \to U$ respectively. Then $(Y, Y^\vee)$ is a mirror pair.

2. Mirror symmetry: the topological test

In this section, we compute the topological Euler characteristic of the double cover Calabi–Yau varieties as well as their Hodge numbers. For an $n$-dimensional
(quasi-projective) variety $W$, the topological Euler characteristic is defined to be the alternating sum
\[
\chi(W) := \sum_{k=0}^{2n} (-1)^k \dim H^k(W) = \sum_{k=0}^{2n} (-1)^k \dim H^k(W).
\]
Let $\pi: Y \to X$ be a double cover branched along $D$. Then $Y \setminus \pi^{-1}(D) \to X \setminus D$ is a finite étale cover of degree 2. We then have
\[
\chi(Y) = \chi(\pi^{-1}(D)) + \chi(Y \setminus \pi^{-1}(D))
= \chi(D) + 2 \cdot \chi(X \setminus D)
= \chi(D) + 2(\chi(X) - \chi(D)).
\]
(2.1)

To compute the topological Euler characteristic of the branch divisor, we need a result by Danilov and Khovanskii. To this end, let us introduce some terminologies and notation. Let $Z_1, \ldots, Z_r$ be nef torus invariant divisors on a projective toric manifold $X$ and $\Delta_{Z_i}$ be the polytope of $Z_i$. Put
\[
\Delta_{Z_1} \ast \cdots \ast \Delta_{Z_r} := \text{Conv}(e_1 \times \Delta_{Z_1}, \ldots, e_r \times \Delta_{Z_r})
\]
in $\mathbb{R}^r \times M_\mathbb{R}$. Here $\{e_1, \ldots, e_r\}$ is the standard basis of $\mathbb{R}^r$. This is called the Cassley polytope of $\Delta_{Z_1}, \ldots, \Delta_{Z_r}$. For a nonempty subset $I \subset \{1, \ldots, r\}$, we similarly put
\[
\Delta^{\ast I} := \ast_{i \in I} \Delta_{Z_i} \subset \mathbb{R}^{|I|} \times M_\mathbb{R}.
\]
Let $\Lambda_I$ be the pyramids with vertex 0 and base $\Delta^{\ast I}$ in $\mathbb{R}^{|I|} \times M_\mathbb{R}$. Now we can state the result by Danilov and Khovanskii.

**Theorem 2.1** (cf. [DK86, §6]). For general $D_i$ in the linear system $|Z_i|$, we have
\[
\chi(D_1 \cap \cdots \cap D_r \cap T) = -\sum_I (-1)^{|I|+1} \text{vol}_{n+|I|}(\Lambda_I),
\]
where the summation runs over all nonempty subsets $I \subset \{1, \ldots, k\}$ and $\text{vol}_d$ is the normalized volume in $d$-dimensional spaces.

Applying the theorem to the case $Z_i = E_i$ and $X \to \mathbb{P}_\Delta$ is a MPCP desingularization, we can compute the topological Euler characteristic of the branch divisor which turns out to be
\[
\chi(D) = \chi(X) - \chi(T \cap (D_1 \cup \cdots \cup D_r))
= \chi(X) + (-1)^n \text{vol}_{n+r}(\Lambda)
\]
where $D_i$ is a general element in $|E_i|$ and $\Lambda = \Lambda_{\{1, \ldots, r\}}$. When $X^\vee$ is smooth, together with the fact that $F_i$ are nef, one can show that $\text{vol}_{n+r}(\Lambda) = \chi(X^\vee)$. Denote by $Y$ (resp. $Y^\vee$) a fiber of $\mathcal{Y} \to V$ (resp. $\mathcal{Y}^\vee \to U$). It follows that
\[
\chi(Y) = \chi(X) + (-1)^n \chi(X^\vee)
\]
and therefore $\chi(Y) = (-1)^n \chi(Y^\vee)$. To summarize, we obtain

**Theorem 2.2** (cf. [HLLY20, Theorem 2.2]). *Let notation be as above. We have*

$$\chi(Y) = (-1)^n \chi(Y^\vee).$$

We can further determine the Hodge numbers $h^{p,q}(Y)$ for $p + q \neq n$. As observed by Baily [Bai57] and Steenbrink [Ste77], most statements in Hodge theory generalize to orbifolds. Combined with the affine vanishing theorem, we can prove the following proposition.

**Proposition 2.3.** *Let notation be as above and $\pi: Y \to X$ be the branched double cover. We have $h^{p,q}(X) = h^{p,q}(Y)$ for $p + q \neq n$. For more details, we refer the reader to [HLLY20, Ara12].*

**Corollary 2.4.** *When $n = 3$, we have $h^{p,q}(Y) = h^{3-p,q}(Y^\vee)$ for all $p, q$.***

**Example 2.1** *(Example 1.2 continued).* Let us retain the notation in Example 1.2. On one hand, $Y$ is a singular $K3$ surface with 12 ODPs singularities. A direct computation shows that $\chi(Y) = 24 - 12 = 12$.

On the other hand, $Y^\vee$ is also a singular $K3$ surface with at worst ODPs singularities. Let us now compute its topological Euler characteristic. $Y^\vee$ is singular over the points on $X^\vee$ where the branch divisor is singular. A generic section in $H^0(X^\vee, -K_{X^\vee})$ only touches the Weil divisors associated with $(-1, -1), (2, -1)$, and $(-1, 2)$ with intersection number 1. Together with the wheel of $P^1$s, we obtain $9 + 3$ ODPs singularities. We conclude that

$$\chi(Y^\vee) = 24 - 12 = 12 = \chi(Y).$$

One can also check that $\text{vol}_4(\Lambda_1) = 3$. (See the paragraph right before Theorem 2.1 for notation.)

**Example 2.2.** Retain the notation in Example 1.3. Let us compute that Euler characteristic of $Y$ directly. Denote by $D$ the scheme-theoretic zero of a generic section in $H^0(X, -K_X)$ and by $D_x, D_y, D_z, D_w$ the coordinate hyperplanes. One can easily compute

$$\chi(D_a \cap D_b) = 2,$$

$$\chi(D_a \cap D_b \cap D_c) = 1,$$

$$\chi(D \cap D_a) = -4,$$

$$\chi(D \cap D_a \cap D_b) = 4,$$
where \(a, b, c \in \{x, y, z, w\}\) are distinct elements. It follows that \(\chi(Y) = -60\). Moreover, by Proposition 2.3

\[
h^{1,1}(Y) = h^{1,1}(X) = 1, \quad h^{1,0}(Y) = h^{0,1}(Y) = h^{2,0}(Y) = h^{0,2}(Y) = 0.
\]

This implies that \(h^{2,1}(Y) = 31\).

On the other hand, since \(\chi(X^\vee) = \text{vol}_3(\Delta) = 64\) and \(X^\vee\) is smooth and toric, we have \(h^{1,1}(X^\vee) = (64 - 2)/2 = 31\). By Proposition 2.3, \(h^{1,1}(Y^\vee) = h^{1,1}(X^\vee) = 31\). Together with \(\chi(Y^\vee) = 60\), we see that \(h^{2,1}(Y^\vee) = 1\).

3. Mirror symmetry: the quantum test

The purpose of this section is to explain the quantum test. Let \(Y^\vee \to U\) be the singular Calabi–Yau family constructed in §1.2. Let \(\Sigma\) be the fan defining \(X\). By its very construction, we see that

\[
\text{H}^0(X^\vee, F_i) \cong \bigoplus_{\rho \in \nabla_k \cap N} \mathbb{C} \cdot t^\rho.
\]

and that the set \(I_k\) in the nef-partition \(\Sigma(1) = I_1 \cup \cdots \cup I_r\) is identified with

\[
\{\rho \in \Sigma(1) \mid \mathbf{0} \neq \rho \in \nabla_k \cap N\}.
\]

Writing \(I_k = \{\rho_{k,1}, \ldots, \rho_{k,n_k}\}\) with \(n_k = \#I_k\), we put \(\nu_{i,j} := (\rho_{i,j}, \delta_{1,i}, \ldots, \delta_{r,i})\) and additionally \(\nu_{i,0} := (\mathbf{0}, \delta_{1,i}, \ldots, \delta_{r,i})\), where \(\delta_{i,j}\) is the Kronecker delta. Let

\[
A = [\nu_{1,0}^\top \cdots \nu_{r,n_r}^\top] \in \text{Mat}_{(n+r) \times (p+r)}(\mathbb{Z}) \text{ where } p = n_1 + \cdots + n_r.
\]

It can be shown that the period integrals for \(Y^\vee \to U\) are governed by the GKZ \(A\)-hypergeometric system associated with the matrix \(A\) and a fractional exponent

\[
\beta = \begin{bmatrix} \mathbf{0} & -1/2 & \cdots & -1/2 \end{bmatrix}^\top \in \mathbb{Q}^{n+r}.
\]

Let \(Y^\vee\) be a reference fiber in \(Y^\vee \to U\) and \(D\) be the branch divisor of the cover \(Y^\vee \to X^\vee\). Then \(\pi: Y^\vee \setminus D \to X^\vee \setminus D\) is an étale double cover. Denote by \(\mathcal{L}\) the unique non-trivial eigensheaf in \(\pi_* \mathbb{C}_{Y^\vee \setminus D}\). The period integrals over \(X^\vee \setminus D\) are of the form

\[
(3.1) \quad \Pi_\gamma(x) := \int_{\gamma} \frac{1}{u_{1,1/2} \cdots u_{r,1/2}} \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n},
\]

where \(\gamma\) is an \(n\)-cycle in \(X^\vee \setminus D\) with coefficient in \(\mathcal{L}\) and

\[
u_{i,2} = x_{i,0} + \sum_{j=1}^{n_i} x_{i,j} t^{\rho_{i,j}} \in \text{H}^0(X^\vee, F_i)
\]
is the universal section. We shall call $\Pi_\gamma(x)$ an affine period integral. It is easy to check that $\Pi_\gamma(x)$ is annihilated by the relevant GKZ $A$-hypergeometric system. For convenience, we also define the normalized affine period integrals to be $\bar{\Pi}_\gamma(x) := \left(\prod_{i=1}^r x_i\right)^{1/2} \Pi_\gamma(x)$.

Since $\bar{\Pi}_\gamma(x)$ is $(\mathbb{C}^*)^{r+n}$-invariant, it descends to a local section of a locally constant sheaf on a suitable open subset in $(\mathbb{C}^*)^{r+p}/(\mathbb{C}^*)^{r+n}$. Regarding $A$ as a map $\mathbb{Z}^{r+p} \to \mathbb{Z}^{r+n}$, we have an identification

$$(\mathbb{C}^*)^{r+p}/(\mathbb{C}^*)^{r+n} \cong \text{Hom}_\mathbb{Z}(L, \mathbb{C}^*),$$

where $L = \ker(A)$. Any complete fan $F$ in $L' \otimes \mathbb{R}$ determines a toric compactification $X_F$. Recall that a smooth boundary point $p \in X_F$ is called a maximal degeneracy point if near $p$ there exists exactly one normalized affine period integral which extends to $p$ holomorphically. Following the ideas in [HL Y97], the algebraic torus $(\mathbb{C}^*)^{r+p}/(\mathbb{C}^*)^{r+n}$ admits a toric compactification via the secondary fan $S\Sigma$ associated to $A$. Moreover, using the toric ideal associated to $A$, we can define another fan $G\Sigma$, called the Gröbner fan, which refines $S\Sigma$. In the present case, we can prove that

**Theorem 3.1.** For every toric resolution $X_{G\Sigma'} \to X_{G\Sigma}$, there exists at least one maximal degeneracy point in $X_{G\Sigma'}$.

We can write down the unique holomorphic normalized period near $p$ explicitly. Pick an $\alpha \in \mathbb{C}^{p+r}$ such that $A(\alpha) = \beta$. In the present case, we can pick $\alpha = (\alpha_{i,j})$ with $\alpha_{i,0} = -1/2$ for $i = 1, \ldots, r$ and $\alpha_{i,j} = 0$ for $i = 1, \ldots, r$ and $j = 1, \ldots, n_i$. One solution to the GKZ system is given by

$$\sum_{\ell \in L} \prod_{i=1}^r \frac{\prod_{i=1}^r \Gamma(-\alpha_{i,0} - \ell_{i,0}) \prod_{i=1}^r \prod_{j=1}^{n_i} \Gamma(\ell_{i,j} + \alpha_{i,j} + 1)}{\prod_{i=1}^r \Gamma(-\alpha_{i,0}) \prod_{i=1}^r \prod_{j=1}^{n_i} \Gamma(\ell_{i,j} + 1)} (-1)^{\sum \ell_{i,0} x_{i,j}}.$$ 

Here the components of $\ell \in L \subset \mathbb{Z}^{p+r}$ are labeled by $(i, j)$ with $1 \leq i \leq r$ and $0 \leq j \leq n_i$ and $x_{i,j}$ are the coordinates for our GKZ $A$-hypergeometric system.

### 3.1. Generalized Frobenius method

Now let us explain the generalized Frobenius method. Let $D_{i,j}$ be the toric divisor associated with $\rho_{i,j}$. Combining (3.2) with these cohomology classes, we define a cohomology-valued power series

$$B^\alpha_x(x) := \left( \sum_{\ell \in \text{NE}(X) \cap L} \mathcal{O}_x^{\alpha_x \cdot \ell + \alpha} \right) \exp \left( \sum_{i=1}^r \sum_{j=0}^{n_i} (\log x_{i,j}) D_{i,j} \right),$$

as the unique holomorphic normalized period near $p$ explicitly.
where $\overline{\text{NE}}(X)$ is the Mori cone of $X$ and
\[
\mathcal{O}_\ell^\alpha := \frac{\prod_{i=1}^r (-1)^{\ell_i,0} \Gamma(-D_{i,0} - \ell_{i,0} - \alpha_{i,0})}{\prod_{i=1}^r \Gamma(-\alpha_{i,0}) \prod_{i=1}^r \prod_{j=1}^{m_i} \Gamma(D_{i,j} + \ell_{i,j} + \alpha_{i,j} + 1)}.
\]

with $D_{i,0} := -\sum_{j=1}^{m_i} D_{i,j}$.

The cohomology-valued series (3.3) was introduced by Hosono et al. in [HLY96] which encodes the information from the $A$ model and the $B$ model for a Calabi–Yau mirror pair. The series is also called a Givental’s $I$-function in the literature. We have the following theorem.

**Theorem 3.2.** The pairings $\langle B_\alpha X(x), h \rangle$ give a complete set of solution to the GKZ $A$-hypergeometric system associated with $\mathcal{Y}^\vee \to U$ when $h \in H^\bullet(X, \mathbb{C})^\vee$ runs through a basis of $H^\bullet(X, \mathbb{C})^\vee$.

**Example 3.1** (Example 2.2 continued). Let us keep the notation. We describe the GKZ $A$-hypergeometric system for the singular family $\mathcal{Y}^\vee \to U$. Note that
\[
H^0(X^\vee, -K_{X^\vee}) \cong \bigoplus_{\rho \in \nabla \cap \mathbb{N}} \mathbb{C} \cdot t^\rho.
\]

The period integrals of $\mathcal{Y}^\vee \to U$ are governed by the GKZ $A$-hypergeometric system with
\[
A = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 1 & -1
\end{bmatrix}
\quad \text{and} \quad
\beta = \begin{bmatrix}
-1/2 \\
0 \\
0 
\end{bmatrix}
\]

Regarding $A$ as a linear map $A: \mathbb{Z}^5 \to \mathbb{Z}^4$ as before, we may take
\[
\alpha = \begin{bmatrix}
-1/2 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

so that $A(\alpha) = \beta$. We can compute the lattice relation of $A$
\[
\ker(A) = \mathbb{Z}((-4, 1, 1, 1, 1)).
\]

We also note that $\text{vol}_1(A) = 4$. Denote by $x_0, x_1, \ldots, x_4$ the variable attached to the columns of $A$ in the GKZ system. We can write down the unique holomorphic series solution
\[
\sum_{n \geq 0} \frac{\Gamma(1/2 + 4n)}{\Gamma(1/2) \Gamma(n + 1)^4} \left( \frac{x_1 x_2 x_3 x_4}{x_0^4} \right)^n.
\]
In the present case, we have \( r = j = 1 \). For simplicity, we will drop the index \( i \) in the above formula and write \( D_j \equiv D_{1,j} \). The cohomology-valued series (3.3) becomes

\[
B^\alpha_X(x) = \left( \sum_{n \geq 0} \mathcal{O}^\alpha_n \left( \frac{x_1 x_2 x_3 x_4}{x_0^4} \right)^n \frac{1}{x_0^{1/2}} \right) \exp \left( \sum_{j=0}^4 (\log x_j) D_j \right)
\]

where

\[
\mathcal{O}^\alpha_n = \frac{\Gamma(-D_0 + 4n + 1/2)}{\Gamma(1/2) \prod_{j=1}^4 \Gamma(D_j + n + 1)}.
\]

Denote by \( H \) the hyperplane class of \( \mathbb{P}^3 \). We have \( D_1 = D_2 = D_3 = D_4 = H \) and \( D_0 = -4H \). We may re-write (3.4) into

\[
B^\alpha_X(x) = \frac{1}{x_0^{1/2}} \left( \sum_{n \geq 0} \mathcal{O}^\alpha_n \left( \frac{x_1 x_2 x_3 x_4}{x_0^4} \right)^n \right) \exp \left( \log \left( \frac{x_1 x_2 x_3 x_4}{x_0^4} \right) \cdot H \right)
\]

with

\[
\mathcal{O}^\alpha_n = \frac{\Gamma(4H + 4n + 1/2)}{\Gamma(1/2) \Gamma(H + n + 1)^4}.
\]

If we expand the series (3.5) according to the cohomology basis \( \{ 1, H, H^2, H^3 \} \), then the coefficients give a complete set of the solutions to the relevant GKZ A-hypergeometric system.

### 3.2. The quantum test.

In this subsection, we will consider the double cover over \( \mathbb{P}^3 \) branched along four hyperplanes and one quartic (see Example 1.3) and carry out the quantum test in this case. Let us keep the notation in Example 1.3 and Example 2.2.

#### 3.2.1. Crepant resolutions.

According to the results in [SXZ13], there exists a family of crepant resolutions \( \tilde{Y}^\vee \to Y^\vee \to U \) without modifying \( h^{p,q} \) of each fiber for \( p \neq q \). Fix a reference fiber \( Y^\vee \) of \( Y^\vee \to U \) as before and denote by \( \tilde{Y}^\vee \to Y^\vee \) the crepant resolution. One can easily see that \( \tilde{Y}^\vee \) is smooth Calabi–Yau with \( h^{2,1}(\tilde{Y}^\vee) = h^{2,1}(Y^\vee) = 1 \). Moreover, we have \( \text{H}^3(\tilde{Y}^\vee, \mathbb{Q}) \cong \text{H}^3(Y^\vee, \mathbb{Q}) \).

#### 3.2.2. Picard–Fuchs equations.

Recall that the GKZ A-hypergeometric system for the family \( Y^\vee \to U \) is given by the data

\[
A = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & -1
\end{bmatrix}
\quad \text{and} \quad
\beta = \begin{bmatrix}
-1/2 \\
0 \\
0 \\
0
\end{bmatrix}.
\]
Let us carry out the Picard–Fuchs equations on the one-dimensional moduli. Set \( \ell := (-4, 1, 1, 1) \in \ker(A) \). It is a generator of \( \ker(A) \) and define

\[
z = x^\ell := x_1^4 x_2 x_3 x_4 / x_4^4.
\]

Here \( z \) can be regarded as the coordinate on the quotient \((\mathbb{C}^*)^5 / \ker(A) \otimes \mathbb{C}^* \cong \mathbb{C}^*\). Consider the box operator

\[
\square_{\ell} = \partial_1 \partial_2 \partial_3 \partial_4 - \partial_0^4, \text{ where } \partial_j := \frac{\partial}{\partial x_j}.
\]

To carry out the Picard–Fuchs equation, we consider the conjugate operator

\[
x^{-\alpha} x_1 x_2 x_3 x_4 \square_{\ell} x^{\alpha} = x_1^{1/2} x_2 x_3 x_4 \partial_1 x_0^{-1/2}
\]

\[
= x_1 x_2 x_3 x_4 \partial_1 \partial_2 \partial_3 \partial_4 - z x_1^{1/2} x_0^{-1/2} \partial_0^4 x_0^{-1/2}
\]

\[
= \theta_1 \theta_2 \theta_3 \theta_4 - z \left( \theta_0 - \frac{7}{2} \right) \left( \theta_0 - \frac{5}{2} \right) \left( \theta_0 - \frac{3}{2} \right) \left( \theta_0 - \frac{1}{2} \right)
\]

\[
(3.6)
\]

Here \( \theta_i = x_i \partial_i \) and \( \theta_z = z \partial_z \).

**Remark 3.2.** We note that the equation (3.6) is the same as the Picard–Fuchs equation for the mirror of a smooth degree 8 hypersurface in the weighted projective space \( \mathbb{P}(1, 1, 1, 1, 4) \).

**Remark 3.3.** From Batyrev’s duality construction, \( X^\vee \) is a resolution of the toric variety \( \mathbb{P}_\nu \cong \mathbb{P}^3 / G \) for some finite abelian group \( G \). Indeed,

\[
\nabla = \text{Conv}\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (-1, -1, -1)\}.
\]

The normal fan of \( \nabla \) is the face fan of its dual

\[
\nabla^\vee = \Delta = \text{Conv}\{(3, -1, -1), (-1, 3, -1), (-1, -1, 3), (-1, -1, -1)\}.
\]

The group \( G \) can be identified with the quotient group \( \mathbb{Z}^3 / N_0 \) where \( N_0 \) is the sublattice generated by \( (3, -1, -1), (-1, 3, -1), (-1, -1, 3), (-1, -1, -1) \).

The singular mirror \( Y^\vee \) is a double cover branched along the union of toric divisors on \( X^\vee \) together with a smooth anti-canonical section. The universal family for \( \tilde{Y}^\vee \) can be realized as a partial resolution of the double cover over \( \mathbb{P}_\nu \) branched along the union of toric divisors and

\[
\{z_1^4 + z_2^4 + z_3^4 + z_4^4 + x_0 \cdot z_1 z_2 z_3 z_4 = 0\} / G, \text{ where } [z_1: z_2: z_3: z_4] \in \mathbb{P}^3.
\]

The variable \( z := x_0^{-4} \) is the coordinate on the moduli space of \( \tilde{Y}^\vee \) (or \( Y^\vee \)).
3.2.3. The instanton prediction. Mimicking the classical case, we can compute the unique holomorphic period around $z = 0$

$\omega_0(z) := \sum_{n \geq 0} \frac{\Gamma(4n + 1/2)}{\Gamma(1/2) \Gamma(n + 1)^4} z^n.$

Denote by $\Omega(z)$ a relative holomorphic top form on the moduli space of $\tilde{Y}^\vee$. Let us compute the Yukawa coupling

$\langle \theta_z, \theta_z, \theta_z \rangle^\Omega := \int_{\tilde{Y}^\vee} \Omega(z) \wedge \theta_z^3 \Omega(z).$

By Griffiths transversality,

$\int_{\tilde{Y}^\vee} \Omega(z) \wedge \theta_z^2 \Omega_z = 0.$

Differentiating the displayed equation twice yields

$\int_{\tilde{Y}^\vee} \theta_z \Omega(z) \wedge \theta_z^3 \Omega(z) + \theta_z \langle \theta_z, \theta_z, \theta_z \rangle^\Omega = 0.$

By chain rule, we then have

$\theta_z \left( \int_{\tilde{Y}^\vee} \Omega(z) \wedge \theta_z^3 \Omega(z) \right) - \int_{\tilde{Y}^\vee} \Omega(z) \wedge \theta_z \langle \theta_z, \theta_z, \theta_z \rangle^\Omega = 0; \quad \text{in other words,}$

$2 \theta_z \langle \theta_z, \theta_z, \theta_z \rangle^\Omega - \int_{\tilde{Y}^\vee} \Omega(z) \wedge \theta_z^4 \Omega(z) = 0.$

Manipulating the Picard–Fuchs equation (3.6), we obtain

$\theta_z \langle \theta_z, \theta_z, \theta_z \rangle^\Omega = \frac{z}{1 - z} \langle \theta_z, \theta_z, \theta_z \rangle^\Omega.$

We can solve the above equation and get

$\langle \theta_z, \theta_z, \theta_z \rangle^\Omega = \frac{C}{1 - z} \quad \text{for some constant } C.$

One can check the normalized Yukawa coupling

$\langle \theta_z, \theta_z, \theta_z \rangle := \int_{\tilde{Y}^\vee} \Omega(z) \wedge \theta_z^3 \left( \frac{\Omega(z)}{\omega_0(z)} \right)$

is given by

$\langle \theta_z, \theta_z, \theta_z \rangle = \frac{C}{(1 - z) \omega_0(z)^2},$}

where $\omega_0(z)$ is the holomorphic series solution (3.7). Consider the deformed series

$\omega_0(z; \rho) := \sum_{n \geq 0} \frac{\Gamma(4n + 4\rho + 1/2)}{\Gamma(1/2) \Gamma(n + \rho + 1)^4} z^{n+\rho}$
and its derivative with respect to \( \rho \)

\[
\omega_1(z) := \frac{d}{d\rho} \bigg|_{\rho=0} \omega_0(z; \rho).
\]

Recall that the mirror map is given by

\[
q = \exp \left( 2\pi \sqrt{-1} t \right), \quad t = \frac{1}{2\pi \sqrt{-1} \omega_0(z)} \omega_1(z).
\]

Using the classical product, one finds

\[
C = 2 \text{ in (3.9)} \quad \text{and the “mirror map” is}
\]

\[
q = \frac{z}{256} + \frac{247 z^2}{1024} + \frac{1336851 z^3}{524288} + \cdots
\]

whose inverse is given by

\[
z = 256 q - \frac{4046848 q^2}{18282602496 q^3} + \cdots
\]

The (expected) A model correlation function is

\[
\langle H, H, H \rangle(q) = 2 + 29504 q + 1030708800 q^2 + 38440454795264 q^3 + \cdots
\]

3.2.4. An instanton calculation. To complete the quantum test, we will compute the oribifold Gromov–Witten invariants of \( Y \) and compare them with (3.11).

Let \([z_1: \ldots : z_4]\) be the homogeneous coordinates on \( X = P^3 \) as before and \( f \) be a degree 4 polynomial in \( P^4 \) such that \( \{f = 0\} \cup \bigcup_{i=1}^4 \{z_i = 0\} \) is the branch locus of the double cover \( Y \to X \). Consider the graph map

\[
\Gamma_f: X \to P(1, 1, 1, 1, 4), \quad [z_1: \ldots : z_4] \mapsto [z_1: \ldots : z_4 : f(z)].
\]

This is well-defined since \( f \) is of degree 4 and the branch divisor is SNC. Obviously, \( \Gamma_f \) defines an embedding \( X \hookrightarrow P(1, 1, 1, 1, 4) \).

Let \([y_1: \ldots : y_5]\) be the homogeneous coordinate on \( P(1, 1, 1, 1, 4) \). Consider the covering map

\[
\Phi: P(1, 1, 1, 1, 4) \to P(1, 1, 1, 1, 4), \quad [y_1: \ldots : y_5] \mapsto [y_1^2: \ldots : y_5^2].
\]

Let \( Y' \subset P(1, 1, 1, 4) \) be the subvariety defined by the degree 8 polynomial \( y_5^2 - f(y_1^2,\ldots,y_4^2) \). It is clear that \( Y' \) is a smooth Calabi–Yau hypersurface.

Look at the diagram

\[
\begin{array}{ccc}
P(1, 1, 1, 1, 4) & \xrightarrow{\Phi} & P(1, 1, 1, 1, 4) \\
\downarrow \phi & & \\
X & \xrightarrow{\Gamma_f} & P(1, 1, 1, 1, 4)
\end{array}
\]
Taking the fibred product, we obtain a cover $Y' \to X$ branched along

$$\{f = 0\} \cup \bigcup_{i=1}^{4}\{z_i = 0\}.$$ 

Let $\mu_2 = \{-1, 1\} \subset \mathbb{C}^*$. We define an action of $\mu_2^5$ on $\mathbb{P}(1, 1, 1, 1, 4)$

$$g \cdot [y_1 : \ldots : y_5] := [g_1 \cdot y_1 : \ldots : g_5 \cdot y_5] \text{ where } g = (g_1, \ldots, g_5) \in \mu_2^5.$$ 

Notice that the subgroup $K := \langle (-1, -1, -1, -1, 1) \rangle$ acts trivially on $\mathbb{P}(1, 1, 1, 1, 4)$. Then $G = \mu_2^5/K$ is the Galois group of the cover $Y' \to X$. The map

$$\mu_2^5 \to \mu_2, \quad (g_1, \ldots, g_5) \mapsto \prod_{i=1}^{5} g_i$$

induces a map $G \to \mu_2$. Let $G'$ be its kernel. Explicitly,

$$G' = \left\{ (g_1, \ldots, g_5) \in \mu_2^5 \mid \prod_{i=1}^{5} g_i = 1 \right\} / K.$$ 

One can prove the following lemma.

**Lemma 3.3.** The map

$$\mathbb{P}(1, 1, 1, 1, 4)/G' \to \mathbb{P}(1, 1, 1, 1, 4)/G \cong \mathbb{P}(1, 1, 1, 1, 4)$$

is a double cover branched along the union of all toric divisors.

**Corollary 3.4.** We have $Y \simeq Y'/G'$.

**Proof.** Both $Y$ and $Y'/G'$ are double covers over $\mathbb{P}^3$ having the same branch locus. Since the Picard group of $\mathbb{P}^3$ is torsion free, $Y$ and $Y'/G'$ must be isomorphic. \qed

We can regard $Y'/G'$ as a Calabi–Yau hypersurface in $\mathbb{P}(1, 1, 1, 1, 4)/G'$. The orbifold Gromov–Witten invariants of $Y \cong Y'/G'$ can be computed by applying the orbifold quantum hyperplane section theorem \cite[Theorem 5.2.3]{Tse10}. We will prove the following theorem.

**Theorem 3.5.** The equation (3.11) is the generating series of the untwisted genus zero orbifold Gromov–Witten invariants of $Y$ with all insertions $H$, where $H$ is the pullback of the hyperplane class of $X$.

Now let us prove Theorem 3.5. We have the following commutative diagram:

$$\begin{array}{ccc}
\mathbb{P}(1, 1, 1, 1, 4) & \xrightarrow{q} & \mathbb{P}(1, 1, 1, 1, 4)/G' \\
\xrightarrow{\Phi} & & \xleftarrow{p} \\
\mathbb{P}(1, 1, 1, 1, 4) & & \mathbb{P}(1, 1, 1, 1, 4)/G'
\end{array}$$ (3.14)
In the above diagram, $\Phi$ is defined in (3.12), $q$ is the quotient map and $p$ is the induced double cover.

**Remark 3.4.** $\mathbb{P}(1, 1, 1, 4)/G'$ is a toric variety. We can describe its fan structure as follows. Consider a rank four lattice $\mathbb{N} := \mathbb{Z}^4$ and integral vectors

$$
\rho_1 = (1, 1, -1, 1),
\rho_2 = (-1, 1, -1, 1),
\rho_3 = (1, -1, 1, 1),
\rho_4 = (-1, 1, 1, -1),
\rho_5 = (3, -5, -3, 1).
$$

Note that $\rho_1 + \rho_2 + \rho_3 + 4\rho_4 + \rho_5 = 0$ and that $\rho_1, \ldots, \rho_5$ generate a sublattice $\mathbb{N}'$ of index 8 in $\mathbb{Z}^4$. Put

$$
\sigma_j := \text{Cone}\{\rho_1, \ldots, \hat{\rho}_j, \ldots, \rho_5\}, \ j = 1, \ldots, 5.
$$

All the $\sigma_j$ together with all their faces form a complete fan $\Xi$ in $\mathbb{N}' \otimes \mathbb{R}$. With respect to $\mathbb{N}'$, the toric variety $X_{\Xi, \mathbb{N}'}$ is isomorphic to $\mathbb{P}(1, 1, 1, 4)$. Moreover, we can show that $G' \cong \mathbb{N}'/\mathbb{N}$ and $X_{\Xi, \mathbb{N}} \cong \mathbb{P}(1, 1, 1, 4)/G'$.

Let $\mathfrak{X} = [\mathbb{P}(1, 1, 1, 4)/G']$ be the quotient stack. The coarse moduli space is denoted by $|\mathfrak{X}|$ ($\cong \mathbb{P}(1, 1, 1, 4)/G'$). Let us write down the (non-extended) $I$-function for $\mathfrak{X}$. Denote by $D_i$ the toric divisor associated to the 1-cone $\mathbb{R}_{\geq \rho_i}$ defined in Remark 3.4. One can easily prove that $D_1 \equiv D_2 \equiv D_3 \equiv D_5$ and $D_4 \equiv 4D_5$ and $D_1. \ell = 1/4$. Write $H = D_1$. In the present case, $H^2(|\mathfrak{X}|; \mathbb{C}) = \mathbb{C} \cdot H$ and $H_2(|\mathfrak{X}|; \mathbb{Z}) = \mathbb{Z}(\ell)$ where $\ell$ is the curve class coming from a wall in $\Xi$. Then $8H$ is a Cartier divisor on $|\mathfrak{X}|$. The non-extended $I$-function (on the very small parameter space $H^{\leq 2}(|\mathfrak{X}|; \mathbb{C}) \subset H_{\text{CR}}(\mathfrak{X}; \mathbb{C})$) is given by

$$
I_{\mathfrak{X}}(t; z) = z \cdot \exp(Ht/z) \sum_{g \in C(G')} \sum_{d \in \mathbb{N}^g} q^d \prod_{j=1}^5 \frac{\prod_{(d) = (m), m \leq 0} (D_j + mz)}{\prod_{(d) = (m), m \leq d} (D_j + mz)} \cdot 1_g
$$

$$
= z \cdot \exp(Ht/z) \sum_{g \in C(G')} \sum_{d \in \mathbb{N}^g} q^d \prod_{(d) = (m)} \frac{1}{(H + mz)^4} \prod_{0 < m \leq 4d} (4H + mz) \cdot 1_g
$$

where $C(G')$ denotes the set of conjugacy classes of $G'$ and $1_g$ is the unit in the cohomology ring of the component associated to $g$ (cf. [CCIT15]). In our case, $G'$ is a finite abelian group and $C(G') = G'$.

Now we can apply the orbifold quantum Lefschetz hyperplane theorem to compute the orbifold Gromov–Witten invariants for $Y$. Recall that $Y \cong Y'/G'$ is an anticanonical hypersurface in $\mathfrak{X}$. Applying the hypergeometric modification trick to $I_{\mathfrak{X}}$...
and restricting the result to the *untwisted sector* $1_e$, we obtain

$$
(3.15) \quad I^\text{untw}_X(t; z) = z \cdot \exp(HT/z) \sum_{d \in \mathbb{Z}_{\geq 0}} q^d \frac{\prod_{1 \leq m \leq 8d} (8H + mz)}{\prod_{1 \leq m \leq d} (H + mz)^4 \prod_{1 \leq m \leq 4d} (4H + mz)} \cdot 1_e
$$

The series (3.15) is almost identical to

$$
\tilde{I}_Y(t; z) = z \cdot \exp(HT/z) \sum_{d \in \mathbb{Z}_{\geq 0}} q^d \frac{\prod_{1 \leq m \leq 8d} (8h + mz)}{\prod_{1 \leq m \leq d} (h + mz)^4 \prod_{1 \leq m \leq 4d} (4h + mz)}
$$

the hypergeometric modification of the $I$-function for the Calabi–Yau hypersurface $Y'$ in $P(1,1,1,1,4)$. Here $h$ is the hyperplane class of $P(1,1,1,1,4)$. The only difference is the hyperplane classes $h$ and $H$.

**Corollary 3.6.** The mirror maps for $8h \cdot \tilde{I}_Y(t; z)$ and $8H \cdot I^\text{untw}_X(t; z)$ are identical if we treat $H$ and $h$ as formal variables such that $h^5 = H^5 = 0$.

Now we investigate the Poincaré pairing on $Y'$ and the orbifold Poincaré pairing $Y$. Let $h$ and $H$ be the hyperplane classes on $P(1,1,1,1,4)$ and $|X|$ as before. By abuse of notation, the restriction of $h$ and $H$ to $Y'$ and $Y$ are also denoted by $h$ and $H$. From (3.14), we see that $H = p^*h$ and $q^*H = 2h$ (note that $\Phi^*h = 2h$). Therefore,

$$
\int_Y H^3 = \frac{1}{8} \int_{Y'} (2h)^3 = \int_Y h^3 = \int_{P(1,1,1,1,4)} 8h^4 = 2.
$$

We see that $\{1, H, H^2/2, H^3/2\}$ is a symplectic basis of $H^{\text{even}}(Y; \mathbb{C})$ with respect to the orbifold Poincaré pairing. On the other hand, we know that $\{1, h, h^2/2, h^3/2\}$ is a symplectic basis of $H^{\text{even}}(Y'; \mathbb{C})$ with respect to the Poincaré pairing on $Y'$. This shows that

$$
(3.16) \quad H^{\text{even}}(Y'; \mathbb{C}) \rightarrow H^{\text{even}}(Y; \mathbb{C}), \ h^k \mapsto H^k
$$

is an isomorphism between normed linear spaces (with respect to Poincaré paring and orbifold Poincaré pairing). Since the $J$-function of $Y'$ is identical to the restriction of the untwisted part of the $J$-function of $Y$ to the very small parameter space, we conclude the proof of Theorem 3.5.

**Remark 3.5.** We can also consider the double cover $Y \rightarrow X := P^3$ branched along eight hyperplanes in general position. In this case, we have $r = 4$ and the nef-partition is $-K_X = H + H + H + H$. Denote by $Y'$ the singular mirror. As before, we can compute $h^{2,1}(Y') = 1$. We can carry out the Picard–Fuchs equation, the
unique holomorphic (affine) period as well as the mirror map. The Picard–Fuchs equation turns out to be

$$\theta_{z}^4 - z \left( \theta_{\frac{1}{2}} + \frac{1}{2} \right)^4 = 0.$$ 

We obtain the predicted $A$ model correlation function

$$\langle H, H, H \rangle(q) = 2 + 64q + 9792q^2 + 14049280q^3 + 30593496064q^5 + \cdots.$$ 

The series was also obtained by E. Sharpe in [Sha13] using the technique of gauged linear sigma models (GLSMs). We can prove that it is the generating series of the untwisted genus zero orbifold Gromov–Witten invariants of $Y$ with all insertions $H$, where $H$ is the pullback of the hyperplane class of $X = \mathbb{P}^3$.

3.2.5. Geometric transitions. A geometric transition is a complex degeneration $W \rightsquigarrow W_0$ followed by a resolution $W_0 \leftarrow Z$. In [Mor99], Morrison conjectured that geometric transitions are reversed under mirror symmetry. To be precise, let $W$ and $Z$ be Calabi–Yau manifolds and $W^\vee$ and $Z^\vee$ be their mirror. Suppose that $W \rightsquigarrow W_0 \leftarrow Z$ is a geometric transition. Then the conjecture asserts that there exists a geometric transition $Z^\vee \rightsquigarrow W_0^\vee \leftarrow W^\vee$ connecting $W^\vee$ and $Z^\vee$. In this paragraph, we shall see that our singular mirror construction fits into the picture nicely.

Our singular double cover $Y$ admits a smoothing to $W$ by deforming the branch locus into a smooth degree 8 hypersurface in $\mathbb{P}^3$ and $W$ can be realized as a smooth Calabi–Yau hypersurface in $\mathbb{P}(1, 1, 1, 1, 4)$. If we put

$$\Delta_1 = \text{Conv}\{(6, -2, -2, -1), (-2, 6, -2, -1), (-2, -2, 6, -1), (0, 0, 0, 1), (-2, -2, -2, -1)\},$$

then $\mathbb{P}(1, 1, 1, 1, 4) = \mathbb{P}_{\Delta_1}$.

The matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \in \text{GL}_4(\mathbb{Z})$$

takes $\{\rho_1, \rho_2, \rho_3, \rho_4, \rho_5\}$ into

$$\begin{align*} 
\nu_1 &= (2, 0, 0, -1), \\
\nu_2 &= (0, 2, 0, -1), \\
\nu_3 &= (0, 0, 2, -1), \\
\nu_4 &= (0, 0, 0, 1), \\
\nu_5 &= (-2, -2, -2, -1). 
\end{align*}$$
Let $\nabla_2 := \text{Conv}\{\nu_1, \ldots, \nu_5\}$. It is easy to check that $\nabla_2$ is reflexive and

$$\Delta_2 := \nabla_2^\vee = \text{Conv}\{(3, -1, -1, -1), (-1, 3, -1, -1), (-1, -1, 3, -1), (0, 0, 0, 1), (-1, -1, -1, -1)\}.$$ 

In other words, $Y$ is a Calabi–Yau hypersurface in $\mathbb{P}_{\Delta_2}$. Denote by $\hat{\mathbb{P}}_{\Delta_2} \to \mathbb{P}_{\Delta_2}$ an MPCP desingularization and $Z \to Y$ be the induced partial resolution. We obtain the following diagram.

$$Z \subset \hat{\mathbb{P}}_{\Delta_2} \quad \text{(3.17)}$$

$$\mathbb{P}_{\Delta_1} \supset W \xrightarrow{\sim} Y \subset \mathbb{P}_{\Delta_2}.$$ 

One can obtain a MPCP desingularization $\hat{\mathbb{P}}_{\Delta_1} \to \mathbb{P}_{\Delta_1}$ by taking weighted blow-up at the singular point. $W$ can be also regarded as a Calabi–Yau hypersurface in $\hat{\mathbb{P}}_{\Delta_1}$.

On the other hand, we can construct the mirror of $W$ and $Z$ by taking their dual polytope. Note that

$$\nabla_1 := \Delta_1^\vee = \text{Conv}\{(1, 0, 0, -1), (0, 1, 0, -1), (0, 0, 1, -1), (0, 0, 0, 1), (-1, -1, -1, -1)\}.$$ 

By our construction, $Y^\vee$ is a double cover over an MPCP desingularization $X^\vee \to \mathbb{P}_{\nabla}$. It turns out that $Y^\vee$ can be realized as a Calabi–Yau hypersurface in some toric variety which we now describe.

$Y^\vee \to X^\vee$ is a double cover branched along the union of toric divisors and a general section $s \in H^0(X^\vee, -K_{X^\vee})$. We have an embedding $\Gamma_s: X^\vee \hookrightarrow \mathbb{P}_{X^\vee}(\mathbb{L} \oplus \mathbb{C})$ via the graph of $s$. Here $\mathbb{L}$ is the total space of the anticanonical bundle of $X^\vee$. The fan of the toric variety $\mathbb{P}_{X^\vee}(\mathbb{L} \oplus \mathbb{C})$ is easy to describe. Let $\tau$ be a maximal cone in the fan defining $X^\vee$ generated by $\eta_1, \eta_2, \eta_3 \in \Delta \cap M$. We put

$$\tau_0 = \text{Cone}\{(\eta_1, -1), (\eta_2, -1), (\eta_3, -1), (0, 0, 0, 1)\},$$

$$\tau_{\infty} = \text{Cone}\{(\eta_1, -1), (\eta_2, -1), (\eta_3, -1), (0, 0, 0, -1)\}.$$ 

The collection of $\tau_0$ and $\tau_{\infty}$ together with all their faces with $\tau$ running through all maximal cones in the fan for $X^\vee$ is the fan defining $\mathbb{P}_{X^\vee}(\mathbb{L} \oplus \mathbb{C})$. We denote it by $\Theta$. Then $\Theta$ is a fan in $\overline{M}_{\mathbb{R}}$, where $\overline{M} := M \times \mathbb{Z}$. Consider three sublattices

$$\overline{M}_1 := 2M \times 2\mathbb{Z} \subset \overline{M}_2 := M \times 2\mathbb{Z} \subset \overline{M} := M \times \mathbb{Z}.$$ 

**Proposition 3.7.** $Y^\vee$ can be realized as a Calabi–Yau hypersurface in $X_{\Theta, \overline{M}_2}$.

**Proof.** Note that

$$X_{\Theta, \overline{M}} \cong X_{\Theta, \overline{M}_1} \cong \mathbb{P}_{X^\vee}(\mathbb{L} \oplus \mathbb{C})$$
and the inclusion $\overline{M}_1 \subset \overline{M}$ induces a finite cover $X_{\Theta, M_1} \rightarrow X_{\Theta, M}$. We obtain the following diagram:

$$
\begin{array}{ccc}
X_{\Theta, M_1} & \xrightarrow{q} & X_{\Theta, M_2} \\
\downarrow \Phi & & \downarrow \Phi \\
X_{\Theta, \overline{M}} & \xrightarrow{p} & X_{\Theta, \overline{M}}
\end{array}
$$

Put $G := \overline{M}_2/\overline{M}_1$. We have

$$X_{\Theta, \overline{M}_1}/G \cong X_{\Theta, \overline{M}_2}.$$ 

Moreover, $p$ is a double cover branched along the union of toric divisors in $X_{\Theta, \overline{M}}$. Let $S$ be the fibred product

$$
\begin{array}{ccc}
S & \rightarrow & X_{\Theta, \overline{M}_1} \\
\downarrow & & \downarrow \Phi \\
X^\vee & \rightarrow & X_{\Theta, \overline{M}_1}
\end{array}
$$

It then follows that $Y^\vee \cong S/G \subset X_{\Theta, \overline{M}_2}$.

One can check that $\Theta$ (with respect to the integral structure $\overline{M}_2$) is a refinement of the normal fan of $\nabla_1$. In other words, there exists a MPCP desingularization $\hat{P}_{\nabla_1}$ which dominants $X_{\Theta, \overline{M}_2}$. On the other hand, one can again deform the branch locus of $Y^\vee \rightarrow X^\vee$ to obtain a smooth Calabi–Yau hypersurface $Z^\vee$ in the toric variety $X_{\Theta, \overline{M}}$ which can be viewed as an MPCP desingularization of $P_{\nabla_2}$. We thus obtain the following diagram mirror to (3.17)

$$
\begin{array}{ccc}
Z^\vee & \subset & \hat{P}_{\nabla_2} \\
\hat{P}_{\nabla_1} \supset W^\vee & \rightarrow & Y^\vee \subset X_{\Theta, \overline{M}_2}
\end{array}
$$

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