Standard versus Non-Standard CP Phases in Neutrino Oscillation in Matter with Non-Unitarity

Ivan Martinez-Soler\textsuperscript{a,b,c,d} Hisakazu Minakata\textsuperscript{a,e,f}

\textsuperscript{a}Instituto Física Teórica, UAM/CSIC, Calle Nicola’s Cabrera 13-15, Cantoblanco E-28049 Madrid, Spain
\textsuperscript{b}Theoretical Physics Department, Fermi National Accelerator Laboratory, P.O. Box 500, Batavia IL 60510, USA
\textsuperscript{c}Department of Physics and Astronomy, Northwestern University, Evanston, IL 60208, USA
\textsuperscript{d}Colegio de Física Fundamental e Interdisciplinaria de las Américas (COFI), 254 Norzagaray street, San Juan, Puerto Rico 00901
\textsuperscript{e}Center for Neutrino Physics, Department of Physics, Virginia Tech, Blacksburg, Virginia 24061, USA
\textsuperscript{f}Research Center for Cosmic Neutrinos, Institute for Cosmic Ray Research, University of Tokyo, Kashiwa, Chiba 277-8582, Japan

E-mail: ivan.martinezsoler@northwestern.edu, minakata71@vt.edu

Abstract:
To derive a structure revealing expression of the neutrino oscillation probability in matter with non-unitarity, we formulate a perturbative framework with the expansion parameters, the ratio $\epsilon$ of the solar $\Delta m^2_{sol}$ to atmospheric $\Delta m^2_{atm}$ and the parameters which describe unitarity violation (UV). Using the $\alpha$ parametrization for non-unitary mixing matrix and to first order in our perturbation theory, we show that there is a universal correlation between the $\nu$SM CP phase $\delta$ and the three UV complex $\alpha$ parameter phases. Using a phase convention of the flavor mixing matrix $U_{\text{MNS}}$ in which $e^{\pm i\delta}$ is attached to $\sin \theta_{23}$, it is expressed as $e^{-i\delta} \alpha_{\mu e}$, $\alpha_{\tau e}$, and $e^{i\delta} \alpha_{\tau \mu}$, always the same combination in all the oscillation channels. We also show that in a different $U_{\text{MNS}}$ phase convention with $e^{\pm i\delta}$ attached to $\sin \theta_{12}$, the $\delta - \alpha$ parameter phase correlation is absent. We discuss the meaning of the phase-convention dependence. Finally, we argue that the three-flavor neutrino evolution has to be unitary in the presence of non-unitary mixing matrix, and discuss how it can be reconciled with the non-unitarity of the whole system.

\textsuperscript{1}First version released while the authors were at: Instituto Física Teórica, UAM/CSIC, Calle Nicola’s Cabrera 13-15, Cantoblanco E-28049 Madrid, Spain
Contents

1 Introduction 1

2 Formulating the helio-unitarity violation (UV) perturbation theory 3
  2.1 Unitary evolution of neutrinos in the mass eigenstate basis 3
  2.2 $\alpha$ parametrization of the non-unitary mixing matrix and its convention dependence
    2.2.1 Neutrino evolution with general convention of the MNS matrix 4
    2.2.2 The three useful conventions of the MNS matrix 6
  2.3 Preliminary step toward perturbation theory: Tilde-basis 6
  2.4 Unperturbed and perturbed Hamiltonian in the tilde basis 8
  2.5 Diagonalization of zeroth-order Hamiltonian and the hat basis 9
  2.6 Flavor basis, the tilde and hat bases, the $S$ and $\hat{S}$ matrices, and their relations 10
  2.7 Calculation of $\hat{S}$ matrix 11
  2.8 Recapitulating the leading order $\tilde{S}$ matrix and the first order helio corrections 12
  2.9 Calculation of $\hat{S}^{(1)}$ and $\tilde{S}^{(1)}$ for the UV part 12

3 Neutrino oscillation probability to first order: $\nu_e - \nu_\mu$ sector 13
  3.1 $P(\nu_e \rightarrow \nu_e)^{(0+1)}_{\text{helio}}$ and $P(\nu_\mu \rightarrow \nu_e)^{(0+1)}_{\text{helio}}$: the “simple and compact” formulas 14
  3.2 $P(\nu_e \rightarrow \nu_e)^{(1)}$ and $P(\nu_\mu \rightarrow \nu_e)^{(1)}$: Both intrinsic and extrinsic UV contributions 14

4 Unitarity of neutrino evolution with first order UV corrections: $\nu_e$ row 16
  4.1 The oscillation probabilities for proving perturbative unitarity in $\nu_e$ row 17
  4.2 Perturbative unitarity of intrinsic UV contribution: $\nu_e$ row 18
  4.3 No perturbative unitarity of extrinsic UV contribution: $\nu_e$ row 18

5 How accurate are the first order formulas for $P(\nu_\beta \rightarrow \nu_\alpha)^{\text{UV}}$? 19

6 Canonical phase combination: Stability and phase convention dependence 21
  6.1 Mechanism for generating the canonical phase combination 21
  6.2 Phase convention dependence of the canonical phase combination 23
  6.3 Meaning of convention dependent phase correlation 23

7 Some additional remarks 24
  7.1 Vacuum limit 24
  7.2 Non-unitarity and Non-standard interactions (NSI) 24

8 Concluding remarks 25

A Expression of $H$ and $\Phi$ matrix elements 27
B Expressions of $\tilde{S}_{\text{UV}}^{(1)}$ matrix elements

C The oscillation probabilities in $\nu_\mu - \nu_\tau$ sector

C.1 $P(\nu_\mu \rightarrow \nu_\mu)^{(0+1)}_{\text{helio}}$ and $P(\nu_\mu \rightarrow \nu_\tau)^{(0+1)}_{\text{helio}}$: “the simple and compact” formula

C.2 $P(\nu_\mu \rightarrow \nu_\mu)^{(1)}_{\text{int-UV}}$ and $P(\nu_\mu \rightarrow \nu_\tau)^{(1)}_{\text{int-UV}}$: Intrinsic UV contribution

C.3 $P(\nu_\mu \rightarrow \nu_\mu)^{(1)}_{\text{ext-UV}}$ and $P(\nu_\mu \rightarrow \nu_\tau)^{(1)}_{\text{ext-UV}}$: Extrinsic UV contribution

C.4 Perturbative unitarity yes or no of intrinsic and extrinsic UV contributions:

\[ \nu_\mu \text{ row} \]

D Identifying the relevant variables

1 Introduction

It appears that by now the three flavor lepton mixing [1] is well established after the long term best endeavor by the experimentalists, which are recognized in an honorable way [2, 3]. Though we do not know the value of CP phase $\delta$, which we call the lepton KM phase [4], and the neutrino mass ordering, there appeared some hints toward identifying these unknowns. That is, the long-baseline (LBL) neutrino experiment T2K sees with a continuously improving confidence level (CL) that the phase $\delta$ is around the value $\sim \frac{3\pi}{2}$ [5].

This is the best place for the determination of the mass ordering, as can be seen clearly by the bi-probability plot introduced in ref. [8]. The preference of the normal mass ordering over the inverted one has been seen in the atmospheric neutrino observation by Super-Kamiokande [9], which is modestly strengthened by the ongoing LBL experiments [5, 6]. A recent global analysis [10] shows that it can be claimed at $3\sigma$ CL. Also, a reanalysis of NO$\nu$A data seem to confirm the so far dominant result that $\theta_{23}$ is near maximal [6].

The apparent convergence of various results from dozens of experiments suggests that we may reach a stage of knowing the remaining unknowns at a time earlier than we thought. It will allow us to confirm or reject the important phenomenon of lepton CP violation in a definitive way, for example, by Hyper-K [11], T2HKK [12], and DUNE [13]. Yet, it prompts us to think about how to conclude the era of discovery of neutrino mass and the lepton flavor mixing. One of the most important key elements is the paradigm test, that is, to verify the standard three flavor mixing scheme of neutrinos. As in the quark sector, unitarity test is the most popular, practical way of carrying this out.

A favourable way of performing a leptonic unitarity test is to formulate a model independent generic framework in which unitarity is violated, and confront it to the experimental data. It was attempted in a pioneering work by Antusch et al. [14], which indeed provided such a framework in the context of high-scale unitarity violation (UV). In low-scale UV,
on the other hand, the currently available model is essentially unique, the 3 active plus $N_s$ sterile model, see refs. [15, 16] for a partial list of the early references. In the present context, low and high scales imply, typically, energy scales of new physics much lower and higher than the electroweak scale, respectively. Recently, within the $(3 + N_s)$ model, a model-independent framework is created to describe neutrino propagation in vacuum [17] and in matter [18] in such a way that the observable quantities are insensitive to details of the sterile sector, e.g., its mass spectrum and active-sterile mixing.

In this paper, we construct a perturbative framework by which we can derive a simple expression of the neutrino oscillation probability in matter in the presence of UV. The framework has the two kind of expansion parameters, the ratio $\epsilon \approx \Delta m^2_{21}/\Delta m^2_{31}$ (precise definition is in eq. (2.15)), and the UV parameters, hence dubbed as the “helio-UV perturbation theory” in this paper. It can be regarded as an extension of the “renormalized helio perturbation theory” in matter to include non-unitarity [19]3, which allows us to discuss UV flavor transition of neutrinos with sizeable matter effect. It would be useful for analyzing experiments such as Super-K, Hyper-K, T2HKK, DUNE, IceCube-Gen2/PINGU, and KM3NeT-ORCA [9, 11–13, 20–22].

Introduction of non-unitary mixing matrix, which replaces the unitary MNS flavor mixing matrix in the neutrino-mass embedded standard model ($\nu$SM), brings nine additional parameters in the neutrino oscillation probability. Unraveling the correlations between the MNS and the UV parameters, as well as among the UV parameter themselves would be necessary to analyse the system with non-unitarity. It turns out that our helio-UV perturbation theory is extremely structure-revealing. That is, we will see that the lepton KM phase $\delta$ and the complex UV parameters come in into the oscillation probability in a certain fixed combination. If we use so called the $\alpha$ parametrization [23], and use a phase convention of the flavor mixing matrix $U_{\text{MNS}}$ in which $e^{\pm i\delta}$ is attached to $\sin \theta_{23}$, it is expressed as $e^{-i\delta}\alpha_{\mu e}, \alpha_{\tau e}$, and $e^{i\delta}\alpha_{\tau \mu}$, a universal feature in all the oscillation channels. It will be referred to as the “canonical phase combination” in this paper. We will see, however, the form of CP phase correlation is $U_{\text{MNS}}$ phase convention dependent.

A few remarks on high-scale vs. low-scale UV are in order: As recapitulated in [17], the notable differences between them are presence (high-scale) or absence (low-scale) of flavor non-universality and zero-distance flavor transition. In an effort toward formulating model-independent framework for testing low-scale UV the two more criteria are uncovered for distinguishing low-scale from high-scale UV. That is, presence of the probability leaking term in the oscillation probability and possible detection of UV perturbative corrections which testifies for the low-scale UV [17, 18]. See refs. [24, 25] for the current constraints on unitarity violation in low-scale UV scenario.

High-scale unitarity violation is a well studied subject with many references, only part of which is quoted here [14, 23, 25–40]. In the context of the present paper, we want to remark that the evolution equation of the three flavor active neutrinos in the mass eigenstate basis in high-scale UV, see e.g., [25], is the same as the leading order one in low-scale UV, \footnote{Having the same zeroth order Hamiltonian as the one in [19] was not expected because the neutral current reaction is involved in the Hamiltonian. But, it came out quite naturally, as will be explained in section 2.}
which can be singled out by vanishing limit of active-sterile transition elements \cite{18}. This property allows us to discuss both high-scale and the leading-order low-scale UV in the same footing.

Finally, in this paper we give a pedagogical discussion to clarify the point of how non-unitarity of the flavor mixing matrix leads to non-unitarity of the observable, the oscillation probability \( P(\nu_\beta \to \nu_\alpha) \). The answer is not totally trivial, because the neutrino evolution has to be unitary in high-scale UV, and to our knowledge this point has never been discussed explicitly in the literature. By integrating out heavy new physics sector at high scale, only the three active neutrinos remains as the neutral leptons in low energy effective theory. That is, they span the complete state space of neutral leptons at low energies.\(^4\) The completeness implies that neutrino evolution must be unitary, because there is no way to go outside of the complete neutral lepton state space during propagation, assuming absence of inelastic scattering, absorption, etc. Then, the question is: how and why the oscillation probability does not respect unitarity? We will answer these questions in the next section.

In section 2, we construct our perturbative framework with UV in matter from scratch in a step-by-step manner, and address its \( U_{\text{MNS}} \) phase convention dependence. In section 3, we compute the neutrino oscillation probabilities in the \( \nu_e - \nu_\mu \) sector to first order in the helio-UV expansion. A universal correlation between the complex \( \alpha \) parameters and the \( \nu_{\text{SM}} \) CP phase is demonstrated. An explicit proof of unitarity in neutrino evolution is given in in sections 4, and accuracy of the helio-UV expansion is examined in section 5. The stability and phase convention dependence of the phase correlation are discussed in section 6. The two miscellaneous topics, the vacuum limit and the relation with NSI, are addressed in section 7 before giving our conclusion in section 8. The oscillation probabilities in the \( \nu_\mu - \nu_\tau \) sector are calculated in appendix C. Table 1 summarizes the equation numbers of all the oscillation probabilities.

2 Formulating the helio-unitarity violation (UV) perturbation theory

Following the observation in ref. \cite{18}, we work with the neutrino evolution in 3 \( \times \) 3 active neutrino space in the vacuum mass eigenstate basis, see eq. (2.2) below. It describes both high-scale UV as well as low-scale UV in the leading (zeroth) order expansion in terms of the active-sterile transition elements (denoted as \( W \)). See, e.g., \cite{25} for the equivalent evolution equation in high-scale UV.

2.1 Unitary evolution of neutrinos in the mass eigenstate basis

The three active neutrino evolution in matter in the presence of non-unitary flavor mixing can be described by the Schrödinger equation in the vacuum mass eigenstate basis \cite{18, 25}

\[
\frac{d}{dx} \tilde{\nu} = \frac{1}{2E} \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Delta m^2_{21} & 0 \\ 0 & 0 & \Delta m^2_{31} \end{bmatrix} + N^\dagger \begin{bmatrix} a - b & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & -b \end{bmatrix} N \right\} \tilde{\nu}.
\]

\(^4\) Since the structure of neutral lepton state space is the same as in the low-scale UV at its leading order in the \( W \) expansion \cite{18}, the above and subsequent discussions equally applies to the case of low-scale UV at the leading order as well.
In this paper, we denote the vacuum mass eigenstate basis as the "check basis". In eq. (2.1), $N$ denotes the $3 \times 3$ non-unitary flavor mixing matrix which relates the flavor neutrino states to the vacuum mass eigenstates as

$$\nu_\alpha = N_{\alpha i} \tilde{\nu}_i.$$  \hspace{1cm} (2.2)

Hereafter, the subscript Greek indices $\alpha, \beta$, or $\gamma$ run over $e, \mu, \tau$, and the Latin indices $i, j$ run over the mass eigenstate indices 1, 2, and 3. $E$ is neutrino energy and $\Delta m^2_{ji} \equiv m^2_j - m^2_i$. The usual phase redefinition of neutrino wave function is done to leave only the mass squared differences.

The functions $a(x)$ and $b(x)$ in eq. (2.1) denote the Wolfenstein matter potential \[41\] due to charged current (CC) and neutral current (NC) reactions, respectively.

$$a = 2\sqrt{2}G_F N_e E \approx 1.52 \times 10^{-4} \left( \frac{Y_e \rho}{\text{g cm}^{-3}} \right) \left( \frac{E}{\text{GeV}} \right) \text{eV}^2,$$

$$b = \sqrt{2}G_F N_n E = \frac{1}{2} \left( \frac{N_n}{N_e} \right) a.$$  \hspace{1cm} (2.3)

Here, $G_F$ is the Fermi constant, $N_e$ and $N_n$ are the electron and neutron number densities in matter. $\rho$ and $Y_e$ denote, respectively, the matter density and number of electrons per nucleon in matter. For simplicity and clarity we will work with the uniform matter density approximation in this paper. But, it is not difficult to extend our treatment to varying matter density case if adiabaticity holds.

By writing the evolution equation as in eq. (2.1) with the hermitian Hamiltonian, the neutrino evolution is obviously unitary, which is in agreement with our discussion given at the end of section 1. Then, the answer to the remaining question, "how the effect of non-unitarity comes in into the observables as a consequence of non-unitary mixing matrix" is given in section 2.6.

### 2.2 $\alpha$ parametrization of the non-unitary mixing matrix and its convention dependence

To parametrize the non-unitary $N$ matrix we use the so-called $\alpha$ parametrization \[23\], $N = (1 - \alpha) U$, where $U \equiv U_{\text{MNS}}$ denotes the SM $3 \times 3$ unitary flavor mixing matrix.\[^5\] To define the $\alpha$ matrix, however, we must specify the phase convention by which $U$ matrix is defined.

We start from the most commonly used form, the Particle Data Group (PDG) \[46\] convention of the MNS matrix,

$$U_{\text{PDG}} = \begin{bmatrix}
1 & 0 & 0 \\
0 & c_{23} & s_{23} \\
0 & -s_{23} & c_{23}
\end{bmatrix} \begin{bmatrix}
\begin{array}{c}
c_{13} \\
0 \\
-s_{13} e^{-i\delta}
\end{array} & \begin{array}{c}
s_{13} e^{-i\delta} \\
0 \\
c_{13}
\end{array}
\end{bmatrix} \begin{bmatrix}
\begin{array}{c}
c_{12} & s_{12} \\
-s_{12} & c_{12}
\end{array} & 0 \\
0 & 1
\end{bmatrix},$$  \hspace{1cm} (2.4)

\[^5\] For early references for parametrizing the UV effect, see e.g., \[26, 42–45\]. It must be remarked that the authors of ref. \[25\] made an important point in explaining why the $\alpha$ parametrization is more superior than their traditional way of using the hermitian $\eta$ matrix.
with the obvious notations $s_{ij} \equiv \sin \theta_{ij}$ etc. and $\delta$ being the CP violating phase. Then, we define the non-unitary mixing matrix $N_{\text{PDG}}$ as

$$N_{\text{PDG}} = (1 - \bar{\alpha}) U_{\text{PDG}} = \left\{ 1 - \begin{bmatrix} \bar{\alpha}_{ee} & 0 & 0 \\ \bar{\alpha}_{\mu e} & \bar{\alpha}_{\mu\mu} & 0 \\ \bar{\alpha}_{\tau e} & \bar{\alpha}_{\tau\mu} & \bar{\alpha}_{\tau\tau} \end{bmatrix} \right\} U_{\text{PDG}}. \quad (2.5)$$

By inserting $N = N_{\text{PDG}}$ in (2.5) to eq. (2.1), we define the neutrino evolution equation in the vacuum mass eigenstate basis.

**2.2.1 Neutrino evolution with general convention of the MNS matrix**

After reducing the standard three-flavor mixing matrix to $U_{\text{PDG}}$, which has four degree of freedom, we still have freedom of phase redefinition

$$\tilde{\nu} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{i\beta} & 0 \\ 0 & 0 & e^{i\gamma} \end{bmatrix} \tilde{\nu} \equiv \Gamma (\beta, \gamma) \tilde{\nu} \quad (2.6)$$

without affecting physics of the system. Then, the evolution equation in the $\Gamma (\beta, \gamma)$ transformed basis,

$$\frac{d}{dx} \tilde{\nu} = \frac{1}{2E} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Delta m_{21}^2 & 0 \\ 0 & 0 & \Delta m_{31}^2 \end{bmatrix} \tilde{\nu} + \frac{1}{2E} U(\beta, \gamma)^\dagger \begin{bmatrix} a - b & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & -b \end{bmatrix} \{1 - \alpha(\beta, \gamma)\} U(\beta, \gamma) \tilde{\nu}, \quad (2.7)$$

describes the same physics. In (2.7), $U(\beta, \gamma)$ and $\alpha(\beta, \gamma)$ denote, respectively, the $\Gamma (\beta, \gamma)$ transformed MNS matrix and $\bar{\alpha}$ matrix:

$$U(\beta, \gamma) \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-i\beta} & 0 \\ 0 & 0 & e^{-i\gamma} \end{bmatrix} \quad U_{\text{PDG}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{i\beta} & 0 \\ 0 & 0 & e^{i\gamma} \end{bmatrix}$$

$$\alpha(\beta, \gamma) \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-i\beta} & 0 \\ 0 & 0 & e^{-i\gamma} \end{bmatrix} \quad \bar{\alpha} \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{i\beta} & 0 \\ 0 & 0 & e^{i\gamma} \end{bmatrix} \quad (2.8)$$

That is, we can use different convention of the MNS matrix $U(\beta, \gamma)$, but then our $\alpha$ matrix has to be changed accordingly, as in (2.8).
The reason for our naming of the MNS matrix. In addition to $U_{\text{PDG}}$ in (2.4), they are $U(0, \delta)$ and $U(\delta, \delta)$:

\[
U_{\text{ATM}} \equiv U(0, \delta) = \begin{pmatrix}
1 & 0 & 0 \\
0 & c_{23} & s_{23} e^{-i\delta} \\
0 & -s_{23} e^{i\delta} & c_{23}
\end{pmatrix}
= \begin{pmatrix}
c_{13} & 0 & s_{13} \\
0 & 1 & 0 \\
-s_{13} & 0 & c_{13}
\end{pmatrix}
\begin{pmatrix}
c_{12} & s_{12} e^{i\delta} & 0 \\
-s_{12} e^{-i\delta} & c_{12} & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

\[
U_{\text{SOL}} \equiv U(\delta, \delta) = \begin{pmatrix}
1 & 0 & 0 \\
0 & c_{23} & s_{23} \\
0 & -s_{23} & c_{23}
\end{pmatrix}
= \begin{pmatrix}
c_{13} & 0 & s_{13} \\
0 & 1 & 0 \\
-s_{13} & 0 & c_{13}
\end{pmatrix}
\begin{pmatrix}
c_{12} & s_{12} e^{i\delta} & 0 \\
-s_{12} e^{-i\delta} & c_{12} & 0 \\
0 & 0 & 1
\end{pmatrix}. \tag{2.9}
\]

The reason for our naming of $U_{\text{ATM}}$ and $U_{\text{SOL}}$ in (2.9) is because CP phase $\delta$ is attached to the “atmospheric angle” $s_{23}$ in $U_{\text{ATM}}$, and to the “solar angle” $s_{12}$ in $U_{\text{SOL}}$, respectively. Whereas in $U_{\text{PDG}}$, $\delta$ is attached to $s_{13}$.

Accordingly, we have the three different definition of the $\alpha$ matrix. In addition to $N_{\text{PDG}} = (1 - \tilde{\alpha}) U_{\text{PDG}}$ as in (2.5), we have $N_{\text{ATM}} = (1 - \alpha_{\text{ATM}}) U_{\text{ATM}}$, and $N_{\text{SOL}} = (1 - \alpha_{\text{SOL}}) U_{\text{SOL}}$. The latter two and their relations to $\tilde{\alpha}$ are given by

\[
\alpha_{\text{ATM}} = \alpha(0, \delta) \equiv \begin{pmatrix}
\alpha_{ee} & 0 & 0 \\
\alpha_{\mu e} & \alpha_{\mu\mu} & 0 \\
\alpha_{\tau e} & \alpha_{\tau\mu} & \alpha_{\tau\tau}
\end{pmatrix}
= \begin{pmatrix}
\tilde{\alpha}_{ee} & 0 & 0 \\
\tilde{\alpha}_{\mu e} & \tilde{\alpha}_{\mu\mu} & 0 \\
\tilde{\alpha}_{\tau e} e^{-i\delta} & \tilde{\alpha}_{\tau\mu} e^{-i\delta} & \tilde{\alpha}_{\tau\tau}
\end{pmatrix},
\]

\[
\alpha_{\text{SOL}} = \alpha(\delta, \delta) \equiv \begin{pmatrix}
\tilde{\alpha}_{ee} & 0 & 0 \\
\tilde{\alpha}_{\mu e} & \tilde{\alpha}_{\mu\mu} & 0 \\
\tilde{\alpha}_{\tau e} e^{i\delta} & \tilde{\alpha}_{\tau\mu} e^{i\delta} & \tilde{\alpha}_{\tau\tau}
\end{pmatrix}. \tag{2.10}
\]

In this paper, for convenience of the calculations, we take the “ATM” convention with $U_{\text{ATM}}$ and $\alpha_{\text{ATM}}$. But, the translation of our results to the PDG or the “SOL” conventions can be done easily by using eq. (2.10).

Notice that, because of the structure $N_{\text{ATM}} = (1 - \alpha_{\text{ATM}}) U_{23} U_{13} U_{12}$, the $\alpha$ matrix is always attached to $U_{23}$. Then, the correlation between the lepton KM phase $\delta$ and the UV parameter phases becomes more transparent if $e^{\pm i\delta}$ is attached to $U_{23}$. This is the reason why we take the MNS matrix convention $U_{\text{ATM}}$ in (2.9) in our following calculation.

2.3 Preliminary step toward perturbation theory: Tilde-basis

Taking the $U_{\text{ATM}}$ convention with $\alpha_{\text{ATM}}$ matrix, we formulate our helio-UV perturbation theory. We assume that deviation from unitarity is small, so that $\alpha_{\beta\gamma} \ll 1$ hold for all flavor indices $\beta$ and $\gamma$ including the diagonal ones. Therefore, we are able to use the two kind of expansion parameters, $\epsilon \approx \Delta m_{21}^2 / \Delta m_{31}^2$ (see eq. (2.15) below) and the $\alpha$ parameters in our helio-UV perturbation theory.

We define the following notations for simplicity to be used in the discussions hereafter in this paper:

\[
\Delta_{ji} \equiv \frac{\Delta m_{ji}^2}{2E}, \quad \Delta_{a} \equiv \frac{a}{2E}, \quad \Delta_{b} \equiv \frac{b}{2E}. \tag{2.11}
\]
For convenience in formulating the helio-UV perturbation theory, we move from the check basis to an intermediate basis, which we call the “tilde basis”, 6 \( \tilde{\nu} = (U_{13}U_{12})\bar{\nu} \), with Hamiltonian

\[
\tilde{H} = (U_{13}U_{12})\bar{H}(U_{13}U_{12})^\dagger = (U_{13}U_{12}) \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Delta_{21} & 0 \\ 0 & 0 & \Delta_{31} \end{bmatrix} (U_{13}U_{12})^\dagger + U_{23}^\dagger \begin{bmatrix} 1 - \begin{bmatrix} \alpha_{ee} & \alpha_{\mu e} & \alpha_{\tau e} \\ 0 & \alpha_{\mu\mu} & \alpha_{\tau\mu} \\ 0 & 0 & \alpha_{\tau\tau} \end{bmatrix} \end{bmatrix} \Delta_{a} - \Delta_{b} 0 0 \begin{bmatrix} 1 - \begin{bmatrix} \alpha_{ee} & 0 & 0 \\ \alpha_{\mu e} & \alpha_{\mu\mu} & 0 \\ \alpha_{\tau e} & \alpha_{\tau\mu} & \alpha_{\tau\tau} \end{bmatrix} \end{bmatrix} U_{23} \]

\[
= \tilde{H}_{\text{vac}} + \tilde{H}_{\text{UV}}. \tag{2.12}
\]

In the last line, we have denoted the first and the second terms in eq. (2.12) as \( \tilde{H}_{\text{vac}} \) and \( \tilde{H}_{\text{UV}} \), respectively. The explicit form of the \( \tilde{H}_{\text{vac}} \) in a form decomposed into the unperturbed and perturbed parts is given by

\[
\tilde{H}_{\text{vac}}^{(0)}(x) = \Delta_{\text{ren}} \begin{bmatrix} s_{13}^2 & 0 & c_{13}s_{13} \\ 0 & 0 & 0 \\ c_{13}s_{13} & 0 & c_{13}^2 \end{bmatrix} + \epsilon \begin{bmatrix} s_{12}^2 & 0 & 0 \\ 0 & c_{12}^2 & 0 \\ 0 & 0 & s_{12}^2 \end{bmatrix}, \tag{2.13}
\]

\[
\tilde{H}_{\text{vac}}^{(1)}(x) = \epsilon c_{12}s_{12}\Delta_{\text{ren}} \begin{bmatrix} 0 & c_{13} & 0 \\ c_{13} & 0 & -s_{13} \\ 0 & -s_{13} & 0 \end{bmatrix}, \tag{2.14}
\]

where

\[
\Delta_{\text{ren}} \equiv \frac{\Delta m_{\text{ren}}^2}{2E}, \quad \Delta m_{\text{ren}}^2 \equiv \Delta m_{31}^2 - s_{12}^2 \Delta m_{21}^2, \\
\epsilon \equiv \frac{\Delta m_{21}^2}{\Delta m_{\text{ren}}^2}. \tag{2.15}
\]

The superscripts (0) and (1) in eqs. (2.13) and (2.14), respectively, show that they are zeroth and first order in \( \epsilon \). After transforming to the tilde basis, as we expected, we recover the same Hamiltonian as used in the “renormalized helio perturbation theory” without unitarity violation [19]. An order \( \epsilon \) term is intentionally absorbed into the zeroth-order term in \( \tilde{H}_{\text{vac}}^{(0)} \) as in eq. (2.13) to make the formulas of the oscillation probabilities simple and compact.

We note that the matter term \( \tilde{H}_{\text{UV}} \) in eq. (2.12) can be decomposed into the zeroth, first and the second order terms in \( \alpha \) (or \( \tilde{\alpha} \)) matrix elements as \( \tilde{H}_{\text{UV}} = \tilde{H}_{\text{matt}}^{(0)} + \tilde{H}_{\text{UV}}^{(1)} + \)

\[+ \tilde{H}_{\text{matt}}^{(2)} \]

\[= (U_{23}H_{\text{flavor}}U_{23})^\dagger \]

\[= \tilde{H}_{\text{vac}} + \tilde{H}_{\text{UV}}. \tag{2.12}\]

From the flavor basis, the tilde basis is \( U_{23} \) transformed basis, \( \tilde{\nu}_\alpha = (U_{23})_{\alpha\beta}\bar{\nu}_\beta \) and \( \tilde{H} = U_{23}^\dagger H_{\text{flavor}}U_{23} \), which is commonly used in various treatments of neutrino propagation in matter.

\[\text{– 7 –}\]
\[ 
\hat{H}_{UV}^{(2)} : \\
\hat{H}_{\text{matt}}^{(0)} = \begin{bmatrix}
\Delta_a - \Delta_b & 0 & 0 \\
0 & -\Delta_b & 0 \\
0 & 0 & -\Delta_b 
\end{bmatrix}, \\
\hat{H}_{UV}^{(1)} = U_{23}^{\dagger} \begin{bmatrix}
\Delta_b & 2\alpha_{ee} \left(1 - \frac{\Delta_a}{\Delta_b}\right) & \alpha_{ee}^* \alpha_{\tau\tau} \\
\alpha_{ee} & 2\alpha_{\mu\mu} & \alpha_{\mu\mu}^* \alpha_{\tau\tau} \\
\alpha_{\tau\tau} & 2\alpha_{\tau\tau} & 2\alpha_{\tau\tau} 
\end{bmatrix} U_{23}, \\
\hat{H}_{UV}^{(2)} = -U_{23}^{\dagger} \begin{bmatrix}
\Delta_b & \alpha_{ee}^2 \left(1 - \frac{\Delta_a}{\Delta_b}\right) + |\alpha_{ee}|^2 + |\alpha_{\tau\tau}|^2 & \alpha_{ee}^* \alpha_{\tau\tau} \\
\alpha_{ee} \alpha_{\mu\mu} + \alpha_{\tau\tau} \alpha_{\tau\tau}^* & \alpha_{\mu\mu}^2 + |\alpha_{\mu\mu}|^2 & \alpha_{\tau\tau}^* \alpha_{\tau\tau} \\
\alpha_{\tau\tau} \alpha_{\tau\tau} & \alpha_{\tau\tau}^2 & \alpha_{\tau\tau}^2 
\end{bmatrix} U_{23}.
\]

The total Hamiltonian in the tilde basis is, therefore, given by \( \hat{H} = \hat{H}_{\text{vac}} + \hat{H}_{UV} \), where \( \hat{H}_{\text{vac}} = \hat{H}_{\text{vac}}^{(0)} + \hat{H}_{\text{vac}}^{(1)} \).

2.4 Unperturbed and perturbed Hamiltonian in the tilde basis

To formulate the helio-UV perturbation theory, we decompose the tilde basis Hamiltonian in the following way:

\[ \hat{H} = \hat{H}^{(0)} + \hat{H}^{(1)}. \] (2.17)

The unperturbed (zeroth-order) Hamiltonian is given by \( \hat{H}^{(0)} = \hat{H}_{\text{vac}}^{(0)} + \hat{H}_{\text{matt}}^{(0)} \). We make a phase redefinition

\[ \tilde{\nu} = \exp \left[ i \int x^\prime dx^\prime \Delta_b(x^\prime) \right] \nu^\prime \] (2.18)

which is valid even for non-uniform matter density. Then, the Schrödinger equation for \( \nu^\prime \) becomes the form in eq. (2.2) with unperturbed part of the Hamiltonian \( (\hat{H}^{(0)})^\prime \) as given in

\[ (\hat{H}^{(0)})^\prime = \Delta_{\text{ren}} \begin{bmatrix}
\frac{a(x)}{\Delta m_{\text{sol}}} + s_{13}^2 & 0 & c_{13} s_{13} \\
0 & 0 & 0 \\
c_{13} s_{13} & 0 & c_{13}^2 
\end{bmatrix} + \epsilon \begin{bmatrix}
s_{12}^2 & 0 & 0 \\
0 & c_{12}^2 & 0 \\
0 & 0 & s_{12}^2 
\end{bmatrix}, \] (2.19)

namely, without NC matter potential terms. It is evident that the phase redefinition does not affect the physics of flavor change. Hereafter, we omit the prime symbol and use the zeroth-order Hamiltonian eq. (2.19) without NC term. This is nothing but the zeroth order Hamiltonian used in [19], which led to the “simple and compact” formulas of the oscillation probabilities in the standard three-flavor mixing.

The perturbed Hamiltonian is then given by

\[ \hat{H}^{(1)} = \hat{H}_{\text{vac}}^{(1)} + \hat{H}_{UV}^{(1)} + \hat{H}_{UV}^{(2)} \] (2.20)

where each term in eq. (2.20) is defined in eqs. (2.14) and (2.16). In the actual computation, we drop the second-order term (the last term) in eq. (2.20) because we confine ourselves into the zeroth and first order terms in the UV parameters in this paper.
2.5 Diagonalization of zeroth-order Hamiltonian and the hat basis

To carry out perturbative calculation, it is convenient to transform to a basis which diagonalizes $\tilde{H}^{(0)}$, which we call the “hat basis”. $\tilde{H}^{(0)}$ is diagonalized by the unitary transformation as follows:

$$\hat{H}_0 = U_\phi^\dagger \tilde{H}_0 U_\phi = \begin{bmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{bmatrix},$$

(2.21)

where the eigenvalues $h_i$ are given by

$$h_1 = \frac{1}{2} \left[ (\Delta_{\text{ren}} + \Delta_a) - \text{sign}(\Delta_{\text{ren}}^2) \sqrt{(\Delta_{\text{ren}} - \Delta_a)^2 + 4s_{13}^2 \Delta_{\text{ren}} \Delta_a} \right] + \epsilon \Delta_{\text{ren}} s_{12}^2,$$

$$h_2 = \epsilon c_{12} \Delta_{\text{ren}},$$

(2.22)

$$h_3 = \frac{1}{2} \left[ (\Delta_{\text{ren}} + \Delta_a) + \text{sign}(\Delta_{\text{ren}}^2) \sqrt{(\Delta_{\text{ren}} - \Delta_a)^2 + 4s_{13}^2 \Delta_{\text{ren}} \Delta_a} \right] + \epsilon \Delta_{\text{ren}} s_{12}^2.$$

See eqs. (2.11) and (2.15) for the definitions of $\Delta_{\text{ren}}$, $\Delta_a$ etc. By the convention with $\text{sign}(\Delta_{\text{ren}}^2)$, we can treat the normal and the inverted mass orderings in a unified way. The foregoing and the following treatment of the system without the UV $\alpha$ parameters in this section, which recapitulates the one in ref. [19], is to make description in this paper self-contained.

$U_\phi$ is parametrized as

$$U_\phi = \begin{bmatrix} \cos \phi & 0 & \sin \phi \\ 0 & 1 & 0 \\ -\sin \phi & 0 & \cos \phi \end{bmatrix}.$$  \hspace{1cm} (2.23)

where $\phi$ is nothing but the mixing angle $\theta_{13}$ in matter. With the definitions of the eigenvalues eq. (2.22), the following mass-ordering independent expressions for cosine and sine $2\phi$ are obtained:

$$\cos 2\phi = \frac{\Delta_{\text{ren}} \cos 2\theta_{13} - \Delta_a}{h_3 - h_1},$$

$$\sin 2\phi = \frac{\Delta_{\text{ren}} \sin 2\theta_{13}}{h_3 - h_1}.$$  \hspace{1cm} (2.24)

The perturbing Hamiltonian in vacuum in the tilde basis, $\tilde{H}_{\text{vac}}^{(1)}$, has a simple form such that the positions of “zeros” are kept after transformed into the hat basis:

$$\hat{H}_{\text{vac}}^{(1)} = U_\phi^\dagger \tilde{H}_{\text{vac}}^{(1)} U_\phi = \epsilon c_{12} s_{12} \Delta_{\text{ren}} \begin{bmatrix} 0 & \cos (\phi - \theta_{13}) & 0 \\ \cos (\phi - \theta_{13}) & 0 & \sin (\phi - \theta_{13}) \\ 0 & \sin (\phi - \theta_{13}) & 0 \end{bmatrix}.$$  \hspace{1cm} (2.25)
In fact, $\hat{H}_1$ is identical to $\tilde{H}_1$ with $\theta_{13}$ replaced by $(\theta_{13} - \phi)$. However, the form of $\hat{H}^{(1)}_{UV}$ is somewhat complicated,

$$\hat{H}^{(1)}_{UV} = U_\phi^\dagger \tilde{H}^{(1)}_{UV} U_\phi \equiv \Delta_b U_\phi^\dagger H U_\phi$$  \hspace{1cm} (2.26)

where we have defined $H$ matrix

$$H \equiv \begin{pmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{pmatrix} = U_{23}^\dagger \begin{pmatrix} 2\alpha_{ee} \left(1 - \frac{\Delta_a}{\Delta_b}\right) & \alpha_{\mu e}^* & \alpha_{\tau e}^* \\ \alpha_{\mu e} & 2\alpha_{\mu \mu} & \alpha_{\tau \mu}^* \\ \alpha_{\tau e} & \alpha_{\tau \mu} & 2\alpha_{\tau \tau} \end{pmatrix} U_{23}.  \hspace{1cm} (2.27)$$

The explicit expressions of the elements $H_{ij}$ are given in appendix A.

### 2.6 Flavor basis, the tilde and hat bases, the $S$ and $\hat{S}$ matrices, and their relations

We summarize the relationship between the flavor basis, the check (vacuum mass eigenstate) basis, the tilde, and the hat (zeroth order diagonalized Hamiltonian) basis.

Only the unitary transformations are involved in changing from the hat basis to the tilde basis, and from the tilde basis to the check basis:

$$\hat{H} = U_\phi^\dagger \hat{H} U_\phi, \quad \text{or} \quad \hat{H} = U_\phi \hat{H} U_\phi^\dagger, \quad H = (U_{13} U_{12}) \tilde{H} (U_{13} U_{12})^\dagger, \quad \text{or} \quad \tilde{H} = (U_{13} U_{12})^\dagger \tilde{H} (U_{13} U_{12}).  \hspace{1cm} (2.28)$$

The non-unitary transformation is involved from the check basis to the flavor basis:

$$\nu_\alpha = N_{\alpha i} \tilde{\nu}_i = \{(1 - \alpha)U\}_\alpha \tilde{\nu}_i.  \hspace{1cm} (2.29)$$

The relationship between the flavor basis Hamiltonian $H_{\text{flavor}}$ and the hat basis one $\hat{H}$ is

$$H_{\text{flavor}} = \{(1 - \alpha)U\} \hat{H} \{(1 - \alpha)U\}^\dagger = (1 - \alpha)U_{23} U_\phi \hat{H} U_\phi^\dagger U_{23}^\dagger (1 - \alpha)^\dagger.  \hspace{1cm} (2.30)$$

Then, the flavor basis $S$ matrix is related to $\hat{S}$ and $\tilde{S}$ matrices as

$$S = (1 - \alpha) U_{23} U_\phi \hat{S} U_\phi^\dagger U_{23}^\dagger (1 - \alpha)^\dagger = (1 - \alpha) U_{23} \tilde{S} U_{23}^\dagger (1 - \alpha)^\dagger.  \hspace{1cm} (2.31)$$

Notice that both $\hat{S}$ and $\tilde{S}$ are unitary, but $S$ is not because of non-unitarity of the $(1 - \alpha)$ matrix.

This is the answer to the question we posed in section 1. Namely, the non-unitarity of $S$ matrix in the flavor basis, whose square is the observable, comes from the initial projection from the flavor- to mass-basis and the final projection back from the mass- to flavor-eigenstate. Notice that there is no other way, because neutrino evolution has to be unitary, as discussed in section 1.\footnote{We do not assume that our discussion affects the formulas used so far in the treatment of high-scale UV, and it is perfectly consistent with that in ref. \cite{25}, for example. Our discussion just aims at serving for a transparent understanding of the point, how neutrino’s unitary evolution is reconciled with non-unitary nature of the observable $P(\nu_\beta \rightarrow \nu_\alpha)$. We will see below and in the next section that at first order in the UV parameter expansion a clear separation between the unitary and non-unitaly part of the oscillation probability occurs.}
Because of the reasoning above, we denote \( U_{23} \tilde{S} U_{23}^\dagger \) as the “propagation-\( S \) matrix”. For convenience, we write down explicitly all the pieces in eq. (2.31), the propagation-\( S \) matrix in terms of the \( \tilde{S} \) elements [19]:

\[
(U_{23} \tilde{S} U_{23}^\dagger)_{ee} = \tilde{S}_{ee},
\]
\[
(U_{23} \tilde{S} U_{23}^\dagger)_{e\mu} = c_{23} \tilde{S}_{e\mu} + s_{23} e^{-i\delta} \tilde{S}_{e\tau},
\]
\[
(U_{23} \tilde{S} U_{23}^\dagger)_{e\tau} = c_{23} \tilde{S}_{e\tau} - s_{23} e^{i\delta} \tilde{S}_{e\mu},
\]
\[
(U_{23} \tilde{S} U_{23}^\dagger)_{\mu\mu} = c_{23} \tilde{S}_{\mu\mu} + s_{23} e^{i\delta} \tilde{S}_{\mu\tau},
\]
\[
(U_{23} \tilde{S} U_{23}^\dagger)_{\mu\tau} = c_{23} \tilde{S}_{\mu\tau} - s_{23} e^{-i\delta} \tilde{S}_{\mu\mu}.
\]

We note that \( \tilde{S} \), which can be expanded by the small parameters, \( \epsilon \) and \( \alpha \), as

\[
\tilde{S} = \tilde{S}^{(0)} + \tilde{S}^{(1)}_{\text{helio}} + \tilde{S}^{(1)}_{\text{UV}}.
\]

To first order in these small parameters, we obtain

\[
S = U_{23} \tilde{S}^{(0)} U_{23}^\dagger + U_{23} \left( \tilde{S}^{(1)}_{\text{helio}} + \tilde{S}^{(1)}_{\text{UV}} \right) U_{23}^\dagger - \alpha U_{23} \tilde{S}^{(0)} U_{23}^\dagger - U_{23} \tilde{S}^{(0)} U_{23}^\dagger \alpha^\dagger.
\]

We shall call the \( \tilde{S}^{(1)}_{\text{UV}} \) piece in the second term “intrinsic” UV contribution, and the last two terms in eq. (2.34) as “extrinsic” UV contribution. The intrinsic UV contribution is in unitary part, and the extrinsic UV contribution represents non-unitary effect. Finally, the oscillation probabilities are simply given by

\[
P(\nu_\beta \rightarrow \nu_\alpha; x) = |S_{\alpha\beta}(x)|^2.
\]

### 2.7 Calculation of \( \tilde{S} \) matrix

To calculate \( \tilde{S}(x) \) we define \( \Omega(x) \) as

\[
\Omega(x) = e^{i\tilde{H}_0 x} \tilde{S}(x).
\]

Then, \( \Omega(x) \) obeys the evolution equation

\[
i \frac{d}{dx} \Omega(x) = H_1 \Omega(x)
\]

where

\[
H_1 \equiv e^{i\tilde{H}_0 x} \hat{H}_1 e^{-i\tilde{H}_0 x}.
\]

where \( \hat{H}_1 = \hat{H}^{(1)}_{\text{vac}} + \hat{H}^{(1)}_{\text{UV}}. \) See eqs. (2.25) and (2.26). Then, \( \Omega(x) \) can be computed perturbatively as

\[
\Omega(x) = 1 + (-i) \int_0^x dx' H_1(x') + (-i)^2 \int_0^x dx' H_1(x') \int_0^{x'} dx'' H_1(x'') + \cdots,
\]

and the \( \tilde{S} \) matrix is given by

\[
\tilde{S}(x) = e^{-i\tilde{H}_0 x} \Omega(x).
\]
2.8 Recapitulating the leading order $\tilde{S}$ matrix and the first order helio corrections

Since all the relevant quantities are computed for the leading order and the helio corrections in ref. [19], we just recapitulate them in below. The zeroth order result of $\tilde{S}$ matrix is given by

$$\tilde{S}^{(0)} = U_\phi e^{-iH_0x}U_\phi^\dagger = \begin{bmatrix} c_\phi^2 e^{-ih_{1x}} + s_\phi^2 e^{-ih_{3x}} & c_\phi s_\phi (e^{-ih_{3x}} - e^{-ih_{1x}}) \\ 0 & c_\phi^2 e^{-ih_{3x}} + s_\phi^2 e^{-ih_{1x}} \end{bmatrix}$$ (2.41)

where $c_\phi \equiv \cos \phi$ and $s_\phi \equiv \sin \phi$. The non-vanishing first order helio corrections (order $\sim \epsilon$) to $\tilde{S}$ matrix are given by

$$\left( \tilde{S}^{(1)}_{\text{helio}} \right)_{\epsilon \mu} = \left( \tilde{S}^{(1)}_{\text{helio}} \right)_{\mu \epsilon} = \epsilon \Delta_{\text{ren}} C_{12} s_{12} \begin{bmatrix} c_\phi^{-1} (\phi - \theta_{13}) e^{-ih_{2x}} - e^{-ih_{1x}} & e^{-ih_{3x}} - e^{-ih_{2x}} \\ c_\phi s_\phi (\phi - \theta_{13}) & e^{-ih_{1x}} - e^{-ih_{2x}} \end{bmatrix},$$

$$\left( \tilde{S}^{(1)}_{\text{helio}} \right)_{\mu \tau} = \left( \tilde{S}^{(1)}_{\text{helio}} \right)_{\tau \mu} = \epsilon \Delta_{\text{ren}} C_{12} s_{12} \begin{bmatrix} -s_\phi (\phi - \theta_{13}) & c_\phi s_\phi (\phi - \theta_{13}) e^{-ih_{1x}} - e^{-ih_{2x}} \\ h_2 - h_1 & \end{bmatrix},$$ (2.42)

and all the other elements vanish. The elements of the propagation-$S$ matrix $U_{23} \tilde{S} U_{23}^\dagger$ can be obtained from $\tilde{S}$ matrix elements by using eq. (2.32).

2.9 Calculation of $\tilde{S}^{(1)}$ and $\tilde{S}^{(2)}$ for the UV part

In this paper, we restrict ourselves to the perturbative calculation to first order in $\epsilon \equiv \Delta m_{32}^2/\Delta m_{\text{ren}}^2 \approx \Delta m_{21}^2/\Delta m_{\text{ren}}^2$ and to first order in the UV parameters $\alpha_{\beta\gamma}$. Then, the form of $S$ matrix and the oscillation probability in zeroth and the first-order helio corrections are identical with those computed in ref. [19]. Therefore, we only calculate, in the rest of this section, the matter part which produces the UV contributions.

By inserting $U_\phi^\dagger U_\phi$, $H_1$ (hereafter the matter part only) can be written as

$$H_1 = \Delta_b U_\phi^\dagger U_\phi e^{iH_0x} U_\phi^\dagger U_\phi e^{-iH_0x} U_\phi$$

$$= \Delta_b U_\phi \begin{bmatrix} c_\phi^2 e^{ih_{1x}} + s_\phi^2 e^{ih_{3x}} & 0 \\ 0 & c_\phi^2 e^{ih_{3x}} + s_\phi^2 e^{ih_{1x}} \end{bmatrix} \begin{bmatrix} h_1 & h_2 \\ h_2 & h_3 \end{bmatrix}$$

$$\times \begin{bmatrix} c_\phi^2 e^{-ih_{1x}} + s_\phi^2 e^{-ih_{3x}} & 0 \\ 0 & c_\phi^2 e^{-ih_{3x}} + s_\phi^2 e^{-ih_{1x}} \end{bmatrix} U_\phi$$

$$\equiv \Delta_b U_\phi \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} \\ \Phi_{21} & \Phi_{22} & \Phi_{23} \\ \Phi_{31} & \Phi_{32} & \Phi_{33} \end{bmatrix} U_\phi \equiv \Delta_b U_\phi \Phi U_\phi,$$ (2.43)

where we have introduced another simplifying matrix notation $\Phi$ and its elements $\Phi_{ij}$. The explicit expressions of $\Phi_{ij}$ are given in appendix A.
Assuming the uniform matter density, we obtain

\[
\hat{S}(x)^{(1)}_{\text{matt}} = e^{-iH_0x}U(x)^{(1)}_{\text{matt}} = \Delta SU^\dagger \phi U \phi e^{-iH_0x}U^\dagger \phi \left[ (-i) \int_0^x dx' \Phi(x') \right] U \phi
\]

\[
= \Delta SU^\dagger \phi \begin{bmatrix} c^2_\phi e^{-ih_1x} + s^2_\phi e^{-ih_3x} & 0 & c_\phi s_\phi (e^{-ih_3x} - e^{-ih_1x}) \\ 0 & e^{-ih_2x} & 0 \\ c_\phi s_\phi (e^{-ih_3x} - e^{-ih_1x}) & 0 & c^2_\phi e^{-ih_3x} + s^2_\phi e^{-ih_1x} \end{bmatrix}
\]

\[
\times (-i) \int_0^x dx' \begin{bmatrix} \Phi_{11}(x') & \Phi_{12}(x') & \Phi_{13}(x') \\ \Phi_{21}(x') & \Phi_{22}(x') & \Phi_{23}(x') \\ \Phi_{31}(x') & \Phi_{32}(x') & \Phi_{33}(x') \end{bmatrix} U \phi \quad (2.44)
\]

Since \( \tilde{S}_{UV}^{(1)} = U^\dagger \phi \tilde{S}_{\text{matt}}^{(1)} U^\dagger \phi \), one can obtain \( \tilde{S}_{UV}^{(1)} \) by removing \( U^\dagger \phi \) and \( U \phi \) from eq. (2.44). Their explicit forms are given in appendix B. Then, the first order UV contribution to the flavor basis \( S \) matrix can be readily calculated as

\[
S_{UV}^{(1)} = U_{23} \tilde{S}_{UV}^{(1)} U^\dagger_{23} \quad (2.45)
\]

because the extrinsic factors \((1 - \alpha)\) and/or \((1 - \alpha)^\dagger\) only yield higher order terms.

3 **Neutrino oscillation probability to first order: \( \nu_e - \nu_\mu \) sector**

In this section, we calculate the expressions of the oscillation probabilities. For clarity, we concentrate on \( \nu_e \rightarrow \nu_\mu \) and \( \nu_\mu \rightarrow \nu_e \) channels. The other oscillation probabilities which are required to discuss unitarity in the \( \nu_e \) row will be obtained in section 4. The oscillation probabilities in the \( \nu_\mu - \nu_\tau \) sector are given in appendix C. Table 1 at the end of this section summarizes the locations and the equation numbers of all the probability formulas.

We have obtained \( S \) matrix elements in the zeroth order and first order helio collections using eq. (2.32) with the \( \tilde{S}^{(0)} \) and \( \tilde{S}^{(1)}_{\text{helio}} \) matrix elements in eqs. (2.41) and (2.42), respectively, and the first order matter correction \( S_{UV}^{(1)} \) in eq. (2.45). Therefore, we know the whole \( S \) matrix to first order in the helio and the UV parameters

\[
S = S^{(0)} + S^{(1)}_{\text{helio}} + S_{UV}^{(1)} - \alpha S^{(0)} - S^{(0)} \alpha^\dagger. \quad (3.1)
\]

Then, we are ready to calculate the expressions of the oscillation probabilities using the formula \( P(\nu_\beta \rightarrow \nu_\alpha; x) = |S_{\alpha\beta}|^2 \) to first order in the expansion parameters. Since all the building elements are known, we just present the final expressions of the oscillation probabilities.

We categorize \( P(\nu_\beta \rightarrow \nu_\alpha) \) into the three types of terms:

\[
P(\nu_\beta \rightarrow \nu_\alpha) = P(\nu_\beta \rightarrow \nu_\alpha)^{(0+1)}_{\text{helio}} + P(\nu_\beta \rightarrow \nu_\alpha)^{(1)}_{\text{int-UV}} + P(\nu_\beta \rightarrow \nu_\alpha)^{(1)}_{\text{ext-UV}}, \quad (3.2)
\]

where

\[
P(\nu_\beta \rightarrow \nu_\alpha)^{(0+1)}_{\text{helio}} = |S_{\alpha\beta}^{(0)}|^2 + 2\text{Re} \left[ \left( S_{\alpha\beta}^{(0)} \right)^* \left( S_{\text{helio}}^{(1)} \right)_{\alpha\beta} \right],
\]

\[
P(\nu_\beta \rightarrow \nu_\alpha)^{(1)}_{\text{int-UV}} = 2\text{Re} \left[ \left( S_{\alpha\beta}^{(0)} \right)^* \left( S_{UV}^{(1)} \right)_{\alpha\beta} \right],
\]

\[
P(\nu_\beta \rightarrow \nu_\alpha)^{(1)}_{\text{ext-UV}} = -2\text{Re} \left[ \left( S_{\alpha\beta}^{(0)} \right)^* \left( \alpha S^{(0)} + S^{(0)} \alpha^\dagger \right)_{\alpha\beta} \right]. \quad (3.3)
\]
The first term in eq. (3.2), $P(\nu_\beta \to \nu_\alpha)_{\text{helio}}^{(0+1)}$, is nothing but the “simple and compact” formulas for the probability derived in ref. [19] which is based on the standard unitary three-flavor mixing and is valid to first order in $\epsilon$.

### 3.1 $P(\nu_e \to \nu_e)^{(0+1)}_{\text{helio}}$ and $P(\nu_\mu \to \nu_e)^{(0+1)}_{\text{helio}}$: the “simple and compact” formulas

Since all the calculations for $P(\nu_e \to \nu_e)^{(0+1)}_{\text{helio}}$ and $P(\nu_\mu \to \nu_e)^{(0+1)}_{\text{helio}}$ are done in [19] and described in detail in this reference we just present here the result:

$$P(\nu_e \to \nu_e)^{(0+1)}_{\text{helio}} = 1 - \sin^2 2\phi \sin^2 \left(\frac{h_3 - h_1}{2}\right), \quad (3.4)$$

$$P(\nu_\mu \to \nu_e)^{(0+1)}_{\text{helio}}$$

$$= \left[ s_{23}^2 \sin^2 \theta_{13} + 4\epsilon J_r \cos \delta \left( \frac{h_3 - h_1}{h_3 - h_2} \right) \right] \sin^2 \left(\frac{h_3 - h_1}{2}\right) \sin^2 \left(\frac{h_2 - h_1}{2}\right) \cos \left(\frac{h_3 - h_2}{2}\right), \quad (3.5)$$

where $x$ is the baseline and $J_r$, the reduced Jarlskog factor [47], is defined as

$$J_r \equiv c_{13} s_{12} c_{23} s_{13} s_{13}^2. \quad (3.6)$$

For simplified notations such as $\Delta_{\text{ren}} = \frac{\Delta m_{12}^2}{2E}$, and $\Delta_a = \frac{a}{2E}$, see sections 2.1 and 2.3. $h_i$ ($i = 1, 2, 3$) denote the eigenvalues of $\tilde{H}^{(0)}$ as defined in eq. (2.21).

Because the matter potential due to the NC interaction is removed from the zeroth-order Hamiltonian $\tilde{H}^{(0)}$ by the phase redefinition (see section 2.4), the unitary part $P(\nu_\beta \to \nu_\alpha)^{(0+1)}_{\text{helio}}$ is free from NC matter potential $\Delta_b \equiv \frac{b}{2E}$.

### 3.2 $P(\nu_e \to \nu_e)^{(1)}$ and $P(\nu_\mu \to \nu_e)^{(1)}$: Both intrinsic and extrinsic UV contributions

The first order intrinsic and extrinsic UV contributions to $P(\nu_e \to \nu_e)$ read

$$P(\nu_e \to \nu_e)^{(1)}_{\text{int-UV}} = \sin^2 2\phi \left[ \cos 2\phi \left[ \sin^2 \left(\frac{h_3 - h_1}{2}\right) \right] - \cos 2\phi \left[ \sin^2 \left(\frac{h_2 - h_1}{2}\right) \right] \right],$$

$$P(\nu_e \to \nu_e)^{(1)}_{\text{ext-UV}} = -4\epsilon_{ee} \left[ 1 - \sin^2 2\phi \sin^2 \left(\frac{h_3 - h_1}{2}\right) \right]. \quad (3.8)$$
Similarly, the first order intrinsic and extrinsic UV contributions to \( P(\nu_\mu \to \nu_e) \) are given by

\[
P(\nu_\mu \to \nu_e)^{(1)}_{\text{int-UV}} = 2\text{Re} \left[ S_{e\mu}^{(0)} \ast S_{UV}^{(1)} \right]_{e\mu}
\]

\[
- s_{23}^2 \sin^2 2\phi \cos 2\phi \left[ \alpha_{ee} \left( 1 - \frac{\Delta_a}{\Delta_b} \right) - s_{23}^2 \alpha_{\mu\mu} - c_{23}^2 \alpha_{\tau\tau} - c_{23} s_{23} \text{Re}(e^{i\delta} \alpha_{\tau\mu}) \right] \left( \Delta_b x \right) \sin(h_3 - h_1)x
+ s_{23}^2 \sin^3 2\phi \left[ s_{23} \text{Re}(e^{-i\delta} \alpha_{\mu\mu}) + c_{23} \text{Re}(\alpha_{\tau\tau}) \right] \left( \Delta_b x \right) \sin(h_3 - h_1)x
+ \sin 2\theta_{23} \sin 2\phi
\]

\[
\times \left\{ c_\phi^2 \left[ s_{23} \text{Re}(e^{-i\delta} \alpha_{\mu\mu}) - s_{23} \text{Re}(\alpha_{\tau\tau}) \right] - \cos 2\theta_{23} \csc \phi \text{Re}(e^{i\delta} \alpha_{\tau\mu}) - \sin 2\theta_{23} \csc \phi \text{Re}(\alpha_{\mu\mu} - \alpha_{\tau\tau}) \right\}
\times \frac{\Delta_b}{(h_2 - h_1)} \left\{ -\sin^2 \left( \frac{h_3 - h_2}{2} \right) + \sin^2 \left( \frac{h_3 - h_1}{2} \right) + \frac{(h_2 - h_1)x}{2} \right\}
+ \sin 2\theta_{23} \sin 2\phi
\]

\[
\times \left\{ s_\phi^2 \left[ s_{23} \text{Re}(e^{-i\delta} \alpha_{\mu\mu}) + s_{23} \text{Re}(\alpha_{\tau\tau}) \right] + \cos 2\theta_{23} \csc \phi \text{Re}(e^{i\delta} \alpha_{\tau\mu}) + \sin 2\theta_{23} \csc \phi \text{Re}(\alpha_{\mu\mu} - \alpha_{\tau\tau}) \right\}
\times \frac{\Delta_b}{(h_3 - h_2)} \left\{ -\sin^2 \left( \frac{h_2 - h_1}{2} \right) + \sin^2 \left( \frac{h_2 - h_3}{2} \right) + \frac{(h_3 - h_1)x}{2} \right\}
+ 4s_{23}^2 \sin 2\phi \cos 2\phi \left\{ \sin 2\phi \left[ \alpha_{ee} \left( 1 - \frac{\Delta_a}{\Delta_b} \right) - s_{23}^2 \alpha_{\mu\mu} - c_{23}^2 \alpha_{\tau\tau} - c_{23} s_{23} \text{Re}(e^{i\delta} \alpha_{\tau\mu}) \right]
+ \cos 2\phi \left[ s_{23} \text{Re}(e^{-i\delta} \alpha_{\mu\mu}) + c_{23} \text{Re}(\alpha_{\tau\tau}) \right] \right\}
\times \frac{\Delta_b}{(h_3 - h_1)} \left\{ -\sin^2 \left( \frac{h_3 - h_1}{2} \right) + \sin^2 \left( \frac{h_3 - h_2}{2} \right) + \frac{(h_3 - h_1)x}{2} \right\}
+ 2 \sin 2\theta_{23} \sin 2\phi \left\{ c_\phi^2 \left[ s_{23} \text{Im}(e^{-i\delta} \alpha_{\mu\mu}) - s_{23} \text{Im}(\alpha_{\tau\tau}) \right] + c_\phi \text{Im}(e^{i\delta} \alpha_{\tau\mu}) \right\}
\times \frac{\Delta_b}{(h_2 - h_1)} \left\{ -\sin^2 \left( \frac{h_2 - h_1}{2} \right) + \sin^2 \left( \frac{h_2 - h_3}{2} \right) + \frac{(h_2 - h_1)x}{2} \right\}.
\]

\[
P(\nu_\mu \to \nu_e)^{(1)}_{\text{ext-UV}} = 2\text{Re} \left[ S_{e\mu}^{(0)} \ast S_{e\mu}^{(1)} \right]_{\text{ext}}
\]

\[
= 2s_{23} \sin 2\phi \left[ \cos 2\phi \text{Re}(e^{-i\delta} \alpha_{\mu\mu}) - s_{23} \sin 2\phi \left( \alpha_{ee} + \alpha_{\mu\mu} \right) \right] \sin^2 \left( \frac{h_3 - h_1}{2} \right)x
- s_{23} \sin 2\phi \text{Im}(e^{-i\delta} \alpha_{\mu\mu}) \sin(h_3 - h_1)x.
\]

The expressions of the oscillation probabilities in eqs. (3.7), (3.8), (3.9), and (3.10) are the first explicit demonstration of the canonical phase combination \( e^{-i\delta} \alpha_{\mu\mu}, \alpha_{\tau\tau}, \) and \( e^{i\delta} \alpha_{\tau\mu} \) in this paper. Non-association of \( e^{\pm i\delta} \) to \( \alpha_{\tau\tau} \) must be understood as a particular “correlation”, and naturally there is no association of \( \delta \) in the diagonal \( \alpha \) parameters, \( \alpha_{\beta\beta} \) \((\beta = e, \mu, \tau)\). We will see in the rest of this paper that the canonical phase combination
is always realized in both the first order intrinsic and extrinsic UV correction terms in the oscillation probabilities in all the channels.\textsuperscript{8}

We should mention that the phase correlation $e^{-i\delta_{\alpha\mu\epsilon}}$, a part of our canonical phase combination, has been observed in ref. [23] but only in vacuum, and in ref. [39] in matter but only as an outcome of numerical study in the particular channels.\textsuperscript{9}

We defer our presentation of the oscillation probabilities in the $\nu_\mu - \nu_\tau$ sector to appendix C. The generic features of them are very similar to the ones presented in this section, with slightly more complicated expressions. Importantly, the canonical phase combination prevails in the oscillation probabilities in the $\nu_\mu - \nu_\tau$ sector, as we will see in appendix C. A possibility of reducing the number of parameters by expanding in another small parameter is briefly discussed in appendix D.

The expressions of the oscillation probabilities are scattered into various places in this paper. Therefore, for the readers’ convenience, we tabulate in table 1 the equation numbers for $P(\nu_\beta \to \nu_\alpha)_{\text{int-UV}}^{(1)}$ and $P(\nu_\beta \to \nu_\alpha)_{\text{ext-UV}}^{(1)}$ in various channels.

| Channel | $P(\nu_\beta \to \nu_\alpha)_{\text{int-UV}}^{(1)}$ | $P(\nu_\beta \to \nu_\alpha)_{\text{ext-UV}}^{(1)}$ |
|---------|-------------------------------------------------|-------------------------------------------------|
| $\nu_\mu \to \nu_e$ | eq. (3.9) in section 3.2 | eq. (3.10) in section 3.2 |
| $\nu_e \to \nu_e$ | eq. (3.7) in section 3.2 | eq. (3.8) in section 3.2 |
| $\nu_\epsilon \to \nu_\mu$ | T transformation from eq. (3.9) | eq. (4.5) in section 4.2 |
| $\nu_\mu \to \nu_\mu$ | eq. (4.3) in section 4.2 | eq. (4.6) in section 4.3 |
| $\nu_\mu \to \nu_\tau$ | eq. (C.3) in appendix C.2 | eq. (C.5) in appendix C.3 |
| $\nu_\mu \to \nu_\tau$ | eq. (C.4) in appendix C.2 | eq. (C.6) in appendix C.3 |

4 Unitarity of neutrino evolution with first order UV corrections: $\nu_\epsilon$ row

In section 2 we have shown that, to first order in the helio-UV perturbation theory, the oscillation probability $P(\nu_\beta \to \nu_\alpha)$ can be decomposed into the two parts, the unitary part denoted as $P(\nu_\beta \to \nu_\alpha)_{\text{helio}}^{(0+1)} + P(\nu_\beta \to \nu_\alpha)_{\text{int-UV}}^{(1)}$, and the non-unitary part $P(\nu_\beta \to \nu_\alpha)_{\text{ext-UV}}^{(1)}$. It is the unique form of the probability which is consistent with the unitarity of the propagation-S matrix $U_{23}SU_{23}^\dagger$ before the initial and final projection to and from the mass eigenstate basis, respectively, are applied. As discussed in section 2.6, the property is in complete harmony with the reasonings for unitary evolution even with non-unitary mixing matrix, which is spelled out in section 1.

\textsuperscript{8} A perturbative treatment using the similar expansion parameters is presented in ref. [48] within the framework of 3 + 3 model, in which the calculation of the oscillation probabilities of the first order are carried out. However, due to different implementation of UV, it is essentially impossible to compare our formulas to theirs. As a consequence, none of the points of our emphasis, the canonical phase combination and unitarity of neutrino propagation in matter is not reached in their paper.

\textsuperscript{9} Our result is consistent with theirs if the authors of refs. [23] and [39] have used the $U_{PDG}$ or $U_{ATM}$ phase conventions because the correlation $e^{-i\delta_{\alpha\mu\epsilon}}$ holds in the both conventions. See section 6.2.
In this section, we give an explicit proof of unitarity of $P(\nu_e \to \nu_\alpha)^{(0+1)}_{\text{helio}} + P(\nu_e \to \nu_\alpha)^{(1)}_{\text{int-UV}}$ in $\nu_e$ row. The similar explicit proof of unitarity in $\nu_\mu$ row will be given in appendix C. To our knowledge it is the first explicit proof at the probability level that neutrino propagation is unitary in the presence of non-unitary mixing matrix. Since we already know that the oscillation probability to first-order helio corrections is unitary [19],

$$
\sum_{\alpha=e,\mu,\tau} P(\nu_\beta \to \nu_\alpha)^{(0+1)}_{\text{helio}} = 1 + \mathcal{O}(\epsilon^2),
$$

(4.1)

it is sufficient to show

$$
\sum_{\alpha=e,\mu,\tau} P(\nu_\beta \to \nu_\alpha)^{(1)}_{\text{int-UV}} = \mathcal{O}(\epsilon^2)
$$

(4.2)

to prove perturbative unitarity. In (4.2) we have assumed that all $\alpha,\beta,\gamma \sim \epsilon$.

4.1 The oscillation probabilities for proving perturbative unitarity in $\nu_e$ row

To discuss perturbative unitarity in $\nu_e$ row, therefore, we need to prepare the following three oscillation probabilities at first order, $P(\nu_e \to \nu_e)$, $P(\nu_e \to \nu_\mu)$, and $P(\nu_e \to \nu_\tau)$. $P(\nu_e \to \nu_e)^{(1)}_{\text{int-UV}}$ and $P(\nu_e \to \nu_\mu)^{(1)}_{\text{int-UV}}$ are given in eqs. (3.7) and (3.8), respectively. $P(\nu_e \to \nu_\tau)^{(1)}_{\text{int-UV}}$ and $P(\nu_e \to \nu_\mu)^{(1)}_{\text{ext-UV}}$, can be obtained by generalized T transformation [18] of eqs. (3.9) and (3.10), respectively. More generally, the transformation of the probabilities from $\nu_\beta \to \nu_\alpha$ to $\nu_\alpha \to \nu_\beta$ channels can be done by taking the complex conjugate of all the complex parameters, that is, to flip the sign of the imaginary part of $e^{\pm i\delta}$ and the UV parameters. Therefore, what we need to compute here are $P(\nu_e \to \nu_\tau)^{(1)}_{\text{int-UV}}$ and $P(\nu_e \to \nu_\tau)^{(1)}_{\text{ext-UV}}$. 

– 17 –
4.2 Perturbative unitarity of intrinsic UV contribution: $\nu_e$ row

To examine unitarity of intrinsic UV contribution we compute $P(\nu_e \rightarrow \nu_\tau)^{(1)}_{\text{int-UV}}$ with the result

$$
P(\nu_e \rightarrow \nu_\tau)^{(1)}_{\text{int-UV}} = -c_{23}^2 \sin^2 2\phi \cos 2\phi \left[ \alpha_{ee} \left( 1 - \frac{\Delta_a}{\Delta_b} \right) - s_{23}^2 \alpha_{\mu \mu} - c_{23}^2 \alpha_{\tau \tau} - c_{23} s_{23} \text{Re} \left( e^{i\delta \alpha_{\tau \mu}} \right) \right] (\Delta_b x) \sin (h_3 - h_1) x \\
+ c_{23}^2 \sin^3 2\phi \left[ s_{23} \text{Re} \left( e^{-i\delta \alpha_{\mu \mu}} \right) + c_{23} \text{Re} (\alpha_{\tau \tau}) \right] (\Delta_b x) \sin (h_3 - h_1) x \\
+ 4 c_{23} \sin 2\phi \left[ \alpha_{ee} \left( 1 - \frac{\Delta_a}{\Delta_b} \right) - s_{23}^2 \alpha_{\mu \mu} - c_{23}^2 \alpha_{\tau \tau} - c_{23} s_{23} \text{Re} \left( e^{i\delta \alpha_{\tau \mu}} \right) \right] \\
+ \cos 2\phi \left[ s_{23} \text{Re} \left( e^{-i\delta \alpha_{\mu \mu}} \right) + c_{23} \text{Re} (\alpha_{\tau \tau}) \right] \frac{\Delta_b}{h_3 - h_1} \sin^2 \left( \frac{(h_3 - h_1) x}{2} \right) \\
- \sin 2\theta_{23} \sin 2\phi \left[ c_{23} \text{Re} \left( e^{-i\delta \alpha_{\mu \mu}} \right) - s_{23} \text{Re} (\alpha_{\tau \tau}) \right] - c_{\phi} s_{\phi} \left[ \sin 2\theta_{23} (\alpha_{\mu \mu} - \alpha_{\tau \tau}) + \cos 2\theta_{23} \text{Re} \left( e^{i\delta \alpha_{\tau \mu}} \right) \right] \\
+ \frac{\Delta_b}{(h_3 - h_2)} \left[ - \sin \left( \frac{(h_2 - h_1) x}{2} \right) + \sin \left( \frac{(h_3 - h_1) x}{2} \right) + \sin \left( \frac{(h_2 - h_1) x}{2} \right) \right] \\
+ 2 \sin 2\theta_{23} \sin 2\phi \left[ c_{\phi} s_{\phi} \left[ c_{23} \text{Im} \left( e^{-i\delta \alpha_{\mu \mu}} \right) - s_{23} \text{Im} (\alpha_{\tau \tau}) \right] \right] + c_{\phi} s_{\phi} \text{Im} \left( e^{i\delta \alpha_{\tau \mu}} \right) \\
+ \frac{\Delta_b}{(h_3 - h_1)} \sin \left( \frac{(h_3 - h_1) x}{2} \right) \sin \left( \frac{(h_1 - h_2) x}{2} \right) \sin \left( \frac{(h_2 - h_3) x}{2} \right) \\
- 2 \sin 2\theta_{23} \sin 2\phi \left[ s_{\phi}^2 \left[ c_{23} \text{Im} \left( e^{-i\delta \alpha_{\mu \mu}} \right) - s_{23} \text{Im} (\alpha_{\tau \tau}) \right] - c_{\phi} s_{\phi} \text{Im} \left( e^{i\delta \alpha_{\tau \mu}} \right) \right] \\
+ \frac{\Delta_b}{(h_3 - h_2)} \sin \left( \frac{(h_3 - h_1) x}{2} \right) \sin \left( \frac{(h_1 - h_2) x}{2} \right) \sin \left( \frac{(h_2 - h_3) x}{2} \right). \quad (4.3)
$$

Given the expressions of the oscillation probabilities in $\nu_e$ row, one can readily prove perturbative unitarity for neutrino evolution without extrinsic UV corrections

$$
P(\nu_e \rightarrow \nu_e)^{(1)}_{\text{int-UV}} + P(\nu_e \rightarrow \nu_\mu)^{(1)}_{\text{int-UV}} + P(\nu_e \rightarrow \nu_\tau)^{(1)}_{\text{int-UV}} = 0. \quad (4.4)
$$

where "0" in the right-hand side implies absence of first order terms in the UV parameters. This completes our proof of perturbative unitarity in $\nu_e$ row of the intrinsic UV contributions to first order in the $\alpha$ parameters. The similar result will be shown to hold in $\nu_\mu$ row in appendix C.4.

4.3 No perturbative unitarity of extrinsic UV contribution: $\nu_e$ row

For completeness, we explicitly verify that $P(\nu_\beta \rightarrow \nu_\alpha)^{(1)}_{\text{ext-UV}}$ gives raise to non-unitary contribution. Among the relevant three probabilities, $P(\nu_e \rightarrow \nu_e)^{(1)}_{\text{ext-UV}}$ is given in eq. (3.8)
in section 3.2. \( P(\nu_e \to \nu_\mu)^{(1)}_{\text{ext-UV}} \) can be obtained by generalized T transformation of eq. (3.10)

\[
P(\nu_e \to \nu_\mu)^{(1)}_{\text{ext-UV}} = 2s_{23} \sin 2\phi \left[ \cos 2\phi \Re (e^{-i\delta} \alpha_{\mu e}) - s_{23} \sin 2\phi (\alpha_{ee} + \alpha_{\mu \mu}) \right] \sin^2 \frac{(h_3 - h_1)x}{2} \\
+ s_{23} \sin 2\phi \Im (e^{-i\delta} \alpha_{\mu e}) \sin(h_3 - h_1)x.
\]

(4.5)

Finally, \( P(\nu_e \to \nu_\tau)^{(1)}_{\text{ext-UV}} \) can be easily computed as

\[
P(\nu_e \to \nu_\tau)^{(1)}_{\text{ext-UV}} = -2c_{23} \sin 2\phi \left\{ \sin 2\phi \left[ s_{23} \Re (e^{i\delta} \alpha_{\tau \mu}) + c_{23} (\alpha_{ee} + \alpha_{\tau \tau}) \right] - \cos 2\phi \Re (\alpha_{\tau e}) \right\} \sin^2 \frac{(h_3 - h_1)x}{2} \\
+ c_{23} \sin 2\phi \Im (\alpha_{\tau e}) \sin(h_3 - h_1)x.
\]

(4.6)

It is evident that they do not add up to zero, as there is no way for imaginary part cancels when \( P(\nu_e \to \nu_\mu)^{(1)}_{\text{ext-UV}} \) and \( P(\nu_e \to \nu_\tau)^{(1)}_{\text{ext-UV}} \) are added up. We find no indication of even partial cancellation between the various terms. Therefore, there is no perturbative unitarity of extrinsic UV contribution in \( \nu_e \) row, as expected. We will see the same result in the \( \nu_\mu \) row in section C.4.

5 How accurate are the first order formulas for \( P(\nu_\beta \to \nu_\alpha)^{\text{UV}} \)?

The principal objective of constructing our helio-UV perturbation theory is to understand the qualitative features of oscillation probability with UV. Yet, it may be better to have an idea of how good is the approximation it can offer. In particular, we are interested in the UV part, \( P(\nu_\beta \to \nu_\alpha)^{(1)}_{\text{UV}} \equiv P(\nu_\beta \to \nu_\alpha)^{(1)}_{\text{int-UV}} + P(\nu_\beta \to \nu_\alpha)^{(1)}_{\text{ext-UV}} \), because the accuracies of \( P(\nu_\beta \to \nu_\alpha)^{\text{helio}} \) have been examined in [19]. Notice that it corresponds to the quantity

\[
\Delta P(\nu_\beta \to \nu_\alpha) \equiv P(\nu_\beta \to \nu_\alpha)^{\text{standard}} - P(\nu_\beta \to \nu_\alpha)^{(0)}_{\text{non-unitary}}
\]

(5.1)

which is numerically computed with high precision and is plotted in the lower panels of figures 1-3 in [18] for \((\beta\alpha) = \mu e, \mu \tau, \text{and } \mu \mu\). Then, we confront our first order formulas of \( P(\nu_\beta \to \nu_\alpha)^{(1)}_{\text{UV}} \) to \( \Delta P(\nu_\beta \to \nu_\alpha) \) in [18].

In figure 1, plotted are the iso-contours of \(-P(\nu_\mu \to \nu_\alpha)^{(1)}_{\text{UV}} \) as a function of energy \( E \) and baseline \( L \) for \( \alpha = e \) (upper panel) and \( \alpha = \mu \) (lower panel). We have used the same values for the \( \nu SM \) mixing parameters as well as the UV \( \alpha \) parameters as in ref. [18]. We see overall agreement, not only qualitatively but also quantitatively to a certain level, between the iso-contours in the upper and lower panels in figure 1 and the ones given in figures 1 (for \( \nu_\mu \to \nu_e \)) and 3 (for \( \nu_\mu \to \nu_\mu \)) in ref. [18], respectively.\(^{10}\) If the numerical accuracy

\(^{10}\) The only exception might be in a relatively small region with shape of oblique ellipse centered around \( L = 2000 \text{ km} \) and \( E = 100 \text{ MeV} \), which extends to a few 100 MeV, the region of solar MSW enhancement [41, 49]. But, it was understood that the disagreement is largely due to a difference in mesh between our figure 1 and figures 1 and 3 in ref. [18]. Notice that an extremely fine mesh is required to display the contours accurately, given the feature of the probability in this region due to superimposed high-frequency atmospheric-scale oscillations on long-wavelength solar-scale oscillations. But, we did not try to elaborate this point, because a numerical accuracy of the first order formula is not the main point of this paper.
Figure 1. The iso-contour of $-P(\nu_\mu \rightarrow \nu_\alpha)^{(1)}_{\text{UV}} \equiv -\left[ P(\nu_\mu \rightarrow \nu_\alpha)^{(1)}_{\text{int-UV}} + P(\nu_\mu \rightarrow \nu_\alpha)^{(1)}_{\text{ext-UV}} \right]$ is presented in space of neutrino energy $E$ and baseline $L$ for $\alpha = e$ (upper panel) and $\alpha = \mu$ (lower panel). It corresponds to the difference $\Delta P(\nu_\mu \rightarrow \nu_\alpha) \equiv P(\nu_\mu \rightarrow \nu_\alpha)^{\text{standard}} - P(\nu_\mu \rightarrow \nu_\alpha)^{(0)}_{\text{non-unitary}}$ plotted in the lower panel of figures 1 ($\alpha = e$) and 3 ($\alpha = \mu$) of ref. [18]. In this calculation, the same values for the standard mixing parameters as well as the UV $\alpha$ parameters as in [18] are used: $\alpha_{ee} = 0.01$, $\alpha_{\mu e} = 0.0141$, $\alpha_{\mu \mu} = 0.005$, $\alpha_{\tau e} = 0.0445$, $\alpha_{\tau \mu} = 0.0316$, $\alpha_{\tau \tau} = 0.051$. The matter density is taken to be $\rho = 3.2$ g cm$^{-3}$ over the entire baseline.
of the first order formula is affected by the treatment of the solar level crossing [19], it
would be worthwhile to re-examine this problem with a different framework developed in
refs. [50, 51].

6 Canonical phase combination: Stability and phase convention dependence

In this section we clarify the two aspects of the canonical phase combination, the particular
way how the $\alpha$ parameter phases come in in the special combination with the $\nu$SM CP
phase. They are to answer the questions on the canonical phase combination: (1) Why
so stable over the oscillation channels as well as quite different nature of the intrinsic and
and extrinsic contributions, and (2) Are they independent of phase convention of the MNS
matrix?

6.1 Mechanism for generating the canonical phase combination

Knowing the universal phase correlation between the lepton KM phase and the ones associated
with the UV parameters may simplify the analyses, e.g., to constrain non-unitarity.
Therefore, it is important to understand how the phase correlation comes about and why
it is so stable.

To make the discussion concrete, let us ask a question: Observe that the phase fac-
tor $e^{\pm i\delta}$ is distributed in the $S$ matrix elements in a quite nontrivial fashion (as will be
diagnosed below) which is inherited from those of $H_{ij}$ in eq. (A.1) in the first order am-
plitudes. The first order oscillation probability is given by the interference between the
two different amplitudes. Then, what is the reason why such canonical phase combination
appears systematically in both $P(\nu_\mu \rightarrow \nu_\tau)_{\text{int-UV}}$ and $P(\nu_\mu \rightarrow \nu_e)_{\text{ext-UV}}$ simultaneously, and
throughout all the oscillation channels? In this section, we answer this question.

Toward the goal we first note that the flavor basis $S$ matrix at zeroth order has a
characteristic form of $e^{\pm i\delta}$, in a lozenge positions, as

$$
S^{(0)} = U_{23} \tilde{S}^{(0)} U_{23}^{\dagger} =
\begin{bmatrix}
  c_{23}^2 e^{-i\theta_{13}} + s_{23}^2 e^{-i\theta_{13}} & c_{23}s_{23} \phi (e^{-i\theta_{13}} - e^{-i\theta_{13}}) & s_{23}c_{23} \phi (e^{-i\theta_{13}} - e^{-i\theta_{13}}) \\
  s_{23}^2 e^{-i\theta_{13}} - e^{-i\theta_{13}} & c_{23}^2 e^{-i\theta_{13}} + s_{23}^2 (c_{23}^2 e^{-i\theta_{13}} + s_{23}^2 e^{-i\theta_{13}}) & s_{23}c_{23} \phi (e^{-i\theta_{13}} - e^{-i\theta_{13}}) \\
  c_{23}s_{23} e^{-i\phi} (c_{23}^2 e^{-i\theta_{13}} + s_{23}^2 e^{-i\theta_{13}} - e^{-i\theta_{13}}) & s_{23}^2 e^{-i\theta_{13}} + c_{23}^2 (c_{23}^2 e^{-i\theta_{13}} + s_{23}^2 e^{-i\theta_{13}})
\end{bmatrix}
$$

(6.1)

which can be written in an abbreviated form as

$$
X =
\begin{bmatrix}
  X_{ee} & X_{e\mu} e^{-i\delta} & X_{e\tau} \\
  X_{\mu e} e^{i\delta} & X_{\mu\mu} & X_{\mu\tau} e^{i\delta} \\
  X_{\tau e} & X_{\tau\mu} e^{-i\delta} & X_{\tau\tau}
\end{bmatrix}
$$

(6.2)

where $X_{\alpha\beta}$ is independent of any CP phases.

\(^{11}\) Here, we refer $S$ matrices in flavor basis by the notations $X$ and $Y$ not to trigger confusion with $S$
matrix elements $S_{\alpha\beta}$ which describe the neutrino flavor transformation.
Then, the obvious (and probably unique) possibility to realize the canonical phase combination is that the interfering amplitude, $S_{\text{int-UV}}^{(1)}$ and $S_{\text{ext-UV}}^{(1)}$, has the same structure

$$\begin{bmatrix}
Y_{ee} & Y_{e\mu}e^{-i\delta} & Y_{e\tau} \\
Y_{\mu e}e^{i\delta} & Y_{\mu\mu} & Y_{\mu\tau}e^{i\delta} \\
Y_{\tau e} & Y_{\tau\mu}e^{-i\delta} & Y_{\tau\tau}
\end{bmatrix},$$

(6.3)

where $Y_{\alpha\beta}$ contain the lepton KM and the UV phases, but in the form of canonical phase combination, $e^{-i\delta}Y_{ee}$, $e^{i\delta}Y_{\mu\mu}$, and $e^{i\delta}Y_{\tau\tau}$. It is obvious that the extra phase factors $e^{\pm i\delta}$ cancel out in $P(\nu_\beta \to \nu_\alpha) \propto (X_{\alpha\beta})^*Y_{\alpha\beta}$, leaving the canonical phase combination in the oscillation probabilities.

The rest of the task that remains needed to answer the question we posed above is to show that the both $S_{\text{int-UV}}^{(1)}$ and $S_{\text{ext-UV}}^{(1)}$ have the structure in eq. (6.3). Let us start from the simpler case, $S_{\text{ext-UV}}^{(1)}$. The last two terms of the $S$ matrix in eq. (3.2) are the form $\alpha X$ and $X\alpha^\dagger$, respectively. For generality and possible use in wider context, we use $Y$, instead of $X$:

$$\alpha Y = \begin{bmatrix}
\alpha_{ee} & 0 & 0 \\
\alpha_{\mu e} & \alpha_{\mu\mu} & 0 \\
\alpha_{\tau e} & \alpha_{\tau\mu} & \alpha_{\tau\tau}
\end{bmatrix} \begin{bmatrix}
Y_{ee} & Y_{e\mu}e^{-i\delta} & Y_{e\tau} \\
Y_{\mu e}e^{i\delta} & Y_{\mu\mu} & Y_{\mu\tau}e^{i\delta} \\
Y_{\tau e} & Y_{\tau\mu}e^{-i\delta} & Y_{\tau\tau}
\end{bmatrix}
$$

$$= \begin{bmatrix}
\alpha_{ee}Y_{ee} \\
\alpha_{ee}Y_{ee} + e^{i\delta}\alpha_{\mu e}Y_{ee} + \alpha_{\mu\mu}Y_{\mu\mu} \\
\alpha_{ee}Y_{ee} + e^{i\delta}\alpha_{\tau e}Y_{ee} + \alpha_{\tau\mu}Y_{\mu\mu} + \alpha_{\tau\tau}Y_{\tau\tau}
\end{bmatrix} \begin{bmatrix}
Y_{ee} & Y_{e\mu}e^{-i\delta} & Y_{e\tau} \\
Y_{\mu e}e^{i\delta} & Y_{\mu\mu} & Y_{\mu\tau}e^{i\delta} \\
Y_{\tau e} & Y_{\tau\mu}e^{-i\delta} & Y_{\tau\tau}
\end{bmatrix}
$$

$$= \begin{bmatrix}
Y_{ee} & Y_{e\mu}e^{-i\delta} & Y_{e\tau} \\
Y_{\mu e}e^{i\delta} & Y_{\mu\mu} & Y_{\mu\tau}e^{i\delta} \\
Y_{\tau e} & Y_{\tau\mu}e^{-i\delta} & Y_{\tau\tau}
\end{bmatrix}
$$

(6.4)

In eqs. (6.4) and (6.5), the square parentheses imply that inside them only the canonical phase combination is contained. Therefore, the canonical phase structure of $S_{\text{ext-UV}}^{(1)}$ matrix, $e^{\pm i\delta}$ located in lozenge positions attached to functions only with CP phases with the canonical phase combination, is maintained in both $\alpha S^{(0)}$ and $S^{(0)}\alpha^\dagger$ (or, more generically for $\alpha Y$ and $Y\alpha^\dagger$). It guarantees that the first order extrinsic UV correction terms in the oscillation probability respect the canonical phase combination.

We now examine the structure of $S_{\text{int-UV}}^{(1)} = U_{23}\tilde{S}_{\text{UV}}U_{23}^\dagger$, as the final task to understand the canonical phase structure. A close examination of the expressions of $\tilde{S}_{\text{UV}}$ matrix elements given in appendix B reveals that they possess the canonical phase structure, the form in eq. (6.3). This structure can be recognized in the process of computing the $S$ matrix and the oscillation probability, which is left as an exercise for the readers.
6.2 Phase convention dependence of the canonical phase combination

It must be obvious from our discussion in section 2.2 that the $\alpha$ matrix, and hence the $\alpha_{\beta\gamma}$ parameters, depend on the phase convention of the MNS matrix. Then, the form of the canonical phase combination also depends on the phase convention. But, since the relationship between the $\alpha$ parameters belonging to the three different phase conventions is explicitly given in eq. (2.10), it is straightforward to translate the form of canonical phase combination from one convention to another.

We have obtained the canonical phase combination with the $U_{ATM}$ phase convention as

$$e^{-i\delta_{\mu\epsilon}}, \alpha_{\tau\epsilon}, e^{i\delta_{\tau\mu}}.$$  \hfill (6.6)

It can be translated into the one with the $U_{PDG}$ phase convention

$$e^{-i\delta_{\bar{\alpha}_{\mu\epsilon}}}, e^{-i\delta_{\bar{\alpha}_{\tau\epsilon}}}, \bar{\alpha}_{\tau\mu},$$  \hfill (6.7)

for the $\bar{\alpha}$ parameters and the one with the $U_{SOL}$ phase convention

$$\bar{\alpha}_{\mu\epsilon}, \bar{\alpha}_{\tau\epsilon}, \bar{\alpha}_{\tau\mu},$$  \hfill (6.8)

for the $\tilde{\alpha}$ parameters. See eqs. (2.5) and (2.10) for the definitions of $\bar{\alpha}$ and $\tilde{\alpha}$ parameters, and their relationship with the $U_{ATM}$ convention $\alpha$ parameters. That is, under the $U_{SOL}$ phase convention, no correlation between $\nu_{SM}$ CP phase $\delta$ and the UV $\bar{\alpha}$ parameter phases exists. Conversely, one can easily show that the $U_{SOL}$ phase convention is the unique case without phase correlations.

6.3 Meaning of convention dependent phase correlation

To summarize, we have observed that, to first-order in the helio-UV perturbation theory, that the UV $\alpha$ parameter phases come into the oscillation probabilities in a fixed combination of $\nu_{SM}$ CP phase $\delta$. However, the relation is $U_{MNS}$ phase-convention dependent, as we saw in eqs. (6.6), (6.7), and (6.8). The convention dependence is perfectly legitimate because a change in phase convention for the $U_{MNS}$ matrix translates into the one of the $\alpha$ parameters, as shown in section 2.2.

What would be the consequence and the interpretation of these features? The clearest message we can convey to the readers is:

- A natural suggestion in analyzing data, e.g. to place constraints on UV, is to utilize the $U_{SOL}$ convention (2.9) with the $\bar{\alpha}$ parameters (2.10). In this way, one can avoid unwanted correlations between the physical parameters, $\nu_{SM}$ CP phase $\delta$ and the $\bar{\alpha}$ parameter phases.

Notice that this prescription is independent on our interpretation below of the phase correlation at a “deeper level”. It appears to us that the following two conflicting views are possible:

- The $\delta - \alpha$ parameter phases correlation in the $U_{ATM}$ and $U_{PDG}$ conventions suggests that the way how UV new physics effect is implemented into the low energy effective theory is dictated by the framework of UV itself.
The fact that the $\delta - \alpha$ parameter phases correlation is absent by taking the $U_{\text{SOL}}$ convention implies that the phase correlation is artificial without physical significance.

Despite temptation for the latter view, we note that the phase correlation vanishes only at the fine tuned $U_{\text{SOL}}$ convention. In every other phase convention, there is $\delta - \alpha$’s phase correlation which is universal in all the oscillation channels. That is, existence of the phase correlation is generic. It would imply that a certain consistency condition must be met when the effect of new physics is introduced into the low energy effective theory, $\nu_{\text{SM}}$. At this moment, we are unable to make a definitive choice from the two alternative views above, partly because our discussion is based on a particular perturbative framework, whose region of validity is quite limited.

### 7 Some additional remarks

#### 7.1 Vacuum limit

The vacuum limit in our helio-UV perturbation theory can be taken in a straightforward manner. With vanishing matter potentials, the Hamiltonian in eq. (2.1) in the vacuum mass eigenstate basis reduces to the free Hamiltonian. Then, all the intrinsic UV contributions $\mathcal{P}(\nu_\beta \rightarrow \nu_\alpha)^{(1)}_{\text{int-UV}}$ vanish, and the neutrino oscillation probability coincides with the vacuum limit of $\mathcal{P}(\nu_\beta \rightarrow \nu_\alpha)^{(0+1)}_{\text{helio}} + \mathcal{P}(\nu_\beta \rightarrow \nu_\alpha)^{(1)}_{\text{ext-UV}}$ to first order in the $\alpha$ parameters. Notice that the vacuum limit of the probabilities implies to take the following limits:

$$\phi \rightarrow \theta_{13}, \quad h_1 \rightarrow 0, \quad h_2 \rightarrow \frac{\Delta m^2_{21}}{2E}, \quad h_3 \rightarrow \frac{\Delta m^2_{31}}{2E}. \quad (7.1)$$

See eq. (2.24) to understand the first one. Since it is straightforward to take the vacuum limit in the expressions of the helio and the extrinsic UV contributions, $\mathcal{P}(\nu_\beta \rightarrow \nu_\alpha)^{(0+1)}_{\text{helio}} + \mathcal{P}(\nu_\beta \rightarrow \nu_\alpha)^{(1)}_{\text{ext-UV}}$, we do not write the explicit forms of the oscillation probabilities in vacuum.

#### 7.2 Non-unitarity and Non-standard interactions (NSI)

A question is often raised: What is the relationship between non-unitarity and non-standard interactions (NSI) [52–54]? A short answer is that starting from a generic situation which include not only NSI in propagation, but also the ones in production and detection our framework could be reproduced by placing appropriate relations between the propagation, production, and detection NSI. Notice that the latter two introduce non-unitarity [55]. However, it implies a huge reduction of number of parameters, 27 to 9, excluding $\nu_{\text{SM}}$ ones. In addition, the statement is true only if the ratio of neutron number density to electron number density is constant over the entire environment we deal with. Clearly, the condition is not valid in the sun, and is broken even inside the Earth. Assuming $N_e = r N_n$ (r is a constant) a more detailed correspondence may be established for propagation NSI. Notice that neutrino propagation with NSI is usually formulated by implementing unitarity (see e.g., [56, 57]). Since our intrinsic UV part of neutrino evolution is unitary, it is possible
to establish one to one correspondence between our $\alpha$ parameters and the propagation NSI elements $\epsilon_{\alpha\beta}$ ($\alpha, \beta = e, \mu, \tau$), as shown in [25] for the case of $r = 1$.$^{12}$

8 Conclusion remarks

In this paper, we have formulated a perturbative framework which is called the “helio-UV (unitarity violation) perturbation theory”. It utilizes the two kind of expansion parameters, $\epsilon \approx \Delta m^2_{21}/\Delta m^2_{31}$ and the UV $\alpha$ parameters. To our knowledge it is one of the first trials to formulate perturbation theory of neutrino oscillation with UV in matter. As an outcome of first-order computation of the oscillation probability, we were able to obtain the following interesting results:

- The phases of the complex UV parameters always come in into the observable in the particular combination with the $\nu$SM CP phase $\delta$, $[e^{-i\delta} \alpha_{\mu e}, \alpha_{\tau e}, e^{i\delta} \alpha_{\tau \mu}]$, under the phase convention of $U_{\text{MNS}}$ in which $e^{\pm i\delta}$ is attached to $s_{23}$.

- We have also observed that the way the complex $\alpha$ parameters are correlated with $\delta$ is $U_{\text{MNS}}$ convention dependent, which stems from convention dependence of the $\alpha$ matrix. It is $[e^{-i\delta} \alpha_{\mu e}, e^{-i\delta} \alpha_{\tau e}, \alpha_{\tau \mu}]$ in the PDG convention, and no correlation between $\alpha$ parameters and $\delta$ in the $U_{\text{MNS}}$ phase convention called $U_{\text{SOL}}$ with $e^{\pm i\delta}$ attached to $s_{12}$.

We would like to emphasize that this is a rare occasion in which the correlation between $\nu$SM and the new physics parameters is explicitly discussed and elucidated.

From these results, the most importance message, which could be relevant to the readers, is that usage of the $U_{\text{SOL}}$ phase convention and the associated $\tilde{\alpha}$ parameters (see (2.10) for definition) may be preferable for a merit of avoiding unwanted correlations between the physically different two groups of phases. It would simplify analyses of data to constrain UV with clearer interpretation of the results, and makes discussions of the parameter correlation and degeneracy more transparent.

Now, the important question is: “What do the above features of the phase correlation mean? Our particular concern is about the $U_{\text{MNS}}$ phase convention dependence uncovered in our study. One can argue that existence of the phase convention in which the UV $\alpha$ parameters do not have correlation with $\delta$ implies that it is of superficial nature. However, the correlation exists in all the phase convention except for $U_{\text{SOL}}$. The correlation is universal, i.e., the identical combinations in all the channels for a given $U_{\text{MNS}}$ phase convention. Then, an alternative interpretation which is natural in this line of thought is that the way $\nu$SM CP phase $\delta$ couples with the complex UV parameters is dictated by the framework of UV itself. In this paper, we are not able to make a definite choice from these two interpretations. One of the key obstacles is that nothing is known about whether the similar phase correlation

$^{12}$ The structure corresponding to $\epsilon_{\alpha\alpha} - \epsilon_{\beta\beta}$ for diagonal NSI elements, which is due to re-phasing freedom, is not visible in our oscillation probability formulas which are written by the $\alpha$ parameters. But, it must exist at the level of elements $H_{ij}$ defined in eq. (2.27). For an explicit demonstration of the former structure for NSI, see e.g., arXiv version 1 of ref. [56].
exists in regions outside validity of our perturbative framework. Furthermore, even within the current framework we must be able to give an all-order proof of the canonical phase combination for a firmer statement. To carry it out, however, one has to deal with the situation where “helio” and “UV” amplitudes interfere in a fully mixed way. We hope that we can return to these issues in the future.

As being a consistent framework, perturbation theory is often useful for finding answers to such qualitative questions as above, even though low order calculations may not be so accurate numerically. Yet, we have observed that our formulas for the first order UV corrections agree reasonably well with the exact results. Utility of first order formula must increase even more in the precision measurement era in which constraints on UV would reach to $|\alpha| \lesssim 10^{-3}$.

We have given a clarifying discussion on how (must be) unitary nature of neutrino evolution in high-scale UV is reconciled with non-unitarity of the whole system. Unitarity of neutrino evolution must be true even with the non-unitary mixing matrix, given the Schrödinger equation (2.1) in the mass eigenstate basis with hermitian Hamiltonian. Our formulation indicates explicitly that the propagation in matter of three flavor active neutrinos is indeed unitary with the propagation-$S$ matrix $U_{23}\tilde{S}U_{23}^{\dagger}$. Then, non-unitarity of the $S$ matrix, or of the oscillation probability, occurs only when initial and final projections of flavor states from/onto the mass eigenstates come into play. When understood in this way, the property holds to all orders in our helio-UV perturbation theory.

**Acknowledgments**

We thank Enrique Fernandez-Martinez for illuminating discussions about theories with non-unitarity, and Hiroshi Nunokawa for raising the issue discussed in appendix D. H.M. expresses a deep gratitude to Instituto Física Teórica, UAM/CSIC in Madrid, for its support via “Theoretical challenges of new high energy, astro and cosmo experimental data” project, Ref: 201650E082. I.M.S. acknowledges support from the Spanish grant FPA2015-65929-P (MINECO/FEDER, UE) and the Spanish Research Agency (“Agencia Estatal de Investigacion”) grants IFT “Centro de Excelencia Severo Ochoa” SEV2012-0249 and SEV-2016-0597. This work has received funding/support from the European Union’s Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie grant agreement No 674896 and No 690575.
A Expression of $H$ and $\Phi$ matrix elements

The expressions of the elements $H_{ij}$ defined in eq. (2.27) are given by

\[
H_{11} = 2\alpha e e \left(1 - \frac{\Delta_{2}}{\Delta_{1}}\right),
\]
\[
H_{12} = e^{-i\delta} \left\{ c_{23} \left(e^{-i\delta} \alpha_{\mu e} \right)^{*} - s_{23} \alpha_{e e} \right\},
\]
\[
H_{13} = s_{23} \left( e^{-i\delta} \alpha_{\mu e} \right)^{*} + c_{23} \alpha_{e e}^*,
\]
\[
H_{21} = e^{i\delta} \left( c_{23} e^{-i\delta} \alpha_{\mu e} - s_{23} \alpha_{e e} \right),
\]
\[
H_{22} = 2 \left[ c_{23}^{2} \alpha_{\mu \mu} + s_{23}^{2} \alpha_{e \tau} - c_{23} s_{23} \text{Re} \left(e^{i\delta} \alpha_{\mu e} \right) \right],
\]
\[
H_{23} = e^{i\delta} \left( 2 c_{23} s_{23} (\alpha_{\mu \mu} - \alpha_{e \tau}) + c_{23}^{2} \left( e^{i\delta} \alpha_{\mu e} \right)^{*} - s_{23}^{2} e^{i\delta} \alpha_{\mu e} \right),
\]
\[
H_{31} = s_{23} e^{-i\delta} \alpha_{\mu e} + c_{23} \alpha_{e e},
\]
\[
H_{32} = e^{-i\delta} \left[ 2 c_{23} s_{23} (\alpha_{\mu \mu} - \alpha_{e \tau}) + c_{23}^{2} e^{i\delta} \alpha_{\mu e} - s_{23}^{2} \left( e^{i\delta} \alpha_{\mu e} \right)^{*} \right],
\]
\[
H_{33} = 2 \left[ s_{23}^{2} \alpha_{\mu \mu} + c_{23}^{2} \alpha_{e \tau} + c_{23} s_{23} \text{Re} \left(e^{i\delta} \alpha_{\mu e} \right) \right].
\]

The expressions of $\Phi_{ij}$ defined by eq. (2.43) in section 2.9 is given by

\[
\Phi_{11} = \left\{ H_{11} - 2 c_{23}^{2} s_{33}^{2} (H_{11} - H_{33}) - c_{23} s_{23} \cos 2\phi (H_{13} + H_{31}) \right\}
\]
\[
+ e^{i(h_{3} - h_{1})x} \left\{ c_{23}^{2} s_{33}^{2} (H_{11} - H_{33}) + c_{23} s_{23} \left( -s_{23}^{2} H_{13} + c_{23}^{2} H_{31} \right) \right\},
\]
\[
+ e^{-i(h_{3} - h_{1})x} \left\{ c_{23}^{2} s_{33}^{2} (H_{11} - H_{33}) + c_{23} s_{23} \left( c_{23}^{2} H_{13} - s_{23}^{2} H_{31} \right) \right\},
\]
\[
\Phi_{12} = e^{-i(h_{2} - h_{1})x} \left( c_{23}^{2} H_{12} - c_{23} s_{23} H_{32} \right) + e^{i(h_{3} - h_{2})x} \left( s_{23}^{2} H_{12} + c_{23} s_{23} H_{32} \right),
\]
\[
\Phi_{13} = \left\{ -c_{23} s_{33} \cos 2\phi (H_{11} - H_{33}) + 2 c_{23}^{2} s_{33}^{2} (H_{31} + H_{13}) \right\}
\]
\[
+ e^{i(h_{3} - h_{1})x} \left\{ -c_{23} s_{33}^{2} (H_{11} - H_{33}) + s_{23}^{2} H_{13} - c_{23} s_{23}^{2} H_{31} \right\},
\]
\[
+ e^{-i(h_{3} - h_{1})x} \left\{ c_{23}^{2} s_{33}^{3} (H_{11} - H_{33}) + c_{23}^{4} H_{13} - c_{23}^{2} s_{23}^{2} H_{31} \right\},
\]
\[
\Phi_{21} = e^{i(h_{2} - h_{1})x} \left( c_{23}^{2} H_{21} - c_{23} s_{23} H_{23} \right) + e^{-i(h_{3} - h_{2})x} \left( s_{23}^{2} H_{21} + c_{23} s_{23} H_{23} \right),
\]
\[
\Phi_{22} = H_{22},
\]
\[
\Phi_{23} = e^{i(h_{2} - h_{1})x} \left\{ -c_{23} s_{23} H_{21} + s_{23}^{2} H_{23} \right\} + e^{-i(h_{3} - h_{2})x} \left\{ c_{23} s_{23} H_{21} + c_{23}^{2} H_{23} \right\},
\]
\[
\Phi_{31} = \left\{ -c_{23} s_{33} \cos 2\phi (H_{11} - H_{33}) + 2 c_{23}^{2} s_{33}^{2} (H_{13} + H_{31}) \right\}
\]
\[
+ e^{i(h_{3} - h_{1})x} \left\{ c_{23}^{2} s_{33}^{2} (H_{11} - H_{33}) - c_{23}^{2} s_{23}^{2} H_{13} + c_{23}^{4} H_{31} \right\},
\]
\[
+ e^{-i(h_{3} - h_{1})x} \left\{ -c_{23}^{3} s_{33}^{2} (H_{11} - H_{33}) - c_{23}^{2} s_{33}^{2} H_{13} + s_{23}^{4} H_{31} \right\},
\]
\[
\Phi_{32} = e^{-i(h_{2} - h_{1})x} \left\{ -c_{23} s_{23} H_{12} + s_{23}^{2} H_{32} \right\} + e^{i(h_{3} - h_{2})x} \left\{ c_{23} s_{23} H_{12} + c_{23}^{2} H_{32} \right\},
\]
\[
\Phi_{33} = \left\{ H_{33} + 2 c_{23}^{2} s_{33}^{2} (H_{11} - H_{33}) + c_{23} s_{23} \cos 2\phi (H_{13} + H_{31}) \right\}
\]
\[
+ e^{i(h_{3} - h_{1})x} \left\{ -c_{23}^{2} s_{33}^{2} (H_{11} - H_{33}) + c_{23} s_{33}^{3} H_{13} - c_{23}^{3} s_{23} H_{31} \right\}
\]
\[
+ e^{-i(h_{3} - h_{1})x} \left\{ -c_{23}^{2} s_{33}^{2} (H_{11} - H_{33}) - c_{23}^{3} s_{33} H_{13} + c_{23} s_{23}^{2} H_{31} \right\}.
\]
B Expressions of $\tilde{S}_{UV}^{(1)}$ matrix elements

The expressions of $\tilde{S}_{UV}^{(1)}$ matrix elements computed with $\tilde{S}_{UV}^{(1)} = U_\phi \tilde{S}_{\text{matt}}^{(1)} U_\phi^\dagger$ where $\tilde{S}_{\text{matt}}^{(1)}$ is defined by eq. (2.44) in section 2.9 is given by

$$
\begin{align*}
\left( \tilde{S}_{UV}^{(1)} \right)_{11} &= (-i)(\Delta_b x) \left[ \left( s_\phi^2 e^{-ih_3 x} + c_\phi^2 e^{-ih_1 x} \right) H_{11} - c_\phi s_\phi \left( e^{-ih_3 x} + e^{-ih_1 x} \right) \right] (H_{11} - H_{33}) \\
&+ c_\phi s_\phi \left( s_\phi^2 e^{-ih_3 x} - c_\phi^2 e^{-ih_1 x} \right) (H_{13} + H_{31}) \\
&+ \frac{\Delta_b}{h_3 - h_1} \left( e^{-ih_3 x} - e^{-ih_1 x} \right) \left[ 2c_\phi^2 s_\phi (H_{11} - H_{33}) + c_\phi s_\phi \cos 2\phi (H_{13} + H_{31}) \right]. \\
&B.1
\end{align*}
$$

$$
\begin{align*}
\left( \tilde{S}_{UV}^{(1)} \right)_{21} &= \Delta_b \left[ \frac{e^{-ih_2 x} - e^{-ih_1 x}}{(h_2 - h_1)} \left( c_\phi^2 H_{21} - c_\phi s_\phi H_{23} \right) + \frac{e^{-ih_3 x} - e^{-ih_2 x}}{(h_3 - h_2)} \left( s_\phi^2 H_{21} + c_\phi s_\phi H_{23} \right) \right]. \\
&B.2
\end{align*}
$$

$$
\begin{align*}
\left( \tilde{S}_{UV}^{(1)} \right)_{31} &= (-i)(\Delta_b x) \left[ c_\phi s_\phi \left( e^{-ih_3 x} - e^{-ih_1 x} \right) H_{11} - c_\phi s_\phi \left( 2c_\phi e^{-ih_3 x} - s_\phi^2 e^{-ih_1 x} \right) \right] (H_{11} - H_{33}) \\
&+ c_\phi^2 s_\phi \left( e^{-ih_3 x} + e^{-ih_1 x} \right) (H_{13} + H_{31}) \\
&+ \frac{\Delta_b}{h_3 - h_1} \left( e^{-ih_3 x} - e^{-ih_1 x} \right) \left[ c_\phi s_\phi \cos 2\phi (H_{11} - H_{33}) + H_{31} - 2c_\phi^2 s_\phi^2 (H_{13} + H_{31}) \right]. \\
&B.3
\end{align*}
$$

$$
\begin{align*}
\left( \tilde{S}_{UV}^{(1)} \right)_{12} &= \Delta_b \left[ \frac{e^{-ih_2 x} - e^{-ih_1 x}}{(h_2 - h_1)} \left( c_\phi^2 H_{12} - c_\phi s_\phi H_{32} \right) + \frac{e^{-ih_3 x} - e^{-ih_2 x}}{(h_3 - h_2)} \left( s_\phi^2 H_{12} + c_\phi s_\phi H_{32} \right) \right]. \\
&B.4
\end{align*}
$$

$$
\begin{align*}
\left( \tilde{S}_{UV}^{(1)} \right)_{22} &= (-i)(\Delta_b x) e^{-ih_2 x} H_{22}. \\
&B.5
\end{align*}
$$

$$
\begin{align*}
\left( \tilde{S}_{UV}^{(1)} \right)_{32} &= \Delta_b \left[ \frac{e^{-ih_2 x} - e^{-ih_1 x}}{(h_2 - h_1)} \left( c_\phi s_\phi H_{12} - s_\phi^2 H_{32} \right) + \frac{e^{-ih_3 x} - e^{-ih_2 x}}{(h_3 - h_2)} \left( c_\phi s_\phi H_{12} + c_\phi^2 H_{32} \right) \right]. \\
&B.6
\end{align*}
$$

$$
\begin{align*}
\left( \tilde{S}_{UV}^{(1)} \right)_{13} &= (-i)(\Delta_b x) \left[ c_\phi s_\phi \left( e^{-ih_3 x} - e^{-ih_1 x} \right) H_{33} + c_\phi s_\phi \left( s_\phi^2 e^{-ih_3 x} - c_\phi^2 e^{-ih_1 x} \right) \right] (H_{11} - H_{33}) \\
&+ c_\phi^2 s_\phi \left( e^{-ih_3 x} + e^{-ih_1 x} \right) (H_{13} + H_{31}) \\
&+ \frac{\Delta_b}{h_3 - h_1} \left( e^{-ih_3 x} - e^{-ih_1 x} \right) \left[ c_\phi s_\phi \cos 2\phi (H_{11} - H_{33}) + H_{13} - 2c_\phi^2 s_\phi^2 (H_{13} + H_{31}) \right]. \\
&B.7
\end{align*}
$$

$$
\begin{align*}
\left( \tilde{S}_{UV}^{(1)} \right)_{23} &= \Delta_b \left[ \frac{e^{-ih_2 x} - e^{-ih_1 x}}{(h_2 - h_1)} \left( c_\phi s_\phi H_{21} - s_\phi^2 H_{23} \right) + \frac{e^{-ih_3 x} - e^{-ih_2 x}}{(h_3 - h_2)} \left( c_\phi s_\phi H_{21} + c_\phi^2 H_{23} \right) \right]. \\
&B.8
\end{align*}
$$
In this section, we discuss defined in eqs. (3.2) with (3.3), we present the oscillation probabilities in $C$. The oscillation probabilities in $\nu$ ones which may be convenient to verify unitarity: $P$ Similarly, if necessary, one can compute $P(\nu_\tau \to \nu_\mu)$ in the same way. The former can be used to verify unitarity in $\nu_\tau$ row with the other probabilities $P(\nu_\tau \to \nu_\mu)$ and $P(\nu_\tau \to \nu_\mu)$ which can be obtained by generalized $T$ transformation from eqs. (4.3) and (C.4), respectively.

The formulas written here may be more reader friendly compared to the ones in ref. [19] which are presented in a condensed and abstract fashion.

$$\left( \tilde{S}_{UV}^{(1)} \right)_{33} = (-i)(\Delta_{bw}) \left[ c_\phi^2 e^{-i\theta_1 x} + s_\phi^2 e^{-i\theta_3 x} \right] H_{33} + c_\phi^2 s_\phi^2 \left( e^{-i\theta_3 x} + e^{-i\theta_1 x} \right) (H_{11} - H_{33}) + c_\phi s_\phi \left( c_\phi^2 e^{-i\theta_3 x} - s_\phi^2 e^{-i\theta_1 x} \right) (H_{13} + H_{31})$$

$$- \frac{\Delta_b}{h_3 - h_1} \left[ e^{-i\theta_3 x} - e^{-i\theta_1 x} \right] \left[ 2c_\phi^2 s_\phi^2 (H_{11} - H_{33}) + c_\phi s_\phi \cos 2\phi (H_{13} + H_{31}) \right]. \quad (B.9)$$

C The oscillation probabilities in $\nu_\mu \to \nu_\tau$ sector

In this section, we discuss $\nu_\mu \to \nu_\mu$ and $\nu_\mu \to \nu_\tau$ channels in parallel. Using the notations defined in eqs. (3.2) with (3.3), we present the oscillation probabilities in $\nu_\mu \to \nu_\tau$ sector. We start from the zeroth-order and the “helio contributions”, by just copying the “simple and compact” formula in [19] for self-containment.

C.1 $P(\nu_\mu \to \nu_\mu)^{(0+1)}_{\text{helio}}$ and $P(\nu_\mu \to \nu_\tau)^{(0+1)}_{\text{helio}}$: “the simple and compact” formula

There are many ways to write $P(\nu_\mu \to \nu_\mu)^{(0+1)}_{\text{helio}}$ and $P(\nu_\mu \to \nu_\tau)^{(0+1)}_{\text{helio}}$. We present here the ones which may be convenient to verify unitarity:

$$P(\nu_\mu \to \nu_\mu)^{(0+1)}_{\text{helio}}$$

$$= 1 - \left[ s_{23}^4 \sin^2 2\phi + 8\epsilon J_r \cos \delta \sin^2 2\phi \Delta_{\text{ren}}^2 \frac{(h_3 - h_1) - (\Delta_{\text{ren}} - \Delta_\mu)}{(h_3 - h_1)^2(h_3 - h_2)} \right] \frac{\sin 2(h_3 - h_1)x}{2}$$

$$- \left[ \sin^2 2\theta_{23} c_\phi^2 - 4\epsilon \left( J_r \cos \delta / c_{13}^2 \right) \cos 2\theta_{23} \Delta_{\text{ren}} \frac{(h_3 - h_1) - (\Delta_{\text{ren}} + \Delta_\mu)}{(h_3 - h_1)(h_3 - h_2)} \right] \frac{\sin 2(h_3 - h_2)x}{2}$$

$$- \left[ \sin^2 2\theta_{23} s_\phi^2 - 4\epsilon \left( J_r \cos \delta / c_{13}^2 \right) \cos 2\theta_{23} \Delta_{\text{ren}} \frac{(h_3 - h_1) + (\Delta_{\text{ren}} + \Delta_\mu)}{(h_3 - h_1)(h_1 - h_2)} \right] \frac{\sin 2(h_1 - h_2)x}{2}$$

$$- 16\epsilon J_r \cos \delta \frac{\sin^2 2\phi \Delta_{\text{ren}}^3}{(h_3 - h_1)(h_3 - h_2)(h_1 - h_2)} \frac{\sin (h_3 - h_1)x}{2} \frac{\sin (h_1 - h_2)x}{2} \frac{\cos (h_3 - h_2)x}{2}.$$ \quad (C.1)
\[
\begin{align*}
P(\nu_\mu \to \nu_\tau)^{(0+1)}_{\text{helio}} &= -\left[ \frac{2 \Delta_\nu}{(h_3 - h_1)} \sin^2 2\phi + 4 \epsilon J_{\nu_c} \cos \delta \cos 2\theta_{23} \frac{(\Delta_{\text{ren}})^2 \{ (h_3 - h_1) - (\Delta_{\text{ren}} - \Delta_n) \}}{(h_3 - h_1)^2 (h_3 - h_2)} \right] \sin^2 \frac{(h_3 - h_1)x}{2} \\
+ &\left[ \frac{\sin^2 2\theta_{23} c_\phi^2 - 4 \epsilon (J_{\nu_c} \cos \delta / c_{13})}{\cos 2\theta_{23}} \frac{\Delta_{\text{ren}} \{ (h_3 - h_1) + (\Delta_{\text{ren}} + \Delta_n) \}}{(h_3 - h_1)(h_3 - h_2)} \right] \sin^2 \frac{(h_3 - h_2)x}{2} \\
+ &\left[ \frac{\sin^2 2\theta_{23} c_\phi^2 - 4 \epsilon (J_{\nu_c} \cos \delta / c_{13})}{\cos 2\theta_{23}} \frac{\Delta_{\text{ren}} \{ (h_3 - h_1) + (\Delta_{\text{ren}} + \Delta_n) \}}{(h_3 - h_1)(h_3 - h_2)} \right] \sin^2 \frac{(h_1 - h_2)x}{2}
\end{align*}
\]

\[\times \left[ \cos 2\theta_{23} \cos \delta \cos \frac{(h_3 - h_2)x}{2} - \sin \delta \frac{(h_3 - h_2)x}{2} \right]. \quad (C.2)\]

C.2 \(P(\nu_\mu \to \nu_\mu)^{(1)}_{\text{int-UV}}\) and \(P(\nu_\mu \to \nu_\tau)^{(1)}_{\text{int-UV}}\): Intrinsic UV contribution

The first order intrinsic UV correction to the oscillation probability \(P(\nu_\mu \to \nu_\mu)\) reads

\[
P(\nu_\mu \to \nu_\mu)^{(1)}_{\text{int-UV}}
= -\sin^2 2\theta_{23} \left[ \cos 2\theta_{23} (\alpha_{\tau\tau} - \alpha_{\mu\mu}) + \sin 2\theta_{23} \text{Re} \left( e^{i\delta} \alpha_{\tau\mu} \right) \right]
\]

\[
\times (\Delta_\phi) \left\{ c_\phi^2 \sin(h_3 - h_2)x - s_\phi^2 \sin(h_2 - h_1)x \right\}
\]

\[
- 4s_{23}^2 c_\phi s_\phi \left[ \frac{(\Delta_{\text{ren}})^3}{(h_3 - h_1)(h_3 - h_2)} \right] \sin \frac{(h_3 - h_1)x}{2} \sin \frac{(h_1 - h_2)x}{2}
\]

\[
\times \left[ \cos 2\theta_{23} \cos \delta \cos \frac{(h_3 - h_2)x}{2} - \sin \delta \sin \frac{(h_3 - h_2)x}{2} \right]. \quad (C.2)\]
While the first order intrinsic UV correction to the appearance oscillation probability $P(\nu_\mu \rightarrow \nu_\tau)$ reads

\[
P(\nu_\mu \rightarrow \nu_\tau)^{(1)}_{\text{int-UV}} = \sin^2 2\theta_{23} \cos 2\theta_{23} (\alpha_{\tau \tau} - \alpha_{\mu \mu}) + \sin 2\theta_{23} \text{Re} \left( e^{i\delta_{\alpha_{\mu \mu}}} \right) \left( \Delta_{\nu} \right) \left\{ c_\phi^2 \sin(h_3 - h_2)x - s_\phi^2 \sin(h_2 - h_1)x \right\} \\
+ \sin^2 2\theta_{23} c_\phi^2 \sin^2 \left[ e^{i\alpha_{\mu}} \left( 1 - \frac{\Delta_n}{\Delta_b} \right) - s_{23}^2 \cos 2\phi \sin(h_3 - h_1)x \right] \\
\times \left( \Delta_{\nu} \right) \left\{ s_{23} \text{Re} \left( e^{-i\delta_{\alpha_{\mu}}} \right) + c_{23} \text{Re} \left( \alpha_{\tau e} \right) \right\} \\
+ \sin^2 2\theta_{23} c_\phi s_\phi \left[ c_{23} \text{Re} \left( e^{-i\delta_{\alpha_{\mu}}} \right) - s_{23} \text{Re} \left( \alpha_{\tau e} \right) \right] - \cos 2\theta_{23} \left( \frac{\Delta_n}{\Delta_b} \right) \left\{ \frac{1}{2} \left( c_\phi^2 \sin^2 \left( h_3 - h_2 \right)x - s_\phi^2 \sin^2 \left( h_3 - h_1 \right)x \right) + \frac{1}{2} \left( h_3 - h_1 \right) \right\} \\
- 2\sin 2\theta_{23} \cos 2\theta_{23} \left\{ c_\phi s_\phi \left[ e^{i\delta_{\alpha_{\mu}}} - s_{23} \text{Re} \left( \alpha_{\tau e} \right) \right] - \sin^2 2\phi \left[ \frac{1}{2} \left( h_3 - h_1 \right) \right] - \sin^2 \left( h_3 - h_1 \right) \right\} \\
+ 2\sin 2\theta_{23} \cos 2\theta_{23} \left\{ \frac{1}{2} \left( h_3 - h_2 \right) \right\} \left\{ \frac{1}{2} \left( c_\phi^2 \sin^2 \left( h_3 - h_2 \right)x - s_\phi^2 \sin^2 \left( h_3 - h_1 \right) \right) \right\} \\
- 4 \sin 2\theta_{23} c_\phi^2 \left\{ c_\phi s_\phi \left[ e^{-i\delta_{\alpha_{\mu}}} - s_{23} \text{Im} \left( \alpha_{\tau e} \right) \right] + s_\phi^2 \text{Re} \left( e^{i\delta_{\alpha_{\mu}}} \right) \right\} \\
- 4 \sin 2\theta_{23} s_\phi^2 \left\{ c_\phi s_\phi \left[ e^{-i\delta_{\alpha_{\mu}}} - s_{23} \text{Im} \left( \alpha_{\tau e} \right) \right] - c_\phi^2 \text{Im} \left( e^{i\delta_{\alpha_{\tau}}} \right) \right\} \\
- 4 \sin 2\theta_{23} s_\phi^2 \left\{ c_\phi s_\phi \left[ e^{-i\delta_{\alpha_{\mu}}} - s_{23} \text{Im} \left( \alpha_{\tau e} \right) \right] - c_\phi^2 \text{Im} \left( e^{i\delta_{\alpha_{\tau}}} \right) \right\} \\
\times \frac{\Delta_b}{h_3 - h_2} \frac{\sin \left( h_3 - h_1 \right)x}{\sin \left( h_2 - h_3 \right)x} \frac{\sin \left( h_1 - h_2 \right)x}{\sin \left( h_2 - h_3 \right)x} \frac{\sin \left( h_2 - h_1 \right)x}{\sin \left( h_3 - h_1 \right)x} \tag{C.4}
\]

The expressions of $P(\nu_\mu \rightarrow \nu_\mu)^{(1)}_{\text{int-UV}}$ in eq. (C.3), and $P(\nu_\mu \rightarrow \nu_\tau)^{(1)}_{\text{int-UV}}$ in eq. (C.4) are another explicit demonstration of the canonical phase combination, $e^{-i\delta_{\alpha_{\mu}}} \alpha_{\tau e}$, and $e^{i\delta_{\alpha_{\mu}}} \alpha_{\tau e}$, with no correlation between $\delta$ and the diagonal $\alpha$ parameters. The exposition of the mechanism which leads to the canonical phase combination is given in section 6.1.
C.3 \( P(\nu_\mu \to \nu_\mu)^{(1)}_{\text{ext-UV}} \) and \( P(\nu_\mu \to \nu_\tau)^{(1)}_{\text{ext-UV}} \): Extrinsic UV contribution

The first order extrinsic UV contributions \( P(\nu_\mu \to \nu_\mu)^{(1)}_{\text{ext-UV}} \) and \( P(\nu_\mu \to \nu_\tau)^{(1)}_{\text{ext-UV}} \) are given by

\[
P(\nu_\mu \to \nu_\mu)^{(1)}_{\text{ext-UV}} &= 4s_{23}\sin 2\phi \Re \left( e^{-i\delta \alpha_{\mu e}} \right) \left[ -s_{23}^2 \cos 2\phi \sin^2 \left( \frac{h_3 - h_1}{2} \right) + c_{23}^2 \left\{ \sin^2 \left( \frac{h_3 - h_2}{2} \right) - \sin^2 \left( \frac{h_2 - h_1}{2} \right) \right\} \right] \\
&+ 4\alpha_{\mu \tau} \left[ -1 + s_{23}^4 \sin^2 2\phi \sin^2 \left( \frac{h_3 - h_1}{2} \right) + \sin^2 2\theta_{23} \left\{ c_{\phi}^2 \sin^2 \left( \frac{h_3 - h_2}{2} \right) + s_{\phi}^2 \sin^2 \left( \frac{h_2 - h_1}{2} \right) \right\} \right] \quad \text{(C.5)}
\]

\[
P(\nu_\mu \to \nu_\tau)^{(1)}_{\text{ext-UV}} &= -\sin 2\theta_{23} \sin \phi \left[ c_{23} \Re \left( e^{-i\delta \alpha_{\mu e}} \right) + s_{23} \Re \left( \alpha_{\tau e} \right) \right] \\
&\times \left\{ \cos 2\phi \sin^2 \left( \frac{h_3 - h_1}{2} \right) + \sin^2 \left( \frac{h_3 - h_2}{2} \right) - \sin^2 \left( \frac{h_2 - h_1}{2} \right) \right\} \\
+ \sin 2\theta_{23} \sin^2 2\phi \left[ s_{23}^2 \Re \left( e^{i\delta \alpha_{\mu \tau}} \right) + c_{23}s_{23} \left( \alpha_{\mu \tau} + \alpha_{\tau \mu} \right) \right] \sin^2 \left( \frac{h_3 - h_1}{2} \right) \\
+ 2\sin 2\theta_{23} \left[ \cos 2\phi \Re \left( e^{i\delta \alpha_{\tau e}} \right) \right] \left\{ c_{\phi}^2 \sin^2 \left( \frac{h_3 - h_2}{2} \right) + s_{\phi}^2 \sin^2 \left( \frac{h_2 - h_1}{2} \right) \right\} \right\} \\
- 2\sin 2\theta_{23} \sin 2\phi \left[ c_{23} \Im \left( e^{-i\delta \alpha_{\mu e}} \right) - s_{23} \Im \left( \alpha_{\tau e} \right) \right] \sin \left( \frac{h_3 - h_1}{2} \right) \sin \left( \frac{h_1 - h_2}{2} \right) \sin \left( \frac{h_2 - h_3}{2} \right) \\
+ \sin 2\theta_{23} \Im \left( e^{i\delta \alpha_{\tau e}} \right) \left\{ c_{\phi}^2 \sin(h_3 - h_2)x - s_{\phi}^2 \sin(h_2 - h_1)x \right\} \quad \text{(C.6)}
\]

C.4 Perturbative unitarity yes or no of intrinsic and extrinsic UV contributions: \( \nu_\mu \) row

Given the expressions of the oscillation probabilities in \( \nu_\mu \) row in eqs. (3.9), (C.3), and (C.4), it is straightforward to prove perturbative unitarity for neutrino evolution only with intrinsic UV corrections to first order in \( \alpha \)’s

\[
P(\nu_\mu \to \nu_\epsilon)^{(1)}_{\text{Int-UV}} + P(\nu_\mu \to \nu_\mu)^{(1)}_{\text{Int-UV}} + P(\nu_\mu \to \nu_\tau)^{(1)}_{\text{Int-UV}} = 0. \quad \text{(C.7)}
\]

On the other hand, the extrinsic UV corrections in first order in \( \alpha \) parameters in the oscillation probabilities in \( \nu_\mu \) row, eqs. (3.10), (C.5), and (C.6), do not cancel out as in the case of \( \nu_\epsilon \) row, giving no indication of even for a partial cancellation. Therefore, clearly the extrinsic UV corrections do not respect unitarity.

D Identifying the relevant variables

When non-unitarity is introduced the number of parameters increases from six (\( \nu \text{SM} \)) to fifteen (adding nine \( \alpha \) parameters), a growth by a factor of 2.5. We look for a possibility of reducing the number of parameters by finding an extra small parameter by which the oscillation probability can be expanded. By the estimate \( a/\Delta m^2_{\text{ren}} \lesssim 0.1 \) (assuming \( Y_\epsilon = 0.5 \)) for \( E = 1 \text{ GeV} \) and \( \rho = 3 \text{ g/cm}^3 \), \( \sin \phi \) can be approximated as \( s_{13} \). Since the measured
value of $\theta_{13}$ is small, $s_{13} = 0.148$, which is the one from the largest statistics measurement [58].\textsuperscript{15} it can be used as another expansion parameter. Then, we can expand the probability formulas in terms of $s_\phi \equiv \sin \phi \simeq s_{13}$ to first order, assuming $\rho E \ll 10 \text{ GeV g/cm}^3$.\textsuperscript{16}

Given the oscillation probability formulas tabulated in table 2, it is easy to expand $P(\nu_\beta \to \nu_\alpha)$ to first order in $s_\phi$. Then, we count the $\alpha$ parameters that remain in the zeroth- and the first-order formulas. The results of this exercise are presented in table 2.

Table 2. The UV $\alpha$ parameters which are present in $P(\nu_\beta \to \nu_\alpha)^{(1)}_{UV}$ to zeroth (second column) and to the first order (third column) in $\sin \phi$. The results for anti-neutrino channels are the same as the corresponding neutrino channels.

| channel | parameters in $P(\nu_\beta \to \nu_\alpha)^{(1)}_{UV}$ in zeroth order in $s_\phi$ | parameters in $P(\nu_\beta \to \nu_\alpha)^{(1)}_{UV}$ to first order in $s_\phi$ |
|---------|----------------------------------|----------------------------------|
| $\nu_e \to \nu_e$ | $\alpha_{ee}$ | $\text{left col. plus Re}(e^{-i\delta_{\mu e}}), \text{Re} (\alpha_{\tau e})$ |
| $\nu_e \to \nu_\mu$, $\nu_\mu \to \nu_e$ | does not apply | $\text{Re} (e^{-i\delta_{\mu e}}), \text{Im} (e^{-i\delta_{\mu e}}), \text{Re} (\alpha_{\tau e}), \text{Im} (\alpha_{\tau e})$ |
| $\nu_e \to \nu_\tau$, $\nu_\tau \to \nu_e$ | $\alpha_{\mu \mu}, \alpha_{\tau \tau}, \text{Re} (e^{i\delta_{\mu e}}) \alpha_{\tau e}$ | $\text{left col. plus Re} (e^{-i\delta_{\mu e}}), \text{Re} (\alpha_{\tau e})$ |
| $\nu_\mu \to \nu_\mu$ | $\alpha_{\mu \mu}, \alpha_{\tau \tau}, \text{Re} (e^{i\delta_{\mu e}}) \alpha_{\tau e}$ | $\text{left col. plus Re} (e^{-i\delta_{\mu e}}), \text{Re} (\alpha_{\tau e})$ |
| $\nu_\mu \to \nu_\tau$, $\nu_\tau \to \nu_\mu$ | $\alpha_{\mu \mu}, \alpha_{\tau \tau}, \text{Re} (e^{i\delta_{\mu e}}) \alpha_{\tau e}$ | $\text{left col. plus Re} (e^{-i\delta_{\mu e}}), \text{Re} (\alpha_{\tau e})$ |

A few remarks are in order: First of all, we should note that in the appearance channels, $\nu_\mu \to \nu_e$ and $\nu_\mu \to \nu_\tau$, all the nine UV parameters come in in propagation in matter if we do not expand in terms of $\sin \phi$. When expended by $\sin \phi$ to first order, reduction of number of parameters is effective for $\nu_\mu \to \nu_e$ and $\nu_e \to \nu_\tau$ channels, only four parameters out of nine. On the other hand, reduction of number of parameters to first order in $\sin \phi$ is not so effective for $\nu_\mu \to \nu_\mu$ and $\nu_\mu \to \nu_\tau$ channels, missing only a single parameter $\alpha_{ee}$.

References

[1] Z. Maki, M. Nakagawa and S. Sakata, “Remarks on the unified model of elementary particles,” Prog. Theor. Phys. 28 (1962) 870. doi:10.1143/PTP.28.870

[2] T. Kajita, “Nobel Lecture: Discovery of atmospheric neutrino oscillations,” Rev. Mod. Phys. 88 (2016) no.3, 030501. doi:10.1103/RevModPhys.88.030501

[3] A. B. McDonald, “Nobel Lecture: The Sudbury Neutrino Observatory: Observation of flavor change for solar neutrinos,” Rev. Mod. Phys. 88 (2016) no.3, 030502. doi:10.1103/RevModPhys.88.030502

[4] M. Kobayashi and T. Maskawa, “CP Violation in the Renormalizable Theory of Weak Interaction,” Prog. Theor. Phys. 49 (1973) 652. doi:10.1143/PTP.49.652

\textsuperscript{15} See some recent global fits [10, 59, 60] for the similar values of $s_{13}$.

\textsuperscript{16} Though we take a short cut here, one can formulate a systematic expansion by $s_{13}$, called “$\sqrt{e}$ perturbation theory” [57, 61].
[5] K. Abe et al. [T2K Collaboration], “Search for CP Violation in Neutrino and Antineutrino Oscillations by the T2K Experiment with $2.2 \times 10^{21}$ Protons on Target,” Phys. Rev. Lett. 121 (2018) no.17, 171802 doi:10.1103/PhysRevLett.121.171802 [arXiv:1807.07891 [hep-ex]].

[6] M. A. Acero et al. [NOvA Collaboration], “New constraints on oscillation parameters from $\nu_e$ appearance and $\nu_\mu$ disappearance in the NOvA experiment,” Phys. Rev. D 98 (2018) 032012 doi:10.1103/PhysRevD.98.032012 [arXiv:1806.00096 [hep-ex]].

[7] K. Abe et al., “Proposal for an Extended Run of T2K to $20 \times 10^{21}$ POT,” arXiv:1609.04111 [hep-ex].

[8] H. Minakata and H. Nunokawa, “Exploring neutrino mixing with low-energy superbeams,” JHEP 0110 (2001) 001 doi:10.1088/1126-6708/2001/10/001 [hep-ph/0108085].

[9] K. Abe et al. [Super-Kamiokande Collaboration], “Atmospheric neutrino oscillation analysis with external constraints in Super-Kamiokande I-IV,” Phys. Rev. D 97 (2018) no.7, 072001 doi:10.1103/PhysRevD.97.072001 [arXiv:1710.09126 [hep-ex]].

[10] F. Capozzi, E. Lisi, A. Marrone and A. Palazzo, “Current unknowns in the three neutrino framework,” Prog. Part. Nucl. Phys. 102 (2018) 48 doi:10.1016/j.ppnp.2018.05.005 [arXiv:1804.09678 [hep-ph]].

[11] K. Abe et al. [Hyper-Kamiokande Proto- Collaboration], “Physics potential of a long-baseline neutrino oscillation experiment using a J-PARC neutrino beam and Hyper-Kamiokande,” PTEP 2015 (2015) 053C02 doi:10.1093/ptep/ptv061 [arXiv:1502.05199 [hep-ex]].

[12] K. Abe et al. [Hyper-Kamiokande proto- Collaboration], “Physics potentials with the second Hyper-Kamiokande detector in Korea,” PTEP 2018 (2018) no.6, 063C01 doi:10.1093/ptep/pty044 [arXiv:1611.06118 [hep-ex]].

[13] R. Acciarri et al. [DUNE Collaboration], “Long-Baseline Neutrino Facility (LBNF) and Deep Underground Neutrino Experiment (DUNE) : Volume 2: The Physics Program for DUNE at LBNF,” arXiv:1512.06148 [physics.ins-det].

[14] S. Antusch, C. Biggio, E. Fernandez-Martinez, M. B. Gavela and J. Lopez-Pavon, “Unitarity of the Leptonic Mixing Matrix,” JHEP 0610 (2006) 084 doi:10.1088/1126-6708/2006/10/084 [hep-ph/0607020].

[15] J. Schechter and J. W. F. Valle, “Neutrino Masses in SU(2) x U(1) Theories,” Phys. Rev. D 22 (1980) 2227. doi:10.1103/PhysRevD.22.2227

[16] V. D. Barger, P. Langacker, J. P. Leveille and S. Pakvasa, “Consequences of Majorana and Dirac Mass Mixing for Neutrino Oscillations,” Phys. Rev. Lett. 45 (1980) 692. doi:10.1103/PhysRevLett.45.692

[17] C. S. Fong, H. Minakata and H. Nunokawa, “A framework for testing leptonic unitarity by neutrino oscillation experiments,” JHEP 1702 (2017) 114 doi:10.1007/JHEP02(2017)114 [arXiv:1609.08623 [hep-ph]].

[18] C. S. Fong, H. Minakata and H. Nunokawa, “Non-unitary evolution of neutrinos in matter and the lepton unitarity test,” JHEP 1902 (2019) 015 doi:10.1007/JHEP02(2019)015 [arXiv:1712.02798 [hep-ph]].

[19] H. Minakata and S. J. Parke, “Simple and Compact Expressions for Neutrino Oscillation Probabilities in Matter,” JHEP 1601 (2016) 180 doi:10.1007/JHEP01(2016)180 [arXiv:1505.01826 [hep-ph]].
[20] R. Abbasi et al. [IceCube Collaboration], “The Design and Performance of IceCube DeepCore,” Astropart. Phys. 35 (2012) 615 doi:10.1016/j.astropartphys.2012.01.004 [arXiv:1109.6096 [astro-ph.IM]].

[21] M. G. Aartsen et al. [IceCube Collaboration], “PINGU: A Vision for Neutrino and Particle Physics at the South Pole,” J. Phys. G 44 (2017) no.5, 054006 doi:10.1088/1361-6471/44/5/054006 [arXiv:1607.02671 [hep-ex]].

[22] S. Adrián-Martínez et al., “Intrinsic limits on resolutions in muon- and electron-neutrino charged-current events in the KM3NeT/ORCA detector,” JHEP 1705 (2017) 008 doi:10.1007/JHEP05(2017)008 [arXiv:1612.05621 [physics.ins-det]].

[23] F. J. Escrihuela, D. V. Forero, O. G. Miranda, M. Tórtola and J. W. F. Valle, “On the description of non-unitary neutrino mixing,” Phys. Rev. D 92 (2015) no.5, 053009 doi:10.1103/PhysRevD.92.053009 [arXiv:1503.08879 [hep-ph]].

[24] S. Parke and M. Ross-Lonergan, “Unitarity and the three flavor neutrino mixing matrix,” Phys. Rev. D 93 (2016) no.11, 113009 doi:10.1103/PhysRevD.93.113009 [arXiv:1508.05095 [hep-ph]].

[25] M. Blennow, P. Coloma, E. Fernandez-Martinez, J. Hernandez-Garcia and J. Lopez-Pavon, “Non-Unitarity, sterile neutrinos, and Non-Standard neutrino Interactions,” JHEP 1704 (2017) 153 doi:10.1007/JHEP04(2017)153 [arXiv:1609.08637 [hep-ph]].

[26] E. Fernandez-Martinez, M. B. Gavela, J. Lopez-Pavon and O. Yasuda, “CP-violation from non-unitary leptonic mixing,” Phys. Lett. B 649 (2007) 427 doi:10.1016/j.physletb.2007.03.069 [hep-ph/0703098].

[27] S. Goswami and T. Ota, “Testing non-unitarity of neutrino mixing matrices at neutrino factories,” Phys. Rev. D 78 (2008) 033012 doi:10.1103/PhysRevD.78.033012 [arXiv:0802.1434 [hep-ph]].

[28] S. Antusch, M. Blennow, E. Fernandez-Martinez and J. Lopez-Pavon, “Probing non-unitary mixing and CP-violation at a Neutrino Factory,” Phys. Rev. D 80 (2009) 033002 doi:10.1103/PhysRevD.80.033002 [arXiv:0903.3986 [hep-ph]].

[29] S. Antusch, S. Blanchet, M. Blennow and E. Fernandez-Martinez, “Non-unitary Leptonic Mixing and Leptogenesis,” JHEP 1001 (2010) 017 doi:10.1007/JHEP01(2010)017 [arXiv:0910.5957 [hep-ph]].

[30] S. Antusch and O. Fischer, “Non-unitarity of the leptonic mixing matrix: Present bounds and future sensitivities,” JHEP 1410 (2014) 094 doi:10.1007/JHEP10(2014)094 [arXiv:1407.6607 [hep-ph]].

[31] E. Fernandez-Martinez, J. Hernandez-Garcia and J. Lopez-Pavon, “Global constraints on heavy neutrino mixing,” JHEP 1608 (2016) 033 doi:10.1007/JHEP08(2016)033 [arXiv:1605.08774 [hep-ph]].

[32] O. G. Miranda, M. Tortola and J. W. F. Valle, “New ambiguity in probing CP violation in neutrino oscillations,” Phys. Rev. Lett. 117 (2016) no.6, 061804 doi:10.1103/PhysRevLett.117.061804 [arXiv:1604.05690 [hep-ph]].

[33] S. F. Ge, P. Pasquini, M. Tortola and J. W. F. Valle, “Measuring the leptonic CP phase in neutrino oscillations with nonunitary mixing,” Phys. Rev. D 95 (2017) no.3, 033005 doi:10.1103/PhysRevD.95.033005 [arXiv:1605.01670 [hep-ph]].
[34] D. Dutta and P. Ghoshal, “Probing CP violation with T2K, NOνA and DUNE in the presence of non-unitarity,” JHEP 1609 (2016) 110 doi:10.1007/JHEP09(2016)110 [arXiv:1607.02500 [hep-ph]].

[35] D. Dutta, P. Ghoshal and S. Roy, “Effect of Non Unitarity on Neutrino Mass Hierarchy determination at DUNE, NOνA and T2K,” Nucl. Phys. B 920 (2017) 385 doi:10.1016/j.nuclphysb.2017.04.018 [arXiv:1609.07094 [hep-ph]].

[36] H. Päs and P. Sickling, “Discriminating sterile neutrinos and unitarity violation with CP invariants,” Phys. Rev. D 95 (2017) no.7, 075004 doi:10.1103/PhysRevD.95.075004 [arXiv:1611.08450 [hep-ph]].

[37] F. J. Escribuela, D. V. Forero, O. G. Miranda, M. Tórtola and J. W. F. Valle, “Probing CP violation with non-unitary mixing in long-baseline neutrino oscillation experiments: DUNE as a case study,” New J. Phys. 19 (2017) no.9, 093005 doi:10.1088/1367-2630/aa79ec [arXiv:1612.07377 [hep-ph]].

[38] J. Rout, M. Masud and P. Mehta, “Can we probe intrinsic CP and T violations and nonunitarity at long baseline accelerator experiments?,” Phys. Rev. D 95 (2017) no.7, 075035 doi:10.1103/PhysRevD.95.075035 [arXiv:1702.02163 [hep-ph]].

[39] Y. Abe, Y. Asano, N. Haba and T. Yamada, “Heavy neutrino mixing in the T2HK, the T2HKK and an extension of the T2HK with a detector at Oki Islands,” Eur. Phys. J. C 77 (2017) no.12, 851 doi:10.1140/epjc/s10052-017-5294-7 [arXiv:1705.03818 [hep-ph]].

[40] Y. F. Li, Z. Z. Xing and J. Y. Zhu, “Indirect unitarity violation entangled with matter effects in reactor antineutrino oscillations,” Phys. Lett. B 782 (2018) 578 doi:10.1016/j.physletb.2018.05.079 [arXiv:1802.04964 [hep-ph]].

[41] L. Wolfenstein, “Neutrino Oscillations in Matter,” Phys. Rev. D 17 (1978) 2369. doi:10.1103/PhysRevD.17.2369

[42] J. W. F. Valle, “Resonant Oscillations of Massless Neutrinos in Matter,” Phys. Lett. B 199 (1987) 432. doi:10.1016/0370-2693(87)90947-6

[43] A. Broncano, M. B. Gavela and E. E. Jenkins, “The Effective Lagrangian for the seesaw model of neutrino mass and lepton genesis,” Phys. Lett. B 552 (2003) 177 Erratum: [Phys. Lett. B 636 (2006) 332] doi:10.1016/j.physletb.2006.04.003, 10.1016/S0370-2693(02)03130-1 [hep-ph/0210271].

[44] Z. Z. Xing, “Correlation between the Charged Current Interactions of Light and Heavy Majorana Neutrinos,” Phys. Lett. B 660 (2008) 515 doi:10.1016/j.physletb.2008.01.038 [arXiv:0709.2220 [hep-ph]].

[45] Z. Z. Xing, “A full parametrization of the 6 X 6 flavor mixing matrix in the presence of three light or heavy sterile neutrinos,” Phys. Rev. D 85 (2012) 013008 doi:10.1103/PhysRevD.85.013008 [arXiv:1110.0083 [hep-ph]].

[46] M. Tanabashi et al. [Particle Data Group], “Review of Particle Physics,” Phys. Rev. D 98 (2018) no.3, 030001. doi:10.1103/PhysRevD.98.030001

[47] C. Jarlskog, “Commutator of the Quark Mass Matrices in the Standard Electroweak Model and a Measure of Maximal CP Violation,” Phys. Rev. Lett. 55 (1985) 1039. doi:10.1103/PhysRevLett.55.1039

[48] Y. F. Li and S. Luo, “Neutrino Oscillation Probabilities in Matter with Direct and Indirect
Unitarity Violation in the Lepton Mixing Matrix,” Phys. Rev. D 93 (2016) no.3, 033008
doi:10.1103/PhysRevD.93.033008 [arXiv:1508.00052 [hep-ph]].

[49] S. P. Mikheev and A. Y. Smirnov, “Resonance Amplification of Oscillations in Matter and
Spectroscopy of Solar Neutrinos,” Sov. J. Nucl. Phys. 42 (1985) 913 [Yad. Fiz. 42 (1985)
1441].

[50] P. B. Denton, H. Minakata and S. J. Parke, “Compact Perturbative Expressions For Neutrino
Oscillations in Matter,” JHEP 1606 (2016) 051 doi:10.1007/JHEP06(2016)051
[arXiv:1604.08167 [hep-ph]].

[51] S. K. Agarwalla, Y. Kao and T. Takeuchi, “Analytical approximation of the neutrino
oscillation matter effects at large $\theta_{13}$,” JHEP 1404 (2014) 047 doi:10.1007/JHEP04(2014)047
[arXiv:1302.6773 [hep-ph]].

[52] T. Ohlsson, “Status of non-standard neutrino interactions,” Rept. Prog. Phys. 76 (2013)
044201 doi:10.1088/0034-4885/76/4/044201 [arXiv:1209.2710 [hep-ph]].

[53] O. G. Miranda and H. Nunokawa, “Non standard neutrino interactions: current status and
future prospects,” New J. Phys. 17 (2015) no.9, 095002 doi:10.1088/1367-2630/17/9/095002
[arXiv:1505.06254 [hep-ph]].

[54] Y. Farzan and M. Tortola, “Neutrino oscillations and Non-Standard Interactions,” Front. in
Phys. 6 (2018) 10 doi:10.3389/fphy.2018.0010 [arXiv:1710.09360 [hep-ph]].

[55] M. C. Gonzalez-Garcia, Y. Grossman, A. Gusso and Y. Nir, “New CP violation in neutrino
oscillations,” Phys. Rev. D 64 (2001) 096006 doi:10.1103/PhysRevD.64.096006
[hep-ph/0105159].

[56] T. Kikuchi, H. Minakata and S. Uchinami, “Perturbation Theory of Neutrino Oscillation
with Nonstandard Neutrino Interactions,” JHEP 0903 (2009) 114
doi:10.1088/1126-6708/2009/03/114 [arXiv:0809.3312 [hep-ph]].

[57] K. Asano and H. Minakata, “Large-Theta(13) Perturbation Theory of Neutrino Oscillation
for Long-Baseline Experiments,” JHEP 1106 (2011) 022 doi:10.1007/JHEP06(2011)022
[arXiv:1103.4387 [hep-ph]].

[58] D. Adey et al. [Daya Bay Collaboration], “Measurement of the Electron Antineutrino
Oscillation with 1558 Days of Operation at Daya Bay,” Phys. Rev. Lett. 121 (2018) no.24,
241805 doi:10.1103/PhysRevLett.121.241805 [arXiv:1809.02261 [hep-ex]].

[59] I. Esteban, M. C. Gonzalez-Garcia, M. Maltoni, I. Martinez-Soler and T. Schwetz, “Updated
fit to three neutrino mixing; exploring the accelerator-reactor complementarity,” JHEP 1701
(2017) 087 doi:10.1007/JHEP01(2017)087 [arXiv:1611.01514 [hep-ph]].

[60] S. Gariazzo, M. Arcidiacono, P. F. de Salas, O. Mena, C. A. Ternes and M. Tórtola,
“Neutrino masses and their ordering: Global Data, Priors and Models,” JCAP 1803 (2018)
no.03, 011 doi:10.1088/1475-7516/2018/03/011 [arXiv:1801.04946 [hep-ph]].

[61] H. Minakata, “Large-Theta(13) Perturbation Theory of Neutrino Oscillation,” Acta Phys.
Polon. B 40 (2009) 3023 [arXiv:0910.5545 [hep-ph]].