Classical Supersymmetric Mechanics

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Abstract

We analyse a supersymmetric mechanical model derived from (1 + 1)-dimensional field theory with Yukawa interaction, assuming that all physical variables take their values in a Grassmann algebra $\mathcal{B}$. Utilizing the symmetries of the model we demonstrate how for a certain class of potentials the equations of motion can be solved completely for any $\mathcal{B}$. In a second approach we suppose that the Grassmann algebra is finitely generated, decompose the dynamical variables into real components and devise a layer-by-layer strategy to solve the equations of motion for arbitrary potential. We examine the possible types of motion for both bosonic and fermionic quantities and show how symmetries relate the former to the latter in a geometrical way. In particular, we investigate oscillatory motion, applying results of Floquet theory, in order to elucidate the role that energy variations of the lower order quantities play in determining the quantities of higher order in $\mathcal{B}$.

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1 Introduction

Classical supersymmetry sets out to extend the unified treatment of bosonic and fermionic quantities in the usual QFT framework to the classical level. Normally, in semiclassical treatments the fermionic variables are set to zero as soon as the supersymmetric theory has been constructed. The usual argument goes that since we cannot find classical fermions in nature, fermionic quantities should be omitted altogether at the classical level.

However, this is far from necessary. In fact, a consistent approach to classical supersymmetry has long been available – for a review see e.g. the book by de Witt [3]. Fermionic quantities are then treated as anticommuting variables taking values in a Grassmann algebra $\mathcal{B}$. Grassmann-valued mechanics has been analysed in the works of Berezin and Marinov [1] and Casalbuoni [2] and later by Junker and Matthiesen [6]. A main difference to our work is that both [1] and [2] do not distinguish clearly between generators of the algebra and dynamical quantities and thus define the Grassmann algebra $\mathcal{B}$ rather implicitly. The fact that the bosonic variables take values in the even part of the same algebra $\mathcal{B}$ is not apparent in these works, although both recognize that the bosonic variables cannot be real functions anymore – without, however, elaborating on this fact. A central aim of this paper is therefore to make sense of the general Grassmann-valued equations of motion, including the fermionic ones, and to find ways to their solution, which is done in [1] and [2] only in very special cases. Junker and Matthiesen, who investigate a similar mechanical model, achieve a more general solution than in [1] and [2], but again under the (implicit) assumption that the Grassmann algebra is spanned by only two generators identified with the fermionic dynamical variables. We can confirm most of their results (in different form, though, due to a different choice of variables) as special cases of our solutions. However, we disagree about some details, in particular, concerning the case of zero energy.

The mechanical model that we study here is the supersymmetric motion of a particle in a one-dimensional potential, derived by dimensional reduction from the usual $N = 2$ supersymmetric (1 + 1)-dimensional field theory with Yukawa interaction. A slightly different version of this model was investigated in [4], where a different concept of reality was used that led to a negative potential in the Lagrangian. The approach taken here stays closer to the usual case with the positive potential.

An important result of [4] was that a complete solution for the particle motion could be found on the assumption that the underlying algebra $\mathcal{B}$ has only two generators. This led to relatively simple results, however is unnecessarily restrictive.

Here we show first that for a large class of potentials the solution to the equations of motion can be found for any $\mathcal{B}$ and depends only on a small number of $\mathcal{B}$-valued constants of integration, one of which is a Grassmann energy $E$.

To deal with essentially arbitrary potentials we adopt a second method which is closer to that of [4], although we need not restrict ourselves to two generators: Choosing the Grassmann algebra to be finitely generated, with $n$ generators, we split all dynamical quantities and equations into their real components, named according to the number of generators involved in the corresponding monomial. Then, beginning from the zeroth order equation, which can be seen as a form of Newton’s equation, we subsequently work our way up to higher and higher orders, utilizing the solutions already found for the lower levels. This layer-by-layer strategy allows us to solve the equations of motion for any potential with reasonable mathematical properties.

The existence of a complete solution to the coupled system of equations of motion looks
surprising in view of the increasingly large number of equations involved for large $n$. However, on second thoughts it is not so unexpected: Due to our first solution method we know that a full Grassmann solution can be found in many cases, the decomposition of which should give us exactly the component solutions obtained by the second method – which it does indeed as we shall demonstrate.

A final word has to be said about the assumption of only a finite number of generators since it has been claimed that this must necessarily lead to contradictions: Emphasizing that our paper deals with the classical theory we do not find this to be true.

We begin our analysis in section 2 by presenting the Lagrangian and the equations of motion that we will be concerned with in this paper. Essential for solving these equations are the symmetries and associated Noether charges of the Lagrangian which we therefore examine in section 3. For a certain class of potential functions, namely those for which a particular integral can be calculated analytically, we describe in section 4 how the equations of motion can be solved completely and illustrate this method for two exemplary potentials, the harmonic potential $U(x) = kx$ and the hyperbolic potential $U(x) = c \tanh kx$.

Sections 5 and 6 are devoted to the description of our layer-by-layer method which is then explicitly carried out up to fourth order, and illustrated by the harmonic oscillator case in section 6.1.

Next, we investigate the symmetries in component form in section 7: While all component charges can be simply derived by decomposing the original charges, they also reflect a huge number of symmetries of the highest order component Lagrangian by which they can be found using Noether’s procedure. In addition to the symmetries known from the original Lagrangian there appear extra classes of symmetries which cannot be related to the former. It is possible to trace them back to invariance under certain changes of base in the Grassmann algebra, viewed as a $2^n$-dimensional vector space.

One question that immediately arises from the decomposition of the physical quantities into components is addressed in section 8: How must we interpret the numerous new functions that arise from this procedure? Utilizing the new symmetries found in section 7 we offer a geometric interpretation for the lowest order bosonic and fermionic functions and describe the three basic types of motion possible for them. The higher order quantities are interpreted here from a different point of view: We see them as variations of the lower order quantities with respect to the integration constants involved in these functions, namely an initial time $t_0$ and the energy $E_0$.

In section 9 we apply our results to study general oscillatory motion, by which we mean that the lowest order bosonic function is periodic. The characteristic appearance of linear-periodic terms is explained from both a physical and mathematical perspective employing results of Floquet theory. Again, we choose a particular potential to illustrate this in section 9.1, which allows us also to show that the two solution methods presented in this paper coincide.

A common restriction on both methods is that the energy $E_0$ of the system must be positive, therefore the zero energy case has to be discussed separately. We do this in section 10.

Some ideas for further generalization and analysis conclude this paper.
2 Supersymmetric Mechanics

We start our discussion with the standard Lagrangian density for (1 + 1)-dimensional supersymmetric field theory with Yukawa interaction:

\[ L = \frac{1}{2} \partial_+ \phi \partial_- \phi - \frac{1}{2} U(\phi)^2 + \frac{i}{2} \psi_+ \partial_- \psi_+ + \frac{i}{2} \psi_- \partial_+ \psi_- + i \frac{dU}{d\phi} \psi_+ \psi_-, \]

containing a real bosonic scalar field \( \phi \), a real two-component fermionic spinor field \( \psi \) and a potential function \( U(\phi) \); \( \partial_{\pm} \) are the light cone derivatives \( \partial_t \pm \partial_x \).

We assume that \( \phi \) and the two components of \( \psi \) take their values in the real even and odd part, respectively, of an arbitrary Grassmann algebra \( B \), and that the potential function \( U(\phi) \) can be expanded into a power series in \( \phi \) with real coefficients. Complex conjugation is defined such that \((z_1 z_2)^* = z_2^* z_1^*)\ in accordance with the conventions in [3] so that the Lagrangian density is a real function despite the \( i \)-factors that occur in front of the fermion terms. As we are dealing with the classical case we assume further that all fields and their derivatives commute or anticommute, depending in the usual way on the bosonic or fermionic nature of the fields.

In this paper we shall be interested only in spatially independent fields, i.e. \( \partial_x \phi = \partial_x \psi_+ = \partial_x \psi_- = 0 \). As the fields are then functions of time only this leads us directly from field theory to mechanics. It is therefore sensible to think of the bosonic field \( \phi \) as describing the one-dimensional motion of a particle in the potential \( U^2 \). To support this notion we will change the variable \( \phi \) to \( x \) for the rest of this paper. We then obtain the following Lagrange function:

\[ L = \frac{1}{2} \dot{x}^2 - \frac{1}{2} U(x)^2 + i \frac{1}{2} \psi_+ \dot{\psi}_+ + \frac{i}{2} \psi_- \dot{\psi}_- + i U'(x) \psi_+ \psi_-; \]

where the dot denotes a time derivative and \( U' \) means the derivative of \( U \) with respect to \( x \). Performing a formal variation of this Lagrangian with respect to the variables \( x, \psi_+ \) and \( \psi_- \) and neglecting total time derivatives we derive the equations of motion for the system:

\[ \ddot{x} = -U(x)U'(x) + iU''(x)\psi_+ \psi_- \]
\[ \dot{\psi}_+ = -U'(x)\psi_- \]
\[ \dot{\psi}_- = U'(x)\psi_+. \]

There is a slight ambiguity in these equations if the Grassmann algebra is finitely generated and has an odd number of generators. Then the two equations for the fermion variables will be determined only up to an arbitrary function of highest order in the Grassmann algebra. We can think of this as a gauge degree of freedom and will come back to this point later.

In order to solve the equations it is advisable to understand the symmetries of the system first. This we will do in the next section.

3 The symmetries of the model

The first thing to notice is that the Lagrangian has no explicit time dependence. We therefore have invariance under time translation, leading to a conserved Hamiltonian as the corresponding Noether charge. This we calculate to be

\[ H = \frac{1}{2} \dot{x}^2 + \frac{1}{2} U(x)^2 - iU'(x)\psi_+ \psi_-; \]
its conserved value we call the energy, denoted by $E$. Note, however, that this Hamiltonian is an even Grassmann-valued function and that therefore the conserved energy $E$ is an even element of $\mathcal{B}$.

The operator $\frac{d}{dt}$ generates time translations, and acts on the dynamical variables in the obvious way:

$$\Delta x = \eta \dot{x}, \quad \Delta \psi_1 = \eta \dot{\psi}_1, \quad \Delta \psi_2 = \eta \dot{\psi}_2,$$

where $\eta$ is an infinitesimal even Grassmann parameter.

In addition to time translation invariance there are two further independent symmetries of the Lagrangian, relating fermions and bosons. These can be written in infinitesimal form as:

$$\delta x = i \epsilon \psi_+, \quad \delta \psi_+ = - \epsilon \dot{x}, \quad \delta \psi_- = - \epsilon U(x),$$

$$\tilde{\delta} x = i \epsilon \psi_-, \quad \tilde{\delta} \psi_+ = \epsilon U(x), \quad \tilde{\delta} \psi_- = - \epsilon \dot{x},$$

where $\epsilon$ is an arbitrary infinitesimal odd Grassmann parameter. These transformations lead only to a change in the Lagrangian by a total time derivative. We can therefore apply Noether’s procedure and find the following charges:

$$Q = \dot{x} \psi_+ + U(x) \psi_-, \quad \tilde{Q} = \dot{x} \psi_- - U(x) \psi_+,$$

the conservation of which can be easily shown using the equations of motion. Note that the charges are odd elements of the Grassmann algebra.

As for time translation invariance we now define two operators $Q$ and $\tilde{Q}$ generating the two symmetry transformations. From (7) we can read off the action of these operators on the dynamical variables:

$$Q x = \psi_+, \quad Q \psi_+ = i \dot{x}, \quad Q \psi_- = i U(x)$$

$$\tilde{Q} x = \psi_-, \quad \tilde{Q} \psi_+ = - i U(x), \quad \tilde{Q} \psi_- = i \dot{x}.$$  

Using the action of the operators on $x$, $\psi_+$ and $\psi_-$ we find that $Q$, $\tilde{Q}$ and $\frac{d}{dt}$ form a closed algebra with the relations

$$Q^2 = \frac{d}{dt}, \quad \tilde{Q}^2 = \frac{d}{dt}, \quad \{Q, \tilde{Q}\} = 0$$

as long as the equations of motion are satisfied. Notice that $\frac{d}{dt}$ commutes with everything.

This is the usual $N = 2$ supersymmetry algebra in 0 space dimensions, and with (10) we have an on-shell representation of this algebra. Thus we will from now on speak of supersymmetry transformations and call the associated charges supercharges.

Beside time-translation invariance and the two supersymmetries there exists still a further invariance, this time for the fermionic functions only. In infinitesimal form it is given by the transformation

$$\tilde{\Delta} \psi_+ = \eta \psi_-, \quad \tilde{\Delta} \psi_- = - \eta \psi_+,$$

where $\eta$ again denotes an infinitesimal even Grassmann parameter. We can think of this as an internal rotation of the fermionic variables. This invariance leads to a further conserved Noether charge:

$$R = i \psi_+ \psi_-.$$
Comparison with (1) shows that the fermionic functions enter the interaction term in the Lagrangian only via $R$. This leads directly to the fact that the $x$-motion depends on the fermionic functions only through this one constant.

In addition, the Hamiltonian (5) reveals that the only fermion contribution to the total energy $E$ is through $R$. This is an important simplification, and it will allow us to solve the equations of motion completely for a number of potential functions $U$ without restriction on the nature of the Grassmann algebra $B$.

4 General solutions via Grassmann integrals

We have already mentioned that the $x$-equation of motion (2) nearly decouples from the other two equations because the fermion functions in the coupling term $i U'(x) \psi_+ \psi_-$ form a Grassmann constant. It is therefore sensible to begin with this equation.

From the conserved Hamiltonian we know that

$$\dot{x}^2 = 2E - U(x)^2 + 2RU'(x).$$

For the next step we have to use that every Grassmann number $z$ can be split into two parts, its 'body' and its 'soul': $z = z_b + z_s$. The body is just the real number content of $z$, the soul is the remaining linear combination of products of (odd) Grassmann generators and will be nilpotent if the Grassmann algebra is finitely generated. Note, however, that we do not need such a restriction for body and soul to be well-defined. A square root for a Grassmann quantity can be defined by its power series as long as the body is positive. Since $R$ is the product of two odd Grassmann terms, its body and therefore the body of the whole third term on the right hand side of (13) is zero, leaving the restriction $2E_b - U_b^2(x) > 0$ if $\dot{x}$ is to be well-defined. This just means that the kinetic energy of the classical particle moving in the potential $U^2$ has to be positive.

The resulting first order differential equation for $x$ is

$$\frac{dx}{dt} = \pm \sqrt{2E - U(x)^2 + 2RU'(x)}.$$  \hspace{1cm} (13)

Provided $B$ is finitely generated, we may regard $x$ as lying in the vector space $B$ and apply the standard theory of systems of ODE’s to show that this equation has a unique solution for any given initial data $x(t_0)$.

The equation can formally be solved by separating variables:

$$t - t_0 = \pm \int_{x(t_0)}^{x(t)} \frac{dx}{\sqrt{2E - U(x)^2 + 2RU'(x)}}.$$ \hspace{1cm} (14)

Note that while the left-hand side is just a real expression, the right-hand side is a Grassmann integral of a Grassmann integrand. Such an integral is defined as a line integral in the Grassmann algebra, thought of as a finite or infinite-dimensional vector space spanned by products of generators. As is shown in [3] such an integral is independent of the actual path relating start- and endpoint of the integral since the integrand is Grassmann-even.

Now the integrand comes from an ordinary real function extended into the full Grassmann algebra using its power series. We can therefore sensibly ask whether there is an indefinite integral $F(x)$ to this real function. If so, we can extend this function back into the Grassmann
algebra thus gaining an indefinite integral for the Grassmann integrand. Because the integrand is even it then follows [3] that the integral is given by the difference of $F$ evaluated at the start and endpoints of the path:

$$t - t_0 = \pm (F(x(t)) - F(x(t_0)))$$

If the function $F$ has an inverse, we finally get $x$ as a function of $t$ and therefore the solution of (2). We will illustrate this method below for two potential functions $U$.

The solution to the fermion equations (3) and (4) is now easy. From the solution for $x(t)$ we can immediately calculate $\dot{x}(t)$ and $U(x(t))$. Using the explicit formulae (8) and (9) for the two conserved supercharges $Q$ and $\tilde{Q}$ we can evaluate the linear combinations

$$Q\dot{x} - \tilde{Q}U(x) = 2H\psi_+,$$

$$QU(x) + \tilde{Q}\dot{x} = 2H\psi_-,$$

where we have used that $R\psi_+ = R\psi_- = 0$ which follows from (12). Since the Hamiltonian $H$ has the constant value $E$ we deduce that

$$\psi_+ = \frac{Q\dot{x} - \tilde{Q}U(x)}{2E},$$

$$\psi_- = \frac{QU(x) + \tilde{Q}\dot{x}}{2E}.$$ (15)

This effectively solves (3) and (4), as the values of $Q$ and $\tilde{Q}$ are determined by the data at the initial time $t_0$. There is one subtle point here, however, since the division by $2E$ is only possible as long as $E_b$, the body of $E$, which can be interpreted as the classical energy of the particle, is non-zero. Indeed, when we analyse the solution for finitely generated Grassmann algebras later, we will find again that the ($E_b = 0$)-case is special.

A second point which has to be mentioned here is that we have treated $R$ as an independent parameter. However, $R$ is determined by the two fermionic quantities $\psi_+$ and $\psi_-$ and, in effect, by the two supercharges $Q$ and $\tilde{Q}$:

$$R = \frac{iQ\tilde{Q}}{2E},$$

which can be verified using the definition of $R$ in (12). This does not render our solution invalid but it shows that $R$ is not a parameter that can be chosen independently. But let us now turn to some examples:

### 4.1 The harmonic potential $U(x) = kx$

We start with one of the easiest problems, the harmonic oscillator with potential $U(x) = kx$, $k$ being real. Here the integrand in (14) is given by:

$$\frac{1}{\sqrt{(2E + 2kR) - (kx)^2}}$$

where $E$ is the constant Grassmann energy. Note that due to the special nature of the potential we can combine the $U'(x)$-term $2kR$ with the energy $E$ into one overall constant. If we treat the integrand as an ordinary real function it has an indefinite integral:

$$\int \frac{1}{\sqrt{a^2 - b^2x^2}} \, dx = \frac{1}{b} \arcsin \left( \frac{b}{a} x \right),$$

\[6\]
so that we can write
\[ t - t_0 = \pm \left[ \frac{1}{k} \arcsin \left( \frac{k}{\sqrt{2E + 2kR}} x \right) \right]_{x(t_0)}^{x(t)}. \]

This formula can be inverted to yield the solution
\[ x(t) = x(t_0) \cos k(t - t_0) \pm \frac{v(t_0)}{k} \sin k(t - t_0), \quad (17) \]
where we have denoted the constant \( \sqrt{2E + 2kR - k^2x(t_0)^2} \) by \( v(t_0) \). This looks formally like the usual harmonic oscillator solution, and it is periodic with period \( \frac{2\pi}{k} \), but all terms are Grassmann-valued and not just real functions.

For the two fermion terms we need only calculate \( \dot{x} \) and \( U(x) \) and insert them into equations (15) and (16), which leaves us with:
\[ \psi_+ = p_+ \cos k(t - t_0) - p_- \sin k(t - t_0), \quad (18) \]
\[ \psi_- = p_- \cos k(t - t_0) + p_+ \sin k(t - t_0), \quad (19) \]
where
\[ p_+ = \frac{\pm Qv(t_0) - \bar{Q}kx(t_0)}{2E}, \quad p_- = \frac{\pm \bar{Q}v(t_0) + Qkx(t_0)}{2E}. \]

With (17), (18) and (19) we have found the complete solution to the equations of motion for \( E_b > 0 \), independent of the nature of the Grassmann algebra \( \mathcal{B} \).

### 4.2 The hyperbolic potential \( U(x) = c \tanh kx \)

As a second example we choose the hyperbolic potential \( U(x) = c \tanh kx \), with \( c \) and \( k \) both real. The integrand in (14) is
\[ \frac{1}{\sqrt{(2E + 2Rck) - (c^2 + 2Rck) \tanh^2 kx}}, \]
and – viewed as a real function – has an indefinite integral, the form of which depends on the constants involved:
\[ \int \frac{1}{\sqrt{a - b \tanh^2 kx}} dx = \frac{1}{k} \begin{cases} \frac{1}{\sqrt{b-a}} \arcsin \left( \frac{\sqrt{b} - 1}{\sqrt{a}} \sinh x \right) & ; \frac{b}{a} > 1 \\ \frac{1}{\sqrt{a-b}} \arcsinh \left( \frac{1 - \sqrt{a}}{\sqrt{b}} \sinh x \right) & ; \frac{b}{a} < 1 \\ \frac{1}{\sqrt{a}} \sinh x & ; \frac{b}{a} = 1 \end{cases} \quad (20) \]

Because the Grassmann-valued function is completely determined by the corresponding real function we have to look at the body of
\[ \frac{b}{a} = \frac{c^2 + 2ckR}{2E + 2ckR} \]
to decide which integral to use. Since \( R \) as product of two odd Grassmann constants has no body, the crucial term is \( \frac{c^2}{2E_b} \).
All three integrals \((21)\) can be inverted to give \(x\) as a function of \(t\):

\[
x(t) = \frac{1}{k} \begin{cases} \text{arcsinh} \left( \sqrt{\frac{c^2-2E}{2E-2c^2}} \sinh(\omega_I(t-t_0)+\kappa_I) \right) & ; c_b^2 > 2E_b \\
\text{arcsinh} \left( \sqrt{\frac{c^2-2E}{2E-2c^2}} \sin(\omega_{II}(t-t_0)+\kappa_{II}) \right) & ; c_b^2 < 2E_b \\
\text{arcsinh} \left( \omega_{III}(t-t_0)+\kappa_{III} \right) & ; c_b^2 = 2E_b
\end{cases}
\]

where

\[
\begin{align*}
\omega_I &= k\sqrt{c^2-2E}, \quad \kappa_I = \text{arcsinh} \left( \sqrt{\frac{c^2-2E}{2E-2c^2}} \sinh(kx(t_0)) \right) \\
\omega_{II} &= k\sqrt{2E-c^2}, \quad \kappa_{II} = \text{arcsinh} \left( \sqrt{\frac{2E-c^2}{2E-2c^2}} \sinh(kx(t_0)) \right) \\
\omega_{III} &= k\sqrt{2E+2ckR}, \quad \kappa_{III} = \sinh kx(t_0).
\end{align*}
\]

From equations \((15)\) and \((16)\) we can calculate the fermion solutions:

\[
\begin{align*}
\psi_+ &= \frac{1}{2E} \sqrt{\frac{2E+2ckR}{1+c^2+2ckR \tan^2 y_{II}}} \begin{cases} \pm Q - \tilde{Q} \sqrt{\frac{c^2}{c^2-2E}} \tan y_I & ; c_b^2 > 2E_b, \\
\pm Q + \tilde{Q} \sqrt{\frac{c^2}{c^2-2E}} \tan y_I & ; c_b^2 < 2E_b \end{cases} \\
\psi_- &= \frac{1}{2E} \sqrt{\frac{2E+2ckR}{1+c^2+2ckR \tan^2 y_{II}}} \begin{cases} \pm Q - \tilde{Q} \sqrt{\frac{c^2}{c^2-2E}} \tanh y_{II} & ; c_b^2 > 2E_b, \\
\pm Q + \tilde{Q} \sqrt{\frac{c^2}{c^2-2E}} \tanh y_{II} & ; c_b^2 < 2E_b \end{cases}
\end{align*}
\]

with \(y_I = \pm \omega_I (t-t_0) + \kappa_I\), etc. Again, we have found the complete solution for the system without using any information about the Grassmann algebra \(B\) itself. We will return to this example later, after we have investigated a special class of Grassmann algebras.

## 5 Finitely generated Grassmann algebras

We will now make a choice for the underlying Grassmann algebra \(B\) of the system, namely that it is generated by a finite number of elements \(\xi_i, i = 1, \ldots, n\) with the property

\[\xi_i \xi_j = -\xi_j \xi_i.\]

Every element \(z\) of \(B\) can be written in the form

\[z = \sum_{k=0}^{n} z_{i_1\ldots i_k} \xi_{i_1} \cdots \xi_{i_k}\]

with complex coefficients \(z_{i_1\ldots i_k}\). \(z_0\) is the body of \(z\). The requirement for \(z\) to be real (in the sense explained earlier) fixes each coefficient to be either purely real or purely imaginary, depending on the number of generators involved. For practical reasons we will usually restrict ourselves to the case of four generators, although the general aspects of our discussion should be true for an arbitrary number \(n\).

In the case of four generators it is useful to define the following real monomials:

\[\xi_{ij} = i \xi_i \xi_j; \quad \xi_{ijk} = i \xi_i \xi_j \xi_k; \quad : \xi_{1234} = -\xi_1 \xi_2 \xi_3 \xi_4.\]

Note that due to antisymmetry there are only six linearly independent monomials involving two generators, four monomials involving three generators, and just one monomial of highest order, i.e. involving all four generators.
We can decompose the dynamical quantities $x, \psi_+$ and $\psi_-$ as follows:

\[
\begin{align*}
  x(t) &= x_0(t) + x_{ij}(t)\xi_{ij} + x_{1234}(t)\xi_{1234} \\
  \psi_+(t) &= \lambda_i(t)\xi_i + \lambda_{ijk}(t)\xi_{ijk} \\
  \psi_-(t) &= \rho_i(t)\xi_i + \rho_{ijk}(t)\xi_{ijk},
\end{align*}
\]

where it is implied from here on that the summation over indices is with $i < j < k$. In total there are 24 independent real functions which we call of zeroth, first, second, third or fourth order according to the number of generators involved in the corresponding monomials.

Every analytic function involving $x, \psi_+$ or $\psi_-$ can be decomposed as well using its Taylor expansion. Applied to the potential function $U$ this yields:

\[
U(x) = U(x_0) + U'(x_0)x_{ij}\xi_{ij} + \left(U''(x_0)x_{1234} + \frac{1}{2}U'''(x_0)x_{1234}\right)\xi_{1234},
\]

where the brackets denote antisymmetrization. For example

\[
x_{[1234]} = x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} + x_{23}x_{14} - x_{24}x_{13} + x_{34}x_{12}.
\]

Note that $U(x_0)$ and its derivatives are ordinary real functions. Inserting these results into (1) yields eight component Lagrangians of order zero, two and four, respectively:

\[
\begin{align*}
  L_0 &= \frac{1}{2}\ddot{x}_0^2 - \frac{1}{2}U^2 \\
  L_{ij} &= \ddot{x}_{0ij} - UU'x_{ij} + \lambda_i\dot{x}_i + \rho_i\dot{\rho}_i + U'(\lambda_i\rho_j - \rho_i\lambda_j) \\
  L_{1234} &= \frac{1}{2}\ddot{x}_{1234} + x_0\ddot{x}_{1234} - \frac{1}{2}(UU')'x_{[1234]} -UU'x_{1234} \\
  &+ \lambda_{123}\dot{\lambda}_4 + \rho_{123}\dot{\rho}_4 + U'\left(\lambda_{123}\rho_4 - \rho_{123}\lambda_4\right) + U''x_{12}\left(\lambda_3\rho_4 - \lambda_4\rho_3\right)
\end{align*}
\]

where the argument of $U, U'$ and $U''$ is always $x_0(t)$, here and below.

From the highest order Lagrangian $L_{1234}$, which is a functional of 24 generally different component functions, we get the following set of Euler-Lagrange equations:

\[
\begin{align*}
  \ddot{\lambda}_i &= -U'\rho_i \\
  \dot{\rho}_i &= U''\lambda_i \\
  \ddot{\lambda}_{ijk} &= -U'\rho_{ijk} - U''x_{[ij}\rho_k] \\
  \dot{\rho}_{ijk} &= U'\lambda_{ijk} + U''x_{[ij}\lambda_k]
\end{align*}
\]

The same equations can also be obtained by splitting the original equations of motion (2)–(4) into their components.

It is remarkable that all equations can be derived from just the one Lagrangian. The other seven component Lagrangians are completely redundant and yield no new equations: From (23) we can only obtain equation (26) whereas from the Lagrangians of type (24) we can derive equations (26), (27) and (29).
Now we can also see where the ambiguity mentioned earlier comes from when there is an odd number \( n \) of generators: Because the Lagrangian is an even functional, none of its components can contain any function of highest order \( n \), so these functions cannot be governed by any equations of motion. E.g. for three generators the functions \( \lambda_{123} \) and \( \rho_{123} \) will stay completely undetermined. Formally, we can derive evolution equations similar to \((30)\) by decomposing \((3)\) into components but they are meaningless since an arbitrary function can be added to both sides. Thus we can regard these highest order functions as non-physical and treat them as trivially separated gauge degrees of freedom. However, if there is an even number of generators this problem cannot occur, since there will always be one Lagrangian of highest order containing component functions of all orders which are thus determined by equations of motion.

6 Layer-by-layer solutions

We now proceed to derive the solutions to the equations of motion \((26)–(30)\) for an arbitrary, sufficiently differentiable potential function \( U \) adopting a layer-by-layer strategy. That means that we will start with the lowest order or 'body' equation \((26)\) and then use its solution to work our way up to the higher order and more complex equations, including also the fermionic ones. We do this here up to fourth order, although there is no obstruction in principle to continue the strategy for an algebra with more than four generators. In fact, in every layer the equations and their solutions are the same irrespective of how many generators there are; there will only be a different number of them.

Before we start with the bosonic equations, it is useful to look at the decomposition of the Hamiltonian \( H = H_0 + H_{ij} \xi_{ij} + H_{1234} \xi_{1234} \) and of the Noether charge \( R = R_{ij} \xi_{ij} + R_{1234} \xi_{1234} \) (note that \( R \) does not have a body):

\[
H_0 = \frac{1}{2} \dot{x}_0^2 + \frac{1}{2} U^2(x_0) \quad (31)
\]

\[
H_{ij} = \dot{x}_0 \dot{x}_{ij} + UU' x_{ij} - U'R_{ij} \quad (32)
\]

\[
H_{1234} = \frac{1}{2} \dot{x}_{[123} \dot{x}_{4]} + \dot{x}_0 \dot{x}_{1234} + \frac{1}{2} (UU')' x_{[12}x_{34]} + UU' x_{1234} - U'R_{1234} - U''x_{[12}R_{34]} \quad (33)
\]

\[
R_{ij} = \lambda_i \rho_j - \rho_i \lambda_j \quad (34)
\]

\[
R_{1234} = \lambda_{[123} \rho_{4]} - \rho_{[123} \lambda_{4]} \quad (35)
\]

Since these are components of Grassmann conserved quantities, they are conserved too, as can be checked easily using the equations of motion. All components can be derived as Noether charges from the highest order Lagrangian \( L_{1234} \) and we will do this in section \( 7 \) but for now we only need that they are constant in time. The respective values of the Hamiltonian functions \( H_0, H_{ij} \) and \( H_{1234} \) are denoted in the following by \( E_0, E_{ij} \) and \( E_{1234} \).

We notice first that \((26)\) is just the standard Newtonian equation of motion for a particle moving in a potential \( \frac{1}{2} U^2 \). This is the bottom layer of the system.

The solution to \((26)\) using the constancy of the Hamiltonian \( H_0 \) is well-known:

\[
t - t_0 = \pm \int_{x_0(t_0)}^{x_0(t)} \frac{1}{\sqrt{2E_0 - U^2(x')}} dx'. \quad (36)
\]
The sign has to be chosen carefully to comply with the direction of motion of the particle; if the particle motion changes direction, the integral will be multi-valued and has to be glued together from pieces with unique sign to ensure that the overall result for \( t(x_0) \) is monotonically growing. The implicit function theorem then allows us to locally invert \( t(x_0) \) and obtain the required \( x_0(t) \). Since (23) is a second order equation there are two constants of integration, \( x_0(t_0) \) (or equivalently \( t_0 \)) and the energy \( E_0 \).

Whereas (26) is a non-linear equation, (27) is an inhomogeneous linear equation (which can be simplified using (34)). It is now convenient to use the Hamiltonian (32): Solving the equation \( H_{ij} = E_{ij} \) for \( \dot{x}_{ij} \) we can reduce the problem to a first order differential equation. This can be solved by standard methods to yield:

\[
x_{ij} = c_{ij} \dot{x}_0 + E_{ij} \dot{x}_0 \int_{t_0}^{t} \frac{1}{x_0^2} dt' + R_{ij} \dot{x}_0 \int_{t_0}^{t} U' \frac{1}{x_0^2} dt'.
\]

where \( c_{ij} \) is an arbitrary integration constant related to the initial value of \( x_{ij} \). The second integration constant of (27) is the energy variable \( E_{ij} \). This result was also found in (3) with \( R_{ij} \) set to \(-1\). However, using equation (26) and \( H_0 = E_0 \), the last term on the right hand side can be rewritten:

\[
R_{ij} \dot{x}_0 \int_{t_0}^{t} U' \frac{1}{x_0^2} dt' = R_{ij} \dot{x}_0 \int_{t_0}^{t} \frac{U'(x_0^2 + U^2)}{2E_0 x_0^2} dt' = \frac{R_{ij} \dot{x}_0}{2E_0} \int_{t_0}^{t} \frac{\dot{U} x_0 - U \dot{x}_0}{x_0^2} dt' = \frac{R_{ij} \dot{x}_0}{2E_0} U.
\]

so that we end up with the simpler expression

\[
x_{ij} = c_{ij} \dot{x}_0 + E_{ij} \dot{x}_0 \int_{t_0}^{t} \frac{1}{x_0^2} dt' + \frac{R_{ij} \dot{x}_0}{2E_0} U.
\]

All three functions which occur in (38) can be calculated directly from \( x_0(t) \).

The same procedure can be applied to solve (28) which simplifies when we use (34) and (35). We rearrange the equation \( H_{1234} = E_{1234} \) to isolate \( \dot{x}_{1234} \) and then solve this first order equation. The result can be written as:

\[
x_{1234} = c_{1234} \dot{x}_0 + E_{1234} \dot{x}_0 \int_{t_0}^{t} \frac{1}{x_0^2} dt' + \frac{R_{1234} \dot{x}_0}{2E_0} U \]

Again, there are two additional integration constants, \( c_{1234} \) and \( E_{1234} \). Note that the solution depends only on functions that we already know from the lower layers, namely \( x_0 \) and \( x_{ij} \). In fact, the solution can be expressed as a function of \( x_0 \) and its derivatives only by inserting (38) into (39):

\[
x_{1234} = c_{1234} \dot{x}_0 + E_{1234} \dot{x}_0 \int_{t_0}^{t} \frac{1}{x_0^2} dt' + \frac{R_{1234} \dot{x}_0}{2E_0} U \]

\[
+ \frac{1}{2} \dot{x}_0 \int_{t_0}^{t} \frac{x_{1234} - \dot{x}_{12} \dot{x}_{34}}{x_0^2} dt' + \frac{1}{2} \dot{x}_0 \int_{t_0}^{t} U' x_{12} R_{34} \frac{1}{x_0^2} dt'.
\]

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This complicated-looking expansion is particularly useful when we come to the interpretation of the solution in section 3.

There remain the fermion equations (29) and (30) to be solved. Both are systems of two homogeneous first order equations and therefore have two independent solutions. Just as we used the decomposition of $H$ to solve the bosonic equations, so it is now appropriate to decompose the conserved supercharges $Q = Q_i \xi_i + Q_{ijk} \xi_{ijk}$ and $\tilde{Q} = \tilde{Q}_i \chi_i + \tilde{Q}_{ijk} \xi_{ijk}$ to solve the fermionic equations. We find:

\[
\begin{align*}
Q_i &= \lambda_i \dot{x}_0 + \rho_i U, \\
\dot{Q}_i &= \rho_i \dot{x}_0 - \lambda_i U, \\
Q_{ijk} &= \lambda_{ijk} \dot{x}_0 + \rho_{ijk} U + \lambda_{[i} \dot{x}_{jk]} + \rho_{[i} U' x_{jk]}, \\
\dot{Q}_{ijk} &= \rho_{ijk} \dot{x}_0 - \lambda_{ijk} U + \rho_{[ij} \dot{x}_{k]} - \lambda_{[ij} U' x_{k]}.
\end{align*}
\]

Using suitable linear combinations of $\dot{x}_0$ and $U$ with coefficients $Q_i$ and $\dot{Q}_i$ and applying $H_0 = E_0$ we find that

\[
\lambda_i = l_i \dot{x}_0 - r_i U, \quad \rho_i = r_i \dot{x}_0 + l_i U,
\]

where $l_i = \frac{Q_i}{2 E_0}$ and $r_i = \frac{\dot{Q}_i}{2 E_0}$, hence the two integration constants of the solution are basically given by the first order supercharges. A similar result was obtained in [3], although the restrictions placed there on the shape of the potential $U$ are unnecessary.

Note, however, that this solution does not work if the particle energy $E_0$ equals zero. This has to do with the fact that both $\dot{x}_0$ and $U$ must vanish in this case. We will return to this in section 10.

The same procedure can be used to find the solution to equation (30) and we get the result:

\[
\begin{align*}
\lambda_{ijk} &= l_{ijk} \dot{x}_0 - r_{ijk} U - \frac{E_{[ij} \lambda_{k]}}{2 E_0} + \frac{U \dot{x}_{[ij} \rho_{k]} - U' x_{[ij} \rho_{k]}}{2 E_0} \\
\rho_{ijk} &= r_{ijk} \dot{x}_0 + l_{ijk} U - \frac{E_{[ij} \rho_{k]}}{2 E_0} - \frac{U \dot{x}_{[ij} \lambda_{k]} - U' x_{[ij} \lambda_{k]}}{2 E_0},
\end{align*}
\]

where we have used the equations $H_0 = E_0$ and $H_{ij} = E_{ij}$ and the identities $R_{[ij} \lambda_{k]} - R_{[ij} \rho_{k]} = 0$. The new integration constants $l_{ijk}$ and $r_{ijk}$ are related to the third order supercharges $Q_{ijk}$ and $\dot{Q}_{ijk}$ in the same way as $l_i$ and $r_i$ are to $Q_i$ and $\dot{Q}_i$. As was the case for the second and fourth order bosonic solutions we find that $\lambda_{ijk}$ and $\rho_{ijk}$ depend only on functions we know already from the lower order layers, namely, $x_0$, $\lambda_i$, $\rho_i$ and $x_{ij}$, and again we can simplify the solution even further, inserting (38) and (42) into equations (43) and (44):

\[
\begin{align*}
\lambda_{ijk} &= l_{ijk} \dot{x}_0 + c_{[ij} l_{k]} \dot{x}_0 + E_{[ij} l_{k]} \left( \dot{x}_0 \int_{t_0}^t \frac{1}{\dot{x}_0} dt' + \frac{1}{\dot{x}_0} - \frac{\dot{x}_0}{E_0} \right) \\
&\quad - r_{ijk} U - c_{[ij} r_{k]} U' \dot{x}_0 - E_{[ij} r_{k]} \left( U' \dot{x}_0 \int_{t_0}^t \frac{1}{\dot{x}_0} dt' - \frac{U}{E_0} \right) \\
\rho_{ijk} &= r_{ijk} \dot{x}_0 + c_{[ij} r_{k]} \dot{x}_0 + E_{[ij} r_{k]} \left( \dot{x}_0 \int_{t_0}^t \frac{1}{\dot{x}_0} dt' - \frac{1}{\dot{x}_0} - \dot{x}_0 \frac{E_0}{E_0} \right) \\
&\quad + l_{ijk} U + c_{[ij} l_{k]} U' \dot{x}_0 + E_{[ij} l_{k]} \left( U' \dot{x}_0 \int_{t_0}^t \frac{1}{\dot{x}_0} dt' - \frac{U}{E_0} \right).
\end{align*}
\]

We have now derived all solutions to equations (29)–(30), i.e. we have explicitly solved the equations of motion up to fourth order for an arbitrary potential function $U$. They are all
functions of only four quantities and their time derivatives:

\[ x_0(t), U, \int_{t_0}^{t} \frac{1}{x_0'} dt' \text{ and } \int_{t_0}^{t} \frac{1}{x_0'} dt'. \]

To illustrate the method we choose the harmonic oscillator as our first example. This has the advantage that we can compare the solution with our earlier result (17)–(19).

### 6.1 The harmonic potential \( U(x) = kx \)

The solution for \( x_0 \) could be found using (38) but here it is easy to solve equation (24) directly, with the familiar result:

\[ x_0(t) = x_0(t_0) \cos k(t-t_0) + \frac{v_0}{k} \sin k(t-t_0); \quad v_0 = \pm \sqrt{2E_0-k^2x_0^2(t_0)}. \] (47)

The functions \( \dot{x}_0 \) and \( U \) can be calculated easily and we find:

\[ \lambda_i(t) = L_i \cos k(t-t_0) - R_i \sin k(t-t_0), \] (48)

\[ \rho_i(t) = L_i \sin k(t-t_0) + R_i \cos k(t-t_0), \] (49)

where \( L_i \) and \( R_i \) are the constants

\[ L_i = \frac{Q_i}{2E_0}v_0 - \frac{\dot{Q}_i}{2E_0}kx_0(t_0), \] (50)

\[ R_i = \frac{Q_i}{2E_0}kx_0(t_0) + \frac{\dot{Q}_i}{2E_0}v_0. \] (51)

Next we have to calculate \( x_{ij} \) and thus we need

\[ \int_{t_0}^{t} \frac{1}{x_0'} dt' = \int_{x_0(t_0)}^{x_0(t)} \frac{1}{2E_0-k^2x_0^2} dx_0' = \frac{1}{2E_0} \left( \frac{x_0(t)}{x_0(t_0)} - \frac{x_0(t_0)}{v_0} \right). \] (52)

On using (38) we find that

\[ x_{ij}(t) = \left( \frac{R_{ij}}{2E_0}kx_0(t_0)+c_{ij}v_0 \right) \cos k(t-t_0) + \left( \frac{R_{ij}}{2E_0}v_0-c_{ij}kx_0(t_0)+\frac{E_{ij}}{kv_0} \right) \sin k(t-t_0). \] (53)

For \( \lambda_{ijk} \) and \( \rho_{ijk} \) we need not calculate any new terms except \( \dot{x}_0 \). After collecting various constants into \( L_{ijk} \) and \( R_{ijk} \) we end up with

\[ \lambda_{ijk} = L_{ijk} \cos k(t-t_0) - R_{ijk} \sin k(t-t_0) \]

\[ \rho_{ijk} = L_{ijk} \sin k(t-t_0) + R_{ijk} \cos k(t-t_0). \]

Finally we have to calculate \( x_{1234} \). Therefore we need the integral

\[ \int_{t_0}^{t} \frac{1}{x_0'} dt' = \left[ \frac{1}{6E_0} \frac{x_0}{x_0'} + \frac{1}{6E_0} \frac{x_0}{x_0'} \right] \frac{x_0(t)}{x_0(t_0)}. \]

There seems to be a singularity when \( \dot{x}_0 \) approaches zero but fortunately this term has to be multiplied by \( \dot{x}_0 \) and combined with two other terms which share the same coefficient \( E_{[12E_{34}]} \).
One finds that all terms involving $\dot{x}_0$ in the denominator cancel each other, so that the overall result is just a linear combination of $x_0$ and $\dot{x}_0$, i.e. of cos- and sin-terms. This holds also for all other contributions to $x_{1234}$, so that we can write:

$$x_{1234} = C_1 \cos k(t-t_0) + C_2 \sin k(t-t_0),$$

where $C_1$ and $C_2$ are rather complicated functions of $E_{1234}$, $c_{1234}$ and all lower order integration constants.

The most interesting observation is that all bosonic and fermionic functions are linear combinations of cos- and sin-terms only. As we will see later, this similarity between the solutions of different orders is the exception rather than the rule and a special feature of the harmonic oscillator. We also find that the fermionic functions oscillate with the same period as the bosonic functions and differ only in amplitude and phase. Once the motion of the lowest order bosonic function $x_0$ is fixed, the principal motion for $\lambda_i$, $\rho_i$ is almost completely determined. This is a general feature of our system, and it can be fully understood by investigating the symmetries of our model.

Finally, it is easy to verify that the complete component solution is compatible with the general solution found in section 4.1.

7 Symmetries in component form

We have already dealt with the decomposition of the Hamiltonian $H$, the constant $R$ and the two supercharges $Q$ and $\tilde{Q}$. Now we want to explain the origin of the component charges from the underlying component symmetries. The Lagrangian $L_{1234}$ is all we need to look at, since the non-trivial symmetries of all lower order Lagrangians form subgroups of the full symmetry group of $L_{1234}$. One would not guess the extraordinary number of symmetries of $L_{1234}$ if one did not know their supersymmetric origin.

To start with, $L_{1234}$ is invariant under time translations, and this leads to the conservation of the highest order Hamiltonian $H_{1234}$. This result can be generalized by looking at the full Grassmann transformation. Decomposition into components gives us back the time translation symmetry mentioned above when $\eta = \eta_0$ is real; the choices $\eta = \eta_{ij} \xi_{ij}$ (no summation) or $\eta = \eta_{1234} \xi_{1234}$ lead to seven extra charges which correspond to the Hamiltonians $H_{ij}$ and $H_0$, respectively.

Apart from the Hamiltonians, sixteen Noether charges $Q_i$, $\tilde{Q}_i$, $Q_{ijk}$ and $\tilde{Q}_{ijk}$ derive from the supercharges $Q$ and $\tilde{Q}$ and belong to a set of symmetry transformations which can be obtained from $L_{1234}$ by the choices $\epsilon = \epsilon_{i} \xi_{i}$ and $\epsilon = \epsilon_{ijk} \xi_{ijk}$ (no summation both times), respectively. As an example, $Q_i$ and $\tilde{Q}_i$ are the charges that belong to:

$$\delta_i x_{1234} = \epsilon_i \lambda_i, \quad \delta_{i} \lambda_{jkl} = -\epsilon_{i} \dot{x}_0, \quad \delta_{i} \rho_{jkl} = -\epsilon_{i} U,$$

$$\tilde{\delta}_i x_{1234} = \epsilon_{i} \rho_i, \quad \tilde{\delta}_i \lambda_{jkl} = \epsilon_i U, \quad \tilde{\delta}_i \rho_{jkl} = -\epsilon_{i} \dot{x}_0,$$

where $\{ijk\}$ is a cyclic permutation of $\{1234\}$ and no summation over $i$ is implied. The term supersymmetry transformations should be avoided here, though, since the dynamical quantities in component form are just real functions – therefore they can be termed bosonic or fermionic only by convention. Notwithstanding this fact the component transformations and the corresponding charges reflect the underlying supersymmetry.
Further component Noether charges derived from the Lagrangian (24) are $R_{ij}$ and $R_{1234}$. The transformations can be read off from (11) choosing $\eta = \eta_{ij} \xi_{ij}$ (no summation) and $\eta = \eta_0$ (real), respectively. In the latter case we get, for example

$$
\begin{align*}
\delta_{1234} \lambda_i & = \eta_0 \rho_i, \\
\delta_{1234} \lambda_{ijk} & = \eta_0 \rho_{ijk}, \\
\delta_{1234} \rho_i & = -\eta_0 \lambda_i, \\
\delta_{1234} \rho_{ijk} & = -\eta_0 \lambda_{ijk}.
\end{align*}
$$

So far all transformations have just resembled those found in section 3. There are however further sets of symmetry transformations which only occur for the component Lagrangian $L_{1234}$. The easiest of these is given by

$$
\delta_i \lambda_{jkl} = -\eta_i \lambda_i, \quad \delta_i \rho_{jkl} = -\eta_i \rho_i,
$$

where again $\{ijkl\}$ is a cyclic permutation of $\{1234\}$. This leads to four charges

$$
S_i = \frac{1}{2} (\lambda_i^2 + \rho_i^2).
$$

The next easiest is

$$
\begin{align*}
\delta_{ij} \lambda_{ikl} & = \eta_{ij} \lambda_i, \\
\delta_{ij} \lambda_{jkl} & = -\eta_{ij} \lambda_j, \\
\delta_{ij} \rho_{ikl} & = \eta_{ij} \rho_i, \\
\delta_{ij} \rho_{jkl} & = -\eta_{ij} \rho_j
\end{align*}
$$

and leads to six independent charges

$$
S_{ij} = \lambda_i \lambda_j + \rho_i \rho_j.
$$

We shall exploit these charges later but for now we are more interested in where they come from. It turns out that these and two other groups of transformations not mentioned yet reflect invariance of the original Lagrangian under change of basis in the Grassmann algebra. A Grassmann algebra with $n$ generators can be viewed as a $2^n$-dimensional vector space in the obvious way, but the choice of the generators $\xi_i$ and their products as basis vectors is somewhat arbitrary. A change of basis, however, has to be compatible with the multiplicative structure of the algebra. This means firstly that we only have to consider transformations of the $n$ generators and secondly that we can change an (odd) generator only by another odd element of the algebra. The final constraint is that the highest order monomial, e.g. $\xi_{1234}$ in the case of four generators, remains unchanged by the transformation so that the highest order Lagrangian, here $L_{1234}$, remains invariant.

In the four generator case, this leaves only a relatively small group of acceptable linear transformations. There are eight independent odd basis elements, namely $\xi_i$ and $\xi_{ijk}$. Correspondingly, there are the following independent infinitesimal transformations that fulfil our conditions:

1. $\xi_i \mapsto \xi_i + \eta \xi_j$ ($i \neq j$)

2. $\xi_i \mapsto \xi_i - \eta \xi_{jkl}$ ($\{ijkl\}$ an even permutation of $\{1234\}$)

3. $\xi_i \mapsto \xi_i + \eta \xi_i$, $\xi_j \mapsto \xi_j - \eta \xi_j$ ($i \neq j$): There are just three independent scaling transformations of this type.

4. $\xi_i \mapsto \xi_i + \eta \xi_{ikl}$, $\xi_j \mapsto \xi_j - \eta \xi_{jkl}$ ($\{ijkl\}$ an even permutation of $\{1234\}$): There are six independent Grassmann scaling transformations of this type.
In order for the Grassmann quantities $x$, $\psi_+$ and $\psi_-$ to be invariant, corresponding to these transformations of the basis vectors of $\mathcal{B}$ there have to be transformations of the real components. It is these component transformations that we have found in (53) and (57), which belong to the basis transformations $\mathcal{A}$ and $\mathcal{B}$, respectively.

The component transformations belonging to $\mathcal{A}$ and $\mathcal{B}$ are also symmetries of the Lagrangian, but since they are more complicated to write down and we will not use them later we refrain from giving them here. It should be mentioned though that they generate a full $SL_4$ symmetry group.

8 Bosonic and fermionic motion – general properties

We finally come to the physical interpretation of the results we have so far obtained concerning the component dynamical variables. Starting with the lowest order bosonic function $x_0$ we first recall that this always describes the motion of a particle in one dimension in the potential $\frac{1}{2}U^2(x_0)$. There are three possible types of motion in one dimension:

1. Movement with no turning point: The particle velocity is always positive (or negative), the motion can be bounded, approaching finite values of $x_0$ as $t$ goes to $\pm\infty$, or unbounded. Example: Flat potential function $U(x) = c$.

2. Movement with one turning point: The particle velocity changes sign once, when $x_0$ reaches a maximal (minimal) value. Again, the motion can be bounded or unbounded. Example: Reciprocal potential $U(x) = \frac{c}{x}$.

3. Oscillatory motion: The particle velocity changes sign infinitely often, thus the motion is always restricted to a finite interval. Example: Harmonic oscillator $U(x) = kx$.

There is one conserved quantity, namely the lowest order Hamiltonian $H_0$, given in (51). We can interpret this as half the squared length of the two-dimensional bosonic vector

$$\left( \begin{array}{c} \dot{x}_0 \\ U \end{array} \right). \quad (59)$$

The conservation of $H_0$ means that the motion of this vector is restricted to a circle with squared radius $2E_0$.

Corresponding to the three types of motion for $x_0$ there are three types of motion for this vector:

1. Movement with no turning point: The vector moves on the right (left) semicircle in the $\dot{x}_0-U$-plane. The number of direction changes depends on the number of potential extrema of $U$.

2. Movement with one turning point: Motion starts on the right (left) semicircle of the $\dot{x}_0-U$-plane, then changes when the turning point is reached to the other semicircle where it then mirrors the previous motion.

3. Oscillatory motion: There are two subcases, depending on whether the number of sign changes of $U(x)$ between the two extremal points $x_{\text{min}}$ and $x_{\text{max}}$ is odd or even.
(a) When there is an odd number the vector will continually wind around the circle; direction changes depending on the number of extrema of $U$ can be superimposed. This is the only case where the motion covers the complete circle. The most important example is the harmonic oscillator where we can see directly the circular motion of the bosonic vector.

(b) When the number of sign changes of $U$ is even, the bosonic vector will oscillate on the circle between two points lying symmetrically on either side of the $U$-axis. An example of this case is the potential function $U(x) = \sqrt{(kx)^2 + c}$ with $c > 0$, which leads to a harmonic oscillator potential shifted upwards by $c$.

The bosonic vector is important because it largely determines the behaviour of the $n$ first order fermionic functions. To see this, we look at the $n$ two-dimensional vectors

\[
\begin{pmatrix}
\lambda_i \\
\rho_i 
\end{pmatrix}; \ i = 1, \ldots, n. \tag{60}
\]

The charges $S_i$ introduced in (56) guarantee that the lengths of these vectors are conserved, thus their motions are also restricted to circles. The squared radii are given by $2S_i$, providing us with a nice geometrical interpretation of these constants. Furthermore, the charges $S_{ij}$, given by (58), can be interpreted as scalar products between the $i$th and $j$th vectors, thus effectively specifying the absolute value of the angle between them. Hence we can see already that all $n$ vectors must corotate. This can be confirmed further by the charges $R_{ij}$ calculated in (34). They can be seen as two-dimensional determinants, fixing the area of the parallelogram defined by the above-mentioned two vectors. It then follows that the sign of the angle between these is determined, too.

The other conserved quantities that involve only terms of first or lower order are the supercharges $Q_i$ and $\tilde{Q}_i$, given in (11). They couple the fermionic vectors with the bosonic vector: The charges $Q_i$ fix the scalar products, the constants $\tilde{Q}_i$ the determinants between $(\lambda_i, \rho_i)$ and $(\dot{x}_0, U)$, so that all angles between these vectors, including their signs, are determined and constant in time. This means that once the bosonic motion is calculated and the fermionic initial data is given, we can read off the time evolution of all fermionic functions since their vectors are rigidly corotating with the bosonic one.

Having now dealt with the lowest order bosonic and fermionic functions we have to ask how to interpret the higher order functions. Unfortunately, the geometric interpretation of these can not be seen as clearly. Instead we will interpret the higher order functions as variations of those of lower order.

To start with, $x_{ij}$ as given in (38) has two terms coming from the two homogeneous solutions of the $x_{ij}$-equation with coefficients $c_{ij}$ and $E_{ij}$ and the term with coefficient $R_{ij}$ which constitutes a particular solution. We will now show that the first two terms can be written as variations of the lowest order motion $x_0$ with respect to the two free parameters $t_0$ and $E_0$ of that motion.

The variation of $x_0(t)$ with $t_0$ is clearly proportional to the velocity $\dot{x}_0$:

\[
\frac{\delta x_0}{\delta t_0} = -\dot{x}_0
\]
which gives us the term with coefficient $c_{ij}$. Next we vary the equation $H_0 = E_0$:

$$2\dot{x}_0 \delta x_0 + 2UU' \delta x_0 = 2 \delta E_0.$$  

Using equation (26), dividing by $\dot{x}_0^2$ and integrating between $t_0$ and $t$ we obtain

$$\int_{t_0}^{t} \frac{\dot{x}_0 \delta x_0 - \ddot{x}_0 \delta x_0}{\dot{x}_0^2} dt' = \delta E_0 \int_{t_0}^{t} \frac{1}{\dot{x}_0} dt',$$

hence

$$\left[ \frac{\delta x_0}{\dot{x}_0} \right]_{t_0}^{t} = \delta E_0 \int_{t_0}^{t} \frac{1}{\dot{x}_0^2} dt',$$

and finally

$$\frac{\delta x_0}{\delta E_0} = \dot{x}_0 \int_{t_0}^{t} \frac{1}{\dot{x}_0^2} dt',$$

which is exactly the term with coefficient $E_{ij}$ in (38). So we end up with:

$$x_{ij} = -c_{ij} \frac{\delta x_0}{\delta t_0} + E_{ij} \frac{\delta x_0}{\delta E_0} + \frac{R_{ij}}{2E_0} U.$$

In the same way we can analyse the third order fermion solutions (13) and (16). Both have parts proportional to $\dot{x}_0$ and $U$. But in addition there are two further terms which have to be interpreted as variations of these parts with $t_0$ and $E_0$. The $t_0$-variations can be written down immediately:

$$\frac{\delta \dot{x}_0}{\delta t_0} \propto \ddot{x}_0, \quad \frac{\delta U}{\delta t_0} \propto U' \ddot{x}_0$$

and yield the terms with coefficients $c_{i[k]}$ and $c_{i[r]k}$. The $E_0$-variations give:

$$\frac{\delta \dot{x}_0}{\delta E_0} = \frac{d}{dt} \frac{\delta x_0}{\delta E_0} = \dot{x}_0 \int_{t_0}^{t} \frac{1}{\dot{x}_0} dt' + \frac{1}{\dot{x}_0} \frac{\delta U}{\delta E_0} = U' \frac{\delta x_0}{\delta E_0} = U' \dot{x}_0 \int_{t_0}^{t} \frac{1}{\dot{x}_0} dt'.$$
They explain the first part of the terms with coefficients $E_{[ij]k}$ and $E_{[ij]r_k}$; note that the second part can be absorbed into the $l_{ijk}$ and $r_{ijk}$-terms.

So the vectors formed by $\lambda_{ijk}$ and $\rho_{ijk}$ consist of three parts: A vector rigidly fixed to the familiar first order fermionic vectors and two parts which are proportional to the variation of these vectors with $t_0$ and $E_0$.

Finally we want to explain the parts of the highest order bosonic function $x_{1234}$. The first three terms of (40) are familiar: They are the variations of $x$ with respect to $t_0$; for the term with coefficient $c_{[12]E_{34}}$ we have already shown that it is the time derivative of $\frac{\delta E_0}{\delta E_0}$; hence it is proportional to $\frac{\delta^2 x_0}{\delta E_0^2}$.

The last two terms in (40) are easy again: They are given by the two first variations of the inhomogeneity term proportional to $U$. The next three terms can be explained as second variations. So $\dot{x}_0$ gives the second variation of $x_0$ with respect to $t_0$; for the term with coefficient $c_{[12]E_{34}}$ we have already shown that it is the time derivative of $\frac{\delta E_0}{\delta E_0}$, hence it is proportional to $\frac{\delta^2 x_0}{\delta E_0^2}$.

The last term can be rewritten as

$$\dot{x}_0 \int_{t_0}^t \frac{2 \delta \dot{x}_0}{\delta E_0^2 \delta E_0} dt' = -2 \dot{x}_0 \int_{t_0}^t \left( \frac{1}{\dot{x}_0^2} + \frac{\ddot{x}_0}{\dot{x}_0^3} \int_{t_0}^t \frac{1}{\dot{x}_0^2} dt'' \right) dt'$$

$$= -2 \ddot{x}_0 \int_{t_0}^t \frac{1}{\dot{x}_0^2} dt' + \ddot{x}_0 \left( \frac{1}{\dot{x}_0^2} \int_{t_0}^t \frac{1}{\dot{x}_0^2} dt'' - \int_{t_0}^t \frac{1}{\dot{x}_0^2} dt' \right).$$

Combining this with the expanded first term

$$\ddot{x}_0 \left( \int_{t_0}^t \frac{1}{\dot{x}_0^2} dt' \right)^2 + \frac{1}{\dot{x}_0} \int_{t_0}^t \frac{1}{\dot{x}_0^2} dt'$$

we find exactly the component in (40) with coefficient $E_{[12]E_{34}}$.

The last two terms in (40) are easy again: They are given by the two first variations of the inhomogeneity term $\frac{R_{E_0}}{E_0} U$ with $t_0$ and $E_0$, which we have already calculated above. Note that the term $-\frac{U}{E_0}$ with coefficient $R_{[12]E_{34}}$ can be absorbed into the $R_{1234}$ term.

In summary:

1. The lowest order bosonic function $x_0$ describes the one-dimensional motion of a point particle with energy $E_0$ in the potential $\frac{1}{2}U^2(x_0)$. It is completely unaffected by all higher order bosonic and all fermionic functions and not influenced by any higher order charges. There are two free parameters involved, an initial time $t_0$ and the energy $E_0$.

2. The first order fermionic functions comprise two-dimensional vectors $(\lambda_i, \rho_i)$. The lengths of these vectors and the angles between them are fixed by charges arising from symmetry under change of basis in the Grassmann algebra and from the second order charge $R_{ij}$, and so are constant in time, i.e. all fermionic vectors rigidly corotate. Their motion is in turn determined by the bosonic vector $(\dot{x}_0, U)$ which has constant squared length $2E_0$ and is rigidly coupled to the fermion vectors by the supercharges $Q_i$ and $\tilde{Q}_i$. For every generator there is a pair of free parameters which specify the length and the phase of the corresponding vector.
3. The second order bosonic function $x_{ij}$ has terms proportional to the (first) variation of $x_0$ with $t_0$ and $E_0$. There is one additional term proportional to $U$ stemming from the inhomogeneity of the $x_{ij}$-equation of motion and ultimately from the Yukawa interaction term.

4. The third order fermionic functions again comprise two-dimensional vectors $(\lambda_{ijk}, \rho_{ijk})$. They can be divided into three additive parts. The first part is rigidly rotating with the first order fermionic vectors, the second and third parts are proportional to the variations of this first order vectorial motion with initial time $t_0$ and $E_0$.

5. Finally, the fourth order bosonic function involves all first and second order variations of $x_0$ with respect to $t_0$ and $E_0$ plus a further term proportional to $U$ and its first order variations.

9 Oscillatory Motion

In this section we apply our results of the previous section to oscillatory motion, i.e. we assume that the lowest order bosonic function $x_0$ is periodic with period $T$. We have found already that in this case the first order fermionic motion is described by two-dimensional vectors that either continuously wind around a circle or oscillate between two symmetrically lying points on it, depending on the number of sign changes of $U$.

When we look at the next bosonic level, i.e. the functions $x_{ij}$, we can see immediately that the first part of the solution (38), which is proportional to $\dot{x}_0$, is a periodic function with the same period $T$. The last term proportional to $U(x_0(t))$ is periodic, too, but it can have period $\frac{T}{2}$ if the potential function $U(x_0)$ has reflection symmetry, i.e. if $U(a + x) = U(a - x)$ for some constant $a$. Let therefore $\hat{T}$, which is either $T$ or $\frac{T}{2}$, denote the period of $U(x_0(t))$. The most interesting term is the remaining second one containing the integral. We will analyse this term from a mathematical point of view using Floquet theory but before we do that we would like to understand it from a more physical perspective.

From equation (61) in the previous section we know that the integral term can be interpreted as the variation of the solution $x_0$ with respect to the energy $E_0$. Because the variation is infinitesimally small, we can make the assumption that the functional form of the motion remains unchanged. We treat the motion as determined by only two parameters, the period $T$ and a characteristic amplitude $A$, defined e.g. as the distance between the two turning points of the motion. In the generic case $T$ and $A$ will be related by a non-trivial function $T(A)$, therefore a change in energy means not only a change in the amplitude $A$ but also in the period $T$, so:

$$\frac{dx_0}{dE_0} = \frac{dx_0}{dA} \frac{dA}{dE_0} = \left( \frac{\partial x_0}{\partial A} + \frac{\partial x_0}{\partial T} \frac{dT}{dA} \right) \frac{dA}{dE_0}.$$

Since $x_0$ is a $T$-periodic function we can use its Fourier series $\sum_j c_j e^{i\omega_j t}$, where $\omega_j = \frac{2\pi}{T}j$ and the coefficients $c_j$ are regarded as functions of $A$, to find:

$$\frac{\partial x_0}{\partial T} = -\frac{t}{T} \sum_j c_j i \omega_j e^{i\omega_j t} = -\frac{\dot{x}_0}{T}.$$

Thus

$$\frac{dx_0}{dE_0}(t) = \left( \frac{\partial x_0}{\partial A}(t) - \frac{\dot{x}_0(t)}{T} \frac{dT}{dA} \right) \frac{dA}{dE_0}.$$
Both $\frac{\partial x_0}{\partial A}$ and $\dot{x}_0$ are $T$-periodic functions, but $\frac{dE_0}{dA}$ itself is clearly not since the second term is linear-periodic, i.e. the product of the linear term $t$ with a periodic function. The extra linear-periodic term diverges with time, thus seemingly spoiling the oscillatory nature of the solution. However, this problem only arises because the period of the oscillation depends on the amplitude $A$ and thus ultimately on the energy $E_0$. Expressing a solution with slightly changed period $T + dT$ in terms of $T$-periodic functions inevitably leads to non-periodic terms. The effect is well-known from celestial mechanics where linear-periodic functions appear as secular terms in the study of stability problems of planetary orbits (see e.g. [5]).

There is one case where no linear-periodic term occurs, and this is when

$$\frac{dT}{dA} = 0,$$

i.e. when the period is independent of the amplitude. This is true for the harmonic oscillator, and consequently there is no non-periodic term in equation (53).

The existence of non-periodic terms in the bosonic function $x_{ij}$ can be derived in a mathematically more stringent way from Floquet theory, the theory of linear differential equations with periodic coefficients. Equation (27), with $\lambda_i \rho_j - \rho_i \lambda_j$ replaced by the constant $R_{ij}$ according to (34), determines the motion of $x_{ij}$. Because we know that the particular solution proportional to $U$ is periodic it suffices to treat the homogeneous equation which can be written in the form

$$\ddot{x}_{ij} + p(t)x_{ij} = 0,$$

(62)

where $p$ is a $\hat{T}$-periodic function. This allows us to apply Floquet theory which states that [4]:

1. There exists a non-zero constant $\alpha$, called the characteristic multiplier, and a non-trivial solution $x_{ij}(t)$ such that

$$x_{ij}(t + \hat{T}) = \alpha x_{ij}(t),$$

(63)

from which one deduces

2. There exist linearly independent solutions $x_{ij,1}$ and $x_{ij,2}$ to (62), such that either

$$x_{ij,1}(t) = e^{m_1 t}P_1(t), \quad x_{ij,2}(t) = e^{m_2 t}P_2(t)$$

(64)

or

$$x_{ij,1}(t) = e^{mt}P_1(t), \quad x_{ij,2}(t) = e^{mt}(tP_1(t) + P_2(t)),$$

(65)

where in both cases $P_1, P_2$ are $\hat{T}$-periodic functions and $m_1, m_2, m$, called characteristic exponents, are – not necessarily distinct – constants.

Whether the solutions take the form (64) or (63) depends on whether there are two independent solutions of (22) with the property (63) or just one. If we denote by $X_1, X_2$ the two linearly independent solutions of (22) with

$$X_1(0) = 1, \dot{X}_1(0) = 0, X_2(0) = 0, \dot{X}_2(0) = 1, (t_0 = 0 \text{ for simplicity})$$

and by $M(t)$ the matrix

$$\begin{pmatrix}
X_1(t) & X_2(t) \\
\dot{X}_1(t) & \dot{X}_2(t)
\end{pmatrix},$$
we get a solution of type (64) if $M(\hat{T})$ is diagonalizable and a solution of type (65) if $M(\hat{T})$ has Jordan normal form.

To determine the characteristic exponents $m_i$ it is useful to notice that our equation is not an arbitrary differential equation with periodic coefficients but an example of Hill’s equation which takes the form

$$F(t)\dddot{x}_{ij} + F'(t)\ddot{x}_{ij} + G(t)x_{ij} = 0,$$

$F$ and $G$ being $\hat{T}$-periodic functions. For equations of Hill’s type the Floquet multipliers are completely determined by the trace $D$ of the fundamental matrix $M(\hat{T})$. This trace can be determined indirectly, using the fact that we know one periodic solution of (62), namely $\dot{x}_0$, which has either period $\hat{T}$ or $2\hat{T}$, depending on whether $U(x_0)$ is reflection-symmetric or not.

Now we can use another theorem of Floquet theory which states:

1. Hill’s equation has non-trivial solutions with period $\hat{T}$ if and only if $D = 2$. Then either $m_1 = m_2 = m = 0$ and

$$x_{ij,1}(t) = P_1(t), \ x_{ij,2}(t) = P_2(t),$$

or $m = 0$ and

$$x_{ij,1}(t) = P_1(t), \ x_{ij,2}(t) = tP_1(t) + P_2(t).$$

2. Hill’s equation has non-trivial solutions with period $2\hat{T}$ if and only if $D = -2$. Then either $m_1 = m_2 = \frac{i\pi}{\hat{T}}$ and

$$x_{ij,1}(t) = e^{\frac{i\pi}{\hat{T}}t}P_1(t), \ x_{ij,2}(t) = e^{\frac{i\pi}{\hat{T}}t}P_2(t),$$

or $m = \frac{i\pi}{\hat{T}}$ and

$$x_{ij,1}(t) = e^{\frac{i\pi}{\hat{T}}t}P_1(t), \ x_{ij,2}(t) = e^{\frac{i\pi}{\hat{T}}t}(tP_1(t) + P_2(t)).$$

Using these theorems we find that the second independent solution (61) has to be either periodic or linear-periodic. The period is in both cases $T$, the period of the first independent solution $\dot{x}_0$. This means that we can formally confirm our earlier result derived from physical arguments. Additionally, we have that any solution must be bounded in finite intervals. This is not immediately obvious from the explicit form of (61), which could be singular when $\dot{x}_0 = 0$.

Before we proceed to analyse a particular potential function which admits oscillatory motion we want to comment briefly on the higher order bosonic and fermionic quantities $\lambda_{ijk}, \rho_{ijk}$ and $x_{1234}$. From equations (45) and (46) we can see that all terms that appear in the third order fermionic solution are either periodic or linear-periodic. For the term

$$\dddot{x}_0 \int_{t_0}^t \frac{1}{\dot{x}_0} dt' + \frac{1}{\dot{x}_0},$$

this follows e.g. from the fact that the derivative of a linear-periodic term is again linear-periodic. We can expect this behaviour physically since $\lambda_{ijk}$ and $\rho_{ijk}$ are given by the first variation of the motion of the first order fermionic vectors with energy and initial time. (There are further periodic contributions to $\lambda_{ijk}$ and $\rho_{ijk}$ with coefficients $l_{ijk}$ and $r_{ijk}$.)

Most terms in the fourth order bosonic solution $x_{1234}$ can be easily seen to be periodic or linear-periodic in the same manner. However, there are the three terms which have been
interpreted as the second variation of $x_0$ with energy $E_0$. These should contain a quadratic term in $t$, because
\[
\frac{d^2 x_0}{dE_0^2} = \left[ \frac{\partial^2 x_0}{\partial A^2} + 2 \frac{\partial^2 x_0}{\partial A \partial T} \frac{dT}{dA} + \frac{\partial^2 x_0}{\partial T^2} \left( \frac{dT}{dA} \right)^2 + \frac{\partial x_0}{\partial A} \frac{d^2 T}{dA^2} \right] \left( \frac{dA}{dE_0} \right)^2 + \left[ \frac{\partial x_0}{\partial A} + \frac{\partial x_0}{\partial T} \frac{d^2 T}{dA^2} \right] \frac{d^2 A}{dE_0^2}
\]
and
\[
\frac{\partial^2 x_0}{\partial T^2} = \frac{\partial}{\partial T} \left( -\frac{x_0 t}{T} \right) = \frac{1}{T^2} \left( 2 \ddot{x}_0 t + \dot{x}_0 t^2 \right),
\]
all other terms being (linear-)periodic. Again, the quadratic term does not occur when period $T$ and amplitude $A$ are independent, i.e. when $\frac{dT}{dA} = 0$. This is confirmed by the harmonic oscillator example where $x_{1234}$ is completely periodic.

We can also understand the existence of a quadratic-periodic term directly from the solution (40) which contains the term $\dddot{x}_0 \left( \int_{t_0}^{t} \dot{x}_0 dt' \right)^2$. We know already that the integral will yield a term proportional to $t$; this is squared to give a quadratic term in $t$, and then multiplied by the periodic function $\dddot{x}_0$. This quadratic term cannot cancel out because the two remaining terms with coefficient $E_{1234}$ in (40) are at most linear-periodic.

### 9.1 The hyperbolic potential $U(x) = c \tanh kx$

We illustrate our results by the hyperbolic potential $U(x) = c \tanh kx$, already discussed in section 4.2 for general Grassmann algebra. Since we are only interested in oscillatory motion here, we make the assumption that $c^2 > 2E_0$.

Inverting the result obtained from (36) we can calculate that
\[
x_0(t) = \pm \frac{1}{k} \arcsinh \left( \sqrt{\frac{2E_0}{c^2 - 2E_0}} \sin(\omega (t-t_0) + \kappa) \right), \tag{66}
\]
with
\[
\omega = k \sqrt{c^2 - 2E_0} \quad , \quad \kappa = \arcsin \left( \sqrt{\frac{c^2}{2E_0}} - 1 \sinh kx_0(t_0) \right).
\]
So $x_0$ is indeed a periodic function. For $\dot{x}_0$ and $U$ we immediately find that
\[
\dot{x}_0(t) = \pm \frac{\sqrt{c^2 - 2E_0}}{\sqrt{2E_0} \sec^2 y - 1}, \tag{67}
\]
\[
U(t) = \pm \frac{c \tan y}{\sqrt{2E_0} \sec^2 y - 1}, \tag{68}
\]
where we have abbreviated $(\omega (t-t_0) + \kappa)$ to $y$. Therefore the motion of the bosonic vector (59) is a non-uniform rotation, which confirms our comments in section 8 since the potential function $U$ changes sign only once. All fermionic vectors corotate with the bosonic one.

To determine $x_{ij}$ we need the integral
\[
\int_{t_0}^{t} \frac{1}{x_0^2} dt' = \frac{1}{k \sqrt{(c^2 - 2E_0)^3}} \left( -\omega (t-t_0) + \frac{c^2}{2E_0} \tan y \right); \tag{69}
\]
we then obtain for the $E_{ij}$-related part of $x_{ij}$:

$$
\pm \frac{E_{ij}}{k(c^2 - 2E_0)} \left( \frac{c^2 \tan y}{\sqrt{2E_0}} - \frac{\omega(t-t_0)}{\sqrt{2E_0}} \right).
$$

This result clearly shows a linear periodic term which means that the period-amplitude relation is non-trivial. In fact, period and amplitude of $x_0(t)$, the latter defined as distance between the two turning points, can be easily calculated using (66):

$$
T = \frac{2\pi}{k\sqrt{c^2 - 2E_0}}, \quad A = \frac{2}{k} \text{arcsinh} \sqrt{\frac{2E_0}{c^2 - 2E_0}}.
$$

Thus

$$
T(A) = \frac{2\pi}{k|c|} \cosh \frac{kA}{2}
$$

and $\frac{dT}{dA} \neq 0$. We point out that all terms in the solution are clearly bounded, which is not the case for the integral itself.

Figure 2 gives an example of both the bosonic quantities $x_0$ and $x_{ij}$ and the fermionic quantities $\lambda_i$ and $\rho_i$.

Figure 2: $x_0$ (continuous), $x_{ij}$ (dotted), $\lambda_i$ (long dashes) and $\rho_i$ (short dashes) as functions of time for $c = k = 1$, $x_0(0) = c_{ij} = 0$, $E_0 = \frac{1}{4}$, $E_{ij} = \frac{1}{2}$ and $Q_i = \tilde{Q}_i = \frac{3}{2}$.

We do not present the higher order solutions in detail here since their explicit form, although not complicated to find, is lengthy and does not provide new insights. The only interesting exception is the part of $x_{1234}$ which has been characterized as a second variation with energy. This part can be calculated to be

$$
\frac{1}{2}E_{1234} \left[ \frac{(\alpha^3 - 3\alpha^2 + 6\alpha) \sin y \sec^3 y + (2\alpha^2 - 4\alpha) \sin y \sec y}{k(c^2 - 2E_0)^2 \sqrt{(\alpha \sec^2 y - 1)^3}} \right]
$$

$$(t - t_0) \left( \frac{(-2\alpha^2 + \alpha) \sec^2 y + 1}{\sqrt{(c^2 - 2E_0)^3} \sqrt{(\alpha \sec^2 y - 1)^3}} \right) + (t - t_0)^2 \left( \frac{-k\alpha \sin y \sec^3 y}{c^2 - 2E_0 \sqrt{(\alpha \sec^2 y - 1)^3}} \right)
$$

where we have abbreviated $\frac{c^2}{2E_0}$ to $\alpha$. Again, there are two noteworthy points here: Firstly, the occurrence of a quadratic-periodic term, which has been predicted in section 8 and secondly the
complete boundedness of the whole solution, which is not immediately obvious from the functional form \([10]\), especially regarding the term where the velocity appears in the denominator.

Next we want to show that the two solution methods presented in this paper deliver the same result. Therefore we have to decompose our previous solution, the first of equations \([21]\), from section \([1,2]\). Recall that any real analytical function \(f\) is extended to a Grassmann-valued function by the formula

\[
f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^n(z_b) z_s^n,
\]

where \(z_b\) and \(z_s\) are body and soul of the Grassmann number \(z\), respectively \([3]\). To keep the calculation short we will expand here only up to second order, i.e. the following equations have to be understood modulo a component proportional to \(\xi_{1234}\).

We begin with the frequency \(\omega\):

\[
\omega = k \sqrt{(c^2 - 2E_0) - 2E_0 a_{ij} \xi_{ij}} = k \sqrt{c^2 - 2E_0} - \frac{k E_{ij}}{\sqrt{c^2 - 2E_0}} \xi_{ij}.
\]

Then

\[
\sin(\omega(t-t_0) + \kappa) = \sin(k \sqrt{c^2 - 2E_0}(t-t_0) + \kappa_0) + \cos(k \sqrt{c^2 - 2E_0}(t-t_0) + \kappa_0) \left( -\frac{k E_{ij}(t-t_0)}{\sqrt{c^2 - 2E_0}} + \kappa_{a,ij} \right) \xi_{ij}.
\]

Since

\[
\sqrt{\frac{2E+2ckR}{c^2-2E}} = \sqrt{\frac{2E_0}{c^2-2E}} + \left( \frac{c^2 E_{ij} + c k R_{ij} (c^2 - 2E_0)}{\sqrt{2E_0 (c^2 - 2E_0)^3}} \right) \xi_{ij}
\]

we find

\[
\sqrt{\frac{2E+2ckR}{c^2-2E}} \sin(\omega(t-t_0) + \kappa) = \sqrt{\frac{2E_0}{c^2-2E_0}} \sin y + \sqrt{\frac{2E_0}{c^2-2E_0}} \left[ R_{ij} \frac{ck}{2E_0} \sin y + \kappa_{ij} \cos y \right]
\]

\[
+ \frac{c^2}{2E_0} \xi_{ij} \left( \frac{c^2}{c^2 - 2E_0} \sin y - \frac{k(t-t_0)}{\sqrt{c^2 - 2E_0}} \cos y \right)
\]

where again \(y = k \sqrt{c^2 - 2E_0}(t-t_0) + \kappa_0\). This result equals \(\sinh kx\); decomposition yields:

\[
\sinh kx = \sinh(kx_0 + kx_0 \xi_{ij}) = \sinh kx_0 + kx_0 \cosh kx_0 \xi_{ij},
\]

thus by comparison

\[
\sinh kx_0 = \sqrt{\frac{2E_0}{c^2-2E_0}} \sin y,
\]

which gives the same solution for \(x_0\) as \([36]\). Similarly, we can read off the result for \(kx_0 \cosh kx_0\) and then calculate \(x_{ij}\):

\[
x_{ij}(t) = \frac{R_{ij}}{2E_0} \frac{c \tan y}{\sqrt{\frac{c^2}{2E_0} \sec^2 y - 1}} - \frac{\kappa_{ij}}{k} \frac{1}{\sqrt{\frac{c^2}{2E_0} \sec^2 y - 1}}
\]

\[
+ \frac{E_{ij}}{k(c^2-2E_0)} \frac{\tan y}{\sqrt{\frac{c^2}{2E_0} \sec^2 y - 1}} + \frac{1}{\sqrt{c^2-2E_0} \sqrt{\frac{c^2}{2E_0} \sec^2 y - 1}} -(t-t_0).
\]
The term with coefficient $R_{ij}$ can be readily identified as the potential function $U(t)$, derived in (68). The second term with coefficient $\kappa_{ij}$ equals the velocity $\dot{x}_0$, which we have calculated in (67), when we make the identification $\kappa_{ij} = -\omega c_{ij}$. Finally, the term proportional to $E_{ij}$ is identical with our result (69).

So the solution found by using our layer-by-layer approach can also be obtained by the decomposition of the full Grassmann solution calculated through direct integration in Grassmann space. We have demonstrated this feature here using the bosonic quantity $x(t)$ but we could equally well have chosen one of the fermionic quantities $\psi_+(t)$ or $\psi_-(t)$. This means that our two solution methods are compatible with each other.

10 The Zero Energy Solutions

We have already mentioned in sections 4 and 6 that the case where the energy $E_0$, the body of the full Grassmann energy $E$, equals zero has to be treated differently. All the solutions to the equations of motion (26)–(30) were given under the restriction $E_0 \neq 0$ so that we cannot use our previous results. It will turn out, however, that for a finitely generated Grassmann algebra we can derive all solutions in explicit functional form for arbitrary potential.

Starting with equation (31) we can see that if $E_0 = 0$ both $\dot{x}_0$ and $U(x_0)$ have to vanish:

$$\dot{x}_0 = 0, \quad U(x_0) = 0.$$  

This means that the particle stays permanently at rest at a minimum $x_{0,\text{min}}$ of the potential $U^2$. One could assume at this point that all higher order bosonic and fermionic functions are trivial as well, but as we shall soon see this is far from being necessary, in contradiction to [6].

Because $x_0(t)$ is constant, all the (spatial) derivatives of $U$ are constant functions of time too; we denote their values by

$$U'(x_0) = k_1, \quad U''(x_0) = k_2, \quad U'''(x_0) = k_3, \ldots;$$

we assume in the following that $k_1 \neq 0$.

When we now look at the first order fermionic equations (29) for $\lambda_i$ and $\rho_i$, we find that they are easily solved by the harmonic oscillator ansatz:

$$\lambda_i = l_i \cos k_1(t - t_0) - r_i \sin k_1(t - t_0) \quad (70)$$
$$\rho_i = l_i \sin k_1(t - t_0) + r_i \cos k_1(t - t_0). \quad (71)$$

Unlike the true harmonic oscillator, the constants $l_i$ and $r_i$ are, however, not linked to the first order supercharges $Q_i$ and $\tilde{Q}_i$ which vanish completely as can be verified from (41).

To understand the first order fermionic motion we assume that the energy $E_0$ is small but non-zero, restricting the particle motion to a small neighbourhood of the stability point $x_{0,\text{min}}$ where the potential function $U$ is approximately linear. The result is an almost harmonic oscillation with frequency $k_1$, mirrored by the fermionic quantities – as follows from (48) and (49). When we now let $E_0$ approach zero, this fermionic motion seems to diverge as one can see from the formulae for the coefficients $L_i$ and $R_i$ in (40) and (44). However, there is a subtle point here: While the supercharges $Q_i$ and $\tilde{Q}_i$ are arbitrarily chosen constants for $E_0 \neq 0$, they have to vanish for $E_0 = 0$. Thus to avoid any discontinuities they have to smoothly approach...
zero as the energy decreases. Exactly how they approach zero finally determines which value
the harmonic oscillator coefficients $L_i$ and $R_i$ take in the limit $E_0 = 0$, which justifies the two
remaining degrees of freedom in our solution, $l_i$ and $r_i$.

Next we want to analyse the second order bosonic quantity $x_{ij}$. Because $U'$ and $U''$ are
constants the equation of motion (27) describes formally a harmonic oscillator with frequency
$k_1$ subject to a constant external force $k_2R_{ij}$, so the solution (for $k_1 \neq 0$) is

$$x_{ij} = v_{ij} \cos k_1(t-t_0) + \tilde{v}_{ij} \sin k_1(t-t_0) + \frac{k_2}{k_1^2} R_{ij},$$

(72)

where $v_{ij}$ and $\tilde{v}_{ij}$ are integration constants. To interpret this result we recall that in case $E_0 \neq 0$
the homogeneous part of $x_{ij}$ consists of two terms, the two variational derivatives of $x_0$ with
respect to $E_0$ and $t_0$. When $E_0$ is zero, a small change in energy will result in oscillatory
motion around the stability point $x_{0,\text{min}}$. This explains the functional form and one of the free
parameters of the solution (72). The second parameter, though, cannot be connected to the
variation with $t_0$ anymore since this variation is zero for a constant $x_0$.

An interesting observation is connected to the second order energy which can be calculated
from (52) to be $E_{ij} = -k_1 R_{ij}$. Thus the second order energy is not independent any more but
determined by the four first order constants $l_i, r_i, l_j, r_j$. Note the fact that $E_{ij}$ is not connected
to the parameters $v_{ij}$ and $\tilde{v}_{ij}$.

The third order fermion equations (80) are again equations describing a harmonic oscillator
with a driving term. The homogeneous part of the solution looks therefore the same as (74) and (71)
with integration constants $l_{ijk}$ and $r_{ijk}$. To find the particular solution we have to
investigate the driving terms $-k_2 x_{[ij] \rho_k}$ and $k_2 x_{[ij] \lambda_k}$ further. Using equations (74), (71) and (72) we find

$$x_{[ij] \rho_k} = C_{ijk} + D_{1,ijk} \sin 2k_1(t-t_0) - D_{2,ijk} \cos 2k_1(t-t_0),$$

$$x_{[ij] \lambda_k} = \tilde{C}_{ijk} + D_{2,ijk} \sin 2k_1(t-t_0) + D_{1,ijk} \cos 2k_1(t-t_0),$$

where $C_{ijk}, \tilde{C}_{ijk}, D_{1,ijk}$ and $D_{2,ijk}$ are constants built from $l_i, r_i, v_{ij}$ and $\tilde{v}_{ij}$. So the driving terms
are not constant but have an oscillating part that oscillates with double the basic frequency $k_1$.
The particular solutions are thus

$$\lambda_{ijk,\text{part.}} = -\frac{k_2}{k_1} C_{ijk} + \frac{k_2}{k_1} D_{1,ijk} \cos 2k_1(t-t_0) + \frac{k_2}{k_1} D_{2,ijk} \sin 2k_1(t-t_0)$$

$$\rho_{ijk,\text{part.}} = -\frac{k_2}{k_1} C_{ijk} + \frac{k_2}{k_1} D_{1,ijk} \sin 2k_1(t-t_0) - \frac{k_2}{k_1} D_{2,ijk} \cos 2k_1(t-t_0).$$

The fermionic vectors $(\lambda_{ijk}, \rho_{ijk})$ consist of three parts: A uniform rotation with natural frequency
$k_1$, a forced uniform rotation with double this frequency that we can interpret as the first excited
mode and a constant shift away from the origin. Note that there is no double frequency term if $U''(x_0) = 0$, and in particular for the harmonic oscillator potential. This agrees with our earlier result in section 6.1.

Calculating the supercharges $Q_{ijk}$ and $\tilde{Q}_{ijk}$ from (14) we find that $Q_{ijk} = 2k_1 C_{ijk}$ and
$\tilde{Q}_{ijk} = -2k_1 \tilde{C}_{ijk}$ which gives us a neat interpretation of the displacement constants $C_{ijk}$ and
$\tilde{C}_{ijk}$. It further means that the third order supercharges are not related to the free parameters
$l_{ijk}$ and $r_{ijk}$ as is the case for $E_0 \neq 0$ but instead are completely determined by first and
second order parameters. This is the same phenomenon already encountered for the second order energy $E_{ij}$ and again it means that we should not think of $Q_{ijk}$ and $\tilde{Q}_{ijk}$ as independent constants when $E_0 = 0$.

Finally we have to investigate the fourth order bosonic quantity $x_{1234}$. The equation of motion \((28)\) simplifies in the case $E_0 = 0$ to

$$\ddot{x}_{1234} + k_1^2 x_{1234} = -\frac{3}{2} k_1 k_2 x_{[12][34]} + k_2 R_{1234} + k_3 x_{[12]R_{34}},$$

so it describes a harmonic oscillator with three driving terms. The homogeneous part is analogous to \((22)\) with two integration constants $c_{1234}$ and $c_{1234}$; the comments made about $x_{ij}$ apply here, too. Analyzing the three driving terms we first note that $k_2 R_{1234}$ specifies a constant external force, playing the same role as $k_2 R_{ij}$ for the variables $x_{ij}$. The first and third terms can be calculated using the explicit formulae for $x_{ij}$, resulting in

$$-\frac{3}{2} k_1 k_2 x_{[12][34]} + k_3 x_{[12]R_{34}} = -\frac{3}{4} k_1 k_2 \left( C_{1234} + D_{1234} \cos 2k_1 (t-t_0) + \tilde{D}_{1234} \sin 2k_1 (t-t_0) \right)$$

$$+ \left( k_3 - \frac{k_2^2}{k_1} \right) \left( F_{1234} \cos k_1 (t-t_0) + \tilde{F}_{1234} \sin k_1 (t-t_0) \right),$$

where $C_{1234}$, $D_{1234}$, $\tilde{D}_{1234}$, $F_{1234}$ and $\tilde{F}_{1234}$ are fixed constants built from $v_{ij}$, $\tilde{v}_{ij}$ and $R_{ij}$. So besides a further constant term we find four oscillating ones, two with frequency $k_1$ and two with frequency $2k_1$. The latter ones will cause forced harmonic motion with double the natural frequency as we found for the third order fermionic terms. Now, however, are the oscillating terms which have the same frequency as the homogenous solution. This means that the oscillator is driven in resonance. Therefore the solution is not bounded anymore but includes linear-periodic terms like $t \sin k_1 (t-t_0)$ and $t \cos k_1 (t-t_0)$. Combining all contributions we find the particular solution

$$x_{1234, \text{part.}} = \frac{k_2}{k_1} \left( \frac{R_{1234}}{k_1} - \frac{3}{4} C_{1234} \right) + \frac{1}{4} k_2 \left( D_{1234} \cos 2k_1 (t-t_0) + \tilde{D}_{1234} \sin 2k_1 (t-t_0) \right)$$

$$+ \frac{1}{2} \left( \frac{k_2^2}{k_1} - \frac{k_3}{k_1} \right) \left( F_{1234} t \cos k_1 (t-t_0) - \tilde{F}_{1234} t \sin k_1 (t-t_0) \right).$$

Thus the fourth order bosonic solution consists of harmonic oscillation with the natural frequency $k_1$, an excited mode with twice this frequency, linear-periodic motion and a constant shift.

Especially interesting are the linear-periodic terms. They can be explained physically as in section 4, although we have to see them as the second variation with energy here: When the particle stays at rest in a potential minimum the first variation gives rise to harmonic oscillation, independently of the shape of the potential function $U$ (as long as $k_1 \neq 0$). Only when we look at the second order variation do non-harmonic terms come into play: The period of oscillation will in general depend on the amplitude and the energy, therefore a variation in energy results in a change of period; expressing the new solution in terms of the old period then generally leads to secular, non-periodic terms.

Notice the fascinating fact that although the physical interpretation is similar, the way the linear-periodic terms arise mathematically is completely different: For non-zero energy oscillations they come as solutions of a homogenous differential equation with periodic coefficients; in the zero energy case they are particular solutions to an inhomogenous differential equation with constant coefficients.
11 Discussion

In this work we have analysed a supersymmetric mechanical model from two different viewpoints: Either we make no specification with regard to the nature of the underlying Grassmann algebra $B$. Then for a range of potential functions $U$ the model can be explicitly solved. Or we regard $B$ as finitely generated thus reducing all quantities to a set of real functions and their interrelationships. Then we are able to solve the system completely, without any restrictions on $U$. The methods have been shown to be compatible with each other here.

An open question with our first point of view is what meaning we can give to the full, i.e. non-decomposed Grassmann solutions $x(t), \psi_+(t)$ and $\psi_-(t)$. Interpretation may be aided by our second approach, where we have shown that the component solutions include the purely classical motion dependent on $E_0$ and all its variations with respect to this (real) constant. Apparently, supersymmetric dynamics captures information about a whole range of energies of the mechanical system, so in a sense we can say that the Grassmann energy $E$ corresponds to some fuzzy classical energy. This suggests possibly that supersymmetric classical dynamics is closely related to the quantum dynamics. A study of the quantized version of our model would therefore be an important step to complement this research.

We also hope to apply our methods to the full, i.e. space- and time-dependent field theory.

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