Knot soliton solutions for the one-dimensional non-linear Schrödinger equation

Rahul O R and S Murugesh

Department of Physics, Indian Institute of Space Science and Technology, Thiruvananthapuram-695 547, India.

E-mail: murugesh@iist.ac.in

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Abstract

We identify that a breather soliton solution for the one-dimensional non-linear Schrödinger equation, presented here, is characteristically distinct when one studies the associated space curve, specifically that this space curve is knotted. The significance of these solutions with such a non-trivial geometrical element is pre-eminent on two counts: it is a one-dimensional model wherein structures with such non-trivial geometry are unexpected, and that the nonlinear Schrödinger equation is well known to model a plethora of physical systems.

1. Introduction

The one-dimensional nonlinear Schrödinger equation (NLSE) is a fundamental model naturally and frequently arising in a variety of physical systems such as fluid dynamics, dynamics of polymeric fluids, ferromagnetic spin chains, fiber optics and vortex dynamics in superfluids, to name a few [1–7]. Besides its physical importance, it also is a very important model in soliton theory, owing to its rich mathematical structure [8]. It is completely integrable with soliton solutions, and presents itself amenable to nearly every method available in the study of nonlinear systems, making it a perfect pedagogical model. Its complete integrability was first established in a classic paper by Zakharov and Shabat in 1972, which also brought about a deeper clarity, from a geometric point of view, on the method of inverse scattering transform being developed around that period [8, 9]. For these aforementioned reasons it remains one of the most studied, and among the most understood of nonlinear integrable systems. In spite of its rich history and continued interest, investigations on NLSE have often thrown up novel results and physical behavior never anticipated or intuited earlier. One such is the integrable systems. In spite of its rich history and continued interest, investigations on NLSE have often thrown up novel results and physical behavior never anticipated or intuited earlier. One such is the integrable systems.

Furthermore, a natural connection exists between the complex field described by NLSE in one dimension and a space curve in 3-d, whose time evolution spans a surface. This inherent relationship also provides a deeper geometrical interpretation of integrability and integrable systems. In the case of systems integrable and endowed with a Lax pair the soliton surfaces thus obtained are of much interest in soliton theory [16, 17]. Often the space curves thus obtained are by themselves physically realizable, adding to their significance [18]. In particular, the space curves for the NLSE field can be thought of as approximating the motion of thin filament vortices in fluids, or superfluids [1–3]. This association leads to a connected question whether the one dimensional NLSE can possibly possess a soliton solution that can be related to a curve with non-trivial geometry, such as one with a knot. In fact, Kleckner and Irvine experimentally showed that such knotted vortices can indeed be created in fluids [19], although short lived, which further raises curiosity concerning the existence of such solutions to the one-dimensional NLSE.
The space curves associated with the breather solitons were obtained explicitly by Cieśliński et al.

The breather soliton is a localized traveling wave of the usual soliton, the corresponding space curve is a closed loop, carrying in it a smaller traveling loop that folds around the larger loop as it travels. But, to our knowledge, no solutions of the NLSE have been reported thus far that is associated with a knotted curve.

A variant of the Akhmediev breather solitons, owing to a certain Galilean gauge admissibility of the NLSE, have been studied by various authors [21, 22]. Further, the associated space curves were also constructed numerically by Salman [22]. However, a more detailed investigation of these breather solitons, particularly an explicit expression for these curves in its full generality, proves more worthwhile. As a case in point we show that for a wide range of parameters, these soliton solutions are characteristically different in that the associated space curve is knotted—a simple overhand knot that occurs periodically (as is expected of a breather) as the curve evolves in time. Although localized stable knots have been encountered in the study of complex nonlinear systems, especially NLSE like systems (see for instance [23]), what separates the solitons presented here is that these exact solutions are associated with the one-dimensional NLSE, wherein structures with a non-trivial geometry are not expected.

2. Knotted breathers for the NLSE

The one-dimensional NLSE for a complex field $\psi(x, t)$ is given by

\[ i\psi_t + \psi_{xx} + 2|\psi|^2 \psi = 0, \tag{1} \]

wherein the subscripts $t$ and $x$ indicate derivatives with respect to time and space, respectively. The variable $\psi(x, t)$ could refer to the complex amplitude of the electric field, in the context of light transmitted through an optical fiber with quadratic nonlinearity [5], or a complex function of the curvature and torsion in the case of thin vortices in a fluid, etc [18]. $\psi_0 = 0$ is clearly a trivial solution for the NLSE, equation (1). Starting from this seed solution, one may proceed with any of the standard techniques of obtaining solitons, such as the direct method due to Hirota, or a Darboux transformation, to obtain a soliton solution—a localized traveling wave of the secant hyperbolic type [17, 24].

Alternately, if we start with a seed solution

\[ \psi_0 = \kappa_0 e^{i\delta t}, \tag{2} \]

for some real constant $\kappa_0$, one may derive a periodic breather solution [10–13]. The breather soliton is a localized moving wave, much like the normal soliton, but can show temporal, or spatial, periodicity.

In this paper we investigate in detail the soliton solutions obtained if one starts with another non-trivial seed solution

\[ \psi_0 = \kappa_0 e^{i\pi \kappa a x} \tag{3} \]

—a constant field of uniform magnitude with a spatially periodic phase, where, as in equation (2), $\kappa_0$ is a real constant. Proceeding by the Darboux method, one can obtain, after some detailed algebra, the three parameter one-soliton solution

\[ \psi_1 = e^{i\pi \kappa a x} \left( \kappa_0 - 2 \lambda_0 \frac{(\zeta - i \eta)}{\chi} \right) \tag{4} \]

wherein

\[
\begin{align*}
\zeta &= \zeta_1 \cos(2 \Omega_0 x) + \zeta_2 \cosh(2 \Omega_0 t) \\
\eta &= \eta_1 \sin(2 \Omega_0 x) - \eta_2 \sinh(2 \Omega_0 t) \\
\chi &= \chi_1 \cos(2 \Omega_0 x) + \chi_2 \cosh(2 \Omega_0 t) \\
\Omega_0 &= \Omega_{0v} + i\Omega_{0l} = f_0 (x - \sqrt{2} \mu t) \\
f_0 &= f_{0r} + if_{0l} = \frac{1}{\sqrt{2}} \sqrt{\nu_0^2 + 2 \kappa_0^2} \\
\mu &= \mu_{0r} + i\mu_{0l} = \kappa_0 - \sqrt{2} \lambda_0 \\
\nu_0 &= \nu_{0r} + i\nu_{0l} = \kappa_0 + \sqrt{2} \lambda_0 \\
\zeta_1 &= 2(4 \kappa_0^2 + 2 |\nu_0|^2 + 4 \sqrt{2} \kappa_0 \nu_0 - 4 |f_0|^2) \\
\zeta_2 &= 2(4 \kappa_0^2 + 2 |\nu_0|^2 + 4 \sqrt{2} \kappa_0 \nu_0 - 4 |f_0|^2) \\
\zeta_3 &= 2(8 \kappa_0 \nu_{0r} + 4 \sqrt{2} (\nu_{0l} f_{0r} - \nu_{0r} f_{0l})) \\
\zeta_4 &= -2(8 \kappa_0 \nu_{0r} + 4 \sqrt{2} (\nu_{0l} f_{0r} - \nu_{0r} f_{0l}))
\end{align*}
\]
and \( \lambda_0 = \lambda_{0q} + i \lambda_{0t} \) is an arbitrary complex spectral parameter associated with the one-soliton (the scattering parameter in the framework of inverse scattering transforms). Two more parameters indicating the initial position and phase of the soliton are taken to be zero, without any loss of generality.

It may be noted that the seed solution for the breather, equation (2), differs from the seed we have chosen in equation (3) only by a phase factor. Under a Galielian transformation to the NLSE, the field parameter \( \psi \) gets phase shifted. More specifically, the NLSE is invariant under the transformation

\[
\begin{align*}
  x &\to x - \nu t, \quad t \to t, \\
  \psi &\to \psi e^{i(\kappa t/2 + \nu t/4)}.
\end{align*}
\]

(6)

Thus, starting from the spatially periodic Akhmediev breather, one may directly obtain a Galilean gauge related counterpart simply by effecting such a transformation \([21, 22]\). Indeed, for the choice \( \lambda_{0q} = -\kappa_0/\sqrt{2} \) and \( \kappa_0^2 > \lambda_{0q}^2 \), the general solution we have presented in equation (4) reduces to such a breather obtained by Salman \([22]\). Alternately, when \( \bar{f}_0 \bar{g}_0 \mu_0 + \bar{f}_0 \bar{g}_0 \mu_0 = 0 \) it reduces to the Galilean transformed version of the temporally periodic Kuznetsov-Ma breather. Being a complex function in one dimension, the profile of the breather does not fully reveal its intricacies. But, as pointed out earlier, the complex field \( \psi \) can be inherently and systematically related to a curve in three dimensional euclidean space. Such a curve can display non-trivial geometry, and in this case forms an overhand knot in the process of its time evolution.

3. Knotted breather space curves

The complex field of the NLSE, \( \psi(x, t) \), can be linked systematically to a moving non-stretching curve in three dimensions. Thus, one way of describing such a curve through NLSE is to relate \( \psi \) to the intrinsic curvature \( \kappa \) and torsion \( \tau \) of the curve \([25]\), such as

\[
\psi = \frac{\kappa}{2} e^{i\kappa\tau}, \quad \sigma_x = \tau,
\]

(7)

where, importantly, \( x \) now represents the arc-length parameter of the curve. Often referred as the Hasimoto transformation, this form arises quite naturally in certain cases—for instance, in studying the motion of a thin vortex filament in a fluid \([1]\). If \( \mathbf{R}(x, t) \) were such a space-curve, the NLSE, equation (1), can be rewritten in terms of \( \mathbf{R} \) as

\[
\mathbf{R}_t = \mathbf{R}_x \times \mathbf{R}_{xx},
\]

(8)

— the localized induction approximation (LIA) \([26]\). It should be noted that, whereas in the NLSE \( x \) represented the spatial coordinate, in the LIA though the same is the arc-length parameter of a non-stretching curve. Consequently, while a Galilean transformation to the NLSE may be countered by an appropriate phase transformation to \( \psi \) (see equation (6)), it is not as straightforward in the case of the LIA. For instance, the transformation equation (6) amounts to changing the torsion of the curve by a constant factor while retaining its curvature \( (\kappa \to \kappa, \tau \to \tau + \nu/2) \). In the language of the curve vector \( \mathbf{R}(x, t) \) this is non-trivial, and certainly not achieved by the co-ordinate transformation given in equation (6).

The fundamental theorem of curves guarantees the existence of a unique curve, given \( \kappa \) and \( \tau \) (upto a global shift, or rotation). While the general solution for such a curve for a given \( \kappa \) and \( \tau \) is unknown, for special cases such as when the complex function \( \psi \) is a soliton soliton obtainable by inverse scattering, the associated curve and surface can indeed be systematically found \([16]\). To a reasonable approximation, thin line vortices in incompressible fluids and superfluids are often interpreted as such curves associated with the NLSE \([2, 3]\). The curves associated with the Kuznetsov-Ma and Akhmadiev breathers have been investigated by Cieśliński, et al in \([20]\), while that for the Galilean transformed Akhmediev breather has been numerically studied in \([22]\).

The seed solution in equation (3) corresponds to a moving helix with a constant intrinsic curvature \( 2\kappa_0 \) and a torsion \( \tau = \sqrt{2} \kappa_0 \):

\[
\mathbf{R}_0 = \frac{1}{3} \left[ (\sqrt{3} (x + 2 \sqrt{2} \kappa_0 t)) \mathbf{i} + \left( \frac{1}{\kappa_0} \sin \theta \right) \mathbf{j} - \left( \frac{1}{\kappa_0} \cos \theta \right) \mathbf{k} \right],
\]

(9)

where \( \theta = \sqrt{6} \kappa_0 (x - \sqrt{2} \kappa_0 t) \), having both a global translation along its axis with velocity \( 2 \sqrt{3} \kappa_0 \), and a rotation about its axis with period

\[
T_0 = \pi/(\sqrt{3} \kappa_0),
\]

(10)

effectively constituting a screw motion. The helix has a pitch \( \sqrt{2} \pi/(3 \kappa_0) \) and radius \( 1/(3 \kappa_0) \).
For the one-soliton solution in equation (4), the associated curve can be shown to be

\[ R_i = R_0 + \frac{\lambda_{0i}}{|\lambda_{0i}|^2} \chi \left[ -\left( \sqrt{2} \eta + \xi \right) \hat{i} + \left( -\zeta \sin \theta + \cos \theta \left( \eta - \sqrt{2} \xi \right) \right) \hat{j} + \left( \zeta \cos \theta + \sin \theta \left( \eta - \sqrt{2} \xi \right) \right) \hat{k} \right] \]  

(11)

where

\[ \xi = c_4 \sin (2 \Omega_{0i}) + c_5 \sinh (2 \Omega_{0i}), \]  

(12)

and \( \zeta, \eta, \chi \), and the constants \( c_i, i = 1 - 4 \) were defined in equation (5). They also obey the conditions

\[ \xi^2 + \eta^2 + \xi^2 = \chi^2, \]
\[ c_1^2 + c_2^2 + c_4^2 = c_1^2. \]  

(13)

What makes this soliton solution, equation (4), really distinct is that, for a range of values, the filament self intersects, and then developing into an overhand knot, and back into an un-knotted loop, as it evolves in time (figure 1). The overall period of the breather, \( T_{\text{total}} \), is thus divided into two phases—the knot phase with period \( T_{\text{knot}} \) and the loop phase with period \( T_{\text{loop}} \). While \( \kappa_0 \) determines both the radius and pitch of the helical backbone (see equation (9)), the radius of the soliton loop is determined by both \( \kappa_0 \) and \( \lambda_0 \). When the loop is larger than the pitch of the backbone, folding results in self intersections with the helical backbone, with the outcome being periodic knot formation. In general, this behavior is multiply periodic, both temporally and spatially, involving periods that are generally incommensurate. To see this, we first note that the helical backbone in itself has a periodicity \( T_c \) decided by \( \kappa_0 \), equation (10). Besides, due to the conditions in equation (13), the terms in the curve equation, equation (11), can be re-written as:
for appropriate real quantities $A$ and $B$, determined by $\kappa_0$ and $\lambda_0$. Thus $R_1$ is a function of three generally incommensurate periodic terms. However, for carefully chosen values of $\kappa_0$ and $\lambda_0$, the periodicity could be exclusively temporal, or spatial.

For a specific choice of the parameters $\kappa_0$ and $\lambda_0$ (for ease of analysis, henceforth we shall choose $\lambda_0 = 0$, and freeze the value of $\kappa_0$, varying only $\lambda_0$), snapshots of this curve through its period are shown in figure 1 (a)-(e)—a loop traveling along a helical backbone of the seed, folding around the axis of the helix as it travels (see supplementary material available online at stacks.iop.org/JPCO/2/055033/mmedia for detailed animation). In the process, it also encounters self intersections, leading to formation of a overhand knot. The behavior is not generic, however. For certain choices of $\lambda_0$, given $\kappa_0$, the two self intersections are simultaneous at two different points on the curve, with a vanishing $T_{\text{knot}}$ (see figure 1 (f), and supplementary material for detailed animation). The dependence of $T_{\text{knot}}$ and $T_{\text{loop}}$ on $\lambda_0$ is cumbersome to be analyzed analytically from equation (11). In figure 2 we show this dependence following a numerical study. Generally, $T_{\text{loop}} > T_{\text{knot}}$. The overall period of evolution $T_{\text{total}}$ is better elucidated if one transforms to a new varying arc-length parameter

$$x' = x - \sqrt{2} \frac{\eta + \xi}{\sqrt{3} \chi} (f_{0l r} f_{0l} + f_{0l l} \mu_{0}) t,$$

and choosing $x' = 0$. Recalling the role of $x$ in LIA as the arc-length parameter, this transformation amounts to moving along the curve at a constant speed such that $\Omega_{0l} = 0$. Consequently, $\zeta$, $\eta$, $\xi$ and $\chi$ in equation (11) are periodic functions of time with period

$$T_{\text{total}} = \frac{\pi f_{0l}}{\sqrt{2} \mu_{0 l} f_{0l}^2} = \frac{\pi f_{0l}}{2 \lambda_{0l} f_{0l}^2}$$

(15)

and choosing $x' = 0$. Recalling the role of $x$ in LIA as the arc-length parameter, this transformation amounts to moving along the curve at a constant speed such that $\Omega_{0l} = 0$. Consequently, $\zeta$, $\eta$, $\xi$ and $\chi$ in equation (11) are periodic functions of time with period

$$T_{\text{total}} = \frac{\pi f_{0l}}{\sqrt{2} \mu_{0 l} f_{0l}^2} = \frac{\pi f_{0l}}{2 \lambda_{0l} f_{0l}^2}$$

(16)

determining the total period of the loop evolution. From a simple dimensional argument one can conclude that the size of the loop inversely varies with the value of $\lambda_0$. Beyond a certain value of $\lambda_0$, intersections lead only to loops. Heuristically, this can be attributed to the size of the loop being smaller than the pitch of the helical backbone for large values of $\lambda_0$.

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1 Knot.avi.
2 2-intersections.avi.
It is indeed tempting to interpret the knotted breather curves as vortex filaments in inviscible fluids, owing to the equivalence of NLSE to the LIA. In fact, helical fluid vortices have been experimentally demonstrated in the laboratory, further prompting a likelihood \cite{27, 28}. However, we hasten to clarify that the LIA is not an appropriate model when non-local interactions are predominant, as in this case. For instance, self intersections are permissible under LIA, as evidenced in figure 1, but in reality such intersections would only lead to bifurcation of the filament at the point of intersection (followed by possible recombinations) \cite{19, 22, 29}. However, there are also bands of intermediate $\lambda_0 I$ values for which no self intersections are noted, and hence no knots appear (gray bands in figure 2). Furthermore, as the value of $\lambda_0 I$ is lowered the loop also carries in it windings, or petals, such that with decreasing $\lambda_0 I$ a flowering scenario occurs, wherein the number of windings increases while the overall loop size decreases (figure 3). This is in fact qualitatively similar to winding petal forms reported in \cite{20} in closed vortex loops, wherein however, a self intersection was indicated as a constant presence, preempting any possibility of a knot formation. As a whole the loop evolves, rotating about the helix axis. The bands in figure 2 represent the regions of $\lambda_0 I$ wherein the size of the sub windings of the loops are smaller than the pitch of the backbone helix, so that intersections do not happen (see supplementary material\footref{supplementary} for detailed animation). Consequently, this knot-less evolution is perhaps more relevant in the context of helical vortex motion in fluids than the knotted phase. But being a fundamental model for a variety of physical systems (optical wave propagation in nonlinear media, or 1-d ferromagnetic chains, for instance) it is fair to surmise at this point that these knotted breathers for the NLSE, howsoever short lived, bear a wider scope and significance.

ORCID iDs

Rahul O R @ https://orcid.org/0000-0002-2309-9475
S Murugesh @ https://orcid.org/0000-0002-3143-7286

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