Abelian and Non-Abelian States in $\nu = 2/3$ Bilayer Fractional Quantum Hall Systems

Michael R. Peterson, Yang-Le Wu, Meng Cheng, Maissam Barkeshli, Zhenghan Wang, and Sankar Das Sarma

$^1$Department of Physics & Astronomy, California State University Long Beach, Long Beach, California 90840, USA
$^2$Joint Quantum Institute and Condensed Matter Theory Center, Department of Physics, University of Maryland, College Park, MD 20742
$^3$Station Q, Microsoft Research, Santa Barbara, California 93106-6105, USA
$^4$Department of Mathematics, University of California, Santa Barbara, California 93106, USA

(Dated: February 11, 2015)

There are several possible theoretically allowed non-Abelian fractional quantum Hall (FQH) states that could potentially be realized in one- and two-component FQH systems at total filling fraction $\nu = n + 2/3$, for integer $n$. Some of these states even possess quasiparticles with non-Abelian statistics that are powerful enough for universal topological quantum computation, and are thus of particular interest. Here, we initiate a systematic numerical study, using both exact diagonalization and variational Monte Carlo, to investigate the phase diagram of FQH systems at total filling fraction $\nu = n + 2/3$, including in particular the possibility of the non-Abelian $Z_4$ parafermion state. In $\nu = 2/3$ bilayers, we determine the phase diagram as a function of interlayer tunneling and repulsion, finding only three competing Abelian states, without the $Z_4$ state. On the other hand, in single-component systems at $\nu = 8/3$, we find that the $Z_4$ parafermion state has significantly higher overlap with the exact ground state than the Laughlin state, together with a larger gap, suggesting that the experimentally observed $\nu = 8/3$ state may be non-Abelian. Our results from the two complementary numerical techniques agree well with each other qualitatively.

PACS numbers: 73.43.-f, 71.10.Pm

Introduction—Multi-component fractional quantum Hall (FQH) states appear in a wide variety of two-dimensional electron systems (2DES) [1], such as in multilayer or multiusbband quantum wells [2], in systems with small Zeeman energy where the electron spin plays an active role, and in systems with multiple valley degrees of freedom, such as graphene [3–7], Silicon [8], AlAs [9, 10]. These systems typically offer several tunable parameters, which allow for the observation of rich zero temperature phase diagrams involving topologically distinct FQH states even at a fixed total filling fraction, and indeed novel FQH phases of multicomponent systems have been occasionally experimentally observed. Most notable perhaps is the observation of the so-called 331 Abelian even-denominator FQH state in half-filled bilayer systems [11–13]. However in many cases, little is known about the myriad possible FQH phases and phase transitions that can be experimentally realized in multi-component 2D systems.

In recent years, motivated by the possibility of a non-Abelian state at $\nu = 5/2$ in GaAs quantum wells [14, 15], there have been detailed numerical studies at total filling fraction $\nu = n + 1/2$, for integer $n$, in two-component systems [16]. While $n + 1/2$ has been studied in detail, the analogous problem at $\nu = n + 2/3$ has received little attention from numerical studies. Such systems were first studied experimentally over twenty years ago, where a two-component to single-component phase transition was observed in both monolayer (presumably due to spin) and bilayer systems [17–20]. There are three simple Abelian FQH states that can be realized at $\nu = 2/3$: (1) the 330 state, which consists of two decoupled $1/3$ Laughlin states in each layer, (2) a pseudo-spin singlet Abelian state, referred to as the 112 state, and (3) the particle-hole conjugate of the $1/3$ Laughlin state. Early numerical work on $\nu = 2/3$ bilayers considered the overlap of model wave functions for these states with the exact ground state of the system for up to $N = 6$ electrons on a torus [21], finding these three phases in the two-component 2D system (for the monolayer spin-full system, the 330 state is unlikely to be present).

More recently, a variety of different theoretical studies have suggested five possible exotic non-Abelian FQH states that can occur in principle at $\nu = 2/3$, but which have not been numerically investigated to date (see Table I). These include: (1) the $Z_4$ parafermion FQH state [22], (2) a Fibonacci state based on $SU(3)_2$ Chern-Simons theory [23, 24], inter- and intra-layer Pfaffian states [25, 26], and a Bonderson-Slingerland hierarchy state [27]. Of these, the $Z_4$ and Fibonacci states have recently been shown theoretically to exhibit continuous phase transitions from the 330 state [24, 28], suggesting that

| Possible States at $\nu = 2/3$ | Shift, $S$ |
|---------------------------------|-----------|
| 1. 330                          | 3         |
| 2. Pseudo-spin singlet 112      | 1         |
| 3. P-H conjugate of $1/3$-Laughlin | 0        |
| 4. $Z_4$ Parafermion            | 3         |
| 5. Bonderson-Slingerland Hierarchy | 4    |
| 6. Interlayer Pfaffian          | 3         |
| 7. Intralayer Pfaffian          | 3         |
| 8. Bilayer Fibonacci            | 3         |

Table I. Candidate Abelian and non-Abelian FQH states at total filling fraction $\nu = 2/3$. On the sphere, the ground states for these phases occur at different shift, $S \equiv \frac{2}{3}N - N_\Phi$, where $N_\Phi$ is the number of flux quanta. The Fibonacci state, if viewed as a single-component system, has a shift of 6; in a two-component system, it would have a shift of 3 per layer.
these exotic states might be stabilized nearby more conventional ones if the appropriate microscopic parameters can be found and tuned experimentally. The goal of the current work is to investigate numerically the possible existence (or not) of exotic non-Abelian 2/3 (or more generally, \(n + 2/3\)) FQH states in realistic 2DES.

The Fibonacci FQH state contains the non-Abelian Fibonacci quasiparticle, whose braiding statistics is well-known to be powerful enough to be utilized for universal topological quantum computation (TQC) [29]. The \(Z_4\) parafermion FQH state is based on the \(SU(2)_4\) topological quantum field theory, which recently has been discovered to also allow for the possibility of universal TQC [30, 31]. The Bonderson-Slingerland hierarchy state at \(\nu = n + 2/3\), and the inter-layer Pfaffian state, can also be used for universal TQC if they can be realized on topologically non-trivial spaces with certain topological operations known as Dehn twists [32, 33]. Recently it has been shown these Dehn twists can be effectively realized in a physically realistic experimental setup [34, 35]. It is thus timely to revisit the \(\nu = 2/3\) bilayer phase diagram through numerical study in order to investigate the possibility of realizing these exotic non-Abelian states.

In this paper, we carry out a systematic study of two-component FQH systems at total filling fraction \(\nu = n + 2/3\). Here, we mainly analyze the relative stability of the three Abelian states mentioned above and the non-Abelian \(Z_4\) state through exact diagonalization and variational Monte Carlo studies. In the LLL, our results are consistent with the phase diagram proposed previously in Ref. 21 and we find that the \(Z_4\) state is not competitive relative to the other Abelian states. However, in the limit of large interlayer tunneling in the second Landau level (SLL), at \(\nu = 8/3\), our numerical results suggest that the \(Z_4\) state is preferable relative to the possible Abelian states. This is an unexpected new finding suggesting that the already experimentally observed 8/3 FQH state may very well be the exotic \(Z_4\) non-Abelian state, rather than the usual Abelian Laughlin \(2 + 2/3\) state. Given the existence of the 5/2 FQH state in the SLL, which is thought to be the non-Abelian Moore-Read state, the possibility that the SLL 8/3 FQH state might also be a (different) non-Abelian state seems plausible and consistent with the fact that the experimental 8/3 state typically is considerably weaker than the 5/2 state as manifested in the measured energy gaps [36].

**Model Hamiltonian and wave functions**—We consider the following Hamiltonian, which describes two quantum Hall layers with \(N\) total spin-polarized electrons, separated by a distance \(d\), with interlayer electron tunneling strength \(\Delta\):

\[
H = \sum_{i<j}^{N} \left[ V_{\text{intra}}(|r_i - r_j|) + V_{\text{intra}}(|r'_i - r'_j|) \right] + V_{\text{inter}}(|r_i - r'_j|) - \frac{2}{e} \frac{\Delta}{d} S_{x,i},
\]

where \(r_i\) and \(r'_i\) are the position of the \(i^{th}\) electron in the right or left layer, respectively, \(l = \sqrt{\hbar c/eB}\) is the magnetic length and \(B\) is the applied magnetic field. The intra-layer Coulomb interaction is given by \(V_{\text{intra}}(r) = \frac{e^2}{\pi \epsilon r^2}\), while the inter-layer interaction is given by \(V_{\text{inter}}(r) = \frac{e^2}{e l^2} \frac{1}{\sqrt{r^2 + 4d^2}}\), where \(\epsilon\) is the dielectric of the host semiconductor. The interlayer tunneling term can be written in terms of the total pseudo spin \(S_x\) operator, with \(\Delta\) a dimensionless measure of the interlayer tunneling energy scale, in units of \(\frac{e^2}{\epsilon l}\).

At filling fraction \(\nu = 2/3\), there are a wide variety of distinct FQH states that might be realized. We consider two Abelian two-component states: One of these is the 330 state, which is described by the following model wave function:

\[
\Psi_{mnn} = \prod_{i<j}(z_i - z_j)^m(w_i - w_j)^n\prod_{i,j}(z_i - w_j)^n, \quad (2)
\]

for \(m = 3, n = 0\), and where \(z_i, w_i\), for \(i = 1, \ldots, N/2\), are the complex coordinates of the electrons in the two layers. The other Abelian two-component state is the pseudo-spin singlet 112 state:

\[
\Psi_{\text{singlet}} = P_{\text{LLL}} \prod_{i<j} |z_i - z_j|^2 |w_i - w_j|^2 \Psi_{330}^{112} \quad (3)
\]

where \(P_{\text{LLL}}\) indicates the projection onto the lowest Landau level (LLL).

The single-component states we consider include the particle-hole conjugate of the 1/3-Laughlin state, which can be obtained using composite fermion theory [37]

\[
\Psi_{\text{P-H}} = P_{\text{LLL}} (z_i - z_j)^2 \Phi_{\nu = -2}, \quad (4)
\]

where \(\Phi_{\nu = -2}\) is the wave function for the \(\nu = -2\) IQH state. In what follows, we will refer to this state as the 2/3 Laughlin state. The \(Z_4\) parafermion FQH state can be written in terms of an anti-symmetrization over the electron coordinates of the two-component 330 state [38, 39]:

\[
\Psi_{Z_4} = A[\Psi_{330}]. \quad (5)
\]

The above wave functions also have a Gaussian factor (\(\sim \exp(-\sum_i |z_i|^2/4l^2)\)) which we have not shown.

In addition to the FQH states described by these model wave functions, there are a number of other possible non-Abelian FQH states as well, as summarized in Table I. We leave a detailed comparison with model wave functions for these states for future work.

**Bilayer phase diagram in LLL**—Here we consider the phase diagram at \(\nu = 2/3\) in a bilayer system in the LLL. The Hamiltonian, Eq. (1), has two important dimensionless parameters: \(d/l\), which sets the ratio of the inter- and intra-layer Coulomb interactions, and \(\Delta\), which sets the ratio of the interlayer tunneling to the intra-layer Coulomb interaction. The relative stability of the three Abelian states was studied through wave function overlap calculations in [21] for \(N = 6\) electrons (3 per layer) on the torus. Here, we revisit this problem for larger system sizes using exact diagonalization (ED) studies for up to \(N = 12\) electrons and variational Monte Carlo studies for up to \(N = 50\) electrons for the full bilayer problem in the spherical geometry. In the single component
limit we performed ED on systems with up to \( N = 16 \) electrons with the goal of considering the relative stability of the competing non-Abelian \( Z_4 \) state.

Fig. 1 displays our numerical results for the overlaps of the model wave functions for the 330, singlet 112, and 2/3-Laughlin states, with the exact Coulomb ground state at shifts \( S = 3, 1, \) and 0, respectively, together with the energy gap of these states. The energy gap is defined to be the difference between the angular momentum \( L = 0 \) ground state and the first excited state; if the ground state has \( L \neq 0 \), the gap is taken to be zero. We note that this gap is not necessarily the transport gap that would be measured in experiment but instead the gap connected to the robustness of the phase, however in many cases these two gaps are known to be qualitatively similar. Similar results are obtained for \( N = 6 \) and 10 (not shown). We do not compute the overlap with the \( Z_4 \) parafermion state for \( S = 3 \) and \( N = 8 \) for three reasons. One reason is that the \( Z_4 \) state is a single-component state and for \( N = 8 \) electrons and \( S = 3 \) there is only one possible \( L = 0 \) state with these quantum numbers. The second reason is the \( Z_4 \) state has 4-electron clustering properties that cause it to vanish exactly unless \( N \ mod 4 = 0 \). Hence, one must consider at least \( N = 12 \) electrons at \( S = 3 \), see Fig. 1(g), (h). The third reason is that the energy gap at \( S = 3 \) in the single-component limit is significantly below the energy gap at \( S = 0 \) corresponding to the 2/3 Laughlin state.

We define a phase diagram for the bilayer system by creating a single energy gap function \( \delta(d/l, \Delta) \) by taking all three shifts into account and choosing the value of the gap to be the value for the largest shift. For each system size, we take the topological order to be identified by the wave functions with the highest overlap with the ground state. We find that this gives results that are consistent with the states that give the highest values of the gap \( \delta \). Fig. 1(i) shows a plot of the gap \( \delta(d/l, \Delta) \) for \( N = 8 \), together with contour lines that show the wave function overlaps. Our results for \( N = 6, 10 \) electrons (not shown) are consistent with this phase diagram. We emphasize that this approximate phase diagram matches remarkably well with that of Ref. 21 determined by wavefunction overlap alone.

To further corroborate this phase diagram, we perform variational Monte Carlo calculations\(^{40, 41} \) for up to \( N = 50 \) electrons, with sample size \( 10^7 \). The relative statistical error in the energy estimate for each finite-size system is on the order of \( 10^{-5} \), negligible for our purposes. We use linear extrapolation in \( 1/N \) towards \( N \rightarrow \infty \) to estimate the energy per particle for each wave function at each value of \( (d/l, \Delta) \). We determine the phase diagram in the \( (d/l, \Delta) \) plane according to the wave function with the lowest energy per particle, and we characterize the phase stability using the energy difference \( \delta E \) between the dominant wave function and its lowest energy competitor. Fig. 2 shows a contour plot of \( \delta E(d/l, \Delta) \) that qualitatively agrees with the results obtained from exact diagonalization, except that the singlet now occupies a much smaller corner of the phase diagram. These calculations assume a width \( w = 0 \) for each quantum well; qualitatively similar results have been found upon varying \( w \).

In order to investigate the relative stability of the \( Z_4 \) parafermion state, we consider the system for \( N = 12 \) particles. In Fig. 1(g), (h), we show the overlap of the exact ground state with the \( Z_4 \) parafermion state, together with the value of the energy gap above the ground state at shift \( S = 3 \).
While the $Z_4$ state has a maximum overlap of $\approx 0.93$ with the exact ground state in the single-component limit, the Laughlin state has a much higher overlap of $\approx 0.99$. Furthermore, the system possesses a much larger energy gap at the shift of the Laughlin state relative to that at the shift of the $Z_4$ parafermion state. We have investigated the $Z_4$ state using our Monte Carlo simulations also (not shown), finding that $Z_4$ remains energetically unfavorable throughout the phase diagram, thus providing additional support for the conclusion that the $Z_4$ non-Abelian state is unlikely to occur in the bilayer 2/3 2DES.

Finally, we consider the single-component limit, $\Delta \gg 1$, $d/l < 1$ for $N = 16$ electrons to more definitively determine the possibility of the $Z_4$ state in the strong tunneling and small $d/l$ regime and study a particularly realistic model that includes the effects of Landau level mixing and finite width of the single quantum well [42]. These two effects are parametrized by well width $w/l$ the Landau level mixing parameter $\kappa = (\hbar \omega_c)/(e^2/\ell c)$ which is the ratio of the cyclotron energy to the Coulomb energy. We restrict our attention to $w/l < 4$; wide quantum well systems, with $w/l > 4$, are often better described as bilayers. We display our results for the LLL in Fig. 3, where the Laughlin state is clearly shown to be preferable. The $Z_4$ overlap at $S = 3$ is large ($\approx 0.82$) and essentially decreases monotonically with $\kappa$ and is robust to width $w/l$. At $S = 0$ the Laughlin state has an overlap of nearly unity ($\approx 0.99$) and it is robust to both $\kappa$ and $w/l$. The bottom panels of the LLL plots show the gap. Both $S = 0$ and 3 have non-zero gaps, but the gap at $S = 0$ is nearly three times larger than it is at $S = 3$. Both overlaps and gaps are robust to varying $\kappa$ and $w/l$.

By studying both the bilayer problem and its single-component limit, we do not expect the $Z_4$ state in the bilayer system at $\nu = 2/3$ in the LLL—this is consistent with our Monte Carlo simulations (not shown).

**Second Landau level (SLL)—** We now consider the analogous problem, using pseudopotential interactions that are suitable for the second Landau level—note this corresponds to the experimental $\nu = 8/3$ system. Repeating the overlap and gap calculations with the SLL pseudopotentials, we obtain results that are quite different from the case of the LLL. In particular, we find that in the single-component limit, where $d/l \ll 1$ and $\Delta \gg 1$, the state at shift $S = 3$ has a significantly larger gap than the state at $S = 0$. This suggests that the ground state in this limit might not be the 2/3 Laughlin state, but rather an alternative state with $S = 3$.

To investigate this further, we focus on the single-component limit and compare the $Z_4$ parafermion state with the Laughlin state for system sizes up to $N = 16$ electrons. Surprisingly, in the SLL, we find that the $Z_4$ state appears to be favored over the Laughlin state according to both our overlap and gap calculations, as shown in Fig. 4. In the SLL we see that the overlap with the $Z_4$ state is qualitatively similar to the LLL, i.e., it is nearly $0.83 - 0.84$ for small $\kappa$ and decreases to zero as $\kappa$ is increased. The Laughlin state at $S = 0$ has a smaller overlap of about $0.64 - 0.8$, increases with $w/l$, and monotonically decreases with $\kappa$. Perhaps more importantly, the gap is approximately 1.5 times larger at $S = 3$ than it is at $S = 0$. Our results for $N = 12$ electrons are qualitatively similar, but with quantitatively higher overlaps for both the $Z_4$ and Laughlin states. We have also computed the energy of the $Z_4$ state for up to $N = 28$ particles [43], which indicate that the $Z_4$ state is highly competitive with the Laughlin state.

**Conclusion**—Based on our exact diagonalization and Monte Carlo studies, we find that $\nu = 2/3$ bilayers in the LLL, in the limit of weak Landau level mixing, most likely do not realize the non-Abelian $Z_4$ parafermion state. Perhaps most remarkably, in the single-component limit of the SLL, our studies point to the conclusion that the non-Abelian $Z_4$ state is highly competitive with the Laughlin state. 

![FIG. 3. Overlap and gap calculations for $N = 16$ electrons in the LLL in the single-component limit. Top left (a) shows overlaps with the $Z_4$ state, at $S = 3$; top right (b) shows overlaps with the Laughlin state, at $S = 0$. Lower panels, (c) and (d), show the gaps at $S = 3$ and $S = 0$.](image)

![FIG. 4. Overlap and gap calculations for $N = 16$ electrons in the SLL in the single-component limit as a function of $\kappa$ and $w/l$. (a) and (b) show the overlaps with the $Z_4$ state at $S = 3$ and the 2/3 Laughlin state at $S = 0$, respectively. Panels (c) and (d) show the energy gaps.](image)
phase may be favorable for the $8/3$ FQH state relative to the Laughlin state. Indeed, previous detailed studies of the experimentally obtained energy gaps of FQH states in the SLL have already indicated the possibility that the electron correlations are sufficiently different from those of the LLL that novel exotic states might be realized in the SLL [44]. The implication of our finding that the observed $8/3$ SLL FQH state may be the parafermionic $Z_4$ non-Abelian phase is enormous since this particular state can be used for universal topological quantum computation.

Acknowledgements – M.R.P. thanks the Office of Research and Sponsored Programs at California State University Long Beach and Microsoft Station Q. We thank M. Zaletel, P. Bonderson, and N. Regnault for useful discussions. As this manuscript was being completed, we became aware of a related manuscript [45].

[1] S. D. Sarma and A. Pinczuk, Perspectives in Quantum Hall Effects (Wiley, 1996).
[2] Y. Liu, S. Hasdemir, J. Shabani, M. Shayegan, L. Pfeiffer, K. West, and K. Baldwin (2015), arXiv:1501.06958, and references therein.
[3] X. Du, I. Skachko, F. Duerr, A. Luican, and E. Y. Andrei, Nature 462, 192 (2009).
[4] K. I. Bolotin, F. Gahari, M. D. Shulman, H. L. Stormer, and P. Kim, Nature 462, 196 (2009).
[5] C. R. Dean, A. F. Young, P. Cadden-Zimansky, L. Wang, H. Ren, K. Watanabe, T. Taniguchi, P. Kim, J. Hone, and K. L. Shepard, Nature Phys. 7, 693 (2011).
[6] B. E. Feldman, B. Krauss, J. H. Smet, and A. Yacoby, Science 337, 1196 (2012).
[7] B. E. Feldman, A. J. Levin, B. Krauss, D. A. Abanin, B. I. Halperin, J. H. Smet, and A. Yacoby, Phys. Rev. Lett. 111, 076802 (2013).
[8] T. M. Kott, B. Hu, S. H. Brown, and B. E. Kane, Phys. Rev. B 89, 041107 (2014).
[9] N. C. Bishop, M. Padmanabhan, K. Vakili, Y. P. Shkolnikov, E. P. De Poortere, and M. Shayegan, Phys. Rev. Lett. 98, 266404 (2007).
[10] M. Padmanabhan, T. Gokmen, and M. Shayegan, Phys. Rev. B 80, 035423 (2009).
[11] J. P. Eisenstein, G. S. Boebinger, L. N. Pfeiffer, K. W. West, and S. He, Phys. Rev. Lett. 68, 1383 (1992).
[12] Y. W. Suen, L. W. Engel, M. B. Santos, M. Shayegan, and D. C. Tsui, Phys. Rev. Lett. 68, 1379 (1992).
[13] S. He, S. Das Sarma, and X. C. Xie, Phys. Rev. B 47, 4394 (1993).
[14] G. Moore and N. Read, Nuclear Physics B 360, 362 (1991).
[15] R. Willett, J. P. Eisenstein, H. L. Störmer, D. C. Tsui, A. C. Gossard, and J. H. English, Phys. Rev. Lett. 59, 1776 (1987).
[16] M. R. Peterson and S. Das Sarma, Phys. Rev. B 81, 165304 (2010).
[17] J. P. Eisenstein, H. L. Stormer, L. N. Pfeiffer, and K. W. West, Phys. Rev. B 41, 7910 (1990).
[18] Y. W. Suen, H. C. Manoharan, X. Ying, M. B. Santos, and M. Shayegan, Phys. Rev. Lett. 72, 3405 (1994).
[19] H. C. Manoharan, Y. W. Suen, T. S. Lay, M. B. Santos, and M. Shayegan, Phys. Rev. Lett. 79, 2722 (1997).
[20] T. S. Lay, T. Jungwirth, L. Smrčka, and M. Shayegan, Phys. Rev. B 56, R7092 (1997).
[21] I. A. McDonald and F. D. M. Haldane, Phys. Rev. B 53, 15845 (1996).
[22] N. Read and E. Rezayi, Phys. Rev. B 59, 8084 (1999).
[23] X. G. Wen, Phys. Rev. B 44, 2664 (1991).
[24] A. Vaezi and M. Barkeshli, Phys. Rev. Lett. 113, 236804 (2014).
[25] E. Ardonne, F. J. M. v. Lankvelt, A. W. W. Ludwig, and K. Schoutens, Phys. Rev. B 65, 041305 (2002).
[26] M. Barkeshli and X.-G. Wen, Phys. Rev. B 82, 245301 (2010).
[27] P. Bonderson and J. K. Slingerland, Phys. Rev. B 78, 125323 (2008).
[28] M. Barkeshli and X.-G. Wen, Phys. Rev. Lett. 105, 216804 (2010).
[29] M. H. Freedman, M. Larsen, and Z. Wang, Communications in Mathematical Physics 227, 605 (2002), ISSN 0010-3616.
[30] S. X. Cui and Z. Wang (2014), arXiv:1405.7778.
[31] C. Levailiann, B. Bauer, P. Bonderson, M. Freedman, and Z. Wang, to appear (2015).
[32] S. Bravyi and A. Y. Kitaev, unpublished (2000).
[33] P. Bonderson, private communication (2012).
[34] M. Barkeshli, C.-M. Jian, and X.-L. Qi, Phys. Rev. B 87, 045130 (2013), arXiv:1208.4834.
[35] M. Barkeshli and X.-L. Qi, Phys. Rev. X 4, 041035 (2014).
[36] C. R. Dean, B. A. Piot, P. Hayden, S. Das Sarma, G. Gervais, L. N. Pfeiffer, and K. W. West, Phys. Rev. Lett. 100, 146803 (2008).
[37] J. K. Jain, Composite Fermions (Cambridge University Press, 2007).
[38] E. Rezayi, X.-G. Wen, and N. Read (2010), arXiv:1007.2022.
[39] M. Barkeshli and X.-G. Wen, Phys. Rev. B 81, 045323 (2010), arXiv:0909.4882.
[40] R. Morf and B. Halperin, Z. Phys. B 68, 391 (1987).
[41] S. C. Davenport and S. H. Simon, Phys. Rev. B 85, 245303 (2012).
[42] M. R. Peterson and C. Nayak, Phys. Rev. B 87, 245129 (2013).
[43] B. A. Bernevig and N. Regnault, Phys. Rev. Lett. 103, 206801 (2009).
[44] H. C. Choi, W. Kang, S. Das Sarma, L. N. Pfeiffer, and K. W. West, Phys. Rev. B 77, 081301 (2008).
[45] S. Geraedts, M. P. Zaletel, Z. Papic, and R. S. K. Mong (2015), arXiv:1502.01340.