Sharp lower bounds on the spectral radius of uniform hypergraphs concerning degrees

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Abstract

Let $A(H)$ and $Q(H)$ be the adjacency tensor and signless Laplacian tensor of an $r$-uniform hypergraph $H$. Denote by $\rho(H)$ and $\rho(Q(H))$ the spectral radii of $A(H)$ and $Q(H)$, respectively. In this paper, we present a lower bound on $\rho(H)$ in terms of vertex degrees and we characterize the extremal hypergraphs attaining the bound, which solves a problem posed by Nikiforov [V. Nikiforov, Analytic methods for uniform hypergraphs, Linear Algebra Appl. 457 (2014) 455-535]. Also, we prove a lower bound on $\rho(Q(H))$ concerning degrees and give a characterization of the extremal hypergraphs attaining the bound.

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1 Introduction

Let $G = (V(G), E(G))$ be a simple undirected graph with $n$ vertices, and $A(G)$ be the adjacency matrix of $G$. Let $\rho(G)$ be the spectral radius of $G$, and $d_i$ be the degree of vertex $i$ of $G$, $i = 1, 2, \ldots, n$. In 1988, Hofmeister [6] obtained a lower bound on $\rho(G)$ in terms of degrees of vertices of $G$ as follows:

$$\rho(G) \geq \left( \frac{1}{n} \sum_{i=1}^{n} d_i \right)^{\frac{1}{2}} . \quad (1)$$

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Furthermore, if \( G \) is connected, then equality holds if and only if \( G \) is either a regular graph or a semiregular bipartite graph (see details in [6] and [19]). The inequality (1) has many important applications in spectral graph theory (see [4, 8, 9]).

In recent years the research on spectra of hypergraphs via tensors have drawn increasingly extensive interest, accompanying with the rapid development of tensor spectral theory. A hypergraph \( H = (V, E) \) consists of a (finite) set \( V \) and a collection \( E \) of non-empty subsets of \( V \) (see [1]). The elements of \( V \) are called vertices and the elements of \( E \) are called hyperedges, or simply edges of the hypergraph. If there is a risk of confusion we will denote the vertex set and the edge set of a hypergraph \( H \) explicitly by \( V(H) \) and \( E(H) \), respectively. An \( r \)-uniform hypergraph is a hypergraph in which every edge has size \( r \). Throughout this paper, we denote by \( V(H) = [n] := \{1, 2, \ldots, n\} \) the vertex set of a hypergraph \( H \). For a vertex \( i \in V(H) \), the degree of \( i \), denoted by \( d_H(i) \) or simply by \( d_i \), is the number of edges containing \( i \). If each vertex of \( H \) has the same degree, we say that the hypergraph \( H \) is regular. For different \( i, j \in V(H) \), \( i \) and \( j \) are said to be adjacent, written \( i \sim j \), if there is an edge of \( H \) containing both \( i \) and \( j \). A walk of hypergraph \( H \) is defined to be an alternating sequence of vertices and edges \( i_1 e_1 i_2 e_2 \cdots i_{\ell} e_{\ell} i_{\ell + 1} \) satisfying that \( \{i_j, i_{j+1}\} \subseteq e_j \in E(H) \) for \( 1 \leq j \leq \ell \). A walk is called a path if all vertices and edges in the walk are distinct. A hypergraph \( H \) is called connected if for any vertices \( i, j \), there is a walk connecting \( i \) and \( j \). For positive integers \( r \) and \( n \), a real tensor \( \mathcal{A} = (a_{i_1 i_2 \cdots i_r}) \) of order \( r \) and dimension \( n \) refers to a multidimensional array (also called hypermatrix) with entries \( a_{i_1 i_2 \cdots i_r} \in \mathbb{R} \) for all \( i_1, i_2, \ldots, i_r \in [n] \). We say that tensor \( \mathcal{A} \) is symmetric if its entries \( a_{i_1 i_2 \cdots i_r} \) are invariant under any permutation of its indices.

Recently, Nikiforov [10] presented some analytic methods for studying uniform hypergraphs, and posed the following question (see [10, Question 11.5]):

**Question 1.1** ([10]). Suppose that \( H \) is an \( r \)-uniform hypergraph on \( n \) vertices \((r \geq 3)\). Let \( d_i \) be the degree of vertex \( i \), \( i \in [n] \), and \( \rho(H) \) be the spectral radius of \( H \). Is it always true

\[
\rho(H) \geq \left( \frac{1}{n} \sum_{i=1}^{n} \frac{d_i^{r-1}}{d_i} \right)^{\frac{r-1}{r}}?
\]

In this paper, we focus on the above question, and give a solution to Question 1.1. Our main results can be stated as follows.

**Theorem 1.1.** Suppose that \( H \) is an \( r \)-uniform hypergraph on \( n \) vertices \((r \geq 3)\). Let \( d_i \) be the degree of vertex \( i \) of \( H \), and \( \rho(H) \) be the spectral radius of \( H \). Then

\[
\rho(H) \geq \left( \frac{1}{n} \sum_{i=1}^{n} \frac{d_i^{r-1}}{d_i} \right)^{\frac{r-1}{r}}.
\]

If \( H \) is connected, then the equality holds if and only if \( H \) is regular.
Theorem 1.2. Let $H$ be a connected $r$-uniform hypergraph on $n$ vertices ($r \geq 3$). Suppose that $d_i, i \in [n]$, is the degree of vertex $i$, and $\rho(Q(H))$ is the spectral radius of the signless Laplacian tensor $Q(H)$. Then
\[
\rho(Q(H)) \geq 2 \left( \frac{1}{n} \sum_{i=1}^{n} \frac{d_i^{r-1}}{r} \right),
\]
with equality if and only if $H$ is regular.

2 Preliminaries

In this section we review some basic notations and necessary conclusions. Denote the set of nonnegative vectors (positive vectors) of dimension $n$ by $\mathbb{R}^n_+$ ( $\mathbb{R}^n_+$). The unit tensor of order $r$ and dimension $n$ is the tensor $I_n = (\delta_{i_1i_2\cdots i_r})$, whose entry is 1 if $i_1 = i_2 = \cdots = i_r$ and 0 otherwise.

The following general product of tensors was defined by Shao [15], which is a generalization of the matrix case.

Definition 2.1 ([15]). Let $A$ (and $B$) be an order $r \geq 2$ (and order $k \geq 1$), dimension $n$ tensor. Define the product $AB$ to be the following tensor $C$ of order $(r-1)(k-1)+1$ and dimension $n$
\[
c_{i\alpha_1\cdots \alpha_{r-1}} = \sum_{i_2,\ldots,i_r=1}^{n} a_{i_2\cdots i_r} b_{i_2\alpha_1} \cdots b_{i_r\alpha_{r-1}} \quad (i \in [n], \alpha_1, \ldots, \alpha_{r-1} \in [n]^{k-1}).
\]

From the above definition, let $x = (x_1, x_2, \ldots, x_n)^T$ be a column vector of dimension $n$. Then $Ax$ is a vector in $\mathbb{C}^n$, whose $i$-th component is as the following
\[
(Ax)_i = \sum_{i_2,\ldots,i_r=1}^{n} a_{i_2\cdots i_r} x_{i_2} \cdots x_{i_r}, \quad i \in [n]
\]
and
\[
x^T(Ax) = \sum_{i_1,i_2,\ldots,i_r=1}^{n} a_{i_1i_2\cdots i_r} x_{i_1} x_{i_2} \cdots x_{i_r}.
\]

In 2005, Lim [7] and Qi [12] independently introduced the concepts of tensor eigenvalues and the spectra of tensors. Let $A$ be an order $r$ and dimension $n$ tensor, $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{C}^n$ be a column vector of dimension $n$. If there exists a number $\lambda \in \mathbb{C}$ and a nonzero vector $x \in \mathbb{C}^n$ such that
\[Ax = \lambda x^{[r-1]},\]
then $\lambda$ is called an eigenvalue of $A$, $x$ is called an eigenvector of $A$ corresponding to the eigenvalue $\lambda$, where $x^{[r-1]} = (x_1^{r-1}, x_2^{r-1}, \ldots, x_n^{r-1})^T$. The spectral radius $\rho(A)$ of $A$ is the
maximum modulus of the eigenvalues of $A$. It was proved that $\lambda$ is an eigenvalue of $A$ if and only if it is a root of the characteristic polynomial of $A$ (see details in [16]).

In 2012, Cooper and Dutle [3] defined the adjacency tensors for $r$-uniform hypergraphs.

**Definition 2.2 ([3, 14]).** Let $H = (V(H), E(H))$ be an $r$-uniform hypergraph on $n$ vertices. The adjacency tensor of $H$ is defined as the order $r$ and dimension $n$ tensor $A(H) = (a_{i_1i_2 \cdots i_r})$, whose $(i_1i_2 \cdots i_r)$-entry is

\[ a_{i_1i_2 \cdots i_r} = \begin{cases} \frac{1}{(r-1)!}, & \text{if } \{i_1, i_2, \ldots, i_r\} \in E(H), \\ 0, & \text{otherwise}. \end{cases} \]

Let $D(H)$ be an order $r$ and dimension $n$ diagonal tensor with its diagonal element $d_{ii \cdots i}$ being $d_i$, the degree of vertex $i$, for all $i \in [n]$. Then $L(H) = D(H) - A(H)$ is the Laplacian tensor of $H$, and $Q(H) = D(H) + A(H)$ is the signless Laplacian tensor of $H$.

For an $r$-uniform hypergraph $H$, denote the spectral radius of $A(H)$ by $\rho(H)$. It should be announced that spectral radius defined in [10] differ from this paper, while for an $r$-uniform hypergraph $H$ the spectral radius defined in [10] equals to $(r - 1)! \rho(H)$. This is not essential and does not effect the result.

In [5], the weak irreducibility of nonnegative tensors was defined. It was proved that an $r$-uniform hypergraph $H$ is connected if and only if its adjacency tensor $A(H)$ is weakly irreducible (see [5] and [18]). Clearly, this shows that if $H$ is connected, then $A(H)$, $L(H)$ and $Q(H)$ are all weakly irreducible. The following result for nonnegative tensors is stated as a part of Perron-Frobenius theorem in [2].

**Theorem 2.1 ([2]).** Let $A$ be a nonnegative tensor of order $r$ and dimension $n$. Then we have the following statements.

1. $\rho(A)$ is an eigenvalue of $A$ with a nonnegative eigenvector corresponding to it.
2. If $A$ is weakly irreducible, then $\rho(A)$ is the unique eigenvalue of $A$ with the unique eigenvector $x \in \mathbb{R}^n_{++}$, up to a positive scaling coefficient.

**Theorem 2.2 ([13]).** Let $A$ be a nonnegative symmetric tensor of order $r$ and dimension $n$. Then we have

\[ \rho(A) = \max \{ x^T Ax \mid x \in \mathbb{R}^n_{++}, \|x\|_r = 1 \} . \]

Furthermore, $x \in \mathbb{R}^n_{++}$ with $\|x\|_r = 1$ is an optimal solution of the above optimization problem if and only if it is an eigenvector of $A$ corresponding to the eigenvalue $\rho(A)$.

The following concept of direct products (also called Kronecker product) of tensors was defined in [15], which is a generalization of the direct products of matrices.
Definition 2.3 ([15]). Let $A$ and $B$ be two order $r$ tensors with dimension $n$ and $m$, respectively. Define the direct product $A \otimes B$ to be the following tensor of order $r$ and dimension $mn$ (the set of subscripts is taken as $[n] \times [m]$ in the lexicographic order):

$$(A \otimes B)_{(i_1,j_1)(i_2,j_2)\cdots(i_r,j_r)} = a_{i_1i_2\cdots i_r}b_{j_1j_2\cdots j_r}.$$ 

In particular, if $x = (x_1, x_2, \ldots, x_n)^T$ and $y = (y_1, y_2, \ldots, y_m)^T$ are two column vectors with dimension $n$ and $m$, respectively. Then

$$x \otimes y = (x_1y_1, \ldots, x_1y_m, x_2y_1, \ldots, x_2y_m, \ldots, x_ny_1, \ldots, x_ny_m)^T.$$ 

The following basic results can be found in [15].

Proposition 2.1 ([15]). The following conclusions hold.

1. $(A_1 + A_2) \otimes B = A_1 \otimes B + A_2 \otimes B$.
2. $(\lambda A) \otimes B = A \otimes (\lambda B) = \lambda(A \otimes B)$, $\lambda \in \mathbb{C}$.
3. $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$.

Theorem 2.3 ([15]). Suppose that $A$ and $B$ are two order $r$ tensors with dimension $n$ and $m$, respectively. Let $\lambda$ be an eigenvalue of $A$ with corresponding eigenvector $u$, and $\mu$ be an eigenvalue of $B$ with corresponding eigenvector $v$. Then $\lambda \mu$ is an eigenvalue of $A \otimes B$ with corresponding eigenvector $u \otimes v$.

3 Proof of Theorem 1.1

In this section, we shall give a proof of Theorem 1.1.

Proof of Theorem 1.1. Let $H$ be an $r$-uniform hypergraph with spectral radius $\rho(H)$ and vertex set $V(H) = [n]$, and denote by $d_i$ the degree of vertex $i$ of $H$, $i = 1, 2, \ldots, n$.

We now define an $r$-uniform hypergraph $\tilde{H}$ as follows. Hypergraph $\tilde{H}$ has vertex set $V(H) \times [r]$, and $\{(i_1,j_1), (i_2,j_2), \ldots, (i_r,j_r)\} \in E(\tilde{H})$ is an edge of $\tilde{H}$ if and only if $\{i_1, i_2, \ldots, i_r\} \in E(H)$ and $j_1, j_2, \ldots, j_r$ are distinct each other. Let $A(H) = (a_{i_1i_2\cdots i_r})$ be the adjacency tensor of $H$. We define an order $r$ and dimension $r$ tensor $B = (b_{j_1j_2\cdots j_r})$ as follows:

$$b_{j_1j_2\cdots j_r} = \begin{cases} 1, & \text{if } j_1, j_2, \ldots, j_r \text{ are distinct each other}, \\ 0, & \text{otherwise}. \end{cases}$$

We denote the adjacency tensor of $\tilde{H}$ by $A(\tilde{H})$, in which the set of subscripts is taken as $[n] \times [r]$ in the lexicographic order.
Claim 1. If $H$ is connected, then $\tilde{H}$ is connected.

Proof of Claim 1. It suffices to show that for any $(i, j)$, $(s, t) \in V(\tilde{H})$, there exists a walk connecting them. We distinguish the following two cases.

Case 1. $i \neq s, j \neq t$.

Since $H$ is connected, there exists a path $i = i_1 e_1 i_2 \cdots i_\ell e_\ell i_{\ell+1} = s$. Since $r \geq 3$, there exist $j'$ such that $j' \neq j$, $j' \neq t$. From the definition of $\tilde{H}$, if $\ell$ is odd, we have

$$
\begin{align*}
(i_h, j) &\sim (i_{h+1}, j'), & h &= 1, 3, \ldots, \ell - 2, \\
(i_k, j') &\sim (i_{k+1}, j), & k &= 2, 4, \ldots, \ell - 1, \\
(i_\ell, j) &\sim (s, t).
\end{align*}
$$

If $\ell$ is even, we obtain

$$
\begin{align*}
(i_h, j) &\sim (i_{h+1}, j'), & h &= 1, 3, \ldots, \ell - 1, \\
(i_k, j') &\sim (i_{k+1}, j), & k &= 2, 4, \ldots, \ell - 2, \\
(i_\ell, j') &\sim (s, t).
\end{align*}
$$

Hence there exists a walk connecting $(i, j)$ and $(s, t)$.

Case 2. $i = s, j \neq t$.

Since $r \geq 3$, there exist $i'$ and $j'$ such that $i' \neq i$, $j' \neq j$, $j' \neq t$. According to Case 1 we know that there is a path connecting $(i, j)$ and $(i', j')$. Noting that $i' \neq s$ and $j' \neq t$, there is a path connecting $(i', j')$ and $(s, t)$ by Case 1. So there exists a walk connecting $(i, j)$ and $(s, t)$, as desired. The proof of the claim is completed.

Claim 2. $A(\tilde{H}) = A(H) \otimes B$.

Proof of Claim 2. From the definition of $\tilde{H}$, it follows that

$$(A(\tilde{H}))(i_1, j_1)(i_2, j_2)\cdots(i_r, j_r) = \begin{cases} 
\frac{1}{(r-1)!}, & \text{if } \{i_1, i_2, \ldots, i_r\} \in E(H), b_{j_1 j_2 \cdots j_r} = 1, \\
0, & \text{otherwise}.
\end{cases}$$

According to Definition 2.3, $A(H) \otimes B$ is an order $r$ and dimension $rn$ tensor, whose entries are given by

$$(A(H) \otimes B)(i_1, j_1)(i_2, j_2)\cdots(i_r, j_r) = a_{i_1 i_2 \cdots i_r} b_{j_1 j_2 \cdots j_r}.$$  

If $\{i_1, i_2, \ldots, i_r\} \in E(H)$ and $b_{j_1 j_2 \cdots j_r} = 1$, then

$$(A(H) \otimes B)(i_1, j_1)(i_2, j_2)\cdots(i_r, j_r) = \frac{1}{(r-1)!},$$

and 0 otherwise. Hence $A(\tilde{H}) = A(H) \otimes B$, as desired.
Claim 3. $\rho(B) = (r - 1)!$.

Proof of Claim 3. Let $e = (1, 1, \ldots, 1)^T \in \mathbb{R}^r$. It follows from Theorem 2.2 that

$$\rho(B) \geq \frac{e^T (Be)}{||e||_r^2} = \frac{r!}{r} = (r - 1)!.$$ 

On the other hand, let $z = (z_1, z_2, \ldots, z_r) \in \mathbb{R}^r_+$ be a nonnegative eigenvector corresponding to $\rho(B)$ with $||z||_r = 1$. By AM-GM inequality, we have

$$\rho(B) = z^T (Bz) = r!z_1z_2 \cdots z_r \leq r! \left( \frac{z_1^r + z_2^r + \cdots + z_r^r}{r} \right) = (r - 1)!,$$

with equality holds if and only if $z_1 = z_2 = \cdots = z_r = \frac{1}{\sqrt[r]{r}}$.

Therefore, $\rho(B) = (r - 1)!$. The proof of the claim is completed.

Claim 4. $\rho(\tilde{H}) = (r - 1)\rho(H)$.

Proof of Claim 4. By Claim 2, $\mathcal{A}(\tilde{H}) = \mathcal{A}(H) \otimes B$. We consider the following two cases depending on whether or not $H$ is connected.

Case 1. $H$ is connected.

Since $H$ is connected, we have $\mathcal{A}(H)$ is weakly irreducible. From Theorem 2.1, let $u$ be the positive eigenvector corresponding to the eigenvalue $\rho(H)$. Then by Theorem 2.3, $\rho(H)\rho(B)$ is an eigenvalue of $\mathcal{A}(H) \otimes B$ with a positive eigenvector $u \otimes e$. By Claim 1, $\tilde{H}$ is connected. It follows from Theorem 2.1 and Claim 3 that $(r - 1)!\rho(H)$ must be the spectral radius of $\mathcal{A}(H) \otimes B$, i.e., $\rho(\tilde{H}) = (r - 1)!\rho(H)$.

Case 2. $H$ is disconnected.

Let $\varepsilon > 0$ and $\mathcal{A}_\varepsilon = \mathcal{A}(H) + \varepsilon J_1$, $\mathcal{B}_\varepsilon = B + \varepsilon J_2$, where $J_1$ and $J_2$ are order $r$ tensors with all entries 1 with dimension $n$ and $r$, respectively. Then $\mathcal{A}_\varepsilon$ and $\mathcal{B}_\varepsilon$ are both positive tensors, and therefore are weakly irreducible. Using the similar arguments as Case 1, we have

$$\rho(\mathcal{A}_\varepsilon \otimes \mathcal{B}_\varepsilon) = \rho(\mathcal{A}_\varepsilon)\rho(\mathcal{B}_\varepsilon).$$

Notice that the maximal absolute value of the roots of a complex polynomial is a continuous function on the coefficients of the polynomial. Take the limit $\varepsilon \to 0$ on both sides of the above equation, we obtain the desired result. The proof of the claim is completed.

It is clear that $\tilde{H}$ is an $r$-partite hypergraph with partition

$$V(\tilde{H}) = \bigcup_{i=1}^{r} (V(H) \times \{i\}).$$

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We define a vector \( x \in \mathbb{R}^{rn} \) as follows:
\[
x_{(i,j)} = \begin{cases} 
\frac{a_i}{\sqrt{rn}}, & \text{if } i = 1, 2, \ldots, n, j = 1, \\
\frac{1}{\sqrt{rn}}, & \text{otherwise}, 
\end{cases}
\]
where \( a_1, a_2, \ldots, a_n \geq 0 \) and \( a_1^r + a_2^r + \cdots + a_n^r = n \). It is obvious that \( d_{\tilde{\mathcal{H}}}(i, j) = (r - 1)!d_i \) for any \( i \in [n], j \in [r] \). By Theorem 2.2, we deduce that
\[
\rho(\tilde{\mathcal{H}}) \geq x^T(A(\tilde{\mathcal{H}})x) = r \sum_{\{(i_1, j_1), \ldots, (i_r, j_r)\} \in E(\tilde{\mathcal{H}})} x_{(i_1, j_1)}x_{(i_2, j_2)} \cdots x_{(i_r, j_r)}
\]
\[
= r \left[ \sum_{i=1}^{n} \frac{a_i}{\sqrt{rn}} \cdot \left( \frac{1}{\sqrt{rn}} \right)^{r-1} \cdot (r - 1)!d_i \right] 
\]
\[
= \frac{(r - 1)!}{n} \sum_{i=1}^{n} a_i d_i.
\]
It follows from \( a_1^r + a_2^r + \cdots + a_n^r = n \) and Hölder inequality that
\[
\sum_{i=1}^{n} a_i d_i \leq \left( \sum_{i=1}^{n} a_i^r \right)^{\frac{1}{r}} \cdot \left( \sum_{i=1}^{n} d_i^{r-1} \right)^{\frac{r-1}{r}}
\]
\[
= \sqrt{n} \left( \sum_{i=1}^{n} d_i^{r-1} \right)^{\frac{r-1}{r}}
\]
with equality if and only if
\[
a_i = \frac{\sqrt{nd_i^{r-1}}}{\left( \sum_{i=1}^{n} d_i^{r-1} \right)^{\frac{1}{r}}}, \quad i = 1, 2, \ldots, n.
\]
(5)

Now we set \( a_i \) as (5). In the light of (4) and (5) we have
\[
\rho(H) = \rho(\tilde{\mathcal{H}}) = \frac{\rho(\tilde{\mathcal{H}})}{(r - 1)!} \geq \frac{1}{n} \sum_{i=1}^{n} a_i d_i = \left( \frac{1}{n} \sum_{i=1}^{n} d_i^{r-1} \right)^{\frac{r-1}{r}}.
\]
(6)

Now we give a characterization of extremal hypergraphs achieving the equality in (6). Suppose first the equality holds in (6). Then the vector \( x \in \mathbb{R}^{rn} \) defined by (3) is an eigenvector corresponding to \( \rho(\tilde{\mathcal{H}}) \) by Theorem 2.2. Note that \( H \) is connected, by Theorem 2.1, we let \( u = (u_1, u_2, \ldots, u_n)^T \in \mathbb{R}^{r+} \) be a positive eigenvector corresponding to \( \rho(H) \). From Theorem 2.3 and Claim 4, it follows that \( u \otimes e \) is a positive eigenvector to \( \rho(\tilde{\mathcal{H}}) \). By Claim 1 and Theorem 2.1, \( \tilde{\mathcal{H}} \) is connected, and we see that \( x \) and \( u \otimes e \) are linear dependence. Notice that
\[
x = \frac{1}{\sqrt{rn}}(a_1, 1, \ldots, 1, a_2, 1, \ldots, 1, \ldots, a_n, 1, \ldots, 1)^T \in \mathbb{R}^{rn}_{++}
\]
and
\[ u \otimes e = (u_1, u_1, \ldots, u_1, u_2, u_2, \ldots, u_2, \ldots, u_n, u_n, \ldots, u_n)^T \in \mathbb{R}^{rn}_{++}. \]

Consequently, \( u_1 = u_2 = \cdots = u_n \), which implies that \( H \) is regular.

Conversely, if \( H \) is a connected regular hypergraph, it is easy to see that the equality (6) holds.

**Remark 3.1.** It is known that the spectral radius \( \rho(H) \) of \( H \) is greater than or equal to the average degree \([3]\), i.e.,
\[ \rho(H) \geq \frac{\sum_{i=1}^{n} d_i}{n}. \]

It follows from PM inequality that
\[ \left( \frac{1}{n} \sum_{i=1}^{n} \frac{d_i}{r} \right)^{r-1} \geq \frac{\sum_{i=1}^{n} d_i}{n}. \]

Therefore Theorem 1.1 has a better estimation for spectral radius of \( H \).

## 4 Proof of Theorem 1.2

In this section, we shall give a proof of Theorem 1.2.

**Proof of Theorem 1.2.** Let \( H \) be a connected \( r \)-uniform hypergraph with vertex set \( V(H) = [n] \). Let \( \tilde{H} \) be the \( r \)-uniform hypergraph as defined in Theorem 1.1. Suppose that \( \mathcal{B} \) is the order \( r \) and dimension \( r \) tensor given by (2), and \( \mathcal{I}_r \) is the unit tensor of order \( r \) and dimension \( r \).

We have the following claims.

**Claim 5.** \( Q(\tilde{H}) = (r-1)!(\mathcal{D}(H) \otimes \mathcal{I}_r) + A(H) \otimes \mathcal{B}. \)

**Proof of Claim 5.** Recall that \( d_{H}((i, j)) = (r-1)!d_i \) for any \( i, j \in [r] \), we have
\[ \mathcal{D}(\tilde{H})(i_1, j_1)(i_2, j_2)\cdots(i_r, j_r) = \begin{cases} (r-1)!d_i, & \text{if } i_1 = \cdots = i_r = i, j_1 = \cdots = j_r = j, \\ 0, & \text{otherwise}. \end{cases} \]

On the other hand, by Definition 2.3 we obtain that
\[ [(r-1)!(\mathcal{D}(H) \otimes \mathcal{I}_r)](i_1, j_1)(i_2, j_2)\cdots(i_r, j_r) = (r-1)!\mathcal{D}(H)_{i_1i_2\cdots i_r}(\mathcal{I}_r)_{j_1j_2\cdots j_r}. \]

If \( i_1 = i_2 = \cdots = i_r = i, j_1 = j_2 = \cdots = j_r = j \), then
\[ [(r-1)!(\mathcal{D}(H) \otimes \mathcal{I}_r)](i_1, j_1)(i_2, j_2)\cdots(i_r, j_r) = (r-1)!d_i \]
and 0 otherwise. It follows that
\[ D(\tilde{H}) = (r - 1)!(D(H) \otimes I_r). \]

Therefore, we have
\[ Q(\tilde{H}) = D(\tilde{H}) + A(\tilde{H}) = (r - 1)!(D(H) \otimes I_r) + A(H) \otimes B. \]

The proof of the claim is completed.

Claim 6. \( \rho(Q(\tilde{H})) = (r - 1)!\rho(Q(H)). \)

Proof of Claim 6. Since \( H \) is connected, \( Q(H) \) is weakly irreducible. From Theorem 2.1, we let \( u \) be the positive eigenvector to \( \rho(Q(H)) \). Let \( e = (1, 1, \ldots, 1)^T \in \mathbb{R}^r_{++} \). By Proposition 2.1, we deduce that
\[
Q(\tilde{H})(u \otimes e) = [(r - 1)!(D(H) \otimes I_r)](u \otimes e) + [A(H) \otimes B](u \otimes e)
= (r - 1)![(D(H)u) \otimes (I_r e)] + (A(H)u) \otimes (Be)
= (r - 1)!(D(H)u) \otimes e + (r - 1)![A(H)u] \otimes e
= (r - 1)!(D(H)u + A(H)u) \otimes e
= (r - 1)!(Q(H)u) \otimes e.
\]

It follows from \( Q(H)u = \rho(Q(H))u^{[r-1]} \) that
\[
Q(\tilde{H})(u \otimes e) = (r - 1)!\rho(Q(H))(u \otimes e)^{[r-1]},
\]
which yields that \( u \otimes e \) is a positive eigenvector of \( Q(\tilde{H}) \) corresponding to \( (r - 1)!\rho(Q(H)) \). Note that \( H \) is connected, then \( \tilde{H} \) is connected by Claim 1. Therefore \( Q(\tilde{H}) \) is weakly irreducible. By Theorem 2.1, \( (r - 1)!\rho(Q(H)) \) is the spectral radius of signless Laplacian tensor \( Q(\tilde{H}) \), as claimed.

Let \( x \in \mathbb{R}^n \) be the column vector defined by (3). By Theorem 2.2, we have
\[
\rho(Q(\tilde{H})) \geq x^T(Q(\tilde{H})x) = x^T(D(\tilde{H})x) + x^T(A(\tilde{H})x)
= \sum_{j=1}^r \sum_{i=1}^n D(\tilde{H})_{(i,j)(i,j)}x_{(i,j)}^r + x^T(A(\tilde{H})x)
= \frac{(r - 1)!}{nn} \sum_{i=1}^n (a^r_i + r - 1)d_i + x^T(A(\tilde{H})x). \tag{7}
\]

Furthermore, by AM-GM inequality, we have
\[
a^r_i + r - 1 = a^r_i + 1 + 1 + \cdots + 1 \geq ra_i, \quad i \in [n]. \tag{8}
\]
So it follows from (4), (7) and (8) that

\[
\rho(Q(\tilde{H})) \geq \frac{(r - 1)!}{rn} \sum_{i=1}^{n} (a_i^2 + r - 1)d_i + \frac{(r - 1)!}{n} \sum_{i=1}^{n} a_id_i
\]

\[
\geq 2\frac{(r - 1)!}{n} \sum_{i=1}^{n} a_id_i.
\]

By (5), we obtain

\[
\rho(Q(H)) = \frac{\rho(Q(\tilde{H}))}{(r-1)!} \geq 2\frac{1}{n} \sum_{i=1}^{n} a_id_i = 2 \left( \frac{1}{n} \sum_{i=1}^{n} d_i^{\frac{r-1}{r}} \right)^{\frac{r-1}{r}}.
\]  \tag{9}

Suppose that the equality holds in (9). Then the vector \( \mathbf{x} \in \mathbb{R}^{rn} \) defined by (3) is an eigenvector corresponding to \( \rho(Q(\tilde{H})) \) by Theorem 2.2 and \( a_i = 1, \ i \in [n] \) by (8). Recall that \( Q(\tilde{H}) \) is weakly irreducible. By Claim 6, \( \mathbf{u} \otimes \mathbf{e} \) is a positive eigenvector to \( \rho(Q(\tilde{H})) \). We see that \( \mathbf{x} \) and \( \mathbf{u} \otimes \mathbf{e} \) are linear dependence by Theorem 2.1. Therefore, \( d_1 = d_2 = \cdots = d_n \), which implies that \( H \) is regular.

Conversely, if \( H \) is a regular connected hypergraph, it is straightforward to verify that the equality (9) holds.

\[\square\]

**Remark 4.1.** Recently, Nikiforov introduces the concept of odd-colorable hypergraphs \([11]\), which is a generalization of bipartite graphs. Let \( r \geq 2 \) and \( r \) be even. An \( r \)-uniform hypergraph \( H \) with \( V(H) = [n] \) is called odd-colorable if there exists a map \( \varphi : [n] \to [r] \) such that for any edge \( \{i_1, i_2, \ldots, i_r\} \) of \( H \), we have

\[
\varphi(i_1) + \varphi(i_2) + \cdots + \varphi(i_r) \equiv \frac{r}{2} \mod r.
\]

It was proved that if \( H \) is a connected \( r \)-uniform hypergraph, then \( \rho(L(H)) = \rho(Q(H)) \) if and only if \( r \) is even and \( H \) is odd-colorable \([17]\). Therefore we have

\[
\rho(L(H)) \geq 2 \left( \frac{1}{n} \sum_{i=1}^{n} d_i^{\frac{r-1}{r}} \right)^{\frac{r-1}{r}}
\]

for a connected odd-colorable hypergraph \( H \), which generalizes the result in \([19]\).

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