Randomly stopped sums with consistently varying distributions

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Abstract Let \( \{ \xi_1, \xi_2, \ldots \} \) be a sequence of independent random variables, and \( \eta \) be a counting random variable independent of this sequence. We consider conditions for \( \{ \xi_1, \xi_2, \ldots \} \) and \( \eta \) under which the distribution function of the random sum \( S_\eta = \xi_1 + \xi_2 + \cdots + \xi_\eta \) belongs to the class of consistently varying distributions. In our consideration, the random variables \( \{ \xi_1, \xi_2, \ldots \} \) are not necessarily identically distributed.

Keywords Heavy tail, consistently varying tail, randomly stopped sum, inhomogeneous distributions, convolution closure, random convolution closure

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1 Introduction

Let \( \{ \xi_1, \xi_2, \ldots \} \) be a sequence of independent random variables (r.v.s) with distribution functions (d.f.s) \( \{ F_{\xi_1}, F_{\xi_2}, \ldots \} \), and let \( \eta \) be a counting r.v., that is, an integer-valued, nonnegative, and nondegenerate at zero r.v. In addition, suppose that the r.v. \( \eta \) and r.v.s \( \{ \xi_1, \xi_2, \ldots \} \) are independent. Let \( S_0 = 0 \), \( S_n = \xi_1 + \xi_2 + \cdots + \xi_n \) for \( n \in \mathbb{N} \),

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and let

\[ S_\eta = \sum_{k=1}^{\eta} \xi_k \]

be the randomly stopped sum of r.v.s \( \{\xi_1, \xi_2, \ldots\} \).

We are interested in conditions under which the d.f. of \( S_\eta \),

\[ F_{S_\eta}(x) = \mathbb{P}(S_\eta \leq x) = \sum_{n=0}^{\infty} \mathbb{P}(\eta = n) \mathbb{P}(S_n \leq x), \tag{1} \]

belongs to the class of consistently varying distributions.

Throughout this paper, \( f(x) = o(g(x)) \) means that \( \lim_{x \to \infty} f(x)/g(x) = 0 \), and \( f(x) \sim g(x) \) means that \( \lim_{x \to \infty} f(x)/g(x) = 1 \) for two vanishing (at infinity) functions \( f \) and \( g \). Also, we denote the support of a counting r.v. \( \eta \) by

\[ \text{supp}(\eta) := \{ n \in \mathbb{N}_0 : \mathbb{P}(\eta = n) > 0 \} \].

Before discussing the properties of \( F_{S_\eta} \), we recall the definitions of some classes of heavy-tailed d.f.s, where \( \overline{F}(x) = 1 - F(x) \) for all real \( x \) and a d.f. \( F \).

- A d.f. \( F \) is heavy-tailed (\( F \in \mathcal{H} \)) if for every fixed \( \delta > 0 \),

  \[ \lim_{x \to \infty} \overline{F}(x)e^{\delta x} = \infty. \]

- A d.f. \( F \) is long-tailed (\( F \in \mathcal{L} \)) if for every \( y \) (equivalently, for some \( y > 0 \)),

  \[ \overline{F}(x+y) \sim \overline{F}(x). \]

- A d.f. \( F \) has a dominatedly varying tail (\( F \in \mathcal{D} \)) if for every fixed \( y \in (0, 1) \) (equivalently, for some \( y \in (0, 1) \)),

  \[ \limsup_{x \to \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} < \infty. \]

- A d.f. \( F \) has a consistently varying tail (\( F \in \mathcal{C} \)) if

  \[ \lim_{y \uparrow 1} \limsup_{x \to \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = 1. \]

- A d.f. \( F \) has a regularly varying tail (\( F \in \mathcal{R} \)) if

  \[ \lim_{x \to \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\alpha} \]

  for some \( \alpha \geq 0 \) and any fixed \( y > 0 \).
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• A d.f. $F$ supported on the interval $[0, \infty)$ is subexponential ($F \in S$) if

$$\lim_{x \to \infty} \frac{F \ast F(x)}{F(x)} = 2. \tag{2}$$

If a d.f. $G$ is supported on $\mathbb{R}$, then we suppose that $G$ is subexponential ($G \in S$) if the d.f. $F(x) = G(x) \mathbb{1}_{[0,\infty)}(x)$ satisfies relation (2).

It is known (see, e.g., [4, 11, 13], and Chapters 1.4 and A3 in [8]) that these classes satisfy the following inclusions:

$$\mathcal{R} \subset \mathcal{C} \subset \mathcal{L} \cap \mathcal{D} \subset S \subset \mathcal{L} \subset \mathcal{H}, \quad \mathcal{D} \subset \mathcal{H}.$$  

These inclusions are depicted in Fig. 1 with the class $\mathcal{C}$ highlighted.

There exist many results on sufficient or necessary and sufficient conditions in order that the d.f. of the randomly stopped sum (1) belongs to some heavy-tailed distribution class. Here we present a few known results concerning the belonging of the d.f. $F_{S\eta}$ to some class. The first result on subexponential distributions was proved by Embrechts and Goldie (see Theorem 4.2 in [9]) and Cline (see Theorem 2.13 in [5]).

**Theorem 1.** Let $\{\xi_1, \xi_2, \ldots\}$ be independent copies of a nonnegative r.v. $\xi$ with subexponential d.f. $F_\xi$. Let $\eta$ be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$. If $\mathbb{E}(1 + \delta)^\eta < \infty$ for some $\delta > 0$, then the d.f. $F_{S\eta} \in S$.

Similar results for the class $\mathcal{D}$ can be found in Leipus and Šiaulys [14]. We present the statement of Theorem 5 from this work.

**Theorem 2.** Let $\{\xi_1, \xi_2, \ldots\}$ be i.i.d. nonnegative r.v.s with common d.f. $F_\xi \in \mathcal{D}$ and finite mean $\mathbb{E}\xi$. Let $\eta$ be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$ with d.f. $F_\eta$ and finite mean $\mathbb{E}\eta$. Then d.f. $F_{S\eta} \in \mathcal{D}$ iff $\min\{F_\xi, F_\eta\} \in \mathcal{D}$.

We recall only that the d.f. $F$ belongs to the class $\mathcal{D}$ if and only if the upper Matuszewska index $J_F^+ < \infty$, where, by definition,

$$J_F^+ = -\lim_{y \to \infty} \frac{1}{\log y} \log \left(\liminf_{x \to \infty} \frac{F(xy)}{F(x)}\right).$$

The random convolution closure for the class $\mathcal{L}$ was considered, for instance, in [1, 14, 16, 17]. We now present a particular statement of Theorem 1.1 from [17].
Theorem 3. Let \( \{\xi_1, \xi_2, \ldots\} \) be independent r.v.s, and \( \eta \) be a counting r.v. independent of \( \{\xi_1, \xi_2, \ldots\} \) with d.f. \( F_\eta \). Then the d.f. \( F_{S_\eta} \in L \) if the following five conditions are satisfied:

(i) \( \mathbb{P}(\eta \geq \kappa) > 0 \) for some \( \kappa \in \mathbb{N} \);

(ii) for all \( k \geq \kappa \), the d.f. \( F_{S_k} \) of the sum \( S_k \) is long tailed;

(iii) \( \sup_{k \geq 1} \sup_{x \in \mathbb{R}} (F_{S_k}(x) - F_{S_k}(x - 1)) \sqrt{k} < \infty \);

(iv) \( \limsup_{z \to \infty} \sup_{k \geq \kappa} \sup_{x \geq (z-1)+z} \frac{F_{S_k}(x - 1)}{F_{S_k}(x)} = 1 \);

(v) \( F_{\eta}(ax) = o(\sqrt{x} F_{S_k}(x)) \) for each \( a > 0 \).

We observe that the case of identically distributed r.v.s is considered in Theorems 1 and 2. In Theorem 3, r.v.s \( \{\xi_1, \xi_2, \ldots\} \) are independent but not necessarily identically distributed. A similar result for r.v.s having d.f.s with dominatedly varying tails can be found in [6].

Theorem 4 ([6], Theorem 2.1). Let r.v.s \( \{\xi_1, \xi_2, \ldots\} \) be nonnegative independent, not necessarily identically distributed, and \( \eta \) be a counting r.v. independent of \( \{\xi_1, \xi_2, \ldots\} \). Then the d.f \( F_{S_\eta} \) belongs to the class \( D \) if the following three conditions are satisfied:

(i) \( F_{\xi_\kappa} \in D \) for some \( \kappa \in \text{supp}(\eta) \),

(ii) \( \limsup_{x \to \infty} \sup_{n \geq \kappa} \frac{1}{n F_{\xi_\kappa}(x)} \sum_{i=1}^{n} \frac{F_{\xi_i}(x)}{F_{\xi_\kappa}(x)} < \infty \),

(iii) \( \mathbb{E}\eta^{p+1} < \infty \) for some \( p > J_{F_{\xi_\kappa}}^+ \).

In this work, we consider randomly stopped sums of independent and not necessarily identically distributed r.v.s. As noted before, we restrict ourselves on the class \( C \). If r.v.s \( \{\xi_1, \xi_2, \ldots\} \) are not identically distributed, then different collections of conditions on \( \{\xi_1, \xi_2, \ldots\} \) and \( \eta \) imply that \( F_{S_\eta} \in C \). We suppose that some r.v.s from \( \{\xi_1, \xi_2, \ldots\} \) have distributions belonging to the class \( C \), and we find minimal conditions on \( \{\xi_1, \xi_2, \ldots\} \) and \( \eta \) for the distribution of the randomly stopped sum \( S_\eta \) to remain in the same class. It should be noted that we use the methods developed in [6] and [7].

The rest of the paper is organized as follows. In Section 2, we present our main results together with two examples of randomly stopped sums \( S_\eta \) with d.f.s having consistently varying tails. Section 3 is a collection of auxiliary lemmas, and the proofs of the main results are presented in Section 4.
2 Main results

In this section, we present three statements in which we describe the belonging of a randomly stopped sum to the class $C$. In the conditions of Theorem 5, the counting r.v. $\eta$ has a finite support. Theorem 6 describes the situation where no moment conditions on the r.v.s $\{\xi_1, \xi_2, \ldots\}$ are required, but there is strict requirement for $\eta$. Theorem 7 deals with the opposite case: the r.v.s $\{\xi_1, \xi_2, \ldots\}$ should have finite means, whereas the requirement for $\eta$ is weaker. It should be noted that the case of real-valued r.v.s $\{\xi_1, \xi_2, \ldots\}$ is considered in Theorems 5 and 6, whereas Theorem 7 deals with non-negative r.v.s.

Theorem 5. Let $\{\xi_1, \xi_2, \ldots, \xi_D\}, D \in \mathbb{N}$, be independent real-valued r.v.s, and $\eta$ be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots, \xi_D\}$. Then the d.f. $F_{S_\eta}$ belongs to the class $C$ if the following conditions are satisfied:

(a) $\mathbb{P}(\eta \leq D) = 1$,

(b) $F_{\xi_1} \in C$,

(c) for each $k = 2, \ldots, D$, either $F_{\xi_k} \in C$ or $F_{\xi_k}(x) = o(F_{\xi_1}(x))$.

Theorem 6. Let $\{\xi_1, \xi_2, \ldots\}$ be independent real-valued r.v.s, and $\eta$ be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$. Then the d.f. $F_{S_\eta}$ belongs to the class $C$ if the following conditions are satisfied:

(a) $F_{\xi_1} \in C$,

(b) for each $k \geq 2$, either $F_{\xi_k} \in C$ or $F_{\xi_k}(x) = o(F_{\xi_1}(x))$,

(c) $\limsup_{x \to \infty} \sup_{n \geq 1} \frac{1}{nF_{\xi_1}(x)} \sum_{i=1}^{n} F_{\xi_i}(x) < \infty$,

(d) $\mathbb{E}\eta^{p+1} < \infty$ for some $p > J_{F_{\xi_1}}$.

When $\{\xi_1, \xi_2, \ldots\}$ are identically distributed with common d.f. $F_\xi \in C$, conditions (a), (b), and (c) of Theorem 6 are satisfied obviously. Hence, we have the following corollary.

Corollary 1 (See also Theorem 3.4 in [3]). Let $\{\xi_1, \xi_2, \ldots\}$ be i.i.d. real-valued r.v.s with d.f. $F_\xi \in C$, and $\eta$ be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$. Then the d.f. $F_{S_\eta}$ belongs to the class $C$ if $\mathbb{E}\eta^{p+1} < \infty$ for some $p > J_{F_{\xi}}$.

Theorem 7. Let $\{\xi_1, \xi_2, \ldots\}$ be independent nonnegative r.v.s, and $\eta$ be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$. Then the d.f. $F_{S_\eta}$ belongs to the class $C$ if the following conditions are satisfied:

(a) $F_{\xi_1} \in C$,

(b) for each $k \geq 2$, either $F_{\xi_k} \in C$ or $F_{\xi_k}(x) = o(F_{\xi_1}(x))$,

(c) $\mathbb{E}\xi_1 < \infty$,
(d) $\overline{F}_\eta(x) = o(\overline{F}_{\xi_1}(x))$, 

(e) $\limsup_{x \to \infty} \sup_{n \geq 1} \frac{1}{n} \sum_{i=1}^{n} \overline{F}_{\xi_i}(x) < \infty$, 

(f) $\limsup_{u \to \infty} \sup_{n \geq 1} \frac{1}{n} \sum_{k=1}^{n} \mathbb{E}\xi_k = 0$.

Similarly to Corollary 1, we can formulate the following statement. We note that, in the i.i.d. case, conditions (a), (b), (e), and (f) of Theorem 7 are satisfied.

**Corollary 2.** Let $\{\xi_1, \xi_2, \ldots\}$ be i.i.d. nonnegative r.v.s with common d.f. $F_\xi \in C$, and $\eta$ be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$. Then the d.f. $F_{S_\eta}$ belongs to the class $C$ under the following two conditions: $\mathbb{E}\xi < \infty$ and $\overline{F}_\eta(x) = o(\overline{F}_\xi(x))$.

Further in this section, we present two examples of r.v.s $\{\xi_1, \xi_2, \ldots\}$ and $\eta$ for which the random sum $F_{S_\eta}$ has a consistently varying tail.

**Example 1.** Let $\{\xi_1, \xi_2, \ldots\}$ be independent r.v.s such that $\xi_k$ are exponentially distributed for all even $k$, that is,

$$\overline{F}_{\xi_k}(x) = e^{-x}, \quad x \geq 0, \; k \in \{2, 4, 6, \ldots\},$$

whereas, for each odd $k$, $\xi_k$ is a copy of the r.v.

$$(1 + U) 2^G,$$

where $U$ and $G$ are independent r.v.s, $U$ is uniformly distributed on the interval $[0, 1]$, and $G$ is geometrically distributed with parameter $q \in (0, 1)$, that is,

$$\mathbb{P}(G = l) = (1 - q) q^l, \quad l = 0, 1, \ldots.$$ 

In addition, let $\eta$ be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$ and distributed according to the Poisson law.

Theorem 6 implies that the d.f. of the randomly stopped sum $S_\eta$ belongs to the class $C$ because:

(a) $F_{\xi_1} \in C$ due to considerations in pp. 122–123 of [2],

(b) $F_{\xi_k} \in C$ for $k \in \{3, 5, \ldots\}$, and $\overline{F}_{\xi_k}(x) = o(\overline{F}_{\xi_1}(x))$ for $k \in \{2, 4, 6, \ldots\}$,

(c) $\limsup_{x \to \infty} \sup_{n \geq 1} \frac{1}{n} \sum_{i=1}^{n} \overline{F}_{\xi_i}(x) \leq 1$,

(d) all moments of the r.v. $\eta$ are finite.

Note that $\xi_1$ does not satisfy condition (c) of Theorem 7 in the case $q \geq 1/2$. Hence, Example 1 describes the situation where Theorem 6 should be used instead of Theorem 7.
Example 2. Let \( \{\xi_1, \xi_2, \ldots\} \) be independent r.v.s such that \( \xi_k \) are distributed according to the Pareto law (with tail index \( \alpha = 2 \)) for all odd \( k \), and \( \xi_k \) are exponentially distributed (with parameter equal to 1) for all even \( k \), that is,

\[
\overline{F}_{\xi_k}(x) = \frac{1}{x^2}, \quad x \geq 1, \ k \in \{1, 3, 5, \ldots\},
\]

\[
\overline{F}_{\xi_k}(x) = e^{-x}, \quad x \geq 0, \ k \in \{2, 4, 6, \ldots\}.
\]

In addition, let \( \eta \) be a counting r.v independent of \( \{\xi_1, \xi_2, \ldots\} \) that has the Zeta distribution with parameter 4, that is,

\[
P(\eta = m) = \frac{1}{\zeta(4) (m + 1)^4}, \quad m \in \mathbb{N}_0,
\]

where \( \zeta \) denotes the Riemann zeta function.

Theorem 7 implies that the d.f. of the randomly stopped sum \( S_\eta \) belongs to the class \( C \) because:

(a) \( F_{\xi_1} \in C \),

(b) \( F_{\xi_k} \in C \) for \( k \in \{3, 5, \ldots\} \), and \( \overline{F}_{\xi_k}(x) = o(\overline{F}_{\xi_1}(x)) \) for \( k \in \{2, 4, 6, \ldots\} \),

(c) \( \mathbb{E}\xi_1 = 2 \),

(d) \( \overline{F}_\eta(x) = o(\overline{F}_{\xi_1}(x)) \),

(e) \( \limsup_{x \to \infty} \sup_{n \geq 1} \frac{1}{n} \sum_{i=1}^{n} \overline{F}_{\xi_i}(x) \leq 1 \),

(f) \( \max_{k \in \mathbb{N}} \mathbb{E}\xi_k = 2 \).

Regarding condition (d), it should be noted that the Zeta distribution with parameter 4 is a discrete version of Pareto distribution with tail index 3.

Note that \( \eta \) does not satisfy the condition (d) of Theorem 6 because \( J_{\overline{F}_{\xi_1}}^+ = 2 \) and \( \mathbb{E}\eta^3 = \infty \). Hence, Example 2 describes the situation where Theorem 7 should be used instead of Theorem 6.

3 Auxiliary lemmas

This section deals with several auxiliary lemmas. The first lemma is Theorem 3.1 in [3] (see also Theorem 2.1 in [15]).

Lemma 1. Let \( \{X_1, X_2, \ldots X_n\} \) be independent real-valued r.v.s. If \( F_{X_k} \in C \) for each \( k \in \{1, 2, \ldots, n\} \), then

\[
P\left( \sum_{i=1}^{n} X_i > x \right) \sim \sum_{i=1}^{n} F_{X_i}(x).
\]
The following statement about nonnegative subexponential distributions was proved in Proposition 1 of [10] and later generalized to a wider distribution class in Corollary 3.19 of [12].

**Lemma 2.** Let \( \{X_1, X_2, \ldots, X_n\} \) be independent real-valued r.v.s. Assume that \( \frac{F_{X_i}/F(x)}{b_i} \xrightarrow{x \to \infty} b_i \) for some subexponential d.f. \( F \) and some constants \( b_i \geq 0, i \in \{1, 2, \ldots, n\} \). Then

\[
\frac{F_{X_1} \ast F_{X_2} \ast \cdots \ast F_{X_n}(x)}{F(x)} \xrightarrow{x \to \infty} \sum_{i=1}^{n} b_i.
\]

In the next lemma, we show in which cases the convolution \( F_{X_1} \ast F_{X_2} \ast \cdots \ast F_{X_n} \) belongs to the class \( C \).

**Lemma 3.** Let \( \{X_1, X_2, \ldots, X_n\}, n \in \mathbb{N}, \) be independent real-valued r.v.s. Then the d.f. \( F_{\Sigma_n} \) of the sum \( \Sigma_n = X_1 + X_2 + \cdots + X_n \) belongs to the class \( C \) if the following conditions are satisfied:

(a) \( F_{X_1} \in \mathcal{C} \),

(b) for each \( k = 2, \ldots, n \), either \( F_{X_k} \in \mathcal{C} \) or \( F_{X_k}(x) = o(F_{X_1}(x)) \).

**Proof.** Evidently, we can suppose that \( n \geq 2 \). We split our proof into two parts.

**First part.** Suppose that \( F_{X_k} \in \mathcal{C} \) for all \( k \in \{1, 2, \ldots, n\} \). In such a case, the lemma follows from Lemma 1 and the inequality

\[
\frac{a_1 + a_2 + \cdots + a_m}{b_1 + b_2 + \cdots + b_m} \leq \max \left\{ \frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_m}{b_m} \right\}
\]

for \( a_i \geq 0 \) and \( b_i > 0, i = 1, 2, \ldots, m \).

Namely, using the relation of Lemma 1 and estimate (3), we get that

\[
\limsup_{x \to \infty} \frac{F_{\Sigma_n}(xy)}{F_{\Sigma_n}(x)} = \limsup_{x \to \infty} \frac{\sum_{k=1}^{n} F_{X_k}(xy)}{\sum_{k=1}^{n} F_{X_k}(x)} \leq \max_{1 \leq k \leq n} \limsup_{x \to \infty} \frac{F_{X_k}(xy)}{F_{X_k}(x)}
\]

for arbitrary \( y \in (0, 1) \).

Since \( F_{X_k} \in \mathcal{C} \) for each \( k \), the last estimate implies that the d.f. \( F_{\Sigma_n} \) has a consistently varying tail, as desired.

**Second part.** Now suppose that \( F_{X_k} \notin \mathcal{C} \) for some of indexes \( k \in \{2, 3, \ldots, n\} \). By the conditions of the lemma we have that \( F_{X_k}(x) = o(F_{X_1}(x)) \) for such \( k \). Let \( K \subset \{2, 3, \ldots, n\} \) be the subset of indexes \( k \) such that

\[
F_{X_k} \notin \mathcal{C} \quad \text{and} \quad F_{X_k}(x) = o(F_{X_1}(x)).
\]

By Lemma 2,

\[
F_{\Sigma_n}(x) \sim F_{X_1}(x),
\]

where

\[
\Sigma_n = X_1 + \sum_{k \in K} X_k.
\]
Hence,
\[
\limsup_{x \to \infty} \frac{\hat{F}_{\Sigma_n}(xy)}{F_{\Sigma_n}(x)} = \limsup_{x \to \infty} \frac{F_{X_1}(xy)}{F_{X_1}(x)}
\]
for every \(y \in (0, 1)\).

Equality (4) implies immediately that the d.f. \(F_{\Sigma_n}\) belongs to the class \(\mathcal{C}\). Therefore, the d.f. \(F_{\Sigma_n}\) also belongs to the class \(\mathcal{C}\) according to the first part of the proof because
\[
\Sigma_n = \hat{\Sigma}_n + \sum_{k \notin \mathcal{K}} X_k
\]
and \(F_{X_k} \in \mathcal{C}\) for each \(k \notin \mathcal{K}\). The lemma is proved.

The following two statements about dominatedly varying distributions are Lemma 3.2 and Lemma 3.3 in [6]. Since any consistently varying distribution is also dominatedly varying, these statements will be useful in the proofs of our main results concerning the class \(\mathcal{C}\).

**Lemma 4.** Let \(\{X_1, X_2, \ldots\}\) be independent real-valued r.v.s, and \(F_{X_v} \in \mathcal{D}\) for some \(v \geq 1\). Suppose, in addition, that
\[
\limsup_{x \to \infty} \sup_{n \geq v} n \frac{1}{F_{X_v}(x)} \sum_{i=1}^{n} F_{X_i}(x) < \infty.
\]
Then, for each \(p > J^{+}_{F_{X_v}}\), there exists a positive constant \(c_1\) such that
\[
\hat{F}_{S_n}(x) \leq c_1 n^{p+1} F_{X_v}(x)
\]
for all \(n \geq v\) and \(x \geq 0\).

In fact, Lemma 4 is proved in [6] for nonnegative r.v.s. However, the lemma remains valid for real-valued r.v.s. To see this, it suffices to observe that \(P(X_1 + X_2 + \cdots + X_n > x) \leq P(X_1^+ + X_2^+ + \cdots + X_n^+ > x)\) and \(P(X_k > x) = P(X_k^+ > x)\), where \(n \in \mathbb{N}, k \in \{1, 2, \ldots, n\}, x \geq 0,\) and \(a^+\) denotes the positive part of \(a\).

**Lemma 5.** Let \(\{X_1, X_2, \ldots\}\) be independent real-valued r.v.s, and \(F_{X_v} \in \mathcal{D}\) for some \(v \geq 1\). Let, in addition,
\[
\lim_{u \to \infty} \sup_{n \geq v} \frac{1}{n} \sum_{k=1}^{n} \mathbb{E}(|X_k| \mathbb{1}_{(X_k \leq -u)}) = 0,
\]
\[
\limsup_{x \to \infty} \sup_{n \geq v} \frac{1}{n F_{X_v}(x)} \sum_{i=1}^{n} F_{X_i}(x) < \infty,
\]
and \(\mathbb{E}X_k = \mathbb{E}X_k^+ - \mathbb{E}X_k^- = 0\) for \(k \in \mathbb{N}\). Then, for each \(\gamma > 0\), there exists a positive constant \(c_2 = c_2(\gamma)\) such that
\[
P(S_n > x) \leq c_2 n \hat{F}_{X_v}(x)
\]
for all \(x \geq \gamma n\) and all \(n \geq v\).
4 Proofs of the main results

Proof of Theorem 5. It suffices to prove that

\[
\limsup_{y \uparrow 1} \limsup_{x \to \infty} \frac{F_{S_n}(xy)}{F_{S_n}(x)} \leq 1. \quad (6)
\]

According to estimate (3), for \( x > 0 \) and \( y \in (0, 1) \), we have

\[
\frac{F_{S_n}(xy)}{F_{S_n}(x)} = \frac{\sum_{n=1}^{D} \mathbb{P}(S_n > xy) \mathbb{P}(\eta = n)}{\sum_{n=1}^{D} \mathbb{P}(S_n > x) \mathbb{P}(\eta = n)} \leq \max_{1 \leq n \leq D} \frac{\mathbb{P}(S_n > xy)}{\mathbb{P}(S_n > x)}.
\]

Hence, by Lemma 3,

\[
\limsup_{y \uparrow 1} \limsup_{x \to \infty} \frac{F_{S_n}(xy)}{F_{S_n}(x)} \leq \limsup_{y \uparrow 1} \limsup_{x \to \infty} \max_{1 \leq n \leq D} \frac{F_{S_n}(xy)}{F_{S_n}(x)} \leq \max_{1 \leq n \leq D} \limsup_{y \uparrow 1} \limsup_{x \to \infty} \frac{F_{S_n}(xy)}{F_{S_n}(x)} = 1,
\]

which implies the desired estimate (6). The theorem is proved. \( \square \)

Proof of Theorem 6. As in Theorem 5, it suffices to prove inequality (6). For all \( K \in \mathbb{N} \) and \( x > 0 \), we have

\[
\mathbb{P}(S_\eta > x) = \left( \sum_{n=1}^{K} + \sum_{n=K+1}^{\infty} \right) \mathbb{P}(S_n > x) \mathbb{P}(\eta = n).
\]

Therefore, for \( x > 0 \) and \( y \in (0, 1) \), we have

\[
\frac{\mathbb{P}(S_\eta > xy)}{\mathbb{P}(S_\eta > x)} = \frac{\sum_{n=1}^{K} \mathbb{P}(S_n > xy) \mathbb{P}(\eta = n)}{\mathbb{P}(S_\eta > x)} + \frac{\sum_{n=K+1}^{\infty} \mathbb{P}(S_n > xy) \mathbb{P}(\eta = n)}{\mathbb{P}(S_\eta > x)} =: J_1 + J_2. \quad (7)
\]

The random variable \( \eta \) is not degenerate at zero, so there exists \( a \in \mathbb{N} \) such that \( \mathbb{P}(\eta = a) > 0 \). If \( K \geq a \), then using inequality (3), we get

\[
J_1 \leq \frac{\sum_{n=1}^{K} \mathbb{P}(S_n > xy) \mathbb{P}(\eta = n)}{\sum_{n=1}^{K} \mathbb{P}(S_n > x) \mathbb{P}(\eta = n)} \leq \max_{1 \leq n \leq K} \frac{\mathbb{P}(S_n > xy)}{\mathbb{P}(S_n > x)}.
\]

Similarly as in the proof of Theorem 5, it follows that

\[
\limsup_{y \uparrow 1} \limsup_{x \to \infty} J_1 \leq \max_{1 \leq n \leq K} \limsup_{y \uparrow 1} \limsup_{x \to \infty} \frac{F_{S_n}(xy)}{F_{S_n}(x)} = 1. \quad (8)
\]
Since $C \subset D$, we can use Lemma 4 for the numerator of $J_2$ to obtain

$$\sum_{n=K+1}^{\infty} \Pr(S_n > xy) \Pr(\eta = n) \leq c_3 \overline{F}_{\xi_1}(xy) \sum_{n=K+1}^{\infty} n^{p+1} \Pr(\eta = n)$$

with some positive constant $c_3$. For the denominator of $J_2$, we have that

$$\Pr(S_\eta > x) = \sum_{n=1}^{\infty} \Pr(S_n > x) \Pr(\eta = n) \geq \Pr(S_a > x) \Pr(\eta = a).$$

The conditions of the theorem imply that

$$S_a = \xi_1 + \sum_{k \in K_a} \xi_k + \sum_{k \notin K_a} \xi_k,$$

where $K_a = \{k \in \{2, \ldots, a\} : F_{\xi_k} \notin C, \overline{F}_{\xi_k}(x) = o(\overline{F}_{\xi_1}(x))\}$. By Lemma 2

$$\overline{F}_{\hat{S}_a}(x)/\overline{F}_{\xi_1}(x) \to 1,$$

where $F_{\hat{S}_a}$ is the d.f. of the sum

$$\hat{S}_a = \xi_1 + \sum_{k \in K_a} \xi_k$$

In addition, by Lemma 3 we have that the d.f. $F_{\hat{S}_a}$ belongs to the class $C$.

If $k \notin K_a$, then $F_{\xi_k} \in C$ by the conditions of the theorem. This fact and Lemma 1 imply that

$$\liminf_{x \to \infty} \frac{\Pr(S_a > x)}{\overline{F}_{\xi_1}(x)} \geq 1 + \sum_{k \notin K_a} \liminf_{x \to \infty} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)}.$$

Hence,

$$\Pr(S_\eta > x) \geq \frac{1}{2} \overline{F}_{\xi_1}(x) \Pr(\eta = a) \tag{9}$$

for $x$ sufficiently large. Therefore,

$$\limsup_{y \uparrow 1} \limsup_{x \to \infty} J_2 \leq \frac{2 c_3}{\Pr(\eta = a)} \left( \limsup_{y \uparrow 1} \limsup_{x \to \infty} \frac{\overline{F}_{\xi_1}(xy)}{\overline{F}_{\xi_1}(x)} \right) \sum_{n=K+1}^{\infty} n^{p+1} \Pr(\eta = n). \tag{10}$$

Estimates (7), (8), and (10) imply that

$$\limsup_{y \uparrow 1} \limsup_{x \to \infty} \frac{\Pr(S_\eta > xy)}{\Pr(S_\eta > x)} \leq 1 + \frac{2 c_3}{\Pr(\eta = a)} \Pr(\eta = a) \mathbb{1}_{\{\eta > K\}} \mathbb{1}_{\{\eta > a\}}$$

for arbitrary $K \geq a$.

Letting $K$ tend to infinity, we get the desired estimate (6) due to condition (d). The theorem is proved. □
Proof of Theorem 7. Once again, it suffices to prove inequality (6).

By condition (e) we have that there exist two positive constants \( c_4 \) and \( c_5 \) such that

\[
\sum_{i=1}^{n} \overline{F}_{\xi_i}(x) \leq c_5n\overline{F}_{\xi_1}(x), \quad x \geq c_4, \ n \in \mathbb{N}.
\]

Therefore,

\[
\mathbb{E}S_n = \sum_{j=1}^{n} \mathbb{E}\xi_j = \sum_{j=1}^{n} \left( \int_{0}^{c_4} + \int_{c_4}^{\infty} \right) \overline{F}_{\xi_j}(u)du \leq c_4n + c_5n\mathbb{E}\xi_1 =: c_6n \tag{11}
\]

for a positive constant \( c_6 \) and all \( n \in \mathbb{N} \).

If \( K \in \mathbb{N} \) and \( x > 4Kc_6 \), then we have

\[
\mathbb{P}(S_{\eta} > x) = \mathbb{P}(S_{\eta} > x, \eta \leq K) + \mathbb{P}(S_{\eta} > x, K < \eta \leq \frac{x}{4c_6}) + \mathbb{P}(S_{\eta} > x, \eta > \frac{x}{4c_6}).
\]

Therefore,

\[
\frac{\mathbb{P}(S_{\eta} > xy)}{\mathbb{P}(S_{\eta} > x)} = \frac{\mathbb{P}(S_{\eta} > xy, \eta \leq K)}{\mathbb{P}(S_{\eta} > x)} + \frac{\mathbb{P}(S_{\eta} > xy, K < \eta \leq \frac{xy}{4c_6})}{\mathbb{P}(S_{\eta} > x)} + \frac{\mathbb{P}(S_{\eta} > xy, \eta > \frac{xy}{4c_6})}{\mathbb{P}(S_{\eta} > x)} =: I_1 + I_2 + I_3 \tag{12}
\]

if \( xy > 4Kc_6 \), \( x > 0 \), and \( y \in (0, 1) \).

The random variable \( \eta \) is not degenerate at zero, so \( \mathbb{P}(\eta = a) > 0 \) for some \( a \in \mathbb{N} \). If \( K \geq a \), then

\[
\limsup_{y \uparrow 1} \limsup_{x \to \infty} I_1 \leq 1 \tag{13}
\]

similarly to estimate (8) in Theorem 6.

For the numerator of \( I_2 \), we have

\[
I_{2,1} := \mathbb{P}\left( S_{\eta} > xy, K < \eta \leq \frac{xy}{4c_6} \right) = \sum_{K < n \leq \frac{xy}{4c_6}} \mathbb{P}\left( \sum_{i=1}^{n} (\xi_i - \mathbb{E}\xi_i) > xy - \sum_{j=1}^{n} \mathbb{E}\xi_j \right) \mathbb{P}(\eta = n)
\]

\[
\leq \sum_{K < n \leq \frac{xy}{4c_6}} \mathbb{P}\left( \sum_{i=1}^{n} (\xi_i - \mathbb{E}\xi_i) > \frac{3}{4}xy \right) \mathbb{P}(\eta = n) \tag{14}
\]

by inequality (11).
The random variables $\xi_1 - E\xi_1, \xi_2 - E\xi_2, \ldots$ satisfy the conditions of Lemma 5. Namely, $E(\xi_k - E\xi_k) = 0$ for $k \in \mathbb{N}$ and $F_{\xi_k} - E\xi_k \in \mathcal{C} \subset \mathcal{D}$ obviously. In addition,

$$\limsup_{x \to \infty} \sup_{n \geq 1} \frac{1}{n} \sum_{k=1}^{n} P(\xi_k - E\xi_k > x) < \infty$$

by conditions (a), (c) and (e). Finally,

$$\limsup_{u \to \infty} \sup_{n \geq 1} \frac{1}{n} \sum_{k=1}^{n} E(|\xi_k - E\xi_k| \mathbb{1}_{\{\xi_k - E\xi_k \leq -u\}}) = \limsup_{u \to \infty} \sup_{n \geq 1} \frac{1}{n} \sum_{k=1}^{n} E((E\xi_k - \xi_k) \mathbb{1}_{\{\xi_k - E\xi_k \leq -u\}}) \leq \limsup_{u \to \infty} \sup_{n \geq 1} \frac{1}{n} \sum_{1 \leq k \leq n} E\xi_k = 0$$

because of condition (f). So, applying the estimate of Lemma 5 to (14), we get

$$I_{2,1} \leq c_7 \sum_{k<n \leq \frac{3}{4}xy} n F_{\xi_1} \left( \frac{3}{4}xy + E\xi_1 \right) P(\eta = n)$$

$$\leq c_7 F_{\xi_1} \left( \frac{3}{4}xy \right) E\eta \mathbb{1}_{\{\eta > K\}}$$

with a positive constant $c_7$. For the denominator of $I_2$, we can use the inequality

$$P(S_\eta > x) = \sum_{n=1}^{\infty} P(S_n > x) P(\eta = n)$$

$$\geq \sum_{n=1}^{\infty} P(\xi_1 > x) P(\eta = n)$$

$$\geq F_{\xi_1}(x) P(\eta = a)$$

(15)

since the r.v.s $\{\xi_1, \xi_2, \ldots\}$ are nonnegative by assumption. Hence,

$$I_2 \leq \frac{c_7}{P(\eta = a)} E\eta \mathbb{1}_{\{\eta > K\}} \frac{F_{\xi_1} \left( \frac{3}{4}xy \right)}{F_{\xi_1}(x)}.$$

If $y \in (1/2, 1)$, then the last estimate implies that

$$\limsup_{x \to \infty} I_2 \leq \frac{c_7}{P(\eta = a)} E\eta \mathbb{1}_{\{\eta > K\}} \limsup_{x \to \infty} \frac{F_{\xi_1} \left( \frac{3}{8}x \right)}{F_{\xi_1}(x)} \leq c_8 E\eta \mathbb{1}_{\{\eta > K\}}$$

(16)

with some positive constant $c_8$ because $F_{\xi_1} \in \mathcal{C} \subset \mathcal{D}$. 
Using inequality (15) again, we obtain
\[ I_3 \leq \frac{P(\eta > \frac{xy}{4c_6})}{P(S_\eta > x)} \leq \frac{1}{P(\eta = a)} \frac{F_\eta(\frac{xy}{4c_6})}{F_{\xi_1}(\frac{xy}{4c_6})}. \]

Therefore, for \( y \in (1/2, 1) \), we get
\[ \limsup_{x \to \infty} I_3 \leq \frac{1}{P(\eta = a)} \limsup_{x \to \infty} \frac{F_\eta(\frac{xy}{4c_6})}{F_{\xi_1}(\frac{xy}{4c_6})} \limsup_{x \to \infty} \frac{F_{\xi_1}(\frac{xy}{4c_6})}{F_{\xi_1}(x)} = 0 \] (17)

by condition (d).

Estimates (12), (13), (16), and (17) imply that
\[ \limsup_{y \uparrow 1} \limsup_{x \to \infty} \frac{P(S_\eta > xy)}{P(S_\eta > x)} \leq 1 + c_8 E\eta \mathbb{1}_{\{\eta > K\}} \] for \( K \geq a \).

Letting \( K \) tend to infinity, we get the desired estimate (6) because \( E\eta < \infty \) by conditions (c) and (d). The theorem is proved.

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