Entanglement and interference between different degrees of freedom of photons states

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In this paper, photonic entanglement and interference are described and analyzed with the language of quantum information process. Correspondingly, a photon state involving several degrees of freedom is represented in a new expression based on the permutation symmetry of bosons. In this expression, each degree of freedom of a single photon is regarded as a qubit and operations on photons as qubit gates. The two-photon Hong-Ou-Mandel interference is well interpreted with it. Moreover, the analysis reveals the entanglement between different degrees of freedom in a four-photon state from parametric down conversion, even if there is no entanglement between them in the two-photon state. The entanglement will decrease the state purity and photon interference visibility in the experiments on a four-photon polarization state.

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I. INTRODUCTION

Photons have been widely applied to quantum information process (QIP), and photon interference is the kernel of these quantum operations. The optical part of the experimental quantum key distribution with phase modulation is the single photon interference. The two-photon Hong-Ou-Mandel interference (HOMI) [1] has been the heart of linear optical quantum computation [2]. Quantum state preparation [3], manipulation [4, 5] and measurement [6, 7] can be well realized with HOMI. Based on these, quantum teleportation was successfully demonstrated in experiment [8]. Besides, HOMI is well applied to the purification of quantum state [9] and the analysis of photon resource properties [10]. It can be used in demonstrations of quantum cloning and NOT gate [11, 12]. Moreover, the multi-photon interference has been used in the demonstration of the multi-photon de Broglie wavelength and high resolution quantum measurement [13, 14, 15, 16, 17].

In the above QIP protocols, the degree of freedom (DOF) of polarization is the main role. However, photons involving other DOF are also investigated. For example, spatial transverse DOF and time-energy DOF can form a high-dimensional system and also have wide applications in QIP [18, 19]. High-dimensional entanglement with spatial transverse DOF has been observed experimentally. The DOF of time energy has been well applied to the long distance quantum communication [21]. Recently, there appears a hyper-entangled two-photon state which is entangled in several DOF. States entangled in polarization, spatial mode and time energy [22], and in polarization and momentum (path) [23, 24] have been successfully generated and well discussed in experiment. However, they do not involve the entanglement between different DOF.

Usually, photon pairs are produced in the nonlinear optical processes of parametric down conversion (PDC). With different types of phase-matching conditions, experimenters can get photon states entangled in different DOF. However, the entanglement between different DOF is ignored when one of the DOF is investigated emphatically. There could be two reasons for legally ignoring this. One reason is that different types of filters are used in experiments. For the polarization state, narrow band interference filter and single-mode fiber act as the filters on frequency and spatial DOF, respectively. These filters erase the possible correlations that may exist between polarization DOF and frequency or spatial DOF. Thus, the polarization DOF is purified. The other reason is that there may be no correlations between these DOF. For the two-photon state from PDC, such as the hyper-entangled state [22, 23, 24], there is no evidence that the correlations exist between those DOF. However, for the multi-photon states from PDC, there will be entanglement between different DOF even when no entanglement is in the two-photon state from the lower order PDC. The entanglement will decrease the state purity and photon interference visibility if the correlated modes are neglected illegally. It is also the reason for decoherence. Several experiments discussed this kind of decoherence [25, 26]. However, the full description of the entanglement and decoherence is seldom introduced when the photon states contain several DOF. It is more complicated for the multi-DOF multi-photon state.

In this paper, we will give the demonstration of photon interference on different DOF with the language of QIP. A photon state containing several DOF is rewritten in an expression based on the permutation symmetry of bosonic
particles. In this state description and interference demonstration, each degree of freedom of a single photon is regarded as a qubit and an operation on photons is regarded as a single-qubit gate or two-qubit gate. As an application, two-photon HOMI is well re-interpreted. Those are the main contents of Section II. In Section III, we will discuss the entanglement between different DOF in a multi-photon state. It reveals the entanglement between different DOF in the four-photon state from PDC, even when there is no entanglement between them in the two-photon state. It is the entanglement that makes decoherence in polarization DOF in the experiments on the four-photon state. Our analysis is coincident with the results of experiments in Ref. [25, 26] which discussed the coherence of polarization DOF of the four-photon state. The last section is the conclusion.

II. PHOTON STATE REPRESENTATION AND TWO-PHOTON INTERFERENCE

Photon interference can be regarded as the result of QIP and can be well described with the language of QIP. Usually, there is more than one DOF involved in the multi-photon interference. Firstly, the photon system is a boson, which satisfies the permutation symmetric principle. Naturally, the state description should be permutation symmetric. From the photon creation operators, the multi-photon state can be written as:

$$\prod_{i=1}^{N} a_i^\dagger |\text{vac}\rangle = \frac{1}{\sqrt{N!}} \sum_P P(|a_1\rangle |a_2\rangle ... |a_N\rangle),$$

(1)

where $P$ is the permutation of the $N$ single-photon states. When all the $a_i^\dagger$ are the same, it degenerates to

$$a^\dagger N |\text{vac}\rangle = \sqrt{N!} |a\rangle \otimes N,$$

(2)

where $\sqrt{N!}$ shows the photon bunching. Next, if there is no coupling between different DOF, the single photon state is described as:

$$a^\dagger (\alpha, \beta, ..., \gamma) |\text{vac}\rangle = |\alpha, \beta, ..., \gamma\rangle = |\alpha\rangle |\beta\rangle ... |\gamma\rangle,$$

(3)

where $\alpha, \beta, ..., \gamma$ are for different DOF. With Eqs.(1) and (3), any photon state can be described in a new expression. Now, we will apply it to the two-photon state to re-interpret the HOMI.

FIG. 1: Illustration of two-photon Hong-Ou-Mandel interference on the Beamsplitter. The $u$ and $d$ describe the two different spatial mode.

The two-photon state involving different DOF is described as:

$$|\Phi_2\rangle = c \sum_{\alpha_i, \beta_i, ..., \gamma_i} \phi(\alpha_1, \beta_1, ..., \gamma_1, \alpha_2, \beta_2, ..., \gamma_2)$$

$$\times a^\dagger (\alpha_1, \beta_1, ..., \gamma_1)a^\dagger (\alpha_2, \beta_2, ..., \gamma_2) |\text{vac}\rangle,$$

(4)

where $c$ is the normalization number. In the usual cases, one DOF is path mode and the other is polarization, frequency, time-energy, or transverse spatial mode. Here, we consider the two-photon states from non-collinear PDC. In this case, the two photons are in two different paths, which are labeled as $u$ and $d$, as shown in Fig.1. For simplicity,
we here only consider two DOF \[^{27}\]. Thus the state is written as

$$|\Phi_2\rangle = c \sum_{\alpha_1, \alpha_2} \phi(\alpha_1, u, \alpha_2, d) a^\dagger(\alpha_1, u) a(\alpha_2, d) |\text{vac}\rangle$$

$$= \frac{c}{4\sqrt{2}} \sum_{\alpha_1, \alpha_2} \{[\phi(\alpha_1, u, \alpha_2, d) + \phi(\alpha_2, u, \alpha_1, d)](|\alpha_1, \alpha_2\rangle + |\alpha_2, \alpha_1\rangle)(|u, d\rangle + |d, u\rangle) + [\phi(\alpha_1, u, \alpha_2, d)$$

$$- \phi(\alpha_2, u, \alpha_1, d)](|\alpha_1, \alpha_2\rangle - |\alpha_2, \alpha_1\rangle)(|u, d\rangle - |d, u\rangle)\}. \quad (5)$$

In the above expressions, $\alpha_1$ and $\alpha_2$ are in the same basis. For convenience, we gather the notations which describe the same DOF. From the above state, we will consider some special cases in the HOMI where two photons are injected into the two input ports of the beamsplitter (BS) simultaneously.

The lossless BS can be regarded as the transformation on the path DOF while it has no effect on polarizations \[^{28}\]. From the above discussion, any single qubit transformation collectively on one DOF of the particles does not change the symmetry of that DOF. Especially for the anti-symmetric state $|\Psi^\rangle$, it is the decoherence-free entangled state \[^{29}\].

For symmetric BS ($R = T = 50\%$), the two photon state is transformed to

$$|\Phi_2\rangle_{\text{out}} = \frac{c}{4\sqrt{2}} \sum_{\alpha_1, \alpha_2} \{[\phi(\alpha_1, u, \alpha_2, d) + \phi(\alpha_2, u, \alpha_1, d)](|\alpha_1, \alpha_2\rangle$$

$$+ |\alpha_2, \alpha_1\rangle)(|u, d\rangle + |d, u\rangle) + [\phi(\alpha_1, u, \alpha_2, d)$$

$$- \phi(\alpha_2, u, \alpha_1, d)](|\alpha_1, \alpha_2\rangle - |\alpha_2, \alpha_1\rangle)(|u, d\rangle - |d, u\rangle)\}. \quad (7)$$

When $\phi(\alpha_1, u, \alpha_2, d) = \phi(\alpha_2, u, \alpha_1, d)$, the two photons are in the symmetric states, including the polarized Bell triplet states

$$|\Phi^\pm\rangle = \frac{1}{2}(|HH\rangle \pm |VV\rangle)(|u, d\rangle + |d, u\rangle),$$

$$|\Psi^+\rangle = \frac{1}{2}(|HV\rangle + |VH\rangle)(|u, d\rangle + |d, u\rangle), \quad (8)$$

and superpositions of the three states. Photons in these states will cause the photon bunching in the same output port of BS. If $\phi(\alpha_1, u, \alpha_2, d) = -\phi(\alpha_2, u, \alpha_1, d)$, it is the antisymmetric state or singlet state

$$|\Psi^-\rangle = \frac{1}{2}(|HV\rangle - |VH\rangle)(|u, d\rangle - |d, u\rangle). \quad (9)$$

Two photons will be in different path DOF after the BS. It shows the photon anti-bunching in the output ports of the BS. The above deduction is also the principle of partial discrimination of four Bell states with the BS.

From the above discussion, any single qubit transformation collectively on one DOF of the particles does not change the symmetry of that DOF. Especially for the anti-symmetric state $|\Psi^\rangle$, the state will still not change after passing through any other BSs with arbitrary reflection and transmission. It is also valid for the polarization DOF. Actually, any operation collectively on the polarization DOF of the two photons state $|\Psi^\rangle$ will not change the state description for there is only one form in the antisymmetric subspace. It is the rotation invariance property of the singlet state $|\Psi^\rangle$. That is:

$$\rho = U \otimes U \rho U^\dagger \otimes U^\dagger, \quad (10)$$

where $\rho = |\Psi^-\rangle \langle \Psi^-|$. Generally, the operations in operator sum representation also keep the permutation symmetry. So the state $|\Psi^-\rangle$ will still be unchanged under those operations and it is the decoherence-free entangled state \[^{28}\].

The BS also has no effect on the other DOF such as frequency, time-energy and is still a single qubit gate on the path DOF. For those DOF, the system can be multi-dimensional and can be dealt with similar to the polarization DOF. After the HOMI, two photons will also show the photon bunching if $\phi$ is permutation symmetric and show the photon anti-bunching if $\phi$ is permutation antisymmetric.

However, it is more complicated for the BS that has effect on different DOF simultaneously, such as the transverse spatial mode. For example, the HOMI of the two-photon state containing polarization DOF and transverse spatial
mode DOF in Ref. [30] shows different results according to both the symmetry of polarization DOF and the parity of the transverse spatial mode DOF. It also can be successfully interpreted with the above method by considering the BS as a two-qubit gate on transverse spatial mode DOF and path DOF.

It is the HOMI that we must consider the permutation symmetry of the path DOF. When the two paths never interact, it becomes simple, since photons can be labeled with the path DOF. For example, Eq. (9) is simplified to normal form:

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}} (|H_u V_d\rangle - |V_u H_d\rangle),$$

(11)

However, for the multi-photon state, especially the photons from process of parametric down-conversion, there are more cases of more than one photon are in one path DOF. It is not convenient to label each photon with path DOF. The state description based on permutation symmetry is more natural.

Moreover, the state description reveals the correlations between different photons and different DOF. It is not difficult to find that there is no entanglement between the polarization DOF and the path DOF in Eq. (9), while both polarization and path entanglement exists between two photons. Along with Eq. (10), when $\phi$ is symmetric or antisymmetric, there is no entanglement between different DOF. For a single DOF, it is a pure entangled state when both polarization and path entanglement exists between two photons. The decoherence will happen in one single DOF and the visibility of photon interference will dropped if the other DOF are neglected. Especially in the multi-photon system, this state description can conveniently be used to deal with the entanglement and coherence between photons and DOF. Next, we will show that entanglement exists between different DOF in the four-photon state.

III. ENTANGLEMENT BETWEEN DIFFERENT DOF IN THE FOUR-PHOTON STATE

As shown above, for the two photons from the PDC, even in the hyper-entangled state [22, 23, 24], there is no evidence that correlations exist between these DOF. However, there will be entanglement between different DOF in the four-photon state from the PDC.

The state from the process of the PDC pumped by a coherent pulse can be described as [31]:

$$|\Psi\rangle = (1 - \eta^2 / 2) |\text{vac}\rangle + \eta |\Phi_2\rangle + \eta^2 |\Phi_4\rangle + ...$$

(12)

$|\eta^2|$ is the probability of the two-photon conversion in a single pump pulse. The two-photon state $|\Phi_2\rangle$ has the form of the Schmidt decomposition:

$$|\Phi_2\rangle = \sum_i \phi(\alpha_i, \beta_i) a^\dagger(\alpha_i) b^\dagger(\beta_i) |\text{vac}\rangle.$$

(13)

The real variables $\phi(\alpha_i, \beta_i)$ satisfy the normalization condition $\sum_i |\phi(\alpha_i, \beta_i)|^2 = 1$. Correspondingly, the four-photon state is:

$$|\Phi_4\rangle = \frac{1}{2} \sum_{i,m} \phi(\alpha_i, \beta_i) \phi(\alpha_m, \beta_m) a^\dagger(\alpha_i) b^\dagger(\beta_i) a^\dagger(\alpha_m) b^\dagger(\beta_m)$$

$$\times |\text{vac}\rangle.$$

(14)

$\alpha_i, \beta_i$ denote the DOF of the photon, such as polarization, spatial or temporal mode, etc.

Let us introduce a measurable coefficient $K$ which is defined as

$$K = \sum_i \phi^2(\alpha_i, \beta_i).$$

(15)

$K$ indicates the entanglement of the two-photon state. The less $K$, the more entanglement between the two photons [32]. It is not difficult to get $K$ with the method of quantum state tomography. Also, $K$ can be directly measured in experiment from the single-photon rate ($P_1$) and two-photon rate ($P_2$) in one path mode: $K = P_2 / P_1^2 - 1$ [33].

In details, we consider a hyper-entangled state from non-collinear PDC,

$$|\Phi_2\rangle = \sum_i \phi(a_i, b_i) |a_i, b_i\rangle \sum_j \phi(\alpha_j, \beta_j) |\alpha_j, \beta_j\rangle$$

$$= \sum_{i,j} \phi(a_i, b_i) \phi(\alpha_j, \beta_j) a^\dagger(a_i, \alpha_j) b^\dagger(b_i, \beta_j) |\text{vac}\rangle.$$  

(16)
where $a_u^\dagger$ and $b_d^\dagger$ are the photon creation operators for two path modes $u$ and $d$. The italic letters $(a, b)$ are for different states of one DOF and the greek letters ($\alpha, \beta$) are for the other DOF. There is no entanglement between two DOF in the two-photon state. The four-photon from the second order PDC is

$$ |\Phi_4\rangle = \frac{1}{2} \sum_{i,j,m,n} \phi(a_i,b_i)\phi(a_j,\beta_j)\phi(a_m,b_m)\phi(\alpha_n,\beta_n)$$

$$\times a_u^\dagger(a_i,\alpha_j)b_d^\dagger(b_i,\beta_j)a_u^\dagger(a_m,\alpha_n)b_d^\dagger(b_m,\beta_n)|\text{vac}\rangle$$

$$= [\sqrt{K_A K_B} |A_1\rangle |B_1\rangle + \sqrt{K_A(1-K_B)/2} |A_1\rangle |B_2\rangle$$

$$+ \sqrt{(1-K_A)K_B/2} |A_2\rangle |B_1\rangle + \sqrt{(1-K_A)(1-K_B)/2} |A_2\rangle |B_2\rangle$$

$$\times |B_2\rangle + \sqrt{(1-K_A)(1-K_B)/4} |A_3\rangle |B_3\rangle]/\sqrt{(1+K_A K_B)/2},$$

(17)

where the bases are:

$$|A_1\rangle = \frac{1}{\sqrt{K_A}} \sum_i \phi^2(a_i,b_i)(|i,i\rangle_u |i,i\rangle_d)_A,$$

$$|B_1\rangle = \frac{1}{\sqrt{K_B}} \sum_j \phi^2(\alpha_j,\beta_j)(|j,j\rangle_u |j,j\rangle_d)_B,$$

$$|A_2\rangle = \frac{1}{\sqrt{(1-K_A)/2}} \sum_{i<m} \phi(a_i,b_i)\phi(a_m,b_m)(|i,m\rangle$$

$$+ |m,i\rangle_u (|i,m\rangle + |m,i\rangle_d)_A,$$

$$|B_2\rangle = \frac{1}{\sqrt{(1-K_B)/2}} \sum_{j<n} \phi(\alpha_j,\beta_j)\phi(\alpha_n,\beta_n)(|j,n\rangle$$

$$+ |n,j\rangle_u (|j,n\rangle + |n,j\rangle_d)_B,$$

$$|A_3\rangle = \frac{1}{\sqrt{(1-K_A)/2}} \sum_{i<m} \phi(a_i,b_i)\phi(a_m,b_m)(|i,m\rangle$$

$$- |m,i\rangle_u (|i,m\rangle - |m,i\rangle_d)_A,$$

$$|B_3\rangle = \frac{1}{\sqrt{(1-K_B)/2}} \sum_{j<n} \phi(\alpha_j,\beta_j)\phi(\alpha_n,\beta_n)(|j,n\rangle$$

$$- |n,j\rangle_u (|j,n\rangle - |n,j\rangle_d)_B.$$  

(18)

(19)

The indices $i, m (j, n)$ are for the different photon states in $A (B)$ DOF. $K_A = \sum_i \phi^4(a_i,b_i)$ and $K_B = \sum_j \phi^4(\alpha_j, \beta_j)$. So, $K_{A(B)} \leq 1$ can be measured with the method mentioned above by breaking the entanglement with the DOF $B$ (A). In the above four-photon state description, we neglect the permutation symmetry between the different path DOF for they never meet and label photons by $u$ and $d$. Also, we write the two-photon state with the same path DOF in one ket and describe them in permutation symmetric or anti-symmetric form.

For simplicity, we make

$$|A(B)_{12}\rangle = (\sqrt{K_{A(B)}} |A(B)\rangle_1$$

$$+ \sqrt{(1-K_{A(B)}/2) |A(B)\rangle_2})/\sqrt{(1+K_{A(B)})/2}. $$

(20)

Then the four-photon state is:

$$|\Phi_4\rangle = (\sqrt{(1+K_A)(1+K_B)|A_{12}\rangle |B_{12}\rangle + \sqrt{(1-K_A)(1-K_B)|A_3\rangle}$$

$$\times |B_3\rangle)/\sqrt{2(1+K_A K_B}).$$

(21)

In Eq. (21), states $|A(B)_{12}\rangle$ and $|A(B)_3\rangle$ are the permutation symmetric and antisymmetric forms for four photons in one DOF. The product keeps the permutation symmetry of bosonic particles. It is easy to get

$$\rho_A = tr_B \rho_{AB} = [(1+K_A)(1+K_B)|A_{12}\rangle \langle A_{12}| + (1-K_A)(1-K_B)$$

$$\times |A_3\rangle \langle A_3|]/(2+2K_A K_B),$$

(22)
According to the two-photon cases, the measurement is rewritten as

\[ tr\rho_A^2 = tr\rho_B^2 = \frac{1 + 4K_AK_B + K_A^2 + K_B^2 + K_A^2K_B^2}{2(1 + K_AK_B)^2}. \]  

(23)

When there is no entanglement in DOF \( A (B) \) of the two-photon state, then \( K_{A(B)} = 1 \), \( tr\rho_A^2 = tr\rho_B^2 = 1 \). This implies that there is no entanglement between the two DOF in the four-photon state. In this case, all photons are in the same state of DOF \( A (B) \). The antisymmetric part \( |A_3\rangle (|B_3\rangle) \) will not appear for it violates the permutation symmetry of photons. However, if entanglement exists on both two single DOF of the two-photon state, i.e., hyper-entangled state, to make \( K_{A(B)} < 1 \), the antisymmetric part \( |Eqs.(18) and (19) \) will appear. There is the entanglement between the two DOF in four-photon state, although there is no entanglement between the two DOF in the two-photon hyper-entangled state. As shown in Fig. 2, when \( K_A = K_B \to 0 \), \( tr\rho_A^2 (tr\rho_B^2) \) approaches the minimal value of 0.5, which is the maximal entanglement between the two modes.

As an example, we will discuss the system containing polarization DOF and another DOF besides the path DOF. In Refs. [25, 26, 34], there exists other DOF entangling with the polarization DOF which makes interference of the polarization state less than 100\%. Here we briefly interpret it by considering two DOF in the system. We also suppose there is no entanglement between different DOF in the two-photon state for the high visibility of two-photon interference. There are two reasons to induce the result. One is that the polarization state is a mixed state as mentioned above. The two-photon Einstein-Podolsky-Rosen state entangled in polarization DOF is described as:

\[ |\Psi_2\rangle = (|H\rangle_u |V\rangle_d - |V\rangle_u |H\rangle_d)/\sqrt{2}. \]  

(24)

\( K_A = 1/2 \), it is the maximally entangled state. If there is no other modes correlated with polarization mode, then the four-photon state will be [25]:

\[ |\Psi_4\rangle = (|HH\rangle_u |VV\rangle_d - |HV\rangle_u |HV\rangle_d + |VV\rangle_u |HH\rangle_d)/\sqrt{3} \]

\[ = (\sqrt{2}|\pi_1\rangle + |\pi_2\rangle)/\sqrt{3}, \]  

(25)

where \( |\pi_1\rangle = \frac{1}{\sqrt{2}}(|HH\rangle_u |VV\rangle_d + |VV\rangle_u |HH\rangle_d) \) and \( |\pi_2\rangle = -(|HV\rangle_u |VH\rangle_u |HV\rangle_d) \). \( |\Psi_4\rangle \) is the symmetric state for polarization DOF according to \( |A_{12}\rangle \) in Eq.(20). However, if there is another DOF, the two photon state will be \( |\Phi_2\rangle = |\Psi_2\rangle \otimes |\Phi_2\rangle \). By tracing the mode \( B \), the four-photon state in the polarization DOF will be in the form of

\[ \rho_A = [3(1 + K_B)] |\Psi_4\rangle (\langle \Psi_4| + (1 - K_B) |\Phi_4\rangle (\langle \Phi_4|)/(4 + 2K_B), \]  

(26)

where \( |\Phi_4\rangle = (|HV\rangle - |VH\rangle)_u (|HV\rangle - |VH\rangle)_d/2 \) is the anti-symmetric state for polarization DOF according to \( |A_3\rangle \) in Eq.(18). It is clear that the polarization DOF is entangled with the other DOF and the four-photon polarization state is in a mixed state if \( K_B \neq 1 \).

The other reason comes from the measurement. On one path DOF, the two-photon state measurement can be

\[ M = \sum_{i,j,m,n} f(a_i, \alpha_j) g(a_m, \alpha_n) \times a^\dagger(a_i, \alpha_j)a^\dagger(a_m, \alpha_n)a(a_m, \alpha_n)a(a_i, \alpha_j). \]  

(27)

According to the two-photon cases, the measurement is rewritten as
\[ M_{u(d)} = \sum_{i,j,m,n} f(a_i,\alpha_j)g(a_m,\alpha_n)[|im\rangle + |mi\rangle][|jn\rangle + |nj\rangle]\]

\[ +(|im\rangle - |mi\rangle)[(jn\rangle - |nj\rangle)] + (|im\rangle - |mi\rangle)[(|jn\rangle + |nj\rangle)]/8. \]

(28)

The result of the measurement will be \( P = \text{tr}(\rho_{AB} M_u \otimes M_d) \). Actually, the antisymmetric part will be invariant under any collective operation and gives unchanged result. So the antisymmetric part of \( |\Phi_4\rangle \) will contribute a "background" in the measurement result if there is entanglement between the two DOF both in the initial state and the measurement part. This will make the visibility of the four-photon interference less than 100% or induce the distinguishability of the state.

In Refs. [25, 26], a four-photon polarization state is described by the sum of two parts with a variable \( \alpha \). It is a mixed state as shown above in Eq. (26), or Eq. (5) in Ref. [26]. If we set \( \alpha = 3K_B/(2 + K_B) \), they describe the same state. However, the two parts in the state described in Ref. [25] are not mutually orthogonal.

For the polarization DOF correlated with the frequency DOF case, it has been successfully discussed in the language of multi-mode description [31, 34, 35]. Comparing with Ref. [34], we find \( K_B = E/A \). However, the descriptions in the above references did not reveal the relations between different DOF while Eq. (21) gives a clear description of the existence of entanglement between different DOF.

IV. CONCLUSION

In conclusion, we described the photon interference with the language of QIP. Together with the state expression based on the permutation symmetry of boson nature, the two-photon HOMI is re-interpreted and the multi-photon state involving different DOF is discussed. It reveals the existence of entanglement between different DOF in the four-photon state even there is no entanglement in the two-photon state. For a special case, the polarization DOF entangled with other DOF is analyzed. It reveals the existence of entanglement between different DOF in the measurement result if there is entanglement between the two DOF both in the initial state and the measurement part.

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[20] A. Mair, A. Vaziri, G. Weihs, and A. Zeilinger, Nature (London) 412, 313 (2001).
[21] I. Marcikic, H. de Riedmatten, W. Tittel, H. Zbinden, and N. Gisin, Nature (London) 421, 509 (2003).
[22] J. T. Barreiro, N. K. Langford, N. A. Peters, and P. G. Kwiat, Phys. Rev. Lett. 95, 260501 (2005).
[23] C. Cinelli, M. Barbieri, R. Perris, P. Mataloni, and F. De Martini, Phys. Rev. Lett. 95, 240405 (2005).
[24] M. Barbieri, C. Cinelli, P. Mataloni, and F. De Martini, Phys. Rev. A 72, 052110 (2005).
[25] K. Tsujino, H. F. Hofmann, S. Takeuchi, and K. Sasaki, Phys. Rev. Lett. 92, 153602 (2004).
[26] H. S. Eisenberg, J. F. Hodelin, G. Khoury, and D. Bouwmeester, Phys. Rev. Lett. 96, 160404 (2006).

Actually, different DOF can be combined into one DOF in mathematics just to span the basis.

Here we neglect the extra $\pi$ phase shift when the horizontally polarized photon is reflected by the BS. It can be compensated by adding phase shifters on the outports of the BS.

[27] P. Zanardi and M. Rasetti, Phys. Rev. Lett. 79, 3306 (1997).
[28] S. P. Walborn, A. N. de Oliveira, S. Pádua, and C. H. Monken, Phys. Rev. Lett. 90, 143601 (2003).
[29] Z. Y. Ou, J. -K. Rhee, and L. J. Wang, Phys. Rev. A 60, 593 (1999).
[30] C. K. Law and J. H. Eberly, Phys. Rev. Lett. 92, 127903 (2004); K. Rzażewski and J. H. Eberly, J. Phys. B 27, L503 (1994). For consistency with $E/A$, we choose the sum form but not the reciprocal form.
[31] F. W. Sun, et. al, Phys. Rev. A 76, 052303 (2007).
[32] Z. Y. Ou, Phys. Rev. A 72, 053814 (2005).
[33] F. W. Sun, Z. Y. Ou, and G. C. Guo, Phys. Rev. A 73, 023808 (2006).