SCHUR–WEYL DUALITY, VERMA MODULES, AND ROW QUOTIENTS OF ARIKI–KOIKE ALGEBRAS

ABEL LACABANNE AND PEDRO VAZ

Abstract. We prove a Schur–Weyl duality between the quantum enveloping algebra of \( \mathfrak{gl}_m \) and certain quotient algebras of Ariki-Koike algebras, which we give explicitly. The duality involves several algebraically independent parameters and is realized through the tensor product of a parabolic universal Verma module and a tensor power of the natural representation of \( \mathfrak{gl}_m \). We also give a new presentation by generators and relations of the generalized blob algebras of Martin and Woodcock as well as an interpretation in terms of Schur–Weyl duality by showing they occur as a particular case of our algebras.

1. Introduction

Schur–Weyl duality is a celebrated theorem connecting the finite-dimensional representations of the general linear and the symmetric groups. It states that, over a field \( \mathbb{k} \) of characteristic 0, the actions of \( GL_m(\mathbb{k}) \) and \( \mathfrak{S}_n \) on \( V = (\mathbb{k}^m)^{\otimes n} \) commute and form double centralizers. Its way into the quantum world was made by Jimbo [13] who established that \( \mathcal{U}_q(\mathfrak{gl}_m) \) and the Hecke algebra form a Schur–Weyl pair. Several variants of Schur–Weyl duality were later found, resulting in other Schur–Weyl type pairs (see for example [3, 5, 8, 20, 4]). One particular generalization consists of looking at representations of \( \mathfrak{gl}_m \) but allowing \( V \) to be infinite-dimensional. For example, in [12] it is established a Schur–Weyl duality between \( \mathcal{U}_q(\mathfrak{sl}_2) \) and the blob algebra\(^1\) of Martin and Saleur [16] by letting them act on the tensor product of a projective Verma module with several copies of the natural representation of \( \mathcal{U}_q(\mathfrak{sl}_2) \).

1.1. In this paper. We consider the tensor product of a parabolic universal Verma module with the \( m \)-folded tensor product of the natural representation for \( \mathcal{U}_q(\mathfrak{gl}_m) \) to establish a Schur–Weyl duality with a quotient of Ariki–Koike algebras. Ariki–Koike algebras were first considered by Cherednik in [9] as a cyclotomic quotient of the affine Hecke algebra of type \( \mathfrak{A} \). These algebras were later rediscovered and studied by Ariki and Koike [2] from a representation theoretic point of view. Independently, Brou and Malle attached in [6] a Hecke algebra to any complex reflection group. By definition (see [6 Definition 4.1]) Ariki–Koike algebras turned out to be the Hecke algebras associated to the complex reflection groups \( G(d, 1, n) \): these algebras are deformations of the group algebra of \( G(d, 1, n) \).

\(^1\)In [12] the blob algebra was called the Temperley–Lieb algebra of type \( B \) (see [15] for further explanations).
Recall that the Ariki-Koike algebra $\mathcal{H}(d, n)$ with parameters $q$ and $u = (u_1, \ldots, u_d)$ is the $k$-algebra with generators $T_0, T_1, \ldots, T_{n-1}$, where $T_1, \ldots, T_{n-1}$ generate a finite-dimensional Hecke algebra of type $A$ and $T_0$ satisfies $T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0$, $T_0 T_i = T_i T_0$ for $i > 1$, and $\prod_{i=1}^d (T_0 - u_i) = 0$. We consider the semisimple case, where the irreducible modules $V_\mu$ of $\mathcal{H}(d, n)$ are indexed by $d$-partitions of $n$.

In this paper we introduce the row-quotient algebra $\mathcal{H}_m(d, n)$, that depends on a $d$-tuple $m = (m_1, \ldots, m_d)$ of positive integers, as the quotient of $\mathcal{H}(d, n)$ by the kernel of the surjection

$$\mathcal{H}(d, n) \to \text{End}_{\mathcal{H}(d, n)} \left( \bigoplus_\mu V_\mu \right),$$

the sum being over all $d$-partitions $\mu = (\mu^{(1)}, \ldots, \mu^{(d)})$ such that $l(\mu^{(i)}) \leq m_i$ for all $1 \leq i \leq d$.

Let $M^p(\Lambda)$ be a parabolic Verma module and $V$ the natural representation for $\mathcal{U}_q(\mathfrak{gl}_m)$. In our conventions, $p$ is standard and has Levi factor $\mathfrak{l} = \mathfrak{gl}_{m_1} \times \cdots \times \mathfrak{gl}_{m_d}$, with $m_i \geq 1$ and $m_1 + m_2 + \cdots + m_d = m$ and $\Lambda$ depends on $d$ algebraically independent parameters $\lambda_1, \ldots, \lambda_d$ (see Section 3.2 for more details). Our main result connects $\mathcal{H}_m(d, n)$ to Schur–Weyl duality,

**Theorem A** (Theorem 4.1 and Lemma 4.2). Let $p$ be a standard parabolic subalgebra of $\mathfrak{gl}_m$ with Levi subalgebra $\mathfrak{l} = \mathfrak{gl}_{m_1} \times \cdots \times \mathfrak{gl}_{m_d}$. The $\mathcal{H}(d, n) \otimes \mathcal{U}_q(\mathfrak{gl}_m)$-module $M^p(\Lambda) \otimes V^{\otimes n}$ is decomposed as

$$M^p(\Lambda) \otimes V^{\otimes n} \simeq \bigoplus_{\mu \in \mathcal{P}_n^d} V_\mu \otimes M^p(\Lambda, \mu),$$

the parameters of the Ariki-Koike algebra being $u_i = (\lambda_i q^{-(m_1 + \cdots + m_{i-1})})^2$ and where $\mathcal{P}_n^d$ denotes the set of $d$-partitions of $n$ with the $i$-th component of length at most $m_i$. Moreover,

$$\text{End}_{\mathcal{U}_q(\mathfrak{gl}_m)}(M^p(\Lambda) \otimes V^{\otimes n}) = \mathcal{H}_m(d, n).$$

This has several particular specializations (Corollaries 4.3–4.7), some of them yielding well-known algebras:

- If $p = \mathfrak{gl}_m$ and $m \geq n$, then $\text{End}_{\mathcal{U}_q(\mathfrak{gl}_m)}(M^p(\Lambda) \otimes V^{\otimes n})$ is isomorphic to the Hecke algebra of type $A$.
- If $p = \mathfrak{gl}_m$ and $m = 2$, then $\text{End}_{\mathcal{U}_q(\mathfrak{gl}_m)}(M^p(\Lambda) \otimes V^{\otimes n})$ is isomorphic to the Temperley–Lieb algebra of type $A$.
- For $p$ such that $m \geq nd$ and $m_i \geq n$ for all $1 \leq i \leq d$, then $\text{End}_{\mathcal{U}_q(\mathfrak{gl}_m)}(M^p(\Lambda) \otimes V^{\otimes n})$ is isomorphic to the Ariki-Koike algebra $\mathcal{H}(d, n)$.
- If $p$ is such that $d = 2$ and $m_1, m_2 \geq n$, then $\text{End}_{\mathcal{U}_q(\mathfrak{gl}_m)}(M^p(\Lambda) \otimes V^{\otimes n})$ is isomorphic to the Hecke algebra of type $B$ with unequal and algebraically independent parameters (see [11, Example 5.2.2, (c)])
- If the parabolic subalgebra $p$ coincides with the standard Borel subalgebra of $\mathcal{U}_q(\mathfrak{gl}_m)$ then $\text{End}_{\mathcal{U}_q(\mathfrak{gl}_m)}(M^p(\Lambda) \otimes V^{\otimes n})$ is isomorphic to Martin–Woodcock’s [17] generalized blob algebra $\mathcal{B}(d, n)$. This generalizes the case of $\mathcal{U}_q(\mathfrak{sl}_2)$ covered in [12].
In the last case, this gives a new interpretation of the generalized blob algebras $B_{p,d,n;\mathbf{q}}$ in terms of Schur–Weyl duality. We also give a new presentation of $B_{p,d,n;\mathbf{q}}$ as a quotient of Ariki–Koike algebras:

**Theorem B** (Theorem 2.14). Suppose that $\mathcal{H}(d,n)$ is semisimple and that for every $i,j,k$ we have $(1+q^{-2})u_k \neq u_i + u_j$. The generalized blob algebra $B_{p,d,n}^{q;\mathbf{u}}$ is isomorphic to the quotient of $\mathcal{H}(d,n)$ by the two-sided ideal generated by the element

$$\tau = \prod_{1 \leq i < j \leq d} \left( (T_1 - q) \left( T_0 - q \frac{u_i + u_j}{q + q^{-1}} \right) (T_1 - q) \right).$$

We hope that the explicit presentations of the algebras in this paper will turn out useful for their study.

1.2. **Connection to other works.** The idea of writing this note originated when we started thinking of possible extensions of our work in [15] to more general Kac–Moody algebras and were not able to find the appropriate generalizations of [12] in the literature. When we were finishing writing this note Peng Shan informed us about [19]. We expect our results to be connected to [19, 8] using a braided equivalence of categories between a category of modules for the quantum group $\mathcal{U}_q(\mathfrak{gl}_m)$ and a category of modules over the affine Lie algebra $\widehat{\mathfrak{gl}}_m$, which is due to Kazhdan and Lusztig [14]. However, the description of the kernel of the map $\psi_{R,d}^s$ of [19, Proposition 8.36] does not seem to appear anywhere in [19], except in the particular case of our Corollary 4.5.

Another motivation for the results presented here resides in the potential applications to low-dimensional topology, as indicated in [18]. We find that it would be also interesting to investigate the use of several Verma modules in a tensor product as suggested in [10].

**Acknowledgments.** We would like to thank Steen Ryom-Hansen for comments on an earlier version of this paper. The authors were supported by the Fonds de la Recherche Scientifique - FNRS under Grant no. MIS-F.4536.19.

2. **Ariki-Koike algebras, row quotients and generalized blob algebras**

We recall the notion of Ariki-Koike algebras and define some quotients which will appear as endomorphism algebras of modules over a quantum group. As a particular case, we recover the generalized blob algebras of Martin and Woodcock [17] and give them a presentation in terms of generators and relations which, up to our knowledge is new.

2.1. **Reminders on Ariki-Koike algebras.** Fix once and for all a field $\mathbb{k}$ and two positive integers $d$ and $n$ and choose $q,u_1,\ldots,u_d$ invertible elements in $\mathbb{k}$. We recall the definition of the Ariki-Koike algebra introduced in [2], which we view as a quotient of the group algebra of the Artin-Tits braid group of type $B$.

**Definition 2.1.** The *Ariki-Koike algebra* $\mathcal{H}(d,n)$ with parameters $q$ and $\mathbf{u} = (u_1,\ldots,u_d)$ is the $\mathbb{k}$-algebra with generators $T_0,T_1,\ldots,T_{n-1}$, the quadratic relation

$$(T_i - q)(T_i + q^{-1}) = 0,$$
the cyclotomic relation
\[ \prod_{i=1}^{d} (T_0 - u_i) = 0, \]
and the braid relations
\[ T_i T_j = T_i T_j \text{ if } |i - j| > 1, \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \text{ for } 1 \leq i \leq n - 2, \]
\[ T_0 T_1 T_0 = T_1 T_0 T_1. \]

**Remark 2.2.** We use different conventions than in [2]. In order to recover their definition, one should replace \( q \) by \( q^2 \), \( T_0 \) by \( a_1 \), and \( qT_{i-1} \) by \( a_i \).

Since the generators \( T_i \) for \( i \geq 1 \) satisfy the braid relations, we define \( T_w = T_{s_1} \cdots T_{s_r} \) for any reduced expression of an element \( w = s_{i_1} \cdots s_{i_r} \) of the symmetric group \( S_n \). Thanks to Matsumoto lemma, this is independent of the chosen reduced expression.

It is shown in [2] that the algebra \( \mathcal{H}(d, n) \) is of dimension \( d^nn! \) and a basis is given in terms of Jucys-Murphy elements. One can define inductively these elements by \( X_1 = T_0 \) and \( X_{i+1} = T_i X_i T_i \).

**Theorem 2.3** ([2, Theorem 3.10, Theorem 3.20]). A basis of \( \mathcal{H}(d, n) \) is given by the set
\[ \{ X_1^{r_1} \cdots X_d^{r_d} T_w \mid 0 \leq r_i < d, w \in S_n \}. \]
Moreover, the center of \( \mathcal{H}(d, n) \) is generated by the symmetric polynomials in \( X_1, \ldots, X_d \).

We end this section with a semisimplicity criterion due to Ariki [1].

**Theorem 2.4.** The algebra \( \mathcal{H}(d, n) \) is semisimple if and only if
\[ \left( \prod_{-n \leq l \leq n \atop 1 \leq i < j \leq d} (q^{2l} u_i - u_j) \right) \left( \prod_{1 \leq i \leq n} (1 + q^2 + q^4 \cdots + q^{2(i-1)}) \right) \neq 0. \]

**Remark 2.5.** The appearance of \( q^2 \) instead of \( q \) in [1] is due to our conventions.

2.2. **Representations of Ariki-Koike algebras.** In this section, we suppose that the algebra \( \mathcal{H}(d, n) \) is semisimple. In [2], Ariki and Koike gave a construction of the irreducible representations of \( \mathcal{H}(d, n) \), using the combinatorics of multipartitions.

2.2.1. **d-partitions and Young’s lattice.** A partition \( \mu \) of \( n \) of length \( l(\mu) = k \) is a non-increasing sequence \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_k > 0 \) of integers summing to \( |\mu| = n \). A \( d \)-partition of \( n \) is a \( d \)-tuple of partitions \( \mu = (\mu^{(1)}, \ldots, \mu^{(d)}) \) such that \( \sum_{i=1}^{d} |\mu^{(i)}| = n \). Given a \( d \)-partition \( \mu \) its Young diagram is the set
\[ [\mu] = \{(a, b, c) \in \mathbb{N} \times \mathbb{N} \times \{1, \ldots, d\} \mid 1 \leq a \leq l(\mu), 1 \leq b \leq \mu_a^{(c)} \}, \]
whose elements are called boxes. We usually represents a Young diagram as a \( d \)-tuple of sequences of left-aligned boxes, with \( \mu_a^{(c)} \) boxes in the \( a \)-th row of the \( c \)-th component.
Example 2.6. The Young diagram of the 3-partition \((2,1,\emptyset,3)\) of 6 is

\[
\begin{pmatrix}
\begin{array}{cccc}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\end{array}
\end{pmatrix}, \emptyset, \begin{pmatrix}
\begin{array}{cccc}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\end{array}
\end{pmatrix}.
\]

A box \(\gamma\) of \([\mu]\) is said to be removable if \([\mu] \setminus \{\gamma\}\) is the Young diagram of a \(d\)-partition \(\nu\), and in this case the box \(\gamma\) is said to be addable to \(\nu\).

Example 2.7. The removable boxes of the 3-partition \((2,1,\emptyset,3)\) below are depicted with a cross

\[
\begin{pmatrix}
\begin{array}{cccc}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\times & \times & \times & \times \\
\end{array}
\end{pmatrix}, \emptyset, \begin{pmatrix}
\begin{array}{cccc}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\times & \times & \times & \times \\
\end{array}
\end{pmatrix}.
\]

We consider the Young lattice for \(d\)-partitions and some sublattices. It is a graph with vertices consisting of \(d\)-partitions of any integers, and there is an edge between two \(d\)-partitions if and only if one can be obtained from the other by adding a box.

Example 2.8. The beginning of the Young lattice for 2-partitions is the following

\[
\begin{array}{cccc}
(\emptyset, \emptyset) & (\emptyset, \{\}) & (\{\}, \emptyset) & (\{\}, \{\}) \\
(\emptyset, \{\}) & (\{\}, \emptyset) & (\{\}, \{\}) & (\{\}, \{\}) \\
(\{\}, \{\}) & (\{\}, \{\}) & (\{\}, \{\}) & (\{\}, \{\})
\end{array}
\]

If we fix integers \(m_1, \ldots, m_d\), we will encounter the set of \(d\)-partitions \(\mu\) such that \(l(\mu^{(i)}) \leq m_i\), together with the corresponding sublattice of the Young lattice.

Example 2.9. With \(m_1 = 1\) and \(m_2 = 2\), the beginning of the Young lattice for 2-partitions \(\mu\) with \(l(\mu^{(1)}) \leq 1\) and \(l(\mu^{(2)}) \leq 2\) is the following

\[
\begin{array}{cccc}
(\emptyset, \emptyset) & (\emptyset, \{\}) & (\{\}, \emptyset) & (\{\}, \{\}) \\
(\emptyset, \{\}) & (\{\}, \emptyset) & (\{\}, \{\}) & (\{\}, \{\}) \\
(\{\}, \{\}) & (\{\}, \{\}) & (\{\}, \{\}) & (\{\}, \{\})
\end{array}
\]
We end this subsection with the notion of a standard tableau of shape \( \mu \) where \( \mu \) is a \( d \)-partition of \( n \). It is a bijection \( t: [\mu] \to \{1,\ldots,n\} \) such that for all boxes \( \gamma = (a,b,c) \) and \( \gamma' = (a',b',c) \) we have \( t(\gamma) < t(\gamma') \) if \( a = a' \) and \( b < b' \) or \( a < a' \) and \( b = b' \). Giving a standard tableau of shape \( \mu \) is equivalent to giving a path in the Young lattice from the empty \( d \)-partition to the \( d \)-partition \( \mu \).

**Example 2.10.** The standard tableau

\[
\begin{array}{|c|}
\hline
1 & 2 & 3 \\
\hline
4 & & \\
\hline
\end{array}
\]

of shape \(((1,1),\emptyset,(2))\) correspond to the path

\[
(\emptyset,\emptyset,\emptyset) \longrightarrow (\square \emptyset,\emptyset) \longrightarrow (\square \emptyset,\square) \longrightarrow (\square \emptyset,\square) \longrightarrow (\square \emptyset,\square) .
\]

2.2.2. **Constructing the irreducible representations.** For \( \mu = (\mu^{(1)},\ldots,\mu^{(d)}) \) a \( d \)-multipartition of \( n \), we set

\[
V_{\mu} = \bigoplus_{t} k v_t ,
\]

where the sum is over all the standard tableaux of shape \( \mu \). Ariki and Koike gave an explicit action of the generators on the basis of \( V_{\mu} \) given by the standard tableaux. The action of \( T_0 \) is diagonal with respect to this basis:

\[
T_0 v_t = u_c v_t ,
\]

where \( c \) is such that \( t(1,1,c) = 1 \). The action of \( T_i \) is more involved and depends on the relative positions of the numbers \( i \) and \( i+1 \) in the tableau \( t \):

1. if \( i \) and \( i+1 \) are in the same row of the standard tableau \( t \), then \( T_i v_t = q v_t \),
2. if \( i \) and \( i+1 \) are in the same column of the standard tableau \( t \), then \( T_i v_t = -q^{-1} v_t \),
3. if \( i \) and \( i+1 \) neither appear in the same row nor the same column of the standard tableau \( t \), then \( T_i \) will act on the two dimensional subspace generated by \( v_t \) and \( v_s \), where \( s \) is the standard tableau obtained from \( t \) by permuting the entries \( i \) and \( i+1 \). The explicit matrix is given in [2] and we will not need it.

**Proposition 2.11** ([2 Theorem 3.7]). If \( \mu \) is any \( d \)-multipartition of \( n \), the space \( V_{\mu} \) is a well-defined representation of \( \mathcal{H}(d,n) \) and it is absolutely simple. A set of isomorphism classes of simple representations of \( \mathcal{H}(d,n) \) is moreover given by \( \{V_{\mu}\}_{\mu} \), for \( \mu \) running over the set of \( d \)-partitions of \( n \).

The action of the Jucys-Murphy elements is also diagonal in the basis of standard tableaux:

1. \( X_i v_t = u_c q^{2(b-a)} v_t \),

where \( t(a,b,c) = i \). A useful consequence of Proposition 2.11 is the following: if \( V \) is an irreducible \( \mathcal{H}(d,n) \)-module and \( v \in V \) is a common eigenvector for \( X_1,\ldots,X_d \) with eigenvalues as in (1) for some standard tableau \( t \) of shape \( \mu \), then \( V \) is isomorphic to \( V_{\mu} \).
From the explicit description of the representations $V_\mu$, using the standard inclusion $H_p(n, d) \hookrightarrow H_p(n + 1, d)$, we see that for any $d$-partition of $n + 1$ we have

$$\text{Res}_{H_p(n, d)}^{H_p(n + 1, d)}(V_\mu) = \bigoplus_{\nu} V_\nu,$$

where the sum is over all $d$-partition $\nu$ of $n$ whose Young diagram is obtained by deleting one removable box from the Young diagram of $\mu$. The branching rule of the inclusions $H_p(1, d) \subset H_p(2, d) \subset \cdots \subset H_p(n, d)$ is therefore governed by the Young lattice of $d$-partitions.

### 2.3. Row quotients of $H_p(d, n)$ and generalized blob algebras.

We now define the row quotients of $H_p(d, n)$ which will appear later as endomorphism algebras of a tensor product of modules for $U_q(\mathfrak{gl}_m)$.

**Definition 2.12.** Let $m_1, \ldots, m_d$ be positive integers. The $m$-row quotient of $H_p(d, n)$, denoted $H_m(d, n)$, is the quotient of $H_p(d, n)$ by the kernel of the surjection

$$H_p(d, n) \twoheadrightarrow \text{End}_{H_p(d, n)} \left( \bigoplus_\mu V_\mu \right),$$

where the sum is over all $d$-partitions $\mu$ such that $l(\mu(i)) \leq m_i$ for all $1 \leq i \leq d$.

**Remark 2.13.** If $m_i \geq n$ for all $1 \leq i \leq d$ then $H_m(d, n) = H_p(d, n)$.

Similar to the case of $H_p(d, n)$, we have inclusions $H_m(1, d) \subset H_m(2, d) \subset \cdots \subset H_m(n, d)$ and the branching rule for representations is governed by the corresponding truncation of the Young lattice of $d$-partitions.

#### 2.3.1. Generalized blob algebras.

In the particular case where $m_i = 1$ for all $1 \leq i \leq d$, we recover the generalized blob algebras defined in [17], which we denote $B(d, n)$.

**Theorem 2.14.** Suppose that $H(d, n)$ is semisimple and that for every $i, j, k$ we have $(1 + q^{-2})u_k \neq u_i + u_j$. The generalized blob algebra $B(d, n)$ is isomorphic to the quotient of $H(d, n)$ by the two-sided ideal generated by the element

$$\tau = \prod_{1 \leq i < j \leq d} \left( (T_1 - q) \left( T_0 - q \frac{u_i + u_j}{q + q^{-1}} \right) (T_1 - q) \right).$$

This relation may look cumbersome, but can be better understood thanks to the following lemma:

**Lemma 2.15.** The two-sided ideal of $H(d, n)$ generated by $\tau$ is equal to the two-sided ideal generated by

$$(T_1 - q) \prod_{1 \leq i < j \leq d} \left( X_1 + X_2 - (u_i + u_j) \right).$$

**Proof.** A simple computation in $H(d, n)$ shows that

$$\tau = q (X_1 + X_2 - (u_i + u_j)) (T_1 - q).$$
We therefore conclude using the fact that \((T_1 - q)^2 = - (q + q^{-1})(T_1 - q)\) and that \(T_1\) commutes with \(X_1 + X_2\).

Let us denote by \(b(d, n)\) the quotient of \(\mathcal{H}(d, n)\) by the two sided ideal generated by \(\tau\). We now investigate which representations \(V_\mu\) of \(\mathcal{H}(d, n)\) factor through the quotient \(b(d, n)\). Theorem \(2.14\) will follow immediately from the following proposition.

**Proposition 2.16.** The representation \(V_\mu\) factors through \(b(d, n)\) if and only if \(l(\mu^{(k)}) \leq 1\) for every \(k\) such that \((1 + q^{-2})u_k \neq u_i + u_j\) for all \(i, j\).

**Proof.** Suppose that \(\mu\) and \(k\) are such that \(l(\mu^{(k)}) \geq 2\) with \((1 + q^{-2})u_k \neq u_i + u_j\) for all \(i, j\). Then there exist a tableau \(t\) of shape \(\mu\) such that 1 and 2 are in the first two columns of the \(k\)-th component of the Young diagram of \(\mu\). By definition of \(V_\mu\), the generator \(T_1\) acts on \(v_1\) by multiplication by \(-q^{-1}\). The Jucys-Murphy element \(X_1\) acts on \(v_1\) by multiplication by \(u_k\) whereas the Jucys-Murphy element \(X_2\) acts on \(v_1\) by multiplication by \(q^{-2}u_k\). Therefore, thanks to Lemma \(2.15\) the representation \(V_\mu\) does not factor through the quotient \(b(d, n)\).

It remains to check that the defining relation of \(b(d, n)\) acts by zero on \(V_\mu\) with \(l(\mu^{(k)}) \leq 1\) whenever \((1 + q^{-2})u_k \neq u_i + u_j\) for all \(i, j\). Let \(t\) be a standard tableau of shape \(\mu\). If 1 and 2 are in the same component of the tableau \(t\), then either 1 and 2 are in the same row and \(T_1\) acts on \(v_1\) by multiplication by \(q\), either 1 and 2 are in the same column and \(X_1 + X_2\) acts on \(t\) by multiplication by \((1 + q^{-2})u_k\). The second case is possible only if there exists \(i, j\) such that \((1 + q^{-2})u_k = u_i + u_j\) and the extra relation of \(b(d, n)\) is indeed satisfies. If 1 and 2 are in two different Young diagrams and \(X_1 + X_2\) acts on \(t\) by \(u_k + u_l\), where \(k\) (resp. \(l\)) is such that \(\tau(1, 1, k) = 1\) (resp \(\tau(1, 1, l) = 2\)). In both cases, the relation in Lemma \(2.15\) is satisfied.

3. Quantum \(\mathfrak{g}_m\), parabolic Verma modules and tensor products

We review the definition of the quantum enveloping algebra of \(\mathfrak{g}_m\), together with the study of parabolic Verma modules and tensor products of modules.

3.1. The quantum enveloping algebra of \(\mathfrak{g}_m\). Let \(q\) be an indeterminate. We recall the definition of the quantum enveloping algebra over \(\mathbb{Q}(q)\), but will define some representations over a bigger field.

**Definition 3.1.** The quantum enveloping algebra \(\mathcal{U}_q(\mathfrak{g}_m)\) is the \(\mathbb{Q}(q)\)-algebra with generators \(L_i^\pm, E_j\) and \(F_j\), for \(1 \leq i \leq m\) and \(1 \leq j \leq m - 1\) with the following relations:

\[
L_i^\pm L_j^\pm = 1, \quad L_i L_j = L_j L_i, \\
L_i E_j = q^{\delta_{i,j} - \delta_{i,j+1}} E_j L_i, \quad L_i F_j = q^{-\delta_{i,j} + \delta_{i,j+1}} F_j L_i, \\
[E_i, F_j] = \delta_{i,j} \frac{L_i L_{i+1}^\pm - L_i^\pm L_{i+1}}{q - q^{-1}},
\]

and the quantum Serre relations

\[
E_i E_j = E_j E_i \text{ if } |i - j| > 1, \quad E_i^2 E_{i+1} - (q + q^{-1}) E_i E_{i+1} E_i + E_{i+1} E_i^2 = 0, \\
F_i F_j = F_j F_i \text{ if } |i - j| > 1, \quad F_i^2 F_{i+1} - (q + q^{-1}) F_i F_{i+1} F_i + F_{i+1} F_i^2 = 0.
\]
It is a Hopf algebra, with comultiplication $\Delta$, counit $\varepsilon$ and antipode $S$ given on generators by the following:

\[
\begin{align*}
\Delta(L_i) &= L_i \otimes L_i, & \varepsilon(L_i) &= 1, & S(L_i) &= L_i^{-1}, \\
\Delta(E_i) &= E_i \otimes 1 + L_i L_{i+1}^{-1} \otimes E_i, & \varepsilon(E_i) &= 0, & S(E_i) &= -L_i^{-1} L_{i+1} E_i, \\
\Delta(F_i) &= F_i \otimes L_i^{-1} L_{i+1} + 1 \otimes F_i, & \varepsilon(F_i) &= 0, & S(F_i) &= -F_i L_i L_{i+1}^{-1}.
\end{align*}
\]

We denote by $\mathcal{U}_q(\mathfrak{gl}_m)$ the subalgebra generated by $(L_i)_{1 \leq i \leq m}$, by $\mathcal{U}_q(\mathfrak{gl}_m)^{\geq 0}$ the subalgebra generated by $(L_i, E_j)_{1 \leq i \leq m, \ 1 \leq j \leq m-1}$.

In order to set some notations, we denote by $P = \bigoplus_{i=1}^m \mathbb{Z} \varepsilon_i$ the weight lattice of $\mathfrak{gl}_m$ with $\mathbb{Z}$-basis given by the fundamental weights $(\varpi_i)_{1 \leq i \leq m}$ with $\varpi_i = \varepsilon_1 + \cdots + \varepsilon_i$. We denote by $Q$ the root lattice with $\mathbb{Z}$-basis given by the simple roots $(\alpha_i)_{1 \leq i \leq d-1}$ with $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$. Denote by $\Phi^+$ the set of positive roots, by $P^+$ the set of dominant weights for $\mathfrak{gl}_m$, that is $\mu = \sum_{i=1}^m \mu_i \varepsilon_i$ with $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$. We also endow $P$ with the standard non-degenerate bilinear form: $\langle \varepsilon_i, \varepsilon_j \rangle = \delta_{i,j}$. The symmetric group $S_m$ acts on $P$ by permuting the coordinates and leaves the bilinear form $\langle \cdot, \cdot \rangle$ invariant. Finally, let $\rho$ be the half-sum of positive roots.

We will often work with extensions $\mathbb{Z}[\beta_1, \ldots, \beta_k] \otimes P$, where the $\beta_i$'s are indeterminates and we also extend the bilinear form $\langle \cdot, \cdot \rangle$ to $\mathbb{Z}[\beta_1, \ldots, \beta_k] \otimes P$.

### 3.2. Weights and parabolic Verma modules.

Suppose that our field $\mathbb{k}$ contains the field $\mathbb{Q}(q)$ and let $M$ be an $\mathcal{U}_q(\mathfrak{gl}_m)$-module over the ground field $\mathbb{k}$. An element $v \in M$ is said to be a weight vector if $L_i v = \varphi(\varepsilon_i)v$, where $\varphi : P \to \mathbb{k}$ is the corresponding weight. The module $M$ is said to be a weight module if the action of the elements $L_1, \ldots, L_m$ is simultaneously diagonalizable. A highest weight module is a weight module $M$ such that $M = \mathcal{U}_q(\mathfrak{gl}_m)v$, where $v$ is a weight vector such that $E_i v = 0$ for $1 \leq i \leq m-1$.

It is well known that finite dimensional weight representations of $\mathcal{U}_q(\mathfrak{gl}_m)$ are parametrized by the set $P^+$ of dominant weights.

In this paper, we will be interested in modules over the field $\mathbb{Q}(q, \lambda_1, \ldots, \lambda_k)$, where $\lambda_i = q^{\beta_i}$ is an indeterminate. Moreover we consider only type 1 modules, where the weights are of the form

\[
\varphi(\nu) = q^{\langle \mu, \nu \rangle},
\]

for some $\mu \in \mathbb{Z}[\beta_1, \ldots, \beta_k] \otimes P$ and for all $\nu \in P$.

We now turn to parabolic Verma modules. Let $\mathfrak{p}$ be a standard parabolic subalgebra of $\mathfrak{gl}_m$ with Levi factor $\mathfrak{l} = \mathfrak{gl}_{m_1} \times \cdots \times \mathfrak{gl}_{m_d}$ with $m_i \geq 1$ and $\sum_{i=1}^d m_i = m$. Denote by $I$ the set $\{ m_i \mid 1 \leq i \leq d-1 \}$, where $\tilde{m}_i = m_1 + \cdots + m_i$, so that $\mathcal{U}_q(\mathfrak{l})$ is generated by $L_i, E_j$ and $F_j$ for $1 \leq i \leq m$ and $j \notin I$ and $\mathcal{U}_q(\mathfrak{p})$ is generated by $L_i, E_j$ and $F_k$ for $1 \leq i \leq m$, $1 \leq j \leq m-1$ and $k \notin I$. We identify the set $P_1^+ \times \cdots \times P_d^+$, where $P_i^+$ is the set of
dominant weights for $\mathfrak{gl}_{m_1}$, with the dominant weights $P_1^+$ of $I$ by the following map

$$(\mu^{(1)}, \ldots, \mu^{(d)}) \to \sum_{i=1}^d \left( \sum_{j=1}^{m_i} \mu_j^{(i)} \varepsilon_{m_i-1+j} \right).$$

We work over the field $\mathbb{Q}(q, \lambda_1, \ldots, \lambda_d)$ with $\lambda_i = q^{\beta_i}$. For a dominant weight $\mu \in P_1^+$, we have an irreducible integrable finite dimensional representation $V^I(\Lambda, \mu)$ of $\mathcal{U}_q(\mathfrak{gl}_n)$ of highest weight

$$\Lambda_\mu = \sum_{i=1}^d \left( \sum_{j=1}^{m_i} (\beta_i + \mu_j^{(i)}) \varepsilon_{m_i-1+j} \right).$$

Indeed, we check that $\langle \Lambda_\mu, \alpha_i \rangle \in \mathbb{N}$ for any $i \notin I$. We turn this representation into a representation of $\mathcal{U}_q(\mathfrak{gl}_m)$ by setting $E_i V^I(\Lambda, \mu) = 0$ for all $i \in I$. Then the parabolic Verma module $M^p(\Lambda, \mu)$ is

$$M^p(\Lambda, \mu) = \mathcal{U}_q(\mathfrak{gl}_m) \otimes_{\mathcal{U}_q(\mathfrak{gl}_n)} V^I(\Lambda, \mu).$$

It is clearly a highest weight module with highest weight $\Lambda_\mu$. If $\mu = 0$ we will simply denote this module by $M^p(\Lambda)$ and its highest weight by $\Lambda$.

**Lemma 3.2.** For any $\mu \in P_1^+$, the parabolic Verma module $M^p(\Lambda, \mu)$ is simple.

**Proof.** Since for any $i \in I$ the scalar product $\langle \Lambda_\mu, \alpha_i \rangle$ is not an integer, it follows from usual arguments that $M^p(\Lambda, \mu)$ is simple. \hfill $\square$

**Remark 3.3.** If the parabolic subalgebra $\mathfrak{p}$ is the Borel subalgebra $\mathfrak{b}$ of upper triangular matrices, we have $\mathcal{U}_q(\mathfrak{p}) = \mathcal{U}_q(\mathfrak{gl}_m)_{\geq 0}$ and the parabolic Verma module $M^p(\Lambda)$ is the universal Verma module.

In the rest of this article, all dominant weights $\mu \in P_1^+$ will moreover satisfy $\mu_m^{(i)} \geq 0$ for all $1 \leq i \leq d$. Therefore it will be convenient to identify such a weight $\mu$ with the corresponding $d$-partition satisfying $l(\mu^{(i)}) \leq m_i$. We then denote by $\mathcal{P}_n^d$ the set of $d$-partitions $\mu$ of $n$ such that $l(\mu^{(i)}) \leq m_i$. We will use the same notation $\mu$ to denote the $d$-partition or the corresponding dominant weight.

We also denote by $V$ the standard representation of $\mathfrak{gl}_m$ of dimension $m$. It is a highest weight module with highest weight $\varepsilon_1$, it has as a basis $v_1, \ldots, v_m$ and the action of $\mathcal{U}_q(\mathfrak{gl}_m)$ is given by

$$L_i \cdot v_j = q^{\delta_{i,j}} v_j, \quad E_i \cdot v_j = \delta_{i+1,j} v_{j-1} \quad \text{and} \quad F_i \cdot v_j = \delta_{i,j} v_{j+1}.$$

### 3.3. Tensor products and branching rule

As $\mathcal{U}_q(\mathfrak{gl}_m)$ is a Hopf algebra, its category of representations is endowed with a tensor product. Given $M$ and $N$ two modules over a ground ring $R$, the action of the generators on $M \otimes_R N$ is given using the comultiplication: for all $v \in M$ and $w \in N$, one have

$$L_i \cdot (v \otimes w) = L_i \cdot v \otimes L_i \cdot w, \quad E_i \cdot (v \otimes w) = E_i \cdot v \otimes w + L_i L_{i+1}^{-1} \cdot v \otimes E_i \cdot w$$

and

$$F_i \cdot (v \otimes w) = F_i \cdot v \otimes L_i^{-1} L_{i+1}^{-1} \cdot w + v \otimes F_i \cdot w.$$
We will write $\otimes$ instead of $\otimes_R$ to simplify the notations. Since we will be interested in the endomorphism algebra of $M^p(\Lambda) \otimes V^\otimes n$, we first start by understanding the decomposition of this module.

**Proposition 3.4.** For any $\mu \in \mathcal{P}_1^n$, there is an isomorphism of $\mathcal{U}_q(\mathfrak{gl}_m)$-modules

$$M^p(\Lambda, \mu) \otimes V \simeq \bigoplus_{\nu \in \mathcal{P}_1^{n+1}} M^p(\Lambda, \nu),$$

where the sum is over all $\nu \in \mathcal{P}_1^{n+1}$ whose Young diagram is obtained from the Young diagram of $\mu$ by adding one addable box.

**Proof.** We start by showing that $M^p(\Lambda, \mu) \otimes V$ has a filtration given by the $M^p(\Lambda, \nu)$ as in the lemma. First, we have the following tensor identity:

$$\left( \mathcal{U}_q(\mathfrak{gl}_m) \otimes \mathcal{U}_q(\mathfrak{p}) \right) V^l(\Lambda, \mu) \otimes V \simeq \mathcal{U}_q(\mathfrak{gl}_m) \otimes \mathcal{U}_q(\mathfrak{p}) \left( V^l(\Lambda, \mu) \otimes V \right).$$

Noticing that $L \mapsto \mathcal{U}_q(\mathfrak{gl}_m) \otimes \mathcal{U}_q(\mathfrak{p}) L$ is an exact functor from the category of finite dimensional $\mathcal{U}_q(\mathfrak{p})$-modules to the category of $\mathcal{U}_q(\mathfrak{gl}_m)$-modules, it remains to show that

$$V^l(\Lambda, \mu) \otimes V \simeq \bigoplus_{\nu \in \mathcal{P}_1^{n+1}} V^l(\Lambda, \nu),$$

where the sum is over all $\nu \in \mathcal{P}_1^{n+1}$ whose Young diagram is obtained from the Young diagram of $\mu$ by adding one addable box. This follows from the usual branching rule for $\mathcal{U}_q(\mathfrak{gl}_m)$-modules.

To show that the sum is direct, we will look at these modules as $\mathcal{U}_q(\mathfrak{sl}_m)$-modules and show that each $M^p(\Lambda, \nu)$ lie in a different block. We know that $M^p(\Lambda, \nu)$ and $M^p(\Lambda, \nu')$ are in the same block if and only if their highest weights are in the same orbit for the action of the symmetric group shifted by $\rho$. The highest weight for $\mathfrak{sl}_m$ of $M^p(\Lambda, \nu)$ is given by

$$\Lambda + \nu - \frac{|\Lambda| + |\nu|}{m}(\varepsilon_1 + \ldots + \varepsilon_m).$$

Since the dot action satisfies $w \cdot (\eta + \gamma) = w \cdot \eta + w(\gamma)$, we obtain that $M^p(\Lambda, \nu)$ and $M^p(\Lambda, \nu')$ are in the same block if and only if there exists $w \in \mathfrak{S}_m$ such that

$$w \cdot \left( \nu - \frac{|\nu|}{m}(\varepsilon_1 + \ldots + \varepsilon_m) \right) = \nu' - \frac{|\nu'|}{m}(\varepsilon_1 + \ldots + \varepsilon_m).$$

Since both $\nu - \frac{|\nu|}{m}(\varepsilon_1 + \ldots + \varepsilon_m)$ and $\nu' - \frac{|\nu'|}{m}(\varepsilon_1 + \ldots + \varepsilon_m)$ are dominants, we conclude that $w = 1$ and then $\nu = \nu'$ since $|\nu| = |\nu'|$. \hfill $\square$

Using the previous lemma and induction, one show the following proposition.

**Corollary 3.5.** There is an isomorphism

$$M^p(\Lambda) \otimes V^\otimes n \simeq \bigoplus_{\mu \in \mathcal{P}_1^n} M(\Lambda, \mu)^{n_{\mu}},$$

where $n_{\mu}$ is the number of paths from the empty $d$-partition to $\mu$ in the Young lattice of $d$- multipartitions.
3.4. Braiding and an action of the Artin-Tits group of type $B$. The quantized enveloping algebra (or rather a completion of the tensor product with itself) contains an element, called the quasi-$R$-matrix, which is a crucial tool in defining a braiding on a subcategory of the representations of $\mathcal{U}_q(\mathfrak{gl}_m)$. Since there are several possible braidings, we make our choice explicit and refer to [7][10.1.D] for more details.

In a completion of $\mathcal{U}_q(\mathfrak{gl}_m) \otimes \mathcal{U}_q(\mathfrak{gl}_m)$, we define an element $\Theta$ by

$$\Theta = \prod_{\alpha \in \Phi^+} \left( \sum_{n=0}^{+\infty} q^{\frac{n(n-1)}{2}} (q - q^{-1})^n \frac{\alpha^n \otimes F(\alpha^n)}{[n]!} \right),$$

where $[n] = \prod_{i=1}^{n} \frac{q^i - q^{-i}}{q^i - 1}$ and $E_\alpha, F_\alpha$ being the root vectors associated to a positive root $\alpha$. If $M$ and $N$ are two $\mathcal{U}_q(\mathfrak{gl}_m)$ type 1 weight modules over the ground ring $\mathbb{Q}(q, \lambda_1, \ldots, \lambda_{d-1})$ where $\mathcal{U}_q(\mathfrak{gl}_m)^{\geq 0}$ act locally nilpotently, $\Theta$ induces an isomorphism of vector spaces $\Theta_{M,N}: M \otimes N \rightarrow M \otimes N$. We then define a morphism of $\mathcal{U}_q(\mathfrak{gl}_m)$-modules

$$c_{M,N}: M \otimes N \rightarrow N \otimes M,$$

by $c_{M,N} = \tau \circ f \circ \Theta_{M,N}$, where $\tau$ is the flip $v \otimes w \mapsto w \otimes v$ and $f$ is the map $v \otimes w \mapsto q^{\langle \mu, \nu \rangle} v \otimes w$ if $v$ and $w$ are of respective weights $\mu$ and $\nu$. This endows the category of type 1 weight modules on which $\mathcal{U}_q(\mathfrak{gl}_m)^{> 0}$ act locally nilpotently with a braiding, which satisfy the so-called hexagon axioms:

$$c_{L \otimes M, N} = (c_{L,N} \otimes \text{Id}_M) \circ (\text{Id}_L \otimes c_{M,N}) \quad \text{and} \quad c_{L,M \otimes N} = (\text{Id}_M \otimes c_{L,N}) \circ (c_{L,M} \otimes \text{Id}_N).$$

Let $\mathcal{B}_n$ be the Artin-Tits braid group of type $B_n$. It has the following presentation in terms of generators and relations:

$$\mathcal{B}_n = \left\langle \tau_0, \tau_1, \ldots, \tau_{n-1} \middle| \begin{array}{l}
\tau_0 \tau_1 \tau_0 \tau_1 = \tau_1 \tau_0 \tau_1 \tau_0, \\
\tau_i \tau_j = \tau_j \tau_i, \quad \text{if } |i - j| > 1, \\
\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}, \quad \text{for } 1 \leq i \leq n - 2
\end{array} \right\rangle.$$

Using the braiding, we define the following endomorphisms of $M \otimes N^{\otimes n}$:

$$R_0 = (c_{N,M} \circ c_{M,N}) \otimes \text{Id}_{N^{\otimes n-1}},$$

$$R_i = \text{Id}_{M^{\otimes n-1} \otimes \otimes c_{N,N} \otimes \text{Id}_{N^{\otimes n-1}}} \otimes \text{Id}_{N^{\otimes n-i-1}}, \text{ for } 1 \leq i \leq n - 1.$$

**Proposition 3.6.** The assignment $\tau_i \mapsto R_i$ defines an action of $\mathcal{B}_n$ on the module $M \otimes N^{\otimes n}$ which commutes with the $\mathcal{U}_q(\mathfrak{gl}_m)$ action.

**Proof.** The fact that $R_i$ is a $\mathcal{U}_q(\mathfrak{gl}_m)$-morphism follows by definition of $R_i$. We only have to check that the defining relations of $\mathcal{B}_n$ are satisfied. Clearly, if $|i - j| > 1$ then $R_i$ and $R_j$ commute. For $1 \leq i \leq n - 2$, the relation $R_i R_{i+1} R_i = R_{i+1} R_{i+1} R_{i+1}$ follows from the Yang-Baxter equation. Finally, it remains to show that $R_0 R_1 R_0 R_1 = R_1 R_0 R_1 R_0$ and we
may, and will, suppose that \( n = 2 \). Using graphical calculus, we simply have to show that
\[
N \rightarrow N \rightarrow M \rightarrow N = N \rightarrow M \rightarrow N
\]
which follows from applying several Reidemeister III moves (we leave the details to the reader). \( \square \)

Finally, we end this section with a lemma due to Drinfeld computing the action of the double braiding on highest weight modules, which is related with the action of the ribbon element.

**Lemma 3.7.** Let \( L, M \) and \( N \) be highest weight modules of respective highest weight \( \lambda, \mu \) and \( \nu \) such that \( L \subset M \otimes N \). Then the double braiding \( c_{N,M} \circ c_{M,N} \) restricted to \( N \) acts by multiplication by the scalar
\[
q^{\langle \lambda, \lambda + 2\rho \rangle - \langle \mu, \mu + 2\rho \rangle - \langle \nu, \nu + 2\rho \rangle}.
\]

4. **The endomorphism algebra of \( M^p(\Lambda) \otimes V^\otimes n \)**

The aim of this section is to prove the following:

**Theorem 4.1.** Let \( p \) be a standard parabolic subalgebra of \( \mathfrak{gl}_m \) with Levi subalgebra \( \mathfrak{l} = \mathfrak{gl}_{m_1} \times \cdots \times \mathfrak{gl}_{m_d} \). The \( \mathcal{H}(d, n) \otimes \mathcal{U}_q(\mathfrak{gl}_m) \)-module \( M^p(\Lambda) \otimes V^\otimes n \) is decomposed as
\[
M^p(\Lambda) \otimes V^\otimes n \cong \bigoplus_{\mu \in \mathfrak{h}_l^+} V_\mu \otimes M^p(\Lambda, \mu),
\]
the parameters of the Ariki-Koike algebra being given by Lemma 4.2 below. Moreover,
\[
\text{End}_{\mathcal{U}_q(\mathfrak{gl}_m)}(M^p(\Lambda) \otimes V^\otimes n) = \mathcal{H}_l(d, n).
\]

We first explain why \( M^p(\Lambda) \otimes V^\otimes n \) inherits an action of the Ariki-Koike algebra from the action of the braid group of type \( B_n \). It is a classical result that the eigenvalues of \( R_i \) are \( q \) and \(-q^{-1}\): the action of the braiding on \( V \otimes V \) is
\[
v_i \otimes v_j \mapsto \begin{cases} 
qv_j \otimes v_i & \text{if } i = j, \\
v_j \otimes v_i & \text{if } i > j, \\
v_j \otimes v_i + (q - q^{-1})v_i \otimes v_j & \text{if } i < j.
\end{cases}
\]
Moreover, using Lemma 3.4 we easily compute the eigenvalues of the endomorphism $R_0$ in order to show that the action of $\mathcal{B}_n$ factors through the Ariki-Koike algebra.

**Lemma 4.2.** The eigenvalues $u_1, \ldots, u_d$ of $R_0$ on $M^p(\Lambda) \otimes V$ are equal to

$$u_i = (\lambda_i q^{-\bar{m}_{i-1}})^2.$$

**Proof.** Let $\Lambda$ be the highest weight of $M^p(\Lambda)$. The decomposition of $M^p(\Lambda) \otimes V$ is given in Proposition 3.4

$$M^p(\Lambda) \otimes V \simeq \bigoplus_{i=1}^{d} M^p(\Lambda, \mu_i),$$

where $\mu_i$ is the $d$-partition of 1 whose only non-zero component is the $i$-th one and is equal to (1). The highest weight of $M^p(\Lambda, \mu_i)$ being $\Lambda + \varepsilon_{\bar{m}_{i-1}+1}$, the action of $R_0$ on $M^p(\Lambda, \mu_i)$ is given by

$$q^{\langle \Lambda + \varepsilon_{\bar{m}_{i-1}+1}, \Lambda + \varepsilon_{\bar{m}_{i-1}+1} + 2\rho \rangle - \langle \Lambda + 2\rho \rangle - \langle \varepsilon_1 + 2\rho \rangle},$$

and we check that

$$\langle \Lambda + \varepsilon_{\bar{m}_{i-1}+1}, \Lambda + \varepsilon_{\bar{m}_{i-1}+1} + 2\rho \rangle - \langle \Lambda + 2\rho \rangle - \langle \varepsilon_1 + 2\rho \rangle = 2(\beta_i - \bar{m}_{i-1}).$$

Therefore, the assignment $T_i \rightarrow R_i$ defines a morphism of algebras

$$\mathcal{H}(d, n) \rightarrow \text{End}_{U_q(\mathfrak{gl}_n)}(M^p(\Lambda) \otimes V^\otimes n),$$

the parameters of the Ariki-Koike algebra being $u_i = (\lambda_i q^{-\bar{m}_{i-1}})^2$.

**Proof of Theorem 4.1.** Using Corollary 3.3 and the fact that $\mathcal{H}(d, n)$ acts on $M^p(\Lambda) \otimes V^\otimes n$ by $U_q(\mathfrak{gl}_n)$-linear endomorphisms, we see that

$$M^p(\Lambda) \otimes V^\otimes n \simeq \bigoplus_{\mu \in \mathcal{P}^p_i} \tilde{V}_\mu \otimes M^p(\Lambda, \mu),$$

for some representations $\tilde{V}_\mu$ of $\mathcal{H}(d, n)$. Since the multiplicity of $M^p(\Lambda, \mu)$ in $M^p(\Lambda) \otimes V^\otimes n$ is given by the number of paths in the Young’s lattice from the empty $d$-partition to the $d$-partition $\mu$, we have $\dim(\tilde{V}_\mu) = \dim(V_\mu)$. It then remains to show that $V_\mu$ is a subrepresentation of $\tilde{V}_\mu$.

Let $t$ be a standard Young tableau of shape $\mu$ and denote by $(a_i, b_i, c_i) = t^{-1}(i)$. Denote by $\mu[i]$ the $d$-partition of $i$ obtained by adding to the empty $d$-partition the boxes labeled by 1 to $i$ in the chosen standard tableau $t$. We now choose a highest weight vector $v \in M^p(\Lambda) \otimes V^\otimes n$ of weight $\Lambda_\mu$ such that for all $1 \leq i \leq n$ we have

$$v \in M^p(\Lambda, \mu[i]) \otimes V^\otimes (n-i) \subset M^p(\Lambda) \otimes V^\otimes n.$$

Using the branching rule, one see that such a vector exists and is unique up to a scalar. Let us show that this vector $v$ is a common eigenvector of the Jucys-Murphy elements $X_i$. It is easy to see that the action of the Jucys-Murphy element $X_i$ on $M^p(\Lambda) \otimes V^\otimes n$ is given
by the double braiding \((c_{V,M^p(\Lambda)} \otimes V^{\otimes (n-i)}) \circ c_{M^p(\Lambda) \otimes V^{\otimes (i-1)}, V}) \otimes \text{Id}_{V^{\otimes (n-i)}}\). By Lemma 3.7, we obtain that \(X_i\) acts on \(v\) by multiplication by
\[
g^q(\Lambda_{\mu[i]} \Lambda_{\mu[i]+2\rho} - (\Lambda_{\mu[i]-1} \Lambda_{\mu[i-1]+2\rho} - \varepsilon_{\varepsilon_1} \varepsilon_{\varepsilon_1+2\rho})\).

Indeed, \(v\) lies in the summand \(M^p(\Lambda, \mu[i]) \otimes V^{\otimes (n-i)} \subseteq M^p(\Lambda, \mu[i-1]) \otimes V \otimes V^{\otimes (n-i)}\) of \(M^p(\Lambda) \otimes V^{\otimes n}\). But \(\Lambda_{\mu[i]} = \Lambda_{\mu[i]+1} + \varepsilon_{k_i}\) where \(k_i = \tilde{m}_{\varepsilon_i-1} + a_i\) so that
\[
\langle \Lambda_{\mu[i]}, \Lambda_{\mu[i]} + 2\rho \rangle - \langle \Lambda_{\mu[i]-1}, \Lambda_{\mu[i]-1} + 2\rho \rangle - \langle \varepsilon_{\varepsilon_1}, \varepsilon_{\varepsilon_1+2\rho} \rangle = 2\langle \Lambda_{\mu[i]-1}, \varepsilon_{k_i} \rangle + 2(1 - k_i)
\]
\[
= 2(\beta_{\varepsilon_i} + b_i - k_i),
\]
since the component of \(\Lambda_{\mu[i]-1}\) on \(\varepsilon_{k_i}\) is \(\beta_{\varepsilon_i} + (b_i - 1)\). Therefore \(X_i\) acts on \(v\) by multiplication by
\[
(\lambda_{\varepsilon_i} g^{b_i - k_i})^2 = u_{\varepsilon_i} g^{2(b_i - a_i)}.
\]
Therefore the \(\mathcal{H}(d, n)\) submodule spanned by \(v\) is isomorphic to \(V_{\mu}\) and then \(V_{\mu}\) is a subrepresentation of \(V_{\mu}\). 

4.1. Some particular cases. We finish by giving some special cases of Theorem 4.1 in order to recover various well-known algebras. The two first special cases involve the well-known situation without a parabolic Verma module: it suffices to note that if \(p = \mathfrak{gl}_m\) then \(M^p(\Lambda)\) is the trivial module.

**Corollary 4.3.** If the parabolic subalgebra \(p\) is \(\mathfrak{gl}_m\) and \(m \geq n\), then the endomorphism algebra of \(M^p(\Lambda) \otimes V^{\otimes n}\) is isomorphic to Hecke algebra of type A.

**Corollary 4.4.** If the parabolic subalgebra \(p\) is \(\mathfrak{gl}_m\) and \(m = 2\), then the endomorphism algebra of \(M^p(\Lambda) \otimes V^{\otimes n}\) is isomorphic to Temperley–Lieb algebra of type A.

We now turn to special cases where \(p\) is a strict subalgebra of \(\mathfrak{gl}_m\). The following corollary follows from the Remark 2.13.

**Corollary 4.5.** For \(p\) such that \(m \geq nd\) and \(m_i \geq n\) for all \(1 \leq i \leq d\), the endomorphism algebra of \(M^p(\Lambda) \otimes V^{\otimes n}\) is isomorphic to the Ariki-Koike algebra \(\mathcal{H}(d, n)\).

The Hecke algebra of type \(B\) with unequal parameters appears when we work with a standard parabolic subalgebra \(p\) with Levi factor \(\mathfrak{gl}_{m_1} \times \mathfrak{gl}_{m_2}\).

**Corollary 4.6.** If the parabolic subalgebra \(p\) is such that \(d = 2, m_1 \geq n\) and \(m_2 \geq n\), then the endomorphism algebra of \(M^p(\Lambda) \otimes V^{\otimes n}\) is isomorphic to the Hecke algebra of type \(B\) with unequal and algebraically independent parameters.

Finally, the last special case is a generalization of the \(\mathfrak{gl}_2\) case of [12], where we recover the generalized blob algebra.

**Corollary 4.7.** If the parabolic subalgebra \(p\) is the standard Borel subalgebra \(b\) of \(\mathfrak{gl}_m\), that is \(d = m\) and \(m_i = 1\) for \(1 \leq i \leq d\), then the endomorphism algebra of \(M(\Lambda) \otimes V^{\otimes n}\) is isomorphic to the generalized blob algebra \(B(d, n)\).
References

[1] S. Ariki. On the Semi-simplicity of the Hecke Algebra of $(\mathbb{Z}/r\mathbb{Z}) \wr \mathfrak{S}_n$. *J. Algebra*, 169(1):216–225, 1994.

[2] S. Ariki and K. Koike. A Hecke Algebra of $(\mathbb{Z}/r\mathbb{Z}) \wr \mathfrak{S}_n$ and Construction of its Irreducible Representations. *Adv. Math.*, 106(2):216–243, 1994.

[3] S. Ariki, T. Terasoma, and H. Yamada. Schur–Weyl reciprocity for the Hecke algebra of $(\mathbb{Z}/r\mathbb{Z}) \wr \mathfrak{S}_n$. *J. Algebra*, 178(2):374–390, 1995.

[4] M. Balagović, Z. Daugherty, I. Entova-Aizenbud, I. Halacheva, J. Hennig, M. S. Im, G. Letzter, E. Norton, V. Serganova, and C. Stroppel. The affine VW supercategory. *Sel. Math. New Ser.*, 26, 2020.

[5] H. Bao, W. Wang, and H. Watanabe. Multiparameter quantum Schur duality of type $B$. *Proc. Amer. Math. Soc.*, 146(8):3203–3216, 2018.

[6] M. Broué and G. Malle. Zyklotomische Heckealgebren. Number 212, pages 119–189. 1993. Représentations unipotentes génériques et blocs des groupes réductifs fins.

[7] V. Chari and A. Pressley. *A guide to quantum groups*. Cambridge University Press, Cambridge, 1994.

[8] V. Chari and A. Pressley. Quantum affine algebras and affine Hecke algebras. *Pacific J. Math.*, 174(2):295–326, 1996.

[9] I. V. Cherednik. A new interpretation of Gelfand-Tzetlin bases. *Duke Math. J.*, 54(2):563–577, 1987.

[10] Z. Daugherty and A. Ram. Two boundary Hecke Algebras and combinatorics of type $C$. 2018, arXiv: 1804.10296v1.

[11] M. Geck and N. Jacon. *Representations of Hecke algebras at roots of unity*, volume 15 of *Algebra and Applications*. Springer-Verlag London, Ltd., London, 2011.

[12] K. Iohara, G. I. Lehrer, and R. B. Zhang. Schur-Weyl duality for certain infinite dimensional $U_q(\mathfrak{sl}_2)$-modules. 2018, arXiv: 1811.01325v2.

[13] M. Jimbo. A $q$-analogue of $U(\mathfrak{gl}(N + 1))$, Hecke algebra, and the Yang-Baxter equation. *Lett. Math. Phys.*, 11(3):247–252, 1986.

[14] D. Kazhdan and G. Lusztig. Tensor structures arising from affine Lie algebras i-iv. *J. Amer. Math. Soc.*, 6-7:905–947, 949–1011,335–383,383–453, 1993-1994.

[15] A. Lacabanne, G. Naisse, and P. Vaz. Tensor product categorifications, Verma modules, and the blob algebra. 2020, in preparation.

[16] P. Martin and H. Saleur. The blob algebra and the periodic Temperley-Lieb algebra. *Lett. Math. Phys.*, 30(3):189–206, 1994.

[17] P. Martin and D. Woodcock. Generalized blob algebras and alcove geometry. *LMS J. Comput. Math.*, 6:249–296, 2003.

[18] D. Rose and D. Tubbenhauer. HOMFLYPT homology for links in handlebodies via type A Soergel bimodules. 2019, arXiv: 1908.06878v1.

[19] R. Rouquier, P. Shan, M. Varagnolo, and E. Vasserot. Categorifications and cyclotomic rational double affine Hecke algebras. *Invent. Math.*, 204(3):671–786, 2016.

[20] M. Sakamoto and T. Sakamoto. Schur-Weyl reciprocity for Ariki-Koike algebras. *J. Algebra*, 221(1):293–314, 1999.

Institut de Recherche en Mathématique et Physique, Université Catholique de Louvain, Chemin du Cyclotron 2, 1348 Louvain-la-Neuve, Belgium

E-mail address: abel.lacabanne@uclouvain.be

Institut de Recherche en Mathématique et Physique, Université Catholique de Louvain, Chemin du Cyclotron 2, 1348 Louvain-la-Neuve, Belgium

E-mail address: pedro.vaz@uclouvain.be