Kato-Nakayama’s comparison theorem and analytic log etale topoi

Yukiyoshi Nakkajima

Abstract: In this paper we study topics related to one of Kato-Nakayama’s comparison theorems in [KN] using analytic log etale topoi.

Key words: Kato-Nakayama’s comparison theorem, log Kummer sequences, log exponential sequences, analytic log etale topoi.

1 Introduction

In [KN] Kato and Nakayama have proved the following comparison theorem (cf. [Il, (5.9)]):

Theorem 1.1 ([KN, (0.2) (1); log etale vs. log Betti]). Let $X$ be an fs (fine and saturated) log scheme over $\mathbb{C}$ whose underlying scheme is locally of finite type over $\mathbb{C}$. Let $X_{\mathrm{an}}^{\log}$ be the real blow up of the analytification $X_{\mathrm{an}}$ of $X$ ([KN]). Let $D_{c,\mathrm{tor}}^{+}(X_{\mathrm{et}}^{\log})$ be the derived category of bounded below complexes of abelian sheaves in $X_{\mathrm{et}}^{\log}$ whose cohomology sheaves are constructible ([KN]) and torsion. Let $K^{\bullet}$ be an object of $D_{c,\mathrm{tor}}^{+}(X_{\mathrm{et}}^{\log})$. Let $K^{\bullet}_{\mathrm{log}}$ be the inverse image of $K^{\bullet}$ in $X_{\mathrm{an}}^{\log}$. Then there exists a canonical isomorphism

$$H^{h}(X_{\mathrm{et}}^{\log}, K^{\bullet}) \sim H^{h}(X_{\mathrm{an}}^{\log}, K^{\bullet}_{\mathrm{log}}) \quad (h \in \mathbb{Z}).$$

In this paper, for an fs log analytic space $Y$ over $\mathbb{C}$, we introduce a new topos $\bar{Y}_{\mathrm{et}}^{\log}$ of $Y$ ($\bar{Y}_{\mathrm{et}}^{\log}$ is an analytic analogue of $X_{\mathrm{et}}^{\log}$) and we prove the following:

Theorem 1.2. (1) (analytically log etale vs. log Betti) Let $D_{\mathrm{cl},\mathrm{tor}}^{+}(Y_{\mathrm{et}}^{\log})$ be the derived category of bounded below complexes of abelian sheaves in $Y_{\mathrm{et}}^{\log}$ whose cohomology sheaves are locally classical and torsion (see [EL] below for the definition of a locally classical abelian sheaf). Let $K^{\bullet}$ be an object of $D_{\mathrm{cl},\mathrm{tor}}^{+}(Y_{\mathrm{et}}^{\log})$. Let $K^{\bullet}_{\mathrm{log}}$ be the inverse image of $K^{\bullet}$ in $Y_{\mathrm{an}}^{\log}$. Then there exists a canonical isomorphism

$$H^{h}(Y_{\mathrm{et}}^{\log}, K^{\bullet}) \sim H^{h}(Y_{\mathrm{log}}^{\log}, K^{\bullet}_{\mathrm{log}}) \quad (h \in \mathbb{Z}).$$

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(2) (GAGA: algebraically log etale vs. analytically log etale) Let the notations be as in §1. Let $K^\an$ be the inverse image of $K^\bullet$ in $(X^\an_{\log})_{\et}$. Then there exists a canonical isomorphism

$$H^h(X^\log_\et, K^\bullet) \sim H^h((X^\an_{\log})_{\et}, K^\an_{\et}) \quad (h \in \mathbb{Z}).$$

Let $D^+_c$ $(Y^\log_{\et})$ be the derived category of bounded below complexes of abelian sheaves in $Y^\log_{\et}$ whose cohomology sheaves are constructible and torsion (see §4 below for the definition of a constructible abelian sheaf in $Y^\log_{\et}$). Then we shall see that we have a natural functor $D^+_c(X^\et) \rightarrow D^+_c((X^\an_{\log})_{\et})$ (if $X$ is quasi-compact and quasi-separated) and an inclusion $D^+_c(Y^\log_{\et}) \subset D^+_c(Y^\log_{\et})$. Consequently, by (1) and (2), we immediately reobtain §1, which is another proof of $\text{(2)}$ (1).

Theorem (1) prompts us to say roughly that the abstract topos $\tilde{Y}^\log_{\et}$ can replace the topological space $Y^\log_{\et}$ and conversely that the concrete topological space $Y^\log$ represents the topos $\tilde{Y}^\log_{\et}$. The theorem (1.2) (2) is a theorem of GAGA type. Though (1.2) (1) and (2) tempt us to say that $X^\log_{\et}$, $(X^\an_{\log})_{\et}$ and $\tilde{X}^\log_{\et}$ are the same topoi for the calculations of the cohomologies of bounded below complexes of abelian sheaves whose cohomology sheaves are constructible and torsion, I think that $(X^\an_{\log})_{\et}$ is more closely connected with $\tilde{X}^\log_{\et}$ than $\tilde{X}^\log_{\et}$ in general by taking §1.10 (1) below (see also §1.10 (3) and §1.11 below) into account.

The contents of this paper are as follows.

In §2 we introduce $\tilde{Y}^\log_{\et}$. The topos $\tilde{Y}^\log_{\et}$ plays a key role in almost all parts of this paper. In particular, a key abelian sheaf $\mathcal{M}_{Y, \log}$ defined in §2 $(\mathcal{M}_{Y, \log}$ is an analytic analogue of $\mathcal{M}_{X, \log}$ defined in $\text{[KN]}$) lives in $\tilde{Y}^\log_{\et}$ (not in $Y^\log_{\et}$). In §3 we give a proof of §2 (1) by using the analytic log Kummer sequence; in §4 we give a proof of §2 (2). In fact we prove a base change theorem which is a generalization of §2 (2). Motivated by §2 (2), we give a logarithmic version of Grauert-Riemenschneider's theorem in the end of §4. In §5 and §6 using $\mathcal{M}_{X, \an, \log}$ we obtain a commutative diagram (5.5.1) below which compares three calculations of certain two higher direct images by the use of the analytic and algebraic log Kummer sequences and by the use of the log exponential sequence in $\text{[KN]}$. The starting purpose in writing this paper was to give the commutative diagram because it is necessary in $\text{[Nakk]}$ for the determination of the delicate sign before the Čech-Gysin morphism appearing in the boundary morphism of the $E_1$-terms of the $l$-adic weight spectral sequence in $\text{[Nak2]}$ from the analogous determination in $\text{[Nakk]}$ for the $\infty$-adic weight spectral sequence essentially obtained in $\text{[FN]}$.

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Notation. (1) For a log scheme (resp. log analytic space over $\mathbb{C}$) $X$ in the sense of Fontaine-Illusie-Kato (K), (resp. $\text{[KN]}$), we denote by $\tilde{X}$ the underlying scheme (resp. underlying analytic space) of $X$ and by $\mathcal{M}_X$ the log structure of $X$. For a morphism $f: X \rightarrow Y$ of log schemes (resp. log analytic spaces over $\mathbb{C}$), $\tilde{f}: \tilde{X} \rightarrow \tilde{Y}$ denotes the underlying morphism of $f$ between schemes (resp. analytic spaces over $\mathbb{C}$).
respectively. Then we have natural morphisms

\[ \mu_{\text{Y log}}^Y \text{real blow up of} \]

a Grothendieck topology on the category \( T \) of the images of \( U \).

First we recall a well-known method ([SGA 4-3, XI 4]) quickly to fix our ideas.

2 Analytic log etale topoi

\[ (1.2.1) \quad d(x^ts) = \sum_{i=0}^{t+1} (-1)^i \delta_i(x^ts) + (-1)^r d_M(x^ts) \quad (x^ts \in M^ts), \]

where \( d_M : M^ts \to M^{t+1,s} \) is the boundary morphism arising from the boundary morphism of the complex \( M \) and \( \delta_i : M^ts \to M^{t+1,s} \) is a standard coface morphism. The convention on signs in (1.2.1) is different from that in [D] (5.1.9) IV].

Let \( (T, A) \) be a ringed topos. Let \( (T_t, A^t) := (T_t, A^t) \in \mathbb{N} \) be a constant simplicial ringed topos defined by \( (T, A) : T_t = T, A^t = A \). Let \( M \) be a complex of \( A^t \)-modules. The complex \( M \) defines a double complex \( M^{**} = (M^{ts})_{t \in \mathbb{N}} \) of \( A^t \)-modules whose boundary morphisms will be fixed in (1.2.1) below (Our convention on the place of cosimplicial degrees is different from that in [D, (5.1.9) IV]). Let \( s(M) \) be the single complex \( \bigoplus_{t+s=n} M^{ts} \) with the following boundary morphism:

\[ (2.0.3) \quad A \text{ morphism in } T \text{ of topological spaces.} \]

2 Analytic log etale topoi

First we recall a well-known method ([SGA 4-3 XI 4]) quickly to fix our ideas.

Let \( T \) be a topological space. Let \( T_{cl} \) be a site defined by the following:

2.0.1 An object of \( T_{cl} \) is a local isomorphism \( U \to T \) of topological spaces.

2.0.2 A morphism in \( T_{cl} \) is a morphism of topological spaces over \( T \).

2.0.3 A family \( \{U_\lambda \to U\}_\lambda \) of morphisms in \( T_{cl} \) is called a covering if the union of the images of \( U_\lambda \)'s is \( U \); the coverings define a Grothendieck pretopology and hence a Grothendieck topology on the category \( T_{cl} \).

Let \( Y \) be an fs log analytic space over \( \mathbb{C} \) in the sense of [KN §1]. Let \( Y^\text{log} \) be the real blow up of \( Y \) ([KN (1.2)]). Let \( Y_{cl}^\text{log} \) be the site above for the topological space \( Y^\text{log} \). Let \( \tilde{Y}^\text{log} \) and \( \tilde{Y} \) be the topos defined by the classical topologies of \( Y^\text{log} \) and \( Y \), respectively. Then we have natural morphisms \( \epsilon_{cl} : \tilde{Y}_{cl}^\text{log} \to \tilde{Y}_{cl}, \epsilon_{\text{top}} : \tilde{Y}^\text{log} \to \tilde{Y}, \mu_{\text{log}} : \tilde{Y}_{cl}^\text{log} \to \tilde{Y}^\text{log} \) and \( \mu : Y_{cl} \to Y \) of topos fitting into the following commutative
We sometimes denote $\epsilon_{cl}$ by $\epsilon_{Y \log}$.

Let $M_{Y_{cl}} := \mu^{-1}(M_{Y})$ (resp. $O_{Y_{cl}} := \mu^{-1}(O_{Y})$) be the sheaf of log structures (resp. the structure sheaf) in $\tilde{Y}_{cl}$. Henceforth, in this paper, we consider $\tilde{Y}_{cl}, \tilde{Y}_{log\ cl}, M_{Y_{cl}}$ and $O_{Y_{cl}}$ in almost all cases, and we denote them simply by $\tilde{Y}, \tilde{Y}_{log}, M_{Y}$ and $O_{Y}$, respectively, as in [KN] unless stated otherwise.

Next, let us introduce a topos $\tilde{Y}_{et}$.

As in [K, (3.1)], we can define the exact closed immersion of fine log analytic spaces over $C$. Let $f : U \rightarrow V$ be a commutative diagram of fine log analytic spaces over $C$, where the left vertical morphism is an exact closed immersion defined by a nilpotent quasi-coherent ideal sheaf of $O_{T}$. As in [K, (3.3)], we say that $f$ is log smooth (resp. log etale) if there exists a (resp. unique) morphism $T \rightarrow U$ locally which makes the resulting two diagrams commutative. (In the (log) analytic case, we do not assume that $f$ is locally of finite presentation.) As usual, the properties of the log smoothness and the log etaleness are stable under the composition of morphisms and the base change in the category of fine log analytic spaces over $C$ (We can prove that the fiber product exists in the category of fine log analytic spaces over $C$ using the classical result of the existence of the fiber product in the category of analytic spaces over $C$; we can also prove the existence of the fiber product in the category of fs log analytic spaces over $C$).

The proof below for local descriptions of log smooth and etale morphisms in the (log) analytic case are slightly different from the proof of [K (3.5)] in the log algebraic case (we need to give care to the convergence):

**Proposition 2.1.** Let $f : U \rightarrow V$ be a morphism of fine log analytic spaces over $C$. Then the following conditions (1) and (2) are equivalent:

1. $f$ is log smooth (resp. log etale).
2. There exists a local chart $Q \rightarrow P$ of $f$ satisfying the following conditions (a) and (b):
   a. The morphism $Q^{gp} \otimes \mathbb{Z} \rightarrow P^{gp} \otimes \mathbb{Z}$ is injective (resp. isomorphic).
   b. The locally induced morphism $\tilde{U} \rightarrow \tilde{V} \times_{\text{Spec}(\mathbb{C}[Q])_{an}} \text{Spec}(\mathbb{C}[P])_{an}$ is locally isomorphic as analytic spaces over $C$.

**Proof.** (2) $\implies$ (1): Because the morphism $U \rightarrow V \times_{\text{Spec}(\mathbb{C}[Q])_{an}, Q^{a}}(\text{Spec}(\mathbb{C}[P])_{an}, P^{a})$ is log etale in our sense by the easy implication of (2) below, we have only to prove that the morphism $(\text{Spec}(\mathbb{C}[P])_{an}, P^{a}) \rightarrow (\text{Spec}(\mathbb{C}[Q])_{an}, Q^{a})$ is log smooth (resp. log etale). Consider the commutative diagram (2.0.5) for this morphism and let $t_{0} : T_{0} \rightarrow (\text{Spec}(\mathbb{C}[P])_{an}, P^{a})$ be the morphism. By the same proof as that of [K]

\[\begin{array}{c}
\tilde{Y}^{\log}_{cl} & \xrightarrow{\mu} & \tilde{Y}^{\log} \\
\epsilon_{cl} \downarrow & & \downarrow \epsilon_{top} \\
\tilde{Y}_{cl} & \xrightarrow{\tilde{\phi}} & \tilde{Y}.
\end{array}\]
(3.4)], we have an (resp. unique) extension homomorphism \( P \to M_T \) of \( P \to M_{T_0} \) and \( Q \to M_T \). For a point \( x \in T_0 \), let \( m_x \) and \( m_{0,x} \) be the maximal ideals of \( O_{T,x} \) and \( O_{T_0,x} \), respectively. Since we have the composite morphism \( T_0 \to \text{Spec}(\mathbb{C}[P])_{\text{an}} \to \text{Spec}(\mathbb{C}[P]) \) of local ringed spaces and since \( O_{T,x}/m_x = O_{T_0,x}/m_{0,x} \), the induced morphism \( (\pi_{T_0})^{-1}(\mathbb{C}[P]) \to O_T \) by the morphism \( P \to M_T \) extends to a morphism \( T \to \text{Spec}(\mathbb{C}[P])_{\text{an}} \) by the universality of the analytification (\text{SGA 1} XII (1.1))). Consequently we have a (resp. unique) desired morphism \( T \to (\text{Spec}(\mathbb{C}[P])_{\text{an}}, P^n) \).

(1)\(\Rightarrow\)(2): Assume that \( f \) is log smooth. We may assume that there exists a global chart \( Q \to O_V \) of \( V \).

Let \( \Lambda_{U/V}^1 \) be the sheaf of log differential forms on \( U/V \) defined similarly in [K] (1.7) (see also [E]T for the semistable case and [K1, (3.5)] for the absolute case). Let \( \mathcal{L} \) be an \( O_U \)-module of finite type. Let \( |U| \) be the underlying topological space of \( U \).

Let \( \tilde{U}(\mathcal{L}) := (|U|, O_U \oplus \mathcal{L}) \) be an analytic space over \( \mathbb{C} \), where we endow \( O_U \oplus \mathcal{L} \) with a natural structure of a sheaf of rings by defining \( \mathcal{L}^2 = 0 \). Let \( \iota: \tilde{U} \to \tilde{U}(\mathcal{L}) \) be a closed immersion and endow \( \tilde{U}(\mathcal{L}) \) with the log structure \( \iota_* (\mathcal{M}_U) \). Let \( U(\mathcal{L}) \) be the resulting fine log analytic space over \( \mathbb{C} \). For a surjective morphism \( \mathcal{L} \to \mathcal{L}_0 \) of \( O_U \)-modules of finite type, we have a closed immersion \( U(\mathcal{L}) \to U(\mathcal{L}_0) \) of fine log analytic spaces over \( \mathbb{C} \). Then, using the definition of the smoothness and the standard deformation theory (cf. \text{SGA 1} III (5.1))), we easily see that the natural morphism \( \text{Hom}_{O_U}(\Lambda_{U/V}^1, \mathcal{L}) \to \text{Hom}_{O_U}(\Lambda_{U/V}^1, \mathcal{L}_0) \) is locally surjective. Hence \( \Lambda_{U/V}^1 \) is a locally projective \( O_U \)-module of finite type. Let \( x \in \tilde{U} \) be a point and let \( d \log t_1, \ldots, d \log t_r \) \((t_1, \ldots, t_r \in M_{U,x}, r \in \mathbb{N})\) be a basis of \( \Lambda_{U/V,x}^1 \). If \( f \) is log etale, we easily see that \( \Lambda_{U/V}^1 = 0 \) by the standard log deformation theory (cf. \text{SGA 1} III (5.1))). Then, as in the proof of [K] (3.5), by using a natural morphism \( \mathbb{N}^r \oplus Q \to M_{U,x} \) and a well-defined surjective morphism

\[
\Lambda_{U/V,x}^1 \ni d \log a \mapsto 1 \otimes a \in \kappa(x) \otimes \mathbb{Z} \left( M_{U,x}^{\text{gp}}/O_{U,x}^{\text{gp}} \right) \left( f^{-1}( \mathcal{M}_{U,V,f(x)} ) \right) (a \in M_{U,x}),
\]

we have a surjective morphism \( \mathbb{C} \otimes \mathbb{Z} (\mathbb{Z} \oplus Q^{\text{gp}}) \to \mathbb{C} \otimes \mathbb{Z} (M_{U,x}^{\text{gp}}/O_{U,x}^{\text{gp}}) \). Because \( M_{U,x}^{\text{gp}}/O_{U,x}^{\text{gp}} \) is torsion-free, there exists a surjective homomorphism \( A \to M_{U,x}^{\text{gp}}/O_{U,x}^{\text{gp}} \), where \( A \) is an abelian group which has \( \mathbb{Z} \oplus Q^{\text{gp}} \) as a subgroup of finite index. Let \( P \subset A \) be the inverse image of \( M_{U,x}/O_{U,x} \). Then \( P \) is a local chart of \( M_U \) at a neighborhood of \( x \) (cf. [K] (2.10))). It is clear that the morphism \( \mathbb{N}^r \otimes Q \to \mathbb{N}^r ) \) \( a \otimes m \mapsto a d \log m \in \Lambda_{U/V,x}^1 \) is an isomorphism of \( O_{U,x} \)-modules.

Set \( W := (V \times_{\text{Spec}(\mathbb{C}[Q])_{\text{an}}} \text{Spec}(\mathbb{C}[P])_{\text{an}}, P^n) \). Then the natural morphism \( W \to V \) is log smooth by the implication (2)\(\Rightarrow\)(1). Let \( \pi: U \to W \) be also the natural morphism. We claim that the following morphism

\[
\mathcal{O}_{W,\tilde{z}(x)} \otimes \mathbb{Z} (\mathbb{Z} \oplus Q^{\text{gp}}) \ni a \otimes m \mapsto a d \log m \in \Lambda_{W/V,\tilde{z}(x)}^1 \quad (a \in \mathcal{O}_{W,\tilde{z}(x)}, m \in P)
\]
of \( \mathcal{O}_{W,\tilde{z}(x)} \)-modules is an isomorphism. (For a homomorphism \( R \to S \) of commutative monoids with unit elements and for a point \( y \) of \( \text{Spec}(\mathbb{C}[S])_{\text{an}} \), I do not know whether

\[
\Lambda_{\text{Spec}(\mathbb{C}[S])_{\text{an}} , S^m}/(\text{Spec}(\mathbb{C}[R])_{\text{an}}, R^n) \otimes \mathbb{Z} (S^{\text{gp}}/\text{Im}(R^{\text{gp}} \to S^{\text{gp}})) \]

because I do not know whether there exists a derivation

\[
\mathcal{O}_{\text{Spec}(\mathbb{C}[S])_{\text{an}}, y} \to \mathcal{O}_{\text{Spec}(\mathbb{C}[S])_{\text{an}}, y} \otimes \mathbb{Z} (S^{\text{gp}}/\text{Im}(R^{\text{gp}} \to S^{\text{gp}}))
\]

5
which extends the algebraic derivation

$$\mathbb{C}[S] \ni m \mapsto m \otimes m \in \mathbb{C}[S] \otimes \mathbb{Z} (S^{\text{sp}}/\text{Im}(R^{\text{sp}} \to S^{\text{sp}})) \quad (m \in S).$$

Set $P' := N' \oplus Q$ and $W' := (V \times \text{Spec}(\mathbb{C}[Q])_{an} \text{Spec}(\mathbb{C}[P'])_{an}, P'^{\sigma})$. Then the natural morphism $P'^{\text{sp}} \otimes \mathbb{Q} \to P^{\text{sp}} \otimes \mathbb{Q}$ is an isomorphism. Hence the natural morphism $p: W \to W'$ is log etale by the implication (2) $\Rightarrow$ (1). Furthermore, Spec$(\mathbb{C}[P'])_{an} = (A^1_{\mathbb{C}}, (N \ni 1 \mapsto z \in O_{A^1_{\mathbb{C}}})^r \times (\text{Spec}(\mathbb{C}[Q])_{an}, Q^{\sigma})$, where $A^1_{\mathbb{C}} (= \mathbb{C})$ is the line and $z$ is the holomorphic function $id: C \to C$. Hence $A^1_{W'/V, \eta(x)} \cong O_{W'/V, \eta(x)} \otimes \mathbb{Z}$ $(P'^{\text{sp}}/Q^{\text{sp}})$. As in the algebraic case ([K (3.12)]), for morphisms $g: X \to Y$ and $h: Y \to Z$ of fine log analytic spaces over $\mathbb{C}$, we have the following exact sequence

$$(2.1.2) \quad g^*(A^1_{Y/Z}) \to A^1_{X/Z} \to A^1_{X/Y} \to 0,$$

As in the algebraic case ([K (3.12)]), as to the following conditions (i) and (ii),

(i) $g$ is log smooth (resp. log etale),

(ii) $g^*(A^1_{Y/Z})$ is a local direct factor of $A^1_{X/Z}$ (resp. $g^*(A^1_{Y/Z}) \sim A^1_{X/Z}$), the following implications hold: (i) $\Rightarrow$ (ii); if $h \circ g$ is log smooth, then (ii) $\Rightarrow$ (i). Since $p$ is log etale, $A^1_{W'/V, \eta(x)} = p^*(A^1_{W'/V, \eta(x)}) = O_{W', \eta(x)} \otimes (P'^{\text{sp}}/Q^{\text{sp}})$. Therefore we have proved that the morphism (2.1.1) is an isomorphism.

Now it is clear that $A^1_{W'/V, x} = \pi^*(A^1_{W/V, \eta(x)})$ and consequently the morphism $U \to W$ is log etale by the implication (ii) $\Rightarrow$ (i). Since $\mathcal{M}_U$ is the pull-back of $\mathcal{M}_W$, the morphism $(U, O_U) \to (W, O_W)$ is etale in our sense. The rest that we have to prove is that the morphism $\tilde{U} \to \tilde{W}$ is locally isomorphic, which follows from the following lemma (2.2) (2) whose proof is not trivial. (Though the following lemma may be well-known (see [EGA IV-4] and [SGA 1] for the algebraic case), we give the proof of it because we cannot find an appropriate reference in the analytic case.)

**Lemma 2.2.** Let the notations be as above. Assume that the log analytic spaces $T_0$, $T$, $U$ and $V$ are trivial. Then the following hold:

(1) Assume that $f$ is smooth (resp. log smooth in our sense). Then $f$ is flat.

(2) The morphism $f$ is etale (resp. log etale in our sense) if and only if $f$ is locally isomorphic as a morphism of analytic spaces.

**Proof.** (1): (The proof of (1) is similar to that in [EGA IV-4, 17.5.1].) By the proof of (2.1.1), $\Omega^1_{U/V}$ is a locally projective $O_U$-module of finite type. Let $x$ be a point of $U$ and let $y \in V$ be the image of $x$ by $f$. Let $\mathbb{C}\{u_1, \ldots, u_n\}$ be the ring of convergent series in $n$-variables over $\mathbb{C}$. Then it is well-known that $\mathbb{C}\{u_1, \ldots, u_n\}$ $(n \in \mathbb{N})$ is a noetherian regular local ring. By the definition of an analytic space over $\mathbb{C}$, there exist nonnegative integers $m$ and $n$ such that $O_{U, x} = \mathbb{C}\{u_1, \ldots, u_m\}/I_x$ and $O_{V, y} = \mathbb{C}\{v_1, \ldots, v_n\}/I_y$ for some ideals $I_x$ and $I_y$ of $\mathbb{C}\{u_1, \ldots, u_m\}$ and $\mathbb{C}\{v_1, \ldots, v_n\}$, respectively. The morphism $O_{V, y} \to O_{U, x}$ induces a surjection

$$p: \mathbb{C}\{u_1, \ldots, u_m, v_1, \ldots, v_n\}/I_y \mathbb{C}\{u_1, \ldots, u_m, v_1, \ldots, v_n\} \to O_{U, x}.$$ 

Let $I$ be the kernel of this surjection. Set

$$O_{V, y}(\mathbb{Z}) := \mathbb{C}\{u_1, \ldots, u_m, v_1, \ldots, v_n\}/I_y \mathbb{C}\{u_1, \ldots, u_m, v_1, \ldots, v_n\}.$$
Then we claim that the natural morphism
\[ (2.2.1) \quad I/I^2 \ni t \mapsto dt \otimes 1 \in \Omega^1_{O_{V,y}/O_{V,y}} \otimes_{O_{V,y}} O_{U,x} \]
has a left inverse. Indeed, the following exact sequence
\[ 0 \longrightarrow I/I^2 \longrightarrow O_{V,y} \{ u \} - \longrightarrow O_{U,x} - \longrightarrow 0 \]
of \( O_{V,y} \)-modules is split since \( U \longrightarrow V \) is smooth and since \( O_{V,y} \{ u \} / I^2 \) defines a closed analytic space \( V \times (\mathbb{A}^n_{\mathbb{C}})_{\text{an}} \) in a neighborhood of \( (x,O) \). Let \( s: O_{U,x} - \longrightarrow O_{V,y} \{ u \} / I^2 \) be a section of \( p \). It is easy to check that \( (\text{id} - s \circ p): O_{V,y} \{ u \} / I^2 \longrightarrow I/I^2 \) is a derivation over \( O_{V,y} \). Since \( (\text{id} - s \circ p)|_{I/I^2} = \text{id}_{I/I^2} \), the derivation gives a desired left inverse of the morphism \( (2.2.1) \). Therefore, by [EGA IV-1, Chapitre 0 (19.1.12)], there exist a generator \( \{ f_j \}_{j \in J} \) of finite elements of \( I \) which generate \( I/I^2 \) and a subset \( \{ u_j \}_{j \in J} \subset \{ u_1, \ldots, u_m \} \) such that \( \det(\partial f_i/\partial u_j)_{i,j \in J} \in \mathfrak{m} \), where \( \mathfrak{m} \) is the maximal ideal of \( O_{V,y} \{ u \} \).

Remark 2.3. As in the algebraic case [K, (3.6)], we can replace (a) in (2.1) by the following condition (a):'

(a') the morphism \( Q^\text{sp} \rightarrow P^\text{sp} \) is injective (resp. and the morphism \( Q^\text{sp} \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow P^\text{sp} \otimes_{\mathbb{Z}} \mathbb{Q} \) is surjective).

As in [Nak1, (2.1.2) (iii)], we can define a Kummer morphism of fs log analytic spaces over \( \mathbb{C} \). The composite morphisms of two Kummer morphisms is of Kummer type.

Proposition 2.4 (cf. [V, (1.2)])]. Let \( f: U \longrightarrow V \) be a morphism of fine log analytic spaces over \( \mathbb{C} \). Then the following conditions (1) and (2) are equivalent:

1. \( f \) is log etale and of Kummer type.
2. There exists a local chart \( Q \rightarrow P \) of \( f \) satisfying the following conditions (a), (b) in (2.1):
   (a) \( P \) and \( Q \) are saturated.
   (b) The morphism \( Q \rightarrow P \) is injective and there exists a positive integer \( n \) such that \( P^n \subset \text{Im}(Q \rightarrow P) \).

Proof. By the proof of (2.1) and by the analytic analogue of [K, (2.10)], we easily obtain (2.4).
**Corollary 2.5.** For a morphism \( f: U \rightarrow V \) of fs log analytic spaces over \( \mathbb{C} \), \( f \) is log etale and of Kummer type if and only if \( f \) is log etale and exact. Consequently, the log etaleness of Kummer type is stable under the base change in the category of fs log analytic spaces over \( \mathbb{C} \).

**Proof.** (2.5) immediately follows from (2.4).

Let \( Y \) be an fs log analytic space over \( \mathbb{C} \). By (2.5), we obtain a topos \( \widetilde{Y}_{\text{et}}^{\text{log}} \) which is the obvious analogue of the log etale topos of an fs log scheme ([Nak1 (2.2)]). In particular, for the trivial log analytic space \((Y, \mathcal{O}_Y^*)\), we obtain the topos \( \widetilde{O}_{\text{et}}^{\text{log}} \). We call \( \widetilde{Y}_{\text{et}}^{\text{log}} \) the (analytic) log etale topos of \( Y \). As in [Nak1 (2.5)], we obtain the notion of the log geometric point of the topos \( \widetilde{Y}_{\text{et}}^{\text{log}} \).

By using the local description of a Kummer log etale morphism \( f: U \rightarrow V \) of fs log analytic spaces over \( \mathbb{C} \) (2.4) and using [KN (1.3) (3)] and [KN (1.2.1.1)], the associated morphism \( f^{\log}: U^{\log} \rightarrow V^{\log} \) is a local isomorphism of topological spaces by the same proof as that of [KN (2.2)]. Hence we have a natural morphism

\[
(2.5.1) \quad \beta: \widetilde{Y}_{\text{et}}^{\text{log}} \rightarrow \widetilde{Y}_{\text{et}}^{\text{log}}
\]

of topoi. We often denote \( \beta \) by \( \beta_Y \). We also have a natural morphism

\[
(2.5.2) \quad \epsilon_{\text{an}}: \widetilde{Y}_{\text{et}}^{\text{log}} \rightarrow \widetilde{O}_{\text{et}}^{\text{log}}
\]

of topoi. We sometimes denote \( \epsilon_{\text{an}} \) by \( \epsilon_Y \). We have the following commutative diagram

\[
\begin{array}{ccc}
\widetilde{Y}_{\text{et}}^{\text{log}} & \xrightarrow{\beta_Y} & \widetilde{Y}_{\text{et}}^{\text{log}} \\
\epsilon_{\text{an}} \downarrow & & \downarrow \epsilon_{\text{an}} \\
\widetilde{Y} & \xrightarrow{\beta_Y} & \widetilde{O}_{\text{et}}^{\text{log}}
\end{array}
\]

(2.5.3)

The direct image \( \beta_Y^* \) is an exact functor from the category of abelian sheaves in \( \widetilde{Y} \) to the category of abelian sheaves in \( \widetilde{Y}_{\text{et}}^{\text{log}} \) by [M III (3.3)] and commutes with the tensor product over \( \mathbb{Z} \). It is trivial to check that \( R\beta_Y^* \beta_Y^{-1} \sim id \). More generally,

\[
(2.5.4) \quad \beta_Y^* \beta_Y^{-1} = id \quad \text{and} \quad \beta_Y^{-1} \beta_Y^* = id
\]

on the category of sheaves of sets in \( \widetilde{Y}_{\text{et}}^{\text{log}} \) and \( \widetilde{Y} \), respectively. Let \( \mathcal{M}_{Y,\text{log}} \) be a sheaf of monoids in \( \widetilde{Y}_{\text{et}}^{\text{log}} \) which is associated to the presheaf \( U \mapsto \Gamma(U, \mathcal{M}_U) \) \((U \in Y_{\text{et}}^{\text{log}})\). Let \( \mathcal{O}_{Y,\text{log}} \) be the structure sheaf in \( \widetilde{Y}_{\text{et}}^{\text{log}} \). Then we have a natural commutative diagram

\[
\begin{array}{ccc}
\epsilon_{\text{an}}^{-1} \beta_Y^* (\mathcal{M}_Y) & \xrightarrow{} & \mathcal{M}_{Y,\text{log}} \\
\downarrow & & \downarrow \\
\epsilon_{\text{an}}^{-1} \beta_Y^* (\mathcal{O}_Y) & \xrightarrow{} & \mathcal{O}_{Y,\text{log}}.
\end{array}
\]

(2.5.5)

Denote \( (\mathcal{M}_{Y,\text{log}})^{\text{sp}} \) simply by \( \mathcal{M}_{Y,\text{log}}^{\text{sp}} \).
3 Proof of (1.2) (1)

In this section we give the proof of (1.2) (1).

Let the notations be as in [2]. The following is an analogue of [KN] (2.3):

Lemma 3.1 (Analytic log Kummer sequence). For a positive integer \( m \), the following sequence

\[
\begin{align*}
0 & \longrightarrow (\mathbb{Z}/m)(1) \longrightarrow \mathcal{M}^\text{gp}_{Y, \log}^\text{m,x} \longrightarrow \mathcal{M}^\text{gp}_{Y, \log} \longrightarrow 0
\end{align*}
\]

is exact in \( \overset{\circ}{Y} \text{et} \).

Proof. The obvious analytic analogue of the proof of [KN] (2.3) works. \( \square \)

The natural morphism \( \epsilon_\text{an}^{-1} \beta_{Y_*} (\mathcal{M}_Y) \longrightarrow \mathcal{M}_{Y, \log} \) induces a morphism

\[
\epsilon_\text{cl}^{-1}(\mathcal{M}_Y) \longrightarrow \beta_{Y_*}^{-1}(\mathcal{M}_{Y, \log})
\]

by (2.5.3) and (2.5.4).

Proposition 3.2. Let \( m \) be a positive integer. Let \( E \) be an \( m \)-torsion abelian sheaf in \( \overset{\circ}{Y} \text{et} \). Then the canonical morphism

\[
\begin{align*}
\bigwedge^k (\mathcal{M}_{Y, \log}^\text{gp}/\mathcal{O}_Y^\text{gp}) \otimes_{\mathbb{Z}/m} (-k) \otimes_{\mathbb{Z}} E \longrightarrow \epsilon_\text{cl}^{-1} E \end{align*}
\]

is an isomorphism.

Proof. The proof is the same as that of [KN] (1.5)]. \( \square \)

Proposition 3.3. Let \( m \) be a positive integer. Let \( E \) be an \( m \)-torsion abelian sheaf in \( \overset{\circ}{Y} \text{et} \). Then there exists a canonical isomorphism

\[
\begin{align*}
\beta_{Y_*} \big( \bigwedge^k (\mathcal{M}_{Y, \log}^\text{gp}/\mathcal{O}_Y^\text{gp}) \otimes_{\mathbb{Z}/m} (-k) \otimes_{\mathbb{Z}} E \big) \cong \epsilon_\text{cl}^{-1} E \end{align*}
\]

(3.3.1) \( k \in \mathbb{Z}_{\geq 0} \).

Proof. As we said before, we have the notion of the log geometric point of the topos \( \overset{\circ}{Y} \text{et} \). By using the analytic log Kummer sequence (3.1.1), the obvious analytic analogue of the proof of [KN] (2.4)] works. \( \square \)

We say that a log analytic space \( U \) over \( \mathbb{C} \) is affine if \( \overset{\circ}{U} \) is isomorphic to a closed analytic space of a polydisk over \( \mathbb{C} \).

The following is an analytic analogue of an algebraic constructible sheaf in a log etale topos in [Nak1] (3.3) 8.

Definition 3.4. (1) Let \( f : Z \longrightarrow Y \) be a morphism of fs log analytic spaces over \( \mathbb{C} \). We say that \( f \) is of log finite type if, for any affine open log analytic space \( U \) of \( Y \) which has a global chart \( P \) of \( \mathcal{M}_U \), \( f^{-1}(U) \) is the union of a finite number of open log analytic spaces \( V_i \)'s of \( Z \) which have global charts \( P \longrightarrow Q_i \) of \( f|_{V_i} : V_i \longrightarrow U \).

(2) Let \( A \) be a commutative ring with unit element. Let \( K \) be a sheaf of \( A \)-modules in \( \overset{\circ}{Y} \text{et} \). We say that \( K \) is constructible if, for any affine open log analytic subspace \( U \) of \( Y \), there exist objects \( V \) and \( W \) of \( U_{\text{et}} \) such that the structural morphisms \( V \longrightarrow U \) and \( W \longrightarrow U \) are of log finite type and such that \( K|_U \) is isomorphic to the cokernel of a morphism \( A_{V,U} \longrightarrow A_{W,U} \), where \( A_{V,U} := j_{V!*}(A_V) \) and \( A_{W,U} := j_{W!*}(A_W) \) for the structural morphisms \( j_V : V \longrightarrow U \) and \( j_W : W \longrightarrow U \), respectively.
Definition 3.5. Let $K$ be an abelian sheaf in $\tilde{Y}_{\text{et}}^{\log}$. We say that $K$ is classical (resp. classically constructible) if there exists an abelian sheaf (resp. constructible abelian sheaf) $E$ in $\tilde{Y}_{\text{et}}$ such that $K$ is isomorphic to $\epsilon^{-1}_Y(E)$. We say that $K$ is locally classical (resp. locally classically constructible) if, for any point $y$ of $Y$, there exists an object $U \to Y$ of $\tilde{Y}_{\text{et}}$ whose image in $Y$ contains $y$ and if $K|_U$ is classical (resp. classically constructible). Let $D^{+}_{\text{lcl-tor}}(\tilde{Y}_{\text{et}}^{\log})$ (resp. $D^{+}_{\text{lclc-tor}}(\tilde{Y}_{\text{et}}^{\log})$) be the derived category of bounded below complexes of abelian sheaves in $\tilde{Y}_{\text{et}}^{\log}$ whose cohomology sheaves are locally classical and torsion (resp. locally classically constructible and torsion). We obtain the similar notions in the algebraic case.

Proposition 3.6. Let $A$ be a commutative ring with unit element. Let $K$ be a constructible sheaf of $A$-modules in $\tilde{Y}_{\text{et}}^{\log}$. Then $K$ is locally classically constructible. In particular, $D^{+}_{\text{c-tor}}(\tilde{Y}_{\text{et}}^{\log}) \subset D^{+}_{\text{lclc-tor}}(\tilde{Y}_{\text{et}}^{\log})$. The obvious algebraic analogue also holds.

Proof. By using (2.4), the proof is the same as that of [KN, (2.5.2)].

Theorem 3.7. The adjunction morphism

$K^\bullet \to R\beta_{Y*}(\beta^{-1}_Y(K^\bullet))$

for an object $K^\bullet$ in $D^{+}_{\text{lcl-tor}}(\tilde{Y}_{\text{et}}^{\log})$ is an isomorphism. Consequently (2.7) (1) holds.

Proof. We may assume that $K^\bullet$ is a locally classical torsion abelian sheaf $K$ in $\tilde{Y}_{\text{et}}^{\log}$. By the lemma (3.8) (3) below, we may assume that $K$ is killed by a positive integer $m$. Because the question is local, we may furthermore assume that $K = \epsilon^{-1}_\text{cl}(E)$ for some $m$-torsion abelian sheaf $E$ in $\tilde{Y}_{\text{et}}$. By (3.8) and (3.2), we see that the adjunction morphism $id \to R\beta_{Y*}(\beta^{-1}_Y(K^\bullet))$ induces an isomorphism

$R\epsilon_\text{cl}(\epsilon^{-1}_\text{cl}(E)) \to R\beta_{Y*}(\beta^{-1}_Y(\epsilon^{-1}_\text{cl}(E)))$.

The same argument as that in [KN, p. 172, l. 1–7] tells us that (3.7.2) shows (3.7).

The following (3) is not included in a general theorem [SGA 4-2, VI (5.1)] since $\tilde{Y}_{\text{et}}^{\log}$ is not algebraic in general.

Lemma 3.8. (1) Let $\{F_\lambda\}_{\lambda \in \Lambda}$ be an inductive system of abelian sheaves in $\tilde{Y}^{\log}$. Then

$\lim_{\lambda \in \Lambda} R\epsilon_\text{cl}(F_\lambda) = R\epsilon_\text{cl}(\lim_{\lambda \in \Lambda} F_\lambda)$.

(2) Let $\{F_\lambda\}_{\lambda \in \Lambda}$ be an inductive system of abelian sheaves in $\tilde{Y}_{\text{et}}^{\log}$. Then

$\lim_{\lambda \in \Lambda} R\epsilon_\text{cl}(F_\lambda) = R\epsilon_\text{cl}(\lim_{\lambda \in \Lambda} F_\lambda)$.

(3) Let $\{F_\lambda\}_{\lambda \in \Lambda}$ be an inductive system of abelian sheaves in $\tilde{Y}^{\log}$. Then

$\lim_{\lambda \in \Lambda} R\beta_{Y*}(F_\lambda) = R\beta_{Y*}(\lim_{\lambda \in \Lambda} F_\lambda)$. 

10
Proof. (1): By the proper base change theorem for locally compact spaces ([Go II (4.11.1)]), we may assume that \( Y \) is a point. Then \( Y^{\log} = (S^1)^r \) for some \( r \in \mathbb{N} \) ([KN (1.3)]). Since \( S^1 \) is compact, the cohomology of an abelian sheaf with compact support on \( Y^{\log} \) and the usual cohomology of an abelian sheaf on \( Y^{\log} \) are the same. Because to take the direct limit of abelian sheaves and to take the cohomology with compact support are commutative ([Go, II (4.11.2)], [Iv, III (5.1)]), we obtain (1).

(2): The question is local on \( Y_{et} \). Hence we may assume that \( Y \) has a global chart \( P \rightarrow O_Y \). In this case, we can calculate the higher direct image \( R^h\epsilon_{an}^* \) for some \( h \in \mathbb{N} \) by the sheafed version of the cohomology of the group Hom\((P^\delta, \hat{\mathbb{Z}}(1))\) as in the algebraic case in [Nak1, (4.7.1)]. Hence (2) follows.

(3): As in the argument in [KN, p. 172, l. 1–7], we have only to prove that

\[
R\epsilon_{an}^\ast \lim_{\lambda \in \Lambda} R\beta_{Y^\ast}(F_\lambda) = R\epsilon_{an}^\ast R\beta_{Y^\ast}(\lim_{\lambda \in \Lambda} F_\lambda).
\]

By (1) and (2) and by the quasi-equivalence of \( \beta_Y^\ast \), the left hand side of (3.8.4) is equal to

\[
\lim_{\lambda \in \Lambda} R\epsilon_{an}^\ast R\beta_{Y^\ast}(F_\lambda) = \lim_{\lambda \in \Lambda} R\beta_{Y^\ast}(\lim_{\lambda \in \Lambda} F_\lambda) = R\epsilon_{an}^\ast R\beta_{Y^\ast}(\lim_{\lambda \in \Lambda} F_\lambda).
\]

Because I do not know whether a log exponential sequence in \( Y_{et}^{\log} \) exists when \( M_Y \) is nontrivial, I do not know the answer of the following problem:

Problem 3.9. Let \( D^+_{lcl}(Y_{et}^{\log}) \) be the derived category of bounded below complexes of abelian sheaves in \( Y_{et}^{\log} \) whose cohomology sheaves are locally classical. Is the adjunction morphism (3.7.1) isomorphic for an object \( K^\bullet \) in \( D^+_{lcl}(Y_{et}^{\log}) \)?

4 Proof of (1.2) (2)

In this section we give a base change theorem below which is a generalization of (1.2) (2).

Let \( X \) be an fs log scheme over \( \mathbb{C} \) whose underlying scheme is locally of finite type over \( \mathbb{C} \). Let \( O_X \) and \( O_{X_{an}} \) be the structure sheaves in \( X_{et} \) and \( X_{an} \), respectively. Let \( M_X \) be the log structure in \( X_{et} \). Let \( \eta: X_{an} \rightarrow X_{et} \) be the natural morphism of topoi. Let \( \eta^!(M_X) \) be the log structure on \( X_{an} \) which is associated to the composite morphism \( \eta^{-1}(O_X) \rightarrow O_{X_{an}} \). As in [KN], we call the fs log analytic space \( X_{an} := (X_{an}, \eta^!(M_X)) \) the log analytic space associated to \( X \). We call \( \eta^!(M_X) \) the analytification of \( M_X \). Also, as in [KN], \( X_{an}^{\log} \) denotes the real blow up of \( X_{an} \).
Let $\eta_{\log}: \tilde{X}_{\log}^{\an} \to \tilde{X}_{\log}^{\et}$ be the natural morphism of topoi ([KN (2.1), (2.2)]). Then we have the following commutative diagram of topoi ([KN, p. 171]):

$$\begin{array}{ccc}
\tilde{X}_{\log}^{\an} & \xrightarrow{\eta_{\log}} & \tilde{X}_{\log}^{\et} \\
\epsilon_{\cl} \downarrow & & \downarrow \epsilon_{\et} \\
\tilde{X}_{\an}^{\cl} & \xrightarrow{\eta} & \tilde{X}_{\et}^{\cl}.
\end{array}$$

(4.0.1)

We sometimes denote $\epsilon_{\et}$ by $\epsilon_X$, and $\eta_{\log}$ and $\eta_{\et}$ by $\eta_{X,\log}$ and $\eta_X$, respectively.

**Lemma 4.1.** Let $f: U \to V$ be a morphism of fs log schemes over $\mathbb{C}$ whose underlying schemes are locally of finite type over $\mathbb{C}$. Let $f_{\an}: U_{\an} \to V_{\an}$ be the associated morphism of fs log analytic spaces over $\mathbb{C}$. If $f$ is log etale and of Kummer type, then $f_{\an}$ is so.

**Proof.** By the local descriptions of algebraic and analytic log etale morphisms of Kummer type ([K, (3.5)] (cf. [Nak1, p. 369], [V, (1.2)], [Z]), immediately follows.

Because a functor $U \mapsto \tilde{X}_{\log}^{\an}(U \in (X_{\log}^{\et}))$ defines a continuous functor $X_{\log}^{\et} \to (X_{\an}^{\log})_{\log}^{\et}$ by (4.1), we have a morphism

$$\eta_{\et}: (X_{\an}^{\log})_{\et} \to \tilde{X}_{\et}^{\log}$$

of topoi. We often denote $\eta_{\et}$ by $\eta_{X,\et}$. Then we have the following commutative diagram:

$$\begin{array}{ccc}
\tilde{X}_{\log}^{\an} & \xrightarrow{\beta_{X_{\an}}} & (X_{\an}^{\log})_{\et} \\
\epsilon_{\cl} \downarrow & & \downarrow \epsilon_{\et} \\
\tilde{X}_{\an}^{\cl} & \xrightarrow{\beta_{X_{\an}}} & (X_{\an}^{\log})_{\et} \\
\downarrow \tilde{\circ} & & \downarrow \tilde{\circ} \\
X_{\an}^{\cl} & \xrightarrow{\beta_{X_{\an}}} & (X_{\an}^{\log})_{\et} \\
\downarrow \tilde{\circ} & & \downarrow \tilde{\circ} \\
\tilde{X}_{\et}^{\log} & \xrightarrow{\tilde{\circ}} & \tilde{X}_{\et}^{\log}.
\end{array}$$

(4.1.2)

By the definition of $\eta_{\log}$ and $\eta$, we have two equalities:

$$\eta_{\log}^{\cl} = \eta_{X,\et} \circ \beta_{X_{\an}}, \quad \eta = \eta_{X,\et} \circ \beta_{X_{\an}}^{\et}.$$ 

(4.1.3)

**Proposition 4.2.** Let $A$ be a commutative ring with unit element. Assume that $\tilde{X}$ is quasi-compact and quasi-separated. Then, for a constructible sheaf $K$ of $A$-modules in $X_{\log}^{\et}$, $\eta^{-1}_{X,\et}(K)$ is a constructible sheaf of $A$-modules in $(X_{\an}^{\log})_{\et}$. Consequently $\eta^{-1}_{X,\et}$ induces a functor $D^+_{c,tor}(X_{\log}^{\et}) \to D^+_{c,tor}((X_{\an}^{\log})_{\et})$.

**Proof.** The proof is easy.

For a log scheme $X$ and a log analytic space $Y$ over $\mathbb{C}$, $D^+(X_{\log}^{\et})$ and $D^+(Y_{\log}^{\et})$ denote the derived categories of bounded below complexes of abelian sheaves in $X_{\log}^{\et}$ and $Y_{\log}^{\et}$, respectively.

**Lemma 4.3.** Let $g: Z \to W$ be a morphism of analytic spaces over $\mathbb{C}$. As in §2, $\tilde{Z}$ and $\tilde{W}$ denote $\tilde{Z}_{\cl}$ and $\tilde{W}_{\cl}$, respectively. Let $\beta_{\tilde{Z}}: \tilde{Z} \to \tilde{Z}_{\et}$ and $\beta_{W}: \tilde{W} \to \tilde{W}_{\et}$ be the morphisms of topoi defined in [Z] for the trivial log analytic spaces $Z$ and $W$. 

Since $\beta$ rem ([SGA 4-3, XVI (4.1)]) this complex is equal to (4.4.2).

Consider the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{Z} & \longrightarrow & \mathcal{W} \\
g\downarrow & & \downarrow g \\
\mathcal{Z} & \longrightarrow & \mathcal{W}
\end{array}
\]

By (4.3) we see that the target of the morphism (4.4.3) is an isomorphism. Hence it suffices to prove that the morphism

\[
(4.3.2) \quad \beta_{W, Rg}(G) \longrightarrow \beta_{W, Rg_{\text{cl}}(\beta^{-1}_Z(G))}
\]

is an isomorphism. The left hand side on (4.3.2) is equal to $Rg_*(G)$. On the other hand, the target of the morphism (4.3.2) is equal to

\[
R\beta_{W, Rg_{\text{cl}}(\beta^{-1}_Z(G))} = Rg_*(\beta_{Z, (\beta^{-1}_Z(G))}) = Rg_*(G).
\]

Now we complete the proof.

The following is a variant of Artin-Grothendieck's base change theorem ([SGA 4-3 XVI (4.1)]).

**Corollary 4.4.** Let $g: Z \longrightarrow W$ be a morphism of schemes which are locally of finite type over $C$. Let $\eta_{\mathcal{Z}, \mathcal{W}}: (\mathcal{Z}_{\text{an}})_{\text{et}} \longrightarrow \mathcal{Z}_{\text{et}}$ and $\eta_{\mathcal{W}, \mathcal{W}}: (\mathcal{W}_{\text{an}})_{\text{et}} \longrightarrow \mathcal{W}_{\text{et}}$ be the morphisms of topoi defined in (4.3.1) for the trivial log schemes $\mathcal{Z}$ and $\mathcal{W}$, respectively. Let $g_{\text{an}}: (\mathcal{Z}_{\text{an}})_{\text{et}} \longrightarrow (\mathcal{W}_{\text{an}})_{\text{et}}$ and $g: \mathcal{Z} \longrightarrow \mathcal{W}$ be the induced morphisms of topoi by $g: Z \longrightarrow W$. Assume that $g$ is of finite type. Let $K^\bullet$ be an object of $D^+_C(\mathcal{Z}_{\text{et}})$. Then the following base change morphism

\[
(4.4.1) \quad \eta_{\mathcal{W}, \mathcal{W}}^{-1} Rg_*(K^\bullet) \longrightarrow Rg_{\text{an}}(\eta_{\mathcal{Z}, \mathcal{W}}^{-1}(K^\bullet)).
\]

is an isomorphism.

**Proof.** We may assume that $K^\bullet$ is a constructible torsion abelian sheaf $K$ in $\mathcal{Z}_{\text{et}}$. Consider the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{Z}_{\text{an}} & \longrightarrow & (\mathcal{Z}_{\text{an}})_{\text{et}} \\
(g_{\text{an}})_{\text{et}} \downarrow & & \downarrow g \\
\mathcal{W}_{\text{an}} & \longrightarrow & (\mathcal{W}_{\text{an}})_{\text{et}}
\end{array}
\]

Since $\beta^{-1}_{W, \text{an}}$ is exact, it suffices to prove that the morphism

\[
(4.4.3) \quad \beta^{-1}_{W, \text{an}} \eta^{-1}_{W, \text{et}} Rg_*(K) \longrightarrow \beta^{-1}_{W, \text{an}} Rg_{\text{an}}(\eta_{\mathcal{Z}, \mathcal{W}}^{-1}(K))
\]

is an isomorphism. By (4.3) we see that the target of the morphism (4.4.3) is equal to $R(g_{\text{an}})_{\text{cl}}((\eta_{\mathcal{Z}, \mathcal{W}}^{-1}(K))$. By Artin-Grothendieck's base change theorem ([SGA 4-3 XVI (4.1)]), this complex is equal to $(\eta_{\mathcal{W}, \mathcal{W}}\beta^{-1}_{W, \text{an}}) Rg_*(K)$. Now it is clear that the morphism (4.4.3) is an isomorphism.
**Theorem 3.5.1** (in our proof, we do not need to assume that $\eta$ is an isomorphism. The obvious analogue for the analytic log etale topoi of fs log analytic spaces over $\mathbb{C}$ also holds.

**Proof.** We may assume that $E^\bullet$ is an abelian sheaf $E$ in $\widetilde{X}_{et}^\log$. Because the question is local on $Y$, we may assume that $Y$ has a global chart $P$, where $P$ is an fs monoid. Since $f$ is strict, $X$ also has a global chart $P$. Set $I(X) := \text{Hom}_{\text{gp}}(P_{\text{rep}}, \mathbb{Z}(1))$ as in [Nak1, p. 376]. Then, by [Nak1, (4.6.1)], the category of abelian sheaves in $\widetilde{X}_{et}^\log$ is equivalent to the category of $I(X)$-modules in $\widetilde{X}_{et}$. The sheaf $\epsilon_X^{-1}(E)$ corresponds to $E$ with trivial $I(X)$-action ([Nak1 (4.7 (i))]). Let $\epsilon_X^{-1}(E) \rightarrow \Gamma$ be an injective resolution of $\epsilon_X^{-1}(E)$. Because it is clear that $f^\bullet$ can be considered as an injective resolution of $E$, the morphism (4.5.1) is an isomorphism by [Nak1 (4.7 (ii))].

The proof for the analytic case is the same.

The following is a main result of this section.

**Theorem 4.6 (Base change theorem).** Let $f: X \rightarrow Y$ be a morphism of fs log schemes over $\mathbb{C}$ whose underlying schemes are locally of finite type over $\mathbb{C}$. Assume that $f$ is of finite type. Let $f_{\text{an}}: (\tilde{X}_{an})^\log_{\text{et}} \rightarrow (\tilde{Y}_{an})^\log_{\text{et}}$ be the associated morphism to $f$. Consider the following commutative diagram:

$$
\begin{array}{ccc}
\tilde{X}_{et}^\log & \xrightarrow{\eta_{X,et}} & \tilde{X}_{et}^\log \\
\downarrow f & & \downarrow f \\
\tilde{Y}_{et}^\log & \xrightarrow{\eta_{Y,et}} & \tilde{Y}_{et}^\log
\end{array}
$$

(4.6.1)

Let $K^\bullet$ be an object of $D^+_c(\text{tor}(X_{et}^\log))$. If $f$ is log injective ([Nak1 (5.5.1)]), then the following base change morphism

$$
\eta_{Y,et}^{-1}Rf_{\text{an}}^\circ(K^\bullet) \rightarrow Rf_{\text{an}}(\eta_{X,et}^{-1}(K^\bullet))
$$

(4.6.2)

is an isomorphism.

**Proof.** We may assume that $K^\bullet$ is a constructible torsion abelian sheaf $K$ in $\widetilde{X}_{et}^\log$. We decompose the morphism $f$ by the composite morphism $X \rightarrow (\tilde{X}, f^\circ(M_Y)) \rightarrow Y$. Set $Z := (\tilde{X}, f^\circ(M_Y))$ and let $g: Z \rightarrow Y$ be the morphism above.

First we claim that the base change morphism (4.6.2) for $g$ is an isomorphism. Indeed, let $\{Z_i\}_{i \in I}$ be a Kummer log etale covering of $Z$ such that the pull-backs of $K$ to $Z_i$ for all $i \in I$ are classically constructible and torison (5.6). Set $Z_0 := \coprod_{i \in I} Z_i$ and $Z_\bullet := \text{cosk}_0^\ast(Z_0) (\bullet \in \mathbb{N})$. By the cohomological descent in [SGA 4-2, $\text{V}^\text{bis}$] and by the same proof as that of the crystalline base change theorem [B, Chapitre V Théorème 3.5.1] (in our proof, we do not need to assume that $f$ is quasi-separated nor that the index set $I$ is finite because we do not need to consider the derived functor $\varprojlim$ in [B], we have only to prove that the base change morphism (4.6.2) for each morphism $Z_n \rightarrow Y (n \in \mathbb{N})$ is an isomorphism. (Here we have used the
Convention (3) implicitly in the cohomological descent.) Consequently we may assume that $K = \epsilon_Z^{-1}(G)$ for a constructible torsion abelian sheaf $G$ in $\tilde{Z}_{et}$. Then we have the following formulae

$$(4.6.3) \quad \eta_{Y,et}^{-1} Rg_*(\epsilon_Z^{-1}(G)) = \eta_{Y,et}^{-1} \epsilon_Y^{-1} R\eta_Y^0(G) = \epsilon_{Y,et}^{-1} \eta_{Y,et}^{-1} Rg_*(G)$$

$$= \epsilon_{Y,et}^{-1} R\eta_{et,an}(\epsilon_{Z,et}^{-1}(G)) = R\eta_{et,an}(\epsilon_{Z,et}^{-1}(G))$$

Here the numbers above equalities mean that the equalities follow from the statements numbered. Hence we have proved the claim.

Let $h: X \to Z$ be the morphism above. Secondly, we claim that the base change morphism (4.6.2) for $h$ is an isomorphism. In fact we claim that the morphism (4.6.2) for $h$ is an isomorphism for a bounded below complex of abelian sheaves in $\tilde{X}_{et}$. Indeed, let $z$ be a closed point of $\tilde{Z} (= \tilde{X})$. Because the problem is local on $\tilde{Z}$, we may assume that there exists a global chart $P_{\tilde{Z}} \to P_X$ of $h$, where $P_{\tilde{Z}}$ and $P_X$ are fs monoids. Set $I_X := \text{Hom}(P_{\tilde{Z}}^{et}, \tilde{Z}(1))$, $I_Z := \text{Hom}(P_{\tilde{Z}}^{et}, \tilde{Z}(1))$ and $I_h := \text{Ker}(I_X \to I_Z)$. Let $K$ be an abelian sheaf in $\tilde{X}_{et}$; $K$ corresponds to an abelian sheaf $\tilde{K}$ in $\tilde{X}_{et}$ with continuous $I_X$-action ([Nak1] (4.6.1)). Then, by [Nak1] (4.7.1), it is easy to check that $R^\theta h_*(K)_{z(\text{log})} = \text{Map}_{c,I_X/I_h}(I_Z, H^q(I_h, \tilde{K}_z))$. On the other hand, by the analytic analogues of [Nak1] (4.6.1) and [Nak1] (4.7.1), $R^\theta h_{an}(K)_{z(\text{log})} = \text{Map}_{c,I_X/I_h}(I_Z, H^q(I_h, \eta_{Z,et}^{-1}(\tilde{K}_{z_h}))) = \text{Map}_{c,I_X/I_h}(I_Z, H^q(I_h, \tilde{K}_z))$, where $z_h$ is the point of $\tilde{Z}_{an}$ corresponding to $z$.

Finally (4.6.4) follows from [Nak1] (5.5.2). Indeed, because $f$ is log injective, so is $h$. Hence [loc. cit.] tells us that $R^\theta h_*(K)$ is constructible, which permits us to use the base change theorem (4.6) for $g$ as follows:

$$(4.6.4) \quad Rf_{\eta_{et,an}}(\eta_{X,et}^{-1}(K)) = Rg_{an,\eta_{et,an}}(\eta_{X,et}^{-1}(K)) = Rg_{an}(\tilde{K}) = Rf_{\eta_{et}}(K)$$

Corollary 4.7. Let $K^\bullet$ be an object of $D^+_{c,\text{tor}}(X_{et})$. Then the adjunction morphism

$$K^\bullet \to R\eta_{et}^{-1}(K^\bullet)$$

is an isomorphism. Consequently (1.2) holds.

Proof. We may assume that $K^\bullet$ is a constructible torsion abelian sheaf $K$ in $\tilde{X}_{et}$. We may assume that $\tilde{X}$ is quasi-compact. By (4.6), for any object $U \in X_{et}$,

$$H^p((U_{an})_{et}^{\log}, \eta_{et}^{-1}(K[U])) = H^p(U_{et}^{\log}, K[U]) \quad (p \in \mathbb{N}).$$

Hence $\eta_{et}(\eta_{et}^{-1}(K)) = K$ and $R^p\eta_{et}^{-1}(K[U]) = 0$ for $p > 0$ (cf. [M] III (2.11) (a)).

Corollary 4.8 ([KN] (2.6)). Let $K^\bullet$ be an object of $D^+_{c,\text{tor}}(X_{et})$. Then the adjunction morphism

$$K^\bullet \to R\eta_{et}^{-1}(K^\bullet)$$

is an isomorphism. Consequently (1.1) holds.
Proof. (4.8.1) immediately follows from the following equality

\[ \eta_{et} \circ \beta, \]

and from (4.2), (3.6), (3.7) and (4.7). \( \square \)

Though we do not use the log proper base change theorem ([Nak1, (5.1)]) nor the following analytic log proper base change theorem in the proof of (4.6), we may use them (however we shall not use them in this paper).

**Theorem 4.9 (Analytic log proper base change theorem).** Let

\[
\begin{array}{ccc}
X_4 & \xrightarrow{g'} & X_1 \\
\downarrow f' & & \downarrow f \\
X_3 & \xrightarrow{g} & X_2
\end{array}
\]

be a cartesian diagram of fs log analytic spaces over \( \mathbb{C} \). Assume that \( g \) is strict. Let \( g' \) be a morphism of fs log analytic spaces over \( \mathbb{C} \). Consider the following commutative diagram

\[
\begin{array}{ccc}
\tilde{X}_4 & \xrightarrow{\tilde{g}'} & \tilde{X}_1 \\
\downarrow \tilde{f}' & & \downarrow \tilde{f} \\
\tilde{X}_3 & \xrightarrow{\tilde{g}} & \tilde{X}_2
\end{array}
\]

Then the example [KN, (2.7)] tells us that the base change morphism

\[ \beta_W^{-1} Rg_* (K) \rightarrow Rg_*^\log (\beta_Z^{-1}(K)) \]

for a constructible torsion abelian sheaf \( K \) in \( \tilde{Z}_{et}^{\log} \) is not an isomorphism in general.

**Remark 4.10.** (1) Let \( g: Z \rightarrow W \) be a morphism of fs log analytic spaces over \( \mathbb{C} \). Consider the following commutative diagram

\[
\begin{array}{ccc}
\tilde{Z}_{et}^{\log} & \xrightarrow{\beta_Z} & \tilde{Z}_{et}^{\log} \\
\downarrow g^{\log} & & \downarrow g \\
\tilde{W}_{et}^{\log} & \xrightarrow{\beta_W} & \tilde{W}_{et}^{\log}
\end{array}
\]

Then the example [KN, (2.7)] tells us that the base change morphism

\[ \beta_W^{-1} Rg_* (K) \rightarrow Rg_*^\log (\beta_Z^{-1}(K)) \]
Let \( f : X \to Y \) be a morphism of fs log schemes whose underlying schemes are locally of finite type. The difference between the higher direct images of
\[
f^\log_{\text{an}} : X_{\text{an}}^\log \to Y_{\text{an}}^\log \quad \text{and} \quad f : X_{\et}^\log \to Y_{\et}^\log
\]
pointed out in [loc. cit.] arises from the difference between those of
\[
f^\log_{\text{an}} : X_{\text{an}}^\log \to Y_{\text{an}}^\log \quad \text{and} \quad f : (X_{\text{an}})_{\text{et}}^\log \to (Y_{\text{an}})_{\text{et}}^\log.
\]

(2) C. Nakayama has informed me that he and T. Kajiwara have proved that the base change morphism in \([KN, (2.7.1)]\) is an isomorphism if \( f \) is log injective. It is reasonable to expect that the obvious analytic analogue of this also holds.

(3) I expect that the base change morphism (4.6.2) is an isomorphism even if \( f \) is not necessarily log injective.

(4) Because we introduce the topos \( Y_{\text{et}}^\log \) in §2, we can give a proof of \([KN, (2.6)]\) in (4.8) by using Artin-Grothendieck’s base change theorem ([SGA 4-3, XVI (4.1)]) in a full form and only by using the analytic log Kummer sequence; we have not used the algebraic log Kummer sequence nor the log exponential sequence in \([KN]\) to obtain \([KN, (2.6)]\), though we have used the same proof as that of \([KN, (1.5)]\) in the proof of (3.2).

We conclude this section by giving a logarithmic version of Grauert-Remmert’s theorem ([GR], [SGA 4-3, XI (4.3)]), which is of independent interest.

Let \( f : X \to Y \) be a morphism of fine log schemes (resp. fine log analytic spaces) over \( \mathbb{C} \). We say that \( f \) is finite if the morphism \( \circ f : \circ X \to \circ Y \) is finite.

Let \( M \) be a finite abelian group. Let \( X \) be an fs log scheme over \( \mathbb{C} \) whose underlying scheme is locally of finite type over \( \mathbb{C} \). Then, by (1.2) (2), we have
\[
H^1(X_{\text{et}}^\log, M) = H^1((X_{\text{an}})_{\text{et}}^\log, M),
\]
which makes us expect the following (4.10.2) immediately follows from the following:

**Theorem 4.11 (Log Grauert-Remmert’s theorem).** Let \( X \) be a fine log scheme over \( \mathbb{C} \) whose underlying scheme is locally of finite type over \( \mathbb{C} \). Let \( \text{Fet}(X) \) be the category of finite log etale coverings over \( X \) and \( \text{Fet}(X_{\text{an}}) \) the similar category for \( X_{\text{an}} \). Then the functor of analytifications
\[
\text{Fet}(X) \ni X' \mapsto X'_{\text{an}} \in \text{Fet}(X_{\text{an}})
\]
gives an equivalence of categories.

Assume furthermore that \( \mathcal{M}_X \) is saturated. Let \( \text{Fket}(X) \) be the category of finite log etale coverings of Kummer types over \( X \) and \( \text{Fket}(X_{\text{an}}) \) the similar category for \( X_{\text{an}} \). Then the functor of analytifications
\[
\text{Fket}(X) \ni X' \mapsto X'_{\text{an}} \in \text{Fket}(X_{\text{an}})
\]
gives an equivalence of categories.

**Proof.** Because the proof for (1.1.2) is similar to that for (1.1.1), we give only the proof for (4.11.1).

Let \( X' \) and \( X'' \) be two objects of \( \text{Fet}(X) \). For simplicity of notation, set \( Y := X_{\text{an}}, Y' := X'_{\text{an}} \) and \( Y'' := X''_{\text{an}} \). Then it is easy to check that the natural map
\[
\text{Hom}_X(X', X'') \to \text{Hom}_Y(Y', Y'')
\]
is injective. Let \( h \) be an element of \( \text{Hom}_Y(Y', Y'') \). By \([SGA 4-3, XI (4.3)]\) the morphism \( \bar{h} : \bar{Y}' \to \bar{Y}'' \) is associated to a morphism \( \bar{g} : \bar{X}' \to \bar{X}'' \) of schemes over
be two charts, where \( \tilde{U}' \) and \( \tilde{U}'' \) are etale neighborhoods of \( \tilde{\pi} \) and \( \tilde{g}(x) \). Then we have two composite surjective homomorphisms

\[
P'^{\mathrm{gp}} \to \mathcal{M}_{X'}^{\mathrm{gp}} / \mathcal{O}_{X',\pi} \xrightarrow{\sim} \mathcal{M}_{X'}^{\mathrm{gp}} / \mathcal{O}_{X',\tilde{\pi}}
\]

and

\[
P''^{\mathrm{gp}} \to \mathcal{M}_{X''}^{\mathrm{gp}} / \mathcal{O}_{X'',\tilde{\pi}} \xrightarrow{\sim} \mathcal{M}_{X''}^{\mathrm{gp}} / \mathcal{O}_{X'',\tilde{\pi}}.
\]

Let \( K' \subset P'^{\mathrm{gp}} \) and \( K'' \subset P''^{\mathrm{gp}} \) be the kernels of the homomorphisms above and let \( P'_1 \subset P'^{\mathrm{gp}} / K' \) and \( P''_1 \subset P''^{\mathrm{gp}} / K'' \) be the inverse images of \( \mathcal{M}_{Y',\pi} / \mathcal{O}_{Y',\pi} \) and \( \mathcal{M}_{Y'',\tilde{\pi}} / \mathcal{O}_{Y'',\tilde{\pi}} \), respectively. Then \( P'_1 \) and \( P''_1 \) are also the inverse images of \( \mathcal{M}_{X',\pi} / \mathcal{O}_{X',\pi} \) and \( \mathcal{M}_{X'',\tilde{\pi}} / \mathcal{O}_{X'',\tilde{\pi}} \). The morphism \( h \) induces a homomorphism \( P'_1 \to P'_1 \) and this gives a local chart of \( h \) at \( \tilde{\pi} \) and \( \tilde{h}(x) \). By \([\mathbf{K}] \ (2.10)\), \( (P'_1)^{\downarrow}_{\tilde{U}'} = \mathcal{M}_{Y',\pi}^{\downarrow}_{\tilde{U}'_1} \) and \( (P''_1)^{\downarrow}_{\tilde{U}''_1} = \mathcal{M}_{Y'',\tilde{\pi}}^{\downarrow}_{\tilde{U}''_1} \) for some etale neighborhoods \( \tilde{U}'_1 \) and \( \tilde{U}''_1 \) of \( \tilde{\pi} \) and \( \tilde{g}(x) \), respectively, with a morphism \( \tilde{g}_1 : \tilde{U}'_1 \to \tilde{U}''_1 \) over \( \tilde{g} \). Hence we have a morphism \( \tilde{g}_1^{-1}(\mathcal{M}_{Y',\pi}^{\downarrow}_{\tilde{U}'_1}) \to \mathcal{M}_{Y'',\tilde{\pi}}^{\downarrow}_{\tilde{U}''_1} \). Therefore the map \([\mathbf{K}] \ (2.11)\) is etale-locally surjective. In fact, it is surjective since it is injective. Now we have proved that the functor \([\mathbf{K}] \ (2.11)\) is fully faithful.

The rest is to prove that the functor \([\mathbf{K}] \ (2.11)\) is essentially surjective. Let \( Z \) be an object of \( \text{Fet}(Y) \). Since the problem is local, we may assume that there exists a chart \( P \to Q \) of the morphism \( Z \to Y \) such that the induced morphism \( P^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q} \to Q^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q} \) is an isomorphism and we may assume that the induced morphism \( \tilde{Z} \to Y \times_{\text{Spec}(\mathbb{C}[P])_{\text{an}}} \text{Spec}(\mathbb{C}[Q])_{\text{an}} \) is etale \((2.1)\). Then this morphism is finite. Indeed, by the assumption, the composite morphism \( \tilde{Z} \to Y \times_{\text{Spec}(\mathbb{C}[P])_{\text{an}}} \text{Spec}(\mathbb{C}[Q])_{\text{an}} \to Y \) is finite. Furthermore, the morphism \( Y \times_{\text{Spec}(\mathbb{C}[P])_{\text{an}}} \text{Spec}(\mathbb{C}[Q])_{\text{an}} \to Y \) is separated by the stability of the separation under the base change in the algebraic case and by the GAGA of the separation \( ([\mathbf{SGA}] \ XII \ (3.1))\). Hence, by the obvious analytic analogue of \( ([\mathbf{EGA}] \ V \ (6.1.5)) \), we see that the morphism \( \tilde{Z} \to \tilde{Y} \times_{\text{Spec}(\mathbb{C}[P])_{\text{an}}} \text{Spec}(\mathbb{C}[Q])_{\text{an}} \) is finite. By \((2.2)\) \((2)\), \( \tilde{Z} \) is a finite covering space of \( \tilde{Y} \times_{\text{Spec}(\mathbb{C}[P])_{\text{an}}} \text{Spec}(\mathbb{C}[Q])_{\text{an}} \) as topological spaces. Now, by \( ([\mathbf{SGA}] \ IV \ (4.3)) \), there exists a finite etale covering \( \tilde{X}' \to \tilde{X} \times_{\text{Spec}(\mathbb{C}[P])_{\text{an}}} \text{Spec}(\mathbb{C}[Q])_{\text{an}} \) such that \( \tilde{X}'_{\text{an}} = \tilde{Z} \). Endow \( \tilde{X}' \) with the log structure associated to the morphism \( Q \to \mathcal{O}_{\tilde{X}} \), and let \( X' \) be the resulting log scheme. Then \( X'_{\text{an}} = Z \).

\[ \square \]

**Remark 4.12.** Let \( X \) be an fs log scheme over \( \mathbb{C} \) whose underlying scheme is locally of finite type over \( \mathbb{C} \). Though a finite abelian Galois covering of \( X_{\text{log}} \) as topological spaces is obtained by a finite abelian Galois covering of \( X \) by \([\mathbf{L}] \ (4.4)\), I know nothing about the nonabelian case.

### 5 Analytic vs. algebraic log Kummer sequences

The aim in this section is to give a commutative diagram \([\mathbf{Z}] \ (2.1)\) below which compares the calculations of certain higher direct images by the use of the analytic and algebraic
log Kummer sequences.

Let $X$ be an fs log scheme over $\mathbb{C}$ whose underlying scheme $\hat{X}$ is locally of finite type over $\mathbb{C}$.

Let $\mathcal{M}_{X, \log}$ be a sheaf of monoids which is associated to the presheaf $U \mapsto \Gamma(U, \mathcal{M}_U) \ (U \in X^\text{et}^\log)$.

Let $\mathcal{O}_{X, \log}$ be the structure sheaf in $\tilde{X}_{\text{et}}^\log$. Then we have natural commutative diagrams

\[
\begin{array}{ccc}
\epsilon_\text{et}^{-1}(\mathcal{M}_X) & \longrightarrow & \mathcal{M}_{X, \log} \\
\downarrow & & \downarrow \\
\epsilon_\text{et}^{-1}(\mathcal{O}_X) & \longrightarrow & \mathcal{O}_{X, \log}
\end{array}
\]

and

\[
\begin{array}{ccc}
\eta_\text{et}^{-1}(\mathcal{M}_{X, \log}) & \longrightarrow & \mathcal{M}_{\text{an}, \log} \\
\downarrow & & \downarrow \\
\eta_\text{et}^{-1}(\mathcal{O}_{X, \log}) & \longrightarrow & \mathcal{O}_{\text{an}, \log}
\end{array}
\]

Let $\eta^*_\text{et}(\mathcal{M}_{X, \log}) \in (\tilde{X}_{\text{an}, \log})^\log$ be the associated log structure to the composite morphism $\eta_\text{et}^{-1}(\mathcal{M}_{X, \log}) \longrightarrow \eta_\text{et}^{-1}(\mathcal{O}_{X, \log}) \longrightarrow \mathcal{O}_{\text{an}, \log}$. Let

\[
\begin{array}{ccc}
\eta_\text{et}^{-1}(\mathcal{M}_{X, \log}) & \longrightarrow & \eta^*_\text{et}(\mathcal{M}_{X, \log})
\end{array}
\]

be the natural morphism. Set

\[
\eta^\log*(\mathcal{M}_{X, \log}) := \beta_{X, \text{et}}^{-1}(\eta^*_\text{et}(\mathcal{M}_{X, \log}))
\]

by abuse of notation.

**Definition 5.1.** We call $\eta^*_\text{et}(\mathcal{M}_{X, \log})$ the *analytification* of $\mathcal{M}_{X, \log}$.

The upper horizontal morphism in (5.0.1) induces a morphism $\epsilon_\text{an}^{-1} \eta^{-1}_X(\mathcal{M}_X) \longrightarrow \eta^{-1}_\text{et}(\mathcal{M}_{X, \log})$. Composing this morphism with the morphism (5.0.3), we have a morphism

\[
\begin{array}{ccc}
\epsilon_\text{an}^{-1} \eta^{-1}_X(\mathcal{M}_X) & \longrightarrow & \eta^*_\text{et}(\mathcal{M}_{X, \log})
\end{array}
\]

In fact we have a natural morphism

\[
\begin{array}{ccc}
\epsilon_\text{an}^{-1} \eta^*_X(\mathcal{M}_X) & \longrightarrow & \eta^*_\text{et}(\mathcal{M}_{X, \log})
\end{array}
\]

By the upper morphism of (5.0.2), we have a natural morphism

\[
\begin{array}{ccc}
\eta_\text{et}(\mathcal{M}_{X, \log}) & \longrightarrow & \mathcal{M}_{\text{an}, \log}
\end{array}
\]

By the definition of $\eta^\log*(\mathcal{M}_{X, \log})$, by (5.0.3) and by (4.8.1), we have a natural morphism

\[
\begin{array}{ccc}
\eta^{-1}\log(\mathcal{M}_{X, \log}) & \longrightarrow & \eta^\log*(\mathcal{M}_{X, \log})
\end{array}
\]

19
By the definition of \( \eta^\log\ast(\mathcal{M}_{X, \log}) \), we have the following formula

\[(5.1.5) \quad \eta^\log\ast(\mathcal{M}_{X, \log})/\beta_{X_{an}}^{-1}(\mathcal{O}_{X_{an}, \log}) = \eta^\log\ast(\mathcal{M}_{X, \log}/\mathcal{O}_{X, \log}).\]

Pulling back the morphism \((5.1.11)\) by the functor \(\beta_{X_{an}}^{-1}\), we have a morphism

\[(5.1.6) \quad \epsilon_{cl}^{-1} \eta^{-1}(\mathcal{M}_X) \to \eta^\log\ast(\mathcal{M}_{X, \log}).\]

Let the notation be as in the beginning of [2]. Let \(\gamma_E: Y_{et} \to Y(\neq Y_{cl})\) be the natural morphism of topoi. Let \(\mathcal{O}_Y\) be the structure sheaf of the topos \(Y(\neq Y_{cl})\). Because there exists a natural morphism \(\epsilon_{an}^{-1} \eta^{-1}_E(\mathcal{O}_E) \to \mathcal{O}_{X_{an}, \log}\), the morphism \((5.1.6)\) induces a morphism

\[(5.1.7) \quad \epsilon_{cl}^{-1} \eta^\ast(\mathcal{M}_X) \to \eta^\log\ast(\mathcal{M}_{X, \log}).\]

Composing the morphism \(\beta_{X_{an}}^{-1}(5.1.3)\) with the morphism above, we obtain the following morphism

\[(5.1.8) \quad \epsilon_{cl}^{-1} \eta^\ast(\mathcal{M}_X) \to \beta_{X_{an}}^{-1}(\mathcal{M}_{X_{an}, \log}).\]

Let \(m\) be a positive integer. Recall the algebraic log Kummer sequence

\[(5.1.9) \quad 0 \to (\mathbb{Z}/m)(1) \to \mathcal{M}_{X, \log}^{gp} \xrightarrow{m} \mathcal{M}_{X, \log}^{gp} \to 0\]

in \(\tilde{X}_{et}^{log}\) ([KN (2.3)]). Then we have the following commutative diagram

\[(5.1.10) \quad \begin{array}{ccc}
0 & \to & (\mathbb{Z}/m)(1) \\
\phantom{0} & \phantom{\to} & \mathcal{M}_{X_{an}, \log}^{gp} \xrightarrow{m} \mathcal{M}_{X_{an}, \log}^{gp} \\
\phantom{0} & \phantom{\to} & (5.1.5) \phantom{\to} \\
\phantom{0} & \phantom{\to} & \mathcal{M}_{X_{an}, \log}^{gp} \\
0 & \to & (\mathbb{Z}/m)(1) \phantom{\to} \mathcal{M}_{X_{an}, \log}^{gp} \xrightarrow{m} \mathcal{M}_{X_{an}, \log}^{gp} \to 0 \\
\end{array}\]

of exact sequences (the exactness of the middle sequence is easy to check).

Let \(E\) be an \(m\)-torsion abelian sheaf in \(\tilde{X}_{et}\). Then Kato and Nakayama have proved that the log Kummer sequence \((5.1.9)\) gives the following canonical isomorphism ([KN (2.4)]):

\[(5.1.11) \quad \bigwedge^k(\mathcal{M}_{X, \log}^{gp}/\mathcal{O}_{X}^{\ast}) \otimes_{\mathbb{Z}} E(-k) \sim R^k_{et\ast}(\epsilon_{et}^{-1}(E)) \quad (k \in \mathbb{Z}_{\geq 0}).\]

Here, in \((5.1.11)\), we change the turn of the tensor product in [loc. cit.] because the cup product is usually taken by the left cup product of a fundamental section (see [RZ] for example).

Let \(K^{\bullet}\) be an object of \(D^+(\tilde{X}_{et}^{log})\). Then we have the following base change morphism

\[(5.1.12) \quad \eta^{-1}R_{et\ast}(K^{\bullet}) \to R_{et\ast}(\eta^\log\ast^{-1}(K^{\bullet})).\]
In particular, for a nonnegative integer \( k \), we have the following morphism
\[
(5.1.13) \quad \eta^{-1} R^k \epsilon_{et,*}(K^\bullet) \rightarrow R^k \epsilon_{cl,*}(\eta^{log,-1}(K^\bullet)).
\]
Hence we have a canonical morphism
\[
(5.1.14) \quad R^k \epsilon_{et,*}(K^\bullet) \rightarrow R \eta_* (R^k \epsilon_{cl,*}(\eta^{log,-1}(K^\bullet))).
\]

**Proposition 5.2.** Let \( m \) be a positive integer and let \( E \) be an \( m \)-torsion abelian sheaf in \( \tilde{X}_{et} \). Let \( k \) be a nonnegative integer. Then there exists the following commutative diagram
\[
(5.2.1) \quad R \eta_* (\bigwedge^k (\eta^*(\mathcal{M}_{X}^{\text{gp}})/\mathcal{O}_{X_{\text{ss}}}) \otimes_{Z} \eta^{-1}(E)(-k)) \xrightarrow{\text{R} \eta_* (\text{5.2.1})} R \eta_* (R^k \epsilon_{cl,*}(\epsilon_{cl}^{-1} \eta^{-1}(E))) \\
\bigwedge^k (\mathcal{M}_{X}^{\text{gp}}/\mathcal{O}_{X}) \otimes_{Z} E(-k) \xrightarrow{\sim} R^k \epsilon_{et,*}(\epsilon_{et}^{-1}(E)),
\]
where the left vertical morphism above is induced by the adjunction morphism \( \eta^{-1} \rightarrow R \eta_* \eta^{-1} \). Furthermore, if \( E \) is constructible, then the left vertical morphism is an isomorphism.

**Proof.** As to the commutativity of the diagram \textbf{5.2.1}, it suffices to prove that the following diagram is commutative:
\[
(5.2.2) \quad \bigwedge^k (\eta^*(\mathcal{M}_{X}^{\text{gp}})/\mathcal{O}_{X_{\text{ss}}}) \otimes_{Z} \eta^{-1}(E)(-k) \xrightarrow{\sim} R^k \epsilon_{cl,*}(\epsilon_{cl}^{-1} \eta^{-1}(E)) \\
\eta^{-1} (\bigwedge^k (\mathcal{M}_{X}^{\text{gp}}/\mathcal{O}_{X}) \otimes_{Z} E(-k)) \xrightarrow{\eta^{-1}} \eta^{-1} R^k \epsilon_{et,*}(\epsilon_{et}^{-1}(E)),
\]
Using \textbf{5.1.14}, we have the following commutative diagram of triangles
\[
(5.2.3) \quad R \epsilon_{cl,*}((\mathbb{Z}/m)(1)) \longrightarrow R \epsilon_{cl,*}(\beta^{-1}(\mathcal{M}^{\text{gp}}_{X_{\text{ss},\text{log}}})) \xrightarrow{\text{m} \times} R \epsilon_{cl,*}(\beta^{-1}(\mathcal{M}^{\text{gp}}_{X_{\text{ss},\text{log}}})) \xrightarrow{+1} \\
\bigwedge \quad \bigwedge \quad \bigwedge \\
R \epsilon_{cl,*}((\mathbb{Z}/m)(1)) \longrightarrow R \epsilon_{cl,*}(\eta^{log,-1}(\mathcal{M}^{\text{gp}}_{X_{\text{log}}})) \xrightarrow{\text{m} \times} R \epsilon_{cl,*}(\eta^{log,-1}(\mathcal{M}^{\text{gp}}_{X_{\text{log}}})) \xrightarrow{+1} \\
\eta^{-1} R \epsilon_{et,*}((\mathbb{Z}/m)(1)) \longrightarrow \eta^{-1} R \epsilon_{et,*}(\mathcal{M}^{\text{gp}}_{X_{\text{log}}}) \xrightarrow{\text{m} \times} \eta^{-1} R \epsilon_{et,*}(\mathcal{M}^{\text{gp}}_{X_{\text{log}}}) \xrightarrow{+1}
\]
(Here we have used the Convention \textbf{(2).} In particular, we have the commutativity of the diagram \textbf{5.2.2} for the case \( k = 1 \) and \( E = \mathbb{Z}/m \). We leave the reader to the detail of the rest of the proof of the commutativity of the diagram \textbf{5.2.2} because it is a routine work by using the Godement resolution of an abelian sheaf in a topos with enough points and using the definition of the cup product.

Assume now that \( E \) is constructible. Since \( (\mathcal{M}_{X}^{\text{gp}}/\mathcal{O}_{X}) \otimes_{Z} \mathbb{Z}/m \) is a constructible torsion abelian sheaf in \( \tilde{X}_{et} \), the left vertical morphism in \textbf{5.2.1} is an isomorphism by Artin-Grothendieck’s comparison theorem \cite{SGA 4-3 XVI (4.1)} as used in the proof of \cite{KN} (2.6)).
Remark 5.3. Using some results and some arguments in [KN] (2.6), using (5.2) but without using the log exponential sequence in [KN], we can give a proof of [loc. cit.] again. But we omit it because it resembles the proof in [loc. cit.].

Let

\[(5.3.1) \quad R^k \epsilon_{et*}(K^\bullet) \rightarrow R\eta_{X,et*}(R^k \epsilon_{an*}(\eta^{-1}_{X,et}(K^\bullet)))\]

be an analogous morphism to (5.1.14).

Proposition 5.4. Let m be a positive integer and let E be an m-torsion abelian sheaf in X_{et}. Let k be a nonnegative integer. Then the following diagram is commutative:

\[(5.4.1) \quad R\eta_{X,et*}(\bigwedge^k (\eta^{-1}_{X,et}(M_{\text{gp}}^Y/O_X) \otimes_{\mathbb{Z}} \eta^{-1}_{X,et}(E)(-k))) \xrightarrow{R\eta_{X,et*}(5.3.1)} R\eta_{X,et*}(R^k \epsilon_{an*}(\epsilon^{-1}_{X,et}(E))))\]

\[\bigwedge^k (M_{\text{gp}}^\text{cl} / O_X) \otimes_{\mathbb{Z}} E(-k) \xrightarrow{5.3.1} R^k \epsilon_{et*}(\epsilon^{-1}_{et}(E)).\]

Furthermore, if E is constructible, then the left vertical morphism is an isomorphism.

Proof. The proof of the commutativity of (5.4.1) is the same as that of the commutativity of (5.3.1). By (5.1.14) for \(\hat{X}\), the left vertical morphism is an isomorphism if E is constructible. \qed

6 Log exponential sequences

Let the notations be as in [22]. Let Y be an fs log analytic space over \(\mathbb{C}\). In this section, introducing a new abelian sheaf \(L^\log_Y\) in \(\hat{Y}\) which is a variant of the sheaf of logarithms \(L_Y^\log\) defined in [KN] and using the commutative diagram (5.2.1), we give a commutative diagram (6.5.1) below which compares the calculations of certain higher direct images by the use of the log exponential sequence in [loc. cit.] and by the use of the algebraic log Kummer sequence in [loc. cit.]. We also introduce a new log exponential sequence which corrects the commutative diagram in [II] (5.9.1) (see 6.3.1 and 6.9 (1) below).

First let us recall the sheaf \(L_Y^\log\) of logarithms of local sections of \(\epsilon^{-1}_{cl}(M_{Y}^\text{gp})\) in [KN] (1.4).

Let \(\text{Cont}_{Y^\log}(,T)\) be a sheaf in \(\hat{Y}\) of continuous functions to a commutative topological group \(T\). For a morphism \(S \rightarrow T\) of commutative topological groups, we have a natural morphism \(\text{Cont}_{Y^\log}(,S) \rightarrow \text{Cont}_{Y^\log}(,T)\) of abelian sheaves in \(Y^\log\).

The sheaf \(L_Y^\log\) is, by definition, the following fiber product

\[(6.0.1) \quad L_Y^\log := \text{Cont}_{Y^\log}(,\sqrt{-1}\mathbb{R}) \times_{\text{exp,Cont}_{Y^\log}(,S)} \epsilon^{-1}_{cl}(M_{Y}^\text{gp}).\]

Then we have an exponential sequence

\[(6.0.2) \quad 0 \rightarrow \mathbb{Z}(1) \rightarrow L_Y^\log \rightarrow \epsilon^{-1}_{cl}(M_{Y}^\text{gp}) \rightarrow 0\]

in \(\hat{Y}\) ([loc. cit., (1.4)]).
By [KN (1.5)], for an abelian sheaf $E$ on $\tilde{Y}$, we have a canonical isomorphism

$$\bigwedge^k (\mathcal{M}_Y^\text{gp}/\mathcal{O}_Y^*)(-k) \otimes_\mathbb{Z} E \xrightarrow{\sim} R^k_{\text{et}!}(\epsilon^{-1}_!(E)) \quad (k \in \mathbb{Z}_{\geq 0}).$$

Next let us define an abelian sheaf $L^\dagger_{Y,\log}$ in $\tilde{Y}^\log$.

We have a natural morphism

$$\beta^{-1}(\mathcal{M}_Y^\text{gp}) \rightarrow \text{Cont}_{Y,\log}(-, S^1)$$

of abelian sheaves in $\tilde{Y}^\log$ induced by the natural morphism

$$\Gamma(V, \mathcal{M}_Y^\text{gp}) \rightarrow \text{Cont}_{Y,\log}(Y^\log, S^1) \quad (V \in Y^\log)$$

of presheaves on $Y^\log$.

Set

$$L^\dagger_{Y,\log} := \text{Cont}_{Y,\log}(-, \sqrt{-1}\mathbb{R}) \times_{\exp, \text{Cont}_{Y,\log}(-, S^1)} \beta^{-1}(\mathcal{M}_Y^\text{gp}).$$

Then we have an exponential sequence

$$0 \rightarrow \mathbb{Z}(1) \rightarrow L^\dagger_{Y,\log} \xrightarrow{\exp} \beta^{-1}(\mathcal{M}_Y^\text{gp}) \rightarrow 0.$$

**Definition 6.1.** We call $L^\dagger_{Y,\log}$ the sheaf of logarithms of local sections of $\beta^{-1}(\mathcal{M}_Y^\text{gp})$.

**Lemma 6.2.** The natural composite morphism

$$\mathbb{C}^* \xrightarrow{\sim} \beta^{-1}(\mathcal{M}_Y^\text{gp}) \rightarrow \text{Cont}_{Y,\log}(-, S^1)$$

of abelian sheaves in $\tilde{Y}^\log$ is induced by the map $c \mapsto c/|c|$ ($c \in \mathbb{C}^*$).

**Proof.** The proof of (6.2) is clear because, for any object $V$ of $Y^\log$, $\mathbb{C}^* \subset \mathcal{O}^*_V$ for any point $x$ of $V$. $\square$

**Lemma 6.3.** For a positive integer $m$, the multiplication morphism

$$m \times : L^\dagger_{Y,\log} \rightarrow L^\dagger_{Y,\log}$$

is an isomorphism.

**Proof.** First we show the injectivity of (6.3.1). Let $(a, s)$ ($a \in \sqrt{-1}\mathbb{R}$, $s \in \beta^{-1}(\mathcal{M}_Y^\text{gp})$) be a local section of $L^\dagger_{Y,\log}$ such that $m(a, s) = 0$. Then $a = 0$ and $s \in (\mathbb{Z}/m)(1)$. By (6.2), we see that $s = 1$. Hence the morphism (6.3.1) is injective.

Next we show the surjectivity of (6.3.1). For an object $U$ of $Y^\log$, let $(a, u)$ ($a \in \sqrt{-1}\mathbb{R}$, $u \in \Gamma(U, \beta^{-1}(\mathcal{M}_Y^\text{gp}))$) be a section of $\Gamma(U, L^\dagger_{Y,\log})$. We may assume that $u \in \Gamma(U, \beta^{-1}(\mathcal{M}_Y^\text{gp}))$. Since $\mathcal{M}_Y^\text{log}$ is $m$-divisible by (6.1) and since the functor $\beta^{-1}$ is right-exact, there exists a section $v_\lambda$ of $\Gamma(U_\lambda, \beta^{-1}(\mathcal{M}_Y^\text{log}))$ for some covering $(U_\lambda \rightarrow U)^\dagger$ of $U$ in $Y^\log$ such that $v_\lambda^m = u|_{U_\lambda}$. Let $w_\lambda$ be the image of $v_\lambda$ in $\text{Cont}_{Y^\log}(U_\lambda, S^1)$. Then $\zeta_\lambda := w_\lambda \exp(-m^{-1}a)$ is an $m$-th root of unity. Hence $(m^{-1}a, v_\lambda \zeta_\lambda^{-1})$ is indeed an element of $\Gamma(U_\lambda, L^\dagger_{Y,\log})$ by (6.2), and $m(m^{-1}a, v_\lambda \zeta_\lambda^{-1}) = (a, u)|_{U_\lambda}$. Hence the morphism (6.3.1) is surjective. $\square$
Proposition 6.4. Let $E$ be an abelian sheaf in $\tilde{Y}$. Then the following diagram
\[
\begin{array}{c}
\Lambda^k(M_{\text{gp}}^{\text{et}}/\mathcal{O}_Y)(-k) \otimes_{\mathbb{Z}} E \\
\downarrow \text{id} \otimes \exp(m^{-1} \times) \otimes \text{proj} \\
\Lambda^k(M_{\text{gp}}^\text{et}/\mathcal{O}_Y)(-k) \otimes_{\mathbb{Z}} (\mathbb{Z}/m)(-k) \otimes_{\mathbb{Z}} (E/mE) \\
\downarrow \cong \\
R^k\epsilon_{\text{cl}}(\epsilon^{-1}_e(E))
\end{array}
\]
(6.4.1)

is commutative for $k \in \mathbb{Z}_{\geq 0}$.

Proof. By (6.3) we obtain a well-defined morphism
\[
\exp(m^{-1} \times) : L^t_{Y,\log} \to \beta^{-1}(M_{Y,\log}^{\text{et}}).
\]
(6.4.2)

Since $(2\pi \sqrt{-1}n/m, \exp(2\pi \sqrt{-1}n/m))$ $(n \in \mathbb{Z})$ is a section of $L^t_{Y,\log}$ by (6.2),
\[
m^{-1}(2\pi \sqrt{-1}n, 1) = (2\pi \sqrt{-1}n/m, \exp(2\pi \sqrt{-1}n/m)).
\]

Hence we obtain the following commutative diagram
\[
\begin{array}{c}
0 \to \mathbb{Z}(1) \to L^t_{Y,\log} \to \epsilon^{-1}_e(M_{Y,\log}^{\text{et}}) \to 0 \\
\downarrow \exp(m^{-1} \times) \downarrow \exp(m^{-1} \times) \\
0 \to (\mathbb{Z}/m)(1) \to \beta^{-1}(M_{Y,\log}^{\text{et}}) \to 0
\end{array}
\]
(6.4.3)

of exact sequences. (6.4.3) immediately follows from the commutative diagram (6.4.3) and from the definitions of the isomorphisms (6.0.3) and (3.2.1). \(\square\)

Corollary 6.5. Let $X$ be an fs log scheme over $\mathbb{C}$ whose underlying scheme $\tilde{X}$ is locally of finite type over $\mathbb{C}$. Let $m$ be a positive integer. Let $E$ be an $m$-torsion abelian sheaf in $X_{\text{et}}$. Then the following diagram is commutative:

\[
\begin{array}{c}
R\eta_*(\Lambda^k(\eta^*(M_{\text{gp}}^\text{et})/\mathcal{O}_{X_{\text{et}}}^\text{et}) \otimes_{\mathbb{Z}} \eta^{-1}(E)(-k)) \\
R\eta_*(\text{id} \otimes \exp(m^{-1} \times)) \\
R\eta_*(\Lambda^k(\eta^*(M_{\text{gp}}^\text{et})/\mathcal{O}_{X_{\text{et}}}^\text{et}) \otimes_{\mathbb{Z}} \eta^{-1}(E)(-k)) \\
\downarrow \cong \\
R^k\epsilon_{\text{cl}}(\epsilon^{-1}_e(\eta^{-1}(E)))
\end{array}
\]
(6.5.1)

Proof. By (6.2) we obtain the upper commutative diagram; the lower one is nothing but the commutative diagram (6.2.1). \(\square\)
Lastly we point out the mistakes in the proof of [II (5.9)] and we correct them.

In the proof of [II (5.9)], it is claimed that $L^\log_{X_{\text{log}}}$ is uniquely divisible by a positive integer $m$ for an fs log scheme $X$ over $\mathbb{C}$ whose underlying scheme is locally of finite type over $\mathbb{C}$. However the divisibility does not hold for $m \geq 2$. Indeed, if $L^\log_{X_{\text{log}}}$ were $m$-divisible, then $\epsilon_{\text{cl}}^{-1}(M^\log_{X_{\text{log}}})$ would also be by the exponential sequence [6.5.7]. Consequently $\epsilon_{\text{cl}}^{-1}(M^\log_{X_{\text{log}}}/O^\log_{X_{\text{log}}})$ would also be $m$-divisible. Clearly this does not hold in general. Indeed, let $X$ be the log point $s = (\text{Spec } \mathbb{C}, \mathbb{N} \oplus \mathbb{C}^*)$. Then $\epsilon_{\text{cl}}^{-1}(M^\log_{X_{\text{log}}}/O^\log_{X_{\text{log}}}) = \epsilon_{\text{cl}}^{-1}(\mathbb{Z}) = \mathbb{Z}$. Therefore the proof of [II (5.9)] is mistaken.

I think that there does not exist the natural morphism $\tau^{-1}(M^\text{gp}) \rightarrow \eta^{-1}(M^\log_{X_{\text{log}}})$ in the commutative diagram [II (5.9.1)] (I think that there does not exist a useful direct relation between the log exponential sequence in [KN] and the algebraic log Kummer sequence in [loc. cit.]): consider the trivial log case. Then $M^\text{gp}$ in [II (5.9.1)] is the multiplicative group of the sheaf of germs of invertible holomorphic functions on $X_{\text{an}}$ (not the sheaf of invertible algebraic functions on $X$) and the morphism $\tau^{-1}(M^\text{gp}) \rightarrow \eta^{-1}(M^\log_{X_{\text{log}}})$ is a morphism from an analytic sheaf to an algebraic sheaf. Usually such a morphism does not naturally exist except the trivial morphism.

Let us give a right commutative diagram in [6.8.1] below (see also [6.9] below).

Let the notations be as in [6.8.1]. Then we have a natural morphism

\begin{equation}
\eta_{\text{log}}^{-1}(M^\log_{X_{\text{log}}}) \rightarrow \text{Cont}_{X_{\text{log}}}(\cdot, S^1)
\end{equation}

induced by the natural morphisms $\Gamma(U, M^\log_{X_{\text{log}}}) \rightarrow \text{Cont}_{X_{\text{log}}}(U^\log_{\text{an}}, S^1)$ of abelian presheaves for objects $U$'s of $X_{\text{log}}$. Set

\begin{equation}
L^{-1}_{X_{\text{log}}} = \text{Cont}_{X_{\text{log}}}(\cdot, \sqrt{-1}\mathbb{R}) \times_{\exp, \text{Cont}_{X_{\text{log}}}} S^1 \eta_{\text{log}}^{-1}(M^\log_{X_{\text{log}}}).
\end{equation}

Then we have an exponential sequence

\begin{equation}
0 \rightarrow \mathbb{Z}(1) \rightarrow L^{-1}_{X_{\text{log}}} \rightarrow \eta_{\text{log}}^{-1}(M^\log_{X_{\text{log}}}) \rightarrow 0.
\end{equation}

Let us also recall $\eta^{\log*}(M^\log_{X_{\text{log}}})$ in [4.0.4]. We have a natural morphism

\begin{equation}
\eta_{\text{log}}^*(M^\log_{X_{\text{log}}}) \rightarrow \text{Cont}_{X_{\text{log}}}(\cdot, S^1)
\end{equation}

induced by the natural morphism

\begin{equation}
\Gamma(U, M^\log_{X_{\text{log}}}) \rightarrow \text{Cont}_{X_{\text{log}}}(U^\log_{\text{an}}, S^1)
\end{equation}

of abelian presheaves for objects $U$'s of $X_{\text{log}}$ and by the following morphism

\begin{equation}
\Gamma(V, O^\log_{X_{\text{an}}}) \ni f \mapsto ((x, h) \mapsto f(x)/|f(x)|) \in \text{Cont}_{X_{\text{an}}}(V^\log, S^1)
\end{equation}

of abelian presheaves for objects $V$'s of $(X_{\text{an}})^{\log}_{\text{et}}$. Here $x$ is a point of $V$ and $h : M^\log_{V,x} \rightarrow S^1$ is a morphism of groups such that $h(g) = g(x)/|g(x)|$ for any $g \in O^\log_{V,x}$. Set

\begin{equation}
L^\log_{X_{\text{log}}} := \text{Cont}_{X_{\text{log}}}(\cdot, \sqrt{-1}\mathbb{R}) \times_{\exp, \text{Cont}_{X_{\text{log}}}} \eta_{\text{log}}^*(M^\log_{X_{\text{log}}}).
\end{equation}

Then we have an exponential sequence

\begin{equation}
0 \rightarrow \mathbb{Z}(1) \rightarrow L^\log_{X_{\text{log}}} \rightarrow \eta_{\text{log}}^*(M^\log_{X_{\text{log}}}) \rightarrow 0.
\end{equation}
Definition 6.6. Let $\flat$ be $-1$ or $\ast$. We call $\mathcal{L}^{\flat}_{X_{\operatorname{an}}^\log}$ the sheaf of logarithms of local sections of $\eta_{\log}^{\flat}(\mathcal{M}_{X_{\log}}^\operatorname{gp})$.

Lemma 6.7. (1) The natural composite morphism
\[
\mathbb{C}^* \overset{\iota}{\rightarrow} \eta_{\log}^{-1}(\mathcal{M}_{X_{\log}}^\operatorname{gp}) \rightarrow \operatorname{Cont}_{X_{\operatorname{an}}^\log}(\mathcal{S}^1)
\]
of abelian sheaves in $\tilde{X}_{\operatorname{an}}^\log$ is induced by the map $c \mapsto c/|c|$ ($c \in \mathbb{C}^*$).

(2) For a positive integer $m$, the multiplication morphism
\[
m \times : \mathcal{L}^{\flat}_{X_{\operatorname{an}}^\log} \rightarrow \mathcal{L}^{\flat}_{X_{\operatorname{an}}^\log}
\]
is an isomorphism.

Proof. The proof of (1) (resp. (2)) is the same as that of (6.2) (resp. (6.3)).

By (6.7) (2) we obtain a well-defined morphism
\[
\exp(m^{-1} \times) : \mathcal{L}^{\flat}_{X_{\operatorname{an}}^\log} \rightarrow \eta_{\log}^{\flat}(\mathcal{M}_{X_{\log}}^\operatorname{gp})
\]
and the following commutative diagram
\[
\begin{array}{c}
0 \rightarrow \mathbb{Z}(1) \rightarrow \mathcal{L}^{\flat}_{X_{\operatorname{an}}^\log} \rightarrow \eta_{\log}^{\flat}(\mathcal{M}_{X_{\log}}^\operatorname{gp}) \rightarrow 0 \\
\exp(m^{-1} \times) \downarrow \quad \exp(m^{-1} \times) \downarrow \\
0 \rightarrow (\mathbb{Z}/m)(1) \rightarrow \eta_{\log}^{\flat}(\mathcal{M}_{X_{\log}}^\operatorname{gp}) \rightarrow 0
\end{array}
\]
of exact sequences.

As a summary, we obtain the following:

Proposition 6.8. There exists the following commutative diagram
\[
\begin{array}{c}
0 \rightarrow \mathbb{Z}(1) \rightarrow \mathcal{L}_{X_{\operatorname{an}}^\log} \rightarrow \epsilon_{\operatorname{cl}}^{-1}(\eta^*(\mathcal{M}_{X_{\log}}^\operatorname{gp})) \rightarrow 0 \\
\| \quad \| \\
0 \rightarrow \mathbb{Z}(1) \rightarrow \mathcal{L}^1_{X_{\operatorname{an}}^\log} \rightarrow \beta^{-1}(\mathcal{M}_{X_{\log}}^\operatorname{gp}) \rightarrow 0 \\
\| \quad \| \\
0 \rightarrow \mathbb{Z}(1) \rightarrow \mathcal{L}^*_{X_{\operatorname{an}}^\log} \rightarrow \eta_{\log}^*(\mathcal{M}_{X_{\log}}^\operatorname{gp}) \rightarrow 0 \\
\| \quad \| \\
0 \rightarrow \mathbb{Z}(1) \rightarrow \mathcal{L}^{-1}_{X_{\operatorname{an}}^\log} \rightarrow \eta_{\log}^{\ast,-1}(\mathcal{M}_{X_{\log}}^\operatorname{gp}) \rightarrow 0
\end{array}
\]
of exact sequences.

Remark 6.9. (1) Using the morphism (5.1.7), we can delete the second exact sequence in (6.8.1).

(2) Using the commutative diagram (6.8.1) (or the commutative diagram obtained in (1)), we can give a proof of [KN, (2.6)].
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Department of Mathematics, Tokyo Denki University, 2–2 Kanda Nishiki-cho Chiyoda-ku, Tokyo 101–8457, Japan.

E-mail address: nakayuki@cck.dendai.ac.jp