Analytic Preconditioners for Decoupled Potential Integral Equations and Wideband Analysis of Scattering From PEC Objects

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Abstract—Many integral equations used to analyze scattering, such as the standard combined field integral equation (CFIE), are not well-conditioned for a wide range of frequencies and multiscale geometries. There has been significant effort to alleviate this problem. A more recent one is using a set of decoupled potential integral equations (DPIEs). These equations have been shown to be robust at low frequencies and immune to topology breakdown. But they mimic the ill-conditioning behavior of CFIE at high frequencies. This article addresses this deficiency through new Calderón-type identities derived from the vector potential integral equation (VPIE). We construct novel analytic preconditioners for the VPIE and scalar potential integral equation (SPIE) constrained to perfect electric conductors (PECs). These new formulations are wideband well-conditioned and converge rapidly for multiscale geometries. This is demonstrated through a number of examples that use analytic and piecewise basis sets.

Index Terms—Calderón, decoupled potential, integral equation, preconditioning, wideband.

I. INTRODUCTION

Researchers have developed various boundary integral equation approaches to predict scattered magnetic and electric fields from perfect electric conductor (PEC) objects of arbitrary shape. When applicable, the computation time of this approach is less than that of other approaches such as the finite element method (FEM) [1], and the Silver–Müller radiation condition is automatically enforced. One classic PEC boundary integral equation is the combined field integral equation (CFIE) [2], [3], where the electric field integral equation (EFIE) and magnetic field integral equation (MIE) are combined to fashion a provably unique formulation avoiding the spurious modes latent in other nonunique formulations [1].

The EFIE and MIE components of the CFIE formulation suffer from a variety of issues such as low-frequency breakdown [4], [5], [6], [7], catastrophic cancellation [8], dense mesh breakdown [9], static nullspaces or topology breakdown [10], and a poor approximation of the identity operator in the MIE with Rao–Wilton–Glisson (RWG) testing and basis functions [11]. These issues have been addressed in various ways. Buffa–Christiansen testing sets better approximate the identity operator of the MIE when the basis set is composed of RWG functions [12]. These functions have also been used in conjunction with the Calderón identities to precondition the ill-conditioned EFIE operator [13]. Other suggested basis sets to alleviate breakdown are the so-called loop-star and loop-tree functions [14], [15] or the related basis-free quasi-Helmholtz projection matrices [16], subdivision surfaces [17], and manifold harmonics [18]. Yet another option is solving for current and charge densities in the current and charge integral equation (CCIE) [19].

The focus of this article is to use decoupled potentials instead of fields. The resulting equations are the scalar potential integral equation (SPIE) and the vector potential integral equation (VPIE) that are solved independently. For PEC objects analyzed in the frequency domain, [20] suggested various combinations of the potential integral equations to construct well-conditioned formulations at low frequency. Eris et al. [21] suggested a unique combined potential formulation for dense discretizations. The low-frequency behavior of the VPIE was further analyzed for PEC in [22]. For dielectric objects analyzed in the frequency domain, [20] have suggested well-conditioned potential formulations at low frequency. A potential integral formulation for solving lossy conductors is detailed in [23] as well. The time-domain variant of these integral equations has been analyzed in [24]. Specifically, the decoupled potential integral equation (DPIE) approach has several niceties: no low-frequency breakdown, no dense mesh breakdown, no topological low-frequency breakdown, and well-conditioned dielectric and PEC formulations at low to medium range frequencies [20], [25]. Recently, the DPIE has been implemented on arbitrary dielectric objects [26] with pulse and RWG functions, and the results therein demonstrate

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low singular value conditioning and a low iteration count to converge. However, fast convergence with iterative solvers has yet to be demonstrated in the high-frequency region for arbitrary objects using the DPIE.

This article extends the DPIE property of well-conditioned to the high-frequency region for arbitrary PEC objects by constructing a new formulation through novel Calderón-type identities. Several salient contributions are made along the way. First, new integral equation identities are derived in a similar manner to the derivation of the Calderón identities in [27], and the process is applicable to other scattering problems whose forward matrices are also projectors. Second, we show that the new Calderón-type identities are effective in preconditioning a combined VPIE (CVPIE) and SPIE formulation while maintaining uniqueness. Third, the new combined formulations map a div-conforming space to a curl-conforming space and a curl-conforming space to a curl-conforming space. And finally, the new formulation quickly converges in practice for tessellations of multiscale geometries to converge. However, fast convergence with iterative solvers takes a different approach to solve for scattering fields in $\mathbb{D}$, as opposed to the usual approach of $\mathbb{S}$, and the process is applicable to other scattering fields in $\mathbb{D}$.

In [27], and the process is applicable to other scattering fields in $\mathbb{D}$, as opposed to the usual approach of $\mathbb{S}$, and the process is applicable to other scattering fields in $\mathbb{D}$.

In what follows, all the quantities associated to the high-frequency region for arbitrary PEC objects by

\[ \nabla(\mathbf{r}) \mathbf{A}_+(\mathbf{r}) + [\nabla(\mathbf{r}) - 1] \mathbf{A}_-(\mathbf{r}) = - \int_{\mathbb{S}} G(\mathbf{r}, \mathbf{r'}) \hat{n'} \times \mathbf{a}_\pm(\mathbf{r'}) dS' + \nabla \times \int_{\mathbb{S}} G(\mathbf{r}, \mathbf{r'}) \mathbf{b}_\pm(\mathbf{r'}) dS' \]

\[ - \nabla \cdot \int_{\mathbb{S}} G(\mathbf{r}, \mathbf{r'}) \gamma_\pm(\mathbf{r'}) dS' - \int_{\mathbb{S}} G(\mathbf{r}, \mathbf{r'}) \hat{n} \sigma_\pm(\mathbf{r'}) dS' \]

(2)

where the source terms are defined as

\[ \mathbf{a}(\mathbf{r}) = \hat{n} \times \hat{n} \times \nabla \times \mathbf{A}(\mathbf{r}) \]

\[ \mathbf{b}(\mathbf{r}) = \hat{n} \times \mathbf{A}(\mathbf{r}) \]

\[ \gamma(\mathbf{r}) = \hat{n} \cdot \mathbf{A}(\mathbf{r}) \]

\[ \sigma(\mathbf{r}) = \nabla \cdot \mathbf{A}(\mathbf{r}) \]

(3)

In a similar manner, the scalar potential can be written as

\[ \nabla(\mathbf{r}) \phi_+(\mathbf{r}) + [\nabla(\mathbf{r}) - 1] \phi_-(\mathbf{r}) = - \nabla \cdot \int_{\mathbb{S}} G(\mathbf{r}, \mathbf{r'}) \hat{n'} \alpha_\pm(\mathbf{r'}) dS' - \int_{\mathbb{S}} G(\mathbf{r}, \mathbf{r'}) \beta_\pm(\mathbf{r'}) dS' \]

(4)

where the source terms are $\alpha(\mathbf{r}) = \phi(\mathbf{r})$ and $\beta(\mathbf{r}) = \hat{n} \cdot \nabla \phi(\mathbf{r})$

\[ G(\mathbf{r}, \mathbf{r'}) = \frac{e^{-j\kappa |\mathbf{r} - \mathbf{r}'|}}{4\pi |\mathbf{r} - \mathbf{r}'|} \]

(5)

is the free-space Green’s function. We note that $\gamma(\mathbf{r})$ and $\mathbf{b}(\mathbf{r})$ are Hölder continuous, and $\sigma(\mathbf{r})$ and $\hat{n} \times \mathbf{a}(\mathbf{r})$ are continuous.

In the context of scattering, we associate quantities denoted by $\pm$, with scattered fields and incident fields, respectively. That is, $\mathbf{A}_-(\mathbf{r}) \rightarrow \mathbf{A}_+^{i}(\mathbf{r})$ and $\mathbf{A}_+(\mathbf{r}) \rightarrow \mathbf{A}_-^{s}(\mathbf{r})$. Using the classical jump conditions, adding the two forms $\mathbf{A}_\pm$, and denoting $\mathbf{A}(\mathbf{r}) = \mathbf{A}_+^{i}(\mathbf{r}) + \mathbf{A}_-^{s}(\mathbf{r})$, we have

\[ \frac{1}{2} \mathbf{A}(\mathbf{r}) - \mathbf{A}_+^{i}(\mathbf{r}) = \frac{1}{2} \int_{\mathbb{S}} \left[ - G(\mathbf{r}, \mathbf{r'}) \hat{n'} \times \mathbf{a}(\mathbf{r'}) + G(\mathbf{r}, \mathbf{r'}) \nabla \times \mathbf{b}(\mathbf{r'}) - \nabla G(\mathbf{r}, \mathbf{r'}) \gamma(\mathbf{r'}) - G(\mathbf{r}, \mathbf{r'}) \hat{n} \sigma(\mathbf{r'}) \right] dS'. \]

(6)

It can be shown that the quantities associated with the scalar potential, namely, $\hat{n} \mathbf{a}(\mathbf{r})$ and $\hat{n} \mathbf{b}(\mathbf{r})$, are Hölder continuous and continuous, respectively. As before, we associate the scattered scalar potential with scalar potential in the exterior region and the incident scalar potential with the interior region $\phi_-(\mathbf{r}) \rightarrow \phi_+^{s}(\mathbf{r})$ and $\phi_+^{s}(\mathbf{r}) \rightarrow \phi_+^{i}(\mathbf{r})$. Using classical jump conditions, adding the two forms $\phi_\pm$ and denoting $\phi(\mathbf{r}) = \phi_+^{i}(\mathbf{r}) + \phi_+^{s}(\mathbf{r})$, we obtain

\[ \frac{1}{2} \phi(\mathbf{r}) - \phi_+^{s}(\mathbf{r}) = \frac{1}{4\pi} \int_{\mathbb{S}} \left[ - \nabla G(\mathbf{r}, \mathbf{r'}) \cdot \hat{n'} \alpha(\mathbf{r'}) - \nabla G(\mathbf{r}, \mathbf{r'}) \beta(\mathbf{r'}) \right] dS'. \]

(7)

While these equations hold in general, they simplify considerably when $\mathbb{S}$ is a PEC. $\mathbf{b}(\mathbf{r})$, $\sigma(\mathbf{r})$, and $\alpha(\mathbf{r})$ are zero [20]. This is a point that we save for later in the article.

With (6) and (7) and the decoupled potential boundary conditions discussed in [20], one can obtain a set of integral equations for both the scalar and vector potentials through
the boundary conditions. As reported in [20] and [26], these
equations that can be succinctly written as
\[
(I - Z^{\text{SPIE}}) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} \quad (8a)
\]
and
\[
(I - Z^{\text{VPIE}}) \begin{pmatrix} a \\ b \\ \gamma \\ \sigma \end{pmatrix} = \begin{pmatrix} a' \\ b' \\ \gamma' \\ \sigma' \end{pmatrix} \quad (8b)
\]
where
\[
Z^{\text{SPIE}} = \begin{pmatrix} D & -S \\ N & -D' \end{pmatrix} \quad (8c)
\]
and
\[
Z^{\text{VPIE}} = \begin{pmatrix} K^i & \kappa^2 L' & 0 & -Q^{(1)} \\ -J^{(2)} & K & -P^{(2)} & -Q^{(2)} \\ -J^{(3)} & M^{(3)} & -D' & -Q^{(3)} \\ -J^{(4)} & 0 & \kappa^2 S & D \end{pmatrix} \quad (8d)
\]
The terms SPIE and VPIE denote the scalar and VPIEs, respectively, and the operators are defined in Appendix A. Note that we have not separated out the principal value components in the diagonal elements of \(Z^{\text{SPIE/VPIE}}\).

IV. NOVEL CALDERÓN-TYPE IDENTITIES

Next, we derive Calderón-type identities for both the SPIE and VPIE. To do so, we follow a similar procedure outlined in [27] but stress the perspective of projectors, namely, we use the projector and complement projector properties of \(Z^{\text{SPIE/VPIE}}\) and \(I - Z^{\text{SPIE/VPIE}}\). Indeed, Calderón-type identities may be derived for any scattering problem provided the forward matrix is a projector.

In the absence of incident potentials in both (8a) and (8b), one can show that
\[
\begin{pmatrix} 0 \\ 0 \end{pmatrix} = (I - Z^{\text{SPIE}}) Z^{\text{SPIE}} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad (9a)
\]
and
\[
\begin{pmatrix} 0 \\ 0 \end{pmatrix} = (I - Z^{\text{VPIE}}) Z^{\text{VPIE}} \begin{pmatrix} a \\ b \\ \gamma \\ \sigma \end{pmatrix} \quad (9b)
\]
Note that \((I - Z^{\text{SPIE/VPIE}})\) and \(Z^{\text{SPIE/VPIE}}\) commute. As the sources are arbitrary, it follows that the cascaded operators have to be zero. This results in the well-known identities
\[
\begin{align*}
(I - D)D + SN &= 0 \quad (10a) \\
-(I - D)S - SD' &= 0 \quad (10b) \\
-ND + (I + D')N &= 0 \quad (10c) \\
NS - (I + D')D' &= 0 \quad (10d)
\end{align*}
\]
for the SPIE; note that these have been previously derived in [28] for scalar potentials and acoustics. As before, these operator equations are meant to be read from right to left. For instance, \(SN\alpha = S(N\alpha)\). Following a similar procedure for the VPIE system, we have
\[
\begin{align*}
(I - K^i)K^i + \kappa^2 L' J^{(2)} - Q^{(1)} J^{(4)} &= 0 \quad (11a) \\
J^{(2)} K^i - (I - K) J^{(2)} - P^{(2)} J^{(3)} - Q^{(2)} J^{(4)} &= 0 \quad (11b) \\
J^{(3)} K^i + M^{(3)} J^{(2)} - (I + D') J^{(3)} - Q^{(3)} J^{(4)} &= 0 \quad (11c) \\
J^{(4)} K^i + \kappa^2 S J^{(3)} - (I - D) J^{(4)} &= 0 \quad (11d) \\
(I - K^i) L' - L' K &= 0 \quad (11e) \\
J^{(2)} \kappa^2 L' + (I - K) K + P^{(2)} M^{(3)} &= 0 \quad (11f) \\
J^{(3)} \kappa^2 L' - M^{(3)} K + (I + D') M^{(3)} &= 0 \quad (11g) \\
J^{(4)} L' - SM^{(3)} &= 0 \quad (11h) \\
L' P^{(2)} + Q^{(3)} S &= 0 \quad (11i) \\
-(I - K) P^{(2)} - P^{(2)} D' + Q^{(2)} \kappa^2 S &= 0 \quad (11j) \\
M^{(3)} P^{(2)} - (I + D') D' + Q^{(3)} \kappa^2 S &= 0 \quad (11k) \\
SD' + (I - D) S &= 0 \quad (11l) \\
-(I - K^i) Q^{(1)} + \kappa^2 L' Q^{(2)} + Q^{(1)} D &= 0 \quad (11m) \\
-J^{(2)} Q^{(1)} - (I - K) Q^{(2)} - P^{(2)} Q^{(3)} + Q^{(2)} D &= 0 \quad (11n) \\
-J^{(3)} Q^{(1)} + M^{(3)} Q^{(2)} - (I + D') Q^{(3)} + Q^{(3)} D &= 0 \quad (11o) \\
-J^{(4)} Q^{(1)} + \kappa^2 S Q^{(3)} + (I - D) D &= 0. \quad (11p)
\end{align*}
\]
We will leverage these identities to precondition SPIE and VPIE systems as applied to the analysis of scattering from perfect electrically conducting objects. Parenthetically, we note the following: Given the similarity of operators to those for the decoupled field integral equation (DFIE) [29], the same approach can be used for these as well, and we will report the results in a future paper.

Next, we present the reduction necessary for the analysis of scattering from PEC objects. Note that the boundary conditions should enforce that the tangential component of the total electric field is zero. For potentials, this is tantamount to setting \(\alpha(r) = 0 = \sigma(r)\) and \(b(r) = 0\). These result in equations of the form
\[
\begin{align*}
\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} &= \begin{pmatrix} S & I + D' \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\
\begin{pmatrix} a' \\ b' \\ \gamma' \\ \sigma' \end{pmatrix} &= \begin{pmatrix} I - K^i & 0 \\ J^{(2)} & I + D' \\ J^{(3)} & -\kappa^2 S \\ J^{(4)} & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ \gamma \\ \sigma \end{pmatrix} \quad (12a)
\end{align*}
\]
As is to be expected, the equations are overdetermined. But as alluded to in [20] and discussed in [21] and [24], and demonstrated in this article using analytical basis sets, one can combine potential integral formulations to remove interior resonances. This article uses combinations that are akin to the EFIE, MFIE, and CFIE. Next, we discuss these equations and preconditioners.

V. LOCALIZED CALDERÓN-TYPE COMBINED SPIE AND VPIE FORMULATIONS

A. SYSTEMS OF OPERATORS

From (12a), it is apparent that one can use either of the two equations but both will have null spaces at irregular
frequencies. The null spaces of \( S \) operator correspond to solution of a cavity with the Dirichlet boundary conditions, whereas those of \( I + D' \) correspond to those of an interior cavity with the Neumann boundary conditions. To avoid these null spaces, we follow the usual combined field approach, i.e., the two formulations are weighted with a coefficient \( \delta \) and \( 1 - \delta \), where \( 0 \leq \delta \leq 1 \). The combined SPIE (CSPIE) is written as

\[
\delta \mathbf{a}^{i} + (1 - \delta) \mathbf{b}^{i} = \delta Z_{1}^{SPIE} \mathbf{a} + (1 - \delta) Z_{2}^{SPIE} \mathbf{b} \tag{13}
\]

where

\[
Z_{1}^{SPIE} = S \nonumber
\]

\[
Z_{2}^{SPIE} = I + D'. \tag{14}
\]

Next, we consider the SPIE system. It can be demonstrated that rows 1 and 3 of (12b) are spectrally akin to those of an MFIE (denoted by \( Z_{1}^{VPIE} \)), whereas rows 2 and 4 are similar to those of an EFIE (denoted by \( Z_{2}^{VPIE} \))

\[
Z_{1}^{VPIE} = (I - K^{i} - J^{(3)}) \nonumber
\]

\[
Z_{2}^{VPIE} = (J^{(2)} P^{(2)} - k^{2} S). \tag{15}
\]

The null spaces of these equations are more complex than those of either the MFIE or EFIE operators, but they share a number of null spaces. Take \( Z_{1}^{VPIE} \) for instance. Given the lower triangular nature of the operators, it follows that the null spaces fall into two categories: (a) the null spaces of \( I - K^{i} \) and (b) null spaces of \( I + D' \) for rotational \( \mathbf{a} \). While similar analysis is possible for \( Z_{2}^{VPIE} \) and can be done with the analytical framework given later in the article, the above is sufficient for the ensuing discussion.

A linear combination of the two systems does not share a null space. As a result, the CVPIE is prescribed as follows:

\[
\delta \mathbf{a}^{i} + (1 - \delta) \mathbf{b}^{i} = \delta Z_{1}^{VPIE} \mathbf{a} + (1 - \delta) Z_{2}^{VPIE} \mathbf{a}. \tag{16}
\]

Next, we discuss preconditioners that can be chosen from the identities derived in Section IV.

B. Choosing Analytic Preconditioners

To derive a preconditioner, our approach is to start with the identities presented in (10) and (11). We start with the SPIE. As was alluded to earlier, both \( Z_{1}^{SPIE} \) and \( Z_{2}^{SPIE} \) are not unique at irregular frequencies. It is also apparent that (10) can potentially be used to derive well-conditioned systems. To wit, one can readily recognize that \( N \) operating on \( S \) results in an operator that is second kind, but it is also apparent that such a composition shares null spaces, i.e., \( N S \) and \( I + D' \) share resonances due to identity (10d). To overcome this conundrum, we choose a complexification of the wavenumber included in \( N \). Specifically, we let \( \tilde{N} = -\kappa^{2} \) such that instead of \( \kappa \), we use \( \tilde{\kappa} = \kappa - 0.4H^{(2/3)}k^{\alpha} \) [30] where \( \alpha = (1/3) \).

The resulting local Calderón-type combined SPIE (LC-CSPIE) formulation is resonance-free and second-kind

\[
\delta \tilde{\mathbf{a}}^{i} + (1 - \delta) \tilde{\mathbf{b}}^{i} = \delta \tilde{Z}_{1}^{SPIE} \mathbf{a} + (1 - \delta) \tilde{Z}_{2}^{SPIE} \mathbf{a}. \tag{17}
\]

Proceeding in a similar manner, we observe that both \( Z_{1}^{VPIE} \) and \( Z_{2}^{VPIE} \) are nonunique at different irregular frequencies. Furthermore, \( Z_{1}^{VPIE} \) is well-conditioned and \( Z_{2}^{VPIE} \) is ill-conditioned [20]. As before, (11) provides the necessary fodder for developing a preconditioner. Note that \( Z_{1}^{VPIE} \) is well-conditioned [20], whereas \( Z_{2}^{VPIE} \) is ill-conditioned [20]. Using (11) consider the following preconditioning operator and its action on \( Z_{2}^{VPIE} \):

\[
\left( \kappa^{2} L^{i} - Q^{(1)} \right) \left( J^{(2)} P^{(2)} - k^{2} S \right). \tag{18}
\]

It can be shown using (11a), (11c), (11i), and (11k) that (18) reduces to

\[
\left( -\mathbf{K}^{i} \mathbf{K}^{i} + (I + D') J^{(3)} \right) \left( I + D' D' \right). \tag{19}
\]

It is apparent that (19) includes an identity plus compact operator along the diagonal. The off-diagonal operators are compact operators acting on compact operators or compact operators acting on bounded operators, the result of both being compact [31]. As a result, the entire system of operators can be partitioned into identity plus compact operators, which are well-conditioned by definition. While this is a desirable outcome, (19) has null spaces that pose challenges.

Examining (19) in a manner similar to what was done for \( Z_{1}^{VPIE} \), the lower triangular nature of the operators dictate that the null spaces fall into two categories: (a) those of \( (I - K^{i}) \mathbf{K}^{i} \) and (b) for any rotational \( \mathbf{a} \) the null spaces of \( (I + D') D' \). It is apparent that these null spaces are shared with \( Z_{2}^{VPIE} \). Furthermore, the below factorization demonstrates that \( Z_{1}^{VPIE} \) and (19) share null spaces

\[
\left( -\mathbf{K}^{i} \mathbf{K}^{i} + (I + D') J^{(3)} \right) \left( I + D' D' \right). \tag{20}
\]

An alignment of these operators’ null spaces is numerically depicted in Fig. 4.

As before, to ensure uniqueness at all the frequencies, the preconditioned operators are complexified. Specifically, \( \kappa \) from [30] is again selected to localize the preconditioner and interlace the null spaces of the preconditioned \( Z_{1}^{VPIE} \) and \( Z_{2}^{VPIE} \) operators. The new local Calderón-type combined VPIE (LC-CVPIE) formulation is

\[
\delta \tilde{\mathbf{a}}^{i} + (1 - \delta) \tilde{\mathbf{b}}^{i} = \delta \tilde{Z}_{1}^{VPIE} \mathbf{a} + (1 - \delta) \tilde{Z}_{2}^{VPIE} \mathbf{a}. \tag{21}
\]

where

\[
\tilde{Z}_{2}^{VPIE} = \left( \kappa^{2} L^{i} - Q^{(1)} \right) \left( M^{(3)} - Q^{(3)} \right). \tag{22}
\]
Next, we discuss the implementation of these equations using both an analytical framework and basis functions defined on piecewise tesselation. Of course, the LC-CVPIE and LC-CSPIE together constitute the local Calderón combined decoupled potential integral equation (LC-CDPIE) whose unknowns are the concatenation of the unknowns for the VPIE and SPIE constrained to PEC objects.

C. Left and Right Preconditioners

The left and right preconditioners are used to scale the various operators in the suggested VPIE formulation such that all the unknowns are of the same units. The left and right scaling matrices are detailed in [20] and [26].

For the LC-CVPIE in (21), these matrices are

\[ P_l^{(1)} = \text{diag}(1, -j\kappa) \]
\[ P_l^{(2)} = \text{diag}(-j\kappa, 1) \]
\[ P_r^{(1)} = \left( P_l^{(1)} \right)^{-1} \]
\[ P_r^{(2)} = \left( P_l^{(2)} \right)^{-1} \]

After scaling (21) with (23), the new well-conditioned formulation implemented in the results section is

\[ \delta\bar{p}_l^{(1)} = (1 - \delta)\bar{p}_l^{(1)}Z^{VPIE}P_r^{(2)}P_l^{(2)}\left( \begin{array}{c} \alpha \\ \beta \end{array} \right) \]
\[ = \delta\bar{p}_l^{(1)}Z^{VPIE}P_r^{(1)} \left( \begin{array}{c} \bar{a} \\ \bar{y} \end{array} \right) \]
\[ + (1 - \delta)\bar{p}_l^{(1)}Z^{VPIE}P_r^{(2)}Z^{SPIE}P_l^{(1)} \left( \begin{array}{c} \bar{a} \\ \bar{y} \end{array} \right) \]

where

\[ \left( \begin{array}{c} \bar{a} \\ \bar{y} \end{array} \right) = P_l^{(1)} \left( \begin{array}{c} \alpha \\ \beta \end{array} \right). \]

For the LC-CSPIE, the scaling parameters are

\[ S_l^{(1)} = -j\kappa \]
\[ S_l^{(2)} = 1 \]

which results in the following LC-CSPIE formulation:

\[ \delta S_l^{(2)}Z^{SPIE}S_l^{(1)}\beta \]
\[ = \delta S_l^{(2)}Z^{SPIE}S_l^{(1)}Z^{SPIE}S_l^{(2)}[\beta] \]
\[ + (1 - \delta)S_l^{(2)}Z^{SPIE}S_l^{(2)}[\beta] \]

where

\[ \beta = S_l^{(2)}\beta. \]

D. Mapping Properties

The DPIE operators in (8d) and (8c) are a map between function spaces. We now summarize these mapping properties using vector and scalar spherical harmonic functions defined in Appendix B for a collection of domains and associate ranges

\[ Z^{VPIE} : \left( \begin{array}{c} \Psi \\ 0 \\ 0 \end{array} \right) \rightarrow \left( \begin{array}{c} \Psi \text{ or } 0 \\ 0 \\ 0 \end{array} \right) \]

These mapping properties demonstrate that div-conforming and curl-conforming spaces map to div-conforming and curl-conforming spaces, respectively. Therefore, the Gram matrix of a method of moments (MOM) system with RWG and hat testing and basis functions is well-conditioned for the DPIE operators and nondegenerate.

VI. DISCRETE IMPLEMENTATION

The crux of what follows is a means of discretizing (17) and (21). It is apparent that to do so one needs both the scalar and vector bases. For analytical analysis, the bases are spherical and vector spherical harmonics. For piecewise tesselation, these are \( W^0 \) or hat functions and \( W^2 \) or RWG functions. It is important to keep in mind that for (21), we are not representing the vector potentials; instead, we are representing either traces of or operators acting on the vector potential. Likewise, for (17), we represent the potential and its normal derivative on the manifold.

A. Analytic Basis Sets

Using the notation in [26], the spherical harmonic basis functions are denoted by \( B_{\ell m} \), where \( n \in [0, N_s] \) and the vector harmonic basis functions are denoted by \( B_{\ell m} \), where \( n \in [0, N_v] \). The number of basis functions in (17) is \( N_s \), and the span is denoted by \( Y_{\text{SPIE}} = \sum_{m=0}^{N_s-1} B_{\ell m}Y_{\ell m} \), where \( Y_{\ell m} \) are the basis function coefficients. The number of basis functions in (21) is \( N_v + N_s \). Likewise, the span of basis functions is denoted by \( \mathcal{Y}_{\text{SPIE}} = \sum_{n=0}^{N_s+N_v-1} F_nY_{\ell m} \), where \( Y_{\ell m} \) is a list of basis function coefficients and \( F_n = \text{diag}(B_{\ell n}, B_{\ell n}) \).

The sets of harmonics are denoted by

\[ B_{\ell} = \left( \begin{array}{ccc} Y_{\ell 0} & \cdots & Y_{\ell m} \end{array} \right) \]
\[ B_{\ell} = \left( \begin{array}{ccc} \Psi_{\ell 0} & \cdots & \Psi_{\ell m} \end{array} \right) \]

and the scalar and vector basis functions are then

\[ B^s = B_{\ell} \]
\[ B^v = (B_{\ell}, B_{\ell}) \]

where \( N_s = (N_h + 1)^2, N_v = 2(N_h + 1)^2 \), and \( N_h = \kappa a + 2 \) where \( a \) is the radius of the sphere.

B. Piecewise Basis Sets

The SPIE and VPIE discrete scalar and vector basis functions are hat and RWG functions on a mesh with \( N_f \) faces.
and \(N_e\) edges and \(N_n\) nodes. The hat functions are defined by

\[
h_{n_e}(r) = \begin{cases} \frac{l_{n_e}}{2A_{n_e}} \hat{u}_{n_e} \cdot (e_{n_e} - \rho_{n_e}^\pm), & r \in T_{n_e} \\ 0, & \text{else} \end{cases}
\]

(32a)

and the RWG functions are defined by

\[
f_{n_e}(r) = \begin{cases} \frac{l_{n_e}}{2A_{n_e}} \rho_{n_e}^\pm, & r \in T_{n_e}^\pm \\ 0, & \text{else} \end{cases}
\]

(33a)

where

\[
\rho^\pm(r) = \begin{cases} \pm (r - p_{n_e}^\pm), & r \in T_{n_e}^\pm \\ 0, & \text{else} \end{cases}
\]

(34a)

and \(A_{n_e}\) is the area of the face connected to node \(n_e\); \(A_{n_e}^\pm\) is the area of the faces sharing edge \(n_e\); \(\hat{u}_{n_e}\) is the planar normal of the edge opposite of node \(n_e\) on face \(T_{n_e}\), and pointing away from node \(n_e\); \(l_{n_e}\) is the length of the edge opposite of node \(n_e\) and on face \(T_{n_e}\); \(\rho_{n_e}^\pm\) is the length of the edge \(n_e\); \(\rho_{n_e}^\pm\) originates in node \(n_e\) and terminates in face \(T_{n_e}\); \(e_{n_e}\) are the coordinates of either node on the edge opposite of node \(n_e\); \(T_{n_e}^\pm\) are the faces sharing edge \(n_e\); \(T_{n_e}^\pm\) are either nodal or edge faces; \(\rho_{n_e}^\pm\) are the coordinates for the nodes opposite of the selected edge on the appropriate face sharing the selected edge; and \(p_{n_e}^\pm\) are the appropriate nodal coordinates for the RWG or hat basis.

The scalar and vector basis functions are then

\[
\begin{align*}
\mathbf{B}^V &= (h_0 \cdots h_{n_e} \cdots h_{N_n}) \\
\mathbf{B}^V &= (f_0 \cdots f_{n_e} \cdots f_{N_n})
\end{align*}
\]

(35a, 35b)

C. Discrete System

The MOM system is constructed with Galerkin testing using inner products defined as

\[
\langle g(r), f(r) \rangle = \int g^*(r)f(r)dS
\]

(36a)

\[
\langle g(r), f(r) \rangle = \int g^*(r) \cdot f(r)dS
\]

(36b)

The LC-CVPIE discrete system is

\[
Z^{VPIE} \gamma^{VPIE} = b^{VPIE}
\]

(37)

where the elements are defined by

\[
Z_{kn}^{VPIE} &= \begin{cases} \mathcal{F}^V, & k = 0 \\ \delta G^{-1} P_i (1) Z_i^{VPIE} p_i^{(1)} \mathcal{F}^V_n, & \text{else} \end{cases} \\
(38a)
\]

\[
\gamma_i^{VPIE} = \begin{cases} \mathcal{F}^V_i, & \text{else} \end{cases}
\]

(38c)

\[
b_k^{VPIE} &= \begin{cases} \mathcal{F}^V, & k = 0 \\ \delta G^{-1} P_i (1) \hat{\mathcal{P}}^{VPIE} p_i^{(1)} G^{-1} \mathcal{P}^{(2)} \mathcal{F}^V, & \text{else} \end{cases}
\]

(38d)

\[
G_{kn} = \{ \mathcal{F}^V, \mathcal{F}^V_n \}
\]

(38f)

The LC-CSPIE discrete system is

\[
Z_{kn}^{SPIE} \gamma_{\sigma}^{SPIE} = b_{\sigma}^{SPIE}
\]

(39)

where the elements are defined by

\[
Z_{kn}^{SPIE} &= \begin{cases} \mathcal{F}^V_i, & k = 0 \\ \delta G^{-1} S_i^{(2)} \mathcal{F}^{SPIE} S_i^{(1)} G^{-1} S_i^{(1)} Z_i^{SPIE} S_i^{(2)} \mathcal{F}^V_n, & \text{else} \end{cases} \\
(40a)
\]

\[
\gamma_{\sigma}^{SPIE} = \begin{cases} \mathcal{F}^V_i, & \text{else} \end{cases}
\]

(40c)

\[
b_k^{SPIE} &= \begin{cases} \mathcal{F}^V_i, & k = 0 \\ \delta G^{-1} S_i^{(2)} \mathcal{F}^{SPIE} S_i^{(1)} G^{-1} S_i^{(1)} \alpha, & \text{else} \end{cases}
\]

(40d)

\[
G_{kn} = \{ \mathcal{F}^V_i, \mathcal{F}^V_n \}
\]

(40e)

A few notes are in order. First, the above integrations use singularity subtraction to handle the singularity region where the testing function is within 0.15\(\lambda\) of the basis function as detailed in [26].

D. Zero-Mean Constraint

Both \(\gamma = \hat{n} \cdot A(r)\) and \(\beta = \hat{n} \cdot \nabla \phi(r)\) have a zero-mean constraint (ZMC) [20]. This is enforced with two different methods. Method 1 is a rank-one update technique embedding the Lagrange multiplier in the forward Galerkin tested system by appending a list of basis function areas (entries corresponding to nonzero-mean constrained variables are zero) for \(\gamma\) and \(\beta\) unknowns to the rows and columns of the forward matrix [26]. Method 2 solves the same VPIE and SPIE Galerkin tested system with an RHS of incident potentials and an RHS of basis function areas (entries corresponding to nonzero-mean constrained variables are zero) as discussed in [32].

E. Fine-Grain Localization

Localizing (17) and (21) with a single complex wavenumber determined by the geometry’s max-mean curvature is not necessary and, in some cases, suboptimal. A simple generalization of this localization approach is computing the max-mean curvature of each edge and defining the max-mean curvature of a patch as the maximum of the mean curvatures of the patch’s edges. Note that when computing the discrete preconditioner, the RWG and hat interactions are constituted by the interactions of the patches composing the basis and testing functions. Then, for each source–observer patch pair, the localizing complex wavenumber is generated from the maximum of the max-mean curvatures of the sources and observers. Here, this is referred to as fine-grain localization, and we use this approach unless otherwise stated.

VII. RESULTS

In what follows, we will demonstrate that (a) the new formulations (17) and (21) have flat, low eigenvalue conditioning from low frequency to high-frequency, (b) the suggested SPIE in (17) also has flat, low singular value conditioning across the band, while (21) increases at low frequency, and (c) the preconditioned VPIE and SPIE formulations rapidly converge to potential quantities which generate the correct RCS plots for spherical, multiscale, and
arbitrary geometries in comparison to the standard CFIE. In all that examples that follow, \( \delta \) is chosen to be 0.5. This is very similar to the choice of weight for constructing the CFIE.

### A. Spectral Properties With Analytic Basis Sets

We now analyze the generalized eigenvalues of the MOM system for the various, suggested formulations. The eigenvalue condition number is denoted by \( \kappa(\lambda, Z) \) and defined as the ratio of maximum and minimum eigenvalue moduli. The singular value condition number is denoted by \( \kappa(Z) \). Figs. 1-3 show the eigenvalues of

\[
G^{-1}P^{(1)} \mathcal{V PIE} P^{(2)} G^{-1}P^{(2)} Z \mathcal{V PIE} P^{(1)} + (1 - \delta)G^{-1}P^{(1)} \mathcal{Z} \mathcal{V PIE} P^{(1)}
\]

and, respectively, for a unit sphere illuminated at 30 GHz. Each operator has bounded eigenvalues but some eigenvalues are near the origin of the complex plane.

Fig. 4 depicts the conditioning for frequencies between 1e-5 Hz and 1e12 Hz for formulations (41b) and (41c). The sharp increases in condition number align at all the shown frequencies. This alignment indicates that the two formulations share the same resonances, and, therefore, Fig. 4 numerically verifies the properties of new Calderón-type identities developed in Section V-B. Also, the condition numbers of formulations (41b) and (41c) increase \( \propto \omega \) in the high-frequency region. The more common Calderón preconditioning approach for the CFIE and the dielectric implementation of the DPIE show similar results [13], [26], [27].

Next, Figs. 5 and 6 show the eigenvalues for the LC-CVPIE and LC-CSPIE formulations, respectively, for a unit sphere illuminated at 30 GHz. The eigenvalues are bounded and collected away from the origin unlike the spectra in Fig. 1–3. The LC-CSPIE has an eigenvalue condition number of 4 and singular value condition number of 6. The spectrum of the LC-CVPIE has an eigenvalue condition number of 4 and singular value condition number of 4.

Fig. 7 is a eigenvalue condition number plot of the LC-CVPIE and LC-CSPIE formulations for frequencies between 1e-5 Hz and 1e12 Hz. The condition number is flat at low
frequency and does not increase \( \propto \omega \) in the high-frequency region unlike the formulations in Fig. 4.

Finally, Fig. 8 is a singular value condition number plot of the LC-CVPIE and LC-CSPIE for frequencies ranging from \( 1 \text{e-5} \) to \( 1 \text{e12} \) Hz. On one hand, the singular value conditioning for the LC-CVPIE formulation increases at low frequencies. Due to the complex wavenumber, the upper right operator of (18) is not equal to zero

\[
\lim_{\omega \to 0} \tilde{\mathcal{C}}' \mathcal{P}^{(2)} + \tilde{Q}^{(1)} S \neq 0. \tag{42}
\]

On the other hand, the LC-CSPIE possesses flat, wideband singular value conditioning. Fortunately, LC-CVPIE quickly converges using QMR, GMRES, and LGMRES to the Mie RCS at low frequency as is demonstrated in the next section. In other words, there is no practical cost to LC-CVPIE at low frequency.

B. Analysis of Spheres Using Piecewise Basis

Next, we predict RCS and benchmark the performance of LC-CVPIE, LC-CSPIE, and LC-CDPIE against the CVPIE, CSPIE, and CFIE formulations using the Rao–Wilton–Glisson (RWG) functions for vector quantities and hat functions for scalar quantities. Fig. 9 shows the RCS of a PEC sphere with radius 1 m illuminated at 10 \( \mu \)Hz with a plane wave of \( \hat{z} \) propagation axis and \( \hat{x} \) polarization, and the sphere is discretized with a mean edge length of \( \lambda/1\text{e14} \). The LC-CVPIE and LC-CSPIE converged in four and three iterations, respectively, using QMR with a tolerance of \( 1\text{e-5} \) and Method 1 to enforce the ZMC. The predicted RCS using the LC-CDPIE solution agrees with the predicted RCS of the Mie solution.
Furthermore, the LC-CDPIE formulation was tested with a PEC sphere of radius 0.5 m excited by a 1500-MHz frequency plane wave with \( \hat{z} \) propagation axis and \( \hat{x} \) polarization. The sphere discretization is \( \lambda/10 \), and the electrical length is 5\( \lambda \). The MOM system was solved using QMR with a tolerance of 1e-12 and ZMC Method 1. The LC-CVPIE converged in 32 iterations, while the LC-CSPIE converged in 24 iterations. The RCS predicted by the LC-CDPIE agrees with that of the Mie solution in Fig. 10.

Next, the iteration count of each formulation to converge using QMR with a tolerance of 1e-12 and ZMC Method 1 for spheres whose electrical lengths range from 1e-12\( \lambda \) to 5\( \lambda \) is shown in Fig. 11. Four different unit spheres are used in this test. For frequencies below 100 MHz, a sphere of average edge wavelength \( \lambda/10 \) at 100 MHz is used. For frequencies between 100 and 200 MHz, a sphere of average wavelength \( \lambda/10 \) at 200 MHz is used. For frequencies between 200 and 400 MHz, a sphere of average wavelength \( \lambda/10 \) at 400 MHz is used. Finally, for frequencies between 400 and 750 MHz, a sphere of average wavelength \( \lambda/10 \) at 750 MHz is used. The LC-CVPIE and LC-CSPIE formulations converge in less iterations than the CFIE, CVPIE, and CSPIE for all the tested electrical lengths.

The ZMC methods (Method 1 or 2) are specified because they affect the conditioning of the forward matrix. For example, the Lagrange multiplier in Method 2 increases the conditioning at higher frequencies, while the Lagrange multiplier removes an eigenvalue near the origin of the complex plane at low frequencies. The formulation has constant eigenvalue conditioning for the recorded frequency range, but the singular value conditioning increases as the frequency lowers for reasons discussed above. However, large low-frequency singular value condition numbers for the LC-CVPIE have no practical impact on the convergence or predicted RCS; the LC-CVPIE quickly converges in four iterations using QMR at 10 \( \mu \)Hz to the RCS of the Mie solution for a unit sphere.

The LC-CDPIE is also tested with a multiscale sphere of unit radius (as shown in Fig. 12) illuminated with a 400-MHz plane wave that propagates along the \( \hat{z} \) axis and is polarized along \( \hat{x} \) polarization. The electrical length of the mesh edges ranges from \( \lambda/11 \) to \( \lambda/406 \). The LC-VPIE and LC-SPIE formulations converge more quickly than the CFIE formulation using QMR, TFQMR, and GMRES. Converging to a residue of 1e-12 requires less than 100 iterations for the LC-CSPIE (Group 1 in Fig. 13) and less than 210 iterations for...
C. Analysis of Noncanonical Geometries Using Piecewise Basis Sets

The next set of benchmarking tests uses noncanonical geometries. We select a bumpy cube, a NASA Geographos asteroid as shown in Fig. 14, and a sharp pencil as shown in Fig. 19. The NASA Geographos asteroid is illuminated by a 240-MHz plane wave that propagates along $\hat{y}$ and polarized along $\hat{x}$. The electrical lengths of the mesh edges range from $\lambda/13$ to $\lambda/64$, and the electrical length of the object is $4\lambda$. The MOM systems are solved using QMR with a tolerance of 1e-12, ZMC Method 1, and the predicted RCS of the LC-CPIE and CFIE agrees in Fig. 15. The LC-CVPIE and LC-CSPIE formulations converge in 258 and 112 iterations, respectively, while the CFIE converges in 2027 iterations. The plot of residues and iteration counts in Fig. 16 shows that the LC-CVPIE and LC-CSPIE formulations converge more quickly to an arbitrary residue than the CFIE for the asteroid.

The bumpy cube is illuminated by a 400-MHz plane wave propagating along $\hat{y}$ and polarized along $\hat{x}$. The electrical lengths of the mesh edges for the bumpy cube range from $\lambda/12$ to $\lambda/50$. Furthermore, the object fits in a cube of size $1.67\lambda$. Again, the MOM systems are solved using QMR with a tolerance of 1e-12 and ZMC Method 1, and the predicted RCS of the LC-CDPIE and CFIE agrees in Fig. 17. The LC-CVPIE and LC-CSPIE formulations converge in 305 and 120 iterations, respectively, while the CFIE converges in 3761 iterations. Also, the plot of residues and iteration counts in Fig. 18 shows that the LC-CVPIE and LC-CSPIE formulations converge more quickly to an arbitrary residue than the CFIE for the bumpy cube and the asteroid.
The last and most challenging test is the analysis of scattering from a sharp pencil as shown in Fig. 19. This geometry’s curvature max-mean curvature is 2585 due to the sharp, narrow tip. The sharp pencil is illuminated by a 269-MHz plane wave with $\hat{y}$ propagation axis and $\hat{x}$ polarization. The electrical lengths of the mesh edges range from $\lambda/13$ to $\lambda/5648$. This range is greater than the multiscale sphere, bumpy cube, and asteroid. Also, the pencil’s electrical length is $9\lambda$. The MOM systems are solved using QMR with a tolerance of $1 \times 10^{-12}$ and ZMC Method 1. The LC-CDPIE, CFIE, and MFIE RCS plots agree in Fig. 20, but the LC-CDPIE and MFIE agree most closely while the CFIE departs from both near $\phi = -\pi/2$ and $\pi/2$.

The LC-CVPIE and LC-CSPIE formulations converge in 159 and 90 iterations, respectively, while the CFIE converges in 47,045 iterations and the MFIE in 1761 iterations. If the fine-grain localization approach is not used but rather the global max-mean curvature localization approach discussed in Section VI-E is used, the LC-CVPIE and LC-CSPIE converge in 34,764 and 91 iterations, respectively. Again, the plot of residues and iteration counts in Fig. 21 shows that the LC-CVPIE and LC-CSPIE formulations converge more quickly to any residue than both the CFIE and MFIE.

VIII. CONCLUSION

This article has presented new Calderón-type identities and used these identities to construct the LC-CVPIE, LC-CVPIE, and LC-CDPIE formulations for solving the PEC scattering problem with RWG and hat functions. The spectra of these formulations were shown to be bounded and nonzero, and the increase in singular value conditioning at low frequency for the LC-CVPIE was shown to be of no practical importance for iterative solvers. Indeed, these novel formulations are well-suited for efficiently predicting the scattering from multiscale, real-world PEC geometries.

APPENDIX A

DEFINITION OF OPERATORS

\[ S \left( \frac{x}{x} \right) = \int G(\mathbf{r}, \mathbf{r}^{'}) \left( \frac{x}{x} \right) dS' \] (43a)

\[ D(\mathbf{x}) = -\nabla \cdot S(\hat{x} \times \mathbf{x}) \] (43b)

\[ N(\mathbf{x}) = -\hat{n} \cdot \nabla S(\hat{x} \times \mathbf{x}) \] (43c)

\[ D'(\mathbf{x}) = \hat{n} \cdot \nabla G(\mathbf{x}, \mathbf{x}) \] (43d)

\[ K'(\mathbf{x}) = -\nabla \times S(\hat{n} \times \mathbf{x}) \] (43e)

\[ J^{(2)}(\mathbf{x}) = \hat{n} \times S(\hat{n} \times \mathbf{x}) \] (43f)

\[ J^{(3)}(\mathbf{x}) = \hat{n} \cdot S(\hat{n} \times \mathbf{x}) \] (43g)

\[ J^{(4)}(\mathbf{x}) = \nabla \cdot S(\hat{n} \times \mathbf{x}) \] (43h)

\[ L(\mathbf{x}) = \frac{1}{k^2} \nabla \times \nabla \times S(\mathbf{x}) \] (43i)

\[ K(\mathbf{x}) = \hat{n} \times \nabla \times S(\mathbf{x}) \] (43j)

\[ M^{(3)}(\mathbf{x}) = \hat{n} \cdot \nabla \times S(\mathbf{x}) \] (43k)

\[ P^{(2)}(\mathbf{x}) = \hat{n} \cdot \nabla S(\mathbf{x}) \] (43l)

\[ Q^{(1)}(\mathbf{x}) = \hat{n} \times \hat{n} \times \nabla \times S(\hat{n} \times \mathbf{x}) \] (43m)

\[ Q^{(2)}(\mathbf{x}) = \hat{n} \times S(\hat{n} \times \mathbf{x}) \] (43n)

\[ Q^{(3)}(\mathbf{x}) = \hat{n} \cdot S(\hat{n} \times \mathbf{x}) \] (43o)

Prime denotes an adjoint operator, and superscript $t$ is shorthand for $O' = \hat{n} \times \hat{n} \times O$. 

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APPENDIX B

SPHERICAL HARMONICS

\[
Y_n^m(\hat{r}) = \frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!} P_n^m(\cos \theta)e^{im\phi}
\]

(44a)

\[
\mathbf{\Psi}_n^m(\hat{r}) = -\hat{r} \times \mathbf{\Phi}_n^m(\hat{r}) = c_n \hat{r} \times \nabla Y_n^m(\hat{r})
\]

(44b)

\[
\mathbf{\Phi}_n^m(\hat{r}) = \hat{r} \times \mathbf{\Psi}_n^m(\hat{r}) = c_n \hat{r} \times \nabla Y_n^m(\hat{r})
\]

(44c)

\[
c_n = \begin{cases} 
1, & n = 0 \\
\frac{1}{\sqrt{n(n+1)}}, & n \neq 0 
\end{cases}
\]

(44d)

with \( P_n^m(\cos \theta) \) being the associated Legendre polynomials and \( n \geq 0 \) and \( |m| \leq n \).

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