Geometric phase accumulated in a driven quantum system coupled to an structured environment

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We study the role of driving in a two-level system evolving under the presence of a structured environment. We find that adding a periodical modulation to the two-level system can greatly enhance the survival of the geometric phase for many time periods. We show that this effect is mainly important for a markovian regimen but can still be found when the environment exhibits non-markovian behavior as time revivals. This knowledge can aid the search for physical set-ups that best retain quantum properties under dissipative dynamics.

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The state of a point like discrete energy level quantum system interacting with a quantum field acquires a geometric phase (GP) that is independent of the state of the field [1]. The phase depends only on the system’s path in parameter space, particularly the flux of some gauge field enclosed by that path. Due to its topological properties and close connection with gauge theories of quantum fields, the GP has recently become a fruitful venue of investigation to infer features of the quantum system. For pure field states, the GP is said to encode information about the number of particles in the field [2].

If the field is in a thermal state, the GP encodes information about its temperature, and so it has been used in a proposal to measure the Unruh effect at low accelerations [3]. Furthermore, in [4], it has been proposed as a high precision thermometer in order to infer the temperature of two atoms interacting with a known hot source and an unknown temperature cold cavity. In this context, the study of the GP in open quantum systems has been a subject of investigation lately. The definition of the geometric phase for nonunitary evolution was first stated in [5]. This definition has been used to measure the corrections of the GP in a non-unitary evolution [6] and to explain the noise effects in the observation of the GP in a superconducting qubit [7, 8].

The geometric phase of a two-level system under the influence of an external environment has been studied in a wide variety of scenarios [9]. It has further been used to track traces of quantum friction in a very simplistic model of an atom coupled to a scalar quantum field [10].

The coupling of the quantum system to the environment is described by the spectral density function. If the system couples to all modes of the environment in an equal way the spectrum of the reservoir is flat. If, otherwise, the spectral density function strongly varies with the frequency of the environmental oscillators, the environment is said to be structured. In this type of environment the memory effects induce a feedback of information from the environment into the system. They are therefore called non-markovian [11].

Numerous works have investigated the presence of non-markovianity in a variety of scenarios in quantum open systems so as to determine whether non-markovianity is a useful resource for quantum technologies. It has been studied how the presence of a driving field affects the non-markovian features of a quantum open system. For instance, studies which assessed the effectiveness of optimal control methods [12, 13] in open quantum system evolutions showed that non-markovianity allowed for an improved controllability [14–16]. Likewise, the non-markovian effects were associated to the reduction of efficiency in dynamical decoupling schemes [17] and accounted for corrections to the GP acquired [18–20].

In this work, we investigate to what extent external driving acting solely on the system can increase non-markovianity (and therefore modify the geometric phase) with respect to the undriven case. To this end, we consider a two-level system described by a time-periodic Hamiltonian interacting with a structured environment. It has been recently shown that the driving has a peculiar effect on the non-markovian character of the system dynamics: it can generate a large enhancement of the degree of non-markovianity with respect to the static case for a weak coupling between the system and environment [21].

The importance of the driven two-state model is especially pronounced in quantum computation and quantum technologies, where one or more driven qubits constitute the basic building block of quantum logic gates [22]. Different implementations of qubits for quantum logic gates are subjected to different types of environmental noise, i.e., to different environmental spectra. We focus only in the weak coupling since we try to track traces of the geometric phase, which is literally destroyed under a strong influence of the environment. This means that while there are non-markovian effects that induce a correction to the unitary GP, the system maintains its purity for several cycles, which allows the GP to be observed. It is important to note that if the noise effects induced on the system are of considerable magnitude, the coherence terms of the quantum system are rapidly destroyed and...
the GP literally disappears [9]. This knowledge can aid the search for physical set-ups that best retain quantum properties under dissipative dynamics.

I. THE MODEL

We shall consider a two-level system described by a time-periodic Hamiltonian interacting with an environment. The total Hamiltonian which describes this model reads (we set $\hbar = 1$ from here on)

$$H = \tilde{\omega}_0(t)\sigma_+\sigma_- + V \sum_k (g_k b_k + g_k^* b_k^\dagger) + \sum_k \tilde{\omega}_k b_k^\dagger b_k,$$

where $V = \sigma_x$ and $b_k, b_k^\dagger$ the annihilation and creation operators corresponding to the $k$-th mode of the bath. The coupling constant is $g_k$ and $\tilde{\omega}_0(t)$ is the time-dependent energy difference between the states $|0\rangle$ and $|1\rangle$ of the two-level system. We shall assume it has the following form:

$$\tilde{\omega}_0(t) = \tilde{\Omega} + \tilde{\Delta} \cos(\tilde{\omega} DT).$$

The exact dynamics of the system in the interaction picture has been derived in [23]. If the qubit and the bath are initially in a separable state, i.e. $\rho(0) = \rho_s(0) \otimes \rho_b$, the formal solution is:

$$\rho_S^{(f)}(t) = \mathcal{T} \exp \left( -\int_0^t dt_2 \int_0^{t_2} dt_1 V^\times (t_2) \right) \left[ C^R(t_2-t_1)V^\times (t_1) + iC^I(t_2-t_1)V^\circ (t_1) \right],$$

where $\mathcal{T}$ is the chronological time-ordering operator and we have introduced the following notation $A^\times B = [A, B] = AB - BA$ and $A^\circ B = A, B = AB + BA$. $C^R(t_2-t_1)$ and $C^I(t_2-t_1)$ are the real and imaginary parts of the bath time-correlation function, defined as

$$C(t_2-t_1) = \langle B(t_2)B(t_1) \rangle = \text{Tr}[B(t_2)B(t_1)\rho_B] = \int_0^\infty d\omega J(\omega) e^{-i\omega(t_2-t_1)}$$

and

$$B(t) = \sum_k \left( g_k b_k \exp(-i\omega_k t) + g_k^* b_k^\dagger \exp(i\omega_k t) \right)$$

Eq.(4) is difficult to solve directly. An effective method for obtaining a solution has been developed by defining a set of hierarchy equations [23–25]. The key condition in deriving the hierarchy equations is that the correlation function can be decomposed into a sum of exponential functions of time. At finite temperatures, the system-bath coupling can be described by the Drude spectrum, however, if we consider qubit devices, they are generally prepared in nearly zero temperatures. Then we shall consider a Lorentz type spectral density $J(\omega)$,

$$J(\omega) = \frac{\gamma_0}{2\pi} \frac{\lambda^2}{(\omega - \tilde{\Omega})^2 + \lambda^2},$$

and the hierarchy method can also be applied [26]. As has been stated in [21], this method can be used if i) the initial state of the system plus bath is separable, ii) the interaction Hamiltonian is bilinear, and iii) if the environmental correlation function can be cast in multi-exponential form. In this case, $\gamma_0$ is the coupling strength between the system and the bath and $\lambda$ characterizes the broadening of the spectral peak, which is connected to the bath correlation time $\tau_c = \lambda^{-1}$. The relaxation time scale at which the state of the system changes is determined by $\tau_r = \gamma_0^{-1}$. At zero-temperature, if we consider the bath in a vacuum state, the correlation function can be expressed as

$$C(t_2-t_1) = \frac{\lambda\gamma_0}{2} \exp[-(\lambda + i\Omega)|t_2-t_1|]$$

which is the exponential form required for the hierarchy method. The advantage of solving the dynamics of the system by this method is that we can take an insight of different regimes of the dynamics. For example, in the limiting case $\gamma_0 \ll \lambda$, i.e. $\tau_r \ll \tau_c$, we have a flat spectrum and the correlation tends to $C(t_2-t_1) \rightarrow \gamma_0\delta(t_2-t_1)$. This is the so called markovian limit. Therefore, we can study the full spectrum of behavior by solving the hierarchy method, which can be expressed as

$$\frac{d}{d\tau} \rho_\tau(\tau) = -iH_s[\tau]^\times + \tilde{n}\tilde{\nu}^\dagger \rho_\tau(\tau) - i \sum_{k=1}^{2} \left[ V^\times \rho_{\tilde{n}+\tilde{e}_k}(\tau) - i\frac{\gamma_0}{2} \sum_{k=1}^{2} n_k [V^\times + (-1)^k V^\circ] \rho_{\tilde{n}-\tilde{e}_k}(\tau) \right],$$

where we have defined dimensionless parameters vari-
with units of energy in the model described. The subscript $\mathbf{n} = (n_1, n_2)$ with integers numbers $n_1(2) \geq 0$, and $\rho_\mathbf{n}(t) \equiv \rho_{(0,0)}(t)$. This means that the “physical” solution is encoded in $\rho_{(0,0)}(t)$ and all other $\rho_\mathbf{n}(\tau)$ with $\mathbf{n} \neq (0,0)$ are auxiliary operators implemented for the sake of computation. The vectors $\mathbf{e}_1 = (1,0)$, $\mathbf{e}_2 = (0,1)$ and $\mathbf{b} = (\nu_1, \nu_2) = (1 - i\Omega, 1 + i\Omega)$. The hierarchy equations are a set of linear differential equations, that can be solved by using a Runge Kutta routine. For numerical computations, the hierarchy equations must be truncated for large $\mathbf{n}$. The hierarchy terminator equation is similar to that of Eq.(7) for the term $\mathbf{n}$, and the corresponding term related to $\rho_{\mathbf{N}+\mathbf{e}_k}$ is dropped [23]. The numerical results in this paper have been all tested and converged, using a maximum value of $\mathbf{N} = (25,25)$. We shall take advantage of this model, whose non-markovian properties has been studied in [21], and set the scenario to study the corrections to the GP for a driven two-level system.

II. ENVIRONMENTALLY INDUCED DYNAMICS

We begin by studying the environmentally induced dynamics by considering a qubit with no driving at all ($\Delta = 0$). In this case, we must consider a qubit and a dipolar coupling to the cavity mode, for example. This means, that the dynamics of the system contemplates decoherence and dissipation as well as variation of the population numbers (in contrast to the spin boson model). The density matrix for this case has a formal expression as

$$\rho_\mathbf{n}(\tau) = \begin{pmatrix} \rho_{11}|G(\tau)|^2 & \rho_{12}G(\tau) \\ \rho_{21}G^*(\tau) & 1 - \rho_{11}|G(\tau)|^2 \end{pmatrix}$$

where $G(\tau)$ is a single-complex valued function that characterizes the dynamics of the system. We herein do not write its explicit form since we shall solve the problem numerically through the hierarchy approach.

Qualitatively, decoherence can be thought of as the deviation of probabilities measurements from the ideal intended outcome. Therefore decoherence can be understood as fluctuations in the Bloch vector $R$ induced by noise. Since decoherence rate depends on the state of the qubit, we will represent decoherence by the change of $R$ in time, starting from $R = 1$ for the initial pure state, and decreasing as long as the quantum state loses purity. The contributions of the bath to the dynamics of the system, including both dissipation and Lamb shift, are fully contained in the hierarchy equation. In Fig. 1 we present the absolute value of the Bloch vector of the state system $R = |R| = \sqrt{x^2 + y^2 + z^2}$ as a function of time measured in natural cycles $\omega_0 \tau = N2\pi$ for different values of $\gamma_0$. In this case, we can note that the trajectory differs substantially from the unitary one, meaning the system's dynamics is affected by the noise effects. In the case the unitary dynamics is considered, $\gamma_0 = 0$ and $R = 1$ for all times.

![FIG. 1](color online) We plot the loss of the quantum state purity $R$ as function of time (Number of cycles). We can see that as the coupling constant with the bath increases for a fixed value of $\lambda$ ($\gamma_0 = \gamma_0/\lambda$), the dynamical behavior is modified. Orange dashed line is $\gamma_0 = 0.001$ and dot dashed purple line for $\gamma_0 = 0.0007$ represent situations of $\tau_f > \tau_c$. Dotted magenta line is for $\gamma_0 = 0.01$, blue solid line for $\gamma_0 = 0.005$ are situations of $\tau_f \sim \tau_c$. Parameters used: $\Delta = 0$, $\mathbf{n} = (25,25)$, $\Omega = 20$.

We can notice that the dynamical behavior is modified as the coupling constant $\gamma_0$ is increased. It is interesting to see the interplay between time and $\gamma_0$: a stronger bath can initially produce less damage on the dynamics but has a stronger effect in the renormalization of the frequency. A weak-coupling has a more “adiabatic” modification of the dynamics in an equal period of time. In Fig.1, we have set $\lambda$ fixed. As $\gamma_0$ increases, the relaxation time $\tau_f$ of the system decreases and $\tau_f \sim \tau_c$. The presence of oscillations in the Bloch vector $R$ for short times, as $\gamma_0$ becomes similar to $\lambda$ indicates non-markovian dynamics induced by the reservoir memory and describing the feedback of information and/or energy from the reservoir into the system [17]. We can see that as long as $\gamma_0/\lambda < 1$, the systems exhibits a markovian dynamics (orange-dashed line and purple dotdashed line). On the other side, if $\gamma_0/\lambda \geq 1$, there are non-markovian features in the systems’ dynamics. We can notice that as long as $\gamma_0/\lambda < 1$ and $\lambda < 1$, the behavior remains similar to that of $\lambda \rightarrow 0$ (markovian since $\tau_c \rightarrow \infty$). However, as $\lambda$ increases, the environmentally-induced dynamics is considerably modified, introducing oscillations again. So, with this kind of environment we can simulate different regimes by the solely selection of the $\gamma_0$ and $\lambda$ parameters.

In Fig.2, we compare the dynamics of two different environmental situations: the left column is for $\gamma_0/\lambda < 1$, and the right one for $\gamma_0/\lambda \geq 1$, both evolutions are sim-
updated for fixed \( \bar{\Omega} \) and zero driving (\( \Delta = 0 \)). In this example, we can see that when \( \tau_c < \tau_r \), the system presents a markovian evolution. On the other hand, if \( \tau_c > \tau_r \), non-markovian effects can be seen, for example by accelerating the transition between quantum states and revivals for longer times. For initial short times, the spontaneous decay of the atom can not only be suppressed or enhanced, but also partly reversed, when non-markovian oscillations induced by reservoir memory effects are present. As has been shown, by choosing the right set of parameters, we can simulate different type of environments.

III. CORRECTION TO THE GEOMETRIC PHASE

In this section, we shall compute the geometric phase for the central spin and analyze its deviation from the unitary geometric phase for a two-level driven system. A proper generalization of the geometric phase for unitary evolution to a non unitary evolution is crucial for practical implementations of geometric quantum computation. In [5], a quantum kinematic approach was proposed and the geometric phase (GP) for a mixed state under non-unitary evolution has been defined as

\[
\Phi = \text{arg}(\sum_k \sqrt{\epsilon_k(0)\epsilon_k(\tau)}\langle \psi_k(0) | \psi_k(\tau) \rangle) \times e^{\int_0^\tau dt \langle \dot{\psi}_k | \dot{\psi}_k \rangle},
\]

(8)

where \( \epsilon_k(t) \) are the eigenvalues and \( | \psi_k \rangle \) the eigenstates of the reduced density matrix \( \rho_t \) (obtained after tracing over the reservoir degrees of freedom). In the last definition, \( \tau \) denotes a time after the total system completes a cyclic evolution when it is isolated from the environment. Taking the effect of the environment into account, the system no longer undergoes a cyclic evolution. However, we will consider a quasi cyclic path \( \tau \in [0, \tau_S] \) with \( \tau_S = 2\pi/\omega_0 \) (the system’s dimensionless frequency).

To this end, we shall start by choosing a pure state of the form:

\[
|\Psi(0)\rangle = \cos(\theta_0/2)|0\rangle + \sin(\theta_0/2)|1\rangle.
\]

We shall solve the master equation and then, compute the geometric phase acquired by the quantum system. If the environment is strong, then the unitary evolution is destroyed in a decoherence time \( \tau_D \). Otherwise, we can imagine an scenario where the effect of the environment is not so drastic. In the following, we shall focus on how driving can affect (or even benefit) the measurement of the geometric phase under different regimes, both markovian and non-markovian. In particular, we shall investigate to what extent external driving acting solely on the system can correct the geometric phase with respect to the undriven or unitary case.

A. Geometric phase under a markovian environment

As we are dealing with a structured environment, we shall start studying a markovian regime (\( \bar{\gamma}_0/\lambda < 1 \)) with a unitary evolution to see how different the evolution is and decide whether the geometric phase can be measured in such scenario. In Fig. 3, we show the total geometric phase accumulated for the non-unitary (red circled line) and unitary (blue asterisk line) evolution as time evolves, being the number of cycles \( N = \tau/\tau_S \). Therein, it is possible to see that initially the geometric phases are similar. As time evolves, the difference among both lines increases as expected, since for long times the loss of purity of the system would be considerable. The dynamics of the driven two-level system comprises of three different dynamical effects, occurring each at a different timescale. Dissipation and decoherence occur at the relaxation timescale \( \tau_\text{r} \), non-markovian memory effects occur at times shorter or similar to the reservoir correlation timescale \( \tau_c \) [17]. Finally, nonsecular terms cause oscillations in a timescale of the system \( \tau_S = (\Omega^2 + \Delta^2)^{-1/2} \). Generally, this nonsecular terms can be neglected when
$\gamma_0 \ll \lambda$ (circled red dotted line) (corresponding to a similar situation to that of the left column of Fig. 2).

$\tau_c \ll \tau_S$. We shall consider the secular regime, by assuming $\tau_S \ll \tau_c$, and in the markovian regime $\tau_S \ll \tau_c \ll \tau_R$. As we are interested in studying the role of driving, we shall first consider the effect of adding $\Delta$ to the model.

In Fig. 4, we show the geometric phase acquired by the two level compared to the case when $\Delta = 0$. As expected, if $\Delta$ is added to the system, then the geometric phase acquired would be different from that with $\Delta = 0$, modifying the timescales involved. We plot the normalized geometric phase ($\phi_g/\phi_u$) so as to make the difference noticeable. The difference becomes relevant as the number of cycles increases. We can see the $\Delta = 0$ geometric phase represented by a dotted black line, $\Delta = 0.3$ magenta asterisk line, $\Delta = 0.5$ blue triangle dotdashed line and $\Delta = 1$ circle orange line. Parameters used: $\gamma_0 = 0.0001$, $\Omega = 20$ and $\omega_D = 0$.

In Fig. 5, we show the geometric phase acquired when low-frequency driving is added: blue asterisk line correspond to $\omega_D = 0.1$ and $\Delta = 0.3$, Magenta line is for $\Delta = 0$, red dotted line is for $\Delta = 0.5$ and $\omega_D = 0.3$, the green circled solid line for $\Delta = 0.2$ and $\omega_D = 0.8$ and the orange circled dotdashed line for $\Delta = 0.1$ and $\omega_D = 0.5$. Black dotted unitary geometric phase is included for a reference. Low-frequency driving corrects the geometric phase accumulated for very short times.

We shall therefore study the interplay of adding driving to the two level system. In particular, we shall focus on the effect of the driving in the possibility of measuring the geometric phase acquired by the two-state particle. In Fig. 5, we show the geometric phase acquired when low-frequency driving is added: blue asterisk line correspond to $\omega_D = 0.1$ and $\Delta = 0.3$. Magenta line is for $\Delta = 0$, red dotted line is for $\Delta = 0.5$ and $\omega_D = 0.3$, the green circled solid line for $\Delta = 0.2$ and $\omega_D = 0.8$ and the orange circled dotdashed line for $\Delta = 0.1$ and $\omega_D = 0.5$. Black dotted unitary geometric phase is included for a reference. In the zoom plot we show the geometric phase acquired for $\Delta = 0.3$ and $\omega_D = 0$, and compared it to...
\[ \Delta = 0.3 \text{ and } \omega_D = 0.1. \] We can see that the driven system acquires a geometric phase closer to the unitary one for longer periods of time. This is an exciting result. In the main plot of Fig. 5, we can see that other driven systems are closer to the unitary geometric phase for ten periods as well. Therefore, we can see that for some parameters, driving “preserves purity”. The geometric phase acquired is more similar to the unitary geometric phase acquired when there is no a low frequency driving added for some values of \( \Delta \).

In Fig. 6, we explore this result by representing the normalized geometric phase \( \phi_D/\phi_u \) as function of \( \Delta \) and \( \omega_D \), for different periods evolved \( (N) \): in (a) \( N = 4 \), (b) \( N = 8 \) and (c) \( N = 12 \). It is easy to note that adding a low frequency driving for short time evolutions renders a geometric phase similar to the unitary geometric phase, which leads to a good scenario of measuring the geometric phase in structure environments. For low-frequency driving and small value of \( \Delta \), geometric phases acquired are very similar to the non-driven isolated geometric phases. In fact, a similar result has already been stated in [21] for a driven two-level system in open quantum systems. It has been shown that for low-frequency driving and low driven frequency, the driving fails to increase the degree of non-markovianity with respect to the static case. The use of driven systems can help the measurement of geometric phases under some set of parameters. This knowledge can aid the search for physical set-ups that best retain quantum properties under dissipative dynamics.

\[ \begin{array}{c}
\text{FIG. 6: (Color online)} \, \phi_D/\phi_u \text{ for different values of } \Delta \text{ and } \omega_D \text{ for different times: (a) } N = 4, \text{ (b) } N = 8 \text{ and (c) } N = 12. \text{ For low-frequency driving and low driven frequency, geometric phases acquired are very similar to the non-driven isolated geometric phases.}
\end{array} \]

B. Geometric phase under non markovian environment

In the above section we show the geometric phase acquired by the two-level driven system in a structured environment. The above selection of parameters renders a markovian situation where one can still find evidence of a quasi-cyclic evolution, since the degradation of the pure state is done slowly and there are no revivals. Herein, we shall show what happens if the parameters are chosen so as to model a non-markovian environment \( \gamma_0/\lambda > 0.25 \), as it has been said in [21]. In such a case, the evolution is wildly modify and one can find revivals after a given number of periods. This shall help us to understand the role of driving in this type of environments. If we set parameters so as to see non-markovian behavior, then we must mention that finding tracks of the geometric phase can be much more difficult. In Fig. 7, we show two different scenarios by setting the model parameters. On top of Fig. 7, we show the temporal evolution of the Bloch vector for different driven frequencies with \( \gamma_0 \) and fixed \( \lambda \). As can be seen, this type of environment starts to exhibit non-markovian environment though revivals are small in amplitude: the magenta line represents \( \Delta = 0.0 = \omega_D \); orange line \( \Delta = 0.5 \); red dotted line \( \omega_D = 0.1 \) and \( \Delta = 0.5 \); and dotted green line \( \omega_D = 0.5 \) and \( \Delta = 0.5 \) and dotted cyan line \( \omega_D = 0.9 \) and \( \Delta = 0.5 \). We can easily note that the amount of driving changes considerably the evolution of the initial quantum state (modifies \( \tau_e \)). On the top right corner we show the populations probability for different lines: solid line, represents the \( \Delta = 0.0 = \omega_D \), the orange dotted line \( \Delta = 0.5 \), and the cyan dashed line \( \omega_D = 0.1 \) and \( \Delta = 0.5 \); and dotted green line \( \omega_D = 0.5 \) and \( \Delta = 0.5 \). We can see that by adding a frequency \( \Delta \) and a driving frequency \( \omega_D \), revivals disappear, recuperating the opportunity to track
a geometric phase. This fact can be easily observed in the Bloch sphere. At the bottom of Fig. 7, we represent the evolution in Bloch sphere of the initial state of the three different sets of parameters for the same number of cycles evolved. We can see that the transition among states is done in a short time for the magenta line. The revivals stimulate the exploration of the south pole of the Bloch sphere, for another period of time until it finally decays. In such an evolution, one can only achieve a geometric phase during the revivals and compare it to the one the system would have acquired if it has started at that latitude of the Bloch sphere. In the case of the orange line, transition among states is delayed by the frequency change of the system’s period \( \tau_s = 2\pi/(\Omega + \Delta) \). In such case, the geometric phase can be measured for very short initial periods. Finally, for the cyan curve we can observe that initially the state remains at its original latitude for almost 3 cycles before starting the transition among states.

We can therefore compute the geometric phase for this different situations in order to see if it is possible to track traces of an accumulated geometric phase during the evolutions. In Fig. 8 we show the geometric phase accumulated for different set of parameters (as those considered in Fig. 7). The colors of the lines in Fig. 8 correspond to the same values of Fig. 7. The magenta line (with dots) is the temporal evolution of an initial state under a structured environment in a non-markovian regime with \( \Delta = 0 \). In this case after 4 periods, the evolution presents some revivals after having made a transition between states (therefore revivals are done in the south pole sphere). This is easily understand with the information given in Fig. 7 where we see that transition is done at very short times. Therefore, the geometric phase acquired for \( \Delta = 0 \) is very different to that the system would have acquired in a markovian regime (black squared solid line) or an isolated evolution (black dotted line). However, in Fig. 8, we also present the geometric phase for driven systems under non-markovian regime. The diamond orange line represents a driven case of \( \omega_D = 0.5 \) and \( \Delta = 0.5 \). In such situation, we see that the evolution of the system initially recovers some “unitarity”, acquiring a geometric phase very similar to that of the unitary case; an after some periods finally makes
When designing experimental set-ups to measure the geometric phase, it is important to control the geometric phase. This fact contributes to the observation of geometric phases in various situations, including those where driving enhances the “robustness” condition of the geometric phase when Ω delays the revivals and ω_D is small. We have seen that for a low frequency driving, the system’s dynamic tends to be corrected towards the undriven one, leading to a scenario where the geometric phase can still be found. It has been argued that the observation of GPs should be done in times long enough to obey the adiabatic approximation but short enough to prevent decoherence from deleting all phase information. This means that while there are dissipative and diffusive effects that induce a correction to the unitary GP, the system maintains its purity for several cycles, which allows the GP to be observed. It is important to note that if the noise effects induced on the system are of considerable magnitude, the coherence terms of the quantum system are rapidly destroyed and the geometric phase literally disappears. For a markovian regime, it becomes a useful situation so as to get control of the geometric phase. This fact contributes to the result obtained in [21] and should be taken into consideration when designing experimental set-ups to measure geometric phases.

**IV. CONCLUSIONS**

In this manuscript, we have focused on the hierarchy equations of motion method in order to study the interplay between driving and geometric phases. This method can be used if (i) the initial state of the system plus the bath is separable and (ii) the interaction Hamiltonian is bilinear. The results obtained herein can be compared to those derived by obtaining the equation of motion under the rotating wave approximation. It results in an advantageous method since it provides a tool to simulate markovian and non markovian behavior in the structured spectrum.

We have therefore studied the dynamics of the system and computed the geometric phase for different environment regimes. In all cases we have focused on the effect of adding driving to the two-state system. We have seen that for a low frequency driving, the system’s dynamic tends to be corrected towards the undriven one, leading to a scenario where the geometric phase can still be found. It has been argued that the observation of GPs should be done in times long enough to obey the adiabatic approximation but short enough to prevent decoherence from deleting all phase information. This means that while there are dissipative and diffusive effects that induce a correction to the unitary GP, the system maintains its purity for several cycles, which allows the GP to be observed. It is important to note that if the noise effects induced on the system are of considerable magnitude, the coherence terms of the quantum system are rapidly destroyed and the GP literally disappears. For a markovian regime, it...
has been shown that for a low-frequency driving and small value of $\Delta$, geometric phases acquired are very similar to the non-driven isolated geometric phases. For a non-markovian regime, a similar result has been obtained, taking into account that the dynamics of the system is more complicated. However, we have seen that for low-frequency driving, the driving fails to increase the degree of non-markovianity with respect to the static case, recuperating in some cases a scenario where a geometric phase can still be measured ($\phi_g = \phi_u + \delta\phi$). This knowledge can aid the search for physical set-ups that best retain quantum properties under dissipative dynamics.

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