FOURIER COEFFICIENTS OF AUTOMORPHIC FORMS AND INTEGRABLE DISCRETE SERIES

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Abstract. Let $G$ be the group of $\mathbb{R}$–points of a semisimple algebraic group $\mathcal{G}$ defined over $\mathbb{Q}$. Assume that $G$ is connected and noncompact. We study Fourier coefficients of Poincaré series attached to matrix coefficients of integrable discrete series. We use these results to construct explicit automorphic cuspidal realizations, which have appropriate Fourier coefficients $\neq 0$, of integrable discrete series in families of congruence subgroups. In the case of $G = Sp_{2n}(\mathbb{R})$, we relate our work to that of Li [16]. For $\mathcal{G}$ quasi–split over $\mathbb{Q}$, we relate our work to the result about Poincaré series due to Khare, Larsen, and Savin [13].

1. Introduction

Let $G$ be the group of $\mathbb{R}$–points of a semisimple algebraic group $\mathcal{G}$ defined over $\mathbb{Q}$. Assume that $G$ is connected and noncompact. Let $K$ be its maximal compact subgroup, $\mathfrak{g}$ be the real Lie algebra of $G$, and $\mathcal{Z}(\mathfrak{g}_\mathbb{C})$ the center of the universal enveloping algebra of the complexification of $\mathfrak{g}$. In this paper we assume that $\text{rank}(K) = \text{rank}(G)$ so that $G$ admits discrete series. Let $\Gamma$ be a discrete subgroup of finite covolume in $G$. The problem of (mostly asymptotic) realization of discrete series as subrepresentations in the discrete part of the spectrum of $L^2(\Gamma\backslash G)$ has been studied extensively using various methods such as cohomology of discrete subgroups and adelic Arthur trace formula [25], [7], [24], [11], [8]. But Fourier coefficients of such realizations are not well understood. On the other hand, assuming that $\mathcal{G}$ is a quasi-split almost simple algebraic group over $\mathbb{Q}$, for any appropriate generic integrable discrete series $\pi$ of $G$, Khare, Larsen, and Savin ([13] Theorem 4.5) construct globally generic automorphic cuspidal representation $W$ of $\mathcal{G}(\mathbb{A})$, where $\mathbb{A}$ is the ring of adeles of $\mathbb{Q}$, such that its Archimedean component is $\pi$ i.e., $W_\infty = \pi$. They use classical theory of Poincaré series attached to matrix coefficients of $\pi$ which are $\mathcal{Z}(\mathfrak{g}_\mathbb{C})$–finite and $K$–finite on the right extended to adelic settings ([4], [3], [2]). A detailed treatment of such series and criteria for being $\neq 0$ in the adelic settings can be found in [19]. In ([13] Theorem 4.5), Khare, Larsen, and Savin in fact prove an analogue of a well–known result of Vignéras, Henniart and Shahidi [20] for generic supercuspidal representations of semisimple $p$–adic groups. ([13] Theorem 4.5) is used to study problems in inverse Galois theory. On the other hand, (possibly degenerate) Fourier coefficients of automorphic forms are important for the theory of automorphic $L$–functions [26], [9], [27]. So, it is important to study (possibly degenerate) Fourier coefficients of Poincaré series attached to matrix coefficients of integrable discrete series which are $K$–finite. This is the goal of the present paper. The techniques

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used in this paper are refinements of those of Khare, Larsen, and Savin used in the proof of ([13] Theorem 4.5), and those of (for adelic or for real groups) used in ([20], [18], [21], [23]. We do not improve only on results in generic case (see Theorems 6-7 and 6-9) but we also construct very explicit degenerate cuspidal automorphic models of integrable discrete series [15] for $Sp_{2n}(\mathbb{R})$ (see Theorem 6-12).

The paper actually has two parts: preliminary local Archimedean part (Sections 2, 3, 7), and main cuspidal automorphic part (Sections 4, 5, 6). Integrable discrete series for $G$ are analogues of supercuspidal representations for semisimple $p$–adic groups. In Section 3 we refine and generalize ([13], Proposition 4.2) discussing the analogue of the following simple result for $p$–adic groups which we explain in detail. Assume for the moment that $G$ is a semisimple $p$–adic group, $U$ closed unimodular subgroup of $G$, and $\chi : U \rightarrow \mathbb{C}^\times$ a unitary character. Let $(\pi, \mathcal{H})$ be an irreducible supercuspidal representation acting on the Hilbert space $\mathcal{H}$. Let us write $\mathcal{H}^\infty$ for the space of smooth vectors (i.e., the subspace of all vectors in $\mathcal{H}$ which have open stabilizers). The space $\mathcal{H}^\infty$ is usual smooth irreducible algebraic representation of $G$. We say that $\pi$ is $(\chi, U)$–generic if there exists a non–zero (algebraic) functional $\lambda : \mathcal{H}^\infty \rightarrow \mathbb{C}$ which satisfies $\lambda(\pi(u)h) = \chi(u)\lambda(h)$, $u \in U$, $h \in \mathcal{H}^\infty$. Let $\varphi$ be a non–zero matrix coefficient of $(\pi, \mathcal{H}^\infty)$. Then, $\varphi \in C_\infty^c(G)$. Next, a simple argument (see Lemma 3-7), first observed by Miličić in his unpublished lecture notes about $SL_2(\mathbb{R})$, Schur orthogonality relations show that for each $h \in \mathcal{H}^\infty$, we can select a matrix coefficient $\varphi$ such that $\pi(\varphi)h = h$. Now, since $h = \pi(\varphi)h = \int_G \varphi(g)\pi(g)hdg$ is essentially a finite sum, we have

$$\lambda(h) = \lambda(\pi(\varphi)h) = \int_G \varphi(g)\lambda(\pi(g)h)dg = \int_{U \setminus G} \left( \int_U \varphi(ug)\chi(u)du \right) \lambda(\pi(g)h)dg.$$  

So, selecting $h$ such that $\lambda(h) \neq 0$, we have that $\varphi$ has a non–zero Fourier coefficient

\[ F_{(\chi, U)}^{loc}(\varphi)(g) \equiv \int_U \varphi(ug)\overline{\chi(u)}du, \quad g \in G. \]  

Going back to our original meaning for $G$ and assuming that $(\pi, \mathcal{H})$ is an integrable discrete series acting on a Hilbert space $\mathcal{H}$, we try to adjust these consideration where now $\varphi$ is a $K$–finite matrix coefficient of $\pi$ and $\lambda$ is continuous in appropriate Fréchet topology on $\mathcal{H}^\infty$ ([31]; see Section 3 for details).

Since $\varphi$ is no longer compactly supported, the main obstacle to the extension of above result is how to exchange $\lambda$ and $\int_G$. This boils down to absolute convergence of the integral $\int_G \varphi(g)\lambda(\pi(g)h)dg$ (see Lemma 3-9). We attack this problem using results of Casselman [9] and Wallach [30] on representations of moderate growth, and the estimates on the growth of $K$–finite matrix coefficients of discrete series due to Miličić [16]. For general $U$ and $\chi$, Theorem 3-11 contains the result.

If we in addition assume that $U = N$ is the unipotent radical of a minimal parabolic subgroup of $G$, and $\chi$ is a generic character of $N$ (i.e., the differential $d\chi$ is non–trivial on any simple root subgroup $n_\alpha \subset Lie(N)$), Wallach ([30], Theorem 15.2.5) shows that the asymptotics of the function $g \mapsto \lambda(\pi(g)h)$ is similar to the asymptotics of a $K$–finite matrix coefficient. This implies that $g \mapsto \lambda(\pi(g)h)$ is bounded on $G$ (see the proof of Corollary 3-12). Since $\varphi \in L^1(G)$ being a $K$–finite matrix coefficient of an integrable discrete series,
the absolute convergence $\int_G \varphi(g) \lambda(\pi(g) h) dg$ is obvious. So, in this case there exists a non–zero $K$–finite matrix coefficient $\varphi$ of $\pi$ such that $F_{(\chi,x,N)}^{loc}(\varphi) \neq 0$ (see Corollary 3-12). In (13, Proposition 4.2), Khare, Larsen, and Savin obtain slightly weaker result (for some matrix coefficient of $\pi$ $K$–finite on the right only) for general discrete series relying again on results of Wallach but they use more advanced results of the same chapter in [31].

Going back to general $U$ and $\chi$, the existence of non–zero $K$–finite matrix coefficients $\varphi$ of $\pi$ such that $F_{(\chi,x)}^{loc}(\varphi) \neq 0$ (see Corollary 3-12). In ([13], Proposition 4.2), Khare, Larsen, and Savin obtain slightly weaker result (for some matrix coefficient of $\pi$ $K$–finite on the right only) for general discrete series relying again on results of Wallach but they use more advanced results of the same chapter in [31].

In the appendix to the paper (see Section 7) we describe large integrable representations in usual $L$–packets of discrete series. The results of this section were communicated to us by Miličić. It shows that many discrete series are large and integrable.

In the second part of the paper (Sections 4, 5, 6), we consider automorphic forms and prove the main results of the paper. Let $\Gamma \subset G$ be congruence subgroup with respect to the arithmetic structure given by the fact that $G$ defined over $\mathbb{Q}$ (see [5], or Section 6). In Section 4, we review basic notions of automorphic forms, and introduce a classical construction of automorphic forms via Poincaré series

$$P(\varphi)(x) = P_{\Gamma}(\varphi)(x) = \sum_{\gamma \in \Gamma} \varphi(\gamma x), \quad x \in G,$$

attached to $K$–finite matrix coefficients $\varphi$ of integrable discrete series $\pi$ of $G$ ([4], [3], [2]). We recall the criteria for $P(\varphi)$ being non–zero ([19], Theorem 4-1 and 4-9).

In Sections 5 and 6 we assume that $\chi$ and $U$ are as above, with addition that $U$ is the group of $\mathbb{R}$–points of the unipotent radical of a proper $\mathbb{Q}$–parabolic $P \subset G$. Then, $U \cap \Gamma$ is cocompact in $U$. We define the $(\chi,U)$–Fourier coefficient of a function $f \in C^\infty(\Gamma \backslash G)$ as usual

$$F_{(\chi,U)}(f)(x) = \int_{U \cap \Gamma \backslash U} f(ux) \overline{\chi(u)} du,$$

where we use a normalized measure on a compact topological space $U \cap \Gamma \backslash U$. If $W$ is a $(g,K)$–submodule of the space of automorphic forms, then we say that $W$ is $(\chi,U)$–generic if there exists $f \in W$ such that $F_{(\chi,U)}(f) \neq 0$.

The main point of Section 5 is to compute the Fourier coefficient $F_{(\chi,U)}(P(\varphi))$ of $P(\varphi)$, and use the results to study when $P(\varphi) \neq 0$ for some new cases not covered by the results of [19], and for construction of $(\chi,U)$–generic automorphic realizations of $\pi$. The main result of Section 5 is the following proposition (see Proposition 5-6):
Proposition 1-2. Let $\pi$ be an integrable discrete series of $G$. Assume that there exists a $K$–finite matrix coefficient $\varphi$ of $\pi$ such that the following holds:

$$W^\Gamma_{(\chi,U)}(\mathcal{F}_{(\chi,U)}^{loc}(\varphi)) \neq 0.$$ 

Then, there is a realization of $\pi$ as a $(\chi,U)$–generic cuspidal automorphic representation.

As we explained above $\psi \overset{\text{def}}{=} \mathcal{F}_{(\chi,U)}^{loc}(\varphi) \in I^1(G,U,\chi)$. Next, the function $W_{(\chi,U)}(\psi)$ is an automorphic form for $\Gamma$ defined via the following series (see Lemma 5-3):

$$W_{(\chi,U)}(\psi)(x) = W^\Gamma_{(\chi,U)}(\psi)(x) \overset{\text{def}}{=} \sum_{\gamma \in U \cap \Gamma \setminus \Gamma} \psi(\gamma x), \quad x \in G.$$ 

This statement is especially important for construction $(\chi,U)$–generic automorphic cuspidal realization of $\pi$ in view of a criterion which gives a sufficient condition that the series of this form (and more general) are not identically zero ([23], Lemma 2-1, recalled as Lemma [6-1] in Section 6 and its formulation given by Lemma [6-2] for series $W_{(\chi,U)}(\psi)$). We prove the following result (see Theorem [6-7]):

**Theorem 1-3.** We fix an embedding $\mathcal{G} \hookrightarrow SL_M$ over $\mathbb{Q}$, and define Hecke congruence subgroups $\Gamma_1(n)$, $n \geq 1$, using that embedding (see (6-1)). Assume that $U$ is a subgroup of all upper triangular unipotent matrices in $G$ considered as $G \subset SL_M(\mathbb{R})$. Let $\chi$ be a unitary character $U \longrightarrow \mathbb{C}^\times$ trivial on $U \cap \Gamma_1(l)$ for some $l \geq 1$. Let $\pi$ be an integrable discrete series of $G$ such that there exists a $K$–finite matrix coefficient $\varphi$ of $\pi$ such that $\mathcal{F}_{(\chi,U)}^{loc}(\varphi) \neq 0$. Then, there exists $n_0 \geq 1$ such that for $n \geq n_0$ we have a realization of $\pi$ as a $(\chi,U)$–generic cuspidal automorphic representation for $\Gamma_1(ln)$.

The reader may observe that $U \cap \Gamma_1(ln)$ is independent of $n \geq 1$. So, the same $\chi$ can be used for all $\Gamma_1(ln)$. Also, the reader may observe that the congruence subgroups $\Gamma_1(ln)$ ($n \geq 1$) are neither linearly ordered by inclusion nor their intersection is trivial as it is usually required (see for example [23] and [11]).

As a next application, we reprove the result of Khare, Larsen, and Savin ([13], Theorem 4.5) that large integrable discrete series are local components of globally generic automorphic cuspidal forms (see Theorem [6-9]):

**Theorem 1-4.** Let $\mathcal{A}$ be the ring of adeles of $\mathbb{Q}$. We assume that $\mathcal{G}$ is quasi–split over $\mathbb{Q}$. Let $\mathcal{N}$ be the unipotent radical of Borel subgroup defined over $\mathbb{Q}$. We assume that $G$ poses representations in discrete series. Let $L$ be the $L$–packet of discrete series for $G$ such that there is a large representation in that packet which is integrable (then all are integrable by Proposition [7-3]). Let $\eta : N(\mathbb{Q}) \backslash N(A) \longrightarrow \mathbb{C}^\times$ be a unitary generic character. By the change of splitting we can select an $(\eta_{\infty},N(\mathbb{R}))$–generic discrete series $\pi$ in the $L$–packet $L$. Then, there exists a cuspidal automorphic module $W$ for $G(A)$ which is $\eta$–generic and its Archimedean component is infinitesimally equivalent to $\pi$ (i.e., considered as a $(\mathfrak{g},K)$–module only it is a copy of (infinitely many) $(\mathfrak{g},K)$–modules infinitesimally equivalent to $\pi$).

In the very last part of Section 6 (see Theorem 6-12), we consider a sort of a converse to approach described above. We use criteria for $P(\varphi) \neq 0$ given by [19], where $\varphi$ is a $K$–finite
matrix coefficient of an integrable discrete series \( \pi = \text{Sp}_{2n}(\mathbb{R}) \), and show that there exists matrix coefficients \( \varphi \) with non–zero Fourier coefficients \( \mathcal{F}^{loc}_{(\chi,U)}(\varphi) \) for certain unitary characters \( \chi \) where \( U \) is the unipotent radical of a Siegel parabolic subgroup of \( \text{Sp}_{2n}(\mathbb{R}) \). In this way, any integrable discrete series \( \pi = \text{Sp}_{2n}(\mathbb{R}) \) is infinitesimally equivalent to a closed irreducible subrepresentation of a Banach representation \( I^1(G,U,\chi) \) for certain \( \chi \) (see Section 2). This has relation to the work of Li [15], and it is related to the models discussed by Gourevitch [10].

In closing the introduction, we would like to say that purpose of this paper is to write down typical applications in sufficient generality and to convince the audience how the methods of our earlier papers such as [19], [21], and [23], are also useful for proving existence or constructing explicit examples of automorphic cuspidal representations with non–zero Fourier coefficients. We do not write down an exhaustive list of applications. For example, using methods of [21], in "Whittaker model case" we can do much more than it is stated in Theorem [6-9].

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2. On Certain Banach Representations

In this section we assume that \( G \) is a connected semisimple Lie group with finite center, \( U \) is a closed unimodular subgroup of \( G \), and \( \chi : U \to \mathbb{C}^\times \) unitary character. We state some simple results about the following Banach representation that will be used in the remainder of the paper.

We consider the Banach representation \( I^1(G,U,\chi) \) (by right translations) on the space of classes of all measurable functions \( f : G \to \mathbb{C} \) which satisfies the following two conditions:

(i) \( f(ux) = \chi(u)f(x) \) for all \( u \in U \), and for almost all \( x \in G \),

(ii) \( \|f\|_{U,1} \overset{\text{def}}{=} \int_{U \backslash G} |f(x)| \, dx < \infty \).

The following is a folklore lemma, which might be useful for unexperienced reader. We include it with a complete proof.

**Lemma 2-1.** The function \( \varphi \in C_c^\infty(G) \) acts on \( I^1(G,U,\chi) \) as follows: \( \varphi.f(x) = \int_G \varphi(y)f(xy)dy \) i.e., in terms of convolution \( \varphi.f = f \ast \varphi^\vee \) (see below for the notation).

**Proof.** Indeed, the function \( x \mapsto \int_G \varphi(y)f(xy)dy \) is in \( I^1(G,U,\chi) \) since

\[
\int_{U \backslash G} \left| \int_G \varphi(y)f(xy)dy \right| \, dx \leq \int_{U \backslash G} \int_G |\varphi(y)f(xy)| \, dy \, dx = \int_G |\varphi(y)| \left( \int_{U \backslash G} |f(xy)| \, dx \right) \, dy \\
\leq ||\varphi||_{\infty} \text{vol(supp(\varphi))} ||f||_{U,1}.
\]
For $g \in C^\infty_c(G)$, we have a continuous functional on $I^1(G,U,\chi)$ given by
\[
f \mapsto - \int_{U \setminus G} f(x) \left( \int_{U} g(u x) \chi(u) du \right) dx = \int_{G} f(x) g(x) dx.
\]
So, by definition of the action of $\varphi$, we have
\[
\int_{G} \varphi.f(x) g(x) dx = \int_{U \setminus G} \varphi.f(x) \left( \int_{U} g(u x) \chi(u) du \right) dx
\]
\[
= \int_{G} \varphi(y) \left( \int_{U \setminus G} f(xy) \left( \int_{U} g(u x) \chi(u) du \right) dx \right) dy
\]
\[
= \int_{U \setminus G} \left( \int_{G} \varphi(y) f(xy) dy \right) \left( \int_{U} g(u x) \chi(u) du \right) dx
\]
\[
= \int_{G} \left( \int_{G} \varphi(y) f(xy) dy \right) g(x) dx
\]
which proves the desired formula for $\varphi.f$.

Now, we recall fundamental theorem of Harish–Chandra ([12], Section 8, Theorem 1):

**Lemma 2-2.** Assume that $f \in C^\infty(G)$ is $\mathcal{Z}(\mathfrak{g}_c)$–finite and $K$–finite on the right. Let $W \subset G$ be a neighborhood of 1 invariant under conjugation of $K$. Then, there exists a function $\alpha \in C^\infty_c(G)$, $\text{supp}(\alpha) \subset W$, and invariant under conjugation of $K$ such that $f = f \ast \alpha$.

We recall the formula for convolution
\[
f \ast \alpha(x) = \int_{G} f(xy^{-1}) \alpha(y) dy = \int_{G} f(xy) \alpha^\vee(y) dy = \alpha^\vee.f(x),
\]
in our previous notation. Here $\alpha^\vee(x) = \alpha(x^{-1})$. Obviously, $\alpha^\vee$ is invariant under the conjugation by $K$ if $\alpha$ satisfies the same.

Thus, as a corollary, we obtain the following standard result:

**Corollary 2-3.** Assume that $f \in I^1(G,U,\chi)$, is $\mathcal{Z}(\mathfrak{g}_c)$–finite, and $K$–finite on the right. Then, $f$ is equal to a smooth function almost everywhere, therefore can be taken to be smooth. Moreover, there exists $\beta \in C^\infty_c(G)$ invariant under the conjugation by $K$ such that $f = \beta.f$.

**Proof.** Since $f$ is $\mathcal{Z}(\mathfrak{g}_c)$–finite and $K$–finite on the right, $f$ satisfies the same in the sense of distributions on $G$. By the usual theory it is then real analytic which proves the first claim. The second claim follows from above discussion.

The following result is also standard ([29], Corollary 3.4.7, Theorem 4.2.1).

**Lemma 2-4.** Assume that $f \in I^1(G,U,\chi)$, is $\mathcal{Z}(\mathfrak{g}_c)$–finite, and $K$–finite on the right. Then, $(\mathfrak{g},K)$–module generated by $f$ is admissible and it has a finite length.
Next, we prove the following lemma:

**Lemma 2-5.** Assume that \( \varphi \in C^\infty(G) \) is \( \mathcal{Z}(\mathfrak{g}_c) \)-finite, \( K \)-finite on the right, and in \( L^1(G) \). Then, the following holds:

(i) The integral \( \int_U \varphi(ux) \chi(u) du \) converges absolutely for almost all \( x \in G \).

(ii) The function \( x \mapsto \int_U \varphi(ux) \chi(u) du \) is \( \mathcal{Z}(\mathfrak{g}_c) \) and \( K \)-finite vector in the Banach representation \( I^1(G, U, \overline{\chi}) \).

**Proof.** Let \( f \in L^1(G) \). Then,

\[
\int_{U \setminus G} \left( \int_U |f(ux)| du \right) dx = \int_G |f(x)| dx < \infty.
\]

Thus, \( f^U(x) = \int_U f(ux) \chi(u) du \) converges absolutely for almost all \( x \in G \). Obviously, we have \( f^U \in I^1(G, U, \overline{\chi}) \). Since also

\[
\int_{U \setminus G} |f^U(x)| dx \leq \int_{U \setminus G} \left( \int_U |f(ux)| du \right) dx = \int_G |f(x)| dx,
\]

we obtain a continuous intertwining map \( f \mapsto f^U \) between Banach representations of \( G: L^1(G) \to I^1(G, U, \overline{\chi}) \) (given by right translations). The Garding space of \( L^1(G) \), which by definition consists of functions \( f \star \alpha, f \in L^1(G), \alpha \in C_c^\infty(G) \), is mapped onto a (similarly defined) Garding space of \( I^1(G, U, \overline{\chi}) \). The restriction to the Garding space is \( (\mathfrak{g}, K) \)-equivariant map as one easily check.

Above discussion shows (i) and \( \varphi^U \in I^1(G, U, \overline{\chi}) \). Also, Lemma \[2-2\] shows that \( \varphi \) is in the Garding space of \( L^1(G) \). Now, above discussion and Corollary \[2-3\] imply (ii). \( \square \)

The existence of \( \mathcal{Z}(\mathfrak{g}_c) \)-finite and \( K \)-finite on the right functions in \( L^1(G) \) is standard and explained in the proof of (19, Theorem 3-10). They exists if and only if \( G \) admits discrete series i.e., if and only if \( \text{rank}(G) = \text{rank}(K) \). Furthermore, if for example \( U \) is the unipotent radical of a minimal parabolic subgroup of \( G \), then the integral \( \int_U \varphi(ux) \chi(u) du \) converges absolutely for all \( x \in G \) (29, Theorem 7.2.1) if for example \( \varphi \) is a \( K \)-finite matrix coefficient of an integrable discrete series of \( G \).

Where we use the following terms. Let \( (\pi, \mathcal{H}) \) be an irreducible unitary representation of \( G \). Let \( (\ , \ ) \) be the invariant inner product on \( \mathcal{H} \). Let \( \mathcal{H}_K \) be the space of \( K \)-finite inside \( \mathcal{H} \). A matrix coefficient of \( \pi \) is a function on \( G \) of the form \( x \mapsto (\pi(x)h, h') \), where \( h, h' \in \mathcal{H} \). Obviously, \( \varphi \neq 0 \) if and only if \( h, h' \neq 0 \). The matrix coefficient is \( K \)-finite on the right (resp., on the left and on both sides) if and only if \( h \in \mathcal{H}_K \) (resp., \( h' \in \mathcal{H}_K \) and \( h, h' \in \mathcal{H}_K \)).

Let \( \varphi \) be any \( K \)-finite matrix coefficient of an integrable discrete series \( \pi \). Then, we let

\[
(2-6) \quad \mathcal{F}_{(x, U)}^{loc}(\varphi)(x) \overset{df}{=} \int_U \varphi(ux) \overline{\chi(u)} du \quad x \in G.
\]

By Lemma \[2-3\](i), converges absolutely for almost all \( x \in G \). By part (ii) of the same lemma and Corollary \[2-3\] is equal to a smooth function almost everywhere, therefore can be taken to be a smooth. We assume that in what follows. The proof of Corollary \[2-3\] show that it is
\( Z(g_C) \)-finite and \( K \)-finite on the right. The meaning of (2-6) is contained in the following result:

**Corollary 2-7.** Assume that \( \mathcal{F}_{(\chi, U)}^{\text{loc}}(\varphi) \neq 0 \). Then, there exists a closed irreducible subscape of the Banach representation \( I^1(G, U, \chi) \) which is infinitesimally equivalent to \( (\pi, \mathcal{H}) \).

**Proof.** On can use the argument from the proof of Lemma 2-5 to check that the map \( H_K \rightarrow I^1(G, U, \chi) \) which maps \( h' \in H_K \) onto \( x \mapsto \int_U (\pi(ux) h', h) \chi(u) du \) gives us the required infinitesimal equivalence. \( \square \)

3. On Certain Generic Representations of \( G \)

The goal of the present section is to put Corollary 2-7 in a effective form exploring when \((\chi, U)\)-generic and integrable discrete series \( \pi \) poses a \( K \)-finite matrix coefficient \( \varphi \) such that \( \mathcal{F}_{(\chi, U)}^{\text{loc}}(\varphi) \neq 0 \). (See below for the definition of the terms used.)

We start introducing some notation. Let \( \theta \) be the Cartan involution on \( G \) with the fixed point equal to \( K \). Let \( g = k \oplus p \) be the decomposition into +1 and \(-1\)-eigenspace of \( d\theta \). Let \( a \subset p \) be the maximal subspace subject to the condition that is Abelian Lie subalgebra. Let \( A \subset G \) be the vector group with Lie algebra \( a \).

In this section we fix a minimal parabolic subgroup \( P = MAN \) of \( G \) in the usual way (see [29], Section 2). We have the Iwasawa decomposition \( G = NAK \).

We recall the notion of a norm on the group (see [29], 2.A.2). A norm \( || \cdot || \) is a function \( G \rightarrow [1, \infty[ \) satisfying the following properties:

1. \( ||x^{-1}|| = ||x|| \), for all \( x \in G \);
2. \( ||x \cdot y|| \leq ||x|| \cdot ||y|| \), for all \( x, y \in G \);
3. the sets \( \{ x \in G; \ ||x|| \leq r \} \) are compact for all \( r \geq 1 \);
4. \( ||k_1 \exp (tX)k_2|| = || \exp (X)||^t \), for all \( k_1, k_2 \in K, X \in p, \ t \geq 0 \).

Any two norms \( || \cdot ||_i, i = 1, 2 \), are equivalent: there exist \( C, r > 0 \) such that \( ||x||_1 \leq C||x||_2 \), for all \( x \in G \).

Let \( \Phi(g, a) \) be the set of all roots of \( a \) in \( g \). Let \( \Phi^+(g, a) \) be the set of positive roots with respect to \( n = \text{Lie}(N) \). Set

\[
\rho(H) = \frac{1}{2} \text{tr}(\text{ad}(H)|_a), \quad H \in a.
\]

We set

\[
m(\alpha) = \dim g_\alpha, \quad \alpha \in \Phi^+(g, a).
\]

For \( \mu \in a^* \), we let

\[
a^\mu = \exp(\mu(H)), \quad a = \exp(H).
\]

We define \( A^+ \) to be the set of all \( a \in A \) such that \( a^\alpha > 1 \) for all \( \alpha \in \Phi^+(g, a) \). Finally, we let

\[
D(a) = \prod_{\alpha \in \Phi^+(g, a)} \sinh(\alpha(H))^{m(\alpha)}, \quad a = \exp(H).
\]

Then, we may define a Haar measure on \( G \) by the following formula:
\[ \int_{G} f(g) dg = \int_{K} \int_{A^+} \int_{K} D(a) f(k_1 ak_2) dk_1 dk_2 da, \quad f \in C_\infty^c(G). \]

Let \( \{\alpha_1, \ldots, \alpha_r\} \) be the set of simple roots in \( \Phi^+(\mathfrak{g}, \mathfrak{a}) \). Since \( G \) is semisimple, we have that this set spans \( \mathfrak{a}^* \). We define the dual basis \( \{H_1, \ldots, H_r\} \) of \( \mathfrak{a}^* \) in the standard way: \( \alpha_i(H_j) = \delta_{ij} \). By (29, Lemma 2.A.2.3), there exists, \( \mu, \eta \in \mathfrak{a}^* \) such that \( \mu(H_i), \eta(H_j) > 0 \), for all \( j \), and constants \( C, D > 0 \) such that

\[ C a^\mu \leq ||a|| \leq D a^\eta, \quad a \in \text{Cl}(A^+). \]

We remark that \( \rho(H_j) > 0 \) for all \( j \). So, we can find \( c, d > 0 \) such that

\[ a^c \rho \leq a^\mu \leq a^d \rho, \quad a \in \text{Cl}(A^+). \]

We record this in the next lemma:

**Lemma 3-1.** There exists real constants \( c, C, d, D > 0 \) such that

\[ C a^c \rho \leq ||a|| \leq D a^d \rho, \quad a \in \text{Cl}(A^+). \]

Let \( (\pi, \mathcal{H}) \) be a unitary irreducible representation of \( G \) acting on the Hilbert space \( \mathcal{H} \). We write \( \langle \cdot, \cdot \rangle \) for the invariant inner product on \( \mathcal{H} \). We denote by \( \mathcal{H}^\infty \) the subspace of smooth vectors in \( \mathcal{H} \). It is a complete Fréchet space under the family of semi–norms:

\[ ||h||_T = ||d\pi(T)h||, \quad T \in \mathcal{U}(\mathfrak{g}_C), \]

where \( || \cdot || \) is the norm on \( \mathcal{H} \).

Let \( U \subset G \) be a closed subgroup. Let \( \chi : U \to \mathbb{C}^\times \) be a unitary character. Consider a continuous functional \( \lambda : \mathcal{H}^\infty \to \mathbb{C} \) which satisfies

\[ \lambda(\pi(u)h) = \chi(u) \lambda(h), \quad u \in U, \ h \in \mathcal{H}^\infty. \]

The fact that \( \lambda \) is continuous means that there exists a constant \( C > 0 \), and \( T_1, \ldots, T_k \in \mathcal{U}(\mathfrak{g}_C) \) such that

\[ ||\lambda(h)|| \leq C \left( \sum_{k=1}^{l} ||h||_{T_k} \right), \quad h \in \mathcal{H}^\infty. \]

By (30, Lemma 11.5.1), since \( |\lambda| \) is a continuous semi–norm on \( \mathcal{H}^\infty \), there exist \( d \in \mathbb{R} \) and a continuous semi–norm \( \kappa \) on \( \mathcal{H}^\infty \) such that

\[ ||\lambda(\pi(g)h)|| \leq ||g||^d \kappa(h), \quad g \in G, \ h \in \mathcal{H}^\infty. \]

We define

\[ d_{\pi, \lambda} = d_{\pi, \chi, U, \lambda}. \]
to be the infimum of all \( d \in \mathbb{R} \) for which there exist a continuous semi–norm \( \kappa \) on \( \mathcal{H}_\infty \) such that (3-3) holds. We remark that for each \( d > d_{\pi, \lambda} \) there exists a continuous semi–norm \( \kappa \) on \( \mathcal{H}_\infty \) such that (3-3) holds. Obviously, the function \( g \mapsto \lambda(\pi(g)h) \) (for fixed \( h \in \mathcal{H}_K \), \( \neq 0 \)) is bounded provided that \( d_{\pi, \lambda} < 0 \).

We define the notion of generic representations that we are interested in. It generalizes the usual notion of generic representation (i.e., the one having a Whittaker model) for quasi–split \( G \) \(^{[14]} \). In all cases that we consider \( U \) is (conjugate) to a closed subgroup of \( N \) (see Section \( \ref{sec:generic} \)).

**Definition 3-5.** Let \( U \subset G \) be a closed subgroup and \( \chi : U \to \mathbb{C}^\times \) be a unitary character. Let \( (\pi, \mathcal{H}) \) be an unitary irreducible representation of \( G \) acting on the Hilbert space \( \mathcal{H} \). We say that \( (\pi, \mathcal{H}) \) is \((\chi, U)\)–generic if there exists a continuous functional \( \lambda : \mathcal{H}_\infty \to \mathbb{C} \) which satisfies

\[
\lambda(\pi(u)h) = \chi(u)\lambda(h), \quad u \in U, \quad h \in \mathcal{H}_\infty.
\]

Now, we assume that \( G \) admits discrete series and let \( \pi \) be an integrable discrete series. A complete characterization of them can be found in \(^{[16]} \) (see also Section \( \ref{sec:discrete} \)). Let \( h \in \mathcal{H}_K \), \( h \neq 0 \). Consider the matrix coefficient given by

\[
(3-6) \quad \varphi(x) = (\pi(x)h, h).
\]

Since \( \pi \) is integrable and \( \varphi \) is \( K \)–finite, we have \( \varphi \in L^1(G) \). This implies that its complex conjugate \( \overline{\varphi} \) acts on any unitary representation of \( G \). In particular, it acts on \( \pi \):

\[
\pi(\overline{\varphi})h'' = \int_G \overline{\varphi(x)}\pi(x)h''dx, \quad h'' \in \mathcal{H}.
\]

We compute this action in the following standard and well–known lemma. We have learned this lemma from Savin which attributes the result to Milićić.

**Lemma 3-7.** (Milićić) Assume that \( \pi \) is integrable discrete series. Let \( d(\pi) > 0 \) be its formal degree. Then, \( \pi(\overline{\varphi}) \) is a rank–one operator given by

\[
\pi(\overline{\varphi})h'' = \frac{1}{d(\pi)}(h'', h)h, \quad h'' \in \mathcal{H}.
\]

**Proof.** The Schur orthogonality relation implies

\[
\int_G (\pi(x)h, h')(\pi(x)h'')dx = \frac{1}{d(\pi)}(h, h'')(h', h), \quad h', h'' \in \mathcal{H}.
\]

Written differently

\[
(\pi(\overline{\varphi})h'', h') = \frac{1}{d(\pi)}(h'', h)(h, h').
\]

Equivalently

\[
\pi(\overline{\varphi})h'' = \frac{1}{d(\pi)}(h'', h)h, \quad h'' \in \mathcal{H}.
\]
We continue with a simple lemma:

**Lemma 3-8.** Let \( \psi \in L^1(G) \cap C^\infty(G) \) be the \( \mathcal{Z}(\mathfrak{g}_c) \)-finite and \( K \)-finite on the left function. Then, we have the following:

(i) \( \pi(\psi)\mathcal{H} \subset \mathcal{H}^\infty \).

(ii) There exists a sequence \( \psi_n \in C^\infty_c(G) \), \( n \geq 1 \), such that \( l(T)\psi_n \xrightarrow{L^1} l(T)\psi \), for all \( T \in \mathcal{U}(\mathfrak{g}_c) \), where \( l \) denotes the action derived from the action of \( G \) by left translations.

**Proof.** By the assumption on \( \psi \), there exists \( \alpha \in C^\infty_c(G) \) such that \( \psi = \alpha \ast \psi \) (see [12], Section 8, Theorem 1). Then, \( \pi(\psi) = \pi(\alpha)\pi(\psi) \). So, we have

\[
\pi(\psi)\mathcal{H} = \pi(\alpha)\pi(\psi)\mathcal{H} \subset \pi(\alpha)\mathcal{H} \subset \mathcal{H}^\infty.
\]

This proves (i). By the standard measure theory, there exists a sequence \( \eta_n \in C^\infty_c(G) \), \( n \geq 1 \), such that \( \eta_n \xrightarrow{L^1} \psi \). Then, for any \( \beta \in C^\infty_c(G) \), \( \beta \ast \eta_n \xrightarrow{L^1} \beta \ast \psi \). Also, for \( \beta \in C^\infty_c(G) \) and \( \eta \in L^1(G) \), the function \( \beta \ast \eta \) is in \( C^\infty(G) \). Moreover, for \( T \in \mathcal{U}(\mathfrak{g}_c) \), we have by direct computation

\[
l(T) (\beta \ast \eta) = (l(T)\beta) \ast \eta.
\]

Finally, the sequence in (ii) can be taken to be \( \psi_n = \alpha \ast \eta_n, n \geq 1 \), as the reader can easily check. \( \square \)

**Lemma 3-9.** Assume that \( \pi \) is integrable and \( (\chi, U) \)-generic. Assume that there exists \( \lambda : \mathcal{H}^\infty \rightarrow \mathbb{C} \) satisfying \([3-2]\) such that for some \( h \in \mathcal{H}_K \) we have

\[
\lambda(h) \neq 0, \text{ and, } \int_G \left| \overline{\phi(x)}\lambda(\pi(x)h) \right| dx < \infty,
\]

\( \phi \) is given by \([3-0]\). Then, \( \mathcal{F}^{loc}_{(\chi, U)}(\phi) \neq 0 \).

**Proof.** We let

\[
\psi = \frac{d(\pi)}{(h, h)} \phi.
\]

Then, by Lemma \([3-7]\) we have

\[
\pi(\psi)h = h.
\]

Also, \( \psi \) satisfies the assumptions of Lemma \([3-8]\) let us fix a sequence \( \psi_n \in C^\infty_c(G) \), \( n \geq 1 \), as in (ii) of that lemma. Then the sequence \( \pi(\psi_n)h, n \geq 1 \), converges to \( \pi(\psi)h = h \) in the Fréchet space \( \mathcal{H}^\infty \) (by Lemma \([3-8]\)(ii)). Since, \( \lambda \) is continuous we obtain the following:

\[
\lambda(h) = \lambda(\pi(\psi)h) = \lim_n \pi(\psi_n)h = \lim_n \lambda(\pi(\psi_n)h) = \lim_n \int_G \psi_n(x)\lambda(\pi(x)h)dx,
\]

where in the last step we have used the fact that the action of \( C^\infty_c(G) \) on Hilbert space \( \mathcal{H} \) and locally convex space \( \mathcal{H}^\infty \) coincide which is easy to see by acting with continuous linear functional on \( \mathcal{H} \) which are also continuous functionals on \( \mathcal{H}^\infty \). So, we get

\[
\lim_n \int_G \psi_n(x)\lambda(\pi(x)h)dx = \lambda(h) \neq 0.
\]
We would like to interchange the limit and the integral. For this, we recall that Lemma 3-8 (ii) implies that \( \psi_n \xrightarrow{L^1} \psi \). This implies that \( \psi = (a.e.) \lim_n \psi_n \). Also, by our assumption, the integral \( \int_G \psi(x)\lambda(\pi(x)h)dx \) converges absolutely, and, therefore, we are able to apply the Dominated convergence theorem. We obtain

\[
\int_G \psi(x)\lambda(\pi(x)h)dx = \lambda(h) \neq 0.
\]

Combining with (3-2), again because of absolute convergence of the integral, this implies

\[
\int_{U \setminus G} \left( \int_U \psi(ux)\chi(u)du \right) \lambda(\pi(x)h)dx = \int_G \psi(x)\lambda(\pi(x)h)dx = \lambda(h) \neq 0.
\]

Hence

\[
\int_U \psi(ux)\chi(u)du \neq 0, \quad (a.e.) \ x \in G.
\]

□

We are left studying the absolute convergence of the integral \( \int_G |\varphi(x)\lambda(\pi(ux)h)| dx \), where \( \varphi(x) = (\pi(x)h, h) \) with \( h \in \mathcal{H}_k, h \neq 0 \) as before. The function \( \varphi \) is not just \( L^1 \) but by Miličić [16] it satisfies the following estimate:

**Lemma 3-10.** There exists \( \epsilon_\pi > 0 \) (see Definition 7-2) which depends on equivalence class of \( \pi \) only such that for each \( \epsilon \in [0, \epsilon_\pi] \) there exist \( M > 0 \) and \( k > 0 \) such that

\[
|\varphi(k_1ak_2)| \leq Ma^{-(2+\rho)}(1 + \log ||a||)^k, \quad k_1, k_2 \in K, \ a \in Cl(A^+).
\]

**Proof.** By ([16], the theorem in the introduction), for each \( \epsilon \in [0, \epsilon_\pi] \) there exist \( M > 0 \) and \( k > 0 \) such that

\[
|\varphi(x)| \leq M \cdot \Xi(x)^{2+\epsilon} (1 + \log ||x||))^k,
\]

where \( \Xi \) is the zonal spherical function ([29], 3.6). We recall ([29], Theorem 4.5.3) the following classical estimate:

\[
a^{-\rho} \leq \Xi(a) \leq E \cdot a^{-\rho} (1 + \log ||a||)^l, \quad a \in Cl(A^+),
\]

for some constants \( E, l > 0 \). Thus, we obtain

\[
|\varphi(k_1ak_2)| \leq ME^{2+\epsilon}a^{-(2+\epsilon)\rho} (1 + \log ||a||)^{k+(2+\epsilon)l}, \quad k_1, k_2 \in K, \ a \in Cl(A^+).
\]

This obviously proves the claim of the lemma. □

Now, we prove the following result:

**Theorem 3-11.** Assume that \( \pi \) is integrable. Let \( \lambda : \mathcal{H}^\infty \to \mathbb{C} \) be a non-zero continuous linear functional such that (3-2) holds. Assume that \( d_{\pi,\lambda} < \epsilon_\pi/d \), where constant \( d > 0 \) is given by Lemma 3-1. Then, there exists a non-zero \( K \)-finite matrix coefficient \( \varphi \) of \( \pi \) such that \( \mathcal{F}_{(\chi,U)}(\varphi) \neq 0 \).
Proof. We keep the notation introduced in Lemma 3-9 and immediately before the statement of Lemma 3-10. It is clear from Lemma 3-9 and comments after (3-4) that the lemma is true provided that \( d_{\pi,\lambda} < 0 \) since then the function \( g \mapsto \lambda(\pi(g)h) \) is bounded. So, assume that \( d_{\pi,\lambda} \geq 0 \). By our assumption \( d \cdot d_{\pi,\lambda} < \epsilon_{\pi} \). We select any real number \( \epsilon' \in ]d \cdot d_{\pi,\lambda}, \epsilon_{\pi}[ \).

Then, by definition of \( d_{\pi,\lambda} \), we have

\[
|\lambda(\pi(g)h)| \leq M' \cdot ||g||^{\epsilon'/d}, \quad g \in G,
\]

for some constant \( M' > 0 \). Hence, using the property (4) of the norm and Lemma 3-1, there exists a constant \( M'' > 0 \) such that

\[
|\varphi(k_1ak_2) \cdot \lambda(\pi(k_1ak_2)h)| \leq Ma^{-(2+\epsilon)\rho} (1 + \log ||a||)^k, \quad k_1, k_2 \in K, \ a \in Cl(A^+).
\]

Thus, we obtain

\[
|\varphi(k_1ak_2) \cdot \lambda(\pi(k_1ak_2)h)| \leq M_1a^{-2\rho+(\epsilon'-\epsilon)\rho} (1 + \log ||a||)^k, \quad k_1, k_2 \in K, \ a \in Cl(A^+),
\]

for some constant \( M_1 > 0 \). Finally, by above recalled formula for Haar measure, using normalized measure on \( K \) and \( D(a) \leq a^{2\rho}, \ a \in A^+ \),

we obtain

\[
\int_{G} \left| \varphi(g) \lambda(\pi(g)h) \right| \, dg = \int_{K} \int_{A^+} \int_{K} D(a) \left| \varphi(k_1ak_2) \cdot \lambda(\pi(k_1ak_2)h) \right| \, dk_1 \, da \, dk_2
\]

\[
\leq M_1 \int_{A^+} a^{(\epsilon'-\epsilon)\rho} (1 + \log ||a||)^k < \infty.
\]

We expect that in all reasonable cases i.e., \( U \subset N \), the function \( x \mapsto \lambda(\pi(x)h) \) is bounded for all \( h \in H^\infty \). This is what happens in the most important case.

**Corollary 3-12.** Assume that \( \pi \) is integrable and \((\chi, N)-\text{generic}\) for generic character \( \chi \) (by definition, the differential \( d\chi \) is non-trivial on any simple root subgroup \( n_\alpha \), where \( \alpha = \alpha_i, 1 \leq i \leq r \)). Then, there exists a non-zero \( K\)-finite matrix coefficient \( \varphi \) of \( \pi \) such that \( \mathcal{F}_{(\chi, U)}^{loc}(\varphi) \neq 0 \).

**Proof.** This follows from Theorem 3-11 as soon as we show that \( x \mapsto \lambda(\pi(x)h) \) is bounded for all \( h \in H^\infty \). First, we define

\[
\Lambda = \Lambda_{N_h} \in a^*
\]

as in (29), 4.3. Since \( \pi \) is in the discrete series, there exists real numbers \( k_1, k_2, \ldots, k_r > 0 \) such that

\[
\Lambda = -\rho - \sum_{i=1}^{r} k_i \alpha_i,
\]
where \(\{\alpha_1, \ldots, \alpha_r\}\) is the set of simple roots in \(\Phi^+(g, a)\) as we denoted above ([29], Proposition 5.1.3 and Theorem 5.5.4). Furthermore, by ([30], Lemma 15.2.3 and Theorem 15.2.5), there exists a continuous seminorm \(q_A\) on \(\mathcal{H}\) and a real number \(d \geq 0\) such that
\[
|\lambda(\pi(a)h')| \leq a^d (1 + \log ||a||)^d q_A(h'), \quad h' \in \mathcal{H}, \quad a \in Cl(A^+).
\]

By above considerations, for \(k \in K\) and \(h' = \pi(k)h\) we have
\[
|\lambda(\pi(ak)h)| \leq a^{-\rho} (1 + \log ||a||)^d q_A(\pi(k)h), \quad a \in Cl(A^+)
\]
which implies that the function
\[
(a, k) \in A \times K \mapsto |\lambda(\pi(ak)h)| \in \mathbb{C}
\]
is bounded on \(Cl(A^+) \times K\). But the analogous holds when \(Q\) ranges over a finite set of parabolic subgroups with split component \(A\): the function is bounded on \(Cl(A_Q^+) \times K\) for analogously defined \(Cl(A_Q^+)\). But then the function itself is bounded on \(A \times K\) since \(A\) is union of all \(Cl(A_Q^+)\) when \(Q\) ranges over all parabolic subgroups with a split component \(A\).

This is essentially the fact that \(a\) is the union of closures of Weyl chambers.

Now, since \(\chi\) is unitary, ([3-2]) holds, the Iwasawa decomposition \(G = NAK\) implies that the function \(g \mapsto |\lambda(\pi(g)h)|\) is bounded for all \(h \in \mathcal{H}\). To complete the proof, we apply Lemma 3-9. \(\square\)

4. Preliminaries on Automorphic Forms and Poincaré Series

From now on, in this paper we assume that \(G\) is a group of \(\mathbb{R}\)-points of a semisimple algebraic group \(\mathcal{G}\) defined over \(\mathbb{Q}\). Assume that \(G\) is not compact and connected. Let \(\Gamma \subseteq G\) be congruence subgroup with respect to arithmetic structure given by the fact that \(\mathcal{G}\) defined over \(\mathbb{Q}\) (see [5], or Section 6). Then, \(\Gamma\) is a discrete subgroup of \(G\) and it has a finite covolume.

An automorphic form (for \(\Gamma\)) is a function \(f \in C^\infty(G)\) satisfying the following three conditions ([31] or [5]):

(A-1) \(f\) is \(\mathcal{G}(\mathbb{Q})\)-finite and \(K\)-finite on the right;

(A-2) \(f\) is left–invariant under \(\Gamma\) i.e., \(f(\gamma x) = f(x)\) for all \(\gamma \in \Gamma, x \in G\);

(A-3) there exists \(r \in \mathbb{R}, r > 0\) such that for each \(u \in U(\mathcal{G})\) there exists a constant \(C_u > 0\) such that \(|u.f(x)| \leq C_u \cdot ||x||^r\), for all \(x \in G\).

We write \(\mathcal{A}(\Gamma\backslash G)\) for the vector space of all automorphic forms. It is easy to see that \(\mathcal{A}(\Gamma\backslash G)\) is a \((\mathfrak{g}, K)\)-module. An automorphic form \(f \in \mathcal{A}(\Gamma\backslash G)\) is a cuspidal automorphic form if for every proper \(\mathbb{Q}\)-proper parabolic \(\mathcal{P} \subset \mathcal{G}\) we have
\[
\int_{U \cap \Gamma \backslash U} f(ux)dx = 0, \quad x \in G,
\]
where \(U\) is the group of \(\mathbb{R}\)-points of the unipotent radical of \(\mathcal{P}\). We remark that the quotient \(U \cap \Gamma \backslash U\) is compact. We use normalized \(U\)-invariant measure on \(U \cap \Gamma \backslash U\). The space of all cuspidal automorphic forms for \(\Gamma\) is denoted by \(\mathcal{A}_{\text{cusp}}(\Gamma\backslash G)\).

For the sake of completeness, we state the following standard result:

**Lemma 4-1.** Under above assumptions, we have the following:
(a) If $f \in C^\infty(G)$ satisfies (A-1), (A-2), and there exists $r \geq 1$ such that $f \in L^r(\Gamma \backslash G)$, then $f$ satisfies (A-3), and it is therefore an automorphic form. We speak about $r$–integrable automorphic form, for $r = 1$ (resp., $r = 2$) we speak about integrable (resp., square–integrable) automorphic form.

(b) Let $r \geq 1$. Every $r$–integrable automorphic form is integrable.

(c) Bounded integrable automorphic form is square–integrable.

(d) If $f$ is square integrable automorphic form, then the minimal $G$–invariant closed subspace of $L^2(\Gamma \backslash G)$ is a direct is of finitely many irreducible unitary representations.

(e) Every cuspidal automorphic form is square–integrable.

Proof. For the claims (a) and (e) we refer to [5] and reference there. Since the volume of $\Gamma \backslash G$ is finite, the claim (b) follows from Hölder inequality (as in [19], Section 3). The claim (c) is obvious. The claim (d) follows from ([29], Corollary 3.4.7 and Theorem 4.2.1).

\[ \text{□} \]

In the generality that we consider in this section, the constant function is one example of square–integrable automorphic form. The other one are provided by the standard Poincaré Series (see [4], Theorem 5.4, and [3], Theorem 6.1). The case of $SL_2(\mathbb{R})$ is sharpen in ([20], Lemma 2.9). The adelic version of the following lemma (i) and (ii) is contained in ([19], Theorem 3.10).

Lemma 4-2. Assume $\varphi \in C^\infty(G)$ that is $\mathcal{Z}(\mathfrak{g}_C)$–finite, $K$–finite, and in $L^1(G)$. Then, we have the following:

(i) The Poincaré series $P(\varphi)(g) = \sum_{\gamma \in \Gamma} \varphi(\gamma g)$ converges absolutely and uniformly on compact sets, and it is a cuspidal automorphic form.

(ii) (Non–vanishing criterion of [19]) Let $\pi$ be an integrable discrete series and $\varphi$ be a non–zero $K$–finite matrix coefficient of $\pi$. Assume that there exists a compact neighborhood $C$ in $G$ such that

\[ \int_C |\varphi(g)| dg > \int_{G-C} |\varphi(g)| dg \quad \text{and} \quad \Gamma \cap C \cdot C^{-1} = \{1\}. \]

Then, $P(\varphi) \neq 0$.

(iii) Let $\Gamma_1 \supset \Gamma_2 \supset \ldots$ be a sequence of discrete subgroups of $G$ such that $\cap_{n \geq 1} \Gamma_n = \{1\}$. Let $\pi$ be an integrable discrete series and $\varphi$ be a non–zero $K$–finite matrix coefficient of $\pi$. Put $P_n(\varphi)(x) = \sum_{\gamma \in \Gamma_n} \varphi(\gamma x)$, for all $n \geq 1$. Then, there exists $n_0$ depending on $\varphi$ such that $P_n(\varphi) \neq 0$ for $n \geq n_0$.

Proof. (i) has a proof similar to that of ([3], Theorem 6.1) where the case $G = SL_2(\mathbb{R})$ is considered (see also the proof of ([19], Theorem 3.10) in adelic settings). The claim (ii) is ([19], Theorem 4-1). Finally, the claim (iii) is ([19], Corollary 4-9). \[ \text{□} \]

5. ON A CONSTRUCTION OF CERTAIN POINCARÉ SERIES

In this section we maintain assumptions of Section 4. We assume that $U$ is the group of $\mathbb{R}$–points of the unipotent radical of a proper $\mathbb{Q}$–proper parabolic $\mathcal{P} \subset \mathcal{G}$. Then, $U \cap \Gamma$ is cocompact in $U$. Let $\chi$ be a character of $U$ trivial on $U \cap \Gamma$. 
The non-vanishing of Poincaré series discussed in Lemma 4-2 is a subtle problem (see applications of sufficient criterion Lemma 4-2 (vi) in adelic settings [19] and in the case of $SL_2(\mathbb{R})$ [20]). In this section, we discuss a different approach based on Fourier coefficients.

**Definition 5-1.** The $(\chi, U)$–Fourier coefficient of a function $f \in C^\infty(\Gamma \backslash G)$ is defined as follows:

$$F_{(\chi, U)}(f)(x) = \int_{U \cap \Gamma \backslash U} f(ux) \chi(u) du,$$

where we use a normalized measure on a compact topological space $U \cap \Gamma \backslash U$. We say that an automorphic form $f$ (see Section 4 for definition) is $(\chi, U)$–generic if $F_{(\chi, U)}(f) \neq 0$.

A $(g, K)$–submodule $V \subset A(\Gamma \backslash G)$ is generic if there exists at least one $f \in W$ such that $F_{(\chi, U)}(f) \neq 0$.

Now, we compute the Fourier coefficients of Poincaré series described by Lemma 4-2. We remind the reader that $\varphi^\vee(x) = \varphi(x^{-1})$, and $l(x) \varphi(y) = \varphi(x^{-1}y)$ is a left translation.

**Lemma 5-2.** Assume $\varphi \in C^\infty(G)$ that is $Z(g_C)$–finite, $K$–finite, and in $L^1(G)$. Then, the Fourier coefficient $F_{(\chi, U)}(P(\varphi))$ of the Poincaré series $P(\varphi)$ is given by the following expression:

$$F_{(\chi, U)}(P(\varphi))(x) = \sum_{\gamma \in \Gamma \backslash U} \int_{U} l(x) \varphi^\vee(u\gamma) \chi(u) du, \quad x \in G.$$

**Proof.** Since the series $P(\varphi)(g) = \sum_{\gamma \in \Gamma} \varphi(\gamma g)$ converges absolutely and uniformly on compact sets (see Lemma 4-2(i)), and $U \cap \Gamma \backslash U$ is compact, we have a standard unfolding

$$F_{(\chi, U)}(P(\varphi))(x) = \int_{U \cap \Gamma \backslash U} \left( \sum_{\gamma \in \Gamma} \varphi(\gamma ux) \right) \overline{\chi(u)} du$$

$$= \int_{U \cap \Gamma \backslash U} \left( \sum_{\gamma \in \Gamma \cap U} \sum_{\delta \in \Gamma \cap U} \varphi(\gamma \delta ux) \right) \overline{\chi(u)} du$$

$$= \sum_{\gamma \in \Gamma \cap U} \int_{U \cap \Gamma \backslash U} \left( \sum_{\delta \in \Gamma \cap U} \varphi(\gamma \delta ux) \right) \overline{\chi(u)} du.$$
\[
\int_{U \cap \Gamma \setminus U} \left( \sum_{\delta \in U \cap \Gamma} \varphi(\gamma \delta ux) \right) \chi(u) du = \int_{U \cap \Gamma \setminus U} \left( \sum_{\delta \in U \cap \Gamma} \varphi(\gamma \delta ux) \right) \chi(\delta u) du = \int_U \varphi(\gamma ux) \chi(u) du = \int_U l(x) \varphi^\gamma(u^{-1} \gamma^{-1}) \chi(u) du = \int_U l(x) \varphi^\gamma(u \gamma^{-1}) \chi(u) du.
\]

Finally, we have
\[
\mathcal{F}_{(\chi,U)}(P(\varphi))(x) = \sum_{\gamma \in \Gamma \setminus U \cap \Gamma} \int_U l(x) \varphi^\gamma(u \gamma^{-1}) \chi(u) du = \sum_{\gamma \in U \cap \Gamma \setminus \Gamma} \int_U l(x) \varphi^\gamma(u \gamma) \chi(u) du.
\]

Lemma 5-2 suggest that we consider the following class of automorphic forms:

**Lemma 5-3.** Assume \( \varphi \in C^\infty(G) \cap I^1(G,U,\chi) \) (see Section 2 for notation) is \( Z(g_C) \)-finite and \( K \)-finite on the right. Then, the series
\[
W_{(\chi,U)}(\varphi)(x) = W_{(\chi,U)}^\Gamma(\varphi)(x) \overset{\text{def}}{=} \sum_{\gamma \in \Gamma \setminus U \cap \Gamma} \varphi(\gamma x)
\]
converges absolutely and uniformly on compacta in \( G \) to an integrable automorphic form i.e., \( W_{(\chi,U)}(\varphi) \in \mathcal{A}(\Gamma \setminus G) \cap L^1(\Gamma \setminus G) \).

**Proof.** The idea of the proof that it converges absolutely on compacta in \( G \) is based on a slight improvement of classical argument as explained in (19, Theorem 3-10). We adapt the argument to the present situation.

Since \( \Gamma \) is discrete, there exists a compact neighborhood \( W \) of \( 1 \in G \) such that
\[
\Gamma \cap W = \{1\}.
\]

To prove that the series converges absolutely and uniformly on compacta in \( G \), it is enough to prove that for each \( g \in G \) there exists a compact neighborhood \( g \in D \subset G \) such that the series converges absolutely and uniformly. So, let \( g \in G \) be fixed. We select and compact
neighborhoods \( g \in D \subset G \) and \( 1 \in V \subset G \) such that \( DV^{-1}VD^{-1} \subset W \). By (5-4), we have the following:

\[(5-5) \text{ if } (U \cap \Gamma) \gamma DV^{-1} \cap (U \cap \Gamma) \delta DV^{-1} \neq \emptyset, \text{ for } \gamma, \delta \in \Gamma, \text{ then } \gamma \in (U \cap \Gamma) \delta.\]

Since \( \varphi \) is \( Z(g_c) \)-finite and \( K \)-finite on the right, there exists \( \alpha \in C_c^\infty(G) \), \( \text{supp}(\alpha) \subset V \), such that \( \varphi = \varphi \ast \alpha \) (see Lemma 2-2). This can be rewritten as follows:

\[
\varphi(x) = \varphi \ast \alpha(x) = \int_G \varphi(xy^{-1})\alpha(y)dy = \int_{V^{-1}} \varphi(xy)\alpha^\gamma(y)dy,
\]

where as usual, we let \( \alpha^\gamma(y) = \alpha(y^{-1}) \). Now, since we have the obvious equality of sup–norms \( |\alpha^\gamma| = |\alpha| \), for \( x \in D \) and \( \gamma \in \Gamma \), we have the following:

\[
|\varphi(x)| \leq |\alpha| \cdot \int_{V^{-1}} |\varphi(xy)|dy = |\alpha| \cdot \int_{\gamma V^{-1}} |\varphi(y)|dy
\]

\[
\leq |\alpha| \cdot \int_{\gamma DV^{-1}} |\varphi(y)|dy = |\alpha| \cdot \int_G \gamma DV^{-1} \varphi(y)dy
\]

\[
= |\alpha| \cdot \int_{U \cap \Gamma \setminus G} \left( \sum_{\delta \in U \cap \Gamma} 1_{\gamma DV^{-1}}(\delta y)|\varphi(\delta y)| \right)dy
\]

\[
= |\alpha| \cdot \int_{U \cap \Gamma \setminus (U \cap \Gamma) \gamma DV^{-1}} |\varphi(y)| \left( \sum_{\delta \in U \cap \Gamma} 1_{\gamma DV^{-1}}(\delta y) \right)dy
\]

\[
= |\alpha| \cdot \int_{U \cap \Gamma \setminus (U \cap \Gamma) \gamma DV^{-1}} |\varphi(y)| \left( \sum_{\delta \in U \cap \Gamma} 1_{\gamma DV^{-1}}(\delta y) \right)dy
\]

\[
\leq |\alpha| \cdot \int_{U \cap \Gamma \setminus (U \cap \Gamma) \gamma DV^{-1}} |\varphi(y)| \left( \sum_{\delta \in U \cap \Gamma} 1_{\gamma DV^{-1}VD^{-1}} \right)dy
\]

\[
\leq |\alpha| \cdot \int_{U \cap \Gamma \setminus (U \cap \Gamma) \gamma DV^{-1}} |\varphi(y)| \left( \sum_{\delta \in U \cap \Gamma} 1_{\gamma W \gamma^{-1}} \right)dy
\]

\[
= |\alpha| \cdot \int_{U \cap \Gamma \setminus (U \cap \Gamma) \gamma DV^{-1}} |\varphi(y)|dy.
\]
The last equality holds since $\delta \in \gamma W\gamma^{-1}$ implies that $\gamma^{-1}\delta \gamma \in W \subset \Gamma = \{1\}$ by (5-4).

In view of (5-5) and the fact that $U \cap \Gamma$ is cocompact in $U$, we get

$$\left|\alpha\right|_\infty \cdot \sum_{\gamma \in U \cap \Gamma \setminus \Gamma} \int_{U \cap \Gamma \setminus (U \cap \Gamma)} |\varphi(y)|dy \leq \left|\alpha\right|_\infty \cdot \int_{U \cap \Gamma \setminus \Gamma} |\varphi(y)|dy < \infty$$

which shows that the series converges absolutely and uniformly on $D$.

Since

$$\int_{\Gamma \setminus G} \left( \sum_{\gamma \in U \cap \Gamma \setminus \Gamma} |\varphi(\gamma x)| \right) dx = \int_{U \cap \Gamma \setminus G} |\varphi(x)| dx < \infty,$$

we have $W_{(\chi,U)}(\varphi) \in L^1(\Gamma \setminus G)$. The argument used in the proof of ([19], Theorem 3-10) shows that $Z(g_c)$–finite and $K$–finite on the right. So, it is an automorphic form by Lemma 4-1 (b).

Now, we come to the main result of the present section.

**Proposition 5-6.** Let $\pi$ be an integrable discrete series of $G$. Assume that there exists a $K$–finite matrix coefficient $\varphi$ of $\pi$ such that the following holds:

$$W^\Gamma_{(\chi,U)}(\mathcal{F}^{loc}_{(\chi,U)}(\varphi)) \neq 0.$$

Then, there is a realization of $\pi$ as a $(\chi,U)$–generic cuspidal automorphic representation.

**Proof.** By the assumption

(i) $\psi \overset{def}{=} \mathcal{F}^{loc}_{(\chi,U)}(\varphi) \neq 0$, and

(ii) $W^\Gamma_{(\chi,U)}(\psi) \neq 0$.

By the comment in the paragraph containing (2-6), we may apply Lemma 5-3 to construct automorphic form $W_{(\chi,U)}(\psi)$. In particular, $W_{(\chi,U)}(\psi)$ is real–analytic. So, we have the Taylor expansion

$$W_{(\chi,U)}(\psi)(\exp(X)) = \sum_{n=0}^{\infty} \frac{1}{n!} X^n W_{(\chi,U)}(\psi)(1),$$

where $X$ belongs to a convenient neighborhood $\mathcal{V}$ of 0 in $\mathfrak{g}$. Since $G$ is assumed to be connected, $W_{(\chi,U)}(\psi) \neq 0$ if and only if there exists $X \in \mathcal{V}$, and $n \geq 1$ such that

(5-7) $X^n W_{(\chi,U)}(\psi)(1) \neq 0$.

Let us fix $X \in \mathfrak{g}$, and $n \geq 1$ satisfying (5-7), and make this condition explicit. Let us write $\langle \, , \, \rangle$ for the invariant inner product on the Hilbert space $\mathcal{H}$ realizing the representation $\pi$. Let us write $\mathcal{H}_K$ for the set of $K$–finite vectors in $\mathcal{H}$.

For $v, v' \in \mathcal{H}_K$, we let

$$\varphi_{v,v'}(g) = \langle \pi(g)v, v' \rangle, \quad g \in G.$$

Then

$$\overline{\varphi_{v,v'}}(g) = \langle v', \pi(g)v \rangle, \quad g \in G.$$
Furthermore, for $X \in \mathfrak{g}$, $g \in G$, we have

$$X \varphi_{v,v'}(g) = \left. \frac{d}{dt} \varphi_{v,v'}(\exp(tX)v) \right|_{t=0} = (v', \pi(g)d\pi(X)v) = \varphi_{d\pi(X)v,v'}(g).$$

We also let (see (2-6))

$$\psi_{v,v'}(g) = F_{\text{loc}}(\chi, U) \left( \varphi_{v,v'}(g) \right) = \int_{U} \varphi_{v,v'}(ug) \chi(u) du, \quad g \in G.$$

Then, by (5-8) and the discussion in the proof of Lemma 2-5, we have

$$\psi_{d\pi(X)v,v'} = X \psi_{v,v'}, \quad v, v' \in \mathcal{H}_K, \quad X \in \mathfrak{g}.$$

Next, by our assumption, there exists $h, h' \in \mathcal{H}_K$ such that

$$\varphi = \varphi_{h,h'}.$$

Then, (5-7) implies

$$W(\chi, U) \left( \psi_{d\pi(X)^n h, h'} \right) (1) = W(\chi, U) \left( X^n \psi_{h,h'} \right) (1) = X^n W(\chi, U) \left( \psi_{h,h'} \right) (1) = X^n W(\chi, U) (\psi) (1) \neq 0.$$

We remark that one can see that $W(\chi, U)$ commutes with the action of $U(\mathfrak{g}_C)$ as in the proof of (19, Theorem 3-10). Thus, by Lemma 5-2 we have

$$F(\chi, U) \left( P(\psi_{d\pi(X)^n h, h'}) \right) (1) = W(\chi, U) \left( F_{\text{loc}}(\chi, U) \left( \varphi_{d\pi(X)^n h, h'} \right) \right) (1) = W(\chi, U) \left( \psi_{d\pi(X)^n h, h'} \right) (1) \neq 0.$$

But

$$\varphi_{d\pi(X)^n h, h'}(x) = (\pi(x^{-1})d\pi(X)^n h, h') = (\pi(x)h', d\pi(X)^n h),$$

is a $K$–finite matrix coefficient of $\pi$. Now, we use Lemma 4-2 to complete the proof. \hfill \square

In the case of $U = N$ and $\chi$ is a generic character, the condition (i) is fulfilled by at least one matrix coefficient by Corollary 3-12.

We end the following section with the following proposition.

**Proposition 5-9.** We assume that $U \cap \Gamma \setminus U$ is Abelian. Assume that $\pi$ is an integrable discrete series of $G$. Then, if there exists a $K$–finite matrix coefficient $\varphi$ of $\pi$ such that $P(\varphi) \neq 0$, then $\pi$ is infinitesimally equivalent to a closed subrepresentation of $I^1(G, U, \chi)$ for some $\chi \in (U \cap \Gamma \setminus U)$.

**Proof.** Since $U \cap \Gamma \setminus U$ is a compact Abelian group, we can compute the Fourier expansion of $P(\varphi)$. We recall that we have normalized the Haar measure on $U \cap \Gamma \setminus U$ such that the total volume is equal to one. In $L^2(U \cap \Gamma \setminus U)$, we have the following Fourier expansion:

$$P(\varphi)(ux) = \sum_{\chi \in (U \cap \Gamma \setminus U)} c_\chi(\varphi)(x) \chi(u), \quad x \in G,$$
where the Fourier coefficients are given by
\[ c_\chi(P(\varphi))(x) = \int_{U \cap \Gamma \backslash U} P(\varphi)(ux)\overline{\chi(u)}du = \mathcal{F}_{(\chi,U)}(P(\varphi))(x). \]

Now, if \( P(\varphi) \neq 0 \), then the Fourier coefficient \( \mathcal{F}_{(\chi,U)}(P(\varphi)) \neq 0 \), for some \( \chi \in (U \cap \Gamma \backslash U) \). But this Fourier coefficient is real-analytic function of \( x \in G \). Hence, since \( G \) is connected, as in the proof of Proposition 5-6 there exists \( X \in \mathfrak{g} \), and \( n \geq 1 \) such that
\[ X^n \mathcal{F}_{(\chi,U)}(P(\varphi))(1) \neq 0. \]

We can write this as follows:
\[ \mathcal{F}_{(\chi,U)}(P(X^n \cdot \varphi))(1) = X^n \mathcal{F}_{(\chi,U)}(P(\varphi))(1) \neq 0. \]

Using Lemma 5-2, this can be written as follows:
\[ W_{(\chi,U)}(\mathcal{F}_{(\chi,U)}((X^n \cdot \varphi)'))(1) = \mathcal{F}_{(\chi,U)}(P(X^n \cdot \varphi))(1) \neq 0. \]

This implies
\[ \mathcal{F}_{(\chi,U)}^\text{loc}((X^n \cdot \varphi)') \neq 0. \]

is not equal to zero almost everywhere. As in the proof of Proposition 5-6 we can check that \( (X^n \cdot \varphi)' \) is a \( K \)-finite matrix coefficient of \( \pi \). Hence, \( \pi \) is infinitesimally equivalent to a closed subrepresentation of \( I^1(G,U,\chi) \) by Corollary 2-7.

6. Applications

We maintain the assumptions of Sections 4 and 5 (see the first paragraphs in those sections). We need to study when \( W_{(\chi,U)}(\varphi) \neq 0 \) for \( \varphi \in C^\infty(G) \cap I^1(G,U,\chi) \) which is \( Z(\mathfrak{g}_C) \)-finite and \( K \)-finite on the right (see Lemma 5-3). In this case the criterion Lemma 4-2(ii) (see ([19], Theorem 4-1)) is not applicable here. We use a generalization of ([19], Theorem 4-1) given by ([23], Lemma 2-1):

**Lemma 6-1.** Let \( G \) be a locally compact unimodular (Hausdorff) topological group. Let \( \Gamma \subset G \) be its discrete subgroup and \( \Gamma_1 \subset \Gamma \) an arbitrary subgroup. We let \( \eta : \Gamma \to \mathbb{C}^\times \) be an unitary character of \( \Gamma \) trivial on \( \Gamma_1 \). Let \( \varphi \in L^1(\Gamma_1 \backslash G) \). Assume that there exists a subgroup \( \Gamma_2 \subset \Gamma \) such that \( \Gamma_1 \) is normal subgroup of \( \Gamma_2 \) of finite index and there exists a compact set \( C \subset G \) such that the following conditions hold:

1. \( \varphi(\gamma g) = \eta(\gamma)\varphi(g) \), for all \( \gamma \in \Gamma_2 \) and almost all \( g \in G \),
2. \( \Gamma \cap C \cdot C^{-1} \subset \Gamma_2 \), and
3. \( \int_{\Gamma_1 \backslash \Gamma} |\varphi(g)| \, dg > \frac{1}{2} \int_{\Gamma_1 \backslash G} |\varphi(g)| \, dg \).

Then the Poincaré series defined by
\[ P_{\Gamma_1 \backslash \Gamma}^{(\chi)}(\varphi)(g) \overset{\text{def}}{=} \sum_{\gamma \in \Gamma_1 \backslash \Gamma} \eta(\gamma)\varphi(\gamma \cdot g) \]

converges absolutely almost everywhere to a non-zero element of \( L^1(\Gamma \backslash G) \).

In our case, the lemma immediately implies the following result:
Lemma 6-2. Assume that $\varphi \in C^\infty(G) \cap I^1(G,U,\chi)$ is $Z(G) \cap \mathcal{I}_G$–finite and $K$–finite on the right. Then, the series (see Lemma 5-3)

$$W_{(\chi,U)}(\varphi)(x) = \sum_{\gamma \in U \cap \Gamma \setminus \Gamma} \varphi(\gamma x)$$

is non–zero provided that there exists a compact set $C \subset G$ such that

1. $\Gamma \cap \frac{C}{C} \subset U \cap \Gamma$, and
2. $\int_{(U \cap \Gamma) \setminus (U \cap \Gamma) \cdot C} |\varphi(g)| \, dg > \frac{1}{2} \int_{(U \cap \Gamma) \setminus G} |\varphi(g)| \, dg$.

Proof. Since $U \cap \Gamma$ is cocompact in $U$ and $\chi$ is trivial on $U \cap \Gamma$, we see that $\varphi \in L^1(U \cap \Gamma \setminus G)$. Now, we apply Lemma [6-1].

We now introduce some congruence subgroups. Let $\mathbb{A}$ (resp., $\mathbb{A}_f$) be the ring of adeles (resp., finite adeles) of $\mathbb{Q}$. For each prime $p$, let $\mathbb{Z}_p$ be the maximal compact subring of $\mathbb{Q}_p$. Then, for almost all primes $p$, the group $G$ is unramified over $\mathbb{Q}_p$; in particular, $G$ is a group scheme over $\mathbb{Z}_p$, and $G(\mathbb{Z}_p)$ is a hyperspecial maximal compact subgroup of $G(\mathbb{Q}_p)$ (32, 3.9.1). Let $G(\mathbb{A}_f)$ be the restricted product of all groups $G(\mathbb{Q}_p)$ with respect to the groups $G(\mathbb{Z}_p)$ where $p$ ranges over all primes $p$ such that $G$ is unramified over $\mathbb{Q}_p$:

$$G(\mathbb{A}_f) = \prod_p G(\mathbb{Q}_p).$$

Note that

$$G(\mathbb{A}) = G(\mathbb{R}) \times G(\mathbb{A}_f).$$

The group $G(\mathbb{Q})$ is embedded into $G(\mathbb{R})$ and $G(\mathbb{Q}_p)$. It is embedded diagonally in $G(\mathbb{A}_f)$ and in $G(\mathbb{A})$.

The congruence subgroups of $G$ are defined as follows (see [5]). Let $L \subset G(\mathbb{A}_f)$ be an open compact subgroup. Then, considering $G(\mathbb{Q})$ diagonally embedded in $G(\mathbb{A}_f)$, we may consider the intersection

$$\Gamma_L = L \cap G(\mathbb{Q}).$$

Now, we consider $G(\mathbb{Q})$ as subgroup of $G = G(\mathbb{R})$. One easily show that the group $\Gamma_L$ is discrete in $G$ and it has a finite covolume. The group $\Gamma_L$ is called a congruence subgroup attached to $L$.

We introduce two families of examples of congruence groups (which depend on an embedding over $\mathbb{Q}$ of $G$ into some $SL_M$). So, in order to define a family of principal congruence subgroups $\mathcal{G}$, we fix an embedding over $\mathbb{Q}$

$$G \hookrightarrow SL_M$$

with a image Zariski closed in $SL_M$. Then there exists $N \geq 1$ such that

(6-4) $\mathcal{G}$ is a group scheme over $\mathbb{Z}[1/N]$ and the embedding (6-3) is defined over $\mathbb{Z}[1/N]$. 
We fix such $N$.

As usual, we let $\mathcal{G}_Z = \mathcal{G}(\mathbb{Q}) \cap SL_M(\mathbb{Z})$, and $\mathcal{G}_{Z_p} = \mathcal{G}(\mathbb{Q}_p) \cap SL_M(\mathbb{Z}_p)$ for all prime numbers $p$. We remark that $\mathcal{G}$ is a group scheme over $\mathbb{Z}_p$ and the embedding (6-3) is defined over $\mathbb{Z}_p$ when $p$ does not divide $N$. Then $\mathcal{G}_{Z_p} = \mathcal{G}(\mathbb{Z}_p)$ but $\mathcal{G}$ may not be unramified over such $p$. In general, $\mathcal{G}_{Z_p}$ is just an open compact subgroup of $\mathcal{G}(\mathbb{Q}_p)$.

Now, we are ready to define the principal congruence subgroups with respect to the embedding (6-3).

\begin{equation}
\Gamma(n) = \{ x = (x_{i,j}) \in \mathcal{G}_Z : x_{i,j} \equiv \delta_{i,j} \pmod{n} \}, \quad n \geq 1.
\end{equation}

Obviously, $\Gamma(1) = \mathcal{G}_Z$, and when $n|m$ we have $\Gamma(m) \subset \Gamma(n)$. One easily check that $\Gamma(n)$ correspond to an open compact subgroup $L(n)$ defined as follows. For each prime number $p$ and $l \geq 1$, we define the open–compact subgroup

$$L(p^l) = \mathcal{G}(\mathbb{Q}_p) \cap \{ x = (x_{i,j}) \in SL_M(\mathbb{Z}_p) : x_{i,j} - \delta_{i,j} \in p^l\mathbb{Z}_p \}.$$ 

We decompose $n$ into primes numbers $n = p_1^{l_1} \cdots p_t^{l_t}$. Then, one easily sees that

$$\Gamma(n) = \left( L(p_1^{l_1}) \times \cdots \times L(p_t^{l_t}) \times \prod_{p \notin \{p_1, \ldots, p_t\}} \mathcal{G}_{Z_p} \right) \cap \mathcal{G}(\mathbb{Q}).$$

We also define Hecke congruence subgroups which are of a rather different nature than principal congruence subgroups.

\begin{equation}
\Gamma_1(n) = \{ x = (x_{i,j}) \in \mathcal{G}_Z : x_{i,j} \equiv \delta_{i,j} \pmod{n} \text{ for } i \geq j \}, \quad n \geq 1.
\end{equation}

Obviously, $\Gamma_1(1) = \mathcal{G}_Z$, and when $n|m$ we have $\Gamma_1(m) \subset \Gamma_1(n)$. We leave as an exercise to the reader to write down corresponding open–compact subgroups.

We do not want to write down an exhaustive list of all applications but to give some typical results. First, we prove the following theorem:

**Theorem 6-7.** We fix an embedding $\mathcal{G} \hookrightarrow SL_M$ over $\mathbb{Q}$, and define Hecke congruence subgroups $\Gamma_1(n)$, $n \geq 1$, using that embedding. Assume that $U$ is a subgroup of all upper triangular unipotent matrices in $G$ considered as $G \subset SL_M(\mathbb{R})$. Let $\chi$ be a unitary character $U \to \mathbb{C}^\times$ trivial on $U \cap \Gamma_1(l)$ for some $l \geq 1$. Let $\pi$ be an integrable discrete series of $G$ such that there exists a $K$–finite matrix coefficient $\varphi$ of $\pi$ such that $\mathcal{F}_{(\chi,U)}^{loc}(\varphi) \neq 0$. Then, there exists $n_0 \geq 1$ such that for $n \geq n_0$ we have a realization of $\pi$ as a $(\chi,U)$-generic cuspidal automorphic representation for $\Gamma_1(ln)$.

**Proof.** We remark that $\Gamma_1(ln) \subset \Gamma_1(l)$ for all $n \geq 1$. Since $U$ is a subgroup of upper–triangular unipotent matrices in $SL_M(\mathbb{R})$, we see that $U \cap \Gamma_1(ln)$ is independent of $n \geq 1$. Thus, the same $\chi$ can be used for all $\Gamma_1(ln)$. Furthermore, since $\mathcal{F}_{(\chi,U)}^{loc}(\varphi) \neq 0$ is integrable on $U \cap \Gamma_1(ln) \setminus G$, there exists a compact set $C \subset G$ such that

$$\int_{(U \cap \Gamma_1(ln) \setminus (U \cap \Gamma_1(ln)) \setminus C} |\mathcal{F}_{(\chi,U)}^{loc}(\varphi)(g)| \, dg > \frac{1}{2} \int_{U \cap \Gamma_1(ln) \setminus G} |\mathcal{F}_{(\chi,U)}^{loc}(\varphi)(g)| \, dg,$$

for all $n \geq 1$. This is (2) in Lemma [6-2].
For \( x = (x_{ij}) \in \Gamma_1(ln) \), we have either \( x_{ij} = \delta_{ij} \) or \( |x_{ij} - \delta_{ij}| \geq ln \) for all \( i \geq j \). But matrix entries of \( C \cdot C^{-1} \) are bounded. Thus, we see that there must exist \( n_0 \geq 1 \) such that for \( n \geq n_0 \) and \( x \in \Gamma_1(ln) \cap C \cdot C^{-1} \) we have \( x_{ij} = \delta_{ij} \) for all \( i \geq j \). Hence, \( x \) is an upper–triangular unipotent matrix. Thus, by definition of \( U \), \( x \in U \cap \Gamma_1(ln) \). This is (1) in Lemma 6-2.

Now, Lemma 6-2 implies that
\[
W_{(\chi,U)}^{\Gamma_1(ln)} \left( \mathcal{F}_{(\chi,U)}(\varphi) \right) \neq 0, \quad n \geq n_0.
\]
Thus, Proposition 5-6 implies that there exists a realization of \( \pi \) as a \((\chi,U)\)-generic cuspidal automorphic representation for \( \Gamma_1(ln) \) for all \( n \geq n_0 \). 

Now, we give a global application (see [19]). We briefly recall the notion of a cuspidal automorphic form [5], or ([19], Section 1). A cuspidal automorphic form is a function \( W \) is not identically equal to zero. A submodule

For any open–compact subgroup \( L \) of \( \Gamma_1(ln) \cap C \cdot C^{-1} \) we have \( x_{ij} = \delta_{ij} \) for all \( i \geq j \). Hence, \( x \) is an upper–triangular unipotent matrix. Thus, by definition of \( U \), \( x \in U \cap \Gamma_1(ln) \). This is (1) in Lemma 6-2.

Now, Lemma 6-2 implies that
\[
W_{(\chi,U)}^{\Gamma_1(ln)} \left( \mathcal{F}_{(\chi,U)}(\varphi) \right) \neq 0, \quad n \geq n_0.
\]
Thus, Proposition 5-6 implies that there exists a realization of \( \pi \) as a \((\chi,U)\)-generic cuspidal automorphic representation for \( \Gamma_1(ln) \) for all \( n \geq n_0 \). 

Now, we give a global application (see [19]). We briefly recall the notion of a cuspidal automorphic form [5], or ([19], Section 1). A cuspidal automorphic form is a function \( W \) is not identically equal to zero. A submodule

The space \( \mathcal{A}_{cusp}(\mathcal{G}(\mathbb{Q}) \backslash \mathcal{G}(\mathbb{A})) \) is a \((\mathfrak{g},K) \times \mathcal{G}(\mathbb{A})\)-module. If \( f \in \mathcal{A}_{cusp}(\mathcal{G}(\mathbb{Q}) \backslash \mathcal{G}(\mathbb{A})) \) is a non–zero, then \((\mathfrak{g},K) \times \mathcal{G}(\mathbb{A})\)-module generated by \( f \) is a direct sum of finitely many irreducible modules (they consists of cuspidal automorphic forms) (see [5] for details).

Now, we assume that \( \mathcal{G} \) is quasi–split over \( \mathbb{Q} \). Let \( \mathcal{N} \) be the unipotent radical of a Borel subgroup defined over \( \mathbb{Q} \). We assume that \( G \) posess representations in discrete series. Let \( L \) be the \( L \)-packet of discrete series for \( G \) such that some large representation in that packet is integrable. Let \( \eta : \mathcal{N}(\mathbb{Q}) \backslash \mathcal{N}(\mathbb{A}) \rightarrow \mathbb{C}^\times \) be a unitary and non–degenerate character (i.e., it is not trivial by restriction to a root subgroup of simple root). For any open–compact subgroup \( L \subset \mathcal{G}(\mathbb{A}_f) \), we have \( \mathcal{N}(\mathbb{A}_f) = \mathcal{N}(\mathbb{Q}) \cdot (L \cap \mathcal{N}(\mathbb{Q})) \) (where again \( \mathcal{N}(\mathbb{Q}) \) is diagonally embedded in \( \mathcal{N}(\mathbb{A}_f) \)) by strong approximation. Assume that \( \eta \) is right–invariant under \( \mathcal{N}(\mathbb{A}_f) \cap L \) (at least one such \( L \) exists), then by writing \( \mathcal{N}(\mathbb{A}) = \mathcal{N}(\mathbb{R}) \times \mathcal{N}(\mathbb{A}_f) \), we see that a character \( \eta_\infty \) defined by \( \eta_\infty(n) = \eta(n,1) \) is trivial on \( \Gamma_L \cap \mathcal{N}(\mathbb{R}) \). The obtained character is generic in sense of definition introduced in the statement of Corollary 3-12.

A cuspidal automorphic form \( f \) is \( \eta \)-generic (or \( \eta,\mathcal{N} \)-generic) if \( \eta,\mathcal{N} \)-Fourier coefficient defined by
\[
\int_{\mathcal{N}(\mathbb{Q}) \backslash \mathcal{N}(\mathbb{A})} f(nx)\overline{\eta(n)}dn, \quad x \in \mathcal{G}(\mathbb{A}),
\]
is not identically equal to zero. A submodule \( W \) of \( \mathcal{A}_{cusp}(\mathcal{G}(\mathbb{Q}) \backslash \mathcal{G}(\mathbb{A})) \) is \( \eta \)-generic if there exists a cuspidal automorphic form \( f \in W \) such that it is \( \eta \)-generic.
Let $f$ be a cuspidal automorphic form. Let us select an open–compact subgroup $L \subset \mathcal{G}(\mathbb{A}_f)$ such that $f$ is right–invariant under $L$ in the second variable (see (C-2)), and such that $\eta$ is right–invariant under $\mathcal{N}(\mathbb{A}_f) \cap L$ in the second variable also. Then, by ([21], Lemma 3.3), we have

$$\int_{\mathcal{N}(\mathbb{Q}) \setminus \mathcal{N}(\mathbb{A})} f(nx)\overline{\eta(n)}dn = \operatorname{vol}_{\mathcal{N}(\mathbb{A}_f)}(\mathcal{N}(\mathbb{A}_f) \cap L) \int_{\mathcal{N}(\mathbb{R}) \setminus \Gamma_L \setminus \mathcal{N}(\mathbb{R})} f(nx)\overline{\eta_\infty(n)}dn,$$

for all $x \in G$.

Now, we are ready to state and prove the following global result concerning the realization of generic integrable discrete series in the space of generic cuspidal automorphic forms.

**Theorem 6.9.** Let $\mathcal{G}$ be a semisimple algebraic group defined over $\mathbb{Q}$. Assume that $G = \mathcal{G}(\mathbb{R})$ is connected. In addition, we assume that $\mathcal{G}$ is quasi–split over $\mathbb{Q}$. Let $\mathcal{N}$ be the unipotent radical of Borel subgroup defined over $\mathbb{Q}$. We assume that $G$ poses representations in discrete series. Let $L$ be the $L$–packet of discrete series for $G$ such that there is a large representation in that packet which is integrable (then all are integrable by Proposition 7.4). Let $\eta : \mathcal{N}(\mathbb{Q}) \setminus \mathcal{N}(\mathbb{A}) \rightarrow \mathbb{C}^\times$ be a unitary generic character. By the change of splitting we can select an $(\eta_\infty, \mathcal{N}(\mathbb{R}))$–generic discrete series $\pi$ in the $L$–packet $L$. Then, there exists a cuspidal automorphic module $W$ for $\mathcal{G}(\mathbb{A})$ which is $\eta$–generic and its Archimedean component is infinitesimally equivalent to $\pi$ (i.e., considered as a $(\mathfrak{g}, K)$–module only it is a copy of (infinitely many) $(\mathfrak{g}, K)$–modules infinitesimally equivalent to $\pi$).

**Proof.** Let $\varphi$ be a non–zero $K$–finite matrix coefficient of $\pi$. Let $L \subset \mathcal{G}(\mathbb{A}_f)$ be an open–compact subgroup such that $\eta$ is right–invariant under $\mathcal{N}(\mathbb{A}_f) \cap L$. Let $1_L$ be the characteristic function of $L$ in $\mathcal{G}(\mathbb{A}_f)$. By ([19], Theorem 3.10), the Poincaré series

$$P(\varphi \otimes 1_L)(x) = \sum_{\gamma \in \mathcal{G}(\mathbb{Q})} (\varphi \otimes 1_L)(\gamma \cdot x), \quad x \in \mathcal{G}(\mathbb{A})$$

converges absolutely and uniformly on compact sets to a function in $\mathcal{A}_{cusp}(\mathcal{G}(\mathbb{Q}) \setminus \mathcal{G}(\mathbb{A}))$.

Obviously $P(\varphi \otimes 1_L)$ is right–invariant under $L$. Hence, (6-8) is applicable and we obtain

$$\int_{\mathcal{N}(\mathbb{Q}) \setminus \mathcal{N}(\mathbb{A})} P(\varphi \otimes 1_L)(nx)\overline{\eta(n)}dn =$$

$$= \operatorname{vol}_{\mathcal{N}(\mathbb{A}_f)}(\mathcal{N}(\mathbb{A}_f) \cap L) \int_{\mathcal{N}(\mathbb{R}) \setminus \Gamma_L \setminus \mathcal{N}(\mathbb{R})} P(\varphi \otimes 1_L)(nx, 1)\overline{\eta_\infty(n)}dn$$

$$= \operatorname{vol}_{\mathcal{N}(\mathbb{A}_f)}(\mathcal{N}(\mathbb{A}_f) \cap L) \int_{\mathcal{N}(\mathbb{R}) \setminus \Gamma_L \setminus \mathcal{N}(\mathbb{R})} P_{\Gamma_L}(\varphi)(nx)\overline{\eta_\infty(n)}dn, \quad x \in G,$$

since for $x \in G$ using $\mathcal{G}(\mathbb{A}) = G \times \mathcal{G}(\mathbb{A}_f)$ we have

$$P(\varphi \otimes 1_L)(x, 1) = \sum_{\gamma \in \mathcal{G}(\mathbb{Q})} (\varphi \otimes 1_L)(\gamma x, \gamma) = \sum_{\gamma \in \Gamma_L} \varphi(\gamma x) \overset{\text{def}}{=} P_{\Gamma_L}(\varphi)(x).$$
We will show that we can choose \( \varphi \) such that after shrinking \( L \) we have (see Definition 3-1)

\[
\mathcal{F}_{(\eta_{\infty, N(\mathbb{R})})}(P_{\Gamma_L}(\varphi))(1) = \int_{N(\mathbb{R}) \cap \Gamma_L \setminus N(\mathbb{R})} P_{\Gamma_L}(\varphi)(n) \eta_{\infty}(n) \, dn \neq 0.
\]

Then, in view of (6-10) we can find required \( W \) among finitely many irreducible submodules that make in a direct sum the module generated by \( P(\varphi \otimes 1_L) \).

So, let us prove that there exists a \( K \)-finite matrix coefficient \( \varphi \) such that (6-11) hold. We may assume that \( L \) is factorizable. Then, there exists primes \( p_1, \ldots, p_l \), and open–compact subgroups \( L_i \subset G(Q_{p_i}) \), for \( i = 1, \ldots, l \), such that

\[
L = L_1 \times \cdots \times L_t \times \prod_{p \notin \{p_1, \ldots, p_t\}} G_{\mathbb{Z}_p}.
\]

We fix prime \( p_1 \) and shrink \( L_1 \) without altering other groups in the decomposition of \( L \). We fix a sufficiently large natural number \( t \) such that

\[
L'_1 \overset{\text{def}}{=} G(Q_{p_1}) \cap \{ x = (x_{ij}) \in SL_M(\mathbb{Z}_{p_1}); \; x_{i,j} - \delta_{i,j} \in p_1^t \mathbb{Z}_{p_1} \} \subset L_1.
\]

Next, for each \( u \geq t \) we define open–compact subgroups (we "shrink" the lower part of \( L'_1 \))

\[
L'_1(u) \overset{\text{def}}{=} G(Q_{p_1}) \cap
\cap \{ x = (x_{ij}) \in SL_M(\mathbb{Z}_{p_1}); \; x_{i,j} \in p_1^u \mathbb{Z}_{p_1} \text{ for } j > i, \; x_{i,j} - \delta_{i,j} \in p_1^t \mathbb{Z}_{p_1} \text{ for } i \geq j \}.
\]

We consider the following family of open–compact subgroups:

\[
L(u) = L'_1(u) \times L_2 \times \cdots \times L_t \times \prod_{p \notin \{p_1, \ldots, p_t\}} G_{\mathbb{Z}_p}, \; u \geq t.
\]

Next, we may assume that embedding (6-3) such that \( N \) embeds into unipotent upper triangular matrices. Then, \( N(\mathbb{R}) \) consist of all unipotent upper triangular matrices in \( G \). Also, as in the proof of Theorem 6-7 we have that \( N(\mathbb{R}) \cap \Gamma_{L(u)} \) is independent of \( u \geq t \). This will be applied in the following way. By Corollary 3-12 there exists a \( K \)-finite matrix coefficient \( \varphi_1 \) of \( \pi \) which satisfies

\[
\mathcal{F}_{(\eta_{\infty, U})}^{\text{loc}}(\varphi_1) \neq 0
\]

Now, as in the proof of Theorem 6-7 there exists \( u_0 \geq t \) such that the series (see Lemma 3-3)

\[
W_{(\eta_{\infty, N(\mathbb{R})})}^{\Gamma_{L(u)}} \left( \mathcal{F}_{(\eta_{\infty, U})}^{\text{loc}}(\varphi_1) \right)(x) = \sum_{\gamma \in N(\mathbb{R}) \cap \Gamma_{L(u)}} \mathcal{F}_{(\eta_{\infty, U})}^{\text{loc}}(\varphi_1)(\gamma x), \; x \in G,
\]

is non–zero for \( u \geq u_0 \). Now, we let \( u = u_0 \), and as in the proof of Proposition 5-6 we construct required \( \varphi \) (i.e., the one satisfying (6-11)) exists. This completes the proof of the theorem. \( \Box \)
In particular, \( \pi \) for all \( k \geq 1 \). Since \( n_1|n_2|n_3 \cdots \), we obtain that

\[
\Gamma(n_1) \supset \Gamma(n_2) \supset \cdots,
\]

and

\[
\cap_{k \geq 1} \Gamma(n_k) = \{1\}.
\]

Hence, Lemma (iii) implies that there exists \( k_0 \) such that for each \( k \geq k_0 \) we have \( P_k(\varphi) \neq 0 \). Now, we apply Proposition to see that \( \pi \) is infinitesimally equivalent to a subrepresentation of \( \Gamma^1(G, N, \chi_k) \), for some character \( \chi_k \) of \( N \cap \Gamma(n_k) \setminus N \).

In fact, the set of all such characters is exactly the set of characters of \( N \cap \Gamma(n_k) \setminus N \) for which the corresponding Fourier coefficients of \( P_k(\varphi) \) are non–zero. Some of them must be
non–degenerate since the same holds for the following cuspidal automorphic form in global adelic settings [15]:

$$P(\varphi \otimes 1_{L(n_k)})(x) = \sum_{\gamma \in Sp_{2n}(Q)} (\varphi \otimes 1_{L(n_k)})(\gamma \cdot x), \ x \in Sp_{2n}(\mathbb{A})$$

(see ([19], Theorem 3-10) for the treatment of such forms). Here $L(n_k)$ is an open compact subgroup $Sp_{2n}(\mathbb{A}_f)$ such that

$$\Gamma(n_k) = \Gamma_{L(n_k)}.$$

Finally, we transfer this result to $P_k(\varphi)$ using analogue of (6-10) (see the proof of Theorem 6-9).

\[\square\]

7. Appendix: Some Remarks on Integrable Discrete Series

This section is based on the correspondence with Milić [17]. The main purpose of this section to explain that many discrete series are integrable and are $(\chi, N)$–generic for some generic unitary character $\chi$ of the unipotent radical $N$ of a minimal parabolic subgroup (see Definition 3-5 and the definition included in the statement of Corollary 3-12).

Assume that $G$ admits discrete series. By the well–known result of Harish–Chandra this is equivalent to $rank(G) = rank(K)$. We recall their classification following the introduction of [16] with almost the same notation. We remind the reader that $G$ is a connected semisimple Lie group with finite center.

Let $H$ be the compact Cartan subgroup of $G$. We write $\mathfrak{h} = Lie(H)$. In Section 3, we have introduce $\mathfrak{k} = Lie(K)$. We denote by $\Phi$ the set of roots of $\mathfrak{h}_C$ inside $\mathfrak{g}_C$. The root $\alpha \in \Phi$ is compact if the corresponding (one–dimensional) root subspace belong to $\mathfrak{k}_C$. Otherwise, it is non–compact.

Let $W$ be the Weyl group of $\mathfrak{h}_C$ in $\mathfrak{g}_C$ and $W_K$ its subgroup generated by the reflections with respect to the compact roots. The Killing form of $\mathfrak{g}_C$ induces an inner product $(\cdot | \cdot)$ on $\sqrt{-1}\mathfrak{h}^*$, the space of all linear forms on $\mathfrak{h}_C$ which assume imaginary values of $\mathfrak{h}$. An element $\lambda$ of $\sqrt{-1}\mathfrak{h}^*$ is singular if it is orthogonal to at least one root in, and nonsingular otherwise. The differentials of the characters of $H$ form a lattice $\Lambda$ in $\sqrt{-1}\mathfrak{h}^*$. Let $\rho$ be the half-sum of positive roots in $\Phi$, with respect to some ordering on $\sqrt{-1}\mathfrak{h}^*$. Then $\Lambda + \rho$ does not depend on the choice of this ordering.

Harish-Chandra has shown that to each nonsingular $\lambda \in \Lambda + \rho$ we can attach a discrete series representation $\pi_\lambda$, of infinitesimal character $\mu_\lambda$ (usual parameterization) corresponding to $\lambda$, with $\pi_\lambda \simeq \pi_\mu$ if and only if $\lambda \in W_K \nu$. In this way, all discrete series are obtained up to equivalence.

By Langlands, discrete series are divided into $L$–packets according to their infinitesimal characters: to infinitesimal character $\mu$, we define the $L$–packet by $L_\mu = \{\pi_\lambda; \ \mu = \mu_\lambda\}$. By above, if we fix $\pi_\lambda \in L_\mu$, then $L_\mu = \{\pi_{w(\lambda)}; \ w \in W\}$ has $#(W/W_K)$–elements.

Following [16], we define

$$\kappa(\alpha) = \frac{1}{4} \sum_{\beta \in \Phi} |(\alpha | \beta)|, \ \alpha \in \Phi.$$
By the paragraph after the statement of the theorem in the Introduction of [16], the discrete series $\pi_\lambda$ is integrable if and only if

$$\text{(7-1)} \quad |(\lambda|\alpha)| > \kappa(\alpha),$$

for all non-compact roots $\alpha \in \Phi$.

We remark that $W_K$ transforms non-compact roots onto non-compact roots. Therefore, if $\lambda$ satisfies (7-1), then $w(\lambda)$ satisfies the same. In particular, the condition (7-1) is independent of the way we write $\pi$ in the form $\pi_\lambda$.

Roughly speaking, in the language of [16], $\pi_\lambda$ is integrable if and only if $\lambda$ is far enough from non-compact walls.

**Definition 7-2.** Let $\pi$ be an integrable discrete series representation, let us write $\pi = \pi_\lambda$. We let

$$\epsilon_\pi = \min_{\alpha \text{ is non-compact}} \frac{|(\lambda|\alpha)|}{\kappa(\alpha)}.$$

Again, this is independent of the way we write $\pi$ as $\pi_\lambda$.

Since we are working with integrable representations, we include the following lemma which shows to some extent how big is the set of integrable representations in terms of $L$-packets. The lemma is an easy consequence above results of Milićić. We leave the proof as an exercise to the reader.

**Lemma 7-3.** Let $\lambda \in \Lambda + \rho$ be such that $|(\lambda|\alpha)| > \kappa(\alpha)$, for all roots $\alpha \in \Phi$. Then, all representations in the $L$-packet $L_{\mu_\lambda}$ are integrable.

We remark that the following is an easy way to see that $\lambda$’s as in Lemma 7-3 exists. Let $C$ be the Weyl chamber used to determine positive roots of $\Phi$ used to compute $\Lambda$. We select any $\lambda_0 \in (\Lambda + \rho) \cap C$. Then, for sufficiently large odd integer $l$, we have that $\lambda = l\lambda_0$ satisfies the requirements of the lemma.

Let $(\pi, \mathcal{H})$ be an irreducible unitary representation on the Hilbert space $\mathcal{H}$. We say that $(\pi, \mathcal{H})$ is large if the annihilator of $\mathcal{H}_K$ in $U(\mathfrak{g}_C)$ is of maximal Gelfand–Kirillov dimension (i.e., a minimal primitive ideal) [28]. This implies that $G$ is quasi-split ([28], Corollary 5.8). From now on until the end of this section, we assume that $G$ is quasi-split.

The reformulation of largeness in the analytic language is due to Kostant [14]. The representation $(\pi, \mathcal{H})$ is large if and only if it is $(\chi, N)$-generic for some generic unitary character $\chi$ of the unipotent radical $N$ of minimal parabolic subgroup (see Definition 3-5 and the definition included in the statement of Corollary 3-12).

By ([28], Theorem 6.2) (or [17] for more conceptual proof). The discrete series representation $\pi$ is large if and only if the following holds. Let us write $\pi = \pi_\lambda$. Since $\lambda$ is non-singular, it determines the set of positive roots $\Phi_\lambda^+$ by $\alpha \in \Phi_\lambda^+$ if and only if $(\lambda|\alpha) > 0$. Now, $\pi_\lambda$ is large if and only if the basis of $\Phi_\lambda^+$ consists of non-compact roots. Clearly, this claim does not depend on choice of $\lambda$ used to write $\pi = \pi_\lambda$. Every $L$-packet of discrete series contains large representations [28] (or [17] which shows that this statement reduces to certain facts from the structure theory of Lie algebras). By above mentioned test for largeness we can...
easily describe all large representations in the $L$–packet. The details are left to the reader as an exercise. In [17], Miličić has observed the following:

**Proposition 7-4.** A large representation $\pi$ is integrable if and only if its $L$–packet is of the form considered in Lemma 7-3. If this is so, all representations in the $L$–packet are integrable.

**Proof.** It remains to see that ’only if’ part. Let $\pi_\lambda$ be large and integrable. Then, by Miličić’s criterion, $\pi_\lambda$ is integrable if and only if $| (\lambda|\alpha_i) | > \kappa(\alpha_i)$, for all non–compact roots $\alpha \in \Phi$. On the other hand, $\pi_\lambda$ is large, so by the criterion, this is equivalent to the fact that the basis $\{\alpha_1, \ldots, \alpha_l\}$ of $\Phi^\lambda_\vee$ consists of non–compact roots. So, in particular, $(\lambda|\alpha_i) > \kappa(\alpha_i)$, for $i = 1, \ldots, l$. Then, if $\alpha = \sum_{i=1}^l k_i \alpha_i$ is any root in $\Phi^\lambda_\vee$, then

$$
(\lambda|\alpha) = \sum_{i=1}^l k_i (\lambda|\alpha_i) > \sum_{i=1}^l k_i \kappa(\alpha_i) = \frac{1}{4} \sum_{\beta \in \Phi} \left( \sum_{i=1}^l k_i |(\alpha_i|\beta)| \right) \geq \frac{1}{4} \sum_{\beta \in \Phi} \left( \sum_{i=1}^l k_i |\alpha_i|\beta) \right) = \kappa(\alpha).
$$

So, we are in the assumptions of Lemma 7-3. $\Box$

In closing this section, we mention that Proposition 7-4 shows that the great many representations in discrete series are large and integrable. In view of results of Kostant [14] mentioned above, this shows that the results that we present in this paper are for a non–trivial number of discrete series.

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