THE POISSON SATURATION OF COREGULAR SUBMANIFOLDS

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Abstract. This paper is devoted to coregular submanifolds in Poisson geometry. We show that their local Poisson saturation is an embedded Poisson submanifold, and we give a normal form for this Poisson submanifold around the coregular submanifold. This result recovers the normal form around Poisson transversals, and it yields Poisson versions of some normal form/rigidity results around constant rank submanifolds in symplectic geometry. As an application, we prove a uniqueness result concerning coisotropic embeddings of Dirac manifolds in Poisson manifolds. We also show how our results generalize to the setting of coregular submanifolds in Dirac geometry.

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Introduction

A well-known result in symplectic geometry is Weinstein’s generalized Darboux theorem, which states that for any embedded submanifold $X$ of a symplectic manifold $(M, \omega)$, the restriction of $\omega$ to $TM|_X$ determines the symplectic form $\omega$ on a neighborhood of $X$ up to symplectomorphism [We2]. By contrast, given a Poisson manifold $(M, \pi)$ and any embedded submanifold $X \subset M$, one should not expect $\pi$ to be determined, up to neighborhood equivalence, by its restriction $\pi|_X$. For instance, the origin in $\mathbb{R}^2$ is a fixed point for both the zero Poisson structure and the Poisson structure $\pi = (x^2 + y^2)\partial_x \wedge \partial_y$, which are clearly not diffeomorphic around $(0,0)$.

In order for the restriction $\pi|_X$ to determine $\pi$ around $X$, the ambient Poisson manifold needs to satisfy a minimality condition with respect to $X$. Since $\pi|_X$ only contains information in the leafwise direction along $X$, we are led to consider the saturation of $X \subset (M, \pi)$, i.e. the union of the symplectic leaves that intersect $X$. Clearly, the saturation of $X$ fails to be smooth in general; the purpose of this note is to single out a class of embedded submanifolds $X \subset (M, \pi)$ whose saturation is smooth near $X$, in a sense that will be made precise later. Since the saturation $Sat(X)$ of $X \subset (M, \pi)$ is traced out by following Hamiltonian flows starting at points of $X$ in directions normal to $X \subset M$, it is natural to impose the following regularity condition on $X$. 

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**Definition.** We call an embedded submanifold $X$ of a Poisson manifold $(M, \pi)$ **coregular** if the map $\text{pr} \circ \pi^2 : T^*M|_X \to TM|_X/TX$ has constant rank.

It is equivalent to require that the $\pi$-orthogonal $TX^{\perp_{\pi}} := \pi^*(TX^0)$ has constant rank. Extreme examples are transversals and Poisson submanifolds of $(M, \pi)$, and we show that any coregular submanifold $X \subset (M, \pi)$ is obtained by intersecting such submanifolds. Note that if $\pi$ is symplectic, then any submanifold of $(M, \pi)$ is coregular.

The main result of Section 1 is the fact that the saturation of a coregular submanifold $X \subset (M, \pi)$ is smooth around $X$, in the following sense.

**Theorem A.** If $X \subset (M, \pi)$ is a coregular submanifold, then there exists a neighborhood $V$ of $X$ such that the saturation of $X$ inside $(V, \pi|_V)$ is an embedded Poisson submanifold.

We will refer to this Poisson submanifold as the **local Poisson saturation** of $X$. The proof of Theorem A relies on some contravariant geometry and some results concerning dual pairs in Poisson geometry.

Sections 2 and 3 are devoted to the construction of a normal form for the local Poisson saturation of a coregular submanifold $X \subset (M, \pi)$.

**Theorem B.** Let $X \subset (M, \pi)$ be a coregular submanifold. A neighborhood of $X$ in its local Poisson saturation is Poisson diffeomorphic with the local model $(U, \pi(W, \eta))$.

The proof of this result goes along the same lines as the proof of the normal form around Poisson transversals (PM1), using dual pairs in Dirac instead of Poisson geometry.

Since the local model $(U, \pi(W, \eta))$ is constructed out of the restriction $\pi|_X$, Theorem B shows that the local Poisson saturation of a coregular submanifold $X$ is determined by the restriction $\pi|_X$, up to Poisson diffeomorphism around $X$. We thus obtain a Poisson version of Weinstein’s generalized Darboux theorem in symplectic geometry. In general, one needs the full information of $\pi|_X$ in order to determine the local Poisson saturation of $X$. However, there are distinguished coregular submanifolds $X$ for which only part of this information is required, as we show in Section 4.
In Section 4, we specialize our normal form to some particular classes of coregular submanifolds. These allow for a good choice of complement $W$ and/or closed extension $\eta$, and as such our normal form becomes more explicit. Most notably, we obtain statements concerning the following types of submanifolds, i) and ii) being particular instances of iii):

i) Poisson transversals: We recover the normal form theorem around Poisson transversals, which was established in [FM1], [BLM].

ii) Coregular coisotropic submanifolds: We obtain a Poisson version of Gotay’s normal form theorem from symplectic geometry [G], which shows that the local Poisson saturation of a coregular coisotropic submanifold $\iota : X \hookrightarrow (M, \pi)$ is determined, up to Poisson diffeomorphism around $X$, by the pullback Dirac structure $\iota^!L_\pi$.

iii) Coregular pre-Poisson submanifolds: We obtain a Poisson analog of Marle’s constant rank theorem from symplectic geometry [Ma]. Loosely speaking, the result shows that the local Poisson saturation of a coregular pre-Poisson submanifold $\iota : X \hookrightarrow (M, \pi)$ is determined, up to Poisson diffeomorphism around $X$, by the pullback Dirac structure $\iota^!L_\pi$ and the restriction of $\pi$ to $(TX^\perp)^*/(TX^\perp \cap TX)^*$.

In Section 5, we present an application of our normal form specialized to the case of coregular coisotropic submanifolds. We address the problem of embedding a Dirac manifold $(X, L)$ coisotropically into a Poisson manifold $(M, \pi)$, which was considered before in [CZ2] and [Wa]. Existence of coisotropic embeddings is settled in [CZ2], where one shows that such an embedding exists exactly when $L \cap TX$ has constant rank. An explicit construction of the Poisson manifold $(M, \pi)$ is given in that case; another construction appears in [Wa]. The uniqueness of such embeddings was conjectured in [CZ2], but only proved under additional regularity assumptions on $(X, L)$. Using our normal form result, we can show that any coisotropic embedding of $(X, L)$ factors through the model $(M, \pi)$ constructed in [CZ2], which in turn proves the conjecture concerning the uniqueness of coisotropic embeddings.

In Section 6, we discuss how our results can be generalized to the setting of coregular submanifolds in Dirac geometry. The Appendix contains a result in differential topology for which we could not find a proof in the literature.

Terminology and notation. We freely use notions from Dirac geometry throughout the text, adopting terminology and notation from [FM2]. For more background on Dirac structures, see e.g. [B]. We also mention here that the recent work [BFM] addresses coregular submanifolds $X \subset (M, \pi)$ for which additionally $TX^\perp \cap TX$ is trivial, ensuring that $X$ has an induced Poisson structure. We warn the reader that these submanifolds are also referred to as “coregular” in [BFM]. In the book [CFM], these are called “coregular Poisson-Dirac”.

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1. The saturation of a coregular submanifold

In this section, we discuss the saturation of embedded submanifolds $X$ in a Poisson manifold $(M, \pi)$. Our aim is to give sufficient conditions on $X$ that ensure smoothness of its saturation in a neighborhood of $X$. We introduce the class of coregular submanifolds $X \subset (M, \pi)$, and we show that such a submanifold $X$ has a neighborhood $U$ in $M$ such that the saturation of $X$ in $(U, \pi|_U)$ is an embedded Poisson submanifold.
**Definition 1.1.** The saturation of a submanifold $X$ of a Poisson manifold $(M, \pi)$ is the union of all the leaves of $(M, \pi)$ that intersect $X$. We denote the saturation of $X$ by $Sat(X)$.

Recall that a Poisson submanifold $P \subset (M, \pi)$ is said to be complete if the inclusion $(P, \pi_P) \to (M, \pi)$ is a complete Poisson map [CW §6.2]. It is clear that, given a submanifold $X \subset (M, \pi)$, the saturation $Sat(X)$ is the smallest complete Poisson submanifold of $(M, \pi)$ containing $X$, provided it is smooth. Indeed, a complete Poisson submanifold $P \subset (M, \pi)$ is saturated [CW Prop. 6.1], so if $X \subset P$ then $Sat(X) \subset Sat(P) = P$.

The saturation of a submanifold can be very wild; in general it does not have a submanifold structure. For instance, consider the log-symplectic manifold $(\mathbb{R}^2, x\partial_x \wedge \partial_y)$; its saturation is $\{x < 0\} \cup \{(0,0)\} \cup \{x > 0\}$. Clearly, this saturation doesn’t even contain a Poisson submanifold around the $x$-axis.

We now single out classes of submanifolds $X \subset (M, \pi)$ that do satisfy this property, i.e. whose saturation contains a Poisson submanifold which contains $X$. Examples of such submanifolds are transversals (whose saturation is open, and therefore a Poisson submanifold) and Poisson submanifolds. These are extreme cases of what we call coregular submanifolds.

**Definition 1.2.** Given a Poisson manifold $(M, \pi)$, we call an embedded submanifold $X \subset M$ coregular if the map $pr \circ \pi^i : T^*M|_X \to TM|_X/TX$ has constant rank.

Note indeed that transversals and Poisson submanifolds are exactly those submanifolds $X \subset (M, \pi)$ for which the map $pr \circ \pi^i$ is respectively of full rank or identically zero.

We will now list some more observations about coregular submanifolds. For any submanifold $X \subset (M, \pi)$, we denote its $\pi$-orthogonal by $TX^\perp := \pi^i(TX^\circ)$. If $x \in X$ and $L$ is the symplectic leaf through $x$, then $T_xX^\perp$ is the symplectic orthogonal of $T_xX \cap T_xL$ in the symplectic vector space $(T_xL, (\pi|_{L})_x^{-1})$. Various types of submanifolds in Poisson geometry are defined in terms of their $\pi$-orthogonal; see [CFM] and [Z] for a systematic overview.

a) Given a submanifold $X \subset (M, \pi)$, we get an exact sequence at points $x \in X$:

$$0 \longrightarrow (T_xX^\perp)^\circ \longrightarrow T_xM \overset{pr^i\circ}{\longrightarrow} T_xM/T_xX. \quad (1)$$

Hence, $X \subset (M, \pi)$ is coregular exactly when $TX^\perp$ has constant rank. This observation explains the name “coregular” because, as mentioned above, $TX^\perp$ consists of the symplectic orthogonal of the (singular) distribution $TX \cap \pi^i(T^*M|_X)$.

b) We give an alternative characterization of coregular submanifolds $X \subset (M, \pi)$ in Dirac geometric terms. Denote by $L_\pi$ the Dirac structure $L_\pi = \{\pi^i(\alpha) + \alpha : \alpha \in T^*M\}$ defined by $\pi$, and let $i : X \hookrightarrow (M, \pi)$ be any submanifold. Then $X$ is coregular exactly when $L_\pi \cap \ker((di)^*)$ has constant rank. Indeed, $L_\pi \cap \ker((di)^*) = \ker(\pi^i) \cap TX^\circ$, so for any $x \in X$, we have

$$\dim (L_\pi \cap \ker((di)^*))_x = \dim (T_xX^\circ) - \dim (T_xX^\perp).$$

Hence, coregular submanifolds are exactly those $X \subset (M, \pi)$ for which the Dirac structure $L_\pi$ automatically pulls back to a smooth Dirac structure on $X$ [B Prop. 1.10].

We mention here that $X$ being coregular is not a necessary condition for $Sat(X)$ to contain a Poisson submanifold around $X$, as illustrated by the following examples.

**Examples 1.3.** i) Take the Lie-Poisson structure $\{\mathfrak{so}(3)^*, z\partial_z \wedge \partial_y + x\partial_y \wedge \partial_z + y\partial_z \wedge \partial_x\}$ and let $X$ be the plane defined by $z = 0$. The symplectic foliation of $\mathfrak{so}(3)^*$ consists of concentric spheres of radius $r \geq 0$ centered at the origin, so that $Sat(X) = \mathfrak{so}(3)^*$. However, $TX^\perp = \text{Span}\{y\partial_x - x\partial_y\}$ vanishes at the origin, so $X$ is not coregular.
ii) Consider the regular Poisson manifold \((\mathbb{R}^3, \partial_x \wedge \partial_y)\) and let \(X\) be defined by the equation \(z = x^3\). Then the saturation \(\text{Sat}(X)\) is all of \(\mathbb{R}^3\), but \(X\) is not coregular. Indeed, we have \(TX^{\perp_x} = \text{Span}\{-3x^2 \partial_y\}\), which drops rank at points of the form \((0, y, 0) \in X\).

To construct a Poisson submanifold around the coregular submanifold \(X \subset (M, \pi)\), we use some notions from contravariant geometry and some theory of dual pairs [CM,FM1].

**Definition 1.4.** A Poisson spray on a Poisson manifold \((M, \pi)\) is a vector field \(\chi\) on the cotangent bundle \(T^*M\) satisfying:

i) \(\text{spray exponential} \ \exp_{\chi} : \Sigma \subset T^*M \to M : \xi \mapsto \exp_{\chi}(\xi)\).

This neighborhood \(\Sigma \subset T^*M\) also supports a closed two-form \(\Omega\), which is defined by averaging the canonical symplectic form \(\omega_{\text{can}}\) with respect to the flow \(\phi_{\chi}^t\) of the Poisson spray \(\chi \in \mathfrak{x}(T^*M)\):

\[
\Omega_{\chi} := \int_0^1 (\phi_{\chi}^t)^* \omega_{\text{can}} \, dt.
\]

It was proved in [CM] that \(\Omega_{\chi}\) is non-degenerate along the zero section \(M \subset T^*M\), so shrinking \(\Sigma \subset T^*M\) if necessary, we can assume that \(\Omega_{\chi}\) is symplectic on \(\Sigma\). By [FM1, Lemma 25], the symplectic manifold \((\Sigma, \Omega_{\chi})\) fits in a full dual pair

\[
(M, \pi) \xleftarrow{\text{pr}} (\Sigma, \Omega_{\chi}) \xrightarrow{\exp_{\chi}} (M, -\pi).
\]

That is, denoting by \(\pi_{\chi} := \Omega_{\chi}^{-1}\) the Poisson structure corresponding with \(\Omega_{\chi}\), the maps \(\text{pr} : (\Sigma, \pi_{\chi}) \to (M, \pi)\) and \(\exp_{\chi} : (\Sigma, \pi_{\chi}) \to (M, -\pi)\) are surjective Poisson submersions with symplectically orthogonal fibers: \((\ker \text{spray})^{\perp_{\chi}} = \ker \text{d} \exp_{\chi}\).

Both legs in the diagram (2) are symplectic realizations. We will need the following lemma, which concerns the interplay between symplectic realizations and coregular submanifolds of a Poisson manifold.

**Lemma 1.5.** Let \(X \subset (M, \pi)\) be a coregular submanifold and let \(\mu : (\Sigma, \Omega) \to (M, \pi)\) be a symplectic realization. Then \((\ker \text{d} \mu)^{\perp_\Omega} \cap T(\mu^{-1}(X))\) has constant rank, equal to the corank of \(TX^{\perp_x} \subset TM|_X\).

**Proof.** Denote by \(\pi_{\Omega} : = \Omega^{-1}\) the Poisson structure corresponding with \(\Omega\). First note that, for \(\xi \in \mu^{-1}(X)\), we have

\[
(\ker \text{d} \mu)^{\perp_\Omega} = (d\mu)^{\perp_\Omega}(T_{\mu(\xi)}M).
\]

Since for any \(\beta \in T_{\mu(\xi)}M\), we have that \((d\mu)^{\perp_\Omega}(T_{\mu(\xi)}M)\) belongs to \(T_{\mu(\xi)}M\) exactly when \(\beta \in (T_{\mu(\xi)}M)^{\perp_x}\), we obtain

\[
(\ker \text{d} \mu)^{\perp_\Omega} \cap T_{\xi}(\mu^{-1}(X)) = (\ker \text{d} \mu)^{\perp_{\xi}} \cap (d\mu)^{\perp_{\xi}}(T_{\mu(\xi)}M) = \pi_{\Omega}( (d\mu)^{\perp_\Omega}(T_{\mu(\xi)}X^{\perp_x})^0 ) .
\]

Hence, the rank of \((\ker \text{d} \mu)^{\perp_\Omega} \cap T(\mu^{-1}(X))\) is constant, equal to \(\dim M - \text{rk}(TX^{\perp_x})\). \(\Box\)
We now prove that for a coregular submanifold $X \subset (M, \pi)$, there exists an embedded Poisson submanifold of $(M, \pi)$ containing $X$ that lies in the saturation $Sat(X)$. This Poisson submanifold is in fact the saturation of $X$ in a neighborhood $(U, \pi|_U)$ of $X$.

**Theorem 1.6.** Let $X \subset (M, \pi)$ be a coregular submanifold.

1. There exists an embedded Poisson submanifold $(P, \pi_P) \subset (M, \pi)$ containing $X$ that lies inside the saturation $Sat(X)$.

2. Shrinking $P$ if necessary, there exists a neighborhood $U$ of $X$ in $M$ such that $(P, \pi_P)$ is the saturation of $X$ in $(U, \pi|_U)$.

**Proof.** We divide the proof into four steps.

**Step 1:** Construction of the embedded submanifold $P \subset M$.

Choose a Poisson spray $\chi \in \mathfrak{X}(T^*M)$ and denote by $\exp_{\chi} : \Sigma \subset T^*M \to M$ the corresponding spray exponential. Notice that the restriction $\exp_{\chi} : \Sigma|_X \to M$ takes values in $Sat(X)$, because sprays trace cotangent paths [CF §1]. Indeed, for $\xi \in \Sigma|_X$, the curve $\gamma(t) := \text{pr}(\phi^t_\chi(\xi))$ satisfies

$$\gamma'(t) = (d\text{pr})\phi^t_\chi(\xi) = (d\text{pr})\phi^t_\chi(\xi) \left( \chi(\phi^t_\chi(\xi)) \right) = \pi_\gamma^\sharp(t)(\phi^t_\chi(\xi)),$$

showing that $t \mapsto (\phi^t_\chi(\xi), \gamma(t))$ is a cotangent path between $\gamma(0) = \text{pr}(\xi)$ and $\gamma(1) = \exp_{\chi}(\xi)$. Consequently, $\text{pr}(\xi)$ and $\exp_{\chi}(\xi)$ lie in the same leaf of $(M, \pi)$, hence $\exp_{\chi}(\Sigma|_X) \subset Sat(X)$.

Choosing a complement to $TX^{\perp_\pi}$ in $TM|_X$, we get an inclusion $(TX^{\perp_\pi})^* \subset T^*M|_X$. The restriction of the spray exponential $\exp_{\chi} : (TX^{\perp_\pi})^* \cap \Sigma \to M$ fixes points of $X$, and its differential along $X$ reads [FM1 Lemma 24]:

$$d\exp_{\chi} : T_xX \oplus (T_xX^{\perp_\pi})^* \to T_xM : (v, \xi) \mapsto v + \pi^\sharp(\xi).$$

This map is injective, for if $\pi^\sharp(\xi) = -v \in T_xX$, then $\xi \in (\pi^\sharp)^{-1}(T_xX) = (T_xX^{\perp_\pi})^0$ and therefore $\xi \in (T_xX^{\perp_\pi})^* \cap (T_xX^{\perp_\pi})^0 = \{0\}$. Proposition 7.1 and Remark 7.2 in the Appendix imply that, shrinking $\Sigma$ if necessary, the map $\exp_{\chi} : (TX^{\perp_\pi})^* \cap \Sigma \to M$ is an embedding. Setting

$$P := \exp_{\chi}((TX^{\perp_\pi})^* \cap \Sigma),$$

this is an embedded submanifold of $M$ containing $X$ that lies inside $Sat(X)$.

**Step 2:** Shrinking $\Sigma$ if necessary, we have $P = \exp_{\chi}(\Sigma|_X)$.

To see this, let us denote for short $\Sigma_X := \Sigma|_X \subset T^*M|_X$ and $\widehat{\Sigma_X} := (TX^{\perp_\pi})^* \cap \Sigma$.

First, we claim that the restriction $\exp_{\chi}|_{\Sigma_X}$ has constant rank, equal to the rank of $\exp_{\chi}|_{\widehat{\Sigma_X}}$. Indeed, using the self-dual pair [2], we have that

$$\ker \left( d \left( \exp_{\chi}|_{\Sigma_X} \right) \right) = \ker(d\exp_{\chi}) \cap T(pr^{-1}(X)) = \ker(d\text{pr}) \cap T(pr^{-1}(X)),$$

which has constant rank equal to $\dim M - rk(TX^{\perp_\pi})$ by Lemma 1.5. Consequently, the rank of $\exp_{\chi}|_{\Sigma_X}$ is equal to $\dim M + rk(TX^{\perp_\pi})$, which is the rank of $\exp_{\chi}|_{\widehat{\Sigma_X}}$.

Using the claim just proved, we will now show that $\exp_{\chi}(\widehat{\Sigma_X}) = \exp_{\chi}(\Sigma_X)$, shrinking $\Sigma$ if necessary. It is enough to prove that every point $\xi \in \widehat{\Sigma_X}$ has a neighborhood $V^\xi \subset \Sigma_X$
such that \( \exp_{\chi}(V^\xi) \subset \exp_{\chi}(\Sigma_X) \). We keep in mind the diagram
\[
\begin{array}{ccc}
\Sigma_X \subset (TX^\perp)^* & \xrightarrow{\exp_X|_{\Sigma_X}} & \Sigma_X \subset T^*M|_X \\
\exp_X|_{\Sigma_X} & ? & \exp_X|_{\Sigma_X} \\
\exp_X(\Sigma_X) \quad \quad \quad & \quad \quad M
\end{array}
\]

Pick \( \xi \in \Sigma_X \subset X \). Since \( \exp_X|_{\Sigma_X} \) has constant rank, there is an open \( U^\xi \subset \Sigma_X \) around \( \xi \) such that \( \exp_X(U^\xi) \subset M \) is an embedded submanifold. As \( \exp_X \) is an embedding on \( \Sigma_X \), also \( \exp_X(U^\xi \cap \Sigma_X) \subset M \) is an embedded submanifold. Since \( \dim \exp_X(U^\xi \cap \Sigma_X) = \dim \exp_X(U^\xi) \) by the previous claim, the inverse function theorem implies that \( \exp_X(U^\xi \cap \Sigma_X) \) is open in \( \exp_X(U^\xi) \). Since \( \exp_X|_{U^\xi} : U^\xi \to \exp_X(U^\xi) \) is continuous, the set \( \exp_X|_{U^\xi}(\exp_X(U^\xi \cap \Sigma_X)) \) is open in \( U^\xi \), hence in \( \Sigma_X \). Setting \( V^\xi := \exp_X|_{U^\xi}(\exp_X(U^\xi \cap \Sigma_X)) \) proves the assertion.

Step 3: \( P \) is a Poisson submanifold of \( (M, \pi) \).
We use the previous step, which states that \( P = \exp_X(\Sigma_X) \). Pick a point \( x \in P \) and let \( \xi \in \Sigma_X \) be such that \( \exp_X(\xi) = x \). We have to show that \( \pi^2(T_xP^\circ) = \{0\} \). Making use of the dual pair \([2]\), we have
\[
\pi^2(T_xP^\circ) = \left( (d \exp_{\chi})_\xi \circ \pi^2_{\chi} \circ (d \exp_{\chi})_\xi \right)(T_xP^\circ),
\]
so it is enough to show that
\[
\pi^2_{\chi}(T_xP^\circ) \subset \ker (d \exp_{\chi})_\xi = \pi^2_{\chi}(\ker \text{dpr})_\xi.
\]
To see that this inclusion holds, note that \( (\ker \text{dpr})_\xi \subset T\Sigma_X \) and \( (d \exp_{\chi})_\xi(T\Sigma_X) \subset T_xP \), which implies that
\[
(d \exp_{\chi})_\xi(T_xP^\circ) \subset (T\Sigma_X)^\circ \subset (\ker \text{dpr})_\xi.
\]
We now showed that \( P \subset (M, \pi) \) is a Poisson submanifold, which finishes Step 3.

Step 4: Construction of the neighborhood \( U \) of \( X \).
The idea is to extend \( \exp_X : (TX^\perp)^* \cap \Sigma \to M \) to a local diffeomorphism, using the same reasoning as in the proof of Proposition [7,1] in the Appendix. Choosing a complement
\[
TM|_X = TX \oplus \pi^2(TX^\perp)^* \subset TP|_X \oplus C,
\]
and a linear connection \( \nabla \) on \( TM \), we obtain a map
\[
\psi : V \subset ((TX^\perp)^* \oplus C) \to M : (\xi, c) \mapsto \exp_{\nabla} (Tr_{\exp_X(t\xi)}c),
\]
which is a diffeomorphism onto a neighborhood of \( X \). Here \( V \) is a suitable convex neighborhood of the zero section, and \( Tr_{\exp_X(t\xi)} \) denotes parallel transport along the curve \( t \mapsto \exp_X(t\xi) \) for \( t \in [0, 1] \). Note that \( \psi \) satisfies \( \psi(\xi, 0) = \exp_X(\xi) \). Consequently, shrinking \( P \) if necessary, we can assume that
\[
P = \psi \left( V \cap ((TX^\perp)^* \oplus \{0\}) \right).
\]
We now set \( U := \psi(V) \), and we check that \( P \) is the Poisson saturation of \( X \) in \( (U, \pi|_U) \).

On one hand, since \( (TX^\perp)^* \) is closed in \( (TX^\perp)^* \oplus C \), also \( P \) is closed in \( U \). It follows that \( P \) is saturated, being a properly embedded Poisson submanifold. Hence, the saturation of \( X \) in \( (U, \pi|_U) \) is contained in \( P \). On the other hand, if \( \exp_X(\xi) = \psi(\xi, 0) \in P \subset U \), then also
exp_\chi(t_\xi) \in U \text{ for } t \in [0, 1] \text{ since } V \text{ is convex. Consequently, the path } t \mapsto (\phi_1^t(\xi), \exp_\chi(t_\xi)) \text{ is a cotangent path covering a path in } U \text{ that connects } \exp_\chi(\xi) \text{ with a point in } X. \text{ This shows that } \exp_\chi(\xi) \text{ is contained in the Poisson saturation of } X \text{ in } (U, \pi|_U).

The theorem above shows that the saturation of a coregular submanifold \( X \subset (M, \pi) \) in some neighborhood \((U, \pi|_U)\) of \( X \) is an embedded Poisson submanifold. Clearly, one cannot take \( U \) to be all of \( M \) in general. In this respect, we have the following sufficient condition.

**Corollary 1.7.** Let \( X \subset (M, \pi) \) be a coregular submanifold. If the submanifold \( P \) constructed in Theorem 1.6 is open in \( \text{Sat}(X) \) for the induced topology, then \( \text{Sat}(X) \) is an embedded submanifold of \( M \).

**Proof.** Recall the following general fact [CFM, Lemma 4.5]: if \( \{N_i\}_{i \in I} \) is a collection of embedded submanifolds of \( M \), all of the same dimension, such that \( N_i \cap N_j \) is open in \( N_i \) for all \( i, j \in I \), then \( N := \bigcup_{i \in I} N_i \) is a smooth manifold, possibly not second countable, for which the inclusion \( N \hookrightarrow M \) is an immersion. The smooth structure is uniquely determined by the condition that the maps \( N_i \hookrightarrow N \) are smooth open embeddings.

We want to apply this fact to the collection \( \{\phi_1^t(P) : f \in C_c^\infty([0, 1] \times M)\} \), where \( \phi_1^t \) denotes the time-1-flow of the Hamiltonian vector field associated with the compactly supported function \( f \in C_c^\infty([0, 1] \times M) \). We have to check that \( \phi_1^t(P) \cap \phi_1^s(P) \) is open in \( \phi_1^t(P) \). To this end, note that both \( \phi_1^t(P) \) and \( \phi_1^s(P) \) are open in \( \text{Sat}(X) \), since by assumption \( P \) is open in \( \text{Sat}(X) \) and \( \phi_1^t, \phi_1^s \) are diffeomorphisms preserving \( \text{Sat}(X) \). Hence, also \( \phi_1^t(P) \cap \phi_1^s(P) \) is open in \( \text{Sat}(X) \), so there exists an open \( V \subset M \) such that

\[
\phi_1^t(P) \cap \phi_1^s(P) = V \subset \text{Sat}(X).
\]

Since also \( \phi_1^t(P) = U \cap \text{Sat}(X) \) for some open \( U \subset M \), we obtain

\[
\phi_1^t(P) \cap \phi_1^s(P) = \phi_1^t(P) \cap \phi_1^s(P) \cap U = V \cap (U \cap \text{Sat}(X)) = V \cap \phi_1^t(P),
\]

which shows that \( \phi_1^t(P) \cap \phi_1^s(P) \) is open in \( \phi_1^t(P) \).

So we can apply the fact mentioned above, which gives \( \text{Sat}(X) \) a smooth manifold structure, a priori not necessarily second countable, for which \( \text{Sat}(X) \hookrightarrow M \) is an immersion. But since the topology of this smooth structure is generated by open subsets of the submanifolds \( \phi_1^t(P) \), it coincides with the induced topology on \( \text{Sat}(X) \). In particular, it is second countable, and \( \text{Sat}(X) \) is an embedded submanifold of \( M \).

The proof of Corollary 1.7 breaks down if \( P \) is not open in \( \text{Sat}(X) \), because the technical requirements of [CFM, Lemma 4.5] are no longer met. See for instance Example 1.9 below.

**Remark 1.8.** We comment on the condition in Corollary 1.7 that \( P = \exp_\chi(\Sigma|_X) \) needs to be open in \( \text{Sat}(X) \) for the induced topology. This occurs exactly when we are able to find a small transversal \( \tau \subset (M, \pi) \) to the leaves such that \( \tau \cap \text{Sat}(X) = X \).

To see that then \( \exp_\chi(\Sigma|_X) \) is indeed open in \( \text{Sat}(X) \) with respect to the induced topology, we note that \( \exp_\chi : \Sigma|_\tau \to M \) is a submersion, shrinking \( \Sigma \) if necessary. Indeed, at points \( p \in \tau \), the differential

\[
\exp_\chi : T_p\tau \oplus T_p^*M \to T_pM : (v, \xi) \mapsto v + T_\chi^*(\xi)
\]

is surjective since \( \tau \subset (M, \pi) \) is a transversal. Hence \( \exp_\chi \) is of maximal rank in a neighborhood of \( \tau \subset \Sigma|_\tau \). In particular, shrinking \( \Sigma \) if needed, we have that \( \exp_\chi(\Sigma|_\tau) \subset M \) is open. It now suffices to remark that \( \exp_\chi(\Sigma|_X) = \exp_\chi(\Sigma|_\tau) \cap \text{Sat}(X) \). The forward
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Inclusion is clear, since \( X \subset \tau \) and \( \exp_X(\Sigma_{|X}) \subset \text{Sat}(X) \). For the backward inclusion, assume that \((p, \xi) \in \Sigma_{|\tau}\) is such that \( \exp_X(\xi) \in \text{Sat}(X) \). Since \( p \) lies in the same leaf as \( \exp_X(\xi) \in \text{Sat}(X) \) and \( \text{Sat}(X) \) is saturated, it follows that \( p \in \text{Sat}(X) \). Consequently, \( p \in \tau \cap \text{Sat}(X) = X \). This shows that \( \exp_X(\Sigma_{|X}) = \exp_X(\Sigma_{|\tau}) \cap \text{Sat}(X) \) is open in \( \text{Sat}(X) \) for the induced topology.

In the particular case where \( X \) is a point, then \( \text{Sat}(X) \) is just the leaf through \( X \), which is well-known to possess a natural smooth structure. Indeed, each leaf of a Poisson manifold is an initial submanifold, so in particular it possesses a unique smooth structure that turns it into an immersed submanifold. For an arbitrary coregular submanifold \( X \), its saturation does not have a natural smooth structure, as illustrated in the following example.

**Example 1.9.** We look at the manifold \((\mathbb{R}^3 \times S^1, x, y, z, \theta)\) with Poisson structure \( \pi = \partial z \wedge \partial \theta \). Consider the curve \( \beta: \mathbb{R} \to \mathbb{R}^3 : t \mapsto (\sin(2t), \sin(t), t) \), which is a “figure eight” coming out of the \( xy \)-plane. Denote its image by \( C \subset \mathbb{R}^3 \), and let \( C_{\text{base}} \) be the projection of \( C \) onto the \( xy \)-plane. The submanifold \( X := C \times S^1 \subset \mathbb{R}^3 \times S^1 \) is embedded, and we claim that it is coregular. To see this, we only have to check that \( \dim(T_pX \cap T_pL) \) is constant for \( p \in X \), where \( L \) denotes the leaf through \( p \). Since at a point \( p = (\beta(t_0), \theta_0) \) we have

\[
T_pX = \text{Span}\{ \partial_{\theta}|_p, 2 \cos(2t_0) \partial_x|_p + \cos(t_0) \partial_y|_p + \partial_z|_p \},
\]

it is clear that \( T_pX \cap T_pL = \text{Span}\{ \partial_{\theta}|_p \} \), since \( \cos(t_0) \) and \( \cos(2t_0) \) cannot both be zero. Hence, \( X \subset (\mathbb{R}^3 \times S^1, \pi) \) is coregular. Its saturation is given by \( \text{Sat}(X) = C_{\text{base}} \times \mathbb{R} \times S^1 \).

The saturation has two obvious smooth structures that turn it into an immersed submanifold, coming from those on the “figure eight”. But neither of them can be called natural, because for both smooth structures the inclusion \( X \hookrightarrow \text{Sat}(X) \) is not even continuous.

Let us also come back to the proof of Corollary 1.7 and see why it fails in this case. We refer to the figure below, where we removed the \( S^1 \)-factor, which is not essential to the spirit of the example. The embedded submanifold \( P \) in this case is obtained by slightly thickening the curve in vertical direction. One can take a Hamiltonian flow \( \phi^1_Xf \) such that \( \phi^1_Xf(P) \cap P \) consists of vertical segments of the line in which the cylinder intersects itself. This is not an open subset of \( P \), so we can no longer apply [CFM, Lemma 4.5]. And indeed, the conclusion of Corollary 1.7 fails in this example.

**Figure 1.** The coregular submanifold \( X \) and its saturation \( \text{Sat}(X) \). This is the picture in \( \mathbb{R}^3 \); the \( S^1 \)-factor is omitted for the sake of depiction.
As a consequence of Theorem 1.6 we obtain an alternative characterization of coregular submanifolds. It turns out that the two extreme examples – Poisson submanifolds and transversals – are the building blocks of any coregular submanifold.

**Proposition 1.10.** A submanifold \( X \subset (M, \pi) \) is coregular if and only if \( X \) is the intersection of a Poisson submanifold \( P \subset (M, \pi) \) with a transversal \( \tau \subset (M, \pi) \).

A transversal \( \tau \subset (M, \pi) \) is also transverse to any Poisson submanifold \( P \subset (M, \pi) \), since the intersection of \( P \) with any leaf of \( (M, \pi) \) is open in the leaf. Indeed, if \( p \in P \) and \( L \) is the leaf through \( p \), then

\[
T_p M = T_p \tau + T_p L = T_p \tau + T_p (P \cap L) \subset T_p \tau + T_p P,
\]

which shows that \( \tau \cap P \). In particular, the intersection \( \tau \cap P \) is smooth.

*Proof of Prop. 1.10.* First assume that \( X \subset (M, \pi) \) is a coregular submanifold. Then Theorem 1.6 then gives a Poisson submanifold \( P \subset (M, \pi) \) containing \( X \), and the proof shows that

\[
TP|_X = TX \oplus \pi^\sharp(TX^{\perp_\pi})^*.
\]

Choose a complement \( E \) to this subbundle of \( TM|_X \), i.e. \( TM|_X = TX \oplus \pi^\sharp(TX^{\perp_\pi})^* \oplus E \). Using a (metric) exponential map, we can construct a submanifold \( \tau \subset M \) containing \( X \) such that \( T\tau|_X = TX \oplus E \). For small enough \( \tau \), we have \( \tau \cap P = X \), and moreover

\[
TM|_X = TX \oplus \pi^\sharp(TX^{\perp_\pi})^* \oplus E = (TX + \text{Im}(\pi^\sharp|_X)) \oplus E = \text{Im}(\pi^\sharp|_X) + T\tau|_X,
\]

which shows that \( \tau \) is a transversal along \( X \). Shrinking \( \tau \) if necessary, this implies that \( \tau \) is a transversal in \( (M, \pi) \). This proves the forward implication.

For the converse, assume that \( X = \tau \cap P \) is a submanifold of \( M \), where \( P \subset (M, \pi) \) is a Poisson submanifold and \( \tau \subset (M, \pi) \) is a transversal. Then \( TX = T\tau|_X \cap TP|_X \), so that \( TX^\circ = (T\tau|_X)^\circ + (TP|_X)^\circ \). Since \( P \) is a Poisson submanifold, we get \( TX^{\perp_\pi} = \pi^\sharp((T\tau|_X)^\circ) \). Since \( \tau \) is a transversal, the restriction \( \pi^\sharp|_{T\tau^\circ} \) is injective, hence \( X \) is coregular. \( \square \)

In what follows, we denote by \((P, \pi_P)\) the Poisson submanifold containing \( X \) that was constructed in Theorem 1.6. We refer to \((P, \pi_P)\) as the *local Poisson saturation* of \( X \). In the next two sections, we prove a normal form theorem for \((P, \pi_P)\) around \( X \).

### 2. The Local Model

This section introduces the local model for the local Poisson saturation \((P, \pi_P)\) of a coregular submanifold \( X \subset (M, \pi) \). The local model is defined on the vector bundle \((TX^{\perp_\pi})^*\), which is indeed isomorphic with the normal bundle of \( X \) in \( P \). An explicit isomorphism is obtained by choosing an embedding \((TX^{\perp_\pi})^* \hookrightarrow T^*M|_X \) and then applying the bundle map \( \pi^\sharp \), see equation 4. The local model involves some extra choices, which we now explain.

Let \( X \subset (M, \pi) \) be a coregular submanifold, and choose a *complement* \( W \) to \( TX^{\perp_\pi} \) inside \( TM|_X \). We obtain correspondingly an inclusion map \( j : (TX^{\perp_\pi})^* \hookrightarrow T^*M|_X \). Define skew-symmetric bilinear forms \( \sigma \in \Gamma(\wedge^2TX^{\perp_\pi}) \) and \( \tau \in \Gamma(T^*X \otimes TX^{\perp_\pi}) \) on the restricted tangent bundle \( T((TX^{\perp_\pi})^*|_X) = TX \oplus (TX^{\perp_\pi})^* \) by the formulas

\[
\sigma(\xi_1, \xi_2) = \pi(j(\xi_1), j(\xi_2)), \quad \tau((v_1, \xi_1), (v_2, \xi_2)) = \langle v_1, j(\xi_2) \rangle - \langle v_2, j(\xi_1) \rangle,
\]

(5)
for $\xi_1, \xi_2 \in (T_xX^{\perp_{\pi}})^*$ and $v_1, v_2 \in T_xX$. Denote by $E_W(-\sigma - \tau)$ the set of all closed two-forms $\eta$, defined on a neighborhood of $X \subset (TX^{\perp_{\pi}})^*$, whose restriction to the zero section $X \subset (TX^{\perp_{\pi}})^*$ equals

$$\eta|_X = -\sigma \oplus -\tau \oplus 0 \in \Gamma(\Lambda^2TX^{\perp_{\pi}}) \oplus \Gamma(T^*X \otimes TX^{\perp_{\pi}}) \oplus \Gamma(\Lambda^2T^*X).$$

We refer to a two-form $\eta \in E_W(-\sigma - \tau)$ as a **closed extension** of $-\sigma - \tau$. Closed extensions of $-\sigma - \tau$ exist, see for instance [We1, Extension Theorem].

The local model for the local Poisson saturation of a coregular submanifold $X \xrightarrow{i} (M, \pi)$ is now defined as follows: pull back the Dirac structure $i^*L_\pi$ on $X$ to $(TX^{\perp_{\pi}})^*$ under the submersion $pr : (TX^{\perp_{\pi}})^* \rightarrow X$ and gauge transform by a closed extension $\eta \in E_W(-\sigma - \tau)$. The obtained Dirac structure $(pr^*(i^*L_\pi))^\eta$ indeed defines a Poisson structure in a neighborhood of $X \subset (TX^{\perp_{\pi}})^*$, as we now show.

**Proposition 2.1.** Let $X \subset (M, \pi)$ be a coregular submanifold. Fix a complement $W$ to $TX^{\perp_{\pi}}$ in $TM|_X$, define $\sigma \in \Gamma(\Lambda^2TX^{\perp_{\pi}})$ and $\tau \in \Gamma(T^*X \otimes TX^{\perp_{\pi}})$ by the formulas [3] and let $\eta \in E_W(-\sigma - \tau)$ be any closed extension. The Dirac structure $(pr^*(i^*L_\pi))^\eta$ is Poisson on a neighborhood $U$ of $X \subset (TX^{\perp_{\pi}})^*$.

**Proof.** It suffices to show that $(pr^*(i^*L_\pi))^\eta$ is transverse to $T(TX^{\perp_{\pi}})^*$ along $X$. By the sequence [1], we have $i^*L_\pi = \{\pi^2(\alpha) + (di)^*\alpha : \alpha \in (TX^{\perp_{\pi}})^0\}$, and therefore

$$\left.\left(pr^*(i^*L_\pi)\right)^\eta\right|_X = \left\{\pi^2(\alpha) + \xi + (dpr)^*((di)^*\alpha) + \iota_{\pi^2(\alpha) + \xi}\eta : \alpha \in (TX^{\perp_{\pi}})^0, \xi \in (TX^{\perp_{\pi}})^*\right\}.$$

Assume that $\pi^2(\alpha) + \xi \in T(TX^{\perp_{\pi}})^*|_X \cap \left(pr^*(i^*L_\pi)\right)^\eta|_X$ for $\alpha \in (TX^{\perp_{\pi}})^0, \xi \in (TX^{\perp_{\pi}})^*$. Then $(dpr)^*((di)^*\alpha) + \iota_{\pi^2(\alpha) + \xi}\eta = 0$, which implies the following:

- For all $v \in TX$, we get
  $$\alpha(v) + \eta(\pi^2(\alpha) + \xi, v) = 0 \Rightarrow \alpha(v) + \langle j(\xi), v \rangle = 0.$$

  So $\alpha + j(\xi) \in TX^0$, and therefore $\pi^2(\alpha + j(\xi)) \in TX^{\perp_{\pi}}$.

- For all $\beta \in (TX^{\perp_{\pi}})^*$, we get
  $$\eta(\pi^2(\alpha) + \xi, \beta) = 0 \Rightarrow \pi(j(\xi), j(\beta)) + \langle \pi^2(\alpha), j(\beta) \rangle = 0$$
  $$\Rightarrow \langle \pi^2(\alpha) + j(\xi), j(\beta) \rangle = 0.$$

Since $j((TX^{\perp_{\pi}})^*) = W^0$, this shows that $\pi^2(\alpha + j(\xi))$ lies in $W$.

We now proved that $\pi^2(\alpha + j(\xi)) \in TX^{\perp_{\pi}} \cap W = \{0\}$. So $\pi^2(j(\xi)) = -\pi^2(\alpha) \in TX$, which implies that $j(\xi) \in (TX^{\perp_{\pi}})^0$, again using exactness of the sequence [1]. But then $j(\xi) \in W^0 \cap (TX^{\perp_{\pi}})^0 = \{0\}$, so that $\xi = 0$, which in turn implies that also $\pi^2(\alpha) = 0$. □

We denote the Poisson manifold from Proposition 2.1 by $(U, \pi(W, \eta))$, and we refer to it as the **local model corresponding with $W$ and $\eta$**. A priori, the construction depends on a choice of complement $W$ and a choice of closed extension $\eta$. We now show that different choices produce isomorphic local models.

**Proposition 2.2.** Any two local models for the local Poisson saturation of a coregular submanifold $X \subset (M, \pi)$ are isomorphic around $X$, through a diffeomorphism that restricts to the identity along $X$. 

\[\]
Similarly, we obtain

Step 2: Let \((U, \pi(W_0, \eta_0))\) and \((V, \pi(W_1, \eta_1))\) be two local models for the local Poisson saturation of \(X\). The idea of the proof is to construct a diffeomorphism between them in two stages, where each stage relies on a Moser argument. We first map the local model \((U, \pi(W_0, \eta_0))\) to an intermediate local model \((V', \pi(W_1, \eta'_1))\), which is defined in terms of the complement \(W_1\). Then we pull \((V', \pi(W_1, \eta'_1))\) to the second local model \((V, \pi(W_1, \eta_1))\). Throughout, we shrink the neighborhoods on which the models are defined, when necessary.

We interpolate smoothly between the complements \(W_0, W_1\) to \(TX^{1-s}\) in \(TM|_X\), as follows. Decomposing \(W_1\) in the direct sum \(TM|_X = TX^{1-s} \oplus W_0\), we find \(A \in \Gamma(\text{Hom}(W_0, TX^{1-s}))\) such that \(W_1 = \text{Graph}(A)\). If we define \(W_t := \text{Graph}(tA)\) for \(t \in [0, 1]\), then the family \(\{W_t\}_{t \in [0, 1]}\) consists of complements to \(TX^{1-s}\), i.e. \(TM|_X = TX^{1-s} \oplus W_t\), and it interpolates between \(W_0\) and \(W_1\). Denote by \(q_t : TM|_X \to TX^{1-s}\) and \(j_t : (TX^{1-s})^* \to T^* M|_X\) the projection and inclusion, respectively, induced by the complement \(W_t\). We first determine the bilinear forms \(\sigma_t \in \Gamma(\wedge^2 TX^{1-s})\) and \(\tau_t \in \Gamma(T^* X \otimes TX^{1-s})\), which are defined by the formulas \(\Box\) using the inclusion \(j_t\), in terms of \(\sigma_0\) and \(\tau_0\).

Step 1: We compute \(\sigma_t\) and \(\tau_t\).

For \(e + w \in TX^{1-s} \oplus W_0 = TM|_X\), we have

\[
q_t(e + w) = q_t(e - tA(w) + w + tA(w))
= e - tA(w)
= q_0(e + w) - tA(\text{Id} - q_0)(e + w).
\]

This shows that \(q_t = q_0 - tA(\text{Id} - q_0)\) and therefore \(j_t = j_0 - t(\text{Id} - j_0)A^*\). We now compute for \(v_1, v_2 \in T_x X\) and \(\xi_1, \xi_2 \in (T_x X^{1-s})^*\):

\[
\tau_t((v_1, \xi_1), (v_2, \xi_2)) = \langle v_1, j_t(\xi_2) \rangle - \langle v_2, j_t(\xi_1) \rangle
= \langle v_1, j_0(\xi_2) \rangle - t\langle v_1, (\text{Id} - j_0)A^*\xi_2 \rangle
- \langle v_2, j_0(\xi_1) \rangle + t\langle v_2, (\text{Id} - j_0)A^*\xi_1 \rangle
= \tau_0((v_1, \xi_1), (v_2, \xi_2)) + t\langle A(\text{Id} - q_0)v_1, \xi_1 \rangle - t\langle A(\text{Id} - q_0)v_2, \xi_2 \rangle.
\]

Similarly, we obtain

\[
\sigma_t(\xi_1, \xi_2) = \langle q_t(\pi^*(j_t(\xi_1))), \xi_2 \rangle
= \langle (q_0 - tA(\text{Id} - q_0))\pi^*(j_0 - t(\text{Id} - j_0)A^*) \rangle (\xi_1), \xi_2 \rangle
= \sigma_0(\xi_1, \xi_2) - t\langle (q_0\pi^*(\text{Id} - j_0)A^*) (\xi_1), \xi_2 \rangle
- t\langle (A(\text{Id} - q_0)\pi^*j_0)(\xi_1), \xi_2 \rangle + t^2 \langle (A(\text{Id} - q_0)\pi^*(\text{Id} - j_0)A^*) (\xi_1), \xi_2 \rangle.
\]

Step 2: Get closed extensions, smoothly varying in \(t \in [0, 1]\), of

\[-\sigma_t \oplus -\tau_t \oplus 0 \in \Gamma(\wedge^2 TX^{1-s}) \oplus \Gamma(T^* X \otimes TX^{1-s}) \oplus \Gamma(\wedge^2 T^* X)\]

Thanks to [We1] Extension Theorem and [We1] Relative Poincaré Lemma, we find a one-form \(\beta_1\), defined on a neighborhood of \(X \subset (TX^{1-s})^*\), such that

\[
\left\{
\begin{array}{l}
\beta_1|_X = 0, \\
d\beta_1|_X \in \Gamma(T^* X \otimes TX^{1-s}), \\
d\beta_1|_X ((v_1, \xi_1), (v_2, \xi_2)) = \langle A(\text{Id} - q_0)v_1, \xi_2 \rangle - \langle A(\text{Id} - q_0)v_2, \xi_1 \rangle,
\end{array}
\right.
\]
for \((v_1, \xi_1), (v_2, \xi_2) \in T_x X \oplus (T_x X^\perp)^*\). Similarly, we find one-forms \(\beta_2, \beta_3\) defined around \(X \subset (T X^\perp)^*\) satisfying

\[
\begin{align*}
\beta_2|_X &= 0, \\
\gamma \beta_2|_X &\in \Gamma(\wedge^2 T X^\perp), \\
\gamma \beta_3|_X &\in \Gamma(\wedge^2 T X^\perp), \\
d\beta_3|_X &\in \Gamma(\wedge^2 T X^\perp),
\end{align*}
\]

and

\[
\begin{align*}
\beta_3|_{\xi_1, \xi_2} &= \langle (\gamma \beta_3|_X) (\xi_1, \xi_2) \angle, \\
\beta_3|_{\xi_1, \xi_2} &= \langle \gamma \beta_3|_X (\xi_1, \xi_2) \rangle,
\end{align*}
\]

for \(\xi_1, \xi_2 \in (T_x X^\perp)^*\). Using (7) and (8), we see that

\[
(\gamma_l + \beta_3|_{\xi_1, \xi_2})|_X = -\gamma_l - \gamma_2 + \beta_3|_{\xi_1, \xi_2}.
\]

Step 3: A Moser argument pulls \((U, \pi(W_0, \eta_0))\) to \((U', \pi(W_1, \eta_0 + (d_1 + d_2 - d_3))\).

By Proposition 2.1, we get a path of Dirac structures

\[
\pi_t := (pr'(t) L_\pi)|_{\eta_0 + d_1 + d_2 - d_3}
\]

for \(t \in [0, 1]\), where \(\pi_t\) is Poisson on a neighborhood \(U_t\) of \(X\) in \((T X^\perp)^*\). Note that the set \(\bigcup_{t \in [0, 1]} \{t\} \times U_t\) is open, since it consists of the points \((t, x)\) for which \((\pi_t)_x\) is Poisson. The Tube Lemma implies that \(U' := \bigcap_{t \in [0, 1]} U_t\) is a neighborhood of \(X\) on which \(\pi_t\) is Poisson for all \(t \in [0, 1]\). Now, these Poisson structures are related by gauge transformations:

\[
\pi_t = \gamma_t d_1 + d_2 - d_3,
\]

where

\[
\frac{d}{dt}(d_1 + d_2 - d_3) = -d(2d_1 - d_2 - d_3).
\]

A Poisson version of Moser's theorem (e.g. [Mc, Theorem 2.11]) shows that the flow \(\Phi_t\) of the time-dependent vector field \(\pi_t(2d_1 - d_2 - d_3)\) satisfies \((\pi_t)_\pi = 0\), whenever it is defined. Moreover, since the primitive \(2d_1 - d_2 - d_3\) vanishes along \(X\), the flow \(\Phi_t\) fixes all points in \(X\). Now set \(\phi := \Phi_t^{-1}\). Shrinking \(U\) if necessary, we can assume that \(\phi : U \to U'\) where \(V' := \phi(U)\). We then have

\[
\phi : (U, \pi(W_0, \eta_0)) \simeq (U', \pi(W_1, \eta_0 + d_1 + d_2 - d_3)), \quad \phi|_X = Id.
\]

Step 4: Another Moser argument pulls \((V', \pi(W_1, \eta_0 + d_1 + d_2 - d_3))\) to \((V, \pi(W_1, \eta_1))\).

Both \(\eta_1\) and \(\eta_0 + d_1 + d_2 - d_3\) are closed extensions of

\[
-\gamma_1 - \gamma_2 + \gamma_3 = \Gamma(\wedge^2 T X^\perp) \oplus \Gamma(T^* X \otimes T X^\perp) \oplus \Gamma(\wedge^2 T^* X),
\]

see equation (9). So their difference \(\gamma_1 - (\eta_0 + d_1 + d_2 - d_3)\) is exact around \(X\) with a primitive \(\gamma\) that vanishes along \(X\), by the Relative Poincaré Lemma. Denote

\[
\pi_0' := (pr'(t) L_\pi)|_{\eta_0 + d_1 + d_2 - d_3}, \quad \pi_t' := (\pi_0')^t(\gamma).
\]

for \(t \in [0, 1]\). Since \(\pi_0'\) is Poisson on \(V'\) and \(d(\gamma)|_X = 0\), we see that \(\pi_t'\) is Poisson on a neighborhood \(V'_t\) of \(X\) in \((T X^\perp)^*\). Using the Tube Lemma as in Step 3, we find a neighborhood \(O\) of \(X\) in \((T X^\perp)^*\) such that \(\pi_t'\) is Poisson on \(O\) for all \(t \in [0, 1]\). The Moser Theorem [Mc, Theorem 2.11] implies that the flow \(\Psi_t\) of the time-dependent vector field \(-\gamma(\pi_t)'(\gamma)\) satisfies \((\Psi_t)_\pi = \pi_0\), whenever it is defined. Moreover, since \(\gamma|_X = 0\), the flow...
\( \Psi \) fixes all points of \( X \). Now set \( \psi := \Psi^{-1} \). Shrinking both \( V' \) and \( V \) if necessary, we can assume that \( \psi : V' \to V \). We then have

\[
\psi : (V', \pi(W, \eta_0 + d\beta_1 + d\beta_2 - d\beta_3)) \sim (V, \pi(W_1, \eta_1)), \quad \psi|_X = \text{Id}.
\]

The diffeomorphism \( \psi \circ \phi \) now satisfies the criteria: it fixes points in \( X \) and defines a Poisson diffeomorphism

\[
\psi : (U, \pi(W_0, \eta_0 + d\beta_1 + d\beta_2 - d\beta_3)) \sim (V, \pi(W_1, \eta_1)).
\]

It is now justified to call \( (U, \pi(W, \eta)) \) the local model for the local Poisson saturation of the coregular submanifold \( X \subset (M, \pi) \).

### 3. The normal form

We now show that the local Poisson saturation of a coregular submanifold \( X \subset (M, \pi) \) is isomorphic around \( X \) to the local model \( (U, \pi(W, \eta)) \) constructed in Proposition 2.1. We will use the theory of dual pairs in Dirac geometry, as developed in [FM2]. We first need a lemma, which describes how to obtain a weak Dirac dual pair out of the self-dual pair \( (M, \pi) \) whenever a coregular submanifold \( X \subset (M, \pi) \) is given. Recall from the proof of Theorem 1.6 that the local Poisson saturation \( (P, \pi_P) \) of \( X \subset (M, \pi) \) is given by \( \exp(\Sigma|_X) \).

**Lemma 3.1.** Let \( i : X \hookrightarrow (M, \pi) \) be a coregular submanifold with local Poisson saturation \( (P, \pi_P) \). Then the following is a weak Dirac dual pair, in the sense of [FM2]:

\[
(M, \pi) \xleftarrow{\text{pr}} (\Sigma, \Omega_\chi) \xrightarrow{\exp_\chi} (M, -\pi)
\]

whenever a coregular submanifold \( X \subset (M, \pi) \) is given. Recall from the proof of Theorem 1.6 that the local Poisson saturation \( (P, \pi_P) \) of \( X \subset (M, \pi) \) is given by \( \exp_\chi(\Sigma|_X) \).

This means that \( \Omega_\chi|_X \) is a closed two-form on \( \Sigma|_X \), that \( \text{pr} \) and \( \exp_\chi \) are surjective forward Dirac submersions, and that

\[
\begin{align*}
(\Omega_\chi|_X)(S_1, S_2) &= 0, \quad (10) \\
\text{rk}(S_1 \cap K \cap S_2) &= \dim \Sigma|_X - \dim X - \dim P, \quad (11)
\end{align*}
\]

where \( S_1 := \ker \text{dpr} \), \( S_2 := \ker \text{dexp}_\chi \) and \( K := \ker (\Omega_\chi|_X) \).

**Proof.** It is clear that \( \text{pr} \) is a surjective submersion. The fact that \( \exp_\chi \) is a surjective submersion follows from the proof of Theorem 1.6. Because \( (M, \pi) \) is a dual pair, property \( (10) \) is automatic. To see that \( \text{pr} : (\Sigma|_X, \text{Gr}(\Omega_\chi|_X)) \to (X, i^!L_\pi) \) is forward Dirac, consider the following commutative diagram of Dirac manifolds and smooth maps:

\[
\begin{array}{ccc}
(\Sigma|_X, \text{Gr}(\Omega_\chi|_X)) & \xrightarrow{\text{pr}} & (X, i^!L_\pi) \\
\downarrow i' & & \downarrow i \\
(\Sigma, \text{Gr}(\Omega_\chi)) & \xrightarrow{\text{pr}} & (M, L_\pi)
\end{array}
\]

The maps \( i' \) on the left and \( i \) on the right are backward Dirac by definition, and the bottom map \( \text{pr} \) is forward Dirac because of the dual pair \( (2) \). Since the bottom map \( \text{pr} \) is a submersion, we can apply [FM3, Lemma 3] to obtain that also the map at the top \( \text{pr} : (\Sigma|_X, \text{Gr}(\Omega_\chi|_X)) \to (X, i^!L_\pi) \) is forward Dirac.
Similarly, we get that \( \exp_x : (\Sigma \mid X, \text{Gr}(\Omega_x \mid X)) \to (P, -L_{\pi_P}) \) is forward Dirac considering the diagram

\[
\begin{array}{ccc}
(\Sigma \mid X, \text{Gr}(\Omega_x \mid X)) & \xrightarrow{\exp_x} & (P, -L_{\pi_P}) \\
\downarrow i' & & \downarrow i \\
(\Sigma, \text{Gr}(\Omega_x)) & \xrightarrow{\exp_x} & (M, -L_\pi)
\end{array}
\]

Here the map \( i' \) is backward Dirac, the map \( i \) is backward (and forward) Dirac, and the bottom map \( \exp_x \) is forward Dirac because of the dual pair \([2]\). Again, the map \( \exp_x \) on the bottom is a submersion, so we can apply [FM3, Lemma 3] to obtain that also the map \( \exp_x : (\Sigma \mid X, \text{Gr}(\Omega_x \mid X)) \to (P, -L_{\pi_P}) \) at the top is forward Dirac.

It remains to check that property (11) holds. For \((x, \xi) \in \Sigma \mid X\), we have

\[
K_{(x, \xi)} = \pi_x^\sharp \left( (dpr)_{(x, \xi)}^* \left( T_x X^\circ \right) \cap T_{(x, \xi)} (T^* M \mid X) \right),
\]

where \( \pi_x := \Omega_x^{-1} \). Consequently, we obtain

\[
(S_1)_{(x, \xi)} \cap K_{(x, \xi)} = \pi_x^\sharp \left( (dpr)_{(x, \xi)}^* \left( T_x X^\circ \cap \ker \pi_x^\sharp \right) \right),
\]

using that the left leg of the dual pair \([2]\) is a Poisson map. The equality \([3]\) in the proof of Lemma 1.5 shows that

\[
(S_2)_{(x, \xi)} = \pi_x^\sharp \left( (dpr)_{(x, \xi)}^* \left( T_x X^\perp \right) \right),
\]

so we obtain

\[
(S_1)_{(x, \xi)} \cap K_{(x, \xi)} \cap (S_2)_{(x, \xi)} = \pi_x^\sharp \left( (dpr)_{(x, \xi)}^* \left( T_x X^\circ \cap \ker \pi_x^\sharp \right) \right).
\]

Consequently,

\[
\text{rk}(S_1 \cap K \cap S_2)_{(x, \xi)} = \dim \left( T_x X^\circ \cap \ker \pi_x^\sharp \right) \\
= \dim(T_x X^\circ) - \dim(T_x X^\perp) \\
= (\dim M - \dim X) - (\dim P - \dim X) \\
= \dim \Sigma \mid X - \dim X - \dim P.
\]

So also property (11) holds, and this finishes the proof. \( \square \)

We are now ready to state the main results of this section.

**Theorem 3.2.** Let \( X \subset (M, \pi) \) be a coregular submanifold with local Poisson saturation \((P, \pi_P)\). Choose a complement \( W \) to \( T X^\perp \) in \( TM \mid X \) and denote by \( j : (TX^\perp)^* \hookrightarrow T^* M \mid X \) the corresponding inclusion. Then \(-j^*(\Omega_x \mid X) \in \mathcal{E}_W(-\sigma - \tau)\). Moreover, the corresponding local model \((U, \pi(W, -j^*(\Omega_x \mid X)))\) is isomorphic with \((P, \pi_P)\) around \( X \). Explicitly, a Poisson diffeomorphism onto a neighborhood of \( X \) is given by

\[
\exp_x \circ j : (U, \pi(W, -j^*(\Omega_x \mid X))) \xrightarrow{\sim} (P, \pi_P).
\]

We will denote by \( \text{pr}_M \) and \( \text{pr}_X \) the bundle projections \( T^* M \mid X \to X \) and \((TX^\perp)^* \to X\), respectively. So \( \text{pr}_M \circ j = \text{pr}_X \).
Proof. We first check that $-j^*(\Omega_X|_X) \in \mathcal{E}_W(-\sigma - \tau)$. The fact that $-j^*(\Omega_X|_X)$ restricts along $X \subset (TX^{1*})^*$ as required in (6) is an immediate consequence of the following equality [FM1] Lemma 24:

$$\Omega_X((v_1, \xi_1), (v_2, \xi_2)) = \langle v_1, \xi_2 \rangle - \langle v_2, \xi_1 \rangle + \pi(\xi_1, \xi_2),$$

where $(v_1, \xi_1), (v_2, \xi_2) \in T_x(T^*M) = T_xM \oplus T^*_xM$ for $x \in M$.

To prove the second statement, we apply [FM2] Proposition 6 to the weak dual pair constructed in Lemma 3.1

$$\begin{align*}
(X, i^!L_\pi) & \overset{pr_M}{\iff} (\Sigma|_X, Gr(\Omega_X|_X)) \overset{\exp}{\exp} (P, -L_{\pi_P}),
\end{align*}$$

and we get the following equality of Dirac structures on $\Sigma|_X$:

$$(pr^!_M(i^!L_\pi))|_X = \exp^!_X L_{\pi_P}. \quad (12)$$

Since the map $j : (TX^{1*})^* \hookrightarrow T^*M|_X$ is transverse to this Dirac structure, we can pull back the equality (12) to $j^{-1}(\Sigma|_X) \cong j(TX^{1*})^* \cap \Sigma$, which yields

$$(pr^!_X(i^!L_\pi)) \overset{\exp}{\cong} j^*(\Omega_X|_X) = (\exp_X \circ j)^! L_{\pi_P}. \quad (13)$$

The left hand side of (13) is Poisson on a neighborhood $U \subset j^{-1}(\Sigma|_X) \subset (TX^{1*})^*$, where it defines the local model $(U, \pi(W, -j^*(\Omega_X|_X)))$. Moreover, by the proof of Theorem 1.6 we know that $\exp_X \circ j$ takes $j^{-1}(\Sigma|_X)$ diffeomorphically onto $P$. So we obtain that

$$\exp_X \circ j : (U, \pi(W, -j^*(\Omega_X|_X))) \rightarrow (P, \pi_P)$$

is a Poisson diffeomorphism onto a neighborhood of $X \subset (P, \pi_P)$, as desired. \hfill \Box

We now combine Proposition 2.2 and Theorem 3.2. Also noticing that any local model $(U, \pi(W, \eta))$ for the local Poisson saturation of $X$ is constructed out of the restriction $\pi|_X$, we obtain the following.

**Corollary 3.3.** Let $X \subset (M, \pi)$ be a coregular submanifold with local Poisson saturation $(P, \pi_P)$. For any choice of complement $W$ to $TX^{1*}$ and closed extension $\eta \in \mathcal{E}_W(-\sigma - \tau)$, the corresponding local model $(U, \pi(W, \eta))$ is Poisson diffeomorphic around $X$ with $(P, \pi_P)$. In particular, up to Poisson diffeomorphism, the local Poisson saturation is determined by the restriction of $\pi$ along $X$.

In general, one needs the full information of $\pi|_X$ in order to determine the local Poisson saturation of $X$. We will see in the next section that for certain coregular submanifolds $X$, only part of this information is required.

**Remark 3.4.** We outline an alternative proof for our normal form result, relying on the fact that a coregular submanifold $X \subset (M, \pi)$ is a transversal in its local Poisson saturation. This allows one to use the normal form around Dirac transversals [FLM1 Thm. 5.1], [FM2 §7] instead of Theorem 3.2 which in combination with Proposition 2.2 yields Corollary 3.3. We now elaborate on this, using explicitly the normal form in [FM2 §7] because it is the closest in spirit to the arguments in this note.

A choice of complement $TM|_X = TX^{1*} \oplus W$ gives an inclusion $j : (TX^{1*})^* \hookrightarrow T^*M|_X$, and an identification of the normal bundle

$$\pi^* \circ j : (TX^{1*})^* \rightarrow \pi^*(j(TX^{1*})^*) \cong TP|_X/TX.$$ 

According to [FM2 §7], the Dirac manifold $(P, L_{\pi_P})$ is isomorphic around $X$ with

$$\left(U \subset (TX^{1*})^*, (pr^!(i^!(L_{\pi_P})))^{-\rho^*\omega|_X}\right). \quad (14)$$

...
Here \( \rho \) is a splitting of the exact sequence
\[
0 \to \imath^* L_{\pi P} \to L_{\pi P} \to (TX^{1*})^* \to 0,
\]
where the last arrow is the anchor map \( \text{pr}_T : L_{\pi P} \to TP|_X \) composed with the projection to the normal bundle \( TP|_X/TX \cong (TX^{1*})^* \). The two-form \( \omega \) appearing in [14] is defined choosing a spray on \( L_{\pi P} \), see [20] for the precise formula. Hence, to prove our normal form for one specific choice of local model, as we did in Theorem 3.2, we just have to show that there is a splitting \( \rho \) of the sequence (15) satisfying
\[
\rho^* \omega|_X = \sigma + \tau,
\]
where \( \sigma \) and \( \tau \) were defined in (5). We claim that such a splitting is given by
\[
\rho : (TX^{1*})^* \to L_{\pi P}|_X : \xi \mapsto \pi^\sharp(j(\xi)) + (d\iota)^*(j(\xi)),
\]
where \( \iota : P \hookrightarrow M \) is the inclusion. Indeed, choosing \((v_1, \xi_1), (v_2, \xi_2) \in T_x(TX^{1*})^*\), we get
\[
\rho^* \omega|_X((v_1, \xi_1), (v_2, \xi_2))
= \omega|_X \left((v_1, \pi^\sharp(j(\xi_1)) + (d\iota)^*(j(\xi_1))), (v_2, \pi^\sharp(j(\xi_2)) + (d\iota)^*(j(\xi_2)))\right)
= (d\iota)^*(j(\xi_2)) \left(v_1 + \frac{1}{2} \pi^\sharp(j(\xi_1))\right) - (d\iota)^*(j(\xi_1)) \left(v_2 + \frac{1}{2} \pi^\sharp(j(\xi_2))\right)
= \langle v_1, j(\xi_2) \rangle - \langle v_2, j(\xi_1) \rangle + \pi(\iota(\xi_1), \iota(\xi_2))
= (\sigma + \tau)((v_1, \xi_1), (v_2, \xi_2)),
\]
using [FM2, eq. 6] in the second equality. This proves that the local Poisson saturation \((P, \pi P)\) is isomorphic around \(X\) with the local model \((U, \pi(W, -\rho^* \omega|_X))\). Along with Proposition 2.2 this gives an alternative proof for our normal form in Corollary 3.3.

4. Some particular cases

We proved that the local model \((U, \pi(W, \eta))\) described in Proposition 2.1 depends neither on the choice of complement \(W\) to \(TX^{1*}\) in \(TM|_X\), nor on the choice of closed extension \(\eta\). We now show that, for certain classes of coregular submanifolds \(X \subset (M, \pi)\), a good choice of complement and/or closed extension simplifies the normal form considerably. Some of our results recover well-known normal form and rigidity statements around distinguished submanifolds in symplectic and Poisson geometry.

4.1. Submanifolds in symplectic geometry.

Recall that, if \((M, \omega)\) is a symplectic manifold and \(N \subset M\) is any submanifold, then the restriction of \(\omega\) to \(TM|_N\) determines the symplectic form \(\omega\) on a neighborhood of \(N\) (see [We2, Theorem 4.1]). We can recover this result from our normal form, as follows.

First note that, in case \(\pi = \omega^{-1}\) is symplectic, any submanifold \(X \subset (M, \pi)\) is coregular since \(TX^{1*} = TX^{1\omega}\), where \(TX^{1\omega} = \{v \in TM|_X : \omega(v, w) = 0 \forall w \in TX\}\) denotes the symplectic orthogonal of \(X\). Next, the local Poisson saturation \((P, \pi P)\) of \(X\) is an embedded submanifold of \(M\) of dimension \(\dim X + \text{rk}(\pi^!(TX^{1*})^*)\), by the equality (4). So if \(\pi\) is symplectic, then \(P \subset M\) is a neighborhood of \(X\). Finally, the Poisson structure \(\pi(W, \eta) = (\text{pr}^!(\imath^* L_{\pi}))^\eta\) from the local model is determined by the restriction \(\pi|_X\).

In conclusion, our normal form shows that, for any submanifold \(X\) of the symplectic manifold \((M, \pi)\), the restriction \(\pi|_X\) determines \(\pi\) on a neighborhood of \(X \subset M\), which recovers the aforementioned rigidity result in symplectic geometry.
4.2. Poisson transversals.

A submanifold \( X \) of a Poisson manifold \((M, \pi)\) is called a Poisson transversal if it meets each symplectic leaf transversally and symplectically, that is

\[
TX \oplus TX^\perp = TM|_X.
\]

In the local model of Proposition 2.1, we can take \( TX \) as a canonical complement to \( TX^\perp \) in \( TM|_X \). Then the associated embedding \( j : (TX^\perp)^* \hookrightarrow T^*M|_X \) identifies \( (TX^\perp)^* \) with \( TX^\circ \). The following simplifications occur in the local model:

- The pullback \( i^!L_\pi \) of the Dirac structure \( L_\pi \) to \( X \) defines a Poisson structure on \( X \) [FMI, Lemma 3], which we denote by \( \pi_X \in \Gamma(\wedge^2 TX) \).
- Consider \( \sigma \in \Gamma(\wedge^2 TX^\perp) \) and \( \tau \in \Gamma(T^*X \otimes TX^\perp) \) defined in \([5]\):
  \[
  \sigma(\xi_1, \xi_2) = \pi(j(\xi_1), j(\xi_2)),
  \tau((v_1, \xi_1), (v_2, \xi_2)) = \langle v_1, j(\xi_2) \rangle - \langle v_2, j(\xi_1) \rangle,
\]
  for \( \xi_1, \xi_2 \in (T_xX^\perp)^* \) and \( v_1, v_2 \in T_xX \). Since \( j((TX^\perp)^*) = TX^\circ \), we get that \( \tau \equiv 0 \), and since the restriction of \( \pi \) to the conormal bundle \( TX^\circ \) is fiberwise non-degenerate, we get a symplectic vector bundle \( ((TX^\perp)^*, \sigma) \).

Moreover, since \( X \) is a transversal, its local Poisson saturation \((P, \pi_P)\) is in fact a neighborhood of \( X \) in \( M \). In conclusion, our normal form shows that a neighborhood of \( X \) in \((M, \pi)\) is Poisson diffeomorphic with a neighborhood of \( X \) in \((TX^\perp)^*\), endowed with the Poisson structure

\[
(pr^!(L_{\pi_X}))^\eta,
\]
where \( \eta \) is a closed extension of \(-\sigma\). This is exactly the normal form established in [FMI].

4.3. Coregular coisotropic submanifolds.

Recall that a submanifold \( N \) of a symplectic manifold \((M, \omega)\) is called coisotropic if its symplectic orthogonal \( TN^\perp\omega \) is contained in \( TN \). Gotay’s theorem [G] provides a normal form for \( \omega \) around \( N \), which is obtained as follows. Choose a complement to \( TN^\perp\omega \) inside \( TN \), and denote by \( j : (TN^\perp\omega)^* \hookrightarrow T^*N \) the induced inclusion. On the total space of the vector bundle \( pr : (TN^\perp\omega)^* \to N \), one gets a closed two-form

\[
pr^*(i^*\omega) + j^*\omega_{can},
\]
where \( i^*\omega \) is the pullback of \( \omega \) to \( N \) and \( \omega_{can} \) is the canonical symplectic form on \( T^*N \).

This two-form is non-degenerate on a neighborhood of the zero section \( N \subset (TN^\perp\omega)^* \), and \((M, \omega)\) is isomorphic with \( ((TN^\perp\omega)^*, pr^*(i^*\omega) + j^*\omega_{can}) \) around \( N \). In particular, the pullback \( i^*\omega \in \Gamma(\wedge^2 T^*N) \) determines \( \omega \) on a neighborhood of \( N \subset M \).

More generally, recall that a submanifold \( X \) of a Poisson manifold \((M, \pi)\) is coisotropic if \( TX^\perp \subset TX \). In this subsection, we prove a Poisson version of Gotay’s theorem by specializing our normal form to coregular submanifolds \( i : X \hookrightarrow (M, \pi) \) that are coisotropic.

Mimicking Gotay’s construction, we choose a complement \( TX = TX^\perp \oplus G \) to get an inclusion \( j : (TX^\perp)^* \hookrightarrow T^*X \), and we obtain a Dirac structure

\[
(pr^!(i^!L_\pi))^j\omega_{can}
\]
(16)
on \((TX^\perp)^*\), where \( \omega_{can} \) denotes the canonical symplectic form on \( T^*X \).

**Corollary 4.1** (Poisson version of Gotay’s Theorem). Let \( X \subset (M, \pi) \) be coregular coisotropic. The Dirac structure \([16]\) defines a Poisson structure on a neighborhood \( U \subset (TX^\perp)^* \) of \( X \), which is Poisson diffeomorphic around \( X \) with the local Poisson saturation of \( X \).
Proof. It suffices to show that the Dirac structure (16) is diffeomorphic around $X$ with a local model for the local Poisson saturation of $X$. By Lemma 4.2 in the next subsection, the splitting $TX = TX^\perp + G$ induces a splitting $TM|_X = TX^\perp + W_G$, where

$$\pi^\perp((W_G)^\circ) \subset W_G \quad \text{and} \quad W_G \cap TX = G.$$ 

Denote by $\tilde{j} : (TX^\perp)^* \hookrightarrow T^* M|_X$ the inclusion induced by the complement $W_G$; it embeds $(TX^\perp)^*$ into $T^* M|_X$ as $(W_G)^\circ$. Consider $\sigma \in \Gamma(\wedge^2 TX^\perp)$ and $\tau \in \Gamma(T^* X \otimes TX^\perp)$ as defined in (5):

$$\sigma(\xi_1, \xi_2) = \pi((\tilde{j}(\xi_1), \tilde{j}(\xi_2))),$$

$$\tau((v_1, \xi_1), (v_2, \xi_2)) = \langle v_1, \tilde{j}(\xi_2) \rangle - \langle v_2, \tilde{j}(\xi_1) \rangle,$$

for $\xi_1, \xi_2 \in (T_x X^\perp)^*$ and $v_1, v_2 \in T_x X$. Since $\pi^\perp((W_G)^\circ) \subset W_G$, we have $\sigma \equiv 0$, and since $W_G \cap TX = G$, we have

$$\tau((v_1, \xi_1), (v_2, \xi_2)) = \langle v_1, \tilde{j}(\xi_2) \rangle - \langle v_2, \tilde{j}(\xi_1) \rangle = \langle j^* \omega_{can} \rangle_{\pi}(\xi_1, (v_2, \xi_2))$$

for $\xi_1, \xi_2 \in (T_x X^\perp)^*$ and $v_1, v_2 \in T_x X$. This shows that $(U, \pi(W_G, -j^* \omega_{can}))$ is a local model for the local Poisson saturation of $X$, where $U \subset (TX^\perp)^*$ is a suitable neighborhood of $X$. Note however that the Dirac structure (16) still differs by a sign from this model; we now remedy this. Shrinking $U$ if necessary, we can assume that $U$ is invariant under fiberwise multiplication by $-1$. Denoting this map by $m_{-1}$, we have

$$m_{-1}^* (pr^i(i^1 L_\pi))_{\pi}^* \omega_{can} = (pr^i m_{-1}^i i^1 L_\pi)_{\pi}^{(j m_{-1})^* \omega_{can}} = (pr^i(i^1 L_\pi))_{\pi}^* - j^* \omega_{can},$$

the latter being the Poisson structure $(U, \pi(W_G, -j^* \omega_{can}))$. This shows that the Dirac structure (16) is in fact Poisson on $U$, and that it is Poisson diffeomorphic around $X$ with the local Poisson saturation of $X$. \hfill \Box

In particular, the pullback Dirac structure $i^1 L_\pi$ determines a neighborhood of $X$ in its local Poisson saturation, up to Poisson diffeomorphism. Indeed, if one knows $i^1 L_\pi$, then one also knows $TX^\perp = i^1 L_\pi \cap TX$, hence one can construct the local model (16) which recovers the local Poisson saturation up to Poisson diffeomorphism around $X$. In the next subsection, we generalize this result to the class of coregular pre-Poisson submanifolds.

4.4. Coregular pre-Poisson submanifolds.

Recall that, given a symplectic manifold $(M, \omega)$, a submanifold $i : N \hookrightarrow (M, \omega)$ is said to be of constant rank if the pullback $i^* \omega$ has constant rank. Marle’s constant rank theorem [Ma] states that a neighborhood of a constant rank submanifold $i : N \hookrightarrow (M, \omega)$ is determined by the pullback $i^* \omega$ and the symplectic vector bundle $(TN^\perp) / (TN^\perp \cap TN), \omega$.

Generalizing this notion to Poisson geometry, a submanifold $X$ of a Poisson manifold $(M, \pi)$ is called pre-Poisson if $TX + TX^\perp$ has constant rank [CZ1]. It is equivalent to ask that the bundle map $pr \circ \pi^2 : TX^\circ \to TX^\perp \to TM|_X / TX$ has constant rank. Examples include Poisson transversals (in which case $pr \circ \pi^2$ is an isomorphism) and coisotropic submanifolds (in which case $pr \circ \pi^2$ is the zero map). If $X$ is coregular pre-Poisson, i.e. $TX^\perp$ has constant rank, then its characteristic distribution $TX^\perp \cap TX$ also has constant rank.

In this subsection, we prove a Poisson version of Marle’s theorem by specializing our normal form to coregular pre-Poisson submanifolds. We need the following auxiliary result.
Lemma 4.2. Let \(X \subset (M, \pi)\) be a coregular pre-Poisson submanifold. For any choice of splittings \(TX = (TX_{\perp H} \cap TX) \oplus G\) and \(TX_{\perp H} = (TX_{\perp X} \cap TX) \oplus H\), there exists a complement \(TM|_X = (TX_{\perp X} \cap TX) \oplus H \oplus W_{G,H}\) such that
\[
\pi^*(\{(H + W_{G,H})^\circ\} \subset W_{G,H} \quad \text{and} \quad W_{G,H} \cap TX = G.
\]

Proof. We have in particular that
\[
TX + TX_{\perp H} = (TX_{\perp X} \cap TX) \oplus G \oplus H. \tag{17}
\]

The proof is divided into four steps.

Step 1: \(\pi^*((G + H)^\circ)\) has constant rank, equal to twice the rank of \(TX_{\perp X} \cap TX\).

Since \(\ker \pi^* \subset (TX_{\perp X})^\circ \subset (TX_{\perp X} \cap TX)^\circ\), we have
\[
\ker \pi^* \cap (G + H)^\circ = \ker \pi^* \cap (TX_{\perp X} \cap TX)^\circ + (G + H)^\circ
\]
\[
= \ker \pi^* \cap (TX_{\perp X} + G + H)^\circ
\]
\[
= \ker \pi^* \cap (TX + TX_{\perp X})^\circ
\]
\[
= \ker \pi^* \cap TX^\circ \cap (TX_{\perp X})^\circ
\]
\[
= \ker \pi^* \cap TX^\circ.
\]

Since \(X\) is coregular, the latter has constant rank, which shows that also \(\pi^*((G + H)^\circ)\) has constant rank. Explicitly,
\[
\text{rk} \left( \pi^*((G + H)^\circ) \right) = \dim M - \text{rk}(G + H) - \text{rk}(\ker \pi^* \cap (G + H)^\circ)
\]
\[
= \dim M - \text{rk}(TX + TX_{\perp X}) + \text{rk}(TX_{\perp X} \cap TX) - \text{rk}(\ker \pi^* \cap TX^\circ)
\]
\[
= \dim M - \text{rk}(TX + TX_{\perp X}) + \text{rk}(TX_{\perp X} \cap TX)
\]
\[
- (\dim M - \dim X - \text{rk}(TX_{\perp X}))
\]
\[
= \text{rk}(TX) + \text{rk}(TX_{\perp X}) - \text{rk}(TX + TX_{\perp X}) + \text{rk}(TX_{\perp X} \cap TX)
\]
\[
= 2\text{rk}(TX_{\perp X} \cap TX).
\]

Step 2: \((\pi^*((G + H)^\circ), \omega)\) is a symplectic vector bundle, where
\[
\omega(\pi^*(\alpha), \pi^*(\beta)) := \pi(\alpha, \beta).
\]

We first show that \(\pi^*((G + H)^\circ) \cap (G + H) = \{0\}\). Assume that \(\gamma \in (G + H)^\circ\) is such that \(\pi^*(\gamma) = g + h \in G + H\). Since \(h \in TX_{\perp X}\), we can write \(h = \pi^*(\beta)\) for some \(\beta \in TX^\circ\), and we obtain that \(\pi^*(\gamma - \beta) = g \in TX\). The exact sequence \((1)\) then implies that \(\gamma - \beta \in (TX_{\perp X})^\circ\), and therefore \(\gamma \in TX^\circ + (TX_{\perp X})^\circ = (TX \cap TX_{\perp X})^\circ\). Hence,
\[
\gamma \in (TX \cap TX_{\perp X})^\circ \cap (G + H)^\circ = (TX + TX_{\perp X})^\circ = TX^\circ \cap (TX_{\perp X})^\circ,
\]

using \(\text{(17)}\) in the first equality. This implies that \(\pi^*(\gamma) \in TX_{\perp X} \cap TX\), so we obtain that \(\pi^*(\gamma) \in (TX_{\perp X} \cap TX) \cap (G + H) = \{0\}\). This shows that \(\pi^*((G + H)^\circ) \cap (G + H) = \{0\}\).

It now follows that \(\omega\) is non-degenerate: if \(\pi^*(\alpha) \in \ker \omega\) for \(\alpha \in (G + H)^\circ\), then for all \(\beta \in (G + H)^\circ\) we get \(\langle \pi^*(\alpha), \beta \rangle = 0\), which shows that \(\pi^*(\alpha) \in G + H\). By what we just proved, we then get \(\pi^*(\alpha) \in \pi^*((G + H)^\circ) \cap (G + H) = \{0\}\), which shows that \(\omega\) is non-degenerate.

Step 3: \(TX_{\perp X} \cap TX \subset (\pi^*((G + H)^\circ), \omega)\) is a Lagrangian subbundle.
Since $G + H \subset TX + TX^\perp$, we have $(TX + TX^\perp)^\circ \subset (G + H)^\circ$ and therefore
\[
TX^\perp \cap TX = \pi^\perp((TX^\circ \cap (TX^\perp)^\circ) = \pi^\perp((TX + TX^\perp)^\circ) \subset \pi^\perp((G + H)^\circ).
\]

By Step 1, we know that the rank of $\pi^\perp((G + H)^\circ)$ is twice the rank of $TX^\perp \cap TX$, so we only have to check that $TX^\perp \cap TX \subset (\pi^\perp((G + H)^\circ), \omega)$ is an isotropic subbundle. This is clearly the case, for if $\alpha, \beta \in TX^\circ \cap (TX^\perp)^\circ$ then
\[
\omega(\pi^\perp(\alpha), \pi^\perp(\beta)) = \langle \pi^\perp(\alpha), \beta \rangle = 0.
\]

Here we use that $\pi^\perp(\alpha) \in TX$ since $\alpha \in (TX^\perp)^\circ$, and that $\beta \in TX^\circ$.

**Step 4:** Let $C \subset (\pi^\perp((G + H)^\circ), \omega)$ be a Lagrangian complement of $TX^\perp \cap TX$, and choose any subbundle $Y \subset TM|_X$ such that
\[
TM|_X = (TX^\perp \cap TX) \oplus (H \oplus G \oplus C \oplus Y).
\]

Then the subbundle $W_{G,H} := G \oplus C \oplus Y$ satisfies the criteria.

If $\alpha \in (H + G + C + Y)^\circ$, then $\alpha \in (G + H)^\circ$ and $\alpha \in C^\circ$. So for all $c \in C$, we get
\[
0 = \langle \alpha, c \rangle = \omega(c, \pi^\perp(\alpha)),
\]
which implies that $\pi^\perp(\alpha) \in C^\perp = C \subset G + C + Y$. Therefore, $\pi^\perp((H + W_{G,H})^\circ) \subset W_{G,H}$. The fact that $W_{G,H} \cap TX = G$ follows immediately from the decomposition
\[
TM|_X = (TX^\perp \cap TX) \oplus H \oplus W_{G,H} = TX \oplus H \oplus C \oplus Y.
\]

Lemma 4.2 implies that there is a splitting $TM|_X = (TX^\perp \cap TX) \oplus H \oplus W$, where $TX^\perp = (TX^\perp \cap TX) \oplus H$ and $\pi^\perp((H + W)^\circ) \subset W$.

Since $\pi((H + W)^\circ, W^\circ) = 0$, a local model for the local Poisson saturation of $X$ defined in terms of the complement $W$ can be constructed out of the data
\[
(H, W, i^L_{\pi_\perp}, (W^\circ/(H + W)^\circ, \pi)).
\]
Interpreting the vector bundle $W^\circ/(H + W)^\circ$ as a well-defined version of the “quotient” $(TX^\perp)^*/(TX^\perp \cap TX)^*$, we regard this fact as a Poisson analog of Marle’s theorem.

**Corollary 4.3** (Poisson version of Marle’s theorem). If $X \subset (M, \pi)$ is a coregular pre-Poisson submanifold, then a quadruple as in (18) determines a neighborhood of $X$ in its local Poisson saturation, up to Poisson diffeomorphism.

The corollary shows that the local Poisson saturation of a coregular pre-Poisson submanifold is determined by less data than that of a general coregular submanifold, since it uses $\pi$ on a quotient of $W^\circ$ rather than on all of $W^\circ$. The exception are those pre-Poisson submanifolds $X$ for which $TX^\perp \cap TX = 0$; these are the coregular Poisson-Dirac submanifolds of $(M, \pi)$ (see [CF] §8.2 or [CFM] §8.3). They are studied in the recent work [BFM].

**Remark 4.4.** For the classes of coregular submanifolds $X \subset (M, \pi)$ considered in this section, we summarize loosely the data that determine the local Poisson saturation $(P, \pi_P)$ near $X$. 

| Type of submanifold                  | $(P, \pi_P)$ locally determined by |
|--------------------------------------|------------------------------------|
| $X \subset (M, \pi)$ Poisson transversal | $i^L_{\pi_\perp}$ and $\pi_{|(TX^\perp)^*}$ |
| $X \subset (M, \pi)$ coregular coisotropic | $i^L_{\pi_\perp}$ |
| $X \subset (M, \pi)$ coregular pre-Poisson | $i^L_{\pi_\perp}$ and $\pi_{|(TX^\perp)^*/(TX^\perp \cap TX)^*}$ |
5. Coisotropic embeddings of Dirac manifolds in Poisson manifolds

As an application of Corollary 4.1, we look at the following question, which was considered
by Cattaneo and Zambon [CZ2] and by Wade [Wa]: Given a Dirac manifold \((X, L)\), can it be embedded coisotropically into a Poisson manifold \((M, \pi)\)? That is, when does there exist an embedding \(i : X \hookrightarrow (M, \pi)\) such that \(i^!L_\pi = L\) and \(i(X)\) is coisotropic in \((M, \pi)\)? Moreover, to what extent is such an embedding unique?

The question on the existence of coisotropic embeddings \((X, L) \hookrightarrow (M, \pi)\) is settled in [CZ2] Theorem 8.1: such an embedding exists exactly when \(L \cap TX^\perp\) has constant rank. The construction of \((M, \pi)\) in that case is carried out as follows: a choice of complement \(V\) to \(L \cap TX\) in \(TX\) gives an inclusion \(j : (L \cap TX)^* \hookrightarrow T^*X\), one takes \(M\) to be the total space of the vector bundle \(pr : (L \cap TX)^* \to X\) and one shows that the Dirac structure \((pr^!L)^\pi^*\omega_{\text{can}}\) on \(M\) is in fact Poisson on a neighborhood of \(X \subset M\). A different proof of the existence result is given in [Wa] Theorem 4.1.

The question on the uniqueness of coisotropic embeddings \((X, L) \hookrightarrow (M, \pi)\) is still open. In [Wa], it is claimed (without proof) that uniqueness can be obtained if \(L \cap TX\) defines a simple foliation on \(X\). In [CZ2] it is conjectured that, if \((X, L)\) is embedded coisotropically in two different Poisson manifolds, then these must be neighborhood equivalent around \(X\), provided that they are of minimal dimension \(\dim X + \text{rk}(L \cap TX)\). However, a proof of this uniqueness statement is only given under the additional regularity assumption that the presymplectic leaves of \((X, L)\) have constant dimension [CZ2] Proposition 9.4.

We now show that this extra assumption can be dropped. Using Corollary 4.1, we prove that the model \((U, (pr^!L)^\pi^*\omega_{\text{can}})\) constructed in [CZ2] is minimal, thereby obtaining the uniqueness result in full generality. In the proof below, given an embedding \(i : X \hookrightarrow (M, \pi)\), we may assume that it is the inclusion map by identifying \(X\) with \(i(X)\).

**Proposition 5.1.** Let \((X, L)\) be a Dirac manifold for which \(L \cap TX\) has constant rank, and denote by \(pr : (L \cap TX)^* \to X\) the bundle projection.

i) Any coisotropic embedding \(i : (X, L) \hookrightarrow (M, \pi)\) into a Poisson manifold \((M, \pi)\) factors through the local model \((U, (pr^!L)^\pi^*\omega_{\text{can}})\). That is, we have a diagram

\[
\begin{array}{ccc}
(X, L) & \xrightarrow{i} & (M, \pi) \\
\downarrow & & \downarrow \psi \\
(U, (pr^!L)^\pi^*\omega_{\text{can}}) & \xrightarrow{\psi} & (M, \pi)
\end{array}
\]

where \(\psi : (U, (pr^!L)^\pi^*\omega_{\text{can}}) \hookrightarrow (M, \pi)\) is a Poisson embedding.

ii) In particular, if \((M_1, \pi_1)\) and \((M_2, \pi_2)\) are Poisson manifolds of minimal dimension \(\dim X + \text{rk}(L \cap TX)\) in which \((X, L)\) embeds coisotropically, then \((M_1, \pi_1)\) and \((M_2, \pi_2)\) are Poisson diffeomorphic around \(X\).

**Proof.** i) The assumptions imply that \(X \subset (M, \pi)\) is a coregular coisotropic submanifold, since

\[
TX^{\perp_{\pi}} = \pi^!(TX^0) = (i^!L_\pi)^\perp = (i^!L_\pi) \cap TX = L \cap TX.
\] (19)

Denote by \((P, \pi_P)\) the local Poisson saturation of \(X \subset (M, \pi)\). By Corollary 4.1, there is a neighborhood \(U \subset (L \cap TX)^*\) of \(X\) and a Poisson embedding

\[
\phi : (U, (pr^!L)^\pi^*\omega_{\text{can}}) \to (P, \pi_P).
\]

Since \((P, \pi_P)\) is an embedded submanifold of \((M, \pi)\), this proves the statement.
ii) By what we just proved, there exist a neighborhood $U \subset (L \cap TX)^*$ of $X$ and two Poisson embeddings

$$\phi_1 : (U, (\text{pr}^1 L)^\ast \omega_{\text{can}}) \to (P_1, \pi_{P_1}),$$

$$\phi_2 : (U, (\text{pr}^1 L)^\ast \omega_{\text{can}}) \to (P_2, \pi_{P_2}),$$

where $(P_1, \pi_{P_1})$ and $(P_2, \pi_{P_2})$ denote the local Poisson saturations of $X$ in $(M_1, \pi_1)$ and $(M_2, \pi_2)$, respectively. The assumption implies that, for $l = 1, 2$:

$$\dim P_l = \dim TX^{\perp_{(t_l \circ L)}} = \dim X + rk(L \cap TX) = \dim M_l,$$

where we used (19). Since $P_l \subset M_l$ is an embedded submanifold, this shows that $P_l \subset M_l$ is a neighborhood of $X$, for $l = 1, 2$. So the composition $\phi_2 \circ \phi_1^{-1}$ is a Poisson diffeomorphism between neighborhoods of $X$ in $(M_1, \pi_1)$ and $(M_2, \pi_2)$.

\[\square\]

6. Coregular submanifolds in Dirac geometry

We now discuss how the results that we obtained in Sections 1, 2 and 3 can be generalized to the setting of Dirac manifolds. The relevant tools are developed in [FM2], from which we adopt the terminology and notation. For background on Dirac geometry, see e.g. [B].

**Definition 6.1.** We call an embedded submanifold $X$ of a Dirac manifold $(M, L)$ coregular if the map $\text{pr}_T : L|_X \to TM|_X/TX$, which is obtained composing the anchor $\text{pr}_T : L \to TM$ with the projection to the normal bundle, has constant rank.

Given any submanifold $i : X \hookrightarrow (M, L)$, we have at points $x \in X$ that

$$\text{pr}_T(L_x) = \frac{\text{pr}_T(L_x) + T_x X}{T_x X},$$

and therefore

$$X \subset (M, L) \text{ is coregular} \iff \text{pr}_T(L) + TX \text{ has constant rank}$$

$$\iff \ker((di)^\ast) \cap L \text{ has constant rank},$$

using that $\ker((di)^\ast) \cap L = (\text{pr}_T(L) + TX)^\circ$. In particular, the Dirac structure $L$ automatically induces a Dirac structure on a coregular submanifold $X \subset (M, L)$ [B, Prop. 1.10].

We recall some results about sprays and dual pairs in Dirac geometry [FM2].

**Definition 6.2.** Let $L \subset TM \oplus T^*M$ be a Dirac structure on $M$, and let $s : L \to M$ denote the bundle projection. A **spray** for $L$ is a vector field $V \in \mathfrak{X}(L)$ satisfying

i) $ds(V_a) = \text{pr}_T(a)$ for all $a \in L$,

ii) $m_t^* V = t V$, where $m_t : L \to L$ denotes fiberwise multiplication by $t \neq 0$.

Sprays exist on any Dirac structure. Condition ii) implies that the spray $V$ vanishes along the zero section $M \subset L$, and therefore there exists a neighborhood $\Sigma \subset L$ of $M$ on which the flow $\varphi_a$ of $V$ is defined for all times $a \in [0, 1]$. We can then define the **spray exponential** associated with $V$ as

$$\exp_V : \Sigma \to M : a \mapsto s(\varphi_1(a)).$$

Moreover, this neighborhood $\Sigma \subset L$ supports a two-form $\omega$ defined by

$$\omega := \int_0^1 \varphi_1^*((\text{pr}_T \ast)^{\ast} \omega_{\text{can}}) \, da,$$

(20)
where \( \text{pr}_T : L \to T^* M \) is the projection and \( \omega_{can} \) is the canonical symplectic form on \( T^* M \).

It is proved in [FM2] that, shrinking \( \Sigma \subset L \) if necessary, these data fit into a Dirac dual pair:

\[
(M, L) \xrightarrow{s} (\Sigma, \text{Gr}(\omega)) \xrightarrow{\exp_V} (M, -L).
\]

This means that both legs in the diagram (21) are surjective, forward Dirac submersions, and we have the additional requirements that \( \omega(V, W) = 0 \) and \( V \cap K \cap W = 0 \), where \( V = \ker ds, W = \ker d\exp_V \) and \( K = \ker \omega \).

We need the following lemma, which is a Dirac substitute for Lemma 1.5. The statement is not exactly the Dirac analog of Lemma 1.5; we address this in Remark 6.4 below.

**Lemma 6.3.** Consider a Dirac dual pair

\[
(M_0, L_0) \xrightarrow{s} (\Sigma, \text{Gr}(\omega)) \xrightarrow{t} (M_1, -L_1),
\]

and let \( X \subset (M_0, L_0) \) be a coregular submanifold. We denote \( V := \ker ds, W := \ker dt \) and \( K := \ker \omega \). Then \( W \cap ds^{-1}(TX) \) has constant rank, equal to the rank of \( \text{pr}_T^{-1}(TX) \subset L_0|X \).

**Proof.** Consider the following diagram of vector bundle maps:

\[
\begin{array}{ccc}
W|_{s^{-1}(X)} & \xrightarrow{\overline{ds}} & TM_0|X/TX \\
R_\omega \downarrow & & \uparrow \text{pr}_T \\
R_\omega(W|_{s^{-1}(X)}) & \xrightarrow{\psi} & L_0|X
\end{array}
\]

Here \( R_\omega \) is an injective bundle map defined by \( R_\omega : W \to T\Sigma \oplus T^*\Sigma : w \mapsto w + \iota_w \omega \). The map \( \psi : R_\omega(W) \to L_0 \) is defined by setting \( \psi(w + \iota_w \omega) := \overline{ds}(w) + \beta \), where \( \beta \) is uniquely determined by the relation \( \overline{ds}^*(\beta) = \iota_w \omega \). Note that \( \psi \) is well-defined: existence of \( \beta \) follows from the fact that \( \omega(V, W) = 0 \), and \( \beta \) is unique since \( s \) is a submersion. Since the map \( s : (\Sigma, \text{Gr}(\omega)) \to (M_0, L_0) \) is forward Dirac, \( \psi(w + \iota_w \omega) = \overline{ds}(w) + \beta \) is contained in \( L_0 \).

Moreover, we claim that the map \( \psi \) is an isomorphism. To see that \( \psi \) is injective, assume that \( \psi(w + \iota_w \omega) = \overline{ds}(w) + \beta = 0 \) for some \( w \in W \). Then \( \beta = 0 \), and therefore \( \iota_w \omega = \overline{ds}^*(\beta) = 0 \), so that \( w \in W \cap K \). But also \( \overline{ds}(w) = 0 \), so that \( w \in V \). Hence \( w \in V \cap K \cap W = 0 \), which shows that \( \psi \) is injective. Since the rank of \( R_\omega(W) \) is given by

\[
\text{rk}(R_\omega(W)) = \text{rk}(W) = \dim \Sigma - \dim M_1 = \dim M_0 = \text{rk}(L_0),
\]

it follows that \( \psi : R_\omega(W) \to L_0 \) is a vector bundle isomorphism. Since the diagram (22) commutes, it follows that

\[
\text{rk}(\overline{ds}) : W|_{s^{-1}(X)} \to TM_0|X/TX = \text{rk}(\overline{\text{pr}}_T : L_0|X \to TM_0|X/TX) = \dim M_0 - \text{rk}(\overline{\text{pr}}_T^{-1}(TX)).
\]

Consequently, we obtain that

\[
\text{rk}(W \cap ds^{-1}(TX)) = \text{rk}(W) - \dim M_0 + \text{rk}(\text{pr}_T^{-1}(TX)) = \text{rk}(\text{pr}_T^{-1}(TX)),
\]

which finishes the proof of the lemma.

**Remark 6.4.** For completeness, we state here the Dirac geometric analog of Lemma 1.5. Recall that a forward Dirac map \( \varphi : (M_0, L_0) \to (M_1, L_1) \) is strong if \( L_0 \cap \ker d\varphi = 0 \). When \( L_0 \) is the graph of a closed 2-form, then the map \( \varphi \) is called a presymplectic realization of \( (M_1, L_1) \). One can show that the following is true:
“Let $s : (\Sigma, \text{Gr}(\omega)) \to (M, L)$ be a strong forward Dirac submersion, and assume that $X \subset (M, L)$ is a coregular submanifold. If $V := \ker ds$, then $V^{\perp} \cap ds^{-1}(TX)$ has constant rank, equal to the rank of $\text{pr}^{-1}_T(TX)$.”

We won’t address this in more detail, since we want to use the legs of the diagram (21) and these are in general not presymplectic realizations. Indeed, using expressions for $\omega|_M$ that appear in [FM2], one can check that

$$(\text{Gr}(\omega) \cap \ker ds)|_M = 0 \oplus L \cap TM \subset TM \oplus L,$$

$$(\text{Gr}(\omega) \cap \ker d \exp_V)|_M = \{(-v, v) : v \in L \cap TM\} \subset TM \oplus L,$$

so that both legs are presymplectic realizations only when the Dirac structure $L$ is Poisson. In that case, $\omega$ is non-degenerate along $M \subset \Sigma$, so that shrinking $\Sigma$ if necessary, the diagram (21) is a full dual pair. In particular, the legs of the diagram (21) are symplectic realizations.

We obtain the following generalization of Theorem 1.6

**Theorem 6.5.** Let $X \subset (M, L)$ be a coregular submanifold.

1. There exists an embedded invariant submanifold $(P, L_P) \subset (M, L)$ containing $X$ that lies inside the saturation $\text{Sat}(X)$.

2. Shrinking $\Sigma$ if necessary, there exists a neighborhood $U$ of $X$ in $M$ such that $(P, L_P)$ is the saturation of $X$ in $(U, L|_U)$.

**Proof.** The proof is divided into four steps, just like the proof of Theorem 1.6

**Step 1:** Construction of the submanifold $P \subset M$.

Choose a spray $V \in \mathfrak{X}(L)$ and denote by $\exp_V : \Sigma \subset L \to M$ the corresponding spray exponential. Let $s : L \to M$ denote the bundle projection. Note that $\exp_V(a)$ and $s(a)$ lie in the same presymplectic leaf of $(M, L)$, for all $a \in L$. Indeed, the path $t \mapsto \varphi_t(a)$ is an $A$-path for the Lie algebroid $A = (L, [, , ], \text{pr}_T)$, covering the path $t \mapsto s(\varphi_t(a))$ which connects $s(a)$ with $\exp_V(a)$. In particular, we have that $\exp_V(\Sigma|_X) \subset \text{Sat}(X)$.

Since $X \subset (M, L)$ is coregular, we have that $\text{pr}^{-1}_T(TX)$ is a subbundle of $L|_X$, being the kernel of the constant rank bundle map $\text{pr}_T : L|_X \to TM|_X/TX$. Choose a complement $L|_X = \text{pr}^{-1}_T(TX) \oplus C$ and consider the restriction $\exp_V : C \cap \Sigma \to M$. It fixes points of $X$, and its differential along $X$ reads [FM2, Lemma 7]:

$$d\exp_V : T_xX \oplus C_x \to T_xM : (u, a) \mapsto u + \text{pr}_T(a).$$

This map is injective and therefore, shrinking $\Sigma$ if necessary, the map $\exp_V : C \cap \Sigma \to M$ is an embedding by Prop. 7.1. We set $P := \exp_V(C \cap \Sigma)$.

**Step 2:** Shrinking $\Sigma$ if necessary, we have that $P = \exp_V(\Sigma|_X)$.

It is enough to show that the restriction of $\exp_V$ to $\Sigma|_X$ has constant rank, equal to the rank of $\exp_V|_{C \cap \Sigma}$. To see this, we apply Lemma 6.3 to the self-dual pair (21), and we obtain that

$$\ker (d(\exp_V|_{\Sigma|_X})) = \ker (d\exp_V) \cap ds^{-1}(TX)$$

has constant rank, equal to the rank of $\text{pr}^{-1}_T(TX) \subset L|_X$. This implies that the rank of $\exp_V|_{\Sigma|_X}$ is constant, equal to

$$rk(\exp_V|_{\Sigma|_X}) = \dim X + rk(L) - rk(\text{pr}^{-1}_T(TX))$$

$$= \dim X + rk(C)$$

$$= rk(\exp_V|_{C \cap \Sigma}).$$
Step 3: The submanifold $P \subset (M, L)$ is invariant.

We have to check that the characteristic distribution $\text{pr}_T(L)$ of $L$ is tangent to $P$, i.e.
that $\text{pr}_T(L_{\exp_V(a)}) \subset (d\exp_V)_a(T_a\Sigma|_X)$ for all $a \in \Sigma|_X$. We will first show that
\[\text{pr}_T(L_{\exp_V(a)}) = (d\exp_V)_a(W^\perp),\]
where $W$ denotes $\ker d\exp_V$ as before. To see this, first pick $u + \xi \in L_{\exp_V(a)}$. Then
$v = u - \xi \in -L$, and since the map $\exp_V : (\Sigma, \text{Gr}(\omega)) \to (M, -L)$ is forward Dirac, there exists
$v \in T_a\Sigma$ such that $v + \iota_v\omega$ is $\exp_V$-related with $u - \xi$, i.e.
\[
\begin{aligned}
\iota_v\omega &= (d\exp_V)_a^*(-\xi), \\
u &= (d\exp_V)_a(v).
\end{aligned}
\]
This implies that $v \in W^\perp$, so $\text{pr}_T(u+\xi) = u = (d\exp_V)_a(v)$ is contained in $(d\exp_V)_a(W^\perp)$. Conversely, assume $v \in T_a\Sigma$ lies in $W^\perp$. Then $\iota_v\omega = (d\exp_V)_a^*(\xi)$ for some $\xi \in T^*_{\exp_V(a)}M$. This implies that $(d\exp_V)_a(v) + \xi$ is $\exp_V$-related with $v + \iota_v\omega \in \text{Gr}(\omega)$, and since the map $\exp_V : (\Sigma, \text{Gr}(\omega)) \to (M, -L)$ is forward Dirac, we get that $(d\exp_V)_a(v) + \xi \in -L$, i.e.
$(d\exp_V)_a(v) - \xi \in L$. It follows that $(d\exp_V)_a(v) \in \text{pr}_T(L_{\exp_V(a)})$.

Consequently, we obtain that
\[
\text{pr}_T(L_{\exp_V(a)}) = (d\exp_V)_a(W^\perp) = (d\exp_V)_a(V + W \cap K) \subset (d\exp_V)_a(ds^{-1}(TX)) = (d\exp_V)_a(T_a\Sigma|_X),
\]
where the second equality uses [FM2] Lemma 3], and the third equality holds because $W = \ker d\exp_V$ and $V = \ker ds \subset ds^{-1}(TX)$. This proves Step 3.

Step 4: Construction of the neighborhood $U$ of $X$.

The proof is completely analogous to the proof of Step 4 in Theorem 1.6. We want to extend
the map $\exp_V : C \cap \Sigma \to M$ to a local diffeomorphism. To do so, we choose a complement
\[TM|_X = TX \oplus (\text{pr}_T(C) \oplus E)\]
and a linear connection $\nabla$ on $TM$. We obtain a map
\[
\psi : O \subset (C \oplus E) \to M : (a, e) \mapsto \exp_V(Tr_{\exp_V(ta)}e),
\]
which is a diffeomorphism onto a neighborhood of $X$. Here $O$ is a suitable convex neighbor-
hood of the zero section, and $Tr_{\exp_V(ta)}$ denotes parallel transport along the curve $t \mapsto \exp_V(ta)$ for $t \in [0, 1]$. Note that $\psi(a, 0) = \exp_V(a)$, so shrinking $P$, we can assume that $P = \psi(O \cap (C \oplus \{0\}))$. Setting $U := \psi(O)$ finishes the proof. \qed

In the following, we denote by $(P, L_P)$ the Dirac manifold constructed in Thm. 6.5, we refer to it as the local Dirac saturation of $X$. Since $X$ is a Dirac transversal in $(P, L_P)$, the normal form theorem around Dirac transversals [BLM], [FM2] gives a normal form for
the local Dirac saturation around $X$. We will reprove this result, continuing the argument from Theorem 6.5. We need the following Dirac version of Lemma 3.1

**Lemma 6.6.** Let $i : X \hookrightarrow (M, L)$ be a coregular submanifold with local Dirac saturation $(P, L_P)$. Then the following is a weak Dirac dual pair, in the sense of [FM2]:
\[
(X, i^!L) \overset{s}{\leftrightarrow} (\Sigma|_X, \text{Gr}(\omega|_X)) \overset{\exp_V}{\rightarrow} (P, -L_P).
\]
This means that \( \omega|_X \) is a closed two-form on \( \Sigma|_X \), that \( s \) and \( \exp_V \) are surjective forward Dirac submersions, and that
\[
\omega|_X(S_1, S_2) = 0, \tag{25}
\]
\[
\text{rk}(S_1 \cap \tilde{K} \cap S_2) = \dim \Sigma|_X - \dim X - \dim P, \tag{26}
\]
where \( S_1 := \ker ds, S_2 := \ker d \exp_V \) and \( \tilde{K} := \ker(\omega|_X) \).

**Proof.** The only non-trivial part is that equality (26) holds. The other claims are proved exactly like in Lemma 3.1, so we don’t address them here.

To prove (26), note that \( S_1 \cap \tilde{K} \cap S_2 = V \cap (ds^{-1}(TX))^{1\omega} \cap W \), where \( V, W \) are the vertical distributions of the original dual pair (21). Note that for any subspace \( U \subset (T_a \Sigma, \omega_a) \), we have
\[
\dim(U^{1\omega}_a) = \dim(T_a \Sigma) - \dim(U_a) + \dim(U_a \cap K_a),
\]
where \( K := \ker \omega \). It follows that a family \( U \subset T \Sigma \) of linear subspaces has constant rank if both \( U^{1\omega} \) and \( U \cap K \) have constant rank. On one hand, we have
\[
(V \cap (ds^{-1}(TX))^{1\omega} \cap W) \cap K = 0,
\]
since \( V \cap K \cap W = 0 \). On the other hand, we consider \((V \cap (ds^{-1}(TX))^{1\omega} \cap W)^{1\omega}\). Using that \( K \subset (ds^{-1}(TX))^{1\omega} \), one checks that
\[
(V \cap (ds^{-1}(TX))^{1\omega} \cap W)^{1\omega} = (V \cap W)^{1\omega} + ((ds^{-1}(TX))^{1\omega})^{1\omega}.
\]
Moreover, using that \( V \) and \( W \) are the vertical distributions of the dual pair (21), one proves that \((V \cap W)^{1\omega} = V^{1\omega} + W^{1\omega}\). Altogether, we obtain
\[
(V \cap (ds^{-1}(TX))^{1\omega} \cap W)^{1\omega} = V^{1\omega} + ((ds^{-1}(TX))^{1\omega})^{1\omega} + W^{1\omega} = V^{1\omega} + ds^{-1}(TX) + K + W^{1\omega} = V^{1\omega} + ds^{-1}(TX) + W^{1\omega} = W + V \cap K + ds^{-1}(TX) + V + W \cap K = W + ds^{-1}(TX) + V = W + ds^{-1}(TX).
\]
In the fourth equality, we use [FM2, Lemma 3]. Using Lemma 6.3, we have now proved that \( S_1 \cap \tilde{K} \cap S_2 = V \cap (ds^{-1}(TX))^{1\omega} \cap W \) has constant rank. The rank is given by
\[
\text{rk}(V \cap (ds^{-1}(TX))^{1\omega} \cap W) = \text{rk}(T \Sigma) - \text{rk}(W + ds^{-1}(TX)) = \text{rk}(T \Sigma) - \text{rk}(W) - \text{rk}(ds^{-1}(TX)) + \text{rk}(W \cap ds^{-1}(TX)) = \dim(L) - \text{rk}(W) - \dim(X) - \text{rk}(V) + \text{rk}(pr_X^{-1}(TX)) = \dim(M) - \dim(\exp_V(\Sigma|_X)) + \text{rk}(L) - \text{rk}(V) = \dim(\Sigma) - \text{rk}(V) - \dim(\exp_V(\Sigma|_X)) = \dim(\Sigma|_X) - \dim(X) - \dim(\exp_V(\Sigma|_X)).
\]
This is exactly the rank condition (26), so the proof is finished. \( \square \)
Corollary 6.7. Let \( i : X \hookrightarrow (M, L) \) be a coregular submanifold. We choose a complement \( L|_X = \text{pr}^{-1}_T(TX) \oplus C \) and denote by \( j : C \hookrightarrow L|_X \) the inclusion. The local Dirac saturation \( (P, L_P) \) of \( X \) is diffeomorphic with
\[
(C \cap \Sigma, (s^* i^! L)^{-j^! \omega|_X}).
\]
In particular, \( (P, L_P) \) is determined by the pullback Dirac structure \( i^! L \), up to diffeomorphisms and exact gauge transformations.

Proof. Applying [FM2, Prop.6] to the diagram (24), we have the following equality of Dirac structures on \( \Sigma|_X \):
\[
(s^* i^! L)^{-\omega|_X} = (\exp V)^! L_P.
\]
Taking the pullback under the map \( j \), which is transverse to this Dirac structure, we obtain
\[
(s^* i^! L)^{-j^* \omega|_X} = (\exp V \circ j)^! L_P,
\]
which is an equality of Dirac structures on \( C \cap \Sigma \). We showed in Theorem 6.5 that \( \exp V \circ j \) is a diffeomorphism from \( C \cap \Sigma \) onto \( P \), which proves the first statement. Moreover, since \( -j^* \omega|_X \) is closed and its pullback to \( X \subset C \cap \Sigma \) vanishes, it is exact on a neighborhood of \( X \), by the relative Poincaré lemma. This implies the second statement of the corollary. \( \square \)

Remark 6.8. As mentioned before, the submanifold \( X \) is a Dirac transversal in \( (P, L_P) \). Corollary 6.7 agrees with the normal form around Dirac transversals proved in [FM2], upon identifying the normal bundle \( TP|_X/ TX \) with \( C \).

7. Appendix

We prove a result in differential topology that may be of independent interest. It should be standard, but we could not find a reference in the literature. The statement is well-known under the stronger assumption that the derivative of the map is an isomorphism along the zero section [MM, Lemma 6.1.3]. Our strategy is to reduce the proof to this case.

Proposition 7.1. Let \( E \to N \) be a vector bundle, and let \( \varphi : E \to M \) be a smooth map satisfying
\[
\begin{align*}
&\varphi|_N \text{ is an embedding} \\
&(d\varphi)_p \text{ is injective } \forall p \in N
\end{align*}
\]
(27)

Then there is a neighborhood \( U \subset E \) of \( N \) such that \( \varphi|_U \) is an embedding.

Proof. We get a vector subbundle \( d\varphi|_N (E) \subset TM|_{\varphi(N)} \) which has trivial intersection with \( T\varphi(N) \). Choose a complement \( C \) to \( d\varphi|_N (E) \oplus T\varphi(N) \) in \( TM|_{\varphi(N)} \), i.e.
\[
TM|_{\varphi(N)} = T\varphi(N) \oplus d\varphi|_N (E) \oplus C.
\]

Fix a linear connection \( \nabla \) on \( TM \), and define a map
\[
\psi : E \oplus (\varphi|_N)^* C \to M : (e, c) \to \exp_\nabla (Tr_{\varphi(te)} c),
\]
where \( Tr_{\varphi(te)} \) denotes parallel transport along the curve \( t \mapsto \varphi(te) \) for \( t \in [0, 1] \). We slightly abuse notation, since the map \( \psi \) is only defined on a small enough neighborhood of the zero section \( N \). Clearly, \( \psi \) satisfies the following properties:

- \( \psi \) restricts to \( \varphi|_N \) along the zero section \( N \).
• For $p \in N$ and a vertical tangent vector $(e, c) \in T_p(E \oplus (\varphi|_N)^*C)$, we have
\[
(d\psi)_p(e, c) = \left. \frac{d}{ds} \right|_{s=0} \psi(se, 0) + \left. \frac{d}{ds} \right|_{s=0} \psi(0, sc)
\]
\[
= \left. \frac{d}{ds} \right|_{s=0} \exp_{\nabla}(0_{\varphi(se)}) + \left. \frac{d}{ds} \right|_{s=0} \exp_{\nabla}(sc)
\]
\[
= \left. \frac{d}{ds} \right|_{s=0} \varphi(se) + c
\]
\[
= (d\varphi)_p(e) + c,
\]
which shows that $d\psi$ is an isomorphism at points of the zero section.

• We have that $\psi(e, 0) = \varphi(e)$, i.e. the following diagram commutes:

\[
\begin{array}{ccc}
E & \xrightarrow{\varphi} & M \\
\downarrow{\psi} & & \\
E \oplus (\varphi|_N)^*C
\end{array}
\]

Using the first and second bullet point above, the inverse function theorem for submanifolds (e.g. [Mu, Lemma 6.1.3]) shows that $\psi$ is an embedding on a neighborhood of $N$. Since also the inclusion $E \hookrightarrow E \oplus (\varphi|_N)^*C$ on the left in (28) is an embedding, it follows that $\varphi$ is an embedding on a neighborhood of $N$ in $E$.

\[\square\]

Remark 7.2. If a map $\varphi : U \subset E \to M$ satisfying the assumptions (27) of Proposition 7.1 is only defined on a neighborhood $U \subset E$ of $N$, then the conclusion of the proposition still holds. This can be obtained, for instance, by constructing a smooth map $\mu : E \to E$ such that $\mu(E) \subset U$ and $\mu = \text{Id}$ near $N$ (see [H, Chapter 4, §5]). Then Proposition 7.1 implies that the composition $\varphi \circ \mu : E \to M$ is an embedding on a neighborhood of $N$, hence the same holds for $\varphi$.

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