Hawking Radiation of Massive Vector Particles From Warped AdS$_3$ Black Hole

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Hawking radiation (HR) of massive vector particles from a rotating Warped Anti-de Sitter black hole in 2+1 dimensions (WAdS$_3$BH) is studied in detail. The quantum tunneling approach with the Hamilton-Jacobi method (HJM) is applied in the Proca equation (PE), and we show that the radial function yields the tunneling rate of the outgoing particles. Comparing the result obtained with the Boltzmann factor, we satisfactorily reproduce the Hawking temperature (HT) of the WAdS$_3$BH.

I. INTRODUCTION

General relativity (GR) predicts a mysterious object, which is a spacetime whose attractive gravitational force is so intense that no matter, light, or information of any kind can escape. This mysterious object consists of a point of no return known as the event horizon and a singularity with no volume. In classical approach, unless the particle can move backwards in time, which yet does not seem to be possible, it will not be able to get out. Having John Wheeler [1] named these mysterious objects as ‘black holes (BHs)’, physicists are forced to ask this question to themselves; are BHs really black? So far, HR [2,3] seems to be only calculable theoretical application of GR and quantum mechanics to a BH. It analyzes the evaporation of a BH, which lasts for an infinitely long period of time in an empty space. In addition to the HR, a wide range of studies in theoretical and experimental physics shows that BHs emit strong gravitational waves that is being hoped to prove their existence [4].

In this paper, Section 2 is devoted to the introduction of WAdS$_3$BH. In Sec. 3, we analyze PE for a massive gauge boson field in this geometry. We also show the derivation of HR of the vector particles tunneling from the WAdS$_3$BH. In Sec. 4, we conclude our investigations.

Throughout the paper, we use units wherein $c = G = k_B = 1$.

II. WAdS$_3$BH SPACETIME

Conformal field theory (CFT) [15] plays a vital role on the derivation of WAdS$_3$BHs due to the AdS/CFT correspondence [16]. Its metric is the result of the Arnowitt-Deser-Misner (ADM) formalism with integer spin bosons [17,18]. ADM is a topological trick based on the decomposition of spacetime into space and time [19]. During this process, a shift does not only occur in time but also in spatial coordinates which, in our case, arises as the lapse function $N(r)$ and the shift vector $N^\phi(r)$. Furthermore, the geometry of WAdS$_3$BH has a symmetry breaking of $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R \rightarrow SL(2,\mathbb{R}) \times U(1)$ [5].

Einstein-Hilbert action, which describes topologically massive gravity, is given by [19]

$$I = \frac{1}{16\pi} \int_M d^3x \sqrt{-g} \left( R - 2\Lambda \right) + I_{\text{matter}},$$

where $\Lambda$, $I_{\text{matter}}$, and $\mathbb{R}$ represent cosmological constant, gravitational Chern-Simons action and Ricci tensor trace, respectively. In Schwarzschild coordinates, the line-element of WAdS$_3$BH takes the following form [5]

$$ds^2 = -N^2 dt^2 + R^2 \left[ d\phi + N^\phi dt \right]^2 + \frac{\ell^2 dr^2}{4R^2 N^2},$$
where

\[ R^2 = \frac{r}{4} \left[ 3 (\nu^2 - 1) r + (\nu^2 + 3) (r_+ + r_-) - 4 \nu \sqrt{r_+ r_- (\nu^2 + 3)} \right], \]
\[ N^2 = \frac{(\nu^2 + 3) (r - r_+) (r - r_-)}{4R(r)^2}, \]
\[ N^\phi = \frac{2\nu r - \sqrt{r_+ r_- (\nu^2 + 3)}}{2R(r)^2}. \]

Wick rotation \[20\] being applied to two-dimensional spacetime predicts holomorphic and anti-holomorphic functions that are stated as left and right moving central charges. The corresponding right and left moving temperatures, respectively, are given by \[5\]

\[ T_R = \frac{(\nu^2 + 3)}{8\pi \ell} (r_+ - r_-), \]
\[ T_L = \frac{(\nu^2 + 3)}{8\pi \ell} \left( r_+ - r_- - \sqrt{(\nu^2 + 3) r_+ r_-} \right). \]

Moreover,

\[ T_H = \frac{\kappa}{2\pi} = \left( \frac{T_R + T_L}{4\pi \nu \ell} \right) \left( \frac{4\nu^2 + 3}{r_+ - r_-} \right), \]
\[ \Omega_H = \frac{2}{2\nu r_+ - \sqrt{r_+ r_- (\nu^2 + 3)}} \ell. \]

where \( T_H, \kappa, \) and \( \Omega_H \) denote the HT, surface gravity and angular velocity of WAdS\(_3\)BH. For each \( \nu \) value, we have a different case to be considered; having \( \nu < 1 \) suggests squashed AdS\(_3\), whereas \( \nu > 1 \) implies AdS\(_3\) to be stretched. Both cases are spacelike and if the chiral point \( \nu = 1 \) is met, we can no longer consider warpage \[5\]. These BHs have very interesting applications in the literature (see for instance \[21 – 24\] and references therein).

The special linear group \( SL(n, \mathbb{R}) \) includes \( n \times n \) matrices having determinants that are equal to one.

### III. QUANTUM TUNNELING OF VECTOR PARTICLES FROM WADS\(_3\)BHS

For a massive vector boson (spin-1) field, PE in a curved spacetime is given by \[7\]

\[ D_\mu \Psi^{\beta \mu} + \frac{m^2}{\hbar^2} \Psi^{\beta} = 0, \]

in which

\[ \Psi^{\beta \mu} = D_\beta \Psi_\mu - D_\mu \Psi_\beta = \partial_\beta \Psi_\mu - \partial_\mu \Psi_\beta, \]

where \( \Psi^{\beta}, D_\mu, m \) and \( \hbar \) represent three component spinor field, covariant derivatives, mass and reduced Planck constant, respectively. Since \( \beta = \{ 0, 1, 2 \} \) and \( x^\beta = \{ t, r, \phi \} \), PEs can be expressed as a set of triad equations:

\[ m^2 \ell^2 R^2 \Xi + \hbar^2 \ell^2 \partial_\phi (\Psi_{t \phi}) + 4\hbar^2 R^2 N^2 \partial_\phi \xi = 0, \]
\[ m^2 R^2 N^2 \Psi_r + \hbar^2 (N^\phi)^2 R^2 \lambda + \xi \hbar^2 = 0, \tag{13} \]

\[ m^2 \ell^2 R^2 \Xi + \hbar^2 \ell^2 \partial_\phi (\Psi_{t\phi}) + 4 \hbar^2 R^2 N^2 \delta = 0, \tag{14} \]

where

\[ \Xi = \Psi_t - N^\phi \Psi_\phi, \]
\[ \xi = R^2 \partial_\Psi_{tr} + N^2 \partial_\Phi_{r\phi}, \]
\[ \lambda = \partial_{\xi r}(N^\phi \Psi) + \partial_{\xi r}(N^\phi R^2 \Psi_{r\phi}), \]
\[ \delta = \partial_{r}(R^2 \Psi_{tr}) + \partial_{r}(N^\phi R^2 \Psi_{r\phi}). \tag{15} \]

Let us assume an ansatz of the form

\[ \Psi_\nu = \alpha_\nu \exp \left[ i \frac{\hbar}{\hbar} S(t, r, \phi) \right], \tag{16} \]

where \( \alpha_\nu = \{ \alpha_0, \alpha_1, \alpha_2 \} \) represents some arbitrary constants, and the action \( S \) is defined as

\[ S(t, r, \phi) = S_0(t, r, \phi) + \hbar S_1(t, r, \phi) + \hbar^2 S_2(t, r, \phi) + ... \tag{17} \]

Furthermore, we may set

\[ S_0(t, r, \phi) = -Et + W(r) + j\phi + \mathbb{C}, \tag{18} \]

where \( E \) and \( j \) denote the energy and angular momentum of the spin-1 particle, respectively, and \( \mathbb{C} \) is a complex constant. After applying the WKB approximation, we obtain the elements of the coefficient matrix \( H \) for the Eqs. (12)-(14), to the lowest order in \( \hbar \), as follows

\[ H_{00} = -4R^4N^2W'^2 - \ell^2 \left( m^2 R^2 + j^2 \right), \]
\[ H_{01} = H_{10} = -4R^4N^2W'E_{net}, \]
\[ H_{02} = H_{20} = -\ell^2 jE + m^2 \ell^2 R^2 N^\phi + 4R^4N^2N^\phi W'^2, \]
\[ H_{11} = -4N^2R^2 \left[ (-m^2 N^2 + E_{net}^2) R^2 - N^2 j^2 \right], \]
\[ H_{12} = H_{21} = 4W' \left[ N E_{net} R^2 - N^2 j \right] N^2 R^2, \]
\[ H_{22} = \left[ -4R^4N^2 \left( N^\phi \right)^2 + 4R^4N^4 \right] W'^2 - \ell^2 \left[ E^2 + m^2 R^2 \left( N^\phi \right)^2 - m^2 N^2 \right]. \tag{19} \]

where \( W' = \frac{d}{dr} W(r) \), and

\[ E_{net} = E + jN^\phi. \tag{20} \]

Having non-trivial solution for the radial function \( W_{\pm}(r) \) is conditional on \( \det(H) = 0 \):

\[ \det(H) = -4R^2N^2m^2 - 4R^4N^4W'^2 + \ell^2 \left[ (-m^2 N^2 + E_{net}^2) R^2 - N^2 j^2 \right]^2 = 0, \tag{21} \]

and it is worthwhile to mention that \( W_+ \) stands for the radial function of outgoing (emitted) particles, and conversely \( W_- \) corresponds to the ingoing (absorbed) solution. Therefore, the solution for the radial function is
\[ W_\pm (r) = \pm \int \frac{\ell}{2RN^2} \sqrt{E_{\text{net}}^2 - N^2 \left( m^2 + \frac{j^2}{R^2} \right)} dr. \] (22)

In the vicinity of the event horizon, \( N \) approaches to zero, hence the second term inside the square root vanishes, and the integral becomes

\[ W_\pm (r) \approx \pm \frac{\ell E_{\text{net}}}{2RN^2} \int \sqrt{E_{\text{net}}^2 - N^2 \left( m^2 + \frac{j^2}{R^2} \right)} dr. \] (23)

where

\[ \Upsilon = (\nu^2 + 3) (r_+ + r_-) - 4\nu \sqrt{r_+ r_- (\nu^2 + 3)}. \] (24)

As it can be seen from Eq. (23), the integrand has a simple pole at the event horizon, \( r_+ \). Using the Feynman’s prescription \[25\], we obtain

\[ W_+ (r) \approx \frac{i\pi \ell E_{\text{net}}}{(\nu^2 + 3) (r_+ - r_-)} \sqrt{3r_+^2 (\nu^2 - 1) r + \Upsilon r_+}. \] (25)

Therefore, one can deduce that imaginary parts of the action can only come about due to the pole at the horizon or from the imaginary part of \( C \). The probabilities of the particles crossing the event horizon each way are given by \[8\]

\[ \Gamma_{\text{emission}} = \exp \left( -\frac{2}{\hbar} \text{Im} S \right) = \exp \left[ -\frac{2}{\hbar} (\text{Im} W_+ + \text{Im} C) \right], \] (26)

and

\[ \Gamma_{\text{absorption}} = \exp \left( -\frac{2}{\hbar} \text{Im} S \right) = \exp \left[ -\frac{2}{\hbar} (\text{Im} W_- + \text{Im} C) \right]. \] (27)

According to the semi-classical approach, there exists a 100\% chance for the ingoing spin-1 particles to enter the BH, i.e., \( \Gamma_{\text{absorption}} = 1 \). This implies that \( \text{Im} C = -\text{Im} W_- \). Since \( W_- = -W_+ \), whence we can read the probability of the outgoing (tunneling) particles as

\[ \Gamma_{\text{emission}} = \exp \left( -\frac{4}{\hbar} \text{Im} W_+ \right). \] (28)

Thus, the tunneling rate becomes

\[ \Gamma_{\text{rate}} = \frac{\Gamma_{\text{emission}}}{\Gamma_{\text{absorption}}} = \exp \left( -\frac{4}{\hbar} \text{Im} W_+ \right), \] (29)

which is also equivalent to the Boltzmann factor: \( \exp(-E_{\text{net}}/T) \) \[8, 11, 26\]. From this point on, we can compute the surface temperature of the event horizon of \( \text{WAdS}_3 \)BH as

\[ T_s = \frac{\hbar}{2\pi} \left( \frac{\nu^2 + 3}{(\nu^2 + 3) \left( r_+ + r_- \right)} \right) \frac{1}{\sqrt{3r_+^2 (\nu^2 - 1) + \Upsilon r_+}}. \] (30)
Henceforth, we set $\hbar = 1$, and Eq. (30) can be rewritten as

$$T_H = \frac{(\nu^2 + 3) (r_+ - r_-)}{4\pi l \sqrt{4\nu^2 r_+^2 + \nu^2 r_+ r_- + 3r_+ r_- - 4\nu r_+ \sqrt{r_+ r_- (\nu^2 + 3)}}}.$$  

Equation (31) is fully agree with Eq. (8), which is the HT of WAdS$_3$BH.

IV. CONCLUSION

The present study has constituted a valid process of evaluating HR by promoting the application of PEs together with HJM. As an application, we have shown how the quantum tunneling of vector particles from WAdS$_3$BH results in the standard HT, which is the combination of holomorphic and anti-holomorphic functions; $T_R$ and $T_L$. We plan to extend our work to higher dimensional rotating BHs. This is going to be our next work in the near future.

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