HOLED CONE STRUCTURES ON 3-MANIFOLDS

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Abstract. We introduce holed cone structures on 3-manifolds to generalize cone structures. In the same way as a cone structure, a holed cone structure induces the holonomy representation. We consider the deformation space consisting of the holed cone structures on a 3-manifold whose holonomy representations are irreducible. This deformation space for positive cone angles is a covering space on a reasonable subspace of the character variety.

1. Introduction

A cone-manifold is a generalization of a Riemannian manifold of constant sectional curvature, allowed to have cone singularity. The rotational angle around cone singularity is not equal to $2\pi$, and it is called the cone angle. While a finite volume complete hyperbolic structure on a 3-manifold admits no continuous deformation by the Mostow rigidity, it can often be deformed via hyperbolic cone-manifolds. For example, a hyperbolic Dehn surgery (in the strong sense) gives continuous deformation from a cusped hyperbolic 3-manifold to a hyperbolic 3-manifold obtained by gluing solid tori via hyperbolic cone-manifolds whose singular locus consists of the cores of glued solid tori. If the cone angles of a cone-manifold are equal to $2\pi/n_i$ for $n_i \in \mathbb{Z}_{>0}$, this cone-manifold can be regarded as an orbifold. Proofs of the geometrization of 3-orbifolds in \cite{3, 6} are based on this fact.

Local and global rigidities of hyperbolic cone-manifolds are known as follows. For an oriented 3-manifold $X$ and an $n$-component link $\Sigma$ in $X$, let $C_{[0,\theta]}(X, \Sigma)$ denote the space of hyperbolic cone structures on $(X, \Sigma)$ with cone angles at most $\theta$. Let $\Theta: C_{[0,\theta]}(X, \Sigma) \to [0, \theta]^n$ denote the map assigning the cone angles. The local rigidity by Hodgson and Kerckhoff \cite{14} states that $\Theta: C_{[0,2\pi]}(X, \Sigma) \to [0, 2\pi]^n$ is a local homomorphism. The global rigidity by Kojima \cite{19} states that $\Theta: C_{[0,\pi]}(X, \Sigma) \to [0, \pi]^n$ is injective. The global rigidity is not known if cone angles exceed $\pi$. Izmestiev \cite{16} gave examples that the global rigidity does not hold and cone angles exceed $2\pi$. The proof in \cite{19} is based on the fact that two cone loci with cone angles less than $\pi$ are not close. A cone structure in $C_{[0,\pi]}(X, \Sigma)$ can be continuously deformed to the cusped hyperbolic structure on $X \setminus \Sigma$. However, this argument does not work if cone angles exceed $\pi$. Cone structures may degenerate by meeting cone loci even if the cone angles decrease \cite{23}.

In this paper, we introduce the notion of holed cone structures to generalize cone structures. A holed cone structure on $(X, \Sigma)$ is defined as an equivalence class of cone metrics on outside balls in $X$. In the same way as a non-holed one, a holed cone structure induces the holonomy representation of $\pi_1(X \setminus \Sigma)$ to $\text{Isom}^+(\mathbb{H}^3)$ up to conjugate. We consider the deformation space $HC_{\text{irr}}(X, \Sigma)$ consisting of the holed cone structures on $(X, \Sigma)$ whose holonomy representations are irreducible. In Theorem 3.13, we will show that $HC_{\text{irr}}(X, \Sigma)$ is Hausdorff. We will define a subspace $X^{\text{irr}}(X, \Sigma)$ of the character variety $X(\pi_1(X \setminus \Sigma))$. 

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$\hat{X}^\text{cone}(X, \Sigma)$ consists of the pairs of elements in $X^\text{cone}(X, \Sigma)$ and compatible cone angles. The map $\hat{\text{hol}} : \mathcal{H}^\text{irr}(X, \Sigma) \rightarrow \hat{X}^\text{cone}(X, \Sigma)$ assigns the holonomy representation and the cone angles. In Theorem 3.12 we will show that the map $\hat{\text{hol}} : \mathcal{H}^\text{irr}(X, \Sigma) \rightarrow \hat{X}^\text{cone}(X, \Sigma)$ is a regular covering map to each path-connected component of $\hat{X}^\text{cone}(X, \Sigma)$ that contains an image, where the map is restricted to the elements with positive cone angles. Lemma 3.8 implies that the map $\hat{\text{hol}} : \mathcal{H}^\text{irr}(X, \Sigma) \rightarrow \hat{X}^\text{cone}(X, \Sigma)$ is not injective. Consequently, global rigidity for holed cone structures does not hold even if $X \setminus \Sigma$ admits a hyperbolic structure.

Section 4 concerns (non-holed) cone structures in the space of holed cone structures. In Theorem 4.1, we will show that cone metrics $g$ and $g'$ with $\hat{\text{hol}}(g) = \hat{\text{hol}}(g')$ are equivalent. By Corollary 4.4, we may regard a cone structure as a holed cone structure. We expect that the notion of holed cone structures is useful to consider global rigidity for cone-manifolds. It should be worthwhile to ask whether a cone structure can be deformed to the cusped hyperbolic structure via holed cone structures.

In Section 5, we will introduce the volume of a holed cone structure. This is defined as the sum of the volume of a holed cone metric and the volume enclosed by the holes. After all, this volume is equal to the volume of the holonomy representation by Theorem 5.3.

In Section 6, we will give an explicit example of holed cone structures. This is consistent with the construction of cone structures by the author [23]. Holed cone structures enable us to avoid degeneration with meeting cone loci.

Euclidean and spherical holed cone structures can be defined in the same manner. To show the corresponding results, it is necessary to take care of the topologies of quotients of representation spaces.

2. Definition of holed cone structures

Hyperbolic metrics on a manifold are Riemannian metrics with constant sectional curvature $-1$. Equivalently, they are modeled by the hyperbolic space $\mathbb{H}^n$ of constant sectional curvature $-1$. According to [6], a (hyperbolic) cone-manifold is a topological manifold with a complete path metric (called a cone metric) which can be triangulated into hyperbolic simplices. (Although it does not matter in 3 dimensions, the link of each simplex in this triangulation is needed to be piecewise linearly homeomorphic to a standard sphere.)

The singular locus of a cone-manifold consists of the points with no neighborhood isometric to a hyperbolic ball. We consider 3-dimensional cone-manifolds whose singular locus consists of closed geodesics. Then locally a cone metric has the form

$$dr^2 + \sinh^2 r d\theta^2 + \cosh^2 r dz^2$$

in cylindrical coordinates around an axis, where $r$ is the distance from the axis, $z$ is the distance along the axis, and $\theta$ is the angle measured modulo the cone angle $\theta_0 > 0$. If a cone angle is equal to $2\pi$, then the metric is smooth around the point. By generalizing the notion, a cusp of a hyperbolic 3-manifold is regarded as a component of the singular locus with cone angle zero.

Let $X$ be an oriented 3-manifold, and let $\Sigma$ be a union of disjoint circles in $X$. A cone metric on $(X, \Sigma)$ is a metric on $X$ such that $X$ is a cone-manifold with singular locus $\Sigma$. We allow that a cone angle is equal to 0 or $2\pi$. We call a component of $\Sigma$ a cone locus. A cone structure on $(X, \Sigma)$ is an isometry class of a cone metric on $(X, \Sigma)$, where two metrics are equivalent if there is an isometry isotopic to the identity between them.

We introduce holed cone structures on $(X, \Sigma)$ as a generalization of cone structures.
Definition 2.1. Let $B$ be a union of finitely many (possibly empty) disjoint closed 3-balls in $X \setminus \Sigma$. A holed cone metric on $(X, \Sigma)$ is a cone metric on $(X \setminus \text{Int}(B), \Sigma)$ with smooth boundary $\partial B$. We call each component of $B$ a hole. We call the metric space $(X \setminus \text{Int}(B), \Sigma; g)$ a holed cone-manifold.

Definition 2.2. Let $g$ and $g'$ be holed cone metrics on $(X, \Sigma)$ respectively with holes $B$ and $B'$. The metrics $g$ and $g'$ are equivalent if there are holed cone metrics $g_i$ with holes $B_i$ on $(X, \Sigma)$ for $0 \leq i \leq n$ such that $g_0 = g$, $g_n = g'$, and for each $0 \leq i \leq n - 1$ either

1. there is a map $f : (X, \Sigma) \to (X, \Sigma)$ isotopic to the identity (preserving $\Sigma$) such that the restriction of $f$ to $(X \setminus \text{Int}(B_i), \Sigma; g_i)$ is an isometry onto $(X \setminus \text{Int}(B_{i+1}), \Sigma; g_{i+1})$,
2. $B_i \subset B_{i+1}$, and $g_{i+1}$ is the restriction of $g_i$ to $X \setminus \text{Int}(B_{i+1})$, or
3. $B_{i+1} \subset B_i$, and $g_i$ is the restriction of $g_{i+1}$ to $X \setminus \text{Int}(B_i)$.

We call an equivalent class $[g]$ a holed cone structure. The relation (1) will not be mentioned explicitly.

A cone metric is a holed cone metric by definition. Moreover, we can say that a cone structure is a holed cone structure. We will prove it in Corollary 4.4.

We give some elementary examples of holed cone metrics. If we remove an embedded ball disjoint from the cone loci in a holed cone-manifold, then we obtain an equivalent holed cone metric. Conversely, if a lift of the boundary of a hole is embedded by the developing map (which we will define in Section 3) and neighborhood of this boundary extends outward, then we can fill the hole by gluing the bounded ball in $\mathbb{H}^3$. In general, however, the boundary of a hole may not be embedded by the developing map.

Cone loci may prevent a hole from expanding to a fillable one as indicated in Figure 1. This enables us to deform holed cone structures when cone structures degenerate with meeting cone loci. An explicit example will be given in Section 6.

![Figure 1. A non-fillable hole between two cone loci](image)

Let $(X, \Sigma; g)$ and $(X', \Sigma'; g')$ be holed cone-manifolds. By making holes in $(X, \Sigma)$ and $(X', \Sigma')$ and attaching a 1-handle, we obtain a holed cone metric on the connected sum $(X \# X', \Sigma \sqcup \Sigma')$. By making two holes in $(X, \Sigma)$ attaching a 1-handle, we obtain a holed cone metric on the connected sum $(X \# S^2 \times S^1, \Sigma)$. Since metrics on a 1-handle can be arbitrarily deformed, the local rigidity for holed cone structures does not hold in general.
3. Deformation space of holed cone structures

Let $\Sigma = \bigsqcup_{i=1}^n \Sigma_i \subset X$ be a link in an oriented 3-manifold. Let $g$ be a (hyperbolic) holed cone metric on $(X, \Sigma)$ with holes $B$. Let $\Gamma = \pi_1(X \setminus \Sigma) = \pi_1(X \setminus (\Sigma \cup \text{int}(B)))$. Suppose that $\Gamma$ is finitely generated. Note that the holes do not affect the fundamental group. Let $\tilde{M}$ denote the universal cover of the incomplete hyperbolic 3-manifold $M = (X \setminus (\Sigma \cup \text{int}(B)); g)$. The developing map $\text{dev}_g: \tilde{M} \to \mathbb{H}^3$ is defined in an ordinary way as in [3, 5, 6].

Let $G = \text{Isom}^+(\mathbb{H}^3)$ denote the group consisting of the orientation-preserving isometries of $\mathbb{H}^3$. The holonomy representation $\rho_g: \Gamma \to G$ is also defined so that $\text{dev}_g$ is equivariant, i.e. $\text{dev}_g(\gamma \cdot x) = \rho_g(\gamma) \cdot \text{dev}_g(x)$ for any $\gamma \in \Gamma$ and $x \in \tilde{M}$. The map $\text{dev}_g$ and the representation $\rho_g$ are unique up to conjugate in $G$. Since the restriction and extension in Definition 2.2 do not affect the holonomy representation, $\text{dev}_g$ does not affect the holonomy representation, the holonomy representation $\rho_g$ is well-defined for a holed cone structure $[g]$. In other words, we have $\rho_g = \rho_{g'}$ for equivalent holed cone metrics $g$ and $g'$. The developing map is an immersion. Conversely, an equivariant immersion induces a metric. The restriction of the developing map to the boundary of a hole is an immersion, but it is not an embedding in general.

Consider the representation space $\text{Hom}(\Gamma, G)$ endowed with the compact-open topology. The group $G$ acts on $\text{Hom}(\Gamma, G)$ by conjugation. Let $[\rho]$ denote the conjugacy class of $\rho \in \text{Hom}(\Gamma, G)$. The topological quotient $\text{Hom}(\Gamma, G)/G$ consisting of such $[\rho]$ is not Hausdorff in general. For instance, if $\rho_0: \Gamma \to \mathbb{Z} \to G$ whose non-trivial images are parabolic, conjugate elements of $\rho_0$ accumulate the trivial representation. To obtain a manageable deformation space, we need to restrict the space of holonomy representations.

The group $G = \text{Isom}^+(\mathbb{H}^3) \simeq \text{PSL}(2, \mathbb{C}) \simeq \text{SO}(3, \mathbb{C})$ is a complex algebraic group. Hence the space $\text{Hom}(\Gamma, G)$ admits a structure of a complex affine algebraic set. The character variety $\mathcal{X}(\Gamma) = \text{Hom}(\Gamma, G)/G$ is defined as the GIT quotient in the category of algebraic varieties. Consider the Euclidean topology of the affine variety $\mathcal{X}(\Gamma)$, which is induced as a subset in $\mathbb{C}^N$ for some $N$. It is known that the space $\mathcal{X}(\Gamma)$ is the largest Hausdorff quotient of $\text{Hom}(\Gamma, G)/G$ (see [21]).

A representation in $\text{Hom}(\Gamma, G)$ is irreducible if there is no proper $\Gamma$-invariant subspace of $\mathbb{C}^2$ in the action as $G = \text{PSL}(2, \mathbb{C})$. Let $\text{Hom}^{\text{irr}}(\Gamma, G)$ denote the set of irreducible representations. Let $t: \text{Hom}(\Gamma, G) \to \mathcal{X}(\Gamma)$ denote the projection. Any representation $\rho \in \text{Hom}^{\text{irr}}(\Gamma, G)$ satisfies that the set $t^{-1}(t(\rho))$ consists of the conjugates of $\rho$ (see [1] [8] [12] [22] for details). Hence we may regard $\mathcal{X}^{\text{irr}}(\Gamma) = \text{Hom}^{\text{irr}}(\Gamma, G)/G$ as a subset of $\mathcal{X}(\Gamma)$. In particular, $\mathcal{X}^{\text{irr}}(\Gamma)$ is Hausdorff. We write $[\rho] = t(\rho) \in \mathcal{X}^{\text{irr}}(\Gamma)$.

We define the deformation space of holed cone structures. Let $\mathcal{HC}(X, \Sigma)$ denote the space of holed cone metrics on $(X, \Sigma)$ endowed with the $C^\infty$-topology induced by the metric tensors. Let $\mathcal{HC}(X, \Sigma)$ denote the space of holed cone structures on $(X, \Sigma)$ endowed with the quotient topology from $\mathcal{HC}(X, \Sigma)$. The continuous map $\text{hol}: \mathcal{HC}(X, \Sigma) \to \mathcal{X}(\Gamma)$ is defined by $\text{hol}([\rho]) = [\rho]$. Let $\mathcal{HC}^{\text{irr}}(X, \Sigma) \subset \mathcal{HC}(X, \Sigma)$ and $\mathcal{HC}^{\text{irr}}(X, \Sigma) \subset \mathcal{HC}(X, \Sigma)$ denote the preimages of $\mathcal{X}^{\text{irr}}(\Gamma)$ by $\text{hol}$. Note that a quotient space of a Hausdorff space is not Hausdorff in general. We show that $\mathcal{HC}^{\text{irr}}(X, \Sigma)$ is Hausdorff in Theorem 3.13.

Let us consider a condition for the holonomy of a neighborhood of cone loci. For each $1 \leq i \leq n$, let $N(\Sigma_i)$ be a regular neighborhood of the cone locus $\Sigma_i$ in $X$. The meridian for $\Sigma$ is (the homotopy class of) a simple closed curve on $\partial N(\Sigma_i)$ that is contractible in $N(\Sigma_i)$. Fix a longitude for $\Sigma$, which is a simple closed curve on $\partial N(\Sigma_i)$ intersecting the meridian at exactly once. Fix orientations of the meridian and longitude compatible with the orientation of $X$. Fix commuting
elements \( \mu_i, \lambda_i \in \Gamma \) which respectively correspond to the meridian and longitude for \( \Sigma_i \). Let \( g \in \mathcal{HC}(X, \Sigma) \). Suppose that the cone angle at \( \Sigma_i \) is not zero. Then the isometry \( \rho_\mu(\lambda_i) \in G \) is a loxodromic transformation along a geodesic axis \( \ell_i \). (This is not identity.) The isometry \( \rho_\mu(\mu_i) \in G \) is a rotation about \( \ell_i \) of the cone angle at \( \Sigma_i \). (This is identity if the cone angle is \( 2n\pi \), which we allow.) In the metric completion of \( g \), the cone locus \( \Sigma_i \) is the image of the geodesic \( \ell_i \). If \( \Sigma_i \) is a cusp, then \( \rho(\lambda_i) \) and \( \rho(\mu_i) \) are parabolic. Moreover, they are conjugate to elements so that their actions on \( \mathbb{C} \subset \partial \mathbb{H}^3 \) are linearly independent translations. For this condition, we say that \( \rho(\lambda_i) \) and \( \rho(\mu_i) \) are parabolic with rank two.

Let \( \mathcal{X}^{cone}(X, \Sigma) \) denote the set of \( [\rho] \in \mathcal{X}^{irr}(\Gamma) \) satisfying the following conditions for each \( 1 \leq i \leq n \): either \( \rho(\lambda_i) \) and \( \rho(\mu_i) \) are parabolic with rank two, or

- \( \rho(\lambda_i) \) is a loxodromic transformation along an axis, and
- \( \rho(\mu_i) \) is a (possibly trivial) rotation about this axis.

Then the image of \( \mathcal{HC}^{irr}(X, \Sigma) \) by \( \text{hol} \) is contained in \( \mathcal{X}^{cone}(X, \Sigma) \). Note that a rotation angle has an arbitrariness modulo \( 2\pi \). We define the space of representations equipped with compatible cone angles. Let \( \tilde{\mathcal{X}}^{cone}(X, \Sigma) \) denote the space consisting of the elements \( (\rho_\mu(\lambda_1), \theta_1, \ldots, \theta_n) \in \mathcal{X}^{cone}(X, \Sigma) \times [0, \infty)^n \) such that

- \( \theta_i = 0 \) if \( \rho(\lambda_i) \) and \( \rho(\mu_i) \) are parabolic,
- otherwise \( \rho(\mu_i) \) is a rotation of angle \( \theta_i \) for the fixed orientation.

Let \( \tilde{\pi} : \tilde{\mathcal{X}}^{cone}(X, \Sigma) \rightarrow \mathcal{X}^{cone}(X, \Sigma) \) denote the natural projection. A fiber of \( \tilde{\pi} \) consists of elements of the form \( ([\rho], \theta_1 + 2\pi k_1, \ldots, \theta_n + 2\pi k_n) \) for some \( \theta_i \in [0, 2\pi] \) where \( k_i = 0 \) if \( \theta_i = 0 \), and otherwise \( k_i \) is a non-negative integer. Define the continuous map \( \text{hol} : \mathcal{HC}^{irr}(X, \Sigma) \rightarrow \tilde{\mathcal{X}}^{cone}(X, \Sigma) \) by \( \text{hol}([g]) = (\text{hol}(g), \Theta(g)) \), where \( \Theta(g) \) is \( n \)-tuple of the cone angles for the metric \( g \). Then \( \text{hol} = \tilde{\pi} \circ \tilde{\text{hol}} \).

Let \( \mathcal{H}_+^{irr}(X, \Sigma) \subset \mathcal{H}^{irr}(X, \Sigma) \), \( \mathcal{H}_+^{cone}(X, \Sigma) \subset \mathcal{H}^{cone}(X, \Sigma) \), \( \mathcal{X}^{cone}(X, \Sigma) \subset \mathcal{X}^{cone}(X, \Sigma) \), and \( \tilde{\mathcal{X}}^{cone}(X, \Sigma) \subset \tilde{\mathcal{X}}^{cone}(X, \Sigma) \) denote the subspaces with positive cone angles.

The spaces \( \mathcal{X}^{cone}(X, \Sigma) \) and \( \tilde{\mathcal{X}}^{cone}(X, \Sigma) \) may not be topological manifolds, but their topologies are not very wild.

**Lemma 3.1.** The spaces \( \mathcal{X}^{cone}(X, \Sigma) \) and \( \tilde{\mathcal{X}}^{cone}(X, \Sigma) \) are locally contractible.

**Proof.** For any \( [\rho] \in \mathcal{X}(\Gamma) \) and \( \gamma \in \Gamma \), the trace of \( \rho(\gamma) \) is determined up to sign. The map \( \rho \mapsto (\text{tr}(\rho(\gamma)))^2 \) is a regular map on the affine variety \( \mathcal{X}(\Gamma) \). The subspace \( \mathcal{X}^{irr}(\Gamma) \) is open in \( \mathcal{X}(\Gamma) \).

The condition for the \( i \)-th meridian to define \( \mathcal{X}^{cone}(X, \Sigma) \) is that the trace of \( \rho(\mu_i) \) belongs to \([-2, 2]\). Since \( \mu_i \) and \( \lambda_i \) commute, the condition for the \( i \)-th longitude is that

- \( \rho(\lambda_i) \) and \( \rho(\mu_i) \) are parabolic with rank two if \( \rho(\mu_i) \) is parabolic,
- the trace of \( \rho(\lambda_i) \) belongs to \( \mathbb{C} \setminus [-2, 2] \) if \( \rho(\mu_i) \) is elliptic or the identity.

This is an open condition under the supposition of the condition for the \( i \)-th meridian.

Therefore the space \( \mathcal{X}^{cone}(X, \Sigma) \) is an open subspace of a real semi-algebraic subset of the affine algebraic set \( \mathcal{X}(X, \Sigma) \). Each point of a real semi-algebraic set has a neighborhood homeomorphic to a cone (in the sense of topology) (see [2] Theorem 5.48). Hence \( \mathcal{X}^{cone}(\Gamma) \) is locally contractible. The space \( \tilde{\mathcal{X}}^{cone}(X, \Sigma) \) is locally homeomorphic to \( \mathcal{X}^{cone}(X, \Sigma) \).

We consider a problem: which holonomy representations are realized by holed cone structures? For elements in \( \tilde{\mathcal{X}}^{cone}(X, \Sigma) \), we try to construct holed cone structures. For this purpose, we introduce the notion of a “handle decomposition”
for a holed cone metric. We define a handle decomposition of $(X, \Sigma)$ with holes $B$ as a filtration $X_0 \subset X_1 \subset X_2 \subset X_3 = X$ of smooth submanifolds satisfying that
- $X_0$ is the disjoint union of regular neighborhoods of the cone loci and 0-handles,
- $X_i$ is obtained by attaching $i$-handles to $X_{i-1}$ for $i = 1, 2, 3,$ and
- $X_2 \subset X \setminus B$.

If $g$ is a holed cone metric on $(X, \Sigma)$ with holes $B$, then the restriction of $g$ to $X_2$ is also a holed cone metric on $(X, \Sigma)$. In this case, we call it a handle decomposition for $g$.

We show that if a continuous deformation of holed cone metrics preserves the holonomy representation, it also preserves the holed cone structure.

**Lemma 3.2.** Let $g_t \in \hat{HC}^{\text{irr}}(X, \Sigma)$ be a continuous family for $0 \leq t \leq 1$. Suppose that $\text{hol}(g_t)$ are constant. Then $[g_t] \in HC^{\text{irr}}(X, \Sigma)$ are constant.

**Proof.** Let $I_0$ be the set consisting of points $s \in [0, 1]$ satisfying that $[g_t] = [g_0]$ for any $t \in [0, s]$. It is sufficient to show that $I_0 = [0, 1]$. Clearly $0 \in I_0$.

Let $s \in [0, 1]$. We consider small deformation of holed cone metrics $g_t$ for $s - \epsilon < t < s + \epsilon$ and some $\epsilon > 0$. Fix a basepoint in $X \setminus \Sigma$. We can take a common handle decomposition for $g_t$ with a single 0-handle such that the basepoint lies in the 0-handle. By compositing isometries isotopic to the identity, we may assume that the holonomy representation is constant in the space $\text{Hom}(\Gamma, G)$. Since the holonomy of the longitudes of the cone loci is fixed, the cone loci are isometrically determined.

Hence we may assume that the developing maps have identical restrictions to the preimage of the space $X_0$. By replacing smaller handles, we may assume that the restriction of the developing maps for $g_t$ to the preimage of $X_2$ are contained in that for $g_s$. By taking the quotients, we obtain that the restriction of the metrics $g_t$ to $X_2$ are isometrically contained in $g_s$. Thus each $g_t$ is equivalent to $g_s$.

The argument for $s \in I_0$ implies that $I_0$ is open. Let $t_1 = \sup I_0$. The argument for $s = t_1$ implies that $t_1 = 1 \in I_0$. Therefore $I_0 = [0, 1]$. \hfill \square

We will show that any fiber of the map $\hat{\text{hol}}: HC^{\text{irr}}(X, \Sigma) \to \hat{X}^{\text{cone}}(X, \Sigma)$ is discrete. For this purpose, we introduce the framing invariant for a holed cone structure with respect to a fixed handle decomposition on a holed cone metric of reference.

We focus on immersed annuli in $\mathbb{R}^3$. We use the notation $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$. We say that an immersed annulus $f: S^1 \times [0, 1] \to \mathbb{R}^3$ bounds an immersed 2-handle if there is an immersion of $D^2 \times [0, 1]$ in $\mathbb{R}^3$ whose restriction to $\partial D^2 \times [0, 1]$ coincides with the immersion $f$. Here we fix the normal direction on the annulus.

**Lemma 3.3.** Let $f_t: S^1 \times [0, 1] \to \mathbb{R}^3$ be continuous family of immersions of an annulus for $0 \leq t \leq 1$, i.e. a regular homotopy between immersed annuli $f_0$ and $f_1$. If $f_0$ bounds an immersed 2-handle, there is a continuous family of immersed 2-handles bounded by $f_t$ which starts from this 2-handle.

**Proof.** For manifolds $M$ and $N$, let $\text{Imm}(M, N)$ denote the space of immersions from $M$ to $N$ with the $C^\infty$-topology. The Smale-Hirsch fibration lemma (see [1 Prop.3.8]) implies that $\text{Imm}([D^2 \times [0, 1], \mathbb{R}^3] \to \text{Imm}(\partial D^2 \times [0, 1], \mathbb{R}^3)$ induced by the inclusion $\partial D^2 \hookrightarrow D^2$ is a Serre fibration. Hence the path in the base space can be lifted. \hfill \square

The following lemma is a well-known obstruction by the Smale-Hirsch Theorem. The regular homotopy class of 2-handles with a fixed boundary annulus is unique by $\pi_2(SO(3)) \cong 0$ (cf. the sphere eversion).
Lemma 3.4. It holds that $\pi_0(\text{Imm}(S^1 \times [0,1], \mathbb{R}^3)) \cong \pi_1(\text{SO}(3)) \cong \mathbb{Z}_2$. An immersed annulus bounds an immersed 2-handle if and only if it belongs to the component corresponding to $0 \in \mathbb{Z}_2$. In this case, the regular homotopy class of 2-handles is unique.

Similarly, the following lemma for framings of a 1-handle holds (cf. the belt trick).

Lemma 3.5. Let $\mathcal{I}$ be a space of immersions of the 1-handle $[0,1] \times D^2$ to $\mathbb{R}^3$ such that the ends $\{0\} \times D^2$ and normal directions are fixed pointwise. Then $\pi_0(\mathcal{I}) \cong \pi_1(\text{SO}(3)) \cong \mathbb{Z}_2$. In other words, there are precisely two regular homotopy classes of immersed 1-handles fixing the ends. Moreover, a continuous family of immersions from $\{0\} \times D^2$ to $\mathbb{R}^3$ extends to a continuous family of immersions $[0,1] \times D^2$ to $\mathbb{R}^3$.

We introduce the framing invariant. Fix a holed cone metric $g_0 \in \mathcal{HC}_{\text{irr}}^+ (X, \Sigma)$ and a handle decomposition for $g_0$ satisfying that
- there is a single 0-handle,
- each cone loci is joined to the 0-handle by a single 1-handle, and
- both ends of the other 1-handles are attached to the 0-handle.

Recall that $n$ is the number of cone loci. Let $m$ denote the number of 1-handles both of whose ends are attached to the 0-handle. The space $X_1$ induces $2n + m$ generators of the fundamental group $\Gamma$, and the 2-handles induce relations. Let $g \in \mathcal{HC}_{\text{irr}}^+ (X, \Sigma)$ such that $\text{hol}(g) = \text{hol}(g_0) \in \mathcal{X}_{\text{cone}}^+ (X, \Sigma)$. We define $\text{Fr}(g; g_0) = (x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_m) \in \mathbb{Z}^n \times (\mathbb{Z}_2^{n+1} \times \mathbb{Z}_2^m)$, called the framing of $g$ with respect to $g_0$, as follows. Take a handle decomposition for $g$ which is topologically equivalent to that for $g_0$. Consider the restriction of the developing maps for $g_0$ and $g$ to the space $X_1$. We deform the metric $g$ continuously preserving the holonomy representation. The holed cone structure $[g]$ is not changed by Lemma 3.2. Then we may assume that the restriction of the developing map for $g$ to the 0-handle is equal to that for $g_0$. Moreover, we may assume that the restriction of the developing map for $g$ to the neighborhood $N(\Sigma_i)$ of cone loci $\Sigma_i$ has the same image as that for $g_0$. However, these restrictions may not be homotopic. The developing maps induce a map $f_i : (N(\Sigma_i); g) \to (N(\Sigma_i); g_0)$ for each $i$. Its homotopy class of $\text{dev}_g|_{N(\Sigma_i)}$ is determined by a framing around $\Sigma_i$. Fix a framing around $\Sigma_i$. Let $x_i \in \mathbb{Z}$ denote the twist number of $f_i$ measured by counting the rotation of the image of the framing by $f_i$. Note that the cone angles are positive. The numbers $y_i, z_i \in \mathbb{Z}_2$ are defined as the regular homotopy classes of the restrictions to the developing maps to the 1-handles by Lemma 3.5 where they are equal to zero if $g = g_0$. However, a rotation of the 0-handle corresponding the generator of $\pi_1(\text{SO}(3)) \cong \mathbb{Z}_2$ changes the framings in the way that $(y_1, \ldots, y_n) \mapsto (y_1 + 1, \ldots, y_n + 1)$. Hence we need to take the quotient by this $\mathbb{Z}_2$-action on $\mathbb{Z}_2^n$. Then the framing of $g$ is well-defined in $\mathbb{Z}^n \times (\mathbb{Z}_2^{n+1} / \mathbb{Z}_2) \times \mathbb{Z}_2^m$. Clearly $\text{Fr}(g_0; g_0) = 0$. By the above argument, the framing $\text{Fr}(g; g_0)$ is well defined for the holed cone structure $[g] \in \mathcal{HC}_{\text{irr}}^+ (X, \Sigma)$.

Remark 3.6. If the cone angle at $\Sigma_i$ is equal to zero, there is no arbitrariness of the twist number $x_i$ of the framing around $\Sigma_i$. The frame invariant of such a holed cone structure is defined by setting $x_i = 0$.

Conversely, the framing determines the holed cone structure.

Lemma 3.7. Let $g_0, g, g' \in \mathcal{HC}_{\text{irr}}^+ (X, \Sigma)$ such that $\text{hol}(g_0) = \text{hol}(g) = \text{hol}(g')$. Suppose that $\text{Fr}(g; g_0) = \text{Fr}(g'; g_0)$. Then $[g] = [g']$.

Proof. Fix handle decompositions for $g$ and $g'$ which are topologically equivalent to that of $g_0$. We show that there is an equivariant regular homotopy between $\text{dev}_g|_{X_2}$ and...
and \( \text{dev}_{g'}|_{X_0} \). This corresponds to a continuous deformation of the metrics \( g \) to \( g' \). Then Lemma 3.2 implies that \( [g] = [g'] \).

Since \( \text{Fr}(g; g_0) = \text{Fr}(g'; g_0) \), we may assume that \( \text{dev}_g|_{X_0} = \text{dev}_{g'}|_{X_0} \). Moreover, the restrictions of the developing maps to the 1-handles are regularly homotopic. This regular homotopy extends to the 2-handles by Lemma 3.3. Hence we may assume that the restrictions of the developing maps to the 1-handles are regularly homotopic. Lemma 3.4 implies that \( \text{dev}_g|_{X_2} \) and \( \text{dev}_{g'}|_{X_2} \) are regularly homotopic. \( \square \)

We consider whether elements are realized as the framings of holed cone structures. Let \( g_0 \in \mathcal{H}^{\text{irr}}(X, \Sigma) \). Let \( F(g_0) \) denote the set consisting of \( \text{Fr}(g; g_0) \) for \( g \) with \( \text{hol}(g) = \text{hol}(g_0) \). Lemma 3.7 implies that the set \( F(g_0) \) is identified with the fiber of the map \( \text{hol}: \mathcal{H}^{\text{irr}}(X, \Sigma) \to \hat{X}^{\text{cone}}(X, \Sigma) \) over \( \text{hol}(g_0) \).

**Lemma 3.8.** The set \( F(g_0) \) is a subgroup of \( \mathbb{Z}^n \times (\mathbb{Z}_2^n / \mathbb{Z}_2) \times \mathbb{Z}_2^m \cong \mathbb{Z}^n \times \mathbb{Z}_2^{n+m-1} \). Moreover, the group structure of \( F(g_0) \) does not depend on the choice of a handle decomposition for \( g_0 \). The group \( F(g_0) \) is infinite unless all the cone angles for \( g_0 \) are equal to zero.

**Proof.** For simplicity, we suppose that \( g_0 \in \mathcal{H}^{\text{irr}}_+(X, \Sigma) \). Any element in \( \mathbb{Z}^n \times (\mathbb{Z}_2^n / \mathbb{Z}_2) \times \mathbb{Z}_2^m \) is realized as a framing by a metric on the space \( X_1 \). However, this metric is not extended to the 2-handles in general. The obstruction is given by Lemma 3.3.

Let \( D_1, \ldots, D_r \) denote the 2-handles. We define a map \( f: \mathbb{Z}^n \times (\mathbb{Z}_2^n / \mathbb{Z}_2) \times \mathbb{Z}_2^m \to \mathbb{Z}_2^r \) as follows. For \( 1 \leq i \leq n \), \( 1 \leq j \leq m \), and \( 1 \leq k \leq r \), let \( a_{ki}, b_{ki}, c_{kj} \) denote the intersection numbers of the attaching annulus of \( D_k \) and the meridians of \( N(\Sigma_i) \) or 1-handles. Then \( b_{ki} \) is an even number. For \( \mathbf{x} = (x_1, \ldots, x_n, y_1, \ldots, y_m, z_1, \ldots, z_m) \in \mathbb{Z}^n \times (\mathbb{Z}_2^n / \mathbb{Z}_2) \times \mathbb{Z}_2^m \), we define \( u_k \in \mathbb{Z}_2 \) as \( u_k \equiv \sum_i a_{ki} x_i + \sum_i b_{ki} y_i + \sum_j c_{kj} z_j \equiv \sum_i a_{ki} x_i + \sum_j c_{kj} z_j \). The homomorphism \( f \) is defined by \( f(\mathbf{x}) = (u_1, \ldots, u_r) \). Since the element \( u_k \) represents the twist of the attaching annulus of \( D_k \) measured from that for \( g_0 \), it is the obstruction to extend the metric to \( D_k \). Therefore \( F(g_0) \) is the kernel of \( f \). Thus it is an infinite subgroup of \( \mathbb{Z}^n \times (\mathbb{Z}_2^n / \mathbb{Z}_2) \times \mathbb{Z}_2^m \).

The set \( F(g_0) \) is the fiber of \( \text{hol} \), which is determined as a set. Take another handle decomposition \( X_0 \subset X'_0 \subset X'_1 \subset X_3 = X \) for \( g_0 \). We may assume that \( X_0 \) is the same as above. However, 1-handles in \( X'_1 \) may not be homotopic to those in \( X_1 \). Similarly to the case for ordinary handle decompositions (see [18, Theorem 1.1]), two handle decompositions are connected with a sequence of the following moves and their inverses:

- slide of 1-handles and 2-handles,
- addition of a 2-handle along a null-homotopic annulus, and
- stabilization adding a 1-handle and a 2-handle.

Since slides of handles merely exchange the component of the framing, it does not change the group structure of \( F(g_0) \). Any addition of a 2-handle does not affect the map \( f \). Hence we may assume that the new handle decomposition is obtained by stabilization adding a 1-handle and a 2-handle. Then the obstruction map \( f': \mathbb{Z}^n \times (\mathbb{Z}_2^n / \mathbb{Z}_2) \times \mathbb{Z}_2^m \to \mathbb{Z}_2^{r+1} \) for the new handle decomposition is obtained by \( f'(\mathbf{x}, z_{m+1}) = (f(\mathbf{x}), z_{m+1}) \), where \( \mathbf{x} = (x_1, \ldots, x_n, y_1, \ldots, y_m, z_1, \ldots, z_m) \). Therefore the group structures of \( F(g_0) \) for the two handle decomposition coincide. \( \square \)

Since the set of framings is discrete, the framing invariant is constant under continuous deformation as in the following lemma.
Lemma 3.9. Fix a handle decomposition of \((X, \Sigma)\). Let \(g_t, g'_t \in \widehat{HC}^{irr}(X, \Sigma)\) for \(0 \leq t \leq 1\) be continuous families of metrics on \(X^2\) such that \(\hat{\text{hol}}(g_t) = \hat{\text{hol}}(g'_t)\). Then \(\text{Fr}(g'_t; g_t)\) are constant for \(0 \leq t \leq 1\).

We can deform the structures along representations. Note that the existence of small deformation is a general result (see [13, 15, 16]).

**Lemma 3.10.** Let \([g_0] \in \hat{HC}^{irr}(X, \Sigma)\). Let \(\hat{\rho}_t \in \hat{X}_+^{cone}(X, \Sigma)\) for \(0 \leq t \leq 1\) be a continuous family of representations with positive cone angles such that \(\hat{\text{hol}}(g_0) = \hat{\rho}_0\). Then there exists a unique continuous family of holed cone structures \([g_t] \in \hat{HC}^{irr}(X, \Sigma)\) starting from \([g_0]\) such that \(\hat{\text{hol}}(g_t) = \hat{\rho}_t\).

**Proof.** Take a handle decomposition for \(g_0\). Lemma 3.5 implies that there is a continuous deformation of metrics \(g_t\) on \(X^1\) such that \(\hat{\text{hol}}(g_t) = \hat{\rho}_t\). Note that the cone angles are compatible with the representations by definition of \(\hat{X}_+^{cone}\). The developing maps induce a continuous family of the attaching annuli of the 2-handles. Then Lemma 3.9 implies that the metrics extend to the 2-handles. Thus we obtain a continuous family of holed cone metrics \(g_t \in \hat{HC}^{irr}(X, \Sigma)\) such that \(\hat{\text{hol}}(g_t) = \hat{\rho}_t\). The uniqueness of the structures \([g_t] \in \hat{HC}^{irr}(X, \Sigma)\) follows from Lemmas 3.7 and 3.9.

**Remark 3.11.** The assertion of Lemma 3.10 holds even if \([g_0] \in \hat{HC}^{irr}(X, \Sigma)\), the cone angle at \(\Sigma_t\) for \([g_0]\) is equal to zero, and \(\hat{\rho}_t \in \hat{X}_+^{cone}\) for \(0 < t \leq 1\). In other words, there is a deformation from a cusp to a cone locus along prescribed representations. However, there may not be a deformation from a cone locus to a cusp. Recall Remark 3.6. Let \([g_1] \in \hat{HC}^{irr}(X, \Sigma)\) for \(0 \leq t \leq 1\) be a continuous family of holed cone structures. Suppose that \([g_1] \in \hat{HC}^{irr}(X, \Sigma)\) for \(0 \leq t < 1\), and the cone angle at \(\Sigma_t\) for \([g_1]\) is equal to zero. Let \([g'_t] \in \hat{HC}^{irr}(X, \Sigma)\) for \(0 \leq t < 1\) such that \(\hat{\text{hol}}(g'_t) = \hat{\text{hol}}(g_0)\). Suppose that \(\text{Fr}(g'_t; g_t)\) has a non-zero entry at \(x_t\). Then \([g'_t]\) do not converge at \(t = 1\). For \(t\) near \(1\), the metric \(g'_t\) is highly twisted around \(\Sigma_t\) with respect to \(g_t\).

Finally, we obtain the main theorems.

**Theorem 3.12.** The map \(\hat{\text{hol}}: \hat{HC}^{irr}(X, \Sigma) \to \hat{X}_+^{cone}(X, \Sigma)\) is a regular covering map to each path-connected component of \(\hat{X}_+^{cone}(X, \Sigma)\) that contains an image.

**Proof.** For any \([g_0] \in \hat{HC}^{irr}(X, \Sigma)\), we can take a contractible neighborhood \(U\) of \(\hat{\text{hol}}(g_0)\) in \(\hat{X}_+^{cone}(X, \Sigma)\) by Lemma 3.1. The unique lifting property by Lemma 3.10 implies that there is a section of \(\text{hol}\) on \(U\) containing \([g_0]\). Moreover, Each fiber of \(\text{hol}\) is discrete by Lemma 3.8. Hence the image \(U_0\) of this section is a neighborhood of \([g_0]\). The group \(F(g_0)\) acts on \(\hat{\text{hol}}^{-1}(U)\) by Lemma 3.9. For \(\varphi \in F(g_0)\), let \(U_{\varphi} = \varphi \cdot U_0\) by this action. Then \(\hat{\text{hol}}^{-1}(U)\) is the disjoint union \(\bigsqcup_{\varphi \in F(g_0)} U_{\varphi}\). Hence the map \(\hat{\text{hol}}\) is a regular covering map to each path-connected component of \(\hat{X}_+^{cone}(X, \Sigma)\) containing an image.

**Theorem 3.13.** The deformation space \(\hat{HC}^{irr}(X, \Sigma)\) is Hausdorff.

**Proof.** By the argument in Remark 3.11 there is a space \(\overline{\hat{HC}^{irr}(X, \Sigma)}\) such that the map \(\hat{\text{hol}}: \hat{HC}^{irr}(X, \Sigma) \to \hat{X}_+^{cone}(X, \Sigma)\) extends to a map \(\hat{\text{hol}}: \overline{\hat{HC}^{irr}(X, \Sigma)} \to \hat{X}_+^{cone}(X, \Sigma)\), which is a regular covering map to each path-connected component of \(\hat{X}_+^{cone}(X, \Sigma)\) that contains an image. The space \(\hat{HC}^{irr}(X, \Sigma)\) is an open subset of \(\overline{\hat{HC}^{irr}(X, \Sigma)}\), which is Hausdorff.
Proposition 3.15. There are holed cone structures with a common holonomy representation and attaching a 1-handle, we obtain a holed cone metric $g_0$ on the connected sum $(X \# X', \Sigma \sqcup \Sigma')$. By twisting the 1-handle, we obtain a continuous family of holed cone metrics $g_t$ on $(X \# X', \Sigma \sqcup \Sigma')$ for $0 \leq t \leq 1$ such that $\text{hol}(g_0) = \text{hol}(g_1)$ and $\text{Fr}(g_1; g_0) \neq 0$. Nonetheless, the question remains considerable in the case that $X \setminus \Sigma$ admits a hyperbolic structure.

The map $\text{hol}: \mathcal{HC}^{\text{irr}}(X, \Sigma) \to \hat{X}^{\text{cone}}(X, \Sigma)$ is not injective by Lemma 3.8. Moreover, there are holed cone structures with a common holonomy representation and different cone angles.

Proposition 3.15. Let $[g_0] \in \mathcal{HC}^{\text{irr}}(X, \Sigma)$ with cone angles $\theta_i > 0$ at $\Sigma_i$. Suppose that $\theta_i' = \theta_i + 4\pi n_i > 0$ for some $n_i \in \mathbb{Z}$. Then there is $[g] \in \mathcal{HC}^{\text{irr}}(X, \Sigma)$ with cone angles $\theta_i'$ at $\Sigma_i$ such that $\text{hol}(g) = \text{hol}(g_0)$.

Proof. Take a handle decomposition for $g_0$. There is a metric $g$ on $X_1$ with cone angles $\theta_i'$ at $\Sigma_i$ which are the same as $g_0$ outside the neighborhoods of the cone loci. Then the attaching annulus of 2-handles for $g$ are obtained by adding an even number of twists (by $2\pi n_i$-rotations) to that for $g_0$. Since their framing obstruction in Lemma 3.4 vanishes, the metric $g$ extends to $X$. We obtain $[g] \in \mathcal{HC}^{\text{irr}}(X, \Sigma)$ with cone angles $\theta_i'$ at $\Sigma_i$ such that $\text{hol}(g) = \text{hol}(g_0)$. □

Remark 3.16. The map $\text{hol}: \mathcal{HC}^{\text{irr}}(X, \Sigma) \to \hat{X}^{\text{cone}}(X, \Sigma)$ is not surjective in general.

Since an oriented 3-manifold is parallelizable, any (possibly incomplete) metric on it can be lifted to a spin structure. For a hyperbolic metric $g$ on an oriented 3-manifold $M$, the holonomy representation $\rho_g: \pi_1(M) \to \text{PSL}(2, \mathbb{C})$ has a lift to $\tilde{\rho}_g: \pi_1(M) \to \text{SL}(2, \mathbb{C})$, which is the holonomy representation of a spin structure (see [7]). For a fixed metric $g$, there is a one-to-one correspondence between the spin structures over $g$ and the group $H^1(\Gamma, \mathbb{Z}_2) = \text{Hom}(\Gamma, \mathbb{Z}_2)$. Due to Heusener and Porti [12, Theorem 1.4], however, there exists an oriented 3-manifold $M$ with boundary consisting of a torus such that there are components of the character variety $\mathcal{X}(\Gamma)$ whose elements do not lift to $\text{SL}(2, \mathbb{C})$. An element in $\hat{X}^{\text{cone}}(X, \Sigma)$ with a representation in such a component is not contained in the image of $\text{hol}$.

For instance, let $M$ be a bundle over $S^1$ whose fiber is a once-punctured torus $S$. Suppose that $M$ is obtained by the action of the monodromy on $H_1(S, \mathbb{Z})$ given by the matrix \( \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} \). Then there are presentations $\pi_1(S) = \langle \alpha, \beta \rangle$ and $\pi_1(M) = \langle \alpha, \beta, \mu \mid \mu \alpha \mu^{-1} = \alpha, \mu \beta \mu^{-1} = \beta, \alpha \beta^2 \rangle$. Let

\[
A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in \text{PSL}(2, \mathbb{C}).
\]

By setting $\rho(\alpha) = A$, $\rho(\beta) = B$, and $\rho(\mu) = I$, we obtain a representation $\rho: \pi_1(M) \to \text{PSL}(2, \mathbb{C})$, which does not lift to $\text{SL}(2, \mathbb{C})$. The element $\alpha \beta \alpha^{-1} \beta^{-1}$ is peripheral in $S$. Its image $ABA^{-1}B^{-1} = \begin{pmatrix} -3 & -2 \\ -4 & -3 \end{pmatrix}$ is loxodromic. Let $X$ be the manifold obtained by the Dehn filling on $M$ along the slope corresponding to $\mu$. Let $\Sigma$ be the core of the attached solid torus. Then $([\rho], 2\pi) \in \hat{X}^{\text{cone}}(X, \Sigma)$.

Question 3.17. Let $\hat{\rho} = (\rho, \theta_1, \ldots, \theta_n) \in \hat{X}^{\text{cone}}(X, \Sigma)$. Suppose that $\rho$ has a lift to a representation to $\text{SL}(2, \mathbb{C})$. Then is $\hat{\rho}$ contained in the image of the map $\text{hol}: \mathcal{HC}^{\text{irr}}(X, \Sigma) \to \hat{X}^{\text{cone}}(X, \Sigma)$?
4. Cone structures in the deformation space

In this section, we consider (non-holed hyperbolic) cone structures in the space of holed cone structures. Recall that a cone structure is an equivalence class of cone metrics, where the equivalence relation is induced by isometry isotopic to the identity. Suppose that there are cone structures on $(X, \Sigma)$. By Kojima [19], this is equivalent to that $X \setminus \Sigma$ admits a hyperbolic structure. In particular, the 3-manifold $X \setminus \Sigma$ is irreducible.

**Theorem 4.1.** Let $g$ and $g'$ be cone metrics on $(X, \Sigma)$. Suppose that there are cone structures on $(X_0 \setminus \Sigma; g)$ and $(X_0 \setminus \Sigma; g')$. By deforming $g'$ by isotopy, we extend this isometry to $X_1 \setminus \Sigma$. Then $g$ and $g'$ are equivalent as cone metrics.

To show the theorem, we need the following lemmas. Take a handle decomposition $X_0 \subset X_1 \subset X_2 \subset X_3 = X$ of $(X, \Sigma)$.

**Lemma 4.3.** Suppose that there is an isometry $\iota$ from $(X_1 \setminus \Sigma; g)$ to $(X_1 \setminus \Sigma; g')$, then $\iota$ extends to an isometry from $(X_1 \setminus \Sigma; g)$ to $(X \setminus \Sigma; g)$.

**Proof.** Let $M = X \setminus \Sigma$ and $M_1 = X_1 \setminus \Sigma$. Let $\widetilde{M}$ denote the universal covering of $M$, and let $\widetilde{M}_1 \subset \widetilde{M}$ denote the preimage of $M_1 \subset M$. Since the natural homomorphism $\pi_1(M_1) \to \pi_1(X \setminus \Sigma)$ is surjective, the space $\widetilde{M}_1$ is simply connected. The metrics $g$ and $g'$ induce the holonomy representations $\rho_g$ and $\rho_{g'}$. We have $\rho_{g'} = \rho_g \circ \tau$. The developing maps $\text{dev}_g, \text{dev}_{g'}: M \to \mathbb{H}^3$ satisfies that $\text{dev}_{g'}|_{\widetilde{M}_1} = \text{dev}_g|_{\widetilde{M}_1} \circ \iota$, where $\iota: (\widetilde{M}_1; g) \to (\widetilde{M}_1; g')$ is the isometry induced by $\iota$. We extend $\iota$ to an isometry from $(\widetilde{M}; g)$ to $(\widetilde{M}; g')$ by connecting local isometries along a path from $\widetilde{M}_1$. Since $\widetilde{M}$ is simply connected, the extension is well-defined, that is, it does not depend on the ways of taking paths. The subset of $\widetilde{M}$ consisting of the points to which the isometry extends is closed and open. Hence we can extend $\iota$ to an isometry from whole of $\widetilde{M}$. The uniqueness of the extension implies that the isometry $\iota: (M; g) \to (M; g')$ is equivariant with respect to $\rho_g$ and $\rho_{g'}$. By taking the quotient, we obtain an isometry from $(M; g)$ to $(M; g')$ which is an extension of $\iota$. 

**Lemma 4.4.** Suppose that there is an isometry between cone metrics $(X, \Sigma; g)$ and $(X, \Sigma; g')$ which induces the identity of the fundamental group. Then $g$ and $g'$ are equivalent as cone metrics.

**Proof.** Let $\text{Diff}(X \setminus \Sigma)$ denote the space of diffeomorphisms of $X \setminus \Sigma$ with the $C^\infty$ topology. Let $\text{Out}(\Gamma)$ denote the outer automorphism group of $\Gamma = \pi_1(X \setminus \Sigma)$. The action of diffeomorphisms on the fundamental group $\Gamma$ induces the natural map $j: \text{Diff}(X \setminus \Sigma) \to \text{Out}(\Gamma)$. Since $X \setminus \Sigma$ admits a hyperbolic structure, the map $j$ is a homotopy equivalence [11, 15]. In particular, the subspace of $\text{Diff}(X \setminus \Sigma)$ consisting the elements mapped to $1 \in \text{Out}(\Gamma)$ is path-connected. Hence the isometry is isotopic to the identity.

**Proof of Theorem 4.1.** Since $\widetilde{\text{hol}}(g) = \widetilde{\text{hol}}(g')$, we may assume that there is an isometry $(X_0 \setminus \Sigma; g)$ to $(X_0 \setminus \Sigma; g')$. By deforming $g'$ by isotopy, we extend this isometry to $X_1 \setminus \Sigma$. Then we obtain an isometry from $(X_1 \setminus \Sigma; g)$ to $(X_1 \setminus \Sigma; g')$. Hence there is an isometry from $(X \setminus \Sigma; g)$ to $(X \setminus \Sigma; g')$ by Lemma 4.2. Since $\widetilde{\text{hol}}(g) = \widetilde{\text{hol}}(g')$, this isometry induces the identity of the fundamental group. Hence $g$ and $g'$ are equivalent as cone metrics by Lemma 4.3.

**Corollary 4.4.** Two cone metrics $g$ and $g'$ on $(X, \Sigma)$ are equivalent as holed cone metrics if and only if they are equivalent as cone metrics.
Corollary 4.4 implies that we may regard $\hat{C}$ by Theorem 4.1. □

Then the map $\hat{\text{hol}} : C(X, \Sigma) \to \hat{X}^{\text{cone}}(\Gamma)$ is injective.

**Question 4.5.** Is the map $\text{hol} : C(X, \Sigma) \to \hat{X}^{\text{cone}}(\Gamma)$ injective? In other words, are there cone-manifolds with a common holonomy representation and distinct cone angles?

The generalized hyperbolic Dehn surgery around the cusped hyperbolic structure $[g] \in \hat{C}(X, \Sigma)$ induces a neighborhood of $\hat{X}^{\text{cone}}(\Gamma)$ of the discrete faithful representation, which is parametrized by the small cone angles.

Thus a holed cone structures is a generalization of a cone structure. To consider how much the generalization is meaningful, we ask the following questions.

**Question 4.6.** Can a cone structure in $C(X, \Sigma)$ be deformed to the cusped hyperbolic structure on $X \setminus \Sigma$ via holed cone structures? Equivalently, is $C(X, \Sigma)$ contained in a single component of $\hat{C}^{\text{int}}(X, \Sigma)$?

5. Volumes of holed cone structures

In this section, we introduce the volume of a holed cone structure. To obtain a well-defined value, we add the volume enclosed by the holes to the volume of a holed cone metric.

Let $f : S^2 \to \mathbb{H}^3$ be an immersion. Fix a normal orientation on $f(S^2)$. We define the signed enclosed volume $V(f) \in \mathbb{R}$ as follows. Suppose that $f$ is generic. Let $R_0, R_1, \ldots, R_N$ denote the connected components of the complement $\mathbb{H}^3 \setminus f(S^2)$, where $R_0$ is the unbounded component. Each component $R_i$ is assigned an integer $\text{deg}(R_i)$ so that

- $\text{deg}(R_0) = 0$, and
- $\text{deg}(R_i) = \text{deg}(R_j) + 1$ if $R_i$ and $R_j$ are adjacent and the normal orientation on $f(S^2)$ has direction from $R_i$ to $R_j$.

Then we define $V(f) = \sum_{i=1}^N \text{deg}(R_i) \text{vol}(R_i)$. The function $V : \text{Imm}(S^2, \mathbb{H}^3) \to \mathbb{R}$ is defined by continuous extension. Recall that $\text{Imm}(S^2, \mathbb{H}^3)$ is the space of immersions from $S^2$ to $\mathbb{H}^3$ with the $C^\infty$ topology. The following lemma verifies the definition.

**Lemma 5.1.** Let $F$ be a smooth map from the 3-ball $B^3$ to $\mathbb{H}^3$ whose restriction to $\partial B^3 = S^2$ is a generic immersion $f \in \text{Imm}(S^2, \mathbb{H}^3)$. The orientation on $B^3$ induces the normal inward orientation on $f(S^2)$. Write $\mathbb{H}^3 \setminus f(S^2) = R_0 \cup R_1 \cup \cdots \cup R_N$ as above. Then the map $F$ has the mapping degree $\text{deg}(R_i)$ on each $R_i$ for $0 \leq i \leq N$. Moreover, we have $\int_{B^3} F^* \omega = V(f)$, where $\omega$ is the volume form on $\mathbb{H}^3$.

**Proof.** Note that any immersion $f$ has such an extension $F$ since the space $\mathbb{H}^3$ is contractible. The mapping degree of $F$ on a regular value $p$ in $\mathbb{H}^3 \setminus f(S^2)$ is defined as the sum of $\pm 1$’s over $F^{-1}(p)$, where the sign is positive if and only if $F$
preserves the orientation around the point. The mapping degree of $F$ is constant on each $R_i$. Since the image of $F$ is bounded, we have $\deg(R_0) = 0$. The choice of the orientation implies that the mapping degrees of $F$ satisfy the same relation for $\deg(R_i)$. Hence they coincide. The definition of induced form implies that

$$\int_{B^3} F^* \omega = \sum_{i=1}^{N} \int_{R_i} \deg(R_i) \omega = \sum_{i=1}^{N} \deg(R_i) \text{vol}(R_i) = V(f).$$

An explicit formula for $V(f)$ may be useful for differential geometric approaches.

**Question 5.2.** Is there an explicit formula for $V(f)$? Is it possible to express it by the curvatures of $f$?

Let $g \in \overline{HC}^{\text{irr}}(X, \Sigma)$ and $\sigma = [g] \in \overline{HC}^{\text{irr}}(X, \Sigma)$. Let $B_1, \ldots, B_m$ denote the holes of $g$. For $1 \leq i \leq m$, define an immersion $f_i: S^2 \to \mathbb{H}^3$ as the restriction of the developing map $\text{dev}_g$ to a lift of $\partial B_i$. Fix the normal orientation of $f_i(S^2)$ towards the interior of $B_i$. We define the volume of the holed cone structure $\sigma$ as $\text{vol}(\sigma) = \text{vol}(g) + \sum_{i=1}^{m} V(f_i)$, where $\text{vol}(g)$ is the volume of the holed cone metric $g$ as a Riemannian metric. Let $g' \in \overline{HC}^{\text{irr}}(X, \Sigma)$ be a restriction of $g$ with holes $B'_1, \ldots, B'_m$. Let $f'_j$ be immersions for $B'_j$ as above. Then $\text{vol}(g) - \text{vol}(g') = \sum_{j=1}^{m} V(f'_j) - \sum_{i=1}^{m} V(f_i)$ by the definition of the signed enclosed volume. Hence $\text{vol}(\sigma)$ is well defined.

Let $\rho: \Gamma = \pi_1(X \setminus \Sigma) \to \text{PSL}(2, \mathbb{C})$ be a representation. Due to Dunfield [9], the volume of the representation $\rho$ is defined as follows. Let $\hat{M} \to M = X / \Sigma$ denote the universal covering. Let $\hat{M}$ and $\hat{M}$ respectively denote the compactifications of $M$ and $\hat{M}$ such that each end is compactified by adding one point. The added points are called ideal points. Let $\overline{\mathbb{H}^3} = \mathbb{H}^3 \cup \partial \mathbb{H}^3$ denote the natural compactification of $\mathbb{H}^3$ by adding the sphere at infinity.

The singular locus $\Sigma_i$ corresponds to an ideal point $v_i$ of $\hat{M}$. We fix a product structure of a neighborhood $N_i$ of $v_i$ which is obtained from $T_i \times [0, \infty]$ by collapsing $T_i \times \{\infty\}$ to $v_i$. We lift this product structure to the ends of $\hat{M}$. Then an ideal point $\hat{v}$ of $\hat{M}$ has a neighborhood $\hat{N}_{\hat{v}} = P_0 \times [0, \infty]/(P_0 \times \{\infty\})$, where $P_0$ covers a torus $T_i$ for some $i$. For a set $A$, let $\check{A} = A \times [c, \infty]/(A \times \{\infty\})$ by collapsing $A \times \{\infty\}$ to a point $\infty$. A map $f: C \to \overline{\mathbb{H}^3}$ is a cone map if

- $f(C) \cap \partial \mathbb{H}^3 = \{f(\infty)\}$, and
- for any $a \in A$, the map $f|_{a \times [c, \infty]}$ is the geodesic ray from $f(a, c)$ to $f(\infty)$ parametrized by the arc length.

A pseudo-developing map for $\rho$ is a piecewise smooth map $D_\rho: \hat{M} \to \overline{\mathbb{H}^3}$ satisfying the following conditions:

- $D_\rho$ is equivariant for the $\Gamma$-actions, which are the deck transformation on $\hat{M}$ and the action on $\overline{\mathbb{H}^3}$ via $\rho$.
- $D_\rho$ extends continuously to a map $\hat{D}_\rho: \hat{M} \to \overline{\mathbb{H}^3}$.
- There exists $c \geq 0$ such that the restriction of $\hat{D}_\rho$ to each end $P_0 \times [c, \infty]/(P_0 \times \{\infty\})$ is a cone map.

Let $\hat{v}_i$ an ideal point of $\hat{M}$ which is a lift of $v_i \in \hat{M}$. Recall that $\mu_i, \lambda_i \in \Gamma$ are commuting elements corresponding to the meridian and the longitude for $\Sigma_i$. The stabilizer $\text{Stab}(\hat{v}_i) < \Gamma$ of $\hat{v}_i$ is generated by conjugates of $\mu_i$ and $\lambda_i$. The fixed point set $\text{Fix}(\rho(\text{Stab}(\hat{v}_i))) \subset \partial \overline{\mathbb{H}^3}$ consists of one or two points which depends on
whether the cone angle at \( \Sigma_i \) is positive. If it is positive, \( \text{Fix}(\rho(\text{Stab}(\hat{v}_i))) \) consists of the endpoints of a lift of \( \Sigma_i \). Then \( \hat{D}_{\rho}(\hat{v}_i) \) is contained in \( \text{Fix}(\rho(\text{Stab}(\hat{v}_i))) \).

Let \( \omega \) denote the volume form of \( \mathbb{H}^3 \). Since the 3-form \( D_{\rho}^* \omega \) on \( \hat{M} \) is equivariant, it projects to a 3-form (also denoted by \( D_{\rho}^* \omega \)) on \( M \). Define the volume of the representation \( \rho \) as \( \text{vol}(\rho) = \int_M D_{\rho}^* \omega \). Francaviglia [10] proved that \( \text{vol}(\rho) \) does not depend on the choice of a pseudo-developing map. We refer Kim [17] for various equivalent definitions of the volume of a representation. We show that the volume of a holed cone structure coincides with the volume of the holonomy representation. The assertion for cone-manifolds was shown by Porti [20].

**Theorem 5.3.** Let \( \sigma \in \mathcal{HC}^{\text{irr}}(X, \Sigma) \). Then \( \text{vol}(\sigma) = \text{vol}(\text{hol}(\sigma)) \).

**Proof.** Take \( g \in \mathcal{HC}^{\text{irr}}(X, \Sigma) \) such that \([g] = \sigma\). Let \( \rho = \text{hol}(\sigma) \). For simplicity, we assume that the hole \( B \) of \( g \) is a single ball. Let \( f \) be the restriction of \( \text{dev}_g \) to a lift of \( \partial B \). We extend \( f \) to \( F: B \to \mathbb{H}^3 \) as in Lemma 5.1.

Let \( N(\Sigma_i) \) be a regular neighborhood of \( \Sigma_i \), which is a standard tube or cusp with respect to \( g \). Let \( \tilde{N}_i \) denote the one-point compactification of the universal cover of \( N(\Sigma_i) \setminus \Sigma_i \). The restriction \( \text{dev}_g \) to the preimage of \( \partial N(\Sigma_i) \) extends to a cone map from \( \tilde{N}_i \), obtained by joining each point of the boundary to a point in \( \text{Fix}(\rho(\text{Stab}(\hat{v}_i))) \). This cone map is equivariant and projected to a self-homeomorphism on \( N(\Sigma_i) \setminus \Sigma_i \).

We construct a pseudo-developing map \( D_{\rho} \) by combining the restriction \( \text{dev}_g \) to the preimage of \( X \setminus (\bigcup_i N(\Sigma_i) \cup B) \), the cone maps from \( \tilde{N}_i \), and copies of \( F \). Then we have \( \int_{X \setminus B} D_{\rho}^* \omega = \text{vol}(g) \) and \( \int_B D_{\rho}^* \omega = V(f) \) by Lemma 5.1. Hence \( \text{vol}(\rho) = \int_{X \setminus \Sigma} D_{\rho}^* \omega = \text{vol}(g) + V(f) = \text{vol}(\sigma) \). \( \square \)

The Schlafli formula for holed cone structures is expressed as follows. This also follows from Hodgson’s form for representations [13].

**Proposition 5.4.** Let \( \sigma_i \in \mathcal{HC}^{\text{irr}}(X, \Sigma) \) be a continuous family for \( -\epsilon < t < \epsilon \). Then

\[
\frac{d}{dt} \text{vol}(\sigma_i) = -\frac{1}{2} \sum_{i=1}^n \ell_i(t) \frac{d}{dt} \theta_i(t),
\]

where \( \ell_i(t) \) is the length of \( \Sigma_i \) for \( \sigma_i \) and \( \theta_i(t) \) is the cone angle at \( \Sigma_i \) for \( \sigma_i \).

**Proof.** There are continuous family \( g_t \in \mathcal{HC}^{\text{irr}}(X, \Sigma) \) for \( -\epsilon < t < \epsilon \) with a single hole \( B \) such that \( \sigma_t = [g_t] \). By taking sufficiently small \( \epsilon \), we may assume that the restrictions of \( \text{dev}_{g_t} \) to each lift of \( \partial B \) are deformed by isometries of \( \mathbb{H}^3 \). Then

\[
\frac{d}{dt} \text{vol}(\sigma_t) = \frac{d}{dt} \text{vol}(g_t).
\]

Take a triangulation of \( X \setminus \text{int}(B) \) so that \( \Sigma \) is contained in the 1-skeleton. We may assume that the simplices adjacent to \( \partial B \) have constant metrics for \( t \), and the other simplices are geodesic. The formula for a hyperbolic tetrahedron is expressed by the lengths of edges and dihedral angles in the same form as the assertion (see [20]). The assertion follows by taking the sum of terms for the tetrahedra. Since the sum of dihedral angles about each internal edge is \( 2\pi \), the corresponding terms in the formula cancel. \( \square \)

6. Example

Let \( L = L_1 \sqcup \cdots \sqcup L_4 \) be a link in \( X = T^2 \times I \) as indicated on the left of Figure 2, where \( I \) is an open interval. This is a quotient of the “plain weave.” The author [24] described hyperbolic cone structures on \( (X, L) \), and gave an example of
their degeneration with decreasing cone angles. Cone loci meet in this degeneration. Nonetheless, holed cone structures enable us to avoid such degeneration.

\[ L_4 \]
\[ L_1 \]
\[ L_3 \]
\[ L_2 \]
\[ I \]
\[ ^L_1 \]
\[ ^L_2 \]
\[ ^L_3 \]
\[ ^L_4 \]
\[ ^L_4 \]
\[ ^L_1 \]
\[ ^L_3 \]
\[ ^L_2 \]

Figure 2. Decomposition of \((X, L)\) into trapezohedra

The space \((X, L)\) is decomposed into four polyhedra, called \((tetragonal)\) trapezohedra, as indicated in Figure 2. Conversely, we can construct a hyperbolic cone structure on \((X, L)\) by gluing four copies of a hyperbolic trapezohedron. For each \(i = 1, \ldots, 4\), the cone locus \(L_i\) derives from the edge \(\hat{L}_i\). Suppose that the dihedral angle at \(\hat{L}_i\) is \(\alpha_i\). The dihedral angle at any other edge needs to be a right angle. If \(\alpha_i = 0\), then \(L_i\) degenerates to an ideal vertex. Nonetheless, we continue to call it a trapezohedron. Then the cone angle at \(L_i\) is \(2\alpha_i\). It is not known whether every cone structure on \((X, L)\) is obtained by this construction. However, this is true if the global rigidity for the cone structures on \((X, L)\) holds.

Extending this, we construct a holed hyperbolic cone metric on \((X, L)\) by gluing “holed trapezohedra.” As a holed trapezohedron, we consider the complement of holes in a trapezohedron endowed with a hyperbolic metric such that

- each hole is disjoint from the edges \(\hat{L}_i\) and the vertices,
- each face is totally geodesic, and
- the boundary of holes is orthogonal to faces.

A holed trapezohedron is isometrically immersed in \(\mathbb{H}^3\) as indicated in Figure 3. The boundary of a hole is drawn in blue.

Figure 3. A holed trapezohedron

Let us consider a holed trapezohedron such that the cone angles at the edges other than \(L_i\) are right angles. We can glue four copies of such a holed trapezohedron in the same manner as for non-holed ones. Then we obtain a holed cone metric on \((X, L)\).
According the description in [23], let

\[ B = \{(q_1, \ldots, q_4, t) \in \mathbb{R}^4_{>0} \times \mathbb{R}_{\geq 0} \mid \prod_{i=1}^{4} q_i = 1, \ t \geq \frac{1}{2}(q_i - q_i^{-1})\}, \]

\[ B_0 = \{(q_1, \ldots, q_4, t) \in B \mid (1 - q_i q_{i+1})t < q_i + q_{i+1}\} \]

with the indices \( i = 1, \ldots, 4 \) modulo 4. Define \( f : B \to \mathbb{R}^4 \) by

\[ f(q_1, \ldots, q_4, t) = \left( \frac{q_1 - t}{\sqrt{1 + t^2}}, \ldots, \frac{q_4 - t}{\sqrt{1 + t^2}} \right). \]

Then \( f \) is a homeomorphism onto \((-1, 1)^4\). There exists a tetrahedron such that the cone angle at \( L_i \) is \( \alpha_i \) and the cone angle at any other edge is a right angle if and only if \((\cos \alpha_1, \ldots, \cos \alpha_4) \in f(B_0)\).

Let \((q_1, \ldots, q_4, t) \in B\). Let \( \alpha_i \in [0, \pi) \) satisfy \( \cos \alpha_i = \frac{q_i - t}{\sqrt{1 + t^2}} \). We show that \((q_1, \ldots, q_4, t)\) corresponds to a holed trapezohedron even if \((1 - q_i q_{i+1})t < q_i + q_{i+1}\) for some \( i \). Because the construction is consistent with that in [23], we briefly sketch it.

We use the upper half-space model of \( \mathbb{H}^3 \). Regard \( \partial \mathbb{H}^3 = \mathbb{R}^2 \cup \{\infty\} \). Let \( p_i > 0 \) satisfy \( q_i = \frac{p_i + 1}{p_i} \). Let

\[ P_1 = (p_1, p_2), P_2 = (-p_3, p_2), P_3 = (-p_3, -p_4), P_4 = (p_1, -p_4), \]

\[ R_1 = (p_1, t_1), R_2 = (-t_2, p_2), R_3 = (-p_3, -t_3), R_4 = (t_4, -p_4). \]

Let \( C_i \) denote the circle with the center \( R_i \) which contains \( O = (0, 0) \). Let \( S_i \) denote the intersectional point of \( C_i \) and \( C_{i+1} \) other than \( O \). Let \( Q_i \) denote the intersectional point of the lines \( OS_i \) and \( P_i P_{i+1} \). The point \( Q_i \) may not be contained in the interior of the segment \( P_i P_{i+1} \).
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Let $\tilde{C}_i$ denote the totally geodesic plane in $\mathbb{H}^3$ bounded by $C_i$. Let $\tilde{P}_i = \infty P_i \cap \tilde{C}_i$, $\tilde{Q}_i = \infty Q_i \cap \tilde{C}_i$ are the vertices of holed polyhedron. The edge $\tilde{L}_i$ is the geodesic path between $\tilde{P}_i$ and $\tilde{Q}_i$. If $Q_i$ is not contained in the interior of the segment $P_i P_{i+1}$, we make a hole between $\tilde{Q}_i$ and $\tilde{P}_{i+1}$ as indicated in Figure 4. Then the faces $\infty \tilde{P}_i \tilde{Q}_i P_{i+1}$ and $\infty \tilde{Q}_{i+1} \tilde{P}_{i+1} \tilde{Q}_i$ are “holed polygons.” Thus $(q_1, \ldots, q_t, t) \in B$ corresponds to a holed trapezohedron with the desired dihedral angles. Hence any 4-tuple of the angles $\alpha_i$ in $[0, \pi)$ is realized. Moreover, the holed polyhedra depend continuously on $\alpha_i$.

Let $H_{\text{sym}}(X, L)$ denote the set of holed cone structures obtained by the above construction. Then we obtain the following theorem.

**Theorem 6.1.** There exists a subspace $H_{\text{sym}}^c(X, L) \subset H_{\text{sym}}(X, L)$ such that the map $\Theta : H_{\text{sym}}^c(X, L) \to [0, 2\pi)^4$ assigning the cone angles is a homeomorphism. In particular, any 4-tuple of cone angles in $[0, 2\pi)$ is realized by a holed cone structure on $(X, L)$.

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