ON ENDOMORPHISM RINGS AND DIMENSIONS OF LOCAL COHOMOLOGY MODULES

PETER SCHENZEL

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ABSTRACT. Let \((R, \mathfrak{m})\) denote an \(n\)-dimensional complete local Gorenstein ring. For an ideal \(I\) of \(R\) let \(H^i_I(R), i \in \mathbb{Z}\), denote the local cohomology modules of \(R\) with respect to \(I\). If \(H^i_I(R) = 0\) for all \(i \neq c = \text{height} I\), then the endomorphism ring of \(H^c_I(R)\) is isomorphic to \(R\). Here we prove that this is true if and only if \(H^i_I(R) = 0\) for \(i = n, n-1\), provided \(c \geq 2\) and \(R/I\) has an isolated singularity, resp. if \(I\) is set-theoretically a complete intersection in codimension at most one. Moreover, there is a vanishing result of \(H^i_I(R)\) for all \(i > m, m\) a given integer, and an estimate of the dimension of \(H^i_I(R)\).

1. Main results

Let \((R, \mathfrak{m})\) denote a local Noetherian ring with \(n = \dim R\). For the ideal \(I \subset R\) let \(H^i_I(\cdot), i \in \mathbb{Z}\), denote the local cohomology functor with respect to \(I\); see [2] for its definition and basic results. It is a difficult question to describe \(\sup\{i \in \mathbb{Z} | H^i_I(R) \neq 0\}\), the cohomological dimension \(\text{cd} I\) of \(I\) with respect to \(R\). Recall that \(\text{height} I \leq \text{cd} I\). Recently some interesting results for ideals with \(\text{height} I = \text{cd} I\), the so-called cohomologically complete intersections have been proved. If \((R, \mathfrak{m})\) is a complete local ring, Hellus and Stückrad [5] have shown that the endomorphism ring \(\text{Hom}_R(H^c_I(R), H^c_I(R))\) is isomorphic to \(R\). See also [4, Lemma 2.8] for a more functorial proof and a slight extension in the case where \((R, \mathfrak{m})\) is a Gorenstein ring.

The first aim of the consideration here is a characterization of when the endomorphism ring of \(H^c_I(R)\) is isomorphic to \(R\). To this end we call \(I\) locally a cohomologically complete intersection provided \(\text{cd} IR_p = \text{height} I\) for all prime ideals \(p \in V(I) \setminus \{\mathfrak{m}\}\). For instance, if \(I\) has an isolated singularity, it is locally a cohomologically complete intersection.

**Theorem 1.1.** Let \((R, \mathfrak{m})\) denote a complete local Gorenstein ring with \(n = \dim R\). Let \(I\) be an ideal of height \(I = c\). Suppose that \(I\) is locally a cohomologically complete intersection. Then the following conditions are equivalent:

(i) The natural homomorphism \(R \to \text{Hom}_R(H^c_I(R), H^c_I(R))\) is an isomorphism.

(ii) \(H^i_I(R) = 0\) for \(i = n - 1, n\).
In a certain sense, condition (ii) of Theorem 1.1 provides a numerical condition for the property that the endomorphism ring of $H^j_I(R)$ is $R$. In the case of $R$ being a regular local ring containing a field, Huneke and Lyubeznik [6, Theorem 2.9] have given a topological characterization of the above condition (ii).

**Theorem 1.2.** Let $(R, m)$ be a local Gorenstein ring. Let $J \subset I$ denote two ideals of height $c$.

(a) There is a natural homomorphism

$$\text{Hom}_R(H^j_I(R), H^j_I(R)) \to \text{Hom}_R(H^j_I(R), H^j_I(R)).$$

(b) Suppose that $\text{Rad} JR_p = \text{Rad} IR_p$ for all $p \in V(I)$ with $\dim R_p \leq c + 1$. Then the homomorphism in (a) is an isomorphism.

(c) Let $R$ be in addition complete. Let $J$ denote a cohomologically complete intersection contained in $I$ and satisfying the assumptions of (b). Then $R \to \text{Hom}_R(H^j_I(R), H^j_I(R))$ is an isomorphism.

Our results are based on a certain estimate of $\dim H^j_I(R)$, $i > c$; see Theorem 3.1. In the case of a regular local ring, partial results of this type have been used by Ken-Ichiro Kawasaki [7] for the study of Lyubeznik numbers (see [8] for their definition). Here we use the truncation complex as invented in [4, Section 2] (see Definition 2.1). Moreover it provides some technical statements about the endomorphism ring of $H^j_I(R)$, $c = \text{height} I$; see Lemma 2.2.

In terminology the author follows the paper [4].

## 2. The truncation complex

Let $(R, m, k)$ denote a local Gorenstein ring with $n = \dim R$. First of all we will recall the truncation complex as it was introduced in [4, Section 2] and in a different context in [9, §4]. Let $R \to E'$ denote a minimal injective resolution of $R$ as an $R$-module. It is a well-known fact that

$$E^i \simeq \bigoplus_{p \in \text{Spec } R, \text{height } p = i} E_R(R/p),$$

where $E_R(R/p)$ denotes the injective hull of $R/p$ (see [1] for these and related results about Gorenstein rings).

Now let $I \subset R$ denote an ideal and let $c = \text{height } I$. Then $d = \dim R/I = n - c$. The local cohomology modules $H^j_I(R), i \in \mathbb{Z}$, are, by definition, the cohomology modules of the complex $\Gamma_I(E')$. Because $\Gamma_I(E_R(R/p)) = 0$ for all $p \notin V(I)$, it follows that $\Gamma_I(E')^i = 0$ for all $i < c$. Therefore $H^j_I(R) = \text{Ker}(\Gamma_I(E')^c \to \Gamma_I(E')^{c+1})$. This observation provides an embedding $H^j_I(R)[-c] \to \Gamma_I(E')$ of complexes of $R$-modules.

**Definition 2.1.** The cokernel of the embedding $H^j_I(R)[-c] \to \Gamma_I(E')$ is defined as $C^*_R(I)$, the truncation complex with respect to $I$. So there is a short exact sequence of complexes of $R$-modules

$$0 \to H^j_I(R)[-c] \to \Gamma_I(E') \to C^*_R(I) \to 0.$$

In particular it follows that $H^i(C^*_R(I)) = 0$ for $i < c$ or $i > n$ and $H^i(C^*_R(I)) \simeq H^i_I(R)$ for $c < i \leq n$.

First we need to establish some basic results about the truncation complex. For more details we refer to the exposition in [4, Section 2].
Lemma 2.2. With the previous notation, the following results hold:

(a) There exist an exact sequence

\[ 0 \to H_m^{n-1}(C_R(I)) \to H_m^i(H_I^j(R)) \to E \to H_m^n(C_R(I)) \to 0 \]

and isomorphisms \( H_m^{i-\epsilon}(H_I^j(R)) \simeq H_m^{n-1}(C_R(I)) \) for \( i < n \).

(b) \( H_m^i(H_I^j(R)) \neq 0 \) and \( H_m^{n-\epsilon}(H_I^j(R)) = 0 \) for \( i > n \).

c) Let \( \mathfrak{p} \in V(I) \) denote a prime ideal. Then there is an isomorphism

\[ C_R(I) \otimes_R R_\mathfrak{p} \simeq C_{R_\mathfrak{p}}(IR_\mathfrak{p}) \]

provided height \( I = \text{height } IR_\mathfrak{p} \).

(d) There is a natural isomorphism \( \text{Hom}_R(H_I^j(R), H_I^j(R)) \simeq \text{Ext}_R^c(H_I^j(R), R) \).

Proof. For the proof of (a) apply the derived functor \( R\Gamma_m(\cdot) \) to the short exact sequence given in Definition 2.1. Then \( R\Gamma_m(\Gamma_I(E)) \simeq E[-n] \). So the long exact cohomology sequence of the corresponding exact sequence of complexes provides what is claimed (see [4, Lemma 2.2] for the details). Statement (b) is shown in [4, Corollary 2.9] and [4, Lemma 1.2].

For the proof of (c) localize the exact sequence in Definition 2.1 at \( \mathfrak{p} \). Then there is a short exact sequence of complexes

\[ 0 \to H_I^j(R)[\mathfrak{p}][-c] \to \Gamma_I(R_\mathfrak{p})(E_{R_\mathfrak{p}}) \to C_R(I) \otimes_R R_\mathfrak{p} \to 0. \]

To this end recall first that \( c = \text{height } IR_\mathfrak{p} = \text{height } I \) and that the local cohomology commutes with localization. Furthermore \( E' \otimes R_\mathfrak{p} \) is isomorphic to the minimal injective resolution \( E_{R_\mathfrak{p}} \) of \( R_\mathfrak{p} \). Then the definition of the truncation complex proves the claim.

Finally, we prove (d). As shown at the beginning of this section, there is an exact sequence \( 0 \to H_I^j(R) \to \Gamma_I(E)^c \to \Gamma_I(E)^{c+1} \). This induces a natural commutative diagram with exact rows

\[
\begin{array}{cccc}
0 & \to & \text{Hom}_R(H_I^j(R), H_I^j(R)) & \to & \text{Hom}_R(H_I^j(R), \Gamma_I(E)^c) & \to & \text{Hom}_R(H_I^j(R), \Gamma_I(E)^{c+1}) \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \text{Ext}_R^c(H_I^j(R), R) & \to & \text{Hom}_R(H_I^j(R), E)^c & \to & \text{Hom}_R(H_I^j(R), E)^{c+1}
\end{array}
\]

because \( \Gamma_I(E)^c \) is a subcomplex of \( E' \). The last two vertical homomorphisms are isomorphisms because \( \text{Hom}_R(X, E(R/\mathfrak{p})) = 0 \) for an \( R \)-module \( X \) with \( \text{Supp}_R X \subset V(I) \) and \( \mathfrak{p} \notin V(I) \). Therefore the first vertical map is also an isomorphism. □

In order to compute the local cohomology of the truncation complex \( C_R(I) \), there is the following spectral sequence for computation of the hyper cohomology of a complex.

Proposition 2.3. With the notation of Definition 2.1, there is the spectral sequence

\[ E_2^{p,q} = H_m^p(H^q(C_R(I))) \Rightarrow E_\infty^{p+q} = H_m^{p+q}(C_R(I)), \]

where \( H^q(C_R(I)) = 0 \) for \( i < c \) and \( i > n \), and \( H^q(C_R(I)) \simeq H_I^j(R) \) for \( c < i \leq n \).

Proof. The spectral sequence is a particular case of the spectral sequence of hyper cohomology (cf. [10]). For the initial terms check the definition of the truncation complex. □
In the following we shall use the notion of the dimension $\dim X$ for $R$-modules $X$ which are not necessarily finitely generated. This is defined by $\dim X = \dim \text{Supp}_R X$, where the dimension of the support is understood in the Zariski topology of $\text{Spec} R$. In particular, $\dim X < 0$ means $X = 0$.

**Lemma 2.4.** With the notation above, we have the following results:

(a) $\dim H^i_j(R) \leq n - i$ for all $i \geq c = \text{height } I$.

(b) $\dim H^i_j(R) = \dim R/I$.

(c) If $\dim H^i_j(R) < n - i$ for all $i > c$, then $R/I$ is unmixed, i.e. $c = \text{height } IR_p$ for all minimal $p \in V(I)$.

**Proof.** (a): This result is well-known (see for instance [7]).

(b): Let $p \in V(I)$ denote a minimal prime ideal in $V(I)$ such that $\dim R_p = c$. Then $H^i_j(R) \otimes_R R_p \simeq H^i_{pR_p}(R_p) \neq 0$ by the Grothendieck non-vanishing result. So, $p \in \text{Supp } H^i_j(R)$ and $\dim R/p = d$. Together with (a) this proves the claim.

(c): Let $p \in V(I)$ be minimal with $h := \text{height } IR_p > c$. Then $h = \dim R_p$ and

$$0 \neq H^h_{pR_p}(R_p) \simeq H^h_j(R) \otimes_R R_p.$$

This implies that $p \in \text{Supp } H^h_j(R)$ with $\dim R/p + h = n$, a contradiction. □

**Proof of Theorem 1.2.** Let $\alpha \geq 1$ denote an integer. The inclusion $J \subset I$ induces a short exact sequence $0 \to I^\alpha/J^\alpha \to R/J^\alpha \to R/I^\alpha \to 0$. By applying the long exact cohomology sequence with respect to $\text{Ext}_R(\cdot, R)$ and passing to the direct limit, we get the following exact sequence:

$$0 \to H^\alpha_j(R) \to H^\alpha_j(R) \xrightarrow{\phi} \lim_{\to} \text{Ext}_R^\alpha(I^\alpha/J^\alpha, R).$$

Recall that grade $I^\alpha/J^\alpha \geq c$ for all $\alpha$. Let $X = \text{Im } \phi$. The short exact sequence $0 \to H^\alpha_j(R) \to H^\alpha_j(R) \to X \to 0$ provides (after applying $\text{Ext}_R(\cdot, R)$) a natural homomorphism

$$\text{Ext}_R^\alpha(H^\alpha_j(R), R) \to \text{Ext}_R^\alpha(H^\alpha_j(R), R).$$

By Lemma 2.2 (d) this proves the statement in (a).

In order to prove (b) we may assume that $JR_p = IR_p$ for all $p \in V(I)$ with $\dim R_p \leq c + 1$. This is possible because local cohomology does not change by passing to the radical. Next we claim that $\dim X \leq d - 2$. This follows because $\dim_R I^\alpha/J^\alpha \leq d - 2$ for all $\alpha \in \mathbb{N}$ under the additional assumption of $J \subset I$. Moreover, $\dim X \leq d - 2$ is true by a localization argument and the embedding $X \to \lim_{\to} \text{Ext}_R^\alpha(I^\alpha/J^\alpha, R)$.

By passing to the completion and because of the Matlis duality (see [3] Lemma 1.2), it will be enough to show that the natural homomorphism $H^d_m(H^\alpha_j(R)) \to H^d_m(H^\alpha_j(R))$ is an isomorphism. Now this is true by virtue of the local cohomology with respect to the maximal ideal applied to the short exact sequence $0 \to H^\alpha_j(R) \to H^\alpha_j(R) \to X \to 0$ and the fact that $\dim X \leq d - 2$.

For the proof of (c) recall that for a cohomologically complete intersection $J$ it is known that the endomorphism ring of $H^\alpha_j(R)$ is isomorphic to $R$ (see [5] or [4] Lemma 3.3). □
3. Dimensions of local cohomology

As before let \((R, \mathfrak{m})\) denote a \(n\)-dimensional Gorenstein ring. Let \(I \subset R\) be an ideal with \(c = \text{height } I\) and \(\dim R/I = n - c\). We prove the following theorem in order to estimate the dimension of local cohomology modules. To this end let us fix the abbreviation \(h(p) = \dim R_p - c\) for a prime ideal \(p \in V(I)\).

**Theorem 3.1.** Let \(l \geq 1\) denote an integer. With the previous notation the following conditions are equivalent:

(i) \(\dim H^i_I(R) \leq n - l - i\) for all \(i > c\).

(ii) For all \(p \in V(I)\) the natural map

\[
H_{pR_p}^h(\mathcal{C}^c_{IR_p}(R_p)) \rightarrow E(k(p))
\]

is bijective (resp. surjective if \(l = 1\)) and

\[
H_{pR_p}^i(\mathcal{C}^c_{IR_p}(R_p)) = 0
\]

for all \(h(p) - l + 1 < i < h(p)\).

**Proof.** (i) \(\Rightarrow\) (ii): From Lemma 2.2 it follows that \(R/I\) is unmixed; i.e. \(c = \text{height } I = \text{height } IR_p\) for all minimal prime ideals \(p \in V(I)\). In particular this implies that \(h(p) = \dim R_p/IR_p \) for all prime ideals \(p \in V(I)\). Moreover it will be enough to prove the statement in (ii) for \(p = \mathfrak{m}\), the maximal ideal of \((R, \mathfrak{m})\).

From Lemma 2.2 (a) it will be enough to show the vanishing of \(H^I_{\mathfrak{m}}(C^c_{IR}(I))\) for all \(i > n - l\). To this end consider the spectral sequence of Proposition 2.3. By our assumption we have for the initial terms \(E_2^{p,q} = H^p_m(H_I^q(R)) = 0\) for all \(p+q > n-l\), where \(q \neq c\). This provides the vanishing of the limit terms \(H^I_{\mathfrak{m}}(C^c_{R}(I)) = 0\) for all \(i > n-l\), as required.

(ii) \(\Rightarrow\) (i): Because \(l \geq 1\) the first statement in (ii) provides that \(H_{pR_p}^{h(p)}(\mathcal{C}^c_{IR_p}(R_p))\) does not vanish. By virtue of Lemma 2.2 (b) it follows that \(\dim R_p/IR_p \geq h(p)\) for all \(p \in V(I)\), whence \(c = \text{height } IR_p\) for all \(p \in V(I)\). As a consequence (cf. Lemma 2.2 (c)) we see that \(C_R(I) \otimes_R R_p \cong C_{R_p}(IR_p)\) for all \(p \in V(I)\).

Now we proceed by induction on \(d = \dim R/I\). In the case of \(d = 0\) the ideal \(I\) is \(\mathfrak{m}\)-primary. Therefore the statement is true because \(R\) is a Gorenstein ring. So let \(d > 0\). First we show that the inductive hypothesis implies

\[
\dim H^i_I(R) \leq \max\{n - l - i, 0\}
\]

for all \(c < i \leq n\).

To this end assume that \(\dim H^i_I(R) > 0\) for a certain \(i \geq n - l\). Choose a prime ideal \(p \in \text{Supp } H^i_I(R) \setminus \{\mathfrak{m}\}\). Therefore \(H_{pR_p}^i(\mathcal{C}^c_{IR_p}(R_p)) \neq 0\) and \(i + l \leq \dim R_p\) by the induction hypothesis. On the other hand, \(l + i \leq \dim R_p < n \leq l + i\), a contradiction. Second, suppose that \(\dim H^i_I(R) > n - l - i\) for a certain \(c < i < n - l\). Choose a prime ideal \(p \in \text{Supp } H^i_I(R)\) such that \(\dim R/p = \dim H^i_I(R)\). Therefore \(\dim R/p > n - l - i\) and \(l + i > \dim R_p\). Moreover \(H_{pR_p}^i(\mathcal{C}^c_{IR_p}(R_p)) \neq 0\) and \(i + l \leq \dim R_p\), which is again a contradiction.
With this information in mind, the spectral sequence (as given in Proposition 3.3) degenerates to isomorphisms $H^*_p(C_R(I)) \simeq H^i(C_R(I))$ for all $i > n - l$. Finally the assumption in (ii) for $p = m$ implies that $H^*_p(C_R(I)) = 0$ for all $i > n - l$ (by Lemma 2.2). This finishes the proof because $H^i(C_R(I)) \simeq H^i_1(R)$ for $i > c$. \hfill □

For $l \geq \dim R/I$ the previous result yields, as a particular case, the equivalence of the conditions (i) and (ii) of [4, Theorem 3.1]. Another corollary is the following:

**Corollary 3.2.** Suppose that $c \geq 2$. With the above notation suppose that

\[ \hat{R}_p \to \text{Hom}_{\hat{R}}(H^c_{I_{\hat{R}^c}}(\hat{R}_p), H^c_{I_{\hat{R}^c}}(\hat{R}_p)) \]

is an isomorphism for all $p \in V(I) \setminus \{m\}$ (e.g. this is satisfied in the case where $I$ is locally a cohomologically complete intersection). Then the following conditions are equivalent:

(i) $H^i_1(R) = 0$ for $i = n - 1, n$.

(ii) The natural homomorphism $\hat{R} \to \text{Hom}_{\hat{R}}(H^c_{I_{\hat{R}^c}}(\hat{R}), H^c_{I_{\hat{R}^c}}(\hat{R}))$ is an isomorphism.

**Proof.** By the Local Duality Theorem the assumption is equivalent to the isomorphisms

\[ H^{h(p)}_{pR_p}(H^c_{I_{R_p}}(R_p)) \to E(k(p)) \]

for all $p \in V(I) \setminus \{m\}$. By a localization argument and Theorem 3.1 this is equivalent to $\dim H^i_1(R) \leq \max\{n - 2 - i, 0\}$ for all $i > c$. Therefore, by Theorem 3.1 the statement in (ii) holds if and only if $H^i_1(R) = 0$ for $i = n - 1, n$. \hfill □

Note that Corollary 3.2 proves Theorem 1.1 of the Introduction. Another corollary of Theorem 3.1 is the following vanishing result.

**Corollary 3.3.** Fix the notation as above. Suppose that $I$ is locally a cohomologically complete intersection. For an integer $l \geq 1$ the following conditions are equivalent:

(i) $\text{cd} I \leq \min\{n - l, c\}$, i.e. $H^i_1(R) = 0$ for all $i > \max\{n - l, c\}$.

(ii) The natural homomorphism $H^{d}_{d}(H^{c}_{I_{\hat{R}^c}}(\hat{R})) \to E$ is bijective (resp. surjective if $l = 1$) and $H^{d}_{m}(H^{c}_{I_{\hat{R}^c}}(\hat{R}))$ is isomorphic to $E$ for all $d - l + 1 < i < d$.

**Proof.** Note that the ideal $I$ is locally a cohomologically complete intersection if and only if $\dim H^i_1(R) \leq 0$ for all $i > c$. This follows from localization and the fact that $H^i_{R_p}(R_p) = 0$ for all $i \neq c$ and all $p \in V(I) \setminus \{m\}$. Therefore, as a consequence of Theorem 3.1 the statement is true. \hfill □

4. Problems and examples

The first example shows that the assumptions in Corollary 3.2 are not necessary for the equivalence of the two statements. Moreover, it shows that the isomorphism $R \simeq \text{Hom}_{\hat{R}}(H^c_1(R), H^c_1(R))$ does not localize.

**Example 4.1** (cf. [4, Example 4.1]). Let $k$ be an arbitrary field. Let $R = k[[x_0, \ldots, x_4]]$ denote the formal power series ring in five variables over $k$. Let

\[ I = (x_0, x_1) \cap (x_1, x_2) \cap (x_2, x_3) \cap (x_3, x_4). \]
Then $c = \text{height } I = 2$ and $H^i_I(R) = 0$ for all $i \neq 2,3$, by use of the Mayer-Vietoris sequence for local cohomology. Moreover (see [4] Example 4.1) it can be shown that $H^3_I(R) = E_R(R/p), p = (x_0,x_1,x_3,x_4)$. The spectral sequence

$$E^{p,q}_2 = H^p_m(H^q_I(R)) \Rightarrow E^{p+q}_\infty = H^{p+q}_m(R)$$

provides an isomorphism $H^3_m(H^2_I(R)) \simeq E$. Recall that $H^2_m(H^3_I(R)) = 0$ for all $p \in \mathbb{N}$. By Local Duality it follows that the natural homomorphism $R \to \text{Hom}_R(H^3_I(R),H^2_I(R))$ is an isomorphism. On the other hand, it is easily seen that this is not true for $R$ because

$$H^2_{I_R(R_p)} \simeq H^2_{I, R_p}(R_p) \oplus H^2_{I_2 R_p}(R_p), I_1 = (x_0,x_1), I_2 = (x_3,x_4)$$

decomposes into two non-zero direct summands. This is seen by using the Mayer-Vietoris sequence for local cohomology.

The following example shows that the endomorphism ring $\text{Hom}_R(H^2_I(R),H^2_I(R))$, $c = \text{height } I$, is in general not a finitely generated $R$-module.

**Example 4.2** (cf. [3] §3). Let $k$ denote a field and let $R = k[x,y,u,v]/(xu-yv)$, where $k[x,y,u,v]$ denotes the power series ring in four variables over $k$. Let $I = (u,v)R$. Then dim $R = 3, \dim R/I = 2$ and $c = 1$. It follows that $H^1_I(R) = 0$ for $i \neq 1,2$. Moreover $\text{Supp } H^2_I(R) \subset \{m\}$. The truncation complex with the short exact sequence

$$0 \to H^2_I(R)[-c] \to \Gamma_I(E') \to C^*_R(I) \to 0$$

(of Definition 2.1) induces a short exact sequence on local cohomology

$$0 \to H^2_I(R) \to H^2_m(H^1_I(R)) \to E \to 0$$

(see Lemma 2.2). Hartshorne [3] §3 has shown that the socle of $H^2_I(R)$ is not a finite dimensional $k$-vector space. Therefore, the socle of $H^2_m(H^1_I(R))$ is infinite. Moreover there are the following isomorphisms

$$\text{Hom}_R(H^1_I(R),H^1_I(R)) \simeq \text{Ext}^1_R(H^1_I(R),R) \simeq \text{Hom}_R(H^2_m(H^1_I(R)),E)$$

(from Lemma 2.2 (d) and [4] Lemma 1.2). By the Nakayama Lemma this means that $\text{Hom}_R(H^1_I(R),H^1_I(R))$ is not a finitely generated $R$-module.

So, one might ask for a characterization of the finiteness of the endomorphism ring of $H^1_I(R), c = \text{height } I$.

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Institut für Informatik, Martin-Luther-Universität Halle-Wittenberg, 06099 Halle (Saale), Germany

E-mail address: peter.schenzel@informatik.uni-halle.de