In the pioneering paper [FF], Feigin and Fuchs have constructed intertwining operators between "Fock-type" modules over the Virasoro algebra via contour integrals of certain operator-valued one dimensional local systems over top homology classes of a configuration space. Similar constructions exist for affine Lie algebras. Key ingredients in such a construction are the so called "screening operators". The main observation of the present paper is that the screening operators contain more information. Specifically, at the chain level, the screening operators provide a certain canonical cocycle of the Virasoro (resp. affine) Lie algebra with coefficients in the de Rham complex of an operator-valued local system on the configuration space. This way we obtain canonical morphisms from higher homology spaces of the above local systems to appropriate higher Ext-groups between the Fock space representations.

The screening operators that we are interested in this paper are linear maps $S : M_1 \rightarrow M_2[[z, z^{-1}]]$, where $M_1$ and $M_2$ are certain modules over the Lie algebra $\mathfrak{g}$ in question, e.g., an affine Lie algebra. We think of such an operator as
a "function" of the formal variable $z$, and write it as $S(z)$. The key property of screening operators is that they satisfy an equation of the type

$$[x, S(w)] = \frac{\partial S(x, w)}{\partial w}, \quad \forall x \in \mathfrak{g},$$

(0.1)

where $S(x, w) : M_1 \to M_2[[z, z^{-1}]]$ is some other operator, a companion of $S(w)$. The equation above clearly says that the pair $S(w), S(\bullet, w)$ is a cocycle in the total complex associated with a double-complex with Koszul differential, $d'$, and de Rham differential, $d''$. This observation is a starting point of our analysis, and the rest of our results is just an elaboration of that observation. We would like to emphasize that screenings still present a mystery in the sense that we do not know any kind of general mechanism that would give rise to operators $S$ satisfying (0.1). In all known cases the screening operators are constructed by hand, using an explicit description of the modules $M_1$ and $M_2$ in terms of "creation" and "annihilation" operators. Such an explicit description is known as a bosonisation procedure, and the bulk of the paper is devoted to writing out the bosonisation procedure for the modules we need, since the corresponding formulas are spread over the (mostly physics) literature (see [FeFr1], [FeFr2] for mathematical results).

Our construction is motivated by, and in a special case reduces to the construction of [BMP1], [BMP2]. In fact, as the results of loc. cit and [FS] suggest, an explanation of our construction should lie in some equivalence of (derived) categories of representations of quantum groups, and the corresponding affine algebras, sending (contragradient) Verma modules to Wakimoto modules.

On the other hand, we believe that the cocycles that we study can be most adequately interpreted as de Rham cohomology classes of the chiral algebras considered by Beilinson-Drinfeld [BD].

Recently A. Sebbar, [S], was able to obtain a "$q$-deformation" of most of the constructions of this paper, where the de Rham cohomology is deformed to Aomoto-Jackson $q$-de Rham cohomology. It is interesting to note that, as was expected, the operators from Lemma 3.3 are deformed to Kashiwara operators (see [Ka] or [L], n.1.213).

The main results of this paper have been obtained back in 1991, and written up in February 1996. We are deeply indebted to Ed Frenkel for the numerous enlightening discussions on bosonisation during 1990-1991. We thank Jim Stasheff for useful remarks.

Chapter 1. Toy examples.

§ 2. The toy example

In this work, everything will be over the base field $\mathbb{C}$.

2.1. Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{sl}(2)$, with the standard generators $E$, $F$, $H$. For $\lambda \in \mathbb{C}$, let $M(\lambda)$ denote the Verma module over $\mathfrak{g}$ generated by the vacuum vector $v_\lambda$ subject to the relations $Ev_\lambda = 0$, $ Hv_\lambda = \lambda v_\lambda$. We shall use the formulas

$$EF^av_\lambda = a(\lambda - a + 1)F^{a-1}v_\lambda, \quad HF^av_\lambda = (\lambda - 2a)F^av_\lambda.$$

(In the sequel, in our formulas we shall use the agreement $F^bv_\lambda = 0$ for $b < 0$.)
Let us pick \( \lambda, \lambda' \in \mathbb{C} \). For an integer \( n \geq 0 \), we consider the \( \mathbb{C} \)-linear operator
\[
V_n: M(\lambda' - 1) \to M(\lambda - 1), \quad V_n(F^a v_{\lambda'-1}) = F^{a+n} v_{\lambda-1}.
\]
These operators satisfy the following commutation relations.

(a) \([E, V_n](F^a v_{\lambda'-1}) = (-n^2 + (\lambda - 2a)n + a(\lambda - \lambda'))F^{a+n-1} v_{\lambda-1},\)

(b) \([H, V_n](F^a v_{\lambda'-1}) = (\lambda - \lambda' - 2n)F^a v_{\lambda-1},\)

(c) \([F, V_n] = 0.\)

2.2. Consider the operator-valued formal power series
\[
V(z) = \sum_{n \geq 0} V_n z^{-n-1} dz: \quad M(\lambda' - 1) \to M(\lambda - 1)[[z^{-1}]] dz/z
\]
(here \( z \) is a formal variable). Let us try to find a number \( \alpha \in \mathbb{C} \) and an operator
\[
V(E; z) = \sum_{n \geq 0} V_n(E) z^{-n}: \quad M(\lambda' - 1) \to M(\lambda - 1)[[z^{-1}]]
\]
such that

(a) \([E, V(z)] = (d + \alpha dz/z) V(E; z).\)

The equation (a) is equivalent to the system of equations

(b) \([E, V_n] = (-n + \alpha) V_n(E) \quad (n \geq 0).\)

So, we have
\[
(-n + \alpha) V_n(E)(F^a v_{\lambda'-1}) = (-n^2 + (\lambda - 2a)n + a(\lambda - \lambda'))F^{a+n-1} v_{\lambda-1},
\]
therefore \( V_n(E)(F^a v_{\lambda'-1}) = (n + \beta(a))F^{a+n-1} v_{\lambda-1} \) for some function \( \beta(a) \) such that
\[
(-n + \alpha)(n + \beta(a)) = -n^2 + (\lambda - 2a)n + a(\lambda - \lambda'),
\]
that is, \( \beta(a) - \alpha = -\lambda + 2a \), i.e., \( \beta(a) = 2a + \alpha - \lambda \), and \( \beta(a)\alpha = a(\lambda - \lambda') \) for all \( a \).

Suppose that \( \lambda \neq \lambda' \). Then we must have \( \beta(a) = 2a \), \( \alpha = \lambda \), hence from the second equation we obtain \( \lambda' = -\lambda \).

2.3. From now on we suppose that \( \lambda' = -\lambda \). Thus, we have

(a) \([E, V_n] = (-n + \lambda) V_n(E) \quad \text{where} \quad V_n(E)(F^a v_{-\lambda-1}) = (n+2a)F^{a+n-1} v_{\lambda-1}.\)

From 2.1 (b) we obtain

(b) \([H, V_n] = (-n + \lambda) V_n(H),\)

where \( V_n(H) = 2V_n \). Finally,

(c) \([F, V_n] = 0.\)
Therefore, we come to the following conclusion.

2.4. The operator

\[ V(z) = \sum_{n \geq 0} V_n z^{-n-1} \frac{dz}{z} \]

defined by \( V_n(F^a v_{-\lambda-1}) = F^{a+n} v_{\lambda-1} \), satisfies the following relation

(a) \[ [X, V(z)] = (d + \lambda \frac{dz}{z}) V(X; z) \quad (X \in \mathfrak{g}), \]

where the operators

\[ V(X; z) = \sum_{n \geq 0} V_n(X) z^{-n} : M(-\lambda - 1) \to M(\lambda - 1)[[z]], \]

linearly depending on \( X \in \mathfrak{g} \), are defined by

\[ V_n(E)(F^a v_{-\lambda-1}) = (n + 2a) F^{a+n-1} v_{\lambda-1}, \quad V_n(H) = 2V_n, \quad V_n(F) = 0. \]

2.5. We have

\[
[H, V_n(E)](F^a v_{-\lambda-1}) = -2(n + 2a)(n - \lambda - 1) F^{a+n-1} v_{\lambda-1}, \\
[F, V_n(E)] = -V_n(H), \\
[E, V_n(H)](F^a v_{-\lambda-1}) = -2(n + 2a)(n - \lambda) F^{a+n-1} v_{\lambda-1}, \\
[F, V_n(H)] = 0. 
\]

It follows that for any \( X, Y \in \mathfrak{g} \) and \( n \geq 0 \), we have

(a) \[ V_n([X, Y]) = [X, V_n(Y)] - [Y, V_n(X)]. \]

2.6. Let us consider the complex (of length 1)

(a) \[ \Omega^\prime: 0 \to \Omega^0 \xrightarrow{d_\lambda} \Omega^1 \to 0, \]

where \( \Omega^0 = \mathbb{C}[[z^{-1}]] \), \( \Omega^1 = \mathbb{C}[[z^{-1}]] \frac{dz}{z} \), \( d_\lambda = d + \lambda \frac{dz}{z} \).

We will always write Hom for Hom_\mathbb{C} unless specified otherwise. For any two integers \( i, j \geq 0 \) set

\[ C^{ij}(\mathfrak{g}; M(-\lambda - 1), M(\lambda - 1)) = \text{Hom}(\Lambda^i \mathfrak{g} \otimes M(-\lambda - 1), M(\lambda - 1) \otimes \Omega^j) \]

\[ = \text{Hom}(\Lambda^i \mathfrak{g}, \text{Hom}(M(-\lambda - 1), M(\lambda - 1) \otimes \Omega^j)). \]

The bigraded space \( C^\ast\ast(\mathfrak{g}; M(-\lambda - 1), M(\lambda - 1)) \) has the natural structure of a bicomplex. The first differential

\[ d' : C^{ij}(\mathfrak{g}; M(-\lambda - 1), M(\lambda - 1)) \to C^{i+1,j}(\mathfrak{g}; M(-\lambda - 1), M(\lambda - 1)) \]

is induced by the standard Koszul differential in the cochain complex of the Lie algebra \( \mathfrak{g} \) with coefficients in the module \( \text{Hom}(M(-\lambda - 1), M(\lambda - 1)) \).
The second differential

\[ d'' : \ C^{ij}(g; M(-\lambda - 1), M(\lambda - 1)) \to C^{i,j+1}(g; M(-\lambda - 1), M(\lambda - 1)) \]

is induced by the differential in the complex \( \Omega' \).

Let \( C'(g; M(-\lambda - 1), M(\lambda - 1)) \) denote the associated total complex.

The operator \( V(z) \) is an element of the space \( C^{01}(g; M(-\lambda - 1), M(\lambda - 1)) \).

Let us denote this element by \( V^{01}(z) \). The operators \( V(X; z) \) \( (X \in g) \) define an element \( V^{10}(z) \) of the space \( C^{10}(g; M(-\lambda - 1), M(\lambda - 1)) \).

Property 2.4 (a) means that \( d'(V^{01}(z)) = d''(V^{10}(z)) \). Property 2.5 (a) means that \( d''(V^{10}(z)) = 0 \). Therefore, the element \( (V^{01}(z), V^{10}(z)) \) is a 1-cocycle of the total complex \( C'(g; M(-\lambda - 1), M(\lambda - 1)) \).

2.7. Suppose that \( \lambda \) is a nonnegative integer. In this case, the complex \( \Omega' \) has two one-dimensional cohomology spaces. The space \( H^0(\Omega') \) is generated by the function \( z^{-\lambda} \); and the space \( H^1(\Omega') \) generated by the class of the form \( z^{-\lambda} \frac{dz}{z} \).

(If \( \lambda \notin \mathbb{N} \), the complex \( \Omega' \) is acyclic.)

Consider the dual spaces \( H_i = H^i(\Omega')^* \). The space \( H_1 \) is generated by the functional \( \Omega^1 \to \mathbb{C} \) which assigns to a form \( \omega \) the residue \( \text{res}_{z=0}(\omega z^\lambda) \). The space \( H_0 \) is generated by the (restriction of) the functional \( \Omega^0 \to \mathbb{C} \) which assigns to a function \( f(z) \) the residue \( \text{res}_{z=0}(f(z) z^\lambda \frac{dz}{z}) \).

The previous discussion implies the following.

(a) The operator \( \text{res}_{z=0}(V^{01}(z) z^\lambda) \in \text{Hom}_\mathbb{C}(M(-\lambda - 1), M(\lambda - 1)) \) is an intertwiner.

(b) The operator \( \text{res}_{z=0}(V^{10}(z) z^\lambda \frac{dz}{z}) \in \text{Hom}_\mathbb{C}(g, \text{Hom}(M(-\lambda - 1), M(\lambda - 1))) \) is a 1-cocycle of the Lie algebra \( g \) with coefficients in the \( g \)-module \( \text{Hom}(M(-\lambda - 1), M(\lambda - 1)) \).

Therefore, this operator defines certain element of the space \( \text{Ext}^1_g(M(-\lambda - 1), M(\lambda - 1)) \).

§3. Generalization of the toy example

3.1. Let \( A = (a_{ij})_{i,j=1}^r \) be a symmetrizable generalized Cartan matrix, and let \( g \) be the corresponding Kac-Moody Lie algebra defined by the Chevalley generators \( E_i, F_i, H_i \) \( (i = 1, \ldots, r) \) and relations (see [K], 0.3)

\[ [H_i, H_j] = 0; \quad [H_i, E_j] = a_{ij} E_i; \quad [H_i, F_i] = -a_{ij} F_i; \quad [E_i, F_j] = \delta_{ij} H_i; \]

\[ \text{ad}(E_i)^{-a_{ij}+1}(E_j) = \text{ad}(F_i)^{-a_{ij}+1}(F_j) = 0. \]

Let \( h \subset g \) be the Cartan subalgebra spanned by the elements \( H_1, \ldots, H_r \). For \( i = 1, \ldots, r \), let \( \alpha_i \in h^* \) be the corresponding simple root; let \( r_i : h^* \to h^* \) be the corresponding simple reflection,

\[ r_i \lambda = \lambda - (H_i, \lambda) \alpha_i. \]

Let \( \rho \in h^* \) be the element defined by \( (H_i, \rho) = 1 \) \( (i = 1, \ldots, r) \). Let \( n_- \subset g \) be the Lie subalgebra generated by the elements \( F_1, \ldots, F_r \).

For \( \lambda \in h^* \), let \( M(\lambda) \) denote the Verma module over \( g \), with one generator \( v_\lambda \) and relations \( E_i v_\lambda = 0; \ H_i v_\lambda = (H_i, \lambda - \rho) v_\lambda \).
3.3. **Lemma-definition.** There exists a unique linear operator $\partial_i : U_n \rightarrow U_n$ such that $\partial_i(F_j) = \delta_{ij} \cdot 1$ $(j = 1, \ldots, r)$ and for any $x,y \in U_n$, $\partial_i(xy) = \partial_i(x)y + x\partial_i(y)$.

**Proof.** The uniqueness is clear. Let us prove the existence. Let $A$ be the free associative $\mathbb{C}$-algebra with generators $\theta_j$, $j = 1, \ldots, r$. It is clear that there exists a unique linear operator $\partial_i : A \rightarrow A$ such that $\partial_i(\theta_j) = \delta_{ij} \cdot 1$, and such that, for any $x, y \in A$, one has $\partial_i(xy) = \partial_i(x)y + x\partial_i(y)$.

For an integer $a \geq 1$ and $j \neq k$ in $\{1, \ldots, r\}$, define the following element in the algebra $A$

$$C(j,k;a) = \text{ad}(\theta_j)^a(\theta_k) = \sum_{p=0}^{a} (-1)^p \binom{a}{p} \theta_j^{a-p} \theta_k \theta_j^p.$$ 

We claim that

(a) for any $a, j, k$, $C(j,k;a) \in \text{Ker}(\partial_i)$.

Indeed, the claim is clear for $(j,k)$ such that $i \neq j$ and $i \neq k$. We have

$$\partial_i(C(j,i;a)) = \sum_{p=0}^{a} (-1)^p \binom{a}{p} \theta_i^p = (1 - 1)^a \theta_i^a = 0.$$ 

Let us prove that $\partial_i(C(i,k;a)) = 0$ by induction on $a$. For $a = 1$ this is obvious. We have

$$\partial_i(C(i,k;a)) = \partial_i(\theta_i C(i,k;a-1) - C(i,k;a-1) \theta_i)$$

(by induction)

$$= \partial_i(\theta_i) C(i,k;a-1) - C(i,k;a-1) \partial_i(\theta_i)$$

$$= C(i,k;a-1) - C(i,k;a-1) = 0.$$ 

The claim is proved.

It follows from (a) that the operator $\partial_i : A \rightarrow A$ induces an operator $\partial_i : U_n \rightarrow U_n$, since $U_n$ is the quotient of $A$ by the two-sided ideal generated by the elements $C(j,k;a-jk+1)$. The lemma is proven. □

3.2 **Remark.** The operators $\partial_i$ are classical limits of the Kashiwara operators, see [Ka], on the quantized universal enveloping algebra $U_q(n_-)$. In fact most of the constructions of this and the subsequent sections have been recently extended to the quantized setup in [S].

3.3. Pick an element $\lambda \in \mathfrak{h}^*$ and $i \in \{1, \ldots, r\}$. Set $\lambda' = r_i \lambda$. For each integer $n \geq 0$, we introduce linear operators $V_{i;n}$, and $V_{i;n}(E_j) : M(\lambda') \rightarrow M(\lambda)$ defined for $x \in U_n$ by

$$V_{i;n} : xv_{\lambda'} \mapsto x F_i^n v_{\lambda}, \quad V_{i;n}(E_j) : xv_{\lambda'} \mapsto a_{ji} \partial_j(x) F_i^n v_{\lambda} + \delta_{ij} nx F_i^{n-1} v_{\lambda}.$$ 

We further form the following elements of $\text{Hom}(M(r_i \lambda), M(\lambda)[[z^{-1}]] [dz])$:

$$V_i(z) = \sum_{n=0}^{\infty} V_{i;n} z^{-n-1} \quad V_i(E_j; z) = \sum_{n=0}^{\infty} V_{i;n}(E_j) z^{-n}.$$
\section{Proposition.} For any $i, j$ and $n \geq 0$, we have $[E_j, V_{i;n}] = (-n + \langle H_i, \lambda \rangle) \cdot V_{i;n}(E_j)$, equivalently, there is a power series identity

\[ [E_j, V_i(z)] = (d + \langle H_i, \lambda \rangle \frac{dz}{z}) V_i(E_j; z). \]

\textbf{Proof.} For any $i$ and $x \in \mathfrak{u}_-$ we can write by definition

\[ V_i(z) : x \cdot v_\lambda' \mapsto x e^{z \cdot F_i} \cdot v_\lambda. \]

Let $E_j$ be a simple Chevalley generator. We will frequently use the following formulas:

\begin{align*}
[E_j, x] &= \partial_j(x) \cdot H_j + x_1, \quad x_1 \in \mathfrak{u}_- . \quad (a) \\
\exp^{-z \cdot F_i} \cdot H_j \cdot \exp^{z \cdot F_i} &= H_j + \alpha_i(H_j) \cdot z \cdot F_i \quad (b) \\
E_j e^{z \cdot F_i} - e^{z \cdot F_i} E_j &= P(z) \in z \cdot \mathfrak{u}_- \cdot \mathbb{Z}, \text{ and } P = 0 \text{ if } i \neq j \quad (c)
\end{align*}

Thus, $P$ is a polynomial in $z$ without constant term with values in $\mathfrak{u}_-$. Using formulas (a)-(b) above, we find:

\[ [E_j, x e^{z \cdot F_i}] = [E_j, x] \cdot e^{z \cdot F_i} + x \cdot [E_j, e^{z \cdot F_i}] = (\partial_j(x) \cdot H_j + x_1) \cdot e^{z \cdot F_i} + x \cdot P(z). \]

Hence, \setcounter{equation}{1}

\begin{equation*}
(* \quad E_j \left( V_i(z)(x v_\lambda') \right) = E_j(x e^{z \cdot F_i} v_\lambda) = (\partial_j(x) \cdot H_j + x_1) \cdot e^{z \cdot F_i} v_\lambda x \cdot P(z) v_\lambda
\end{equation*}

\[ = \partial_j(x) \cdot e^{z \cdot F_i}(H_j + z \alpha_i(H_j) F_i) v_\lambda + x_1 e^{z \cdot F_i} v_\lambda + x \cdot P(z) v_\lambda. \]

On the other hand, from (a) we get

\[ E_j x v_\lambda' = [E_j, x] \cdot v_\lambda' = \partial_j(x) \cdot H_j \cdot v_\lambda' + x_1 v_\lambda'. \]

Hence, \setcounter{equation}{2}

\begin{equation*}
(** \quad V_i(z)(E_j x v_\lambda') = \partial_j(x) \cdot H_j e^{z \cdot F_i} v_\lambda + x_1 e^{z \cdot F_i} v_\lambda \quad \text{by (b)}
\end{equation*}

\[ = \partial_j(x) \cdot e^{z \cdot F_i}(H_j + \alpha_i(H_j) \cdot z \cdot F_i) + x_1 e^{z \cdot F_i} v_\lambda. \]

From (*) and (**) we obtain

\[ [E_j, V_i(z)](x v_\lambda') = \langle \lambda' - \lambda, H_j \rangle \partial_j(x) \cdot e^{z \cdot F_i} v_\lambda + \langle \alpha, H_j \rangle z \cdot F_i e^{z \cdot F_i} v_\lambda + x \cdot P(z) v_\lambda. \]

It is easy to deduce from the last formula and formula (c) above, that if $i \neq j$ the RHS of takes the form

\[ (d + \langle H_i, \lambda \rangle \frac{dz}{z}) V_i(E_j; z) \]

The case $i = j$ is treated similarly. \hfill \Box

\section{For each $n \geq 0$ we define the operators $V_{i;n}(H_j) : M(\lambda') \to M(\lambda)$ by $V_{i;n}(H_j) = a_{ji} V_{i;n}$. One checks easily that}

\begin{itemize}
  \item[(a)] $[H_j, V_{i;n}] = (-n + \langle H_i, \lambda \rangle)V_{i;n}(H_j)$.
  \item[(b)] $[F_j, V_{i;n}] = 0$ for all $j$.
  \item[(c)] $V_{i;n}(F_j) = 0$.
\end{itemize}
3.7. **Lemma-definition.** For each \( n \geq 0 \), there exists a unique element

\[
V_{i,n}(\cdot) \in \operatorname{Hom}(\mathfrak{g}, \operatorname{Hom}(M(\lambda'), M(\lambda)))
\]

such that \( V_{i,n}(E_j), V_{i,n}(H_j), V_{i,n}(F_j) \) are the elements defined above and the following cocycle condition holds:

(a) For any \( X,Y \in \mathfrak{g} \), \( V_{i,n}([X,Y]) = [X,V_{i,n}(Y)] - [Y,V_{i,n}(X)] \). \( \square \)

The previous considerations may be reformulated in terms of the generating functions

\[
V_i(X;z) = \sum_{n=0}^{\infty} V_{i,n}(X)z^{-n} \in \operatorname{Hom}(M(r_i \lambda), M(\lambda)[[z^{-1}]]), \quad X \in \mathfrak{g}.
\]

as the follows.

3.9. **Theorem.** For any \( X,Y \in \mathfrak{g} \), we have

\[
[X, V_i(z)] = (d + \langle H_i, \lambda \rangle dz/z)V_i(X; z),
\]

\[
V_i([X,Y]; z) = [X, V_i(Y; z)] - [Y, V_i(X; z)]. \quad \square
\]

3.10. Suppose an element \( w \) of the Weyl group of \( \mathfrak{g} \) together with its reduced decomposition \( w = r_{i_1} \cdots r_{i_k} \), and an element \( \lambda \in \mathfrak{h}^* \) is given. For \( p = 1, \ldots, a \), set \( \lambda_p = r_{i_p-1} \cdots r_{i_1} \lambda \).

Define a complex

\[
\Omega^\bullet: \quad 0 \rightarrow \Omega^0 \rightarrow \ldots \rightarrow \Omega^a \rightarrow 0
\]

as follows. Set \( A = \mathbb{C}[z_1^{-1}, \ldots, z_a^{-1}] \). By definition, \( \Omega^p \) is the free \( A \)-module with the basis \( \{(dz_{j_1}/z_{j_1}) \wedge \cdots \wedge (dz_{j_p}/z_{j_p}) \mid 1 \leq j_1 < \ldots < j_p \leq a\} \). The differential is defined by

\[
d\eta = d_{DR}(\eta) + \left( \sum_{p=1}^{a} \langle H_{i_p}, \lambda_p \rangle z_p^{-1}dz_p \right) \wedge \eta
\]

where \( d_{DR} \) is the de Rham differential.

3.11. For each \( p = 1, \ldots, a \), consider the operators \( \omega_p = V_{i_p}(z_p) : M(\lambda_{p+1}) = M(r_{i_p} \lambda_p) \rightarrow M(\lambda_p)z_p^{-1}[[z_p^{-1}]] \) and \( \tau_p(X) = V_{i_p}(X; z_p) : M(\lambda_{p+1}) \rightarrow M(\lambda_p)[[z_p^{-1}]] \). \( \square \)

For each \( m, 0 \leq m \leq a \), we define the operators

\[
V^{m,a-m} \in \operatorname{Hom}(\Lambda^m \mathfrak{g}, \operatorname{Hom}(M(\lambda), M(\lambda) \otimes \Omega^{a-m}))
\]

as follows. We set

\[
V^{m,a-m}(X_1 \wedge \cdots \wedge X_m) = \sum_{1 \leq p_1 < \cdots < p_m \leq a} (-1)^{\operatorname{sgn}(p_1, \ldots, p_m)}
\]

\[
\times \left( \sum_{\sigma \in \Sigma_m} (-1)^{\operatorname{sgn}(\sigma)} \omega_1 \cdots \tau_{p_1}(X_{\sigma(1)}) \cdots \tau_{p_m}(X_{\sigma(m)}) \cdots \omega_a \right)
\]

\[
\times dz_1 \wedge \cdots \wedge \hat{dz}_{p_1} \wedge \cdots \wedge \hat{dz}_{p_m} \wedge \cdots \wedge dz_a.
\]
Here for each sequence $1 \leq p_1 < \ldots < p_m \leq a$, the corresponding summands are obtained by replacing the operators $\omega_{p_j}$ in the product $\omega_1 \cdots \omega_a$, by $\tau_{p_j}(X_{\sigma(j)})$. 

$\Sigma_m$ denotes the group of all bijections $\{1, \ldots, m\} \to \{1, \ldots, m\}$. The power $\text{sgn}(p_1, \ldots, p_m)$ is defined by induction on $m$ as follows. We set

$$\text{sgn}(\emptyset) = 0; \text{sgn}(p_1, \ldots, p_m) = \text{sgn}(p_1, \ldots, p_{m-1}) + p_m + m.$$ 

For example,

$$V^{0a}_0 = V_{i_1}(z_1) \cdots V_{i_a}(z_a) dz_1 \wedge \ldots \wedge dz_a.$$ 

Consider the double complex $C^\bullet(g; \text{Hom}(M(w\lambda), M(\lambda) \otimes \Omega^*))$. Here the first differential is the Koszul differential in the complex of cochains of the Lie algebra $g$ with coefficients in the complex of $g$-modules $\text{Hom}(M(w\lambda), M(\lambda) \otimes \Omega^*)$. The action of $g$ is induced by the standard action (by the commutator) of $g$ on $\text{Hom}(M(w\lambda), M(\lambda))$. The second differential is induced by the differential in $\Omega^*$.

By definition, we have

$$V^{m,a-m}_m \in C^m(g; \text{Hom}(M(w\lambda), M(\lambda) \otimes \Omega^{a-m})).$$

3.12. **Theorem.** The element $V = (V^{0a}_0, \ldots, V^{a0}_0)$ is an $a$-cocycle in the total complex associated with the double complex $C^\bullet(g; \text{Hom}(M(w\lambda), M(\lambda) \otimes \Omega^*))$. \hfill $\square$

3.13. **Corollary.** The cocycle $V$ induces linear maps

$$f_m: H^m(\Omega^*) \to \text{Ext}^a_{g}(M(w\lambda), M(\lambda)), \quad m = 0, \ldots, a.$$ 

3.14. **Example.** Assume that all numbers $\langle H_p, \lambda_p \rangle$, $p = 1, \ldots, a$, are nonnegative integers. The highest homology space $H^a(\Omega^*)$ is one-dimensional, generated by the (image of) the functional $r \in \Omega^{\text{a+}}$ defined by

$$r(\eta) = \text{res}_{z_a = 0} \cdots \text{res}_{z_1 = 0} (z_1^{\langle H_{i_1}, \lambda_1 \rangle} \cdots z_a^{\langle H_{i_a}, \lambda_a \rangle} \eta).$$

The image $f_0(r) \in \text{Hom}_g(M(w\lambda), M(\lambda))$ is the unique (up to proportionality) intertwiner sending $v_{w\lambda}$ to $F^{\langle H_{i_1}, \lambda_1 \rangle + 1}_{i_1} \cdots F^{\langle H_{i_a}, \lambda_a \rangle + 1}_{i_a} v_{\lambda}$.

**Chapter 2. Virasoro algebra.**

§4. **Cartan cocycle**

4.1. Let $g$ be a Lie algebra. Consider the differential graded Lie algebra $g^* = g^{-1} \oplus g^0$ defined as follows. We set $g^{-1} = g^0 = g$; the differential $d: g^{-1} \to g^0$ is the identity map; the bracket $\Lambda^2 g^0 \to g^0$ coincides with the bracket in $g$; the bracket $g^{-1} \otimes g^0 \to g^0$ coincides with the bracket in $g$.

For $x \in g$, we denote by the same letter $x$ the corresponding element of $g^0$, and by $i_x$ the corresponding element of $g^{-1}$.

As a dg Lie algebra, the algebra $g^*$ is generated by the Lie algebra $g$ (in degree 0) and by the elements $i_x \in g^{-1}$ ($x \in g$) subject to the relations (a)–(c) below.

(a) $d(i_x) = x$ \quad ($x \in g$),
(b) $[x, i_y] = i_{[x, y]}$ \quad ($x, y \in g$),
(c) $[i_x, i_y] = 0$ \quad ($x, y \in g$).
4.2. Let $M^*$ be a complex of vector spaces. A $g^*$-module structure on $M^*$ is the same thing as a collection of data (a), (b) below satisfying properties (c)–(e) below.

(a) Morphisms of complexes $x: M^* \to M^*$, ($x \in g$) that define an action of the Lie algebra $g$.

(b) Morphisms of graded spaces $i_x: M^* \to M^*[−1]$, ($x \in g$).

(c) (Cartan formula) $[d, i_x] = x$ ($x \in g$).

Here (and below) the commutators are understood in the graded sense, i.e., $[d, i_x] = d \circ i_x + i_x \circ d$.

(d) $[x, i_y] = i_{[x, y]}$ ($x, y \in g$).

(e) $[i_x, i_y] = 0$ ($x, y \in g$).

4.3. Let us consider the enveloping algebra $Ug^*$. It is a dg associative algebra. We have the canonical embedding

$$\Lambda'(g^{-1}) \hookrightarrow Ug^*.$$ 

We shall use the notation $i_{x_1\ldots x_a}$ for the elements $i_{x_1} \cdots i_{x_a} \in (Ug^*)^{-a}x_1, \ldots, x_a \in g$, where $(Ug^*)^k$ stands for grade degree $k$ component of $Ug^*$.

4.4. Lemma. For any $x_1, \ldots, x_a \in g$, we have

$$di_{x_1\ldots x_a} = \sum_{p=1}^{a}(-1)^{p-1}i_{x_p}i_{x_1\ldots \hat{x_p}\ldots x_a} + \sum_{1 \leq p < q \leq a}(-1)^{p+q+i_{[x_p, x_q]}x_1\ldots \hat{x_p}\ldots \hat{x_q}\ldots x_a}$$

$$= \sum_{p=1}^{a}(-1)^{p-1}i_{x_1\ldots x_p\ldots x_a} x_p + \sum_{1 \leq p < q \leq a}(-1)^{p+q+1+i_{[x_p, x_q]}x_1\ldots \hat{x_p}\ldots \hat{x_q}\ldots x_a}.$$ 

Proof. Induction on $a$. For $a = 1$ it is the Cartan formula. Suppose that $a > 1$.

We have

$$di_{x_1\ldots x_a} = d(i_{x_1} \cdot i_{x_2\ldots x_a}) = di_{x_1} \cdot i_{x_2\ldots x_a} - i_{x_1} \cdot di_{x_2\ldots x_a}$$

(by induction)

$$= i_{x_1} \cdot i_{x_2\ldots x_a} - i_{x_1} \cdot \left( \sum_{p=2}^{a}(-1)^{p}i_{x_p}i_{x_1\ldots \hat{x_p}\ldots x_a} + \sum_{2 \leq p < q \leq a}(-1)^{p+q+i_{[x_p, x_q]}x_1\ldots \hat{x_p}\ldots \hat{x_q}\ldots x_a} \right)$$

We use $i_{x_1}x_p = x_pi_{x_1} + i_{[x_1, x_p]}$ and the anticommutation of the various elements $i_x$.

$$= i_{x_1} \cdot i_{x_2\ldots x_a} + \sum_{p=2}^{a}(-1)^{p-1}i_{x_p}i_{x_1\ldots \hat{x_p}\ldots x_a} + \sum_{p=2}^{a}(-1)^{1+p+i_{[x_1, x_p]}x_2\ldots \hat{x_p}\ldots x_a}$$

$$+ \sum_{2 \leq p < q \leq a}(-1)^{p+q+i_{[x_p, x_q]}x_1\ldots \hat{x_p}\ldots \hat{x_q}\ldots x_a}$$

$$= \sum_{p=1}^{a}(-1)^{p-1}i_{x_p}i_{x_1\ldots \hat{x_p}\ldots x_a} + \sum_{1 \leq p < q \leq a}(-1)^{p+q+i_{[x_p, x_q]}x_1\ldots \hat{x_p}\ldots \hat{x_q}\ldots x_a}.$$
This proves the first equality. The second equality is proved in the same manner, or may be deduced from the first one. \hfill \Box

4.5. Let $M$ be a $\mathfrak{g}$-module. Recall that the complex $C^\ast(\mathfrak{g}; M)$ of cochains of $\mathfrak{g}$ with coefficients in $M$ is defined by $C^a(\mathfrak{g}; M) = \text{Hom}(\Lambda^a \mathfrak{g}, M)$; the differential $d$: $C^{a-1}(\mathfrak{g}; M) \to C^a(\mathfrak{g}; M)$ acts as

$$d\phi(x_1 \wedge \cdots \wedge x_a) = \sum_{p=1}^{a} (-1)^{p-1} x_p \phi(x_1 \wedge \cdots \wedge \hat{x}_p \wedge \cdots \wedge x_a) + \sum_{1 \leq p < q \leq a} (-1)^{p+q} \phi(x_1 \wedge \cdots \wedge \hat{x}_p \wedge \cdots \hat{x}_q \wedge \cdots \wedge x_a).$$

The differential in $C^\ast(\mathfrak{g}; M)$ is called the Koszul differential.

4.6. \textit{Remark.} Consider $U\mathfrak{g}^\ast$ as a $\mathfrak{g}$-module by means of the left multiplication, where $\mathfrak{g}$ is identified with $\mathfrak{g}^0$. Consider the complex of cochains of $\mathfrak{g}$ with coefficients in $U\mathfrak{g}^\ast$, $C^\ast(\mathfrak{g}, U\mathfrak{g}^\ast)$. It is a double complex in which the first differential is the Koszul differential, and the second one is induced by the differential in $U\mathfrak{g}^\ast$. Let $C^\ast$ be the associated total complex.

For each $a \geq 0$, we define an element $c^{a,-a} \in C^a(\mathfrak{g}, U\mathfrak{g}^{-a})$ by

$$c^{a,-a}(x_1 \wedge \cdots \wedge x_a) = i_{x_1 \cdots x_a}, \quad c^{00} = 1.$$

One can reformulate the previous lemma as the following statement.

(a) The element $c = \sum_{a \geq 0} c^{a,-a}$ is a 0-cocycle in $C^\ast$.

4.7. Suppose we are given the collection of data (a)–(d) below.

(a) A Lie algebra $\mathfrak{g}$;

(b) a $\mathfrak{g}$-module $M$;

(c) a dg $\mathfrak{g}$-module $\Omega$;

(d) an element $\omega \in M \otimes \Omega^n$, for some $n \geq 0$.

Assume that the element $\omega$ satisfies the properties (i), (ii) below.

(i) $d\omega = 0$.

Here $d = \text{Id}_M \otimes d\Omega$: $M \otimes \Omega^n \to M \otimes \Omega^{n+1}$.

(ii) $\omega \in (M \otimes \Omega^n)^0$.

Here the superscript $(\cdot)^0$ denotes the subspace of $\mathfrak{g}$-invariants. We consider $M \otimes \Omega^n$ as a $\mathfrak{g}$-module equal to the tensor product of the two $\mathfrak{g}$-modules $M$ and $\Omega^n$, the last one being the $\mathfrak{g}$-module obtained via the identification $\mathfrak{g} = \mathfrak{g}^0$.

Consider the double complex $C^\ast(\mathfrak{g}; M \otimes \Omega^\ast)$ defined by

$$C^a(\mathfrak{g}; M \otimes \Omega^b) = \text{Hom}(\Lambda^a \mathfrak{g}, M \otimes \Omega^b).$$

The first differential is the Koszul differential in the standard cochain complex of $\mathfrak{g}$ with coefficients in $M \otimes \Omega^\ast$. Here the action of $\mathfrak{g}$ on $M \otimes \Omega^\ast$ is defined \textit{through the first factor} (i.e., as $x \cdot (m \otimes \alpha) = xm \otimes \alpha$). The second differential is induced by the differential in $\Omega^\ast$. Let $C^\ast(\mathfrak{g}; M, \Omega^\ast)$ denote the associated total complex.

For $0 \leq a \leq n$, we define the elements $w^a \in C^a(\mathfrak{g}; M \otimes \Omega^{n-a})$ by

$$\omega^0 = 0, \quad \omega^a(x_1 \wedge \cdots \wedge x_a) = i_{x_1 \cdots x_a} \omega.$$

Here the action of $U\mathfrak{g}^\ast$ is defined through the second factor. Consider the element $\tilde{\omega} = \sum_{a=0}^n \omega^a \in C^n(\mathfrak{g}; M, \Omega^\ast)$. 
4.8. Lemma. The element $\hat{\omega}$ is an $n$-cocycle in $C^\ast(\mathfrak{g}; M, \Omega)$. We shall call $\hat{\omega}$ the Cartan cocycle associated with $\omega$.

Proof. Let $d'$ (resp., $d''$) denote the first (resp., second) differential in $C^\ast(\mathfrak{g}; M \otimes \Omega)$. We have $d''(\omega^0) = 0$ by property 4.7(i). We have

$$d'w^{a-1}(x_1 \wedge \cdots \wedge x_n) = \sum_{p=1}^a (-1)^{p-1} x_p \cdot \omega^{a-1}(x_1 \wedge \cdots \wedge \hat{x}_p \wedge \cdots \wedge x_n) + \sum_{1 \leq p < q \leq a} (-1)^{p+q} \omega^{a-1}([x_p, x_q] \wedge x_1 \wedge \cdots \wedge \hat{x}_p \wedge \cdots \wedge \hat{x}_q \wedge \cdots \wedge x_n).$$

Here $\omega \mapsto x^{(1)} \cdot \omega$, (resp., $\omega \mapsto x^{(2)} \cdot \omega$), denotes the action of $\mathfrak{g}$ on $M \otimes \Omega^n$ through the first (resp., second) factor.

Using the identity

$$x_p \cdot (i x_1 \ldots \hat{x}_p \ldots x_n \omega) = i x_1 \ldots \hat{x}_p \ldots x_n (x_p^{(1)} \omega) = (\text{by } 4.7(ii)) = -i x_1 \ldots \hat{x}_p \ldots x_n (x_p^{(2)} \omega) = -(i x_1 \ldots \hat{x}_p \ldots x_n x_p) \omega.$$

by Lemma 4.4 = $(di x_1 \ldots x_n) \omega = (by \ (i)) = d(i x_1 \ldots x_n \omega) = (d'' \omega^a) (x_1 \wedge \cdots \wedge x_n).$ □

4.9. Consider the dual complex $(\Omega')^\ast$. Note that the cocycle $\hat{\omega}$ defines a morphism of complexes

$$\hat{\omega}: (\Omega^{n-\ast})^\ast \to C^\ast(\mathfrak{g}; M).$$

Let us denote $H_i(\Omega) := H^{-i}(\Omega^\ast)$. The morphism $\hat{\omega}$ induces the following maps

$$H^{-i}(\omega): H_i(\Omega^\ast) \to H^{n-i}(\mathfrak{g}; M) \quad (0 \leq i \leq n).$$

4.10. The construction of this Section should be compared with [Br].

§5. Bosonization for the Virasoro algebra

Our account of the bosonization for the Virasoro algebra essentially follows the paper [F].

5.1. Heisenberg algebra

The Heisenberg algebra is the Lie algebra $\mathcal{H}$ defined by the generators $b_n$ ($n \in \mathbb{Z}$) and $1$, and relations

(a) $[b_n, b_m] = 2n\delta_{n+m, 0} 1, \quad [1, b_n] = 0$ ($n, m \in \mathbb{Z}$).

The elements $1$ and $b_0$ lie in the center of $\mathcal{H}$. The elements $b_n$, $n > 0$, are called annihilation operators.

We shall denote by $\mathcal{H}^+$ (resp., $\mathcal{H}^-$, $\mathcal{H}^0$) the Lie subalgebra of $\mathcal{H}$ generated by the elements $b_n$, $n > 0$ (resp., by $b_n$, $n < 0$, by $b_0$ and $1$).

5.2. Given two generators $b_n$, $b_m$, define their normal ordered product $:b_n b_m: \in \mathcal{U}\mathcal{H}$ as follows. If $n > 0$ and $m \leq 0$, we set $:b_n b_m: = b_n b_m$; otherwise set $:b_n b_m: := b_n b_m$. We define

$$\{b_n b_m\} = b_n b_m - :b_n b_m:.$$
(not to be confused with the Lie bracket!).

Similarly, the normal ordering of an arbitrary monomial \( b_{n_1} \cdots b_{n_k} \) is defined: one should pull all the annihilation operators to the right, not changing their order. (The last requirement does not matter very much since all the annihilation operators commute.)

We have by definition

\[
  b_nb_m = :b_nb_m: + \{b_nb_m\}.
\]

The following identity generalizes this equality.

5.3. **Wick theorem.** Let \( c_1, \ldots, c_a, c_1', \ldots, c_b' \) be arbitrary elements of the set \( \{b_n, n \in \mathbb{Z}\} \). We have

\[
  :c_1 \cdots c_a: \cdot :c_1' \cdots c_b': = :c_1 \cdots c_a' : + \sum_{p,q} \{c_pc_q'\} :c_1 \cdots \hat{c}_p \cdots \hat{c}_q' \cdots c_a' : + \ldots. \quad \square
\]

5.4. We introduce a "free bosonic field", the formal expression:

\[
  \phi(z) = -b_0 \log(z) + \sum_{n \neq 0} \frac{b_n}{n} z^{-n},
\]

where \( z \) is a formal variable. Let \( \phi'(z) \) be the formal derivative of \( \phi(z) \), that is the generating function

\[
  \phi'(z) = -\sum_{n \in \mathbb{Z}} b_n z^{-n-1}.
\]

Here \( z \) is a formal variable.

Commutation relations 5.1 (a) are equivalent to the identity

(a) \( \{\phi'(z)\phi'(w)\} = \frac{2}{(z-w)^2} \).

Let us deduce (a) from 5.1 (a). We have by definition

\[
  \{\phi'(z)\phi'(w)\} = \phi'(z)\phi'(w) - :\phi'(z)\phi'(w): = \sum_{n,m} (b_nb_m - :b_nb_m:) z^{-n-1}w^{-m-1}
\]

\[
  = \sum_{n>0} 2nz^{-n-1}w^{-n} = 2 \frac{\partial}{\partial w} \left( \sum_{n \geq 0} z^{-n-1}w^n \right) = 2z^{-1} \frac{\partial}{\partial w} \left( \frac{1}{1-w/z} \right)
\]

\[
  = 2 \frac{\partial}{\partial w} \left( \frac{1}{z-w} \right) = \frac{2}{(z-w)^2}.
\]

5.5. Given a complex number \( \alpha \in \mathbb{C} \), define the Fock representation \( F_\alpha \) of the Heisenberg algebra \( \mathcal{H} \) as an \( \mathcal{H} \)-module with one generator \( v_\alpha \) and the relations

\[
  b_nv_\alpha = 0 \quad (n > 0), \quad b_0v_\alpha = 2\alpha v_\alpha, \quad 1v_\alpha = v_\alpha.
\]
The mapping \( x \mapsto x \cdot v_\alpha \) gives an isomorphism of \( \mathcal{H}^- \)-modules \( U\mathcal{H}^- \xrightarrow{\sim} F_\alpha \).

Define the shift operator
\[
T_\beta : F_\alpha \rightarrow F_{\alpha + \beta}
\]
as the unique \( \mathcal{H}^- \)-linear operator sending \( v_\alpha \) to \( v_{\alpha + \beta} \).

Define the category \( \mathcal{O} \) to be the full subcategory of the category of \( \mathcal{H} \)-modules whose objects are representations \( M \) having the properties
\begin{enumerate}[(a)]  
  \item \( 1 \) acts as the identity on \( M \);
  \item \( M \) is \( b_0 \)-diagonalizable. All \( b_0 \)-weight spaces are finite-dimensional;
  \item \( M \) is \( \mathcal{H}^+ \)-locally finite. This means that for any \( x \in M \), the space \( U\mathcal{H}^+ \cdot x \) is finite-dimensional.
\end{enumerate}

All Fock representations belong to the category \( \mathcal{O} \).

5.6. Recall that the Witt algebra \( \mathcal{L} \) is the Lie algebra of algebraic vector fields on \( \mathbb{C}^* = \mathbb{C} - \{0\} \). It can be defined by the generators \( e_n = -z^{n+1} d/dz \) (\( n \in \mathbb{Z} \)) and the relations
\[
[e_n, e_m] = (n - m) e_{n+m} \quad (n, m \in \mathbb{Z}).
\]
The Virasoro algebra \( \hat{\mathcal{L}} \) is the Lie algebra defined by the generators \( L_n \) (\( n \in \mathbb{Z} \)) and \( c \), and relations
\[
[L_n, L_m] = (n - m) L_{n+m} + \frac{n^3 - n}{12} \delta_{n+m,0} \cdot c, \quad [c, L_n] = 0 \quad (n, m \in \mathbb{Z}).
\]
We have the morphism of Lie algebras \( \hat{\mathcal{L}} \rightarrow \mathcal{L} \) sending \( L_n \) to \( e_n \), and \( c \) to 0. This morphism identifies \( \mathcal{L} \) with \( \hat{\mathcal{L}}/\mathbb{C} \cdot c \). It identifies \( \hat{\mathcal{L}} \) with a universal central extension of \( \mathcal{L} \).

Let \( \hat{\mathcal{L}}^+ \) (resp., \( \hat{\mathcal{L}}^- \), \( \hat{\mathcal{L}}^0 \)) denote the Lie subalgebras of \( \hat{\mathcal{L}} \) generated by the elements \( L_n \), \( n > 0 \) (resp., by \( L_n \), \( n < 0 \), by \( L_0 \), \( c \)).

The generating function
\[
T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}
\]
is called the stress-energy tensor.

5.7. Let \( \alpha_0 \in \mathbb{C} \). Consider the expressions
\begin{enumerate}[(a)]  
  \item \[
L_n = \frac{1}{4} \sum_{p+q = n} :b_p b_q: - \alpha_0 (n + 1) b_n \quad (n \in \mathbb{Z}).
\]
\end{enumerate}
Although they are infinite sums, these expressions are well defined as operators on modules from the category \( \mathcal{O} \). We can rewrite them as follows.
\begin{enumerate}[(b)]  
  \item \[
L_n = \frac{1}{4} \sum_{p \in \mathbb{Z}} b_{n-p} b_p - \alpha_0 (n + 1) b_n \quad \text{if} \ n \neq 0,
\]
  \item \[
L_0 = \frac{1}{2} \sum_{p \geq 1} b_{-p} b_p + \frac{1}{4} b_0^2 - \alpha_0 b_0.
\]
\end{enumerate}
In terms of generating functions, (a) reads
\begin{enumerate}[(c)]  
  \item \[
T(z) = \frac{1}{4} :\phi'(z)^2: - \alpha_0 \phi''(z).
\]
\end{enumerate}
5.8. **Theorem.** The expressions 5.7 (a) define the action of the Virasoro algebra on modules from the category $\mathcal{O}$, with the central charge $1 - 24\alpha_0^2$.

5.9. **Lemma.** The operators $L_n$, 5.7 (a), satisfy the following commutation relations:

(a) \[ [b_n, L_m] = nb_{n+m} + 2n(n-1)\alpha_0\delta_{n,-m}. \]

**Proof.** This can be checked easily using the definitions. Let us give an alternative proof, using some simple chiral calculus. We claim that (a) is equivalent to

(b) \[ \phi'(z)T(w) = -\frac{4\alpha_0}{(z-w)^3} + \frac{\phi'(w)}{(z-w)^2} + \cdots. \]

Here (and below) dots denote the expression regular at $z = w$. Let us prove that (b) implies (a). According to Cauchy formula of the chiral calculus, we have

\[ [b_n, T(w)] = -\res_{z=w} (z^n \phi'(z)T(w)). \]

for any $n$. By the binomial formula,

\[ \res_{z=w} \frac{z^n}{(z-w)^a} = \left( \frac{n}{a-1} \right) w^{n-a+1}. \]

Therefore, (b) implies

(c) \[ [b_n, T(w)] = 2n(n-1)\alpha_0w^{n-2} - n\phi'(w)w^{n-1} \]

which is equivalent to (a).

Let us prove (b). We have, by Wick’s theorem,

\[ \phi'(z)\phi'(w)^2 = 2\{\phi'(z)\phi'(w)\}\phi'(w) + \cdots = \frac{4\phi'(w)}{(z-w)^2} + \cdots, \]

\[ \phi'(z)\phi''(w) = \frac{\partial}{\partial w}(\phi'(z)\phi'(w)) = \frac{4}{(z-w)^3} + \cdots. \]

This implies (b) and proves the lemma. □

5.10. **Proof of Theorem 5.8.** The Virasoro commutation relations can be verified directly, using the previous lemma and 5.7 (b), by a tedious computation.

Let us give a proof using the chiral calculus. We must prove the following.

(a) \[ T(z)T(w) = \frac{1 - 24\alpha_0^2}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{T'(w)}{z-w} + \cdots. \]

Let us derive the Virasoro commutation relations from (a). By the Cauchy formula of the chiral calculus, we have

\[ [L_n, T(w)] = \res_{z=w} (z^{n+1}T(z)T(w)). \]
As in the proof of the previous lemma, (a) implies that

\[(b) \quad [L_n, T(w)] = T'(w) w^{n+1} + 2(n+1) T(w) w^n + (1 - 24\alpha_0^2) \frac{n^3 - n}{12} w^{n-2}\]

which is equivalent to the Virasoro commutation relations with the central charge \(1 - 24\alpha_0^2\).

Let us prove (a). We have, by Wick’s theorem,

\[
: \phi'(z)^2: \phi'(w)^2 : = 2 \{ \phi'(z) \phi'(w) \}^2 + 4 \{ \phi'(z) \phi'(w) \} : \phi'(z) \phi'(w) : =
\]

\[
\frac{8}{(z-w)^2} + \frac{8}{(z-w)^2} : \phi'(w)^2 : + \frac{8}{z-w} : \phi'(w) \phi''(w) : + \cdots,
\]

\[
\phi''(z) : \phi'(w)^2 = 2 \{ \phi''(z) \phi'(w) \} \phi'(w) + \cdots
\]

\[
\frac{8}{(z-w)^3} \phi'(w) - \frac{8}{(z-w)^3} \phi'(w) + \cdots,
\]

\[
: \phi'(z)^2 : \phi''(w) = 2 \{ \phi'(z) \phi''(w) \} \phi'(z) = 2 \frac{\partial}{\partial w} \{ \phi'(z) \phi'(w) \} \phi'(z)
\]

\[
= \frac{8}{(z-w)^3} \left( \phi'(w) + (z-w) \phi''(w) + \frac{(z-w)^2}{2} \phi'''(w) + \cdots \right)
\]

\[
= \frac{8}{(z-w)^3} \phi'(w) + \frac{8}{(z-w)^3} \phi''(w) + \frac{4}{z-w} \phi'''(w) + \cdots,
\]

\[
\phi''(z) \phi''(w) = \{ \phi''(z) \phi''(w) \} + \cdots = \frac{12}{(z-w)^4} + \cdots.
\]

Therefore we obtain, after adding,

\[
T(z) T(w) = \left( \frac{1}{4} : \phi'(z)^2 : - \alpha_0 \phi'''(z) \right) \left( \frac{1}{4} : \phi'(w)^2 : - \alpha_0 \phi'''(w) \right)
\]

\[
= \frac{1 - 24\alpha_0^2}{2(z-w)^4} + \frac{(1/2) : \phi'(w)^2 : - 2\alpha_0 \phi''(w)}{(z-w)^2}
\]

\[
+ \frac{(1/2) : \phi'(w) \phi''(w) : - \alpha_0 \phi'''(w)}{z-w} + \cdots
\]

\[
= \frac{1 - 24\alpha_0^2}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{T'(w)}{z-w} + \cdots.
\]

This proves (a) and the theorem. \(\square\)

5.11. We define the representation \(F_{\alpha,\alpha_0}\) of the Virasoro algebra as the \(H\)-module \(F_{\alpha}\), regarded as an \(\hat{L}\)-module by means of the formulas in the previous theorem.

The representations \(F_{\alpha,\alpha_0}\) will be called Feigin–Fuchs modules.

§6. Bosonic vertex operators

6.1. We need first to recall some formulas for ”bosonization”, see [FF, TK].

Let \(\alpha, \beta \in \mathbb{C}\). Define the operators

\[
V_n(\beta) : F_{\alpha} \rightarrow F_{\alpha+\beta} \quad (n \in \mathbb{Z})
\]
by means of the generating function

\[ V(\beta; z) = \sum_{n \in \mathbb{Z}} V_n(\beta) z^{-n}. \]

We set by definition

\[
V(\beta; z) = T_\beta \exp \left( -\beta \sum_{n \neq 0} \frac{b_n}{n} z^{-n} \right) = T_\beta \exp \left( -\beta \sum_{n<0} \frac{b_n}{n} z^{-n} \right) \exp \left( -\beta \sum_{n>0} \frac{b_n}{n} z^{-n} \right).
\]

This expression is an operator acting as follows

\[ V(\beta; z): F_\alpha \to F_\alpha + \beta((z^{-1})) \).

These operators are called the \textit{bosonic vertex operators}. We also define the operators

\[
V_-(\beta; z) = \exp \left( -\beta \sum_{n<0} \frac{b_n}{n} z^{-n} \right): F_\alpha \to F_\alpha[[z]],
\]

\[
V_+(\beta; z) = \exp \left( -\beta \sum_{n>0} \frac{b_n}{n} z^{-n} \right): F_\alpha \to F_\alpha[z^{-1}].
\]

Thus, \( V(\beta; z) = T_\beta V_-(\beta; z) V_+(\beta; z) \).

6.2. \textbf{Lemma.} We have

\[
[b_n, V_-(\beta; z)] = \begin{cases} 
2\beta z^n V_-(\beta; z) & \text{if } n > 0, \\
0 & \text{otherwise},
\end{cases}
\]

\[
[b_n, V_+(\beta; z)] = \begin{cases} 
2\beta z^n V_+(\beta; z) & \text{if } n < 0, \\
0 & \text{otherwise}.
\end{cases}
\]

We leave the easy proof to the reader.

6.3. \textbf{Theorem.} For every \( n \in \mathbb{Z} \), we have \([b_n, V(\beta; z)] = 2\beta z^n V(\beta; z)\).

\textbf{Proof.} Follows at once from the previous lemma. \(\square\)

6.4. Let us give an alternative proof, which uses the chiral calculus. Let us introduce the expression

\[
\tilde{V}(\beta; z) = z^{2\beta\alpha} V(\beta; z).
\]

At this point \( z^{\beta\alpha} \) is a formal symbol. It will play its role in the sequel, when we start differentiating. More precisely, \( \tilde{V}(\beta; z) \) is the operator \( V(\beta; z) \) considered as a section of a certain De Rham complex with a nontrivial connection.

Our formula is equivalent to \([b_n, \tilde{V}(\beta; z)] = 2\beta \tilde{V}(\beta; z)\). We must prove that

(a) \[ \phi'(z) \tilde{V}(\beta; w) = -\frac{2\beta}{z-w} \tilde{V}(\beta; w) + \cdots. \]
Recall the “free bosonic field”

$$\phi(z) = q - b_0 \log(z) + \sum_{n \neq 0} \frac{b_n}{n} z^{-n}. $$

where now we added a $z$-independent term, an operator $q$ that satisfies the following commutation relations

$$[q, b_n] = 2\delta_{n,0}. $$

It follows that

$$[b_0, e^{-\beta q}] = 2\beta e^{-\beta q}. $$

We identify the operator $e^{-\beta q}$ with $T_\beta$. Thus,

$$\tilde{V}(\beta; z) = \exp(-\beta \phi(z)), $$

where the only nontrivial normal ordering with the operator $q$ is defined as \( :b_0 q: = q b_0 \).

We have

(b) \( \phi(z) \phi(w) = 2 \log(z-w) + :\phi(z)\phi(w):. \)

It follows that

$$\phi'(z) \phi(w) = \frac{2}{z-w} + \cdots, $$

hence, from the Wick theorem

$$\phi'(z):\phi(w)^n: = \frac{2n}{z-w} :\phi(w)^{n-1}: + \cdots $$

\((n \geq 0)\), so if $f(\phi)$ is any power series in $\phi$, 

$$\phi'(z):f(\phi(w)): = \frac{2}{z-w} :f'(\phi(w)): + \cdots. $$

In particular, we have

$$\phi'(z):\exp(-\beta \phi(w)): = -\frac{2\beta}{z-w} :\exp(-\beta \phi(w)): + \cdots, $$

which proves (a) and the theorem. □

6.5. We regard the operators $V_n(\beta)$ as operators acting on Feigin–Fuchs modules

$$V_n(\beta): F_{\alpha;\alpha_0} \to F_{\alpha+\beta;\alpha_0}. $$

The generating function $V(\beta; z)$ will be understood in the same sense.
6.6. **Theorem.** For any \( n \in \mathbb{Z} \), we have

\[
[L_n, V(\beta; z)] = \left( z^{n+1} \frac{d}{dz} + (\beta^2 - 2\alpha_0\beta)(n+1) + 2\alpha\beta z^n \right) V(\beta; z).
\]

In other words,

\[
[L_n, \tilde{V}(\beta; z)] = \left( z^{n+1} \frac{d}{dz} + (\beta^2 - 2\alpha_0\beta)(n+1) z^n \right) \tilde{V}(\beta; z).
\]

**Proof.** We must prove that

\[
(a) \quad T(z)\tilde{V}(\beta; w) = \frac{\beta^2 - 2\alpha_0\beta}{(z-w)^2} \tilde{V}(\beta; w) + \frac{1}{z-w} \tilde{V}'(\beta; w) + \cdots.
\]

Let us prove \((a)\). We have, by the Wick theorem,

\[
: \phi'(z)^2 : \phi(w)^n = \frac{4n(n-1)}{(z-w)^2} : \phi(w)^{n-2} : + \frac{4n}{z-w} : \phi'(w)\phi(w)^{n-1} : + \cdots
\]

for any \( n \geq 0 \), hence

\[
: \phi'(z)^2 : \tilde{V}(\beta; w) = \frac{4\beta^2}{(z-w)^2} \tilde{V}(\beta; w) + \frac{4}{z-w} \tilde{V}'(\beta; w) + \cdots.
\]

It follows from 6.4 \((a)\) that

\[
\phi''(z) \tilde{V}(\beta; w) = \frac{2\beta}{(z-w)^2} \tilde{V}(\beta; w) + \cdots.
\]

Summing, we get \((a)\). This proves the theorem. \(\square\)

6.7. **Lemma.** We have

\[
V_+(\beta_1; z_1) V_-(\beta_2; z_2) = (1 - z_1^{-1} z_2)^{2\beta_1\beta_2} V_-(\beta_2; z_2) V_+(\beta_1; z_1)
\]

\((equality~in~\text{Hom}(F_{\alpha}, F_{\alpha}[z_1^{-1}, z_2]))\).

Here we understand \((1 - z_1^{-1} z_2)^{2\beta_1\beta_2}\) as the formal power series

\[
\exp \left( -2\beta_1\beta_2 \sum_{n>0} z_1^{-n} z_2^n \frac{z_2^n}{n} \right).
\]

**Remark.** The operator \( V_-(\beta_2; z_2) V_+(\beta_1; z_1) \) belongs to the space \( \text{Hom}(F_{\alpha}, F_{\alpha}[z_1^{-1}, z_2]) \).

**Proof.** By Lemma 6.2, we have

\[
V_+(\beta_1; z_1) b_{-n} = (b_{-n} - 2\beta_1 z_1^{-n}) V_+(\beta_1; z_1)
\]
for \( n > 0 \), hence

\[
V_+(\beta_1; z_1) \exp \left( \beta_2 \frac{b_{-n}}{n} z_2^n \right) = \exp \left( \beta_2 \frac{b_{-n}}{n} - 2\beta_1 z_1^{-n} z_2^n \right) V_+(\beta_1; z_1)
\]

\[
= \exp \left( -2\beta_1 \beta_2 \frac{1}{n} z_1^{-n} z_2^n \right) \exp \left( \beta_2 \frac{b_{-n}}{n} z_2^n \right) V_+(\beta_1; z_1)
\]

Therefore,

\[
V_+(\beta_1; z_1) V_-(\beta_2; z_2) = \exp \left( -2\beta_1 \beta_2 \sum_{n>0} \frac{1}{n} z_1^{-n} z_2^n \right) V_-(\beta_2; z_2) V_+(\beta_1; z_1)
\]

\[
= \exp(2\beta_1 \beta_2 \log(1 - z_1^{-1} z_2)) V_-(\beta_2; z_2) V_+(\beta_1; z_1)
\]

\[
= (1 - z_1^{-1} z_2)^{2\beta_1 \beta_2} V_-(\beta_2; z_2) V_+(\beta_1; z_1). \quad \Box
\]

6.8. Choose complex numbers \( \alpha, \beta_1, \ldots, \beta_p \). Consider the generating function for all compositions of the form

\[
F_\alpha \xrightarrow{V_\alpha(\beta_p)} F_{\alpha+\beta_p} \rightarrow \cdots \xrightarrow{V_{n_1}(\beta_1)} F_{\alpha+\beta_p+\cdots+\beta_1};
\]

We have an equation

\[
\sum_{(n_1, \ldots, n_p) \in \mathbb{Z}_p} V_{n_1}(\beta_1) \cdots V_{n_p}(\beta_p) z_1^{-n_1} \cdots z_p^{-n_p} = V(\beta_1; z_1) \cdots V(\beta_p; z_p)
\]

as a formal power series. The previous lemma shows that

(a) \( V(\beta_1; z_1) \cdots V(\beta_p; z_p) = \prod_{1 \leq i < j \leq p} (1 - z_i^{-1} z_j)^{2\beta_i \beta_j} V(\beta_1; z_1) \cdots V(\beta_p; z_p); \)

6.9. Let us look more closely at the operator \( :V(\beta_1; z_1) \cdots V(\beta_p; z_p): \).

We have

\[
:V(\beta_1; z_1) \cdots V(\beta_p; z_p): = T_{\beta_1 + \cdots + \beta_p} \exp \left( - \sum_{n<0} \beta_1 z_1^{-n} + \cdots + \beta_p z_p^{-n} \right) b_n
\]

\[
\times \exp \left( - \sum_{n>0} \beta_1 z_1^{-n} + \cdots + \beta_p z_p^{-n} \right) b_n.
\]

It follows that

(a) the operator \( :V(\beta_1; z_1) \cdots V(\beta_p; z_p): \) belongs to the space \( \text{Hom}(F_\alpha, F_{\alpha+\sum_{\beta_i} [z_1^{-1}, \ldots, z_p, z_p^{-1}]} \). \)

In particular, this operator is a holomorphic operator-valued function on the space \( \mathbb{C}^p - \bigcup_{i=1}^p \{ z_i = 0 \} \).

Also, the above formula shows that

(b) for any bijection \( \sigma: \{ 1, \ldots, p \} \xrightarrow{\sim} \{ 1, \ldots, p \}, \) we have
In other words, the theorem states that:

\[ V(\beta_1; z_1) \cdots V(\beta_p; z_p) = :V(\beta_{\sigma(1)}; z_{\sigma(1)}) \cdots V(\beta_{\sigma(p)}; z_{\sigma(p)}) :. \]

6.10. Formula 6.8 (a) shows that the formal power series \( V(\beta_1; z_1) \cdots V(\beta_p; z_p) \) defines the germ of a holomorphic multivalued function in the domain \(|z_1| > \cdots > |z_p| > 0\), where \((1 - z_i^{-1}z_j)^{2\beta_i \beta_j}\) is understood as \(\exp(-2\beta_i \beta_j \sum_{n>0} z_i^{-n}z_j^n/n)\).

6.11. Let us introduce the tilded operators. We set

\[ \tilde{V}(\beta_1; z_i) = :\exp(-\beta_i \phi(z_i)) = T_{\beta_i} z_i^{\beta_i b_0} :\exp(-\beta_i \sum_{n \neq 0} \frac{b_n}{n} z_i^{-n}) :. \]

Since \( z_i^{\beta_i b_0} T_{\beta_j} = z_i^{2\beta_i \beta_j} T_{\beta_j} z_i^{\beta_i b_0} \), we have

\[ \tilde{V}(\beta_1; z_1) \cdots \tilde{V}(\beta_p; z_p) = T_{\sum \beta_i} \prod \left( z_i^{\beta_i b_0} \prod_{i<j} z_i^{2\beta_i \beta_j} V(\beta_1; z_1) \cdots V(\beta_p; z_p) \right) \]

\[ = \prod z_i^{2\beta_i \alpha} \prod_{i<j}(z_i - z_j)^{2\beta_i \beta_j} V(\beta_1; z_1) \cdots V(\beta_p; z_p) :. \]

Recall that the operator \( V(\beta_1; z_1) \cdots V(\beta_p; z_p) :\) belongs to the space

\[ \text{Hom}(F_{\alpha}, F_{\alpha + \sum \beta_i [z_1, z_1^{-1}, \ldots, z_p, z_p^{-1}]}). \]

This defines the product \( \tilde{V}(\beta_1; z_1) \cdots \tilde{V}(\beta_p; z_p) \) as the germ of the multivalued holomorphic function in the domain \( \{(z_1, \ldots, z_p) \in \mathbb{C}^p \mid z_i \notin \mathbb{R}_{\leq 0} \text{ for all } i; |z_1| > \cdots > |z_p|\} \).

Here \( z^\gamma \) is understood as \(\exp(\gamma \text{log}(z))\), where \(\text{log}(z)\) is the branch of the logarithm that takes real values for \( z \in \mathbb{R}_{>0} \).

Let us regard the previous compositions as operators acting on the Feigin–Fuchs modules

\[ \tilde{V}(\beta_1; z_1) \cdots \tilde{V}_p(\beta_p; z_p): F_{\alpha;\alpha_0} \to F_{\alpha + \sum \beta_i;\alpha_0}. \]

6.12. Theorem. For any \( n \in \mathbb{Z}, \)

\[ [L_n, \tilde{V}(\beta_1; z_1) \cdots \tilde{V}(\beta_p; z_p)] \]

\[ = \left( \sum_{i=1}^p z_i^{n+1} \partial_{z_i} + (\beta_i^2 - 2\alpha_0 \beta_i)(n + 1) z_i^n \right) \tilde{V}(\beta_1; z_1) \cdots \tilde{V}(\beta_p; z_p). \]

In other words,

\[ [L_n, :V(\beta_1; z_1) \cdots V(\beta_p; z_p):] \]

\[ = \left( \sum_{i=1}^p (z_i^{n+1} \partial_{z_i} + ((\beta_i^2 - 2\alpha_0 \beta_i)(n + 1) + 2\beta_i \alpha) z_i^n) \right) :V(\beta_1; z_1) \cdots V(\beta_p; z_p):. \]
Proof. By the Cauchy formula,

$$[L_n, \tilde{V}(\beta_1; z_1) \cdots \tilde{V}(\beta_p; z_p)] = \sum_{i=1}^{p} \text{res}_{z=z_i} (z^{n+1}T(z) \tilde{V}(\beta_1; z_1) \cdots \tilde{V}(\beta_p; z_p)).$$

Note that

$$\tilde{V}(\beta_1; z_1) \cdots \tilde{V}(\beta_p; z_p) = \prod_{i<j} (z_i - z_j)^{2\beta_i\beta_j}.\tilde{V}(\beta_1; z_1) \cdots \tilde{V}(\beta_p; z_p).$$

We claim that

(a) \( T(z) : \tilde{V}(\beta_1; z_1) \cdots \tilde{V}(\beta_p; z_p) : = \int_{-\infty}^{\infty} \frac{1}{2\pi i} \left( \sum_{i=1}^{p} \frac{\beta_i^2 - 2\alpha_0 \beta_i}{(z - z_i)^2} + \sum_{i<j} \frac{2\beta_i \beta_j}{(z - z_i)(z - z_j)} \right) : \tilde{V}(\beta_1; z_1) \cdots \tilde{V}(\beta_p; z_p) : + \cdots.

Let us prove (a). To shorten the formulas, assume that \( p = 2 \); the general case is completely similar. By Wick’s theorem, we have

\[
: \phi'(z)^2 : = \phi(z_1)^{n_1} \phi(z_2)^{n_2} = \frac{4n_1(n_1 - 1)}{(z - z_1)^2} : \phi(z_1)^{n_1 - 1} \phi(z_2)^{n_2} : + \frac{4n_2(n_2 - 1)}{(z - z_2)^2} : \phi(z_1)^{n_1} \phi(z_2)^{n_2 - 2} : + \frac{8n_1n_2}{(z - z_1)(z - z_2)} : \phi(z_1)^{n_1 - 1} \phi(z_2)^{n_2 - 1} : + \frac{4n_2}{z - z_2} : \phi(z_1)^{n_1} \phi'(z_2) \phi(z_2)^{n_2 - 1} : + \cdots
\]

for any \( n_1, n_2 \geq 0 \), hence

\[
: \phi'(z)^2 : = \frac{4}{(z - z_1)^2} \phi(z_1)^{n_1} \phi(z_2)^{n_2} : \tilde{V}(\beta_1; z_1) \tilde{V}(\beta_2; z_2) : = 4 \left( \frac{\beta_1}{z - z_1} + \frac{\beta_2}{z - z_2} \right)^2 : \tilde{V}(\beta_1; z_1) \tilde{V}(\beta_2; z_2) : + \frac{4}{(z - z_1)^2} \phi(z_1)^{n_1} \phi(z_2)^{n_2 - 1} : \tilde{V}(\beta_1; z_1) \tilde{V}(\beta_2; z_2) : + \cdots.
\]

Similarly, the application of the Wick theorem gives

\[
\phi'(z) : \tilde{V}(\beta_1; z_1) \tilde{V}(\beta_2; z_2) : = \left( - \frac{2\beta_1}{z - z_1} - \frac{2\beta_2}{z - z_2} \right) : \tilde{V}(\beta_1; z_1) \tilde{V}(\beta_2; z_2) : + \cdots,
\]

whence

\[
\phi''(z) : \tilde{V}(\beta_1; z_1) \tilde{V}(\beta_2; z_2) : = \left( \frac{2\beta_1}{(z - z_1)^2} + \frac{2\beta_2}{(z - z_2)^2} \right) : \tilde{V}(\beta_1; z_1) \tilde{V}(\beta_2; z_2) : + \cdots.
\]

Summing, we get formula (a).

The theorem follows from formula (a) by the above mentioned Cauchy residue formula. One should take into account that

$$\left( \text{res}_{z=z_i} + \text{res}_{z=z_j} \right) \frac{z^{n+1}}{(z - z_i)(z - z_j)} = \frac{z_i^{n+1} - z_j^{n+1}}{z_i - z_j}$$

and

$$\left( z_i^{n+1} \partial_{z_i} + z_j^{n+1} \partial_{z_j} \right) (z_i - z_j)^{2\beta_i\beta_j} = 2\beta_i\beta_j \frac{z_i^{n+1} - z_j^{n+1}}{z_i - z_j} (z_i - z_j)^{2\beta_i\beta_j}.$$

This completes the proof. \( \square \)
§7. Screening charges

7.1. We fix the parameters \( \alpha, \alpha_0 \in \mathbb{C} \) as in the previous section. Let \( \beta_+, \beta_- \) be the complex numbers defined by

\[
\beta_{\pm} = \alpha_0 \pm \sqrt{\alpha_0 + 1}.
\]

Thus, \( \beta_{\pm} \) are the two roots of the equation \( \beta^2 - 2\alpha_0 \beta = 1 \).

The vertex operators \( V(\beta_+; z), V(\beta_-; z) \) are called the screening charges. By theorem 6.6, for any \( n \in \mathbb{Z} \),

\[
(L_n, V(\beta_{\pm}; z)) = \left( \frac{d}{dz} + \frac{2\alpha \beta_{\pm}}{z} \right)(z^{n+1}V(\beta_{\pm}; z)).
\]

In other words,

\[
T(z)\tilde{V}(\beta_{\pm}; z) = \frac{\tilde{V}(\beta_{\pm}; w)}{(z - w)^2} + \frac{\tilde{V}'(\beta_{\pm}; w)}{z - w} + \cdots + \frac{\partial}{\partial w} \left( \frac{\tilde{V}(\beta_{\pm}; w)}{z - w} \right) + \cdots.
\]

7.2. Let us introduce the operators

\[
T(L_n; z) = z^{n+1}T(z), \quad V_{\pm}(e_n; z) = z^n\tilde{V}(\beta_{\pm}; z) \quad (L_n \in \hat{L}, e_n \in L).
\]

It follows from (b) above that these operators satisfy the following cocycle property

\[
(L_n; z)V_{\pm}(e_m; w) - T(L_m; z)V_{\pm}(e_n; w) = \frac{V_{\pm}(e_n, e_m; w)}{z - w} + \cdots.
\]

7.3. More generally, suppose that \( \beta_1, \ldots, \beta_p \) is a sequence of complex numbers, each \( \beta_i \) being equal to \( \beta_- \) or \( \beta_+ \). Let us call such a sequence a screening sequence.

Let us consider the operator :\( V(\beta_1; z_1) \cdots V(\beta_p; z_p) : \). Note that by the symmetry property 6.9 (b), we may actually assume that \( \beta_1 = \cdots = \beta_{p'} = \beta_- \) and \( \beta_{p'+1} = \cdots = \beta_p = \beta_+ \).

It follows from Theorem 6.12, that

\[
[L_n; :V(\beta_1; z_1) \cdots V(\beta_p; z_p):] = \sum_{i=1}^{p} \left( \partial_{z_i} + \frac{2\alpha \beta_i}{z_i} + \frac{2\beta_i \beta_j}{z_i - z_j} \right)(z_i^{n+1}V(\beta_1; z_1) \cdots V(\beta_p; z_p))
\]

for any screening sequence \( \beta_1, \ldots, \beta_p \), and for all \( n \in \mathbb{Z} \).

In the sequel we fix the screening sequence \( \beta_1 = \cdots = \beta_p = \beta_+ \). However, all the constructions below are valid, with the obvious modifications, for an arbitrary screening sequence.

7.4. Consider the ring

\[
A_p = \mathbb{C}[\![z_1, \ldots, z_p]\!]\left[ \prod_{i=1}^{p} z_i^{-1}, \prod_{1 \leq i < j \leq p} (z_i - z_j)^{-1} \right]
\]
(thus $A_p$ is the ring of functions on the formal variety $X_p$, the $p$th power of the formal punctured disk without diagonals).

For $1 \leq a \leq p$, let $\Omega^a$ denote the space of algebraic differential $a$-forms on $X_p$. Thus, $\Omega^0 = A_p$, and the elements of $\Omega^a$ have the form

$$\sum f(z_1, \ldots, z_p) dz_1 \wedge \cdots \wedge dz_a \quad (f(z_1, \ldots, z_p) \in A_p).$$

Consider the complex

$$\Omega^* : 0 \to \Omega^0 \overset{d}{\to} \Omega^1 \overset{d}{\to} \cdots \overset{d}{\to} \Omega^p \to 0,$$

where the differential is defined by

$$d\eta = d_{DR}(\eta) + \left( \sum_{i=1}^{p} 2\alpha^i_+ \frac{dz_i}{z_i} + \sum_{1 \leq i < j \leq p} 2\beta^i_j \frac{dz_i - dz_j}{z_i - z_j} \right) \wedge \eta$$

($\eta \in \Omega^a$), $d_{DR}$ being the usual de Rham differential.

Each vector field $\tau = \mu(z) \partial_z \in \mathcal{L}$, $\mu(z) \in \mathbb{C}[z, z^{-1}]$, defines a series of morphisms $i_\tau : \Omega^a \to \Omega^{a-1}$, by

$$i_\tau(f(z_1, \ldots, z_p) dz_1 \wedge \cdots \wedge dz_a)$$

$$= \sum_{b=1}^{a} (-1)^{b-1} \mu(z_b) f(z_1, \ldots, z_p) dz_1 \wedge \cdots \wedge \hat{dz}_b \wedge \cdots \wedge dz_a.$$

Let us define the morphisms of the Lie derivative $\text{Lie}_\tau : \Omega^a \to \Omega^a$ by

$$\text{Lie}_\tau(\eta) = di_\tau(\eta) + i_\tau(d\eta).$$

The morphisms $i_\tau, \text{Lie}_\tau$ define the action of the dg Lie algebra $\mathcal{L}^*$ associated with the Witt algebra $\mathcal{L}$ (cf. 4.2), on the complex $\Omega^*$ (cf. 4.2).

7.5. Consider the complex $\text{Hom}(F_{\alpha; \alpha_0}, F_{\alpha+p_+; \alpha_0} \otimes \Omega^*).$

Let us note that the action of the Virasoro algebra $\widehat{\mathcal{L}}$ on the modules $F_{\alpha; \alpha_0}$, $F_{\alpha+p_+; \alpha_0}$ induces the action of $\widehat{\mathcal{L}}$ on the space $\text{Hom}(F_{\alpha; \alpha_0}, F_{\alpha+p_+; \alpha_0})$ by the usual commutator formula, which factors through $\mathcal{L}$, since the Feigin–Fuchs modules have the same central charge. This in turn induces the action of $\mathcal{L}$ on the complex $\text{Hom}(F_{\alpha; \alpha_0}, F_{\alpha+p_+; \alpha_0} \otimes \Omega^*),$ which we shall call the first action. We have also the second action of $\mathcal{L}$ on $\text{Hom}(F_{\alpha; \alpha_0}, F_{\alpha+p_+; \alpha_0} \otimes \Omega^*)$ which is induced by the action of $\mathcal{L}$ on $\Omega^*$ through the Lie derivative. The first and the second actions commute.

We also have the operators

$$i_\tau : \text{Hom}(F_{\alpha; \alpha_0}, F_{\alpha+p_+; \alpha_0} \otimes \Omega^*) \to \text{Hom}(F_{\alpha; \alpha_0}, F_{\alpha+p_+; \alpha_0} \otimes \Omega^*)[-1] \quad (\tau \in \mathcal{L})$$

induced by the operators of the same name on $\Omega^*$. These operators, together with the second action, define the action of the dg Lie algebra $\mathcal{L}^*$ on the complex $\text{Hom}(F_{\alpha; \alpha_0}, F_{\alpha+p_+; \alpha_0} \otimes \Omega^*).$

Let us define the element $V^{0p} = V^{0p}(z_1, \ldots, z_p) \in \text{Hom}(F_{\alpha; \alpha_0}, F_{\alpha+p_+; \alpha_0} \otimes \Omega^p)$ by

$$V^{0p}(z_1, \ldots, z_p) = :V(\beta_+; z_1) \cdots V(\beta_+; z_p): dz_1 \wedge \cdots \wedge dz_p.$$

The formula in 7.3 may be reformulated as the following theorem.
7.6. **Theorem.** The element $V^0p$ lies in the subspace of $L$-invariants

$$\text{Hom}(F_{\alpha;\alpha_0}, F_{\alpha+p\beta_+;\alpha_0} \otimes \Omega^p)^L.$$ 

Here the action of the Lie algebra $L$ on the space $\text{Hom}(F_{\alpha;\alpha_0}, F_{\alpha+p\beta_+;\alpha_0} \otimes \Omega^p)$ is the sum of the first and the second actions. $\square$

7.7. Now we apply the construction of §4. Consider the double complex

$$C^\cdot(L; \text{Hom}(F_{\alpha;\alpha_0}, F_{\alpha+p\beta_+;\alpha_0} \otimes \Omega^p)).$$

Here the first differential is the Koszul differential of the cochain complex of the Lie algebra $L$ with coefficients in the module $\text{Hom}(F_{\alpha;\alpha_0}, F_{\alpha+p\beta_+;\alpha_0} \otimes \Omega^p)$, with the first $L$-action. The second differential is induced by the differential in $\Omega^p$.

Define the elements

$$V^{a,p-a} \in C^a(L; \text{Hom}(F_{\alpha;\alpha_0}, F_{\alpha+p\beta_+;\alpha_0} \otimes \Omega^{p-a})) \quad (0 \leq a \leq p)$$

by

$$V^{a,p-a}(\tau_1 \wedge \cdots \wedge \tau_a) = i_{\tau_1} \cdots i_{\tau_a}(V^0p).$$

7.8. **Theorem.** The element $V = (V^0p, \ldots, V^p0)$ is a $p$-cocycle in the total complex associated with the double complex $C^\cdot(L; \text{Hom}(F_{\alpha;\alpha_0}, F_{\alpha+p\beta_+;\alpha_0} \otimes \Omega^p))$.

The proof is the same as that of Lemma 4.8.

7.9. **Corollary.** The cocycle $V$ induces the maps

$$f_i: H^i(\Omega^p)^* \to \text{Ext}_{L}^{p-i}(F_{\alpha;\alpha_0}, F_{\alpha+p\beta_+;\alpha_0}), \quad i = 0, \ldots, p. \quad \square$$

The map

$$f_p: H^p(L)^* \to \text{Hom}(F_{\alpha;\alpha_0}, F_{\alpha+p\beta_+;\alpha_0})$$

is the Feigin–Fuchs intertwiner, [FF, Chapter 4].

**Chapter 3. \widehat{\mathfrak{sl}}(2)-case**

The main construction of this and the next chapters was inspired by [BMP1, BMP2].

§8. **Wakimoto realization**

8.1. Let $\mathfrak{g}$ be a Lie algebra and $B(\cdot, \cdot)$ an invariant bilinear form on $\mathfrak{g}$. The corresponding affine Lie algebra $\widehat{\mathfrak{g}}$ is defined by the generators $X_n$ ($X \in \mathfrak{g}$, $n \in \mathbb{Z}$) and $1$, and the relations

(a) $[X_n, X_m] = [X, Y]_{m+n} + nB(X, Y)\delta_{m+n, 0} \cdot 1 \quad (X, Y \in \mathfrak{g}, m, n \in \mathbb{Z}).$

The element $1$ will act as the identity on all our representations.

Let us introduce the generating functions (currents) $X(z) = \sum_{n \in \mathbb{Z}} X_n z^{-n-1}$ ($X \in \mathfrak{g}$). Formula (a) is equivalent to

(b) $X(z)Y(w) = \frac{B(X, Y)}{(z-w)^2} + \frac{[X, Y](w)}{z-w} + \cdots.$
Here as usual the dots stands for the part regular at \( z = w \). One deduces (a) from (b) at once, using the chiral Cauchy formula
\[
[X_n, Y(w)] = \text{res}_{z=w} (z^n X(z) Y(w)).
\]

8.2. In this chapter we assume that \( g = \mathfrak{sl}(2) \) with the standard generators \( E, F, H \). For \( X, Y \in g \), we set \( \langle X, Y \rangle = \text{tr}(XY) \). Thus, \( (E, F) = (F, E) = 1, (H, H) = 2 \). We fix a complex number \( k \), and set \( B(X, Y) = k\langle X, Y \rangle \).

The bosonization formulas for the algebra \( \hat{g} \) presented below were discovered by M. Wakimoto, [W].

8.3. Let \( a \) denote the Lie algebra defined by the generators \( b_n, a_n, a_n^* (n \in \mathbb{Z}) \), \( 1 \) and the relations
(a) \([b_n, b_m] = 2n\delta_{n+m,0} \cdot 1;\)
(b) \([a_n, a_n^*] = \delta_{n+m,0} \cdot 1, [a_n, a_m] = [a_n^*, a_m^*] = 0;\)
(c) \([b_n, a_m] = [b_n, a_n^*] = 0, 1 \) commutes with everything.

Let \( a_+ \) denote the Lie subalgebra of \( a \) generated by the elements \( b_n, a_n^* (n > 0) \), \( a_n (n > 0) \). These generators are called annihilation operators. One introduces the normal ordering of a monomial in \( Ua \) in the usual way: all annihilation operators should be pulled to the right.

All \( a \)-modules \( M \) that we consider be \( a_+ \)-locally finite, i.e., will have the property that for every \( x \in M \), the space \( (Ua_+)x \) is finite-dimensional. The operator \( 1 \) will act as the identity.

For computational purposes, we shall also use one more operator \( q \), with the only nontrivial commutation relation
\[
[q, b_0] = 1.
\]

8.4. For \( \lambda \in \mathbb{C} \), let \( F_\lambda \) denote the \( a \)-module defined by one generator \( v_\lambda \) and relations \( a_+ v_\lambda = 0, b_0 v_\lambda = 2\lambda v_\lambda, 1 v_\lambda = v_\lambda \).

8.5. Let us introduce the generating functions
\[
\phi(z) = q - b_0 \log(z) + \sum_{n \neq 0} \frac{b_n}{n} z^{-n}, \quad p(z) = \phi'(z) = -\sum_n b_n z^{-n-1},
\]
\[
\beta(z) = \sum_n a_n z^{-n-1}, \quad \gamma(z) = \sum_n a_n^* z^{-n}.
\]

We have
\[
\phi(z) \phi(w) = 2 \log(z - w) + \cdots, \quad p(z) \phi(w) = \frac{2}{z - w} + \cdots,
\]
\[
p(z) p(w) = \frac{2}{(z - w)^2} + \cdots, \quad \gamma(z) \beta(w) = \frac{1}{z - w} + \cdots,
\]
all other products being trivial (do not have a singular part).

8.6. Let us define the currents
(a) \( E(z) = \beta(z) \),
(b) \( H(z) = 2 :\gamma(z) \beta(z) : + \nu p(z) \),
(c) \( F(z) = - :\gamma(z)^2 \beta(z) : - \nu :\gamma(z) p(z) : - k \gamma'(z) \),

where \( \nu^2 = k + 2 \).
8.7. **Theorem.** The previous formulas define the bosonization for \( \hat{g} \), i.e., the Fourier components of the currents \( E(z), H(z), F(z) \) satisfy the commutation relations of \( \hat{g} \).

**Proof.** We must check the relations (a)–(f) below.

(a) \[ H(z)H(w) = \frac{2k}{(z-w)^2} + \cdots. \]

Indeed, we have (using Wick’s theorem)

\[
H(z)H(w) = (2 :\gamma(z)\beta(z) : + \nu p(z))(2 :\gamma(w)\beta(w) : + \nu p(w))
= 4 :\gamma(z)\beta(z) : \gamma(w)\beta(w) : + \nu^2 p(z)p(w) + \cdots
\]

(the terms of the first order cancel out)

\[
= 4\{\gamma(z)\beta(w)\}\{\beta(z)\gamma(w)\} + \nu^2 p(z)p(w) + \cdots
= -4 + 2\nu^2
\]

(b) \[ H(z)E(w) = \frac{2E(w)}{z-w} + \cdots. \]

Indeed,

\[ H(z)E(w) = (2 :\gamma(z)\beta(z)+\nu p(z))\beta(w) = 2\{\gamma(z)\beta(w)\} \beta(w) + \cdots = \frac{2\beta(w)}{z-w} + \cdots. \]

(c) \[ H(z)F(w) = -\frac{2F(w)}{z-w} + \cdots. \]

Indeed,

\[ H(z)F(w) = -(2 :\gamma(z)\beta(z)+\nu p(z))\cdot :\gamma(w)^2\beta(w) : + \nu :\gamma(w)p(w) : + k\gamma'(w)). \]

Let us compute all the nontrivial products. We have

\[
:\gamma(z)\beta(z) : \gamma(w)^2\beta(w) : = -\frac{2}{(z-w)^2} \gamma(w) - \frac{1}{z-w} :\beta(w)\gamma(w)^2 : + \cdots,
\]

\[
:\gamma(z)\beta(z) : \gamma(w)p(w) : = -\frac{1}{z-w} :\gamma(w)p(w) : + \cdots,
\]

\[
:\gamma(z)\beta(z) : \gamma'(z) = -\frac{1}{(z-w)^2} \gamma(z) + \cdots
= -\frac{1}{(z-w)^2} \gamma(w) - \frac{1}{z-w} \gamma'(w) + \cdots,
\]

\[ p(z) :\gamma(w)p(w) : = \frac{2}{(z-w)^2} \gamma(w) + \cdots. \]
Summing, we get (c).

(d) \[ E(z)E(w) = 0 + \cdots. \]

This is obvious.

(e) \[ E(z)F(w) = \frac{k}{(z-w)^2} + \frac{H(w)}{z-w} + \cdots. \]

Indeed,

\[
E(z)F(w) = -\beta(z)(\gamma(w)^2\beta(w): + \nu\gamma(w)p(w): + k\gamma'(w)) \\
= \frac{2}{z-w} \gamma(w)\beta(w): + \frac{\nu}{z-w} p(w): + \frac{k}{(z-w)^2} + \cdots \\
= \frac{k}{(z-w)^2} + \frac{H(w)}{z-w} + \cdots.
\]

(f) \[ F(z)F(w) = 0 + \cdots. \]

Indeed,

\[
F(z)F(w) = (\gamma(z)^2\beta(z): + \nu\gamma(z)p(z): + k\gamma'(z)) \\
	\times (\gamma(w)^2\beta(w): + \nu\gamma(w)p(w): + k\gamma'(w)).
\]

Let us compute the nontrivial products.

\[
\gamma(z)^2\beta(z): \gamma(w)^2\beta(w): = -\frac{4}{(z-w)^2} \gamma(z)\gamma(w): + \frac{2}{z-w} \gamma(z)\beta(z)\gamma(w)^2: \\
- \frac{2}{z-w} \gamma(z)^2\gamma(w)\beta(w): + \cdots \\
= -\frac{4}{(z-w)^2} \gamma(w)^2: - \frac{4}{z-w} \gamma(w)\gamma'(w): + \cdots,
\]

\[
\gamma(z)p(z): \gamma(w)p(w): = \frac{2}{(z-w)^2} \gamma(z)\gamma(w): + \cdots \\
= \frac{2}{(z-w)^2} \gamma(w)^2: + \frac{2}{z-w} \gamma(w)\gamma'(w): + \cdots,
\]

\[
\gamma(z)^2\beta(z): \gamma(w)p(w): = -\frac{1}{z-w} \gamma(w)^2p(w): + \cdots,
\]

\[
\gamma(z)p(z): \gamma(w)^2\beta(w): = \frac{1}{z-w} p(w)\gamma(w)^2: + \cdots,
\]

\[
\gamma(z)^2\beta(z): \gamma'(w) = -\frac{1}{(z-w)^2} \gamma(z)^2: + \cdots \\
= -\frac{1}{(z-w)^2} \gamma(w)^2: - \frac{2}{z-w} \gamma'(w)\gamma(w): + \cdots,
\]

\[
\gamma'(z): \gamma(w)^2\beta(w): = -\frac{1}{(z-w)^2} \gamma(w)^2: + \cdots.
\]

After summing, we get a zero singular part. This proves (f) and completes the proof of the theorem. \(\square\)
§9. Screening current

9.1. We keep the assumptions of the previous section, in particular \( g = \mathfrak{sl}(2) \). We will assume throughout that \( \nu \neq 0 \) (the level is noncritical). The results of this section are due essentially to Feigin-Frenkel [FeFr1], [FeFr2].

For \( \chi \in \mathbb{C} \), we shall denote by \( W_{\chi; \nu} \) the \( \hat{g} \)-module \( F_{-\chi/2\nu} \), the \( \hat{g} \)-module structure being defined by formulas 8.6 (a)–(c). Such \( \hat{g} \)-modules are called \emph{Wakimoto modules}.

9.2. For \( \alpha \in \mathbb{C} \), define the operator

\[
V(\alpha; z) = : \exp(-\alpha \phi(z)) : = T_{\alpha} z^{\alpha b_0} \exp \left( -\alpha \sum_{n<0} b_n z^{-n} \right) \exp \left( -\alpha \sum_{n>0} b_n z^{-n} \right)
\]

acting from \( F_\lambda \) to \( F_{\lambda+\alpha} \otimes z^{2\alpha \lambda} \mathbb{C}((z^{-1})) \).

9.3. \textbf{Lemma.} We have

\[
F(z)(-\beta(w)V(\alpha; w)) = -\frac{2\alpha \nu + k}{(z-w)^2} V(\alpha; w) + \frac{2-2\alpha \nu}{z-w} \gamma(w) \beta(w) : V(\alpha; w) : + \frac{\nu}{z-w} : p(w) V(\alpha; w) : + \cdots .
\]

\textit{Proof.} The right hand side is equal to

\[
( : \gamma(z)^2 \beta(z) : + \nu : \gamma(z) p(z) : + k \gamma'(z))(\beta(w) V(\alpha; w)) .
\]

Let us compute all the products using the Wick theorem. We have

\[
: \gamma(z)^2 \beta(z) : \beta(w) V(\alpha; w) = \frac{2}{z-w} : \gamma(w) \beta(w) : V(\alpha; w) + \cdots ;
\]

recalling that by 6.4 (a)

\[
p(z) V(\alpha; w) = -\frac{2\alpha}{z-w} V(\alpha; w) + \cdots ,
\]

we have

\[
: \gamma(z) p(z) : \beta(w) V(\alpha; w) = -\frac{2\alpha}{(z-w)^2} V(\alpha; w) + \frac{1}{z-w} : p(w) V(\alpha; w) : - \frac{2\alpha}{z-w} : \gamma(w) \beta(w) : V(\alpha; w) + \cdots .
\]

Finally,

\[
\gamma'(z) \beta(w) V(\alpha; w) = -\frac{1}{(z-w)^2} V(\alpha; w) + \cdots .
\]

By summing all up, we get the statement of the lemma. \( \square \)

9.4. Let us introduce the operators called \emph{screening currents} by

\[
S(z) = -\beta(z) V(\nu^{-1}; z) : W_{\chi; \nu} \rightarrow W_{\chi-2; \nu} \otimes z^{-\chi/\nu^2} \mathbb{C}((z^{-1})) \quad (\chi \in \mathbb{C}) .
\]

Set

\[
S(F; z) = -\nu^2 V(\nu^{-1}; z) .
\]
9.5. **Theorem.** We have

\[ F(z) S(w) = \frac{\partial}{\partial w} \left( \frac{S(F;w)}{z-w} \right) + \cdots, \quad E(z) S(w) = 0 + \cdots, \quad H(z) S(w) = 0 + \cdots. \]

**Proof.** The first formula follows from the previous lemma after the substitution \( \alpha = \nu^{-1} \), taking into account that

\[ \frac{\partial}{\partial w} V(\alpha;w) = -\alpha : p(w) V(\alpha;w). \]

The second formula is obvious. Let us prove the third formula. We have

\[ H(z) S(w) = -2(2 : \gamma(z) \beta(z) + \nu p(z)) \beta(w) V(\nu^{-1};w). \]

Now,

\[ \gamma(z) \beta(z) V(\nu^{-1};w) = \frac{1}{z-w} \beta(w) V(\nu^{-1};w) + \cdots, \]

\[ p(z) \beta(w) V(\nu^{-1};w) = -\frac{2\nu^{-1}}{z-w} \beta(w) V(\nu^{-1};w) + \cdots. \]

By adding the two terms, we get the third formula. \( \square \)

9.6. **Corollary.** For every \( n \in \mathbb{Z} \),

\[ [F_n, S(w)] = \frac{\partial}{\partial w} (w^n S(F;w)), \quad [E_n, S(w)] = [H_n, S(w)] = 0. \]

9.7. We define the operators \( S(X; z) (X \in \mathfrak{g}) \) as follows. First, \( S(X; z) \) linearly depends on \( X \in \mathfrak{g} \). Second, \( S(F; z) \) is as above, and we put \( S(E; z) = S(H; z) = 0 \).

Now, we define the operators \( S(x; z) (x \in \hat{\mathfrak{g}}) \) as follows. First, they depend linearly on \( x \in \hat{\mathfrak{g}} \). Second, we set

\[ (a) \quad S(X_n; z) = z^n S(X; z) \quad (X \in \mathfrak{g}, \ n \in \mathbb{Z}), \quad S(1; z) = 0. \]

In this notation, we can rewrite the previous theorem and corollary as follows.

9.8. **Theorem.**

(a) For all \( X \in \mathfrak{g} \),

\[ X(z) S(w) = \frac{\partial}{\partial w} \left( \frac{S(X;w)}{z-w} \right) + \cdots. \]

(b) For all \( x \in \hat{\mathfrak{g}} \),

\[ [x, S(w)] = \frac{\partial}{\partial w} S(x; w). \]

9.9. **Lemma.** We have

\[ E(z) S(F;w) = 0 + \cdots, \]

\[ H(z) S(F;w) = -2 \frac{S(F;w)}{z-w} + \cdots, \quad F(z) S(F;w) = \frac{\gamma(w) S(F;w)}{z-w} + \cdots. \]

This is proved by a simple direct computation.
9.10. **Theorem.** (a) For any \( X, Y \in \mathfrak{g} \), we have

\[
X(z) S(Y; w) - Y(z) S(X; w) = \frac{S([X,Y]; w)}{z-w} + \ldots.
\]

(b) For any \( x, y \in \hat{\mathfrak{g}} \), we have

\[
[x, S(y; w)] - [y, S(x; w)] = S([x,y]; w).
\]

**Proof.** (a) follows from the previous lemma. Alternatively, it follows from 9.8 (a) and the facts (c), (d) below, using the associativity of the operator products.

(b) follows from (a) and the fact that

(c) in the products \( X(z) S(Y; w) \) at most first order poles are present.

(This claim is a weaker version of the previous lemma.)

Alternatively, (b) follows from 9.8 (b), if we notice that

(d) for generic \( \chi \) the operator \( \frac{\partial}{\partial w} \) is an isomorphism.

This proves the theorem. \( \square \)

9.11. Set \( A = \mathbb{C}((z^{-1})) \). Consider the twisted de Rham complex

\[
\Omega^\prime: \ 0 \to \Omega^0 \to \Omega^1 \to 0,
\]

where \( \Omega^0 = A, \ \Omega^1 = A \, dz \), the differential being equal to

\[
d_{DR} = -\frac{\chi}{\nu^2} \cdot \frac{dz}{z}.
\]

As before, we form the double complex

\[
C^\prime(\hat{\mathfrak{g}}; \text{Hom}(W_{\chi,\nu}, W_{\chi-2,\nu} \otimes \Omega^\prime)).
\]

Let us define a one-cochain \( V = (V^{01}, V^{10}) \) in the associated total complex as follows. We set

\[
V^{01} = S(z) z^{\chi/\nu^2} dz, \quad V^{10}(x) = S(x; z) z^{\chi/\nu^2} \quad (x \in \hat{\mathfrak{g}}).
\]

The next theorem is the reformulation of Theorems 9.8 (b) and 9.10 (b).

9.12. **Theorem.** The cochain \( V \) is a 1-cocycle. \( \square \)

9.13. For an integer \( p \geq 1 \), consider the normally ordered product

\[
:S(z_1) \cdots S(z_p):.
\]

To simplify the notations, we regard it as an element of

\[
\text{Hom}(W_{\chi,\nu}, W_{\chi-2p,\nu} \otimes A_p)
\]

(we ignore the powers of \( z_i \)). The same concerns the operators \( S(x; z_i) \). Recall that

\[
A_p = \mathbb{C}[[z_1, \ldots, z_p]] \left[ \prod_i z_i^{-1}, \prod_{i \neq j} (z_i - z_j)^{-1} \right] = \Omega^0(X_p)
\]
Consider the twisted de Rham complex
\[ \Omega' : 0 \to \Omega^0 \to \cdots \to \Omega^p \to 0, \]
where \( \Omega^i = \Omega^i(X_p) \). The differential is
\[
d_{DR} - \sum_i \chi_{\nu}^2 \frac{dz_i}{z_i} + \sum_{i<j} \frac{2}{\nu^2} \cdot \frac{dz_i - dz_j}{z_i - z_j}. \]

9.14. For each \( a = 0, \ldots, p \), we define the operators
\[ V^{a,p-a} \in \text{Hom}(\Lambda^a \hat{g}; \text{Hom}(W_{\chi;\nu}, W_{\chi-2p;\nu} \otimes \Omega^{p-a})) \]
as follows. By definition,
\[
V^{a,p-a}(x_1 \wedge \cdots \wedge x_a) = \sum_{1 \leq i_1 < \cdots < i_a \leq p} (-1)^{\text{sgn}(i_1, \ldots, i_a)} \times \left( \sum_{\sigma \in \Sigma_n} (-1)^{\text{sgn}(\sigma)} :S(z_1) \cdots S(x_{\sigma(1)}; z_1) \cdots S(x_{\sigma(a)}; z_{i_a}) \cdots S(z_p); \right) \\
\times dz_1 \wedge \cdots \wedge dz_{i_1} \wedge \cdots \wedge dz_{i_a} \wedge \cdots \wedge dz_p.
\]
Here the sign \( \text{sgn}(i_1, \ldots, i_a) \) is defined by induction on \( a \) as follows.
\[
\text{sgn}(\ ) = 0, \quad \text{sgn}(i_1, \ldots, i_a) = \text{sgn}(i_1, \ldots, i_{a-1}) + i_a + a.
\]
For example,
\[
V^{0p} = :S(z_1) \cdots S(z_p); dz_1 \wedge \cdots \wedge dz_p, \\
V^{p0}(x_1 \wedge \cdots \wedge x_p) = \sum_{\sigma \in \Sigma_p} (-1)^{\text{sgn}(\sigma)} :S(x_{\sigma(1)}; z_1) \cdots S(x_{\sigma(p)}; z_p);.
\]

9.15. Consider the double Koszul complex \( C^*(\hat{g}; \text{Hom}(W_{\chi;\nu}, W_{\chi-2p;\nu} \otimes \Omega^p)) \).
We have a \( p \)-cochain \( V = (V^{0p}, \ldots, V^{p0}) \) in the associated total complex.

9.16. **Theorem.** The cochain \( V \) is a \( p \)-cocycle. \( \square \)

**Chapter 4. Affine Lie algebras (general case)**

**§10. Bosonization**

The basic mathematical results on bosonization for arbitrary affine Lie algebras are due to B. Feigin and E. Frenkel, see [FeFr1], [FeFr2], [Fr], and also [FFR] and references therein. For an original physical approach see [BMP2].

10.1. Let \( \mathfrak{g} \) be the finite-dimensional complex Lie algebra\(^1\) corresponding to a Cartan matrix \( A = (a_{ij})_{i,j=1}^r \) and with the Chevalley generators \( E_i, H_i, F_i \)

\[ ^1 \text{We suspect that most of what appears below is true for the Kac–Moody algebra corresponding to an arbitrary symmetrizable generalized Cartan matrix.} \]
Let $\mathfrak{h} \subset \mathfrak{g}$ be the Cartan subalgebra generated by $H_1, \ldots, H_r$. Let $\alpha_1, \ldots, \alpha_r \in \mathfrak{h}^*$ be the simple roots; let $(\cdot, \cdot)$ be the symmetric nondegenerate bilinear form on $\mathfrak{h}^*$ such that $a_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$. This bilinear form defines an isomorphism $\mathfrak{h}^* \sim \rightarrow \mathfrak{h}$; using this isomorphism, we carry over the bilinear form to $\mathfrak{h}$; this last bilinear form will also be denoted by $(\cdot, \cdot)$. Finally, $\Delta_+$ will denote the set of positive roots.

Let $g$ be the dual Coxeter number of the root system of $\mathfrak{g}$. We fix a complex parameter $\nu \neq 0$ and set $k = \nu^2 - g$.

10.2. Let $\mathfrak{a}$ be the Lie algebra defined by the generators $b_n^i$ ($i = 1, \ldots, r$, $n \in \mathbb{Z}$), $a_n^\alpha$, $a_n^{\alpha x}$ ($\alpha \in \Delta_+$, $n \in \mathbb{Z}$), $1$ and the relations

(a) $[b_n^i, b_m^j] = (H_i, H_j) n\delta_{ij}\delta_{n+m, 0} \cdot 1$;

(b) $[a_n^\alpha, a_m^{\beta x}] = \delta_{\alpha\beta}\delta_{n+m, 0} \cdot 1$;

(c) all other commutators between the generators vanish.

Let $\mathfrak{a}_+$ denote the Lie subalgebra of $\mathfrak{a}$ generated by the elements $b_n^i$, $a_n^\alpha$ ($n > 0$, $\alpha \in \Delta_+$), $a_n^{\alpha x}$ ($n \geq 0$, $\alpha \in \Delta_+$). These generators are called annihilation operators. One introduces the normal ordering of a monomial in $U\mathfrak{a}$ in the usual way: all annihilation operators should be pulled to the right.

All the $\mathfrak{a}$-modules $M$ that we shall consider will be $\mathfrak{a}_+$-locally finite, i.e., have the property that for every $x \in M$, the space $U\mathfrak{a}_+ x$ is finite-dimensional. The operator $1$ will act as the identity.

For computational purposes, we shall also use the operators $q^i$ ($i = 1, \ldots, r$), with the only nontrivial commutation relations

$$[q^i, b_n^j] = (H_i, H_j) \cdot 1.$$  

10.3. For $\lambda \in \mathfrak{h}^*$, let $\mathcal{F}_\lambda$ denote the $\mathfrak{a}$-module defined by one generator $v_\lambda$ and the relations $\mathfrak{a}_+ v_\lambda = 0$, $b_0^i v_\lambda = (H_i, \lambda) \lambda v_\lambda$, $1 v_\lambda = v_\lambda$.

10.4. Let us introduce the generating functions

$$\phi^i(z) = q^i - b_0^i \log(z) + \sum_{n \neq 0} \frac{b_n^i}{n} z^{-n},$$

$$p^i(z) = \phi^{i\prime}(z) = -\sum_n b_n^i z^{-n-1} \quad (i = 1, \ldots, r),$$

$$\beta^\alpha(z) = \sum_n a_n^\alpha z^{-n-1}, \quad \gamma^\alpha(z) = \sum_n a_n^{\alpha x} z^{-n} \quad (\alpha \in \Delta_+).$$

We have

$$\phi^i(z) \phi^j(w) = (H_i, H_j) \log(z - w) + \cdots, \quad \gamma^\alpha(z) \beta^\alpha(w) = \frac{\delta_{\alpha\alpha'}}{z - w} + \cdots,$$  

all other products being trivial.

To shorten the notation, we shall write $\beta^i(z)$, $\gamma^i(z)$ instead of $\beta^\alpha(z)$, $\gamma^\alpha(z)$.
10.5. The bosonization formulas for the affine Lie algebra \( \hat{g} \) with central charge \( k \) have the following form.

(a) \( E_i(z) = :E_i(\gamma^\alpha(z), \beta^{\alpha'}(z)):\),

(b) \( H_i(z) = \sum_{\alpha \in \Delta_+} \langle H_i, \alpha \rangle :\gamma^\alpha(z)\beta^{\alpha}(z): + \nu p^i(z), \)

(c) \( F_i(z) = :F_i(\gamma^\alpha(z), \beta^{\alpha'}(z)):- \nu \frac{\langle \alpha_i, \alpha_i \rangle}{2} :\gamma^i(z)p^i(z) - c_i(\nu) \frac{d\gamma^i(z)}{dz}. \)

Here \( E_i(\gamma^\alpha, \beta^{\alpha'}) \), \( F_i(\gamma^\alpha, \beta^{\alpha'}) \) are certain polynomials, linear in the \( \beta \)'s. They depend on the Poincaré–Birkhoff–Witt isomorphism for the enveloping algebra \( \mathcal{U} \), which identifies this algebra with the symmetric algebra on root vectors \( F_\alpha \) (\( \alpha \in \Delta_+ \)). For their definition, see [BMP2],[FeFr1],[FeFr2]. The coefficients \( c_i(\nu) \) are certain numbers.

\[ \text{§11. Screening currents} \]

11.1. For \( \chi \in \mathfrak{h}^* \), we define the Wakimoto module \( W_{\chi;\nu} \) as the \( \hat{g} \)-module \( F_{-\chi/\nu} \), the \( \hat{g} \)-module structure being defined by the bosonization formulas 10.5 (a)–(c).

11.2. For \( \mu = \sum_i \mu_i \alpha_i \in \mathfrak{h}^* \), define the bosonic vertex operator

\[
V(\mu; z) = :\exp \left( -\sum_i \mu_i \frac{\langle \alpha_i, \alpha_i \rangle}{2} \phi^i(z) \right):.
\]

This operator acts from \( \mathcal{F}_\lambda \) to \( \mathcal{F}_{\lambda+\mu} \otimes z^{(\lambda,\mu)} \mathbb{C}((z^{-1})) \).

11.3. We have the product formula (cf. 6.11)

\[
V(\mu_1; z_1)V(\mu_2; z_2) = (z_1 - z_2)^{(\mu_1,\mu_2)} :V(\mu_1; z_1)V(\mu_2; z_2):.
\]

11.4. By definition, the screening currents are defined by

\[
S_i(z) = S_i(\gamma^\alpha(z), \beta^{\alpha'}(z))V(\nu^{-1}\alpha_i; z): W_{\chi;\nu} \rightarrow W_{\chi-\alpha_i;\nu} \otimes z^{-(\chi,\alpha_i)/\nu^2} \mathbb{C}((z^{-1})).
\]

Here \( S_i(\gamma^\alpha, \beta^{\alpha'}) \) are certain polynomials depending on a PBW decomposition for \( \mathcal{U}\mathfrak{n}_- \), see [BMP2].

We set

\[
S_i(F_j; z) = -\delta_{ij}\nu^2 V(\nu^{-1}\alpha_i; z).
\]

11.5. The following operator expansion formulas which were proved in [BMP1], [FeFr1] and [FeFr2] for all classical simple Lie algebras, essentially on a case by case basis, will play a key role below

\[
F_i(z)S_j(w) = \frac{\partial}{\partial w} \left( \frac{S_j(F_i; w)}{z-w} \right) + \cdots,
\]

\[
E_i(z)S_j(w) = 0 + \cdots, \quad H_i(z)S_j(w) = 0 + \cdots.
\]

11.6. Let \( g \) be the free Lie algebra with generators \( E_i, F_i \) and \( H_i, i = 1, \ldots, r \). To each \( i = 1, \ldots, r \) and \( X \in g \) we associate an operator \( S_i(X; z) \) as follows. On
generators we put $S_t(E_j; z) = S_t(H_j; z) = 0$, and let $S_t(F_j; z)$ be as above. Then define the $S_t(X; z)$ inductively by the formula

$$[X, S_t(Y; w)] - [Y, S_t(X; w)] = S_t([X, Y]; w), \quad \forall X, Y \in \mathfrak{g}$$

and by $\mathbb{C}$-linearity. We further extend this definition to the loop Lie algebra $\mathfrak{g}(z)$ by setting

$$(a) \quad S_t(X_n; z) = z^n S_t(X; z), \quad i = 1, \ldots, r, X \in \mathfrak{g}, n \in \mathbb{Z}.$$  

11.7. Let $\mathfrak{g} \to \hat{\mathfrak{g}}$ be the canonical projection from the free Lie algebra to the semisimple Lie algebra. Write $\pi: \mathfrak{g}(z) \to \hat{\mathfrak{g}}$ for the induced map of on loops. It is straightforward to deduce by induction from formulas of n.11.5, that

For all $x \in \mathfrak{g}(z)$, $i = 1, \ldots, r$, we have

$$[\pi(x), S_t(w)] = \frac{\partial}{\partial w} S_t(x; w). \quad (11.7.1)$$

11.8. Theorem. The assignment $x \mapsto S_t(x; w)$ descends, for any $i = 1, \ldots, r$, to a well-defined map $\hat{\pi}: \text{Hom}(W_{X; w}, W_{X-\alpha_i; w} \otimes z^{-\frac{1}{2} \pi^2 \mathbb{C}(z^{-1})})$. The following equations hold for any $x, y, w \in \hat{\mathfrak{g}}, i = 1, \ldots, r$

$$[x, S_t(y; w)] - [y, S_t(x; w)] = S_t([x, y]; w), \quad [x, S_t(w)] = \frac{\partial}{\partial w} S_t(x; w).$$

Proof. The only statement that is not immediate from construction is that the map $x \mapsto S_t(x; w)$ descends from $\mathfrak{g}(z)$ to $\hat{\mathfrak{g}}$. To prove this, let $x, x' \in \mathfrak{g}(z)$ be such that $\pi(x) = \pi(x') \in \hat{\mathfrak{g}}$. Then equation (11.7.1) yields

$$\frac{\partial}{\partial w} S_t(x; w) = \frac{\partial}{\partial w} S_t(x'; w).$$

But for generic $\chi$ the operator $\frac{\partial}{\partial w}$ is an isomorphism on the de Rham complex of our local system. Hence, for generic $\chi$, we can conclude that $S_t(x; w) = S_t(x'; w)$. Now, for arbitrary $\chi$, the equation $S_t(x; w) = S_t(x'; w)$ follows by continuity. \hfill \Box

11.9. Conjecture. For any $X, Y \in \mathfrak{g}, i = 1, \ldots, r$, one has

$$X(z) S_t(Y; w) - Y(z) S_t(X; w) = \frac{S_t([X, Y]; w)}{z - w} + \cdots.$$  

11.10. Let us fix a sequence $i_1, \ldots, i_p$, where $1 \leq i_j \leq r$ for all $j$.

Consider the twisted de Rham complex

$$\Omega: \quad 0 \to \Omega^0 \to \cdots \to \Omega^p \to 0,$$

where $\Omega^i = \Omega^i(X_p)$ are the same spaces as in 9.13. The differential is by definition equal to

$$d_{DR} = -\sum_{j=1}^p \frac{\chi_i \alpha_{i_j}}{\nu^2} \cdot \frac{d z_j}{z_j} + \sum_{1 \leq j' < j'' \leq p} \frac{(\alpha_{i_j'}, \alpha_{i_{j''}})}{\nu^2} \cdot \frac{d z_{j'}}{z_{j'}} - \frac{d z_{j''}}{z_{j''}}.$$
11.11. Set $\alpha = \sum_{j=1}^{p} \alpha_{ij}$. For each $a = 0, \ldots, p$, we define the operators

$$\psi^{a:p-a} \in \text{Hom}(\Lambda^p \hat{\mathfrak{g}}; \text{Hom}(W_{\chi;\nu}, W_{\chi-a;\nu} \otimes \Omega^{p-a}))$$

as follows. By definition,

$$\psi^{a:p-a}(x_1 \wedge \cdots \wedge x_a) = \sum_{1 \leq j_1 < \cdots < j_a \leq p} (-1)^{\text{sgn}(j_1, \ldots, j_a)} \times \left( \sum_{\sigma \in \Sigma_m} (-1)^{\text{sgn}(\sigma)} \cdot S_{i_{1}}(z_{1}) \cdots S_{i_{j_1}}(x_{\sigma(1)}; z_{j_1}) \cdots S_{i_{j_a}}(x_{\sigma(a)}; z_{j_a}) \cdots S_{i_{p}}(z_{p}) : \right)$$

$$\times d z_{1} \wedge \cdots \wedge d z_{j_1} \wedge \cdots \wedge d z_{j_a} \wedge \cdots \wedge d z_{p}$$

(cf. 9.14). The sign $\text{sgn}(j_1, \ldots, j_a)$ is defined in 9.14.

For example,

$$\psi^{0:p} = : S_{i_1}(z_1) \cdots S_{i_p}(z_p) : d z_1 \wedge \cdots \wedge d z_p,$$

$$\psi^{p:0}(x_1 \wedge \cdots \wedge x_p) = \sum_{\sigma \in \Sigma_p} (-1)^{\text{sgn}(\sigma)} \cdot S_{i_1}(x_{\sigma(1)}; z_1) \cdots S_{i_p}(x_{\sigma(p)}; z_p) : .$$

11.12. Consider the double Koszul complex

$$C^{\ast} (\hat{\mathfrak{g}}; \text{Hom}(W_{\chi;\nu}, W_{\chi-a;\nu} \otimes \Omega^{\ast})) .$$

We have a $p$-cochain $\psi = (\psi^{0:p}, \ldots, \psi^{p:0})$ in the associated total complex.

11.13. **Theorem.** The cochain $\psi$ is a $p$-cocycle. 

\[ \square \]

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