Abstract. A wide class of boundary problems in quantum mechanics is discussed by using path integrals. This includes motion in half-spaces, radial boxes, rings, and moving boundaries. As a preparation the formalism for the incorporation of δ-function perturbations is outlined, which includes the discussion of multiple δ-function perturbations, δ-function perturbations along perpendicular lines and planes, and moving δ-function perturbations. The limiting process, where the strength of the δ-function perturbations gets infinite repulsive, has the effect of producing impenetrable walls at the locations of the δ-function perturbations, i.e. a consistent description for boundary problems with Dirichlet boundary-condition emerges. Several examples illustrate the formalism.
1. Introduction

Boundary problems, respectively boundary-value problems, appear at almost every stage of quantum physics. In a conventional treatment boundary-conditions are usually used to define the solution of a particular solution, say of the Schrödinger equation, in an unambiguous way. Such boundary-conditions, e.g. vanishing wave-functions at infinity, are the basic requirement for the very set-up of the relevant Hilbert space.

Boundary-conditions at infinity, however, are a very specific idealization and for many more physical situation not appropriate. Typical experimental situations require boundary-conditions at a finite distance from the origin, for instance in electrodynamics Dirichlet or Neumann boundary-conditions are the most discussed. Analogous considerations in quantum mechanics lead to Dirichlet and Neumann boundary-conditions, respectively, for the wave functions. In the following we will consider such quantum mechanical problems with Dirichlet boundary-conditions.

To deal with these problems, there have been only few attempts in the literature. For instance, numerical studies can be found in Refs. [2, 14, 54, 62] for the problem of an oscillator in a box.

Studies of magnetic systems in euclidean and radial boxes in the context a solid state physics are due to Falkovsky and Klama [22, 23], and recently by Klama and Rössler [46]. In these studies the authors succeeded in the explicit construction of the relevant Green function by exploiting the given boundary-conditions for a particular system. However, their approach lacks a systematic approach to the whole problem of boundary-conditions, in particular for multidimensional boundary problems with several boundaries. Also, path integral considerations were not made.

Path integral studies for boundary problems have been done by several authors. First, by the explicit construction of the propagator by using the “mirror” principle (see below), c.f. in the classical textbooks on path integrals of Feynman and Hibbs [25] and Schulman [57], and more recently by Inomata and Singh [44], and Janke and Kleinert [45], and second by an attempt to build in general boundary conditions into the path integral, see e.g. Clark et al. [13], and Carreau et al. [9, 10]. The best known examples are that of bounded, but otherwise free motion, i.e. motion in half-spaces, euclidean and radial potential wells, which are all standard examples.

The explicit construction of the propagator for boundary problems with, say, Dirichlet boundary-conditions is generally quite involved. For simple systems, however, the mirror “principle” can be applied. Let us consider the boundary at $x = 0$ as totally reflecting, and denote the reflecting action by the operator $\gamma$. The propagator corresponding to a quantum Hamiltonian $H_0$ ($x \in \mathbb{R}$)

$$K_0(x'', x'; t'', t') = \left< x'' \right| e^{-iH_0(t'' - t')/\hbar} \left| x' \right> \Theta(t'' - t')$$

(1.1)
of the time-evolution equation

\[ \Psi(x'', t'') = \int_{\mathbb{R}} K_0(x'', x'; t'', t') \Psi(x', t') dx' \] (1.2)

then can be used to construct the propagator in the half-space \( \mathbb{R}^+ \) in the following way:

\[ K_{\mathbb{R}^+}(x'', x'; T) = K_0(x'', x'; T) - K_0(x'', \gamma x'; T) \] (1.3)

Here it must be provided that \( K_0(T) \) has the invariance property \( K_0(\gamma x'', \gamma x'; T) = K_0(x'', x'; T) \), respectively the corresponding Hamiltonian is invariant under the action of \( \gamma \): \( \gamma H_0 = H_0 \). Examples, in the case of the half-space \( \mathbb{R}^+ \), are the free particle, the harmonic oscillator, or the potential \( V(x) = V_0 / \cosh^2 x \) [32, 34].

For a (free) particle in a box, there is an infinite number of reflections and translations which must be summed over, which gives the propagator for a particle in a box in terms of Jacobi \( \Theta \)-functions, a well-known result [25, 57]. That is, the propagator \( K(T) \) is then constructed as

\[ K(x'', x'; T) = \sum_{\gamma \in \Gamma} K_0(x'', \gamma x'; T) \] (1.4)

and \( \Gamma \) denotes the set of all actions \( \gamma \). This general “mirror” principle which also includes group actions plays an important rôle in the theory of semiclassical periodic orbit theory and quantum chaos (see e.g. [36, 58, 60] and references therein).

However, the conceptual simplicity of Eq. (1.4) may be striking, but in most cases, \( K(T) \) cannot be explicitly evaluated, or even Eq. (1.4) cannot be applied because \( H_0 \) and \( K_0(T) \) are not invariant with respect to \( \gamma \).

In this paper I want to consider quantum mechanical problems with boundary-conditions for a wide class of systems. I will consider only Dirichlet boundary-conditions and do not discuss generalizations of more general local boundary-conditions along the lines of Refs. [9, 10, 13]. The idea starts by first considering a \( \delta \)-function perturbation in the path integral [34]. The path integral can be expanded into a perturbation series, the \( \delta \) functions allow to integrate over all intermediate coordinate positions, and the convolution theorem of the Fourier transformation decouples the time-convolutions, giving a geometric power series in the perturbation which can be exactly summed, yielding finally a closed expression for the (energy-dependent) Green function of the perturbed problem (c.f. the monograph by Albeverio et al. [1], as the main reference for a sound mathematical definition of such \( \delta \)-function perturbations in quantum mechanics).

Of course, by repeating this procedure, an arbitrary number of \( \delta \)-function perturbations can be taken into account.
Dirichlet boundary-conditions at the location of the $\delta$-function perturbation are now generated by making the strength of the perturbation infinitely repulsive [13, 52], thus yielding a closed expression for the Green function of the boundary problem. Having introduced the first boundary, we obtain by repeating this procedure with a second $\delta$-function perturbation the Green function in a box with Dirichlet boundary-conditions on both boundaries.

As I will show, it is possible to generalize this formalism to higher dimensions. Here an appropriate combination of $\delta$-function perturbations along perpendicular lines, planes and hyperplanes can be introduced, which produce by the same limiting procedure infinitely repulsive walls at the location of the $\delta$-function perturbations. It is obvious that arbitrarily shaped boundaries cannot be taken into account by our formalism.

The further contents of this paper is as follows.

In the next section I shortly sketch the theory of incorporating $\delta$-function perturbations in the path integral. The whole theory was already presented in some length in Ref. [34] and is not repeated here in all details. However, I will develop the formalism further and will discuss

1) Multiple $\delta$-function perturbations. Closed formulæ for the Green function for such arrangements were were not presented in [34] (compare also Ref. [24], however with the restriction to the wave-functions only),
2) $\delta$-function perturbations along perpendicular lines and planes, and
3) Moving $\delta$-function perturbations.

These results are used in the third section to construct by a limiting procedure Green functions for boundary problems with Dirichlet boundary-conditions. I consider half-lines, one-dimensional boxes, half-spaces and two-dimensional boxes (with obvious generalizations to higher dimensions), and moving boundaries.

In the fourth section I present several examples to illustrate the formalism. The examples will include

1) Boundary problems in half-spaces,
2) Boundary problems in boxes,
3) Boundary problems in radial boxes,
4) Boundary problems in radial rings, and
5) Moving Boundaries.

Numerous examples for $\delta$-function perturbations in the path integral were already given in Ref. [34] and are not repeated here.

The fifth section is devoted to a summary and discussion of the results; in Appendix A, the derivation of the Green function for multiple $\delta$-function perturbations is given, and in Appendix B the Green function for the linear potential is constructed.
2. Path Integrals with $\delta$-Function Perturbations

2.1. Summation of the Perturbation Expansion

The general method for the time-ordered perturbation expansion is quite simple. We assume that we have a potential $W(x) = V(x) + \tilde{V}(x)$ in the path integral, where it is assumed that $W$ is so complicated that a direct path integration is not possible. However, the path integral corresponding to $V(x)$ is assumed to be known. We expand the path integral containing $\tilde{V}(x)$ in a perturbation expansion about $V(x)$ in the following way. The initial kernel corresponding to $V$ propagates in $\Delta t$-time unperturbed, then it is interacting with $\tilde{V}$, propagates again in another $\Delta t$-time unperturbed, a.s.o, up to the final state. This gives the series expansion [25] $(x \in \mathbb{R})$

$$K(x'', x'; T) = \int_{x(t') = x'}^{x(t'') = x''} \mathcal{D}x(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \dot{x}^2 - V(x) - \tilde{V}(x) \right] dt \right\}$$

$$= K^{(V)}(x'', x'; T) + \sum_{n=1}^{\infty} \left( -\frac{i}{\hbar} \right)^n \frac{1}{n!} \left( \prod_{j=1}^{n} \int_{-\infty}^{\infty} dx_j \int_{t_j}^{t_{j+1}} dt_j \right)$$

$$\times K^{(V)}(x_1, x'; t_1 - t') \tilde{V}(x_1) K^{(V)}(x_2, x_1; t_2 - t_1) \times \ldots$$

$$\ldots \times \tilde{V}(x_{n-1}) K^{(V)}(x_n, x_{n-1}; t_n - t_{n-1}) \times \ldots$$

$$\ldots \times \tilde{V}(x_n) K^{(V)}(x'', x_n; t'' - t_n)$$

$$= K^{(V)}(x'', x'; T) + \sum_{n=1}^{\infty} \left( -\frac{i}{\hbar} \right)^n \left( \prod_{j=1}^{n} \int_{t_j}^{t_{j+1}} dt_j \int_{-\infty}^{\infty} dx_j \right)$$

$$\times K^{(V)}(x_1, x'; t_1 - t') \tilde{V}(x_1) K^{(V)}(x_2, x_1; t_2 - t_1) \times \ldots$$

$$\ldots \times \tilde{V}(x_{n-1}) K^{(V)}(x_n, x_{n-1}; t_n - t_{n-1}) \times \ldots$$

$$\ldots \times \tilde{V}(x_n) K^{(V)}(x'', x_n; t'' - t_n) \right) . \quad (2.1)$$

In the second step I have ordered the time as $t' = t_0 < t_1 < t_2 < \ldots < t_{n+1} = t''$ and paid attention to the fact that $K(t_j - t_{j-1})$ is different from zero only if $t_j > t_{j-1}$, ([34], see also e.g. Bauch [6], Lawande and Bhagwat [50]), and we set $T = t'' - t'$.

We consider now an arbitrary potential $V(x)$ in one dimension with an additional $\delta$-function perturbation so that [34]

$$W(x) = V(x) - \gamma \delta(x - a) . \quad (2.2)$$
The path integral for this potential problem reads
\[ K(\delta)(x'', x'; T) = \int_{x(t') = x'}^{x(t'') = x''} \mathcal{D}x(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \dot{x}^2 - V(x) + \gamma \delta(x - a) \right] dt \right\} . \] (2.3)

We have assumed that the path integral (Feynman kernel, respectively) for the potential \( V \) is known, i.e. we set
\[ K(V)(x'', x'; T) = \int_{x(t') = x'}^{x(t'') = x''} \mathcal{D}x(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \dot{x}^2 - V(x) \right] dt \right\} , \] (2.4)

including, of course, the (energy-dependent) Green function
\[ G(V)(x'', x'; E) = i \int_0^\infty dT \ e^{iET/\hbar} K(V)(x'', x'; T) \]
\[ K(V)(x'', x'; T) = \int_{-\infty}^\infty dE \frac{e^{-iET/\hbar}}{2\pi i} G(V)(x'', x'; E) . \] (2.5)

Introducing the Green function \( G(\delta)(E) \) of the perturbed system similarly to (2.5), it is easy to sum up the emerging geometric power series, and we obtain due to the convolution theorem of the Fourier transformation
\[ G(\delta)(x'', x'; E) = G(V)(x'', x'; E) - \frac{G(V)(x'', a; E)G(V)(a, x'; E)}{G(V)(a, a; E) - 1/\gamma} , \] (2.6)

where it is assumed that \( G(V)(a, a; E) \) actually exists. The energy levels \( E_n \) of the perturbed problem \( W(x) \) are therefore determined in a unique way by the equation
\[ G(V)(a, a; E_n) = \frac{1}{\gamma} . \] (2.7)

The wave-functions are given by the residua of \( G(E) \) at \( E = E_n \), i.e.
\[ \Psi_n(x) = \lim_{E \to E_n} \left[ \frac{E_n - E}{1/\gamma - G(V)(a, a; E)} \right]^{1/2} G(V)(x, a; E) . \] (2.8)

Here and in the following the index \( n \) denotes the level index of a corresponding energy level \( E_n \), with \( n = 0, 1, \ldots, N_M \), and \( N_M \) may be finite or infinite.

Let us note that an implicit equation for the time-dependent propagator is due to Gaveau and Schulman [27]. They obtained
\[ K(\delta)(x'', x'; T) = K(V)(x'', x'; T) + \frac{\gamma}{\hbar} \int_{t'}^{t''} K(V)(x'', a; t)K(\delta)(a, x'; T - t) dt . \] (2.9)
Similarly as for one-dimensional problems, we can consider in \( D \) dimensions a radial potential according to

\[
W(r) = V(r) - \gamma \delta(r - a),
\]

(2.10)
i.e. a spherical shaped \( \delta \)-function (also called “shell”-, “surface”- \( \delta \)-interaction [1, 3]) perturbation of the potential \( V(r) \). We must require \( a \neq 0 \) because a point perturbation leads to the evaluation of Green functions, where both arguments are equal (and zero), an expression which in general does not exist. Of course, we are using the usual \( D \)-dimensional polar coordinates. For a radial problem we can separate variables in the path integral:

\[
K(W)(x'', x'; T) = \int_{x(t') = x'}^{x(t'') = x''} Dx(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \dot{x}^2 - W(r) \right] dt \right\}
\]

\[
= \sum_{l=0}^{\infty} K(l)(r'', r'; T) S_{\mu}^l(\Omega'') S_{\mu}^*(\Omega'),
\]

(2.11)

where \( S_{\mu}^l(\Omega) \) are the real hyper-spherical harmonics of degree \( l \) with unit-vector \( \Omega \) [20] and the radial path integral \( K(l)(T) \) is given by

\[
K(l)(r'', r'; T)
\]

\[
= (r'^{2})^{\frac{1-D}{2}} \int_{r(t')=r'}^{r(t'')=r''} Dr(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \dot{r}^2 - W(r) \right] dt \right\}
\]

\[
\equiv (r'^{2})^{\frac{1-D}{2}} \int_{r(t')=r'}^{r(t'')=r''} \mu_{l+\frac{D-2}{2}}[r^2] Dr(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \dot{r}^2 - W(r) \right] dt \right\}.
\]

(2.12)

The functional weight \( \mu_{\nu}[r^2] \) is defined by by [39, 61]:

\[
\mu_{\nu}[r^2] = \lim_{N \to \infty} \prod_{j=1}^{N} \mu_{\nu}[r_{j-1}r_j]
\]

(2.13a)

\[
= \lim_{N \to \infty} \prod_{j=1}^{N} \left( \frac{2\pi mr_{j-1}r_j}{i e \hbar} \right)^{1/2} \exp \left( - \frac{mr_{j-1}r_j}{i e \hbar} \right) I_{\nu} \left( \frac{mr_{j-1}r_j}{i e \hbar} \right),
\]

(2.13b)

in order to guarantee a well-behaved boundary behaviour at initial at final points. \( I_{\nu} \) describes a modified Bessel function. Proceeding similar to the previous section and
expanding the perturbed problem into a perturbation series yields:

\[
\begin{align*}
\frac{i}{\hbar} \int_0^\infty dt \ e^{i ET/\hbar} \int_{r(t')=r'}^{r(t'')=r''} \mu_{l+\frac{D-2}{2}}[r^2] \mathcal{D}r(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \dot{x}^2 - V(r) + \gamma \delta(r - a) \right] dt \right\} = \\
= \sum_{l=0}^{\infty} G_l^{(\delta)}(r'', r'; E) S^\mu_l(\Omega'') S^{\mu*}_l(\Omega'),
\end{align*}
\]

(2.14)

where the radial Green function is given by

\[
G_l^{(\delta)}(r'', r'; E) = G_l^{(V)}(r'', r'; E) - \frac{G_l^{(V)}(r'', a; E) G_l^{(V)}(a, r'; E)}{G_l^{(V)}(a, a; E) - 1/a^{D-1}}.
\]

(2.15)

Therefore the energy-levels \( E_n \) are determined by the equation

\[
\frac{1}{a^{D-1}} = G_l^{(V)}(a, a; E_n).
\]

(2.16)

The corresponding wave-functions are given in a similar way as in the one-dimensional case.

2.2. Multiple \( \delta \)-Function Perturbations

Starting from Eq. (2.6) we first consider the problem of two \( \delta \)-function perturbations (such problems play e.g. a rôle in the study of the Casimir effect between to perfectly conducting planes [7])

\[
K^{(\delta_2)}(x'', x'; T)
\]

\[
= \int_{x(t'=x')}^{x(t''=x'')} \mathcal{D}x(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \dot{x}^2 - V(x) + \gamma_1 \delta(x - a_1) + \gamma_2 \delta(x - a_2) \right] dt \right\}.
\]

(2.17)

We obtain

\[
G^{(\delta_2)}(x'', x'; E) = G^{(\delta_1)}(x'', x'; E) - \frac{G^{(\delta_1)}(x'', a_2; E) G^{(\delta_1)}(a_2, x'; E)}{G^{(\delta_1)}(a_2, a_2; E) - \frac{1}{\gamma_2}},
\]

(2.18)

where \( G^{(\delta_1)}(E) \) in turn is given by

\[
G^{(\delta_1)}(x'', x'; E) = G^{(V)}(x'', x'; E) - \frac{G^{(V)}(x'', a_1; E) G^{(V)}(a_1, x'; E)}{G^{(V)}(a_1, a_1; E) - \frac{1}{\gamma_1}}.
\]

(2.19)
Insertion of $G^{(\delta_1)}(E)$ into $G^{(\delta_2)}(E)$ gives
\begin{align*}
G^{(\delta_2)}(x'', x'; E) &= G^{(V)}(x'', x'; E) \\
&+ \left[ \left( G^{(V)}(a_2, a_2; E) - \frac{1}{\gamma_2} \right) \left( G^{(V)}(a_1, a_1; E) - \frac{1}{\gamma_1} \right) - G^{(V)}(a_2, a_1; E)G^{(V)}(a_1, a_2; E) \right]^{-1} \\
&\times \left\{ G^{(V)}(a_1, x'; E) \left[ G^{(V)}(x'', a_2; E)G^{(V)}(a_2, a_1; E) - \left( G^{(V)}(a_2, a_2; E) - \frac{1}{\gamma_2} \right)G^{(V)}(x'', a_1; E) \right] \\
&+ G^{(V)}(a_2, x'; E) \left[ G^{(V)}(x'', a_1; E)G^{(V)}(a_1, a_2; E) - \left( G^{(V)}(a_1, a_1; E) - \frac{1}{\gamma_1} \right)G^{(V)}(x'', a_2; E) \right] \right\}
\end{align*}
\hspace{1cm}
(2.20)

\begin{align*}
\begin{vmatrix}
G^{(V)}(x'', x'; E) & G^{(V)}(x'', a_1; E) & G^{(V)}(x'', a_2; E) \\
G^{(V)}(a_1, x'; E) & G^{(V)}(a_1, a_1; E) - \frac{1}{\gamma_1} & G^{(V)}(a_1, a_2; E) \\
G^{(V)}(a_2, x'; E) & G^{(V)}(a_2, a_1; E) & G^{(V)}(a_2, a_2; E) - \frac{1}{\gamma_2}
\end{vmatrix}
&= \frac{G^{(V)}(a_1, a_1; E) - \frac{1}{\gamma_1} G^{(V)}(a_1, a_2; E) \\
&- G^{(V)}(a_2, a_1; E) G^{(V)}(a_1, a_2; E) - \frac{1}{\gamma_2}}{G^{(V)}(a_2, a_2; E) - \frac{1}{\gamma_2}}
\end{align*}

Bound states are determined by
\begin{align*}
\begin{vmatrix}
G^{(V)}(a_1, a_1; E_n) - \frac{1}{\gamma_1} & G^{(V)}(a_1, a_2; E_n) \\
G^{(V)}(a_2, a_1; E_n) & G^{(V)}(a_2, a_2; E_n) - \frac{1}{\gamma_2}
\end{vmatrix}
&= 0 \\
\end{align*}
\hspace{1cm}
(2.21)

The bound state wave-functions are given by
\begin{align*}
\Psi_n^*(x')\Psi_n(x'') &= \lim_{E \to E_n} \left( (E_n - E)G^{(\delta_2)}(x'', x'; E) \right)
\end{align*}
\hspace{1cm}
(2.22)

Eq. (2.20) has an obvious generalization to the problem of $N$ $\delta$-functions.
We obtain

\[ G^{(\delta N)}(x'', x'; E) = \frac{i}{\hbar} \int_0^\infty dt \ e^{i E t/\hbar} \int_{x(t')=x'} Dx(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \dot{x}^2 - V(x) + \sum_{k=1}^N \gamma_k \delta(x - a_k) \right] dt \right\} \]

The quantization condition for the of bound states has the form

\[ \begin{vmatrix} G^{(V)}(x'', x'; E) & G^{(V)}(x'', a_1; E) & \cdots & G^{(V)}(x'', a_N; E) \\ G^{(V)}(a_1, x'; E) & G^{(V)}(a_1, a_1; E) - \frac{1}{\gamma_1} & \cdots & G^{(V)}(a_1, a_N; E) \\ \vdots & \vdots & \ddots & \vdots \\ G^{(V)}(a_N, x'; E) & G^{(V)}(a_N, a_1; E) & \cdots & G^{(V)}(a_N, a_N; E) - \frac{1}{\gamma_N} \end{vmatrix} = 0 . \]  

This can be written for short by

\[ \det(a_{ij}) = 0 , \quad \text{with the } N \times N \text{ matrix } \quad (a_{ij}) = \left( G^{(V)}(a_i, a_j; E) - \frac{\delta_{ij}}{\gamma_i} \right) . \]

The bound state wave-functions are given by

\[ \Psi_n^*(x') \Psi_n(x'') = \lim_{E \to E_n} \left[ (E_n - E) G^{(\delta N)}(x'', x'; E) \right] . \]  

Eq. (2.23) is proven in appendix A. Let us note that in the limit \( N \to \infty \) Eq.(2.23) can be used to describe periodic \( \delta \)-functions in solid state physics, see e.g. Goovaerts and Broeckx [29] for such a discussion, including the determination of energy bands.
2.3. δ-Function Perturbation Along a Line

The case of an arbitrary located two-dimensional δ-function perturbation on a line can now also be treated. We consider a δ-function perturbation \( -\gamma \delta(ax - by - c) \). This can be alternatively interpreted as two particles interacting pointwise with coupling \( \gamma \). By a simple shift of variables and introducing center of mass and relative coordinates \( R \) and \( r \), respectively

\[ r = ax - by - c, \quad \mu = \frac{m}{a^2 + b^2}, \quad (2.27) \]

\[ R = \frac{ab^2 x + a^2(by + c)}{a^2 + b^2}, \quad M = m \frac{a^2 + b^2}{a^2 b^2}, \quad (2.28) \]

we obtain by using the known exact propagator for the δ-function perturbation (e.g. [34] and references therein) the path integral identity

\[ x(t'') = x'' \int_{x(t') = x'}^x \mathcal{D}x(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} (\dot{x}^2 + \dot{y}^2) + \gamma \delta(ax - by - c) \right] dt \right\} \]

\[ y(t'') = y'' \int_{y(t') = y'}^y \mathcal{D}x(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} (\dot{x}^2 + \dot{y}^2) + \gamma \delta(ax - by - c) \right] dt \right\} \]

\[ = ab \int_{R(t') = R'}^{R(t'') = R''} \mathcal{D}R(t) \mathcal{D}R(t') \exp \left\{ \frac{i M}{2 \hbar} \int_{t'}^{t''} \dot{R}^2 \right\} \]

\[ \times \int_{r(t') = r'}^{r(t'') = r''} \mathcal{D}r(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{\mu}{2} r^2 + \gamma \delta(r) \right] dt \right\} \]

\[ = \sqrt{\frac{M}{2 \pi i \hbar T}} \exp \left[ - \frac{M}{2 \hbar i T} (R'' - R')^2 \right] \]

\[ \times \left\{ \frac{\mu \gamma}{\hbar^2} \exp \left[ - \frac{\mu \gamma}{\hbar^2} (r' + r'') + \frac{i \mu \gamma^2}{\hbar 2 \hbar^2 T} \right] + \frac{1}{2 \pi} \int_{-\infty}^{\infty} dp \exp \left[ - \frac{p^2 \hbar T}{2 \mu} \right] \right. \]

\[ \times \left[ \sin pr' \sin pr'' + \cos pr' \cos pr'' - \frac{e^{i p (r' + r'')}}{1 + i \frac{p \hbar^2}{\mu \gamma}} \right]. \quad (2.29) \]

There seems no obvious way to incorporate more than one arbitrary line in \( \mathbb{R}^2 \) equipped with δ-function perturbations. However, the two-dimensional result can be generalized to \( D \) dimensions. We introduce a δ-function perturbation on a hyperplane \( \vec{a} \cdot \vec{x} - b = 0 \) (\( \vec{a}, \vec{x} \in \mathbb{R}^D \)). Introducing appropriate center of mass and relative coordinates [51, 52], we obtain similarly as in the two-dimensional case a \((D - 1)\)-dimensional product of free particle kernels and one δ-function potential kernel in the relative coordinate.
2.4. δ-Function Perturbations along Perpendicular Lines and Planes

Let us consider first a two-dimensional system. Obviously we have the identity

\[
\begin{align*}
\int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \int_{y(t')=y'}^{y(t'')=y''} \mathcal{D}y(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} (\dot{x}^2 + \dot{y}^2) - V(x) - V(y) + \sum_{k=1}^{N_1} \gamma_{1,k} \delta(x - a_k) + \sum_{k=1}^{N_2} \gamma_{2,k} \delta(y - b_k) \right] dt \right\} \\
= \int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \dot{x}^2 - V(x) + \sum_{k=1}^{N_1} \gamma_{1,k} \delta(x - a_k) \right] dt \right\} \\
\times \int_{y(t')=y'}^{y(t'')=y''} \mathcal{D}y(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \dot{y}^2 - V(y) + \sum_{k=1}^{N_2} \gamma_{2,k} \delta(y - b_k) \right] dt \right\} \\
\equiv K_x^{(N_1)}(x'', x'; T) \cdot K_y^{(N_2)}(y'', y'; T) = K^{(N_1, N_2)}(x'', x', y'', y'; T) .
\end{align*}
\]

The solution of each kernel in terms of its corresponding Green function is given by Eq. (2.23). Now call these Green functions \( G_x^{(N_1)}(E) \) and \( G_y^{(N_2)}(E) \), respectively. The convolution theorem of the Fourier transformation now states that the Green function corresponding to the entire kernel \( G^{(N_1, N_2)}(x'', x', y'', y'; E) \) is given by the convolution of \( G_x^{(N_1)}(E) \) and \( G_y^{(N_2)}(E) \), i.e.

\[
G^{(N_1, N_2)}(x'', x', y'', y'; E) = \int_{-\infty}^{\infty} dz \ G_x^{(N_1)}(x'', x'; z) G_y^{(N_2)}(y'', y'; E - z) .
\]

Obviously, we can generalize this result to the \( D \)-dimensional case. By separating the \( D \) path integrations we obtain a \( D \)-fold product of one-dimensional multiple δ-function potential kernels. The Green function of the entire kernel is given by a \((D-1)\)-fold convolution of the corresponding one-dimensional multiple δ-function potential Green function, each of them given by Eq. (2.23). As complicated as this may look, for practical purposes only the case \( D = 3 \) is relevant, and what remains is a two-fold convolution. Because the number of δ-functions we take into account is not restricted, nor their variability in strength, we can take this model in the large \( N_1, N_2 \) limit of perpendicular δ-function perturbations as a model for a rigid body lattice. The various entries in the determinants are simple, and all what one has to do, is the evaluation of the determinant and some numerical integration. Of course, the whole system can also be put into a \((D\)-dimensional\) box (see next section) which only slightly increases the complexity of the system. A numerical investigation should yield a band structure of the energy levels.
2.5. Moving δ-Function Perturbation

Explicitly time-dependent problems in quantum mechanics turn out to be usually quite difficult and involved. In concrete physical situations one encounters mostly scattering problems and it is necessary to study the quantum mechanical properties of a system with a moving background, e.g. electrons scattered by moving ions. One useful model of such set-ups are δ-function perturbations, simulating point interactions (e.g. [49, 64]). In this Subsection we study the quantum motion of a particle in the field of a moving δ-function perturbation [64]:

\[ V(x,t) = -\gamma \delta(x - vt) \tag{2.32} \]

where \( v \) denotes the constant velocity of the δ-function. I present an alternative treatment as given by Duru [19] (compare also [36]) based directly on the summation of the corresponding perturbation expansion. The path integral now has the form

\[
K^{(\delta)}(x'', x'; t'', t') = \int_{x(t') = x'}^{x(t'') = x''} Dx(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \dot{x}^2 + \gamma \delta(x - vt) \right] dt \right\} \\
= \lim_{N \to \infty} \left( \frac{m}{2 \pi i \hbar} \right)^{N/2} \prod_{j=1}^{N-1} \int_{-\infty}^{\infty} dx_j \exp \left\{ \frac{i}{\hbar} \sum_{j=1}^{N} \left[ \frac{m}{2\epsilon} (x_j - x_{j-1})^2 + \epsilon \gamma \delta(x_j - vt_j) \right] \right\} ,
\tag{2.33}
\]

\( (x_j = x(t_j), t_j = t' + \epsilon_j, \text{ with } \epsilon = (t'' - t')/N = T/N \text{ in the limit } N \to \infty) \). We first perform the coordinate substitution

\[
y = x - vt , \quad \text{with} \quad \begin{cases} y'' = x'' - vt'' , \\ y' = x' - vt' . \end{cases}
\tag{2.34}
\]

This yields

\[
K^{(\delta)}(x'', x'; t'', t') = \exp \left( \frac{i}{\hbar} \frac{mv^2 T}{2} \right) \int_{y(t') = y'}^{y(t'') = y''} Dy(t) \exp \left\{ \frac{i}{\hbar} \int_{y'}^{y''} \left[ \frac{m}{2} \dot{y}^2 + m v \dot{y} + \gamma \delta(y) \right] \right\} \bigg|_{y' = x' - vt'}^{y'' = x'' - vt''} ,
\]

\[
= \sqrt{\frac{m}{2 \pi i \hbar T}} \exp \left[ - \frac{m}{2 \pi i \hbar T} (x'' - x')^2 \right] + \exp \left( \frac{i}{\hbar} \frac{mv^2 T}{2} \right) \tilde{K}^{(\delta)}(y'', y'; t'', t') \bigg|_{y' = x' - vt'}^{y'' = x'' - vt''} ,
\tag{2.35}
\]
where $\widetilde{K}^{(\delta)}$ is given by

$$
\widetilde{K}^{(\delta)}(y'', y'; t'', t') = \sum_{n=1}^{\infty} \left( \frac{i\gamma}{h} \right)^n \int_{t''}^{t'} dt_n \cdots \int_{t'}^{t_2} dt_1 \prod_{j=1}^{n} \int_{-\infty}^{\infty} dy_j \delta(y_j) \times K^{(v)}(y_1, y'; t_1) K^{(v)}(y_2, y_1; t_2, t_1) \cdots K^{(v)}(y_n, y_n; t'', t_n)
$$

$$
= \sum_{n=1}^{\infty} \left( \frac{i\gamma}{h} \right)^n \int_{t''}^{t'} dt_n \cdots \int_{t'}^{t_2} dt_1 \times K^{(v)}(0, y'; t_1) \cdot \prod_{j=2}^{n-1} K^{(v)}(0, 0; t_j, t_j-1) \cdot K^{(v)}(y'', 0; t'', t_n).
$$

(2.36)

Here $\widetilde{K}^{(v)}(t'', t')$ denotes the path integral

$$
\widetilde{K}^{(v)}(y'', y'; t'', t') = \int_{y(t')=y'}^{y(t'')=y''} \mathcal{D}y(t) \exp \left[ \frac{i}{h} \int_{t'}^{t''} \left( \frac{m}{2} y^2 + mv \dot{y} \right) dt \right]
$$

$$
= \sqrt{\frac{m}{2\pi i hT}} \exp \left[ - \frac{m}{2i hT} (y'' - y')^2 + i \frac{mv}{h} (y'' - y') \right].
$$

(2.37)

We find

$$
G^{(v)}(y'', y'; E) = \frac{1}{h} \sqrt{-\frac{m}{2E}} \exp \left[ - \frac{|y'' - y'|}{h} \sqrt{-2mE} + i \frac{mv}{h} (y'' - y') \right],
$$

(2.38)

and we can apply the convolution theorem of the Fourier transformation yielding

$$
\tilde{G}^{(\delta)}(y'', y'; E) = \frac{m\gamma}{2h^2} \frac{\exp \left[ - \frac{|y'| + |y''|}{h} \sqrt{-2mE} + i \frac{mv}{h} (y'' - y') \right]}{\sqrt{-E} \left( \sqrt{-E} - \frac{\gamma}{h} \sqrt{\frac{m}{2}} \right)}.
$$

(2.39)

$K(t'', t')$ can be easily calculated by inverse Laplace transformation [21, Vol.I. p.247]

$$
\mathcal{L}^{-1} \left[ p^{-1/2}(p^{1/2} + \beta)^{-1} e^{-\alpha \sqrt{p}} \right] (t) = e^{\alpha \beta + \beta^2 t} \text{erfc} \left( \frac{\alpha}{2\sqrt{t}} + \beta \sqrt{t} \right),
$$

(2.40)

yielding finally (re-inserting $x'$ and $x''$)
\( K^{(5)}(x'', x'; t'', t') \)
\[
= \sqrt{\frac{m}{2\pi i\hbar T}} \exp \left[ \frac{im}{2\hbar T} (x'' - x')^2 \right] \\
+ \frac{m\gamma}{2\hbar^2} \exp \left\{ \frac{i}{\hbar} \left[ v(x'' - xt'') - v(x' - vt') + m\frac{v^2}{2} T \right] \right\} \\
\times \exp \left[ - \frac{m\gamma}{\hbar^2} \left( |x'' - vt''| + |x' - vt'| \right) + \frac{i m\gamma^2}{2\hbar} T \right] \\
\times \text{erfc} \left[ \sqrt{\frac{m}{2\hbar T}} \left( |x'' - vt''| + |x' - vt'| - \frac{i \hbar \gamma T}{2} \right) \right] \\
= \frac{m\gamma}{\hbar^2} \exp \left[ - \frac{m\gamma}{\hbar^2} \left( |x'' - vt''| + |x' - vt'| \right) \right] \\
- \frac{imv}{\hbar} (x' - vt') + \frac{imv}{\hbar} (x'' - vt'') + \frac{im}{2\hbar} \left( \frac{\gamma^2}{\hbar^2} + v^2 \right) T \\
+ \frac{1}{2\pi} \int_{-\infty}^{\infty} dp \exp \left( - \frac{ip^2\hbar T}{2m} \right) \left\{ \frac{e^{ip(x' - x')}}{1 + i \frac{ph^2}{m\gamma}} \right\} \right. \\
\left. \exp \left[ - \frac{ip(|x'' - vt''| + |x' - vt'|) + \frac{imv}{\hbar} \left( x'' - x' - \frac{vT}{2} \right)}{1 + i \frac{ph^2}{m\gamma}} \right] \right\} . \\
(2.41)
\]

We obtain one bound state
\[
\Psi^{(bound)}(x, t) = \frac{\sqrt{m\gamma}}{\hbar} \exp \left[ - \frac{m\gamma}{\hbar^2} |x - vt| + \frac{imv}{\hbar} (x - vt) + \frac{im}{2\hbar} \left( \frac{\gamma^2}{\hbar^2} + v^2 \right) t \right], \\
(2.42)
\]

which is the result of Ref.[64], and the continuous states have the form
\[
\Psi^{(cont.)}(x, t) = \frac{1}{\sqrt{2\pi}} \exp \left( - \frac{ip^2\hbar t}{2m} \right) \\
\times \left\{ e^{ipx} - \frac{\exp \left[ i p |x - vt| + \frac{imv}{\hbar} (x - vt) + \frac{imv^2}{2\hbar} t \right]}{1 + i \frac{ph^2}{m\gamma}} \right\} . \\
(2.43)
\]

For \( v = 0 \) the usual result is recovered.
3. Path Integrals for Boundary Problems

3.1. Motion in Half-Spaces and Boxes

In Eq. (2.6) we consider now the limit \( \gamma \to -\infty \) which has the effect that an impenetrable wall appears at \( x = a \) \([13, 52]\). We set \( \lim_{\gamma \to -\infty} G^{(\delta)}(E) \equiv G^{(Wall)}(E) \)
i.e. we obtain

\[
G^{(Wall)}(x'', x'; E) = \frac{i}{\hbar} \int_0^\infty dT e^{iET/\hbar} \int_{x(t')=x'}^{x(t'')=x''} D_{Wall}x(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \dot{x}^2 - V(x) \right] dt \right\}
\]

\[
= G^{(V)}(x'', x'; E) - \frac{G^{(V)}(x'', a; E)G^{(V)}(a, x'; E)}{G^{(V)}(a, a; E)} . \tag{3.1}
\]

Bound states are determined by the equation

\[
G^{(V)}(a, a; E_n) = 0 . \tag{3.2}
\]

Again, the wave-functions are given by the residua of \( G(E) \) at \( E = E_n \), i.e.

\[
\Psi_n(x) = \lim_{E \to E_n} \left[ -\frac{E_n - E}{G^{(V)}(a, a; E)} \right]^{1/2} G^{(V)}(x, a; E) . \tag{3.3}
\]

The Green functions \( G^{(V)}(E) \) usually can be written as a product of two linearly independent solutions of the corresponding homogeneous Schrödinger equation, i.e. we can consider in general the Ansatz for \( G^{(V)}(E) \)

\[
G^{(V)}(x'', x'; E) = f_E A_E(x_>) B_E(x_<) , \tag{3.4}
\]

where \( f_E \) is a prefactor which depend only on \( E \), and \( A_E(x_>) \) and \( B_E(x_<) \) are functions of the larger (smaller) of \( x', x'' \), and depending, of course, on \( E \) as an parameter. We set the barrier at \( x = R \) and consider the motion for the (right-) half-space \( x > R \). This yields

\[
G^{(x > R)}(x'', x'; E) = f_E A_E(x_>) B_E(x_<) - \frac{f_E B_E(R) A_E(x') A_E(x'')}{A_E(R)} \tag{3.5}
\]

and the quantization condition has the form

\[
A_{E_n}(R) = 0 . \tag{3.6}
\]
Similarly, if we put the barrier at $x = a$ and consider the motion in the (left-) half-space $x < a$, we obtain

$$
G^{(x<a)}(x'', x'; E) = f_E A_E(x_>) B_E(x_<) - \frac{f_E A_E(a) B_E(x') B_E(x'')}{B_E(a)}
$$

(3.7)

and the quantization condition has the form

$$
B_{E_n}(a) = 0 .
$$

(3.8)

Repeating the procedure for the double $\delta$-function perturbation, we consider the limit $\lim_{\gamma_1, \gamma_2 \to -\infty} G^{(\delta_2)}(E) \equiv G^{(Box)}(E)$ and obtain for the motion in the box $a < x < b$

$$
G^{(Box)}(x'', x'; E) = \frac{1}{\hbar} \int_0^\infty \!dT \; e^{iET/\hbar} \int_{x(t')=x'} D_{Box} x(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \dot{x}^2 - V(x) \right] dt \right\}
$$

$$
\left| \begin{array}{ccc}
G^{(V)}(x'', x'; E) & G^{(V)}(x'', b; E) & G^{(V)}(x'', a; E) \\
G^{(V)}(b, x'; E) & G^{(V)}(b, b; E) & G^{(V)}(b, a; E) \\
G^{(V)}(a, x'; E) & G^{(V)}(a, b; E) & G^{(V)}(a, a; E)
\end{array} \right| .
$$

(3.9)

The quantization condition for the (infinite number of) bound states has the form

$$
\left| \begin{array}{cc}
G^{(V)}(b, b; E_n) & G^{(V)}(b, a; E_n) \\
G^{(V)}(a, b; E_n) & G^{(V)}(a, a; E_n)
\end{array} \right| = 0 .
$$

(3.10)

The bound state wave-functions are given by

$$
\Psi^*_n(x') \Psi_n(x'') = \lim_{E \to E_n} \left[ (E_n - E) G^{(Box)}(x'', x'; E) \right] .
$$

(3.11)

Taking the same Ansatz (3.4) as before for the Green function $G^{(V)}(E)$ we can simplify Eq. (3.9) into

$$
G^{(Box)}(x'', x'; E) = f_E A_E(x_>) B_E(x_<) + \frac{f_E}{A_E(a) B_E(b) - A_E(b) B_E(a)}
$$

$$
\times \left\{ A_E(x'') B_E(a) \left[ B_E(x') A_E(b) - A_E(x') B_E(b) \right] + B_E(x'') A_E(b) \left[ A_E(x') B_E(a) - B_E(x') A_E(a) \right] \right\} ,
$$

(3.12)
and the quantization condition has the form

\[ A_{E_n}(a)B_{E_n}(b) - A_{E_n}(b)B_{E_n}(a) = 0 \ . \]  

(3.13)

For radial problems with a boundary at \( r = R \) we obtain similarly

\[
\frac{i}{\hbar} \int_0^\infty dT \ e^{i ET/\hbar} \int_{x'(t')}^{x''(t'')} D_{Box} x(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \dot{x}^2 - V(r) \right] dt \right\} = \sum_{l=0}^\infty G_l^{(Box)}(r'', r'; E) S_l^\mu(\Omega'') S_l^{\mu*}(\Omega') ,
\]

(3.14)

where the radial Green function is given by

\[
G_l^{(Box)}(r'', r'; E) = G_l^{(V)}(r'', r'; E) - \frac{G_l^{(V)}(r', R; E)G_l^{(V)}(R, r'; E)}{G_l^{(V)}(R, R; E)} .
\]

(3.15)

The energy levels are determined by

\[
G_l^{(V)}(R, R; E_n) = 0 .
\]

(3.16)

For a radial ring with motion constraint by \( a < r < b \) we obtain

\[
G_l^{(Ring)}(r'', r; E) = \begin{vmatrix}
G_l^{(V)}(x'', x'; E) & G_l^{(V)}(x'', b; E) & G_l^{(V)}(x'', a; E) \\
G_l^{(V)}(b, x'; E) & G_l^{(V)}(b, b; E) & G_l^{(V)}(b, a; E) \\
G_l^{(V)}(a, x'; E) & G_l^{(V)}(a, b; E) & G_l^{(V)}(a, a; E)
\end{vmatrix} ,
\]

(3.17)

and the bound states energy levels are determined by

\[
\begin{vmatrix}
G_l^{(V)}(b, b; E_n) & G_l^{(V)}(b, a; E_n) \\
G_l^{(V)}(a, b; E_n) & G_l^{(V)}(a, a; E_n)
\end{vmatrix} = 0 .
\]

(3.18)

The bound state wave-functions can be obtained from

\[
\Psi_{l_n}(x')\Psi_{l_n}(x'') = \lim_{E \to E_n} \left[ (E_n - E)G_l^{(Ring)}(x'', x'; E) \right] .
\]

(3.19)
3.2. Boundaries Along Perpendicular Lines and Planes

Because the Green functions for problems with Dirichlet boundary-conditions can be constructed from the corresponding Green functions for \( \delta \)-function perturbations we can also discuss the case of multidimensional boxes. We consider Eq. (2.30). Since we are only interested in one domain which emerges from the limit \( \gamma_{1/2,k} \to -\infty \), it is sufficient to consider the case \( N_1 = N_2 = 2 \). According to the convolution theorem this gives the Green function for a two-dimensional system in a box \( a_1 < x < b_1 \), \( a_2 < y < b_2 \)

\[
\frac{i}{\hbar} \int_0^\infty dT \ e^{i \frac{E T}{\hbar}} \left. \int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}_{Box} x(t) \int_{y(t')=y'}^{y(t'')=y''} \mathcal{D}_{Box} y(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} (\dot{x}^2 + \dot{y}^2) - V(x) - V(y) \right] dt \right\} \right.
\]

\[
= \int_{-\infty}^\infty dz \ G_x^{(Box)}(x'', x'; z) G_y^{(Box)}(y'', y'; E - z) .
\]  

(3.20)

This results can also be generalized in an obvious way to higher dimensions with the restriction that no more perpendicular planes are allowed than twice the dimension of the space, i.e. \( \#(\text{boundaries}) \leq 2D \). The entire Green function is then given by a \( (D-1) \)-fold convolution of Green functions (3.9). \( G^{(V)}(E) \) itself can include \( \delta \)-function perturbations, c.f. the remark at the end of the last section.

3.3. Moving Boundaries

Actually, the problem of a moving \( \delta \)-function perturbation belongs to a larger class of explicitly time-dependent potentials. One possibility is to change a usual potential \( V(x) \) according to \( V(x) \mapsto V(x/\zeta)/\zeta^2 \), where \( \zeta = \zeta(t) = (at^2 + 2bt + c)^{1/2} \), and one derives the path integral identity \([15, 37]\)

\[
\int_{x(t')=x'}^{x(t'')=x''} \mathcal{D} x(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \dot{x}^2 - \frac{1}{\zeta^2(t)} V\left( \frac{x}{\zeta(t)} \right) \right] dt \right\}
\]

\[
= (\zeta''')^{-D/2} \exp \left[ \frac{i m}{2\hbar} \left( x'' \frac{\dot{z}''}{\zeta'''} - x' \frac{\dot{z}'}{\zeta''} \right) \right] K_{\omega^2, V} \left( \frac{x''}{\zeta''}, \frac{x'}{\zeta'}; \tau(t'') \right) .
\]

(3.21)

with \( \zeta' = \zeta(t') \), \( \zeta'' = \zeta(t'') \), etc., and we have set \( \tau(t'') = \int_{t'}^{t''} dt/\zeta^2(t) \). Furthermore \( \omega^2 = ac - b^2 \) and \( K_{\omega^2, V} \) denotes the path integral

\[
K_{\omega^2, V} (z'', z'; s'') = \int_{z(0)=z'}^{z(s'')=z''} \mathcal{D} z(s) \exp \left\{ \frac{i}{\hbar} \int_0^{s''} \left[ \frac{m}{2} \dot{z}^2 - \frac{m}{2} \omega^2 z^2 - V(z) \right] ds \right\} .
\]

(3.22)

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Another class of time-dependent problems has a time-dependence according to \( V(x) \mapsto V(x - f(t)) \). Here one gets \([19]\) \((q' = x' - f', f' = f(t'), \text{etc.})\)

\[
\begin{align*}
x'(t'') &= x'' \\
\int_{x(t') = x'}^x \mathcal{D}x(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \dot{x}^2 - V(x - f(t)) \right] dt \right\} \\
&= \exp \left\{ \frac{i m}{\hbar} \left[ \dot{f}'(x'' - f'') - \dot{f}''(x' - f') + \frac{1}{2} \int_{t'}^{t''} \dot{f}^2(t) dt \right] \right\} K_{\dot{f},V}(q'', q'; T)
\end{align*}
\]

with the path integral \( K_{\dot{f},V}(T) \) given by

\[
K_{\dot{f},V}(q'', q'; T) = \int_{q(t') = q'}^{q(t'') = q''} \mathcal{D}q(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \dot{q}^2 - V(q) - m \ddot{f}(t) q \right] dt \right\} , \quad (3.24)
\]

Hence, in both cases the time-dependent problem can be rewritten in terms of a time-independent problem, however, additional terms appear in the Lagrangian. Considering now time-dependent \( \delta \)-function perturbations, the time dependence can be transformed away and one is left with the path integrals \( K_{\omega',\delta}(\tau(t'')) \) and \( K_{f,\delta}(T) \), respectively, with the \( \delta \)-function perturbation located at the coordinate origin. Let us denote the corresponding Green functions by \( G_{\omega',\delta}(E) \) and \( G_{f,\delta}(E) \). Considering the limiting process of making the strength of the \( \delta \)-function perturbation infinite repulsive, gives the corresponding Green functions \( G_{\omega',\text{wall}}(E) \) and \( G_{f,\text{wall}}(E) \) according to Eq. (3.1), with the totally reflecting boundary located at the origin, which describe still time-independent problems. The time-dependent problem can be obtained by applying Eqs. (3.21,3.23), provided we can explicitly evaluate \( K_{\omega',\text{wall}}(\tau(t'')) \) and \( K_{f,\text{wall}}(T) \). It is obvious that this last step is a rather strong restriction which allows only for a limited number of explicitly soluble examples, say if we consider \( \omega' = 0 \) and \( \dot{f} = 0 \), respectively. One then obtains [(MB) - moving boundary, \( D = 1 \)]

\[
K^{(MB)}(x'', x'; t'', t') = (\zeta'' \xi')^{-1/2} \exp \left\{ \frac{im}{2\hbar} \left( x'' \zeta'' - x' \xi' \right) \right\} K_{\omega',\text{wall}} \left( \frac{x''}{\zeta''}, \frac{x'}{\xi'}; \tau(t'') \right) , \quad (3.25)
\]

and similarly as for the case of Eq. (3.23).

4. Examples

In this section I am going to discuss several examples of the formalism presented in the previous section. I do not discuss examples with \( \delta \)-function perturbations, this has been already done in Ref. [34].

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4.1. Boundary Problems in Half-Space

4.1.1. Free Motion in Half-Space. As the first trivial example we consider for completeness the free motion in the half-space \( x > 0 (\mathbb{R}^+) \). We have for the free motion in \( \mathbb{R}^+ \)

\[
\frac{d^2}{dt^2} x(t) \exp \left( \frac{im}{2\hbar} \int_t^{t'} \dot{x}^2 \, dt \right) = \sqrt{\frac{m}{2\pi i \hbar T}} \exp \left[ - \frac{m}{2i \hbar T} (x'' - x')^2 \right]
\]

(4.1)

\[
G^{(\mathbb{R})}(x'', x'; E) = \frac{1}{\hbar} \sqrt{-\frac{m}{2E}} \exp \left( -\frac{\sqrt{-2mE}}{\hbar} |x'' - x'| \right).
\]

(4.2)

Insertion of \( G^{(\mathbb{R})}(E) \) into Eq. (3.1) gives the Green function of the motion in the half space \( \mathbb{R}^+ \), which can be transformed into the time-dependent propagator yielding

\[
K^{(\mathbb{R}^+)}(x'', x'; T) = \sqrt{\frac{m}{2\pi i \hbar T}} \left[ \exp \left( -\frac{m}{2i \hbar T} (x'' - x')^2 \right) - \exp \left( -\frac{m}{2i \hbar T} (x'' + x')^2 \right) \right],
\]

(4.3)

which is the classical result, e.g. [25, 57].

4.1.2. The Harmonic Oscillator. In order to apply our formalism we have to know the Green function of the harmonic oscillator (HO). It is not difficult to derive form the propagator

\[
\frac{d^2}{dt^2} x(t) \exp \left[ \frac{im\omega}{2\hbar} \int_t^{t'} \left( \dot{x}^2 - \omega^2 x^2 \right) dt \right]
\]

\[
= \sqrt{\frac{m\omega}{2\pi i \hbar \sin \omega T}} \exp \left\{ \frac{im\omega}{2\hbar} \left( (x'' + x'^2) \cot \omega T - \frac{2x' x''}{\sin \omega T} \right) \right\}.
\]

(4.4)

the corresponding Green function \( G^{(HO)}(E) \), which has the form [4]

\[
G^{(HO)}(x'', x'; E) = -\sqrt{\frac{m}{\pi \hbar^3 \omega}} \Gamma \left( \frac{1}{2} - \frac{E}{\hbar \omega} \right) \times D^{-\frac{1}{2} + \frac{E}{\hbar \omega}} \left( \sqrt{\frac{2m\omega}{\hbar}} x > \right) D^{-\frac{1}{2} + \frac{E}{\hbar \omega}} \left( -\sqrt{\frac{2m\omega}{\hbar}} x < \right).
\]

(4.5)

Here one can make use of the integral representation [30, p.729], \( a_1 > a_2 \)

\[
\int_0^\infty \coth^{2\nu} \frac{x}{2} \exp \left[ -\frac{a_1 + a_2}{2} \cosh x \right] I_{2\mu} \left( \sqrt{a_1 a_2} \sinh x \right) dx
\]

\[
= \frac{\Gamma \left( \frac{1}{2} + \mu - \nu \right)}{\sqrt{a_1 a_2} \Gamma (1 + 2\mu)} W_{\nu, \mu}(a_1) M_{\nu, \mu}(a_2)
\]

(4.6)
and exploit some relations between the Whittaker functions $W_{\nu,\mu}(z)$ and $M_{\nu,\mu}(z)$, and the parabolic cylinder-functions $D_{\nu}(z)$, respectively. The Green functions of the harmonic oscillator in the half-spaces $x > a$ and $x < a$ are then given by Eq. (3.1) (compare also the case of the so-called double oscillator [4]), respectively Eqs. (3.5, 3.7), and the bound state energy levels are determined by

\begin{align}
D_{-\frac{1}{2}+\frac{\beta}{2\omega}}\left(-\sqrt{\frac{2m\omega}{\hbar}}a\right) &= 0 \quad \text{for } x < a \ , \quad (4.7) \\
D_{-\frac{1}{2}+\frac{\beta}{2\omega}}\left(\sqrt{\frac{2m\omega}{\hbar}}a\right) &= 0 \quad \text{for } x > a \ . \quad (4.8)
\end{align}

### 4.1.3. Morse Potential and Liouville Quantum Mechanics

As the next example for motion in a half-space we consider the Morse potential (M)

$$V^{(M)}(x) = \frac{V_0^2\hbar^2}{2m}(e^{2x} - 2\alpha e^x)$$

($V_0 > 0$, $\alpha \in \mathbb{R}$ constants). This potential has $N_{Max} < \alpha V_0 - \frac{1}{2}$ bound states. The Green function can be explicitly calculated by the path integral formalism [8, 16, 31, 55] and has the form

\begin{align}
\frac{i}{\hbar} \int_0^\infty dT \ e^{iET/\hbar} \int_{x(t')=x'}^{x(t'')=x''} Dx(t) \exp\left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \dot{x}^2 - \frac{V_0^2\hbar^2}{2m}(e^{2x} - 2\alpha e^x) \right] dt \right\} \\
= \frac{m}{V_0\hbar^2} \frac{\Gamma\left(\frac{1}{2} + \sqrt{-2mE/\hbar} - \alpha V_0\right)}{\Gamma(1 + \sqrt{-8mE/\hbar})} e^{(x' + x'')/2} W_{\alpha V_0, \sqrt{-2mE/\hbar}}^\left(2V_0 e^{x'}\right) M_{\alpha V_0, \sqrt{-2mE/\hbar}}^\left(2V_0 e^{x''}\right) . \quad (4.10)
\end{align}

The Green functions in the half-spaces $x > a$ and $x < a$ are then given by Eq. (3.1), respectively Eqs. (3.5, 3.7), and the bound state energy levels are determined by

\begin{align}
M_{\alpha V_0, \sqrt{-2mE_n/\hbar}}^\left(2V_0 e^{a}\right) &= 0 \quad \text{for } x < a \ , \quad (4.11) \\
W_{\alpha V_0, \sqrt{-2mE_n/\hbar}}^\left(2V_0 e^{a}\right) &= 0 \quad \text{for } x > a \ . \quad (4.12)
\end{align}

Liouville (L) quantum mechanics can be studied by setting $\alpha = 0$ in $V^{(M)}$. We have

\begin{align}
G^{(L)}(x'', x'; E) = \frac{2m}{\hbar^2} I_{\sqrt{-2mE/\hbar}}(V_0 e^{x'}) K_{\sqrt{-2mE/\hbar}}(V_0 e^{x''}) . \quad (4.13)
\end{align}

Again, the bound state energy levels $E_n$ in the half-space $x > a$ are determined by

$$K_{\sqrt{-2mE_n/\hbar}}(V_0 e^{a}) = 0 . \quad (4.14)$$

Note that the function $K_{i\nu}(z)$ for large values of $\nu$ is an oscillating sin($z$)-like function (e.g. [53], p.142).
4.1.4. The Wood Saxon Potential. We consider the Wood-Saxon potential. It is defined as \[ V(r) = -\frac{V_0}{1 + e^{(r-b)/R}}, \quad r > 0. \]
\[ (4.15) \]

The corresponding radial path integral formulation has the form \((D = 3)\)
\[ K_l(r'', r', T) = \int_{r(t')=r''}^{r(t')=r'} \mu_l + \frac{1}{2} [r^2] Dr(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \dot{r}^2 + \frac{V_0}{1 + e^{(r-b)/R}} \right] dt \right\}. \]
\[ (4.16) \]

This potential is used to describe the potential trough for the strong interaction near the nucleus for radial symmetric nuclei. There are only a finite number of bound states. Putting \(l = 0\) makes this path integral looking like the “smooth step” potential in \(\mathbb{R}\) with a barrier at \(r = 0\). The path integral solution of the “smooth step” potential in the entire \(\mathbb{R}\) in turn is given by \([33, 48]\)
\[ \frac{i}{\hbar} \int_0^\infty dT \ e^{i ET/\hbar} \int_{x(t')=x''}^{x(t')=x'} \mathcal{D} x(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \dot{x}^2 + \frac{V_0}{1 + e^{(x-b)/R}} \right] dt \right\} = \frac{2mR}{\hbar^2} \frac{\Gamma(m_1)\Gamma(m_1 + 1)}{\Gamma(m_1 + m_2 + 1)\Gamma(m_1 - m_2 + 1)} \times \left( \frac{1 - \tanh \frac{x_{<}-b}{2R}}{2} \right)^{m_1-m_2} \left( \frac{1 + \tanh \frac{x_{<}-b}{2R}}{2} \right)^{m_1+m_2} \times \left( \frac{1 - \tanh \frac{x_{>}-b}{2R}}{2} \right)^{m_1-m_2} \left( \frac{1 + \tanh \frac{x_{>}-b}{2R}}{2} \right)^{m_1+m_2} \times 2F_1 \left( \frac{m_1, m_1 + 1; m_1 - m_2 + 1; 1 - \tanh \frac{x_{>}-b}{2R}}{2} \right) \times 2F_1 \left( \frac{m_1, m_1 + 1; m_1 + m_2 + 1; 1 + \tanh \frac{x_{<}-b}{2R}}{2} \right). \]
\[ (4.17) \]

Here denote \(m_{1,2} = \sqrt{2mR(\sqrt{-E-V_0} \pm \sqrt{-E})}/\hbar\). With a barrier at \(x = a\), such that we consider motion in the half-space \(x > a\), we obtain that the Green function of the Wood-Saxon potential is given by Eq. (3.1), and the bound state energy levels are determined by (with \(0 < |E_n| < V_0\))
\[ 2F_1 \left( \beta + i \lambda, \beta + i \lambda + 1; 1 + 2\beta; \frac{1 - \tanh \frac{a-b}{2R}}{2} \right) = 0 \]
\[ (4.18) \]

Here denote \(\beta^2 = -2mE_nR^2/\hbar^2, \lambda^2 = 2m(E_n + V_0)R^2/\hbar^2\). The claim of Duru [18] of having solved this problem cannot be seen as correct due to an improper handling of the boundary-conditions.
4.1.5. The Linear Potential. The problem of the linear potential \( L \) \( V(x) = kx \) in the half-space \( x > 0 \) has a well-known result, i.e. the infinite number of bound state energy levels are determined by the zeros of the Airy-function [26]. The propagator of the linear potential in the entire \( \mathbb{R} \) is quite easy to obtain due to the fact that it belongs to the class of quadratic Lagrangians whose path integral solution are determined by the classical action. It is given by [25, 57] (\( x \in \mathbb{R} \))

\[
\frac{x(t'') = x''}{\int_{x(t') = x'}^{x(t'') = x''}} Dx(t) \exp \left[ \frac{i}{\hbar} \int_{t'}^{t''} \left( \frac{m}{2} \dot{x}^2 - kx \right) dt \right] = \left( \frac{m}{2 \pi i \hbar T} \right)^{1/2} \exp \left[ \frac{i}{\hbar} \left( \frac{m}{2} \frac{(x'' - x')^2}{T} - \frac{kT}{2} (x' + x'') - \frac{k^2 T^3}{24m} \right) \right]. \tag{4.19}
\]

The corresponding energy dependent Green function cannot be evaluated in an obvious direct way by means of a Fourier transformation from the propagator due to the rather nasty \( T^3 \)-dependence. However, as it is shown in Appendix B, a space-time transformation does the job. For the Green function we obtain

\[
G^{(L)}(x'', x'; E) = \frac{4 m}{3 h^2} \left[ \left( \frac{x'}{E} - \frac{E}{k} \right) \left( x'' - \frac{E}{k} \right) \right]^{1/2} \times K_{1/3} \left[ \frac{2}{3} \sqrt{\frac{2mk}{h^2}} (x_+ - \frac{E}{k})^{3/2} \right] I_{1/3} \left[ \frac{2}{3} \sqrt{\frac{2mk}{h^2}} (x_- - \frac{E}{k})^{3/2} \right]. \tag{4.20}
\]

The Green functions in the half-spaces \( x > a \) and \( x < a \) are then given by Eq. (3.1), respectively Eqs. (3.5,3.7), and the bound state energy levels are determined by (\( n \in \mathbb{N}_0, x > a \))

\[
K_{1/3} \left[ \frac{2}{3} \sqrt{\frac{2mk}{h^2}} \left( a - \frac{E_n}{k} \right)^{3/2} \right] = \text{Ai} \left[ \left( a - \frac{E_n}{k} \right) \left( \frac{2mk}{h^2} \right)^{1/3} \right] = 0, \tag{4.21}
\]

4.2. Boundary Problems in Boxes

4.2.1. The Infinite Potential Well. Again we start with a simple example, i.e. with the infinite potential well (IPW). We need the Green function of the free particle, and insert it into Eq.(3.9). Performing the Fourier transformation with respect to \( E \) then gives the well-known result [57] (\( -b < x < b \))

\[
K^{(IPW)}(x'', x'; T) = \sqrt{\frac{m}{2 \pi i \hbar T}}.
\]

23
\[ \sum_{n=-\infty}^{\infty} \left\{ \exp \left[ -\frac{m}{2 i \hbar T} (x'' - x' + 4nb)^2 \right] - \exp \left[ -\frac{m}{2 i \hbar T} (x'' + x' + 2(2n + 1)b)^2 \right] \right\} \]

\[ = \frac{1}{4b} \left[ \Theta_3 \left( \frac{x'' - x'}{4b}, -\frac{\pi \hbar T}{8mb^2} \right) - \Theta_3 \left( \frac{x'' + x'}{4b} + \frac{1}{2}, -\frac{\pi \hbar T}{8mb^2} \right) \right] \]

\[ = \frac{1}{b} \sum_{n=1}^{\infty} \exp \left( -i \frac{\pi n^2}{8mb^2} \right) \cos \left( \frac{\pi n}{2b} x' \right) \cos \left( \frac{\pi n}{2b} x'' \right). \] (4.22)

Here \( \Theta_3 \) denotes a Jacobi \( \Theta \)-function.

4.2.2. The Harmonic Oscillator. This particular problem has attracted some attention by Aguilera-Navarro et al. [2], Consortini and Frieden [14], Marin and Cruz [54], and Vawter [62]. By means of the Green function for the harmonic oscillator we obtain for the Green function in the box \( a < x < b \) Eq. (3.15), and the bound state energy-levels are determined by

\[ D_{-\frac{1}{2} + \frac{i}{8\sqrt{m\omega}}} \left( \sqrt{\frac{2m\omega}{\hbar} a} \right) D_{-\frac{1}{2} + \frac{i}{8\sqrt{m\omega}}} \left( -\sqrt{\frac{2m\omega}{\hbar} b} \right) \]

\[ = D_{-\frac{1}{2} + \frac{i}{8\sqrt{m\omega}}} \left( \sqrt{\frac{2m\omega}{\hbar} b} \right) D_{-\frac{1}{2} + \frac{i}{8\sqrt{m\omega}}} \left( -\sqrt{\frac{2m\omega}{\hbar} a} \right). \] (4.23)

4.3. Boundary Problems in Radial Boxes

4.3.1. The Radial Box. In order to treat the problem of the simple radial box we consider the radial free Green function which is given by

\[ (r'r'')^{\frac{3-D}{2}} \frac{i}{\hbar} \int_{0}^{\infty} dT \ e^{iET/\hbar} \int_{r(t')=r'}^{r(t'')=r''} D_{r}(t) \exp \left[ \frac{i}{\hbar} \int_{t'}^{t''} \left( \frac{m}{2} r'^2 - \hbar^2 l(l + D - 2) \right) dt \right] \]

\[ = (r'r'')^{\frac{3-D}{2}} \frac{i}{\hbar} \int_{0}^{\infty} dT \ e^{iET/\hbar} \int_{r(t')=r'}^{r(t'')=r''} \mu_{l+\frac{D-2}{2}} [r'^2]D_{r}(t) \exp \left( \frac{im}{2\hbar} \int_{t'}^{t''} r'^2 dt \right) \]

\[ = \frac{2m}{\hbar^2 (r'r'')}^{\frac{3-D}{2}} K_{l+\frac{D-2}{2}} \left( \sqrt{-2mE} \frac{T_>}{\hbar} \right) I_{l+\frac{D-2}{2}} \left( \sqrt{-2mE} \frac{T_<}{\hbar} \right). \] (4.24)

Therefore the Green function for the simple radial box \( 0 < r < R \) is given by Eq. (3.15) and the energy levels are determined by

\[ I_{l+\frac{D-2}{2}} \left( \sqrt{-2mE} \frac{R}{\hbar} \right) = 0, \] (4.25)
i.e. by the zeros $z_{\lambda,n}$ of the modified Bessel function $I_\lambda(z)$. The corresponding problem with $r > R$ describes the free motion in $D$ dimensions with a impenetrable (hard) sphere at the origin. There are no bound states.

This result allows us to describe the problem of the radial potential

$$V_\lambda(r) = \frac{\hbar^2 \lambda^2 - \frac{1}{4}}{2m 2mr^2}$$

(4.26)

within, respectively outside, a radial box. The Green function of the problem without boundary is given by [it can be considered as an analytic continuation of Eq (4.24)]:

$$G^{(V_\lambda)}(r'', r'; E) = \frac{2m}{\hbar^2} K_\lambda \left( \sqrt{-2mE} \frac{r''}{\hbar} \right) I_\lambda \left( \sqrt{-2mE} \frac{r'}{\hbar} \right),$$

(4.27)

and the Green function with boundary in the space $0 < r < R$ is given by Eq. (3.15), and the bound state energy levels are determined by

$$I_\lambda \left( \sqrt{-2mE_n \frac{R}{\hbar}} \right) = 0 .$$

(4.28)

This kind of problems appear e.g. by considering a sector of angel $\alpha$ which has a boundary at $r = R$, where $\lambda = l\pi/\alpha$ ($l \in \mathbb{N}$), i.e. we have motion inside a region which looks like a piece of a cake [11].

Vice versa, we obtain for the Green function for the potential $V_\lambda$ in the space $r > R$

$$G^{(r>R)}(r'', r'; E) = \frac{2m}{\hbar^2} K_\lambda \left( \sqrt{-2mE} \frac{r''}{\hbar} \right) I_\lambda \left( \sqrt{-2mE} \frac{r'}{\hbar} \right) - \frac{2m}{\hbar^2} I_\lambda \left( \sqrt{-2mE} \frac{R}{\hbar} \right) K_\lambda \left( \sqrt{-2mE} \frac{r''}{\hbar} \right) K_\lambda \left( \sqrt{-2mE} \frac{r'}{\hbar} \right).$$

(4.29)

Hence, we describe the problem of the potential $V_\lambda$ outside a hard sphere (hard disc) of radius $R$. A model for such a Green function is the Green function for a Bohm-Aharonov solenoid outside a hard disc of radius $R$ with flux $\Phi$ inside. Then $\lambda = |\nu + q\Phi/2\pi c\hbar|$ ($\nu \in \mathbb{Z}$), with $q$ the charge of the particle and $c$ the velocity of light (e.g. [42, 43, 63] and references therein).

4.3.2. The Radial Harmonic Oscillator. The model of the radial harmonic oscillator (RHO) can either be seen as a radial harmonic oscillator inside a radial box, or a radial harmonic oscillator with a hard sphere at the origin. We have for the path integral solution for the radial harmonic oscillator [17, 28, 56]

$$\int_{r(t')=r'}^{r(t'')=r''} D\mathbf{r}(t) \exp \left[ \frac{i}{\hbar} \int_{t'}^{t''} \left( \frac{m}{2} \dot{r}^2 - \hbar^2 \frac{l(l+D-2)}{2mr^2} - \frac{m}{2} \omega^2 r^2 \right) dt \right]$$
The corresponding Green function is given by [4, 5] (c.f. Eq. (4.6))

\[
G^{(RHO)}_l(r'', r'; E) = \frac{\Gamma\left[\frac{1}{2}(l + \frac{D-1}{2}) - \frac{p}{2}\right]}{\hbar \omega (r''r')^{D/2} \Gamma(l + D/2)} \times W_n r''_{\frac{D-2}{2}(l+\frac{D-2}{2})} \left(\frac{m\omega}{\hbar} r''^2\right) M_n r''_{\frac{D-2}{2}(l+\frac{D-2}{2})} \left(\frac{m\omega}{\hbar} r''^2\right) .
\]

With a barrier at \(x = R\) we obtain the Green function of the radial harmonic oscillator for a radial box by Eq. (3.15), and the bound state energy levels are determined by

\[
M_n r''_{\frac{D}{2}(l+\frac{D}{2})} \left(\frac{m\omega}{\hbar} R^2\right) = 0 \quad \text{for } r < R ,
\]

\[
W_n r''_{\frac{D}{2}(l+\frac{D}{2})} \left(\frac{m\omega}{\hbar} R^2\right) = 0 \quad \text{for } r > R .
\]

4.3.3. The Coulomb Potential. We consider the Coulomb potential (C). The path integral reads (\(\vec{x} \in \mathbb{R}^D\))

\[
K^{(C)}(\vec{x''}, \vec{x'}; T) = \int_{\vec{x}(t') = \vec{x'}} D\vec{x}(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \dot{\vec{x}}^2 + \frac{q_1 q_2}{\vec{x}} \right] dt \right\} .
\]

Here only the Green function can be explicitly calculated. The radial Coulomb Green function has the form (see e.g. [4, 12, 35, 40, 41, 59] and references therein)

\[
G^{(C)}(r'', r'; E) = (r'r'')^{-\frac{1-D}{2}} \frac{1}{\hbar} \sqrt{-\frac{m}{2E}} \frac{\Gamma\left(l + \frac{D-1}{2} + ip\right)}{(2l + D - 2)!} \times W_{-ip,l+\frac{D-2}{2}} \left(\sqrt{-\frac{8mE}{\hbar^2} r''}\right) M_{-ip,l+\frac{D-2}{2}} \left(\sqrt{-\frac{8mE}{\hbar^2} r''}\right) ,
\]

with \(p = (q_1 q_2/\hbar) \sqrt{m/2E}\). With a barrier at \(x = R\) we obtain the Green function of the Coulomb potential for a radial box by Eq. (3.15), and the bound state energy levels are determined by

\[
M_{\frac{q_1 q_2}{\hbar}} r''_{\frac{D}{2}(l+\frac{D}{2})} \left(\sqrt{-\frac{8mE}{\hbar^2} R}\right) = 0 \quad \text{for } r < R ,
\]

\[
W_{\frac{q_1 q_2}{\hbar}} r''_{\frac{D}{2}(l+\frac{D}{2})} \left(\sqrt{-\frac{8mE}{\hbar^2} R}\right) = 0 \quad \text{for } r > R .
\]
4.4. Boundary Problems in Radial Rings

It is not difficult to generalize the preceding results to the case of radial rings. The corresponding Green function in a radial ring \(a < r < R\) are given by Eq. (3.17) and the bound state energy levels are determined by:

4.4.1. Simple Radial Ring.

\[
I_{l + \frac{d-2}{2}} \left( \sqrt{-2mE_n \frac{a}{\hbar}} \right) K_{l + \frac{d-2}{2}} \left( \sqrt{-2mE_n \frac{R}{\hbar}} \right)
= I_{l + \frac{d-2}{2}} \left( \sqrt{-2mE_n \frac{R}{\hbar}} \right) K_{l + \frac{d-2}{2}} \left( \sqrt{-2mE_n \frac{a}{\hbar}} \right). \tag{4.38}
\]

4.4.2. Radial Harmonic Oscillator in a Ring.

\[
M_{\frac{E_n}{2m\omega}, \frac{1}{2}(l + \frac{d-2}{2})} \left( \frac{m\omega}{\hbar} a^2 \right) W_{\frac{E_n}{2m\omega}, \frac{1}{2}(l + \frac{d-2}{2})} \left( \frac{m\omega}{\hbar} R^2 \right)
= W_{\frac{E_n}{2m\omega}, \frac{1}{2}(l + \frac{d-2}{2})} \left( \frac{m\omega}{\hbar} a^2 \right) M_{\frac{E_n}{2m\omega}, \frac{1}{2}(l + \frac{d-2}{2})} \left( \frac{m\omega}{\hbar} R^2 \right). \tag{4.39}
\]

4.4.3. Coulomb Potential in a Ring.

\[
M_{\frac{2l+2}{2}, \frac{1}{2}(l + \frac{d-2}{2})} \left( \sqrt{-8mE_n \frac{a}{\hbar}} \right) W_{\frac{2l+2}{2}, \frac{1}{2}(l + \frac{d-2}{2})} \left( \sqrt{-8mE_n \frac{R}{\hbar}} \right)
= W_{\frac{2l+2}{2}, \frac{1}{2}(l + \frac{d-2}{2})} \left( \sqrt{-8mE_n \frac{a}{\hbar}} \right) M_{\frac{2l+2}{2}, \frac{1}{2}(l + \frac{d-2}{2})} \left( \sqrt{-8mE_n \frac{R}{\hbar}} \right). \tag{4.40}
\]

4.5. Moving Boundary Problems

4.5.1. The Infinite Potential Well. We consider the example of the infinite potential well with one boundary fixed at \(x = 0\), and the other moving uniformly in time according to \(L(t) = L_0\zeta(t)\) [15]. According to Eq. (3.21), the result then has for \(\omega' = 0\) the form

\[
K^{(IPW)}(x'', x'; t'', t') = \frac{(\zeta'\zeta'')^{-1/2}}{2L_0} \exp \left[ \frac{i}{2\hbar} \left( x''^2 \frac{\zeta''}{\zeta'} - x'^2 \frac{\zeta'}{\zeta''} \right) \right]
\times \left[ \Theta_3 \left( \frac{x''/\zeta'' - x'/\zeta'}{2L_0}, -\frac{\pi\hbar \tau(t'')}{2mL_0^2} \right) - \Theta_3 \left( \frac{x''/\zeta'' + x'/\zeta'}{2L_0}, -\frac{\pi\hbar \tau(t'')}{2mL_0^2} \right) \right]. \tag{4.41}
\]
4.5.2. Moving Boundary in the Half-Space. We consider the half-space with the boundary moving uniformly in time according to Eq. (3.23) and we consider \( f(t) = vt \). Hence \( \ddot{f} = 0 \), and for a free particle we obtain

\[
K^{(MB)}(x'', x'; t'', t') = \sqrt{\frac{m}{2\pi i\hbar T}} \exp \left\{ \frac{im}{\hbar} \left[ v(x'' - vt'') - v(x' - vt') + \frac{v^2}{2} T \right] \right\} \\
\times \left[ \exp \left( \frac{im}{2\hbar T} |x'' - x' - vT|^2 \right) - \exp \left( \frac{im}{2\hbar T} |x'' + x' - v(t'' + t')|^2 \right) \right].
\]

(4.42)

Generally, if the “mirror” principle can be applied, say for the harmonic oscillator, the result reads

\[
K^{(MB)}(x'', x'; t'', t') = \exp \left\{ \frac{im}{\hbar} \left[ v(x'' - vt'') - v(x' - vt') + \frac{v^2}{2} T \right] \right\} \\
\times \left[ K_0(x'' - vt'', x' - vt'; T) - K_0(x'' - vt'', -(x' - vt'); T) \right].
\]

(4.43)

5. Summary

In this paper I have presented a comprehensive approach to the theory of boundary problems with Dirichlet boundary-conditions derived by an appropriate limiting process from \( \delta \)-function perturbations in path integrals.

First, I could derive from a path integral formulation and an exact summation of a perturbation expansion, closed formulae for the problem of multiple \( \delta \)-function perturbations in one dimension. This result could be used to derive an integral representation for the problem of multiple perpendicular \( \delta \)-function perturbations along lines, planes and hyperplanes in \( D \) dimensions, in the form of \( (D - 1) \)-fold convolutions of the corresponding one-dimensional problem.

I could also discuss the problems of a \( \delta \)-function perturbation along a line, respectively its equivalent of two particles interacting via a \( \delta \)-function, and a moving \( \delta \)-function perturbation.

Second, I could consider boundary problems with Dirichlet boundary-conditions. They were derived form the ones with \( \delta \)-function perturbations by letting the strength of the \( \delta \)-function perturbations be infinite repulsive. This gave closed expressions for the one-dimensional motions in half-spaces, boxes, radial boxes, rings and moving boundaries. The advanced problem of multidimensional perpendicular boundaries along perpendicular lines and planes was given similarly as for the \( \delta \)-function perturbations, i.e. by convolutions of the corresponding Green functions of the one-dimensional case, however, only two boundaries are required for each dimension, thus simplifying
the corresponding one-dimensional Green functions, and the number of boundaries must not exceed \((2 \times \text{dimension})\).

These models of multiple \(\delta\)-function perturbations and boundaries, respectively, along perpendicular lines and planes should serve as a numerically simple approach for the investigation of band structures.

Several examples put our results in a better context and illustrated the formalism. Here some long unsolved problems in path integration could be treated successfully, for instance the Green function of the linear potential, and the Wood-Saxon potential in the half-space \(x > a\). For the former, there are an infinite number of levels determined by the zeros of the Airy-function, whereas for the latter there are only a finite number of levels. For the linear potential it was required to calculate the Green function in the entire \(\mathbb{R}\) which was done in Appendix B.

However, the question arises how to deal with other than Dirichlet boundary-conditions \([9, 10, 13]\), which present only one possibility to define a self-adjoint extension of the (free) Hamiltonian with boundaries. It is known that an appropriate chosen \(\delta\)-function perturbation can do the job; at least in case of the one-dimensional free particle in \(\mathbb{R}^+\) the propagator with general boundary-conditions at \(x = 0\) can be explicitly computed \([13]\). It is derived from two perturbed constructively interfering kernels, i.e. a modified “mirror” principle is applied. As we know, this simple method does not work in the general case. All what we know in the general case is the corresponding Green function for the perturbed-, respectively boundary-problem. Of course, the same difficulties arise for the case of more than one boundary and in higher dimensions. These problems, however, will be discussed elsewhere.

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Appendix 1. Green Function for Multiple $\delta$-Function Perturbations

In this appendix I want to show Eq. (2.23). The proof goes by induction. It is obvious that it is sufficient to show the validity of Eq. (2.23) for the denominator. Let us consider now the determinant $(n \in \mathbb{N}) \det(a_{ij}) = 0$, with the $(n-1) \times (n-1)$ matrix

$$
(a_{ij}) = \left( G^{(V)}(a_i, a_j; E) - \frac{\delta_{ij}}{\gamma_i} \right).
$$

(A1.1)

We know that Eq. (2.23) is valid for $n = 1, 2$. We make the step form $n - 1$ to $n$. Doing this, every matrix entry is changed in the following way

$$
a_{ij} \mapsto \tilde{a}_{ij} = \frac{a_{nn}a_{ij} - a_{in}a_{nj}}{a_{nn}}.
$$

(A1.2)

Therefore we obtain by expanding the determinant

$$
\det \left( \tilde{a}_{ij} \right)_{i,j=1}^{n-1}
= \frac{1}{a_{nn}^{n-1}} \left| \begin{array}{cccc}
a_{11}a_{nn} - a_{1n}a_{n1} & \ldots & a_{1,n-1}a_{nn} - a_{1n}a_{n,n-1} \\
\vdots & \ddots & \vdots \\
a_{n-1,1}a_{nn} - a_{n-1,n}a_{n1} & \ldots & a_{n-1,n-1}a_{nn} - a_{n-1,n}a_{n,n-1} \\
\end{array} \right|
= \frac{1}{a_{nn}^{n-1}} \left\{ \begin{array}{cccc}
a_{11} & a_{12} & \ldots & a_{1,n-1} \\
a_{21} & a_{22} & \ldots & a_{2,n-1} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n-1,1} & a_{n-1,2} & \ldots & a_{n-1,n-1} \\
\end{array} \right|
- a_{nn}^{n-2} \left| \begin{array}{ccc}
a_{11} & \ldots & a_{1,n-2} \\
a_{21} & \ldots & a_{2,n-2} \\
\vdots & \ddots & \vdots \\
a_{n-1,1} & \ldots & a_{n-1,n-2} \\
\end{array} \right|
+ \ldots + (-1)^n a_{nn}^{n-2} \left| \begin{array}{ccc}
a_{11} & a_{1,2} & \ldots & a_{1,n-1} \\
a_{21} & a_{2,2} & \ldots & a_{2,n-1} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n-1,1} & a_{n-1,2} & \ldots & a_{n-1,n-1} \\
\end{array} \right|
+ a_{nn}^{n-3} \left| \begin{array}{ccc}
a_{11} & \ldots & a_{1,n} \\
a_{21} & \ldots & a_{2,n} \\
\vdots & \ddots & \vdots \\
a_{n-1,1} & \ldots & a_{n-1,n} \\
\end{array} \right|
+ \ldots + (-1)^{n-1} a_{n-1} \left| \begin{array}{ccc}
a_{11} & \ldots & a_{1,j} \\
a_{21} & \ldots & a_{2,j} \\
\vdots & \ddots & \vdots \\
a_{n-1,1} & \ldots & a_{n-1,j} \\
\end{array} \right| \right\}.
$$

(A1.3)
The last terms with the power of the $a_{nn}$ factor less than $n - 2$ are all zero because two columns are always equal. The last non-vanishing term can be rearranged by a permutation where the first column is put into the last with increasing second index of the $a_{1,j}$, etc. The emerging sum is now the definition of the expansion of a $n \times n$ matrix with the required feature, i.e. we have

$$\det \left( \widetilde{a}_{ij} \right)_{i,j=1}^{n-1} = \det \left( a_{ij} \right)_{i,j=1}^{n} , \quad (A1.4)$$

which shows the induction.

**Appendix 2. Green Function for the Linear Potential**

We consider the path integral formulation for the linear potential

$$K(x'', x'; T) = \int_{x(t') = x'}^{x(t'') = x''} Dx(t) \exp \left[ \frac{i}{\hbar} \int_{t'}^{t''} \left( \frac{m}{2} \dot{x}^2 - kx \right) dt \right] . \quad (A2.1)$$

We perform the combined space-time transformation (c.f. Refs. [39, 47, 60] and references therein)

$$y = \left( x - \frac{E}{k} \right)^{3/2} , \quad dt = \frac{4}{9} y^{-2/3} ds , \quad (A2.2)$$

and introduce a new pseudo-time $s''$

$$s'' = \frac{9}{4} \int_{t'}^{t''} y^{2/3}(t) dt , \quad (A2.3)$$

in this path integral, where the lattice implementation is given by $\Delta t_j = \frac{4}{9} y_j^{-1/3} y_{j-1}^{-1/3}$ to guarantee a symmetric transformation with respect to initial and final coordinates. This gives the transformation formulæ

$$K(x'', x'; T) = \int_{-\infty}^{\infty} \frac{dE}{2\pi i} e^{-iET/\hbar} G(x'', x'; E) \quad (A2.4)$$

$$G(x'', x'; E) = \frac{i}{\hbar} \frac{2}{3} (y'y'')^{-1/6} \int_{0}^{\infty} ds'' e^{-4i s'' k/9h} \tilde{K}(y'', y'; s'') , \quad (A2.5)$$

with the transformed path integral $\tilde{K}(s'')$ given by

$$\tilde{K}(y'', y'; s'') = \int_{y(0) = y'}^{y(s'') = y''} Dy(s) \exp \left[ \frac{i}{\hbar} \int_{0}^{s''} \left( \frac{m}{2} y'^2 + \frac{\hbar^2}{8my'^2} \frac{5}{9} \right) ds \right].$$
\[
\begin{align*}
\left( s'' \right) &= y'' \\
= & \int y^{(1)}[s''] \, \mathrm{d}y(s) \exp \left( \frac{im}{2\hbar} \int_{0}^{s''} y^2 \, \mathrm{d}s \right) \\
&= \sqrt{y'y''} \frac{m}{\hbar s''} \exp \left[ - \frac{m}{2\hbar s''} \left( y^2 + y''^2 \right) \right] I_{1/3} \left( \frac{my'y''}{\hbar s''} \right) .
\end{align*}
\] (A2.6)

Here use has been made of the radial path integral formulation (2.12) with the functional weight \( \mu_\nu \) together with the path integral solution (4.30) for inverse square radial potentials. Making use of the integral representation [30, p.719]

\[
\int_{0}^{\infty} e^{-a/x} J_\nu(cx) \, \mathrm{d}x = 2J_\nu \left[ \sqrt{2a} \left( \sqrt{b^2 + c^2} - b \right) \right] K_\nu \left[ \sqrt{2a} \left( \sqrt{b^2 + c^2} + b \right) \right]
\] (A2.7)

we obtain for \( G(E) \)

\[
G(x'', x'; E) = 4 \frac{m}{3 \hbar^2} \left[ \left( x' - \frac{E}{k} \right) \left( x'' - \frac{E}{k} \right) \right]^{1/2} \\
\times K_{1/3} \left[ \frac{2m}{3 \hbar^2} \left( \frac{2}{3} \right)^{3/2} \right] I_{1/3} \left[ \frac{2m}{3 \hbar^2} \left( \frac{2}{3} \right)^{3/2} \right],
\] (A2.8)

(compare Steiner [60] for the case \( E = 0 \)). The modified Bessel functions \( I_{1/3} \) and \( K_{1/3} \) can be rewritten as Airy functions. One has to make use of the following relations [53, p.75]

\[
\begin{align*}
\text{Ai}(z) &= \left( \frac{z}{3} \right)^{1/2} \left[ I_{-1/3} \left( \frac{2}{3} z^{3/2} \right) - I_{1/3} \left( \frac{2}{3} z^{3/2} \right) \right] = \frac{1}{\pi} \left( \frac{z}{3} \right)^{1/2} K_{1/3} \left( \frac{2}{3} z^{3/2} \right) \\
\text{Bi}(z) &= \left( \frac{z}{3} \right)^{1/2} \left[ I_{-1/3} \left( \frac{2}{3} z^{3/2} \right) + I_{1/3} \left( \frac{2}{3} z^{3/2} \right) \right].
\end{align*}
\] (A2.9)

Then we obtain

\[
G(x'', x'; E) = \frac{\pi}{h} \left( \frac{4m^2}{h k} \right)^{1/3} \text{Ai} \left[ \left( x' - \frac{E}{k} \right) \left( \frac{2mk}{\hbar^2} \right)^{1/3} \right] \\
\times \left\{ \text{Bi} \left[ \left( x' - \frac{E}{k} \right) \left( \frac{2mk}{\hbar^2} \right)^{1/3} \right] - \sqrt{3} \text{Ai} \left[ \left( x' - \frac{E}{k} \right) \left( \frac{2mk}{\hbar^2} \right)^{1/3} \right] \right\}.
\] (A2.11)
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