SPATIAL ANALYTICITY OF SOLUTIONS FOR A COUPLED SYSTEM OF GENERALIZED KDV EQUATIONS

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Abstract. The solution of a coupled system consisting of generalized Korteweg-de Vries-type equations is obtained for all time where the initial data are analytic on a band in the complex plane. We show that the width of this band decreases algebraically with time.

Keywords. Generalized Korteweg–de Vries, well-posedness, Radius of spatial analyticity, Analytic space.

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1. INTRODUCTION

This paper deals with the initial-value problem for a coupled system of generalized Korteweg–de Vries (gKdV) equation

\[
\begin{align*}
  u_t + \partial_x^3 u + \partial_x (u^p v^{p+1}) &= 0, \\
  v_t + \partial_x^3 v + \partial_x (u^{p+1} v^p) &= 0, \\
  u(x,0) &= u_0(x), \\
  v(x,0) &= v_0(x),
\end{align*}
\]

where the unknown \( u = u(x,t) \), \( v = v(x,t) \) and the initial data \((u_0(x), v_0(x))\) are real-valued.

This type of equation is a special case of an important vast class of nonlinear evolution equations which was studied by M. Ablowitz [1], and it has applications in physical problems, which describes the strong interaction of two dimensional long internal gravity waves.

For \( p = 1 \), the system can be reduce to a coupled system of modified KdV (mKdV) equations

\[
\begin{align*}
  u_t + \partial_x^3 u + \partial_x (uv^2) &= 0, \\
  v_t + \partial_x^3 v + \partial_x (u^2 v) &= 0, \\
  u(x,0) &= u_0(x), \\
  v(x,0) &= v_0(x).
\end{align*}
\]

Here, the author proved the local well posdness in in \( H^s, s \geq \frac{1}{4} \). For \( s \geq 1 \), it is proved that the global well posdness is assured. In addition, M. Panthee improved it to extend solution to be in any time interval \([0, T] \) for \( s > \frac{4}{7} \).
The authors in [14] studied the local well-posedness in \((H^s \times H^s)\) with \(s > -\frac{1}{2}\) for system consisting modified Korteweg–de Vries-type equations
\[
\begin{aligned}
& \begin{cases}
  u_t + \partial_x^3 u + \partial_x (u^2 v^3) = 0, \\
  v_t + \alpha \partial_x^3 v + \partial_x (u^3 v^2) = 0, \\
  u(x,0) = u_0(x), \
  v(x,0) = v_0(x),
\end{cases}
\end{aligned}
\]
where \(0 < \alpha < 1\) and \((u_0,v_0)\) is given in low regularity Sobolev spaces \((H^s \times H^s)\), but if \(\alpha = 1\) the authors obtained the local well posedness for \(s \geq \frac{1}{4}\).

In [2], the problem (1.1) is studied and the local and global well-posedness results with \((u_0,v_0) \in H^s \times H^s, s \geq 1\) and \(p \geq 1\) is shown. The global well-posedness was obtained by using the next conserved quantities satisfied by the flow of (1.1)
\[
\int_{\mathbb{R}} udx \quad \int_{\mathbb{R}} vdx \quad \frac{1}{2} \int_{\mathbb{R}} u^2 + v^2 dx \quad \text{and} \quad \frac{1}{2} \int_{\mathbb{R}} u_x^2 + v_x^2 - \frac{2}{p+1} u^{p+1} v^{p+1} dx.
\]
In addition, the authors showed the existence and nonlinear stability of the solitary wave solution. The study of stability for solitary wave solution is followed from the abstract results of Grillakis, for more details, please see [3, 4, 17, 21].

For \(p = 2\), the system is turn out to a coupled system of modified Korteweg–de Vries (gKdV) equation
\[
\begin{aligned}
& \begin{cases}
  u_t + \partial_x^3 u + \partial_x (u^2 v^3) = 0, \\
  v_t + \partial_x^3 v + \partial_x (u^3 v^2) = 0, \\
  u(x,0) = u_0(x), \
  v(x,0) = v_0(x),
\end{cases}
\end{aligned}
\]
Panthee and Scialom [19], investigated some well-posedness issues for eq (1.4) in \(H^s \times H^s\), which proved local and global will posdness for \(s \geq 0\).

For related problems in analytic Gevrey spaces, we review the results in 2D by M. Shan, L. Zhang [20], where the authors proved that the following problem (the Cauchy problem associated with the 2D generalized Zakharov-Kuznetsov equation)
\[
\begin{aligned}
& \begin{cases}
  u_t + (\partial_x^3 + \partial_y^3) u + (\partial_x + \partial_y) u^{p+1} = 0, \\
  u(0,x,y) = u_0(x,y),
\end{cases}
\end{aligned}
\]
has an analytic solutions in a strip the width, and they gave an algebraic lower bounds.

Bona and Grujić [6] showed the well-posedness of a KdV-type Boussinesq system
\[
\begin{aligned}
& \begin{cases}
  u_t + v_x + uu_x + v_{xxx} = 0, \\
  v_t + u_x + (uv)_x + u_{xxx} = 0.
\end{cases}
\end{aligned}
\]
There is another method in this direction, we mention the works by A. Boukarou et al. in the next series of papers [7, 8, 9, 10, 11, 12, 22].

Motivated by the previuos results, we consider our main problem with initial data are analytic on a band in the complex plane and obtained solution for all time. We also showed that the width of this band decreases algebraically with time.

This paper is continuation of our previuos results and it is structured as follows. In section 1, we give some historical review and motivate this paper to further strengthened, and innovate the main contributions and introduce our main results which we will prove later (local and global well posedness of equation (1.1)). In section 2, we present some definition and the necessary function spaces such as the analytic function spaces \(G_{p,s}\), analytic Bourgain space \(X_{p,s,b}\) which will be used. In section 3, we prove
the Linear and Bilinear Estimates which needed to prove the main results. In section 4, we prove the
d-local and global well-posdness and then obtained lower bound.
We provide a clear, sober and well-written analysis of the problem.

**Theorem 1.1.** Let \( s > \frac{3}{2} \) and \( p \geq 1 \) and for initial data \((u_0, v_0) \in \mathcal{G}_{p,s} \times \mathcal{G}_{p,s}, \) \( p > 0, \) there exists a positive
time \( T, \) such that the initial -value problem (1.1) is well-posed in the space

\[
C([0, T]; \mathcal{G}_{p,s}) \times C([0, T]; \mathcal{G}_{p,s}).
\]

**Theorem 1.2.** Let \( \rho_0 > 0 \) and \( s > \frac{3}{2} \) and let \( T \geq t_0 \) suppose that the solution \( u, v \) given by Theorem (1.1)
extends globally in time. Then, we have

\[
(u, v) \in C([0, 2T]; \mathcal{G}_{p(T)/2,s}) \times C([0, 2T]; \mathcal{G}_{p(T)/2,s}),
\]

where \( \rho(T) \) is given by

\[
\rho(t) = \min \{ \rho_1, KT^{-2p^2-6p-1} \}.
\]

for some constant \( K > 0. \)

### 2. Preliminary estimates and Function spaces

The \( \hat{u} \) is denote the Fourier transform of \( u \) which is defined as

\[
\hat{u}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x) e^{-ix\xi} dx.
\]

For a function \( u(x,t) \) of two variable we have

\[
\hat{u}^x(\xi, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t) e^{-ix\xi} dx,
\]

and

\[
\hat{u}(\xi, \eta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x,t) e^{-ix\xi} e^{-iy\eta} dx d\eta.
\]

We note that the operators \( A, \Lambda \) and \( F_\rho \) are defined as

\[
A\hat{u}(\xi, \eta) = (1 + |\xi|) \hat{u}(\xi, \eta);
\]

\[
\Lambda\hat{u}(\xi, \eta) = (1 + |\eta|) \hat{u}(\xi, \eta);
\]

\[
F_\kappa(\xi, \eta) = \frac{f(\xi, \eta)}{(1 + |\eta - \xi^3|)^{\kappa}}.
\]

The mixed \( L^p - L^q \) -norm is defined by

\[
\|u\|_{L^p L^q} = \left( \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} |u(x,t)|^q dx \right)^{\frac{p}{q}} dt \right)^{\frac{1}{p}}.
\]

The analytic Gevrey class \( \mathcal{G}_{p,s} \) is defined by Foias and Temam [15] as

\[
\|u_0\|_{\mathcal{G}_{p,s}}^2 = \| e^{\rho(1+|\xi|)} (1 + |\xi|)^s \hat{u}_0(\xi) \|_{L^2_\xi}.
\]

For \( s, b \in \mathbb{R}, b \in [-1, 1] \) and \( \rho > 0, \) we denote \( X_{p,s,b} \) by \( \| \cdot \|_{\rho,s,b} \) with respect to the norm

\[
\|u\|_{X_{p,s,b}} = \| e^{\rho(1+|\xi|)} (1 + |\xi|)^s (1 + |\eta - \xi^3|)^b \hat{u}(\xi, \eta) \|_{L^2_\xi L^2_\eta}.
\]
For $\rho = 0, X_{\rho,s,b}$ coincides with the space $X_{s,b}$ introduced by Bourgain [13], and Kenig, Ponce and Vega [18]. The norm of $X_{s,b}$ is denoted by $\| \cdot \|_{s,b}$, as follow

$$\| u \|_{X_{s,b}} = \left\| (1 + |\zeta|)^s (1 + |\eta - \zeta|^3) \hat{u}(\zeta, \eta) \right\|_{L^2_{\zeta, \eta}}.$$ 

3. Linear and Multilinear Estimates

In this section, we shall deduce several estimates to be used in the proof of Theorem (1.1).

**Lemma 3.1.** Let $0 < \sigma < \rho$ and $n \in \mathbb{N}$. Then, we have

$$\sup_{x + iy \in S_{\rho - \sigma}} |\partial^n_{x} u(x + iy)| \leq C \| u \|_{\mathcal{G}_\sigma},$$

where $C$ is constant depending on $\zeta$ and $n$.

**Lemma 3.2.** Let $b > \frac{1}{2}, s \in \mathbb{R}$ and $\rho \geq 0$, then for all $T > 0$, we have

$$X_{\rho,s,b} \hookrightarrow C([0,T], G^{p,s}).$$

**Proof.** We define the operator $\Theta$

$$\Theta u (\zeta, t) = e^{\rho (1 + |\zeta|)} \hat{u} (\zeta, t),$$

satisfy

$$\| u \|_{X_{\rho,s,b}} = \| \Theta u \|_{X_{\rho,s,b}},$$

and

$$\| u \|_{\mathcal{G}_\rho} = \| \Theta u \|_{H^s}.$$ 

We observe that $\Theta u$ belongs to $C([0,T], H^s)$ and for some $C > 0$ we have

$$\| \Theta u \|_{C([0,T], H^s)} \leq C \| \Theta u \|_{X_{\rho,s,b}}.$$

Thus, it follows that $u \in C([0,T], G^{p,s})$ and

$$\| u \|_{C([0,T], G^{p,s})} \leq C \| u \|_{X_{\rho,s,b}}.$$ 

By using Duhamel’s formula (1.1), we may write the solution

$$\begin{cases} u(x, t) = W(t)u_0(x) - \int_0^t W(t - t')w_1(x, t')dt', \\
v(x, t) = W(t)v_0(x) - \int_0^t W(t - t')w_2(x, t')dt', \end{cases}$$

where $W(t) = e^{-i\partial^3 t}, w_1 = \partial_x (u^p v^{p+1})$ and $w_2 = \partial_x (u^{p+1}v^p)$.

Next, we localize in time variable by using a cut-off function $\psi(t) \in C_0^\infty (-2, 2)$ with
We define $\psi_T(t) = \psi\left(\frac{t}{T}\right)$, where
\[
\begin{align*}
\psi &\in C_0^\infty, \quad \psi = 1 \text{ in } [-1; 1] \\
supp \psi &\subset [-2; 2] \\
\psi_T(t) &= \psi\left(\frac{t}{T}\right). 
\end{align*}
\]

We consider the operator $\Xi$, $\Gamma$ given by the following
\[
\begin{align*}
\Xi(t) &= \psi(t)W(t)u_0 - \psi_T(t)\int_0^t W(t-t')w_1(t')dt' \\
\Gamma(t) &= \psi(t)W(t)v_0 - \psi_T(t)\int_0^t W(t-t')w_2(t')dt'.
\end{align*}
\]

We start with the following useful Lemma.

**Lemma 3.3.** [18, 16] Let $\rho \geq 0$, $b > \frac{1}{2}$, $b - 1 < b' < 0$, and $T \geq 1$. Then there exist a constant $c$ such that the following estimates holds
\[
\begin{align*}
\|\psi(t)W(t)u_0\|_{p,s,b} &\leq cT^{\frac{1}{2}}\|u_0\|_{\varphi_{p,r}}, \\
\|\psi(t)W(t)v_0\|_{p,s,b} &\leq cT^{\frac{1}{2}}\|v_0\|_{\varphi_{p,r}},
\end{align*}
\]

and
\[
\begin{align*}
\|\psi_T(t)u\|_{p,s,b} &\leq c\|u\|_{p,s,b}, \\
\|\psi_T(t)v\|_{p,s,b} &\leq c\|v\|_{p,s,b},
\end{align*}
\]

and
\[
\|\psi_T(t)\int_0^t W(t-s)w(s)ds\|_{p,s,b} \leq cT\|w\|_{p,s,b'}.
\]

**Lemma 3.4.** ([16, 4]) Let $s$ and $\kappa$ be given. There is a constant $c$ depending on $s$ and $\kappa$ such that

If $\kappa > \frac{1}{4}$, then
\[
\|A^\frac{s}{2}F_\kappa\|_{L_x^2L_t^\infty} \leq C\|f\|_{L_x^\infty L_t^\infty},
\]

If $\kappa > \frac{1}{4}$, then
\[
\|AF_\kappa\|_{L_x^2L_t^\infty} \leq C\|f\|_{L_x^2L_t^\infty},
\]

If $\kappa > \frac{1}{2}$, and $s > 3\kappa$, then
\[
\|A^{-s}F_\kappa\|_{L_x^2L_t^\infty} \leq C\|f\|_{L_x^2L_t^\infty},
\]

If $\kappa > \frac{1}{2}$, and $s > \frac{1}{2}$, then
\[
\|A^{-s}F_\kappa\|_{L_x^2L_t^\infty} \leq C\|f\|_{L_x^2L_t^\infty},
\]

If $\kappa > \frac{1}{2}$, and $s > \frac{1}{2}$, then
\[
\|A^{-s}F_\kappa\|_{L_x^2L_t^\infty} \leq C\|f\|_{L_x^2L_t^\infty}.
\]

**Lemma 3.5.** Let $b > \frac{1}{2}$, $b' < -\frac{1}{4}$, and $s \geq 3b$. Let $p \in \mathbb{N}$ and suppose $u_1, \ldots, u_{p+1}, v_1, \ldots, v_{p+1} \in X_{p,s,b}$. Then there exists a constants $c$ such that

\[
\|\partial_x^j \prod_{i=1}^p u_i \prod_{j=1}^{p+1} v_j\|_{p,s,b'} \leq C \prod_{i=1}^p \|u_i\|_{p,s,b} \prod_{j=1}^{p+1} \|v_j\|_{p,s,b'},
\]

and

\[
\|\partial_x^j \prod_{i=1}^{p+1} u_i \prod_{j=1}^p v_j\|_{p,s,b'} \leq C \prod_{i=1}^{p+1} \|u_i\|_{p,s,b} \prod_{j=1}^p \|v_j\|_{p,s,b'}.
\]
Proof. First of all, for $i = 1, 2, \ldots, p + 1$ and $j = 1, 2, \ldots, p + 1$, we define

\[ f_i(\xi, \eta) = (1 + |\xi|)^s (1 + |\eta - \xi^3|)^b e^{p(1+|\xi|)} |\tilde{g}_j(\xi, \eta)| \]
\[ g_j(\xi, \eta) = (1 + |\xi|)^s (1 + |\eta - \xi^3|)^b e^{p(1+|\xi|)} |\tilde{v}_j(\xi, \eta)|. \]

The proof is first given for the case $p = 1$, after which the proof for a general $2p + 1$ will be more transparent, that means we prove

\[ \|\partial_\xi u_1 v_1\|_{p,s,b'} \leq C \|u_1\|_{p,s,b} \|v_1\|_{p,s,b} \]
\[ \|\partial_\xi u_1 u_2 v_1\|_{p,s,b'} \leq C \|u_1\|_{p,s,b} \|u_2\|_{p,s,b} \|v_1\|_{p,s,b}. \]

We have

\[
\|\partial_\xi u_1 v_1 v_2\|_{p,s,b'} = \left\| (1 + |\xi|)^s (1 + |\eta - \xi^3|)^b e^{p(1+|\xi|)} |\tilde{u}_1 v_1 v_2(\xi, \eta)| \right\|_{L^2_x L^p_\eta}
\]
\[
= \left\| (1 + |\xi|)^s e^{p(1+|\xi|)} (1 + |\eta - \xi^3|)^b |\tilde{u}_1 v_1 v_2(\xi, \eta)| \right\|_{L^2_x L^p_\eta}
\]
\[
= \left\| (1 + |\xi|)^s e^{p(1+|\xi|)} (1 + |\eta - \xi^3|)^b |\tilde{u}_1 * \tilde{v}_1 * \tilde{v}_2(\xi, \eta)| \right\|_{L^2_x L^p_\eta}
\]
\[
= \left\| (1 + |\xi|)^s e^{p(1+|\xi|)} (1 + |\eta - \xi^3|)^b |\tilde{u}_1(\xi, \eta)\tilde{v}_1(\xi - \xi_2, \eta - \eta_2) \tilde{v}_2(\xi_2 - \xi_1, \eta_2 - \eta_1) | \right\|_{L^2_x L^p_\eta}
\]
\[
= \left\| (1 + |\xi|)^s e^{p(1+|\xi|)} (1 + |\eta - \xi^3|)^b |\int_{\mathbb{R}^4} \tilde{u}_1(\xi, \eta) \tilde{v}_1(\xi - \xi_2, \eta - \eta_2) \tilde{v}_2(\xi_2 - \xi_1, \eta_2 - \eta_1) d\xi_1 d\eta_1 d\xi_2 d\eta_2 \right\|_{L^2_x L^p_\eta}
\]
\[
= \left\| (1 + |\xi|)^s e^{p(1+|\xi|)} (1 + |\eta - \xi^3|)^b \right\|_{L^2_x L^p_\eta} \int_{\mathbb{R}^4} \left( \frac{(1 + |\xi_1|)^{-s} e^{-p(1+|\xi_1|)} \tilde{f}_1(\xi_1, \eta_1)}{(1 + |\eta - \xi^3|)^b} \right)
\]
\[
\times \left( \frac{(1 + |\xi - \xi_2|)^{-s} e^{-p(1+|\xi - \xi_2|)} \tilde{g}_1(\xi - \xi_2, \eta - \eta_2)}{(1 + |(\eta - \eta_2) - (\xi - \xi_2)^3|)^b} \right)
\]
\[
\times \left( \frac{(1 + |\xi_2 - \xi_1|)^{-s} e^{-p(1+|\xi_2 - \xi_1|)} \tilde{g}_2(\xi_2 - \xi_1, \eta_2 - \eta_1)}{(1 + |\eta_2 - \eta_1 - (\xi_2 - \xi_1)^3|)^b} \right) d\mu_{L^2_x L^p_\eta},
\]
where $d\mu = d\xi_1 d\eta_1 d\xi_2 d\eta_2 d\xi d\eta$.

By using the duality, we proof this estimate, where $m(\xi, \eta)$ is a positive function in $L^2(\mathbb{R}^2)$ with norm...
\[ \|m\|_{L^2(\mathbb{R}^2)} = 1, \text{ then} \]
\[
\|\partial_{1} u_{1} v_{2}\|_{\rho, s, b'} \leq \int_{\mathbb{R}^b} \frac{e^{\rho(1 + |\xi|)}(1 + |\xi|)^s f_1(\xi, \eta_1)}{(1 + |\eta - \xi|^{-3})^{-b'}} \frac{e^{-\rho(1 + |\eta_1|)}(1 + |\eta_1|)^{-s} f_1(\xi_1, \eta_1)}{(1 + |\eta_1 - \xi_1|)^b} \frac{e^{-\rho(1 + |\xi - \xi_2|)}(1 + |\xi - \xi_2|)^{-s} g_1(\xi - \xi_2, \eta - \eta_2)}{(1 + |\eta - \eta_2|)^b} \frac{e^{-\rho(1 + |\xi_2 - \xi_1|)}(1 + |\xi_2 - \xi_1|)^{-s} g_2(\xi_2 - \xi_1, \eta_2 - \eta_1)}{(1 + |\eta_2 - \eta_1|)^b} \] 

Using the inequality
\[
|\xi| \leq |\xi_1| + |\xi - \xi_2| + |\xi_2 - \xi_1| \text{ then } e^{\rho(1 + |\xi|)} \leq e^{\rho(1 + |\xi_1|)} \times e^{\rho(1 + |\xi - \xi_2|)} \times e^{\rho(1 + |\xi_2 - \xi_1|)}. \]

Then
\[
\|\partial_{1} u_{1} v_{2}\|_{\rho, s, b'} \leq \int_{\mathbb{R}^b} \frac{(1 + |\xi|)^s m(\xi, \eta)(1 + |\eta_1|)^{-s} f_1(\xi_1, \eta_1)(1 + |\xi_2 - \xi_1|)^{-s} g_1(\xi - \xi_2, \eta - \eta_2)}{(1 + |\eta - \xi_1|)^{-b'}} \frac{(1 + |\xi_2 - \xi_1|)^{-s} g_2(\xi_2 - \xi_1, \eta_2 - \eta_1)}{(1 + |\eta_2 - \eta_1|)^b} \] 

Now, split the Fourier space into six regions as follow

1. \(|\xi - \xi_2| \leq |\xi_2 - \xi_1| \leq |\xi_1| |
2. \(|\xi - \xi_2| \leq |\xi_1| \leq |\xi_2 - \xi_1| |
3. \(|\xi_1| \leq |\xi_2 - \xi_1| \leq |\xi - \xi_2| |
4. \(|\xi_1| \leq |\xi - \xi_2| \leq |\xi_2 - \xi_1| |
5. \(|\xi_2 - \xi_1| \leq |\xi - \xi_2| \leq |\xi_1| |
6. \(|\xi_2 - \xi_1| \leq |\xi_1| \leq |\xi - \xi_2| |

We begin by the case (1)
\[
|\xi - \xi_2| \leq |\xi_2 - \xi_1| \leq |\xi_1|. \]

Then
\[
(1 + |\xi - \xi_2|)^{-s} \geq (1 + |\xi_2 - \xi_1|)^{-s} \geq (1 + |\xi_1|)^{-s}, \quad (3.12) \]

and, we assume that \(|\xi_1| \leq 1 \text{ or } |\xi_1| \geq 1 \).

Firstly, by \(|\xi_1| \geq 1 \), then
\[
(1 + |\xi_1|)^s \leq (|\xi_1| + |\xi|)^s = 2^s(|\xi|)^s = C(|\xi|)^s. \]
By the last inequality and (3.12), we obtain

\[
\|\partial_u v_1 v_2\|_{\rho, s, b} \leq C \int_{\mathbb{R}^6} \frac{(|\xi|^s |m(\xi, \eta)| + |\xi|^s f_1(\xi, \eta_1)}{(1 + |\eta - \xi^3|)^{s-\beta} (1 + |\eta_1 - \xi^3|)^{b-\beta}} d\mu
\]

\[
\leq C \int_{\mathbb{R}^6} \frac{(|\xi|^s |m(\xi, \eta)| + |\xi|^s f_1(\xi, \eta_1)}{(1 + |\eta - \xi^3|)^{s-\beta} (1 + |\eta_1 - \xi^3|)^{b-\beta}} d\mu,
\]

then

\[
\|\partial_u v_1 v_2\|_{\rho, s, b} \leq C \int_{\mathbb{R}^6} \frac{(|\xi|^s |m(\xi, \eta)| + |\xi|^s f_1(\xi, \eta_1)}{(1 + |\eta - \xi^3|)^{s-\beta} (1 + |\eta_1 - \xi^3|)^{b-\beta}} d\mu.
\]

By

\[
|\xi - \xi_2| \leq |\xi_2 - \xi_1| \leq |\xi_1|,
\]

and

\[
|\xi|^s (1 + |\xi_1|)^{-s} = |\xi|^s + |\xi_1|^s (1 + |\xi_1|)^{-s} \leq |\xi|^s + |\xi_1|^s (1 + |\xi_1|)^{-s} \leq |\xi|^s |\xi_1|^s (1 + |\xi_1|)^{-s} \leq |\xi|^s |\xi_1|^s (1 + |\xi_1|)^{-s},
\]

and

\[
|\xi|^s |\xi_1|^{-s} = |\xi|^s |\xi_1|^s |\xi|^s |\xi_1|^{-s} - \frac{1}{2}
\]

\[
\leq C |\xi|^s |\xi_1|^s (|\xi - \xi_2| + |\xi_2 - \xi_1| + |\xi_1|) |\xi_1|^{-s} - \frac{1}{2}
\]

\[
\leq C |\xi|^s |\xi_1|^s (|\xi_1|^s + |\xi|^s + |\xi_1|) |\xi_1|^{-s} - \frac{1}{2}
\]

\[
\leq C |\xi|^s |\xi_1|^s (|\xi_1|^s + |\xi|^s) |\xi_1|^{-s} - \frac{1}{2}
\]

\[
\leq C |\xi|^s |\xi_1|^s |\xi_1|^{-s}.
\]

We suppose that

\[
\tilde{A}_M = \frac{(|\xi|^s |m(\xi, \eta)|}{(1 + |\eta - \xi^3|)^{s-\beta}}
\]

\[
\tilde{A}_F = \frac{(|\xi|^s f_1(\xi, \eta_1)}{(1 + |\eta_1 - \xi^3|)^{b-\beta}}
\]

\[
\tilde{A}^{-s} G_s(\xi - \xi_2, \eta - \eta_2) = \frac{(1 + |\xi - \xi_2|)^{-s} g_1(\xi - \xi_2, \eta - \eta_2)}{(1 + |\eta - \eta_2 - (\xi - \xi_2)|)^{b-\beta}}
\]

\[
\tilde{A}^{-s} G_s(\xi_2 - \xi_1, \eta_2 - \eta_1) = \frac{(1 + |\xi_2 - \xi_1|)^{-s} g_2(\xi_2 - \xi_1, \eta_2 - \eta_1)}{(1 + |\eta_2 - \eta_1 - (\xi_2 - \xi_1)|)^{b-\beta}}
\]
and

\[
\int_{R^2} \frac{(|\zeta|)^{\frac{1}{2}} m(\zeta, \eta)}{(1 + |\eta - \zeta|^3)^{\beta}} \frac{1 + |\zeta_1|^2 f_1(\zeta_1, \eta_1)}{(1 + |\eta_1 - \zeta|^3)^{\beta}} \frac{(1 + |\zeta - \zeta_2|^3) g_1(\zeta - \zeta_2, \eta - \eta_2)}{(1 + |\eta_2 - (\zeta - \zeta_2)^3|^\beta} \frac{(1 + |\zeta_2 - \zeta_1|^3) g_2(\zeta_2 - \zeta_1, \eta_2 - \eta_1)}{(1 + |\eta_2 - \eta_1 - (\zeta_2 - \zeta_1)^3|^\beta} d\mu
\]

\[
\int_{R^2} (b M^G_b(\xi, \eta) F_b(\zeta, \eta) A^{-G_b^1}(\xi - \zeta, \eta - \eta_2) A^{-G_b^2}(\zeta_2 - \zeta_1, \eta_2 - \eta_1)) d\mu
\]

\[
= \int_{R^2} (b M^G_b(\xi, \eta) (\int_{R^2} (b M^G_b(\zeta, \eta) A^{-G_b^1}(\xi - \zeta, \eta - \eta_2) A^{-G_b^2}(\zeta_2 - \zeta_1, \eta_2 - \eta_1)) d\zeta d\eta)
\]

\[
= \int_{R^2} (b M^G_b(\xi, \eta) (\int_{R^2} (b F_b(\zeta, \eta) A^{-G_b^1} A^{-G_b^2}(\xi - \zeta, \eta - \eta_2) A^{-G_b^2}(\zeta_2 - \zeta_1, \eta_2 - \eta_1)) d\zeta d\eta)
\]

\[
= \int_{R^2} (b M^G_b(\xi, \eta) (\int_{R^2} (b F_b(\zeta, \eta) A^{-G_b^1} A^{-G_b^2}(\xi - \zeta, \eta - \eta_2) A^{-G_b^2}(\zeta_2 - \zeta_1, \eta_2 - \eta_1)) d\zeta d\eta)
\]

\[
= \int_{R^2} (b M^G_b(\xi, \eta) (\int_{R^2} (b F_b(\zeta, \eta) A^{-G_b^1} A^{-G_b^2}(\xi - \zeta, \eta - \eta_2) A^{-G_b^2}(\zeta_2 - \zeta_1, \eta_2 - \eta_1)) d\zeta d\eta)
\]

\[
= \int_{R^2} A^G_b M^G_b(x, t) (A^G_b F_b A^{-G_b^1} A^{-G_b^2}(\xi, \eta) d\zeta d\eta.
\]

We suppose that

\[
h_1(x, t) = A^G_b M^G_b(x, t)
\]

\[
h_2(x, t) = A^G_b F_b(x, t)
\]

\[
h_3(x, t) = A^{-G_b^1}(x, t)
\]

\[
h_4(x, t) = A^{-G_b^2}(x, t),
\]

then

\[
\left| \int_{R^2} A^G_b D^G_b(x, t) A^G_b F_b A^{-G_b^1} A^{-G_b^2}(x, t) d\zeta d\eta \right| = \int_{R^2} h_1(x, t) h_2(x, t) h_3(x, t) h_4(x, t) d\zeta d\eta
\]

\[
\leq \int_{R^2} h_1(x, t) h_2(x, t) \sup_{t \in [0, T]} h_3(x, t) \sup_{t \in [0, T]} h_4(x, t) d\zeta d\eta
\]

By using Cauchy-Schwarz's inequality for the variables \(x\) and \(t\)

\[
\left| \int_{R^2} (h_1(x, t) h_2(x, t)) \left( \sup_{t \in [0, T]} h_3(x, t) \sup_{t \in [0, T]} h_4(x, t) \right) d\zeta d\eta \right|
\]

\[
\leq \|h_1(x, t)\|_{L^2_{t,x}} \|h_2(x, t)\|_{L^2_{t,x}} \|h_3(x, t)\|_{L^2_{t,x}} \|h_4(x, t)\|_{L^2_{t,x}}
\]

\[
= \|A^G_b M^G_b\|_{L^2_{t,x}} \|A^G_b F_b\|_{L^2_{t,x}} \|A^{-G_b^1}\|_{L^2_{t,x}} \|A^{-G_b^2}\|_{L^2_{t,x}}
\]

Then

\[
\|\partial_{x_1} u_1 v_2\|_{\rho, s, b} \leq c\|A^G_b M^G_b\|_{L^2_{t,x}} \|A^G_b F_b\|_{L^2_{t,x}} \|A^{-G_b^1}\|_{L^2_{t,x}} \|A^{-G_b^2}\|_{L^2_{t,x}}
\]

Hence by Lemma 3.4

\[
\|\partial_{x_1} u_1 v_2\|_{\rho, s, b} \leq c\|m\|_{L^2_{i,x}} \|f\|_{L^2_{i,x}} \|g_1\|_{L^2_{i,x}} \|g_2\|_{L^2_{i,x}}
\]

\[
\leq c\|u_1\|_{\rho, s, b} \|v_1\|_{\rho, s, b} \|v_2\|_{\rho, s, b}.
\]
Secondly for the case $|\zeta| \leq 1$, then

$$(1 + |\zeta|)^{s} |\zeta|(1 + |\zeta|)^{-s} = (1 + |\zeta|)^{\frac{1}{2}} (1 + |\zeta_1|)^{\frac{1}{2}} (1 + |\zeta|)^{-\frac{s}{2}} (1 + |\zeta_1|)^{\frac{s}{2}} |\zeta|$$

$$\leq (1 + |\zeta|)^{\frac{1}{2}} (1 + |\zeta_1|)^{\frac{1}{2}} 1 + |\zeta_1|)^{-\frac{1}{2}} (1 + |\zeta|)^{\frac{s}{2}} (1 + |\zeta_1|)^{-\frac{s}{2}}$$

$$\leq (1 + |\zeta|)^{\frac{1}{2}} (1 + |\zeta_1|)^{\frac{1}{2}} (1 + |\zeta_1|)^{-\frac{1}{2}} (1 + |\zeta|)^{s+\frac{1}{2}}$$

$$\leq (1 + |\zeta|)^{\frac{1}{2}} (1 + |\zeta_1|)^{\frac{1}{2}} (1 + |\zeta_1|)^{-\frac{1}{2}} (1 + |\zeta - \zeta_2| + |\zeta_2 - \zeta_1|)^{s+\frac{1}{2}}$$

$$\leq (1 + |\zeta|)^{\frac{1}{2}} (1 + |\zeta_1|)^{\frac{1}{2}} (1 + |\zeta|)^{s+\frac{1}{2}} (3(1 + |\zeta_1|))^{s+\frac{1}{2}}$$

$$\leq C (1 + |\zeta|)^{\frac{1}{2}} (1 + |\zeta_1|)^{\frac{1}{2}},$$

then

$$\|\partial_x v_{1} v_{2}\|_{\rho,s,b'} \leq \int_{\mathbb{R}^n} \frac{(1 + |\zeta|)^{s} m(\zeta, \eta) (1 + |\zeta_1|)^{-s} f_1(\zeta_1, \eta_1) (1 + |\zeta - \zeta_1|)^{-s} g_1(\zeta - \zeta_2, \eta - \eta_2)}{(1 + |\zeta_1 - \zeta_1|)^{b}} \frac{(1 + |\zeta - \zeta_1|)^{-s} g_1(\zeta - \zeta_2, \eta - \eta_2)}{(1 + |\zeta - \zeta_2|)^{b}} \frac{(1 + |\zeta_1 - \zeta_1|)^{b}}{(1 + |\zeta_1 - \zeta_1|)^{b}} \frac{(1 + |\zeta_2 - \zeta_1|)^{-s} g_2(\zeta_2 - \zeta_1, \eta_2 - \eta_1)}{(1 + |\eta_2 - \eta_1|)^{b}} \frac{(1 + |\eta_2 - \eta_1|)^{b}}{(1 + |\eta_2 - \eta_1|)^{b}} \frac{(1 + |\eta_2 - \eta_1|)^{b}}{(1 + |\eta_2 - \eta_1|)^{b}} d\mu$$

Then, by the inner product, we have

$$\|\partial_x u_{1} v_{1} v_{2}\|_{\rho,s,b'} \leq c \|A^{\frac{1}{2}} M_{-b'} A^{\frac{1}{2}} F_{b} \| A^{-\frac{1}{2}} G_{b} \| A^{-\frac{1}{2}} G_{b} \|$$

$$\leq c \|A^{\frac{1}{2}} M_{-b'} A^{\frac{1}{2}} F_{b} \| A^{-\frac{1}{2}} G_{b} \| A^{-\frac{1}{2}} G_{b} \|$$

$$\leq c \|A^{\frac{1}{2}} M_{-b'} A^{\frac{1}{2}} F_{b} \| A^{-\frac{1}{2}} G_{b} \| A^{-\frac{1}{2}} G_{b} \|$$

$$\|\partial_x u_{1} v_{1} v_{2}\|_{L^2 L^2} \|A^{\frac{1}{2}} F_{b} \| A^{-\frac{1}{2}} G_{b} \| A^{-\frac{1}{2}} G_{b} \| L^2 L^2 \| A^{-\frac{1}{2}} G_{b} \| L^2 L^2 \| A^{-\frac{1}{2}} G_{b} \| L^2 L^2.$$

Hence by Lemma 3.4

$$\|\partial_x u_{1} v_{1} v_{2}\|_{L^2 L^2} \leq c \|m\|_{L^2 L^2} \|f\|_{L^2 L^2} \|g_1\|_{L^2 L^2} \|g_2\|_{L^2 L^2} \|g_2\|_{L^2 L^2} \|g_2\|_{L^2 L^2} \|g_2\|_{L^2 L^2} \|g_2\|_{L^2 L^2} \|g_2\|_{L^2 L^2} \|g_2\|_{L^2 L^2} \|g_2\|_{L^2 L^2} \|g_2\|_{L^2 L^2} \|g_2\|_{L^2 L^2} \|g_2\|_{L^2 L^2} \leq c \|u_{1}\|_{\rho,s,b'} \|v_{1}\|_{\rho,s,b'} \|v_{2}\|_{\rho,s,b'} \|v_{2}\|_{\rho,s,b'}.$$

By the same way, we prove the inequality in the five region.

For the case $p \geq 2$ is virtually identical. The only difference is that we need to split the Fourier space in $(2p+1)!$.

We prove that

$$\|\partial_x \prod_{i=1}^{p} u_i \prod_{j=1}^{p+1} v_j\|_{\rho,s,b'} \leq C \prod_{i=1}^{p} \|u_i\|_{\rho,s,b'} \prod_{j=1}^{p+1} \|v_j\|_{\rho,s,b'},$$
We have:

\[
\| \partial_t \prod_{j=1}^{p+1} u_j \|_{\rho,s,b'} = \| (1 + |\xi|)^s (1 + |\eta - \xi^3|)^b \rho^{(1 + |\xi|)} \partial_t \prod_{j=1}^{p+1} v_j(\xi,\eta) \|_{L^2_t L^2_x},
\]

\[
= \| (1 + |\xi|)^s (1 + |\eta - \xi^3|)^b \rho^{(1 + |\xi|)} \|_{L^2_t L^2_x}.
\]

By the same way, by the inner product, we have

\[
\| \partial_t \prod_{j=1}^{p} u_j \prod_{j=1}^{p+1} v_j \|_{\rho,s,b'} \leq c \langle A^{2M-p';}b A^{2f_b} A^{S-s} G_1, \partial_j^p \prod_{j=1}^{p} A^{-s} F_b + \prod_{j=1}^{p} A^{-s} G_b \rangle
\]

\[
\leq c \langle A^{2M-p';}b A^{2f_b} A^{S-s} G_1, \partial_j^p \prod_{j=1}^{p} A^{-s} F_b + \prod_{j=1}^{p} A^{-s} G_b \rangle
\]

\[
\leq c \| A^{2f} b \|_{L^2_t L^2_x} \| A^{2f} b \|_{L^2_t L^2_x} \| A^{-s} G_1, \partial_j^p \prod_{j=1}^{p} A^{-s} F_b + \prod_{j=1}^{p} A^{-s} G_b \|_{L^2_t L^2_x}
\]

\[
\leq c \sum_{j=1}^{p} \| u_j \|_{\rho,s,b'} \prod_{j=1}^{p+1} \| v_j \|_{\rho,s,b'}.
\]

\[\square\]

**Lemma 3.6.** Let \( p > 0, s \geq 3b, b > \frac{1}{2}, \) and \( b' < -\frac{1}{4}. \) Let \( p \in \mathbb{N} \) and suppose that \( u_1, \ldots, u_{p+1}, v_1, \ldots, v_{p+1} \in X_{p,s,b}. \) Then there exists a constants \( c \) such that

\[
\| \partial_t \prod_{j=1}^{p} u_j \prod_{j=1}^{p+1} v_j \|_{\rho,s,b'} \leq C \sum_{j=1}^{p} \| u_j \|_{\rho,s,b'} \prod_{j=1}^{p+1} \| v_j \|_{\rho,s,b'} + c \prod_{j=1}^{p} \| u_j \|_{\rho,s,b'} \prod_{j=1}^{p+1} \| v_j \|_{\rho,s,b'}.
\]

\[
\| \partial_t \prod_{j=1}^{p+1} u_j \prod_{j=1}^{p+1} v_j \|_{\rho,s,b'} \leq C \sum_{j=1}^{p+1} \| u_j \|_{\rho,s,b'} \prod_{j=1}^{p+1} \| v_j \|_{\rho,s,b'} + c \prod_{j=1}^{p+1} \| u_j \|_{\rho,s,b'} \prod_{j=1}^{p+1} \| v_j \|_{\rho,s,b'}.
\]

**Proof.** We begin by the case \( p = 1, \) that's mean we prove that

\[
\| \partial_t (u_1 v_1 v_2) \|_{\rho,s,b'} \leq C \| u_1 \|_{\rho,s,b'} \| v_1 \|_{\rho,s,b'} \| v_2 \|_{\rho,s,b'} + c \| u_1 \|_{\rho,s,b'} \| v_1 \|_{\rho,s,b'} \| v_2 \|_{\rho,s,b'}. \quad (3.13)
\]

We define

\[
f_i(\xi,\eta) = (1 + |\xi|)^s (1 + |\eta - \xi^3|)^b e^{p(1 + |\xi|)} | \hat{u}_i(\xi,\eta) |,
\]

\[
g_j(\xi,\eta) = (1 + |\xi|)^s (1 + |\eta - \xi^3|)^b e^{p(1 + |\xi|)} | \hat{v}_j(\xi,\eta) |.
\]
Then
\[
\|\partial_\xi u_1 v_1 v_2\|_{L^2} = \|(1 + |\xi|) e^{(1 + |\xi|)} \| \partial_\xi u_1 v_1 v_2(\xi, \eta)\|_{L^2}^{\eta}
\]
\[
= \|(1 + |\xi|) e^{(1 + |\xi|)} (1 + |\eta - \xi^3|) \| \partial_\xi u_1 v_1 v_2(\xi, \eta)\|_{L^2}^{\eta}
\]
\[
= \|(1 + |\xi|) e^{(1 + |\xi|)} (1 + |\eta - \xi^3|) \| \partial_\xi u_1 v_1 v_2(\xi, \eta)\|_{L^2}^{\eta}
\]
\[
\times \gamma_2(\xi - \xi_1, \eta_2 - \eta_1) d\xi_1 d\eta_1 d\xi_2 d\eta_2 \|_{L^2}^{\eta}
\]
\[
= \|(1 + |\xi|) e^{(1 + |\xi|)} (1 + |\eta - \xi^3|) \| \partial_\xi u_1 v_1 v_2(\xi, \eta)\|_{L^2}^{\eta}
\]
\[
\times \frac{(1 + |\xi - \xi_1|) e^{(1 + |\xi - \xi_1|)} \| \gamma_1(\xi - \xi_2, \eta_2 - \eta_2)\|_{L^2}^{\eta}}{(1 + |\eta - \eta_2| - (\xi - \xi_2)^3)}
\]
\[
\times \frac{(1 + |\xi - \xi_1|) e^{(1 + |\xi - \xi_1|)} \| \gamma_2(\xi_2 - \xi_2, \eta_2 - \eta_1)\|_{L^2}^{\eta}}{(1 + |\eta - \eta_2| - (\xi - \xi_2)^3)}
\]
\[
\times \| \mu \|_{L^2}^{\eta}.
\]
We proof this estimate by the duality. Let \(m(\xi, \eta)\) be a positive function in \(L^2(\mathbb{R}^3)\) with norm \(\|m\|_{L^2(\mathbb{R}^3)} = 1\), then
\[
\int_{\mathbb{R}^6} \frac{e^{(1 + |\xi|)} (1 + |\xi|) e^{(1 + |\xi|)} m(\xi, \eta) e^{(1 + |\xi|)} (1 + |\xi|)}{(1 + |\eta - \xi^3|)}
\]
\[
\int_{\mathbb{R}^6} \frac{e^{(1 + |\xi - \xi_1|)} (1 + |\xi - \xi_1|) e^{(1 + |\xi - \xi_1|)} m(\xi, \eta) e^{(1 + |\xi - \xi_1|)} (1 + |\xi - \xi_1|)}{(1 + |\eta - \xi_2|)}
\]
\[
\int_{\mathbb{R}^6} \frac{e^{(1 + |\xi - \xi_2|)} (1 + |\xi - \xi_2|) e^{(1 + |\xi - \xi_2|)} m(\xi, \eta) e^{(1 + |\xi - \xi_2|)} (1 + |\xi - \xi_2|)}{(1 + |\eta - \xi_2|)}
\]
\[
\leq I + I',
\]
where

\[
I + I' = \sup_{m \in B} \int_{\mathbb{R}^6} \frac{(1 + |\xi|^{1+s}m(\xi, \eta) - e^{-p(1+|\xi|)}(1 + |\xi_1|)^{-s}f_1(\xi_1, \eta_1)}{(1 + |\eta - \xi^3|)^{1-b}} d\mu \\
\times \frac{e^{-p(1+|\xi - \xi_1|)}(1 + |\xi - \xi_2|)^{-s}g_1(\xi - \xi_2, \eta - \eta_2)}{(1 + |\eta - \eta_2 - (\xi - \xi_2)^3|)^{1-b}} \frac{e^{-p(1+|\xi - \xi_1|)}(1 + |\xi - \xi_1|)^{-s}g_2(\xi - \xi_1, \eta - \eta_1)}{(1 + |\eta_2 - \eta_1 - (\xi - \xi_1)^3|)^{1-b}} d\mu.
\]

Now, split the Fourier space into six regions (the same division as before 3). We begin by the case \((|\xi - \xi_1| \leq |\xi_2 - \xi_1| \leq |\xi_1|)\). The integral of \(I\) corresponding to the particular region just delineated can be dominated by the supremum over all \(m\) in \(B\) of the duality relation the integral can be dominated by the inner product.

\[
I \leq c(A^A_{\Lambda}M_{\Lambda}A^F_{\Lambda} \ast e^{-pA}A^{-s}G^1_{\Lambda} \ast e^{-pA}A^{-s}G^2_{\Lambda})
\]

\[
\leq c(A^A_{\Lambda}M_{\Lambda}A^F_{\Lambda} \ast e^{-pA}A^{-s}G^1_{\Lambda} \ast e^{-pA}A^{-s}G^2_{\Lambda})
\]

\[
\leq c(A^A_{\Lambda}M_{\Lambda}A^F_{\Lambda} \ast e^{-pA}A^{-s}G^1_{\Lambda} \ast e^{-pA}A^{-s}G^2_{\Lambda})
\]

\[
\leq c(A^A_{\Lambda}M_{\Lambda}A^F_{\Lambda} \ast e^{-pA}A^{-s}G^1_{\Lambda} \ast e^{-pA}A^{-s}G^2_{\Lambda})
\]

Hence by Lemma 3.4

\[
I \leq c(|m|_{L^2_{\xi_{x,q}}} \|e^{-pA}f_1\|_{L^2_{\zeta_1L^q}} \|e^{-pA}g_1\|_{L^2_{\zeta_1L^q}} \|e^{-pA}g_2\|_{L^2_{\zeta_1L^q}} \leq c \|u_1\|_{s,b} \|v_1\|_{s,b} \|v_2\|_{s,b}.
\]

By the same way, we treat the second part, that is, the integration \(I'\) and we use the following inequality

\[
e^{p(1+|\xi|)} \leq e^{p(1+|\xi|)} \times e^{p(1+|\xi - \xi_1|)} \times e^{p(1+|\xi - \xi_1|)},
\]

we find

\[
I' \leq cP^{i} \sup_{m \in B} \|A^A_{\Lambda}M_{\Lambda}A^F_{\Lambda} \|_{L^2_{\zeta_1L^q}} \|A^A_{\Lambda}M_{\Lambda}A^F_{\Lambda} \|_{L^2_{\zeta_1L^q}} \|A^{-s}G^1_{\Lambda} \|_{L^2_{\zeta_1L^q}} \|A^{-s}G^2_{\Lambda} \|_{L^2_{\zeta_1L^q}}
\]

\[
\leq c \|m\|_{L^2_{\zeta_1L^q}} \|f_1\|_{L^2_{\zeta_1L^q}} \|g_1\|_{L^2_{\zeta_1L^q}} \|g_2\|_{L^2_{\zeta_1L^q}} \|u_1\|_{\rho,s,b} \|v_1\|_{\rho,s,b} \|v_2\|_{\rho,s,b}.
\]

The other five cases, follow by symmetry.

For the case \(p > 2\), the same scheme of estimation will yield for \((p - 2)\) with additional factors of the form

\[
\|A^{-s}(G^1_{\Lambda})A^{-s}(F^2_{\Lambda})\|_{L^2_{\zeta_1L^q}}.
\]

We deal with the rest of the parts in the same way.
4. Proof of Theorem 1.1

Existence of solution. We define
\[ \mathcal{B}_{p,s,b} = X_{p,s,b} \times X_{p,s,b}, \quad \mathcal{N}^{p,s} = \mathcal{G}_{p,s} \times \mathcal{G}_{p,s}, \]
\[ \| (u,v) \|_{\mathcal{B}_{p,s,b}} = \max \{ \| u \|_{p,s,b}; \| v \|_{p,s,b} \} \quad \text{and} \quad \| (u_0,v_0) \|_{\mathcal{N}^{p,s}} = \max \{ \| u_0 \|_{\mathcal{G}_{p,s}}; \| v_0 \|_{\mathcal{G}_{p,s}} \}. \]

Lemma 4.1. Let \( s \geq 0, p \geq 0, b \geq \frac{1}{2} \) and \( T \in (0;1) \). Then, for all \( (u_0,v_0) \in \mathcal{N}^{p,s} \), the map \( \Xi \times \Gamma : B(0,R) \rightarrow B(0,R) \) is a contraction, where \( B(0,R) \) is given by
\[ B(0,R) = \{ (u,v) \in \mathcal{B}_{p,s,b}; \quad \| (u,v) \|_{\mathcal{B}_{p,s,b}} \leq R \} \quad \text{where} \quad R = 2C \| (u_0,v_0) \|_{\mathcal{N}^{p,s}}. \]

Proof. First it is proved that \( \Xi \times \Gamma \) is mapping on \( B(0,R) \)
\[ \| \Xi[u,v](t) \|_{p,s,b} = \| \Psi(t)W(t)u_0 - \Psi(t) \int_0^T W(t-t')w_1(t')dt' \|_{p,s,b} \]
\[ \leq \| \Psi(t)W(t)u_0 \|_{p,s,b} + \| \Psi(t) \int_0^T W(t-t')w_1(t')dt' \|_{p,s,b} \]
\[ \leq C \| u_0 \|_{\mathcal{G}_{p,s}} + CT^{1-b+b'} \| w_1(t') \|_{p,s,b} \]
\[ = C \| u_0 \|_{\mathcal{G}_{p,s}} + CT^{1-b+b'} \| \partial_u (u^p v^{p+1}) \|_{p,s,b}. \]

We use Lemma 3.5 to have
\[ \| \partial_u u^p v^{p+1} \|_{p,s,b} \leq C \| u \|_{\mathcal{G}_{p,s}}^p \| v \|_{\mathcal{G}_{p,s}}^{p+1}. \]
Then
\[ \| \Xi[u,v](t) \|_{p,s,b} \leq C \| u_0 \|_{\mathcal{G}_{p,s}} + CT^{1-b+b'} \| u \|_{\mathcal{G}_{p,s}}^p \| v \|_{\mathcal{G}_{p,s}}^{p+1} \]
\[ \leq C \max \left( \| u_0 \|_{\mathcal{G}_{p,s}}, \| v_0 \|_{\mathcal{G}_{p,s}} \right) + CT^{1-b+b'} \max \left( \| u \|_{\mathcal{G}_{p,s}}, \| v \|_{\mathcal{G}_{p,s}} \right)^p \]
\[ \times \max \left( \| u \|_{\mathcal{G}_{p,s}}, \| v \|_{\mathcal{G}_{p,s}} \right)^{p+1} \]
\[ \leq C \max \left( \| u_0 \|_{\mathcal{G}_{p,s}}, \| v_0 \|_{\mathcal{G}_{p,s}} \right) + CT^{1-b+b'} \max \left( \| u \|_{\mathcal{G}_{p,s}}, \| v \|_{\mathcal{G}_{p,s}} \right)^{2p+1}. \]
The estimates for the second term \( \Gamma \) are similar.
\[ \| \Gamma[u,v](t) \|_{p,s,b} \leq C \| (u_0,v_0) \|_{\mathcal{N}^{p,s}} + CT^{1-b+b'} \left( \| (u,v) \|_{\mathcal{B}_{p,s,b}} \right)^{2p+1}. \]
Then we have
\[ \| \Xi[u,v](t), \Gamma[u,v](t) \|_{\mathcal{B}_{p,s,b}} \leq C \| (u_0,v_0) \|_{\mathcal{N}^{p,s}} + CT^{1-b+b'} \left( \| (u,v) \|_{\mathcal{B}_{p,s,b}} \right)^{2p+1}. \]
Then
\[ \| \Xi[u,v](t), \Gamma[u,v](t) \|_{\mathcal{B}_{p,s,b}} \leq C \| (u_0,v_0) \|_{\mathcal{N}^{p,s}} + CT^{1-b+b'} \left( \| (u,v) \|_{\mathcal{B}_{p,s,b}} \right)^{2p+1} \]
\[ \leq \frac{R}{2} + T^e CR^{2p+1}. \]
We choose sufficiently small \( T \) such that
\[ T^e \leq \frac{1}{4CR^2}. \]
Hence
\[ \| \Xi[u,v](t), \Gamma[u,v](t) \|_{\mathcal{B}_{p,s,b}} \leq R, \quad \forall (u,v) \in B(0,R). \]
Secondly we prove that the map $\mathcal{E} \times \Gamma : \mathbb{B}(0,R) \to \mathbb{B}(0,R)$ is a contraction. For this end, let $(u,v) \in \mathbb{B}(0,R)$ and $(u^*,v^*) \in \mathbb{B}(0,R)$ such that

$$
\| \mathcal{E}[u,v](t) - \mathcal{E}[u^*,v^*](t) \|_{\rho,s,b} = C \| \psi_T(t) \int_0^t W(t-t') \partial_x \left( u^p u^{p+1} + u^p v^{p+1} \right) dt' \|_{\rho,s,b}
$$

$$
= C \| \psi_T(t) \int_0^t W(t-t') \partial_x \left[ (u^p + u^p v^{p+1} + u^p v^{p+1}) \right] dt' \|_{\rho,s,b}.
$$

We use the Lemma 3.5 to have

$$
\| \partial_x (u^p - u^p v^{p+1}) \|_{\rho,s,b} \leq C \| u^p - u^p v^{p+1} \|_{\rho,s,b},
$$

$$
\| \partial_x u^p (v^{p+1} - v^{p+1}) \|_{\rho,s,b} \leq C \| u^p \|_{\rho,s,b} \| v^{p+1} - v^{p+1} \|_{\rho,s,b}.
$$

According to Lemma 3.5, we have

$$
\| (u^p - u^p) \|_{\rho,s,b} \leq C \| u - u^p \|_{\rho,s,b},
$$

$$
\| (v^{p+1} - v^{p+1}) \|_{\rho,s,b} \leq C \| v - v^p \|_{\rho,s,b}.
$$

Then

$$
\| \partial_x (u^p - u^p v^{p+1}) \|_{\rho,s,b} \leq C \| u^p - u^p \|_{\rho,s,b} \| v \|_{\rho,s,b}^{p+1},
$$

$$
\| \partial_x u^p (v^{p+1} - v^{p+1}) \|_{\rho,s,b} \leq C R^{2p} \| u - u^p \|_{\rho,s,b},
$$

and

$$
\| \mathcal{E}[u,v](t) - \mathcal{E}[u^*,v^*](t) \|_{\rho,s,b} \leq 2CT^{1-b+b'R^2p} \| u - u^*, v - v^* \|_{B_{\rho,s,b}},
$$

$$
\| \Gamma[u,v](t) - \Gamma[u^*,v^*](t) \|_{\rho,s,b} \leq 2CT^{1-b+b'R^2p} \| u - u^*, v - v^* \|_{B_{\rho,s,b}}.
$$

By the same way we prove that $\Gamma[u,v](t)$ is contraction, so we have

$$
\| \mathcal{E}[u,v](t) - \mathcal{E}[u^*,v^*](t) \|_{\rho,s,b} \leq 2CT^{1-b+b'R^2p} \| u - u^*, v - v^* \|_{B_{\rho,s,b}}.
$$

Since $T^\epsilon \leq \frac{1}{4C^{2p}}$, we have

$$
\| \mathcal{E}[u,v] - \mathcal{E}[u^*,v^*](t), \Gamma[u,v] - \Gamma[u^*,v^*](t) \|_{B_{\rho,s,b}} \leq \frac{1}{2} \| (u - u^*), v - v^* \|_{B_{\rho,s,b}}.
$$

Since the map $\mathcal{E} \times \Gamma : \mathbb{B}(0,R) \to \mathbb{B}(0,R)$ is a contraction, it follows that has a unique fixed point $(u,v)$ in $B(0,R)$.

The rest of the proof follows a standard argument.
5. Large time estimates on the radius of analyticity.

Lemma 5.1. Let $s > \frac{3}{2}, \rho > 0, T \geq 1$ and $b \in [-1, 1]$. We suppose that $(u, v)$ is solution of (1.1) on the time interval $[0, 2T]$. Then there exists a constants $C$ such that

$$
\| \psi_T(t)u(\cdot, t), \psi_T(t)v(\cdot, t)\|_{\mathcal{B}_{b,s}} \leq CT^\frac{s}{2} (1 + \lambda_T(u, v))^{2p + 1},
$$

(5.1)

and

$$
\| \psi_T(t)u(\cdot, t), \psi_T(t)v(\cdot, t)\|_{\mathcal{B}_{b,s}} \leq CT^\frac{s}{2} (1 + \kappa_T(u, v))^{2p + 1},
$$

(5.2)

with

$$
\lambda_T(u, v) = \sup_{t \in [0, 2T]} \left( \|u, v\|_{L^{p+1}} \right) \quad \text{and} \quad \kappa_T(u, v) = \sup_{t \in [0, 2T]} \left( \|u, v\|_{L^{p+1}} \right),
$$

where $\mathcal{N}^s = H^s \times H^s$ and $\mathcal{B}_{b,s} = X_{b,s} \times X_{b,s}$.

Proof. We have

$$
\| \psi_T(t)u(x, t)\|_{s,b}^2 = \int_{-\infty}^{+\infty} \left| \Lambda^b \left( e^{-it\xi^3} \psi_T(t) \hat{u}^X(\xi, t) \right) \right|^2 \, dt \, d\xi.
$$

By using the inquality

$$
|\Lambda^b v(x, t)| \leq c|v(x, t)| + |\partial_v v(x, t)|,
$$

we get

$$
\| \psi_T(t)u(\cdot, t)\|_{s,b}^2 \leq c \int_{-\infty}^{+\infty} (1 + |\xi|)^2 \int_{-\infty}^{+\infty} \left| e^{-it\xi^3} \psi_T(t) \hat{u}^X(\xi, t) \right|^2 \, dt \, d\xi
$$

$$
+ c \int_{-\infty}^{+\infty} (1 + |\xi|)^2 \int_{-\infty}^{+\infty} \left| \partial_t \left( e^{-it\xi^3} \psi_T(t) \hat{u}^X(\xi, t) \right) \right|^2 \, dt \, d\xi.
$$

We have

$$
\partial_t \left( e^{-it\xi^3} \psi_T(t) \hat{u}^X(\xi, t) \right) = \frac{1}{T} \psi_T(t) e^{-it\xi^3} \hat{u}^X(\xi, t) + \psi_T(t) (-i\xi^3 e^{-it\xi^3}) \hat{u}^X(\xi, t)
$$

$$
+ \psi_T(t) e^{-it\xi^3} \partial_t \hat{u}^X(\xi, t),
$$

and

$$
u_t = -\partial^3 \xi u - \partial_t (u^\rho v^{p+1}).
$$

Then

$$
\bar{u}_t^X(\xi, t) = -\partial^3 \xi u(\xi, t) - \partial_t (u^\rho v^{p+1})^X(\xi, t)
$$

$$
= i\xi^3 \bar{u}^X(\xi, t) - i\xi (u^\rho v^{p+1})^X(\xi, t).
$$

So

$$
\partial_t \left( e^{-it\xi^3} \psi_T(t) \hat{u}^X(\xi, t) \right) = \frac{1}{T} \psi_T(t) e^{-it\xi^3} \hat{u}^X(\xi, t) + \psi_T(t) e^{-it\xi^3} i\xi (u^\rho v^{p+1})^X(\xi, t),
$$

and
\[
\|\Psi_T(t)u(.,t)\|_{s,b}^2 = \int_{-\infty}^{+\infty} (1 + |\xi|)^2 s \int_{-\infty}^{+\infty} \left| \Lambda^b \left( e^{-it\xi^3} \Psi_T(t)\hat{u}(\xi,t) \right) \right|^2 dt d\xi
\]
\[
\leq c \int_{-\infty}^{+\infty} (1 + |\xi|)^2 s \int_{-\infty}^{+\infty} \left| \left( e^{-it\xi^3} \Psi_T(t)\hat{u}(\xi,t) \right) + c \int_{-\infty}^{+\infty} (1 + |\xi|)^2s \int_{0}^{2T} \left| (\hat{\xi}(\xi,t)) \right|^2 dt d\xi + c \int_{-\infty}^{+\infty} (1 + |\xi|)^2s \int_{0}^{2T} \left| (\hat{\xi}(\xi,t)) \right|^2 dt d\xi,
\]
and
\[
\|\Psi_T(t)u(.,t)\|_{s,b}^2 \leq 4cT \sup_{t \in [0,2T]} \|u(.,t)\|_{H^s}^2 + 2cT \sup_{t \in [0,2T]} \|u^p\|_{H^{s+1}}^2
\]
\[
\leq 4cT \sup_{t \in [0,2T]} \|u(.,t)\|_{H^s}^2 + 2cT \sup_{t \in [0,2T]} \|u^p\|_{H^{s+1}}^2
\]
\[
\leq 4cT \sup_{t \in [0,2T]} (\|(u,v)\|_{H^{s+1}})^2 + 2cT \sup_{t \in [0,2T]} (\|(u,v)\|_{H^{s+1}})^2
\]
and
\[
\|\Psi_T(t)u(.,t)\|_{s,b} \leq cT^{\frac{1}{2}} (1 + \lambda_T(u,v))^{2p+1},
\]
where
\[
\lambda_T(u,v) = \sup_{t \in [0,2T]} (\|(u,v)\|_{H^{s+1}}).
\]
Similarity,
\[
\|\Psi_T(t)v(.,t)\|_{s,b} \leq cT^{\frac{1}{2}} (1 + \lambda_T(u,v))^{2p+1},
\]
and
\[
\|(\Psi_T(t)u(.,t), \Psi_T(t)v(.,t))\|_{s,b} \leq 2cT^{\frac{1}{2}} (1 + \lambda_T(u,v))^{2p+1}
\]
\[
\leq CT^{\frac{1}{2}} (1 + \lambda_T(u,v))^{2p+1}.
\]
This completes the proof.

To prove the Theorem 1.2, we need to define a sequence of approximations to (1.1) as follows
\[
\begin{align*}
  u^p &+ \partial_x^2 u^p = -\partial_t \left( (\rho_n * \Psi_T u^p)(\rho_n * \Psi_T v^p)^{p+1} \right), \\
  v^p &+ \partial_x^3 v^p = -\partial_t \left( (\rho_n * \Psi_T u^p)(\rho_n * \Psi_T v^p)^{p+1} \right), \\
  u^p(x,0) = u_0(x), &\quad v^p(x,0) = v_0(x),
\end{align*}
\]
where \(T > 0, n \in \mathbb{N}\) and \(\rho_n\) is defined as
\[
\tilde{\rho}_n(\xi) = \begin{cases} 
  0, & |\xi| \geq 2n \\
  1, & |\xi| \leq n,
\end{cases}
\]
where \(\tilde{\rho}_n\) is smooth and monotone on \((n,2n)\).
Lemma 5.2. Let $s \geq 0$ and $(u_0, v_0) \in \mathcal{M}^s$ and we assume that $(u, v)$ is solution of (1.1) with $(u_0, v_0)$. Then for $n \in \mathbb{N}$, we have

- $(u^n, v^n)$ is in $C([0, 2T], H^s) \times C([0, 2T], H^s)$. The sequence $\{(u^n, v^n)\}$ converge to $(u, v)$ in $C([0, T], H^s) \times C([0, T], H^s)$.
- The estimate in Lemma 5.1 holds for $(u^n, v^n)$ uniformly in $n$.
- If $(u_0, v_0) \in \mathcal{M}^{P, s}$ for $\rho > 0$, then the result is also given for $C([0, T], \mathcal{G}_{\rho, s}) \times C([0, T], \mathcal{G}_{\rho, s})$.

Lemma 5.3. Let $(u, v)$ be solution of (1.1) with the initial data $(u_0, v_0) \in \mathcal{M}^{P, 0}$ for $\rho_0 > 0$ and $s > \frac{3}{2}$ and $\eta > 0$, then

$$\sup_{n \in [0, 2\eta]} \|(u(\cdot, t), v(\cdot, t))\|_{\mathcal{M}^{P, 0}, +1} \leq \|(u_0, v_0)\|_{\mathcal{M}^{P, 0}, +1} + C \eta \sup_{n \in [0, 2\eta]} \|(u(\cdot, t), v(\cdot, t))\|_{\mathcal{M}^{P, 0}, +1},$$

with $\rho(t) = \rho_0 e^{-\gamma(t)}$ and $\gamma(t)$ is defined as

$$\gamma(t) = \int_0^t \left(k_1 + k_2 \int_0^t \|(u(\cdot, t), v(\cdot, t))\|_{\mathcal{M}^{P, 0}, +1}^{2p + 2} dt\right) dt,$$

where

$$k_1 = \|(u_0, v_0)\|_{\mathcal{M}^{P, 0}, +1}^2,$$

and $k_2$ is a constant.

Proposition 5.4. Let $\rho_0 > 0$, $p \geq 1, T \geq 1$ and $s > 3b$, we assume that $(u, v)$ is solution of (1.1) in $C([0, 2T], H^{s+1}) \times C([0, 2T], H^{s+1})$ with $(u_0, v_0) \in \mathcal{G}_{\rho_0, +1} \times \mathcal{G}_{\rho_0, +1}$, then there exist $\rho_1 < \rho_0$ such that

$$\{\Psi_T u^n, \Psi_T v^n\} \text{ bounded in } \mathcal{G}_{\rho(\cdot), s, b},$$

with

$$\rho(t) \leq \min\{\rho_1, KT^{-2p^2 - 6p - 1}\}.$$

Proof. We have

$$\Psi_T(t) u^n = \Psi_T(t) W(t) u_0 - \int_0^t \partial_x \left( (\rho_n * \Psi_T u^n)^{p+1}(\rho_n * \Psi_T v^n)^p \right) dx,$$ (5.4)

where $t \in (0, \infty)$. This will show that $\Psi_T u^n \in X_{\rho, s, b}$ for all $n \in \mathbb{N}$.

We have

$$\|(\Psi_T(t) u^n)\|_{\rho, s, b} \leq \|(\Psi_T(t) W(t) u_0)\|_{\rho, s, b} + \|(\Psi_T(t) \int_0^t \partial_x \left( (\rho_n * \Psi_T u^n)^{p+1}(\rho_n * \Psi_T v^n)^p \right) dx, (5.4)\|_{\rho, s, b}$$

$$\leq cT^{\frac{s}{2}} \|u_0\|_{\rho, s, b} + cT \|\partial_x \left( (\rho_n * \Psi_T u^n)^{p+1}(\rho_n * \Psi_T v^n)^p \right) \|_{\rho, s, b}$$

$$\leq cT^{\frac{s}{2}} \|u_0\|_{\rho, s, b} + cT \left( \|\Psi_T u^n\|_{\rho, s, b}^{p+1} \|\Psi_T v^n\|_{\rho, s, b}^p + \rho^{\frac{s}{2}} \|\Psi_T u^n\|_{\rho, s, b}^{p+1} \|\Psi_T v^n\|_{\rho, s, b}^p \right).$$

For $0 < \rho < \rho_0$ and $b' = b - 1 + \varepsilon'$ where $\varepsilon' > 0$ and we use the Lemma 5.1 to obtain

$$\|(\Psi_T(t) u^n)\|_{\rho, s, b} \leq cT^{\frac{s}{2}} (1 + \alpha_T(u^n, v^n))^{2p+1} \leq 2cT^{\frac{s}{2}} (1 + \alpha_T(u, v))^{2p+1},$$

and

$$\|(\Psi_T(t) v^n)\|_{\rho, s, b} \leq cT^{\frac{s}{2}} (1 + \alpha_T(u^n, v^n))^{2p+1} \leq 2cT^{\frac{s}{2}} (1 + \alpha_T(u, v))^{2p+1}.$$
Thus
\[
\| \varphi_T(t) u^n \|_{\rho(t), s, b} \leq c T^{\frac{1}{2}} \| u_0 \|_{\rho(t), s} + c T^{\frac{2p+1}{2}} (1 + \alpha_T(u, v))^{(2p+1)2} + c T^{\frac{1}{2}} \| \varphi_T u^n \|_{\rho(t), s, b}^{p+1} \| \varphi_T v^n \|_{\rho(t), s, b}^{p},
\]
holds for \( T \geq 1 \).

In the case \( T = 1 \), and by using Lemma 5.1 and Lemma 5.3, we have
\[
\| (\varphi_T(t) u(., t), \varphi_T(t) v(., t)) \|_{\rho(t), s, b} \leq c T^{\frac{1}{2}} (1 + \kappa_T(u, v))^{2p+1},
\]
where
\[
\kappa_T(u, v) = \sup_{r \in [0, 2]} (\| (u, v) \|_{\rho(t), s+1})^{2p+1},
\]
\[
\| \varphi_1(t) u^n \|_{\rho(t), s, b} \leq c \left( 1 + \sup_{r \in [0, 2]} (\| (u^n, v^n) \|_{\rho(t), s+1}) \right)^{2p+1}
\]
\[
\leq 2c \left( 1 + \sup_{r \in [0, 2]} (\| (u, v) \|_{\rho(t), s+1}) \right)^{2p+1}
\]
\[
\leq 2c c_1 \left( 1 + (\| (u_0, v_0) \|_{\rho(t), s+1}^{2p+1} + \sup_{r \in [0, 2]} (\| (u, v) \|_{\rho(t), s+1})^{(2p+2)(2p+1)/2} \right).
\]
We assume that
\[
M^* = 2c c_1 \left( 1 + (\| (u_0, v_0) \|_{\rho(t), s+1}^{2p+1} + \sup_{r \in [0, 2]} (\| (u, v) \|_{\rho(t), s+1})^{(2p+2)(2p+1)/2} \right).
\]
Then
\[
\| \varphi_T(t) u^n \|_{\rho(t), s, b} \leq M^* + c T^{\frac{1}{2}} (\| (u_0, v_0) \|_{\rho(t), s} + c T^{\frac{2p+1}{2}} (1 + \alpha_T(u, v))^{(2p+1)2}
\]
\[
+ c T^{\frac{1}{2}} \| (\varphi_T u^n, \varphi_T v^n) \|_{\rho(t), s, b}^{2p+1}
\]
\[
\| (\varphi_T(t) u^n, \varphi_T(t) v^n) \|_{\rho(t), s, b} \leq M^* + c T^{\frac{1}{2}} (\| (u_0, v_0) \|_{\rho(t), s} + c T^{\frac{2p+1}{2}} (1 + \alpha_T(u, v))^{(2p+1)2}
\]
\[
+ c T^{\frac{1}{2}} \| (\varphi_T u^n, \varphi_T v^n) \|_{\rho(t), s, b}^{2p+1}
\]
For \( T \geq 1 \), \( \rho(t) \leq \rho_1 \leq \rho_0 \), and for large enough \( n \), we define the new variables
\[
y = y(T) = \| \varphi_T(t) u^n, \varphi_T(t) v^n \|_{\rho(t), s, b}
\]
\[
x = x(T) = M^* + c T^{\frac{1}{2}} (\| (u_0, v_0) \|_{\rho(t), s} + c T^{\frac{2p+1}{2}} (1 + \alpha_T(u, v))^{(2p+1)2}
\]
\[
d = d(T) = c T^{1/2}.
\]
Then
\[
y \leq x + d \rho(T)^{\frac{1}{2}} y^{2p+1}.
\]
If define
\[
\rho(T) = \frac{a^2}{d^2 x^{p/2} y^{2p+1}}.
\]
Then
\[ y \leq x + d(T)^{\frac{1}{2}} y^{2p+1} \leq x + d\left( \frac{a^2}{d^2 x^2 p^2} \right)^{\frac{1}{2}} y^{2p+1} \leq x + \left( \frac{a}{(2x)^2} \right) y^{2p+1} \]
\[ \implies y \leq x + a \left( \frac{y}{2x} \right)^{2p} y \implies \frac{y}{2x} \leq \frac{1}{2} + a \left( \frac{y}{2x} \right)^{2p+1}. \]

We define \( h(t) = \frac{y(t)}{2(x(t))}. \) Then
\[ h(1 - ah^{2p}) \leq \frac{1}{2}. \]

We can choose small \( a \) for all \( p \), then there is \( M' \) and \( m' \) such that
\[ \frac{1}{2} < m' < 1 < M', \]
and
\[ h \leq m' \quad \text{or} \quad h \geq M'. \]

As \( \| \psi_T(t) u^p, \psi_T(t) v^p \|_{\rho(t), \lambda, h} \) is a continuous function of \( T \geq 1 \), then
\[ h(t) \geq m' < 1 \implies y(t) \leq 2x(t), \]
which means that
\[ \| \psi_T(t) u^p, \psi_T(t) v^p \|_{\rho(t), \lambda, h} \leq 2x. \]

Then
\[ \{ \Psi_T u^p \} \text{ and } \{ \Psi_T v^p \} \text{ bounded in } X_{\rho(t), \lambda, b}. \]

On the other hand, we have
\[
\begin{cases}
\rho(t) < \rho_1 \\
\rho(t) = \frac{a^2}{d^2 x^2 p^2}.
\end{cases}
\tag{5.5}
\]

Since
\[ x^{2p} = (x(T))^{2p} = \left( M^* + cT^{\frac{p+1}{2}} \| (u_0, v_0) \|_{A, \rho_0, s} + cT^\frac{p+3}{2} (1 + \alpha_T(u, v)) \right)^{2p} \]
\[ \geq \left( cT^{\frac{p+1}{2}} \| (u_0, v_0) \|_{A, \rho_0, s} + cT^\frac{p+3}{2} (1 + \alpha_T(u, v)) \right)^{2p} \]
\[ \geq T^{\frac{4p}{2}} \left( c \| (u_0, v_0) \|_{A, \rho_0, s} + cT^\frac{p+2}{2} (1 + \alpha_T(u, v)) \right)^{4p} \]
\[ \geq T^{2p} \left( cT^\frac{p+2}{2} (1 + \alpha_T(u, v)) \right)^{4p} \]
\[ = T^{2p^2 + 6p} (1 + \alpha_T(u, v))^{4p}. \]

Then
\[ x^{-4p} \leq (T)^{-2p^2 - 6p} (1 + \alpha_T(u, v))^{-4p}, \tag{5.6} \]

and
\[ \rho(t) = \frac{a^2}{d^2 x^{2p} 2^{4p}} = \frac{a^2}{c^2 T^{3p} 2^{4p}} = \frac{a^2 T^{-1}}{c^2 x^2 p^2 2^{4p}} \leq \frac{a^2 T^{-1} T^{-2p^2 - 6p}}{c^2 (1 + \alpha_T(u, v))^{4p} 2^{4p}} \]
\[ \rho(t) \leq \frac{a^2}{c^2 (1 + \alpha_T(u, v))^{4p} 2^{4p}} T^{-2p^2 - 6p - 1} = KT^{-2p^2 - 6p - 1}, \]
where
\[
K = \frac{a^2}{C^2((1 + \alpha_T(u, v))^4)^{2p} 2^{4p}}.
\]
and
\[
\rho(t) = \min \left\{ \rho_1, KT^{-2(p^2 - 6p - 1)} \right\}.
\]

We are now in potion to prove Theorem 1.2.

**Proof.** Of Theorem 1.2. We have \((u_0, v_0) \in \mathcal{M}^{\rho_0, s+1}\), then by Theorem 1.1, we obtain
\[
(u, v) \in C([0, T^*], \mathcal{G}_{\rho_0, s+1}) \times C([0, T^*], \mathcal{G}_{\rho_0, s+1}).
\]
We prove that
\[
(u, v) \in C \left( [0, T], \mathcal{G}_{\rho(t), s+1} \right) \times C \left( [0, T], \mathcal{G}_{\rho(t), s+1} \right).
\]
If \(T^* = \infty\), it is done.
If \(T^* < \infty\), it remains to prove that
\[
(u, v) \in C \left( [0, T], \mathcal{G}_{\rho(t), s+1} \right) \times C \left( [0, T], \mathcal{G}_{\rho(t), s+1} \right), \quad \forall \, T \geq T^*.
\]
From the Proposition 5.4, we obtain that the sequence \(\{(u^n, v^n)\}\) is solution of (5.3) where \((u_0, v_0)\) is bounded in \(\mathcal{G}_{\rho(t), s}\) uniformly on \([0, T]\).

By using Lemma 3.2, with \((u^n, v^n)\) satisfies (5.3) then, we obtain
\[
(\partial_t u^n, \partial_t v^n) \quad (\partial_x u^n, \partial_x v^n) \quad (\partial_x^3 u^n, \partial_x^3 v^n)\] are uniformly bounded on the strip \(G_{\rho(t), s}\).

Then
\[
(\partial_t u^n, \partial_t v^n) \quad (\partial_x u^n, \partial_x v^n) \quad (\partial_x^3 u^n, \partial_x^3 v^n)\] are equicontinuous families on strip \(G_{\rho(t), s}\).

Then, we can extract a subsequence (without changing symbol of \(\{(u^n, v^n)\}\) ) converging uniformly on compact subsets of \((0, T) \times G_{\rho(t), s}\) to smooth function \((\bar{u}, \bar{v})\) and
\[
(\partial_t u^n, \partial_t v^n) \quad (\partial_x u^n, \partial_x v^n) \quad (\partial_x^3 u^n, \partial_x^3 v^n)\] is converging uniformly on compact subsets of \((0, T) \times G_{\rho(t), s}\).

Next we passe to the limit in (5.3), we obtain that \((\bar{u}, \bar{v})\) is a smooth extension of \((u, v)\).

Since, \((u^n, v^n)\) is analytic \(G_{\rho(t), s}\) to the \((\bar{u}, \bar{v})\), so \((\bar{u}, \bar{v})\) is analytic in \(G_{\rho(t), s}\), on the other hand, since \(\{(u^n, v^n)\}\) is bounded in \(G_{\rho(t), s}\) uniformly on \([0, T]\), then
\[
\bar{u} \equiv u \in L^{\infty}(0, T, \mathcal{G}_{\rho(t)}) \quad , \quad \bar{v} \equiv v \in L^{\infty}(0, T, \mathcal{G}_{\rho(t)}).
\]
then
\[
u \in C((0, T), \mathcal{G}_{\rho(t)}),
\]
\[
u \in C((0, T), \mathcal{G}_{\rho(t)}).
\]
\]
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