Geometry of the Smallest 1-form Laplacian Eigenvalue on Hyperbolic Manifolds

Michael Lipnowski and Mark Stern

1 Introduction

Let $M^n$ be a closed Riemannian manifold. The geometry of the smallest positive eigenvalue $\lambda_1^0(M)$ of the 0-form Laplace operator is well studied. Work of Cheeger [17] and Buser [8] proves that $\lambda_1^0(M)$ is comparable to the square of the Cheeger isoperimetric constant of $M$.

Much less is known about the smallest positive eigenvalue $\lambda_1^q(M)$ of the $q$-form Laplace operator for $0 < q < n$. In this paper, motivated by questions arising in the study of torsion cohomology of closed arithmetic hyperbolic manifolds $M$, we prove geometric upper and lower bounds for $\lambda_1^1(M)$.

1.1 Main results

Let $M$ be a closed hyperbolic $n$-manifold. Its fundamental group $\pi_1(M)$ acts by isometries on $\mathbb{H}^n$. For $\gamma \in \pi_1(M)$, let $\ell(\gamma)$ denote the translation length of $\gamma$.

Fix a basepoint $q_0 \in M$. For $x, y \in \mathbb{H}^n$, let $\alpha_{x,y}$ denote the oriented geodesic segment from $x$ to $y$. If $\gamma$ bounds, define the area of $\gamma$ to be

$$\text{Area}(\gamma) := \inf_{\partial S = \alpha_{q_0, \gamma q_0}} \text{area}(S). \quad (1.1)$$

Define the stable area of $\gamma$, denoted $\text{sArea}(\gamma)$, to be

$$\text{sArea}(\gamma) := \inf \left\{ \frac{\text{area}(\gamma^m)}{m} : \gamma^m \text{ bounds} \right\}, \quad (1.2)$$

assuming that $\gamma^k$ bounds for some integer $k$. The quantity $\text{sArea}(\gamma)$ is independent of the basepoint $q_0$.

Under the latter assumption, recall that the stable commutator length of $\gamma$, denoted $\text{scl}(\gamma)$, is defined by

$$\text{scl}(\gamma) = \inf_m \inf_{\text{connected } S: \partial S = m\gamma} \frac{\max\{-\chi(S), 0\}}{m}. \quad (1.3)$$

On hyperbolic manifolds, stable area is always bounded above by $4\pi$ times stable commutator length [11].

Let $\lambda_1^q(M)_{d^*}$ (resp. $\lambda_1^q(M)_d$) denote the smallest positive eigenvalue of the $q$-form Laplacian acting on $d^*\Omega^{q+1}(M)$ (resp. $d\Omega^{q-1}(M)$). Then

$$\lambda_1^q(M) = \min\{\lambda_1^q(M)_d, \lambda_1^q(M)_{d^*}\},$$

by the Hodge decomposition.

---

These results are supported by the National Science Foundation (NSF) under grant DMS 1005761.

Department of Mathematics, Duke University and University of Toronto.

E-mail: stern@math.duke.edu, malipnow@math.utoronto.ca
Theorem 1.3 (Geometric Upper Bound for $\frac{1}{\lambda_1(M)_{d^*}}$). Let $M_0$ be a closed hyperbolic $n$-manifold. Let $M$ be an arbitrary finite cover of $M_0$ with first betti number $0$. Then

$$\frac{1}{\sqrt{\lambda_1^2(M)_{d^*}}} \leq C^2 \left( 2\pi V + D \sup_{\gamma \in \pi_1(M); \ell(\gamma) \leq D} \frac{\text{sArea}(\gamma)}{\ell(\gamma)} \right) + \frac{C}{2} \sqrt{\text{vol}(M)}.$$ 

The quantity $C$ is defined in Proposition 2.2, and it is uniformly bounded above when the injectivity radius of $M$ is bounded below and $\lambda_1^2(M)$ is bounded above.

The quantities $V$ and $D$ are defined in Theorem 1.1, they respectively satisfy

$$V \leq V_{M_0} \cdot \text{vol}(M) \quad \text{and} \quad D \leq D_{M_0} \cdot \text{diam}(M)$$

for constants $V_{M_0}, D_{M_0}$ depending only on $M_0$ (in an explicit manner to be described later).

Let $\gamma \rightarrow [\gamma]$ denote the quotient map from $\pi_1(M) \rightarrow H_1(M, \mathbb{Q})$. We can extend the bounds of Theorem 1.3 to the case $n = 3, b_1(M) = 1$:

Theorem 1.4 (“Regulator-independent” Geometric Upper Bound for $\frac{1}{\lambda_1^2(M)_{d^*}}$ when $n = 3, b_1(M) = 1$). Let $M_0$ be a closed hyperbolic 3-manifold. Let $M$ be an arbitrary finite cover of $M_0$ with $b_1(M) = 1$. Fix $\delta > 0$. Then

$$\left( 1 - E \right) \frac{1}{\sqrt{\lambda_1^2(M)_{d^*}}} \leq C^2 \left( 3\pi V + 2\sqrt{2}D^2 \cdot \text{vol}(M)^{3+1/2} \sup_{\gamma \in \pi_1(M); [\gamma] = 0} \frac{\text{sArea}(\gamma)}{\ell(\gamma)} + 5\pi \right) + \frac{C}{2} \sqrt{\text{vol}(M)}.$$ 

The quantities $V, C, D$ are as in Theorem 1.3. The quantity $E$ satisfies

$$0 \leq E \leq C \cdot \text{vol}(M)^{-\delta}.$$ 

Notably, the upper bound in Theorem 1.4 does not depend on $A'$, the maximum over a generating set $\{ [S] \}$ of $H_2(M, \mathbb{Z})$ of the least area representative of $[S]$.

When we don’t impose restrictions on $b_1(M)$, Theorem 1.1 proves an upper bound for $\frac{1}{\lambda_1^2(M)_{d^*}}$ in terms of $\sup_{\gamma \in \pi_1(M); [\gamma] = 0} a_{\text{sArea}(\gamma)}{\ell(\gamma)}$, which depends a priori on $A'$.

Let $K_0$ be a triangulation of $M_0$. If $M$ is a finite cover, let $K$ denote the pullback triangulation of $K_0$. The cochain complex $C^*(M; K)$ maps into $\Omega^*(M)$ by the Whitney map [29]. Endow $C^*(M; K)$ with the norm induced from the $L^2$-norm on $\Omega^*(M)$ via the Whitney map. Let $\lambda^{q}_{1, \text{Whitney}}(M)$ denote the smallest positive eigenvalue of the associated Whitney Laplacian on $C^q(M; K)$. Let $\lambda^{q}_{1, \text{Whitney}}(M)_{d^*}$ (resp. $\lambda^{q}_{1, \text{Whitney}}(M)_{d^*}$) denote the smallest positive eigenvalue of the Whitney Laplacian acting on $d_{\text{Whitney}}^*C^{q+1}(M; K)$ (resp. $d_{\text{Whitney}}C^q(M; K)$).

Then

$$\lambda^{q}_{1, \text{Whitney}}(M) = \min \{ \lambda^{q}_{1, \text{Whitney}}(M)_{d^*}, \lambda^{q}_{1, \text{Whitney}}(M)_{d^*} \}$$

by the Hodge decomposition.

Theorem 1.5 (Geometric Lower Bound for $\frac{1}{\lambda_{1, \text{Whitney}}(M)_{d^*}}$). Let $M_0$ be a closed hyperbolic $n$-manifold. Let $M$ be an arbitrary finite cover of $M_0$. If some multiple of $[\gamma] \in \pi_1(M)$ bounds, then

$$\left( \frac{\text{scl}(\gamma)}{\ell(\gamma)} \right)^2 \leq W_{M_0} \cdot \frac{\text{vol}(M) \cdot \text{diam}(M)^2}{\lambda^{q}_{1, \text{Whitney}}(M)_{d^*}},$$

for some constant $W_{M_0}$ depending only on $M_0$ (described in Theorem 6.7).
Theorem 1.10. Let \( N_0 \) be a closed hyperbolic \( n \)-manifold, \( n > 3 \). Let \( M_0 \subset N_0 \) be a totally geodesic submanifold. Suppose \( N \xrightarrow{\pi} N_0 \) is an arbitrary finite cover. Let \( M = (\text{a connected component of}) \; \pi^{-1}(M_0) \). Suppose that there is a covering \( p : N' \to N \) of degree \( d \) for which

- the submanifold \( M \) lifts to \( N' \)
- \( N' \) retracts onto \( M \).
Then

\[
\frac{1}{\lambda_1(M)d^2} \leq F_{M_0} \cdot \text{diam}(M)^2 \cdot (d^2 \cdot \text{vol}(N) \cdot \text{diam}(N))^2 \cdot \frac{1}{\lambda_1(N)d^2}
\]

for a constant \(F_{M_0}\) depending only on \(M_0\) (in a manner to be described explicitly later).

A rich family of (arithmetic) examples satisfying the hypotheses of Theorem 1.10 is provided by the work of Bergeron, Haglund, and Wise \[2\]. The 1-form spectra of hyperbolic 3-manifolds converging geometrically to \(H\) are typically much easier to bound away from 0 than the 1-form spectrum of hyperbolic 3-manifolds because \(\lambda_1^2(H^n) > 0\) if \(n > 3\) while \(\lambda_1^2(H^3) = 0\); see \[8, 2\] for further discussion. Theorem 1.10 may therefore be useful for proving good lower bounds for the 1-form spectra of closed hyperbolic 3-manifolds.

This work began as an attempt to prove that \(\frac{1}{\lambda_1^2(M)} \ll \text{vol}(M)^C\), for some constant \(C\). This bound is motivated by applications to estimating growth of \(H_1(M,\mathbb{Z})_{\text{tors}}\), as we’ll describe in \[1, 2\]. Focusing attention on \(\lambda_1(M)d^2\) may appear to miss “half the story”: bounding \(\lambda_1(M)d = \lambda_1^2(M)\) from below. But \(\lambda_1(M)d\) is much simpler to control; see \[1, 3\] for further discussion.

### 1.2 Motivation: Relationship to torsion cohomology growth

This paper began as an attempt to prove growth of torsion in \(H_1(M,\mathbb{Z})\) for towers of closed hyperbolic 3-manifolds \(M\).

For every closed Riemannian manifold \(M\), the Cheeger-Müller Theorem \[18, 27\] relates torsion cohomology to analytic invariants of Riemannian manifolds:

\[
\sum_{q=0}^{\dim M} (-1)^q \left( \log |H^q(M,\mathbb{Z})_{\text{tors}}| + \log(R^{\dim M-q}(M)) + \frac{q}{2} \sum_{\lambda \in \text{spectrum}(\Delta_q) \setminus \{0\}} \log \left( \frac{1}{\lambda} \right) \right) = 0.
\]

(1.11)

The regulator \(R^q(M)\) measures the volume of \(H^q(M,\mathbb{Z}) \setminus H^q(M,\mathbb{R})\) with \(L^2\)-metric induced from harmonic forms. In particular, \(\log R^q(M) = 0\) if \(H^q(M,\mathbb{R}) = 0\). The notation \(\sum_{\text{reg}}\) means zeta-regularized sum.

Under favorable circumstances, one hopes that for many sequences of hyperbolic 3-manifolds \(M\) “geometrically converging to \(\mathbb{H}^3\),”

\[
\frac{1}{\text{vol}(M)} \sum_{q=0}^{\dim M} (-1)^q \frac{q}{2} \sum_{\lambda \in \text{spectrum}(\Delta_q) \setminus \{0\}} \log \left( \frac{1}{\lambda} \right) =: \frac{1}{\text{vol}(M)} \cdot \log T_{\text{an}}(M)
\]

\[\rightarrow \log T_{\text{an}}^{(2)}(\mathbb{H}^3) = -\frac{1}{6\pi},\]

where \(T_{\text{an}}^{(2)}\) is the \(L^2\)-analytic torsion of \(\mathbb{H}^3\) \[25, \S 3\]. Nonetheless, convergence of analytic torsion to its expected \(L^2\)-limit has not been proven for even a single sequence of closed hyperbolic 3-manifolds converging geometrically to \(\mathbb{H}^3\).

Equation (1.11) shows that small \(q\)-form Laplacian eigenvalues and large \(R^{\dim M-q}(M)\) suppress torsion cohomology in degree \(q\).

Bergeron and Venkatesh \[3\] found many interesting examples of non-trivial unimodular metrized local systems \(L\) of free abelian groups for which \(R^{\dim M-q}(M;L) = 1\) and without

\[\footnote{Large \(R^{\dim M-q}(M)\) is corresponds to homology classes in \(H_q(M,\mathbb{Z})\) whose minimal complexity is very large. See \[3\] for further details.}\]
small eigenvalues. In the absence of these two torsion suppressors, Bergeron and Venkatesh prove a limit multiplicity formula showing that the analytic torsion \( \frac{\log T_{an}(M,L)}{\text{vol}(M)} \) approaches its expected \( L^2 \)-limit. Upon applying Müller’s generalization of the Cheeger-Müller theorem to metrized unimodular local systems \( L \), this proves growth of torsion in the cohomology \( H^*(M,L) \).

For the trivial local system \( \mathbb{Z} \) and many others, the analytic obstructions of small eigenvalues and complicated cycles alluded to above are genuine and present interesting geometric problems.

Bergeron, Venkatesh, and Sengün [4, Theorem 1.2] have codified the obstruction to proving the torsion cohomology growth theorems via the methods of [3]:

**Theorem 1.12** ([4, Theorem 1.2]). Let \( M_0 \) be a closed hyperbolic 3-manifold and \( M_n \to M_0 \) normal coverings for which \( \bigcap \pi_1(M_n) = \{1\} \). Suppose that

- \( M_n \) has “small betti numbers”, i.e.
  \[
  b_1(M_n) = o \left( \frac{\text{vol}(M_n)}{\log \text{vol}(M_n)} \right).
  \]

- \( M_n \) has “few small 1-form eigenvalues”, i.e.
  \[
  \lim_{n \to \infty} \frac{1}{\text{vol}(M_n)} \sum_{\lambda \in \text{spectrum}(\Delta_1) \cap (0, \text{vol}(M_n)^{-\delta})} \log \left( \frac{1}{\lambda} \right) = 0 \text{ for every } \delta > 0.
  \]

- \( M_n \) has “simple cycles”, i.e.
  \[
  H_2(M_n, \mathbb{R}) \text{ is spanned by cycles of area } \ll \text{vol}(M_n)^C \text{ for some constant } C.
  \]

Then it follows that

- \[
  \lim_{n \to \infty} \frac{\log R^q(M_n)}{\text{vol}(M_n)} = 0
  \]

- \[
  \lim_{n \to \infty} \frac{-\log |H_1(M_n, \mathbb{Z})_{\text{tors}}|}{\text{vol}(M_n)} = \lim_{n \to \infty} \frac{\log T_{an}(M_n)}{\text{vol}(M_n)} = -\frac{1}{6\pi}.
  \]

The main focus of [4] was on understanding the simple cycles condition (1.15). Throughout the present paper, we focus on the small eigenvalue condition (1.14).

Under the simplifying assumption \( b_1(M_n) = 0 \), we state different set of sufficient conditions that emphasizes the connection between small eigenvalues and geometry; this is a consequence of Theorem 1.3.

**Theorem 1.18.** Let \( M_0 \) be a closed hyperbolic 3-manifold and \( M_n \to M_0 \) normal coverings for which \( \bigcap \pi_1(M_n) = \{1\} \) and which satisfy \( b_1(M_n) = 0 \). Suppose that

- Bergeron and Venkatesh construct strongly acyclic metrized local systems, for which the \( q \)-form Laplace operators for a tower of hyperbolic 3-manifolds admit a uniform spectral gap for every \( q \). See [3, §4] for details and [3, §8] for constructions.
- [4, Theorem 1.2] makes the further assumption that \( M_n \) be arithmetic congruence. Under this assumption, the 0-form Laplacian admits a uniform spectral gap and the analogous condition for “few small 0-form eigenvalues” is automatically satisfied. However, Lemma [A] together with know bounds on almost-betti numbers in degree 0 show that “few small 0-form eigenvalues” is satisfied even without assuming every \( M_n \) is arithmetic congruence.
- We do not believe the simple cycle condition (1.15) and the small eigenvalue condition (1.14) should be regarded as independent. We will return to the connection between regulators and small eigenvalues in future work.
• $M_n$ has “small almost-betti numbers”, i.e.
\[
\sum_{\lambda \in \text{spectrum}(\Delta_1) \cap [0, \text{vol}(M_n)^{-\delta}]} 1 = o\left( \frac{\text{vol}(M_n)}{\log \text{vol}(M_n)} \right) \text{ for all } \delta > 0.
\] (1.19)

• $M_n$ has “simple almost-cycles”, i.e. for some constant $D_{M_0}$ depending only on $M_0$,
\[
\text{If } \ell(\gamma) \leq D_{M_0} \cdot \text{diam}(M_n), \text{ then } s\text{Area}(\gamma) \leq \text{vol}(M_n)^C \text{ for some constant } C.
\] (1.20)

Then it follows that
\[
\lim_{n \to \infty} \frac{-\log |H_1(M_n, \mathbb{Z})_{\text{tors}}|}{\text{vol}(M_n)} = \lim_{n \to \infty} \frac{\log T_{an}(M_n)}{\text{vol}(M_n)} = -\frac{1}{6\pi}.
\] (1.21)

The simple almost-cycle condition (1.20) in Theorem 1.18 replaces the small eigenvalue condition (1.14) from Theorem 1.12. Additionally, the simple almost-cycle condition (1.20) from Theorem 1.18 distinctly resembles the simple cycle condition from Theorem 1.12.

Remark 1.22. Let $M_0$ be a closed hyperbolic 3-manifold and $M \to M_0$ a finite cover. Make the same assumptions as in the statement of Theorem 1.18. The simple almost-cycle condition (1.20) implies that
\[
\frac{1}{\lambda_1(M)} = O_{M_0} \left( \frac{\text{vol}(M)^C}{\text{vol}(M)} \right).
\] (1.23)

(1.23) is not implied by the small eigenvalue condition (1.14). However, given the best progress to date on the small almost-betti number problem [32], it is difficult to imagine proving (1.14) without also proving a spectral gap of quality similar to (1.23). Unfortunately, multiplicity of the 1-form eigenvalue $\lambda_1(M) \cdot \log \left( \frac{1}{\lambda_1(M)} \right) = O_{M_0} \left( \text{vol}(M) \right)$
(1.24)
gives the best currently known lower bound for $\lambda_1(M)$. The gulf between (1.23) and (1.24) is enormous. We will revisit the issue of deriving improved lower bounds for $\lambda_1(M)$, or equivalently constructing simple almost cycles, in § 8.2 and in future work.

1.3 Outline

Let $M_0$ be a closed hyperbolic manifold and $M \to M_0$ an arbitrary finite cover satisfying $b_1(M) = 0$. Let $K_0$ be a triangulation of $M_0$ and $K$ the pullback triangulation of $M$.

• In § 2 we recall standard Sobolev estimates needed in § 3, 4 and 7

• § 3 is the heart of this paper, building toward the key Corollary 3.13. We construct almost-primitives for Laplacian eigen 1-forms on the image of $d^*$ on $M$ when $b_1(M) = 0$. If the eigenvalue is extremely small, this almost succeeds. On the other hand, the image of $d$ and the image of $d^*$ are orthogonal. This tension results in lower bounds for $\lambda_1^1(M)_{d^*}$.

In § 3.1 we control the geometry of two types of fundamental domains for $M$ insofar as necessary to estimate terms arising in Corollary 3.13.

• § 4 describes how to extend the results of § 3 when $b_1(M) > 0$. In particular, our lower bounds for $\lambda_1^1(M)_{d^*}$ are as good when $b_1(M) = 1$ as they are when $b_1(M) = 0$, independent of the 1-form regulator of $M$. 
• In §5, we compare combinatorial and Riemannian $L^p$-norms on the cochain complex $C^\bullet(M; K)$.

• In §6, we prove that $\frac{1}{\lambda_1(M)_d} = O_{M_0}(\text{vol}(M)^{-1})$ or there is a comparison, $\lambda_1(M)_d = O_{M_0}(\text{vol}(M) \cdot \lambda_1(M)_d)$.

• In §7, we prove that either $\frac{1}{\lambda_1(M)_d} = O_{M_0}(\text{vol}(M))$ or there is a comparison, $\lambda_1(M)_d = O_{M_0}(\text{vol}(M) \cdot \lambda_1(M)_d)$.

• In §8, we show how our main results imply an exponential upper bound $\frac{1}{\lambda_1(M)_d} \leq \exp(O_{M_0}(\text{vol}(M)))$.

As we explain, it is often possible to prove a polynomial upper bound for $\frac{1}{\lambda_1(N)_d}$ for hyperbolic $n$-manifolds $N$ when $n > 3$. This gives a new prospect for proving useful upper bounds for $\lambda_1(M)_d$, for hyperbolic 3-manifolds $M$, by geodesically embedding $M$ in a higher dimensional hyperbolic manifold $N$ and applying retraction theorems such as those from [2].

• In §A, we show that if $M$ is a closed hyperbolic $n$-manifold, then $\frac{1}{\lambda_1(M)_d} \leq C \cdot \text{vol}(M) \cdot \text{diam}(M)^2$ for some constant $C$ depending only on a lower bound for the injectivity radius of $M$.

1.4 Acknowledgements

We would like to thank Nicolas Bergeron, Danny Calegari, Nathan Dunfield, Aurel Page, Peter Sarnak, Akshay Venkatesh, Alden Walker, and Alex Wright for stimulating discussions related to the present work.

2 $L^\infty$ Estimates

The Sobolev inequality for $\mathbb{H}^n$ gives for all $\phi \in C_0^\infty(\mathbb{H}^n)$,

$$\int_{\mathbb{H}^n} |d\phi|^2 dv \geq \kappa_n \left( \int_{\mathbb{H}^n} |\phi|^{\frac{2n}{n-2}} dv \right)^{\frac{n-2}{n}},$$

where

$$\kappa_n := \frac{n(n-2)\text{vol}(S^n)^{2/n}}{4}.$$

See, for example, [22 Section 8.2].

Let $M$ be a compact hyperbolic $n$-manifold with injectivity radius $D$.

**Proposition 2.2.** Let $f$ be a $q$-form on $M$ with $\Delta f = \lambda f$. For all $L < \frac{D}{2}$, there exists $C(n, q, L, \lambda) > 0$ such that

$$\|f\|_{L^\infty(B(p, L))} \leq C(n, q, L, \lambda) \|f\|_{L^2(B(p, 2L))}.$$  \hspace{1cm} (2.3)

For every fixed $X, Y$, the constant $C(n, q, L, \lambda)$ is uniformly bounded above for $\lambda \leq X$ and $L \geq Y$. 

Proof. The existence of such an estimate is an immediate consequence of the Sobolev embedding theorem. For the convenience of the reader and to determine the dependence of $C(n,q,L,\lambda)$ on $L$ and $\lambda$, we recall a standard Moser iteration argument leading to (2.4). Let $\eta : \mathbb{R} \rightarrow [0,1]$ be a smooth function identically 1 on $(-\infty,L]$ and supported on $(-\infty,2L]$, with $|d\eta| \leq \frac{2}{L}$. Set $\eta_k(t) = \eta(2^k(t-L))$. Observe $\eta_k(t) = 1$ on $(-\infty,L(1+2^{-k}))$ and supported on $(-\infty,L(1+2^{1-k}))$. Let $\chi_k$ denote the characteristic function of $B(p,L(1+2^{-k}))$. Then

$$|d\eta_k| \leq \frac{2^{k+1}}{L} \eta_{k-1}.$$  

Consider the Bochner formula for $f$ :

$$\Delta f = \nabla^* \nabla f - q(n-q)f = \lambda f. \quad (2.4)$$

Taking the $L^2$-inner product of $\Delta f$ with $\psi^2 f$ for a smooth function $\psi$ in (2.4) and integrating by parts gives

$$\|\nabla(\psi f)\|^2 - \langle (q(n-q) + \lambda)\psi f, \psi f \rangle = \|d\psi |f|^2. \quad (2.5)$$

Recall Kato’s inequality:

$$\|\nabla F\|^2 \geq \|dF\|^2. \quad (2.6)$$

Inserting this into (2.5) yields

$$\|d\psi |f|^2 \leq (q(n-q) + \lambda)\|\psi f\|^2 + \|d\psi |f|^2. \quad (2.7)$$

Now we choose

$$\psi = \psi_k = \eta_k(r)|f|^\gamma_k^{-1},$$

$\gamma_k > 1$ to be determined. Substituting this choice of $\psi$ into (2.7) gives

$$\|d(\psi_k |f|)\|^2 \leq (q(n-q) + \lambda)\|\psi_k f\|^2 + \|d\psi_k |f|^2 \leq (q(n-q) + \lambda)\|\psi_k f\|^2 + \|d\eta_k \gamma_k^{-1}|f|^\gamma_k + \frac{1}{\gamma_k} |f|^\gamma_k d\eta_k\|^2.$$ 

Hence

$$\frac{2\gamma_k - 1}{\gamma_k^2} \|d(\psi_k |f|)\|^2 \leq (q(n-q) + \lambda)\|\psi_k f\|^2 + \frac{1}{\gamma_k} |f|^\gamma_k d\eta_k\|^2 + 2\frac{\gamma_k - 1}{\gamma_k} |f|^\gamma_k d\eta_k $$

$$\leq (q(n-q) + \lambda)\|\psi_k f\|^2 + \frac{1}{\gamma_k} \|f|^\gamma_k d\eta_k\|^2 + \frac{\gamma_k - 1}{\gamma_k^2} \|d(\eta_k |f|\gamma_k)\|^2.$$ 

Therefore

$$\|d(\psi_k |f|)\|^2 \leq (q(n-q) + \lambda)\gamma_k \|\psi_k f\|^2 + \left(\frac{2^{k+1}}{L} |f|^\gamma_k \chi_{k-1}\right)^2. \quad (2.8)$$

Applying (2.4) to the left side of (2.8) gives

$$\kappa_n \|\eta_k |f|^\gamma_k \frac{\partial}{\partial x^n} \frac{|dx|}{n} \leq (q(n-q) + \lambda)\gamma_k \|\psi_k f\|^2 + \left(\frac{2^{k+1}}{L} |f|^\gamma_k \eta_{k-1}\right)^2. \quad (2.9)$$

Set $\gamma := \frac{n}{\gamma_k}$ and $\gamma_k := \gamma_k - \gamma = \gamma^k \gamma_0$, then take $\gamma^k$ roots in (2.9) to get

$$\|\chi_k f\|^2_{L^{2^k+1}} \leq \kappa_n \gamma^k \left( (q(n-q) + \lambda)\gamma_k + \frac{2^{k+1}}{L^2}\right)^{\frac{1}{\gamma_k}} \|\chi_{k-1} f\|^2_{L^{2^{k+1}}}. \quad (2.10)$$
Taking the product of \( (2.10) \) from \( k = 0, \ldots, K \) and letting \( K \to \infty \) gives
\[
\| f \|_{L^\infty(B(p, L))}^2 \leq C(n, q, L, \lambda) \| f \|_{L^2(B(p, 2L))}^2,
\]
where
\[
C(n, q, L, \lambda) = \prod_{k=0}^{\infty} \kappa_n^{\frac{1}{\gamma_k}} \left[ (q(n - q) + \lambda) \gamma_k + \frac{4^{k+1}}{L^2} \right]^{\frac{1}{\gamma_k}}.
\]

More generally, the same proof shows that if \( \sigma \) is a section of a vector bundle \( E, W a \) section of \( \text{End}(E) \), and \( (\nabla^* \nabla + W) \sigma = 0 \), then
\[
\| \sigma \|_{L^\infty(B(p, L))}^2 \leq C(n, L, W) \| \sigma \|_{L^2(B(p, 2L))}^2,
\]
where
\[
C(n, L, W) = \prod_{k=0}^{\infty} \kappa_n^{\frac{1}{\gamma_k}} \left[ ||W||_{L^\infty} \gamma_k + \frac{4^{k+1}}{L^2} \right]^{\frac{1}{\gamma_k}}.
\]

When \( n \) is understood, set
\[
C(\lambda) := C(n, 1, \text{Inj}(M), \lambda).
\]

3 Constructing almost-primitives to bound \( \lambda_1^1(M) \) below

Let \( M \) be a closed hyperbolic \( n \)-manifold. Let \( \Gamma \) denote \( \pi_1(M) \) realized as a group of deck transformations of \( \mathbb{H}^n \). Let \( F \) denote a fundamental domain for \( M \) in \( \mathbb{H}^n \). By this, we mean
(a) \( F \) is a closed domain in \( \mathbb{H}^n \) with piecewise smooth boundary.
(b) \( \Gamma F = \mathbb{H}^n \)
(c) The open sets \( \gamma \cdot \text{int}(F), \gamma \in \Gamma \), are pairwise disjoint.
(d) We can partition \( \partial F \) into oriented smooth submanifolds with corners \( \{ \Sigma_1, \Sigma_1', \ldots, \Sigma_J, \Sigma_J' \} \)
   (i) whose union is \( \partial F \) and whose interiors are pairwise disjoint,
   (ii) the interior of each \( \Sigma_j \) and \( \Sigma_j' \) projects isometrically to \( M \) under the quotient map \( \mathbb{H}^n \to \Gamma \setminus \mathbb{H}^n = M \), and
   (iii) there exist \( \{ \gamma_1, \ldots, \gamma_J \} \subset \Gamma \) with \( \gamma_j \Sigma_j = -\Sigma_j' \) for \( j = 1, \ldots, J \).

We will refer to the \( \Sigma_i \), and \( \Sigma_j' \) as faces. We will refer to each \( \gamma_j \) as a face-pairing element. Set
\[
P_F := \{ \gamma_1, \ldots, \gamma_J \}.
\]

For \( x, y \in \mathbb{H}^n \), let \( \alpha_{x, y} \) denote the oriented geodesic segment from \( x \) to \( y \).

**Lemma 3.2.** Let \( b(q) = \int_{\alpha_{p, q}} f \). For every vector \( v \in T_q \mathbb{H}^n \),
\[
| db_q(v) - f_q(v) | \leq \frac{1}{2} || df ||_{L^\infty} \cdot || v^\perp ||,
\]
where \( v^\perp \) denotes the component of \( v \) perpendicular to the tangent vector to \( \alpha_{p, q} \) at \( q \). In particular,
\[
|| db - f ||_{L^\infty(F)} \leq \frac{1}{2} || df ||_{L^\infty(M)}.
\]
Proof. Let \( \ell \subset T_q\mathbb{H}^n \) be the line tangent \( \alpha_{p,q} \) at \( q \).

Suppose \( v \in \ell \). Then \( \text{db}_q(v) - f_q(v) = 0 \).

Suppose \( v \in \ell^\perp \). Let \( \Delta \) be the geodesic triangle with vertices \( p, q, \exp_q(\epsilon v) \) and oriented boundary \( \alpha_{p,q}, \alpha_{q,\exp_q(\epsilon v)}, \alpha_{\exp_q(\epsilon v),p} \). By Stokes,

\[
\frac{1}{\epsilon} (b(\exp_q(\epsilon v)) - b(q)) + \frac{1}{\epsilon} \int_{\alpha_{q,\exp_q(\epsilon v)}} f = \int_{\partial \Delta} f
\]

\[
= \frac{1}{\epsilon} \int_{\Delta} df. \tag{3.3}
\]

The area of a geodesic hyperbolic triangle with edges of lengths \( a, b \) meeting at a right angle equals \( \arctan \left( \tanh \frac{a}{2} \cdot \tanh \frac{b}{2} \right) \). Letting \( \epsilon \to 0 \) in (3.3) gives

\[
|\text{db}_q(v) - f_q(v)| \leq \|df\|_\infty \cdot \|v\| \cdot \tanh \left( \frac{d(p,q)}{2} \right) \leq \frac{1}{2} ||df||_\infty \cdot ||v||.
\]

The result follows. \( \square \)

**Proposition 3.4.** Let \( f \) be a 1-form on \( M \) with \( d^*f = 0 \). For every face \( \Sigma_j \) in the aforementioned partition of \( \partial F \), fix some \( q_j \in \Sigma_j \). Then

\[
||f||_2^2 \leq \text{vol}(\partial F)\|f\|_{L^\infty} \left( 3\pi \|df\|_{L^\infty} + \sup_j \left| \int_{\alpha_{q_j,\gamma_j q_j}} f \right| \right) + \frac{5}{2} \|df\|_{L^\infty} \|f\|_{L^2(M)} \sqrt{\text{vol}(M)}.
\]

Proof. Lift \( f \) to \( \mathbb{H}^n \). For \( q \in \mathcal{F} \), define

\[
b(q) := \int_{\alpha_{p,q}} f, \tag{3.5}
\]

as in Lemma 3.2

Hence

\[
\|f\|^2 = \int_F f \wedge *f
\]

\[
= \int_F \text{db} \wedge *f + \int_F (f - \text{db}) \wedge *f
\]

\[
= \int_{\partial F} b \ast f + \int_F (f - \text{db}) \wedge *f \tag{3.6}
\]

Write

\[
\int_{\partial F} b \ast f = \sum_{j=1}^J \int_{\Sigma_j - \gamma_j \Sigma_j} b \ast f = \sum_{j=1}^J \int_{\Sigma_j} \left( \int_{\alpha_{p,q} + \alpha_{q,\gamma_j p}} f \right) \ast f. \tag{3.7}
\]

Let \( \Delta_{a,b,c} \) denote the oriented hyperbolic triangle with vertices \( a, b, c \) and orientation such that \( \partial \Delta_{a,b,c} = \alpha_{ab} + \alpha_{bc} + \alpha_{ca} \). For \( q \in \Sigma_j \), let

\[
R_{j,q} := \Delta_{p,q,q_j} \cup \Delta_{p,\gamma_j q,\gamma q} \cup \Delta_{p,q_j,\gamma q_j}. \tag{3.8}
\]

Then

\[
\int_{\alpha_{\gamma_j q,p} + \alpha_{p,q}} f = - \int_{\alpha_{q_j,\gamma_j q_j}} f + \int_{R_{j,q}} df. \tag{3.9}
\]
Here we have used \( \int_{\alpha_q, q_j + \alpha_j, \gamma_q} f = 0 \). This gives

\[
\int_{\partial F} b \ast f = \sum_{j=1}^{J} \left( - \int_{\alpha_q, \gamma_j, q_j} f \right) \left( \int_{\Sigma_j} \ast f + \sum_{j=1}^{J} \int_{\Sigma_j} \left( \int_{R_j, q_j} \ast f \right) \right).
\]

(3.10)

Since hyperbolic triangles have area at most \( \pi \), we have

\[
\left| \int_{R_j, q_j} \ast f \right| \leq 3 \pi \| df \|_{L^\infty}.
\]

(3.11)

Substituting (3.10) into (3.6) and estimating gives

\[
\| f \|_{L^2} \leq 3 \pi \| df \|_{L^\infty} \sum_{j=1}^{J} \int_{\Sigma_j} |f| dA + \sum_{j=1}^{J} \left( \int_{\alpha_q, \gamma_j, q_j} f \right) \cdot \left( \int_{\Sigma_j} \ast f \right) + \| f - db \wedge \ast f \|_{L^1(F)}
\]

\[
\leq \text{vol}(\partial F) \| f \|_{L^\infty} \left( 3 \pi \| df \|_{L^\infty} + \| f - db \wedge \ast f \|_{L^2(F)} \cdot \sqrt{\text{vol}(M)} \right)
\]

\[
\leq \text{vol}(\partial F) \| f \|_{L^\infty} \left( 3 \pi \| df \|_{L^\infty} + \| f - db \|_{L^\infty(F)} \cdot \| f \|_{L^2(M)} \cdot \sqrt{\text{vol}(M)} \right)
\]

\[
\leq \text{vol}(\partial F) \| f \|_{L^\infty} \left( 3 \pi \| df \|_{L^\infty} + \| f - db \|_{L^\infty(M)} \cdot \| f \|_{L^2(M)} \cdot \sqrt{\text{vol}(M)}, \right)
\]

(3.12)

where the last line follows from Lemma 3.2.

\[\square\]

**Corollary 3.13.** Let \( f \) be a 1-form on \( f \) satisfying \( df = 0 \). Suppose that \( f \) is a linear combination of eigen 1-forms of eigenvalue at most \( \lambda \). Let \( q_j \in \Sigma_j \) be the fixed reference points chosen in Proposition 3.4. Then

\[
\| f \|_2 \leq \sqrt{\lambda} \cdot \text{vol}(\partial F) \cdot C(\lambda)^2 \cdot 3 \pi \cdot \| f \|_2 + C(\lambda) \cdot \sup_j \left| \int_{\alpha_q, \gamma_j, q_j} f \right|
\]

\[
+ \sqrt{\lambda} \cdot C(\lambda) \cdot \frac{1}{2} \cdot \sqrt{\text{vol}(M)} \cdot \| f \|_2.
\]

(3.14)

In particular, if some multiple of \( \gamma_j \) bounds for every \( j \), then

\[
\frac{1}{\sqrt{\lambda}} \leq C(\lambda)^2 (3 \pi \cdot \text{vol}(\partial F) + \sup_j \text{sArea}(\gamma_j)) + C(\lambda) \cdot \frac{1}{2} \cdot \sqrt{\text{vol}(M)}.
\]

The quantity \( \text{sArea}(\gamma) \) is defined in (1.2), and \( C(\lambda) \) is defined in (2.12).

**Proof.** The first part follows upon applying the Sobolev inequality from Proposition 2.2 to Proposition 3.4.

11
For the second part: suppose $\gamma^n$ is bounded by a surface $S$. By Stokes theorem,

$$\left| \int_{\alpha_q, \gamma^q} f \right| = \frac{1}{m} \left| \int_{\alpha_q, \gamma^{m_q}} f \right| = \frac{1}{m} \left| \int_S df \right| \leq ||df||_\infty \frac{\text{area}(S)}{m}.$$ 

Applying this inequality to all period integrals appearing on the right side of the inequality from Proposition 3.4 together with the Sobolev inequality 2.2 gives the second part of the Corollary.

3.1 The geometry of two types of fundamental domains

In this section, we analyze the geometry of two types of fundamental domains $F$. Understanding this geometry is necessary to control the $\text{vol}(\partial F)$ terms occurring in the estimates in Corollary 3.13. We also need to control $d(q_j, \gamma_j q_j)$ in order to bound the periods $\int q_j, \gamma_j q_j f$. Upper bounds on $\text{diam}(F)$ suffice.

3.1.1 Type 1: tree-type fundamental domains induced from a covering map

Let $M_0$ be a closed hyperbolic $n$-manifold. Let $M \to M_0$ be a covering. Let $F_0$ be a (closed) Dirichlet fundamental domain for $M_0$ relative to a fixed center $p_0 \in \mathbb{H}^n$. Let $\Gamma$ and $\Gamma_0$ respectively denote $\pi_1(M)$ and $\pi_1(M_0)$. Let

$$S_0 = \{ \gamma \in \Gamma_0 : \gamma F_0 \cap F_0 \neq \emptyset \}.$$ 

Note that $S_0$ is a symmetric generating set for $\pi_1(M_0)$. The fundamental domain $F_0$ induces a tiling of $\mathbb{H}^n$. Let $\mathcal{G}(M_0)$ denote the dual graph of this tiling. Quotienting this tiling by $\Gamma$ induces a tiling of $M$. The dual graph of the induced tiling of $M$, which equals $\Gamma \backslash \mathcal{G}(M_0)$, is isomorphic to the Schreier graph $\mathcal{G}(M, M_0)$ of $M$ relative to $M_0$: vertices are given by elements of $\Gamma \backslash \Gamma_0$ and two vertices are connected by an edge if they differ by $s \in S_0$ [9 Corollary 0.9].

Let $T$ be a spanning tree in $\mathcal{G}(M, M_0)$. Fix a vertex $v_0 \in T$. Associated with the unique geodesic in $T$ from $v_0$ to $v$ is a corresponding ordered sequence of elements $s_1, \ldots, s_n \in S_0$; these are the Schreier graph edge labels in the ordered edge sequence determined by the geodesic from $v$ to $v_0$. Let $\gamma_{v_0} = s_n s_{n-1} \cdots s_1$.

**Definition 3.15.** The *tree-type fundamental domain* $F_T$ associated with $F_0$ and $T$ is

$$F_T := \bigcup_{\text{vertices } v \text{ of } T} F_v,$$ 

where $F_v := \gamma_{v_0,v} F_0$.

The boundary $\partial F_T$ is a union of $\Gamma$-translates of codimension-1 faces of $F_0$ which project isometrically to $M$ and which have disjoint interiors. Because $M$ is closed, these boundary faces can be identified in pairs; i.e. there exists a decomposition of the boundary $F_T$ into $F_0$-faces $\Sigma_1, \Sigma'_1, \ldots, \Sigma_l, \Sigma'_l$ and corresponding $\gamma_1, \ldots, \gamma_l \in \Gamma$ for which $\gamma_j \Sigma_j = -\Sigma'_j$.

Thus, $F_T$ is a fundamental domain as defined at the beginning of 3. It is not in general convex, but we do not require convexity for the arguments of this section.

**Lemma 3.16.** The boundary volume $\text{vol}(\partial F_T)$ is bounded above by

$$\text{vol}(\partial F_T) < \text{vol}(\partial F_0) \cdot \frac{\text{vol}(M)}{\text{vol}(M_0)},$$ 

12
Lemma 3.17. The diameter of $F_T$ is bounded above by

$$\text{diam}(F_T) \leq \text{diam}(F_0) \cdot (\text{diam}(T) + 1),$$

where $\text{diam}(T)$ denotes the combinatorial diameter of the tree $T$.

**Proof.** Let $p \in F_a$ and $q \in F_b$, where $a, b$ are vertices of $T$. Let $a = v_0 \to v_1 \to \cdots \to v_n = b$ be the unique shortest path from $a$ to $b$. Let $F_{v_0}, \ldots, F_{v_n}$ be the corresponding chain of $F_0$-tiles in $F_T$. Every point $p_i \in F_{v_i}$ lies within $\text{diam}(F_{v_i}) = \text{diam}(F_0)$ of some point $p_{i+1}$ of $F_{v_{i+1}}$. The broken geodesic path $p = p_0 \to p_1 \to \cdots \to p_n = q$ therefore has total length at most $(n+1) \cdot \text{diam}(F_0) \leq (\text{diam}(T)+1) \cdot \text{diam}(F_0)$. Therefore, $d(p, q) \leq (\text{diam}(T)+1) \cdot \text{diam}(F_0)$. \[\square\]

For a special choice of tree $T$, the diameter $\text{diam}(T)$ can be bounded above in terms of $\text{diam}(G(M, M_0))$.

Lemma 3.18. Let $G$ be an arbitrary finite, connected graph. There exists a spanning subtree $T_{\text{fat}} \subset G$ for which $\text{diam}(T_{\text{fat}}) \leq 2 \text{diam}(G)$.

**Proof.** Fix a vertex $v \in G$. A shortest path subtree relative to $v$ is a spanning subtree, $T_{\text{fat}}$, such that for all $w \in T_{\text{fat}}$, $d_{T_{\text{fat}}}(w, v) = d_G(w, v)$. There exists at least one shortest path subtree relative to $v$. For the reader’s convenience, we recall one such construction [10 §3]: to every vertex $w \in G \setminus \{v\}$, assign a neighboring vertex $p_w \in G$ for which $d_G(p_w, v) = d_G(w, v) - 1$. The subgraph with full vertex set and edge set the edges connecting $w$ and $p_w$ for every $w \neq v$ is a shortest path tree relative to $v$.

In particular, for every $a, b \in T_{\text{fat}}$,

$$d_{T_{\text{fat}}}(a, b) \leq d_{T_{\text{fat}}}(a, v) + d_{T_{\text{fat}}}(v, b) = d_G(a, v) + d_G(v, b) \leq 2 \text{diam}(G).$$

\[\square\]

**Definition 3.19.** Let $v$ be a vertex of $G(M_0)$. Define the combinatorial ball of radius $r$ centered at $F_v$, denoted $B_{\text{comb}, r}(F_v)$, to be

$$B_{\text{comb}, r}(F_v) = \bigcup_{d_G(M_0, w, v) \leq r} F_w \subset \mathbb{H}^n.$$ 

Define the combinatorial sphere of radius $r$ centered at $F_v$, denoted $S_{\text{comb}, r}(F_v)$, to be

$$S_{\text{comb}, r}(F_v) := \partial B_{\text{comb}, r}(F_v).$$

The next lemma bounds $\text{diam}(G(M, M_0))$ above in terms of $\text{diam}(M)$.

**Lemma 3.20.** Suppose that $B_{\text{comb}, 1}(F_v)$ projects isometrically to $M$, for all $v$. Let $r_0 := d(F_0, S_{\text{comb}, 1}(F_0))$. Let $k_0$ denote the number of $F_0$-tiles in $B_{\text{comb}, 1}(F_0)$. Suppose that $a \in \text{int}(F_{v_1}), b \in \text{int}(F_{v_2})$. Suppose $d(a, b) < r_0$. Then

$$d_G(M, M_0)(v_1, v_2) \leq 2k_0.$$ 

In particular, for every $p, q \in M$, if $p \in F_{w_1}, q \in F_{w_2}$, then

$$d_G(M, M_0)(w_1, w_2) \leq \frac{d(p, q)}{r_0} \cdot 2k_0 + 2k_0.$$
and hence

\[ \text{diam}(G(M, M_0)) \leq \frac{\text{diam}(M)}{r_0} \cdot 2k_0 + 2k_0. \]

**Proof.** For all \( v \), \( B_{\text{comb},1}(F_v) \) is isometric to \( B_{\text{comb},1}(F_0) \). Hence \( r_0 \) and \( k_0 \) are the same for all balls. Because \( d(a, b) < r_0 \), the geodesic segment \( \alpha_{a,b} \) intersects only those \( F_0 \)-tiles of \( M \) contained in \( B_{\text{comb},1}(F_v) \). Perturbing \( a, b \) as necessary, we may assume that \( \alpha_{a,b} \) intersects only codimension-1 faces of the \( F_0 \)-tiles.

The combinatorial distance \( d_{\text{comb}}(v_1, v_2) \) is at most the number of intersection points between codimension 1 faces of tiles of \( B_{\text{comb},1}(F_v) \) and the segment \( \alpha_{a,b} \). By convexity, \( \alpha_{a,b} \) intersects the boundary of every tile at most twice. Therefore,

\[ d_{\text{comb}}(v_1, v_2) \leq 2k_0 \]

as claimed.

For the second claim, divide a length minimizing geodesic from \( p \) to \( q \) into \( m = \left[ \frac{d(p,q)}{r_0 - \epsilon} \right] \) equal segments of length \( r_0 - \epsilon \) together with one terminal segment of length \( < r_0 - \epsilon \). Applying the above argument to all \( m + 1 \) segments gives

\[
d_{\text{comb}}(w_1, w_2) \leq \left\lfloor \frac{d(p,q)}{r_0 - \epsilon} \right\rfloor \cdot 2k_0 + 2k_0 \leq \frac{d(p,q)}{r_0 - \epsilon} \cdot 2k_0 + 2k_0, \forall \epsilon > 0,
\]

and the result follows. \( \square \)

**Corollary 3.21.** There exists a spanning subtree \( T \subset G(M, M_0) \) for which

\[ \text{diam}(F_T) \leq \text{diam}(F_0) \cdot \left( \frac{4k_0}{r_0} \cdot \text{diam}(M) + 4k_0 + 1 \right), \]

where \( r_0 \) and \( k_0 \) are the constants from Proposition \( \PageIndex{3.20} \).

**Proof.** Let \( T = T_{\text{fat}} \) be the spanning subtree from Lemma \( \PageIndex{3.18} \). The Corollary follows upon combining Lemma \( \PageIndex{3.18} \) and Lemma \( \PageIndex{3.20} \). \( \square \)

### 3.1.2 Type 2: Dirichlet fundamental domains

Let \( N \) be a closed hyperbolic \( n \)-manifold with fundamental group \( G = \pi_1(N) \) acting by deck transformations in \( \mathbb{H}^n \).

**Definition 3.22.** The **Dirichlet fundamental domain domain** \( F \) associated to \( N \) and \( p_0 \in \mathbb{H}^n \) is \( F := \{ x \in \mathbb{H}^n : d(x, p_0) \leq d(x, \gamma p_0) \text{ for all } 1 \neq \gamma \in G \} \).

If \( F \) is a Dirichlet domain for \( N \), the group \( G \) is generated by the finite symmetric set

\[ S = S_G := \{ \gamma \in G : \gamma F \cap F \neq \emptyset \}. \]

**Lemma 3.23.** There is an upper bound

\[ \text{diam}(F) \leq 2 \cdot \text{diam}(M). \]

**Proof.** Suppose \( x \in \mathbb{H}^n \) satisfies \( d(x, p_0) > \text{diam}(M) \). The projection of \( B_{\text{diam}(M)}(x) \) to \( M \) covers all of \( M \). Therefore, there is some \( \gamma \in \Gamma \) for which \( d(x, \gamma p_0) < \text{diam}(M) \). But by assumption, \( d(x, p_0) > \text{diam}(M) \). Thus \( x \notin F \). It follows that \( F \) is contained in \( B_{\text{diam}(M)}(p_0) \), implying that \( \text{diam}(F) \leq 2\text{diam}(M) \). \( \square \)
The following Proposition proves an upper bound on the number of codimension-1 faces of $F$. Combined with the diameter upper bound from Lemma 3.23, this yields an upper bound for $\text{vol}(\partial F)$.

**Proposition 3.24.**

$$\text{vol}(\partial F) \leq E_n \cdot e^{(2n-3) \cdot 4 \text{diam}(M)},$$

for some constant $E_n$ depending only on $n$.

**Proof.** The cells of $\partial F$ are formed by intersections of bisectors: $B_\gamma := \{x : d(x, p_0) = d(x, \gamma p_0)\}$. However, $d(x, p_0) \leq \text{diam} F$ for every $x \in B_\gamma$. Therefore, if the bisector $B_\gamma$ intersects $F$, we must have

$$d(p_0, \gamma p_0) \leq d(p_0, x) + d(x, \gamma p_0) = 2d(x, p_0) \leq 2 \text{diam}(F).$$

Therefore,

$$\#(n-1)\text{-dimensional faces of } \partial F \leq \#\{\gamma \in \Gamma : d(p_0, \gamma p_0) \leq 2 \text{diam}(F)\} \leq \frac{\text{vol}(B_{2 \text{diam}(F) + 4 \text{diam}(M)})}{\text{vol}(B_{\frac{4}{2} \text{diam}(M)})} \leq D_n e^{(n-1)2 \text{diam}(F)}.$$

Hence

$$\text{vol}(\partial F) \leq \#(n-1)\text{-dimensional faces of } \partial F \cdot \max_{C = \text{codimension-1 face of } \partial F} \text{vol}(C) \leq D_n e^{(n-1)2 \text{diam}(F)} \cdot \text{vol}(B_{n-1}(2 \text{diam}(F))) \leq E_n \cdot e^{(n-1)2 \text{diam}(F)} \cdot e^{(n-2)2 \text{diam}(F)} \leq E_n \cdot e^{(2n-3)4 \text{diam}(M)},$$

where the last line follows from Proposition 3.23. \qed

**Remark 3.26.** There is an upper bound for the diameter of a closed hyperbolic $n$-manifold $M$ of $\frac{1}{\lambda_1(M)} \log \text{vol}(M)$. So in everything that preceded, upper bounds of $\exp(\text{diam}(M))$ may all be replaced by $\text{vol}(M)^{1/\lambda_1(M)}$.

### 4 Almost-primitives and regulators when $b_1(M) > 0$

Theorem 4.11 proves a general upper bound for $\frac{1}{\lambda_1(M)}$, where $M$ is a closed hyperbolic $n$-manifold, in terms of stable area of an explicit subset of $\Gamma = \pi_1(M)$ whose projection to $H_1(M, \mathbb{Q})$ is trivial. The key is to bound the “period integrals” of a 1-form $f$ in the image of $d^*$ with smallest positive eigenvalue. In the notation of Theorem 4.11, we bound the period integral over $\gamma$, where $\gamma$-pairs two faces of a fundamental domain for $M$, in terms of the stable area of $\gamma' = \gamma \cdot \gamma_1^{-n_1(\gamma)} \cdots \gamma_k^{-n_k(\gamma)}$.

However, if $\gamma'$ is long, then its stable area should be bounded below by constant $\ell/\log \ell$, where $\ell = |n_1(\gamma)| + \cdots + |n_k(\gamma)|$, with very high probability (cf. Conjecture A.10). It is thus imperative that we prove good upper bounds on the integers $n_i(\gamma)$, which are intersection numbers between $\gamma$ and surfaces $S_i$ generating $H_2(M, \mathbb{Z})$. When $n = 3$, Proposition 4.14 uses known facts about minimal surfaces in hyperbolic 3-manifolds to represent every $S_i$ by homologous surfaces with “bounded geometry.” Proposition 4.18
bounds the intersection numbers $S_i \cap \gamma$ above in terms of $A'$, an upper bound for the minimal area representatives of every class $S_i$.

When $n = 3$ and $b_1(M) = 1$, Proposition 4.23 estimates the damping factor $\frac{\|f\|_{\mathcal{L}}}{\|f\|_{\mathcal{L}} + \|f\|_{\mathcal{L}}'}$ occurring in Theorem 4.11. The upshot: this damping factor is smaller than the inverse of the translation length of $\gamma'$. As a result, Proposition 4.23 shows “regulator-independent” upper bounds on $\frac{1}{\lambda_1(M)_{d^*}}$, i.e. upper bounds in terms of $s\text{Area}(\gamma')/\ell(\gamma')$, for an explicit finite collection of $\gamma' \in \Gamma$, which is independent of $A'$.

**Remark 4.1.** Though Proposition 4.23 is proven only when $n = 3$ and $b_1(M) = 1$, “regulator-independent” upper bounds on $\frac{1}{\lambda_1(M)_{d^*}}$ are to be expected in general. Indeed, the Cheeger-Müller Theorem suggests that small eigenvalues and regulators suppress each other. See (1.2) for further discussion.

### 4.1 General upper bounds on $\frac{1}{\lambda_1(M)_{d^*}}$ when $b_1(M) > 0$

Let $M \to M_0$ be a finite cover of the closed hyperbolic $n$-manifold $M_0$. Suppose that $H_1(M, \mathbb{Z}) = \mathbb{Z}\langle \gamma_1 \rangle \oplus \cdots \oplus \mathbb{Z}\langle \gamma_k \rangle \oplus \text{finite}$. We may take $\gamma_1, \ldots, \gamma_k \subset P_F$ for some fundamental domain $F$ of $M$ in $\mathbb{H}^n$; we take this fundamental domain to be either a tree-type domain or a Dirichlet domain in the sense of §3.1. By Corollary 3.21 and Lemma 3.23 in the case of tree-type fundamental domains and Lemma 3.23 in the case of Dirichlet fundamental domains, we have

\[
\ell(\gamma_i) \leq \begin{cases} 
\text{diam}(F_0) \cdot \left( \frac{4k_0}{r_0} \cdot \text{diam}(M) + 4k_0 + 1 \right) & \text{if the fundamental domain is tree-type} \\
2 \cdot \text{diam}(M) & \text{if the fundamental domain is Dirichlet}
\end{cases}
\]

for every $i$.

**Proposition 4.2.** Let $n_i = n_i(\gamma) \in \mathbb{Z}$ be the unique integers for which $\gamma - n_1\gamma_1 - \cdots - n_k\gamma_k$ is torsion in $H_1(M, \mathbb{Z})$.

Suppose $f$ is a 1-form on $M$ satisfying $d^* f = 0$. Fix a base point $q_0 \in M$. Let $h$ be a harmonic 1-form satisfying

\[
\int_{\alpha^*_{q_0}} f = \int_{\alpha^*_{q_0}} h \quad \text{for } i = 1, \ldots, k.
\]

Then

\[
\left| \int_{\alpha^*_{q_0}} (f - h) \right| \leq \|d f\|_{\infty} \cdot \left( s\text{Area}(\alpha_{q_0, n_1 \gamma_1 \cdots n_k \gamma_k q_0}) + 5b_1(M)\pi \right)
\]

**Proof.** Suppose that $\alpha_{q_0, n_1 \gamma_1 \cdots n_k \gamma_k q_0}$ bounds $S$. The geodesic triangle $\Delta$ with positively oriented vertex set $q_0, \gamma_1^{n_1} \cdots \gamma_k^{n_k} q_0, \gamma q_0$ has oriented boundary $\alpha_{q_0, \gamma_1^{n_1} \cdots \gamma_k^{n_k} q_0} + \alpha_{\gamma_1 \cdots \gamma_k^{n_k} q_0, \gamma q_0}$ $\alpha_{\gamma q_0, q_0}$. By Stokes,
\[
\int_{\alpha_{q_0}} (f - h) \leq \int_{\alpha_{q_0} \gamma_1 \cdots \gamma_k q_0} (f - h) + \int_{\alpha \gamma_1 \cdots \gamma_k q_0} (f - h) + \int_{\Delta} \alpha d(f - h)
\]

\[
= \int_{\alpha_{q_0} \gamma_1 \cdots \gamma_k q_0} (f - h) + \int \frac{1}{m} \left( \int_{\alpha \gamma_k \cdots \gamma_1 q_0} m \right) (f - h) + \int_{\Delta} \alpha d(f - h)
\]

\[
= \int_{\alpha_{q_0} \gamma_1 \cdots \gamma_k q_0} (f - h) + \frac{1}{m} \int_{\Delta} S d(f - h) + \int_{\Delta} \alpha d(f - h)
\]

\[
\leq \int_{\alpha_{q_0} \gamma_1 \cdots \gamma_k q_0} (f - h) + \|d\|_{\infty} \cdot \left( \frac{\text{area}(S)}{m} + \text{area}(\Delta) \right) . \quad (4.3)
\]

As (4.3) holds for arbitrary \( m, S \) for which \( \alpha_{q_0} \gamma_1 \cdots \gamma_k q_0 \) bounds \( S \), it follows that

\[
\int_{\alpha_{q_0}} (f - h) \leq \int_{\alpha_{q_0} \gamma_1 \cdots \gamma_k q_0} (f - h) + \|d\|_{\infty} \cdot \left( \text{sArea}(\alpha_{q_0} \gamma_1 \cdots \gamma_k q_0) + \pi \right) . \quad (4.4)
\]

Let

\[
\eta = \gamma_k^{n_k+1} \cdots \gamma_1^{n_1} .
\]

The geodesic segments \( \alpha_{q_0}, \gamma_1 q_0, \alpha_{n_1 q_0}, \gamma_k q_0, \cdots, \alpha_{n_{k-1} q_0}, \gamma_{k-1} q_0, \alpha_{q_0} \gamma_1 \cdots \gamma_k q_0 \) form the oriented boundary of a "broken geodesic \( k + 1 \)-gon" \( P \), the union of \( k - 1 \) geodesic triangles meeting at kinks. This broken polygon has area at most \((k - 1)\pi\). By Stokes,

\[
\int_{\alpha_{q_0} \gamma_1 \cdots \gamma_k q_0} (f - h) \leq \sum_i \int_{\alpha_{q_0} \gamma_1 \cdots \gamma_k q_0} (f - h) + \int_{\Delta} \alpha d(f - h)
\]

\[
= \sum_i \int_{\alpha_{q_0} \gamma_1 \cdots \gamma_k q_0} (f - h) + \int_{\Delta} \alpha d(f - h)
\]

\[
\leq \sum_i \int_{\alpha_{q_0} \gamma_1 \cdots \gamma_k q_0} (f - h) + \|d\|_{\infty} \cdot (k - 1)\pi . \quad (4.5)
\]

For arbitrary \( a, b \in \Gamma \), the geodesic segments \( \alpha_{a^{-1} ba q_0} + \alpha_{a^{-1} ba q_0} + \alpha_{b a q_0} \) form the oriented boundary of a geodesic triangle \( \Delta' \). By Stokes,

\[
\int_{\alpha_{q_0}} (f - h) \leq \int_{\alpha_{q_0} \gamma_1 \cdots \gamma_k q_0} (f - h) + \int_{\alpha_{a^{-1} ba q_0} \gamma_0} (f - h) + \int_{\Delta'} \alpha d(f - h)
\]

\[
= \int_{\alpha_{q_0} \gamma_1 \cdots \gamma_k q_0} (f - h) + \int_{\alpha_{a^{-1} ba q_0} \gamma_0} (f - h) + \int_{\Delta'} \alpha d(f - h) \quad \text{where } \gamma_0 = a^{-1} ba q_0 . \quad (4.6)
\]
Next, consider the oriented broken geodesic \( \alpha_{p_0, a^{-1}b^{-1}ap_0} + \alpha_{a^{-1}b^{-1}ap_0, b^{-1}ap_0} + \alpha_{b^{-1}ap_0, ap_0} + \alpha_{ap_0, [b, a^{-1}]p_0} + \alpha_{[b, a^{-1}]p_0, p_0} \). It is the boundary of a broken geodesic pentagon \( P' \) which projects to a surface in \( M \) with boundary \(-\alpha_{p_0, [b, a^{-1}]q_0} \). Therefore,

\[
\left| \int_{\alpha_{p_0, [b, a^{-1}]p_0}} (f - h) \right| \leq \left| \int_{P'} df \right|.
\]

Substituting back into (4.7) gives

\[
\left| \int_{\alpha_{q_0, a^{-1}b=0}} (f - h) \right| \leq \left| \int_{\alpha_{q_0, b=0}} (f - h) \right| + \left| \int_{P'} df \right| + \left| \int_{\Delta'} df \right|
\leq \left| \int_{\alpha_{q_0, b=0}} (f - h) \right| + ||df||_\infty \cdot (\text{area}(P') + \text{area}(\Delta'))
\leq \left| \int_{\alpha_{q_0, b=0}} (f - h) \right| + ||df||_\infty \cdot 4\pi. \tag{4.7}
\]

Substituting (4.7) back into (4.5) gives

\[
\left| \int_{\alpha_{q_0, \gamma_{1}^{-n_1} \cdots \gamma_{k}^{-n_k} q_0}} (f - h) \right| \leq \sum_{i} \left( \left| \int_{\alpha_{q_0, \gamma_{k-i}^{-1} q_0}} (f - h) \right| + ||df||_\infty \cdot 4\pi \right) + ||df||_\infty \cdot (k - 1)\pi
\leq \sum_{i} n_{k-i} \left| \int_{\alpha_{q_0, \gamma_{k-i}^{-1} q_0}} (f - h) \right| + ||df||_\infty \cdot (5k - 1)\pi
= \sum_{i} n_{k-i} \cdot 0 + ||df||_\infty \cdot (5k - 1)\pi
= ||df||_\infty \cdot (5k - 1)\pi. \tag{4.8}
\]

Finally, substituting (4.8) back into (4.4) gives

\[
\left| \int_{\alpha_{q_0, \gamma q_0}} (f - h) \right| \leq ||df||_\infty \cdot (5k - 1)\pi + ||df||_\infty \cdot (\text{sArea}(\gamma \cdot \gamma_{1}^{-n_1} \cdots \gamma_{k}^{-n_k}) + \pi)
= ||df||_\infty \cdot \left(\text{sArea}(\alpha_{q_0, \gamma_{1}^{-n_1} \cdots \gamma_{k}^{-n_k} q_0}) + 5b_1(M)\pi\right)
\]

\[\square\]

**Proposition 4.9.** Notation as in Corollary 3.13. Let \( M_0 \) be a closed hyperbolic \( n \)-manifold. Let \( M \to M_0 \) be a finite cover. Suppose \( H_1(M, \mathbb{Z}) = \mathbb{Z} \langle \gamma_1 \rangle \oplus \cdots \oplus \mathbb{Z} \langle \gamma_k \rangle \oplus \text{finite} \). For every \( \gamma \in \Gamma, \) let \( n_{i}(\gamma) \in \mathbb{Z} \) be the unique integers for which \( \gamma - \sum_{i=1}^{k} n_{i}(\gamma) \gamma_i \in H_1(M, \mathbb{Z}) \) has finite order. Fix a basepoint \( q_0 \in M \).

Suppose \( f \) is a 1-form on \( M \) contained in the image of \( d^* \). Suppose that \( h \) is a harmonic 1-form satisfying

\[
\int_{\alpha_{q_0, \gamma q_0}} h = \int_{\alpha_{q_0, \gamma q_0} \gamma_i q_0} f \text{ for } i = 1, \ldots, k.
\]
Then

\[
\frac{1}{\sqrt{\lambda}} \leq 3\pi \cdot \text{vol}(\partial F) \cdot C(\lambda)^2 + C(\lambda) \cdot \frac{1}{2} \cdot \sqrt{\text{vol}(M)}
\]

\[
+ C(\lambda)^2 \cdot \frac{\|f\|_2}{(\|f\|^2 + \|h\|^2)^{1/2}} \left( \sup_j \text{sArea} \left( \alpha_{q_0,q_j^{-1n_j}} - n_k(\gamma_j) \right) + (5b_1(M) + 2\pi) \right).
\]

\textbf{Proof.} Let \( q_j \) be the reference points used in Corollary 3.13. The broken geodesic \( \alpha_{q_1,q_2,q_3} + \alpha_{q_2,q_3,q_4} + \alpha_{q_3,q_4,q_5} + \alpha_{q_4,q_5,q_6} \) is the oriented boundary of a broken geodesic quadrilateral \( Q \) of area at most \( 2\pi \). The projection of this quadrilateral to \( M \) has boundary \( \alpha_{q_1,q_2} + \alpha_{q_3,q_5,q_6} \). By Stokes and Proposition 4.2

\[
\sup_j \left| \int_{\alpha_{q_j,q_j}} (f - h) \right| \leq \sup_j \left| \int_{\alpha_{q_0,q_j}} (f - h) \right| + \left| \int_Q df \right|,
\]

\[
\leq \sup_j \left| \int_{\alpha_{q_0,q_j}} (f - h) \right| + ||df||_\infty \cdot 2\pi \leq ||df||_\infty \cdot \left( \sup_j \text{sArea} \left( \alpha_{q_0,q_j^{-1n_j}} - n_k(\gamma_j) \right) + (5b_1(M) + 2\pi) \right).
\]

Substituting (4.11) into the first inequality stated in Corollary 3.13 (and remembering that \( f - h \) here plays the role of \( f \) in Corollary 3.13) followed by the Sobolev inequality from Proposition 2.2 gives

\[
(\|f\|^2 + \|h\|^2)^{1/2} \leq \sqrt{\lambda} \cdot \text{vol}(\partial F) \cdot C(\lambda)^2 \cdot 3\pi \cdot (\|f\|^2 + \|h\|^2)^{1/2}
\]

\[
+ \sqrt{\lambda} \cdot C(\lambda)^2 \cdot \|f\|_2 \cdot \left( \sup_j \text{sArea} \left( \alpha_{q_0,q_j^{-1n_j}} - n_k(\gamma_j) \right) + (5b_1(M) + 2\pi) \right)
\]

\[
+ \sqrt{\lambda} \cdot C(\lambda) \cdot \frac{1}{2} \cdot \sqrt{\text{vol}(M)} \cdot (\|f\|^2 + \|h\|^2)^{1/2}
\]

Upon rearranging:

\[
\frac{1}{\sqrt{\lambda}} \leq 3\pi \cdot \text{vol}(\partial F) \cdot C(\lambda)^2 + C(\lambda) \cdot \frac{1}{2} \cdot \sqrt{\text{vol}(M)}
\]

\[
+ C(\lambda)^2 \cdot \frac{\|f\|_2}{(\|f\|^2 + \|h\|^2)^{1/2}} \left( \sup_j \text{sArea} \left( \alpha_{q_0,q_j^{-1n_j}} - n_k(\gamma_j) \right) + (5b_1(M) + 2\pi) \right).
\]

\[\square\]

\textbf{Theorem 4.11 (Geometric Upper Bound for \( \frac{1}{\sqrt{\lambda(M)}} \))}. Let \( M_0 \) be a closed hyperbolic \( n \)-manifold. Let \( M \to M_0 \) be a finite cover. Suppose \( H_1(M,\mathbb{Z}) = \mathbb{Z}(\gamma_1) \oplus \cdots \oplus \mathbb{Z}(\gamma_k) \oplus \text{finite} \). Let \( n_i(\gamma) \) be the unique integers for which \( \gamma - \sum_{i=1}^k n_i(\gamma) \gamma_i \in H_1(M,\mathbb{Z}) \) has finite order. Fix a basepoint \( q_0 \in M \).
Let $f$ be a 1-form in the image of $d^*$ satisfying $\Delta f = \lambda f$, where $\lambda = \lambda_1(M,d^*)$. Let $h$ be the unique harmonic 1-form satisfying
\[ \int_{\alpha_{q_0,\gamma_0}} h = \int_{\alpha_{q_0,\gamma_0}} f \text{ for } i = 1, \ldots, k. \]

Then
\[
\frac{1}{\sqrt{\lambda}} \leq 3\pi \cdot C(\lambda)^2 \cdot V + C(\lambda) \cdot \frac{1}{2} \cdot \sqrt{\text{vol}(M)} + C(\lambda)^2 \cdot \frac{||f||_2}{(||f||_2^2 + ||h||_2^2)^{1/2}} \left( \sup_{\ell(\gamma) \leq D} \text{Area} \left( \alpha_{q_0,\gamma_1,\ldots,n_i(\gamma),\ldots,n_k(\gamma),q_0} \right) + (5b_1(M) + 2)\pi \right)
\]

where
\[ V := \text{vol}(\partial F) \leq \begin{cases} \text{vol}(\partial F) \cdot \frac{\text{vol}(M)}{\text{vol}(M_0)} & \text{if } F \text{ is tree-type} \\ E_n \cdot e^{(2n-3) \cdot 4\text{diam}(M)} & \text{if } F \text{ is Dirichlet} \end{cases} \]

and
\[ D := \max_j \ell(\gamma_j) \leq \begin{cases} \text{diam}(F_0) \cdot \left( \frac{4k_0}{\gamma_0} \cdot \text{diam}(M) + 4k_0 + 1 \right) & \text{if } F \text{ is tree-type} \\ 2 \cdot \text{diam}(M) & \text{if } F \text{ is Dirichlet}. \end{cases} \]

Here, $F_0$ is a Dirichlet fundamental domain for $M_0$, $r_0$ and $k_0$ are the constants from Proposition 3.24, $E_n$ is the constant from Proposition 3.24 and $C(\lambda)$ is the Sobolev constant from (2.12).

Proof. This follows immediately from Proposition 4.9. The upper bounds on $V$ follow from Propositions 3.10 and 3.24 The upper bounds on $D$ follow from Propositions 3.21 and 3.23. \qed

The upper bound for $\frac{1}{\lambda(\gamma_0,\gamma_0)}$ from Theorem 4.11 is only useful if $\gamma - \sum_{i=1}^k n_i(\gamma)\gamma_i$ is a short loop for $\gamma$ appearing in the sum on the right side of the inequality featured therein. We turn next to controlling the size of the $n_i(\gamma)$.

### 4.1.1 Controlling the free part of $\gamma \in \pi_1(M)$ via regulators

Let $M_0$ be a closed hyperbolic 3-manifold and let $M \to M_0$ be a finite cover. Let $H_1(M,\mathbb{Z}) = \mathbb{Z}\langle \gamma_1 \rangle \oplus \cdots \oplus \mathbb{Z}\langle \gamma_k \rangle \oplus \text{finite}$ and $H_2(M,\mathbb{Z}) = \mathbb{Z}\langle S_1 \rangle \oplus \cdots \oplus \mathbb{Z}\langle S_k \rangle \oplus \text{finite}$.

Every $S_i$ is represented by a stable, properly embedded minimal surface of least area in its homology class [20 5.1.6 and 5.4.6] [26 10.2] [34, p. 28]. Let $\Sigma = \Sigma_i$ denote such a surface. Schoen’s theorem [33 Theorem 3] proved that the second fundamental form of $\Sigma$ is bounded by some constant independent of $M$ and $\Sigma$. Using the Gauss equations for the curvature of submanifolds and the vanishing of the mean curvature of $\Sigma$, Schoen’s bound implies that the curvature $K_\gamma$ of $\Sigma$ with respect to the induced metric $g$ is bounded between $[-C, -1]$ for some constant $C \geq 1$ independent of $M, \Sigma$.

With the help of these curvature bounds, we construct a triangulation of $(\Sigma, g)$ of bounded geometry.

**Lemma 4.12.** Let $M$ be a compact hyperbolic 3-manifold. Then there exists $\mu_M > 0$ depending only on the injectivity radius of $M$ so that for every compact embedded stable minimal surface $S$ in $M$ the injectivity radius of $S$ is greater than or equal to $\mu_M$. 

20
Proof. By [33, Theorem 3] (See also [31, Corollary 11]), there exists $\mu_1 > 1$ depending only on the injectivity radius of $M$ so that the second fundamental form $\sigma$ of $S$ satisfies $|\sigma| < \mu_1$. Let $\gamma$ be a closed geodesic in $S$, parameterized by arclength. Then

$$\nabla^M_{\dot{\gamma}} \dot{\gamma} = a(t) \nu(t),$$

for some scalar function $a$ and unit normal vector field $\nu$. Because $\gamma$ is parametrized by arclength,

$$0 = \frac{d^2}{dt^2}|\dot{\gamma}|^2,$$

and

$$0 = |\nabla^M_{\dot{\gamma}} \dot{\gamma}|^2 + \langle (\nabla^M_{\dot{\gamma}})^2 \dot{\gamma}, \dot{\gamma} \rangle = a^2 - a \sigma(\dot{\gamma}, \dot{\gamma}).$$

Hence

$$|\nabla^M_{\dot{\gamma}} \dot{\gamma}| \leq \mu_1. \tag{4.13}$$

In a geodesic ball $B_{\text{inj}(M)} \subset M$ centered at $\gamma(0)$, $\lim_{t \to 0} \frac{d}{dt}(\dot{\gamma}(t)) = 1$. Thus,

$$r(\gamma(t)) = \int_0^t \left\langle \dot{\gamma}(s), \frac{\partial}{\partial r} \right\rangle ds = \int_0^t \left( 1 + \int_0^s \frac{d}{du} \left\langle \dot{\gamma}(u), \frac{\partial}{\partial r} \right\rangle du \right) ds$$

$$= t + \int_0^t \int_0^s \left\langle \nabla^M_{\dot{\gamma}} \dot{\gamma}(u), \frac{\partial}{\partial r} \right\rangle dus + \int_0^t \int_0^s \left\langle \dot{\gamma}(u), \nabla^N_{\dot{\gamma}} \frac{\partial}{\partial r} \right\rangle dus$$

$$\geq t - \mu_1 \frac{t^2}{2}. \tag{4.14}$$

Hence, $r$ achieves a max greater than or equal to $\min\{\frac{\text{diam}}{\mu_1}, \text{inj}(M)\}$. The lower bound on $r$ gives a lower bound on the length of $\gamma$ and inj$(S)$. The injectivity radius of $S$ is therefore greater than or equal to $\min\{\frac{\text{diam}}{\mu_1}, \text{inj}(M)\}$. \qed

Let $V_r(-K)$ denote the volume of a geodesic ball of radius $r$ in the hyperbolic space of constant curvature $-K$.

**Proposition 4.15** (triangulations of $\Sigma$ with bounded geometry). Suppose $\Sigma$ is a stable, properly embedded, minimal surface in hyperbolic 3-manifold $M$. Suppose $\Sigma$ with its induced metric $g$ has curvature $K_g \in [-C, -1]$. Let $\mu_M$ denote the injectivity bound of Lemma 4.12. There is a triangulation of $(\Sigma, g)$ by at most

$$\frac{\text{vol}_g(\Sigma)}{\text{V}_M(-C)} \cdot \frac{\text{V}_{\mu_M}(-1)}{\text{V}_M(-1)}$$

triangles such that every triangle $T$ contains a vertex with distance at most $\frac{2\mu_M}{3}$ from all other points of $T$.

**Proof.** Fix $\delta < \frac{2\mu_M}{3}$. Geodesic balls on $(\Sigma, g)$ of radius $5\delta/2$ are embedded. Let $\mathcal{P}$ be a maximal subset of $(\Sigma, g_0)$ for which the pairwise distances between all points of $\mathcal{P}$ are at least $\delta$. For $p \in \mathcal{P}$, define the Dirichlet polygons

$$D_p := \{ x \in \Sigma : \text{dist}_g(x, p) \leq \text{dist}_g(x, p') \text{ for all } p' \in \mathcal{P} \setminus \{p\} \}.$$ 

The $\{D_p\}_{p \in \mathcal{P}}$ tile $\Sigma$. For every $q \in \Sigma$, $B_\delta(q)$ contains some $p' \in \mathcal{P}$ by maximality of $\mathcal{P}$. Therefore, if $q$ lies in $D_p$,

$$\text{dist}_g(p, q) \leq \text{dist}_g(p', q) \leq \delta.$$

Therefore,

$$\text{diam}_g(D_p) \leq \delta. \tag{4.16}$$
Every edge on the boundary of $D_\sigma$ is a segment $F_{p,p'}$ of some bisector $C_{p,p'} = \{x \in \Sigma : \text{dist}_{g_0}(x,p) = \text{dist}_{g_0}(x,p')\}$ for $p' \in \mathcal{P}$. Therefore, if $q$ lies on $F_{p,p'}$,

$$\text{dist}_g(p,p') \leq \text{dist}_g(p,q) + \text{dist}_g(q,p') = 2\text{dist}(p,q) \leq 2\delta.$$ 

Thus, the number of faces of $D_\sigma$ is at most the number of $p' \in \mathcal{P}$ contained in $B_{2\delta}(p)$. For $F_{p,p'}$ and $F_{p,p''}$ nonempty, $B_{\frac{\delta}{2}}(p')$ and $B_{\frac{\delta}{2}}(p'')$ are disjoint subsets of $B_{\delta}(p)$. Therefore, there are at most $\frac{\text{vol}(B_{\delta/2}(p))}{V}$ such points $p'$, where $V := \inf_{q \in \Sigma} \text{vol}(B_{\delta/2}(q))$. Because the curvature $K_\sigma \in [-C, -1]$,

$$\# \text{ faces of } D_\sigma \leq \frac{V_{\delta/2}(-C)}{V_{\delta/2}(-1)}, \quad (4.17)$$

Joining $p$ to the vertices of $D_\sigma$ using geodesics, we may thus triangulate $D_\sigma$ by at most $\frac{V_{\delta/2}(-C)}{V_{\delta/2}(-1)}$ triangles of diameter at most $\delta$.

Covering $\mathcal{P}$ by disjoint balls of radius $\delta/2$ gives

$$\# \mathcal{P} \leq \frac{\text{vol}_g(\Sigma)}{V_{\delta/2}(-1)}.$$

Therefore, we may triangulate $(\Sigma, g)$ by at most

$$\frac{\text{vol}_g(\Sigma)}{V_{\delta/2}(-1)} \cdot \frac{V_{\delta/2}(-C)}{V_{\delta/2}(-1)}$$

triangles of diameter at most $\delta$.

\[\square\]

**Proposition 4.18.** Suppose $\Sigma$ is a stable, properly embedded, minimal surface in the closed hyperbolic 3-manifold $M$. Suppose $\Sigma$ with its induced metric $g$ has curvature $K_\sigma \in [-C, -1]$. Let $\mu_M$ denote the injectivity radius lower bound of Lemma 4.12. Let $\gamma$ be a geodesic in $M$ intersecting $\Sigma$ transversely. Let $\#(\Sigma \cap \gamma)$ denote the absolute value of the topological intersection number of $\Sigma$ and $\gamma$. Then

$$\#(\Sigma \cap \gamma) \leq \frac{\text{vol}_g(\Sigma)}{V_{\mu_M}(-1)} \cdot \frac{V_{\mu_M}(-C)}{V_{\mu_M}(-1)} \cdot \frac{\ell(\gamma)}{6 \mu_M / \delta}.$$ 

**Proof.** By Proposition 4.15, there is a triangulation of $(\Sigma, g)$ having at most $\frac{\text{vol}_g(\Sigma)}{V_{\mu_M}(-1)} \cdot \frac{V_{\mu_M}(-C)}{V_{\mu_M}(-1)}$ triangles $T$, each containing a vertex $p_T$ of distance at most $\frac{2\mu_M}{\delta}$ from all other points of the triangle. Apply the “straightening map” $\sigma_\ell$ [39] §11.6 to $\Sigma$, the linear homotopy deforming every triangle $T$ from the above triangulation to a geodesic triangle in $M$ with the same vertices. Every geodesic triangle $T' = \sigma_1(T)$ contains a vertex $p_{T'}$ of distance at most $\frac{2\mu_M}{\delta}$ from all other points of $T'$. Because $\sigma_1(\Sigma)$ and $\Sigma$ are homotopic,

$$\#(\Sigma \cap \gamma) = \#(\sigma_1(\Sigma) \cap \gamma). \quad (4.19)$$

We bound the right side of (4.19) by the cardinality of $\sigma_1(\Sigma) \cap \gamma$.

Let $p$ be a point at which $\gamma$ intersects $T' = \sigma_1(T)$. The geodesic ball $B_{\mu_M - \epsilon}(p_T')$ is embedded in $M$. The distance from $p$ to the boundary of the ball $B_{\mu_M - \epsilon}(p)$ is at least $\mu_M - \epsilon - \frac{2\mu_M}{\delta} = \frac{3\mu_M - 2\epsilon}{\delta}$. The geodesic enters the ball, intersects $T'$ at $p$, then exits the ball without intersecting any other points of $T'$. Each component of $\gamma \cap B_{\mu_M - \epsilon}(p_T')$ which
intersects \( T' \) therefore has length greater than or equal to \( 2(\frac{3\mu M}{\delta} - \epsilon) = \frac{6\mu M}{\delta} - 2\epsilon \), and contains only one intersection point with \( T' \). Therefore,

\[
\#(T' \cap \gamma) \leq \frac{\ell(\gamma)}{6\mu M/5}.
\]

Summing (4.20) over all cells of the cellulation proves the desired result. \( \Box \)

**Lemma 4.21.** Let \( M \) be a closed 3-manifold. Suppose \( H_1(M, \mathbb{Z}) = \mathbb{Z}\langle \gamma_1 \rangle \oplus \cdots \oplus \mathbb{Z}\langle \gamma_k \rangle \oplus \text{finite} \) and \( H_2(M, \mathbb{Z}) = \mathbb{Z}\langle S_1 \rangle \oplus \cdots \oplus \mathbb{Z}\langle S_k \rangle \oplus \text{finite} \). Then some multiple of \( \gamma - \sum_{i=1}^{k} n_i(\gamma_i) \gamma_i \) bounds, where

\[
n_r = \frac{\det(A_r)}{\det(S_i \cap \gamma_j)} = \frac{\det(A_r)}{\pm 1},
\]

where \( A_r \) is the intersection matrix \( A = (S_i \cap \gamma_j) \) with \( r \)th column replaced by the column \( (S_i \cap \gamma) \).

**Proof.** This follows by Cramer’s rule, upon observing that \( \vec{x} = \begin{pmatrix} n_1(\gamma) \\ \vdots \\ n_k(\gamma) \end{pmatrix} \) is the unique solution to the system of equations

\[Ax = (S_i \cap \gamma).\]

Also, \( \det(A) = \pm 1 \) because \( A \) is an invertible integer matrix. \( \Box \)

**Theorem 4.22.** Let \( M_0 \) be a closed hyperbolic 3-manifold. Let \( M \to M_0 \) be a finite cover. Let \( \gamma \in \Gamma = \pi_1(M) \). Let \( H_2(M, \mathbb{Z}) = \mathbb{Z}\langle S_1 \rangle \oplus \cdots \oplus \mathbb{Z}\langle S_k \rangle \oplus \text{finite} \). Suppose that the minimal area surface \( \Sigma_i \) in the homology class of \( S_i \) is at most \( A_i \) for \( i = 1, \ldots, k \). Let

\[
D := \begin{cases} 
\text{diam}(F_0) \cdot \left( \frac{4k_0}{\tau_0} \cdot \text{diam}(M) + 4k_0 + 1 \right) & \text{if } F \text{ is tree-type} \\
2 \cdot \text{diam}(M) & \text{if } F \text{ is Dirichlet}.
\end{cases}
\]

Then there exists \( \gamma_0 \in \Gamma \) satisfying

\[
\ell(\gamma_0) \leq \left( A \cdot D \cdot \sqrt{b_1(M)} \right)^{b_1(M)} \cdot D \cdot b_1(M)
\]

for which some multiple of \( \gamma - \gamma_0 \) bounds.

**Proof.** We can find a basis \( \gamma_i, i = 1, \ldots, k \), for \( H_1(M, \mathbb{Z})/\text{torsion} \) with \( \gamma_i \in P_F \), where \( F \) is either a tree-type or Dirichlet fundamental domain. By Propositions 3.21 and 3.23

\[
\ell(\gamma_i) \leq D \text{ for } i = 1, \ldots, k.
\]

Use Proposition 4.18 to bound the entries of the matrix \( A_r \) obtained by replacing the \( r \)th column of \( A = (S_i \cap \gamma_j) \) with \( (S_i \cap \gamma) \). The determinant is bounded above by the products of the norms of its columns. The result follows. \( \Box \)

### 4.1.2 Regulator-independent upper bound for \( \frac{1}{\Lambda(M)\mu} \) when \( b_1(M) = 1 \)

The next Proposition proves a “regulator-independent” upper bound for \( \frac{1}{\Lambda(M)\mu} \) for closed hyperbolic 3-manifolds \( M \) satisfying \( b_1(M) = 1 \) in terms of the stable area of loops which project trivially to \( H_1(M, \mathbb{Q}) \).

Let \( \gamma \to [\gamma] \) denote the quotient map from \( \pi_1(M) \) (or even closed curves) to \( H_1(M, \mathbb{Q}) \).
Proposition 4.23. Let $M_0$ be a closed hyperbolic 3-manifold. Let $M \to M_0$ be a finite cover. Suppose $H_1(M, \mathbb{Z}) = \mathbb{Z} \langle \gamma_0 \rangle \oplus \text{finite}$, where $\gamma_0 \in P_F$, for $F$ either a tree-type or Dirichlet fundamental domain of $M$. Let $H_2(M, \mathbb{Z}) = \mathbb{Z} \langle S_0 \rangle \oplus \text{finite}$. Fix a basepoint $q_0 \in M$.

Let $\lambda = \lambda^1(M)_\gamma$. Fix $\delta > 0$. There is an explicit subset $\mathcal{C}_f \subset \pi_1(M)$ which projects trivially to $H_1(M, \mathbb{Q})$ relative to which the following upper bound on $\frac{1}{h}$ holds:

\[
\frac{1}{\sqrt{\lambda}} \cdot (1 - E) \leq 3\pi \cdot C(\lambda)^2 \cdot V + C(\lambda) \cdot \frac{1}{2} \cdot \sqrt{\text{vol}(M)} + C(\lambda)^2 \cdot \left(2\sqrt{2}D^2 \cdot \text{vol}(M)^{\delta + 1/2} \cdot \sup_{\gamma \in \mathcal{C}_f} \frac{s\text{Area}(\alpha_{\gamma q_0 q_0})}{\ell(\gamma)} + 5\pi \right). \tag{4.24}
\]

In (4.24), $V$ and $D$ are the quantities from Theorem 4.11 and they satisfy the upper bounds therein. The number $E$ satisfies $0 \leq E \leq C(\lambda) \cdot \text{vol}(M)^{-\delta}$.

**Proof.** Let $f$ be coexact and satisfy $\Delta f = \lambda f$. If

\[
\left| \int_{\alpha_{\gamma_j} \gamma \gamma_j} f \right| \leq \text{vol}(M)^{-\delta} \|f\|_2
\]

for every $\gamma \in P_F$ which is non-trivial in $H_1(M, \mathbb{Z})$/torsion, then we subtract this contribution from both sides of (3.14). Then (4.24) follows directly from Corollary 3.13 with $\mathcal{C}_f = \{ \gamma \in \Gamma : |\gamma| = 0 \text{ and } \ell(\gamma) \leq D \}$.

Otherwise, choose $\gamma_{\text{big period}} \in P_F$ from among those $\gamma$ occurring on the right side of (3.14) which is non-zero in $H_1(M, \mathbb{Q})$ and which satisfies

\[
\left| \int_{\alpha_{\gamma_0 \gamma_{\text{big period}} q_0}} f \right| \geq \text{vol}(M)^{-\delta} \|f\|_2 \tag{4.25}
\]

Let $h$ be a harmonic 1-form satisfying

\[
\int_{\alpha_{\gamma_0 \gamma_{\text{big period}} q_0}} f = \int_{\gamma_{\text{big period}}} h. \tag{4.26}
\]

Let $[\gamma_{\text{big period}}] = m[\gamma_0]$. Let $A = \frac{1}{\text{reg}_1(M)}$. By definition of the regulator on 1-forms,

\[
\|h\|_2 = \left| \int_{\gamma_0} h \right| A = \frac{1}{m} \left| \int_{\gamma_{\text{big period}}} h \right| \cdot A \geq \frac{A}{m} \cdot \text{vol}(M)^{-\delta} \cdot \|f\|_2. \tag{4.27}
\]

Let $\gamma \in P_F$ be non-zero in $H_1(M, \mathbb{Q})$. Let $n_0 = n_0(\gamma) = S_0 \cap \gamma$ be the unique integer for which $\gamma - n_0 \gamma = 0 \in H_1(M, \mathbb{Q})$. Then $m\gamma - n_0 \gamma_{\text{big period}} = 0 \in H_1(M, \mathbb{Q})$ too. Define the following three broken geodesic polygons:

\[
(\Delta) \quad \text{The broken geodesic } \alpha_{\gamma_0 \gamma_{\text{big period}} q_0} + \alpha_{\gamma_{\text{big period}} \gamma_{\text{big period}} q_0} - n_0 - n_0 - n_0 - n_0 \text{ is the oriented boundary of a geodesic triangle } \Delta \text{ of area at most } \pi.
\]
(Q) The broken geodesic $\alpha_{\gamma^m q_0, \gamma_{\text{big period}}} - n_0 + \alpha_{\gamma, -n_0} + \alpha_{\gamma^m q_0, \gamma_{\text{big period}}} - n_0 + \alpha_{\gamma, -n_0} + \alpha_{\gamma^m q_0, \gamma_{\text{big period}}} - n_0 + \alpha_{\gamma, -n_0}$ forms the oriented boundary of a broken geodesic quadrilateral $Q$, of area at most $2\pi$, whose projection to $M$ has boundary $\alpha_{\gamma^m q_0, \gamma_{\text{big period}}} - n_0 + \alpha_{\gamma, -n_0} + \alpha_{\gamma^m q_0, \gamma_{\text{big period}}} - n_0 + \alpha_{\gamma, -n_0}$.

(Q') Let $\gamma = \gamma_j$ and let $q_j \in \Sigma_j$ be the reference point from Corollary 3.13. The broken geodesic $\alpha_{\gamma_j, \gamma q_0} - \alpha_{\gamma_j, \gamma q_0} + \alpha_{\gamma q_0, \gamma q_0} + \alpha_{\gamma q_0, \gamma q_0}$ forms the oriented boundary of a broken geodesic quadrilateral $Q'$ of area at most $2\pi$, whose projection to $M$ has boundary $\alpha_{\gamma_j, \gamma q_0} - \alpha_{\gamma q_0, \gamma q_0}$.

By three applications of Stokes,

$$\left| \int_{\alpha_{\gamma_j, \gamma q_0}} (f-h) \right| \leq \left| \int_{\alpha_{\gamma_j, \gamma q_0}} (f-h) \right| + \int_{\partial Q} d(f-h)$$

$$= \frac{1}{m} \left( \int_{\alpha_{\gamma_j, \gamma q_0}} (f-h) + \int_{\partial Q} d(f-h) \right)$$

$$\leq \frac{1}{m} \left( \int_{\alpha_{\gamma_j, \gamma q_0}} (f-h) + \int_{\alpha_{\gamma_j, \gamma q_0}} (f-h) + \int_{\partial Q} d(f-h) \right) + \int_{\partial Q} d(f-h)$$

$$\leq \frac{1}{m} \left( \int_{\alpha_{\gamma_j, \gamma q_0}} (f-h) + n_0 \cdot \int_{\alpha_{\gamma_j, \gamma q_0}} (f-h) \right) + ||df||_{\infty} \cdot \left( \frac{\text{area}(Q)}{m} + \frac{\text{area}(\Delta)}{m} + \text{area}(Q') \right)$$

$$= \frac{1}{m} \left| \int_{\alpha_{\gamma_j, \gamma q_0}} (f-h) \right| + ||df||_{\infty} \cdot \left( \frac{\text{area}(Q)}{m} + \frac{\text{area}(\Delta)}{m} + \text{area}(Q') \right)$$

$$\leq \frac{1}{m} \left| \int_{\alpha_{\gamma_j, \gamma q_0}} (f-h) \right| + ||df||_{\infty} \cdot 5\pi$$

$$\leq \frac{1}{m} ||df||_{\infty} \cdot \text{sArea}(\alpha_{\gamma^m q_0, \gamma_{\text{big period}}} - n_0) + ||df||_{\infty} \cdot 5\pi$$

$$\leq \sqrt{\lambda} \cdot ||f||_{2} \cdot C(\lambda) \cdot \left( \frac{\ell(\gamma_{\text{big period}}, \gamma^m)}{m} \cdot \text{sArea}(\alpha_{\gamma^m q_0, \gamma_{\text{big period}}} - n_0) \right) \cdot \ell(\gamma_{\text{big period}}, \gamma^m) + 5\pi \right). \quad (4.28)$$

By Proposition 4.18

$$\frac{\ell(\gamma_{\text{big period}}, \gamma^m)}{m} \leq \frac{1}{m} \cdot (n \cdot \ell(\gamma) + \#(S_n) \cdot \ell(\gamma_{\text{big period}}))$$

$$= \ell(\gamma) + \frac{\#(S_n) \cap \gamma}{m} \cdot \ell(\gamma_{\text{big period}})$$

$$\leq \ell(\gamma) + \frac{\inf_{S} (\text{area}(S)) \cdot \ell(\gamma) \cdot V_{\text{M}}}{m V_{\text{M}}(-1)^2 \mu_M} \cdot \ell(\gamma_{\text{big period}}), \quad (4.29)$$
where $S$ runs over all stable, minimal surfaces representing the homology class $[S_0]$.  

Let $\Sigma$ be a stable minimal surface representing $[S_0]$. Let $\gamma_0^\vee \in H^1(M, \mathbb{R})$ be Poincaré dual to $[S_0]$; $\gamma_0^\vee$ may be viewed as a linear map $H_1(M, \mathbb{R}) \to \mathbb{R}$ and is uniquely determined by the condition $\gamma_0^\vee(\gamma_0) = 1$. Then

$$
\text{area}(\Sigma) = \inf\{||\alpha||_{L^1(M)} : \alpha \in \Omega^1(M) \text{ represents } \gamma_0^\vee\}, \quad \text{by [6, Lemma 3.1]}
$$

By Propositions 3.21 and 3.23,

$$
S_d\text{ual to } [\gamma_0^\vee] \quad \text{by the condition } \gamma_0^\vee(\lambda) = 1
$$

Combining (4.28), (4.32), and (4.27) implies that

Let $\Sigma$ be a stable minimal surface representing $[S_0]$. Let $\gamma_0^\vee \in H^1(M, \mathbb{R})$ be Poincaré dual to $[S_0]$; $\gamma_0^\vee$ may be viewed as a linear map $H_1(M, \mathbb{R}) \to \mathbb{R}$ and is uniquely determined by the condition $\gamma_0^\vee(\gamma_0) = 1$. Then

$$
\text{area}(\Sigma) = \inf\{||\alpha||_{L^1(M)} : \alpha \in \Omega^1(M) \text{ represents } \gamma_0^\vee\}, \quad \text{by [6, Lemma 3.1]}
$$

By Propositions 3.21 and 3.23,

$$
\ell(\gamma), \ell(\gamma_{\text{big period}}) \leq D := \begin{cases} \text{diam}(F_0) \cdot \left(\frac{4k_0}{\ell_0} \cdot \text{diam}(M) + 4k_0 + 1\right) & \text{if } F \text{ is tree-type} \\ 2 \cdot \text{diam}(M) & \text{if } F \text{ is Dirichlet.} \end{cases}
$$

Inserting (4.30) and (4.31) into (4.29) yields

$$
\ell(\gamma) \leq \ell(\gamma_{\text{big period}}) \leq \ell(\gamma) + \frac{A \cdot \sqrt{\text{vol}(M)} V^{\mathbb{R}} \cdot (-C) \cdot \ell(\gamma)}{m V^{\mathbb{R}} (-1)^2 \mu_M} \cdot \ell(\gamma_{\text{big period}})
$$

Combining (4.28), (4.32), and (4.27) implies that

$$
\int_{\alpha_{qj} \cdot \gamma_{qj}} (f - h) \bigg|_{\text{area}(\alpha_{qj} \cdot \gamma_{qj})} = \frac{1}{(||f||_2 + ||h||_2)^{1/2}} \int_{\alpha_{qj} \cdot \gamma_{qj}} (f - h)
$$

$$
\leq \sqrt{A} \cdot C(\lambda) \cdot \frac{||f||_2}{(||f||_2 + ||h||_2)^{1/2}} \cdot \frac{\ell(\gamma_{\text{big period}} \cdot \gamma^m)}{m} \cdot \frac{\text{sArea}(\alpha_{qj} \cdot \gamma_{qj})}{\ell(\gamma_{\text{big period}} \cdot \gamma^m)} + 5\pi
$$

$$
\leq \sqrt{A} \cdot C(\lambda) \cdot \frac{||f||_2}{(||f||_2 + ||h||_2)^{1/2}} \cdot \frac{D \cdot \ell(\gamma_{\text{big period}} \cdot \gamma^m)}{m} \cdot \frac{\text{sArea}(\alpha_{qj} \cdot \gamma_{qj})}{\ell(\gamma_{\text{big period}} \cdot \gamma^m)} + 5\pi
$$
Using this final inequality to bound the period integrals from Corollary 3.13, the Proposition follows for
\[ C_f = \{ \gamma \in \Gamma : \ell(\gamma) \leq D, \gamma \text{ projects trivially to } H_1(M, \mathbb{Q}) \} \]
\[ \bigcup \{ \gamma_{\text{big period}}^m(\gamma) : \ell(\gamma) \leq D, \gamma \text{ projects non-trivially to } H_1(M, \mathbb{Q}) \}. \]

\[ \square \]

5 Comparing combinatorial and Riemannian $L^p$-norms on the Whitney complex

5.0.3 Notation and setup

Let $M_0$ be a closed hyperbolic $n$-manifold and $M \xrightarrow{\pi} M_0$ an arbitrary finite cover. Let $K_0$ be a triangulation of $M_0$. We define an integer valued distance function on the $n$-simplices by defining for $\sigma \neq \tau$, $d(\sigma, \tau) = 1$ if $\sigma \cap \tau \neq \emptyset$. The triangle inequality yields a unique minimal integer valued extension. For every top degree simplex $\sigma$, let $B_r(\sigma)$ denote those simplices at distance at most $r$ from $\sigma$. Assume that $K_0$ is fine enough so that $B_2(\sigma)$ is contained in an embedded geodesic ball for every top degree simplex $\sigma$.

Let $C^q(M; K)$ denote the space of real-valued cochains of the triangulation $K$. Let $C_q(M; K)$ denote the space of real chains. We denote cochains by Greek letters $\alpha, \beta, \ldots$ and chains by Roman letters $a, b, \ldots$.

Let $K = \pi^{-1}(K_0)$ - the pulled-back triangulation of $M$. For every $q$-cell $c \in K$, let $1_c$ denote the dual cochain. Let $W$ denote the Whitney map [29]

\[ W : C^q(M; K) \rightarrow \Omega^q(M). \]

By construction, for each cell $c$,

\[ \int_c W(1_c) = 1. \quad (5.1) \]

Hence, the Whitney map respects the duality between chains and cochains.

5.0.4 Norms on cochains

The cochain spaces $C^q(M; K)$ have two natural families of norms:

\[ \| \gamma \|_{p,M} := \| W(\gamma) \|_{p,M} \]
\[ \left\| \sum_i a_i 1_{c_i} \right\|_{p,\text{comb}} := \left( \sum_i |a_i|^p \right)^{1/p} \]

Above, $\| \cdot \|_{p,M}$ denotes the Riemannian $L^p$ norm on $\Omega^q(M)$.

Because all norms are equivalent on finite dimensional vector spaces, for $1 \leq p, m \leq \infty$, there exists $A_{p,m,M_0}$ so that

\[ A_{p,m,M_0}^{-1} \| \gamma \|_{m,M_0} \leq \| \gamma \|_{p,\text{comb}} \leq A_{p,m,M_0} \| \gamma \|_{m,M_0} \quad \forall \gamma \in C^q(M_0; K_0). \quad (5.2) \]
Definition 5.4. The combinatorial support of a cochain $\sigma$ is the $\mathbb{E}$xpected $V$ with $\|\cdot\|$.

Lemma 5.5. Combinatorial localization is bounded.

5.1 Comparing the Riemannian $L^2$ and $L^\infty$ norms on the Whitney complex

Definition 5.4. The combinatorial support of a cochain $c = \sum_i a_i 1_{c_i}$ is $\cup_{a_i \neq 0} c_i$.

The Whitney map satisfies the property that if $c_j$ is a $q$ cell contained in the $n$ cell $\sigma$, then the support of $W(1_{c_j})$ is contained in $B_1(\sigma)$. The following lemma shows that combinatorial localization is bounded.

Lemma 5.5. Let $\sigma$ be a top degree cell of $K$. Define the projection

$$P_{\text{comb}} : (C^q(M;K), \|\cdot\|_{2,M}) \rightarrow (C^q(M;K), \|\cdot\|_{2,M})$$

$$\sum_i x_i 1_{c_i} \mapsto \sum_{i: c_i \cap \sigma \neq \emptyset} x_i 1_{c_i},$$

which is orthogonal with respect to the combinatorial-$L^2$ inner product. There exists $D_{M_0} > 0$ (depending only on $M_0$) such that $\|P_{\text{comb}}\| < D_{M_0}$.

Proof.

$\|P_{\text{comb}}\| = \sup_{0 \neq z} \frac{\|W(P_{\text{comb}}(z))\|_{2,M}}{\|W(z)\|_{2,M}}.$

Write $z = z_1 + z_2 + z_3$, where $z_1 = P_{\text{comb}}(z)$, $z_2$ has combinatorial support in $B_2(\sigma) \setminus B_1(\sigma)$, and $z_3$ has combinatorial support in $B_2(\sigma)^c$. Then

$$\|W(z)\|_{2,M} = \|W(z_1) + \chi_{B_2(\sigma)}W(z_2)\|_{2,M}^2 + \|W(z_3) + (1 - \chi_{B_2(\sigma)})W(z_2)\|_{2,M}^2.$$
where $\chi_A$ denotes the characteristic function of $A$. Hence

\[
\|W(z)\|_{2,M}^2 \geq \inf \{ \|W(z_1) + \chi_{B_2(\sigma)} W(z_2)\|_{2,M}^2 : z_2 \text{ has combinatorial support in } B_2(\sigma) \setminus B_1(\sigma) \}.
\]

The right hand side of the preceding inequality is nonzero for $z_1 \neq 0$ because $W(z_1)$ and $\chi_{B_2(\sigma)} W(z_2)$ are linearly independent. Hence, it defines a new norm $\| \cdot \|_{\text{quot}}$ (and associated inner product structure) on the cochains with combinatorial support in $B_1(\sigma)$. Since the vector space is finite dimensional, there exists $C > 0$ such that

\[
\|W(z_1)\|_{2,M} \leq C \|z_1\|_{\text{quot}}.
\]

The assumption that $B_2(\sigma)$ is contained in an embedded geodesic ball implies that $\frac{1}{C} := D_{M_0}$ depends only on $K_0$. Thus,

\[
\|P_{\text{comb}}\| \leq \sup_{0 \neq P_{\text{comb}}(z)} \frac{\|W(P_{\text{comb}}(z))\|_{2,M}}{\|P_{\text{comb}}(z)\|_{\text{quot}}} \leq D_{M_0}.
\]

(5.7)

**Proposition 5.8.** There exists $c_{M_0}, C_{M_0} > 0$ such that for all cochains $\gamma$,

\[
\|\gamma\|_{\infty,M} \leq c_{M_0} \|\gamma\|_{\infty,\text{comb}}.
\]

(5.9)

\[
\|\gamma\|_{\infty,M} \leq C_{M_0} \|\gamma\|_{2,M}.
\]

(5.10)

**Proof.** Let $\gamma = \sum_i a_i 1_{c_i} \in C^q(M; K)$ be a cochain. Suppose that $|W(\gamma)|$ attains its sup at $p \in s$, for some $n$-simplex $s$. Then

\[
\|\gamma\|_{\infty,M} = \left\| \sum_{i : c_i \cap s \neq \emptyset} a_i W(1_{c_i}) \right\|_{\infty,M}.
\]

Hence

\[
\|\gamma\|_{\infty,M} \leq c_{M_0} \|\gamma\|_{\infty,\text{comb}},
\]

(5.11)

with

\[
c_{M_0} := \sup_s \left\| \sum_{i : c_i \cap s \neq \emptyset} W(1_{c_i}) \right\|_{\infty,M}.
\]

(5.12)

By our assumption that $B_2(\pi(s))$ is embedded,

\[
\left\| \sum_{i : c_i \cap s \neq \emptyset} a_i W(1_{c_i}) \right\|_{\infty,M} = \left\| \sum_{i : \pi(c_i) \cap \pi(s) \neq \emptyset} a_i W(1_{\pi(c_i)}) \right\|_{\infty,M_0}
\]

\[
\leq A_{\infty,2,M_0} \left\| \sum_{i : \pi(c_i) \cap \pi(s) \neq \emptyset} a_i W(1_{\pi(c_i)}) \right\|_{2,M_0}
\]

\[
= A_{\infty,2,M_0} \left\| \sum_{i : c_i \cap s \neq \emptyset} a_i W(1_{c_i}) \right\|_{2,M}
\]

\[
\leq D_{M_0} A_{\infty,2,M_0} \|\gamma\|_{2,M}.
\]

(5.13)

(5.14)

Here we have used equation (5.2) and Lemma 5.5. \qed
5.2 Comparing the Riemannian and combinatorial \(L^p\)-norms on the Whitney complex

**Proposition 5.15.** There are inequalities

\[
A_{\infty, M_0}^{-1} ||\gamma||_{\infty, \text{comb}} \leq ||\gamma||_{\infty, M} \leq c_{M_0} ||\gamma||_{\infty, \text{comb}} \text{ for all } \gamma \in C^q(M; K)
\]

(5.16)

and

\[
c_{M_0}^{-1} ||c||_{1, \text{comb}} \leq ||c||_{1, M} \leq A_{\infty, M_0} ||c||_{1, \text{comb}} \text{ for all } c \in C_q(M; K).
\]

(5.17)

Above, \(c_{M_0}\) denotes the constant from Proposition 5.8.

**Proof.** Let \(\gamma = \sum a_i c_i \in C^q(M, K)\). Arguing as in the previous subsection:

\[
||\gamma||_{\infty, M} = \max_{\text{top degree cells } s} \left| \sum_{i : c_i \cap s \neq \emptyset} a_i \ell(1_{c_i}) \right|_{\infty, M}
\]

\[
= \max_{\text{top degree cells } s} \left| \sum_{i : \pi(c_i) \cap \pi(s) \neq \emptyset} a_i \ell(1_{\pi(c_i)}) \right|_{\infty, M_0}
\]

\[
\geq A_{\infty, M_0}^{-1} \max_{\text{top degree cells } s} \left| \sum_{i : \pi(c_i) \cap \pi(s) \neq \emptyset} a_i \right|
\]

\[
= A_{\infty, M_0}^{-1} ||\gamma||_{\infty, \text{comb}}.
\]

Hence

\[
A_{\infty, M_0}^{-1} ||\gamma||_{\infty, \text{comb}} \leq ||\gamma||_{\infty, M} \leq c_{M_0} ||\gamma||_{\infty, \text{comb}}.
\]

(5.18)

The statement relating combinatorial and de Rham \(L^1\)-norms on chains follows by duality. \(\square\)

6 The Whitney 2-chain Laplacian spectral gap controls stable commutator length

Fix a triangulation \(K_0\) of a closed hyperbolic \(n\)-manifold \(M_0\). Let \(M \xrightarrow{p} M_0\) be a finite cover of \(M_0\) with triangulation \(K = p^{-1}(K_0)\).

**Proposition 6.1.** Let \(f \in C_1(M; K)\) be an integral 1-chain, some multiple of which bounds. Let \(\ell(f)\) denote the Riemannian length of \(f\). Then there exists \(B_{M_0} > 0\), depending only on \(M_0\), an integer \(m\), and a surface \(S_m \in C_2(M; K)\) bounding \(nf\) satisfying

\[
\left( \frac{|\chi(S_m)|/m}{\ell(f)} \right)^2 \leq \frac{B_{M_0} \text{vol}(M)^2}{\lambda_1^{(f)} \text{Whitney}(M)}.
\]

**Proof.** Equip the chain groups \(C_q(M; K)\) with the norm \(|| \cdot ||_{2,M}\) dual to the norm \(|| \cdot ||_{2,M}\) on \(C^q(M; K)\) induced from the \(L^2\)-norm on \(\Omega^q(M)\) via the Whitney embedding.

Let \(\partial\) denote the boundary map on \(C_*(M; K)\). Let \(g\) be a real 2-chain satisfying \(\partial g = f\) and \(g \perp \ker(\partial)\). Because the kernel and the image of \(\partial\) have rational bases (and so the existence of a real solution to \(\partial c = f\) implies the existence of a rational solution), we can find a rational 2-chain \(g'\) with \(\partial g' = f\) and \(||g' - g||_{2, M}\) arbitrarily small. So we may assume for any \(\delta > 0\) there exists \(g'\) so that:

\(5\)If \(f = \sum a_i c_i\), its Riemannian length is defined to equal \(\sum |a_i| \ell(c_i)\).
(1') $\partial g' = f$

(2') $\|g'\|_{2,M}^2 \leq \frac{1+\delta}{\lambda_1^{\text{Whitney}}(M) d^*} \|f\|_{2,M}^2$

(3') $g'$ is a rational 2-chain.

Condition (2') can be guaranteed because $\lambda_1(M)_{\text{Whitney},d^*}$ equals the smallest eigenvalue of $\partial^* \partial$ acting on 2-chains perpendicular to $\ker \partial$. Choose an integer $m$ so that

$$mg' = \sum_{j=1}^k a_j c_{2,j}, \text{ for } a_q \in \mathbb{Z} \text{ and } \{c_{2,j}\}_j \text{ the 2-cells of } K. \quad (6.2)$$

The number of faces of $mg'$ equals $\sum_{i=1}^k |a_i| = \|mg'\|_{1,\text{comb}}$. Counting incident pairs $\{\text{vertex} \in \text{face}\}$ and $\{\text{edge} \subset \text{face}\}$, we see that

$$3F \geq V, \quad 3F \geq E,$$

where $V, E, F$ denote the number of vertices, edges, and faces on the surface bounding $mf$. Therefore, the absolute value of the euler characteristic $= \|V + F - E\|$ is at most $\max \{V + F, E\} \leq 4F \leq 4\|mg'\|_{1,\text{comb}}$. Thus, letting $S_m = mg'$,

$$|\chi(S_m)|^2 \leq 16\|mg'\|_{2,\text{comb}}^2 \leq c_{M_0} \cdot 16 \cdot \|mg'\|_{1,M}^2 \leq 16 c_{M_0} \cdot \text{vol}(M) \cdot \|mg'\|_{2,M}^2 \leq m^2 \cdot 16 c_{M_0} \cdot \text{vol}(M) \cdot \frac{1+\delta}{\lambda_1^{\text{Whitney}}(M) d^*} \|f\|_{2,M}^2 \text{ by (2').} \quad (6.3)$$

Also,

$$\|f\|_{2,M} = \sup_{\gamma \in C^1(M;K):\|\gamma\|_{2,M} \leq 1} \left| \int_f W(\gamma) \right| \leq \sup_{\gamma \in C^1(M;K):\|\gamma\|_{2,M} \leq 1} \|\gamma\|_{\infty,M} \text{ length}(f) \leq C_{M_0} \sup_{\gamma \in C^1(M;K):\|\gamma\|_{2,M} \leq 1} \|\gamma\|_{2,M} \text{ length}(f) \text{ by Proposition 5.8} = C_{M_0} \cdot \text{length}(f). \quad (6.4)$$

Together (6.3) and (6.4) yield

$$\left(\frac{|\chi(S_m)|^2}{\text{length}(f)}\right)^2 \leq B_{M_0} \frac{\text{vol}(M)}{\lambda_1^{\text{Whitney}}(M) d^*}. \quad (6.5)$$

where $B_{M_0} = C_{M_0} \cdot 16 c_{M_0} \cdot (1 + \delta)$.

**Remark 6.6.** The same result is true, replacing the Whitney Laplacian with respect to $K$ and $\lambda_1^{\text{Whitney}}(M)_{d^*}$ by the combinatorial Laplacian with respect to $K$ and $\lambda_1^{\text{comb}}(M)_{d^*}$. The proof actually simplifies, because comparing the combinatorial $L^1$ and $L^2$-norms is easier than comparing the combinatorial $L^1$-norm and Riemannian $L^2$-norm.
Theorem 6.7. Let $\gamma \in \pi_1(M)$ have translation length $\ell(\gamma)$. Suppose that some multiple of $\gamma$ bounds. Then

$$\left(\frac{\text{scl}(\gamma)}{\ell(\gamma)}\right)^2 \leq W_{M_0} \cdot \frac{\text{vol}(M) \cdot \text{diam}(M)^2}{\lambda_1(M)_{\text{Whitney,d}}^2}$$

for some constant $W_{M_0}$ depending only on $M_0$.

Proof. Pull back the triangulation $K$ of $M$ to a triangulation $\tilde{K}$ on $\mathbb{H}^n$. Let $p$ be a vertex of $\tilde{K}$ whose distance to the minimum translation set of $\gamma$ is minimal; this choice of $p$ depends on $\gamma$. The distance from $p$ to the minimum translation set of $\gamma$ is at most $\text{diam}(M)$.

Let $\alpha$ be a geodesic segment in $\mathbb{H}^n$ from $p$ to $\gamma p$. Suppose $s_0, \ldots, s_k$ are the top degree simplices of $\tilde{K}$ whose interiors contain $\alpha$ and the above expression should be read in “left to right order”)

(satisfying

\begin{itemize}
  \item every $c_j^i$ is an edge of $s_i$,
  \item no edge is repeated (implying that $j_0, \ldots, j_k \leq \binom{n+1}{2}$),
  \item $c_1^i$ begins at $p$ and $c_{j_k}^k$ ends at $\gamma p$.
\end{itemize}

Let $f \in C^1(\mathbb{H}^n; \tilde{K})$ denote the chain induced by $\alpha_{\text{comb}}$; i.e.

$$f = (c_1^0 + \cdots + c_{j_0}^0 + \cdots + (c_1^k + \cdots + c_{j_k}^k)),$$

and let $\overline{f}$ denote its projection to $M$. By Theorem 6.7 there is an integer $m$, a surface $S_m$ bounding $m\overline{f}$, and a constant $B_{M_0}$ for which

$$\left(\frac{\text{vol}(S_m)}{m}\right)^2 \leq \frac{B_{M_0} \text{vol}(M)}{\lambda_1(M)_{\text{Whitney,d}}} \cdot \left(\frac{\text{length}(\overline{f})}{\ell(\gamma)}\right)^2.$$

The projection $\pi_{\text{comb}}$ of $\alpha$ to $M$ is homotopic to $\gamma$ in $\pi_1(M)$. Therefore,

$$\left(\frac{\text{scl}(\gamma)}{\ell(\gamma)}\right)^2 \leq \left(\frac{\text{scl}(\pi_{\text{comb}})}{\text{length}(\overline{f})}\right)^2 \cdot \left(\frac{\text{length}(\overline{f})}{\ell(\gamma)}\right)^2 \leq \left(\frac{\text{vol}(S_m)}{m}\right)^2 \cdot \left(\frac{\text{length}(\overline{f})}{\ell(\gamma)}\right)^2 \leq \frac{B_{M_0} \text{vol}(M)}{\lambda_1(M)_{\text{Whitney,d}}} \cdot \left(\frac{\text{length}(\overline{f})}{\ell(\gamma)}\right)^2,$$

where passage to the last line follows from Proposition 6.1. The second bullet point above implies that $\text{length}(f)$ is at most $\binom{n+1}{2} \cdot k \cdot e_0$, where $k$ is the combinatorial distance from $s_0$ to $s_k$ in the dual graph to the triangulation $\tilde{K}$ and $e_0$ is the length of the longest edge in $K_0$. By the argument from Lemma 3.20 there are constants $a_0, b_0$, depending only on $M_0$, for which

$$k \leq a_0 \cdot d(p, \gamma p) + b_0.$$

Because the distance from $p$ to the minimum translation set of $\gamma$ is at most $\text{diam}(M)$, the latter inequality implies

$$\text{length}(f) \leq \binom{n+1}{2} \cdot e_0 \leq [a_0 \cdot (2 \text{diam}(M) + \ell(\gamma)) + b_0] \cdot \binom{n+1}{2} \cdot e_0,$$

where $e_0$ denotes the maximum edge length in $K_0$. The result follows.
7 Comparing $\lambda_1^1(M)d^*$ to $\lambda_{1,\text{Whitney}}^1(M)d^*$

Proposition 7.1. Let $M_0$ be a closed hyperbolic manifold with triangulation $K_0$. Let $M \xrightarrow{\pi} M_0$ be an arbitrary finite cover with pullback triangulation $K = \pi^{-1}(M_0)$. Either

$$\lambda_{1,\text{Whitney}}^1(M)d^* \geq \frac{1}{4G_{M_0}^2C_{M_0}^2\text{vol}(M)},$$

or

$$\lambda_1^1(M)d^* \leq 4G_{M_0}^2\text{vol}(M) \cdot \lambda_{1,\text{Whitney}}^1(M)d^*, $$

where $C_{M_0}$ is defined in Proposition 5.8 and $G_{M_0}$ is defined in the proof body.

Remark 7.2. If the first alternative in Proposition 7.1 holds, Theorem 6.7 implies that

$$\text{scl}(\gamma) \leq \ell(\gamma) \leq \sqrt{D_{M_0} \cdot 2G_{M_0}C_{M_0} \cdot \text{vol}(M) \cdot \text{diam}(M)}.$$

In particular, if $b_1(M) = 0$ or $n = 3$ and $b_1(M) = 1$, then Corollary 3.13 and Proposition 4.23 respectively imply that

$$\lambda_1^1(M)d^* \leq \begin{cases} E_{M_0} \cdot \text{vol}(M)^2 \cdot \text{diam}(M)^4 & \text{if } b_1(M) = 0 \\ E_{M_0,\delta} \cdot \text{vol}(M)^{3+2\delta} \cdot \text{diam}(M)^3 & \text{if } n = 3 \text{ and } b_1(M) = 1 \end{cases}.$$

Proof. Let $f \in d_{\text{Whitney}}^*C^2(M; K)$ satisfy

$$\lambda_{1,\text{Whitney}}^1(M)d^* = \frac{||dW(f)||^2_{2,M}}{||W(f)||^2_{2,M}}.$$

There is an orthogonal decomposition in $\Omega^1(M)$

$$W(f) = z + \epsilon,$$

with $\epsilon$ coclosed and $z$ closed. We will show that $\epsilon$ and $W(f)$ have comparable $L^2$-norms. Equip the chain group $C_q(M; K)$ with the norm $|| \cdot ||_{2,M}$ dual to the $L^2$-Whitney norm on $C^q(M; K)$; see §5.0.5 for further discussion.

Because $f \in \text{Im}(d_{\text{Whitney}}^*) = \ker(d)^\perp_{\text{Whitney}} = (\text{annihilator of } \text{Im}(d))^{\perp_{\text{Whitney}}}$, we have

$$||W(f)||_{2,M} = \sup_{||\partial\sigma||_{2,M} = 1} \left| \int_{\partial\sigma} W(f) \right| = \sup_{||\partial\sigma||_{2,M} = 1} \left| \int_{\partial\sigma} \epsilon \right|. \quad (7.3)$$

The inequality

$$|| \cdot ||_{2,M} \leq \text{vol}(M)^{1/2}|| \cdot ||_{\infty,M} \quad \text{on Whitney cochains}$$

implies the dual inequality

$$\text{vol}(M)^{1/2}|| \cdot ||_{2,M} \geq || \cdot ||_{1,M} \quad \text{on Whitney chains}. \quad (7.4)$$

The inequality

$$c_{M_0}^{-1}|| \cdot ||_{\infty,M} \leq || \cdot ||_{\infty,\text{comb}} \quad \text{on Whitney cochains}$$

implies the dual inequality

$$|| \cdot ||_{1,\text{comb}} \leq c_{M_0}|| \cdot ||_{1,M} \quad \text{on Whitney chains}. \quad (7.5)$$
Proposition 8.1. Let $\partial \sigma = \sum a_i c_i$ for cells $c_i$ of the triangulation $K$. Let $e_0$ denote the maximum length among all 1-cells of $K$. Combining (7.3), (7.4), and (7.5) gives

$$||W(f)||_{2,M} \leq ||e||_{\infty,M} \cdot e_0 \cdot \sup_{||\partial \sigma||_{2,M} = 1} |a_i|$$

$$= ||e||_{\infty,M} \cdot e_0 \cdot \sup_{\partial \sigma \neq 0} \frac{||\partial \sigma||_{1,\text{comb}}}{||\partial \sigma||_{2,M}}$$

$$\leq c_{M_0} \cdot ||e||_{\infty,M} \cdot e_0 \cdot \sup_{\partial \sigma \neq 0} \frac{||\partial \sigma||_{1,M}}{||\partial \sigma||_{2,M}}$$

$$\leq c_{M_0} \cdot ||e||_{\infty,M} \cdot e_0 \cdot |\text{vol}(M)|^{1/2}. \quad (7.6)$$

Furthermore, let $|e|$ achieve its supremum at $p \in M$. Then we have for some constant $S_{B_0}$ determined by Garding’s inequality for the elliptic operator $d + d^*$ on $B_0 = B_{4\text{inj}(M_0)}(p)$ and Sobolev constants for $B_0 \subset H^n$,

$$||e||_{\infty,M} \leq S_{B_0} (||e||_{2,M} + ||(d + d^*)e||_{\infty,M})$$

$$= S_{B_0} (||e||_{2,M} + ||dW(f)||_{\infty,M})$$

$$\leq S_{B_0} (||e||_{2,M} + C_{M_0} ||dW(f)||_{2,M}) \quad \text{by Proposition 5.8}$$

$$= S_{B_0} \left(||e||_{2,M} + C_{M_0} \sqrt{\lambda_{1,\text{Whitney}}(M)d^*} \cdot ||W(f)||_{2,M}\right). \quad (7.7)$$

Inserting inequality (7.7) into (7.6) yields

$$\left(1 - G_{M_0} \cdot C_{M_0} \cdot |\text{vol}(M)|^{1/2} \sqrt{\lambda_{1,\text{Whitney}}(M)}\right) ||W(f)||_{2,M} \leq G_{M_0} \cdot |\text{vol}(M)|^{1/2} ||e||_{2,M},$$

$$\quad \text{where } G_{M_0} := c_{M_0} \cdot e_0 \cdot S_{B_0}. \text{ If } \lambda_{1,\text{Whitney}}(M)d^* \leq \frac{1}{4G_{M_0}^2 C_{M_0}^2 |\text{vol}(M)|}, \text{ then}$$

$$||W(f)||_{2,M}^2 \leq 4G_{M_0}^2 |\text{vol}(M)| \cdot ||e||_{2,M}^2.$$ 

Therefore,

$$\lambda_{1}(M)d^* \leq \frac{||dW(f)||_{2,M}^2}{||e||_{2,M}^2}$$

$$= \frac{||dW(f)||_{2,M}^2}{||e||_{2,M}^2}$$

$$\leq 4G_{M_0}^2 |\text{vol}(M)| \cdot \frac{||dW(f)||_{2,M}^2}{||W(f)||_{2,M}^2}$$

$$= 4G_{M_0}^2 |\text{vol}(M)| \cdot \lambda_{1,\text{Whitney}}(M)d^*.$$

\[\square\]

8 Applications

8.1 Naive lower bounds on $\lambda_{1}(M)$

Proposition 8.1. Let $M_0$ be a closed hyperbolic $n$-manifold. Let $M \to M_0$ be an arbitrary finite cover with $b_1(M) = 0$. Then

$$\frac{1}{\lambda_{1}(M)} \leq \exp(H_{M_0}|\text{vol}(M))$$

for some constant $H_{M_0}$ depending only on $M$. 

34
Remark 8.2. By the Cheeger-Müller theorem, under the assumption $b_1(M) = 0$,

$$\limsup_M \frac{\log \frac{1}{\lambda_1(M)}}{\text{vol}(M)} \leq \frac{1}{6\pi},$$

as $M$ varies through any sequence of closed hyperbolic 3-manifold Benjamini-Schramm converging to $\mathbb{H}^3$. However, we are unaware of any upper bound for $\frac{1}{\lambda_1(M)}$ for higher dimensional hyperbolic manifolds in the literature.

Proof. By Lemma A.1

$$\frac{1}{\lambda_1(M)^d} \leq C \cdot \text{diam}(M)^2 \cdot \text{vol}(M)$$

for some constant $C$ depending only on a lower bound for the injectivity radius of $M$. So, we focus our attention on $\lambda_1(M)^d$.

Let $K_0$ be a triangulation of $M_0$. Let $K$ be the pullback triangulation of $M$. Consider the operator

$$A := \partial^*_1 \partial_2 : C_2(K) \to C_2(K).$$

$A$ is a sparse-integer matrix, i.e. every column has a bounded number of entries (upper bound depending only on $M_0$). By Hadamard’s inequality, every $k \times k$ minor has determinant of absolute value at most $\exp(O_{M_0}(k))$. Let $N = \dim C_2(K) \approx M_0 \text{vol}(M)$. If the characteristic polynomial of $A$ equals $x^N + a_{N-1}x^{N-1} + \cdots + a_{k+1}x^{k+1} + a_kx^k$ (where $a_k$ is the last non-zero coefficient), then

$$\sum_{\lambda = \text{non-zero e.value of } A} \frac{1}{\lambda} = \frac{|a_{k+1}|}{|a_k|} \leq |a_{k+1}|$$

because $|a_k|$ is an integer $\geq 1$.

But $a_{k+1}$ is the sum of the $\binom{N}{k+1} \leq 2^N$ principal $(k+1) \times (k+1)$ principal minors of $A$, all of which have absolute value at most $\exp(O_{M_0}(k))$ by our earlier remark. Therefore,

$$\frac{1}{\lambda_1(M)^d} \leq \sum_{\lambda = \text{non-zero e.value of } A} \frac{1}{\lambda} \leq \exp(O_{M_0}\text{vol}(M)).$$

By (the proof of) Theorems 6.1, 6.7, and Remark 6.6, there is an upper bound

$$\frac{scl(\gamma)}{\ell(\gamma)} \ll M_0 \frac{1}{\lambda_1(M)^d} = \exp(O_{M_0}\text{vol}(M)).$$

In particular, by Corollary 3.13 and the diameter bounds from Propositions 3.20 and 3.23 there is an upper bound

$$\frac{1}{\lambda_1(M)^d} = \exp(O_{M_0}\text{vol}(M)).$$

8.2 Improved lower bounds on $\lambda_1(M^n)$ for hyperbolic $n$-manifolds, $n > 3$

Proposition 8.3. Let $M_0$ be a closed hyperbolic $n$-manifold, $n > 3$. Fix a constant $C > 0$. Suppose $M \to M_0$ is an arbitrary finite cover satisfying $\lambda_1(M) \gg \text{vol}(M)^{-C}$. Suppose some multiple of $\gamma \in \pi_1(M)$ bounds. Then

$$\frac{scl(\gamma)}{\ell(\gamma)} \ll M_0 \text{vol}(M)^{1 + \frac{3}{2n}} \cdot \text{diam}(M).$$
Proof. This follows immediately from Theorem 6.7 and Proposition 7.1.

Remark 8.4. The bottom of the 1-form spectrum $\lambda_1^1(\mathbb{H}^n)$ for the Laplacian $\Delta_1$ acting on smooth compactly supported 1-forms on $\mathbb{H}^n$ equals $(\frac{n-1}{2})^2$ for $n \geq 3$ [19, Theorem 1]. In particular, 1-form eigenvalues less than $\lambda_1^1(\mathbb{H}^n)$ are exceptional. There are natural families of closed hyperbolic $n$-manifolds $M$, $n > 3$, such as arithmetic congruence hyperbolic $n$-manifolds, for which for which $\lambda_1^1(M)$ is uniformly bounded below [1]. For such families, we may set $C = 0$ in Proposition 8.3. More generally, it seems plausible to us that if $M_0$ is a closed hyperbolic $n$-manifold, $n > 3$, and $M \to M_0$ is an arbitrary finite cover, then $\lambda_1^1(M) \geq M_0 \cdot \text{vol}(M)^{-C}$, for some constant $C$.

8.3 Lower bounds on $\lambda_1^1(M^3)$ using retractions from hyperbolic $n$-manifolds, $n > 3$

Proposition 8.5. Let $N_0$ be a closed hyperbolic $n$-manifold, $n > 3$. Let $M_0 \subset N_0$ be a totally geodesic submanifold. Suppose $N \to N_0$ is an arbitrary finite cover. Let $M = (a connected component of) \pi^{-1}(M_0).$ Suppose that there is a covering $p : N' \to N$ of degree $d$ for which

- the submanifold $M$ lifts to $N'$
- $N$ retracts onto $M$.

Suppose some integer multiple of $\gamma \in \pi_1(M)$ bounds. Then

$$\frac{\text{scl}(\gamma)}{\ell(\gamma)} \leq N_0 \cdot d^2 \cdot \text{vol}(N) \cdot \text{diam}(N) \cdot \sqrt{\frac{1}{\lambda_1^1(N')^d}}.$$

Remark 8.6. The work of Bergeron-Haglund-Wise [2] produces many interesting examples satisfying the hypotheses of Proposition 8.5. The main theorems of the present paper relate $\text{scl}$ and $\frac{1}{\lambda_1^1}$. Proposition 8.5 punts the difficulty of bounding $\lambda_1^1(M)$ below to that of bounding $\lambda_1^1(N')$ below. This should be regarded as a significant gain, since the 1-form spectrum of $N'$ should be much easier to bound away from 0 than the 1-form spectrum of $M$; see Remark 8.4. In particular, modulo the hope expressed in Remark 8.4 and assuming that $d$ can be taken polynomial in $\text{vol}(N)$, Proposition 8.5 will produce a rich family of examples of $M$ for which $\lambda_1^1(M) \gg \text{vol}(M)^{-C}$ for some constant $C$.

Proof. Let $\gamma \in \pi_1(M) \subset \pi_1(N)$ be as in the proposition statement. Let $N'$ be the covering realizing the retraction onto $M$. By Theorem 6.7 and Proposition 7.1,

$$\frac{\text{scl}_{N'}(\gamma)}{\ell_{N'}(\gamma)} \leq N_0 \cdot \text{vol}(N')^{1/2} \cdot \text{diam}(N') \cdot \sqrt{\frac{1}{\lambda_1^1(\text{Whitney}(N'))^d}} \leq N_0 \cdot \text{vol}(N')^{1/2} \cdot \text{diam}(N') \cdot \text{vol}(N')^{1/2} \cdot \sqrt{\frac{1}{\lambda_1^1(N')^d}} \leq N_0 \cdot d^2 \cdot \text{vol}(N) \cdot \text{diam}(N) \cdot \sqrt{\frac{1}{\lambda_1^1(N')^d}}.$$

Also,

$$\text{scl}_M(\gamma) \leq \text{scl}_{N'}(\gamma) \text{ and } \ell_M(\gamma) = \ell_{N'}(\gamma),$$

the latter because $M$ is geodesically embedded in $N'$ and the former because the retraction $p_* : \pi_1(N') \to \pi_1(M)$ reduces commutator length. The conclusion follows. \qed
Remark 8.7. We emphasize that Proposition 8.5 does not require any cohomology vanishing hypothesis. The two key inputs for Proposition 8.5 are Propositions 6.1 and 7.1. And indeed,

- Proposition 6.1 upper bounds \( \text{scl}_{N^*} (\gamma) \) in terms of \( \frac{1}{\lambda_1(N')_{\text{Whit}}} \) provided some multiple of \( \gamma \in \pi_1(N') \) bounds; no supplementary cohomology vanishing hypothesis is required.

- Proposition 7.1 proves \( \frac{1}{\lambda_1,\text{Whitney}(N^*)_{\text{Whit}}} \ll_{N_0} \frac{1}{\lambda_1,\text{Whitney}(N^*)_{\text{Whit}}} \); no supplementary cohomology vanishing hypothesis is required.

Corollary 8.8. Same notation and hypotheses as Proposition 8.5. Suppose in addition that \( b_1(M) = 0 \).

Then

\[
\frac{1}{\lambda_1(M)} \ll_{M_0} \text{diam}(M)^2 \cdot \left( d^2 \cdot \text{vol}(N) \cdot \text{diam}(N) \right)^2 \cdot \frac{1}{\lambda_1(N^*)}.
\]

Proof. This follows directly from Proposition 8.5 and Corollary 3.13. \( \square \)

A Estimating \( \lambda^0_1(M) \)

In this section we give a weak lower bound for the first nonzero eigenvalue of the Laplacian acting on functions on a hyperbolic \( n \)-manifold. With more work, the bound can be considerably improved, but the easy given bound suffices for our purposes.

Lemma A.1. There exists \( C > 0 \), depending only on the minimum of 1 and the injectivity radius of \( M \), so that

\[
\lambda_1^0 \geq \frac{C}{\text{diam}(M)^2 \cdot \text{vol}(M)}.
\]

Proof. Let \( u \in C^\infty(M) \) with \( \|u\|_{L^2} = 1 \) and \( \Delta u = \lambda_1^0(M)u \). Then \( \|du\|_{L^2} = \lambda_1^0(M) \).

By Proposition 2.2,

\[
\|du\|_{L^\infty} \leq \sqrt[4]{\lambda_1^0} \cdot C \left( n, 1, \frac{\text{inj}(M)}{2}, \lambda \right).
\]

Since \( u \perp_{L^2} 1 \) and has \( L^2 \) norm one, there exist \( p_1, p_2 \in M \) so that \( u(p_1) = 0 \) and \( u(p_2) = \frac{1}{\sqrt{\text{vol}(M)}} \). Then

\[
\frac{1}{\sqrt{\text{vol}(M)}} = |u(p_2)| \leq d(p_1, p_2) \cdot \sqrt[4]{\lambda} \cdot C \left( n, 1, \frac{\text{inj}(M)}{2}, \lambda \right).
\]

Hence

\[
\frac{C \left( n, 1, \frac{\text{inj}(M)}{2}, \lambda \right)^{-2}}{\text{diam}(M)^2 \cdot \text{vol}(M)} \leq \lambda.
\]

Since \( C(n, 1, L, \lambda) \) is a decreasing function of \( L \), the result follows. \( \square \)
References

[1] N. Bergeron and L. Clozel. Spectre automorphe des variétés hyperboliques et applications topologiques. Astérisque No. 303 (2005).

[2] N. Bergeron, F. Haglund, and D. Wise. Hyperplane sections in arithmetic hyperbolic manifolds. J. Lond. Math. Soc. (2) 83 (2011), no. 2, 431-448.

[3] N. Bergeron and A. Venkatesh. The asymptotic growth of torsion homology in arithmetic groups. J. Inst. Math. Jussieu 12 (2013), no. 2, 391-447.

[4] N. Bergeron, M. Sengün, and A. Venkatesh. Torsion homology growth and cycle complexity of arithmetic manifolds. Duke Math. J. 165 (2016), no. 9, 1629-1693.

[5] J. Brock and N. Dunfield. Injectivity radii of hyperbolic integer homology 3-spheres. Geom. Topol. 19 (2015), no. 1, 497-523.

[6] J. Brock and N. Dunfield. Norms on the cohomology of hyperbolic 3-manifolds. arXiv:1510.06292v1. Preprint.

[7] R. Brooks. The first eigenvalue in a tower of coverings. Bull. Amer. Math. Soc. (N.S.) 13 (1985), no. 2, 137-140.

[8] P. Buser. A note on the isoperimetric constant. Ann. Sci. École Norm. Sup. (4) 15 (1982), no. 2, 213-230.

[9] E. Breuillard. Property T, expander graphs, and approximate groups. PCMI 12 Lecture Notes.

[10] F. Calegari and N. Dunfield. Automorphic forms and rational homology 3-spheres. Geom. Topol. 10 (2006), 295-329.

[11] D. Calegari. scl. MSJ Memoirs, 20. Mathematical Society of Japan, Tokyo, 2009.

[12] D. Calegari. Stable commutator length is rational in free groups. J. Amer. Math. Soc. 22 (2009), no. 4, 941-961.

[13] D. Calegari and K. Fujiwara. Stable commutator length in word-hyperbolic groups. Groups Geom. Dyn. 4 (2010), no. 1, 59-90.

[14] D. Calegari and J. Maher. Statistics and compression of scl. Ergodic Theory Dynam. Systems 35 (2015), no. 1, 64-110.

[15] D. Calegari and C. Walker. Random rigidity in the free group. Geom. Topol. 17 (2013), no. 3, 1707-1744.

[16] B. Chao and K. Wu. Spanning trees and optimization problems. Discrete Mathematics and its Applications. Chapman & Hall/CRC, Boca Raton, FL, 2004.

[17] J. Cheeger. A lower bound for the smallest eigenvalue of the Laplacian. Problems in analysis (Papers dedicated to Salomon Bochner, 1969), pp. 195-199. Princeton Univ. Press, Princeton, N. J., 1970.

[18] J. Cheeger. Analytic torsion and the heat equation. Ann. of Math. (2) 109 (1979), no. 2, 259-322.
[19] H. Donnelly. *The differential form spectrum of hyperbolic space*. Manuscripta Math. 33 (1980/81), no. 3-4, 365-385.

[20] H. Federer. *Geometric Measure Theory*. Die Grundlehren der mathematischen Wissenschaften, Band 153 Springer-Verlag New York Inc., New York 1969.

[21] M. Freedman, J. Hass, and P. Scott. *Least area incompressible surfaces in 3-manifolds*. Invent. Math. 71 (1983), no. 3, 609-642.

[22] E. Hebey. *Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities*. Courant Lecture Notes in Mathematics. 5. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1999.

[23] S. Hoory, N. Linial, and A. Wigderson. *Expander graphs and their applications*. Bull. Amer. Math. Soc. (N.S.) 43 (2006), no. 4, 439-561.

[24] A. Horvath. *Hyperbolic plane geometry revisited*. J. Geom. 106 (2015), no. 2, 341-362.

[25] W. Lück. *L2-invariants: Theory and Applications to Geometry and K-theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3, 44. Springer-Verlag, Berlin, 2002.

[26] F. Morgan. *Geometric Measure Theory*. 3rd ed., Academic Press, San Diego, 2000.

[27] W. Müller. *Analytic torsion and R-torsion of Riemannian manifolds*. Adv. in Math. 28 (1978), no. 3, 233-305.

[28] W. Müller. *Analytic torsion and R-torsion for unimodular representations*. J. Amer. Math. Soc. 6 (1993), no. 3, 721-753.

[29] A. Ranicki and D. Sullivan. *A semi-local combinatorial formula for the signature of a 4k-manifold*. J. Differential Geometry 11 (1976), no. 1, 23-29.

[30] J. Ratcliffe. *Foundations of hyperbolic manifolds. Second edition*. Graduate Texts in Mathematics, 149. Springer, New York, 2006.

[31] A. Ros, *One-sided complete stable minimal surfaces*, J. Differential Geom. 74 (2006), no. 1, 69-92.

[32] P. Sarnak. *Letter to Rudnick*. October, 2002.

[33] R. Schoen. *Estimates for stable minimal surfaces in three-dimensional manifolds*. Seminar on minimal submanifolds, 111-126, Ann. of Math. Stud., 103, Princeton Univ. Press, Princeton, NJ, 1983.

[34] L. Simon. *Survey Lectures on Minimal Submanifolds*. Seminar on minimal submanifolds, 3-52, Ann. of Math. Stud., 103, Princeton Univ. Press, Princeton, NJ, 1983.