AN EXPONENTIAL-TYPE UPPER BOUND FOR FOLKMAN NUMBERS

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For given integers $k$ and $r$, the Folkman number $f(k;r)$ is the smallest number of vertices in a graph $G$ which contains no clique on $k+1$ vertices, yet for every partition of its edges into $r$ parts, some part contains a clique of order $k$. The existence (finiteness) of Folkman numbers was established by Folkman (1970) for $r=2$ and by Nešetřil and Rödl (1976) for arbitrary $r$, but these proofs led to very weak upper bounds on $f(k;r)$.

Recently, Conlon and Gowers and independently the authors obtained a doubly exponential bound on $f(k;2)$. Here, we establish a further improvement by showing an upper bound on $f(k;r)$ which is exponential in a polynomial of $k$ and $r$. This is comparable to the known lower bound $2^{Ω(rk)}$. Our proof relies on a recent result of Saxton and Thomason (or, alternatively, on a recent result of Balogh, Morris, and Samotij) from which we deduce a quantitative version of Ramsey’s theorem in random graphs.

1. Introduction

For two graphs, $G$ and $F$, and an integer $r ≥ 2$ we write $G \rightarrow (F)_r$ if every $r$-coloring of the edges of $G$ results in a monochromatic copy of $F$. By a copy we mean here a subgraph of $G$ isomorphic to $F$. Let $K_k$ stand for the complete graph on $k$ vertices and let $R(k;r)$ be the $r$-color Ramsey number.

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that is, the smallest integer $n$ such that $K_n \rightarrow (K_k)_r$. As it is customary, we suppress $r=2$ and write $R(k):=R(k;2)$ as well as $G \rightarrow F$ for $G \rightarrow (F)_2$.

In 1967 Erdős and Hajnal [8] asked if for some $l$, $k+1 \leq l < R(k)$, there exists a graph $G$ such that $G \rightarrow K_k$ and $G \not\rightarrow K_l$. Graham [12] answered this question in positive for $k=3$ and $l=6$ (with a graph on eight vertices), and Pósa (unpublished) for $k=3$ and $l=5$. Folkman [10] proved, by an explicit construction, that such a graph exists for every $k \geq 3$ and $l= k+1$. He also raised the question to extend his result for more than two colors, since his construction was bound to two colors.

For integers $k$ and $r$, a graph $G$ is called $(k;r)$-Folkman if $G \rightarrow (K_k)_r$ and $G \not\rightarrow K_{k+1}$. We define the $r$-color Folkman number for $K_k$ by

$$f(k;r) = \min\{n \in \mathbb{N} : \exists G \text{ such that } |V(G)| = n \text{ and } G \text{ is } (k;r)\text{-Folkman}\}.$$  

For $r=2$ we set $f(k) := f(k;2)$. It follows from [10] that $f(k)$ is well defined for every integer $k$, i.e., $f(k) < \infty$. This was extended by Nešetřil and Rödl [17], who showed that $f(k;r) < \infty$ for an arbitrary number of colors $r$.

Already the determination of $f(3)$ is a difficult, open problem. In 1975, Erdős [7] offered max(100 dollars, 300 Swiss francs) for a proof or disproof of $f(3) < \! \!< 10^{10}$. For the history of improvements of this bound see [5], where a computer assisted construction is given yielding $f(3) < 1000$. For general $k$, the only previously known upper bounds on $f(k)$ come from the constructive proofs in [10] and [17]. However, these bounds are tower functions of height polynomial in $k$. On the other hand, since $f(k) \geq R(k)$, it follows by the well known lower bound on the Ramsey number that $f(k) \geq 2^{k/2}$, which for $k=3$ was improved to $f(3) \geq 19$ [19].

We prove an upper bound on $f(k;r)$ which is exponential in a polynomial of $k$ and $r$. Set $R:= R(k;r)$ for the $r$-color Ramsey number for $K_k$. It is known that there exists some $c > 0$ such that for every $r \geq 2$ and $k \geq 3$ we have

$$2^{ck^r} < R < r^{rk}.$$ 

The upper bound already appeared in the work of Skolem [25]. The lower bound obtained from a random $r$-coloring of the complete graphs is of the form $r^{k/2}$. However, Lefmann [14] noted that the simple inequality $R(k;s+t) \geq (R(k;s) - 1)(R(k;t) - 1) + 1$ yields a lower bound of the form $2^{kr/4}$. Using iteratively random 3-colorings in this “product-type” construction yields a slightly better lower bound of the form $3^{rk/6}$. Our main result establishes an upper bound on the Folkman number $f(k;r)$ of similar order of magnitude.

**Theorem 1.** For all integers $r \geq 2$ and $k \geq 3$,

$$f(k;r) \leq k^{400k^3} R^{40k^2} \leq 2^{c(k^4 \log k + k^3 r \log r)},$$

for some $c > 0$ independent of $r$ and $k$. 
To prove Theorem 1, we consider a random graph $G(n,p)$, $p=Cn^{-2(k+1)}$, where $n=n(k,r)$ and $C=C(n,k,r)$ and carefully estimate from below the probabilities $P(G(n,p) \to (K_k)_r)$ and $P(G(n,p) \not\supseteq K_{k+1})$, so that their sum is strictly greater than 1. The latter probability is easily bounded by the FKG inequality. However, to set a bound on $P(G(n,p) \to (K_k)_r)$ we rely on a recent general result of Saxton and Thomason [24], elaborating on ideas of Nenadov and Steger [15] (see Remark 3).

Remark 1. Instead of the Saxton-Thomason theorem, we could have used a concurrent result of Balogh, Morris, and Samotij [1], which, by using our method, yields only a slightly worse upper bound on the Folkman numbers $f(k;r)$ than Theorem 1 (the $k^4$ in the exponent has to be replaced $k^6$).

Remark 2. In a related paper [23], we combined ideas from [9,20,22] and, for $r = 2$, obtained another proof of the Ramsey threshold theorem that yields a self-contained derivation of a double-exponential bound for the two-color Folkman numbers $f(k)$. Independently, a similar double-exponential bound for $f(k;r)$, for $r \geq 2$, was obtained by Conlon and Gowers [2] by a different method.

Motivated by the original question of Erdős and Hajnal, one can also define, for $r = 2$, $k \geq 3$, and $k+1 \leq l \leq R(k)$, a relaxed Folkman number as

$$f(k,l) = \min \{ n : \exists G \text{ such that } |V(G)| = n, \ G \to K_k \text{ and } G \not\supseteq K_l \}.$$ 

Note that $f(k,k+1) = f(k)$. As mentioned above, Graham [12] found out that $f(3,6) = 8$, while Nenov [16] and Piwakowski, Radziszowski and Urbański [18] determined that $f(3,5) = 15$ (see also [26]).

Of course, the problem is easier when the difference $l-k$ is bigger. Our final result provides an exponential bound of the form $f(k,l) \leq \exp\{-ck\}$, when $l$ is close to but bigger than $4k$ (the constant $c$ is proportional to the reciprocal of the difference between $l/k$ and 4).

**Theorem 2.** For every $0 < \alpha < \frac{1}{4}$ there exists $k_0$ such that for $k$ and $l$ satisfying $k \geq k_0$ and $k \leq \alpha l$,

$$f(k;l) \leq 2^{4k/(1-4\alpha)}.$$ 

It would be interesting to decide if the true order of the logarithm of $f(k,k+1) = f(k)$ is also linear in $k$.

The paper is organized as follows. In the next section we prove our main result, Theorem 1, while Theorem 2 is proved in Section 3. Finally, a short Section 4 offers a brief discussion of the analogous problem for hypergraphs.
Most logarithms in this paper are binary and are denoted by \( \log \). Only occasionally, when citing a result from [24] (Theorem 5 in Section 2 below), we will use the natural logarithms, denoted by \( \ln \).

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2. Proof of Theorem 1

We will prove Theorem 1 by the probabilistic method. Let \( G(n,p) \) be the binomial random graph, where each of the \( \binom{n}{2} \) possible edges is present, independently, with probability \( p \). We are going to show that for every \( n \geq 40k^4R^10k^2 \) and a suitable function \( p = p(n) \), with positive probability, \( G(n,p) \) has simultaneously two properties: \( G(n,p) \to (K_k)_r \) and \( G(n,p) \not\supset K_{k+1} \). Of course, this will imply that there exists a \((k;r)\)-Folkman graph on \( n \) vertices. We begin with a simple lower bound on \( \mathbb{P}(G(n,p) \not\supset K_{k+1}) \).

Lemma 3. For all \( k,n \geq 3 \), and \( C > 0 \), if \( p = Cn^{-2/(k+1)} \leq \frac{1}{2} \), then
\[
\mathbb{P}(G(n,p) \not\supset K_{k+1}) > \exp\left(-C\left(\frac{k}{2}\right)^2n\right).
\]

Proof. By applying the FKG inequality (see, e.g., [13, Theorem 2.12 and Corollary 2.13]), we obtain the bound
\[
\mathbb{P}(G(n,p) \not\supset K_{k+1}) \geq \left(1 - p^{\left(\frac{k+1}{2}\right)}\right)^{\binom{n}{k+1}} \\
\geq \exp\left(-2C\left(\frac{k}{2}\right)^2n^{-k}\binom{n}{k+1}\right) > \exp\left(-C\left(\frac{k}{2}\right)^2n\right),
\]
where we also used the inequalities \( \binom{n}{k+1} < n^{k+1}/2 \) and \( 1 - x > e^{-2x} \) for \( 0 < x < \frac{1}{2} \).

The main ingredient of the proof of Theorem 1 traces back to a theorem from [20] establishing edge probability thresholds for Ramsey properties of \( G(n,p) \). A special case of that result states that for all integers \( k \geq 3 \) and \( r \geq 2 \) there exists a constant \( C \) such that if \( p = p(n) \geq Cn^{-2/k+1} \), then
\[
\lim_{n \to \infty} \mathbb{P}(G(n,p) \to (K_k)_r) = 1.
\]

Adapting an idea of Nenadov and Steger [15] (see Remark 3 for more on that), and based on a result of Saxton and Thomason [24], we obtain the
following quantitative version of the above random graph theorem. Recall our notation \( R = R(k; r) \) for the \( r \)-color Ramsey number and notice an easy lower bound

\[
R(k; r) > 2r
\]

valid for all \( r \geq 2 \) and \( k \geq 3 \) (just consider a factorization of \( K_{2r} \)).

**Lemma 4.** For all integers \( r \geq 2, \; k \geq 3, \) and

\[
n \geq k^{400k^4}R^{40k^2},
\]

the following holds. Set

\[
\begin{align*}
&b = \frac{1}{2R^2}, \quad C = 2^{5\sqrt{\log n \log k}}R^{16}, \quad \text{and} \quad p = Cn^{-\frac{2}{k+1}}. \\
&\text{Then} \quad \mathbb{P}(G(n, p) \to (K_k)_r) \geq 1 - \exp\left(-bp\left(\frac{n}{2}\right)\right).
\end{align*}
\]

We devote the next two subsections to the proof of Lemma 4. Now, we deduce Theorem 1 from Lemmas 3 and 4.

**Proof of Theorem 1.** For given \( r \) and \( k \), let \( n \) be as in (2), and let \( b, C \), and \( p \) be as in (3). Below we will show that these parameters satisfy not only the assumptions of Lemma 4, but also the assumption \( p \leq \frac{1}{2} \) of Lemma 3, as well as an additional inequality

\[
n \geq (3/b)^{\frac{k+1}{k+1}} C^{\frac{k+2}{2}}.
\]

With these two inequalities at hand, we may quickly finish the proof of Theorem 1. Indeed, (4) implies that

\[
bp\left(\frac{n}{2}\right) \geq \frac{1}{3}bpn^2 = (b/3)Cn^{1+\frac{k-1}{k+1}} \geq C^{\left(k^2\right)}n,
\]

which, by Lemma 3, implies in turn that

\[
\mathbb{P}(G(n, p) \not\supset K_{k+1}) > \exp\left(-bp\left(\frac{n}{2}\right)\right).
\]

Since, by Lemma 4,

\[
\mathbb{P}(G(n, p) \to (K_k)_r) \geq 1 - \exp\left(-bp\left(\frac{n}{2}\right)\right),
\]

we conclude that

\[
\mathbb{P}(G(n, p) \to (K_k)_r \text{ and } G(n, p) \not\supset K_{k+1}) > 0.
\]
Thus, there exists a \((k; r)-\text{Folkman graph}\) on \(n\) vertices, and thus, \(f(k) \leq k^{400k^4}R^{40k^2}\).

It remains to show that \(p \leq \frac{1}{2}\) and that \(4\) holds. The first inequality is equivalent to

\[(6)
 n \geq (2C)^{\frac{k+1}{2}}.\]

We will now show that this inequality is a consequence of \(4\) and then establish \(4\) itself. Since \(C > 2\) and \(3/b^{(3)} = 6R^2 \geq 1\), we infer that

\[(3/b)^{\frac{k+1}{k+1}}C^{(k+2)} \geq C^{(k+2)} \geq (2C)^{\frac{k+1}{2}},\]

and hence, \(6\) indeed follows from \(4\).

Finally, we establish \(4\). In doing so we will use again the identity \(3/b^{(3)} = 6R^2\), as well as the inequalities \(36 \leq C\), which follows from \(2\) and \(3\), \((k+2) \leq k^2+1 \leq 2k^2-1\), and \(\frac{k+1}{k+1} \leq 2\), valid for all \(k \geq 3\). The R-H-S of \(4\) can be bounded from above by

\[(6R^2)^{\frac{k+1}{k+1}}C^{(k+2)} \leq 36R^4C^{(k+2)} \leq R^4C^{k+2} \leq 2^{10k^2\sqrt{\log n \log k}}R^{20k^2}.\]

Hence, it suffices to show that

\[(7)
 n \geq 2^{10k^2\sqrt{\log n \log k}}R^{20k^2}.\]

Observe that, by \(2\), \(\frac{1}{2}\log n \geq 20k^2\log R\), and thus, it remains to check that

\(\frac{1}{2}\log n \geq 10k^2\sqrt{\log n \log k},\)

or equivalently that

\[\log n \geq 400k^4\log k.\]

This, however, follows trivially from \(2\).

\[\square\]

\section{The proof of Lemma 4 – preparations}

In this and the next subsection we present a proof of Lemma 4, which is inspired by the work of Nenadov and Steger [15] and is based on a recent general result of Saxton and Thomason [24] on the distribution of independent sets in hypergraphs. For a hypergraph \(H\), a subset \(I \subseteq V(H)\) is independent if the subhypergraph \(H[I]\) induced by \(I\) in \(H\) has no edges.

For an \(h\)-graph \(H\), the degree \(d(J)\) of a set \(J \subset V(H)\) is the number of edges of \(H\) containing \(J\). (Since in our paper letter \(r\) is reserved for the
number of colors, we will use $h$ for hypergraph uniformity.) We will write $d(v)$ for $d(\{v\})$, the ordinary vertex degree. We further define, for a vertex $v \in V(H)$ and $j = 2, \ldots, h$, the maximum $j$-degree of $v$ as

$$d_j(v) = \max \left\{ d(J) : v \in J \subset \binom{V(H)}{j} \right\}.$$

Finally, the co-degree function of $H$ with a formal variable $\tau$ is defined in [24] as

$$\delta(H, \tau) = \frac{2^{(h)_2} - 1}{nd} \sum_{j=2}^{h} \frac{2^{(j-1)}_2}{\tau^{j-1}} \sum_v d_j(v),$$

where the inner sum is taken over all vertices $v \in V(H)$ and $d$ is the average vertex degree in $H$, that is, $d = \frac{1}{n} \sum_v d(v)$.

Theorem 5 below is an abridged version of [24, Corollary 3.6], where we suppress part of conclusion (a) (about the sets $T_i$), as well as the “Moreover” part therein, since we do not use this additional information here. In part (c) of the theorem below, for convenience, we switch from $\ln$ to $\log$, but only on the R-H-S of the upper bound on $\ln |C|$.

Theorem 5 (Saxton & Thomason, [24]). Let $H$ be an $h$-graph on vertex set $[n]$ and let $\varepsilon$ and $\tau$ be two real numbers such that $0 < \varepsilon < 1/2$,

$$\tau \leq 1/(144(h!)^2h) \quad \text{and} \quad \delta(H, \tau) \leq \varepsilon/(12(h!)).$$

Then there exists a collection $\mathcal{C}$ of subsets of $[n]$ such that the following three properties hold.

(a) For every independent set $I$ in $H$ there exists a set $C \in \mathcal{C}$ such that $I \subset C$.

(b) For all $C \in \mathcal{C}$, we have $e(H[C]) \leq \varepsilon e(H)$.

(c) $\ln |\mathcal{C}| \leq c \log(1/\varepsilon) \tau \log(1/\tau)n$, where $c = 800(h!)^3h$.

We will now tailor the above result to our application. The hypergraphs we consider have a very symmetric structure. Given $k$ and $n$, let $H(n, k)$ be the hypergraph with vertex set $\binom{[n]}{2}$, the edges of which correspond to all copies of $K_k$ in the $K_n$ with vertex set $[n]$. Thus, $H(n, k)$ has $\binom{n}{2}$ vertices, $\binom{n}{k}$ edges, and is $\left(\frac{k}{2}\right)$-uniform and $\left(\frac{n-2}{k-2}\right)$-regular.

For $J \subseteq \binom{[n]}{2}$, the degree of $J$ in $H(n, k)$ is $d(J) = \binom{n-v_J}{k-v_J}$, where $v_J$ is the number of vertices in $J$ treated as a graph on $[n]$ rather than a subset of vertices of $H(n, k)$. Thus, over all $J$ with $|J| = j$, $d(J)$ is maximized by
the smallest possible value of \( v_J \), that is, when \( v_J = l_j \), the smallest integer \( l \) such that \( j \leq \binom{l}{2} \). Consequently, for every vertex \( v \) of \( H(n,k) \) (that is, an edge of \( K_n \) on \([n]\)) and for each \( j = 2, \ldots, \binom{k}{2} \), we have
\[
d_j(v) = \binom{n - l_j}{k - l_j}.
\]

Clearly, \( l_j \geq 3 \) for \( j \geq 2 \), which will be used later.

Let
\[
\delta(n, k, \tau) := \sum_{j=2}^{\binom{k}{2}} \frac{2^{k^4}k^{k-2}}{\tau^{j-1}n^{l_j-2}}.
\]

The co-degree function of \( H(n,k) \) can be bounded by \( \delta(n, k, \tau) \).

**Claim 6.**
\[
\delta(H(n,k), \tau) \leq \delta(n, k, \tau).
\]

**Proof.** By the definition of \( \delta(H, \tau) \) in (8) with \( h \) replaced by \( \binom{k}{2} \), \( n \) by \( \binom{n}{2} \), \( d \) by \( \binom{n-2}{k-2} \), \( d_j(v) \) by \( \binom{n-l_j}{k-l_j} \), and with \( 2^{\binom{j-1}{2}} \) dropped out from the denominator, we have
\[
\delta(H(n,k), \tau) \leq 2^{k^4} \sum_{j=2}^{\binom{k}{2}} \frac{\binom{n-l_j}{k-l_j}}{\tau^{j-1}\binom{n-2}{k-2}}.
\]

Now, observe that \( \frac{\binom{n-l_j}{k-l_j}}{\binom{n-2}{k-2}} \leq (k/n)^{l_j-2} \) and \( l_j \leq k \).

The most important property of hypergraph \( H(n,k) \) is that a subset \( S \) of the vertices of \( H \) corresponds to a graph \( G \) with vertex set \([n]\) and edge set \( S \), and \( S \) is an independent set in \( H(n,k) \) if and only if the corresponding graph \( G \) is \( K_k \)-free. We now apply Theorem 5 to \( H(n,k) \).

**Corollary 7.** Let \( k \geq 3, n \geq 3 \), and let \( \epsilon \) and \( \tau \) be two real numbers such that \( 0 < \epsilon < 1/2 \),
\[
\tau \leq (k^2!)^{-2} \quad \text{and} \quad \delta(n, k, \tau) \leq \frac{\epsilon}{k^2!}.
\]

Then there exists a collection \( C \) of subgraphs of \( K_n \) such that the following three properties hold.

(a) For every \( K_k \)-free graph \( G \subseteq K_n \) there exists a graph \( C \in C \) such that \( G \subset C \).

(b) For all \( C \in C \), \( C \) contains at most \( \epsilon \binom{n}{k} \) copies of \( K_k \).

(c) \( \ln |C| \leq (2k^2)! \log(1/\epsilon) \tau \log(1/\tau) \binom{n}{2} \).
Proof. Note that for \( k \geq 3 \),
\[
k^2! > 12 \binom{k}{2}! \quad \text{and, consequently,} \quad (k^2!)^2 > 144 \binom{k}{2}! \binom{k}{2},
\]
and that, by Claim 6, \( \delta(H(n,k),\tau) \leq \delta(n,k;\tau) \). Thus, the assumptions of Theorem 5 hold for \( H := H(n,k) \) with \( h = \binom{k}{2} \), and its conclusions (a-c) translate into the corresponding properties (a-c) of Corollary 7. Finally, notice that
\[
(2k^2)! > c = 864 \binom{k}{2}!^3 \binom{k}{2}.
\]

In the next subsection we deduce Lemma 4 from Corollary 7. First, however, we make a simple observation about the number of monochromatic copies of \( K_k \) in every coloring of \( K_n \). Recall that \( R = R(k;r) \) is the \( r \)-color Ramsey number for \( K_k \) and set
\[
(10) \quad \alpha = \binom{R}{k}^{-1}.
\]

**Proposition 8.** Let \( n \geq R \). For every \((r+1)\)-coloring of the edges of \( K_n \) either there are more than \( \frac{1}{2} \binom{n}{k} \) monochromatic copies of \( K_k \) colored by the first \( r \) colors, or more than \( \frac{1}{R^2} \binom{n}{2} \) edges receive color \( r+1 \).

**Proof.** Consider an \((r+1)\)-coloring of the edges of \( K_n \). Let \( x \binom{n}{R} \) be the number of the \( R \)-element subsets of the vertices of \( K_n \) with no edge colored by color \( r+1 \). By the definition of \( R \), each of these subsets induces in \( K_n \) a monochromatic copy of \( K_k \). Thus, counting repetitions, there are at least
\[
x \cdot \binom{n}{R} - x \cdot \binom{n-k}{R-k} = x \alpha \binom{n}{k}
\]
monochromatic copies of \( K_k \) colored by one of the first \( r \) colors. Suppose that their number is at most
\[
\alpha \binom{n}{k}.
\]
Then \( x \leq \frac{1}{2} \), that is, at least a half of the \( R \)-element subsets of \( V(K_n) \) contain at least one edge colored by \( r+1 \). Hence, color \( r+1 \) appears on at least
\[
\frac{1}{2} \binom{n}{R} - \frac{1}{2} \binom{n-2}{R-2} = \frac{1}{2} \binom{n}{2} \binom{R}{2} > \frac{1}{R^2} \binom{n}{2}
\]
edges of \( K_n \). This completes the proof.
2.2. Proof of Lemma 4 – details

Let \( r \geq 2, k \geq 3 \), and let \( n, b, C, \) and \( p \) be as in Lemma 4, see (3) and (2). We have to show that

\[
P(G(n, p) \rightarrow (K_k)_r) \geq 1 - \exp(-bp(n/2))\]

First we set up a few auxiliary constants required for the application of Corollary 7. Recalling that \( \alpha \) is defined in (10), let

\[
\varepsilon = \frac{\alpha}{2r},
\]

\[
C_0 = 2^{4\sqrt{\log n}} R^{10/k}, \quad \text{and} \quad \tau = C_0 n^{-\frac{2}{k+1}}.
\]

We will now prove that the above defined constants \( \varepsilon \) and \( \tau \) satisfy the assumptions of Corollary 7.

**Claim 9.** Inequalities (9) hold true for every \( k \geq 3 \).

**Proof.** In order to verify the first inequality in (9), note that by the definitions of \( \tau \) and \( C_0 \) in (12) and the obvious bound \( x! < x^x \),

\[
(k^2!)^2 \tau \leq k^{4k^2} 2^{4\sqrt{\log n}} R^{10/k} n^{-2}.
\]

It remains to show that the R-H-S of (13) is smaller than one, or, by taking logarithms, that

\[
4k^2 \log k + 4\sqrt{\log n} + \frac{10}{k} \log R < \frac{2}{k+1} \log n.
\]

This, however, follows from

\[
4\sqrt{\log n} < \frac{1}{k+1} \log n,
\]

or equivalently,

\[
16(k+1)^2 < \log n,
\]

and from

\[
4k^2(k+1) \log k + \frac{10}{k} (k+1) \log R < \log n,
\]

both of which are true by the lower bound on \( n \) in (2).

To prove the second inequality in (9), note that since \( \tau \leq 1 \) and \( j \leq \binom{l_j}{2} \), the quantity \( \tau^{j-1} n^{l_j-2} \) is minimized when \( j = \binom{l_j}{2} \). Thus, we have

\[
\tau^{j-1} n^{l_j-2} \geq \tau^{\binom{l_j}{2}-1} n^{l_j-2} = C_0^{\binom{l_j}{2}-1} n^{(l_j-2)(l_j+1)+l_j-2} = C_0^{\binom{l_j}{2}-1} n^{(l_j-2)(k-l_j)} n^{-\frac{k+1}{k+1}}.
\]
Recall that for \( j \geq 2 \) we have \( l_j \geq 3 \). In what follows we obtain a lower bound on the R-H-S of (14) by distinguishing two cases: \( l_j < k \) and \( l_j = k \). If \( l_j < k \), then \((l_j - 2)(k - l_j)\) is minimized for \( l_j = 3 \) and \( l_j = k - 1 \) and owing to \( C_0 > 1 \) we infer

\[
\tau^{j-1} \cdot n^{l_j-2} \geq C_0 \frac{(l_j-2)(k-l_j)}{n^2} > n^{k-3} \geq n^{k^4 R^2 k^2},
\]

where we also used the bound \( \frac{k+1}{k-3} \leq 5 \) for all \( k \geq 4 \), which holds due to \( 3 \leq l_j < k \). If, on the other hand, \( l_j = k \), then, by the definition of \( C_0 \) in (12) and the bound on \( n \) in (2),

\[
(15) \quad C_0 \geq 2^{80k^2} R^{10/k}.
\]

Hence, in view of (15), and the fact that \( \binom{k}{2} - 1 \geq \frac{1}{5} k^2 \) for \( k \geq 3 \), we have that

\[
\tau^{j-1} \cdot n^{l_j-2} \geq C_0 \frac{(k-2)(k-1)}{n^2} \geq \left( 2^{80k^2} R^{10/k} \right)^{2k/5} = 2^{16k^4} R^{2k}.
\]

Consequently, using the trivial bounds \( k^k \cdot k^2! < 2^{15k^4} \), \( R^k \), and \( R^k > 2r \), we conclude that

\[
\sum_{j=2}^{(k)} 2^{k^4} k^{k-2} \leq \sum_{j=2}^{(k)} 2^{16k^4} R^{2k} \leq \frac{k^k}{2^{15k^4} R^{2k}} \leq \frac{1}{2r (R^k) \cdot k^2!} = \varepsilon \frac{1}{k^2!},
\]

which concludes this proof.

In view of Claim 9, the conclusions of Corollary 7 hold true with \( \varepsilon \) and \( \tau \) defined in, resp., (11) and (12). That is, there exists a collection \( \mathcal{C} \) of subgraphs of \( K_n \) such that Properties (a), (b), and (c) of Corollary 7 are satisfied for these specific values of \( \varepsilon \) and \( \tau \).

To continue with the proof of Lemma 4 consider a random graph \( G(n,p) \) and let \( \mathcal{E} \) be the event that \( G(n,p) \not\rightarrow (K_k)_r \). For each \( G \in \mathcal{E} \), there exists an \( r \)-coloring \( \varphi: E(G) \rightarrow [r] \) yielding no monochromatic copy of \( K_k \). (Further on we will call such a coloring proper.) In other words, there are \( K_k \)-free graphs \( G_1, \ldots, G_r \), defined by \( G_i = \varphi^{-1}(i) \), such that \( G_1 \cup \ldots \cup G_r = G \). According to Property (a) of Corollary 7, for every \( i \in [r] \) there exists a graph \( C_i \in \mathcal{C} \) such that \( G_i \subseteq C_i \). Consequently,

\[
G \cap \left( K_n \setminus \bigcup_{i=1}^{r} C_i \right) = \emptyset.
\]

Notice that there are only at most \( |\mathcal{C}|^r \) distinct graphs \( K_n \setminus \bigcup_{i=1}^{r} C_i \). Moreover, we next show that all these graphs are dense (see Claim 10). Hence,
as it is extremely unlikely for a random graph $G(n,p)$ to be completely disjoint from one of the few given dense graphs, it will ultimately follow that $\mathbb{P}(E) = o(1)$.

**Claim 10.** For all $C_1, \ldots, C_r \in \mathcal{C}$,

$$\left| K_n \setminus \bigcup_{i=1}^{r} C_i \right| \geq \left( \frac{n}{2} \right) / R^2.$$  

**Proof.** The graphs $C_i$, $i \in [r]$, together with $K_n \setminus \bigcup_{i=1}^{r} C_i$, form an $(r+1)$-coloring of $K_n$, more precisely, an $(r+1)$-coloring where, for each $i = 1, \ldots, r$, the edges of color $i$ are contained in $C_i$, while all edges of $K_n \setminus \bigcup_{i=1}^{r} C_i$ are colored with color $r+1$. (Note that this coloring may not be unique, as the graphs $C_i$ are not necessarily mutually disjoint.) By Proposition 8, this $(r+1)$-coloring yields either more than $(\alpha/2) \binom{n}{k}$ monochromatic copies of $K_k$ in the first $r$ colors or more than $\binom{n}{2} / R^2$ edges in the last color. Since for each $i \in [r]$, the $i$-th color class is contained in $C_i$, it follows from Property (b) that there are at most

$$r \cdot \varepsilon \binom{n}{k} = \alpha \frac{n}{2} \binom{n}{k}$$

monochromatic copies of $K_k$ in the first $r$ colors. Consequently, we must have

$$\left| K_n \setminus \bigcup_{i=1}^{r} C_i \right| > \frac{1}{R^2} \binom{n}{2},$$

which concludes the proof. \hfill \Box

Based on Claim 10 we can now bound $\mathbb{P}(E) = \mathbb{P}(G(n,p) \not\to (K_s)_r)$ from above.

**Claim 11.**

$$\mathbb{P}(G(n,p) \not\to (K_s)_r) \leq |\mathcal{C}|^r \exp \left\{ -\frac{p(n)}{R^2} \right\}$$

**Proof.** Let $\mathcal{F}$ be the event that $G(n,p) \cap (K_n \setminus \bigcup_{i=1}^{r} C_i) = \emptyset$ for at least one $r$-tuple of graphs $C_i \in \mathcal{C}$, $i = 1, \ldots, r$. We have $E \subseteq \mathcal{F}$. Indeed, if $G \in \mathcal{E}$ then there is a proper coloring $\varphi$ of $G$ and graphs $C_1, \ldots, C_r \in \mathcal{C}$ such that $G \subseteq \bigcup_{i=1}^{r} C_i$ and, by Claim 10, $K_n \setminus \bigcup_{i=1}^{r} C_i$ has at least $\frac{1}{R^2} \binom{n}{2}$ edges and is disjoint from $G$. Thus, $G \in \mathcal{F}$. Consequently,

$$\mathbb{P}(G(n,p) \not\to (K_k)_r) \leq \mathbb{P}(\mathcal{F}).$$
To estimate \( \mathbb{P}(\mathcal{F}) \) we write
\[
\mathcal{F} = \bigcup_{(C_1, \ldots, C_r)} \mathcal{F}(C_1, \ldots, C_r),
\]
where the summation runs over all collections \((C_1, \ldots, C_r)\) with \(C_i \in \mathcal{C}, \ i = 1, \ldots, r\), and the event \(\mathcal{F}(C_1, \ldots, C_r)\) means that \(G(n,p) \cap (K_n \setminus \bigcup_{i=1}^r C_i) = \emptyset\). Clearly,
\[
\mathbb{P}(\mathcal{F}(C_1, \ldots, C_r)) = (1 - p)^{|K_n \setminus \bigcup_{i=1}^r C_i|} \leq (1 - p)\binom{n}{2}/R^2,
\]
where the last inequality follows by Claim 10. Finally, applying the union bound, we have
\[
\mathbb{P}(G(n,p) \not\rightarrow (K_s)_r) \leq \mathbb{P}(\mathcal{F}) \leq |\mathcal{C}|^r (1 - p)\binom{n}{2}/R^2 \leq |\mathcal{C}|^r \exp \left\{ -\frac{p\binom{n}{2}}{R^2} \right\}.
\]

Observe that by Property (c) of Corollary 7,
\[
|\mathcal{C}|^r \leq \exp \left\{ r(2k^2)! \log(1/\varepsilon) \tau \log(1/\tau) \binom{n}{2} \right\}.
\]

In view of Claim 11 and inequality (17), to complete the proof of Lemma 4, it suffices to show that
\[
r(2k^2)! \log(1/\varepsilon) \tau \log(1/\tau) \binom{n}{2} \leq \frac{p\binom{n}{2}}{2R^2},
\]
or, equivalently, after applying the definitions of \(p\) and \(\tau\) ((3) and (12), resp.) and dividing sidewise by \(n^{-\frac{2}{k+1}} \binom{n}{2}\), that
\[
r(2k^2)! \log(1/\varepsilon) C_0 \log(1/\tau) \leq C/(2R^2).
\]
To this end, observe that, since \(C_0 \geq 1\) and, by (1), \(R > 2r\), we have
\[
\log(1/\tau) \leq \frac{2}{k+1} \log n \tag{12}
\]
and
\[
\log(1/\varepsilon) = \log(2r\binom{R}{k}) \leq (k + 1) \log R. \tag{11}
\]
Hence, the L-H-S of (18) can be upper bounded by \(2r(2k^2)!C_0 \log R \log n\). Consequently, using also the bounds \((2k^2)! < (2k)^{4k^2}\) and, again, \(R > 2r\), we realize that (18) will follow from
\[
2R^3 \log R \cdot (2k)^{4k^2} \log n \leq C/C_0.
\]
On the other hand,
\[
C/C_0 \stackrel{(3),(12)}{=} 25^{\sqrt{\log n \log k - 4\sqrt{\log n} R^{16-10/k}}} \geq 2^{\sqrt{\log n \log k + 4\sqrt{\log n} (\sqrt{\log k - 1}) R^{12}}. 
\]
Thus, (19) is an immediate consequence of the following two inequalities, which are themselves easy consequences of (2):

\[
2^{\sqrt{\log n \log k}} \geq 2^{20k^2 \log k} \geq (2k)^{4k^2}
\]

and

\[
2^4\sqrt{\log n (\sqrt{\log k} - 1)} > 2^{\sqrt{\log n}} \geq \log n.
\]

For the latter inequality we first used \( k \geq 3 \) and \( \sqrt{\log 3} > \frac{5}{4} \), and then the fact that \( 2^{\sqrt{x}} \geq x \) for all \( x \geq 16 \), which can be easily verified by checking the first derivative (note that by (2), \( \log n \geq 16 \)). This completes the proof of Lemma 4.

**Remark 3.** The idea of utilizing hypergraph containers for Ramsey properties of random graphs comes from a recent paper by Nenadov and Steger [15] (see also [11], Ch. 7) where the authors give a short proof of the main theorem from [20] establishing an edge probability threshold for the property \( G(n,p) \rightarrow (F)_r \). Let us point to some similarities and differences between their and our approach. For clarity of the comparison, let us restrict ourselves to the case \( F = K_k \) considered in our paper (the generalization to an arbitrary graph \( F \) is quite straightforward).

In [15] the goal is to prove an asymptotic result with \( n \to \infty \) and all other parameters fixed. Consequently, they do not optimize, or even specify constants. Our task is to provide as good as possible upper bound on \( n \) in terms of \( k \) and \( r \), so there is no asymptotics.

The observation that a \( K_k \)-free coloring of the edges of \( G(n,p) \) yields \( r \) independent sets in the hypergraph \( H(n,k) \), and therefore, by the Saxton-Thomason Theorem there are \( r \) graphs \( C_i, i = 1, \ldots, r \), each with only a few copies of \( K_k \), whose union contains all the edges of \( G(n,p) \), was made in [15]. Also there one can find a statement similar to our Proposition 8 (Corollary 3 in [15].) These two facts lead to similar estimates of the probability that \( G(n,p) \) is not Ramsey. However, Nenadov and Steger, assuming that \( C \) is a constant, are forced to use Theorem 2.3 from [24] which involves the sequences of sets \( T_i \). In our setting, we choose \( C = C(n) \) in a balanced way, allowing us to go through with the estimates of \( P(G(n,p) \nrightarrow (K_k)_r) \) without introducing the \( T_i \)'s, while, on the other hand, keeping the upper bound on \( n \) exponential in \( k \). In fact, as observed by Conlon and Gowers [2], the approach via random graphs cannot yield a better than double-exponential upper bound on \( n \) if one assumes that \( p \) is at the Ramsey threshold, i.e., if \( C \) is a constant.
3. Relaxed Folkman numbers

In this section we prove Theorem 2. We will need an elementary fact about Ramsey properties of quasi-random graphs. For constants \( \varrho \) and \( d \) with \( 0 < d, \varrho \leq 1 \), we say that an \( n \)-vertex graph \( \Gamma \) is \((\varrho, d)\)-dense if every induced subgraph on \( m \geq \varrho n \) vertices contains at least \( d (m^2/2) \) edges. It follows by an easy averaging argument that it suffices to check the above inequality only for \( m = \lceil \varrho n \rceil \). Note also that every induced subgraph of a \((\varrho, d)\)-dense \( n \)-vertex graph on at least \( cn \) vertices is \((\varrho c, d)\)-dense.

It turns out that for a suitable choice of the parameters, \((\varrho, d)\)-dense graphs are Ramsey.

**Proposition 12.** For every \( k \geq 2 \) and every \( d \in (0, 1) \), if \( n \geq (2/d)^{2k-4} \) and \( 0 < \varrho \leq (d/2)^{2k-4} \), then every two-colored \( n \)-vertex \((\varrho, d)\)-dense graph \( \Gamma \) contains a monochromatic copy of \( K_k \).

**Proof.** For a two-coloring of the edges of a graph \( \Gamma \) we call a sequence of vertices \((v_1, \ldots, v_l)\) canonical if for each \( i = 1, \ldots, l-1 \) all the edges \( \{v_i, v_j\} \), for \( j > i \), are of the same color.

We will first show by induction on \( l \) that for every \( l \geq 2 \) and \( d \in (0, 1) \), if \( n \geq (2/d)^{l-2} \) and \( 0 < \varrho \leq (d/2)^{l-2} \), then every two-colored \( n \)-vertex \((\varrho, d)\)-dense graph \( \Gamma \) contains a canonical sequence of length \( l \).

For \( l = 2 \), every ordered pair of adjacent vertices is a canonical sequence. Assume that the statement is true for some \( l \geq 2 \) and consider an \( n \)-vertex \((\varrho, d)\)-dense graph \( \Gamma \), where \( \varrho \leq (d/2)^{l-1} \) and \( n \geq (2/d)^{l-1} \). As observed above, there is a vertex \( u \) with degree at least \( dn \). Let \( M_u \) be a set of at least \( dn/2 \) neighbors of \( u \) connected to \( u \) by edges of the same color. Let \( \Gamma_u = \Gamma[M_u] \) be the subgraph of \( \Gamma \) induced by the set \( M_u \). Note that \( \Gamma_u \) has \( n_u \geq dn/2 \geq (2/d)^{l-2} \) vertices and is \((\varrho_u, d)\)-dense with \( \varrho_u \leq (d/2)^{l-2} \). Hence, by the induction assumption, there is a canonical sequence of length \( l \) in \( \Gamma_u \). This sequences preceded by the vertex \( u \) makes a canonical sequence of length \( l + 1 \) in \( \Gamma \).

To complete the proof of Proposition 12, set \( l = 2k - 2 \) above and observe that every canonical sequence \((v_1, \ldots, v_{2k-2})\) contains a monochromatic copy of \( K_k \). Indeed, among the vertices \( v_1, \ldots, v_{2k-3} \), some \( k-1 \) have the same color on all the “forward” edges. These vertices together with vertex \( v_{2k-2} \) form a monochromatic copy of \( K_k \).

**Proof of Theorem 2.** Let \( n = 2^{4k/(1-4\alpha)} \). Consider a random graph \( G(n, p) \) where

\[
p = 2n^{-7+4\alpha/16k} = 2^{-20\alpha+3/(1-4\alpha)} \cdot n^{-1} p^{l-1/2} \cdot \left( \frac{e n}{l p^{l-1/2}} \right) ^l.
\]

By elementary estimates one can bound the expected number of \( l \)-cliques in \( G(n, p) \) by

\[
\left( \frac{en}{lp^{l-1}} \right)^l.
\]
Thus, if
\[
\frac{l - 1}{2} \geq \frac{\log n}{\log(1/p)} = \frac{16k}{20\alpha + 3},
\]
then, as \(k \to \infty\), a.a.s. there are no \(l\)-cliques in \(G(n,p)\). By assumption,
\[
\frac{l - 1}{2} \geq \frac{k - \alpha}{2\alpha} \geq \frac{16k}{20\alpha + 3},
\]
where the last inequality, equivalent to \((3 - 12\alpha)k \geq 20\alpha^2 + 3\alpha\), holds if \(k \geq \frac{2}{3(1 - 4\alpha)}\) (we used here the assumption that \(\alpha < \frac{1}{4}\)).

Further, by a straightforward application of Chernoff’s bound (see, e.g., [13, ineq. (2.6)]), a.a.s. \(G(n,p)\) is \((\varrho, p - o(p))\)-dense, where \(\varrho = \frac{\log^2 n}{n}\), say. Indeed, setting \(t = g n = \log^2 n\), \(\epsilon = \epsilon(n) = (\log n)^{-1/3}\), and \(d = (1 - \epsilon)p\), the probability that a fixed set \(T\) of \(t\) vertices spans in \(G(n,p)\) fewer than \(dt^2/2\) edges is at most
\[
P(e(T) \leq (1 - \epsilon)pt^2/2) \leq P\left( e(T) \leq (1 - \epsilon/2)p\left(\frac{t}{2}\right) \right) 
\leq \exp \left\{ -\frac{\epsilon^2}{8}p\left(\frac{t}{2}\right) \right\} \leq \exp \left\{ -\frac{\epsilon^2}{24}pt^2 \right\}.
\]

Finally, note that the above bound, even multiplied by \(\binom{n}{t}\), the number of all \(t\)-element subsets of vertices in \(G(n,p)\), still converges to zero (recall that \(p\) is a constant).

Using that \(\epsilon k = O(\log^{2/3} n)\) one can easily verify that both assumptions of Proposition 12, that is, \(n \geq (2/d)^{2k-4}\) and \(\varrho \leq (d/2)^{2k-4}\), hold true. Indeed, dropping the subtrahend 4 for simplicity,
\[
(d/2)^{2k} = (1 - \epsilon)^{2k}n^{-1+\delta} \geq \varrho \geq \frac{1}{n},
\]
for \(n\) large enough, that is, for \(k\) large enough.

In conclusion, a.a.s. \(G(n,p)\) is such that

- it contains no \(K_l\), and
- for every two-coloring of its edges, there is a monochromatic copy of \(K_k\).

Hence, there exists an \(n\)-vertex graph with the above two properties and, consequently, \(f(k,l) \leq n = 2^{4k/(1-4\alpha)}\).
4. Hypergraph Folkman numbers

Hypergraph Folkman numbers are defined in an analogous way to their graph counterparts. Given three integers $h$, $k$, and $r$, the $h$-uniform Folkman number $f_h(k;r)$ is the minimum number of vertices in an $h$-uniform hypergraph $H$ such that $H \rightarrow (K^{(h)}_k)_r$ but $H \not\supseteq K^{(h)}_{k+1}$. Here $K^{(h)}_k$ stands for the complete $h$-uniform hypergraph on $k$ vertices, that is, one with $\binom{k}{h}$ edges. The finiteness of hypergraph Folkman numbers was proved by Nešetřil and Rödl in [17, Colloary 6, page 206] and besides the gigantic upper bound stemming from their construction, no reasonable bounds have been proven so far. Much better understood are the vertex-Folkman numbers (where instead of edges, the vertices are colored), which for both, graphs and hypergraphs, are bounded from above by an almost quadratic function of $k$, while from below the bound is only linear in $k$ (see [6,4]).

The study of Ramsey properties of random hypergraphs began in [21] where a threshold was found for $K^{(3)}_4$, the 3-uniform clique on 4 vertices. Also there a general conjecture was stated that a theorem analogous to that in [20] holds for hypergraphs too. This was confirmed for $h$-partite $h$-uniform hypergraphs in [22], and, finally, for all $h$-uniform hypergraphs in [9] and, independently, in [3].

As remarked by Nenadov and Steger in [15], the Saxton-Thomason (or the Balogh-Morris-Samotij) theorem should also yield a much simpler proof of the hypergraph Ramsey threshold theorem from [9,3]. We believe that, similarly, our quantitative approach should also provide an upper bound on the hypergraph Folkman numbers $f_h(k;r)$, exponential in a polynomial of $k$ and $r$.

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