SUBDIVIDING THE POLAR OF A FACE

DIRK OLIVER THEIS

ABSTRACT. Let $S$ be a convex polytope. The set of all valid inequalities carries the structure of a convex polytope $S^\Delta$, called the polar (polytope) of $S$. The facial structure of the polar provides information for each of its points: two points $a$ and $b$ are in the same face of $S^\Delta$ if and only if the faces of $S$ obtained by intersecting it with the hyperplanes given respectively by $a$ and $b$ coincide. Suppose now that $S$ is a face of another polyhedron $P$. Then the points of $S^\Delta$ carry some additional information: the set of faces of $P$ which one can obtain by “rotating” the hyperplane given by a point. This additional information is captured by the structure of a polyhedral complex subdividing $S^\Delta$.

In this paper, we study this subdivision for the following examples: The Birkhoff polytope as a face of the matching polytope; the permutahedron as a facet of another permutahedron; the Symmetric Traveling Salesman Polytope, also known as Hamiltonian Cycle polytope, as a face of the connected Eulerian multi-graph polyhedron, also known as Graphical Traveling Salesman Polyhedron.

1. INTRODUCTION & MOTIVATION

Let $S$ be a convex polytope. The set of all (normalized) valid linear inequalities carries the structure of a convex polytope $S^\Delta$, commonly called the polar (or dual) polytope of $S$. The facial structure of the polar provides information for each of its points: two points are in the same face of $S^\Delta$ if and only if the faces of $S$ they define as inequalities coincide. This means that the respective hyperplanes given by these points have the same intersection with $S$.

Suppose now that $S$ is a face of another polyhedron $P$. A valid inequality for $S$ defines a whole “fiber” of valid inequalities for $P$ by “rotating” the hyperplane given by the inequality arbitrarily without changing its intersection with the affine subspace spanned by $S$ and such that the resulting inequality is valid for $P$. Hence, the points of $S^\Delta$ carry some additional information: the set of faces of $P$ which one can obtain by rotating the hyperplane given by a point in this way.

Just as the facial structure of $S^\Delta$ provides the information which faces of $S$ are defined by its points, the information which faces of $P$ can be defined by points in $S^\Delta$ is captured by the structure of a polyhedral complex subdividing $S^\Delta$. It can be defined mirroring the property of the polar polytope: two points of $S^\Delta$ are in the
same face of $S$ if and only if the two sets of faces of $P$ obtainable by rotating the respective hyperplanes coincide. We call $S$ a rotation complex for $S \subset P$.

In this paper, we study examples of these polyhedral complexes. We are interested in giving “local” or “global” descriptions of such a complex. By a local description we mean a statement of the form: “Two points $a$ and $b$ in $S^\Delta$ are contained in the relative interior of the same face of the rotation complex $S$ if and only if they satisfy some common property.” Of course, such a property ought to be natural and allow us to understand $S$ in a non-trivial kind of way (a trivial property is already known: defining the same set of faces of $P$). A global description consists in the identification of another polyhedral complex $T$ such that the rotation complex $S$ is equal to the common refinement of $T$ and the face complex of $S^\Delta$. Again, a trivial complex is not desired, but could always be named: We will see in Section 3 that there is a natural projective mapping $\pi: P^\Delta \to S^\Delta$, and that $S$ is equal to the image complex of the deletion $\text{dl}(S^\Phi, C(P^\Delta))$ of the conjugate face $S^\Phi$ of $S$ in $P^\Delta$ from the complex of $P^\Delta$. (The mapping $\pi$ can be visualized as a projection onto a face figure $P^\Delta/S^\Phi$.)

We allow for the possibility that these descriptions cover only a sub-complex of $S^\Delta$. If, for example, $S$ is any instance of the family of Symmetric Traveling Salesman Polytopes, a.k.a. Hamiltonian Cycle polytopes, this allows us to steer around questions of Hamiltonicity of graphs by disregarding a clearly defined set of vertices of the polar $S^\Delta$. For an integer $n$, the corresponding Symmetric Traveling Salesman Polytope has as its vertices the cycles on a fixed $n$-element vertex set. The corresponding polyhedron $P$ we use in this example, whose vertices are connected Eulerian multi-graphs, has been used frequently in the context of polyhedral combinatorics for the Symmetric Traveling Salesman Polytope (see the references in Section 2.3). For this pair of polyhedra, we obtain both a local and a global description of $S$. For the local descriptions, to decide whether two points are in the same face of $S$, nothing more is necessary but to compare values of expressions of the form $a_k + a_i - a_j$ for points $a \in S^\Delta$. For the global description, we identify a sub-complex of the face complex of the metric cone (the cone consisting of all metrics on a fixed $n$-element set), which serves as a refining complex $T$ in the above sense.

For the Traveling Salesman polyhedra, we will also show that the above projective mentioned mapping $\pi$ maps the corresponding subcomplex of $P^\Delta$ homeomorphically onto the subcomplex of $S^\Delta$. In other words, $\pi$ is a refinement map defining a combinatorial equivalence of these two complexes. Generally speaking, the property that $\pi$, probably restricted to a subcomplex of the complex of $P^\Delta$, becomes injective, is a strong fact. It means that the complex of $P^\Delta$ can be “flattened” to obtain a refinement of the complex of $S^\Delta$. For the Traveling Salesman polyhedra we will undertake some effort to prove injectivity. This property is used in [38] (see also [50]) to prove the completeness of a description by linear inequalities of the Graphical Traveling Salesman Polyhedron for nine cities.

Apart from Traveling Salesman polyhedra, we describe rotation complexes in the following two cases: $S$ a Birkhoff polytope and $P$ a corresponding matching polytope; $S$ and $P$ permutahedra and $S$ a facet of $P$. 
Since we refer to points of the polar polytope $S^\Delta$ of $S$, we require to construct $S^\Delta$ as a point set, rather than just a combinatorial type of polytopes. Hence we must identify a relative interior point of $S$ which plays the role of zero. It will be seen that the rotation complex does not depend on the choice of the zero point. Though, to our knowledge, the geometric questions we raise have not yet been considered before, the rotation process which defines the complexes we are interested in is folklore. In particular, it is deployed in some well-known concepts in polyhedral combinatorics. For example, it is used as a trick to speed up machine computations of complete descriptions by facet-defining inequalities of polyhedra given as convex hulls of finite sets of points and rays. The computational effort of this transformation is high (see [22] for the theoretical complexity). If it is performed on the basis of Fourier-Motzkin elimination, then the so-called adjacency decomposition [9] decreases the sizes of the inputs by first computing the list of facets of a face (usually a facet) $S$ of $P$ and then computing the rotated facets of $P$.

In a very similar way, rotation is used in the so-called local cuts approach for the machine computation of facet-defining inequalities in the so-called separation problem for combinatorial optimization polyhedra, see [1].

Another, well-established concept in practical polyhedral combinatorics which relies on rotation is that of sequential lifting. Here, the aim is to obtain facet-defining inequalities for a polyhedron $P$ out of facet-defining inequalities for a face $S$ of $P$ which is the intersection of $P$ with a number of coordinate hyperplanes. This is done by dealing with the involved coordinate hyperplanes one after another (whence the term “sequential”). The facet of $P$ one obtains may depend on the order in which the hyperplanes were dealt with.

It should be noted that, while relationships to applications are visible, the geometric questions we raise are not believed or meant to be directly applicable to, say, sequential lifting questions, machine computations, or polyhedral combinatorics of Traveling Salesman Polytopes. Instead, we believe them to be of interest in their own right, an opinion which we find to be vindicated by the apparent applicability to diverse polytopes and the beauty of the results we obtain for the very non-trivial case of the Traveling Salesman Polytope.

The remainder of this paper is organized as follows. In the next section, we describe the results we obtain for our examples, before considering the definition of the rotation complex $S$ more rigorously in Section 3. The fourth section contains the proofs for the result on Birkhoff polytopes and permutahedra, the sixth is dedicated to proving the results on the Traveling Salesman Polytopes.

We would like to emphasize that we do not assume the reader to be familiar with the example polyhedra involved beyond what can be found in [3, 40]. The results in this paper are geometric, and enjoying them requires familiarity with the general theory of convex polytopes in the sense of [17] or [40].

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1Some of the results of this paper have been applied, though: Building on them in an essential way, it was possible to prove a complete description of the Graphical Traveling Salesman Polyhedron on 9 cities, see [38].
2. Exposition of Results

We now give descriptions of rotation complexes for three (families of) pairs of polytopes. Only the results are described here; the proofs are deferred to later sections.

2.1. Birkhoff polytopes. As a first illustrative example, we will carry out the construction of the rotation complex $S$ in the case when $S$ is a Birkhoff polytope and $P$ a corresponding matching polytope.

Fix a positive integer $n$. We abbreviate $[n] := \{1, \ldots, n\}$. Let $S$ denote the Birkhoff polytope of dimension $2n$, i.e., the set of all real $n \times n$ matrices $X$ which satisfy the following system of inequalities and equations:

$$X_{k,l} \geq 0 \quad \forall k, l \in [n]$$  \hfill (1a)

$$\sum_{l=1}^{n} X_{k,l} = 1 \quad \forall k \in [n]$$  \hfill (1b)

$$\sum_{k=1}^{n} X_{k,l} = 1 \quad \forall l \in [n]$$

It is well known that the vertices of $S$ are the $n \times n$ permutation matrices \cite{5,39}. For the polytope $P$, we take matchings in the complete bipartite graph $K_{n,n}$. They correspond to 0/1-matrices $X$ satisfying

$$X_{k,l} \geq 0 \quad \forall k, l \in [n]$$  \hfill (2a)

$$\sum_{l=1}^{n} X_{k,l} \leq 1 \quad \forall k \in [n]$$  \hfill (2b)

$$\sum_{k=1}^{n} X_{k,l} \leq 1 \quad \forall l \in [n]$$

The bipartite matching polytope, $P$, is the convex hull of all these 0/1-matrices. It is a well-known fact that $P$ is equal to the set of solutions to (2). (Unfortunately, the original reference for this fact is not known to the author.) Clearly, $S$ is a face of $P$. As mentioned in the introduction, we must fix a point $z \in \text{relint } S$ which serves as $0$ in the definition of $S^\Delta$. The barycenter of the vertices $S$ is a natural choice for $z$, because is also the means of all the coefficient matrices corresponding to equations (1b). We will show the following.

**Proposition 1.** For $n \geq 3$, the rotation complex $S$ is the finest subdivision of $S^\Delta$ without new vertices, i.e., with $\text{vert } S = \text{vert } S^\Delta$. For two points $a, b$ in $S^\Delta$, the following are equivalent (local description):

- $a$ and $b$ are in the relative interior of the same face of $S$
- $a$ and $b$ are in the relative interior of the same face of $S$
- The two points can be written as convex combinations of the same sets of vertices. In other words, for every set of vertices $V$ of $S^\Delta$, we have $a \in \text{conv } V$ if and only if $b \in \text{conv } V$ holds.

The proof in Section 4.1 relies entirely on combinatorial properties of $S$. 
2.2. Permutahedra. Let $n \geq 1$ and $w \in \mathbb{R}^n$, with not all entries equal. We also assume that $\sum_k w_k = (n+1)/2$. Denote by $\Pi^{n-1}(w)$ the orbit polytope of the point $w$ under the operation of the group of all permutations of the set $[n]$ acting on $\mathbb{R}^n$ by permutation of variables, i.e., $\sigma.x = (x_{\sigma^{-1}(j)})_j$. In other words,

$$\Pi^{n-1}(w) := \text{conv}\{(w_{\sigma^{-1}(j)})_{j=1,\ldots,n} | \sigma \text{ permutation of } [n]\}. \quad (3)$$

The dimension of the orbit polytope is $n-1$, an equation defining the affine hyperplane $L$ containing $\Pi^{n-1}(a)$ is

$$\sum_k x_k = \binom{n+1}{2}. \quad (4)$$

We think of $L$ as a vector space with zero vector $(\frac{n+1}{2}, \ldots, \frac{n+1}{2})^\top$, i.e., the barycenter of the vertices of $\Pi^{n-1}(w)$, and denote it by $L$. The polar $\Pi^{n-1}(w)^\circ$ is then constructed as a point set in $L$.

The permutahedron is $\Pi^{n-1} := \Pi^{n-1}((1, \ldots, n)^\top)$ is centrally symmetric and simple. Its facets correspond bijectively to non-empty proper subsets of $[n]$, where the correspondence is realized by associating to a set $U \subseteq [n]$ the facet of $\Pi^{n-1}$ defined by the inequality $\sum_{k \in U} x_k \leq \binom{n+1}{2} - \binom{|U|}{2}$.

For every $n' > n$, the permutahedron $S := \Pi^{n-1}$ is a face of the permutahedron $P := \Pi^{n'-1}$, defined by the inequalities

$$\sum_{k=l}^{n'} x_k \leq \binom{n'+1}{2} - \binom{l}{2} \quad \text{for } l = n+1, \ldots, n', \quad (5)$$

If $n' = n + 1$, the rotation complex $S$ for $S \subset P$ can be characterized in a nice way. For this, we define a family of orbit polytopes $S_r := \Pi^{n-1}(w(r))$, by

$$w(r) := (1, 2, \ldots, r-1, 2r-n, r+2, r+3, \ldots, n+1)^\top \quad (6)$$

for $r = 1, \ldots, n$. Note that $\sum_j w_j(r) = \binom{n+1}{2}$. Thus $S_n = \Pi^{n-1} = S$, and it can be easily verified that $S_j \subset S_{j-1}$ for $j = 2, \ldots, n$. Hence we have

$$S_{\Delta} = S_n \supset S_{n-1} \supset \cdots \supset S_1.$$

Using these “onion skins”, we can now describe the rotation complex.

**Theorem 2 (a).** The facets of the rotation complex for $\Pi^{n-1} \subset \Pi^n$ are the sets $C \cap S_r^\Delta \setminus \text{relint}(S_{r+1}^\Delta)$, with $C$ a facet of the face fan of $\Pi^{n-1}$ in $L$.

We can write this differently. It is an elementary exercise to check that the vertices of $S_r^\Delta$ are of the form $\nu_{r,l}a_U$, with $\nu_{r,l} \in [0,1]$, where $a_U$ is the vertex of $S_{\Delta}$ corresponding to the set $U \subset [n]$. (Not all of these points are vertices.) Moreover, it can be verified that the face fan of $S_{\Delta}$ is a refinement of the face fan of $S_r$, $r = 1, \ldots, n$. Now define the split face fan $F_r$ of $S_r^\Delta$ to be the common refinement of the face fan of $S_r^\Delta$ and $S_r^\Delta$ itself; in other words, each facet $C$ of the face fan is replaced by the two polyhedra $C \cap S_r^\Delta$ and $C \setminus \text{relint}(S_r^\Delta)$. We have the following global description of the rotation complex.
Theorem 2 (b). The rotation complex for $\Pi^{n-1} \subset \Pi^n$ is the common refinement of $(\Pi^{n-1})^\Delta$ and the split faces $\mathcal{F}_r$, $r = 1, \ldots, n$. This subdivides each $(n-1)$-dimensional cone in the face fan of $(\Pi^{n-1})^\Delta$ into $n$ simplices and one unbounded polyhedron. In particular, the subdivision of $(\Pi^{n-1})^\Delta$ is a diagram.

We note that the subdivision of $S^\Delta$ is a diagram whenever $P$ is simple (this follows easily from Corollary 10 in Section 3). Theorem 2(a) immediately gives a local description of the rotation complex:

**Corollary 3.** Let $a, b \in (\Pi^{n-1})^\Delta \subset L$. The two points are in the same face of the rotation complex, if and only if the following properties hold:

1. They are on the same side of each of the hyperplanes $x_k - x_l = 0$, and
2. Let $\sigma$ be a permutation with $a_{\sigma(1)} \leq \cdots \leq a_{\sigma(n)}$. For each $r$, $a$ and $b$ are on the same side of the hyperplane defined by $(\sigma \cdot w(r)) \cdot x = 1$.

2.3. Symmetric Traveling Salesman Polytopes. The proofs in the previous section make use of known properties of the facial structures of the respective polytopes in a fundamental way. The question arises whether non-trivial local or global descriptions of rotation complexes can be obtained in cases when one is not so lucky as to have such information at his disposal. We now see an example of a family of pairs of polyhedra $S$ and $P$ for which complete descriptions in terms of linear inequalities are elusive. Nonetheless, both local and global descriptions of (the important) parts of the rotation complex can be derived just from the “relationship” between the polyhedra involved.

Fix an integer $n \geq 3$. Let $E_n$ be the set of all two-element subsets of the set $[n] := \{1, \ldots, n\}$, in other words, $E_n$ is the set of all possible edges in graphs with vertex set $[n]$. The Symmetric Traveling Salesman Polytope is defined as the convex hull in $\mathbb{R}^{E_n}$ of all edge sets of circles with vertex set $[n]$ (or Hamiltonian cycles in the complete graph $K_n$):

$$S = S_n := \text{conv}\{\chi^{E(C)} \mid C \text{ is the circle with } V(C) = [n]\},$$

where $\chi^F$ denotes the characteristic vector of a set $F$, i.e., $\chi^F_e = 1$ if $e \in F$ and 0 otherwise. Ever since the mid nineteen-fifties, when a series of short communications and papers initiated the study of this family of polytopes [18, 19, 20, 23, 29], it has received steady research attention. Apart from being of importance in combinatorial optimization for solving the famous Traveling Salesman Problem, which consists in finding a shortest Hamilton cycle in a complete graph with “lengths” assigned to the edges (see, e.g., [2, 11, 16, 21, 24, 33]), their combinatorial properties have been an object of research. For example, questions about aspects of the graph (1-skeleton) have been addressed [34], particularly focusing on its diameter [31, 32, 35, 36], which is conjectured to be equal to two by Grötschel & Padberg [16].
The second polyhedron which we will consider is defined to be the convex hull of all edge multi-sets of connected Eulerian multi-graphs on the vertex set \([n]\):

\[
P = P_n := \text{conv}\{x \in \mathbb{Z}_{+}^{E_n} \mid x \text{ defines a connected Eulerian multi-graph with vertex set } [n]\},
\]

identifying sub-multi-sets of \(E_n\) with vectors in \(\mathbb{Z}_{+}^{E_n}\). This polyhedron was introduced in \([10]\) under the name of “Graphical Traveling Salesman Polyhedron” and has since frequently occurred in the literature on Traveling Salesman Polyhedra. It is particularly important in the study of properties, mainly facets, of Symmetric Traveling Salesman Polytopes (e.g., \([13, 25, 26, 27, 28]\), see \([2, 24]\) for further references).

With few exceptions (for example \([12, 29]\) for the case \(n \leq 5\); \([6]\) for \(n = 6, 7\); \([7, 8, 9]\) for \(n = 8, 9\)), no complete characterization of the facets of \(S\) or \(P\) are known. A particularly noteworthy argument for the complexity of these polytopes is a result of Billera & Saranarajan \([4]\): For every 0/1-polytope \(P\), there exists an \(n\) such that \(P\) is affinely isomorphic to a face of \(S\).

\(P\) has been called the “Graphical Relaxation” of \(S\) by Naddef & Rinaldi \([26, 27]\) who discovered and made use of the fact that \(S\) is a face of \(P\): While the latter is a full-dimensional unbounded polyhedron in \(\mathbb{R}^{E_n}\) \([10]\), the former is a polytope of dimension \((\binom{n}{2}) - n\) \([29]\), and the inequality \(\sum_{e \in E_n} x_e \geq n\) is valid for \(P\) and satisfied with equality only by cycles, certifying the face relation.

We now explain our results. As mentioned in the introduction, we describe \(S\) only on a sub-complex of \(C(S^\Delta)\) instead of on the whole polytope \(S^\Delta\).

More precisely, we consider the set of faces of \(S^\Delta\) which do not contain a vertex corresponding to a non-negativity inequality \(x_e \geq 0\) for \(e \in E_n\): if \(N\) denotes the set of these vertices of \(S^\Delta\), we write this as

\[
\text{dl}(N, S^\Delta) := \{F \text{ face of } S^\Delta \mid F \cap N = \emptyset\},
\]

(avoiding the slightly more cumbersome notation \(\text{dl}(N, C(S^\Delta))\)). We obtain both a global and a local description of the restriction of \(S\) to \(\text{dl}(N, S^\Delta)\).

For the global description, the complex \(T\) is essentially a sub-complex of the metric cone. The metric cone, \(C = C_n\), consists of all (semi-)metrics on \([n]\), i.e., points \(d \in \mathbb{R}^{E_n}\) which satisfy the triangle inequality, i.e.,

\[
d_{vu} + d_{uw} - d_{vw} \geq 0
\]

for all distinct \(u, v, w\). If we now let \(F_{u,vw}\) denote the face of \(C\) defined by inequality \((10)\) we define the TT-fan as follows:

\[
T' := \bigcap_{u \in [n]} \bigcup_{v, w \neq u} C(F_{u,vw}) \subset C(C),
\]

where “TT” stands for “tight triangular”, a term coined by Naddef & Rinaldi \([27]\) for a point’s property of being in \(T'\). However, we are not aware of any reference to this fan in the literature. Heuristically, the elements of \(T'\) are metrics on \([n]\) satisfying the following: for every point \(u \in [n]\), there exist two other points \(v, w \in [n]\) such that \(u\) lies in the middle between \(v\) and \(w\).
After a natural orthogonal projection onto the space of $S$, we obtain the fan $\mathcal{T}$ isomorphic to $\mathcal{T}'$, which we shall call the flat $\mathcal{T}$-fan. Hence, we have the following global description.

**Theorem 4 (a).** The restriction of the rotation complex $S$ to $\text{dl}(N, S^\Delta)$ is the common refinement of the latter and the flat $\mathcal{T}$-fan $\mathcal{T}$.

We come to the local description of $S$. Let $a \in S^\Delta$. For every $u \in [n]$, we let $E_u(a)$ be the set of edges on which the slack of the triangle inequality (10) is minimized:

$$E_u(a) := \left\{ vw \in E_n \mid u \neq v, w, \text{ and } a_{vu} + a_{vw} - a_{vw} = \min_{v',w' \neq u} a_{v'u} + a_{uw'} - a_{v'w'} \right\}. \quad (12)$$

The minimum in (12) may be negative. We will prove the following local description of the rotation complex.

**Theorem 4 (b).** Two points $a, b$ in $|\text{dl}(N, S^\Delta)|$ are in the relative interior of the same face of the rotation complex $S$ if, and only if, they are in the relative interior of some face of $S^\Delta$ and $E_u(a) = E_u(b)$ for all $u \in [n]$.

As mentioned in the introduction, the rotation complex can be defined as the image complex of a subcomplex of $P^\Delta$ under a projective mapping $\pi$.

The property that $\pi$, probably restricted to a subcomplex of the complex of $P^\Delta$, becomes injective, is a strong fact. It means that the complex of $P^\Delta$ can be “flattened” to obtain a refinement of the complex of $S^\Delta$. In Section 5 we will undertake some effort to prove injectivity for the case of Traveling Salesman Polyhedra, a fact which is used to prove the completeness of a description by linear inequalities of the Graphical Traveling Salesman Polyhedron for nine cities [30, 38].

In the case of the Traveling Salesman polyhedra, this subcomplex is what remains of the complex $\bar{C}(P^\Delta)$ of bounded faces of $P^\Delta$ after deleting the conjugate face of $S$ in $P^\Delta$, in symbols $\text{dl}(S^\Diamond, \bar{C}(P^\Delta))$.

**Theorem 5.** $\pi: |\text{dl}(S^\Diamond, \bar{C}(P^\Delta))| \to |\text{dl}(N, S^\Delta)|$ is a homeomorphism and induces a combinatorial equivalence between the former polyhedral complex and the rotation complex restricted to $\text{dl}(N, S^\Delta)$.

3. **General Definition and Basic Properties of a Rotation Complex**

Studying rotation complexes in concrete examples requires genuine definitions and reasoning, particularly when only sub-complexes of $S^\Delta$ are involved. None the less, we find it useful to make some elementary remarks about the general case, the intention being to provide geometric intuition and introduce some technical machinery. Everything in this section ought to be folklore in the polyhedral theory community; surprisingly, though, I was unable to find it written down anywhere.

Some notation is required. We define polar objects in the same spaces as the original objects, using as inner product the obvious choice in each case. Denote the inner product of $x$ and $y$ by $x \cdot y$. For a set of points $X$, we let $\text{aff } X$ be its
affine hull and denote the corresponding linear space by \( \text{lin } X := \text{span}(X - x) \), for an arbitrary \( x \in \text{aff } X \).

In what follows, \( S \) will be a polyhedron which is a face of a polyhedron \( P \) in \( \mathbb{R}^m \). We let \( L := \text{lin } S \), and \( L^\perp \) the orthogonal complement of \( L \), i.e., the vector space of all left-hand sides of equations satisfied by all points of \( S \).

3.1. Rotation complexes for cones. We first discuss the rotation complex for the case when \( P \) is a convex polyhedral cone and \( S \) a proper unbounded face of \( P \). We define \( S \) as sketched in the introduction. More precisely, for an \( a \in S^\triangle := \{ a \in L \mid a \cdot x \leq 0 \ \forall x \in S \} \), we define the set \( \mathcal{F}(a) \) of all faces of \( P \) which are defined by inequalities obtained from \( a \) by rotation: a face \( F \) of \( P \) is in \( \mathcal{F}(a) \) if and only if there exists a \( q \in L^\perp \) such that the inequality

\[
(a + q) \cdot x \leq 0
\]

is satisfied by all \( x \in P \) and equality holds if and only if \( x \in F \). Clearly, the face of \( S \) defined by \( a \) is in \( \mathcal{F}(a) \).

Now, let \( S^\circ \) be the partition of \( S^\triangle \) given by the equivalence relation \( a \sim b :\iff \mathcal{F}(a) = \mathcal{F}(b) \), and define \( S \) as the set of all closures of sets in \( S^\circ \). We note the following fact for easy reference.

**Lemma 6.** With the above notations, \( S \) is a fan which subdivides \( S^\triangle \).

**Proof.** The partition \( S^\circ \) can be defined by the orthogonal projection \( p: \mathbb{R}^m \to L: \) it is readily verified that \( p(P^\triangle) = S^\triangle \). Moreover, it is an easy folklore fact that a projection of polyhedra \( p: Q \to R \) defines a subdivision of \( R \), where two points \( a, b \) are in the relative interior of the same face if and only if their fibers \( p^{-1}(a) \) and \( p^{-1}(b) \) intersect the same set of faces of \( Q \). The subdivision of \( S^\triangle \) defined by this projection clearly coincides with \( S \). \( \square \)

3.2. Rotation complexes for bounded faces. Now let \( S \) be a non-empty proper, bounded face of a polyhedron \( P \). Fixing a relative interior point \( z \in \text{relint } S \), we can define the polar of \( S \) as the polar of the polytope \( S' := S - z \) in the ambient space \( L \). By a slight abuse of notation, we let

\[
S^\triangle := S'^\triangle = \{ a \in L \mid a \cdot x \leq 1 \ \forall x \in S' \} = \\
= \{ a \in L \mid a \cdot x \leq 1 + a \cdot z \ \forall x \in S \}
\]

As in Section 3.1, for a \( a \in S^\triangle \), we define the set \( \mathcal{F}(a) \) of all faces of \( P \) which are defined by inequality obtained from \( a \) by rotation. Here, a face \( F \) of \( P \) is in \( \mathcal{F}(a) \) if and only if there exists a \( q \in L^\perp \) such that the inequality

\[
(a + q) \cdot x \leq 1 + a \cdot z + q \cdot z
\]

is satisfied by all \( x \in P \) and equality holds if and only if \( x \in F \). Again, the face of \( S \) defined by \( a \) is in \( \mathcal{F}(a) \).

Let \( S^\circ \) be the partition of \( S^\triangle \) given by the equivalence relation \( a \sim b :\iff \mathcal{F}(a) = \mathcal{F}(b) \), and denote by \( S \) the set of all closures of sets in \( S^\circ \).

Note that, if \( a \in F \) for some face \( F \) of \( S^\triangle \), then the closure of the equivalence class containing \( a \) is contained in \( F \).
Lemma 7. With the above notations, \( S \) is a subdivision of \( S^\triangle \).

Proof. This is readily gained as a corollary of Lemma 6 by homogenization, invoking a linear mapping \( T: \mathbb{R} \times \mathbb{R}^m \to \mathbb{R} \times \mathbb{R}^m \) defined by the matrix \( \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \). We omit the technicalities. \(\square\)

It is of course important to know that \( S \) does not depend on the choice of \( z \). The following lemma gives the technical statement, whose prove is elementary.

Lemma 8. Let \( z' \) be another relative interior point of \( S \), let \( S' \triangle \) be defined as in (13), but with \( z \) replaced by \( z' \), and let \( T: S^\triangle \to S'^\triangle \) be the canonical projective isomorphism. For \( a \in S^\triangle \), the set of all inequalities of the form (14) is equal to the set of all inequalities of the form (14) with \( a \) replaced by \( T(a) \) and \( z \) replaced by \( z' \). \(\square\)

In the case of cones in Section 3.1, the subdivision \( S \) is defined as intersections of images of faces under a projection \( P^\triangle \to S^\triangle \). When \( S \) and \( P \) are polytopes, this is still true, with the projection now being projective. Moreover, we may restrict to a subcomplex of \( P^\triangle \). We will discuss this now.

3.3. Alternative definition of the rotation complex \( S \). We now assume that \( P \) is bounded. Observe that \( S \) as we have just defined it, is invariant under translations of \( P \): If, for some \( y \in \mathbb{R}^m \), \( P \) is replaced by \( P + y \), \( S \) by \( S + y \) and \( z \) by \( z + y \), then the sets \( S^\triangle \) and \( S^\circ \) are the same. We can thus assume that \( P \) contains 0 as an interior point, and we can define the polar of \( P \) as a point set:

\[
P^\triangle := \{ a \in \mathbb{R}^m \mid a \cdot x \leq 1 \quad \forall x \in P \}.
\]

Since we are now dealing with multiple polar polytopes (and even polar cones) at the same time, if \( F \) is a face of \( S \) (and \( S \) a face of \( P \) ), we need to make clear what we mean by

- the conjugate face of \( F \) in \( S^\triangle \), namely the set
  \[
  F^\diamond_S := \{ a \in S^\triangle \mid a \cdot x = 1 + a \cdot z \quad \forall x \in F \} \subset L;
  \]
- and the conjugate face of \( F \) in \( P^\triangle \), respectively, which is the set
  \[
  F^\diamond_P := \{ a \in P^\triangle \mid a \cdot x = 1 \quad \forall x \in F \} \subset \mathbb{R}^m.
  \]

We will usually omit the indices on the \( \diamond \).

Given an inequality \( a \cdot x \leq 1 \) valid for \( P \), we can project it into \( L \) and obtain a valid inequality for \( S \):

\[
p(a) \cdot x \leq 1 - (a - p(a)) \cdot z \quad \forall x \in S 
\]

where \( p: \mathbb{R}^m \to L \) denotes the orthogonal projection, as for cones in Section 3.1 above (this also coincides with the \( p \) used in Section 2.3 on page 8). If (15) is not satisfied with equality by all points in \( S \), i.e., if the point \( a \) of \( P^\triangle \) is not contained in the conjugate face \( S^\diamond \) of \( S \) in \( P^\triangle \), then this inequality defines a point in \( S^\diamond \), namely

\[
\pi(a) := \frac{1}{1 - a \cdot z} p(a).
\]

In other words, we have a projective mapping \( \pi: \mathbb{R}^m \to L \) which maps \( P^\triangle \setminus S^\circ \) to \( S^\diamond \). The way to proceed now is to restrict \( \pi \) to a sub-complex of the face complex
that $x \in \mathcal{D}$ be the set of all faces of $P^\Delta$ which have empty intersection with $S^\Delta$, in other words: $\mathcal{D} := \text{dl}(S^\Delta, \mathcal{C}(P))$ the deletion of $S^\Delta$ in $\mathcal{C}(P)$.

As usual, for a polyhedral complex $C$, we denote by $|C|$ the underlying point-set of $C$. As a crucial fact, the mapping

$$\pi : |\mathcal{D}| \rightarrow S^\Delta$$

is onto; we defer the proof of this fact to Lemma 11 below. In 3.4, we will show that $S$ can equivalently be defined as the image complex of $\mathcal{D}$ under $\pi$, i.e., the polytopal complex whose faces are intersections of images under $\pi$ of faces of $\mathcal{D}$. More precisely, we define a partition $S^\circ_\pi$ of $S^\Delta$ by letting two points $a$, $b$ be in the same set if and only if their fibers $\pi^{-1}(a) := \{c \in \mathbb{R}^m \mid \pi(c) = a\}$ and $\pi^{-1}(b)$ intersect the same set of faces of $\mathcal{D}$.

Below we will show that the two definitions $S_\pi$ and $S$ coincide. For now, assuming this is prove, it appears sensible to visualize what $\pi$ does geometrically. Since $S^\Delta$ is combinatorially equal to $P^\Delta/S^\circ$, we might want to think of $S^\Delta$ as the intersection of $P^\Delta$ with a suitable $(\dim L)$-dimensional affine subspace of $\mathbb{R}^m$. Since the definition based on equation (14) is commutes with affine isomorphisms of $\mathbb{R}^m$, we may assume, for the moment, the following ideal situation. Letting $m_1 := \dim S^\circ_\pi$, write $\mathbb{R}^m = \mathbb{R} \times \mathbb{R}^{m_1} \times L$, and assume $z = (1,0,0)^T$, and that $S^\circ$, the face of $P^\Delta$ defined by $z$, is equal to $(1 \times \mathbb{R}^{m_1} \times 0) \cap P^\Delta$. Intersecting $(0 \times 0 \times L)$ with the cone generated by the point $z$ and the rays $a - z$, for $x \in P^\Delta$, we obtain a face figure of $S^\circ$. Then for $(\xi, a, b) \in P^\Delta \setminus S^\circ$, we have that $\pi(\xi, a, b) = b/(1-\xi)$ is the orthogonal projection of the point onto $\mathbb{R} \times 0 \times L$ followed by radial projection with respect to the point $(1,0,0)$ onto the face figure. Thus, the rotation complex may be understood as a projection onto a face figure. The reader be warned, however, that non-canonical constructions of the face figure and projection may yield subdivisions of $S^\Delta$ which are not isomorphic to the rotation complex.

**3.4. Equivalence of the two definitions.** Apart from technicalities, which include the fact that $S^\circ$ is defined using the whole of $\mathcal{C}(P^\Delta)$ while $S$ is defined using a sub-complex $\mathcal{D}$ of $\mathcal{C}(P^\Delta)$, it should be obvious that $S^\circ = S^\circ_\pi$. For the sake of completeness, we go through the technicalities.

**Lemma 9.** For a fixed $a \in S^\Delta$, the intersection of the fiber $\pi^{-1}(a)$ with $|\mathcal{D}|$ is the set of points of $|\mathcal{D}| \subset P^\Delta$ corresponding to valid inequalities of the form (14).

**Proof.** We prove the following.

(a) Let $q \in L^\perp$ such that the inequality (14) is valid for $P$. Then $\pi$ maps the point $b := (a+q)/(1+(a+q) \cdot z)$ of $P^\Delta$ corresponding to the inequality to $a$.

(b) If $\pi(b) = a$ for a $b \in P^\Delta \setminus S^\circ$, then there exists a $\beta > 0$ such that the inequality $\beta b \cdot x \leq \beta$ is of the form (14).

To make sense of the first statement, we must make sure that the right hand side $1 + (a + q) \cdot z$ of inequality (14) is non-zero. Assume the contrary. Since the
inequality is valid, we must then have \( a + q = 0 \). But since \( a \in L \) and \( q \in L^\perp \), this implies that \( a = q = 0 \), whence the right hand side of (14) is 1, a contradiction.

The rest of the proof is totally straightforward, we could just as well omit it. The computation of the value of \( \pi \) for the first statement runs like this:

\[
\pi \left( \frac{1}{1+(a+q) \cdot z} (a + q) \right) = \frac{1}{1 - \frac{1}{1+(a+q) \cdot z}} p \left( \frac{1}{1+(a+q) \cdot z} (a + q) \right) =
\]

\[
\frac{1}{1 + (a + q) \cdot z - (a + q) \cdot z} a = a
\]

For the second statement, we note that \( \pi(b) = a \) implies that there exists an \( \alpha > 0 \) such that \( p(b) = \alpha a \) and \( 1 - b \cdot z = \alpha \). There exists a \( q \in L^\perp \) such that \( b = \alpha (a + q) \). Moreover, solving \( \alpha = 1 - (a + q) \cdot z \) yields \( \alpha = 1/(1+(a+z) \cdot z) \).

Letting \( \beta := 1/\alpha \) yields the claim. \( \square \)

We derive the following intermediate fact from Lemma 9.

**Corollary 10.** If \( F \in \mathcal{D} \) and \( b \in \text{relint} F \), then \( \pi(b) \in \text{relint}(F^{\alpha_p} \cap S)^{\alpha_S} \). In particular, \( \pi \) maps \( \text{lk}(S^{\alpha}, P) \) into the boundary of \( S^\beta \).

**Proof.** \( b \) defines the face \( F^{\alpha_p} \) of \( P \). But all inequalities of the form (14) with \( a := \pi(b) \) define faces of \( P \) whose intersection with \( S \) is equal to the face of \( S \) defined by \( \pi(b) \). So \( \pi(b) \) defines the face \( F^{\alpha_p} \cap S \) of \( S \). \( \square \)

**Lemma 11.** \( \pi : |\mathcal{D}| \rightarrow S^\beta \) is onto.

Before we prove this, we remark that \( \pi \) is onto in the following “combinatorial” sense. It is elementary to see that for every face \( F \) of \( P \) there exists a face \( F' \) of \( P \) which is complementary to \( S \) in the polytope \( P/F \) in the sense that \( S \cap F' = F \) and \( S \sqcup F' = P/F \) (for example, apply Theorem 1.3.8 in [17] to \( P/F \) and \( S \)). This implies that the conjugate face \( F^{\alpha} \) of \( F' \) in \( P^\Delta \) is in \( \text{dl}(S^{\alpha}, C(P^\Delta)) \) and, by Corollary 10, that points in \( F^{\alpha} \) are mapped by \( \pi \) to points in the conjugate face \( F^\alpha \) of \( F \) in \( S^\Delta \). Hence, the relative interior of every face of \( S^\beta \) has non-empty intersection with the image under \( \pi \). The question is whether we obtain all points of \( S^\Delta \).

For the fact that \( \pi \) is onto, we give a genuinely geometric proof which makes use of the following nice elementary geometric fact.

**Fact 12.** Let \( C \) be a \( d \)-dimensional cone with apex 0 in some \( d \)-dimensional vector space \( V \), and \( A \) an affine subspace of \( V \) not containing 0, and having non-empty intersection with \( C \). Then \( A \) intersects a face of \( C \) of dimension at least 1 and at most \( d - \dim A \).

Moreover, if \( b_0 \in C \cap A \) is a point of a face \( F_0 \) of \( C \), then the intersecting face can be chosen among the faces of \( F_0 \).

**Proof.** The intersection of \( A \) with \( C \) is a non-empty polyhedron \( Q \), which is pointed because \( C \) is. Let \( v \) be one of its vertices. Then \( v \) is a relative interior point of some \( k \)-dimensional face of \( C \). Since \( k + \dim A \leq d \), the claim follows. \( \square \)

Fact 12 is the geometric content of Lemma 11, the rest is just technical:
**Proof of Lemma 11** Let \( a \in S^\Delta \), let \( F \) be the face of \( S \) defined by the corresponding inequality, and \( k := \dim F \). Let \( F^\Diamond \) be the conjugate face of \( F \) in \( P^\Delta \). We now consider the polar cone \( \{ (\beta, b) \in \mathbb{R} \times \mathbb{R}^m \mid \beta + b \cdot x \leq 0 \forall x \in P \} = \mathbb{R}_+ (1 \times P^\Delta) = \text{homog } P^\Delta \) of \( P \) in the space \( \mathbb{R} \times \mathbb{R}^m \) for which we use the canonical inner product. Here homog designates homogenization.

In the face \( C := \text{homog } F^\Diamond = \mathbb{R}_+ (1 \times F^\Diamond) \) of the polar cone, consisting of all inequalities valid for \( P \) defining a face containing \( F \), we need to find a point \((\beta, b)\) which is contained in a face of \( C \) of dimension at most \( S - k \), as will be made clear below. To achieve \( \pi(-b/\beta) = a \), by Lemma 9, the point \((\beta, b)\) must also be in the set \( A := \{ (-1 - (a + q) \cdot z, a + q) \mid q \in L^\perp \} \) of all inequalities of the form \((14)\).

To apply Fact 12, we need to make sure that \((a)\) \( A \subset \text{span } C \), \((b)\) \( 0 \not\in A \) and \((c)\) \( A \) contains a point of \( C \). For the first item, \((a)\), we note that \( \text{span } C \) is the set of all points which are orthogonal to \( \text{homog } F \). Thus we have to show that \( A \subset \text{homog } F^\perp \), or, stronger, \( A^\perp \supset \text{span}(1 \times F) \), for which it suffices to show \( 1 \times F \subset A^\perp \), which is obvious. Item \((b)\) is clear. For \((c)\), we just note that there exists a \( q_0 \in L^\perp \) such that the inequality \( (a + q_0) \cdot x \leq 1 + (a + q_0) \cdot z \) is valid for \( P \) and defines the face \( F \). Hence \((\beta_0, b) := (-1 - (a + q_0) \cdot z, a + q_0) \) is in \( C \).

Why does a \((\beta, b)\) with these properties solve the problem? It defines a face \( G \) of dimension \( m - \dim S + k \) of \( P \), such that \( G \cap S = F \). This implies that \( G \cap S \) has dimension at least \( \dim G + \dim S - \dim(G \cap S) = m \), so, if \( G^\Diamond \) is the conjugate face of \( G \) in \( P^\Delta \), we have \( G^\Diamond \cap S^\Diamond = \emptyset \) and \(-b/\beta \in G^\Diamond \). All in all, we have \(-b/\beta \in |D| \) and \( \pi(-b/\beta) = a \). \( \square \)

From the proof we extract the following sharpening of Lemma 11.

**Remark 13.** Suppose there is a point \((\beta_0, b_0) \in C \) with \( \pi(-b_0/\beta_0) = a \), such that \((\beta_0, b_0) \) is a relative interior point of some face \( \text{homog } F_0^\Diamond \) of \( \text{homog } F^\Diamond \), where \( F_0 \) is a face of \( P \) containing \( F \). Then, using the second statement in Fact 12 we can obtain a \((\beta, b)\) in the fiber defining a face \( G \) of \( P \) with \( G^\Diamond \in D \) and \( G \supset \triangle F_0 \). Then we compute the following
\[
G \cap (F_0 \setminus S) = G \cap F_0 \setminus G \cap S = F_0 \setminus F = F_0.
\]

We come to the remaining fact which needs to be addressed.

**Lemma 14.** \( S^\Diamond = S_0^\Diamond \).

**Proof:** We have to show the following: For two points \( a \) and \( b \) in \( S^\Delta \), we have \( \mathcal{F}(a) = \mathcal{F}(b) \) if and only if \( \pi^{-1}(a) \) and \( \pi^{-1}(b) \) intersect the same set of faces of \( D \). We denote by \( \mathcal{F}'(a) \) the set of all conjugate faces in \( P^\Diamond \) of elements of \( \mathcal{F}(a) \), for \( a \in S^\Delta \). By Lemma 9, we have \( F \in \mathcal{F}(a) \) if and only if \( \pi^{-1}(a) \) contains a relative interior point of the conjugate face \( F^\Diamond \) of \( F \) in \( P^\Delta \). In other words, for all \( a, b \in S^\Delta \), equality \( \mathcal{F}(a) = \mathcal{F}(b) \) is equivalent to \( \mathcal{F}'(a) = \mathcal{F}'(b) \). Again by Lemma 9, \( \pi^{-1}(a) \) and \( \pi^{-1}(b) \) intersect the same set of faces of \( D \) if and only if \( \mathcal{F}'(a) \cap D = \mathcal{F}'(b) \cap D \). It only remains to be shown that this latter equation implies \( \mathcal{F}'(a) = \mathcal{F}'(b) \). In other words, we have to show that the set of faces of \( P^\Delta \) which
intersect a given fiber is completely determined by the set of faces of \( \mathcal{D} \) which intersect it.

For every \( G^\Diamond \in \mathcal{F}'(a) \cap \mathcal{D} \) and every face \( H^\Diamond \) of \( S^\Diamond \subset P^\Diamond \), we have \( F_0^\Diamond := G^\Diamond \lor H^\Diamond \in \mathcal{F}'(a) \). To see this, let \( q \) be a relative interior point of \( H^\Diamond \). Noting that \( S^\Diamond \subset L^\perp \), we see that the inequality \( (a + q) \cdot x \leq 2 \) is of the form \( 14 \) and defines the face \( G \cap H = (G^\Diamond \lor H^\Diamond)^\Diamond \) of \( P \). This means \( G^\Diamond \lor H^\Diamond \in \mathcal{F}'(a) \).

This motivates us to verify that for every \( F_0^\Diamond \in \mathcal{F}'(a) \) with \( H^\Diamond := F_0^\Diamond \cap S^\Diamond \neq \emptyset \), we can find a face \( G^\Diamond \in \mathcal{F}'(a) \cap \mathcal{D} \) with \( G^\Diamond \lor H^\Diamond = F_0^\Diamond \). Remark \( 13 \) implies this by letting \( b_0 \) be a relative interior point of \( F_0^\Diamond \) with \( \pi(b) = a \), and \( \beta_0 := -1 \).

### 3.5. Some general properties of rotation complexes

We summarize a number of basic properties of rotation complexes.

**Proposition 15.**

(a) If \( F \in \mathcal{D} \) and \( b \in \text{relint}(F) \), then \( \pi(b) \in \text{relint}(F^\Diamond \cap S)^\Diamond \). (This is Corollary \( 10 \))

(b) If \( P \) is simple, then \( S \) is a diagram.

(c) If \( S \) is a facet of \( P \), then \( \pi : |\mathcal{D}| \to S^\Diamond \) is injective.

**Proof.** (b). Follows immediately from (a). Since we are making a combinatorial statement, we may identify \( S^\Diamond \) with \( P^\Diamond / S^\Diamond \). If \( G \) is a proper face of \( S^\Diamond \), then there exists a face \( F \) of \( \mathcal{D} \) with \( G = F \lor S^\Diamond \), which is unique because \( P^\Diamond \) is simplicial.

(c). If \( S \) is a facet, then \( L^\perp \) is one-dimensional, so there is a unique \( q \) such that the inequality \( 14 \) defines a non-empty face of \( P \). The claim now follows from Lemma \( 9 \).

Clearly, if \( S \) is not a face of \( P \), then \( \pi \) is not in general injective, but it may still be.

4. Proofs for Birkhoff polytopes and Permutahedra

#### 4.1. Proof of Proposition 1 (Birkhoff polytope)

The canonical inner product in the matrix space \( \mathbb{M}(n) \) of all real \( n \times n \) matrices is \( X \cdot Y := \sum_{k,l} X_{k,l} Y_{k,l} \).

Let \( E_{k,l} \) the matrix with zeroes everywhere except \( E_{k,l} = 1 \), and \( Q^k := \sum_{l=1}^n E_{k,l}, Q^l := \sum_{k=1}^n E_{k,l}, \) i.e., \( Q^k \) and \( Q^l \) matrices which all zero entries except for a single row or column of ones. The vector space \( L \) is the set of all \( X \in \mathbb{M}(n) \) for which equations \( 11b \) with the right hand side replaced by 0, i.e.,

\[
L := \{ X \in \mathbb{M}(n) | Q^j \cdot X = 0, Q^k \cdot X = 0 \forall k, l \}.
\]

For a polar polytope \( P^\Diamond \) of \( P \) we take the intersection of the polar cone \( \{ (\alpha, A) \in \mathbb{R} \times \mathbb{M}(n) \ | \ \alpha + A \cdot X \leq 0 \forall X \in P \} \) of \( P \) with the hyperplane defined by

\[
(n + 1)\alpha - \sum_{k,l} A_{k,l} = 1.
\]

This way, \( P^\Diamond \) is projectively isomorphic to what one would obtain by translating \( P \). Assuming \( n \geq 2 \), it is well-known and easy to see that the vertices of \( P^\Diamond \) are
• the points \((0, E^{k,l})\), for \(k, l \in [n]\), which are the vertices of the face of \(P^\Delta\) defined by the inequality \(\alpha \leq 0\); and
• the points \((-1, Q^{k})\), \(k \in [n]\), and \((-1, Q^{l})\), \(l \in [n]\), which are the vertices of the face of \(P^\Delta\) defined by \(\alpha \geq -1\).

The conjugate face \(S^\Diamond\) of \(S\) in \(P^\Delta\) is the latter face, and \(\mathcal{D} := \text{dl}(S^\Diamond, C(P^\Delta))\) is just the former face defined by \(\alpha \leq 0\). This is an \((n^2 - 1)\)-dimensional simplex. We also assume that \(n \geq 3\), which implies that the vertices of \(S^\Delta\) correspond precisely to the matrices \(E^{kl}\). In accordance with this bijection, we denote the vertices of \(S^\Delta\) by \(v_{k,l}, k, l \in [n]\).

Writing down \(S^\Delta\) and \(\pi\) is somewhat awkward, but we do not need to: we can construct the subdivision \(S\) of \(S^\Delta\) from what we just said. Let \(F \in \mathcal{D}\). Then, as we said, \(F\) is a simplex, with vertices, say \(\text{vert} F = \{E^{k_j,l_j} \mid j = 0, \ldots, \dim F\}\). The image of \(F\) under \(\pi\) is thus the convex hull of the corresponding vertices \(v_{k_j,l_j}\), \(j = 0, \ldots, \dim F\) of \(S^\Delta\).

Hence, \(S\) is the set of all intersections of polytopes spanned by subsets of the vertex set of \(S^\Delta\), and as such the finest subdivision of \(S^\Delta\) whose 0-dimensional faces are vertices of \(S^\Delta\).

4.2. **Proof of Theorems 2 (Permutahedron).** First of all, recall that a linear function \(x \mapsto a \cdot x\) is maximized over \(P = \Pi^n\) by a vertex \((\sigma^{-1}(1), \ldots, \sigma^{-1}(n + 1))^\top\) if and only if \(a_{\sigma(1)} \leq a_{\sigma(2)} \leq \ldots \leq a_{\sigma(n)} \leq a_{\sigma(n+1)}\).

To prove Theorem(s) 2 first note that by Proposition 15, the subdivision \(S\) of \(S^\Delta\) is in fact a diagram. Now let \(a \in \text{relint} S^\Delta\). In view of Lemma 9, we want to characterize all inequalities of the form (14) which define non-empty faces of \(P\) containing only vertices corresponding to permutations \(\sigma\) with \(\sigma(n + 1) \leq n\).

The linear space \(L\) defined in Section 2.2 embeds as \(L \times 0\) into \(\mathbb{R}^{n+1}\). We denote \(L \times 0\) again by \(L\). As a basis for \(L^\perp\), In view of (5) and (4), we may use the vector

\[
q := (0, \ldots, 0, 1)^\top \in \mathbb{R}^{n+1}.
\]

Hence, in this example, inequalities of the form (14) read

\[
\sum_{k=1}^{n} a_k x_k + \mu x_{n+1} \leq 1 + \mu(n + 1) \tag{17}
\]

(note that \(a \cdot z = 0\)).

In characterizing the \((n-1)\)-dimensional faces of \(S\), we may assume that \(a\) is in general position, i.e., it has \(n\) pairwise distinct entries. Now let \(\sigma\) be a permutation of \([n]\) such that \(a_{\sigma(1)} < \cdots < a_{\sigma(n)}\), and take \(r\) to be an integer satisfying

\[
a_{\sigma(r-1)} \leq \mu \leq a_{\sigma(r)} \tag{18}
\]

with \(a_0 \colonequals -\infty\) and \(a_{n+1} \colonequals \infty\) temporarily. The maximum of the left hand side of (17) is now maximized by the vertex of \(\Pi^n\) corresponding to the permutation

\[
\begin{array}{cccccc}
1 & \ldots & r-1 & r & r+1 & \ldots \n+1 \\
\sigma(1) & \ldots & \sigma(r-1) & \sigma(r+1) & \sigma(r) + 1 & \ldots \sigma(n) + 1
\end{array}
\]

(19)
As mentioned above, we may restrict to inequalities (17) defining non-empty faces, and thus rewrite (17) as an equation in the following form:

\[ \sum_{k=1}^{r-1} ka_{\sigma(k)} + \sum_{k=r}^{n} (k+1)a_{\sigma(k)} - 1 = \mu(n+1-r). \quad (20) \]

From (18), we conclude that such an \( a \) must satisfy two inequalities:

\[ (\sigma.w(r)) \cdot a = \sum_{k=1}^{r-1} ka_{\sigma(k)} + (2r-n)a_{\sigma(r)} + \sum_{k=r+1}^{n} (k+1)a_{\sigma(k)} \leq 1 \]

\[ (\sigma.w(r-1)) \cdot a = \sum_{k=1}^{r-2} ka_{\sigma(k)} + (2r-2-n)a_{\sigma(r-1)} + \sum_{k=r}^{n} (k+1)a_{\sigma(k)} \geq 1 \]

In other words, the image of the facet of \( P^\Delta \) corresponding to the permutation (19) is the set of all points in \( S^\Delta \) with \( a_{\sigma(1)} \leq \cdots \leq a_{\sigma(n)} \) satisfying \( (\sigma.w(r)) \cdot a \leq 1 \) and \( (\sigma.w(r-1)) \cdot a \geq 1 \). This completes the argument for the Permutahedron.

5. PROOFS FOR TRAVELING SALESMAN POLYHEDRA

In 5.1 we will need to discuss some properties of Symmetric and Graphical Traveling Salesman polyhedra. Most of them are generalizations of facts in the seminal papers by Naddef & Rinaldi [26, 27]. The proof of Theorems 4a, 4b and 5 then takes up Sections 5.3 and 5.4.

Let \( S := S_n \) and \( P := P_n \) as in Section 2.3. We assume in the whole section that \( n \geq 5 \), because, technically, we need the non-negativity inequalities \( x_e \geq 0 \), for an \( e \in E_n \), to define facets of \( S \), which is true if and only if \( n \geq 5 \), see [14, 15].

5.1. Preliminaries on connected Eulerian multi-graph polyhedra. As mentioned above, \( P \) is full-dimensional in \( \mathbb{R}^{E_n} \), and \( S \) has dimension \( \binom{n}{2} - n \) [29]. The set of facets of \( P \) containing \( S \) is known. For \( u \in [n] \), let \( \delta_u \) be the point in \( \mathbb{R}^{E_n} \) which is \( 1/2 \) on all edges incident to \( u \) and zero otherwise. It is proven in [10] that the inequalities \( \delta_u \cdot x \geq 1, u \in [n] \), define facets of \( P \), the so-called degree facets. Clearly, \( S \) is the intersection of all the degree facets. Naddef & Rinaldy [27] proved that every facet of \( S \) is contained in precisely \( n + 1 \) facets of \( P \); the \( n \) degree facets and one additional facet. This fact and its generalizations prove powerful for our purposes. We will prove it below, for the sake of completeness.

A triangle rooted at \( u \) is a pair \( u, vw \) consisting of a vertex \( u \in [n] \) and an edge \( vw \in E_n \) not incident to \( u \). Let \( a \in \mathbb{R}^{E_n} \). We say that \( a \) is metric, if it satisfies the triangle inequality, i.e., \( t_{u,vw}(a) := a_{vu} + a_{uw} - a_{vw} \geq 0 \) for all rooted triangles \( u, vw \). Note that this implies \( a_e \geq 0 \) for all \( e \). We follow [27] in calling a tight triangular (TT), if it is metric and for each \( u \in [n] \) there exists \( v, w \) such that the triangle inequality for this rooted triangle is satisfied with equation: \( t_{u,vw}(a) = 0 \). Abusively, we say that a linear inequality is metric, or TT, if the left hand side vector has the property. Given a vertex \( u \) and an edge \( vw \) not incident to \( u \), a shortcut is a vector \( s_{u,vw} := \chi^{uw} - \chi^{vu} - \chi^{vw} \in \mathbb{R}^{E_n} \).
In the following lemma, we summarize basic facts about tight triangularity. Let $D$ denote the $[n] \times E_n$-matrix whose rows are the $\delta^u_v$, $u \in [n]$. Define the linear space $L := \ker D$, and note that the orthogonal complement $L^\perp$ of $L$ is equal to $\text{im } D^\top = \{ D^\top \xi \mid \xi \in \mathbb{R}^{|n|} \}$. This definition of $L$ coincides with that of Section 3.

For brevity, we say that a face $F$ of $P$ is good if it is not contained in a non-negativity facet, i.e., a facet defined by $x_e \geq 0$ (these inequalities do define facets of $P$\cite{10}).

**Lemma 16.**

(a) A metric inequality which is valid for $S$ is also valid for $P$.

(b) An inequality defining a good face of $P$ is metric.

(c) An inequality defining a good face $F$ of $P$ is TT if and only if $F$ is not contained in a degree facet.

(d) If a face $F$ of $P$ is good, then $S \cap F$ is also good.

(e) Let the TT inequality $a \cdot x \geq 1$ be valid for $P$. If it defines a face of co-dimension $c$ of $S$, then it defines a face of co-dimension at most $c$ of $P$.

(f) For every $a \in \mathbb{R}^{E_n}$ there is a unique TT representative in the coset $a + \mathbb{R}^{|n|}$, i.e., $a + \mathbb{R}^{|n|} = \{ a + D^\top \xi \mid \xi \in \mathbb{R}^{|n|} \}$. More precisely, we can obtain a unique $\lambda(a) \in \mathbb{R}^{|n|}$ for which $a - D^\top \lambda(a)$ is TT by letting

$$\lambda_u(a) := \min_{v, w \neq u} t_{u, vw}(a)$$

**Proof (sketches).** The proofs of these facts are easy generalizations of arguments which can be found in\cite{27}.

The key ingredient in (a–c) is the shortcut argument. Let $x \in \mathbb{Z}_{+}^{E_n}$ represent the edge multi-set of a connected Eulerian multi-graph $H$ with vertex set $[n]$. If there $H$ is not a cycle, i.e., if $H$ has a vertex $u$ of degree four or more, then one can find an edge $vw$ such that $vu$ and $wu$ are in $H$, and $H' := H \cup \{ vw \} \setminus \{ vu, uw \}$ is still a connected Eulerian multi-graph; cf. the picture on the right. If $y$ represents its edge multi-set, then $y = x + s_{u,vw}$. This gives (a), the implication $\Rightarrow$ in (c), and by carefully selecting the edge $vw$, (d). Similarly, one can subtract a shortcut from an $x$, which gives (b), the other direction in (c), and, by taking for each vertex $u$ a shortcut $s_{u,vw}$, implies (e).

Item (f) is straightforward computation.

We now prove the important theorem of Naddef & Rinaldi.

**Theorem 17\cite{27}**.

(a) If a facet $G$ of $P$ contains $S$, then $G$ is a degree facet.

(b) Let $F$ be a good facet of $S$. There exists a unique facet $G$ of $P$ with $F = G \cap S$.

**Proof.** (a). If $G \supset S$, then $G$ is good by definition. If $G$ is not equal to a degree facet, then, by Lemma\cite{16}, it is defined by a TT inequality, which contradicts Lemma\cite{16}.
Clearly, $G$ exists because $S$ is a face of $P$. Let $G$ be defined by an inequality $a \cdot x \geq \alpha$. Then $a$ is TT by Lemma 16, hence, by Lemma 16f, unique in the set $a + L^\perp$ of all left hand sides of inequalities defining the facet $F$ of $S$. □

5.2. Definition of polar polyhedra. As we have just seen, left hand sides of inequalities $a \cdot x \geq \alpha$ valid for $P$ have nice properties. In order to avoid cumbersome notation (like having to say that $-a$ satisfies the triangle inequality) we define the polars using “$\geq$” inequalities. Thus

$$S^\vee := \{a \in L \mid a \cdot x \geq -1 + a \cdot z \ \forall x \in S\}; \quad (13)$$

replaces the definition (13) on page 9 (we will define $z$ in (22) below). The inequalities (14) then take the form

$$(a + q) \cdot x \geq -1 + a \cdot z + q \cdot z \quad (14)$$

As in Section 3, let $p : \mathbb{R}^m \to L = \text{span} S^\vee$ be the orthogonal projection. The definition of $\pi$ corresponding to (16a) now reads

$$\pi(a) := \frac{1}{a \cdot z - 1} p(a), \quad (16a)$$

whenever $a \cdot x \geq 1$ is a valid inequality for $P$.

We now construct a kind of polar for $P$. We might intersect the polar cone $C := \{(\alpha, a) \in \mathbb{R} \times \mathbb{R}^E_n \mid a \cdot x \geq \alpha \ \forall x \in P\}$ with the hyperplane $\alpha + \sum a_e = 1$. From the observation [10] that $P$ is the Minkowski sum of $\mathbb{R}^E_n + \text{a finite set of points in } \mathbb{R}^E_n$, we see that this hyperplane intersects all extreme rays of $C$ except for $\mathbb{R}_+^E \cdot (0,0)$ which does not correspond to a facet of $P$. However, to better suit the definition of $\pi$, we will actually define $P^\vee$ to be a polyhedron which is projectively isomorphic to the one we have just described:

$$P^\vee := \{a \in \mathbb{R}^m \mid a \cdot x \geq 1 \ \forall x \in P\}.$$

The following are basic facts about this polar construction which can be easily verified.

Lemma 18.

(a) Let $a \in \mathbb{R}^E_n \setminus \{0\}$ and $d \geq -1$. Then $a$ is a relative interior point of a non-trivial face of $P^\vee$ with codimension $d + 1$ if and only if $(a, 1)$ is valid for $P$ and defines a face of dimension $d$ of $P$.

(b) Let $N \subset C(P)$ be the set of intersections of non-negativity facets $P$, and similarly $N' \subset C(P^\vee)$ be the set of all intersections of non-negativity faces $P^\vee$. Then conjugation of faces $C(P) \setminus N \to C(P^\vee) \setminus N'$, $F \mapsto F^\circ := \{a \in P^\vee \mid a \cdot x = 1 \ \forall x \in F\}$ defines an inclusion reversing bijection.

(c) A face $F$ of $P$ is good if and only if $F^\circ$ is bounded. □

The points $\delta_u$ defined above are vertices of $P^\vee$, more precisely, they are the vertices of $S^\circ$. Moreover, if $a$ is a vertex of $P^\vee$ such that the inequality $a \cdot x \geq 1$ defines a facet of $S$, then $a$ and $\delta_u$, $u \in [n]$, are the vertices of an $n$-simplex which is a face of $P^\vee$, by Theorem 33b.
As explained in Section 2.3, we want to describe the rotation complex restricted to $\text{dl}(N,S^\triangle)$, with
\begin{equation}
z := \frac{2}{n-1} - 1 = \frac{2}{n-1} \sum_{u=1}^{n} \delta_u = \frac{1}{(n-1)!} \sum_{C} \chi_C, \tag{22}
\end{equation}
where the last sum ranges over all cycles with vertex set $[n]$; thus $z$ is at the same time the barycenter of the vertices of $S$ and the barycenter of the vertices of $S^\triangle$.

By Lemma 16, a point $a$ in the complex $\bar{C}(P^\triangle)$ of bounded faces of $P^\triangle$, defines a good face of $S$, so we have $\pi(a) \in |\text{dl}(N,S^\triangle)|$, if $\pi(a)$ is defined, i.e., $a \notin S^\triangle$. Hence, we have the mapping
\[\pi : |\text{dl}(S^\triangle,\bar{C}(P^\triangle))| \rightarrow |\text{dl}(N,S^\triangle)| \tag{16b}\]
Later we will argue that this is a homeomorphism, as stated in Theorem 5.

Remark 19. By Lemma 18b and Lemma 16c, the points of the complex $\text{dl}(S^\triangle,\bar{C}(P^\triangle))$ are precisely the points in $|\bar{C}(P^\triangle)|$ which are tight triangular.

5.3. Descriptions of the rotation complex. We will now prove Theorems 4. We start by giving the precise definition for the flat TT-fan drafted in Section 2.3 and then prove that the two refinements of $\text{dl}(N,S^\triangle)$ defined in Theorem 4a and Theorems 4b are identical.

We prove that $p$ maps the TT-fan $|T'|$ homeomorphically onto $L$ by giving an inverse mapping, which we define using (21):
\[\lambda : \mathbb{R}^E \rightarrow \mathbb{R}^{[n]} : a \mapsto (\lambda_1(a), \ldots, \lambda_n(a))^\top, \tag{23}\]
\[\vartheta : \mathbb{R} \times \mathbb{R}^E \rightarrow \mathbb{R} \times \mathbb{R}^E : (\alpha, a) \mapsto (\alpha - \frac{1}{\lambda_1(a), \ldots, \lambda_n(a)} \vartheta(a)).\]

Lemma 20. The mappings $p : |T'| \rightarrow L$ and $\vartheta_L : L \rightarrow |T'|$ are inverses of each other.

Proof. By Lemma 16 every coset $a + L^\perp$ of $L^\perp$ contains a unique TT point, namely $\vartheta(a)$. The coset also contains a unique point of $L$, namely the orthogonal projection $p(a)$ of $a$ onto $L$. Hence, the two mappings are inverses of each other. \qed

In view of Lemma 20, $p$ transports the polyhedral fan $T'$ into a complete polyhedral fan $T := p(T')$ in $L$, the flat TT-fan defined in Section 2. It is a complete fan. The next lemma states that the refinements of $\text{dl}(N,S^\triangle)$ defined in Theorem 4a and Theorems 4b respectively are identical. The proof is a direct verification using the definitions of $E_u(\cdot)$ and $\vartheta$.

Lemma 21. For two points $a, b \in L$, the following are equivalent:
(i) $E_u(a) = E_u(b)$ for all $u \in [n]$
(ii) $a$ and $b$ are in the relative interior of the same face of the flat TT-fan $T$. \qed
This shows that Theorems 4a and 4b are equivalent. To prove that the refinements of $\text{dl}(N, S^\Delta)$ which we have defined in Theorem 4b using $E_u(\cdot)$ coincide with the rotation complex restricted to $\text{dl}(N, S^\Delta)$, we need to descent a bit deeper into the properties of $P$. If $F$ is a face of $P$, then a shortcut is said to be feasible for $F$, if it is contained in the space $\text{lin} F$. We note the following for easy reference.

**Lemma 22.** If $F$ is a good face of $P$, then a shortcut $s_{u,vw}$ is feasible for $F$ if and only if $a \cdot s_{u,vw} = 0$ for one (and hence for all) $a \in F^\circ$. □

Recall that $a \cdot s_{u,vw} \geq 0$ for all shortcuts $s_{u,vw}$ by Lemma 16b. The following lemma highlights the importance of shortcuts in the relationship between $S$ and $P$.

**Lemma 23.** A good face $F$ of $P$ is uniquely determined by

- the set of cycles whose characteristic vectors are contained in $F$,
- the set of its feasible shortcuts.

**Proof.** By the shortcut argument, every vertex of $F$ is either itself a cycle, or it can be constructed from a cycle by successively subtracting feasible shortcuts. Further, $\mathbb{R}_+ \chi_{uv}$ is a ray of $F$ iff, for any $a \in \text{relint} F^\circ$, we have $a_{uw} = 0$ (by Lemma 16b).

By Lemma 22 this is equivalent to the property that for every $w \neq u, v$, both $s_{u,vw}$ and $s_{v,uw}$ are feasible shortcuts. □

We can now prove Theorem 4b.

**Proof of Theorem 4b.** Let $a \in |\text{dl}(N, S^\Delta)|$. The inequalities of the form (14) all define good faces of $P$, because $a$ defines a face of $S$ not contained in a non-negativity facet of $S$. Moreover, since every inequality of the form (14) defines the same face of $S$, Lemma 23 implies that every member of the set $\mathcal{F}(a)$ of faces of $P$ defined by inequalities of the form (14) is uniquely determined by its set of feasible shortcuts.

We claim that the set $\mathcal{F}(a)$ is in bijection with the set of all subsets of $[n]$, where the bijection is accomplished in the following way: To a subset $I \subset [n]$, there is a face in $\mathcal{F}(a)$ whose set of feasible shortcuts is precisely

$$
\bigcup_{u \in I} \{s_{u,e} \mid e \in E_u(a)\}.
$$

(\*)

The faces obtainable in this way are clearly pairwise distinct by what we have just said. We have to construct a corresponding inequality for every set $I$, and we have to show that all faces in $\mathcal{F}(a)$ can be reached in this way.

For the former issue, for $I \subset [n]$ we define $q := \sum_{u \notin I} \delta_u$, and consider the inequality

$$(\vartheta(a) + q) \cdot x \geq -1 + a \cdot z - 1 \cdot \lambda(a) + q \cdot z,$$

which is of the form (14) because $1 = Dz$, and defines a good face of $P$ whose set of feasible shortcuts is easily verified to be (\*), by Lemma 22.

To see that every face in $\mathcal{F}(a)$ can be obtained in this way, it is easy to check, invoking Lemma 16 and the definition of $E_u(a)$, that, if there exists an edge $vw$ such that $s_{u,vw}$ is feasible for a face $F$ in $\mathcal{F}(a)$, then $vw \in E_u(a)$ and $s_{u,e}$ is feasible for $F$ for every $e \in E_u(a)$. 

This completes the proof of Theorem 4b.

5.4. The inverse mapping of \( \pi \). In this subsection, we will prove that \( \pi \) as given in (16b) is a homeomorphism, and show that it induces a combinatorial equivalence between \( \text{dl}(S^0, \tilde{C}(P^\Delta)) \) and the restriction of the rotation complex \( S \) to \( \text{dl}(N, S^\Delta) \); i.e., we prove Theorem 5. We will explicitly construct the inverse mapping \( \pi^{-1} \), which essentially transforms a point into its TT representative in the sense of Lemma 16f.

When we write the projective mapping \( \pi \) as a linear mapping from \( \mathbb{R} \times \mathbb{R}^E \rightarrow \mathbb{R} \times L \), it has the following form:

\[
\tilde{\pi} := \begin{pmatrix}
-1 & z \cdot \square \\
0 & p
\end{pmatrix}.
\]

As a technical intermediate step in the construction of \( \pi^{-1} \), we define a linear mapping \( I : \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R} \times \mathbb{R}^m \) taking points in \( \mathbb{R} \times L \) to points in \( \mathbb{R} \times \mathbb{R}^E \) by the matrix

\[
I := \begin{pmatrix}
-1 & z \cdot \square \\
0 & \text{id}
\end{pmatrix},
\]

Now we let \( (\gamma, c) := \tilde{\vartheta} \circ I(1, \square) \); in long:

\[
(\gamma, c) : a \mapsto (\gamma(a), c(a)) := \tilde{\vartheta}(I(1, a)) = (-1 + a \cdot z - 1 \cdot \lambda(a), a - D^\top \lambda(a)). \tag{24}
\]

Clearly, for all \( a \in L \), the point \( c(a) \) is TT. If \( a \in S^\Delta \), i.e., if the inequality \( a \cdot x \geq -1 + a \cdot z \) is valid for \( S \), then the inequality \( c(a) \geq \gamma(a) \) is of the form (14) (cf. the corresponding statement in the proof of Theorem 4b above). We note the following fact as a lemma for the sake of easy reference.

**Lemma 24.** The two inequalities \( a \cdot x \geq -1 + a \cdot z \) and \( c(a) \cdot x \geq \gamma(a) \) define the same face of \( S \).

Finally, we define

\[
\varphi : \text{dl}(N, S^\Delta) \rightarrow \text{dl}(S^0, \tilde{C}(P^\Delta)) : a \mapsto \frac{1}{\gamma(a)} c(a). \tag{25}
\]

**Proof of Theorem 5** In the remainder of this section, we will discuss the following issues:

- (a) \( \varphi \) is well-defined (in 5.4.a)
- (b) \( \varphi \) is a left-inverse of \( \pi \) (in 5.4.b)
- (c) \( \pi : \text{dl}(S^0, \tilde{C}(P^\Delta)) \rightarrow \text{dl}(N, S^\Delta) \) is onto (in 5.4.c)

Items (b) and (c) imply that

\[
\varphi \circ \pi = \text{id}_{\text{dl}(S^0, \tilde{C}(P^\Delta))}, \quad \text{and} \quad \pi \circ \varphi = \text{id}_{\text{dl}(N, S^\Delta)}.
\]

From this Theorem 5 follows.
Lemma 25. For all $a \in |d\ell(N, S^\Delta)|$ we have $\gamma(a) > 0$.

Proof. Assume to the contrary that $\gamma(a) = 0$. Since $c(a)$ is metric, $c(a) \geq 0$ holds. We distinguish two cases: $c(a) = 0$ and $c(a) \geq 0$. In the first case, the hyperplane defined by $c(a) \cdot x = \gamma(a)$ contains $S$, while $a \cdot x \geq -1 + a \cdot z$ defines a proper face of $S$, a contradiction to Lemma 24. On the other hand, if $c(a) \geq 0$, then the inequality $c(a) \cdot x \geq \gamma(a)$ is a non-negative linear combination of non-negativity inequalities, and hence the face defined by $c(a) \cdot x = \gamma(a)$ is contained in a non-negativity facet of $P$. But since $a \in |d\ell(N, S^\Delta)|$, i.e., $a$ is not a relative interior point of a face of $S^\Delta$ which contains a vertex of $S^\Delta$ corresponding to a non-negativity facet of $S$, the face of $S$ defined by $a \cdot x \geq -1 + a \cdot z$ is not contained in a non-negativity facet of $S$. Thus Lemma 24 yields a contradiction.

It remains to be shown that the image of $|d\ell(N, S^\Delta)|$ under $\varphi$ is really contained in the target space given in (25): For all $a \in |d\ell(N, S^\Delta)|$ we have $\varphi(a) \in |d\ell(S^\varphi, \tilde{C}(P^\Delta))|$. This also follows from Lemma 24. The inequality $\varphi(a) \cdot x \geq 1$ is valid for $P$, and the face it defines is good. Since $\varphi(a)$ is TT, the conclusion follows from Remark 19.

5.4.b. We show: $\varphi$ is a left-inverse of $\pi$, i.e., for all $a \in |d\ell(S^\varphi, \tilde{C}(P^\Delta))|$ the identity $\varphi(\pi(a)) = a$ holds.

Lemma 26. For all $a \in |d\ell(S^\varphi, \tilde{C}(P^\Delta))|$ we have $(\gamma, c)(\tilde{\pi}(1, a)) = (1, a)$. In particular, we have that $\varphi \circ \pi$ restricted to $|d\ell(S^\varphi, \tilde{C}(P^\Delta))|$ is equal to the identity mapping on this set.

Proof. To see this we compute

$$I(\tilde{\pi}(1, a)) = I(-1 + a \cdot z, p(a)) = \left(1 - a \cdot z - z \cdot p(a), p(a)\right) = \left((p(a) - a) \cdot z + 1, p(a)\right)$$

Using that $a$ is TT (Remark 19), we conclude

$$\tilde{\varphi}(I(\tilde{\pi}(1, a))) = \left((p(a) - a) \cdot z + 1 - \lambda(p(a)) \cdot 1, a\right).$$

Since $a$ is TT, by Lemma 16 $\lambda(p(a))$ is a solution to $p(a) - a = D^\top \lambda$. Thus, using $1 = Dz$, it follows that

$$(p(a) - a) \cdot z + 1 - \lambda(p(a)) \cdot 1 = (p(a) - a) \cdot z + 1 - D^\top \lambda(p(a)) \cdot z = 1.$$
mappings \( h_1: a \mapsto a - D^\top \lambda(a) \) and \( h_2: a \mapsto a \cdot z + \lambda(a) \cdot 1 \) are positive homogeneous, i.e., \( h_i(\eta a) = \eta h_i(a) \) for \( \eta \geq 0 \), \( i = 1, 2 \), which follows directly from the definition of \( \lambda \). □

5.4.c. We show: \( \varphi \) is one-to-one. Since we already know that \( \varphi \circ \pi = \text{id} \), surjectivity of \( \pi \) is equivalent to injectivity of \( \varphi \). Because of the particular definition of the polar of \( P \), this does not follow from Lemma 11 in Section 3. It is actually easier to prove the following slightly stronger statement.

Lemma 27. Let \( a, b \in L \). If there exists an \( \eta \in \mathbb{R}_+ \) such that \( (\gamma(a), c(a)) = \eta(\gamma(b), c(b)) \) then \( \eta = 1 \) and \( a = b \). In particular, \( \varphi \) is injective.

Proof. Let such \( a, b, \eta \) be given. We have

\[
0 = c(a) - \eta c(b) = a - D^\top \lambda(a) - \eta b - D^\top \lambda(b) = a - \eta b - D^\top \lambda(a) - \eta \lambda(b).
\]

Since \( a, b \in L \) and \( D^\top [\lambda(a) - \eta \lambda(b)] \in L^\perp \) we have

\[
a - \eta b = 0 = D^\top \lambda(a) - \eta D^\top \lambda(b) \tag{*}
\]

Applying \( z \cdot \square \) to the second equation, we obtain

\[
0 = 1 \cdot \lambda(a) - \eta 1 \cdot \lambda(b)
\]

Applying this to the \( \gamma \)s, we have

\[
0 = \gamma(a) - \eta \gamma(b) = -1 + a \cdot z - 1 \cdot \lambda(a) - \eta [-1 + b \cdot z - 1 \cdot \lambda(b)] = -1 + \eta + (a - \eta b) \cdot z.
\]

Since \( z \in L^\perp \) we have \((a - \eta b) \cdot z = 0\), whence \( \eta = 1 \). Now \( a = b \) follows from \( (*) \). □

6. Outlook

Naturally one asks, whether the rotation complex for \( \Pi^{n-1} \subset \Pi^{n'-1} \) with \( n' - n \geq 2 \) can be described. More generally, situations in which an interesting polytope is a face of another interesting polytope abound, though the existence of an interesting characterization of the rotation complex is not guaranteed.

In some cases, rotation complexes might help to understand Minkowski sums of polytopes or polyhedra. The results in Section 2.3 are in a way a special case of a variant of this question: it can be shown that, with \( S \) a Symmetric Traveling Salesman Polytope and \( P \) the corresponding Graphical polyhedron, \( P = (S + C) \cap \mathbb{R}_m^p \), where \( C \) is the polar of the metric cone \([37]\). It should be possible to find conditions on a polytope \( S \) and a polyhedron \( Q \) such that the rotation complex for \( S \subset P := S + Q \) can be understood.

In view of Theorem 5, the author is intrigued to know the homotopy type of the complex \( dL(N, C(S^\Delta)) \), with \( S \) a Symmetric Traveling Salesman Polytope.
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**DIRK OLIVER THEIS, SERVICE DE GÉOMÉTRIE COMBINATOIRE ET THÉORIE DES GROUPES, DéPARTEMENT DE MATHÉMATIQUE, UNIVERSITÉ LIBRE DE BRUXELLES, BRUSSELS, BELGIUM**

_E-mail address:_ Dirk.Theis@ulb.ac.be