ABSTRACT. One of the highlights of this note is that the author presents the relativistic gravity field that Einstein was looking for. The field is a byproduct of the matter in motion. This field can include both the discrete and continuous components. In free space the waves produced in this field propagate with velocity of light.

Another highlight is the proof of amended Feynman’s formulas for electromagnetic potentials. This makes the formulas mathematically complete and precise.

The main result can be stated as follows. In a fixed Lorentzian frame given is a trajectory $r_2(t, r_0)$ of flow of matter. The parameter $r_0$ changes in a compact set $F$ representing the position of the matter at some initial time $t_0$. The flow must satisfy certain conditions of regularity.

Given any signed measure $q(Q)$ of finite variation defined on Borel subsets of $F$, representing total charge contained in the set $Q \subset F$, such a flow determines the scalar $\phi$ and the vector $A$ potentials for a pair $(E, B)$ of fields satisfying Maxwell’s equations and as a byproduct the field $\rho$, representing density of charges, and the field $j$, representing the density of currents. All these fields are represented in terms of generalized functions.

1. Uniqueness of time delay field for flows of matter

The purpose of this note is to present formulas for gravitational and electromagnetic field in terms of potentials generated by flows of matter. We want to consider initial distributions of charges and mass that include both continuous and discrete components.

For the sake of simplicity in notation we select units so that the speed of light is $c = 1$, and the electrostatic constant satisfies the condition $4\pi\epsilon_0 = 1$, and the universal constant of gravity $G = 1$. We are working here in a fixed Lorentzian frame.

This means that if we keep meter as our unit of length the unit of time is approximately 3.3 nanoseconds and the unit of mass is about 3.871 metric tons and the unit of charge is $4\pi$ of coulombs.
Definition 1.1 (Flow of matter). Let $F \subset R^3$ be a compact set representing the position of matter at some initial time $t_0$.

By a flow of matter we shall understand a continuous function $(t, r_0) \mapsto r_2(t, r_0)$ from the product $R \times F$ into $R^3$, infinitely differentiable with respect to $t$ and such that the velocity $v_2(t, r_0) = \dot{r}_2(t, r_0)$ and the acceleration $a_2(t, r_0) = \ddot{r}_2(t, r_0)$ and the derivative $\dot{a}_2(t, r_0)$ and so on, are continuous on the product $R \times F$.

Moreover the following two conditions are satisfied:

- For every time $t_1$ there is a velocity $v_1 < c = 1$ such that

$$|v_2(t, r_0)| \leq v_1 \quad \text{for all } \quad t \leq t_1 \text{ and } r_0 \in F.$$  

- For every time $t \in R$ the map $P_t$ given by the formula

$$P_t(r_0) = r_2(t, r_0) \quad \text{for all } \quad t \in R \text{ and } r_0 \in F$$

represents a one-to-one mapping of $F$ onto $P_t(F)$.

Clearly we have

$$P_{t_0}(r_0) = r_2(t_0, r_0) = r_0 \quad \text{for all } \quad r_0 \in F.$$  

The function $t \mapsto r_2(t, r_0)$ will be called a line of flow corresponding to the index $r_0$.

**Remark**

The above definition of flow of matter depends on the Lorentzian frame. To make it frame independent define the set $F$ as a subset of a hyperplane of codimension 3 consisting of space like points.

Define the lines of flow as time like trajectories of point mass satisfying certain continuity conditions and conditions on the initial parts of flow to be able to find the representation of the flow in any Lorentzian frame as defined above.

We leave the details to the reader.

Let $T = T(r_1, t, r_0)$ denote the time delay required to reach point $r_1 \in R^3$ at time $t$ from the line of flow corresponding to index $r_0$. Its value must satisfy the Lorentz [11] time delay equation

$$T = |r_1 - r_2(t - T, r_0)|.$$  

**Theorem 1.2** (Time delay is unique and continuous). For every point $r_1 \in R^3$, and time $t \in R$, and index $r_0 \in F$ there exists one and only one solution $T$ of equation (1.3). Moreover the function $T = T(r_1, t, r_0)$ is continuous on its entire domain $R^3 \times R \times F$.

For a proof of an analogous theorem see [3]. Notice the following relations

$$T = 0 \iff \{ r_1 = r_2(t, r_0) \text{ for some } r_0 \in F \} \iff r_1 \in P_t(F).$$

The set $P_t(F)$ represents the position of matter at time $t$.

Now define set

$$G = \{(r_1, t, r_0) \in R^3 \times R \times F : T(r_1, t, r_0) > 0 \}.$$  

The set $G$ represents the natural domain of the field $u = T^{-1}$. 


2. Fundamental fields corresponding to the flow

**Definition 2.1** (Fundamental fields). Introduce the retarded time function
\[ \tau = \tau(r_1, t, r_0) = t - T(r_1, t, r_0) \quad \text{for all} \quad (r_1, t, r_0) \in \mathbb{R}^3 \times R \times \mathbb{R}^3, \]
retarded velocity
\[ v = v_2(\tau(r_1, t, r_0), r_0) \quad \text{for all} \quad (r_1, t, r_0) \in \mathbb{R}^3 \times R \times \mathbb{R}^3, \]
and retarded acceleration
\[ a = a_2(\tau(r_1, t, r_0), r_0) \quad \text{for all} \quad (r_1, t, r_0) \in \mathbb{R}^3 \times R \times \mathbb{R}^3, \]
and vector field \( r_{12} \) by
\[ r_{12} = r_1 - r_2(\tau(r_1, t, r_0), r_0) \quad \text{for all} \quad (r_1, t, r_0) \in \mathbb{R}^3 \times R \times \mathbb{R}^3. \]

Introduce the unit vector field \( e \), and the fields \( u \) and \( z \) by the formulas
\begin{equation}
(2.1) \quad \text{and} \quad e = \frac{r_{12}}{T} \quad \text{and} \quad u = \frac{1}{T} \quad \text{and} \quad z = \frac{1}{(1 - \langle e, v \rangle)} \quad \text{on} \quad G.
\end{equation}
These functions will be called the fundamental fields associated with the flow \( r_2(t, r_0) \), where \( t \in R \) and \( r_0 \in F \).

Notice that by definition of flow of matter the velocities are smaller in magnitude than the speed of light \( c = 1 \). Thus we must have for the dot product \( |\langle e, v \rangle| \leq |v| < 1 \). So the field \( z \) is well defined.

All the above functions consist of compositions of continuous functions, therefore each of them is continuous on its respective domain and thus all of them are continuous on their common domain, the set \( G \).

We would like to stress here that the fundamental fields depend on the Lorentzian frame, in which we consider the flow. It is important to find expressions involving fundamental fields that yield fields invariant under Lorentzian transformations.

Lorentz and Einstein [7], Part II, section 6, established that fields satisfying Maxwell equations are invariant under Lorentzian transformations.

Our main goal is to prove that fields constructed for flows of matter will satisfy Maxwell equations. We shall do this by showing that these fields are representable by means of fundamental fields and using the formulas for partial derivatives of the fundamental fields prove that such fields generate fields satisfying Maxwell equations.

Introduce operators \( D = \frac{\partial}{\partial t} \) and \( D_i = \frac{\partial}{\partial x^i} \) for \( i = 1, 2, 3 \) and \( \nabla = (D_1, D_2, D_3) \).

Observe that \( \delta_i \) in the following formulas denotes the i-th unit vector of the standard base in \( \mathbb{R}^3 \) that is \( \delta_1 = (1, 0, 0) \), \( \delta_2 = (0, 1, 0) \), \( \delta_3 = (0, 0, 1) \).

The symbols \( e_i, v_i, a_i \), denote the corresponding component of the vector fields \( e, v, a \), respectively.
Theorem 2.2 (Partial derivatives of fundamental fields). Assume that in some Lorentzian frame we are given a plasma flow \((t, r_0) \mapsto r_2(t, r_0)\). For partial derivatives with respect to coordinates of the vector \(r_1\) we have the following identities on the set \(G\)

\[
\begin{align*}
\text{(2.2)} & \quad D_i T = ze_i, \\
\text{(2.3)} & \quad D_i u = -zu^2 e_i, \\
\text{(2.4)} & \quad D_i v = -e_iz a, \\
\text{(2.5)} & \quad D_i \tau = -ze_i, \\
\text{(2.6)} & \quad D_i e = -uhe_i + u\delta_i + uze_i v \quad \text{where} \quad \delta_i = (\delta_{ij}), \\
\text{(2.7)} & \quad D_i z = -z^3 e_i(e, a) - uz^3 e_i + uz^2 v_i + uz^3 e_i(v, v) \\
\text{(2.8)} & \quad \nabla T = ze, \\
\text{(2.9)} & \quad \nabla u = -zu^2 e, \\
\text{(2.10)} & \quad \nabla z = -z^3 (e, a)e - uz^3 e + uz^2 e + uz^2 v + uz^3 (v, v)e.
\end{align*}
\]

and for the partial derivative with respect to time we have

\[
\begin{align*}
\text{(2.11)} & \quad DT = 1 - z, \\
\text{(2.12)} & \quad Du = zu^2 - u^2, \\
\text{(2.13)} & \quad D\tau = z, \\
\text{(2.14)} & \quad Dv = za, \\
\text{(2.15)} & \quad De = -ue + uze - uzv, \\
\text{(2.16)} & \quad Dz = uz - 2uz^2 + z^3 (e, a) + uz^3 - uz^3 (v, v).
\end{align*}
\]

Since the expression on the right side of each formula represents a continuous function, the fundamental fields are at least of class \(C^1\) on the set \(G\). Since the lines of flow \(t \mapsto r_2(t, r_0)\) of the matter are of class \(C^\infty\), we can prove by induction that the fundamental fields are of class \(C^\infty\) on \(G\).

The proof of the above theorem is similar to the proof of analogous theorem in Bogdan [5].

3. Integration with respect to a signed measure

Let \(V\) be a prereg of subsets of \(F\) consisting of sets of the form \(Q \cap B\) where \(Q\) is compact and \(B\) is open. See Bogdanowicz [1, page 498].

Assume that the set functions \(q^+(A)\) and \(q^-(A)\) represent, respectively, the total positive and total negative charge contained in the body covered by the set \(A \in V\). We shall assume that these functions are countably additive.

Remark

A heuristic argument relying on assumption that charge of an electron is indivisible can be presented as follows: Take a decomposition of a set \(A \in V\) into a countable union of disjoint sets

\[A = A_1 \cup A_2 \cup \ldots A_n \cup \ldots\]

Since every charge comes in the form of finite number of indivisible unit charges, that are all equal to the charge of a single electron, only a
finite number of the sets may contain a charge. Thus starting from a sufficiently large index $n_0$ all sets $A_n$ will have charge zero. Thus

$$q^+(A) = \sum_{n \leq n_0} q^+(A_n) + \sum_{n > n_0} 0 = \sum_{n=1}^{\infty} q^+(A_n).$$

Similarly we can get countable additivity of $q^-$. Put $q(A) = q^+(A) + q^-(A)$ and $\eta(A) = q^+(A) - q^-(A)$. The value $q(A)$ represents the total charge in the body covered by the set $A$ and $\eta(A)$ represents a non-negative countably additive set function on $V$ such that $|q(A)| \leq \eta(A)$.

Such a function satisfies the requirements of a volume function as defined in [1, page 492]. Observe that the function $q$ belongs to the space $M$, defined on page 492, and its norm $\|q\| \leq 1$. Therefore we can use the trilinear integral $\int u(f, dq)$ developed there. In our case for the bilinear operator $u(y, r) = ry = yr$ defined for $y \in Y$ and $r \in R$, where $Y$ stands for either the vector space $R^3$ or the space of $R$ of reals.

Thus we can use the theory developed in the papers Bogdanowicz [1] and [2]. Both papers are available on the web.

The classical theory of measure based on sigma rings of sets, and the Lebesgue theory of integration and theory of Bochner integral follow from these two papers, making all the classical tools of measure and integration available if needed in applications.

Concerning notation: We are using the symbol $\int u(f, dq)$ to denote the integral over the entire space $F$ of integration. When it is desirable to indicate the variable of integration we shall write $\int u(f(r_0), q(dr_0))$.

If we have a set $A \subset F$ and a function $f : F \mapsto Y$ such that the product $\chi_A f$ yields an $\eta$-summable function, where $\chi_A$ denotes the characteristic function of the set $A$, then we shall say that the function $f$ is summable on the set $A$ and by its integral over the set we shall understand the following

$$\int_A u(f, dq) = \int u(\chi_A f, dq).$$

Since $\chi_F f = f$ for all functions defined on $F$, the two notions for the set $F$ coincide, that is

$$\int u(f, dq) = \int_F u(f, dq).$$

In the case when the bilinear is of the form $u(r, \lambda) = r\lambda = \lambda r$, where $r$ is a vector and $\lambda$ is a scalar, we shall write the integral with respect to $u$ just as $\int f \, dq$.

4. Basic notions from the theory of generalized vector fields

It will be convenient here to use the theory of generalized functions, called also distributions, originally introduced heuristically by Dirac in his works on Quantum Mechanics and put on precise mathematical footing by L. Schwartz [12] and Gelfand and Shilov [10].

Assume that $G$ represents an open set in $R^4$. Let $D_k$ denote the space $C^\infty_0(G, R^k)$ of infinitely differentiable functions having compact supports contained in $G$ and values in $R^k$. 
Definition 4.1 (Sequential topology on $D_k$). On the space $D_k$ we introduce a sequential topology. We shall say that a sequence of functions $g_n \in D_k$ converges to a function $g \in D_k$ in the sequential topology if it converges uniformly together with all its partial derivatives $D^\alpha g$, on every compact subset $K$ of $G$, to the function $g$, that is for every $\alpha$ the following sequence converges uniformly on $K$

$$D^\alpha g_n(x) \to D^\alpha g(x)$$

where $\alpha = (k_1, \cdots, k_4)$ denotes the multi-index of a partial derivative

$$D^\alpha = D^{k_1}_1 \cdots D^{k_4}_4$$

with $k_j = 0, 1, 2, \ldots$ and $D_j = \frac{\partial}{\partial x_j}$ where $j = 1, 2, 3, 4$.

Definition 4.2 (Generalized vector valued functions). Let $D'_k$ denote the space of all linear continuous real functionals on $D_k$. Convergence in $D'_k$ will be understood as pointwise convergence. The space $D'_k$ will be called the space of generalized vector valued functions or the space of vector distributions.

We shall use the following equivalent notation for such a functional

$$f(g) = \langle f, g \rangle = \int_G f(x) \cdot g(x) \, dx \quad \text{for all } g \in D_k.$$ 

In the above $y \cdot x$ denotes the dot product, also called the scalar product, of two vectors $y, x \in \mathbb{R}^k$.

Any continuous function $f$ on the set $G$ generates by means of the above integral formula a vector distribution.

Notice that the space $D'_k$ is linearly and topologically isomorphic with the Cartesian product of $k$ copies of the space $D'_1$

$$D'_1 \times \cdots \times D'_1 = (D'_1)^k.$$ 

Indeed, if for every $m = 1, \ldots, k$ we define maps $P_m : R \to R^k$ by the condition

$$P_m(t) = x \iff \{x_m = t \text{ and } x_j = 0 \text{ if } j \neq m\},$$ 

where $x = (x_1, \ldots, x_k) \in R^k$, then the functionals

$$f_m(g) = \int_G f(x) \cdot P_m(g(x)) \, dx \quad \text{for all } g \in D_1$$

are well defined and represent elements of the space $D'_1$. Thus the element

$$(f_1, \cdots, f_k) \in (D'_1)^k.$$ 

Conversely define maps $P'_m : R^k \to R$ for $m = 1, \ldots, k$ by the formula

$$P'_m(x) = x_m \quad \text{for all } x \in R^k, \ m = 1, \ldots, k.$$ 

Then the formula

$$f(g) = \sum_m \int_G f_m P'_m(g(x)) \, dx \quad \text{for all } g \in D_k$$

yields a linear continuous functional on the space $D_k$. It is easy to verify that the transformation $Q : D'_k \to (D'_1)^k$ defined by

$$f \mapsto (f_1, \ldots, f_k)$$

is indeed a linear and topological isomorphism of the two spaces.
Now notice the following fact.

**Proposition 4.3** (Imbedding of continuous functions into $D'_k$ is one-to-one). Given two functions $f_1$ and $f_2$. Assume that they are continuous on the open set $G$ with exception perhaps of points lying on an admissible trajectory of a point mass. If they generate the same vector distribution, that is

$$\int_G f_1(x)g(x)\,dx = \int_G f_2(x)g(x)\,dx \quad \text{for all} \quad g \in D_k,$$

then they coincide

$$f_1(x) = f_2(x)$$

at every point $x \in G$ with exception perhaps of the points on the trajectory.

**Proof.** Indeed, from linearity of the map $f \mapsto \langle f, g \rangle$ and the previous isomorphism it is sufficient to prove that for any real function $f$ continuous at every point except perhaps points lying on the admissible trajectory such that

$$\int_G f(x)g(x)\,dx = 0 \quad \text{for all} \quad g \in D_1$$

follows that $f = 0$ on $G$ with the exception of points on the trajectory.

Assuming that this is not true then at some point $x_0 \in G$ we have $f(x_0) \neq 0$. We may assume without loss of generality that $2\delta = f(x_0) > 0$ otherwise we would consider the function $-f$. From continuity of $f$ follows that there is a rectangular neighborhood $V \subset G$ of $x_0$ such that

$$f(x) \geq \delta \quad \text{for all} \quad x \in V.$$

There exists a nonnegative function $g$ of class $C^\infty$ with support in the set $V$ with integral $\int_G g(x)\,dx = 1$. Thus for such a function we would get

$$\int_G f(x)g(x)\,dx = \int_V f(x)g(x)\,dx \geq \int_V \delta g(x)\,dx = \delta > 0.$$

A contradiction. So all we have to do is to show that there exists a function $g$ having the above properties. To this end consider a nonnegative function $g_0(t)$ defined by the formula

$$g_0(t) = \alpha e^{-1/(1-t^2)} \quad \text{if} \quad |t| < 1; \quad \text{and} \quad g_0(t) = 0 \quad \text{if} \quad |t| \geq 1,$$

where the constant is selected so that $\int_{-1}^1 g_0(t)\,dt = 1$. It follows from the above formula that the function $g_0$ is infinitely differentiable at every point except perhaps at $t = +1$ or $t = -1$. One can prove that at these points the one-sided derivatives exist and they are equal. Thus the function $g_0$ is of class $C^\infty$.

Now consider the function

$$g_1(x) = g_0(x_1)g_0(x_3)g_0(x_2)g_0(x_4) \quad \text{for all} \quad x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$$

with the integral, over the cube in $\mathbb{R}^4$ representing its support, equal to 1. For the fixed $x_0 \in G$ and sufficiently large $n$ we see that the function given by the formula

$$g_n(x) = n^4g(n(x - x_0)) \quad \text{for all} \quad x \in \mathbb{R}^4$$

will satisfy our requirements. \qed
Any linear continuous operator \( H : \mathcal{D}_k \rightarrow \mathcal{D}_k \) generates a linear continuous dual operator \( H' : \mathcal{D}'_k \rightarrow \mathcal{D}'_k \) by the formula
\[
\langle H'f, g \rangle = \langle f, Hg \rangle \quad \text{for all} \quad g \in \mathcal{D}_k.
\]
The dual operator corresponding to scalar multiplication \( g \mapsto \lambda g \) is scalar multiplication \( f \mapsto \lambda f \) as follows from the above definition.

**Definition 4.4** (Generalized differential operator). By a generalized partial derivative \( D_i = \frac{\partial}{\partial x_i} \), acting onto the \( m \)-th component of \( f \in \mathcal{D}'_k \)
\[
D_{i,m}(f_1, \ldots, f_k) = (f_1, \ldots, D_if_m, \ldots, f_k)
\]
we shall understand the dual operator to the operator acting onto the \( m \)-th component of \( g \in \mathcal{D}_k \) by the formula
\[
(-1)D_{i,m}(g_1, \ldots, g_k) = (g_1, \ldots, (-1)D_ig_m, \ldots, g_k).
\]

In the case when the set \( G \) represents a Cartesian product of open bounded intervals and the vector function \( f \) is continuous together with \( D_if_m \), the above formula can be easily verified through iterated integral and integration by parts. In this case the ordinary partial derivative will produce the derivative in the sense of distributions. So it is natural to extend this property to vector distributions.

**Definition 4.5** (Weak and strong partial derivatives). Now every distribution and in particular every continuous function is differentiable in the sense of distributions. If it happens that the distributional partial derivative is representable by means of a continuous function, such a function is called a weak derivative. If a vector function has a continuous partial derivative such a derivative is called a strong derivative.

It is not obvious that weak and strong derivatives coincide in the general case of an arbitrary open set \( G \) in \( \mathbb{R}^4 \). This calls for the following theorem.

**Theorem 4.6** (Weak and strong partial derivatives coincide). Assume that \( f = (f_1, \ldots, f_k) \) is a vector-valued function on an open set \( G \subset \mathbb{R}^4 \) and that its \( m \)-th component has a continuous partial derivative \( D_if_m \).

Then this derivative coincides with the derivative in the sense of distribution, that is the weak derivative coincides with the strong one.

**Proof.** To prove this fact for general open sets in \( \mathbb{R}^4 \) notice that for every point \( x \in G \) there exists a neighborhood \( V(x) \subset G \) in the form of the Cartesian product of open intervals. Restricting our test functions to functions with support in \( V(x) \) will yield that on such a domain the weak and strong partial derivatives coincide.

Since every open set in \( \mathbb{R}^4 \) is a union of a countable number of compact sets, from the collection \( V(x) (x \in G) \) one can extract a locally finite countable cover of the set \( G \).

Now using partition of unity theorem, (for reference concerning this theorem see for instance Gelfand and Shilov [10], vol. 1, Appendix to Chapter 1, Section 2,) we can prove this theorem for any open set \( G \subset \mathbb{R}^4 \). \( \square \)
5. Wave with gauge equations imply Maxwell equations for generalized vector fields

**Theorem 5.1** (Wave and gauge imply Maxwell equations). Let on the set $\mathbb{R}^4$ be given two generalized scalar fields $\phi$ and $S$ and two generalized vector fields $A$ and $J$.

If these fields satisfy the following wave equations

\[ \nabla^2 \phi - \frac{\partial^2 \phi}{\partial t^2} = -S, \quad \nabla^2 A - \frac{\partial^2 A}{\partial t^2} = -J, \]

with Lorentz gauge formula

\[ \nabla \cdot A + \frac{\partial}{\partial t} \phi = 0 \]

then the generalized fields $E$ and $B$ defined by the formulas

\[ E = -\nabla \phi - \frac{\partial}{\partial t} A \quad \text{and} \quad B = \nabla \times A \]

will satisfy the following Maxwell equations

\begin{align*}
(5.1) \quad (a) \quad \nabla \cdot E &= S, \\
(b) \quad \nabla \times E &= -\frac{\partial}{\partial t} B, \\
(c) \quad \nabla \cdot B &= 0, \\
(d) \quad \nabla \times B &= \frac{\partial}{\partial t} E + J,
\end{align*}

and the equation of continuity

\[ DS + \nabla \cdot J = 0. \]

**Theorem 5.2** (Continuity on parameter $r_0$). For any flow of matter $r_2(t,r_0)$ the functions

\[ r_0 \mapsto uz \quad \text{and} \quad r_0 \mapsto u z v \]

are continuous from the set $F$ into the space $\mathcal{D}'_1$ and $\mathcal{D}'_3$, respectively.

**Theorem 5.3** (Commutativity of differential and integral operators). Assume that $r_0 \mapsto h$ is a continuous function from the set $F$ into the space $\mathcal{D}'_k$ of generalized functions on the open set $G$.

Then the generalized function

\[ H = \int_F h(r_0) q(dr_0) \]

is well defined and we have the following formulas

\[ D \int_F h \, dq = \int_F Dh \, dq \quad \text{and} \quad D_i \int_F h \, dq = \int_F D_i h \, dq. \]
Definition 5.4 (Scalar and vector potentials). For any flow of matter and any countably additive measure \( q(Q) \), where \( q(Q) \) represents the charge contained in the space covered by the set \( Q \subset F \), define the **scalar potential** \( \phi \), and the **vector potential** \( A \), by the formulas

\[
\phi(r_1, t) = \int_F [(uz)(r_1, t, r_0)] q(dr_0),
\]

\[ A(r_1, t) = \int_F [(uzv)(r_1, t, r_0)] q(dr_0), \]

The functions under the integral are treated as generalized functions of variable \((r_1, t)\).

Theorem 5.5 (Potentials are well defined). For any flow of matter and any measure \( q(Q) \) over \( F \) the scalar and vector potentials are well defined as generalized functions of \((r_1, t) \in \mathbb{R}^4\).

Introduce the D’Alembertian operator by the formula

\[
\Box^2 = \nabla^2 - D^2.
\]

Theorem 5.6 (Potentials provide solution to Maxwell’s equations). For any flow of matter and any measure \( q(Q) \) of finite variation over \( F \) define generalized fields \( S \) and \( J \) by the formula

\[
S = -\Box^2 \phi \quad \text{and} \quad J = -\Box^2 A
\]

Then the fields defined by \( E = -\nabla \phi - \frac{\partial}{\partial t} A \) and \( B = \nabla \times A \) will satisfy the Maxwell equations

\[
\nabla \cdot E = S, \quad \nabla \times E = -\frac{\partial}{\partial t} B, \quad \nabla \cdot B = 0, \quad \nabla \times B = \frac{\partial}{\partial t} E + J
\]

and can be represented by means of the integral formulas

\[
E = \int_F \left( u^2 e + u^{-1} \frac{\partial}{\partial t} (u^2 e) + \frac{\partial^2}{\partial t^2} e \right) dq,
\]

\[
B = \int_F e \times \left( u^2 e + u^{-1} \frac{\partial}{\partial t} (u^2 e) + \frac{\partial^2}{\partial t^2} e \right) dq.
\]

Moreover the field \( S \) represents the generalized density of charges and the field \( J \) represents the generalized density of currents. They satisfy the equation of continuity

\[
\nabla \cdot J + \frac{\partial}{\partial t} S = 0
\]

of flow of charge. Here \( F \) represents the initial position of the plasma in \( \mathbb{R}^3 \), and the scalar field \( u \) and the vector field \( e \) are defined in formula (2.1).
Einstein using general theory of relativity proved that waves in the gravity field have to propagate with velocity of light. For reference see Einstein and Rosen [8]. Since in any field satisfying the homogeneous wave equation, waves propagate with velocity of light, the fields $E$ and $B$ in free space have this property. The field should represent the dynamics of matter and should have both continuous and discrete parts. Clearly the fields we just investigated satisfy these conditions.

For fields which represent purely discrete fields we have not only their form but also the dynamics of their evolution in the form of n-body problems. For references in this regard see Bogdan [3], [4], and [5].

Now we shall consider the gravity field. Assume that at the time $t = t_0$, we know the measure $m_0(Q)$ representing the distribution of the rest mass of the system.

Since from the previous considerations follows that the Lorentzian frame with the formula of the flow of mass determine the potentials $\phi$ and $A$ and the fields $E$ and $B$, the physical nature of the fields is of no importance. Properties of these fields we derived just from the geometrical part of the nature of the special theory of relativity.

So the gravity field also should be representable by means of these potentials. Again, we remind the reader that we are working in a Lorentzian frame with units selected so that the speed of light $c = 1$ the electrostatic constant in free space satisfies the condition $4\pi\varepsilon_0 = 1$ and the unit of mass is selected so that the gravitational constant $G = 1$.

This means that if we keep meter as our unit of length the unit of time is approximately 3.3 nanoseconds and the unit of mass is about 3.871 metric tons and the unit of charge is $4\pi$ of coulombs.

**Definition 6.1 (Generalized gravity field).** Let $(t, r_0) \mapsto r_2(t, r_0)$ represent a flow of matter and $m_0(Q)$ represent the rest mass of the part of space covered by the set $Q \subset F$ at time $t = t_0$.

Then the intensity $E$ of the gravity field is given by the formula

$$E(r_1, t) = \int_F [\nabla \phi + \frac{\partial}{\partial t} \phi](r_1, t, r_0) m_0(dr_0)$$

where all the the fields represent generalized fields. The associated field $B$ is given by the formula

$$B(r_1, t) = -\int_F [e \times (\nabla \times A)](r_1, t, r_0) m_0(dr_0)$$

The pair of fields $(E, B)$ should be treated as one relativistic entity since Lorentz [11] and, independently, Einstein [7], see Part 2, section 6, have established that Maxwell equations are invariant under Lorentzian transformations. To be more precise a pair of fields

$$E = (E_1, E_2, E_3) \quad \text{and} \quad B = (B_1, B_2, B_3)$$
that satisfies Maxwell equations transforms as a part of an antisymmetric tensor of second rank. The matrix of this tensor looks as follows

\[
\begin{bmatrix}
0 & +E_1 & +E_2 & +E_3 \\
-E_1 & 0 & +B_3 & -B_2 \\
-E_2 & -B_3 & 0 & +B_1 \\
-E_3 & +B_2 & -B_1 & 0
\end{bmatrix}
\]

7. Completing Feynman formulas for potentials of the electromagnetic field

Again we are working in a Lorentzian frame in which we have given a flow \((t, r_0) \mapsto r_2(t, r_0)\) of matter. The fields \(T, r_{12}, e, v\) are the fundamental fields (2.1) associated with the flow.

We consider the case when the measure \(q(Q)\) has Lebesgue summable density \(\rho(r_0)\). In this case since the field \(u_z\) is

\[
u_z = \frac{1}{T} \frac{1}{(1 - \langle e, v \rangle)} = \frac{1}{(|r_{12}| - \langle r_{12}, v \rangle)}
\]

the potential fields \(\phi\) and \(A\) have the following form

\[
\begin{align*}
\phi(r_1, t) &= \int_{F} \frac{1}{(|r_{12}| - \langle r_{12}, v \rangle)}(r_1, t, r_0) \rho(r_0) \, dr_0, \\
A(r_1, t) &= \int_{F} \frac{v}{(|r_{12}| - \langle r_{12}, v \rangle)}(r_1, t, r_0) \rho(r_0) \, dr_0.
\end{align*}
\]

(7.1)

The above formulas complete the formulas for potentials of the electromagnetic field obtained by Feynman by a heuristic argument. For details see Feynman-Leighton-Sands [9], vol. 2, chapter 15, page 15.15.

Similar representation is valid for the fields corresponding to gravity.

8. The independence of the fields from initial measure

Assume that \(q(Q)\) represents as before the total charge contained in the body covered by a set \(Q \subset F\) at time \(t_0\). Assume that at some later time \(\tilde{t}_0\) the position of the matter is in the set \(\tilde{F}\) and

\[
P(r) = r_2(\tilde{t}_0, r) \quad \text{for all} \quad r \in F
\]

represents transformation of points in \(F\) at time \(t_0\) to points in \(\tilde{F}\) at time \(\tilde{t}_0\). Since by definition of a flow of matter the transformation \(P\) is homeomorphism the set \(\tilde{F}\) is compact since \(F\) is such.

Let \(\tilde{V}\) be the prering consisting of intersections of compact sets with open sets of the space \(\tilde{F}\). Sets of this prering can be represented as set differences of two compact sets. Define set function

\[
\tilde{q}(\tilde{Q}) = q(P^{-1}(\tilde{Q})) \quad \text{for all} \quad \tilde{Q} \in \tilde{F}.
\]

Since the transformation \(P^{-1}\) preserves compact sets and set differences, the set function \(\tilde{q}\) is well defined. We shall prove that it represents distribution of charges at time \(\tilde{t}_0\).
The following theorem shows that the formulas for potentials in integral form do not depend on transition from one initial time $t_0$ to a later time $\tilde{t}_0$.

**Theorem 8.1.** Let $h(r)$ be a continuous function on the set $F$ with values the space $D'_k$ generalized vector fields. Let

$$\tilde{h}(r) = h(P^{-1}(r)) \quad \text{for all} \quad r \in \tilde{F}.$$  

Then we have the equality

$$\int_{\tilde{F}} \tilde{h} \, dq = \int_F h \, dq. \quad (8.1)$$

**Corollary 8.2.** The scalar potential $\phi$ and the vector potential $A$ are independent of the initial time $t_0$ when the distribution $q$ of charges was observed, and as a consequence the field $E$ and the field $B$ also do not depend on the initial time when the distribution was observed.

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