GLOBAL STABILITY OF THE PREDATOR-PREY MODEL WITH A SIGMOID FUNCTIONAL RESPONSE

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Abstract. A predator-prey model with Sigmoid functional response is studied. The main purpose is to investigate the global stability of a positive (coexistence) equilibrium, whenever it exists. A recent developed approach shows that, associated with the model, there is an implicitly defined function which plays an important role in determining the global stability of the positive equilibrium. By performing an analytic and geometrical analysis we demonstrate that a crucial property of this implicitly defined function is governed by the local stability of the positive equilibrium. With this crucial property we are able to show that the global and local stability of the positive equilibrium, whenever it exists, is equivalent. We believe that our approach can be extended to study the global stability of the positive equilibrium for predator-prey models with some other types of functional response.

1. Introduction. The predator-prey systems described by differential equations have served as one of important tools in the studies of dynamical interaction of predator-prey species in ecological systems. The mathematical results on the predator-prey systems provide very useful information and insight of the mechanism that govern the evolution of ecological systems. Hence the research on the predator-prey systems has been one of long and lasting efforts in the theoretical studies of ecology. Let us consider the following type of predator-prey systems

\begin{align}
x' &= rx \left(1 - \frac{x}{K}\right) - f(x)y, \\
y' &= y \left[-\mu + \beta f(x)\right],
\end{align}

where \(x\) and \(y\) are respectively the prey and predator populations; \(rx(1 - x/K)\) is a logistic growth of the prey population with \(r\) the intrinsic growth rate and \(K\) the carrying capacity; \(f(x)\) is the functional response of predator species on prey population; \(\mu\) stands for the predator death rate and \(\beta\) is the predator’s conversion
rate after its consumption of prey species. The simplest functional response is \( f(x) = ax \), to which the model (1) is called the Lotka-Volterra predator-prey system with a logistic growth and its global dynamics has been completely understood by the Liapunov’s direct method. The most commonly used functional response in literature is the Holling Type - II functional response [9], i.e.,

\[
f(x) = \frac{ax}{b + cx},
\]

Many other types of functional response \( f \) have also been introduced to model the different nature of environment, including the general Holling Type functional response [13]

\[
f(x) = \frac{x^n}{a + cx^n}, \quad n > 0,
\]

which is the Holling type - III functional response when \( n = 2 \), the Sigmoid functional response [8]

\[
f(x) = \frac{x^2}{a + bx + cx^2},
\]

the Monod-Haldane function response, or Holling type - IV function response [1]

\[
f(x) = \frac{mx}{a + bx + x^2},
\]

and a simplified Monod-Haldane functional response [15, 22]

\[
f(x) = \frac{mx}{a + x^2}.
\]

After intensive studies by many researchers for years, we now have a complete understanding of the global dynamical structure of the models with Holling types - II and III functional response [3, 6, 10, 11, 16]. All these models share the same global dynamical properties as follows:

Suppose that the positive equilibrium \( E_* \) exists. Then

1. If \( E_* \) is unstable, then there is a unique positive closed orbit that attracts all positive solutions except \( E_* \);
2. If \( E_* \) is locally stable, then it is globally stable.

The above properties have also been confirmed to be true for large range of \( n \) for the general Holling type functional response (3) [4, 12, 19, 20]. It is interesting to mention that the above properties do not hold for the model (1) with the simplified Monod-Haldane functional response (6). It has been shown in [15, 22] that, with the simplified Monod-Haldane functional response (6), the model can exhibit complicated dynamics, such as the subcritical Hopf bifurcation that leads to the co-existence of a stable positive equilibrium and multiple limit cycles, a phenomenon also observed in [17, 18] for certain types of functional response.

Our main interest of this paper is to investigate the global dynamical behavior of the model (1) with the Sigmoid functional response (4). To the best of our knowledge, little has been done in literature on the analysis of the global dynamics of the systems with Sigmoid functional response. A special case of (4), where

\[
f(x) = \frac{mx^2}{(a + x)(b + x)},
\]

was considered in [5], in which the authors obtained some sufficient conditions on the global stability of the positive equilibrium by the application of the Dulac’s criterion and by the construction of a Liapunov function.
In this paper, by using a recently introduced approach [2], we shall establish the following main theorem about the global stability of the positive equilibrium of the system (1) with the Sigmoid functional response (4). We believe that our approach can be extended to the investigation of the global stability of the positive equilibrium for some other predator-prey systems, including the system with Monod-Haldane functional response for certain range of parameters. This will be our future research effort.

**Theorem 1.1.** For the predator-prey model (1) with the Sigmoid functional response given by (4), suppose that the positive equilibrium exists. Then it is globally asymptotical stable if and only if it is locally stable.

This paper is organized as follows. The local stability of the positive equilibrium is studied in Section 2. In Section 3 we prove the equivalence between the global stability and local stability of the positive equilibrium whenever it exists.

2. **Local stability of positive equilibrium.** Without loss of generality, let us suppose that \( r = K = 1 \) in the model (1), for otherwise we can achieve it by a scaling if necessary. Hence in what follows we consider the system

\[
\begin{align*}
x' &= x(1-x) - f(x)y, \\
y' &= y[-\mu + \beta f(x)]
\end{align*}
\]

with

\[
f(x) = \frac{x^2}{a + bx + cx^2}.
\]

Since the carrying capacity is 1 and \( x' < 0 \) when \( x > 1 \), any positive solution will eventually enter the region \( \{(x, y) : 0 \leq x \leq 1, 0 \leq y\} \). Hence we only need to study the properties of solutions that stay in the above region. Notice that the function \( f \) given in (9) is monotone increasing. If \( \beta f(1) \leq \mu \), then the model (8) has no positive equilibrium and one easily sees that the model has trivial dynamical behavior. That is, all positive solutions converge to the boundary equilibrium \((1, 0)\). This is less interesting biologically. Thus, throughout this paper we assume that

\[
\beta f(1) = \frac{\beta}{a + b + c} > \mu.
\]

Then the equation

\[
-\mu + \frac{\beta x^2}{a + bx + cx^2} = 0
\]

has a unique solution \( x_* \in (0, 1) \). Hence the system (8) has a unique positive equilibrium \( E_* = (x_*, y_*) \) with

\[
y_* = \frac{(1-x_*)(a + bx_* + cx_*^2)}{x_*}.
\]

The purpose of this paper is to study the equivalency of the local stability and global stability of the equilibrium \( E_* = (x_*, y_*) \).

To this end let us write the system (8) in the form

\[
\begin{align*}
x' &= f(x)[g(x) - y], \\
y' &= [\beta f(x) - \mu]y,
\end{align*}
\]
where
\[
g(x) = \frac{x(1-x)}{f(x)} = \frac{(1-x)(a+bx+cx^2)}{x}
\] (11)

By applying the result in [14] or by a straightforward computation of the Jacobian matrix at \(E^*\) one immediately has the following

**Lemma 2.1.** The positive equilibrium \(E^* = (x^*, y^*)\) is locally asymptotically stable if \(g'(x^*) < 0\), and \(E^*\) is unstable if \(g'(x^*) > 0\).

By the expression of \(g(x)\) in (11) we obtain that
\[
g'(x) = -\frac{a}{x^2} - b + c(1-2x), \quad g''(x) = \frac{2a}{x^3} - 2c.
\] (12)

(12) implies that
\[
\lim_{x \to 0^+} g'(x) = -\infty, \quad g'(x) < 0, \quad \text{for } x \geq \frac{1}{2}.
\] (13)

Moreover it is obvious that there is an \(x_0 > 0\) such that \(g''(x_0) = 0\) if and only if
\[x_0 = \left[\frac{a}{c}\right]^{1/3}.
\]

Hence if \(x_0 < 1\), then \(g'(x)\) has a maximum at \(x_0\) in the interval \([0, 1]\) with
\[
g'(x_0) = -\frac{a}{x_0^2} - b + c(1-2x_0).
\] (14)

We shall study the global stability of the positive equilibrium \(E^*\) for the following two separate cases.

**Case 1.** \(x_0 \geq \frac{1}{2}\) or \(x_0 < \frac{1}{2}\) but \(g'(x_0) \leq 0\).

**Case 2.** \(x_0 < \frac{1}{2}\) and \(g'(x_0) > 0\).

3. **Global stability analysis.** In this section we investigate the global stability of the positive equilibrium \(E^*\). First we have the following

**Theorem 3.1.** If the assumptions in Case 1 are satisfied, then the equilibrium \(E^*\) is globally asymptotically stable whenever it exists.

We shall omit the proof of this theorem. Note that, under the assumptions in Case 1, the prey nullcline, \(g(x)\) is decreasing for all \(x \in (0; 1]\). It was proved early in [21] using the Poincaré criterion (see Lemma 3.2 and lemma 3.3), that any nontrivial periodic orbit of a system of the form (1.1) must surround a local maximum or a local minimum of the prey nullcline, and hence no such periodic orbits exist in this case. The proof can also be done using the standard Liapunov function introduced by Harrison [7]. Actually Theorem 3.1 can also be treated as a direct consequence of Theorem 3.2 since the decreasing property of \(g(x)\) implies that Inequality (17) holds.

Now let us turn to the Case 2. Note that \(g''(x) > 0\) for \(x \in (0, x_0)\) and \(g''(x) < 0\) for \(x \in (x_0, 1]\). This implies that \(g'(x)\) is increasing for \(x \in [0, x_0)\) and decreasing for \(x \in (x_0, 1]\). It follows that there are exactly two numbers \(x_1 \in (0, x_0)\) and \(x_2 \in (x_0, 1)\) such that [see Figure 3.1]
\[
g'(x) < 0 \quad \text{for } x \in (0, x_1) \cup (x_2, 1], \quad g'(x) > 0 \quad \text{for } x \in (x_1, x_2).\] (14)

From the above discussion we have the following proposition.
Proposition 1. Under the assumptions in Case 2, the positive equilibrium $E_\ast = (x_\ast, y_\ast)$ is locally asymptotically stable if $x_\ast \in (0, x_1) \cup (x_2, 1)$, and $E_\ast$ is unstable if $x_\ast \in (x_1, x_2)$.

To investigate the global stability of the positive equilibrium $E_\ast$, let us first rewrite the system (10) as

$$
\begin{align*}
x' &= f(x)[g(x) - y] \\
y' &= f(x)h(x) y
\end{align*}
$$

(15)

where

$$h(x) = \beta - \frac{\mu}{f(x)} = \beta - \frac{\mu(a + bx + cx^2)}{x^2}.$$ 

It is clear that $h(x_\ast) = 0$ and

$$h(x) < 0 \quad \text{for} \quad x < x_\ast \quad h(x) > 0 \quad \text{for} \quad x > x_\ast.$$ 

Let

$$\xi = G(x) = \int_{x_\ast}^{x} |h(s)| ds.$$ 

(16)

Then it is obvious that $G(x)$ is a strictly monotone increasing function of $x$. Hence (16) defines a unique inverse function $x = G^{-1}(\xi)$ with $G^{-1}(0) = x_\ast$. Moreover, it is evident that $G^{-1}(\xi)$ is monotone increasing on $\xi$ and

$$\lim_{x \to 0^+} G(x) = -\infty, \quad \lim_{x \to \infty} G(x) = \infty.$$ 

Hence the function $G^{-1}(\xi)$ is well defined for $\xi \in (-\infty, \infty)$.

For the global stability of $E_\ast$ of the system (15), we have the following theorem established in [2].

Theorem 3.2. The equilibrium $E_\ast$ of System (15) is globally stable if for all $\xi > 0$,

$$g(G^{-1}(-\xi)) > g(G^{-1}(\xi)).$$

(17)
Remark. The proof of this theorem can be found in the proofs of Theorem 2.1, Proposition 3.2 and Lemma 3.3 in [2]. Since the functional response $C(N)$ in [2] and the functional response $f(x)$ in this paper (where both $N$ and $x$ denote the prey population) do not have the same form, some explanations of why the proofs in [2] are applicable to our case will be helpful. First, the Proposition 3.2 in [2], which claims the non-existence of a positive closed orbit, is proved under the assumption $g(G^{-1}(-\xi)) > g(G^{-1}(\xi))$ for $\xi > 0$ (it uses $\theta$ in [2] instead of $G^{-1}$). It is easy to see that the proof of Proposition 3.2 in [2] only requires that the function $C(N)$ has the properties of $C(0) = 0$, $C(N) > 0$ for $N > 0$, and the existence of a unique positive equilibrium $E_*$. These properties are obviously satisfied by the function $f(x)$ in this paper. Secondary, it is not difficult to show that any positive solution to our model (15) is bounded. Finally, one is also able to see that our model cannot possess any nonnegative homoclinic orbit or graphic chain (a closed curve consisting of orbits and equilibrium points) since there are in total three nonnegative equilibria: $(0,0)$, $(1,0)$, and $E_*$, where $E_*$ is locally asymptotically stable; the stable manifold of the equilibrium point $(0,0)$ is the $y$-axis; while the stable manifold of the equilibrium point $(1,0)$ is the $x$-axis. Hence Theorem 3.3 follows from the Poincaré-Bendixson Theorem.

By the definition of $\xi = G(x)$ and $x = G^{-1}(\xi)$, we have
\[ \xi = G(G^{-1}(\xi)) = \int_{x_*}^{G^{-1}(\xi)} |h(s)| \, ds. \] (18)

Lemma 3.3. For each $\xi > 0$, $G^{-1}(-\xi) < x_* < G^{-1}(\xi)$ and
\[ x_* - G^{-1}(-\xi) < G^{-1}(\xi) - x_* . \]

Proof. Note that $G(x)$ is strictly increasing and so is for the function $G^{-1}(\xi)$. The definition of $G(x)$ yields that $G(x_*) = 0$, i.e., $x_* = G^{-1}(0)$. It follows that
\[ G^{-1}(-\xi) < G^{-1}(0) = x_* < G^{-1}(\xi) \quad \text{for} \quad \xi > 0 . \]

Moreover, we have $h(x_*) = 0$ and
\[ h'(x) = \mu \frac{(2a + bx)}{x^3} > 0, \quad h''(x) = -\frac{\mu(6a + 2bx)}{x^4} < 0 . \]

Hence $h(x)$ is monotone increasing and concave downward. Notice that $\xi$ is the area of the region between the $x$-axis and the graph of $h(x)$ on the interval $[x_*, G^{-1}(\xi)]$ and $-\xi$ is the area of the (negative) region between the graph of $h(x)$ and the $x$-axis on the interval $[G^{-1}(-\xi), x_*]$. By the downward concavity of $h(x)$ one therefore easily concludes that for each $\xi > 0$, $x_* - G^{-1}(-\xi) < G^{-1}(\xi) - x_*$. \qed

Now we proceed to show that Condition (17) holds if $g'(x_*) \leq 0$ (i.e., $E_*$ is locally stable). Recall that $h(x_*) = 0$, so that
\[ h(x) = h(x) - h(x_*) \]
\[ = -\mu \frac{a+bx+cx^2}{x^2} + \mu \frac{a+bx_*+cx_*^2}{x_*^2} \]
\[ = -\mu \left( \frac{a}{x_*^2} - \frac{a}{x^2} \right) - \mu \left( \frac{b}{x_*} - \frac{b}{x} \right) \]
\[ = \mu \frac{a(x+x_*)(x-x_*)}{x_*^2x^2} + \mu \frac{b(x-x_*)}{x_*x} . \] (19)
Lemma 3.4. For \( x > 0 \), let
\[
\phi(x) = \frac{1}{2} \frac{(x + x_*)|x - x_*|}{x^2}, \quad \psi(x) = \frac{|x - x_*|}{x}.
\]
If \( 0 < x_- < x_* \) and \( x_+ > x_* \) such that
\[
\int_{x_-}^{x_*} \phi(s)ds \leq \int_{x_*}^{x_+} \phi(s)ds,
\]
then
\[
\int_{x_-}^{x_*} \psi(s)ds < \int_{x_*}^{x_+} \psi(s)ds.
\]

Proof. By the definition of \( \phi(x) \) and \( \psi(x) \) both \( \phi(x) \) and \( \psi(x) \) are positive. Moreover we have
\[
\phi(x) = \left( \frac{1}{2} + \frac{x_*}{2x} \right) \psi(x).
\]
Hence it is apparent that
\[
\phi(x) \begin{cases} > \psi(x) & \text{if } x < x_* \\ < \psi(x) & \text{if } x > x_* \end{cases}
\]
It immediately follows that
\[
\int_{x_-}^{x_*} \psi(s)ds < \int_{x_-}^{x_*} \phi(s)ds \leq \int_{x_*}^{x_+} \phi(s)ds < \int_{x_*}^{x_+} \psi(s)ds.
\]

Corollary 1. For any given \( \xi > 0 \),
\[
\int_{x_*}^{G^{-1}(\xi)a(s + x_*)(s - x_*)} ds > \int_{x_*}^{G^{-1}(\xi)a(s + x_*)(s - x_*)} ds. \tag{20}
\]
Proof. Let \( x_- = G^{-1}(-\xi) \) and \( x_+ = G^{-1}(\xi) \). Then by the definition of \( G^{-1} \) we have
\[
-\xi = \int_{x_*}^{G^{-1}(-\xi)} |h(s)|ds = \int_{x_*}^{x_-} |h(s)|ds.
\]
Hence
\[
\int_{x_-}^{x_*} |h(s)|ds = \xi = \int_{x_*}^{G^{-1}(\xi)} |h(s)|ds = \int_{x_*}^{x_+} |h(s)|ds. \tag{21}
\]
Note that by (19)
\[
|h(x)| = \mu \left[ \frac{2a}{x_*^2} \phi(x) + \frac{b}{x_*} \psi(x) \right].
\]
It therefore follows from the equality (21) that
\[
\frac{2a}{x_*^2} \int_{x_-}^{x_*} \phi(s)ds + \frac{b}{x_*} \int_{x_*}^{x_-} \psi(s)ds = \frac{2a}{x_*^2} \int_{x_*}^{x_+} \phi(s)ds + \frac{b}{x_*} \int_{x_*}^{x_+} \psi(s)ds. \tag{22}
\]
We claim that
\[
\int_{x_-}^{x_*} \phi(s)ds > \int_{x_*}^{x_+} \phi(s)ds. \tag{23}
\]
Suppose on the contrary that $\int_{x}^{x^+} \phi(s)ds \leq \int_{x}^{x^+} \phi(s)ds$. Then by Lemma 3.4 we have $\int_{x}^{x^+} \psi(s)ds < \int_{x}^{x^+} \psi(s)ds$. Thus we deduce that

$$\frac{2a}{x^2} \int_{x}^{x^+} \phi(s)ds + \frac{b}{x} \int_{x}^{x^+} \psi(s)ds < \frac{2a}{x^2} \int_{x}^{x^+} \phi(s)ds + b \int_{x}^{x^+} \psi(s)ds,$$

which contradicts equality (22). Inequality (20) therefore follows from equalities (23) and

$$\int_{x}^{x^+} \phi(s)ds = \int_{x}^{x^+} \frac{(s + x_\ast)(x - x_\ast)}{s^2} ds$$

$$\int_{x}^{x^+} \phi(s)ds = \int_{x}^{x^+} \frac{(s + x_\ast)(x - x_\ast)}{s^2} ds.$$

\[ \square \]

**Theorem 3.5.** Suppose that $g'(x_\ast) \leq 0$. Then the positive equilibrium $E_\ast$ of the system (8) is globally stable.

**Proof.** For any $\xi > 0$, let $x_- = G^{-1}(-\xi)$ and $x_\ast = G^{-1}(\xi)$. Then $x_- < x_\ast < x_+$ and $x_\ast - x_- < x_+ - x_\ast$ by Lemma 3.3. From the expression $g'(x)$ in (12) it follows that

$$g'(x) - g'(x_\ast) = - \frac{a}{x^2} + \frac{a}{x^2} - 2c(x - x_\ast) = \frac{a(x + x_\ast)(x - x_\ast)}{x^2 x^2} - 2c(x - x_\ast).$$

Thus by Corollary 1 and the above equality we deduce that

$$\int_{x_\ast}^{G^{-1}(\xi)} g'(s)ds$$

$$= \int_{x_\ast}^{G^{-1}(\xi)} [g'(s) - g'(x_\ast)] ds + \int_{x_\ast}^{G^{-1}(\xi)} g'(x_\ast)ds$$

$$= \int_{x_\ast}^{x_-} \frac{a(s + x_\ast)(s - x_\ast)}{x^2 x^2} ds - c(x_- - x_\ast)^2 + \int_{x_\ast}^{G^{-1}(\xi)} g'(x_\ast)ds$$

$$> \int_{x_\ast}^{x_-} \frac{a(s + x_\ast)(s - x_\ast)}{x^2 x^2} ds - c(x_- - x_\ast)^2 + \int_{x_\ast}^{G^{-1}(\xi)} g'(x_\ast)ds$$

$$= \int_{x_\ast}^{G^{-1}(\xi)} [g'(s) - g'(x_\ast)] ds + \int_{x_\ast}^{G^{-1}(\xi)} g'(x_\ast)ds$$

$$= \int_{x_\ast}^{G^{-1}(\xi)} g'(s)ds$$

It follows that for any $\xi > 0$,

$$g(G^{-1}(\xi)) = g(x_\ast) + \int_{x_\ast}^{G^{-1}(\xi)} g'(s)ds$$

$$> g(x_\ast) + \int_{x_\ast}^{G^{-1}(\xi)} g'(s)ds$$
Theorem 3.2 and the inequality (24) immediately imply that the positive equilibrium $E^*$ is globally stable.

Proof of Theorem 1.1. It is obvious that Theorem 1.1 is a direct consequence of Lemma 2.1, Theorems 3.1 and 3.5.

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