About robust hyperstability and dissipativity of linear time-invariant dynamic systems subject to hyperstable controllers and unstructured delayed state and output disturbances

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Abstract: This paper considers the robust asymptotic closed-loop hyperstability of a nominal time-invariant plant with an associate strongly positive real transfer function subject to unstructured disturbances in the state and output. Such disturbances are characterized by upper-bounding growing laws of the state and control. It is assumed that the controller is any member within a class which satisfies a Popov’s type integral inequality. The continuous-time nonlinear and perhaps time-varying feedback controllers belong to a certain class which satisfies a discrete-type Popov’s inequality. The robust closed-loop hyperstability property is proved under certain explicit conditions of smallness of the coefficients of the upper-bounding functions of the norms of the unstructured disturbances related to the absolute stability abscissa of the modelled part of the nominal feed-forward transfer function.

Keywords: hyperstability; asymptotic hyperstability; Popov’s inequalities; positive real transfer function; strictly positive real transfer function; robustness

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Professor M. De la Sen serves currently as a Professor of systems engineering and automatic control at the University of Basque Country, where he also serves as the Head of the Institute of Research and Development of Processes. He has coauthored about eight hundred papers in scientific journals and proceedings of conferences. He is currently serving as an Associate Editor of the following journals: Applied Mathematical Sciences; Nonlinear Analysis, Modeling and Control; Fixed Point Theory and Applications; Heliyon; Frontiers in Applied Mathematics and Statistics; and others. His main interest research areas include discrete and sampled-data control systems, nonperiodic and adaptive sampling, adaptive control, fixed point theory, positive systems, stability, models for ecology, epidemic models, time-delay systems, artificial intelligence and heuristic tools for dynamic systems, and ordinary differential equations. The material of this paper relies on the properties of dissipativity and hyperstability which are particular aspects of the stability of dynamical systems.

PUBLIC INTEREST STATEMENT

Generally speaking, dissipative systems are open systems which interchange energy and matter with the environment. In the framework of Control Theory, those systems satisfy the dissipation inequality in terms that the time-derivative of the storage function is less than or equal to the supply rate given by the instantaneous input-output power. Dissipative is related to stability. Hyperstable closed-loop systems are systems which are stable not under just one stabilizing controller but for any member of a family satisfying a Popov’s-type inequality. This allows the achievement of the stability property under certain tolerance to controller parametrical uncertainties among the various control elements got from a production chain independently of each particular controller device. The hyperstability theory is relevant, for instance, in military and aeronautical targeting applications in which high precision is required in spite of potential dispersion in the fabrication procedure of the control devices.
1. Introduction

If both the state and output of a dynamic system possess the positivity property under given non-negative initial conditions and controls, its positivity is said to be internal or, simply, that the system is positive. Continuous-time and discrete-time positive dynamic systems have been studied in detail in recent years as well as their positivity properties, (Bru, Coll, Sánchez, 2002; De la Sen, 1996, 2007b; Farina Rinaldi, 2000; Kaczorek, 2011; Najson, 2013; Rami Napp, 2016; Rantzer, 2016; Tanaka Langbort, 2011). If the output possesses such a positivity property, the system is said to be externally positive. On the other hand, time-invariant dynamic linear systems which are externally positive with positive real or strictly positive real transfer matrices are also hyperstable or, even, asymptotically hyperstable, i.e. globally Lyapunov stable for any nonlinear and/or time-varying feedback device satisfying a Popov’s-type inequality for all time. See, for instance, (De la Sen, 1997; De la Sen, 1986) and references therein. Such a property of asymptotic hyperstability generalizes that of absolute stability and standard stability. See, for instance, (Alberdi et al., 2011; Ali Yogambigai, 2016; Brogliato, Lozano, Maschke, Egeland, 2007; De La Sen, 1997, 2004, 2015; Farina Rinaldi, 2000; Heath, Carrasco, De la Sen, 2015; Kaczorek, 2011; Landau Mendel, 1979; Li, 2016; Liu Stechlinski, 2016; Marchenko, 2012, 2013; Popov, 1973; Rajchakit, Rojsiraphisal, Rajchakit, 2012; Yuan Yang, 2015) and references therein. A further background is given in (De la Sen, 1996, 1997, 2007; Kabamba Hara, 1994; Kailath, 1980; Marchenko, 2012, 2013, 2015; Wu Lam, 2008) and some of the references therein. On the other hand, some conditions have been given in the background literature to guarantee the positive realness of discrete transfer functions from related conditions on their continuous-time counterparts which are then maintained under discretization. See, for instance, (De La Sen, 2002; de la Sen, 1998; Mahmoud, 1998; Marquez Damaren, 1995; Mahmoud Ismail, 2004, 2006; Rasvan, Niculescu, Lozano, 2000; Shatyrko Khusainov, 2016). The main novelties of this paper related to the background literature are as follows. The current feed-forward system is under the influence of unstructured disturbed and eventually delayed unmodeled dynamics subject to noise while the nominal one is delay-free. The delays are bounded functions of time but not necessarily smooth while the unmodeled input-output energy contribution is upper-bounded for all time by a weighted input energy contribution. The unstructured disturbances can be interpreted as joint effects of the unmodeled dynamics, eventual noise, potential internal delayed dynamics and uncertain parameters of the nominal part. In the most general obtained hyperstability results, it is not assumed that the disturbance dynamics is either bounded or square-integrable while those conditions are got as results by assuming growing laws for such a disturbance which depend on the control input and state time-evolution. The formal closed-loop hyperstability results are obtained by proving that the input-output energy is positive for any time interval of nonzero measure while the controller hyperstability through the Popov’s-type inequality implies that the above energy measure is bounded for all time. The advantages of the proposed framework are that the hyperstability of closed-loop system is formulated by considering the positivity of the input-output energy of the feed-forward block as that of the time-invariant part with a certain allowed tolerance to the disturbances part which allows keeping the input-output energy measure positivity. In particular, the feed-forward loop consists of a time-invariant strongly nominal positive real transfer function plus an extra disturbance dynamics eventually subject to time-varying delays. The disturbances contribution to the dynamics is characterized by upper-bounding functions which depend on the state and input related to a certain maximum tolerance which guarantees that the input-output energy measure of the feed-forward plant is still positive for all time interval as it is the nominal time-invariant part. On the other hand, Popov’s inequality which characterizes the class of stabilizing controllers (referred to as the hyperstability of the controller class) guarantees that the above input-output energy is, furthermore, uniformly bounded for all time.

This paper is organized as follows. Firstly, a notation and terminology subsection is allocated in the subsequent section. Section 2 presents and discusses a linear and time-invariant single-input single-output system of the feed-forward uncontrolled part of the closed-loop feedback system. This controlled object is subject to certain unstructured disturbances which might include unmodeled dynamics effects, noise and delays influences and contributions to the dynamics due to parametrical errors. The modeled part is assumed to possess an associated strongly positive real transfer
function. The feedback output-control input is generated by a nonlinear and, in general, time-varying device which satisfies a Popov’s type integral inequality which defines a class of hyperstable controllers. Section 2 formulates closed-loop robust hyperstability results for the closed-loop disturbed system under a family if unstructured disturbances under the assumption that such state/output disturbances grow through time at a sufficiently slow rate mainly compared to the absolute value of the stability abscissa of the nominal transfer function of the feed-forward loop. Section 3 formulates some further results which are linked to the more general concept of dissipativity of the feed-forward loop which includes as a particular case that of its hyperstability. In energetic terms, the property of dissipativity refers to the non-negativity of the contributed terms of input, output and input-output energy characterized by a related power supply function while the hyperstability property (or, equivalently, the passivity property) only considers, as a particular case, the input-output energy terms or its associated supply function to characterize the positivity property. Finally, conclusions end the paper.

1.1. Notation

\[ R_+ = \{ r \in R : r > 0 \}; R_{0+} = R_+ \cup \{ 0 \} \]

\[ C_+ = \{ z \in C : \text{Re} z > 0 \}; C_{0+} = C_+ \cup \{ 0 \} \]

\( \text{cl}R = [-\infty, +\infty] \) is the closure of the real set, i.e. \( R \cup \{ \pm \infty \} \), the set of real numbers plus the infinity points.

Accordingly defined are the closures \( \text{cl}R_+ = \text{cl}R_{0+} \equiv [0, +\infty) \) and \( \text{cl}C_+ = \text{cl}C_{0+} \). \( i = \sqrt{-1} \) is the complex unity, \( I \) is the identity matrix and the matrix and vector transposition operation is denoted with the superscript \( T \).

The Fourier transform of \( v(t) \), if it exists, is denoted by \( \hat{v}(i\omega) \).

\[ p = \{ 1, 2, \ldots, p \} \), and the symbols \( \land \) and \( \lor \) denote the conjunction and disjunction of logic propositions and \( \| \cdot \|_2 \) denotes the spectral norm.

If \( v: R_+ \cup [-h(t), 0) \to R^n \) then \( \nu_h(t) \) denotes the strip \( v: [t-h(t), t] \to R^n ; \forall t \in R_{0+} \) and \( \| \nu_h \|_2 = \sup_{r \in [t-h, t]} \| v(r) \|_2 \)

\( \{ \text{SSPR} \} \) and \( \{ \text{SPR} \} \) denote, respectively, the sets of strongly positive real and strictly positive real transfer functions.

\( RH_\infty \) is the space of rational \( H_\infty \) functions of norm \( \| \cdot \|_\infty \)

\( L_1, L_2 \) and \( L_\infty \) are the respective sets of bounded, absolutely integrable and square-integrable real or complex functions on \([0, \infty]\). An added superscript in those notations refers to square-integrable vectors of the corresponding dimension.

The notation \( M > 0 \) (respectively, \( M \succeq 0 \)) stands for a square positive definite (respectively, positive semidefinite) real matrix \( M \).

\( \lambda_{\text{min}}(M) \) and \( \lambda_{\text{max}}(M) \) stand, respectively, for the minimum and maximum eigenvalue of the symmetric real matrix \( M \).

\( \mu_2(M) = \lambda_{\text{max}} \left( \frac{M + M^T}{2} \right) \) is the matrix measure of the square matrix \( M \) which belongs to \([\| M \|_2, \| M \|_2] \).

\( I_p \) is the identity matrix of order \( p \).
2. Problem statement

Consider the dynamic system:

\[ \dot{x}(t) = Ax(t) + bu(t) + \eta(t) \]  
\[ y(t) = c^T x(t) + du(t) + \eta_0(t) \]

\[ u(t) = -\phi(y(t), t) \]

where \( x \in \mathbb{R}^n \) is the state vector and \( u(t) \) and \( y(t) \) are the scalar input and output, where \( A \) is a real \( n \times n \) matrix and \( b \) and \( c \) are real \( n \times 1 \) vectors and \( d \) is a real scalar, which is subject to initial conditions \( x(t) = \varphi(t) \) for \( t \in [-h(0), 0] \) where \( h: [0, +\infty) \to [h_M, h_m] \) is a, in general, bounded piecewise-continuous time-varying internal delay and \( \varphi: [-h(0), 0] \to \mathbb{R}^n \) is an absolutely continuous \( n \)-vector function with eventual finite isolated jump discontinuities, \( \eta(t) = \eta(x(t), u(t), t) \) and \( \eta_0(t) = \eta(x(t), u(t), t) \) are, in general, unstructured real \( n \)-vector disturbance functions which take account of parammetrical and unmodeled dynamics uncertainties and noise disturbances being eventually subjected to an internal delay \( h(t) \geq h_m \) where \( \tilde{x}_i(t) \) is the strip \( x_i[t - h(t)], t \to \mathbb{R}^n; \forall t \in \mathbb{R}_+ \).

It is assumed that the disturbance functions are subject to certain regularity conditions such as Lipschitz continuous with respect to all the arguments in order to ensure the existence and uniqueness of the solution for any bounded admissible initial conditions.

The nominal system is defined as the disturbance-free one, that is

\( \eta(t) = \eta(\tilde{x}_i(t), u(t), t) \equiv 0, \eta_0(t) = \eta(x(t), u(t), t) \equiv 0 \).

The function \( \phi: \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R} \) defining the, in general, eventually non-linear and time-varying controller is any member of a class \( \{ \Phi \} \) satisfying a Popov’s-type integral inequality of the form:

\[ \Gamma(t_0, t) = \int_{t_0}^{t} y(\tau) \varphi(y(\tau), \tau) d\tau \geq -\gamma_0 \]

for some finite \( \gamma_0 \in \mathbb{R}_+; \forall t_0 \in \mathbb{R}_+ \), \( t \geq t_0 \) \( \in \mathbb{R}_+ \). Such a class \( \{ \Phi \} \) is said to be a hyperstable class of controllers and any \( \varphi \in \{ \Phi \} \) is said to be \( \varphi \)-hyperstable.

If the closed-loop system (1)–(3) is globally stable for any controller \( \varphi \in \{ \Phi \} \) with the forward-time invariant block having a positive real transfer function then it is said to be \( \varphi \)-hyperstable so that the state and the output are uniformly bounded for all time for any given finite initial conditions. If, in addition, the feed-forward block has a strictly positive real transfer function then the closed-loop system is \( \varphi \)-asymptotically hyperstable, so that the state and output are uniformly bounded and converge asymptotically to zero as time tends to infinity, that is, it is \( \text{globally asymptotically stable for any given finite initial conditions and any controller of class } \{ \Phi \}. \) Thus:

(1) the feed-forward controlled plant is asymptotically hyperstable (or strictly positive, or strictly passive) if its transfer function is strictly positive real,

(2) the class \( \Phi \) of feedback controllers is hyperstable if it satisfies (4),

(3) any closed-loop configuration of the controlled \( \varphi \)-asymptotically hyperstable (respectively, \( \varphi \)-hyperstable) plant with a controller of class \( \{ \Phi \} \) (i.e. with a \( \varphi \)-hyperstable controller) is \( \varphi \)-asymptotically hyperstable (respectively, \( \varphi \)-hyperstable).

Now, define truncate functions and \( h \)-delay truncated functions for any \( 0 \leq T \leq \infty \), respectively, as

\[ \nu_{\tau}(t) = \begin{cases} v(t) & \text{for } t \in [0, T] \\ 0 & \text{for } t > T \end{cases} \]
\[ \nu_{\eta}(t) = \begin{cases} \nu(t) & \text{for } t \in [-h(t), T] \\ 0 & \text{for } t > T \end{cases} \]

It is assumed that \( \eta: R^{n+1} \times R^m_0 \rightarrow R^n \), \( \eta: R^{n+1} \times R^m_0 \rightarrow R \) and \( \varphi: R \times R^m_0 \rightarrow R \) satisfy regularity conditions (for example, uniform Lipschitz continuity with respect to their arguments) to guarantee the uniqueness of the state and output trajectory solutions from (1)–(2) for any given finite initial conditions on \([-h(t), 0]\). Such solutions can be expressed via (5) as follows by assuming that \( \eta(t) = 0 \), \( \eta(-t - h(t)) = 0 \) and \( \eta_0(-t - h(t)) = 0 \); \( \forall t \in R^+ \):

\[
x(t) = 0; t \in (-\infty, -h(0)]; x(t) = \Phi(t); t \in [-h(0), 0) \tag{6} \]

\[
x(t) = e^{At}(x(0) + \int_0^t e^{-A\tau} [bu_x(\tau) + \eta_{h\tau}(\tau)]d\tau); \quad \forall t \in R^+_0 \tag{7.0a} \]

\[
y(t) = c^T e^{At}(x(0) + \int_0^t e^{-A\tau} [bu_x(\tau) + \eta_{h\tau}(\tau)]d\tau) + \int_0^t du(t) + \eta_0(t) \]

\[
y_f(t) = c^T e^{At}(x(0) + \int_0^t e^{-A\tau} [bu_x(\tau) + \eta_{h\tau}(\tau)]d\tau) + \eta_0(t); \quad \forall t \in R^+_0 \tag{7.b} \]

where \( y_f(t) \) is the forced output given by:

\[
y_f(t) = c^T \int_{-\infty}^{\infty} e^{At-\tau} bu_x(\tau)d\tau + du(t); \quad \forall t \in R^+_0 \tag{8} \]

It is well-known that the Fourier transform exist of any absolutely-integrable vector real function on \((\infty, +\infty)\). Truncated functions with no escape time instants over finite time intervals always fulfil this property so that for any finite \( t \in R^+_0 \), the Fourier transforms \( \hat{x}(i\omega), \hat{y}(i\omega), \hat{\eta}(i\omega), \hat{\eta}_0(i\omega) \) and \( \hat{\eta}_0(i\omega) \) exist for all \( \omega \in R \). By using Parseval’s theorem, it follows that the following input-output energy measure \( E(t) \) on the time interval \([0, t]\) fulfils the subsequent associated relationships for any \( t \in R^+_0 \):

\[
E(t) = \frac{1}{2\pi} \int_{0}^{\infty} |\hat{y}_f(i\omega)\hat{u}_x(-i\omega)|d\omega 
\]

\[
E_0(t) = \frac{1}{2\pi} \int_{0}^{\infty} |\hat{y}_f(i\omega)\hat{u}_x(-i\omega)|d\omega 
\]

\[
\hat{E}(t) = \frac{1}{2\pi} \int_{0}^{\infty} [c^T e^{At} x(0) + \int_{-\infty}^{\infty} e^{-A\sigma} \eta_x(\sigma) d\sigma + \eta_0(\tau) + \eta_0(\tau)]d\tau 
\]

\[
\hat{E}(t) = \frac{1}{2\pi} \int_{0}^{\infty} [c^T e^{At} x(0) + \int_{-\infty}^{\infty} e^{-A\sigma} \eta_x(\sigma) d\sigma + \eta_0(\tau)]d\tau \quad \forall t \in R^+_0 \tag{10.b} \]

The main result follows below:

**Theorem 1.** Assume that

(a) the system (1) is under a feedback control generated by any controller \( \varphi \in \{\Phi\} \), where the class \( \{\Phi\} \) is defined by the Popov’s-type inequality (4),

(b) the nominal transfer function, namely, that obtained when \( \eta(t) = \eta(\tilde{x}_\tilde{f}(t), u(t), t) \equiv 0 \) and \( \eta_0(t) = \eta(x(t), u(t), t) \equiv 0 \), satisfies \( \hat{g}_0 \in \{SSPR\} \) such that \( \min \hat{g}(i\omega) \geq d \) for some real constant \( \lambda \in (0, 1) \), where the direct input-output interconnection \( d \) in (2) is positive,

(c) \( \hat{E}(t) \leq \hat{\gamma}(t) \leq +\infty; \forall t \in R^+_0 \) for some real function \( \hat{\gamma}: R^+_0 \rightarrow R \) subject to the constraint:
\[ \dot{y}(t) \leq \varepsilon \int_{0}^{t} u^2(\tau)d\tau; \forall t \in R_{+}. \quad (11) \]

Then, the following properties hold:

(i) \( u \in L_{2}, u(t) \to 0 \) as \( t \to \infty \), \( \text{ess sup}_{t \in R_{+}} |u(t)| < +\infty \), \( \sup_{t \in R_{+}} |\dot{E}(t)| < +\infty \), \( 0 < E(t) < +\infty \) and \( 0 < E_{0}(t) < +\infty \); \( \forall t \in R_{+} \).

(ii) There exist constants \( K_{i} \in R_{+} \) for \( i \leq 4 \) of which \( K_{i} \) and \( K_{j} \) might depend, in general, on the control and initial conditions on \( \frac{[\text{h}]}{[\text{0}]} \) such that:

\[
\lim_{t \to \infty} \sup_{t \in R_{+}} \left( \|x(t)\|_{2} - K_{1} - K_{2} \right) \leq 0 \sup_{t \in R_{+}} \left( \|y(t) - \eta_{0}(t)\|_{2} - K_{3} - K_{4} \right) \leq 0.
\]

(iii) If \( \|\eta\|_{2} \) and \( \eta_{0} \) are in \( L_{\infty} \cup L_{1} \cup L_{2} \) then \( x(0, +\infty) \times ([\text{h}](0, 0) \times R^{n}) \to R^{n} \) and \( y(0, +\infty) \times ([\text{h}](0, 0) \times R^{n}) \to R \) are uniformly bounded for each given vector function of initial conditions and the closed-loop system is \( \Phi \)-hyperstable for the class \( \Phi \) of output feedback controllers defined by the Popov's inequality \( (4) \). In addition, if the uncertainties are identically zero then the closed-loop system is \( \Phi \)-asymptotically hyperstable.

**Proof.** Rewrite \((8)\), via \((1)-(2)\) for the case \( \eta(t) = \eta(\tilde{x}(t), u(t), t) \equiv 0 \) and \( \eta_{0}(t) = \eta(x(t), u(t), t) \equiv 0 \); equivalently, in the Laplace framework since it has a linear and time-invariant structure leading to \( y_{s}(s) = \hat{g}(s)\tilde{u}(s) \), where \( s \) is the Laplace transform argument, and \( \hat{g}(s) = y_{s}(s)/\tilde{u}(s) \) is the nominal transfer function defined by \( \hat{g}(s) = C'(sI - A)^{-1}b + d \) so that its associate frequency response obtained for \( s = j\omega \), which is the Fourier transform of the impulse response of \((1)-(2)\) \( g(t) \), is:

\[ \hat{g}(j\omega) = C'((j\omega I - A)^{-1}b + d); \forall \omega \in clR \quad (12) \]

Note that the hodograph \( \hat{g}(j\omega) \) satisfies the symmetry constraints \( \text{Re}(\hat{g}(j\omega)) = \text{Re}(\hat{g}(-j\omega)) \) and \( \text{Im}(\hat{g}(j\omega)) = -\text{Im}(\hat{g}(-j\omega)) \); \( \forall \omega \in clR \). Now, one has for any given finite constant \( \gamma \geq \gamma_{0} \in R_{+} \) from \((4)\), \((9)\), \((11)\) and \((12)\), since \( \delta_{\gamma} - \delta = 0 \) for \( \delta \in \{\text{SSPR}\} \), with \( \min_{\omega \in clR} \hat{g}(j\omega) = \min_{\omega \in clR} \hat{g}(j\omega) \geq \lambda \delta \) that

\[
0 \leq \left( \lambda d - \varepsilon \right) \int_{0}^{t} u^2(\tau)d\tau \leq \lambda d \int_{0}^{t} u^2(\tau)d\tau + \dot{E}(t) = \lambda d \int_{0}^{t} u^2(\tau)d\tau + \dot{E}(t) \leq \min_{\omega \in clR} \text{Re}(\hat{g}(j\omega)) \int_{0}^{t} u^2(\tau)d\tau + \dot{E}(t) \leq E_{0}(t) + \dot{E}(t) = E(t) = -\Gamma(0, t) \leq \gamma < +\infty \forall t \in R_{+}. \quad (13) \]

The constraints \((13)\) imply directly that \( \lim_{t \to \infty} u(t) = \lim_{t \to \infty} u_{i}(r) = 0 \) and \( u \in L_{2}. \) From Lebesgue integration theory, it follows that \( u(t) \) can have a finite number of finite jump discontinuities or a set of infinity jump impulsive isolated discontinuities on a set \( D_{l} \subset R_{-} \), which should have a finite cardinal \( (D_{l} \text{had an infinite cardinal then the above properties on the control input following from (13) would be violated}). \) As a result, ess sup \( \|u(t)\| < +\infty \). Also, \( \dot{E}(t) \leq \dot{E}(t) \leq \int_{0}^{t} u^2(\tau)d\tau + \dot{E}(t) \leq E_{0}(t) + \dot{E}(t) = E(t) + \dot{E}(t) \leq \Gamma(0, t) \leq \gamma < +\infty \forall t \in R_{+}. \) Property (i) has been proved. On the other hand, since \( g \in \{\text{SSPR}\} \), then \( A \) is a stability matrix so that there exist real constants \( K \geq 1 \) and \( d > \rho_{0} \) in \( R_{+} \) being the stability abscissa of \( A \) such that \( \|e^{A\tau}\|_{2} \leq Ke^{-\rho_{0}\tau} \); \( \forall t \in R_{+}. \) Then, since \( u \in L_{2} \), and the use of the Cauchy-Bunyakovsky-Schwarz inequality for the integrals below:

\[
\int_{0}^{t} e^{-A\tau}(bu(\tau) + \eta_{0}(\tau))d\tau = \int_{0}^{t} e^{-A\tau}(bu(\tau) + \eta_{0}(\tau))d\tau
\]

one has from \((6)-(7)\) that

\[
\|x(t)\|_{2} \leq Ke^{-\xi\tau} \|x(0)\|_{2} + \frac{K}{\sqrt{2}p} \left( \|b\|_{2} \int_{0}^{t} u^2(\tau)d\tau + \int_{0}^{t} \|\eta_{0}(\tau)\|_{2}d\tau \right)
\]

\[ (14) \]
\[
|y(t) - n_y(t)| \leq K\|c\|_2 \left[ e^{-\rho t} \|x(0)\|_2 + \frac{1}{\sqrt{2\rho}} \left( \|b\|_2 \int_0^t u^2(\tau) d\tau + \int_0^t \|\eta(\tau)\|_2^2 d\tau \right) \right]
\] (15)

\[K_1 = K_1(u, \Phi) = K_2 \|b\|_2 \int_0^\infty u^2(\tau) d\tau = \frac{K \|b\|_2}{\sqrt{2\rho}} \int_0^\infty u^2(\tau) d\tau < +\infty; K_2 = \frac{K}{\sqrt{2\rho}}\] (16)

\[K_3 = K_3(u, \Phi) = K_3 \|c\|_2; K_4 = K_2 \|c\|_2\] (17)

Property (iii) follows from (6)-(7) and Property (ii) since: (a) \(e^{\delta t}\) is uniformly bounded and integrable (then square-integrable as well) on \([0, +\infty]\) since \(A\) is a stability matrix leading to closed-loop \(\Phi\)-hyperstability for any controller of class \(\Phi\); (b) the \(\Phi\)-asymptotic hyperstability follows for the whole class of controllers \(\Phi\) in the absence of uncertainties such that \(\|x(t)\|_p\) is uniformly bounded and integrable (then square-integrable as well) on \([0, +\infty]\) since \(A\) is a stability matrix leading to closed-loop \(\Phi\)-hyperstability for any controller of class \(\Phi\).

Remarks 2

(1) Note that the assumption on the contribution due to uncertainties to the input-output energy \(\dot{E}(t) \leq \dot{\eta}(t)\) in (11) does not imply “a priori” the assumption that \(\dot{\eta}(t)\) is finitely upper-bounded as a given assumption while this boundedness from above is a proved result of Theorem 1 (i).

(2) Note also that the uniform boundedness of the control input of Theorem 1 (ii) does not guarantee “a priori” that of the state and output even under the boundedness from above of \(\dot{E}(t)\) since the state disturbances depend on the state vector itself.

(3) Even if the boundedness of the incremental input/output energy \(\dot{E}(t)\) associated to disturbances is proved in Theorem 1 (i) and that of the input is proved in Theorem 1 (ii), these features do not imply the boundedness of the state and output without some additional assumptions on the model uncertainty functions as given in Theorem 1 (iii). The reason is that the boundedness of \(\dot{E}(t)\) under that of the control input does not imply that of the state and, even, that of the output since the integrand generating \(\dot{E}(t)\) can, for instance, be unbounded if certain changes of sign in the integrator happen.

(4) Note that we add to the conditions of Theorem 1 (iii) the extra constraint that the disturbances are nonzero but converge to zero exponentially fast according to \(\|\eta(t)\|_2 = o(e^{-\gamma t})\) and \(n_y(t) = o(e^{-\gamma t})\) with \(\min(\rho_1, \rho_2) > \rho_0\) (\(-\rho_0\) being the stability abscissa of the delay-free nominal transfer function of the controlled plant in the feed-forward block) then the closed-loop system is still \(\Phi\)-asymptotically hyperstable.

The next result extends some results of Theorem 1 in a more general way, concerning the essential boundedness of the input and its square-integrability by replacing the hypothesis \(\dot{E}(t) \leq \dot{\eta}(t)\) by some alternative one on growing rate properties of the evolution through time of the upper-bounding functions of the uncertainties related to the control input and their contribution to the output. It is assumed that the uncertainties can be affected by a delay which can exceed the nominal one. Now, it is not either assumed that \(\|\eta\|_2\) and \(n_y\) are in \(L_\infty \cup L_1 \cup L_2\) but that they can grow with the input and the state evolutions according to a delay size memory.

**Theorem 3** Under the assumptions a-b of Theorem 1, assume also that the function of uncertainties \(\eta(t) = \eta(x(t), u(t), t)\) and \(n_y(t) = \eta(x(t), u(t), t)\) satisfy the following upper-bounding constraints;
\[
\|y(t)\|_2 \leq \theta_1 \sup_{r \in [t-2\theta_0, t]} |u(r)| + \theta_2 \sup_{r \in [t-2\theta_0, t]} \|u^2(r)\| + \theta_3 \left(\left\|x_{\theta_0}(t)\right\|^2_2, t\right) \sup_{r \in [t-2\theta_0, t]} \|x(r)\|_2 + \theta_4(t) \tag{18}
\]

\[
\left|y(t)\right| \leq \lambda_1 |u(t)| + \lambda_2 u^2(t) + \lambda_3 \left(\left\|x(t)\right\|^2_2, t\right) \|x(t)\|_2 + \lambda_4(t) \tag{19}
\]

where

\[
\theta_3 \left(\left\|x_{\theta_0}(t)\right\|^2_2, t\right) = \theta_3 + \bar{\theta} \left(\left\|x_{\theta_0}\right\|^2_2, p \in \Omega \subseteq R, t\right)
\]

\[
\lambda_3 \left(\left\|x(t)\right\|^2_2, t\right) = \lambda_3 + \tilde{\lambda} \left(\left\|x\right\|^2_2, q \in \Omega \subseteq R, t\right)
\]

for \( t \in R_0 \), where \( \Omega \) is some finite discrete subset of the positive real numbers, and \( \theta_1, \theta_2, \theta_3, \theta_4 \in R_0 \) and \( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in R_0 \) are uniformly bounded functions and independent of the initial conditions and controls such that \( \theta_4(t) \to 0 \) and \( \lambda_4(t) \to 0 \) as \( t \to \infty \).

Then, the following properties hold:

(i) The output satisfies the subsequent upper-bounding constraint:

\[
\left|y(t) + t - y_{\theta_0}(t) + t - y_{\theta_0}(t)\right| \leq \rho_1(t_\theta, t) + \rho_2 \sup_{r \in [t-2\theta_0, t]} |u(r)| + \rho_3 \sup_{r \in [t-2\theta_0, t]} \|u^2(r)\| + \rho_4(t); \forall t \in R_0 \tag{20}
\]

where

\[
\rho_1(t_\theta, t) = K e^{-\theta_1 \|c\|_2 \left\|x_\theta(t)\right\|_2}
\]

\[
\rho_2 = \rho_2(\theta_1 \|c\|_2 \left\|x_\theta(t)\right\|_2) = (\|c\|_2 + \lambda_1 + \theta_4(t)) \left(1 + \frac{\|c\|_2}{\rho} \left(\theta_1 + \frac{\|c\|_2}{\rho}\right) + \lambda_1\right)
\]

\[
\rho_3(\theta_1 \|c\|_2 \left\|x_\theta(t)\right\|_2) = \left[\frac{K}{\rho} \left(\frac{\rho}{\rho - (\theta_1 + \lambda_1)}\right)\right] \rho_2 + \lambda_2 + \lambda_4
\]

\[
\rho_4(t) = \lambda_4(t_\theta, t)
\]

for some existing finite \( t_\theta = t_\theta(\mu, \nu, \epsilon) \in R \), \( t_\theta(\mu, \nu, \epsilon) \in R \) provided that

\[
\bar{\theta}_3 = \theta_3 + \sup_{t \in [t-2\theta_0, t]} \bar{\theta}_3(t, t) \leq \frac{\rho}{K} - \mu - \nu = \bar{\theta}_3 < +\infty
\]

\[
\tilde{\lambda}_3 = \lambda_3 + \sup_{t \in [t-2\theta_0, t]} \tilde{\lambda}(\|x(t)\|^2_2, q \in \Omega \subseteq R, t) \leq \tilde{\lambda}_3 < +\infty
\]

for some real constants \( \mu \in (0, \rho/K), \nu \in (0, \rho/K - \mu) \), and

\[
\nu_0 \geq \max\left(0, \tilde{\lambda}(\|x(t)\|^2_2, q \in \Omega \subseteq R, t) - \lim_{t \to +\infty} \sup_{t \in [t-2\theta_0, t]} \tilde{\lambda}(\|x(t)\|^2_2, q \in \Omega \subseteq R, t)\right)
\]

(ii) Assume that \( \rho_3 = \rho_3(|u|) \) is subject to the constraint:

\[
\lim_{t \to +\infty} \sup_{t \in [t-2\theta_0, t]} \rho_3^{-1}(t) \leq K_{\rho_0} \rho_3 + K_{\rho_3} \lim_{t \to +\infty} \sup_{t \in [t-2\theta_0, t]} |u(t)|
\]

for some constants \( K_{\rho_0} \in R_0, K_{\rho_3} \in R_0 \). If \( \rho_3 + K_{\rho_3} < \lambda \) then:


\[ u \in L_2, \ u(t) \to 0 \ as \ t \to \infty, \ \text{ess sup} \ |u(t)| < +\infty, \ \sup_{t \in [t_0, t]} |\dot{E}(t)| < +\infty \ \text{and} \ 0 < E(t) < +\infty; \ \forall t \in R_+ \ \text{for any non-identically zero control on a time interval} [0, t_0] \ \text{of nonzero measure.}\]

**Proof** One gets from (18) via (7.a) that

\[
\|\eta(t)\|_2 \leq \theta_1 \sup_{r \in [t_0, t]} |u(r)| + \theta_2 \sup_{r \in [t_0, t]} u^2(r)
+ \left[ \theta_3 + \left( \frac{p}{|\theta|} \right) \sup_{r \in [t_0, t]} \|x(r)\|_2 + \theta_4(t) \right] \sup_{r \in [t_0, t]} \|x(r)\|_2 + \theta_5(t)
\leq \theta_1 \sup_{r \in [t_0, t]} |u(r)| + \theta_2 \sup_{r \in [t_0, t]} u^2(r) + \theta_6(t)
+ \left[ \theta_3 + \left( \frac{p}{|\theta|} \right) \sup_{r \in [t_0, t]} \|x(r)\|_2 + \theta_7(t) \right] \sup_{r \in [t_0, t]} \|x(r)\|_2
\]

(25)

with \( \rho > 0 \) and \( K \geq 1 \) from assumption b of Theorem 1. Thus, since \( e^{-\rho t} \to 0 \) and \( \theta_3(t) \to 0 \) as \( t \to \infty \), one has

\[
\lim_{t \to \infty} \left( \frac{1}{\rho} \theta_3(t) K \right) = 0
\]

(26)

Define \( t_0 = t_0(t) \) as the maximum \( \tau \in [t - 2h(t), t] \) such that

\[
\eta(t, \tau) \leq \sup_{\tau \in [t - 2h(t), t]} \|\eta(\tau)\|_2
\]

Then,

\[
\lim_{t \to \infty} \left( \frac{1}{\rho} \theta_3(t) K \right) \eta(t_0(t), \tau) \leq 0
\]

(27)

and, since \( \theta_3 \to 0 \) as \( t \to \infty \), there exists some real \( \mu \in (0, \rho/K) \) and \( t_0 = t_0(\mu, \epsilon, v) \) such that \( \theta_3 + \mu = \theta_3 + \theta_3(t) \leq \epsilon - \mu \).

Define \( \tau \geq t_0 \) and \( t_0 \to \infty \) as \( \epsilon \to 0 \) for such a \( v \) and \( \mu \), and one gets directly from (27) that

\[
\lim_{t \to \infty} \left( \frac{1}{\rho} \theta_3(t) K \right) \eta(t_0(t), \tau) \leq 0
\]

(28)

\[
\|\eta(t)\|_2 \leq \theta_1 \sup_{t \in [t_0, t]} |u(r)| + \theta_2 \sup_{t \in [t_0, t]} u^2(r)
+ \theta_3(t) \sup_{t \in [t_0, t]} \|x(r)\|_2 + \theta_4(t) \sup_{t \in [t_0, t]} \|x(r)\|_2
\]

(29)

and using again (7.a), the following three relationships hold:

\[
\|x(t_0 + t)\|_2 \leq Ke^{-\rho t} \|x(t_0)\|_2 + \left( \frac{\|x(t_0)\|_2}{\rho} \right) \sup_{t \in [t_0, t_0 + t]} |u(r)| + \left( \frac{\|x(t_0)\|_2}{\rho} \right)^2 \theta_2 \sup_{t \in [t_0, t_0 + t]} u^2(r)
\]

(30)
\[
\lim_{t_0 \to \infty} \sup_{t \in [t_0, t_0+\epsilon]} \left( \|x(t_0 + t)\|_2 - Ke^{-\rho d} \|x(t_0)\|_2 - \frac{k \|b\|}{\rho} \left( 1 + \frac{\rho + \lambda}{\rho} \right) \theta_1 \sup_{r \in [t_0, t_0+t]} |u(r)| \right)
\]
\[
- \frac{k}{\rho} \left( \frac{\rho + \lambda}{\rho} \right) \theta_1 \sup_{r \in [t_0, t_0+t]} u^2(r) \leq 0; \forall t \in R_d
\]
\[
\lim_{t_0 \to \infty} \sup_{t \in [t_0, t_0+\epsilon]} \left( \|x(t_0 + t)\|_2 - \frac{k \|b\|}{\rho} \left( 1 + \frac{\rho + \lambda}{\rho} \right) \theta_1 \sup_{r \in [t_0, t_0+t]} |u(r)| \right)
\]
\[
- \frac{k}{\rho} \left( \frac{\rho + \lambda}{\rho} \right) \theta_1 \sup_{r \in [t_0, t_0+t]} u^2(r) \leq 0
\]  
(31)

In the same way, we can easily obtain from (19) via (7.b)–(8) that (20) subject to (21)–(23) holds.

\[
\left| y(t_0 + t) - yf(t_0 + t) \right| \leq \rho_0(t_0, t) \rho_1 \rho_2 \sup_{r \in [t_0, t_0+t]} |u(r)| + \rho_3 \sup_{r \in [t_0, t_0+t]} u^2(r) + \rho_4(t)
\]  
(32)

since it turns out from (25) that for any prefixed \( \alpha , \nu_0 \in R_d \), there exists some finite

\( t_0 ( \geq t_j ) = t_j (\mu, \nu, \nu_0, \epsilon) \) such that \( \lambda_0 + \nu_0 \geq \lambda_0 + \lambda \left( \|x(t)\|_2 \right) q \in \Omega_0 \subset R_d , t_j \geq t_0 \). Property (i) has been proved. On the other hand, note that

\[
y(t) = yf(t) + \left( y(t) - yf(t) \right) \geq yf(t) - \left| y(t) - yf(t) \right|
\]  
(33)

\[
y(t) \geq -\Gamma(t, t) = E(t_0, t) = \int_{t_0}^{t} y(r) u(r) dr
\]
\[
\geq \left( \lambda d - \rho_2 - K_{\rho_3} \sup_{t \in [t_0, t_0+t]} |u(r)| \right) \lim_{t_0, t_\to\infty} \int_{t_0}^{t} u^2(r) dr - \epsilon (t_0) - |a(t(t_0))|
\]  
(34)

since \( \phi \in \Phi \), \( \theta_1(t) \to 0 \) and \( \lambda_0(t) \to 0 \) as \( t \to \infty \). If \( \rho_2 + K_{\rho_3} < \lambda d \), since \( \rho_3 \sup_{t \in [t_0, t_0+t]} |u(r)| \leq K_{\rho_3} t \to \infty \) as \( \epsilon (t) \to 0 \) and \( \lambda d > 0 \), since \( \delta \in \{ \text{SSPR} \} \) and \( t_\delta \to \infty \), \( t - t_\delta \to \infty \), then for non-zero control:

\[
y(t) \geq \left( \lambda d - \rho_2 - K_{\rho_3} \sup_{t \in [t_0, t_0+t]} |u(r)| \right) \lim_{t_0, t_\to\infty} \int_{t_0}^{t} u^2(r) dr > 0
\]  
(35)

Then, \( \lim_{t_0 \to \infty} \sup_{t \in [t_0, t_0+t]} u(t) = 0 \) and \( \lim_{t_0 \to \infty} \sup_{t \in [t_0, t_0+t]} |u(t)| < +\infty \) for any \( t_0 < +\infty \). On the other hand, one has from (17) to (22) that the state and output are essentially bounded for all time since \( \delta \in \{ \text{SSPR} \} \), then it is strictly stable and the state and output converge asymptotically to zero for any initial conditions in the absence of unstructured disturbances of the given class. Property (ii) has been proved.

Note that the assumption that \( \rho_2 + K_{\rho_3} < \lambda d \) of Theorem 3 (ii) is satisfied if \( \rho_2 \) and \( \rho_3 \) in (21) are small enough according to the size of the absolute stability abscissa of the nominal feed-forward time-invariant dynamics. This holds under corresponding sufficiency-type conditions on the corresponding parameters which define the upper-bounds of (18)–(19). Theorem 3 does not guarantee the boundedness of the state and input since that of the unstructured disturbances from the nominal system is not proved. It is now proved that they are bounded under certain additional conditions leading to the hyperstability stability of the system under explicit upper-bounding functions for those uncertainty contributed terms.

**Theorem 4** If all the hypothesis of Theorem 3 hold, \( \rho \) is large enough related so that \( \rho \geq K \left( e^{\rho (t_0 - t_0)} - e^{-\rho d} \right) \) and \( e^{\rho d} \left( e^{\rho d} - 1 \right) \geq \mu + \nu - 2 \) then the closed-loop system is \( \Phi \)-hyperstable.

**Proof** From (7) and (22) and (28)–(29), one gets:
\[ \| x(t + h(t)) \|_2 \leq Ke^{-\rho h_0} \]
\[ \times \left( \| x(t) \|_2 + \frac{\rho^{n-1}}{\rho} \sup_{t \in [t-h_0, t+h_0]} \| u(\tau) \| + \sup_{t \in [t-h_0, t+h_0]} \| \tilde{\theta}_3(t) \| \sup_{t \in [t-h_0, t+h_0]} \| x(\tau) \|_2 \right) \]
\[ \leq Ke^{-\rho h_0} \frac{\rho^{n-1}}{\rho} \left( \sup_{t \in [t-h_0, t+h_0]} \| u(\tau) \| + \left( 1 + \sup_{t \in [t-h_0, t+h_0]} \| \tilde{\theta}_3(t) \| \right) \sup_{t \in [t-h_0, t+h_0]} \| x(\tau) \|_2 \right) \]

and then, since \( u(t) \to 0 \) as \( t \to \infty \), one gets:
\[ \lim_{t \to \infty} \sup \left( \| x(t + h(t)) \|_2 - Ke^{(\nu - \rho) \frac{\rho^{n-1}}{\rho} (1 + \tilde{\theta}_3_0)} \sup_{t \in [-h_0, t+h_0]} \| x(\tau) \|_2 \right) < 0 \]

Assume that there is a positive real sequence \( \{ t_i \} \to +\infty \) with
\[ \| x(t_i + h(t_i)) \|_2 = \sup_{t \in [t_i-h_0, t_i+h_0]} \| x(\tau) \| \to \infty \] as \( t_i \to +\infty \). Then,
\[ \lim_{t_i \to \infty} \left( 1 - Ke^{(\nu - \rho) \frac{\rho^{n-1}}{\rho} (1 + \tilde{\theta}_3_0)} \right) \| x(t_i + h(t_i)) \|_2 < 0 \]

so that \( \tilde{\theta}_3_0 = \frac{\mu}{\nu} > 1 \), by taking into account (22.a), which is violated under the given hypothesis (hence, a contradiction) if \( 0 \leq \tilde{\theta}_3_0 \leq \frac{\rho^{n-1}}{\rho} (1 - e^{\nu - \rho}) \) and a stronger sufficiency-type conditions which satisfies both constraints is \( 2 + e^{\rho h_0} (e^{\nu h_0} - 1) \geq \mu + \nu \). This constraints guarantees the uniform boundedness for all time and any admissible initial conditions of the disturbance state and output functions as well as that of the state and output from Theorem 3.

\[ \square \]

3. A more general framework for dissipative systems

Through this section, we consider a more general formalism including multivariable systems, i.e. multi-input and/or multi-output systems under a more general framework of dissipativity which includes as a particular case the hyperstability property. Consider a dynamic system (Rasvan et al., 2000) with \( p \) outputs and \( m \) inputs which is identified with the input-output pairs \((u, y)\), where \( u \in U_2^m \) and \( y \in Y_2^p \) are the spaces of truncated input and output complex vector functions defined for any arbitrary truncation time instant \( t \in \mathbb{R}_0 \), which are extended spaces of the pre-Hilbertian spaces \( U^m \) and \( Y^p \). In the following, we identify \( U^m \) and \( Y^p \) being the spaces of vector functions \( u: \mathbb{R} \to \mathbb{R}^m \) and \( y: \mathbb{R} \to \mathbb{R}^p \) being in \( L_2^m \) and \( L_2^p \) respectively, as commonly supposed in the background literature.

Note that truncated functions can also be defined on any closed intervals of the whole real set, so as to define the extended spaces of the pre-Hilbertian spaces of inputs and outputs and to be formally ready to define the corresponding Fourier transforms and associate Parseval’s theorem, as follows:

\[ v(t) = v_{[0, T]} = \begin{cases} v(t) & \text{for } t \in [0, T] \\ 0 & \text{for } t \in (-\infty, 0) \cup (T, \infty) \end{cases} \]  

(37.a)

\[ v(t) = v_{[t_1, t_2]} = \begin{cases} v(t) & \text{for } t \in [t_1, t_2] \\ 0 & \text{for } t \in (-\infty, t_1) \cup (t_2, \infty) \end{cases} \]  

(37.b)
Those truncated function will be then used. Consider now the $n$-th multivariable system of inputs and $p$ outputs of similar structure to the single-input single-output (1)–(3), given by:

$$\dot{x}(t) = Ax(t) + Bu(t) + \eta(t)$$

(38)

$$y(t) = Cx(t) + Du(t) + \eta_0(t)$$

(39)

The nominal version of the system (37)–(38) is that which is disturbance-free, i.e. $\eta \equiv 0$ and $\eta_0 \equiv 0$. Define the following integral inequality which is an “ad hoc” generalization of (4):

$$\chi(t_0, t_1) = \int_{t_0}^{t_1} w(u(\tau), y(\tau))d\tau; \forall t_0, \ t_1 \geq t_0 \quad (40)$$

where

$$w(u(t), y(t)) = u^T(t)Ku(t) + 2u^T(t)L^T y(t) + y^T(t)M y(t); \forall t \in \mathbb{R}_0$$

(41)

where $K$, $L$, and $M$ are real matrices of appropriate orders for (41) to be the well-posed.

**Definition 5** Rasvan et al. (2000) is said that the system (37)–(38) is $(K, L, M)$-dissipative with respect to the supply rate $w(u(t), y(t))$ if $\chi(t_0, t_1) \geq 0; \forall t_0, \ t_1 \geq t_0$ and all input-output pairs of the system.

Note that in order that (40) be well-posed according to (41) for any given triple $(K, L, M)$, it is necessary that any pairs $(u, y)$ in (37)–(38) be locally square-integrable on $c \mathbb{R}$.

Definition 5 is extended in a natural way as follows:

**Definition 6** It is said that the system (37)–(38) is $(K, L, M)$-strictly dissipative with respect to the supply rate $w(u(t), y(t))$ if $\chi(t_0, t_1) > 0; \forall t_0, \ t_1 > t_0$ and all input-output pairs of the system.

Note that in accordance with the concepts of Section 2, the nominal system (37)–(38) is hyperstable if its transfer matrix is positive real which implies that $\int_{t_0}^{t_1} u^T(\tau)y(\tau)d\tau \geq 0; \forall t_0, \ t_1 \geq t_0$ and asymptotically hyperstable if the transfer matrix is strictly positive real which implies $\int_{t_0}^{t_1} u^T(\tau)y(\tau)d\tau > 0; \forall t_0, \ t_1 > t_0$ for a control which is nonzero on a subset of nonzero measure of $[t_0, t_1]$.

The following result is direct:

**Proposition 7** Assume that $p = m$. The nominal system (37)–(38) is $(0, L, 0)$-strictly dissipative for any $L > 0$ with respect to the supply rate $w(u(t), y(t))$ if and only if it is asymptotically hyperstable. It is $(0, L, 0)$-dissipative for any $L \geq 0$ if and only if it is hyperstable.

Furthermore, the subsequent result holds:

**Proposition 8** Assume that $p = m$. The following properties hold:

(i) The nominal system in (37)–(38), assumed to be asymptotically hyperstable with an associate strongly positive real transfer matrix, is $(K, L, M)$-strictly dissipative with respect to the supply rate $w(u(t), y(t))$ if and only if $x_{d_0} \geq \lambda_{\max} \left( K + M + L^T I_p \right)$, where $d_0 = \lambda_{\min} \left( \frac{\alpha M}{\beta} \right) > 0$. 


(ii) If the associate positive real transfer matrix is non-strongly strictly positive real (i.e. \( \text{Re}\tilde{G}(j\omega) \) for \( |\omega| < \infty \) and \( \tilde{G}(j\omega) \to 0 \) as \( |\omega| \to \infty \)) then the nominal system (37)–(38) is \((K, L, M)\)-strictly dissipative with respect to the supply rate \( w(u(t), y(t)) \) if and only if \( K + M + L + L^T = I_p \).

(iii) If the associate positive real transfer matrix is positive real then the nominal system (37)–(38) is \((K, L, M)\)-dissipative with respect to the supply rate \( w(u(t), y(t)) \) if and only if \( K + M + L + L^T = I_p \).

**Proof** Since \( p = m \) and \( \text{Re}\tilde{G}(j\omega) + \tilde{G}^*(j\omega) \geq 2\lambda d_0 \) for some real \( \lambda \in \mathbb{R} \) and all \( \omega \in \mathbb{R} \), where \( \tilde{G}(s) = C(sI - A)^{-1}B + D \) is the strongly positive real transfer matrix of (37)–(38). Since

\[
\int_0^t u^T(\tau)y(\tau) d\tau \geq \lambda d_0 \int_0^t u^T(\tau)u(\tau) d\tau > 0 \text{ for } t > t_0 \text{ then }
\]

and the sufficiency part of the proof of strict dissipativity follows. The necessity follows since if \( \lambda d_0 \leq \lambda \max(K + M + L + L^T - I_p) \) it is always possible to find a control and a pair \( (t_0, t) \subseteq cI \mathbb{R}_+ \) such that \( \chi(t_0, t) < 0 \). Properties (ii)–(iii) follows in the same way taking into account that \( d_0 = 0 \). \( \square \)

**Theorem 9** Define real square matrices \( Q = Q_0 + \hat{Q} \) and \( Q_0 \) of order \( p = m \), where

\[
Q = \begin{bmatrix} KL^T \\ LM \end{bmatrix}; Q_0 = \begin{bmatrix} KL^T - I_p/2 \\ L - I_p/2M \end{bmatrix}; \hat{Q} = \begin{bmatrix} 0p/2 \\ Ip/2 \end{bmatrix}
\]

and consider the zero-state nominal system in (38)–(39); i.e. the one under zero initial conditions and being disturbance-free system by assuming that \( p = m \). The following properties hold:

(i) The system is \((K, L, M)\)-strictly dissipative (respectively, \((K, L, M)\)-dissipative) with respect to the supply rate \( w(u, y) \) if and only if \( Q > 0 \) (respectively, if and only if \( Q \succeq 0 \)).

(ii) If, in addition, \( \hat{G} \in \{\text{SSPR}\} \) with \( d_0 = \lambda \min \left( \frac{D_0^2}{2} \right) > 0 \) then the nominal zero-state system is asymptotically hyperstable and then \( \left( 0, I_p/2, 0 \right) \)-strictly dissipative with respect to the supply rate \( w(u, y) \). Also, there exist matrices \( \hat{K}, \hat{L} \) and \( \hat{M} \) such that the nominal zero-state system is \((K, L + I_p/2, M)\)-strictly dissipative for all \( K \in [-\hat{K}, \hat{K}], L \in [-\hat{L}, \hat{L}] \) and \( M \in [-\hat{M}, \hat{M}] \).

(iii) Assume, in addition, that a feedback controller \( u(t) = -\phi(y(t), t) \) of \( p = m \) components belonging to an hyperstable class \( \{\phi\} \) defined by the Popov’s inequality:

\[
\Gamma(t_0, t) = \int_{t_0}^t y^T(\tau) \phi(y(\tau), t) d\tau \geq -\gamma_0
\]

for some finite \( \gamma_0 \in \mathbb{R}_+ \) \( \forall t_0 \in \mathbb{R}_+ \) \( t \geq t_0 \in \mathbb{R}_+ \) is incorporated to generate the control input. Then, the closed-loop system is asymptotically hyperstable and \((K, L + I_p/2, M)\)-strictly dissipative if

\[
\lambda d_0 > 2 \sup_{\omega \in \mathbb{R}_+} \left| \frac{1}{2} \|K + \left(L^T - I_p/2\right) \tilde{G}(j\omega)\right| + \|M\|_2 \sup_{\omega \in \mathbb{R}_+} \left| \tilde{G}(j\omega) \right|_2^2
\]

**Proof** Property (i) follows directly from Definitions 5–6 and (43). Define \( u^*(t) = (u^T(t), y^T(t)) \) for any \( t \in \mathbb{R}_+ \) and assume that \( \hat{G} \in \{\text{SSPR}\} \) with \( d_0 = \lambda \min \left( \frac{D_0^2}{2} \right) > 0 \) so that one gets from (40)–(41) for any real constant \( \gamma \geq \gamma_0 \) that
\[
\left( \lambda d_{\nu} - \frac{1}{2} \sup_{0 < \nu \leq \tilde{\nu}} \left[ \mu_0 \left( K + \left( L^T - I_p / 2 \right) \hat{G}(\nu) \right) \right] + \| M \|_2 \sup_{\omega \in \mathbb{R}_+} \left\| \hat{G}(\nu) \right\|_2^2 \right) \int_{\nu}^{\tilde{\nu}} \| u(\tau) \|_2^2 d\tau \\
\leq \lambda d_{\nu} \int_{\nu}^{\tilde{\nu}} \mu_0 \left( L^T - I_p / 2 \right) Q_{\nu} u_{\nu}^T(\tau) \| u(\tau) \|_2 d\tau \\
- \int_{\nu}^{\tilde{\nu}} \mu_0 \left( L^T - I_p / 2 \right) Q_{\nu} u_{\nu}^T(\tau) Q_{\nu} u_{\nu}(\tau) d\tau \\
\leq \chi(t_0, t) = \int_{\nu}^{\tilde{\nu}} u_{\nu}^T(\tau) Q_{\nu} u_{\nu}(\tau) d\tau = \int_{\nu}^{\tilde{\nu}} u_{\nu}^T(\tau) Q_{\nu} u_{\nu}(\tau) d\tau
\]

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}_{\nu}^T(\tau) \left( -i\nu \right) Q_{\nu} \hat{u}_{\nu}(\tau) d\nu
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}_{\nu}^T(\tau) \left( -i\nu \right) \left( \hat{L} + \hat{M} \right) \hat{G}(\nu) d\nu
\]

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}_{\nu}^T(\tau) \left( -i\nu \right) \left( \hat{L} - \hat{M} \right) \hat{G}(\nu) d\nu
\]

\[
\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}_{\nu}^T(\tau) \left( -i\nu \right) Q_{\nu} \hat{u}_{\nu}(\tau) d\tau
\]

\[
\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}_{\nu}^T(\tau) \left( -i\nu \right) \left( \hat{L} - \hat{M} \right) \hat{G}(\nu) d\nu
\]

\[
\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}_{\nu}^T(\tau) \left( -i\nu \right) \left( \hat{L} + \hat{M} \right) \hat{G}(\nu) d\nu
\]

\[
\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}_{\nu}^T(\tau) \left( -i\nu \right) \left( \hat{L} + \hat{M} \right) \hat{G}(\nu) d\nu
\]

\[
\leq \mu_0 \left( \hat{G}(\nu) \right) \left[ \int_{\nu}^{\tilde{\nu}} \| u(\tau) \|_2^2 d\tau \right] \int_{\nu}^{\tilde{\nu}} \| u(\tau) \|_2^2 d\tau
\]

\[
\leq \gamma \left( \| Q_{\nu} \|_2 \left[ \int_{\nu}^{\tilde{\nu}} \| u(\tau) \|_2^2 d\tau \right] \int_{\nu}^{\tilde{\nu}} \| u(\tau) \|_2^2 d\tau \right)
\]

\[
\leq \gamma \left( \| Q_{\nu} \|_2 \left[ \int_{\nu}^{\tilde{\nu}} \| u(\tau) \|_2^2 d\tau \right] \int_{\nu}^{\tilde{\nu}} \| u(\tau) \|_2^2 d\tau \right)
\]

Then, \( \chi(t_0, t) > 0; \forall t_0, t > t_0 \in \mathbb{R}_+ \), from the lower-bounding expression for \( \chi(t_0, t) > 0 \) in (46) and the nominal zero-state system is asymptotically hyperstable and \( \left( 0, I_p / 2, 0 \right) \) strictly dissipative, which are, in fact, equivalent concepts, with respect to the supply rate \( \omega_u(\cdot), \omega_y(\cdot) \). Now, by the continuity of the eigenvalues of \( Q \) as mappings of the entries of the matrices \( K, L, M \), one concludes the existence of \( K, L \) such that the nominal zero-state system is \( \left( K, L + I_p / 2, M \right) \) strictly dissipative for all \( K \in [-K], L + I_p / 2, M \) strictly dissipative for all \( K \in [-K], L, M \) and \( M \in [-M, M] \). Property (ii) is proved. Property (iii) follows from (45) and the upper-bounding expression for \( \chi(t_0, t) \) in (46) since

\[
0 < \chi(t_0, t) = -\Gamma(t_0, t) + \tilde{\gamma} \leq \gamma_0 + \tilde{\gamma} < +\infty; \forall t > t_0 \in \mathbb{R}_+
\]

where

\[
\tilde{\gamma} = \max \left( \| Q_{\nu} \|_2 \left[ \int_{\nu}^{\tilde{\nu}} \| u(\tau) \|_2^2 d\tau \right] \int_{\nu}^{\tilde{\nu}} \| u(\tau) \|_2^2 d\tau \right)
\]

\[
\tilde{\gamma} = \max \left( \| Q_{\nu} \|_2 \left[ \int_{\nu}^{\tilde{\nu}} \| u(\tau) \|_2^2 d\tau \right] \int_{\nu}^{\tilde{\nu}} \| u(\tau) \|_2^2 d\tau \right)
\]
Note that Theorem 9 (i) still holds, that is, the \((K, L, M)\) dissipativity and strict dissipativity of Definitions 5 and 6 hold if \(Q > 0\) and, respectively, \(Q > 0\) irrespective of the system being nominal or not since the properties depend only of the measured input and output.

If the system is subject to disturbances, then the following robustness- type extended result from Theorem 9 holds inspired by previous results in Theorem 3:

**Theorem 10** Assume that:

1. \(\hat{G} \in \{\text{SSPR}\} \) with \(d_0 = \lambda_{\min}\left(\frac{\partial u}{\partial x}\right) > 0\), and that a feedback controller \(u(t) = -\phi(y(t), t)\) of \(p = m\) components belonging to an hyperstable class \((\Phi)\) of controllers defined by the integral Popov’s inequality (44) is incorporated to generate the control input.

2. unstructured disturbances can be present in the system (38)–(39) which satisfy the upper-bounding constraints of the forms (18)–(19), where \(c, b\) and \(d\) are replaced with \(C, B\) and \(D\), respectively, and the absolute values of signals are replaced by the corresponding \(l_1\)-norms on the space \(R^p\), and such that the parameters (21) are subject to (22)–(24) hold. Assume also that bounded time-varying delays can also exist satisfying \(h_\gamma > \sup_{t \in \mathbb{R}_+} h(t)\) with \(\theta_1, \theta_2, \theta_{3\gamma}, \lambda_1, \lambda_2, \lambda_{3\gamma} \in R_{\gamma}\), and \(\theta_\alpha, \lambda_\alpha : R_{\alpha} \to R_{\gamma}\), are uniformly bounded and independent of the initial conditions and controls with \(\theta_\gamma(t) \to 0\) and \(\lambda_\gamma(t) \to 0\) as \(t \to \infty\).

3. Then, the closed-loop system is \(\Phi\)-asymptotically hyperstable and \((K, L + I_{p/2}, M)\) strictly dissipative if

\[
\lambda d_0 > \left[ 2 \sup_{\omega \in \mathbb{R}_{\gamma}} \left| \mu_2 \left( K + \left( L^T - I_{p/2} \right) \hat{G}(j\omega) \right) + \|M\|_2 \sup_{\omega \in \mathbb{R}_{\gamma}} \left\| \hat{G}(j\omega) \right\|_2 \right]^2 + \rho_\delta \|M\|_2 + 2K_{\gamma} \|L\|_2 \right] (49)
\]

**Outline of proof:** The proof follows by combining Theorem 9 (iii) with the results of Theorems 3–4 and the corresponding change of the constraint (45) by the constraint (49).

4. **Concluding remarks**

This paper has studied the stability and asymptotic hyperstability property of a linear and time- invariant continuous-time system with an associated strongly positive real transfer function and a wide class of controllers satisfying a Popov’s type inequality referred to as being an hyperstable class of controllers. It is also assumed that there are unstructured uncertainties in the state and output which are interpreted as being jointly generated by unmodeled dynamics under a potential internal delay, uncertain parameterizations and eventual noise. It is proved that global closed-loop stability is robust in the sense that it is achievable under some smallness explicit conditions on the absolute values of the parameters defining some bounding terms through time of the norms of the uncertain contributions to the state and output. Since the stability property holds for any member belonging to the whole class of hyperstable controllers acting on a strongly positive real transfer function in the feed-forward loop, the closed-loop system is referred to as being hyperstable. In the particular case that the unstructured disturbances on the state and output converge asymptotically to zero at exponential higher rate than that defined by the stability abscissa of the transfer function the closed-loop system is, in addition, asymptotically hyperstable. A second group of results which can be discussed refer to the general dissipativity framework of the controlled plant. The property of dissipativity is defined related to a triple of design matrices which define a quadratic form of the supply rate as a function of the input and output vectors and whose time- integral defines the passivity of the controlled plant related to the above triple in terms that such an integral is non-negative for any interval of time. The asymptotic hyperstability of the controlled plant, or its equivalent strict passivity, is a particular case of its dissipative characteristic property. Such a property, together with the hyperstability of the class of controllers, lead to the asymptotic hyperstability of the closed-loop system under any controller of an hyperstable class for certain restrictions in terms of smallness related to the stability abscissa of the nominal controlled plant on the unstructured disturbances.
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