Quantum resources covariance

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The developments of special relativity and quantum mechanics marked the beginning of the modern physics age. The former has taught us that while space and time are frame dependent notions, there is a quantity—the space-time interval—whose value all inertial observers agree upon. This reveals, so to speak, a genuine “fact” of the universe, a relativistic invariant. On the other hand, since the dawn of quantum mechanics, there is no consensus on what the theory is all about. The situation is admittedly subtler: quantum theory is grounded on a complex vector space and the very notions of observer and reference frame are controversial. Here we construct a theoretical framework within which a given combination of quantum resources is shown to be a Galilean invariant. To this end, we postulate a principle of relational symmetry between “the observer” and “the observed” and employ the notion of quantum reference frame. Unitary transformations then follow that allow us to perceive the physical resources seen from the viewpoint of any quantum system. Interestingly, we find that one needs more than quantum coherence and quantum correlations to prove quantum resources covariance. Finally, we show that the notion of physical reality implied by quantum mechanics is not absolute.

I. INTRODUCTION

Physics is a deeply relational model of nature. Its fundamental laws are expected to be the same for all inertial observers. Even the very act of producing scientific information is a relational physical process. An observer system, R, typically provided with a brain, physically interacts with an observed system, A, collects information about it, and sends processed information to further brain-endowed systems. R does not interact with itself and cannot describe its own physical state; it is, in effect, a reference for the establishment of relational statements about A. Information itself is physical [1, 2], since it can be transformed via physical interactions and encoded in the states of physical systems. Remarkably, even though our physical theories are in full harmony with relativity fundamentals, be them Galilean or Lorentzian, none of them is so adapted to deal with the physics of information as quantum mechanics.

Once we recognize this feature and the fact that reference frames can be treated quantum mechanically [3], then complex physical problems can be solved. It is by now well understood, for instance, that the flow of information from the system to operationally inaccessible degrees of freedom—the environment—is a key mechanism for the emergence of classical behavior from the quantum substratum [4–6]. Even the long-lasting foundational dichotomy “collapse vs unitary evolution”, as posed by Everett [7], can be addressed [8].

The perception that information is stored, manipulated, and communicated through quantum devices inaugurated promising research fields such as quantum computing [9], quantum cryptography [10], and quantum thermodynamics [11]. The huge amount of conceptual and technological developments achieved so far reveals that information is, if not the whole, a significant part of the story that quantum mechanics can tell us about nature. Among the many nonclassical mechanisms through which quantum information can be encoded and distributed, entanglement [12–15] and coherence [14–13] are distinctive ones, especially because of their roles as quantum resources [20].

Interestingly enough, it has been recognized that entanglement and coherence are frame-dependent resources [21, 22]. We are then inevitably induced to query about the existence of some informational invariance upon changes of reference frames. Is it the case that different observers may not agree on their diagnosis about entanglement and coherence while agreeing about some combination of these resources, much like one has in special relativity, where different observers do not agree with respect to space and time intervals but do agree on the combination $\Delta x^2 - (c\Delta t)^2$? In what follows we show how one can identify the invariance of quantum information and, as a nontrivial consequence, the covariance of quantum resources. For one to achieve this fundamental result, two ingredients prove mandatory, namely, (i) reference frames need to be treated quantum mechanically and (ii) a nonclassical resource further than coherence and correlations must be regarded.

The literature of quantum reference frames is by now well developed, at least with regard to the nonrelativistic regime [21, 24], within which time is an absolute notion. Formally, one can construct a quantum reference frame by starting with the quantum description of a system with respect to some classical reference frame and then “jumping” to the perspective of one of the particles, as we do, for instance, when solving the hydrogen atom from the perspective of the proton. This mathematical procedure can be shown to be accomplished by means of unitary transformations [22, 25, 26] henceforth denoted $T$. They allow us to obtain the transition $\rho \mapsto \rho' = T\rho T\dagger$ to the perspective of a quantum reference frame. With that, we can trivially conclude, by virtue of the unitary invariance of the von Neumann entropy, $S(\rho) = S(T\rho T\dagger)$, that the information

$$I(\rho) = \ln d - S(\rho)$$

associated with the state $\rho$ is an invariant, that is $I(\rho) = I(\rho')$. Here, $d$ stands for the dimension of the Hilbert space on which $\rho$ acts. We see that, whichever the quantum reference frame one decides to adopt to describe nature, the in-
formation content $I(\rho)$ of the state $\rho$ is always the same. This means, in particular, that a pure state will be the maximally resourceful one for all observers, while $1/d$ will be always resourceless. Despite the indisputable status of $I(\rho)$ as a meaningful measure of information [27], and the proven role of information as a fundamental quantum resource (see Ref. 28 and references therein), it is by no means clear how to factorize this quantity in terms of coherence and correlations. In fact, it is an open question whether these resources suffice to expand information in all frames of reference. We now address these questions.

II. INFORMATION DECOMPOSITION

The Lorentz invariant $ds^2$ can only be experimentally accessed through frame-dependent measurements of $d\tau^2$ and $dt^2$. Here we show that the quantum invariant $I(\rho)$ is likewise decomposable in resources specific to each particular reference frame. To this end, we devise a measurement-oriented procedure through which one guarantees that the entire information encoded in $\rho$ is erased in all reference frames. This is so because, in our approach, measurements are Galilean events, that is, they occur in every reference frame at the same time, although with respect to distinct observables.

Since the generalization of our approach to multipartite systems is straightforward, we restrict our analysis to the bipartite case to keep the presentation simpler. Consider a quantum state $\rho \in \mathcal{B}(\mathcal{H})$, where $\mathcal{B}(\mathcal{H})$ is the set of bounded operators acting on the Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. The informational content of this state, which has been prepared within a quantum reference frame $\mathcal{F}$, is $I(\rho) = \ln d - S(\rho)$, where $d = d_A d_B$ and $d_{AB} = \text{dim}(\mathcal{H}_{AB})$. Let $A = \sum_i a_i A_i$ be a discrete-spectrum nondegenerate observable acting on $\mathcal{H}_A$, with corresponding projectors $A_i = |a_i\rangle \langle a_i|$. After a measurement of this observable, the state collapses to $A_i \otimes \rho_{B|i}$, where $\rho_{B|i} = \langle a_i | \rho | a_i \rangle / p_i$ and $p_i = \text{Tr}(A_i \otimes 1_B)$. If, however, the outcome $a_i$ is not revealed, then the post-measurement state is given by

$$\sum_i p_i A_i \otimes \rho_{B|i} = \sum_i (A_i \otimes 1_B) \rho (A_i \otimes 1_B) =: \Phi_A(\rho).$$

(2)

$\Phi_A$ denotes a completely positive trace-preserving map which indicates that an unrevealed measurement of $A$ has been performed in the reference frame $\mathcal{F}$. Known as the dephasing operation in the quantum resource theory of coherence [18,19], $\Phi_A$ removes both coherence in the $A$ basis and entanglement. Most importantly, this map manifests itself here as a key tool for our purposes, since it helps us to build the well known quantifiers of quantum coherence [16],

$$C_A(\rho_A) := S(\Phi_A(\rho_A)) - S(\rho_A),$$

(3)

one-way quantum discord [29,31],

$$D_A(\rho) := I_{AB}(\rho) - I_{A|B}(\Phi_A(\rho)),$$

(4)

and symmetric quantum discord

$$D_{AB}(\rho) := I_{AB}(\rho) - I_{A|B}(\Phi_{AB}(\rho)),$$

(5)

where $I_{AB}(\rho) = S(\rho_A) + S(\rho_B) - S(\rho)$ is the mutual information between the parts $A$ and $B$, $\rho_{A|B} = \text{Tr}_{B}(\rho_{AB})$ are reduced states, $\Phi_{AB}(\rho) \equiv \Phi_A \Phi_B(\rho)$ is a joint local map, and $\Phi_B$ is an unrevealed-measurement map for another observable $B = \sum_j b_j B_j \in \mathcal{B}(\mathcal{H}_B)$. Being basis dependent, the above measures are henceforth referred to as $A$-coherence, $A$-discord, and $AB$-discord, respectively, with a similar terminology for measures related to observables acting on $\mathcal{H}_B$.

Now, given the above, it is not difficult to check that, upon a measurement of $A$, we have

$$I(\Phi_A(\rho)) - I(\rho) = - \left[ C_A(\rho_A) + D_A(\rho) \right],$$

(6)

which shows that the informational content of $\rho$ decreases by a value that precisely corresponds to the amount of $A$-coherence and $A$-discord (correlations) that are removed from $\rho$ by the $A$ measurement. Via direct calculations, we verify that $C_A(\Phi_A(\rho)) = D_A(\Phi_A(\rho)) = 0$, confirming that the post-measurement state $\Phi_A(\rho) = \sum_i p_i a_i A_i \otimes \rho_{B|i}$ no longer has such resources. On the other hand, there is some quantum resource in the form of $B$-coherence and $B$-discord. Now, performing another unrevealed measurement, this time on part $B$, yields the resulting state $\Phi_{BA}(\rho)$ and decreases the information by

$$I(\Phi_{BA}(\rho)) - I(\Phi_A(\rho)) = - \left[ C_B(\text{Tr}_A \Phi_A(\rho) + D_B(\Phi_A(\rho)) \right].$$

(7)

where $\text{Tr}_A \Phi_A(\rho) = \rho_B$ (meaning that $\Phi_A$ is nonsignaling). The above expression shows that $B$-coherence and $B$-discord are removed upon the measurement of $B$, as expected. Noting that $D_A(\rho) + D_B(\Phi_A(\rho)) = D_{AB}(\rho)$, one verifies that the total resource suppressed so far, which follows from adding the expressions (6) and (7), can be written as $I(\Phi_{BA}(\rho)) - I(\rho) = - \left[ C_A(\rho_A) + C_B(\rho_B) + D_{AB}(\rho) \right]$. Clearly, $\{A, B\}$-related coherences and quantum correlations have been removed from the initial preparation. This motivates us to introduce the notion of quantumness underlying the set $\mathcal{O} \equiv \{A \otimes 1_B, 1_A \otimes B\}$ ($\mathcal{O} = \{A, B\}$, for short) as

$$\Xi_{\mathcal{O}}(\rho) := C_A(\rho_A) + C_B(\rho_B) + D_{AB}(\rho) = I(\rho) - I(\Phi_{BA}(\rho)).$$

(8)

Evidently, $\Xi_{\mathcal{O}}$ can be interpreted as the amount of information that is removed from $\rho$ via a $\{A, B\}$ measurements. The resulting state, $\Phi_{AB}(\rho) = \sum_{ij} p_{ij} A_i \otimes B_j$, still encodes an amount $I(\Phi_{AB}(\rho)) = \mathcal{H}(\{p_{ij}\})$ of information, where $\mathcal{H}(\{p_{ij}\})$ is the Shannon entropy of the distribution $p_{ij}$. Interestingly, this suggests that some quantum feature remains, meaning that the information in $\rho$ is not entirely encoded in the form of coherence and correlations. To appreciate this point, let us consider the set $\mathcal{O}$ of observables maximally noncommuting with $\mathcal{O} = \{A, B\}$, that is, $[A, A] \neq 0$ with their corresponding eigenbases satisfying $|a_i \langle \bar{a}_i|)^2 = 1/d_A$ (similarly for $B$ and $\bar{B}$). In this sense, we shall say that $\mathcal{O}$ and $\bar{\mathcal{O}}$ are maximally unbiased (MU), in direct reference to the concept of maximally unbiased bases [32]. It follows that $\Phi_{AB}(\rho_{AB}(\rho)) = 1/d$ and $I(\Phi_{AB}(\rho_{AB}(\rho))) = 0$. That is, to erase all the information encoded in $\rho$, we still have to submit the system to measurements of the maximally noncommuting set $\mathcal{O}$. This analysis suggests that the remaining quantum element
is quantum incompatibility. To rigorously make this point, let us briefly review some recent developments on this topic.

Usually the notion of quantum incompatibility is related to the noncommutativity of two observables, say $A_1$ and $A_2$, acting on $\mathcal{H}_A$. In effect, if $[A_1, A_2] \neq 0$, then one cannot find simultaneous eigenvectors for these observables and, therefore, one cannot obtain simultaneous eigenvalues through a single measurement. There has been a significant effort to frame this notion as a quantum resource, giving its operational interpretation and mathematical support [53–59]. Recently, the two of us and a collaborator introduced the notion of context incompatibility [60], whose quantifier reads

$$\mathcal{J}_{(\rho_A, Q_A)} = I(\Phi_{A_1}(\rho_A)) - I(\Phi_{A_2}(\rho_A)),$$

where the referred context is $\{\rho_A, Q_A\} \subset \mathfrak{B}(\mathcal{H}_A)$, with $Q_A = \{A_1, A_2\}$. The above measure vanishes if and only if (i) $\Phi_{A_1}(\rho_A) = \mathbb{I}_A/\mathcal{D}_A$ or (ii) $[A_1, A_2] = 0$, and possesses an operational interpretation in a communication protocol. It admits a natural extension to bipartite scenarios, where the context is now given by $\{\rho, Q_1, Q_2\} \subset \mathfrak{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$, with $Q_1 = \{A_1, B_1\}$, $Q_2 = \{A_2, B_2\}$, $A_{1,2} \in \mathfrak{B}(\mathcal{H}_A)$, and $B_{1,2} \in \mathfrak{B}(\mathcal{H}_B)$. In this case, we have

$$\mathcal{J}_{(\rho, Q_1, Q_2)} = I(\Phi_{D_1}(\rho)) - I(\Phi_{D_2}(\rho)),$$

where $\Phi_{D_1} \equiv \Phi_{A_1}B_{1} = \Phi_{A_1B_1}$ and $\Phi_{D_2}\Phi_{D_1} \equiv \Phi_{D_2}\Phi_{D_1}$. It can be readily checked that either for commuting observables $\{A_1, A_2\} = \{B_1, B_2\} = 0$ or $\rho = \mathbb{I}_A/\mathcal{D}_A \otimes \mathbb{I}_B/\mathcal{D}_B$, one has $\mathcal{J}_{(\rho, Q_1, Q_2)} = 0$. On the other hand, if $Q_2 = Q_1$, meaning that $A_1(B_1)$ and $A_2 = 1(B_2 = 1)$ are MU, then $I(\Phi_{D_2}(\rho)) = 0$ and the inequality $\mathcal{J}_{(\rho, Q_1, Q_2)} \geq I(\Phi_{D_1}(\rho))$ saturates. If, in addition, $\rho = \mathbb{I}_{A_1} \otimes \mathbb{I}_{B_1}$, a product of $A_1$ and $B_1$ projectors, then $\mathcal{J}_{(\rho, Q_1, Q_2)}$ reaches the maximum value of $\ln(d_A d_B)$. Moreover, if $\rho = \rho_A \otimes \rho_B$, then the bipartite incompatibility splits as $\mathcal{J}_{(\rho_A, \rho_B, Q_1, Q_2)} = \mathcal{J}_{(\rho_A, Q_1, Q_2)} + \mathcal{J}_{(\rho_B, Q_1, Q_2)}$. All this demonstrates that the generalized measure [60] has all the features that ects the original formulation [49] as a faithful quantifier of context incompatibility.

Then, with $\tilde{Q} = \{\tilde{A}, \tilde{B}\}$ being MU to $Q = \{A, B\}$, we see that $\mathcal{J}_{(\rho, \tilde{Q})} = I(\Phi_{\tilde{Q}}(\rho)) = I(\Phi_{AB}(\rho))$, which is the precise amount of information that remained in the state in our previous discussion. This legitimates us to return to the Eq. (\ref{eq:invariant}) to obtain the decomposition

$$I(\rho) = \mathcal{S}_Q(\rho) + \mathcal{J}_{(\rho, \tilde{Q})}.$$  

Note that the notion of quantumness ($\mathcal{S}_Q$) encompasses coherence and correlations associated with the set $Q$, whereas $\mathcal{J}_{(\rho, \tilde{Q})}$, which is a nonclassical resource as well, is linked with both MU sets $Q$ and $\tilde{Q}$. Now, even though we can find, in general, more than one set $\tilde{Q}$ for each given context $Q$, the choice of the former is constrained to its algebraic relation with the latter, so that the information decomposition can ultimately be related to $Q$ solely. To see that this is indeed the case, we note that for $\rho = \Phi_Q(\gamma)$ ($\Phi_Q(\gamma)$ one has $\mathcal{J}_{(\Phi_Q(\gamma), Q, \tilde{Q})} = 0$ and $I(\Phi_Q(\gamma)) = \mathcal{S}_Q(\Phi_Q(\gamma))$. Then, referring to Eq. (\ref{eq:invariant}), one has $\mathcal{J}_{(\rho, \tilde{Q})} = I(\Phi_{\tilde{Q}}(\rho)) = \mathcal{S}_Q(\Phi_{\tilde{Q}}(\rho))$. This result leads us to introduce the incompatible quantumness $\mathcal{S}$ such that

$$\mathcal{J}_{(\rho, \tilde{Q})}(\rho) = \mathcal{S}_{\tilde{Q}}(\Phi_{\tilde{Q}}(\rho)) = \tilde{\mathcal{S}}_{\tilde{Q}}(\rho).$$

(12)

With that, we finally arrive at the desired decomposition

$$I(\rho) = \mathcal{S}_Q(\rho) + \tilde{\mathcal{S}}_{\tilde{Q}}(\rho).$$

(13)

In this very compact form we can appreciate the quantum contents of information, namely, the quantumness $\mathcal{S}_Q$, which encompasses quantum coherence and quantum correlations with respect to $Q$, and the incompatible quantumness $\tilde{\mathcal{S}}_{\tilde{Q}}$, which pinpoints the fundamental role of incompatibility for quantum information. Note that, as we have for the Lorentz invariant $\delta s^2 = d\tau^2 - (cdt)^2$, expression (\ref{eq:invariant}) puts on the left-hand side the absolute quantity and, on the right-hand side, the frame-dependent objects, the ones that are accessed via measurements.

### III. COVARIANCE

The above discussion suggests that, while quantum coherence, quantum correlations, and quantum incompatibility are not absolute resources individually, they add up to an invariant one. However, to definitely prove information covariance, we need to be concrete. This demands showing the existence of meaningful sets of observables and the unitary transformation $T$ that yields the jump from $R$’s to $A$’s perspective. We consider a scenario wherein two systems, $A$ and $B$, are described by an observer $R$ through the context $\mathcal{C} = \{\rho, Q\}$, where $Q = \{A, B\}$. The change to $A$’s perspective, which reduces $R$ and $B$ to observed systems, cannot be implemented by effectively transforming both the observables and the state. In fact, there are two recipes that lead $R$’s description, $\mathcal{C} = \{\rho, Q\} \subset \mathfrak{B}(\mathcal{H})$, to $A$’s description, $\mathcal{C}' = \{\rho', Q'\} \subset \mathfrak{B}(\mathcal{H}')$, namely,

Active Picture (AP) :

\begin{equation}
\text{Passive Picture (PP) :}
\end{equation}

\begin{equation}
\end{equation}

where $T : \mathcal{H} \mapsto \mathcal{H}'$, with $\mathcal{H}' = \mathcal{H}_R' \otimes \mathcal{H}_B'$ being the Hilbert space adopted in $A$’s perspective. That these pictures are equivalent is readily checked through the expectation value formula $(\langle Q'\rangle') = \text{Tr}[\rho' T^T] = \text{Tr}[\rho T^T]$, except by the change of vector spaces, the AP could be thought of as deriving from a Hamiltonian interaction in the R frame, while the PP refers to free alterations in the coordinate system. It follows from the above recipes and $\Phi_Q(T \rho^T) = \Phi_{Q'}(\rho')$ (for any set $Q$) that $\mathcal{S}_Q(T \rho^T) = \mathcal{S}_{Q'}(\rho')$, and similarly for $\tilde{\mathcal{S}}$. Then, given the decomposition (\ref{eq:invariant}) and the invariance $I(\rho) = I(T \rho^T)$ one can guarantee that

$$\mathcal{S}_Q(\rho) + \tilde{\mathcal{S}}_{\tilde{Q}}(\rho) = \mathcal{S}_Q'(\rho') + \tilde{\mathcal{S}}_{\tilde{Q}}'(\rho')$$

in both AP and PP. This formula, which constitutes the main result of this work, points the form invariance (covariance) of quantum information upon its measurement factorization in...
different reference frames, where distinct sets of observables, \{O, O\}' and \{O', O\}' are used for the actual access of information. In other words, it ensures that the total amount of quantum resources available is the same in all quantum reference frames. It is very difficult to imagine a self-consistent way to decompose \( l(\rho) \) in terms of measurement-based quantities of a purely classical information theory. This suggests that information covariance may be a fundamental principle of quantum mechanics.

We now specialize our approach to the Galilean relativity. More specifically, we consider the set \( \mathcal{O} = \{X_A, X_B\} \) formed by the position operators of A and B relative to R. Here we restrict our discussion to one-dimensional systems, for simplicity. Although different transformations \( T \) may exist that involve relative coordinates, one of particular interest for us yields, in the PP, \( \mathcal{O}' = \{X'_R, X'_B\} \equiv \{-X_A, X_B - X_A\} \),

\[
\mathcal{O}' = \{X'_R, X'_B\} \equiv \{-X_A, X_B - X_A\},
\]

where \( X'_R = T^\dagger X_A T \) gives the position of R relative to A and \( X'_B = T^\dagger X_B T \) gives the position of B relative to A. The notation \( \mathcal{O}' \) refers to an operator \( X'_R \) that acts on \( \mathcal{H}_A \) respecting the same algebra with which \( -X_A \) acts on \( \mathcal{H}_A \). In addition, we have \( \langle X'_B \rangle_{\rho'} = \langle -X_A \rangle_{\rho} \), thus confirming that the application of \( T \) correctly promotes the system A to the role of reference frame. Most importantly, this approach tacitly admits that there is nothing special about the initial reference frame R. To better emphasize this point, let us restrict our attention, for a while, to the systems R and A. Suppose that R prepares A in a position eigenstate \( |\chi\rangle \). Because physics is deeply relational, there is no reason preventing us to believe that A has just prepared R in a state \( \langle -\chi | \). (The appearance of the minus sign "-" is not really fundamental here, but it helps to point the change of reference.) As far as momentum eigenstates are concerned, the statement "R prepares A in \( \mu |\mu_{AR}\rangle \)" is equivalent to "A prepares \( \mu_{AR} \) in \( \langle -\mu_{AR} | \)\), where \( \mu_{AR} = m_A m_R / (m_A + m_R) \) is the reduced mass of the system. This sort of kinematical relationality is understood here as a fundamental premise that ought to be obeyed even when the mechanics is quantum and no matter how big or fast the involved bodies are. Although we have restricted our analysis to spatial degrees of freedom, we postulate that this must hold for every physical state.

**Postulate.** (No privileged quantum reference frame.) Every physical entity is allowed "to observe" or "to be observed". Therefore, whenever R prepares A in a state \( |\psi\rangle \), A automatically prepares R in a counterpart state \( |\psi'\rangle \).

For variables such as position and momentum it is clear that \( \psi' = -\psi \), so that the unitary transformation that implements the change is the usual parity operator \( \Pi \). The aforementioned preparation process in such two-particle universe is assumed to be consistent with all physical interactions and conservation laws [41], and is not constrained to the absoluteness of time. For instance, if R prepares A in a linear-momentum superposition \( |-p\rangle + |p\rangle \), then A prepares R in \( |p\rangle + |-p\rangle \). It is crucial to note that this preparation is critically different from \( |-p\rangle |p\rangle + |p\rangle |-p\rangle \), which is feasible only in the presence of a third system, say S, that can make sense of the motion of both A and R, and hence can prepare them in a momentum-conserving entangled state. In S’s viewpoint, because of the existence of correlations neither A or R is individually in a superposition of momentum states. In the two-particle universe, however, only a single degree of freedom exists (A’s position relative to R, or vice-versa), so there is no "informer" able to encode "which-way information" about the observed system. As a consequence, a fundamental wave-like behavior, associated with a quantum superposition, manifests itself (see Ref. [42] for a related discussion).

Having identified the relevance of the parity operator to the formalism, let us return to the earlier discussion involving the systems A, B, and R. To produce the relative coordinate \( X_B - X_A \), we use the shift operator \( e^{i X_A P_B / \hbar} \), which yields a displacement of B conditioned to the position of A relative to R. Then, the unitary transformation we are looking for is

\[
T := \Pi_A e^{i X_A P_B / \hbar}.
\]

The generalization for many particles is simple, requiring just the replacement \( P_B \rightarrow \sum_S P_S \), with \( S = B, C, D \cdots \). Also, via the subindex replacement \( A(B) \rightarrow B(A) \) we find the transformation that promotes B to the role of reference frame. In terms of the AP and PP, one can straightforwardly check that \( T |u\rangle |v\rangle = |-u\rangle |v - u\rangle \) and \( T^\dagger T = \mathcal{O}' \) [see Eq. (16)], respectively. Another well known feature of quantum reference frames is the fact that the transformation that gives relative coordinates does not give relative momenta \( \mathcal{P}'(P'_R, P'_B) \equiv \{P'_R, P'_B\} \) whereby no connection with relative momenta is found. The transformation that produces the correct relative momenta reads

\[
T := \Pi_A \exp \left( \frac{i a \langle X_B, P_B \rangle}{2\hbar} \right) \exp \left( -\frac{i}{\hbar} m_B X_B P_A \right),
\]

with \( a = \ln(\mu/m_B) \) and \( \{X_B, P_B\} \equiv X_B P_B + P_B X_B \). This transformation gives \( T^\dagger \{P_A, P_B\} T = \{ -P_A, P_B / (\mu_{m_B} - \mu_{m_B}) \} \), which is the desired result. On the other hand, the new coordinates are \( T^\dagger \{X_A, X_B\} T = \{ -X_A + m_B X_B, \mu_{m_B} X_B \} \), having no link with the expected relative positions.

We have seen, therefore, that there exist some variables, namely, positions and momenta, for which one can exhibit an explicit unitary transformation that switches the description to the viewpoint of the quantum particle A. Of course, it is implicit in the treatment presented that R itself is a quantum system, so that no privileged observer needs to be conceived. The fact that these variables are continuous does not forbid the application of the formalism we introduced for the decomposition of information. In fact, an operational discretization method has been recently developed to treat
position and momentum as discrete variables \[43\]. In Appendix A, the whole idea of our approach is applied and a particular solution is found to be the problem of the breakdown, in the nonrelativistic quantum domain, of the universality of free fall. Also noteworthy is the fact that quantum mechanics admits as well a Galilean space-time invariant, which turns out to be the trivial counterpart of \(d x^2 = d t^2 - (cdt)^2\).

To see this, let us consider the parts R, A, B, and C. In R’s perspective, the position of C relative to B is computed as \((X_C - X_B)_r\). The generalization of transformation (17) to this case is \(T = \Pi_A \exp[i \lambda (P_B + P_C)/\hbar]\). By direct application of \(T\) we find \(X'_C = X_C - X_A\) and \(X'_B = X_B - X_A\). It then follows that \((X'_C - X'_B)_{rp} = (X_C - X_B)_r\), which proves invariance in the Galilean space-time (see Ref. \[24\] for a thorough discussion involving dynamics).

IV. RELATIVITY OF REALITY

Recently, a criterion of physical reality was introduced \[44\] which has been shown to be rather enlightening with respect to both foundational and applied issues such as (i) the discovery of an information-reality complementarity \[8\], (ii) the definition of bipartite \[45, 46\] and tripartite \[47\] aspects of nonlocality that are fundamentally different from those deriving from Bell inequality violations, (iii) the discussion of realism violations and nonlocality in a two-walker system \[48\], and (iv) the proposition of an alternative solution to Hardy’s paradox \[49\]. The key premise behind the aforementioned criterion is that after a measurement is conducted of an observable \(A \in \mathcal{B}(\mathcal{H}_A)\), there must be an element of reality associated with \(A\) (or, \(A\) is an element of reality), even when the measurement outcome is kept in secret. In this context, the unrevealed-measurement state \(\Phi_A(\rho)\) can thus be taken as a state of reality for \(A\) (henceforth referred to as an A-reality state). Accordingly, if \(\Omega = \Phi_A(\rho)\), then \(\Phi_A(\Omega) = \Omega\), meaning that further unrevealed measurements of \(A\) on an A-reality state \(\Omega\) does not alter the fact that \(A\) is already an element of reality. It then follows that \(\rho = \Phi_A(\rho)\) can be adopted as a criterion of A-reality and, as a consequence,

\[ \mathcal{I}_A(\rho) := S(\Phi_A(\rho)) - S(\rho) \]

emerges as a quantifier of the degree with which the A-reality criterion is violated for a given \(\rho\). Called \textit{irreality}, \(\mathcal{I}_A(\rho)\) is nonnegative and vanishes if and only if \(\rho = \Phi_A(\rho)\). Interestingly, it is easy to verify that \(\mathcal{I}_A(\rho) = \mathcal{C}_A(\rho) + \mathcal{D}_A(\rho)\), implying that \(\mathcal{C}_D(\rho) = \mathcal{I}_A(\rho) + \mathcal{I}_B(\Phi_A(\rho))\). This shows that the notion of quantumness and its incompatible counterpart \(\mathcal{C}\) can be entirely rephrased in terms of irreality.

Now, via the AP-PP equivalence one can easily check that \(S(\Phi_A(TPT^\dagger)) = S(\Phi_TAT(\rho))\), but none of these versions can be equated to \(S(\Phi_A(\rho))\). This is to say that \(\mathcal{I}_A(TPT^\dagger) \neq \mathcal{I}_A(\rho)\), which implies that quantum irreality as diagnosed via Eq. (19) is not absolute. A simple illustration of this fact can be given by use of the discrete approach (see Ref. \[43\] and Appendix A for further details), within which a position eigenstate is written as \(|x_k\rangle = \delta q |k\rangle\), where \(x_k = k\delta q\) (\(k \in \mathbb{Z}\)) and \(\delta q\) is the experimental resolution for a position measurement. Suppose that \(R\) prepares the state

\[ |\psi\rangle = \delta q \left( \frac{|j\rangle |j\rangle + |j\rangle |j + k\rangle + |j + k\rangle |j\rangle}{\sqrt{2}} \right), \]

where \(i, j, k \in \mathbb{Z}^\ast\). Since \(|\psi\rangle\rangle = \delta_{ij} \langle i|\langle j|\delta q\), one has \(\langle \psi |\psi\rangle = 1\).

For concreteness, we can imagine the situation where a diatomic molecule, with atoms \(A\) and \(B\), has just passed through a double-slit setup, with \(k\delta q\) being the spatial separation of the slits. Using the projectors \(\Pi_i = \delta q \langle i|\langle i|\delta q\), one can apply standard techniques of discrete spaces algebra to obtain \(\rho_B = (|j\rangle \langle j| + |j\rangle |j + k\rangle + |j + k\rangle |j\rangle)/\sqrt{2}\), \(\mathcal{C}_X_B(\rho)\) = \(\langle |i\rangle \langle i| + |i\rangle |i + k\rangle + |i + k\rangle |i\rangle\rangle)/\sqrt{2}\), and \(\mathcal{I}_X_B(\rho) = \mathcal{I}_B(\rho)\). From direct calculations it follows that \(C_X_B(\rho_B) = 0\) and \(\mathcal{I}_X_B(\rho) = \mathcal{I}_B(\rho) = \ln 2\). Hence, from R’s perspective, B’s position is not an element of reality. On the other hand, form A’s perspective we have

\[ |\psi'\rangle = T |\psi\rangle = \delta q \left( \frac{|j - i\rangle + |j - i - k\rangle}{\sqrt{2}} \right) |j - i\rangle. \]

We see that B’s position relative to A is now well defined (atom A always “sees” atom B alongside). It is not difficult to show that \(\mathcal{I}_X_B(\rho'\rho) = 0\) and \(\mathcal{I}_X_B(\rho') = \mathcal{I}_X_B(\rho) = 0\), which confirms that B’s position is an element of reality for A. Therefore, R and A do not agree on the degree of realism underlying B’s position. As a side remark, we note that, incidentally, in this example we have \(\mathcal{I}_X_B(\rho) = \mathcal{I}_X_A(\rho')(\rho')\) (see Appendix A for a counter-example).

V. DISCUSSION

The theory of special relativity tells us that while the space-time quantities \(d \vec{r}\) and \(d t\) are frame dependent, there is an invariant \(d x^2 = d t^2 - (cdt)^2\) whose value is the same for all observers. In this sense, \(d x^2\) is a “fact” of the universe—an absolute physical quantity. Here we have shown that within the nonrelativistic quantum paradigm, besides the trivial length element, \(d s = \text{Tr}[\rho(X_C - X_B)]\), one has that the total amount of quantum resources also is a Galilean invariant. This means that while two distinct quantum reference frames may not agree upon the amount of some specific resource they have at disposal, they will never disagree on the total amount. To make this point, we have shown how information—a fundamental quantum invariant upon changes of quantum reference frames—can be decomposed in terms of experimentally accessible resources. Interestingly, we have found that its complete factorization involves not only quantum coherence and quantum correlations, but also quantum incompatibility. Our findings highlight the roles played by quantum information and quantum references frames in allowing quantum theory to obey the fundamental tenet that physics must be fundamentally relative.

Although we have all along been concerned with Galilean relativity, our approach paves firm grounds for similar discussions within the Lorentzian framework. First, the crucial
tool we have employed for the information decomposition—the unrevealed measurements—have already been shown to be equivalent to procedures involving the unitary establishment of correlations with an ancilla followed by its discard [8]. This means that we do not need to deal with the instantaneous state reduction, which would eventually introduce some difficulties in Lorentzian treatments of the measurement process. Second, it is widely accepted that reference-frame changes are implemented via the application of unitary transformations either to observables or states. Even though we already have consistent clues for devising the correct Lorentz transformations, the missing link still is a definitive relativistic quantum theory able to encompass time as a canonical conjugate operator in its algebraic structure.

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Appendix A: Free fall in different reference frames

To illustrate our findings, we now discuss the so-called breakdown of the universality of free fall in the quantum framework [51–55], a problem that seems to preclude a pacific coexistence between quantum mechanics and the weak equivalence principle. Before starting, let us notice that the solution of the Schrödinger problem with initial state \( |\Psi_0 \rangle \) and Hamiltonian \( H = P^2/(2m) + mgX \) describing a particle of mass \( m \) under a uniform gravitational field \( g \) can be written in the form

\[
|\Psi_t \rangle = e^{-i\frac{pt^2}{2m}} e^{-i\frac{mgx}{2}} e^{-i\frac{pt}{2m}} e^{-i\frac{gt^2p}{2m}} |\Psi_0 \rangle .
\]

The two \( P \)-dependent unitary transformations have clear interpretations: one of them imposes a spatial shift \(-gt^2/2 \) on the wave function \( \Psi_0(x) \) and the other introduces the typical free-evolution spreading. The others yield phases.

Let us consider two particles, A and B, both in free fall motion from the perspective of an inertial observer R on Earth. Classically, if the particles depart from rest at heights \( x_{A,B} \) such that \( x_B - x_A = d \), then they fall under the same uniform gravitational acceleration \( g \), the relative height \( d \) does not change, and the times of flight do not depend on the masses of the particles. Quantum mechanically, however, R cannot simultaneously prepare well defined positions and momenta for the particles, so that the time-of-flight statistics are seen to depend on the masses \( m_{A,B} \) of the falling particles. Associated with the typical spreading of moving wave packets, this result precludes the universality of free fall. Incidentally, one may still conceive an instance where it is restored, at least within the short-time approximation. In terms of the center-of-mass and relative canonical operators,

\[
X_{cm} = \frac{m_A X_A + m_B X_B}{M}, \quad X_t = X_B - X_A, \quad (A2a)
\]

\[
P_{cm} = P_A + P_B, \quad P_t = \mu \left( \frac{P_B}{m_B} - \frac{P_A}{m_A} \right), \quad (A2b)
\]

where \( M = m_A + m_B \) and \( \mu = m_A m_B / M \), the original two-particle Hamiltonian \( H \) is mapped onto \( H \) as

\[
H = \sum_{i=A,B} \left( \frac{p_i^2}{2m_i} + m_i g X_i \right) \xlongmapsto{T} H = \frac{p_{cm}^2}{2M} + Mg X_{cm} + \frac{p_t^2}{2\mu}. \quad (A3)
\]

The unitary transformation \( T \) is exhibited in Ref. [25]. It is clear that the center of mass is in free fall whereas the relative coordinate is in free motion. With the help of Eq. (A1), we propose, in terms of the \( \{X_{cm}, X_t\} \) eigenbases, the uncoupled solution

\[
|\psi_t \rangle = \int du \varphi_t (u - \frac{gt^2}{2}) |u \rangle \int dv \varphi_t (v - d) |v \rangle, \quad (A4)
\]

where \( |\varphi_t \rangle = |G_{\sigma_t} (u - gt^2/2) \rangle \) and \( |\varphi_t \rangle = |G_{\sigma_t} (v - d) \rangle \), with

\[
G_\sigma (x - x_\sigma) := \exp \left[ - \frac{(x - x_\sigma)^2}{2\pi \sigma^2} \right] \quad (A5)
\]

The function \( \sigma_t^2 = \sigma_t^2 (1 + t^2/t_c^2)^{1/2} \) gives the uncertainty associated with the degree of freedom \( s \) \( \in \{ \text{cm}, r \} \), where \( t_{cm} = 2M \sigma_c^2 / \hbar \) and \( t_r = 2\sigma_r^2 / \hbar \). This particular solution suggests how quantum mechanics restores the universality of free fall in the semiclassical limit: for small \( \sigma \) and large \( t \) (attainable for large masses and \( m_A \gg m_B \)), the \( \sigma_t^2 \) spreading remains negligible for very long times, and thus the relative distance between the particles does not fluctuate.

We now consider the map \( |u \rangle \langle v| \xlongmapsto{T} |u - \frac{m_B}{m_A} v \rangle \langle u + \frac{m_A}{m_B} v| \), which implements the transition from \( \{x_{cm}, x_t\} \) back to the coordinates \( \{x_A, x_B\} \) relative to R. With that, we find

\[
|\psi_t \rangle = \int \int du \, dv \varphi_t (r_{u,v} - \chi_t) \varphi_t (s_{u,v} - d) |u \rangle \langle v|, \quad (A6a)
\]

\[
r_{u,v} = \frac{m_A u + m_B v}{M}, \quad (A6b)
\]

where \( \chi_t = gt^2/2 \). To move to A’s perspective, we apply \( T \) as given by Eq. (17), which yields \( x_t' = -x_A \) and \( x_B' = x_B - x_A \). We obtain \( T \) \( |u \rangle \langle v| \xlongmapsto{T} |v \rangle \langle -u| \) and

\[
|\psi_t' \rangle = \int \int du \, dv \varphi_t (r_{u,v} - \chi_t) \varphi_t (s_{u,v} - d) |u \rangle \langle v|, \quad (A7a)
\]

\[
r_{u,v} = r_{u,-v}, \quad s_{u,v} = s_{-v,u}. \quad (A7b)
\]

To compute the quantenesses of the states \( |A6 \rangle \) and \( |A7 \rangle \), we adopt the discretization formalism developed in Ref. [43]. This amounts to setting \( (u, v) = (i, j)\delta \theta \), with \( \delta \theta \) the spatial resolution (which is the same in every reference frame), and replacing the integrals with summations running over integers \( i, j \in \{-L, L\} \), where \( L = (\xi - 1)/2 \) and \( \xi = 2n\hbar \delta \theta / (\delta x \delta p) \) (space dimension). Here, \( \delta \theta \) denotes the momentum resolution. In addition, projectors are given by \( \Pi_i = \delta \theta |i \rangle \langle i| \),

\[
\Pi_i = \delta \theta |i \rangle \langle i|.
\]
with \( \Pi_i \Pi_i' = \delta_{ii'}\Pi_i \) and \( \sum_{i=1}^{L} \Pi_i = 1 \), \( \langle i'|i' \rangle = \delta_{ii'}/\delta q \), and \( \langle j|j \rangle = \delta_{jj'}/\delta q \). Then, we have

\[
|\psi_t\rangle = \sum_{i,j} \delta q \bar{\varphi}_t(\bar{\sigma}_{i,j} - \bar{X}) \tilde{\varphi}_t(\bar{\sigma}_{i,j} - \bar{d}) |i\rangle |j\rangle , \tag{A8a}
\]

\[
|\psi'_t\rangle = \sum_{i,j} \delta q \bar{\varphi}_t(\bar{\sigma}_{i,j} - \bar{X}) \tilde{\varphi}_t(\bar{\sigma}_{i,j} - \bar{d}) |i\rangle |j\rangle , \tag{A8b}
\]

where \( (\bar{\varphi}_t, \bar{\varphi}_t) \equiv (\varphi_t, \varphi_t)/\sqrt{\delta q} \) are dimensionless amplitudes such that \( |\bar{\varphi}_t|^2 = \mathcal{G}_\varphi(\bar{\sigma}_{i,j} - \bar{X}) \) and \( |\tilde{\varphi}_t|^2 = \mathcal{G}_\phi(\bar{\sigma}_{i,j} - \bar{d}) \). The set \( (\bar{\sigma}_{i,j}, \bar{X}_t, \bar{d}, \bar{\sigma}_{cm}, \bar{\sigma}'_t) \) are formed with quantities normalized with \( \delta q \).

Now, in the regime where \( \eta \equiv \frac{dt}{\delta q} \to 0 \), one readily sees that \( (\bar{\sigma}_{i,j}, \bar{X}_t, \bar{d}, \bar{\sigma}_{cm}) = (i, j, t) \) and \( (\bar{\sigma}_{i,j}, \bar{X}_t, \bar{d}, \bar{\sigma}_{cm}, \bar{\sigma}'_t) = (-i, j, t) \), which renders \( \rho'_t = |\psi'_t\rangle \langle \psi'_t| \) separable. This proves that

\[
0 < D_{X}\mathcal{X}_{A}(\rho'_t) - D_{X}\mathcal{X}_{A}(\rho'_t) = 0 , \tag{A9}
\]

illustrating that quantum correlations are not generally preserved under changes of quantum reference frames. With respect to the state \( \rho_A = \mathcal{T}_R(\rho_t) \), where \( \rho_t = |\psi_t\rangle \langle \psi_t| \), we introduce \( i(t) \equiv \bar{\varphi}_t(i - \bar{X}) \) and \( g(j - i) \equiv \tilde{\varphi}_t(j - \bar{d}) \) to compactly write

\[
\rho_A = \sum_{i,j} \delta q^2 f(i) f'(i') \gamma_{ij} |i\rangle |i\rangle , \tag{A10}
\]

where the parameter \( \gamma_{ij} = \sum_j g(j-i)g'(j-i') \), which depends on \( \sigma_i \), regulates the quantum coherence of \( \rho_A \) but plays no role for \( \rho'_t \). In particular, for \( \sigma_t \to 0 \), we have \( \gamma_{ij} = \delta_{ij} \) and

\[
0 = C_{X}\mathcal{X}_{A}(\rho'_t) - C_{X}\mathcal{X}_{A}(\rho'_t) > 0 . \tag{A11}
\]

This shows that quantum coherence is not an invariant resource either. Returning to Eqs. (A8), one analytically finds, via direct calculations, \( \Phi_{X}\mathcal{X}_{A}(\rho_t) = \sum_{i,j} \delta q^2 \varphi_{ij} |i,j\rangle |i,j\rangle \) and \( \Phi'_{X}\mathcal{X}_{A}(\rho'_t) = \sum_{i,j} \delta q^2 \varphi'_{ij} |i,j\rangle |i,j\rangle \), where

\[
\varphi_{ij} = \mathcal{G}_{\varphi}(\bar{\sigma}_{i,j} - \bar{X}) \mathcal{G}_{\sigma}(\bar{\sigma}_{i,j} - \bar{d}) , \tag{A12a}
\]

\[
\varphi'_{ij} = \mathcal{G}_{\varphi}(\bar{\sigma}_{i,j} - \bar{X}) \mathcal{G}_{\sigma}(\bar{\sigma}_{i,j} - \bar{d}) . \tag{A12b}
\]

With these expressions, which hold for arbitrary \( \eta \), we obtain, via Eq. (A8), \( \mathcal{G}_{X}\mathcal{X}_{A}(\rho_t) = \mathcal{G}(\Phi_{X}\mathcal{X}_{A}(\rho_t)) = \mathcal{H}((\varphi_{ij})) \) and its counterpart \( \mathcal{G}_{X}\mathcal{X}_{A}(\rho'_t) = \mathcal{H}((\varphi'_{ij})) \) in \( A \)’s frame. It can be readily inferred that, in general, \( \varphi_{ij} \neq \varphi'_{ij} \) and

\[
\mathcal{G}_{X}\mathcal{X}_{A}(\rho_t) \neq \mathcal{G}_{X}\mathcal{X}_{A}(\rho'_t) , \tag{A13}
\]

meaning that not even quantunness is invariant. In what follows we present the results of a simulation for the case involving equal-mass particles. The discretized model adopted here is such that \( L = 15 \), implying the space dimension \( d_{A}(\mathcal{B}) = \xi = 31 \) and \( \delta q \delta p \equiv h/5 \) (roughly, \( \delta q \sim \delta p \sim 0.45 \sqrt{\hbar} \), with pertinent SI units). Here, the time evolution is analyzed in terms of the dimensionless time \( \bar{t} = t/\tau \), where \( \tau = 2m_{A}\delta q^2/h = 10^{-10} s \) with \( m_{A}(\mathcal{B}) = 2.4 \times 10^{-10} \) kg. Also, we use \( \bar{d} = 3, \sigma_{cm} = 7, \) and \( \bar{\sigma}_t = 3. \) It is worth mentioning that, to ensure the physical validity of the discretized model, the probability distributions \( \varphi_{ij} \) and \( \varphi'_{ij} \) were monitored and, when necessary, suitably renormalized for all times of the simulation. Figure III shows the behavior of the quantity

\[
\Delta(\bar{t}) := \frac{\mathcal{G}_{X}\mathcal{X}_{A}(\rho_t) - \mathcal{G}_{X}\mathcal{X}_{A}(\rho'_t)}{I(\rho_0)} \times 100\% , \tag{A14}
\]

where \( I(\rho_0) = \ln \xi^2 \). It gives the percentage difference, with respect to the invariant information \( I(\rho_0) \), between the quantunnesses available to the quantum reference frames \( R \) and \( A \). This result illustrates that quantum coherence and quantum correlations do not form an invariant; to this end, the incompatible quantunness \( \mathcal{G}_{X}\mathcal{X}_{A}(\rho_t) = I(\rho_0) - \mathcal{G}_{X}\mathcal{X}_{A}(\rho'_t) \) is an indispensable parcel.

**Appendix B: Intrinsic entanglement**

There is an important aspect that is sometimes overlooked with respect to the transformation \( |\rangle \langle \rangle \). Here we show that the map \( |a\rangle |b\rangle \to |\alpha a\rangle |\beta b\rangle \) does not directly apply to the centers of Gaussian states, not even for very sharp ones. Let us consider the state

\[
|\psi\rangle = \int du dv G^{\alpha}_{\mathcal{A}}(u - a) G^{\beta}_{\mathcal{B}}(v - b) |u\rangle |v\rangle , \tag{B1}
\]

describing the physics of systems \( A \) and \( B \) relative to \( R \), with \( G_{\sigma} \) given by Eq. (A5). Using the map given above and performing a change of dummy variables, we find

\[
|\psi'\rangle = \int du dv G^{\alpha}_{\mathcal{A}}(-u - a) G^{\beta}_{\mathcal{B}}(v - b) |u\rangle |v\rangle , \tag{B2}
\]

which gives the physics relative to \( A \). The product of Gaussian functions in the integrand is proportional to

\[
e^{-\frac{(m_{\mathcal{A}}+\frac{m_{\mathcal{B}}}{2})^2}{2\sigma_{cm}^2}} e^{-\frac{m_{\mathcal{A}}^2}{2\sigma_t^2}} e^{-\frac{m_{\mathcal{B}}^2}{2\sigma_t^2}}, \tag{B3}
\]
with \( a = a(\zeta / \Lambda)^2 \) and \( \zeta = \delta / \sqrt{\delta^2 + \Lambda^2} \). In contrast with the case where center-of-mass and relative coordinates are used \([21]\), the above transformation does not yield any reasonable regime where the crossing term can be neglected. Thus, the expansion \([31]\), which could be denoted \( |\psi\rangle = |a\rangle |b\rangle \) in reference to a product of sharp states centered at \( a \) and \( b \), is not mapped onto \( |\psi'\rangle = | -a\rangle |b - a\rangle \), not even approximately, because \( |\psi'\rangle \) is strongly entangled. That is, the transformation rules for position eigenstates do not trivially apply to sharp Gaussian states.

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