Topological gravity and Wess-Zumino-Witten term

Patricio Salgado A.,¹,* Patricio Salgado-Rebolledo,¹,²,* and Omar Valdivia¹,³,⁴,†

¹Departamento de Física, Universidad de Concepción, Casilla 160-C, Concepción, Chile
²Centro de Estudios Científicos, Casilla 1469, Valdivia, Chile.‡
³Department of Mathematics, Heriot-Watt University, Riccarton, Edinburgh EH14 4AS, U.K.
⁴Maxwell Institute for Mathematical Sciences, The Tait Institute, Edinburgh, U.K.

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Abstract

It is shown that the action for topological gravity in even dimensions found by A. Chamseddine in ref.[1] is, except a multiplicative constant, a gauged Wess-Zumino-Witten Term.
I. INTRODUCTION

Lovelock theory of gravity [2], [3], [4] and its very interesting Chern–Simons subclasses have been the subject of an intensive study during the last two decades [1], [5], [6], [7], [8]. Chern–Simons gravity theories have been extended by using transgression forms instead of Chern–Simons forms as actions [9], [10], [11], [12], [13], [14], [15]. Chern–Simons and Transgression theories of gravity are valid only in odd-dimensions and in order to have a well defined even-dimensional theory it would be necessary some kind of dimensional reduction or compactification. In ref.[16], subsequently ref.[17], [18], [19] and most recently ref.[20], it was pointed out that Chern–Simons theories are connected with some even-dimensional structures known as gauged Wess-Zumino-Witten (gWZW) terms.

The connection between this even-dimensional structure and the Chern–Simons gravity theories suggest that this mechanism could be regarded as an alternative to compactification or dimensional reduction.

On the other hand, in ref.[1], Chamseddine constructed topological actions for gravity in all dimensions. The odd-dimensional theories are based on the Chern–Simons forms. Even-dimensional theories use, in addition to the gauge fields, a scalar field $\phi^a$ in the fundamental representation of the gauge group.

In this work it is shown that the action for topological gravity in even dimensions found by Chamseddine in ref.[1] is a gWZW.

This article is organized as follows. In section II, we review some aspects of the topological gravity theory, the so called Stelle-West formalism and of the gWZW term. In section III, it is shown that the action for the topological gravity studied in ref.[1] corresponds to a gWZW term. Section IV concludes the work with some comments and conclusions. The details of some calculations are summarized in an Appendix A and B.

II. TOPOLOGICAL GRAVITY, STELLE-WEST-FORMALISM AND THE gWZW TERMS

A. Topological gravity

Some time ago A.H. Chamseddine constructed actions for topological gravity in 1 + 1 and in $(2n - 1) + 1$ dimensions [1], [5], [6]. These actions were constructed from the product of
field strengths, $F^{ab}$, and a scalar field $\phi^a$ in the fundamental representation of the gauge group.

In (1 + 1)-dimensions the action is given by

$$S^{(1+1)}[A, \phi] = k \int_{M^2} \epsilon_{abc} \phi^a F^{bc}, \quad (1)$$

and in $(2n-1) + 1$ dimensions the corresponding action can be written in the form [6, 1]

$$S^{(2n)}[A, \phi] = k \int_{M_{2n}} \epsilon_{a_1...a_{2n+1}} \phi^{a_1} F^{a_2 a_3}...F^{a_{2n}a_{2n+1}}, \quad (2)$$

where $F^{ab} = dA^{ab} + A^{ac} A^b_c$ and $A$ is a one-form gauge connection. This action was obtained from a Chern–Simons form using a dimensional reduction method.

In ref.[21] a related approach to this problem was discussed. In this reference it was shown that the action (2) can be obtained from the $(2n + 1)$-dimensional Chern–Simons gravity genuinely invariant under the Poincaré group with suitable boundary conditions. Now we will show that the action (2) corresponds to a $g_{WZW}$ term.

**B. The Stelle-West-formalism**

The basic idea of the Stelle-West formalism is founded on the non-linear realizations studied in refs.[22], [23], [24]. Following these references, we consider a Lie group $G$ and its stability subgroup $H$. The Lie group $G$ has $n$ generators. Let us call $\{X_i\}_{i=1}^{n-d}$ the generators of $H$. We shall assume that the remaining generators $\{Y_l\}_{l=1}^d$ are chosen so that they form a representation of $H$. In other words, the commutator $[X_i, Y_l]$ should be a linear combination of $Y_l$ alone. If the elements of $G/H$ are denoted by $z$, and if the independent fields needed to parametrize $z$, are denoted by $\phi^i$, i.e., the $\phi^i$ parametrize the coset space $G/H$, then a group element $g \in G$ can be uniquely represented in the form $g = z h$ where $h$ is an element of $H$ and $z = e^{-\phi^i Y_i}$.

When $G$ is the group associated to the AdS Lie algebra $[P_a, P_b] = m^2 J_{ab}$, $[J_{ab}, P_c] = (\eta_{bc} P_a - \eta_{ac} P_b)$; $[J_{ab}, J_{cd}] = (\eta_{ac} J_{bd} - \eta_{bc} J_{ad} + \eta_{bd} J_{ac} - \eta_{ad} J_{bc})$ whose generators are $P_a, J_{ab}$ and if the subalgebra $H$ is the Lorentz algebra $SO(3, 1)$ whose generators are $J_{ab}$, then [25] (see also [26], [27], [28], [29])

$$\frac{1}{2} W^{ab} J_{ab} + V^a P_a = e^{\phi^a P_a} \left[ d + \frac{1}{2} \omega^{ab} J_{ab} + e^a P_a \right] e^{-\phi^a P_a}. \quad (3)$$
Using the commutation relation of the AdS algebra, we find that the nonlinear fields $V^a$ and $W^{ab}$ are given by

\[
V^a = e^a + (\cosh x - 1) \left( \delta^a_b - \frac{\phi_b \phi^a}{\phi^2} \right) e^b \\
+ \frac{\sinh x}{x} D\phi^a - \left( \frac{\sinh x}{x} - 1 \right) \left( \frac{\phi_c D\phi_c}{\phi^2} - \phi^a \right) e^b,
\]

(4)

\[
W^{ab} = \omega^{ab} - m^2 \frac{\sinh x}{x} (\phi^a e^b - \phi^b e^a) \\
- m^2 \left[ \phi^a D\phi^b - \phi^b D\phi^a \right] \left( \frac{\cosh x - 1}{x^2} \right).
\]

(5)

with $x = m (\phi^a \phi_a)^{1/2} = m \phi$.

Taking the limit $m \to 0$ in the commutation relation of the AdS Lie algebra and in (4) and (5) we find that the AdS Lie algebra takes the form of the Poincare Lie algebra where now the nonlinear fields are given by

\[
V^a = e^a + D\phi^a, \quad W^{ab} = \omega^{ab}.
\]

(6)

C. The gauged Wess-Zumino-Witten Term

1. Chern Simons Form and WZ Term

Consider the gauge transformed field [30]

\[
A^g = g^{-1} Ag + g^{-1} dg = g^{-1} (A + V) g
\]

(7)

with $V = dgg^{-1}$ and the transformed curvature

\[
F^g = dA^g + (A^g)^2 = g^{-1} F g
\]

(8)

where $g = g(x)$ denotes the gauge element. Let us choose a homotopy such as

\[
A^g_t = tg^{-1} Ag + g^{-1} dg = g^{-1} (A_t + V) g
\]

(9)

with $A_t = tA$, $t \in [0,1]$. The corresponding homotopic curvature is

\[
F^g_t = dA^g_t + (A^g_t)^2 = g^{-1} F_t g
\]

(10)

with

\[
F_t = dA_t + A_t^2 = tF + (t^2 - t)A^2
\]

(11)
Both homotopies (9) and (10) interpolate continuously between \( A_{t=1}^g = A^g, \quad F_{t=1}^g = F^g \) and 
\[ A_{t=0}^g = g^{-1} dg = g^{-1} V g, \quad F_{t=0}^g = 0. \] 
(See Appendix B.)

Applying the Cartan homotopy formula \((B7)\) to a Chern-Simons form\((B13)\) containing the homotopies (9) and (10), we have

\[
Q_{2n+1}(A^g, F^g) = (n + 1) \int_0^1 dt \left\langle A^g \left( \hat{F}_{t}^g \right)^n \right\rangle
\] 
(12)

\[
Q_{2n+1}(A^g, F^g) - Q_{2n+1}(A_0^g, F_0^g) = (k_{01} d + d k_{01}) Q_{2n+1}(A_t^g, F_t^g)
\]

From \((B13)\) we can see that the gauge transformed Chern-Simons term is given by \([30]\)

\[
Q_{2n+1}(g^{-1} dg, 0) = (k_{01} d + d k_{01}) Q_{2n+1}(A_t^g, F_t^g)
\]

\[
Q_{2n+1}(A^g, F^g) = (n + 1) \int_0^1 dt \left\langle g^{-1} (A + V) g \left[ g^{-1} \hat{F}_t g \right]^n \right\rangle
\]
(14)

where

\[
\hat{F}_t^g = t F^g + (t^2 - t) (A^g)^2 = g^{-1} \hat{F}_t g
\]

\[
\hat{F}_t = t F + (t^2 - t) (A + V)^2
\]

so that

\[
Q_{2n+1}(A^g, F^g) = (n + 1) \int_0^1 dt \left\langle g^{-1} (A + V) g \left[ g^{-1} \hat{F}_t g \right]^n \right\rangle
\]
(16)

\[
= (n + 1) \int_0^1 dt \left\langle g^{-1} (A + V) \hat{F}_t^n g \right\rangle = Q_{2n+1}(A + V, F)
\]

Analogously, from \((14)\) we can see that

\[
Q_{2n+1}(A_{t}^g, F_{t}^g) = (n + 1) \int_0^1 ds \left\langle A_{t}^g \left( (F_{t}^g)_{s} \right)^n \right\rangle
\]
(17)

with

\[
(F_{t}^g)_{s} = s F_{t}^g + (s^2 - s) (A_{t}^g)^2 = g^{-1} F_{ts}^g
\]

\[
F_{ts} = s F_{t} + (s^2 - s) (A_{t} + V)^2
\]

so that

\[
Q_{2n+1}(A_t^g, F_t^g) = (n + 1) \int_0^1 ds \left\langle (A_t + V) F_{ts}^n \right\rangle = Q_{2n+1}(A_t + V, F_t)
\]

To calculate \((k_{01} d + d k_{01}) Q_{2n+1}(A_t^g, F_t^g)\), remember that the results \((B12)\)
\[
dQ_{2n+1}(A, F) = \left\langle F^{n+1} \right\rangle
\]
can be generalized to

\[
dQ_{2n+1}(A_{t}^g, F_{t}^g) = \left\langle g^{-1} F_{t}^{n+1} g \right\rangle = \left\langle F_{t}^{n+1} \right\rangle
\]
(21)
so that

\[ k_{01} dQ_{2n+1}(A_t^g, F_t^g) = k_{01} \langle F_{t}^{n+1} \rangle = \int_{0}^{1} l_{t} \langle F_{t}^{n+1} \rangle = (n + 1) \int_{0}^{1} dt \langle AF_{t}^{n} \rangle = Q_{2n+1}(A, F) \]  

(22)

Defining the 2n-form

\[ \alpha_{2n} = k_{01} Q_{2n+1}(A_t^g, F_t^g) = k_{01} Q_{2n+1}(A_t + \mathcal{V}, F_t) \]  

(23)

we find that the Cartan homotopy formula (13) takes the form

\[ Q_{2n+1}(A^g, F^g) = Q_{2n+1}(A, F) + Q_{2n+1}(\mathcal{V}, 0) + d\alpha_{2n} \]  

(24)

The second term in the r.h.s. of (24) corresponds to the so called Wess-Zumino term and since it represents a winding number, it will be total derivative, unless \( G \) has non-trivial homotopy group \( \pi_3(G) \) and large gauge transformations are performed.

We now consider the terms \( Q_{2n+1}(\mathcal{V}, 0) \) and \( \alpha_{2n} \). From (16) with \( A = 0, F = 0 \), we find

\[ Q_{2n+1}(\mathcal{V}, 0) = (n + 1) \int_{0}^{1} dt \langle \mathcal{V}F_{t}^{n} \rangle \]  

(25)

where \( \tilde{F}_t = t(t - 1)\mathcal{V}^2 = -t(1 - t)\mathcal{V}^2 \), so that \( \tilde{F}_{t}^{n} = (-1)^n t^n (1 - t)^n \mathcal{V}^{2n} \) and therefore \[30\]

\[ Q_{2n+1}(\mathcal{V}, 0) = (-1)^n \frac{n! (n + 1)!}{(2n + 1)!} \langle \mathcal{V}^{2n+1} \rangle \]  

(26)

which corresponds to the generalization of the Wess-Zumino term.

2. **Cartan homotopy formula and transgression form**

Applying the Cartan homotopy formula (B7) to the Chern-Simons form (B13) we have \[30\]

\[ k_{01} dQ_{2n+1}(A_t, F_t) = Q_{2n+1}(A_1, F_1) - Q_{2n+1}(A_0, F_0) - d k_{01} Q_{2n+1}(A_t, F_t) \]  

(27)

where now \( A_t = A_0 + t (A_1 - A_0) \). From (27) we can see that

\[ Q_{2n+1}(A_1, A_0) = Q_{2n+1}(A_1, F_1) - Q_{2n+1}(A_0, F_0) - d [k_{01} Q_{2n+1}(A_t, F_t)] \]  

(28)

where

\[ Q_{2n+1}(A_1, A_0) = k_{01} \langle F_{t}^{n+1} \rangle = (n + 1) \int_{0}^{1} dt \langle \theta F_{t}^{n} \rangle \]  

(29)
is known as the transgression form.

Defining

\[ B_{2n} = k_{01} dQ_{2n+1}(A_t, F_t) \]  

we find that (28) takes the form

\[ Q_{2n+1}(A_1, A_0) = Q_{2n+1}(A_1, F_1) - Q_{2n+1}(A_0, F_0) - dB_{2n}(A_1, A_0) \]  

where

\[ B_{2n}(A_1, A_0) = n(n + 1) \int_0^1 dt \int_0^t ds \langle (A_1 - A_0) A_0 F_{st}^{n-1} \rangle \]

with \( F_{ts} = sF_t + s(s - 1)A_t^2, A_{ts} = sA_t = sA_0 + st(A_1 - A_0) \), \( A_t = A_0 + t(A_1 - A_0) \).

3. **Gauged Wess-Zumino-Witten Term**

If \( A_1 \) is related to \( A_0 \) by a gauge transformation and if \( A_1, A_0 \) are denoted by \( A^g, A \) respectively, we can write

\[ Q_{2n+1}(A^g, A) = Q_{2n+1}(A^g, F^g) - Q_{2n+1}(A, F) - dB_{2n}(A^g, A) \]  

From (24) and (33) we can see that the transgression form for two gauge equivalent connections correspond to the \( gWZW \) term

\[ Q_{2n+1}(A^g, A) = Q_{2n+1}(\mathcal{V}, 0) + d\alpha_{2n} - dB_{2n}(A^g, A). \]

In the particular case \( n = 1 \), i.e., in the \( (2 + 1) \)-dimensional case, we have that \( \alpha_2 \) takes the form

\[ \alpha_2 = \int_0^1 l_t Q_{2n+1}(A^g_t, F^g_t) = \int_0^1 l_t \left\langle (A_t + \mathcal{V}) \left( F_t - \frac{1}{3} A_t^2 \right) \right\rangle \]

\[ = - \left\langle \int_0^1 dt (tA + \mathcal{V}) \frac{\partial A_t}{\partial t} \right\rangle = - \langle \mathcal{V}A \rangle, \]

where we have used (B1) and (B2). On the other hand, from equation (32) we see that

\[ B_2(A^g, A) = \langle A^g A \rangle. \]

From (26), (35) and (36) we can see that the \( gWZW \) term in \( 2+1 \) dimensions can be written as [20]

\[ Q_{2n+1}(A^g, A) = -\frac{1}{3} \langle \mathcal{V}^3 \rangle - d(\langle \mathcal{V}A + A^g A \rangle). \]
III. TOPOLOGICAL GRAVITY AS A GAUGED WESS-ZUMINO-WITTEN TERM

In this section we show that even-dimensional topological gravity is a $gWZW$ term.

A. Topological Gravity in $(1 + 1)$-dimensions

From (3) we can see that the nonlinear and the linear connection, $A^Z$ and $A = e + \omega$ respectively, are related by a gauge transformation given by

$$A^Z = z^{-1} (d + A) z,$$

where $z = e^{-\phi^a P_a}$ and $A^Z = \frac{1}{2} W^{ab} J_{ab} + V^a P_a = V + W$. This means that the linear and nonlinear curvatures $F^Z$ and $F$ are related by

$$F^Z = z^{-1} F_z.$$

In the $(2 + 1)$-dimensional case, the only non-vanishing component is given by

$$\langle J_{ab} P_c \rangle = \epsilon_{abc}$$

so that the Wess-Zumino term (26) vanishes

$$Q_{2+1}(\mathcal{V}, 0) = -\frac{1}{3} \langle \mathcal{V}^3 \rangle = 0.$$

Hence, the $gWZW$ term (37) takes the form

$$Q_3(A^Z, A) = d \left( \alpha_2 - B_2(A^Z, A) \right) = -d \left( \langle \mathcal{V} A + A^z A \rangle \right),$$

and defines a Lagrangian in $(1 + 1)$-dimensions. Consider first the term $\langle \mathcal{V} A \rangle$

$$\langle \mathcal{V} A \rangle = \langle \mathcal{V} (e + \omega) \rangle = -\left\langle \left( e^a P_a + \frac{1}{2} \omega^{ab} J_{ab} \right) d\phi^c P_c \right\rangle$$

$$= -\frac{1}{2} \omega^{ab} d\phi^c \langle J_{ab} P_c \rangle$$

$$= -\frac{1}{2} \epsilon_{abc} \omega^{ab} d\phi^c$$

which can be rewritten as

$$\langle \mathcal{V} A \rangle = -\epsilon_{abc} d\omega^{ab} \phi^c + d \left( \frac{1}{2} \epsilon_{abc} \omega^{ab} \phi^c \right) + \frac{1}{2} \epsilon_{abc} \omega^{ab} d\phi^c$$

$$= -\epsilon_{abc} \left( d\omega^{ab} \phi^c + \frac{1}{2} \omega^{ab} \omega^{cd} d\phi^d \right) + \frac{1}{2} \epsilon_{abc} \omega^{ab} D\phi^c + d \left( \frac{1}{2} \epsilon_{abc} \omega^{ab} \phi^c \right)$$

$$= -\epsilon_{abc} R^{ab} \phi^c + \frac{1}{2} \epsilon_{abc} \omega^{ab} D\phi^c + d \left( \frac{1}{2} \epsilon_{abc} \omega^{ab} \phi^c \right)$$
where we have used the identity $\varepsilon_{abc}\omega^{ab}\omega^{cd}\phi^d = 2\varepsilon_{abc}\omega^{a}d\omega^{db}\phi^c$. On the other hand, the term $\langle A^z A \rangle$ is given by

$$\langle A^z A \rangle = \langle (A^z - A) A \rangle = \langle D\phi (e + \omega) \rangle = -\frac{1}{2}\varepsilon_{abc}\omega^{ab}D\phi^c.$$  \hspace{1cm} (42)

Substituting (42) and (41) in (40) we obtain

$$Q_3(A^Z, A) = d \left[ \varepsilon_{abc}R^{ab}\phi^c - d \left( \frac{1}{2}\varepsilon_{abc}\omega^{ab}\phi^c \right) \right] = d \left( \varepsilon_{abc}R^{ab}\phi^c \right)$$

which proves that the action for Topological gravity in (1+1)-dimensions, found in ref.[1], [6], is a gWZW term given by,

$$S_{gWZW}^{(2+1)}[A^Z, A] = k \int_M Q_3(A^Z, A) = k \int_{\partial M} \varepsilon_{abc}R^{ab}\phi^c$$  \hspace{1cm} (43)

**B. Topological Gravity in 2n−dimensions**

From (34) and (24) we can see that

$$Q_{2n+1}(A^Z, A) = Q_{2n+1}(V, 0) + d\alpha_{2n} - dB_{2n}(A^Z, A)$$  \hspace{1cm} (44)

Similarly to the three-dimensional case, the only non-vanishing component of the invariant tensor is

$$\langle J_{a_1a_2}...J_{a_{2n−1}a_{2n}}P_{a_{2n+1}} \rangle = \frac{2^n}{n+1}\varepsilon_{a_1a_2...a_{2n+1}}$$  \hspace{1cm} (45)

so that the Wess-Zumino term (26) vanishes

$$Q_{2n+1}(V, 0) = (-1)^n\frac{n!(n+1)!}{(2n+1)!}\left( dzz^{-1} \right)^{2n+1} = 0,$$

and the gWZW term defines a Lagrangian for a 2n−dimensional manifold given by

$$Q_{2n+1}(A^Z, A) = d \left( \alpha_{2n} - B_{2n}(A^Z, A) \right).$$  \hspace{1cm} (46)

From (24), $d\alpha_{2n}$ can be derived in a straightforward way

$$d\alpha_{2n} = Q_{2n+1}(A^Z, F^Z) - Q_{2n+1}(A, F)$$  \hspace{1cm} (47)

In fact, the term $Q_{2n+1}(A, F)$ corresponding to the Lagrangian for (2n + 1)-dimensional Chern-Simons gravity for the one-form connection $A$, is given by (see Appendix A.)

$$Q_{2n+1}(A, F) = \varepsilon_{a_1a_2...a_{2n+1}}R^{a_1a_2}...R^{a_{2n-1}a_{2n}}e^{a_{2n+1}}$$

$$- n(n+1)d \left\{ \int_0^1 dt t^{n} \langle R_t^{n-1}\omega e \rangle \right\}$$  \hspace{1cm} (48)
where $R_t = d\omega + t\omega^2$.

If $A^z$ and $A$ are given by $A = e^a P_a + \frac{1}{2} \omega^{ab} J_{ab} = e + \omega$ and $A^Z = V^a P_a + \frac{1}{2} W^{ab} J_{ab} = V + W$, where $V^a = e^a + D_\omega \phi^a$ and $W^{ab} = \omega^{ab}$, then $Q_{2n+1}(A^Z, F^Z)$ is given by

$$Q_{2n+1}(A^Z, F^Z) = \varepsilon_{a_1\ldots a_{2n+1}} R^{a_1 a_2 \ldots a_{2n-1} a_{2n}} V^{a_{2n+1}}$$

$$- n(n+1)d \left\{ \int_0^1 dt \, t^n \, \langle R_t^{n-1} \omega^V \rangle \right\}$$

$$+ \varepsilon_{a_1 a_2 \ldots a_{2n+1}} R^{a_1 a_2 a_3 \ldots a_{2n-1} a_{2n} d\phi^{a_{2n+1}}}$$

$$- n(n+1)d \left\{ \int_0^1 dt \, t^n \, \langle R_t^{n-1} \omega^{D\phi} \rangle \right\}.$$

Introducing (48) and (49) in (47) we have

$$d\alpha_{2n} = \varepsilon_{a_1\ldots a_{2n+1}} R^{a_1 a_2 \ldots a_{2n-1} a_{2n} D\phi^{a_{2n+1}}}$$

$$- n(n+1)d \left\{ \int_0^1 dt \, t^n \, \langle R_t^{n-1} \omega^{D\phi} \rangle \right\}.$$

On the other hand, from equation (32) we can see that $B_{2n}(A, A^Z)$ is given by

$$B_{2n}(A^Z, A) = n(n+1) \int_0^1 dt \, \int_0^t ds \, \langle (A^Z - A) A F_{st}^{n-1} \rangle.$$

Since $A = e + \omega$, $A^Z = A + D\phi$, $F_{st} = dA_{st} + A_{st} A_{st}$ and $A_{st} = tA + s(A^Z - A)$ we have

$$F_{st} = tR_t + tT_t + sD_t D\phi.$$

where $R_t = d\omega + t\omega^2$, $T_t = de + t[\omega, e]$ and $D_t D\phi = d(D\phi) + t[\omega, D\phi]$. Introducing (52) into (51) we find

$$B_{2n}(A^Z, A) = n(n+1) \int_0^1 dt \, \int_0^t ds \, \langle D\phi (e + \omega) (tR_t + tT_t + sD_t D\phi)^{n-1} \rangle.$$

Since the only nonvanishing component of the invariant tensor is (45), the only nonzero term in (53) is $\langle D\phi \omega (tR_t)^{n-1} \rangle$. So that

$$B_{2n}(A^Z, A) = -n(n+1) \int_0^1 dt \, t^n \, \langle R_t^{n-1} \omega^{D\phi} \rangle.$$
Introducing (50) and (54) into (44) we have

\[ Q_{2n+1}(A^Z, A) = \varepsilon_{a_1...a_{2n+1}} R^{a_1a_2}...R^{a_{2n-1}a_{2n}} D \phi^{a_{2n+1}} \]  

(55)

using the Bianchi identity \( DR^{ab} = 0 \) we can write

\[ Q_{2n+1}(A^Z, A) = d \left[ \varepsilon_{a_1...a_{2n+1}} R^{a_1a_2}...R^{a_{2n-1}a_{2n}} \phi^{a_{2n+1}} \right] \]  

(56)

which proves that the action for Topological gravity in 2\( n \)–dimensions, found in Ref. [1], [6], is a \( \mathbf{gWZW} \) term given by

\[ S^{(2n+1)}_{\mathbf{gWZW}} [A^Z, A] = k \int_{\mathcal{M}_{2n+1}} Q_{2n+1}(A^Z, A) = k \int_{\partial \mathcal{M}_{2n+1}} \varepsilon_{a_1...a_{2n+1}} R^{a_1a_2}...R^{a_{2n-1}a_{2n}} \phi^{a_{2n+1}} \]  

(57)

IV. COMMENTS

We have shown in this work that the action for topological gravity in 2\( n \)-dimensions, introduced in ref.[1][6], is a gauged Wess-Zumino-Witten term. This means that the 2\( n \)-dimensional topological gravity is described by the dynamics of the boundary of a (2\( n + 1 \)) Chern-Simons gravity.

The field \( \phi^a \), which is necessary to construct this type of topological gravity in even dimensions [1], is identified by the coset field associated with non-linear realizations of the Poincare group ISO(2\( n, 1 \)). This shows a clear geometric interpretation of this field originally introduced in an "ad-hoc" manner.

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Appendix A: Lagrangian for (2\( n + 1 \))-dimensional Chern-Simons gravity

A Chern–Simons form \( Q_{2n+1}(A) \) is a differential form defined for a connection \( A \), whose exterior derivative yields a Chern class. Although Chern classes are gauge invariant, the Chern–Simons forms are not; under gauge transformations they change by a closed form. A transgression form \( Q_{2n+1}(A, \tilde{A}) \) on the other hand, is an invariant differential form whose
exterior derivative is the difference of two Chern classes. It generalizes the Chern–Simons form and has the additional advantage that it is gauge invariant.

To obtain the Lagrangian for \((2n + 1)\)-dimensional Chern–Simons gravity we use the so called Triangle equation \([31]\)

\[
Q_{2n+1}(A, \bar{A}) = Q_{2n+1}(A, \bar{A}) - Q_{2n+1}(\bar{A}, \bar{A}) - dQ_{2n}(A, \bar{A}, \bar{A})
\]

with \(\bar{A} = 0\), and the method of separation in subspaces. Let us recall that the method of separation in subspaces consists of the following steps \([14], [12]\): The first step is to decompose the algebra into subspaces. In our case \(G = V_1 \oplus V_2\), where \(V_1\) corresponds to the Lorentz subalgebra generated by \(\{J_{ab}\}\) and \(V_2\) corresponds to the subspace spanned by \(\{P_a\}\). The second step is to write the connection as a sum of pieces valued in each subspace. This means \(A = a_1 + a_2\), where \(a_1 = \omega\) and \(a_2 = e\). The third step is to use the triangular equation with \(\tilde{A} = 0\), \(\bar{A} = \omega\) and \(A = \omega + e\).

Since \(Q_{2n+1}(A, 0) = Q_{2n+1}(A)\) and \(Q_{2n+1}(\bar{A}, 0) = Q_{2n+1}(\bar{A})\), we can write

\[
Q_{2n+1}(A) = Q_{2n+1}(A, \bar{A}) + Q_{2n+1}(\bar{A}, 0) + dQ_{2n}(A, \bar{A}, 0)
\]

\((A1)\)

\((a)\) To determine the first term we consider

\[
Q_{2n+1}(A, \bar{A}) = (n + 1) \int_0^1 dt \langle \theta F_t^n \rangle
\]

\((A2)\)

where \(\theta = A - \bar{A} = e\), \(A_t = \bar{A} + t\theta\) and \(F_t = dA_t + A_tA_t = R + tT\), where \(R = d\omega + \omega^2\) and \(T = de + [\omega, e]\). So that

\[
L_T^{(2n+1)}(A, \bar{A}) = (n + 1) \int_0^1 dt \langle e (R + tT)^n \rangle
\]

\((A3)\)

Using Newton's binomial theorem and taking into account that the only component non-zero, of the invariant tensor is given by

\[
\langle J_{a_1} a_2 \cdots J_{a_{2n-1}} a_{2n} P_{a_{2n+1}} \rangle = \frac{2^n}{n + 1} \varepsilon_{a_1 \cdots a_{2n+1}}
\]

\((A4)\)

it is straightforward to see that

\[
Q_{2n+1}(A, \bar{A}) = (n + 1) \int_0^1 dt \langle R^n e \rangle = \varepsilon_{a_1 \cdots a_{2n+1}} R^{a_1 a_2} \cdots R^{a_{2n-1} a_{2n}} e^{a_{2n+1}}
\]

\((A5)\)

\((b)\) To determine the second term we consider
\[ Q_{2n+1}(\tilde{A}, 0) = (n + 1) \int_0^1 dt \langle \tilde{A} F^n_t \rangle \]

where now \( A_t = t \tilde{A} = t \omega \) and \( F_t = dA_t + A_tA_t = t (d\omega + t\omega^2) = t R_t \), with \( R_t = \frac{1}{2} R_t^{ab} J_{ab} \). So that

\[ Q_{2n+1}(\tilde{A}, 0) = (n + 1) \int_0^1 dt \langle \omega R^n_t \rangle = 0 \] (A7)

because the only nonzero invariant tensor is given by \((A4)\).

(c) To determine the third term we consider

\[ Q_{2n}(A, \tilde{A}, 0) = n(n + 1) \int_0^1 dt \int_0^t ds \langle (A - \tilde{A}) \tilde{A} F^{n-1}_{st} \rangle = n(n + 1) \int_0^1 dt \int_0^t ds \langle e\omega F^{n-1}_{st} \rangle \]

where \( A_{st} = t \omega + se_1, F_{st} = dA_{st} + A_{st}A_{st} = t R_t + s T_t \) with \( R_t = d\omega + t\omega^2 \) and \( T_t = de + t[\omega, e] \). So that

\[ Q_{2n}(A, \tilde{A}, 0) = n(n + 1) \int_0^1 dt \int_0^t ds \langle e\omega (t R_t + s T_t)^{n-1} \rangle. \] (A9)

Using Newton’s binomial theorem and taking into account that the only component nonzero, of the invariant tensor is given by \((A4)\), we find

\[ Q_{2n+1}(A) = \epsilon_{a_1...a_{2n+1}} R^{a_1 a_2}...R^{a_{2n-1} a_{2n}} e^{a_{2n+1}} - n(n + 1)d \left\{ \int_0^1 dt \langle R_t^{n-1} \omega e \rangle \right\} \]

If \( n = 1 \) we have

\[ Q_3(\tilde{A}) = \epsilon_{a_1 a_2 a_3} R^{a_1 a_2} e^{a_3} - \frac{1}{2} d [\epsilon_{a_1 a_2 a_3} \omega^{a_1 a_2} e^{a_3}] \].

**Appendix B: Cartan homotopy formula**

We start with the homotopic connection \( A_t = A_0 + t(A_1 - A_0), \quad t \in [0, 1] \) and its curvature \( F_t = dA_t + A_t^2 \). The homotopy derivation operator \( l_t \) is defined by

\[ l_t A_t = 0 \] (B1)

\[ l_t F_t = d_t A_t = dt \frac{\partial}{\partial t} A_t = dt (A_1 - A_0) \] (B2)

acting on polynomials in \( A_t \) y \( F_t \). It is direct to see that

\[ (l_t d + dl_t) A_t = d_t A_t = dt \frac{\partial}{\partial t} A_t \] (B3)
\[(l_t d + dl_t) F_t = d_t F_t = dt \frac{\partial}{\partial t} F_t \]  
(B4)

which implies for any polynomial \( S \) in \( A_t \) and \( F_t \)
\[(l_t d + dl_t) S(A_t, F_t) = d_t S(A_t, F_t) = dt \frac{\partial}{\partial t} S(A_t, F_t). \]  
(B5)

Defining the homotopy operator by
\[ k_{01} = \int_0^1 l_t \]  
(B6)
i.e., as the \( t \)-integrated version of the derivation \( l_t \), we find that integrating equation (B5) with respect to \( t \) we arrive at the Cartan homotopy formula [30]
\[ S(A_1, F_1) - S(A_0, F_0) = (k_{01} d + dk_{01}) S(A_t, F_t). \]  
(B7)

In the particular case where the arbitrary polynomial \( S(A_t, F_t) \) is an invariant polynomial \( P_{n+1}(F_t) = \langle F_t^{n+1} \rangle \) we have \( dP_{n+1}(F_t) = 0 \) and the Cartan homotopy formula takes the form
\[ P_{n+1}(F_1) - P_{n+1}(F_0) = dk_{01} P_{n+1}(F_t). \]  
(B8)

Since
\[ k_{01} P_{n+1}(F_t) = \int_0^1 l_t P_{n+1}(F_t) = (n + 1) \int_0^1 dt P_{n+1}((A_1 - A_0), F_t^n) = Q_{2n+1}(A_1, A_0) \]  
(B9)

we find that the Cartan homotopy formula supplies the Chern–Weil theorem:
\[ P_{n+1}(F_1) - P_{n+1}(F_0) = dQ_{2n+1}(A_1, A_0). \]  
(B10)

Finally, choosing the case \( A_0 = 0, A_1 = A \) the homotopy operator \( k_{01} \) is usually denoted by \( k \), i.e.,
\[ k = \int_0^1 l_t \]  
(B11)

Applying Cartan homotopy formula to this special case we find
\[ P_{n+1}(F) = dk P_{n+1}(F_t) = dQ_{2n+1}(A, F) \]  
(B12)

and equation (B9) provides the Chern–Simons form
\[ Q_{2n+1}(A, F) = k P_{n+1}(F_t) = (n + 1) \int_0^1 dt P((A, F_t^n) = (n + 1) \int_0^1 dt \langle AF_t^n \rangle \]  
(B13)
with $A_t = tA$ and $F_t = dA_t + A_t^2 = tF + (t^2 - t)A^2$.

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