Weyl-Gauging and Conformal Invariance

A. Iorio, L. O’Raifeartaigh, I. Sachs and C. Wiesendanger

*Dublin Institute for Advanced Studies*
*School of Theoretical Physics*
*10 Burlington Road, Dublin 4, Ireland*

**Abstract**

Scale-invariant actions in arbitrary dimensions are investigated in curved space to clarify the relation between scale-, Weyl- and conformal invariance on the classical level. The global Weyl-group is gauged. Then the class of actions is determined for which Weyl-gauging may be replaced by a suitable coupling to the curvature (Ricci gauging). It is shown that this class is exactly the class of actions which are conformally invariant in flat space. The procedure yields a simple algebraic criterion for conformal invariance and produces the improved energy-momentum tensor in conformally invariant theories in a systematic way. It also provides a simple and fundamental connection between Weyl-anomalies and central extensions in two dimensions. In particular, the subset of scale-invariant Lagrangians for fields of arbitrary spin, in any dimension, which are conformally invariant is given. An example of a quadratic action for which scale-invariance does not imply conformal invariance is constructed.
1 Introduction

It has been known for a long time that the symmetric (Belinfante) energy-momentum tensor may be obtained by writing the flat-space action $A$ in a diffeomorphic-invariant manner $A$ and by computing its derivative with respect to the metric tensor $g$:

$$T_{\mu\nu} = \frac{4\pi}{\sqrt{g}} \frac{\delta A}{\delta g_{\mu\nu}}. \quad (1)$$

For scale-invariant theories the trace $T_{\mu}^{\mu}$ of the symmetric energy-momentum tensor should be zero in the flat-space limit but in practice this is not always the case and the symmetric $T_{\mu\nu}$ has to be replaced by a so-called 'improved' energy-momentum tensor $\tilde{T}_{\mu\nu}$ in a rather ad hoc manner but it has been observed that for some scale-invariant scalar theories the improved energy-momentum tensor may also be obtained using (1) provided that a suitable extra coupling of the fields to the Ricci scalar $R$ is introduced $\tilde{T}_{\mu\nu}$. Furthermore, in certain two-dimensional conformal field theories this procedure allows the Virasoro centre to be computed in a simple manner using the identity $\langle T_{\mu}^{\mu} \rangle = -\frac{c}{12}R$. (2)

To our knowledge it has however not been established so far why, and under what conditions this procedure works, and what form the coupling to $R_{\nu\rho\sigma}$ should take for fields of arbitrary spin. The purpose of the present paper is to fill this gap.

Our strategy is as follows: First we convert the rigid scale-invariance to rigid Weyl-invariance and then promote the latter to local Weyl-invariance by gauging in the standard manner. This permits the question to be formulated more precisely as to how and when the Weyl gauge-potential $W_\mu$ may be replaced by curvature-terms. We find that the necessary and sufficient condition for such a replacement is that the theory should be conformally invariant and that the general coupling is to the Ricci tensor $R_{\mu\nu}$ rather than the Ricci scalar $R$.

Hence, we obtain a new criterion for conformal invariance. In the infinitesimal limit it reduces to the criterion for conformal invariance which was given in [1, 2], namely that the so-called virial current $j_\mu$ be the divergence of a tensor, $j_\mu = \partial_\nu J_\mu^{\nu}$. However, we can go further considering finite local Weyl-transformations. This allows us to identify the tensor $J_\mu^{\nu}$ and thus to reduce the condition for conformal invariance to an algebraic condition which permits a systematic analysis. We then obtain the constraints for conformal invariance and the coupling to the Ricci-tensor for Lagrangians for arbitrary spin in any dimensions. For spin zero we also exhibit an action which is scale-invariant and quadratic in the derivatives of the fields but is not conformally invariant.

The paper is organized as follows. In section 2 we illustrate the coupling to the Ricci scalar by means of the simplest non-trivial conformally invariant theory, namely the Liouville theory, in both the classical and quantum cases. In section 3 we establish the equivalence of rigid scale-invariance and rigid Weyl-invariance in the diffeomorphic context. The latter is then gauged in the standard manner by introducing a Weyl-potential $W_\mu$. In sections 4 and 5 we analyse under what conditions $W_\mu$ may be replaced by a particular combination of the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar $R$ and show that these conditions are equivalent to conformal invariance in flat space. The
relation between our results and those of [4, 5] is also discussed. In section 6 we discuss in detail the role of central terms in the conformal variation of the Lagrangian and relate them to gravitational anomalies. The extension of our approach to quantum systems is also explained in that section. Finally, in section 7 we find the set of scale-invariant Lagrangians which are conformally invariant for fields of arbitrary spin and construct the abovementioned counter example. Section 8 contains the conclusions. Our conventions are those of Birrell and Davies [6].

2 A Two-dimensional Illustration

We first illustrate the use of non-minimal coupling to the curvature (Ricci gauging) by means of the two-dimensional Liouville theory with action

$$A = \frac{1}{4\pi} \int \sqrt{g} \left( \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - m^2 e^\phi + R \phi \right),$$

where we have discarded a purely geometrical Polyakov term which means that the action (3) will be Weyl-invariant only up to field-independent terms. Varying (3) as in (1) one then obtains

$$T_{\mu\nu} = \frac{1}{2} \partial_\nu \phi \partial_\mu \phi - \frac{1}{2} g_{\mu\nu} \left( \frac{1}{2} \partial^\rho \phi \partial_\rho \phi - m^2 e^\phi \right) + \left( g_{\mu\nu} \nabla^2 - \nabla_\mu \nabla_\nu \right) \phi.$$  (4)

Note that $\left( g_{\mu\nu} \nabla^2 - \nabla_\mu \nabla_\nu \right) \phi$ is the well-known improvement term [2]. It is clear that it originates from the $R \phi$-term in the action. This term survives in the flat space limit even though $R \phi$ itself vanishes. Hence, the processes of varying with respect to $g^{\mu\nu}$ and taking the flat limit do not commute. The trace of the energy-momentum tensor (4) is then

$$T^\mu_\mu = m^2 e^\phi + \nabla^2 \phi,$$  (5)

which reduces to $R$ on the mass-shell and vanishes in the flat limit.

In the quantum case the situation is a little more subtle. Because of renormalization the classical coefficient of the coupling-term $R \phi$ must be modified. That is, we write $\alpha R \phi$ with $\alpha$ a constant to be fixed according to the trace condition (2). For arbitrary $\alpha$ the variation of $Z[g] = \int D\phi e^{\frac{i}{\hbar}A}$ then leads to the exact result

$$\langle T^\mu_\mu \rangle = \frac{\sqrt{g}}{Z[g]} \delta Z[g] = -\frac{1}{12} \left( \hbar R + \alpha^2 R + 12 (\alpha - 1 - \frac{\hbar}{2}) \nabla^2 \langle \phi \rangle \right),$$  (6)

where the first term is due to the scale-anomaly in the measure. Since scale-invariance requires that $\langle T^\mu_\mu \rangle$ vanish in the flat limit the constant $\alpha$ must be $\alpha = 1 + \frac{\hbar}{2}$. Then the right-hand side of (6) reduces to $cR$ where $c = \hbar + 12 (1 + \frac{\hbar}{2})^2$, which is just the expression for the Virasoro centre obtained earlier by canonical methods [7]. We note in passing that variations of the model described above have been discussed extensively in connection with dilaton gravity and string theory [8, 9].
3 Rigid Scale- and Rigid Weyl-Invariance

In this section we convert rigid scale-invariance into rigid Weyl-invariance which we then promote to a local symmetry by gauging in the standard manner. For this we first consider the diffeomorphic version of Poincaré-invariant flat-space theories. For simplicity we shall consider actions \( A \) for which the Lagrangian contains only the fields, denoted generically by \( \phi \), and their first derivatives

\[
A = \int d^n x L(\phi, \partial_a \phi). \tag{7}
\]

We make \( A \) diffeomorphic invariant by letting

\[
A \rightarrow A = \int d^n x \sqrt{g} L(\phi, e_a^\mu \nabla_\mu \phi), \tag{8}
\]

where \( \nabla_\mu \) is the diffeomorphism-covariant derivative, \( e_a^\mu \) is the Vielbein defined by

\[
e_a^\mu e^a_\nu = g_{\mu \nu} \quad \text{and} \quad g_{\mu \nu} \quad \text{is the metric. More explicitly, for a field } \phi^m \text{ with spin indices } m \text{ the covariant derivative may be written as}
\]

\[
(\nabla_\mu \phi)^m = \partial_\mu \phi^m + S_{\mu,n}^m \phi^n, \tag{9}
\]

where the spin connection \( S_{\mu}^m \) is given by

\[
S_{\mu,n}^m = -S_{\mu}^a \left( \Sigma_{ab} \right)_n^m, \quad S_{\mu}^a = \epsilon^a_\tau (g^\lambda \tau \partial_\mu + \Gamma^\lambda_\mu_\tau) e^\tau a \tag{10}
\]

and the \( \Sigma_{ab} \) are the generators of the Lorentz transformations.

Rigid Scale-Invariance vs. Rigid Weyl-Invariance:
By a rigid scale-transformation in flat space we mean a transformation of the Cartesian coordinates and of the fields

\[
x^a \rightarrow e^\omega x^a \quad \phi \rightarrow e^{d_\phi} \phi, \tag{11}
\]

where \( \omega \) is a real constant and \( d_\phi \) is the dimension of the field. In practice, the dimension of the fields are canonical, ie. determined by the kinetic part of the Lagrangian. Now, because of diffeomorphism invariance the transformations (11) are equivalent to the transformations

\[
e_\mu^m \rightarrow e^{e^\omega_\mu} e_\mu^m \quad \text{and} \quad \phi \rightarrow e^{d_\phi} \phi \tag{12}
\]

with no change of coordinates. In other words, the scaling can be transferred to the Vielbein and thence to the metric and to \( \sqrt{g} \), which scales as \( e^{n_\omega} \). The transformations (12) are called rigid Weyl-transformations and thus any action which is invariant with respect to rigid scale-transformations in the flat limit is rigid Weyl-invariant in curvilinear coordinates. Rigid Weyl-invariance is therefore the natural way of generalizing rigid scale-invariance to curvilinear coordinates.

Weyl-gauging:
To see under what conditions a theory can be made locally Weyl-invariant by a non-minimal coupling to curvature-terms we first consider standard Weyl-gauging. Because rigid Weyl-invariance is an inner symmetry it can always be converted to a local symmetry by gauging in the usual manner. The purpose of Weyl-gauging is to compensate
for the inhomogeneous terms that arise in the derivatives of the fields for local Weyl-transformations, \( \phi(x) \rightarrow e^{d_\omega(x)} \phi(x) \).

Let us first consider scalar fields. In that case we have

\[
\partial_\mu (e^{d_\omega(x)} \phi(x)) = e^{d_\omega(x)}(\partial_\mu + d_\phi \partial_\mu \omega(x)) \phi(x)
\]

and hence Weyl-gauging consists of letting

\[
\partial_\mu \phi \rightarrow D_\mu \phi, \quad \text{where} \quad D_\mu = \partial_\mu + d_\phi W_\mu \quad \text{and} \quad W_\mu \rightarrow W_\mu - \partial_\mu \omega.
\]

The situation is quite analogous to that in electrodynamics and indeed historically Weyl-gauging preceded and motivated electrodynamic gauging. However, it differs from electrodynamics in that there is no imaginary factor \( i \). An important consequence of this is that the current associated with \( W_\mu \) is not of the electromagnetic form \( i(\phi^* \partial_\mu \phi - \phi \partial_\mu \phi^*) \) but \( (\phi^* \partial_\mu \phi + \phi \partial_\mu \phi^*) = \partial_\mu (\phi^* \phi) \). Thus, it is not zero for real fields and is a total derivative.

The situation for higher spin fields is a little more complicated because the ordinary derivative \( \partial_\mu \) is replaced by the diffeomorphic covariant derivative \( \nabla_\mu \) and the latter varies with respect to local Weyl-transformations. To see how it varies we recall that in the Vielbein formalism it takes the form \( \nabla_\mu = \partial_\mu + S_\mu \) where \( S_\mu \) is given in (10).

Since the Vielbein \( e^b_\tau \) is a covariant vector it has scale-dimension \( d_e = 1 \) and hence the Weyl-variation of \( \nabla_\mu \) is

\[
\Delta \nabla_\mu = -2\Sigma^{\nu \mu} \partial_\nu \omega.
\]

Taking the variations of \( \nabla_\mu \) and of the field into account we see that for general spin fields we must Weyl-gauge according to

\[
\nabla_\mu \rightarrow \nabla_\mu + \Lambda^\nu_\mu W_\nu,
\]

where

\[
\Lambda^\nu_\mu = d_\phi g^\nu_\mu + 2\Sigma^{\nu \mu} \quad \text{and} \quad W_\nu \rightarrow W_\nu - \partial_\nu \omega.
\]

The operator \( \Lambda_{\nu \mu} \) is the same as that entering in the virial current \([4]\). Its spin component \( \Sigma_{\mu \nu} \) is due to the spin connection. In the next section we will see under what conditions the Weyl-gauging is equivalent to a non-minimal coupling to curvature.

4 Ricci Gauging

Since the Weyl-potential \( W_\mu \) is a vector and the Ricci curvature is a second rank tensor it is clear that we can only find a relation between them if we can construct a second rank tensor from \( W_\mu \). There are two local second rank tensors that we can construct, namely \( W_\mu W_\nu \) and \( \nabla_\mu W_\nu \). Let us now see how they transform under finite Weyl scaling. From \([17]\) we have

\[
\Delta(W_\mu W_\nu) = \omega_\mu \omega_\nu - (W_\mu \omega_\nu + W_\nu \omega_\mu), \quad \text{where} \quad \omega_\mu = \partial_\mu \omega
\]

has been introduced as not to clutter the notation. To compute the finite Weyl-variation of \( \Delta(\nabla_\mu W_\nu) \) we need the variation of \( \Gamma^r_{\mu \nu} \) \([6]\)

\[
\Delta \Gamma^r_{\mu \nu} = g^r_\sigma (g_{\mu \sigma} \omega_\nu + g_{\nu \sigma} \omega_\mu - g_{\mu \nu} \omega_\sigma),
\]
which leads to
\[ \Delta(\nabla_\mu W_\nu) = \nabla_\mu \omega_\nu - g_{\mu\nu} \omega_\mu \omega_\nu + 2 \omega_\mu \omega_\nu W \cdot \omega - (W_\mu \omega_\nu + W_\nu \omega_\mu). \] (20)

Each of the above variations (18) and (20) depend on \( W_\mu \). However, there is a combination whose variation is independent of \( W_\mu \). For this we first note that contracting (18) we obtain
\[ \Delta(g_{\mu\nu} W \cdot W) = g_{\mu\nu}(\omega \cdot \omega - 2W \cdot \omega). \] (21)

It is then easy to see that the variation of the tensor
\[ \Omega_{\mu\nu}[W] = \nabla_\mu W_\nu - W_\mu W_\nu + \frac{1}{2} g_{\mu\nu} W \cdot W \] (22)
is independent of \( W_\mu \) and symmetric. More precisely,
\[ \Delta \Omega_{\mu\nu}[W] = - (\nabla_\mu \omega_\nu - \omega_\mu \omega_\nu + \frac{1}{2} g_{\mu\nu} \omega \cdot \omega) = -\Omega_{\mu\nu}[\omega]. \] (23)

Note the appearance of the same operator \( \Omega_{\mu\nu} \) on both sides of the equation. For \( n=2 \) we have
\[ \nabla^\mu W_\mu = -\nabla^2 \omega = -\Omega^\mu_\mu[\omega]. \] (24)

Because \( \Omega_{\mu\nu}[\omega] \) depends only on the scale parameter \( \omega \) it must have a geometrical significance. To see what geometrical object it corresponds to we recall the scale variation of the Ricci tensor [3]
\[ \Delta R_{\mu\nu} = R_{\mu\nu}[e^{2\omega} g_{\mu\nu}] - R_{\mu\nu}[g_{\mu\nu}] = g_{\mu\nu} \nabla^2 \omega + (n - 2) \left( \nabla_\mu \omega_\nu - \omega_\mu \omega_\nu + g_{\mu\nu} \omega \cdot \omega \right). \] (25)

From (24) we see that the tensor
\[ S_{\mu\nu} = R_{\mu\nu} - \frac{1}{2(n-1)} g_{\mu\nu} R \] (26)
transforms under Weyl-scalings in the same way as \( \Omega_{\mu\nu} \), ie.
\[ \Delta S_{\mu\nu} = (n - 2) \Omega_{\mu\nu}[\omega]. \] (27)

This shows for \( n \neq 2 \) that \( \Omega_{\mu\nu}[\omega] \) is proportional to the variation of \( S_{\mu\nu} \). For \( n=2 \) we have from (23)
\[ R[e^{2\omega} g_{\mu\nu}] - e^{-2\omega} R[g_{\mu\nu}] = 2\nabla^2 \omega = -2\Omega^\mu_\mu[\omega]. \] (28)

We now can see when and why Weyl-gauging can be replaced by a non-minimal coupling to the curvature: Weyl-gauging was introduced to compensate the inhomogeneous part of the Weyl-variation of the kinetic part of the action. But since, according to (20) and (27), the Weyl-variations of \( \Omega_{\mu\nu}[W] \) and \( \Omega^\mu_\mu[W] \) are proportional to the variations of \( S_{\mu\nu} \) and \( R \) we see that whenever \( W_\mu \) appears in the action only in the combination \( \Omega_{\mu\nu}[W] \) or \( \Omega^\mu_\mu[W] \) then it can be replaced by \( S_{\mu\nu} \) or \( R \). From now on we refer to the compensation of the inhomogeneous part of the kinetic terms by \( S_{\mu\nu} \) as Ricci gauging and we say that an action for which this is possible is Ricci gaugeable.
What remains to be understood is under what conditions an action can be converted to a form in which \( W_\mu \) appears only in the combination \( \Omega_{\mu\nu} \).
5 The Condition for Ricci Gauging

It turns out that the necessary properties of the theory to be Ricci gaugeable are closely connected to conformal invariance. We therefore elaborate first on the connection between local Weyl-scalings and the conformal transformations, before obtaining the criterion for Ricci gauging in the second part of this section.

**Connection with Conformal Invariance**:
Within the diffeomorphic context the equation defining the conformal transformations $x^\mu \to y^\mu(x)$ is

$\frac{\partial x^\mu}{\partial y^\alpha} \frac{\partial x^\nu}{\partial y^\beta} g_{\mu\nu}(x) = \hat{g}_{\alpha\beta}(y) \quad \text{and} \quad \hat{g}_{\alpha\beta}(x) = e^{\hat{\omega}(x)} g_{\alpha\beta}(x). \quad (29)$

The $\hat{\omega}$ in (29) form a subgroup of the group of local Weyl-transformations that is induced by conformal transformations. We will call it the **conformal Weyl-group**. We can characterize $\hat{\omega}$ using that we have from (29)

$\frac{\partial x^\mu}{\partial y^\alpha} \frac{\partial x^\nu}{\partial y^\beta} R_{\mu\nu}(x) = \hat{R}_{\alpha\beta}(y) = R_{\alpha\beta} \left[ e^{2\hat{\omega}} g_{\mu\nu} \right](y), \quad (30)$

which leads, in virtue of (25) and (27), to the differential equation

$(n-2)\Omega_{\alpha\beta}[\hat{\omega}] = \hat{S}_{\mu\nu} - S_{\mu\nu}, \quad (31)$

for $n \neq 2$ and

$\nabla^2 \hat{\omega} = 0, \quad (32)$

for $n = 2$. Next, we make a few comments on the solutions. The two-dimensional condition, which is a consequence of the conformal Killing equation, is solved by all harmonic functions. In higher dimensions the existence of global solutions of (31) is a non-trivial problem in general [10]. In the flat-space limit $S_{\alpha\beta}$ vanishes and the condition (31) reduces to

$\nabla_\mu \hat{\omega}_\nu - \hat{\omega}_\mu \hat{\omega}_\nu + \frac{1}{2} g_{\mu\nu} \hat{\omega} \cdot \hat{\omega} = 0. \quad (33)$

In Cartesian coordinates the general solution of (33) is

$\hat{\omega}(x) = \log(1 - 2c \cdot x + c^2 x^2), \quad n > 2, \quad (34)$

where $c$ is an arbitrary constant vector.

**The Criterion for Ricci Gauging**:
After this digression we now obtain the criterion for Ricci gauging. Suppose that the Weyl-gauged action $\mathcal{A}$ permits Ricci gauging. We then have for the gauged action (with obvious modifications for $n=2$)

$\mathcal{A}(\phi, S_{\mu\nu}) = \mathcal{A}(\phi^\omega, S_{\mu\nu} + \Omega_{\mu\nu}[\omega]), \quad (35)$

where $\phi^\omega$ denotes the transformed fields. From this it follows in particular that if $\omega$ fulfills $\Omega_{\mu\nu}[\omega] = 0$ then the action is invariant even without gauging. But $\Omega_{\mu\nu}[\omega] = 0$ is
just the condition satisfied by the conformal Weyl-group \((31)\) in flat space. Thus we have the result: A necessary condition for a Weyl-gauged action to admit Ricci gauging is that it be conformally invariant in the flat-space limit.

We now show that for actions which contain only first derivatives of the conformally variant fields this condition is also sufficient. Suppose an ungauged action \(A_0\) is conformal invariant in flat space. For infinitesimal conformal transformations we then have in Cartesian coordinates

\[
\Delta A_0 = \int \hat{\omega}_\mu j^\mu = c_\mu \int j^\mu = 0, \tag{36}
\]

where \(j^\mu\) is the virial current

\[
j^\mu = \pi_\nu A^{\mu\nu} \phi; \quad \pi^\mu = \frac{\delta A}{\delta (\partial_\mu \phi)} \tag{37}
\]

introduced in section 3 and \(\hat{\omega} = c_\mu x^\mu\) solves the linearized equation \((33)\). Because invariance is required off-shell, ie. for arbitrary field configurations, \((36)\) implies that \(j^\mu\) is a divergence

\[
j^\mu = \partial_\nu J^{\mu\nu}, \tag{38}
\]

where \(J^{\mu\nu}\) is a tensor local in the fields, which we call the virial tensor.

For \(n = 2\) the condition is stronger because the conformal Weyl-group contains all harmonic functions. Using this larger group we have shown in the appendix that for \(n = 2\)

\[
J^{\mu\nu} = g^{\mu\nu} J \quad \text{and hence} \quad j^\mu = \partial^\mu J, \tag{39}
\]

where \(J\) is a scalar. In other words for \(n = 2\) the virial current must not only be a divergence but a gradient.

So far our results match those of Refs. \([4, 5]\). However, we can go farther because eqns. \((38)\) and \((39)\) are off-shell identities. As the actions considered contain only first derivatives of the conformally variant fields the same is true for \(j^\mu\). Therefore if \(J^{\mu\nu}\) contained any derivatives of these fields, then \((38)\) and \((39)\) would imply a relation between the second derivatives of the fields and lower ones, which is impossible for arbitrary field configurations. Hence, we conclude that \(J^{\mu\nu}\) depends only on the conformally variant fields themselves and not on their derivatives. Now we reverse the argument: since \((38)\) and \((39)\) involve only one derivative, they imply that \(j^\mu\) is at most linear in the first derivatives of the conformally variant fields. Thus, conformal invariance is possible only for actions which are at most quadratic in the derivatives of the conformally variant fields. Furthermore, in the case where the action is linear in the first derivatives the same argument implies that the virial current be zero. This situation occurs for the Dirac action. Note that the dependence on the conformally invariant fields is not restricted by this argument. Examples where the Lagrangian is of higher order in the derivatives of the invariant fields are given by the Skyrme- and WZW-models.

Because conformal invariance requires the action to be at most quadratic in the first derivatives of the fields its variation under finite conformal transformations is then simply

\[
\Delta A_0 = \int \left( \hat{\omega}_\mu j^\mu + \hat{\omega}_\mu \hat{\omega}_\nu T^{\mu\nu} \right), \tag{40}
\]
where $T^{\mu\nu}$ does not contain any derivatives of the fields. Using furthermore that $j^{\mu}$ is a total divergence we have after a partial integration

$$\Delta A_0 = \int \left( -J^{\mu\nu} \partial_\mu \hat{\omega}_\nu + \hat{\omega}_\mu \hat{\omega}_\nu T^{\mu\nu} \right).$$

For $n \neq 2$ we may use (33) to recast (11) in the form

$$\Delta A_0 = \int \hat{\omega}_\mu \hat{\omega}_\nu \left( T^{\mu\nu} - J^{\mu\nu} + \frac{1}{2} g^{\mu\nu} J \right),$$

where $J$ denotes the trace of $J^{\mu\nu}$. Although $\hat{\omega}$ is not a completely arbitrary function, it depends on an arbitrary four-vector $c_\mu$. This allows us to conclude from the vanishing of the variation for all field configurations that the integrand in (12) must be zero (see appendix). Thus,

$$T^{\mu\nu} = J^{\mu\nu} - \frac{1}{n-2} g^{\mu\nu} T.$$

Accordingly, if we Weyl-gauge the action we obtain

$$\mathcal{A} = \mathcal{A}_0 + \int \sqrt{g} \left( W_\mu j^{\mu} + W_\mu W_\nu T^{\mu\nu} \right)$$

and hence by using the same manipulations as were used above we find that

$$\mathcal{A} = \mathcal{A}_0 + \int \sqrt{g} \left( -J^{\mu\nu} \nabla_\mu W_\nu + W_\mu W_\nu T^{\mu\nu} \right),$$

which, using (13), can be written as

$$\mathcal{A} = \mathcal{A}_0 + \int \sqrt{g} J^{\mu\nu} \Omega_{\mu\nu}[W].$$

For $n = 2$ the expression in (17) then reduces to

$$\mathcal{A} = \mathcal{A}_0 - \int \sqrt{g} J \nabla \cdot W = \mathcal{A}_0 - \int \sqrt{g} J \Omega_{\mu\nu}[W].$$

Thus, for theories which are conformally invariant in the flat-space limit the Weyl-potential only appears in the form $\Omega_{\mu\nu}[W]$ which is just the condition for Ricci gauging. Therefore we have proved the following:

A necessary and sufficient condition for a scale-invariant action $\mathcal{A}$ to allow for Ricci gauging is that the flat-space limit of the ungauged action $\mathcal{A}_0$ is conformally invariant. Furthermore the Ricci-gauging is achieved by (17) respectively (18).

The improved energy-momentum tensor is then derived from (17) and (18) respectively in the usual way by a metric variation.
6 Gravitational Anomalies in Two Dimensions

We have seen in the last section that for $n = 2$ the tensor $T^{\mu \nu}$ is of the form $T^{\mu \nu} = Kg^{\mu \nu}$, with $K$ fixed by the group property of the conformal Weyl-transformat ions. We now have a closer look at this particular contribution to the conformal variation of the action. Since the linear term vanishes separately for $n = 2$ we have

$$\Delta_\omega A = K \int \sqrt{g} \hat{\omega}_\mu \hat{\omega}^\mu, \quad (49)$$

with positive definite integrand, so that $\Delta_\omega A$ is non-zero unless $\hat{\omega}$ is constant. We then conclude that, for $K \neq 0$, the action is conformally invariant only up to central (ie. field independent) terms. On the other hand, for arbitrary Weyl-transformations $g \rightarrow e^{2\omega} g$, falling off at infinity, the variation becomes after a partial integration

$$\Delta_\omega A = K \int \sqrt{g} \omega R - K \int \omega \nabla^2 \omega. \quad (50)$$

Hence, if we Ricci-gauge the action according to (48), then the resulting action is still Weyl-invariant up to a field independent contribution (50). The reader can easily convince himself that taking a derivative of (50) with respect to the metric leads to the trace formula (2) which in turn determines the central charge of the Virasoro algebra [3]. Now we see the connection between the restricted conformal invariance (49), the centre of the Virasoro algebra and the breaking (50) of Weyl-invariance of partially Ricci-gauged conformal theories. The Liouville theory described in section 2 nicely illustrates this connection. The ungauged model, obtained from (3) by removing the coupling to the curvature scalar, is totally invariant under rigid scale transformations, but is invariant under arbitrary conformal Weyl-transformations only up to the central term $\frac{1}{2} \int \hat{\omega}_\mu \hat{\omega}^\mu$. Discarding this term in the Weyl-gauging then leads to the partially Ricci-gauged action (3) and the corresponding improved energy-momentum tensor satisfies the trace formula (2). So much for the classical theory. At the quantum level there is an additional central term due to the scale anomaly, which can be avoided only at the expense of reducing the diffeomorphism invariance [11]. Although our approach so far was purely classical it naturally extends to the quantum theory replacing the currents by their expectation values. In particular, the operator analog of (40)

$$\Delta W = \int \hat{\omega}_\mu \langle j^\mu \rangle + \hat{\omega}_\mu \hat{\omega}_\nu \langle T^{\mu \nu} \rangle, \quad (51)$$

is obtained by varying the Schwinger functional $W$. It takes into account the scale anomaly of the measure which contributes a central term to (51).

7 Conformal Condition for Fields of Arbitrary Spin

Scalar Fields

The standard rigid scale-invariant action for one scalar field in $n \geq 3$ dimensions takes the form

$$A = \int \left( \frac{1}{2} \partial^\rho \phi \partial_\rho \phi - f \phi^{\frac{2n}{n-2}} \right), \quad (52)$$
where the scale-dimension of the field is \( d_\phi = (2 - n)/2 \) and \( f \) is a coupling constant. It is easy to see that the virial current in this case is \( j_\mu = d_\phi \partial_\mu \phi^2 \). Thus the action is conformally invariant, and if we Weyl-gauge it we obtain

\[
A = \int \sqrt{g} \left( \frac{1}{2} \delta^{\mu
u} (\partial_\mu + d_\phi W_\mu) \phi (\partial_\nu + d_\phi W_\nu) \phi - f \phi^{2n/(n-2)} \right),
\]

where \( \Omega_{\mu\nu} \) is as in (22) and can therefore be replaced by a multiple of \( R \).

For \( n = 2 \) scalar fields are dimensionless and no invariant polynomial potential can be constructed. However, definite scale dimensions can be assigned to the exponentials of scalar fields and these exponentials can be used to construct invariant potentials. For example the action

\[
A = \int \left( \frac{1}{2} \eta^{ab} [\partial_a \theta \partial_b \theta + h_{\alpha\beta}(\hat{\phi}) \partial_a \hat{\phi}^\alpha \partial_b \hat{\phi}^\beta] - e^\theta V(\hat{\phi}) \right),
\]

where the fields \( \hat{\phi} \) are conformal scalars, is scale-invariant provided \( e^\theta \) has scale dimension \( d_{e^\theta} = -2 \). The action (53) is just the Liouville action modified by the addition of some scalar fields \( \hat{\phi} \) whose potential is arbitrary. In this case the virial current is \( j_\mu = \partial_\mu \theta \) and hence the full action is conformally invariant. If we Weyl-gauge it we obtain

\[
A = \int \sqrt{g} \left( \frac{1}{2} \delta^{\mu\nu} \left[ (\partial_\mu \theta - 2W_\mu)(\partial_\nu \theta - 2W_\nu) + h_{\alpha\beta}(\hat{\phi}) \partial_\mu \hat{\phi}^\alpha \partial_\nu \hat{\phi}^\beta \right] - e^\theta V(\hat{\phi}) \right)
\]

This is of the form of (13) given in section 5. The last term in (53) is an example of a central term discussed in the last section. Dropping it the (partial) Ricci gauging then amounts to replacing the term \( \nabla \cdot W \theta \) by \( R \theta \).

**Counter Example:**

It is interesting to see what happens if we allow the kinetic term of the Liouville field to be multiplied by scalar fields. Consider for example

\[
A = \int \left( \frac{1}{2} \gamma^{ab} (h(\hat{\phi}) \partial_a \theta \partial_b \theta + h_{\alpha\beta}(\hat{\phi}) \partial_a \hat{\phi}^\alpha \partial_b \hat{\phi}^\beta] - e^\theta V(\hat{\phi}) \right),
\]

where as before the \( \phi \)-fields are conformal scalars and \( e^\theta \) has scale dimension \( d_{e^\theta} = -2 \). In this case the virial current is \( j_\mu = h(\hat{\phi}) \partial_\mu \theta \) and thus is not a total derivative. It follows that although the action is rigid scale-invariant it is not conformally invariant, and if it is Weyl-gauged the Weyl-gauging cannot be replaced by Ricci gauging. Thus, even for actions which are quadratic in the derivatives of scalar fields rigid scale-invariance does not necessarily imply conformal invariance.

The case of spin \( \frac{1}{2} \) and spin 1 fields has been widely treated in the literature, starting with [4]. Below we consider fields of any spin in \( n \)-dimensions.
Fermions

The Dirac Lagrangian for spin $\frac{1}{2}$ fermions is found to be not only conformally invariant but locally Weyl invariant, i.e. has a virial current which is identically zero. We wish to show that the same is true for the Rarita-Schwinger Lagrangian for fermions of arbitrary spin. The Rarita-Schwinger fields describing fermions of spin $s + \frac{1}{2}$ are of the form $\psi_{\alpha ij...k}(x)$, where $\alpha$ is a Dirac index and the $i, j, k = 1...s$ are vector indices with respect to which $\psi$ is completely symmetric and traceless. The fields also satisfy the condition

$$\gamma^i \psi_{ij...k}(x) = 0$$  \hspace{1cm} (57)

Condition (57) is not a mass-shell condition but an irreducibility condition that ensures that the spin of the field is exactly $s + \frac{1}{2}$, and not a mixture of this and lower spins. The Rarita-Schwinger Lagrangian for these fields is

$$L(\psi) = \bar{\psi}^{ij...k} \gamma^\mu \nabla_\mu \psi_{ij...k}$$  \hspace{1cm} (58)

and is the natural generalization of the Dirac Lagrangian. The scale-dimension is evidently $d_\psi = (1 - n)/2$ and the virial current is

$$j_\mu = \bar{\psi}^{ij...k} \left( \gamma^\nu (d_\psi g_{\mu\nu} - \sigma_{\mu\nu}) \right) \psi_{ij...k} - \sum \bar{\psi}^{i...j...k} \left( \gamma^\nu (\tau_{\mu\nu})_j^{j'} \right) \psi_{i...j...k}$$  \hspace{1cm} (59)

where the sum is over all the vector indices in turn and

$$\sigma_{\mu\nu} = \frac{1}{4} [\gamma_\mu, \gamma_\nu] \quad (\tau_{\mu\nu})_j^{j'} = g_{j\mu}g_{\nu j'} - g_{j\nu}g_{\mu j'}.$$  \hspace{1cm} (60)

The first term in (60) is the term that would be obtained in spin $\frac{1}{2}$ case and vanishes in the usual manner because

$$\gamma^\nu \sigma_{\mu\nu} = \frac{(1 - n)}{2} \gamma_\nu = d_\psi \gamma_\nu$$  \hspace{1cm} (61)

The interesting point, however, is that the other spin terms also vanish because

$$\gamma^\mu (g_{j\mu}g_{\nu j'} - g_{j\nu}g_{\mu j'}) = \gamma_j g_{\nu j'} - \gamma_j' g_{j\nu}$$  \hspace{1cm} (62)

and these matrices annihilate the fields on account of the irreducibility condition (57). Thus, like the Dirac Lagrangian, the Rarita-Schwinger Lagrangian (58) is not only conformally invariant but local-Weyl invariant.

Bosons

The most general Lagrangian that is quadratic in the derivatives of the fields is

$$L(\phi) = \frac{a}{2} \nabla^\mu \phi^{ij...k} \nabla_\mu \phi_{ij...k} + \frac{b}{2} \nabla^r \phi^{ij...k} \nabla_s \phi_{rj...k} + \frac{c}{2} \nabla_p \phi^{ij...k} \nabla_t \phi_{ij...k},$$  \hspace{1cm} (63)

where $\phi_{ij...k}$ is a totally symmetric, traceless tensor and we discard possible potential terms which play only a passive role for the conformal conditions. The scale-dimension is evidently $d_\phi = (2 - n)/2$. The case $a + b = c = 0$ is the generalization of the Maxwell Lagrangian to arbitrary spin. Note that the last two terms in (63) differ only by a total divergence. The virial current is by (37)

$$j_\nu = d_\phi \bar{\pi}_\nu^{ij...k} \phi_{ij...k} - \sum \bar{\pi}_\nu^{ij...k} (\tau_{\nu} \phi_{ij...k})$$  \hspace{1cm} (64)
where \((\tau_{\nu\mu})_{jj}\) is given in (60) and

\[
\tau_{\nu\mu}^{ij...k} = a \nabla_\nu \phi_{ij...k}^{jj'} + b \nabla^i \phi_{ij...k}^{jj'} + c \nabla_\rho \phi^{jj...k} g^{ij}_{\nu}
\] (65)

is the canonical momentum. Substitution into (63) yields after rearranging of the terms

\[
j_\nu = \frac{1}{2} (d_\phi a - b) \nabla_\nu (\phi_{ij...k}^{jj'} \phi_{ij...k}) + \left[ b (d_\phi + 1 - s) - sa \right] \nabla_i (\phi_{ij...k}^{jj'} \phi_{ij...k}) + \left[ 2sa + b (s - d_\phi) + c (d_\phi + n + s - 2) \right] \nabla_\rho \phi^{jj...k} \phi_{ij...k}
\] (66)

Since the first two terms in (66) are divergences and the last term is not it follows that the condition for conformal invariance is

\[
2sa + b (s - d_\phi) + c (d_\phi + n + s - 2) = 0
\] (67)

in which case the virial current is of the form (63) with \(J^{\mu\nu}\) given by

\[
J^{\mu\nu} = \frac{1}{2} (d_\phi a - b) g^{\mu\nu} \phi_{ij...k}^{jj'} \phi_{ij...k} + \left[ b (d_\phi + 1 - s) - sa \right] \phi_{ij...k}^{jj'} \phi_{j...k}^{\nu} .
\] (68)

Thus conformal invariance reduces the number of parameters from three to two and if the total divergence (ratio of \(b\) to \(c\)) is fixed in advance it reduces them from two to one. Local Weyl invariance requires that the virial current vanishes and thus requires

\[
d_\phi a - b = 0 \quad \text{and} \quad b (d_\phi + 1 - s) - sa = 0,
\] (69)

which, for \(a, b \neq 0\) becomes

\[
(s - d) (d + 1) = 0.
\] (70)

Because \(d \leq 0\) and \(s > 0\), (70) then implies \(d = -1\) or, equivalently \(n = 4\). Hence, in 4-dimensions a Weyl invariant action exists for any integer spin while in any other dimension there is no Weyl-invariant Lagrangian of the type (63)! In the general case the Weyl-gauged action is given by (47) which includes coupling to the Ricci scalar as well as Ricci tensor.

We have seen in the previous section that the conformal properties of the Lagrangian depend on total divergence terms, i.e. they depend on the parameters \(b\) and \(c\) separately and not on the combination \(b + c\). This may be surprising, but can be understood by noting that conformal transformations depend explicitly on the coordinates and so conformal variations of local quantities may produce divergent terms. This property was used in [4] to construct a 4-dimensional spin 1 Lagrangian that was conformally invariant but not locally Weyl invariant, namely the Lagrangian (63) with \(n = 4\), \(s = 1\).

### Quadratic Rarita-Schwinger Fields

Finally we consider Lagrangians that are quadratic in Rarita-Schwinger fields. In analogy to (63) the most general such Lagrangian is

\[
L(\psi) = \frac{a}{2} \nabla_\rho \bar{\psi}^{ij...k} \nabla_p \psi_{ij...k} + \frac{b}{2} \nabla_p \bar{\psi}^{ij...k} \nabla^i \psi_{ij...k} + \frac{c}{2} \nabla_p \bar{\psi}^{pj...k} \nabla^t \psi_{ij...k}.
\] (71)

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where the last two terms on the right-hand side differ only by a total divergence. The scale-dimension is \( d_\psi = (2 - n)/2 \). Due to the irreducibility condition, the generator \( \sigma_{\nu\mu} \) in \( \Lambda^{\nu\mu} \) can be absorbed by substituting the integer spin \( s \) by \( s + \frac{1}{2} \) in the corresponding formulas for bosons in the last section, except for the term

\[
\frac{a}{2} \left[ \nabla^\mu \bar{\psi}^{j...k} \sigma_{\nu\mu} \psi_{i...k} - \bar{\psi}^{j...k} \sigma_{\nu\mu} \nabla^\mu \psi_{i...k} \right].
\]

(72)

Conformal invariance then requires

\[
a = 0 \quad \text{and} \quad b(s + \frac{1}{2} - d_\psi) + c(d_\psi + n + s - \frac{3}{2}) = 0.
\]

(73)

Furthermore

\[
J^{\nu\mu} = \frac{b}{2}(d_\psi + \frac{1}{2} - s) \left[ \bar{\psi}^{j...k} \psi^{j...k} - \bar{\psi}^{j...k} \psi^{j...k} \right] - \frac{b}{2} g^{\nu\mu} \bar{\psi}^{j...k} \psi_{j...k}.
\]

(74)

To summarize, the conformal condition puts two constraints on the three parameters \( a, b, c \) and in general, for a fixed total divergence there is no Lagrangian that is conformally invariant. Only if the total divergence is chosen so that (73) holds do we get conformal invariance. Furthermore, there is no non-zero value of the parameters for which the Lagrangian (71) is locally Weyl invariant. The Weyl-gauging is then given by (47) with \( J^{\nu\mu} \) from (74).

8 Conclusions

Exploiting the equivalence between the conformal group and the conformal Weyl-group we have reformulated a previous criterion for conformal invariance in flat space and obtained a purely algebraic criterion which permits a systematic analysis. In particular, any conformally invariant action without higher derivatives in the conformal variant fields must be at most quadratic in the first derivatives of variant fields. For such actions we have obtained the conditions for scale-invariance to imply conformal invariance for any spin in arbitrary dimensions.

We have shown that the flat space conformal theories are the only ones whose diffeomorphic versions can be made Weyl-invariant (Weyl-gauged) by non-minimal coupling to the curvature. In \( n \) dimensions the coupling is to the Ricci-tensor as well as to the Ricci-scalar. The Dirac action for half-integer spin fields is Weyl-invariant even without gauging due to the vanishing of the virial current. An interesting situation arises for integer spin \( s \neq 0 \): For \( n = 4 \) a local Weyl-invariant action exists for arbitrary \( s \), but for \( n \neq 4 \) no such action exists for any \( s \).

In two dimensions the non-minimal coupling is related to the central term in the conformal variation of the Lagrangian and to the centre of the Virasoro algebra. Furthermore, the Weyl-gauged action leads automatically to the improved energy-momentum tensor upon variation with respect to the metric. Because the scale parameter is parity invariant, the improvement terms induced by Weyl-gauging are parity even. Parity violating improvements which are compatible with the conformal group in two dimensions are therefore not induced by Weyl-gauging. But they can be obtained by coupling each chiral component to gravity separately.
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Appendix

Supplementary Condition on the Virial Current in Two Dimensions:

Using the general divergence condition (44) we see that for $n = 2$ the condition on the virial current may be written as

$$\int j^\mu \hat{\omega}_\mu = -\int J^{\mu\nu} \nabla_\nu \hat{\omega}_\mu = 0,$$  \hspace{1cm} (1)

where $\hat{\omega}$ is any harmonic function. Furthermore, by the arguments given earlier, if we assume that the action depends only on the fields and their first derivatives the virial tensor $J^{\mu\nu}$ cannot depend on the derivatives. Now, choosing $\hat{\omega} = \hat{\omega}(x^+)$ we see that (1) becomes

$$\int dx^+ dx^- J^{++} \nabla_+^2 \hat{\omega}(x^+) = 0$$  \hspace{1cm} (2)

which implies that

$$\int dx^- J^{++} = 0 \quad \text{or} \quad J^{++} = \partial_- S^{++}$$  \hspace{1cm} (3)

for some local function $S$. But since $J^{++}$ does not depend on the derivatives of the fields (3) cannot hold for arbitrary field configurations unless $J^{++} = 0$. Similarly $J^{--} = 0$. It follows that

$$J^{\mu\nu} = g^{\mu\nu} J$$  \hspace{1cm} (4)

as required. Note that (1) imposes no further condition on $J$ because $\hat{\omega}$ is harmonic.

Vanishing of the Integrand in (42): 

Let $\hat{\omega}$ be as in (34). Then

$$\int \hat{\omega}_\mu \hat{\omega}_\nu S_{\mu\nu} d^n x = \int \frac{x_\mu x_\nu}{x^4} S_{\mu\nu}(x + e) d^n x \quad \text{where} \quad e_\mu = \frac{c_\mu}{c^2}.$$  \hspace{1cm} (5)

Multiplying (5) by $e^{ikx}$ and integrating over $e$ then leads to the identity

$$S_{\mu\nu}(k) \int \frac{x_\mu x_\nu}{x^4} e^{-ikx} d^n x = 0.$$  \hspace{1cm} (6)

Performing the integration over $x$ leads to

$$\frac{2}{n} k^2 S^\mu_\mu + (2 - n) k_\mu k^\nu \hat{S}_{\mu\nu} = 0,$$  \hspace{1cm} (7)

where $S_{\mu\nu} = \frac{1}{n} S^{\mu\nu} g_{\mu\nu} + \hat{S}_{\mu\nu}$. Because (7) is an off-shell identity we conclude that $\hat{S}_{\mu\nu} = 0$ and $S^\mu_\mu$ can at most be constant, which is excluded for $n > 2$ on dimensional grounds.
The case $n=2$ :

For $n=2$ we obtain a relation between $j^\mu$ and $T^{\mu\nu}$ from (40). Because the linear term in (40) vanishes separately, the tensor $T^{\mu\nu}$ can be at most constant i.e. $T^{\mu\nu} = K g^{\mu\nu}$. Then, using the group property $\Delta_{\hat{\omega}_1 + \hat{\omega}_2} = \Delta_{\hat{\omega}_2} \Delta_{\hat{\omega}_1}$ we obtain the relation

$$\int \hat{\omega}_\mu^1 (\Delta_{\hat{\omega}_2} j)^\mu = 2K \int \hat{\omega}_\mu^1 \hat{\omega}_\nu^2, \tag{8}$$

which is the analog of (4) for $n=2$ and implies

$$\frac{\delta j^\mu(x)}{\delta \hat{\omega}_\mu(y)} = Kg^{\mu\nu} \delta(x-y). \tag{9}$$

Note that (9) requires the virial current $j^\mu$ to be linear in the conformally variant fields with constant coefficients. This property is not automatic, even for quadratic Lagrangians as is shown by the counter example in section 7.

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