ON AN ISOPERIMETRIC INEQUALITY FOR
A SCHRÖDINGER OPERATOR DEPENDING
ON THE CURVATURE OF A LOOP

Almut Burchard, Lawrence E. Thomas
University of Virginia
Department of Mathematics
Charlottesville, Virginia 22904

{burchard,let}@virginia.edu

May 6, 2005

Abstract
Let $\gamma$ be a smooth closed curve of length $2\pi$ in $\mathbb{R}^3$, and let $\kappa(s)$ be its curvature, regarded as a function of arc length. We associate with this curve the one-dimensional Schrödinger operator $H_\gamma = -\frac{d^2}{ds^2} + \kappa^2(s)$ acting on the space of square integrable $2\pi$-periodic functions. A natural conjecture is that the lowest spectral value $e_0(\gamma)$ of $H_\gamma$ is bounded below by 1 for any $\gamma$ (this value is assumed when $\gamma$ is a circle). We study a family of curves $\{\gamma\}$ that includes the circle and for which $e_0(\gamma) = 1$ as well. We show that the curves in this family are local minimizers; i.e., $e_0(\gamma)$ can only increase under small perturbations leading away from the family. To our knowledge, the full conjecture remains open.

1 Introduction
Let $\gamma$ be a smooth closed curve of length $2\pi$ in $\mathbb{R}^3$, parametrized by arclength $s$. We associate with this curve a Schrödinger operator $H_\gamma$ on the space of square integrable, $2\pi$-periodic functions by

$$H_\gamma \Phi(s) = -\frac{d^2 \Phi(s)}{ds^2} + \kappa^2(s) \Phi(s),$$

where $\kappa(s)$ is the curvature of $\gamma$ at $s$. Let

$$e_0(\gamma) = \inf \text{ spec } H_\gamma = \inf_{\Phi \neq 0} \frac{\int_0^{2\pi} (\Phi')^2 + \kappa^2 \Phi^2 \, ds}{\int_0^{2\pi} |\Phi|^2 \, ds}$$

(1.1)
be the smallest eigenvalue of $H_\gamma$. It has been conjectured that $e_0(\gamma)$ achieves its minimum
\[
e_{\text{min}} = \inf_{\gamma} e_0(\gamma)
\]
when $\gamma$ is a circle. In that case, $\kappa^2 \equiv 1$, the minimizing eigenfunction $\Phi$ is constant, and $e_0(\gamma) = 1$. But the functional assumes the same value for an entire family $F$ of curves given by translations, rotations and dilations of planar loops which have tangent vector $U(s)$ proportional to $(\cos(s), \beta \sin(s), 0)$ for some constant $\beta$ with $0 < \beta \leq 1$. So if indeed circles are minimizers, they certainly are not the only minimizers.

In this article, we show that loops in the family $F$ locally minimize the functional $e_0(\gamma)$ given in Eq. (1.1). Small deformations about any one of these loops cause $e_0$ to strictly increase, provided the the loop is not simply deformed to another loop of the same family. This result is a first step towards understanding the landscape in the space of curves $\{\gamma\}$ defined by the values of $e_0$. We emphasize that the conjecture itself remains open; our results only add credibility to it.

That $e_0(\gamma) \geq 1$ with the circle as a minimizer seems to have been implicitly conjectured by a number of people. The conjecture was articulated by Benguria and Loss [1], who showed it to be equivalent to establishing the best constant for a one-dimensional Lieb-Thirring inequality for a Schrödinger operator with two bound states. They did show that $e_0(\gamma) \geq 1/2$. We too had made the conjecture in our work on the local existence for a dynamical Euler elastica [2]. There, the issue of the invertibility of $H_\gamma$ arises in determining the tension of an elastic loop. We showed that $e_0(\gamma) \geq 1/4$, which is in fact optimal for curves which are possibly open, and for which the tangent vector $U$ is $2\pi$-periodic and each of the components of $U$ vanishes at least once.

In related work, Harrell and Loss [3] showed that Schrödinger operators of the form $-\Delta - d\kappa^2$ on $d$-dimensional hypersurfaces, with $\Delta$ the Laplace-Beltrami operator and $\kappa$ the mean curvature, have at least two negative eigenvalues unless the surface is a sphere (a circle in one dimension). Previously, Harrell [4] had proved a similar result for Schrödinger operators on embedded surfaces in $\mathbb{R}^3$ that are topologically equivalent to $S^2$, with potentials given by arbitrary definite quadratics in the principal curvatures.

Exner, Harrell, and Loss [5] discussed a variety of isoperimetric inequalities related to Schrödinger operators including the operator $H_{\gamma,g} = -d^2/ds^2 + g\kappa^2(s)$ on closed curves, and showed that, for the least eigenvalue of $H_{\gamma,g}$, the circle is a minimizer when $g \leq 1/4$ and not a minimizer for $g > 1$. Friedrich considered the operator with $g = 1/4$ for simple loops on the unit sphere, in connection with the Dirac operator on the region enclosed by such a loop [6]. The significance of the value $g = 1$ is that two natural candidates for minimizing the lowest eigenvalue of $H_{\gamma,g}$ appear to exchange stability there: When $\gamma$ is a circle, $\inf \text{spec } H_{\gamma,g} = g$, whereas for the extreme case of a collapsed curve $\gamma$, consisting of two straight line segments of length $\pi$ joined at their ends, we have $\inf \text{spec } H_{\gamma,g} = 1$. Such collapsed curves are limiting points of the family $F$.

The functional $e_0$ has no obvious convexity properties, and it is not amenable to standard symmetrization techniques. One difficulty is that $\kappa^2$ cannot be varied freely, since the condition that $\kappa$ be the curvature of a closed curve in $\mathbb{R}^3$ is a complicated, nonlocal condition. Technically, we show that the second variation of $e_0(\gamma_\mu)$ is non-negative for one-parameter families $\gamma_\mu$, leaving away from a loop $\gamma = \gamma_\mu|_{\mu=0}$ in $F$; this second variation is strictly positive if the perturbation is transversal to the family. For the case of the $\gamma$ a circle, where the eigenfunctions and eigenvalues
of $H_\gamma$ are known, one can simply perform second order perturbation theory to show this positivity. For other curves in the family, the higher eigenvalues and eigenfunctions of $H_\gamma$ are not explicitly available, and different methods are needed to show the positivity.

We find it useful to rewrite the variational problem as follows. Let $U(s)$ be the unit tangent vector to the curve, again parametrized by arclength $s$, let $\Phi(s)$ be the minimizing eigenfunction, and set

$$X(s) = \Phi(s)U(s),$$

so that $X'(s) = \Phi'(s)U(s) + \Phi(s)U'(s)$. Since $|U(s)| \equiv 1$, $U(s) \cdot U'(s) \equiv 0$, and $|U'(s)| \equiv \kappa(s)$, we can rewrite Eq. (1.1) as

$$e_0(\gamma) = \frac{\int_0^{2\pi} |X'(s)|^2 ds}{\int_0^{2\pi} |X(s)|^2 ds}. \tag{1.3}$$

It follows that

$$e_{\min} = \inf \frac{\int_0^{2\pi} |X'(s)|^2 ds}{\int_0^{2\pi} |X(s)|^2 ds},$$

where the infimum is taken over all $2\pi$-periodic, vector-valued functions $X$, vanishing only on a set of measure zero, with

$$\int_0^{2\pi} \frac{X(s)}{|X(s)|} ds = 0, \tag{1.4}$$

guaranteeing that the curve $\gamma$ with unit tangent $U(s) = X(s)/|X(s)|$ is closed. We will refer to the vector function $X(s)$ as an orbit. Given a vector-valued function $X(s)$ that satisfies Eq. (1.4), the curve $\gamma$ can be reconstructed up to a translation as a function $Y_\gamma(s) \in \mathbb{R}^3$ by computing

$$Y_\gamma(s) = \int_0^s U(\tilde{s}) d\tilde{s}.$$  

It is apparent that for any choice of vectors $v_1 \neq 0$ and $v_2$, the orbits

$$X_0(s) = \cos(s)v_1 + \sin(s)v_2 \tag{1.5}$$

all satisfy the constraint in Eq. (1.4), and all give the same value ($e_0(\gamma) = 1$) for the functional in Eq. (1.3). When $v_1$ and $v_2$ are linearly independent, these orbits correspond to curves in $F$. When $v_1$ and $v_2$ are linearly dependent, we obtain the collapsed curves mentioned above. Our results imply the following:

**Theorem 1.1** Let $U_0$ be the tangent vector to a curve $\gamma_0 \in F$, and assume that, for each $\mu$ sufficiently close to 0, $U(\mu, s)$ describes the tangent vector of a closed curve of length $2\pi$ parametrized by arc length, i.e.,

$$|U(\mu, s)| \equiv 1, \quad \int_0^{2\pi} U(\mu, s) ds = 0.$$  

If $U(\mu, ds)$ has an expansion

$$U(\mu, s) \equiv U_0(s) + \mu u_1(s) + \mu^2 u_2(s) + o(\mu^2)$$


in $H^1$, then there exists a positive number $c$ such that

$$e(\gamma_\mu) \geq e(\gamma_0)$$

for $|\mu| < c$. The inequality is strict unless $\gamma_\mu$ belongs again to the family $\mathcal{F}$.

To prove the theorem, we will show that the orbits in Eq. (1.5) corresponding to loops in $\mathcal{F}$ locally minimize the functional

$$\mathcal{L}(X) = \frac{1}{2} \int_0^{2\pi} \left\{ |X'(s)|^2 - |X(s)|^2 \right\} ds \quad (1.6)$$

subject to the constraint in Eq. (1.4). This implies that they locally minimize the functional in Eq. (1.3). We note in passing that the Euler-Lagrange equation for this minimization problem is given by

$$X''(s) + X(s) = \frac{|X(s)|^2 b - (X(s) \cdot b) X(s)}{|X(s)|^2} =: A(s)b, \quad (1.7)$$

where $b \in \mathbb{R}^3$ is a vector of Lagrange multipliers, and the $3 \times 3$ matrix $A(s)$ is computed by differentiating the constraint in Eq. (1.4). These equations are easily seen to have first integrals, an energy

$$\frac{1}{2} |X'(s)|^2 + \frac{1}{2} |X(s)|^2 - \frac{b \cdot X(s)}{|X(s)|^2}$$

and an angular momentum

$$b \cdot X(s) \times X'(s).$$

We are unaware of another constant of integration which would make them an integrable system.

In Section 2, we consider deformations around orbits of the form given in Eq. (1.5) for the generic case where $v_1$ and $v_2$ are linearly independent. These elliptical orbits are critical points for the functional in Eq. (1.6) even without the constraint, since they satisfy Eq. (1.7) with $b = 0$. We show that to second order in a parameter $\mu$ this functional can only increase for deformations of the orbit that do not simply transform the orbit into another elliptical orbit new choices of $v_1$ and $v_2$. The proof relies on an identity of elliptic integrals which is not transparent (to us). The section ends with the proof of Theorem 1.1.

In Section 3, we consider deformations about collapsed orbits given by Eq. (1.5) where $v_1$ is nonzero and $v_2$ is a constant multiple of $v_1$. We show that the functional again increases for nondegenerate perturbations. Unfortunately, the analysis of these collapsed curves is somewhat vexing. Their curvature is zero along the line segments and infinite at the end points. This forces the minimizing eigenfunctions to vanish at these endpoints and results in a ground state of multiplicity two so that the curve corresponds to a two-parameter family of orbits. We relegate the expansion of the constraint in Eq. (1.4) about a collapsed critical orbit to the following Section 4, the reason being that the computations are somewhat gruesome, and their presentation would break the flow of the main arguments showing positivity of $\mathcal{L}$.

Curiously, the analysis of the second variation about the collapsed orbits relies in part on the explicit diagonalization of the Schrödinger operator $K_g = -d^2/ds^2 + g \sec^2(s)$, acting in $L^2[-\pi/2, \pi/2]$ by Gegenbauer polynomials. This is discussed in the Appendix.
2 Elliptical orbits

We expand an orbit \( X \) in terms of a small parameter \( \mu \) as

\[
X(\mu, s) \equiv X_0(s) + \mu x_1(s) + \mu^2 x_2(s) + o(\mu^2) .
\]  

(2.1)

Here, \( X_0 \) is a nondegenerate elliptical orbit given by Eq. (1.5), \( x_1 \) and \( x_2 \) are vector-valued functions in \( H^1 \), and the error estimate is understood with respect to the \( H^1 \)-norm. Since the functional \( L \) in Eq. (1.3) and the constraint in Eq. (1.4) are symmetric under rotations, we may assume that

\[
X_0(s) = \begin{pmatrix} \alpha \cos(s) \\ \beta \sin(s) \\ 0 \end{pmatrix},
\]

(2.2)

where \( \alpha \geq \beta > 0 \) represent the major and minor semi-axes of the ellipse. The curvature of the corresponding loop \( \gamma \) is given by

\[
\kappa(s) = \left| \frac{d}{ds} \left( \frac{X_0(s)}{|X_0(s)|} \right) \right| = \frac{\alpha \beta}{|X_0|^2}.
\]

The principal eigenvalue and eigenfunction of the Schrödinger operator \( H_\gamma \) are

\[
e_0(\gamma) = 1, \quad \Phi(s) = |X_0(s)| = \sqrt{\alpha^2 \cos^2(s) + \beta^2 \sin^2(s)},
\]

and the eigenvalue-eigenvector equation reads

\[
H_\gamma \Phi = -\Phi'' + \frac{\alpha^2 \beta^2}{\Phi^3} = \Phi.
\]

(2.3)

Expanding the functional \( L \) defined by Eq. (1.6) in powers of \( \mu \),

\[
L(X) \equiv L(X_0) + \mu L_1 + \mu^2 L_2 + o(\mu^2),
\]

(2.4)

we see that \( L_1 = 0 \) since \( X_0 \) satisfies the Euler-Lagrange equation in Eq. (1.7). The second variation is given by

\[
L_2 = \frac{1}{2} \int_0^{2\pi} \left\{ |x'_1(s)|^2 - |x_1(s)|^2 \right\} ds + \int_0^{2\pi} \left\{ X'_0(s) \cdot x'_2(s) - X_0(s) \cdot x_2(s) \right\} ds
\]

\[
= L(x_1);
\]

the contribution of \( x_2 \) vanishes after an integration by parts since \( X''_0 + X_0 = 0 \). The constraint Eq. (1.4) expanded to first order in \( \mu \) implies that \( x_1 \) satisfies the condition

\[
\int_0^{2\pi} A(s) x_1(s) ds = 0 ,
\]

(2.5)
where
\[
A(s) = \frac{1}{|X_0|^3} \begin{pmatrix}
\beta^2 \sin^2(s) & -\alpha \beta \cos(s) \sin(s) & 0 \\
-\alpha \beta \cos(s) \sin(s) & \alpha^2 \cos^2(s) & 0 \\
0 & 0 & \alpha^2 \cos^2(s) + \beta^2 \sin^2(s)
\end{pmatrix}
\] (2.6)
is the matrix appearing in Eq. (1.7).

Consider for a moment the special case where the orbit is a circle, \( \alpha = \beta > 0 \). Denote the components of \( x_1 \) by
\[
x_1(s) = \begin{pmatrix} x_1(s) \\ y_1(s) \\ z_1(1) \end{pmatrix}.
\]
The constraints in Eq. (2.5) can be expressed with the double-angle formula as
\[
\begin{align*}
\int_0^{2\pi} & \frac{1}{2} (1 - \cos(2s)) x_1(s) - \frac{1}{2} \sin(2s) y_1(s) \, ds = 0 \\
\int_0^{2\pi} & \frac{1}{2} (1 + \cos(2s)) y_1(s) \, ds = 0 \\
\int_0^{2\pi} & z_1(s) \, ds = 0.
\end{align*}
\]
In other words, the zeroth and second Fourier coefficients of the components of \( x_1 \) satisfy
\[
\hat{x}_1(\pm 2) + \mp i\hat{y}_1(\pm 2) = \hat{x}_1(0) + i\hat{y}_1(0), \quad \hat{z}_1(0) = 0.
\]
Since \( x_1 \) is real-valued, \( \hat{x}_1(0) \) is real as well. By the triangle inequality, \( |\hat{x}_1(0)|^2 \leq 2|\hat{x}_1(\pm 2)|^2 \), which implies \( L_2 \geq 0 \) by Parseval’s identity. The following proposition shows the corresponding statement for perturbations about general elliptical orbits.

**Proposition 2.1** The elliptical orbits in Eq. (2.2) locally minimize Eq. (1.6) under the constraint in Eq. (1.4) for each \( \alpha \geq \beta > 0 \). More precisely, there exists a positive constant \( c = c(\alpha, \beta) \) such that for every perturbation \( X(\mu, s) \) given by Eq. (2.1) which satisfies the constraint in Eq. (1.4) to order \( o(\mu) \), we have
\[
L_2 = L(x_1) \geq c(\alpha, \beta) \|P_{n \neq \pm 1} x_1\|^2,
\] (2.7)
where \( P_{n \neq \pm 1} \) is the projection onto the space of functions whose first order Fourier coefficients vanish.

**Remark:** Variations of the form \( x_1(s) = 2\text{Re} e^{is} \hat{x}(0) \) are of course along the line of critical orbits, and give zero second variation.

**Proof of Proposition 2.1** For notational convenience, we drop the subscript on \( x_1 \) and simply write \( x(s) \) instead of \( x_1(s) \). For the Fourier coefficients of \( x \) and \( A \), we use the convention
\[
\hat{x}(s) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-ins} x(s) \, ds, \quad \hat{A}(s) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-ins} A(s) \, ds.
\]
By Parseval’s identity, the functional \( \mathcal{L} \) can be expressed as
\[
\mathcal{L}(x) = \frac{1}{2} \sum_n (n^2 - 1)|\hat{x}(n)|^2 .
\]

When \( \hat{x}(0) = 0 \), the claim in Eq. (2.7) holds with \( c = 3/2 \), so we assume without loss of generality that \( \hat{x}(0) \neq 0 \). The Fourier coefficients of \( A \) are nonzero only for even \( n \), since \( A \) is \( \pi \)-periodic. Using Parseval’s identity again, we write the constraint in Eq. (2.5) as
\[
\hat{A}(0)\hat{x}^*(0) = -\sum_{n \neq 0} \hat{A}(n)\hat{x}^*(n) ,
\]
where \( ^* \) denotes complex conjugation. Since the first order Fourier coefficients of \( x \) contribute neither to the constraint nor to the claim, we may assume that \( \hat{x}(\pm 1) = 0 \).

The matrix \( \hat{A}(0) \) is invertible, since the off-diagonal elements of \( A(s) \) are odd in \( s \) and its diagonal elements are strictly positive, see Eq. (2.6). Multiplying by \( \hat{A}(0)^{-1} \) and taking the inner product with \( \hat{x}(0) \) yields
\[
|\hat{x}(0)|^2 = -\sum_{n \neq 0} (\hat{A}(0)^{-1}\hat{x}(0)) \cdot \hat{A}(n)\hat{x}^*(n) = \langle -\hat{A}^*(n)\hat{A}(0)^{-1}\hat{x}(0), P\hat{x}(n)^* \rangle_{\ell^2} ,
\]
where \( P \) is the projection onto the nonzero Fourier modes and \( \ell^2 \) denotes the space of vector-valued sequences whose sequence of norms is square summable. Since \( \hat{A}(n) = 0 \) and \( \hat{x}(n) = 0 \) for \( n = \pm 1 \), and \( n^2 - 1 > 0 \) for \( n \neq 0, \pm 1 \), we can apply the Cauchy-Schwarz inequality to obtain
\[
|\hat{x}(0)|^2 \leq \|(n^2 - 1)^{-1/2}P\hat{A}(n)\hat{A}(0)^{-1}\hat{x}(0)\|_{\ell^2} \|(n^2 - 1)^{1/2}P\hat{x}(n)\|_{\ell^2} .
\] (2.8)

This yields the lower bound
\[
\mathcal{L}(x) = \frac{1}{2} \left( \|(n^2 - 1)^{1/2}P\hat{x}(n)\|_{\ell^2}^2 - |\hat{x}(0)|^2 \right) \geq \frac{1}{2} \left( \frac{|\hat{x}(0)|^2}{\|(n^2 - 1)^{-1/2}P\hat{A}(n)\hat{A}(0)^{-1}\hat{x}(0)\|_{\ell^2}^2} - 1 \right) |\hat{x}(0)|^2 \geq \frac{\eta}{2(1 - \eta)}|\hat{x}(0)|^2 ,
\] (2.9)

where \( \eta \) is the lowest eigenvalue of the \( 3 \times 3 \) matrix
\[
D = \hat{A}(0)^{-1}\left\{ \sum_{n \neq 0, \pm 1} \frac{1}{1 - n^2}\hat{A}(n)\hat{A}(n)^* \right\} \hat{A}(0)^{-1} .
\] (2.10)

Note that the identity matrix is included as the \( n = 0 \) term in the definition of \( D \). Clearly \( \eta < 1 \) since \( D \) is the identity minus a positive definite matrix. We will show that \( \eta > 0 \) by verifying that the sum inside the braces of Eq. (2.10) is a positive definite matrix.
We express this sum as a convolution integral. In order to invert the Fourier multiplication operator \( 1 - n^2 \) on the space of functions whose odd Fourier coefficients vanish, we need to solve the equation

\[
y'' + y = f
\]
on the space of \( \pi \)-periodic functions. Since \( K(s) = \frac{1}{4} |\sin(s)| \) satisfies \( K''(s) + K(s) = \frac{1}{2}(\delta_0 + \delta_\pi) \), the unique \( \pi \)-periodic solution is given by

\[
K * f(s) = \int_0^{2\pi} K(s-t)f(t) \, dt ,
\]
and so

\[
\sum_{n \neq \pm 1} \frac{1}{1 - n^2} \hat{A}(n) \hat{A}(n)^* = \frac{1}{4} \int_0^{2\pi} \int_0^{2\pi} A(s)A(t)|\sin(s-t)| \, dsdt .
\] (2.11)

From the expression for \( A(s) \) in Eq. (2.6) it is apparent that the off-diagonal terms in \( A(s)A(t) \) change sign if \( (s, t) \) is replaced by \( (-s, -t) \) and hence integrate to zero. Thus the expression in Eq. (2.11) is actually diagonal with diagonal entries given by

\[
I_1 = \left< A_{11}, K * A_{11} \right>_{L^2} + \left< A_{12}, K * A_{12} \right>_{L^2}
\]
\[
I_2 = \left< A_{22}, K * A_{22} \right>_{L^2} + \left< A_{12}, K * A_{12} \right>_{L^2}
\]
\[
I_3 = \left< A_{33}, K * A_{33} \right>_{L^2},
\]
(2.12)

where \( A_{ij} \) is the \( ij \)-th entry of \( A \). It just remains to show positivity of these \( I_j \)'s. Clearly,

\[
I_3 = \frac{1}{4} \int_0^{2\pi} \int_0^{2\pi} |X_0(s)|^{-1} |X_0(t)|^{-1} |\sin(s-t)| \, dsdt > 0 ,
\]

and we note that

\[
I_1 + I_2 = \int_0^{2\pi} \int_0^{2\pi} \frac{(a^2 \cos(s) \cos(t) + b^2 \sin(s) \sin(t))^2}{|X_0(s)|^3 |X_0(t)|^3} |\sin(t-s)| \, dsdt > 0
\]
since the integrands are nonnegative. It follows from Lemma 2.2 which is proved below, that \( I_1 = \frac{\beta^2}{\alpha^2 + \beta^2} (I_1 + I_2) \) and \( I_2 = \frac{\alpha^2}{\alpha^2 + \beta^2} (I_1 + I_2) \) are both positive. Since \( \hat{A}(0) \) is a diagonal matrix with positive entries, we conclude from Eq. (2.10) that \( \eta > 0 \), and hence \( L_2 > 0 \).

In the proof of Proposition 2.1 we used that \( I_1 \) and \( I_2 \) are positive multiples of \( I_1 + I_2 \). This is a consequence of the following identity which we state as a lemma. We have no geometric insight why this identity should hold; it was discovered numerically.

**Lemma 2.2** The integrals in Eq. (2.12) satisfy \( \alpha^2 I_1 = \beta^2 I_2 \).
Proof. The lemma clearly holds for $\alpha = \beta > 0$, since then $I_2$ can be obtained from $I_1$ by replacing $(s, t)$ with $(s + \pi/2, t + \pi/2)$. For $\alpha > \beta > 0$, we write

$$A_{11}(s) = \frac{\beta^2 \sin^2(s)}{|X_0(s)|^3} = -\frac{\beta^2}{(\alpha^2 - \beta^2)}|X_0(s)|^{-1} + \frac{\alpha^2 \beta^2}{(\alpha^2 - \beta^2)}|X_0(s)|^{-3}.$$

Since $\alpha^2 \beta^2 K * |X_0(s)|^{-3} = |X_0(s)|$ by Eq. (2.3) and the definition of $K$, we have

$$\left\langle A_{11}, K * A_{11} \right\rangle_{L^2} = \frac{\beta^4}{(\alpha^2 - \beta^2)^2} \left\langle |X_0|^{-1}, K * |X_0|^{-1} \right\rangle_{L^2} - 2 \frac{\alpha^2 \beta^2}{(\alpha^2 - \beta^2)^2} \left\langle |X_0|^{-1}, |X_0| \right\rangle_{L^2} (2.13)$$

For the second term in $I_1$, we compute

$$A_{12}(s) = -\frac{\alpha \beta \cos(s) \sin(s)}{|X_0(s)|^3} = -\frac{\alpha \beta}{\alpha^2 - \beta^2} \frac{d}{ds} |X_0(s)|^{-1},$$

which gives

$$\frac{d}{ds} K * A_{12} = -\frac{\alpha \beta}{\alpha^2 - \beta^2} \frac{d^2}{ds^2} K * |X_0|^{-1} = \frac{\alpha \beta}{\alpha^2 - \beta^2} K * |X_0|^{-1} - \frac{\alpha \beta}{\alpha^2 - \beta^2} |X_0|^{-1}.$$

by the definition of $K$. With an integration by parts, we see that

$$\left\langle A_{12}, K * A_{12} \right\rangle_{L^2} = \frac{\alpha^2 \beta^2}{(\alpha^2 - \beta^2)^2} \left\langle |X_0|^{-1}, K * |X_0|^{-1} \right\rangle_{L^2} - \frac{\alpha^2 \beta^2}{(\alpha^2 - \beta^2)^2} \left\langle |X_0|^{-1}, |X_0|^{-1} \right\rangle_{L^2} (2.14)$$

Adding Eqs. (2.13) and (2.14), we obtain

$$I_1 = \beta^2 \left\{ \frac{\alpha^2 + \beta^2}{(\alpha^2 - \beta^2)^2} \left\langle |X_0|^{-1}, K * |X_0|^{-1} \right\rangle_{L^2} - 2 \frac{1}{(\alpha^2 - \beta^2)^2} \right\}$$

In the same way, we compute

$$I_2 = \alpha^2 \left\{ \frac{\alpha^2 + \beta^2}{(\alpha^2 - \beta^2)^2} \left\langle |X_0|^{-1}, K * |X_0|^{-1} \right\rangle_{L^2} - 2 \frac{1}{(\alpha^2 - \beta^2)^2} \right\},$$

which proves the lemma.

The lower bound on $|L_2|$ in Proposition 2.1 deteriorates when the elliptical orbit $X_0$ collapses. Fix $\alpha = 1$, and let $\beta \to 0$. By an analysis of the integrands in Eq. (2.12), particularly near $s, t = \pm \pi/2$, we find that

$$I_1 \sim \beta^2 \ln(1/\beta),$$

$$I_2 \sim \ln(1/\beta),$$

$$I_3 \sim \ln(1/\beta),$$

9
and similarly

\[ \hat{A}(0) \sim \begin{pmatrix} 1 & 0 \\ 0 & \ln(1/\beta) & 0 \\ 0 & 0 & \ln(1/\beta) \end{pmatrix} \]

It follows that the lowest eigenvalue of the diagonal matrix \( D \) in Eq. (2.10) is given by the entry involving \( I_1 \), and so, by Eq. (2.9),

\[ L_2 \geq \frac{\eta}{2(1-\eta)} \sim \beta^2 \ln(1/\beta) |\hat{x}_1(0)|^2. \]

On the other hand,

\[ L_2 \geq \frac{3}{2} ||P_{n \neq 0, \pm 1}^1 x_1||^2 - \frac{1}{2} |\hat{x}_1(0)|^2, \]

using the first line of Eq. (2.9). Interpolating between these two inequalities we obtain

\[ L_2 \geq c\beta^2 \ln(1/\beta) ||P_{n \neq 0, \pm 1}^1 x_1||^2 \quad (\alpha = 1, \beta \to 0), \]

where \( c \) is an absolute constant. Since Eq. (2.8) can hold with equality, the lowest eigenvalue of \( L \) on the space of functions whose first order Fourier coefficients vanish is also bounded above by a constant multiple of \( \beta^2 \ln(1/\beta) \).

**Proof of Theorem 1.1** Let \( U(\mu, s) \) be as in the statement of the theorem, and let \( \Phi(\mu, s) \) be the normalized minimizing eigenfunction for the corresponding curve \( \gamma(\mu) \). Since the ground state of \( H_\gamma \) is simple, we may expand \( \Phi(\mu, s) \) in \( H^1 \) as

\[ \Phi(\mu, s) \equiv \Phi_0(s) + \mu \phi_1(s) + \mu^2 \phi_2(s) + o(\mu^2). \]

The corresponding orbit is given by \( X(\mu, s) = \Phi(\mu, s) U(\mu, s) \), see Eq. (1.2), which has an expansion as in Eq. (2.1) with

\[ X_0(s) = \Phi_0(s) U_0(s) \]
\[ x_1(s) = \phi_1(s) U_0(s) + \Phi_0(s) u_1(s) \]
\[ x_2(s) = \phi_2(s) U_0(s) + \phi_1(s) u_1(s) + \Phi_0(s) u_2(s). \]

Since the unperturbed curve \( U_0 \) belongs to the family \( \mathcal{F} \), we may assume by performing a suitable rotation and translation that \( X_0(s) \) satisfies Eq. (2.2). By Proposition 2.1), there exists a constant \( c > 0 \) such that \( L(X(\mu, s)) \geq 0 \) for \( |\mu| < c \), with strict inequality if the variation is transversal to the family \( \mathcal{F} \). The claim now follows from the definition of \( L \) in Eq. (1.3).

3 Collapsed orbits

If the vectors \( v_1 \) and \( v_2 \) defining the elliptical orbits in Eq. (1.5) are linearly dependent, then the corresponding curve collapses into a pair of straight line segments joined at the ends. The associated Schrödinger operator is just the second derivative operator acting on \( 2\pi \)-periodic functions.
in $H^1$ which vanish at $\pi/2$ and $3\pi/2$. The lowest eigenvalue of this operator is $\epsilon_0 = 1$, and has multiplicity two, and the eigenfunctions are multiples of

$$
cos_{\alpha \beta}(s) = \begin{cases} 
\alpha \cos(s), & \text{if } -\pi/2 \leq s \leq \pi/2 \\
\beta \cos(s), & \pi/2 \leq s \leq 3\pi/2,
\end{cases}
$$

where $\alpha$ and $\beta$ are constants. The corresponding orbits are given by

$$
X_0(s) = \begin{pmatrix} 
cos_{\alpha \beta}(s) \\
0 \\
0
\end{pmatrix}.
$$

(3.1)

In this section, we show that these collapsed orbits also locally minimize the functional $L$. We consider perturbations around an orbit $X_0$ given by Eq. (3.1) with $\alpha > 0$ and $0^+ \leq \beta \leq \alpha$. We expand the perturbation to order $o(\mu^2)$ in $H^1$ as in Eq. (2.1). Expanding $L$ as in Eq. (2.4), we obtain for the first variation

$$
L_1 = \int_0^{2\pi} \{ X'_0(s) \cdot x'_1(s) - x_0(s) \cdot x_1(s) \} \, ds 
= - (\alpha - \beta) \{ x_1(\pi/2) + x_1(3\pi/2) \}.
$$

(3.2)

We have integrated by parts on each of the intervals $[-\pi/2, \pi/2]$ and $[\pi/2, 3\pi/2]$ and used that $X''_0 + X_0 = 0$ in the interior of these intervals. Note that $L_1$ vanishes when $\alpha = \beta$. For $\alpha \neq \beta$ the boundary terms can be of either sign, indicating that these orbits are not critical for $L$ without constraints. We will show that $L$ can only increase under small non-degenerate deformations that respect the constraint in Eq. (1.4).

**Proposition 3.1** Let $X_0$ be an orbit defined by by Eq. (3.1) with $\alpha > 0$ and $0^+ \leq \beta \leq \alpha$. Consider perturbations of $X_0$ given by

$$
X(\mu, s) \equiv X_0(s) + \mu X_1(s) + o(\mu)
$$

in $H^1$, and let the corresponding expansion of $L$ be given by

$$
L(X) = L(X_0) + \mu L_1 + o(\mu).
$$

If the first component of the constraint in Eq. (1.4) is satisfied to order $o(\mu)$, then $\mu L_1 \geq 0$. It is strictly positive unless either $\alpha = \beta > 0$ or $x_1(\pi/2) = x_1(3\pi/2) = 0$.

**Proof.** As mentioned in the introduction, we will need an expansion of the constraint in Eq. (1.4). This expansion is provided by Lemma 4.1 in the next section.

Consider the first case where $\alpha > 0$ and $\beta = 0^+$. Denote the components of the perturbed orbit by

$$
X(\mu, s) = \begin{pmatrix} X(\mu, s) \\
Y(\mu, s) \\
Z(\mu, s)
\end{pmatrix}, \quad x_1(s) = \begin{pmatrix} x_1(s) \\
y_1(s) \\
z_1(s)
\end{pmatrix}.
$$

(3.3)
By Lemma 4.1, the contribution of the interval \([-\pi/2, \pi/2]\) to the first component of the integral in Eq. (1.4) has an expansion
\[
\int_{-\pi/2}^{\pi/2} \frac{X(\mu, s)}{|X(\mu, s)|} \, ds = \pi + O(\mu) .
\] (3.4)

The contribution of \([\pi/2, 3\pi/2]\) is given by
\[
\int_{\pi/2}^{3\pi/2} \frac{X(\mu, s)}{|X(\mu, s)|} \, ds = \int_{\pi/2}^{3\pi/2} \frac{\mu x_1(s) + o(\mu)}{|\mu x_1(s) + o(\mu)|} \, ds \geq -\pi .
\] (3.5)

If \(\mu x_1(\pi/2) > 0\), then \(X(\mu, s)\) is greater than zero on a set whose measure does not go to zero as \(\mu \to 0\). The same is true if \(x_1(3\pi/2) > 0\). Similarly, if \(y_1\) or \(z_1\) is nonzero for some \(s \in [\pi/2, 3\pi/2]\), then by the continuity of these functions, the integrand differs from \(-1\) by at least some fixed positive value on a set whose measure does not go to zero as \(\mu \to 0\). In either case, the integral then would strictly exceed \(-\pi + \varepsilon\) for some \(\varepsilon > 0\) for all sufficiently small values of \(\mu\). Adding Eqs. (3.4) and (3.5), we see that if the constraint in Eq. (1.4) is satisfied to order \(\mu\), then
\[
\mu L_1 = |\mu| \frac{(\alpha - \beta)^2}{\alpha + \beta} (|x_1(\pi/2)| + |x_1(3\pi/2)|) \geq 0 ,
\]

as claimed.

If \(\alpha = \beta\) or \(x_1(\pi/2) = x_1(3\pi/2) = 0\), we must work to higher order in \(\mu\) to detect positivity of \(L\). Expanding \(L\) to second order in \(\mu\) yields with a similar computation as in Eq. (3.2)
\[
L_2 = \frac{1}{2} \int_0^{2\pi} \left\{|x_1'(s)|^2 - |x_1(s)|^2\right\} \, ds - (\alpha - \beta) \left\{x_2(\pi/2) + x_2(3\pi/2)\right\} .
\] (3.7)

Our next result is that the second variation of the functional is nonnegative whenever the first variation vanishes.
Proposition 3.2  Let $X_0$ be given by Eq. (3.7), and let $X(\mu, s)$ be an $H^1$-perturbation of $X_0$, given by an expansion as in Eq. (2.7). Assume that the first component of the constraint in Eq. (1.4) is satisfied to order $o(\mu^2)$, and the second and third components of Eq. (1.4) are satisfied to order $o(\mu)$. Consider the corresponding expansion of $\mathcal{L}$ given by Eq. (2.4). If $\mathcal{L}_1 = 0$, then $\mathcal{L}_2 \geq 0$. If the perturbation is transversal to the family of collapsed orbits, then $\mathcal{L}_2 > 0$.

PROOF. Let $\alpha \geq \beta \geq 0^+$, $X$, and $X_0$ be as in the statement of the theorem. Denote the components of the vector-valued functions appearing in the Eq. (2.1) by

$$X(\mu, s) = \begin{pmatrix} X(\mu, s) \\ Y(\mu, s) \\ Z(\mu, s) \end{pmatrix}, \quad x_1(s) = \begin{pmatrix} x_1(s) \\ y_1(s) \\ z_1(s) \end{pmatrix}, \quad x_2(s) = \begin{pmatrix} x_2(s) \\ y_2(s) \\ z_2(s) \end{pmatrix}. \quad (3.8)$$

Since $\mathcal{L}_1 = 0$, we have by Proposition 3.1 that either $\alpha = \beta$ or $x_1(\pi/2) = x_1(3\pi/2) = 0$. When $\alpha = \beta$, we invoke the first component of the constraint to order $o(\mu)$ and the second and third components to order $o(1)$ and use Lemma 4.1 to conclude that $x_1(\pi/2) = x_1(3\pi/2) = 0$ as well. In either case, the integral involving $x_1$ in Eq. (3.7) is strictly positive, unless the restrictions of $x_1$ to $[-\pi/2, \pi/2]$ and $[\pi/2, 3\pi/2]$ are multiples of $\cos(s)$. Expanding the second and third component of the constraint in Eq. (1.4) to order $o(1)$ and using Lemma 4.1 we see that then $y_1$ and $z_1$ are multiples of $\cos s \beta$, i.e., the variation is in the direction of the family of collapsed orbits. When $\alpha = \beta > 0$, this concludes the argument. For $\alpha > \beta$, the terms containing $y_1$ and $z_1$ will be used to balance the terms containing $x_2$.

Consider first the case where $\alpha > 0$ and $\beta = 0$. By Lemma 4.2 the contribution of the interval $[-\pi/2, \pi/2]$ to the integral in Eq. (1.4) satisfies

$$\int_{-\pi/2}^{\pi/2} \frac{X(\mu, s)}{|X(\mu, s)|} \, ds = \pi + O(\mu^2) \quad (3.9)$$

If $x_2(\pi/2) > 0$, then it follows from the continuity estimate in Eq. (4.7) that $X(\mu, s) = \mu x_1(s) + \mu^2 x_2(s) + o(\mu^2)$ is nonnegative on an interval $[\pi/2, s^*(\mu)]$, where $s^*(\mu) - \pi/2 = \mu^2/\alpha(1)$ as $\mu \to 0$. It follows that the contribution of the interval $[\pi/2, 3\pi/2]$ satisfies

$$\int_{\pi/2}^{3\pi/2} \frac{X(\mu, s)}{|X(\mu, s)|} \, ds = \int_{\pi/2}^{3\pi/2} \frac{\mu x_1(s) + \mu^2 x_2(s) + o(\mu^2)}{|\mu x_1(s) + \mu^2 x_2(s) + o(\mu^2)|} \geq -\pi + \frac{\mu^2}{o(1)}. \quad (3.10)$$

Adding Eqs. (3.9) and (3.10), we see that then the constraint in Eq. (1.4) cannot be satisfied to order $o(\mu^2)$. Therefore $x_2(\pi/2)$ and similarly $x_2(3\pi/2)$ cannot be positive. The claim now follows directly from Eq. (3.7).

When $\alpha \geq \beta > 0$, we use Lemma 4.2 to expand the first component of the constraint in Eq. (1.4) over the entire interval $[0, 2\pi]$,

$$\int_0^{2\pi} \frac{X(\mu, s)}{|X(\mu, s)|} \, ds = \mu^2 \left\{ -\frac{1}{2} \int_0^{2\pi} \text{sign}(\cos(s)) \left| \cos^{2\beta}(s) \right| \, ds \right. \left. + \left( \frac{1}{\alpha} + \frac{1}{\beta} \right) \left( x_2(\pi/2) + x_2(3\pi/2) \right) - \left( \frac{1}{\alpha} - \frac{1}{\beta} \right) \left( |x_2(\pi/2)| + |x_2(3\pi/2)| \right) \right\} + o(\mu^2). \quad (3.11)$$
The integral on the right hand side is well-defined by Lemma A.1 of the Appendix. To enforce the constraint in Eq. (1.4), we set the leading term in Eq. (3.11) equal to zero and solve for \( x_2(\pi/2) + x_2(3\pi/2) \). Inserting the resulting expression into Eq. (3.7) yields

\[
\mathcal{L}_2 = \frac{1}{2} \int_0^{2\pi} \left\{ |x'_1(s)|^2 - |x_1(s)|^2 + g_{\alpha\beta}(s) \sec^2(s)(y_1^2(s) + z_1^2(s)) \right\} \, ds ,
\]

where

\[
g_{\alpha\beta}(s) \equiv \begin{cases} -\frac{\beta(\alpha-\beta)}{\alpha(\alpha+\beta)} & -\pi/2 \leq s < \pi/2 \\ \frac{\alpha-\beta}{\beta(\alpha+\beta)} & \pi/2 \leq s < 3\pi/2 . \end{cases}
\]

The terms involving \( x_2 \) in Eq. (3.12) are clearly nonnegative. The part of the integral involving the first component \( x_1 \) is nonnegative because \( x_1 \) vanishes at \( \pi/2 \) and \( 3\pi/2 \).

To analyze the contribution of \( y_1 \) to the integral in Eq. (3.12), we invoke the second component of the constraint in Eq. (1.4) to order \( o(\mu) \). By Lemma 4.2,

\[
\int_0^{2\pi} \frac{Y(\mu, s)}{|X(\mu, s)|} \, ds = \mu \int_0^{2\pi} \frac{y_1(s)}{|\cos\alpha\beta(s)|} \, ds + o(\mu) .
\]

The corresponding statements hold for the third component, \( z_1 \). Thus, we minimize

\[
\int_0^{2\pi} \left\{ (w'(s))^2 + g_{\alpha\beta}(s) \sec^2(s)w^2(s) \right\} \, ds \tag{3.14}
\]

on the space of \( 2\pi \)-periodic functions in \( H^1 \)-functions that vanish at \( \pi/2 \) and \( 3\pi/2 \) subject to the constraints that

\[
||w||^2 = 1 , \quad \int_0^{2\pi} \frac{w(s)}{|\cos\alpha\beta(s)|} \, ds = 0 . \tag{3.15}
\]

We will prove that the minimum is 1, thereby showing that the total contributions of \( y_1 \) and \( z_1 \) to Eq. (3.12) are nonnegative.

The Euler-Lagrange equation for the minimization problem in Eqs. (3.14)-(3.15) is given by

\[
Kw(s) := -\frac{d^2w(s)}{ds^2} + g_{\alpha\beta}(s) \sec^2(s)w(s) = \frac{\nu}{|\cos\alpha\beta(s)|} + \eta w , \tag{3.16}
\]

where \( \eta = (w, Kw) \) is the value of the functional, and \( \nu \) is a Lagrange multiplier. We verify by direct computation that

\[
w_0(s) = -\nu \frac{\alpha(\alpha+\beta)}{\beta(\alpha-\beta)} \cos\alpha\beta(s)
\]

solves Eq. (3.16) with \( \eta = 1 \). This shows that \( \eta = 1 \) is a critical value of the functional.
Since $g_{\alpha\beta} > -1/4$ by Eq. (3.13), we can apply Lemma A.1 from the appendix to see that the operator $K$ is bounded below and has compact resolvent. The spectrum of $K$ consists of an increasing sequence of eigenvalues $\lambda_0, \lambda_1, \ldots$ with $\lambda_n \to \infty$. The spectrum of $K$ is the union of the spectra of its restrictions to $[-\pi/2, \pi/2]$ and $[\pi/2, 3\pi/2]$, which are determined explicitly in the appendix. It follows from Eq. (A.3) that $\lambda_0 > 1/4$ and $\lambda_1 > 1$.

Furthermore, a solution of the minimization problem in Eqs. (3.14)-(3.15) exists. In fact, the constrained functional has an infinite sequence of critical values $\eta_0 \leq \eta_1 \leq \ldots$, for which the Euler-Lagrange equation in Eq. (3.16) has a nontrivial solution. If $P$ is the projection onto the orthogonal complement of $1/|\cos_{\alpha\beta}|$ in $L^2$, then these critical values are just the eigenvalues of the operator $PKP$. By the minimax characterization of eigenvalues of self-adjoint operators, the second-lowest critical value $\eta_1$ satisfies

$$\eta_1 \geq \min \{D : D \perp [\cos_{\alpha\beta}] \} \max \{w \in D : \|w\| = 1\} \langle w, Kw \rangle_{L^2}$$

Here, $D$ runs over two-dimensional subspaces of $L^2$, see Theorem 12.1 of [7], Eq. (5).

We conclude that $w_0$ is indeed the minimizer, and $\eta_0 = 1$ is the minimum value. Since $\eta_1$ can also be characterized by

$$\eta_1 = \min \{(w, Kw) : \|w\|^2 = 1, w \perp w_0, w \perp 1/|\cos_{\alpha\beta}|\},$$

the functional in Eq. (3.14) is bounded below on the subspace of functions perpendicular to $1/|\cos_{\alpha\beta}|$ by

$$\langle w, Kw \rangle_{L^2} \geq \|w\|^2_{L^2} + (\eta_1 - 1) \left\{ \|P_{w_0^\perp} w_{\perp 0}\|^2 \right\},$$

where $P_{w_0^\perp}$ is the projection onto the subspace orthogonal to $w_0$.

4 The constraint integrals near a collapsed orbit

In this section we consider two expansions for $X(\mu, s)$ about a singular orbit $X_0$, as given in Eq. (3.1). The calculations are summarized in the following two lemmas.

**Lemma 4.1** Assume that a vector-valued function $X$ on the interval $[-\pi/2, \pi/2]$ satisfies

$$X(\mu, s) = \begin{pmatrix} \alpha \cos(s) \\ 0 \\ 0 \end{pmatrix} + \mu x_1(s) + o(\mu), \quad (4.1)$$

in $H^1$. Then, using the notation of Eq. (3.8),

$$\int_{-\pi/2}^{\pi/2} \frac{X(\mu, s)}{|X(\mu, s)|} \, ds = \pi \text{sign}(\alpha) + \frac{\mu}{|\alpha|} \left( x_1(-\pi/2) + x_1(\pi/2) \right) - \frac{|\mu|}{|\alpha|} \left( |x_1(-\pi/2)| + |x_1(\pi/2)| \right) + o(\mu) \quad (4.2)$$
and
\[ \int_{-\pi/2}^{\pi/2} \frac{1}{|X(\mu, s)|} \left( \begin{array}{c} Y(\mu, s) \\ Z(\mu, s) \end{array} \right) \, ds = \frac{\mu}{|\alpha|} \ln(1/|\mu|) \left\{ \left( \frac{y_1(-\pi/2) + y_1(\pi/2)}{z_1(-\pi/2) + z_1(\pi/2)} \right) + o(1) \right\} \] (4.3)

On the interval \([\pi/2, 3\pi/2]\), the corresponding formulae hold with \(\alpha\) replaced by \(-\alpha\) on the right hand sides.

The appearance of the absolute values of \(\mu\) and \(\alpha\) plays a crucial role in the analysis of the first variation of \(\mathcal{L}\) in Proposition [3.1]. We also need the following higher order expansion:

**Lemma 4.2** Assume that a vector-valued function \(X(s)\) on \([-\pi/2, \pi/2]\) satisfies
\[ X(\mu, s) = \left( \begin{array}{c} \alpha \cos(s) \\ 0 \\ 0 \end{array} \right) + \mu x_1(s) + \mu^2 x_2(s) + o(\mu^2) \] (4.4)
in \(H^1\), with \(x_1(-\pi/2) = x_1(\pi/2) = 0\). Then, in the notation of Eq. (3.3),
\[ \int_{-\pi/2}^{\pi/2} \frac{X(\mu, s)}{|X(\mu, s)|} \, ds = \pi \text{sign}(\alpha) + \frac{\mu^2}{|\alpha|} \left( x_2(-\pi/2) + x_2(\pi/2) \right) \] (4.5)
\[ -\frac{\mu^2}{\alpha} \left( |x_2(-\pi/2)| + |x_2(\pi/2)| \right) - \text{sign}(\alpha) \frac{\mu^2}{2\alpha^2} \int_{-\pi/2}^{\pi/2} \frac{1}{\cos^2(s)} \left( y_1^2(s) + z_1^2(s) \right) \, ds + o(\mu^2), \]
and
\[ \int_{-\pi/2}^{\pi/2} \frac{1}{|X(\mu, s)|} \left( \begin{array}{c} Y(\mu, s) \\ Z(\mu, s) \end{array} \right) \, ds = \frac{\mu}{|\alpha|} \int_{-\pi/2}^{\pi/2} \frac{1}{\cos(s)} \left( \begin{array}{c} y_1(s) \\ z_1(s) \end{array} \right) \, ds + o(\mu). \] (4.6)

On the interval \([\pi/2, 3\pi/2]\), the corresponding formulae hold with \(\alpha\) replaced by \(-\alpha\) on the right hand sides.

**Remark:** Since \(X(s)\) is an \(H^1\)-function with \(x_1(-\pi/2) = x_1(\pi/2) = 0\), the integrals in Eq. (4.5) and Eq. (4.6) are finite by Lemma [A.1].

The proofs rely on the well-known fact that \(H^1\)-functions on the circle are bounded and Hölder continuous with exponent 1/2. We will need the slightly stronger estimate
\[ |x(s) - x(t)| \leq \int_t^s |dx/dt(s')| \, ds' \leq \left( \int_t^s ds' \right)^{1/2} \left( \int_t^s \left( \frac{dx}{dt}(s') \right)^2 ds' \right)^{1/2} = |s-t|^{1/2} o(1). \] (4.7)

Since \(F(t) \equiv \int_0^t (dx/dt)^2(s') \, ds'\) is uniformly continuous in \(t\), the \(o(1)\) estimate holds uniformly in \(s\) and \(t\).

**Proof of Lemma 4.1** Let \(X(\mu, s)\) be of the form given in Eq. (4.1), and use the notation in Eq. (3.3) for the component functions. By the scaling invariance of the integrand, we may replace...
\(\alpha\) with 1 and \(\mu\) with \(\mu/\alpha\) without changing the values of the integrals. We also assume that \(\mu > 0\), replacing \(\mu\) with \(-\mu\) and \(x_1\) with \(-x_1\) if necessary.

Let us consider the resulting integral in the half-interval \([0, \pi/2]\), beginning with a neighborhood of \(\pi/2\) where the denominators are small. For \(s \in [\pi/2 - \mu/\delta(\mu), \pi/2]\) and with \(\delta = \delta(\mu) = o(1)\) to be further specified below, we see with the Taylor expansion of the cosine and the Hölder continuity of the \(H^1\)-function \(x_1\) that

\[
X(\mu, s) = \left( \begin{array}{c} \pi/2 - s + O(s - \pi/2)^3 \\ \mu(\pi/2) \\ 0 \end{array} \right) + \mu(\mu(\pi/2) + O(s - \pi/2)^{3/2}) + o(\mu)
\]

\[
= \left( \begin{array}{c} \pi/2 - s + \mu x_1(\pi/2) \\ \mu y_1(\pi/2) \\ \mu z_1(\pi/2) \end{array} \right) + O(\mu^3\delta^{-3}) + o(\mu(\mu^3\delta^{-1/2}) + o(\mu)
\]

\[
= v(\pi/2 - s) + \{O(\mu^3\delta^{-3}) + o(\mu(\mu^3\delta^{-1/2}) + o(\mu)\}
\]

In the second step, we have used that \(|s - \pi/2| \leq \mu/\delta\). We may neglect contributions to the integrals over the set

\[
\Delta = \Delta(\mu) := \{s \in [0, \pi/2] : |\pi/2 - s + \mu x_1(\pi/2)| \leq \mu \delta(\mu)\}
\]

because the integrands are bounded, and the measure of \(\Delta\) is \(o(\mu)\). On the complement of \(\Delta\) we use the inequality that for any pair of vectors \(v, w\) with \(|v| \geq 2|w| > 0\),

\[
\left| \frac{v + w}{|v + w|} - \frac{v}{|v|} \right| \leq 4 \left| \frac{w}{|v|} \right|.
\]

We apply this to \(v(\pi/2 - s)\) and \(w(\mu, s) = O(\mu^3\delta^{-3}) + o(\mu(\mu^3\delta^{-1/2}) + o(\mu)\) outside of \(\Delta\) with \(\delta = \delta(\mu)\) now chosen so that \(\|w\|_\infty/(\mu\delta^2) = o(1)\), which is the case if \(\delta(\mu)\) exceeds \(\mu^{1/5}\) and \(o(\mu)/\mu\delta^2 = o(1)\) where the \(o(\mu)\)-term refers to that in the expansion in Eq. (4.11) and \(\delta(\mu)\) itself is still \(o(1)\). We obtain

\[
\int_{[\pi/2 - \mu/\delta(\mu), \pi/2]} \frac{X(\mu, s)}{|X(\mu, s)|} - \left( \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right) \, ds
\]

\[
= \int_{[\pi/2 - \mu/\delta(\mu), \pi/2]} \left\{ \frac{v(\pi/2 - s)}{|v(\pi/2 - s)|} - \left( \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right) + \delta(\mu)o(1) \right\} \, ds + o(\mu)
\]

\[
= \int_0^{\mu/\delta(\mu)} \left\{ \frac{v(s)}{|v(s)|} - \left( \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right) \right\} \, ds + o(\mu).
\]

(4.9)
The \( x \)-component of the integral in the last line of Eq. (4.9) is elementary and equals
\[
\int_{0}^{\mu/\delta(\mu)} \frac{s + \mu x_1(\pi/2)}{(s + \mu x_1(\pi/2))^2 + \mu^2 y_1(\pi/2)^2 + \mu^2 z_1(\pi/2)^2} ds
\]
\[
= \sqrt{(s + \mu x_1(\pi/2))^2 + \mu^2 y_1(\pi/2)^2 + \mu^2 z_1(\pi/2)^2} - s \bigg|_{s=0}^{\mu/\delta(\mu)}
\]
\[
= \mu \left(x_1(\pi/2) - |x_1(\pi/2)|\right) + o(\mu). \tag{4.10}
\]

The \( y \)-and \( z \)-components of the integral in Eq. (4.9) are computed similarly, e.g.,
\[
\int_{0}^{\mu/\delta(\mu)} \frac{\mu y_1(\pi/2)}{\mu^2 y_1(\pi/2)^2 + \mu^2 z_1(\pi/2)^2} ds
\]
\[
= \mu y_1(\pi/2) \ln \left(s + \mu x_1(\pi/2) + (s + \mu x_1(\pi/2))^2 + \mu^2 y_1(\pi/2)^2 + \mu^2 z_1(\pi/2)^2\right) \bigg|_{0}^{\mu/\delta(\mu)}
\]
\[
= \mu \ln \left(\frac{1}{\delta(\mu)}\right) y_1(\pi/2) + O(\mu). \tag{4.11}
\]

The error of order \( O(\mu) \) reflects the shift of the zero in the denominator by \( \mu x_1(\pi/2) \). For the remaining part of the interval, the cosine dominates the denominator, and one finds for the \( x \)-component that
\[
\int_{0}^{\pi/2 - \mu/\delta(\mu)} \frac{\cos(s) + \mu x_1(s) + o(\mu)}{|X(s)|} - 1 \bigg| ds
\]
\[
= \int_{0}^{\pi/2 - \mu/\delta(\mu)} \frac{\mu y_1(s)}{\cos(s)} \left(\frac{\mu y_1(s)}{\cos(s) + \mu x_1(s) + o(\mu)}\right)^2 ds
\]
\[
= O(\mu \delta(\mu)) = o(\mu). \tag{4.12}
\]

We have used that \( x_1(s) \) is uniformly bounded. For the \( y \)-component, we have
\[
\int_{0}^{\pi/2 - \mu/\delta(\mu)} \frac{\mu y_1(s) + o(\mu)}{|X(s)|} ds
\]
\[
= \int_{0}^{\pi/2 - \mu/\delta(\mu)} \frac{\mu y_1(\pi/2) + \mu o((\pi/2 - s)^{1/2}) + o(\mu)}{\cos(s)} (1 + o(1)) ds
\]
\[
= \mu y_1(\pi/2) \ln(\sec(s) + \tan(s)) \bigg|_{0}^{\pi/2 - \mu/\delta(\mu)} + o(\mu \ln(1/\mu))
\]
\[
= -\mu \ln(\mu/\delta(\mu)) y_1(\pi/2) + o(\mu \ln(1/\mu)), \tag{4.13}
\]

where we have again exactly evaluated the integral and expanded the result. The \( z \)-component is analyzed in the same way.

Adding Eqs. (4.10) and (4.11) to Eqs. (4.12) and (4.13) respectively, we get that
\[
\int_{0}^{\pi/2} \left\{ \frac{X(\mu, s)}{|X(\mu, s)|} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\} ds = \begin{pmatrix} \mu \left\{ x_1(\pi/2) - |x_1(\pi/2)| \right\} + o(\mu) \\ \mu \ln(1/\mu) y_1(\pi/2) + o(\mu \ln(1/\mu)) \\ \mu \ln(1/\mu) z_1(\pi/2) + o(\mu \ln(1/\mu)) \end{pmatrix}.
\]
To obtain Eq. (4.2), we repeat the computation for the interval \([-\pi/2, 0]\) and add the results. The claim for the interval \([\pi/2, 3\pi/2]\) follows by replacing \(X(s)\) with \(-X(s - \pi)\).

\[\text{PROOF OF LEMMA 4.2}\] Here, we assume that \(X(\mu, s)\) has the expansion in Eq. (4.4) and \(x_1(-\pi/2) = x(\pi/2) = 0\). We may assume by scaling that \(\alpha = 1\) and \(\mu > 0\). Let us use again the notation in Eq. (3.8) to denote the components of the various vector-valued functions.

We will expand the integrand and partition the interval of integration as in the proof of Lemma 4.1. By Eq. (4.7), there is a function \(m(s) = o(s^{1/2})\) such that \(|x_i(s) - x_i(t)| \leq m(|s - t|)\), for \(i = 1, 2\). Let \(\delta = \delta(\mu) = o(1)\) to be further specified below. On \([\pi/2 - \mu^2/\delta, \pi/2]\) we expand

\[
X(\mu, s) = \begin{pmatrix} \pi/2 - s + O(s - \pi/2)^3 \\ 0 \\ 0 \end{pmatrix} + \mu^2x_2(\pi/2) + \mu O(m(s - \pi/2)) + o(\mu^2)
\]

\[
= \begin{pmatrix} \pi/2 - s + \mu^2x_2(\pi/2) \\ \mu^2y_2(\pi/2) \\ \mu^2z_2(\pi/2) \end{pmatrix} + O(\mu^6/\delta^3) + \mu m(\mu^2/\delta) + o(\mu^2)
\]

\[
=: \begin{pmatrix} \pi/2 - s \end{pmatrix} + \{O(\mu^6/\delta^3) + \mu m(\mu^2/\delta) + o(\mu^2)\}.
\]

At this point we choose \(\delta = \delta(\mu)\) so that \(\mu^4\delta^{-5} = o(1)\), \(m(\mu^2/\delta)\mu^{-1}\delta^{-2} = o(1)\) and that \(\mu^{-2}\delta^{-2}o(\mu^2) = o(1)\), still keeping \(\delta(\mu) = o(1)\). This will ensure that the sum of the last three terms of Eq. (4.14) divided by \(|v(\pi - s)|\), is no bigger than \(\delta(\mu) \times o(1)\) outside of \(\Delta\) defined by

\(\Delta = \Delta(\mu) = \{s \in [0, \pi/2] : |s - \pi/2 + \mu^2x_2(\pi/2)| \leq \delta \mu^2\}\).

We again neglect the integral over \(\Delta\), since

\[
\left| \int_{\Delta} \left\{ \frac{X(\mu, s)}{|X(\mu, s)|} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\} ds \right| \leq 8\mu^2\delta.
\]

We also apply the vector inequality Eq. (4.8) again; we obtain

\[
\int_{\pi/2 - \mu^2/\delta}^{\pi/2} \left\{ \frac{X(\mu, s)}{|X(\mu, s)|} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\} ds = \int_{[\pi/2 - \mu^2/\delta, \pi/2] \setminus \Delta} \left\{ \frac{v(\pi/2 - s)}{|v(\pi/2 - s)|} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\} ds + o(\mu^2).
\]

where the last \(o(\mu^2)\)-term is simply \(\delta(\mu) \times o(1) \times \mu^2/\delta(\mu)\) coming from the integral of the vector inequality, and from neglecting the integral over \(\Delta\). The integral on the right side of this last expression is done explicitly and then estimated as in the proof of the previous lemma, giving

\[
\int_{\pi/2 - \mu^2/\delta}^{\pi/2} \frac{X(\mu, s)}{|X(\mu, s)|} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} ds = \begin{pmatrix} \mu^2(x_2(\pi/2) - |x_2(\pi/2)|) + o(\mu^2) \\ \mu^2\ln(1/\delta)y_2(\pi/2) + O(\mu^2) \\ \mu^2\ln(1/\delta)z_2(\pi/2) + O(\mu^2) \end{pmatrix}.
\]

(4.15)
When \( s \in [0, \pi/2 - \mu^2/\delta] \), the cosine dominates both the numerator and denominator,

\[
\frac{\mu x_1(s) + \mu^2 x_2(s) + o(\mu^2)}{\cos(s)} = \frac{\mu (x_1(s) - x_1(\pi/2)) + \mu^2 x_2(s) + o(\mu^2)}{\cos(s)} = O(\delta^{1/2}) = o(1).
\]

For the \( x \)-component of the integral, we have

\[
\int_0^{\pi/2 - \mu^2/\delta} \left\{ \frac{X(\mu, s)}{|X(\mu, s)|} - 1 \right\} ds = -\frac{1}{2} \int_0^{\pi/2 - \mu^2/\delta} (\mu y_1(s) + O(\mu^2))^2 + (\mu z_1(s) + O(\mu^2))^2 \cos^2(s) (1 + o(1)) ds
\]

\[
= -\frac{\mu^2}{2} \int_0^{\pi/2} \frac{y_1^2(s) + z_1^2(s)}{\cos^2(s)} ds + o(\mu^2).
\]

(4.16)

In the last line we used Lemma [A.1] to see that \( y_1(s)/\cos(s) \) and \( z_1(s)/\cos(s) \) are square integrable over the entire interval \([0, \pi/2]\), so that extending the interval of integration introduces only an additional \( \mu^2 \times o(1) = o(\mu^2) \) error. For the \( y \)-component of the integral, we get that

\[
\int_0^{\pi/2 - \mu^2/\delta} \frac{Y(\mu, s)}{|X(\mu, s)|} ds = \mu \int_0^{\pi/2 - \mu^2/\delta} \frac{y_1(s) + O(\mu)}{\cos(s)} (1 + o(1)) ds
\]

\[
= \mu \int_0^{\pi/2} \frac{y_1(s)}{\cos(s)} ds + o(\mu)
\]

(4.17)

and a similar expression for the \( z \)-component, where again extension of the interval of integration introduces only an \( o(\mu) \) error. Collecting the results of Eqs. (4.15)-(4.17), we obtain

\[
\int_0^{\pi/2} \left( \frac{X(\mu, s)}{|X(\mu, s)|} - 1 \right) ds = \mu^2 \left\{ x_2(\pi/2) - |x_2(\pi/2)| - \frac{1}{2} \int_0^{\pi/2} \frac{y_1^2(s)}{\cos^2(s)} ds \right\} + o(\mu^2)
\]

and

\[
\int_0^{\pi/2} \frac{1}{|X(\mu, s)|} \left( \frac{Y(\mu, s)}{Z(\mu, s)} \right) ds = \mu \int_0^{\pi/2} \frac{1}{\cos(s)} \left( \frac{y_1(s)}{z_1(s)} \right) ds + o(\mu).
\]

To arrive at Eqs. (4.5) and (4.6), we repeat the computations on the interval \([-\pi/2, 0]\) and add the results. The claim for the interval \([\pi/2, 3\pi/2]\) follows by replacing \( X(s) \) with \(-X(s - \pi)\).
Appendix

A Eigenvalues of a Sturm-Liouville operator

We provide an overview of the spectral theory for the operator

$$K_g = -\frac{d^2}{ds^2} + g \sec^2(s)$$

on $[-\pi/2, \pi/2]$, with Dirichlet boundary conditions at the endpoints (cf. Methods of Theoretical Physics [8], P.M. Morse and H. Feshbach, Part I, p.388 and the discussion there of hypergeometric functions.) Here, $g$ is a constant. We first show that $K_g$ is bounded below for $g \geq -\frac{1}{4}$.

**Lemma A.1** Suppose that $w(t)$ is an $H^1$ function on $[0,a]$, vanishing at $t = 0$ and $t = a > 0$. Then

$$\frac{1}{4} \int_0^a \frac{w(s)^2}{s^2} ds \leq \int_0^a (w'(s))^2 ds.$$ 

**Proof.** By scale invariance, it suffices to consider the case $a = 1$. We have that

$$0 \leq \int_0^1 \left(w'(s) - \frac{w(s)}{2s}\right)^2 ds = \int_0^1 \left(\frac{dw}{ds}(s)\right)^2 ds - \frac{1}{2} \int_0^1 \frac{d/ds w^2(s)}{s} ds + \frac{1}{4} \int_0^1 \left(\frac{w(s)}{s}\right)^2 ds.$$ 

Integrating by parts in the second integral and collecting terms, we get

$$\frac{1}{4} \int_0^1 \left(\frac{w(s)}{s}\right)^2 ds \leq \int_0^1 (w'(s))^2 ds - \frac{w^2(s)}{s}\bigg|_{s=0}^{s=1}.$$ 

By assumption, $w(1) = 0$, and by Eq. (4.7), $w(t) = o(t^{1/2})$. The desired conclusion follows by taking $t \to 0$.

The lemma implies that $K_g$ is bounded below for $g \geq -1/4$, because

$$\inf_{s \in [0,2\pi]} \left\{ \frac{1}{(\pi/2 - s)^2} + \frac{1}{(3\pi/2 - s)^2} - \sec^2(s) \right\} > -\infty.$$ 

Furthermore, $K_g$ has compact resolvent when $g > -1/4$, since $K_g \geq -c_1(g)d^2/ds^2 - c_2(g)I$ for some constants $c_1(g), c_2(g) > 0$, and the positive operator $-d^2/ds^2$ has compact resolvent. Consequently, the spectrum of $K_g$ consists of a nondecreasing sequence of eigenvalues $\lambda_0 < \lambda_1 \leq \ldots$ with $\lambda_n \to \infty$. The ground state $\lambda_0$ is simple by a Perron-Frobenius argument.

To solve the eigenvalue-eigenvector equation

$$K_g w(s) = \lambda w(s),$$

we consider the Dirichlet problem

$$\frac{d^2}{ds^2} w(s) + \frac{g}{\sec^2(s)} w(s) = \frac{1}{s^2} \int_0^a \frac{(w')^2}{s} ds,$$

satisfying $w(0) = w(a) = 0$. By Lemma A.1, this problem has a nonnegative solution $w(s)$ by the Fredholm alternative, and we choose $w(s)$ to be the solution that vanishes at both endpoints. Moreover, $w(s)$ is nonnegative, and the normalizing condition $\int_0^a w(s)^2 ds = 1$ determines $w(s)$ up to a constant factor. Thus, the eigenfunctions $w(s)$ are orthonormal, and it is easy to see that the spectrum $\lambda_0 < \lambda_1 \leq \ldots$ consists of a nondecreasing sequence of eigenvalues.
one can write $w = \cos^a(s)\phi(s)$ with

$$a = \frac{1}{2} \left(1 + \sqrt{1 + 4g}\right) \quad (A.1)$$

and obtain a second order differential equation for $\phi$. A substitution $\xi = (1 + \sin(s))/2$ results in the hypergeometric equation for $\phi$ regarded now with a slight abuse of notation as a function of $\xi$

$$-\xi(1 - \xi)\frac{d^2\phi(\xi)}{d\xi^2} + 2(a + \frac{1}{2})(\xi - \frac{1}{2})\frac{d\phi(\xi)}{d\xi} + (a^2 - \lambda)\phi(\xi) = 0.$$

Expanding $\phi$ in a power series about $\xi = 0$, one obtains a hypergeometric series,

$$\phi(\xi) = \sum_{n=0}^{\infty} b_n \xi^n$$

with the coefficients $b_n$ satisfying a two-term recursion relation,

$$b_{n+1} = \frac{(n + a)^2 - \lambda}{(n + a + \frac{1}{2})(n + 1)} b_n;$$

(The indicial equation gives that the series indeed should begin with the $n = 0$ term. The other solution leads to a function $w$ which is not locally $H^1$ at $-\pi/2$, i.e., $\frac{dw}{ds}$ is not locally square-integrable there). One finds that

$$b_n = \frac{\Gamma(a + \frac{1}{2})}{\Gamma(r_1)\Gamma(r_2)} \times \frac{\Gamma(r_1 + n)\Gamma(r_2 + n)}{\Gamma(a + n + \frac{1}{2}) n!},$$

where $-r_1$ and $-r_2$ are the roots of the equation $n^2 + 2an + a^2 - \lambda = 0$. Via Stirling’s approximation, one can infer from the expression for the $b_n$’s that $b_n \sim n^{a-3/2}(1 + O(1/n))$ for $n$ large further implying that $\phi(\xi) \sim (1 - \xi)^{1/2-a}$ or that $w(s)$ would not be locally square integrable in a neighborhood of $s = \pi/2$. (Alternatively this conclusion can be arrived at through well-known integral representations for hypergeometric functions.) Thus $b_n$ must be eventually zero. It follows from the recursion relation that the eigenvalues $\lambda_n$ satisfy the quantization condition

$$\lambda_n = (n + a)^2, \quad n = 0, 1, .. \quad (A.2)$$

In particular, the ground state satisfies $\lambda_0 = a^2 \geq 1/4$ for all $g > -1/4$.

The function $\phi_n(\xi)$ corresponding to $\lambda_n$ is a polynomial of degree $n$. In fact, with the further transformation $z = 2\xi - 1$, the equation for $\phi_n$ as a function of $z$ is that of a Gegenbauer polynomial,

$$(z^2 - 1)\frac{d^2\phi_n(z)}{dz^2} + (2a + 1)z\frac{d\phi_n(z)}{dz} - (2an + n^2)\phi_n(z) = 0$$

with solution $\phi_n(z) = T_n^{a-\frac{1}{2}}(z)$, with well-known orthogonality and normalization properties. The resulting functions $\{w_n(s) = \cos^a(s)T_n^{a-\frac{1}{2}}(\sin(s))\}$ are complete.◼
Remark: (1) Recalling the relationship between the parameters $a$ and $g$ from Eq. (A.1), we see that Eq. (A.2) implies the lower bounds

$$\begin{cases} 
\lambda_0 > \frac{1}{4}, & \lambda_1 > 1, & g > -\frac{1}{4} \\
\lambda_0 > 1, & g > 0 . 
\end{cases} (A.3)$$

(2) When $g \leq -1/4$, the function $\cos^a(s)\phi(\frac{1+\sin(a)}{2})$ appearing in the change of variables is no longer locally in $H^1$ and the above construction of the eigenfunctions and eigenvalues does not apply. For $g = -1/4$ we have the sharp inequality

$$\frac{1}{4} \int_{-\pi/2}^{\pi/2} \sec^2(s) w^2(s) \, ds \leq \int_{-\pi/2}^{\pi/2} \left( \frac{dw(s)}{ds} \right)^2 \, ds - \frac{1}{4} \int_{-\pi/2}^{\pi/2} w^2(s) \, ds$$

for functions $w$ satisfying Dirichlet conditions at $\pm \pi/2$: Our above analysis gives this result with the $1/4$ on the left side replaced by $-g < 1/4$, and taking $g \downarrow -1/4$ completes the argument.

References

[1] R. D. Benguria and M. Loss, Connection between the Lieb-Thirring conjecture for Schrödinger operators and an isoperimetric problem for ovals on the plane. Contemporary Math. 362:53-61 (2004).

[2] A. Burchard and L. E. Thomas, On the Cauchy problem for a dynamical Euler’s elastica. Commun. Partial Diff. Equations 28:271-300 (2003).

[3] E. M. Harrell and M. Loss, On the Laplace operator penalized by mean curvature. Commun. Math. Phys. 195:643-650 (1998).

[4] E. M. Harrell, On the second eigenvalue of the Laplacian penalized by curvature. Differential Geom. Appl. 6:397-400 (1996).

[5] P. Exner, E. M. Harrell, and M. Loss, Optimal eigenvalues for some Laplacians and Schrödinger operators depending on curvature. Oper. Theory Adv. Appl. 108:47-58 (1999).

[6] T. Friedrich, A geometric estimate for a periodic Schrödinger operator. Colloq. Math. 83:209–216 (2000).

[7] E. H. Lieb and M. Loss, Analysis. Second edition, Graduate Studies in Mathematics 14, Providence, RI., American Mathematical Society (AMS), (2001).

[8] P. M. Morse and H. Feshbach, Methods of Theoretical Physics, Part I McGraw-Hill Book Company, Inc., New York, (1953). See page 388.