INTEGRAL GEOMETRY OF TENSOR FIELDS ON A CLASS OF NON-SIMPLE RIEMANNIAN MANIFOLDS

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Abstract. We study the geodesic X-ray transform $I_{\Gamma}$ of tensor fields on a compact Riemannian manifold $M$ with non-necessarily convex boundary and with possible conjugate points. We assume that $I_{\Gamma}$ is known for geodesics belonging to an open set $\Gamma$ with endpoints on the boundary. We prove generic $s$-injectivity and a stability estimate under some topological assumptions and under the condition that for any $(x, \xi) \in \mathcal{T}^* M$, there is a geodesic in $\Gamma$ through $x$ normal to $\xi$ without conjugate points.

1. Introduction and statement of the main results

Let $(M, \partial M)$ be a smooth compact manifold with boundary, and let $g \in C^k(M)$ be a Riemannian metric on it. We can always assume that $(M, \partial M)$ is equipped with a real analytic atlas, while $\partial M$ and $g$ may or may not be analytic. We define the geodesic X-ray transform $I$ of symmetric 2-tensor fields by

$$I f(\gamma) = \int_0^{l_\gamma} \langle f(\gamma(t)), \gamma^2(t) \rangle \, dt,$$

where $[0, l_\gamma] \ni t \mapsto \gamma$ is any geodesic with endpoints on $\partial M$ parameterized by its arc-length. Above, $\langle f, \theta^2 \rangle$ is the action of $f$ on the vector $\theta$, that in local coordinates is given by $f_{ij} \theta^i \theta^j$. The purpose of this work is to study the injectivity, up to potential fields, and stability estimates for $I$ restricted to certain subsets $\Gamma$ (that we call $I_{\Gamma}$), and for manifolds with possible conjugate points. We require however that the geodesics in $\Gamma$ do not have conjugate points. We also require that $\Gamma$ is an open sets of geodesics such that the collection of their conormal bundles covers $\mathcal{T}^* M$. This guarantees that $I_{\Gamma}$ resolves the singularities. The main results are injectivity up to a potential field and stability for generic metrics, and in particular for real analytic ones.

We are motivated here by the boundary rigidity problem: to recover $g$, up to an isometry leaving $\partial M$ fixed, from knowledge of the boundary distance function $\rho(x, y)$ for a subset of pairs $(x, y) \in \partial M \times \partial M$, see e.g., [MR, Shi, CDS, SU4, PU]. In presence of conjugate points, one should study instead the lens rigidity problem: a recovery of $g$ from its scattering relation restricted to a subset. Then $I_{\Gamma}$ is the linearization of those problems for an appropriate $\Gamma$. Since we want to trace the dependence of $I_{\Gamma}$ on perturbations of the metric, it is more convenient to work with open $\Gamma$’s that have dimension larger than $n$, if $n \geq 3$, making the linear inverse problem formally overdetermined. One can use the same method to study restrictions of $I$ on $n$ dimensional subvarieties but this is behind the scope of this work.

Any symmetric 2-tensor field $f$ can be written as an orthogonal sum of a solenoidal part $f^s$ and a potential one $dv$, where $v = 0$ on $\partial M$, and $d$ stands for the symmetric differential of the 1-form $v$, see Section 2. Then $I(dv)(\gamma) = 0$ for any geodesic $\gamma$ with endpoints on $\partial M$. We say that $I_{\Gamma}$ is $s$-injective, if $I_{\Gamma} f = 0$ implies $f = dv$ with $v = 0$ on $\partial M$, or, equivalently, $f = f^s$. This problem

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has been studied before for simple manifolds with boundary, i.e., under the assumption that \( \partial M \) is strictly convex, and there are no conjugate points in \( M \) (then \( M \) is diffeomorphic to a ball). The book [Sh1] contains the main results up to 1994 on the integral geometry problem considered in this paper. Some recent results include [Sh2], [Ch], [SU3], [D], [Pe], [SSU], [ShU]. In the two dimensional case, following the method used in [PU] to solve the boundary rigidity problem for simple 2D manifolds, injectivity of the solenoidal part of the tensor field of order two was proven in [Sh3]. In [SU4], we considered \( I \) on all geodesics and proved that the set of simple metrics on a fixed manifold for which \( I \) is s-injective is generic in \( C^k(M) \), \( k \gg 1 \). Previous results include s-injectivity for simple manifolds with curvature satisfying some explicit upper bounds [Sh1, Sh2, Pe]. A recent result by Dairbekov [D] proves s-injectivity for non-trapping manifolds (not-necessarily convex) satisfying similar bounds, that in particular prevent the existence of conjugate points.

Fix another compact manifold \( M_1 \) with boundary such that \( M_{1} \supset M \), where \( M_{1} \) stands for the interior of \( M_1 \). Such a manifold is easy to construct in local charts, then glued together.

**Definition 1.** We say that the \( C^k(M) \) (or analytic) metric \( g \) on \( M \) is regular, if \( g \) has a \( C^k \) (or analytic, respectively) extension on \( M_1 \), such that for any \((x, \xi) \in T^*M\) there exists \( \theta \in T_xM \setminus 0 \) with \((\xi, \theta) = 0\) such that there is a geodesic segment \( \gamma_{x, \theta} \) through \((x, \theta)\) such that

(a) the endpoints of \( \gamma_{x, \theta} \) are in \( M_{1} \setminus M \).

(b) there are no conjugate points on \( \gamma_{x, \theta} \).

Any geodesic satisfying (a), (b) is called a simple geodesic.

Note that we allow the geodesics in \( \Gamma \) to self-intersect.

Since we do not assume that \( M \) is convex, given \((x, \theta)\) there might be two or more geodesic segments \( \gamma_j \) issued from \((x, \theta)\) such that \( \gamma_j \cap M \) have different numbers of connected components. Some of them might be simple, others might be not. For example for a kidney-shaped domain and a fixed \((x, \theta)\) we may have such segments so that the intersection with \( M \) has only one, or two connected components. Depending on which point in \( T^*M \) we target to recover the singularities, we may need the first, or the second extension. So simple geodesic segments through some \( x \) (that we call simple geodesics through \( x \)) are uniquely determined by an initial point \( x \) and a direction \( \theta \) and its endpoints. In case of simple manifolds, the endpoints (of the only connected component in \( M \), unless the geodesics does not intersect \( M \)) are not needed, they are a function of \((x, \theta)\).

Another way to determine a simple geodesic is by parametrizing it with \((x, \eta) \in T(M_{1} \setminus M)\), such that \( \exp_x \eta \in M_{1} \setminus M \) then

\[
\gamma_{x, \eta} = \{\exp_x(t\eta), 0 \leq t \leq 1\}.
\]

This parametrization induces a topology on the set \( \Gamma \) of simple geodesics through points of \( M_{1} \).

**Definition 2.** The set \( \Gamma \) of geodesics is called complete, if

(a) \( \forall (x, \xi) \in T^*M \) there exists a simple geodesic \( \gamma \in \Gamma \) through \( x \) such that \( \dot{\gamma} \) is normal to \( \xi \) at \( x \).

(b) \( \Gamma \) is open.

In other words, a regular metric \( g \) is a metric for which a complete set of geodesics exists. Another way to express (a) is to say that

\[
N^*\Gamma := \{N^*\gamma; \gamma \in \Gamma\} \subset T^*M,
\]

where \( N^*\gamma \) stands for the conormal bundle of \( \gamma \).

We always assume that all tensor fields defined in \( M \) are extended as 0 to \( M_1 \setminus M \). Notice that \( f f \) does not change if we replace \( M \) by another manifold \( M_{1/2} \) close enough to \( M \) such that \( M \subset M_{1/2} \subset M_1 \) but keep \( f \) supported in \( M \). Therefore, assuming that \( M \) has an analytic
structure as before, we can always extend $M$ a bit to make the boundary analytic and this would keep $(M, \partial M, g)$ regular. Then $s$-injectivity in the extended $M$ would imply the same in the original $M$, see [SU4 Prop. 4.3]. So from now on, we will assume that $(M, \partial M)$ is analytic but $g$ does not need to be analytic. To define correctly a norm in $C^K(M)$, respectively $C^K(M_1)$, we fix a finite analytic atlas.

The motivation behind Definitions 1, 2 is the following: if $g$ is regular, and $\Gamma$ is any complete set of geodesics, we will show that $I_\Gamma f = 0$ implies that $f^s \in C^l(M)$, where $l = l(k) \to \infty$, as $k \to \infty$, in other words, the so restricted X-ray transform resolves the singularities.

The condition of $g$ being regular is an open one for $g \in C^K(M)$, i.e., it defines an open set. Any simple metric on $M$ is regular but the class of regular metrics is substantially larger if $\dim M \geq 3$ and allows manifolds not necessarily diffeomorphic to a ball. For regular metrics on $M$, we do not impose convexity assumptions on the boundary; conjugate points are allowed as far as the metric is regular; $M$ does not need to be non-trapping. In two dimensions, a regular metric can not have conjugate points in $M$ but the class is still larger than that of simple metrics because we do not require strong convexity of $\partial M$.

**Example 1.** To construct a manifold with a regular metric $g$ that has conjugate points, let us start with a manifold of dimension at least three with at least one pair of conjugate points $u$ and $v$ on a geodesic $[a, b] \ni t \mapsto \gamma(t)$. We assume that $\gamma$ is non-selfintersecting. Then we will construct $M$ as a tubular neighborhood of $\gamma$. For any $x_0 \in \gamma$, define $S_{x_0} = \exp_{x_0} \{ v; \langle v, \dot{\gamma}(x_0) \rangle = 0, |v| \leq \varepsilon \}$, and $M := \bigcup_{x_0 \in \gamma} S_{x_0}$ with $\varepsilon \ll 1$. Then there are no conjugate points along the geodesics that can be loosely described as those “almost perpendicular” to $\gamma$ but not necessarily intersecting $\gamma$; and the union of their conormal bundles covers $T^*M$. More precisely, fix $x \in M$, then $x \in S_{x_0}$ for some $x_0 \in \gamma$. Let $0 \neq \xi \in T^*_x M$. Then there exists $0 \neq v \in T_x M$ that is both tangent to $S_{x_0}$ and normal to $\xi$. The geodesic through $(x, v)$ is then a simple one for $\varepsilon \ll 1$, and the latter can be chosen in a uniform way independent of $x$. To obtain a smooth boundary, one can perturb $M$ so that the new manifold is still regular.

**Example 2.** This is similar to the example above but we consider a neighborhood of a periodic trajectory. Let $M = \{(x^1)^2 + (x^2)^2 \leq 1\} \times S^1$ be the interior of the torus in $\mathbb{R}^3$, with the flat metric $(dx^1)^2 + (dx^2)^2 + d\theta^2$, where $\theta$ is the natural coordinate on $S^1$ with period $2\pi$. All geodesics perpendicular to $\theta = \text{const.}$ are periodic. All geodesics perpendicular to them have lengths not exceeding 2 and their conormal bundles cover the entire $T^*M$ (to cover the boundary points, we do need to extend the geodesics in a neighborhood of $M$). Then $M$ is a regular manifold that is trapping, and one can easily show that a small enough perturbation of $M$ is also regular, and may still be trapping.

The examples above are partial cases of a more general one. Let $(M', \partial M')$ be a simple compact Riemannian manifold with boundary with $\dim M' \geq 2$, and let $M''$ be a Riemannian compact manifold with or without boundary. Let $M$ be a small enough perturbation of $M' \times M''$. Then $M$ is regular.

Let $g$ be a fixed regular metric on $M$. The property of $\gamma$ being simple is stable under small perturbations. The parametrization by $(x, \eta)$ as in (2) clearly has two more dimensions that what is needed to determine uniquely $\gamma|_M$. Indeed, a parallel transport of $(x, \eta)$ along $\gamma_{x, \eta}$, close enough to $x$, will not change $\gamma|_M$, similarly, we can replace $\eta$ by $(1 + \varepsilon)\eta$, $|\varepsilon| \ll 1$.

We assume throughout this paper that $M$ satisfies the following.

**Topological Condition:** Any path in $M$ connecting two boundary points is homotopic to a polygon $c_1 \cup \gamma_1 \cup c_2 \cup \gamma_2 \cup \cdots \cup \gamma_k \cup c_{k+1}$ with the properties:
(i) \( c_j \) are paths on \( \partial M \);
(ii) For any \( j \), \( \gamma_j = \tilde{c}_j|_M \) for some \( \tilde{c}_j \in \Gamma \); \( \gamma_j \) lie in \( M^{\text{int}} \) with the exception of its endpoints and is transversal to \( \partial M \) at both ends.

**Theorem 1.** Let \( g \) be an analytic, regular metric on \( M \). Let \( \Gamma \) be a complete complex of geodesics. Then \( I_\Gamma \) is s-injective.

The proof is based on using analytic pseudo-differential calculus, see [Sj, Tre]. This has been used before in integral geometry, see e.g., [BQ, Q], see also [SU4].

To formulate a stability estimate, we will parametrize the simple geodesics in a way that will remove the extra two dimensions. Let \( H_m \) be a finite collection of smooth hypersurfaces in \( M^{\text{int}} \). Let \( H_m \) be an open subset of \( \{ (z, \theta) \in SM; z \in H_m, \theta \not\in T_z H_m \} \), and let \( \pm l^\pm_m(z, \theta) \geq 0 \) be two continuous functions. Let \( \Gamma(H_m) \) be the set of geodesics

\[
\Gamma(H_m) = \left\{ \gamma_z,\theta(t); l^-_m(z, \theta) \leq t \leq l^+_m(z, \theta), (z, \theta) \in H_m \right\},
\]

that, depending on the context, is considered either as a family of curves, or as a point set. We also assume that each \( \gamma \in \Gamma(H_m) \) is a simple geodesic.

If \( g \) is simple, then one can take a single \( H = \partial M \) with \( l^- = 0 \) and an appropriate \( l^+(z, \theta) \). If \( g \) is regular only, and \( \Gamma \) is any complete set of geodesics, then any small enough neighborhood of a simple geodesic in \( \Gamma \) has the properties listed above and by a compactness argument on can choose a finite complete set of such \( \Gamma(H_m) \)'s, that is included in the original \( \Gamma \), see Lemma 1.

Given \( \mathcal{H} = \{ H_m \} \) as above, we consider an open set \( \mathcal{H}' = \{ H'_m \} \), such that \( H'_m \subset H_m \), and let \( \Gamma(\mathcal{H}'_m) \) be the associated set of geodesics defined as in (4), with the same \( l^*_m \). Set \( \Gamma(\mathcal{H}) = \cup \Gamma(\mathcal{H}') \), \( \Gamma(\mathcal{H}') = \Gamma(\mathcal{H}') \).

The restriction \( \gamma \in \Gamma(\mathcal{H}_m) \subset \Gamma(\mathcal{H}_m) \) can be modeled by introducing a weight function \( \alpha_m \) in \( \mathcal{H}_m \), such that \( \alpha_m = 1 \) on \( \mathcal{H}'_m \), and \( \alpha_m = 0 \) otherwise. More generally, we allow \( \alpha_m \) to be smooth but still supported in \( \mathcal{H}'_m \). We then write \( \alpha = \{ \alpha_m \} \), and we say that \( \alpha \in C^k(\mathcal{H}) \), if \( \alpha_m \in C^k(\mathcal{H}_m) \), \( \forall m \).

We consider \( I_{\alpha_m} = \alpha_m I \), or more precisely, in the coordinates \( (z, \theta) \in H_m \).

\[
I_{\alpha_m} f = \alpha_m(z, \theta) \int_0^{l^+_m(z, \theta)} \langle f(\gamma_z,\theta), \gamma_z^\theta \rangle \, dt, \quad (z, \theta) \in H_m.
\]

Next, we set

\[
I_\alpha = \{ I_{\alpha_m} \}, \quad N_{\alpha_m} = I_{\alpha_m}^* I_{\alpha_m} = I^* |\alpha_m|^2 I, \quad N_\alpha = \sum N_{\alpha_m},
\]

where the adjoint is taken w.r.t. the measure \( d\mu := |(\nu(z), \theta)| \, dS_z \, d\theta \) on \( \mathcal{H}_m \), \( dS_z \, d\theta \) being the induced measure on \( SM \), and \( \nu(z) \) being a unit normal to \( H_m \).

S-injectivity of \( N_\alpha \) is equivalent to s-injectivity for \( I_{\alpha_m} \), which in turn is equivalent to s-injectivity of \( I \) restricted to \( \text{supp} \alpha \), see Lemma 2. The space \( \tilde{H}^2 \) is defined in Section 2; see (5).

**Theorem 2.**

(a) Let \( g = g_0 \in C^k \), \( k \gg 1 \) be regular, and let \( \mathcal{H}' \subset \mathcal{H} \) be as above with \( \Gamma(\mathcal{H}') \) complete. Fix \( \alpha = \{ \alpha_m \} \in C^\infty \) with \( \mathcal{H}_m \subset \text{supp} \alpha_m \subset \mathcal{H}_m \). Then if \( I_\alpha \) is s-injective, we have

\[
\| I_\alpha f \|_{L^2(M)} \leq C \| N_\alpha f \|_{\tilde{H}^2(M_1)}.
\]

(b) Assume that \( \alpha = \alpha_g \) in (a) depends on \( g \in C^k \), so that \( C^k(M_1) \ni g \rightarrow C^l(\mathcal{H}) \ni \alpha_g \) is continuous with \( l \gg 1, k \gg 1 \). Assume that \( I_{g_0, \alpha_{g_0}} \) is s-injective. Then estimate (7) remains true for \( g \) in a small enough neighborhood of \( g_0 \) in \( C^k(M_1) \) with a uniform constant \( C > 0 \).
In particular, Theorem 2 proves a locally uniform stability estimate for the class of non-trapping manifolds considered in [D].

Theorems 1, 2 allow us to formulate generic uniqueness results. One of them is formulated below. Given a family of metrics \( \mathcal{G} \subset C^k(M_1) \), and \( U_g \subset T(M_1^{\text{int}} \setminus M) \), depending on the metric \( g \in \mathcal{G} \), we say that \( U_g \) depends continuously on \( g \), if for any \( g_0 \in \mathcal{G} \), and any compact \( K \subset U_{g_0}^{\text{int}} \), we have \( K \subset U_g^{\text{int}} \) for \( g \) in a small enough neighborhood of \( g_0 \) in \( C^k \). In the next theorem, we take \( U_g = \Gamma_g \), that is identified with the corresponding set of \((x, \eta)\) as in (2).

**Theorem 3.** Let \( \mathcal{G} \subset C^k(M_1) \) be an open set of regular metrics on \( M \), and let for each \( g \in \mathcal{G} \), \( \Gamma_g \) be a complete set of geodesics related to \( g \) and continuously depending on \( g \). Then for \( k \gg 0 \), there is an open and dense subset \( \mathcal{G}_s \) of \( \mathcal{G} \), such that the corresponding X-ray transform \( I_{\Gamma_g} \) is \( s \)-injective.

Of course, the set \( \mathcal{G}_s \) includes all real analytic metrics in \( \mathcal{G} \).

**Corollary 1.** Let \( \mathcal{R}(M) \) be the set of all regular \( C^k \) metrics on \( M \) equipped with the \( C^k(M_1) \) topology. Then for \( k \gg 1 \), the subset of metrics for which the X-ray transform \( I \) over all simple geodesics is \( s \)-injective, is open and dense in \( \mathcal{R}(M) \).

The results above extend the generic results in [SU4], see also [SU3], in several directions: the topology of \( M \) may not be trivial, we allow conjugate points but we use only geodesics without conjugate points; the boundary does not need to be convex; and we use incomplete data, i.e., we use integrals over subsets of geodesics only.

In Section 6, we discuss versions of those results for the X-ray transform of vector fields and functions, where the proofs can be simplified. Our results remain true for tensors of any order \( m \), the necessary modifications are addressed in the key points of our exposition. To keep the paper readable, we restrict ourselves to orders \( m = 2, 1, 0 \).

**2. Preliminaries**

We say that \( f \) is analytic in some subset \( U \) of an analytic manifold, not necessarily open, if \( f \) can be extended analytically to some open set containing \( U \). Then we write \( f \in \mathcal{A}(U) \). Let \( g \in \mathcal{C}^k(M) \), \( k \gg 2 \) or \( g \in \mathcal{A}(M) \) be a Riemannian metric in \( M \). We work with symmetric 2-tensors \( f = \{f_{ij}\} \) and with 1-tensors/differential forms \( v_j \) (the notation here and below is in any local coordinates). We use freely the Einstein summation convention and the convention for raising and lowering indices.

We think of \( f_{ij} \) and \( f^{ij} = f_{kl}g^{ki}g^{lj} \) as different representations of the same tensor. If \( \xi \) is a covector at \( x \), then its components are denoted by \( \xi_j \), while \( \xi^i \) is defined as \( \xi^i = g^{ij}\xi_j \). Next, we denote \( |\xi|^2 = \xi_i\xi^i \), similarly for vectors that we usually denote by \( \theta \). If \( \theta_1, \theta_2 \) are two vectors, then \( \langle \theta_1, \theta_2 \rangle \) is their inner product. If \( \xi \) is a covector, and \( \theta \) is a vector, then \( \langle \xi, \theta \rangle \) stands for \( \xi(\theta) \). This notation choice is partly justified by identifying \( \xi \) with a vector, as above.

The geodesics of \( g \) can be also viewed as the \( x \)-projections of the bicharacteristics of the Hamiltonian \( E_g(x, \xi) = \frac{1}{2}g^{ij}(x)\xi_i\xi_j \). The energy level \( E_g = 1/2 \) corresponds to parametrization with the arc-length parameter. For any geodesic \( \gamma \), we have \( f^{ij}(x)\xi_i\xi_j = f_{ij}(\gamma(x))\dot{x}^i(t)\dot{x}^j(t) \), where \( (x, \xi) = (x(t), \xi(t)) \) is the bicharacteristic with \( x \)-projection equal to \( \gamma \).

**2.1. Semigeodesic coordinates near a simple geodesic and boundary normal coordinates.** Let \( \gamma \) be a simple geodesic through \( x_0 = \gamma_{x_0, \theta_0}(0) \) in \( M_1 \) with \( \theta_0 \in S_{x_0}M_1 \). The map \( t\theta \mapsto \exp_{x_0}(t\theta) \) is a local diffeomorphism for \( \theta \) close enough to \( \theta_0 \) and \( t \in [t^-, t^+] \) by our simplicity assumption but may not be a global one, since \( \gamma_{x_0, \theta_0} \) may self-intersect. On the other hand, there can be finitely many intersections only and we can assume that each subsequent intersection happens on a different copy of \( M \). In other words, we think of
where, in local coordinates, \( \langle f, h \rangle = \int_M \langle f, \bar{h} \rangle (\det g)^{1/2} \, dx \),

where, in local coordinates, \( \langle f, h \rangle = f_{ij} \bar{h}^{ij} \). Similarly, we define the \( L^2 \) space of 1-tensors (vector fields, that we identify with 1-forms) and the \( L^2 \) space of functions in \( M \). Also, we will work in Sobolev \( H^s \) spaces of 2-tensors, 1-forms and functions. In order to keep the notation simple, we will use the same notation \( L^2 \) (or \( H^s \)) for all those spaces and it will be clear from the context which one we mean.

The space \( H^2(M_1) \) has the property that for each \( f \in H^1(M) \) (extended as zero outside \( M \)), we have \( Nf \in H^2(M_1) \). This is not true if we replace \( H^2(M_1) \) by \( H^2(M_1) \).

**Lemma 1.** Let \( \Gamma_g \) and \( \mathcal{G} \) be as in Theorem 3. Then for \( k \gg 1 \), for any \( g_0 \in \mathcal{G} \), there exist \( \mathcal{H}' = \{ \mathcal{H}'_m \} \in \mathcal{H} = \{ \mathcal{H}_m \} \) such that \( \Gamma(H) \in \Gamma_{g_0} \) and \( \mathcal{H}', \mathcal{H} \) satisfy the assumptions of Theorem 3.
Moreover, \( H' \) and \( H \) satisfy the assumptions of Theorem 2 for \( g \) in a small enough neighborhood of \( g_0 \) in \( C^k \).

**Proof.** Fix \( g_0 \in \mathcal{G} \) first. Given \((x_0, \xi_0) \in T^* M\), there is a simple geodesic \( \gamma : [-1, 1] \to M_1 \) in \( \Gamma g_0 \) through \( x_0 \) normal to \( \xi_0 \) at \( x_0 \). Choose a small enough hypersurface \( H \) through \( x_0 \) transversal to \( \gamma \in \Gamma g_0 \), and local coordinates near \( x_0 \) as in Section 2.1 above, so that \( x_0 = 0, H \) is given by \( x^n = 0, \dot{\gamma}(0) = (0, \ldots, 0, 1) \). Then one can set \( \mathcal{H}_0 = \{x; x^n = 0; |x'| < \varepsilon \} \times \{\theta; |\theta'| < \varepsilon \}, \) and \( \mathcal{H}'_0 \) is defined in the same way by replacing \( \varepsilon \) by \( \varepsilon/2 \). We define \( \Gamma(\mathcal{H}'_0) \) as in [11] with \( l^\pm(z, \theta) = l^\pm \). Then the properties required for \( \mathcal{H}_0 \), including the simplicity assumption are satisfied when \( 0 < \varepsilon \ll 1 \). Choose such an \( \varepsilon \), and replace it with a smaller one so that those properties are preserved under a small perturbation of \( g \). Any point in \( SM \) close enough to \((x_0, \xi_0)\) still has a geodesic in \( \Gamma(\mathcal{H}'_0) \) normal to it. By a compactness argument, one can find a finite number of \( \mathcal{H}'_m \) so that the corresponding \( \Gamma(\mathcal{H}'_m) = \cup \Gamma(\mathcal{H}'_m) \) is complete.

The continuity property of \( \Gamma g \) w.r.t. \( g \) guarantees that the construction above is stable under a small perturbation of \( g \).

Similarly to [SU3], one can see that the map \( I_{\alpha_m} : L^2(M) \to L^2(\mathcal{H}_m, \mu) \) defined by [5] is bounded, and therefore the normal operator \( N_{\alpha_m} \) defined in [6] is a well defined bounded operator on \( L^2(M) \). Applying the same argument to \( M_1 \), we see that \( N_{\alpha_m} : M \to M_1 \) is also bounded. By [SU3], at least when \( f \) is supported in the local chart near \( x_0 = 0 \) above, and \( x \) is close enough to \( x_0 \),

\[
[N_{\alpha_m} f]_{i;j'}(x) = \int_0^\infty \int_{S_{\varepsilon}} \left| \alpha_{m}(x, \theta) \right|^2 \theta^i \theta^{j'} f_{ij}(\gamma_{x,\theta}(t)) \dot{\gamma}^i_{x,\theta}(t) \dot{\gamma}^j_{x,\theta}(t) d\theta dt,
\]

where \( \left| \alpha_m(x, \theta) \right|^2 = |\tilde{\alpha}_m(x, \theta)|^2 + |\tilde{\alpha}_m(x, -\theta)|^2 \), and \( \tilde{\alpha}_m \) is the extension of \( \alpha_m \) as constant along the geodesic through \((x, \theta) \in \mathcal{H}_m \); and equal to 0 for all other points not covered by such geodesics. Formula (9) has an invariant meaning and holds without the restriction on supp \( f \). On the other hand, if supp \( f \) is small enough (but not necessarily near \( x_0 \)), \( y = \exp_p(t\theta) \) defines a local diffeomorphism \( t\theta \mapsto y \in \text{supp } f \), therefore after making the change of variables \( y = \exp_p(t\theta), \) see [SU3], this becomes

\[
N_{\alpha_m} f(x) = \frac{1}{\sqrt{\det g}} \int A_m(x, y) \frac{f_{ij}(y)}{\rho(x, y)^{n-1}} \frac{\partial \rho}{\partial y^i} \frac{\partial \rho}{\partial y^j} \frac{\partial \rho}{\partial x^k} \frac{\partial \rho}{\partial x^l} \det \frac{\partial^2 (\rho^2/2)}{\partial x \partial y} dy,
\]

where

\[
A_m(x, y) = \left| \alpha_m^*(x, \text{grad}_x \rho(x, y)) \right|^2,
\]

\( y \) are any local coordinates near \( \text{supp } f \) and \( \rho(x, y) = |\exp^{-1}_x y| \). Formula (46) can be also understood invariantly by considering \( \text{d}_x \rho \) and \( \text{d}_y \rho \) as tensors. For arbitrary \( f \in L^2(M) \) we use a partition of unity in \( TM_1^{\text{aff}} \) to express \( N_{\alpha_m} f(x) \) as a finite sum of integrals as above, for \( x \) near any fixed \( x_0 \).

We get in particular that \( N_{\alpha_m} \) has the pseudolocal property, i.e., its Schwartz kernel is smooth outside the diagonal. As we will show below, similarly to the analysis in [SU3] [SU4], \( N_{\alpha_m} \) is a \( \Psi^{-1} \)DO of order \(-1\).

We always extend functions or tensors defined in \( M \) as outside \( M \). Then \( N_{\alpha} f \) is well defined near \( M \) as well and remains unchanged if \( M \) is extended such that it is still in \( M_1 \), and \( f \) is kept fixed.
2.3. Decomposition of symmetric tensors. For more details about the decomposition below, we refer to [Sh1]. Given a symmetric 2-tensor $f = f_{ij}$, we define the 1-tensor $\delta f$ called divergence of $f$ by
\[ [\delta f]_i = g^{kj} \nabla_k f_{ij}, \]
in any local coordinates, where $\nabla_k$ are the covariant derivatives of the tensor $f$. Given an 1-tensor (a vector field or an 1-form) $v$, we denote by $dv$ the 2-tensor called symmetric differential of $v$:
\[ [dv]_{ij} = \frac{1}{2} (\nabla_i v_j + \nabla_j v_i). \]
Operators $d$ and $-\delta$ are formally adjoint to each other in $L^2(M)$. It is easy to see that for each smooth $v$ with $v = 0$ on $\partial M$, we have $I(dv)(\gamma) = 0$ for any geodesic $\gamma$ with endpoints on $\partial M$. This follows from the identity
\[ \frac{d}{dt} \langle v(\gamma(t)), \dot{\gamma}(t) \rangle = \langle dv(\gamma(t)), \dot{\gamma}^2(t) \rangle. \]
If $\alpha = \{\alpha_m\}$ is as in the Introduction, we get
\[ I_\alpha(dv) = 0, \quad \forall v \in C^1_0(M), \]
and this can be extended to $v \in H^1_0(M)$ by continuity.

It is known (see [Sh1] and (15) below) that for $g$ smooth enough, each symmetric tensor $f \in L^2(M)$ admits unique orthogonal decomposition $f = f^s + dv$ into a solenoidal tensor $Sf := f^s$ and a potential tensor $Pf := dv$, such that both terms are in $L^2(M)$, $f^s$ is solenoidal, i.e., $\delta f^s = 0$ in $M$, and $v \in H^1_0(M)$ (i.e., $v = 0$ on $\partial M$). In order to construct this decomposition, introduce the operator $\Delta^s = \delta d$ acting on vector fields. This operator is elliptic in $M$, and the Dirichlet problem satisfies the Lopatinskii condition. Denote by $\Delta^s_D$ the Dirichlet realization of $\Delta^s$ in $M$. Then
\[ v = (\Delta^s_D)^{-1} \delta f, \quad f^s = f - d (\Delta^s_D)^{-1} \delta f. \]
Therefore, we have
\[ P = d (\Delta^s_D)^{-1} \delta, \quad S = \text{Id} - P, \]
and for any $g \in C^1(M)$, the maps
\[ (\Delta^s_D)^{-1} : H^{-1}(M) \to H^1_0(M), \quad P, S : L^2(M) \to L^2(M) \]
are bounded and depend continuously on $g$, see [SU4] Lemma 1 that easily generalizes for manifolds. This admits the following easy generalization: for $s = 0, 1, \ldots$, the resolvent above also continuously maps $H^s$ into $H^{s+1} \cap H^1_0$, similarly, $P$ and $S$ are bounded in $H^s$, if $g \in C^k, k \gg 1$ (depending on $s$). Moreover those operators depend continuously on $g$.

Notice that even when $f$ is smooth and $f = 0$ on $\partial M$, then $f^s$ does not need to vanish on $\partial M$. In particular, $f^s$, extended as 0 to $M_1$, may not be solenoidal anymore. To stress on the dependence on the manifold, when needed, we will use the notation $v_M$ and $f_M^s$ as well.

Operators $S$ and $P$ are orthogonal projectors. The problem about the $s$-injectivity of $I_\alpha$ then can be posed as follows: if $I_\alpha f = 0$, show that $f^s = 0$, in other words, show that $I_\alpha$ is injective on the subspace $\mathcal{S}L^2$ of solenoidal tensors. Note that by (13) and (6),
\[ N_\alpha = N_\alpha S = SN_\alpha, \quad PN_\alpha = N_\alpha P = 0. \]

Lemma 2. Let $\alpha = \{\alpha_m\}$ with $\alpha_m \in C^\infty_0(\mathcal{H}_m)$ be as in the Introduction. The following statements are equivalent:
(a) $I_\alpha$ is $s$-injective on $L^2(M)$;
(b) $N_\alpha : L^2(M) \to L^2(M)$ is $s$-injective;
(c) $N_\alpha : L^2(M) \to L^2(M_1)$ is $s$-injective;
(d) If $\Gamma^\alpha_m$ is the set of geodesics issued from $(\text{supp } \alpha_m)^{\text{int}}$ as in (4), and $\Gamma^\alpha = \cup \Gamma^\alpha_m$, then $I_{\Gamma^\alpha}$ is s-injective.

Proof. Let $I_\alpha$ be s-injective, and assume that $N_\alpha f = 0$ in $M$ for some $f \in L^2(M)$. Then

$$0 = (N_\alpha f, f)_{L^2(M)} = \sum \|\alpha_m I f\|^2_{L^2(\mathcal{H}_m, d\mu)} \implies f^s = 0.$$ 

This proves the implication $(a) \Rightarrow (b)$. Next, $(b) \Rightarrow (c)$ is immediate. Assume $(c)$ and let $f \in L^2(M)$ be such that $I_\alpha f = 0$. Then $N_\alpha f = 0$ in $M_1$, therefore $f^s = 0$. Therefore, $(c) \Rightarrow (a)$. Finally, $(a) \Leftrightarrow (d)$ follows directly from the definition of $I_\alpha$. \hfill $\square$

Remark. Lemma 2 above, and Lemma 4(a) in next section show that $(\text{supp } \alpha_m)^{\text{int}}$ in (d) can be replaced by $\text{supp } \alpha_m$ if $\Gamma^\alpha$ is a complete set of geodesics.

3. Microlocal Parametrix of $N_\alpha$

**Proposition 1.** Let $g = g_0 \in C^k(M)$ be a regular metric on $M$, and let $\mathcal{H}' \subset \mathcal{H}$ be as in Theorem 2.

(a) Let $\alpha$ be as in Theorem 2(a). Then for any $t = 1, 2, \ldots$, there exists $k > 0$ and a bounded linear operator

$$Q : \tilde{H}^2(M_1) \mapsto SL^2(M),$$

such that

$$Q N_\alpha f = f^s_M + K f, \quad \forall f \in H^1(M),$$

where $K : H^1(M) \rightarrow \mathcal{S}H^{1+t}(M)$ extends to $K : L^2(M) \rightarrow \mathcal{S}H^t(M)$. If $t = \infty$, then $k = \infty$.

(b) Let $\alpha = \alpha_g$ be as in Theorem 2(b). Then, for $g$ in some $C^k$ neighborhood of $g_0$, (a) still holds and $Q$ can be constructed so that $K$ would depend continuously on $g$.

Proof. A brief sketch of our proof is the following: We construct first a parametrix that recovers microlocally $f^s_M$ from $N_\alpha f$. Next we will compose this parametrix with the operator $f^s_M \mapsto f^s_M$ as in [SU3]. Part (b) is based on a perturbation argument for the Fredholm equation (17). The need for such two step construction is due to the fact that in the definition of $f^s$, a solution to a certain boundary value problem is involved, therefore near $\partial M$, our construction is not just a parametrix of a certain elliptic $\Psi DO$. This is the reason for losing one derivative in (7). For tensors of orders 0 and 1, there is no such loss, see [SU3] and (61), (62).

As in [SU4], we will work with $\Psi DO$s with symbols of finite smoothness $k \geqslant 1$. All operations we are going to perform would require finitely many derivatives of the amplitude and finitely many seminorm estimates. In turn, this would be achieved if $g \in C^k$, $k \geqslant 1$ and the corresponding $\Psi DO$s will depend continuously on $g$.

Recall [SU3, SU4] that for simple metrics, $N$ is a $\Psi DO$ in $M^{\text{int}}$ of order $-1$ with principal symbol that is not elliptic but $N + [D]^{-1}P$ is elliptic. This is a consequence of the following. We will say that $N_\alpha$ (and any other $\Psi DO$ acting on symmetric tensors) is elliptic on solenoidal tensors, if for any $(x, \xi)$, $\xi \neq 0$, $\sigma_p(N_\alpha)^{ijkl}(x, \xi)f_{kl} = 0$ and $\xi^i f_{ij} = 0$ imply $f = 0$. Then $N$ is elliptic on solenoidal tensors, as shown in [SU3]. That definition is motivated by the fact that the principal symbol of $\delta$ is given by $f_{ij} \mapsto i\xi^i f_{ij}$, and s-injectivity is equivalent to the statement that $N f = 0$ and $\delta f = 0$ in $M$ imply $f = 0$. Note also that the principal symbol of $d$ is given by $\nu_j \mapsto (\xi_i \nu_j + \xi_j \nu_i)/2$, and $\sigma_p(N)$ vanishes on tensors represented by the r.h.s. of the latter. We will establish similar properties of $N_\alpha$ below.

Let $N_{\alpha_m}$ be as in Section 2.2 with $m$ fixed.
Lemma 3. \( N_{\alpha_m} \) is a classical \( \Psi DO \) of order \(-1\) in \( M^{\text{int}} \). It is elliptic on solenoidal tensors at \((x_0, \xi_0)\) if and only if there exists \( \theta_0 \in T_{x_0}M_1 \setminus 0 \) with \( \langle \xi_0, \theta_0 \rangle = 0 \) such that \( \alpha_0(x_0, \theta_0) \neq 0 \). The principal symbol \( \sigma_p(N_{\alpha_m}) \) vanishes on tensors of the kind \( f_{ij} = (\xi_1 v_i + \xi_2 v_i)/2 \) and is non-negative on tensors satisfying \( \xi'_{f_{ij}} = 0 \).

Proof. We established the pseudolocal property already, and formulas (9), (36) together with the partition of unity argument following them imply that it is enough to work with \( \alpha_0 \). Then \( M \) is an amplitude of order \( p \), positively homogeneous of order \( n \). Since \( M \) is an amplitude of order \( p \), positively homogeneous of order \( n \), \( M \) is the Fourier transform of a positively homogeneous distribution in the \( z \) variable, of order \( n-1 \). Therefore, \( M_{ijkl} \) itself is positively homogeneous of order \(-1\) in \( \xi \). Write

\[
M(x, y, \xi) = \int e^{-i\xi \cdot z} \left( G^{(1)}(z) \cdot z \right)^{\frac{n+1}{2}} \left| \alpha_m^z(x, g^{-1}G^{(2)} z) \right|^2 \times [G^{(2)} z]_i [G^{(2)} z]_j [G^{(2)} z]_k [G^{(2)} z]_l \det G^{(3)} \sqrt{det g(z)} \, dz,
\]

where \( G^{(2)}(x, y) = G^{(2)}(y, x) \). As in [SU4], we note that \( M_{ijkl} \) is the Fourier transform of a positively homogeneous distribution in the \( z \) variable, of order \( n-1 \). Therefore, \( M_{ijkl} \) itself is positively homogeneous of order \(-1\) in \( \xi \). Write

\[
M(x, y, \xi) = \int e^{-i\xi \cdot z} |z|^{-n+1} m(x, y, \theta) \, dz, \quad \theta = z/|z|,
\]

where

\[
m_{ijkl}(x, y, \theta) = \left( G^{(1)}(\theta) \cdot \theta \right)^{\frac{n+1}{2}} \left| \alpha_m^z(x, g^{-1}G^{(2)} \theta) \right|^2 \times [G^{(2)} \theta]_i [G^{(2)} \theta]_j [G^{(2)} \theta]_k [G^{(2)} \theta]_l \det G^{(3)} \sqrt{det g(\theta)} ,
\]

and pass to polar coordinates \( z = r\theta \). Since \( m \) is an even function of \( \theta \), smooth w.r.t. all variables, we get (see also [H, Theorem 7.1.24])

\[
M(x, y, \xi) = \pi \int_{|\theta|=1} m(x, y, \theta) \delta(\theta \cdot \xi) \, d\theta.
\]

This proves that \( M \) is an amplitude of order \(-1\).

To obtain the principal symbol, we set \( x = y \) above (see also [SU3, sec. 5] to get

\[
\sigma_p(N_{\alpha_m})(x, \xi) = M(x, x, \xi) = \pi \int_{|\theta|=1} m(x, x, \theta) \delta(\theta \cdot \xi) \, d\theta,
\]

where \( m(x, x, \theta) \) is the amplitude of order \(-1\) in \( \xi \).
where
\[ m^{ijkl}(x, x, \theta) = |\alpha_m^\sharp(x, \theta)|^2 \sqrt{\det g(x)} \left( g_{ij}(x) \theta^i \theta^j \right)^{-\frac{n+1}{2}} \theta^i \theta^j \theta^k \theta^l. \]

To prove ellipticity of \( M(x, \xi) \) on solenoidal tensors at \((x_0, \xi^0)\), notice that for any symmetric real \( f_{ij} \), we have
\[ m^{ijkl}(x_0, x_0, \theta) f_{ij} f_{kl} = |\alpha_m^\sharp(x_0, \theta)|^2 \sqrt{\det g(x_0)} \left( g_{ij}(x_0) \theta^i \theta^j \right)^{-\frac{n+1}{2}} \left( f_{ij} \theta^i \theta^j \right)^2 \geq 0. \]

This, \( (22) \), and the assumption \( \alpha_m(x_0, \theta_0) \neq 0 \) imply that \( M^{ijkl}(x_0, x_0, \xi^0) f_{ij} f_{kl} = 0 \) yields \( f_{ij} \theta^i \theta^j = 0 \) for \( \theta \) perpendicular to \( \xi^0 \), and close enough to \( \theta_0 \). If in addition \( (\xi^0)^t f_{ij} = 0 \), then this implies \( f_{ij} \theta^i \theta^j = 0 \) for \( \theta \in \text{neigh}(\theta_0) \), and that easily implies that it vanishes for all \( \theta \). Since \( f \) is symmetric, this means that \( f = 0 \).

The last statement of the lemma follows directly from \( (22), (23), (24) \).

Finally, we note that \( (23), (24) \) and the proof above generalizes easily for tensors of any order. \( \square \)

We continue with the proof of Proposition 1. Since (b) implies (a), we will prove (b) directly. Notice that \( \mathcal{H}' \) and \( \mathcal{H} \) satisfy the properties listed in the Introduction, right before Theorem 2 if \( g = g_0 \). On the other hand, those properties are stable under small \( C^\infty \) perturbation of \( g_0 \). We will work here with metrics \( g \) close enough to \( g_0 \).

By Lemma 3, since \( \Gamma(\mathcal{H}') \) is complete, \( N_\alpha \) defined by \( (6) \) is elliptic on solenoidal tensors in \( M \).

The rest of the proof is identical to that of \([SU4, Proposition 4]\). We will give a brief sketch of it. To use the ellipticity of \( N_\alpha \) on solenoidal tensors, we complete \( N_\alpha \) to an elliptic \( \Psi \)DO as in \([SU4]\).

Set
\[ W = N_\alpha + |D|^{-1} P_{M_1}, \]
where \( |D|^{-1} \) is a properly supported parametrix of \( (-\Delta_g)^{1/2} \) in \( \text{neigh}(M_1) \). The resolvent \( (-\Delta_{M_1,D})^{-1} \) involved in \( P_{M_1} \) and \( S_{M_1} \) can be expressed as \( R_1 + R_2 \), where \( R_1 \) is any parametrix near \( M_1 \), and \( R_2 : L^2_{\text{comp}}(M_1) \to C^l(M_1), R_2 : H^l(M_1) \to H^{l+2}(M_1) \), where \( l = l(k) \gg 1 \), if \( k \gg 1 \). Then \( W \) is an elliptic \( \Psi \)DO inside \( M_1 \) of order \(-1\) by Lemma 3.

Let \( P \) be a properly supported parametrix for \( W \) of finite order, i.e., \( P \) is a classical \( \Psi \)DO in the interior of \( M_1 \) of order 1 with amplitude of finite smoothness, such that
\[ PW = \text{Id} + K_1, \]
and \( K_1 : L^2_{\text{comp}}(M_1) \to H^l(M_1) \) with \( l \) as above. Then
\[ P_1 := S_{M_1} P \]
satisfies
\[ P_1 N_\alpha = S_{M_1} + K_2, \]
where \( K_2 \) has the same property as \( K_1 \). To see this, it is enough to apply \( S_{M_1} \) to the left and right of \( (26) \) and to use \( (10) \).

Next step is to construct an operator that recovers \( f_{ij}^\sharp \), given \( f_{ij}^M \), and to apply it to \( P_1 N_\alpha - K_2 \).

In order to do this, it is enough first to construct a map \( P_2 \) such that if \( f_{ij}^M \) and \( v_{M_1} \) are the solenoidal part and the potential, respectively, corresponding to \( f \in L^2(M) \) extended as zero to \( M_1 \setminus M \), then \( P_2 : f_{ij}^M \mapsto v_{M_1} |_{\partial M} \). This is done as in \([SU3]\) and \([SU4, Proposition 4]\). We also have
\[ P_2 P_1 : \tilde{H}^2(M_1) \to H^{1/2}(\partial M). \]

Then we showed in \([SU4, Proposition 4]\) that one can set
\[ Q = (\text{Id} + dRP_2) P_1, \]
where $R : h \mapsto u$ is the Poisson operator for the Dirichlet problem $\Delta u = 0$ in $M$, $u|_{\partial M} = h$.

As explained above, we work with finite asymptotic expansions that require finite number of derivatives on the amplitudes of our PDOs. On the other hand, these amplitudes depend continuously on $g \in C^k$, $k \gg 1$. As a result, all operators above depend continuously on $g \in C^k$, $k \gg 1$.

The first part of next lemma generalizes similar results in [SU3 Thm 2], [Ch] [SSU] to the present situation. The second part shows that $I_f f = 0$ implies that a certain $\tilde{f}$, with the same solenoidal projection, is flat at $\partial M$. This $\tilde{f}$ is defined by the property (29) below.

**Lemma 4.** Let $g \in C^k(M)$ be a regular metric, and let $\Gamma$ be a complete set of geodesics. Then
(a) $\ker I_\Gamma \cap \mathcal{S}L^2(M)$ is finite dimensional and included in $C^l(M)$ with $l = l(k) \rightarrow \infty$, as $k \rightarrow \infty$.
(b) If $I_f f = 0$ with $f \in L^2(M)$, then there exists a vector field $v \in C^l(M)$, with $v|_{\partial M} = 0$ and $l$ as above, such that for $\tilde{f} := f - dv$ we have
\begin{equation}
\partial^\alpha \tilde{f}|_{\partial M} = 0, \quad |\alpha| \leq l,
\end{equation}
and in boundary normal coordinates near any point on $\partial M$ we have
\begin{equation}
\tilde{f}_{ni} = 0, \quad \forall i.
\end{equation}

**Proof.** Part (a) follows directly from Proposition [1].

Without loss of generality, we may assume that $M_1$ is defined as $M_1 = \{x, \text{ dist}(x, M) \leq \epsilon\}$, with $\epsilon > 0$ small enough. By Proposition [1] applied to $M_1$,
\begin{equation}
f_\ast_{M_1} \in C^l(M_1),
\end{equation}
where $l \gg 1$, if $k \gg 1$.

Let $x = (x', x^n)$ be boundary normal coordinates in a neighborhood of some boundary point. We recall how to construct $v$ defined in $M$ so that (29) holds, see [SU2] for a similar argument for the non-linear boundary rigidity problem, and [E] [Sh2] [SU3] [SU4] for the present one. The condition $(f - dv)_{in} = 0$ is equivalent to
\begin{equation}
\nabla_n v_i + \nabla_i v_n = 2 f_{in}, \quad v|x^n=0 = 0, \quad i = 1, \ldots, n.
\end{equation}
Recall that $\nabla_i v_j = \partial_i v_j - \Gamma^k_{ij} v_k$, and that in those coordinates, $\Gamma^k_{nn} = \Gamma^n_{kn} = 0$. If $i = n$, then (31) reduces to $\nabla_n v_n = \partial_n v_n = f_{in}, \quad v_n = 0$ for $x^n = 0$; we solve this by integration over $0 \leq x^n \leq \epsilon \ll 1$; this gives us $v_n$. Next, we solve the remaining linear system of $n - 1$ equations for $i = 1, \ldots, n - 1$ that is of the form $\nabla_n v_i = 2 f_{in} - \nabla_i v_n$, or, equivalently,
\begin{equation}
\partial_n v_i - 2 \Gamma^n_{ni} v_\alpha = 2 f_{in} - \partial_i v_n, \quad v|x^n=0 = 0, \quad i = 1, \ldots, n - 1,
\end{equation}
(recall that $\alpha = 1, \ldots, n-1$). Clearly, if $g$ and $f$ are smooth enough near $\partial M$, then so is $v$. If we set $f = f^s$ above (they both belong to $\ker I_\Gamma$), then by (a) we get the statement about the smoothness of $v$. Since the condition (29) has an invariant meaning, this in fact defines a construction in some one-sided neighborhood of $\partial M$ in $M$. One can cut $v$ outside that neighborhood in a smooth way to define $v$ globally in $M$. We also note that this can be done for tensors of any order $m$, see [Sh2], then we have to solve consecutively $m$ ODEs.

Let $\tilde{f} = f - dv$, where $v$ is as above. Then $\tilde{f}$ satisfies (29), and let
\begin{equation}
\tilde{f}^s_{M_1} = \tilde{f} - d\tilde{v}_{M_1}
\end{equation}
be the solenoidal projection of $\tilde{f}$ in $M_1$. Recall that $\tilde{f}$, according to our convention, is extended as zero in $M_1 \setminus M$ that in principle, could create jumps across $\partial M$. Clearly, $\tilde{f}^s_{M_1} = f^s_{M_1}$ because $f - \tilde{f} = dv$ in $M$ with $v$ as in the previous paragraph, and this is also true in $M_1$ with $\tilde{f}, f$ and
v extended as zero (and then \( v = 0 \) on \( \partial M_1 \)). In (33), the l.h.s. is smooth in \( M_1 \) by (30), and \( \tilde{f} \) satisfies (29) even outside \( M \), where it is zero. Then one can get \( \tilde{v}_{M_1} \) by solving (31) with \( M \) replaced by \( M_1 \), and \( f \) there replaced by \( \tilde{f}_{M_1} \in C^l(M_1) \). Therefore, one gets that \( \tilde{v}_{M_1} \), and therefore \( \tilde{f} \), is smooth enough across \( \partial M \), if \( g \in C^k, k \gg 1 \), which proves (28).

One can give the following alternative proof of (28): Let \( N_\alpha \) be related to \( \Gamma \), as in Theorem 2. One can easily check that \( N_\alpha \), restricted to tensors satisfying (29), is elliptic for \( \xi_n \neq 0 \). Since \( N_\alpha \tilde{f} = 0 \) near \( M \), with \( \tilde{f} \) extended as zero, as above, we get that this extension cannot have conormal singularities across \( \partial M \). This implies (28), at least when \( g \in C^\infty \). The case of \( g \) of finite smoothness can be treated by using parametrices of finite order in the conormal singularities.

\[ \square \]

4. S-injectivity for analytic regular metrics

In this section, we prove Theorem 1. Let \( g \) be an analytic regular metrics in \( M \), and let \( M_1 \supset M \) be the manifold where \( g \) is extended analytically according to Definition 1. Recall that there is an analytic atlas in \( M \), so that \( g \) can be assumed to be analytic, too. In other words, in this section, \((M, \partial M, g)\) is a real analytic manifold with boundary.

We will show first that \( J f = 0 \) implies \( f^* \in \mathcal{A}(M) \). We start with interior analytic regularity. Below, \( WF_\Lambda(f) \) stands for the analytic wave front set of \( f \), see [Sj] [Tre].

**Proposition 2.** Let \( (x_0, \xi^0) \in T^* M \setminus 0 \), and let \( \gamma_0 \) be a fixed simple geodesic through \( x_0 \) normal to \( \xi^0 \). Let \( I f(\gamma) = 0 \) for some 2-tensor \( f \in L^2(M) \) and all \( \gamma \in \text{neigh}(\gamma_0) \). Let \( g \) be analytic in \( \text{neigh}(\gamma_0) \) and \( \delta f = 0 \) near \( x_0 \). Then

\[
(x_0, \xi^0) \notin WF_\Lambda(f).
\]

**Proof.** As explained in Section 2.1, without loss of generality, we can assume that \( \gamma_0 \) does not self-intersect. Let \( U \) be a tubular neighborhood of \( \gamma_0 \) with \( x = (x', x^n) \) analytic semigeodesic coordinates in it, as in the second paragraph of Section 2.1. We can assume that \( x_0 = 0 \), \( g_{ij}(0) = \delta_{ij} \), and \( x' = 0 \) on \( \gamma_0 \). In those coordinates, \( U \) is given by \( |x'| < \varepsilon, l^- < x^n < l^+ \), with some \( 0 < \varepsilon \ll 1 \), and we can choose \( \varepsilon \ll 1 \) so that \( \{x^n = l^\pm; |x'| \leq \varepsilon \} \) lie outside \( M \). Recall that the lines \( x' = \text{const.} \) in \( U \) are geodesics.

Then \( \xi^0 = ((\xi^0)'_0, 0) \) with \( \xi^n_0 = 0 \). We need to show that

\[
(0, \xi^0) \notin WF_\Lambda(f).
\]

We choose a local chart for the geodesics close to \( \gamma_0 \). Set first \( Z = \{x^n = 0; |x'| < 7\varepsilon/8 \} \), and denote the \( x' \)-variable on \( Z \) by \( z' \). Then \( z', \theta' \) with \( |\theta'| \ll 1 \) are local coordinates in \( \text{neigh}(\gamma_0) \) determined by \((z', \theta') \rightarrow \gamma((z', 0), (\theta', 1)) \). Each such geodesic is assumed to be defined on \( l^- \leq t \leq l^+ \), the same interval on which \( \gamma_0 \) is defined.

Let \( \chi_N(z'), N = 1, 2, \ldots, \) be a sequence of smooth cut-off functions equal to 1 for \( |z'| \leq 3\varepsilon/4 \), supported in \( Z \), and satisfying the estimates

\[
|\partial^\alpha \chi_N| \leq (CN)^{|\alpha|}, \quad |\alpha| \leq N,
\]

see [Tre] Lemma 1.1. Set \( \theta = (\theta', 1) \), \( |\theta'| \ll 1 \), and multiply

\[
I f(\gamma((z', 0), \theta)) = 0
\]

by \( \chi_N(z') e^{\lambda z' \cdot \xi'} \), where \( \lambda > 0 \), \( \xi' \) is in a complex neighborhood of \( (\xi^0)' \), and integrate w.r.t. \( z' \) to get

\[
\iint e^{\lambda z' \cdot \xi'} \chi_N(z') f_{ij} \left( \gamma((z', 0), \theta(t)) \right) \gamma_{ij}((z', 0), \theta(t))^2 \gamma((z', 0), \theta(t)) dt dz' = 0.
\]
For $|\theta'| \ll 1$, $(\zeta', t) \in Z \times (l^-, l^+)$ are local coordinates near $\gamma_0$ given by $x = \gamma_{(\zeta', 0), \theta}(t)$.

If $\theta' = 0$, we have $x = (\zeta', t)$. By a perturbation argument, for $\theta'$ fixed and small enough, $(t, \zeta')$ are analytic local coordinates, depending analytically on $\theta'$. In particular, $x = (\zeta' + t\theta', t) + O(|\theta'|)$ but this expansion is not enough for the analysis below. Performing a change of variables in (37), we get
\begin{equation}
\int e^{i\lambda(x, \theta') \xi} a_N (x, \theta') f_{ij} (x) b^i (x, \theta') b^j (x, \theta') \, dx = 0
\end{equation}
for $|\theta'| \ll 1$, $\forall \lambda$, $\forall \xi'$, where, for $|\theta'| \ll 1$, the function $(x, \theta') \mapsto a_N$ is analytic and positive for $x$ in a neighborhood of $\gamma_0$, vanishing for $x \not\in U$, and satisfying (36). The vector field $b$ is analytic on $\text{supp} a_N$, and $b(0, \theta') = 0$, $a_N (0, \theta') = 1$.

To clarify the arguments that follow, note that if $g$ is Euclidean in neigh($\gamma_0$), then (38) reduces to
\begin{equation}
\int e^{i\lambda(x, -\theta' \xi)} \chi_N f_{ij} (x) \delta^i (x) \delta^j (x) \, dx = 0,
\end{equation}
where $\chi_N = \chi_N (x' - x^\theta \theta')$. Then $\xi = (\xi', -\theta' \cdot \xi')$ is perpendicular to $\theta = (\theta', 1)$. This implies that
\begin{equation}
\int e^{i\lambda \xi \cdot \xi} \chi_N f_{ij} (x) \delta^i (\xi) \delta^j (\xi) \, dx = 0
\end{equation}
for any function $\theta(\xi)$ defined near $\xi^0$, such that $\theta(\xi) \cdot \xi = 0$. This has been noticed and used before if $g$ is close to the Euclidean metric (with $\chi_N = 1$), see e.g., [SU2]. We will assume that $\theta(\xi)$ is analytic. A simple argument (see e.g. [Sh1] [SU2]) shows that a constant symmetric tensor $f_{ij}$ is uniquely determined by the numbers $f_{ij} \delta^i \delta^j$ for finitely many $\xi$'s (actually, for $N' = (n + 1)n/2$ $\xi$'s); and in any open set on the unit sphere, there are such $\xi$'s. On the other hand, $f$ is solenoidal near $x_0$. To simplify the argument, assume for a moment that $f$ vanishes on $\partial M$ and is solenoidal everywhere. Then $\xi^i \hat{f}_{ij} (\xi) = 0$. Therefore, combining this with (39), we need to choose $N = n(n - 1)/2$ vectors $\theta(\xi)$, perpendicular to $\xi$, that would uniquely determine the tensor $\hat{f}$ on the plane perpendicular to $\xi$. To this end, it is enough to know that this choice can be made for $\xi = \xi^0$, then it would be true for $\xi \in \text{neigh}(\xi^0)$. This way, $\xi^i \hat{f}_{ij} (\xi) = 0$ and the $N$ equations (39) with the so chosen $\theta_p (\xi)$, $p = 1, \ldots, N$, form a system with a tensor-valued symbol elliptic near $\xi = \xi^0$. The $C^\infty$ $\Psi$DO calculus easily implies the statement of the lemma in the $C^\infty$ category, and the complex stationary phase method below, or the analytic $\Psi$DO calculus in [TrC] with appropriate cut-offs in $\xi$, implies the lemma in this special case ($g$ locally Euclidean).

We proceed with the proof in the general case. Since we will localize eventually near $x_0 = 0$, where $g$ is close to the Euclidean metric, the special case above serves as a useful guideline. On the other hand, we work near a “long geodesic” and the lack of points conjugate to $x_0 = 0$ along it will play a decisive role in order to allow us to localize near $x = 0$.

Let $\theta(\xi)$ be a vector analytically depending on $\xi$ near $\xi = \xi^0$, such that
\begin{equation}
\theta(\xi) \cdot \xi = 0, \quad \theta^0 (\xi) = 1, \quad \theta (\xi^0 ) = e_n.
\end{equation}
Here and below, $e_j$ stand for the vectors $\partial / \partial x^j$. Replace $\theta = (\theta', 1)$ in (38) by $\theta(\xi)$ (the requirement $|\theta'| \ll 1$ is fulfilled for $\xi$ close enough to $\xi^0$), to get
\begin{equation}
\int e^{i\lambda \varphi (x, \xi)} \tilde{a}_N (x, \xi) \tilde{f}_{ij} (x) \tilde{b}^i (x, \xi) \tilde{b}^j (x, \xi) \, dx = 0,
\end{equation}
where $\tilde{a}_N$ is analytic near $\gamma_0 \times \{ \xi^0 \}$, and satisfies (36) for $\xi$ close enough to $\xi^0$ and all $x$. Next, $\varphi$, $\tilde{b}$ are analytic on $\text{supp} \tilde{a}_N$ for $\xi$ close to $\xi^0$. In particular,
\begin{equation}
\tilde{b} = \tilde{\gamma}_{(\zeta', 0), (\theta(\xi), 1)} (t), \quad t = t(x, \theta(\xi)), \quad z' = z' (x, \theta(\xi)),
\end{equation}
and
\[ \tilde{b}(0, \xi) = \theta(\xi), \quad \tilde{a}_N(0, \xi) = 1. \]
The phase function is given by
\[ \varphi(x, \xi) = \theta'(x, \theta(\xi)) \cdot \xi'. \]
To verify that \( \varphi \) is a non-degenerate phase in \( \text{neigh}(0, \xi^0) \), i.e., that \( \det \varphi_{xxx}(0, \xi^0) \neq 0 \), note first that \( z' = x' \) when \( x_n = 0 \), therefore, \( (\partial z'/\partial x')(0, \theta(\xi)) = \text{Id} \). On the other hand, linearizing near \( x_n = 0 \), we easily get \( (\partial z'/\partial x_n)(0, \theta(\xi)) = -\theta'(\xi) \). Therefore,
\[ \varphi_x(0, \xi) = (\xi', -\theta'(\xi) \cdot \xi') = \xi \]
by (40). So we get \( \varphi_{xx}(0, \xi) = \text{Id} \), which proves the non-degeneracy claim above. In particular, we get that \( x \mapsto \varphi_x(x, \xi) \) is a local diffeomorphism in \( \text{neigh}(0) \) for \( \xi \in \text{neigh}(\xi^0) \), and therefore injective. We need however a semiglobal version of this along \( \gamma_0 \) as in the lemma below. For this reason we will make the following special choice of \( \theta(\xi) \). Without loss of generality we can assume that
\[ \xi^0 = e^{n-1}. \]
Set
\[ \theta(\xi) = \left( \xi_1, \ldots, \xi_{n-2}, -\frac{\xi_1^2 + \cdots + \xi_{n-2}^2 + \xi_n}{\xi_{n-1}}, 1 \right). \]
If \( n = 2 \), this reduces to \( \theta(\xi) = (-\xi_2/\xi_1, 1) \). Clearly, \( \theta(\xi) \) satisfies (40). Moreover, we have
\[ \frac{\partial \theta}{\partial \xi_\nu}(\xi^0) = e_\nu, \quad \nu = 1, \ldots, n-2, \quad \frac{\partial \theta}{\partial \xi_{n-1}}(\xi^0) = 0, \quad \frac{\partial \theta}{\partial \xi_n}(\xi^0) = -e_{n-1}, \]
In particular, the differential of the map \( S^{n-1} \ni \xi \mapsto \theta'(\xi) \) is invertible at \( \xi = \xi^0 = e^{n-1} \).

**Lemma 5.** Let \( \theta(\xi) \) be as in (43), and \( \varphi(x, \xi) \) be as in (42). Then there exists \( \delta > 0 \) such that if
\[ \varphi_x(x, \xi) = \varphi_x(y, \xi) \]
for some \( x \in U, |y| < \delta, |\xi - \xi^0| < \delta, \xi \) complex, then \( y = x \).

**Proof.** We will study first the case \( y = 0, \xi = \xi^0, x' = 0 \). Since \( \varphi_x(0, \xi^0) = 0 \), we need to show that \( \varphi_x((0, x_n), \xi^0) = 0 \) for \( (0, x_n) \in U \) (i.e., for \( -l < x_n < l^+ \)) implies \( x_n = 0 \).

To compute \( \varphi_x(x, \xi^0) \), we need first to know \( \partial z'(x, \theta')/\partial \theta' \) at \( \theta' = 0 \). Differentiate \( \gamma'(z', \theta'), \theta = \gamma(0, \theta) \), to get
\[ \frac{\partial \theta_v}{\partial \gamma'(x, \theta')} + \frac{\partial \gamma'(x, \theta')}{\partial \theta_v} \cdot \frac{\partial \gamma'(x, \theta')}{\partial \theta_v} = 0. \]
Plug \( \theta' = 0 \). Since \( \partial \theta/\partial \theta' = 0 \) at \( \theta' = 0 \), we get
\[ \frac{\partial z'}{\partial \theta_v} = -\frac{\partial \gamma'(x', \theta')}{\partial \theta_v} \bigg|_{(x', \theta') = (x_n, 0)} = -J'_{\nu}(x_n), \]
where the prime denotes the first \( n - 1 \) components, as usual; \( J_{\nu}(x_n) \) is the Jacobi field along the geodesic \( x_n \mapsto \gamma_0(x_n) \) with initial conditions \( J_{\nu}(0) = 0 \), \( DJ_{\nu}(0) = e_\nu \); and \( D \) stands for the covariant derivative along \( \gamma_0 \). Since \( z'((0, x_n), \theta'(\xi^0)) = 0 \), by (42) we then get
\[ \frac{\partial \gamma'(x, \xi^0)}{\partial \xi_l}((0, x_n), \xi^0) = -\frac{\partial \theta^\mu}{\partial \xi_l}(\xi^0) J_{\mu}(x_n) \cdot (\xi^0)' . \]
By (43), (recall that $\xi^0 = e^{n-1}$),
\begin{equation}
\frac{\partial \varphi}{\partial \xi_l}(0, x^n, \xi^0) = \begin{cases}
-J^n_{l}(x^n), & l = 1, \ldots, n-2, \\
0, & l = n-1, \\
J^n_{n-1}(x^n), & l = n,
\end{cases}
\end{equation}
where $J^n_{l}$ is the $(n-1)$-th component of $J^{n}_{l}$. Now, assuming that the l.h.s. of (45) vanishes for some fixed $x^n = t_0$, we get that $J^n_{l}(t_0) = 0$, $l = 1, \ldots, n-1$. On the other hand, $J^n_{l}$ are orthogonal to $e_n$ because the initial conditions $J^n_{0}(0) = 0$, $D J^n_{l}(0) = e_n$ are orthogonal to $e_n$, too. Since $g_n = \delta_{in}$, this means that $J^n_{n} = 0$. Therefore, $J^n_{l}(t_0)$, $l = 1, \ldots, n-1$, form a linearly dependent system of vectors, thus some non-trivial linear combination $a^* J^n_{l}(t_0)$ vanishes. Then the solution $J^n_{l}(t)$ of the Jacobi equation along $\gamma_0$ with initial conditions $J^n_{l}(0) = 0$, $D J^n_{l}(0) = a^* e_n$ satisfies $J^n_{l}(t) = 0$. Since $D J^n_{0}(0) \neq 0$, $J^n_{0}$ is not identically zero. Therefore, we get that $x_0 = 0$ and $x = (0, t_0)$ are conjugate points. Since $\gamma_0$ is a simple geodesic $x_0$, we must have $t_0 = 0 = x^n$.

The same proof applies if $x' \neq 0$ by shifting the $x'$ coordinates.

Let now $y$, $\xi$ and $x$ be as in the Lemma. The lemma is clearly true for $x$ in the ball $B(0, \varepsilon_1) = \{|x| < \varepsilon_1\}$, where $\varepsilon_1 \ll 1$, because $\varphi(0, \xi^0)$ is non-degenerate. On the other hand, $\varphi_{\xi}(x, \xi) \neq \varphi_{\xi}(y, \xi)$ for $x \in U \setminus B(0, \varepsilon_1)$, $y = 0$, $\xi = \xi^0$. Hence, we still have $\varphi_{\xi}(x, \xi) \neq \varphi_{\xi}(y, \xi)$ for a small perturbation of $y$ and $\xi$.

The arguments that follow are close to those in [KSU, Section 6]. We will apply the complex stationary phase method [S]. For $x$, $y$ as in Lemma [S] and $|\eta - \xi| \leq \delta/C$, $C \gg 2$, $\delta \ll 1$, multiply (41) by
\begin{equation}
\bar{\chi}(\xi - \eta) e^{i\lambda i(\xi - \eta^2/2 - \varphi(y, \xi))},
\end{equation}
where $\bar{\chi}$ is the characteristic function of the ball $B(0, \delta) \subset \mathbb{C}^n$, and integrate w.r.t. $\xi$ to get
\begin{equation}
\int e^{i\lambda \Phi(y, x, \eta, \xi)} \bar{a}_N(x, \xi, \eta) f_{ij}(x) \bar{b}^i(x, \xi, \eta) \bar{b}^j(x, \xi) \, \text{d}x \, \text{d}\xi = 0.
\end{equation}
Here $\bar{a}_N = \bar{\chi}(\xi - \eta) \bar{a}_N$ is another amplitude, analytic and elliptic for $x$ close to 0, $|\xi - \eta| < \delta/C$, and
\begin{equation}
\Phi = -\varphi(y, \xi) + \varphi(x, \xi) + \frac{i}{2}(\xi - \eta)^2.
\end{equation}
We study the critical points of $\xi \mapsto \Phi$. If $y = x$, there is a unique (real) critical point $\xi_c = \eta$, and it satisfies $\Im \Phi_{\xi \xi} > 0$ at $\xi = \xi_c$. For $y \neq x$, there is no real critical point by Lemma [S]. On the other hand, again by Lemma [S], there is no (complex) critical point if $|x - y| > \delta/C_1$ with some $C_1 > 0$, and there is a unique complex critical point $\xi_c$ if $|x - y| < \delta/C_2$, with some $C_2 > C_1$, still non-degenerate if $\delta \ll 1$. For any $C_0 > 0$, if we integrate in (46) for $|x - y| > \delta/C_0$, and use the fact that $|\Phi_{\xi}|$ has a positive lower bound (for $\xi$ real), we get
\begin{equation}
\int e^{i\lambda \Phi(y, x, \eta, \xi)} \bar{a}_N(x, \xi, \eta) f_{ij}(x) \bar{b}^i(x, \xi, \eta) \bar{b}^j(x, \xi) \, \text{d}x \, \text{d}\xi \leq C_2(C_3 N/\lambda) N + C N e^{-\lambda/C_0}.
\end{equation}
Estimate (47) is obtained by integrating $N$ times by parts, using the identity
\begin{equation}
L e^{i\lambda \Phi} = e^{i\lambda \Phi}, \quad L := \frac{\Phi_{\xi} \cdot \partial \xi}{i\lambda |\Phi_{\xi}|^2}
\end{equation}
as well as using the estimate (36), and the fact that $\Phi_{\xi}$ on the boundary of integration in $\xi$, the $e^{i\lambda \Phi}$ is exponentially small. Choose $C_0 \gg C_2$. Note that $\Im \Phi > 0$ for $\xi \in \partial(\text{supp} \bar{\chi}(\cdot - \eta))$, and $\eta$ as above, as long as $\dot{C} \gg 1$, and by choosing $C_0 \gg 1$, we can make sure that $\xi_c$ is as close to $\eta$, as we want.
To estimate (46) for \(|x - y| < \delta/C_0\), set
\[
\psi(x, y, \eta) := \Phi|_{\xi = \xi_c}.
\]
Note that \(\xi_c = -i(y - x) + \eta + O(\delta)\), and \(\psi(x, y, \eta) = \eta \cdot (x - y) + \frac{1}{2}|x - y|^2 + O(\delta)\). We will not use this to study the properties of \(\psi\), however. Instead, observe that at \(y = x\) we have
\[
\psi_y(x, x, \eta) = -\varphi_x(x, \eta), \quad \psi_x(x, x, \eta) = \varphi_x(x, \eta), \quad \psi(x, x, \eta) = 0.
\]
We also get that
\[
\exists \psi(y, x, \eta) \geq |x - y|^2/C.
\]

The latter can be obtained by setting \(h = y - x\) and expanding in powers of \(h\). The stationary complex phase method [Sj], see Theorem 2.8 there and the remark after it, gives
\[
\int_{|x - y| \leq \delta/C_0} e^{i\lambda \psi(x, \alpha)} f_{ij}(x) B^{ij}(x, \alpha; \lambda) \, dx = O \left( \lambda^{n/2} (C_3 N/\lambda)^N + N e^{-\lambda/C} \right), \quad \forall N,
\]
where \(\alpha = (y, \eta)\), and \(B\) is a classical analytic symbol [Sj] with principal part equal to \(\tilde{b} \otimes \tilde{b}\), up to an elliptic factor. The l.h.s. above is independent of \(N\), and choosing \(N\) so that \(N \leq \lambda/(C_3 \delta) \leq N + 1\) to conclude that the r.h.s. above is \(O(e^{-\lambda/C})\).

In preparation for applying the characterization of an analytic wave front set through a generalized FBI transform [Sj], define the transform
\[
\alpha \mapsto \beta = (\alpha_x, \nabla_{\alpha_x} \varphi(\alpha)),
\]
where, following [Sj], \(\alpha = (\alpha_x, \alpha_\xi)\). It is a diffeomorphism from \(\text{neigh}(0, \xi^0)\) to its image, and denote the inverse one by \(\alpha(\beta)\). Note that this map and its inverse preserve the first (\(n\)-dimensional) component and change only the second one. This is equivalent to setting \(\alpha = (y, \eta)\), \(\beta = (y, \zeta)\), where \(\zeta = \varphi_y(y, \eta)\). Note that \(\zeta = \eta + O(\delta)\), and at \(y = 0\), we have \(\zeta = \eta\).

Plug \(\alpha = \alpha(\beta)\) in (50) to get
\[
\int e^{i\lambda \psi(x, \beta)} f_{ij}(x) B^{ij}(x, \beta; \lambda) \, dx = O \left( e^{-\lambda/C} \right),
\]
where \(\psi\), \(B\) are (different) functions having the same properties as above. Then
\[
\psi_y(x, x, \xi) = -\zeta, \quad \psi_x(x, x, \xi) = \zeta, \quad \psi(x, x, \xi) = 0.
\]
The symbols in (51) satisfy
\[
\sigma_p(B)(0, 0, \zeta) \equiv \theta(\zeta) \otimes \theta(\zeta) \quad \text{up to an elliptic factor},
\]
and in particular, \(\sigma_p(B)(0, 0, \xi^0) \equiv e_n \otimes e_n\), where \(\sigma_p\) stands for the principal symbol.

Let \(\theta_1 = e_n , \theta_2, \ldots, \theta_N\) be \(N = n(n - 1)/2\) unit vectors at \(x_0 = 0\), normal to \(\xi^0 = e^{n-1}\) such that any constant symmetric 2-tensor \(f\) such that \(f^{n-1} = 0\), \(\forall i\) (i.e., \(f_i^l \xi^0_j = 0\)) is uniquely determined by \(f_{ij} \theta^i \theta^j, \theta = \theta_p, p = 1, \ldots, N\). Existence of such vectors is easy to establish, as mentioned above, and one can also see that such a set exists in any open set in \((\xi^0)^\perp\). We can therefore assume that \(\theta_p\) belong to a small enough neighborhood of \(\theta_1 = e_n\) such that the geodesics \([-t^-, t^+] \ni t \mapsto \gamma_0, \theta_p(t)\) through \(x_0 = 0\) are all simple. Then we can rotate a bit the coordinate system such that \(\xi^0 = e^{n-1}\) again, and \(\theta_p = e_n\), and repeat the construction above. This gives us \(N\) phase functions \(\psi_{(p)}\), and as many symbols \(B_{(p)}\) in (51) such that (52) holds for all of them, i.e., in the coordinate system related to \(\theta_1 = e_n\), we have
\[
\int e^{i\lambda \psi_{(p)}(x, \beta)} f_{ij}(x) B_{(p)}^{ij}(x, \beta; \lambda) \, dx = O \left( e^{-\lambda/C} \right), \quad p = 1, \ldots, N,
\]
and by \((54)\),
\[
\sigma_p(B_{(p)})(0,0,\xi_0) \equiv \theta_p \otimes \theta_p, \quad p = 1, \ldots, N.
\]

Recall that \(\delta f = 0\) near \(x_0 = 0\). Let \(\chi_0 = \chi_0(x)\) be a smooth cutoff close enough to \(x = 0\), equal to 1 in \(\text{neigh}(0)\). Integrate \(\frac{1}{\chi} \exp \left( i \lambda \psi_1(x, \beta) \right) \chi_0 \delta f = 0\) w.r.t. \(x\), and by \((49)\), after an integration by parts, we get
\[
\int e^{i \lambda \psi_1(x, \beta)} \chi_0(x) f_{ij}(x) C^j(x, \beta; \lambda) \, dx = O(e^{-\lambda/C}), \quad i = 1, \ldots, n,
\]
for \(\beta_x = y\) small enough, where \(\sigma_p(C^j)(0,0,\xi_0) = (\xi_0)^j\).

Now, the system of \(N + n = (n + 1)n/2\) equations \((51), (56)\) can be viewed as a tensor-valued operator applied to the tensor \(f\). Its symbol, an elliptic factor at \((0,0,\xi_0)\), has “rows” given by \(\theta_p^i \theta_p^j, p = 1, \ldots, N\); and \(\delta^i_k (\xi_0)^j, k = 1, \ldots, n\). It is easy to see that it is elliptic; indeed, the latter is equivalent to the statement that if for some (constant) symmetric 2-tensor \(f\), in Euclidean geometry (because \(g_{ij}(0) = \delta_{ij}\)), we have \(f_{ij} \theta_p^i \theta_p^j = 0, p = 1, \ldots, N\); and \(f_{ij}^{n+1} = 0, i = 1, \ldots, n\), then \(f = 0\). This however follows from the way we chose \(\theta_p\). Therefore, \((35)\) is a consequence of \((51), (56)\), see \([5]\) Definition 6.1. Note that in \([5]\), it is required that \(f\) must be replaced by \(\tilde{f}\) in \((51), (56)\). If \(f\) is complex-valued, we could use the fact that \(I(\Re f)(\gamma) = 0\), and \(I(\Im f)(\gamma) = 0\) for \(\gamma \nearrow 0\) and then work with real-valued \(f\)’s only.

Since the phase functions in \((51)\) depend on \(p\), we need to explain why the characterization of the analytic wave front sets in \([5]\) can be generalized to this vector-valued case. The needed modifications are as follows. We define \(h^{ij}_{(p)}(x, \beta; \lambda) = B^{ij}_{(p)}, p = 1, \ldots, N; \) and \(h^{ij}_{(N+k)}(x, \beta; \lambda) = C^j \delta_k^i, k = 1, \ldots, n\). Then \(\{h^{ij}_{(p)}\}, p = 1, \ldots, N + n\), is an elliptic symbol near \((0,0,\xi_0)\). In the proof of \([5]\) Prop. 6.2], under the conditions \((49), (52)\), the operator \(Q\) given by
\[
[Qf]_p(x, \lambda) = \int \int e^{i \lambda (\psi(x, \beta) - \psi(y, \beta))} f_{ij}(y, \lambda) h^{ij}_{(p)}(x, \beta; \lambda) \, dy \, d\beta
\]
is a \(\Psi DO\) in the complex domain with an elliptic matrix-valued symbol, where we view \(f\) and \(Qf\) as vectors in \(\mathbb{R}^{N+n}\). Therefore, it admits a parametrix in \(H_{\psi,x_0}\) with a suitable \(\psi\) (see \([5]\)). Hence, one can find an analytic classical matrix-valued symbol \(r(x, \beta, \lambda)\) defined near \((0,0,\xi_0)\), such that for any constant symmetric \(f\) we have
\[
\left[ Q \left( r(\cdot, \beta, \lambda) e^{i \lambda \psi_1 f} \right) \right]_p = e^{i \lambda \psi_1 f}, \quad \forall p.
\]
The rest of the proof is identical to that of \([5]\) Prop. 6.2] and allows us to show that \((51)\) is preserved with a different choice of the phase functions satisfying \((49), (52)\), and elliptic amplitudes; in particular,
\[
\int e^{i \lambda \psi_1(x, \beta)} \chi_2(x) f_{ij}(x) \, dx = O(e^{-\lambda/C}), \quad \forall i, j
\]
for \(\beta \in \text{neigh}(0,\xi_0)\) and for some standard cut-off \(\chi_2\) near \(x = 0\). This proves \((35)\), see \([5]\) Definition 6.1.

This concludes the proof of Proposition \((2)\). Notice that the proof works in the same way, if \(f\) is a distribution valued tensor field, supported in \(M\).

\textbf{Lemma 6.} Under the assumptions of Theorem \((4)\) let \(f\) be such that \(I f = 0\). Then \(f^s \in \mathcal{A}(M)\).

\textbf{Proof.} Proposition \((2)\) combined with the completeness of \(\Gamma\), imply that \(f^s\) is analytic in the interior of \(M\). To prove analyticity up to the boundary, we do the following.
We can assume that $M_1 \setminus M$ is defined by $-\varepsilon_1 \leq x^n \leq 0$, where $x^n$ is a boundary normal coordinate. Define the manifold $M_{1/2} \supset M$ by $x^n \geq -\varepsilon_1/2$, more precisely, $M_{1/2} = M \cup \{-\varepsilon_1/2 \leq x^n \leq 0\} \subset M_1$.

We will show first that $f^s_{M_{1/2}} \in \mathcal{A}(M_{1/2})$. Let us first notice, that in $M_{1/2} \setminus M$, $f^s_{M_{1/2}} = -dv_{M_{1/2}}$, where $v_{M_{1/2}}$ satisfies $\Delta v_{M_{1/2}} = 0$ in $M_{1/2} \setminus M$, $v|_{\partial M_{1/2}} = 0$. Therefore, $v_{M_{1/2}}$ is analytic up to $\partial M_{1/2}$ in $M_{1/2} \setminus M$, see [MN, SU4]. Therefore, we only need to show that $f^s_{M_{1/2}}$ is analytic in some neighborhood of $M$. This however follows from Proposition 2 applied to $M_{1/2}$. Note that if $\varepsilon_1 \ll 1$, simple geodesics through some $x \in M$ would have endpoints outside $M_{1/2}$ as well, and by a compactness argument, we need finitely many such geodesics to show that Proposition 2 implies that $f^s_{M_{1/2}}$ is analytic in, say, $M_{1/4}$, where the latter is defined similarly to $M_{1/2}$ by $x^n \geq -\varepsilon_1/4$. To compare $f^s_{M_{1/2}}$ and $f^s = f^s_M$, see also [SU3, SU4], write $f^s_{M_{1/2}} = f - dv_{M_{1/2}}$ in $M_{1/2}$, and $f^s = f - dv_M$ in $M$. Then $dv_{M_{1/2}} = -f^s_{M_{1/2}}$ in $M_{1/2} \setminus M$, and is therefore analytic there, up to $\partial M$. Given $x \in \partial M$, integrate $\langle dv_{M_{1/2}}, \dot{\gamma}^2 \rangle$ along geodesics in $M_{1/2} \setminus M$, close to ones normal to the boundary, with initial point $x$ and endpoints on $\partial M_{1/2}$. Then we get that $v_{M_{1/2}}|_{\partial M} \in \mathcal{A}(\partial M)$. Note that $v_{M_{1/2}} \in H^1$ near $\partial M$, and taking the trace on $\partial M$ is well defined, and moreover, if $x^n$ is a boundary normal coordinate, then $v_{M_{1/2}}(\cdot, x^n)$ is continuous. Now,

\begin{equation}
(57) \quad f^s_M = f - dv_M = f^s_{M_{1/2}} + dw \quad \text{in } M, \quad \text{where } w = v_{M_{1/2}} - v_M.
\end{equation}

The vector field $w$ solves

\begin{equation}
\Delta^a w = 0, \quad w|_{\partial M} = v_{M_{1/2}}|_{\partial M} \in \mathcal{A}(\partial M).
\end{equation}

Therefore, $w \in \mathcal{A}(M)$, and by (57), $f^s_M \in \mathcal{A}(M)$.

This completes the proof of Lemma 6. \hfill \square

**Proof of Theorem 1.** Let $I^f \neq 0$. We can assume first that $f = f^s$, and then $f \in \mathcal{A}(M)$ by Lemma 6. By Lemma 4 there exists $h \in S^{-1}Sf$ such that $\partial^a h = 0$ on $\partial M$ for all $\alpha$. The tensor field $h$ satisfies (29), i.e., $h_{ni} = 0$, $\forall i$, in boundary normal coordinates, which is achieved by setting $h = f - dv_0$, where $v_0$ solves (31) near $\partial M$. Then $v_0$, and therefore, $h$ is analytic for small $x^n \geq 0$, up to $x^n = 0$. Lemma 4 then implies that $h = 0$ in $\text{neigh}(\partial M)$. So we get that

\begin{equation}
(58) \quad f = dv_0, \quad 0 \leq x^n < \varepsilon_0, \quad \text{with } v_0|_{x^n=0} = 0,
\end{equation}

where $x^n$ is a global normal coordinate, and $0 < \varepsilon_0 \ll 1$. Note that the solution $v_0$ to (58) (if exists, and in this case we know it does) is unique, as can be easily seen by integrating $\langle f, \dot{\gamma}^2 \rangle$ along paths close to normal ones to $\partial M$ and using (12).

We show next that $v_0$ admits an analytic continuation from a neighborhood of any $x_1 \in \partial M$ along any path in $M$.

Fix $x \in M$. Let $c(t), 0 \leq t \leq 1$ be a path in $M$ such that $c(0) = x_0 \in \partial M$ and $c(1) = x$. Given $\varepsilon > 0$, one can find a polygon $x_0x_1\ldots x_kx$ consisting of geodesic segments of length not exceeding $\varepsilon$, that is close enough and therefore homotopic to $c$. One can also assume that the first one is transversal to $\partial M$, and if $x \in \partial M$, the last one is transversal to $\partial M$ as well; and all other points of the polygon are in $M^{\text{int}}$. We choose $\varepsilon \ll 1$ so that there are no conjugate points on each geodesic segment above. We also assume that $\varepsilon \leq \varepsilon_0$. Then $f = dv$ near $x_0x_1$ with $v = v_0$ by (58). As in the second paragraph of Section 2.3, one can choose semigeodesic coordinates $(x', x^n)$ near $x_1x_2$, and a small enough hypersurface $H_1$ through $x_1$ given locally by $x^n = 0$. As in Lemma 4 one can find an analytic 1-form $v_1$ defined near $x_1x_2$, so that $(f - dv_1)_m = 0$, $v_1|_{x^n=0} = v_0(x', 0)$. Close enough to $x_1$, we have $v_1 = v_0$ because $v_0$ is also a solution, and the solution is unique, see also (32). Since $v_1$ is analytic, we get that it is an analytic extension of $v_0$ along $x_1x_2$. Since $f$ and
Proof of Corollary 1. It is enough to notice that the set of all simple geodesics related to
\( g \) has the properties required for \( \alpha \) and \( \beta \) being close enough to \( g \). So we extended \( v_0 \) along \( x_0 x_1 x_2 \), let us call this extension \( v \). Then we do the same thing near \( x_2 x_3 \), etc.,
until we reach \( \text{neigh}(x) \), and then \( f = dv \) there.

This defines \( v \) in \( \text{neigh}(x) \), where \( x \in M \) was chosen arbitrary. It remains to show that this
definition is independent of the choice of the path. Choose another path that connects some
\( y_1 \in \partial M \) and \( x \). Combine them both to get a path that connects \( x_1 \in \partial M \) and \( y_1 \in \partial M \). It
suffices to prove that the analytic continuation of \( v_0 \) from \( x_1 \) to \( y_1 \) equals \( v_0 \) again. Let \( c_1 \cup \gamma_1 \cup c_2 \cup \gamma_2 \cup \cdots \cup \gamma_k \cup c_{k+1} \) be the polygon homotopic to the path above. Analytic continuation along \( c_1 \)
coincides with \( v_0 \) again by \( (58) \). Next, let \( p_1, p_2 \) be the initial and the endpoint of \( \gamma_1 \), respectively,
where \( p_1 \) is also the endpoint of \( c_1 \). We continue analytically \( v_0 \) from \( \text{neigh}(p_1) \) to \( \text{neigh}(p_2) \) along
\( \gamma_1 \), let us call this continuation \( v \). By what we showed above, \( f = dv \) near \( \gamma_1 \). Since \( I f(\gamma_1) = 0 \),
and \( v(p_1) = 0 \), we get by \( (12) \), that \( \langle v(p_2), \gamma_1(l) \rangle = 0 \) as well, where \( l \) is such \( \gamma_1(l) = p_2 \). Using
the assumption that \( \gamma_1 \) is transversal to \( \partial M \) at both ends, one can perturb the tangent vector
\( \gamma_1(l) \) and this will define a new geodesic through \( p_2 \) that hits \( \partial M \) transversely again near \( p_1 \), where
\( v = v_0 = 0 \). Since \( \Gamma \) is open, integral of \( f \) over this geodesic vanishes again, therefore \( \langle v(p_2), \xi_2 \rangle = 0 \)
for \( \xi_2 \) in an open set. Hence \( v(p_2) = 0 \). Choose \( q_2 \in \partial M \) close enough to \( p_2 \), and \( q_2 \) close enough
to \( \xi_2 \) (in a fixed chart). Then the geodesic through \( (q_2, q_2) \) will hit \( \partial M \) transversally close to \( p_1 \),
and we can repeat the same arguments. We therefore showed that \( v = 0 \) on \( \partial M \) near \( p_2 \). On the
other hand, \( v_0 \) has the same property. Since \( f = dv = dv_0 \) there, by the remark after \( (58) \), we get
that \( v = v_0 \) near \( p_2 \). We repeat this along all the legs of the polygon until we get that the analytic
continuation \( v \) of \( v_0 \) along the polygon, from \( x_1 \) to \( y_1 \), equals \( v_0 \) again.

As a consequence of this, we get that \( f = dv \) in \( M \) with \( v = 0 \) on \( \partial M \). Since \( f = f^s \), this implies
\( f = 0 \).

This completes the proof of Theorem 1.

5. Proof of Theorems 2 and 3

Proof of Theorem 2. Theorem 2(b), that also implies (a), is a consequence of Proposition 1 as shown in \( \text{SU}4 \), see the proof of Theorem 2 and Proposition 4 there. Part (a) only follows more
directly from \( \text{[Ta1]} \) Prop. V.3.1 and its generalization, see \( \text{SU3} \) Thm 2.

Proof of Theorem 3. First, note that for any analytic metric in \( G \), \( I_{\gamma_g} \) is s-injective by Theorem 1. We build \( G_s \) as a small enough neighborhood of the analytic metrics in \( G \). Then \( G_s \) is dense in \( G \)
in the \( C^k(M_1) \) topology since it includes the analytic metrics. To complete the definition of \( G_s \),
fix an analytic \( g_0 \in G \). By Lemma 1 one can find \( H' \subset H \) related to \( g = g_0 \) and \( \Gamma_g \), satisfying
the assumptions of Theorem 2 and they have the properties required for \( g \) close enough to \( g_0 \).

Let \( \alpha \) be as in Theorem 2 with \( \alpha = 1 \) on \( H' \). Then, by Theorem 2 \( I_{\alpha,g} \) is s-injective for \( g \)
close enough to \( g_0 \) in \( C^k(M_1) \). By Lemma 2 for any such \( g \), \( I_{\Gamma_g} \) is s-injective, where \( \Gamma^\alpha = \Gamma(H^\alpha) \),
\( H^\alpha = \text{supp} \alpha \). If \( g \) is close enough to \( g_0 \), \( \Gamma^\alpha \subset \Gamma_g \) because when \( g = g_0 \), \( \Gamma^\alpha \subset \Gamma(H) \subset \Gamma_g \),
and \( \Gamma_g \) depends continuously on \( g \) in the sense described before the formulation of Theorem 3.
Those arguments show that there is a neighborhood of each analytic \( g_0 \in G \) with an s-injective \( \Gamma_g \).
Therefore, one can choose an open dense subset \( G_s \) of \( G \) with the same property.

Proof of Corollary 1. It is enough to notice that the set of all simple geodesics related to \( g \) depends
continuously on \( g \) in the sense of Theorem 3. Then the proof follows from the paragraph above.
6. X-ray transform of functions and 1-forms/vector fields

If \( f \) is a vector field on \( M \), that we identify with a 1-form, then its X-ray transform is defined quite similarly to (1) by

\[
I_{\Gamma} f(\gamma) = \int_0^{l_\gamma} \langle f(\gamma(t)), \dot{\gamma}(t) \rangle \, dt, \quad \gamma \in \Gamma.
\]

If \( f \) is a function on \( M \), then we set

\[
I_{\Gamma} f(\gamma) = \int_0^{l_\gamma} f(\gamma(t)) \, dt, \quad \gamma \in \Gamma.
\]

The latter case is a partial case of the X-ray transform of 2-tensors; indeed, if \( f = \alpha g \), where \( f \) is a 2-tensor, \( \alpha \) is a function, and \( g \) is the metric, then \( I_{\Gamma} f = I_{\Gamma} \alpha \), where in the l.h.s., \( I_{\Gamma} \) is as in (1), and on the right, \( I_{\Gamma} \) is as in (60). The proofs for the X-ray transform of functions are simpler, however, and in particular, there is no loss of derivatives in the estimate (7), as in [SU3]. This is also true for the X-ray transform of vector fields and the proofs are more transparent than those for tensors of order 2 (or higher). Without going into details (see [SU3] for the case of simple manifolds), we note that the main theorems in the Introduction remain true. In case of 1-forms, estimate (7) can be improved to

\[
\| f \|_{L^2(M)} / C \leq \| \alpha f \|_{H^1(M)} \leq C \| f \|_{L^2(M)},
\]

while in case of functions, we have

\[
\| f \|_{L^2(M)} / C \leq \| \alpha f \|_{H^1(M)} \leq C \| f \|_{L^2(M)}.
\]

If \((M, \partial M)\) is simple, then the full X-ray transform of functions and 1-forms (over all geodesics) is injective, respectively s-injective, see [Mu2, MuR, EG, AR].
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