Physics of the interior of a spherical, charged black hole with a scalar field

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We analyse the physics of nonlinear gravitational processes inside a spherical charged black hole perturbed by a self-gravitating massless scalar field. For this purpose we created an appropriate numerical code. Throughout the paper, in addition to investigation of the properties of the mathematical singularities where some curvature scalars are equal to infinity, we analyse the properties of the physical singularities where the Kretschmann curvature scalar is equal to the planckian value. Using a homogeneous approximation we analyse the properties of the spacetime near a spacelike singularity in spacetimes influenced by different matter contents namely a scalar field, pressureless dust and matter with ultrarelativistic isotropic pressure. We also carry out full nonlinear analyses of the scalar field and geometry of spacetime inside black holes by means of an appropriate numerical code with adaptive mesh refinement capabilities. We use this code to investigate the nonlinear effects of gravitational focusing, mass inflation, matter squeeze, and these effects dependence on the initial boundary conditions. It is demonstrated that the position of the physical singularity inside a black hole is quite different from the positions of the mathematical singularities. In the case of the existence of a strong outgoing flux of the scalar field inside a black hole it is possible to have the existence of two null singularities and one central $r = 0$ singularity simultaneously.

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I. INTRODUCTION

The problems of the internal structure of black holes are a real great challenge and has been the subject of very active analytical and numerical researches during the last decades [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24]. There has been a great progress in these researches in the last few years and we now know many important properties of the realistic black hole’s interior, but some details and crucial problems are still the subject of much debate.

Many important results have been obtained under simplifying assumptions. One of the most widely used test toy models is a spherical, charged, non-rotating black hole, nonlinerly perturbed by a minimally coupled and self-gravitating massless, uncharged, scalar field. While this toy model is not very realistic, it share many properties, including causal structure, with the more realistic rotating black holes (e.g. [25] page 5 and [26]). There has been a great progress in these researches in the last few years and we now know many important properties of the realistic black hole’s interior, but some details and crucial problems are still the subject of much debate.

The purpose of this paper is to continue the analysis of the physical processes in the interior of black holes in the framework of this toy model.

Inside a black hole the main sights are the singularity. A number of rigorous theorems (see references in [27]) imply that singularities in the structure of spacetime develops inside black holes. Unfortunately these theorems tell us practically nothing about the locations and the nature of the singularities. It has been found that in principle two types of singularities can exist inside black holes corresponding to the toy model: A strong spacelike singularity and a weak null singularity (instead of the inner horizon of a Reissner-Nordström black hole). Probably both types of singularities can exist simultaneously in the same black hole and probably it is possible to have cases where only the strong spacelike singularity exists.

In the works [21] and references therein, some physical and geometrical properties of the singularities have been investigated. Numerical simulations of the fully nonlinear evolution of the scalar field and the geometry inside the spherical charged black hole has been carried out in [10, 14, 15, 25].

Near the strong spacelike singularity, one can use a homogeneous approximation in which it is supposed that temporal gradients are much greater then the spatial gradients. Hence it may be assumed that all processes near the singularity depend on the time coordinate only. This uniform approximation has been used in [25] (page 212) and [11, 26] to gain important new knowledge about the processes near the singularity. We will use the same homogeneous approximation to extend these analyses and clarify some fundamental physical processes near spacelike singularities under the influence of three different
matter contents, namely for the case of pressureless dust, a massless scalar field and matter with ultrarelativistic isotropic pressure. This investigation is done by means of a suitable numerical code which we develop for this purpose.

Subsequently we will study the nonlinear processes in large regions inside the toy model black hole, not just limited to the homogeneous approximation near the singularity. This will be done by using a stable and second order accurate, numerical code with adaptive mesh refinement capabilities. We will perturb the black hole with initial infalling scalar fields of different forms and strengths to further investigate the behaviour of the singularities and physical processes near them. We will also investigate the influence of outgoing scalar fluxes on the interior regions and the singularity. Such outgoing fluxes will unavoidably appear as a result of the scattering of ingoing scalar field flux by the curvature of the spacetime and will also be emitted from the surface of a star collapsing to a black hole.

Today it is widely believed that in the singularity of a realistic black hole, the curvature of the spacetime tends to infinity. Close to the singularity, where the curvature approaches the Planck value \( (\frac{\hbar c}{G})^2 \approx 1.5 \times 10^{131} \text{ cm}^{-4} \), classical General Relativity is not applicable. There is not yet a final version of the quantum theory of gravity, thus any extension of the discussion of physics in this region would be highly speculative and we will consider these regions as singularities from the classical point of view throughout the paper.

The paper is organised as follows; In section II we present our model of the spherically symmetric, charged black hole. In section III we discuss the mass function and some important nonlinear effects which are fundamental for understanding the physical processes inside black holes. In section IV we use a homogeneous approximation to analyse the (spacelike) singularity for three different matter contents: dust with zero pressure, a massless scalar field and matter with relativistic isotropic pressure. In section V we analyse the full nonlinear equations of the model from section II using a numerical approach. Finally we summarize our conclusions in section VI. Details of the numerical code used to obtain the results in section V and analysis of it are given in appendices A and B.

II. THE MODEL

We wish to study the geometry inside a spherically symmetric black hole with a fixed electrical charge \( q \) (i.e. Reissner-Nordström metric), which is nonlinearly perturbed by a selfgravitating, minimally coupled, massless scalar field. While astrophysical black holes are more likely to be described by the Kerr metric, it is believed that this toy model captures the essential physics, since the causal and horizon structures of the Reissner-Nordström and Kerr black holes are known to be very similar \( 30 \) (page 5). However, it is much simpler to make a numerical model of the toy model since this can be simulated in a two-dimensional spacetime. In constructing the toy model, we follow here the approach of Burko and Ori \( 11 \) who have done similar investigations.

A. Field equations

In spherical symmetry, the general line element in double null-coordinates can be written as:
\[
ds^2 = -2e^{2\sigma(u,v)}du dv + r^2(u,v)d\Omega^2
\]
where \( d\Omega^2 = d\theta^2 + \sin^2(\theta) d\phi^2 \) is the line element on the unit two-sphere and \( r \) is a function of the null coordinates \( u \) and \( v \) (in- and outgoing respectively).

With this metric the non-zero components of the Einstein tensor are:
\[
G_{uu} = \frac{4}{r} \sigma_u \sigma_u - 2 \sigma_{uu} \quad (2a)
\]
\[
G_{vv} = \frac{4}{r} \sigma_v \sigma_v - 2 \sigma_{vv} \quad (2b)
\]
\[
G_{uv} = \frac{e^{2\sigma}}{r} + 2 \sigma_u \sigma_v + 2 r \sigma_{uv} \quad (2c)
\]
\[
G_{\theta\theta} = -2e^{-2\sigma} r (\sigma_{uv} + r \sigma_{uv}) \quad (2d)
\]
\[
G_{\phi\phi} = -2e^{-2\sigma} \sin^2(\theta) (\sigma_{uv} + r \sigma_{uv}) \quad (2e)
\]

The energy-momentum tensor can be written as a sum of contributions from electromagnetic and scalar fields:
\[
T_{\mu\nu} = T_{\mu\nu}^s + T_{\mu\nu}^m \quad (3)
\]

The energy-momentum tensor of a massless scalar field \( \Phi \) is \( 30 \):
\[
T_{\mu\nu}^s = \frac{1}{4\pi} \left( \Phi_{,\mu} \Phi_{,\nu} - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \Phi_{,\alpha} \Phi_{,\beta} \right) \quad (4)
\]
whose non-zero components for the metric \( 4 \) are:
\[
T_{\mu\nu}^s = \frac{1}{4\pi} \Phi^2_{,\mu} \quad (5a)
\]
\[
T_{\nu\nu}^s = \frac{1}{4\pi} \Phi^2_{,\nu} \quad (5b)
\]
\[
T_{\theta\theta}^s = \frac{1}{4\pi} r^2 e^{-2\sigma} \Phi_{,\nu} \Phi_{,\nu} \quad (5c)
\]
\[
T_{\phi\phi}^s = \frac{1}{4\pi} r^2 \sin^2(\theta) e^{-2\sigma} \Phi_{,\nu} \Phi_{,\nu} \quad (5d)
\]

The energy-momentum tensor of an electric field in spherical symmetry and null coordinates is \( 30 \):
\[
T_{\mu\nu}^m = F_{\mu\alpha} F_{\nu}^{\alpha} + \frac{1}{4} g_{\mu\nu} F_{\mu\nu} F_{\mu\nu} \quad (6)
\]
whose non-zero components for the metric (1) are:

\[
\begin{align*}
T_{uu}^m &= \frac{q^2}{8\pi r^4}e^{2\sigma} \\
T_{vv}^m &= \frac{q^2}{8\pi r^4}r^2 \\
T_{\phi\phi}^m &= \frac{q^2}{8\pi r^4}r^2\sin^2(\theta)
\end{align*}
\]  

(7a) - (7c)

From the Einstein and energy-momentum tensors we can write up the Einstein equations, \(G_{\mu\nu} = 8\pi T_{\mu\nu}\) (with \(c = 1, G = 1\), governing the spacetime. The \(u - u, v - v, \ u - v \) and \(\theta - \theta\) components of the Einstein equations respectively are:

\[
\begin{align*}
\sigma_{uu} - 2r_u\sigma_u + r(\Phi_u)^2 &= 0 \\
\sigma_{vv} - 2r_v\sigma_v + r(\Phi_v)^2 &= 0 \\

\sigma_{uv} + \frac{r_{uv}}{r} + \frac{e^{2\sigma}}{2r^2} \left(1 - \frac{q^2}{r^2}\right) &= 0 \\
\sigma_{uv} &= \frac{r_{uv}}{r^2} - \frac{e^{2\sigma}}{2r^2} \left(1 - \frac{q^2}{r^2}\right) + \Phi_{,u}\Phi_{,v} = 0
\end{align*}
\]

(8) - (11)

Lastly, the scalar field must satisfy the Gordon-Klein equation (note that Gordon-Klein is a consequence of the Einstein equations for the scalar field (31), \(\nabla^\nu\nabla_\nu\Phi = 0\), which in the metric (1) becomes:

\[
\Phi_{,uv} + \frac{1}{r}(r_u\Phi_{,u} + r_v\Phi_{,v}) = 0
\]

(12)

Equations (10) - (12) are evolution equations which are supplemented by the two constraint equations (8) and (9). It is noted that none of these equations depends on the scalar field \(\Phi\) itself, but only on the derivatives of \(\Phi\), i.e. the derivative of the scalar field is a physical quantity, while the absolute value of the scalar field itself is not. Specifically we note the \(T_{uu} = (\Phi_u)^2/4\pi\) and \(T_{vv} = (\Phi_v)^2/4\pi\) components of the energy-momentum tensor which are part of the constraint equations. Physically \(T_{uu}\) and \(T_{vv}\) represents the flux of the scalar field through a surface of constant \(v\) and \(u\) respectively. These fluxes play an important role in our interpretation of the numerical results in section (S) and (G) on the initial segments. Consistency of the evolving fields with the constraint equations is then ensured via the contracted Bianchi identities (29), but we use the constraint equations throughout the domain of integration to check the accuracy of the numerical simulation.

On the initial null segments, the constraint equations reduces the number of unknowns by one on \(v = v_0\) and \(u = u_0\) respectively. The remaining two unknowns expresses only one degree of physical freedom, while the other unknown expresses the gauge freedom associated with the transformation \(u \to \tilde{u}(u), v \to \tilde{v}(v)\), (the line element (1) and eqs. (8)-(12) are invariant to such a transformation). We choose a standard gauge in which \(r\) is linear in \(v\) and \(u\) on the initial null segments. Specifically we choose:

\[
r(u_0, v) = v, \quad r(u, v_0) = r_0 + u r_{u0}
\]

(13)

We also choose that the outgoing segment should run along \(u_0 = 0\).

We can now use (8) and (9) to find:

\[
\sigma_{,u}(u_0, v) = \frac{1}{2r_v}\left(r_{vv} + r(\Phi_v)^2\right) = \frac{v}{2}(\Phi_v)^2
\]

(14a)

\[
\sigma_{,u}(u, v_0) = \frac{1}{2r_u}\left(r_{uu} + r(\Phi_u)^2\right) = \frac{r_0 + u}{2r_{u0}}(\Phi_u)^2
\]

(14b)

which can be readily integrated to find \(\sigma(u, v)\) on the initial null segments if \(\Phi\) and the constants \(r_0, r_{u0}\) and \(\sigma(u_0, v_0)\) are specified on these.

Following (12) we choose \(\sigma(u_0, v_0) = -\ln(2)\) and \(r_0 = 5\). The parameter \(r_{u0}\), can be related to the initial mass and charge of the black hole via the mass function (the mass function is further discussed in section (H) in which the metric (1) has the form:

\[
m(u, v) = \frac{r}{2}\left(1 + \frac{q^2}{r^2} + \frac{4r_{u0}r_v}{2e^{2\sigma}}\right)
\]

(15)

which in our choice of gauge, at the point of intersection of the initial null segments, take the form:

\[
m_0 = m(u_0, v_0) = \frac{r_0}{2}\left(1 + \frac{q^2}{r_0^2} + 4r_{u0}\right)
\]

(16)

hence \(r_{u0}\) can be determined by \(r_0, m_0\) and \(q\) as:

\[
r_{u0} = \frac{1}{4}\left(\frac{2}{r_0}\left(m_0 - \frac{q^2}{2r_0}\right) - 1\right).
\]

(17)

Hence, by specifying a distribution of the scalar field \(\Phi\) on the initial null segments, choosing a gauge and initial charge and mass of the black hole we can specify complete initial conditions on the initial null segments. Using a numerical code (described in appendix (A)) we can then use the evolution equations, eqs. (10) - (12) to evolve the unknown functions throughout the computational domain.

B. Initial value problem

We wish to numerically evolve the unknown functions \(r(u, v), \sigma(u, v)\) and \(\Phi(u, v)\) throughout some computational domain. We do this by following the approach of (10, 15, 29) to numerically integrate the three evolution equations (10) - (12). These equations form a well-posed initial value problem in which we can specify initial values of the unknowns on two initial null segments, namely an outgoing \((u = u_0 = \text{constant})\) and an ingoing \((v = v_0 = \text{constant})\) segment. We impose the constraint equation (S) and (G) on the initial segments.
III. NONLINEAR EFFECTS; INTERNAL MASS FUNCTION

We will in the next sections consider the evolution of the scalar field together with the geometry of the interior of a black hole. This evolution is highly nonlinear. One of the main parameters of this evolution is the mass function which represents the total effective mass in a sphere of radius \( r(u,v) \). We give here different expressions for the mass function, which emphasize its different characteristics.

In the metric:
\[
\begin{align*}
    ds^2 &= g_{tt} dt^2 + g_{rr}(r) dr^2 + r^2 d\Omega^2 \\
    d\Omega^2 &= d\theta^2 + \sin^2 \theta d\phi^2
\end{align*}
\]
the mass function can be written in the following forms:
\[
m = \frac{r}{2} \left( 1 + \frac{q^2}{r^2} - g_{rr}^{-1} \right)
\] (19)
or
\[
m = 4\pi \int_{r_1}^{r_2} T^t_t r^2 dr + m_0
\] (20)

In the metric
\[
ds^2 = -\alpha^2 dudv + r^2 d\Omega^2
\] (21)
it has the form \[16, 28\]:
\[
m = \frac{r}{2} \left( 1 + \frac{q^2}{r^2} + 4 \frac{ru\nu}{\alpha^2} \right)
\] (22)
or (for the scalar field, \( \Phi \)) \[10\]:
\[
m_{,uv} = 2 \frac{\alpha^3}{\alpha^2} \Phi_{,u} \Phi_{,v} - r \left( 1 - \frac{2m}{r} + \frac{q^2}{r^2} \right) \Phi_{,u} \Phi_{,v}
\] (23)

There are two important physical processes which can lead to a nonlinear change of the mass parameter:

1. The mass \( m \) inside a sphere can change because of the work of pressure forces on the surface of the sphere. A clear manifestation of this squeeze effect is the change of the mass of a spherical volume in a homogeneous model of the Universe filled with relativistic gas (see \[32\], page 13). In section IV we will consider another example, namely for the case of the imitation of the interior of a black hole. For the description of this process, it is most appropriate to use the form \[20\] for the mass function. Remember that inside the event horizon, \( r \) is a time-like coordinate.

2. Mass inflation \[2\]. This process inside the black hole exists if near the Cauchy horizon (CH) there are simultaneous ingoing and outgoing fluxes of a massless field (for example scalar field). Actually the existence of the outgoing flux together with the ingoing is inevitable because of backscattering of part of the ingoing flux by the spacetime curvature. The simplest exact model of the mass inflation process inside of a black hole has been constructed by Ori \[3\]. For the description of this process it is most useful to use formula \[23\] from which it can be seen that evolution of \( m \) with both \( u \) and \( v \) is possible only if there are both \( \Phi_{,u} \) and \( \Phi_{,v} \) fluxes simultaneously.

One can often observe the simultaneous manifestation of both these processes.

Another important nonlinear effect is the focusing effect by the gravity of beams of opposite fluxes of radiation. A particular manifestation of the focusing effect is the contraction of the CH under the gravity of transverse irradiation by the outgoing radiation. Eventually the CH singularity shrinks down to a point-like size and meets a central (probably spacelike) singularity. This process inside of a black hole has been constructed by Ori \[3\]. For the description of this process it is most useful to use formula \[23\] from which it can be seen that evolution of \( m \) with both \( u \) and \( v \) is possible only if there are both \( \Phi_{,u} \) and \( \Phi_{,v} \) fluxes simultaneously.

IV. HOMOGENEOUS APPROXIMATION

In the close vicinity of the spacelike singularity of a black hole all processes, as a rule, have high temporal gradients, much higher than the spatial gradients along the singularity, and the processes depend on a very restricted space region. So for clarification of some physical processes one can use a homogeneous approximation and assume that all processes depend on the time coordinate only. This approach has been proposed by Burko \[4, 26\] and we will use it at the beginning of our analysis to clarify some main properties of the singularity before coming to the full analysis of the spherical model in section V.

1. Leading order analysis

The general homogeneous, spherically symmetric line element has the form:
\[
\begin{align*}
    ds^2 &= g_{tt}(r) dt^2 + g_{rr}(r) dr^2 + r^2 d\Omega^2 \\
    d\Omega^2 &= d\theta^2 + \sin^2 \theta d\phi^2
\end{align*}
\] (24)
Inside a black hole in the region between the event horizon and the Cauchy horizon (or the spacelike singularity) $r$ is timelike and $t$ is spacelike. To describe the contraction of the CH, we should thus consider the variation of the time coordinate $r$ from bigger to smaller values. The $r-r, t-t$ and $\theta-\theta$ components of the Einstein equations (with $c = 1, G = 1$) are then given by:

\[
\begin{align*}
\frac{g_{tt} - g_{rr}g_{tt} + r g_{tr}^2}{r^2 g_{rr}g_{tt}} &= 8\pi (T_r^r + E_r^r) \quad (25) \\
\frac{g_{rr} - g_{rr}^2 - r g_{tr}^2}{r^2 g_{rr}} &= 8\pi (T_t^t + E_t^t) \quad (26)
\end{align*}
\]

where the primes denotes differentiation with respect to $r$ (the $\phi-\phi$ component of the Einstein equations again yields equation (27)). The tensor $E$ represents here contribution from a free electric field corresponding to a charge $q$, which we will assume to be constant:

\[
E_r^r = E_t^t = -E_\theta^\theta = -\frac{q^2}{8\pi r^2}, \quad (28)
\]

while the tensor $T$ represent contributions from other matter contents.

To clarify the meaning of different processes we will consider three different physical matter contents (in addition to the electric field) with different equations of state. Namely, we will consider:

A) Dust

B) A massless scalar field

C) Ultrarelativistic gas

From the Einstein equations one can find the following expressions for the non-zero components of $T$ for these matter contents:

A) Dust (with pressure $P = 0$):

\[
\begin{align*}
T_r^r &= -\epsilon = \epsilon_0 \left(\frac{g_{tt,init}}{g_{tt}}\right)^{\frac{1}{2}} \left(\frac{r_{init}}{r}\right)^2 \quad (29)
\end{align*}
\]

where $\epsilon_0, g_{tt,init}$ and $r_{init}$ are constants.

B) Massless scalar field $q$:

\[
\begin{align*}
T_r^r &= -\epsilon \quad (30a) \\
T_t^t &= \epsilon \quad (30b) \\
T_\theta^\theta &= \frac{\epsilon}{g_{rr}} \quad (30c)
\end{align*}
\]

\[
\begin{align*}
\epsilon &= \epsilon_0 \left(\frac{g_{tt,init}}{g_{tt}}\right) \left(\frac{r_{init}}{r}\right)^4 = -\frac{1}{8\pi g_{rr}} \left(\frac{d\phi}{dr}\right)^2 \quad (30d)
\end{align*}
\]

\[
\epsilon_0 = \frac{d^2}{8\pi g_{tt,init}^2} \quad (30e)
\]

where $d$ is a constant which were used in $q$.

C) Ultrarelativistic gas (with isotropic pressure $P = \frac{\epsilon}{3}$, $\epsilon$ being matter density):

\[
\begin{align*}
T_r^r &= -\epsilon \quad (31a) \\
T_t^t &= T_\theta^\theta = \frac{\epsilon}{3} \quad (31b) \\
\epsilon &= \epsilon_0 \left(\frac{g_{tt,init}}{g_{tt}}\right)^{\frac{1}{2}} \left(\frac{r_{init}}{r}\right)^{\frac{1}{2}} \quad (31c)
\end{align*}
\]

Substitution of (28) into (25) and (26) enables us to find the unknown functions $g_{rr} = g_{rr}(r)$ and $g_{tt} = g_{tt}(r)$ and hence solve the problem for each of the three cases. Equation (27) can be used as a control of the calculations.

Formally metric (22) corresponds to a metric of a special class of the “homogeneous cosmological models” considered by Zeldovich and Novikov [32] (page 535), Grishchuk [34] and others.

A. Dust, $P = 0$

Let us start the discussion with the simplest case, namely the case of dust with pressure $P = 0$.

In this case it is possible to have a singularity at $r = r_{sing} \neq 0$ with $r_{CH} < r_{sing} < r_{EH}$, where $r_{CH}$ and $r_{EH}$ are the positions of the Cauchy Horizon and the Event Horizon in the absence of dust.

Let us consider the leading order terms in a series expansion for the metric functions and leading order terms in the Einstein equations, near the singularity. Close to the singularity, where $g_{tt} \to 0$, a leading order expansion of eqs. (28) gives:

\[
\frac{dg_{tt}}{dx} = \frac{1}{2} g_{tt}^{-1} = -8\pi \left(\frac{g_{rr}(r_{sing})}{g_{tt,init}}\right)^{\frac{1}{2}} \left(\frac{r_{sing}}{r_{init}}\right)^{2} \quad (32)
\]

where ($g_{rr})_{sing} = g_{rr}(r_{sing})$ and $g_{tt,init} = g_{tt,init}(r_{init})$ and where we assume that $g_{tt} = Ax^{\alpha}$ and $x = r - r_{sing}$. Also $\alpha$ are constants and ($g_{rr})_{sing} = g_{rr}(r_{sing})$ is the value of $g_{rr}$ at the singularity $r = r_{sing}$.

From (28) one find:

\[
\alpha = 2 \quad (33)
\]

which leads in turn to (remember that ($g_{rr})_{sing} = g_{rr}(r_{sing})$ is negative for $r_{CH} < r < r_{EH}$):

\[
r_{sing} = -\frac{4\pi}{2} \left(\frac{g_{rr}(r_{sing})}{g_{tt,init}}\right)^{\frac{1}{2}} \left(\frac{r_{sing}}{r_{init}}\right)^{2} \quad (34)
\]

Using the proper time $\tau : d\tau = \sqrt{|g_{rr}|} dr$ we have for the vicinity of the singularity:

\[
\begin{align*}
g_{tt} &\propto \tau^{2}, \\
r &\propto \tau^{\alpha} = const., \\
\tau &= 0 \text{ at the singularity} \quad (35)
\end{align*}
\]
For the Kretschmann scalar $K \equiv R_{iklm}R^{iklm}$ we have

$$K = \frac{12}{r^4}$$

(36)

Thus this spacelike singularity does not correspond to $r = 0$

1. Numerical analysis

To understand the behaviours of the model as functions of the parameters of the model we perform use a simple numerical code to numerically solve (25)+(26) substituting $\rho_{\text{m}} + \rho_{\text{q}}$ for the stress-energy tensor. According to the remark in the introduction we consider the region with $K = K_{\text{planck}}$ as a physical singularity and will consider only the region with $r > r_c$, where $r_c$ corresponds to the critical value of $r$ at which the Kretschmann scalar is equal to the planckian value. We will take this restriction into account in all our subsequent analyses.

We note that for the case of dust $P = 0$ there are no nonlinear effects causing an increase of the mass function $m$. This is seen from eq. (24) because $T^I_0 = P = 0$.

To analyse the change of $r_c$ with variation of the matter contents we numerically integrate eqs. (25), (26), (29). As initial values we use $r_{\text{init}} = 0.95 \cdot r_{\text{EH}} \approx 1.25$ and set $g_{tt,\text{init}}$ and $g_{rr,\text{init}}$ equal to their values at $r_{\text{init}}$ for the Reissner-Nordström solution (with initial mass $m_0 = 1$ and charge $q = 0.95$) and vary the initial matter density $\epsilon_0$.

Figure 1(a) shows an example of the variation of the metric functions with $r$ for the case $\epsilon_0 = 0.03$. It is seen that $g_{tt} \to 0$ as $r \to r_c \approx 1.149$. As $g_{tt}(r) \to 0$, the density and curvature increases rapidly, which can easily be understood from eq. (24). This is indicated in fig. 1(b) which shows the variation of $K(r)$ for the same case. The line in this figure does not visibly reach $K = K_{\text{planck}} \approx 1.5 \cdot 10^{131}$, however this is solely due to limitations in numerical resolution because of the catastrophic blowup of $K(r)$ as indicated by the vertical line in the figure.

The dependence of $r_c$ on the initial matter density $\epsilon_0$ can be seen in fig. 2. Near the mathematical singularity, $r_{\text{sing}}$, the scalar $K$ increases very rapidly with decreasing $r$, so approximately $r_{\text{sing}} \approx r_c$ (physical singularity). As one can see from the figure, for the case of dust, $r_c$ decreases with decreasing $\epsilon_0$ until $r_c \to r_{\text{CH}}$ at $\epsilon_0 \to 0$. This behaviour is easily understood: for smaller matter contents it takes a longer time to compress the dust to the critical density at $r_c$. On the other hand, in the Reissner-Nordström solution without additional matter, the volume of the uniform reference frame tends to zero in fig. 1(b) which shows the variation of $K(r)$ for the same case. The line in this figure does not visibly reach $K = K_{\text{planck}} \approx 1.5 \cdot 10^{131}$, however this is solely due to limitations in numerical resolution because of the catastrophic blowup of $K(r)$ as indicated by the vertical line in the figure.

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Finally we note that the laws has been confirmed by numerical calculations.

B. Massless scalar field

The case of a scalar field has been analysed by Burko in [9]. Here we extend his analysis.

This case differs drastically from the case of dust. In
this case a mathematical singularity \( r_{\text{sing}} \) can exist only at \( r_{\text{sing}} = 0 \). In the vicinity of this singularity the solution (25), (26), (30) can be written in the first approximation as follows:

\[
\begin{align*}
g_{tt} &= 2 m C r^2 \\
g_{rr} &= -(\beta + 2) \frac{1}{2m} r^{\beta+2} \\
\Phi &= \sqrt{\beta + 1} \ln r,
\end{align*}
\]  

\((37a)\)\((37b)\)\((37c)\)

where \( m, C \) and \( \beta \) are constants. We note that we use constants \( m \) and \( C \) which are different from Burko’s constants. Our constants have direct physical meanings: \( m \) is the black hole mass, \( C \) is a gauge parameter, related to the possibility of changing the scale of measurement of the \( t \) space coordinate. Also we have \( d^2 = \frac{(\beta+1)}{(\beta+2)} 4m^2C \) where \( d \) is a constant used by Burko in [9].

The exponent \( \beta \) depends on the amplitude of the scalar field. As Burko demonstrated \( \beta > 0 \) if \( q \neq 0 \). So in the vicinity of the singularity the value of \( \left( \frac{d}{dr} \right)^2 \), which is the only term in the equations (25), (26), (30), which determines the strength of the scalar field, can not be smaller then \( \frac{1}{\beta} \) (unless it is equal to zero identically). To understand the behaviour of the singularity in this case let us note the following:

In the metric (24) the scalar field can be represented as a sum of two equal fluxes moving in opposite directions along the (spacelike) \( t \)-axis with the fundamental velocity \( c \). Indeed, let us suppose that in \( u,v \) coordinates there are everywhere and always two equal, opposite directed, fluxes along these coordinates, hence \( \frac{du}{dr} = \frac{dv}{dr} \) which depend on \( r = u + v \), but not on \( t = u - v \). Then there is a coordinate transformation:

\[
u = R - t, v = R + t
\]  

\((38)\)

which corresponds to a transformation to the metric (24) but with another time coordinate \( R : dR = \sqrt{-g_{tt}/g_{rr}} \, dr \).

The transformation (38) corresponds to a transformation of the tensor of the scalar field:

\[
\begin{align*}
T_{rr} &= T_{tt} = \tilde{T}_{uu} + \tilde{T}_{uv} \quad (39a) \\
T_{\theta\theta} &= \tilde{T}_{\theta\theta} \quad (39b) \\
T_{\phi\phi} &= \tilde{T}_{\phi\phi} \quad (39c) \\
\text{All other } T_{ik} &= 0 \quad (39d)
\end{align*}
\]

Applying transformation (38) to (30), the new energy-momentum tensor depends on the timelike coordinate \( r \) but not on \( t \). The existence of two opposite fluxes near the Cauchy Horizon should lead to two nonlinear effects: mass inflation and shrinking of the CH down to \( r = 0 \). The uniformity and equality of the two fluxes lead to the situation where both effects manifest themself simultaneously and there are not any gradients in space. To see these effects we perform the numerical integration of the system (25), (26), (30). Here, as for the case of dust, we start the computation from \( r_{\text{init}} = 0.95 \cdot R_{\text{EH}} \approx 1.25 \), put the initial values of \( g_{tt} \) and \( g_{rr} \) equal to their values for the zero matter content Reissner-Nordström solution (with \( q = 0.95, m = 1.0 \)) at \( r_{\text{init}} \) and vary the characteristic of the initial amplitude of the scalar field, \( \epsilon_0 \).

In fig. 3 one can see the propagation (\( r \) vs. \( t \)) of the ingoing signal with the velocity \( c (c = 1) \) in models with different \( \epsilon_0 \). Fig. 4 shows the mass function as a function of \( r \) for the same choices of \( \epsilon_0 \) and fig. 5 presents examples of the evolution of the metric functions, \( g_{tt} \) and \( g_{rr} \), and \( K \) in the models with different \( \epsilon_0 \). Also we refer to fig. 2 (line b) depicting \( r_c \) as a function of \( \epsilon_0 \).

From these figures it is clearly seen that in models with very small \( \epsilon_0 \) (e.g. \( \epsilon_0 = 0.0001 \)) there is a manifestation of a mass inflation at \( r \) close to the CH. First of all we see that the light signal propagates along \( r \approx r_{CH} \) during a long period (line “a” in fig. 3(a)). This is a necessary condition for mass inflation to occur. Secondly, we see more directly that the massfunction, which was small at large \( r \), starts to manifest mass inflation at \( r \) close to...
The metric functions $g_{tt}$ and $g_{rr}$ behaves like the case of the pure Reissner-Nordström solution at larger $r$, but in the vicinity of $r_{CH}$ they start to collapse (fig. 5(a)). We also see that $K$ demonstrates a sudden sharp increases at $r$ close to $r_{CH}$ (fig. 5(d), line “a”), and it reaches the $K_{\text{planck}}$ value at $r_0$ close to $r_{CH}$ before the shrinkage of the CH manifests itself strongly (fig. 5(d)). Thus here we have the physical singularity at $r$ close to $r_{CH} \neq 0$.

At larger $\epsilon_0$, the term associated with scalar matter in the Einstein equations starts to dominate over a term which represents the electric charge much earlier, hence the manifestation of the electric field (which is responsible for the origin of the CH) is not so essential in this case. Functions $g_{tt}$ and $g_{rr}$ differ from the case of the Reissner-Nordström at $r$ essentially larger than $r_{CH}$ (see figs. 5(b), 5(c) and compare with fig. 5(a) which essentially behaves like the Reissner-Nordström solution for $r > r_{CH}$). For the cases $\epsilon_0 \geq 0.001$, the light signal propagates at $r$ close to $r_{CH}$ for a very short period of time (line “b,c,d” on Fig. 3(a)). For $\epsilon_0 \geq 0.01$ the light signal does not feel the presence of $r_{CH}$ at all (see fig. 3(b)). For $0.01 \leq \epsilon_0 \leq 0.03$, we observe the shrinkage of the model to $r$ close to $r = 0$ before $K$ reaches $K_{\text{planck}}$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.45\textwidth]{fig3.png}
\caption{$r$ versus $t$ for the scalar case for various $\epsilon_0$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.45\textwidth]{fig4.png}
\caption{Mass function versus $r$ for the scalar case for various $\epsilon_0$.}
\end{figure}
FIG. 5: Metric functions (a-c) and Kretschmann scalar (d) versus $r$ for the scalar case for various $\epsilon_0$. (see line “b” on fig. 4). So for these values of $\epsilon_0$ the physical singularity is at $r$ close to $r = 0$.

C. Ultrarelativistic gas, $P = \frac{\dot{\epsilon}}{\ddot{\epsilon}}$

At big values of $\epsilon_0$ (for example $\epsilon_0 \geq 0.050$) there is not any manifestation of the effects near $r = r_{CH}$ because in this case the light signal does not propagate long enough along $r \approx r_{CH}$ for mass inflation to occur. The mass function nevertheless increases impetously with decreasing $r$ due to the compression of the model and $K$ reaches the critical value $K_{planck}$ at rather big $r$ (see figs. 5(c) and 5(d)).

The case $P = \frac{\dot{\epsilon}}{\ddot{\epsilon}}$ is in some sense intermediate between the cases of pressureless dust and scalar field as it is seen in fig. 4. In fig. 6 one can see the contraction of the model and corresponding increase of the Kretschmann scalar $K$. There is not any manifestation of the mass inflation near $r \approx r_{CH}$, but only the nonlinear effect of the increase of the mass function because of the matter squeeze.
V. PHYSICS OF THE INTERIOR

In this section we analyse the fully nonlinear processes inside a spherical, charged black hole with a scalar field, as described in section III using results from numerical simulations. Our numerical code is described in appendix A and tested in appendix B. As mentioned in the introduction, some parts of this problem have been discussed in works [10, 14, 15]. In this section we extend these analyses and reveal new aspects of the problem. In subsection V A we investigate a simple compact pulse. In subsection V B we investigate a somewhat more complicated compact pulse and in subsection V C we investigate the influence of the $T_{uu}$ flux on the singularities.

To perform this analysis we specify different boundary conditions along some initial $u = u_0 = 0.00$ and $v = v_0 = 5.00$ to imitate some physical fluxes of energy into the charged black hole, perform numerical simulations and analyse the results. Throughout this section, the black hole, prior to influence from scalar pulses, has initial mass $m_0 = 1.00$ and charge $q = 0.95$. Also, our domain of integration is from $5.0 < v < 20.0$ and $0.0 < u < 30.0$. For all simulations the gauge is chosen as described in section III.

A. Simple compact pulse

We start from the simplest case when the flux of the scalar field into the charged black hole is specified along initial $u = u_0$ outside of the black hole in the following way:

$$\Phi_v(u_0, v) = A \sin^2\left(\frac{v - v_0}{v_1 - v_0}\right)$$

where $v_0$ and $v_1$ marks the beginning and end of the ingoing scalar pulse, respectively (i.e. we set the beginning of the pulse equal to the beginning of our computational domain) and $A$ measures the amplitude of the pulse. This can readily be integrated to give:

$$\Phi(u_0, v) = \frac{A}{4\pi} \left(2\pi(v - v_0) - (v_1 - v_0)\sin\left(\frac{2\pi(v - v_0)}{v_1 - v_0}\right)\right)$$

After the pulse, at $v > v_1$, the flux through $u = u_0$ is set equal to zero, i.e. $\Phi_v(u_0, v) = 0$. The flux of the scalar field through initial ingoing segment $v = v_0$ is set equal to zero: $\Phi_v(u,v_0) = 0$. This means that there is no flux of energy from the surface of a collapsing charged star into the computational domain.

Note that we formulate the initial condition directly for the flux $T_{vu} = (\Phi^2)/4\pi$ of the scalar field through the surface $u = u_0$, rather than for $\Phi$ itself since $T_{vu}$ has the direct physical meaning. Also remember from section III that once the flux through the two initial surfaces has been chosen, all other initial conditions are determined by our choice of gauge and the constraint equations.

In our computations we vary the width of the signal $\Delta = (v_1 - v_0)$, and its amplitude $A$, in a broad range. In fig. 7 is seen a typical example of the evolution of the scalar field $\Phi$ for the case of $\Delta = 1.00$, $A = 0.05$. Fig. 7(a) and 7(b) represents the evolution of the flux $T_{uu}$ of the scalar energy into the black hole. Fig. 7(c) and 7(d) shows the $T_{vu}$ flux which arises as a result of $T_{uu}$ being scattered by the spacetime curvature. In fig. 7(a) and 7(b) the initial pulse (between $5.0 < v < 6.0$) and subsequent tails with resonances are clearly seen. In different regions, $T_{uu}$ and $T_{vu}$ are converted into one another due to curvature and resonances. In some regions, $T_{vu}$ is locally greater than $T_{uu}$, it is especially noted that the highest local flux is $T_{vu}$ inside the pulse (between $5.0 < v < 6.0$, fig. 7(b).
We will now consider some direct effects related to these fluxes.

1. Focusing effects

One of the first noticeable effects is that the initial pulse $T_{vv}$ leads to an initial change of the outer apparent horizon (OAH) and inner apparent horizon (IAH) in the region within the pulse itself, e.g. $5 < v < 6$ for fig. 7(a). In fig. 8(a) it can be seen that the mass function, $m$ (eq. 22) near the IAH increases correspondingly as the IAH moves from $u \approx 27.12$ to $u \approx 27.51$. A similar change can be seen for the OAH in the same region (e.g. fig. 9(a)). The increase of the mass function and the change of the apparent horizons in this region is the trivial effect of mass being pumped into the black hole by the $T_{vv}$-flux of the initial pulse.

The dramatic change of the IAH at $v \approx 7 - 8$, however, is related with other nonlinear effects. We remember that
worldlines of imaginary test photons along $u = \text{const}$ and $v = \text{const}$ are under action of the gravity of the radiation $T_{vv}$ and $T_{uu}$, which leads to a focusing effect. For example, in the absence of scalar radiation, outgoing photons along $u = \text{const}$ slightly above of the IAH will go to greater $r$ as $v \to \infty$ in the Reissner-Nordstr"om solution. With the existence of scalar radiation, a similar outgoing ray initially slightly above the IAH will now, because of the focusing effect of the $T_{vv}$ and $T_{uu}$ radiation, go to smaller $r$ and generate a maxima $\left( \frac{\partial u}{\partial v} = 0 \right)_{r = \text{const}}$ which correspond to the position of the IAH. This is seen in fig. 8(a) and more clearly in fig. 9(b). This effect leads to a drastic change of the shape of the IAH. It is seen by lines c) and d) in figure 8(a) that this change of the IAH in this region ($v \approx 7 - 8$) is not accompanied by any significant change of the mass function. About the increase of the mass function at $v > 8$ see below.

In the case of smaller initial amplitude of the pulse $A$, 

![Diagram](image-url)
FIG. 10: Various contour plots for the simple compact pulse, case: $\Delta = 1.0, A = 0.20$

(a) Lines of constant $\log_{10}(T_{uu})$. Lines are from $\log_{10}(T_{uu}) = -10.0$ to $\log_{10}(T_{uu}) = 0.00$ in $\Delta \log_{10}(T_{uu}) = 0.50$ intervals. Thick dotted line marks $\log_{10}(T_{uu}) = -1.50$ (decreasing "outwards"). Fully drawn thick line marks apparent horizon.

(b) $\log_{10}(T_{uu})$ along $v = 15.00$

(c) Lines of constant $r$. Lines are from $r = 0.520$ (bottom left line) to $r = 0.475$ (top right line) in intervals of $\Delta r = 0.005$.

(d) Lines of constant $\log_{10}(T_{\theta \theta})$. Lines are from $\log_{10}(T_{\theta \theta}) = -2.0$ (bottom right line) to $\log_{10}(T_{\theta \theta}) = 32.0$ (top right line) in intervals of $\Delta \log_{10}(T_{\theta \theta}) = 2.0$.

the change of the shape of IAH due to focusing starts later. For example, for the case $\Delta = 1, A = 0.01$ (see figure 9(b), the change starts at $v \approx 10$. In this case the change of the OAH and IAH in the region $5 < v < 6$, related to the initial pulse of the scalar field, is so small that it is invisible in the figure.

In figs. 10(a), 10(c) (case: $\Delta = 1.0, A = 0.2$) one can see another manifestation of the focusing effect related with the change of the flux $T_{uu}$. The figures shows close correspondence between $T_{uu}$ and the rate of focusing of lines of constant $r$. Especially we note that along the line $u \approx 22.5$ for $v > 8$ there is a minimum of $T_{uu}$ (seen by the collection of very closely spaced lines in fig. 10(a) and as the local minima in fig. 10(b)). Comparing with figure 10(c) we see that the lines of $r = const$ shows minimal focusing along this minima, compared to the focusing at $u < 22.5$ and $24 < u < 28$. At $u > 28$ there is a decrease of $T_{uu}$ and we see a corresponding decrease of the focusing effect.

We should also remember that in the dynamic equations (equations 10-12) and the expression for $T_{\theta \theta}$, (eqs. 5 + 7) the scalar field appears only in the form
of the product $\Phi_u \Phi_v$ and the same nonlinear effects can be described in terms of $T_{u\theta}$ component which is presented in fig. 10(d). For example, where $T_{uu}$ has its minima, a corresponding effect is clearly visible in fig. 10(d). We also note from fig. 10(d) that for this case the initial pulse is so strong that its flux changes the IAH inside the pulse and there are no double turns as it was seen on fig. 10(d).

2. Mass function

Let us come now to the discussion of the behaviour of the mass function $m$. As we remember, the (modest) increase of $m$ in the region $5 < v < 6$ in fig. 8(a) is related to the input of the energy in the initial pulse. The increase of $m$ seen in the region $8 < V < 15$ in fig. 8(a) is related partly with the compression-effect (see section III), but still it is very difficult to separate this effect from
the beginning of the mass inflation. The essentially faster increase of $m$ at $v > 15$ (fig. 8(b)) is the manifestation of the mass inflation when we come to the CH.

Mass inflation depends mainly on the $T_{vv}$ flux but also on the $T_{uu}$ flux and, as described in section III, the co-existence of both fluxes is essential for mass inflation to occur. An example of the dependence on $T_{vv}$ is seen in fig. 11 (case $\Delta = 1$, $A = 0.40$). It is seen that where $T_{vv}$ has a minimum (the narrowly spaced lines at $v \approx 15$) the increase of $m$ almost stops. Fig. 12 demonstrates the dependence of mass inflation on $T_{uu}$ for the stronger pulse: $\Delta = 2.0$, $A = 0.20$. This pulse is so strong that a $r = 0$ singularity is formed in the domain (this is further discussed in the next subsection). At the minimum of $T_{uu}$ at $u \approx 23.8$ the increase of $m$ also stops and at $u > 24$ where $T_{uu}$ increases rapidly, $m$ also has similar rapid increase. However, the line plotted terminates at the $r = 0$ singularity, thus the final rapid increase of mass is a combination of the effects of compression and mass inflation.

The compression effect can be more clearly seen in fig. 13(a), which shows $m$, along lines of constant $u$, again for the case $\Delta = 2.0$, $A = 0.2$. When one comes to the $r = 0$ singularity, compression tends to infinity and we see catastrophic infinite increase of $m$. This can be seen by the near vertical lines in the lower left in the figure, which represents lines of high $u$. These lines experience a catastrophic infinite increase of mass as they approach $r = 0$, as indicated by these lines being near vertical.

The remaining right hand side lines which run in all the range $5 < v < 20$, represents lines of constant $u$ which reach the CH and the mass increase along those lines are due to the mass inflation. The line marked by thick dashes represent a line that comes close to the point where the $r = 0$ and CH singularities meet. The structure of this line is more complicated as it is influenced by both processes.

Finally, in fig. 13(b) is seen the Kretschmann scalar for the same lines.

3. The singularity

Now we will discuss the singularity. When the initial pulse is rather weak we cannot see the manifestation of the spacelike singularity in our computational domain. However, we can see the asymptotic approach of $u = \text{constant}$ test photons to the CH singularity. In fig. 14(a) we see that for the case $\Delta = 2.0$, $A = 0.01$, all our test photons come asymptotically to the same value at $r \approx 0.69$, corresponding to the analytical value for $r$ at the CH for the pure Reissner-Nordström solution, i.e. the CH singularity itself does not show any tendency to shrink down (within our computational domain).

With an increase of the amount of energy in the initial pulse one can observe a nonlinear effect of shrinkage of the CH-singularity under the action of the gravity of the irradiating flux $T_{uu}$ together with the $T_{vv}$ flux. In fig. 14(b) (case $\Delta = 2.0$, $A = 0.125$) the test photons with greater $u = \text{constant}$ come asymptotically to smaller values of $r$ and in fig. 14(c) (case $\Delta = 2.0$, $A = 0.180$) the pulse is so strong that the CH almost (but not quite) shrinks down to $r = 0$.

In fig. 14(d) (case $\Delta = 2.0$, $A = 0.200$) one can see both the manifestation of the shrinkage of the CH singularity (photons with higher $u$ come asymptotically to

![Fig. 13: Mass function and Kretschmann scalar along lines of constant $u$ for the simple compact pulse, case: $\Delta = 2.0$, $A = 0.2$. Separation between lines is $\Delta u = 0.40$. Lines are from $u = 19.40$ (lowest right line) to $u = 29.80$ (near vertical line in lower left corner). Thick dotted line is along $u = 24.60$.](image-url)
(a) Case: $\Delta = 2, A = 0.010$. From $u = 23.00$ (rightmost line) to $u = 29.80$ (leftmost line).

(b) Case: $\Delta = 2, A = 0.125$. From $u = 21.40$ (rightmost line) to $u = 29.80$ (leftmost line).

(c) Case: $\Delta = 2, A = 0.180$. From $u = 20.20$ (rightmost line) to $u = 29.80$ (leftmost line).

(d) Case: $\Delta = 2, A = 0.200$. From $u = 19.80$ (rightmost line) to $u = 29.80$ (bottom leftmost line).

FIG. 14: $v$ versus $r$ along lines of constant $u$ for amplitudes for the simple compact pulse. Separation between lines of constant $u$ is $\Delta u = 0.40$. smaller $r$) and existence of the $r = 0$ singularity (photons with the highest $u$ come to $r = 0$).

Figure 15 shows lines of constant $r$ and the position of $K = K_{\text{planck}}$ (marked by the thick dotted line) for the two strongest cases from fig. 14. We remember that this line and places with higher $K$ should be considered as a singularity from the point of view of classical physics. Thus, for both these cases the physical singularity is placed at finite values of $v$ and is not a null singularity. In fig. 15(b) we furthermore see the $r = 0$ spacelike singularity inside of our computational domain. This singularity can be considered as a result of mutual gravitational focusing of $T_{vv}$ and $T_{uu}$ fluxes in the region between inner and outer apparent horizons. At small $v < 15$ the physical singularity practically coincide with $r = 0$, but for $v > 17$ we see that the spacetime structure of the physical singularity is quite different from the structure of the mathematical singularity. The physical singularity here depends mainly on the true CH-singularity, but its position in the $u - v$ diagram is quite different from the position of the true CH-singularity which is at $v = \infty$.

This can also be compared with fig. 13(b) from which we see different behaviours of $K$ for the test photons for the strong case of $\Delta = 2.0, A = 0.200$. The lines in the lower left hand corner, sharply increasing to near vertical, are the lines which come to $r = 0$. The thick dotted line is the line which comes to a point at the singularity close to the meeting of the $r = 0$ and CH singularities. The
FIG. 15: Lines of constant \( r \) for two different simple compact pulses. Thick dotted lines in right part of the figures represents line of planckian curvature. Bottom fully drawn thick line marks OAH.

FIG. 16: Contour lines for double sine pulse of form of eq. (42), case \( \Delta = 2.0, A = 0.25 \). Thick bottom line is AH, thick upper line is \( r = 0 \).

remaining lines in the right hand side are the lines which come to the CH-singularity.

B. Double sine pulse

In this subsection we choose the ingoing flux to be of the form:

\[
\Phi_{uv}(u_0, v) = \sqrt{6} A \cos \left( \pi \frac{v - v_0}{v_1 - v_0} \right) \sin \left( \pi \frac{v - v_0}{v_1 - v_0} \right)^2
\]

(42)
(a) Lines of constant \( \log_{10}(T_{uu}) \). Lines are from \( \log_{10}(T_{uu}) = -10.0 \) to \( \log_{10}(T_{uu}) = 3.00 \) in \( \Delta \log_{10}(T_{uu}) = 0.25 \) intervals. Thick dotted line marks \( \log_{10}(T_{uu}) = 1.50 \), decreasing leftwards.

(b) Lines of constant \( \log_{10}(T_{vv}) \). Lines are from \( \log_{10}(T_{vv}) = -10.0 \) to \( \log_{10}(T_{vv}) = 3.00 \) in \( \Delta \log_{10}(T_{vv}) = 0.25 \) intervals. Thick dotted line marks \( \log_{10}(T_{vv}) = 1.50 \), decreasing downwards.

FIG. 17: Contours for the double-flux case based on the simple compact pulse: \( \Delta = 2.0, A = 0.25 \). Thick top right line marks \( r = 0 \).

FIG. 18: Lines of constant \( r \) for the double-flux case based on the simple compact pulse: \( \Delta = 2.0, A = 0.25 \). Thick top right line marks \( r = 0 \). Lines are from \( r = 2.50 \) (bottom left line) to \( r = 0.10 \) (top right thin line) in intervals of \( \Delta r = 0.10 \). Thick dotted line marks \( K = K_{\text{planck}} \).

\[
\Phi(u_0, v) = \sqrt{\frac{2}{\pi}} A (v_1 - v_0) \sin\left(\frac{\pi (v_1 - v_0)}{v_1 - v_0}\right)^3
\]

(43)

where \( v_0 \) and \( v_1 \) as before, marks the beginning and end of the pulse respectively and \( A \) is the amplitude of the pulse. Also, as before we set \( \Phi_v(u_0, v) = 0 \) for \( v > v_1 \). The pulse is scaled in such a way that for a given width \( \Delta = v_1 - v_0 \) and amplitude \( A \), the integral of the initial flux, \( \int_{v_0}^{v_1} T_{vv} dv \), is equal for pulses of the form (40) and (42). Eq. (42) has the shape of a double pulse, rather than (40) which is the shape of a single initial pulse. This pulse is more complicated than (40), but is similar in shape to the pulse shapes used in some other papers (e.g., [28, 29]).

We have performed a series of computations based on eq. (42) (all other initial conditions equal to those in the previous subsection). The results of these computations demonstrate a more complex picture of interplay between the scalar fluxes than in the case of subsection VA. This is natural because of the more complex shape of the initial pulse. But the main physics and principal properties of the singularities are the same in the two cases. An example illustrating the increased complexity in structure of the fluxes can be seen in figure 16, which illustrates the fluxes for the case: \( \Delta = 2.0, A = 0.25 \). The shapes of the apparent horizons and the central singularity \( r = 0 \) are now more complex as well as the distribution of the \( T_{vv} \) field. Still the general characters of the \( r = 0 \) and the physical \( K = K_{\text{planck}} \) singularities are the same.

C. The influence of the \( T_{uu} \) flux

So far in all our analyses we have assumed that the flux of the scalar field \( T_{uu} \), through \( v = v_0 \) was zero and that any \( T_{uu} \)-flux arised only as a result of scattering of the \( T_{vv} \)-flux by the curvature of the spacetime. Now we would like to consider the influence of a \( T_{uu} \)-flux through the surface \( v = v_0 \). To do this we will consider an ex-
treme case when $T_{uu}$ through $v = v_0$ is equal exactly to $T_{vv}$ through $u = \text{const}$ just inside of a black hole. More concretely we do the following: We specify some initial $\Phi(u_0, v)$ along $u = u_0$ with a pulse width $\Delta = v_1 - v_0$ and amplitude $A$ and set all initial conditions equal to those in subsection \[\text{V}A\] including $\Phi_{,u}(u, v_0) = 0$ along $v = v_0$. These initial data are then simulated as usual, however only in the computational domain of $5 < v < 20$ and $0 < u \leq u_{AH}$, where $u_{AH}$ is the first computational point which is inside of the outer apparent horizon along $v = v_1$. Because of scattering of the initial pulse there is now a $T_{vv}$ flux into the black hole along the line $u = u_{AH}$ for $v > v_1$. We then stop the computation and start a new with the following domain of integration: $v_1 \leq v \leq 20$ and $u_{AH} \leq u \leq u_{\text{max}}$ (where $u_{\text{max}} - u_{AH} = (20 - v_1)$). Along the (new) outgoing initial hypersurface, $u = u_{AH}$, all the variables are kept as they were in the original simulation, while along the (new) ingoing initial hypersurface, $v = v_1$, initial data are set equal to the data along the outgoing hypersurface, hence we have completely symmetrical initial conditions. Subsequently we performed a computation for the new
computational domain. It is obvious that all conditions in this domain are symmetrical with respect to \( u \) and \( v \) and hence the boundary fluxes \( T_{uu} \) along \( v = v_1 \) and \( T_{vv} \) along \( u = u_{AH} \) must be exactly equal.

We performed our computations for the compact simple pulses (eq. (40)) for different parameters \( \Delta \) and \( A \). An example of our results can be seen in figs. 17-21 for the case \( \Delta = 2.0, A = 0.25 \). All pictures are symmetrical with respect to \( u \) and \( v \), as they should be. The outer apparent horizon is naturally outside of our computational domain. Figs. 17 and 18 shows the \( T_{uu} \) and \( T_{vv} \) distributions and \( r \) contour lines. On these figures there is no inner apparent horizon because it coincide with another CH singularity at \( u \to \infty \) (see below). But on fig. 18 one can see the mathematical strong singularity \( r = 0 \) and the physical singularity (with \( K = K_{\text{planck}} \)) which is reached long before the CH singularity at \( u \to \infty \) and \( v \to \infty \) in most of the computational domain.

Fig. 19 shows the huge increase of the mass function with growing \( u \) and \( v \), and fig. 20 shows the increase of the Kretschmann scalar as function of \( u \) and \( v \) up to \( K = K_{\text{planck}} \). Finally fig. 21 shows light signals along \( v = const \) and \( u = const \) respectively. This figure shows both the existence of null singularities of the CH singularity-type and \( r = 0 \) singularity. Indeed for example in the case fig. 21(a) one can see that for some \( v = const \) (the rightmost curves) for \( u \to \infty \), signals goes asymptotically to practically \( r = const \) (asymptotically approach to the null singularity at \( u \to \infty \)). Smaller asymptotic values \( r \) are seen for the signals at bigger \( v \) corresponding to the focusing effect. Finally the signals with \( v \) big enough, come to the central singularity \( r = 0 \). The symmetrical picture for the signals with \( u = const \) is shown in fig. 21(b). Here we see the asymptotic approach to another null singularity at \( v \to \infty \).

Note also the different character of the increase of mass function on fig. 19 for lines of big and small \( v \) and \( u \) respectively. For example on fig. 19(a) the topmost lines (big \( v = const \)) come to the central singularity but lines with small \( v = const \) (bottom lines) go to the null singularity at \( u \to \infty \). The different behaviour of these lines correspond to the different nonlinear processes influencing the mass function for lines terminating at the \( r = 0 \) singularity and reaching the CH singularity.

Thus in this case there are three different singularities inside the black hole: two null singularities at \( u \to \infty \) and \( v \to \infty \) and the physical singularity at \( K = K_{\text{planck}} \). In addition there is also a central (mathematical) singularity at \( r = 0 \).

VI. CONCLUSIONS

In this paper we investigated the physics of nonlinear processes inside of the spherical charged black hole, nonlinearly perturbed by a selfgravitating, minimally coupled, massless scalar field. For this purpose we created and tested a numerical code which is stable and second-order accurate. For our computations we used an adaptive mesh refinement approach in ingoing \( u \)-direction.

The following nonlinear physical processes are important inside the black hole: Scattering of radiation by the curvature of the spacetime, gravitational focusing effect, mass inflation and squeeze of matter with pressure.

At the beginning of our analysis we used a homogeneous approximation to clarify some physical processes near a spacelike singularity. In this approximation one supposes that near the singularity temporal gradients are much higher than the spatial gradients, so one assumes that all processes depend on the time coordinate only.

FIG. 21: \( u \) and \( v \) versus \( r \) for the double-flux case double-flux case based on the simple compact pulse: \( \Delta = 2.0, A = 0.25 \)
(uniform approximation). We used both analytical analyses and a numerical approach to analyse three different physical matter contents: dust, a massless scalar field and an ultrarelativistic gas.

For the case $P = 0$ (dust) we found that the singularity is at $r = r_{\text{sing}} \neq 0$, $r_{CH} < r_{\text{sing}} < r_{EH}$, where $r_{CH}$ and $r_{EH}$ are the positions of the Cauchy Horizon and the Event Horizon when there are no dust at all. The value $r_{\text{sing}}$ decreases monotonically with a decrease of the matter contents and tends to $r_{\text{sing}} = r_{CH}$ when the matter contents goes to zero.

In the case of the scalar field, the uniform approximation demonstrates more a complex behaviour. Here the scalar field can be represented as a sum of two equal fluxes moving in opposite directions. One can for this case see the manifestation of both the effect of mass inflation and the effect of shrinkage of the CH down to $r = 0$. For very small matter contents ($\epsilon_0 \ll 0.01$) the Kretschmann scalar, $K$, becomes equal to the Planck value at $r$ close to $r_{CH}$. So in this case the physical singularity (when $K = K_{\text{planck}}$) is at $r \approx r_{CH}$. For larger values of $\epsilon_0$, (e.g. $0.01 \lesssim \epsilon_0 \lesssim 0.03$), the CH does not manifest itself and the model squeezes to $r$ very close to $r = 0$ before $K$ reaches $K_{\text{planck}}$. It means that for these values of $\epsilon_0$, the physical singularity practically coincides with $r = 0$. For rather big $\epsilon_0$, $K$ reaches $K_{\text{planck}}$ at rather big $r$ as it was for the case of dust, $P = 0$.

In the case of matter with isotropic relativistic pressure, $P = \epsilon/3$, we have the situation intermediate between $P = 0$ and the scalar field. The physical singularity, in this case, is located at $r$ essentially greater then $r = 0$.

We performed the analysis of the full nonlinear processes inside the spherical charged black hole with a scalar field using the numerical approach. This analysis extends the analysis of the earlier works [11, 14, 15] and reveal new aspects of the problem. The detailed description of the results is given in section [11]. We analysed nonlinear gravitational interaction of the fluxes of the scalar field, the dependence of the effects on the boundary conditions, analysed the focusing effects, mass inflation and squeeze effect and the behaviours of the Kretschmann scalar $K$. We payed special attention to the analysis of the singularity in subsection [VA2]. We investigated the focusing of the CH singularity and its dependence on the boundary conditions. We determined the position of the physical singularity (where $K = K_{\text{planck}}$) inside the black hole and demonstrated that this position is quite different from the positions of the mathematical $r = 0$ singularity and CH singularity.

Finally we investigated the influence of the boundary $T_{uv}$-flux on the physics of the singularity. We demonstrated that it is possible to have the existence of three different singularities inside the black hole: two null singularities at $u \to \infty$ and $v \to \infty$ and the physical singularity $K = K_{\text{planck}}$ in addition to a central mathematical singularity $r = 0$.

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APPENDIX A: THE NUMERICAL CODE

In this appendix we describe the structure of the numerical code used to obtain the numerical results presented in section [11].

1. The numerical scheme

Our approach to numerically integrating the field equations is similar to that of other authors [11, 14, 15]. Denoting the three unknowns, $r, \sigma, \Phi$ as $h_j$ with $j = 1, 3$, the three evolution equations [10, 12] are each of the form:

$$h_j,uv = F_j(h_k, h_{k,v}, h_{k,u}), k = 1, 3 \quad (A1)$$

We denote four grid points in the $(u,v)$-domain by $p_1 \equiv (u,v)$, $p_2 \equiv (u, v + \Delta v)$, $p_3 \equiv (u + \Delta u, v)$ and $p_4 \equiv (u + \Delta u, v + \Delta v)$, where $\Delta v$ and $\Delta u$ are finite increments in the $v$ and $u$ directions (see fig. [22]).

By combining a Taylor expansion $h_j$ at these four points (evaluated around an intermediate point $p_5 \equiv (u + \Delta u/2, v + \Delta v/2)$) with eq. [A1] we find that:

$$h_j(p_4) = h_j(p_3) + h_j(p_2) - h_j(p_1) + \Delta u \Delta v [F_j(h_k(p_5), h_{k,u}(p_5), h_{k,v}(p_5))] + \frac{\Delta u^2}{24} h_j,uuuu(p_5) + \frac{\Delta v^2}{24} h_j,vvvv(p_5) + O(\Delta u^3, \Delta v^3)], k = 1, 3 \quad (A2)$$

i.e. we can evaluate $h_j$ at $p_4$ with second order accuracy if we know $h_j$ at the four points $p_1, p_2, p_3, p_5$ and $h_{j,v}, h_{j,u}$ at $p_5$. Initially we only know the values of $h_j$ along in- and outgoing null segments (see section [11]) i.e. for example at $p_1, p_2$ and $p_3$. However, by employing eq. [A2] twice in a predictor-corrector style we can evaluate $h_j, h_{j,v}, h_{j,u}$ at $p_5$ with the desired accuracy and hence subsequently find $h_j$ at $p_4$. 

Ingoing ray reach the end of our computational domain at the last grid point at infinity, i.e. regardless of how close the two rays were initially, their distance will diverge as their advanced time grows. Integration across such great gradients using fixed unknowns along the next ingoing ray along v starting at the point \( u = u_0 + \Delta u \), then solving for \( u = u_0 + 2\Delta u \) etc. until we reach the last grid point at \( u = u_{final} \). Using the (now known) values of \( h_j \) along \( v = v_0 + \Delta v \), we can then calculate the unknowns along the next ingoing ray along \( v = v_0 + 2\Delta v \) and so on throughout our computational domain until we reach the end of our computational domain at \( v = v_{final} \).

2. Splitting algorithm

When outside the event horizon (EH), the scheme described above works reasonably good as it is, but when the domain of integration crosses the EH, the unknown functions exhibits extremely great gradients. This can be seen by considering two outgoing rays, one just inside and one just outside of the EH. One ray is destined to be trapped inside the EH, while the other must escape to infinity, i.e. regardless of how close the two rays were initially, their distance will diverge as their advanced time \( v \) grows. Integration across such great gradients using fixed increments in \( u \) and \( v \) would cause the numerical errors to explode unless the initial increments were chosen to be unrealistically small. To overcome this difficulty an adaptive mesh refinement (AMR) approach is crucial for an accurate integration of the field equations. However, the strong gradients are mainly in the ingoing \( u \)-direction, hence AMR is most important in this direction. AMR in both \( v \) and \( u \) directions would be desirable, implementation of AMR in only one direction, while much less complicated, still produces good results. This is supported by other authors who also employ uni-directional AMR.

To determine whether AMR is needed and a cell should be split (or possibly desplit), we focus on the truncation error in the numerical scheme. As can be seen from equation (A2), the primary error introduced by the scheme comes mainly from the terms:

\[
\epsilon_j^u(p_5) = \frac{\Delta u^2}{24} h_{j,uuuu}(p_5) \quad (A3a)
\]

\[
\epsilon_j^v(p_5) = \frac{\Delta v^2}{24} h_{j,vvvv}(p_5) \quad (A3b)
\]

Since we use AMR only in the ingoing \( u \) direction, thus keeping \( \Delta v \) constant, we can only control \( \epsilon_j^v(p_5) \) by splitting, hence we will focus on this term. By introducing four new points: \( p_k \equiv (u - \Delta u, v), p_7 \equiv (u - \Delta u, v + \Delta v), p_8 \equiv (u - 2\Delta u, v), p_9 \equiv (u - 2\Delta u, v + \Delta v) \), (see fig. 22) we can estimate \( h_{j,uuuu} \) (and hence \( \epsilon_j^u \)) at \( p_5 \) with the following finite difference operator:

\[
h_{j,uuuu}(p_5) = \Delta v \Delta u^3 \left( 3h_j(p_1) - 3h_j(p_3) 
- 3h_j(p_6) + 3h_j(p_7) - h_j(p_2) + h_j(p_4) 
+ h_j(p_8) - h_j(p_9) \right) + O(\Delta u) \quad (A4)
\]

In practice we proceed as follows; Following a calculation of \( h_j \) at \( p_4 \) we use eq. (A3a) + (A3b) to estimate the relative error \( \epsilon_j^u(p_5) \) involved in the calculation and compare it to \( F_j(h_k(p_5), h_{k,u}(p_5), h_{k,v}(p_5)) \) in eq. (A2) for each of the variables, \( j = 1, 3 \). We now require that the relative error should be within some fixed interval:

\[
S_{desplit} < \frac{\epsilon_j^u(p_5)}{F_j(h_k(p_5), h_{k,u}(p_5), h_{k,v}(p_5))} < S_{split} \quad (A5)
\]

If the relative error is greater then the threshold parameter \( S_{split} \) we perform a split. Conversely, if the relative error is less than the threshold parameter \( S_{desplit} \) a desplit is made. Note that because we need knowledge of the points \( p_6 \) to \( p_9 \) to calculate the relative error, we don’t use splitting for the first two cells \( u < u_0 + 2\Delta u \) on a given ingoing ray.

If a split is required a new point is introduced at \( p'_5 \equiv (u + \Delta u/2, v) \). The values of \( h_j \) at \( p'_5 \) are then found by a four-point interpolation scheme (using data from the two points above and the two below \( p'_5 \)). The numerical scheme is then used to calculate \( h_j \) at \( p'_4 \equiv (u + \Delta u/2, v + \Delta v) \) using the points \( p_4, p_5 \) and \( p'_5 \). If a desplitting is needed the reverse process takes place, i.e. \( p_4 \) is deleted and the point above is used in its place.

To obtain the results presented in section IV the splitting thresholds were set to \( S_{split} = 10^{-7} \), \( S_{desplit} = 10^{-9} \). The resolutions along the initial outgoing null segment were set to \( \Delta u = 1/100 \) and \( \Delta v = 1/12800 \). The initial
value of $\Delta u$ can be set rather low, since the splitting algorithm will assure that $\Delta u$ always have an appropriate value. The value of $\Delta v$ on the other hand, is constant throughout the computational domain and must initially be chosen to have a suitable value everywhere. The high value of $\Delta v$ has been chosen such that $\Delta v$ is as comparable to $\Delta u$ in large parts of the interior of the black hole. This has been verified by numerical tests to give the best results.

APPENDIX B: ANALYSIS OF THE CODE

We have tested the code and found it to be stable and second-order accurate. Here we presents tests to demonstrate this.

1. Basic convergence tests

We test the basic convergence properties of the code by performing simulations with the same initial condi-
tions, but with varying resolutions and with the splitting algorithm disabled. As initial conditions we use a Reissner-Nordström spacetime (with initial mass $m_0 = 1$ and charge $q = 0.95$) which is perturbed by a massless scalar field. We set the specific form of the scalar field along the initial outgoing null segment in the interval $v_0 \leq v \leq v_1$ to be:

$$
\Phi(u_0, v) = \frac{A}{4\pi} \left( 2\pi (v - v_0) - (v_1 - v_0) \sin \left( \frac{2\pi v - v_0}{v_1 - v_0} \right) \right)
$$

(B1)

with $A = 0.05$ and $v_0 = 5, v_1 = 6$. For $v > v_1$ we set $\Phi$ constant to $\Phi(u_0, v) = \Phi(u_0, v_1)$. Along the ingoing null segment $v = v_0$ we specify $\Phi_a(u, v_0) = 0$ (these initial conditions are the same as those used in section V. As domain of integration we used $5 < v < 100$ and $0 < u < 23.05$ and with the gauge choice described in section II.

We make simulations with varying resolutions $N = 1/\Delta u = 1/\Delta v$ and examine data along an outgoing ray $u = 23.00$ which is just outside of the event horizon. Along this ray we calculate the average absolute difference at each data point between the unknown functions for two resolutions along the outgoing ray. We define:

$$
\psi_N(x) = \frac{1}{n} \sum_{i} |x_N^i - x_{2N}^i|
$$

(B2)

where $x_N^i$ denotes the value of variable $x$ at the $i$'th data point (out of a total of $n$ points) from a simulation with resolution $N$. I.e. we basically calculate the normalized $L_1$-norm of the difference between a simulation with resolution $N$ and one with resolution $2N$ at each point for the three unknowns, which is similar to the convergence tests used in [28].

In figure 23(a) are shown $\psi_N(r), \psi_N(\Phi)$ and $\psi_N(\sigma)$ for $N = 50, 100, 200, 400, 800$. We see that all the lines have a slope of approximately minus two (compare with line 'd') indicating that the functions converge with second order. However, we also need to show that the converging solution is a physical solution of our field equations. We have tested the code against the Reissner-Nordström and Schwarzschild solutions and reproduced all known analytical features (e.g. location of the event horizon as a function of initial mass etc.) of these spacetimes. If a scalar field is present there are no suitable analytic solutions against which we can compare the numerical results, but we can use the constraint equations [8] and [9] to test that our code is converging to a physical solution.

Denoting the left hand side of the constraint equations [8] and [9] by $C_1$ and $C_2$ respectively, we calculate the average absolute value of $C_1$ and $C_2$ at each point:

$$
\chi(x) = \frac{1}{n} \sum_{i} |x^i|
$$

(B3)

defined similar to eq. (B2), with $x$ being either $C_1$ or $C_2$. In figure 23(b) are shown $\chi(C_1)$ and $\chi(C_2)$ along $u = 23.00$. It is noted that $\chi(C_2)$ is much greater than $\chi(C_1)$. This is mainly because the line $u = 23.00$ is very close to the event horizon and the gradients along ingoing rays are here very large. This results in great inaccuracy for coarse resolutions when calculating the derivatives used to calculate $C_2$. Related to this is of course the fact that we are not using AMR for these basic convergence tests and hence the large average value for $C_2$ is also an indication that splitting would be most desirable in this region. However, $\chi(C_2)$ is decreasing with increasing resolution, which is the important result in this context. We note also that the typical absolute value of the biggest term in eq. (4) is approximately ten times greater than $C_2$ even for rough resolutions. This means that the constraint equation is roughly satisfied for this case even though $C_2$ has very large values. For the highest resolution ($N = 800$), $\chi(C_1)$ shows indications of a slightly decreasing convergence rate. This can also be traced to roundoff errors when calculating the $r_{\nu\nu}$-term of $C_1$.

Figure 24(a) shows $C_1(v)$ for the resolutions $N = 100, 200, 400, 800$. It is seen that $C_1$ is clearly converging to zero for increasing resolution. It is also clear that numerical noise play a dominant role for $N = 800$, especially at large $v$, causing the mentioned decreasing convergence rate effect in fig. 23(b). As noted, this numerical noise can be traced to the numerical calculation of the $r_{\nu\nu}$-term in eq. (4). The reason that this particular term exhibits large roundoff errors is mainly that $r_{\nu\nu} \approx 0$ along the line at which we look. This results in the subtraction of two nearly equal numbers which results in large roundoff errors. Figure 24(b) shows the convergence for $C_2$ again for $N = 100, 200, 400, 800$. Clearly $C_2$ is also converging to a zero value, but for a given resolution, $C_2$ is much larger than $C_1$ due to the proximity to the event horizon and the absence of AMR in these simulations as explained above.

It is also noted that although $C_1$ and $C_2$ seem to be
converging with second order accuracy, the convergence rates of $C_1$ and $C_2$ are not by themself as important as the convergence rates of the unknown functions, since $C_1$ and $C_2$ are functions of the derivatives of the unknown function. The unknown functions, however, directly measures the convergence rate of the code as these are the actual variables being evolved.

2. Convergence with AMR

Another test to show that the numerical solution is converging to a physical solution is to monitor whether the code is mass conserving. Figure 25 shows the mass function along the outgoing ray $u = 23.00$ in the interval $5 < v < 100$ for the resolutions $N = 200, 400, 800, 1600$. We see that the mass function is converging to a constant value for high resolutions, further indicating that the code is converging to a physical solution of the field equations.

In the previous subsection we verified that with the
fixed increments $\Delta u = \Delta v$, our code converges to a physical solution with second order accuracy. In this subsection we will demonstrate the convergence properties of our code with the splitting algorithm turned on, as a function of the splitting criterias used.

We wish to confirm that our splitting algorithm works as desired and that the numerical solution converges to a physical solution. For this purpose we make a number of simulations with identical initial conditions, but vary the splitting thresholds. We use the same pulse shape as in the previous subsection (eq. (B1)), but with a wider and stronger pulse with $A = 0.20$ and $v_0 = 5.0$ and $v_1 = 7.0$ which is strong enough to generate a $r = 0$ singularity within our computational domain. Our computational domain is in the range $5 < v < 20$ and $0 < u < 25$. We examine an outgoing ray ($u = 24.60$) which is inside the event horizon and in fact comes very close to the $r = 0$ singularity.

To measure the convergence rates we use eq. (B2) and (B3) only in this subsection they are functions of $S_{\text{split}}$ (see section A2) instead of the resolution $N$.

Figure 26(a) shows $\chi(C_1)$ and $\chi(C_2)$ as functions of the splitting threshold $S_{\text{split}}$ (see section A2), that we require the relative numerical error to be below for all computational cells, i.e. the left hand side of the figure denotes a more strict splitting policy. The threshold for desplitting is here and throughout the paper set to two orders of magnitude lower then the splitting threshold, $S_{\text{split}} = 10^{-2}.S_{\text{split}}$. We see that as the splitting threshold $S_{\text{split}}$ is decreased, so is the average absolute of $\chi$. We see that for smaller $S_{\text{split}}$, the unknown functions themselves continue to benefit from lower splitting thresholds.

For the lowest splitting threshold at the left side of the figure we see that lower splitting thresholds does not result in significant smaller values of $C_1$ and $C_2$. This arises because $C_1$ and $C_2$ becomes so small that numerical noise (which is little affected by resolution) begins to dominate the calculations of $C_1$ and $C_2$.

Figure 26(b) illustrates the convergence of the functions $\Phi$ and $\sigma$ along the same outgoing ray $u = 24.60$. The figure, which is qualitatively similar to figure 23(a) shows the difference between simulations ($\psi$) with different the splittings thresholds $S_{\text{split}} = 10^{-7}, 10^{-6}, 10^{-5}, 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}$. Since $\psi$ in this subsection is a function of $S_{\text{split}}$ instead of $N$, a point in fig. 26(b) illustrates the average absolute difference between the simulation a $S_{\text{split}}$ and the simulation with the next higher splittings threshold. For example the leftmost points in fig. 26(b) (at $S_{\text{split}} = 10^{-7}$) illustrates the difference between simulations with $S_{\text{split}} = 10^{-7}$ and $S_{\text{split}} = 10^{-6}$.

As the splitting thresholds decreases we expect to see $\psi$ converge to zero. We see that for smaller $S_{\text{split}}$, the $\psi$ decrease indication a convergence of the unknown functions. This convergence is further demonstrated in figure 27 which shows $\Phi$ and $\sigma$ as functions of $r$ close to the $r = 0$ singularity (in the interval $0.02 < r < 0.0201$) along the same outgoing ray that was used for previous plots. From these figures it is clear that even in the vicinity of the $r = 0$ singularity, the unknown functions are converging. It is also important to notice that even though $C_1$ and $C_2$ cease to decrease at rather high splitting thresholds, the unknown functions themselves continue to benefit from lower splitting thresholds.

[1] D. Goldwirth and T. Piran, Phys. Rev. D 36, 3575 (1987).
[2] E. Poisson and W. Israel, Phys. Rev. D 41, 1796 (1990).
[3] A. Ori, Phys. Rev. Lett. 67, 789 (1991).
[4] A. Ori, Phys. Rev. Lett. 68, 2117 (1992).
[5] M. Gnedin and N. Gnedin, Class. Quant. Grav. 10, 1083 (1993).
[6] A. Bonanno, S. Droz, W. Israel, and S. Morsink, Proc. Roy. Soc. London A 450, 553 (1994).
[7] P. Brady and J. Smith, Phys. Rev. Lett. 75, 1256 (1995).
[8] S. Droz, Helv.Phys.Acta 69, 257 (1996).
[9] L. M. Burko and A. Ori, eds., Internal structure of black holes and spacetime singularities, vol. 13 of the Annals of the Israel Physical Society (Israel Physical Society, Jerusalem, 1997), ISBN 0-7503-05487.
[10] L. M. Burko, Phys. Rev. Lett. 79, 4958 (1997).
[11] L. M. Burko, Phys. Rev. D. 59, 024011 (1999).
[12] L. M. Burko and A. Ori, Phys. Rev. D 57, 7084 (1998).
[13] L. M. Burko, Phys. Rev. D 60, 104033 (1999).
[14] L. M. Burko, Phys. Rev. D 66, 024046 (2002).
[15] L. M. Burko, Phys. Rev. Lett. 90, 121101 (2003), erratum in Phys. Rev. Lett. 90, 249902 (E).
[16] Y. Oren and T. Piran, Phys. Rev. D 68, 044013 (2003).
[17] S. Hod and T. Piran, Phys. Rev. D 55, 3485 (1997).
[18] S. Hod and T. Piran, Gen. Rel. Grav. 30, 1555 (1998).
[19] S. Hod and T. Piran, Phys. Rev. Lett. 81, 1554 (1998).
[20] A. Ori, Phys. Rev. Lett. 83, 5423 (1999).
[21] A. Ori, Phys. Rev. D 61, 024001 (2000).
[22] B. K. Berger, Living Reviews Relativity (2002), http://www.livingreviews.org/ - gr-qc/0201056.
[23] A Hamilton and S. Pollack, gr-qc/0411061 (2004).
[24] A Hamilton and S. Pollack, gr-qc/0411062 (2004).
[25] M. Dafermos, Proceedings of the Seventh Hungarian Relativity Workshop (2004), to appear, gr-qc/0401121.
[26] V. P. Frolov and I. D. Novikov, Black Hole Physics (Kluwer Academic Publishers, 1998).
[27] O. Gurtug and M. Halilsoy, gr-qc/0203019 (2002).
[28] L. M. Burko and A. Ori, Phys. Rev. D 56, 7820 (1997).
[29] L. M. Burko, Phys. Rev. D 58, 084013 (1998).
[30] C. W. Misner, K. S. Thorne, and J. A. Wheeler, Gravitation (W. H. Freeman, San Francisco, 1973).
[31] E. Poisson and W. Israel, Phys. Rev. Lett. 63, 1663 (1989).
[32] Y. B. Zeldovich and I. D. Novikov, Relativistic Astrophysics, vol. II, The Structure and Evolution of the Universe (The University of Chicago press, 1983).
[33] W. B. Bonnor and P. C. Vaidya, Gen. Relativ. Gravit. 1 (1970).
[34] L. P. Grishchuk, Sov. Phys. Doklady **190**, 1066 (1970).