A Hölder Stability Estimate for a 3D Coefficient
Inverse Problem for a Hyperbolic Equation With a
Plane Wave

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Abstract

A 3D coefficient inverse problem for a hyperbolic equation with non-overdetermined
data is considered. The forward problem is the Cauchy problems with the initial
condition the delta function concentrated at a single plane (i.e. the plane wave). A
certain associated operator is written in finite differences with respect to two out of
three spatial variables, i.e. “partial finite differences”. The grid step size is bounded
from the below by a fixed number. A Carleman estimate is applied to obtain, for
the first time, a Hölder stability estimates for this problem. Another new result is
an estimate from the below of the amplitude of the first term of the expansion of
the solution of the forward problem near the characteristic wedge.

Key Words: coefficient inverse problem, hyperbolic equation, geodesic lines, Carle-
man estimate, Hölder stability estimate.

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1 Introduction

We consider a forward Cauchy problem for a wave-like PDE in \( \mathbb{R}^3 \). In this problem,
the source is the \( \delta \)-function concentrated at a plane, which models the incident plane
wave. For this problems, we consider a Coefficient Inverse Problem (CIP). Applications
of this CIP is discussed in section 2. We obtain a Hölder stability estimate for our CIP.
Uniqueness theorem follows immediately from this estimate.

We assume that a certain 2 × 2 system of non local PDEs associated with the original
Cauchy problem is written in partial finite differences, i.e. finite differences with respect
to two out of three spatial variables. The derivatives with respect to both the third
spatial variable and time are written in the conventional manner. The grid step size of
finite differences is assumed to be bounded from the below by a certain small positive
constant. We point out that such a bound always takes place in computations. Thus,
this assumption has an applied meaning.
The data for our CIP are non-overdetermined ones, i.e. the number \( m = 3 \) of free variables in the data equals the number \( n = 3 \) of free variables in the unknown coefficient. It is well known that uniqueness and stability results for multidimensional CIPs with non-overdetermined data are currently proven only by the method of [6], which is based on Carleman estimates, see, e.g. [3, 4, 11, 17, 30] and references cited therein for some samples of publications, which use this method.

Applications of the idea of [6] to the developments of the so-called “convexification” globally convergent numerical method for CIPs can be found in, e.g. [7, 8, 13, 16, 17]. The originating publications about the convexification are [9, 10]. Numerical studies by the convexification method of a similar CIP for the same hyperbolic PDE as the one of this paper and with the point source in the initial condition can be found in [13] and [17] Chapter 8]. The second generation of the convexification method was developed in [1, 2, 19].

However, in terms of uniqueness theorems and stability estimates, the method of [6] works only under the assumption that one of initial conditions of a corresponding PDE does not equal zero in the entire domain of interest. The only exceptions are two recent works of Rakesh and Salo [22, 23] as well as some follows up publications of these authors. In [22] a stability result was proven for a CIP for the PDE \( u_{tt} = \Delta u + q(x)u, x \in \mathbb{R}^3 \) with two initializing plane waves and with non-overdetermined data. In [23] uniqueness is proven for a CIP with non-overdetermined data for the analog of this equation in the frequency domain.

In this paper, we consider the case of the hyperbolic PDE with a non constant unknown coefficient in the principal part of the hyperbolic operator with a single incident plane wave. Uniqueness and stability results were not proven for this CIP in the past. Our case is more complicated than the one in [22]. This is basically because the geodesic lines in our case are curves rather than straight lines of the above PDE. The above mentioned assumption of partial finite differences for boundary value problems for \( 2 \times 2 \) systems of non-local PDEs is imposed to avoid the assumption of [6] of the non-vanishing initial condition.

To prove our target result, we use the expansion of the solution of our forward problem near the characteristic wedge. Such an expansion is known only for the case of the point source [24, Theorem 4.1], [25, Lemma 2.2.1]. Then we combine this expansion with the Carleman estimate for the above mentioned \( 2 \times 2 \) systems of non-local PDEs.

There are two new results of this paper:

1. An estimate from the below of the amplitude of the first term of the expansion of the solution of the forward problem near the characteristic wedge (Theorem 3.1).

2. A Hölder stability estimate for the above mentioned CIP (Theorem 6.1).

**Remark 1.1.** It seems to be convenient to formulate our main theorems in the next section 2. However, prior to these formulations, we need to apply some transformations to the solution of our Forward Problem. Thus, we postpone those formulations until sections 3, 5 and 6.

The paper is organized as follows. In section 2 we pose forward and inverse problems and describe some applications of our CIP. In section 3, we derive the structure of the solution for the forward problem. In section 4, we change variables and reduce our CIP to a boundary value problem for a \( 2 \times 2 \) system of non-local PDEs and reformulate in
partial finite differences a boundary value problem, which was derived in this section. In section 5 we formulate and prove two Carleman estimates. In section 6 we prove a stability estimate for the CIP.

2 Statements of Forward and Inverse Problems

Below \( x = (x, y, z) \in \mathbb{R}^3 \). Denote

\[
\mathbb{R}_+^3 = \{(x, y, z) : z > 0\}, \quad \mathbb{R}_-^3 = \{(x, y, z) : z < 0\}, \quad \mathbb{R}_T^4 = \mathbb{R}^3 \times (0, T).
\]

Let \( X, T > 0 \) be two numbers. We define our domain of interest \( \Omega \subset \mathbb{R}^3 \) and related surfaces as:

\[
\Omega = \{x: -X < x, y < X, z \in (0, 1)\}, \quad (2.1)
\]

\[
\Gamma = \{x = (x, y, z): -X < x, y < X, z = 0\} \subset \partial \Omega, \quad \Gamma_T = \Gamma \times (0, T), \quad (2.2)
\]

\[
\Gamma' = \{x = (x, y, z): -X < x, y < X, z = 1\} \subset \partial \Omega, \quad \Gamma'_T = \Gamma' \times (0, T), \quad (2.3)
\]

\[
\Theta = \{x, y = \pm X, z \in (0, 1)\} \subset \partial \Omega, \quad \Theta_T = \Theta \times (0, T). \quad (2.4)
\]

We assume that the function \( n(x) \) be defined in \( \mathbb{R}^3 \) and

\[
n(x) \in C^{16}(\mathbb{R}^3). \quad (2.5)
\]

Even though the smoothness requirement (2.5) seems to be excessive, we are unaware how to decrease the smoothness. Indeed, we pose the Forward Problem in section 2. Next, we prove in section 3 a certain representation of the solution of this problem. This representation uses (2.5). In addition, it is worth to point out that the issue of the minimal smoothness is traditionally a minor concern in the field of Coefficient Inverse Problems, see, e.g. [20, 21], [24, Theorem 4.1].

Consider two arbitrary numbers \( n_0, n_{00} \) such that \( 1 < n_0 < n_{00} \). We assume everywhere below that

\[
1 \leq n(x) \leq n_0, \quad x \in \mathbb{R}^3, \quad \|n\|_{C^2(\mathbb{R}^3)} \leq n_{00}, \quad (2.6)
\]

\[
n(x) = 1 \text{ in } \mathbb{R}_-^3 \cup \{ |x| \geq X, |y| \geq X \}. \quad (2.7)
\]

In the case of an electric wave field propagation, \( n(x) \) is the refractive index, \( n^2(x) \) is the spatially distributed dielectric constant of the medium, and the function \( u(x, t) \), which we introduce below, is a component of that electric field. Therefore, the CIP, which we study here, has an application in the problem of the determination of the dielectric constant of the medium using the data of the scattered electric wave. These data are measured at a part \( (\Gamma \cup \Theta) \subset \partial \Omega \) of the boundary \( \partial \Omega \) of the domain \( \Omega \). It was shown numerically on microwave experimental data in [7, 8, 17] and analytically in [28] that in some applications, one wave-like PDE can govern the process of electric waves propagations equally well with the full system of Maxwell’s equations. In fact, this is also claimed heuristically in the classic textbook of Born and Wolf [5, pages 695,696]. Another area of applications of the CIP of this paper is acoustics, in which case the function \( 1/n(x) \) is the speed of sound waves propagation in the medium.

Forward Problem. Solve the following Cauchy problem:

\[
n^2(x) u_{tt} - \Delta u = \delta(z) \delta(t), \quad (x, t) \in \mathbb{R}^4, \quad (2.8)
\]
\[ u_{t=0} = 0. \]  

Since by (2.7) \( n(x) = 1 \) in \( \mathbb{R}^3 \), then the incident wave in (2.8), (2.9) is the plane wave, which is incident at the plane \( \Sigma \),

\[ \Sigma = \{ x \in \mathbb{R}^3 : z = 0 \} \]  

and propagates in the direction parallel to the \( z \)-axis. The function \( u(x, t) \) in the CIP is the solution of the Forward Problem.

**Coefficient Inverse Problem (CIP).** Suppose that the following functions \( f_0, f_1, f_2 \) are given:

\[ u|_{\Gamma_T} = f_0(x, t), \quad u_z|_{\Gamma_T} = f_1(x, t), \quad u|_{\Theta_T} = f_2(x, t), \]  

(2.11)

where \( u(x, t) \) is the solution of Forward Problem (2.8), (2.9). Determine the function \( n(x) \) for \( x \in \Omega \), assuming that this function satisfies conditions (2.6), (2.7).

It follows from (2.1), (2.2) that the data at \( \Gamma_T \) in (2.11) are the backscattering data.

**Remark 2.1.** Let \( S_T = \partial \Omega \times (0, T) \). Note that we do not need the data at \( S_T \setminus (\Gamma_T \cup \Theta_T) \) in our CIP, i.e. we do not need the data on the transmitted side of the rectangular prism \( \Omega \).

3 **The Structure of the Solution of the Forward Problem**

3.1 **Geodesic lines**

Let \( \tau(x) \) be the travel time, which the above plane wave needs to travel from the plane (2.10) to the point \( x \in \mathbb{R}^3 \). The function \( \tau(x) \) satisfies the eikonal equation and it equals zero at the plane \( \Sigma \), i.e.

\[ |\nabla \tau(x)|^2 = n^2(x), \quad x \in \mathbb{R}^3; \quad \tau|_{z=0} = 0. \]  

(3.1)

We obtain from (3.1)

\[ \tau(x) = \pm \int_0^z \sqrt{n^2(x, y, s) - \left( \tau_x^2 + \tau_y^2 \right)(x, y, s)} \, ds. \]

However, since Physics tells us that \( \tau(x) > 0 \), then the correct formula for \( \tau(x) \) is

\[ \tau(x) = \int_0^z \sqrt{n^2(x, y, s) - \left( \tau_x^2 + \tau_y^2 \right)(x, y, s)} \, ds > 0, \quad x \in \mathbb{R}^3. \]  

(3.2)

In particular,

\[ \partial_z \tau(x) > 0, \quad x \in \mathbb{R}^3. \]  

(3.3)

The function \( \tau(x) \) is generated by the conformal Riemannian metric with the element of its length

\[ d\tau = n(x) |dx|. \]  

(3.4)
Geodesic lines of this metric are orthogonal at any point \( x \) to the corresponding wave front passing through this point. We assume everywhere below that the following assumption holds:

**Assumption 3.1.** Geodesic lines of metric (3.4) satisfy the regularity condition in \( \mathbb{R}^3_+ \), i.e., for each point \( x \in \mathbb{R}^3_+ \) there exists a single geodesic line \( L(x) \) connecting \( x \) with the plane \( \Sigma \) in (2.10) and such that \( L(x) \) intersects the plane \( \Sigma \) orthogonally.

Below \( L(x) \) denotes geodesic lines specified in Assumption 3.1. The function \( \tau(x) \) has the form

\[
\tau(x) = \int_{L(x)} n(x') d\sigma, \quad x \in \mathbb{R}^3_+, \tag{3.5}
\]

where \( x' = x'(\sigma) \) and \( d\sigma \) is the element of the Euclidean length. Denote

\[
p(x) = \nabla \tau(x). \tag{3.6}
\]

To find \( L(x) \) we need to solve the Cauchy problem for the following system of ordinary differential equations:

\[
\frac{dx}{ds} = \frac{p(x)}{n^2(x)}, \quad \frac{dp(x)}{ds} = \nabla \ln n(x), \quad \frac{d\tau(x)}{ds} = 1, \quad s > 0, \tag{3.7}
\]

\[
x|_{s=0} = x^0, \quad p|_{s=0} = p^0, \quad \tau|_{s=0} = 0, \tag{3.8}
\]

where \( x^0 = (x^0, y^0, 0) \in \Sigma \). Since the vector \( p^0 \) is orthogonal to the plane \( \Sigma \) and \( |p^0| = 1 \) on \( \Sigma \), then \( p^0 = (0, 0, 1) \).

For each point \( x^0 \in \Sigma \), the solution of Cauchy problem (3.7), (3.8) defines the geodesic line of the Riemannian metric (3.4), which is orthogonal to the plane \( \Sigma \) at the point \( x^0 \). By Assumption 3.1, there exists a one-to-one correspondence between the points \( x \in \mathbb{R}^3_+ \) and the pairs \((x^0, s) \in \Sigma \times (0, \infty)\). The equation of \( L(x) \) is given in the form \( x = \xi(s, x^0) \). It follows from the last equation (3.7) and initial conditions (3.8) that, to find the function \( \tau(x) \), we need to solve the equation \( x = \xi(s, x^0) \) with respect to the vector \((s, x^0, y^0)\). Then we find \( s = s(x) \). Next, recalling that by (3.7) and (3.8) the parameter \( s \) coincides with \( \tau(x) \), we set \( \tau(x) = s(x) \). The smoothness of functions \( \xi(s, x^0), p(s, x^0), s(x) \) is determined by the smoothness of the function \( n(x) \). Since \( n \in C^{16}(\mathbb{R}^3) \) by (2.5), then the function \( \tau \in C^{16}(\mathbb{R}^3_+) \) (see, for instance, [25], pp. 26-27, for the similar derivation).

Introduce the non-negative function \( \varphi(x) \) as

\[
\varphi(x) = \begin{cases} 
-z & \text{for } z \leq 0, \\
\tau(x) & \text{for } z > 0.
\end{cases} \tag{3.9}
\]

Then the function \( \varphi \in C^{16}(\mathbb{R}^3_+) \). The equation \( t = \varphi(x) \) defines the characteristic wedge in \( \mathbb{R}^4 \) for the plane wave originated on the plane \( \{z = 0\} \) while it travels inside space \( \mathbb{R}^3 \).

Let \( T > 0 \) be a number. Define the domain \( D_T \) as

\[
D_T = \{(x, t) : 0 \leq \varphi(x) \leq t \leq T\}. \tag{3.10}
\]

By (2.6) the speed \( 1/n(x) \leq 1 \). Hence, (3.2) and (3.9) imply that

\[
D_T \subset \{(x, y) \in \mathbb{R}^2, z \in [-T, T] \times [0, T]\}. \tag{3.11}
\]
3.2 The structure of the solution of the forward problem

Lemma 3.1. Let conditions (2.1)-(2.7) be in place and let $L(x)$ be the geodesic line corresponding to $\tau(x)$. Then the following inequality holds along this geodesic line:

$$\frac{d}{ds} \Delta \tau(x) \leq 6n_{00}^2, \; x \in \mathbb{R}^3_+,$$

(3.12)

where constant $n_{00}$ is defined in (2.6).

Proof. Only in this proof we introduce the following notations:

$$x = (x_1, x_2, x_3), \; p(x) = \nabla \tau(x), \; \tau_{x_ix_j}(x) = \kappa_{ij}(x), \; i, j = 1, 2, 3,$$

$$\Delta \tau(x) = \kappa(x) = \sum_{i=1}^{3} \kappa_{ii}(x).$$

Denote $p_i = \tau_{x_i}, \; i = 1, 2, 3$ and use the eikonal equation $|\nabla \tau(x)|^2 = n^2(x)$. Hence,

$$\sum_{i=1}^{3} p_i^2 = n^2(x).$$

Differentiate this equation with respect to $x_j$ and use $(p_i)_{x_j} = (p_j)_{x_i} = \kappa_{ij}$. We obtain

$$\sum_{i=1}^{3} p_i \kappa_{ij} = nn_{x_j}, \; j = 1, 2, 3.$$

Differentiating this equality with respect to $x_\ell$, we get

$$\sum_{i=1}^{3} p_i(\kappa_{\ell j})_{x_i} + \sum_{i=1}^{3} \kappa_{ij} \kappa_{i\ell} = n_{x_j} n_{x_\ell} + nn_{x_\ell x_j}, \; j, \ell = 1, 2, 3.$$

Along the geodesic line $\Gamma(x)$ we can rewrite the latter equation as

$$\frac{dk_{j\ell}}{ds} + \frac{1}{n^2} \sum_{i=1}^{3} \kappa_{ij} \kappa_{i\ell} = \frac{1}{n^2} (n_{x_j} n_{x_\ell} + nn_{x_\ell x_j}), \; j, \ell = 1, 3.$$

Take here $\ell = j$ and consider the summation with respect to $j$ from 1 to 3. We obtain

$$\frac{dk}{ds} + \frac{1}{n^2} \sum_{i,j=1}^{3} \kappa_{ij}^2 = \frac{1}{n^2(x)} |\nabla n(x)|^2 + \frac{1}{n(x)} \Delta n(x).$$

(3.13)

It follows from (2.6) that

$$\left| \frac{1}{n^2(x)} |\nabla n(x)|^2 + \frac{1}{n(x)} \Delta n(x) \right| \leq 6n_{00}^2, \; x \in \mathbb{R}^3_+.$$

(3.14)

The target estimate (3.12) of Lemma 3.1 follows from (3.13) and (3.14). □

Let $H(t)$ be the Heaviside function,

$$H(t) = \begin{cases} 1 \text{ for } t \geq 0, \\ 0 \text{ for } t < 0. \end{cases}$$
Theorem 3.1. Let conditions (2.1)-(2.7) hold. Then:

1. There exists unique solution of problem (2.8), (2.9), which can be represented as

\[
    u(x, t) = H(t - \varphi(x)) [A(x) + \hat{u}(x, t)], \ x \in \mathbb{R}^3, \ t \in (0, T],
\]

where the function \( \hat{u} \in C^2(D_T) \), is compactly supported in the domain \( D_T \) and

\[
    \lim_{t \to \varphi(x)^+} \hat{u}(x, t) = 0,
\]

\[
    A(x) > 0 \ in \ \mathbb{R}^3.
\]  

The function \( A(x) \in C^{14}(\mathbb{R}^3) \) and has the form:

\[
    A(x) = \frac{1}{2} \left\{ \begin{array}{cl}
    1 & \text{for } z \leq 0, \\
    \exp \left( -\frac{1}{2} \int \frac{\Delta r(x')}{n^2(x')} ds \right) & \text{for } z > 0,
    \end{array} \right.
\]  

where \( x' \) is the variable point along \( L(x) \).

2. The inequality \( A(x) > 0 \) in (3.17) can be replaced with the following stronger estimate from the below:

\[
    A(x) \geq A_0 = \frac{1}{2} \exp \left( -3n^2_{\text{min}}n_0^2/2 \right), \ x \in \Omega.
\]

Proof. Problem (2.8), (2.9) is equivalent with the following one:

\[
    Lu = n^2(x) u_{tt} - \Delta u = 0, \ (x, t) \in \mathbb{R}^4, \ z \neq 0, \ u|_{t<0} = 0,
\]

\[
    u|_{z=+0} - u|_{z=-0} = -\delta(t), \ u|_{z=+0} - u|_{z=-0} = 0, \ (x, y, t) \in \mathbb{R}^3.
\]

Let \( r > 1 \) be an integer, which will be chosen later. We represent the solution of problem (3.20), (3.21) in the form

\[
    u(x, t) = \sum_{k=0}^{r} \alpha_k(x) H_k(t - \varphi(x)) + u_r(x, t),
\]

where

\[
    H_k(t) = \frac{t^k}{k!} H(t), \ k = 0, 1, 2, \ldots, r.
\]

Also, denote \( H_{-1}(t) = \delta(t) \) and \( H_{-2}(t) = \delta'(t) \).

Recall that the function \( \varphi(x) \) is continuous in \( \mathbb{R}^3 \), \( \varphi(x) = -z \) for \( z \leq 0 \) and \( \varphi \in C^{16}(\mathbb{R}^3_+) \). But the derivative \( \varphi_z(x) \) is discontinuous across \( z = 0 \), namely, \( \varphi_z(x) = -1 \) for \( z \leq 0 \) and \( \varphi_z(x) = 1 \) for \( z = +0 \). Thus,

\[
    \varphi|_{z=0} = 0, \ \varphi_z|_{z=+0} = 1, \ \varphi_z|_{z=-0} = -1.
\]

Taking into account (3.24), we need to consider representation (3.22) separately for \( z < 0 \) and \( z \geq 0 \).

We seek functions \( \alpha_k(x) \) in the form:

\[
    \alpha_k(x) = \left\{ \begin{array}{ll}
    \alpha_k^-(x), & z < 0, \\
    \alpha_k^+(x), & z \geq 0.
    \end{array} \right.
\]
Substituting representation \((3.22)\) in \((3.21)\) and equating coefficients at \(H_k(t)\), we obtain
\[
\begin{align*}
-\alpha_k & - \alpha_k |_{z=+0} \alpha_k |_{z=-0} = -\delta_{k0}, \\
\delta_{k0} & \text{ is the Kronecker’s delta. Then function } u_r(x, t) \text{ satisfies the following conjugate conditions}
\end{align*}
\]
\[
\begin{align*}
(u_r) |_{z=+0} - (u_r) |_{z=-0} = 0, \quad u_r |_{z=+0} - u_r |_{z=-0} = 0.
\end{align*}
\]
Using \((3.24)\), we find from equations \((3.22)\)
\[
\alpha_k^- |_{z=0} = \alpha_k^+ |_{z=0} = \frac{\delta_{k0}}{2}, \quad k = 0, 1, \ldots, r.
\]
Apply the operator \(L u\) for \(z \neq 0\) to both sides of \((3.22)\). First,
\[
\partial_t^2 \left[n^2(x) \alpha_k(x) H_k(t - \varphi(x))\right] = \alpha_k(x) n^2(x) H_{k-2}(t - \varphi(x)).
\]
Second,
\[
-\Delta \left[\alpha_k(x) H_k(t - \varphi(x))\right] = -H_k(t - \varphi(x)) \Delta \alpha_k(x) - 2\nabla \alpha_k(x) \cdot \nabla H_k(t - \varphi(x))
\]
\[
\begin{align*}
&= -H_k(t - \varphi(x)) \Delta \alpha_k(x) + 2\nabla \alpha_k(x) \cdot \nabla \varphi(x) H_{k-1}(t - \varphi(x)) + \alpha_k(x) \Delta \varphi(x) H_{k-1}(t - \varphi(x)) \quad (3.30)
\end{align*}
\]
Since by \((3.1)\) \(n^2(x) - |\nabla \varphi(x)|^2 = 0\), then \((3.29)\) and \((3.30)\) imply:
\[
\begin{align*}
-\Delta \left[\alpha_k(x) H_k(t - \varphi(x))\right] &= -H_k(t - \varphi(x)) \Delta \alpha_k(x) + 2\nabla \alpha_k(x) \nabla \varphi(x) H_{k-1}(t - \varphi(x)) + \alpha_k(x) \Delta \varphi(x) H_{k-1}(t - \varphi(x)).
\end{align*}
\]
Hence,
\[
\begin{align*}
\sum_{k=0}^r \alpha_k(x) H_k(t - \varphi(x)) &= \sum_{k=0}^r \left[2\nabla \alpha_k(x) \nabla \varphi(x) + \alpha_k(x) \Delta \varphi(x)\right] H_{k-1}(t - \varphi(x)) \\
&\quad - \sum_{k=0}^r [\Delta \alpha_k(x)] H_k(t - \varphi(x)).
\end{align*}
\]
Next,
\[
\begin{align*}
- \sum_{k=0}^r [\Delta \alpha_k(x)] H_k(t - \varphi(x)) &= - \sum_{k=1}^{r+1} [\Delta \alpha_{k-1}(x)] H_{k-1}(t - \varphi(x)) = \\
&\quad - \sum_{k=0}^r [\Delta \alpha_{k-1}(x)] H_{k-1}(t - \varphi(x)) - [\Delta \alpha_r(x)] H_r(t - \varphi(x)),
\end{align*}
\]
where we formally set \(\alpha_{-1}(x) \equiv 0\). Hence, we obtain
\[
\mathcal{L} u = \sum_{k=0}^r \left[2\nabla \alpha_k(x) \cdot \nabla \varphi(x) + \alpha_k(x) \Delta \varphi(x) - \Delta \alpha_{k-1}(x)\right] H_{k-1}(t - \varphi(x)).
\]
Equating here to zero terms at \( H_{k-1}(t - \varphi(x)) \) and taking into account conditions (3.28), we obtain equations for \( \alpha_k^- \) and \( \alpha_k^+ \), \( k = 0, 1, \ldots, r \),

\[
2 \nabla \alpha_k^- \cdot \nabla \varphi(x) + \alpha_k^-(x) \Delta \varphi(x) = \Delta \alpha_{k-1}^-(x), \quad z < 0, \quad \alpha_k^-|_{z=-0} = \frac{1}{2} \delta_{k0}, \quad (3.31)
\]

\[
2 \nabla \alpha_k^+ \cdot \nabla \tau(x) + \alpha_k^+(x) \Delta \tau(x) = \Delta \alpha_{k-1}^+(x), \quad z > 0, \quad \alpha_k^+|_{z=0} = \frac{1}{2} \delta_{k0}, \quad (3.32)
\]

where \( \alpha_{-1}^- = \alpha_{-1}^+ = 0 \).

Note that in equations (3.31) \( \varphi(x) = -z \). Therefore it can be written as follows

\[
-2(\alpha_k^-)z = \Delta \alpha_{k-1}^-(x), \quad z < 0, \quad \alpha_k^-|_{z=0} = \frac{1}{2} \delta_{k0}, \quad k = 0, 1, \ldots, r. \quad (3.33)
\]

It follows from (3.33) that

\[
\alpha_k^-(x) = \frac{1}{2} \delta_{k0}, \quad k = 0, 1, \ldots, r. \quad (3.34)
\]

Since \( L u(x, t) = 0 \) for \( z \neq 0 \) and conditions (3.27) hold, the equation for the function \( u_r(x, t) \) is:

\[
n^2(x)(u_r)_{tt} - \Delta u_r = F_r(x, t), \quad (x, t) \in \mathbb{R}^4, \quad u_r|_{t<0} = 0, \quad (3.35)
\]

where

\[
F_r(x, t) = H_r(t - \varphi(x)) \Delta \alpha_r(x). \quad (3.36)
\]

Integrate now equation (3.32) along the geodesic line \( L(x) \). By (3.36) and (3.37) we have along this line \( \nabla \tau(x) = p(x) = n^2(x) dx/\alpha \). Therefore,

\[
\nabla \alpha_k^-(x) \cdot \nabla \tau(x) = n^2(x) \nabla \alpha_k^-(x) \cdot \frac{dx}{ds} = n^2(x) \frac{d}{ds} \alpha_k^-(x).
\]

Hence, (3.32) is equivalent with:

\[
2n^2(x) \frac{d}{ds} \alpha_k^-(x) + \alpha_k^-(x) \Delta \tau(x) = \Delta \alpha_{k-1}^-(x), \quad \alpha_k^-|_{z=0} = \frac{1}{2} \delta_{k0}. \quad (3.37)
\]

The solution of the Cauchy problem

\[
2n^2(x) \frac{d}{ds} \alpha_0^+(x) + \alpha_0^+(x) \Delta \tau(x) = 0, \quad \alpha_0^+|_{z=0} = \frac{1}{2}
\]

is given by the formula

\[
\alpha_0^+(x) = \frac{1}{2} \exp \left( -\frac{1}{2} \int_{L(x)} \frac{\Delta \tau(x')}{n^2(x')} ds \right), \quad (3.38)
\]

where \( x' \) is a variable point along \( L(x) \). Next, dividing both sides of equation (3.37) by \( 2n^2(x) \) we can rewrite (3.37) in the form

\[
\exp \left( -\frac{1}{2} \int_{L(x)} \frac{\Delta \tau(x')}{n^2(x')} ds \right) \frac{d}{ds} \left[ \alpha_k^+(x) \exp \left( \frac{1}{2} \int_{L(x)} \frac{\Delta \tau(x')}{n^2(x')} ds \right) \right] = \quad (3.39)
\]
\[
\alpha_k^+(x) = \frac{\Delta \alpha_{k-1}^+(x)}{2n^2(x)}, \quad \alpha_k^+|_{z=0} = 0, \ k = 1, \ldots, r.
\]

It follows from (3.37) and (3.38) that (3.39) is equivalent with
\[
\frac{d}{ds} \left( \frac{\alpha^+_k(x)}{\alpha^+_0(x)} \right) = \frac{\Delta \alpha^+_{k-1}(x)}{2n^2(x)\alpha^+_0(x)}, \quad \alpha_k^+|_{z=0} = 0, \ k = 1, \ldots, r. \tag{3.40}
\]

Integrating (3.40) with respect to \(s\) along \(L(x)\), we obtain
\[
\alpha_k^+(x) = \frac{1}{2} \alpha_0^+(x) \int_{L(x)} \frac{\Delta \alpha^+_{k-1}(x')}{n^2(x')\alpha^+_0(x')} ds, \ k = 1, \ldots, r. \tag{3.41}
\]

Let \(m\) be a sufficiently large integer, which we will choose below. If the function \(n \in C^m(\mathbb{R}^d)\), then
\[
\tau \in C^m(\mathbb{R}^3_+), \quad \alpha_k^+ \in C^{m-2k-2}(\mathbb{R}^3_+), \quad \Delta \alpha^+_r \in C^{m-2r-4}(\mathbb{R}^3_+).
\]

Define the domain \(G_X\) as
\[
G_X = \{x \in \mathbb{R}^3_+ : |x| \geq X, |y| \geq X, z \geq 0\}. \tag{3.42}
\]

It follows from (3.38) and (3.41) that \(\alpha_0^+(x) = 1/2\) and functions \(\alpha_k^+(x) = 0, \ k = 1, \ldots, r,\) for \(x \in G_X\). Indeed, since by (2.7) \(n(x) = 1\) in \(G_X\), then Assumption 3.1 implies that if \(x \in G_X\), then the geodesic line \(L(x)\) is a segment of the straight line orthogonal to the plane \(z = 0\), and this line does not intersect \(\Omega\).

Setting \(A(x) = \alpha_0(x)\) and using (3.22), (3.25), (3.34) and (3.38), we obtain (3.18).

It follows from (3.39) and the above arguments that the function \(F_r(x,t)\) possesses the following properties:

1. For any \(T > 0\)

\[
\text{support} (F_r(x,t)) \subset D^+_T = \{(x,t) \in D_T : z \geq 0\},
\]

where the set \(D_T\) is defined in (3.10).

2. \(F_r(x,t) = 0\) for \(x \in G_X\).

Furthermore, the projection of the set \(D^+_T\) on the space \(\mathbb{R}^3\) coincides with the set \(Y_T = \{x \in \mathbb{R}^3 : 0 \leq \tau(x) \leq T\}\). It follows from (3.11) that this set is bounded with respect to \(z\), i.e., \(Y_T \subset \{(x,y) \in \mathbb{R}^2, z \in [0,T]\}\). Hence, the set \((Y_T \setminus G_X) \subset \{|x|, |y| < X, 0 \leq z \leq T\}\) is bounded, where \(G_X\) is the set defined in (3.42). Thus, the function \(F_r(x,t) = 0\) outside of the finite domain \([|x|, |y| < X, 0 \leq z \leq T]\), i.e., it is compactly supported in \(\mathbb{R}^4_T\). Moreover, \(F_r \in H^d(\mathbb{R}^4_T)\), where \(d = \min(m - 2r - 4, r)\). Using the general theory of hyperbolic equations [18, Chapter 4], we conclude that the unique solution \(u_r \in H^{d+1}(\mathbb{R}^4_T)\) of the Cauchy problem (3.39), (3.40) exists and this solution is also compactly supported in \(\mathbb{R}^4_T\). Moreover, since \(F_r(x,t) = 0\) for \((x,t) \notin D_T\), then \(u_r(x,t) = 0\) for \(t < \varphi(x)\). Embedding theorem implies \(u_r \in C^2(\mathbb{R}^4_T)\) if \(d + 1 > 4\). Choose \(d = r = 4\) and \(m = 16\).

Then \(u_r \in C^2(\mathbb{R}^4_T)\). Since \(u_r(x,t) = 0\) for \(t < \varphi(x)\), we conclude that \(u_r \in C^2(\overline{D_T})\) and \(u_r(x,t) = 0\) for \(t = \varphi(x)\) together with derivatives up to the second order. This explains our smoothness condition (2.5).
Finally, setting in (3.22)

\[ A(\mathbf{x}) = \alpha_0(\mathbf{x}), \quad \tilde{u}(\mathbf{x}, t) = \sum_{k=1}^{r} \alpha_k(\mathbf{x}) \frac{(t - \varphi(\mathbf{x}))^k}{k!} + u_r(\mathbf{x}, t), \]

we finish the proof of (3.15) and (3.16).

We now want to prove that \( A(\mathbf{x}) = \alpha_0(\mathbf{x}) \geq A_0 \) with the positive number \( A_0 \) defined in (3.19). We use Lemma 3.1 for this purpose.

The function \( \alpha_0(\mathbf{x}) = 1/2 \) for \( z \leq 0 \). Using (3.12) and (3.38), estimate now the function \( \alpha_0^+(\mathbf{x}) \). Note first that

\[ \Delta \tau(\mathbf{x}) \big|_{z=0} = 0. \quad (3.43) \]

Indeed, eikonal equation (3.1) implies that

\[ \nabla \partial_z \tau(\mathbf{x}) \cdot \nabla \tau(\mathbf{x}) = n(\mathbf{x}) n_z(\mathbf{x}). \quad (3.44) \]

It follows from the condition \( \tau(\mathbf{x}) \big|_{z=0} = 0 \) in (3.1) that, at \( z = 0 \), \( (\tau)_{xx} = (\tau)_{yy} = 0 \), \( (\tau)_z = 1 \) and \( (\tau)_{xz} = (\tau)_{yz} = 0 \). Therefore (3.44) at \( z = 0 \) becomes

\[ (\tau)_{zz}(x, y, 0) = n(x, y, 0) n_z(x, y, 0) = 0. \]

Here the equality \( n_z(x, y, 0) = 0 \) follows from (2.5) and (2.7). Thus, (3.43) holds. Hence, integrating inequality (3.12) of Lemma 3.1 with respect to \( s \in (0, s') \), we conclude that along \( L(\mathbf{x}) \)

\[ \frac{\Delta \tau(\mathbf{x}')}{2 n^2(\mathbf{x}')} \leq 3 n_0^2 s', \quad \mathbf{x}' \in \mathbb{R}^3_+, \quad (3.45) \]

Formulae (3.38) and (3.45) imply:

\[ \alpha_0^+(\mathbf{x}) \geq \frac{1}{2} \exp \left( - \int_{\Gamma(\mathbf{x})} 3 n_0^2 s' ds' \right) = \frac{1}{2} \exp \left( - \frac{3 n_0^2}{2} \tau^2(\mathbf{x}) \right), \quad \mathbf{x} \in \mathbb{R}^3_+. \]

It follows from (2.6) and eikonal equation (3.1) that

\[ \partial_z \tau \leq n_0 \quad \text{in} \quad \mathbb{R}^3_+. \quad (3.46) \]

Since \( \tau(x, y, 0) = 0 \), then

\[ \tau(\mathbf{x}) = \int_{0}^{z} \partial_r \tau(x, y, r) dr \leq n_0, \quad \mathbf{x} \in \overline{\Omega}. \]

Hence,

\[ A(\mathbf{x}) \geq \alpha_0^+(\mathbf{x}) \geq \frac{1}{2} \exp \left( - \frac{3 n_0^2}{2} n_0^2 \right) = A_0, \quad \mathbf{x} \in \overline{\Omega}. \quad \square \]
4 A Boundary Value Problem in Partial Finite Differences

4.1 The boundary value problem

Consider the function \( v(x,t) \),

\[ v(x,t) = \int_0^t u(x,s) ds. \]  

(4.1)

Recall that by (3.9) \( \varphi(x) = \tau(x) \) for \( x \in \Omega \). Hence, by (3.15), (3.16) and (4.1)

\[ v(x,t) = (t - \tau(x)) H(t - \tau(x)) [A(x) + \hat{v}(x,t)] , \ x \in \Omega, \]  

(4.2)

where \( \hat{v}(x, \tau(x) + 0) = 0 \).

Estimate \( \max_{x \in \Omega} \tau(x) \). Since \( \tau|_{z=0} = 0 \), then, using (3.46), we obtain

\[ \tau (x, y, z) = \int_0^z \partial_r \tau(x, y, r) dr \leq n_0 z \leq n_0, \ x = (x, y, z) \in \Omega. \]

Hence,

\[ \max_{x \in \Omega} \tau(x) \leq n_0 \]  

(4.3)

We assume that \( T > n_0 \). Denote

\[ T_1 = T - n_0 > 0. \]  

(4.4)

Also, denote \( Q_{T_1} = \Omega \times (0, T_1) \). Let

\[ P(x,t) = v(x, t + \tau(x)) \) for \( (x, t) \in Q_{T_1}. \]  

(4.5)

This function is defined for all \( (x, t) \in Q_{T_1} \) since by (4.3)

\[ 0 < t + \tau(x) < T_1 + \max_{x \in \Omega} \tau(x) \leq T_1 + n_0 = T \) in \( Q_{T_1}. \)

Hence, by (4.2), (4.4) and (4.5)

\[ P(x,t) = t \left[ A(x) + \hat{P}(x,t) \right] , \ (x, t) \in Q_{T_1}, \]  

(4.6)

\[ \lim_{t \to 0^+} \hat{P}(x,t) = 0. \]  

(4.7)

Using (2.8), (3.1), (4.1) and (4.5), we obtain

\[ \Delta P - 2 \nabla_x P_t \cdot \nabla \tau - P_t \Delta \tau = 0, \ (x,t) \in Q_{T_1}. \]  

(4.8)

Denote

\[ w(x,t) = P_t (x, t). \]  

(4.9)
Note that by (3.19), (4.6) and (4.7)
\[ w(x, 0) = A(x) \geq A_0 = \frac{1}{2} \exp \left( -3n_0^2n_0^2/2 \right) > 0, \ x \in \Omega. \] (4.10)

Setting in (4.8) \( t = 0 \), using the fact that by (4.6) and (4.7) \( \Delta P(x, 0) = 0 \) and also using (4.9) and (4.10), we obtain \( \Delta \tau + 2(\nabla \ln w(x, 0)) \cdot \nabla \tau = 0 \) for \( x \in \Omega \). Hence, using this equation and (4.6), we obtain the 2 × 2 system of non local nonlinear PDEs:
\[
\begin{aligned}
\Delta w - 2\nabla_x w_1 \cdot \nabla \tau + 2w_1 (\nabla \ln w(x, 0)) \cdot \nabla \tau = 0, & \quad (x, t) \in Q_{T_1}, \\
\Delta \tau + 2(\nabla \ln w(x, 0)) \cdot \nabla \tau = 0, & \quad x \in \Omega.
\end{aligned}
\] (4.11)

To find boundary conditions for system (4.11), we use functions \( f_0(x, t), f_1(x, t), f_2(x, t) \) in (2.11). Let
\[
g_0(x, t) = f_0(x, \tau(x) + t), \quad (x, t) \in \Gamma_{T_1}, \\
g_1(x, t) = f_1(x, \tau(x) + t) + \partial_\tau g_0(x, t) \partial_\tau \tau(x), \quad (x, t) \in \Gamma_{T_1}, \\
g_2(x, t) = f_2(x, \tau(x) + t), \quad (x, t) \in \Theta_{T_1}.
\]

Since by (2.2) \( \Gamma \subset \{ z = 0 \} \) and since by (2.7) \( n(x) |_{\Gamma} = 1 \) and also since by the second condition in (3.1) \( \tau |_{\Gamma} = 0 \), then \( \partial_\tau \tau |_{\Gamma} = \partial_\tau \tau |_{\Gamma} = 0 \). Hence, (3.1) and (3.3) imply \( \partial_\tau \tau |_{\Gamma} = n(x) |_{\Gamma} = 1 \). Also, it obviously follows from (2.4) and (2.7) that \( \tau(x) |_{\Theta} = z \). Thus, the boundary conditions for system (4.11) are:
\[
w |_{\Gamma_{T_1}} = g_0(x, t), \quad w_z |_{\Gamma_{T_1}} = g_1(x, t), \quad w |_{\Theta_{T_1}} = g_2(x, t),
\] (4.12)
\[
\tau |_{\Gamma} = 0, \quad \partial_\tau \tau |_{\Gamma} = 1, \quad \tau |_{\Theta} = z.
\] (4.13)

Therefore, we arrive at the following Boundary Value Problem for the 2 × 2 system (4.11) of nonlinear and non local PDEs:

**Boundary Value Problem (BVP):** Find the pair of functions \( (w, \tau) \in C^2(Q_{T_1}) \times C^2(\overline{\Omega}) \) satisfying equations (4.11) and boundary conditions (4.12), (4.13), assuming that (4.10) holds.

As it was pointed out in Introduction, we cannot prove stability estimates for this BVP. However, we can prove the desired stability estimates if we rewrite this BVP in the form of partial finite differences, in which the derivatives with respect to \( x \) and \( y \) are written in finite differences, whereas the derivatives with \( z \) and \( t \) are written in the conventional continuous way. In doing so, we assume that the step size \( h \) of the finite difference scheme is bounded from the below by a fixed positive number. The latter assumption is a quite natural one in computations. Thus, we rewrite in subsection 4.2 the above BVP in partial finite differences.

### 4.2 Partial finite differences

For brevity, we use the same grid step size \( h \) in both \( x \) and \( y \) directions. Choose a small number \( h_0 \in (0, 1) \). We assume everywhere below in this paper that
\[
h \geq h_0.
\] (4.14)
Consider two partitions of the interval \([-X, X]\),

\(-X = x_0 < x_1 < \ldots < x_N < x_{N+1} = X, \quad x_i - x_{i-1} = h,\)

\(-X = y_0 < y_1 < \ldots < y_N < y_{N+1} = X, \quad y_i - y_{i-1} = h.\)

Thus, the interior grid points are \(\{(x_i, y_j)\}_{i,j=1}^N\) and other grid points are located on the part of the boundary \(\Theta \subset \partial \Omega\), see (2.4). Let \(x^h = (x_i, y_j)_{i,j=0}^{N+1}\) denotes semi-discrete points. For any function \(s(x, t)\) defined on the set \(Q_{T_1}\), we denote by \(s(x^h, t)\) the corresponding semi-discrete vector function defined at points \((x^h, t) \in \Omega \times [0, T_1]\). We also introduce the following notations:

\[\Omega_h = \left\{ x^h = (x_i, y_j, z)_{i,j=1}^N, z \in (0, 1) \right\},\]

\[\Omega_{h,T_1} = \Omega_h \times (0, T_1), \quad \Omega_{h,T_1} = \Omega_{h,T_1} \times [0, T_1],\]

\[\Gamma_h = \left\{ x^h = ((x_i, y_j, 0)_{i,j=1}^N, \Gamma_{h,T_1} = \Gamma_h \times (0, T_1),\right\}\]

\[\Gamma'_h = \left\{ x^h = ((x_i, y_j, 1)_{i,j=1}^N, \Gamma_{h,T_1} = \Gamma'_h \times (0, T_1),\right\}\]

\[\Theta_h = \left\{ x^h \in \Theta \right\}, \quad \Theta_{h,T_1} = \Theta_h \times (0, T_1).\]

The derivatives in finite differences with respect to \(x\) are defined as:

\[\partial_x s^h_{i,j}(z, t) = \frac{s^h_{i-1,j}(z, t) - s^h_{i+1,j}(z, t)}{2h}, \quad i, j = 1, \ldots, N, \quad (4.15)\]

\[\partial_x s^h(x^h, t) = \left\{ \partial_x s^h_{i,j}(z, t) \right\}_{i,j=1}^N, \quad (4.16)\]

\[\partial_x^2 s^h_{i,j}(z, t) = \frac{s^h_{i+1,j}(z, t) - 2s^h_{i,j}(z, t) + s^h_{i-1,j}(z, t)}{h^2}, \quad i, j = 1, \ldots, N, \quad (4.17)\]

\[\partial_x^2 s^h(x^h, t) = \left\{ \partial_x^2 s^h_{i,j}(z, t) \right\}_{i,j=1}^N. \quad (4.18)\]

Derivatives \(s^h_{xy}, s^h_{yy}\) are defined completely similarly. Formulas (4.13)-(4.18) as well as their analogs for \(s^h_{yx}, s^h_{yy}\) fully define finite difference versions of \(x, y\) derivatives involved in equations (4.11). Everywhere below the corresponding Laplace operator in the partial finite differences as well as the gradient vector are given by:

\[\Delta^h s^h = s^h_{xx} + s^h_{yy} + s^h_{zz} \nabla^h s^h = (s^h_{x}, s^h_{y}, s^h_{z})^T, \quad (4.19)\]

where the \(z\)-derivatives are understood in the regular manner, and the same for the \(t\)-derivatives in follow up formulas.

We introduce the semi-discrete analogs of conventional function spaces of real valued functions as:

\[H^{2,h}(Q_{h,T_1}) = \left\{ s^h(x^h, t) : \left\| s^h \right\|^2_{H^{2,h}(Q_{h,T_1})} = \sum_{i,j=1}^N h^2 \int_0^{T_1} \int_0^1 \sum_{m=0}^2 \left( \partial_x s^h(z, t) \right)^2 + \left( \partial_x^2 s^h \right)^2 \right] (x_i, y_j, z, t) \, dz \, dt < \infty, \right\} \]
In addition, \((4.12)\) and \((4.13)\) lead to the following boundary conditions for the vector functions

\[
H^{1,h}(\Omega_h) = \left\{ s^h(x^h) : \|s^h\|_{H^{1,h}(\Omega_h)}^2 = \sum_{i,j=1}^n h^2 \int_0^1 \left[ (\partial_x s^h)^2 + (s^h)^2 \right] (x_i, y_j, z) \, dz < \infty \right\},
\]

\[
L^h_2(\Omega_h) = \left\{ s^h(x^h) : \|s^h\|_{L^2(\Omega_h)}^2 = \sum_{i,j=1}^n h^2 \int_0^1 (s^h)^2 (x_i, y_j, z) \, dz \right\} < \infty,
\]

\[
H^{1,h}(\Gamma_h, T_1) = \left\{ s^h(x^h, t) : \|s^h\|_{H^{1,h}(\Gamma_h, T_1)} = \sum_{i,j=1}^n \int_{T_1}^1 (s^h)^2 (x_i, y_j, 0, t) \, dt < \infty \right\},
\]

\[
L^h_2(\Gamma_h, T_1) = \left\{ s^h(x^h, t) : \|s^h\|_{L^2(\Gamma_h, T_1)}^2 = \sum_{i,j=1}^n h \int_0^1 (s^h)^2 (x_i, y_j, 0, t) \, dt < \infty \right\},
\]

\[
L^h_2(\Theta_h, T_1) = \left\{ s^h(x^h, t) : \|s^h\|_{L^2(\Theta_h, T_1)}^2 = \sum_{i,j=1}^n h \int_0^1 \left[ (s^h)^2 (x_0, y_j, z, t) + (s^h)^2 (x_{N+1}, y_j, z, t) \right] \, dxdz < \infty \right\},
\]

\[
C^{2,h}(\Omega_h, T_1) = \left\{ s^h(x^h, t) : \|s^h\|_{C^{2,h}(\Omega_h, T_1)} = \max_{m=0,1,2} \left( \max_{Q_{h,T_1}} |\partial^m s^h|, \max_{\Gamma_{h,T_1}} |\partial^2_{x^h} s^h| \right) < \infty, \right\}
\]

\[
C^{n,h}(\Omega_h) = \left\{ s^h(x^h) : \|s^h\|_{C^{n,h}(\Omega_h)} = \max_{m\in[0,n]} \left( \max_{Q_{h,T_1}} |\partial^m s^h|, \max_{\Gamma_{h,T_1}} |\partial^2_{x^h} s^h| \right) < \infty, \right\}
\]

### 4.3 The Boundary Value Problem in Partial Finite Differences

We now rewrite the BVP \((4.11)\)-(\(4.13)\) as the BVP in partial finite differences with respect to the vector functions \((w^h(x^h, t), \tau^h(x^h))\):

\[
\Delta^h w^h - 2\tau^h w^h_{zt} - 2w^h_{ty} \tau^h_x - 2w^h_{ty} \tau^h_y + 2w^h_t (\nabla^h \ln w^h(x^h, 0)) \cdot \nabla^h \tau^h = 0, \quad (x^h, t) \in Q^h_{T_1},
\]

\[
\Delta^h \tau^h + 2 (\nabla^h \ln w^h(x^h, 0)) \cdot \nabla^h \tau^h = 0, \quad (x^h) \in \Omega^h.
\]

In addition, \((4.12)\) and \((4.13)\) lead to the following boundary conditions for the 2 \times 2 system \((4.20), (4.21)\)

\[
w^h|_{\Gamma_{h,T_1}} = g^0_h, \quad w^h_z|_{\Gamma_{h,T_1}} = g^1_h, \quad w^h|_{\Theta_{h,T_1}} = g^2_h.
\]
Also, using (4.10), we impose the following condition on the function \( w^h(x^h, 0) \):

\[
w^h(x^h, 0) = A^h(x^h) \geq A_0 = \frac{1}{2} \exp\left(-3n_0^2n_0^2/2\right) > 0 \text{ for } x^h \in \Omega^h.
\] (4.24)

**Boundary Value Problem** \(^h\) (BVP\(^h\)). Find the pair of functions \((w^h, \tau^h) \in C^{2,h}(\Omega^h) \times C^{2,h}(\Omega^h)\) satisfying conditions (4.20)-(4.23), assuming that (4.24) holds.

## 5 Two Carleman Estimates

In this section, we prove two Carleman estimates for operators written in the above partial finite differences. Let the function \( \xi(x^h) \in C^{1,h}(\Omega^h) \). Consider three numbers \( \xi_0, \xi_1, \xi_2 \) such that \( 0 < \xi_0 < \xi_1 \) and \( \xi_2 > 0 \). We assume that

\[
0 < \xi_0 \leq \xi^h(x^h) \leq \xi_1, \quad \text{for all } x^h \in \Omega^h,
\] (5.1)

\[
\xi_2 = \max_{\Omega^h} |\xi^h(x^h)|.
\] (5.2)

For functions \( v^h \in H^{2,h}(Q_{h,T_1}) \), we define the linear operator \( L^h \) as:

\[
L^h v^h = \Delta^h v^h - \xi^h(x^h) v^h_{2t}, \text{ in } Q_{h,T_1},
\] (5.3)

where the operator \( \Delta^h \) is defined in (4.19).

**Theorem 5.1** (the first Carleman estimate). Let the function \( \xi(x^h) \in C^{1,h}(\Omega^h) \) satisfies conditions (5.1), (5.2). Consider the number \( \alpha_0 \),

\[
\alpha_0 = \alpha_0(\xi_1) = \frac{2}{3\xi_1} > 0.
\] (5.4)

Then there exists a sufficiently large number \( \lambda_0 = \lambda_0(T_1, \xi_0, \xi_1, \xi_2, h_0, X) \geq 1 \) depending only on listed parameters, such that for all \( \alpha \in (0, \alpha_0] \), all \( \lambda \geq \lambda_0 \) and for all functions \( v \in H^{2,h}(Q_{h,T_1}) \) the following Carleman estimate is valid:

\[
\sum_{i,j=1}^{N} h^2 \int_{0}^{T_1} \int_{0}^{1} (L^h v^h)^2(x_i, y_j, z, t) e^{-2\lambda(z+\alpha t)} dz dt \\
\geq C \lambda \sum_{i,j=1}^{N} h^2 \int_{0}^{T_1} \int_{0}^{1} \left[ (v_{z}^h)^2 + (v^h)^2 + \lambda^2 (v^h)^2 \right] (x_i, y_j, z, t) e^{-2\lambda(z+\alpha t)} dz dt \\
+ C \lambda \sum_{i,j=1}^{N} h^2 \int_{0}^{T_1} \left[ (v_{z}^h)^2 + \lambda^2 (v^h)^2 \right] (x_i, y_j, z, 0) e^{-2\lambda z} dz \\
- C \lambda e^{-2\lambda \alpha T_1} \left( \| v_{z}^h(x^h, T_1) \|^2_{L^2_{h}(\Omega_h)} + \lambda^2 \| v^h(x^h, T_1) \|^2_{L^2_{h}(\Omega_h)} \right) \\
- C \| v^h \|^2_{L^2_{h}(\theta_{h,T_1})} - C \lambda \left( \| v_{z}^h \|^2_{H^1,1,1} + \lambda^2 \| v^h \|^2_{L^2_{h}(\Gamma_{h,T_1})} \right),
\] (5.5)

end
where the constant \( C = C(T_1, \xi_0, \xi_1, \xi_2, h_0, X, \alpha) > 0 \) depends only on listed parameters.

**Proof.** Here and below in this paper \( C = C(T_1, \xi_0, \xi_1, \xi_2, h_0, X, \alpha) > 0 \) denotes different constants depending only on listed parameters. By (4.17), (4.19), (5.3) and Young’s inequality

\[
(L^h v^h)^2(x^h, t) \geq \frac{1}{2} \left( \partial_x^2 v^h - \xi^h v_{x^h}^h \right)^2(x^h, t) - \left( \partial_x^2 v^h + \partial_y^2 v^h \right)^2(x^h, t).
\]  

(5.6)

Obviously

\[
\sum_{i,j=1}^{N} h^2 \int_0^{T_1} \int_0^1 \left( \partial_x^2 v^h + \partial_y^2 v^h \right)^2(x_i, y_j, z, t) e^{-2\lambda(z+\alpha t)} dt dz \leq C \|v^h\|^2_{L^2(\Omega_{h,T_1})} + C \sum_{i,j=1}^{N} h^2 \int_0^{T_1} \int_0^1 (v^h)^2(x_i, y_j, z, t) e^{-2\lambda(z+\alpha t)} dt dz.
\]

Hence, (5.6) implies

\[
\sum_{i,j=1}^{N} h^2 \int_0^{T_1} \int_0^1 \left( L^h v^h \right)^2(x_i, y_j, z, t) e^{-2\lambda(z+\alpha t)} dt dz \geq \frac{1}{2} \left( \partial_x^2 v^h - \xi^h v_{x^h}^h \right)^2(x_i, y_j, z, t) e^{-2\lambda(z+\alpha t)} dt dz \]

\[
\geq \frac{1}{2} \sum_{i,j=1}^{N} h^2 \int_0^{T_1} \int_0^1 \left( \partial_x^2 v^h - \xi^h_v \right)^2(x_i, y_j, z, t) e^{-2\lambda(z+\alpha t)} dt dz - C \|v^h\|^2_{L^2(\Omega_{h,T_1})} - C \sum_{i,j=1}^{N} h^2 \int_0^{T_1} \int_0^1 (v^h)^2(x_i, y_j, z, t) e^{-2\lambda(z+\alpha t)} dt dz.
\]

Theorem 3.1 of [16], Theorem 3.1] implies that, given (5.1) and (5.4), the following Carleman estimate holds:

\[
\sum_{i,j=1}^{N} h^2 \int_0^{T_1} \int_0^1 \left( \partial_x^2 v^h - \xi^h v_{x^h}^h \right)^2(x_i, y_j, z, t) e^{-2\lambda(z+\alpha t)} dt dz \geq C\lambda \sum_{i,j=1}^{N} h^2 \int_0^{T_1} \left( (v_x^h)^2 + (v_y^h)^2 + \lambda^2 (v^h)^2 \right) e^{-2\lambda(z+\alpha t)} dt dz \]

\[
- C\lambda e^{-2\lambda T_1} \left( \|v^h(x^h, T_1)\|^2_{L^2(\Omega_h)} + \lambda^2 \|v^h(x^h, T_1)\|^2_{H^1(\Gamma_{h,T_1})} \right) - C\lambda \left( \|v^h\|^2_{L^2(\Gamma_{h,T_1})} + \lambda^2 \|v^h\|^2_{H^1(\Gamma_{h,T_1})} \right).
\]  

(5.7)

Choosing a sufficiently large \( \lambda_0 = \lambda_0(T_1, \xi_0, \xi_1, \xi_2, h_0, X) \geq 1 \), setting \( \lambda \geq \lambda_0 \) and combining (5.7) and (5.8), we obtain (5.5), which is the target estimate of this theorem. □

**Remarks 5.1:**

1. We now explain why the terms reflecting boundary conditions at \( \Gamma_{h,T_1} \) are absent in the right hand side of (5.5) and why the condition \( \alpha \in (0, \alpha_0) \), with \( \alpha_0 \) as in (5.4) is
imposed. The point here is that the condition $\alpha \in (0, \alpha_0]$ ensures that those terms are non-negative. This follows immediately from the combination of (5.1) with the formula (3.1) of [10] as well as with the following formulas in the proof of Theorem 3.1 of [10]: the formula (3.14) (third and fourth lines), the inequality just below (3.14) and the formula (3.15).

2. We also note that Carleman estimate (5.2) is valid for any value $T_1 > 0$. This is because of the presence of the negative term in (5.2),

$$-C\lambda e^{-2\lambda \alpha T_1} \left( \| v^h (x^h, T_1) \|_{L^2(\Omega_h)}^2 + \lambda^2 \| v^h (x^h, T_1) \|_{L^2(\Omega_h)}^2 \right).$$

**Theorem 5.2** (the second Carleman estimate). Let the parameter $\alpha$ be the same as in Theorem 5.1. Then there exists a sufficiently large number $\lambda_1 = \lambda_1 (h_0, X) \geq 1$ such that for all $\lambda \geq \lambda_1$ and for all functions $v^h \in H^{2,h} (Q_{h,T_1})$ the following Carleman estimate is valid

$$\sum_{i,j=1}^N h^2 \int_0^{T_1} \int_0^1 (\Delta h v^h)^2 (x_i, y_j, z, t) e^{-2\lambda(z+\alpha t)} dz dt \geq C_1 \lambda \sum_{i,j=1}^N h^2 \int_0^{T_1} \int_0^1 \left[ (v^h_z)^2 + \lambda^2 (v^h)^2 \right] (x_i, y_j, z, t) e^{-2\lambda(z+\alpha t)} dz dt$$

$$-C_1 \| v^h \|^2_{L^2(\Theta_{h,T_1})} - C_1 \lambda \left[ \| v^h \|^2_{L^2(\Gamma_{h,T_1})} + \lambda^2 \| v^h \|^2_{L^2(\Gamma_{h,T_1})} \right].$$

Both constants $\lambda_1 (h_0, X) \geq 1$ and $C_1 = C_1 (h_0, X) > 0$ depend only on listed parameters.

**Proof.** Here and below in this paper $C = C (h_0, X) > 0$ denotes different constants depending only on listed parameters. We obtain similarly with (5.1)

$$\sum_{i,j=1}^N h^2 \int_0^{T_1} \int_0^1 (\Delta h v^h)^2 (x_i, y_j, z, t) e^{-2\lambda(z+\alpha t)} dz dt \geq \frac{1}{2} \sum_{i,j=1}^N h^2 \int_0^{T_1} \int_0^1 \left( v^h_{zz} \right)^2 (x_i, y_j, z, t) e^{-2\lambda(z+\alpha t)} dz dt$$

$$-C \| v^h \|^2_{L^2(\Theta_{h,T_1})} - C \sum_{i,j=1}^N h^2 \int_0^{T_1} \int_0^1 (v^h)^2 (x_i, y_j, z, t) e^{-2\lambda(z+\alpha t)} dz dt.$$

It follows from [15] Theorem 7.1 that the following Carleman estimate holds for sufficiently large $\lambda \geq \lambda_0 \geq 1$:

$$\sum_{i,j=1}^N h^2 \int_0^{T_1} \int_0^1 (v^h_{zz})^2 (x_i, y_j, z, t) e^{-2\lambda(z+\alpha t)} dz dt \geq C \lambda \sum_{i,j=1}^N h^2 \int_0^{T_1} \int_0^1 \left[ (v^h_z)^2 + \lambda^2 (v^h)^2 \right] (x_i, y_j, z, t) e^{-2\lambda(z+\alpha t)} dz dt$$

(5.11)
\[-C\lambda \left[ \| v^h \|_{L_h^2(T_{\Gamma_h, T})}^2 + \lambda^2 \| v^h \|_{L_h^2(T_{\Gamma_h, T})}^2 \right].\]

Combining (5.10) and (5.11), we obtain (5.9), which is the target estimate of this theorem. □

6 The stability Estimate for the CIP

Consider the set \( \mathcal{N} \) of functions \( n(\mathbf{x}) \) defined as:

\[
\mathcal{N} = \{ n(\mathbf{x}) \text{ satisfies (2.3)-(2.7) and } n_z(\mathbf{x}) \geq 0 \text{ in } \mathbb{R}^3 \}. \quad (6.1)
\]

Lemma 6.1. The following inequality holds:

\[
\partial_z \tau(\mathbf{x}) \geq 1, \quad \mathbf{x} \in \Omega, \quad \forall n(\mathbf{x}) \in \mathcal{N}.
\]

Proof. It follows from (3.6)-(3.8) that, along the geodesic line \( L(\mathbf{x}) \),

\[
\frac{d}{ds} (\partial_z \tau) = \frac{n_z}{n}, \quad \text{for } s > 0
\]

and \( \tau_{|s=0} = 0 \). At \( s = 0 \) the geodesic line \( L(\mathbf{x}) \) intersects with the plane \( \Sigma = \{ z = 0 \} \) defined in (2.10). Hence, \( \partial_z \tau_{|s=0} = \partial_z \tau_{|s=0} = 0 \). By (2.7) \( n(\mathbf{x}) |_{\Sigma} = 1 \). Hence, (3.2) implies that \( \partial_z \tau_{|s=0} = 1 \). Hence, using (6.1), we obtain

\[
\partial_z \tau(\mathbf{x}) = 1 + \int_{L(\mathbf{x})} \frac{n_z}{n} (\mathbf{\xi}(s)) ds \geq 1. \quad \square
\]

Our CIP is an ill-posed problem. Therefore, to prove the desired stability estimate, it is necessary to assume, in accordance with the well known Tikhonov’s concept of conditional correctness for ill-posed problems [29], that some \textit{a priori} known bounds are imposed on the functions \( w^h, \tau^h \).

Thus, let \( M > 0 \) be a positive number. Introduce the set of semi-discrete functions \( S^h = S^h(M, X, n_0, n_{00}) \) as:

\[
S^h = S^h(M, n_0, n_{00}) = \left\{ (w^h, \tau^h) \in C^{2,h}(\overline{Q}_{h,T_1}) \times C^{2,h}(\overline{\Omega}_h) : \begin{align*}
\nabla^h \tau^h(\mathbf{x}^h) &\leq n_0, \quad \mathbf{x}^h \in \overline{\Omega}_h, \\
\| w^h \|_{C^{2,h}(\overline{Q}_{h,T_1})}, \| \tau^h \|_{C^{2,h}(\overline{\Omega}_h)} &\leq M, \\
1 &\leq \partial_z \tau^h(\mathbf{x}) \leq n_0, \quad \mathbf{x}^h \in \overline{\Omega}_h, \\
w^h(\mathbf{x}^h, 0) &\geq A_0,
\end{align*} \right\}
\]

where the vector function \( (w^h, \tau^h) \) is generated by the above procedure and the number \( A_0 \) is defined in (3.19).

Conditions in the second, fourth and fifth lines of (6.2) are imposed due to (2.6), (3.1), Lemma 6.1 and Theorem 3.1, respectively. We have

\[
S^h = S^h_w \times S^h_{\tau} = \left\{ (w^h(\mathbf{x}^h, t), \tau^h(\mathbf{x}^h)) \right\}. \quad (6.3)
\]
Using (3.1), define the function $n^h(x)$ as:
\[
|\nabla x\tau^h(x)|^2 = (n^h)^2(x), \quad \forall \tau^h \in S^r. \tag{6.6}
\]

**Theorem 6.1** (Hölder stability estimate). Let two vector functions $(w^h_1, \tau^h_1), (w^h_2, \tau^h_2) \in S^h$ be solutions of BVP with two sets of boundary data at $\Gamma_{h,T_1}, \Gamma_{h,T_2}$, and $\Theta_h$,
\[
w^h_j |_{\Gamma_{h,T_1}} = g^h_{j,0}, \quad \partial_\nu w^h_j |_{\Gamma_{h,T_1}} = g^h_{j,1}, \quad w^h_j |_{\Theta_h,T_1} = g^h_{j,2}, \quad j = 1, 2, \tag{6.7}
\]
\[
\tau^h_j |_{\Gamma_{h}} = 0, \quad \partial_\nu \tau^h_j |_{\Gamma_{h}} = 1, \quad \tau^h_j |_{\Theta_h} = z, \quad j = 1, 2. \tag{6.8}
\]

Denote
\[
\begin{align*}
\tilde{w}^h &= w^h_1 - w^h_2, \\
\tilde{\tau}^h &= \tau^h_1 - \tau^h_2, \\
\tilde{g}^h_0 &= g^h_{1,0} - g^h_{2,0}, \\
\tilde{g}^h_1 &= g^h_{1,1} - g^h_{2,1}, \\
\tilde{g}^h_2 &= g^h_{1,2} - g^h_{2,2}, \\
\tilde{n}^h &= n^h_1 - n^h_2.
\end{align*} \tag{6.9}
\]
where functions $n^h_1$ and $n^h_2$ are obtained from functions $\tau^h_1$ and $\tau^h_2$ respectively via (6.6). Assume that
\[
\|\tilde{g}^h_0\|_{H^{1,h}(\Gamma_{h,T_1})}, \quad \|\tilde{g}^h_1\|_{H^1(\Gamma_{h,T_1})}, \quad \|\tilde{g}^h_2\|_{L^2(\Theta_h,T_1)} < \delta, \tag{6.10}
\]
where $\delta \in (0, 1)$ is a number. Define the number $\alpha_0 = \alpha_0(n_0) = 2/(3n_0)$. Consider an arbitrary number $\alpha \in (0, \alpha_0]$ and set $T_1 = 3/\alpha$. Then there exists a sufficiently small number $\delta_0 = \delta_0(M, n_0, n_{00}, h_0, X, \alpha) \in (0, 1)$ and a number $C_2 = C_2(M, n_0, n_{00}, h_0, X, \alpha) > 0$, both numbers depending only on listed parameters, such that if $\delta \in (0, \delta_0)$, then the following stability estimates are valid for the functions $\tilde{w}^h, \tilde{\tau}^h, \tilde{n}^h$
\[
\|\tilde{w}^h\|_{H^{1,h}(\Omega_{h}\setminus S^r)} , \quad \|\tilde{\tau}^h\|_{H^{1,h}(\Omega_{h})} , \quad \|\tilde{n}^h\|_{L^2(\Theta_h,T_1)} \leq C_2 \delta^{1/3} \ln(\delta^{-1}). \tag{6.11}
\]

**Proof.** Below $C_2 = C_2(M, n_0, n_{00}, h_0, X, \alpha) > 0$ denotes different constants depending only on listed parameters. Let $a_1, a_2, b_1, b_2 \in \mathbb{R}$ be arbitrary numbers. Let $a = a_1 - a_2$ and $b = b_1 - b_2$. Then
\[
a_1 b_1 - a_2 b_2 = \tilde{a} b_1 + \tilde{b} a_2. \tag{6.12}
\]
Using (3.8), (6.2), (6.3) and (6.12), we obtain
\[
|\nabla^h \ln w^h_1(x^h, 0) - \nabla^h \ln w^h_2(x^h, 0)| \leq C_2 \left(|\nabla^h \tilde{w}^h(x^h, 0)| + |\tilde{w}^h(x^h, 0)|\right). \tag{6.13}
\]
It is well known that, when applying a Carleman estimate, one can replace differential equations with appropriate differential inequalities, see, e.g. [17]. Hence, subtract two equations (4.20), (4.21) for the pair $(w^h_1, \tau^h_1)$ from two equations (4.20), (4.21) for the pair $(w^h_2, \tau^h_2)$, use the first line of (6.9) and (6.12). Then, using (6.13), turn resulting equations in inequalities with respect to functions $\tilde{w}^h$ and $\tilde{\tau}^h$. We obtain two differential inequalities:
\[
|\Delta^h \tilde{w}^h - \partial_\tau^h \tilde{w}^h_{zt}| \tag{6.14}
\]
Using Cauchy-Schwarz inequality, we obtain  
\[
\sum_{i,j=1}^{N} h^2 \int_0^1 \int_0^1 I(x_i, y_j, z, t) e^{-2\lambda(z+\alpha t)} dz dt 
\leq C_1 \sum_{i,j=1}^{N} h^2 \int_0^1 \int_0^1 (\tilde{\omega}_i^h(x_i, y_j, z, t))^2 e^{-2\lambda(z+\alpha t)} dz dt 
+ C_1 \sum_{i,j=1}^{N} h^2 \int_0^1 \int_0^1 \left(\left(\frac{w^h}{2} + (w^h)^2\right)(x_i, y_j, z, 0)\right) e^{-2\lambda(z+\alpha t)} dz dt 
+ C_1 \sum_{i,j=1}^{N} h^2 \int_0^1 \int_0^1 \left[\left(\partial_x \tilde{\omega}_i^h\right)^2 + (\tilde{\omega}_i^h)^2\right](x_i, y_j, z) e^{-2\lambda(z+\alpha t)} dz dt.
\]

And also  
\[
\sum_{i,j=1}^{N} h^2 \int_0^1 \int_0^1 \left[\left(\frac{w^h}{2} + (w^h)^2\right)(x_i, y_j, z, 0) + |\nabla^h \tilde{\omega}_i^h| (x_i, y_j, z, 0) + |\nabla^h \tilde{\omega}_i^h| (x_i, y_j, z)\right]^2 e^{-2\lambda(z+\alpha t)} dz dt 
\leq C_1 \sum_{i,j=1}^{N} h^2 \int_0^1 \int_0^1 \left[\left(\partial_x \tilde{\omega}_i^h\right)^2 + (\tilde{\omega}_i^h)^2\right](x_i, y_j, z) e^{-2\lambda(z+\alpha t)} dz dt 
+ C_1 \sum_{i,j=1}^{N} h^2 \int_0^1 \int_0^1 \left[\left(\frac{w^h}{2} + (w^h)^2\right)(x_i, y_j, z, 0)\right] e^{-2\lambda(z+\alpha t)} dz dt.
\]

Square both sides of both inequalities (6.14) and (6.15). Then multiply the results by the function $e^{-2\lambda(z+\alpha t)}$, construct sums combined with integrals like in the left hand sides of Carleman estimates (5.5) and (5.9) and then use (6.18)-(6.20). We obtain  
\[
\sum_{i,j=1}^{N} h^2 \int_0^1 \int_0^1 (\Delta^h \tilde{\omega}_i^h - \partial_x \tilde{\omega}_i^h \tilde{\omega}_z^h)^2(x_i, y_j, z, t) e^{-2\lambda(z+\alpha t)} dz dt
\]
Theorem 5.1 is applicable here. Indeed, it follows from (6.2) and Lemma 6.1 that we can replace in (5.5) 
$sides of (6.21)$ and (6.22) respectively. Then sum up resulting inequa lities. In doing so, 
By (5.4) and the second line of (6.2) Set in Theorem 5.1

And also

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Theorem 5.1 is applicable here. Indeed, it follows from (6.2) and Lemma 6.1 that we can now take in (5.1)
+C_2 \lambda \sum_{i,j=1}^N h^2 \int_0^1 \left[ (\tilde{w}_z^h)^2 + \lambda^2 (\tilde{w}_t^h)^2 \right] (x_i, y_j, z, t) e^{-2\lambda z} \, dz \\
-C_2 \lambda^3 e^{-6\lambda} \|\tilde{w}^h (x^h, T_1)\|^2_{H^{1,h}(\Omega_n)} - C_2 \lambda^3 \left( \|\tilde{w}^h\|^2_{H^{1,h}(\Gamma_n, T_1)} + \|\tilde{w}^h\|^2_{L^2(\Gamma_n, T_1)} \right) \\
-C_2 \left( \mathcal{H} \left( \tilde{w}^h \right) \bigg|_{\subset \subset \Omega_n} \right)^2.

The multiplier $e^{-6\lambda}$ in the sixth line of (6.23) is due to the fact that $T_1 = 3/\alpha$ and $e^{-2\lambda(z+\alpha t)} \leq e^{-6\lambda}$ for $z \in (0, 1)$. Choose $\lambda_1 = \lambda_1 (M, n_0, n_00, h_0, X, \alpha) \geq \lambda_0 \geq 1$ so large that $C_2 \lambda_1 / 2 > 1$. Then (6.9), (6.16) and (6.23) imply

$$C_2 \lambda^3 e^{-6\lambda} \|\tilde{w}^h (x^h, T_1)\|^2_{H^{1,h}(\Omega_n)} + C_2 \lambda^3 \left( \|\tilde{w}^h\|^2_{H^{1,h}(\Gamma_n, T_1)} + \|\tilde{g}^h\|^2_{L^2(\Gamma_n, T_1)} \right)$$

$$+ C_2 \left( \mathcal{H} \left( \tilde{w}^h \right) \bigg|_{\subset \subset \Omega_n} \right)^2 \geq \lambda \sum_{i,j=1}^N \int_0^1 \left( \left( \tilde{w}_z^h \right)^2 + \left( \tilde{w}_t^h \right)^2 + \lambda^2 \left( \tilde{w}_h^h \right)^2 \right) (x_i, y_j, z, t) e^{-2\lambda(z+\alpha t)} \, dz \, dt \quad (6.24)$$

$$+ \lambda \sum_{i,j=1}^N h^2 \int_0^1 \left( \left( \tilde{w}_z^h \right)^2 + \lambda^2 \left( \tilde{w}_h^h \right)^2 \right) (x_i, y_j, z, t) e^{-2\lambda z} \, dz \, dt$$

$$+ \lambda \sum_{i,j=1}^N h^2 \int_0^1 \left( \left( \tilde{w}_z^h \right)^2 + \lambda^2 \left( \tilde{w}_h^h \right)^2 \right) (x_i, y_j, z, 0) e^{-2\lambda z} \, dz, \forall \lambda \geq \lambda_1.$$ 

Introduce the number $t_1 = 1/\alpha \in (0, T_1) = (0, 3/\alpha)$. Then

$$e^{-2\lambda(z+\alpha t)} > e^{-2\lambda(1+\alpha t)} = e^{-4\lambda} \text{ for } (z, t) \in (0, 1) \times (0, t_1). \quad (6.25)$$

Divide (6.24) by $\lambda e^{-4\lambda}$ and replace $\lambda^2 \geq 1$ with 1 in the right hand side of the resulting inequality, thus, making it stronger. Using (6.25), we obtain

$$C_2 \lambda^2 e^{4\lambda} \left( \|\tilde{w}_0^h\|^2_{H^{1,h}(\Gamma_n, T_1)} + \|\tilde{g}_0^h\|^2_{L^2(\Gamma_n, T_1)} \right)$$

$$+ C_2 e^{4\lambda} \left( \|\tilde{w}^h\|^2_{L^2(\Omega_n, T_1)} + \|\tilde{w}^h\|^2_{L^2(\Gamma_n, T_1)} \right) + C_2 \lambda^2 e^{-2\lambda} \|\tilde{w}^h (x^h, T_1)\|^2_{H^{1,h}(\Omega_n)}$$

$$\geq \sum_{i,j=1}^N h^2 \int_0^{t_1} \int_0^1 \left( \left( \tilde{w}_z^h \right)^2 + \left( \tilde{w}_t^h \right)^2 + \left( \tilde{w}_h^h \right)^2 \right) (x_i, y_j, z, t) \, dz \, dt \quad (6.26)$$

$$+ \sum_{i,j=1}^N h^2 \int_0^{t_1} \int_0^1 \left( \left( \tilde{w}_z^h \right)^2 + \lambda^2 \left( \tilde{w}_h^h \right)^2 \right) (x_i, y_j, z) \, dz \, dt$$

$$+ \sum_{i,j=1}^N h^2 \int_0^{t_1} \int_0^1 \left( \left( \tilde{w}_z^h \right)^2 + \lambda^2 \left( \tilde{w}_h^h \right)^2 \right) (x_i, y_j, z, 0) e^{-2\lambda z} \, dz, \forall \lambda \geq \lambda_1.$$
Choose $\delta_0 = \delta_0(M, n_0, h_0, X, \alpha) \in (0, 1)$ such that $\lambda_1 = \ln \left( \frac{1}{3\delta_0} \right)$. For every $\delta \in (0, \delta_0)$ we choose $\lambda = \lambda(\delta) > \lambda_1$ such that

$$e^{4\lambda} \delta^2 = e^{-2\lambda}. \quad (6.27)$$

Hence, $\lambda = \lambda(\delta) = \frac{1}{3} \ln (\delta^{-1})$. It follows from the third line of (6.2) and the first line of (6.9) that $\| \tilde{w}^h (x^h, T_1) \|^2_{H^1, h(\Omega_h)} \leq C_2$. Hence, using (6.10), (6.26) and (6.27), we obtain

$$\| \tilde{w}^h \|_{H^1, h(Q_h, 1/\alpha)} \leq C_2 \delta^{1/3} \ln (\delta^{-1}) \quad (6.28)$$

Next, by the fourth line of (6.2) and (6.9)

$$| \tilde{n}^h | = \left| \frac{(n_1^h)^2 - (n_2^h)^2}{n_1^h + n_2^h} \right| \leq \frac{1}{2} \left| (n_1^h)^2 - (n_2^h)^2 \right| = \frac{1}{2} \left| |\nabla \tau_1^h|^2 - |\nabla \tau_1^h|^2 \right| \leq \frac{1}{2} \left| \nabla \tilde{\tau} \cdot (\nabla \tau_1^h + \nabla \tau_2^h) \right| \leq n_0 \left| \nabla \tilde{\tau} \right|.$$

Hence, by (6.28)

$$\| \tilde{n}^h \|_{L^2(\Omega_h)} \leq C_2 \delta^{1/3} \ln (\delta^{-1}). \quad (6.29)$$

Estimates (6.28) and (6.29) imply the target estimates (6.11). \(\Box\)

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