Two asymptotic distributions related to Rényi-type continued fraction expansions

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Abstract
We attempt to investigate a two-dimensional Gauss–Kuzmin theorem for Rényi-type continued fraction expansions. More precisely, our focus is to obtain specific lower and upper bounds for the error term considered which imply the convergence rate of the distribution function involved to its limit. To achieve our goal, we exploit the significant properties of the Perron–Frobenius operator of the Rényi-type map under its invariant measure on the Banach space of functions of bounded variation. Finally, we give some numerical calculations to conclude the paper.

Keywords Rényi continued fractions · Gauss–Kuzmin-problem · Natural extension · Perron–Frobenius operator

Mathematics Subject Classification Primary 11J70, 11K50; Secondary 28D05, 60J20

1 Introduction

The present paper continues and completes our series of papers dedicated to Rényi-type continued fraction expansions [8,9,16]. Our aim here is to contribute a solution to the two-dimensional Gauss–Kuzmin theorem for Rényi-type continued fraction expansions. In order to do this, we present a short survey of the topic in discussion.
1.1 Gauss’s problem and current developments

The first basic result in the metrical theory of continued fractions is the Gauss–Kuzmin theorem. C.F. Gauss stated in 1812 that, in modern notation,

$$\lim_{n \to \infty} \lambda \left( G^n \leq x \right) = \frac{\log(1 + x)}{\log 2}, \quad x \in I := [0, 1]. \quad (1.1)$$

Here $\lambda$ is the Lebesgue measure and $G : [0, 1) \to [0, 1)$ is the Gauss map defined by

$$G(x) := \begin{cases} \frac{1}{x} - \left[ \frac{1}{x} \right] & x \in (0, 1], \\ 0, & x = 0, \end{cases} \quad (1.2)$$

where $[\cdot]$ stands for integer part. For each $x \in (0, 1)$ put $a_n = a_n(x) = a_1 \left( G^{n-1}(x) \right)$, $n \in \mathbb{N}_+ := \{1, 2, \ldots\}$ with $G^0(x) = x$ and $a_1 = a_1(x) = \lfloor 1/x \rfloor$, $x \neq 0$. Any irrational $x \in (0, 1)$ can be written as the infinite regular continued fraction (RCF)

$$x = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ddots}}} =: [a_1, a_2, a_3, \ldots]. \quad (1.3)$$

The positive integers $a_n$ are called the digits of $x$. Gauss asked for an estimate of the $n$th error term

$$e_n(x) = \lambda \left( G^n \leq x \right) - \frac{\log(1 + x)}{\log 2}, \quad n \in \mathbb{N}_+, x \in I, \quad (1.4)$$

and this has been the first problem of the metrical theory of continued fractions. The first solution was given by Kuzmin [7] who showed in 1928 that $e_n(x) = O\left(q^{\sqrt{n}}\right)$ as $n \to \infty$, uniformly in $x$ with some unspecified $0 < q < 1$. This has been called the Gauss–Kuzmin theorem. Ramifications of this problem are extensively studied in [6].

Gauss’s measure $\gamma([0, x]) = \frac{\log(1 + x)}{\log 2}$ is the invariant measure of the map $G$ underlying the RCF. Moreover, if $B_I$ denotes the $\sigma$-algebra of all Borel subsets of $I$, the measure-preserving dynamical system

$$(I, B_I, \gamma, G) \quad (1.5)$$

is ergodic.

From that time on, a great number of Gauss–Kuzmin theorems followed. Generalizations of these problems for non-RCFs are also known as the Gauss–Kuzmin problem. The remarkable fact is that in the last 25 years, such an investigation has been made in the two-dimensional case. After Nakada [10] gave an exact definition of natural extensions for a class of continued fraction transformations, this type of study was initiated by Dajani and Kraaikamp [1]. To be more precise they obtained a Gauss–Kuzmin theorem related to the natural extension of the dynamical system in (1.5), i.e.,

$$(I^2, B_I^2, \overline{\gamma}, \overline{G}), \quad (1.6)$$
where $\overline{\gamma}$ is the extended Gauss measure on the square space $(I^2, B^2_I) := (I, B_I) \times (I, B_I)$ with density $\frac{1}{\log 2} \frac{1}{(1 + xy)^2}$ and $\overline{G} : I^2 \rightarrow I^2$ is defined by

$$\overline{G}(\xi, \eta) = \left( \frac{G(\xi)}{a_1(\xi) + \eta}, \frac{1}{a_1(\xi) + \eta} \right), \quad (\xi, \eta) \in I^2. \quad (1.7)$$

Consider the $n$th error term

$$e_n(x, y) = \overline{x} \left( G^n \in [0, x] \times [0, y] \right) - \frac{\log(1 + xy)}{\log 2}, \quad n \in \mathbb{N}_+, (x, y) \in I^2, \quad (1.8)$$

where $\overline{x}$ is the Lebesgue measure on the square space and

$$G^n(\xi, \eta) = \left( [a_{n+1}(\xi), a_{n+2}(\xi), \ldots], [a_n(\xi), \ldots a_2(\xi), a_1(\xi) + \eta] \right), \quad (\xi, \eta) \in I^2, \quad n \in \mathbb{N}_+ \quad (1.9)$$

and $G^0(\xi, \eta) = (\xi, \eta)$. Thus, the result stated in [1] that for all $n \geq 2$ and all $(x, y) \in I^2$, $e_n(x, y) = O(g^n)$, where $g = (\sqrt{5} - 1)/2 = 0.6180\ldots$, gained several modifications and acquired considerable development in [6].

There is also a whole family $(\gamma_a)_{a \in I}$ of probability measures on $B_I$ defined by their distribution functions

$$\gamma_a([0, x]) = \frac{(a + 1)x}{ax + 1}, \quad x \in I, \quad a \in I, \quad (1.10)$$

which plays an important part. In particular, $\gamma_0 = \lambda$. Beside $\gamma$, these probability measures, which we call conditional, are the most natural ones associated with the RCF expansion. This idea goes back to W. Doeblin [3] and hopefully, was fully capitalized in the following context. For any $a \in I$, put

$$s_{n+1}^a = \frac{1}{a_{n+1} + s_n^a}, \quad a \in I, \quad n \in \mathbb{N} := \mathbb{N}_+ \cup \{0\}, \quad (1.11)$$

with $s_0^a = a$. Noting that $G^n(\xi) = [a_{n+1}(\xi), a_{n+2}(\xi), \ldots], n \in \mathbb{N}, \xi \in I$, it follows that

$$\gamma_a(a_{n+1} = i | a_1, a_2, \ldots, a_n) = \frac{s_{n+1}^a + 1}{(s_n^a + i)(s_n^a + i + 1)} =: P_i(s_n^a) \quad (1.12)$$

for any $a \in I$, and $i, n \in \mathbb{N}_+$. Hence for any $a \in I$ the sequence $(s_n^a)_{n \in \mathbb{N}}$ on $(I, B_I, \gamma_a)$ is an $I$-valued Markov chain which starts at $s^a_0 = a$ and has the following transition mechanism: from state $s \in I$ the possible transitions are to any state $1/(s + i)$ with corresponding transition probability $P_i(s), i \in \mathbb{N}_+$.

For the convenience of comparative analysis, (1.8) needs to be expressed as the following equation:

$$e_n(x, y) = \overline{x} \left( G^n \leq x, s_n^\eta \leq y \right) - \frac{\log(1 + xy)}{\log 2} \quad (1.13)$$

for any $\eta \in I$, $n \in \mathbb{N}_+, (x, y) \in I^2$.

We may synthesize the most relevant results for RCFs as follows [6, Chapter 2].

For any $a \in I$, $a \neq a_0$, with $a_0$ very close to 0.4, the exact convergence rate to 0 as $n \rightarrow \infty$ of

$$\sup_{x \in I} \left| \gamma_a \left( G^n \leq x \right) - \frac{\log(x + 1)}{\log 2} \right| \quad (1.14)$$
is $O(\lambda_0^n)$, where $\lambda_0 = 0.303663007 \ldots$ (Wirsing’s constant). For $a = a_0$ the rate is $O(\beta^n)$, with $\beta = 0.100884509 \ldots$. Note that these results needed pretty nearly 160 years to be reached.

The study of the joint distribution function of $G^n$ and $s_n^a$, $n \in \mathbb{N}$, under $\gamma_a$, $a \in I$ leads to a very interesting result. The asymptotic distribution function, lower and upper bounds for the error as well as its optimal convergence rate to 0 as $n \to \infty$ are obtained. Consider the $n$th error term

$$e_n^a(x, y) = \gamma_a \left(G^n \leq x, s_n^a \leq y\right) - \frac{\log(1 + xy)}{\log 2}, \quad n \in \mathbb{N}, \ (x, y) \in I^2.$$  \hspace{1cm} (1.15)

The following result holds [6, Chapter 2]. For any $a \in I$, $(x, y) \in I^2$ and $n \in \mathbb{N}$

$$\frac{a + 1}{2(F_n + aF_{n-1})(F_{n+1} + aF_n)} \leq \sup_{(x, y) \in I^2} \left| e_n^a(x, y) \right| \leq \frac{k_0}{F_nF_{n+1}}.$$  \hspace{1cm} (1.16)

Here the $F_n$, $n \in \mathbb{N}$, are the Fibonacci numbers defined recursively by $F_{-1} = 0$, $F_0 = 1$, $F_{n+1} = F_n + F_{n-1}$, $n \in \mathbb{N}$ and $k_0$ is a constant not exceeding 14.8.

Letting $x = 1$ in (1.16) we obtain the same lower and upper bounds as above for

$$\sup_{x \in I} \left| \gamma_a \left(s_n^a \leq y\right) - \frac{\log(y + 1)}{\log 2}\right|.$$  \hspace{1cm} (1.17)

These facts imply that for any $a \in I$ the exact convergence rate to 0 as $n \to \infty$ of both $\sup_{(x, y) \in I^2} \left| e_n^a(x, y) \right|$ and the supremum in (1.17) are $O(g^{2n})$, with $g^2 = (3 - \sqrt{5})/2 = 0.38196 \ldots$.

The explorations of the two-dimensional Gauss–Kuzmin theorem for other continued fractions such as Hurwitz’s singular continued fractions and grotesque continued fractions were fulfilled in [2,13] and [14]. Recently, we accomplished our research on the Gauss–Kuzmin problem for $N$-continued fractions [15].

### 1.2 Recent advances on Rényi-type continued fractions

The subject of Rényi-type continued fractions has relations to $u$-backward continued fractions studied by Gröchenig and Haas [4].

As is known, in 1957 Rényi [11] showed that every irrational number $x \in [0, 1)$ has an infinite continued fraction expansion of the form

$$x = 1 - \frac{1}{n_1 - \frac{1}{n_2 - \frac{1}{n_3 - \ddots}}} =: [n_1, n_2, n_3, \ldots],$$  \hspace{1cm} (1.18)

where each $n_i$ is an integer greater than 1. The expansion in (1.18) is called the backward continued fraction expansion of $x$.

The underlying dynamical system is the Rényi map $R$ defined from $[0, 1)$ to $[0, 1)$ by

$$R(x) := \begin{pmatrix} 1 & 1 \\ 1 - x & 1 \end{pmatrix}.$$  \hspace{1cm} (1.19)
Rényi showed that the infinite measure $dx/x$ is invariant for $R$. This map does not possess a finite absolutely continuous invariant measure and the usual trick to investigate its thermodynamic formalism does not work [12].

Starting from the expansion (1.18) and the Rényi transformation $R$, Gröchenig and Haas [4] define the family of maps $T_u(x) := \frac{1}{u(1-x)} - \frac{1}{u(1+x)}$, where $u > 0$, $x \in [0, 1)$. Observe that $T_1(x) = R(x)$, $x \in [0, 1)$.

Given $u \in (0, 4)$ and $x \in [0, 1)$, $x$ has the $u$-backward continued fraction expansion

$$x = 1 - \frac{1}{un_1 - \frac{1}{n_2 - \frac{1}{un_3 - \frac{1}{n_4 - \ldots}}}} =: [a_1, a_2, a_3, \ldots]_u,$$

(1.20)

where the integers $n_i = 1 + a_i$ are at least 2 and the coefficient of $n_i$ is 1 or $u$, depending on the parity of $i$. In the particular case $u = 1/N$, for a positive integer $N \geq 2$, they have identified a finite absolutely continuous invariant measure for $T_u$, namely $dx/(x + N - 1)$. For $u = 1/N$, where $N \geq 2$ is an integer, we will call $T_u$ the Rényi-type continued fraction transformation and denote it by $R_N$.

In [8] we started an approach to the metrical theory of the Rényi-type continued fraction expansions via dependence with complete connections. We obtained a version of the Gauss–Kuzmin theorem for these expansions by applying the theory of random systems with complete connections, due to Iosifescu [5]. Briefly, we showed that the associated random Gauss–Kuzmin theorem for these expansions by applying the theory of random systems with contraction and their transition operators are regular with respect to the Banach space of Lipschitz functions.

In [16] using a Wirsing-type approach [18] we obtained upper and lower bounds of the error which provide a refined estimate of the convergence rate. For example, in case $N = 100$, the upper and lower bounds of the convergence rate are $O \left( w_{100}^N \right)$ and $O \left( v_{100}^N \right)$ as $n \to \infty$, with $v_{100} > 0.00503350150708559$ and $w_{100} < 0.00503358526129032$, respectively. The strategy in this paper was to restrict the domain of the Perron–Frobenius operator of $R_N$ under its invariant measure $\rho_N$ to the Banach space of functions which have a continuous derivative on $I$.

Recently, in [9] using the method of Szüsz [17], we obtained more information on the convergence rate involved. The main novelty was the explicit expression in terms of Hurwitz zeta functions on $\eta_N$ that appears in [9, Theorem 3.1]. Finally, to enable direct comparisons of the results obtained in the last two methods (Wirsing and Szüsz) we give upper and lower bounds of $\eta_N$ for $N = 100$: $0.00505050495049505 < \eta_N < 0.0050753806723955975$.

The framework of this paper is arranged as follows. In Sect. 2 we gather prerequisites needed to prove our results in Sects 3 and 4. In Sect. 3 we treat the Perron–Frobenius operator of $R_N$ under its invariant measure on the Banach space of functions of bounded variation and study the significant properties of this operator.

Section 4 is devoted to a very special two-dimensional Gauss–Kuzmin theorem concerning the natural extension of corresponding interval maps $R_N$, $N \geq 2$. Using the same strategy as the one presented at the end of Sect. 1.1, we consider a one-parameter family $\{ \rho_N^i \}$ of conditional probability measures on $(I, \mathcal{B}(I))$ and we study the joint distribution function of $R_N$ and $s_N^i$ under $\rho_N^i$, $i \in I$. In the main result, Theorem 4.5, we derive the asymptotic distribution function, lower and upper bounds for the error term as well its convergence rate to 0 as $n \to \infty$. Here the specific lower and upper bounds for the error term considered
are approached via the characteristic properties of the associated transfer operator in Sect. 3. Finally, we give some remarks and numerical calculations to conclude the paper.

2 Prerequisites

In this section we briefly present known results about Rényi-type continued fractions (see, e.g., [8]).

2.1 Rényi-type continued fraction expansions as dynamical systems

For a fixed integer \(N \geq 2\), we define the Rényi-type continued fraction transformation \(R_N : I \to I\) by (Fig. 1)

\[
R_N(x) := \begin{cases} 
\frac{N}{1-x} - \left\lfloor \frac{N}{1-x} \right\rfloor, & x \in [0, 1), \\
0, & x = 1.
\end{cases}
\] 

(2.1)

For any irrational \(x \in I\), \(R_N\) generates a new continued fraction expansion of \(x\) of the form

\[
x = 1 - \frac{N}{1 + a_1 - \frac{N}{1 + a_2 - \frac{N}{1 + a_3 - \ddots}}} =: [a_1, a_2, a_3, \ldots]_{1/N},
\]

(2.2)

where \(a_n\)s are positive integers greater than or equal to \(N\) defined by

\[
a_1 := a_1(x) = \left\lfloor \frac{N}{1-x} \right\rfloor, \quad x \neq 1, \quad a_1(1) = \infty,
\]

(2.3)
and
\[ a_n := a_n(x) = a_1 \left( R_N^{n-1}(x) \right), \quad n \geq 2, \] (2.4)
with \( R_N^0(x) = x. \)

The Rényi-type continued fraction in (2.2) can be viewed as a measure-preserving dynamical system \((I, \mathcal{B}_I, R_N, \rho_N)\), and for any integer \( N \geq 2 \)
\[ \rho_N(A) := \frac{1}{\log \left( \frac{N}{N-1} \right)} \int_A \frac{dx}{x + N - 1}, \quad A \in \mathcal{B}_I, \] (2.5)
is the absolutely continuous probability measure invariant under \( R_N \) [4].

The Rényi-type continued fraction expansion (2.2) is convergent. To prove this, define the real functions \( p_n(x) \) and \( q_n(x) \), for \( n \in \mathbb{N} \), by
\[ p_0 = 1, \quad p_1 = 1 + a_1 - N, \quad p_n = (1 + a_n) p_{n-1} - N p_{n-2}, \quad n \geq 2, \] (2.6)
\[ q_0 = 1, \quad q_1 = 1 + a_1, \quad q_n = (1 + a_n) q_{n-1} - N q_{n-2}, \quad n \geq 2. \] (2.7)

As in the case of RCFs [6], it follows that \( p_n(x)/q_n(x) = [a_1, a_2, \ldots, a_n]_1/N \) which is called the \( n \)th order convergent of \( x \) in \( I \). A simple inductive argument gives
\[ p_{n-1}q_n - p_nq_{n-1} = N^n, \quad n \in \mathbb{N}_+. \] (2.8)

Put \( \Lambda := \{N, N + 1, \ldots\} \). An \( n \)-block \((a_1, a_2, \ldots, a_n)\) is said to be admissible for the expansion in (2.2) if there exists \( x \in [0, 1) \) such that \( a_i(x) = a_i \) for all \( 1 \leq i \leq n \). If \((a_1, a_2, \ldots, a_n)\) is an admissible sequence, we call the set
\[ I(a_1, a_2, \ldots, a_n) = \{x \in I : a_1(x) = a_1, a_2(x) = a_2, \ldots, a_n(x) = a_n\} \] (2.9)
the \( n \)th order cylinder. As we mentioned above, \((a_1, a_2, \ldots, a_n) \in \Lambda^n \). For example, for any \( a_1 = i \in \Lambda \) we have
\[ I(a_1) = \{x \in I : a_1(x) = a_1\} = \left[ 1 - \frac{N}{i}, 1 - \frac{N}{i+1} \right). \] (2.10)

By induction, it can be shown that
\[ I(a_1, a_2, \ldots, a_n) = \left[ \frac{p_n - p_{n-1}}{q_n - q_{n-1}}, \frac{p_n}{q_n} \right) \] (2.11)
for all \( n \geq 1 \). Hence using (2.8) we have
\[ \left| x - \frac{p_n}{q_n} \right| \leq \lambda(I(a_1, a_2, \ldots, a_n)) = \frac{N^n}{q_n(q_n - q_{n-1})}. \] (2.12)

From [4, Lemma 3], there exist \( C > 0 \) and \( \delta \), \( 0 < \delta < 1 \) such that
\[ \lambda(I(a_1, a_2, \ldots, a_n)) \leq C\delta^n \] (2.13)
for all \( n \geq 1 \). Hence, \( \lim_{n \to \infty} \frac{p_n}{q_n} = x. \)
2.2 Natural extension of $R_N$

Let $(I, B_I, R_N)$ be as in Sect. 2.1. Define $(u_N^i)_{i \geq N}$ by

$$u_N^i : I \to I; \quad u_N^i(x) := 1 - \frac{N}{x + i}, \quad x \in I.$$  \hspace{1cm} (2.14)

For each $i \geq N$, $u_N^i$ is a right inverse of $R_N$, that is,

$$\left( R_N \circ u_N^i \right)(x) = x, \quad \text{for any } x \in I.$$  \hspace{1cm} (2.15)

Furthermore, if $a_1(x) = i$, then $\left( u_N^i \circ R_N \right)(x) = x$ where $a_1$ is as in (2.3).

**Definition 2.1** The natural extension $(I^2, B_I^2, \overline{R}_N)$ of $(I, B_I, R_N)$ is the transformation $\overline{R}_N$ of the square space $(I^2, B_I^2)$ defined as follows [10]:

$$\overline{R}_N : I^2 \to I^2; \quad \overline{R}_N(x, y) := \left( R_N(x), u_N^{a_1(y)}(y) \right), \quad (x, y) \in I^2.$$  \hspace{1cm} (2.16)

From (2.15), we see that $\overline{R}_N$ is bijective on $I^2$ with the inverse

$$\left( \overline{R}_N \right)^{-1}(x, y) = (u_N^{a_1(y)}(x), R_N(y)), \quad (x, y) \in I^2.$$  \hspace{1cm} (2.17)

For $\rho_N$ in (2.5), we define its extended measure $\overline{\rho}_N$ on $(I^2, B_I^2)$ as

$$\overline{\rho}_N(B) := \frac{1}{\log \left( \frac{N}{N-1} \right)} \int_B \frac{N \, dx \, dy}{(N - (1-x)(1-y))^2}, \quad B \in B_I^2.$$  \hspace{1cm} (2.18)

Then $\overline{\rho}_N(A \times I) = \overline{\rho}_N(I \times A) = \rho_N(A)$ for any $A \in B_I$. The measure $\overline{\rho}_N$ is preserved by $\overline{R}_N$ [8], i.e., $\overline{\rho}_N(\left( \overline{R}_N \right)^{-1}(B)) = \overline{\rho}_N(B)$ for any $B \in B_I^2$.

2.3 Extended random variables

Define the projection $E : I^2 \to I$ by $E(x, y) := x$. With respect to $\overline{R}_N$ in (2.16), define the extended incomplete quotients $\overline{a}_l(x, y), l \in \mathbb{Z} := \{-, \ldots, -2, -1, 0, 1, 2, \ldots\}$ at $(x, y) \in I^2$ by

$$\overline{a}_l(x, y) := (a_1 \circ E) \left( \left( \overline{R}_N \right)^{l-1}(x, y) \right), \quad l \in \mathbb{Z}.$$  \hspace{1cm} (2.19)

Note that $\overline{a}_l(x, y)$ in (2.19) is also well defined for $l \leq 0$ because $\overline{R}_N$ is invertible. By (2.14) and (2.17) we have

$$\overline{a}_n(x, y) = a_n(x), \quad \overline{a}_0(x, y) = a_1(y), \quad \overline{a}_{-n}(x, y) = a_{n+1}(y),$$  \hspace{1cm} (2.20)

for any $n \in \mathbb{N}_+$ and $(x, y) \in I^2$.

Since the measure $\overline{\rho}_N$ is preserved by $\overline{R}_N$, the doubly infinite sequence $(\overline{a}_l(x, y))_{l \in \mathbb{Z}}$ is strictly stationary (i.e., its distribution is invariant under a shift of the indices) under $\overline{\rho}_N$. The stochastic property (2.22) of $(\overline{a}_l(x, y))_{l \in \mathbb{Z}}$ follows from the fact that

$$\overline{\rho}_N([0, x] \times I \mid \overline{a}_0, \overline{a}_{-1}, \ldots) = \frac{N x}{N - (1-x)(1-a)} \overline{\rho}_N \text{-a.s.},$$  \hspace{1cm} (2.21)

for any $x \in I$, where $a := [\overline{a}_0, \overline{a}_{-1}, \ldots]_{1/N}$ with $\overline{a}_l := \overline{a}_l(x, y)$ and $(x, y) \in I^2$. 

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If $I_n$ denotes the cylinder $I(\bar{a}_0, \bar{a}_{-1}, \ldots, \bar{a}_{-n})$ for $n \in \mathbb{N}$ for $l \in \mathbb{Z}$ and $(\bar{a}_1 = i) = I(i) \times I$, $i \in \Lambda$, it follows that

$$\varphi_N(\bar{a}_1 = i | \bar{a}_0, \bar{a}_{-1}, \ldots) = P^i_N(a) \varphi_N \text{-a.s.}$$

(2.22)

where $a = [\bar{a}_0, \bar{a}_{-1}, \ldots]_{1/N}$ and

$$P^i_N(x) := \frac{x + N - 1}{(x + i)(x + i - 1)}.$$  

(2.23)

The strict stationarity of $(\bar{a}_l)_{l \in \mathbb{Z}}$ under $\varphi_N$ implies

$$\varphi_N(\bar{a}_{l+1} = i \mid \bar{a}_l, \bar{a}_{l-1}, \ldots) = P^i_N(a) \varphi_N \text{-a.s.}$$

(2.24)

for any $i \geq N$ and $l \in \mathbb{Z}$, where $a = [\bar{a}_1, \bar{a}_{-1}, \ldots]_{1/N}$.

Motivated by (2.21), we shall consider the one-parameter family $\{\rho^t_N : t \in I\}$ of (conditional) probability measures on $(I, B_I)$ defined by their distribution functions

$$\rho^t_N([0, x]) := \frac{Nx}{N - (1 - x)(1 - t)}, \quad x, t \in I.$$  

(2.25)

Note that $\rho^0_N = \lambda$.

For $n \geq 2$ let $a_n = a_n(x)$ be as in (2.4). For any $t \in I$ put

$$s^i_{N,0} := t, \quad s^i_{N,n} := 1 - \frac{N}{a_n + s^i_{N,n-1}}, \quad n \in \mathbb{N}.$$  

(2.26)

Note that by the very definition of $s^i_{N,n}$, we have

$$s^i_{N,n} = [a_n, \ldots, a_2, a_1 + t - 1]_{1/N}, \quad n \geq 2,$$

(2.27)

while $s^i_{N,1} = 1 - N/(a_1 + t)$, $t \in I$. These facts lead us to the random system with complete connections [8] $(I, B_I, \Lambda, u, P)$, where $u : I \times \Lambda \to I$ is defined as

$$u(x, i) := u_i(x) = u^i_N(x)$$

(2.28)

with $u^i_N$ as in (2.14) and $P : I \times \Lambda \to I$ is defined as

$$P_i(x) := P^i_N(x)$$

(2.29)

with $P^i_N$ as in (2.23), for all $x \in I$ and $i \in \Lambda$.

Then $(s^i_{N,n})_{n \in \mathbb{N}+}$ is an $I$-valued Markov chain on $(I, B_I, \rho^t_N)$ which starts from $s^i_{N,0} = t$, $t \in I$, and has the following transition mechanism: from state $s \in I$ the only possible transitions are those to states $1 - N/(s + i)$ with the corresponding transition probability $P^i_N(s) = i \in \Lambda$.

Let $B(I)$ denote the Banach space of all bounded $I$-measurable complex-valued functions defined on $I$ which is a Banach space under the supremum norm. The transition operator of $(s^i_{N,n})_{n \in \mathbb{N}+}$ takes $f \in B(I)$ into the function defined by

$$E_{\rho^t_N}\left(f(s^i_{N,n+1}) \mid s^i_{N,n} = s\right) = \sum_{i \in \Lambda} P^i_N(s) f\left(u^i_N(s)\right) = U_N f(s)$$

(3.20)

for any $s \in I$, where $E_{\rho^t_N}$ stands for the mean-value operator with respect to the probability measure $\rho^t_N$, whatever $t \in I$, and $U_N$ is the Perron–Frobenius operator of $(I, B_I, \rho_N, R_N)$ defined as in (3.2).
Note that for any \( t \in I \) and \( n \in \mathbb{N}_+ \) we have
\[
\rho_N^t (A | a_1, \ldots, a_n) = \rho_N^{s^t}_{n} (R_N^{-n} (A)),
\]
whichever the set \( A \) belonging to the \( \sigma \)-algebra generated by the random variables \( a_{n+1}, a_{n+2}, \ldots \), that is, \( \sigma (a_{n+1}, a_{n+2}, \ldots) = R_N^{-n} (B_I) \). In particular, it follows that the Brodén–Borel–Lévy formula \([6, 8]\) holds under \( \rho_N^t \) for any \( t \in I \), that is,
\[
\rho_N^t (R_N^{-n} ([0, x]) | a_1, \ldots, a_n) = \frac{N x}{N - (1 - x)} \left( 1 - s^t_{n} \right), \quad x \in I, n \in \mathbb{N}_+.
\]

### 3 The Perron–Frobenius operator of \( R_N \)

We shall discuss the relevant properties of the Perron–Frobenius operator of \( R_N \) under the invariant measure \( \rho_N \) and related problems in terms of specified operator domains.

Let \( \mu \) be a probability measure on \((I, B_I)\) such that \( \mu \left( R_N^{-1} (A) \right) = 0 \) whenever \( \mu (A) = 0 \) for \( A \in B_I \). For example, this condition is satisfied if \( R_N \) is \( \mu \)-preserving, that is, \( \mu R_N^{-1} = \mu \).

Let \( L^1 (I, \mu) := \{ f : I \to \mathbb{C} : \int_I |f| \, d\mu < \infty \} \). The Perron–Frobenius operator of \((I, B_I, R_N, \mu)\) can be defined by the Radon–Nikodym theorem as the unique linear and positive operator \( U_N \) on the Banach space \( L^1 (I, \mu) \) satisfying
\[
\int_A U_N f \, d\mu = \int_{R_N^{-1} (A)} f \, d\mu \quad \text{for all } A \in B_I, \; f \in L^1 (I, \mu).
\]

If \((I, B_I, \rho_N, R_N)\) is as in (2.1) and (2.5), then the following holds \([8]\):
\[
U_N f (x) = \sum_{i \geq N} P_N^i (x) f \left( u_N^i (x) \right), \quad f \in L^1 (I, \rho_N),
\]
where \( P_N^i \) and \( u_N^i \) are as in (2.23) and (2.14), respectively.

For a function \( f : I \to \mathbb{C} \), define the variation \( \var_A f \) of \( f \) on a subset \( A \) of \( I \) by
\[
\var_A f := \sup \sum_{i=1}^{k-1} \left| f(y_{i+1}) - f(y_i) \right|,
\]
where the supremum is taken over \( y_1 < \cdots < y_k, y_i \in A, i = 1, \ldots, k \) and \( k \geq 2 \). We write simply \( \var f \) for \( \var_\mathcal{A} f \). Let \( BV (I) := \{ f : I \to \mathbb{C} | \var f < \infty \} \) under the norm \( \| f \|_\var := \var f + |f| \), where \( |f| := \sup_{x \in I} |f (x)| \).

Next, we calculate the variation of the Perron–Frobenius operator \( U_N \).

**Proposition 3.1** For any \( f \in BV (I) \) we have
\[
\var U_N f \leq \frac{1}{N} \cdot \var f + K_N \cdot |f|,
\]
where
\[
K_N := \frac{2}{2N - 1 + 2\sqrt{N(N-1)}}.
\]

(Springer)
Proof Recall that \( P_i^N(x) = \frac{i+1-N}{x+i} - \frac{i-N}{x+i-1}, i \geq N \). We have

\[
\left( P_i^N(x) \right)' = \frac{i-N}{(x+i-1)^2} - \frac{i+1-N}{(x+i)^2} = \frac{L(N,x)}{(x+i-1)^2(x+i)^2}
\]

with \( L(N,x) = -x^2 + 2x(1-N) + i^2 + i(1-2N) + N-1 \), for every \( i \geq N \).

If \( N \leq i \leq 2N-2 \), then \( L(N,x) < 0 \) for all \( x \in I \), i.e., \( \left( P_i^N(x) \right)' < 0 \), \( x \in I \). Hence, the functions \( P_i^N \) are decreasing on \( I \).

If \( i = 2N-1 \), then \( L(N,x) > 0 \) for all \( x \in [0, 1-N + \sqrt{N(N-1)}] \), and \( L(N,x) < 0 \) for all \( x \in (1-N + \sqrt{N(N-1)}, 1] \). Hence \( P_N^{2N-1} \) is increasing on \( [0, 1-N + \sqrt{N(N-1)}] \) and decreasing on \( (1-N + \sqrt{N(N-1)}, 1] \).

If \( i \geq 2N \), then \( L(N,x) > 0 \) for all \( x \in I \), i.e., \( \left( P_i^N(x) \right)' > 0 \), \( x \in I \). Hence, the functions \( P_i^N \) are increasing on \( I \).

Hence

\[
\text{var } P_i^N = \begin{cases} 
P_i^N(0) - P_i^N(1), & N \leq i \leq 2N-2, \\
2P_i^N(1-N + \sqrt{N(N-1)}) - P_i^N(0) - P_i^N(1), & i = 2N-1, \\
P_i^N(1) - P_i^N(0), & i \geq 2N,
\end{cases}
\]

and

\[
|P_i^N| = \sup_{x \in I} P_i^N(x) = \begin{cases} 
P_i^N(0), & N \leq i \leq 2N-2, \\
P_i^N(1-N + \sqrt{N(N-1)}), & i = 2N-1, \\
P_i^N(1), & i \geq 2N.
\end{cases}
\]

Thus,

\[
\sup_{i \geq N} |P_i^N| = \max \left\{ P_N^N(0), P_N^{2N-1}(1-N + \sqrt{N(N-1)}), P_N^{2N}(1) \right\}
= \max \left\{ \frac{1}{N}, \frac{1}{(\sqrt{N} + \sqrt{N-1})^2}, \frac{1}{2(1+2N)} \right\} = \frac{1}{N}.
\]

Also,

\[
\sum_{i \geq N} \text{var } P_i^N = \sum_{N \leq i \leq 2N-2} (P_i^N(0) - P_i^N(1)) + \text{var } P_N^{2N-1} + \sum_{i \geq 2N} (P_i^N(1) - P_i^N(0))
= \frac{1}{2(2N-1)} + \frac{2}{2N-1 + 2\sqrt{N(N-1)}} - \frac{1}{2N-1} + \frac{1}{2(2N-1)}
= \frac{2}{2N-1 + 2\sqrt{N(N-1)}}.
\]

Taking into account

\[
\sum_{i \geq N} \text{var}(f \circ u_i^N) = \sum_{i \geq N} \text{var}_{\left[1-\frac{x}{N}, 1-\frac{N}{N+1}\right]} f = \text{var } f,
\]
we have
\[
\var U_N f = \var \sum_{i \geq N} P^i_N \cdot (f \circ u^i_N) \leq \sum_{i \geq N} \var \left( P^i_N \cdot (f \circ u^i_N) \right)
\]
\[
\leq \sum_{i \geq N} |P^i_N| \var (f \circ u^i_N) + \sum_{i \geq N} |f \circ u^i_N| \var P^i_N
\]
\[
\leq \left( \sup_{i \geq N} |P^i_N| \right) \sum_{i \geq N} \var (f \circ u^i_N) + |f| \sum_{i \geq N} \var P^i_N \leq \frac{1}{N} \cdot \var f + K_N \cdot |f|,
\]
where the constant $K_N$ is as in (3.4). \hfill \Box

If $f \in B(I)$, define the linear functional $U_N^\infty$ by
\[
U_N^\infty : B(I) \to \mathbb{C}; \quad U_N^\infty f = \int_I f(x) \, d\rho_N(x).
\] (3.5)

Then, by the invariance of $R_N$ under $\rho_N$, we have
\[
U_N^\infty U_N^\infty f = U_N^\infty f \quad \text{for any } n \in \mathbb{N}_+.
\] (3.6)

**Corollary 3.2** For any $f \in BV(I)$ and for all $n \in \mathbb{N}$ we have
\[
\var U_N^\infty f \leq \left( \frac{1}{N} + K_N \right)^n \cdot \var f,
\] (3.7)
\[
|U_N^\infty f - U_N^\infty f| \leq \left( \frac{1}{N} + K_N \right)^n \cdot \var f.
\] (3.8)

**Proof** Note that for any $f \in BV(I)$ and $u \in I$, since $\int_I d\rho_N(x) = 1$, we have
\[
|f(u) - \int_I f(x) \, d\rho_N(x)| \leq \left| f(u) - \int_I f(x) \, d\rho_N(x) \right| = \int_I (f(u) - f(x)) \, d\rho_N(x) \leq \var f,
\]
whence
\[
|f| = \sup_{u \in I} |f(u)| \leq \left| \int_I f(x) \, d\rho_N(x) \right| + \var f, \quad f \in BV(I).
\] (3.9)

Finally, (3.5), (3.6) and (3.9) imply
\[
|U_N^\infty f - U_N^\infty f| \leq \left| \int_I (U_N^\infty f - U_N^\infty f) \, d\rho_N(x) \right| + \var (U_N^\infty f - U_N^\infty f)
\]
\[
\leq |U_N^\infty U_N^\infty f - U_N^\infty f| + \var U_N^\infty f = \var U_N^\infty f,
\] (3.10)
for all $n \in \mathbb{N}$ and $f \in BV(I)$.

It follows from Proposition 3.1 that for all $f \in BV(I)$ we have
\[
\var (U_N f - U_N^\infty f) \leq \frac{1}{N} \cdot \var (f - U_N^\infty f) + K_N \cdot |f - U_N^\infty f|.
\]
But
\[
\var (U_N f - U_N^\infty f) = \var U_N f, \quad \var (f - U_N^\infty f) = \var f.
\]
and \( |f - U_N^\infty f| \leq \text{var } f \) which is (3.10) with \( n = 0 \). Thus,
\[
\text{var } U_N f \leq \frac{1}{N} \cdot \text{var } f + K_N \cdot \text{var } f = \left( \frac{1}{N} + K_N \right) \cdot \text{var } f
\]
which leads to (3.7). Next, (3.8) follows from (3.10) and (3.7).

\[\square\]

### 4 A two-dimensional Gauss-Kuzmin theorem

In this section we study the joint distribution function of \( R_N^n \) and \( s_{N,n}^n \), \( n \in \mathbb{N}_+ \), under \( \rho^t_N \), \( t \in I \). We derive the asymptotic distribution function
\[
\lim_{n \to \infty} \rho^t_N \left( R_N^n \in [0, x], s_{N,n}^n \in [0, y] \right)
\]
\[
= \frac{1}{\log(N)} \log \left( \frac{(x + N - 1)(y + N - 1)}{(N - 1)(N - (1 - x)(1 - y))} \right), \quad x, y \in I
\]
and we deliver an estimate of the \( n \)th error term
\[
e^t_{N,n}(x, y) = \rho^t_N \left( R_N^n \in [0, x], s_{N,n}^n \in [0, y] \right) - \frac{1}{\log(N)} \log \left( \frac{(x + N - 1)(y + N - 1)}{(N - 1)(N - (1 - x)(1 - y))} \right)
\]
for any \( t \in I, x, y \in I \) and \( n \in \mathbb{N}_+ \).

The main result of this section, Theorem 4.5, represents a similar result as in (1.16) for Rényi-type continued fractions. We shall derive lower and upper bounds (not depending on \( t \in I \)) of the supremum
\[
\sup_{x, y \in I} |e^t_{N,n}(x, y)|, \quad t \in I,
\]
which provide an estimate of the convergence rate involved. First, to parallel a result for RCFs regarding (1.17), we obtain a lower bound for the error, which suggests the convergence rate of \( \rho^t_N \left( s_{N,n}^n \in [0, y] \right) \) to \( \rho_N ([0, y]) \) as \( n \to \infty \) for all \( t \in I \).

**Proposition 4.1** For any \( t \in I \) and \( n \in \mathbb{N}_+ \) we have
\[
\frac{1}{2} P_N^{N(n)}(1) \leq \sup_{y \in I} \left| \rho^t_N \left( s_{N,n}^n \in [0, y] \right) - \rho_N ([0, y]) \right|
\]
with \( P_N^{N(n)}(t) = \sup_{s \in I} \rho^t_N \left( s_{N,n}^n = s \right) \), where we write \( N(n) \) for \( (i_1, \ldots, i_n) \) with \( i_1 = \cdots = i_n = N, n \in \mathbb{N}_+ \).

**Proof** First, the continuity of the function \( y \mapsto \rho_N ([0, y]), y \in I \), and the equations
\[
\lim_{h \downarrow 0} \rho^t_N \left( s_{N,n}^n \leq y - h \right) = \rho^t_N \left( s_{N,n}^n < y \right)
\]
and
\[
\lim_{h \downarrow 0} \rho^t_N \left( s_{N,n}^n < y + h \right) = \rho^t_N \left( s_{N,n}^n \leq y \right)
\]
Two asymptotic distributions related…

imply
\[
\sup_{y \in I} \left| \rho_N^t (s_{N,n}^t \leq y) - \rho_N (I_0 (y)) \right| = \sup_{y \in I} \left| \rho_N^t (s_{N,n}^t < y) - \rho_N (I_0 (y)) \right|
\]
for all \( t \in I \) and \( n \in \mathbb{N} \). Second, for any \( s \in I \),
\[
\rho_N^t (s_{N,n}^t = s) = \rho_N^t (s_{N,n}^t \leq s) - \rho_N (I_0 (s)) - \left( \rho_N^t (s_{N,n}^t < s) - \rho_N (I_0 (s)) \right)
\leq \sup_{y \in I} \left| \rho_N^t (s_{N,n}^t \leq y) - \rho_N (I_0 (y)) \right| + \sup_{y \in I} \left| \rho_N^t (s_{N,n}^t < y) - \rho_N (I_0 (y)) \right|
= 2 \sup_{y \in I} \left| \rho_N^t (s_{N,n}^t \leq y) - \rho_N (I_0 (y)) \right|.
\]

Hence
\[
\sup_{y \in I} \left| \rho_N^t (s_{N,n}^t \in [0, y]) - \rho_N (I_0 (y)) \right| = \sup_{y \in I} \left| \rho_N^t (s_{N,n}^t \leq y) - \rho_N (I_0 (y)) \right|
\geq \frac{1}{2} \sup_{s \in I} \rho_N^t (s_{N,n}^t = s),
\]
for all \( t \in I \) and \( n \in \mathbb{N} \).

By induction with respect to \( n \in \mathbb{N} \), we get
\[
U^N_n f(x) = \sum_{i_1, \ldots, i_n \in \Lambda} P_{N}^{i_1 \cdots i_n} (x) f(u_{N}^{i_1 \cdots i_n} (x)), \quad x \in I,
\]
where
\[
u_{N}^{i_1} = u_{N}^{i_1} \circ \cdots \circ u_{N}^{i_1}, \quad P_{N}^{i_1 \cdots i_n} (x) = P_{N}^{i_1} (x) P_{N}^{i_2} (u_{N}^{i_1} (x)) \cdots P_{N}^{i_n} (u_{N}^{i_{n-1} \cdots i_1} (x)), \quad n \geq 2,
\]
and the functions \( u_{N}^{i} \) and \( P_{N}^{i} \) are defined in (2.14) and (2.23), respectively, for all \( i \in \Lambda \).

Next, using (2.30), we obtain
\[
U^N_n f(t) = E_{\rho_N^t} \left( f \left( s_{N,n}^t \right) \right), \quad n \in \mathbb{N}, \quad f \in B(I), \quad t \in I.
\]
As \( s_{N,n}^t = u_{N}^{a_1 \cdots a_1} (t), \quad t \in I, \quad n \in \mathbb{N}, \), we derive
\[
U^N_n f(t) = \sum_{i(n) \in \Lambda^n} \rho_N^t \left( (a_1, a_2, \ldots, a_n) = i(n) \right) f \left( u_{N}^{i_1 \cdots i_1} (t) \right)
\]
for any \( n \in \mathbb{N}, \quad f \in B(I), \quad t \in I \) and \( i(n) = (i_1, \ldots, i_n) \in \Lambda^n \). Hence, by (2.9), (4.2) and (4.5) we get
\[
P_{N}^{i_1 \cdots i_n} (t) = \rho_N^t \left( I_N \left( i(n) \right) \right) = \rho_N^t \left( s_{N,n}^t \in [i_1, \ldots, i_2, i_1 + t - 1]_R \right), \quad n \geq 2,
\]
\[
P_{N}^{i_1} (t) = \rho_N^t \left( I_N \left( i_1 \right) \right) = \rho_N^t \left( s_{N,1}^t \in \left[ 1 - \frac{N}{i_1 + t} \right] \right),
\]
for all \( t \in I \) and \( i_1, \ldots, i_n \in \Lambda \).
Since, as can be easily seen,
\[
\max_{i(n) \in \Lambda^n} \rho_N^t \left( I_N \left( i(n) \right) \right) = \rho_N^t \left( I_N \left( N(n) \right) \right),
\]
where we write \( N(n) \) for \( i(n) = (i_1, \ldots, i_n) \) with \( i_1 = \cdots = i_n = N, \quad n \in \mathbb{N}_+ \).
From (2.11) we know that \( I_N \left( i^{(n)} \right) \) is the set of irrationals in the interval with end-points \( \frac{p_n - p_{n-1}}{q_n - q_{n-1}} \) and \( \frac{p_n}{q_n} \). Since

\[
\frac{p_n}{q_n} = [i_1, \ldots, i_n]_{1/N}, \quad n \in \mathbb{N}_+,
\]

we get

\[
\frac{p_n}{q_n} = \begin{cases} 
1 - \frac{N}{i_1 + 1}, & \text{if } n = 1, \\
1 - \frac{N}{i_1 + \frac{p_{n-1}(i_2, \ldots, i_n)}{q_{n-1}(i_2, \ldots, i_n)}}, & \text{if } n \geq 2,
\end{cases}
\]

and

\[
\frac{p_n - p_{n-1}}{q_n - q_{n-1}} = \begin{cases} 
1 - \frac{N}{i_1}, & \text{if } n = 1, \\
\left[ i_1, \ldots, i_{n-1}, i_n - 1 \right]_{1/N}, & \text{if } n \geq 2,
\end{cases}
\]

Next, we can write

\[
P_{N}^{i_1 \ldots i_n}(t) = \frac{(t + N - 1)N^{n-1}}{(t + i_1)q_{n-1}(i_2, \ldots, i_n) - Nq_{n-2}(i_3, \ldots, i_{n-1}, i_n)} \\
\times \frac{1}{(t + i_1)q_{n-1}(i_2, \ldots, i_{n-1}, i_n - 1) - Nq_{n-2}(i_3, \ldots, i_{n-1}, i_n - 1)}
\]

(4.6)

for all \( i_n \in \Lambda, n \geq 2, \) and \( t \in I \). Also by (4.6) we have

\[
P_{N}^{N^{(n)}}(t) = \frac{(t + N - 1)N^{n-1}}{(t + N)q_{n-1}(N, \ldots, N) - Nq_{n-2}(N, \ldots, N)} \\
\times \frac{1}{(t + N)q_{n-1}(N, \ldots, N, N - 1) - Nq_{n-2}(N, \ldots, N, N - 1)}.
\]

It is easy to see that \( P_{N}^{N^{(n)}}(\cdot) \) is a decreasing function. Therefore

\[
\sup_{s \in \mathbb{I}} \rho_N^I (s^I_{N,n} = s) = P_{N}^{N^{(n)}}(t) \geq P_{N}^{N^{(n)}}(1)
\]

for all \( t \in I \). \( \square \)
Two asymptotic distributions related…

Now, Proposition 4.1 allows us to give a very simple proof of a result which provides a lower bound for the supremum (4.1).

**Proposition 4.2 (The lower bound)** For any \( t \in I \) we have

\[
\frac{1}{2} P_N^{N(n)}(1) \leq \sup_{x, y \in I} \left| \rho_N^t \left( R_N^n \in [0, x], s_{N, n}^t \in [0, y] \right) - \frac{1}{\log \left( \frac{N}{N-1} \right)} \log \left( \frac{x + N - 1}{(N - 1)(N - (1 - x)(1 - y))} \right) \right|
\]

for all \( n \in \mathbb{N}_+ \).

**Proof** For any \( t \in I \) and \( n \in \mathbb{N}_+ \), by Proposition 4.1 we have

\[
\sup_{x, y \in I} \left| \rho_N^t \left( R_N^n \in [0, x], s_{N, n}^t \in [0, y] \right) - \frac{1}{\log \left( \frac{N}{N-1} \right)} \log \left( \frac{x + N - 1}{(N - 1)(N - (1 - x)(1 - y))} \right) \right| \\
\geq \sup_{y \in I} \left| \rho_N^t \left( R_N^n \in [0, 1], s_{N, n}^t \in [0, y] \right) - \frac{1}{\log \left( \frac{N}{N-1} \right)} \log \left( \frac{y + N - 1}{N - 1} \right) \right| \\
\geq \sup_{y \in I} |\rho_N^t (s_{N, n}^t \in [0, y]) - \rho_N (\{0, y\})| \geq \frac{1}{2} P_N^{N(n)}(1).
\]

\(\square\)

**Remark 4.3** Since \( q_n(N, \ldots, N, N - 1) = N^n \), we get

\[
P_N^{N(n)}(1) = \frac{1}{q_n(N(n))}, \quad n \in \mathbb{N}_+.
\]

By the recurrence relation (2.7) with \( a_n = i_n \) for all \( n \in \mathbb{N} \), we obtain

\[
q_n(N(n)) = \frac{N^{n+1} - 1}{N - 1}.
\]

It should be noted that Proposition 4.2 in connection with the limit

\[
\lim_{n \to \infty} \left( \frac{1}{2} P_N^{N(n)}(1) \right)^{1/n} = \lim_{n \to \infty} \left( \frac{N - 1}{2(N^{n+1} - 1)} \right)^{\frac{1}{n}} = \frac{1}{N}
\]

leads to an estimate of the order of magnitude of the error term \( e_{N, n}^t(x, y) \).

In what follows we exploit the characteristic properties of the transition operator associated with the random system with complete connections underlying Rényi-type continued fractions. By restricting this operator to the Banach space of functions of bounded variation on \( I \), we derive an explicit upper bound for the supremum (4.1).
Proposition 4.4 (The upper bound) For any $t \in I$ we have
\[
\sup_{x,y \in I} \left| \rho^t_N \left( R^n_N \in [0,x], s^n_{N,n} \in [0,y] \right) - \frac{1}{\log \left( \frac{N}{N-1} \right)} \log \frac{(x+N-1)(y+N-1)}{(N-1)(N-(1-x)(1-y))} \right| \\
\leq \left( \frac{1}{N} + K_N \right)^n
\]
for all $n \in \mathbb{N}$, where $K_N$ is as in (3.4).

Proof Let $F^t_{N,n}(y) = \rho^t_N(s^n_{N,n} \leq y)$ and $H^t_{N,n}(y) = F^t_{N,n}(y) - \rho_N([0,y]), t, y \in I, n \in \mathbb{N}$. Note that $H^t_{N,n}(0) = 0$. As we have noted, $U_N$ is the transition operator of the Markov chain $(s^n_{N,n})_{n \in \mathbb{N}}$. For any $y \in I$ consider the function $f_y$ defined on $I$ as
\[
f_y(t) := \begin{cases} 1 & \text{if } 0 \leq t \leq y, \\ 0 & \text{if } y < t \leq 1. \end{cases}
\]
Hence
\[
U^n_N f_y(t) = E_{\rho^t_N} \left( f_y(s^n_{N,n}) \middle| s^n_{N,0} = t \right) = \rho^t_N(s^n_{N,n} \leq y)
\]
for all $t, y \in I, n \in \mathbb{N}$. As
\[
U^n_N f_y = \int_0^1 f_y(t) \, d\rho_N(t) = \rho_N([0,y]), \quad y \in I,
\]
it follows from Corollary 3.2 that
\[
|H^t_{N,n}(y)| = \left| \rho^t_N(s^n_{N,n} \leq y) - \rho_N([0,y]) \right| = \left| U^n_N f_y(t) - U^n_N f_y \right|
\]
\[
\leq \left( \frac{1}{N} + K_N \right)^n \text{var } f_y = \left( \frac{1}{N} + K_N \right)^n \tag{4.7}
\]
holds for all $t, y \in I, n \in \mathbb{N}$. By the definition of the conditional probability and (2.31), for all $t \in I, x, y \in I$ and $n \in \mathbb{N}$ we have
\[
\rho^t_N \left( R^n_N \in [0,x], s^n_{N,n} \in [0,y] \right) = \rho^t_N \left( R^n_N \in [0,x] \middle| s^n_{N,n} \in [0,y] \right) \cdot \rho^t_N(s^n_{N,n} \in [0,y])
\]
\[
= \rho^t_N \left( R^n_N \in [0,x] \middle| s^n_{N,n} \in [0,y] \right) \cdot F^t_{N,n}(y)
\]
\[
= \int_0^y \rho^t_N \left( R^n_N \in [0,x] \middle| s^n_{N,n} = z \right) \, dF^t_{N,n}(z)
\]
\[
= \int_0^y \frac{N}{N-(1-x)(1-z)} \, dF^t_{N,n}(z)
\]
\[
= \int_0^y \frac{N}{N-(1-x)(1-z)} \, d\rho_N(z) + \int_0^y \frac{N}{N-(1-x)(1-z)} \, dH^t_{N,n}(z)
\]
\[
= \frac{1}{\log \left( \frac{N}{N-1} \right)} \log \frac{(x+N-1)(y+N-1)}{(N-1)(N-(1-x)(1-y))} + \frac{N}{N-(1-x)(1-z)} H^t_{N,n}(z) \big|_0^y
\]
\[
+ \int_0^y \frac{N}{(N-(1-x)(1-z))^2} H^t_{N,n}(z) \, dz.
\]
Hence, by (4.7)

\[
\left| \rho^t_N \left( R^N_n \in [0, x], s^t_{N,n} \in [0, y] \right) - \frac{1}{\log \left( \frac{N}{N-1} \right)} \log \frac{(x + N - 1)(y + N - 1)}{(N - 1)(N - (1 - x)(1 - y))} \right|
\]

\[
\leq \left( \frac{1}{N} + K_N \right)^n \left( \frac{N x}{N - (1 - x)(1 - y)} - \frac{N x}{N - (1 - x)(1 - z)} \right)_{z=y}^{z=0}
\]

\[= \left( \frac{1}{N} + K_N \right)^n \frac{N x}{N - 1 + x} \leq \left( \frac{1}{N} + K_N \right)^n,
\]

where \( K_N \) is as in (3.4), \( t, x, y \in I, n \in \mathbb{N} \).

\[\square\]

Combining Proposition 4.2 with Proposition 4.4 we obtain Theorem 4.5.

**Theorem 4.5** For any \( t \in I \) we have

\[
\frac{1}{2} P_N^{n(n)}(1)
\]

\[
\leq \sup_{x,y \in I} \left| \rho^t_N \left( R^N_n \in [0, x], s^t_{N,n} \in [0, y] \right) \right|
\]

\[
- \frac{1}{\log \left( \frac{N}{N-1} \right)} \log \frac{(x + N - 1)(y + N - 1)}{(N - 1)(N - (1 - x)(1 - y))} \right|
\]

\[
\leq \left( \frac{1}{N} + K_N \right)^n
\]

for all \( n \in \mathbb{N}_+ \).

Actually, Theorem 4.5 implies that the convergence rate is \( O(\alpha^n) \), with

\[
\frac{1}{N} \leq \alpha \leq \frac{1}{N} + \frac{2}{2N - 1 + 2\sqrt{N(N - 1)}}.
\]

For example, we have Also, the graph below suggests that for very large values of \( N \) the lower and upper bounds are very close (Fig. 2).

The graphs for the lower and upper bounds were obtained from the functions

\[ f_{ib} : \mathbb{N}_+ \to \mathbb{R}, \quad f_{ib}(N) = \frac{1}{N}, \]

and

\[ f_{ub} : \mathbb{N}_+ \to \mathbb{R}, \quad f_{ub}(N) = \frac{1}{N} + \frac{2}{2N - 1 + 2\sqrt{N(N - 1)}}. \]
Fig. 2 Graphs of the lower and upper bounds

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