On a different weighted zero-sum constant

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Abstract

For a finite abelian group \((G, +)\), the constant \(C(G)\) is defined to be the smallest natural number \(k\) such that any sequence in \(G\) having length \(k\) will have a subsequence of consecutive terms whose sum is zero. For a subset \(A \subseteq \mathbb{Z}_n\), the constant \(C_A(n)\) is the smallest natural number \(k\) such that any sequence in \(G\) having length \(k\) has an \(A\)-weighted zero-sum subsequence of consecutive terms. We determine the value of \(C_A(n)\) for some particular weight-sets \(A\).

Keywords: Weighted zero-sum constant, Davenport constant, units in \(\mathbb{Z}_n\)

1 Introduction

For a finite set \(S\), we denote the number of elements in \(S\) by \(|S|\). For \(a, b \in \mathbb{Z}\), let \([a, b]\) denote the set \(\{k \in \mathbb{Z} : a \leq k \leq b\}\). We begin with the following well-known result (see [1], for instance).

Theorem 1. Let \((G, \cdot)\) be a finite group with \(|G| = n\) and \(k \geq n\). Then given any sequence \(S = (x_1, \ldots, x_k)\) in \(G\) of length \(k\), there exist \(i, j \in [1, k]\) such that \(i \leq j\) and \(x_i x_{i+1} \ldots x_j = e\) where \(e\) is the identity element of \(G\).

Proof. Let \(S = (x_1, \ldots, x_k)\) be a sequence in \(G\) and \(y_i = x_1 x_2 \ldots x_i\) for each \(i \in [1, k]\). If some \(y_i = e\), we are done. Else by the pigeonhole principle, there exist \(i, j \in [1, k]\) such that \(i < j\) and \(e = y_i^{-1} y_j = x_{i+1} x_{i+2} \ldots x_j\).

Definition 1. A sequence \(S = (x_1, \ldots, x_\ell)\) in \(G\) is called a product-identity sequence if \(x_1 x_2 \ldots x_\ell = e\).

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**Definition 2.** For a finite group $G$, the *Davenport constant* $D(G)$ is defined to be the smallest natural number $k$ such that any sequence of $k$ elements in $G$ has a product-identity subsequence.

From Theorem [1], we see that for a finite group $G$ we have $D(G) \leq |G|$. A weighted generalization of the Davenport constant was introduced in [1] for finite abelian groups. It was earlier introduced in [3] for finite cyclic groups, following a similar generalization in [2]. We give a generalization of the weighted Davenport constant to finite $R$-modules. For the rest of this section, $R$ will be a ring with unity and $A$ will be a non-empty subset of $R$.

**Definition 3.** Given an $R$-module $M$ and $A \subseteq R$, a sequence $S = (x_1, \ldots, x_k)$ in $M$ is called an *$A$-weighted zero-sum sequence* if for each $i \in [1, k]$, there exists $a_i \in A$ such that $a_1x_1 + \cdots + a_kx_k = 0$. When $A = \{1\}$, an $A$-weighted zero-sum sequence is also called a zero-sum sequence.

**Definition 4.** For a finite $R$-module $M$ and $A \subseteq R$, the *$A$-weighted Davenport constant* of $M$ denoted by $D_A(M)$ is defined to be the least positive integer $k$ such that any sequence in $M$ of length $k$ has an $A$-weighted zero-sum subsequence.

**Observation 1.** For a finite $R$-module $M$ and $A \subseteq R$, we claim that both the constants $D_A(M)$ and $C_A(M)$ exist. Let $M$ be a finite $R$-module. Given any sequence $S$ in $M$ of length $|M|$, by a similar argument as in the proof of Theorem [1], we see that we can find a zero-sum subsequence $T$ of $S$ which has consecutive terms. By multiplying the zero-sum by an element $a \in A$, we see that $T$ is an $A$-weighted zero-sum subsequence of $S$. Hence, for any $A \subseteq R$ we have $C_A(M) \leq |M|$. Also, for any $A \subseteq R$ we clearly have $D_A(M) \leq C_A(M)$.

When $A = \{1\}$, we denote the constants $C_A(M)$ and $D_A(M)$ by $C(M)$ and $D(M)$ respectively. We consider the ring $\mathbb{Z}_n$ as a $\mathbb{Z}_n$-module and for $A \subseteq \mathbb{Z}_n$, we denote the constants $C_A(\mathbb{Z}_n)$ and $D_A(\mathbb{Z}_n)$ by $C_A(n)$ and $D_A(n)$ respectively. The next result is an immediate consequence of Theorem [1]. Here we consider an abelian group $G$ as a $\mathbb{Z}$-module.
Corollary 1. For a cyclic group $G$ we have $C(G) = |G|$.

Proof. For any group $G$ we have $D(G) \leq C(G) \leq |G|$. Let $G = \langle a \rangle$ be a cyclic group of order $n \geq 2$. By considering the constant sequence $(a, \ldots, a)$ of length $n - 1$, we see that $D(G) \geq n$. Hence, for a cyclic group $G$ we have $C(G) = |G|$.

Theorem 2. For $A = \mathbb{Z}_n \setminus \{0\}$ we have $C_A(n) = 2$.

Proof. For any $A \subseteq \mathbb{Z}_n \setminus \{0\}$ we have seen that $C_A(n) \geq 2$. Let $S = (x_1, x_2)$ be a sequence in $\mathbb{Z}_n$. We claim that $S$ has an $A$-weighted zero-sum subsequence of consecutive terms. If either $x_1$ or $x_2$ is zero, we get a zero-sum subsequence of length 1. If both $x_1$ and $x_2$ are non-zero, then $a_1 = x_2$, $a_2 = -x_1 \in A$ and we have $a_1x_1 + a_2x_2 = 0$. This shows that our claim is true and hence $C_A(n) \leq 2$. Thus $C_A(n) = 2$.

Let $U(n)$ denote the multiplicative group of units in the ring $\mathbb{Z}_n$. If $p$ is a prime, by Theorem 2 it follows that $C_{U(p)}(p) = 2$. For $j \geq 1$ let $U(n)^j$ denote the set $\{ x^j : x \in U(n) \}$. For $n = p_1p_2 \ldots p_k$ where $p_i$ is a prime for each $i \in [1, k]$, we define $\Omega(n) = k$. For a divisor $m$ of $n$, we define the homomorphism $f_{n, m} : \mathbb{Z}_n \rightarrow \mathbb{Z}_m$ as $f_{n, m}(a + n\mathbb{Z}) = a + m\mathbb{Z}$. In this article the following are among some of the results which we have obtained.

- For any odd natural number $n$, we have $C_{U(n)}(n) = 2^{\Omega(n)}$.
- For any prime $p$, we have $C_{U(p^2)}(p) = 3$ when $p \neq 2$ and $C_{U(2^2)}(2) = 2$.
- If every prime divisor of $n$ is at least 7, we have $C_{U(n)}(n) = 3^{\Omega(n)}$.
- If $p$ is a prime such that $p \equiv 1 \pmod{3}$, we have $C_{U(p^3)}(p) = D_{U(p^3)}(p) = 3$ when $p \neq 7$. Also we have $C_{U(7^3)}(7) = 4$ and $D_{U(7^3)}(7) = 3$.
- For a squarefree number $n$, we have $C_{U(n)}(n) = 2^{\Omega(n_2)}3^{\Omega(n_1)}$ if $n$ is not divisible by 2,7 or 13. (The notation “$n = n_1n_2$” is defined in Section 6)
- For any number $n$, we have $2^{\Omega(n_2)}3^{\Omega(n_1)} \leq C_{U(n)}(n) \leq 2^{\Omega(n_2)}4^{\Omega(n_1)}$ if $n$ is not divisible by 2, 3 or 7. (The notation is as in the previous result.)
- Let $n = m_1m_2$ and $A, A_1, A_2$ be subsets of $\mathbb{Z}_n, \mathbb{Z}_{m_1}, \mathbb{Z}_{m_2}$ respectively. If $f_{n, m_1}(A) \subseteq A_1$ and $f_{n, m_2}(A) \subseteq A_2$, then $C_A(n) \geq C_{A_1}(m_1)C_{A_2}(m_2)$.

Observation 2. Let $A \subseteq \mathbb{Z}_n \setminus \{0\}$ and let $m$ be given. Consider the sequence $(1, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots)$
in $\mathbb{Z}_n$ (of arbitrary length). This sequence does not have any $A$-weighted zero-sum subsequence of consecutive terms of length $m$. This shows that we cannot have a similar definition like that of $C_A(G)$ in which we place a restriction on the length of the $A$-weighted zero-sum subsequence.

2 When $A = U(p)^2$ where $p$ is a prime

The next result follows from the well-known Cauchy-Davenport theorem (8, Theorem 2.3).

**Theorem 3.** Let $p$ be a prime and $X, Y, W$ be subsets of $\mathbb{Z}_p$. Then either $X + Y + W = \mathbb{Z}_p$ or $|X + Y + W| \geq |X| + |Y| + |W| - 2$.

For an odd prime $p$ we denote $U(p)^2$ by $Q_p$. As $Q_p$ is the image of the homomorphism $U(p) \to U(p)$ given by $x \mapsto x^2$ whose kernel is $\{1, -1\}$, it follows that $|Q_p| = (p - 1)/2$. We denote $U(p) \setminus Q_p$ by $N_p$.

For an odd prime $p$, in 8 it was shown that when $A = Q_p$ (or $N_p$) we have $D_A(p) = 3$. By a similar argument, the next result is also true when $A = N_p$.

**Theorem 4.** For a prime $p$, we have $C_{Q_p}(p) = 3$ when $p \neq 2$ and $C_{Q_2}(2) = 2$.

**Proof.** When $p = 2$ or $3$ we have $Q_p = \{1\}$ and so $C_{Q_p}(p) = C(p)$. Hence by Corollary 1 we have $C_{Q_2}(2) = 2$ and $C_{Q_3}(3) = 3$. When $p = 5$ we have $Q_5 = \{1, -1\}$. Suppose $S = (x, y, z)$ is a sequence in $\mathbb{Z}_5$. Consider the set $\{x \pm y, -x \pm y, \pm z\}$ which has six elements of $\mathbb{Z}_5$. As at least two elements from this set are equal, we get a $Q_5$-weighted zero-sum subsequence of consecutive terms of $S$. Hence $C_{Q_5}(5) \leq 3$.

For a prime $p \geq 7$ let $S = (x, y, z)$ be a sequence in $\mathbb{Z}_p$. If some term of $S$ is zero, we get a $Q_p$-weighted zero-sum subsequence of length one. Suppose all the terms of $S$ are non-zero. For $w \in \mathbb{Z}_p$ let $Q_p w = \{aw : a \in Q_p\}$. It follows that $|Q_p x| = |Q_p y| = |Q_p z| = |Q_p| = (p - 1)/2$. So $|Q_p x| + |Q_p y| + |Q_p z| - 2 = 3(p - 1)/2 - 2 = (3p - 7)/2$. When $p \geq 7$ we have $(3p - 7)/2 \geq p$. Thus, by Theorem 4 we get that $Q_p x + Q_p y + Q_p z = \mathbb{Z}_p$ and so $S$ is a $Q_p$-weighted zero-sum sequence. Hence, when $p \geq 7$ we have $C_{Q_p}(p) \leq 3$.

If $x \in N_p$ for a prime $p \geq 5$, we see that $(−1, x)$ is a sequence in $\mathbb{Z}_p$ which does not have any $Q_p$-weighted zero-sum subsequence. Hence, when $p \geq 5$ we have $C_{Q_p}(p) \geq 3$. Thus, from all the above results we get $C_{Q_p}(p) = 3$ for a prime $p \geq 3$. $\square$
3 When $A = U(p)^3$ where $p$ is a prime

When $p \not\equiv 1 \pmod{3}$, there is no element of order three in $U(p)$. So the kernel of $\varphi : U(p) \to U(p)$ given by $x \mapsto x^3$ is trivial and hence $U(p)^3 = U(p)$. In this case, we have seen that $C_{U(p)}(p) = 2$.

When $p \equiv 1 \pmod{3}$, there is an element $c$ which has order three in $U(p)$. So the kernel of $\varphi$ is the cyclic subgroup generated by $c$. As the image of $\varphi$ is $U(p)^3$, it follows that $U(p)^3$ is a subgroup of index 3 in $U(p)$.

We will use the following results which are the first Theorem and Proposition 6.1 from [7].

**Theorem 5.** Let $F$ be a field with $|F| \neq 4, 7, 16$. Suppose $G$ is a subgroup of index 3 in $F^*$. Then we have $G + G = F$.

**Theorem 6.** Let $F$ be a finite field with $|F| \neq 4, 7$. Suppose $G$ is a subgroup of index 3 in $F^*$. If $a \in G + G$ with $a \notin G \cup \{0\}$, then we have $G + aG = F^*$.

**Lemma 1.** Let $p$ be a prime such that $p \equiv 1 \pmod{3}$ and $p \neq 7, 13$. Suppose $S$ is a sequence in $\mathbb{Z}_p$ such that at least three terms of $S$ are in $U(p)$. Then $S$ is a $U(p)^3$-weighted zero-sum sequence.

**Proof.** Let $S$ be a sequence in $\mathbb{Z}_p$ and $x, y, z$ be terms of $S$ which are units. Let $w$ be the sum of the remaining terms of $S$ (if any). The equation $zX^3 = w$ has at most three roots in $\mathbb{Z}_p$. As there are at least four elements in $U(p)$ when $p > 5$, we can find $t \in U(p)$ such that $zt^3 \neq w$. So if $z' = w - zt^3$, then we have $z' \neq 0$. To prove that $S$ is a $U(p)^3$-weighted zero-sum sequence, it is enough to show that the sequence $S' = (x, y, z')$ is a $U(p)^3$-weighted zero-sum sequence as we have $-t^3 \in U(p)^3$.

For any $c \in U(p)$, the sequence $(cx, cy, cz')$ is a $U(p)^3$-weighted zero-sum sequence if and only if the sequence $S'$ is a $U(p)^3$-weighted zero-sum sequence. So we can assume that $x \in U(p)^3$. From Theorems 5 and 6 (depending on whether $y \in U(p)^3$ or $y \notin U(p)^3$), we see that $-z' \in U(p)^3 + U(p)^3$. As $x \in U(p)^3$, we see that $U(p)^3 = U(p)^3x$. Thus, there exist $a \in U(p)^3$ and $b \in U(p)^3$ such that $-z' = ax + by$. Hence, $S'$ is a $U(p)^3$-weighted zero-sum sequence.

**Remark 3.1.** The conclusion of Lemma 4 is not true when $p = 7$ and 13. This is because we can check that the sequence $(1, 1, 1)$ in $\mathbb{Z}_p$ is not a $U(p)^3$-weighted zero-sum sequence when $p = 7$ and 13 as $U(7)^3 = \{\pm 1\}$ and $U(13)^3 = \{\pm 1, \pm 5\}$.

**Theorem 7.** If $p$ is a prime such that $p \equiv 1 \pmod{3}$, we have $D_{U(p)^3}(p) \geq 3$. If in addition $p \neq 7$, we have $D_{U(p)^3}(p) = C_{U(p)^3}(p) = 3$. 

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By multiplying the terms of $S_3$. Let $D$ such that we get that $U \in \{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6\}$. Hence, the sequence will be $(1, S)$. Suppose $x, y, z$ be a sequence in $Z_7$. We want to show that $S$ has a $U(p)^3$-weighted zero-sum subsequence of consecutive terms. We can assume that $x, y, z \in U(p)$. By Lemma 1 we see that $S$ is a $U(p)^3$-weighted zero-sum sequence. Hence $C_{U(p)^3}(p) \leq 3$. As $D_U(p)^3(p) \leq C_{U(p)^3}(p)$, we get that $D_U(p)^3(p) = C_{U(p)^3}(p) = 3$.

As $C_{U(13)^3}(13) \geq D_{U(13)^3}(13) \geq 3$, it now remains to show that $C_{U(13)^3}(13) \leq 3$. Let $S = (x, y, z)$ be a sequence in $Z_{13}$. We may assume that $x, y, z \in U(13)$. By multiplying the terms of $S$ by an element of $U(13)$, we may also assume that $x \in U(13)^3$. Suppose we have that $y \in U(13)^3$. Then $(x, y)$ is a $U(13)^3$-weighted zero-sum subsequence of $S$ as $yx - xy = 0$ and $y, x \in U(13)^3$.

Suppose $y \in B = U(13) \setminus U(13)^3$. As $U(13)^3 = \{\pm 1, \pm 5\}$, we get that $B = \{\pm 2, \pm 3, \pm 4, \pm 6\}$. We can check that $B \subseteq U(13)^3 + U(13)^3$. So by Theorem 2 we have $U(13)^3 x + U(13)^3 y = U(13)$ as $x \in U(13)^3$. Thus, there exist $a, b \in U(13)^3$ such that $-z = ax + by$. So $S$ is a $U(13)^3$-weighted zero-sum sequence. Hence, we get that $C_{U(13)^3}(13) \leq 3$. \hfill $\square$

**Lemma 2.** We have $D_{U(7)^3}(7) = 3$ and $C_{U(7)^3}(7) = 4$.

**Proof.** We observe that $U(7)^3 = \{\pm 1\}$. Let $S = (x, y, z)$ be a sequence in $Z_7$ of length 3. We want to show that $S$ has a $\{\pm 1\}$-weighted zero-sum subsequence. We can assume that $x, y, z$ are non-zero and so are in $U(7) = \{\pm 1, \pm 2, \pm 3\}$. If any two terms of $S$ are equal up to sign, we get a $\{\pm 1\}$-weighted zero-sum subsequence of $S$. Otherwise, up to sign and up to a permutation of the terms, the sequence will be $(1, 2, 3)$ which is a $\{\pm 1\}$-weighted zero-sum sequence. It follows that $D_{\{\pm 1\}}(7) \leq 3$ and so from Theorem 4 we have that $D_{\{\pm 1\}}(7) = 3$.

As the sequence $(1, 3, 1)$ in $Z_7$ does not have any $\{\pm 1\}$-weighted zero-sum subsequence of consecutive terms, it follows that $C_{\{\pm 1\}}(1) \geq 4$. Let $S = (x, y, z, w)$ be a sequence in $Z_7$. Consider the sequence in $Z_7$ having length eight defined as $(x + y, x - y, -x + y, -x - y, z + w, z - w, -z + w, -z - w)$. As at least two terms of this sequence are equal, we get a $\{\pm 1\}$-weighted zero-sum subsequence of consecutive terms of $S$. Thus $C_{\{\pm 1\}}(1) \leq 4$. \hfill $\square$

4 When $A = U(n)$

**Lemma 3.** Let $n = m_1 m_2$ and $A, A_1, A_2$ be subsets of $Z_n, Z_{m_1}, Z_{m_2}$. Suppose $f_{n,m_1}(A) \subseteq A_1$ and $f_{n,m_2}(A) \subseteq A_2$. Then $C_A(n) \geq C_{A_1}(m_1) C_{A_2}(m_2)$.
Proof. Let \( C_{A_1}(m_1) = k \) and \( C_{A_2}(m_2) = l \). Assume that \( k \) and \( l \) are at least 2. There exists a sequence \( S'_1 = (x'_1, \ldots, x'_{k-1}) \) of length \( k-1 \) in \( \mathbb{Z}_{m_1} \) which has no \( A_1 \)-weighted zero-sum subsequence of consecutive terms. Also, there exists a sequence \( S'_2 = (y'_1, \ldots, y'_{l-1}) \) of length \( l-1 \) in \( \mathbb{Z}_{m_2} \) which has no \( A_2 \)-weighted zero-sum subsequence of consecutive terms.

For each \( i \in [1, k-1] \) let \( f_{n,m_1}(x_i) = x'_i \) and \( S_1 = (m_2x_1, \ldots, m_2x_{k-1}) \). For each \( j \in [1, l-1] \) let \( f_{n,m_2}(y_j) = y'_j \) and \( S_2 = (y_1, \ldots, y_{l-1}) \). Define a sequence \( S \) of length \( (k-1)l + l - 1 = kl - 1 \) in \( \mathbb{Z}_n \) as

\[
(m_2x_1, \ldots, m_2x_{k-1}, y_1, m_2x_1, \ldots, m_2x_{k-1}, y_2, \ldots, y_{l-1}, m_2x_1, \ldots, m_2x_{k-1})
\]

Suppose \( S \) has an \( A \)-weighted zero-sum subsequence \( T \) of consecutive terms. If \( T \) contains some term of \( S_2 \), we will get a subsequence \( S_3 \) which has consecutive terms of \( S_2 \) such that the image of the sequence \( S_3 \) under \( f_{n,m_2} \) has an \( A_2 \)-weighted zero-sum subsequence of consecutive terms, as \( f_{n,m_2}(A) \subseteq A_2 \). This is not possible by our choice of \( S'_2 \). Thus, \( T \) does not contain any term of \( S_2 \) and so \( T \) is a subsequence of \( S_1 \).

Let \( T' \) be the sequence in \( \mathbb{Z}_{m_1} \) whose terms are obtained by dividing the terms of \( T \) by \( m_2 \) and taking their images under \( f_{n,m_1} \). As \( f_{n,m_1}(A) \subseteq A_1 \), we will get the contradiction that \( T' \) is an \( A_1 \)-weighted zero-sum subsequence of consecutive terms of \( S'_1 \). Hence, we see that \( S \) does not have any \( A \)-weighted zero-sum subsequence of consecutive terms. As \( S \) has length \( kl - 1 \), it follows that \( C_A(n) \geq kl \) when both \( k \) and \( l \) are at least two.

If \( k = l = 1 \), we are done. Suppose exactly one of \( k \) and \( l \) is equal to one. We can assume that \( l = 1 \) and \( k > 1 \). As the sequence \( S_1 \) which was defined earlier in this proof does not have any \( A \)-weighted zero-sum subsequence of consecutive terms, we see that \( C_A(n) \geq k \).

Corollary 2. For any natural number \( n \), we have \( C_{U(n)}(n) \geq 2^{\Omega(n)} \).

Proof. If \( m \) is a divisor of \( n \), we have \( f_{n,m}(U(n)) \subseteq U(m) \). Also, for a prime \( p \) we have \( U(p) = \mathbb{Z}_p \setminus \{0\} \). Thus, the result follows from Theorem 2 and Lemma 3 by induction on \( \Omega(n) \).

Corollary 3. Let \( n = 2^k \) for some \( k \). Then \( C_{U(n)}(n) = C_{\{\pm 1\}}(n) = n \).

Proof. As \( \{1\} \subseteq \{\pm 1\} \subseteq U(n) \), it follows that \( C_{U(n)}(n) \leq C_{\{\pm 1\}}(n) \leq C(n) \). Thus, from Corollaries 1 and 2 we get \( 2^k \leq C_{U(n)}(n) \leq C_{\{\pm 1\}}(n) \leq n \). So we see that \( C_{U(n)}(n) = C_{\{\pm 1\}}(n) = n \).
Let \( p \) be a prime divisor of \( n \). We use the notation \( v_p(n) = r \) to mean \( p^r \mid n \) and \( p^{r+1} \nmid n \). Let \( S \) be a sequence in \( \mathbb{Z}_n \). Suppose \( p \) is a prime divisor of \( n \) with \( v_p(n) = r \). Let \( S^{(p)} \) be the sequence in \( \mathbb{Z}_{p^r} \) which is the image of the sequence \( S \) under \( f_{n,p^r} \). The following result is Observation 2.2 of [6]. We restate it here using our notation.

**Observation 3.** A sequence \( S \) is a \( U(n) \)-weighted zero-sum sequence in \( \mathbb{Z}_n \) if and only if for every prime divisor \( p \) of \( n \) we have that the sequence \( S^{(p)} \) is a \( U(p^{v_p(n)}) \)-weighted zero-sum sequence in \( \mathbb{Z}_{p^{v_p(n)}} \).

We have the following more general result. For a prime divisor \( p \) of \( n \) with \( v_p(n) = r \), let \( A_p^r = \{ x^j : x \in U(p^r) \} \).

**Observation 4.** A sequence \( S \) is a \( U(n^1) \)-weighted zero-sum sequence in \( \mathbb{Z}_n \) if and only if for every prime divisor \( p \) of \( n \) we have that the sequence \( S^{(p)} \) is an \( A_p^r \)-weighted zero-sum sequence in \( \mathbb{Z}_{p^{v_p(n)}} \).

**Lemma 4.** Let \( p \) be a prime divisor of \( n \) and \( n' = n/p \). Suppose \( c' \in U(n') \). Then there exists \( c \in U(n) \) such that \( f_{n,n'}(c) = c' \).

**Proof.** Let \( n' = n/p \). If \( p \) does not divide \( n' \), by the Chinese remainder theorem we have an isomorphism \( \psi : \mathbb{Z}_n \to \mathbb{Z}_{n'} \times \mathbb{Z}_p \). If \( c \in U(n) \) such that \( \psi(c) = (c',1) \), we have that \( f_{n,n'}(c) = c' \).

If \( p \) divides \( n' \), then \( n \) and \( n' \) have the same prime factors. As \( c' \) is coprime to \( n' \), it follows that \( c' \) is also coprime to \( n \). Thus there exists \( c' \in U(n) \) such that \( f_{n,n'}(c') = c' \).

**Lemma 5.** Let \( S \) be a sequence in \( \mathbb{Z}_n \) and \( p \) be a prime divisor of \( n \) which divides every element of \( S \). Suppose \( n' = n/p \) and \( S' \) is the sequence in \( \mathbb{Z}_{n'} \) whose terms are obtained by dividing the terms of \( S \) by \( p \). If \( S' \) is a \( U(n') \)-weighted zero-sum sequence, then \( S \) is a \( U(n) \)-weighted zero-sum sequence. Also, if \( S' \) is a \( U(n')^2 \)-weighted zero-sum sequence, then \( S \) is a \( U(n)^2 \)-weighted zero-sum sequence.

**Proof.** Let \( S = (x_1, \ldots, x_k) \). Then \( S' = (x'_1, \ldots, x'_k) \) where for each \( i \in [1, k] \) we have \( x'_i = f_{n,n'}(x_i/p) \). Suppose \( S' \) is a \( U(n') \)-weighted zero-sum sequence. Then for each \( i \in [1, k] \) there exist \( a'_i \in U(n') \) such that \( a'_1 x'_1 + \cdots + a'_k x'_k = 0 \). From Lemma 4 we see that for each \( i \in [1, k] \), there exist \( a_i \in U(n) \) such that \( f_{n,n'}(a_i) = a'_i \). As \( a'_1 x'_1 + \cdots + a'_k x'_k = 0 \) it follows that \( f_{n,n'}((a_1 x_1 + \cdots + a_k x_k)/p) = 0 \). As \( n' \) divides \((a_1 x_1 + \cdots + a_k x_k)/p \), we get that \( n \) divides \( a_1 x_1 + \cdots + a_k x_k \) and so \( a_1 x_1 + \cdots + a_k x_k = 0 \) in \( \mathbb{Z}_n \). Thus, \( S \) is a \( U(n) \)-weighted zero-sum sequence.

The other assertion can be proved in a similar manner.
For the next theorem we need the following (6, Lemma 2.1 (ii)), which we restate here using our terminology:

**Lemma 6.** Let $p$ be an odd prime. If a sequence $S$ over $\mathbb{Z}_p^r$ has at least two terms coprime to $p$, then $S$ is a $U(p^r)$-weighted zero-sum sequence.

**Theorem 8.** When $n$ is odd, we have $C_{U(n)}(n) \leq 2^{\Omega(n)}$.

*Proof.* We prove this theorem by induction on $\Omega(n)$. Let $S = (x_1, \ldots, x_k)$ be a sequence in $\mathbb{Z}_n$ of length $k = 2^{\Omega(n)}$. If $\Omega(n) = 1$ then $n$ is prime and so $U(n) = \mathbb{Z}_n \setminus \{0\}$. Hence we are done by using Theorem 2. Let us now assume that $\Omega(n) > 1$.

**Case 8.1.** For any prime divisor $p$ of $n$ at least two terms of $S$ are coprime to $p$.

Let $p$ be a prime divisor of $n$ and let $r = v_p(n)$. Let $S^{(p)}$ be as defined before Observation 3. Then $S^{(p)}$ has at least two units. As $n$ is odd it follows that $p$ is an odd prime. Hence by Lemma 6 we see that $S^{(p)}$ is a $U(p^r)$-weighted zero-sum sequence in $\mathbb{Z}_{p^r}$. As this is true for any prime divisor $p$ of $n$, by Observation 3 we see that $S$ is a $U(n)$-weighted zero-sum sequence.

**Case 8.2.** There is a prime divisor $p$ of $n$ such that at most one term of $S$ is coprime to $p$.

By partitioning $S$ into two equal halves where each half has $k/2$ consecutive terms, we see that there is a subsequence $T$ of consecutive terms of $S$ of length $k/2$ such that $p$ divides every term of $T$.

Let $n' = n/p$ and $T'$ denote the sequence in $\mathbb{Z}_{n'}$ whose terms are obtained by dividing the terms of $T$ by $p$. As $\Omega(n') = \Omega(n) - 1$ and as $T'$ is a sequence of length $2^{\Omega(n')}$ in $\mathbb{Z}_{n'}$, by the induction hypothesis $T'$ has a $U(n')$-weighted zero-sum subsequence of consecutive terms. From Lemma 5 we see that $T$ has a $U(n)$-weighted zero-sum subsequence of consecutive terms. As $T$ is a subsequence of consecutive terms of $S$, it follows that $S$ has a $U(n)$-weighted zero-sum subsequence of consecutive terms.

**Corollary 4.** When $n$ is odd, we have $C_{U(n)}(n) = 2^{\Omega(n)}$.

*Proof.* This follows from Corollary 2 and Theorem 8.

**5 When $A = U(n)^2$**

This section is a generalisation of the results obtained in Section 2. We begin with the following observation.
Corollary 5. If \( n = 2^r m \) where \( m \) is odd, then \( C_{U(n)}(n) \geq 2^r 3^{\Omega(m)} \).

Proof. If \( m \) is a divisor of \( n \), then \( f_{n,m}(U(2)) \subseteq U(m) \). Also, for a prime \( p \) we have \( U(p)^2 = Q_p \). Thus, the result follows from Theorem 4 and Lemma 3 by induction on \( \Omega(n) \).

For the next theorem we need the following result which follows immediately from [5], Lemma 1.

Lemma 7. Let \( n = p^r \) where \( p \) is a prime which is at least seven. Suppose \( S \) is a sequence in \( \mathbb{Z}_n \) such that at least three terms of \( S \) are in \( U(n) \). Then \( S \) is a \( U(n)^2 \)-weighted zero-sum sequence.

Remark 5.1. The conclusion of Lemma 7 may not hold when \( p < 7 \). When \( n \) is 2 or 5, the sequence \((1,1,1)\) in \( \mathbb{Z}_n \) is not a \( U(n)^2 \)-weighted zero-sum sequence.

Theorem 9. If every prime divisor of \( n \) is at least 7, then \( C_{U(n)}(n) \leq 3^{\Omega(n)} \).

Proof. Let \( S = (x_1, \ldots, x_k) \) be a sequence in \( \mathbb{Z}_n \) of length \( k = 3^{\Omega(n)} \). We want to show that \( S \) has a \( U(n)^2 \)-weighted zero-sum subsequence of consecutive terms.

We will use induction on \( \Omega(n) \). From Theorem 4 we see that for any prime \( p \geq 3 \) we have \( C_{Q_p} = 3 \). Let us now assume that \( \Omega(n) > 1 \).

Case 9.1. For any prime divisor \( p \) of \( n \) at least three terms of \( S \) are coprime to \( p \).

Let \( p \) be a prime divisor of \( n \) and \( v_p(n) = r \). Then \( S(p) \) has at least three units, where \( S(p) \) is as defined before Observation 3. As \( p \) is at least 7, by Lemma 7 we see that \( S(p) \) is a \( U(p^r)^2 \)-weighted zero-sum sequence in \( \mathbb{Z}_{p^r} \). As this is true for every prime divisor of \( n \), by Observation 4 it follows that \( S \) is a \( U(n)^2 \)-weighted zero-sum sequence.

Case 9.2. There is a prime divisor \( p \) of \( n \) such that at most two terms of \( S \) are coprime to \( p \).

By partitioning \( S \) into three equal parts where each part has \( k/3 \) consecutive terms, we see that there is a subsequence \( T \) of consecutive terms of \( S \) of length \( k/3 \) such that \( p \) divides every term of \( T \). Let \( n' = n/p \) and \( T' \) denote the sequence in \( \mathbb{Z}_{n'} \) whose terms are obtained by dividing the terms of \( T \) by \( p \). As \( \Omega(n') = \Omega(n) - 1 \) and \( T' \) is a sequence of length \( 3^{\Omega(n')} \) in \( \mathbb{Z}_{n'} \), by the induction hypothesis it follows that \( T' \) has a \( U(n')^2 \)-weighted zero-sum subsequence of consecutive terms. By Lemma 5 we see that \( T \) has a \( U(n)^2 \)-weighted zero-sum subsequence of consecutive terms. As \( T \) is a subsequence of consecutive terms of
S, it follows that S has a $U(n)^2$-weighted zero-sum subsequence of consecutive terms.

\[ \square \]

**Corollary 6.** If every prime divisor of n is at least 7, then $C_{U(n)^2}(n) = 3^\Omega(n)$.

**Proof.** This follows from Corollary 5 and Theorem 4. \[ \square \]

## 6 When $A = U(n)^3$

Let $n = p_1^{r_1}p_2^{r_2} \ldots p_s^{r_s}$ where the $p_i$’s are distinct primes and $7 \nmid n$. Consider the set $I = \{ i : p_i \equiv 1 \pmod{3} \}$. Let $n_1 = \prod \{ p_i^{r_i} : i \in I \}$ and let $n_2 = n/n_1$. We will follow the notation $n = n_1n_2$ throughout this section.

**Corollary 7.** Let $m = 7^r n$ where $7 \nmid n$ and $n = n_1n_2$ where $n_1, n_2$ are as defined above. Then $C_{U(m)^3}(m) \geq 2^{\Omega(n_2)}3^{\Omega(n_1)}4^r$.

**Proof.** In Theorem 7 we have seen that $C_{U(p)^3}(p) = 3$ when $p \equiv 1 \pmod{3}$ and $p \not\equiv 7$. In Lemma 2 we have seen that $C_{U(7)^3}(7) = 4$. If $p \not\equiv 1 \pmod{3}$, then $U(p)^3 = U(p) = \mathbb{Z}_p \setminus \{0\}$ and so by Theorem 2 we have $C_{U(p)^3}(p) = 2$. Thus, this result now follows from Lemma 3 by induction on $\Omega(n)$ and by using the fact that if $d$ is a divisor of $n$, then $f_{n,d}(U(n)^3) \subseteq U(d)^3$. \[ \square \]

**Theorem 10.** Let $n$ be a squarefree number which is not divisible by 2, 7 or 13. Then $C_{U(n)^3}(n) \leq 2^{\Omega(n_2)}3^{\Omega(n_1)}$.

**Proof.** Let $S$ be a sequence in $\mathbb{Z}_n$ of length $k = 2^{\Omega(n_2)}3^{\Omega(n_1)}$. We want to show that $S$ has a $U(n)^3$-weighted zero-sum subsequence of consecutive terms. We now prove this theorem by induction on $\Omega(n)$.

Suppose $n = p$ where $p$ is a prime. If $p \equiv 1 \pmod{3}$ then $n = n_1$, and by using Lemma 7 we can show that $C_{U(n)^3}(n) \leq 3$. If $p \not\equiv 1 \pmod{3}$ then $n = n_2$, and by using Lemma 6 we can show that $C_{U(n)^3}(n) \leq 2$. Let us now assume that $\Omega(n) > 1$.

**Case 10.1.** For any prime divisor $p$ of $n_1$ at least three terms of $S$ are coprime to $p$, and for any prime divisor $p$ of $n_2$ at least two terms of $S$ are coprime to $p$.

Let $p$ be a prime divisor of $n$ and $S(p)$ be as defined before Observation 8. If $p$ divides $n_1$ then $S(p)$ has at least three units. So by Lemma 1 we get that $S(p)$ is a $U(p)^3$-weighted zero-sum sequence in $\mathbb{Z}_p$ as $p \not\equiv 7,13$. If $p$ divides $n_2$ then $S(p)$ has at least two units. So by Lemma 6 we get that $S(p)$ is a $U(p)$-weighted zero-sum sequence in $\mathbb{Z}_p$ as $p \not\equiv 2$.

We have seen that when $p \not\equiv 1 \pmod{3}$ we have $U(p)^3 = U(p)$. Thus, for every prime divisor $p$ of $n$ we get that $S(p)$ is a $U(p)^3$-weighted zero-sum
sequence in $\mathbb{Z}_p$ and so by Observation 4 we see that $S$ is a $U(n)^3$-weighted zero-sum sequence.

**Case 10.2.** There is a prime divisor $p$ of $n_1$ such that at most two terms of $S$ are coprime to $p$.

Let $n' = n/p$. If we write $n'$ as $n'_1 n'_2$ as per the notation given at the beginning of this section, it follows that $n'_1 = n_1/p$ and $n'_2 = n_2$. By partitioning $S$ into three equal parts where each part has $k/3$ consecutive terms, we see that there is a subsequence $T$ which has consecutive terms of $S$ and length $k/3$ such that $p$ divides every term of $T$. Let $T'$ denote the sequence in $\mathbb{Z}_{n'}$ whose terms are obtained by dividing the terms of $T$ by $p$.

Then $T'$ is a sequence of length $k/3 = 2^{\Omega(n_2)}3^{\Omega(n_1)-1} = 2^{\Omega(n'_2)}3^{\Omega(n'_1)}$ in $\mathbb{Z}_{n'}$. As $n'$ is squarefree and is not divisible by 2, 7 or 13 and as $\Omega(n') = \Omega(n) - 1$, by the induction hypothesis we see that $T'$ has a $U(n)^3$-weighted zero-sum subsequence of consecutive terms. By a similar argument as in Lemma 5 (where we replace squares by cubes), we see that $T$ has a $U(n)^3$-weighted zero-sum subsequence of consecutive terms. As $T$ is a subsequence of consecutive terms of $S$, it follows that $S$ has a $U(n)^3$-weighted zero-sum subsequence of consecutive terms.

**Case 10.3.** There is a prime divisor $p$ of $n_2$ such that at most one term of $S$ is coprime to $p$.

Let $n' = n/p$. It follows that $n'_1 = n_1$ and $n'_2 = n_2/p$. By partitioning $S$ into two equal parts where each part has $k/2$ consecutive terms, we see that there is a subsequence $T$ which has consecutive terms of $S$ and length $k/2$ such that $p$ divides every term of $T$. Let $T'$ denote the sequence in $\mathbb{Z}_{n'}$ whose terms are obtained by dividing the terms of $T$ by $p$.

Then $T'$ is a sequence of length $k/2 = 2^{\Omega(n_2)-1}3^{\Omega(n_1)} = 2^{\Omega(n'_2)}3^{\Omega(n'_1)}$ in $\mathbb{Z}_{n'}$. By a similar argument as in the previous case we see that $S$ has a $U(n)^3$-weighted zero-sum subsequence of consecutive terms.

**Corollary 8.** Let $n$ be a squarefree number which is not divisible by 2, 7 or 13. Then $C_{U(n)^3}(n) = 2^{\Omega(n_2)}3^{\Omega(n_1)}$.

**Proof.** This follows from Corollary 7 and Theorem 10.

**Remark 6.1.** By Corollary 7 we see that the conclusions of Theorem 10 and Corollary 8 are not true when $n$ is divisible by 7.

We will now give an upper bound for $C_{U(n)^3}(n)$ when $n$ is not squarefree, for which we need the following ([9], Lemma 4).
Lemma 8. Let \( n = p^r \) where \( p \) is a prime such that \( p \geq 13 \) and \( p \equiv 1 \pmod{3} \). Let \( S \) be a sequence in \( \mathbb{Z}_n \) such that at least four terms of \( S \) are units. Then \( S \) is a \( U(n)^3 \)-weighted zero-sum sequence.

Remark 6.2. Let \( p \) be a prime and \( r \geq 2 \). As \( |U(p^r)| = p^{r-1}(p-1) \), we see that if \( p \equiv 2 \pmod{3} \) then 3 does not divide \( |U(p^r)| \). So the homomorphism from \( U(p^r) \to U(p^r) \) given by \( x \mapsto x^3 \) has trivial kernel and hence it is onto. Thus we have \( U(p^r)^3 = U(p^r) \).

Corollary 9. Let \( n = p^r \) where \( p \) is an odd prime such that \( p \neq 3, 7 \). Let \( S \) be a sequence in \( \mathbb{Z}_n \) such that at least four elements of \( S \) are units. Then \( S \) is a \( U(n)^3 \)-weighted zero-sum sequence.

Proof. When \( p \equiv 2 \pmod{3} \), the result follows from Lemma 8 as \( p \) is odd and we have \( U(n)^3 = U(n) \). When \( p \equiv 1 \pmod{3} \), the result follows from Lemma 8 as \( p \neq 7 \).

Remark 6.3. The conclusion of Corollary 9 is false when \( p = 2, 3, 7 \).

As \( U(7)^3 = \{1, -1\} \), the sequence \((1, 1, 1, 1, 1) \) in \( \mathbb{Z}_7 \) is not a \( U(7)^3 \)-weighted zero-sum sequence.

As \( U(9)^3 = \{1, -1\} \), the sequence \((1, 1, 1, 1, 1) \) in \( \mathbb{Z}_9 \) is not a \( U(9)^3 \)-weighted zero-sum sequence.

As the sequence \((1, 1, 1, 1, 1) \) in \( \mathbb{Z}_2 \) is not a zero-sum sequence and as the image of \( U(2^r)^3 \) under \( f_{2^r, 2} \) is \( \{1\} \), it follows that the sequence \((1, 1, 1, 1, 1) \) in \( \mathbb{Z}_{2^r} \) is not a \( U(2^r)^3 \)-weighted zero-sum sequence.

Theorem 11. If \( n \) is not divisible by 2, 3 or 7, then \( C_{U(n)^3}(n) \leq 2^{\Omega(n_2)4^{\Omega(n_1)}} \).

Proof. Let \( S \) be a sequence in \( \mathbb{Z}_n \) of length \( k = 2^{\Omega(n_2)}4^{\Omega(n_1)} \). We want to show that \( S \) has a \( U(n)^3 \)-weighted zero-sum subsequence of consecutive terms. We now prove this theorem by induction on \( \Omega(n) \). If \( n \) is a prime, we use a similar argument as in the first paragraph of the proof of Theorem 10. Let us now assume that \( \Omega(n) > 1 \).

Case 11.1. For any prime divisor \( p \) of \( n_1 \) at least four terms of \( S \) are coprime to \( p \), and for any prime divisor \( p \) of \( n_2 \) at least two terms of \( S \) are coprime to \( p \).

Let \( p \) be a prime divisor of \( n \), \( v_p(n) = r \) and \( S^{(p)} \) be as defined before Observation 3. If \( p \) divides \( n_1 \), then \( S^{(p)} \) has at least four units. So from Lemma 8 we get that \( S^{(p)} \) is a \( U(p^r)^3 \)-weighted zero-sum sequence in \( \mathbb{Z}_{p^r} \) as \( p \neq 7 \). If \( p \) divides \( n_2 \), then \( S^{(p)} \) has at least two units. So by Lemma 8 we get that \( S^{(p)} \) is a \( U(p^r) \)-weighted zero-sum sequence in \( \mathbb{Z}_{p^r} \) as \( p \neq 2 \).
We have seen that when \( p \equiv 2 \pmod{3} \) we have \( U(p^r)^3 = U(p^r) \). As \( 3 \nmid n \), for every prime divisor \( p \) of \( n \) we get that \( S(p) \) is a \( U(p^r)^3 \)-weighted zero-sum sequence in \( \mathbb{Z}_{p^r} \). So by Observation 4 we see that \( S \) is a \( U(p^r)^3 \)-weighted zero-sum sequence.

Case 11.2. There is a prime divisor \( p \) of \( n_1 \) such that at most three terms of \( S \) are coprime to \( p \), or there is a prime divisor \( p \) of \( n_2 \) such that at most one term of \( S \) is coprime to \( p \).

The proof of the result in this case is very similar to the proofs of the corresponding cases in Theorem 10.

7 Concluding remarks

In Corollary 4 we have determined \( C_A(n) \) for \( A = U(n) \) when \( n \) is odd. The corresponding result for an even integer can be investigated. It will also be interesting to see whether the lower bounds in Corollaries 5 and 7 are the values of \( C_A(n) \) for \( A = U(n)^2 \) and \( A = U(n)^3 \) respectively.

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