Existence and Properties of Certain Critical Points of the Cahn-Hilliard Energy

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Abstract. The Cahn-Hilliard energy landscape on the torus is explored in the critical regime of large system size and mean value close to $-1$. Existence and properties of a “droplet-shaped” local energy minimizer are established. A standard mountain pass argument leads to the existence of a saddle point whose energy is equal to the energy barrier, for which a quantitative bound is deduced. In addition, finer properties of the local minimizer and appropriately defined constrained minimizers are deduced. The proofs employ the $\Gamma$-limit (identified in a previous work), quantitative isoperimetric inequalities, and variational arguments.

1. Introduction

In this paper, we explore the infinite-dimensional energy landscape associated with the Cahn-Hilliard [6] energy

$$E(u) := \int_{\Omega} \frac{1}{2} |\nabla u|^2 + G(u) \, dx,$$

where $G$ is a double-well potential, $\Omega \subset \mathbb{R}^d$ for $d \geq 2$, and the functions $u$ belong to

$$\left\{ u \in H^1 \cap L^4(\Omega) : \int_{\Omega} u \, dx = m \right\},$$

for mean value $m$ strictly between the minima of $G$. For simplicity of presentation, suppose that the minimizers of $G$ are normalized to be $\pm 1$. The energy landscape, which is a fundamental model for phase separation, reflects a competition between the energy and the mean constraint. Indeed, the mean constraint
rules out the absolute energy minimizers \( u \equiv \pm 1 \) and raises the question of the 
lowest achievable energy given the constraint. One may also ask about the 
existence and “shape” of additional local minimizers and the height of the energy 
barriers surrounding them.

The study of energy barriers and the related critical points is driven by the 
issue of nucleation and growth phenomena in physics and other applications. For 
instance, when nearly homogeneous mixtures of alloys, glasses, or polymers are 
quenched, they tend to separate into distinct preferred phases. When the initial 
homogeneous state is a local energy minimizer, the associated parameter regime is 
called the nucleation and growth regime (to be distinguished from the spinodal 
regime). In the Cahn-Hilliard model, which has been widely studied in experi-
ments, numerical simulations, and analysis, the nucleation phenomenon consists 
of the formation and growth of small droplet-like regions of one phase inside a 
early homogeneous bulk phase. The initial formation and growth of droplets is 
often modelled by the stochastic Cahn-Hilliard-Cook equation [13].

Nucleation behavior was described already by Cahn and Hilliard in [7], where 
they discuss the formation of a so-called “critical nucleus,” a droplet-like state 
whose radius is exactly such that an infinitesimal increase in size leads under deter-
ministic forces to growth and relaxation to a similarly droplet-like local minimizer. 
Moreover, they point out the importance of the height of the energy barrier, which 
they define as the energy difference between the homogeneous state and the sad-
dle point. In terms of mathematical analysis, the fact that nucleation events take 
place by way of the saddle point of least energy was put on rigorous ground by 
the theory of large deviations [15]. Deriving accurate information about the critical 
nucleus experimentally is extremely challenging, and there has been a considerable 
effort to study the nucleation problem numerically [11, 14, 21, 27, 30, 35–37].

In terms of analysis, most previous work has studied the Cahn-Hilliard energy

\[
E_\phi(u) := \int_\Omega \frac{\phi}{2} |\nabla u|^2 + \frac{1}{\phi} G(u) \, dx,
\]

for \( \Omega \) and mean value \( m \) fixed and \( \phi \) small. In the so-called critical parameter 
regime studied in [3, 8, 17] and the present paper, the analysis is subtle because 
the energy of the homogeneous state \( u = m \), the energy of a droplet-like local 
minimizer, and the energy barrier in between these two states are all of the same 
order.

Our results include the existence and symmetry properties of a nonuniform lo-
cal minimizer, existence of a saddle point, and quantitative bounds on the droplet 
shape of critical points in the form of their Fraenkel asymmetry and the \( L^2 \)- 
distance to a sharp-interface droplet profile. Our work uses variational arguments 
and \( \Gamma \)-convergence in a fundamental way; indeed, an objective of the paper is to 
explore the use of \( \Gamma \)-limits and error bounds to glean information about the shape 
of the energy landscape. In addition, we make use of quantitative isoperimetric 
inequalities. We will give the results as we go along, once we have introduced
the necessary notation and tools. For the reader who is eager to turn to the main results, we refer to Theorems 1.4, 1.10, and 1.19, along with Proposition 1.8.

1.1. The critical parameter regime. To be concrete, let $\Omega = \mathbb{T}_L$ be the $d$-dimensional flat torus with side length $L$, and consider mean value $m = -1 + \phi$. For simplicity, we set

$$G(u) = \frac{(1 - u^2)^2}{4},$$

but more general nondegenerate double-well potentials are possible (see Subsection 1.6). For $m$ close to $-1$ and periodic boundary conditions, it is easy to see that the uniform state $\bar{u} \equiv m$ is a local energy minimizer. Determining whether it is the global minimizer is more subtle, as we now explain. Define the constant

$$\xi_d := c_0 \frac{d/(d+1)}{\sigma_d^{1/(d+1)}} \frac{d + 1}{4d/(d+1)d/(d+1)},$$

where here and throughout, $c_0$ denotes

$$c_0 = \int_{-1}^{1} \sqrt{2G(s)} \, ds = \frac{2\sqrt{2}}{3} \quad \text{by (1.2)},$$

and $\sigma_d$ stands for the perimeter (surface area) of the $(d - 1)$-unit sphere in $\mathbb{R}^d$. It was shown in [3, 8] that the scaling $\phi \sim L^{-d/(d+1)}$ is critical in the following sense. For

$$\phi = \xi L^{-d/(d+1)}$$

with $\xi < \xi_d$, $\bar{u}$ is the global minimizer of $E$ in $\mathbb{T}_L$ for $\phi$ sufficiently small. For (1.4) with $\xi > \xi_d$, on the other hand, $\bar{u}$ is not the global minimizer, and moreover the global minimizer is close in $L^p(\mathbb{T}_L)$ to a droplet-shaped function (i.e., a function that is close to 1 on a ball $B$ and close to $-1$ on $\mathbb{T}_L \setminus B$).

To look more closely at the energy difference to the uniform state, one would like to analyze the rescaled energy gap

$$\mathcal{E}_\phi^\xi(u) = \frac{E(u) - E(\bar{u})}{\phi^{-d+1}},$$

which, recalling the mean constraint and rescaling space by a factor of $\phi$, can also be written as

$$\mathcal{E}_\phi^\xi(u) = \int_{\mathbb{T}_L} \frac{\phi}{2} |\nabla u|^2 + \frac{1}{\phi} (G(u) - G(-1 + \phi)) \, dx$$

$$= \int_{\mathbb{T}_L} \frac{\phi}{2} |\nabla u|^2 + \frac{1}{\phi} (G(u) - G(-1 + \phi) - G'(1 + \phi)(u - (-1 + \phi))) \, dx.$$. 
We will restrict the space of functions in $H^{1} \cap L^{4}(\mathbb{T}_{\phi L})$ with the norm

$$\|\nabla u\|_{L^{2}(\mathbb{T}_{\phi L})} + \|u\|_{L^{4}(\mathbb{T}_{\phi L})}$$

to the affine subspace

$$X_{\phi} := \left\{ u \in H^{1} \cap L^{4}(\mathbb{T}_{\phi L}) : \int_{\mathbb{T}_{\phi L}} u \, dx = -1 + \phi \right\}.$$ 

In [17], the first and third authors establish $\Gamma$-convergence as $\phi \to 0$ to

$$E^{\xi}_{0}(u) := \begin{cases} c_{0} \text{Per}(\{u = 1\}) - 4|\{u = 1\}| + 4\frac{|\{u = 1\}|^{2}}{\xi^{d+1}} \\
+\infty & \text{otherwise} \end{cases}$$

(1.6)

in the $-1 + L^{p}(\mathbb{R}^{d})$ topology for any $p \in (1, \infty)$ (see [17] for details). Since the torus $\mathbb{T}_{\phi L}$ converges to $\mathbb{R}^{d}$ in this limit, one does not have the compactness that is available in the classical problem of Modica and Mortola (cf. [23, 24, 28]).

The $\Gamma$-convergence result of [17] is based on matching (at leading order) upper and lower bounds for the rescaled energy gap (see Subsection 2.1 for a summary). These bounds in the regime

$$\xi \in (\xi_{d}, \infty), \quad \phi = \xi L^{-d/(d+1)}, \quad \phi \ll 1$$

imply the existence of the minimizer already pointed out in [3, 8], and imply in addition the existence of a minimum energy saddle point "in between" this minimizer and $\bar{u}$.

Because the classical isoperimetric inequality (cf. Theorem 1.14) says that the perimeter functional is minimized on balls, minimization of (1.6) as a function of the measure $|\{u = 1\}|$ leads to the function $f_{\xi} : [0, \infty) \to \mathbb{R}$ defined via

$$f_{\xi}(\nu) := \tilde{C}_{1}\nu^{(d-1)/d} - 4\nu + 4\xi^{-(d+1)}\nu^{2},$$

where

$$\tilde{C}_{1} := c_{0}\sigma_{d}^{1/d} d^{(d-1)/d}.$$
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\[ \nu = \nu_m \). A second important value of \( \xi \) that will play a role in our paper is the (saddle-node) bifurcation point

\[ \xi_d := \xi_d^{d/(d+1)} \sigma_d^{-1/(d+1)} \left( 1 - \frac{1}{d} \right)^{d/(d+1)} \frac{2^{1/(d+1)}(d + 1)}{2d/(d+1)+1}, \]

which is such that, for \( \xi < \xi_d \), the function \( f_\xi \) has no positive local extrema, while for \( \xi > \xi_d \), \( f_\xi \) has a strictly positive local maximum and a strictly positive local minimum (see Figure 1.1 for an illustration).

**Notation 1.1.** For \( \xi > \xi_d \), we denote the strictly positive maximum and minimum of \( f_\xi \) by \( \nu_s, \nu_m \), respectively, and the corresponding function values by

\[ c_s := f_\xi(\nu_s), \quad c_m := f_\xi(\nu_m). \]

It is easy to check that

\[ 1 \leq \nu_s < \nu_m < \frac{\xi_d^{d+1}}{2} \quad \text{for} \ \xi \in (\xi_d, \xi_d]. \]

(Note that \( \lessapprox \) and related notation are explained in Notation 2.1.) Introducing

\[ \gamma_0^2 := \delta(\nu_m - \nu_s), \]

\[ \begin{align*}
0 < \xi < \xi_d & \quad \text{the function} \ f_\xi \ \text{has no positive local extrema. The global minimizer of} \ f_\xi \ \text{is zero for} \ \xi < \xi_d \ \text{and} \ \nu_m > 0 \ \text{for} \ \xi > \xi_d.
\end{align*} \]
we remark for future reference that

\[ \nu_m \text{ is the strict minimizer of } f_{\xi} \]

on the interval \( [\nu_m - \frac{y_0^2}{4}, \nu_m + \frac{y_0^2}{4}] \).

In addition, we define

\[ \Psi(x; \omega) = \begin{cases} +1 & x \in B_\omega(0), \\ -1 & x \in \mathbb{R}^d \setminus B_\omega(0), \end{cases} \]

where \( B_\omega(0) \) is a ball with volume \( \omega \) and center 0. We will abbreviate

\[ \Psi_s := \Psi(\cdot; \nu_s), \quad \Psi_m := \Psi(\cdot; \nu_m). \]

Later (in Lemmas 2.10 and 2.11) we verify that \( \Psi_s \) and \( \Psi_m \) are a saddle point and a local minimum of the limit functional. For this reason we will sometimes refer to \( \Psi_s, \Psi_m \) as the limit saddle point and limit local minimizer.

**Notation 1.2.** We compensate for the translation invariance of the problem by using

\[ |u - \Psi_m|_{\mathbb{R}^d} := \inf_{x_0 \in \mathbb{R}^d} \|u - \Psi_m(\cdot - x_0)\|_{L^2(\mathbb{R}^d)}, \]

\[ |u - \Psi_m|_{\psi L} := \inf_{x_0 \in \psi L} \|u - \Psi_m(\cdot - x_0)\|_{L^2(\psi L)}, \]

where in the second equation, \( \Psi_m \) is understood in the periodic sense; that is, we restrict to \([-\phi L/2, \phi L/2)^d\] and then take the periodic continuation.

**1.2. First result: a droplet-shaped local minimum.** To state our results, we introduce the following “volume” functional \( \nu(\cdot) \), also used in [17], which will play an important role in our analysis.

**Notation 1.3.** Let \( \kappa := \phi^{1/3} \). As in [17], we define a smooth partition of unity \( \chi_1, \chi_2, \text{ and } \chi_3 : \mathbb{R} \to [0, 1] \) such that \( \chi_1 + \chi_2 + \chi_3 = 1 \) and

\[ \chi_1(t) = \begin{cases} 1 & \text{for } t \leq -1 + \kappa, \\ 0 & \text{for } t \geq -1 + 2\kappa, \end{cases} \]

\[ \chi_2(t) = \begin{cases} 1 & \text{for } -1 + 2\kappa \leq t \leq 1 - 2\kappa, \\ 0 & \text{for } t \leq -1 + \kappa \text{ and } t \geq 1 - \kappa, \end{cases} \]

\[ \chi_3(t) = \begin{cases} 1 & \text{for } t \geq 1 - \kappa, \\ 0 & \text{for } t \leq 1 - 2\kappa. \end{cases} \]
We use $\chi_3$ to define the continuous “volume-type” functional $\nu : X_{\phi} \to \mathbb{R}$ via
\begin{equation}
\nu(u) := \int_{\Omega} \chi_3(u) \, dx.
\end{equation}
This functional roughly measures the volume of $u \approx +1$. We will occasionally refer to the sharp-interface analogue
\begin{equation}
\nu_0(u) = |\{u = +1\}|.
\end{equation}

Our first theorem exploits information about the energy and its connection to the $\Gamma$-limit in order to prove the existence of a droplet-shaped local minimizer.

**Theorem 1.4.** Consider $\xi \in (\tilde{\xi}_d, \xi_d]$ and the critical scaling (1.4). For $\phi$ sufficiently small, there exists a nonconstant local minimizer $u_{m,\phi}$ of $E_{\phi}^{\xi}$. This function minimizes $E_{\phi}^{\xi}$ over $|u - \Psi_m|_{1,\phi} \leq y_0$, where $\Psi_m$ is the limit local minimizer defined in (1.9), and $y_0$ is the constant from (1.7).

The local minimizer $u_{m,\phi}$ is well approximated by $\Psi_m$ in the sense that, for every $y \in (0, y_0)$, there exists $\phi_0 > 0$ such that
\begin{equation}
|u_{m,\phi} - \Psi_m|_{1,\phi} \leq y \quad \text{for all } 0 < \phi \leq \phi_0.
\end{equation}

In addition, the closeness of the local minimizer $u_{m,\phi}$ and the limit local minimizer $\Psi_m$ in volume and energy are estimated by
\begin{align}
|E_{\phi}^{\xi}(u_{m,\phi}) - c_m| &\leq \phi^{1/3}, \\
|\nu(u_{m,\phi}) - \nu_m| &\leq C(\xi, d) \phi^{1/6},
\end{align}
where $C(\xi, d)$ is large for $\xi$ near $\tilde{\xi}_d$.

**Remark 1.5 (Case of equality).** We note that the case $\xi = \xi_d$ is not excluded. In [3, 8], where the global minimizer is studied, only $\xi < \xi_d$ and $\xi > \xi_d$ are considered. In our setting, because we are interested in local minimizers, there is no reason to exclude $\xi = \xi_d$, the crossover point at which global minimality is traded from one minimizer to the other. Although our results do not identify the global minimizer, they do imply that any global minimizer is $L^2$ close to $\bar{u}$ or $\Psi_m$.

**Remark 1.6 (Approximate strictness of $u_{m,\phi}$).** One can deduce from the theorem together with Lemma 3.2 below that $u_{m,\phi}$ is “approximately a strict
"minimizer" in the sense that, for every \( \gamma \in (0, \gamma_0] \), there exists \( \delta > 0 \) and \( \phi_0 > 0 \) such that, for all \( \phi \leq \phi_0 \), it holds that

\[
|u - u_{m, \phi}|_{T^\phi L} = \gamma \implies E_\phi^\xi(u) > E_\phi^\xi(u_{m, \phi}) + \delta.
\]

**Remark 1.7 (Quantified closeness to the sharp-interface minimum).** Our estimates improve from (1.11) to the quantified estimate (1.26), below.

One may expect—via a symmetrization argument—to show that \( u_{m, \phi} \) is spherically symmetric. On \( \mathbb{R}^d \) symmetrization leads indeed to spherical symmetry. The periodic setting “frustrates” the system, however, preventing spherical symmetry of sets whose volume is too large. In work in progress, we use Steiner symmetrization on the torus to deduce additional information about local energy minimizers. Meanwhile, the variational arguments that lead to (1.11) and the quantification in Theorem 1.19 provide a measure of the deviation from sphericity that is forced by confinement to the torus \( T^\phi L \).

**1.3. Second result: towards the critical nucleus.** In Proposition 1.8 below, we use Theorem 1.4, a lower bound on the energy barrier around \( u_{m, \phi} \), and a mountain pass argument to deduce the existence of a saddle point with energy equal to this barrier. As mentioned above, energy barriers are a fundamental object in the study of large deviations, where they give the exponential factor in the expected time for a stochastic perturbation to drive the system out of the basin of attraction of a local minimizer in the small noise limit (cf. [15]).

Given Theorem 1.4, it is natural to define the energy barrier around \( u_{m, \phi} \) as

\[
\Delta E^{\phi, \xi}_1 := \inf \max_{\psi} E_\phi^\xi(\psi(t)),
\]

over paths \( \psi \in C([0, 1]; X^\phi) \) such that

\[
E_\phi^\xi(\psi(0)) < E_\phi^\xi(u_{m, \phi}), \quad \psi(1) = u_{m, \phi}.
\]

Our lower bound (cf. Proposition 2.3) bounds this quantity from below and allows for a mountain pass argument. Unfortunately our constructions do not take us all the way to \( u_{m, \phi} \), so that we do not obtain a matching upper bound. Suppose that we are satisfied with reaching the following neighborhood of \( u_{m, \phi} \):

\[
\mathcal{N}_\varepsilon(u_{m, \phi}) := \{ u \in X^\phi : |u - u_{m, \phi}|_{T^\phi L} + E_\phi^\xi(u) - E_\phi^\xi(u_{m, \phi}) < \varepsilon \}.
\]

Then, we can define the modified energy barrier

\[
\Delta E^{\phi, \xi}_2 := \inf \max_{\psi} E_\phi^\xi(\psi(t)),
\]
over paths \( \psi \in C([0, 1]; X_\phi) \) such that

\[
E_\phi^\xi(\psi(0)) < E_\phi(\psi(1)) \in N_\varepsilon(u_{m, \phi}).
\]

The constructions from Proposition 2.4 together with Theorem 1.4 verify that, for fixed, small \( \varepsilon \), and \( \phi \) sufficiently small, there exists at least one such path. Our existence result for saddle points takes the following form.

**Proposition 1.8.** For \( \xi \in (\tilde{\xi}_d, \xi_d) \), the critical scaling (1.4), and \( \phi > 0 \) small enough, there exists a saddle point \( u_{s, \phi} \) of \( E_\phi^\xi \) such that

\[
E_\phi^\xi(u_{s, \phi}) = \Delta E_\phi^{\xi, \phi} \geq c_s + O(\phi^{1/3}).
\]

In addition, there exists a (possibly different) saddle point \( \tilde{u}_{s, \phi} \) such that

\[
E_\phi^\xi(\tilde{u}_{s, \phi}) = \Delta E_\phi^{\xi, \phi} = c_s + O(\phi^{1/3}).
\]

Although \( \Delta E_\phi^{\xi, \phi} \)—and hence also \( \tilde{u}_{s, \phi} \)—depend on \( \varepsilon \), our estimate on the right-hand side of (1.15) is independent of \( \varepsilon \) as \( \phi \to 0 \). For this reason we do not explicitly denote the \( \varepsilon \)-dependence.

We would like to say more about the “droplet-like” shape of the saddle points and the connection to the saddle point of the \( \Gamma \)-limit. Indeed, it is natural to think of \( u_{s, \phi} \) as the so-called critical nucleus [7], which is close in volume and \( L^2 \) to \( \Psi_s \). Unfortunately, we just miss being able to establish these facts. As a partial substitute, we establish closeness in volume and \( L^2 \) of appropriately defined constrained minimizers of the energy.

We define the constrained minimizers as follows. For \( \omega \in [0, \xi_d^{d+1}/2) \), we define the functions \( u_{\omega, \phi} \in X_\phi \) such that

\[
\nu(u_{\omega, \phi}) = \omega, \quad E_\phi^\xi(u_{\omega, \phi}) = \hat{E}_\phi(\omega),
\]

where

\[
\hat{E}_\phi(\omega) := \min\{E_\phi^\xi(u) : u \in X_\phi, \nu(u) = \omega\}.
\]

Because \( \nu(u) \) is a stand-in for the volume of the set \( \{u \approx 1\} \), we refer to such points \( u_{\omega, \phi} \) as volume-constrained minimizers or simply constrained minimizers, when there is no risk of confusion. Existence of the volume-constrained minimizers follows from the direct method of the calculus of variations.

We would like to use these constrained minimizers to define a continuous path over the mountain pass that keeps the energy as small as possible along the way. Indeed, continuity of \( \omega \to u_{\omega, \phi} \) would allow us to deduce information about the “volume” of \( u_{s, \phi} \) and hence also the \( L^2 \) closeness to \( \Psi_s \), using for instance the work of Ghoussoub and Preiss [18], in which they extract additional information about
the location (in phase space) and type of critical points based on a mountain pass argument involving separating sets. Unfortunately, uniqueness of the constrained minimizers is an open question, and continuity of $\omega \mapsto u_{\omega, \phi}$ is not immediately clear.

Although we cannot yet deduce fine properties of $u_s, \phi$ or $\bar{u}_s, \phi$, we can deduce the corresponding properties of an appropriately defined constrained minimizer. We define the “weak” energy barrier surrounding $u_m, \phi$ as

$$\Delta E_{\xi, \omega, \phi} := \sup_{\omega \in [0, \nu_m]} \hat{E}_{\phi}(\omega).$$

From the lower and upper bounds in Propositions 2.3 and 2.4, one can immediately deduce the following result. (We omit the proof.)

**Lemma 1.9.** The weak energy barrier satisfies

$$\Delta E_{\omega, \phi} = \Delta E_{\omega, \phi} + O(\phi^{1/3}) = c_s + O(\phi^{1/3}).$$

For our “approximate saddle point,” we choose (any) $\omega^* \in [0, \nu_m]$ such that

$$\hat{E}_{\phi}(\omega^*) = \Delta E_{\omega, \phi} + O(\phi^{1/3}),$$

and denote by $u_{\omega^*, \phi}$ a corresponding volume-constrained minimizer. In the following theorem, we establish the desired properties of $u_{\omega^*, \phi}$.

**Theorem 1.10.** Consider $\xi \in ([\tilde{\xi}_d, \xi_d])$ and the critical scaling (1.4). For $\phi$ sufficiently small, the volume-constrained minimizer with volume $\omega^*$ satisfying (1.17) is well approximated by the limit saddle point $\Psi_s$ in the following sense. For every $\gamma > 0$ sufficiently small, there exists $\phi_0 > 0$ such that

$$|u_{\omega^*, \phi} - \Psi_s|_{\mathcal{T}_{\phi_d}} \leq \gamma \quad \text{for all } 0 < \phi \leq \phi_0.$$

The constrained minimizer $u_{\omega^*, \phi}$ and the limit saddle point $\Psi_s$ are close in volume and energy in the sense that

$$|\mathcal{E}_{\phi}^\xi(u_{\omega^*, \phi}) - c_s| \leq \phi^{1/3},$$

$$|\nu(u_{\omega^*, \phi}) - v_s| \leq C(\xi, d) \phi^{1/6},$$

where $C(\xi, d)$ is large for $\xi$ near $\tilde{\xi}_d$.

**Remark 1.11 (Approximate mountain pass property).** We do not establish that $u_{\omega^*, \phi}$ is a saddle point, much less a saddle point of mountain pass type. However, $u_{\omega^*, \phi}$ is an approximate mountain pass point in the following sense. For any $\gamma > 0$, there exist $\delta > 0$, $\phi_0 > 0$ such that for $\phi \leq \phi_0$, the following hold:
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(i) For any \( u \in X_\phi \),
\[
\nu(u) = \nu(u_{\omega_s,\phi}), \quad |u - u_{\omega_s,\phi}|_{\Gamma_\phi} > \gamma \implies E_\phi^\xi(u) > E_\phi^\xi(u_{\omega_s,\phi}) + \delta.
\]

(ii) There is a point in the \( \gamma \)-neighborhood of \( u_{\omega_s,\phi} \) with smaller volume and lower energy and a point in the \( \gamma \)-neighborhood of \( u_{\omega_s,\phi} \) with larger volume and lower energy.

The first point follows from Theorem 1.10 together with Lemmas 2.11 and 3.3 below. For the second point, it is convenient to use closeness of \( u_{\omega_s,\phi} \) to \( \Psi_s \), so that it suffices to find a point \( \hat{u}_\phi \in X_\phi \) such that
\[
|\hat{u}_\phi - \Psi_s| < \frac{\gamma}{2} \quad \text{and} \quad E_\phi^\xi(\hat{u}_\phi) < E_\phi^\xi(u_{\omega_s,\phi}).
\]

The constructions from Proposition 2.4 (for volume slightly less or slightly greater than \( \nu_3 \)) do the job.

**Remark 1.12 (Quantified closeness to the sharp-interface saddle).** As for the minimizer, we quantify (1.18) in (1.26) below.

Although we do not manage to show Theorem 1.10 with \( u_{\omega_s,\phi} \) replaced by \( u_{s,\phi} \) or \( \tilde{u}_{s,\phi} \), we do obtain information about any approximately optimal path for \( \Delta E_2^\phi_\xi \). We make this connection precise in Remark 1.20 below after first introducing Theorem 1.19.

### 1.4. Refinement via isoperimetric inequalities.

Since the perimeter functional plays the only geometric role in the \( \Gamma \)-limit, the (classical) isoperimetric inequality suggests that approximately radial functions should be optimal in terms of energy. This idea is a key ingredient in the lower bound of the \( \Gamma \)-convergence argument. We would now like to measure the defect. As mentioned above, although the critical points would be exactly radial on \( \mathbb{R}^d \), they are frustrated in our setting because they are confined to the torus. We are interested in estimating their deviation from sphericity for \( \phi \) small but nonzero, which corresponds to \( \nu_3 \) large but bounded. We achieve this goal by obtaining quantitative bounds on the \( L^2 \) distance and so-called Fraenkel asymmetry as a function of \( \phi \). The Fraenkel asymmetry and sharp isoperimetric inequality of Fusco, Maggi, and Pratelli \[16\] also play an important role in several of our proofs. We recall the definitions and theorems.

It will be useful to define the “isoperimetric function” \( P_\phi \) that associates with a set the perimeter of the ball with the same volume.

**Definition 1.13.** The Euclidean isoperimetric function in \( \mathbb{R}^d \) is defined by
\[
P_\phi(A) := c_d^{1/d} d^{(d-1)/d} |A|^{(d-1)/d},
\]
for any Borel set \( A \subset \mathbb{R}^d \).
Using this notation, we can express the classical isoperimetric inequality on $\mathbb{R}^d$ in the following way.

**Theorem 1.14.** For any Borel set $A \subset \mathbb{R}^d$ it holds that $\text{Per}(A) \geq \text{PE}(A)$, where $\text{Per}(A)$ is the perimeter of $A$ in $\mathbb{R}^d$.

While Theorem 1.14 does not apply to the torus, its conclusion still holds true for sets of small enough measure. This is the content of the following isoperimetric inequality, which we state as in [8], and which is a special case of [25, Theorem 4.4].

**Theorem 1.15 ([8, Theorem 6.1]).** Let $d \geq 2$, and consider the unit $d$-dimensional flat torus $T_1 := [-\frac{1}{2}, \frac{1}{2}]^d$. There exists an $\varepsilon = \varepsilon(d) > 0$ such that, for any Borel set $A \subset T_1$ with $|A| \leq \varepsilon$, the perimeter $\text{Per}_{T_1}(A)$ of $A$ in $T_1$ satisfies

$$\text{Per}_{T_1}(A) \geq \text{PE}(A).$$

We will use the isoperimetric inequality in the setting of the torus $T_{\phi L}$, in which case Theorem 1.15 takes the form

$$\text{Per}_{T_{\phi L}}(A) \geq \text{PE}(A),$$

for any Borel set $A \subset T_{\phi L}$ with $|A| \leq \varepsilon|T_{\phi L}|$. The positive constant $\varepsilon$ is the same as in Theorem 1.15.

The next order correction to the perimeter is probed via so-called quantitative isoperimetric inequalities, which quantify how much the perimeter of a set is increased from that of a ball when the set deviates from spherical. The deviation from sphericity is measured in terms of the Fraenkel asymmetry.

**Definition 1.16 (Fraenkel asymmetry).** The Fraenkel asymmetry in $\mathbb{R}^d$ of a set $A \subset \mathbb{R}^d$ is defined as

$$\lambda(A) := \min_{x \in \mathbb{R}^d} \frac{|A \Delta B(x)|}{|A|},$$

where $B(x) \subset \mathbb{R}^d$ is a ball with center $x$ and volume $|A|$, and $A \Delta B$ denotes the symmetric difference of $A$ and $B$.

Similarly, the Fraenkel asymmetry in the torus of a set $A \subset T_{\phi L}$ whose measure does not exceed that of a ball of radius $\phi L/2$ is defined as

$$\lambda(A) := \min_{x \in T_{\phi L}} \frac{|A \Delta B(x)|}{|A|},$$

where $B(x) \subset T_{\phi L}$ is a ball with center $x$ and volume $|A|$.

We will use both the quantitative isoperimetric inequality on the full space and on the torus. We begin by recalling the sharp result of Fusco, Maggi, and Pratelli [16].
**Theorem 1.17** ([16, Theorem 1.1]). There exists a constant \( C = C(d) \) such that, for any Borel set \( A \subset \mathbb{R}^d \) with \( 0 < |A| < \infty \), the perimeter \( \text{Per}(A) \) of \( A \) in \( \mathbb{R}^d \) satisfies

\[
(1.23) \quad \text{Per}(A) \geq \text{P}_E(A) + C(d)\lambda(A)^2 \text{P}_E(A),
\]

where \( \lambda(A) \) is the Fraenkel asymmetry of \( A \) in \( \mathbb{R}^d \) and \( \text{P}_E(A) \) is given by (1.21).

Following the method of [8, Theorem 6.2] to verify the classical isoperimetric inequality on the torus, we verify the following quantitative isoperimetric inequality on the torus for sets of small measure.

**Corollary 1.18.** Consider the unit \( d \)-dimensional flat torus \( \mathbb{T}_1 := [-\frac{1}{2}, \frac{1}{2}]^d \). There exist constants \( C = C(d) > 0 \) and \( \varepsilon = \varepsilon(d) > 0 \) such that, for any Borel set \( A \subset \mathbb{T}_1 \) with \( |A| < \varepsilon \), the perimeter \( \text{Per}_{\mathbb{T}_1}(A) \) of \( A \) in \( \mathbb{T}_1 \) satisfies the inequality

\[
(1.24) \quad \text{Per}_{\mathbb{T}_1}(A) \geq \text{P}_E(A) + C(d)\lambda(A)^2 \text{P}_E(A) - 4d|A|.
\]

Here, \( \lambda(A) \) is the Fraenkel asymmetry of \( A \) in the torus, and \( \text{P}_E(A) \) is given by (1.21).

The proof of Corollary 1.18 is included in Appendix A. We will use the sharp quantitative isoperimetric inequality in the setting of the torus \( \mathbb{T}_{\phi L} \), in which case Corollary 1.18 takes the form

\[
(1.25) \quad \text{Per}_{\mathbb{T}_{\phi L}}(A) \geq \text{P}_E(A) + C(d)\lambda(A)^2 \text{P}_E(A) - \frac{4d|A|}{\phi L},
\]

for any Borel set \( |A| \subset \mathbb{T}_{\phi L} \) with \( |A| < \varepsilon|\mathbb{T}_{\phi L}| \). Here, \( \text{Per}_{\mathbb{T}_{\phi L}}(A) \) denotes the perimeter of \( A \) in \( \mathbb{T}_{\phi L} \), and \( C(d) \) and \( \varepsilon = \varepsilon(d) \) are the same as in Corollary 1.18.

In addition to using the quantitative isoperimetric inequality in our existence proofs, we use the inequality in order to prove the following theorem, which quantifies the degree to which the critical points are “droplet-like.”

**Theorem 1.19.** Consider \( \xi \in (\mathbb{F}_d, \mathbb{F}_d) \) and the critical scaling (1.4). The minimizer \( u_{m, \phi} \) and any volume-constrained minimizer \( u_{\omega^s, \phi} \) for \( \omega^s \) satisfying (1.17) are approximately spherical in the sense that, for every \( s \in [-1 + 2\Phi^{1/3}, 1 - 2\Phi^{1/3}] \), both \( u_{m, \phi} \) and \( u_{\omega^s, \phi} \) satisfy

\[
(1.26) \quad \lambda(\{u > s\}) \leq \phi^\alpha \quad \text{with} \quad \alpha = \min\{1/6, 1/(2d)\},
\]

and consequently

\[
(1.27) \quad \lambda(\{u_{\omega^s, \phi} > s\}) + \|u_{\omega^s, \phi} - \Psi(\cdot, \omega^s)\|^2_{\mathbb{T}_{\omega^s}} \leq \frac{\phi^\alpha}{\omega^s} \quad \text{with} \quad \alpha = \min\{1/6, 1/(2d)\}.
\]

In fact, for any \( \omega \in (0, \mathbb{F}_d+1/2) \) and \( \phi \) sufficiently small, any associated volume-constrained minimizer \( u_{\omega^s, \phi} \) satisfies

\[
(1.28) \quad \lambda(\{u_{\omega^s, \phi} > s\}) + \|u_{\omega^s, \phi} - \Psi(\cdot, \omega^s)\|^2_{\mathbb{T}_{\omega^s}} \leq \frac{\phi^\alpha}{\omega^s} \quad \text{with} \quad \alpha = \min\{1/6, 1/(2d)\}.
\]

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Remark 1.20 (Near local minimizers and nearly optimal paths). To be concrete, we state Theorems 1.4, 1.10, and 1.19 in terms of the local minimizer $u_{m,\phi}$ and the constrained minimizer $u_{s,\phi}$. However, neither result uses the Euler-Lagrange equation, and a corollary to the proofs may be stated in the following form: any function $\tilde{u} \in X_\phi$ that is a nearly local minimizer in the sense that
\[
\nu(\tilde{u}) = \nu_m + O(\phi^{1/6}), \quad E_\phi^\xi(\tilde{u}) \leq c_m + O(\phi^{1/3})
\]
is droplet-shaped and close to the sharp-interface minimizer in the sense that
\[
(1.28) \quad \lambda(\{\tilde{u} > s\}) \leq \phi^\alpha \quad \text{for every } s \in [-1 + 2\phi^{1/3}, 1 - 2\phi^{1/3}],
\]
and
\[
|\tilde{u} - \Psi_m|^2_{\Gamma_{\text{perl}}} \leq \phi^\alpha,
\]
where, as in Theorem 1.19, $\alpha = \min\{\frac{1}{6}, 1/(2d)\}$. Similarly, any function $\tilde{u} \in X_\phi$ such that
\[
(1.29) \quad \nu(\tilde{u}) = \nu_s, \quad E_\phi^\xi(\tilde{u}) \leq c_s + O(\phi^{1/3})
\]
is droplet-shaped and close to the sharp-interface saddle in the sense of (1.28) and
\[
(1.30) \quad |\tilde{u} - \Psi_s|^2_{\Gamma_{\text{perl}}} \leq \phi^\alpha.
\]
In particular, we use the second fact to make the following observation. Although we do not determine the volume or shape of the saddle points $u_{s,\phi}$ and $\tilde{u}_{s,\phi}$, we do know that any path that is nearly optimal for $\Delta E_2^{\phi,\xi}$ contains a point $\tilde{u}$ satisfying (1.29), and hence contains an approximate droplet-state that is well approximated by the limit saddle, in the sense made precise by (1.28) and (1.30).

1.5. Additional related results in the literature. Previous analysis has focused on (1.1) with fixed mean and $\phi$ small of order one. In dimension one, the global minimizer and saddle point were analyzed in [1, 9], and stochastic nucleation was analyzed in [5]. In $d \geq 2$, some of the first results on the qualitative properties of critical points appear in [19, 26]. More recently, dynamical systems techniques lie at the heart of a series of papers by Wei and Winter [31]–[34] and Bates and Fusco [2], in which so-called spike and bubble solutions of the Cahn-Hilliard energy are analyzed. In particular, [32] establishes existence of critical points that possess an “interior spike,” [2, 34] establish the existence and properties of critical points with multiple interior spikes, and [33] establishes existence of critical points with a spherical interface. While our parameter regime leads (for both the minimum and saddle point) to the scale separation
\[
(\text{lengthscale of } \Omega) \gg (\text{lengthscale of droplet}) \gg (\text{lengthscale of the interface}),
\]
"macroscale" \text{“mesoscale”} \text{“microscale”}

the interior spikes of [2, 32, 34] have no mesoscopic scale, and the bubbles of [33] satisfy

\[
\text{lengthscale of } \Omega \sim \text{lengthscale of droplet} \gg \text{lengthscale of the interface}.
\]

The idea of using $\Gamma$-convergence to establish existence of local (and not just global) minimizers goes back to Kohn and Sternberg [20]. They study the zeroth-order Allen-Cahn energy (i.e., the energy $E_\phi$ defined in (1.1) with no mean constraint) on nonconvex domains, and establish the existence of $L^1$ local minimizers for $\phi$ small. They also comment on the constrained problem for $\phi \ll 1$, although they do not study droplet-type functions. In a similar spirit, Choksi and Sternberg [12] study $E_\phi$ on the unit flat torus for $d = 2$ and with a fixed mean constraint $m$. After establishing that disks and strips always locally minimize perimeter, they deduce the existence of nearby droplet and strip local minimizers of $E_\phi$ for small $\phi$. Another implementation of this idea can be found in Chen and Kowalczyk [10], who, using local maxima of the curvature of $\partial \Omega$, show the existence of local minimizers of the energy (1.1) on a smooth bounded domain $\Omega \subset \mathbb{R}^2$ for small $\phi$ and a fixed mean $m$. In related work, Sternberg and Zumbrun [29] study the Cahn-Hilliard energy landscape for strictly convex domains $\Omega \subset \mathbb{R}^d$, and show that stable critical points have a thin, connected transition layer connecting the pure phases $\approx \pm 1$. To contrast with our paper, we point out that in [10, 20, 29] the geometry of the domain plays a central role, whereas in our work the central role is played by the nonconvexity of the $\Gamma$-limit.

1.6. Generalizations and organization. Working on the torus is not important for the energetic bounds and $\Gamma$-limit; however, periodicity allows us to apply the quantitative isoperimetric inequality of [16]. Rather than taking the standard double-well potential (1.2), it is straightforward to consider more general double-well potentials. Normalizing as usual so that the global minima are $\pm 1$, we may consider any potential $G \in \hat{C}^2(\mathbb{R})$ such that the following hold:

- $G(\pm 1) = 0$ and $G(u) > 0$ for all $u \in \mathbb{R} \setminus \{\pm 1\}$.
- $G'''(\pm 1) > 0$.
- $G$ is convex on $(-\infty, -1)$ and $(1, \infty)$.
- $G(u) \geq u^2$ for $|u| \gg 1$.

If the last assumption is replaced by $G(u) \geq |u|^p$ for $p \in (1, \infty)$, then our results hold for $L^p(\mathbb{T}_\phi L)$.

We begin in Section 2 by recalling and establishing some preliminary estimates. Then, in Section 3, we establish and exploit connections between the $\Gamma$-limit and the original energy. In particular, in Subsection 3.1, we prove Theorem 1.4, establishing existence and initial properties of the local minimizer $u_{m,\phi}$. In Subsection 3.2, we prove the corresponding results for the saddle point, deducing in particular Proposition 1.8 and Theorem 1.10. Deviation from sphericity is quantified in Subsection 3.3, proving Theorem 1.19 and establishing that $u_{m,\phi}$ is a volume-constrained minimizer. Two auxiliary lemmas are proved in Subsection 3.4.
2. Notation and Preliminary Estimates

In this section, we recall some facts from [17] and establish some preliminary estimates. To begin, we introduce some additional notation that will be used throughout the paper.

Notation 2.1. For nonnegative quantities $X$ and $Y$, we write $X \lesssim Y$ to indicate that there exists a constant $C > 0$ that depends at most on $d$ such that $X \leq C Y$ for small enough $\phi > 0$. Writing $X \sim Y$ means that $X \lesssim Y$ and $Y \lesssim X$.

In addition, we use the standard $O(\cdot)$ and $o(\cdot)$ notation (where again dependency on $d$ is permitted) with respect to $\phi \to 0$ (unless otherwise indicated).

We recall for reference below that, in the critical scaling (1.4), the following hold:

\begin{align}
\frac{1}{\phi} \int_{T \phi L} G(-1 + \phi) \, dx &\to \xi^{d+1}, \\
\phi |T \phi L| &\to \xi^{d+1}.
\end{align}

As above, we abbreviate $\kappa := \phi^{1/3}$. Given a function $u : T \phi L \to \mathbb{R}$, we define the partition of $T \phi L$ via

\begin{align*}
\mathcal{A}_-(u) &:= \{ u \leq -1 - \kappa \}, \\
\mathcal{A}(u) &:= \{-1 - \kappa < u \leq -1 + \kappa \}, \\
\mathcal{B}(u) &:= \{-1 + \kappa < u \leq 1 - \kappa \}, \\
\mathcal{C}(u) &:= \{1 - \kappa < u \leq 1 + \kappa \}, \\
\mathcal{C}_+(u) &:= \{ u > 1 + \kappa \}.
\end{align*}

For simplicity, we write $\mathcal{A}_-, \ldots, \mathcal{C}_+$ instead of $\mathcal{A}_-(u), \ldots, \mathcal{C}_+(u)$ when there is no danger of confusion.

Recall the partition of unity and approximate volume functional $\nu(\cdot)$ from Notation 1.3. We will use the partition of unity to decompose the energy (1.5) as

$$T^\xi_{\phi}(u) = \int_{T \phi L} e_\phi(u)(\chi_1(u) + \chi_2(u) + \chi_3(u)) \, dx.$$ 

In order to maintain universal constants in our estimates, we often restrict to a given range of volumes $\nu(u)$. Also, because we are interested in functions of relatively low energy, we will often restrict to

\begin{equation}
T^\xi_{\phi}(u) \leq \max_{\xi \in [\tilde{\xi}_d, \xi_d]} \max \left\{ 2c_s, f_\xi \left( \frac{\xi_{d+1}}{2} \right) \right\} + 1 =: E_M.
\end{equation}

We can instead consider functions with $T^\xi_{\phi}(u) \leq C$ for any given constant $C < \infty$, but then some bounds will depend on $C$. 

2.1. Lower and upper bounds. In this subsection, we summarize the lower and upper bounds from [17] that we will need in the sequel. In addition, we slightly refine the upper bound.

We begin by pointing out that the proof of [17, Proposition 2.4] rules out functions with order-one energy and large volumes of $u \approx 1$, as we summarize in the following lemma.

**Lemma 2.2.** Let $\xi \in (0, \xi_d)$, and consider the critical scaling (1.4). For every $\epsilon_0 > 0$, there exists $\phi_0 > 0$ such that, for all $\phi \leq \phi_0$ and $u \in X_\phi$, it holds that

$$\nu(u) \geq \epsilon_0 (\phi L)^d \Rightarrow \mathcal{E}_\phi^\xi(u) \gg 1 \quad \text{and hence} \quad \mathcal{E}_\phi^\xi(u) > E_M \ (\text{cf. (2.3)}).$$

**Proof.** First, we point out that, according to the scaling (1.4), it holds that

$$\epsilon_0 (\phi L)^d = \frac{\xi_0 \xi^{d+1}}{\phi}.$$ 

Also, we may without loss of generality assume that $\mathcal{E}_\phi^\xi(u) \leq 1$. Invoking [17, Lemma 2.3] and estimating as in [17, Proposition 2.4], we observe

$$\int_{\mathcal{H}_\phi} e_\phi(u) \chi_2(u) \, dx \geq 0,$$

$$\int_{\mathcal{H}_\phi} e_\phi(u) \chi_3(u) \, dx \geq -\nu(u),$$

while convexity, Jensen’s inequality, and the assumption $\nu(u) \geq \epsilon_0 (\phi L)^d$ lead to

$$\int_{\mathcal{H}_\phi} e_\phi(u) \chi_1(u) \, dx \geq \nu(u)^2.$$ 

Choosing $\phi_0$ small enough that $\nu(u)$ (according to (2.4)) is sufficiently large, and adding the energy estimates, leads to $\mathcal{E}_\phi^\xi(u) \geq \nu(u)^2$, and so to $\mathcal{E}_\phi^\xi(u) \gg 1$. \qed

The lower bound follows directly from [17, Proposition 2.4] after rescaling, applying (1.22), and noting that, according to the previous lemma, we may deduce from (2.5) that $\nu(u) \leq \epsilon_0 (\phi L)^d$ for $\epsilon_0 = \epsilon_0(d)$ as in [17, Proposition 2.4].

**Proposition 2.3 (Lower bound [17]).** For any $\omega > 0$ and $\phi > 0$ sufficiently small, the following holds in the critical regime (1.4). If $u \in X_\phi$ satisfies

$$\mathcal{E}_\phi^\xi(u) \leq E_M,$$

then the energy is bounded below by

$$\mathcal{E}_\phi^\xi(u) \geq C_1(\phi) \nu(u)^{(d-1)/d} - C_2(\phi) \nu(u),$$

where $C_1(\phi) = C_1(\phi) > 0$ and $C_2(\phi) = C_2(\phi) > 0$ for $\phi > 0$.
where
\begin{align*}
C_1(\phi) & := (1 - 8\phi^{1/3})^{1/2}(c_0 - 8\sqrt{2}\phi^{2/3})a_d^{1/d} d^{(d-1)/d}, \\
C_2(\phi) & := (2 + \phi^{1/3})(2 - 3\phi).
\end{align*}

Moreover, if \( u \) in addition satisfies \( \nu(u) \geq \omega \), then
\begin{equation}
E_\xi(\hat{\psi}(u)) \geq f_\xi(\nu(u)) + I(u) + O(\phi^{1/3}),
\end{equation}
where
\begin{equation}
I(u) := \int_{1-2\phi^{1/3}}^{1-2\phi^{1/3}} \sqrt{2\hat{G}(t)(\text{Per}_{\nu_{\psi_t}}(\{u > t\}) - P_E(\{u > t\}))} \, dt.
\end{equation}

Here, we have abbreviated \( \hat{G}(t) := (1 - 8\phi^{1/3})G(t) \).

The positive functional \( I(\cdot) \) defined in (2.7) can be thought of as the extra term in the surface tension owing to the deviation from sphericity of the superlevel sets \( \{u > t\} \).

The next proposition provides the upper bound on the energy that we need. It is based on an idea from [4], and the upper bound construction is used explicitly in [8, Lemma 2.2] and later in [17, Lemma 3.5].

**Proposition 2.4 (Upper bound).** Consider \( \xi \in (\xi_d, \xi_d] \) and the critical regime (1.4), and fix \( \omega_1, \omega_2 \) such that
\begin{equation*}
0 < \omega_1 < \nu_s < \omega_2 < \xi_d^{d+1/2}.
\end{equation*}

Then, for small enough \( \phi > 0 \) and any \( t_1 \in (0,1) \), there is a path \( \hat{\psi} \in C([0,1], X_\phi) \) satisfying
\begin{equation*}
\hat{\psi}(0) \equiv \bar{u}, \quad \nu(\hat{\psi}(t_1)) = \omega_1, \quad \nu(\hat{\psi}(1)) = \omega_2,
\end{equation*}
such that the energy on the first part of the time interval is bounded as
\begin{equation*}
\sup_{t \in [0,t_1]} E_\xi(\hat{\psi}(t)) = f_\xi(\omega_1) + o(1),
\end{equation*}
and such that the construction on the second part of the time interval is close in \( L^2 \) and energy to a sharp interface profile in the sense that, for all \( t \in [t_1, 1] \), it holds that
\begin{equation}
|\hat{\psi}(t) - \Psi(\cdot; \nu(\hat{\psi}(t)))|_{1_{t_1 t}}^2 \lesssim \phi |\ln \phi|,
\end{equation}
\begin{equation}
E_\xi(\hat{\psi}(t)) = f_\xi(\nu(\hat{\psi}(t))) + O(\phi |\ln \phi|).\end{equation}
Proof. As in [17, Proposition 3.1], the path is obtained by a construction which—in our current scaling—consists of a linear interpolation between \( \bar{u} \) and a “droplet-like state” of small volume (cf. [17, Lemma 3.2]), followed by a family of droplet-like states such that the volume grows from \( \omega_1 \) at \( t = t_1 \) to \( \omega_2 \) for the final state \( \hat{\psi}(1) \) (cf. [17, Lemma 3.5]).

We begin by recalling the construction of [17, Lemma 3.5]. For \( R \) large and positive, let the function \( v_R : \mathbb{R} \to [-1,1] \) be a smooth, odd function such that

\[
v_R(x) := \begin{cases} 
- \tanh \frac{x}{\sqrt{2}} & \text{for } |x| < R, \\
- \text{sgn}(x) & \text{for } |x| > 2R,
\end{cases}
\]

with a monotone interpolation on \( R \leq |x| \leq 2R \). Now, for \( \eta \in [0,1] \), let the radius \( r_\eta \) be defined by

\[
r_\eta := \eta^{1/d} \left( \frac{\phi d}{2\sigma_d} \right)^{1/d} L.
\]

Then, the construction \( u_\eta : \mathbb{T}_L \to \mathbb{R} \) is defined by

\[
u_R(|x| - r_\eta) + \alpha(\eta),
\]

where \( \alpha(\eta) \) is a constant chosen such that

\[
u(\hat{u}_{\eta,\phi}) \lesssim v(\hat{u}_{\eta,\phi}) + O(\phi |\ln \phi|).
\]

As a consequence of this bound, the volume \( \hat{\nu}_\eta = \sigma_d(\hat{r}_\eta)^d/d \) is related to the “volume-type” functional defined in (1.10) via

\[
(2.10)
\]

\[
u(\hat{u}_{\eta,\phi}) \leq \hat{\nu}_\eta \leq \nu(\hat{u}_{\eta,\phi}) + O(\phi |\ln \phi|)).
\]
We use this fact in two ways. First, we observe that for every $\omega_1 \leq \omega \leq \omega_2$, there is an $\eta \in (0, 1)$ (bounded away from zero and one) such that

$$\nu(\hat{u}_{\eta, \phi}) = \omega. \quad (2.11)$$

We define $\eta := \eta(\omega)$ and $\hat{u}_{\omega, \phi} := \hat{u}_{\eta(\omega), \phi}$ by using this value.

Using the energy bound from [17, Remark 3.6] for all such $\eta$ values, we deduce—in the notation and scaling of the current paper—that

$$\mathcal{E}_\phi^\xi(\hat{u}_{\omega, \phi}) \leq f_\xi(\nu(\hat{u}_{\omega, \phi})) + O(\phi). \quad (2.12)$$

A second application of (2.10) delivers

$$\mathcal{E}_\phi^\xi(\hat{u}_{\omega, \phi}) \leq f_\xi(\nu(\hat{u}_{\omega, \phi})) + O(\phi|\ln \phi|). \quad (2.12)$$

Finally, we observe that (2.9), (2.10), and (2.11) for $\omega_1 \leq \omega \leq \omega_2$ imply

$$|\hat{u}_{\omega, \phi} - \Psi(\cdot; \omega)|^2_{\mathcal{F}_t^{\phi_2}} \leq \phi|\ln \phi|. \quad (2.13)$$

Together, (2.12) and (2.13) yield (2.8).

The estimate

$$\sup_{t \in [0, t_1]} \mathcal{E}_\phi^\xi(\hat{\psi}(t)) = f_\xi(\omega_1) + o(1)$$

follows from the fact that, for that part of the path $t : [0, t_1] \rightarrow \hat{\psi}(t)$ that consists of a convex combination of $\hat{u}$ and a suitably small droplet-like state, the energy stays well below $f_\xi(\omega_1)$ (cf. [17, Lemma 3.2]), while for the rest of this path the energy is given by an estimate similar to (2.12).

2.2. Elementary bounds. In this subsection, we collect several basic but important estimates. The first lemma summarizes $L^2$ and measures bounds for functions of bounded energy. As above, we will abbreviate $\kappa = \phi^{1/3}$ when we bracket the values of $u$.

Lemma 2.5. Consider $\xi \in (0, \xi_d]$ and the critical scaling (1.4). Suppose that $u \in X_\phi$ satisfies

$$\mathcal{E}_\phi^\xi(u) \leq E_M, \quad (2.14)$$

where $E_M$ is defined in (2.3). Then, the following hold true:

$$\int_{A \cup A} (u + 1)^2 \, dx + \int_{C \cup C} (u - 1)^2 \, dx \lesssim \phi, \quad (2.15)$$

$$\int_B (u + 1)^2 \, dx + \int_B (u - 1)^2 \, dx \lesssim |B| \lesssim \phi^{1/3}, \quad (2.16)$$
One consequence of the lemma is that the volume of suitable superlevel sets of bounded energy functions is close to the volume $\nu(u)$.

**Corollary 2.6.** Consider $\xi \in (0, \xi_d]$ and the critical scaling (1.4). Consider $u \in X_\phi$ such that $E^\xi_{\phi}(u) \leq E_M$.

The superlevel sets for $s \in [-1 + \kappa, 1 - \kappa]$ satisfy

$$|\{u > s\}| = \nu(u) + O(\phi^{1/3}).$$

Next, we bound the $L^2$ distance to a sharp-interface function in terms of the Fraenkel asymmetry of $\{u \geq 1 - 2\kappa\}$.

**Lemma 2.7.** Consider $\xi \in (0, \xi_d]$ and also the critical scaling (1.4). For any $u \in X_\phi$ such that

$$\nu(u) \leq \frac{\xi^{d+1}}{2} \quad \text{and} \quad E^\xi_{\phi}(u) \leq E_M,$$

it holds that

$$|u - \Psi(\cdot, \nu(u))|_{\bar{\phi}_k}^2 \leq \lambda(\{u \geq 1 - 2\kappa\}) + O(\phi^{1/3}).$$

Our next lemma establishes that the Fraenkel asymmetry of the superlevel sets is comparable in the following sense.

**Lemma 2.8.** Consider the critical scaling (1.4) with $\xi \in (0, \xi_d]$, and suppose that $u \in X_\phi$ satisfies (2.20). The Fraenkel asymmetry of the intermediate superlevel sets of $u$ is comparable in the sense that

$$\inf_{s \in [-1 + 2\kappa, 1 - 2\kappa]} \lambda(\{u > s\}) \geq \sup_{s \in [-1 + 2\kappa, 1 - 2\kappa]} \lambda(\{u > s\}) + O(\phi^{1/3}) \frac{\nu(u)}{\nu(\phi(u))}.$$

Finally, we note one can show, via a mild adaptation of [17, Lemma 2.3], that the volume-constrained minimizers are bounded.

**Lemma 2.9.** Consider $\xi \in (0, \xi_d]$, the critical scaling (1.4), and $\omega_1 > 0$. For $\phi > 0$ sufficiently small and any $\omega \in [\omega_1, \xi^{d+1}/2]$, the volume-constrained minimizer $u_{\omega, \phi}$ satisfies

$$-1 - \phi^{1/3} \leq \text{ess inf } u_{\omega, \phi} \leq \text{ess sup } u_{\omega, \phi} \leq 1 + \phi^{1/3}.$$

We now present the proofs of these elementary facts, with the exception of Lemma 2.9, whose proof is longer and is included in Subsection 2.4.
Proof of Lemma 2.5. From
\[ T_\phi^g(u) \geq \frac{1}{\phi} \int_{\mathcal{A}_+} G(u) \, dx - \frac{G(-1 + \phi)}{\phi} |\mathcal{A}_+|, \]
together with (2.1) and (2.14), we deduce that
\[ \int_{\mathcal{A}_+} G(u) \, dx \leq \phi, \quad (2.23) \]
To obtain (2.15), we observe that
\[ G(u) \geq (u + 1)^2 \quad \text{on } \mathcal{A}_- \cup \mathcal{A}, \]
\[ G(u) \geq (u - 1)^2 \quad \text{on } C \cup C_+, \]
and combine this with (2.23).
Estimates (2.16), (2.17), and (2.18) follow from (2.23) and
\[ G(u) \geq \phi^{2/3} \quad \text{on } \mathcal{B}, \]
\[ G(u) \geq \phi^{2/3}(u - 1)^2 \quad \text{on } \mathcal{A}_-, \]
\[ G(u) \geq \phi^{2/3}(u + 1)^2 \quad \text{on } C_+, \]
respectively. □

Proof of Corollary 2.6. By definition of the volume functional $\nu$, it follows that
\[ |\{ u \geq 1 - \kappa \} \leq \nu(u) \leq |\{ u \geq -1 + \kappa \}|. \]
Thus, for any $s \in [-1 + \kappa, 1 - \kappa]$ it holds that
\[ -|\{ -1 + \kappa < u \leq s \}| \leq |\{ u > s \}| - \nu(u) \leq |\{ s < u \leq 1 - \kappa \}|, \]
which, as a consequence of (2.16), implies (2.19). □

Proof of Lemma 2.7. We abbreviate
\[ \Psi_u := \Psi(x; \nu(u)), \]
and we denote by $B(x)$ and $\bar{B}(x)$ the balls with center $x$ and volume $\nu(u)$ and $|\{ u \geq 1 - 2\kappa \}|$, respectively. In addition, we recall the “triangle inequality” for the symmetric difference
\[ |A \Delta B| \leq |A \Delta C| + |C \Delta B|, \quad (2.24) \]
According to Lemma 2.5, we have

\[
\|u - \Psi_u(\cdot - x)\|_{L^2(T_{\phi_L})}^2 = \int_{T_{\phi_L}} (u - (-1))21_{u(\cdot - x) = -1} \, dx + \int_{T_{\phi_L}} (u - 1)21_{u(\cdot - x) = 1} \, dx \\
\leq \int_{T_{\phi_L}} (u - (-1))21_{u(\cdot - x) = -1, 1 - \kappa \leq x < 1 + \kappa} \, dx + \int_{T_{\phi_L}} (u - 1)21_{u(\cdot - x) = 1, -1 - \kappa < x < 1 + \kappa} \, dx + O(\phi^{1/3}) \\
\leq |B(x) \triangle \{u > 1 - 2\kappa\}| + O(\phi^{1/3}) \\
\leq |\tilde{B}(x) \triangle \{u > 1 - 2\kappa\}| + O(\phi^{1/3}) \quad \text{by (2.19), (2.24)} \\
\leq \frac{|\tilde{B}(x) \triangle \{u > 1 - 2\kappa\}|}{\nu(u) + |\tilde{B}(x) \triangle \{u > 1 - 2\kappa\}|} + O(\phi^{1/3}) \quad \text{by (2.20),}
\]

where in the last step we also used the fact that

\[
|\{u > 1 - 2\kappa\}| \leq \nu(u) + |\{1 - 2\kappa \leq u < 1 - \kappa\}| \\
\leq \frac{\xi_{d+1}}{2} + O(\phi^{1/3}) \quad \text{by (2.16)} \\
\leq 1.
\]

Infimizing over \(x\) leads to

\[
\|u - \Psi_u\|_{L^2(T_{\phi_L})}^2 \leq \lambda(\{u > 1 - 2\kappa\}) + O(\phi^{1/3}),
\]

as desired. \(\square\)

**Proof of Lemma 2.8.** The proof uses (2.19) and (2.24). Let \(x \in T_{\phi_L}\), and for \(t \in [-1 + 2\kappa, 1 - 2\kappa]\), denote by \(B_t(x) \subset T_{\phi_L}\) the ball with center \(x\) and volume \(||u > t||\). For any \(s \in [-1 + 2\kappa, 1 - 2\kappa]\), we have

\[
\lambda(\{u > t\}) \leq \frac{|B_t(x) \triangle \{u > t\}|}{||u > t||} \\
\leq \frac{|B_s(x) \triangle \{u > s\}|}{||u > s||} + \frac{|B_s(x) \triangle B_t(x)|}{||u > t||} + \frac{|\{u > s\} \triangle \{u > t\}|}{||u > t||} \quad \text{by (2.24)} \\
= \frac{|B_s(x) \triangle \{u > s\}|}{||u > s||} + \frac{|B_s(x) \triangle B_t(x)|}{||u > t||} + \frac{|\{u > s\} \triangle \{u > t\}|}{||u > t||} \quad \text{by (2.19),}
\]

which implies (2.21). \(\square\)
2.3. Structure of the limit energy. Our existence proofs rely on the structure of the limit energy $E_0^\xi$. In the following two lemmas, we analyze $E_0^\xi$ near $\Psi_m$ and $\Psi_s$. The proofs are straightforward, but we include them for completeness and because the proof of Lemma 2.10 serves as the backbone for the proof of Lemma 3.2. Similarly, the proof of Lemma 2.11 could be used to prove a finite-$\phi$ analogue.

Lemma 2.10 (Sharp interface energy near the local minimizer). We fix $\xi \in (\xi_d, \xi_a)$, and define $\gamma_0$ as in (1.7). The function $\Psi_m$ is a strict local minimizer of $E_0^\xi$ in the sense that, for all $\gamma \in (0, \gamma_0)$, the following hold:

- [Local minimizer] The function $\Psi_m$ locally minimizes $E_0^\xi$:
  \[
  \inf_{|u-\Psi_m|_{\mathbb{R}^d} \leq \gamma} E_0^\xi(u) = E_0^\xi(\Psi_m).
  \]

- [Strictness] There exists $\delta > 0$ such that
  \[
  \gamma \leq |u-\Psi_m|_{\mathbb{R}^d} \leq \gamma_0 \implies E_0^\xi(u) - E_0^\xi(\Psi_m) \geq 2\delta.
  \]

Lemma 2.11 (Sharp interface energy that is near the saddle point). We fix $\xi \in (\xi_d, \xi_a)$, and define $\gamma_0$ as in (1.7). The function $\Psi_s$ is a saddle point of $E_0^\xi$ in the sense that, for all $\gamma \in (0, \gamma_0)$, the following hold:

- [Minimizer at volume $\nu_s$] $\Psi_s$ minimizes $E_0^\xi$ subject to a volume constraint:
  \[
  \inf_{|u-\Psi_s|_{\mathbb{R}^d} \leq \gamma} E_0^\xi(u) = E_0^\xi(\Psi_s).
  \]

- [Strictness at volume $\nu_s$] There exists $\delta > 0$ such that
  \[
  |u-\Psi_s|_{\mathbb{R}^d} \geq \gamma, \nu_0(u) = \nu_s \implies E_0^\xi(u) \geq E_0^\xi(\Psi_s) + 2\delta.
  \]

- [Descent direction] There exists $\delta > 0$ such that
  \[
  \inf_{|u-\Psi_s|_{\mathbb{R}^d} \leq \gamma} E_0^\xi(u) \leq E_0^\xi(\Psi_s) - 2\delta.
  \]

The proofs of the lemmas rely on the quantitative isoperimetric inequality in $\mathbb{R}^d$.

Proof of Lemma 2.10. Since (2.25) follows from the fact that (2.26) holds for every $\gamma' \in (0, \gamma_0)$, it suffices to show (2.26). To this end, consider $u$ such that $|u-\Psi_m|_{\mathbb{R}^d} = \tilde{\gamma} \in [\gamma, \gamma_0]$. 

We may, without loss of generality, assume that \( u = \pm 1 \) almost everywhere and \( \text{Per}(\{u = +1\}) < \infty \), since otherwise \( E_0^\xi(u) = \infty \) and (2.26) is trivially satisfied. Hence, we may write \( u = -1 + 2\chi_S \) with \( \text{Per}(S) < \infty \), so that

\[
E_0^\xi(u) = c_0 \text{Per}(S) - 4|S| + 4\xi^{-(d+1)}|S|^2.
\]

Moreover, from \( u = \pm 1 \) and \( |u - \Psi_m|_{\mathbb{R}^d} = \tilde{\gamma} \), it follows that

\[
\frac{\tilde{\gamma}^2}{4} = \min_{x \in \mathbb{R}^d} |B_{\nu_m}(x) \triangle S| = |B_{\nu_m}(x_0) \triangle S|
\]

for some \( x_0 \in \mathbb{R}^d \). From (2.31) and

\[
|B_{\nu_m}(x_0)| - |B_{\nu_m}(x_0) \triangle S| \leq |S| \leq |B_{\nu_m}(x_0)| + |B_{\nu_m}(x_0) \triangle S|,
\]

one observes that

\[
|S| \in \left[ \nu_m - \frac{\tilde{\gamma}^2}{4}, \nu_m + \frac{\tilde{\gamma}^2}{4} \right].
\]

On the one hand, if

\[
|S| - \nu_m \geq \frac{\tilde{\gamma}^2}{8} \geq \frac{\gamma^2}{8},
\]

then we deduce from (2.30) and the isoperimetric inequality (cf. Theorem 1.14) that \( f_\xi(|S|) \geq f_\xi(\nu_m) + \delta_1 \) for some \( \delta_1 > 0 \). It thus suffices to consider

\[
|S| - \nu_m \leq \frac{\gamma^2}{8}.
\]

In this case, we claim that the Fraenkel asymmetry \( \lambda_S \) of \( S \) satisfies

\[
\lambda_S := \min_{x \in \mathbb{R}^d} \frac{|B_{|S|}(x) \triangle S|}{|S|} \geq \lambda_0 \quad \text{for some } \lambda_0 > 0.
\]

Indeed, recalling (2.31), we have for every \( x \in \mathbb{R}^d \) that

\[
|B_{|S|}(x) \triangle S| \geq |B_{\nu_m}(x_0) \triangle S| - |S| - |B_{\nu_m}(x_0)|
\]

\[
= \frac{\tilde{\gamma}^2}{4} - |S| - \nu_m \geq \frac{\tilde{\gamma}^2}{8} \geq \frac{\gamma^2}{8},
\]

which in particular implies

\[
\frac{|B_{|S|}(x) \triangle S|}{|S|} \geq \frac{\gamma^2}{8|S|} \geq \frac{\gamma^2}{8(\nu_m + \gamma^2/8)} \quad \text{by (2.32)}
\]

\[
= \lambda_0.
\]
Minimizing over $x$ yields (2.33). Now, we use the sharp isoperimetric inequality (1.23) and

\[ \text{Per}(B_\nu) \geq \text{Per}(B_{\nu_s}) \quad \text{for all } \nu \in [\nu_m - \gamma_0^2/4, \nu_m + \gamma_0^2/4] \]

to deduce

\[
\begin{align*}
\mathcal{E}_0^\xi(u) &\geq c_0 \text{Per}(S) - 4|S| + 4 \xi^{-(d+1)}|S|^2 + c_0 C(d) \text{Per}(S) \lambda_0^2 \\
&= f_\xi(|S|) + c_0 C(d) \text{Per}(S) \lambda_0^2 \\
&\geq f_\xi(\nu_m) + c_0 C(d) \text{Per}(B_{\nu_s}) \lambda_0^2 \quad \text{by (1.8), (2.34)} \\
&= \mathcal{E}_0^\xi(\Psi_m) + c_0 C(d) \text{Per}(B_{\nu_s}) \lambda_0^2,
\end{align*}
\]

which establishes (2.26) with $\delta = c_0 C(d) \text{Per}(B_{\nu_s}) \lambda_0^2$. □

**Proof of Lemma 2.11.** The proof is analogous to that of Lemma 2.10. We note that (2.27) follows from the fact that (2.28) holds for every $\gamma \in (0, \gamma_0]$. Also note that it is a simple matter to establish (2.29) since it suffices to observe that, for example, for $u_\nu := -1 + 2 \chi_{B_\nu}$ with

\[ \nu = \nu_s + \frac{\gamma^2}{4}, \]

we have $|u_\nu - \Psi_s|_{R^d} = \gamma$ and

\[ \mathcal{E}_0^\xi(u_\nu) = f_\xi(\nu) = f_\xi(\nu_s) - \delta = \mathcal{E}_0^\xi(\Psi_s) - \delta, \]

for some $\delta > 0$.

Hence, it suffices to show (2.28). As above, we may assume, without loss of generality, that $u = -1 + 2 \chi_S$ with $\text{Per}(S) < \infty$ and $|S| = \nu_s$, in which case it holds that

\[ \mathcal{E}_0^\xi(u) = c_0 \text{Per}(S) - 4\nu_s + \frac{4\nu_s^2}{\xi^{d+1}}. \]

Because of the constraint $|S| = \nu_s$, we observe as for (2.33) that $|u - \Psi_s|_{R^d} \geq \gamma$ implies

\[ \lambda(S) \geq \frac{\gamma^2}{8\nu_s} =: \lambda_0. \]

Hence, (2.35), (2.36), and the quantitative isoperimetric inequality (1.23) yield (2.28) with $\delta := c_0 C(d) \text{Per}(B_{\nu_s}) \lambda_0^2$. □
2.4. Constrained minimizers are bounded. Here, we prove Lemma 2.9. The proof builds on [17, Lemma 2.3].

Proof of Lemma 2.9. As usual, we set \( \kappa = \phi^{1/3} \) and assume \( \phi \leq \phi_0 \) for an appropriately chosen value \( \phi_0 \). We proceed in two steps.

Step 1 (Upper bound). The first step is a reformulation of a specific case of [17, Lemma 2.3]. We show, for any \( u \in X_\phi \), if \( |\{ x : u(x) > 1 + \kappa \}| > 0 \), then there exists a function \( \tilde{u} \in X_\phi \) such that the following hold:

(i) \( \nu(\tilde{u}) = \nu(u) \).
(ii) \( \tilde{u} = u \) on \( \{ x : -1 + \phi \leq u(x) \leq 1 + \kappa \} \).
(iii) \( \text{ess sup} \, \tilde{u} \leq 1 + \kappa \).
(iv) \( \mathcal{E}_\phi^\varepsilon(\tilde{u}) < \mathcal{E}_\phi^\varepsilon(u) \).

For completeness, we recount the proof. Recall that \( C_+ := \{ x : u(x) > 1 + \kappa \} \), and let \( \hat{A} := \{ x : u(x) < -1 + \phi \} \). Notice that \( |C_+| > 0 \) and the mean condition imply \( |\hat{A}| > 0 \). For \( \lambda \in [0, 1] \), we define the function

\[
\tilde{u}_\lambda(x) := \begin{cases} 
\min\{u(x), 1 + \kappa\} & \text{for } x \in \mathbb{T}_{\phi L} \setminus \hat{A}, \\
(1 - \lambda)u(x) + \lambda(-1 + \phi) & \text{for } x \in \hat{A}.
\end{cases}
\]

It is easy to see that

\[
\int_{\mathbb{T}_{\phi L}} \tilde{u}_\lambda_0 \, dx < -1 + \phi, \quad \int_{\mathbb{T}_{\phi L}} \tilde{u}_\lambda_1 \, dx > -1 + \phi.
\]

Hence, there exists \( \lambda_\ast \in (0, 1) \) such that

\[
\int_{\mathbb{T}_{\phi L}} \tilde{u}_{\lambda_\ast} \, dx = -1 + \phi.
\]

The function \( \tilde{u} := \tilde{u}_{\lambda_\ast} \) belongs to \( X_\phi \) and satisfies properties (i)–(iii). It remains to check (iv). The energy difference can be written as

\[
(2.37) \quad \mathcal{E}_\phi^\varepsilon(\tilde{u}) - \mathcal{E}_\phi^\varepsilon(u) = \int_{\hat{A} \cup C_+} \frac{\phi}{2} |\nabla \tilde{u}|^2 - \frac{\phi}{2} |\nabla u|^2 + \frac{1}{\phi} (G(\tilde{u}) - G(u)) \, dx \\
\leq \frac{1}{\phi} \int_{\hat{A} \cup C_+} G(\tilde{u}) - G(u) \, dx,
\]

where we have used (ii) and the fact that the gradient energy of \( \tilde{u} \) is smaller than that of \( u \). Since \( u \leq \tilde{u} \leq -1 + \phi \) on \( \hat{A} \), the convexity of \( G \) on \(( -\infty, -1 + \phi] \) implies

\[
G(\tilde{u}) - G(u) \leq G'(\tilde{u})(\tilde{u} - u) \leq G'(-1 + \phi)(\tilde{u} - u).
\]

On \( C_+ \) on the other hand, \( \tilde{u} = 1 + \kappa \) and the convexity of \( G \) on \([1 + \kappa, \infty) \) imply that

\[
G(1 + \kappa) - G(u) \leq -G'(1 + \kappa)(u - \tilde{u}).
\]
Inserting these two inequalities into (2.37) yields
\[
E^\xi_\phi (\tilde{u}) - E^\xi_\phi (u) \leq \frac{G'(1 + \phi)}{\phi} \int_{\tilde{A}} \tilde{u} - u \, dx + \frac{G'(1 + \kappa)}{\phi} \int_{C} \tilde{u} - u \, dx < \frac{G'(1 + \kappa)}{\phi} \int_{\tilde{A} \cup C} \tilde{u} - u \, dx = 0,
\]
where the final equality is a consequence of (ii) and
\[
\int T_{\phi L} \tilde{u} \, dx = \int T_{\phi L} u \, dx.
\]
The last inequality is strict since \( G'(1 + \phi) < G'(1 + \kappa) \) and because the assumption \(|C_+| > 0\) implies
\[
\int C_+ (\tilde{u} - u) \, dx < 0 \quad \text{and therefore} \quad \int \tilde{A} (\tilde{u} - u) \, dx > 0.
\]

**Step 2 (Lower bound).** We show, for any \( u \in X_\phi \), if \(|\{x : u(x) < -1 - \kappa\}| > 0\) and \( E^\xi_\phi (u) \leq E_M \), then there exists a function \( \tilde{u} \in X_\phi \) such that the following hold:

(i) \( \nu(\tilde{u}) = \nu(u) \).
(ii) \( \tilde{u} = u \) on \( \{x : -1 - \kappa \leq u(x) \leq 1 - \kappa\} \).
(iii) \( \text{ess inf} \tilde{u} \geq -1 - \kappa \).
(iv) \( E^\xi_\phi (\tilde{u}) < E^\xi_\phi (u) \).

Recall \( \mathcal{A}_- := \{x : u(x) < -1 - \kappa\} \), and let \( \hat{\mathcal{C}} := \{x : u(x) > 1 - \kappa\} \). For \( \lambda \in [0, 1] \), we define the function
\[
\tilde{u}_\lambda (x) := \begin{cases} 
\text{max}\{u(x), -1 - \kappa\} & \text{for } x \in T_{\phi L} \setminus \hat{\mathcal{C}}, \\
(1 - \lambda)u(x) + \lambda(1 - \kappa) & \text{for } x \in \hat{\mathcal{C}}.
\end{cases}
\]
It is easy to see that
\[
\int_{T_{\phi L}} \tilde{u}_0 \, dx > -1 + \phi.
\]

To check the mean of \( \tilde{u}_1 \), we deduce from (2.17), (2.23), and the Cauchy-Schwarz inequality that
\[
\int_{\mathcal{A}_-} |u - (-1 - \kappa)| \, dx \leq \left( |\mathcal{A}_-| \int_{\mathcal{A}_-} |u - (-1 - \kappa)|^2 \, dx \right)^{1/2} \leq \left( |\mathcal{A}_-| \int_{\mathcal{A}_-} G(u) \, dx \right)^{1/2} \leq \phi^{2/3}.
\]
We use this fact to estimate
\[
\int_{\Omega} (u - \tilde{u}_1) \, dx
\]
\[
= \int_{\Omega} u - (1 - \kappa) \, dx + \int_{\hat{C}} u - (1 - \kappa) \, dx
\]
\[
\geq -O(\phi^{2/3}) + \int_{\{u > 1 - \kappa/2\}} u - (1 - \kappa) \, dx \quad \text{by (2.38)}
\]
\[
\geq -O(\phi^{2/3}) + \frac{\phi^{1/3}}{2} (\omega_1 + O(\phi^{1/3})) > 0,
\]
where we have, as in Lemma 2.5, deduced
\[
\left| \left\{ x : 1 - \kappa \leq u(x) \leq 1 - \frac{\kappa}{2} \right\} \right| \leq \phi^{1/3}.
\]
From (2.39) we obtain
\[
\int_{\Omega} \tilde{u}_1 \, dx < -1 + \phi.
\]
Hence, as in \textit{Step 1}, there exists \( \lambda_* \in (0, 1) \) such that
\[
\int_{\Omega} \tilde{u}_{\lambda_*} \, dx = -1 + \phi.
\]
As in \textit{Step 1}, we define \( \tilde{u} := \tilde{u}_{\lambda_*} \in X_\phi \), for which (i), (ii'), and (iii') follow immediately, and it remains only to check (iv). In a manner analogous to what was done above, we observe that
\[
(2.40) \quad E^\xi_\phi(\tilde{u}) - E^\xi_\phi(u) \leq \frac{1}{\phi} \int_{\overline{A_\phi + \hat{C}}} G(\tilde{u}) - G(u) \, dx,
\]
and deduce from convexity of \( G \) on \((-\infty, 1 - \kappa)\) and on \([1 - \kappa, \infty)\) that
\[
G(-1 - \kappa) - G(u) \leq -G'(-1 - \kappa)(u - \tilde{u}) \quad \text{on} \quad u < 1 - \kappa,
\]
\[
G(\tilde{u}) - G(u) \leq -G'(\tilde{u})(u - \tilde{u}) \quad \text{on} \quad u > 1 - \kappa,
\]
respectively. Note that, since after \textit{Step 1} we can assume \( u \leq 1 + \kappa \) on \( \hat{C} \), it holds that \( 1 - \kappa \leq \tilde{u} \leq 1 + \kappa - 2\lambda_\kappa \) on \( \hat{C} \). Since \( \lambda_* \in (0, 1) \) one can check that \( \sup_{\hat{C}} |G'(\tilde{u})| < -G'(-1 - \kappa) \). Substituting into (2.40), we conclude as in \textit{Step 1} that
\[
E^\xi_\phi(\tilde{u}) - E^\xi_\phi(u) < \frac{-G'(-1 - \kappa)}{\phi} \int_{\overline{A_\phi + \hat{C}}} u - \tilde{u} \, dx = 0.
\]
The combination of \textit{Step 1} and \textit{Step 2} and the characterization of \( u_{\omega, \phi} \) as a minimizer of \( E^\xi_\phi \) subject to \( v(u) = \omega \) imply (2.22).
3. Existence and Properties via the $\Gamma$-limit and Isoperimetric Inequalities

In this section, we use the $\Gamma$-limit in order to deduce existence and properties of critical points. We begin with the proof of Theorem 1.4, using the upper bound constructions from Proposition 2.4, the structure of the limit energy from Lemma 2.10, and Lemma 3.1 below, which establishes a uniform lower bound on the energy of functions that are $y$ away from $\Psi_m$. We will refer to Lemma 3.1 as the finite $\phi$ estimate for the local minimizer. In Subsection 3.2, we prove Proposition 1.8 and Theorem 1.10, also introducing the finite $\phi$ estimate for the saddle point. Finally, we prove Theorem 1.19 in Subsection 3.3, quantifying the sphericity of the constrained minimizers.

### 3.1. Local minimizer

The finite $\phi$ estimate for the local minimum is the following.

**Lemma 3.1 (Finite $\phi$ estimate).** Consider $\xi \in (\tilde{\xi}_d, \xi_d]$ and the critical scaling (1.4). Define $y_0$ as in (1.7). For every $y \in (0, y_0]$ and $\delta > 0$, there exists $\phi_0 > 0$ such that, for $\phi \leq \phi_0$, one has for all $u \in X_\phi$ that

$$y \leq |u - \Psi_m|_{\Gamma, \phi} \leq y_0 \implies \inf_{y \leq |u - \Psi_m|_{\Gamma, \phi} \leq y_0} E_\xi^\phi(u) \geq \inf_{y \leq |u - \Psi_m|_{\Gamma, \phi} \leq y_0} E_0^\phi(u) - \delta. \tag{3.1}$$

As noted in the Introduction, we cannot get compactness in the usual way, since $\Gamma, \phi$ grows to $\mathbb{R}^d$ in the limit. The proof of Lemma 3.1, which is fairly involved, is given in Subsection 3.4. In fact, we use this information in the proof of Theorem 1.4 only in the form of the simpler estimate (3.1). In Lemma 3.2, we point out that (3.1) can be established by a conceptually simpler and much shorter proof. We include Lemma 3.1 because we believe it is interesting in its own right—indeed, it tells us that being bounded away from the limit minimizer creates the same energetic penalization, up to a small error, as it creates in the limit energy. The same remark holds true for the corresponding saddle point estimates (see also Remarks 1.20 and 3.4).

**Proof of Theorem 1.4.** For any $\phi > 0$, the direct method of the calculus of variations yields a function $u_{m, \phi} \in X_\phi$ that minimizes $E_\phi^\xi_m$ subject to the constraint $|u - \Psi_m|_{\Gamma, \phi} \leq y_0$. For $y \in (0, y_0]$, let $\delta > 0$ be the constant that is given in (2.26). Lemma 3.1 says, for $\phi$ sufficiently small, that functions with $y \leq |u - \Psi_m|_{\Gamma, \phi} \leq y_0$ satisfy

$$E_\phi^\xi(u) \geq \inf_{y \leq |u - \Psi_m|_{\Gamma, \phi} \leq y_0} E_0^\phi(u) - \delta \geq c_m + \delta \text{ by (2.26).} \tag{3.1}$$
At the same time, according to (2.8) with \( \omega_2 = \nu_m \) and \( t = 1 \), there exists \( \hat{u}_{\omega, \phi} \) such that \( |\hat{u}_{\omega, \phi} - \Psi_m|_{\mathcal{T}_{\phi L}} \leq \gamma_0 \) and

(3.2) \[ E_\phi^x(\hat{u}_{\omega, \phi}) = c_m + O(\phi|\ln \phi|) < c_m + \frac{\delta}{2} \]

for \( \phi \) sufficiently small. We deduce on the one hand that \( u_{m, \phi} \) is an unconstrained local minimizer of \( E_\phi^x \) (since it belongs to the interior of the \( L_2^2 - \gamma_0 \) ball around \( \Psi_m \)), and on the other hand that (1.11) holds.

To address the estimates (1.12)–(1.13), we begin by establishing rough closeness to \( \nu_m \) in volume. For the lower bound, we use Lemma 2.5, and assume without loss of generality that \( |u_{m, \phi} - \Psi_m|_{\mathcal{T}_{\phi L}} = \|u_{m, \phi} - \Psi_m\|_{L_2^2(\mathcal{T}_{\phi L})} \), to argue

\[ \nu(u_{m, \phi}) \geq 1 \left| \{ u_{m, \phi} \geq 1 - \phi^{1/3} \ \text{and} \ \Psi_m = -1 \} \right| \quad \text{by (1.11)} \]

\[ \geq \nu_m - \left| \{ x \in B_{\nu_m} : u_{m, \phi} < 1 - \phi^{1/3} \ \text{or} \ u_{m, \phi} > 1 + \phi^{1/3} \} \right| \]

\[ = \nu_m - \left| \{ x \in \mathbb{T}_{\phi L} : \Psi_m = 1 \ \text{and} \ u_{m, \phi} < -1 + \phi^{1/3} \} \right| + o(1) \]

\[ \geq \nu_m - \gamma^2 + o(1) \quad \text{by (1.11).} \]

For the corresponding upper bound, note that

\[ \gamma^2 \geq |\{ u_{m, \phi} \geq 1 - \phi^{1/3} \ \text{and} \ \Psi_m = -1 \} | \quad \text{by (1.11)} \]

\[ \geq |\{ u_{m, \phi} \geq 1 - \phi^{1/3} \}| - \nu_m \]

\[ = \nu(u_{m, \phi}) - \nu_m + o(1) \quad \text{by (2.19)} \]

One thus obtains for \( \phi_0 > 0 \) sufficiently small that

(3.3) \[ |\nu(u_{m, \phi}) - \nu_m| \leq \gamma^2 \quad \text{for all} \quad 0 < \phi < \phi_0. \]

In particular, for \( \gamma \) small enough, \( \nu(u_{m, \phi}) \) falls within a neighborhood of \( \nu_m \) on which \( f_\xi \) is convex and has \( c_m \) as a minimum value. We use these facts to derive quantitative estimates. Combining (3.2) and (2.6) leads on the one hand to

(3.4) \[ O(\phi^{1/3}) \geq E_\phi^x(\hat{u}_{\omega, \phi}) - c_m \geq E^x_\phi(u_{m, \phi}) - c_m \]

\[ \geq f_\xi(\nu(u_{m, \phi})) - c_m + O(\phi^{1/3}) \geq O(\phi^{1/3}), \]

which is (1.12). On the other hand, (3.3) justifies a Taylor approximation of \( f_\xi \) on the right-hand side of (3.4), which implies

\[ O(\phi^{1/3}) \geq E_\phi^x(u_{m, \phi}) - c_m \geq \frac{|\nu(u_{m, \phi}) - \nu_m|^2}{C} + O(\phi^{1/3}) \]

for a constant \( C = C(\xi, d) \). From here one deduces (1.13). \( \square \)
Lemma 3.2. Let $y_0$ be as in (1.7). For any $y \in (0, y_0]$, there exists $\delta > 0$ and $\phi_0 > 0$ such that, for all $\phi < \phi_0$, it holds that

$$y \leq |u - \Psi_m|_{\mathbb{T}_{\phi L}} < y_0 \implies \mathcal{E}_{\phi}^E(u) \geq \mathcal{E}_{\phi}^E(\Psi_m) + \delta.$$  

Proof. We prove Lemma 3.2 via an adaptation to $\phi > 0$, $\mathbb{T}_{\phi L}$ of the proof of Lemma 2.10. Fix any $\gamma \in (0, \gamma_0]$. As usual, by translating on the torus, we will assume that

$$\Phi$$

Note that we may assume without loss of generality that $E_{\Phi} \leq E_M$ (cf. (2.3)), so that by Lemma 2.5, the definition of $\nu(u)$, and $\|u - \Psi_m\|_{L^2(\mathbb{T}_{\phi L})} \leq \gamma_0$, we obtain

$$\nu_m - \frac{y_0^2}{4} + O(\phi^{1/3}) \leq \nu(u) \leq \nu_m + \frac{y_0^2}{4} + O(\phi^{1/3}).$$

As in the proof of Lemma 2.10, we observe that $|\nu(u) - \nu_m| \geq \gamma^2/8$ (and the lower bound from Proposition 2.3) would imply for $\phi$ sufficiently small that

$$\mathcal{E}_{\phi}^E(u) \geq f_{\Phi}(\nu(u)) + O(\phi^{1/3}) > f_{\Phi}(\nu_m) + \delta + O(\phi^{1/3})$$

$$\geq f_{\Phi}(\nu_m) + \frac{\delta}{2} = \mathcal{E}_{\phi}^E(\Psi_m) + \frac{\delta}{2}.$$  

Hence, we may assume that

$$|\nu(u) - \nu_m| \leq \frac{\gamma^2}{8}.$$  

As in the proof of Lemma 2.10, this will lead to a positive lower bound for the Fraenkel asymmetry of suitable sets. It remains to establish this fact.

To begin, we note that, since (3.6) implies $f_{\Phi}(\nu(u)) \geq f_{\Phi}(\nu_m)$, it suffices in light of Proposition 2.3 to establish a positive ($\Phi$-independent) lower bound on

$$I(u) = \int_{-1+2\Phi}^{1-2\Phi} \sqrt{2G(t)(\text{Per}_{\mathbb{T}_{\phi L}}(\{u > t\}) - \mathbb{P}_{\mathcal{E}}(\{u > t\}))) \, dt.$$

Given the scaling regime (1.4), quantitative isoperimetric inequality (1.24), and lower bound on $\mathbb{P}_{\mathcal{E}}(\{u > s\})$ implied by (3.6), it suffices to set a positive ($\Phi$-independent) lower bound on $\lambda(\{u > s\})$ for all $s \in [-1 + 2\phi^{1/3}, 1 - 2\phi^{1/3}]$.  

-1+2\Phi  

$\text{Per}_{\mathbb{T}_{\phi L}}\{u > t\}$
According to Lemma 2.8, it in fact suffices to establish a positive (\(\phi\)-independent) lower bound on \(\lambda(\{u > 0\})\). Since Corollary 2.6 implies that

\[
\nu_u := |\{u > 0\}| = \nu(u) + O(\phi^{1/3}),
\]

it suffices to produce a lower bound on \(\min_{x \in \mathbb{T}_{\phi_u}} |B_{\nu_u}(x) \triangle \{u > 0\}|\).

To this end, we observe that (3.5) and Lemma 2.5 imply

\[
|B_{\nu_m}(x) \triangle \{u > 0\}| \geq \frac{\lambda^2}{4} + o(1).
\]

Using the triangle inequality (2.24), we deduce from this fact together with (3.7) and (3.8) that

\[
|B_{\nu_u}(x) \triangle \{u > 0\}| \geq \frac{\lambda^2}{8} + o(1).
\]

Since \(x \in \mathbb{T}_{\phi_u}\) is arbitrary, we have established

\[
\min_{x \in \mathbb{T}_{\phi_u}} |B_{\nu_u}(x) \triangle \{u > 0\}| \geq \frac{\lambda^2}{8} + o(1). \quad \square
\]

3.2. Analogous estimates for the saddle point. In this subsection, we collect our information about the saddle points of \(E^\xi_{\phi}\) that were defined in the Introduction. We begin by establishing Proposition 1.8, which uses the upper and lower bounds on the energy barrier to invoke a mountain pass theorem for the existence of \(u_{s,\phi}\) and \(\tilde{u}_{s,\phi}\).

Proof of Proposition 1.8. Recall from [17, Lemma A.2] that \(E^\xi_{\phi} \in C^1(X_\phi; \mathbb{R})\) and satisfies the Palais-Smale condition. Using these conditions and a standard mountain pass argument (see, e.g., [17, Corollary 1.8]), we can establish the existence of the saddle point \(u_{s,\phi}\) once we show the following:

(i) There exists a function \(u \in X_\phi\) with \(E^\xi_{\phi}(u) < E^\xi_{\phi}(u_{m,\phi})\).

(ii) There exists \(c > 0\) such that, for any such \(u\) and any continuous path \(\psi\) with \(\psi(0) = u, \psi(1) = u_{m,\phi}\), it holds that

\[
\sup_{t \in [0,1]} E^\xi_{\phi}(\psi(t)) \geq E^\xi_{\phi}(u_{m,\phi}) + c.
\]

The first condition is established via the upper bound construction from Proposition 2.4, by using the fact that \(c_m\) is a local but not global minimum value of \(f_{\xi}\) for \(\xi \in (\xi_d, \xi_d)\). To confirm condition (ii), we define the unique point \(\nu_- \neq \nu_m\) via

\[
f_{\xi}(\nu_-) = f_{\xi}(\nu_m),
\]
and observe that according to the lower bound (2.6), it holds that

\[(3.9)\quad \nu(u) \leq \nu_- + O(\phi^{1/3}).\]

On the other hand, recall from (1.13) that \(\nu(u_{m,\phi}) \geq \nu_m + C\phi^{1/6}\). Hence, a second application of the lower bound (and continuity of \(\nu(\cdot)\)) implies

\[(3.10)\quad \sup_{t \in [0,1]} \mathcal{E}_\phi^\xi(\psi(t)) \geq c_s + O(\phi^{1/3})
\geq \mathcal{E}_\phi^\xi(u_{m,\phi}) + (c_s - c_m) + O(\phi^{1/3}) \quad \text{by (1.12)}
\geq \mathcal{E}_\phi^\xi(u_{m,\phi}) + \frac{c_s - c_m}{2}\]

\[(3.11)\]

for \(\phi\) small.

Analogously, the existence of \(\tilde{u}_{s,\phi}\) follows from a standard mountain pass argument once we show the following:

(i) There exists a function \(u \in X_\phi\) with \(\mathcal{E}_\phi^\xi(u) < \mathcal{E}_\phi^\xi(u_{m,\phi})\).

(ii) There exists \(c > 0\) such that, for any such \(u\), any \(v \in \mathcal{N}_\varepsilon(u_{m,\phi})\), and any continuous path \(\psi\) with \(\psi(0) = u, \psi(1) = v\), it holds that

\[\sup_{t \in [0,1]} \mathcal{E}_\phi^\xi(\psi(t)) \geq \mathcal{E}_\phi^\xi(v) + c.\]

Hence, it suffices to check (ii). To this end, we observe that, by the definition of \(\mathcal{N}_\varepsilon(u_{m,\phi})\), (1.11), and Lemma 2.5, there exists \(\bar{\varepsilon} > 0\) such that, for any \(\varepsilon \in (0,\bar{\varepsilon})\), it holds that

\[\nu(v) \geq (\nu_m + \nu_s)/2\]

for small enough \(\phi\). On the other hand, as above, we observe that (3.9) holds for any \(u\) satisfying (i). It follows as above for any \(\varepsilon \leq (c_s - c_m)/4\) that

\[(3.12)\quad \sup_{t \in [0,1]} \mathcal{E}_\phi^\xi(\psi(t)) \geq c_s + O(\phi^{1/3}) \quad \text{by (3.10)}
\geq \mathcal{E}_\phi^\xi(u_{m,\phi}) + \frac{c_s - c_m}{2} \quad \text{by (3.11)}
\geq \mathcal{E}_\phi^\xi(v) - \varepsilon + \frac{c_s - c_m}{2}
\geq \mathcal{E}_\phi^\xi(v) + \frac{c_s - c_m}{4}.
\]

It remains to bound the energy barriers. The lower bound on \(\Delta E_{1}^{\phi,\xi}\) is implied by the first inequality in (3.10). Similarly, the lower bound \(\Delta E_{2}^{\phi,\xi} \geq c_s + O(\phi^{1/3})\) is implied by (3.12). On the other hand, from Theorem 1.4 and from the upper bound constructions in Proposition 2.4, we can deduce the upper bound \(\Delta E_{2}^{\phi,\xi} \leq c_s + O(\phi^{1/3})\). □
As explained in the Introduction, in order to find a function that satisfies the folkloric properties of the critical nucleus, we turn to the volume-constrained minimizer $u_{\omega_*\phi}$. We begin by stating the analogy of Lemma 3.1 for the sharp-interface saddle point $\Psi_s$. We apply the estimate in the proof of Theorem 1.10 and Remark 1.11 (see also Remark 3.4 below).

**Lemma 3.3 (Finite $\phi$ estimate).** Let $\gamma_0 > 0$ be as in (1.7). Then, for every $\gamma \in (0, \gamma_0]$, $\delta > 0$, there exist $\phi_0 > 0$, $\beta_0 > 0$ such that, for all $u \in X_\phi$ that

$$\nu(u) = \omega*$$

and

$$|u|_{L^p} \geq \gamma, \nu_0(u) = \nu_\gamma \Rightarrow E_{\xi \phi}(u) \geq \nu_\gamma - \delta.$$

The proof of the lemma is given in Subsection 3.4.

**Remark 3.4.** Notice that Lemma 3.3 and (2.28) imply that any approximately optimal path for $\Delta E_{\xi \phi}$ stays within a $\gamma$ neighborhood of $\Psi_s$ for all volumes close to $\nu_s$. The mountain pass around $\Psi_s$ is “narrow” in this sense (see also Remark 1.20).

We now turn to the proof of Theorem 1.10, which establishes the existence and properties of the constrained minimizer $u_{\omega_*\phi}$.

**Proof of Theorem 1.10.** First note that the lower and upper bounds in Propositions 2.3 and 2.4, and the fact that $\nu_s$ is the unique maximum of $f_\xi$ over $[0, \nu_m]$, imply that

$$\omega_* = \nu_s + o(1).$$

Next, for given $\gamma \in (0, \gamma_0]$, we use Lemma 2.11 to identify $\delta > 0$ such that

$$|u - \Psi_s|_{L^p} \geq \gamma, \nu_0(u) = \nu_s \Rightarrow E_{\xi \phi}(u) \geq c_s + 2\delta.$$

With these $\gamma$, $\delta$, we extract from (3.13) that, for $\nu(u) = \omega_*$ (which by (3.14) is close to $\nu_s$) and $|u - \Psi_s|_{L^p} \geq \gamma$, there holds

$$E_{\xi \phi}(u) \geq \inf_{|u - \Psi_s|_{L^p} \geq \gamma, \nu_0(u) = \nu_s} E_{\xi \phi}(u) - \delta \geq c_s + \delta$$

by (3.15).

On the other hand, the constructions from Proposition 2.4 and (3.14) yield a function $\hat{u}_{\omega}$ such that $\nu(\hat{u}_{\omega}) = \omega_*$ and

$$E_{\xi \phi}(\hat{u}_{\omega}) \leq c_s + \frac{\delta}{2}.$$

Minimality of $u_{\omega_*\phi}$ yields (1.18).
It remains to deduce (1.19) and (1.20). Combining the estimate of \( \Delta E_{\omega,\phi}^{\xi} \) and the definition of \( u_{\omega,\phi} \) (see (1.16) and (1.17)) leads to (1.19). To obtain (1.20), note that the definition of \( u_{\omega,\phi} \), the bound (1.19), and Proposition 2.4 imply

\[
(3.16) \quad f_\xi(v_s) + O(\phi^{1/3}) \leq E_\phi^{\xi}(u_{\omega,\phi}) \leq f_\xi(\omega_*) + O(\phi|\ln \phi|),
\]

so that

\[
(3.17) \quad f_\xi(\omega_*) \geq f_\xi(v_s) + O(\phi^{1/3}).
\]

Given (3.14), we may apply the Taylor formula and \( f'_\xi(v_s) = 0, f''_\xi(v_s) < 0 \) to deduce from (3.17) that \(|\omega_* - v_s| \leq C\phi^{1/6} \), for \( C = C(\xi,d) \), which is (1.20). ☐

3.3. Deviation from sphericity. In this subsection, we look more closely at the “droplet-like shape” of the local minimizer \( u_{m,\phi} \) and the volume-constrained minimizers \( u_{\omega,\phi} \) using the quantitative isoperimetric inequality. The main ingredient for establishing the droplet-like shape of \( u_{m,\phi} \) and the volume-constrained minimizers is the following observation.

**Lemma 3.5.** Consider \( \xi \in (0,\xi_d] \) and the critical scaling (1.4). For any \( \omega > 0 \) and for \( \phi > 0 \) sufficiently small, the following holds. If \( u \in X_\phi \) satisfies

\[
(3.18) \quad \omega \leq \nu(u) \leq \frac{\xi^{d+1}}{2} \quad \text{and} \quad E_\phi^{\xi}(u) \leq f_\xi(\nu(u)) + O(\phi^{1/3}),
\]

then for every \( s \in [-1 + 2\phi^{1/3}, 1 - 2\phi^{1/3}] \), the Fraenkel asymmetry of the superlevel set \( \{u > s\} \) satisfies

\[
(3.19) \quad \lambda(\{u > s\}) \leq \frac{\phi^\alpha}{\nu(u)} \quad \text{with} \quad \alpha = \min\{1/6, 1/(2d)\}.
\]

**Proof.** The energy bound in (3.18) and the lower bound (2.6) imply

\[
(3.20) \quad I(u) \leq \phi^{1/3},
\]

where \( I(\cdot) \) is the asymmetry cost defined in (2.7). The quantitative isoperimetric inequality (1.24) applied to \( \{u > s\} \) gives

\[
\begin{align*}
\text{Per}_T(u,1) - \text{Per}_T(u,s) &\geq C(d) P_E(\{u > s\}) \lambda(\{u > s\})^2 - \frac{4d|\{u > s\}|}{\phi L} \\
&= C(d) P_E(\{u > s\}) \lambda(\{u > s\})^2 + O(\phi^{1/d}) \quad \text{by (1.4), (2.19)}.
\end{align*}
\]

Substituting into (2.7) and applying the scaling bound (3.20), we obtain

\[
\int_{-1 - 2\phi^{1/3}}^{1 - 2\phi^{1/3}} \sqrt{2G(s)} P_E(\{u > s\}) \lambda(\{u > s\})^2 ds \leq \phi^{1/3} + \phi^{1/d}.
\]
Applying Lemma 2.8 yields

\[ \left( \sup_{s \in [-1+2\phi^{1/3}, 1-2\phi^{1/3}]} \frac{\lambda(s) + O(\phi^{1/3})}{\nu(u)} \right)^2 \times \int_{-1+2\phi^{1/3}}^{1-2\phi^{1/3}} \sqrt{2G(s) P_E(s)} ds \leq \phi^{1/3} + \phi^{1/3}, \]

so that

\[ \sup_{s \in [-1+2\phi^{1/3}, 1-2\phi^{1/3}]} \lambda(s) + O(\phi^{1/3}) + O(\phi^{1/3}) \nu(u) \leq \frac{1}{2} \int_{-1+2\phi^{1/3}}^{1-2\phi^{1/3}} \sqrt{2G(s) P_E(s)} ds \leq \frac{1}{2} \phi^{1/3} + \phi^{1/3}, \]

where we have used \(|\{u > s\}| \geq \nu(u)\) for all \(s \in [-1+2\phi^{1/3}, 1-2\phi^{1/3}].\) The estimate (3.19) follows.

**Proof of Theorem 1.19.** We begin by establishing that \(u_{m,\phi}\) and \(u_{\omega_{m,\phi}}\) satisfy (1.25) and (1.26). Notice that (1.26) follows from (1.25), Lemma 2.7, (1.13), and (1.20). Hence, it suffices to establish (1.25). This will follow from an application of the previous lemma. Indeed, for the minimizer, we have

\[ E(\xi) \phi(u_{m,\phi}) \leq c_m + O(\phi^{1/3}) \text{ by (1.12)} \]

\[ \leq f(\nu(u_{m,\phi})) + O(\phi^{1/3}), \]

since (1.13) implies that \(\nu(u_{m,\phi})\) is within a neighborhood for which \(c_m\) is the minimum value of \(f\). Hence, (3.18) is verified for \(u_{m,\phi}\). On the other hand, (3.16) and (3.14) verify (3.18) for \(u_{\omega_{m,\phi}}\).

We then verify (1.27) for any volume-constrained minimizer with the help of the upper bound (2.8) and Lemmas 2.7 and 3.5.

Finally, we point out that \(u_{m,\phi}\) can be characterized as a constrained minimizer of appropriate volume.

**Lemma 3.6.** The local minimizer \(u_{m,\phi}\) is a constrained minimizer of the energy subject to a volume constraint in the sense that it minimizes \(E(\xi)\) over all functions \(u\) such that \(\nu(u) = \nu(u_{m,\phi}).\)

**Proof.** We define the volume \(\omega_{m,\phi} := \nu(u_{m,\phi}),\) and let \(u_{m,\phi}\) be an associated constrained minimizer, that is, a function that minimizes \(E(\xi)\) subject to \(\nu(u) = \omega_{m,\phi}\). It suffices to show that for some \(\gamma > 0\) and for all \(\phi\) sufficiently small, the constrained minimizer \(u_{m,\phi}\) belongs to a \(\gamma\) neighborhood of \(\Psi_m:\)

\[ |u_{m,\phi} - \Psi_m|_{\phi} < \gamma. \]
Indeed, it then follows from the characterization of $u_{m,\phi}$ as the minimizer over the $\gamma$-neighborhood that

$$E_{\phi}^{\xi}(u_{m,\phi}) \leq E_{\phi}^{\xi}(u_{\omega}^m) \leq E_{\phi}^{\xi}(u) \quad \text{for all } u \text{ with } \nu(u) = \omega_{\phi}^m.$$

We will now show that (3.21) holds. Let $\nu_{m,\phi} := \{|u_{\omega}^m \geq 1 - \phi^{1/3}\}$, and let $\Psi_{m,\phi}$ denote the sharp-interface profile with this droplet volume. By Corollary 2.6, it follows that

$$\nu_{m,\phi} = \omega_{\phi}^m + O(\phi^{1/3}).$$

(3.22)

In light of (1.13) and (3.22), it holds that $||\Psi_m - \Psi_{m,\phi}||_{L^2(\mathbb{T}_\phi\mathbb{T})} < \frac{\gamma}{2}$, so that, by the triangle inequality, it suffices to show that

$$|u_{\omega}^m - \Psi_{m,\phi}|_{\mathbb{T}_\phi\mathbb{T}} < \frac{\gamma}{2},$$

which in turn follows from (1.27) and (3.22).  

3.4. Proofs of lemmas. We now present the proofs of Lemmas 3.1 and 3.3. Lemma 3.1 establishes a link between the finite $\phi$ energy of functions that are $\gamma$ away from $\Psi_m$ and the limit energy of the same set of functions. In contrast to the lower bound in the $\Gamma$-convergence proof in [17], note the following:

(i) We want an estimate that is uniform over $|u - \Psi_m|_{\mathbb{R}^d} = \gamma$ (rather than just an estimate for any given point $u_0$ in this set).

(ii) We do not assume $L^2$ convergence to some function $u_0$ in the sense that

$$\int_{\mathbb{T}_\phi\mathbb{T}} (u_{\phi} - u_0)^2 \, dx \to 0.$$

Roughly speaking, the issue that arises is that, although there is a function $u_0 = \pm 1$ such that $u_{\phi} \to u_0$ in $L^2(K)$ for any compact set $K$ (cf. Lemma 3.7 below), it may be that the $L^2$ distance to $\Psi_m$ drops in the limit, that is, that $||u_0 - \Psi_m||_{L^2(\mathbb{R}^d)} < \gamma$ even though $||u_{\phi} - \Psi_m||_{L^2(\mathbb{T}_\phi\mathbb{T})} = \gamma$ for every $\phi > 0$. The volume costs are straightforward, but we need a good bound on the perimeter cost. What we establish in the proof below is that there is "no free lunch" in the sense that, on the one hand, $E_{\phi}^{\xi}(u_{\phi})$ includes the full perimeter cost of $u_0$ on $K$, and, on the other hand, if some of $\{u_{\phi} \sim 1\}$—roughly speaking, the volume $\beta/4$ in the proof below—has drifted off to infinity in the limit, then $E_{\phi}^{\xi}(u_{\phi})$ also includes the associated perimeter cost of this mass, at least in the sense of

$$\left(\frac{\beta}{4}\right)^{(d-1)/d}.$$
This is enough to conclude.

We begin by establishing \( L^2 \) convergence on compact sets. The argument is standard but we include it for completeness.

**Lemma 3.7.** Fix \( \xi \in (\tilde{\xi}_d, \xi_d) \) and the critical scaling \((1.4)\). Let \( u_\phi \in X_\phi \) be a sequence of functions such that

\[
(3.23) \quad \mathcal{E}_\phi^\xi(u_\phi) \leq 1.
\]

Then, there exist \( u_0 = \pm 1 \) almost everywhere on \( \mathbb{R}^d \) and a subsequence of \( \{ u_\phi \}_{\phi > 0} \) such that, for any compact set \( K \subset \mathbb{R}^d \),

\[
 u_\phi 1_K \rightharpoonup u_0 1_K \quad \text{in} \quad L^2(K).
\]

**Proof.** Consider a compact set \( K \subset \mathbb{R}^d \), and note that, for \( \phi \) small enough so that \( K \subset \mathbb{T}_\phi \), we have

\[
(3.24) \quad \mathcal{E}_\phi^\xi(u_\phi) = \int_{\mathbb{R}^d} \frac{\phi}{2} \cdot |\nabla u_\phi|^2 + \frac{1}{\phi} \cdot G(u_\phi) \, \text{d}x - \frac{G(1 + \phi)}{\phi} \cdot |\mathbb{T}_\phi|,
\]

where \( F(t) = \int_{-1}^{t} \sqrt{2G(s)} \, \text{d}s \). From \((3.24)\) together with \((3.23)\) and \((2.1)\), we deduce

\[
(3.25) \quad \sup_{\phi > 0} \int_{K} |\nabla F(u_\phi)| \, \text{d}x < \infty.
\]

On the other hand, since \( G(s) \sim s^4 \) for large values of \( s \), we have

\[
(3.26) \quad \sup_{\phi > 0} \int_{K} |F(u_\phi)| \, \text{d}x \leq C_K \left( 1 + \sup_{\phi > 0} \int_{K} G(u_\phi) \, \text{d}x \right) < \infty.
\]

By \((3.25)\) and \((3.26)\), it follows that \( \{ F(u_\phi) \} \) is bounded in \( BV(K) \). Consequently, there exists \( w_0 \in L^1(K) \) and a subsequence of \( \{ u_\phi \} \) such that \( F(u_\phi) \rightharpoonup w_0 \) in \( L^1(K) \). By the uniform continuity of \( F^{-1} \) on \( \mathbb{R} \), it follows that \( u_\phi \) converges in measure on \( K \) to \( u_0 := F^{-1}(w_0) \). Moreover, by the second inequality in \((3.26)\), we have

\[
\sup_{\phi > 0} \| u_\phi \|_{L^1(K)} < \infty.
\]
hence, the family \( \{ u_{\phi}^2 \}_{\phi > 0} \) is uniformly integrable on \( K \). We thus deduce that 
\[ \| u_{\phi} - u_0 \|_{L^2(K)} \to 0. \]

Using an expanding sequence of compact sets \( \{ K_n \}_{n \geq 1} \) with \( \bigcup_{n \geq 1} K_n = \mathbb{R}^d \)
and a diagonal argument, one can define \( u_0 \) on all of \( \mathbb{R}^d \) so that \( u_{\phi} 1_K \to u_0 1_K \)
in \( L^2(K) \), for any compact set \( K \subset \mathbb{R}^d \).

Finally, we check that \( u_0 = \pm 1 \) almost everywhere on \( \mathbb{R}^d \). Indeed, for any 
compact set \( K \subset \mathbb{R}^d \), we have

\[
\mathcal{E}_{\phi}^\xi(u_\phi) \geq \frac{1}{\phi_0} \int_K G(u_{\phi}) \, dx \geq \frac{G(-1 + \phi)}{\phi} |\nabla_{\phi} \xi|,
\]

so that by Fatou’s lemma, (2.1), the energy bound (3.23), and the fact that (up to
a subsequence) \( u_{\phi} \) converges almost everywhere to \( u_0 \) in \( K \), we have

\[
\int_K G(u_0) \, dx \leq \liminf_{\phi \to 0} \int_K G(u_{\phi}) \, dx \leq \liminf_{\phi \to 0} \mathcal{E}_{\phi}^\xi(u_\phi) = 0. \]

Proof of Lemma 3.1: Finite \( \phi \) estimate for the local minimum. The proof is by
contradiction. Hence, assume for a contradiction that there exist \( \gamma \in (0, \gamma_0] \) and
\( \delta > 0 \) such that there exist a sequence \( \phi \downarrow 0 \) and a corresponding sequence of
functions \( u_{\phi} \in X_\phi \) such that

\[
|u_{\phi} - \Psi_m|_{\nabla_\phi} = \gamma_{\phi} \in [\gamma, \gamma_0] \quad \text{and} \quad \mathcal{E}_{\phi}^\xi(u_{\phi}) \leq \inf_{\gamma \leq |u - \Psi_m|_{\nabla d} \leq \gamma_0} \mathcal{E}_{\phi}^\xi(u) - 2\delta.
\]

Without loss of generality, we may assume that \( \gamma_{\phi} \to \gamma \in [\gamma, \gamma_0] \). By translating
on the torus, we will always assume that

\[
\inf_{x_0 \in \nabla_{\phi} \xi} \| u_{\phi} - \Psi_m(\cdot - x_0) \|_{L^2(\nabla_{\phi} \xi)} = \| u_{\phi} - \Psi_m(\cdot - 0) \|_{L^2(\nabla_{\phi} \xi)}.
\]

In addition, by density of smooth functions in \( X_\phi \), there exists a sequence of \( C^\infty \)
functions \( \tilde{u}_{\phi} \in X_\phi \) such that

\[
|\tilde{u}_{\phi} - \Psi_m|_{\nabla_{\phi} \xi} = \gamma + o(1)_{\phi \downarrow 0}
\]

and

\[
\mathcal{E}_{\phi}^\xi(\tilde{u}_{\phi}) \leq \inf_{\gamma \leq |u - \Psi_m|_{\nabla d} \leq \gamma_0} \mathcal{E}_{\phi}^\xi(u) - \delta.
\]

In the remainder of the proof, we will work with this smooth sequence (and for
notational simplicity, we write \( u_{\phi} \) instead of \( \tilde{u}_{\phi} \)).
Critical Points of the Cahn-Hilliard Energy

Step 1: Preliminary bounds. By comparison with radial constructions, it is easy to check that
\[
\inf_{y < |u - \Psi_m|_{ad} \leq \gamma_0} \mathcal{E}_\xi \leq \mathcal{E}_{\xi_0}(u) \leq c_s \quad \text{for } \xi \in (\xi_d, \xi_d),
\]
so that in particular our sequence satisfies
\[
(3.29) \quad \mathcal{E}_\phi(u) \leq E_M,
\]
where we recall the definition of $E_M$ in (2.3). Proposition 2.3, Lemma 2.2, and Corollary 2.6 imply
\[
|C(u_\phi)| \lesssim 1.
\]
By applying Lemma 3.7, we deduce that there exists $u_0 = \pm 1$ almost everywhere such that (up to subsequences) $u_\phi \rightharpoonup u_0$ in $L^2(K)$ for any compact set $K$. Arguing as in [17, Theorem 1.9], we obtain for $u_0$ the bounds
\[
|\{u_0 = 1\}| \lesssim 1, \quad \mathrm{Per}(\{u_0 = 1\}) \lesssim 1.
\]
Moreover, since a set with bounded volume and perimeter can be well approximated by a smooth, open, bounded set (cf. [22, Remark 13.12]), an approximation argument similar to the one used in the proof of [17, Theorem 1.9] allows us to assume that $\{u_0 = 1\}$ is bounded.

Step 2: Estimates on a compact set. Let
\[
(3.30) \quad K = [-k, k]^d
\]
be a compact $d$-dimensional cube that compactly contains $\{u_0 = 1\}$ and $B_{\nu_m}(0)$. (Note for future reference that $\Psi_m = -1$ on $K^c$.) Because of $u_\phi \rightharpoonup u_0$ in $L^2(K)$, we have, according to (3.27), that
\[
(3.31) \quad \left\|u_\phi - \Psi_m\right\|_{L^2(K)}^2 - \left\|u_0 - \Psi_m\right\|_{L^2(K)}^2 = y^2 - \beta
\]
for some $\beta \in [0, y^2]$. In addition, obtaining from Lemma 2.5 that
\[
(3.32) \quad |B(u_\phi)|, |C_+(u_\phi)| \to 0,
\]
we observe that
\[
(3.33) \quad \int_{T_{\phi_\lambda}} \chi_3(u_\phi) 1_K \, d\lambda - |\{u_0 = 1\}|.
\]
Step 3 (The deficit). In view of (3.27) and (3.31), it holds that
\[ \|u_\phi - \Psi_m\|_{L^2(T_{\phi L})}^2 = \|u_\phi - (-1)\|_{L^2(T_{\phi L})}^2 = \beta + o(1)_{\phi,0}. \]
On the other hand, in view of Lemma 2.5, we have
\[ \int_{T_{\phi L}} (u_\phi - (-1))^2 1_{T_{\phi L} \setminus K} \chi_{\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}} \, dx = o(1)_{\phi,0}, \]
so that (3.34) improves to
\[ \int_{T_{\phi L}} (u_\phi - (-1))^2 1_{T_{\phi L} \setminus K} \chi_{\mathcal{C}} \, dx = \beta + o(1)_{\phi,0}. \]
From (3.35) we deduce
\[ |C(u_\phi) \cap (T_{\phi L} \setminus K)| = \frac{\beta}{4}, \]
which because of (3.32) we can also express as
\[ \int_{T_{\phi L}} \chi_3(u_\phi) 1_{T_{\phi L} \setminus K} \, dx = \frac{\beta}{4} + o(1)_{\phi,0}. \]

Step 4 (Total energy). We now calculate the cost associated with the sequence \{u_\phi\}. Combining (3.35) and (3.37) gives
\[ \nu(u_\phi) = |\{u_0 = 1\}| + \frac{\beta}{4} + o(1)_{\phi,0}. \]
Estimating the energy as in the proof of [17, Proposition 2.4], we obtain
\[ \int_{T_{\phi L}} e_\phi(u_\phi) \chi_3(u_\phi) \, dx \geq -(4 + o(1)_{\phi,0}) \nu(u_\phi), \]
\[ \int_{T_{\phi L}} e_\phi(u_\phi) \chi_1(u_\phi) \, dx \geq (4 + o(1)_{\phi,0}) \frac{\nu(u_\phi)^2}{\xi^{-d+1}}. \]
The perimeter cost is more involved. For the contribution corresponding to \chi_2, we split the integral over the compact set K (cf. (3.30)) and \( T_{\phi L} \setminus K \). The convexity of G near -1, expressed in the form
\[ 0 = G(-1) \geq G(-1 + \phi) + G'(-1 + \phi)(-1 - (-1 + \phi)) = G(-1 + \phi) - \phi G'(-1 + \phi), \]
implies that
\[ e_{\phi}(u) \geq \frac{1}{\phi}(G(u) - G'(-1 + \phi)(u + 1)) \geq 0 \quad \text{on } B. \]

Hence, we can replace \( \chi_2 \) by \( \chi_2^\eta \) (where the support is on \((-1 + \eta, 1 - \eta)\)) for fixed \( \eta > 0 \). On \( K \), we use the \( L^2(K) \) convergence and argue as in the proof of [17, Theorem 1.9] to deduce

\begin{equation}
\int_{\mathbb{T}^d} e_{\phi}(u_{\phi})\chi_2(u_{\phi})1_K \, dx \geq \int_{\mathbb{T}^d} e_{\phi}(u_{\phi})\chi_2^\eta(u_{\phi})1_K \, dx \\
\geq (c_0 + o(1)\eta(\{u_0 = 1\}) + o(1)\phi(\{u_0 = 1\}), \\
\geq (c_0 + o(1)\eta)\phi(\{u_0 = 1\}) + o(1)\phi(\{u_0 = 1\}),
\end{equation}

where we have recalled that \( \{u_0 = 1\} \) is compactly contained in \( K \).

On \( \mathbb{T}^d \setminus K \), on the other hand, we use the coarea formula as in the proof of [17, Proposition 2.1] to argue that

\begin{equation}
\int_{\mathbb{T}^d} e_{\phi}(u_{\phi})\chi_2(u_{\phi})1_{\mathbb{T}^d \setminus K} \, dx \\
\geq \int_{1-2\eta}^{1-2\eta} \sqrt{2G(s)}H^{d-1}(\{x \in \mathbb{T}^d \setminus K : u_{\phi}(x) = s\}) \, ds \\
\geq \text{ess inf} \, \int_{s \in (-1 + 2\eta, 1 - 2\eta)} H^{d-1}(\{x \in \mathbb{T}^d \setminus K : u_{\phi}(x) = s\}) \, ds \\
\geq (H^{d-1}(\{x \in \mathbb{T}^d \setminus K : u_{\phi}(x) = s_{\phi}\}) - \phi) \int_{1-2\eta}^{1-2\eta} \sqrt{2G(s)} \, ds,
\end{equation}

where we have chosen \( s_{\phi} \in [-1 + 2\eta, 1 - 2\eta] \) to approximate the essential infimum. Notice that we may in addition, without loss of generality, assume that

\[ H^{d-1}(\{x \in \mathbb{T}^d \setminus K : u_{\phi}(x) = s_{\phi}\}) < \infty, \]

since otherwise (3.41), (3.39), and (3.29) lead to a contradiction. We would like to pass from the level surface on the right-hand side of (3.41) to a closed level surface on the torus. To do so, we allow an extra degree of freedom. Specifically, we introduce the hypercubes and corresponding surfaces

\begin{equation}
K_\ell = [-\ell, \ell]^d, \quad S_\ell := \partial K_\ell, \quad \ell \in [k, 2k].
\end{equation}
Trivially, we estimate

\[ H^{d-1}(\{ x \in \mathbb{T}_\Phi L \setminus K : u_\Phi(x) = s_\Phi \}) \]
\[ \geq H^{d-1}(\{ x \in \mathbb{T}_\Phi L \setminus K_\ell : u_\Phi(x) = s_\Phi \}), \quad \ell \in [k, 2k]. \]

On the other hand, for \( \ell \in [k, 2k] \) we can relate \( \{ u_\Phi > s_\Phi \} \cap (\mathbb{T}_\Phi L \setminus K_\ell) \) and the surface measure on the right-hand side of (3.43) via

\[ H^{d-1}(\partial(\{ u_\Phi > s_\Phi \} \cap (\mathbb{T}_\Phi L \setminus K_\ell))) = H^{d-1}(\{ x \in \mathbb{T}_\Phi L \setminus K_\ell : u_\Phi(x) = s_\Phi \}) + H^{d-1}(\{ u_\Phi > s_\Phi \} \cap S_\ell). \]

It remains to argue that the second term on the right-hand side is small. To do so, we will exploit the degree of freedom allowed by \( \ell \) in the form of the following lemma.

**Lemma 3.8.** Consider the cubes and surfaces defined in (3.42). Consider a set \( E \subset \mathbb{R}^d \) such that \( |E \cap (K_{2k} \setminus K)| \leq \varepsilon/2 \). Then, there exists \( \ell_* \in [k, 2k] \) such that

\[ 0 \leq H^{d-1}(E \cap S_{\ell_*}) \leq \frac{\varepsilon}{K}. \]

The proof of Lemma 3.8 follows from writing the volume as an integral of surface area and considering the infimum. We thus apply Lemma 3.8 to the set \( E = \{ u_\Phi > s_\Phi \} \), noting that the strong \( L^2 \) convergence of \( u_\Phi \) to \(-1\) on \( K_{2k} \setminus K \) yields

\[ |\{ u_\Phi > s_\Phi \} \cap (K_{2k} \setminus K)| = o(1). \]

Combining (3.43), (3.44), and Lemma 3.8 with \( E = \{ u_\Phi > s_\Phi \} \), we obtain

\[ H^{d-1}(\{ x \in \mathbb{T}_\Phi L \setminus K : u_\Phi(x) = s_\Phi \}) \]
\[ \geq H^{d-1}(\partial(\{ u_\Phi > s_\Phi \} \cap (\mathbb{T}_\Phi L \setminus K_{\ell_*}))) - H^{d-1}(\{ u_\Phi > s_\Phi \} \cap S_{\ell_*}) \]
\[ \geq H^{d-1}(\partial(\{ u_\Phi > s_\Phi \} \cap (\mathbb{T}_\Phi L \setminus K_{\ell_*}))) + o(1). \]

The final ingredient we need is the isoperimetric inequality on the torus (1.22), which we apply to \( \{ u_\Phi > s_\Phi \} \cap (\mathbb{T}_\Phi L \setminus K_{\ell_*}) \), recalling the bound on this set implied by (3.32) and (3.36). Using (1.22) in (3.45) and substituting the result in (3.41) leads to

\[ \int_{\mathbb{T}_\Phi L} e_\Phi(u_\Phi)X_2(u_\Phi)1_{\mathbb{T}_\Phi L \setminus K} \, dx \]
\[ \geq (\tilde{C}_1 + o(1))(|\{ u_\Phi \} \cap (\mathbb{T}_\Phi L \setminus K_{\ell_*})|)^{(d-1)/d} + o(1). \]
Moreover, we can improve (3.36) to

\[(3.47) \quad |C(u_\phi) \cap (\mathbb{T}_{\phi L} \setminus K_{\ell^*})| \to \frac{\beta}{4},\]

using the $L^2$ convergence of $u_\phi$ to $-1$ on $K_{2K} \setminus K$.

Adding (3.39), (3.40), and (3.46), and using (3.47) and (3.38) to pass to the limit (first in $\phi$ and then in $\eta$) leads to

\[(3.48) \quad \liminf_{\phi \to 0} \mathcal{E}_\phi(u_\phi) \geq c_0 \operatorname{Per}(|u_0 = 1\}) + \bar{C}_1 \left(\frac{\beta}{4}\right)^{(d-1)/d} - 4 \left(|\{u_0 = 1\}| + \frac{\beta}{4}\right) + 4 \left(|\{u_0 = 1\}| + \frac{\beta}{4}\right)^2.\]

**Step 5 (Derivation of a contradiction).** We now observe that the right-hand side of (3.48) is exactly the energy of the function $\tilde{u}$ defined as follows. Let $\tilde{u} = u_0$ on $K$. Setting $K_R := \{x \in \mathbb{R}^d : \text{dist}(x,K) \leq R\}$, let $\tilde{u} = +1$ on a disk of volume $\beta/4$ in $\mathbb{R}^d \setminus K_{2R}$, and $\tilde{u} = -1$ otherwise. Here, we choose $R$ big enough so that $K_{2R} \setminus K$ contains all balls of volume $\nu_m$ whose centers lie on $\partial K_R$. The function $\tilde{u}$ so defined satisfies

\[(3.49) \quad \|\tilde{u} - \Psi_m\|^2_{L^2(\mathbb{R}^d)} = \|u_0 - \Psi_m\|^2_{L^2(K)} + 4 \left(\frac{\beta}{4}\right)^2.\]

Moreover, we claim that $\Psi_m$ is optimal for $\tilde{u}$ in the sense that

\[|\tilde{u} - \Psi_m|_{\mathbb{R}^d} = \|\tilde{u} - \Psi_m\|_{L^2(\mathbb{R}^d)},\]

so that (3.49) improves to

\[(3.50) \quad |\tilde{u} - \Psi_m|_{\mathbb{R}^d} = \gamma.\]

Indeed, we have on the one hand that for any $x_0 \in \mathbb{R}^d$ such that $\Psi_m(\cdot - x_0)$ has $\{\Psi_m(\cdot - x_0) = 1\} \cap K = \emptyset$, it holds that

\[
\begin{align*}
\|\tilde{u} - \Psi_m(\cdot - x_0)\|^2_{L^2(\mathbb{R}^d)} &\geq 4 \left(|\{u_0 = 1\}| + \nu_m - \frac{\beta}{4}\right) \\
&\geq 4 \left(\nu_m - |\{u_0 = -1, \Psi_m = 1\}| + \nu_m - \frac{\beta}{4}\right) \\
&\geq 4 \left(\nu_m - \frac{\gamma^2}{4} + \frac{\beta}{4} + \nu_m - \frac{\beta}{4}\right) \quad \text{by (3.31)} \\
&= 4 \left(2\nu_m - \frac{\gamma^2}{4}\right) \geq 4(2\nu_m - (\nu_m - \nu_s)) \quad \text{by (1.7)} \\
&> \gamma^2_0 \geq \gamma^2,
\end{align*}
\]
so that (3.49) implies that the centered minimizer \( \Psi_m \) beats any such shift. On the other hand, for any \( x_0 \in K_R \), optimality of \( \Psi_m(\cdot - x_0) \) for \( u_0 \) is inherited from \( u_\phi \) because of the strong \( L^2 \) convergence on \( K_{2R} \), that is,

\[
\| u_0 - \Psi_m(\cdot - x_0) \|_{L^2(K_{2R})} \leq \| u_0 - \Psi_m(\cdot) \|_{L^2(K_{2R})},
\]

so that in particular,

\[
\| \tilde{u} - \Psi_m(\cdot - x_0) \|_{L^2(R^d)} \leq \| \tilde{u} - \Psi_m(\cdot) \|_{L^2(R^d)}.
\]

The combination of (3.50), (3.48), and (3.28) leads to

\[
\inf_{\gamma \leq 1} \| u - \Psi_s \|_{R^d} \leq \gamma \nu(u) = \nu_s \inf_{\gamma \leq 1} \| u - \Psi_s \|_{R^d} - \delta.
\]

This contradiction completes the proof.

It remains to prove Lemma 3.3. The proof mirrors (almost exactly) the proof of Lemma 3.1. Hence, we will be brief and highlight only the differences.

**Proof of Lemma 3.3:** Finite \( \phi \) estimate for the saddle point. Assume, for a contradiction, there exists \( \gamma \in (0, \gamma_0] \) and \( \delta > 0 \) such that there exists a sequence \( \phi \downarrow 0 \) and a corresponding sequence of functions \( u_\phi \in X_\phi \) such that

\[
\lim_{\phi \to 0} \nu(u_\phi) = \nu_s, \quad |u_\phi - \Psi_s|_{T_\phi L} \geq \gamma,
\]

and

\[
E_\phi^\xi(u_\phi) \leq \inf_{v \in X_\phi \setminus \Psi_s} E_0^\xi(v) - \delta. \tag{3.51}
\]

From \( E_\phi^\xi(u_\phi) \leq E_M \), Proposition 2.3, and Lemma 2.2, we deduce \( C(u_\phi) \leq 1 \) and consequently that \( |u_\phi - \Psi_s|_{T_\phi L} \leq 1 \). Indeed, we estimate roughly

\[
|u_\phi - \Psi_s|_{T_\phi L} \leq \| u_\phi - \Psi_s \|_{L^2(T_\phi L)}^2 \\
= \int_{B_{\nu_s}(x)} (u_\phi - 1)^2 \, dx + \int_{T_\phi L \setminus B_{\nu_s}(x)} (u_\phi + 1)^2 \, dx \\
\leq \nu_s + C(u_\phi) \leq 1.
\]

Hence we may, without loss of generality, assume that

\[
|u_\phi - \Psi_s|_{T_\phi L} = \gamma \geq \gamma.
\]
(For ease of notation, we drop the tilde.) In the following argument, the condition \( |u_\phi - \psi_m|_{T_\phi L} = \gamma_\phi \) from the proof of Lemma 3.1 is replaced by
\[
|u_\phi - \Psi_s|_{T_\phi L} = \gamma + o(1).
\]
Also, by translating on the torus, we assume as in the proof of Lemma 3.1 that
\[
\inf_{x_0 \in T_\phi L} \|u_\phi - \Psi_s(\cdot - x_0)\|_{L^2(T_\phi L)} = \|u_\phi - \Psi_s(\cdot - 0)\|_{L^2(T_\phi L)}.
\]

The analogues of Step 1–Step 3 of the proof of Lemma 3.1 carry over to our setting. Because \( \nu(u_\phi) = \nu_s + o(1) \), the estimates in (3.39) simplify to
\[
\int_{T_\phi L} e_\phi(u_\phi)X_\delta(u_\phi) \, dx \geq \gamma_2(\phi)(\nu_s + o(1)),
\]
while the perimeter estimate carries over unchanged. We are led to
\[
\liminf_{\phi \to 0} E_\phi^{\xi}(u_\phi) \geq c_0 \operatorname{Per}(\{u_0 = 1\})
\]
\[
+ \tilde{C}_1 \left( \frac{\beta}{4} \right)^{(d-1)/d} - 4\nu_s + 4 \frac{\nu_s^2}{\xi^{d+1}}.
\]
On the other hand, the analogues of (3.33) and (3.37) together with
\[
\nu(u_\phi) = \nu_s + o(1)
\]
imply the relation
\[
\nu_s = |\{u_0 = 1\}| + \frac{\beta}{4},
\]
so that, in analogy to the proof of Lemma 3.1, we recognize the right-hand side of (3.52) as the energy of a sharp-interface function \( \tilde{u} \) that agrees with \( u_0 \) on \( K \) and takes value +1 on a ball of volume \( \beta/4 \) somewhere in \( \mathbb{R}^d \setminus K_{2R} \), and is −1 otherwise. We observe that
\[
\nu(\tilde{u}) = |\{u_0 = 1\}| + \frac{\beta}{4} = \nu_s \quad \text{by (3.53)}.
\]
On the other hand, exactly as in the proof of Lemma 3.1, we observe
\[
|\tilde{u} - \Psi_s|_{\mathbb{R}^d} = \gamma
\]
and obtain
\[
\inf_{|u - \Psi_s|_{\mathbb{R}^d} \geq 1} \mathcal{E}_0^\varepsilon (u) \leq \mathcal{E}_0^\varepsilon (\bar{u}) \quad \text{by (3.54), (3.55)}
\]
\[
\leq \liminf_{\phi \to 0} \mathcal{E}_\phi^\varepsilon (u, u_\phi) \quad \text{by (3.52)}
\]
\[
\leq \inf_{|u - \Psi_s|_{\mathbb{R}^d} \geq 1} \mathcal{E}_0^\varepsilon (u) - \delta \quad \text{by (3.51)},
\]
a contradiction. □

APPENDIX A. ISOPERIMETRY ON THE TORUS

Proof of Corollary 1.18. Our approach is similar to the one used in establishing [8, Theorem 6.2]. Note first that \( \varepsilon > 0 \) can be chosen small enough to ensure that the diameter of a ball of volume \(|A|\) is less than \( \frac{1}{4} \). This is the only restriction on the value of \( \varepsilon \). With that in mind, let \( B \) be a ball that achieves the optimal overlap with \( A \) in the definition of the Fraenkel asymmetry. In other words, consider a ball \( B \) of volume \(|A|\), such that \(|A \triangle B| = \lambda(A)|A|\). Since we are working on the torus, we can, with no loss of generality, assume that \( B \) is centered at the origin.

Next, define
\[
a_1(t) := \mathcal{H}^{d-1}(A \cap \{x_1 = t\}) \quad \text{for } t \in \left[-\frac{1}{2}, \frac{1}{2}\right],
\]
and observe that, with \( I := [-\frac{1}{2}, \frac{1}{2}] \setminus [-\frac{1}{4}, \frac{1}{4}] \), it holds that
\[
|A| = \int_{-\frac{1}{2}}^{\frac{1}{2}} a_1(t) \, dt \geq \int_I a_1(t) \, dt \geq \frac{1}{2} \text{ess inf}_I a_1,
\]
from which it then follows that, for arbitrary \( \delta > 0 \), there exists \( t_1 \in I \) such that \( a_1(t_1) \leq 2|A| + \delta \). Shifting in the torus in the \( x_1 \)-direction by \( t_1 \), if necessary, we may thus assume that
\[
a_1 \left(-\frac{1}{2}\right) = a_1 \left(\frac{1}{2}\right) \leq 2|A| + \delta.
\]
Repeating this process for the other \( d - 1 \) directions, and since the \( d \) successive translations are independent of each other, we may assume that, in fact,
\[
(A.1) \quad a_i \left(-\frac{1}{2}\right) = a_i \left(\frac{1}{2}\right) \leq 2|A| + \delta, \quad \text{for all } i \in \{1, \ldots, d\}.
\]
Notice that the set \( \tilde{B} \) obtained by shifting the ball \( B \) in all \( d \) directions is still a ball of volume \(|A|\) contained in the open set \((-1/2, 1/2)^d \subset \mathbb{R}^d\). Moreover, the
Fraenkel asymmetry of the set \( \tilde{A} \subset \mathbb{R}^d \) obtained by shifting \( A \) is the same as that of \( A \), for if we regard \( \tilde{A} \) and \( \tilde{B} \) as subsets of \( \mathbb{R}^d \), the ball \( \tilde{B} \) is an optimal ball for \( \tilde{A} \), and \( |\tilde{A} \triangle \tilde{B}| = \lambda(A) |\tilde{A}| \).

It thus follows from the quantitative isoperimetric inequality in \( \mathbb{R}^d \) that

\[
\text{Per}_{\mathbb{R}^d}(\tilde{A}) \geq \text{PE}(\tilde{A}) + C(d)\lambda(\tilde{A})^2 \text{PE}(\tilde{B}).
\]

Note that we have

\[
\text{Per}_{T^1}(\tilde{A}) \geq \text{Per}_{\mathbb{R}^d}(\tilde{A}) - 2d(2|\tilde{A}| + \delta),
\]

since, as a consequence of (A.1), dropping the identification of opposite ends of the torus (and thereby regarding \( \tilde{A} \) merely as a subset of \( \mathbb{R}^d \)) increases the perimeter \( \text{Per}_{T^1}(\tilde{A}) \) by at most \( 2d(2|\tilde{A}| + \delta) \). Since \( \delta > 0 \) is arbitrary, it follows from (A.3) that

\[
\text{Per}_{T^1}(\tilde{A}) \geq \text{Per}_{\mathbb{R}^d}(\tilde{A}) - 4d|\tilde{A}|,
\]

Substituting (A.2) into (A.4), and recalling

\[
\text{Per}_{T^1}(\tilde{A}) = \text{Per}_{T^1}(A), \quad |\tilde{A}| = |A|, \quad \lambda(\tilde{A}) = \lambda(A), \quad \text{and} \quad \text{PE}(\tilde{B}) = \text{PE}(A),
\]

we finally obtain

\[
\text{Per}_{T^1}(A) \geq \text{PE}(A) + C(d)\lambda(A)^2 \text{PE}(A) - 4d|A|.
\]

\[\square\]

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