Semi-regular flat modules over strong Prüfer rings

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Abstract

We first introduce and study the notion of semi-regular flat modules, and then show that a ring $R$ is a strong Prüfer ring if and only if every submodule of a semi-regular flat $R$-module is semi-regular flat, if and only if every ideal of $R$ is semi-regular flat, if and only if every $R$-module has a surjective semi-regular flat (pre)envelope.

Key Words: strong Prüfer rings; semi-regular flat modules; semi-regular coherent rings.

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1. INTRODUCTION

In this paper, we always assume $R$ is a commutative ring with identity and $T(R)$ is the total ring of fractions of $R$. Following from [23], an ideal $I$ of $R$ is said to be dense if $(0:_RI) := \{r \in R | Ir = 0\}$ is 0, or be semi-regular if it contains a finitely generated dense sub-ideal, or be regular if it contains a regular element. Let $I$ be an ideal of $R$. Denote by $I^{-1} = \{z \in T(R) | Iz \subseteq R\}$. If an ideal $I$ of $R$ satisfies $II^{-1} = R$, then $I$ is said to be an invertible ideal.

Early in 1932, Prüfer [21] introduced integral domains over which all finitely generated non-zero ideals are invertible, and then they are called Prüfer domains by Krull [17]. Many algebraists have done a lot of work on Prüfer domains. For the system summaries of Prüfer domains, one can refer to [11]. Since Prüfer domains are of great importance to the study of integral domains, many scholars generalized the notion of integral domains to these of commutative rings with zero-divisors. In 1967, Butts and Smith [9] introduced the notion of Prüfer rings over which every finitely generated regular ideal is invertible. And then, Griffin [14] characterized Prüfer rings utilizing multiplicative ideal theory. Recently, Xiao et al. [27] characterized Prüfer rings by module-theoretic viewpoint.

Since the notion of Prüfer rings is very simple, it is very hard to delve deeper (all total rings of quotients are Prüfer rings). For better understanding Prüfer rings,
Anderson et al. [1] introduced the notion of strong Prüfer rings, over which every finitely generated semi-regular ideal is locally principal, and they showed that a ring \( R \) is strong Prüfer if and only if its Nagata ring \( R(x) \) is a Prüfer ring. In 1987, Anderson et al. [3] characterized strong Prüfer rings by lattice-theoretic viewpoint, i.e., a ring \( R \) is strong Prüfer if and only if the lattice \( \mathcal{L}_{sr}(R) \) consisting of all finitely generated semi-regular ideals is a distributive lattice. In 1993, Lucas [19] proved that a ring \( R \) is strong Prüfer if and only if \( R \) is a Prüfer ring and \( T(R) \) is a strong Prüfer ring. The small finitistic dimensions \( \text{fPD}(R) \) (\( \text{fPD}(R) = \sup \{ \text{pd}_R M \mid M \text{ is super finitely presented, } \text{pd}_R M < \infty \} \)) of a strong Prüfer ring \( R \) is also very attractive. In 2020, Wang et al. [26] showed that a ring \( R \) satisfies \( \text{fPD}(R) = 0 \) if and only if \( R \) is a DQ ring, i.e., the only finitely generated semi-regular ideal of \( R \) is \( R \) itself. Subsequently, Wang et al. [24] showed that a connect strong Prüfer ring has \( \text{fPD}(R) \leq 1 \). Recently, The first author of the paper and Wang showed that every strong Prüfer ring has \( \text{fPD}(R) \leq 1 \), and obtained examples of total rings of quotients \( R \) with \( \text{fPD}(R) = n \) for any \( n \in \mathbb{N} \).

The main motivation of this paper is to give some module-theoretic and homology-theoretic characterizations of strong Prüfer rings. First, we introduce the notions of semi-regular flat modules, semi-regular coflat modules and semi-regular cotorsion modules, and show that the classes of semi-regular flat modules and semi-regular cotorsion modules constitute a perfect cotorsion pair. Then we give some characterizations of DQ rings and strong Prüfer rings. More precisely, we show that a ring \( R \) is a DQ ring if and only if every \( R \)-module is semi-regular flat, if and only if every \( R \)-module is semi-regular coflat, if and only if every semi-regular cotorsion module is injective; we also show that a ring \( R \) is strong Prüfer if and only if every submodule of a semi-regular flat module is semi-regular flat modules, if and only if every finitely generated ideal of \( R \) is semi-regular flat. Finally, we introduce and characterize the notion of semi-regular coherent rings. We also show that a ring \( R \) is strong Prüfer if and only if every \( R \)-module has a surjective semi-regular flat envelope, if and only if every \( R \)-module has a surjective semi-regular flat preenvelope.

2. Semi-regular flat modules

Recall from [27], an \( R \)-module \( M \) is called a regular flat module if \( \text{Tor}_1^R(R/I, M) = 0 \) for any finitely generated regular ideal \( I \). Obviously, every flat module is regular flat. The notion of regular flat modules is used to characterize the total rings of quotients (see [27]). For studying DQ-rings and strong Prüfer rings, we introduced the notion of semi-regular flat modules.
Definition 2.1. An $R$-module $M$ is said to be a semi-regular flat module if, for any finitely generated semi-regular ideal $I$, we have $\text{Tor}_1^R(R/I, M) = 0$. The class of all semi-regular flat modules is denoted by $\mathcal{F}_{sr}$.

Obviously, any flat module is semi-regular flat, and any semi-regular flat module is regular flat. Dually, we can give the notion of semi-regular coflat modules.

Definition 2.2. An $R$-module $M$ is said to be a semi-regular coflat module if, for any finitely generated semi-regular ideal $I$, we have $\text{Ext}_1^R(R/I, M) = 0$.

Lemma 2.3. Let $M$ be an $R$-module. Then the following statements are equivalent:

1. $M$ is a semi-regular flat module;
2. for any finitely generated semi-regular ideal $I$, the natural homomorphism $I \otimes M \to R \otimes M$ is a monomorphism;
3. for any finitely generated semi-regular ideal $I$, the natural homomorphism $\sigma_I : I \otimes M \to IM$ is an isomorphism;
4. for any injective module $E$, $\text{Hom}_R(M, E)$ is a semi-regular coflat module;
5. if $E$ is injective cogenerator, then $\text{Hom}_R(M, E)$ is a semi-regular coflat module.

Proof. (1) $\iff$ (2): Let $I$ be a finitely generated semi-regular ideal. Then we have a long exact sequence:

$$0 \to \text{Tor}_1^R(R/I, M) \to I \otimes M \to R \otimes M \to R/I \otimes M \to 0.$$ 

Consequently, $\text{Tor}_1^R(R/I, M) = 0$ if and only if $I \otimes M \to R \to R \otimes M$ is a monomorphism.

(2) $\implies$ (3): Let $I$ be a finitely generated semi-regular ideal. Then we have the following commutative diagram:

$$
\begin{array}{ccc}
0 & \longrightarrow & I \otimes M \\
\downarrow{\sigma_I} & & \downarrow{\sigma_R} \\
0 & \longrightarrow & IM \\
\end{array}
\qquad \cong

\begin{array}{ccc}
0 & \longrightarrow & R \otimes_R M \\
& & \downarrow{\cong} \\
& \longrightarrow & M.
\end{array}
$$

Then $\sigma_I$ is a monomorphism. Since the multiplicative map $\sigma_I$ is an epimorphism, $\sigma_I$ is actually an isomorphism.

(3) $\implies$ (1): Let $I$ be a finitely generated semi-regular ideal. Then we have a long exact sequence:

$$0 \longrightarrow \text{Tor}_1^R(R/I, M) \longrightarrow IM \longrightarrow M.$$ 

Since $f$ is a natural embedding map, we have $\text{Tor}_1^R(R/I, M) = 0$. 

(1) $\Rightarrow$ (4): Let $I$ be a finitely generated semi-regular ideal and $E$ an injective module. Then $\text{Ext}_R^1(R/I, \text{Hom}_R(M, E)) \cong \text{Hom}_R(\text{Tor}_R^1(R/I, M), E)$. Since $M$ is a semi-regular flat module, then $\text{Tor}_R^1(R/I, M) = 0$. Thus $\text{Ext}_R^1(R/I, \text{Hom}_R(M, E)) = 0$, so $\text{Hom}_R(M, E)$ is a semi-regular coflat module.

(4) $\Rightarrow$ (5): Trivial.

(5) $\Rightarrow$ (1): Let $I$ be a finitely generated semi-regular ideal and $E$ an injective cogenerator. Since $\text{Hom}_R(M, E)$ is a regular coflat module and $\text{Ext}_R^1(R/I, \text{Hom}_R(M, E)) \cong \text{Hom}_R(\text{Tor}_R^1(R/I, M), E)$, we have $\text{Hom}_R(\text{Tor}_R^1(R/I, M), E) = 0$. Since $E$ is an injective cogenerator, $\text{Tor}_R^1(R/I, M) = 0$. So $M$ is a semi-regular flat module.

Lemma 2.4. Let $R$ be a ring. Then the class $\mathcal{F}_{sr}$ of all semi-regular flat modules is closed under direct limits, pure submodules and pure quotients.

Proof. For the direct limits, suppose $\{M_i\}_{i \in I}$ is a direct system consisting of semi-regular flat modules. Then, for any finitely generated semi-regular ideal $I$, we have $\text{Tor}_R^1(R/I, \lim_{\rightarrow} M_i) = \lim_{\rightarrow} \text{Tor}_R^1(R/I, M_i) = 0$. So $\lim_{\rightarrow} M_i$ is a semi-regular flat module.

For pure submodules and pure quotients, let $I$ be a finitely generated semi-regular ideal. Suppose $0 \rightarrow M \rightarrow N \rightarrow L \rightarrow 0$ is a pure exact sequence. We have the following commutative diagram with rows exact:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & M \otimes_R I & \longrightarrow & N \otimes_R I & \longrightarrow & L \otimes_R I & \longrightarrow & 0 \\
& & f & & g & & & & \\
0 & \longrightarrow & M \otimes_R R & \longrightarrow & N \otimes_R R & \longrightarrow & L \otimes_R R & \longrightarrow & 0 \\
& & & \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & M \otimes_R R/I & \longrightarrow & N \otimes_R R/I & \longrightarrow & L \otimes_R R/I & \longrightarrow & 0
\end{array}
\]

By the Snake Lemma, the natural homomorphism $f : M \otimes_R I \rightarrow M \otimes_R R$ and $g : L \otimes_R R/I \rightarrow L \otimes_R R$ are all monomorphisms. Consequently, $M$ and $L$ are all semi-regular flat.

Definition 2.5. An $R$-module $N$ is said to be semi-regular cotorsion if, for any semi-regular flat module $M$, we have $\text{Ext}_R^1(M, N) = 0$. The class of all semi-regular cotorsion modules is denoted by $\mathcal{C}_{sr}$.

Obviously, any injective module is semi-regular cotorsion, and any semi-regular cotorsion module is cotorsion. It is well-known that the classes of all flat modules and all cotorsion modules constitute a perfect cotorsion pair (see [8]). Now, we show
that the class of all semi-regular flat modules and all semi-regular cotorsion modules also constitute a perfect cotorsion pair.

**Theorem 2.6.** Let $R$ be a ring. Then $(\mathcal{F}_{sr}, \mathcal{C}_{sr})$ is a perfect cotorsion pair. Consequently, the class $\mathcal{F}_{sr}$ of all semi-regular flat modules is covering and the class $\mathcal{C}_{sr}$ of all semi-regular cotorsion modules is enveloping.

**Proof.** Obviously, $R$ itself is a semi-regular flat module, and the class $\mathcal{F}_{sr}$ is closed under direct summands and extensions. By Lemma 2.4 and [15, Theorem 3.4], we have $(\mathcal{F}_{sr}, \mathcal{C}_{sr})$ is a perfect cotorsion pair. So the class $\mathcal{F}_{sr}$ of all semi-regular flat modules is covering and the class $\mathcal{C}_{sr}$ of all semi-regular cotorsion modules is enveloping. □

**Proposition 2.7.** Let $R$ be a ring. Then the following statements are equivalent:

1. $M$ is a semi-regular cotorsion module;
2. $\text{Hom}_R(F, M)$ is semi-regular cotorsion for any flat module $F$;
3. $\text{Hom}_R(P, M)$ is semi-regular cotorsion for any projective module $P$.

**Proof.**

(1) ⇒ (2): Let $N$ be a semi-regular flat module and $F$ a flat module. There is a short exact sequence $0 \to K \to P \to N \to 0$, where $P$ is projective. So there is an exact sequence $0 \to K \otimes_R F \to P \otimes_R F \to N \otimes_R F \to 0$. Note that $\text{Tor}_1^R(R/I, N \otimes_R F) = \text{Tor}_1^R(R/I, N) \otimes_R F = 0$ for any finitely generated semi-regular ideal $I$, thus $N \otimes_R F$ is a semi-regular flat $R$-module. Since

$$\text{Hom}_R(P \otimes_R F, M) \to \text{Hom}_R(K \otimes_R F, M) \to \text{Ext}_R^1(N \otimes_R F, M) = 0$$

is exact, there is a short exact sequence

$$\text{Hom}_R(P, \text{Hom}_R(F, M)) \to \text{Hom}_R(K, \text{Hom}_R(F, M)) \to 0.$$

On the other hand, the sequence

$$(P, (F, M)) \to (K, (F, M)) \to \text{Ext}_R^1(N, (F, M)) \to \text{Ext}_R^1(P, (F, M)) = 0$$

is exact (Use $(-, -)$ to instead of $\text{Hom}_R(-, -)$). Consequently,

$$\text{Ext}_R^1(N, \text{Hom}_R(F, M)) = 0.$$

Thus $\text{Hom}_R(F, M)$ is a semi-regular cotorsion $R$-module.

(2) ⇒ (3): Trivial.

(3) ⇒ (1): Set $P = R$, then the result holds. □
3. The homology theory of semi-regular flat modules

Recall from [26, Proposition 2.2], a ring \( R \) is called a DQ ring if the only finitely generated semi-regular ideal of \( R \) is \( R \) itself. It is well-known that a ring \( R \) is a von Neumann regular ring if and only if any \( R \)-module is flat.

**Theorem 3.1.** Let \( R \) be a ring. Then the following statements are equivalent:

1. \( R \) is a DQ ring;
2. \( \text{fPD}(R) = 0 \);
3. every \( R \)-module is semi-regular flat;
4. every \( R \)-module is semi-regular coflat;
5. every semi-regular cotorsion module is injective;
6. for every finitely generated semi-regular ideal \( I \) and any finitely generated ideal \( J \), we have \( I \cap J = IJ \);
7. for every finitely generated semi-regular ideal \( I \), \( R/I \) is flat module;
8. for every finitely generated semi-regular ideal \( I \) and \( a \in I \), there is \( c \in I \) such that \((1 - c)a = 0\).

**Proof.** (1) \( \Rightarrow \) (3), (1) \( \Rightarrow \) (4) and (1) \( \Rightarrow \) (7): Trivial.

(1) \( \iff \) (2): See [26, Proposition 2.2]. (3) \( \iff \) (5): It follows form Theorem 2.6

(6) \( \iff \) (7) \( \iff \) (8): See [13, Theorem 1.2.15].

(3) \( \Rightarrow \) (6): Let \( I \) be a finitely generated semi-regular ideal of \( R \) and \( J \) a finitely generated ideal of \( R \). Then \( R/J \) is a semi-regular flat module. So \( \text{Tor}_1^R(R/I, R/J) = 0 \), that is, \( I \cap J = IJ \) (See [23, Exercise 3.20]).

(7) \( \Rightarrow \) (1): Let \( I \) be a finitely generated semi-regular ideal of \( R \) of \( R \)-module. Then we have \( \text{Tor}_1^R(R/I, M) = 0 \).

(8) \( \Rightarrow \) (1): Suppose \( I = \langle a_1, ..., a_n \rangle \) is finitely generated semi-regular ideal. Then, for any \( i = 1, ..., n \), there exists \( c_i \in I \) such that \((1 - c_i)a_i = 0 \). Set \( c = \prod_{i=1}^n (1 - c_i) \). Then \( ca_i = 0 (i = 1, ..., n) \). Thus \( c \in (0 :_R I) = 0 \). Note that \( 1 - c \in I \). Thus \( 1 \in I \) and \( I = R \).

(4) \( \Rightarrow \) (1): Let \( I \) be a finitely generated semi-regular ideal of \( R \). Then \( R/I \) is a projective module by (4). Thus \( I \) is finitely generated idempotent ideal. By [12, Chapter I, Theorem 1.10], \( I = \langle e \rangle \) where \( e \) is an idempotent. Then \( 1 - e \in (0 :_R I) = 0 \). So \( I = R \). \( \square \)

**Remark 3.2.** Now we give examples of non-flat semi-regular flat modules, and non-semi-regular flat modules.

(1) Suppose the non-field ring \( R \) is a Noetherian local ring with Krull dimension equal to 0 (Take \( R = \mathbb{Z}_p^n \) for example). By [6], we have \( \text{fPD}(R) = 0 \). Since
R is a local ring which is not a field, then R is not a von Neumann regular ring. By Theorem 3.1 there exists a semi-regular flat R-module which is not flat.

(2) Following from [27, Theorem 2.13], R is a total ring of quotient if and only if any R-module is regular flat. Wang [25] give an example of total rings of quotients R with fPD(R) > 0. Then there exist regular flat modules which are not semi-regular flat by Theorem 3.1.

Corollary 3.3. Let R be a ring. Then the following statements are equivalent:

1. R is a semi-simple ring;
2. any semi-regular flat R-module is projective;
3. any R-module is semi-regular cotorsion.

Proof. (1) ⇒ (2) and (1) ⇒ (3): Trivial.
(2) ⇒ (1): Since any flat module is semi-regular flat, any flat module is projective by (2). So R is a perfect ring, that is, FPD(R) = 0. So fPD(R) = 0. By Theorem 3.1, any R-module is semi-regular flat module, so is projective by (2) again. Consequently, R is a semi-simple ring.
(2) ⇔ (3): It follows by Theorem 2.6.

Theorem 3.4. Let R be a ring. Then the following statements are equivalent:

1. R is a strong Prüfer ring;
2. any submodule of a semi-regular flat R-module is semi-regular flat;
3. any submodule of a flat R-module is semi-regular flat;
4. any ideal of R is semi-regular flat;
5. any finitely generated ideal of R is semi-regular flat;
6. any finitely generated semi-regular ideal of R is flat;
7. any finitely generated semi-regular ideal of R is projective;
8. any quotient of a semi-regular coflat R-module is semi-regular coflat;
9. any h-divisible R-module is semi-regular coflat.

Proof. (2) ⇒ (3) ⇒ (4) ⇒ (5), (7) ⇒ (6) and (8) ⇒ (9): Trivial.
(5) ⇔ (6): Let I be a finitely generated semi-regular ideal of R and J a finitely generated ideal of R. Then we have Tor_1^R(R/J, I) ≅ Tor_2^R(R/1, R/J) ≅ Tor_1^R(R/I, J). Consequently, J is semi-regular flat if and only if I is flat.
(6) ⇒ (1): Let I be a finitely generated semi-regular ideal of R and m a maximal ideal of R. Then Im is finitely generated flat R_m-ideal. By [13, Lemma 4.2.1] and [20, Theorem 2.5], we have Im is a free R_m-ideal. So the rank of Im is at most 1. Consequently, Im is a principal ideal of R_m.
(1) ⇒ (6): Let \( I \) be a finitely generated semi-regular ideal of \( R \) and \( m \) a maximal ideal of \( R \). Then \( I_m \) is a principal \( R_m \)-ideal. Suppose \( I_m = \langle \frac{a}{s} \rangle \). Then \( (0 :_{R_m} \frac{a}{s}) = (0 :_R I)_m = 0 \) by [23, Exercise 1.72]. Thus \( \frac{a}{s} \) is regular element. So \( I_m \simeq R_m \). Consequently, \( I \) is a flat \( R \)-ideal.

(6) ⇒ (4): Let \( I \) be a finitely generated semi-regular ideal of \( R \) and \( K \) an ideal of \( R \). Then we have \( \text{Tor}_1^R(R/I, K) \cong \text{Tor}_1^R(R/K, I) = 0 \) by (6). So \( K \) is semi-regular flat.

(6) ⇒ (2): Let \( M \) be a semi-regular flat module and \( N \) a submodule of \( M \). Suppose \( I \) is a finitely generated semi-regular ideal, then \( I \) is a flat ideal. Thus \( \text{fd}_R(R/I) \leq 1 \). Consider the exact sequence

\[
\text{Tor}_2^R(R/I, M/N) \rightarrow \text{Tor}_1^R(R/I, N) \rightarrow \text{Tor}_1^R(R/I, M).
\]

Since \( \text{Tor}_2^R(R/I, M/N) = \text{Tor}_1^R(R/I, M) = 0 \), we have \( \text{Tor}_1^R(R/I, N) = 0 \). So \( N \) is a semi-regular flat module.

(6) ⇒ (7): It follows from [22, Corollary 3.1].

(9) ⇒ (7): Let \( I \) be a finitely generated semi-regular ideal of \( R \) and \( M \) an \( R \)-module. Consider the exact sequence \( 0 \rightarrow M \rightarrow E \rightarrow N \rightarrow 0 \), where \( E \) is an injective module. Then \( N \) is an \( h \)-divisible module. Consequently \( \text{Ext}_R^1(I, M) \cong \text{Ext}_R^2(R/I, M) \cong \text{Ext}_R^1(R/I, N) = 0 \). So \( I \) is a projective ideal of \( R \).

(7) ⇒ (8): Let \( I \) be a finitely generated semi-regular ideal of \( R \). Then \( I \) is a projective ideal. Consequently, \( \text{pd}_R(R/I) \leq 1 \). Suppose \( 0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0 \) is a short exact sequence, where \( M \) is a semi-regular coflat module. Then we have an exact sequence \( \text{Ext}_R^1(R/I, M) \rightarrow \text{Ext}_R^1(R/I, N) \rightarrow \text{Ext}_R^2(R/I, L) \). Since \( M \) is a semi-regular coflat module, \( \text{Ext}_R^1(R/I, M) = 0 \). Since \( \text{pd}_R(R/I) \leq 1 \), we have \( \text{Ext}_R^2(R/I, L) = 0 \). Thus \( \text{Ext}_R^1(R/I, N) = 0 \). So \( N \) is a semi-regular coflat module. \( \square \)

4. Semi-regular coherent rings

Recall from [13] that a ring \( R \) is called a coherent ring if any finitely generated ideal is finitely presented. Some important rings are all coherent, such as Noetherian rings, Prüfer domains. However, strong Prüfer rings are not coherent in general. So we introduce the notion of semi-regular coherent rings and give some new characterization of strong Prüfer rings using semi-regular flat (pre)envelopes.

**Definition 4.1.** A ring \( R \) is called a semi-regular coherent ring if any finitely generated semi-regular ideal \( I \) is finitely presented.
Let $R$ be non-semi-hereditary ring with weak global dimension at most 1. Then
$R$ is strong Prüfer ring, but $R$ is not coherent (See [13, Corollary 4.2.19]). The
following result shows that any strong Prüfer ring is semi-regular coherent.

**Proposition 4.2.** Suppose $R$ is a strong Prüfer ring, then $R$ is semi-regular coherent.

**Proof.** Suppose $I$ is a finitely generated semi-regular ideal of $R$. Then, by Theorem
3.4, $I$ is a projective ideal of $R$, and thus is a finitely presented ideal of $R$. □

Some examples of non-integral domains are constructed by idealization $R(+M)$,
where $M$ is an $R$-module (See [16]). Set $R(+M)$ isomorphic to $R\oplus M$ as $R$-modules. Define

(1) $(r,m)+(s,n)=(r+s,m+n),$
(2) $(r,m)(s,n)=(rs,sm+rn).

Under these operations, $R(+M)$ is a commutative ring with the identity $(1,0)$.

**Proposition 4.3.** Let $D$ be a coherent domain and $K$ its quotient field. Set $R = D(+K)$, then $R$ is a semi-regular coherent ring. Moreover, $R$ is a coherent ring if and only if $D$ is a field.

**Proof.** Following from [2, Remark 1], $R$ is a strongly $\phi$-ring, so $\text{Nil}(R) = 0(+K)$ and any ideal of $R$ can compare with $\text{Nil}(R)$. Then, by [4, Corollary 3.4], any ideal of $R$ is of the form $I(+K)$ and $0(+L)$, where $I$ is a nonzero ideal of $D$ and $L$ is a $D$-submodule of $K$. If $I$ is a nonzero ideal of $D$, then $I(+K)$ is a semi-
regular ideal of $R$ obviously. Let $I(+K)$ be finitely generated $R$-ideal generated by
$\{(d_1,x_1),...,(d_n,x_n)\}$. Then it is easy to verify $I$ is generated by $\{d_1,...,d_n\}$. Since $D$ is a coherent ring, then there is a short exact sequence $D^m \rightarrow D^n \rightarrow I \rightarrow 0$ of $D$-modules. Since $R$ is a flat $D$-module, we obtain an exact sequence $R^m \rightarrow R^n \rightarrow I(+K) \rightarrow 0$ of $D$-modules by tensoring $R$. It is easy to verify this is also an $R$-exact sequence. So $I(+K)$ is a finitely presented $R$-ideal. Since any element in the finitely generated $R$-ideal $0(+L)$ is nilpotent, the ideal of the form $0(+L)$ is not semi-regular. Thus $R$ is a semi-regular coherent ring.

Obviously, if $D$ is a field, then $R$ is a coherent ring. Next we will show that if $D$

is not a field, then $R$ is not coherent. Since $(0,1)R$ is a finitely generated $R$-ideal.
Consider the natural short exact sequence $0 \rightarrow L \rightarrow R \rightarrow (0,1)R \rightarrow 0$, we have
$L = \text{Nil}(R) = 0(+K)$. Since $D$ is not a field, then $K$ is not finitely generated over $D$.
By [7, Lemma 2.2], $\text{Nil}(R)$ is not a finitely generated $R$-ideal. Consequently, $(0,1)R$ is not finitely presented. So $R$ is not coherent. □

**Theorem 4.4.** Let $R$ be a ring. Then the following statements are equivalent:
(1) $R$ is semi-regular coherent; 
(2) any direct product of semi-regular flat modules is semi-regular flat; 
(3) any direct product of flat modules is semi-regular flat; 
(4) any direct product of $R$ is semi-regular flat; 
(5) any direct limit of semi-regular coflat module is semi-regular coflat; 
(6) any quotient of semi-regular coflat modules is semi-regular coflat; 
(7) the class of semi-regular coflat modules is precovering; 
(8) the class of semi-regular coflat modules is covering; 
(9) $\text{Hom}_R(N,E)$ is semi-regular flat module for any semi-regular coflat module $N$ and any injective module $E$; 
(10) if $E$ is injective cogenerator, then $\text{Hom}_R(N,E)$ is semi-regular flat for any semi-regular coflat module $N$; 
(11) $\text{Hom}_R(\text{Hom}_R(M,E_1),E_2)$ is semi-regular flat for any semi-regular flat module $M$ and any injective modules $E_1$ and $E_2$; 
(12) if $E_1$ and $E_2$ are injective cogenerators, then $\text{Hom}_R(\text{Hom}_R(M,E_1),E_2)$ is semi-regular flat for any semi-regular flat module $M$.

**Proof.** (2) $\Rightarrow$ (3) $\Rightarrow$ (4): Trivial.

(1) $\Rightarrow$ (2): Let $I$ be a finitely generated semi-regular ideal of $R$ and $\{F_i\}_{i \in I}$ a family of semi-regular flat modules. Consider the following commutative diagram:

$$
\begin{array}{c}
I \otimes \prod_{i \in I} F_i \\ \phi_i \downarrow \downarrow \\
\prod_{i \in I} I \otimes F_i \cong \prod_{i \in I} IF_i.
\end{array}
$$

Since $R$ is a semi-regular coherent ring, $I$ is finitely presented ideal. Then $\phi_i$ is an isomorphism, thus the epimorphism $\sigma$ is actually an isomorphism. Consequently, $\prod_{i \in I} F_i$ is semi-regular flat.

(4) $\Rightarrow$ (1): Let $I$ be a finitely generated semi-regular ideal of $R$. Consider the following commutative diagram:

$$
\begin{array}{c}
I \otimes \prod_{i \in I} R \\ \phi_i \downarrow \downarrow \\
\prod_{i \in I} I \otimes R \cong \prod_{i \in I} I.
\end{array}
$$

Since $I$ is a finitely generated ideal, then $f$ is an isomorphism. Since $\prod_{i \in I} R$ is a semi-regular flat module, the epimorphism $\sigma : I \otimes \prod_{i \in I} R \rightarrow I \prod_{i \in I} R$ is an isomorphism. So $\phi_i$ is an isomorphism, thus $I$ is finitely presented (See [18, Theorem 2]).
(1) \Rightarrow (5): Let I be a finitely generated semi-regular ideal of R and \{M_i\}_{i \in I} a direct system of semi-regular coflat modules. Then \( \lim \text{Ext}^1_R(R/I, M_i) = 0 \). Consider the short exact sequence \( 0 \to I \to R \to R/I \to 0 \), we have the following commutative diagram with rows exact:

\[
\begin{array}{cccccc}
\text{lim} \text{Hom}_R(R, M_i) & \to & \text{lim} \text{Hom}_R(I, M_i) & \to & \text{lim} \text{Ext}^1_R(R/I, M_i) & \to & 0 \\
\varphi_R & \downarrow & \varphi_I & \downarrow & \psi_{R/1} & \\
\text{Hom}_R(R, \text{lim} M_i) & \to & \text{Hom}_R(I, \text{lim} M_i) & \to & \text{Ext}^1_R(R/I, \text{lim} M_i) & \to & 0.
\end{array}
\]

Since R is a semi-regular coherent ring, I is a finitely presented ideal, then \( \varphi_I \) is an isomorphism. Since \( \varphi_R \) is an isomorphism, then \( \psi_{R/1} \) is also an isomorphism. Consequently, \( \text{lim} M_i \) is a semi-regular coflat module.

(5) \Rightarrow (1): Let I be a finitely generated semi-regular ideal, \{M_i\}_{i \in I} a direct limit of R-modules. Suppose \( \alpha : I \to \text{lim} M_i \) is an R-homomorphism. For any \( i \in I \), \( E(M_i) \) is the injective envelope of \( M_i \), then \( E(M_i) \) is a semi-regular coflat module. By (5), \( \alpha \) can be extended to be \( \beta : R \to \text{lim} E(M_i) \). So there exists \( j \in I \) such that \( \beta \) can factor through \( R \to E(M_j) \). Since the composition \( I \to R \to E(M_j) \to E(M_j)/M_j \) becomes to be 0 in the direct limit. We can assume \( I \to R \to E(M_j) \) factors through \( M_j \). Then \( \alpha \) factor through \( M_j \). So the natural epimorphism \( \text{lim} \text{Hom}_R(I, M_i) \to \text{Hom}_R(I, \text{lim} M_i) \). So I is a finitely presented ideal.

(5) \Leftrightarrow (6) \Leftrightarrow (7) \Leftrightarrow (8): Follows from [28, Lemma 3.4].

(1) \Rightarrow (9): Let I be a finitely generated semi-regular ideal of R, E an injective module and N a semi-regular coflat module. Consider the short exact sequence \( 0 \to I \to R \to R/I \to 0 \). Then we have the following commutative diagram with rows exact (Use \(- , -\) instead of \( \text{Hom}_R(- , -)\))

\[
\begin{array}{cccccc}
0 & \to & \text{Tor}^1_R(R/I, (N, E)) & \to & I \otimes (N, E) & \to & R \otimes (N, E) & \to & R/I \otimes (N, E) & \to & 0 \\
\psi_{R/1} & \downarrow & \psi_I & \downarrow & \psi_R & \downarrow & \psi_{R/I} & \\
0 & \to & (\text{Ext}^1_R(R/I, N), E) & \to & (I, N), E) & \to & (R, N), E) & \to & (R/I, N), E) & \to & 0.
\end{array}
\]

Since R is a semi-regular coherent ring, then I is finitely presented ideal. Thus \( \psi_R \) and \( \psi_I \) are all isomorphisms, so \( \psi_{R/1} \) is also an isomorphism. Since N is a semi-regular coflat module, \( \text{Ext}^1_R(R/I, N) = 0 \). So \( \text{Tor}^1_R(R/I, \text{Hom}_R(N, E)) = 0 \), and thus \( \text{Hom}_R(N, E) \) is a semi-regular flat module.

(9) \Rightarrow (10) and (11) \Rightarrow (12): Trivial.

(9) \Leftrightarrow (11) and (10) \Leftrightarrow (12): Follow by Theorem [2.3].
Let \( I \) be a finitely generated semi-regular ideal of \( R \), \( N \) a semi-regular coflat module and \( E \) an injective cogenerator. Then we have the following commutative diagram with rows exact:

\[
\begin{array}{c}
I \otimes \text{Hom}_R(N, E) \\ f \downarrow \quad \psi_I \\
R \otimes \text{Hom}_R(N, E) \\ \psi_R \sim \\
\text{Hom}_R(\text{Hom}_R(I, N), E) \\
g \downarrow \\
\text{Hom}_R(\text{Hom}_R(R, N), E).
\end{array}
\]

Note that \( f \) is a monomorphism by (10). So \( \psi_I \) is a monomorphism. Thus, by [3, Proposition 8.14(1)], we have \( I \) is an \( \{R\} \)-Mittag-Leffler module. Since \( I \) is a finitely generated ideal, \( I \) is finitely presented by [18, Theorem 2]. □

**Lemma 4.5.** Let \( R \) be a ring. Then \( R \) is a semi-regular coherent ring if and only if the class \( \mathcal{F}_{sr} \) of all semi-regular flat modules is preenveloping.

**Proof.** Suppose \( R \) is a semi-regular coherent ring, then \( \mathcal{F}_{sr} \) is closed under direct products. Since \( \mathcal{F}_{sr} \) is closed under pure submodules, the class of semi-regular flat modules is preenveloping by [10, Corollary 6.2.2, Lemma 5.3.12]. On the other hand, Suppose \( \{F_i\}_{i \in I} \) is semi-regular flat modules. Suppose \( \prod_{i \in I} F_i \rightarrow F \) is a semi-regular flat preenvelope, then there is a factorization \( \prod_{i \in I} F_i \rightarrow F \rightarrow F_i \) for any \( i \in I \). So \( \prod_{i \in I} F_i \rightarrow F \rightarrow \prod_{i \in I} F_i \) is an identity map. Thus \( \prod_{i \in I} F_i \) is a direct summand of \( F \). Hence \( \prod_{i \in I} F_i \) is semi-regular flat. Consequently, \( R \) is semi-regular coherent ring by Theorem 4.4. □

**Corollary 4.6.** Suppose ring \( R \) is a semi-regular coherent ring and the class \( \mathcal{F}_{sr} \) of all semi-regular flat modules is closed under inverse limits, then \( \mathcal{F}_{sr} \) is enveloping.

**Proof.** By Lemma 4.5, the class \( \mathcal{F}_{sr} \) is preenveloping. Then \( \mathcal{F}_{sr} \) is enveloping by [10, Corollary 6.3.5]. □

**Theorem 4.7.** Let \( R \) be a ring. Then the following statements are equivalent:

1. \( R \) is a strong Prüfer ring;
2. any \( R \)-module has an epimorphic semi-regular flat envelope;
3. any \( R \)-module has an epimorphic semi-regular flat preenvelope.

**Proof.** (2) \( \Rightarrow \) (3): Trivial.

(3) \( \Rightarrow \) (1): Let \( F \) be a semi-regular flat module, \( i : K \rightarrow F \) an embedding map and \( f : K \rightarrow F' \) a semi-regular flat preenvelope. Then there is a homomorphism \( g : F' \rightarrow F \) such that \( i = gf \). Thus \( f \) is a monomorphism. So \( K \cong F' \) is a semi-regular flat module. Hence \( R \) is a strong Prüfer ring by Theorem 3.4.
Suppose $R$ is a strong Prüfer ring. Then $R$ is a semi-regular coherent ring by Proposition 4.2. Hence the class $\mathcal{F}_{sr}$ of semi-regular flat modules is preenveloping by Lemma 4.5. So $\mathcal{F}_{sr}$ is closed under submodules by Theorem 3.4. Suppose $\{F_i|i\in I\}$ is a family of semi-regular flat modules. Since $R$ is a semi-regular coherent ring, $\prod_{i\in I} F_i$ is semi-regular flat by Proposition 4.2. Since any inverse limit is a submodule of direct products, semi-regular flat module is closed under inverse limit. So $\mathcal{F}_{sr}$ is enveloping by Corollary 4.6.

Claim that the $\mathcal{F}_{sr}$-envelope of any $R$-module is an epimorphism. Indeed, let $f: M \to F$ is an $\mathcal{F}_{sr}$-envelope. Consider the standard factorization of $f$ as $f = h \circ g$, where $g: M \to \text{im} f$ is an epimorphism, $f: \text{im} f \to F$ is a monomorphism. We will show $g$ is an $\mathcal{F}_{sr}$-envelope of $M$, so $f$ is an epimorphism. First, we will show $g$ is an $\mathcal{F}_{sr}$-preenvelope of $M$. Since $\text{im} f \in \mathcal{F}_{sr}$ and for any $f': M \to F'$ with $F' \in \mathcal{F}_{sr}$, there exists a homomorphism $l: F \to F'$ such that $l \circ f = f'$, so $l \circ h \circ g = f'$. Next, we will show $g$ is an $\mathcal{F}_{sr}$-envelope of $M$. Consider the following commutative diagram:

$$
\begin{array}{ccc}
M & \xrightarrow{g} & \text{im} f \\
\downarrow & & \downarrow h \\
M & \xrightarrow{a} & \text{im} f \\
\end{array}
\begin{array}{ccc}
F & \xrightarrow{h} & F \\
\downarrow & & \downarrow b \\
F & \xrightarrow{b} & F \\
\end{array}
$$

Suppose $a: \text{im} f \to \text{im} f$ is a homomorphism satisfying $g = a \circ g$. Then $a$ is an epimorphism. Since $f = h \circ g$ is an $\mathcal{F}_{sr}$-envelope, there is an isomorphism $b: F \to F$ such that $b \circ f = b \circ h \circ g = h \circ a \circ g = h \circ g = f$. So $h \circ a = b \circ h$ as $g$ is an epimorphism. Thus $a$ is a monomorphism, and therefore $a$ is actually an isomorphism. Hence $f$ is an epimorphism. □

References

[1] D. D. Anderson, D. F. Anderson, R. Markanda, The rings $R(X)$and $R(X)$. J. Algebra, 1985, 95:96-115.
[2] D. F. Anderson, A. Badawi, On $\phi$-Dedekind Rings and $\phi$-Krull Rings. Houston J. Math., 2005, 31:1007-1022.
[3] D. D. Anderson, J. Pascual, Ideals in commutative rings, sublattices of regular ideals, and Prüfer rings. J. Algebra, 1987, 111:404-426.
[4] D. D. Anderson, M. Winders, Idealization of a Module. J. Commut. Algebra, 2009, 1(1):3-56.
[5] L. Angeleri Hügel, D. Herbera, Mittag-Leffler conditions on modules. Indiana Univ. Math. J., 2008, 57:2459-2517.
[6] M. Auslander, D. Buchsbaum, Homological dimension in Noetherian rings, II, Trans. Amer. Math. Soc., 1958, 88: 194-206.
[7] A. Badawi, On nonnil-Noetherian rings. Comm. Algebra, 2003, 31: 1669-1677.
[8] L. Bican, R. E. Bahir, E. E. Enochs, *All modules have flat covers*. Bull. Lond. Math. Soc., 2001, 33(4):385-390.
[9] H. S. Butts, W. Smith, Prüfer *rings*. Math. Z., 1967, 95: 196-211.
[10] E. E. Enochs, O. M. G. Jenda *Relative homological algebra*. DeGruyter Expositions in Mathematics, vol.30, Walter de Gruyter, Berlin, 2000.
[11] M. Fontana, J. Huckaba, I. Papick, Prüfer *Domains*. New York: Marcel Dekker Inc., 1997.
[12] L. Fuchs, L. Salce, *Modules over Non-Noetherian Domains*. Providence: AMS, 2001.
[13] S. Glaz, *Commutative Coherent Rings*. Lecture Notes in Mathematics, vol. 1371, Berlin: Springer-Verlag, 1989.
[14] M. Griffin, Prüfer *rings with zero-divisors*. J. Reine Angew. Math., 1969, 239-240:55-67.
[15] H. Holm, P. Jørgensen, *Covers, precovers, and purity*. Illinois J. Math., 2008, 52:691-703.
[16] J. A. Huckaba, *Commutative Rings with Zero Divisors*. Monographs and Textbooks in Pure and Applied Mathematics, 1998, 117, Marcel Dekker, Inc., New York.
[17] W. Krull, Zum Dimensionsbegriff der Idealtheorie, (Beiträge zur Arithmetik kommutativer Integritätsbereiche, III), Math. Z., 1937, 42: 745-766.
[18] H. Lenzing *präsentierbare Moduln*. Arch. Math. (Basel), 1969, 20(3):262-266.
[19] T. Lucas, *Strong Prüfer rings and the ring of finite fractions*. J. Pure Appl. Algebra, 1993, 84:59-71.
[20] H. Matsumura, *Commutative ring theory*. Cambridge studies in advanced mathematics 8, Cambridge University Press, 1989.
[21] H. Prüfer, *Untersuchungen über Teilbarkeitseigenschaften in Körpren*. J. Reine Angew. Math., 1932, 168:1-36. (in German).
[22] W. Vasconcelos, *On finitely generated flat modules*. Trans. Amer. Math. Soc., 1969, 138:505-512.
[23] F. G. Wang, H. Kim, *Foundations of Commutative Rings and Their Modules*. Singapore: Springer, 2016.
[24] F. G. Wang, L. Qiao, D. C. Zhou, *A Homological Characterization of Strong Prüfer rings*. Acta Math. Sinica (Chin. Ser.), 2021, 64(2):311-316. (in Chinese).
[25] F. G. Wang, D. C. Zhou, D. Chen, *Module-theoretic characterizations of the ring of finite fractions of a commutative ring*. J. Commut. Algebra, to appear. https://projecteuclid.org/euclid.jca/1589335712.
[26] F. G. Wang, D. C. Zhou, H. Kim, T. Xiong, X W. Sun *Every Prüfer ring does not have small finitistic dimension at most one*. Commun. Algebra, 2020, 48(12):5311-5320.
[27] X. L. Xiao, F. G. Wang, S. Y. Lin, *The Coherent Study Determined by Regular Ideals*. J. Sichuan Normal Univ., to appear.
[28] X. L. Zhang, F. G. Wang, W. Qi, *On characterizations of w-coherent rings*. Commun. Algebra, 2020, 48(11):4681-4697.
[29] X. L. Zhang, F. G. Wang, *The small finitistic dimensions over commutative rings*. arxiv.org/abs/2103.08807.