1. Introduction

Let $f$ be a modular eigenform of weight at least two and let $F$ be a finite abelian extension of $\mathbb{Q}$. Fix an odd prime $p$ at which $f$ is ordinary in the sense that the $p$th Fourier coefficient of $f$ is not divisible by $p$. In Iwasawa theory, one associates two objects to $f$ over the cyclotomic $\mathbb{Z}_p$-extension $F_{\infty}$ of $F$: a Selmer group $\text{Sel}(F_{\infty}, A_f)$ (where $A_f$ denotes the divisible version of the two-dimensional Galois representation attached to $f$) and a $p$-adic $L$-function $L_p(F_{\infty}, f)$. In this paper we prove a formula, generalizing work of Kida and Hachimori–Matsuno, relating the Iwasawa invariants of these objects over $F$ with their Iwasawa invariants over $p$-extensions of $F$.

For Selmer groups our results are significantly more general. Let $T$ be a lattice in a nearly ordinary $p$-adic Galois representation $V$; set $A = V/T$. When $\text{Sel}(F_{\infty}, A)$ is $\Lambda$-cotorsion with $\mu_{\text{alg}}(F_{\infty}, A) = 0$, then $\text{Sel}(F'_{\infty}, A)$ is $\Lambda$-cotorsion with $\mu_{\text{alg}}(F'_{\infty}, A) = 0$. Moreover, in this case

$$\lambda_{\text{alg}}(F'_{\infty}, A) = \left[ F'_{\infty} : F_{\infty} \right] \cdot \lambda_{\text{alg}}(F_{\infty}, A) + \sum_{w'} m(F'_{\infty, w'}/F_{\infty, w}, V)$$

where the sum extends over places $w'$ of $F'_{\infty}$ which are ramified in $F'_{\infty}/F_{\infty}$.

If $V$ is associated to a cuspform $f$ and $F'$ is an abelian extension of $\mathbb{Q}$, then the same results hold for the analytic Iwasawa invariants of $f$.

Here $m(F'_{\infty, w'}/F_{\infty, w}, V)$ is a certain difference of local multiplicities defined in Section 2.1. In the case of Galois representations associated to Hilbert modular forms, these local factors can be made quite explicit; see Section 4.1 for details.

It follows from Theorem 1 and work of Kato that if the $p$-adic main conjecture holds for a modular form $f$ over $\mathbb{Q}$, then it holds for $f$ over all abelian $p$-extensions of $\mathbb{Q}$; see Section 4.2 for details.

These Riemann-Hurwitz type formulas were first discovered by Kida [5] in the context of $\lambda$-invariants of CM fields. More precisely, when $F'/F$ is a $p$-extension of CM fields and $\mu^-(F_{\infty}/F) = 0$, Kida gave a precise formula for $\lambda^-(F'_{\infty}/F')$ in terms of $\lambda^-(F_{\infty}/F)$ and local data involving the primes that ramify in $F'/F$. (See also [2] for a representation theoretic interpretation of Kida’s result.) This formula was generalized to Selmer groups of elliptic curves at ordinary primes by Wingberg [12] in the CM case and Hachimori–Matsuno [3] in the general case. The analytic
analogue was first established for ideal class groups by Sinnott \[10]\ and for elliptic curves by Matsuno \[7]\.

Our proof is most closely related to the arguments in \[10]\ and \[7]\ where congruences implicitly played a large role in their study of analytic $\lambda$-invariants. In this paper, we make the role of congruences more explicit and apply these methods to study both algebraic and analytic $\lambda$-invariants.

As is usual, we first reduce to the case where $F'/F$ is abelian. (Some care is required to show that our local factors are well behaved in towers of fields; this is discussed in Section 2.1.) In this case, the $\lambda$-invariant of $V$ over $F'$ can be expressed as the sum of the $\lambda$-invariants of twists of $V$ by characters of $\text{Gal}(F'/F)$. The key observation (already visible in both \[10]\ and \[7]\) is that since $\text{Gal}(F'/F)$ is a $p$-group, all of its characters are trivial modulo a prime over $p$ and, thus, the twisted Galois representations are all congruent to $V$ modulo a prime over $p$. The algebraic case of Theorem 1 then follows from the results of \[11]\ which gives a precise local formula for the difference between $\lambda$-invariants of congruent Galois representations.

The analytic case is handled similarly using the results of \[1]\.

The basic principle behind this argument is that a formula relating the Iwasawa invariants of congruent Galois representations should imply a transition formula in $p$-extensions. As an example of this, in Section 4.3, we use results of \[2]\ to prove a Kida formula for the Iwasawa invariants (in the sense of \[8, 6, 9]\) of weight 2 modular forms at supersingular primes.

2. Algebraic invariants

2.1. Local preliminaries. We begin by studying the local terms that appear in our results. Fix distinct primes $\ell$ and $p$ and let $L$ denote a finite extension of the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}_\ell$. Fix a field $K$ of characteristic zero and a finite-dimensional $K$-vector space $V$ endowed with a continuous $K$-linear action of the absolute Galois group $\text{Gal}(L)$ of $L$. Set

$$m_L(V) := \dim_K (V_L)^{G_L},$$

the multiplicity of the trivial representation in the $I_L$-coinvariants of $V$. Note that this multiplicity is invariant under extension of scalars, so that we can enlarge $K$ as necessary.

Let $L'$ be a finite Galois $p$-extension of $L$. Note that $L'$ must be cyclic and totally ramified since $L$ contains the $\mathbb{Z}_p$-extension of $\mathbb{Q}_\ell$. Let $G$ denote the Galois group of $L'/L$. Assuming that $K$ contains all $[L':L]$-power roots of unity, for a character $\chi: G \to K^\times$ of $G$, we set $V_\chi = V \otimes_K K(\chi)$ with $K(\chi)$ a one-dimensional $K$-vector space on which $G$ acts via $\chi$. We define

$$m(L'/L, V) := \sum_{\chi \in G^\vee} m_L(V) - m_L(V_\chi)$$

where $G^\vee$ denotes the $K$-dual of $G$.

The next result shows how these invariants behave in towers of fields.

**Lemma 2.1.** Let $L''$ be a finite Galois $p$-extension of $L$ and let $L'$ be a Galois extension of $L$ contained in $L'$. Assume that $K$ contains all $[L':L]$-power roots of unity. Then

$$m(L''/L, V) = [L'' : L'] \cdot m(L'/L, V) + m(L''/L', V).$$
Proof. Set $G = \text{Gal}(L''/L)$ and $H = \text{Gal}(L''/L')$. Consider the Galois group $G_{L''}/I_{L''}$ over $L$ of the maximal unramified extension of $L''$. It sits in an exact sequence

$$0 \rightarrow G_{L''}/I_{L''} \rightarrow G_{L}/I_{L''} \rightarrow G \rightarrow 0$$

which is in fact split since the maximal unramified extensions of both $L$ and $L''$ are obtained by adjoining all prime-to-$p$ roots of unity.

Fix a character $\chi \in G$. We compute

$$m_L(V_\chi) = \dim_K ((V_\chi)_{I_L})^{G_L}$$

$$= \dim_K \left( \left( ((V_\chi)_{I_{L''}})^{G_{L''}} \right)^G \right)$$

$$= \dim_K \left( \left( ((V_\chi)_{I_{L''}})^{G_{L''}} \right)^G \right) \text{ since } 1 \text{ is split}$$

$$= \dim_K \left( \left( ((V_\chi)_{I_{L''}})^{G_{L''}} \right)^G \right) \text{ since } G \text{ is finite cyclic}$$

$$= \dim_K ((V_\chi)_{I_{L''}})^{G_{L''}} \otimes \chi)^G \text{ since } \chi \text{ is trivial on } G_{L''}.$$ 

The lemma thus follows from the following purely group-theoretical statement applied with $W = (V_{I_{L''}})^{G_{L''}}$: for a finite dimensional representation $W$ of a finite abelian group $G$ over a field of characteristic zero containing $\mu_{\#G}$, we have

$$\sum_{\chi \in G} \langle W, 1 \rangle_G - \langle W, \chi \rangle_G = \#H \cdot \sum_{\chi \in (G/H)'} \langle W, 1 \rangle_G - \langle W, \chi \rangle_G$$

for any subgroup $H$ of $G$; here $\langle W, \chi \rangle_G$ (resp. $\langle W, \chi \rangle_H$) is the multiplicity of the character $\chi$ in $W$ regarded as a representation of $G$ (resp. $H$). To prove this, we compute

$$\sum_{\chi \in G} \langle W, 1 \rangle_G - \langle W, \chi \rangle_G$$

$$= \#G \cdot \langle W, 1 \rangle_G - \langle W, \text{Ind}_1^G 1 \rangle_G$$

$$= \#G \cdot \langle W, 1 \rangle_G - \#H \cdot \langle W, \text{Ind}_1^G 1 \rangle_{G/H} + \#H \cdot \langle W, \text{Ind}_1^G 1 \rangle_{G/H} - \langle W, \text{Ind}_1^G 1 \rangle_G$$

$$= \#H \cdot \sum_{\chi \in (G/H)'} \langle W, 1 \rangle_G - \langle W, \chi \rangle_G + \sum_{\chi H \cap G' \neq \emptyset} \left( \langle W, \text{Ind}_1^G 1 \rangle_{G/H} - \langle W, \text{Ind}_1^G 1 \rangle_{G/H} \right)$$

$$= \#H \cdot \sum_{\chi \in (G/H)'} \langle W, 1 \rangle_G - \langle W, \chi \rangle_G + \sum_{\chi H \cap G' \neq \emptyset} \langle W, 1 \rangle_H - \langle W, \chi \rangle_H$$

by Frobenius reciprocity. \qed

2.2. Global preliminaries. Fix a number field $F$; for simplicity we assume that $F$ is either totally real or totally imaginary. Fix also an odd prime $p$ and a finite extension $K$ of $Q_p$; we write $O$ for the ring of integers of $K$, $\pi$ for a fixed choice of uniformizer of $O$, and $k = O/\pi$ for the residue field of $O$.

Let $T$ be a nearly ordinary Galois representation over $F$ with coefficients in $O$; that is, $T$ is a free $O$-module of some rank $n$ endowed with an $O$-linear action of
the absolute Galois group $G_F$, together with a choice for each place $v$ of $F$ dividing $p$ of a complete flag

$$0 = T^0_v \subseteq T^1_v \subseteq \cdots \subseteq T^n_v = T$$

stable under the action of the decomposition group $G_v \subseteq G_F$ of $v$. We make the following assumptions on $T$:

1. For each place $v$ dividing $p$ we have

$$(T^i_v/T^{i-1}_v) \otimes k \not\cong (T^j_v/T^{j-1}_v) \otimes k$$

as $k[G_v]$-modules for all $i \neq j$;

2. If $F$ is totally real, then $\text{rank } T^\infty_v = 1$ is independent of the archimedean place $v$ (here $c_v$ is a complex conjugation at $v$);

3. If $F$ is totally imaginary, then $n$ is even.

**Remark 2.2.** The conditions above are significantly more restrictive than are actually required to apply the results of [11]. As our main interest is in abelian (and thus necessarily Galois) extensions of $\mathbb{Q}$, we have chosen to include the assumptions (2) and (3) to simplify the exposition. The assumption (1) is also stronger than necessary: all that is actually needed is that the centralizer of $T \otimes k$ consists entirely of scalars and that $\mathfrak{g}_v/b_v$ has trivial adjoint $G_v$-invariants for all places $v$ dividing $p$; here $\mathfrak{g}_v$ denotes the $p$-adic Lie algebra of $\text{GL}_n$ and $b_v$ denotes the $p$-adic Lie algebra of the Borel subgroup associated to the complete flag at $v$. In particular, when $T$ has rank 2, we may still allow the case that $T \otimes k$ has the form

$$\begin{pmatrix} \chi & * \\ 0 & \chi \end{pmatrix}$$

so long as * is non-trivial. (Equivalently, if $T$ is associated to a modular form $f$, the required assumption is that $f$ is $p$-distinguished.)

Set $A = T \otimes K/O$; it is a cofree $O$-module of corank $n$ with an $O$-linear action of $G_F$. Let $c$ equal the rank of $T^\infty_v = 1$ (resp. $n/2$) if $F$ is totally real (resp. totally imaginary) and set

$$A^c := \text{im}(T^c_v \hookrightarrow T \twoheadrightarrow A).$$

We define the Selmer group of $A$ over the cyclotomic $\mathbb{Z}_p$-extension $F_\infty$ of $F$ by

$$\text{Sel}(F_\infty, A) = \ker \left( H^1(F_\infty, A) \rightarrow \left( \bigoplus_{w \nmid p} H^1(F_\infty, w, A) \right) \times \left( \bigoplus_{w \mid p} H^1(F_\infty, w, A/A^c_w) \right) \right).$$

The Selmer group $\text{Sel}(F_\infty, A)$ is naturally a module for the Iwasawa algebra $\Lambda_0 := O[[\text{Gal}(F_\infty/F)]]$. If $\text{Sel}(F_\infty, A)$ is $\Lambda_0$-cotorsion (that is, if the dual of $\text{Sel}(F_\infty, A)$ is a torsion $\Lambda_0$-module), then we write $\mu_{alg}(F_\infty, A)$ and $\lambda_{alg}(F_\infty, A)$ for its Iwasawa invariants; in particular, $\mu_{alg}(F_\infty, A) = 0$ if and only if $\text{Sel}(F_\infty, A)$ is a cofinitely generated $O$-module, while $\lambda_{alg}(F_\infty, A)$ is the $O$-corank of $\text{Sel}(F_\infty, A)$.

**Remark 2.3.** In the case that $T$ is in fact an *ordinary* Galois representation (meaning that the action of inertia on each $T^i_v/T^{i-1}_v$ is by an integer power $e_i$ (independent of $v$) of the cyclotomic character such that $e_1 > e_2 > \ldots > e_n$), then our Selmer group $\text{Sel}(F_\infty, A)$ is simply the Selmer group in the sense of Greenberg of a twist of $A$; see [11] Section 1.3] for details.
2.3. Extensions. Let $F'$ be a finite Galois extension of $F$ with degree equal to a power of $p$. We write $F'_\infty$ for the cyclotomic $\mathbb{Z}_p$-extension of $F'$ and set $G = \text{Gal}(F'_\infty/F_\infty)$. Note that $T$ satisfies hypotheses (1)–(3) over $F'$ as well, so that we may define $\text{Sel}(F'_\infty, A)$ analogously to $\text{Sel}(F_\infty, A)$. (For (1) this follows from the fact that $G_v$ acts on $(T_v^1/T_v^{v-1}) \otimes k$ by a character of prime-to-$p$ order; for (2) and (3) it follows from the fact that $p$ is assumed to be odd.)

Lemma 2.4. The restriction map

\[ \text{Sel}(F_\infty, A) \to \text{Sel}(F'_\infty, A)^G \]

has finite kernel and cokernel.

Proof. This is straightforward from the definitions and the fact that $G$ is finite and $A$ is cofinitely generated; see [3, Corollary 3.3] for details. \[ \square \]

We can use Lemma 2.4 to relate the $\mu$-invariants of $A$ over $F_\infty$ and $F'_\infty$.

Corollary 2.5. If $\text{Sel}(F_\infty, A)$ is $\Lambda$-cotorsion with $\mu^\text{alg}(F_\infty, A) = 0$, then $\text{Sel}(F'_\infty, A)$ is $\Lambda$-cotorsion with $\mu^\text{alg}(F'_\infty, A) = 0$.

Proof. This is a straightforward argument using Lemma 2.4 and Nakayama’s lemma for compact local rings; see [3] Corollary 3.4 for details. \[ \square \]

Fix a finite extension $K'$ of $K$ containing all $[F' : F]$-power roots of unity. Consider a character $\chi : G \to O'^\times$ taking values in the ring of integers $O'$ of $K'$; note that $\chi$ is necessarily even since $[F' : F]$ is odd. We set

\[ A_\chi = A \otimes \mathcal{O}'(\chi) \]

where $\mathcal{O}'(\chi)$ is a free $\mathcal{O}'$-module of rank one with $G_{F_\infty}$-action given by $\chi$. If we give $A_\chi$ the induced complete flags at places dividing $p$, then $A_\chi$ satisfies hypotheses (1)–(3) and we have

\[ A_{\chi,v}^{ct} = A_v^{ct} \otimes \mathcal{O}'(\chi) \subseteq A_\chi \]

for each place $v$ dividing $p$. We write $\text{Sel}(F_\infty, A_\chi)$ for the corresponding Selmer group, regarded as a $\Lambda_{\mathcal{O}'}$-module; in particular, by $\lambda^\text{alg}(F_\infty, A_\chi)$ we mean the $\mathcal{O}'$-corank of $\text{Sel}(F_\infty, A_\chi)$, rather than the $\mathcal{O}$-corank. We write $G'$ for the set of all characters $\chi : G \to O'^\times$.

Note that as $O'[[G_{F'}]]$-modules we have

\[ A \otimes \mathcal{O}' \cong A_\chi \]

from which it follows easily that

\[ (\text{Sel}(F'_\infty, A) \otimes \mathcal{O}'(\chi))^G = \text{Sel}(F'_\infty, A_\chi)^G \]

Moreover, in the case that $G$ is abelian,

\[ \text{Sel}(F'_\infty, A) \otimes \mathcal{O}' \cong \bigoplus_{\chi \in G'} (\text{Sel}(F'_\infty, A) \otimes \mathcal{O}'(\chi))^G. \]

Applying Lemma 2.4 to each twist $A_\chi$, we obtain the following decomposition of $\text{Sel}(F'_\infty, A)$.

Corollary 2.6. Assume that $G$ is an abelian group. Then the map

\[ \bigoplus_{\chi \in G'} \text{Sel}(F_\infty, A_\chi) \to \text{Sel}(F'_\infty, A) \otimes \mathcal{O}' \]

obtained from the maps [3], [3] and [4] has finite kernel and cokernel.
As an immediate corollary, we have the following.

**Corollary 2.7.** If Sel($F_\infty$, $A$) is $\Lambda$-cotorsion with $\mu^{\text{alg}}(F_\infty, A) = 0$, then each group Sel($F_\infty$, $A_\chi$) is $\Lambda\text{O}_v$-cotorsion with $\mu^{\text{alg}}(F_\infty, A_\chi) = 0$. Moreover, if $G$ is abelian, then

$$\lambda^{\text{alg}}(F_\infty', A) = \sum_{\chi \in G^\vee} \lambda^{\text{alg}}(F_\infty, A_\chi).$$

2.4. **Algebraic transition formula.** We continue with the notation of the previous section. We write $R(F''_\infty/F_\infty)$ for the set of prime-to-$p$ places of $F''_\infty$ which are ramified in $F'_\infty/F_\infty$. For a place $w' \in R(F'_\infty/F_\infty)$, we write $w$ for its restriction to $F_\infty$.

**Theorem 2.8.** Let $F'/F$ be a finite Galois $p$-extension with Galois group $G$ which is unramified at all places dividing $p$. Let $T$ be a nearly ordinary Galois representation over $F$ with coefficients in $O$ satisfying (1)–(3). Set $A = T \otimes K/O$ and assume that:

1. $H^0(F, A[\pi]) = H^0(F, \text{Hom}(A[\pi], \mu_p)) = 0$;
2. $H^0(I_v, A/\mathcal{A}_v^\vee)$ is $O$-divisible for all $v$ dividing $p$.

If Sel($F_\infty$, $A$) is $\Lambda$-cotorsion with $\mu^{\text{alg}}(F_\infty, A) = 0$, then Sel($F'_\infty$, $A$) is $\Lambda$-cotorsion with $\mu^{\text{alg}}(F'_\infty, A) = 0$. Moreover, in this case,

$$\lambda^{\text{alg}}(F'_\infty, A) = [F'_\infty : F_\infty] \cdot \lambda^{\text{alg}}(F_\infty, A) + \sum_{w' \in R(F'_\infty/F_\infty)} m(F''_{\infty, w'}/F_{\infty, w}, V)$$

with $V = T \otimes K$ and $m(F''_{\infty, w'}/F_{\infty, w}, V)$ as in Section 2.7.

Note that $m(F''_{\infty, w'}/F_{\infty, w}, V)$ in fact depends only on $w$ and not on $w'$. The hypotheses (4) and (5) are needed to apply the results of [11]; they will not otherwise appear in the proof below. We note that the assumption that $F'/F$ is unramified at $p$ is primarily needed to assure that the condition (5) holds for twists of $A$ as well.

Since $p$-groups are solvable and the only simple $p$-group is cyclic, the next lemma shows that it suffices to consider the case of $Z/pZ$-extensions.

**Lemma 2.9.** Let $F''/F$ be a Galois $p$-extension of number fields and let $F'$ be an intermediate extension which is Galois over $F$. Let $T$ be as above. If Theorem 2.8 holds for $T$ with respect to any two of the three field extensions $F''/F'$, $F'/F$ and $F''/F$, then it holds for $T$ with respect to the third extension.

**Proof.** This is clear from Corollary 2.7 except for the $\lambda$-invariant formula. Substituting the formula for $\lambda(F'_\infty, A)$ in terms of $\lambda(F_\infty, A)$ into the formula for $\lambda(F''_{\infty}, A)$ in terms of $\lambda(F'_\infty, A)$, one finds that it suffices to show that

$$\sum_{w'' \in R(F''_{\infty}/F_\infty)} m(F''_{\infty, w''}/F_{\infty, w}, V) = [F''_{\infty} : F_\infty] \cdot \sum_{w' \in R(F'_\infty/F_\infty)} m(F'_{\infty, w'}/F_{\infty, w}, V) + \sum_{w'' \in R(F''_{\infty}/F'_\infty)} m(F''_{\infty, w''}/F'_{\infty, w''}, V).$$

This formula follows upon summing the formula of Lemma 2.4 over all $w'' \in R(F''_{\infty}/F_\infty)$ and using the two facts:
• \([F''_{\infty} : F_{\infty}'] / [F''_{\infty,w'} : F_{\infty,w'}]\) equals the number of places of \(F''_{\infty}\) lying over \(w'\) (since the residue field of \(F_{\infty,w}\) has no \(p\)-extensions);

• \(m(F''_{\infty,w''} / F'_{\infty,w'}, V) = 0\) for any \(w'' \in R(F''_{\infty} / F_{\infty}) - R(F''_{\infty} / F'_{\infty})\).

\[\Box\]

Proof of Theorem 3.1. By Lemma 3.3 and the preceding remark, we may assume that \(F'_{\infty} / F_{\infty}\) is a cyclic extension of degree \(p\). The fact that \(\text{Sel}(F'_{\infty}, A)\) is cotorsion with trivial \(\mu\)-invariant is simply Corollary 2.4. Furthermore, by Corollary 2.4, we have

\[\lambda_{\text{alg}}(F'_{\infty}, A) = \sum_{\chi \in G^w} \lambda_{\text{alg}}(F_{\infty}, A_{\chi}).\]

For \(\chi \in G^w\), note that \(\chi\) is trivial modulo a uniformizer \(\pi'\) of \(O'\) as it takes values in \(\mu_p\). In particular, the residual representations \(A_{\chi}[\pi']\) and \(A[\pi]\) are isomorphic. Under the hypotheses (1)–(5), the result [11, Theorem 1] gives a precise formula for the relation between \(\lambda\)-invariants of congruent Galois representations. In the present case it takes the form:

\[\lambda_{\text{alg}}(F_{\infty}, A_{\chi}) = \lambda_{\text{alg}}(F_{\infty}, A) + \sum_{w' | p} (m_{F_{\infty,w}}(V \otimes \omega^{-1}) - m_{F_{\infty,w}}(V_{\chi} \otimes \omega^{-1}))\]

where the sum is over all prime-to-\(p\) places \(w'\) of \(F'_{\infty}\), \(w\) denotes the place of \(F_{\infty}\) lying under \(w'\) and \(\omega\) is the mod \(p\) cyclotomic character. The only non-zero terms in this sum are those for which \(w'\) is ramified in \(F'_{\infty} / F_{\infty}\). For any such \(w'\), we have \(\mu_p \subseteq F_{\infty,w}\) by local class field theory so that \(\omega\) is in fact trivial at \(w\); thus

\[\lambda_{\text{alg}}(F_{\infty}, A_{\chi}) = \lambda_{\text{alg}}(F_{\infty}, A) + \sum_{w' \in R(F'_{\infty} / F_{\infty})} (m_{F_{\infty,w}}(V) - m_{F_{\infty,w}}(V_{\chi})).\]

Summing over all \(\chi \in G^w\) then yields

\[\lambda_{\text{alg}}(F'_{\infty}, A) = [F'_{\infty} : F_{\infty}] \cdot \lambda_{\text{alg}}(F_{\infty}, A) + \sum_{w' \in R(F'_{\infty} / F_{\infty})} m(F'_{\infty,w'} / F_{\infty,w}, V)\]

which completes the proof. \[\Box\]

3. Analytic invariants

3.1. Definitions. Let \(f = \sum a_n q^n\) be a modular eigenform of weight \(k \geq 2\), level \(N\) and character \(\varepsilon\). Let \(K\) denote the finite extension of \(\mathbb{Q}_p\) generated by the Fourier coefficients of \(f\) (under some fixed embedding \(\mathbb{Q} \hookrightarrow \mathbb{Q}_p\)), let \(O\) denote the ring of integers of \(K\) and let \(k\) denote the residue field of \(O\). Let \(V_f\) denote a two-dimensional \(K\)-vector space with Galois action associated to \(f\) in the usual way; thus the characteristic polynomial of a Frobenius element at a prime \(\ell \nmid Np\) is

\[x^2 - a_{\ell} x + \ell^{k-1} \varepsilon(\ell)\]

Fix a Galois stable \(O\)-lattice \(T_f\) in \(V_f\). We assume that \(T_f \otimes k\) is an irreducible Galois representation; in this case \(T_f\) is uniquely determined up to scaling. Set \(A_f = T_f \otimes K / O\).

Assuming that \(f\) is \(p\)-ordinary (in the sense that \(a_p\) is relatively prime to \(p\)) and fixing a canonical period for \(f\), one can associate to \(f\) a \(p\)-adic \(L\)-function \(L_p(Q_{\infty} / Q, f)\) which lies in \(\Lambda_O\). This is well-defined up to a \(p\)-adic unit (depending upon the choice of a canonical period) and thus has well-defined Iwasawa invariants.
Let $F/Q$ be a finite abelian extension and let $F_\infty$ denote the cyclotomic $\mathbb{Z}_p$-extension of $F$. For a character $\chi$ of $\text{Gal}(F/Q)$, we denote by $f_\chi$ the modular eigenform $\sum a_n(\chi(n))q^n$ obtained from $f$ by twisting by $\chi$ (viewed as a Dirichlet character). If $f$ is $p$-ordinary and $F/Q$ is unramified at $p$, then $f_\chi$ is again $p$-ordinary and we define

$$L_p(F_\infty/F, f) = \prod_{\chi \in \text{Gal}(F_\infty/Q)} L_p(Q_\infty/Q, f_\chi).$$

If $F/Q$ is ramified at $p$, it is still possible to define $L_p(F_\infty/F, f)$; see [23, pg. 5], for example.

If $F_1$ and $F_2$ are two distinct number fields whose cyclotomic $\mathbb{Z}_p$-extensions agree, the corresponding $p$-adic $L$-functions of $f$ over $F_1$ and $F_2$ need not agree. However, it is easy to check that the Iwasawa invariants of these two power series are equal. We thus denote the Iwasawa invariants of $L_p(F_\infty/F, f)$ simply by $\mu^\text{an}(F_\infty, f)$ and $\lambda^\text{an}(F_\infty, f)$.

### 3.2. Analytic transition formula

Let $F/Q$ be a finite abelian $p$-extension of $Q$ and let $F'$ be a finite $p$-extension of $F$ such that $F'/Q$ is abelian. As always, let $F_\infty$ and $F'_\infty$ denote the cyclotomic $\mathbb{Z}_p$-extensions of $F$ and $F'$. As before, we write $R(F'_\infty/F_\infty)$ for the set of prime-to-$p$ places of $F'_\infty$ which are ramified in $F'_\infty/F_\infty$.

**Theorem 3.1.** Let $f$ be a $p$-ordinary modular form such that $T_f \otimes k$ is irreducible and $p$-distinguished. If $\mu^\text{an}(F_\infty, f) = 0$, then $\mu^\text{an}(F'_\infty, f) = 0$. Moreover, if this is the case, then

$$\lambda^\text{an}(F'_\infty, f) = [F'_\infty : F_\infty] \cdot \lambda^\text{an}(F_\infty, f) + \sum_{w' \in R(F'_\infty/F_\infty)} m(F'_\infty, w'/F'_\infty, w, V_f).$$

**Proof.** By Lemma 2.9, we may assume $[F : Q]$ is prime-to-$p$. Indeed, let $F_0$ be the maximal subfield of $F$ of prime-to-$p$ degree over $Q$. By Lemma 2.9, knowledge of the theorem for the two extensions $F'/F_0$ and $F/F_0$ would then imply it for $F'/F$ as well.

We may further assume that $F$ and $F'$ are unramified at $p$. Indeed, if $F^\text{ur}$ (resp. $F'^\text{ur}$) denotes the maximal subfield of $F_\infty$ (resp. $F'_\infty$) unramified at $p$, then $F^\text{ur} \subseteq F'^\text{ur}$ and the cyclotomic $\mathbb{Z}_p$-extension of $F^\text{ur}$ (resp. $F'^\text{ur}$) is $F_\infty$ (resp. $F'_\infty$). Thus, by the comments at the end of Section 3.1, we may replace $F$ by $F^\text{ur}$ and $F'$ by $F'^\text{ur}$ without altering the formula we are studying.

After making these reductions, we let $M$ denote the (unique) $p$-extension of $Q$ inside of $F'$ such that $MF = F'$. Set $G = \text{Gal}(F/Q)$ and $H = \text{Gal}(M/Q)$, so that $\text{Gal}(F'/Q) \cong G \times H$. Then since $F$ and $F'$ are unramified at $p$ by definition, we have

$$\mu^\text{an}(F_\infty, f) = \sum_{\psi \in \text{Gal}(F/Q)^\vee} \mu^\text{an}(Q_\infty, \psi_f)$$

and

$$\mu^\text{an}(F'_\infty, f) = \sum_{\psi \in \text{Gal}(F'/Q)^\vee} \mu^\text{an}(Q_\infty, \psi_f) = \sum_{\psi \in G^\vee} \sum_{\chi \in H^\vee} \mu^\text{an}(Q_\infty, \psi_f \chi).$$

Since we are assuming that $\mu^\text{an}(F_\infty, f) = 0$ and since these $\mu$-invariants are non-negative, from (5) it follows that $\mu^\text{an}(Q_\infty, \psi_f) = 0$ for each $\psi \in \text{Gal}(F/Q)^\vee$. 

Fix $\psi \in G^\vee$. For any $\chi \in H^\vee$, $\psi \chi$ is congruent to $\psi$ modulo any prime over $p$ and thus $f_\chi$ and $f_{\psi \chi}$ are congruent modulo any prime over $p$. Then, since $\mu_{\text{an}}(Q_\infty, f_\psi) = 0$, by [1, Theorem 1] it follows that $\mu_{\text{an}}(Q_\infty, f_{\psi \chi}) = 0$ for each $\chi \in H^\vee$. Therefore, by [2] we have that $\mu_{\text{an}}(F'_\infty, f) = 0$ proving the first part of the theorem.

For $\lambda$-invariants, we again have

$$\lambda_{\text{an}}(F_\infty, f) = \sum_{\psi \in \text{Gal}(F/Q)^\vee} \lambda_{\text{an}}(Q_\infty, f_\psi),$$

and

$$\lambda_{\text{an}}(F'_\infty, f) = \sum_{\psi \in G^\vee} \sum_{\chi \in H^\vee} \lambda_{\text{an}}(Q_\infty, f_{\psi \chi}).$$

By [1, Theorem 2] the congruence between $f_\chi$ and $f_{\psi \chi}$ implies that

$$\lambda_{\text{an}}(Q_\infty, f_{\psi \chi}) - \lambda_{\text{an}}(Q_\infty, f_\psi) = \sum_{v' \in R(M_{\infty}/Q_\infty)} (m_{Q_\infty, v'}(V_{\psi \chi} \otimes \omega^{-1}) - m_{Q_\infty, v'}(V_{\psi} \otimes \omega^{-1}))$$

where $v$ denotes the place of $Q_\infty$ lying under the place $v'$ of $M_\infty$. Note that in [1] the sum extends over all prime-to-$p$ places; however, the terms are trivial unless $\chi$ is ramified at $v$. Also note that the mod $p$ cyclotomic characters that appear are actually trivial since if $Q_{\infty, v}$ has a ramified Galois $p$-extensions for $v \nmid p$, then $\mu_p \subseteq Q_{\infty, v}$.

Combining this with [1] and the definition of $m(M_{\infty, v}/Q_{\infty, v}, V_{f_\psi})$, we conclude that

$$\lambda_{\text{an}}(F'_\infty, f) = \sum_{\psi \in G^\vee} \left( [F'_\infty : F_\infty] \cdot \lambda_{\text{an}}(Q_\infty, f_\psi) + \sum_{v' \in R(M_{\infty}/Q_\infty)} m(M_{\infty, v}/Q_{\infty, v}, V_{f_\psi}) \right)$$

$$= [F'_\infty : F_\infty] \cdot \lambda_{\text{an}}(F_\infty, f) + \sum_{v' \in R(M_{\infty}/Q_\infty)} m(M_{\infty, v}/Q_{\infty, v}, V_{f_\psi})$$

$$= [F'_\infty : F_\infty] \cdot \lambda_{\text{an}}(F_\infty, f) + \sum_{v' \in R(M_{\infty}/Q_\infty)} g_{v'}(F'_\infty/M_\infty) \cdot m(M_{\infty, v}/Q_{\infty, v}, \mathbb{Z}[\text{Gal}(F_{\infty, w}/Q_{\infty, v})] \otimes V_f)$$

where $g_{v'}(F'_\infty/M_\infty)$ denotes the number of places of $F'_\infty$ above the place $v'$ of $M_\infty$. By Frobenius reciprocity,

$$m(M_{\infty, v}/Q_{\infty, v}, \mathbb{Z}[\text{Gal}(F_{\infty, w}/Q_{\infty, v})] \otimes V_f) = m(F'_{\infty, w}/F_{\infty, w}, V_f)$$

where $w'$ is the unique place of $F'_\infty$ above $v'$ and $w$. It follows that

$$\lambda(F'_\infty, f) = [F'_\infty : F_\infty] \cdot \lambda_{\text{an}}(F_\infty, f) + \sum_{w' \in R(F'_\infty/F_\infty)} m(F'_{\infty, w}/F_{\infty, w}, V_f)$$

as desired. 

4. Additional Results

4.1. Hilbert modular forms. We illustrate our results in the case of the two-dimensional representation $V_f$ associated to a Hilbert modular eigenform $f$ over a totally real field $F$. Although in principle our analytic results should remain true
in this context, we focus on the less conjectural algebraic picture. Fix a $G_F$-stable lattice $T_f \subseteq V_f$ and let $A_f = T_f \otimes K/O$.

Let $F'$ be a finite Galois $p$-extension of $F$ unramified at all places dividing $p$; for simplicity we assume also that $F'$ is linearly disjoint from $F_{\infty}'$. Let $v$ be a place of $F$ not dividing $p$ and fix a place $v'$ of $F'$ lying over $v$. For a character $\varphi$ of $G_v$, we define

$$h(\varphi) = \begin{cases} -1 & \text{\varphi ramified, } \varphi|_{G_v'} \text{ unramified, and } \varphi \equiv 1 \mod \pi; \\ 0 & \varphi \not\equiv 1 \mod \pi \text{ or } \varphi|_{G_v'} \text{ ramified;} \\ e_v(F'/F) - 1 & \varphi \text{ unramified and } \varphi \equiv 1 \mod \pi \end{cases}$$

where $e_v(F'/F)$ denotes the ramification index of $v$ in $F'/F$ and $G_{v'}$ is the decomposition group at $v'$. Set

$$h_v(f) = \begin{cases} h(\varphi_1) + h(\varphi_2) & f \text{ principal series with characters } \varphi_1, \varphi_2 \text{ at } v; \\ h(\varphi) & f \text{ special with character } \varphi \text{ at } v; \\ 0 & f \text{ supercuspidal or extraordinary at } v. \end{cases}$$

For example, if $f$ is unramified principal series at $v$ with Frobenius characteristic polynomial

$$x^2 - a_v x + c_v,$$

then

$$h_v(f) = \begin{cases} 2(e_v(F'/F) - 1) & a_v \equiv 2, c_v \equiv 1 \mod \pi \\ e_v(F'/F) - 1 & a_v \equiv c_v + 1 \not\equiv 2 \mod \pi \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 4.1.** Assume that $f$ is ordinary (in the sense that for each place $v$ dividing $p$ the Galois representation $V_f$ has a unique one-dimensional quotient unramified at $v$) and that

$$H^0(F, A_f[\pi]) = H^0(F, \text{Hom}(A_f[\pi], \mu_p)) = 0.$$ 

If $\text{Sel}(F_{\infty}, A_f)$ is $\Lambda$-cotorsion with $\mualg(F_{\infty}, A_f) = 0$, then also $\text{Sel}(F_{\infty}', A_f)$ is $\Lambda$-cotorsion with $\mualg(F_{\infty}', A_f) = 0$ and

$$\lambdaalg(F_{\infty}', A) = [F'_{\infty} : F_{\infty}] \cdot \lambdaalg(F_{\infty}, A) + \sum_v g_v(F_{\infty}'/F) \cdot h_v(f);$$

here the sum is over the prime-to-$p$ places of $F$ ramified in $F_{\infty}'$ and $g_v(F_{\infty}'/F)$ denotes the number of places of $F_{\infty}'$ lying over such a $v$.

**Proof.** Fix a place $v$ of $F$ not dividing $p$ and let $w$ denote a place of $F_{\infty}'$ lying over $v$. Since there are exactly $g_v(F_{\infty}'/F)$ such places, by Theorem 2.8 it suffices to prove that

$$h_v(f) = m(F_{\infty}'_{\infty}/F_{\infty,w}, V_f) := \sum_{\chi \in \text{Gal}(F_{\infty}'_{\infty}/F_{\infty,w})^\vee} \left( m_{F_{\infty,w}}(V_f) - m_{F_{\infty,w}}(V_{f,\chi}) \right).$$

This is a straightforward case analysis. We will discuss the case that $V_f$ is special associated to a character $\varphi$ at $v$; the other cases are similar. In the special case, we have

$$V_{f,\chi}|_{F_{\infty,w}} = \begin{cases} K'(\chi \varphi) & \chi \varphi|_{G_{F_{\infty,w}}} \text{ unramified;} \\ 0 & \chi \varphi|_{G_{F_{\infty,w}}} \text{ ramified.} \end{cases}$$
Since an unramified character has trivial restriction to $G_{F_{∞,w}}$ if and only if it has trivial reduction modulo $\pi$, it follows that
\[
m_{F_{∞,w}}(V_f,\chi) = \begin{cases} 
1 & \varphi \equiv 1 \mod \pi \text{ and } \chi|_{G_{F_{∞,w}}} \text{ unramified;} \\
0 & \text{otherwise.}
\end{cases}
\]
In particular, the sum in (8) is zero if $\phi \not\equiv 1 \mod \pi$ or if $\phi$ is ramified when restricted to $G_{F_{∞,w}}$, as then $\chi \phi$ is ramified for all $\chi \in G_{F_{∞,w}}$. If $\phi \equiv 1 \mod \pi$ and $\phi$ itself is unramified, then $m_{F_{∞,w}}(V_f) = 1$ while $m_{F_{∞,w}}(V_f,\chi) = 0$ for $\chi \neq 1$, so that the sum in (8) is \([F'_{∞,w} : F_{∞,w}] = 1 = e_\pi(F'/F) - 1\), as desired. Finally, if $\phi \equiv 1 \mod \pi$ and $\phi$ is ramified but becomes unramified when restricted to $G_{F'}$, then $m_{F_{∞,w}}(V_f) = 0$, while $m_{F_{∞,w}}(V_f,\chi) = 1$ for a unique $\chi$, so that the sum is $-1$.

Suppose finally that $f$ is in fact the Hilbert modular form associated to an elliptic curve $E$ over $F$. The only principal series which occur are unramified and we have $c_v \equiv 1 \mod \pi$ (since the determinant of $V_f$ is cyclotomic and $F_{∞}$ has a $p$-extension (namely, $F_{∞}'$) ramified at $v$), so that
\[
h_v(f) \neq 0 \iff a_v = 2 \iff E(F_v) \text{ has a point of order } p
\]
in which case $h_v(f) = 2(e_\pi(F'/F) - 1)$. The only characters which may occur in a special constituent are trivial or unramified quadratic, and we have $h_v(f) = e_\pi(F'/F) - 1$ or 0 respectively. Thus Theorem 3.1 recovers [3, Theorem 3.1] in this case.

4.2. The main conjecture. Let $f$ be a $p$-ordinary elliptic modular eigenform of weight at least two and arbitrary level with associated Galois representation $V_f$. Let $F$ be a finite abelian extension of $Q$ with cyclotomic $Z_p$-extension $F_{∞}$. Recall that the $p$-adic Iwasawa main conjecture for $f$ over $F$ asserts that the Selmer group $\text{Sel}(F_{∞},A_f)$ is $A$-cotorsion and that the characteristic ideal of its dual is generated by the $p$-adic $L$-function $L_p(F_{∞},f)$. In fact, when the residual representation of $V_f$ is absolutely irreducible, it is known by work of Kato that $\text{Sel}(F_{∞},A_f)$ is indeed $A$-cotorsion and that $L_p(F_{∞},f)$ is an element of the characteristic ideal of $\text{Sel}(F_{∞},A_f)$. In particular, this reduces the verification of the main conjecture for $f$ over $F$ to the equality of the algebraic and analytic Iwasawa invariants of $f$ over $F$. The identical transition formulae in Theorems 2.8 and 3.1 thus yield the following immediate application to the main conjecture.

**Theorem 4.2.** Let $F'/F$ be a finite $p$-extension with $F'$ abelian over $Q$. If the residual representation of $V_f$ is absolutely irreducible and $p$-distinguished, then the main conjecture holds for $f$ over $F$ with $\mu(F_{∞},f) = 0$ if and only if it holds for $f$ over $F'$ with $\mu(F'_{∞},f) = 0$.

We note that in Theorem 2.8 it was assumed that $F'/F$ was unramified at all places over $p$. However, in this special case where $F'/Q$ is abelian, this hypothesis can be removed. Indeed, one simply argues in an analogous way as at the start of Theorem 3.1 by replacing $F'$ (resp. $F$) by the maximal sub-extension of $F'_{∞}$ (resp. $F_{∞}$) that is unramified at $p$.

For an example of Theorem 4.2 consider the eigenform
\[
\Delta = q \prod_{n \geq 1} (1 - q^n)^{24}
\]
of weight 12 and level 1. We take \( p = 11 \). It is well known that \( \Delta \) is congruent modulo 11 to the newform associated to the elliptic curve \( X_0(11) \). The 11-adic main conjecture is known for \( X_0(11) \) over \( \mathbb{Q} \); it has trivial \( \mu \)-invariant and \( \lambda \)-invariant equal to 1 (see, for instance, [8] Example 5.3.1). We should be clear here that the non-triviality of \( \lambda \) in this case corresponds to a trivial zero of the \( p \)-adic \( L \)-function; we are using the Greenberg Selmer group which does account for the trivial zero.) It follows from [1] that the 11-adic main conjecture also holds for \( \Delta \) over \( \mathbb{Q} \), again with trivial \( \mu \)-invariant and \( \lambda \)-invariant equal to 1. Theorem 4.2 thus allows us to conclude that the main conjecture holds for \( \Delta \) over any abelian 11-extension of \( \mathbb{Q} \).

For a specific example, consider \( F = \mathbb{Q}(\zeta_{23})^+ \); it is a cyclic 11-extension of \( \mathbb{Q} \). We can easily use Theorem 4.1 to compute its \( \lambda \)-invariant: using that \( \tau(23) = 18643272 \) one finds that \( h_{23}(\Delta) = 0 \), so that \( \lambda(\mathbb{Q}(\zeta_{23})^+, \Delta) = 11 \).

For a more interesting example, take \( F \) to be the unique subfield of \( \mathbb{Q}(\zeta_{1123}) \) which is cyclic of order 11 over \( \mathbb{Q} \). In this case we have

\[
\tau(1123) \equiv 2 \pmod{11}
\]

so that we have \( h_{1123}(\Delta) = 20 \). Thus, in this case, Theorem 4.1 shows that \( \lambda(F, \Delta) = 31 \).

### 4.3. The supersingular case

As mentioned in the introduction, the underlying principle of this paper is that the existence of a formula relating the \( \lambda \)-invariants of congruent Galois representations should imply a Kida-type formula for these invariants. We illustrate this now in the case of modular forms of weight two that are supersingular at \( p \).

Let \( f \) be an eigenform of weight 2 and level \( N \) with Fourier coefficients in \( K \) some finite extension of \( \mathbb{Q}_p \). Assume further than \( p \nmid N \) and that \( a_p(f) \) is not a \( p \)-adic unit. In [8], Perrin-Riou associates to \( f \) a pair of algebraic and analytic \( \mu \)-invariants over \( \mathbb{Q}_\infty \) which we denote by \( \mu^*_\pm(\mathbb{Q}_\infty, f) \). (Here \( * \) denotes either “alg” or “an” for algebraic and analytic respectively.) Moreover, when \( \mu^*_+(\mathbb{Q}_\infty, f) = \mu^*_-(\mathbb{Q}_\infty, f) \) or when \( a_p(f) = 0 \), she also defines corresponding \( \lambda \)-invariants \( \lambda^*_\pm(\mathbb{Q}_\infty, f) \). When \( a_p(f) = 0 \) these invariants coincide with the Iwasawa invariants of [6] and [9]. We also note that in [8] only the case of elliptic curves is treated, but the methods used there generalize to weight two modular forms.

We extend the definition of these invariants to the cyclotomic \( \mathbb{Z}_p \)-extension of an abelian extension \( F \) of \( \mathbb{Q} \). As usual, by passing to the maximal subfield of \( F_\infty \) unramified at \( p \), we may assume that \( F \) is unramified at \( p \). We define

\[
\mu^*_\pm(F_\infty, f) = \sum_{\psi \in \text{Gal}(F/\mathbb{Q})^\vee} \mu^*_\pm(\mathbb{Q}_\infty, f_\psi) \quad \text{and} \quad \lambda^*_\pm(F_\infty, f) = \sum_{\psi \in \text{Gal}(F/\mathbb{Q})^\vee} \lambda^*_\pm(\mathbb{Q}_\infty, f_\psi)
\]

for \( * \in \{\text{alg}, \text{an}\} \).

The following transition formula follows from the congruence results of [2].

**Theorem 4.3.** Let \( f \) be as above and assume further that \( f \) is congruent modulo some prime above \( p \) to a modular form with coefficients in \( \mathbb{Z}_p \). Consider an extension of number fields \( F'/F \) with \( F' \) an abelian \( p \)-extension of \( \mathbb{Q} \). If \( \mu^*_\pm(F_\infty, f) \neq 0 \), then \( \mu^*_\pm(F'_\infty, f) = 0 \). Moreover, if this is the case, then

\[
\lambda^*_\pm(F'_\infty, f) = [F'_\infty : F_\infty] \cdot \lambda^*_\pm(F_\infty, f) + \sum_{w' \in \mathfrak{R}(F'_\infty/F_\infty)} m(F'_{\infty,w'/F_\infty}, V_f).
\]
In particular, if the main conjecture is true for $f$ over $F$ (with $\mu_+(F_\infty,f) = 0$), then the main conjecture is true for $f$ over $F'$ (with $\mu_+(F'_\infty,f) = 0$).

Proof. The proof of this theorem proceeds along the lines of the proof of Theorem 3.1 replacing the appeals to the results of [1, 11] to the results of [2]. The main result of [2] is a formula relating the $\lambda_\pm$-invariants of congruent supersingular weight two modular forms. This formula has the same shape as the formulas that appear in [1] and [11] which allows for the proof to proceed nearly verbatim. The hypothesis that $f$ be congruent to a modular form with $\mathbb{Z}_p$-coefficients is needed because this hypothesis appears in the results of [2].

One difference to note is that in this proof we need to assume that $F$ is a $p$-extension of $\mathbb{Q}$. The reason for this assumption is that in the course of the proof we need to apply the results of [2] to the form $f_\psi$ where $\psi \in \text{Gal}(F/Q)^\vee$. We thus need to know that $f_\psi$ is congruent to some modular form with coefficients in $\mathbb{Z}_p$. In the case that $\text{Gal}(F/Q)$ is a $p$-group, $f_\psi$ is congruent to $f$ which by assumption is congruent to such a form. □

References

[1] M. Emerton, R. Pollack and T. Weston, Variation of Iwasawa invariants in Hida families, to appear in Invent. Math.
[2] R. Greenberg, A. Iovita and R. Pollack, Iwasawa invariants of supersingular modular forms, preprint.
[3] Y. Hachimori and K. Matsuno, An analogue of Kida’s formula for the Selmer groups of elliptic curves, J. Algebraic Geom. 8 (1999), no. 3, 581–601.
[4] K. Iwasawa, Riemann–Hurwitz formula and $p$-adic Galois representations for number fields, Tohoku Math. J. 33 (1981), 263–288.
[5] Y. Kida, $\ell$-extensions of CM-fields and cyclotomic invariants, J. Number Theory 12 (1980), 519–528.
[6] S. Kobayashi, Iwasawa theory for elliptic curves at supersingular primes, Invent. Math. 152 (2003), 1–36.
[7] K. Matsuno, An analogue of Kida’s formula for the $p$-adic $L$-functions of modular elliptic curves, J. Number Theory 84 (2000), 80–92.
[8] B. Perrin-Riou, Arithmétique des courbes elliptiques à réduction supersingulière en $p$, Experiment. Math. 12 (2003), no. 2, 155–186.
[9] R. Pollack, On the $p$-adic $L$-function of a modular form at a supersingular prime, Duke Math. J. 118 (2003), 523–558.
[10] W. Sinnott, On $p$-adic $L$-functions and the Riemann–Hurwitz genus formula, Comp. Math. 53 (1984), 3–17.
[11] T. Weston, Iwasawa invariants of Galois deformations to appear in Manuscripta Math.
[12] K. Wingberg, A Riemann–Hurwitz formula for the Selmer group of an elliptic curve with complex multiplication, Comment. Math. Helv. 63 (1988), 587–592.

(Robert Pollack) Department of Mathematics, Boston University, Boston, MA

(Tom Weston) Dept. of Mathematics, University of Massachusetts, Amherst, MA

E-mail address, Robert Pollack: rpollack@math.bu.edu
E-mail address, Tom Weston: weston@math.umass.edu