Earth rotation and separability of polarizations of continuous gravitational waves from a known pulsar

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We consider effects of the Earth rotation on antenna patterns of a ground-based gravitational wave (GW) detector in a general metric theory that allows at most six polarization states (two spin-0, two spin-1 and two spin-2) in a four-dimensional spacetime. By defining the cyclically averaged antenna matrix for continuous GWs from a known pulsar, we show that different polarization states can be separated out from a given set of the strain outputs at a single detector. The third-generation GW detectors such as the Cosmic Explorer and the Einstein Telescope can place a stringent constraint $\sim 10^{-30}$ on the extra GW polarization amplitudes at $\sim 100$ Hz.

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I. INTRODUCTION

A century after the birth of Einstein’s theory of general relativity (GR) [1, 2], the first direct observation of gravitational waves (GWs) was done for the golden event GW150914.

GR is not perfectly consistent with quantum physics and string theoretical viewpoints. It is thus important to probe new physics beyond GR [3–5]. In a four-dimensional spacetime, general metric theories allow at most six GW polarization states (two spin-0, two spin-1 and two spin-2) [6]. Once the TT polarizations are detected, it will be of great importance to probe the extra polarizations beyond GR. The two scalar modes called Breathing (B) and Longitude (L) are degenerate in interferometry, because the antenna pattern functions for B and L modes take the same form but with the opposite sign [7]. Therefore, a direct test of each polarization state needs five or more ground-based detectors. For a merger event associated with an electromagnetic counterpart, we can know the GW source sky position by the multi-messenger astronomy. For such multi-messenger events in particular sky regions, the minimum requirement becomes four ground-based detectors including KAGRA [8–11].

The GW150914 data fits well with a binary black hole merger in GR [12], though this test is inconclusive because the number of GW polarization states in GR is equal to the number of aLIGO detectors. The addition of Virgo to the aLIGO detectors for GW170814 enabled the first informative test of GW polarizations. According to their analysis, the GW data are described much better by the tensor, the vector or the scalar response functions, though the signal-to-noise ratio in Virgo was much lower than those in the two aLIGO detectors. The prospects for polarization tests were discussed (e.g. [16–19]).

GW signals are a linear combination of different polarization modes, where the coefficients of each mode is called the antenna pattern function that depends on the polarization state as well as the source direction [20–25]. For a merger event so far, the antenna pattern is almost instantaneous. As a result, the required minimum number of detectors must equal to the number of independent polarization states when we wish a direct separation of all the possible polarizations states.

With GEO600 that has recently placed a tight constraint on a scalar dark matter [26], LIGO continues also searching continuous GWs from pulsars [27, 28]. With great efforts on substantial improvement of the detector sensitivity and the data analysis methods, LIGO has placed upper bounds not only on TT modes of GWs from known pulsars [29–33] especially including the Crab pulsar [34] but also on those from unknown pulsars in all sky survey, from which upper bounds are put on small-ellipticity of neutron stars [35].

The present paper assumes a pulsar for which the spin period and sky position are precisely known. The antenna pattern for continuous GWs changes significantly with time owing to the Earth rotation, whereas the GWs from the pulsar are periodic.

How does the Earth rotation affect the separability of GW polarization states from the known pulsar? The main purpose of the present paper is to show that the Earth rotation separates out all the possible polarization states of the pulsar GWs.

This paper is organized as follows. Section II briefly summarizes expressions for the antenna pattern functions and the strain outputs. Section III discusses the cyclically averaging of the antenna patterns in order to demonstrate the separability of different polarization modes. Section IV is devoted to Conclusion.

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II. ANTENNA PATTERNS AND GW SIGNALS

In a four-dimensional spacetime, a general metric theory allows six polarizations at most [6]; \( h_B(t) \) for the spin-0 B mode, \( h_L(t) \) for the spin-0 L mode, \( h_V(t) \) and \( h_W(t) \) for two spin-1 modes, \( h_+ (t) \) for the plus mode and \( h_\times (t) \) for the cross mode. For a laser interferometer, the antenna pattern function to each polarization is denoted as \( F_I (t) \), where \( I = B, L, V, W, +, \times \) [20, 23]. It depends on the GW source direction \( \theta \) and \( \phi \) as well as the polarization angle \( \Psi \). The latitude and longitude of a GW source are functions of time \( \theta (t) \) and \( \phi (t) \), whereas they are almost instantaneous for a merger or burst event. For the brevity, we omit \( \theta (t) \) and \( \phi (t) \) in the notation.

The strain output at the detector is written as [7, 20, 22]

\[
S(t) = F^S (t) h_S (t) + F^V (t) h_V (t) + F^W (t) h_W (t) + F^+ (t) h_+ (t) + + F^\times (t) h_\times (t) + n(t)
\]

where \( F^S (t) \equiv F^B (t) = -F^L (t) \), we denote \( h_S (t) \equiv h_B (t) - h_L (t) \), and \( n(t) \) means noises. In the rest of this paper, \( I \in S, V, W, +, \times \) is denoted simply as \( I \).

For LIGO-Virgo merger events, the duration is roughly \( 1 - 1000 \) milliseconds (\( \ll T_E \)), where \( T_E \) is the Earth rotation period \( \approx 24 \) hours. The time variation of \( F^I (t) \) is negligible enough for us to safely use the instantaneous antenna pattern for the data analysis. The dependence on time is discussed e.g. by Takeda et al. [36].

On the other hand, the antenna pattern changes significant with time in a day.

III. N-CYCLE AVERAGING FOR PERIODIC GWs

We consider periodic GWs with period \( T_P \) (\( \sim 1 - 1000 \) msec.) as

\[
h_I (t) = h_I (t + nT_P), \quad n \in \mathbb{Z}
\]

where \( n \) is an integer. It is sufficient to consider only \( h_I (t) \) for \( t \in [0, T_P) \) because of being periodic.

For the sake of simplicity, we focus on one day as the observational duration, where the number of the GW cycles in one day is \( N \equiv \lfloor T_E / T_P \rfloor \) for the Gauss symbol \( \lfloor \cdot \rfloor \), namely the integer part. Note that \( h_I (t) \) is cyclic with period \( T_P \), while \( F_I (t) \) has another period \( T_E \).

For \( N \) cycles, the signals can be expressed in terms of the periodic function \( h_I (t) \). We divide the total \( N \) cycles into each one cycle of \( t \in [(a - 1)T_P, aT_P) \), where \( a = 1, 2, \cdots, N \) is an integer.

The strain output in each cycle is

\[
S_I (t) \equiv S(t) = \sum_I F^I (t) h_I (t) + n(t),
\]

\[
S_2 (t) \equiv S(t + T_P) = \sum_I F^I (t + T_P) h_I (t + T_P) + n(t + T_P),
\]

\[
\cdots
\]

\[
S_N (t) \equiv S(t + (N - 1)T_P) = \sum_I F^I (t + (N - 1)T_P) h_I (t + (N - 1)T_P) + n(t + (N - 1)T_P),
\]

where we denote \( S_a (t) \equiv S(t + (a - 1)T_P) \).

Using the least square method, therefore, let us define \( A(t) \) by

\[
A(t) = \left( \sum_{a=1}^{N} S_a(t) - \sum_I F^I (t) h_I (t) \right)^2 + \cdots + \left( S_N(t) - \sum_I F^I (t) h_I (t) \right)^2
\]

\[
= \sum_{a=1}^{N} \left( S_a(t) - F_a^I (t) h_I (t) \right)^2,
\]

where Eqs. 2 and 3 are used and \( F_a^I (t) \equiv F^I (t + (a - 1)T_P) \). In the rest of the paper, the \( N \)-cycle sum \( \sum_{a=1}^{N} \) is abbreviated as \( \sum_a \).

In the least square method, the most expected \( h_I (t) \) should satisfy five equations as \( \partial A(t) / \partial h_I (t) = 0 \) for each \( I \). The coupled equations for \( h_I (t) \) are rearranged in a vectorial form as

\[
M(t) \vec{H}(t) = \vec{L}(t),
\]

where we define
\[ \vec{H}(t) \equiv \begin{pmatrix} h_+(t) \\ h_\times(t) \\ h_V(t) \\ h_W(t) \\ h_S(t) \end{pmatrix}, \]  

where \( M^{-1}(t) \) is the inverse matrix of \( M(t) \). We refer to \( M(t)/N \) as the cyclically averaged antenna matrix (CAAM), because the procedure of \( \frac{1}{N} \sum_a \) is the averaging for the \( N \) cycles. One may ask if \( M(t)/N \) corresponds to the covariance matrix. This is not the case, because the averaging of \( F^I(t) \) as \( \frac{1}{N} \sum_a F^I_a(t) \) does not vanish.

The formal solution as Eq. (9) with Eqs. (6)-(8) shows clearly the existence and uniqueness of the solution for the inverse problem. In practical calculations, however, we do not need obtain \( M^{-1}(t) \), for which numerically performing the inverse of a matrix is rather time-consuming. It is sufficient and even convenient to solve Eq. (6) by using a more sophisticated algorithm.

It is an open problem whether or not the solution is globally unique in an inverse problem even for realistic nonlinear noises. In addition, known pulsars show secular changes in their spin period, for which \( dT_P/dt \) and \( d^2T_P/dt^2 \) should be taken into account. A generalization based on such a realistic pulse modeling in pulsar astronomy is interesting. These issues are beyond the scope of the present paper.

Figure 2 shows numerical results of separating out GW polarization states, where one day observation and a pulsar GW period of 1000 milliseconds are assumed, corresponding to \( N \sim 8 \times 10^4 \), because of the limited computational resources. For exaggerations, this figure assumes that the amplitudes of the plus and cross modes are equal to each other, the standard deviation of the noise \( \bar{n} \) is the double of the TT amplitude, and the amplitude of the extra polarizations is equal to the one tenth of that of the TT mode.

The amplitude of \( h_+(t) \) and \( h_\times(t) \) is denoted simply as \( h_{TT} \). By the same way, we denote the amplitudes of \( S \), \( V \) and \( W \) modes as \( h_S, h_V, h_W \), respectively. In Figure 2, \( h_S = h_V = h_W = h_{TT}/10 \) and \( \bar{n} = 2 \times h_{TT} \) are chosen.

For \( N \) cycles, the noise contribution \( n(t) \) can be reduced effectively to \( n_{\text{eff}}(t) \equiv \frac{1}{N} \sum_a n_a(t) \sim \bar{n}/\sqrt{N} \), when the noise obeys a Gaussian distribution, we denote \( n_a(t) \equiv n(t + (a - 1)T_P) \) and \( N \) is large. Namely, \( n_{\text{eff}}(t) \) gets smaller \( \propto N^{-1/2} \), as \( N \) increases. In Figure 2, roughly estimating, the typical size of \( n_{\text{eff}}(t) \) is \( n(t)/140, n(t)/210, n(t)/250, n(t)/290 \), respectively, for \( N \sim 2.1 \times 10^4, 4.3 \times 10^4, 6.5 \times 10^4, 8.5 \times 10^4 \). This is consistent with Figure 2.

\[ M(t) \equiv \begin{pmatrix} \sum_a [F^+_a(t)]^2 & \sum_a F^+_a(t) F^+_a(t) & \sum_a F^+_a(t) F^V_a(t) & \sum_a F^+_a(t) F^W_a(t) & \sum_a F^+_a(t) F^S_a(t) \\ \sum_a F^V_a(t) F^+_a(t) & \sum_a [F^V_a(t)]^2 & \sum_a F^V_a(t) F^V_a(t) & \sum_a F^V_a(t) F^W_a(t) & \sum_a F^V_a(t) F^S_a(t) \\ \sum_a F^W_a(t) F^+_a(t) & \sum_a F^W_a(t) F^V_a(t) & \sum_a [F^W_a(t)]^2 & \sum_a F^W_a(t) F^W_a(t) & \sum_a F^W_a(t) F^S_a(t) \\ \sum_a F^S_a(t) F^+_a(t) & \sum_a F^S_a(t) F^V_a(t) & \sum_a F^S_a(t) F^W_a(t) & \sum_a [F^S_a(t)]^2 & \sum_a F^S_a(t) F^S_a(t) \end{pmatrix}. \]  

IV. FUTURE PROSPECTS AND POSSIBLE SUBDOMINANT EFFECTS

In this section, we briefly discuss subdominant effects on the current method and result.

First, the expected continuous GW signal is much smaller than a current detector noise. Namely, \( \bar{n} \gg h_{TT} \). A large \( N \) is thus required. For three months (\( \sim 100 \) days) and twelve years for example, the effective \( n_{\text{eff}}(t) \) becomes \( n(t)/3000, n(t)/20000 \), respectively. From a twelve-year observation, the upper bound on the extra polarizations can be placed at the \( O(10^{-4}) \) level of \( \bar{n} \), though it is optimistic as mentioned in next paragraph.

The third-generation detectors such as the Cosmic Explorer (CE) and the Einstein Telescope (ET) are aiming
FIG. 2. Polarization separation: From $S(t)$ to $h_1(t)$ by Eq. (9). The unit of the vertical axis is arbitrary. Top left: $N = 21600$, Top right: $N = 43200$, Bottom left: $N = 64800$, Bottom right: $N = 86400$, each of which corresponds to $\sim 6, 12, 18, 24$ hours, respectively. The LIGO-Hanford detector and the sky location of the Crab pulsar are assumed, where GW waveforms follow sine functions, indicated by solid black (in color) lines. For exaggerations, the GW amplitude for the extra polarizations (S, V, W) is chosen as 0.1. The noise $n(t)$ obeys a Gaussian distribution with the standard deviation of 2. For $N = 21600$ cycles ($\sim 6$ hours), the TT modes are well reconstructed, whereas the S, V and W modes are hardly distinguishable from noises. As $N$ increases, the noise is effectively reduced as $n_{eff}(t) \propto 1/\sqrt{N}$. As a result, the S, V and W modes are well separated out for $N = 86400$ for instance. At the sensitivity of $\sim 2 - 8 \times 10^{-25}$ for a range of 100-500 Hz according to their white papers [38, 39]. For a twelve-year observation of a pulsar with $T_p \sim 10$ milliseconds (corresponding to $\sim 10^2$ Hz), $N$ is $\sim 4 \times 10^{10}$. CE and ET will be thus able to put a stringent upper bound $\sim (2 \times 10^{-25})/(2 \times 10^5) = 10^{-30}$ on the extra polarization amplitudes. This means that, if the TT modes are detected at the level $\sim 10^{-26}$, the upper bound on the extra modes is lower by four digits or more. For a millisecond pulsar, $N$ becomes ten times and thereby the upper bound is tighter by a factor $\sim 3$. In any case, this test is robust, because waveform templates are not used. This possible bound will be much stronger than the existing indirect test by the orbital decay observation of the binary pulsar B1913+16, where the radiation energy by extra polarizations has been limited to less than $\sim 0.1\%$ [3, 40]. Very recently, Kramer et al. have reported that the double pulsar PSR J0737–3039A/B validates the prediction of GR more precisely at the level of $\sim 1 \times 10^{-4}$ [41].

A second comment is related with the first one. For a very long-time observation such as three months or twelve years, a simple periodic model in this paper is not sufficient [37]. In addition to the Earth rotation, we have to take account of the orbital motion of the Earth as well as the geophysical disturbance. These subdominant effects do not affect $h_1(t)$ but modify a function of time for $F(t)$. Hence, the existence and uniqueness from Eq. (9) still hold, where $M(t)$ is calculated from the accordingly modified $F(t)$.

On the other hand, we may need to take account of the modulation in the pulsar spin period [27], which affects both the amplitude and period of the GWs. Hence, some modification of Eq. (9) is needed.

Up to this point, we have assumed det $M \neq 0$, where det denotes the determinant of a matrix. What is meant
by $\det M = 0$. The detector cannot distinguish GW polarizations from a pulsar that satisfies $\det M = 0$, describing a curve in the sky, because the CAAM is degenerate.

Finally, we mention the speed of extra GW modes [15]. The possible arrival time difference between the TT and extra modes does not change the current discussion, because only the GW period matters but the time translation does not affect the $N$-cycle averaging.

V. CONCLUSION

We considered a possible daily variation of antenna patterns for a ground-based GW detector due to Earth rotation. By defining the CAAM for continuous GWs from a known pulsar, we showed that different polarization states can be separated out from a given set of the strain outputs at a single detector. By the planned third-generation GW detectors such as the Cosmic Explorer and the Einstein Telescope, a stringent constraint as $\sim 10^{-30}$ can be placed on the extra GW polarization amplitudes at $\sim 100$ Hz.

Further detailed simulations are needed, when we wish to take account of possible subdominant effects including the Earth orbital motion around the barycenter and the geophysical disturbance as well as secular changes in the pulsar period. It is left for future.

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