Is your function low-dimensional?

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Abstract

We study the problem of testing if a function depends on a small number of linear directions of its input data. We call a function $f$ a linear $k$-junta if it is completely determined by some $k$-dimensional subspace of the input space. In this paper, we study the problem of testing whether a given $n$ variable function $f : \mathbb{R}^n \rightarrow \{0, 1\}$, is a linear $k$-junta or $\epsilon$-far from all linear $k$-juntas, where the closeness is measured with respect to the Gaussian measure on $\mathbb{R}^n$. Linear $k$-juntas are a common generalization of two fundamental classes from Boolean function analysis (both of which have been studied in property testing) 1. $k$-juntas which are functions on the Boolean cube which depend on at most $k$ of the variables and 2. intersection of $k$ halfspaces, a fundamental geometric concept class.

We show that the class of linear $k$-juntas is not testable, but adding a surface area constraint makes it testable: we give a $\text{poly}(k \cdot s / \epsilon)$-query non-adaptive tester for linear $k$-juntas with surface area at most $s$. We show that the polynomial dependence on $s$ is necessary. Moreover, we show that if the function is a linear $k$-junta with surface area at most $s$, we give a $(s \cdot k)^{O(k)}$-query non-adaptive algorithm to learn the function up to a rotation of the basis. In particular, this implies that we can test the class of intersections of $k$ halfspaces in $\mathbb{R}^n$ with query complexity independent of $n$.

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1 Introduction

Property testing of Boolean functions was initiated in the seminal work of Blum, Luby and Rubinfeld [BLR93] and Rubinfeld and Sudan [RS96]. The high level goal of property testing is the following: Given (query) access to a Boolean function $f$, the algorithm must distinguish between (i) the case that $f$ belongs to a class $C$ of Boolean functions (i.e., has a property $C$), and (ii) the case that $f$ is $\epsilon$-far from every function belonging to $C$. Here the distance between functions is measured with respect to some underlying distribution $\mathcal{D}$ and is defined as $\text{dist}(f,g) = \Pr_{x \sim \mathcal{D}}[f(x) \neq g(x)]$. Also, the algorithm is randomized and thus only needs to succeed with high probability (as opposed to probability one). The quality of a testing algorithm is measured by the number of oracle calls it makes to $f$ – its query complexity – and the goal is to minimize this query complexity.

Since the works of [BLR93, RS96], property testing of Boolean functions has been a thriving field and by now several classes $C$ have been studied from this perspective. These include classes such as linear functions [BLR93], low-degree polynomials [JPRZ04, BKS10], monotonicity [FLN+02, CS16, KMS15], algebraic properties [KS08, BGS15, BFH13] and juntas [FKR+04, Bla09, CST+17] among many others (see the surveys [R+10, Gol17]).

Special attention has been devoted to the problem of testing juntas. Recall that a Boolean function $f : \{-1,1\}^n \to \{-1,1\}$ is said to be a $k$-junta if $f$ is only dependent on a subset $S \subseteq [n]$ (of size $k$) of the coordinates. Given (query) access to a function $f$, the problem of testing juntas is to decide whether $f$ is a $k$-junta or $\epsilon$-far from every $k$-junta (under the uniform distribution on $\{-1,1\}^n$). Some of the initial motivation [FKR+04] to study this came from the problem of long-code testing [BGS98, PRS02] (related to PCPs and inapproximability). Another motivation comes from the feature selection problem in machine learning. It is well-known (see, e.g., [Blu94, BL97]) that learning a $k$-junta requires at least $\Omega(k \log n)$ samples, however $k$-juntas can be tested with query complexity independent of $n$ [FKR+04].

The most obvious generalization of $k$-juntas to functions $f : \mathbb{R}^n \to \{-1,1\}$ is to consider functions that depend only on $k$ of the $n$ coordinates. However, in many statistical and machine learning models (e.g. PCA, ICA, kernel learning, dictionary learning) the choice of basis is not a priori clear. Therefore, it is natural to consider a notion of junta that is linearly invariant. We define a function $f : \mathbb{R}^n \to \{-1,1\}$ to be a linear $k$-junta if there are $k$ unit vectors $u_1, \ldots, u_k \in \mathbb{R}^n$ and $g : \mathbb{R}^k \to \{-1,1\}$ such that $f(x) = g(\langle u_1, x \rangle, \ldots, \langle u_k, x \rangle)$.

We note that the family of linear $k$-juntas includes important classes of functions that have been studied in the learning and testing literature. Notably it includes:

- Boolean juntas: If $h : \{-1,1\}^n \to \{0,1\}$ is a Boolean junta, then $f(x) : \mathbb{R}^n \to \{0,1\}$ defined as $f(x) = h(\text{sgn}(x_1), \ldots, \text{sgn}(x_n))$ is a linear $k$-junta.

- Functions of halfspaces: Linear $k$-juntas include as a special case both halfspaces and intersections of $k$-halfspaces. The testability of halfspaces was studied in [MORS09, MORS10, RS15].

We consider the scenario where the ambient dimension $n$ is large but the dimension of the relevant subspace, i.e., $k$ is small. In this setting, we consider the following property testing question:

**Question 1.** Given a function $f$ and access to random examples, $(x, f(x))$, is it possible to test in number of queries that depends on $k$ (but not on $n$) if $f$ is a linear $k$-junta or far from all linear $k$-juntas?

The problem of testing linear-juntas is closely related to the problem of model compression in machine learning. The goal of model compression is to take as an input a complex predictor/classifier function and to output a simpler predictor/classifier see e.g. [BCNM06]. The question of model compression is extensively studied in the context of deep nets, see e.g., [BC14], and follow up work, where the models are often rotationally invariant (with the caveat that the regularization often used in optimization might not be).
Thus as a motivating example we may ask: given a complex deep net classifier, is there a classifier that has essentially the same performance and depends only on \( k \) of the features?

To formally state question [1] we need to define what “close” means. The standard definition is to state that \( f \) is close to \( g \) if \( \Pr[f(x) \neq g(x)] \) is small, for some probability measure \( \Pr \). The most natural choice of \( \Pr \) for learning and testing functions \( f : \mathbb{R}^n \to \{-1, 1\} \) is the Gaussian measure \([\text{MORS09}, \text{KNOW14}, \text{Nee14}, \text{BBBY12}, \text{CFSS17}, \text{KOS08}, \text{Vem10a}, \text{DKS18}, \text{BL13}, \text{HKM12}]\). It is particularly natural in our setup since the Gaussian measure is invariant under many linear transformation, e.g., all rotations.

It is possible to show that the answer to question [1] is no even if \( n = 2 \) and \( k = 1 \), since without smoothness assumptions, measurable functions \( f : \mathbb{R}^n \to \{-1, 1\} \) can look arbitrarily random to any finite number of queries (a more formal statement with stronger results will be discussed shortly). Since the groundbreaking work of \([\text{KOS08}]\), it was recognized that the surface area of a function \( f : \mathbb{R}^n \to \{-1, 1\} \) is a natural complexity parameter (see Definition [14] for the definition of surface area – roughly speaking, if \( A = \{x : f(x) = 1\} \), then the surface area of \( f \) is the size of the boundary of \( A \) weighted by the Gaussian measure). We therefore ask the following question:

**Question 2.** Given a function \( f \) and access to random examples, \((x, f(x))\), is it possible to test in number of queries that depends on \( k \) and \( s \) (but not on \( n \)) if \( f \) is close to any linear \( k \)-junta with surface area at most \( s \)?

In our main result we give an affirmative answer to the question above:

**Theorem.** There is an algorithm \textit{Test-linear-junta} which has the following guarantee: Given oracle access to \( f : \mathbb{R}^n \to \{-1, 1\} \), rank parameter \( k \), surface area parameter \( s \) and error parameter \( \epsilon > 0 \), it makes \( \text{poly}(s, \epsilon^{-1}, k) \) queries and distinguishes between the following cases:

1. The function \( f \) is a linear \( k \)-junta whose surface area is at most \( s \).
2. The function \( f \) is \( \epsilon \)-far from any linear \( k \)-junta with surface area at most \( s(1 + \epsilon) \).

This theorem is proven in Section [3]. We note that while the tester allows a slack of \( 1 + \epsilon \) in the surface area between the soundness and completeness cases, such a slack factor is required even for the easier problem of estimating surface area in \( \mathbb{R}^2 \) \([\text{Nee14}]\). It is natural to ask if our dependence on the surface area is optimal. Towards answering this, in Section [5] we prove:

**Theorem.** Any non-adaptive algorithm for testing whether an unknown Boolean function \( f \) is a linear 1-junta with surface area at most \( s \) versus \( \Omega(1) \)-far from a linear 1-junta makes at least \( s^{\Omega(1)} \) queries.

Thus our tester is optimal in the dependence on \( s \) up to polynomial factors.

**Finding the linear-invariant structure**

Given the previous theorem it is natural to ask for more, i.e., not just test if the function is a linear-junta but also find the junta in number of queries that depends only on \( k \) and \( s \) (but not on \( n \)). In other words, could we output \( g : \mathbb{R}^k \to \{-1, 1\} \) such that there exists a projection matrix \( A : \mathbb{R}^n \to \mathbb{R}^k \) and \( f \) is close to \( g(Ax) \) with query complexity independent of \( n \)? We give an affirmative answer to this question:

**Theorem.** Let \( f : \mathbb{R}^n \to \{-1, 1\} \) be a linear \( k \)-junta with surface area at most \( s \). Then, there is an algorithm \textit{Find-invariant-structure} which on error parameter \( \epsilon > 0 \), makes \( (s \cdot k / \epsilon)^{O(k)} \) queries and outputs \( g : \mathbb{R}^k \to [-1, 1] \) so that the following holds: there exists an orthonormal set of vectors \( w_1, \ldots, w_k \in \mathbb{R}^n \) such that

\[
E[|f(x) - g(w_1, x, \ldots, w_k, x)|] = O(\epsilon).
\]

Moreover, for some \( g^* : \mathbb{R}^k \to \mathbb{R} \):

\[
f(x) = g^*(\langle w_1, x \rangle, \ldots, \langle w_k, x \rangle).
\]
Informally, the theorem states that it is possible to find the “linear-invariant” structure (i.e., the structure up to unitary transformation) of $f$ in number of queries that depends on $s$ and $k$. Of course, one cannot hope to output the relevant directions $w_1, \ldots, w_k$ explicitly as even describing these directions will require $\omega(n)$ bits of information and thus, at least those many queries. We note that the number of functions in $k$ dimensions with $O(1)$ surface area (even up to a unitary rotation) is $\exp(\exp(k))$ and thus even our output has to be $\exp(k)$ bits. Thus, it is not possible to significantly improve on our $\exp(k \log k)$ query complexity in finding the linear-invariant structure.

**Testability of linear invariant families of linear $k$-juntas**

Our ability to find the linear-invariant structure of linear $k$-juntas additionally allows us to test subclasses of linear $k$-juntas which are closed under rotation.

**Definition 3.** Let $C$ be any collection of functions mapping $\mathbb{R}^k$ to $\{-1, 1\}$. For any $n \in \mathbb{N}$ let:

$$\text{Ind}(C)_n = \{f : \exists g \in C \text{ and orthonormal vectors } w_1, \ldots, w_k \text{ such that } f(x) = g(\langle w_1, x \rangle, \ldots, \langle w_k, x \rangle)\}$$

Define $\text{Ind}(C) = \bigcup_{n=1}^{\infty} \text{Ind}(C)_n$, and call it the **induced class of $C$**.

The two key properties of $\text{Ind}(C)$ are (i) each function $f \in \text{Ind}(C)$ is a linear $k$-junta, (ii) the class $\text{Ind}(C)$ is closed under unitary transformations. The definition is a continuous analogue of the so-called “induced subclass of $k$-dimensional functions” from [GOS+09] (that paper was about testing functions over $\mathbb{F}^n$). The following theorem shows that for any $C$, $\text{Ind}(C)$ is testable without any dependence on the ambient dimension.

**Theorem.** Let $C$ be a collection of functions mapping $\mathbb{R}^k$ to $\{-1, 1\}$. Further, for every $f \in \text{Ind}(C)$, surf($f$) $\leq s$. Then, there is an algorithm **Test-structure-$C$** which has the following guarantee: Given oracle access to $f : \mathbb{R}^n \to \{-1, 1\}$ and an error parameter $\epsilon > 0$, the algorithm makes $(s \cdot k/\epsilon)^{O(k)}$ queries and distinguishes between the cases (i) $f \in \text{Ind}(C)$ and (ii) $f$ is $\epsilon$-far from every function $g \in \text{Ind}(C)$.

A particularly important instantiation of the above theorem is the following: Let $C_B$ be any collection of functions mapping $\{-1, 1\}^k \to \{-1, 1\}$ and let $C$ be defined as

$$C = \{g : x \mapsto h(\langle w_1, x \rangle - \theta_1, \ldots, \langle w_k, x \rangle - \theta_k) | w_1, \ldots, w_k \in \mathbb{R}^k, \theta_1, \ldots, \theta_k \in \mathbb{R}, h \in C_B\}.$$  

Note that $C$ defined above is the set of functions obtained by composing a function from $C_B$ with $k$-dimensional halfspaces. Consequently, $\text{Ind}(C)$ is the of all functions which can be obtained by composing a function from $C_B$ with halfspaces. As an example, if $C_B$ consists of the AND function on $k$ or fewer bits, then $\text{Ind}(C)$ is the class of “intersections of $k$-halfspaces”. Since the surface area of any Boolean function of $k$-halfspaces is bounded by $O(k)$ it follows that the this class is testable with $(k/\epsilon)^{O(k)}$ queries.

Roughly speaking, the algorithm **Test-structure-$C$** works as follows: we first run the routine **Test-linear-junta** – if the target function $f$ passes this test, we are guaranteed that it is (very close to) a linear $k$-junta with surface area $s$. We then run the routine **Find-invariant-structure**. If the output of this step is $g$, then we can check whether $g$ is close to some function in $\text{Ind}(C)_k$ and accept accordingly. We crucially note here that the last step, namely checking whether $g$ is close to a function in $\text{Ind}(C)_k$ makes no queries to $f$. While the overall intuition of this procedure is obvious, the precise proof is more delicate and is given in Section 4.
1.1 Related Work

Testing Boolean juntas As we have already mentioned, the problem of testing juntas on \( \{-1, 1\}^n \) has already been well-studied. For example, it is known \(^{15}\) \( \Theta(k^{3/2}) \) queries are necessary and sufficient for non-adaptively testing \( k \)-juntas with respect to the uniform distribution, while \( \tilde{\Theta}(k) \) queries are necessary and sufficient in the adaptive setting \(^{13}\) \( \text{BBM12} \). It even turns out to be possible to test \( k \)-juntas with respect to an unknown distribution \(^{14}\) \( \text{CLS}^+18 \), although in that setting the non-adaptive query complexity becomes exponential in \( k \). We emphasize that while the problem of junta testing inspires the problems considered in this paper, junta testing algorithms have no bearing on the problem of testing linear juntas – e.g., unlike \(^ {17}\) \( \text{CLS}^+18 \), there is no reason to believe that distribution-free testing of linear juntas on \( \mathbb{R}^n \) is even possible, given that the space of probability measures on \( \mathbb{R}^n \) is much richer than the space of probability measures on \( \{-1, 1\}^n \).

Learning juntas of half-spaces. There has been extensive work on learning intersections and other functions of \( k \) half-spaces \(^{11}\) \( \text{BK97} \), \(^{12}\) \( \text{Vem10b} \), \(^{13}\) \( \text{VX13} \), \(^ {15}\) \( \text{KOS08} \). Note that these algorithms (necessarily) require time polynomial in \( n \) (whereas our raison d’etre is a query complexity independent of \( n \)). In particular, \(^ {11}\) \( \text{BK97} \) provided conditions under which intersections of halfspaces can be learnt under the uniform distribution on the ball. Vempala \(^ {12}\) \( \text{Vem10b} \) extended their result to arbitrary log-concave distributions. In terms of the expressivity of the function class, \(^ {13}\) \( \text{VX13} \) explicitly considered the problem of learning linear \( k \)-juntas (they called it subspace juntas) and showed that a linear \( k \)-junta of the form \( g(w_1, x), \ldots, g(w_k, x) \) is learnable in polynomial time if the function \( g \) is identified by low moments and robust to small rotations in \( \mathbb{R}^n \). Along a related but different axis, \(^ {15}\) \( \text{KOS08} \) showed that functions of bounded surface area in the Gaussian space are learnable in polynomial time. Finally, we remark that there also has been work in learning intersections and other functions of halfspaces over the Boolean hypercube as well \(^ {14}\) \( \text{KOS02} \), \(^ {16}\) \( \text{GKM12} \).

Linearly Invariant Testing over Finite Fields We note that the set of linear-juntas is linearly invariant. If \( f \) is a linear \( k \)-junta and \( B \) is any \( n \times n \) matrix then \( x \mapsto f(Bx) \) is also a linear \( k \)-junta. Over finite fields, \(^ {14}\) \( \text{KS08} \) studied general criteria for when a linearly invariant property is testable, see also \(^ {18}\) \( \text{BFH}^+13 \). In particular, \(^ {14}\) \( \text{GOS}^+09 \), gave a \( 2^{O(k)} \) query complexity algorithm to test linear juntas over finite fields. Moreover, they also show that an exponential lower bound on \( k \) is necessary. This should be contrasted with our result which shows that linear juntas over the Gaussian space can be tested with poly\((k)\) queries.

Testing (functions) of halfspaces The question of testing halfspaces was first considered in \(^ {18}\) \( \text{MORS10} \) who showed that in the Gaussian space (as well as the Boolean space), halfspaces are testable with \( O(1) \) queries. Subsequently, the second and third authors (Mossel and Neeman \(^ {12}\) \( \text{MN15} \)) gave a different testing algorithm for a single halfspace in the Gaussian space. In fact, Harms \(^ {12}\) \( \text{Har19} \) recently showed that halfspaces over any rotationally invariant distribution can be tested with sublinear number of queries. However, as far as we are aware, prior to our work, no non-trivial bounds were known for even testing the intersection of two halfspaces. As remarked earlier, from our work, it follows that for any arbitrary \( k \), intersection of \( k \)-halfspaces can be tested in the Gaussian space with \( \exp(k \log k) \) queries.

1.2 Techniques

A major difference between linear juntas over finite fields and linear juntas over Gaussian space is the “infinitesimal geometry” that can be used in the latter and does not exist in the former. In particular, the linear part \( \mathcal{V}_1(f) \) of the Hermite expansion of \( f \) is approximately given by \( e^{-t}(P_t f - \mathbb{E}[f]) \) for large \( t \). Here \( P_t f \) is the Ornstein-Uhlenbeck operator. Both the quantities, \( \mathbb{E}[f] \) and \( P_t f \) can be approximated
by sampling a small number of points from the Gaussian distribution and evaluating \( f \) at those points. Moreover, if \( f(x) = g(\langle u_1, x \rangle, \ldots, \langle u_k, x \rangle) \) is a linear junta, then the linear part of its Hermite expansion, \( \mathcal{W}_1(f) \), lies in the span of \( u_1, \ldots, u_k \).

We would like to obtain “many more directions” that lie in the span of \( u_1, \ldots, u_k \). We do so by considering functions of the form \( f_{t,y}(x) = f(e^{-t}y + \sqrt{1-e^{-2t}}x) \), for randomly chosen \( y \) and an appropriate value of \( t \) (the experts will recognize \( f_{t,y} \) as part of the definition of the Ornstein-Uhlenbeck operator). Note that \( f_{t,y} \) is also a linear junta defined by the same direction \( u_1, \ldots, u_k \) and therefore the linear part of the Hermite expansion of \( f_{t,y} \), is also in the span of \( u_1, \ldots, u_k \).

It is now natural to propose the following algorithm to test if a function is a linear \( k \)-junta: choose points \( y_i \) at random and “compute” \( \mathcal{W}_1(f_{t,y_i}) \) at these points. Then if the rank of the matrix spanned by \( \langle \mathcal{W}_1(f_{t,y_i}) \rangle_i \) is at most \( k \), then output YES; otherwise, output NO.

Of course, actually computing \( \mathcal{W}_1(f_{t,y}) \) requires \( \text{poly}(n) \gg \text{poly}(k) \) samples. Instead we will approximately compute the Gram matrix

\[
A_{i,j} = \langle \mathcal{W}_1(f_{t,y_i}), \mathcal{W}_1(f_{t,y_j}) \rangle.
\]

and test if it is close or far from a matrix of rank \( k \). One advantage of using the Gram matrix, is that we can evaluate the entries \( A_{i,j} \) by sampling random inputs to evaluate the expected values

\[
\mathbb{E}[\mathcal{W}_1(f_{t,y_i})(x)\mathcal{W}_1(f_{t,y_j})(x)].
\]

How do we know that \( \mathcal{W}_1(f_{t,y_i})(x) \) are not very close to 0? If \( f \) has a bounded surface area then \( f \) is close to the noise stable function \( P_{t}f \). For such noise stable functions, we prove that with good probability at a random point \( x \), \( \mathcal{W}_1(f_{t,y_i})(x) \) will be of non-negligible size. In fact, one of our main technical lemmas (Lemma\(^{30} \)) proves much more. It shows that if \( f \) is \( \epsilon \) far from any linear-\( k \)-junta then for any subspace \( W \) with co-dimension at most \( k \), it holds that for a random \( y \) with probability at least \( \text{poly}(\epsilon) \), the projection of \( \mathcal{W}_1(f_{t,y_i})(x) \) into \( W \) will have norm at least \( \text{poly}(\epsilon) \). This result is later combined with a perturbation argument to establish to show that if \( f \) is \( \epsilon \)-far from a linear \( k \)-junta then indeed the Gram matrix will have \( k + 1 \) large eigenvalues. Since our analysis relies on the function \( f \) having surface area at most \( s \), the first stage of the algorithm uses the algorithm by the third author [Nee14] to test if the function of interest is of bounded surface area.

The algorithm to identify the linear invariant structure of \( f \) builds up on the ideas in the algorithm to test linear \( k \)-juntas. More precisely, we can show that if \( f \) is a linear \( k \)-junta with surface area \( s \),

1. we can find directions \( y_1, \ldots, y_{\ell} \) such that \( f \) is close to a function on the space spanned by the directions \( \mathcal{W}_1(f_{t,y_1}), \ldots, \mathcal{W}_1(f_{t,y_\ell}) \) (for some \( \ell \leq k \)).

2. While we cannot find \( \mathcal{W}_1(f_{t,y_j}) \) explicitly for any \( j \), we can evaluate \( \langle \mathcal{W}_1(f_{t,y_j}), x \rangle \) at any point \( x \) up to good accuracy.

3. With the above observation, the high level idea is to try out all smooth functions on the subspace spanned by \( \{ \langle \mathcal{W}_1(f_{t,y_1}), x \rangle, \ldots, \langle \mathcal{W}_1(f_{t,y_\ell}), x \rangle \} \}. Perform hypothesis testing for each such function against \( f \) and output the most accurate one.

The crucial part in the above argument is that even if we have \( \mathcal{W}_1(f_{t,y_1}), \ldots, \mathcal{W}_1(f_{t,y_\ell}) \) implicitly, the space of “all smooth functions” on \( \text{span}(\langle \mathcal{W}_1(f_{t,y_1}), x \rangle, \ldots, \langle \mathcal{W}_1(f_{t,y_\ell}), x \rangle) \) has a cover whose size is independent of \( n \). This lets us identify the linear invariant function defining \( f \) with query complexity just dependent on \( k \) and \( s \).

In order to prove lower bounds in terms of surface area, we construct a distribution over linear 1-juntas with large surface area by splitting \( \mathbb{R}^2 \) into many very thin parallel strips (oriented in a random direction)
and assign our function a random ±1 value on each strip. (Note that the surface area of such a function is proportional to the number of strips.) The intuition is that no algorithm that makes non-adaptive queries can tell that such a random function is a 1-junta, because in order to “see” one of these strips, the algorithm would need to have queried multiple far-away points in a single strip. But if the number of queries is small relative to the number of strips then this is impossible – with high probability every pair of far-away query points will end up in different strips. In order to make this intuition rigorous, we also introduce a distribution on linear 2-juntas by randomly “cutting” the thin strips once in the orthogonal direction. We show that for any non-adaptive set of queries, the two distributions induce almost identical query distributions, and Yao’s minimax lemma implies that no algorithm can distinguish between our random 1-juntas and our random 2-juntas.

2 Preliminaries

In this paper, unless explicitly mentioned otherwise, the domain $\mathbb{R}^n$ is always endowed with the measure $\gamma_n$, the standard $n$-dimensional Gaussian measure. Likewise, we will only consider functions $f \in L_2(\gamma_n)$. For such a function, and $t > 0$, we recall that for all $q \geq 0$, we can define the Hermite polynomial $H_q : \mathbb{R} \rightarrow \mathbb{R}$ as

$$H_0(x) = 1; \quad H_q(x) = \frac{(-1)^q}{\sqrt{q!}} \cdot e^{x^2/2} \cdot \frac{d^q}{dx^q} e^{-x^2/2}.$$ 

Further, for the ambient space $\mathbb{R}^n$, let us define the space $\mathcal{W}_q$ to be the linear subspace of $L_2(\gamma_n)$ spanned by $\{H_q((v, x)) : v \in S^n\}$. Here $S^n$ denotes the unit sphere in $n$-dimensions. For a function $g \in L_2(\gamma_n)$, we let $\hat{g}_q : \mathbb{R}^n \rightarrow \mathbb{R}$ denote the projection of $g$ to the subspace $\mathcal{W}_q$. Note that for any $g$, $\hat{g}_q$ will be a degree-$q$ polynomial lying in the subspace $\mathcal{W}_q$. We now recall some standard facts from Hermite analysis which can be found in any standard text on the subject (see the book by O’Donnell [O’D14].

**Proposition 4.**

1. For $q \neq q'$, the subspaces $\mathcal{W}_q$ and $\mathcal{W}_{q'}$ are orthogonal. In other words, if $r \in \mathcal{W}_q$ and $s \in \mathcal{W}_{q'}$, then $\mathbb{E}_{x \sim \gamma_n} [r(x) \cdot s(x)] = 0$.
2. Every function $g \in L_2(\gamma_n)$ can be expressed as $g(x) = \sum_{q \geq 0} \hat{g}_q(x)$ where $\hat{g}_q$ is the projection of $g$ to $\mathcal{W}_q$.
3. For any $t > 0$, $(P_t g)(x) = \sum_{q \geq 0} e^{-t \cdot q} \cdot \hat{g}_q(x)$.

2.0.1 Oracle computation

We now list several useful claims which all fit the same motif: Given oracle access to $f : \mathbb{R}^n \rightarrow \mathbb{R}$, what interesting quantities can be computed?

**Lemma 5.** Given oracle access to $f : \mathbb{R}^n \rightarrow [-1, 1]$, error parameter $\eta > 0$, there is a function $f_{\theta,\eta} : \mathbb{R}^n \rightarrow \mathbb{R}$ such that the following holds for every $\lambda \geq 1$,

$$\Pr_{x \sim \gamma_n} [ |f_{\theta,\eta}(x) - \hat{f}_1(x)| > \lambda \cdot \eta ] \leq \lambda^{-2}.$$ 

Further, for any $x \in \mathbb{R}^n$, we can compute $f_{\theta,\eta}(x)$ to additive error $\pm \epsilon$ with confidence $1 - \delta$ by making poly$(1/\eta, 1/\epsilon, \log(1/\delta))$ queries to the oracle for $f$. 

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Proof. Observe that for any $t > 0, P_t f = \sum_{q \geq 0} e^{-tq} \hat{f}_q(x)$. This implies that

$$
\frac{P_t f - \mathbb{E}[f]}{e^{-t}} = \hat{f}_1(x) + \sum_{q > 1} e^{-t(q-1)} \hat{f}_q(x).
$$

Set $t$ so that $e^{-t} = \eta$ and let us define $f_{\delta, \eta}$ as $f_{\delta, \eta} = \frac{P_t f - \mathbb{E}[f]}{\eta}$. Now, observe that for $h(x) = \sum_{q > 1} e^{-t(q-1)} \hat{f}_q(x), \mathbb{E}[h(x)] = 0$ and $\text{Var}[h(x)] \leq \eta^2$. We now apply Chebyshev’s inequality to obtain

$$
\Pr_{x \sim \gamma_n} \left[ |f_{\delta, \eta}(x) - \hat{f}_1(x)| > \lambda \cdot \eta \right] \leq \lambda^{-2}.
$$

Next, observe that both $P_t f(x)$ and $\mathbb{E}[f(x)]$ can be computed to error $\pm \epsilon \cdot \eta$ with confidence $1 - \delta$ using $\text{poly}(1/\eta, 1/\epsilon, \log(1/\delta))$ queries to the oracle for $f$. This immediately implies that $f_{\delta, \eta}$ can be computed to error $\pm \epsilon$ using $\text{poly}(1/\eta, 1/\epsilon, \log(1/\delta))$ queries to the oracle for $f$.

Lemma 6. Given oracle access to functions $f, g : \mathbb{R}^n \rightarrow [-1, 1]$, error parameter $\epsilon > 0$ and confidence parameter $\delta > 0$, there is an algorithm which makes $\text{poly}(1/\epsilon, \log(1/\delta))$ queries to $f, g$ and computes $(\hat{f}_1, \hat{g}_1)$ up to error $\epsilon$ with confidence $1 - \delta$.

Proof. Consider the function

$$
h(x) = e^{2t} (P_t f(x) - \mathbb{E}[f])(P_t g(x) - \mathbb{E}[g]).
$$

Writing out the Fourier expansions of $P_t f$ and $P_t g$, note that $P_t f = \sum_{q \geq 0} e^{-tq} \hat{f}_q(x)$, and so

$$
h(x) = \hat{f}_1(x) \hat{g}_1(x) + \sum_{q,r \geq 1} e^{-t(q+r-2)} \hat{f}_q(x) \hat{g}_r(x).
$$

Since $\hat{f}_1$ and $\hat{g}_1$ are linear functions, $\mathbb{E}[\hat{f}_1(x) \hat{g}_1(x)] = \hat{f}_1 \cdot \hat{g}_1$. On the other hand, $\mathbb{E}[\sum_{q \geq 0} \hat{f}_q^2(x)] = \mathbb{E}[f^2] \leq 1$, and so the Cauchy-Schwarz inequality implies that

$$
\mathbb{E} \left[ \sum_{q,r \geq 1} e^{-t(q+r-2)} \hat{f}_q(x) \hat{g}_r(x) \right] \leq e^{-t}.
$$

Hence, $|\mathbb{E}[h(x)] - \hat{f}_1 \cdot \hat{g}_1| \leq e^{-t}$. If we choose $t$ so that $e^{-t} = \epsilon/2$, then it only remains to show that we can estimate $\mathbb{E}[h(x)]$ within additive error $\epsilon/2$ with confidence $1 - \delta$.

Let $y$ and $z$ be Gaussian random variables, independent of $x$, and write $P_t f(x) = \mathbb{E}_y [f(e^{-t}x + \sqrt{1 - e^{-2t}y})]$ and $P_t g(x) = \mathbb{E}_z [g(e^{-t}x + \sqrt{1 - e^{-2t}z})]$. In particular, we can express $\mathbb{E}[h(x)]$ in the form $\mathbb{E}[J(x, y, z)]$ where

$$
J(x, y, z) = e^{2t} (f(e^{-t}x + \sqrt{1 - e^{-2t}y}) - f(y))(g(e^{-t}x + \sqrt{1 - e^{-2t}z}) - g(z)).
$$

Recalling that $e^{-2t} = 4/\epsilon^2$, it follows that $J$ takes values in $[-4/\epsilon^2, 4/\epsilon^2]$, and it follows from Hoeffding’s inequality that we can approximate $J$ to additive error $\epsilon/2$ with confidence $1 - \delta$ using $\text{poly}(1/\epsilon, \log(1/\delta))$ samples of $J$. Moreover, each sample of $J$ can be computed using two oracle queries to $f$ and two oracle queries to $g$.

Definition 7. A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be a linear $k$-junta if there are at most $k$ orthonormal vectors $u_1, \ldots, u_k \in \mathbb{R}^n$ and a function $g : \mathbb{R}^k \rightarrow \mathbb{R}$ such that

$$
f(x) = g(\langle u_1, x \rangle, \ldots, \langle u_k, x \rangle).
$$

Further, if $u_1, \ldots, u_k \in W$ (a linear subspace of $\mathbb{R}^n$), then $f$ is said to be a $W$-junta.
2.1 Derivatives of functions

We will use $D$ to denote the derivative operator. In case, there are two sets of variables involved, we will explicitly indicate the variable with respect to which we are taking the derivative.

**Definition 8.** For $f : \mathbb{R}^n \to \mathbb{R}$ ($f \in C^\infty$) and $t \geq 0$, define the function $f_t : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$,

$$f_t(y, x) = f(e^{-t}y + \sqrt{1 - e^{-2t}}x).$$

Further, in the same setting as above, we let $f_{t,y} : \mathbb{R}^n \to \mathbb{R}$,

$$f_{t,y}(x) = f(e^{-t}y + \sqrt{1 - e^{-2t}}x).$$

Let $D_x$ denote the derivative operator with respect to $x$ and let $D_y$ denote the derivative operator with respect to $y$. Then, it is easy to observe that

$$\sqrt{e^{2t} - 1} \cdot D_y f_t(y, x) = D_x f_t(y, x). \tag{1}$$

Next, for a function $g : \mathbb{R}^n \to \mathbb{R}$, define $W_1(g) \in \mathbb{R}^n$ as the degree-1 Hermite coefficients of $g$. In other words, the $i^{th}$ coordinate of $W_1(g)$

$$W_1(g)[i] = E[g(x) \cdot x_i],$$

where $x \sim \gamma_n$, the standard $n$-dimensional Gaussian measure. With respect to our earlier definition of $\widehat{g}_1$, observe that we have: $\widehat{g}_1(x) = \langle W_1(g), x \rangle$. We next prove the following important lemma which connects the gradient of $P_t f$ at $y$ with $W_1(f_{t,y})$. In particular, we have the following lemma.

**Lemma 9.**

$$W_1(f_{t,y}) = \sqrt{e^{2t} - 1} \cdot D(P_t f)(y).$$

*Proof.* First of all, observe that for any function $g : \mathbb{R}^n \to \mathbb{R}$ with bounded derivatives, and for any $i \in [n]$

$$E_{x \sim \gamma_n} \left[ \frac{\partial g(x)}{\partial x_i} \right] = \int_x \frac{\partial g(x)}{\partial x_i} \gamma_n(x) dx = \int_x x_i g(x) \gamma_n(x) dx = E_{x \sim \gamma_n} [x_i \cdot g(x)].$$

While the first and last equalities are trivial, the middle is a consequence of integration by parts. Assuming that $f$ has bounded derivatives, we may apply this identity to $g = f_{t,y}$, yielding

$$W_1(f_{t,y}) = E_x [D_x f_t(y, x)]$$

$$= \sqrt{e^{2t} - 1} \cdot E_x [D_y f_t(y, x)] \text{ (applying (1))}$$

$$= \sqrt{e^{2t} - 1} \cdot D_y (E_x [f_t(y, x)]) = \sqrt{e^{2t} - 1} \cdot D_y (P_t f)(y).$$

This proves the lemma in the case that $f$ has bounded derivatives. In the general case, we approximate choose a sequence of functions that have bounded derivatives and approximate $f_{t,y}$ in $L_2(\gamma)$. Applying the lemma to these functions and taking the limit proves the general case. □

**Lemma 10.** Given oracle access to $f$, noise parameter $t > 0$, error parameter $\epsilon > 0$, confidence parameter $\delta > 0$ and $y_1, y_2 \in \mathbb{R}^n$, there is an algorithm which makes poly($1/\epsilon, 1/\delta, 1/t$) queries to $f$ and computes $\langle D(P_t f)(y_1), D(P_t f)(y_2) \rangle$ up to error $\epsilon$ with confidence $1 - \delta$.

*Proof.* By Lemma 9 we have

$$\langle D(P_t f)(y_1), D(P_t f)(y_2) \rangle = \frac{1}{e^{2t} - 1} \cdot \langle W_1(f_{t,y_1}), W_1(f_{t,y_2}) \rangle.$$

We can now apply Lemma 9 to finish the proof. □
Proposition 11. For any $f : \mathbb{R}^n \to [-1, 1]$, $\|D(P_t f)(y)\|_2 \leq (e^{2t} - 1)^{-\frac{1}{2}}$.

Proof. By Lemma 9 we have $\|W_1(f_t,y)\|_2 = \sqrt{e^{2t} - 1} \cdot \|D(P_t f)(y)\|_2$. Now, observe that the range of $f_t,y$ is $[-1, 1]$ and thus, $\|W_1(f_t,y)\|_2 \leq 1$, implying the stated upper bound. □

Lemma 12. Given oracle access to $f : \mathbb{R}^n \to [-1, 1], y \in \mathbb{R}^n$, noise parameter $t > 0$, error parameter $\eta > 0$, there is a function $f_{\partial,\eta,t,y} : \mathbb{R}^n \to \mathbb{R}$ such that the following holds for every $\lambda \geq 1$,

$$\Pr_{x \sim \gamma_n}[|f_{\partial,\eta,t,y}(x) - \langle D(P_t f)(y), x \rangle| > \lambda \cdot \eta] \leq \lambda^{-2}.$$ 

Further, for an error parameter $\epsilon > 0$, confidence parameter $\delta > 0$, we can compute $f_{\partial,t,\eta,y}$ to additive error $\pm \epsilon$ with confidence $1 - \delta$ using poly$(1/t, 1/\eta, 1/\epsilon, \log(1/\delta))$ queries to $f$.

Proof. We first use Lemma 9 and obtain that

$$DP_t f(y) = \frac{1}{\sqrt{e^{2t} - 1}} \cdot W_1(f_{t,y}).$$

Consequently, we have that

$$\langle DP_t f(y), x \rangle = \frac{1}{\sqrt{e^{2t} - 1}} \cdot \widehat{f_{t,y}}(x).$$

The claim now follows from Lemma 5. □

2.2 Some useful inequalities concerning noise stability

Lemma 13. [Poincaré inequality] Let $f : \mathbb{R}^n \to \mathbb{R}$ be a $C^1$ function. Then, $\Var[f] \leq E[\|Df\|_2^2]$.

Definition 14. For a Borel set $A \subseteq \mathbb{R}^n$, we define its Gaussian surface area $\Gamma(A)$ to be

$$\Gamma(A) = \lim_{\delta \to 0} \frac{\vol(A_{\delta} \setminus A)}{\delta},$$

provided the limit exists. Here, for any body $K$, $\vol(K)$ denotes the Gaussian volume of $K$, i.e., $\int_{x \in K} \gamma_n(x)dx$. Further, $A_{\delta} = \{x : d(x, A) \leq \delta\}$ where $d(x, A)$ denotes the Euclidean distance of $x$ from $A$.

For a function $f : \mathbb{R}^n \to \{-1, 1\}$, we denote its surface area $\Gamma(f) = \Gamma(A_f)$ where $A_f = \{x : f(x) = 1\}$.

Ledoux [Led94] (and implicitly Pisier [Pis86]) proved the following connection between noise sensitivity and surface area of functions.

Lemma 15. [Ledoux [Led94]] For any $t \geq 0$ and $f : \mathbb{R}^n \to \{-1, 1\}$, $x, y \sim \gamma_n$, we have

$$\Pr_{x,y}[f(x) \neq f(e^{-t}x + \sqrt{1 - e^{-2t}}y)] \leq 2\sqrt{t} \cdot \Gamma(f).$$

The following proposition is an immediate consequence of the above lemma.

Proposition 16. Let $f : \mathbb{R}^n \to \{-1, 1\}, t \geq 0$ and $\Gamma(f) \leq s$. Then,

1. $E[(f(x) - P_t f(x))^2] = 8s \sqrt{t}$.
2. For any $\epsilon > 0$ and $T = O(s^2/\epsilon^2)$, $\sum_{q \geq T} E[\hat{f}_q^2] \leq \epsilon$. 

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Proof. Let $E_1(x, y)$ denote the event that $f(x) \neq f(e^{-t}x + \sqrt{1-e^{-2t}}y)$. To prove the first item, observe that for any $x$,

\[
(f(x) - P_t f(x))^2 = (2 \mathbb{E}_{y \sim \gamma_n} [1(E_1(x, y))])^2 = 4(\mathbb{E}_{y \sim \gamma_n} [1(E_1(x, y))])^2 \leq 4(\mathbb{E}_{y \sim \gamma_n} [1(E_1(x, y))])
\]

Thus, we obtain that

\[
\mathbb{E}[(f(x) - P_t f(x))^2] \leq 4 \mathbb{E}_{x,y \sim \gamma_n} [1(E_1(x, y))] \leq 8s \sqrt{t},
\]

where the last inequality is an application of Lemma 15. The second item here is the same as Theorem 15 (full version) of [KOS08]. So, we do not prove it here.

2.3 Inequalities for matrix perturbation

We will require some basic results on matrix perturbations. For this, we adopt the following notation: Let $A \in \mathbb{C}^{n \times n}$ be a Hermitian matrix. Then $\sigma_1(A) \geq \ldots \geq \sigma_n(A)$ denote its singular values in order.

Lemma 17. [Weyl’s inequality] Let $A, E \in \mathbb{R}^{n \times n}$ be real symmetric matrices. Then for any $j$,

\[
|\sigma_j(A + E) - \sigma_j(A)| \leq \|E\|_F.
\]

Fact 18. [Sch92] Let $A_1, A_2$ be two psd matrices. Let $\sigma_{\text{min}}(A_1), \sigma_{\text{min}}(A_2) \geq c$. Then,

\[
\|A_1^{1/2} - A_2^{1/2}\|_2 \leq \|A_2 - A_1\|_2 \cdot \frac{1}{2\sqrt{c}}.
\]

Fact 19. [Ste73] Let $A_1, A_2$ be two psd matrices. Let $\sigma_{\text{min}}(A_1) \geq c$ and $\|A_2 - A_1\|_2 \leq c/100$. Then,

\[
\|A_2^{-1} - A_1^{-1}\|_2 \leq \|A_2 - A_1\|_2 \cdot \frac{1}{c^2}.
\]

Combining these two facts, we have the following corollary.

Corollary 20. Let $0 < c < 1$ and let $A_1$ be a psd matrix such that $\sigma_{\text{min}}(A_1) \geq c$. Let $A_2 - A_1$ be real symmetric that $\|A_2 - A_1\|_2 \leq \xi \cdot c$ for $|\xi| \leq 1/100$. Then, $\|A_1^{1/2} - A_2^{-1/2}\|_2 \leq \frac{\xi}{2\sqrt{c}}$.

Proof. We first apply Fact 18 to obtain that

\[
\|A_1^{1/2} - A_2^{1/2}\|_2 \leq \frac{\xi c^{1/2}}{2}.
\]

Observe that $\sigma_{\text{min}}(A_1^{1/2}) \geq \sqrt{c}$. Since $c < 1$ and $|\xi| \leq \frac{1}{100}$, we apply Fact 19 to obtain that

\[
\|A_2^{-1/2} - A_1^{-1/2}\|_2 \leq \frac{\xi}{2\sqrt{c}}.
\]

This finishes the proof.
3 Algorithm to test \( k \)-juntas

In this section, we will prove the following theorem.

**Theorem 21.** There is an algorithm \( \text{Test-linear-junta} \) which has the following guarantee: Given oracle access to \( f : \mathbb{R}^n \to \{-1,1\} \), rank parameter \( k \), surface area parameter \( s \) and error parameter \( \epsilon > 0 \), it makes \( \text{poly}(s,\epsilon^{-1},k) \) queries and

1. If \( f \) is a linear \( k \)-junta with \( \text{surf}(f) \leq s \), then the algorithm outputs yes with probability at least 0.9.
2. If \( f \) is \( O(\epsilon) \)-far from any linear \( k \)-junta \( g \) with \( \text{surf}(g) \leq (1 + \epsilon) \cdot s \), then the algorithm outputs no with probability at least 0.9.

**Remark 22.** A convention that we shall adopt (to avoid proliferation of parameters) is to sometimes ignore the confidence parameter of the testing algorithm. Typically, whenever we can estimate a parameter within \( \pm \epsilon \) with \( T \) queries with confidence \( 2/3 \), we can do the usual “median trick” and get the same accuracy with confidence \( 1 - \delta \) with a multiplicative \( O(\log(1/\delta)) \) overhead in the query complexity. Since we only need to succeed with probability 0.9 in the final algorithm, it is sufficient for each of the individual subroutines to succeed with probability sufficiently close to 1. So, unless it is crucial, at some places, we shall ignore the confidence parameter in the theorem statements and many of the calculations. It will be implicit that the confidence parameter is sufficiently close to 1.

The algorithm \( \text{Test-linear-junta} \) is described in Figure 1. The algorithm invokes two different subroutines, \( \text{Test-surface-area} \) and \( \text{Test-rank} \) whose guarantees we state now. To do this, we first define the notion of \((\epsilon,s)\) smooth function.

**Definition 23.** A function \( f : \mathbb{R}^n \to \{-1,1\} \) is said to be \((\epsilon,s)\)-smooth if there is a function \( g : \mathbb{R}^n \to \{-1,1\} \) such that \( \mathbb{E}[|f - g|] \leq \epsilon \) and \( \text{surf}(g) \leq s(1 + \epsilon) \).

In other words, a function \( f \) is \((\epsilon,s)\) smooth if \( f \) is \( \epsilon \)-close to some other function \( g \) (in \( \ell_1 \) distance) and \( g \) has surface area which is essentially bounded by \( s \). With this definition, we can now state the guarantee of the routine \( \text{Test-surface-area} \) (due to Neeman [Nee14]).

**Theorem 24.** There is an algorithm \( \text{Test-surface-area} \) which given oracle access to a function \( f : \mathbb{R}^n \to \{-1,1\} \) and error parameter \( \epsilon > 0 \) makes \( T_{\text{test}} = \text{poly}(s/\epsilon) \) queries and has the following guarantee:

1. If \( f \) is a function with surface area at most \( s \), then the algorithm outputs yes with probability at least \( 1 - \epsilon \).
2. Any function \( f \) which passes the test with probability 0.1 is \((\epsilon,s)\)-smooth.

Next, we state the guarantee of the routine \( \text{Test-rank} \).

**Lemma 25.** The routine \( \text{Test-rank} \) has a query complexity of \( \text{poly}(k,s,\epsilon^{-1}) \). Further, we have

1. If the function \( f \) is a linear-\( k \)-junta, then the algorithm \( \text{Test-rank} \) outputs yes with probability \( 1 - \epsilon \).
2. If \( f : \mathbb{R}^n \to \{-1,1\} \) is a \((\epsilon/30)^2, s\)-smooth function which is \( \epsilon \)-far from a linear \( k \)-junta, then the algorithm \( \text{Test-rank} \) outputs no with probability \( 1 - \epsilon \).

In order to prove Theorem 21, we will need the following claim which shows that property of closeness to a linear \( k \)-junta and closeness to a smooth function can be certified using a single function.
Lemma 26. For a function \( f : \mathbb{R}^n \rightarrow \{-1,1\} \), suppose that there is a linear \( k \)-junta \( g : \mathbb{R}^n \rightarrow \{-1,1\} \) and a function \( h : \mathbb{R}^n \rightarrow \{-1,1\} \) of surface area at most \( s \) such that both \( g \) and \( h \) are \( \epsilon \)-close to \( f \). Then there is a function \( \hat{h} : \mathbb{R}^n \rightarrow \{-1,1\} \) that is a linear \( k \)-junta and has surface area at most \( s(1 + \sqrt{\epsilon}) \), and which is \( O(\sqrt{\epsilon}) \)-close to \( f \).

Proof of Theorem 21: If \( f \) is a linear \( k \)-junta with surface area at most \( s \), then it passes both the tests \texttt{Test-surface-area} as well as \texttt{Test-rank} with probability \( 1 - \epsilon \). Thus, any linear \( k \)-junta with surface area at most \( s \) passes with probability at least \( 1 - 2\epsilon \) (so as long as \( \epsilon \leq 0.05 \), the test succeeds with probability 0.9).

On the other hand, suppose \( f \) passes \texttt{Test-linear-junta} with probability 0.9. Then, applying Theorem 24 is \((\epsilon/30)^4, s\) smooth. In other words, there is a function \( h \) such that \( \text{surf}(h) \leq (1 + (\epsilon/30)^4) \cdot s \) which is \( O(\epsilon^4) \)-close to \( f \). Further, since \( f \) passes \texttt{Test-rank} with probability 0.9, Lemma 25 implies that \( f \) is \( \epsilon^2 \)-close to some linear \( k \)-junta \( g \). We now apply Lemma 26 to obtain that \( f \) is \( O(\epsilon) \)-close to some function \( \hat{h} : \mathbb{R}^n \rightarrow \{-1,1\} \) which is a linear \( k \)-junta and \( \text{surf}(h) \leq (1 + O(\epsilon))s \). This concludes the proof.

We now turn to describing the routine \texttt{Test-rank} and prove Lemma 25.

Proof of Lemma 25: The bound on the query complexity of Lemma 25 is immediate from the settings of our parameters and query complexity of Lemma 10.

The first item (i.e., the completeness of \texttt{Test-rank}) follows from the fact that if \( f \) is a linear \( k \)-junta, \( P_t f \) is also a linear \( k \)-junta. Consequently, \( A \) is a rank-\( k \) matrix. Then, \( A \) has at most \( k \) non-zero singular values. Thus, if \( \sigma_1 \geq \sigma_2 \geq \ldots \) are the singular values of \( A \) (in order), then \( \sigma_{k+1} = 0 \). By invoking Weyl’s inequality (Lemma 17), the \((k+1)^{th}\) singular value of \( B \) is at most \( \epsilon^2/10 \). This finishes the proof of the first item.

The proof of the second item (i.e., the soundness of \texttt{Test-rank}) is more involved. In particular, we can restate the second item as proving the following lemma.

Lemma 27. Let \( f : \mathbb{R}^n \rightarrow \{-1,1\} \) be a \(((\epsilon/30)^2, s)\)-smooth function which is \( \epsilon \)-far from a linear \( k \)-junta, then the algorithm \texttt{Test-rank} outputs \texttt{no} with probability \( 1 - \epsilon \).

The task of proving this lemma shall be the agenda for the rest of this section.

In order to prove Lemma 27 we will need a few preliminary lemmas. The following lemma says that if a function’s gradient is almost always orthogonal to a subspace \( V \). Then, the function is close to a \( V \)-junta.

Lemma 28. Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) be a \( C^1 \) function and let \( V \) be a subspace of rank \( k \) and let \( W = V^\perp \). Let us assume that \( \mathbb{E}[\|Df\|_W] = \epsilon \). Then there is a \( V \)-junta \( g : \mathbb{R}^n \rightarrow \mathbb{R} \) such that \( \mathbb{E}[(g(x) - f(x))^2] \leq \epsilon \).
### Parameters

| Parameter | Description |
|-----------|-------------|
| $t$       | $\frac{\epsilon^4}{900\gamma^2}$ |
| $r$       | $\frac{\epsilon^2}{\kappa^2}$ |
| $\kappa$  | $\frac{1}{2\epsilon}$ |

#### Testing algorithm

1. Sample directions $y_1, \ldots, y_r \sim \gamma_n$.
2. Let $A_{i,j} = \langle DP_i f(y_i), DP_j f(y_j) \rangle$.
3. For all $1 \leq i, j \leq r$, compute $A_{i,j}$ up to error $\kappa$ using Lemma 10. Call the estimates $B_{i,j}$.
4. For the matrix $B \in \mathbb{R}^{r \times r}$, compute the top $k + 1$ singular values of $B$.
5. Output yes if and only if the $(k + 1)^{st}$ singular value is at most $\frac{\epsilon^2}{\kappa^2}$.

Figure 2: Description of the Test-rank algorithm

**Proof.** Let us rotate the space so that $V = \{(x_1, \ldots, x_k, 0, \ldots, 0) : x_1, \ldots, x_k \in \mathbb{R}\}$. Let us now define $g : \mathbb{R}^n \to \mathbb{R}$ as

$$g(x) = \mathbb{E}_{z \sim \gamma_{n-k}} [f(x_1, \ldots, x_k, z_1, \ldots, z_{n-k})].$$

Observe that $g$ is a $V$-junta. Now, for every choice $X = (x_1, \ldots, x_k)$, consider the function $h_X : \mathbb{R}^{n-k} \to \mathbb{R}$ as

$$h_X(z_1, \ldots, z_{n-k}) = f(x_1, \ldots, x_k, z_1, \ldots, z_{n-k}) - g(x).$$

Observe that $\mathbb{E}_{(z_1, \ldots, z_{n-k}) \sim \gamma_{n-k}} [h_X(z_1, \ldots, z_{n-k})] = 0$. By applying Lemma 13,

$$\mathbb{E}[h_X^2(z_1, \ldots, z_{n-k})] = \text{Var}[h_X(z_1, \ldots, z_{n-k})] \leq \mathbb{E}[\|Dh_X\|_2^2].$$

Observe that $Dh_X(z_1, \ldots, z_{n-k}) = Df(x_1, \ldots, x_k, z_1, \ldots, z_{n-k})$. Thus, we get

$$\mathbb{E}[(f(x) - g(x))^2] = \mathbb{E}_{X \sim \gamma_k} \mathbb{E}_{Z \sim \gamma_{n-k}} [h_X^2(Z)] \leq \mathbb{E}_{X \sim \gamma_k} \mathbb{E}_{Z \sim \gamma_{n-k}} [\|Dh_X\|_2^2] = \mathbb{E}_{X \sim \gamma_k} \mathbb{E}_{Z \sim \gamma_{n-k}} [\|Df(X, Z)\|_2^2] = \epsilon.$$  

This finishes the proof. $\square$

For the rest of this section, when we use the value $t$, it will bear the same relation as stated in the description of the algorithm Test-rank (see Figure 2).

**Proposition 29.** Let $f : \mathbb{R}^n \to \{-1, 1\}$ which is $((\epsilon/30)^2, s)$-smooth. Then, $\mathbb{E}[\|P_t f - f\|^2] \leq \frac{\epsilon^2}{5}$.  

**Proof.** Since $f$ is $((\epsilon/30)^2, s)$ smooth, we know that there is a function $g$ such that $\mathbb{E}[|f - g|] \leq (\frac{\epsilon}{30})^2$ and $\text{surf}(g) \leq s(1 + (\frac{\epsilon}{30})^2)$. By using the fact that the operator $P_t$ is contractive, we have,

$$\mathbb{E}[\|P_t f - P_t g\|^2] \leq \mathbb{E}[|f - g|^2] \leq 4\mathbb{E}[|f - g|_1] \leq \frac{\epsilon^2}{200}.$$
Next, we use Proposition 16 to get that $\mathbb{E}[|P_{t}g - g|^2] \leq \frac{\epsilon^2}{25}$. We can now combine these to get
\[
\mathbb{E}[|P_{t}f - f|^2] = 3 \left( \mathbb{E}[|P_{t}f - P_{t}g|^2] + \mathbb{E}[|P_{t}g - g|^2] + \mathbb{E}[|f - g|^2] \right) \leq \frac{\epsilon^2}{5}.
\]

**Lemma 30.** Let $f : \mathbb{R}^n \to \{-1, 1\}$ be a $((\epsilon/30)^2, s)$-smooth function which is $\epsilon$-far from any linear $k$-junta. For any subspace $W$ of co-dimension at most $k$,
\[
\mathbb{P}_{y \sim \gamma_n} \left[ \left\|DP_{t}f(y)\right\|_W^2 \geq \frac{\epsilon^2}{8} \right] \geq \Omega \left( \frac{\epsilon^6}{s^2} \right).
\]

**Proof.** Applying Proposition 29, we have that $\mathbb{E}[|P_{t}f - f|^2] \leq \frac{\epsilon^2}{25}$. By applying Jensen’s inequality, we have $\mathbb{E}[|P_{t}f - f|] \leq \frac{\epsilon}{\sqrt{5}}$. Thus, $P_{t}f$ is $0.5 \cdot \epsilon$-far from any linear $k$-junta (in $\ell_1$ distance). Consequently, we can say that for any $W$-junta $h$, $\mathbb{E}[|P_{t}f - h|_{\frac{1}{2}}] > 0.25\epsilon^2$. By contrapositive of Lemma 28, we have that
\[
\mathbb{E}[|DP_{t}f(y)|_W^2] > 0.25 \cdot \epsilon^2.
\]

Next, observe that Lemma 9 implies that
\[
\left\|DP_{t}f(y)\right\|_W^2 \leq \frac{1}{e^{2t}t - 1} \cdot \left\|W_{1}(ft,y)\right\|_2^2 \leq \frac{1}{e^{2t}t - 1} \leq O(1/t) \leq O \left( \frac{s^2}{\epsilon^4} \right).
\]
The second inequality follows immediately from that $f_{t,y}$ has range bounded between $[-1, 1]$. Combining this with (2), this implies that
\[
\mathbb{P} \left[ \left\|DP_{t}f(y)\right\|_W^2 \geq \frac{\epsilon^2}{8} \right] \geq \Omega \left( \frac{\epsilon^6}{s^2} \right).
\]
Proof. We will prove this claim by induction. So, assume that the top \( j \) singular values of \( G_j \) are all at least \( \epsilon^2/8 \). Now, for \( \ell = \tau_{j+1} \), let \( w \) be the unit vector in the direction of the component of \( Dh(y_\ell) \) orthogonal to \( W_j \). Let \( \Gamma \) be the linear span of \( W_j \) and \( w \). Now, consider any unit vector \( v \in \Gamma \) and express it as \( v = v_1 + v_2 \) where \( v_1 \) lies in \( W_j \) and \( v_2 \) is parallel to \( w \). Next, observe that
\[
v^T \cdot \left( G_j + Dh(y_\ell) \cdot Dh(y_\ell)^T \right) \cdot v = v^T \cdot G_j \cdot v + v^T \cdot Dh(y_\ell) \cdot Dh(y_\ell)^T \cdot v.
\]
The first term \( v^T \cdot G_j \cdot v \) is at least as large as \( v_1^T \cdot G_j \cdot v_1 \) and the second term \( v^T \cdot Dh(y_\ell) \cdot Dh(y_\ell)^T \cdot v \) is the same as \( v_2^T \cdot Dh(y_\ell) \cdot Dh(y_\ell)^T \cdot v_2 \). Next, note that
\[
v_1^T \cdot G_j \cdot v_1 \geq \frac{\epsilon^2}{8} \cdot ||v_1||_2^2; \quad v_2^T \cdot Dh(y_\ell) \cdot Dh(y_\ell)^T \cdot v_2 \geq \frac{\epsilon^2}{8} \cdot ||v_2||_2^2.
\]
Consequently,
\[
v^T \cdot \left( G_j + Dh(y_\ell) \cdot Dh(y_\ell)^T \right) \cdot v \geq \frac{\epsilon^2}{8} \cdot (||v_1||_2^2 + ||v_2||_2^2) = \frac{\epsilon^2}{8}.
\]
Observe that
\[
v^T \cdot G_{j+1} \cdot v \geq v^T \cdot \left( G_j + Dh(y_\ell) \cdot Dh(y_\ell)^T \right) \cdot v \geq \frac{\epsilon^2}{8}.
\]
The first inequality is immediate from the fact that \( ||v||_2^2 \) are all at least \( \epsilon^2 \), \( v \in \Gamma \). Let \( \ell = \tau_{j+1} \) be the linear span of \( v \), where \( v_2 \), \( v_3 \), \ldots , \( v_{\ell-1} \) are all at least \( \epsilon^2 \), \( v_2 \in \Gamma \). Let \( \ell = \tau_{j+1} \) be the unit vector in the direction of the component of \( Dh(y_\ell) \) orthogonal to \( \Gamma \). Now, consider any unit vector \( v \in \Gamma \) and express it as \( v = v_1 + v_2 \). Next, observe that
\[
v_1^T \cdot G_j \cdot v_1 \geq \frac{\epsilon^2}{8} \cdot ||v_1||_2^2; \quad v_2^T \cdot Dh(y_\ell) \cdot Dh(y_\ell)^T \cdot v_2 \geq \frac{\epsilon^2}{8} \cdot ||v_2||_2^2.
\]
Consequently,
\[
v^T \cdot G_j \cdot v_1 \geq \frac{\epsilon^2}{8} \cdot ||v_1||_2^2; \quad v_2^T \cdot Dh(y_\ell) \cdot Dh(y_\ell)^T \cdot v_2 \geq \frac{\epsilon^2}{8} \cdot ||v_2||_2^2.
\]
Thus, with probability \( 1 - \epsilon \), \( \sigma_{k+1} \) is a geometric random variable with parameter \( (\ell, \epsilon^2/8) \). From this, it is not difficult to see that with probability at least \( 1 - \epsilon \), \( \ell = O(s^2 \cdot k/\epsilon^7) \). Thus, with probability \( 1 - \epsilon \), we can assume that the top \( k + 1 \) singular values of \( M \cdot M^\dagger \) are all at least \( \epsilon^2/8 \).

This finishes the proof.

Now applying Lemma \[ \ref{lem:singular_values} \] we have that conditioned on \( \tau_j \), \( \tau_{j+1} - \tau_j \) is a geometric random variable with parameter \( (\Omega((\epsilon^8/s^2)) \). From this, it is not difficult to see that with probability at least \( 1 - \epsilon \), \( \tau_{k+1} = O(s^2 \cdot k/\epsilon^7) \). Thus, with probability \( 1 - \epsilon \), we can assume that the top \( k + 1 \) singular values of \( M \cdot M^\dagger \) are all at least \( \epsilon^2/8 \).

Consequently, we get that the top \( k + 1 \) singular values of \( A = M^\dagger \cdot M \) are all at least \( \epsilon^2/8 \). Now, the algorithm computes a matrix \( B \) such that \( ||A - B||_2 \leq \epsilon^2/10 \). By Weyl’s inequality (Lemma \[ \ref{lem:weyl} \]), we get that the top \( k + 1 \) singular values of \( B \) are all at least \( \epsilon^2/16 \). This proves the lemma.

We finally give the proof of Lemma \[ \ref{lem:co-area} \] The proof relies on the so-called co-area formula.
Lemma 32. Let $f : \mathbb{R}^n \to [-1,1]$ be smooth and $\psi : [-1,1] \to \mathbb{R}_+$ be bounded and measurable. Then

$$\int_{-1}^1 \psi(s) \text{surf}(\{x : f(x) \leq s\}) \, ds = \int_{\mathbb{R}^n} \psi(f(x)) |\nabla f(x)| \, d\gamma(x).$$

Proof of Lemma 26. By [Mag12], there is a smooth function $h_1 : \mathbb{R}^n \to [-1,1]$ with bounded gradient such that $\|h_1 - h\|_2 \leq \epsilon$ and $\mathbb{E}[|\nabla h_1|] \leq 2\epsilon$. Let $E$ be a $k$-dimensional subspace for which $g$ is an $E$-junta, and let $z$ be a standard Gaussian vector on $E^\perp$. Let $\Pi_E$ be the projection operator for subspace $E$ and define $h_2 : \mathbb{R}^n \to [-1,1]$ by $h_2(x) = E_z(h_1(\Pi_E x + z))$. By Jensen’s inequality, $\mathbb{E}[|\nabla h_2|] \leq \mathbb{E}[|\nabla h_1|] \leq 2\epsilon$. Let $t$ be uniformly distributed in $[-1+\eta, 1-\eta]$, and define $h = \tilde{h}_t$ by

$$\tilde{h}_t(x) = \tilde{h}(x) = \begin{cases} -1 & \text{if } h_2(x) \leq t \\ 1 & \text{otherwise} \end{cases}.$$ 

Note that $\tilde{h}_t$ is an $E$-junta (because $h_2$ is an $E$-junta). In expectation over $t$, the surface area of $\tilde{h}$ is

$$\frac{1}{2-2\eta} \int_{-1+\eta}^{1-\eta} \text{surf}(\{x : h_2 \leq s\}) \, ds,$$

which by the co-area formula is equal to

$$\frac{1}{2-2\eta} \int_{\mathbb{R}^n} 1_{\{h_2(x) \in [-1+\eta, 1-\eta]\}} |\nabla h_2(x)| \, d\gamma(x) \leq \frac{1}{2-2\eta} \mathbb{E}_{x \sim \gamma} |\nabla h_2(x)| \leq \frac{s}{1-\eta}.$$ 

In particular, there exists some $t \in [-1+\eta, 1-\eta]$ such that the surface area of $\tilde{h}_t$ is at most $\frac{s}{1-\eta}$.

Next, we will estimate the distance of $\tilde{h}$ from $h$. By the triangle inequality, $\|h - g\|_2 \leq 2\epsilon$ and so $\|h_1 - g\|_2 \leq 3\epsilon$. On the other hand, Pythagoras’ theorem implies that $h_2$ minimizes $\|h_1 - h_2\|_2$ among all $E$-juntas; hence, $\|h_1 - h_2\|_2 \leq 3\epsilon$ and so $\|h - h_2\|_2 \leq 4\epsilon$. Now, $h$ takes values in $\{-1,1\}$ and so $|h(x) - h_2(x)| \geq \eta 1_{\{h_2(x) \in [-1+\eta, 1-\eta]\}}$. On the other hand, the definition of $\tilde{h}$ ensures that

$$|\tilde{h}(x) - h_2(x)| \leq \begin{cases} 2 & \text{if } h_2(x) \in [-1+\eta, 1-\eta] \\ \eta & \text{otherwise} \end{cases}.$$ 

If $p$ is the probability that $h_2(x) \in [1+\eta, 1-\eta]$, it follows that $\eta \sqrt{p} \leq \|h - h_2\|_2$ and so

$$\|\tilde{h} - h_2\|_2 \leq 2\sqrt{p} + \eta \leq \frac{8\epsilon}{\eta} + \eta.$$ 

By the triangle inequality $\|\tilde{h} - h\|_2 \leq 4\epsilon + \frac{8\epsilon}{\eta} + \eta$. Choosing $\eta = \sqrt{\epsilon}$ completes the proof. \hfill \qed

4 Algorithm to find hidden linear invariant structure

In this section, we will prove the following main theorem.

Theorem 33. Let $f : \mathbb{R}^n \to \{-1, 1\}$ be a linear-$k$-junta with surface area $s$. Then, there is an algorithm Find-invariant-structure which for any error parameter $\epsilon > 0$, makes $O(s \cdot k/\epsilon)^{O(k)}$ queries to $f$ and with probability $1 - \epsilon$ outputs (for some $\ell \leq k$) a function $g : \mathbb{R}^\ell \to [-1,1]$ so that the following holds: there is an orthonormal set of vectors $w_1, \ldots, w_{\ell} \in \mathbb{R}^n$ such that

$$\mathbb{E}[|f(x) - g(w_1, x, \ldots, w_{\ell}, x)|] = O(\epsilon).$$

Further, there is a set $V = \{v_1, \ldots, v_k\}$ of orthonormal vectors such that for $1 \leq j \leq \ell$, $v_j = w_j$ and span$\{v_1, \ldots, v_k\}$ is a relevant subspace of $f$. 

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Our algorithm is quite naïve. First, we "identify" – in some implicit sense – the \( k \)-dimensional subspace on which the linear \( k \)-junta acts. We take a fine net of functions defined on that space, and we test them all until we fine the best one. Obviously, this algorithm is not computationally efficient, and it is also not particularly efficient in terms of the query complexity. However, the crucial feature of this algorithm is that its query complexity does not depend on the ambient dimension \( n \). The main difficulty in constructing and analyzing this algorithm is that we cannot explicitly identify even a single vector in the interesting \( k \)-dimensional subspace – that would require a number of queries that depends on \( n \). One consequence of this is that we do not know how to apply an off-the-shelf learning algorithm (such as the one from [KOS08]).

**Definition 34.** A set of vectors \( v_1, \ldots, v_\ell \in \mathbb{R}^n \) is said to be \((\eta, \gamma)\)-linearly independent if the following conditions hold:

1. For all \( 1 \leq i \leq \ell \), \( \|v_i\|_2 \leq \eta \).
2. For all \( 1 < i \leq \ell \), \( \text{dist}(v_i, \text{span}(v_1, \ldots, v_{i-1})) \geq \gamma \).

**Definition 35.** For \( f : \mathbb{R}^n \to [-1, 1] \) and \( t > 0 \), we say that a set of directions \( (y_1, \ldots, y_\ell) \) is \( \gamma \)-linearly independent, if the following holds: For \( 1 \leq i \leq \ell \), let \( v_i = \text{DP}_tf(y_i) \). If for all \( i \), \( \text{dist}(v_i, \text{span}(v_1, \ldots, v_{i-1})) \geq \gamma \).

By Proposition 11 it is immediate that as long as \( t \leq 1/4, \|\text{DP}_tf(y)\|_2 \leq t^{-1/2} \). Thus, if \((y_1, \ldots, y_\ell)\) is \( \gamma \)-linearly independent, then the directions \((v_1, \ldots, v_\ell)\) are \((t^{-1/2}, \gamma)\) linearly independent.

| Inputs |
|--------|
| \( t \) | := noise parameter |
| \( y_1, \ldots, y_\ell \) | := \( \frac{\gamma}{2} \)-linearly independent directions |
| \( \{\beta_{i,j}\} \) | := \( \lambda \)-accurate estimates of \( \langle \text{DP}_tf(y_i), \text{DP}_tf(y_j) \rangle \) where \( \lambda = \lambda(\ell, \nu, t^{-\frac{1}{2}}, \gamma/2) \) and \( \nu = \frac{\gamma^2 t^{1/2}}{100\ell^2} \) (from Lemma 57) |
| \( y_{\ell+1} \) | := candidate direction in \( \mathbb{R}^n \). |

**Testing algorithm**

1. Find the numbers \( \{\alpha_{1 \leq i,j \leq \ell}\} \) from Lemma 57.
2. Estimate \( \langle \text{DP}_tf(y_{\ell+1}), \text{DP}_tf(y_{\ell+1}) \rangle \) up to \( \pm \frac{\gamma^2 \eta}{\sqrt{2}} \). Call the estimate \( \tilde{\beta}_{\ell+1,\ell+1} \).
3. Estimate \( \langle \text{DP}_tf(y_{\ell+1}), \text{DP}_tf(y_j) \rangle \) (for \( 1 \leq j \leq \ell \)) up to accuracy \( \frac{1}{\xi(\ell, t^{-1/2}, \gamma/2)} \cdot \gamma \cdot \sqrt{\eta} \) using Lemma 10 where \( \xi \) is the function from Lemma 57. Call the estimates \( \tilde{\beta}_{j,\ell+1} \).
4. Compute quantity \( \zeta_i = \sum_{1 \leq j \leq \ell} \alpha_{i,j} \cdot \tilde{\beta}_{j,\ell+1} \) for all \( 1 \leq i \leq \ell \).
5. If the quantity \( \tilde{\beta}_{\ell+1,\ell+1}^2 - \sum_{i=1}^\ell \zeta_i^2 > (\frac{3\gamma}{4})^2 \), then output yes. Else output no.

Figure 3: Description of the algorithm Test-candidate-direction

**Lemma 36.** The algorithm Test-candidate-direction described in Figure 3 has the following properties: For noise parameter \( t \), directions \( y_1, \ldots, y_\ell \in \mathbb{R}^n \), \( \{\beta_{i,j}\} \) and candidate direction \( y_{\ell+1} \) (where \( y_1, \ldots, y_\ell \) as well as \( \{\beta_{i,j}\} \) meet the requirements described in Figure 3), the algorithm satisfies

1. The query complexity of the algorithm is \( T_{tc}(t, \gamma, \ell) = \left( \frac{\ell}{\sqrt{\gamma \ell}} \right)^O(\ell) \).
2. If the Euclidean distance of $DP_t f(y_{t+1})$ is at least $\gamma$ from the subspace $\text{span}(DP_t f(y_1), \ldots, DP_t f(y_t))$, then the algorithm outputs \text{yes}. Conversely, if the algorithm outputs \text{no}, then the Euclidean distance must be less than $\frac{\gamma}{2}$.

\textbf{Proof.} The query complexity bound is just immediate from Lemma 10 and plugging in the value of $\xi(\ell, t^{-1/2}, \gamma)$ from Lemma 57. To prove the second guarantee, let us use $v_j$ to denote $DP_t f(y_j)$. Since $(v_1, \ldots, v_j)$ are $(1/t^{-1/2}, \frac{\gamma}{2})$-linearly independent, hence by Lemma 57 we obtain that there are orthonormal vectors $(w_1, \ldots, w_\ell)$ (which span $v_1, \ldots, v_\ell$) such that

$$
\|w_i - \sum_j \alpha_{i,j} v_j\|_2 \leq \frac{\gamma^2 \cdot t}{100\ell^2}.
$$

This implies that if we let $v_{\ell+1} = DP_t f(y_{\ell+1})$ (using $\|v_{\ell+1}\| \leq t^{-1/2}$), then

$$
|\langle w_i, v_{\ell+1} \rangle - \sum_j \alpha_{i,j} \langle v_j, v_{\ell+1} \rangle| \leq \frac{\gamma^2 \sqrt{t}}{100\ell^2}.
$$

Consequently, we have

$$
|\langle w_i, v_{\ell+1} \rangle - \sum_j \alpha_{i,j} \tilde{\beta}_{j,\ell+1} | \leq \frac{\gamma^2 \cdot \sqrt{t}}{100\ell^2} + \sum_j |\alpha_{i,j}| \cdot |\langle v_j, v_{\ell+1} \rangle - \tilde{\beta}_{j,\ell+1} |
$$

$$
\leq \frac{\gamma^2 \cdot \sqrt{t}}{100\ell^2} + \sum_j \xi(\ell, t^{-1/2}, \gamma/2) \cdot \frac{1}{\xi(\ell, t^{-1/2}, \gamma/2)} \leq \frac{\gamma^2 \sqrt{t}}{50\ell^2} \cdot \frac{\gamma^2 \sqrt{t}}{100\ell^2} \leq \frac{\gamma^2 \sqrt{t}}{50\ell^2}.
$$

The penultimate inequality follows from the bound on $|\alpha_{i,j}|$ from Lemma 57 and the accuracy of estimates $\tilde{\beta}_{j,\ell+1}$. This implies that for any $i$,

$$
\left|\left| \langle w_i, v_{\ell+1} \rangle \right| - \sum_j \alpha_{i,j} \tilde{\beta}_{j,\ell+1} \right|^2 \leq \frac{\gamma^2 \sqrt{t}}{50\ell^2} \left|\langle w_i, v_{\ell+1} \rangle \right| + \sum_j \alpha_{i,j} \tilde{\beta}_{j,\ell+1} \leq \frac{\gamma^2 \sqrt{t}}{50\ell^2} \cdot 2 \cdot t^{-\frac{\ell}{2}} \leq \frac{\gamma^2}{50\ell^2}, \quad \text{(4)}
$$

The second inequality uses that fact that $w_i$ is a unit vector whereas $\|v_{\ell+1}\|_2 \leq t^{-\frac{\ell}{2}}$. Thus,

$$
dist^2(DP_t f(y_{\ell+1}), \text{span}(DP_t f(y_1), \ldots, DP_t f(y_\ell))) = \|DP_t f(y_{\ell+1})\|_2^2 - \sum_{j=1}^{\ell} \langle DP_t f(y_{\ell+1}), w_j \rangle^2
$$

$$
= \|DP_t f(y_{\ell+1})\|_2^2 - \sum_{j=1}^{\ell} \xi_j^2 + \theta
$$

where $|\theta| \leq \frac{\gamma^2}{25\ell^2}$ (from [4]). Using the fact that $|\tilde{\beta}_{\ell+1,\ell+1}^2 - \|DP_t f(y_{\ell+1})\|_2^2| \leq \frac{\gamma^2}{50\ell^2}$, we can conclude that

$$
\left|\left| \text{dist}^2(DP_t f(y_{\ell+1}), \text{span}(DP_t f(y_1), \ldots, DP_t f(y_\ell))) - \tilde{\beta}_{\ell+1,\ell+1}^2 \right| - \sum_{i=1}^{\ell} \xi_i^2 \right| \leq \frac{\gamma^2}{25}.
$$

Item 2 in the claim is now an immediate consequence. \hfill \Box

We now give an algorithm which finds out directions $\{y_1, \ldots, y_\ell\}$ such that for $t$ defined before (as $t := \frac{e^4}{900\pi^2}$), $P_t f$ is close to a junta on the directions $\{DP_t f(y_1), \ldots, DP_t f(y_\ell)\}$.
Proof. The bound on the query complexity of this algorithm is immediate by just plugging in the query complexity of Step 4 (Lemma 10).

Next, observe that by the guarantee of Test-candidate-direction, the set $S$ output by the algorithm consists of $\gamma/2$-linearly independent directions. Finally, assume that $f$ is a $W$-junta where $\dim(W) \leq k$. Then, note that for any $y \in \mathbb{R}^n$, $DP_t f(y) \in W$. Now, there are two possibilities: (For the rest of this proof, we will use $v_i$ as a shorthand for $DP_t f(y_i)$)

(a) If $\text{count} = k$, then note that we have found $k$ directions $y_1, \ldots, y_k$ such that $v_i \in W$. Further, the directions $(v_1, \ldots, v_k)$ are $(t^{-1/2}, \gamma)$-linearly independent. Thus, span$v(v_1, \ldots, v_k) = W$. So, in this case, $P_t f$ is indeed a junta on span$(v_1, \ldots, v_k)$ (where $S = \{y_1, \ldots, y_k\}$).

(b) If $\text{count} < k$, then we are in one of the two situations: either $f$ is $\epsilon$-close to a junta on span$(v_1, \ldots, v_\ell)$
With this, we can now apply Lemma 38 and obtain that with probability at least $\tau_{\text{suc}}$, a randomly chosen direction $z$ will be at least $\gamma = \epsilon^2/8$-far from the subspace $\text{span}(v_1, \ldots, v_\ell)$ and will thus pass the algorithm Test-candidate-direction. Thus, over $T_{\text{suc}}$ trials, with probability at least $1 - \frac{\epsilon}{10k}$, the set $S$ will increase in size and we will continue inductively. Since the outer loop (i.e., the loop for count will run at most $k$ times), the total probability that $P_t f$ is not $\epsilon$-close to a $W$-junta for $W = \text{span}(v_1, \ldots, v_\ell)$ but the algorithm terminates is at most $1 - \frac{\epsilon}{10}$. This finishes the proof.

\[ \square \]

With the aid of the algorithm Find-candidate-directions, we are able to find implicitly find directions $\{v_1, \ldots, v_\ell\}$ such that $P_t f$ is close to a junta on $\text{span}(v_1, \ldots, v_\ell)$. In the next subsection, we essentially do a hypothesis testing over a set of functions which form a cover for all juntas on $\text{span}(v_1, \ldots, v_\ell)$.

4.1 Hypothesis testing against subspace juntas

The following lemma says how given the directions $y_1, \ldots, y_\ell$ and an error parameter $\tau$, we can implicitly find directions which form an orthonormal basis of $\text{span}(v_1, \ldots, v_\ell)$ (as before, we are using $v_1, \ldots, v_\ell$ as a shorthand for $DP_t f(y_1), \ldots, DP_t f(y_\ell)$ respectively). All the symbols below will have the same value as Lemma 37 unless mentioned otherwise.

**Lemma 38.** Choose any error parameter $\tau > 0$ and let $y_1, \ldots, y_\ell$ be $\gamma/2$-linearly independent directions for $P_t f$. Then, there is a procedure Compute-ortho-transform which makes $T_{\text{ortho}} = \text{poly}(1/\tau) \cdot \left( \frac{\ell}{\tau} \right)^{O(\ell)}$ queries to $f$, we can obtain numbers $\{\alpha_{i,j}\}_{1 \leq i,j \leq \ell}$ such that the following holds:

1. For $\Lambda(\ell, t, \gamma) = \left( \frac{\ell}{\tau} \right)^{O(\ell)}$, all the numbers $|\alpha_{i,j}| \leq \Lambda(\ell, t, \gamma)$.

2. There exists an orthonormal basis $(w_1, \ldots, w_\ell)$ of $\text{span}(v_1, \ldots, v_\ell)$ such that for all $1 \leq i \leq \ell$,

$$\|w_i - \sum_j \alpha_{i,j}v_j\|_2 \leq \tau.$$  

**Proof.** Let $\lambda(\cdot)$ be the function defined in Lemma 37. Now, observe that

$$\lambda(\ell, \tau, t^{-1/2}, \gamma) = \tau \cdot \left( \frac{\gamma \cdot t}{2 \cdot \ell} \right)^{O(\ell)}.$$  

Thus, using Lemma 10 we can use $T_{\text{ortho}}$ queries to $f$ to obtain numbers $\{\beta_{i,j}\}_{1 \leq i,j \leq \ell}$ such that

$$|\beta_{i,j} - \langle D_y h(y_i), D_y h(y_j) \rangle| \leq \lambda(\ell, \tau, t^{-1/2}, \gamma).$$

As $(y_1, \ldots, y_\ell)$ are $\gamma$-linearly independent, hence the vectors $(v_1, \ldots, v_\ell)$ are $(t^{-\frac{1}{2}}, \gamma)$-linearly independent. With this, we can now apply Lemma 37 to obtain numbers $\{\alpha_{i,j}\}$ such that there is an orthonormal basis $(w_1, \ldots, w_\ell)$ of $\text{span}(DP_t f(y_1), \ldots, DP_t f(y_\ell))$ with the property that (a) $\|w_i - \sum_j \alpha_{i,j}v_j\|_2 \leq \tau$ and (b) $|\alpha_{i,j}| \leq \Lambda(\ell, t, \gamma)$ where $\Lambda(\ell, t, \gamma) = \left( \frac{\ell}{\tau} \right)^{O(\ell)}$.

\[ \square \]

Let us now again set the parameters $t$ and $\gamma$ exactly the same as Lemma 37. Namely, we set $t = \frac{\epsilon^4}{900s^2}$ and $\gamma = \frac{\epsilon^2}{8}$. With this setting of parameters, we state the following lemma.

**Lemma 39.** There is an algorithm Estimate-closest-hypothesis (described in Figure 5) which takes as input oracle access to $f: \mathbb{R}^n \rightarrow \{-1, 1\}$, directions $(y_1, \ldots, y_\ell)$ which are $\gamma/2$-linearly independent, error parameter $\epsilon$, surface area parameter $s$. The algorithm has the following guarantee:
Inputs

\( s \) := surface parameter
\( \epsilon \) := error parameter
\( y_1, \ldots, y_\ell \) := \( \frac{2}{\epsilon} \)-linearly independent directions for \( P_t f \)

Parameters

\( t \) := \( \frac{\epsilon^4}{900 s^3} \)
\( \gamma \) := \( \frac{\epsilon^2}{8} \)
\( \tau \) := \( \frac{\epsilon^2}{100} \cdot \ell^{3/2} \)
\( \delta \) := \( \frac{\epsilon}{10} \)
\( K \) := \( \ell^2 \cdot \Lambda(\ell, t, \gamma) \) where \( \Lambda(\cdot) \) is defined in Lemma 38
\( \xi \) := \( \frac{\epsilon^2}{10} \cdot \sqrt{t} \cdot K \cdot \ell^3 \)
\( \mu \) := \( \frac{\epsilon}{10} \cdot \log(1/\mu) \)
\( J \) := \( 10 \cdot \frac{\epsilon^2}{\log(1/\mu)} \)

Testing algorithm

1. Run the procedure Compute-ortho-transform with directions \( (y_1, \ldots, y_\ell) \) and \( \gamma, t \) and \( \tau \) as set above.
2. Let the output be parameters \( \{\alpha_{i,j}\}_{1 \leq i, j \leq \ell} \).
3. Sample \( J \) points from \( \gamma_n \). Call the points \( x_1, \ldots, x_J \).
4. For each of the points \( x_i \) and each direction \( y_j \),
5. Compute the function \( f_{\partial, \xi, t, y_j}(x_i) \) up to error \( \xi \) using Lemma 12. Call this \( \zeta_{i,j} \).
6. Compute \( x_i, j' = \sum_j \alpha_{j', j} \cdot \zeta_{i,j} \).
7. For all \( g \in \text{Cover}(t, \ell, \delta) \), compute \( O_g = \frac{1}{s} \cdot \sum_{i=1}^{s} |P_t f(x_i) - g(\langle w_1, x_i \rangle, \ldots, \langle w_\ell, x_i \rangle)| \).
8. Return the \( g \) which has the smallest value of \( O_g \).

Figure 5: Description of the algorithm Estimate-closest-hypothesis

1. It makes \( O\left(\frac{s \sqrt{\ell}}{\epsilon}\right)^{O(\ell)} \) queries to \( f \).

2. There is an orthonormal basis \( (w_1, \ldots, w_\ell) \) of \( \text{span}(DP_t f(y_1), \ldots, DP_t f(y_\ell)) \) (which is independent of \( g \)) such that with probability \( 1 - \epsilon \), outputs a function \( g : \mathbb{R}^\ell \to [-1, 1] \) with the following guarantee: Let \( \text{Cover}(t, \ell, \delta) \) be the set of functions from Theorem 54 where the parameters \( t, \delta \) are set as in Figure 5. Then,

\[
\mathbb{E}[|P_t f(x) - g(\langle w_1, x \rangle, \ldots, \langle w_\ell, x \rangle)|] \leq \min_{g^* \in \text{Cover}(t, \ell, \delta)} \mathbb{E}[|P_t f(x) - g^*(\langle w_1, x \rangle, \ldots, \langle w_\ell, x \rangle)|] + 5\epsilon.
\]

Proof. As usual, the query complexity of the procedure is easily seen to be \( O\left(\frac{s \sqrt{\ell}}{\epsilon}\right)^{O(\ell)} \) by just plugging in the values of the parameters along with the guarantees on the query complexity of Compute-ortho-transform (Lemma 38) as well Lemma 12.

To analyze the algorithm, let us now define a point \( x \in \mathbb{R}^n \) to be good if the following two conditions hold:

1. For \( 1 \leq i \leq \ell \),

\[
|f_{\partial, \xi, t, y_i}(x) - \langle DP_t f(y_i), x \rangle| \leq \frac{\ell \cdot \xi}{\epsilon}.
\]
2. For all $1 \leq i \leq \ell$,
\[ | \sum_{j} \alpha_{i,j} \langle DP_{t} f(y_{j}), x \rangle - \langle w_{i}, x \rangle | \leq \frac{\epsilon \cdot \sqrt{t}}{100 \ell^{2}}. \]

**Claim 40.** For $x \sim \gamma_{n}$, $\Pr[x \text{ is good}] \geq 1 - \frac{2\epsilon^{2}}{\ell}$.  

**Proof.** Lemma 12 guarantees that for any specific choice of $i$, $\Pr[|f_{\partial, t, y_{i}}(x) - \langle DP_{t} f(y_{i}), x \rangle| \leq \frac{\ell \epsilon}{\ell}] \leq \frac{\epsilon^{2}}{\ell^{2}}$. Thus, with probability $1 - \frac{\epsilon^{2}}{\ell}$, item 1 holds for $x \sim \gamma_{n}$. Likewise, notice that
\[ \| \sum_{j} \alpha_{i,j} DP_{t} f(y_{j}) - w_{i} \|_{2} \leq \tau. \]

Thus, for any $x_{i} \sim \gamma_{n}$, with probability $1 - \frac{\epsilon^{2}}{\ell}$, item 2 holds. Thus, by a union bound, it holds for all $1 \leq i \leq \ell$ simultaneously, with probability $1 - \frac{\epsilon^{2}}{\ell}$. This proves the claim.  

Next, observe that if a point $x_{i}$ is good, then the following holds for every $j'$:
\[
| \bar{x}_{i,j'} - \langle w_{j'}, x_{i} \rangle | \leq \sum_{j} | \alpha_{j',j} \cdot \zeta_{i,j} - \alpha_{j',j} \cdot f_{\partial, t, y_{j}}(x_{i}) | + | \langle w_{j'}, x_{i} \rangle - \sum_{j} \alpha_{j',j} \cdot f_{\partial, t, y_{j}}(x_{i}) | \\
\leq \xi \cdot \sum_{j} | \alpha_{j',j} | + | \langle w_{j'}, x_{i} \rangle - \sum_{j} \alpha_{j',j} \cdot f_{\partial, t, y_{j}}(x_{i}) | \\
\leq \frac{\epsilon^{2} \sqrt{t}}{2 \ell^{4}} + | \langle w_{j'}, x_{i} \rangle - \sum_{j} \alpha_{j',j} \cdot f_{\partial, t, y_{j}}(x_{i}) | \\
\leq \frac{\epsilon^{2} \sqrt{t}}{2 \ell^{4}} + \frac{\epsilon^{2} \sqrt{t}}{100 \cdot \ell^{2}} + \frac{\epsilon \sqrt{t}}{100 \cdot \ell^{2}} \leq \frac{\epsilon \cdot \sqrt{t}}{\ell^{2}}.  
\]

The penultimate inequalities just follow from the condition that $x_{i}$ is good and the values of the parameters.

Now, observe that
\[
| \mathbf{E}_{x \sim \gamma_{n}}[|P_{t} f(x) - g(\bar{x}_{1}, \ldots, \bar{x}_{\ell})|] - \mathbf{E}_{x \sim \gamma_{n}}[|P_{t} f(x) - g(\langle w_{1}, x \rangle, \ldots, \langle w_{\ell}, x \rangle)|] | \\
\leq \mathbf{E}_{x \sim \gamma_{n}}[|g(\bar{x}_{1}, \ldots, \bar{x}_{\ell}) - g(\langle w_{1}, x \rangle, \ldots, \langle w_{\ell}, x \rangle)|].  
\]

Now, observe that by definition, the term inside the expectation is uniformly bounded by $2$. On the other hand, if a point $x$ is good, then by (5) and exploiting $g$ is $t^{-1/2}$-Lipschitz, then $|g(\bar{x}_{1}, \ldots, \bar{x}_{\ell}) - g(\langle w_{1}, x \rangle, \ldots, \langle w_{\ell}, x \rangle)| \leq \epsilon$. Since the fraction of good points is at least $1 - \frac{\epsilon^{2}}{\ell}$, we get that for any $g \in \text{Cover}(t, \ell, \delta)$,
\[
| \mathbf{E}_{x \sim \gamma_{n}}[|P_{t} f(x) - g(\bar{x}_{1}, \ldots, \bar{x}_{\ell})|] - \mathbf{E}_{x \sim \gamma_{n}}[|P_{t} f(x) - g(\langle w_{1}, x \rangle, \ldots, \langle w_{\ell}, x \rangle)|] | \leq 2\epsilon.  
\]

Now a standard Chernoff bound implies that with for any $g \in \text{Cover}(t, \ell, \delta)$, $O_{g}$ is within $\pm \epsilon/2$ of $\mathbf{E}[|P_{t} f(x) - g(\langle w_{1}, x \rangle, \ldots, \langle w_{\ell}, x \rangle)|]$ with probability $1 - \frac{\epsilon}{10 \cdot \text{Cover}(t, \ell, \delta)}$. Thus, by a union bound, with probability $1 - \frac{\epsilon}{10}$, for all $g \in \text{Cover}(t, \ell, \delta)$, $O_{g}$ is within $\pm \epsilon/2$ of $\mathbf{E}[|P_{t} f(x) - g(\langle w_{1}, x \rangle, \ldots, \langle w_{\ell}, x \rangle)|]$. This finishes the proof.  

\[\square\]
We are now ready to prove Theorem 33.

**Proof of Theorem 33.** Set \( t = \frac{\epsilon}{900s^4} \) (this is the same setting as Lemma 37 and Lemma 39). Observe that with this choice of \( t \), since \( f \) has surface area bounded by \( s \), then by Proposition 29 we get that

\[
\mathbb{E}[|P_t f(x) - f(x)|] \leq \sqrt{\mathbb{E}[|P_t f(x) - f(x)|^2]} \leq \frac{\epsilon}{\sqrt{5}}.
\]

We now run the algorithm **Find-candidate-directions** with noise parameter \( t \), error parameter \( \epsilon \) and surface area parameter \( s \). We are guaranteed that with probability \( 1 - \epsilon \), we will get \( \ell \leq k \) directions \( v_1, \ldots, v_\ell \) which are \( \gamma/2 \)-linearly independent and \( P_t f \) is \( \epsilon \)-close to a junta on the subspace \( \text{span}(v_1, \ldots, v_\ell) \) where \( v_i = DP_t f(y_i) \) (call this event \( \mathcal{E}_1 \)). The query complexity of this (from Lemma 37) is \((s \cdot k/\epsilon)^{O(k)}\).

Next, we run the routine **Estimate-closest-hypothesis** with the directions \( y_1, \ldots, y_\ell \), surface area parameter \( s \), error parameter \( \epsilon \). Observe that the query complexity of **Estimate-closest-hypothesis** is also \((s \cdot k/\epsilon)^{O(k)}\). Thus, the total query complexity remains \((s \cdot k/\epsilon)^{O(k)}\).

By guarantee of **Estimate-closest-hypothesis**, we have the following: there is an orthonormal basis \((w_1, \ldots, w_\ell)\) of \( \text{span}(DP_t f(y_1), \ldots, DP_t f(y_\ell)) \) such that

\[
\mathbb{E}[|P_t f(x) - g(\langle w_1, x \rangle, \ldots, \langle w_\ell, x \rangle)|] \leq \min_{g^* \in \text{Cover}(t, \ell, \delta)} \mathbb{E}[|P_t f(x) - g^*(\langle w_1, x \rangle, \ldots, \langle w_\ell, x \rangle)|] + 5\epsilon.
\]

However, conditioned on \( \mathcal{E}_1 \), \( P_t f \) is \( \epsilon \)-close to a junta on \( \text{span}(DP_t f(y_1), \ldots, DP_t f(y_\ell)) \). By Theorem 54, this implies that the quantity \( \min_{g^* \in \text{Cover}(t, \ell, \delta)} \mathbb{E}[|P_t f(x) - g^*(\langle w_1, x \rangle, \ldots, \langle w_\ell, x \rangle)|] \leq 3\epsilon \). This means that if we output the function \( g \), then \( \mathbb{E}[|P_t f(x) - g(\langle w_1, x \rangle, \ldots, \langle w_\ell, x \rangle)|] = O(\epsilon) \). Consider the subspace \( V \) spanned by vectors \( \{DP_t f(y)\}_{y \in \mathbb{R}^n} \). Note that \( \dim(V) \leq k \) and \( V \) is a relevant subspace for \( f \). Thus, \( w_1, \ldots, w_\ell \) can be extended to a basis for \( V \), finishing the proof.

\[\square\]

**Remark 41.** A crucial point about the routine **Find-invariant-structure**, which will be useful in the next section, is the following: The marginal distribution of all the queries is distributed as the standard n-dimensional Gaussian distribution \( \gamma_n \). To see this, note that

1. In the routine **Find-candidate-directions**, each of the directions \( y_i \) is sampled from \( \gamma_n \). Further, for \( y_i \) and \( y_j \) which are i.i.d. samples from \( \gamma_n \), the queries made to the oracle for \( f \) in computing \( \langle DP_t f(y_i), DP_t f(y_j)\rangle \) are also distributed as \( \gamma_n \) (see Lemma 10).

2. In the routine **Estimate-closest-hypothesis**, the points \( x_i \) are sampled from \( \gamma_n \) as are the directions \( y_j \) (which are output of **Find-candidate-directions**). With this, the queries made to the oracle for \( f \) for computing \( f(0,\xi,t,y_j(x_i)) \) are distributed as \( \gamma_n \) (see Lemma 12).

3. One minor subtlety is that while each sampled \( y_j \) comes from \( \gamma_n \), as stated, our algorithm **Find-invariant-structure** is adaptive. Consequently, the above two items do not imply that the marginal distribution of all queries is coming from \( \gamma_n \). The cause of non-adaptivity is that in the routine **Find-candidate-directions**, while we sample each \( y_j \) from \( \gamma_n \), subsequently, we only use a subset of the sampled \( y_j \)'s (namely, the subset \( S \)). However, we can easily make this algorithm non-adaptive at no asymptotic increase in the sample complexity. This is because the number of candidate directions sampled by the procedure **Find-candidate-directions** is at most \( k \cdot T_{\text{succ}} = \text{poly}(k \cdot s/\epsilon) \). We can run the subsequent routines namely **Compute-ortho-transform** and **Estimate-closest-hypothesis** with all the \( y_j \)'s instead of just those in set \( S \) but only use those which are part of the set \( S \) output by **Find-candidate-directions**, which will only increase the query complexity by a factor of \( \text{poly}(k \cdot s/\epsilon) \).
5 A lower bound in terms of surface area

The query complexity of our testing algorithm depends on the surface area of the set being tested. In this section, we prove that a polynomial dependence on surface area is necessary for non-adaptive tester, by proving a lower bound for distinguishing 1-juntas and 2-juntas in two dimensions. In particular, we show the following theorem.

**Theorem 42.** Any non-adaptive algorithm which can distinguish between a 1-junta with surface area at most \( s \) versus \( \Omega(1) \)-far from a linear 1-junta makes at least \( s^{10} \) queries.

To prove this theorem, as is standard, we will use the Yao’s minimax lemma. More specifically, we will describe a distribution \( D_1 \) over 1-juntas with surface area at most \( \Theta(s) \) and a distribution \( D_2 \) over functions that are far from 1-juntas and have surface area \( \Theta(s) \), such that for any choice of \( x_1, \ldots, x_n \in \mathbb{R}^2 \) with \( n = O(s^{1/10}) \), if \( f \sim D_1 \) and \( g \sim D_2 \) then \( (f(x_1), \ldots, f(x_n)) \) and \( (g(x_1), \ldots, g(x_n)) \) have almost the same distribution.

We begin with the description of \( f \sim D_1 \): let \( \theta \in \mathbb{R}^2 \) be a uniformly random unit vector. Choose \( a_1, \ldots, a_{s-1} \) uniformly from \([-1, 1]\), and then put them in increasing order. We also set \( a_0 = -1 \) and \( a_s = 1 \). Then choose independent random bits \( b_1, \ldots, b_s \) and define \( f \) by

\[
 f(x) = \begin{cases} 
 b_i & \text{if } a_{i-1} < \langle x, \theta \rangle \leq a_i \text{ for some } i \in \{1, \ldots, s\} \\
 1 & \text{otherwise.}
\end{cases}
\]

Clearly, such a function \( f \) is a 1-junta, and its surface area is at most \( s + 1 \) because the boundary of \( \{ f = 1 \} \) is a collection of at most \( s + 1 \) lines, and each line has surface area at most \( 1/\sqrt{2\pi} \).

To describe the construction of \( g \sim D_2 \), we begin with the same collection of random variables as before (i.e., \( \theta, a_1, \ldots, a_{s-1}, b_1, \ldots, b_s \)). Let \( \theta^z \) be a \( 90^\circ \) clockwise rotation of \( \theta \), choose \( z \in [-1, 1] \) independent of the other random variables, and define \( g \) by

\[
 g(x) = \begin{cases} 
 b_i \text{sign}(\langle x, \theta^z \rangle - z) & \text{if } a_{i-1} < \langle x, \theta \rangle \leq a_i \text{ for some } i \in \{1, \ldots, s\} \\
 1 & \text{otherwise.}
\end{cases}
\]

Note that the boundary of \( \{ g = 1 \} \) is contained in at most \( s + 2 \) lines, and so it has surface area at most \( s + 2 \). We will prove below that (with high probability) functions drawn from \( D_2 \) are far from 1-juntas. Then the following Theorem will demonstrate that testing 1-juntas with surface area \( \Theta(s) \) requires \( \text{poly}(1/s) \) queries.

**Theorem 43.** For any query set \( x_1, \ldots, x_n \) with \( n \leq s^{1/10} \), if \( f \sim D_1 \) and \( g \sim D_2 \) then the distributions of \( (f(x_1), \ldots, f(x_n)) \) and \( (g(x_1), \ldots, g(x_n)) \) are \( C s^{-1/10} \)-close in total variation distance.

In order to study the distinguishability of \( D_1 \) and \( D_2 \), we give a slightly different description of \( f \sim D_1 \) and \( g \sim D_2 \): for \( i = 1, \ldots, s \) set

\[
 S_i^+ = \{ x : a_{i-1} < \langle x, \theta \rangle \leq a_i \text{ and } \langle x, \theta^z \rangle \geq z \} \\
 S_i^- = \{ x : a_{i-1} < \langle x, \theta \rangle \leq a_i \text{ and } \langle x, \theta^z \rangle < z \} \\
 S_i = S_i^- \cup S_i^+,
\]

and note that \( f \) was defined by independently assigning a random \( \pm 1 \) value on each set \( S_i \), while \( g \) was defined by independently assigning opposite random \( \pm 1 \) values on each pair \( S_i^+, S_i^- \). Also, \( f \) and \( g \) are both identically one on \( \mathbb{R}^2 \setminus \bigcup S_i \).

Let \( x_1, \ldots, x_n \) be the set of query points, and consider the event that for every \( i \), at least one of \( S_i^+ \) or \( S_i^- \) contains no point in \( x_1, \ldots, x_n \); call this event \( A \). Then \( A \) depends on \( x_1, \ldots, x_n, \theta \), and \( a_1, \ldots, a_{s-1}, \ldots \)
but not on $b_1, \ldots, b_n$. Thanks to the description of $f$ and $g$ above, conditioned on $A$ the random variables $(f(x_1), \ldots, f(x_n))$ and $(g(x_1), \ldots, g(x_n))$ have the same distribution. In particular, we can couple $f$ and $g$ so that $(f(x_1), \ldots, f(x_n)) = (g(x_1), \ldots, g(x_n))$ with probability at least $1 - \Pr[A]$, and so we will prove Theorem 43 by showing that for any choice of $x_1, \ldots, x_n$ with $n \leq s^{1/10}$, $\Pr[A] \leq Cs^{-1/10}$. To do this, we will divide the pairs $(x_i, x_j)$ into "close" pairs and "far" pairs: we say that $x_i$ and $x_j$ are $\delta$-close if $|x_i - x_j| \leq \delta$, and $\delta$-far otherwise.

The following lemma will complete the proof of Theorem 43 because it implies that with high probability no pair of points lies in the same strips $S_i$, but on different sides of the line $\{x : \langle x, \theta^+ \rangle = z\}$.

**Lemma 44.** Suppose that $n \leq s^{1/10}$ and set $\delta = s^{-1/3}$. For any set $x_1, \ldots, x_n$, with probability at least $1 - Cs^{-1/10}$:

1. every pair of points $x_i, x_j$ that are $\delta$-far do not belong to the same set $S_k$ for any $k \in \{1, \ldots, s\}$.
2. every pair of points $x_i, x_j$ that are $\delta$-close lie on the same side of the line $\{x : \langle x, \theta^+ \rangle = z\}$.

The first step of Lemma 44 is the simple observation that far points remain reasonably far even after projecting them in the direction $\theta$.

**Lemma 45.** For all sufficiently small $\delta$ and any $x \in \mathbb{R}^2$, $\Pr(|\langle \theta, x \rangle| \leq \delta|x|) \leq \delta$.

**Proof.** If $\phi$ is the angle between $\theta$ and $x$ then $|\langle \theta, x \rangle| \leq \delta|x|$ exactly when $\cos \phi \leq \delta$, which has probability $\frac{\cos^{-1}(\delta) - \cos^{-1}(0)}{\pi}$. Since $\cos^{-1}$ has derivative 1 at zero, this is approximately $\frac{2}{\pi \delta}$ for small $\delta$. In particular, if $\delta > 0$ is sufficiently small then this probability is at most $\delta$. \qed

**Proof of Lemma 44** Let $\ell_k = \ell_k(\theta, a_k)$ be the line $\{x : \langle x, \theta \rangle = a_k\}$. By Lemma 45 applied to $x_i - x_j$, if $x_i$ and $x_j$ are $\delta$-far then with probability at least $1 - \delta$, $|\langle \theta, x_i - x_j \rangle| \geq \delta^2$. By a union bound, with probability at least $1 - n^2 \delta$, $|\langle \theta, x_i - x_j \rangle| \geq \delta^2$ for every $\delta$-far pair $x_i, x_j$; from now on, we will condition on this event (call it $\Omega_1$) occurring.

Now, if either $\langle \theta, x_i \rangle$ or $\langle \theta, x_j \rangle$ lies outside of the interval $[-1, 1]$ then $x_i$ and $x_j$ do not both lie in any single $S_k$. On the other hand, if both $\langle \theta, x_i \rangle$ and $\langle \theta, x_j \rangle$ lie in $[-1, 1]$, then each line $\ell_k$ has (independently) probability $|\langle \theta, x_i - x_j \rangle| \geq \delta^2$ to "split" $x_i$ from $x_j$. Hence, with probability at least $1 - (1 - \delta^2)s^{-1} \geq 1 - \exp(\delta^2(s - 1))$, there will be a line $\ell_k$ that splits $x_i$ from $x_j$, and so they will not belong to any single set $S_k$. Taking a union bound over all pairs $x_i, x_j$, we see that (conditioned on $\Omega_1$) with probability at least $1 - n^2 \exp(-\delta^2(s - 1))$, no pair of $\delta$-far points lands in the same $S_k$. Removing the conditioning on $\Omega_1$ changes the probability bound to $1 - n^2 \delta - n^2 \exp(-\delta^2(s - 1))$, which with our choice of parameters is at least $1 - Cs^{-1/10}$.

If $x_i$ and $x_j$ are $\delta$-close then $|\langle \theta^+, x_i - x_j \rangle| \leq \delta$, and hence the probability that they land on opposite sides of the line $\{x : \langle x, \theta^+ \rangle = z\}$ is at most $O(\delta)$. By a union bound over all pairs, with probability at least $1 - Cn^2 \delta \geq 1 - Cs^{-1/10}$, every pair of $\delta$-close $x_i, x_j$ land on the same side of that line. \qed

### 5.1 $D_2$ is far from a 1-junta

So far, we have shown that one cannot distinguish $D_1$ from $D_2$ from few samples. It remains to show that functions from $D_2$ are far (with high probability) from 1-juntas, it will follow that one cannot 1-juntas with $O(s)$ surface area with fewer than $s^{1/10}$ queries.

**Theorem 46.** There is a constant $c > 0$ such that with probability at least $1 - \text{poly}(1/s)$ over $g \sim D_2$, $g$ is $c$-far from every 1-junta.
Now recall that the construction of $D_1$ and $D_2$ involved dividing up the strip $\{ x : \langle \theta, x \rangle \in (-1, 1) \}$ into $s$ strips $S_1, \ldots, S_s$ and assigning random values on each strip. Since both the construction of $D_2$ and the notion of distance to a 1-junta are rotationally invariant, we will assume from now on that $\theta = e_1$, which means that the strips $S_1, \ldots, S_s$ are vertically oriented. Let $U^+ = \bigcup_{i: b_i = 1} (a_i, a_{i+1}]$ and let $U^- = [-1, 1] \setminus U^+$.

**Definition 47.** Let $I \subset [-1, 1]$ be an interval. We say that $I$ is $\delta$-balanced if of both $|I \cap U^+|$ and $|I \cap U^-|$ are at least $\delta|I|$, where $| \cdot |$ denotes the one-dimensional Lebesgue measure. We say that $I$ is wide if $|I| \geq \frac{1}{4}$.

We extend these definitions to strips in two dimensions: say that $I \times \mathbb{R}$ is $\delta$-balanced (resp. wide) if $I$ is $\delta$-balanced (resp. wide).

**Definition 48.** For any line $\ell \subset \mathbb{R}^2$, we say that $\ell$ is $\delta$-balanced if both

$$\int_{\ell \cap U^+} e^{-|x|^2/2} \, dx \quad \text{and} \quad \int_{\ell \cap U^-} e^{-|x|^2/2} \, dx$$

are at least

$$\delta \int_\ell e^{-|x|^2/2} \, dx.$$

We will now describe the outline of Theorem 46’s proof: note that if $h$ is a 1-junta then $h(x) = \tilde{h}(\langle \phi, x \rangle)$ for some $\phi$. Now, Fubini’s theorem implies that

$$2\pi \| h - g \|_1 = \int_{\mathbb{R}^2} e^{-|x|^2/2} |h - g| \, dx = \int_{\mathbb{R}} \int_{\{ x : \langle x, \phi^+ \rangle = a \}} e^{-|x|^2/2} |\tilde{h}(a) - g(x)| \, dx \, da.$$

Now, whenever the line $\{ x : \langle x, \phi^+ \rangle \}$ is $\delta$-balanced, the inner integral is at least $\delta \int e^{-|x|^2/2} \, dx$. Therefore, in order to prove Theorem 46 it suffices to show that there is a constant $\delta$ such that at least a constant fraction of the lines $\{ x : \langle x, \phi^+ \rangle = a \}$ are $\delta$-balanced. To be precise, let $L(\phi)$ be the set of lines of the form $\{ x : \langle x, \phi^+ \rangle = a \}$ for $a \in [-10, 10]$. Since $e^{-|x|^2/2}$ is bounded from below on $[-10, 10]$ it suffices to show that there is a constant $\delta > 0$ such that with high probability, for every $\phi$, a constant fraction of $\ell \in L(\phi)$ are $\delta$-balanced. For the remainder of the section, we will focus on proving the preceding statement.

We will consider two cases depending on $\phi$: if the lines in $L(\phi)$ are “steep,” then these lines will be balanced because a constant fraction of them will cross the horizontal line $\{ x : x_2 = z \}$ near the middle of a strip. Since the value of $g$ on a strip changes sign at that horizontal line, this will imply that such a line is balanced. On the other hand, if the lines are not steep, then they will be balanced because they cross many strips, and $g$ will tend to take different values on different strips.

We will first deal with the case of steep lines. In this case, it is deterministically the case that $g$ is far from $h$.

**Lemma 49.** At least half of the points on the line segment from $(-1, 1/2)$ to $(1, 1/2)$ are in a wide strip $S_c$.

**Proof.** There are $s$ strips in total, and so the narrow ones can take up at most a total width of 1, which is only half of the width of the line segment in question. \hfill \qed

**Lemma 50.** There is a constant $c > 0$ such that if the absolute value of the slope of $\{ x : \langle \phi^+, x \rangle = 0 \}$ is at least $s$ then a $c$-fraction of $\ell \in L(\phi)$ are $c$-balanced.

**Proof.** We may assume without loss of generality that $z \leq 0$. By Lemma 49 at least a constant fraction of $\ell \in L(\phi)$ intersect the line $\{ x : x_2 = 1/2 \}$ in the middle third of a wide strip $S_k$. In this case, $\ell$ belongs to $S^+_k$ for a distance of at least $1/3$, and to $S^-_k$ for a distance of at least $1/3$, and it follows that $\ell$ is $c$-balanced for a constant $c$ depending on the minimum and maximum values of $e^{-|x|^2/2}$ for $x \in [-1, 1]^2$. \hfill \qed
For the remainder of the section we will deal with lines that are not steep. For \( k \) with \( 2^{-k} \leq 2/s \), consider an interval of the form \([j2^{-k}, (j + 1)2^{-k}] \subset [-1, 1] \); let \( D_k \) be the set of all such intervals.

**Lemma 51.** There is a constant \( C \) such that with probability at least \( 1 - \text{poly}(1/s) \), for every \( k \) for which \( 2^{-k} \leq 2/s \), at least a \( 1/2 \)-fraction of the intervals \( I \in D_k \) are \( \frac{1}{2} \)-balanced.

**Proof.** For technical convenience, we will consider a slightly different way of generating the strips \( S_i \). Instead of dividing \([-1, 1] \times \mathbb{R} \) using exactly \( s - 1 \) vertical lines, we will take a Poisson number (with mean \( s - 1 \)) of vertical lines. We will prove the claim for this modified model, with a probability estimate of at least \( 1 - \exp(-\Omega(\sqrt{s})) \), and since a Poisson random variable is equal to its mean with probability \( \text{poly}(1/s) \), the claim will also follow for the original model.

Our first claim is that for \( 2^{-k} \leq 1/\sqrt{s} \), each interval in \( D_k \) has a constant probability of being \( \Omega(1) \)-balanced. First, consider the largest \( k \) for which \( 2^{-k} \leq 2/s \). In this case, the width of each \( I \in D_k \) is within a factor 2 of \( s \) (we will call such an interval a primitive interval). It is easy to verify that for each \( I \in D_k \), there is a constant probability that \( I \) will intersect exactly two strips, each taking up at least \( 1/3 \) of the width of \( I \), and that these two strips will receive different labels \( b_i \). Hence, there is a constant probability that \( I \) is \( 1/3 \)-balanced.

Now consider \( k \) for which \( 2^{-k} \leq 1/\sqrt{s} \). Every \( I \in D_k \) is made up of \( \Theta(s2^k) \) primitive intervals, each of which has a constant probability of being balanced. Moreover (thanks to our Poissonized model) the events that different primitive intervals are balanced are independent. By Chebyshev's inequality, there is a constant probability that at least a constant fraction of \( I \)'s primitive intervals are \( 1/3 \)-balanced, and so \( I \) has a constant probability of being \( \Omega(1) \)-balanced. This proves our first claim (that for each \( 2^{-k} \leq 1/\sqrt{s} \), each interval in \( D_k \) has a constant probability of being \( \Omega(1) \)-balanced). Now, for each such \( k \) there are at least \( \Omega(\sqrt{s}) \) such intervals, and so a Chernoff bound implies that with probability at least \( 1 - \exp(-\Omega(\sqrt{s})) \), at least a constant fraction of these intervals are balanced. Taking a union bound over \( k \) proves the claim whenever \( 2^{-k} \leq 1/\sqrt{s} \).

For smaller \( k \), we claim that with high probability, every \( I \in D_k \) is balanced. Indeed, such \( I \in D_k \) contain at least \( \sqrt{s} \) primitive intervals, and so a Chernoff bound implies that with probability \( 1 - \exp(-\Omega(\sqrt{s})) \), at least a constant fraction of those primitive intervals are balanced, and so \( I \) is balanced also. We can take a union bound over all \( k \) and all \( I \).

To complete the proof of Theorem 46 it remains to show that with high probability, every non-steep line is balanced.

**Lemma 52.** There is a constant \( c > 0 \) such that if the absolute value of the slope of \( \{x : \langle \phi^+, x \rangle = 0 \} \) is at most \( s \) then with probability at least \( 1 - \text{poly}(1/s) \), a \( c \)-fraction of \( \ell \in L(\phi) \) are \( c \)-balanced.

**Proof.** Choose \( k \) so that the slope of all lines in \( L(\phi) \) are between \( 2^{k-1} \) and \( 2^k \). Consider a rectangle of the form \( Q = [j2^{-k}, (j + 1)2^{-k}] \times [-2, -1] \), where the interval \([j2^{-k}, (j + 1)2^{-k}] \) is balanced. Since the slope of \( \phi \) is at most \( 2^k \), if the line \( \ell \) intersects the rectangle \( Q \) then it crosses the entire vertical strip \([j2^{-k}, (j + 1)2^{-k}] \times \mathbb{R} \) within the horizontal strip \([-3, 0] \). Since the interval \([j2^{-k}, (j + 1)2^{-k}] \) is balanced, it follows that the line \( \ell \) is also balanced. (We’re assuming here, without loss of generality, that \( z \geq 0 \).

Finally, it is easy to verify that if a constant fraction of the intervals \([j2^{-k}, (j + 1)2^{-k}] \) are balanced then a constant fraction of \( \ell \in L(\phi) \) intersect with some rectangle of the form above. By Lemma 51 this completes the proof.

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A Small net for noise attenuated linear juntas

In this section, we are going to prove the following theorem which essentially shows the existence of a small cover for noise stable linear juntas. To state this theorem, we will require one crucial fact about noise attenuated functions (due to Bakry and Ledoux [Bak94])

**Lemma 53.** Let \( f : \mathbb{R}^n \to [-1, 1] \). Then, \( P_t f \) is \( C_t \)-Lipschitz for \( C_t = O(t^{-1/2}) \).

For the rest of this section, we are going to use \( C_t \) to denote this quantity. We can now state the main theorem of this section.

**Theorem 54.** For any error parameter \( \delta > 0 \), noise parameter \( t > 0 \) and \( k \in \mathbb{N} \), there is a set of functions \( \text{Cover}(t, k, \delta) \) (mapping \( \mathbb{R}^k \) to \([-1, 1]\)) such that the following holds:

1. Let \( f : \mathbb{R}^n \to [-1, 1] \) and \( W \) be a \( k \)-dimensional space such that \( P_t f \) is \( \delta \)-close to a \( W \)-junta. Further, \((w_1, \ldots, w_k)\) be any orthonormal basis of \( W \). Then, \( P_t f \) is \( 3\delta \)-close to \( h((w_1, x), \ldots, (w_k, x)) \) for some \( h \in \text{Cover}(t, k, \delta) \).

2. Every function in \( \text{Cover}(t, k, \delta) \) is \( 2C_t \)-Lipschitz.

3. \( \log |\text{Cover}(t, k, \delta)| \leq \left( \frac{C \sqrt{k} \log^2(1/\delta)}{\delta t} \right)^k \).

The proof of this theorem relies on the following two lemmas.

**Lemma 55.** For any \( L > 0 \), error parameter \( \delta > 0 \) and \( k \in \mathbb{N} \), there is a set \( \text{Cover}_{k, L, \delta} \) consisting of functions mapping \( \mathbb{R}^k \to [-1, 1] \) such that the following holds:

1. For every \( g : \mathbb{R}^k \to [-1, 1] \) which is \( L \)-Lipschitz, there is a function \( h \in \text{Cover}_{k, L, \delta} \) such that \( E[|g(x) - h(x)|] \leq \delta \).

2. Every function in \( \text{Cover}_{k, L, \delta} \) is \( 2L \)-Lipschitz.

3. \( \log |\text{Cover}_{k, L, \delta}| \leq \left( \frac{C L \sqrt{k} \log^2(1/\delta)}{\delta} \right)^k \).

**Proof.** Let \( B = \{ x : \|x\|_2 \leq \sqrt{k} \cdot \log(100/\delta) \} \). Let \( \mathcal{A} \) be a maximal \( \delta/(2L) \)-packing of \( B \) (that is, a maximal subset of \( B \) such that any two distinct points in \( \mathcal{A} \) are at least \( \delta/(2L) \) apart). It is well-known (see, e.g. [LT91]) that \( \mathcal{A} \) is a \( \delta \)-/\( L \)-net of \( B \) and that \( |\mathcal{A}| \leq (C L \sqrt{k} \log(1/\delta))^k \) (the \( \sqrt{k} \log(1/\delta) \) term comes from the diameter of \( B \).

For \( f : \mathbb{R}^n \to [-1, 1] \), we now define \( f_{\text{int}} : \mathcal{A} \to [-1, 1] \) by simply rounding \( f \) to the nearest integer multiple of \( \delta/100 \). To check the Lipschitz constant of \( f_{\text{int}} \), note that if \( x, y \in \mathcal{A} \) then

\[
|f_{\text{int}}(x) - f_{\text{int}}(y)| \leq |f(x) - f(y)| + \delta/50 \leq L \|x - y\| + \frac{L}{25} \|x - y\|,
\]

where the last inequality used the fact that \( f \) is \( L \)-Lipschitz and that every pair of points in \( \mathcal{A} \) is \( \delta/(2L) \)-separated. In particular, \( f_{\text{int}} \) is \( 2L \)-Lipschitz. Let \( \text{Cover} \) be the set of all functions \( f_{\text{int}} \) obtained in this way. Then the size of \( \text{Cover} \) is at most \( \exp((C L \sqrt{k} \log^2(1/\delta))^k) \), because there are at most \( C/\delta \) choices for the value of each point, and there are \( |\mathcal{A}| \) points. Finally, we construct \( \text{Cover}_{k, L, \delta} \) by extending each function in \( \text{Cover} \) to a function \( \mathbb{R}^n \to [-1, 1] \). McShane’s Lemma [McS34] implies that this extension can be done without increasing its Lipschitz constant. Hence, properties 2 and 3 hold.

To check property 1, note that if \( x \in B \) and \( y \in \mathcal{A} \) is the closest point to \( x \) then

\[
|f(x) - f_{\text{int}}(x)| \leq |f(x) - f(y)| + |f(y) - f_{\text{int}}(y)| + |f_{\text{int}}(y) - f_{\text{int}}(x)| \leq 3L \|x - y\| + \delta/100 \leq 4\delta.
\]
It then follows that
\[
\mathbb{E}[|f(x) - f_{\text{int}}(x)|] \leq 2 \cdot \Pr[x \notin B] + \max_{x \in B}[|f(x) - f_{\text{int}}(x)|] \leq \delta + 4\delta \leq 5\delta.
\]

The last inequality just follows from the fact that a $k$-dimensional standard Gaussian is in a ball of radius $\sqrt{\mathbb{E} \log(1/\delta)}$ with probability $1 - \delta/2$. This proves property 1 modulo the constant 5, which can be dropped by redefining $\delta$. \qed

**Lemma 56.** Let $f : \mathbb{R}^n \to [-1, 1]$ be a $C$-Lipschitz function. Further, for $\kappa > 0$, let $g : \mathbb{R}^n \to [-1, 1]$ be a $W$-junta such that $f$ is $\kappa$-close to $g$. Then, there is a function $f_W : \mathbb{R}^n \to [-1, 1]$ which is $C$-Lipschitz and $W$-junta which is $2\kappa$-close to $f$.

**Proof.** Reorient the axes so that $W$ is the space spanned by the first $\ell$-axes. Let us define the $W$-junta $f_W : \mathbb{R}^n \to [-1, 1]$ defined as
\[
f_W(x) = \mathbb{E}_{y_{\ell+1}, \ldots, y_n}[f(x_1, \ldots, x_\ell, y_{\ell+1}, \ldots, y_n)]
\]
For any fixed choice of $x_1, \ldots, x_\ell$, we have
\[
\mathbb{E}_{x_{\ell+1}, \ldots, x_n}[|f(x) - f_W(x)|] \leq \mathbb{E}_{x_{\ell+1}, \ldots, x_n}[|f(x) - g(x)|] + |g(x) - f_W(x)|.
\]
However, the second term can be bounded as
\[
|g(x) - f_W(x)| = |g(x) - \mathbb{E}_{x_{\ell+1}, \ldots, x_n}[f(x_1, \ldots, x_\ell, x_{\ell+1}, \ldots, x_n)]| \leq \mathbb{E}_{x_{\ell+1}, \ldots, x_n}[|g(x) - f(x)|]
\]
The last inequality is simply Jensen’s inequality. Combining these two, we get
\[
\mathbb{E}_{x_{\ell+1}, \ldots, x_n}[|f(x) - f_W(x)|] \leq 2 \cdot \mathbb{E}_{x_{\ell+1}, \ldots, x_n}[|f(x) - g(x)|].
\]
This in turn implies that
\[
\mathbb{E}_{x_1, \ldots, x_n}[|f(x) - f_W(x)|] \leq 2 \cdot \mathbb{E}_{x_1, \ldots, x_n}[|f(x) - g(x)|] \leq 2 \cdot \kappa.
\]
Finally, we see that
\[
|f_W(x) - f_W(y)| = |\mathbb{E}_{x_{\ell+1}, \ldots, x_n}[f(x_1, \ldots, x_\ell, x_{\ell+1}, \ldots, x_n) - f(y_1, \ldots, y_\ell, x_{\ell+1}, \ldots, x_n)]|
\leq \mathbb{E}_{x_{\ell+1}, \ldots, x_n}[|f(x_1, \ldots, x_\ell, x_{\ell+1}, \ldots, x_n) - f(y_1, \ldots, y_\ell, x_{\ell+1}, \ldots, x_n)|]
\leq \mathbb{E}_{x_{\ell+1}, \ldots, x_n}[C \cdot ||(x_1, \ldots, x_\ell) - (y_1, \ldots, y_\ell)||_2] \leq C||x - y||_2.
\]
This finishes the proof. \qed

With these two lemmas, we can now finish the proof of Theorem 54.

**Proof of Theorem 54.** First, we apply Lemma 53 to obtain that $P_t f$ is $C_t = O(t^{-1/2})$-Lipschitz. Since $P_t f$ is $\delta$-close to a $W$-junta, we obtain that $P_t f$ is $2\delta$ close to a $W$-junta $g$ which is $C_t$-Lipschitz (follows from Lemma 56). Let $\text{Cover}(t, k, \delta) = \text{Cover}_{k,C_t \delta}$ (constructed in Lemma 55). By a rotation of the coordinates, it follows from the definition of $\text{Cover}(t, k, \delta)$ that there exists $h \in \text{Cover}(t, k, \delta)$ such that $h((w_1, x), \ldots, (w_k, x))$ is $\frac{\delta}{4}$ close to $g$. The required properties now follow from Lemma 55. \qed
B Some useful results from linear algebra

The next lemma states for any \( v_1, \ldots, v_\ell \) which are \((\eta, \gamma)\)-linearly independent, we can find a set of vectors \((w_1, \ldots, w_\ell)\) (expressed as linear combination of \((w_1, \ldots, w_\ell)\)) which is close to being an orthonormal basis of the \(\text{span}(v_1, \ldots, v_\ell)\) provided we have sufficiently good approximations of \(\{\langle v_i, v_j \rangle\}_{1 \leq i, j \leq \ell}\).

Now, modulo the quantitative estimates, this is essentially just a consequence of a procedure such as the Gram-Schmidt orthogonalization. However, the complexity of our testing algorithm is dependent on the quantitative estimates, so we work out the linear algebra here.

Lemma 57. Let \( v_1, \ldots, v_\ell \) be a \((\eta, \gamma)\)-linearly independent vectors. Then, for any error parameter \( \nu > 0 \) and \( \lambda = \lambda(\ell, \nu, \eta, \gamma) \) defined as

\[
\lambda = 2 \frac{\nu}{\ell^2 \cdot \eta} \cdot \left( \frac{\gamma}{2 \cdot \ell \cdot \eta} \right)^{3 \ell^3 + 3},
\]

given numbers \( \{\beta_{i,j}\}_{1 \leq i, j \leq \ell} \) such that \( |\beta_{i,j} - \langle v_i, v_j \rangle| \leq \lambda \), we can compute numbers \( \{\alpha_{i,j}\}_{1 \leq i, j \leq \ell} \) such that:

1. For \( \xi(\ell, \eta, \gamma) \) defined as

\[
\xi(\ell, \eta, \gamma) = \sqrt{2\ell} \cdot \left( \frac{2\ell \cdot \eta}{\gamma} \right)^{\ell+1},
\]

we have \( |\alpha_{i,j}| \leq \xi(\ell, \eta, \gamma) \).

2. There is an orthonormal basis \((w_1, \ldots, w_\ell)\) of \(\text{span}(v_1, \ldots, v_\ell)\) such that for \( \|w_i - \sum_j \alpha_{i,j} v_j\| \leq \nu \).

Proof. Consider the symmetric matrix \( \Sigma \in \mathbb{R}^{\ell \times \ell} \) defined as \( \Sigma_{i,j} = \langle v_i, v_j \rangle \). By Proposition 58, \( \Sigma \) is non-singular. Define the matrix \( \Gamma = \Sigma^{-1/2} \). It is easy to see that the columns of \( V \cdot \Sigma^{-1/2} \) form an orthonormal basis of \(\text{span}(v_1, \ldots, v_\ell)\). Here \( V = [v_1 \ldots v_\ell] \). Of course, we cannot compute the matrix \( \Sigma \) exactly and consequently, we cannot compute the matrix \( \Sigma^{-1/2} \) either. Instead, if we define the matrix \( \widetilde{\Sigma} \) as \( \widetilde{\Sigma}_{i,j} = \beta_{i,j} \), then observe that \( \widetilde{\Sigma} \) is symmetric. Next, observe that Proposition 58 we have that

\[
\sigma_{\min}(\Sigma) = \sigma_{\min}^2(V) \geq \left( \frac{\gamma}{2 \cdot \ell \cdot \eta} \right)^{2\ell^2 + 2}.
\]

Define a parameter \( \rho \) as

\[
\rho = \frac{2\nu}{\ell \cdot \eta} \cdot \left( \frac{\gamma}{2 \cdot \ell \cdot \eta} \right)^{\ell+1}.
\]

Now, with this setting, observe that

\[
\ell \cdot \lambda = \rho \cdot \left( \frac{\gamma}{2 \cdot \ell \cdot \eta} \right)^{2\ell^2 + 2} \leq \rho \cdot \sigma_{\min}(\Sigma).
\]

Further, since entrywise, \( \Sigma \) and \( \widetilde{\Sigma} \) differ by at most \( \lambda \), hence \( \|\Sigma - \widetilde{\Sigma}\|_F \leq \ell \cdot \lambda \). First, by Weyl’s inequality (Lemma 17), we have that

\[
\sigma_{\min}(\widetilde{\Sigma}) \geq \sigma_{\min}(\Sigma) - \|\Sigma - \widetilde{\Sigma}\|_F \geq (1 - \rho) \cdot \sigma_{\min}(\Sigma).
\]

Thus, \( \widetilde{\Sigma} \) is also psd. Now, we apply the matrix perturbation bound to matrices \( \Sigma \) and \( \widetilde{\Sigma} \) (Corollary 20 with parameter \( c = \left( \frac{\gamma}{2 \cdot \ell \cdot \eta} \right)^{2\ell^2 + 2} \)) to obtain that

\[
\|\Sigma^{-1/2} - \widetilde{\Sigma}^{-1/2}\| \leq \frac{\rho}{2 \left( \frac{\gamma}{2 \cdot \ell \cdot \eta} \right)^{\ell+1}} = \frac{2\nu}{\ell \cdot \eta}.
\]
We now define $\alpha_{i,j} = \tilde{\Sigma}^{-\frac{1}{2}}(j,i)$. We also define $\beta_{i,j} = \Sigma^{-\frac{1}{2}}(i,j)$. Note that the vectors $w_i = \sum_j \beta_{j,i} v_j$ forms an orthonormal basis. As the matrices $\Sigma^{-\frac{1}{2}}$ and $\tilde{\Sigma}^{-\frac{1}{2}}$ are $\frac{2\nu}{\epsilon \eta}$ close in operator norm, this immediately implies item 2. To get item 1, we recall the following basic inequality for Frobenius norm of an inverse matrix. In particular, for a symmetric matrix $A \in \mathbb{R}^{\ell \times \ell}$, $\sigma_{\min}(A) \cdot \|A^{-1}\|_F \leq \sqrt{\ell}$. Thus,

$$\|\tilde{\Sigma}^{-1/2}\|_F \leq \frac{\sqrt{\ell}}{\sigma_{\min}(\Sigma^{1/2})} = \sqrt{\frac{\ell}{\sigma_{\min}(\Sigma)}} \leq \sqrt{2\ell \cdot \left(\frac{2\ell \cdot \eta}{\gamma}\right)^{\ell+1}}.$$ 

The last inequality uses (9) and the fact that $\rho \leq \frac{1}{2}$. This immediately implies the first item.

\[ \Box \]

**Proposition 58.** Let $v_1, \ldots, v_\ell$ be a $(\eta, \gamma)$-linearly independent vectors. Let $V = [v_1 | \ldots | v_\ell]$. Then, the smallest singular value of $V$ is at least $(\frac{2\gamma}{2\ell \cdot \eta})^{\ell+1}$.

**Proof.** Let us set a parameter $\rho = \frac{\gamma}{2\ell \cdot \eta}$. Recall that if $\sigma_{\min}(V)$ is the smallest singular value of $V$, then

$$\sigma_{\min}(V) = \inf_{x : \|x\|_2 = 1} \|V \cdot x\|_2$$

Let us try to lower bound the right hand side. To do this, let $x \in \mathbb{R}^n$ be any unit vector and note that $V \cdot x = \sum_{1 \leq i \leq \ell} v_i \cdot x_i$. Now, let $j$ be the largest coordinate such that $|x_j| \geq \rho^j$ (note that there has to be such a $j$ since $x$ is a unit vector and $\rho < 1/2$). Define $w = \sum_{i \leq j} v_i x_i$. Then, observe that its component in the direction orthogonal to the span of $\{v_1, \ldots, v_{j-1}\}$ is at least $\gamma \cdot \rho^j$ in magnitude. On the other hand, $\|\sum_{i > j} v_i x_i\|_2 \leq \rho^{j+1} \cdot \ell \cdot \eta$. By triangle inequality, we obtain that

$$\|\sum_i v_i x_i\|_2 \geq \|\sum_{i \leq j} v_i x_i\|_2 - \|\sum_{i > j} v_i x_i\|_2 \geq \gamma \cdot \rho^j - \ell \cdot \eta \cdot \rho^{j+1} \geq \frac{\gamma \cdot \rho^j}{2}.$$ 

The last inequality uses the value of $\rho$. This finishes the proof. 

\[ \Box \]