MINIMAL FIXED POINT SET OF MAPS ON TORUS
FIBER BUNDLES OVER THE CIRCLE

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Abstract. The main purpose this work is to study the minimal fixed point set of fiber-preserving maps for spaces which are fiber bundles over the circle and the fiber is the torus. Using the one-parameter fixed point theory is possible to describe these sets in terms of the fundamental group and the induced homomorphism.

1. Introduction

Let $S \to M \overset{p}{\to} B$ be a fiber bundle, where $S, M, B$ are closed manifolds, and $f : M \to M$ be a fiber-preserving map. The minimum number $MF_B[f] = \min\{\#\pi_0(Fix(f'))|f' \sim_B f\}$ of path components of fixed point subspaces of $M$ among all pairs fiberwise homotopic to $f$ is finite, see [6]. The symbol “$\sim_B$” means a fiberwise homotopy.

To determine when the number $MF_B[f]$ is zero, that is, when the fiber-preserving map $f$ can be deformed by a fiberwise homotopy to a fixed point free map is a problem that has been considered by many authors, see for example, [2], [3] and [8]. The study of the minimal fixed point set of a fiber-preserving map is a problem of interest in fixed point theory. These sets have been studied using bordism techniques, that in general are difficult to compute, see [6].

In this paper we present a method to compute $MF_B[f]$, using one-parameter fixed point theory, when the base $B$ is the circle $S^1$. This

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technique allows us to present the minimal fixed point set of a fiber-preserving map in terms of the fundamental group of $M$, and of the induced homomorphism $f_\#$. The one-parameter fixed point theory also allow us to describe each path component of $Fix(f')$ for each fiber-preserving map $f'$ fiberwise homotopic to $f$.

Let $f : M \to M$ be a fiber-preserving map, where $M$ is a fiber bundle over the circle and the fiber is the torus, $T$. Such fiber bundles $M$ are obtained from $T \times [0,1]$ by identifying $(x,0)$ with $(A(x),1)$, where $A$ is a homeomorphism of $T$. We write

$$M(A) = M = \frac{T \times [0,1]}{(x,0) \sim (A(x),1)}$$

The elements of $M(A)$ are denoted by $\langle [(x,y)], t >$. Here $[(x,y)]$ denote a point in $T$. We can identify $A$ with a matrix with integer coefficients and determinant 1 or $-1$, the details are in section 2. The projection map $p : M(A) \to S^1 = I/0 \sim 1$, is given by $p(\langle [(x,y)], t >) = \langle t >$.

Since $f$ is a fiber-preserving map and the base is $S^1$, the fixed point set of $f$ can be seen as the fixed point set of a homotopy of the torus. In this paper we study the minimal fixed point set for homotopies using one-parameter fixed point theory developed by R. Geoghegan and A. Nicas in [4].

This paper is organized into five sections, besides this one. In section 2 we considered fiber-preserving maps in fiber bundles over the circle with fiber torus. In section 3 we present the relation between fixed point sets of fiber-preserving maps and fixed point sets of homotopies. In section 4 we present preliminaries about the one-parameter fixed point theory. In section 5 we prove the main result, which is theorem 5.1.

2. Torus fiber-preserving maps

Let $T$ be, the torus, defined as the quotient space $\mathbb{R} \times \mathbb{R}/\mathbb{Z} \times \mathbb{Z}$. We denote by $(x,y)$ the elements of $\mathbb{R} \times \mathbb{R}$ and by $[(x,y)]$ the elements in $T$.

Let $M(A) = \frac{T \times [0,1]}{([(x,y)],0) \sim ([A(x)],1)}$ be the quotient space, where $A$ is a homeomorphism of $T$ induced by an operator in $\mathbb{R}^2$ that preserves
The space $M(A)$ is a fiber bundle over the circle $S^1$ where the fiber is the torus. For more details on these bundles see [3].

Given a fiber-preserving map $f : M(A) \rightarrow M(A)$, i.e. $p \circ f = p$ we want to compute the number $MF_{S^1}[f]$. More precisely we want to study the path components of $Fix(f')$ for each map $f'$ fiberwise homotopic to $f$.

Consider the loops in $M(A)$ given by: $a(t) =< [(t,0)], 0 >$, $b(t) =< [(0,t)], 0 >$ and $c(t) =< [(0,0)], t >$ for $t \in [0,1]$. We denote by $B$ the matrix of the homomorphism induced on the fundamental group by the restriction of $f$ to the fiber $T$. From [3] we have the following theorem that provides a relationship between the matrices $A$ and $B$.

Theorem 2.1. (1) $\pi_1(M(A), 0) = \langle a, b, c | [a, b] = 1, cac^{-1} = a^2 b^2, cbc^{-1} = a^3 b^3 \rangle$

(2) $B$ commutes with $A$.

(3) If $f$ restricted to the fiber is deformable to a fixed point free map then the determinant of $B - I$ is zero, where $I$ is the identity matrix.

(4) Consider $w = A(v)$ if the pair $v, w$ generators $\mathbb{Z} \times \mathbb{Z}$, otherwise let $w$ be another vector so that $v, w$ span $\mathbb{Z} \times \mathbb{Z}$. Define the linear operator $P : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}$ by $P(v) = (1, 0)$ and $P(w) = (0, 1)$. Consider an isomorphism of fiber bundles, also denoted by $P$, $P : M(A) \rightarrow M(A^1)$ where $A^1 = P \circ A \circ P^{-1}$. Then $M(A)$ is homeomorphic to $M(A^1)$ over $S^1$. Moreover we have one of the cases of the table below with $B^1 = P \circ A \circ P^{-1}$ and $B \neq I$, except in case $I$: 
From [3] we have the following theorem:

**Theorem 2.2.** If \( f : M(A) \to M(A) \) is a fiber-preserving map, then in the case I we have \( MF_{S^1}[f] = 0 \), and in the cases II and III we have \( MF_{S^1}[f] = 0 \) if and only if \( c_1(b_4 - 1) - c_2b_3 = 0 \).

The Theorem 2.2 in [3] provides also conditions for remaining cases. We omit them, since here we will study only II and III.

### 3. Fixed point set of fiber-preserving maps

Given a fiber-preserving map \( f : M(A) \to M(A) \) the set \( Fix(f) \) is given by: \( \{ < [(x, y)], t > \in M(A) \mid f(< [(x, y)], t >) = < [(x, y)], t > \} \).

Since \( f \) is a fiber-preserving map then the map \( f \) is given by formula:

\[
f(< [(x, y)], t >) = < F([(x, y)], t), t >
\]

where \( F : T \times I \to T \) is a homotopy. We call this homotopy \( F \) the homotopy induced by \( f \). If \( f \) has no fixed points in \( t = 0,1 \), then the study of the set \( Fix(f) \) is equivalent to the study of the set \( Fix(F) \), that is,

\[
Fix(f) \approx Fix(F).
\]

This happens since, in the fiber bundle \( M(A) \) the class \( < [(x, y)], t > \) contains only one unique point if \( t \neq 0,1 \). Notice that

| Case | \( A^1 = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \end{pmatrix} \), \( B^1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) |
|------|-------------------------------------------------------------------------------------------------|
| I    | \( a_3 \neq 0 \)                                                                                |
| II   | \( A^1 = \begin{pmatrix} 1 & a_3 \\ 0 & 1 \end{pmatrix} \), \( B^1 = \begin{pmatrix} 1 & b_3 \\ 0 & b_4 \end{pmatrix} \) |
|      | \( a_3(b_4 - 1) = 0 \)                                                                           |
| III  | \( A^1 = \begin{pmatrix} 1 & a_3 \\ 0 & -1 \end{pmatrix} \), \( B^1 = \begin{pmatrix} 1 & b_3 \\ 0 & b_4 \end{pmatrix} \) |
|      | \( a_3(b_4 - 1) = -2b_3 \)                                                                       |
| IV   | \( A^1 = \begin{pmatrix} -1 & a_3 \\ 0 & 1 \end{pmatrix} \), \( B^1 = \begin{pmatrix} 1 & b_3 \\ 0 & b_4 \end{pmatrix} \) |
|      | \( a_3(b_4 - 1) = 0 \)                                                                           |
| V    | \( A^1 = \begin{pmatrix} -1 & a_3 \\ 0 & 1 \end{pmatrix} \), \( B^1 = \begin{pmatrix} 1 & b_3 \\ 0 & b_4 \end{pmatrix} \) |
|      | \( a_3(b_4 - 1) = 2b_3 \)                                                                         |
**Proposition 3.1.** Let \( M(A) \) be a fiber bundle as in theorem 2.1. If \( f : M(A) \to M(A) \) is a fiber-preserving map such the restriction to each fiber \( f_{|T} \) can be deformed to a fixed point free map, then \( f \) can be deformed to a map \( f' \) such that \( f'(\langle [(x, y)], 0 \rangle) : T \to T \) is a fixed point free map.

**Proof.** Let \( f : M(A) \to M(A) \) be a fiber-preserving map given by \( f(\langle [(x, y)], t \rangle) = F(\langle [(x, y)], t \rangle, t >) \). As \( M(A) \) is a locally trivial bundle thus we can choose \( \frac{1}{2} > \epsilon > 0 \) such that \( p^{-1}(\epsilon, 1 - \epsilon) \approx T \times (\epsilon, 1 - \epsilon) \). We take the homotopy \( H : M(A) \times I \to M(A) \) defined by:

\[
H(\langle [(x, y)], t \rangle, s) = \begin{cases}
<F(\langle [(x, y)], 0 \rangle, t) & \text{if } 0 \leq t \leq \epsilon s \\
<F(\langle [(x, y)], \frac{1}{1-2\epsilon}(t - s\epsilon) \rangle, t) & \text{if } \epsilon s \leq t \leq 1 - \epsilon s \\
<F(\langle [(x, y)], 0 \rangle, t) & \text{if } 1 - \epsilon s \leq t \leq 1
\end{cases}
\]

By hypothesis there is one homotopy \( h : T \times I \to T \) satisfying \( h(\langle [(x, y)], 1 \rangle) = F(\langle [(x, y)], 0 \rangle) \) and \( h(\langle [(x, y)], 0 \rangle) \) is a fixed point free map. Therefore we can define the following homotopy:

\[
G(\langle [(x, y)], t \rangle, s) = \begin{cases}
<h(\langle [(x, y)], \frac{t-n}{n} s + 1 \rangle, t) & \text{if } 0 \leq t \leq \epsilon \\
<F(\langle [(x, y)], \frac{1}{1-2\epsilon}(t - \epsilon) \rangle, t) & \text{if } \epsilon \leq t \leq 1 - \epsilon \\
h(\langle [(x, y)], \frac{1}{1-s}\epsilon s + 1 \rangle, t) & \text{if } 1 - \epsilon \leq t \leq 1
\end{cases}
\]

The fiber-preserving homotopy \( J : M(A) \times I \to M(A) \) defined by:

\[
J(\langle [(x, y)], t \rangle, s) = \begin{cases}
H(\langle [(x, y)], t \rangle, 2s) & \text{if } 0 \leq s \leq \frac{1}{2} \\
G(\langle [(x, y)], t \rangle, 2s - 1) & \text{if } \frac{1}{2} \leq s \leq 1
\end{cases}
\]
satisfies the condition of the theorem. \( \square \)

Note that if a fiber-preserving map \( f : M(A) \to M(A) \), where \( M(A) \) is as in Theorem 2.1 has no fixed points in \( t = 0 \) then \( f \) has no fixed points in \( t = 1 \) also. In fact, suppose that \( f \) has one fixed point in \( t = 1 \). We have, \( f(\langle [(x, y)], 0 \rangle) = f(\langle [A(x)], 1 \rangle) \). Using which the matrix \( A \) is invertible in \( \mathbb{Z} \), see section 2 then there should be one point \( \langle [(u, v)], 0 \rangle \), satisfying \( f(\langle [(u, v)], 0 \rangle) = \langle [(u, v)], 0 \rangle \), but this is a contradiction.

**Proposition 3.2.** Let \( F : T \times I \to T \) be the homotopy induced by a fiber-preserving map \( f : M(A) \to M(A) \), i.e, \( f(\langle [(x, y)], t \rangle) = \langle [(x, y)], t \rangle \).
< F([(x, y)], t), t >. If \( P : T \to T \) is an isomorphism and \( g : M(A^1) \to M(A), A^1 = P \circ A \circ P^{-1} \), is a fiber-preserving map defined by \( g(< [(x, y)], t >) = < P \circ F \circ (P^{-1} \times Id) (\tilde{\sigma}, t), t > \), then the numbers \( MF_{S^1} [f] \) and \( MF_{S^1} [g] \) are equals.

\[
\begin{array}{ccc}
M(A) & \xrightarrow{f} & M(A) \\
\downarrow P & & \downarrow P \\
M(A^1) & \xrightarrow{g} & M(A^1)
\end{array}
\]

**Proof.** Note that the homotopy \( G : T \times I \to T \) induces the fiber-preserving map \( g : M(A^1) \to M(A) \). Since that \( G = P \circ F \circ (P^{-1} \times Id) \) then we have \( MF_{S^1} [f] = MF_{S^1} [g] \).

By Proposition 3.1 the study of the minimal fixed point set, over \( S^1 \), of a fiber-preserving map \( f : M(A) \to M(A) \) is equivalent to study of the minimal fixed point set for the homotopy induced by \( f \). In this paper, we applied the one-parameter fixed point theory developed by R. Geoghegan and A. Nicas in [4] to determine the minimal fixed point set of the homotopy \( F \). Since \( T \) is orientable then the fixed point set of \( F \) consists of oriented arcs as Figure 1, see [1], [5] and [9].

![Figure 1. Fixed point set of a homotopy on the torus.](image)

As \( f \) has no fixed points in \( t = 0, 1 \), then in Figure 1 above we have only circles. The one-parameter trace give us the “minimum amount” these circles. In the next section we shall describe some concepts of the one-parameter fixed point theory, for more details see [4].

4. **One-parameter fixed point theory**

4.1. **Hochschild Homology Traces.** Let \( R \) be a ring and \( M \) an \( R - R \) bimodule, that is, a left and right \( R \)-module satisfying \( (r_1m)r_2 =
The Hochschild chain complex \( \{C_*(R, M), d\} \) is given by \( C_n(R, M) = R^\otimes n \otimes M \) where \( R^\otimes n \) is the tensor product of \( n \) copies of \( R \), taken over the intergers, and

\[
d_n(r_1 \otimes \ldots \otimes r_n \otimes m) = r_2 \otimes \ldots r_n \otimes mr_1 \\
+ \sum_{i=1}^{n-1} (-1)^i r_1 \otimes \ldots \otimes r_{i+1} \otimes \ldots \otimes r_n \otimes m \\
+ (-1)^n r_1 \otimes \ldots \otimes r_{n-1} \otimes r_nm.
\]

The \( n - th \) homology of this complex is the Hochschild homology of \( R \) with coefficient bimodule \( M \). It is denoted by \( HH_n(R, M) \). For computed \( HH_1 \) and \( HH_0 \) we have the formulae \( d_2(r_1 \otimes r_2 \otimes m) = r_2 \otimes mr_1 - r_1r_2 \otimes m + r_1 \otimes r_2m \) and \( d_1(r \otimes m) = mr - rm \).

**Lemma 4.1.** If \( 1 \in R \) is the unit element and \( m \in M \) then the 1-chain \( 1 \otimes m \) is a boundary.

**Proof.** \( d_2(1 \otimes 1 \otimes m) = 1 \otimes m - 1 \otimes m + 1 \otimes m = 1 \otimes m. \)

The Hochschild homology will arise in the following situation: let \( G \) be a group and \( \phi : G \rightarrow G \) an endomorphism. Also denote by \( \phi \) the induced ring homomorphism \( \mathbb{Z}G \rightarrow \mathbb{Z}G \). Take the ring \( R = \mathbb{Z}G \) and \( M = (\mathbb{Z}G)^\phi \) the \( \mathbb{Z}G - \mathbb{Z}G \) bimodule whose underlying abelian group is \( \mathbb{Z}G \) and the bimodule structure is given by \( g.m = gm \) and \( m.g = m\phi(g) \).

Two elements \( g_1, g_2 \) in \( G \) are semiconjugate if and only if there exists \( g \in G \) such that \( g_1 = gg_2\phi(g^{-1}) \). We write \( C(g) \) for the semiconjugacy class containing \( g \) and \( G^\phi \) for the set of semiconjugacy classes. Thus, we can decompose \( G \) in the union of its semiconjugacy classes. This partition induces a direct sum decomposition of \( HH_*(\mathbb{Z}G, (\mathbb{Z}G)^\phi) \).

In fact, each generating chain \( \gamma = g_1 \otimes \ldots \otimes g_n \otimes m \) can be written in canonical form as \( g_1 \otimes \ldots \otimes g_n \otimes g_n^{-1} \otimes \ldots g_1^{-1}g \) where \( g = g_1\ldots g_nm \in G \) “marks” a semiconjugacy class. Thus, the decomposition \( (\mathbb{Z}G)^\phi \cong \bigoplus_{C \in G^\phi} \mathbb{Z}C \) as a direct sum of abelian groups determines a decomposition of chains complexes \( C_*(\mathbb{Z}G, (\mathbb{Z}G)^\phi) \cong \bigoplus_{C \in G^\phi} C_*(\mathbb{Z}G, (\mathbb{Z}G)^\phi)_C \) where \( C_*(\mathbb{Z}G, (\mathbb{Z}G)^\phi)_C \) is the subgroup of \( C_*(\mathbb{Z}G, (\mathbb{Z}G)^\phi) \) generated by those generating chains whose markers lie in \( C \). Thus we have the following isomorphism: \( HH_*(\mathbb{Z}G, (\mathbb{Z}G)^\phi) \cong \bigoplus_{C \in G^\phi} HH_*(\mathbb{Z}G, (\mathbb{Z}G)^\phi)_C \).
where the summand $HH_* (\mathbb{Z}G, (\mathbb{Z}G)^\phi)_C$ corresponds to the homology classes marked by the elements of $C$. This summand is called the $C-$component.

Let $Z(h) = \{g \in G | h = gh\phi(g^{-1})\}$ be the semicentralizer of $h \in G$. Choosing representatives $g_C \in C$, then we have the following proposition whose proofs is in \cite{4}:

**Proposition 4.2.** Choosing representatives $g_C \in C$ then we have

$$HH_* (\mathbb{Z}G, (\mathbb{Z}G)^\phi) \cong \bigoplus_{C \in G_\phi} H_*(Z(g_C))_C$$

where $H_*(Z(g_C))_C$ corresponds to the summand $HH_* (\mathbb{Z}G, (\mathbb{Z}G)^\phi)_C$.

Given a $m \times n$ matrix over $R$ and a $n \times m$ matrix over $M$ we define $A \otimes B$ to be the $m \times m$ matrix with entries in $R \otimes M$ given by $(A \otimes B)_{ij} = \sum_{k=1}^{n} A_{ik} \otimes B_{kj}$. The trace of $A \otimes B$, written $\text{trace}(A \otimes B)$, is given by

$$\sum_{i=1}^{m} \sum_{k=1}^{n} A_{ik} \otimes B_{ki} \in C_1(R, M).$$

We have that the 1–chain $\text{trace}(A \otimes B)$ is a cycle if and only if $\text{trace}(AB) = \text{trace}(B\phi(A))$, in which case we denote its homology class by $T_1(A \otimes B) \in HH_1(R, M)$.

### 4.2. One-parameter Fixed Point Theory

Let $X$ be a finite connected CW complex and $F : X \times I \to X$ a cellular homotopy. We consider $I = [0, 1]$ with the usual CW structure and orientation of cells, and $X \times I$ with the product CW structure, where its cells are given the product orientation.

Pick a basepoint $(v, 0) \in X \times I$, and a basepath $\tau$ in $X$ from $v$ to $F(v, 0)$. We identify $\pi_1(X \times I, (v, 0)) \cong G$ with $\pi_1(X, v)$ via the isomorphism induced by projection $p : X \times I \to X$. We write $\phi : G \to G$ for the homomorphism;

$$\pi_1(X \times I, (v, 0)) \xrightarrow{F} \pi_1(X, F(v, 0)) \xrightarrow{\phi} \pi_1(X, v)$$

We choose a lift $\tilde{E}$ in the universal cover, $\tilde{X}$, of $X$ for each cell $E$ and we orient $\tilde{E}$ compatibly with $E$. Let $\tilde{\tau}$ be the lift of the basepath $\tau$ which starts at in the basepoint $\tilde{v} \in \tilde{X}$ and $\tilde{F} : \tilde{X} \times I \to \tilde{X}$ the unique lift of $F$ satisfying $\tilde{F}(\tilde{v}, 0) = \tilde{\tau}(1)$.
We can regard $C_*(\tilde{X})$ as a right $\mathbb{Z}G$ chain complex as follows: if $\omega$ is a loop at $\nu$ which lifts to a path $\tilde{\omega}$ starting at $\tilde{\nu}$ then $\tilde{E}[\omega]^{-1} = h_{[\omega]}(\tilde{E})$, where $h_{[\omega]}$ is the covering transformation sending $\tilde{\nu}$ to $\tilde{\omega}(1)$.

The homotopy $\tilde{F}$ induces a chain homotopy $\tilde{D}_k : C_k(\tilde{X}) \to C_{k+1}(\tilde{X})$ given by $\tilde{D}_k(\tilde{E}) = (-1)^{k+1}F_k(\tilde{E} \times I) \in C_{k+1}(\tilde{X})$, for each cell $\tilde{E} \in \tilde{X}$. This chain homotopy satisfies: $\tilde{D}(\tilde{E}g) = \tilde{D}(\tilde{E})\phi(g)$ and the boundary operator $\tilde{\partial}_k : C_k(\tilde{X}) \to C_{k-1}(\tilde{X})$ satisfies: $\tilde{\partial}(\tilde{E}g) = \tilde{\partial}(\tilde{E})g$.

Define endomorphism of, $\bigoplus_k C_k(\tilde{X})$, by $\tilde{D}_* = \bigoplus_k (-1)^{k+1}D_k$, $\tilde{\partial}_* = \bigoplus_k \tilde{\partial}_k$, $\tilde{F}_0* = \bigoplus_k (-1)^k\tilde{F}_{0k}$ and $F_1* = \bigoplus_k (-1)^k\tilde{F}_{1k}$. We consider trace($\tilde{\partial}_* \tilde{D}_*$) $\in HH_1(\mathbb{Z}G,(\mathbb{Z}G)\phi)$. This is a Hochshcild 1-chain whose boundary is: $\text{trace}(\tilde{D}_*\phi(\tilde{\partial}_*) - \tilde{\partial}_*\tilde{D}_*)$.

We denote by $G_\phi(\tilde{\partial}(F))$ the subset of $G_\phi$ consisting of semiconjugacy classes associated to fixed points of $F_0$ or $F_1$.

**Definition 4.3.** The one-parameter trace of homotopy $F$ is;

$$R(F) \equiv T_1(\tilde{\partial}_* \tilde{D}_*; G_\phi(\tilde{\partial}(F))) \in \bigoplus_{C \in G_\phi - G_\phi(\tilde{\partial}(F))} HH_1(\mathbb{Z}G,(\mathbb{Z}G)\phi)_C$$

$$\cong \bigoplus_{C \in G_\phi - G_\phi(\tilde{\partial}(F))} H_1(Z(g_C)).$$

**Definition 4.4.** The $C$-component of $R(F)$ is denoted by $i(F; C) \in HH_1(\mathbb{Z}G,(\mathbb{Z}G)\phi)_C$. We call it the fixed point index of $F$ corresponding to semiconjugacy class $C \in G_\phi$. The one-parameter Nielsen number, $N(F)$, of $F$ is the number of nonzero fixed point indices.

The one-parameter Lefschetz class, $L(F)$, of $F$ is defined by;

$$L(F) = \sum_{C \in G_\phi - G_\phi(\tilde{\partial}(F))} j_C(i(F; C))$$

where $j_C : H_1(Z(g_C)) \to H_1(G)$ is induced by the inclusion $Z(g_C) \subset G$.

From [4] we have the following theorems:

**Theorem 4.1** (one-parameter Lefschetz fixed point theorem). If $L(F) \neq 0$ then every map homotopic to $F$ relative to $X \times \{0, 1\}$ has a fixed point not in the same fixed point class as any fixed point in $X \times \{0, 1\}$. In particular, if $F_0$ and $F_1$ are fixed point free, every map homotopic to $F$ relative to $X \times \{0, 1\}$ has a fixed point.
Theorem 4.2 (one-parameter Nielsen fixed point theorem). Every map homotopic to $F$ relative to $X \times \{0,1\}$ has at least $N(F)$ fixed point classes other than the fixed point classes which meet $X \times \{0,1\}$. In particular, if $F_0$ and $F_1$ are fixed point free maps, then every map homotopic to $F$ relative to $X \times \{0,1\}$ has at least $N(F)$ path components.

4.3. Semiconjugacy classes in the torus. In this subsection we describe some results about the semiconjugacy classes in the torus.

We take $w = [(0,0)] \in T$ and $G = \pi_1(T,w) = \{u,v|uvu^{-1}v^{-1} = 1\}$, where $u \equiv a$ and $v \equiv b$. Thus, given a homomorphism $\phi : G \to G$ we have $\phi(u) = u^{b_1}v^{b_2}$ and $\phi(v) = u^{b_3}v^{b_4}$. Therefore, $\phi(u^mv^n) = u^{mb_1+nb_3}v^{mb_2+nb_4}$, for all $m, n \in \mathbb{Z}$. We denote this homomorphism by the matrix:

$$[\phi] = \begin{pmatrix} b_1 & b_3 \\ b_2 & b_4 \end{pmatrix}$$

Proposition 4.5. Two elements $g_1 = u^{m_1}v^{n_1}$ and $g_2 = u^{m_2}v^{n_2}$ in $G$ belong to the same conjugacy class, if and only if there are integers $m, n$ satisfying the following equations:

$$\begin{cases} m(b_1 - 1) + nb_3 = m_2 - m_1 \\ mb_2 + n(b_4 - 1) = n_2 - n_1 \end{cases}$$

Proof. If there is $g = u^mv^n \in G$ satisfying $g_1 = gg_2\phi(g)^{-1}$ then we obtain the equation of the proposition. The other direction is analogous. \qed

We take the isomorphism $\Theta : G \to \mathbb{Z} \times \mathbb{Z}$ such that $\Theta(u^mv^n) = (m,n)$. By above proposition two elements $g_1 = u^{m_1}v^{n_1}$ and $g_2 = u^{m_2}v^{n_2}$ in $G$ belong to the same conjugacy class, if and only if there is $z \in \mathbb{Z} \times \mathbb{Z}$ satisfying; $([\phi] - I)z = \Theta(g_2g_1^{-1})$, where $I$ is the identity matrix. If $\det([\phi] - I) \neq 0$ will have an infinite amount of semiconjugacy classes.

Corollary 4.6. The semicentralizer $Z(g)$ of a element $g \in G$ is isomorphic to the kernel of $[\phi] - I$.

Lemma 4.7. The 1-chain, $u^kv^l \otimes u^mv^n$, is a cycle if and only if the element $(k,l) \in \mathbb{Z} \times \mathbb{Z}$ belongs to the kernel of $[\phi] - I$. 

Proof. If $u^kv^l \otimes u^mv^n$ is a cycle, then $0 = d_1(u^kv^l \otimes u^mv^n) = u^m v^n \phi(u^kv^l) - u^kv^l u^m v^n$
$= u^m v_n u^{kb_1 + lb_3 v^{kb_2 + lb_4} - u^k v^l u^m v^n} = u^{m + kb_1 + lb_3 v^{kb_2 + lb_4} + n} - u^{k + m + l + n}$. This implies $k(b_1 - 1) + lb_3 = 0$ and $kb_2 + l(b_4 - 1) = 0$. The other direction is analogous. 

Proposition 4.8. The 1-chain, $u^k \otimes u^m v^n$, is homologous to the 1-chain, $ku \otimes u^{m+k-1} v^n$, for all $k, m, n \in \mathbb{Z}$.

Proof. Note that for $k = 0$ and 1 the proposition is true. We suppose that for some $s > 0 \in \mathbb{Z}$, $u^s \otimes u^m v^n \sim su \otimes u^{m+s-1} v^n$. Considering the to 2-chain $u^s \otimes u \otimes u^m v^n$ then we have

$$d_2(u^s \otimes u \otimes u^m v^n) = u \otimes u^m v^n - u^{s+1} \otimes u^m v^n + u^s \otimes u^{1+m} v^n$$
$$\sim u \otimes u^m v^n - u^{s+1} \otimes u^m v^n + su \otimes u^{1+m+s-1} v^n$$
$$= (s + 1)u \otimes u^{m+(s+1)-1} v^n - u^{s+1} \otimes u^m v^n.$$ 

Therefore $(s + 1)u \otimes u^{m+(s+1)-1} v^n \sim u^{s+1} \otimes u^m v^n$. Using induction, we have the result. The case in which $k < 0$ is analogous. 

Lemma 4.9. Each 1-chain, $\sum_{i = 1}^{t} a_i u^{k_i} v^{l_i} \otimes u^{m_i} v^{n_i}$, is homologous to a 1-chain, $\sum_{i = 1}^{t} \tilde{a}_i u^{\tilde{k}_i} v^{\tilde{l}_i} \otimes u^{\tilde{m}_i} v^{\tilde{n}_i}$, where all elements $\tilde{l}_i, i = 1, ..., t$, are positive.

Proof. We denote by $w_i = a_i u^{k_i} v^{l_i} \otimes u^{m_i} v^{n_i}$ and $\alpha = \sum_{i = 1}^{t} a_i u^{k_i} v^{l_i} \otimes u^{m_i} v^{n_i}$. If there is some $l_i \leq 0$ then considering the to 2-chain $\gamma_i = a_i u^{k_i} v^{l_i} \otimes u^{k_i} v^{-l_i} \otimes u^{m_i - k_i} v^{m_i - l_i}$ we obtain; $d_2(\gamma_i) = w_i - g_i + h_i$, where $g_i = -a_i u^{2k_i} \otimes u^{m_i - k_i} v^{m_i - l_i}$ and $h_i = a_i u^{k_i} v^{-l_i} \otimes u^{m_i + k_i(b_1 - 1) + l_i b_3 v^{l_i + k_i b_2 + l_i(b_4 - 1)}}$.

Thus, $w_i \sim g_i - h_i$, and $g_i$, $h_i$ have the desired form. 

In the following proposition we consider $b_1 = 1$ and $b_2 = 0$. 

Proposition 4.10. If the Hochschild 1-chain; \( \sum_{i=1}^{t} a_i u^{k_i} v^{l_i} \otimes u^{m_i} v^{n_i} \), is a 1-cycle then the 1-chain ; \( \sum_{i=1}^{t} a_i u^{k_i} v^{l_i} \otimes u^{m_i} v^{n_i} \), is a 1-cycle for all \( m, n \in \mathbb{Z} \).

Proof. We take a 1-chain, \( \sum_{i=1}^{t} a_i u^{k_i} v^{l_i} \otimes u^{m_i} v^{n_i} \), with \( d_1(\sum_{i=1}^{t} a_i u^{k_i} v^{l_i} \otimes u^{m_i} v^{n_i}) = \sum_{i=1}^{t} a_i u^{m_i+k_i} v^{l_i+n_i} - a_i u^{m_i+k_i} v^{l_i+n_i} = 0 \). We denote \( e_i = u^{m_i+k_i} v^{l_i+n_i} \) and \( f_i = u^{m_i+k_i} v^{l_i+n_i} \). The last equality implies the following equality on the ring group \( ZG \):

\[
\sum_{i=1}^{t} a_i e_i = \sum_{i=1}^{t} a_i f_i.
\]

Thus, for each \( i, 1 \leq i \leq t \) there is \( j, 1 \leq j \leq t \) such that \( a_i = a_j \) and \( e_i = f_j \), that is, we have

\[
(I) \left\{ \begin{array}{l}
m_i + k_i b_1 + l_i b_3 = k_j + m_j \\
l_i b_4 + k_i b_2 + n_i = l_j + n_j
\end{array} \right.
\]

If \( i = j \) then the equality above says that the vector \( (k_i, l_i) \) satisfies the equation; \( ([\phi] - I) (k_i, l_i) = 0 \), i.e , belongs the to kernel of the \( ([\phi] - I) \).

If \( i \neq j \) then fixing \( j \) there is \( q, 1 \leq q \leq t \) such that \( a_j = a_q \) and \( e_j = f_q \). This implies the following equation:

\[
(II) \left\{ \begin{array}{l}
m_j + k_j b_1 + l_j b_3 = k_q + m_q \\
l_j b_4 + k_j b_2 + n_j = l_q + n_q
\end{array} \right.
\]

Adding the corresponding lines of the \( (I) \) and \( (II) \) we obtain;

\[
\left\{ \begin{array}{l}
(k_i + k_j)(b_1 - 1) + (l_i + l_j)b_3 = k_q - k_i + m_q - m_i \\
(k_i + k_j)b_2 + (l_i + l_j)(b_4 - 1) = l_q - l_i + n_q - n_i
\end{array} \right.
\]

If \( i = q \) then \( (k_i + k_j)(b_1 - 1) + (l_i + l_j)b_3 = 0 \) and \( (k_i + k_j)b_2 + (l_i + l_j)(b_4 - 1) = 0 \), which is equivalent to say that the vector, \( (k_i + k_j, l_i + l_j) \), satisfies the equation; \( ([\phi] - I) (k_i + k_j, l_i + l_j) = 0 \). Thus, we can take a new \( i', 1 \leq i' \leq t \), and do the same process. If \( i \neq q \) then we can do the same process above and to obtain a new equation, \( (III) \), exactly like in the equation \( (II) \), and so forth.
Therefore, after making the process for all indices $1 \leq i \leq t$, just add all vectors and conclude that the vector: $(\sum_{j} k_{j}, \sum_{j} l_{j})$ belongs to the kernel of the $(|\phi| - I)$. Thus, the 1-chain $\sum_{i=1}^{t} a_{i} u^{k_{1}+...+k_{i}} v^{l_{1}+...+l_{i}} \otimes u^{m} v^{n}$, is a cycle, for all $m, n \in \mathbb{Z}$. \hfill \square

Note that if the homomorphism $\phi$ is induced by a homotopy which is induced by a fiber-preserving map as in Theorem 2.1, then the set $\{u \otimes u^{m} v^{n}| m, n \in \mathbb{Z}\}$ is one generating set for $HH_{1}(\mathbb{Z}G, (\mathbb{Z}G)^{\phi}) \cong \bigoplus_{C \in G_{\phi}} H_{1}(Z(g_{C})) \cong \bigoplus_{C \in G_{\phi}} \mathbb{Z}$. Since $u \otimes u^{m-1} v^{n} \sim u^{-1} \otimes u^{m+1} v^{n}$, for all $m, n \in \mathbb{Z}$ we use the generating set $\{u^{-1} \otimes u^{m} v^{n}| m, n \in \mathbb{Z}\}$.

5. Computing the number $MF_{S^{1}}[f]$

In this section we prove the following theorem:

**Theorem 5.1** (Main theorem). If $f : M(A) \rightarrow M(A)$ is a fiber-preserving map then the homomorphism $f_{#} : \pi_{1}(M(A)) \rightarrow \pi_{1}(M(A))$ is given by: $f_{#}(a) = a$, $f_{#}(b) = a^{b_{3}} b_{4}$ and $f_{#}(c) = a^{c_{1}} b_{2} c$ where $a, b, c$ are generators of $\pi_{1}(M(A), 0)$ previously described. If $M(A)$ is one of the fiber bundle given below:

In the case II

\[
A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & n(b_{4} - 1) \\ 0 & b_{4} \end{pmatrix}
\]

\[
A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & b_{3} \\ 0 & -1 \end{pmatrix}
\]

In the case III

\[
A = \begin{pmatrix} 1 & 2k \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & b_{3} \\ 0 & b_{4} \end{pmatrix}
\]

\[
A = \begin{pmatrix} 1 & a_{3} \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & a_{3} \\ 0 & -1 \end{pmatrix}
\]

where $n, k, b_{3}, b_{4}, c_{1}, c_{2}, a_{3} \in \mathbb{Z}$, then the minimal fixed point set of $f$ is composed by $|c_{1}(b_{4} - 1) - c_{2} b_{3}|$ disjoint circles. This implies; $MF_{S^{1}}[f] = |c_{1}(b_{4} - 1) - c_{2} b_{3}|$. 
Given a fiber-preserving map \( f' : M(A) \to M(A) \) fiberwise homotopic to \( f \) then the set \( \text{Fix}(f') \) is composed by circles. The phrase “minimal fixed point set of \( f \)” in the theorem above means that we consider the minimum in terms of the first homology group, that is, we consider; \( \min\{\text{rank}(H_1(\text{Fix}(f')))|f' \sim_B f\} \).

**Proof.** Initially let us consider the case \( b_3 = 0 \) and \( b_4 - 1 \neq 0 \). In this situation we must have \( a_3 = 0 \) in both of cases. We take the homotopy \( F : T \times I \to T \) defined by:

\[
F([[x, y]], t) = \begin{cases} 
[(x + 2c_1t - \frac{1}{2}, b_4y)] & \text{if } 0 \leq t \leq \frac{1}{2} \\
[(x + \frac{2a_3 - 1}{2} - \frac{1}{2}, b_4y + 2c_2t - c_2)] & \text{if } \frac{1}{2} \leq t \leq 1 
\end{cases}
\]

The homotopy \( F \) induces a fiber-preserving map \( f : M(A) \to M(A) \) defined by \( f(< [(x, y)], t >) =< F([(x, y)], t), t > \). Note that the induced homomorphism by \( f \) satisfies; \( f_#(a) = a, f_#(b) = b^{b_4} \) and \( f_#(c) = a^{a_1}b^{b_2}c \). The map \( f \) has not fixed points in \( t = 0, 1 \). This implies that \( \text{Fix}(f) \approx \text{Fix}(F) \).

We use the one-parameter trace of \( F, R(F) \), to compute the minimum number \( MF_{S^1}[f] \).

We choose the cellular decomposition for \( T \) which consist of two 0-cells; \( E_0^0 = \{(0, 0)\}, E_2^0 = \{\{(\frac{1}{2}, 0)\}\} \), four 1-cells; \( E_1^1 = \{(x, 0)||0 \leq x \leq \frac{1}{2}\}, E_2^1 = \{(x, 0)||\frac{1}{2} \leq x \leq 1\}, E_3^1 = \{(0, y)||0 \leq y \leq 1\}, E_4^1 = \{(\frac{1}{2}, y)||0 \leq y \leq 1\} \) and two 2-cells; \( E_5^2 = \{(x, y)||0 \leq x \leq \frac{1}{2}, 0 \leq y \leq 1\}, E_6^2 = \{(x, y)||\frac{1}{2} \leq x \leq 1, 0 \leq y \leq 1\} \). For this decomposition the homotopy \( F \) is cellular.

We orient the cells above as in Figure 2. By Proposition 4.1 of [4] the one-parameter trace \( R(F) \) is independent of the choice of orientation of cells and the choice of lifts to the universal cover.

**Figure 2.** Cellular decomposition, case \( b_4 - 1 \neq 0 \) and \( b_3 = 0 \).
For cellular decomposition above we choose in the universal cover $\mathbb{R}^2$ the lifts which consist of two 0-cells; $\tilde{E}_1^0 = (0, 0)$, $\tilde{E}_2^0 = (\frac{1}{2}, 0)$, four 1-cells; $\tilde{E}_1^1 = \{(x, 0)|0 \leq x \leq \frac{1}{2}\}$, $\tilde{E}_2^1 = \{(x, 0)|\frac{1}{2} \leq x \leq 1\}$, $\tilde{E}_3^1 = \{(0, y)|0 \leq y \leq 1\}$, $\tilde{E}_4^1 = \{(\frac{1}{2}, y)|0 \leq y \leq 1\}$ and two 2-cells; $\tilde{E}_1^2 = \{(x, y)|0 \leq x \leq \frac{1}{2}, 0 \leq y \leq 1\}$, $\tilde{E}_2^2 = \{(x, y)|\frac{1}{2} \leq x \leq 1, 0 \leq y \leq 1\}$.

We consider $w = [(0, 0)]$ the basepoint and $\tau$ basepath, the linear path between $w$ and $F(w, 0)$. We take the lifts $\tilde{w} = (0, 0)$ and $\tilde{\tau}$ the linear path between $\tilde{w}$ and $(-\frac{1}{2}, 0)$. The unique lift $\tilde{F} : \mathbb{R}^2 \times I \to \mathbb{R}^2$ of $F$ mapping $(\tilde{w}, 0)$ to $\tilde{\tau}(1)$ is given by;

$$\tilde{F}(x, y, t) = \begin{cases} (x + 2c_1t - \frac{1}{2}, b_4y) & \text{if } 0 \leq t \leq \frac{1}{2} \\ (x + \frac{2c_1-1}{2}, b_4y + 2c_2t - c_2) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

If $G = \pi_1(T, [(0, 0)]) = \{u, v|uvu^{-1}v^{-1} = 1\}$ then matrices of operators $\tilde{\partial}_1$, $\tilde{\partial}_2$, $\tilde{D}_0$ and $\tilde{D}_1$ are given by;

$$[\tilde{\partial}_1] = \begin{pmatrix} -1 & u^{-1} & v^{-1} - 1 & 0 \\ 1 & -1 & 0 & v^{-1} - 1 \end{pmatrix}$$

$$[\tilde{\partial}_2] = \begin{pmatrix} v^{-1} - 1 & 0 \\ 0 & v^{-1} - 1 \\ 1 & -u^{-1} \\ -1 & 1 \end{pmatrix}$$

$$[\tilde{D}_0] = \begin{pmatrix} -\tilde{X}(c_1) & -\tilde{X}(c_1) \\ -\tilde{Y}(c_1) & -\tilde{X}(c_1) \\ 0 & -u^{-c_1}\tilde{W}(c_2) \\ -u^{1-c_1}\tilde{W}(c_2) & 0 \end{pmatrix}$$

$$[\tilde{D}_1] = \begin{pmatrix} 0 & u^{1-c_1}\tilde{W}(c_2) & \tilde{X}(c_1)\tilde{W}(b_4) & \tilde{X}(c_1)\tilde{W}(b_4) \\ u^{1-c_1}\tilde{W}(c_2) & 0 & \tilde{Y}(c_1)\tilde{W}(b_4) & \tilde{X}(c_1)\tilde{W}(b_4) \end{pmatrix}$$

where

$$\tilde{X}(m) = \begin{cases} \sum_{j=1}^{m} u^{1-j} & \text{if } m > 0 \\ 0 & \text{if } m = 0 \end{cases}$$

$$\tilde{Y}(m) = \begin{cases} \sum_{j=1}^{m} u^{2-j} & \text{if } m > 0 \\ 0 & \text{if } m = 0 \end{cases}$$

$$\sum_{j=1}^{-m} u^j & \text{if } m < 0 \end{cases}$$

$$\sum_{j=1}^{-m} -u^{j+2} & \text{if } m < 0 \end{cases}$$
\[ W(m) = \begin{cases} 
\sum_{j=1}^{m} v^{1-j} & \text{if } m > 0 \\
0 & \text{if } m = 0 \\
\sum_{j=1}^{-m} -v^{j} & \text{if } m < 0 
\end{cases} \]

Thus we have:

\[ R(F) = T_1(\tilde{\partial}_s \otimes \tilde{D}_s) = u^{-1} \otimes \tilde{Y}(c_1) - 2 \otimes \tilde{X}(c_1) + 1 \otimes \tilde{Y}(c_1) + 1 \otimes \tilde{X}(c_1) \tilde{W}(b_4) - u^{-1} \otimes \tilde{Y}(c_1) \tilde{W}(b_4). \]

Two elements \( g_1, g_2 \in G \) belong to the same conjugacy class if and only if there is an element \( g \in G \) satisfying to equation: \( g_1 = gg_2\phi(g^{-1}). \)

In this case two elements \( u^{-1} \otimes u^n v^s \) and \( u^{-1} \otimes u^n v^t, \) \( m, n, s, t \in \mathbb{Z}, \) belong the same semiconjugate class if and only if there is \( k \in \mathbb{Z} \) satisfying:

\[ \begin{cases} 
m = n \\
k = (b_4 - 1) + t
\end{cases} \]

If \( c_1(b_4 - 1) \neq 0 \) then we have \( N(F) = |c_1(b_4 - 1)|. \) Since \( Fix(F) \) consist of \( |c_1(b_4 - 1)| \) circles, \( MF[F] \) is composed of \( |c_1(b_4 - 1)| \) disjoint circles, see Figure 3. Therefore the minimal fixed point set of \( f \) consist of \( |c_1(b_4 - 1)| \) disjoint circles. If \( c_1(b_4 - 1) = 0 \) then \( MF_{S1}[f] = 0. \) Thus the number \( MF_{S1}[f] = |c_1(b_4 - 1)|. \)

\[ \text{Figure 3. The set Fix(F) in the case } c_1 \text{ and } b_4 - 1 \text{ positives.} \]

Now in the case II with \( b_3 = n(b_4 - 1), \) \( n \in \mathbb{Z} \) we take the homotopy \( F : T \to T \) given by \( F([(x,y)], t) = [(x + b_3y + c_1t - \frac{1}{2}, b_4y + c_2t)] \) and the fiber-preserving map \( f : M(A) \to M(A) \) induced by \( F. \) We consider the isomorphism of fiber bundle \( P : M(A) \to M(A^1), \) \( A^1 = \)
$P \circ A \circ P^{-1}$, induced by the isomorphism on torus which also is denoted by $P : T \rightarrow T$ given by the following matrix:

$$P = \begin{pmatrix} 1 & -n \\ 0 & 1 \end{pmatrix}$$

By proposition \ref{prop:homotopy}, the fiber-preserving map $g : M(A^1) \rightarrow M(A^1)$ induced by homotopy $G = P \circ F \circ (P^{-1} \times I)$ has $MF_{S^1}[g] = MF_{S^1}[f]$. Here the homotopy $G$ is given by; $G([(x, y)], t) = [(x + (c_1 - nc_2)t - \frac{1}{2}, b_4y + c_2t)]$.

$$\begin{array}{ccc}
M(A) & \xrightarrow{f} & M(A) \\
p \downarrow & & \downarrow p \\
M(A^1) & \xrightarrow{g} & M(A^1)
\end{array}$$

Note that the homotopy $G$ is homotopic, relative to $T \times \{0, 1\}$, to the homotopy $G'$ given by;

$$G'([(x, y)], t) = \begin{cases} [(x + 2(c_1 - nc_2)t - \frac{1}{2}, b_4y)] , & 0 \leq t \leq \frac{1}{2} \\
[(x + (c_1 - nc_2) - \frac{1}{2}, b_4y + 2c_2t - c_2)] , & \frac{1}{2} \leq t \leq 1
\end{cases}$$

In fact, using the notation $G([(x, y)], t) = [(\alpha(x, y, t), \beta(x, y, t))]$, where $\alpha(x, y, t) = x + (c_1 - nc_2)t - \frac{1}{2}$ and $\beta(x, y, t) = b_4y + c_2t$, then $H : T \times I \times I \rightarrow T$ defined by;

$$H([(x, y)], t, s) = \begin{cases} [(\alpha(x, y, t), \beta(x, y, t))] & \text{if} & 0 \leq t \leq s \\
[(\alpha(x, y, 2t - s), \beta(x, y, s))] & \text{if} & s \leq t \leq \frac{(1+s)}{2} \\
[(\alpha(x, y, 1), \beta(x, y, 2t - 1))] & \text{if} & \frac{(1+s)}{2} \leq t \leq 1
\end{cases}$$

is a homotopy, relative to $T \times \{0, 1\}$, between $G$ and $G'$. Thus, we have $R(G) = R(G')$. Therefore, we can use the previously case and proposition \ref{prop:homotopy} to show that the minimal fixed point set of $f$ over $S^1$ is composed by $|c_1(b_4 - 1) - c_2b_3|$ disjoint circles.

In the case $\text{III}$ we have $a_3(b_4 - 1) = -2b_3$. Therefore if $a_3$ is even then $b_3 = \frac{-a_3}{2}(b_4 - 1)$. Thus we can use a similar argument as in the case above and show that the minimal fixed point set of a fiber-preserving map $f : M(A) \rightarrow M(A)$ in this situation is composed by
\(|c_1(b_4 - 1) - c_2b_3|\) disjoint circles. Note that if \(a_3\) is even, then a fiber-preserving map \(f : M(A) \to M(A)\) in a fiber bundle \(M(A)\) with

\[
A = \begin{pmatrix} 1 & a_3 \\ 0 & -1 \end{pmatrix}
\]

has the minimal fixed point set over \(S^1\) composed by \(|c_1(b_4 - 1) - c_2b_3|\) disjoint circles.

Now, let us consider the cases \(II\) and \(III\) in the following situation; \(b_4 = -1\) and \(b_3 = 2k + 1, k \in \mathbb{Z}\). Note that the case \(b_3\) even has already been solved. First we take \(b_3 = 1\).

Consider the fiber-preserving map \(f : MA \to MA\) induced by homotopy \(F : T \times I \to T\) given by:

\[
F([(x, y)], t) = \begin{cases} 
\left[ (x + y + 2c_1t + \frac{1}{2}, -y + \frac{1}{2}) \right], & 0 \leq t \leq \frac{1}{2} \\
\left[ (x + y + \frac{2a_1 + 1}{2}, -y + 2c_2t - c_2 + \frac{1}{2}) \right], & \frac{1}{2} \leq t \leq 1
\end{cases}
\]

Note that if \(f\) has no fixed point in \(t = 0, 1\). For compute the one-parameter trace \(R(F)\) we consider the cellular decomposition of the torus which consist of four 0-cells; \(E_0^0 = \{(0, 0)\}, E_2^0 = \{(\frac{1}{2}, 0)\}, E_3^0 = \{(0, \frac{1}{2})\}, E_4^0 = \{(\frac{1}{2}, \frac{1}{2})\}\), twelve 1-cells; \(E_1^1 = \{(x, 0)\} 0 \leq x \leq \frac{1}{2}\), \(E_2^1 = \{(x, 0)\} \frac{1}{2} \leq x \leq 1\), \(E_3^1 = \{(0, y)\} 0 \leq y \leq \frac{1}{2}\), \(E_4^1 = \{(y, -y + \frac{1}{7})\} 0 \leq y \leq \frac{1}{7}\), \(E_5^1 = \{(\frac{1}{7}, y)\} 0 \leq y \leq \frac{1}{7}\), \(E_6^1 = \{(y, -y+1)\} \frac{1}{7} \leq y \leq 1\), \(E_7^1 = \{(x, \frac{1}{7})\} 0 \leq x \leq \frac{1}{7}\), \(E_8^1 = \{(x, \frac{1}{7})\} \frac{1}{7} \leq x \leq 1\), \(E_9^1 = \{(0, y)\} \frac{1}{2} \leq y \leq 1\), \(E_{10}^1 = \{(y, -y+1)\} 0 \leq y \leq \frac{1}{7}\), \(E_{11}^1 = \{(\frac{1}{7}, y)\} \frac{1}{2} \leq y \leq 1\), \(E_{12}^1 = \{(y + \frac{1}{2}, -y+1)\} 0 \leq y \leq \frac{1}{2}\), and eight 2-cells; \(E_1^2 = \{((x, y)) 0 \leq x \leq \frac{1}{2}, 0 \leq y \leq -x + \frac{1}{2}\}, E_2^2 = \{((x, y)) 0 \leq x \leq \frac{1}{2}, -x + \frac{1}{2} \leq y \leq \frac{1}{2}\}, E_3^2 = \{((x, y)) \frac{1}{2} \leq x \leq 1, 0 \leq y \leq -x + 1\}, E_4^2 = \{((x, y)) \frac{1}{2} \leq x \leq 1, -x + 1 \leq y \leq \frac{1}{2}\}, E_5^2 = \{((x, y)) 0 \leq x \leq \frac{1}{2}, \frac{1}{2} \leq y \leq -x + 1\}, E_6^2 = \{((x, y)) 0 \leq x \leq \frac{1}{2}, -x + 1 \leq y \leq 1\}, E_7^2 = \{((x, y)) \frac{1}{2} \leq x \leq 1, \frac{1}{2} \leq y \leq -x + \frac{1}{2}\}, E_8^2 = \{((x, y)) \frac{1}{2} \leq x \leq 1, -x + \frac{3}{2} \leq y \leq 1\}.

These cells are oriented as in the figure below. For this cellular decomposition the homotopy \(F\) is cellular.

For the cellular decomposition above we choose in the universal cover \(\mathbb{R}^2\) the lifts which consist of four 0-cells; \(\tilde{E}_0^0 = \{(0, 0)\}, \tilde{E}_2^0 = \{(\frac{1}{2}, 0)\}, \tilde{E}_3^0 = \{(0, \frac{1}{2})\}, \tilde{E}_4^0 = \{(\frac{1}{2}, \frac{1}{2})\}\), twelve 1-cells; \(\tilde{E}_1^1 = \{(x, 0)\} 0 \leq x \leq \frac{1}{2}\),
We take $w = [(0,0)]$ the basepoint and $\tau$ basepath, the linear path between $w$ and $F(w,0)$. We take the lifts $\tilde{w} = (0,0)$ and $\tilde{\tau}$ the linear path between $\tilde{w}$ and $(0,\frac{1}{2})$. The unique lift $\tilde{F} : \mathbb{R}^2 \times I \to \mathbb{R}^2$ of $F$ mapping $(\tilde{w},0)$ to $\tilde{\tau}(1)$ is given by:

$$
\tilde{F}((x, y), t) = \begin{cases}
(x + y + 2c_1 t + \frac{1}{2}, -y + \frac{1}{2}) , & 0 \leq t \leq \frac{1}{2} \\
(x + y + 2c_2 t - c_2 + \frac{1}{2}, -y + 2c_2 t - c_2 + \frac{1}{2}) , & \frac{1}{2} \leq t \leq 1
\end{cases}
$$

If $G = \pi_1(T, [(0,0)]) = \{u, v | uvu^{-1}v^{-1} = 1\}$ then

$$
[\tilde{G}] = \begin{pmatrix}
-1 & u^{-1} & -1 & 0 & 0 & u^{-1} & 0 & 0 & v^{-1} & -v^{-1} & 0 & 0 \\
1 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & v^{-1} & -v^{-1} \\
0 & 0 & 1 & -1 & 0 & 0 & -1 & u^{-1} & -1 & 0 & 0 & u^{-1} \\
0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & 1 & -1 & 0
\end{pmatrix},
$$
\[ [\bar{\partial}_2] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -v^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -v^{-1} \\ -1 & 0 & 0 & u^{-1} & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}, \]

\[ [\bar{D}_0] = \begin{pmatrix} 0 & 0 & -u^{-1}\bar{X}(c_1) & -u^{-2}\bar{X}(c_1) \\ 0 & 0 & -u^{-1}\bar{X}(c_1) & -u^{-1}\bar{X}(c_1) \\ 0 & -v^{-1}\bar{W}(c_2) & -u^{-c_1-1}\bar{W}(c_2) & 0 \\ 0 & 0 & 0 & 0 \\ -u^{-c_1}v^{-1}\bar{W}(c_2) & 0 & 0 & -u^{-c_1-1}\bar{W}(c_2) \\ 0 & 0 & 0 & 0 \\ -u^{-c_1}\bar{X}(c_1) & -u^{-1}\bar{X}(c_1) & 0 & 0 \\ -\bar{X}(c_1) & -u^{-1}\bar{X}(c_1) & 0 & 0 \\ 0 & -u^{-c_1-1}\bar{W}(c_2) & -u^{-c_1-1}\bar{W}(c_2) & 0 \\ 0 & 0 & 0 & 0 \\ -u^{-c_1}\bar{W}(c_2) & 0 & 0 & -u^{-c_1}\bar{W}(c_2) \\ 0 & 0 & 0 & 0 \end{pmatrix} \]

and with the following data:

\[
\begin{align*}
\bar{D}_1(\bar{E}_1^1) &= \tilde{E}_3^2u^{-c_1}v^{-1}\bar{W}(c_2)+\tilde{E}_4^2u^{-c_1}v^{-1}\bar{W}(c_2)+\tilde{E}_5^2u^{-c_1}\bar{W}(c_2)+\tilde{E}_6^2u^{-c_1}\bar{W}(c_2), \\
\bar{D}_1(\bar{E}_2^1) &= \tilde{E}_1^2u^{-c_1-1}v^{-1}\bar{W}(c_2)+\tilde{E}_2^2u^{-c_1-1}v^{-1}\bar{W}(c_2)+\tilde{E}_3^2u^{-c_1-1}\bar{W}(c_2)+\tilde{E}_4^2u^{-c_1-1}\bar{W}(c_2), \\
\bar{D}_1(\bar{E}_3^1) &= \tilde{E}_1^2u^{-c_1}\bar{X}(c_1)+\tilde{E}_2^2u^{-c_1}\bar{X}(c_1)+\tilde{E}_3^2(u^{-c_1}\bar{X}(c_1)+u^{-c_1}v^{-1}\bar{W}(c_2)) \\
&+ \tilde{E}_4^2(\bar{X}(c_1)+u^{-c_1}\bar{W}(c_2))+\tilde{E}_5^2u^{-c_1}\bar{W}(c_2)+\tilde{E}_6^2u^{-c_1}\bar{W}(c_2), \\
\bar{D}_1(\bar{E}_4^1) &= \tilde{E}_1^2u^{-c_1}\bar{X}(c_1)+\tilde{E}_2^2u^{-c_1}\bar{X}(c_1)+\tilde{E}_3^2u^{-c_1}\bar{X}(c_1)+\tilde{E}_4^2u^{-c_1}\bar{X}(c_1), \\
\bar{D}_1(\bar{E}_5^1) &= \tilde{E}_1^2(u^{-2}\bar{X}(c_1)+u^{-c_1-1}v^{-1}\bar{W}(c_2))+\tilde{E}_2^2(u^{-1}\bar{X}(c_1)+u^{-c_1-1}\bar{W}(c_2)) \\
&+ \tilde{E}_3^2u^{-1}\bar{X}(c_1)+\tilde{E}_4^2u^{-1}\bar{X}(c_1)+\tilde{E}_5^2u^{-c_1-1}\bar{W}(c_2)+\tilde{E}_6^2u^{-c_1-1}\bar{W}(c_2), \\
\bar{D}_1(\bar{E}_6^1) &= \tilde{E}_1^2u^{-2}\bar{X}(c_1)+\tilde{E}_2^2u^{-2}\bar{X}(c_1)+\tilde{E}_3^2u^{-1}\bar{X}(c_1)+\tilde{E}_4^2u^{-1}\bar{X}(c_1),
\end{align*}
\]
and proposition 3.2 we can conclude that the minimal fixed point set
\[ \text{Fix}(\tilde{F}) = \tilde{F}(\tilde{x}) = \tilde{x}. \]

Similar to the case \( b \neq 0 \), we obtain;
\[ N(F) = |2c_1 + c_2| = |c_1(b_4 - 1) - c_2b_3|. \]

Since \( Fix(F) \) is composed by \(|c_1(b_4 - 1) - c_2b_3| \) disjoint circles, then
the minimal fixed point set of \( f : M(A) \to M(A) \) induced by \( F : T \times I \to T \) is composed by \(|c_1(b_4 - 1) - c_2b_3| \) disjoint circles. Therefore,
\[ MF_{S^1}[f] = |c_1(b_4 - 1) - c_2b_3|. \]

The case \( b_3 = 2k + 1 \) with \( k \neq 0 \), we take the fiber-preserving map
\( f : M(A) \to M(A) \) induced by \( F : T \times I \to T \) given by \( F([(x, y)], t) = [(x + b_3y + c_1t + \frac{-k+1}{2}, -y + c_2t + \frac{1}{2})] \). Conjugating the homotopy \( F \) by
the isomorphism \( P : T \to T \) given by
\[ [P] = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \]
we obtain the homotopy \( G = P \circ F \circ (P^{-1} \times I) \). The fiber-preserving
map \( g : M(A^1) \to M(A^1), A^1 = P \circ A \circ P^{-1} \), given by \( g(<[(x, y)], t >) = G([(x, y)], t) \), has \( MF_{S^1}[g] = MF_{S^1}[f] \). By the case above
and proposition 3.2 we can conclude that the minimal fixed point set
of $f$ is composed by $|c_1(b_4 - 1) - c_2b_3|$ disjoint circles. This implies $MF_{S^1}[f] = |c_1(b_4 - 1) - c_2b_3|$. □

**Remarks 5.1.** Note that by Theorem 2.2 the number $|c_1(b_4 - 1) - c_2b_3|$ appeared in [3] only to decide when a fiber-preserving map, in the cases II and III, can be deformed by a fiberwise homotopy to a fixed point free map. In Theorem 5.1 we have shown that the number $|c_1(b_4 - 1) - c_2b_3|$ is exactly the number of circles of minimal fixed point set of a fiber-preserving map in the cases II and III.

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MINIMAL FIXED POINT SET OF MAPS ON T-BUNDLES OVER $S^1$

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