Algebro-geometric solutions for the two-component Camassa-Holm Dym hierarchy

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Abstract

This paper is dedicated to provide theta function representations of algebro-geometric solutions and related crucial quantities for the two-component Camassa-Holm Dym (CHD2) hierarchy. Our main tools include the polynomial recursive formalism, the hyperelliptic curve with finite number of genus, the Baker-Akhiezer functions, the meromorphic function, the Dubrovin-type equations for auxiliary divisors, and the associated trace formulas. With the help of these tools, the explicit representations of the algebro-geometric solutions are obtained for the entire CHD2 hierarchy. 

Key words: two-component Camassa-Holm Dym, hyperelliptic curve, algebro-geometric solutions

1 Introduction

In this paper we consider the following integrable two-component Camassa-Holm Dym (CHD2) system:

\[
\begin{align*}
\rho_t + \left( \frac{m}{\rho^2} \right)_x &= 0, \\
m_t - \left( 1 - \partial_x^2 \frac{1}{\rho} \right)_x &= 0,
\end{align*}
\]

(1.1)

where \( m = u - u_{xx} \), which was recently introduced by Holm and Ivanov in [18]. This coupled nonlinear system is at the position in the two component Camassa-Holm hierarchy [5, 6, 10, 15, 16, 17, 21, 22, 27] that corresponds to

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the modified Dym equation, first introduced as a tri-Hamiltonian system in \cite{25}. The CHD2 equation combines to produce the nonlinear wave equation

\[ m_{tt} = (1 - \partial_x^2) \left( \frac{1}{\rho^2} \partial_x \frac{1}{\rho^2} \right) m. \]  

(1.2)

Linearizing this equation around \( m = 0 \) and \( \rho = 1 \) yields the dispersion relation for a plane wave \( \exp(i(\kappa x - wt)) \) with wave number \( \kappa \) and frequency \( w \) as

\[ w^2 = (1 + \kappa^2)^2. \]

Accordingly, the phase speed of the linearized plane waves is \( w/\kappa = \sqrt{1 + \kappa^2} \), so the higher wave numbers travel faster \cite{18}. This type of dispersion relation is the same as for time-dependent Euler-Bernoulli theory for an elastic beam with both bending and vibration response \cite{18,26}.

The CHD2 equation is also a completely integrable system with a bi-Hamiltonian structure, and hence, it possesses a Lax pair and conservation laws \cite{18}. Recently, soliton solutions and travelling wave solutions of the system (1.1) were investigated in \cite{18}. However, within the knowledge of the authors, the algebro-geometric solutions of the entire CHD2 hierarchy are not studied yet.

The principal subject of this paper concerns algebro-geometric quasi-periodic solutions of the whole CHD2 hierarchy, of which (1.1) is just the first of infinitely many members. Algebro-geometric solution, as an important feature of integrable system, is a kind of explicit solution closely related to the inverse spectral theory \cite{1,2,7,8}. In a degenerated case of the algebro-geometric solution, the multi-soliton solution and periodic solution in elliptic function type may be obtained \cite{4,23,24}. A systematic approach, proposed by Gesztesy and Holden to construct algebro-geometric solutions for integrable equations, has been extended to the whole (1+1) dimensional integrable hierarchy, such as the AKNS hierarchy, the Camassa-Holm (CH) hierarchy etc. \cite{11,12,13}. Recently, we investigated algebro-geometric solutions for the modified CH hierarchy and the Degasperis-Procesi hierarchy \cite{19,20}.

The outline of the present paper is as follows.

In section 2, based on the polynomial recursion formalism, we derive the CHD2 hierarchy, associated with the \( 2 \times 2 \) spectral problem. A hyperelliptic curve \( K_n \) of arithmetic genus \( n \) is introduced with the help of the characteristic polynomial of Lax matrix \( V_n \) for the stationary CHD2 hierarchy.

In Section 3, we decompose the stationary CHD2 equations into a system of Dubrovin-type equations. Moreover, we obtain the stationary trace formulas for the CHD2 hierarchy.
In Section 4, we present the first set of our results, the explicit theta function representations of the Baker-Akhiezer function, the meromorphic function, and in particular, that of the potentials \( u, \rho \) for the entire stationary CHD2 hierarchy.

In Sections 5 and 6, we extend the analyses of Sections 3 and 4, respectively, to the time-dependent case. Each equation in the CHD2 hierarchy is permitted to evolve in terms of an independent time parameter \( t_r \). As initial data, we use a stationary solution of the \( n \)th equation and then construct a time-dependent solution of the \( r \)th equation of the CHD2 hierarchy. The Baker-Akhiezer function, the analogs of the Dubrovin-type equations, the trace formulas, and the theta function representations in Section 4 are all extended to the time-dependent case.

Finally, it should perhaps be noted that the system (1.1) was obtained by using the condition \( u\rho = -2 \) in [18]. At this point, we will occasionally use this relation in the following sections.

## 2 The CHD2 hierarchy

In this section, we provide the construction of CHD2 hierarchy and derive the corresponding sequence of zero-curvature pairs using a polynomial recursion formalism. Moreover, we introduce the underlying hyperelliptic curve in connection with the stationary CHD2 hierarchy.

Throughout this section, we make the following hypothesis.

**Hypothesis 2.1.** In the stationary case, we assume that

\[
\begin{align*}
&u, \rho \in C^\infty(\mathbb{R}), \quad u(x) \neq 0, \rho(x) \neq 0, \quad \partial_x^k u, \partial_x^k \rho \in L^\infty(\mathbb{R}), \quad k \in \mathbb{N}_0. \quad (2.1)
\end{align*}
\]

In the time-dependent case, we suppose

\[
\begin{align*}
&u(\cdot, t), \rho(\cdot, t) \in C^\infty(\mathbb{R}), \quad \partial_x^k u(\cdot, t), \partial_x^k \rho(\cdot, t) \in L^\infty(\mathbb{R}), \quad k \in \mathbb{N}_0, \quad t \in \mathbb{R},
&u(x, \cdot), u_{xx}(x, \cdot), \rho(x, \cdot), \rho_x(x, \cdot) \in C^1(\mathbb{R}), \quad x \in \mathbb{R},
&u(x, t) \neq 0, \rho(x, t) \neq 0, \quad (x, t) \in \mathbb{R}^2. \quad (2.2)
\end{align*}
\]

We first introduce the basic polynomial recursion formalism. Define
\{f_l\}_{l \in \mathbb{N}_0}, \{g_l\}_{l \in \mathbb{N}_0}, \text{ and } \{h_l\}_{l \in \mathbb{N}_0} \text{ recursively by }
\begin{align*}
f_0 &= -u, \\
f_l &= \mathcal{G}(-f_{l-2,xxx} + 4(u - u_{xx})f_{l-1,x} + f_{l-2,x} + 2(u_x - u_{xxx})f_{l-1}), \quad l \in \mathbb{N}, \\
g_l &= \frac{1}{2}f_{l,x}, \quad l \in \mathbb{N}_0, \\
h_l &= g_{l-2,x} - \rho^2 f_l + (u - u_{xx})f_{l-1} + \frac{1}{4}f_{l-2}, \quad l \in \mathbb{N}_0,
\end{align*}
\tag{2.3}

where \(\mathcal{G}\) is given by
\begin{align*}
\mathcal{G} : L_\infty(\mathbb{R}) &\to L_\infty(\mathbb{R}), \\
(\mathcal{G}v)(x) &= \frac{1}{4}\rho(x)^{-1} \int_{-\infty}^{x} \rho(y)^{-1}v(y) \, dy, \quad x \in \mathbb{R}, \; v \in L_\infty(\mathbb{R}).
\end{align*}
\tag{2.4}

One observes that
\begin{align*}
\mathcal{G} = (2\partial_x \rho^2 + 2\rho^2 \partial_x)^{-1}.
\end{align*}
\tag{2.5}

Explicitly, one computes
\begin{align*}
f_0 &= -u, \\
f_1 &= \mathcal{G}(-4u_x(u - u_{xx}) - 2u(u_x - u_{xxx})) + c_1(-u), \\
g_0 &= -\frac{1}{2}u_x, \\
g_1 &= \frac{1}{2}\mathcal{G}(-4u_{xx}(u - u_{xx}) - 6u_x(u_x - u_{xxx}) - 2u(u_{xx} - u_{xxxx})) + c_1(-\frac{1}{2}u_x), \\
h_0 &= u\rho^2, \\
h_1 &= -u(u - u_{xx}) - \rho^2 f_1, \text{ etc.,}
\end{align*}
\tag{2.6}

where \(\{c_l\}_{l \in \mathbb{N} \subset \mathbb{C}}\) are integration constants.

Next, it is convenient to introduce the corresponding homogeneous coefficients \(\hat{f}_l, \hat{g}_l, \text{ and } \hat{h}_l\), defined by the vanishing of the integration constants \(c_k, k = 1, \ldots, l,\)
\begin{align*}
\hat{f}_0 &= f_0 = -u, \quad \hat{f}_l = f_l|_{c_k=0, \ k=1,\ldots,l}, \quad l \in \mathbb{N}, \\
\hat{g}_0 &= g_0 = -\frac{1}{2}u_x, \quad \hat{g}_l = g_l|_{c_k=0, \ k=1,\ldots,l}, \quad l \in \mathbb{N}, \\
\hat{h}_0 &= h_0 = u\rho^2, \quad \hat{h}_l = h_l|_{c_k=0, \ k=1,\ldots,l}, \quad l \in \mathbb{N},
\end{align*}
\tag{2.7}
Hence,

\[ f_l = \sum_{k=0}^{l} c_{l-k} \hat{f}_k, \quad g_l = \sum_{k=0}^{l} c_{l-k} \hat{g}_k, \quad h_l = \sum_{k=0}^{l} c_{l-k} \hat{h}_k, \quad l \in \mathbb{N}_0, \] \tag{2.8}

defining

\[ c_0 = 1. \] \tag{2.9}

Now, given Hypothesis 2.1, one introduces the following \(2 \times 2\) matrix \(U\) by

\[ \psi_x = U(z,x)\psi = \begin{pmatrix} 0 & 1 \\ -z^2 \rho^2 + z(u - u_{xx}) + \frac{1}{4} & 0 \end{pmatrix} \psi, \] \tag{2.10}

and for each \(n \in \mathbb{N}_0\), the following \(2 \times 2\) matrix \(V_n\) by

\[ \psi_{t_n} = V_n(z)\psi, \] \tag{2.11}

with

\[ V_n(z) = \begin{pmatrix} -zG_n(z) & zF_n(z) \\ zH_n(z) & zG_n(z) \end{pmatrix}, \quad z \in \mathbb{C} \setminus \{0\}, \quad n \in \mathbb{N}_0, \] \tag{2.12}

assuming \(F_n, G_n,\) and \(H_n\) to be polynomials\(^1\) with respect to \(z\) and \(C^\infty\) in \(x\). The compatibility condition of linear system (2.10) and (2.11) yields the stationary zero-curvature equation

\[ -V_{n,x} + [U, V_n] = 0, \] \tag{2.13}

which is equivalent to

\[ F_{n,x} = 2G_n, \] \tag{2.14}

\[ H_{n,x} = -2 \left(-z^2 \rho^2 + z(u - u_{xx}) + \frac{1}{4}\right) G_n, \] \tag{2.15}

\[ G_{n,x} = -H_n + \left(-z^2 \rho^2 + z(u - u_{xx}) + \frac{1}{4}\right) F_n. \] \tag{2.16}

From (2.14)-(2.16), one infers that

\[ \frac{d}{dx} \det(V_n(z,x)) = -z^2 \frac{d}{dx} \left(G_n(z,x)^2 + F_n(z,x)H_n(z,x)\right) = 0, \] \tag{2.17}

and hence

\[ G_n(z,x)^2 + F_n(z,x)H_n(z,x) = R_{2n+2}(z), \] \tag{2.18}

\(^1F_n, G_n, H_n\) are polynomials of degree \(n, n, n \pm 2\), respectively.
where the polynomial \( R_{2n+2} \) of degree \( 2n + 2 \) is \( x \)-independent. In another way, one can write \( R_{2n+2} \) as

\[
R_{2n+2}(z) = -4 \prod_{m=0}^{2n+1} (z - E_m), \quad \{E_m\}_{m=0,...,2n+1} \in \mathbb{C}.
\] (2.19)

Next, we compute the characteristic polynomial \( \det(yI - z^{-1}V_n) \) of Lax matrix \( z^{-1}V_n \),

\[
\det(yI - z^{-1}V_n) = y^2 - G_n(z)^2 - F_n(z)H_n(z) = y^2 - R_{2n+2}(z) = 0,
\] (2.20)

and then introduce the (possibly singular) hyperelliptic curve \( K_n \) of arithmetic genus \( n \) defined by

\[
K_n : F_n(z, y) = y^2 - R_{2n+2}(z) = 0.
\] (2.21)

In the following, we will occasionally impose further constraints on the zeros \( E_m \) of \( R_{2n+2} \) introduced in (2.19) and assume that

\[
E_m \in \mathbb{C}, \quad E_m \neq E_{m'}, \quad \forall m \neq m', \quad m, m' = 0, \ldots, 2n + 1.
\] (2.22)

The stationary zero-curvature equation (2.13) implies polynomial recursion relations (2.3). Introducing the following polynomials \( F_n(z), G_n(z), \) and \( H_n(z) \) with respect to the spectral parameter \( z \),

\[
F_n(z) = \sum_{l=0}^{n} f_l z^{n-l},
\] (2.23)

\[
G_n(z) = \sum_{l=0}^{n} g_l z^{n-l},
\] (2.24)

\[
H_n(z) = \sum_{l=0}^{n+2} h_l z^{n+2-l}.
\] (2.25)

Inserting (2.23)-(2.25) into (2.14)-(2.16) then yields the recursion relations (2.3) for \( f_l, l = 0, \ldots, n \), and \( g_l, l = 0, \ldots, n \). For fixed \( n \in \mathbb{N}_0 \), we obtain the recursion relations for \( h_l, l = 0, \ldots, n \) in (2.3) and

\[
h_{n+1} = -\frac{1}{2} f_{n-1,xx} + (u - u_{xx}) f_n + \frac{1}{4} f_{n-1},
\]

\[
h_{n+2} = -\frac{1}{2} f_{n,xx} + \frac{1}{4} f_n.
\] (2.26)
Moreover, from (2.15), one infers that
\[
- h_{n+1,x} - (u - u_{xx})f_{n,x} - \frac{1}{4}f_{n-1,x} = 0, \quad n \in \mathbb{N}_0, \\
- h_{n+2,x} - \frac{1}{4}f_{n,x} = 0, \quad n \in \mathbb{N}_0.
\]
Then using (2.26) and (2.27) permits one to write the stationary CHD2 hierarchy as
\[
s\text{-CHD}_2 n (u, \rho) = \left( -2(u - u_{xx})f_{n,x} - (u_x - u_{xxx})f_n - \frac{1}{4}f_{n-1,x} + \frac{1}{2}f_{n-1,xxx} - \frac{1}{2}f_{n,x} + \frac{1}{2}f_{n,xxx} \right) = 0, \quad n \in \mathbb{N}_0.
\]
We record the first equation explicitly,
\[
s\text{-CHD}_2 0 (u, \rho) = \left( 2u_x (u - u_{xx}) + u (u_x - u_{xxx}) \right) = 0.
\]

By definition, the set of solutions of (2.28) represents the class of algebro-geometric CHD2 solutions, with \( n \) ranging in \( \mathbb{N}_0 \) and \( c_l \) in \( \mathbb{C} \), \( l \in \mathbb{N} \). We call the stationary algebro-geometric CHD2 solutions \( u, \rho \) as CHD2 potentials at times.

**Remark 2.2.** Here, we emphasize that if \( u, \rho \) satisfy one of the stationary CHD2 equations in (2.28) for a particular value of \( n \), then they satisfy infinitely many such equations of order higher than \( n \) for certain choices of integration constants \( c_l \). This is a common characteristic of the general integrable soliton equations such as the KdV, AKNS, and CH hierarchies [13].

Next, we introduce the corresponding homogeneous polynomials \( \hat{F}_l, \hat{G}_l, \hat{H}_l \) by
\[
\hat{F}_l(z) = F_l(z)|_{c_k=0, k=1, \ldots, l} = \sum_{k=0}^{l} \hat{f}_k z^{l-k}, \quad l = 0, \ldots, n,
\]
\[
\hat{G}_l(z) = G_l(z)|_{c_k=0, k=1, \ldots, l} = \sum_{k=0}^{l} \hat{g}_k z^{l-k}, \quad l = 0, \ldots, n,
\]
\[
\hat{H}_l(z) = H_l(z)|_{c_k=0, k=1, \ldots, l} = \sum_{k=0}^{l} \hat{h}_k z^{l-k}, \quad l = 0, \ldots, n,
\]
\[
\hat{H}_{n+1}(z) = -\hat{g}_{n-1,x} + (u - u_{xx})\hat{f}_n + \frac{1}{4}\hat{f}_{n-1} + \sum_{k=0}^{n} \hat{h}_k z^{n+1-k}.
\]
\[
\hat{H}_{n+2}(z) = -\hat{g}_{n,x} + \frac{1}{4}\hat{f}_{n} + z\hat{H}_{n+1}(z).
\] (2.34)

In accordance with our notation introduced in (2.7) and (2.30)-(2.34), the corresponding homogeneous stationary CHD2 equations are then defined by

\[
s\cdot\text{CHD}_2(u,\rho) = s\cdot\text{CHD}_2(u,\rho)|_{c_l=0}, \quad l=1,...,n = 0, \quad n \in \mathbb{N}_0.
\] (2.35)

At the end of this section, we turn to the time-dependent CHD2 hierarchy. In this case, \(u, \rho\) are considered as functions of both space and time. We introduce a deformation parameter \(t_n \in \mathbb{R}\) in \(u\) and \(\rho\), replacing \(u(x), \rho(x)\) by \(u(x, t_n), \rho(x, t_n)\), for each equation in the hierarchy. In addition, the definitions (2.10), (2.12), and (2.23)-(2.25) of \(U, V\), and \(F, G, H\), respectively, still apply. The corresponding zero-curvature equation reads

\[
U_{t_n} - V_{n,x} + [U, V_n] = 0, \quad n \in \mathbb{N}_0,
\] (2.36)

which results in the following set of equations

\[
F_{n,x} = 2G_n,
\] (2.37)

\[
G_{n,x} = -H_n + \left(-z^2\rho^2 + z(u - u_{xx}) + \frac{1}{4}\right)F_n,
\] (2.38)

\[
-2z\rho t_n + (u_t - u_{xx}) - H_n - 2\left(-z^2\rho^2 + z(u - u_{xx}) + \frac{1}{4}\right)G_n = 0.
\] (2.39)

For fixed \(n \in \mathbb{N}_0\), inserting the polynomial expressions for \(F_n, G_n, \) and \(H_n\) into (2.37)-(2.39), respectively, first yields recursion relations (2.3) for \(f_l|_{l=0,...,n}, \ g_l|_{l=0,...,n}, \ h_l|_{l=0,...,n}\) and

\[
h_{n+1} = -\frac{1}{2}f_{n-1,xx} + (u - u_{xx})f_n + \frac{1}{4}f_{n-1},
\]

\[
h_{n+2} = -\frac{1}{2}f_{n,xx} + \frac{1}{4}f_n.
\] (2.40)

Moreover, using (2.39), one finds

\[
-2\rho t_n - h_{n+1,x} - (u - u_{xx})f_{n,x} - \frac{1}{4}f_{n-1,x} = 0, \quad n \in \mathbb{N}_0,
\]

\[
u_{t_n} - u_{xx}t_n - h_{n+2,x} - \frac{1}{4}f_{n,x} = 0, \quad n \in \mathbb{N}_0.
\] (2.41)
Hence, using (2.40) and (2.41) permits one to write the time-dependent CHD2 hierarchy as
\[
\text{CHD}_2^n(u, \rho) = \left( -2\rho \rho_t - 2(u - u_{xx})f_{n,x} - (u_x - u_{xxx})f_n - \frac{1}{2}f_{n-1,x} + \frac{1}{2}f_{n-1,xxx}, \right. \\
\left. u_{tt} - u_{xx} + \frac{1}{2}f_{n,x} + \frac{1}{2}f_{n,xxx} \right) = 0, n \in \mathbb{N}_0.
\] (2.42)

For convenience, we record the first equation in this hierarchy explicitly,
\[
\text{CHD}_2^0(u, \rho) = \left( -2\rho \rho_t + 2u_x(u - u_{xx}) + u(u_x - u_{xxx}) \right) = 0. \tag{2.43}
\]

The first equation CHD$_2^0(u, \rho) = 0$ in the hierarchy is equivalent to the CHD2 system as discussed in section 1, taking into account $u\rho = -2$ and the differential relation arising in the $z^{n+2}$ term in (2.39), that is, $-2u_x\rho^2 = 2u\rho\rho_x$. Similarly, one can introduce the corresponding homogeneous CHD2 hierarchy by
\[
\tilde{\text{CHD}}_2^n(u, \rho) = \text{CHD}_2^n(u, \rho)|_{c_l=0}, l=1,...,n=0, \quad n \in \mathbb{N}_0. \tag{2.44}
\]

**Remark 2.3.** The Lenard recursion formalism for the CHD2 hierarchy can be set up as follows. Define the two Lenard operators
\[
K = \begin{pmatrix} \partial^3 - \partial & 0 \\ 0 & -2\partial \rho^2 - 2\rho^2 \partial \end{pmatrix}, \quad J = \begin{pmatrix} 2\partial m + 2m \partial & -2\partial \rho^2 - 2\rho^2 \partial \\ -2\partial \rho^2 - 2\rho^2 \partial & 0 \end{pmatrix},
\]
then the Lenard recursion sequence is given by
\[
K \partial_j = J \partial_{j+1}, \quad J \partial_0 = 0,
\]
where $m = u - u_{xx}$, $\partial_j = (f_j, f_{j+1})^T$, $j = 0, \ldots, n - 1$. Hence, using the zero-curvature equation (2.36), one can obtain the CHD2 hierarchy.

In fact, since the Lenard recursion formalism is almost universally adopted in the contemporary literature, we thought it might be worthwhile to use the Gesztesy’s method, the polynomial recursion formalism, to construct the CHD2 hierarchy.

### 3 The stationary CHD2 formalism

This section is devoted to a detailed study of the stationary CHD2 hierarchy. We first define a fundamental meromorphic function $\phi(P, x)$ on the
hyperelliptic curve $\mathcal{K}_n$, using the polynomial recursion formalism described in section 2, and then study the properties of the Baker-Akhiezer function $\psi(P, x, x_0)$, Dubrovin-type equations, and trace formulas.

For major parts of this section, we assume (2.1), (2.3), (2.6), (2.10)-(2.16), (2.21)-(2.25), and (2.28), keeping $n \in \mathbb{N}_0$ fixed.

Recall the hyperelliptic curve $\mathcal{K}_n$

\[
\mathcal{K}_n : \mathcal{F}_n(z, y) = y^2 - R_{2n+2}(z) = 0,
\]

\[
R_{2n+2}(z) = -4 \prod_{m=0}^{2n+1} (z - E_m), \quad \{E_m\}_{m=0,\ldots,2n+1} \subseteq \mathbb{C}, \tag{3.1}
\]

which is compactified by joining two points at infinity $P_{\infty+}$, with $P_{\infty+} \neq P_{\infty-}$. But for notational simplicity, the compactification is also denoted by $\mathcal{K}_n$. Hence, $\mathcal{K}_n$ becomes a two-sheeted Riemann surface of arithmetic genus $n$. Points $P$ on $\mathcal{K}_n \setminus \{P_{\infty}\}$ are denoted by $P = (z, y(P))$, where $y(\cdot)$ is the meromorphic function on $\mathcal{K}_n$ satisfying $\mathcal{F}_n(z, y(P)) = 0$.

The complex structure on $\mathcal{K}_n$ is defined in the usual way by introducing local coordinates

$\zeta_{Q_0} : P \to (z - z_0)$

near points $Q_0 = (z_0, y(Q_0)) \in \mathcal{K}_n$, which are neither branch nor singular points of $\mathcal{K}_n$; near the branch and singular points $Q_1 = (z_1, y(Q_1)) \in \mathcal{K}_n$, the local coordinates are

$\zeta_{Q_1} : P \to (z - z_1)^{1/2},$

near the points $P_{\infty} \in \mathcal{K}_n$, the local coordinates are

$\zeta_{P_{\infty}} : P \to z^{-1}.$

The holomorphic map $\ast$, changing sheets, is defined by

\[
\ast : \begin{cases} 
\mathcal{K}_n \to \mathcal{K}_n, \\
\quad P = (z, y_j(z)) \to P^\ast = (z, y_{j+1 \text{mod } 2}(z)), \quad j = 0, 1, \\
\quad P^{\ast\ast} := (P^\ast)^\ast, \quad \text{etc.}, \tag{3.2}
\end{cases}
\]

where $y_j(z), \; j = 0, 1$ denote the two branches of $y(P)$ satisfying $\mathcal{F}_n(z, y) = 0$, namely,

\[
(y - y_0(z))(y - y_1(z)) = y^2 - R_{2n+2}(z) = 0. \tag{3.3}
\]
Taking into account (3.3), one easily finds
\[ y_0 + y_1 = 0, \]
\[ y_0 y_1 = -R_{2n+2}(z), \]
\[ y_0^2 + y_1^2 = 2R_{2n+2}(z). \]
\[ (3.4) \]

Moreover, positive divisors on \( K_n \) of degree \( n \) are denoted by
\[ D_{P_1, \ldots, P_n} : \begin{cases} \mathbb{K}_n \to \mathbb{N}_0, \\ P \to D_{P_1, \ldots, P_n} = \left\{ \begin{array}{ll} k & \text{if } P \text{ occurs } k \text{ times in } \{P_1, \ldots, P_n\}, \\ 0 & \text{if } P \notin \{P_1, \ldots, P_n\}. \end{array} \right. \] 
\[ (3.5) \]

Next, we define the stationary Baker-Akhiezer function \( \psi(P, x, x_0) \) on \( \mathbb{K}_n \setminus \{P_{\infty+}, P_{\infty-}, P_0 = (0, y(0))\} \) by
\[ \psi(P, x, x_0) = \left( \begin{array}{c} \psi_1(P, x, x_0) \\ \psi_2(P, x, x_0) \end{array} \right), \]
\[ \psi_x(P, x, x_0) = U(u(x), \rho(x), z(P))\psi(P, x, x_0), \]
\[ z^{-1}V_n(u(x), \rho(x), z(P))\psi(P, x, x_0) = y(P)\psi(P, x, x_0), \]
\[ \psi_1(P, x_0, x_0) = 1; \]
\[ P = (z, y) \in \mathbb{K}_n \setminus \{P_{\infty+}, P_{\infty-}, P_0 = (0, y(0))\}, \quad (x, x_0) \in \mathbb{R}^2. \]
\[ (3.6) \]

Closely related to \( \psi(P, x, x_0) \) is the following meromorphic function \( \phi(P, x) \) on \( \mathbb{K}_n \) defined by
\[ \phi(P, x) = \frac{\psi_1(P, x, x_0)}{\psi_1(P, x, x_0)}, \quad P \in \mathbb{K}_n, \quad x \in \mathbb{R} \]
\[ (3.7) \]

such that
\[ \psi_1(P, x, x_0) = \exp \left( \int_{x_0}^x \phi(P, x') \, dx' \right), \quad P \in \mathbb{K}_n \setminus \{P_{\infty+}, P_{\infty-}, P_0\}. \]
\[ (3.8) \]

Then, based on (3.6) and (3.7), a direct calculation shows that
\[ \phi(P, x) = \frac{y + G_n(z, x)}{F_n(z, x)} = \frac{H_n(z, x)}{y - G_n(z, x)}. \]
\[ (3.9) \]
and

\[ \psi_2(P, x, x_0) = \psi_1(P, x, x_0) \phi(P, x). \] \hfill (3.10)

In the following, the roots of polynomials \( F_n \) and \( H_n \) will play a special role, and hence, we introduce on \( \mathbb{C} \times \mathbb{R} \)

\[ F_n(z, x) = f_0 \prod_{j=1}^{n} (z - \mu_j(x)), \quad H_n(z, x) = h_0 \prod_{l=1}^{n+2} (z - \nu_l(x)). \] \hfill (3.11)

Moreover, we introduce

\[ \hat{\mu}_j(x) = (\mu_j(x), G_n(\mu_j(x), x)) \in \mathcal{K}_n, \quad j = 1, \ldots, n, \quad x \in \mathbb{R}, \] \hfill (3.12)

and

\[ \hat{\nu}_l(x) = (\nu_l(x), -G_n(\nu_l(x), x)) \in \mathcal{K}_n, \quad l = 1, \ldots, n+2, \quad x \in \mathbb{R}. \] \hfill (3.13)

Due to assumption (2.1), \( u \) and \( \rho \) are smooth and bounded, and hence, \( F_n(z, x) \) and \( H_n(z, x) \) share the same property. Thus, one concludes

\[ \mu_j, \nu_l \in C(\mathbb{R}), \quad j = 1, \ldots, n, \quad l = 1, \ldots, n+2, \] \hfill (3.14)

taking multiplicities (and appropriate reordering) of the zeros of \( F_n \) and \( H_n \) into account. From (3.9), the divisor \( (\phi(P, x)) \) of \( \phi(P, x) \) is given by

\[ (\phi(P, x)) = D_{\hat{\nu}_1(x)} \hat{\nu}_2(x)(P) - D_{\hat{\mu}_1(x)} \hat{\mu}_2(x)(P). \] \hfill (3.15)

Here, we abbreviated

\[ \hat{\mu} = \{ \hat{\mu}_1, \ldots, \hat{\mu}_n \}, \quad \hat{\nu} = \{ \hat{\nu}_3, \ldots, \hat{\nu}_{n+2} \} \in \text{Sym}^n(\mathcal{K}_n). \] \hfill (3.16)

Further properties of \( \phi(P, x) \) are summarized as follows.

**Lemma 3.1.** Suppose (2.1), assume the \( n \)th stationary CHD2 equation (2.28) holds, and let \( P = (z, y) \in \mathcal{K}_n \setminus \{ P_{\infty_+}, P_{\infty_-} \}, \quad (x, x_0) \in \mathbb{R}^2 \). Then \( \phi \) satisfies the Riccati-type equation

\[ \phi_x(P) + \phi(P)^2 = -z^2 \rho^2 + z(u - u_{xx}) + \frac{1}{4}, \] \hfill (3.17)

as well as

\[ \phi(P)\phi(P^*) = -\frac{H_n(z)}{F_n(z)}, \] \hfill (3.18)

\[ \phi(P) + \phi(P^*) = \frac{2G_n(z)}{F_n(z)}, \] \hfill (3.19)

\[ \phi(P) - \phi(P^*) = \frac{2y}{F_n(z)}. \] \hfill (3.20)
Proof. Equation (3.17) follows using the definition (3.9) of φ as well as relations (2.14)-(2.16). Relations (3.18)-(3.20) are clear from (2.18), (3.4), and (3.9). □

The properties of ψ(P, x, x₀) are summarized in the following lemma.

Lemma 3.2. Suppose (2.21), assume the nth stationary CHD2 equation (3.28) holds, and let P = (z, y) ∈ Kₙ \ {P∞⁺, P∞⁻, P₀}, (x, x₀) ∈ ℝ². Then ψ₁(P, x, x₀), ψ₂(P, x, x₀) satisfy

\[
ψ₁(P, x, x₀) = \left( \frac{F₀(z, x)}{Fₙ(z, x₀)} \right)^{1/2} \exp \left( y \int_{x₀}^{x} Fₙ(z, x')^{-1} dx' \right), \tag{3.21}
\]

\[
ψ₁(P, x, x₀)ψ₁(P⁺, x, x₀) = \frac{Fₙ(z, x)}{Fₙ(z, x₀)}, \tag{3.22}
\]

\[
ψ₂(P, x, x₀)ψ₂(P⁺, x, x₀) = -\frac{Hₙ(z, x)}{Fₙ(z, x₀)}, \tag{3.23}
\]

\[
ψ₁(P, x, x₀)ψ₂(P⁺, x, x₀) + ψ₁(P⁺, x, x₀)ψ₂(P, x, x₀) = 2\frac{Gₙ(z, x)}{Fₙ(z, x₀)}, \tag{3.24}
\]

\[
ψ₁(P, x, x₀)ψ₂(P⁺, x, x₀) - ψ₁(P⁺, x, x₀)ψ₂(P, x, x₀) = -\frac{2y}{Fₙ(z, x₀)}. \tag{3.25}
\]

Proof. Equation (3.21) is a consequence of (2.14), (3.8), and (3.9). Equation (3.22) is clear from (3.21) and (3.23) is a consequence of (3.10), (3.18), and (3.22). Equation (3.24) follows using (3.10), (3.19), and (3.22). Finally, (3.25) follows from (3.10), (3.20), and (3.22). □

In Lemma 3.2 we denote by

\[
ψ₁(P) = ψ₁⁺, \quad ψ₁(P⁺) = ψ₁⁻, \quad ψ₂(P) = ψ₂⁺, \quad ψ₂(P⁺) = ψ₂⁻,
\]

and then (3.22)-(3.25) imply

\[
(ψ₁⁺ψ₂⁻ - ψ₁⁻ψ₂⁺)^2 = (ψ₁⁺ψ₂⁻ + ψ₁⁻ψ₂⁺)^2 - 4ψ₁⁺ψ₂⁻ψ₁⁻ψ₂⁺, \tag{3.26}
\]

which is equivalent to the basic identity (2.18), \(Gₙ² + FₙHₙ = R_{2n+2}²\). This fact reveals the relations between our approach and the algebraic-geometric solutions of the CHD2 hierarchy.

Remark 3.3. The Baker-Akhiezer function ψ of the stationary CHD2 hierarchy is formally analogous to that defined in the context of KdV or AKNS hierarchies. However, its actual properties in a neighborhood of its essential singularity will feature characteristic differences to standard Baker-Akhiezer functions (cf. Remark 4.2).
Next, we derive Dubrovin-type equations, that is, first-order coupled systems of differential equations that govern the dynamics of $\mu_j(x)$ and $\nu_l(x)$ with respect to variations of $x$.

**Lemma 3.4.** Assume (2.1) and the $n$th stationary CHD2 equation (2.28) holds subject to the constraint (2.22).

(i) Suppose that the zeros $\{\mu_j(x)\}_{j=1,\ldots,n}$ of $F_n(z, x)$ remain distinct for $x \in \Omega_\mu$, where $\Omega_\mu \subseteq \mathbb{R}$ is an open interval, then $\{\mu_j(x)\}_{j=1,\ldots,n}$ satisfy the system of differential equations,

$$\mu_{j,x} = -2\frac{y(\hat{\mu}_j)}{f_0} \prod_{k=1, k \neq j}^{n} (\mu_j(x) - \mu_k(x))^{-1}, \quad j = 1, \ldots, n, \quad (3.27)$$

with initial conditions

$$\{\hat{\mu}_j(x_0)\}_{j=1,\ldots,n} \in \mathcal{K}_n, \quad (3.28)$$

for some fixed $x_0 \in \Omega_\mu$. The initial value problem (3.27), (3.28) has a unique solution satisfying

$$\hat{\mu}_j \in C^\infty(\Omega_\mu, \mathcal{K}_n), \quad j = 1, \ldots, n. \quad (3.29)$$

(ii) Suppose that the zeros $\{\nu_l(x)\}_{l=1,\ldots,n+2}$ of $H_n(z, x)$ remain distinct for $x \in \Omega_\nu$, where $\Omega_\nu \subseteq \mathbb{R}$ is an open interval, then $\{\nu_l(x)\}_{l=1,\ldots,n+2}$ satisfy the system of differential equations,

$$\nu_{l,x} = 2\frac{(\nu_l^2 \rho^2 - (u - u_{xx})\nu_l - \frac{1}{4})y(\hat{\nu}_l)}{h_0} \prod_{k=1, k \neq l}^{n+2} (\nu_l(x) - \nu_k(x))^{-1}, \quad l = 1, \ldots, n+2, \quad (3.30)$$

with initial conditions

$$\{\hat{\nu}_l(x_0)\}_{l=1,\ldots,n+2} \in \mathcal{K}_n, \quad (3.31)$$

for some fixed $x_0 \in \Omega_\nu$. The initial value problem (3.30), (3.31) has a unique solution satisfying

$$\hat{\nu}_l \in C^\infty(\Omega_\nu, \mathcal{K}_n), \quad l = 1, \ldots, n+2. \quad (3.32)$$
Proof. It suffices to prove (3.27) and (3.29) since the proof of (3.30) and (3.32) follow in an identical manner. Differentiating (3.11) with respect to $x$ then yields

$$F_{n,x}(\mu_j) = -f_0\mu_{j,x} \prod_{\substack{k=1\atop k\neq j}}^n (\mu_j(x) - \mu_k(x)).$$  \hspace{1cm} (3.33)

On the other hand, taking into account equation (2.14), one finds

$$F_{n,x}(\mu_j) = 2G_n(\mu_j) = 2y(\hat{\mu}_j).$$  \hspace{1cm} (3.34)

Then combining equation (3.33) with (3.34) leads to (3.27). The proof of smoothness assertion (3.29) is analogous to the KdV case in [13]. □

Next, we turn to the trace formulas of the CHD2 invariants, that is, expressions of $f_l$ and $h_l$ in terms of symmetric functions of the zeros $\mu_j$ and $\nu_l$ of $F_n$ and $H_n$, respectively. For simplicity, we just record the simplest case.

\textbf{Lemma 3.5.} Suppose \textbf{(2.1)}, assume the $n$th stationary CHD2 equation \textbf{(2.28)} holds, and let $x \in \mathbb{R}$. Then

$$u^{-1}G(-4u_x(u - u_{xx}) - 2u(u_x - u_{xxx})) = \sum_{j=1}^n \mu_j(x) - \frac{1}{2} \sum_{m=0}^{2n+1} E_m. \hspace{1cm} (3.35)$$

\textit{Proof.} Equation (3.35) follows by considering the coefficient of $z^{n-1}$ in $F_n$ in (2.23) and (3.11), which yields

$$G(-4u_x(u - u_{xx}) - 2u(u_x - u_{xxx})) - c_1u = -f_0 \sum_{j=1}^n \mu_j. \hspace{1cm} (3.36)$$

The constant $c_1$ can be determined by considering the coefficient of $z^{2n+1}$ in (2.18), which results in

$$c_1 = -\frac{1}{2} \sum_{m=0}^{2n+1} E_m. \hspace{1cm} (3.37)$$

\section{Stationary algebro-geometric solutions of CHD2 hierarchy}

In this section, we obtain explicit Riemann theta function representations for the meromorphic function $\phi$, the Baker-Akhiezer function $\psi$, and especially, for the solutions $u, \rho$ of the stationary CHD2 hierarchy.

We begin with the asymptotic properties of $\phi$ and $\psi_j, j = 1, 2$. 

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Lemma 4.1. Suppose \((2.28)\) holds, and let \(P = (z, y) \in K_n \setminus \{P_{\infty_+}, P_{\infty_-}, P_0\}, (x, x_0) \in \mathbb{R}^2\). Then
\[
\phi(P) = \pm i \rho \zeta^{-1} + \frac{\mp i (u - u_{xx}) - \rho_x}{2 \rho} + O(\zeta), \quad P \to P_{\infty_\pm}, \quad \zeta = z^{-1}, \tag{4.1}
\]
and
\[
\phi(P) = \frac{1}{2} + (u - u_x) \zeta + O(\zeta^2), \quad P \to P_0, \quad \zeta = z, \tag{4.2}
\]
and
\[
\psi_1(P, x, x_0) = \exp \left( \pm i \int_{x_0}^x dx' \rho(x') + O(1) \right), \quad P \to P_{\infty_\pm}, \quad \zeta = z^{-1}, \tag{4.3}
\]
\[
\psi_2(P, x, x_0) = O(\zeta^{-1}) \exp \left( \pm i \int_{x_0}^x dx' \rho(x') + O(1) \right), \quad P \to P_{\infty_\pm}, \quad \zeta = z^{-1}, \tag{4.4}
\]
\[
\psi_1(P, x, x_0) = \exp \left( \frac{1}{2} (x - x_0) \right) (1 + O(\zeta)), \quad P \to P_0, \quad \zeta = z, \tag{4.5}
\]
\[
\psi_2(P, x, x_0) = \left( \frac{1}{2} + O(\zeta) \right) \exp \left( \frac{1}{2} (x - x_0) \right) (1 + O(\zeta)), \quad P \to P_0, \quad \zeta = z. \tag{4.6}
\]

Proof. The existence of the asymptotic expansions of \(\phi\) in terms of the appropriate local coordinates \(\zeta = z^{-1}\) near \(P_{\infty_\pm}\) and \(\zeta = z\) near \(P_0\) is clear from its explicit expression in (3.9). Next, we compute the coefficients of these expansions utilizing the Riccati-type equation (3.17). Indeed, inserting the ansatz
\[
\phi = \phi_{-1} z + \phi_0 + O(z^{-1}) \tag{4.7}
\]
into (3.17) and comparing the same powers of \(z\) then yields (4.1). Similarly, inserting the ansatz
\[
\phi = \phi_0 + \phi_1 z + O(z^2) \tag{4.8}
\]
into (3.17) and comparing the same powers of \(z\) then yields (4.2). Finally, expansions (4.3)-(4.6) follow from (3.8), (3.10), (4.1), and (4.2). \(\square\)

Remark 4.2. We note the fact that \(P_{\infty_\pm}\), are the essential singularities of \(\psi_j, j = 1, 2\). In addition, one easily finds the leading-order exponential term in \(\psi_j, j = 1, 2, \) near \(P_{\infty_\pm}\) is \(x\)-dependent, which makes matters worse. This is in sharp contrast to standard Baker-Akhiezer functions that typically feature a linear behavior with respect to \(x\) in connection with their essential singularities of the type \(\exp(c(x - x_0)\zeta^{-1})\) near \(\zeta = 0\).
Next, we introduce the holomorphic differentials $\eta_l(P)$ on $K_n$

$$\eta_l(P) = \frac{z^{l-1}}{y(P)} dz, \quad l = 1, \ldots, n,$$  \hspace{1cm} (4.9)

and choose a homology basis $\{a_j, b_j\}_{j=1}^n$ on $K_n$ in such a way that the intersection matrix of the cycles satisfies

$$a_j \circ b_k = \delta_{j,k}, \quad a_j \circ a_k = 0, \quad b_j \circ b_k = 0, \quad j, k = 1, \ldots, n.$$  

Associated with $K_n$, one introduces an invertible matrix $E \in GL(n, \mathbb{C})$

$$E = (E_{j,k})_{n \times n}, \quad E_{j,k} = \int_{a_k} \eta_j, \quad (4.10)$$

and the normalized holomorphic differentials

$$\omega_j = \sum_{l=1}^n c_j(l) \eta_l, \quad \int_{a_k} \omega_j = \delta_{j,k}, \quad \int_{b_k} \omega_j = \tau_{j,k}, \quad j, k = 1, \ldots, n.$$  \hspace{1cm} (4.11)

Apparently, the matrix $\tau$ is symmetric and has a positive-definite imaginary part.

We choose a fixed base point $Q_0 \in K_n \setminus \{P_{\infty_+}, P_{\infty_-}\}$. The Abel maps $A_{Q_0}(\cdot)$ and $\omega_{Q_0}(\cdot)$ are defined by

$$A_{Q_0} : K_n \to J(K_n) = \mathbb{C}^n / L_n,$$

$$P \mapsto A_{Q_0}(P) = (A_{Q_0,1}(P), \ldots, A_{Q_0,n}(P)) = \left( \int_{Q_0}^P \omega_1, \ldots, \int_{Q_0}^P \omega_n \right) \pmod{L_n}$$  \hspace{1cm} (4.12)

and

$$\omega_{Q_0} : \text{Div}(K_n) \to J(K_n),$$

$$\mathcal{D} \mapsto \omega_{Q_0}(\mathcal{D}) = \sum_{P \in K_n} \mathcal{D}(P) A_{Q_0}(P),$$  \hspace{1cm} (4.13)

where $L_n = \{z \in \mathbb{C}^n | z = N + \tau M, \quad N, M \in \mathbb{Z}^n\}$.

The following result shows the nonlinearity of the Abel map with respect to the variable $x$, which indicates a characteristic difference between the CHD2 hierarchy and other completely integrable systems such as the KdV and AKNS hierarchies.
Theorem 4.3. Assume \cite{222} and suppose that \( \{ \hat{\mu}_j(x) \}_{j=1,\ldots,n} \) satisfies the stationary Dubrovin equations \cite{327} on an open interval \( \Omega_\mu \subseteq \mathbb{R} \) such that \( \mu_j(x), j = 1, \ldots, n \), remain distinct and nonzero for \( x \in \Omega_\mu \). Introducing the associated divisor \( D_{\hat{\mu}(x)} \in \text{Sym}^n(K_n) \), one computes

\[
\partial_x \varphi_{Q_0}(D_{\hat{\mu}(x)}) = \frac{2}{u(x)} c(n), \quad x \in \Omega_\mu.
\] (4.14)

In particular, the Abel map does not linearize the divisor \( D_{\hat{\mu}(x)} \) on \( \Omega_\mu \).

Proof. Let \( x \in \Omega_\mu \). One finds

\[
\partial_x \varphi_{Q_0}(D_{\hat{\mu}(x)}) = \partial_x \left( \sum_{j=1}^{n} \int_{Q_0} \hat{\mu}_j \right) = \sum_{j=1}^{n} \mu_j x \sum_{k=1}^{n} c(k) \frac{\mu_j^{k-1}}{y(\hat{\mu}_j)}
\]

\[
= \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{2}{f_0} \frac{\mu_j^{k-1}}{\prod_{l \neq j} (\mu_j - \mu_l)} c(k)
\]

\[
= -\frac{2}{f_0} \sum_{k=1}^{n} c(k) \delta_{k,n} = -\frac{2}{f_0} c(n), \quad (4.15)
\]

where we used the notation \( \omega = (\omega_1, \ldots, \omega_n) \), and a special case of Lagrange’s interpolation formula (cf. Theorem E.1 \cite{13}),

\[
\sum_{j=1}^{n} \mu_j^{k-1} \prod_{l \neq j} (\mu_j - \mu_l)^{-1} = \delta_{j,n}, \quad j, k = 1, \ldots, n. \quad (4.16)
\]

The analogous results hold for the corresponding divisor \( D_{\hat{\nu}(x)} \) associated with \( \phi(P, x) \).

Next, given the Riemann surface \( K_n \) and the homology basis \( \{a_j, b_j\}_{j=1,\ldots,n} \), one introduces the Riemann theta function by

\[
\theta(z) = \sum_{\mathbf{n} \in \mathbb{Z}^n} \exp \left( 2\pi i (\mathbf{n}, z) + \pi i (\mathbf{n}, \tau \mathbf{n}) \right), \quad z \in \mathbb{C}^n,
\]

where \( (A, B) = \sum_{j=1}^{n} A_j B_j \) denotes the scalar product in \( \mathbb{C}^n \).

Let

\[
\omega_{P_{\infty}, \nu_1(x)}^{(3)}(P) = \frac{y - G_n(\nu_1)}{z - \nu_1} \frac{dz}{2y} - \frac{1}{2y} \prod_{j=1}^{n} (z - \lambda_j) dz \quad (4.17)
\]
be the normalized differential of the third kind holomorphic on \( K_n \setminus \{ P_{∞+}, \hat{ν}_1(x) \} \) with simple poles at \( P_{∞+} \) and \( \hat{ν}_1(x) \) and residues 1 and \(-1\), respectively,

\[
ω^{(3)}_{P_{∞+}, \hat{ν}_1(x)}(P) = (ζ^{-1} + O(1))dζ, \quad \text{as } P \to P_{∞+},
\]

\[
ω^{(3)}_{P_{∞+}, \hat{ν}_1(x)}(P) = (-ζ^{-1} + O(1))dζ, \quad \text{as } P \to \hat{ν}_1(x),
\]

where \( ζ \) in (4.18) denotes the local coordinate

\[ ζ = z^{-1} \quad \text{for } P \near P_{∞+}, \]

near \( P_{∞+} \), and analogously, \( ζ \) in (4.19) that near \( \hat{ν}_1(x) \). The constants \( \{ λ_j \}_{j=1,...,n} \) in (4.17) are determined by the normalization condition

\[ \int_{a_k} ω^{(3)}_{P_{∞+}, \hat{ν}_1(x)} = 0, \quad k = 1, \ldots, n. \]

Similarly, let \( ω^{(3)}_{P_{∞-}, \hat{ν}_2(x)}(P) \) be another normalized differential of the third kind holomorphic on \( K_n \setminus \{ P_{∞-}, \hat{ν}_2(x) \} \) with simple poles at \( P_{∞-} \) and \( \hat{ν}_2(x) \) and residues 1 and \(-1\), respectively,

\[
ω^{(3)}_{P_{∞-}, \hat{ν}_2(x)}(P) = \frac{y - G_2(ν_2)}{z - ν_2} \frac{dz}{2y} + \frac{1}{2y} \prod_{j=1}^{n} (z - γ_j)dz,
\]

\[
ω^{(3)}_{P_{∞-}, \hat{ν}_2(x)}(P) = (ζ^{-1} + O(1))dζ, \quad \text{as } P \to P_{∞-},
\]

\[
ω^{(3)}_{P_{∞-}, \hat{ν}_2(x)}(P) = (-ζ^{-1} + O(1))dζ, \quad \text{as } P \to \hat{ν}_2(x),
\]

where \( ζ \) in (4.21) denotes the local coordinate

\[ ζ = z^{-1} \quad \text{for } P \near P_{∞-}, \]

near \( P_{∞-} \), and analogously, \( ζ \) in (4.22) that near \( \hat{ν}_2(x) \). The constants \( \{ γ_j \}_{j=1,...,n} \) in (4.20) are determined by the normalization condition

\[ \int_{a_k} ω^{(3)}_{P_{∞-}, \hat{ν}_2(x)} = 0, \quad k = 1, \ldots, n. \]

We define

\[ Ω^{(3)} = ω^{(3)}_{P_{∞+}, \hat{ν}_1(x)} + ω^{(3)}_{P_{∞-}, \hat{ν}_2(x)}. \]
\[
\int_{Q_0}^{P} \Omega^{(3)} = \ln \zeta + d_0 + O(\zeta), \quad \text{as } P \to P_{\infty}, \tag{4.24}
\]

for some constant \(d_0 \in \mathbb{C}\).

Next, let \(\omega_{P_{\infty}}^{(2)}\) be normalized differentials of the second kind, satisfying

\[
\int_{a_k}^{b_k} \omega_{P_{\infty}}^{(2)} = 0, \quad k = 1, \ldots, n, \tag{4.25}
\]

\[
\omega_{P_{\infty}}^{(2)} = (\zeta^{-2} + O(1))d\zeta, \quad \text{as } P \to P_{\infty}. \tag{4.26}
\]

We introduce

\[
\Omega^{(2)}_0 = \omega_{P_{\infty}}^{(2)} - \omega_{P_{\infty}^+}^{(2)}. \tag{4.27}
\]

Then

\[
\int_{Q_0}^{P} \Omega^{(2)}_0 = \pm (\zeta^{-1} + e_{0,0} + O(\zeta)), \quad \text{as } P \to P_{\infty}, \tag{4.28}
\]

for some constant \(e_{0,0} \in \mathbb{C}\). In the following, it will be convenient to introduce the abbreviations

\[
\begin{align*}
\mathbb{z}(P,Q) &= \Xi_Q - A_Q(P) + \omega_{Q_0}(D_Q), \\
P &\in \mathcal{K}_n, \quad Q = (Q_1, \ldots, Q_n) \in \text{Sym}^n(\mathcal{K}_n),
\end{align*}
\]

where \(\Xi_Q\) is the vector of Riemann constants (cf.(A.45) [13]). It turns out that \(\mathbb{z}(\cdot, Q)\) is independent of the choice of base point \(Q_0\) (cf.(A.52), (A.53) [13]).

Based on above preparations, we will give explicit representations for the meromorphic function \(\phi\), the Baker-Akhiezer function \(\psi\), and the stationary CHD2 solutions \(u, \rho\) in terms of the Riemann theta function associated with \(\mathcal{K}_n\).

**Theorem 4.4.** Suppose (2.1), and assume the \(n\)th stationary CHD2 equation (2.28) holds on \(\Omega\) subject to the constraint (2.22). Moreover, let \(P = (z, y) \in \mathcal{K}_n \setminus \{P_{\infty}\}\) and \(x, x_0 \in \Omega\), where \(\Omega \subseteq \mathbb{R}\) is an open interval. In addition, suppose that \(D_\mu(x)\), or equivalently, \(D_\mu(z)\) is nonspecial for \(x \in \Omega\).

Then, \(\phi\), \(\psi\), \(u\), and \(\rho\) admit the following representations

\[
\phi(P, x) = i\rho(x) \frac{\theta(\mathbb{z}(P, \hat{\mu}(x)))\theta(\mathbb{z}(P_{\infty}, \hat{\mu}(x)))}{\theta(\mathbb{z}(P_{\infty}, \hat{\mu}(x)))\theta(\mathbb{z}(P, \hat{\mu}(x)))} \exp \left( d_0 - \int_{Q_0}^{P} \Omega^{(3)} \right), \tag{4.30}
\]

\[
\psi_1(P, x, x_0) = \frac{\theta(\mathbb{z}(P, \hat{\mu}(x)))\theta(\mathbb{z}(P_{\infty}, \hat{\mu}(x)))}{\theta(\mathbb{z}(P, \hat{\mu}(x)))\theta(\mathbb{z}(P_{\infty}, \hat{\mu}(x)))} \exp \left( \int_{x_0}^{x} i\rho(x') \int_{Q_0}^{P} \Omega^{(2)}_0 \right), \tag{4.31}
\]

\[
\psi_1(P, x, x_0) = \frac{\theta(\mathbb{z}(P, \hat{\mu}(x)))\theta(\mathbb{z}(P_{\infty}, \hat{\mu}(x)))}{\theta(\mathbb{z}(P, \hat{\mu}(x)))\theta(\mathbb{z}(P_{\infty}, \hat{\mu}(x)))} \exp \left( \int_{x_0}^{x} i\rho(x') \int_{Q_0}^{P} \Omega^{(2)}_0 \right), \tag{4.31}
\]
\[
\psi_2(P, x, x_0) = i\rho(x) \frac{\theta(z(P, \hat{\mu}(x))) \theta(z(P_{\infty+}, \hat{\mu}(x_0)))}{\theta(z(P_{\infty+}, \hat{\mu}(x))) \theta(z(P, \hat{\mu}(x_0)))} \exp \left( d_0 - \int_{Q_0}^P \Omega^{(3)} \right)
\]
\[
\times \exp \left( \int_{x_0}^x dx' \, i\rho(x') \right) \int_{Q_0}^P \Omega_0^{(2)} \right),
\]

(4.32)

\[
u(x) = -4i \frac{\theta(z(P_0, \hat{\mu}(x))) \theta(z(P_{\infty+}, \hat{\mu}(x)))}{\theta(z(P_{\infty+}, \hat{\mu}(x))) \theta(z(P_0, \hat{\mu}(x )))},
\]

(4.33)

\[
\rho(x) = -\frac{i}{2} \frac{\theta(z(P_0, \hat{\mu}(x))) \theta(z(P_{\infty+}, \hat{\mu}(x)))}{\theta(z(P_{\infty+}, \hat{\mu}(x))) \theta(z(P_0, \hat{\mu}(x )))},
\]

(4.34)

**Proof.** First, we temporarily assume that

\[\mu_j(x) \neq \mu_j'(x), \quad \nu_k(x) \neq \nu_k'(x) \quad \text{for} \quad j \neq j', k \neq k' \quad \text{and} \quad x \in \tilde{\Omega}, \quad (4.35)\]

for appropriate \(\tilde{\Omega} \subseteq \Omega\). Since by (3.15), \(\mathcal{D}_{\mu} \hat{\mu}_{\tilde{\Omega}} \sim \mathcal{D}_{P_{\infty+}, P_{\infty-}, \hat{\mu}}\) and \((P_{\infty+})^* = P_{\infty-} \notin \{\hat{\mu}_1, \ldots, \hat{\mu}_n\}\) by hypothesis, one can use Theorem A.31 [13] to conclude that \(\mathcal{D}_{\hat{\mu}} \in \text{Sym}^2(\mathcal{K}_n)\) is nonspecial. This argument is of course symmetric with respect to \(\hat{\mu}\) and \(\tilde{\mu}\). Thus, \(\mathcal{D}_{\hat{\mu}}\) is nonspecial if and only if \(\mathcal{D}_{\tilde{\mu}}\) is.

Next, we derive the representations of \(\phi, \psi, u,\) and \(\rho\) in terms of the Riemann theta function. A special case of Riemann’s vanishing theorem (cf. Theorem A.26 [13]) yields

\[
\theta(\Xi_{Q_0} - \Delta_{Q_0} (P) + \alpha_{Q_0}(\mathcal{D}_{\hat{\mu}})) = 0 \quad \text{if and only if} \quad P \in \{Q_1, \ldots, Q_n\}. \quad (4.36)
\]

Therefore, the divisor (3.15) shows that \(\phi(P, x)\) has expression of the type

\[
\frac{\theta(\Xi_{Q_0} - \Delta_{Q_0} (P) + \alpha_{Q_0}(\mathcal{D}_{\hat{\mu}(x)}))}{\theta(\Xi_{Q_0} - \Delta_{Q_0} (P) + \alpha_{Q_0}(\mathcal{D}_{\hat{\mu}(x)}))} \exp \left( d_0 - \int_{Q_0}^P \Omega^{(3)} \right),
\]

(4.37)

where \(C(x)\) is independent of \(P \in \mathcal{K}_n\). Then taking into account the asymptotic expansion of \(\phi(P, x)\) near \(P_{\infty+}\) in (4.1), we obtain (4.30).

Now, let \(\Psi\) denote the right-hand side of (4.31). We intend to prove \(\psi_1 = \Psi\) with \(\psi_1\) given by (3.8). For that purpose we first investigate the local zeros and poles of \(\psi_1\). Since they can only come from zeros of \(F_n(z, x')\) in (3.8), one computes using (3.12), the definition (3.9) of \(\phi\), and the Dubrovin equations (3.27),

\[
\phi(P, x) = \frac{y + G_n}{F_n} = \frac{y}{F_n} + \frac{F_{n,x}}{2F_n}
\]

\[
= \frac{1}{2} \mu_{j,x} - \frac{1}{2} \frac{\mu_{j,x}}{z - \mu_j} + O(1)
\]

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\[ y \to y(\hat{\mu}_j(x)) = G_n(\mu_j(x)), \quad \text{as } z \to \mu_j(x). \]

More concisely,
\[ \phi(P, x) = \frac{\partial}{\partial x} \ln(z - \mu_j(x)) + O(1) \quad \text{for } P \text{ near } \hat{\mu}_j(x), \quad (4.39) \]

which together with (3.8) yields
\[
\psi_1(P, x, x_0) = \exp \left( \int_{x_0}^x dx' \left( \frac{\partial}{\partial x'} \ln(z - \mu_j(x')) + O(1) \right) \right) \\
= \frac{z - \mu_j(x)}{z - \mu_j(x_0)} O(1) \\
= \begin{cases} 
(z - \mu_j(x)) O(1) & \text{for } P \text{ near } \hat{\mu}_j(x) \neq \hat{\mu}_j(x_0), \\
O(1) & \text{for } P \text{ near } \hat{\mu}_j(x) = \hat{\mu}_j(x_0), \\
(z - \mu_j(x_0))^{-1} O(1) & \text{for } P \text{ near } \hat{\mu}_j(x_0) \neq \hat{\mu}_j(x), 
\end{cases} 
\]

where \( O(1) \neq 0 \) in (4.40). Consequently, \( \psi_1 \) and \( \Psi \) have identical zeros and poles on \( K_n \setminus \{P_{\infty_+}, P_{\infty_-}\} \), which are all simple by hypothesis (4.35).

It remains to identify the behavior of \( \psi_1 \) and \( \Psi \) near \( P_{\infty_{\pm}} \). Taking into account (3.8), (4.3), (4.28), and the expression (4.31) for \( \Psi \), one observes that \( \psi_1 \) and \( \Psi \) share the same singularities and zeros, and the Riemann-Roch-type uniqueness result (cf. Lemma C.2 [13]) then completes the proof \( \psi_1 = \Psi \).

\[ \alpha Q_0(\hat{\mu}_j(x)) = \Delta + \alpha Q_0(D_{\hat{\mu}}(x)), \quad \Delta = A_{P_{\infty_+}}(\hat{\nu}_1(x)) + A_{P_{\infty_-}}(\hat{\nu}_2(x)). \]
Hence, one can eliminate $\partial_x \hat{A}^\mu$ in (4.30), (4.33), and (4.34), in terms of $\partial_x \hat{A}^\nu$ using
\[ z(P, \hat{A}^\mu) = z(P, \hat{A}^\nu) + \Delta, \quad P \in K_n. \quad (4.43) \]

5 The time-dependent CHD2 formalism

In this section, we extend the algebro-geometric analysis of Section 3 to the time-dependent CHD2 hierarchy.

Throughout this section, we assume (2.2) holds.

The time-dependent algebro-geometric initial value problem of the CHD2 hierarchy is to solve the time-dependent $r$th CHD2 flow with a stationary solution of the $n$th equation as initial data in the hierarchy. More precisely, given $n \in \mathbb{N}_0$, based on the solution $u(0), \rho(0)$ of the $n$th stationary CHD2 equation $s$-CHD2$_n(u(0), \rho(0)) = 0$ associated with $K_n$ and a set of integration constants $\{c_l\}_{l=1}^{n} \subset \mathbb{C}$, we want to construct a solution $u, \rho$ of the $r$th CHD2 flow CHD2$_r(u, \rho) = 0$ such that $u(t_0, r) = u(0), \rho(t_0, r) = \rho(0)$ for some $t_0, r \in \mathbb{R}, \ n, r \in \mathbb{N}_0$.

To emphasize that the integration constants in the definitions of the stationary and the time-dependent CHD2 equations are independent of each other, we indicate this by adding a tilde on all the time-dependent quantities. Hence, we employ the notation $\tilde{V}_r, \tilde{F}_r, \tilde{G}_r, \tilde{H}_r, \tilde{f}_s, \tilde{g}_s, \tilde{h}_s, \tilde{c}_l$ in order to distinguish them from $V_n, F_n, G_n, H_n, f_l, g_l, h_l, c_l$ in the following. In addition, we mark the individual $r$th CHD2 flow by a separate time variable $t_r \in \mathbb{R}$.

Summing up, we are seeking a solution $u, \rho$ of the time-dependent algebro-geometric initial value problem
\[ \text{CHD2}_r(u, \rho) = \begin{pmatrix} -2\rho u_t - 2(u - u_{xx})f_{r,x} - (u_x - u_{xxx})f_r - \frac{1}{2}f_{r-1,x} + \frac{1}{2}f_{r-1,xxx} \\ u_t - u_{xxx} - \frac{1}{2}f_{r,x} + \frac{1}{2}f_{r,xxx} \end{pmatrix} = 0, \quad (5.1) \]

\[ (u, \rho)|_{t_r = t_0, r} = (u(0), \rho(0)), \]

\[ s$-CHD2$_n(u(0), \rho(0)) = \begin{pmatrix} -2(u - u_{xx})f_{n,x} - (u_x - u_{xxx})f_n - \frac{1}{2}f_{n-1,x} + \frac{1}{2}f_{n-1,xxx} \\ -\frac{1}{2}f_{n,x} + \frac{1}{2}f_{n,xxx} \end{pmatrix} = 0, \quad (5.2) \]

for some $t_0, r \in \mathbb{R}, n, r \in \mathbb{N}_0$, where $u = u(x, t_r), \rho = \rho(x, t_r)$ satisfy (2.2), and the curve $K_n$ is associated with the initial data $(u(0), \rho(0))$ in (5.2).

Noticing that the CHD2 flows are isospectral, we further assume that (5.2) holds not only for $t_r = t_0, r$, but also for all $t_r \in \mathbb{R}$. Hence, we start with the
zero-curvature equations

\[
U_t - \tilde{V}_{r,x} + [U, \tilde{V}_r] = 0, \quad (5.3)
\]

\[
- V_{n,x} + [U, V_n] = 0, \quad (5.4)
\]

where

\[
U(z) = \begin{pmatrix}
0 & 1 \\
-z^2 \rho^2 + z(u - u_{xx}) + \frac{1}{4} & 0
\end{pmatrix},
\]

\[
V_n(z) = \begin{pmatrix}
-zG_n(z) & zF_n(z) \\
zH_n(z) & zG_n(z)
\end{pmatrix}, \quad (5.5)
\]

\[
\tilde{V}_r(z) = \begin{pmatrix}
-z\tilde{G}_r(z) & z\tilde{F}_r(z) \\
z\tilde{H}_r(z) & z\tilde{G}_r(z)
\end{pmatrix},
\]

and

\[
F_n(z) = \sum_{l=0}^{n} f_l z^{n-l} = f_0 \prod_{j=1}^{n} (z - \mu_j), \quad (5.6)
\]

\[
G_n(z) = \sum_{l=0}^{n} g_l z^{n-l}, \quad (5.7)
\]

\[
H_n(z) = \sum_{l=0}^{n+2} h_l z^{n+2-l} = h_0 \prod_{l=1}^{n+2} (z - \nu_l), \quad (5.8)
\]

\[
\tilde{F}_r(z) = \sum_{s=0}^{r} \tilde{f}_s z^{r-s}, \quad (5.9)
\]

\[
\tilde{G}_r(z) = \sum_{s=0}^{r} \tilde{g}_s z^{r-s}, \quad (5.10)
\]

\[
\tilde{H}_r(z) = \sum_{s=0}^{r+2} \tilde{h}_s z^{r+2-s}, \quad (5.11)
\]

for fixed \(n, r \in \mathbb{N}_0\). Here, \(\{f_l\}_{l=0,...,n}, \{g_l\}_{l=0,...,n}, \{h_l\}_{l=0,...,n+2}, \{\tilde{f}_s\}_{s=0,...,r}, \{\tilde{g}_s\}_{s=0,...,r}, \text{ and } \{\tilde{h}_s\}_{s=0,...,r+2}\) are defined as in (2.3), with \(u(x), \rho(x)\) replaced by \(u(x, t_r), \rho(x, t_r)\) etc., and with appropriate integration constants.

Explicitly, (5.3) and (5.4) are equivalent to

\[
-2z \rho \rho_{t_r} + (u_{t_r} - u_{xx t_r}) - \tilde{H}_{r,x} - 2 \left( -z^2 \rho^2 + z(u - u_{xx}) + \frac{1}{4} \right) \tilde{G}_r = 0, \quad (5.12)
\]
From (5.15)-(5.17), one finds
\[
det(\psi_{r,x}) = -z^2 \frac{d}{dx} \left( G_n(z)^2 + F_n(z)H_n(z) \right) = 0,
\]
and meanwhile (see Lemma 5.2)
\[
\frac{d}{dt_r} \det(\psi_{r,x}) = -z^2 \frac{d}{dt_r} \left( G_n(z)^2 + F_n(z)H_n(z) \right) = 0.
\]
Hence, \( G_n(z)^2 + F_n(z)H_n(z) \) is independent of variables both \( x \) and \( t_r \), which implies the fundamental identity (2.18) holds,
\[
G_n(z)^2 + F_n(z)H_n(z) = R_{2n+2}(z),
\]
and the hyperelliptic curve \( K_n \) is still given by (2.21).

Next, we define the time-dependent Baker-Akhiezer function \( \psi(P, x, x_0, t_r, t_{0,r}) \) on \( K_n \setminus \{P_{\infty}, P_0\} \) by
\[
\psi(P, x, x_0, t_r, t_{0,r}) = \begin{pmatrix}
\psi_1(P, x, x_0, t_r, t_{0,r}) \\
\psi_2(P, x, x_0, t_r, t_{0,r})
\end{pmatrix},
\]
\[
\psi_x(P, x, x_0, t_r, t_{0,r}) = U(u(x, t_r), \rho(x, t_r), z(P))\psi(P, x, x_0, t_r, t_{0,r}),
\]
\[
\psi_{t_r}(P, x, x_0, t_r, t_{0,r}) = z^{-1} \tilde{V}_r(u(x, t_r), \rho(x, t_r), z(P))\psi(P, x, x_0, t_r, t_{0,r}),
\]
\[
z^{-1}V_n(u(x, t_r), \rho(x, t_r), z(P))\psi(P, x, x_0, t_r, t_{0,r}) = y(P)\psi(P, x, x_0, t_r, t_{0,r}),
\]
\[
\psi_1(P, x_0, x_0, t_{0,r}, t_{0,r}) = 1;
\]
\[
P = (z, y) \in K_n \setminus \{P_{\infty}, P_0\}, \ (x, t_r) \in \mathbb{R}^2.
\]

Closely related to \( \psi(P, x, x_0, t_r, t_{0,r}) \) is the following meromorphic function \( \phi(P, x, t_r) \) on \( K_n \) defined by
\[
\phi(P, x, t_r) = \frac{\psi_{1,x}(P, x_0, x_0, t_{0,r}, t_{0,r})}{\psi_1(P, x, x_0, t_r, t_{0,r})}, \quad P \in K_n \setminus \{P_{\infty}, P_0\}, \ (x, t_r) \in \mathbb{R}^2
\]
such that

\[
\psi_1(P, x, x_0, t_{0,r}) = \exp \left( \int_{t_{0,r}}^{t_r} ds (\tilde{F}_r(z, x_0, s)\phi(P, x_0, s) - \tilde{G}_r(z, x_0, s)) \right) + \int_{x_0}^{x} dx' \phi(P, x', t_r), \quad P = (z, y) \in \mathcal{K}_n \setminus \{P_{\infty+}, P_0\}.
\]

(5.23)

Then, using (5.21) and (5.22), one infers that

\[
\phi(P, x, t_r) = \frac{y + G_n(z, x, t_r)}{F_n(z, x, t_r)} = \frac{H_n(z, x, t_r)}{y - G_n(z, x, t_r)},
\]

and

\[
\psi_2(P, x, x_0, t_{0,r}) = \psi_1(P, x, x_0, t_{0,r})\phi(P, x, t_r).
\]

(5.25)

In analogy to (3.12) and (3.13), we introduce

\[
\hat{\mu}_j(x, t_r) = (\mu_j(x, t_r), G_n(\mu_j(x, t_r), x, t_r)) \in \mathcal{K}_n, \quad j = 1, \ldots, n, \quad (x, t_r) \in \mathbb{R}^2,
\]

(5.26)

\[
\hat{\nu}_l(x, t_r) = (\nu_l(x, t_r), -G_n(\nu_l(x, t_r), x, t_r)) \in \mathcal{K}_n, \quad l = 1, \ldots, n + 2, \quad (x, t_r) \in \mathbb{R}^2.
\]

(5.27)

The regularity properties of \(F_n, H_n, \mu_j, \) and \(\nu_l\) are analogous to those in Section 3 due to assumptions (2.2). Similar to (3.15), the divisor \((\phi(P, x, t_r))\) of \(\phi(P, x, t_r)\) reads

\[
(\phi(P, x, t_r)) = \mathcal{D}_{\hat{\nu}_1(x, t_r)\hat{\nu}_2(x, t_r)\hat{\nu}_3(x, t_r)}(P) - \mathcal{D}_{P_{\infty+}P_{\infty-}\hat{\mu}(x, t_r)}(P)
\]

(5.28)

with

\[
\hat{\mu} = \{\hat{\mu}_1, \ldots, \hat{\mu}_n\}, \quad \hat{\nu} = \{\hat{\nu}_3, \ldots, \hat{\nu}_{n+2}\} \in \text{Sym}^n(\mathcal{K}_n).
\]

(5.29)

The properties of \(\phi(P, x, t_r)\) are summarized as follows.

**Lemma 5.1.** Assume (2.2) and suppose that (5.3), (5.4) hold. Moreover, let \(P = (z, y) \in \mathcal{K}_n \setminus \{P_{\infty+}, P_0\}\) and \((x, t_r) \in \mathbb{R}^2\). Then \(\phi\) satisfies

\[
\phi_x(P) + \phi(P)^2 = -z^2 \rho^2 + z(u - u_{xx}) + \frac{1}{4},
\]

(5.30)

\[
\phi_{t_r}(P) = (-\tilde{G}_r(z) + \tilde{F}_r(z)\phi(P))_x
\]

(5.31)
\[ \phi_{t_r}(P) = \tilde{H}_r(z) + 2\tilde{G}_r(z)\phi(P) - \tilde{F}_r(z)\phi(P)^2, \]  
(5.32)

\[ \phi(P)\phi(P^*) = -\frac{H_n(z)}{F_n(z)}. \]  
(5.33)

\[ \phi(P) + \phi(P^*) = 2\frac{G_n(z)}{F_n(z)}, \]  
(5.34)

\[ \phi(P) - \phi(P^*) = 2y\frac{\nabla_n}{F_n(x)}. \]  
(5.35)

**Proof.** Equations (5.30) and (5.33)-(5.35) can be proved as in Lemma 3.1. Using (5.21) and (5.22), one infers that

\[ \phi_{t_r} = (\ln \psi_1)_{x,t_r} = (\ln \psi_1)_{t,x} = \left( \frac{\psi_{1,t}}{\psi_1} \right)_x \]

\[ = \left( \frac{-\tilde{G}_r\psi_1 + \tilde{F}_r\psi_2}{\psi_1} \right)_x = (-\tilde{G}_r + \tilde{F}_r\phi)_x. \]  
(5.36)

Insertion of (5.14) into (5.36) then yields (5.31). To prove (5.32), one observes that

\[ \phi_{t_r} = \left( \frac{\psi_2}{\psi_1} \right)_{t_r} = \left( \frac{\psi_{2,t_r}}{\psi_1} - \frac{\psi_2\psi_{1,t}}{\psi_1^2} \right) \]

\[ = \left( \tilde{H}_r\psi_1 + \tilde{G}_r\psi_2 - \phi \tilde{G}_r\psi_1 + \tilde{F}_r\psi_2 \right) \psi_1 \]

\[ = \tilde{H}_r + 2\tilde{G}_r\phi - \tilde{F}_r\phi^2, \]  
(5.37)

which leads to (5.32). Alternatively, one can also insert (5.12)-(5.14) into (5.31) to obtain (5.32).  
\[ \square \]

Next, we determine the time evolution of \( F_n, G_n, \) and \( H_n \), using relations (5.12)-(5.14) and (5.15)-(5.17).

**Lemma 5.2.** Assume (2.2) and suppose that (5.3), (5.4) hold. Then

\[ F_{n,t_r} = 2(G_n\tilde{F}_r - \tilde{G}_rF_n), \]  
(5.38)

\[ G_{n,t_r} = \tilde{H}_rF_n - H_n\tilde{F}_r, \]  
(5.39)

\[ H_{n,t_r} = 2(H_n\tilde{G}_r - G_n\tilde{H}_r). \]  
(5.40)

Equations (5.38)-(5.40) are equivalent to

\[ -V_{n,t_r} + [z^{-1}\tilde{V}, V_n] = 0. \]  
(5.41)
Proof. Differentiating (5.33) with respect to $t_r$ naturally yields
\[ (\phi(P) - \phi(P^*))_{t_r} = -2y F_{n,t_r} F_n^{-2}. \] (5.42)

On the other hand, using (5.32), (5.34), and (5.35), the left-hand side of (5.42) can be expressed as
\[ \phi(P)_{t_r} - \phi(P^*)_{t_r} = 2\tilde{G}_{r_r}(\phi(P) - \phi(P^*)) - \tilde{F}_{r_r}(\phi(P)^2 - \phi(P^*)^2) \]
\[ = 4y(\tilde{G}_{r_r} F_n - \tilde{F}_{r_r} G_n) F_n^{-2}. \] (5.43)

Combining (5.42) and (5.43) then proves (5.38). Similarly, differentiating (5.34) with respect to $t_r$, one finds
\[ (\phi(P) + \phi(P^*))_{t_r} = 2(G_{n,t_r} F_n - G_n F_{n,t_r}) F_n^{-2}. \] (5.44)

Meanwhile, the left-hand side of (5.44) also equals
\[ \phi(P)_{t_r} + \phi(P^*)_{t_r} = 2\tilde{G}_{r_r}(\phi(P) + \phi(P^*)) - \tilde{F}_{r_r}(\phi(P)^2 + \phi(P^*)^2) + 2\tilde{H}_r \]
\[ = -2G_n F_n^{-2} F_{n,t_r} + 2F_n^{-1}(\tilde{H}_r F_n - \tilde{F}_{r_r} H_n), \] (5.45)

using (5.32), (5.33), and (5.34). Equation (5.39) is clear from (5.44) and (5.45). Then, (5.40) follows by differentiating (2.18), that is, $G_n^2 + F_n H_n = R_{2n+2}(z)$, with respect to $t_r$, and using (5.33) and (5.39). Finally, a direct calculation shows (5.41) holds. □

Basic properties of $\psi(P, x, x_0, t_r, t_{0_r})$ are summarized as follows.

Lemma 5.3. Assume (2.2) and suppose that (5.3), (5.4) hold. Moreover, let $P = (z, y) \in K_n \setminus \{P_{\infty,^+}, P_0\}$ and $(x, x_0, t_r, t_{0_r}) \in \mathbb{R}^4$. Then, the Baker-Akhiezer function $\psi$ satisfies
\[
\psi_1(P, x, x_0, t_r, t_{0_r}) = \left( \frac{F_n(z, x, t_r)}{F_n(z, x_0, t_{0_r})} \right)^{1/2} \exp \left( y \int_{t_{0_r}}^{t_r} ds \tilde{F}_r(z, x_0, s) F_n(z, x_0, s)^{-1} \right),
\] (5.46)
\[
\psi_1(P, x, x_0, t_r, t_{0_r}) \psi_1(P^*, x, x_0, t_r, t_{0_r}) = \frac{F_n(z, x, t_r)}{F_n(z, x_0, t_{0_r})}, \] (5.47)
\[
\psi_2(P, x, x_0, t_r, t_{0_r}) \psi_2(P^*, x, x_0, t_r, t_{0_r}) = -\frac{H_n(z, x, t_r)}{F_n(z, x_0, t_{0_r})}, \] (5.48)
\[
\psi_1(P, x, x_0, t_r, t_{0_r}) \psi_2(P^*, x, x_0, t_r, t_{0_r}) + \psi_1(P^*, x, x_0, t_r, t_{0_r}) \psi_2(P, x, x_0, t_r, t_{0_r}) = 2 \frac{G_n(z, x, t_r)}{F_n(z, x_0, t_{0_r})}, \] (5.49)
\( \psi_1(P, x, x_0, t_r, t_{0r}) \psi_2(P^*, x, x_0, t_r, t_{0r}) - \psi_1(P^*, x, x_0, t_r, t_{0r}) \psi_2(P, x, x_0, t_r, t_{0r}) = -\frac{2y}{F_n(z, x_0, t_{0r})}. \) 
\( (5.50) \)

**Proof.** To prove (5.46), we first consider the part of time variable in the definition (5.23), that is,
\[
\exp\left(\int_{t_{0r}}^{t_r} ds \left( \widetilde{F}_r(z, x_0, s) \phi(P, x_0, s) - \widetilde{G}_r(z, x_0, s) \right) \right).
\( (5.51) \)

The integrand in the above integral equals
\[
\widetilde{F}_r(z, x_0, s) \phi(P, x_0, s) - \widetilde{G}_r(z, x_0, s)
= \widetilde{F}_r(z, x_0, s) \frac{y + G_n(z, x_0, s)}{F_n(z, x_0, s)} - \widetilde{G}_r(z, x_0, s)
= y \widetilde{F}_r(z, x_0, s) F_n(z, x_0, s)^{-1} + (\widetilde{F}_r(z, x_0, s) G_n(z, x_0, s)
- \widetilde{G}_r(z, x_0, s) F_n(z, x_0, s)) F_n(z, x_0, s)^{-1}
= y \widetilde{F}_r(z, x_0, s) F_n(z, x_0, s)^{-1} + \frac{1}{2} F_{n,s}(z, x_0, s),
\( (5.52) \)

using (5.24) and (5.38). Hence, (5.51) can be expressed as
\[
\left( \frac{F_n(z, x_0, t_r)}{F_n(z, x_0, t_{0r})} \right)^{1/2} \exp\left( y \int_{t_{0r}}^{t_r} ds \widetilde{F}_r(z, x_0, s) F_n(z, x_0, s)^{-1} \right).
\( (5.53) \)

On the other hand, the part of space variable in (5.23) can be written as
\[
\left( \frac{F_n(z, x, t_r)}{F_n(z, x_0, t_r)} \right)^{1/2} \exp\left( y \int_{x_0}^{x} dx' F_n(z, x', t_r)^{-1} \right),
\( (5.54) \)

using the similar procedure in Lemma 3.2. Then combining (5.53) and (5.54) readily leads to (5.46). Evaluating (5.46) at the points \( P \) and \( P^* \) and multiplying the resulting expressions yields (5.47). The remaining statements are direct consequences of (5.25), (5.33), (5.35), and (5.47). \( \square \)

In analogy to Lemma 3.4, the dynamics of the zeros \( \{ \mu_j(x, t_r) \}_{j=1, \ldots, n} \) and \( \{ \nu_l(x, t_r) \}_{l=1, \ldots, n+2} \) of \( F_n(z, x, t_r) \) and \( H_n(z, x, t_r) \) with respect to \( x \) and \( t_r \) are described in terms of the following Dubrovin-type equations.

**Lemma 5.4.** Assume (2.22) and suppose that (5.3), (5.4) hold subject to the constraint (2.22).

\[29\]
(i) Suppose that the zeros \( \{\mu_j(x,t_r)\}_{j=1,\ldots,n} \) of \( F_n(z,x,t_r) \) remain distinct for \( (x,t_r) \in \Omega_\mu \), where \( \Omega_\mu \subseteq \mathbb{R}^2 \) is open and connected, then \( \{\mu_j(x,t_r)\}_{j=1,\ldots,n} \) satisfy the system of differential equations,

\[
\dot{\mu}_j = -2\frac{y(\hat{\mu}_j)}{f_0} \prod_{k=1, k \neq j}^{n} (\mu_j - \mu_k)^{-1}, \quad j = 1, \ldots, n, \tag{5.55}
\]

\[\mu_j(t_r) = -2\frac{\tilde{F}_r(\mu_j)y(\hat{\mu}_j)}{f_0} \prod_{k=1, k \neq j}^{n} (\mu_j - \mu_k)^{-1}, \quad j = 1, \ldots, n, \tag{5.56}\]

with initial conditions

\[\{\hat{\mu}_j(x_0,t_0,r)\}_{j=1,\ldots,n} \in K_n, \tag{5.57}\]

for some fixed \( (x_0,t_0,r) \in \Omega_\mu \). The initial value problem \((5.56), (5.57)\) has a unique solution satisfying

\[\hat{\mu}_j \in C^\infty(\Omega_\mu, K_n), \quad j = 1, \ldots, n. \tag{5.58}\]

(ii) Suppose that the zeros \( \{\nu_l(x,t_r)\}_{l=1,\ldots,n+2} \) of \( H_n(z,x,t_r) \) remain distinct for \( (x,t_r) \in \Omega_\nu \), where \( \Omega_\nu \subseteq \mathbb{R}^2 \) is open and connected, then \( \{\nu_l(x,t_r)\}_{l=1,\ldots,n+2} \) satisfy the system of differential equations,

\[
\dot{\nu}_l = 2\left(\frac{\nu_l^2 \rho^2 - (u - u_{xx})\nu_l - \frac{1}{4}}{h_0} y(\hat{\nu}_l) \prod_{k=1, k \neq l}^{n+2} (\nu_l - \nu_k)^{-1}, \quad l = 1, \ldots, n + 2, \tag{5.59}\]

\[\nu_l(t_r) = -2\frac{\tilde{H}_r(\nu_l)y(\hat{\nu}_l)}{h_0} \prod_{k=1, k \neq l}^{n+2} (\nu_l - \nu_k)^{-1}, \quad l = 1, \ldots, n + 2, \tag{5.60}\]

with initial conditions

\[\{\hat{\nu}_l(x_0,t_0,r)\}_{l=1,\ldots,n+2} \in K_n, \tag{5.61}\]

for some fixed \( (x_0,t_0,r) \in \Omega_\nu \). The initial value problem \((5.60), (5.61)\) has a unique solution satisfying

\[\hat{\nu}_l \in C^\infty(\Omega_\nu, K_n), \quad l = 1, \ldots, n + 2. \tag{5.62}\]
Proof. It suffices to prove (5.56) since the argument for (5.60) is analogous
and that for (5.55) and (5.59) has been given in the proof of Lemma 3.4.
Differentiating (5.6) with respect to \( t_r \) yields
\[
F_{n,t_r}(\mu_j) = - f_0 \mu_j,t_r \prod_{k=1}^{n} (\mu_j - \mu_k). \tag{5.63}
\]
On the other hand, inserting \( z = \mu_j \) into (5.38) and using (5.26), one finds
\[
F_{n,t_r}(\mu_j) = 2 G_n(\mu_j) \tilde{F}_r(\mu_j) = 2 y(\hat{\mu}_j) \tilde{F}_r(\mu_j). \tag{5.64}
\]
Combining (5.63) and (5.64) then yields (5.56). The rest is analogous to the
proof of Lemma 3.4. \( \square \)

Since the stationary trace formulas for CHD2 invariants in terms of symmetric functions of \( \mu_j \) in Lemma 3.5 extend line by line to the corresponding time-dependent setting, we next record the \( t_r \)-dependent trace formulas without proof. For simplicity, we confine ourselves to the simplest one only.

**Lemma 5.5.** Assume (2.2), suppose that (5.3), (5.4) hold, and let \( (x,t_r) \in \mathbb{R}^2 \). Then,
\[
u^{-1} G(-4u_x(u - u_{xx}) - 2u(u_x - u_{xxx})) = \sum_{j=1}^{n} \mu_j(x,t_r) - \frac{1}{2} \sum_{m=0}^{2n+1} E_m. \tag{5.65}
\]

## 6 Time-dependent algebro-geometric solutions of CHD2 hierarchy

In our final section, we extend the results of section 4 from the stationary CHD2 hierarchy, to the time-dependent case. We obtain Riemann theta function representations for the meromorphic function \( \phi \), the Baker-Akhiezer function \( \psi \), and especially, for the algebro-geometric solutions \( u, \rho \) of the whole CHD2 hierarchy.

We first record the asymptotic properties of \( \phi \) in the time-dependent case.

**Lemma 6.1.** Assume (2.2) and suppose that (5.3), (5.4) hold. Moreover, let \( P = (z,y) \in K_n \setminus \{ P_{\infty,0}, P_0 \} \), \( (x,t_r) \in \mathbb{R}^2 \). Then,
\[
\phi(P) \xrightarrow{\zeta \to 0} \pm i \rho \zeta^{-1} + \frac{\mp i(u - u_{xx}) - \rho_x}{2\rho} + O(\zeta), \quad P \to P_{\infty,0}, \quad \zeta = z^{-1}, \tag{6.1}
\]
\[ \phi(P) \equiv \frac{1}{2} (u - u_x) \zeta + O(\zeta^2), \quad P \to P_0, \quad \zeta = z. \quad (6.2) \]

Since the proof of Lemma 6.1 is identical to the corresponding stationary results in Lemma 4.1, we omit the corresponding details.

Next, we investigate the properties of the Abel map. To do this, let \( \underline{\mu} = (\mu_1, \ldots, \mu_n) \in \mathbb{C}^n \), we define the following symmetric functions by

\[ \Psi_k(\mu) = (-1)^k \sum_{\underline{l} \in S_k} \mu_{l_1} \cdots \mu_{l_k}, \quad k = 1, \ldots, n, \quad (6.3) \]

where \( S_k = \{ \underline{l} = (l_1, \ldots, l_k) \in \mathbb{N}^k \mid l_1 < \ldots < l_k \leq n \} \);

\[ \Phi_k^{(j)}(\mu) = (-1)^k \sum_{\underline{l} \in T_k^{(j)}} \mu_{l_1} \cdots \mu_{l_k}, \quad k = 1, \ldots, n-1, \quad (6.4) \]

where \( T_k^{(j)} = \{ \underline{l} = (l_1, \ldots, l_k) \in S_k \mid l_{m_k} \neq j \}, \quad j = 1, \ldots, n \). For the properties of \( \Psi_k(\mu) \) and \( \Phi_k^{(j)}(\mu) \), we refer to Appendix E [13].

Introducing

\[ \tilde{d}_{r,k}(E) = \sum_{s=0}^{r-k} \tilde{c}_{r-k-s}(E) \Phi_k^{(s)}(\mu), \quad k = 0, \ldots, r \wedge n, \quad (6.5) \]

for a given set of constants \( \{ \tilde{c}_s \}_{s=1, \ldots, r} \subset \mathbb{C} \), the corresponding homogeneous and nonhomogeneous quantities \( \hat{F}_r(\mu_j) \) and \( \tilde{F}_r(\mu_j) \) in the CHD2 case are then given by

\[ \hat{F}_r(\mu_j) = f_0 \sum_{s=(r-n)\wedge 0}^{r} \hat{c}_s(E) \Phi_r^{(s)}(\mu), \quad (6.6) \]

\[ \tilde{F}_r(\mu_j) = \sum_{s=0}^{r} \tilde{c}_{r-s} \hat{F}_s(\mu_j) = f_0 \sum_{k=0}^{r \wedge n} \tilde{d}_{r,k}(E) \Phi_k^{(r)}(\mu), \quad r \in \mathbb{N}_0, \quad \tilde{c}_0 = 1, \]

using (D.59) and (D.60) [13]. Here, \( \hat{c}_s(E), \ s \in \mathbb{N}_0, \) is defined by (D.2) [13].

We now state the analog of Theorem 4.3, which indicates marked differences between the CHD2 hierarchy and other completely integrable systems such as the KdV and AKNS hierarchies.

\[^2 m \wedge n = \min\{m, n\}, \ m \vee n = \max\{m, n\} \]
Theorem 6.2. Assume \((2.22)\) and suppose that \(\{\hat{\mu}_j\}_{j=1,\ldots,n}\) satisfies the Dubrovin equations \((5.55)\), \((5.56)\) on an open set \(\Omega_\mu \subseteq \mathbb{R}^2\) such that \(\mu_j, j = 1,\ldots,n\), remain distinct and nonzero on \(\Omega_\mu\) and that \(\tilde{F}_r(\mu_j) \neq 0\) on \(\Omega_\mu, j = 1,\ldots,n\). Introducing the associated divisor \(\hat{D}_{\hat{\mu}}(x,t_r) \in \text{Sym}^n(K_n)\), one computes,

\[
\partial_x \alpha Q_0(\hat{D}_{\hat{\mu}}(x,t_r)) = \frac{2}{u(x,t_r)} c(n), \quad (x,t_r) \in \Omega_\mu, \tag{6.7}
\]

\[
\partial_t r \alpha Q_0(\hat{D}_{\hat{\mu}}(x,t_r)) = -2\left( \sum_{\ell=0}^{n} \tilde{d}_{r,n-\ell}(E)c(\ell) \right), \quad (x,t_r) \in \Omega_\mu. \tag{6.8}
\]

In particular, the Abel map dose not linearize the divisor \(\hat{D}_{\hat{\mu}}(x,t_r)\) on \(\Omega_\mu\).

Proof. Let \((x,t_r) \in \Omega_\mu\). It suffices to prove (6.8), since (6.7) is proved as in the stationary context of Theorem 4.3. Using (5.56), (6.6), and (E.4) \([13]\), one infers that

\[
\partial_t r \left( \sum_{j=1}^{n} \int_{Q_0} \hat{\mu}_j \omega \right) = \sum_{j=1}^{n} \mu_j,t \sum_{k=1}^{n} c(k) \frac{\mu_j^{k-1}}{y(\hat{\mu}_j)}
\]

\[
= -2 \sum_{j=1}^{n} \sum_{k=1}^{n} c(k) \prod_{l=1}^{n} (\mu_j - \mu_l) \frac{\tilde{F}_r(\mu_j)}{f_0}
\]

\[
= -2 \sum_{j=1}^{n} \sum_{k=1}^{n} c(k) \prod_{l=1}^{n} (\mu_j - \mu_l) \left( f_0 \sum_{m=0}^{r \land n} \tilde{d}_{r,m}(E) \Phi_j^{(m)}(\mu) \right)
\]

\[
= -2 \sum_{m=0}^{r \land n} \tilde{d}_{r,m}(E) \sum_{k=1}^{n} c(k) (U_n(\mu))_{k,j} (U_n(\mu))_{j,n-m}^{-1}
\]

\[
= -2 \sum_{m=0}^{r \land n} \tilde{d}_{r,m}(E) c(n - m)
\]

\[
= -2 \sum_{m=0}^{r \land n} \tilde{d}_{r,n-m}(E) c(m), \tag{6.9}
\]

where we used the relations (cf.(E.13), (E.14) \([13]\)),

\[
U_n(\mu) = \left( \frac{\mu_j^{k-1}}{\prod_{l=1}^{n} (\mu_j - \mu_l)} \right)_{j,k=1}^{n}, \quad U_n(\mu)^{-1} = \left( \Phi_j^{(m)}(\mu) \right)_{j,k=1}^{n}. \tag{6.10}
\]

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The analogous results hold for the corresponding divisor \( D_{\hat{\nu}(x,t)} \) associated with \( \phi(P,x,t_r) \).

For subsequent purpose we note the following asymptotic spectral parameter expansion of \( F_n/y \) as \( P \to P_{\infty\pm} \),
\[
\frac{F_n(z)}{y} = \frac{1}{2i} \sum_{l=0}^{\infty} \hat{f}_l \zeta^{l+1}, \quad \text{as } P \to P_{\infty\pm}, \quad \zeta = z^{-1}.
\] (6.11)

Here, \( \hat{f}_l \) denote the homogenous coefficients, satisfying (2.3) with vanishing integration constants.

Next, we shall provide the explicit representations of \( \psi, \phi, u, \) and \( \rho \) in terms of the Riemann theta function associated with \( K_n \), assuming the affine part of \( K_n \) to be nonsingular.

Let \( \omega^{(2)}_{P_{\infty\pm},r} \) be the normalized differentials of the second kind with a unique pole at \( P_{\infty\pm} \), and principal part \( \zeta^{-2-r} d\zeta \) near \( P_{\infty\pm} \), and define
\[
\tilde{\Omega}^{(2)}_r = \sum_{q=0}^{r} (q+1) \tilde{c}_{r-q}(\omega^{(2)}_{P_{\infty\pm},q} - \omega^{(2)}_{P_{\infty\pm},q}), \quad \tilde{c}_0 = 1,
\] (6.12)
where \( \tilde{c}_q, q = 0, \ldots, r, \) are the constants introduced in the definition of \( \tilde{F}_r(z) \). Hence, one infers
\[
\int_{a_k} \tilde{\Omega}^{(2)}_r = 0, \quad k = 1, \ldots, n,
\] (6.13)
\[
\int_{Q_0}^{P} \tilde{\Omega}^{(2)}_r \zeta^{-q-1} + \tilde{c}_{r,0} + O(\zeta), \quad \text{as } P \to P_{\infty\pm}
\] (6.14)
for some constants \( \tilde{c}_{r,0} \in \mathbb{C} \).

Recalling (4.17)-(4.29), the analog of Theorem 4.4 in the stationary case then reads as follows.

**Theorem 6.3.** Assume (2.24) and suppose that (5.3), (5.4) hold on \( \Omega \) subject to the constraint (2.22). In addition, let \( P = (z,y) \in K_n \setminus \{ P_{\infty\pm} \} \) and \( (x,t_r), (x_0,t_{0,r}) \in \Omega \), where \( \Omega \subseteq \mathbb{R}^2 \) is open and connected. Moreover, suppose that \( D_{\hat{\mu}(x,t_r)} \), or equivalently, \( D_{\hat{\mu}(x,t_r)} \) is nonspecial for \( (x,t_r) \in \Omega \). Then, \( \phi, \psi, u, \) and \( \rho \) admit the representations
\[
\phi(P,x,t_r) = i\rho(x,t_r) \frac{\theta(z(P,\hat{\nu}(x,t_r)))\theta(z(P_{\infty\pm},\hat{\mu}(x,t_r)))}{\theta(z(P_{\infty\pm},\hat{\mu}(x,t_r))) \theta(z(P,\hat{\nu}(x,t_r)))} \exp \left( d_0 - \int_{Q_0}^{P} \Omega^{(3)} \right)
\] (6.15)
\[ \psi_1(P, x, x_0, t_r, t_{0,r}) = \frac{\theta(z(P, \hat{\mu}(x, t_r))) \theta(z(P_{\infty +}, \hat{\mu}(x_0, t_{0,r})))}{\theta(z(P_{\infty +}, \hat{\mu}(x, t_r))) \theta(z(P, \hat{\mu}(x_0, t_{0,r})))} \]  

(6.16)

\[ \times \exp \left( \int_{x_0}^{x} dx' i \rho(x', t_r) \int_{Q_0}^{P} \Omega_0^{(2)} + 2i(t_r - t_{0,r}) \int_{Q_0}^{P} \tilde{\Omega}_r^{(2)} \right), \]

\[ \psi_2(P, x, x_0, t_r, t_{0,r}) = i \rho(x, t_r) \frac{\theta(z(P, \hat{\mu}(x, t_r))) \theta(z(P_{\infty +}, \hat{\mu}(x_0, t_{0,r})))}{\theta(z(P_{\infty +}, \hat{\mu}(x, t_r))) \theta(z(P, \hat{\mu}(x_0, t_{0,r})))} \exp \left( d_0 - \int_{Q_0}^{P} \Omega^{(3)} \right), \]  

(6.17)

\[ u(x, t_r) = -4i \frac{\theta(z(P_0, \hat{\mu}(x, t_r))) \theta(z(P_{\infty +}, \hat{\mu}(x, t_r)))}{\theta(z(P_{\infty +}, \hat{\mu}(x, t_r))) \theta(z(P, \hat{\mu}(x, t_r)))}, \]  

(6.18)

\[ \rho(x, t_r) = \frac{i}{2} \frac{\theta(z(P, \hat{\mu}(x, t_r))) \theta(z(P_{\infty +}, \hat{\mu}(x, t_r)))}{\theta(z(P_{\infty +}, \hat{\mu}(x, t_r))) \theta(z(P, \hat{\mu}(x, t_r)))}. \]  

(6.19)

**Proof.** Start with the proof of the theta function representation (6.16) for \( \psi_1 \). Without loss of generality it suffices to treat the homogenous case \( \tilde{c}_0 = 1, \tilde{c}_q = 0, q = 1, \ldots, r \). As in the corresponding stationary case we temporarily assume

\[ \mu_j(x, t_r) \neq \mu_{j'}(x, t_r), \quad \text{for } j \neq j' \text{ and } (x, t_r) \in \tilde{\Omega}, \]  

(6.20)

for appropriate \( \tilde{\Omega} \subseteq \Omega \), and define the right-hand side of (6.16) to be \( \Psi \). We intend to prove \( \psi_1 = \Psi \), with \( \psi_1 \) given by (5.23). For that purpose we first investigate the local zeros and poles of \( \psi_1 \). Using the definition (5.24) of \( \phi \), (5.35), and Dubrovin equations (5.56), one computes

\[ \bar{F}(z, x, t_r) \phi(P, x, t_r) - \bar{G}(z, x, t_r) = \bar{F}(z, x, t_r) \frac{y + G(z, x, t_r)}{F(z, x, t_r)} - \bar{G}(z, x, t_r) \]

\[ = \frac{y}{F(z, x, t_r)} + \frac{1}{2} \frac{F_{x,t_r}(z, x, t_r)}{F(z, x, t_r)} \]

\[ = -\frac{1}{2} \frac{\mu_{j,t_r}}{z - \mu_j} - \frac{1}{2} \frac{\mu_{j,t_r}}{z - \mu_j} + O(1) \]

\[ = -\frac{\mu_{j,t_r}}{z - \mu_j} + O(1), \quad \text{as } z \to \mu_j(x, t_r). \]  

(6.21)

More concisely,

\[ \bar{F}(z, x_0, s) \phi(P, x_0, s) - \bar{G}(z, x_0, s) = \frac{\partial}{\partial s} \ln(z - \mu_j(x_0, s)) + O(1) \]
for $P$ near $\hat{\mu}_j(x_0, s)$. (6.22)

Meanwhile, (5.39) gives

$$\phi(P, x') = \frac{\partial}{\partial x'} \ln (z - \mu_j(x', t_r)) + O(1) \quad \text{for } P \text{ near } \hat{\mu}_j(x', t_r). \quad (6.23)$$

Hence, combining (5.23), (6.22), and (6.23) yields

$$\psi_1(P, x, x_0, t_r, t_{0,r}) = \left\{ \begin{array}{ll}
(z - \mu_j(x, t_r))O(1), & \text{for } P \text{ near } \hat{\mu}_j(x_0, t_{0,r}), \\
O(1), & \text{for } P \text{ near } \hat{\mu}_j(x, t_r) = \hat{\mu}_j(x_0, t_{0,r}), \\
(z - \mu_j(x_0, t_{0,r}))^{-1}O(1), & \text{for } P \text{ near } \hat{\mu}_j(x_0, t_{0,r}) \neq \hat{\mu}_j(x, t_r),
\end{array} \right. \quad (6.24)$$

with $O(1) \neq 0$. Consequently, $\psi_1$ and $\Psi$ have identical zeros and poles on $K_n \setminus \{P_{\infty_+}, P_{\infty_-}\}$, which are all simple by hypothesis (6.20). It remains to study the behavior of $\psi_1$ near $P_{\infty_{\pm}}$. By (5.24), (5.38), (6.1), and (6.11), one infers that

$$\int_{x_0}^{x} dx' \phi(P, x', t_r) + \int_{t_{0,r}}^{t_r} ds \left( \hat{F}_r(\zeta^{-1}, x_0, s) \phi(P, x_0, s) - \hat{G}_r(\zeta^{-1}, x_0, s) \right)$$

$$= \pm i \zeta^{-1} \int_{x_0}^{x} \rho'(x', t_r) dx' + \left\{ \begin{array}{ll}
O(1) & \text{for } P \to P_{\infty_+} \\
O(1) & \text{for } P \to P_{\infty_-}
\end{array} \right. + \int_{t_{0,r}}^{t_r} ds \left( \frac{y}{\hat{F}_n(\zeta^{-1}, x_0, s)} + \frac{1}{2} \frac{F_n(\zeta^{-1}, x_0, s)}{R_n(\zeta^{-1}, x_0, s)} \right)$$

$$= \pm i \zeta^{-1} \int_{x_0}^{x} \rho'(x', t_r) dx' + \left\{ \begin{array}{ll}
O(1) & \text{for } P \to P_{\infty_+} \\
O(1) & \text{for } P \to P_{\infty_-}
\end{array} \right. + \int_{t_{0,r}}^{t_r} ds \left( \pm 2i \zeta^{-r-1} \sum_{m=0}^{r} \hat{f}_{m}(x_0, s) \zeta^m \sum_{l=0}^{m} \hat{f}_{l}(x_0, s) \zeta^l + \frac{1}{2} \frac{u_{t_r}(x_0, s)}{u(x_0, s)} + O(\zeta) \right)$$

$$= \pm i \zeta^{-1} \int_{x_0}^{x} \rho'(x', t_r) dx' + \left\{ \begin{array}{ll}
O(1) & \text{for } P \to P_{\infty_+} \\
O(1) & \text{for } P \to P_{\infty_-}
\end{array} \right. + \int_{t_{0,r}}^{t_r} ds \left( \pm 2i \zeta^{-r-1} \frac{2i \hat{f}_{r+1}(x_0, s)}{\hat{f}_0(x_0, s)} + \frac{1}{2} \frac{u_{t_r}(x_0, s)}{u(x_0, s)} + O(\zeta) \right)$$

$$= \pm i \zeta^{-1} \int_{x_0}^{x} \rho'(x', t_r) dx' \pm 2i \zeta^{-r-1}(t_r - t_{0,r}) \left\{ \begin{array}{ll}
O(1) & \text{for } P \to P_{\infty_+}, \\
O(1) & \text{for } P \to P_{\infty_-},
\end{array} \right. \quad (6.25)$$

where we used $\hat{f}_0 = -u$ and $u_{t_r} = \frac{1}{2} \hat{f}_{r,x}$ (cf. (5.1)) in the homogeneous case $\hat{c}_0 = 1$, $\hat{c}_q = 0$, $q = 1, \ldots, r$. A comparison of $\psi_1$ and $\Psi$ near $P_{\infty_{\pm}}$, 36
taking into account (5.23), (6.14), (6.16), and (6.25), then shows that $\psi_1$ and $\Psi$ have identical exponential behavior up to order $O(1)$ near $P_{\infty \pm}$. Thus, $\psi_1$ and $\Psi$ share the same singularities and zeros, and the Riemann-Roch-type uniqueness result (cf. Lemma C.2 [13]) then proves $\psi_1 = \Psi$. The representation (6.15) for $\phi$ on $\tilde{\Omega}$ follows by combining (5.28), (6.1), and Theorem A.26 [13]. The representation (6.19) for $\rho$ on $\Omega$ is clear from (6.15) and (6.2). The representation (6.18) for $u$ on $\Omega$ follows from (6.19) and the relation $u_\rho = -2$. In fact, since the proofs of (6.15), (6.18), and (6.19) are identical to the corresponding stationary results in Theorem 4.4 which can be extended line by line to the time-dependent setting, here we omit the corresponding details. Finally, the extension of all these results from $(x, t_r) \in \tilde{\Omega}$ to $(x, t_r) \in \Omega$ then simply follows from the continuity of $\alpha_{Q_0}$ and the hypothesis of $D_{\rho(x, t_r)}$ being nonspecial for $(x, t_r) \in \Omega$. □

Remark 4.5 applies in the present time-dependent context as well.

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