On the approximation properties of bivariate \((p, q)\)–Bernstein operators
(Revised)

Ali Karaisa

Department of Mathematics–Computer Sciences, Faculty of Sciences, Necmettin Erbakan University, 42090 Meram, Konya, Turkey

Abstract. In the present study, we have given a corrigendum to our paper on the approximation properties of bivariate \((p, q)\)–Bernstein operators. Recently, we \([2]\) have defined the bivariate \((p, q)\)–Bernstein operators. Later, we have aware of Acar et. al \([1]\) already have given some moments. In this case, we have revised \([2]\) Lemma 2.3.

Keywords: bivariate \((p, q)\)–Bernstein operator, Voronovskaja type theorem, \((p, q)\)–integer

MSC: 41A25, 41A36

1. Introduction

Approximation theory has been used in the theory of approximation of continuous functions by means of sequences of positive linear operators and still remains as a very active area of research. Since Korovkins famous theorem in 1950, the study of the linear methods of approximation given by sequences of positive and linear operators and still remains as a very active area of research. Since Korovkin’s famous approximation theory. The first \([12]\) by applying the idea of \([12]\) and \([15]\), respectively. For some recent works devoted to \((p, q)\)-calculus have emerged as a new area in the field of approximation theory. During the last two decades, the applications of \(q\)–calculus have emerged as a new area in the field of approximation theory. The first \(q\)–analogue of the well-known Bernstein polynomials was introduced by Lupăş \([12]\) by applying the idea of \(q\)–calculus. Since approximation studied by \(q\)–Bernstein polynomials is better than classical one under convenient choice of \(q\), many authors introduced a generalization of various operators and investigated several approximation properties we refer the readers to \([3–5]\).

Recently, Mursaleen et al. used \((p, q)\)-calculus in approximation theory and defined \((p, q)\)–analogue of Bernstein operators \([13]\). They estimated uniform convergence of the operators and rate of convergence, obtained Voronovskaja type as well. Also, \((p, q)\)–analogue of Bernstein-Stancu operators and Bleimann-Butzer-Hahn operators were introduced in \([14]\) and \([15]\), respectively. For some recent works devoted to \((p, q)\)-operators, we can refer the readers to \([6–10]\).

In the present study, we define the bivariate Bernstein operators based on \((p, q)\)–integer. We examine approximation properties of our new operator by the help of Korovkin-type theorem. Further, we present the local approximation properties and establish the rates of convergence by means of the modulus of continuity and the Lipschitz type maximal function. Also, we present a Voronovskaja type asymptotic formula for this operators.

Let us recall some definitions and notations regarding the concept of \((p, q)\)–calculus.

The \((p, q)\)–integer of the number \(n\) is defined by

\[
[n]_{p,q} := \frac{p^n - q^n}{p - q}, \quad n = 1, 2, 3, \ldots, \quad 0 < q < p \leq 1.
\]

The \((p, q)\)–factorial \([n]_{p,q}!\) and the \((p, q)\)–binomial coefficients are defined as:

\[
[n]_{p,q}! := \begin{cases} \binom{n}{0}_{p,q} & n = 0, \\ \binom{n}{k}_{p,q} & n \in \mathbb{N}, \quad n \geq 1, \\ \frac{[n]_{p,q}!}{[n-k]_{p,q}!} & 0 \leq k \leq n. \end{cases}
\]

Further, the \((p, q)\)–binomial expansions are given as

\[
(ax + by)^n_{p,q} = \sum_{k=0}^{n} \binom{n-k}{k}_{p,q} p^{(n-k)}q^{(k)} a^{n-k} b^k x^{n-k} y^k.
\]

and

\[
(x - y)^n_{p,q} = (x - y)(px - qy)(p^2x - q^2y) \cdots (p^{n-1}x - q^{n-1}y).
\]

Further information related to \((p, q)\)–calculus can be found in \([16, 17]\).

*Corresponding author: Tel:+90 332 323 8220; fax:+90 332 323 8245 E-mail: akaraisa@hotmail.com
2. Construction of the operators

Recently, Mursaleen et al. applied $(p, q)$-calculus in approximation theory and introduced revised $(p, q)$-analogue of Bernstein operators as follows:

\[(2.1) \quad B_{n,p,q}(f; x) = \frac{1}{p^n(n-1)/2} \sum_{k=0}^{n} \binom{n}{k}_{p,q} p^{k(k-1)/2} x^k \prod_{s=0}^{n-k-1} \left( p^s - q^s \frac{x}{b_n} \right) f \left( \frac{[k]_{p,q}}{[n]_{p,q}} p^{k-n} \right), \]

On the other hand, another active research area in approximation theory is to approximate the bivariate functions. For example, Barbosu, [19] defined and studied bivariate Bernstein operator. Büyükyazıcı [20] introduced $q$–Bernstein Chlodowsky operator.

Now, we define bivariate Bernstein operator based on $(p, q)$–integers. Let $I = [0, 1] \times [0, 1]$, $f : I \rightarrow R$ and $0 < q_1, q_2 < p_1, p_2 \leq 1$. We define the bivariate extension of the $(p, q)$–Bernstein operator operators as follows:

\[(2.2) \quad B_{n,m}^{(p_1,q_1),(p_2,q_2)}(f; x, y) = \sum_{k=0}^{n} \sum_{j=0}^{m} R_{n,k}(p_1, q_1; x) R_{m,j}(p_2, q_2; y) f \left( \frac{[k]_{p_1,q_1}}{[n]_{p_1,q_1}} p_1^{k-n} + \frac{[j]_{p_2,q_2}}{[m]_{p_2,q_2}} p_2^{j-m} \right), \]

where

\[R_{n,k}(p_1, q_1; x) = p_1^{\frac{k(k-1)-n(n-1)}{2}} \binom{n}{k}_{p_1,q_1} x^k \prod_{s=0}^{n-k-1} \left( p_1^s - q_1^s x \right). \]

**Lemma 2.1.** [14, Lemma 1]

\[B_{n,p,q}(e^0; x) = 1, \quad B_{n,p,q}(e_1; x) = x, \quad B_{n,p,q}(e_2; x) = \frac{p^{n-1}}{[n]_{p,q}} x + \frac{q[n-1]_{p,q}}{[n]_{p,q}} x^2. \]

Also, $(p, q)$–Bernstein operator satisfy following equations:

**Lemma 2.2.**

\[B_{n,p,q}(e_3; x) = \frac{p^{2n-2}}{[n]_{p,q}^2} x + \frac{p^{n-1}(2p + q)q[n-1]_{p,q}}{[n]_{p,q}^2} x^2 + \frac{q^3[n-1]_{p,q}[n-2]_{p,q}}{[n]_{p,q}^2} x^3, \]

\[B_{n,p,q}(e_4; x) = \frac{p^{3n-3}}{[n]_{p,q}^3} + \left( \frac{q(3p^2 + 2pq + q^2)n-1]_{p,q}}{[n]_{p,q}^3} p^{2n-4} \right) x^2 + \left( \frac{q^3(3p^2 + 2pq + q^2)n-1]_{p,q}}{[n]_{p,q}^3} p^{n-3} \right) x^3 + \left( \frac{q^6[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}}{[n]_{p,q}^3} \right) x^4, \]

where $e_i(x) = x^i$, $i = 0, 1, 2, 3, 4$.

**Proof.** Let us we compute $e_3$

\[B_{n,p,q}(e_3; x) = \frac{1}{p^{n(n-1)/2}} \sum_{k=0}^{n} \binom{n}{k}_{p,q} p^{k(k-7)/2} x^k \prod_{s=0}^{n-k-1} \left( p^s - q^s x \right) \frac{[k]_{p,q}^3}{[n]_{p,q}^3}, \]

\[= \frac{1}{p^{n(n-1)/2} [n]_{p,q}^2} \sum_{k=0}^{n-1} \binom{n-1}{k}_{p,q} p^{(k+1)(k-6)/2} [k+1]_{p,q}^2 x^{k+1} \prod_{s=0}^{n-k-2} \left( p^s - q^s x \right). \]
By \( (k+1)^2_{p,q} = p^{2k} + 2qp^k[k]_{p,q} + q^2[k]_{p,q}^2 \), we have

\[
\begin{align*}
&= \frac{1}{p^{n(n-1)/2}[n]_{p,q}^3} \sum_{k=0}^{n-1} \binom{n-1}{k} \prod_{s=0}^{n-k-2} (p^s - q^s x) \\
&\quad + 2q \frac{1}{p^{n(n-1)/2}[n]_{p,q}^3} \sum_{k=0}^{n-1} \binom{n-1}{k} \prod_{s=0}^{n-k-2} (p^s - q^s x) \\
&\quad + \frac{q^2}{p^{n(n-1)/2}[n]_{p,q}^3} \sum_{k=0}^{n-1} \binom{n-1}{k} \prod_{s=0}^{n-k-2} (p^s - q^s x) \\
&= \frac{x}{[n]_{p,q}^2} p^{2n-2} + \frac{2q(n-1)p^2 q^2 [n]_{p,q}^2 p^{n-2}}{[n]_{p,q}^2} + \frac{p^n q^2 [n-1] p q^2 x^2}{[n]_{p,q}^2} + \frac{q^3 [n-1] p q [n-2] p q x^3}{[n]_{p,q}^2}.
\end{align*}
\]

Finally,

\[
B_{n,p,q}(e_4; x) = \frac{1}{p^{n(n-1)/2}[n]_{p,q}^3} \sum_{k=0}^{n-1} \binom{n-1}{k} \sum_{s=0}^{n-k-1} (p^s - q^s x) p^{k(k-1)/2} p^{-4k} [k]_{p,q}^4 \\
= 1 + \frac{3p^3}{p^{n(n-1)/2}[n]_{p,q}^3} \sum_{k=0}^{n-1} \binom{n-1}{k} \prod_{s=0}^{n-k-2} (p^s - q^s x) \\
&\quad + \frac{3pq^2}{p^{n(n-1)/2}[n]_{p,q}^3} \sum_{k=0}^{n-1} \binom{n-1}{k} \prod_{s=0}^{n-k-2} (p^s - q^s x) \\
&\quad + \frac{q^3}{p^{n(n-1)/2}[n]_{p,q}^3} \sum_{k=0}^{n-1} \binom{n-1}{k} \prod_{s=0}^{n-k-2} (p^s - q^s x) \\
&= \frac{x}{[n]_{p,q}^2} p^{3n-3} + \frac{q^3 [3p^2 + 3pq + q^2] [n-1] p q x^2}{[n]_{p,q}^3} p^{2n-4} \\
&\quad + \frac{q^3 [3p^2 + 2pq + q^2] [n-1] p q [n-2] p q x^3}{[n]_{p,q}^3} p^{n-3} + \frac{q^6 [n-1] p [n-2] p q [n-3] p q x^4}{[n]_{p,q}^3}.
\]

Acar et. al. [A] introduced Kantotovich modifications of \((p,q)-\)Bernstein operators for bivariate functions using a new \((p,q)-\)integral and given following moments for bivariate \((p,q)-\)Bernstein operators.

**Lemma 2.3.** [A, Lemma 1]  

\[
\begin{align*}
B_{n,m}^{(p_1,q_1), (p_2,q_2)}(1; x, y) &= 1, \\
B_{n,m}^{(p_1,q_1), (p_2,q_2)}(s; x, y) &= x, \\
B_{n,m}^{(p_1,q_1), (p_2,q_2)}(t; x, y) &= y, \\
B_{n,m}^{(p_1,q_1), (p_2,q_2)}(st; x, y) &= x y, \\
B_{n,m}^{(p_1,q_1), (p_2,q_2)}(s^2; x, y) &= \frac{p_1^{n-1} x + q_1 [n-1] p_1 q_1 x^2}{[n]_{p_1,q_1}}, \\
B_{n,m}^{(p_1,q_1), (p_2,q_2)}(t^2; x, y) &= \frac{p_2^{m-1} y + q_2 [m-1] p_2 q_2 y^2}{[n]_{p_2,q_2}}.
\end{align*}
\]

Using Lemma 2.3 and by linearity of \(B_{n,m}^{(p_1,q_1), (p_2,q_2)}\), we have
Remark 2.4.

\begin{align}
B_{n,m}^{(p_1,q_1),(p_2,q_2)}((t-x)^2;x,y) &= \frac{p_1^{n-1}}{[n]_{p_1,q_1}}(x-x^2), \\
B_{n,m}^{(p_1,q_1),(p_2,q_2)}((s-y)^2;x,y) &= \frac{p_2^{m-1}}{[m]_{p_2,q_2}}(y-y^2).
\end{align}

The Korovkin-type theorem for functions of two variables was proved by Volkov [18].

Theorem 2.5. Let \( q_1 := (q_1,n) \), \( p_1 := (p_1,n) \), \( q_2 := (q_2,m) \), \( p_2 := (p_m,2) \) such that \( 0 < q_1,n,q_2,m < p_1,n,p_m,2 \leq 1 \). If

\begin{equation}
\lim_{n \to \infty} p_{n,m} = 1, \quad \lim_{m \to \infty} p_{n,m} = 1, \quad \lim_{m \to \infty} q_{n,m} = a_1 \quad \text{and} \quad \lim_{m \to \infty} q_{n,m} = a_2,
\end{equation}

\( B_{n,m}^{(p_1,q_1),(p_2,q_2)}(f;x,y) \) convergence uniformly to \( f(x,y) \), on \( [0,1] \times [0,1] \) for each \( f \in C([0,1]^2) \), where \( a_1, a_2 \) be reals numbers and \( C([0,1]^2) \) be the space of all real valued continuous function on \( [0,1]^2 \) with the norm

\[ \| f \|_{C([0,1]^2)} = \sup_{(x,y) \in [0,1]^2} \| f(x,y) \| . \]

Proof. Assume that the equities \( 2.3 \) are holds. Then, we have

\[ \frac{p_1^{n-1}}{[n]_{p_1,q_1}} \to 0, \quad \frac{p_2^{m-1}}{[m]_{p_2,q_2}} \to 0, \quad \frac{q_1,n}{[n]_{p_1,q_1,n}} \to 1 \quad \text{and} \quad \frac{q_2,m}{[m]_{p_2,q_2,m}} \to 1. \]

From Lemma 2.8 we obtain \( \lim_{n,m \to \infty} B_{n,m}^{(p_1,q_1),(p_2,q_2)}(e_{ij};x,y) = e_{ij}(x,y) \) uniformly on \( [0,1]^2 \), where \( e_{ij}(x,y) = x_i y_j, 0 \leq i + j \leq 2 \) are the test functions. By using Korovkin theorem for functions of two variables was presented by Volkov [18], it follows that \( \lim_{n,m \to \infty} B_{n,m}^{(p_1,q_1),(p_2,q_2)}(f;x,y) = f(x,y) \), uniformly on \( [0,1]^2 \), for each \( f \in C([0,1]^2) \).

3. Rate of Convergence

In this section, we compute the rates of convergence of operators \( B_{n,m}^{(p_1,q_1),(p_2,q_2)} \) to \( f(x,y) \) by means of the modulus of continuity. Proceeding further, we provide a summary of the notations and definitions of the modulus of continuity and the Peetre's \( K \)-functional for bivariate real valued functions.

For \( f \in C([0,1]^2) \), the complete modulus of continuity for a bivariate case is defined as follows:

\[ \omega(f, \delta) = \sup \left\{ |f(t,s) - f(x,y)| : \sqrt{(t-x)^2 + (s-y)^2} \leq \delta \right\}. \]

for every \( (t,s),(x,y) \in [0,1]^2 \). Further, partial moduli of continuity with respect to \( x \) and \( y \) are defined as

\[ \omega^1(f, \delta) = \sup \left\{ |f(x_1,y) - f(x_2,y)| : y \in [0,1] \quad \text{and} \quad |x_1 - x_2| \leq \delta \right\}, \]

\[ \omega^2(f, \delta) = \sup \left\{ |f(x,y_1) - f(x,y_2)| : x \in [0,1] \quad \text{and} \quad |y_1 - y_2| \leq \delta \right\}. \]

It is obvious that they satisfy the properties of the usual modulus of continuity [21].

For \( \delta > 0 \), the Peetre-K functional [22] is given by

\[ K(f, \delta) = \inf_{g \in C^2([0,1]^2)} \left\{ \| f - g \|_{C([0,1]^2)} + \delta \| g \|_{C^2([0,1]^2)} \right\}, \]

where \( C^2([0,1]^2) \) is the space of functions of \( f \) such that \( f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \quad (j = 1, 2) \) in \( C([0,1]^2) \). The norm \( \| . \| \) on the space \( C^2([0,1]^2) \) is defined by

\[ \| f \|_{C^2([0,1]^2)} = \| f \|_{C([0,1]^2)} + \sum_{j=1}^2 \left( \left\| \frac{\partial f}{\partial x} \right\|_{C([0,1]^2)} + \left\| \frac{\partial f}{\partial y} \right\|_{C([0,1]^2)} \right). \]

Now, we give an estimate of the rate of convergence of operators \( B_{n,m}^{(p_1,q_1),(p_2,q_2)} \).

Theorem 3.1. Let \( f \in C([0,1]^2) \). For all \( x \in [0,1]^2 \), we have

\[ \left| B_{n,m}^{(p_1,q_1),(p_2,q_2)}(f;x,y) - f(x,y) \right| \leq 2 \omega(f, \delta_{n,m}), \]

where

\[ \delta_{n,m}^2 = \frac{p_1^{n-1}}{[n]_{p_1,q_1}}(x-x^2) + \frac{p_2^{m-1}}{[m]_{p_2,q_2}}(y-y^2). \]
Proof. By definition the complete modulus of continuity of \( f(x,y) \) and linearity and positivity our operator, we can write
\[
|B_{n,m}^{(p_1,q_1),(p_2,q_2)}(f;x,y) - f(x,y)| \leq B_{n,m}^{(p_1,q_1),(p_2,q_2)}(|f(t,s) - f(x,y)|;x,y)
\leq B_{n,m}^{(p_1,q_1),(p_2,q_2)}(\omega(t-x)^2 + (s-y)^2; x,y)
\leq \omega(f,\delta_{n,m}) \left[ \frac{1}{\delta_{n,m}} B_{n,m}^{(p_1,q_1),(p_2,q_2)} \left( \sqrt{(t-x)^2 + (s-y)^2}; x,y \right) \right].
\]

Using Cauchy-Schwartz inequality, from (2.3) and (2.4), one can write following
\[
|B_{n,m}^{(p_1,q_1),(p_2,q_2)}(f;x,y) - f(x,y)| 
\leq \omega(f,\delta_{n,m}) \left[ 1 + \frac{1}{\delta_{n,m}} \left( B_{n,m}^{(p_1,q_1),(p_2,q_2)} \left( (t-x)^2 + (s-y)^2; x,y \right) \right)^{1/2} \right] 
= \omega(f,\delta_{n,m}) \left[ 1 + \frac{1}{\delta_{n,m}} \left( B_{n,m}^{(p_1,q_1),(p_2,q_2)} \left( (t-x)^2; x,y \right) + B_{n,m}^{(p_1,q_1),(p_2,q_2)} \left( (s-y)^2; x,y \right) \right)^{1/2} \right] 
= \omega(f,\delta_{n,m}) \left[ 1 + \frac{1}{\delta_{n,m}} \left( \frac{p_1^{n-1}}{|n|_{p_{1},q_1}} (x-x^2) + \frac{p_2^{m-1}}{|m|_{p_{2},q_2}} (y-y^2) \right)^{1/2} \right].
\]
Choosing \( \delta_{n,m} = \left( \frac{p_1^{n-1}}{|n|_{p_{1},q_1}} (x-x^2) + \frac{p_2^{m-1}}{|m|_{p_{2},q_2}} (y-y^2) \right)^{1/2} \), for all \((x, y) \in [0,1]^2\), we get desired result.

\[\square\]

Theorem 3.2. Let \( f \in C([0,1]^2) \), then the following inequalities satisfy
\[
|B_{n,m}^{(p_1,q_1),(p_2,q_2)} - f(x,y)| \leq \omega^1(f,\delta_n) + \omega^2(f,\delta_m),
\]
where
\[
\begin{align*}
\delta_n^2 &= \frac{p_1^{n-1}}{|n|_{p_{1},q_1}} (x-x^2), \\
\delta_m^2 &= \frac{p_2^{m-1}}{|m|_{p_{2},q_2}} (y-y^2).
\end{align*}
\]

Proof. By definition partial moduli of continuity of \( f(x,y) \) and applying Cauchy-Schwartz inequality, we have
\[
|B_{n,m}^{(p_1,q_1),(p_2,q_2)}(f;x,y) - f(x,y)| \leq B_{n,m}^{(p_1,q_1),(p_2,q_2)}(|f(t,s) - f(x,y)|;x,y)
\leq B_{n,m}^{(p_1,q_1),(p_2,q_2)}(|f(t,s) - f(x,y)|;x,y) + B_{n,m}^{(p_1,q_1),(p_2,q_2)}(|f(x,y) - f(x,y)|;x,y)
\leq B_{n,m}^{(p_1,q_1),(p_2,q_2)}(\omega^1(f;|t-x|);x,y) + B_{n,m}^{(p_1,q_1),(p_2,q_2)}(\omega^2(f;|s-y|);x,y)
\leq \omega^1(f,\delta_n) \left[ 1 + \frac{1}{\delta_n} B_{n,m}^{(p_1,q_1),(p_2,q_2)} \left( |t-x|; x,y \right) \right]
+ \omega^2(f,\delta_m) \left[ 1 + \frac{1}{\delta_m} B_{n,m}^{(p_1,q_1),(p_2,q_2)} \left( |s-y|; x,y \right) \right]
\leq \omega^1(f,\delta_n) \left[ 1 + \frac{1}{\delta_n} \left( B_{n,m}^{(p_1,q_1),(p_2,q_2)} \left( (t-x)^2; x,y \right) \right)^{1/2} \right]
+ \omega^2(f,\delta_m) \left[ 1 + \frac{1}{\delta_m} \left( B_{n,m}^{(p_1,q_1),(p_2,q_2)} \left( (s-y)^2; x,y \right) \right)^{1/2} \right].
\]
Consider (2.3), (2.4) and choosing
\[
\begin{align*}
\delta_n^2 &= \frac{p_1^{n-1}}{|n|_{p_{1},q_1}} (x-x^2), \\
\delta_m^2 &= \frac{p_2^{m-1}}{|m|_{p_{2},q_2}} (y-y^2).
\end{align*}
\]
we reach the result. \(\square\)

For \( \alpha_1, \alpha_1 \in (0,1) \) and \((s,t),(x,y) \in [0,1]^2\), we define the Lipschitz class \( LipM(\alpha_1,\alpha_1) \) for the bivariate case as follows:
\[
|f(s,t) - f(x,y)| \leq M |s - x|^{\alpha_1} |t - y|^{\alpha_2}.
\]
Theorem 3.3. Let $f \in \text{Lip}_M(\alpha_1, \alpha_2)$. Then, for all $(x, y) \in [0, 1]^2$, we have

$$|B_{n,m}^{(p_1,q_1),(p_2,q_2)}(f; x, y) - f(x, y)| \leq M\delta_n^{\alpha_1/2}\delta_m^{\alpha_2/2},$$

where $\delta_n$ and $\delta_m$ defined in (3.1) and (3.2), respectively.

Proof. As $f \in \text{Lip}_M(\alpha_1, \alpha_2)$, it follows

$$|B_{n,m}^{(p_1,q_1),(p_2,q_2)}(f; x, y) - f(x, y)| \leq B_{n,m}^{(p_1,q_1),(p_2,q_2)}(|f(t, s) - f(x, y)| ; q_n ; x, y)
\leq M_{n,m}^{(p_1,q_1),(p_2,q_2)}(|t - x|^\alpha_1|s - y|^\alpha_2 ; x, y)
= M_{n,m}^{(p_1,q_1),(p_2,q_2)}(|t - x|^\alpha_1 ; x)B_{n,m}^{(p_1,q_1),(p_2,q_2)}(|s - y|^\alpha_2 ; y).

For $\hat{p} = \frac{1}{\alpha_1}$, $\hat{q} = \frac{\alpha_1}{2 - \alpha_1}$ and $\hat{p} = \frac{1}{\alpha_2}$, $\hat{q} = \frac{\alpha_2}{2 - \alpha_2}$ applying the Hölder’s inequality, we get

$$|B_{n,m}^{(p_1,q_1),(p_2,q_2)}(f; x, y) - f(x, y)| \leq M\{B_{n,m}^{(p_1,q_1),(p_2,q_2)}(|t - x|^2 ; x)\}^{\alpha_1/2}\{B_{n,m}^{(p_1,q_1),(p_2,q_2)}(1 ; x)\}^{\alpha_1/2}
\times\{B_{n,m}^{(p_1,q_1),(p_2,q_2)}(|s - y|^2 ; y)\}^{\alpha_2/2}\{B_{n,m}^{(p_1,q_1),(p_2,q_2)}(1 ; y)\}^{\alpha_2/2}
= M\delta_n^{\alpha_1/2}\delta_m^{\alpha_2/2}.

Hence, we get desired the result. □

Theorem 3.4. Let $f \in C^1([0, 1]^2)$ and $0 < q_1, q_2, m < p_1, p_2, 1$. Then, we have

$$|B_{n,m}^{(p_1,q_1),(p_2,q_2)}(f; x, y) - f(x, y)| \leq \|f'_x\|_{C([0,1]^2)}\delta_n + \|f'_y\|_{C([0,1]^2)}\delta_m.

Proof. For $(t, s) \in [0, 1]^2$, we obtain

$$f(t) - f(s) = \int^t_s f'(u)du + \int^s_t f'(v)du$$

Applying the our operator on both sides above equation, we deduce

$$|B_{n,m}^{(p_1,q_1),(p_2,q_2)}(f; x, y) - f(x, y)| \leq B_{n,m}^{(p_1,q_1),(p_2,q_2)}\left(\int^t_s f'_u(u)du; x, y\right)
+ B_{n,m}^{(p_1,q_1),(p_2,q_2)}\left(\int^s_t f'_v(v)du; x, y\right).

As

$$\left|\int^t_s f'_u(u)du\right| \leq \|f'_x\|_{C([0,1]^2)}|t - x| \text{ and } \left|\int^s_t f'_v(v)du\right| \leq \|f'_y\|_{C([0,1]^2)}|s - y|,$

we have

$$|B_{n,m}^{(p_1,q_1),(p_2,q_2)}(f; x, y) - f(x, y)| \leq \|f'_x\|_{C([0,1]^2)}B_{n,m}^{(p_1,q_1),(p_2,q_2)}(|t - x|; x, y)
+ \|f'_y\|_{C([0,1]^2)}B_{n,m}^{(p_1,q_1),(p_2,q_2)}(|s - y|; x, y).

Using the Cauchy-Schwarz inequality, we can write following

$$|B_{n,m}^{(p_1,q_1),(p_2,q_2)}(f; x, y) - f(x, y)| \leq \|f'_x\|_{C([0,1]^2)}\left\{B_{n,m}^{(p_1,q_1),(p_2,q_2)}((t - x)^2; x, y)\right\}^{1/2}\left\{B_{n,m}^{(p_1,q_1),(p_2,q_2)}(1; x, y)\right\}^{1/2}
+ \|f'_y\|_{C([0,1]^2)}\left\{B_{n,m}^{(p_1,q_1),(p_2,q_2)}((s - y)^2; x, y)\right\}^{1/2}\left\{B_{n,m}^{(p_1,q_1),(p_2,q_2)}(1; x, y)\right\}^{1/2}.

From (3.3) and (3.4), we get desired the result. □

Theorem 3.5. Let $f \in C([0, 1]^2)$, then we have

$$\left\|B_{n,m}^{(p_1,q_1),(p_2,q_2)}(f; x, y) - f(x, y)\right\|_{C([0,1]^2)} \leq 2M(\delta_n, m)(x/y)/2,$$

where

$$\delta_n, m(x, y) = \frac{1}{2}\max\left(\frac{p_1^{-1}(x - x^2)}{|n|p_1,q_1}, \frac{p_2^{-1}(y - y^2)}{|m|p_2,q_2}\right).$$
Proof. Let \( g \in C^2([0,1]^2) \). By the Taylor’s formula, we get

\[
g(s_1, s_2) - g(x, y) = g(s_1, y) - g(x, y) + g(s_1, s_2) - g(s_1, y) = g(s_1, y) - g(x, y) + g(s_1, s_2) - g(s_1, y) + \int_{x}^{s_1} (s_1 - u) \frac{\partial^2 g(u, y)}{\partial u^2} du + \frac{\partial g(x, y)}{\partial x} (s_1 - y) + \int_{y}^{s_2} (s_2 - v) \frac{\partial^2 g(x, v)}{\partial v^2} dv
\]

Applying \( B_{n,m}^{(p_1,q_1), (p_2,q_2)} \) to both sides of the above equation, we obtain

\[
\left| B_{n,m}^{(p_1,q_1), (p_2,q_2)} (g; x, y) - g(x, y) \right| \leq \left| \frac{\partial g(x, y)}{\partial x} \right| \left| B_{n,m}^{(p_1,q_1), (p_2,q_2)} ((s_1 - x); x, y) \right| + \left| B_{n,m}^{(p_1,q_1), (p_2,q_2)} \left( \int_{0}^{s_1-x} (s_1 - x - u) \frac{\partial^2 g(u, y)}{\partial u^2} du; x, y \right) \right| + \left| \frac{\partial g(x, y)}{\partial y} \right| \left| B_{n,m}^{(p_1,q_1), (p_2,q_2)} ((s_2 - y); x, y) \right| + \left| B_{n,m}^{(p_1,q_1), (p_2,q_2)} \left( \int_{0}^{s_2-y} (s_2 - y - v) \frac{\partial^2 g(x, v)}{\partial v^2} dv; x, y \right) \right|
\]

As \( B_{n,m}^{(p_1,q_1), (p_2,q_2)} ((s_1 - x); x, y) = 0 \) and \( B_{n,m}^{(p_1,q_1), (p_2,q_2)} ((s_2 - y); x, y) = 0 \), one can write following

\[
\left| B_{n,m}^{(p_1,q_1), (p_2,q_2)} (f; x, y) - f(x, y) \right|_{C([0,1]^2)} \leq \frac{1}{2} \left| \frac{\partial g(x, y)}{\partial x} \right|_{C([0,1]^2)} \left| B_{n,m}^{(p_1,q_1), (p_2,q_2)} ((s_1 - x)^2; x, y) \right| + \frac{1}{2} \left| \frac{\partial g(x, y)}{\partial y} \right|_{C([0,1]^2)} \left| B_{n,m}^{(p_1,q_1), (p_2,q_2)} ((s_2 - y)^2; x, y) \right|.
\]

By (2.3), (2.4), we deduce,

\[
\left| B_{n,m}^{(p_1,q_1), (p_2,q_2)} (f; x, y) - f(x, y) \right|_{C([0,1]^2)} \leq \frac{1}{2} \max \left( \frac{p_1^{n-1}(x - x^2)}{p_1^{n}}, \frac{p_2^{n-1}(y - y^2)}{p_2^{n}} \right) \times \left[ \left| \frac{\partial g(x, y)}{\partial x} \right|_{C([0,1]^2)} + \left| \frac{\partial g(x, y)}{\partial y} \right|_{C([0,1]^2)} \right]
\]

(3.3)

By the linearity \( B_{n,m}^{(p_1,q_1), (p_2,q_2)} \), we obtain

\[
\left| B_{n,m}^{(p_1,q_1), (p_2,q_2)} (f; x, y) - f(x, y) \right|_{C([0,1]^2)} \leq \left| B_{n,m}^{(p_1,q_1), (p_2,q_2)} f - B_{n,m}^{(p_1,q_1), (p_2,q_2)} g \right|_{C([0,1]^2)} + \left| B_{n,m}^{(p_1,q_1), (p_2,q_2)} g - g \right|_{C([0,1]^2)} + \left| f - g \right|_{C([0,1]^2)}.
\]

(3.4)

By (3.3) and (3.6), one can see that

\[
\left| B_{n,m}^{(p_1,q_1), (p_2,q_2)} (f; x, y) - f(x, y) \right|_{C([0,1]^2)} \leq 2M (f; \delta_{n,m}(x,y)/2).
\]

This step completes the proof.

\[
\text{Lemma 3.6. Let } 0 < q_n < p_n \leq 1, \text{ be sequence such that } p_n, q_n \rightarrow 1 \text{ and } p_n^a \rightarrow a, q_n^b \rightarrow b \text{ as } n \rightarrow \infty. \text{ Then, we have the following limits:}
\]

(i) \( \lim_{n \rightarrow \infty} [n]q_n B_n^{(p_n,q_n)} ((t-x)^2; x) = ax - ax^2 \)

(ii) \( \lim_{n \rightarrow \infty} [n]^2 q_n B_n^{(p_n,q_n)} ((t-x)^4; x) = 3ax^4 - 6ax^3 + 3ax^2. \)

Proof. From (2.3), we get

\[
[n]q_n B_n^{(p_n,q_n)} ((t-x)^2; x) = -p_n^{n-1}x^2 + xp_n^{n-1}.
\]
Let us take the limit of both sides of the above equality as \( n \to \infty \), then we have
\[
\lim_{n \to \infty} [n]_{p_n,q_n} \left\{ B^{(p_n,q_n)}((t-x)^2, x) \right\} = \lim_{n \to \infty} \left\{ -p_n^{n-1}x^2 + xp_n^{n-1} \right\} = a(x - x^2).
\]

(ii) Using Lemma \[2.1\] and by the linearity of the operators \( B^{(p_n,q_n)}(f; x) \), we obtain
\[
B^{(p_n,q_n)}((t-x)^4; x) = A_{1,n}x^4 + A_{2,n}x^3 + A_{3,n}x^2 + A_{4,n}x
\]
where
\[
A_{1,n} = \frac{p_n^{n-3}[n]_{p_n,q_n}^2}{[n]_{q_n}^2} \left( -p_n^2 + 2p_nq_n - q_n^2 \right) + p_n^{n-5}[n]_{p_n,q_n} \left( -p_n^3 + 3p_nq_n^2 + q_n^3 \right) - p_n^{3n-6}(p_n^2 + p_n^3 + 2p_nq_n^2 + q_n^3)
\]
\[
A_{2,n} = \frac{p_n^{n-3}[n]_{p_n,q_n}^2}{[n]_{q_n}^2} \left( p_n^2 - 2p_nq_n + q_n^2 \right) + p_n^{2n-5}[n]_{p_n,q_n} \left( -q_n^3 - 4p_nq_n^2 - 3p_n^2q_n + 2p_n^3 \right) - p_n^{3n-6}(3p_n^3 + 3p_nq_n^2 + 5p_nq_n) + [n]_{q_n}^3
\]
\[
A_{3,n} = \frac{p_n^{2n-4}[n]_{p_n,q_n}^2}{[n]_{q_n}^2} \left( -p_n^2 + 3p_nq_n + q_n^2 \right) - p_n^{3n-5}(3p_n^2 + q_n^2 + 3p_nq_n)
\]
\[
A_{4,n} = \frac{p_n^{3n-3}}{[n]_{q_n}^3}.
\]

It is clear that
\[
\lim_{n \to \infty} [n]_{q_n}^2 \{ A_{4,n} x \} = 0.
\]

Taking the limit of both sides of \( A_{1,n} \), we get
\[
\lim_{n \to \infty} [n]_{q_n}^2 \{ A_{1,n} \} = \lim_{n \to \infty} \left\{ -p_n^{n-3}[n]_{p_n,q_n} (p_n - q_n)^2 + p_n^{n-5}(-p_n^3 + 3p_nq_n^2 + q_n^3) - p_n^{3n-6}(p_n^2 + p_n^3 + 2p_nq_n^2 + q_n^3) \right\} / [n]_{q_n}^2
\]
\[
= \lim_{n \to \infty} \left\{ -p_p^{n-3}(p_n - q_n)(p_n - q_n) + p_n^{n-5}(-p_n^3 + 3p_nq_n^2 + q_n^3) - p_n^{3n-6}(p_n^2 + p_n^3 + 2p_nq_n^2 + q_n^3) \right\} / [n]_{q_n}^2
\]
\[
= 3a.
\]

Similarly, we can show that;
\[
\lim_{n \to \infty} [n]_{q_n}^2 \{ A_{2,n} \} = -6a \quad \text{and} \quad \lim_{n \to \infty} [n]_{q_n}^2 \{ A_{3,n} \} = 3a.
\]

By combining (3.5), (3.6), (3.7), we reach the desired result. \( \square \)

Now, we give a Voronovskaja type theorem for \( B^{(p_n,q_n)}(f; x, y) \).

**Theorem 3.7.** Let \( f \in C^2([0, 1] \times [0, 1]) \). Then, we have
\[
\lim_{n \to \infty} [n]_{p_n,q_n} B^{(p_n,q_n)}(f; x, y) = \frac{(ax - ax^2)f''_x(x, y)}{2} + \frac{(ay - ay^2)f''_y(x, y)}{2}.
\]

**Proof.** Let \((x, y) \in [0, 1] \times [0, 1]\). Then, write Taylor’s expansion of \( f \) as follows:
\[
f(s, t) = f(x, y) + f'_x(s - x) + f'_y(t - y)
\]
\[
+ \frac{1}{2} f''_x(t - x)^2 + 2f''_y(s - x)(t - y) + f''_y(t - y)^2 + \varepsilon(s, t) ((s - x)^2 + (t - y)^2)
\]
where \((s, t) \in [0, 1]^2\) and \( \varepsilon(s, t) \to 0 \) as \( (s, t) \to (x, y) \).

Applying the operator \( B^{(p_n,q_n)}(f; x, y) \) on (3.8), we get
\[
B^{(p_n,q_n)}(f; s, t) - f(x, y) = f'_x(x, y)B^{(p_n,q_n)}(s - x; x, y) + f'_y(x, y)B^{(p_n,q_n)}(t - y; x, y)
\]
\[
+ \frac{1}{2} \left\{ f''_x B^{(p_n,q_n)}((t - x)^2; x, y) + 2f''_y B^{(p_n,q_n)}((s - x)(t - y); x, y) + f''_y B^{(p_n,q_n)}((t - y)^2; x, y) \right\} + B^{(p_n,q_n)}(\varepsilon(s, t) ((s - x)^2 + (t - y)^2); x, y).
\]

Let us take the limit of both sides of the above equality as \( n \to \infty \), \( \square \)
Hence, one can see that

\[
\lim_{n \to \infty} [n]_{p_n,q_n} B_{n,n}^{(p_n,q_n)}(f; s,t) - f(x,y) = \lim_{n \to \infty} [n]_{p_n,q_n} \frac{1}{2} \left\{ f_{xy}'' B_{n,n}^{(p_n,q_n)}((t-x)^2;x,y) + 2f_{xy} B_{n,n}^{(p_n,q_n)}((s-x)(t-y);x,y) + f_{y}'' B_{n,n}^{(p_n,q_n)}((t-y)^2;x,y) \right\} + \lim_{n \to \infty} [n]_{p_n,q_n} B_{n,n}^{(p_n,q_n)}(\varepsilon(s,t) ((s-x)^2 + (t-y)^2);x,y).
\]

For the last term on the right hand side, using Cauchy-Schwartz inequality, we get

\[
B_{n,n}^{(p_n,q_n)}(\varepsilon(s,t) ((s-x)^2 + (t-y)^2);x,y) \leq \sqrt{\lim_{n \to \infty} B_{n,n}^{(p_n,q_n)}(\varepsilon^2(s,t);x,y)} \times \sqrt{2 \lim_{n \to \infty} [n]_{p_n,q_n}^2 B_{n,n}^{(p_n,q_n)}(\varepsilon(s,t) ((s-x)^4 + (t-y)^4);x,y)}.
\]

As \( \lim_{n \to \infty} B_{n,n}^{(p_n,q_n)}(\varepsilon^2(s,t);x,y) = \varepsilon^2(x,y) = 0 \) and using Lemma 3.6(ii),

\[
\lim_{n \to \infty} [n]_{p_n,q_n}^2 B_{n,n}^{(p_n,q_n)}((s-x)^4 + (t-y)^4);x,y) \text{ is finite, then we obtain}
\]

\[
\lim_{n \to \infty} [n]_{p_n,q_n}^2 B_{n,n}^{(p_n,q_n)}(\varepsilon(s,t) ((s-x)^4 + (t-y)^4);x,y) = 0.
\]

Hence, one can see that

\[
\lim_{n \to \infty} [n]_{p_n,q_n} B_{n,n}^{(p_n,q_n)}(f; x,y) - f(x,y) = \frac{(ax - ax^2)f''_{x}(x,y)}{2} + \frac{(ay - ay^2)f''_{y}(x,y)}{2}.
\]

This step completes the proof. \(\square\)

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