Abstract: Consider a multi-dimensional Brownian motion which models different lines of business of an insurance company. Our main result gives an approximation for the cumulative Parisian ruin probability as the initial capital becomes large. An approximation for the conditional cumulative Parisian ruin time is also derived. As a particular interesting case, the two-dimensional Brownian motion models are discussed in detail. Our results suggest that the company should not merge its two lines of business if we consider the cumulative Parisian ruin probability as a measure of risk. This provides an evidence in supporting the principle of portfolio diversification.

Key Words: multi-dimensional Brownian motion; cumulative Parisian ruin; exact asymptotics; ruin probability; quadratic programming problem.

AMS Classification: 91B30, 60G15, 60G70

1. Introduction

Consider an insurance company which operates simultaneously $d$ ($d \geq 1$) lines of business. It is assumed that the surplus of these lines of business is described by a multi-dimensional risk model:

\begin{equation}
U(t) = u + \mu t - X(t), \quad t \geq 0,
\end{equation}

where $u = (u_1, u_2, \ldots, u_d)^\top$, with $u_i \geq 0$, is a (column) vector of initial capitals for the $i$th business line, $\mu = (\mu_1, \ldots, \mu_d)^\top$, with $\mu_i > 0$, is a vector of net premium income rate, and $X(t) = (X_1(t), X_2(t), \ldots, X_d(t))^\top, t \geq 0$ is a vector of total claim amount processes by time $t$.

In recent years, there has been an increased interest in risk theory in the study of multi-dimensional risk models with different stochastic processes modeling $X(t), t \geq 0$; see, e.g., [1] for an overview. In the literature, there have been mainly two directions of investigation on this topic, namely, ruin probabilities and optimal dividend problem. We refer to [2–13] and references therein for ruin probabilities related studies, and [14–20] and references therein for studies on optimal dividend problem. In comparison with the well-understood 1-dimensional risk models, study of multi-dimensional risk models is more challenging.

More recently, two-dimensional Brownian motion models have drawn a lot of attention due to its tractability and practical relevancy. The optimal dividend problems for the two-dimensional Brownian motion models have been discussed in [14, 18, 19]. The ruin probabilities (which are actually exit problems) for multi-dimensional Brownian motion models have been discussed under different contexts; see [13, 21–27] and the references therein.
We consider in this paper the multi-dimensional Brownian motion model, where

\[ X(t) = AB(t), \quad t \geq 0 \]

denotes the approximated total claim amount process by time \( t \). Here \( A \in \mathbb{R}^{d \times d} \) is a non-singular matrix, and \( B(t) = (B_1(t), \ldots, B_d(t))^\top, t \geq 0 \) is a standard \( d \)-dimensional Brownian motion with independent coordinates.

We shall investigate the cumulative Parisian ruin problem of the model (1) with (2). The cumulative Parisian ruin was introduced by [28] based on the occupation (or sojourn) times of the surplus process and is due to its ties with the cumulative Parisian options; see, e.g., [29]. In our multi-dimensional setup the cumulative Parisian ruin time (at level \( r > 0 \)) is defined as

\[
\tau_r(u) := \inf \left\{ t > 0 : \int_0^t \mathbb{I}(U(s) < 0) ds > r \right\},
\]

where \( \mathbb{I}(\cdot) \) is the indicator function, and the inequality for vectors \( U(s) < 0 \) is meant component-wise. As remarked in [28] "the parameter \( r \) could be interpreted as the length of a clock started at the beginning of the first excursion, paused when the process returns above zero, and resumed at the beginning of the next excursion, and so on.". Clearly, if \( r \) is set to be 0 one obtains the simultaneous ruin time \( \tau_0(u) \) for our multi-dimensional risk model, i.e.,

\[
\tau_0(u) := \inf \{ t > 0 : U(t) < 0 \} = \inf \{ t > 0 : U_i(t) < 0, \ \forall 1 \leq i \leq d \}.
\]

We are interested in the calculation of infinite-time cumulative Parisian ruin probability, i.e.,

\[
P\{\tau_r(u) < \infty\}.
\]

In the 1-dimensional setup, we have from (5) in [30] (see also [28]) that

\[
P\{\tau_r(u) < \infty\} = P \left\{ \int_0^\infty \mathbb{I}(B_1(t) - \mu_1 t > u) dt > r \right\} = \left( 2(1 + \mu_1^2 r) \Psi(\mu_1 \sqrt{r}) - \frac{\mu_1 \sqrt{2\pi} e^{-\mu_1^2 r/2}}{\sqrt{r}} \right) e^{-2\mu_1 u}
\]

holds for all \( u \in \mathbb{R} \), where \( \Psi(s) \) is the standard normal survival function.

It turns out that explicit formula for the cumulative Parisian ruin probability in the multi-dimensional setup is very difficult to obtain. In this case, it is of interest to derive some asymptotic results letting the initial capitals to become large, by resorting to the extreme value theory; see, e.g., [1, 31, 32]. As explained in [32] "... the consideration of large initial capitals is not just a mathematical assumption but also an economic necessity, which is reinforced by the supervisory authorities. ...". To this end, we make the following conventional assumption:

\[
u = \alpha u = (\alpha_1 u, \alpha_2 u, \ldots, \alpha_d u), \quad \alpha_i > 0, \ u \geq 0
\]

For simplicity, hereafter we denote

\[
\tau_r(u) := \tau_r(\alpha u), \quad u \geq 0.
\]

Define the following function

\[
g(t) = \frac{1}{t} \inf_{v \geq \alpha + \mu t} v^\top \Sigma^{-1} v, \quad t \geq 0, \quad \text{with} \ \Sigma = AA^\top,
\]
ON THE CUMULATIVE PARISIAN RUIN OF MULTI-DIMENSIONAL BROWNIAN MOTION MODELS

where \(1/0\) is understood as \(\infty\). Our principal result presented in Theorem 3.1 shows that, for any \(r > 0\),

\[
\mathbb{P}\{\tau_r(u) < \infty\} = \mathbb{P}\left\{\int_0^\infty \mathbb{1}_{\{X(t) - \mu t > \alpha u\}} dt > r\right\}
\sim C_I \mathcal{H}_I(r) u^{\frac{1-m}{2}} e^{-\frac{\inf_{t \geq 0} g(t)}{2} u^2}, \quad u \to \infty,
\]

(7)

where \(C_I > 0, m \in \mathbb{N}\) are known constants and \(\mathcal{H}_I(r)\) is a counterpart of the celebrated Pickands constant; explicit expressions of these constants will be displayed in Section 3.

As in [27], where the case \(r = 0\) is discussed, we shall prove (7) using the celebrated double-sum method combined with the theory of quadratic programming problem. One of the difficulties for our proof comes from the fact that the Bonferroni’s inequality adopted in [27] for the supremum functional cannot be used now, for which new inequalities for sojourn-type probabilities are proposed.

With motivation from [14, 16, 20], it is also of interest to see if merger of two lines of business will benefit the company. Our results for the two-dimensional Brownian motion models discussed in Section 4 suggest that when we consider the cumulative Parisian ruin probability as a measure of risk, it is better to keep operating two lines of business, as merger of two lines of business will make the cumulative Parisian ruin probability larger. This provides an evidence in supporting the principle of portfolio diversification.

As a by-product, we derive in Theorem 3.2 the asymptotic distribution of

\[
\tau_{r_2}(u)|\tau_{r_1}(u) < \infty, \quad u \to \infty
\]

for any \(0 \leq r_1 \leq r_2 < \infty\). The approximation of the above quantity is of interest in risk theory; it will give us some idea of when cumulative Parisian ruin actually occurred at level \(r_2\) knowing that it has occurred at some level \(r_1\). We refer to [1, 31, 33, 34] and references therein for related discussions on different ruin times.

It is worth mentioning that there are some related interesting studies on the asymptotic properties of sojourn times above a high level of 1-dimensional (real-valued) stochastic processes; see, e.g., [35–37]. We refer to [30, 38] for recent developments. The multi-dimensional counterparts of this problem are more challenging, and to the best knowledge of the author there has been no result on this direction. Our study on the cumulative Parisian ruin probability for the multi-dimensional Brownian motion models covers this gap in a sense by giving some asymptotic properties of the sojourn times.

The rest of this paper is organised as follows. In Section 2 we introduce some notation and present some preliminary results, which are extracted from [27]. The main results are presented in Section 3, followed by a detailed discussion on the two-dimensional Brownian motion models. The technical proofs are displayed in Section 5 and Appendix.

2. NOTATION AND PRELIMINARIES

We assume that all vectors are \(d\)-dimensional column vectors written in bold letters with \(d \geq 2\). Operations with vectors are meant component-wise, e.g., \(\lambda x = x \lambda = (\lambda x_1, \ldots, \lambda x_d)^T\) for any \(\lambda \in \mathbb{R}, x \in \mathbb{R}^d\). Further, we denote

\[
0 = (0, \ldots, 0)^T \in \mathbb{R}^d, \quad 1 = (1, \ldots, 1)^T \in \mathbb{R}^d.
\]
If $I \subset \{1, \ldots, d\}$, then for a vector $a \in \mathbb{R}^d$ we denote by $a_I = (a_i, i \in I)$ a sub-block vector of $a$. Similarly, if further $J \subset \{1, \ldots, d\}$, for a matrix $M = (m_{ij})_{i,j \in \{1, \ldots, d\}} \in \mathbb{R}^{d \times d}$ we denote by $M_{I,J} = (m_{ij})_{i \in I, j \in J}$ the sub-block matrix of $M$ determined by $I$ and $J$. Moreover, write $M_{II}^{-1} = (M_{II})^{-1}$ for the inverse matrix of $M_{II}$ whenever it exists.

As a quadratic programming problem is involved in our discussion (see (6)). We introduce the next lemma stated in [39] (see also [27]), which is important for several definitions in the sequel.

**Lemma 2.1.** Let $M \in \mathbb{R}^{d \times d}, d \geq 2$ be a positive definite matrix. If $b \in \mathbb{R}^d \setminus (-\infty, 0]^d$, then the quadratic programming problem

$$
P_M(b) : \text{Minimise } x^\top M^{-1}x \text{ under the linear constraint } x \geq b$$

has a unique solution $\tilde{b}$ and there exists a unique non-empty index set $I \subseteq \{1, \ldots, d\}$ such that

$$\tilde{b}_I = b_I \neq 0_I, \quad M_{II}^{-1}b_I > 0_I,$$

and if $I^c = \{1, \ldots, d\} \setminus I \neq \emptyset$, then $\tilde{b}_{I^c} = M_{I^c,I}M_{II}^{-1}b_I \geq b_{I^c}$.

Furthermore,

$$\min_{x \geq b} x^\top M^{-1}x = \tilde{b}^\top M^{-1}\tilde{b} = b_I^\top M_{II}^{-1}b_I > 0,$$

$$x^\top M^{-1}\tilde{b} = x_I^\top M_{II}^{-1}b_I = x_{I^c}^\top M_{II}^{-1}b_I, \quad \forall x \in \mathbb{R}^d.$$ 

**Definition 2.2.** The unique index set $I$ that defines the solution of the quadratic programming problem in question will be referred to as the essential index set.

For any fixed $t \geq 0$, let $I(t) \subseteq \{1, \ldots, d\}$ be the essential index set of the quadratic programming problem $P_{\Sigma}(b(t))$ with

$$b(t) = \alpha + t\mu,$$

and denote

$$I(t)^c := \{1, \ldots, d\} \setminus I(t).$$

We present next a crucial result concerning the function $g$ defined in (6); the proof of it can be found in [27].

**Lemma 2.3.** We have $g \in C^1(0, \infty)$. Furthermore, $g$ is convex and it achieves its unique minimum at

$$t_0 = \sqrt{\frac{\Sigma_{II}^{-1} \alpha_I}{\mu_I \Sigma_{II}^{-1} \mu_I}} > 0,$$

which is given by

$$g(t_0) = \inf_{t \geq 0} g(t) = \frac{1}{t_0} \frac{b_I^\top \Sigma_{II}^{-1} b_I}{\mu_I \Sigma_{II}^{-1} \mu_I},$$

with

$$b = b(t_0) = \alpha + t_0 \mu.$$
and $I = I(t_0)$ being the essential index set corresponding to $P_{\Sigma}(b)$. Moreover,

$$g(t_0 \pm t) = g(t_0) + \frac{g''(t_0 \pm t)}{2} t^2 (1 + o(1)), \quad t \downarrow 0. \tag{12}$$

Hereafter, we shall use the notation $b = b(t_0)$, and $I = I(t_0)$ for the essential index set of the quadratic programming problem $P_{\Sigma}(b)$. Furthermore, let $\tilde{b}$ be the unique solution of $P_{\Sigma}(b)$. If $I^c = \{1, \ldots, d\} \setminus I \neq \emptyset$, we define (refer to (9))

$$K = \{j \in I^c : \tilde{b}_j = \Sigma_{ij}^{-1} \alpha_I b_I = b_j\}, \tag{13}$$

which will play certain roles in the asymptotic results. Next, define for $t > 0$

$$g_I(t) := \frac{1}{t} b(t) \Sigma_{II}^{-1} b(t) = \frac{1}{t} \alpha_I \Sigma_{II}^{-1} \alpha_I + 2 \alpha_I \Sigma_{II}^{-1} \mu_I + \mu_I \Sigma_{II}^{-1} \mu_I t. \tag{14}$$

Clearly, by Lemma 2.1 and Lemma 2.3 we have

$$\tilde{g} := g(t_0) = g_I(t_0).$$

Furthermore, we have by Taylor expansion

$$g_I(t_0 + t) = \tilde{g} + \frac{\tilde{g}''(t_0)}{2} t^2 (1 + o(1)), \quad t \to 0,$$

with

$$\tilde{g} := g''_I(t_0) = 2t_0^{-3}(\alpha_I \Sigma_{II}^{-1} \alpha_I).$$

3. Main Results

We first introduce some constants that will appear in the main results. First we write

$$m = \sharp I := \sharp\{i : i \in I\} \geq 1$$

for the number of elements of the essential index set $I$. Further, define the following constant (existence is confirmed in Theorem 3.1)

$$H_I(r) = \lim_{T \to \infty} \frac{1}{T} \mathcal{H}_I(r, T), \tag{15}$$

with

$$\mathcal{H}_I(r, T) = \int_{\mathbb{R}^m} e^{-\frac{1}{2} \Sigma_{II}^{-1} b_I} \mathbb{P} \left\{ \int_{t \in [0, T]} I((X(t) - \mu_I) > x_I) dt > r \right\} \, dx_I, \quad r < T.$$

Moreover, set

$$C_I := \frac{1}{\sqrt{(2\pi t_0)^m |\Sigma_{II}|}} \int_{\mathbb{R}} e^{-\frac{1}{2} \frac{x^2}{t_0}} \psi(x) \, dx,$$

where $|\Sigma_{II}|$ denotes the determinant of the matrix $\Sigma_{II}$, and for $x \in \mathbb{R}$

$$\psi(x) = \begin{cases} 
1, & \text{if } K = \emptyset \\
\mathbb{P} \left\{ Y_K > \frac{1}{\sqrt{t_0}} (\mu_K - \Sigma_{II}^{-1} \mu_I) x \right\}, & \text{if } K \neq \emptyset.
\end{cases} \tag{16}$$
Here the non-empty index set $K$ is defined in (13), $Y_K$ is a normally distributed random vector with mean vector $0_K$ and covariance matrix $D_{KK}$ given by

$$D_{KK} = \Sigma_{KK} - \Sigma_{KI} \Sigma_{II}^{-1} \Sigma_{IK}.$$ 

The next theorem constitutes our principal result. Its proof is demonstrated in Section 5.

**Theorem 3.1.** We have, for any $r \geq 0$,

$$\mathbb{P}\{\tau_r(u) < \infty\} \sim C_I \mathcal{H}_I(r) u^{\frac{1-m}{2}} e^{-\frac{\tilde{g}}{2} u}, \quad u \to \infty,$$

where

$$0 < \mathcal{H}_I(r) < \infty, \quad \forall r \geq 0.$$ 

Our next result gives an asymptotic distribution for the conditional cumulative Parisian ruin time.

**Theorem 3.2.** Let $\tau_r(u)$ be defined in (3) and (5), and let the function $\psi$ be defined in (16). We have, for any $0 \leq r_1 \leq r_2 < \infty$ and any $s \in \mathbb{R}$,

$$\lim_{u \to \infty} \mathbb{P}\left\{\frac{\tau_{r_2}(u) - t_0 u}{\sqrt{2u/g}} \leq s \bigg| \tau_{r_1}(u) < \infty\right\} = \frac{\mathcal{H}_I(r_2) \int_{-\infty}^{s} e^{-\frac{x^2}{2}} \psi(\sqrt{2/g} x) dx}{\mathcal{H}_I(r_1) \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \psi(\sqrt{2/g} x) dx}.$$ 

**Remarks 3.3.** (a). If $d = 1$, we have from Theorem 3.1 that

$$\mathbb{P}\{\tau_r(u) < \infty\} \sim \frac{1}{\mu_1} \mathcal{H}_{\{1\}}(r)e^{-2\alpha_1 \mu_1 u}, \quad u \to \infty.$$ 

This together with (4) yields that

$$\mathcal{H}_{\{1\}}(r) = \mu_1 \left(2(1 + \mu_1^2 r) \Psi(\mu_1 \sqrt{r}) - \frac{\mu_1 \sqrt{2r}}{\sqrt{\pi}} e^{-\frac{\mu_1^2 r}{4}}\right).$$

(b). As in [27] we can check that the above two results are both valid under weaker conditions on $\alpha$ and $\mu$. That is, if there exists some $1 \leq i \leq d$ such that

$$\alpha_i > 0, \quad \mu_i > 0,$$

then Theorems 3.1 and 3.2 still hold.

4. Two-dimensional Brownian motion models

In this section we focus on the two-dimensional Brownian motion models, in which we can observe how different entries of the covariance matrix yield different scenarios of asymptotic behaviour. Moreover, by comparing the cumulative Parisian ruin probabilities, we can conclude that merger of two lines of business always make the risk larger, which does not benefit the company.

Without loss of generality we assume

$$\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}, \quad \rho \in (-1, 1).$$
Then we have
\[ X(t) = \left( B_1(t), \rho B_1(t) + \sqrt{1-\rho^2} B_2(t) \right)^\top, \quad t \geq 0. \]
Recall that
\[ g(t) = \frac{1}{t} \inf_{\nu \geq \alpha + \mu} \nu^\top \Sigma^{-1} \nu. \]
In order to apply our main results, we must first solve the quadratic programming problem \( P_{\Sigma}(\alpha + \mu t) \) for \( d = 2 \). To this end, we adopt a direct approach, which is different from that in [27]. It follows from Lemma 2.1 that
\begin{itemize}
  \item [(S1).] On the set \( E_1 = \{ t \geq 0 : \rho(\alpha_2 + \mu_2 t) \geq (\alpha_1 + \mu_1 t) \} \), \( g(t) = \frac{1}{t} (\alpha_2 + \mu_2 t)^2 \);
  \item [(S2).] On the set \( E_2 = \{ t \geq 0 : \rho(\alpha_1 + \mu_1 t) \geq (\alpha_2 + \mu_2 t) \} \), \( g(t) = \frac{1}{t} (\alpha_1 + \mu_1 t)^2 \);
  \item [(S3).] On the set \( E_3 = [0, \infty) \setminus (E_1 \cup E_2) \), \( g(t) = g_0(t) \),
\end{itemize}
where
\[ g_0(t) := \frac{1}{t} (\alpha + \mu)^\top \Sigma^{-1} (\alpha + \mu) = \frac{1}{t} \alpha^\top \Sigma^{-1} \alpha + 2\alpha^\top \Sigma^{-1} \mu + \mu^\top \Sigma^{-1} \mu. \]

4.1. Case \( \rho \leq 0 \). Apparently, from (S1) and (S2) we have \( E_1 = E_2 = \emptyset \), and then
\[ I(t) = \{ 1, 2 \}, \quad g(t) = g_0(t), \quad \forall t \geq 0. \]
Applying Theorems 3.1 and 3.2, we obtain the following result:

**Corollary 4.1.** For the two-dimensional Brownian motion models described above, if \( \rho \leq 0 \), then
\[ \mathbb{P}\{ \tau_1(u) < \infty \} \sim \frac{\mathcal{H}_{(1,2)}(r)}{\sqrt{t_0^2 \pi (1 - \rho^2) g}} u^{-\frac{1}{2}} e^{-\frac{1}{2} u}, \quad u \to \infty, \]
where
\[ t_0 = \left( \frac{\alpha_1^2 + \alpha_2^2 - 2\alpha_1 \alpha_2 \rho}{\mu_1^2 + \mu_2^2 - 2\mu_1 \mu_2 \rho} \right)^{\frac{1}{2}}, \quad \tilde{g} = \frac{2}{t_0^2} (\alpha^\top \Sigma^{-1} \alpha + \alpha^\top \Sigma^{-1} \mu t_0), \quad \tilde{g} = 2t_0^{-3} \frac{\alpha_1^2 + \alpha_2^2 - 2\alpha_1 \alpha_2 \rho}{1 - \rho^2}. \]
Furthermore, for any \( 0 \leq r_1 \leq r_2 < \infty \) and any \( s \in \mathbb{R} \)
\[ \lim_{u \to \infty} \mathbb{P}\left\{ \frac{\tau_{r_2}(u) - t_0 u}{\sqrt{2u/g}} \leq s \left\lfloor \tau_{r_1}(u) < \infty \right\rfloor \right\} = \frac{\mathcal{H}_{(1,2)}(r_2)}{\mathcal{H}_{(1,2)}(r_1)} \Phi(s), \]
where \( \Phi(s) \) is the standard normal distribution function.

Next, if we merge the two lines of business and assume that the merger does not affect the model and its parameters (see [14]), then the merged surplus process is given by
\[ U_0(t) = U_1(t) + U_2(t) = (\alpha_1 + \alpha_2)u + (\mu_1 + \mu_2) t - (X_1(t) + X_2(t)), \quad t \geq 0. \]
Define the cumulative Parisian ruin probability of \( U_0(t), t \geq 0 \) (at level \( r \geq 0 \)) by
\[ P_r(u) := \mathbb{P}\left\{ \int_0^\infty 1_{U_0(t) < 0} dt > r \right\}. \]
We consider the cumulative Parisian ruin probability as a measure of risk for the company. The following result shows that, when \( \rho \leq 0 \), merger of two lines of business always make the risk larger, which does not benefit the company.
Corollary 4.2. Under the conditions of Corollary 4.1, we have
\[ P\{\tau_r(u) < \infty\} < P_r(u) \]
for all large enough \( u \).

Proof. First note that
\[ \{X_1(t) + X_2(t)\}_{t \geq 0} \overset{d}{=} \{\sqrt{2(1 + \rho)}B_1(t)\}_{t \geq 0}, \]
which means that the two stochastic processes have the same finite-dimensional distributions. Then by (4) we have
\[ P_r(u) = \left( 2 + \frac{(\mu_1 + \mu_2)^2}{1 + \rho} \right) \Psi \frac{(\mu_1 + \mu_2)\sqrt{r}}{\sqrt{2(1 + \rho)}} - \frac{\mu_1 + \mu_2}{\sqrt{\pi(1 + \rho)}} e^{-\frac{(\mu_1 + \mu_2)^2}{4(1 + \rho)}} e^{-\frac{(\alpha_1 + \alpha_2)\mu_1 + \mu_2}{1 + \rho}u}. \]  
Comparing the above to (21), we can conclude the claim by showing that
\[ \hat{g}^2 \geq \frac{(\alpha_1 + \alpha_2)(\mu_1 + \mu_2)}{1 + \rho}. \]
In fact, since
\[ \frac{\hat{g}^2}{2} = \sqrt{(\alpha^T\Sigma^{-1}\alpha)(\mu^T\Sigma^{-1}\mu) + \alpha^T\Sigma^{-1}\mu} = \frac{(\alpha_1^2 + \alpha_2^2 - 2\alpha_1\alpha_2\rho)(\mu_1^2 + \mu_2^2 - 2\mu_1\mu_2\rho)}{1 - \rho^2} + \frac{\alpha_1\mu_1 - \alpha_2\mu_2 + \alpha_2\mu_2 - \alpha_1\mu_1\rho}{1 - \rho^2}, \]
we have that (23) is equivalent to
\[ \sqrt{(\alpha_1^2 + \alpha_2^2 - 2\alpha_1\alpha_2\rho)(\mu_1^2 + \mu_2^2 - 2\mu_1\mu_2\rho)} \geq (\alpha_1\mu_2 + \alpha_2\mu_1) - (\alpha_1\mu_1 + \alpha_2\mu_2)\rho, \]
which is further equivalent to
\[ (\alpha_1\mu_1 - \alpha_2\mu_2)^2 \geq \rho^2(\alpha_1\mu_1 - \alpha_2\mu_2)^2. \]
Note that the above is valid for all the possible values of the parameters involved therein. This completes the proof. \( \square \)

4.2. Case \( \rho > 0 \). Below we discuss the case where \( \rho > 0 \), and we aim to see whether merger of two lines of business will still not benefit the company. We could not draw a general conclusion, as now the quadratic programming problem (see (S1)-(S3)) becomes more complex to solve; we have to consider lots of scenarios based on the values of the parameters \( \alpha, \mu, \rho \). But, for the cases considered below, it is indeed that merger of two lines of business will not benefit the company.

First note that in the proof of Corollary 4.2 we have shown that (23) also holds for \( \rho > 0 \). Therefore, we can conclude that if the parameters \( \alpha, \mu, \rho \) can be chosen such that
\[ (\alpha_1\mu_1 - \alpha_2\mu_2)^2 \geq \rho^2(\alpha_1\mu_1 - \alpha_2\mu_2)^2, \]
then (21) in Corollary 4.1 still holds, and thus the same conclusion as in Corollary 4.2 can be drawn. For example, if \( 0 < \rho < \min(\frac{\alpha_1}{\alpha_2}, \frac{\alpha_1}{\alpha_2}, \frac{\mu_1}{\mu_2}) \) then one can show that \( E_1 = E_2 = \emptyset \), and thus (24) is fulfilled.
Hereafter, in order to illustrate our idea we shall discuss an interesting case where

\[(25) \quad \alpha_1 = \alpha_2 = 1, \quad \mu_1 < \mu_2.\]

The discussions below are informative in a sense that it shows a loss of dimension phenomena; see also [27] for more discussions on this phenomena. In principle, using similar arguments one can analyse all the possible cases of the parameters, which due to its complexity will not be included in this paper.

**Corollary 4.3.** Suppose that (25) is fulfilled by the two-dimensional Brownian motion model. For any \( r \geq 0, \) we have:

(i). If \( 0 < \rho < \frac{\mu_1 + \mu_2}{2\mu_2}, \) then as \( u \to \infty \)

\[
\mathbb{P} \{ \tau_r(u) < \infty \} \sim \frac{\mathcal{H}(1,2)(r)}{\sqrt{r_0\pi(1-\rho^2)g}} e^{-\frac{u}{\sqrt{\rho}}},
\]

with

\[
t_0 = \sqrt{\frac{2(1-\rho)}{\mu_1^2 + \mu_2^2 - 2\mu_1\mu_2\rho}}, \quad \tilde{g} = \frac{2}{1+\rho}(\mu_1 + \mu_2 + 2/t_0), \quad \tilde{g} = \frac{4}{1+\rho}t_0^{-3},
\]

\[
\mathcal{H}(1,2)(r) = \lim_{T \to \infty} \frac{1}{T} \int_{\mathbb{R}^2} e^{\frac{\mu_1^2 - \rho^2}{1-\rho^2} t_0(1+\rho)} e^{\frac{\mu_2^2 - \rho^2}{1-\rho^2} t_0(1+\rho)} x_1 + \frac{\mu_3 - \rho^2}{\mu_2^2} e^{\frac{\mu_3 - \rho^2}{\mu_2^2}} x_2 \mathbb{P} \left\{ \int_{t \in [0,T]} \mathbb{I}(X(t)-\mu t > x) dt > r \right\} dx.
\]

Furthermore, for any \( 0 \leq r_1 \leq r_2 < \infty \) and any \( s \in \mathbb{R} \)

\[
\lim_{u \to \infty} \mathbb{P} \left\{ \frac{\tau_{r_2}(u) - t_0 u}{2u/\tilde{g}} \leq s \left| \tau_{r_1}(u) < \infty \right\} = \frac{\mathcal{H}(1,2)(r_2)}{\mathcal{H}(1,2)(r_1)} \Phi(s).
\]

(ii). If \( \rho = \frac{\mu_1 + \mu_2}{2\mu_2} \), then as \( u \to \infty \)

\[
\mathbb{P} \{ \tau_r(u) < \infty \} \sim \frac{\mathcal{H}(1,2)(r)}{\sqrt{2\pi/\mu_2}} \int_{\mathbb{R}} e^{-\frac{u^2}{2\mu_2}} \Psi \left( \frac{\mu_1 - \rho \mu_2}{\sqrt{(1-\rho^2)/\mu_2}} x \right) dx \cdot e^{-2\mu_2 u},
\]

where the explicit expression for \( \mathcal{H}(2)(r) \) is available (cf. (19)). Furthermore, for any \( 0 \leq r_1 \leq r_2 < \infty \) and any \( s \in \mathbb{R} \)

\[
\lim_{u \to \infty} \mathbb{P} \left\{ \frac{\tau_{r_2}(u) - u/\mu_2}{\sqrt{u/\mu_2^2}} \leq s \left| \tau_{r_1}(u) < \infty \right\} = \frac{\mathcal{H}(2)(r_2)}{\mathcal{H}(2)(r_1)} \int_{-\infty}^{s} \Psi \left( \frac{\mu_1 - \rho \mu_2}{\sqrt{(1-\rho^2)/\mu_2}} x \right) dx,
\]

where \( \Psi(s) = 1 - \Phi(s) \) is the standard normal survival function.

(iii). If \( \frac{\mu_1 + \mu_2}{2\mu_2} < \rho < 1 \), then as \( u \to \infty \)

\[
\mathbb{P} \{ \tau_r(u) < \infty \} \sim \mathbb{P} \left\{ \int_{0}^{\infty} \mathbb{I}(B_2(t) - \mu t > u) dt > r \right\} = \frac{1}{\mu_2} \mathcal{H}(2)(r) e^{-2\mu_2 u},
\]

and for any \( 0 \leq r_1 \leq r_2 < \infty \) and any \( s \in \mathbb{R} \)

\[
\lim_{u \to \infty} \mathbb{P} \left\{ \frac{\tau_{r_2}(u) - u/\mu_2}{\sqrt{u/\mu_2^2}} \leq s \left| \tau_{r_1}(u) < \infty \right\} = \frac{\mathcal{H}(2)(r_2)}{\mathcal{H}(2)(r_1)} \Phi(s).
\]
The proof of Corollary 4.3 follows directly from Theorems 3.1 and 3.2, combined with some technical solutions related to the loss of dimension that will be explained in Appendix.

We close this section with a result which also shows that merger of two lines of business will not benefit the company, if we consider the cumulative Parisian ruin probability as a measure of risk for the company.

**Corollary 4.4.** Under the conditions of Corollary 4.3, we have

\[ P\{\tau_r(u) < \infty\} < P_r(u) \]

for all large enough \( u \).

**Proof.** First, since (23) holds for case (i), the claim follows similarly as the proof of Corollary 4.2. For cases (ii) and (iii), it is sufficient to show that (recall (22))

(26) \[ 2\mu_2 > \frac{2(\mu_1 + \mu_2)}{1 + \rho} \]

In fact, for cases (ii) and (iii), we have

\[ \rho \geq \frac{\mu_1 + \mu_2}{2\mu_2} > \frac{\mu_1}{\mu_2} \]

which shows (26), and thus the proof is complete. \( \square \)

5. **Proofs of Main Results**

In this section we present the proofs of Theorems 3.1 and 3.2. We shall focus on the case where \( r > 0 \), since the case with \( r = 0 \) has been included in [27].

In order to convey the main ideas and to reduce complexity of the proof of Theorem 3.1, we shall divide the proof into several steps and then we complete the proof by putting all the arguments together.

By the self-similarity of Brownian motion, for any \( u \) positive we have

\[ P\{\tau_r(u) < \infty\} = P\left\{\int_0^\infty I_{(X(t) - \mu t > \alpha u)} dt > r\right\} = P\left\{u \int_0^\infty I_{(X(t) > \sqrt{u}(\alpha + \mu t))} dt > r\right\}. \]

We have the following sandwich bounds

(27) \[ p_r(u) \leq P\{\tau_r(u) < \infty\} \leq p_r(u) + r_0(u), \]

where

\[ p_r(u) := P\left\{u \int_{t \in \triangle_u} I_{(X(t) > \sqrt{u}(\alpha + \mu t))} dt > r\right\}, \quad r_0(u) := P\left\{u \int_{t \in \tilde{\triangle}_u} I_{(X(t) > \sqrt{u}(\alpha + \mu t))} dt > 0\right\}, \]

with (recall the definition of \( t_0 \) in (10))

\[ \triangle_u = \left[t_0 - \frac{\ln(u)}{\sqrt{u}}, t_0 + \frac{\ln(u)}{\sqrt{u}}\right], \quad \tilde{\triangle}_u = \left[0, t_0 - \frac{\ln(u)}{\sqrt{u}}\right] \cup \left[t_0 + \frac{\ln(u)}{\sqrt{u}}, \infty\right). \]
5.1. **Analysis of** $r_0(u)$. This step is concerned with sharp upper bound for $r_0(u)$ when $u$ is large. Note that

$$r_0(u) = P\left\{ \exists t \in \Delta_u, X(t) > \sqrt{u}(\alpha + \mu t) \right\}.$$  

The following result is Lemma 4.1 in [27] (there was a misprint with $\sqrt{u}$ missing, and in eq.(30) $u$ should be $\sqrt{u}$).

**Lemma 5.1.** For all large $u$ we have

$$r_0(u) \leq C \sqrt{ue} - \frac{\sqrt{u}}{2} \left( \frac{\min(u',(\alpha,\mu))}{2} \right) (\ln(u))^2$$

holds for some constant $C > 0$ and some sufficiently small $\varepsilon > 0$ which do not depend on $u$.

5.2. **Analysis of** $p_r(u)$. We investigate the asymptotics of $p_r(u)$ as $u \to \infty$. Denote, for any fixed $T > 0$ and $u > 0$

$$\Delta_{j,u} = \Delta_{j,u}(T) = [t_0 + jTu^{-1}, t_0 + (j + 1)Tu^{-1}], \hspace{1em} -N_u \leq j \leq N_u,$$

where $N_u = \lceil T^{-1} \ln(u) \sqrt{u} \rceil$ (here $\lceil x \rceil$ denotes the smallest integer larger than $x$).

Denote

$$A_{j,u} = u \int_{t \in \Delta_{j,u}} \mathbb{P}\{X(t) > \sqrt{u}(\alpha + t\mu))\} dt,$$

and define

$$p_{r,j,u} = \mathbb{P}\{A_{j,u} > r\}, \hspace{1em} p_{r,i,j,u} = \mathbb{P}\{A_{i,u} > r, A_{j,u} > r\}.$$  

It follows, using a similar idea as in [30], that

$$p_r(u) \leq \sum_{j=-N_u}^{N_u} A_{j,u} > r\}

= \mathbb{P}\left\{ \sum_{j=-N_u}^{N_u} A_{j,u} > r, \{\text{there exists exactly one } j \text{ such that } A_{j,u} > 0\} \right\}$$

$$+ \mathbb{P}\left\{ \sum_{j=-N_u}^{N_u} A_{j,u} > r, \{\text{there exist } i \neq j \text{ such that } A_{i,u} > 0 \& A_{j,u} > 0\} \right\}$$

$$\leq p_{1,r}(u) + \Pi_0(u),$$

(29)

and by Bonferroni’s inequality

$$p_r(u) \geq \mathbb{P}\left\{ \sum_{j=-N_u+1}^{N_u-1} A_{j,u} > r\right\}$$

$$\geq \mathbb{P}\{3-N_u + 1 \leq j \leq N_u - 1 \text{ such that } A_{j,u} > r\}$$

$$\geq p_{2,r}(u) - \Pi_0(u),$$

(30)

where

$$p_{1,r}(u) = \sum_{j=-N_u}^{N_u} p_{r,j,u}, \hspace{1em} p_{2,r}(u) = \sum_{j=-N_u+1}^{N_u-1} p_{r,j,u}, \hspace{1em} \Pi_0(u) = \sum_{-N_u \leq i < j \leq N_u} p_{0,i,j,u}.$$
We shall focus on the asymptotics of \( p_{1,r}(u) \), which will be easily seen to be asymptotically equivalent to \( p_{2,r}(u) \) as \( u \to \infty \).

First, it is not difficult to see that by the definition in (15) we have \( 0 < \mathcal{H}_1(r,T) \leq \mathcal{H}_1(0,T) < \infty \) (cf. Lemma 4.2 in [27]).

**Lemma 5.2.** For any \( T > 0 \) and \( r \in (0,T) \), we have as \( u \to \infty \)

\[
(31) \quad p_{1,r}(u) \sim p_{2,r}(u) \sim \frac{1}{\sqrt{(2\pi t_0)^m |\Sigma_{II}|}} \mathcal{H}_1(r,T) \frac{1-m}{2} e^{-\frac{t_0^2}{2}} \int_{\mathbb{R}} e^{-\frac{x^2}{2}} \psi(x) \, dx,
\]

where \( \psi(x) \) is given in (16).

**Proof:** First, we fix \( T > 0 \). We show the proof in two steps. In Step I we derive that (31) holds for any \( r \in (0,T) \) at which \( \mathcal{H}_1(r,T) \), as a function of \( r \), is continuous, and then in Step II we show that \( \mathcal{H}_1(r,T), r \in (0,T) \) is actually continuous everywhere, implying that (31) holds for all \( r \in (0,T) \).

**Step I:** The claim follows from the same arguments as in the proof of Lemma 4.3 in [27]; we just need to replace the probabilities of form:

\[ \mathbb{P}\{ \exists t \in D_u, E_{t,u} \} \]

by those of the following form:

\[ \mathbb{P}\left\{ \int_{t \in D_u} u \mathbb{I}_{E_{t,u}} \, dt > r \right\}, \]

where \( D_u \) is some time interval which may depend on \( u \) and \( E_{t,u} \) is some event depending on both \( t \) and \( u \). Note that the appearance of \( u \) before \( I_\tau(t) \) in (32) depends on the context. For example, similarly as in Lemma 4.3 of [27] we have

\[
pr_{r,j}\, u = \mathbb{P}\left\{ u \int_{t \in [t_0 + \frac{jT}{u}, t_0 + \frac{jT}{u} + \frac{b_j(T)}{u}]} \mathbb{I}\left((X(t_0 + \frac{b_j(T)}{u}) - X(t_0 + \frac{b_j(T)}{u}) + \sqrt{\mathbb{E}_z}(\alpha + \mu)) \right) \, dt > r \right\}
= \mathbb{P}\left\{ \int_{t \in [0,T]} \mathbb{I}\left((Z_j,T + \frac{b_j(T)}{u}X(t) - \sqrt{\mathbb{E}_z})(\alpha + \mu)) \right) \, dt > r \right\},
\]

where \( Z_j,T \) is an independent of \( B \) Gaussian random vector with mean \( 0 \) and covariance matrix \( \Sigma_{j,T} = (t_0 + jT/u)\Sigma \), and

\[
b_j(T) = b_j(T) = b(t_0 + \frac{jT}{u}) = b + \frac{jT}{u} - \mu.
\]

With the new form of (32), one could derive, using the same arguments as those of [27], that (31) holds for any \( r \in (0,T) \) at which \( \mathcal{H}_1(r,T) \) is continuous; see also Theorem 5.1 of [38] for related discussions.

**Step II:** We show that \( \mathcal{H}_1(r,T), r \in (0,T) \) is a continuous function. To this end, we shall adopt an idea of [30]. Recall

\[
\mathcal{H}_1(r,T) = \int_{\mathbb{R}} e^{-\frac{t_0^2}{2} \Sigma_{II}^{-1} b_j} \mathbb{P}\left\{ \int_{t \in [0,T]} \mathbb{I}_{\{(X(t) - \mu)^T > x_1\}} \, dt > r \right\} \, dx_1.
\]

Then the claimed continuity at \( r \in (0,T) \) follows if we show

\[
\int_{\mathbb{R}} e^{-\frac{t_0^2}{2} \Sigma_{II}^{-1} b_j} \mathbb{P}\left\{ \int_{t \in [0,T]} \mathbb{I}_{\{(X(t) - \mu)^T > x_1\}} \, dt = r \right\} \, dx_1 = 0.
\]
Next consider the probability space \((C_d([0, T]), \mathcal{F}, \mathbb{P}^*)\) which is induced by the multi-dimensional Brownian motion with drift \(\{B(t) - A^{-1} \mu t, t \in [0, T]\}\), where \(C_d([0, T])\) is the collection of all \(d\)-dimensional continuous vector functions over \([0, T]\) and \(\mathcal{F}\) is the Borel \(\sigma\)-field of \(C_d([0, T])\). With the above notation, it is sufficient to show that for any \(r \in (0, T)\)

\[
(33) \quad \int_{\mathbb{R}^m} e^{ \int_0^r \sum_i b_i^t dt} \mathbb{P}^* \left\{ \int_{t \in [0, T]} \mathbb{I}_{(A u(t))_i > x_I} dt = r \right\} \, dx_I = 0,
\]

where \(u \in C_d([0, T])\). Denote, for any \(r \in (0, T)\),

\[
D_{x_I} = \left\{ u \in C_d([0, T]) : \int_{t \in [0, T]} \mathbb{I}_{(A u(t))_i > x_I} dt = r \right\}, \quad x_I \in \mathbb{R}^m.
\]

By continuity of \(u\), one can easily see that

\[
D_{x_I} \cap D_{x'_I} = \emptyset, \quad x_I \neq x'_I \in \mathbb{R}^m.
\]

Since for any finite number of points \(x_I^{(1)}, \ldots, x_I^{(N)} \in \mathbb{R}^m\) we have

\[
\sum_{i=1}^N \mathbb{P}^* \{D_{x_I^{(i)}}\} = \mathbb{P}^* \{\cup_{i=1}^N D_{x_I^{(i)}}\} \leq 1,
\]

it follows that

\[
\{x_I : x_I \in \mathbb{R}^m \text{ such that } \mathbb{P}^* \{D_{x_I}\} > 0\}
\]

is a countable set, which indicates that (33) holds. Thus \(\mathcal{H}_I(r, T), r \in (0, T)\) is a continuous function. This completes the proof. \(\square\)

**Estimation of double-sum.** In this subsection we shall focus on upper bounds of \(\Pi_0(u)\) for large \(u, T\). Note that

\[
\Pi_0(u) = \sum_{-N_u \leq i \leq j \leq N_u} p_{0, i, j; u} = \sum_{-N_u \leq i \leq j \leq N_u} p_{0, i, j; u} + \sum_{-N_u \leq i \leq j \leq N_u \atop j > i + 1} p_{0, i, j; u} =: \Pi_{0,1}(u) + \Pi_{0,2}(u).
\]

Since

\[
p_{0, i, j; u} = \mathbb{P} \left\{ \exists_{t \in \Delta_{i, u}} X(t) > \sqrt{u}(\alpha + \mu t), \exists_{t \in \Delta_{j, u}} X(t) > \sqrt{u}(\alpha + \mu t) \right\},
\]

we conclude from (52) in \([27]\) that

\[
(34) \quad \lim_{u \to \infty} \frac{\Pi_{0,1}(u)}{u^{1-m/2} e^{\frac{\alpha}{2} u}} = Q_1 \left( \frac{2H_I(0, T)}{T} - \frac{H_I(0, 2T)}{T} \right)
\]

for some constant \(Q_1 > 0\) which does not depend on \(T\). Similarly, as in \([27]\) we have

\[
(35) \quad \lim_{u \to \infty} \frac{\Pi_{0,2}(u)}{u^{1-m/2} e^{\frac{\alpha}{2} u}} \leq Q_2 T \sum_{j \geq 1} \exp \left( -\frac{\alpha}{8T_0} (jT) \right)
\]

holds with some constant \(Q_2 > 0\) which does not depend on \(T\).

**Proof of Theorem 3.1:** We have from (27) - (31), (34) and (35) that, for any \(T_1, T_2 > 0\)

\[
(36) \limsup_{u \to \infty} \frac{\mathbb{P} \{ r_u(t) < \infty \}}{C_I u^{\frac{1-m}{2}} e^{-\frac{\alpha}{2} u}} \leq \frac{\mathcal{H}_I(r, T_1)}{T_1} + Q_1 \left( \frac{2H_I(0, T_1)}{T_1} - \frac{H_I(0, 2T_1)}{T_1} \right) + Q_2 T_1 \sum_{j \geq 1} \exp \left( -\frac{\alpha}{8T_0} (jT_1) \right),
\]

\[
(37) \liminf_{u \to \infty} \frac{\mathbb{P} \{ r_u(t) < \infty \}}{C_I u^{\frac{1-m}{2}} e^{-\frac{\alpha}{2} u}} \geq \frac{\mathcal{H}_I(r, T_2)}{T_2} - Q_1 \left( \frac{2H_I(0, T_2)}{T_2} - \frac{H_I(0, 2T_2)}{T_2} \right) - Q_2 T_2 \sum_{j \geq 1} \exp \left( -\frac{\alpha}{8T_0} (jT_2) \right).
\]
Note that it has been shown in [27] that
\[ \mathcal{H}_T(0) = \lim_{T \to \infty} \frac{\mathcal{H}_T(0, T)}{T} < \infty. \]

Letting \( T_2 \to \infty \) in (37), with \( T_1 \) in (36) fixed, we have
\[ \limsup_{T \to \infty} \frac{\mathcal{H}_T(r, T)}{T} < \infty. \]

Furthermore, letting \( T_1 \to \infty \) we conclude that
\[ \liminf_{T \to \infty} \frac{\mathcal{H}_T(r, T)}{T} = \limsup_{T \to \infty} \frac{\mathcal{H}_T(r, T)}{T} < \infty. \]

Therefore, it is sufficient to prove that
\[ \liminf_{T \to \infty} \frac{\mathcal{H}_T(r, T)}{T} > 0 \]
holds. To this end, first note that
\[
P \{ \tau_r(u) < \infty \} \geq p_r(u) \geq \mathbb{P} \left\{ \sum_{j=-N_u+1}^{N_u-1} A_{j,u} > r \right\}
\[
\geq \mathbb{P} \left\{ \exists \ - N_u + 1 \leq j \leq N_u - 1, j \in \{2k : k \in \mathbb{Z}\} \text{ such that } A_{j,u} > r \right\}
\[
\geq p_{3,r}(u) - \Pi(u),
\]
where
\[
p_{3,r}(u) = \sum_{j=-N_u+1}^{N_u-1} p_{r,j,u}, \quad \Pi(u) = \sum_{-N_u \leq i < j \leq N_u, i,j \in \{2k : k \in \mathbb{Z}\}} p_{0,i,j,u}.
\]

Similar augments as in the derivation of (37) gives that, for some \( T_3 > 0 \),
\[
\liminf_{u \to \infty} \frac{\mathbb{P} \{ \tau_r(u) < \infty \}}{C_T \frac{u^m}{T^m} e^{-\frac{\hat{g}}{2}u}} \geq \frac{\mathcal{H}_T(r, T_3)}{2T_3} - Q_3T_3 \sum_{j \geq 1} \exp\left(\frac{-\hat{g}}{8T_0}(jT_3)\right)
\]
holds with some constant \( Q_3 > 0 \) which does not dependent on \( T_3 \). This together with (36) yields that
\[
\liminf_{T_1 \to \infty} \frac{\mathcal{H}_T(r, T_1)}{T_1} \geq \frac{\mathcal{H}_T(r, T_3)}{2T_3} - Q_3T_3 \sum_{j \geq 1} \exp\left(\frac{-\hat{g}}{8T_0}(jT_3)\right)
\[
\geq \mathcal{H}_T(r, r+1) - Q_3T_3 \sum_{j \geq 1} \exp\left(\frac{-\hat{g}}{8T_0}(jT_3)\right),
\]
holds for all \( T_3 \geq r + 1 \), where the last inequality follows since \( \mathcal{H}_T(r, T) \) as a function of \( T \) is non-decreasing. Since for sufficiently large \( T_3 \) the right-hand side of the above formula is positive, we conclude that (38) is valid.

Thus, the proof is complete. \( \square \)

**Proof of Theorem 3.2:** We have, for any \( s \in \mathbb{R} \)
\[
P \left\{ \frac{\tau_{r_2}(u)-t_{0u}}{\sqrt{u}} \leq s \left| \tau_{r_1}(u) < \infty \right. \right\} = \frac{P \left\{ \tau_{r_2}(u)-t_{0u} \leq s, \tau_{r_1}(u) < \infty \right\}}{P \left\{ \tau_{r_1}(u) < \infty \right\}}
\[
= \frac{P \left\{ \tau_{r_2}(u) \leq ut_0 + \sqrt{us} \right\}}{P \left\{ \tau_{r_1}(u) < \infty \right\}}
\[
= \frac{P \left\{ u \int_{0}^{u \mu} \sqrt{u} 1_{X(t) > (\alpha + \mu)t} \sqrt{u} dt \right\}}{P \left\{ \tau_{r_1}(u) < \infty \right\}}.
\]
Using the same arguments as in the proofs of our Theorem 3.1 and Theorem 3.3 in [27], we have

\[
P \left\{ u \int_{0}^{t_{0}+s/\sqrt{\mu}} E_{(\alpha+\mu t)^{2}/t} dt > r_{2} \right\} \sim P \left\{ u \int_{t_{0}-\ln(u)/\sqrt{\mu}}^{t_{0}+s/\sqrt{\mu}} E_{(\alpha+\mu t)^{2}/t} dt > r_{2} \right\}
\]

\[
\sim \frac{\mathcal{K}_{I}(r_{2})}{\sqrt{(2\pi)^{m} |\Sigma_{II}|}} \int_{-\infty}^{s} e^{-\frac{x^{2}}{2}} \psi(x) dx \frac{1}{\mu_{-2}} e^{-\frac{\hat{\theta}}{\mu_{-2}}}, \quad u \to \infty.
\]

Consequently, the claim follows and thus the proof is complete. \qed

**APPENDIX: PROOF OF COROLLARY 4.3**

We now demonstrate details of the technical proof for Corollary 4.3. Recall that in our notation \( I(t) \) is the essential index set of the quadratic problem \( P_{\Sigma}(\alpha+\mu t) \). If \( I(t)^{c} \neq \emptyset \) we define

\[
K(t) = \{ j \in I(t)^{c} : \Sigma_{I(t)^{c}I(t)} (\alpha+\mu t) I(t) = (\alpha+\mu t)_{j} \}.
\]

It follows from Lemma 2.1 that

(S1). On the set \( E_{1} = \{ t \geq 0 : \rho(1+\mu_{2}t) \geq (1+\mu_{1}t) \}, \inf_{v \geq \mu t+1} v^\top \Sigma^{-1} v = (1+\mu_{2}t)^{2} \)

(S2). On the set \( E_{2} = \{ t \geq 0 : \rho(1+\mu_{1}t) \geq (1+\mu_{2}t) \}, \inf_{v \geq \mu t+1} v^\top \Sigma^{-1} v = (1+\mu_{1}t)^{2} \)

(S3). On the set \( E_{3} = [0, \infty) \setminus (E_{1} \cup E_{2}), \inf_{v \geq \mu t+1} v^\top \Sigma^{-1} v = g_{0}(t) \),

where

\[
g_{0}(t) := (\mu t + 1)^{\top} \Sigma^{-1} (\mu t + 1) = \frac{2}{1+\rho} + \frac{2(\mu_{1} + \mu_{2})}{1+\rho} t + \frac{\mu_{1}^{2} + \mu_{2}^{2} - 2\mu_{1}\mu_{2}\rho}{1-\rho^{2}} t^{2}.
\]

First note that since \( \mu_{1} < \mu_{2} \) we have \( E_{2} = \emptyset \). Furthermore, if \( 0 < \rho \leq \mu_{1}/\mu_{2} \) then \( E_{1} = \emptyset \). In this case, \( I(t) = \{ 1, 2 \}, \ g(t) = g_{1}(t) := \frac{g_{0}(t)}{t}, \ t \geq 0 \).

It follows that for \( t_{0}^{(1)} = \sqrt{\frac{2(1-\rho)}{\mu_{1}^{2} + \mu_{2}^{2} - 2\mu_{1}\mu_{2}\rho}} > 0 \)

we have

\[
\inf_{t \geq 0} g(t) = g_{1}(t_{0}^{(1)}) = \frac{2}{1+\rho} (\mu_{1} + \mu_{2} + 2/t_{0}^{(1)}).
\]

Next we consider the case where \( \mu_{1}/\mu_{2} < \rho < 1 \). Note that in this case

\( E_{1} = \{ t \geq Q \} \neq \emptyset \), with \( Q := \frac{1-\rho}{\rho\mu_{2} - \mu_{1}} \)

and

\[
\inf_{t \geq 0} g(t) = \min \left( \inf_{t \geq Q} g_{2}(t), \inf_{t < Q} g_{1}(t) \right), \ g_{2}(t) := (1+\mu_{2}t)^{2}/t.
\]

Furthermore, referring to (39) we have

\( a \) \ for \( t > Q \), \( I(t) = \{ 2 \}, \ K(t) = \emptyset \)

\( b \) \ for \( t = Q \), \( I(t) = \{ 2 \}, \ K(t) = \{ 1 \} \)

\( c \) \ for \( t < Q \), \( I(t) = \{ 1, 2 \} \).
It is easily checked that \( g_2(t) \) attains its minimum at the unique point \( t_0^{(2)} = 1/\mu_2 \). In order to obtain the value of \( \inf_{t \geq Q} g_2(t) \) we have to check if \( t_0^{(2)} < Q \). We can show that\
\[
  t_0^{(2)} < Q \iff \rho < \frac{\mu_1 + \mu_2}{2\mu_2}.
\]
Thus we have

(a1). If \( \mu_1/\mu_2 < \rho < \frac{\mu_1 + \mu_2}{2\mu_2} \), then
\[
  \inf_{t \geq Q} g_2(t) = g_2(Q) = \frac{(\mu_1 - \mu_2)^2}{(1 - \rho)(\mu_2\rho - \mu_1)}.
\]

(a2). If \( \frac{\mu_1 + \mu_2}{2\mu_2} < \rho < 1 \), then
\[
  \inf_{t \geq Q} g_2(t) = g_2(t_0^{(2)}) = 4\mu_2.
\]
Now consider \( \inf_{t < Q} g_1(t) \). Similarly, we have to check if \( t_0^{(1)} > Q \) or not. We can show that
\[
  t_0^{(1)} > Q \iff \rho > \frac{\mu_1 + \mu_2}{2\mu_2}.
\]
Thus, we have

(b1). If \( \frac{\mu_1 + \mu_2}{2\mu_2} < \rho < 1 \), then \( \inf_{t < Q} g_1(t) = g_1(Q) \).
(b2). If \( \mu_1/\mu_2 < \rho < \frac{\mu_1 + \mu_2}{2\mu_2} \) then \( \inf_{t < Q} g_1(t) = g_1(t_0^{(1)}) \).
Furthermore, by the definitions of \( g_1, g_2 \) and \( Q \) we obtain \( g_1(Q) = g_2(Q) \).

The above findings are summarized in the following lemma:

**Lemma 5.3.** (1). If \( 0 < \rho \leq \mu_1/\mu_2 \), then \( I(t) = \{1, 2\}, t > 0 \) and
\[
  t_0 = t_0^{(1)}, \quad I = \{1, 2\}, \quad \tilde{g} = g_1(t_0^{(1)}), \quad \bar{g} = g_1''(t_0^{(1)}) = \frac{4}{1 + \rho}(t_0^{(1)})^{-3}.
\]

(2). If \( \mu_1/\mu_2 < \rho < \frac{\mu_1 + \mu_2}{2\mu_2} \), then
\[
  I(t) = \{1, 2\}, \quad 0 < t < Q, \quad I(t) = \{2\}, \quad t \geq Q
\]
and
\[
  t_0 = t_0^{(1)} < Q, \quad I = \{1, 2\}, \quad \tilde{g} = g_1(t_0^{(1)}), \quad \bar{g} = g_1''(t_0^{(1)}).
\]

(3). If \( \rho = \frac{\mu_1 + \mu_2}{2\mu_2} \), then
\[
  I(t) = \{1, 2\}, \quad 0 < t < Q, \quad I(t) = \{2\}, \quad t \geq Q
\]
and
\[
  t_0 = t_0^{(1)} = t_0^{(2)} = Q, \quad I = \{2\}, \quad K = \{1\}, \quad \tilde{g} = g_2(t_0^{(2)}) = 4\mu_2, \quad \bar{g} = g_2''(t_0^{(2)}) = 2\mu_2^3.
\]

(4). If \( \frac{\mu_1 + \mu_2}{2\mu_2} < \rho < 1 \), then
\[
  I(t) = \{1, 2\}, \quad 0 < t < Q, \quad I(t) = \{2\}, \quad t \geq Q
\]
and
\[
  t_0 = t_0^{(2)} > Q, \quad I = \{2\}, \quad K = \emptyset, \quad \tilde{g} = g_2(t_0^{(2)}) = 4\mu_2, \quad \bar{g} = 2\mu_2^3.
\]
Consequently, the proof of Corollary 4.3 follows directly from Theorems 3.1 and 3.2.

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