Graphon games

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Abstract

The study of strategic behavior in large scale networks via standard game theoretical methods is a challenging, if not intractable, task. In this paper, we propose a way to approximate games played over networks of increasing size, by using the graph limiting concept of graphon. To this end, we introduce the new class of graphon games for populations of infinite size. As a first contribution, we investigate properties of the Nash equilibrium of this newly defined class of games, including existence, uniqueness and comparative statics. As a second contribution, we illustrate how graphon games can be used to approximate strategic behavior in large but finite network games by assuming that the network is randomly drawn according to the graphon and we derive precise bounds for the distance between graphon and sampled network game equilibria in terms of the population size. Finally, we derive a closed form expression for the Nash equilibrium of linear quadratic graphon games and we illustrate its relation to Bonacich centrality.

Keywords: Network games, graphons, aggregative games, large population games, Nash equilibrium, existence, uniqueness, comparative statics

I. INTRODUCTION

Most complex human interactions, ranging from adversarial behavior to transactions on economic and financial markets to determination of opinions and social influence, involve the simultaneous strategic actions of agents influencing each other in diverse ways. In the last decade, network games emerged as a systematic framework for the formal analysis of such settings, building on the assumption that the interactions among the agents are known and can be represented with a graph [1], [2], [3]. The constant increase of the number of agents involved in networked systems is however making such an assumption more and more difficult to be satisfied. Systems such as the world wide web or social networks, for example, can consist of billions of interconnected agents. For such large networked systems, it is typically not possible to extract a complete and accurate graph-based representation, due to either computational or measurement constraints, errors in the observed data such as false positive or false negative links, or due to the fact that the network itself might be changing over time. Graphons have recently gained popularity as a nonparametric model for large networks that can overcome such difficulties and can represent a rich class of agent interactions [4], [5] (e.g., encoding as special cases both Erdős-Rényi or stochastic block models [6], [7]) while still allowing for consistent estimation from network data [8], [9], [10], [11], [12].

The objective of this paper is to exploit the framework of graphons to study strategic behavior in very large populations, while still accounting for local heterogeneity in agent interactions. To this end, we introduce and study the properties of a new class of infinite population games, which we term “graphon games”, and we show that these games provide a compact and robust representation of strategic behavior for network games of increasing population size. Our method provides a powerful tool for representing large group interactions in a potentially low-dimensional way, thus permitting a systematic analysis of strategic behavior in large-scale networked environments.

Our contributions are as follows. First, we formalize the notion of “graphon game” in terms of a continuum of agents indexed in [0, 1] and a graphon, represented by a bounded symmetric measurable function $W : [0, 1]^2 \to [0, 1]$ with $W(x, y)$ denoting the influence of agent’s $y$ strategy on agent $x$’s cost

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Specifically, we assume that agent \( x \)'s cost function depends on his strategy \( s(x) \in \mathbb{R}^n \) as well as a local aggregate quantity given by the “weighted average” of the other agents’ strategies computed using as weights the values \( W(x, y) \) specified by the graphon model. We define the Nash equilibrium for graphon games as a strategy profile \( s \) at which no agent can unilaterally improve (decrease) its cost function given the fixed local aggregate of the other agents’ strategies.\(^1\) We then study fundamental properties of such equilibrium and derive sufficient conditions in terms of the cost functions, strategy sets and of the underlying graphon to guarantee its existence and uniqueness. Under the same assumptions, we additionally derive a comparative static result quantifying the effect of graphon changes on the equilibrium outcome.

Our next contribution is to connect the infinite population graphon game model introduced above to finite population network games. To this end, we start by showing that any network game can be rewritten as a graphon game, hence graphon games are a generalization of network games. By using such reformulation, we show that the limit of the Nash equilibria of a sequence of network games with graphs \( P[N] \in \mathbb{R}^{N \times N} \) converging to a graphon \( W \) (as the population size \( N \) tends to infinity) converges to the equilibrium of the graphon game with graphon \( W \). This result shows that the graphon equilibrium is an appropriate representation of strategic behavior in the limit of infinite population size.

Most importantly, as another contribution of the paper we show that graphon games can be used to approximate strategic behavior in large but finite population network games. To this end, we draw on the interpretation of graphons as random graph models. In other words, we postulate that the networks of interactions employed by the agents can be described as different stochastic realizations sampled from an underlying graphon model. We then show that the equilibrium of the game associated with such generative graphon (which can be obtained from any of the estimation procedures mentioned above) is a good approximation of the equilibrium of any game played over a sampled network and we provide a precise mathematical bound for the approximation error in terms of the size of the sampled network and the maximum eigenvalue of the generative graphon. This result suggests a departure from the classical approach of network games analysis where, instead of studying the equilibrium associated with a specific network observation (which might be noisy, incomplete or even out-dated), one or more network observations are used to extract the underlying random graph model and equilibrium behavior is then studied over this latent object. In this sense, graphons can be seen as a way to extract the fundamental features of a class of network interactions (that goes beyond a single realization for an observed population size) and the corresponding equilibrium can thus be seen as an abstraction of the fundamental strategic properties arising from such class of network interactions, which is robust to stochastic perturbations or slight variations of population sizes.

To conclude and illustrate our theoretical results, we focus on a specific class of graphon games characterized by linear quadratic cost functions and scalar strategy sets. We show that in this case the graphon equilibrium can be characterized explicitly and exhibits similar properties as in finite linear quadratic network games. Specifically, we distinguish the case of games with strategic complements and substitutes and show that the equilibrium outcome exhibits properties as in finite network games with complements and substitutes, see e.g. [15], [16], [17], [18], [19], [20]. Notably in the case of complements, the strategy of each agent at equilibrium is proportional to its Bonacich centrality in the underlying graphon, as recently defined in [21]. We provide numerical simulations to illustrate the relation between the equilibrium of the infinite population graphon game and the equilibrium of network games sampled from it for different population sizes.

Our work complements previous literature models by incorporating heterogeneous local effects in infinite population games. Specifically, a widely considered infinite population model is that of mean field games [22], [23] which, while focusing on more general dynamic stochastic interactions, assume

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\(^1\)This definition is similar to the notion of Wardrop equilibrium used in nonatomic routing games (with a continuum of agents) where each agent uses routes of least cost given the aggregate congestion level [13], [14].
that each agent is influenced by the same aggregate (i.e. the mean) of the whole population. Another common model is that of population games [24], where a continuum of agents select their strategy among a finite set of options (instead of a continuous set) and the game dynamics are typically described in terms of the total mass of agents playing each strategy. The behavior of infinite but countable populations has also been studied in aggregative games [25], [26], [27], [28], [29], [30], [31], where each agent is influenced by the same aggregate of the strategies of the rest of the population. With respect to all these works, graphon games capture setups that include heterogeneous local interactions while still providing a relatively low dimensional and tractable representation. We finally remark that a different modeling framework to address local interactions in large network games has been described in [32]. Therein a Bayesian version of a linear-quadratic game of strategic complements is used according to which the behavior of each agent can be solved for in closed-form as a function of its type and degree. We stress that the framework proposed here is conceptually different from that of partial information or Bayesian games. In fact, we assume that each agent has perfect knowledge of its local interactions. Consequently, the agents can update their strategies using for example a best response scheme and converge to the equilibrium of the full information sampled network game.

To the best of our knowledge, this work is the first to suggest the use of graphons to model strategic behavior in large populations. We remark that the idea of using graphons as a support for large population analysis has been successfully applied in the last year in different areas such as community detection [33], crowd-sourcing [34], signal processing [35] and optimal control of dynamical systems [36]. Finally, we remark that interpreting observed graphs as random realizations from an underlying random graph model is a core statistical concept and has been recently applied to the study of centrality measures in [37] for stochastic block models and in [21] for graphon models. The authors of these papers study among others Katz-Bonacich centrality, which is known to coincide with the equilibrium of a specific type of network games (with scalar non-negative strategies, quadratic cost functions and strategic complementarities).

The rest of the paper is organized as follows. In Section II we recall basic results from graph and graphon theory. In Section III we start by recalling the framework of network games to then introduce graphon games and define the corresponding Nash equilibrium. In Section IV we derive sufficient conditions for existence and uniqueness of such an equilibrium and for comparative statics. Section V investigates the relation between graphon games and large scale network games. In Section VI we specialize our theoretical results to linear quadratic graphon games and in Section VII we provide supporting numerical simulations.

**Notation:** We denote by $\mathbb{R}^n$ the space of $n$-dimensional vectors, by $L^2([0,1])$ the space of square integrable functions defined on $[0,1]$ and by $[L^2([0,1])]^n$ the space of square integrable vector valued functions defined on $[0,1]$. The norms in these spaces are denoted by $\|v\| := \sqrt{\sum_{h=1}^n |v|^2_h}$, $\|f\|_L := \sqrt{\int_0^1 f(x)^2 dx}$, $\|g\|_{L^2} := \sqrt{\int_0^1 |g(x)|^2 dx}$ for $v \in \mathbb{R}^n$, $f \in L^2([0,1])$, $g \in [L^2([0,1])]^n$, respectively. $[v]^h$ denotes the $h$-th component of the vector $v$. With the exception of $\mathbb{N}$ and $\mathbb{R}$ (that denote the sets of natural and real numbers, respectively), we use blackboard bold symbols (such as $\mathcal{O}$) to denote operators acting on $L^2([0,1])$ or on $[L^2([0,1])]^n$. The induced operator norm are denoted by $\|\cdot\| := \sup_{\|f\|_L = 1} \|\mathcal{O}f\|_L$ and $\|\mathcal{O}\| := \sup_{\|g\|_{L^2} = 1} \|\mathcal{O}g\|_{L^2}$. We denote by $\lambda_{\max}(\mathbb{L})$ and $\rho(\mathbb{L})$ the largest eigenvalue and the spectral radius of the linear integral operator $\mathbb{L}f := \int_0^1 L(x,y)f(y)dy$ with symmetric kernel $L(x,y) = L(y,x)$. We denote sets by using calligraphic symbols (such as $\mathcal{S}$) and the set of subsets of $\mathbb{R}^n$ by $2^{\mathbb{R}^n}$. The symbol $1_N$ denotes the vector of all ones in $\mathbb{R}^N$ and $1_{[0,1]}(x)$ the function constantly equal to one in $[0,1]$. $\mathbb{I}$ is the identity operator and $I$ the identity matrix. $\mathbb{N}[a,b]$ denotes the set of natural numbers between $a$ and $b$.

**II. SUMMARY ON GRAPH AND GRAPHON THEORY**

The objective of this section is to introduce tools related to graph and graphon theory that are used in the rest of the paper and to highlight the connections between the two. These insights allow us to
build on network games to define graphon games in the next section.

A. Graph theory

1) Graph: Graphs are the most prominent tool used to model network interactions in populations with a finite number of agents. In this context, each agent is associated to a vertex indexed with a number \( i \) taking values in the set \( \mathbb{N}[1, N] \), where \( N \) is the population size. Relations between different agents are modeled as (weighted) edges connecting the corresponding vertices.

2) Adjacency matrix: A compact representation of a graph is via its adjacency matrix \( P \in \mathbb{R}^{N \times N} \). Such a matrix is formed by assigning to each element \( P_{ij} \) the weight of the edge connecting vertex \( i \) to vertex \( j \). If agent \( i \) does not interact with agent \( j \), then \( P_{ij} = 0 \). From here on, we use the expressions “a graph \( P \)” and “a graph whose adjacency matrix is \( P \)” interchangeably.

3) Centrality measures for graphs: Node centrality is a measure of the importance of a node within a graph based on its location and connections [38]. A crucial centrality measure in game theory is Bonacich centrality [39], which measures the importance of a given node based on the number of immediate neighbors as well as the number of two-hop neighbors, three-hop neighbors, and so on; discounted at each step by a factor \( \alpha > 0 \). The vector of Bonacich centralities \( b_\alpha \) is therefore computed as

\[
\begin{align*}
    b_\alpha &= (I - \alpha P)^{-1}1_N. \\
    \text{(1)}
\end{align*}
\]

Note that the vector \( b_\alpha \) is element-wise positive since it is the (infinite) sum of positive terms.

B. Graphon theory

1) Graphon: A graphon is a bounded symmetric measurable function \( W : [0, 1]^2 \rightarrow [0, 1] \) encoding interactions among an infinite (continuum) number of agents [4], [5]. In this context agents are indexed with a continuum variable \( x \in [0, 1] \) instead of a finite index \( i \in \mathbb{N}[1, N] \) and \( W(x, y) \) denotes the strength of the interaction between agents \( x \) and \( y \). We denote the space of the graphons as \( \tilde{W} \).

2) Graphon operator: Many properties of a graphon can be obtained by analyzing the spectral properties of an associated linear operator which plays a similar role to the adjacency matrix for graphs of finite size, [4, Section 7.5].

Definition 1 (Graphon operator): For a given graphon \( W \in \tilde{W} \), we define the associated graphon operator \( \mathbb{W} \) as the integral operator \( \mathbb{W} : L^2([0, 1]) \rightarrow L^2([0, 1]) \) given by

\[
    f(x) \mapsto (\mathbb{W}f)(x) = \int_0^1 W(x, y)f(y)dy.
\]

Definition 2 (Eigenvalues and eigenfunctions): A complex number \( \lambda \) is an eigenvalue of the operator \( \mathbb{W} \) if there exists a nonzero function \( \psi \in L^2([0, 1]) \), called the eigenfunction, such that

\[
    (\mathbb{W}\psi)(x) = \lambda\psi(x). \quad \text{(2)}
\]

In the next proposition we summarize the main properties of the graphon operator, which follow from the symmetry of \( W(x, y) \) and will be used in our subsequent analysis.

Proposition 1 (Properties of \( \mathbb{W} \)): The following hold:

1) \( \mathbb{W} \) is a linear, continuous, bounded operator;
2) all the eigenvalues of \( \mathbb{W} \) are real;
3) the spectral radius \( \rho(\mathbb{W}) \) coincides with the maximum eigenvalue \( \lambda_{\text{max}}(\mathbb{W}) > 0 \);
4) \( \|\mathbb{W}\| := \sup_{f \in L^2([0,1]), \|f\|_{L^2}=1} \|\mathbb{W}f\|_L = \lambda_{\text{max}}(\mathbb{W}) \).

We refer to [4, Section 7.5] and [21] for the proof of Proposition 1 and a detailed discussion.
3) **Centrality measures for graphons:** Exactly as in the finite case, one can rank the agents in a graphon according to suitably defined centrality measures. We are particularly interested in the extension of Bonacich centrality to graphons, as suggested in [21] and recalled next.

**Definition 3 (Bonacich centrality for graphons):** Given a graphon $W$, its associated operator $\mathbb{W}$ and a value $\alpha > 0$, consider the linear integral operator $M_{\alpha}$ where $(M_{\alpha}f)(x) := f(x) - \alpha(\mathbb{W}f)(x)$. For any $0 < \alpha < 1/\rho(\mathbb{W})$, we define the Bonacich centrality function $b_\alpha : [0, 1] \to \mathbb{R}_{>0}$ as

$$b_\alpha(x) := \left( M_{\alpha}^{-1}1_{[0,1]} \right)(x).$$

The connection between the Bonacich centrality function defined above and graphon games is highlighted in Section VI. For now, we note that $b_\alpha(x)$ is well-defined (i.e. the operator $M_{\alpha}$ is indeed invertible since $0 < \alpha < 1/\rho(\mathbb{W})$) and it is strictly positive for any $x \in [0, 1]$ since, as in the finite case, equation (3) can be rewritten as an infinite sum of positive functions by using a Neumann series reformulation of the inverse operator.

**C. Relation between graphs and graphons**

The concept of graphons as described in the previous subsection was initially introduced as a natural limiting object of graph sequences [4], [5]. To make this statement precise we recall in the following that graphs can be seen as special cases of graphons and that the set of finite graphs endowed with the cut metric (as described next) gives rise to a metric space whose completion is the space of graphons. Besides their interpretation as a graph limit, we also recall in point 3) below that graphons can be used as random graph models. While these results are known, they are summarized here because they are fundamental to our study of the relation between network and graphon games, which is carried out in Section V.

1) **Graphs are step function graphons:** To show that graphons generalize graphs we start by recalling that a function $W(x, y)$ is called a step function if there is a partition $Q = \{ Q_1, \ldots, Q_N \}$ of $[0, 1]$ into measurable sets such that $W$ is constant on every product set $Q_i \times Q_j$. The sets $Q_i$ are the steps of $W$. For any $N \in \mathbb{N}$ let us consider the uniform partition $\mathcal{U}^{[N]} = \{ U_1, U_2, \ldots, U_N \}$ of $[0, 1]$ obtained by setting $U_k = \left[ \frac{k-1}{N}, \frac{k}{N} \right)$, $k \in \mathbb{N}[1, N - 1]$ and $U_N = \left[ \frac{N-1}{N}, 1 \right]$. Then, for any graph $P^{[N]} \in \mathbb{R}^{N \times N}$ we can define the step function graphon $W^{[N]} \in \mathcal{W}$ corresponding to $P^{[N]}$ by setting

$$W^{[N]}(x, y) := P^{[N]}_{ij}, \quad \forall (x, y) \in U_i \times U_j, \quad \forall i, j \in \mathbb{N}[1, N].$$

Note that $W^{[N]}$ depends on the ordering of the nodes in $P^{[N]}$. Apart from that, the relation between graphs and step function graphons is bijective.

2) **Graphons are the completion of the graph space:** Graphon properties are typically formulated in terms of cut norm. Specifically, the cut norm of a graphon $W \in \mathcal{W}$ is denoted by $\|W\|_\Box$ and is defined as follows

$$\|W\|_\Box := \sup_{M,T \subseteq [0,1]} \left| \int_M \int_T W(x, y)dxdy \right|,$$

where $M$ and $T$ are measurable subsets of $[0, 1]$. The cut metric between two graphons $V, W \in \mathcal{W}$ is given by

$$d_\Box(W, V) := \inf_{\phi \in \Pi^{[0,1]}} \|W^\phi - V\|_\Box,$$

We refer to [21] for a detailed discussion on graphons centrality measures and their relation with graphs centrality measures. Therein, it is proven that Bonacich centrality, as in (3), is the natural limit of the standard Bonacich centrality, as in (1), for sequences of graphs of increasing size.
According to the procedure detailed in Definition 4, for any stochastic matrices. The difference between the two is that corresponding to a graph with graphons provide a natural framework to generate random graphs [4, Chapter 10].

![Illustration of the sampling procedure described in Definition 4. a) The generative graphon. b) The step function graphon corresponding to the realization of the weighted adjacency matrix $P^s[w]$. A linear grayscale colormap is used with white associated to $W = 0$ and black to $W = 1$.]

where $W^\phi(x, y) := W(\phi(x), \phi(y))$ and $\Pi_{[0,1]}$ is the class of measure preserving permutations $\phi : [0, 1] \rightarrow [0, 1]$. Intuitively, the function $\phi$ performs a node relabeling to find the best match between $W$ and $V$. Because of such relabeling, $d_\square(W, V)$ is not a well defined metric in $\mathcal{W}$. In fact, it might happen that $d_\square(W, V) = 0$ even though $W \neq V$. To avoid such a problem, we define the space $\mathcal{W}$ where we identify graphons up to measure preserving permutations, so that $d_\square$ is a well defined metric in $\mathcal{W}$. Intuitively, this means that we consider graphs and graphons with unlabeled nodes. With slight abuse of notation, given a graphon $W$ and a graph $P^N$, we write $d_\square(W, P^N)$ to denote the cut metric $d_\square(W, W^N)$ between the graphon $W$ and the step function graphon $W^N$ corresponding to $P^N$.

In our subsequent analysis, we work also with the operator norm, which is defined as

$$\|\mathbb{W}\| := \sup_{\{f \in L^2([0,1]) \|f\|_{L^1} = 1\} \|\mathbb{W}f\|_{L^1}.$$  

We report in the appendix a lemma showing that cut and operator norms are equivalent, hence they can be used interchangeably. The following proposition is central in graphon theory, [4].

**Proposition 2**: The space $(\mathcal{W}, d_\square)$ is complete.

In particular the previous proposition implies that any Cauchy sequence of graphs (thought as step function graphons) converges in the graphon space.

3) **Graphons as random graph models**: Besides their interpretation as limit of graph sequences, graphons provide a natural framework to generate random graphs [4, Chapter 10].

**Definition 4 (Sampling procedure)**: Uniformly sample $N$ points $\{u_i\}_{i=1}^N$ from $[0, 1]$ and define the weighted adjacency matrix $P^N_w$ as follows

$$(P^N_w)_{ij} = \begin{cases} W(u_i, u_j), & \text{if } i \neq j, \\ 0, & \text{if } i = j. \end{cases}$$

Additionally, starting from $P^N_w$, define the 0-1 adjacency matrix $P^N_s$ as the adjacency matrix corresponding to a graph with $N$ nodes obtained by randomly connecting nodes $i, j \in [1, N]$ with Bernoulli probability $\left(P^N_w\right)_{ij}$.

Figure 1 illustrates the sampling procedure described in Definition 4. Note that both $P^N_w$ and $P^N_s$ are stochastic matrices. The difference between the two is that $P^N_w \in [0, 1]^{N \times N}$ while $P^N_s \in \{0, 1\}^{N \times N}$. According to the procedure detailed in Definition 4, for any $p \in [0, 1]$, the constant graphon $W(x, y) \equiv p$
coincides with the Erdős-Rényi random graph model with edge probability \( p \). Analogously, any step function graphon gives rise to a stochastic block model with \( |Q| \) classes [6], [7]. Interestingly, it can be shown that the distribution of any exchangeable random graph [40] is characterized by a function \( W \) as discussed above [41]. Hence, graphons can be used to encode a rich class of random graph models.

III. GRAPHON GAMES

A. Summary on network games

In a network game, \( N \) players interact over a graph, which we here characterize by its non-negative adjacency matrix \( P \in [0,1]^{N \times N} \). Each player \( i \in \mathbb{N}[1,N] \) aims at selecting a vector strategy \( s^i \in \mathbb{R}^n \) in its feasible set \( S^i \subset \mathbb{R}^n \) to minimize the cost function

\[
J(s^i,z^i(s))
\]

where \( s := \{s^i\}_{i=1}^N \in \mathbb{R}^{Nn} \) and \( z^i(s) := \frac{1}{N} \sum_{j=1}^N P_{ij}s^j \) denotes the weighted average of the neighbors strategies (according to the coefficients of the graph \( P \)).\(^3\) Note that we here assume that each agent has the same cost function \( J \). This assumption simplifies the comparison of populations of different (in the limit infinite) sizes. The Nash equilibria of a network game, compactly denoted as \( \mathcal{G}^{[N]}(\{S^i\}_{i=1}^N,J,P) \), are defined as follows.

**Definition 5 (Nash equilibrium):** A set of strategies \( \{\bar{s}^i\}_{i=1}^N \) is a Nash equilibrium of the network game \( \mathcal{G}^{[N]}(\{S^i\}_{i=1}^N,J,P) \) if for all players \( i \in \mathbb{N}[1,N] \) we have that \( \bar{s}^i \in S^i \) and for all \( s^i \in S^i \)

\[
J(\bar{s}^i,\frac{1}{N} \sum_{j=1}^N P_{ij}\bar{s}^j) \leq J(s^i,\frac{1}{N} P_{ii}s^i + \frac{1}{N} \sum_{j\neq i} P_{ij}\bar{s}^j).
\]

In other words, at a Nash equilibrium each strategy is a best response to the other players strategies

\[
\bar{s}^i \in s^i_{br}(\bar{s}) := \arg \min_{s \in S^i} J(s,\frac{1}{N} P_{ii}s + \frac{1}{N} \sum_{j\neq i} P_{ij}\bar{s}^j).
\]

In the following, we say “a network game \( \mathcal{G}^{[N]} \) with graph \( P \)” if we need to stress the role of the graph.

B. Graphon games: the model

To extend the concept of network games to infinite populations, we consider a set-up where a continuum set of agents interact with each other. Each agent in this context is indexed by the continuum variable \( x \in [0,1] \) instead of the finite index \( i \in \mathbb{N}[1,N] \) and has a strategy vector which is denoted by \( s(x) \in \mathbb{R}^n \) instead of \( s^i \in \mathbb{R}^n \). As in the finite population case, we assume local constraints of the form \( s(x) \in S(x) \), where \( S(x) : [0,1] \rightarrow 2^{\mathbb{R}^n} \) is a set-valued function. In finite network games, each agent computes its best response to the local aggregate \( z^i(s) := \frac{1}{N} \sum_{j=1}^N P_{ij}s^j \) according to the coefficients of the underlying graph \( P \). In the infinite population case, the natural mathematical object to describe the network of interactions is a graphon \( W : [0,1]^2 \rightarrow [0,1] \), as described in Section II-B. Specifically, we define the local aggregate perceived by agent \( x \) in the infinite population case as

\[
z(x | s) := \int_0^1 W(x,y)s(y)dy
\]

\[
= \begin{bmatrix}
\int_0^1 W(x,y)s_1(y)dy \\
\vdots \\
\int_0^1 W(x,y)s_n(y)dy
\end{bmatrix} = (\mathbb{W}s_n)(x)
\]

where \( \mathbb{W} : [L^2([0,1])]^n \rightarrow [L^2([0,1])]^n \) is defined by applying \( \mathbb{W} \) component-wise.

\(^3\)In network games typically there is no factor \( \frac{1}{N} \) in the definition of \( z^i \). Since we study the behavior when \( N \) changes we find it useful to consider this factor explicitly. This is without loss of generality.
Remark 1: Note that for graphon games a strategy profile \( s : [0, 1] \to \mathbb{R}^n \) is a vector valued function. In other words, \( s(x) = [s_1(x), \ldots, s_n(x)]^\top \) for all \( x \in [0, 1] \). In the following, we require that each component of such vector valued function is square integrable, that is, \( s_k(x) \in L^2([0, 1]) \) for all \( k \in \mathbb{N}[1, n] \). We denote this by writing for short \( s(x) \in [L^2([0, 1])]^n \) (see notation in Section I).

As in network games, the aim of each agent is to select the strategy \( s(x) \in S(x) \) that minimizes its cost

\[
J(s(x), z(x \mid s)). \quad (5)
\]

Note that the function \( J \) used in (5) is exactly the same function \( J \) used in (4) for network games. The only difference in the two setups is the network effect which is \( z^i(s) \) in network games and \( z(x \mid s) \) in graphon games. In other words, in a graphon game each agent aims at computing its best response to the local aggregate induced by the strategy profile \( s \) as follows

\[
s_{br}(x \mid s) := \arg \min_{\tilde{s} \in S(x)} J(\tilde{s}, z(x \mid s)). \quad (6)
\]

Note that such a best response might in general be set-valued. Moreover, since there is a continuum of agents, the contribution of agent \( x \) to the aggregate \( z(x \mid s) \) is negligible. Consequently, the decision variable \( \tilde{s} \) affects only the first argument in the cost function in (6). We summarize the previous discussion in the following definition.

Definition 6 (Graphon game): A graphon game \( G \) is defined in terms of a continuum set of agents indexed in \( [0, 1] \), a graphon \( W \), a cost function \( J \) as in (5) and for each agent \( x \) a strategy set \( S(x) \).

In the following, we say “a graphon game \( G \) with graphon \( W \)” if we need to stress the role of the graph and we explicitly write \( G(S, J, W) \) is we want tostress the role of all the game primitives.

Example 1 (Agents competing on a line): Consider an infinite population of agents which are spatially located along a line (e.g., a street). Let \( x \in [0, 1] \) denote the position of each agent along this line. In many real applications, the influence between any two agents is inversely proportional to their spatial distance and agents in more central positions are affected more by their neighbors. This fact can be modeled by using the previously described framework with the “minmax” graphon

\[
W(x, y) = \min(x, y)(1 - \max(x, y)).
\]

Figure 2 illustrates such a graphon. The middle left plot, for example, shows that the agent located at \( x = 0.5 \) is more sensitive to the strategies of the agents around him (i.e. for \( y \approx 0.5 \)) than to the strategies of the corner agents (i.e. for \( y \approx 0 \) or \( y \approx 1 \)).
C. Graphon games: Equilibrium concept

Paralleling the literature on non-atomic routing games [13], [14], one can extend the concept of Nash equilibrium to graphon games.

Definition 7 (Nash equilibrium): A function \( \bar{s} \in [L^2([0, 1])]^n \) with associated local aggregate \( \bar{z}(x) := z(x \mid \bar{s}) = \int_0^1 W(x, y)\bar{s}(y)dy \) is a Nash equilibrium for the graphon game \( G(\bar{s}, J, W) \) if for all \( x \in [0, 1] \), we have \( \bar{s}(x) \in S(x) \) and

\[
J(\bar{s}(x), \bar{z}(x)) \leq J(\bar{s}, \bar{z}(x)) \text{ for all } \bar{s} \in S(x).
\]

In other words, a function \( \bar{s} \) is a Nash equilibrium if, for each agent \( x \), the strategy \( \bar{s}(x) \) is a best response of that agent to the strategies of the other agents. Mathematically,

\[
\bar{s}(x) \in S_{br}(x \mid \bar{s}), \quad \forall x \in [0, 1].
\]

IV. Properties of Nash equilibria

We start by deriving an equivalent characterization of the Nash equilibria of a graphon game as fixed points, under the following assumptions.

Assumption 1: The function \( J(s, z) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \) in (5) is continuously differentiable and strongly convex in \( s \) with uniform constant \( \alpha_J \) for each value of \( z \in \mathbb{R}^n \). Moreover, \( \nabla_s J(s, z) \) is uniformly Lipschitz in \( z \) with constant \( \ell_J \) for all \( s \in \mathbb{R}^n \). For each \( x \in [0, 1] \) the set \( S(x) \) is convex and closed.

Assumption 2: A) There exists \( \hat{z} \in \mathbb{R}^n \) and \( M > 0 \) such that \( \| \arg \min_{\bar{s} \in S(x)} J(\bar{s}, \hat{z}) \| \leq M \) for all \( x \in [0, 1] \). B) There exists a compact set \( S \) such that \( S(x) \subseteq S \) for all \( x \in [0, 1] \).

Remark 2: We note that Assumption 2B implies Assumption 2A). We consider these assumptions separately since some of our results hold under the sole Assumption 2A), which is less restrictive.

To derive a fixed point characterization of the Nash equilibrium, we start by considering any strategy function \( s(x) \in [L^2([0, 1])]^n \). The corresponding local aggregate is \( z(x \mid s) = (\mathbb{W}_ns)(x) \in [L^2([0, 1])]^n \).

Let us now define an operator \( \mathbb{B} : [L^2([0, 1])]^n \rightarrow [L^2([0, 1])]^n \) defined point-wise as follows

\[
(\mathbb{B}z)(x) := \arg \min_{\bar{s} \in S(x)} J(\bar{s}, z(x)),
\]

where \( z(x) \) is any function of \( [L^2([0, 1])]^n \) (i.e., not necessarily \( z(x \mid s) \)). In words, \( (\mathbb{B}z)(x) \) is the best response of agent \( x \) to the fixed local aggregate \( z(x) \). Note that, under Assumption 1, such best response operator is well defined since the minimization problem in (7) has a unique solution. The fact that the codomain of the best response operator \( \mathbb{B} \) is \( [L^2([0, 1])]^n \) will be proven in the next section.

We then see that a strategy profile \( \bar{s} \in [L^2([0, 1])]^n \) is a Nash equilibrium if and only if

\[
\bar{s} = \mathbb{B}\mathbb{W}_n\bar{s},
\]

that is, the function \( \bar{s} \) is a fixed point of the composite operator \( \mathbb{B}\mathbb{W}_n \), which we term the game operator.

A. Properties of the game operator

Existence and uniqueness of a fixed point solving (8) depend on the properties of the composite game operator \( \mathbb{B}\mathbb{W}_n \). Note that both \( L^2([0, 1]) \) and \( [L^2([0, 1])]^n \) are Hilbert spaces. We start by studying the properties of \( \mathbb{B} \) and \( \mathbb{W}_n \) separately.

Lemma 1 (Properties of \( \mathbb{W}_n \)): The following holds:

1) \( \mathbb{W}_n \) is a linear, continuous, bounded operator;  
2) The eigenvalues of \( \mathbb{W}_n \) coincide (besides the multiplicity) with those of \( \mathbb{W} \) and are thus real;  
3) \( \| \mathbb{W}_n \| = \lambda_{\max}(\mathbb{W}) \).

Note that \( s(x) \in [L^2([0, 1])]^n \Rightarrow z(x \mid s) \in [L^2([0, 1])]^n \). In fact \( \|z(x \mid s)\|_2^2 = \sum_k \int_0^1 [z(x \mid s)]_k^2 dx = \sum_k \int_0^1 \int_0^1 W(x, y)s_k(y)dy \|z(x \mid s)\|_2^2 dx \leq \sum_k \int_0^1 \int_0^1 \|W(x, y)\|^2 \|s_k(y)\|^2 dy dx \leq \sum_k \int_0^1 \int_0^1 \|s_k(y)\|^2 dy dx = \|s(x)\|_2^2 \), where we used \( W(x, y)^2 \leq 1 \) and Cauchy-Schwartz.
Proof: This lemma is an immediate consequence of Proposition 1 since $\mathbb{W}_n$ acts independently on each component.

Lemma 2 (Properties of $B$): Suppose that Assumption 1 holds. Then the following holds:
1) $B$ is a Lipschitz operator. That is, for any $f, g \in [L^2([0, 1])]^n$,
\[
\|Bf - Bg\|_{L_n} \leq \frac{\ell_f}{\alpha_J} \|f - g\|_{L_n};
\]
2) $B$ is a continuous operator;
3) Suppose further that Assumption 2A) holds, then the codomain of $B$ is $[L^2([0, 1])]^n$;
4) Suppose further that Assumption 2B) holds and let $s_{\text{max}} := \max_{s \in S} \|s\|$. Then $B$ is a bounded operator and the codomain of $B$ is contained in
\[
L_S := \{f \in [L^2([0, 1])]^n \mid \|f\|_{L_n} \leq s_{\text{max}}\}.
\]

Proof:
1) Take any $f, g \in [L^2([0, 1])]^n$. For any $x \in [0, 1]$ we get
\[
\|Bf(x) - Bg(x)\| = \|\arg \min_{\tilde{s} \in S(x)} J(\tilde{s}, f(x)) - \arg \min_{\tilde{s} \in S(x)} J(\tilde{s}, g(x))\|
\leq \frac{1}{\alpha_J} \|\nabla_s J(Bg(x), f(x)) - \nabla_s J(Bg(x), g(x))\|
\leq \frac{\ell_f}{\alpha_J} \|f(x) - g(x)\|.
\]
The first inequality in (10) can be proven by reformulating the optimization problem in (7) as the variational inequality $VI(S(x), \nabla_s J(\cdot, z(x)))$. In fact, by Assumption 1, the operator $\nabla_s J(\cdot, z)$ is strongly monotone with constant $\alpha_J$ for all $z \in \mathbb{R}^n$, [42, Equation (12)]. The result then follows from a known bound on the distance of the solution of strongly monotone variational inequalities [43, Theorem 1.14]. The second inequality comes from the assumption that $\nabla_s J(s, z)$ is uniformly Lipschitz in $z$ with constant $\ell_f$ for all $s \in \mathbb{R}^n$. Let us now compute $\|Bf - Bg\|_{L_n}^2$:
\[
\|Bf - Bg\|_{L_n}^2 = \sum_{k=1}^n \|Bf_k - Bg_k\|_{L}^2 = \frac{1}{\alpha_J} \|f - g\|_{L_n}^2
\]
2) Lipschitz continuity implies continuity.
3) We need to show that for any $z \in [L^2([0, 1])]^n$, $\|Bz\|_{L_n} < \infty$. Consider the function $\hat{z}(x) := \hat{z}$ for all $x \in [0, 1]$, where $\hat{z}$ is as in Assumption 2A). Note that $\hat{z} \in [L^2([0, 1])]^n$ since
\[
\|B\hat{z}\|_{L_n}^2 = \sum_{k=1}^n \int_0^1 (B\hat{z})_k^2(x)dx = \int_0^1 \sum_{k=1}^n (B\hat{z})_k^2(x)dx
\]
\[
= \int_0^1 \|B\hat{z}(x)\|^2dx = \int_0^1 \|\arg \min_{\hat{s} \in S(x)} J(\hat{s}, \hat{z})\|^2dx \leq M^2.
\]
Consider now any $z \in [L^2([0, 1])]^n$. We have

10
and compact. Consequently, the operator is a contraction (i.e. Lipschitz with constant strictly less than one) in the Hilbert space $[L^2([0, 1])]^n$. The conclusion then follows from Banach fixed point theorem. For any $f, g \in [L^2([0, 1])]^n$, 

Finally, we study the properties of $L_S$ as defined in (9).

**Lemma 3 (Properties of $L_S$):** For any non-empty compact set $S \subset \mathbb{R}^n$, the set $L_S$ in (9) is a non-empty, convex and compact subset of $[L^2([0, 1])]^n$.

**Proof:** Since $S$ is non-empty and compact $s_{\max}$ is well defined. This immediately implies that $L_S$ is non-empty. Given two functions $f, g \in L_S$ and any $\mu \in [0, 1]$

$$
\|\mu f + (1-\mu)g\|^2_{L_n} = \sum_{k=1}^n \left( \int_0^1 (\mu f_k(x) + (1-\mu)g_k(x))^2 dx \right) \leq \sum_{k=1}^n \int_0^1 (\mu f_k(x))^2 + (1-\mu)(g_k(x))^2 dx
$$

where we used that $t \mapsto t^2$ is convex. Hence $\mu f + (1-\mu)g \in L_S$ and $L_S$ is convex. Finally, $L_S$ is bounded by definition and it is closed because the norm is continuous.

**B. Existence and uniqueness**

Existence of a Nash equilibrium is an immediate consequence of the properties of the game operator.

**Theorem 1 (Existence):** Suppose that the graphon game $G(S, J, W)$ satisfies Assumptions 1 and 2B). Then it admits at least one Nash equilibrium.

**Proof:** We aim at applying Schauder fixed point theorem. In Lemma 1 and 2 we have proven that both $\mathbb{W}_n$ and $\mathbb{B}$ are continuous operators. Moreover, the game operator $\mathbb{BW}_n$ maps $L_S$ into $L_S$ since $\mathbb{W}_n : L_S \to [L^2([0, 1])]^n$ and $\mathbb{B} : [L^2([0, 1])]^n \to L_S$. By Lemma 3, $L_S$ is non-empty, convex and compact. Consequently, the operator $\mathbb{BW}_n$ is a continuous mapping from a non-empty, closed and convex set (i.e., $L_S$) to itself and the image $\mathbb{BW}_n L_S$ is contained on a compact set since $\mathbb{BW}_n L_S \subseteq L_S$ and $L_S$ is compact. Schauder fixed point theorem thus guarantees the existence of a fixed point.

Uniqueness on the other hand is not always guaranteed. In fixed point theory it is well known that a sufficient condition for uniqueness is contractiveness. We apply this result to graphon games in the next theorem by exploiting the following additional assumption.

**Assumption 3:** Suppose that

$$
\frac{\ell_{ij}}{\alpha_{ij}} \cdot \lambda_{\max}(\mathbb{W}) < 1,
$$

where $\lambda_{\max}(\mathbb{W})$ is the largest eigenvalue of $\mathbb{W}$.

**Theorem 2 (Uniqueness):** Suppose that the graphon game $G(S, J, W)$ satisfies Assumptions 1, 2A) and 3. Then it admits a unique Nash equilibrium.

**Proof:** We show that under these assumptions the game operator is a contraction (i.e. Lipschitz with constant strictly less than one) in the Hilbert space $[L^2([0, 1])]^n$. The conclusion then follows from Banach fixed point theorem. For any $f, g \in [L^2([0, 1])]^n$, 

$$
\|Bz\|_{L_n} = \|Bz - B\hat{z} + B\hat{z}\|_{L_n} \leq \|Bz - B\hat{z}\|_{L_n} + \|B\hat{z}\|_{L_n}
$$

$$
\leq \left( \frac{\ell_{ij}}{\alpha_{ij}} \right) \|\hat{z} - z\|_{L_n} + M \leq \left( \frac{\ell_{ij}}{\alpha_{ij}} \right) (\|\hat{z}\|_{L_n} + \|z\|_{L_n}) + M < \infty,
$$

where the second inequality follows from statement 1).

4) Under Assumption 2B) for any $x \in [0, 1]$, $(Bz)(x) \in S(x) \subseteq S$ hence

$$
\|Bz\|^2_{L_n} = \int_0^1 \|(Bz)(x)\|^2 dx \leq \int_0^1 s_{\max}^2 dx = s_{\max}^2.
$$

Consequently for any $z \in [L^2([0, 1])]^n$, $Bz \in L_S$. 

\[\square\]
If additionally Assumption 2B) holds then

\[ \|s - s_\beta\|_{L_n} \leq \frac{\ell_{J, f}}{\alpha_J} \|f - g\|_{L_n} = \frac{\ell_{J, f}}{\alpha_J} \|s - s_\beta\|_{L_n}, \]

where we used Lemma 2 for the first inequality, the fact that $W_n$ is linear in the first equality and
the fact that $\|W_n\| = \lambda_{\text{max}}(W)$, as proven in Lemma 1, in the last line. The conclusion follows from
Assumption 3.

Remark 3: Note that in Theorem 2, Assumption 2B) is not needed. In other words the strategy sets
$S(x)$ do not need to be bounded. This is because if the game operator is a contraction then existence
and uniqueness of the fixed point can be guaranteed under the sole assumption that the domain is closed
and convex, without the need of compactness.

C. Comparative statics

Finally, for graphon games satisfying the assumptions of Theorem 2, so that the Nash equilibrium is
unique, we aim at studying the effect of graphon perturbations.

Theorem 3: Suppose that the graphon game $G(S, J, W)$ satisfies Assumptions 1, 2A), 3 and let $\bar{s}$ be
its unique Nash equilibrium. Consider a perturbed graphon $\tilde{W}$ and let $\tilde{s}$ be any Nash equilibrium of
the graphon game $G(S, J, \tilde{W})$. Then it holds

\[ \|s - \tilde{s}\|_{L_n} \leq \frac{\ell_{J, /\alpha_J} ||\tilde{s}\|_{L_n}}{1 - \ell_{J, /\alpha_J} \lambda_{\text{max}}(W)} \|W - \tilde{W}\|. \]  

(11)

If additionally Assumption 2B) holds then

\[ \|s - \tilde{s}\|_{L_n} \leq \frac{\ell_{J, /\alpha_J} s_{\text{max}}}{1 - \ell_{J, /\alpha_J} \lambda_{\text{max}}(W)} \|W - \tilde{W}\| =: \tilde{K} \|W - \tilde{W}\|. \]  

(12)

Proof: By the equivalent characterization of Nash equilibria in terms of fixed points, it holds
$\bar{s} = BWs$ and $\tilde{s} = B\tilde{W}s$ hence

\[ \|s - \tilde{s}\|_{L_n} = \|BW\bar{s} - B\tilde{W}s\|_{L_n} \leq \frac{\ell_{J, /\alpha_J}}{\alpha_J} \|W\bar{s} - \tilde{W}s\|_{L_n} \]

\[ \leq \frac{\ell_{J, /\alpha_J}}{\alpha_J} \|\bar{s} - \tilde{s}\|_{L_n} + \frac{\ell_{J, /\alpha_J}}{\alpha_J} \|W - \tilde{W}\| \|\tilde{s}\|_{L_n}. \]

where we used that $B$ is Lipschitz with constant $\ell_{J, /\alpha_J}$, as proven in Lemma 2. The first conclusion follows
from the fact that $\|W\| = \lambda_{\text{max}}(W)$ and $1 - \ell_{J, /\alpha_J} \lambda_{\text{max}}(W) > 0$ by Assumption 3. The second conclusion
follows by the fact that, under Assumption 2B), $\|\tilde{s}\|_{L_n} \leq s_{\text{max}}$, as proven in Lemma 2.

The result in Theorem 3, besides being of interest on its own, is fundamental for the finite population
analysis performed in the next section.

V. Finite population analysis

The objective of this section is to study the relation between network and graphon games. To this
end, we start by showing that any network game can be equivalently reformulated as a graphon game.
Exploiting this equivalence, we then illustrate how the two main interpretations of graphons (as limit
of graph sequences and as random graph model) can be applied in the context of strategic behavior to
derive useful insights on network games with large populations. Specifically, our first result shows that
the limit (as the graph size goes to infinity) of the Nash equilibria of a sequence of network games
with graphs $P^N$ converging to a graphon $W$ is given by the equilibrium of the graphon game with
limiting graphon $W$. Our second result shows that the Nash equilibria of network games whose graph is randomly sampled from a graphon $W$ (which we term sampled network games) converge almost surely (as the graph size goes to infinity) to the equilibrium of the graphon game with generative graphon $W$. For finite populations, we upper bound the distance between the equilibria of the graphon and of any sampled network game with high probability (i.e., we provide a rate of convergence). This result can be used in two ways. First, one can use it to judge how good of an approximation is the graphon equilibrium for the equilibrium of any (realized) sampled network game. Second, it can be used to judge how different the equilibria of different sampled network games are, thus allowing one to assess the robustness of the realized equilibrium to stochastic variations of number of agents or realized links.

A. Network games are graphon games

In network games Nash equilibria are vectors of $\mathbb{R}^{Nn}$ while in graphon games they are functions of $[L^2([0, 1])]^n$. To compare these two elements, we define a one-to-one correspondence between vector and functions using the same partition $U^{[N]}$ used to compare graph and graphons in Section II-C. Specifically, for any $n, N \in \mathbb{N}$ and any equilibrium $\bar{s}[N] \in \mathbb{R}^{Nn}$ we define the corresponding step function equilibrium $\bar{s}[N](x) \in [L^2([0, 1])]^n$ as follows

$$\bar{s}[N](x) := \bar{s}^i_{|N|}, \quad \forall x \in U_i, \quad \forall i \in \mathbb{N}[1, N].$$

By exploiting this reformulation we can compare the Nash equilibria of graphon and network games (or of network games with different population sizes) by working in the $[L^2([0, 1])]^n$ domain. Since we are working with unlabeled graphs and graphons, to compare equilibria in the $[L^2([0, 1])]^n$ domain we use the following permutation invariant metric

$$d_{\Pi}(f, g) := \inf_{\phi \in \Pi([0, 1])} \|f^\phi - g\|_{L_n},$$

for any $f, g \in [L^2([0, 1])]^n$, where we define $f^\phi(x) := f(\phi(x))$. Note that (13) is consistent with the definition of cut metric $d_{\Box}(W, V)$ given in Section II-C and, similarly, it is a proper metric if we identify the functions of $[L^2([0, 1])]^n$ up to measure preserving permutations.

The following theorem shows that the step function equilibria of any network game with graph $P^{[N]}$ coincide with the Nash equilibria of the graphon game with step function graphon $W^{[N]}$ corresponding to $P^{[N]}$.

**Theorem 4:** A vector $\bar{s}[N] \in \mathbb{R}^{Nn}$ is a Nash equilibrium of the network game $G^{[N]}(\{S^i\}_{i=1}^N, J, P^{[N]})$ with $N$ players, cost function $J$ as in (4), strategy sets $S^i$ and graph $P^{[N]}$ if and only if the corresponding step function equilibrium $\bar{s}[N](x) \in [L^2([0, 1])]^n$ is a Nash equilibrium of the graphon game $G(S^{[N]}, J, W^{[N]})$ with the same cost function, set valued function $S^{[N]}(x) := S^i$ for all $x \in U_i$ and step function graphon $W^{[N]}$ corresponding to $P^{[N]}$.

**Proof:** Suppose that $\bar{s}[N](x)$ is a Nash equilibrium of the graphon game with graphon $W^{[N]}$. Since $W^{[N]}$ is a step function over the partition $U^{[N]}$, the local aggregate $\bar{z}(x) = \int_0^1 W^{[N]}(x, y)\bar{s}[N](y)dy$ is a step function with respect to the same partition. Let $\bar{z}^i$ be the value of $\bar{z}(x)$ in $U_i$. From the definition of Nash equilibrium

$$\bar{s}[N](x) = \arg \min_{s \in S^{[N]}(x)} J(s, \bar{z}(x)) = \arg \min_{s \in S^i} J(s, \bar{z}^i) \text{ for all } x \in U_i.$$ 

Consequently, also $\bar{s}[N](x)$ is a step function with respect to $U^{[N]}$. Let $\bar{s}^i_{|N]}$ be the value of $\bar{s}[N](x)$ in $U_i$. Then $\bar{z}^i = \int_0^1 W^{[N]}(x, y)\bar{s}[N](y)dy = \frac{1}{N} \sum_{j=1}^N P^{[N]}_{ij} \bar{s}^j_{|N]}$ and $\bar{s}[N](x)$ is a Nash equilibrium of the graphon game if and only if for each $i \in \mathbb{N}[1, N]$ it holds

$$\bar{s}^i_{|N]} = \arg \min_{s \in S^i} J(s, \bar{z}^i).$$
The latter is the definition of Nash equilibrium in the network game with network \( P^{[N]} \), thus concluding the proof.

B. Limit of a sequence of network games

Consider a sequence of network games with graphs \( P^{[N]} \) of increasing size \( N \) converging to a graphon \( W \). We next show that, under suitable assumptions, the step function equilibria of the network games with graphs \( P^{[N]} \) “converge” to the equilibrium of the graphon game with graphon \( W \). In other words, the graphon game with graphon \( W \) accurately represents limiting strategic behavior in large networks.

**Theorem 5:** Consider a sequence of network games \( G^{[N]}(\{S\}i=1,...,N,J,P^{[N]}) \) where each player has cost function \( J \) and homogeneous strategy set \( S^i = S \) for all \( N \in \mathbb{N} \) and \( i \in \mathbb{N}[1,N] \). Suppose that there exists a graphon \( W \) such that \( d_\square(P^{[N]},W) \to 0 \) and that the graphon game \( G(S,J,W) \) satisfies Assumptions 1, 2B) and 3, where \( S \) is the set valued function constantly equal to \( S \). Let \( s_{[N]}(x) \) be any step function equilibrium of \( G^{[N]} \) and let \( \bar{s}(x) \) be the unique equilibrium of \( G \). Then

\[
d_\Pi(s_{[N]},\bar{s}) \to 0.
\]

**Proof:** For each \( N \), let \( W^{[N]} \) be the step function graphon corresponding to \( P^{[N]} \), so that by assumption \( d_\square(W^{[N]},W) \to 0 \). By Lemma 5 in the appendix, there exist \( \phi_{[N]} \in \Pi_{[0,1]} \) such that

\[
\sum (W^{[N]}_{\phi_{[N]}} - W_\square) \to 0.
\]

We showed in Theorem 4 that there is a one-to-one correspondence between the equilibria of the network game \( G^{[N]} \) and the equilibria of the graphon game \( G(S,J,W^{[N]}) \). Moreover, it is immediate to show that \( s_{[N]}(x) \) is a Nash equilibrium of \( G(S,J,W^{[N]}) \) if and only if \( s_{[N]}(\phi_{[N]}(x)) \) is a Nash equilibrium of the graphon game \( G(S,J,(W^{[N]})_{\phi_{[N]}}) \) with permuted graphon \( (W^{[N]})_{\phi_{[N]}} \). This operation indeed is just a node relabeling. Theorem 3 yields

\[
d_\Pi(s_{[N]},\bar{s}) = \inf_{\phi \in \Pi_{[0,1]}} \| s_{[N]}(\phi(\cdot)) - s(\cdot) \|_{L_n} \leq \| s_{[N]}(\phi_{[N]}(\cdot)) - s(\cdot) \|_{L_n} \\
\leq \frac{\ell J/\alpha J s_{\text{max}}}{1 - \ell J/\alpha J \lambda_{\text{max}}(W_\square)} \| (W^{[N]})_{\phi_{[N]}} - W_\square \| \to 0.
\]

C. Sampled network games

Suppose now that a graphon \( W \) is given and consider the inverse problem of relating its Nash equilibrium to the Nash equilibria of a network game with graph sampled from the graphon \( W \) according to the standard procedure given in Definition 4. In the following theorem, we derive a bound on the distance between such equilibria that holds for any graphon satisfying Assumption 3. A refinement of such bound is derived under the following additional regularity assumption on the graphon \( W \).

**Assumption 4 (Lipschitz continuity):** There exists a constant \( L \) and a sequence of non-overlapping intervals \( \mathcal{I}_k = [\omega_{k-1}, \omega_k) \) defined by \( 0 = \omega_0 < \cdots < \omega_{\Omega+1} = 1 \), for a (finite) \( \Omega \in \mathbb{N} \) and \( k \in \{1,...,\Omega + 1\} \), such that for any \( k,l \in \mathbb{N}[1,\Omega + 1] \), any set \( \mathcal{I}_{kl} = \mathcal{I}_k \times \mathcal{I}_l \) and pairs \( (x,y) \in \mathcal{I}_{kl} \), \( (x',y') \in \mathcal{I}_{kl} \) we have that

\[
|W(x,y) - W(x',y')| \leq L(|x - x'| + |y - y'|).
\]

Assumption 4 implies that the graphon \( W \) is piecewise Lipschitz (over the intervals \( \mathcal{I}_k \times \mathcal{I}_l \)) and is typically used in the context of graphon estimation, see e.g. [8].

In the following theorem, we use the notation \( w/s \) to compactly write statements that hold for both the weighted and the 0-1 adjacency matrices \( P^{[N]}_w \) and \( P^{[N]}_s \).
Theorem 6: Consider a graphon game $G(S,J,W)$ where each player has cost function $J$ as in (5) and homogeneous strategy set, i.e. $S(x) = S$ for all $x \in [0,1]$. Suppose that $G$ satisfies Assumptions 1, 2B), 3 and let $\bar{s}(x)$ be its unique Nash equilibrium. Then the following statements hold.

1) For any population size $N \in \mathbb{N}$, with probability at least $1 - \exp(-N/(2 \log_2 N))$ it holds

$$d_\Pi(\bar{s}^{[N]}_{w/s}, \bar{s}) \leq 14\bar{K}\left(\frac{1}{(\log_2 N)^{\frac{1}{2}}}\right),$$

where $\bar{s}^{[N]}_{w/s}(x)$ is any step function equilibrium of the sampled network game $G^{[N]}(\{S\}_{i=1}^N, J, P^{[N]}_{w/s})$ and $\bar{K} = \frac{\ell_j}{1 - \ell_j / \alpha_j \max(\{w\})}$ is as in (12). Consequently, $d_\Pi(\bar{s}^{[N]}_{w/s}, \bar{s}) \to 0$ almost surely when $N \to \infty$.

2) Suppose additionally that Assumption 4 holds and fix any sequence $\{\delta_N\}_{N=1}^\infty$ such that $\delta_N \leq e^{-1}$ and $\frac{\log(N/\delta_N)}{N} \to 0$. Then with probability at least $1 - \delta_N$ for $N$ large enough it holds

$$d_\Pi(\bar{s}^{[N]}_{s}, \bar{s}) \leq 2\bar{K} \sqrt{(L^2 - \Omega^2)d_N} + \Omega d_N =: \bar{K}\rho(N)$$

where $d_N := \frac{1}{N} + \sqrt{\frac{8 \log(N/\delta_N)}{N}} \to 0$. Moreover, with probability at least $1 - 2\delta_N$ for $N$ large enough it holds

$$d_\Pi(\bar{s}^{[N]}_{s}, \bar{s}) \leq \bar{K}\left(\rho(N) + \sqrt{\frac{4 \log(2N/\delta_N)}{N}}\right).$$

Remark 4: The previous theorem provides bounds on the distance between the equilibria of graphon and sampled network games that hold with high probability. In the first statement this probability is fixed and is equal to $1 - \exp(-N/(2 \log_2 N))$ (converging to 1 as $N \to \infty$). If additionally Assumption 4 holds, one can instead choose the desired confidence level by tuning the value of $\delta_N$. We note that the choice $\delta_N = \exp(-N/(2 \log_2 N))$ (in line with the first statement) satisfies the constraints on $\delta_N$, but many other possibilities are allowed. For example one can select constant confidence $\delta_N = \delta \in (0, e^{-1})$ or polynomial confidence $\delta_N = \frac{1}{N^k}$ for any $k > 0$, since for $N$ large enough $\frac{1}{N^k} \leq e^{-1}$ and $\frac{\log(N/\delta_N)}{N} = \frac{\log(N+1)}{N} = (k + 1) \frac{\log_2 N}{N} \to 0$.

Proof: Let $W^{[N]}_w$ and $W^{[N]}_s$ be the step function graphons corresponding to $P^{[N]}_w$ and $P^{[N]}_s$, respectively. Then by [4, Lemma 10.16] with probability at least $1 - \exp(-N/(2 \log_2 N))$ it holds

$$d_\Box(W, W^{[N]}_w) \leq \frac{20}{\sqrt{\log_2 N}}, \quad d_\Box(W, W^{[N]}_s) \leq \frac{22}{\sqrt{\log_2 N}}.$$

By definition $d_\Box(W, W^{[N]}_{w/s}) = \inf_{\phi \in \Pi_{[0,1]}} \|(W^{[N]}_{w/s})^{[N]} \phi - W\|_\Box$, hence there exists $\phi^{[N]} \in \Pi_{[0,1]}$ such that $\|(W^{[N]}_{w/s})^{[N]} \phi - W\|_\Box \leq d_\Box(W, W^{[N]}_{w/s}) + \frac{1}{\sqrt{\log_2 N}}$. Consequently, with probability at least $1 - \exp(-N/(2 \log_2 N))$ it holds

$$\|(W^{[N]}_{w/s})^{[N]} \phi - W\|_\Box \leq \frac{23}{\sqrt{\log_2 N}}.$$

As a consequence,

$$d_\Pi(\bar{s}^{[N]}_{w/s}, \bar{s}) \leq \|\bar{s}^{[N]}_{w/s}(\phi^{[N]}(\cdot)) - \bar{s}(\cdot)\|_L_n \leq \bar{K}\|\left((W^{[N]}_{w/s})^{[N]} \phi^{[N]} - W\right)\|$$

$$\leq \bar{K} \sqrt{8\|\left((W^{[N]}_{w/s})^{[N]} \phi^{[N]} - W\right)\|_\Box} \leq \bar{K} \sqrt{\frac{8 \cdot 23}{\log_2 N}} < \frac{14\bar{K}}{(\log_2 N)^{\frac{1}{4}}} =: \frac{K}{(\log_2 N)^{\frac{1}{4}}}.$$
where the first inequality follows from the definition of $d_H$, the second inequality follows by Theorem 3 with $\bar{K} := \frac{\ell_{ij}/\alpha_{ij} \Lambda_{\max}(\mathcal{W})}{1-\ell_{ij}/\alpha_{ij} \Lambda_{\max}(\mathcal{W})}$, the third inequality follows from Lemma 4 in the appendix and we set $\bar{K} := 14 \bar{K}$. To prove almost sure convergence, let us define the infinite sequence of events

$$\mathcal{E}_N := \left\{ d_H(\bar{s}_{u/s}, \bar{s}) > \frac{K}{(\log_2 N)^{\frac{1}{4}}} \right\}.$$  

From (14) it follows that $\Pr[\mathcal{E}_N] < \exp(-N/(2 \log_2 N))$ and $\sum_{N=1}^{\infty} \Pr[\mathcal{E}_N] < \sum_{N=1}^{\infty} \exp(-N/(2 \log_2 N)) < \infty$. By Borel-Cantelli lemma there exists a positive integer $\bar{N}$ such that for all $N \geq \bar{N}$, the complement of $\mathcal{E}_N$, i.e., $d_H(\bar{s}_{u/s}, \bar{s}) \leq \frac{K}{(\log_2 N)^{\frac{1}{4}}}$, holds almost surely. Since $\frac{K}{(\log_2 N)^{\frac{1}{4}}} \to 0$ we obtain $d_H(\bar{s}_{u/s}, \bar{s}) \to 0$ almost surely.

The second statement follows from the first line of equation (14) and [21, Theorem 1]. To illustrate this point, let $\phi_{[N]}^{\phi, s}$ be the permutation that sorts the random points $\{ u_i \}_{i=1}^{N}$ in increasing order (i.e. $\phi_{[N]}^{\phi, s}$ maps the uniform random variables $\{ u_i \}_{i=1}^{N}$ into their ordered statistics $\{ u_{i} \}_{i=1}^{N}$ as defined in [44]). This particular permutation is useful because, by using concentration inequality results for ordered statistics, it is possible to prove that for any $\delta_N \in (Ne^{-N/5}, e^{-1})$ with probability at least $1 - \delta_N$ it holds

$$|u(i) - x| \leq d_N \text{ for any } i \in \mathbb{N}[1, N] \text{ and any } x \in \mathcal{U}_{i}^{[N]} = [\frac{i-1}{N}, \frac{i}{N}).$$  

(15)

By combining the result in (15) with the fact that $(W_w^{[N]})(\phi_{[N]}^{\phi, s}(x, y) = W(u(i), u(j))$ for any $x \in \mathcal{U}_{i}^{[N]}, y \in \mathcal{U}_{j}^{[N]}$ and the fact that $W$ is piecewise Lipschitz continuous by Assumption 4, it is proven in [21, Theorem 1] that with probability at least $1 - \delta_N$

$$\left\| (W_{w}^{[N]})_{\phi_{[N]}^{\phi, s}} - W \right\| \leq \rho(N)$$

for $N$ large enough, since the condition $\frac{\log(N/\delta_N)}{N} \to 0$ implies that $\delta_N \leq Ne^{-N/5}$ for $N$ large enough.

The result on $d_H(\bar{s}_{w/s}^{[N]}, \bar{s})$ then follows from the first line of equation (14) with $\phi_{[N]}^{\phi, s}$ instead of $\phi_{[N]}$. The result on $d_H(\bar{s}_{s}^{[N]}, \bar{s})$ can be proven in a similar fashion upon noting that by triangular inequality

$$\left\| (W_{s}^{[N]})_{\phi_{[N]}^{\phi, s}} - W \right\| \leq \left\| (W_{s}^{[N]})_{\phi_{[N]}^{\phi, s}} - (W_{w}^{[N]})_{\phi_{[N]}^{\phi, s}} \right\| + \left\| (W_{w}^{[N]})_{\phi_{[N]}^{\phi, s}} - W \right\|$$

(16)

where the last inequality holds with probability $1 - \delta_N$ as shown in the previous part. Since both $W_{w}^{[N]}$ and $W_{w}^{[N]}$ are step function graphons (with the same steps) the distance between the corresponding operators can be upper bounded by using a result of matrix concentration inequalities [45, Theorem 1] applied to their corresponding matrices $P_{w}^{[N]}$ and $P_{w}^{[N]}$ (recall that $P_{w}^{[N]} = \mathbb{E}[P_{s}^{[N]}]$), leading to the fact that for $N$ large enough with probability $1 - \delta_N$

$$\left\| (W_{s}^{[N]})_{\phi_{[N]}^{\phi, s}} - (W_{w}^{[N]})_{\phi_{[N]}^{\phi, s}} \right\| \leq \sqrt{\frac{4 \log(2N/\delta_N)}{N}},$$

(17)

as detailed in [21, Theorem 1]. Combining (16) and (17), one can finally show that for $N$ large enough with probability $1 - 2\delta_N$ it holds

$$\left\| (W_{s}^{[N]})_{\phi_{[N]}^{\phi, s}} - W \right\| \leq \sqrt{\frac{4 \log(2N/\delta_N)}{N}} + \rho(N).$$

The result on $d_H(\bar{s}_{s}^{[N]}, \bar{s})$ then follows from the first line of equation (14) with $\phi_{[N]}^{\phi, s}$ instead of $\phi_{[N]}$. ■
Theorem 6 guarantees that for any $\varepsilon > 0$ there exists $N_\varepsilon$ sufficiently large such that for all $N \geq N_\varepsilon$ with high probability the Nash equilibrium of the graphon game is an $\varepsilon$-approximation of the Nash equilibrium of any sampled network game. In many practical contexts, it might also be of interest to quantify the distance between the equilibria of two network games sampled from the same underlying random graph model. Such a result can be used to judge the robustness of the equilibrium outcome to stochastic variations in the realized links or in the number of players. Theorem 6 immediately leads to the following corollary.

**Corollary 1:** Under the assumptions of Theorem 6 consider two graphs $P^{[N_1]}_{w/s}$ and $P^{[N_2]}_{w/s}$ sampled from the graphon $W$ and for $v \in \{1, 2\}$ let $\bar{s}^{[N_v]}_{w/s}(x)$ be any step function equilibrium of the sampled network game $G^{[N_v]}(\{S\}_{i=1}^{N_v}, J, P^{[N_v]}_{w/s})$. Assume without loss of generality that $N_1 \leq N_2$. Then, with probability at least $1 - 2\exp(-N_1/(2\log_2 N_1))$ it holds

$$d_{\Pi}(\bar{s}^{[N_1]}_{w/s}, \bar{s}^{[N_2]}_{w/s}) \leq 28\bar{K}(\log_2 N_1)^{-\frac{1}{2}}.$$ 

If additionally Assumption 4 holds then with probability at least $1 - 4\delta_{N_1}$ for $N_1$ large enough it holds

$$d_{\Pi}(\bar{s}^{[N_1]}_{w/s}, \bar{s}^{[N_2]}_{w/s}) \leq 2\bar{K}\rho(N_1) \quad \text{and} \quad d_{\Pi}(\bar{s}^{[N_1]}_{w/s}, \bar{s}^{[N_2]}_{w/s}) \leq 2\bar{K}\left(\rho(N_1) + \sqrt{4\log(2N_1/\delta_{N_1})}\right).$$

The proof of this corollary is immediate upon noting that $d_{\Pi}(\bar{s}^{[N_1]}_{w/s}, \bar{s}^{[N_2]}_{w/s}) \leq d_{\Pi}(\bar{s}^{[N_1]}_{w/s}, \bar{s}) + d_{\Pi}(\bar{s}, \bar{s}^{[N_2]}_{w/s})$, by triangular inequality.

**VI. LINEAR QUADRATIC GRAPHON GAMES**

In this section we focus on a particular class of graphon games where the strategy of each agent is scalar and non-negative so that $S(x) = \mathbb{R}_{\geq 0}$ for all $x \in [0, 1]$ and the cost function is quadratic in $s$ and linear in $z$

$$J(s, z) = \frac{1}{2}s^2 - s[\alpha z + \beta],$$

for $\alpha \in \mathbb{R}, \beta > 0$. For this case, which nests many economic’s interactions studied in the network games literature, we show that it is possible to derive a closed form expression of the graphon Nash equilibrium. Our findings parallel similar results for the finite dimensional case [16], [19]. To start, we notice that the derivative of the cost function is

$$\frac{\partial J(s, z)}{\partial s} = s - [\alpha z + \beta].$$

This implies that the best response for each agent $x$ is given by

$$s_{br}(x \mid s) = \max\{0, [\alpha z(x \mid s) + \beta]\}. \quad (19)$$

It therefore follows that $J$ satisfies Assumption 1 with $\alpha_J = 1$, $\ell_J = |\alpha|$. Note also that Assumption 2A) is met (take e.g. $\hat{z} = 0$, $M = \beta$). By Theorem 2, we then get that a Nash equilibrium exists and is unique if

$$|\alpha|\lambda_{\max}(\mathcal{W}) < 1. \quad (20)$$

To further study the properties of such equilibrium, we distinguish the two cases $\alpha > 0$ and $\alpha < 0$. 

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A. Games of strategic complements

Suppose that $\alpha > 0$. In this case, we can immediately see from (19) that the best response of each agent is an increasing function of the local aggregate $z(x \mid s)$, i.e., this is a game of strategic complements [16]. We start by proving that the unique Nash equilibrium $\bar{s}(x)$ is internal (i.e., it satisfies $\bar{s}(x) > 0$ for all $x \in [0, 1]$). To this end, note that since $\alpha, \beta > 0$ and $\bar{z}(x) := z(x \mid \bar{s}) \geq 0$, it holds $\frac{\partial J(0, \bar{z}(x))}{\partial s} < 0$. This immediately allows us to conclude that, at the Nash equilibrium, all the agents have a strictly positive strategy. If that was not the case they could in fact lower their cost by slightly increasing their strategy from $0$ to $\bar{s}$, thus obtaining the cost

$$J(\bar{s}, \bar{z}(x)) = J(0, \bar{z}(x)) + \frac{\partial J(0, \bar{z}(x))}{\partial s} \bar{s} + \mathcal{O}(\bar{s})$$

which is strictly less than $J(0, \bar{z}(x))$ for $\bar{s}$ small. Since the Nash equilibrium is internal, from (19) it must hold

$$\bar{s}(x) = \alpha z(x \mid \bar{s}) + \beta \quad \Rightarrow \quad \bar{s}(x) = \alpha (\mathbb{W} \bar{s})(x) + \beta$$

$$\Rightarrow \quad (\mathbb{I} \bar{s})(x) = \alpha (\mathbb{W} \bar{s})(x) + \beta \quad \Rightarrow \quad ((\mathbb{I} - \alpha \mathbb{W}) \bar{s})(x) = \beta 1_{[0,1]}(x)$$

(note that since $n = 1$ we have $\mathbb{W}_n = \mathbb{W}$). The condition $|\alpha|\lambda_{\text{max}}(\mathbb{W}) < 1$ derived in (20) implies, by Proposition 1, that $\alpha < 1/\|\mathbb{W}\|$. Consequently, the operator $(\mathbb{I} - \alpha \mathbb{W})$ is invertible and we obtain

$$\bar{s}(x) = \beta ((\mathbb{I} - \alpha \mathbb{W})^{-1}1_{[0,1]})(x) = \beta b_\alpha(x)$$

(21)

where $b_\alpha$ is the Bonacich centrality function, as defined in Section II-B.3. Recall that the Bonacich centrality is a strictly positive function, hence $\bar{s}(x)$ is indeed an internal equilibrium.

B. Games of strategic substitutes

If $\alpha < 0$ we see that the best response in (19) is decreasing in $z(x \mid s)$, therefore this is a game of strategic substitutes [19]. In this case it is not possible to guarantee a priori that the Nash equilibrium is internal. However, by performing the same steps as in the previous section, we can guarantee that under condition (20) the unique Nash equilibrium is internal if and only if

$$(\mathbb{I} + |\alpha| \mathbb{W}) \bar{s})(x) = \beta 1_{[0,1]}(x)$$

has a feasible solution $\bar{s}(x) > 0$ for all $x \in [0, 1]$. Note that the eigenvalues of $\mathbb{W}$ are real (by Proposition 1). Consequently the minimum eigenvalue of the operator $\mathbb{I} + |\alpha| \mathbb{W}$ is $1 + |\alpha|\lambda_{\text{min}}(\mathbb{W})$. We also know (by Proposition 1) that $\lambda_{\text{min}}(\mathbb{W}) \geq -\lambda_{\text{max}}(\mathbb{W})$. Consequently, $1 + |\alpha|\lambda_{\text{min}}(\mathbb{W}) \geq 1 - |\alpha|\lambda_{\text{max}}(\mathbb{W}) > 0$, where in the last step we used condition (20). Thus $\mathbb{I} + |\alpha| \mathbb{W}$ is invertible and an internal Nash equilibrium exists if and only if

$$\bar{s}(x) = \beta ((\mathbb{I} + |\alpha| \mathbb{W})^{-1}1_{[0,1]})(x)$$

(22)

is strictly positive. Note that it is possible to expand $(\mathbb{I} + |\alpha| \mathbb{W})^{-1}$ by using a Neumann series reformulation (as for the Bonacich centrality in (21)) however the even and odd terms of such series have opposite signs. Consequently, one cannot conclude a priori that (22) is necessarily strictly positive.

VII. Continuation of example 1

Consider again a setup where the agents are located on a line, as discussed in Example 1 and assume that they influence each other according to the minmax graphon $W$ depicted in Figure 2. Furthermore, assume that each agent aims at minimizing the quadratic cost given in (18). In this section we present some numerical results illustrating the Nash equilibrium for different values of $\alpha$ and for $\beta = 1$. 

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Fig. 3: Nash equilibrium for a linear quadratic graphon game with graphon as in Figure 1, for $\beta = 1$ and two different values of $\alpha$. The continuous line is the Nash equilibrium of the graphon game (blue for strategic complements, orange for strategic substitutes). We additionally report the step function equilibria of 4 network games with network $P_{w}^{[N]}$ sampled from $W$ for $N = 10, 50, 200, 2000$ (red, yellow, green, grey), as detailed in Section VII-B. We note that if there were no network effects then each agent would play $\max\{0, \alpha \cdot 0 + \beta\} = \beta = 1$.

A. Infinite population

We start our analysis by noticing that the eigenvalues (ordered such that $\lambda_{\text{max}}(\mathbb{W}) = \lambda_{1} \geq \lambda_{2} \geq \lambda_{3} \ldots$) and eigenfunctions of the minmax graphon operator are known and are

$$
\lambda_{h} := \frac{1}{\pi^{2}h^{2}}, \quad \psi_{h}(x) := \sqrt{2} \sin(h\pi x)
$$

for all $h \in \mathbb{N}$ [21, Section 4.3.1]. Notice that $\mathbb{W}$ has an infinite, but countable, number of nonzero eigenvalues, with an accumulation point at zero. Moreover, $\lambda_{\text{max}}(\mathbb{W}) = \frac{1}{\pi^{2}}$. The discussion in Section VI allows us to conclude that a unique Nash equilibrium exists if $|\alpha| < \frac{1}{\lambda_{\text{max}}(\mathbb{W})} = \pi^{2}$. We consider the following two cases.

1) $\alpha = 0.5$: In this case $\alpha > 0$ and it follows from Section VI-A that the unique Nash equilibrium is proportional to the Bonacich centrality function. This is illustrated in the left plot of Figure 3 (blue line).

2) $\alpha = -0.5$: In this case $\alpha < 0$. It follows from Section VI-B that if an internal equilibrium exists it must satisfy formula (22). For this particular case it turns out that the solution of (22) is strictly positive (see right plot of Figure 3, orange line) hence this is the unique equilibrium.

We note that the structure of the equilibria in cases 1) and 2) are qualitatively different. In particular, the equilibrium strategy of the “more central agents” (i.e., agents whose network effect given by the local aggregate of the other agents strategies is higher according to the minmax graphon) is higher in games of strategic complements and lower in games of strategic substitutes. This is expected since higher network effects lead to agents increasing their strategies in games of complements and decreasing their strategies in games of substitutes (i.e., free riding on other agents efforts in the context of public good games, see [18], [19]).

B. Finite population

Finally, we compare the Nash equilibrium of the graphon game derived in the previous section with the Nash equilibria of network games over sampled graphs, obtained with the sampling procedure described in Definition 4. Note that for Theorem 6 to hold Assumption 2B) is needed. In this example $S(x) = \mathbb{R}_{\geq 0}$ is unbounded; this is however only a technicality since $S(x)$ can be taken as $[0, s_{\text{max}}]$ for a suitable value of $s_{\text{max}}$ without loss of generality.\(^5\)

\(^5\)One can see from (19) that for games of strategic substitutes since $\alpha < 0$ and $z(x | s) \geq 0$ the strategy of each player must be smaller or equal to $\beta$. Hence, one can use $s_{\text{max}} = \beta = 1$. On the other hand, for games of strategic complements the largest strategies are played over the complete network. For any finite $N$, in this case we have $\vec{s} = \beta(I - 0.5 \frac{1}{N} 1_{N} 1_{N})^{-1} 1_{N} = 2$ hence one can use $s_{\text{max}} = 2$. 

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Games with complements: $\alpha = 0.5$

Games with substitutes: $\alpha = -0.5$

Fig. 4: Statistics of the $L^2([0,1])$ distance between the Nash equilibrium of the graphon game and the Nash equilibrium of 500 sampled network games as a function of the population size $N$, for $\beta = 1$ and two different values of $\alpha$. The top plots show network games with graph $P^{[N]}_w$, the bottom ones with $P^{[N]}_s$.

Figure 4 shows the empirical distance between sampled network and graphon game equilibria as a function of the population size $N$. Specifically, for each population size $N$ we constructed 500 different sampled weighted graphs $P^{[N]}_w$ by sampling 500 times $N$ values $\{u_i\}_{i=1}^N$ uniformly at random from $[0,1]$ and then sorting them so that $u_i \leq u_j$ if $i < j$ (which corresponds to the natural measure preserving permutation $\phi^{os}_{[N]} \in \Pi_{[0,1]}$ described in the proof of Theorem 6). The corresponding $P^{[N]}_s$ were computed by Bernoulli sampling. For each of the 500 realizations we computed the corresponding Nash equilibria $\tilde{s}^{[N]}_{w/s}$. Figure 4 shows the whisker plot of $\|s^{[N]}_{w/s} - \tilde{s}\|_L$ (the box denotes the [25÷75] percentiles, the whiskers denote the [0÷95] percentiles, the horizontal line is the median). We additionally report the trend of the theoretical upper bound computed in Theorem 6. Note that for the minmax graphon Assumption 4 holds with $L \equiv 2$ and $\Omega = 0$ (see Lemma 6 in the appendix). Consequently, $d_{\Pi}(\tilde{s}^{[N]}_{w/s}, \tilde{s}) = O\left(\sqrt{\frac{\log(N/\delta_N)}{N}}\right)$. In Figure 4, we used $\delta_N \equiv \delta = 0.05$ (constant in $N$ and consistent with the choice of whiskers) and we report $\gamma\sqrt{\frac{\log(N/\delta)}{N}}$ for an arbitrary scaling factor $\gamma > 0$ because we are interested in comparing the asymptotic rate. Single realizations of $s^{[N]}_{w/s}$ are illustrated in Figure 3 for $N = 10, 50, 200, 2000$.

VIII. DISCUSSION AND CONCLUSION

Motivated by the constant increase in the number of agents involved in networked systems, in this work we introduced the new framework of graphon games and we showed that it can be used in two ways. Firstly, graphon games can be used to model strategic behavior for infinite populations while accounting for local heterogeneity. Secondly, they can be used to approximate strategic behavior in large but finite network games by interpreting the graphon as a random graph model. This latter perspective might be useful from the view point of an external observer (e.g. a system operator or a government agent) that doesn’t take part in the game but is nonetheless interested in having a description of the overall equilibrium, possibly over different repetitions of the game. Our results can be used for example to assess how many nodes should be observed to have an accurate description of the overall strategic behavior (e.g., to decide on the size of a poll) or to assess the effect of stochastic network changes or of slight variations in the number of agents (e.g., modeling day to day variability in network interactions).
On a longer horizon, this new perspective suggests that, to design interventions that are robust to network variations, one should work on the latent graphon game instead of planning interventions for a specific network realization (which might be inaccurate or changing over time).

To the best of our knowledge, graphons have not been used before in game theory. The initial investigation presented in this work can consequently be extended in a number of different directions. First, in this paper we focused on symmetric and dense networks (i.e. undirected networks where the number of neighbors of each agent scales with the population size $N$). It should be possible to derive similar results for more general cases, by exploiting recent progresses in graphon theory. Second, to derive explicit equilibrium formulae, we considered the special class of linear quadratic graphon games. It would be valuable to investigate whether such results can be generalized if the presence of non-linearities and to then validate such theoretical findings over real-world networks. Finally, graphon games as defined in this work are static games. The extension to a dynamic scenario is of future interest.

**APPENDIX**

**Equivalence of graphon norms**

Given a graphon $W$ and the associated graphon operator $\mathbb{W}$ as by Definition 1, we recall that for any $p, q \geq 1$ it holds

$$ \|\mathbb{W}\|_{p,q} := \sup_{f \in L_p, \|f\|_p = 1} \|\mathbb{W}f\|_q. $$

**Lemma 4 ([46]):** For any kernel $W$ it holds

$$ \|W\|_\Box \leq \|\mathbb{W}\|_{\infty,1} \leq 4\|W\|_\Box. $$

Moreover, if $W$ is bounded (i.e. $|W| \leq 1$) then for all $p, q \in [1, \infty]$

$$ \|W\|_\Box \leq \|\mathbb{W}\|_{p,q} \leq \sqrt{2}(4\|W\|_\Box)^{\min(1-1/p,1/q)}. $$

In particular, for $p = q = 2$ we get

$$ \|W\|_\Box \leq \|\mathbb{W}\| \leq \sqrt{8}\|W\|_\Box. $$

The previous lemma is an immediate consequence of [4, Lemma 8.11] and [46, Eq. (4.4), Lemma E.6] (note that therein $\|W\|_\Box$ is denoted by $\|W\|_{\Box,1}$ and it implies that if the function $W$ is bounded (which is always the case for graphons since $W(x,y) \in [0,1]$) then the cut norm $\|W\|_\Box$ is equivalent to the operator norm $\|\mathbb{W}\|_{p,q}$ for any $p > 1$ and $q < \infty$.

**Auxiliary results**

**Lemma 5:** Consider a sequence of graphons $\{W^{[N]}\}_{N=1}^\infty$ and suppose that there exists a graphon $W$ such that $d_\Box(W^{[N]},W) \to 0$. Then there exist $\{\phi^{[N]} \in \Pi_{[0,1]}\}_{N=1}^\infty$ such that

$$ \|((\mathbb{W}^{[N]})^{\phi^{[N]}}) - \mathbb{W}\|_\Box \to 0. $$

**Proof:** By definition for any $\varepsilon > 0$ there exists $\bar{N}_\varepsilon$ such that $d_\Box(W^{[N]},W) \leq \varepsilon$ for any $N \geq \bar{N}_\varepsilon$. Moreover, $d_\Box(W^{[N]},W) = \inf_{\phi \in \Pi_{[0,1]}} \|((W^{[N]})^{\phi} - W\|_\Box$ implies that for any $N$ there exists $\phi^{[N]} \in \Pi_{[0,1]}$ such that $\|((W^{[N]})^{\phi^{[N]}} - W\|_\Box \leq d_\Box(W^{[N]},W) + \frac{1}{N}$. Consequently, for any $N > \max\{\bar{N}_\varepsilon, \frac{2}{\varepsilon}\}$ it holds

$$ \|((W^{[N]})^{\phi^{[N]}} - W\|_\Box \leq d_\Box(W^{[N]},W) + \frac{1}{N} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. $$

Therefore, $\|((W^{[N]})^{\phi^{[N]}} - W\|_\Box \to 0$. The conclusion follows by Lemma 4.
Lemma 6: For the minmax graphon \( W(x, y) = \min(x, y)(1 - \max(x, y)) \) it holds
\[
|W(x, y) - W(x', y')| \leq 2 (|x - x'| + |y - y'|)
\]

for any \( x, x', y, y' \in [0, 1] \).

Proof: Firstly we prove that \( \min(x, y) \) is Lipschitz continuous with constant 1. To this end note that \( 2 \min(x, y) = x + y - |x - y| \). Hence
\[
2|\min(x, y) - \min(x', y')| = |x - x' + y - y' - |x - y| + |x' - y'||
\]
\[
\leq |x - x'| + |y - y'| + |x - y| + |x' - y'|
\]
\[
\leq |x - x'| + |y - y'| + |x + y + x' - y' | \leq 2(|x - x'| + |y - y'|),
\]
where we used \(|a - b| \leq |a - b|\). Similarly, it can be proven that \( (1 - \max(x, y)) \) is Lipschitz continuous with constant 1. Finally,
\[
|W(x, y) - W(x', y')| = |\min(x, y)(1 - \max(x, y)) - \min(x', y')(1 - \max(x', y'))|
\]
\[
\leq |\min(x, y)(1 - \max(x, y)) - \min(x', y')(1 - \max(x, y))| + |\min(x', y')(1 - \max(x, y)) - \min(x', y')(1 - \max(x', y'))|
\]
\[
\leq |\min(x, y) - \min(x', y')(1 - \max(x, y)) + |\min(x', y')(1 - \max(x, y)) - (1 - \max(x', y'))|
\]
\[
\leq |\min(x, y) - \min(x', y')| + (1 - \max(x, y)) - (1 - \max(x', y')) | \leq 2(|x - x'| + |y - y'|).
\]

\[\blacksquare\]

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