POSITION AND MOMENTUM OBSERVABLES ON $\mathbb{R}$
AND ON $\mathbb{R}^3$

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Abstract. We characterize all position and momentum observables on $\mathbb{R}$ and on $\mathbb{R}^3$. We study some of their operational properties and discuss their covariant joint observables.

1. Introduction

In the traditional presentation of quantum mechanics, observables are represented by selfadjoint operators or, equivalently, by spectral measures. It is widely recognized that this concept is too narrow. Indeed, spectral measures correspond to measurements with perfect accuracy, never found in real experiments. A less restrictive mathematical formulation of a quantum mechanical observable is a normalized positive operator measure. This generalization allows one, among many other things, to describe measurements with limited accuracy. (For a review of positive operator measures in quantum mechanics, see \cite{4}, \cite{10}, \cite{15}.)

In this paper we take covariance and invariance with respect to suitable symmetry groups as the defining properties of an observable. For example, a position observable on $\mathbb{R}$ is defined to be an observable which is covariant under space translations and invariant under momentum boosts. We characterize all position and momentum observables on $\mathbb{R}$ and on $\mathbb{R}^3$.

In Section 2 we study position and momentum observables on $\mathbb{R}$. In Subsection 2.1 the definitions are given and in Subsection 2.2 we characterize the structure of position and momentum observables. In Subsection 2.3 we investigate the ability of a position observable to discriminate states, that is, the state distinction power. Another relevant property, the limit of resolution, is studied in Subsection 2.4. In Subsection 2.5 we consider a covariant joint observable of position and momentum in phase space and derive a lower bound for the product of their limits of resolution. Section 3 is devoted to studying position and momentum observables in $\mathbb{R}^3$. The corresponding definitions are
formulated in Subsection 3.1 and a complete classification of position and momentum observables in $\mathbb{R}^3$ is given in Subsection 3.2.

Concluding this section we fix some notations. Let $\mathcal{H}$ be a complex separable Hilbert space and $\mathcal{L}(\mathcal{H})$ the set of bounded operators on $\mathcal{H}$. A positive operator $T \in \mathcal{L}(\mathcal{H})$ of trace one is called a state and the set of all states is denoted by $\mathcal{S}(\mathcal{H})$. A positive operator $A$ bounded from above by the unit operator $I$ is called an effect and the set of all effects is denoted by $\mathcal{E}(\mathcal{H})$. Way say that the null operator $O$ and the unit operator $I$ are trivial effects. Let $\Omega$ be a nonempty set and $\mathcal{A}$ a $\sigma$-algebra of subsets of $\Omega$. A set function $E : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ is an operator measure, if it is $\sigma$-additive with respect to the strong (or equivalently, weak) operator topology.

We call an operator valued measure $E$ an observable if $E(X) \in \mathcal{E}(\mathcal{H})$ for all $X \in \mathcal{A}$ and $E(\Omega) = I$. If an observable $E$ has projections as its values, that is, $E(X)^* = E(X) = E(X)^2$ for all $X \in \mathcal{A}$, it is called a sharp observable. For an observable $E : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ and a state $T \in \mathcal{S}(\mathcal{H})$, we let $p_T^E$ denote the probability measure on $\Omega$ defined by

$$p_T^E(X) = tr[TE(X)], \quad X \in \mathcal{A}.$$  

The number $p_T^E(X)$ is interpreted as the probability of having an outcome in $X$ when the system is in the state $T$ and the observable $E$ is measured.

We denote by $\mathcal{B}(\mathbb{R}^n)$ the Borel $\sigma$-algebra of $\mathbb{R}^n$. The Fourier transform of any $f \in L^1(\mathbb{R}^n)$ is denoted by $\hat{f}$. We set also $\hat{f} = \mathcal{F}(f)$ to denote the Fourier-Plancherel transform of any $f \in L^2(\mathbb{R}^n)$ and $\hat{\mu} = \mathcal{F}(\mu)$ is the Fourier-Stieltjes transform of any complex Borel measure $\mu$ on $\mathbb{R}^n$.

2. Position and momentum observables on $\mathbb{R}$

2.1. Definitions. Let us consider a non-relativistic particle living in the one-dimensional space $\mathbb{R}$ and fix $\mathcal{H} = L^2(\mathbb{R})$. Let $U$ and $V$ be the one-parameter unitary representations on $\mathcal{H}$ related to the groups of space translations and momentum boosts. They act on $\varphi \in \mathcal{H}$ as

$$[U(q)\varphi](x) = \varphi(x - q),$$

$$[V(p)\varphi](x) = e^{ipx}\varphi(x).$$

Let $P$ and $Q$ be the selfadjoint operators generating $U$ and $V$, that is, $U(q) = e^{-iqP}$ and $V(p) = e^{ipQ}$ for every $q, p \in \mathbb{R}$. We denote by $\Pi_P$ and $\Pi_Q$ the spectral decompositions of the operators $P$ and $Q$, respectively.
They have the form
\[
[\Pi_Q(X)\varphi](x) = \chi_X(x)\varphi(x),
\]
\[
\Pi_P(X) = F^{-1}\Pi_Q(X)F.
\]

The sharp observable \(\Pi_Q\) has the property that, for all \(q, p \in \mathbb{R}\) and \(X \in B(\mathbb{R})\),

\[
(1) \quad U(q)\Pi_Q(X)U(q)^* = \Pi_Q(X + q),
\]
\[
(2) \quad V(p)\Pi_Q(X)V(p)^* = \Pi_Q(X).
\]

Equation (1) means that \(\Pi_Q\) is covariant under translations whereas (2) shows that \(\Pi_Q\) is invariant under momentum boosts. Hence, these relations suggest to call \(\Pi_Q\) a position observable. As the kinematical meaning of the observable \(\Pi_Q\) is solely in the relations (1) and (2), we take these symmetry properties as the definition of a general position observable. The observable \(\Pi_Q\) is called the canonical position observable.

**Definition 1.** An observable \(E : B(\mathbb{R}) \to \mathcal{L}(\mathcal{H})\) is a position observable on \(\mathbb{R}\) if, for all \(q, p \in \mathbb{R}\) and \(X \in B(\mathbb{R})\),

\[
(3) \quad U(q)E(X)U(q)^* = E(X + q),
\]
\[
(4) \quad V(p)E(X)V(p)^* = E(X).
\]

We will denote by \(\mathcal{POS}_\mathbb{R}\) the set of all position observables on \(\mathbb{R}\).

In an analogous way we define a momentum observable to be an observable which is covariant under momentum boosts and invariant under translations.

**Definition 2.** An observable \(F : B(\mathbb{R}) \to \mathcal{L}(\mathcal{H})\) is a momentum observable on \(\mathbb{R}\) if, for all \(q, p \in \mathbb{R}\) and \(X \in B(\mathbb{R})\),

\[
(5) \quad U(q)F(X)U(q)^* = F(X),
\]
\[
(6) \quad V(p)F(X)V(p)^* = F(X + p).
\]

Since \(FU(q) = V(-q)F\) and \(FV(p) = U(p)F\), the sharp observable \(\Pi_P = F^{-1}\Pi_QF\) satisfies (5) and (6). It is called the canonical momentum observable. Moreover, an observable \(E\) is a position observable if and only if \(F^{-1}EF\) is a momentum observable. Therefore, in the following we will restrict ourselves to the study of position observables, the results of Sections 2.2, 2.3 and 2.4 being easily converted to the case of momentum observables.

**Remark 1.** In some articles an observable \(E : B(\mathbb{R}) \to \mathcal{L}(\mathcal{H})\) satisfying the covariance condition (1) (and not necessarily the invariance
condition (2) is called a (generalized) position observable. In this paper we say that such an observable is a localization observable. These are characterized in [7], [16]. In Subsection 2.2 it is shown, especially, that every position observable is commutative. However, there exist noncommutative localization observables. Hence, the set $\mathcal{POS}_R$ is a proper subset of all localization observables.

2.2. The structure of position observables. Let $\rho : \mathcal{B}(\mathbb{R}) \to [0,1]$ be a probability measure. For any $X \in \mathcal{B}(\mathbb{R})$, the map $q \mapsto \rho(X - q)$ is bounded and measurable, and hence the equation

$$E_\rho(X) = \int \rho(X - q) \ d\Pi_Q(q)$$

(7)

defines a bounded positive operator. The map

$$\mathcal{B}(\mathbb{R}) \ni X \mapsto E_\rho(X) \in \mathcal{L}(\mathcal{H})$$

is an observable. It is straightforward to verify that the observable $E_\rho$ satisfies the covariance condition (3) and the invariance condition (4), hence it is a position observable on $\mathbb{R}$. Denote by $\delta_t$ the Dirac measure concentrated at $t$. The observable $E_{\delta_0}$ is the canonical position observable $\Pi_Q$. We may also write

$$\Pi_Q(X) = \int \delta_0(X - q) \ d\Pi_Q(q)$$

(8)

and comparing (7) to (8) we note that $E_\rho$ is obtained when the sharply concentrated Dirac measure $\delta_0$ is replaced by the probability measure $\rho$. The observable $E_\rho$ admits an interpretation as an imprecise, or fuzzy, version of the canonical position observable $\Pi_Q$. (See [1], [2], [3] for further details.)

**Proposition 1.** Any position observable $E$ on $\mathbb{R}$ is of the form $E = E_\rho$ for some probability measure $\rho : \mathcal{B}(\mathbb{R}) \to [0,1]$.

The proof of Proposition 1 is given in Appendix 4.1.

Besides covariance (1) and invariance (2), the canonical position observable $\Pi_Q$ has still more symmetry properties. Namely, let $\mathbb{R}_+$ be the set of positive real numbers regarded as a multiplicative group. It has a family of unitary representations $\{A_t \mid t \in \mathbb{R}\}$ acting on $\mathcal{H}$, and given by

$$[A_t(a)f](x) = \frac{1}{\sqrt{a}} f \left( a^{-1}(x - t) + t \right).$$

It is a direct calculation to verify that for all $a \in \mathbb{R}_+, X \in \mathcal{B}(\mathbb{R})$,

$$A_0(a)\Pi_Q(X)A_0(a)^* = \Pi_Q(aX).$$

We adopt the following terminology.
Definition 3. We say that an observable \( E : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{L}(\mathcal{H}) \) is \textit{co-variant under dilations} if there exists a unitary representation \( A \) of \( \mathbb{R}_+ \) such that for all \( a \in \mathbb{R}_+ \) and \( X \in \mathcal{B}(\mathbb{R}) \),
\begin{equation}
A(a)E(X)A(a)^* = E(aX).
\end{equation}

The canonical position observable \( \Pi_Q \) is not the only position observable which is covariant under dilations. An observable \( E_{\delta t}, t \in \mathbb{R} \), is a translated version of \( \Pi_Q \), namely, for any \( X \in \mathcal{B}(\mathbb{R}) \),
\[ E_{\delta t}(X) = \Pi_Q(X - t) = U(t)^*\Pi_Q(X)U(t). \]
Since \( A_{-t}(a) = U(t)^*A_0(a)U(t) \), the observable \( E_{\delta t} \) is covariant under dilations, with, for example, \( A = A_{-t} \).

Proposition 2. Let \( E \) be a position observable on \( \mathbb{R} \). The following conditions are equivalent:
\begin{enumerate}
\item \( E \) is covariant under dilations;
\item \( \| E(U) \| = 1 \) for every nonempty open set \( U \subset \mathbb{R} \);
\item \( E = E_{\delta t} \) for some \( t \in \mathbb{R} \);
\item \( E \) is a sharp observable.
\end{enumerate}

Proof. Let \( E \) be covariant under dilations. In a similar way as in [8, Lemma 3] it can be shown that \( \| E(U) \| = 1 \) for all nonempty open sets \( U \), and so, (a) implies (b). Assume then that (b) holds. For any nonempty open set \( U \) we get
\begin{equation}
1 = \| E(U) \| = \text{ess sup}_{x \in \mathbb{R}} \rho(x + U).
\end{equation}
It follows that \( \text{supp}(\rho) \) contains only one point. Indeed, assume on the contrary that \( \text{supp}(\rho) \) contains two points \( x_1 \neq x_2 \) and denote \( U = \{ x \in \mathbb{R} | x < \frac{1}{4}|x_1 - x_2| \} \). Since \( x_1 + U \) and \( x_2 + U \) are neighborhoods of \( x_1 \) and \( x_2 \), respectively, we have \( m_i := \rho(x_i + U) > 0 \) for \( i = 1, 2 \). Then, for any \( x \in \mathbb{R}, \rho(x + U) \leq 1 - \min(m_1, m_2) \). This is in contradiction with [10]. Hence, (b) implies (c). As previously mentioned, (c) implies (a). Clearly, (c) also implies (d). Since (d) implies (b) the proof is complete.

The dilation covariance means that the observable in question has no scale dependence. A realistic position measurement apparatus has a limited accuracy and hence it cannot define a position observable which is covariant under dilations. Thus, sharp position observables are not suitable to describe nonideal situations.

Remark 2. If \( A \) is a unitary representation of \( \mathbb{R}_+ \) satisfying Eq. [9] with \( E = E_{\delta t} \), then there exists a measurable function \( \beta : \mathbb{R} \rightarrow \mathbb{T} \).
\( \mathbb{T} = \text{the complex numbers of modulus } 1 \) such that
\[
[A(a)f](x) = \frac{1}{\sqrt{a}} \beta(x + t) \overline{\beta(a^{-1}(x + t))} f \left( a^{-1}(x + t) - t \right).
\]
In particular, \( A \) is equivalent to \( A_{-t} \). See Appendix 4.2 for more details.

### 2.3. State distinction power of a position observable.

**Definition 4.** Let \( E_1 \) and \( E_2 \) be observables on \( \mathbb{R} \). The state distinction power of \( E_2 \) is greater than or equal to \( E_1 \) if for all \( T, T' \in S(\mathcal{H}) \),
\[
p^{E_2}_T = p^{E_2}_{T'} \Rightarrow p^{E_1}_T = p^{E_1}_{T'}.
\]
In this case we denote \( E_1 \sqsubseteq E_2 \). If \( E_1 \sqsubseteq E_2 \sqsubseteq E_1 \) we say that \( E_1 \) and \( E_2 \) are informationally equivalent and denote \( E_1 \sim E_2 \). If \( E_1 \sqsubseteq E_2 \) and \( E_2 \not\sqsubseteq E_1 \), we write \( E_1 \ll E_2 \).

**Example 1.** An observable \( E : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H}) \) is trivial if \( p^{E}_T = p^{E}_{T'} \) for all states \( T, T' \in S(\mathcal{H}) \). This implies that a trivial observable \( E \) is of the form \( E(X) = \lambda(X)I, X \in \mathcal{B}(\mathbb{R}) \), for some probability measure \( \lambda \). The state distinction power of any observable \( E' \) is greater than or equal to that of the trivial observable \( E \). Clearly there is no trivial position observable on \( \mathbb{R} \).

**Example 2.** An observable \( E : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H}) \) is called informationally complete if \( p^{E}_T \neq p^{E}_{T'} \) whenever \( T \neq T' \). The state distinction power of an informationally complete observable is greater than or equal to that of any other observable \( E_1 \) on \( \mathbb{R} \). It is easy to see that there is no informationally complete position observable. Namely, let \( \psi \) be a unit vector, \( p \neq 0 \) a real number, and denote \( \psi' = V(p)\psi \). Then the states \( T = |\psi\rangle\langle\psi| \) and \( T' = |\psi'\rangle\langle\psi'| \) are different but for any position observable \( E_\rho, p^{E_\rho}_T = p^{E_\rho}_{T'} \) since \( V(p) \) commutes with all the effects \( E_\rho(X), X \in \mathcal{B}(\mathbb{R}) \).

We will next think of \( \sim \) as a relation on the set \( \mathcal{POS}_\mathbb{R} \). The relation \( \sim \) is clearly reflexive, symmetric and transitive, and hence it is an equivalence relation. We denote the equivalence class of a position observable \( E \) by \([E] \) and the space of equivalence classes as \( \mathcal{POS}_\mathbb{R}/\sim \). The relation \( \sqsubseteq \) induces a partial order in the set \( \mathcal{POS}_\mathbb{R}/\sim \) in a natural way.

Let \( E_\rho \) be a position observable and \( T \) a state. The probability measure \( p^{E_\rho}_T \) is the convolution of the probability measures \( p^{\Pi Q}_T \) and \( \rho \),
\[
p^{E_\rho}_T = p^{\Pi Q}_T * \rho.
\]
It is clear from (11) that for all \( T, T' \in S(\mathcal{H}) \),
\[
p^{\Pi Q}_T = p^{\Pi Q}_{T'} \Rightarrow p^{E_\rho}_T = p^{E_\rho}_{T'}.
\]
and hence $E_\rho \subseteq \Pi_Q$. We conclude that $[\Pi_Q]$ is the only maximal element of the partially ordered set $\mathcal{POS}_R/\sim$.

It is shown in [14, Prop. 5] that a position observable $E_\rho$ belongs to the maximal equivalence class $[\Pi_Q]$ if and only if $\text{supp}(\hat{\rho}) = \mathbb{R}$. The following proposition characterizes the equivalence classes completely.

**Proposition 3.** Let $\rho_1, \rho_2$ be probability measures on $\mathbb{R}$ and $E_{\rho_1}, E_{\rho_2}$ the corresponding position observables. Then

\[ E_{\rho_1} \sqsubseteq E_{\rho_2} \iff \text{supp}(\hat{\rho}_1) \subseteq \text{supp}(\hat{\rho}_2). \]

**Proof.** Taking the Fourier transform of Eq. (11), we get

\[ \mathcal{F}(p_{E_\rho}^\star) = \mathcal{F}(p_{\Pi_Q}^\star) \mathcal{F}(\rho). \]

Since the Fourier transform is injective, it is clear from the above relation that $\text{supp}(\hat{\rho}_1) \subseteq \text{supp}(\hat{\rho}_2)$ implies $E_{\rho_1} \subseteq E_{\rho_2}$.

Conversely, suppose $\text{supp}(\hat{\rho}_1) \not\subseteq \text{supp}(\hat{\rho}_2)$. As $\hat{\rho}_i, i = 1, 2$, are continuous functions and $\hat{\rho}_i(\xi) = \overline{\hat{\rho}_i(-\xi)}$, there exists a closed interval $[2a, 2b]$, with $0 \leq a < b$, such that $[2a, 2b] \cup [-2b, -2a] \subseteq \text{supp}(\hat{\rho}_1)$ and $(2a, 2b) \cup [-2b, -2a]) \cap \text{supp}(\hat{\rho}_2) = \emptyset$. Define the functions

\[ h_1 = \frac{1}{\sqrt{2(b-a)}} \left( \chi_{[a,b]} - \chi_{[-b,-a]} \right), \]
\[ h_2 = \frac{1}{\sqrt{2(b-a)}} \left( \chi_{[a,b]} + \chi_{[-b,-a]} \right), \]

and for $i = 1, 2$, denote

\[ h_i^*(\xi) := \overline{h_i(-\xi)}. \]

Define

\[ f_i = \mathcal{F}^{-1}(h_i), \]

and let $T_i$ be the one-dimensional projection $|f_i\rangle \langle f_i|$. We then have

\[ dp_{T_i}^{\Pi_Q}(x) = |f_i(x)|^2 dx = \left| \left( \mathcal{F}^{-1} h_i \right)(x) \right|^2 dx = \mathcal{F}^{-1}(h_i * h_i^*) (x) dx, \]

and

\[ \mathcal{F}(p_{T_i}^{\Pi_Q}) = \mathcal{F} \mathcal{F}^{-1}(h_i * h_i^*) = h_i * h_i^* \]

\[ = \frac{1}{2(b-a)} \left( 2\chi_{[a,b]} \ast \chi_{[-b,-a]} + (-1)^i \chi_{[-b,-a]} \ast \chi_{[-b,-a]} \right. \]
\[ + \left. (-1)^i \chi_{[a,b]} \ast \chi_{[a,b]} \right). \]
Since

\[ \text{supp } (\chi_{[a,b]} \ast \chi_{[b,-a]}) = [a - b, b - a], \]
\[ \text{supp } (\chi_{[a,b]} \ast \chi_{[a,h]}) = [2a, 2b], \]
\[ \text{supp } (\chi_{[-b,-a]} \ast \chi_{[-b,-a]}) = [-2b, -2a], \]

an application of (13) shows that

\[ \mathcal{F}(p_{E_1}^{F}) \neq \mathcal{F}(p_{E_2}^{F}), \]
\[ \mathcal{F}(p_{E_1}^{F}) = \mathcal{F}(p_{E_2}^{F}), \]

or in other words, \( E_{\rho_1} \not\subseteq E_{\rho_2}. \) \( \square \)

**Remark 3.** It follows from the above proposition that \( E_1 \sqcup E_2 \iff \text{supp } (\hat{\rho}_1) \subset \text{supp } (\hat{\rho}_2), \) and hence the set \( \mathcal{POS}_\rho/ \sim \) has no minimal element. Indeed, if \( \rho_2 \) is a probability measure, there always exists a probability measure \( \rho_1 \) such that \( \text{supp } (\hat{\rho}_1) \subset \text{supp } (\hat{\rho}_2). \) In fact, since \( \hat{\rho}_2 \) is continuous, \( \hat{\rho}_2(\xi) = \hat{\rho}_2(-\xi) \) and \( \hat{\rho}_2(0) = \rho_2(\mathbb{R}) \neq 0, \) there exists \( a > 0 \) such that the closed interval \([-a, a]\) is strictly contained in \( \text{supp } (\hat{\rho}_2). \)

If we define \( h = \frac{1}{\sqrt{\pi}} \chi_{[-\frac{a}{2}, \frac{a}{2}]}, f = \mathcal{F}^{-1} h, \) then \( d\rho_1(x) := |f(x)|^2 dx \) is a probability measure, and \( \text{supp } (\hat{\rho}_1) = \text{supp } (h \ast h) = [-a, a]. \)

2.4. Limit of resolution of a position observable. Let \( \Pi : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H}) \) be a sharp observable. For any nontrivial projection \( \Pi(X), \) there exist states \( T, T' \in \mathcal{S}(\mathcal{H}) \) such that \( p_T^\Pi(X) = 1 \) and \( p_{T'}^\Pi(X) = 0. \) We may say that \( \Pi(X) \) is a **sharp property** and it is **real** in the state \( T. \)

In general, an observable \( E \) has effects as its values which are not projections and, hence, not sharp properties. An effect \( B \in \mathcal{E}(\mathcal{H}) \) is called **regular** if its spectrum extends both above and below \( \frac{1}{2}. \) This means that there exist states \( T, T' \in \mathcal{S}(\mathcal{H}) \) such that \( \text{tr}[TB] > \frac{1}{2} \) and \( \text{tr}[T'B] < \frac{1}{2}. \) In this sense regular effects can be seen as **approximately realizable properties**, see [4] II.2.1. The observable \( E \) is called regular if all the nontrivial effects \( E(X) \) are regular.

It is shown in [4] Prop. 4] that if a probability measure \( \rho \) is absolutely continuous with respect to the Lebesgue measure, then the position observable \( E_\rho \) is not regular. Here we modify the notion of regularity to get a quantification of sharpness, or resolution, of position observables.

For any \( x \in \mathbb{R}, r \in \mathbb{R}_+, \) we denote the interval \([x - \frac{r}{2}, x + \frac{r}{2}]\) by \( I_x,r. \) We also denote \( I_r = I_0,r. \) Let \( E : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H}) \) be an observable and \( \alpha > 0. \) We say that \( E \) is **\( \alpha \)-regular** if all the nontrivial effects \( E(I_{x,r}), x \in \mathbb{R}, r \geq \alpha, \) are regular.
Definition 5. Let \( E : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H}) \) be an observable. We denote
\[
\gamma_E = \inf \{ \alpha > 0 \mid E \text{ is } \alpha\text{-regular} \}
\]
and say that \( \gamma_E \) is the limit of resolution of \( E \).

It follows directly from definitions that the limit of resolution of a regular observable is 0. Especially, the limit of resolution of canonical position observables is 0.

Example 3. Let \( E \) be a trivial observable (see Example 1). For any \( X \in \mathcal{B}(\mathbb{R}) \), we have either \( E(X) \geq \frac{1}{2}I \) or \( E(X) \leq \frac{1}{2}I \). Hence, \( \gamma_E = \infty \).

Proposition 4. A position observable \( E_\rho \) is \( \alpha \)-regular if and only if
\[
\text{ess sup}_{x \in \mathbb{R}} \rho(I_{x,\alpha}) > \frac{1}{2}.
\]

Proof. An effect \( E_\rho(X) \) is regular if and only if \( \| E_\rho(X) \| > \frac{1}{2} \) and \( \| E_\rho(\mathbb{R} \setminus X) \| > \frac{1}{2} \). Since the norm of the multiplicative operator \( E_\rho(X) \) is \( \text{ess sup}_{x \in \mathbb{R}} \rho(X - x) \), we conclude that \( E_\rho(X) \) is regular if and only if
\[
\text{ess sup}_{x \in \mathbb{R}} \rho(X - x) > \frac{1}{2} \quad \text{and} \quad \text{ess inf}_{x \in \mathbb{R}} \rho(X - x) < \frac{1}{2}.
\]

Thus, \( E_\rho \) is \( \alpha \)-regular if and only if, for all \( r \geq \alpha \),
\[
\text{ess sup}_{x \in \mathbb{R}} \rho(I_{x,r}) > \frac{1}{2} \quad \text{and} \quad \text{ess inf}_{x \in \mathbb{R}} \rho(I_{x,r}) < \frac{1}{2}.
\]

The second condition is always satisfied and the first is equivalent to (14).

Corollary 1. A position observable \( E_\rho \) has a finite limit of resolution and
\[
\gamma_{E_\rho} = \inf \{ \alpha > 0 \mid \text{ess sup}_{x \in \mathbb{R}} \rho(I_{x,\alpha}) > \frac{1}{2} \}.
\]

Example 4. Let us consider the case in which the probability measure has Gaussian distribution, that is,
\[
d\rho(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x - \bar{x})^2}{2\sigma^2}} dx.
\]

By Proposition 4, the position observable \( E_\rho \) is \( \alpha \)-regular if, for each \( r \geq \alpha \),
\[
\frac{1}{2} < \int_{I_{x,r}} \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(y - \bar{x})^2}{2\sigma^2}} dy = \frac{1}{\sqrt{2\pi}} \int_{-\frac{\sigma}{2\sigma}}^{\frac{\sigma}{2\sigma}} e^{-\frac{y^2}{2}} dy.
\]

It follows that the limit of resolution \( \gamma_{E_\rho} \) is proportional to the standard deviation \( \sigma \) and \( \gamma_{E_\rho} \approx 1.36 \sigma \).
2.5. Covariant joint observables of position and momentum observables. Let $E_1, E_2 : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{L}(\mathcal{H})$ be two observables. An observable $G : \mathcal{B}(\mathbb{R}^2) \rightarrow \mathcal{L}(\mathcal{H})$ is their joint observable if for all $X, Y \in \mathcal{B}(\mathbb{R})$,

$$
E_1(X) = G(X \times \mathbb{R}), \\
E_2(Y) = G(\mathbb{R} \times Y).
$$

In this case $E_1$ and $E_2$ are the margins of $G$.

For all $(q, p) \in \mathbb{R}^2$, we denote $W(q, p) = e^{-iqP + ipQ} = e^{iqp/2}U(q)V(p)$.

The mapping $W : (q, p) \mapsto W(q, p)$ is an irreducible projective representation of the phase space translation group $\mathbb{R}^2$ in $\mathcal{H}$. An observable $G : \mathcal{B}(\mathbb{R}^2) \rightarrow \mathcal{L}(\mathcal{H})$ is called a covariant phase space observable if for all $(q, p) \in \mathbb{R}^2$ and $Z \in \mathcal{B}(\mathbb{R}^2)$,

$$
W(q, p)G(Z)W(q, p)^* = G(Z + (q, p)).
$$

It is proved in [6, III.A.] that all covariant phase space observables are of the form

$$
G_T(Z) = \frac{1}{2\pi} \int Z W(q, p)TW(q, p)^* \ dq \ dp
$$

for some $T \in \mathcal{S}(\mathcal{H})$.

The margins of a covariant phase space observable $G_T$ are position and momentum observables. Indeed, let $\sum_i \lambda_i |\varphi_i\rangle\langle \varphi_i|$ be the spectral decomposition of $T$. A straightforward calculation shows that

$$
G_T(X \times \mathbb{R}) = \int \rho(X - q) \ d\Pi_Q(q) = E_\rho(X),
$$

where $d\rho(q) = e(q) dq$ and $e(q) = \sum_i \lambda_i |\varphi_i|(-q)|^2$. Similarly,

$$
G_T(\mathbb{R} \times Y) = \int \nu(Y - p) \ d\Pi_P(p) = F_\nu(Y),
$$

where $d\nu(p) = f(p) dp$ and $f(p) = \sum_i \lambda_i |\hat{\varphi}_i(-p)|^2$.

The following proposition is a direct consequence of the previously mentioned results.

**Proposition 5.** A position observable $E_\rho$ [a momentum observable $F_\nu$] is a margin of a phase space observable if and only if the probability measure $\rho$ [prob. measure $\nu$] is absolutely continuous with respect to the Lebesgue measure.
As noted in Example 4, the limit of resolution of a position observable \( E_\rho \) with \( \rho \) having Gaussian distribution is proportional to the standard deviation \( \sigma \) of \( \rho \). This shows, in particular, that there exists a position observable which is a margin of a phase space observable and which has an arbitrary small positive limit of resolution. However, we next show that if position and momentum observables have a covariant phase space observable as their joint observable, then the product of limit of resolutions has a lower bound.

**Proposition 6.** Let \( E_\rho \) be a position observable and \( F_\nu \) a momentum observable. If \( E_\rho \) and \( F_\nu \) have a covariant phase space observable as their joint observable, then

\[
\gamma_{E_\rho} \cdot \gamma_{F_\nu} \geq 3 - 2\sqrt{2}.
\]

**Proof.** Since \( E_\rho \) and \( F_\nu \) have a covariant phase space observable as a joint observable there is a vector valued function \( \theta \in L^2(\mathbb{R}, \mathcal{H}) \) such that \( d\rho(q) = \|\theta(q)\|^2_H dq \) and \( d\nu(p) = \|\hat{\theta}(p)\|^2_H dp \).

By Proposition 4 the observable \( E_\rho \) is \( \alpha \)-regular if and only if

\[
es\sup_{x \in \mathbb{R}} \rho(I_{x;\alpha}) > 1/2.
\]

Since the map \( x \mapsto \rho(I_{x;\alpha}) \) is continuous, this is equivalent to

\[
\sup_{x \in \mathbb{R}} \rho(I_{x;\alpha}) = \sup_{x \in \mathbb{R}} \int_{I_{x;\alpha}} \|\theta(x)\|^2_H dx > 1/2.
\]

By the same argument, \( F_\nu \) is \( \beta \)-regular if and only if

\[
\sup_{x \in \mathbb{R}} \nu(I_{x;\beta}) = \sup_{x \in \mathbb{R}} \int_{I_{x;\beta}} \|\hat{\theta}(x)\|^2_H dx > 1/2.
\]

Using [11, Theorem 2] extended to the case of vector valued functions, we find

\[
\alpha \cdot \beta \geq 3 - 2\sqrt{2},
\]

and hence (19) follows. \( \square \)

### 3. Position and Momentum Observables on \( \mathbb{R}^3 \)

#### 3.1. Definitions.**

In this section \( \mathcal{H} = L^2(\mathbb{R}^3) \). Let \( Q_i, i = 1, 2, 3 \), denote the multiplication operator on \( \mathcal{H} \) given by \( [Q_i f](\vec{x}) = x_i f(\vec{x}) \), where \( x_i \) is the \( i \)th component of \( \vec{x} \). By \( P_i \) we mean the operator \( \mathcal{F}^{-1} Q_i \mathcal{F} \) and we denote \( \vec{Q} = (Q_1, Q_2, Q_3), \vec{P} = (P_1, P_2, P_3) \). The space translation group \( \mathbb{R}^3 \) has a unitary representation \( U(\vec{q}) = e^{-i\vec{q} \cdot \vec{P}} \) and similarly, the momentum boost group has a representation \( V(\vec{p}) = e^{i\vec{P} \cdot \vec{q}} \). It is an immediate observation that the sharp observables \( \Pi_{\vec{Q}} \) and \( \Pi_{\vec{P}} \) on \( \mathbb{R}^3 \), associated to the representations \( V \) and \( U \), respectively,
satisfy the obvious covariance and invariance conditions, analogous to (3)-(6). Let \(D\) be the representation of the rotation group \(SO(3)\) in \(\mathcal{H}\) defined as

\[
[D(R)f](\vec{x}) = f(R^{-1}\vec{x}).
\]

It is straightforward to verify that the sharp observables \(\Pi_{\vec{Q}}\) and \(\Pi_{\vec{P}}\) are covariant under rotations, that is, for all \(R \in SO(3)\) and \(X \in \mathcal{B}(\mathbb{R}^3)\),

\[
D(R)\Pi_{\vec{Q}}(X)D(R)^* = \Pi_{\vec{Q}}(RX),
\]

\[
D(R)\Pi_{\vec{P}}(X)D(R)^* = \Pi_{\vec{P}}(RX).
\]

These observations motivate the following definitions.

**Definition 6.** An observable \(E : \mathcal{B}(\mathbb{R}^3) \to \mathcal{L}(\mathcal{H})\) is a **position observable on** \(\mathbb{R}^3\) if, for all \(\vec{q}, \vec{p} \in \mathbb{R}^3\), \(R \in SO(3)\) and \(X \in \mathcal{B}(\mathbb{R}^3)\),

\[
\begin{align*}
U(\vec{q})E(X)U(\vec{q})^* &= E(X + \vec{q}), \\
V(\vec{p})E(X)V(\vec{p})^* &= E(X), \\
D(R)E(X)D(R)^* &= E(RX).
\end{align*}
\]

We will denote by \(\mathcal{POS}_{\mathbb{R}^3}\) the set of all position observables on \(\mathbb{R}^3\).

**Definition 7.** An observable \(F : \mathcal{B}(\mathbb{R}^3) \to \mathcal{L}(\mathcal{H})\) is a **momentum observable on** \(\mathbb{R}^3\) if, for all \(\vec{q}, \vec{p} \in \mathbb{R}^3\), \(R \in SO(3)\) and \(X \in \mathcal{B}(\mathbb{R}^3)\),

\[
\begin{align*}
U(\vec{q})F(X)U(\vec{q})^* &= F(X), \\
V(\vec{p})F(X)V(\vec{p})^* &= F(X + \vec{p}), \\
D(R)F(X)D(R)^* &= F(RX).
\end{align*}
\]

**3.2. Structure of position observables on** \(\mathbb{R}^3\). We say that a probability measure \(\rho\) on \(\mathbb{R}^3\) is rotation invariant if for all \(X \in \mathcal{B}(\mathbb{R}^3)\) and \(R \in SO(3)\),

\[
(R \cdot \rho)(X) := \rho(R^{-1}X) \equiv \rho(X).
\]

The set of rotation invariant probability measures on \(\mathbb{R}^3\) is denoted by \(M(\mathbb{R}^3)^{+}_{1,inv}\). Using the isomorphism \(\mathbb{R}^3 \setminus \{0\} \simeq \mathbb{R}_+ \times S^2\) and the disintegration of measures, the restriction of any measure \(\rho \in M(\mathbb{R}^3)^{+}_{1,inv}\) to the subset \(\mathbb{R}^3 \setminus \{0\}\) can be written in the form

\[
d\rho|_{\mathbb{R}^3 \setminus \{0\}}(\vec{r}) = d\rho_{\text{rad}}(r) d\rho_{\text{ang}}(\Omega),
\]

where \(d\rho_{\text{rad}}\) is a finite measure on \(\mathbb{R}_+\) with \(d\rho_{\text{rad}}(\mathbb{R}_+) = 1 - \rho(\{0\})\), and \(d\rho_{\text{ang}}\) is the \(SO(3)\)-invariant measure on the sphere \(S^2\) normalized to 1.

Given a rotation invariant probability measure \(\rho\), the formula

\[
E_\rho(X) = \int \rho(X - \vec{q}) d\Pi_{\vec{Q}}(\vec{q}), \quad X \in \mathcal{B}(\mathbb{R}^3),
\]

defines a position observable on \(\mathbb{R}^3\).
Proposition 7. Any position observable $E$ on $\mathbb{R}^3$ is of the form $E = E_\rho$ for some $\rho \in M(\mathbb{R}^3)_1^{+, \text{inv}}$.

Proof. It is shown in Appendix 4.1 that if $E$ satisfies Eqs. (20), (21), then $E$ is given by Eq. (23) for some probability measure $\rho$ in $\mathbb{R}^3$. If $\varphi \in C_c(\mathbb{R}^3)$, let

$$ E(\varphi) = \int_{\mathbb{R}^3} \varphi(\vec{x}) dE(\vec{x}). $$

For all $f \in L^2(\mathbb{R}^3)$, define the measure

$$ d\mu_f(\vec{x}) = |f(\vec{x})|^2 d\vec{x}. $$

We then have

$$ \langle f | E(\varphi) f \rangle = (\mu_f * \rho)(\varphi). $$

From (22) it then follows

$$ (\mu_{D(R)f} * \rho)(\varphi) = (\mu_f * \rho)\left(R^{-1} \cdot \varphi\right), $$

where $(R^{-1} \cdot \varphi)(\vec{x}) = \varphi(R\vec{x}) \forall \vec{x} \in \mathbb{R}^3$. Rewriting explicitly (24), we then find

$$ \int_{\mathbb{R}^3} \varphi(\vec{x} + \vec{y}) |f(R^{-1}\vec{x})|^2 d\vec{x} d\rho(\vec{y}) $$

$$ = \int_{\mathbb{R}^3} (R^{-1} \cdot \varphi)(\vec{x} + \vec{y}) |f(\vec{x})|^2 d\vec{x} d\rho(\vec{y}), $$

With some computations, setting $\psi(\vec{x}) = (R^{-1} \cdot \varphi)(-\vec{x})$, this gives

$$ \int_{\mathbb{R}^3} (\psi * |f|^2) (-\vec{y}) d(R^{-1} \cdot \rho)(\vec{y}) = \int_{\mathbb{R}^3} (\psi * |f|^2)(-\vec{y}) d\rho(\vec{y}). $$

Letting $\psi$ and $f$ vary in $C_c(\mathbb{R}^3)$, the functions $\psi * |f|^2$ span a dense subset of $C_0(\mathbb{R}^3)$. From Eq. (25), it then follows that $R^{-1} \cdot \rho = \rho$. □

Proposition 8. Let $E$ be a position observable on $\mathbb{R}^3$. The following facts are equivalent:

(a) $\|E(U)\| = 1$ for every nonempty open set $U \subset \mathbb{R}$;
(b) $E$ is a sharp observablen;
(c) $E = \Pi_{\vec{Q}}$.

Proof. It is clear that (c)$\Rightarrow$(b)$\Rightarrow$(a). Hence, it is enough to show that (a) implies (c). As in the proof of Proposition 2 it follows from (a) that $\rho = \delta_{\vec{r}}$ for some $\vec{r} \in \mathbb{R}^3$. However, the probability measure $\delta_{\vec{r}}$ is rotation invariant if and only if $\vec{r} = \vec{0}$. This means that $E = \Pi_{\vec{Q}}$. □
4. Appendix

4.1. Translation covariant and boost invariant observables in dimension \( n \). Let \( N = \mathbb{R}^{n+1} \) and \( H = \mathbb{R}^n \), with the usual structure of additive Abelian groups. Denote with \( (p, t), p \in \mathbb{R}^n, t \in \mathbb{R} \), an element of \( N \). Let \( H \) act on \( N \) as

\[
\alpha_q (p, t) = (p, t + q \cdot p) \quad q \in H, \quad (p, t) \in N.
\]

The Heisenberg group is the semidirect product \( G = N \rtimes \alpha H \) (see \[13\]). We will denote an element \( nq \in G \), with \( n = (p, t) \in N \) and \( q \in H \), as \(((p, t), q)\).

Let \( W \) be the following irreducible unitary representation of \( G \) acting in \( L^2 (\mathbb{R}^n) \)

\[
W (((p, t), q) f) (x) = e^{-i(t-p \cdot x)} f (x - q).
\]

Clearly, \( W ((0, 0), q) = U (q) \), \( W ((p, 0), 0) = V (p) \), and \( W ((0, t), 0) = e^{-it} \). The groups \( H \) and \( G / N \) are naturally identified. With such an identification, the canonical projection \( \pi : G \rightarrow G / N \) is

\[
\pi ((p, t), q) = q,
\]

and an element \(((p, t), q) \in G \) acts on \( q_0 \in H \) as

\[
((p, t), q) [q_0] = \pi (((p, t), q) ((0, 0), q_0)) = q + q_0.
\]

An observable \( E \) based on \( \mathbb{R}^n \) and acting in \( L^2 (\mathbb{R}^n) \) then satisfies the analogues of Eqs. (3), (4) in dimension \( n \) if, and only if, for all \( X \in \mathcal{B} (\mathbb{R}) \) and \(((p, t), q) \in G \),

\[
W ((p, t), q) E (X) W ((p, t), q)^* = E (X + q),
\]

i.e. if and only if \( E \) is a \( W \)-covariant observable based on \( G / N \). By virtue of the Generalized Imprimitivity Theorem (see Refs. [5], [9]), \( E \) is \( W \)-covariant if and only if there exists a representation \( \sigma \) of \( N \) and an isometry \( L \) intertwining \( W \) with the induced representation \( \text{ind}^G_N (\sigma) \) such that

\[
E (X) = L^* P (X) L
\]

for all \( X \in \mathcal{B} (\mathbb{R}^n) \), where \( P \) is the canonical projection valued measure of the induced representation. Since \( \text{ind}^G_N (\sigma) \subseteq \text{ind}^G_N (\sigma') \) (as imprimitivity systems) if \( \sigma \subseteq \sigma' \) (as representations), it is not restrictive to assume that such \( \sigma \) has constant infinite multiplicity, so that there exists a positive Borel measure \( \mu_\sigma \) on \( \widehat{N} = \mathbb{R}^{n+1} \) and an infinite dimensional Hilbert space \( \mathcal{H} \) such that \( \sigma \) is the diagonal representation acting in \( L^2 (\mathbb{R}^{n+1}, \mu_\sigma; \mathcal{H}) \), i.e.

\[
[\sigma (p, t) \phi] (h, k) = e^{ip \cdot h} e^{ikt} \phi (h, k).
\]
Denote with $\gamma_{h,k}$, $h \in \mathbb{R}^n$, $k \in \mathbb{R}$ the following character of $N$

$$\gamma_{h,k}(p, t) = e^{ih \cdot p e^{ikt}}.$$ 

The action of $H$ on $\widehat{N}$ is given by

$$(q \cdot \gamma_{h,k})(p, t) = \gamma_{h,k}(\alpha_q(p, t)) = e^{i(h - kq) \cdot p e^{ikt}},$$

or in other words

$$q \cdot \gamma_{h,k} = \gamma_{h - kq, k}.$$ 

If $k \neq 0$, the orbit passing through $\gamma_{h,k}$ is

$$O_{\gamma_{h,k}} = \mathbb{R}^n \times \{k\}$$

and the corresponding stability subgroup is

$$H_{\gamma_{h,k}} = \{0\}.$$ 

From the Mackey Machine it follows that the representations

$$\rho_{h,k} := \text{ind}_N^G (\gamma_{h,k})$$

are irreducible if $k \neq 0$, $\rho_{h,k}$ and $\rho_{h',k'}$ are inequivalent if $k \neq k'$ and, fixed $k \neq 0$, $\rho_{h,k}$ and $\rho_{h',k}$ are equivalent.

The representation $\rho := \text{ind}_N^G (\sigma)$ acts on $L^2(\mathbb{R}^n, dx; L^2(\mathbb{R}^{n+1}, \mu_\sigma; \mathcal{H}))$ according to

$$[\rho ((p, t), q) f](x) = \sigma(p, t - p \cdot x) f(x - q).$$

Using the fact that $\sigma$ acts diagonally in $L^2(\mathbb{R}^{n+1}, \mu_\sigma; \mathcal{H})$ and the identification $L^2(\mathbb{R}^n, dx; L^2(\mathbb{R}^{n+1}, \mu_\sigma; \mathcal{H})) \cong L^2(\mathbb{R}^n \times \mathbb{R}^{n+1}, dx \otimes d\mu_\sigma(x); \mathcal{H})$, we find that $\rho$ acts on $L^2(\mathbb{R}^n \times \mathbb{R}^{n+1}, dx \otimes d\mu_\sigma(x); \mathcal{H})$ as

$$[\rho ((p, t), q) f](x, h, k) = e^{ih \cdot p e^{ikt}(t - p \cdot x)} f(x - q, h, k).$$

Write $\mu_\sigma = \mu_{\sigma_1} + \mu_{\sigma_2}$, where $\mu_{\sigma_1} \perp \mu_{\sigma_2}$ and $\mu_{\sigma_2}(O_{\gamma_0, -1}) = 0$, and let $\sigma = \sigma_1 \oplus \sigma_2$ be the corresponding decomposition of $\sigma$. We then have

$$\text{ind}_N^G(\sigma) = \text{ind}_N^G(\sigma_1) \oplus \text{ind}_N^G(\sigma_2),$$

where the two representations in the sum are disjoint and the sum is a direct sum of imprimitivity systems. Since $W \simeq \text{ind}_N^G (\gamma_{0, -1})$, it is not restrictive to assume $\sigma = \sigma_1$, i.e. that $\mu_\sigma$ is concentrated in the orbit $O_{\gamma_0, -1} = \mathbb{R}^n \times \{-1\} \cong \mathbb{R}^n$. Let $T$ be the following unitary operator in $L^2(\mathbb{R}^n \times \mathbb{R}^n, dx \otimes d\mu_\sigma(x); \mathcal{H})$:

$$[T f](x, h) = f(x + h, h).$$

If we define the representation $\hat{\rho}$, given by

$$[\hat{\rho} ((p, t), q) f](x, h) = e^{-i(t - p \cdot x)} f(x - q, h),$$
then $T$ intertwines $\hat{\rho}$ with $\rho$. Since $\hat{\rho} \simeq W \otimes I_{L^2(\mathbb{R}^n, \mu_\sigma; \mathcal{H})}$ and $W$ is irreducible, every isometry intertwining $W$ with $\hat{\rho}$ has the form

$$[\tilde{L}f](x, h) = f(x) \varphi(h) \quad \forall f \in L^2(\mathbb{R}^n)$$

for some $\varphi \in L^2(\mathbb{R}^n, \mu_\sigma; \mathcal{H})$ with $\|\varphi\|_{L^2} = 1$. The most general isometry $L$ intertwining $W$ with $\rho$ has then the form $L = T\tilde{L}$ for some choice of $\varphi$, and the corresponding observable is given by

$$\langle g \mid E(X)f \rangle = \langle g \mid L^*P(X)Lf \rangle = \left\langle T\tilde{L}g \mid P(X)T\tilde{L}f \right\rangle = \int_{\mathbb{R}^{2n}} \chi_X(x)f(x + h)\overline{g(x + h)} \langle \varphi(h), \varphi(h) \rangle \, dx \, d\mu_\sigma(h).$$

It follows that

$$[E(X)f](x) = f(x) \int_{\mathbb{R}^n} \chi_X(x - h) \|\varphi(h)\|^2 \, d\mu_\sigma(h) = f(x) \int_{\mathbb{R}^n} \chi_X(x - h) \, d\mu(h),$$

where $d\mu(h) = \|\varphi(h)\|^2 \, d\mu_\sigma(h)$ is a probability measure on $\mathbb{R}^n$.

4.2. **Supplement to Remark 2** Let $A'(a) = U(t)A(a)U(t)^*$. Then, $A'(a)\Pi_Q(X)A'(a)^* = \Pi_Q(aX)$. Denote with $\Pi_Q^+$ the restriction of $\Pi_Q$ to the Borel subsets of $\mathbb{R}_+$. Then, $S_0 = (A_0, \Pi_Q^+, L^2(0, +\infty))$ and $S = (A', \Pi_Q^+, L^2(0, +\infty))$ are transitive imprimitivity systems of the group $\mathbb{R}_+$ based on $\mathbb{R}_+$. Using the Mackey Imprimitivity Theorem, there exists a Hilbert space $\mathcal{K}$ such that $S = \text{ind}_{\{1\}}^{\mathbb{R}_+}(I_\mathcal{K})$, where $I_\mathcal{K}$ is the trivial representation of $\{1\}$ acting in $\mathcal{K}$. Since $S_0 = \text{ind}_{\{1\}}^{\mathbb{R}_+}(1)$, we have the isomorphism of intertwining operators $C(1, I_\mathcal{K}) \simeq C(S_0, S)$, and hence there exists an isometry $W_1 : L^2(0, +\infty) \to L^2(0, +\infty)$ intertwining $S_0$ with $S$. In particular, $W_1\Pi_Q^+ = \Pi_Q^+W_1$, and hence there exists a measurable function $\beta_1 : \mathbb{R}_+ \to \mathbb{T}$ such that

$$[W_1f](x) = \beta_1(x)f(x) \quad \forall f \in L^2(0, +\infty).$$

It follows that $W_1$ is unitary. Reasoning as above, one finds a unitary operator $W_2$ intertwining the restrictions of $A_0$ and $A'$ to $L^2(-\infty, 0)$, with

$$[W_2f](x) = \beta_2(x)f(x) \quad \forall f \in L^2(-\infty, 0),$$

for some measurable function $\beta_2 : \mathbb{R}_- \to \mathbb{T}$. Then, $\widehat{W} = W_1 \oplus W_2$ is unitary on $L^2(-\infty, +\infty)$, and $A(a) = U(t)^*\widehat{W}A_0(a)\widehat{W}^*U(t)$ has the claimed form for all $a \in \mathbb{R}_+$. 
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