Equivariant geometric K-homology for compact Lie group actions

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Abstract

Let $G$ be a compact Lie-group, $X$ a compact $G$-CW-complex. We define equivariant geometric K-homology groups $K^G_\ast(X)$, using an obvious equivariant version of the $(M, E, f)$-picture of Baum-Douglas for K-homology. We define explicit natural transformations to and from equivariant K-homology defined via KK-theory (the “official” equivariant K-homology groups) and show that these are isomorphisms.

1 Introduction

K-homology is the homology theory dual to K-theory. For index theory, concrete geometric realizations of K-homology are of relevance, as already pointed out by Atiyah [3]. In an abstract analytical setting, such a definition has been given by Kasparov [13]. About the same time, Baum and Douglas [5] proposed a very geometric picture of K-homology (using manifolds, bordism, and so on), and defined a simple map to analytic K-homology. This map was “known” to be an isomorphism. However, a detailed proof of this was only published in [7].

The relevance of a geometric picture of K-homology extends to equivariant situations. Kasparov’s analytic definition of K-homology immediately does allow for such a generalization, and this is considered to be the “correct” definition. The paper [7] is a spin-off of work on a Baum-Douglas picture for $\Gamma$-equivariant K-homology, where $\Gamma$ is a discrete group acting properly on a $\Gamma$-CW-complex.

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This requires considerable effort because of the difficulty to find equivariant vector bundles in this case. Emerson and Meyer give a very general geometric description even of bivariant equivariant K-theory, provided enough such vector bundles exist — compare [10].

In the present paper, we give a definition of $G$-equivariant K-homology for the case that $G$ is a compact Lie group, in terms of the “obvious” equivariant version of the $(M, E, f)$-picture of Baum and Douglas. Our main result is that these groups indeed are canonically isomorphic to the standard analytic equivariant K-homology groups. The main point of the construction is its simplicity, we were therefore not interested in utmost generality.

In the case of a compact Lie group, equivariant vector bundles are easy to come by, and therefore the work is much easier than in the case of a discrete proper action. We will in part follow closely the work of [7], and actually will omit detailed descriptions of the equivariant generalizations where they are obvious. In other parts, however, we will deviate from the route taken in [7] and indeed give simpler constructions. Much of our theory is an equivariant (and more geometric) version of a general theory of Jakob [13]. These constructions have no generalization to proper actions of discrete groups and therefore are not used in [7]. Moreover, we will use the full force of Kasparov’s KK-theory in some of our analytic arguments. The diligent reader is then asked to supply full arguments where necessary.

2 Equivariant geometric K-homology

Let $G$ be a compact Lie group, $(X, Y)$ be a compact $G$-CW-pair with a $G$-homotopy retraction $(X, Y) \xrightarrow{q} (W, \partial W) \xrightarrow{j} (X, Y)$. We require that $(W, \partial W)$ is a smooth $G$-spin$^c$ manifold with boundary. $G$-homotopy retraction means that $qj$ is $G$-homotopy equivalent to the identity (and the homotopy preserves $Y$).

2.1 Lemma. Every finite $G$-CW-pair, more generally every compact $G$-ENR and in particular every smooth compact $G$-manifold (absolute or relative to its boundary) has the required property, i.e. is such a homotopy retraction of a manifold with boundary.

Proof. This is trivial for a $G$-spin$^c$ manifold.

The following argument is partly somewhat sketchy, we leave it to the reader to add the necessary details.

In general, by [14], every finite $G$-CW-complex $X$ has a (closed) $G$-embedding into a finite dimensional complex linear $G$-space (using [20]) with an open $G$-invariant neighborhood $U$ with a $G$-equivariant retraction $r: U \to X$ onto $X$. Even better, every such $G$-embeddings admits such a neighborhood retraction, using [11]. In other words, a finite $G$-CW-complex is a $G$-ANR. By [1], the converse is true up to $G$-homotopy equivalence.

A complex $G$-representation in particular has a $G$-invariant spin$^c$-structure, and therefore so has $U$. Choose a $G$-invariant metric on $U$, e.g. the metric induced by a $G$-invariant Hermitian metric on the $G$-representation. Let $f$ be the distance to $X$, a $G$-invariant map on $U$. Choose $r > 0$ such that $f^{-1}([0, r])$ is compact. This is possible since $X$ is compact: choose $r$ smaller than the distance from $X$ to the complement of $U$. Choose a smooth $G$-invariant approximation $g$ to $f$, i.e. $g$ has to be sufficiently close to $f$ in the chosen metric. To construct $g$,
we can first choose a non-equivariant approximation and then average it to make it $G$-invariant. Choose a regular value $0 < r' < r$ such that $V := f^{-1}((-\infty, r'])$ is a neighborhood of $X$ and is a compact manifold (necessarily a $G$-manifold) with boundary. Its double $W$ is a $G$-manifold with inclusion $i : X \to W$ (into one of the two copies) and with retraction $W \to X$ obtained as the composition of the “fold map” and the retraction $r$ (restricted to $V$).

This covers the absolute case.

If $(X, Y)$ is a $G$-CW-pair, choose an embedding $j$ of $X$ into some linear $G$-space $E$ of real dimension $n$ (with spin$^c$-structure), and a $G$-invariant distance function. The distance to $Y$ then gives a $G$-invariant function $h : X \to [0, \infty)$ with $h(x) = 0$ if and only if $x \in Y$. Consider $X \cup_Y X$ with the obvious $\mathbb{Z}/2$-action by exchanging the two copies of $X$, and $G$-action by using the given action on both halves. Extend $h$ to a $G \times \mathbb{Z}/2$-equivariant map to $\mathbb{R}$ with $\mathbb{Z}/2$-action given by multiplication with $-1$ (and with trivial $G$-action). Let $q : X \cup_Y X \to X$ be the folding map. Taking the product of $j \circ q$ with $h : X \cup_Y X \to \mathbb{R}$ (with trivial $G$-action on $\mathbb{R}$), we obtain a $G \times \mathbb{Z}/2$-embedding of $X \times_Y X$ into $E \times \mathbb{R}$.

Construct now the $G \times \mathbb{Z}/2$-neighborhood retract $U^+$ and the manifold $W^+$ for this embedding as above. By construction, there is a well defined $\mathbb{R}$-coordinate $r$ for all points in these neighborhoods and also in $W^+$ (a priori only a continuous function). The subset $\{r = 0\}$ consisting exactly of the $\mathbb{Z}/2$-fixed points. The $\mathbb{Z}/2$-action on $W^+$ is smooth. For each $\mathbb{Z}/2$-fixed point $x \in U^+$, (being an open subset of $E \times \mathbb{R}$ with $\mathbb{Z}/2$-action fixing $E$ and acting as $-1$ on $\mathbb{R}$) $T_x U^+ \cong \mathbb{R}^n \oplus \mathbb{R}_-$ as $\mathbb{Z}/2$-representation (where $\mathbb{R}$ denotes the trivial $\mathbb{Z}/2$-representation and $\mathbb{R}_-$ denotes the non-trivial $\mathbb{Z}/2$-representation). The same is then true for any $\mathbb{Z}/2$-submanifold with boundary of codimension 0, and also for a double of such a manifold, like $W^+$.

Because of this special structure of the $\mathbb{Z}/2$-fixed points it follows that $W := W^+ / \mathbb{Z}/2$ obtains the structure of a $G$-manifold with boundary, here homeomorphic to the subset $\{r \geq 0\}$ (as this is a fundamental domain for the action of $\mathbb{Z}/2$). The boundary of $W = W^+ / \mathbb{Z}/2$ is exactly the (homeomorphic) image of the fixed point set $\{r = 0\}$. The $G \times \mathbb{Z}/2$-equivariant retraction of $W^+$ onto $X \cup_Y X$ descends to a $G$-equivariant retraction of $W$ onto $X = X \cup_Y X / \mathbb{Z}/2$; the $\mathbb{Z}/2$-equivariance of the retraction implies that $\partial W$, the image of the fixed point set is mapped under this retraction to $Y$ (the image of the $\mathbb{Z}/2$-fixed point set of $X \cup_Y X$), so we really get a retraction of the pair $(W, \partial W)$ onto $(X, Y)$.

\[\square\]

2.2 Definition. A cycle for the geometric equivariant $K$-homology of $(X, Y)$ is a triple $(M, E, f)$, where

(1) $M$ is a compact smooth $G$-spin$^c$ manifold (possibly with boundary and components of different dimensions)

(2) $E$ is a $G$-equivariant Hermitean vector bundle on $M$

(3) $f : M \to X$ is a continuous $G$-equivariant map such that $f(\partial M) \subset Y$.

Here, a $G$-spin$^c$-manifold is a spin$^c$-manifold with a given spin$^c$-structure — given as in \[2\] Section 4 in terms of a complex spinor bundle for $TM$, now with a $G$-action lifted to and compatible with all the structure.

We define isomorphism of cycles $(M, E, f)$ in the obvious way, given by maps which preserve all the structure (in particular also the $G$-action).
2.3 Definition. If \((M, E, f)\) is a \(K\)-cycle for \((X, Y)\), then its opposite \(- (M, E, f)\) is the \(K\)-cycle \((- M, E, f)\), where \(- M\) denotes the manifold \(M\) equipped with the opposite spin\(^c\)-structure.

2.4 Definition. A bordism of \(K\)-cycles for the pair \((X, Y)\) consists of the following data:

(i) A smooth, compact \(G\)-manifold \(L\), equipped with a \(G\)-spin\(^c\)-structure.

(ii) A smooth, Hermitian \(G\)-vector bundle \(F\) over \(L\).

(iii) A continuous \(G\)-map \(\Phi: L \to X\).

(iv) A smooth map \(G\)-invariant map \(f: \partial L \to \mathbb{R}\) for which \(\pm 1\) are regular values, and for which \(\Phi[f^{-1}[-1, 1]] \subseteq Y\).

The sets \(M_+ = f^{-1}[+1, +\infty)\) and \(M_- = f^{-1}(-\infty, -1]\) are manifolds with boundary, and we obtain two \(K\)-cycles \((M_+, F|_{M_+}, \Phi|_{M_+})\) and \((M_-, F|_{M_-}, \Phi|_{M_-})\) for the pair \((X, Y)\). We say that the first is bordant to the opposite of the second. We follow here [7, Definition 5.5], and as above the role of \(f\) is to be able to talk of bordism of manifolds with boundary without having to introduce manifolds with corners.

2.5 Definition. Let \(M\) be a \(G\)-spin\(^c\)-manifold and let \(W\) be a \(G\)-spin\(^c\)-vector bundle of even dimension over \(M\). Denote by \(1\) the trivial, rank-one real vector bundle (with fiberwise trivial \(G\)-action). The direct sum \(W \oplus 1\) is a \(G\)-spin\(^c\)-vector bundle, and the total space of this bundle is equipped with a \(G\)-spin\(^c\) structure in the canonical way, as in [7, Definition 5.6].

Let \(Z\) be the unit sphere bundle of the bundle \(1 \oplus W\) with bundle projection \(\pi\). Observe that an element of \(Z\) has the form \((t, w)\) with \(w \in W, t \in [-1, 1]\) such that \(t^2 + |w|^2 = 1\). The subset \(\{t = 0\}\) is canonically identified with the unit sphere bundle of \(W, \{t \geq 0\}\) is called the “northern hemisphere”, \(\{t \leq 0\}\) the “southern hemisphere”. The map \(s: M \to Z; m \mapsto (1, z(m))\) is called the north pole section, where \(z: M \to W\) is the zero section. Since \(Z\) is contained in the boundary of the disk bundle, we may equip it with a natural \(G\)-spin\(^c\)-structure by first restricting the given \(G\)-spin\(^c\)-structure on the total space of \(1 \oplus W\) to the disk bundle, and then taking the boundary of this spin\(^c\)-structure to obtain a spin\(^c\)-structure on the sphere bundle.

We construct a bundle \(F\) over \(Z\) via clenching: if \(S_W\) is the spinor bundle of \(W\) (a bundle over \(M\)), then \(F\) is obtained from \(\pi^* S_{W^+}\) over the northern hemisphere of \(Z\) and \(\pi^* S_{W^-}\) over the southern hemisphere of \(Z\) by gluing along the intersection, the unit sphere bundle of \(W\), using Clifford multiplication with the respective vector of \(W\). It follows from the discussion in Appendix A that this bundle is isomorphic to \(S_{W^+}^\vee\), the dual of the even-graded part of the \(Z/2\)-graded spinor bundle \(S_{W^+}\). The latter in turn is obtained from the vertical \((G\text{-spin}^c)\)-tangent bundle of the sphere bundle of \(1 \oplus W\). The modification of a \(K\)-cycle \((M, E, f)\) associated to the bundle \(W\) is the \(K\)-cycle \((Z, F \otimes \pi^* E, f \circ \pi)\).
2.6 Definition. We define an equivalence relation on the set of isomorphism classes of cycles of Definition 2.2 as follows. It is generated by the following three elementary steps:

(1) **direct sum is disjoint union.** Given \((M, E_1, f)\) and \((M, E_2, f)\),
\[(M, E_1, f) + (M, E_2, f) \sim (M, E_1 \oplus E_2, f).\]

(2) **bordism.** If there is a bordism of K-cycles \((L, F, \Phi)\) as in Definition 2.4 with boundary the two parts \((M_1, E_1, f_1)\) and \(-(M_2, E_2, f_2)\), we set
\[(M_1, E_1, f_1) \sim (M_2, E_2, f_2).\]

(3) **modification.** If \((Z, F \otimes \pi^*E, f \circ \pi)\) is the modification of a K-cycle \((M, E, f)\) associated to the \(G\)-spin\(^c\) bundle \(W\), then
\[(Z, F \otimes \pi^*E, f \circ \pi) \sim (M, E, f).\]

2.7 Definition. For a pair \((X, Y)\) as above, we define the **equivariant geometric K-homology**
\[K_{G, \text{geom}}^*(X, Y)\] as the set of isomorphism classes of cycles as in Definition 2.2, modulo the equivalence relation of Definition 2.6.

Disjoint union of K-cycles provides a structure of \(\mathbb{Z}/2\mathbb{Z}\)-graded abelian group, graded by the parity of the dimension of the underlying manifold of a cycle.

2.8 Lemma. Given a compact \(G\)-spin\(^c\)-manifold \(M\) with boundary, a \(G\)-map \(f: (M, \partial M) \to (X, Y)\) and a class \(x \in K_0^G(M)\), we get a well defined element \([M, x, f] \in K_{G, \text{geom}}^*(X, Y)\) by representing \(x = [E] - [F]\) with two \(G\)-vector bundles \(E, F\) over \(M\) and setting

\[[M, x, f] := [M, E, f] - [M, F, f] \in K_{G, \text{geom}}^*(X, Y).\]

In the opposite direction, we can assign to each triple \((M, E, f)\) a triple \((M, [E], f)\) with \([E] = K_0^G(M)\) the K-theory class represented by \(E\).

Proof. We have to check that this construction is well defined, i.e. we have to check that \([E \oplus H] - [F \oplus H]\) gives the same geometric K-homology class, but this follows from the relation “direct sum-disjoint union”.

2.9 Remark. Lemma 2.8 allows to use a geometric picture of equivariant K-homology (for a compact Lie group \(G\)) where the bundle \(E\) is replaced by a K-theory class \(x\); and all other definitions are translated accordingly.

2.10 Definition. \(K_{G, \text{geom}}^*\) is a \(\mathbb{Z}/2\mathbb{Z}\)-graded functor from pairs of \(G\)-spaces to abelian groups. Given \(g: (X, Y) \to (X', Y')\), we define the transformation

\[g_*: K_{G, \text{geom}}^*(X, Y) \to K_{G, \text{geom}}^*(X', Y'); \quad g_*[M, E, f] := [M, E, g \circ f].\]

An inspection of our equivalence relation shows that this is well defined, and it is obviously functorial.

Moreover, we define a boundary homomorphism

\[\partial: K_{G, \text{geom}}^*(X, Y) \to K_{G, \text{geom}}^{*-1}(Y, \emptyset); \quad [M, E, f] \mapsto [\partial M, E|_{\partial M}, f|_{\partial M}].\]

Again, we observe directly from the definitions that this is compatible with the equivalence relation, natural with respect to maps of \(G\)-pairs and a group homomorphism.
Our main Theorem 3.1 shows that we have (for the subcategory of compact 
$G$-pairs which are retracts of $G$-spin$^c$ manifolds) explicit natural isomorphisms 
to $K^{G,an}_*$. In particular, we observe that on this category $K^{G,geom}_*$ with the 
above structure is a $G$-equivariant homology theory.

3 Equivariant analytic K-homology

For $G$ a compact group and $(X, Y)$ a compact $G$-CW-pair, analytic equivariant 
K-homology and analytic equivariant K-theory are defined in terms of bivariant 
KK-theory:

$$K^{G,an}_*(X, Y) := KK^G_0(\mathcal{C}_0(X \setminus Y), \mathbb{C}); \quad K^G_*(X, Y) := KK^G_0(\mathbb{C}, \mathcal{C}_0(X \setminus Y)).$$

Of course, it is well known that $K^G_0(X, Y)$ is naturally isomorphic to the Grothen- 
dieck group of $G$-vector bundle pairs over $X$ with a isomorphism over $Y$. Moreover, 
mmost constructions in equivariant K-homology and K-theory can be de- 
scribed in terms of the Kasparov product in KK-theory.

3.1 Analytic Poincaré duality

The key idea we employ to describe the relation between geometric and ana- 
lYTic K-homology is Poincaré duality in the setting of equivariant KK-theory. 

3.1 Theorem. Given any $G$-spin$^c$-manifold $M$, the Kasparov product with the 
class $[M]$ gives an isomorphism

$$\mathcal{P}D_M: RK^*_G(M) \xrightarrow{\cong} K^{G,\dim(M)}_0(\mathcal{C}_0(M)); \ x \mapsto \iota_M(x) \otimes [M].$$
3.2 Remark.

(1) For a compact space, the equivariant K-theory and the equivariant representable K-theory coincide. In particular, for a compact \( G \)-spin\(^c \)-manifold \( M \), the Poincaré duality can be stated as an isomorphism

\[
P_D^M : K^*_G(M) \xrightarrow{\cong} K^G,an_{\dim M - *}(M),
\]

and moreover, for any complex \( G \)-vector \( E \) on \( M \), \( PD^M(E) \) is the class in \( K^G,an_{\dim M - *}(M) = KK^G_{\dim M - *}(C_0(M), \mathbb{C}) \) associated to the Dirac operator \( D_E \) on \( M \) with coefficient in the complex vector bundle \( E \).

(2) Recall that representable equivariant K-theory is a functor which is invariant with respect to \( G \)-homotopies. In particular, if \( M \) is a compact \( G \)-spin\(^c \)-manifold with boundary \( \partial M \), then \( M \) is \( G \)-homotopy equivalent to its interior \( M \setminus \partial M \) and thus we get a natural identification \( K^*_G(M) \cong RK^*_G(M \setminus \partial M) \) given by restriction to \( M \setminus \partial M \) of the \( C^* \)-structure. In view of this, the Poincaré duality for the pair \( (M, \partial M) \) can be stated in the following way

\[
P_D^M : K^*_G(M) \xrightarrow{\cong} K^G,an_{\dim(M) - *}(M, \partial M),
\]

For a compact \( G \)-space \( X \) and a closed \( G \)-invariant subset \( Y \) of \( X \), let us denote by \( \iota_{X,Y} \), the composition

\[
K^*_G(X) \cong RK^*_G(X) \to RK^*_G(X \setminus Y) \xrightarrow{\iota_{X,Y}} KK^*_G(C_0(X \setminus Y), C_0(X \setminus Y)),
\]

where the first map is induced by the inclusion \( X \setminus Y \hookrightarrow X \). Then, with these notations and under the identification \( K^*_G(M) \cong RK^*_G(M \setminus \partial M) \), we get for any \( x \) in \( K^*_G(M) \) that \( PD^M(x) = i_{M, \partial M}(x) \otimes [M \setminus \partial M] \).

3.3 Definition. We are now in the situation to define the natural isomorphisms

\[
\alpha : K^*_G,geom(X, Y) \to K^*_G,an(X, Y)
\]

\[
\beta : K^*_G,an(X, Y) \to K^*_G,geom(X, Y).
\]

To define \( \alpha \), let \( (M, E, f) \) be a cycle for geometric K-homology, with \( E \) a complex \( G \)-vector bundle on \( M \). Then we set

\[
\alpha([M, E, f]) := f_* (PD^M([E])).
\]

To define \( \beta \), given \( x \in K^G,an_k(X, Y) \), choose a retraction \( (X, Y) \xrightarrow{r} (M, \partial M) \xrightarrow{\pi} (X, Y) \) with \( M \) a compact \( G \)-spin\(^c \)-manifold with \( \dim(M) \equiv k \pmod{2} \) (such a manifold exists by assumption, if the parity is not correct just take the product with \( S^1 \) with trivial \( G \)-action). Then set

\[
\beta(x) := [M, PD_M^{-1}(j_*(x)), p].
\]

3.4 Lemma. The transformation \( \alpha \) is compatible with the relation “direct sum—disjoint union” of the definition of \( K^*_G,geom(X, Y) \). Under the assumption that \( \alpha \) is well defined, it is a homomorphism.
Proof.

\[
α([M, E, f] + [N, F, g]) = α([M \amalg N, E \amalg F, f \amalg g]) = (f \amalg g)_* (PD_M([E]) \oplus PD_N([F])) = f_* (PD_M([E])) + g_* (PD_N([F])).
\]

This implies both assertions, as PD and \(f_*\) are both homomorphisms. \(\Box\)

To prove that both maps are well defined and indeed inverse to each other we need a few more properties of Poincaré duality which we collect in the sequel. These statements are certainly well known, for the convenience of the reader we give proofs of most of them in an appendix.

We first relate Poincaré duality to the Gysin homomorphism, and also describe vector bundle modification in terms of the Gysin homomorphism.

Let \(f: M \to N\) be a smooth \(G\)-map between two compact \(G\)-spin\(^2\)-manifolds without boundary. We use, as a special case of [18, Section 4.3] (see also [22, Section 7.2]), the Gysin element \(f!\) in \(KK^{-\dim M - \dim N}(C(M), C(N))\). It has the functoriality property that if \(f: M \to N\) and \(g: N \to N'\) are two smooth \(G\)-maps between compact \(G\)-spin\(^2\)-manifolds, then \(f! \otimes g! = (g \circ f)!\). We will also need the corresponding construction for manifolds with boundary. We recall all this in the appendix.

3.5 Lemma. An equivalent description of vector bundle modification, using Remark 2.8, is given as follows:

Let \((M, x, φ)\) be a cycle for \(K^G_{\text{geom}}(X, Y)\), with \(x ∈ K^0_G(M)\), and let \(W\) be a \(G\)-spin\(^2\)-vector bundle over \(M\) of even rank. Let \(π: Z → M\) be the underlying \(G\)-manifold of the modification with respect to \(W\). Recall from definition 2.8 that \(s: M → Z\) is the north pole section and that the bundle \(F\) is obtained via clutching. Then the vector bundle modification \((Z, π^*(x) ⊗ [F], φ \circ π)\) of \((M, x, φ)\) along \(W\) is bordant to the cycle \((Z, s!(x), φ \circ π)\) (where we interpret \(s!(x)\) as an element of \(K^0_G(Z)\) using formal Clifford periodicity).

Proof. The \(G\)-vector bundle \(F^∞\) from Proposition A.10 is the topological description of the Thom isomorphism, is pulled back from \(M\), hence \(π^*(x) \otimes [F^∞]\) extends to the disk bundle of \(W \oplus 1\) and we conclude that the first cycle is bordant to \((Z, π^*(x) \otimes ([F] - [F^∞]), φ \circ π)\).

We will now show that its \(K\)-theory class \(π^*(x) \otimes ([F] - [F^∞])\) agrees with

\[
s!(x) = x ⊗_{C(M)} b_W \otimes_{C_0(W)} [θ_M],
\]

where \(θ_M\) is the inclusion \(C_0(W) ⊆ C(Z)\) of \(C^*\)-algebras and \(b_W\) is the “Bott element” (compare Remark B.4). Indeed, using the \(KK\)-picture of the tensor product of vector bundles and commutativity of the exterior Kasparov product we find that, as element of \(KK^{-\dim M}(C(Z), C(Z))\), the tensor product of vector bundles \(π^*(x) \otimes ([F] - [F^∞])\) is given by

\[
x ⊗_{C(M)} ([1_{C(M)}] ⊗ ([F] - [F^∞])) ⊗_{C(M \times Z)} [μ']
\]

where \(μ: C(Z × Z) → C(Z)\) and \(μ' := μ \circ (π \times id_Z): C(M × Z) → C(Z)\) are pointwise multiplication. The claim now follows from comparing the right-hand Kasparov product (which can be computed explicitly) with \(b_W \otimes_{C_0(W)} \).
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\[ \theta_M \] (which is straightforward using the topological description in Proposition A.10 and the fact that off the tubular neighborhood \( W \), the two bundles are isomorphic).

From now on, we will use the following notation: if \( f: M \to N \) is a \( G \)-map between \( G \)-spin\(^c\)-manifolds and \( E \) is a complex vector bundle over \( M \), then \( f!E \) will stand for the element \( f![E] \) of \( K^{*+n}G(\partial M) \). It is well known that the Gysin map and functoriality in K-homology are intertwined by Poincaré duality. This is the key for proving that \( \alpha \) is compatible with vector bundle modification, using the description of the latter given in Lemma 3.5. We will prove the next assertion in B.2.

3.6 Lemma. Let \( f: M \to N \) be a \( G \)-map between \( G \)-spin\(^c\) manifolds with \( m = \dim M \) and \( n = \dim N \), possibly with boundary. Assume that \( f(\partial M) \subset \partial N \).

Then we have the following commutative diagram

\[
\begin{array}{ccc}
K^*_G(M) & \xrightarrow{PD_M} & K^{G,an}_{m-*}(M, \partial M) \\
\downarrow f^* & & \downarrow f_* \\
K^{*+n-m}G(N) & \xrightarrow{PD_N} & K^{G,an}_{n-*}(N, \partial N)
\end{array}
\]

3.7 Lemma. The transformation \( \alpha \) of Definition 3.3 is compatible with vector bundle modification.

Proof. The assertion is a direct consequence of Lemma 3.5 and Lemma 3.6. Explicitly, if \((M, E, f)\) is a cycle for \( KK_G^{geom}(X, Y) \) and \((Z, s!(E), f \circ \pi)\) the result of vector bundle modification according to Lemma 3.5 then

\[
\alpha(Z, s!(E), f \circ \pi) = f_* \pi_* s! PD_Z(s!(E)) \quad \text{[Lemma 3.6]}
\]

\[
= f_* \pi_* s! PD_M(E) \quad \text{[Lemma 3.6]} = f_* PD_M(E)
\]

We now recall that, in the usual long exact sequences in K-homology, the boundary of the fundamental class is the fundamental class, or, formulated more casually: the boundary of the Dirac element is the Dirac element of the boundary. To deal with bordisms of manifolds with boundary, we actually need a slightly more general version as follows, which we prove in Appendix B.4.

3.8 Lemma. Let \( L \) be a \( G \)-spin\(^c\) manifold with boundary \( \partial L \), let \( M \) be a \( G \)-invariant submanifold of \( \partial L \) with boundary \( \partial M \) such that \( \dim M = \dim L - 1 \) and let \( \partial \in KK_G^{geom}(C_0(M \setminus \partial M), C_0(L \setminus \partial L)) \) be the boundary element associated to the exact sequence

\[
0 \to C_0(L \setminus \partial L) \to C_0((L \setminus \partial L) \cup (M \setminus \partial M)) \to C_0(M \setminus \partial M) \to 0.
\]

Then \([\partial] \otimes [L \setminus \partial L] = [M \setminus \partial M] \).
3.9 Corollary. With notation of Lemma 3.8, the following diagram commutes
\[
\begin{array}{ccc}
K^G_0(L) & \longrightarrow & K^G_0(M) \\
\downarrow \mathcal{PD}_L & & \downarrow \mathcal{PD}_M \\
K^G_{\dim L - n}(L, \partial L) & \longrightarrow & K^G_{\dim L - n}(M, \partial M),
\end{array}
\]
where the top arrow is induced by the inclusion \(i: M \hookrightarrow L\).

Proof. Fix \(x \in K^G_0(L)\) and denote by \(x|_M\) the image of \(x\) under the homomorphism \(K^G_0(L) \to K^G_0(M)\) induced by the inclusion \(M \hookrightarrow L\). Then we get
\[
\partial \otimes \mathcal{PD}_L(x) = \partial \otimes \iota_{L, \partial L}(x) \otimes [L, \partial L] = (-1)^{\deg x} \iota_{M, \partial M}(x|_M) \otimes \partial \otimes [M \setminus \partial M] = \mathcal{PD}_M(x|_M),
\]
where the second equality is a well known consequence of the naturality of boundaries and is proved in Lemma 3.8 and where the third equality holds by Lemma 3.8.

3.10 Lemma. The transformation \(\alpha\) is compatible with the bordism relation of \(K^*_g(X, Y)\), i.e. let \((L, F, \Phi, f)\) be a bordism for a \(G\)-\(CW\)-pair \((X, Y)\). Then, with notations of Definition 3.4
\[
\alpha(M^+, F|_{M^+}, \Phi|_{M^+}) = (\Phi|_{M^+}) \cdot \mathcal{PD}_{M^+}([F|_{M^+}]) = -(\Phi|_{M^-}) \cdot \mathcal{PD}_{M^-}([F|_{M^-}]) = \alpha(M^-, F|_{M^-}, \Phi|_{M^-}).
\]

Proof. If we set \(M = M^{-1} M^+\), this amounts to prove that \((\Phi|_{M^+}) \cdot \mathcal{PD}([F|_{M^+}]) = 0\) in \(K^G_{0, an}(X, Y) = KK^G_0(C_0(X \setminus Y), \mathbb{C})\). But this a consequence of Corollary 3.9 together with naturality of boundaries in the following commutative diagram with exact rows
\[
\begin{array}{cccccc}
0 & \longrightarrow & 0 & \longrightarrow & C_0(X \setminus Y) & \longrightarrow & C_0(X \setminus Y) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_0(L \setminus \partial L) & \longrightarrow & C_0(L \setminus f^{-1}([-1, 1])) & \longrightarrow & C_0(M \setminus \partial M) & \longrightarrow & 0
\end{array}
\]
where the middle and right vertical arrows are induced by \(\Phi\).

We are now in the situation to state and prove our main theorem.

3.11 Theorem. The transformations \(\alpha\) and \(\beta\) of Definition 3.5 are well defined and inverse to each other natural transformations for \(G\)-homology theories.

Proof. Lemmas 3.4, 3.7, and 3.8 together imply that \(\alpha\) is a well defined homomorphism. If we fix, for given \((X, Y)\) the manifold \((M, \partial M)\) which retracts to \((X, Y)\) (or rather two such manifolds, one for each parity of dimensions), then \(\beta\) also is well defined. As soon as we show that \(\beta\) is inverse to \(\alpha\) we can conclude that it does not depend on the choice of \((M, \partial M)\).

It is a direct consequence of the construction (and of naturality of K-homology) that \(\alpha\) is natural with respect to maps \(g: (X, Y) \to (X', Y')\).
Corollary \[\text{5.8}\] implies that \( \alpha \) is compatible with the boundary maps of the long exact sequence of a pair, and therefore a natural transformation of homology theories (strictly speaking, we really know that \( K^{G, an}_* \) is a homology theory only after we know that \( \alpha \) is an isomorphism).

We now prove that \( \alpha \circ \beta = \text{id} \). Fix \( x \in K^{G, an}_*(X, Y) \). Then

\[
\alpha(\beta(x)) = \alpha([M, PD^{-1}(j_*(x)), p]) = p_*([PD \circ PD^{-1}(j_*(x))]) = p_*j_*(x) = x
\]

The proof of \( \beta \circ \alpha = \text{id} \) is given in the next section. \[\square\]

4 Normalization of geometric cycles

The goal of this section is to prove that \( \beta \circ \alpha : K^{G, \text{geom}}_*(X, Y) \to K^{G, \text{geom}}_*(X, Y) \) is the identity for a compact \( G \)-pair \((X, Y)\) and for any choice of retraction of \((X, Y)\) (which a priori enters the definition of \( \beta \)).

We prove first the result for a pair \((X, Y) = (N, \partial N)\), where \( N \) is a compact \( G \)-spin\(^c\)-manifold with boundary \( \partial N \).

We start with the construction of \( \beta \) given by the choice of the particular retraction \( \text{id}_N : N \to N \). We will show that with this choice \( \beta \alpha = \text{id} \). This implies of course that \( \alpha \) is invertible. This in turn means that any left inverse is equal to this inverse. As we already know that the a priori different versions of \( \beta \), depending a priori on different retractions of \((N, \partial N)\), are all left inverses of \( \alpha \), they are all equal, and equal to \( \alpha^{-1} \).

Fix now \((M, x, f)\) a cycle for \( K^{\text{geom}}_*(N, \partial N) \) as above, with \( x \) in \( K^0_G(M) \). Then

\[
\beta(\alpha[M, x, f]) = [M, PD^{-1} f_* PD(x), \text{id}] \overset{\text{Lemma}\,\text{1.4}}{=} [M, f!x, \text{id}].
\]

4.1 Theorem. Let \( h : (M, \partial M) \hookrightarrow (N, \partial N) \) be the inclusion of a \( G \)-spin\(^c\) submanifold, \( E \) a complex \( G \)-vector bundle on \( M \) —or more generally an element of \( K^0_G(M) \) — and let \( f : (N, \partial N) \to (X, Y) \) be a \( G \)-equivariant continuous map, where \((X, Y)\) is a \( G \)-space. Let \( \nu \) be the normal bundle of \( h \). Fix the trivial complex line bundle on \( N \). Then the vector bundle modification of \((M, E, f \circ h)\) “along” \( C \oplus \nu \) (with its canonical spin\(^c\)-structure) and of \((N, h!E, f)\) “along” \( C \times N \) are bordant. In particular,

\[
[M, E, f \circ h] = [N, h!E, f] \in K^{G, \text{geom}}_*(X, Y).
\]

Proof. In the situation at hand, we just can write down the bordism between the two cycles. Recall the construction of vector bundle modification (of \( N \) along \( C \times N \)): we consider \( C \times \mathbb{R} \times N \), equip this with the standard Riemannian metric, and consider the unit disc bundle \( D^3 \times N \) with its sphere bundle \( S^2 \times N \) within this bundle. It comes with a canonical “north-pole inclusion” \( i : N \to S^2 \times N \), and the modification is \( (S^2 \times N, i h!E, f \circ \text{pr}_N) \). Observe that, if \( N \) has a boundary, so has \( S^2 \times N \), and \( D^3 \times N \) is a manifold with corners.

Fix \( \epsilon > 0 \) small enough and an embedding of \( \nu \) into \( N \) as tubular neighborhood of \( M \). Fix a \( G \)-invariant Riemannian metric on \( \nu \). Then the \( \epsilon \)-disk bundle and the \( \epsilon \)-sphere bundle of \( C \oplus \mathbb{R} \oplus \nu \) are contained in \( D^3 \times N \), and if we remove the \( \epsilon \)-disk bundle we get a manifold \( W \) with two parts of its boundary being
The claim is proved. □

We now finish the proof that \( \beta(\alpha[M, E, f]) = [N, f!E, \text{id}] \) equals \([M, E, f]\).

For this, choose a finite dimensional \( G \)-representation \( V \) and a \( G \)-embedding \( j_V: M \to V \) (this is possible because \( G \) is a compact Lie group and \( M \) is compact, compare e.g. \([20]\)). Observe that \( j_V \) is \( G \)-homotopic to the constant map with value 0. Embed \( V \) into its one-point compactification \( V^+ \), a sphere (it can also be realized as the unit sphere in \( V \oplus \mathbb{R} \)). By composition we obtain a \( G \)-embedding \( j: M \to V^+ \) which is still homotopic to the constant map \( c: M \to V^+ \) with value 0.

We obtain an embedding \( [f; j] = [N \times V^+, (f, j)!E, \text{pr}_N] \), with \( \text{pr}_N \circ (f, j) = f \).

By Theorem 4.1.1 therefore
\[
[M, E, f] = [N \times V^+, (f, j)!E, \text{pr}_N].
\] (4.2)

On the other hand, \( (f, j): M \to N \times V^+ \) is \( G \)-homotopic to \( (f, c): M \to N \times V^+ \). Lemma 4.1 shows that \( (f, c)!E \) depends only on the homotopy class of the map. Therefore
\[
[N \times V^+, (f, j)!E, \text{pr}_N] = [N \times V^+, (f, c)!E, \text{pr}_N].
\] (4.3)
Finally, \((f, c) = (\id_N, c) \circ f\), and \((\id_N, c): N \to N \times V^+\) is an embedding with \(\pr_N \circ (\id_N, c) = \id_N\). Using functoriality of the Gysin homomorphism and Theorem 4.1 again, we obtain
\[
[N, f!E, \id] = [N \times V^+, (f, c)!E, \pr_N].\tag{4.4}
\]
This finishes the proof of our main theorem for a compact \(G\)-spin\(^c\)-manifold with boundary. Now if \((X, Y)\) is a compact \(G\)-pair with a retraction \((X, Y) \xrightarrow{i} (N, \partial N) \xrightarrow{p} (X, Y)\), let us consider
\[
\begin{align*}
K^G_{*, \mathrm{geom}}(X, Y) &\xrightarrow{j_*} K^G_{*, \mathrm{geom}}(N, \partial N) \\
\downarrow \beta &\downarrow \beta_N \circ \alpha_N = \id
\end{align*}
\]
By functoriality, \(j_*\) is injective (indeed a split injection). Moreover, we just showed that \(\beta_N \circ \alpha_N = \id\). According to the discussion above, the definition of \(\beta_N\) for \(N\) does not depend on the chosen retraction and we choose \(N\) as a retraction of itself. In this case, since \(\alpha\) is an isomorphism, the left square commutes and therefore \(\beta = \id\) also for \(X\), using the equality
\[
\alpha_N j_* \beta_N [M, E, f] = \alpha_N [N, \PD_N^{-1} j_* f, \PD_M[E], j \circ p]
\]
\[
= j_* p_* \PD_N \PD_N^{-1} j_* f, \PD_M[E] = j_* p_* \PD_M[E] = \alpha_N j_* [M, E, f].
\]
Therefore \(\beta\) is inverse to \(\alpha\) in general, proving our main theorem.

### A Bott periodicity and Thom isomorphism in equivariant KK-theory

Bott periodicity and the Thom isomorphism are classical results of \(K\)-theory. It is well-known that these isomorphisms can be implemented by Kasparov multiplication with certain \(KK\)-equivalences called the Bott element and Thom element, respectively. Although one finds many constructions of these elements in the literature, they are often done in a different context. As their relationship is crucial to a proper understanding of vector bundle modification we will sketch the relevant results in this appendix.

Following the usual conventions of analytic \(K\)-homology and the previous articles, \(\Cl_n = \Cl_{0, n}\) is the Clifford algebra of \(\mathbb{C}^n\) that is defined so that \(e_i e_j + e_j e_i = -2\delta_{ij}\) for the standard basis \((e_i)\). We will also need the (isomorphic) Clifford algebra \(\Cl_{-n} = \Cl_{n, 0}\) with respect to the negated quadratic form which is commonly used in \(KK\)-theory. The subgroups \(\Pin_n\) and \(\Spin_n\) are then defined as usual; again for each \(n \in \mathbb{Z}\), the ones for \(n\) are isomorphic to the ones for \(-n\). With these definitions we have \(KK_n(A, B) = KK(A, B \otimes \Cl_n)\) for all \(n \in \mathbb{Z}\).

#### A.1 Equivariant spin\(^c\)-structure of the spheres

A careful analysis of the canonical spin\(^c\)-structure on \(S^n\) is key to the results of this appendix.
Let $\Spin^{c}_{n+1}$ act on the ball $D^{n+1}$ by rotations (i.e. via the canonical homomorphism $\rho: \Spin^{c}_{n+1} \to SO_{n+1}$). Then the natural spin$^c$-structure of $D^{n+1}$ is also $\Spin^{c}_{n+1}$-equivariant (it is trivial and the group acts by rotation on the base and by left multiplication on the fiber). As the boundary $S^n = \partial D^{n+1}$ is invariant under this action, the equivariant version of the usual boundary construction induces a natural $\Spin^{c}_{n+1}$-equivariant spin$^c$-structure on $S^n$. In the following we will use the “outer normal vector first” boundary orientation convention as in [13, p. 90] or [7, 3.2] but still identify $\Spin^{c}_{n} \subseteq \Spin^{c}_{n+1}$ by the natural inclusion $\mathbb{R}^n \subseteq \mathbb{R}^{n+1}$. Hence the north pole $e_{n+1}$ is stabilized by the rotation action of $\Spin^{c}_{n}$ and we get the following lemma.

A.1 Lemma. The $\Spin^{c}_{n+1}$-equivariant principal $\Spin^{c}_{n}$-bundle of $S^n$ is

$$\Spin^{c}_{n+1} \to S^n, \quad g \mapsto (-1)^n \rho(g) e_{n+1}$$

where the left and right actions are given by multiplication.

Let us restrict the left action to $\Spin^{c}_{n} \subseteq \Spin^{c}_{n+1}$. Then the hemispheres (which we will denote by $S^n_\pm$) are invariant and a variation on the argument in [4, §13] yields the following representation:

A.2 Lemma. The $\Spin^{c}_{n}$-equivariant principal $\Spin^{c}_{n}$-bundle of $S^n$ is $\Spin^{c}_{n}$-equivariantly isomorphic to the one obtained by glueing the two bundles

$$S^n_\pm \times \Pin^{c}_{n, \pm} \to S^n_\pm$$

along the equator via the identification $(x, g) \mapsto (x, (-1)^n x g)$. Here $\Pin^{c}_{n, +} = \Spin^{c}_{n}$ and $\Pin^{c}_{n, -}$ is the other component of $\Pin^{c}_{n}$.

For every graded $\mathcal{O}$-module $W = W^+ \oplus W^-$ we have a natural isomorphism $\Pin^{c}_{n, -} \times_{\Spin^{c}_{n}} W^+ \cong W^-$. Hence the even part of the associated spinor bundle on $S^n$ is given by the analogue clutching construction applied to the bundles

$$S^n_\pm \times W^\pm \to S^n_\pm.$$

In particular if $W$ is the standard graded irreducible representation of $\mathcal{O}_{2k}$ this gives a description of the even part $\mathcal{S}^+_{2k}$ of the reduced spinor bundle of an even-dimensional sphere $S^{2k}$. We will later be interested in its dual or, equivalently, its conjugate. It is given by the same clutching construction applied to the conjugate $\mathcal{O}_{2k}$-module $\overline{W}$, and from the action of the complex volume element we see that it is isomorphic to $W$ precisely if $k$ is even.

Let us now turn to computing equivariant indices for even-dimensional spheres. It is clear that the index of its equivariant $\mathcal{O}_{2k}$-linear Dirac operator

$$\text{collapse}_+ [S^{2k}] = \text{collapse}_+ \partial [D^{2k+1}] = \partial \text{collapse}_+ [D^{2k+1}]$$

vanishes as we have factored over $R_{2k+1}(\Spin^{c}_{2k+1}) = 0$ (using naturality of the boundary map). Recall that, for a compact Lie group $G$, $R(G)$ is the complex representation ring, which is canonically isomorphic to $K_G^0(*)$, and $R_n(G) := K_G^n(*)$. This argument in fact only depends on the Dirac bundle over the sphere being induced by the boundary construction from a Dirac bundle over the ball. We conclude that the index still vanishes if we consider instead...
the reduced spinor bundle $S_{S^{2k}}$ twisted with the pullback $E$ of a representation in $R(\text{Spin}^c_{2k+1})$ (which of course extends over the ball).

On the other hand, recall that for every closed even-dimensional spin$^c$-manifold $M$, Clifford multiplication induces isomorphisms of Dirac bundles $\text{Cliff}^c(M) \cong \text{End}(S_M) \cong S_M \otimes \overline{S}_M$. If we identify $\text{Cliff}^c(M)$ with the complexified exterior bundle then an associated Dirac operator is given by the de Rham operator (cf. [12, 11.1.3]). There is a canonical involution on $\text{Cliff}^c(M)$ induced by right Clifford multiplication with $i^kE_1 \cdots E_{2k}$ where $(E_i)$ is any oriented local orthonormal frame; let us designate its positive eigenbundle by $\text{Cliff}^c_+(M)$. It is invariant under the de Rham operator, and the above maps restrict to an isomorphism

$$\text{Cliff}^c_+(M) \cong S_M \otimes \overline{S}_M.$$ 

In particular, the above construction applies to the even-dimensional sphere $M = S^{2k}$ and works equivariantly if we equip the exterior bundle with the action induced by $\rho$. Since the de Rham operator is rotation-invariant we can still use it as our Dirac operator. Its kernel consists precisely of the harmonic forms, hence in view of the cohomology of $S^{2k}$ it is spanned by a 0-form and a $2k$-form (which are rotation-invariant and interchanged by the involution). It follows that after restricting to the positive eigenbundle the kernel is just the one-dimensional trivially-graded trivial representation. In other words,

$$\text{collapse}, [S_{S^{2k}} \otimes \overline{S}_{S^{2k}}] = 1.$$ 

Expressing twisted indices as Kasparov products (cf. [8 24.5.3]) we have

$$(\overline{S}_{S^{2k}} - [E]) \otimes C(S^{2k}) [S_{S^{2k}}] = 1 \in R(\text{Spin}^c_{2k+1}) \quad (A.3)$$

for every pullback $E$ of a representation in $R(\text{Spin}^c_{2k+1})$.

### A.2 Topological Bott periodicity

We will now construct equivariant Bott elements $b_{2k} \in K^0_G(\mathbb{R}^{2k})$ where $G$ is a compact group acting spinorially on $\mathbb{R}^{2k}$ (i.e. the action factors over a continuous homomorphism $G \to \text{Spin}^c_{2k}$). Let us identify $\mathbb{R}^{2k}$ $G$-spin$^c$-structure-preservingly with an open subset of its one-point compactification $S^{2k}$ via stereographic projection from the south pole. If we now use the split short exact sequence

$$0 \longrightarrow K^0_G(\mathbb{R}^{2k}) \longrightarrow K^0_G(S^{2k}) \longrightarrow K^0_G(\ast) = R(G) \longrightarrow 0$$

to pull the south pole fiber of $F_0 := S^{2k}$ back to a bundle $F^{\infty}_0$ over the entire sphere then by exactness there is a unique preimage of $[F_0] - [F^{\infty}_0]$ which we will call the Bott element $b_{2k} \in K^0_G(\mathbb{R}^{2k})$. Using equation (A.3) we get

$$b_{2k} \otimes C_G(\mathbb{R}^{2k}) [S_{S^{2k}}] = b_{2k} \otimes C_G(\mathbb{R}^{2k}) \text{incl}^* [S_{S^{2k}}] = \text{incl}^* b_{2k} \otimes C_G(S^{2k}) [S_{S^{2k}}] = 1.$$ 

The following version of Atiyah’s rotation trick [2] now allows us to establish that $b_{2k}$ is in fact a $KK$-equivalence:
A.4 Lemma. Let \( b \in K^\mathbb{C}_0(\mathbb{R}^n) \) and \( D \in K^\mathbb{C}_0(\mathbb{R}^n) \) satisfy \( b \otimes_{\mathbb{C}_0(\mathbb{R}^n)} D = 1 \). Then they are already KK-equivalences inverse to each other.

Proof. Let, in the following, \( \otimes \) denote the external Kasparov product, and \( \otimes_A \) the composition Kasparov product. Recall first that for \( G-C^\star \)-algebras \( A, A', B, B' \) and \( z \in KK^G(A, A') \), \( z' \in KK^G(B, B') \) we have the following commutativity of the exterior Kasparov product:

\[
\tau_B(z) \otimes_{A \otimes B} \tau_{A'}(z') = \tau_A(z') \otimes_{A \otimes B'} \tau_{B'}(z) \in KK^G(A \otimes B, A' \otimes B'). \tag{A.5}
\]

(where for \( G-C^\star \)-algebras \( A, B \) and \( D, \tau_D : KK^G(A, B) \to KK^G(A \otimes D, B \otimes D) \) is external tensor product with \([1_D])\).

As the rotation \((x, y) \mapsto (y, -x)\) is \( G \)-equivariantly homotopic to the identity we get (the third identity in)

\[
D \otimes_{\mathbb{C}} b = (D \otimes [1_C]) \otimes_C ([1_C] \otimes b)
\]

\[
= ([1_C] \otimes b) \otimes_{\mathbb{C}(\mathbb{R}^{2n})} (D \otimes [1_C] \otimes b)
\]

\[
= (b \otimes \Theta) \otimes_{\mathbb{C}(\mathbb{R}^{2n})} (D \otimes [1_C] \otimes b)
\]

\[
= (b \otimes \Theta) \otimes_{\mathbb{C}(\mathbb{R}^{2n})} [1_C] \otimes \Theta
\]

where \( \Theta \) is the \( KK \)-involution corresponding to \( x \mapsto -x \).

It follows that \( b \) also has a left \( KK \)-inverse (and, in fact, \( \Theta = 1 \)). \( \square \)

A.6 Corollary (Topological Bott periodicity). Let \( G \) be a compact group acting spinorially on \( \mathbb{R}^{2k} \). Then the associated Bott element \( b_{2k} \in K^G_\mathbb{C}(\mathbb{R}^{2k}) \) and the reduced fundamental class \( [S_{\mathbb{R}_{2k}}] \in K^G_0(\mathbb{R}^{2k}) \) are \( KK \)-equivalences inverse to each other.

A.3 Analytical Bott periodicity

We will now derive an analytic cycle for the Bott element. Clearly, the difference bundle \( [F_0] - [F_0^\infty] \) is represented by the \( KK^G(\mathbb{C}(S^{2k})) \)-cycle

\[
(\Gamma(F_0) \oplus \Gamma(F_0^\infty)^{op}, \text{mul}_{\mathbb{C}}, 0). \tag{A.7}
\]

Consider the description of \( F_0 \) from the discussion after Lemma \( \text{A.2} \). Evidently, \( F_0^\infty \) can be described by a similar clutching construction given by using the southern hemisphere representation over both hemispheres and gluing using the identity. We can thus define an operator \( T' \) acting on even (odd) sections by pointwise Clifford multiplication with plus (minus) the first \( n \) coordinates of the respective base point on the upper hemisphere and by the identity on the lower hemisphere, and the linear path gives a homotopy to the cycle

\[
(\Gamma(F_0) \oplus \Gamma(F_0^\infty)^{op}, \text{mul}_{\mathbb{C}}, T').
\]

Now we can restrict to the open upper hemisphere via the homotopy given by the Hilbert \( C([0, 1], C(S^{2k})) \)-module

\[
\{ f \in C([0, 1], \Gamma(F_0) \oplus \Gamma(F_0^\infty)^{op}) : f(0) \in \Gamma_0(F_0|_{S^k_2}) \oplus \Gamma_0(F_0^\infty|_{S^k_2})^{op} \}.\]
Note that all conditions on a Kasparov triple are satisfied because $T'$ is an isomorphism on $S_{2k}$.

Identifying $\mathbb{R}^{2k}$ with the open upper hemisphere via $x \mapsto (x, 1)/\sqrt{1 + ||x||^2}$ (which is equivariantly homotopic to our previous identification) we conclude that the Bott element is given by the cycle $b_{2k} = [C_0(\mathbb{R}^{2k}, \mathcal{W}), \text{mul}_C, T] \in K_G^0(\mathbb{R}^{2k})$ with the obvious Hilbert $C_0(\mathbb{R}^{2k})$-module structure and where $T$ acts by Clifford multiplication with $\pm x/\sqrt{1 + ||x||^2}$ on the conjugate $\mathcal{W}$ of the standard graded irreducible representation of $\mathcal{C}_\mathcal{L}_{2k}$.

Chasing the relevant definitions in [16] Sections 2 and 5 one finds that the Bott element is given by the image of the fundamental class of $[\mathbb{S}_{2k}]$ under formal periodicity of $K$-homology is of course the fundamental class of $\mathbb{R}^{2k}$ we have proved the following result for $n = 2k$.

A.8 Proposition (Analytical Bott periodicity). Let $G$ be a compact group acting spinorly on $\mathbb{R}^n$. Then the Bott element $\beta_n := [C_0([\mathbb{R}^n, \mathcal{C}_\mathcal{L}_{-n}]), \text{mul}_C, T_n] \in K_G^0(\mathbb{R}^n)$ (where $T_n$ is the Clifford multiplication operator defined as above) and the fundamental class $[\mathbb{R}^n] \in K_G^0(\mathbb{R}^n)$ are $KK$-equivalences inverse to each other.

A purely analytic argument can be used to show that it holds in arbitrary dimensions (see e.g. [16] 5.7]). Note that we have $\beta_n = (\beta_1)^n$ (for appropriate group actions); this follows readily from the product formula for fundamental classes.

A.4 Thom isomorphism

Let $G$ be a compact topological group and $\pi_W : W \to X$ be a $G$-spin$^c$-vector bundle of dimension $n$ over a compact $G$-space $X$ with principal Spin$_c^n$-bundle $P$. Then $[P/\text{Spin}^c_n] \cong X$ and $P \times \text{Spin}^c_n \mathbb{R}^n \cong W$ and Kasparov’s induction machinery from [17] 3.4] is applicable. Let us define the Thom element $\beta_W$ as the image of $\beta_n$ under the composition

$$K_{G \times \text{Spin}^c_n}^0(\mathbb{R}^n) = RK_{G \times \text{Spin}^c_n}^0(\ast, C_0(\mathbb{R}^n))$$

$$\cong RK_{G \times \text{Spin}^c_n}^0(P, C_0(\mathbb{R}^n)) = RK_{G \times \text{Spin}^c_n}^0(P_0, C_0(P), C_0(P \times \mathbb{R}^n))$$

$$\cong RK_{G \times \text{Spin}^c_n}^0(\mathcal{C}(X), C_0(W)) = K_{G \times \text{Spin}^c_n}^0(\mathcal{C}(X), C_0(W))$$

Naturality of these operations shows that $\beta_W$ is a $KK$-equivalence; its inverse is given by the image of the fundamental class of $\mathbb{R}^n$ under the analogous composition. Chasing definitions, we find that

$$\beta_W = [C_0(P \times \mathbb{R}^n, \mathcal{C}_\mathcal{L}_{-n} \text{Spin}^c_n), \text{mul}_C(P \times \text{Spin}^c_n), T_n']$$
where $T'_n$ is the equivariant operator acting by Clifford multiplication with the second coordinate. Denoting the Connes-Skandalis spinor bundle $P \times_{\text{Spin}_n} \mathcal{C}l_{-n}$ of $W$ by $S^\text{CS}_W$ (cf. [9]) we get the following result:

**A.9 Proposition** (Analytical Thom isomorphism). Let $G$ be a compact group and $\pi_W: W \to X$ a $G$-spin-$c$-vector bundle over a compact space $X$. Then the Thom element

$$\beta_W = [\Gamma(\pi_W^*(S^\text{CS}_W)), \text{mul}_{C(X)}, T_W] \in KK_{-n}(C(X), C_0(W))$$

(where $T_W$ is the operator of pointwise Clifford multiplication with the base point in $W$) is a $KK$-equivalence.

Let us now consider the even case $n = 2k$. Then we can perform the same construction with the reduced Bott element $b_{2k}$ and the resulting Thom element $b_W$ is just the image of $\beta_W$ under Clifford periodicity. Clearly, $b_W$ is given by the obvious cycle using the reduced spinor bundle $S^\text{CS}_W$. If on the other hand we start with the topological description (A.7) of the Bott element, we find that

$$b_W = [\Gamma(F) \oplus \Gamma(F^\infty)^{\text{op}}, \text{mul}_{C(X)}, 0]$$

where $F := P \times_{\text{Spin}_{2k}} F_0$ is interpreted as a vector bundle over the sphere bundle $Z \cong P \times_{\text{Spin}_{2k}} S^{2k}$. We have seen before that $F_0$ can be described using an equivariant clutching construction. Consequently, the associated bundle $F$ also arises from a clutching construction and it is easy to see that it is precisely the one used for vector bundle modification:

**A.10 Proposition** (Topological Thom isomorphism). The reduced Thom element has the representation

$$b_W = [\Gamma(F) \oplus \Gamma(F^\infty)^{\text{op}}, \text{mul}_{C(X)}, 0] \in KK_0(C(X), C_0(W))$$

where $F$ is the bundle over the sphere bundle $Z$ from Definition 2.5 and where $F^\infty$ is the pullback of the north pole restriction of $F$ back to $Z$.

### B Analytic Poincaré duality and Gysin maps

**B.1 Construction of the Gysin element for closed manifolds**

Let $f: M \to N$ be a smooth $G$-map between two compact $G$-spin-$c$-manifolds without boundary. We describe the construction of the (functorial) Gysin element $f! \in KK_0^{\text{dim} M - \text{dim} N}(C(M), C(N))$ which implement the Gysin maps $f!: K^G_*(M) \to K^G_0^{\text{dim} M - \text{dim} N}(N)$.

By functoriality and since every smooth $G$-map $f: M \to N$ between compact $G$-spin-$c$-manifolds can be written as the composition of the embedding $M \to M \times N, m \mapsto (x, f(x))$ with the canonical projection $\pi_2: M \times N \to N$ it suffices to describe the Gysin element associated to an equivariant embedding and to $\pi_2$.

The Gysin element of the projection $\pi_2$ is just the element $(\pi_2)! = \tau_{C(N)}[M]$ obtained by tensoring the fundamental class of $M$ with $C(N)$. If $f: M \to N$ is
an equivariant embedding of compact $G$-spin$^c$-manifolds then its normal bundle $\nu_M$ is canonically $G$-spin$^c$ and, after fixing a $G$-invariant metric on $N$, can be considered as a $G$-invariant open tubular neighborhood of $N$. The Gysin element of $f$ is then $f! = \beta_{\nu_M} \otimes_{C_0(\nu_M)} [\theta_M]$ where $\theta_M$ is the equivariant inclusion $C_0(\nu_M) \subseteq C(N)$.

### B.2 Gysin and Poincaré duality

**Proof of Lemma [3.7] case $\partial M = \emptyset = \partial N$.** Let us denote by $[f]$ the element of $KK_*^G(C(N), C(M))$ corresponding to the morphism $C(N) \to C(M); h \mapsto h \circ f$. Then the commutativity of the diagram amounts to prove that

$$\iota_N(x \otimes f!) = [f] \otimes \iota_M(x) \otimes f!$$

for all $x$ in $K^*_C(M)$. Namely, using this equality, we have

$$\mathcal{PD}_N(x \otimes f!) = \iota_N(x \otimes f!) \otimes [N] = [f] \otimes \iota_M(x) \otimes f! \otimes [N].$$

Since $[N]$ is the Gysin element corresponding to the map $N \to \{\ast\}$, we get that $f! \otimes [N] = [M]$ and hence that

$$\mathcal{PD}_N(x \otimes f!) = [f] \otimes \iota_M(x) \otimes [M] = f_*(\mathcal{PD}_M(x)).$$

Let us now prove Equation [B.1] Since $f$ can be written as the composition of an embedding and of the projection $\pi_2: M \times N \to N$, it is enough by using the functoriality in K-homology and the composition rule for Gysin elements to check this for an embedding and for $\pi_2$.

We start with $\pi_2$. Fix $x \in K^*_C(M \times N)$. Recall that $\pi_2! = \tau_{C(N)}([M])$ and $[\pi_2] = \tau_{C(N)}([p])$, for $p: M \to \{\ast\}$, and that we can write

$$\iota_{M \times N}(x) = \tau_{C(M \times N)}(x) \otimes \mu_{M \times N},$$

where $\mu_{M \times N}: C(M \times N) \otimes C(M \times N) \to C(M \times N)$ is the multiplication. Then, using also [A.3],

$$[\pi_2] \otimes \iota_{M \times N}(x) \otimes \pi_2! = \tau_{C(N)}([p]) \otimes \tau_{C(M \times N)}(x) \otimes [\mu_{M \times N}] \otimes \pi_2! = \tau_{C(N)}([p] \otimes \tau_{C(M)}(x)) \otimes [\mu_{M \times N}] \otimes \pi_2! = \tau_{C(N)}(x \otimes \tau_{C(M \times N)}([p])) \otimes [\mu_{M \times N}] \otimes \pi_2! = \tau_{C(N)}(x) \otimes \tau_{C(N \times M \times N)}([p]) \otimes [\mu_{M \times N}] \otimes \pi_2! = \tau_{C(N)}(x) \otimes \tau_{C(N \times N)}([M]) \otimes [\mu_N] = \tau_{C(N)}(x \otimes \tau_{C(N)}([M])) \otimes [\mu_N] = \tau_{C(N)}(x \otimes \pi_2!) \otimes [\mu_N] = \iota_N(x \otimes \pi_2!).$$

Recall that, if $x$ is an element in $K^*_C(M)$, then $\iota_M(x)$ is the element of $KK_*^G(C(M), C(M))$ obtained from any K-cycle representing $x$ by noticing that $C(M)$ being commutative, the right action is also a left action. Since $x \otimes f! = [p] \otimes \iota_M(x) \otimes f!$, where $[p]$ is the element of $K^*_C(M)$ corresponding to the inclusion.
\(\mathbb{C} \hookrightarrow C(M)\), we can see that if \((\phi, T, \xi)\) is a K-cycle representing \(\iota_M(x) \otimes f!\), then \(\iota_T(x \otimes f!)\) can be represented by the K-cycle \((\phi', T, \xi)\) where \(\phi'\) is equal to the \((\text{right})\) action of \(C(N)\) on \(\xi\). Thus we only have to check that \((\phi', T, \xi)\) and \((\phi \circ f, T, \xi)\) represent the same class in \(KK^0(C(N), C(N))\).

Since \(f\) is an embedding, the KK-element \(f!\) can be represented by the KK-cycle \((\phi_{\nu_M}, \nu_{\nu_M}^* \xi_{\nu_M}, T_{\nu_M})\) where \(\nu_{\nu_M}^* \xi_{\nu_M}\) is viewed as a \(C(N)\)-Hilbert module via the inclusion \(C_0(\nu_M) \hookrightarrow C(N)\) and where \(\phi_{\nu_M}\) is the representation induced by \(\phi_0: C(M) \to C(\nu_M); h \mapsto h \circ \nu_{\nu_M}\). Thus we can choose the K-cycle \((\phi, T, \xi)\) representing \(\iota_M(x) \otimes f!\) in such a way that

- \(\xi\) is in fact a \(C_0(\nu_M)\)-Hilbert module (by associativity of the Kasparov product);
- \(T\) commutes with the action of \(C_0(\nu_M)\) viewed as the multiplier algebra of \(C_0(\nu_M)\) (use an approximate unit and the continuity of \(T\)), observe that \(The = hT = hTe\) for all \(h \in C_0(\nu_M), e \in C_0(\nu_M)\);
- the \(C(M)\)-structure is induced by \(\phi_0\).

The maps \(\nu_M \to \nu_M; v \mapsto tv\) for \(t\) in \([0, 1]\) provide a homotopy between \(\phi_0 \circ f\) and the restriction map \(C(N) \to C(\nu_M)\) and this homotopy commutes with \(T_{\nu_M}\). But the restriction map corresponds precisely to the \(C(N)\)-Hilbert module structure on \(\nu_{\nu_M}^* \xi_{\nu_M}\), and hence we get the result. \(\square\)

### B.3 Gysin and Poincaré duality if \(\partial M \neq \emptyset\)

Our key tool to study \(G\)-manifolds with boundary is the double.

For a manifold \(X\) with boundary \(\partial X\), let us define the double \(DX\) of \(X\) to be the manifold obtained by identifying the two copies of the boundaries \(\partial X\) in \(X \amalg X\). To distinguish the two copies, we write \(DX = X \cup X^-\). Let \(p_X: DX \to X\) be the map obtained by identifying the two copies of \(X\). Let \(j_X: X \to DX\) be the map induced by the inclusion of the first factor of \(X \amalg X\), and let us set \(g_X = j_X \circ p_X\). It is straightforward to check that if \(X\) is a \(G\)-spin\(^c\) compact manifold with boundary \(\partial X\), then \(DX\) is a \(G\)-spin\(^c\) compact manifold without boundary. The given orientation or \(G\)-spin\(^c\) structure on the first copy of \(X\) and the negative structure on the second copy \(X^-\) together define canonically a \(G\)-spin\(^c\) structure on \(DX\). Note that \(p_X \circ j_X = id_X\). Therefore, the exact sequence

\[
0 \to C_0(X^- \setminus \partial X) \xrightarrow{i_X} C(DX) \xrightarrow{j_X} C(X) \to 0
\]

has a split, and we get induced split exact sequences in K-theory and K-homology. Note that in general there is no split of \(i_X\) by algebra homomorphisms, but the corresponding split in K-theory and K-homology of course exists nonetheless. We use the corresponding sequence and split with the roles of \(X\) and \(X^-\) interchanged. will We now state the workhorse lemma for the extension of the treatment of Gysin homomorphism from closed manifolds to manifolds with boundary.

### B.2 Lemma. Poincaré duality for \(M\) is a direct summand of Poincaré duality for \(DM\), i.e. the following diagram commutes, if \(M\) is a compact \(G\)-spin\(^c\)
Therefore, to be allowed to “cancel” ι from what we have just seen we can conclude that particular since (B.3) is commutative. The relevant groups, namely $K_0(M, \partial M)$, and write $\tilde{M}$ for a compact manifold $M$ (possibly with boundary) it is the composition $KK({\ast}, M) \xrightarrow{\tau C_0(M^\circ)} KK(M^\circ, M^\circ \times M) \xrightarrow{\mu} KK(M^\circ, M^\circ) \xrightarrow{\otimes [M^\circ]} KK(M^\circ, {\ast})$.

Here we abbreviate $KK$ for $KK_0^G$ (and ask the reader to add the correct grading), and write $KK(X, Y) = KK(C_0(X), C_0(Y))$ for two spaces $X, Y$, $\mu$ is the map induced by the multiplication $C_0(M \times M^\circ) = C(M) \otimes C_0(M^\circ) \rightarrow C_0(M^\circ)$. Will naturality of KK-theory and of the fundamental class (under inclusion of open submanifolds) now gives the following commutative diagram, writing $N \equiv \tilde{M}$$$egin{array}{cccccc}
KK({\ast}, M) & \xrightarrow{\tau C_0(M^\circ)} & KK(M^\circ, M^\circ \times M) & \xrightarrow{\mu} & KK(M^\circ, M^\circ) & \xrightarrow{\otimes [M^\circ]} & KK(M^\circ, {\ast}) \\
\uparrow \iota^* & & \uparrow \gamma^* & & \parallel & & \\
KK({\ast}, N) & \xrightarrow{\tau C_0(M^\circ)} & KK(M^\circ, M^\circ \times N) & \xrightarrow{\mu} & KK(M^\circ, M^\circ) & \xrightarrow{\otimes [N]} & KK(M^\circ, {\ast}) \\
\downarrow & & \downarrow \iota^* & & \parallel & & \\
KK(N, N \times N) & \xrightarrow{\iota^*} & KK(M^\circ, N \times N) & \xrightarrow{\mu} & KK(M^\circ, N) & \xrightarrow{\otimes [N]} & KK(N, {\ast}) \\
\parallel & & \parallel & & \parallel & & \\
KK(N, N \times N) & \xrightarrow{\iota^*} & KK(N, N \times N) & \xrightarrow{\mu} & KK(N, N) & \xrightarrow{\otimes [N]} & KK(N, {\ast}) \\
\end{array}$$$

Walking around the boundary of this diagram shows that the right square of (B.3) is commutative.

The commutativity of the left square of (B.3) is more difficult to show, in particular since $s$ is not induced from an algebra homomorphism. However, from what we have just seen we can conclude that $\iota^*PD_Mp^* = PD_Mj^*p^* \circ id = PD_M = \iota^*sPD_M$. (B.4)

The section $s$ is characterized by the properties $\iota^*s = id$ and $sp_* = 0$. Therefore, to be allowed to “cancel” $\iota^*$ in Equation (B.4) we have to show that $PD_Mp^*$ maps to the image of $s$, i.e. to the kernel of $p_*$. We must show that $0 = p_*PD_Np^*: KK({\ast}, M) \rightarrow KK(N, {\ast})$. (B.5)

The relevant groups, namely $K^*(M)$, $K^*(\widetilde{M})$, $K^*(\widetilde{DM})$, $K^*(\widetilde{M^\circ})$ all are $K^*(M)$-modules, and all homomorphisms are $K^*(M)$-module homomorphisms.
The module structure on $K^*(M)$ is induced via the ring structure of $K^*(\mathcal{D}M)$ and the map $p^*$. $K_*(\mathcal{D}M)$ is a $K^*(\mathcal{D}M)$-module via the cap product, and via $p^*$ it therefore also becomes a $K^*(M)$-module; the cap product also gives the $K^*(M)$-module structure on $K_*(M)$. As $K^*(M)$ is generated by 1 as a $K^*(M)$-module, Equation (B.5) follows if

$$0 = p_*\mathcal{P}_D\mathcal{M}p^*1 = p_*[\mathcal{D}M].$$

To see this, remember that every double of a manifold with boundary is canonically a boundary, namely $\mathcal{D}M = \partial(Y := (M \times [-1,1]/\sim))$, where the equivalence relation is generated by $(x,t) \sim (x,s)$ is $x \in \partial M$ and $s,t \in [-1,1]$. Observe that this construction is valid in the world of $G$-spin$^c$ manifolds. Note that $p_M: \mathcal{D}M \to M$ extends to

$$P: Y = (M \times [-1,1]/\sim) \to (M \times [0,1]/\sim); (x,t) \mapsto (x,|t|).$$

From the long exact sequences of the pairs $(Y, \mathcal{D}M)$ and $(M \times [0,1]/\sim, M \times \{1\})$, we have the following commutative diagram

$$
\begin{array}{ccc}
K_{\dim M+1}(Y^\circ) & \xrightarrow{P_*} & K_{\dim M+1}(C_0((M \times [0,1]/\sim) \setminus M \times \{1\})) = \{0\} \\
\downarrow \phi & & \downarrow \phi \\
K_{\dim M}(\mathcal{D}M) & \xrightarrow{p_*} & K_{\dim M}(M).
\end{array}
$$

In this diagram, $[Y^\circ]$ is, according to Lemma (B.8) mapped under the boundary to $[\mathcal{D}M]$. Therefore, by naturality, $p_*[\mathcal{D}M] = \partial p_*([Y^\circ])$. However,

$$(M \times [0,1]/\sim) \setminus M \times \{1\} = M^\circ \times [0,1),$$

and $C_0(M^\circ \times [0,1])$ is $G$-equivariantly contractible, hence its equivariant $K$-homology vanishes. The assertion follows.

**B.6 Definition.** Let now $f: M \to N$ be a $G$-equivariant continuous map between $G$-spin$^c$ manifolds with boundary such that $f(\partial M) \subset \partial N$. Then we define $f!: K^*_G(M) \to K^*_{G+n-m}(N)$ as the composition

$$f! = \mathcal{P}_D\mathcal{M}^{-1}f_*\mathcal{P}_M.$$

**B.7 Remark.** Note that this is consistent with the definition for closed manifolds and smooth maps by the considerations of Section (B.2). Lemma (B.6) holds in the general case by definition.

However, at least in special situations, we can also define the Gysin map geometrically. Let, for example, $M$ be a $G$-spin$^c$ compact manifold with boundary, let $W$ be a $G$-spin$^c$ vector bundle over $M$, let $Z$ be the manifold obtained from vector bundle modification with respect to $W$ and, as above, let $\pi: Z \to M$ and $s: M \to Z$ be the canonical projection or the “north pole” section of $\pi$, respectively. The vector bundle $W$ is the normal vector bundle of $M$ in $Z$ (with respect to the embedding $s$) and is therefore a $G$-invariant tubular open neighbourhood of $M$.

We can then define the Gysin element $s! \in KK^G(C(M), C(Z))$ associated to $s$ as we did for manifold without boundary by $s! = \beta_W \otimes [\theta_M]$, where
θ_M: C_0(W) → C(Z) is the morphism induced by the inclusion of W into Z. The Gysin homomorphism can then be defined correspondingly.

With arguments similar to those of Section B.3 we can show that with this definition Lemma B.6 holds, so that our Definition B.6 is consistent with the geometric one. The proof would also use Lemma B.2, that PD_M is a direct summand of PD_M.

**B.8 Lemma.** If i: M ↪ L is as in Lemma B.8, then

\[ \partial \otimes i_* \partial (x) = (-1)^{\deg x} i_* \partial (i^* x) \otimes \partial \quad \forall x \in K_G(L), \]

where \( \partial \in KK(C_0(M^\circ), C_0(L^\circ)) \) is the boundary element of the exact sequence of \( C_0(L^\circ) ↪ C_0(L^\circ \cup M^\circ) ↪ C_0(M^\circ) \) as in Lemma B.8 (here we abbreviate \( L = L \setminus \partial L \)).

**Proof.** We first recall a KK-description of \( i_* \partial L \). It is given by the composition

\[ KK(\{e\}, L) \xrightarrow{[e_0]} KK(L^\circ, L \times L^\circ) \xrightarrow{\otimes \mu} KK(L^\circ, L^\circ) \]

where \( \mu \) is the multiplication homomorphism. By graded commutativity of the exterior Kasparov product, we therefore get that \( \partial \otimes i_* \partial L \) equals the composition

\[ KK(\{e\}, L) \xrightarrow{\begin{array}{c} \text{nat.} \\ \mu \end{array}} KK(M^\circ, L \times M^\circ) \xrightarrow{\begin{array}{c} \text{nat.} \\ \otimes \partial \end{array}} KK(M^\circ, L^\circ) \xrightarrow{\mu} KK(M^\circ, L^\circ). \] (B.9)

Now observe that we have commutative diagrams of short exact sequences

\[
\begin{array}{cccc}
C_0(L \times L^\circ) & \longrightarrow & C_0(L \times (L^\circ \cup M^\circ)) & \longrightarrow & C_0(L \times M^\circ) \\
\downarrow & & \downarrow & & \downarrow \\
C_0(L \setminus M \times M^\circ \cup L \times L^\circ) & \longrightarrow & C_0(L \times (L^\circ \cup M^\circ)) & \longrightarrow & C_0(M \times M^\circ) \\
\downarrow \mu & & \downarrow \mu & & \downarrow \mu \\
C_0(L^\circ) & \longrightarrow & C_0(L^\circ \cup M^\circ) & \longrightarrow & C_0(M^\circ)
\end{array}
\]

Using naturality of the boundary map, we observe that the composition of the last two arrows of (B.9) coincides with the composition

\[ KK(M^\circ, L \times M^\circ) \xrightarrow{\begin{array}{c} \text{nat.} \\
\mu \end{array}} KK(M^\circ, M \times M^\circ) \xrightarrow{\otimes \partial} KK(M^\circ, M^\circ) \xrightarrow{\otimes \partial} KK(M^\circ, L^\circ). \]

As \( i^* \) commutes with the exterior product with \( C_0(M^\circ) \), this implies the assertion. \( \square \)

**B.4 Proof of Lemma 3.8**

We finish by proving Lemma B.8. Recall that it states

**B.10 Lemma.** Let L be a G-spin manifold with boundary \( \partial L \), let M be a G-invariant submanifold of \( \partial L \) with boundary \( \partial M \) such that \( \dim M = \dim L - 1 \) and let \( \partial \in KK(C_0(M \setminus \partial M), C_0(L \setminus \partial L)) \) be the boundary element associated to the exact sequence

\[ 0 \rightarrow C_0(L \setminus \partial L) \rightarrow C_0((L \setminus \partial L) \cup (M \setminus \partial M)) \rightarrow C_0(M \setminus \partial M) \rightarrow 0. \]

Then \( \partial \otimes [L \setminus \partial L] = [M \setminus \partial M] \).
Proof. Using a $G$-invariant metric on $L$ and a corresponding collar, $(0,1) \times (M \setminus \partial M)$ can be viewed as a $G$-invariant open neighborhood of $\{1\} \times (M \setminus \partial M)$ in $(L \setminus \partial L) \cup (M \setminus \partial M)$. Moreover, the inclusion $C_0((0,1) \times (M \setminus \partial M)) \hookrightarrow C_0(L \setminus \partial L) \cup (M \setminus \partial M)$ gives rise to the following commutative diagram with exact rows (write $L^\circ := L \setminus \partial L$, $M^\circ := M \setminus \partial M$)

\[
\begin{array}{cccccc}
0 & \longrightarrow & C_0(L^\circ) & \longrightarrow & C_0((L^\circ) \cup (M^\circ)) & \longrightarrow & C_0(M^\circ) & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & C_0((0,1) \times (M^\circ)) & \longrightarrow & C_0((0,1) \times (M^\circ)) & \longrightarrow & C_0(M^\circ) & \longrightarrow & 0.
\end{array}
\]

By naturality of the boundary homomorphism and since by [12, Proposition 11.2.12] (for the non-equivariant case, but the equivariant one follows along identical lines) the restriction of $[L^\circ]$ to $(0,1) \times (M^\circ)$ is $[(0,1) \times (M^\circ)]$, the statement of the lemma amounts to show that

$$[\partial'] \otimes [(0,1) \times (M^\circ)] = [M^\circ],$$

where $\partial' \in KK^G(C_0(M^\circ), C_0((0,1) \times (M^\circ)))$ is the boundary element associated to the bottom exact sequence of the diagram above. Viewing $M^\circ$ as an invariant open subset of $D_M$, using naturality of boundaries in the following commutative diagram with exact rows

\[
\begin{array}{cccccc}
0 & \longrightarrow & C_0((0,1) \times D_M) & \longrightarrow & C_0((0,1) \times D_M) & \longrightarrow & C_0(DM) & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & C_0((0,1) \times (M^\circ)) & \longrightarrow & C_0((0,1) \times (M^\circ)) & \longrightarrow & C_0(M^\circ) & \longrightarrow & 0,
\end{array}
\]

and since the elements $[M^\circ]$ of $KK^G(C_0(M^\circ), \mathbb{C})$ and $[(0,1) \times (M^\circ)]$ of $KK^G(C_0((0,1) \times (M^\circ)), \mathbb{C})$ are the restrictions of $[DM]$ to $M^\circ$ and of $[(0,1) \times DM]$ to $(0,1) \times (M^\circ)$, respectively, we can indeed assume without loss of generality that $M$ has no boundary.

Observe now that in the exact sequence in (non-equivariant) K-homology

\[
0 \to C_0((0,1)) \to C_0((0,1)) \to C(\{1\}) \to 0
\]

by the well known principle that “the boundary of the Dirac element is the Dirac element of the boundary” we indeed observe $[\partial'] \otimes [(0,1)] = [(1)]$ in $KK_0(C(\{1\}), \mathbb{C})$, compare [12 Propositions 9.6.7, 11.2.15]. We can now take the exterior Kasparov product of everything with $[M] \in KK^G_{\dim M}(C_0(M), \mathbb{C})$. By naturality of this Kasparov product, we obtain $[\partial'] \otimes [(0,1)] \otimes [M] = [(1)] \otimes [M]$. Finally, we know that the fundamental class of a product is the exterior Kasparov product of the fundamental classes, compare again [12 Proposition 11.2.13]; the equivariant situation follows similarly. This implies the desired relation $[\partial'] \otimes [(0,1) \times M] = [M] \in KK^G_{\dim M}(C_0(M), \mathbb{C})$.

\[
\square
\]

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