Surprising Interactions of Markovian noise and Coherent Driving

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We derive the explicit commutation relations for the generators of quantum dynamical semigroup - Markovian superoperator evolution, allowing the extension of Baker-Campbell-Hausdorff-type relations to general Lindblad-type evolutions. This provides a novel tool for exploring the interaction of time-dependent coherent and Markovian evolutions - a surprisingly rich set of behaviors which include deformation by coherent driving of Markovian terms, generation of new Lindblad terms from time-dependent noise and even a coherent driving term generated by the interaction of time-separated Markovian noises. Finally, we consider the Lindblad superoperators as vectors in a higher space, allowing us to extract the infinitely divisible subspace of a given channel and to recover its Lindblad form.

Introduction

The importance of quantum effects in future technology cannot be overstated, whether desired (quantum computation, metrology, etc) or detrimental (leakage current in MOSFETs).

For some applications, one desires to isolate the quantum system and protect it from dephasing or relaxation resulting from uncontrolled interaction with the environment. To that end, a toolbox of methodologies have been developed to allow the stabilization of quantum states in the presence of decoherence and to implement coherent operations in a noisy environment. The tools range from the design of decoherence-free subspaces \cite{1-3} coupled with a measurement-driven Zeno-type control to keep the state within the protected subspace \cite{4}, to more dynamic methods such as pulsed/dynamical-decoupling \cite{1-3} and optimal control \cite{8-16}. On the other-hand, non-coherent effects resulting from environment interaction may be positively utilized in a variety of tasks, such as state preparation \cite{17-19}, quantum map synthesis \cite{20}, enhancement energy transfer in photosynthetic systems \cite{21-24}. In all these cases, there is strong interplay between coherent and incoherent dynamics.

Surprisingly, even in the most commonly used model for environment interactions, that of Markovian interactions, the algebra governing the compound effect of consecutive periods of coherent driving and Markovian noise has yet to be fully worked-out. It is the goal of this paper to provide algebraic tools, with clear physical interpretation, to allow deep examination of the interplay of coherent and incoherent state evolution.

This letter is organized as follows: First we review the Lindblad-Kossakowski master equation, describing quantum Markovian semigroup evolution, and the Baker-Campbell-Haussdorff (BCH) relations, which allow us to examine the interaction of driving pulses amongst themselves and with the free evolution of the system. We shall introduce a new notation which will allow us to derive explicit commutations relations of the dynamical semigroup generators, thus promoting the BCH relations to superoperator maps. We shall then use this to examine the intricacies and richness of interactions between Markovian noises and coherent driving, culminating in an example where two Markovian noise actions, separated by time, interact to generate a wholly coherent term. Finally, we view superoperators as vectors in a higher (“super-super”) operator Hilbert space, and utilize the language of bi-orthogonal bases to present a method of projecting a general map onto the sub-space of infinitely divisible Markovian superoperators and further phrasing the projected map in its Lindblad form. Some concluding remarks close-out the discussion.

Background Review

The Lindblad-Kossakowski quantum master equation

Coherent evolution in quantum mechanics, as described by the Schrödinger equation $i\hbar\partial_t |\psi\rangle = H|\psi\rangle$, defines a set of continuous, one-parameter, exponential unitary maps $U(t) = \exp\left(-\frac{i}{\hbar}Ht\right)$, identifying the elements of the Lie algebra with the infinitesimal generator of the group, the Hamiltonian.

A general Markovian (memory-less) time-homogeneous, trace-preserving and completely positive evolution of any open quantum system is described by the Lindblad-Kossakowski master equation (henceforth LKME) \cite{25}

\[ \dot{\rho} = \mathcal{H}(H) + \mathcal{L}(\Gamma, L_1 \ldots L_M), \]

\[ \mathcal{H}(H) = -\frac{i}{\hbar} [H, \rho], \]

\[ \mathcal{L}(\Gamma, L_1 \ldots L_M) = \sum_{j,k} \Gamma_{k,j} \left( L_k \rho L_j^\dagger - \frac{1}{2} (\rho L_j^\dagger L_k + L_k^\dagger L_j \rho) \right) \]

where the $\Gamma$ matrix is Hermitian and positive semidefinite.

Master equations of this form are in very common use in a great number of fields, including quantum optics, semiconductor physics, NMR, decay of unstable systems, thermalization, Brownian motion, etc. Such an equation describes the irreversible evolution of quantum system, weakly interacting with a stationary environment in such a way that the timescale of system dynamics and relaxation is much longer than decay time of correlations with and in the environment, so that one may ignore memory (non-Markovian) effects.
One may directly show that leaving all such operators trace-less. Trivially, one may also with being the one-sided Fourier transform of the bath auto-correlation function, $R_{jk}(\omega) = \int_0^\infty dt e^{i\omega t} \langle B_j^\dagger(t) B_k(0) \rangle$.

Note that the Lamb shift Hamiltonian, $H_{L.S.}$, has been explicitly removed from the Lindbladian terms. As we shall see below, it makes a surprise reappearance in the interaction of time-separated noise terms.

The Baker-Campbell-Haussdorf and Wei-Norman equations

In the context of quantum optimal control, one implements the desired unitary linear map by concatenating multiple pulses, intertwined with periods of free evolution. To ascertain analytically the total effect of such a pulse-train, one utilizes the Baker-Campbell-Haussdorf (BCH) relation [34–36], which are essentially a matrix identity

$$\exp(X)\exp(Y) = \exp\left(X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] + \ldots \right).$$

Some conditions are required to ensure the existence of a convergence radius of the exponential map, and are discussed in [44]. When utilizing the BCH relations, we shall restrict ourselves to the finite-dimensional case. One may extend the relations to any number of exponents [37], where the series is represented as a summation over products of the exponent generators (and while by induction it is clear said series can be expressed in terms of commutators, as in eq. (8) above, to our knowledge the general form has not yet been explicitly reformulated). The Wei-Norman relations [33] stipulate the opposite - that an exponential map of a sum may be represented as a product of exponential maps of algebra’s generators.

From a physical perspective, given a series of unitary evolutions each generated by a time-independent Hamiltonian, the BCH relations provide us with a way of phrasing the overall evolution with a single time-independent Hamiltonian. And while often the trajectory of the state in the LHS and RHS of eq. (8) will coincide only at the initial and final times, the explanatory power of such a re-phrasing is significant (see, for example [45, 46] where impulsive driving is intertwined with free evolution, to generate novel interactions by the commutator of the pulse and free Hamiltonian).

Stressing the memory-less nature of the classical Markov process, and following the definitions in [30, 32], one can take the position that a sequence of quantum Markovian evolutions is itself a Markovian evolution. Our goal then is to lift the BCH relations from unitary to semigroup evolution, from operator to superoperator. With this we will be able to understand the unique nature of a time-dependent Markovian process, in that, unlike time-dependent coherent evolution, the compound evolution takes a somewhat different form than the time-independent one.

The Lamb shift

In the derivation of the LKME, one performs several approximations and makes several assumptions: the absence of correlations between system and environment in the initial state, weak coupling (leading to the state remaining a product state; Born), memory-less nature of the evolution (dependent only on the current state of the system and not on past states; Markov) and in some cases the secular approximation (RWA (Rotating Wave Approximation) - averaging over rapidly oscillating terms). In addition, a coherent term, resulting from the system-bath interaction, is added to the Hamiltonian - the Lamb shift term [23].

Explicitly, given a general interaction Hamiltonian,

$$H_I = \sum_k A_k \otimes B_k = \sum_k \left( \sum_\omega A_k(\omega) \right) B_k$$

with $A_k$, $B_k$ being, respectively, system and bath Hermitian operators, and $A_k = \sum_\omega A_k(\omega)$ being the energy-basis decomposition of the system operators. After performing the secular approximation we end up with the LKME in non-canonical form

$$\dot{\rho} = \mathcal{H}(H_0 + H_{L.S.}) + \sum_\omega \mathcal{L}(\Gamma, A_1(\omega), A_2(\omega), \ldots)$$

$$H_{L.S.} = \sum_{j,k,\omega} S_{jk}(\omega) A_j^\dagger(\omega) A_k(\omega)$$

$$\Gamma = \frac{1}{2} \left( R + R^\dagger \right)$$

$$S = \frac{1}{2} \left( R - R^\dagger \right)$$

with $R$ being the one-sided Fourier transform of the bath auto-correlation function, $R_{jk}(\omega) = \int_0^\infty dt e^{i\omega t} \langle B_j^\dagger(t) B_k(0) \rangle$.
Matrix superoperators and vec-ing of the density matrix

To explicitly formulate the commutation relations of the LKME generators, we shall introduce the following notation for superoperators: given a left and right operators acting on a density matrix \( \rho \), one may transform the matrix \( \rho \) into a column vector, row-first,

\[
\vec{\rho} := (\rho_{1,1}, \rho_{1,2} \ldots \rho_{1,N}, \rho_{2,1}, \rho_{2,2} \ldots \rho_{N,N})^T.
\]

(9)

This procedure is known as vec-ing \([28, 29]\). Next, we shall define \( \odot \) in the following manner

\[
\overline{L \rho R} = (L \otimes (R^T))\vec{\rho} =: (L \odot R)\vec{\rho}.
\]

(10)

We define the superoperator equivalents of \( \mathcal{H} \) and \( \mathcal{L} \),

\[
\mathcal{H}^i(H) := -\frac{i}{\hbar} (H \odot I - I \odot H)
\]

and

\[
\mathcal{L}^i(A) := A \odot A^\dagger - \frac{1}{2} I \odot A^\dagger A - \frac{1}{2} A^\dagger A \odot I.
\]

(12)

Allowing us to rewrite the LKME in explicit superoperator notation

\[
\dot{\vec{\rho}} = (\mathcal{H}^i(H) + \sum_k \gamma_k \mathcal{L}^i(A_k))\vec{\rho}
\]

(13)

defining the semigroup

\[
\bar{\rho}(t) = e^{i(H^i(H) + \sum_k \gamma_k \mathcal{L}^i(A_k))t} \rho.
\]

(14)

Explicit form of semigroup generator algebra

Noting that \((A \odot B)(C \odot D) = AC \odot DB\) we can now arrive at the commutation relations for the LKME generators (derivation is straightforward, if somewhat cumbersome):

\[
[\mathcal{H}^i(H), \mathcal{H}^i(G)] = \mathcal{H}^i\left(-\frac{i}{\hbar} [H, G]\right),
\]

\[
[\mathcal{H}^i(H), \mathcal{L}^i(A)] = \mathcal{L}^i\left(\Gamma = \left(\begin{array}{c c} 0 & 1 \\ 1 & 0 \end{array}\right)\right) \{ [A, [H, A]] \}
\]

\[
= -\frac{1}{2\hbar} \mathcal{L}^i(A + i [H, A]) + \frac{1}{2\hbar} \mathcal{L}^i(A - i [H, A])
\]

(15)

and

\[
[\mathcal{L}^i(A), \mathcal{L}^i(B)] =
\]

\[
= \mathcal{L}^i(A^\dagger B - B^\dagger A) +
\]

\[
+ \frac{1}{2} \mathcal{L}^i\left(\Gamma = \left(\begin{array}{c c} 0 & 1 \\ 1 & 0 \end{array}\right)\right) \{ [B, A^\dagger A]\}
\]

\[
+ \frac{1}{2} \mathcal{L}^i\left(\Gamma = \left(\begin{array}{c c} 0 & 1 \\ 1 & 0 \end{array}\right)\right) \{ [B^\dagger B, A], A\}
\]

\[
+ \frac{1}{4} \mathcal{H}^i\left(\frac{i}{\hbar} [A^\dagger A, B^\dagger B]\right)
\]

\[
= \mathcal{L}^i(AB) - \mathcal{L}^i(BA) +
\]

\[
+ \frac{1}{4} \mathcal{L}^i\left(\Gamma = \left(\begin{array}{c c} 0 & 1 \\ 1 & 0 \end{array}\right)\right) \{ [B, A^\dagger A]\}
\]

\[
+ \frac{1}{4} \mathcal{L}^i(AB - BA) +
\]

\[
+ \frac{1}{4} \mathcal{L}^i\left(\Gamma = \left(\begin{array}{c c} 0 & 1 \\ 1 & 0 \end{array}\right)\right) \{ [B, A^\dagger A]\}
\]

\[
+ \frac{1}{4} \mathcal{L}^i\left(\Gamma = \left(\begin{array}{c c} 0 & 1 \\ 1 & 0 \end{array}\right)\right) \{ [B^\dagger B, A], A\}
\]

\[
+ \frac{1}{4} \mathcal{H}^i\left(\frac{i}{\hbar} [A^\dagger A, B^\dagger B]\right).
\]

(16)

Note that the interaction of two noise terms leads to 6 new noise terms, but also to a coherent driving term. This is expected in light of \([27]\), but its explicit form was not known.

Negative pre-factors for Lindbladian terms

Another issue requiring attention is the appearance of negative pre-factors to Lindbladian superoperators in both the Hamiltonian-Lindbladian and Lindbladian-Lindbladian commutators. Such negative rates normally correspond to re-coherence effects - rolling-back of decay processes \([38–40]\), which are clearly non-Markovian. In fact, these negative pre-factors have been used to construct indicators of non-Markovianity \([31, 41, 43]\). Note that the summation of these Lindbladian terms does not remove the difficulty.

The resolution of the dilemma is best viewed by considering the context in which these semigroup structure constants are often used, i.e. the BCH relations. As noted in the background review, the generator provided by the BCH relations matches the overall evolution only in the initial and final times. One therefore concludes, somewhat surprisingly, that superoperator describing the overall effect of a sequence of time-independent Markovian evolutions cannot itself be described by a time-independent Markovian evolution (which requires all-positive rates). This is in stark contrast to the unitary evolution, where the BCH amalgamation of time-independent unitaries, is, in itself, a time-independent unitary. Note that both time-dependent and time-independent Markovian evolutions are infinitesimally divisible quantum channels, as per \([32]\).

Modification of Liouvillian evolution by coherent driving

Consider a system (in the interaction frame) influenced by Markovian noise. We shall drive it using a strong coherent
impulsive drive $\Omega H$ for duration $t$, allow for free evolution for duration $T \gg t$, during which the dominant effect is a Markovian noise $L$ and finally a counter-pulse, $-\Omega H$ of duration $t$ (ignoring Markovian effects during the pulses). Overall evolution is described by the superoperator

$$\exp(t\Omega H)\exp(TL_1(A))\exp(-t\Omega H).$$

(17)

To first order in the generalized BCH series, this may be concatenated, using eq. (15), as

$$\exp\left(T\left(L_1(A) + \frac{1}{2iT}(H, L)\right)\right) =$$

$$\exp\left(T\left(L_1(A) - \frac{i\Omega}{2\hbar}L_1(A + i[H, A]) + \frac{i\Omega}{2\hbar}L_1(A - i[H, A])\right)\right).$$

(18)

Thus, it is clear that driving can shape the noise affecting the system (as opposed to reducing its rate). Note that this is a deeply different phenomena than Dynamical Decoupling \[1, 5\]. In-fact, dynamical decoupling is wholly unrelated to the subject-matter of this report, as it is inherently a non-Markovian phenomena. Even in its most rudimentary bang-form, it requires a re-phasing period, decreasing the limit by Trotterization).

**Pure coherent driving by interaction of Lindbladian terms**

Assume a system is subjected to one noise, $A$, followed by a period with no noise, and later subjected to noise $B$. Furthermore, let us assume both $A$ and $B$ are traceless. From eq. (16) it is clear that the combined effect, as detailed by the BCH series, will include coherent terms. Surprisingly, in the case of single qubit noise, there exist cases where, in the first order correction, all incoherent contributions disappear, and $[L^s(A), L^s(B)] = \mathcal{H}^i(\frac{1}{i\hbar}[A^tA, B^tB])$. Specifically in the following example:

$$A = \begin{pmatrix} 1 & -4 \\ 3 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & 4 \\ 2 & 2 \end{pmatrix},$$

(19)

$$[L^s(A), L^s(B)] = \mathcal{H}^i(4\hbar \sigma_z).$$

(20)

Note that incoherent terms remain in the zeroth order and second and higher orders (the latter may be removed at the limit by Trotterization).

**Comments**

*Comment re. Lamb shift*

Note that in eq. (16) we have terms of the form $A^tA$ and $B^tB$. These are well-known to be the Lamb shift terms. However, their appearance here is surprising, as in the derivation of the MKLE, the Lamb shift terms are explicitly removed from the Lindbladian and added to the Hamiltonian (eq. 7). Specifically, they return in the noise-noise commutation term, as $\mathcal{H}^i(\frac{1}{i\hbar}[A^tA, B^tB])$, which is of the same form as the commutation term of the Lambert-shift factors added to the Hamiltonian, $\mathcal{H}^i\left(-\frac{1}{\hbar}[A^tA, B^tB]\right)$, modifying its magnitude.

**Superoperators as vectors in a higher Hilbert space**

Given a Markovian channel superoperator, one may directly reconstruct the Lindblad operators forming it by considering the superoperators associated with individual Lindblad and Hamiltonian operators as vectors, collectively serving as a non-orthogonal basis for a higher ("super-super") dimensional object.

Going back to the general form of the MKLE in eq. (3), we shall choose the $L_k$ to be the traceless generators of SU($N$), $[S_k]_{i=1}^{N^2-1}$, and, since $H$ can be made traceless, we shall express it as a sum of the same generators, $H = \sum_{j,k} h_{jk} S_{jk}$. Let us denote by $\gamma_{jk}$ a $\Gamma$ matrix (as in eq. (3)) which is zero everywhere, except for the $(j,k)$ element, which is 1. One may show, by virtue of the linear independence of the generators, that the set of superoperators $B := \{\mathcal{H}^i(S_i), L^i(\gamma_{jk}, S_i)\}_{i,j,k=1}^{N^2-1}$ are linearly independent (but not orthogonal). Let us consider the super-operators as vectors in a higher (super-duper) Hilbert space. Now, we can view $B$ as a non-orthogonal basis (and a non-square matrix), to which we can construct the bi-orthogonal basis $G$, such that $G^tB = I$ (via the pseudo-inverse), with $G$ and $B$ spanning the same Hilbert space. Given the logarithm of a linear map $T$ (the issue of branches has been discussed in [41]), one may project it onto the subspace spanned by $B$, via $G \log(T)$. The resultant vector directly maps to the $h_k$ and $\gamma_{jk}$ pre-factors of the operator’s Lindblad-form.

The above projection of the map’s generator onto the space of Lindblad-form operators, which are infinitesimally divisible by construction, allows us to identify what remains in the orthogonal subspace as the non-divisible component.

**Conclusion**

By deriving explicit expressions for the semigroup generator algebra, we have provided a new tool to investigate the interaction of coherent driving and Markovian noises, as well as the interaction of disparate noise terms. Thus, we reveal a rich field of interactions, which may be exploited to improve control of quantum systems.

Two unexpected results emerge. Firstly, that even pure coherent driving may be thus generated from interactions of Markovian terms - a phenomena hitherto unknown. Second, we gain deeper understanding as to why the set of time-dependent Markovian maps is a proper super-set of time-
independent Markovian maps (unlike unitary evolution, where this is not the case). We believe that an extension of this approach to non-Markovian systems will enable new insights into methods of controlling system decoherence, and will result in a richer set of controls than have been known to date.

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[1] D. Lidar, Adv. Chem. Phys. 154, 295-354 (2014)
[2] P. Zanardi and M. Rasetti, Phys. Rev. Lett. 79 3306 (1997)
[3] D. A. Lidar, I. L. Chuang, and B. K. Whaley, Phys. Rev. Lett. 81, 2594 (1998)
[4] P. Facchi and S. Pascazio: Phys. Rev. Lett. 89 080401 (2001)
[5] L. Viola, E. Knill, and S. Lloyd, Phys. Rev. Lett. 82, 2417 (1999)
[6] E.L. Hahn, Physical Review 80 580594 (1950)
[7] J.M. Cai, F. Jelezko, N. Katz, A. Retzker and M.B. Plenio, New J. Phys. 14, 093030 (2012)
[8] R. Kosloff, S. A. Rice, P. Gaspard, S. Tersigni, and D. J. Tannor, Chem. Phys. 139, 201 (1989)
[9] A. Bartana, R. Kosloff, and D. J. Tannor, J. Chem. Phys. 99, 196 (1993)
[10] A. Bartana, R. Kosloff, and D. J. Tannor, J. Chem. Phys. 106, 1435 (1997)
[11] A. P. Peirce, M. A. Dahleh, and H. Rabitz, Phys. Rev. A 37, 4950 (1988)
[12] W. Zhu, J. Botina, and H. Rabitz, J. Chem. Phys. 108, 1953 (1998)
[13] Y. Ohtsuki, W. Zhu, and H. Rabitz, J. Chem. Phys. 110, 9825 (1999)
[14] Y. Ohtsuki, K. Nakagami, Y. Fujimura, W. Zhu, and H. Rabitz., J. Chem. Phys. 114, 8867 (2001)
[15] W. Zhu and H. Rabitz, J. Chem. Phys. 109, 385 (1998)
[16] T. Schulte-Herbrüggen, A. Spoerl, N. Khaneja and S.J. Glaser, J. Phys. B: At. Mol. Opt. Phys. 44 154013 (2011)
[17] M.B. Plenio, S.F. Huelga, A. Beige, and P.L. Knight, Phys. Rev. A 59, 2468–2475 (1999)
[18] M.B. Plenio and S.F. Huelga. Phys. Rev. Lett. 88, 197901 (2002).
[19] V. Bergholm and T. Schulte-Herbrüggen, arXiv:1206.4945
[20] S. Lloyd and L. Viola, Phys. Rev. A 65, 010101 (2001)
[21] M.B. Plenio, S.F. Huelga, New J. Phys. 10, 113019 (2008)
[22] M Mohseni, P Rebentrost, S Lloyd, A Aspuru-Guzik, J. Chem. Phys 129 (17), 174106 (2008)
[23] S.F. Huelga and M.B. Plenio, Contemporary Physics 54, 4 (2013)
[24] Y. Li, F. Caruso, E. Gauger, S.C. Benjamin, arXiv:1405.7914
[25] H. Breuer and F.Petruccione, The Theory of Open Quantum Systems, Oxford (2002)
[26] Hall, Cresser, Li, and Andersson, arXiv:1009.0845v2 (2014)
[27] G. Durr, U. Helmke, I. Kurniawan, and T. Schulte-Herbrüggen, Rep. Math. Phys., 64 93 (2009), also arXiv 0811.3906
[28] R. Horn and C. Johnson , Matrix Analysis, Cambridge University Press (1990)
[29] One may alternatively transform ρ into a column by scanning by columns (as opposed to by rows), in which case \( L^T \rho R = \rho R \rho \)
[30] Á. Rivas and S. Huelga, Open Quantum Systems: An Introduction, Springer Briefs in Physics, also arXiv:1104.5242 (2011)
[31] Á. Rivas, S.F. Huelga, M.B. Plenio arXiv:1405.0303 To appear in Rep Prog Phys
[32] M. M. Wolf and J. I. Cirac, Commun. Math. Phys. 279, 147 (2008)
[33] J. Wei and E. Norman. Proc. of the Amer. Math. Soc., 15 327–334, 1964
[34] H. F. Baker, Proc. London Math. Soc (2) 3, 28 (1905)
[35] J. E. Campbell, Proc. London Math. Soc. (1) 28, 381 (1897), 29, 14 (1897)
[36] SF. Hausdorff, Leipzig. Ber. 58, 19 (1906)
[37] M.W. Reinsch, J. Math. Phys. 41 2434-2442 (2000)
[38] J. Piilo, S. Maniscalco, K. Karkonen, and K.-A. Suominen, Phys. Rev. Lett. 100, 180402 (2008)
[39] E. Andersson, J.D. Cresser and M.J. W. Hall, J. Mod Opt. 54, 1695 (2007)
[40] S. Maniscalco, F. Intravaia, J. Piilo and A. Messina, J. Opt. B: Quantum and Semiclass. Opt. 6, S98 (2004)
[41] M.M. Wolf, J. Eisert, T.S. Cubitt and J.I. Cirac, Phys. Rev. Lett. 101, 150402 (2008)
[42] H.P. Breuer, E.M. Laine and J. Piilo, Phys. Rev. Lett. 103, 210401 (2009)
[43] Á. Rivas, S. F. Huelga and M. B. Plenio, Phys. Rev. Lett. 105, 050403 (2010)
[44] M. Newman, W. So and R.C. Thompson, Linear and Multilinear Algebra, 24, 301-310 (1989)
[45] S. Machnes, M. B. Plenio, B. Reznik, A. M. Steane and A. Retzker, Phys. Rev. Lett. 104, 183001 (2010)
[46] S. Machnes, J. Cerrillo, M. Aspelmeyer, W. Wieczorek, M. B. Plenio, and A. Retzker, Phys. Rev. Lett. 108, 153601 (2012)