SIMPLE BASES FOR QUIVER REPRESENTATIONS ARISING FROM LIMIT LINEAR SERIES

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Abstract. We explore the existence of simple bases for certain special representations of quivers arising from degenerations of linear series on nodal curves. The existence of a simple basis implies that the representation decomposes into representations of dimension one and simplifies the calculus of the multivariate Hilbert polynomial of the quiver Grassmannian associated to the representation. For these representations of quivers, we characterise the existence of a simple basis by a local condition. This paper is the first of two, aimed to study the linked projective space of a limit linear series.

Keywords. Limit Linear Series · Quiver · Representation of Quiver

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This is the first version of a work in progress where we study certain quiver representations arising from degenerations of linear series to singular curves.

1. Introduction

This paper and its sequel [3] aim to study the existence of simple bases for certain representations of quivers associated to degenerations of linear series on nodal curves, and also study geometric properties of the associated quiver Grassmannians of pure dimension 1 subrepresentations, which we call linked projective spaces.

Linear series are vector spaces of sections of a line bundle. A family of linear series on smooth curves degenerating to a nodal curve $X$ gives rise to a limit linear series, a term first coined by Eisenbud and Harris [1] for when $X$ is of compact

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type. For each irreducible component $Y$ of the nodal curve, they considered the limit line bundle with degree 0 on all of the remaining components, and called its restriction and that of the limit sections to $Y$ an aspect of the degeneration. The collection of all the aspects is a limit linear series. Later Osserman [6] argued that for functorial reasons and for allowing work in positive characteristic, more “aspects” should be considered. If $X$ is of compact type with two components, he considered the collection of all limit line bundles of non-negative degrees on each component, together with the limit sections, and called it a limit linear series. Later, Osserman [7] handled general nodal curves.

For a general nodal curve $X$, we obtain many limit line bundles. Assuming the total space of the family is regular, a condition that can be achieved by semistable reduction, each component $Y$ of $X$ becomes an effective divisor on the total space, the zero locus of a section of the associated line bundle. Let $s_Y$ and $L_Y$ be the restrictions of the section and of the line bundle to $Y$. Given two limit line bundles $L_1$ and $L_2$, $L_2 \otimes L_1 \cong \bigotimes_Y L_Y \otimes n_Y$.

As $\otimes_Y L_Y$ is trivial we may assume that $\min\{n_Y\} = 0$ and thus use the $s_Y$ to define a section of $L_2 \otimes L_1^{-1}$ or equivalently a map $L_1 \to L_2$. The map is called simple if $\max\{n_Y\} = 1$ and primitive if in addition $n_Y = 1$ for a single component $Y$. Every map $L_1 \to L_2$ is a composition of primitive maps.

The isomomorphism class of a limit line bundle $L$ is determined by its multidegree, the function $f : V \to \mathbb{Z}$ assigning $Y$ to $\deg (L|_Y)$, where $V$ is the set of components of $X$. We may thus think of a quiver $Q = (G, A)$ whose set of vertices $G$ is the set of multidegrees of the limit line bundles, and whose set of arrows $A$ satisfies $A := \bigcup_Y A_Y \subseteq G \times G$, where $A_Y := \{(f_1, f_2) \in G \times G \mid f_2 - f_1 = u_Y\}$, $u_Y$ being the multidegree of $L_Y$. This way, the collection of all the limit line bundles and the primitive maps between them can be thought as a representation of $Q$ in the category of line bundles. The quiver is special, a $\mathbb{Z}^n$-quiver; see Definition 2.1.

The quiver appeared in [7], Notation 2.11, p. 62. In our paper it is used more proeminently. In fact, given a limit line bundle, its limit sections form a vector space and the maps between limit line bundles carry one space to the other. We have thus a representation $\mathfrak{g}$ of the quiver $Q$ in the category of vector spaces. Not only is $Q$ special: also $\mathfrak{g}$ is accordingly special, an exact linked net of finite support; see Definitions 2.3, 2.4 and 2.5. These representations are not studied in [7].

A linked net has a simple basis if the representation decomposes as a direct sum of irreducible subrepresentations of pure dimension 1; see Definition 5.1. Linked nets arising as above for a curve $X$ with two components have simple bases. Simple bases are, not surprisingly, very useful in arguments, and have appeared in various forms in the literature: Eisenbud and Harris touched on them in their definition of an adapted basis; Osserman used them in his proofs of Lemma A.9 and Lemma A.12 in [6] and later in [7]; they appeared more explicitly in Esteves
and Osserman [2, Lemma 2.3], as well as more subtly in the proofs by Muñoz of [4, Prop. 4.1] and [5, Prop. 3.6].

Our main result is Theorem 7.3, a local characterization of the linked nets that have simple bases. The known fact that linked nets arising from degenerations to a curve with two components have simple bases can be easily derived from it. We use this characterization to give an example of the non-existence of simple bases in the three-component case, our Example 7.4. The example answers a private question made by Osserman to the first author in 2017.

By doing away with the line bundles we focus on the linear aspects of a limit linear series. One of the goals of this paper is to present the basis of the theory of the special quivers and representations that arise. The fundamentals we study here will be applied in our second paper [3], in the study of linked projective spaces.

The linked projective space $\mathbb{LP}(g)$ of a linked net of vector spaces $g$ is simply the quiver Grassmannian of pure 1-dimensional subrepresentations of $g$. It is a subscheme of the product of the projectivized spaces of the vector spaces of $g$ associated to vertices in its support. We borrow our terminology from Osserman [6], who defined “linked Grassmannians” for curves of compact type with two components and proved a few of their properties. One of the key observations by Santana Rocha [8], which asserts the importance of $\mathbb{LP}(g)$, is that the linked projective space of a linked net $g$ arising from a degeneration to a curve $X$ of compact type with two components parameterizes the limits of divisors of the degenerating family of linear series, viewed as subschemes of $X$.

In [3] we study linked nets over $\mathbb{Z}^n$-quivers and show they are pure-dimensional and Cohen–Macaulay. A consequence is flatness of the linked projective spaces under certain additional hypotheses. Flatness shows that the linked projective space $\mathbb{LP}(g)$ is the degeneration of the (small) diagonal in the product of projective spaces where it lies, as long as $g$ is itself a degeneration. In particular, the multivariate Hilbert polynomial of $\mathbb{LP}(g)$ is the same as that of the diagonal.

This paper is organised as follows. In Section 2 we show how $\mathbb{Z}^n$-quivers and their special representations, the linked nets, arise from the degeneration of a linear series to a nodal curve with $n + 1$ components. We show in Theorem 2.2 that linked nets can be very different one from the other, but all $\mathbb{Z}^n$-quivers are isomorphic.

In Section 5 we give the precise definition of a simple basis. We prove a few technical results, Lemma 5.3 and Theorem 5.4, in preparation for the main result of this paper, Theorem 7.3.

In Section 6 we present the definition of the intersection property. The intersection property is local: It is defined at each vertex of the quiver. We also show some results in preparation to prove our main theorem, as Proposition 6.3.

In Section 7 we prove Theorem 7.3: a simple basis exists for a linked net of finite support if and only if the net satisfies the intersection property at each vertex. We finish with an example of an exact linked net which can be realised as the limit of a linear series and does not admit a simple basis.
2. Degenerations of linear series and their quiver representations

For non-negative integers \(d \) and \(n\), define

\[
\mathbb{Z}^{n+1}(d) := \left\{ (d_0, \ldots, d_n) \in \mathbb{Z}^{n+1} \mid \sum d_i = d \right\}.
\]

Let \(v_0, \ldots, v_n\) be elements in \(\mathbb{Z}^{n+1}(0)\) such that their sum is null and any proper subset of them is linearly independent over \(\mathbb{Q}\). To an element \(v\) in \(\mathbb{Z}^{n+1}(d)\) we associate a quiver \(Q(v; v_0, \ldots, v_n)\) whose set of vertices is

\[
G := v + \mathbb{Z}v_0 + \cdots + \mathbb{Z}v_n \subseteq \mathbb{Z}^{d+1}(d)
\]

and whose set of arrows \(A \subseteq G \times G\), is such that \(u \to u'\) is an arrow from \(u\) to \(u'\) if and only if

\[
u' = u + v_i \quad \text{for some} \ i \in \{0, \ldots, n\}.
\]

More generally, we have the following definition:

**Definition 2.1.** Let \(Q\) be a quiver, \(G\) its set of vertices and \(A\) its set of arrows. Let \(n \in \mathbb{N}\). A \(\mathbb{Z}^n\)-structure on \(Q\) is a decomposition of \(A\) in subsets \(A_0, \ldots, A_n\) satisfying the following three properties:

1. For each vertex of \(Q\) and each \(i = 0, \ldots, n\) there is a unique arrow in \(A_i\) leaving the vertex.
2. For each two distinct vertices \(v_1, v_2 \in G\) there is an admissible path \(\gamma\) in \(Q\) connecting \(v_1\) to \(v_2\).
3. Every two paths \(\gamma_1\) and \(\gamma_2\) in \(Q\) with the same initial vertex have the same terminal vertex if and only if \(\gamma_1(i) - \gamma_2(i)\) is constant for \(i \in \{0, \ldots, n\}\).

A quiver with a \(\mathbb{Z}^n\)-structure is called a \(\mathbb{Z}^n\)-quiver.

Notice that the decomposition \(A = \bigcup A_i\) where

\[
A_i := \{u \to u' \in A \mid u' = u + v_i\}
\]

gives the quiver \(Q = Q(v; v_0, \ldots, v_0)\) a \(\mathbb{Z}^n\)-structure.

**Proposition 2.2.** For each \(n\), all \(\mathbb{Z}^n\)-quivers are isomorphic.

*Proof.* Let \(Q\) be the \(\mathbb{Z}^n\)-quiver \(Q(v; v_0, \ldots, v_n)\). Let \(Q'\) be another \(\mathbb{Z}^n\)-quiver, with vertex set \(G'\), arrow set \(A'\) and \(\mathbb{Z}^n\)-structure given by the decomposition \(A' = \bigcup A'_i\). It is enough to show that these two \(\mathbb{Z}^n\)-quivers are equivalent.

We define a bijection \(f_0 : G \to G'\) as follows: Pick any vertex \(w \in G'\) and put \(f_0(v) := w\). Then, for any choice of integers \(\ell_0, \ldots, \ell_n\) with \(\min\{\ell_i\} = 0\), let \(f_0(v + \sum_i \ell_i v_i)\) be the vertex of \(Q'\) obtained as the final point of a path \(\gamma\) in \(Q'\) with initial point \(v\) satisfying \(\gamma(i) = \ell_i\) for each \(i\). That \(\gamma\) exists follows from Property 1 of \(Q'\), and that its final point does not depend on the chosen \(\gamma\) follows from Property 3. Now, any point of \(G\) is expressed as \(v + \sum_i \ell_i v_i\) for unique integers \(\ell_i\) with \(\min\{\ell_i\} = 0\). Thus \(f_0\) is well-defined. It is surjective by Property 2 of \(Q'\), and it is injective by Property 3.
We define now a bijection \( f_1 : A \rightarrow A' \) as follows: For each \( i = 0, \ldots, n \) and each arrow \( a \in A_i \), let \( f_1(a) \) be the arrow in \( A'_i \) leaving the vertex \( f_0(e) \), where \( e \) is the initial point of \( a \); it is unique by Property 1 of \( Q' \). Clearly, \( f_1(A_i) \subseteq A'_i \) for each \( i \). Also, \( f_1 \) has an inverse, mapping for each \( i = 0, \ldots, n \) an arrow \( a' \in A'_i \) to the arrow \( a \in A \) connecting \( e \) to \( e + v_i \), where \( f_0(e) \) is the initial vertex of \( a' \).

Finally, \( f_1 \) is compatible with \( f_0 \). Indeed, let \( a \in A \), with initial vertex \( e \) and terminal vertex \( h \). Then \( a \in A_j \) with \( h = e + v_j \) for some \( j \). Set \( a' := f_1(a) \). By definition, \( a' \in A_j \) and \( w := f_0(e) \) is its initial vertex. Let \( z \) be its terminal vertex. It remains to show that \( z = f_0(h) \). Now,

\[
e = v + \sum_i \ell_i v_i \quad \text{and} \quad h = v + \sum_i m_i v_i
\]

for unique integers \( \ell_i \) and \( m_i \) with \( \min \{ \ell_i \} = \min \{ m_i \} = 0 \). Also, since \( w = f_0(e) \), there is a path \( \gamma \) in \( Q' \) connecting \( v \) to \( w \) satisfying \( \gamma(b) = \ell_b \) for each \( b \). Then the concatenation \( \epsilon \) of \( \gamma \) with \( a' \) is a path in \( Q' \) connecting \( v \) to \( z \). There are two cases:

First, assume that there is \( i \) different from \( j \) with \( \ell_i = 0 \). Then \( m_b = \ell_b \) for each \( b \neq j \) and \( m_j = \ell_j + 1 \). Then \( \epsilon(b) = m_b \) for each \( b \). By definition, \( z = f_0(h) \).

Finally, assume that \( \ell_i > 0 \) for each \( i \) different from \( j \), whence \( \ell_j = 0 \). Then \( m_b = \ell_b - 1 \) for each \( b \neq j \) and \( m_j = \ell_j \). It follows that \( \epsilon(b) = m_b + 1 \) for each \( b \). By Property 3 of \( Q' \), any path \( \rho \) leaving \( v \) with \( \rho(b) = m_b \) for each \( b \) arrives at the same vertex as \( \epsilon \), which is \( z \). Thus, by definition again, \( z = f_0(h) \). \( \square \)

We draw, up to isomorphism, the general \( \mathbb{Z}_1 \)-quiver and \( \mathbb{Z}_2 \)-quiver.

**Figure 1.** \( \mathbb{Z}_1 \)-quiver.

**Figure 2.** \( \mathbb{Z}_2 \)-quiver.
\(\mathbb{Z}^n\)-quivers arise from degenerations of linear series. Indeed, let \(k\) be a field. Let \(X\) be a (connected, projective, reduced, pure-dimensional) nodal curve over \(k\). Assume its irreducible components are geometrically irreducible. Order and denote them by \(X_0, \ldots, X_n\) for \(n \geq 0\). Let \(B := \text{Spec}(k[[T]])\) and \(\pi : \mathcal{X} \to B\) be a flat, projective morphism with smooth generic fiber and special fiber isomorphic to \(X\). We call it a smoothing of \(X\). We say \(\pi\) is regular if \(\mathcal{X}\) is regular. We identify the special fiber with \(X\). In what follows we assume that the smoothing is regular.

Let \(L_\eta\) be a line bundle on the generic fiber \(\mathcal{X}_\eta\); denote by \(d\) its degree. Since \(\mathcal{X}\) is regular, each \(X_i\) is a Cartier divisor and there exists an extension \(L\) of \(L_\eta\) to \(\mathcal{X}\). It is not unique: For each divisor \(D = \sum n_i X_i\), the sheaf \(L(D) := L \otimes \mathcal{O}_X(D)\) is also an extension of \(L_\eta\).

Let \(v_i = (\ldots, \deg \mathcal{O}_X(X_i)|_X, \ldots)\) and \(v = (\ldots, \deg L|_X, \ldots)\) be the multidegrees of \(\mathcal{O}_X(X_i)|_X\) for \(i = 0, \ldots, n\) and \(L|_X\) respectively. Put
\[
G := v + \mathbb{Z}v_0 + \cdots + \mathbb{Z}v_n.
\]
Notice that \(G \subseteq \mathbb{Z}^{n+1}(d)\). For each \(u \in G\) and \(D = \sum n_i v_i\) let \(D \cdot u := u + \sum n_i v_i\). For each divisor \(D = \sum n_i X_i\), the multidegree of \(L(D)|_X\) is \(D \cdot v\), and thus lies in \(G\). Conversely, for each \(u \in G\), let
\[
L_u := L(D).
\]
where \(D = \sum n_i v_i\) satisfies \(u = D \cdot v\). The sheaf \(L_u\) does not depend on the choice of \(D\). Put \(L_u := L_u|_X\).

For each \(u_1, u_2 \in G\) there is a unique \(D = \sum n_i X_i\) with \(\min\{n_i\} = 0\) such that \(u_2 = D \cdot u_1\). Twisting by \(\mathcal{O}_X(D)\) gives us a natural nonzero morphism
\[
\varphi^{u_1}_{u_2} : L_u \to L_{u_2}
\]
The composition \(\varphi^{u_2}_{u_1} \circ \varphi^{u_1}_{u_2}\) is the multiplication by \(T^m\), where \(m := \max\{n_i\}\). The map \(\varphi^{u_1}_{u_2}\) induces (abusing notation)
\[
\varphi^{u_1}_{u_2} : L_{u_1} \to L_{u_2}.
\]
If \(D = \sum n_i X_i\) is a sum of effective divisors \(D_1\) and \(D_2\), then
\[
\varphi^{u_1}_{u_2} = \varphi^{u_2}_{u_3} \circ \varphi^{u_1}_{u_2}
\]
for each \(u_1 \in G\), where \(u_2 := D_1 \cdot u_1\) and \(u_3 := D_2 \cdot u_2\).

Let \(A_i \subseteq G \times G\) be the subset of pairs \((u_1, u_2)\) such that \(u_2 = X_i \cdot u_1\) for each \(i = 0, \ldots, n\) and set \(A := \bigcup A_i\). We obtained the quiver
\[
Q = Q(v; v_0, \ldots, v_n).
\]
The \(L_u\) for \(u \in G\) and the maps \(\varphi^{u_2}_{u_1}\) for \((u_1, u_2) \in A\) give us a representation of \(Q\) in the category of line bundles over \(X\).

In addition, let \(\mathcal{V}_\eta \subseteq H^0(\mathcal{X}_\eta, L_\eta)\) be a vector subspace. For each \(u \in G\) there is an extension \(\mathcal{V}_u\) of \(\mathcal{V}_\eta\) given by
\[
\mathcal{V}_u := \left\{ s \in H^0(\mathcal{X}, L_u) \mid s|_{X_\eta} \in \mathcal{V}_\eta \right\}.
\]
Set $V_u := V_u|_X \subseteq H^0(X, L_u)$. For each $u_1, u_2 \in G$ the map $\varphi_{u_1}^{u_2}$ induces maps (abusing notation)

$$\varphi_{u_1}^{u_2} : V_{u_1} \longrightarrow V_{u_2}$$

and $\varphi_{u_2}^{u_1} : V_{u_2} \rightarrow V_{u_2}$.

The $V_u$ for $u \in G$ and the maps $\varphi_{u_1}^{u_2}$ for $(u_1, u_2) \in A$ give us a representation of $Q$ in the category of vector spaces over $k$, more precisely an \textit{exact linked net of vector spaces with finite support}, as defined below.

Given a representation $g$ of a quiver $Q$, we denote by $V_v$ the vector space associated to each vertex $v$ of $Q$ and by $\varphi_\gamma$ the composition of maps associated to each path $\gamma$ in $Q$.

\textbf{Definition 2.3.} Let $Q$ be a quiver, $G$ its set of vertices and $A$ its set of arrows. Let $A_0, \ldots, A_n$ be a decomposition of $A$ giving $Q$ a $\mathbb{Z}^n$-structure. A \textit{linked net of vector spaces} over $Q$ is a representation of $Q$ of pure positive dimension satisfying the following three properties:

1. If $\gamma_1$ and $\gamma_2$ are two admissible paths connecting the same two vertices then $\varphi_{\gamma_1} = \varphi_{\gamma_2}$.
2. If $\gamma$ is a non-admissible path then $\varphi_\gamma = 0$.
3. If $\gamma_1$ and $\gamma_2$ are two admissible paths leaving the same vertex such that $\gamma_1(i) = 0$ or $\gamma_2(i) = 0$ for each $i$ then $\text{Ker}(\varphi_{\gamma_1}) \cap \text{Ker}(\varphi_{\gamma_2}) = 0$.

If only Properties 1 and 2 are satisfied, we call the representation a \textit{weakly linked net of vector spaces}.

Given a weakly linked net of vector spaces over a $\mathbb{Z}^n$-quiver $Q$, and two vertices $v_1$ and $v_2$ of $Q$, there is an admissible path $\gamma$ connecting $v_1$ to $v_2$, and since $\varphi_\gamma$ does not depend on the choice of $\gamma$, we write $\varphi_{v_1}^{v_2} := \varphi_\gamma$. Because of the independence of the admissible path, given $s \in V_{v_1}$ we put

$$s|_{V_{v_2}} := \varphi_{v_1}^{v_2}(s).$$

\textbf{Definition 2.4.} We say that a weakly linked net of vector spaces $g$ over a $\mathbb{Z}^n$-quiver $Q$ with vertex set $G$ is \textit{supported} in a non-empty subset $H \subseteq G$ if for each $w \in G$ there exists $v \in H$ such that $\varphi_w^v$ is an isomorphism. If $H$ is finite we say $g$ has \textit{finite support}.

An admissible path $\gamma$ in a $\mathbb{Z}^n$-quiver $Q$ is called \textit{simple} if $\gamma(i) \leq 1$ for every $i$. Two distinct vertices of $Q$ that can be connected by an simple admissible path are called \textit{neighbors}. Given a linked net of vector spaces $g$ over $Q$ we have that $\varphi_{u_2}^{u_1} \varphi_{u_1}^{u_2} = 0$ for each two distinct vertices $u_1$ and $u_2$ of $Q$.

\textbf{Definition 2.5.} A weakly linked net of vector spaces $g$ over a $\mathbb{Z}^n$-quiver $Q$ is \textit{exact} if

$$\text{Ker}(\varphi_{u_2}^{u_1}) = \text{Im}(\varphi_{u_1}^{u_2})$$

for each two neighboring vertices $u_1$ and $u_2$ of $Q$.

It can be easily seen that the representation $g$ given by the spaces $V_u$ and the maps $\varphi_{u_1}^{u_2}$ of the $\mathbb{Z}^n$-quiver $Q = Q(v; v_0, \ldots, v_n)$ arising from a degeneration of
linear series is a linked net of vector spaces. It has support in the subset $H$ of $G$
of effective multidegrees, which is finite. Also, $g$ is exact.

3. $\mathbb{Z}^n$-quivers

Lemma 3.1. Let $Q$ be a $\mathbb{Z}^n$-quiver with vertex set $G$. Let $u, v, w \in G$. Then the following first two statements are equivalent and imply the last two:

1. There is an admissible path from $u$ to $w$ through $v$.
2. The concatenation of every admissible path from $u$ to $v$ and every admissible path from $v$ to $w$ is admissible.
3. If $v \neq w$ then all paths from $u$ to $v$ through $w$ are not admissible.
4. If $u \neq v$ then all paths from $v$ to $w$ through $u$ are not admissible.

Furthermore, if $v$ and $w$ are neighbors then the third statement is equivalent to the first two, and if $u$ and $v$ are neighbors then the fourth statement is equivalent to the first two.

Proof. Assume Statement 1. Let $\gamma$ be an admissible path from $u$ to $w$ through $v$. It is the concatenation of a path $\gamma_1$ from $u$ to $v$ with a path $\gamma_2$ from $v$ to $w$, both admissible. If $\beta$ is an admissible path from $u$ to $v$ and $\nu$ is one from $v$ to $w$, then $\beta(i) = \gamma_1(i)$ and $\nu(i) = \gamma_2(i)$, and thus $\beta(i) + \nu(i) = \gamma(i)$ for every $i$. Since $\gamma$ is admissible, it follows that the concatenation of $\beta$ with $\nu$ is admissible.

Assume Statement 2 now. Since there are admissible paths from $u$ to $v$ and from $v$ to $w$, Statement 1 holds. Furthermore, assume $v \neq w$. Let $\gamma$ be a path from $u$ to $w$ and $\mu$ one from $u$ to $v$. Assume by contradiction to Statement 3 that the concatenation of $\gamma$ with $\mu$ is admissible, call it $\beta$. Then $\gamma$ and $\mu$ are admissible. Let $\nu$ be any path leaving $v$ such that $\nu(i) = m - \mu(i)$ for each $i = 0, 1, \cdots, n$, where $m := \max\{\mu(i)\}$. Since $v \neq w$ it follows that $m > 0$. Also, $\nu$ is admissible. Since $\nu(i) + \mu(i)$ is constant for $i = 0, \cdots, n$, the path $\nu$ arrives back to $w$. Now, if $\beta(i) = 0$ then $\gamma(i) = \mu(i) = 0$ and thus $\nu(i) = m > 0$. Then the concatenation of $\beta$ with $\nu$ is not admissible, contradicting Statement 2.

Assume Statement 2 again. Assume $u \neq v$. Let $\gamma$ be a path from $v$ to $u$ and $\mu$ one from $u$ to $w$. Assume by contradiction to Statement 4 that the concatenation of $\gamma$ with $\mu$ is admissible, call it $\beta$. Then $\gamma$ and $\mu$ are admissible. Let $\nu$ be any path leaving $u$ such that $\nu(i) = m - \gamma(i)$ for each $i = 0, 1, \cdots, n$, where $m := \max\{\gamma(i)\}$. Since $u \neq v$ it follows that $m > 0$. Also, $\nu$ is admissible. Since $\nu(i) + \gamma(i)$ is constant for $i = 0, \cdots, n$, the path $\nu$ arrives back to $v$. Now, if $\beta(i) = 0$ then $\gamma(i) = \mu(i) = 0$ and thus $\nu(i) = m > 0$. Then the concatenation of $\nu$ with $\beta$ is not admissible, negating Statement 2.

Let now $\beta$ be an admissible path from $u$ to $v$ and $\nu$ one from $v$ to $w$. Assume that their concatenation $\rho$ is not admissible. Let $\gamma$ be an admissible path connecting $u$ to $w$. Then both $\gamma$ and $\rho$ connect $u$ to $w$. If $\beta$ or $\nu$ is simple, it follows that

$$\gamma(i) = \beta(i) + \nu(i) - 1$$

for every $i$. 

If \( v \) and \( w \) are neighbors (thus \( v \neq w \)), then \( \nu \) is simple. So is \( \mu \), any chosen admissible path from \( w \) to \( v \) satisfying \( \mu(i) + \nu(i) = 1 \) for every \( i \). But then
\[
\beta(i) = \gamma(i) + \mu(i) \quad \text{for every } i.
\]
Since \( \beta \) is admissible, so is the concatenation of \( \gamma \) with \( \mu \), negating Statement 3.

If \( u \) and \( v \) are neighbors (thus \( u \neq v \)), then \( \beta \) is simple. So is \( \alpha \), any chosen admissible path connecting \( v \) to \( u \) satisfying \( \alpha(i) + \beta(i) = 1 \) for every \( i \). But then
\[
\nu(i) = \gamma(i) + \alpha(i) \quad \text{for every } i.
\]
Since \( \nu \) is admissible, so is the concatenation of \( \alpha \) with \( \gamma \), negating Statement 4. \( \square \)

**Definition 3.2.** Let \( Q \) be a \( \mathbb{Z}^n \)-quiver. For each vertex \( v \) of \( Q \) and each proper subset \( I \subseteq \{0, \ldots, n\} \), the \( I \)-cone of \( v \), denoted \( C_I(v) \), is the set of end vertices of all the paths \( \gamma \) leaving \( v \) for which \( \gamma(i) > 0 \) only if \( i \in I \).

**Proposition 3.3.** Let \( Q \) be a \( \mathbb{Z}^n \)-quiver, and \( z_0, \ldots, z_n \) be vertices of \( Q \). For each \( j = 0, \ldots, n \), let \( I_j := \{0, \ldots, n\} - \{j\} \). Then the intersection
\[
\bigcap_{j=0}^{n} C_{I_j}(z_j)
\]
is finite.

**Proof.** Since all \( \mathbb{Z}^n \)-quivers are equivalent by Proposition 2.2, we may assume that \( Q \) is the quiver \( Q(v; v_0, \ldots, v_n) \) with \( v = 0 \), with \( v_1, \ldots, v_n \) forming the canonical basis of \( \mathbb{Z}^n \) and \( v_0 = (-1, \ldots, -1) \). Then the \( j \)th coordinate of a vertex in \( C_{I_j}(z_j) \) is bounded above by the \( j \)th coordinate of \( z_j \) for each \( j = 1, \ldots, n \). And the \( j \)th coordinate of a vertex in \( C_{I_0}(z_0) \) is bounded below by the \( j \)th coordinate of \( z_0 \) for each \( j = 1, \ldots, n \). Thus the vertices of the intersection have bounded coordinates, and hence there are finitely many of them. \( \square \)

**Definition 3.4.** Let \( Q \) be a \( \mathbb{Z}^n \)-quiver with set of vertices \( G \). Let \( H \) be a non-empty subset of \( G \). Let \( P(H) \) be the set of all \( v \in G \) such that for each \( i = 0, \ldots, n \) there are \( z \in H \) and a path \( \gamma \) connecting \( z \) to \( v \) with \( \gamma(i) = 0 \). We call \( P(H) \) the **hull** of \( H \).

**Proposition 3.5.** Let \( Q \) be a \( \mathbb{Z}^n \)-quiver. Let \( H \) be a non-empty set of vertices of \( Q \). Then the following three statements hold:

1. \( H \subseteq P(H) \).
2. If \( H \) is finite, so is \( P(H) \).
3. \( P(P(H)) = P(H) \).

**Proof.** Statement 1 is clear: If \( v \in H \) then the empty path \( \gamma \) connects a vertex of \( H \) to \( v \) and satisfies \( \gamma(i) = 0 \) for every \( i \).

As for Statement 2, observe that
\[
P(H) = \bigcup_{f \in H(0, \ldots, n)} \bigcap_{j=0}^{n} C_{I_j}(f(j)).
\]
If $H$ is finite, so is the set of functions $H^{(0,\ldots,n)}$ from $\{0,\ldots,n\}$ to $H$. Thus $P(H)$ is a finite union of finite sets by Proposition 3.3.

As for Statement 3, the inclusion $P(H) \subseteq P(P(H))$ follows from Statement 1. In addition, for each $v \in P(P(H))$ and $i \in \{0,\ldots,n\}$ there are $w_i \in P(H)$ and a path $\gamma_i$ connecting $w_i$ to $v$ with $\gamma_i(i) = 0$. Since $w_i \in P(H)$, there are $z_i \in H$ and a path $\mu_i$ connecting $z_i$ to $w_i$ with $\mu_i(i) = 0$. The concatenation of $\mu_i$ with $\gamma_i$ is a path $\nu_i$ connecting $z_i$ to $v$ with $\nu_i(i) = 0$. As this holds for each $i = 0,\ldots,n$, it follows that $v \in P(H)$. As this holds for each $v \in P(P(H))$, we have $P(P(H)) \subseteq P(H)$. \hfill $\square$

There are other ways of characterizing $P(H)$:

**Proposition 3.6.** Let $H$ be a non-empty set of vertices of a $\mathbb{Z}^n$-quiver $Q$ and $v$ a vertex of $Q$. Then the following statements are equivalent:

1. $v \not\in P(H)$.
2. There is a vertex $w$ of $Q$ different from $v$ such that there is an admissible path from $z$ to $v$ for each $z \in H$ passing through $w$.
3. There is a vertex $w$ of $P(H)$ different from $v$ such that there is an admissible path from $z$ to $v$ for each $z \in H$ passing through $w$. (If $P(H) = H$ then $w$ is unique.)
4. There are a vertex $w$ of $Q$ and a non-empty proper subset $I$ of $\{0,\ldots,n\}$ such that $v \in C_I(w) - \{w\}$ and for each admissible path $\gamma$ connecting a vertex of $H$ to $w$ there is $i \not\in I$ with $\gamma(i) = 0$.

**Proof.** Assume Statement 1. By definition, there is $i \in \{0,\ldots,n\}$ such that for each $z \in H$ we have $\gamma_z(i) > 0$ for each admissible path $\gamma_z$ connecting $z$ to $v$. Let $a$ be the $i$-arrow arriving at $v$, and $w$ its initial vertex. For each $z \in H$ let $\mu_z$ be a path arriving at $w$ with $\mu_z(j) = \gamma_z(j)$ for each $j \neq i$ and $\mu_z(i) = \gamma_z(i) - 1$. The concatenation of $\mu_z$ with $a$ is an admissible path $\rho_z$ satisfying $\rho_z(i) = \gamma(i)$ for every $i$. Since $\rho_z$ and $\gamma$ arrive at the same vertex, $v$, they leave from the same vertex, $z$. So $\rho_z$ is an admissible path from $z$ to $v$ passing through $w$.

Assume Statement 2. If $w \not\in P(H)$ then we apply the above argument again. As $H$ is non-empty, and as an admissible path from $z \in H$ to $v$ has finite length, the argument cannot be repeated indefinitely. Thus there is a vertex $w \in P(H)$ such that there is an admissible path from $z$ to $v$ for each $z \in H$ passing through $w$. If $w'$ is another vertex of $P(H)$ with the same property, and $P(H) = H$, then there are admissible paths from $w$ to $v$ passing through $w'$ and from $w'$ to $v$ passing through $w$. Then an admissible path from $w'$ to $v$ has length at most that of an admissible path from $w$ to $v$, and length at least that of an admissible path from $w$ to $v$, with equality only if an admissible path from $w$ to $w'$ has length zero, that is, $w' = w$.

Assume Statement 2. Let $\mu$ be an admissible path connecting $w$ to $v$ and put $I := \{i \mid \mu(i) > 0\}$. Then $I$ is a non-empty proper subset of $\{0,\ldots,n\}$ and $v \in C_I(w) - \{w\}$. Also, since there is an admissible path $\gamma_z$ connecting $z$ to $v$ through $w$, for each $z \in H$, then $\gamma_z(i) > 0$ for every $i \in I$, and thus $\gamma_z(i) = 0$ for some $i \not\in I$. Of course, also $\gamma(i) = 0$ for any admissible path connecting $z$ to $v$. 

Assume Statement 4. Let \( \mu \) be an admissible path connecting \( w \) to \( v \). Put \( J := \{ i \mid \mu(i) > 0 \} \). Since \( v \in C_I(w) - \{ w \} \) we have that \( J \) is non-empty and \( J \subseteq I \). For each \( z \in H \) let \( \gamma_z \) be an admissible path from \( z \) to \( w \). Since \( \gamma_z(i) = 0 \) for some \( i \notin I \), it follows that the concatenation \( \rho_z \) of \( \gamma_z \) with \( \mu \) is an admissible path connecting \( z \) to \( v \) such that \( \rho_z(i) > 0 \) for each \( i \in J \). Thus \( v \notin P(H) \). \( \square \)

4. Linked nets

Let \( g \) be a linked net of vector spaces over a \( \mathbb{Z}^n \)-quiver \( Q \). If \( g \) has support in a finite set \( H \), we may remove vertices from \( H \), if necessary, to assume that \( H \) is minimal. Then \( H \) is unique by the next proposition.

**Proposition 4.1.** Let \( g \) be a weakly linked net of vector spaces over a \( \mathbb{Z}^n \)-quiver \( Q \) with vertex set \( G \) and support in \( H \subset G \) and in \( H' \subset G \). If \( H' \) is minimal for this property, then \( H' \subseteq H \). Furthermore, for distinct \( v, w \in H' \) the map \( \varphi_w^v \) is not an isomorphism.

**Proof.** Suppose by contradiction that there exist distinct \( v, w \in H' \) such that the map \( \varphi_w^v \) is an isomorphism. We claim that \( g \) has support on \( H' - \{ w \} \), a contradiction. Indeed, take \( u \in G \). By hypothesis, there exists \( z \in H' \) such that the map \( \varphi_z^u \) is an isomorphism. If \( z \neq w \), we are done. If \( z = w \) then the composition \( \varphi_u^w \varphi_w^v \) is an isomorphism too. It follows that there is an admissible path connecting \( v \) to \( u \) via \( w \), and thus \( \varphi_u^v = \varphi_u^w \varphi_w^v \), also an isomorphism. Hence, we can remove \( w \) from \( H' \).

Now, suppose that \( v \notin H \), for some \( v \in H' \). Since \( g \) is supported on \( H \), there exists \( w \in H \) such that \( \varphi_w^v \) is an isomorphism. Hence, \( \varphi_w^v \) is zero. Since \( g \) is supported on \( H' \) as well, there is \( u \in H' \) such that \( \varphi_w^v \) is an isomorphism. Clearly, \( u \neq v \). But then the composition \( \varphi_u^w \varphi_w^v \) is an isomorphism, and thus agrees with \( \varphi_u^v \). But this contradicts what we have just proved above. Therefore \( H' \subseteq H \). \( \square \)

**Proposition 4.2.** Let \( g \) be a weakly linked net of vector spaces over a \( \mathbb{Z}^n \)-quiver \( Q \) and \( H \) a set of vertices of \( Q \). If \( H \) generates \( g \) then it has support on \( P(H) \). In addition, \( g \) is finitely generated if and only if it has finite support.

**Proof.** If \( g \) has support in a finite set of vertices \( S \) then \( S \) generates \( g \): For each \( v \in S \), let \( s_0^v, \ldots, s_r^v \) be a basis of \( V_v \); then, for each vertex \( w \) of the quiver, there is \( v \in S \) such that \( s_0^v|_{V_w}, \ldots, s_r^v|_{V_w} \) generate \( V_w \).

Because of Proposition 3.3 it remains to prove the first statement. Suppose then that \( H \) generates \( g \). Let \( v \notin P(H) \). Then, by Proposition 3.6 there is a vertex \( w \in P(H) \) different from \( v \) such that there is an admissible path from each \( z \in H \) to \( v \) passing through \( w \). Thus \( s|_{V_v} = \varphi_w^v s|_{V_v} \) for each \( s \in V_z \) for each \( z \in H \). Since \( H \) generates \( g \), it follows that \( \varphi_w^v \) is surjective, whence an isomorphism. \( \square \)

**Lemma 4.3.** Let \( g \) be a weakly linked net over a \( \mathbb{Z}^n \)-quiver \( Q \). Then each of the following statements imply the next:

1. \( g \) is a linked net
(2) \( \ker(\varphi_\nu) \cap \ker(\varphi_\nu) = 0 \) for each two simple admissible paths \( \nu_1 \) and \( \nu_2 \) in \( Q \) leaving the same vertex such that \( \nu_1(i) = 0 \) or \( \nu_2(i) = 0 \) for each \( i \).
(3) For each vertex \( v \) of \( Q \) and each \( i = 0, \ldots, n \), we have
\[
\ker(\varphi_a) \cap \im(\varphi_b) = 0,
\]
where \( a \) is the \( i \)-arrow leaving \( v \) and \( b \) is the \( i \)-arrow arriving at \( v \).
(4) For each admissible path \( \gamma \) in \( Q \),
\[
\ker(\varphi_\gamma) = \ker(\varphi_\nu),
\]
where \( \nu \) is any simple admissible path leaving the same vertex of \( Q \) as \( \gamma \) such that \( \nu(i) > 0 \) if and only if \( \gamma(i) > 0 \) for each \( i = 0, \ldots, n \).

Furthermore, if Statements 2 and 4 hold then Statement 1 holds. Also, if \( g \) is exact all four statements are equivalent.

**Proof.** Statement 1 clearly implies Statement 2, which clearly implies Statement 3, as \( \varphi_\nu \varphi_b = 0 \) for each simple admissible path \( \nu \) leaving \( v \) satisfying \( \nu(j) = 1 \) for \( j \neq i \).

Assume Statement 3. We make a claim. Let \( \mu \) be a nontrivial admissible path and \( i \in \{0, \ldots, n\} \) such that \( \mu(i) > 0 \). Let \( w \) be the final vertex of \( \mu \) and \( a \) an arrow in \( A_i \) leaving \( w \). We claim that
\[
\ker(\varphi_a) \cap \im(\varphi_\mu) = 0.
\]
Indeed, let \( b \) the arrow in \( A_i \) arriving in \( w \). Let \( v \) be its initial vertex. Let \( \beta \) be a path arriving in \( v \) satisfying \( \beta(j) = \mu(j) \) for each \( j \neq i \) and \( \beta(i) = \mu(i) - 1 \). The concatenation of \( \beta \) with \( b \) is an admissible path that leaves and arrives at the same vertex as \( \mu \), whence \( \varphi_\mu = \varphi_b \varphi_\beta \), and so \( \im(\varphi_\mu) \subseteq \im(\varphi_b) \). We may thus assume \( \mu = b \) and apply Statement 3.

We prove Statement 4. Let \( v \) be a vertex of \( Q \) and \( \gamma \) an admissible path in \( Q \) leaving \( v \). We proceed by induction on the length of \( \gamma \). If \( \gamma \) has length 0 or 1, Equation 2 holds trivially. It holds as well if \( \gamma \) is simple, as then \( \varphi_\gamma = \varphi_\nu \). So we may assume there is \( i \in \{0, 1, \ldots, n\} \) such that \( \gamma(i) > 2 \). There is an admissible path \( \beta \) leaving \( v \) whose last arrow \( b \) is of type \( i \) such that \( \gamma(j) = \beta(j) \) for each \( j \neq i \) and \( \gamma(i) = \beta(i) + 1 \). Let \( a \) be an arrow of type \( i \) leaving the final vertex of \( b \). Then \( \varphi_\gamma = \varphi_b \varphi_\beta \). By Statement 3, \( \ker(\varphi_b \varphi_\beta) = \ker(\varphi_\beta) \). By induction, \( \ker(\varphi_\beta) = \ker(\varphi_\nu) \) for any simple admissible path \( \nu \) leaving \( v \) such that \( \nu(j) > 0 \) if and only if \( \beta(j) > 0 \), or equivalently, if and only if \( \gamma(j) > 0 \) for each \( j = 0, \ldots, n \). Statement 4 is proved.

Clearly, Statements 2 and 4 imply Statement 1. Assume now that \( g \) is exact and Statement 4 holds. We need only prove Statement 2. Let \( \nu_1 \) and \( \nu_2 \) be as in Statement 2. Let \( v \) be their initial vertex. We may assume they are both nontrivial. Let \( \mu_2 \) be a simple admissible path arriving at \( v \) such that \( \nu_2(i) + \mu_2(i) = 1 \) for every \( i \). Since \( g \) is exact, \( \ker(\varphi_\nu_2) = \im(\varphi_\mu_2) \). Since \( \nu_1 \) is nontrivial, there is \( i \) such that \( \nu_1(i) = 1 \). Then \( \nu_2(i) = 0 \) and \( \mu_2(i) = 1 \). We may assume that \( \nu_1 \) starts with an arrow of type \( i \) and that \( \mu_2 \) ends with an arrow of type \( i \). Applying Statement 4 we finish the proof. \( \square \)
The dual of a weakly linked net $g$ over a $\mathbb{Z}^n$-quiver denoted by $g^*$ is the representations whose vector spaces are $V_v^*$ are the dual of $V_v$ for each vertex $v$ of the quiver and whose maps, $\varphi_v^* : V_v^* \to V_u^*$, are the dual of $\varphi_v : V_u \to V_v$ for each arrow $\alpha$ of the quiver. Notice that when we consider the dual of a linked net over a $\mathbb{Z}^n$-quiver $Q = (G, A)$ with $\mathbb{Z}^n$-structure $A = \bigcup A_i$ we are inverting the arrows of $Q$. Thus $g^*$ is over the $\mathbb{Z}^n$-quiver $Q^* = (G^*, A^*)$ whose set of vertices are equal to the one of $Q$, but $v \to u$ is an arrow in $A^*$ if and only if $u \to v$ is an arrow in $A$. The $\mathbb{Z}^n$-structure of $Q^*$ is given by $A^* = \bigcup A_i^*$ where $A_i^*: = \{v \to u \in A^* | u \to v \in A_i\}$ for each $i = 0, \ldots, n$.

**Proposition 4.4.** Let $g$ be a weakly linked net of vector spaces over a $\mathbb{Z}^n$-quiver and $h$ a subrepresentation of pure dimension. Then the following statements hold:

1. The dual of $g$ is a weakly linked net.
2. $h$ is a weakly linked net.
3. If $g$ is a linked net so is $h$.
4. If $h \neq g$ then the quotient $g/\mathfrak{h}$ is a weakly linked net.
5. If $g$ has support in a collection of vertices $H$, so have $h$ and $g/\mathfrak{h}$.
6. If $g$ and $h$ are exact, so is $g/\mathfrak{h}$.

**Proof.** All the statements are clearly true, except perhaps the last. Assume $g$ and $h$ are exact. Let $w_1$ and $w_2$ be neighboring vertices and let $t$ be a section of $g$ at $w_1$ such that $\varphi_{w_2}^{w_1}(t)$ is a section of $h$ at $w_2$. Since $\varphi_{w_1}^{w_2}\varphi_{w_2}^{w_1}(t) = 0$ and $h$ is exact, there is a section $s$ of $h$ at $w_1$ such that $\varphi_{w_2}^{w_1}(t) = \varphi_{w_2}^{w_1}(s)$. Since $g$ is exact and $\varphi_{w_2}^{w_1}(t-s) = 0$, there is a section $x$ of $g$ at $w_2$ such that $t-s = \varphi_{w_1}^{w_2}(x)$. This finishes the proof. 

We call $\mathfrak{h}$ above a (weakly) (linked) subnet. Even if $g$ and $h$ are linked nets, the quotient $g/\mathfrak{h}$ need not be; see Proposition 4.2 below though.

5. **Simple bases**

**Definition 5.1.** Let $g$ be a weakly linked net of vector spaces over a $\mathbb{Z}^n$-quiver $Q$. A collection of vertices $v_1, \ldots, v_m$ of $Q$ and vectors $s_i \in V_{v_i}$ for $i = 1, \ldots, m$ such that

$$\{s_1|_{V_w}, \ldots, s_m|_{V_w}\}$$

generates $V_w$ for each vertex $w$ of $Q$

is called a set of generators of $g$. We will also say that $\{v_1, \ldots, v_m\}$ generate $g$ and that $g$ is finitely generated. If $m = \dim g$, it is called a simple basis.

**Proposition 5.2.** If a weakly linked net of vector spaces over a $\mathbb{Z}^n$-quiver admits a simple basis, then it is an exact linked net of finite support.

**Proof.** First of all, $g$ has finite support by Proposition 4.2. Now, let $w_0, \ldots, w_r$ be vertices of the quiver and $s_i \in V_{w_i}$ for $i = 0, \ldots, r$ forming a simple basis for $g$. For each two vertices $u$ and $v$, and each $i = 0, \ldots, r$:

$$\varphi_u^v(s_i|_{V_u}) = \epsilon_is_i|_{V_v},$$
where \( \epsilon_i = 1 \) if there is an admissible path from \( w_i \) to \( v \) passing through \( u \); otherwise \( \epsilon_i = 0 \). It follows that \( \operatorname{Ker}(\varphi^n_u) \) is generated by those \( s_i|_{V_u} \) for which there is no admissible path from \( w_i \) to \( v \) passing through \( u \) and \( \operatorname{Im}(\varphi^n_u) \) is generated by those \( s_i|_{V_u} \) for which there is an admissible path from \( w_i \) to \( u \) through \( v \). By Lemma 3.1, if \( u \) and \( v \) are neighboring vertices the two sets are the same. Thus \( \mathfrak{g} \) is exact.

Also, given two admissible paths \( \gamma_1 \) and \( \gamma_2 \) leaving the same vertex \( u \), the kernels of \( \varphi_{\gamma_1} \) and \( \varphi_{\gamma_2} \) intersect nontrivially if and only if there is \( i \in \{0, \ldots, r\} \) such that the concatenations of every admissible path \( \nu \) connecting \( w_i \) to \( u \) with \( \gamma_1 \) and \( \gamma_2 \) are not admissible. But given \( \nu \) there is \( j \) such that \( \nu(j) = 0 \), and hence \( \gamma_1(j) > 0 \) and \( \gamma_2(j) > 0 \). Property 3 of a linked net is verified. \( \square \)

Let \( \mathfrak{g} \) be a weakly linked net of vector spaces over a \( \mathbb{Z}^n \)-quiver. Let \( v \) be a vertex of the quiver. An element \( s \in V_v \) will be called a section of \( \mathfrak{g} \) at \( v \). The section \( s \) is called primitive if

\[
s \in V_v - \sum_a \operatorname{Im}(\varphi_a),
\]

where \( a \) runs through all arrows arriving at \( v \).

**Lemma 5.3.** Let \( \mathfrak{g} \) be a weakly linked net of vector spaces with finite support over a \( \mathbb{Z}^n \)-quiver \( Q \). Then \( \mathfrak{g} \) has a primitive section at some vertex of \( Q \). If \( \mathfrak{g} \) is an exact linked net and \( s \) is a primitive section of \( \mathfrak{g} \) at a vertex of \( Q \), then there is a unique subnet \( \mathfrak{h} \) of \( \mathfrak{g} \) generated by \( s \). Furthermore, \( s \) is a simple basis for \( \mathfrak{h} \). In particular, \( \mathfrak{h} \) is exact of finite support.

**Proof.** Let \( H \) be a finite set of vertices over which \( \mathfrak{g} \) is supported. We prove the first statement. Suppose by contradiction that

\[
V_v = \sum_a \operatorname{Im}(\varphi_a)
\]

for each vertex \( v \) of the quiver. Let \( v \in H \) and \( s \in V_v - \{0\} \). Then

\[
s = \sum_i \varphi_{\mu_i}(s_i),
\]

where the \( \mu_i \) are certain admissible paths of length 1 and the \( s_i \) satisfy \( \varphi_{\mu_i}(s_i) \neq 0 \).

We claim we can find a sequence of expressions for \( s \) as in (3) for admissible paths \( \mu_i \) with initial vertices \( w_i \) in \( H \) and \( s_i \in V_{w_i} \) satisfying \( \varphi_{\mu_i}(s_i) \neq 0 \), such that the minimum of the lengths of the \( \mu_i \) goes to infinity. The desired contradiction follows from the claim: Since \( H \) is finite the lengths of admissible paths connecting vertices of \( H \) are bounded.

Indeed, consider an expression as in (3) for admissible paths \( \mu_i \) and \( s_i \) such that \( \varphi_{\mu_i}(s_i) \neq 0 \). Let \( w_i \) be the initial vertices of the \( \mu_i \). For each \( i \), there are \( z_i \in H \), an admissible path \( \gamma_i \) connecting \( z_i \) to \( w_i \) and \( t_i \in V_{z_i} \) such that \( s_i = \varphi_{\gamma_i}(t_i) \). Since

\[
\varphi_{\mu_i}(s_i) = \varphi_{\mu_i}(t_i) \neq 0,
\]

it follows that the concatenation of \( \gamma_i \) with \( \mu_i \) is admissible. Up to replacing \( \mu_i \) with this concatenation, which has at least the same length as \( \mu_i \), and \( s_i \) with \( t_i \)
for each $i$, we may assume $w_i \in H$. By contradiction hypothesis, for each $i$ we have $s_i = \sum_\alpha \varphi_\alpha(s_{i,\alpha})$, where the $\alpha$ are arrows arriving at $w_i$. Thus

$$\varphi_{\mu_i}(s_i) = \sum_\alpha \varphi_{\mu_i}\varphi_\alpha(s_{i,\alpha}).$$

Removing from the above sum those $\alpha$ for which $\varphi_{\mu_i}\varphi_\alpha(s_{i,\alpha}) = 0$, in particular, those $\alpha$ whose concatenation with $\mu_i$ is not admissible, we will obtain an expression as in (3) but with paths $\mu_i$ of bigger length, whence the claimed sequence of expressions.

Let us now prove the second and third statements. Assume $g$ is a linked net, let $v$ be a vertex of $Q$ and $s$ a primitive section of $g$ at $v$. Observe first that $\varphi^v_w(s) \neq 0$ for each vertex $w$ of the quiver. Indeed, otherwise $\varphi^v_w(s) = 0$ for a neighboring vertex $w$ by Lemma 5.3 and hence exactness of $g$ yields $s \in \text{Im}(\varphi^v_w)$, contradicting the choice of $s$.

For each vertex $w$ of $Q$ let $W_w \subseteq V_w$ be the subspace generated by $\varphi^v_w(s)$. It has dimension 1. The $W_w$ form a subrepresentation $h$ of $g$, as for each two vertices $w$ and $z$ either $\varphi^w_z\varphi^v_w(s)$ is zero, if there is no admissible path connecting $v$ to $z$ through $w$, or equal to $\varphi^v_z(s)$, at any rate contained in the subspace $W_z$. Of course, $h$ has pure dimension 1. It is a subnet of $g$, as every subrepresentation of pure dimension is. It has $s$ as a generator, in fact as a simple basis. The uniqueness of $h$ is clear. That $h$ is exact and has finite support follows from Proposition 5.2. □

**Theorem 5.4.** An exact linked net of vector spaces of dimension 1 and finite support over a $\mathbb{Z}^n$-quiver admits a simple basis.

*Proof.* Let $g$ be the linked net of the statement. Applying Lemma 5.3 we obtain a subnet $h$ of $g$ which has a simple basis. Since $g$ itself has dimension 1, it is equal to $h$, whence admits a simple basis. □

6. The Intersection Property

Let $Q$ be a $\mathbb{Z}^n$-quiver. Given a vertex $v$ of $Q$ and a proper subset $I$ of $\{0, \ldots, n\}$, all simple admissible paths $\nu$ leaving $v$ and satisfying $\nu(i) > 0$ if and only if $i \in I$ end up at the same vertex $w$. If $g$ is a linked net over $Q$ we let $\varphi^v_\nu := \varphi^v_\nu$ for any such $\nu$. In addition, we put $\varphi^v_{\{0, \ldots, n\}} = 0$. If $v$ is clear from the context, put $\varphi^v_1 := \varphi^v_1$.

**Lemma 6.1.** Let $g$ be a linked net of vector spaces over a $\mathbb{Z}^n$-quiver, and $I_1, I_2$ subsets of $\{0, \ldots, n\}$. Then, for each vertex $v$ of the quiver,

$$\text{Ker}(\varphi^v_{I_1}) \cap \text{Ker}(\varphi^v_{I_2}) = \text{Ker}(\varphi^v_{I_1 \cap I_2}).$$

*Proof.* We can write $\varphi^v_{I_1} = \varphi^v_{I_1 - I_2}\varphi^v_{I_2}$ and $\varphi^v_{I_2} = \varphi^v_{I_2 - I_1}\varphi^v_{I_1}$ for $w$ the end vertex of a simple admissible path $\nu$ leaving $v$ and satisfying $\nu(i) > 0$ if and only if $i \in I_1 \cap I_2$. The inclusion $\text{Ker}(\varphi^v_{I_1}) \cap \text{Ker}(\varphi^v_{I_2}) \subseteq \text{Ker}(\varphi^v_{I_1 \cap I_2})$ follows.

On the other hand, as $(I_1 - I_2) \cap (I_2 - I_1)$ is empty it follows that

$$\text{Ker}(\varphi^v_{I_1 - I_2}) \cap \text{Ker}(\varphi^v_{I_2 - I_1}) = 0.$$
Thus, if $s$ is in \( \ker(\varphi^v_{I_1}) \cap \ker(\varphi^v_{I_2}) \), it is in \( \ker(\varphi^v_{I_1 \cap I_2}) \) as well.  

**Definition 6.2.** A weakly linked net of vector spaces over a \( \mathbb{Z}^n \)-quiver satisfies the intersection property at a vertex \( v \) if for each collection \( I_0, \ldots, I_m \) of subsets of \( \{0, \ldots, n\} \) the following equality holds:

\[
\left( \sum_{\ell=1}^{m} \ker(\varphi^v_{I_\ell}) \right) \cap \ker(\varphi^v_{I_0}) = \sum_{\ell=1}^{m} \ker(\varphi^v_{I_\ell \cap I_0}).
\]

Another way of viewing the intersection property, in light of Lemma 6.1, is by saying that the intersection of kernels distributes with respect to the sum.

In his work [4, def. 2.6] on limit linear series for curves of compact type with three components G. Munõz defines, in a different context, a similar property for the special case when \( m = 2 \).

If a linked net of vector spaces over a \( \mathbb{Z}^n \)-quiver has support in a collection of vertices \( H \), then it satisfies the intersection property at every vertex if it satisfies it at each vertex of \( H \). Indeed, let \( v \) be a vertex of the quiver not in \( H \). Then there is \( w \in H \) such that \( \varphi^w_v \) is an isomorphism. Let \( \gamma \) be an admissible path connecting \( w \) to \( v \) and put \( J := \{ i \mid \gamma(i) > 0 \} \). Let \( I_0, \ldots, I_m \) be a collection of proper subsets of \( \{0, \ldots, n\} \). For each \( i = 1, \ldots, m \), let \( s_i \in \ker(\varphi^v_{I_i}) \). Suppose \( \varphi^w_{I_0}(s_1 + \cdots + s_m) = 0 \). Then \( s_i = \varphi^w_{I_i}(t_i) \) for a certain \( t_i \) for each \( i = 1, \ldots, m \). Set \( J_i := J \cup I_i \) for each \( i = 0, \ldots, m \). It follows from Lemma 6.3 that \( \varphi^w_{J_i}(t_i) = 0 \) for each \( i = 1, \ldots, m \) and \( \varphi^w_{I_0}(t_1 + \cdots + t_m) = 0 \). If the linked net satisfies the intersection property at \( w \), we may write

\[
t_1 + \cdots + t_m = t'_1 + \cdots + t'_m,
\]

where \( t'_i \in \ker(\varphi^w_{J_i \cap I_0}) \) for \( i = 1, \ldots, m \). Set \( s'_i := \varphi^w_v(t'_i) \) for \( i = 1, \ldots, m \). Then \( s'_i \in \ker(\varphi^v_{I_\ell \cap I_0}) \) for each \( i \), and clearly

\[
s_1 + \cdots + s_m = s'_1 + \cdots + s'_m.
\]

**Proposition 6.3.** Let \( g \) be an exact weakly linked net of vector spaces over a \( \mathbb{Z}^n \)-quiver \( Q \) satisfying the intersection property at a vertex \( v \). Let \( h \) be an exact subnet of \( g \) such that each nonzero section of \( h \) at \( v \) is the image of a primitive section of \( h \) at a vertex of \( Q \). Then the quotient \( g/h \) is a weakly linked net satisfying the intersection property at \( v \).

**Proof.** Let \( I_0, \ldots, I_m \) be a collection of subsets of \( \{0, \ldots, n\} \). For each \( i = 1, \ldots, m \) let \( s_i \) be a section of \( g \) at \( v \) such that \( t_i := \varphi^v_{I_i}(s_i) \) is a section of \( h \). Furthermore, assume that \( t := \varphi^v_{I_0}(s_1 + \cdots + s_m) \) is a section of \( h \). We need to show that

\[
s_1 + \cdots + s_m = s'_1 + \cdots + s'_m
\]

for sections \( s'_i \) of \( g \) at \( v \) such that \( \varphi^v_{I_\ell \cap I_0}(s'_i) \) is a section of \( h \) for each \( i = 1, \ldots, m \).

Since \( h \) is exact, for each \( i = 1, \ldots, m \) there is a section \( y_i \) of \( h \) at \( v \) such that \( t_i := \varphi^v_{I_0}(y_i) \). We may thus suppose that \( \varphi^v_{I_i}(s_i) = 0 \) for each \( i \). Also, there is a section \( y \) of \( h \) at \( v \) such that \( \varphi^v_{I_0}(y) = t \). Then \( \varphi^v_{I_0}(s_1 + \cdots + s_m - y) = 0 \).

If \( y = 0 \) we may use that \( g \) satisfies the intersection property at \( v \) to conclude.
Assume now that \( y \neq 0 \). Since \( g \) is exact, we have

\[
y \in \sum_{i=0}^{m} \text{Im}(\varphi_{J_i}),
\]

where \( J_i := \{0, \ldots, n\} - I_i \) for each \( i = 0, \ldots, m \). Write \( y = \varphi_{\gamma}(x) \) for a primitive \( x \). Then \( \varphi_{\gamma} = \varphi_{K_p} \cdots \varphi_{K_1} \) for proper subsets \( K_1, \ldots, K_p \) of \( \{0, \ldots, n\} \) satisfying \( K_1 \subseteq \cdots \subseteq K_p \). Let \( K'_i := \{0, \ldots, n\} - K_i \) for each \( i = 1, \ldots, p \). Applying Lemma 6.4 repeatedly, we obtain that

\[
x \in \sum_{i=1}^{p} \text{Im}(\varphi_{K'_i}) + \sum_{i=0}^{m} \text{Im}(\varphi_{J_i \cap K'_i}).
\]

Since \( x \) is primitive, \( J_j \cap K'_p = \emptyset \) for some \( j \), or equivalently \( I_j \cup K_p = \{0, \ldots, n\} \). It follows that \( \varphi_{I_j}(y) = 0 \). We may thus replace \( s_j \) by \( s_j - y \) and thus assume that \( y = 0 \), the case we have already analyzed. \( \square \)

**Lemma 6.4.** Let \( g \) be an exact weakly linked net of vector spaces over a \( \mathbb{Z}^n \)-quiver satisfying the intersection property at a vertex \( v \). Let \( I_0, I_1, \ldots, I_m \) be subsets of \( \{0, \ldots, n\} \). Let \( I'_0 := \{0, 1, \ldots, n\} - I_0 \). For each \( s \in V_v \):

If \( \varphi_{I'_0}(s) \in \sum_{i=1}^{m} \text{Im}(\varphi_{I_i}) \) then \( s \in \sum_{i=0}^{m} \text{Im}(\varphi_{J_i \cap I_0}). \)

**Proof.** Let \( I'_i \) be the complement of \( I_i \) in \( \{0, \ldots, n\} \) for \( i = 1, \ldots, m \). Since \( g \) is exact,

\[
\varphi_{I'_0}(s) \in \left( \sum_{i=1}^{m} \text{Ker}(\varphi_{I'_i}) \right) \cap \text{Ker}(\varphi_{I_0})
\]

and thus, by the intersection property,

\[
\varphi_{I'_0}(s) \in \sum_{i=1}^{m} \text{Ker}(\varphi_{I_i \cap I_0}).
\]

Using that \( g \) is exact again, we obtain

\[
\varphi_{I'_0}(s) \in \sum_{i=1}^{m} \text{Im}(\varphi_{I_i \cup I'_0}).
\]

In other words, there are \( s_1, \ldots, s_m \) such that

\[
\varphi_{I'_0}\left(s - \sum_{i=1}^{m} \varphi_{I_i \cap I_0}(s_i)\right) = 0,
\]

whence the statement of the lemma, again by the exactness of \( g \). \( \square \)
7. A LOCAL CRITERION FOR THE EXISTENCE OF SIMPLE BASES

In this section we prove the main result of this paper, Theorem 7.3 where we characterize the existence of a simple basis for a linked net of vector spaces over a \( \mathbb{Z}^n \)-quiver with a local condition.

**Proposition 7.1.** If a linked net of vector spaces over a \( \mathbb{Z}^n \)-quiver admits a simple basis, then it satisfies the intersection property at every vertex.

**Proof.** Let \( v \) be a vertex of the quiver. Let \( I_0, \ldots, I_m \) be a collection of subsets of \( \{0, \ldots, n\} \). We need to show (4). We may assume the \( I_\ell \) are proper.

Lemma 6.1 implies that \( \ker(\varphi_0^\mu) \cap \ker(\varphi_i^\mu) = \ker(\varphi_{I_0}^\mu) \cap I_0 \) for each \( \ell \), and hence

\[
\left( \sum_{\ell=1}^m \ker(\varphi_0^\mu) \cap \ker(\varphi_i^\mu) \right) \cap \ker(\varphi_{I_0}^\mu) \cap I_0 \geq \sum_{\ell=1}^m \ker(\varphi_{I_0}^\mu) \cap I_0.
\]

For the opposite inclusion, assume the net has a simple basis, say \( \{s_0, \ldots, s_r\} \) with \( s_j \in V_{v_j} \) for each \( j \). Let \( \mu_j \) be an admissible path from \( v_j \) to \( v \) for each \( j \). For each \( \ell = 0, \ldots, m \), let \( \gamma_\ell \) be a simple admissible path leaving \( v \) such that \( \gamma_\ell(i) > 0 \) if and only if \( i \in I_\ell \). Finally, for each \( \ell = 0, \ldots, m \) and \( j = 0, \ldots, r \), let \( \gamma_\ell \mu_j \) denote the concatenation of \( \mu_j \) with \( \gamma_\ell \). Then

\[
\ker(\varphi_{I_\ell}^\mu) = \langle s_j |_{V_v} \mid \text{the path } \gamma_\ell \mu_j \text{ is not admissible} \rangle.
\]

It follows that the left-hand side of (4) is

\[
\langle s_j |_{V_v} \mid \text{the path } \gamma_0 \mu_j \text{ is not admissible and the } \gamma_\ell \mu_j \text{ is not admissible for some } \ell \rangle.
\]

The last condition on \( j \) is equivalent to

\[
(5) \quad \begin{cases} 
\mu_j(i) + \gamma_0(i) > 0 \text{ for every } i, \\
\text{There is } \ell \in \{1, \ldots, m\} \text{ such that } \mu_j(i) + \gamma_\ell(i) > 0 \text{ for every } i.
\end{cases}
\]

We claim (5) implies

\[
(6) \quad \text{There is } \ell \in \{1, \ldots, m\} \text{ such that } \mu_j(i) + \gamma_{\ell,0}(i) > 0 \text{ for every } i,
\]

where \( \gamma_{\ell,0} \) is a simple admissible path leaving \( v \) such that \( \gamma_{\ell,0}(i) > 0 \) if and only if \( i \in I_\ell \cap I_0 \) for each \( \ell = 1, \ldots, m \). Indeed, suppose (5) holds but (6) does not. Then there are \( i_1, \ldots, i_m \in \{0, \ldots, n\} \) such that

\[
\mu_j(i_\ell) + \gamma_{\ell,0}(i_\ell) = 0 \text{ for each } \ell = 1, \ldots, m.
\]

Equivalently, \( \mu_j(i_\ell) = 0 \) and \( \gamma_{\ell,0}(i_\ell) = 0 \) for \( \ell = 1, \ldots, m \). It follows from (5) that \( \gamma_0(i_\ell) > 0 \) for \( \ell = 1, \ldots, m \). It follows as well that there is \( \ell \) such that \( \gamma_\ell(i_\ell) > 0 \).

For this \( \ell \) we have \( i_\ell \in I_0 \cap I_\ell \), and hence \( \gamma_{\ell,0}(i_\ell) > 0 \), a contradiction.

Thus (5) implies (6) for each \( j = 0, \ldots, r \). Since the right-hand side of (4) is

\[
\langle s_j |_{V_v} \mid \text{the concatenation of } \mu_j \text{ with } \gamma_{\ell,0} \text{ is not admissible for some } \ell \rangle,
\]

we finish the proof.
Lemma 4.3 we need only prove that

\[ a \bowtie b \]

\[ \text{Proof.} \] By Proposition 4.4, the quotient \( g/h \) is an exact weakly linked net. We need only prove that \( g/h \) is a linked net.

Let \( u \) be a vertex of \( Q \) and \( i \in \{0, \ldots, n\} \). Let \( a \) be an \( i \)-arrow leaving \( u \) and \( b \) an \( i \)-arrow arriving at \( u \). Let \( z \) be the initial vertex of \( b \) and \( w \) the final vertex of \( a \). Let \( x \) be a section of \( g \) at \( z \) such that \( \varphi_a \varphi_b(x) \) is a section of \( h \) at \( w \). By Lemma 4.3, we need only prove that \( \varphi_a(x) \) is a section of \( h \) at \( u \).

If \( \varphi_a \varphi_b(x) = 0 \) then \( \varphi_b(x) = 0 \), as \( g \) is a linked net, by Lemma 4.3. We may thus assume \( \varphi_a \varphi_b(x) \neq 0 \).

By Theorem 5.4, there is a section \( s \) of \( h \) at a certain vertex \( v \) that generates \( h \). Then \( \varphi_a \varphi_b(x) = c \varphi_w^v(s) \) for some nonzero scalar \( c \). We may assume \( c = 1 \). Let \( \nu \) be a simple admissible path leaving \( w \) whose concatenation with \( a \) is not admissible. Then \( \varphi_w \varphi_w^v(s) = 0 \). It follows that there is an admissible path from \( v \) to \( w \) passing through \( u \), and hence

\[ \varphi_a(x - \varphi_w^v(s)) = 0. \]

If there is an admissible path from \( v \) to \( u \) passing through \( z \) we would have that

\[ \varphi_a \varphi_b(x - \varphi_w^v(s)) = 0, \]

and hence \( \varphi_b(x - \varphi_w^v(s)) = 0 \) by Lemma 4.3, so \( \varphi_b(x) = \varphi_w^v(s) \) as wished.

Otherwise, \( \gamma(i) = 0 \), where \( \gamma \) is an admissible path connecting \( v \) to \( u \). Let \( J := \{j \mid \gamma(j) = 0\} \) and \( K := \{0, \ldots, n\} \setminus \{i\} \). Then Equation (7) implies that

\[ \varphi_w^v(s) \in \left( \text{Ker}(\varphi_K^s) + \text{Ker}(\varphi_{\{i\}}^s) \right) \cap \text{Ker}(\varphi_j^s). \]

Since \( g \) satisfies the intersection property, it follows that

\[ \varphi_w^v(s) \in \text{Ker}(\varphi_{K \cap J}^s) + \text{Ker}(\varphi_{\{i\}}^s). \]

Now, \( \varphi_w^v = \varphi_{K_1} \cdots \varphi_{K_p} \) for certain proper subsets \( K_1, \ldots, K_p \) of \( \{0, \ldots, n\} \) satisfying \( K_1 \subseteq \cdots \subseteq K_p \). With \( K_p = \{ j \mid \gamma(j) > 0 \} \). In particular, \( i \not\in K_p \). Applying Lemma 6.4 repeatedly, we get

\[ s \in \text{Im}(\varphi_{J \setminus \{i\}}^s) + \text{Im}(\varphi_{\{i\}}^s), \]

which implies that \( J = \{i\} \). But in this case \( \varphi_a \varphi_w^v(s) = 0 \), and hence \( \varphi_a \varphi_b(x) = 0 \) from Equation (7), contradicting our assumption. \( \square \)

Theorem 7.3. An exact linked net of vector spaces over a \( \mathbb{Z}^n \)-quiver with finite support admits a simple basis if and only if it satisfies the intersection property at every vertex.

\[ \text{Proof.} \] Let \( g \) be the linked net of the statement. If \( g \) has a simple basis, then by Proposition 7.1, the representation \( g \) satisfies the intersection property at every vertex.
Conversely, assume $g$ satisfies the intersection property at every vertex. Proceed by induction on $n = \dim g$. If $n = 1$, since $g$ is exact, it has a simple basis by Theorem 5.4. Assume now that $n > 1$ and every exact linked net of vector spaces of dimension $n - 1$ and finite support satisfying the intersection property admits a simple basis. Using Lemma 5.3 we obtain an one-dimensional subnet $h$ of $g$ having a simple basis. The quotient $g/h$ is an exact weakly linked net with finite support satisfying the intersection property at every vertex by Proposition 6.3 and is a linked net by Proposition 7.2. Therefore, by the induction hypothesis, $g/h$ admits a simple basis.

Lift the simple basis from $g/h$ to $g$ and add the simple basis of $h$: we obtain a simple basis for $g$. □

Example 7.4. Here we present an exact linked net $g$ of dimension 2 and finite support over a $\mathbb{Z}^2$-quiver $Q$ which does not satisfy the intersection property at a vertex, and thus does not admit a simple basis by Theorem 7.3. The quiver is $Q = Q(v; v_0, v_1, v_2)$ with

$$v = (2, 2, 2), \quad v_1 = (-2, 1, 1), \quad v_2 = (1, -2, 1), \quad v_3 = (1, 1, -2).$$

In the picture below we describe the representation only over the collection of effective multidegrees, which supports it.

![Diagram of the exact linked net](image)

**Figure 3.** Exact linked net.

At the vertex $(2, 2, 2)$ we can verify that

$$\text{Ker}(\varphi_{(1,4,1)}^{(2,2,2)}) = \langle e_1 + e_2 \rangle, \quad \text{Ker}(\varphi_{(4,1,1)}^{(2,2,2)}) = \langle e_1 \rangle \quad \text{and} \quad \text{Ker}(\varphi_{(1,1,4)}^{(2,2,2)}) = \langle e_2 \rangle.$$
And $\text{Ker}(\varphi_{(3,3,0)}^{(2,2,2)}) = 0$ and $\text{Ker}(\varphi_{(3,0,3)}^{(2,2,2)}) = 0$. Thus

$$\left(\text{Ker}(\varphi_{(1,4,1)}^{(2,2,2)}) + \text{Ker}(\varphi_{(1,1,4)}^{(2,2,2)})\right) \cap \text{Ker}(\varphi_{(4,1,1)}^{(2,2,2)}) \neq \text{Ker}(\varphi_{(3,3,0)}^{(2,2,2)}) \oplus \text{Ker}(\varphi_{(3,0,3)}^{(2,2,2)})$$.

Therefore, $g$ does not satisfy the intersection property at $(2,2,2)$.

Furthermore, the linked net $g$ can be realized as a degeneration of a linear series, as we now explain. Let $X$ be the triangle in $\mathbb{P}^2$, given by $UVW = 0$. It has three components, $X_0$, $X_1$ and $X_2$ given by $U = 0$, $V = 0$ and $W = 0$, respectively. Let $\mathfrak{X} \subset \mathbb{P}^2 \times B$ be the surface defined by $UVW - TF = 0$ for a general cubic $F = 0$. Here $B := \text{Spec}(\mathbb{C}[[T]])$. Since $F$ is general, $\mathfrak{X}$ is regular. Also, the general fiber $\mathfrak{X}_g$ of the projection $\pi : \mathfrak{X} \to B$ is smooth. The special fiber is $X$.

Consider the invertible sheaf $L := \mathcal{O}_\mathfrak{X}(2)$, whose restriction to $X$ has multidegree $(2,2,2)$. The quiver associated to it is the quiver $Q$.

The coordinates $U, V, W$ of $\mathbb{P}^2$ can be thought of as sections of $\mathcal{O}_{\mathbb{P}^2}(1)$ and restrict to sections of $\mathcal{O}_\mathfrak{X}(1)$ which we denote by $u, v, w$, respectively. Consider the linear system $V_\eta$ of sections of $L_\eta := L|_{\mathfrak{X}_g}$ generated by $u(v + w)$ and $w(v - u)$.

For each divisor $D = \sum n_iX_i$ with $\min\{n_i\} = 0$, the sheaf $L(D)$ may be viewed as a subsheaf of $\mathcal{O}_\mathfrak{X}(2 + \sum n_i)$, with $B$-flat quotient. Thus, we may and will think of the limit sections as sections of the larger sheaf $\mathcal{O}_\mathfrak{X}(2 + \sum n_i)$, which may be represented by polynomials of degree $2 + \sum n_i$. In the next picture we describe bases for the spaces of limit sections over the collection of effective multidegrees.

\[
\begin{align*}
\{u^3w^2(v + w), Fuw(x + z)\} & \quad \{u^2(v + w), uw(v - u)\} & \quad \{w^3v^2(v + w), Fuv(v - u)\} \\
\{u^2w^3(v + w), F(u + w)\} & \quad \{u^3v^2(v + w), F(v - u)\} \\
\{uw(v + w), w^2(v - u)\} & \quad \{uv(v + w), vw(v - u)\} \\
\{F(v + w), vw^2(v - u)\} & \\
\{Fvw(v + w), v^2w^3(v - u)\} & \quad \{uv(v + w), vw(v - u)\} \\
\end{align*}
\]

\textbf{Figure 4.} Degeneration of the linear series $(u(v + w), w(v - u))$.

As for the maps between these spaces, the maps labeled $X_0, X_1, X_2$ are multiplicity by $u, v, w$, respectively, as long as they map to a space with sections
represented by polynomials of higher degree. Otherwise, after multiplication, divide by $F$. Keep in mind that $uvw = 0$ on the curve $X$. Using the bases, the maps are represented by the matrices in the first figure of the example.

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