Representations of $\text{Aut}(A(\Gamma))$ Acting on Homogeneous Components of $A(\Gamma)$ and $A(\Gamma)^1$

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Abstract. In this paper we will study the structure of algebras $A(\Gamma)$ associated to two directed, layered graphs $\Gamma$. These are algebras associated with Hasse graphs of $n$-gons and the algebras $Q_n$ (related to pseudoroots of noncommutative polynomials). Namely, we will find the filtration preserving automorphism group of these algebras and then we will find the multiplicities of the irreducible representations of $\text{Aut}(A(\Gamma))$ acting on the homogeneous components of $A(\Gamma)$ and $A(\Gamma)^1$.

0. Preliminaries

$A(\Gamma)$

Certain algebras, denoted $A(\Gamma)$, associated to directed graphs were first defined by Gelfand, Retakh, Serconek, and Wilson [GRSW]. We recall the definitions of $A(\Gamma)$ and $\text{gr}A(\Gamma)$ following the development found in [[RSW], §2]. Let $k$ be a field and for any set $W$ let $T(W)$ be the free associative algebra on $W$ over $k$. Let $\Gamma = (V, E)$ be a directed graph where $V$ is the set of vertices, $E$ the set of edges, and there are functions $t, h : E \to V$ (tail and head of $e$). We say $\Gamma$ is a layered graph if $V = \bigcup_{i=0}^n V_i$, $\bigcup_{i=0}^n E_i$, $t : E_i \to V_i$, and $h : E_i \to V_{i-1}$. If $v \in V_i$ ($e \in E_i$), we say the level of $v$, (respectively $e$) is $i$; denote this by $|v|$, (resp. $|e|$). We will assume throughout this paper that $V_0 = \{\ast\}$ and that for all $v \in V_+ = \bigcup_{i=1}^n V_i$, there exists at least one $e \in E$ such that $t(e) = v$. For each $v \in V_+$, fix some $e_v \in E$ with $t(e_v) = v$; call this a distinguished edge.

A path from $v \in V$ to $w \in V$ is a sequence of edges $\pi = \{e_1, \ldots, e_m\}$ such that $t(e_1) = v$, $h(e_m) = w$, and $t(e_{i+1}) = h(e_i)$, $1 \leq i < m$. We will say $\ell(\pi) = v$, $h(\pi) = w$, and the length of $\pi$, $l(\pi)$, is $m$. Write $v > w$ if there exists a path from $v$ to $w$. For $\pi = \{e_1, \ldots, e_m\}$, define $e(\pi, k) := \sum_{1 \leq i < \cdots < i_k \leq m} e_{i_1} \cdots e_{i_k}$. For each $v \in V$ there is a path $\pi_v = \{e_1, \ldots, e_m\}$, called the distinguished path, from $v$ to $\ast$ defined by $e_1 = e_v$, $e_{i+1} = e h(e_i)$ for $1 \leq i < m$, and $h(e_m) = \ast$. When $\pi_v$ is the distinguished path from $v$ to $\ast$, we will write $e(v, k)$ in lieu of $e(\pi_v, k)$. Let $R$ be the two-sided ideal of $T(E)$ generated by $\{e(\pi_1, k) - e(\pi_2, k) : t(\pi_1) = t(\pi_2), h(\pi_1) = h(\pi_2), 1 \leq k \leq l(\pi_1)\}$.

Definition. $A(\Gamma) = T(E)/R$

Let $\hat{e}(v, k)$ denote the image in $A(\Gamma)$ of $e_1 \cdots e_k$. Finally say that $(v, k)$ covers $(w, l)$ if $v > w$ and $k = |v| - |w|$, write this as $(v, k) > (w, l)$.

Theorem 0.1. [[RSW], Thm 1] - Let $\Gamma = (V, E)$ be a layered graph, $V = \bigcup_{i=0}^n V_i$, $V_0 = \{\ast\}$. Then

$\mathcal{B}(\Gamma) := \{\hat{e}(v_1, k_1) \cdots \hat{e}(v_l, k_l) : l \geq 0, v_1, \ldots, v_l \in V_+, 1 \leq k_i \leq |v_i|, (v_i, k_i) \not> (v_{i+1}, k_{i+1})\}$ is a basis for $A(\Gamma)$. 


There is also a presentation of \(A(\Gamma)\) as a quotient of \(T(V_+)\) \([\text{RSW2}, \S 3]\). Every edge may be expressed as a linear combination of distinguished edges, and the distinguished edge \(e_v\) may be identified with \(v \in V_+\). Define \(S_1(v) := \{w \in V_{|v|-1} : v > w\}\). A layered graph is uniform if for every \(v \in V_j, j \geq 2\), every pair of vertices \(u, w\) in \(S_1(v)\) satisfies \(S_1(u) \cap S_1(w) \neq \emptyset\) ("diamond condition").

**Proposition 0.2.** \([\text{RSW2}, \text{Prop. 3.5}]\) Let \(\Gamma\) be a uniform layered graph. Then \(A(\Gamma) \cong T(V_+)/R_V\) where \(R_V\) is the two-sided ideal generated by \(\{v(u-w)-u^2+w^2+(u-w)x : v \in \bigcup_{i=2}^{n} V_i, u, w \in S_1(v), x \in S_1(u) \cap S_1(w)\}\).

Remark: From now on we will just write \(e(v, k)\) for \(\hat{e}(v, k)\).

\[\text{gr}A(\Gamma)\]

Next we will describe a filtration and grading on \(A(\Gamma)\). Here we will also denote by \(V\) the span of \(V\) in \(T(V)\), and by \(E\) the span of \(E\) in \(T(E)\), when no confusion will arise. \([\text{RSW2}, \S 2]\) Let \(W = \sum_{k \geq 0} W_k\) be a graded vector space (in our case \(W = V\) or \(E\)). Then \(T(W)\) is bigraded. One grading \(T(W) = \sum_{i \geq 0} T(W)_i\) is given by degree in the tensor algebra; i.e., \(T(W)_{[i]} = \text{span}\{w_1 \cdots w_i : w_1, \ldots, w_i \in W\}\). The other grading is given by \(T(W) = \sum_{i \geq 0} T(W)_i\) where \(T(W)_i = \text{span}\{w_1 \cdots w_r : r \geq 0, w_j \in W_{l_j}, l_1 + \ldots + l_r = i\}\).

The second grading induces an increasing filtration on \(T(W)\):

\[T(W)_{(i)} = \text{span}\{w_1 \cdots w_r : r \geq 0, w_j \in W_{l_j}, l_1 + \ldots + l_r \leq i\} = T(W)_0 + \cdots + T(W)_i\].

Because \(T(W)_{(i)}/T(W)_{(i-1)} \cong T(W)_{[i]}\), \(T(W)\) can be identified with its associated graded algebra. Define a map \(\text{gr} : T(W) \setminus \{0\} \to T(W) \setminus \{0\} = \text{gr}(T(W))\) by \(w = \sum_{i=0}^{k} w_i \mapsto w_k\) where \(w_i \in T(W)_{[i]}\), \(w_k \neq 0\). Of course, \(\text{gr}\) is not an additive map.

**Lemma 0.3.** \([\text{RSW2}, \text{Lemma 2.1}]\) Let \(W\) be a graded vector space and \(I\) an ideal in \(T(W)\). Then \(\text{gr}(T(W)/I) \cong T(W)/\text{gr}(I)\).

Thus the associated graded algebra of \(A(\Gamma)\), \(\text{gr}A(\Gamma)\), is isomorphic to \(T(E)/\text{gr}R\). The graded relations, \(\text{gr}R\), are for \(\pi_1 = \{e_1, \ldots, e_m\}\) and \(\pi_2 = \{f_1, \ldots, f_m\}\), \(\{e_1 \cdots e_k = f_1 \cdots f_k, 1 \leq k \leq m\}\) (the leading term of \(e(v, k)\)). Another way to consider this is that \(e(v, k+l) = e(v, k)e(u, l)\) in \(\text{gr}R\) where \(v > u, k = |v| - |u|\). Recalling the definition of \(B(\Gamma)\) from Theorem 0.1, we see that \(\{\text{gr}b : b \in B(\Gamma)\}\) is a basis for \(\text{gr}A(\Gamma)\).

Let us now look at our second description of \(A(\Gamma)\) as isomorphic to \(T(V_+)/R_V\).

**Proposition 0.4.** \([\text{RSW2}, \text{Prop. 3.6}]\) Let \(\Gamma\) be a uniform layered graph. Then \(\text{gr}A(\Gamma) \cong T(V_+)/\text{gr}R_V\) where \(\text{gr}R_V\) is generated by \(\{v(u-w) : v \in \bigcup_{i=2}^{n} V_i, u, w \in S_1(v)\}\).

Also, \(A(\Gamma)_{(i)} = (T(E)_{(i)} + R)/R = (T(V_+)_{(i)} + R_V)/R_V\) and \(A(\Gamma)_{(i)} = (T(E)_{(i)} + R)/R = (T(V_+)_{(i)} + R_V)/R_V\).

\[A(\Gamma^\sigma)\]

We will now define a subalgebra of \(\text{gr}A(\Gamma)\). Let \(\sigma\) be an automorphism of the layered graph \(\Gamma\); i.e., an automorphism that preserves each level of the graph. Define \(\Gamma^\sigma := (V_\sigma, E_\sigma)\) where \(V_\sigma\) is the set of vertices \(v \in V\) such that \(\sigma(v) = v\) and \(E_\sigma\) is the set of edges that connect the vertices minimally. Here minimally means that there is an edge \(e \in E_\sigma\) from \(v\) to \(w\), \(v, w \in V_\sigma\) if and only if \(v \geq u \geq w\), \(u \in V_\sigma\), implies \(u = v\) or \(u = w\).

**Definition** \((A(\Gamma^\sigma))\). \([\text{RW, D}]\) \(A(\Gamma^\sigma)\) has basis \(\{e(v_1, k_1) \cdots e(v_l, k_l) : l \geq 0, v_1, \ldots, v_l \in V_\sigma, 1 \leq k_i \leq |v_i|, e(v_i, k_i) \not\succ e(v_{i+1}, k_{i+1})\}\).
The generators of $A(\Gamma^\sigma)$ are \( \{e(v, k) : v \in V_\sigma, 1 \leq k \leq |v|\} \) and the relations are \( \{e(v, k + l) - e(v, k)e(u, l) : v > u \in V_\sigma, k = |v| - |u|\}\) \([D]\).

Because $\sigma$ preserves the basis of $A(\Gamma)$, the span of its fixed basis elements is a subalgebra.

**Theorem 0.5.** \([D]\) $A(\Gamma^\sigma)$ is a subalgebra of gr$A(\Gamma)$.

**Proof.** Recall that we are working in gr$A(\Gamma)$. Clearly, $A(\Gamma^\sigma)$ is a subset of $A(\Gamma)$. To show $A(\Gamma^\sigma)$ is a subalgebra we need to show it is closed and the relations given above can be derived from the relations in gr$A(\Gamma)$. Let $e(v, k), e(u, l) \in A(\Gamma^\sigma)$; so, $v, u \in V_\sigma$. Then the image of $e(v, k)e(u, l)$ is also in $A(\Gamma^\sigma)$ since $\sigma(e(v, k)e(u, l)) = e(\sigma(v), k)e(\sigma(u), l) = e(v, k)e(u, l)$.

In $A(\Gamma)$ we have

$$e(v, k)e(u, l) - e(v, k + l) = \sum_{i_0, i_r + 1 \geq 0, i_1, \ldots, i_r \geq 1} (-1)^{r+1} e(v, i_0)e(u, i_1) \cdots e(u, i_{r+1})$$

mod $R$ \([GRSW],p6\). However, the elements on the right-hand side are all in a lower grading than those on the left-hand side \([GRSW],Lemma2.2\). Note that the elements on the left-hand side have degree $k + l$

and are in $(T(E)/R)_{(k+l)}|_{e|-(k+l)(k+l+1)/2}$. Therefore, in gr$A(\Gamma)$, the terms on the right-hand side are zero. Hence, we have $e(v, k)e(u, l) - e(v, k + l) = 0$ as desired. \(\Box\)

$A(\Gamma)^{\sigma}$

There are two natural duals relating to $A(\Gamma)$ and its subalgebras to consider. One is to take the dual of $A(\Gamma)$ and then look at fixed points. The other is to take the dual of the subalgebra gr$A(\Gamma^\sigma)$. These two constructions give us different structures. In this paper we will only be concerned with the former. However, more about both can be found in \([D]\).

**Definition** ($A^1$). \([BVW],\S\,2\) Let $A = T(E)/(R)$, $R \subseteq E^\otimes 2$. Then $A^1 = T(E^*/(R^\perp)$ where $E^*$ is the dual vector space of $E$ and $R^\perp$ is the annihilator of $R$; i.e. $R^\perp = \{f \in (E^\otimes 2)^* : f(x) = 0 \forall x \in R\}$ of $(E^\otimes 2)^*$ where $(E^\otimes 2)^*$ is canonically identified with $E^{\ast\otimes 2}$.

In our case, the elements of $A^1$ will be denoted $e(v, k)^\ast$.

**Definition** ($A(\Gamma)^{\sigma}$). \([D]\) $A(\Gamma)^{\sigma} = ((grA(\Gamma)^{\ast})^\sigma$. First take the dual of gr$A(\Gamma)$. Since gr$A(\Gamma) = T(E)/grR$, (gr$A(\Gamma)^{\ast})^1 = T(E^*/(grR)^\perp$. Next we take as generators the elements in the dual fixed by $\sigma$ of degree one.

Thus, $A(\Gamma)^{\sigma} = \langle e(v_1, 1)^\ast \cdots e(v_l, 1)^\ast : l \geq 1, v_1, \ldots, v_l \in V_\sigma, v_i > v_{i+1}, |v_i| - |v_{i+1}| = 1\rangle$.

For $x$ a basis element of $A(\Gamma)$, write $\sigma(x)$ as a linear combination of basis elements and say the coefficient of $x$ in $\sigma(x)$ is $\alpha$. Denote this value $\alpha$ by $t_\sigma(x)$. Then, for finite-dimensional $A(\Gamma)$, $Tr_\sigma(A(\Gamma)) = \sum_{x \in \text{basis}} t_\sigma(x)$.

In this paper we will be looking at the trace of $\sigma$ acting on $A(\Gamma)^{\sigma}$ and $A(\Gamma)^{\sigma}_{\text{id}} = A(\Gamma)^{\sigma}_{\text{id}}$.

1. **Hasse graph of an $n$-gon:** $\Gamma_{D_n}$

A Hasse graph, or Hasse diagram, is a graph which represents a finite poset $P$. The vertices in the graph are elements of $P$ and there is an edge between $x, y \in P$ if $x < y$ and there does not exist a $z \in P$ such that $x < z < y$. Furthermore, the vertex for $x, v_x$, is lower in the picture than that for $y, v_y$ (if we talk about layers in the graph, $|v_x| = |v_y| - 1$).
Consider a polytope. We can put a partial order on the set of k-faces in the polytope by \( x < y \) if \( x \) is an \((n-1)\)-face, \( y \) is an \( n \)-face and \( x \) is a face of \( y \). For example, if there is an edge \( e \) between \( v \) and \( w \) in the polytope, then \( e > v, w \) and \( e \not> u \) for all \( u \neq v, w \).

Thus, the Hasse graph of an n-gon has one vertex in levels 0 and 3 and \( n \) vertices on levels 1 and 2. The top vertex is connected to all vertices in level 2 (all edges are in the 2-dimensional polygon), each vertex in level 2 is connected to the vertex directly below it and the one to that vertex’s right, with wrapping around to the first vertex in level one for the last vertex in level two (each edge connects two adjacent vertices), and each vertex in level 1 is connected to the minimal vertex. Label the vertices by using subscripts in \( \mathbb{Z}/n \mathbb{Z} \).

In level 1 call the vertices \( w_1, \ldots, w_n \), the vertices in level 2 are \( v_{12}, \ldots, v_{n1} \) (where the subscripts indicate to which vertices in level 1 the vertex is connected), and the top vertex is \( u \). See figure 1.

![Figure 1. \( \Gamma_{D_n} \)](image)

We consider the algebra generated by this graph. The construction of this algebra is described in [GRSW] (see § 0). In brief (using the definition given in Proposition 0.2), the generators are the vertices and the relations are that two paths which have the same starting and ending vertices are equivalent. We can write these relations by:

1) \( \forall i, j \in \mathbb{Z}/n \mathbb{Z} \):

\[
\begin{bmatrix}
c^1_1 + c^2_1 & c^2_1 - c^1_2 & c_2^1 - c_2^2 \\
0 & c^1_2 & c_2^2 \\
0 & c^2_2 & c^1_2 + c^2_2 - c_2^2
\end{bmatrix}, \quad c_j^i \in k \forall i, j
\]

**Theorem 1.1.**

a) If \( n \geq 3 \), \( \text{Aut}(A(\Gamma_{D_n})) = k^* \times D_n \), \( k \) the base field

b) If \( n = 2 \),

\[
\text{Aut}(A(\Gamma_{D_2})) = \{ M \in GL(n, k) : M = \begin{bmatrix} c_1 + c_2 & c_2 - c_1 & c_2^2 - c_1^2 \\
0 & c_1 & c_2 \\
0 & c_2 & c_1 + c_2 - c_2^2 \end{bmatrix} , \forall i, j \}
\]

**Proof.** In order to determine the filtration-preserving automorphism group of the algebra, first note that any automorphism \( \sigma \) of the graph induces an automorphism \( \sigma \) of the algebra in that \( \sigma \) will preserve the vertices and their levels and the edges. Because the vertices are the generators of the algebra and the relations come from the paths, the algebra structure will be preserved as well. Thus if \( \sigma \) is an automorphism of the graph sending vertex \( v \) to vertex \( w \), then \( \sigma \) will send \( v \) to \( w \) in the algebra and will be an automorphism. Let us denote both by \( \sigma \).

As mentioned, any automorphism of the graph must preserve the set of vertices at each level and so acts on the set \( \{w_1, ..., w_n\} \) of all \( n \) vertices in level 1. We may say \( \sigma(w_i) = w_{\sigma(i)} \) (again slightly abusing the use of \( \sigma \)). Thus we can think of an automorphism of the graph as being a permutation in \( S_n \) acting on the subscripts/labels of the vertices of level 1. This will uniquely determine what happens on higher levels.
i.e. $\sigma(v_{ij}) = v_{\sigma(i)\sigma(j)}$. Labeling the vertices in level two by the vertices they are connected to in level one ensures that as long as the set of vertices in each level is preserved, the edges will be as well.

Recall that $V_2$ refers to the vertices in level two of the graph. Only permutations which send the set $V_2 = \{(i i+1) : 1 \leq i \leq n\}$ to itself are allowed. Clearly $r = (12...n)$ fixes $V_2$. We may replace $\sigma$ by $r^i\sigma$ for some $i$ and assume $\sigma(1) = 1$. Then $\sigma(12)$ is either $(12)$ or $(1n)$, which implies either $\sigma(2) = 2$ (and thus $\sigma = id$) or $\sigma(2) = n$. In the latter case $\sigma = (2n)(3n-1)(4n-2)\cdots = s$. Thus $r$ and $s$ generate the automorphism group of $\Gamma_{D_n}$; this is the dihedral group on $n$ elements, $D_n$. Note that these automorphisms may be viewed as reflections and rotations of the n-gon.

So far we know that the automorphism group of $\Gamma_{D_n}$ contains $D_n$. Also, for any scalar $\alpha$, multiplication by $\alpha$ is an automorphism because the relations are homogeneous. Thus $Aut(A(\Gamma_{D_n})) \supseteq k^* \times D_n$. Any automorphism of the algebra must preserve the relations; thus, the image of $v_{i+1}(w_i - w_{i+1}) - w_i^2 + w_{i+1}^2$ must equal zero. Let $\sigma \in Aut(A(\Gamma_{D_n}))$ and $\sigma(v_{i+1}) = a_{i}^1 v_{i2} + \ldots + a_{i}^n v_{in} + b_i^1 w_1 + \ldots + b_i^n w_n$, and $\sigma(w_i) = c_i^1 w_1 + \ldots + c_i^n w_n$ for all $i$ with coefficients in the base field $k$.

Now $\sigma(v_{i+1}(w_i - w_{i+1})) = (\sigma(w_i^2 - w_{i+1}^2))$ implies $(a_{i}^1 v_{i2} + \ldots + a_{i}^n v_{in} + b_i^1 w_1 + \ldots + b_i^n w_n)((c_i^1 - c_i^{i+1})w_1 + \ldots + (c_i^n - c_i^{i+1})w_n) = (c_i^1 w_1 + \ldots + c_i^n w_n)^2 - (c_i^{i+1} w_1 + \ldots + c_i^{i+1} w_n)^2$. There are no $\nu$'s on the right-hand side, and so we must use our relations to eliminate them from the left-hand side. Thus, every occurrence of $v_{j+1}$ must be followed by $w_j - w_{j+1}$; and hence, $c_j^i - c_j^{i+1} = -(c_j^{i+1} - c_j^{i+1})$. Therefore, if $a_{j}^i \neq 0$, 

$$ (c_i^1 - c_i^{i+1})w_1 + \ldots + (c_i^n - c_i^{i+1})w_n = \alpha(w_j - w_{j+1}) $$

for some $\alpha \in k$.

This has two consequences. First, at most one $a_{j}^i$ can be nonzero. If all $a_{j}^i$ were zero, the element $v_{i+1} \notin A(\Gamma_{D_n})(1)$ would be sent to an element in $A(\Gamma_{D_n})(1)$, which we cannot allow because then $\sigma$ would not be invertible. Thus $a_{j}^i$ must be nonzero for exactly one $j$. Let us denote this value of $j$ by $\tau(i)$. Then $\sigma(v_{i+1}) = a_{\tau(i)}^1 v_{\tau(i)\tau(i)+1} + b_{\tau(i)}^1 w_1 + \ldots + b_{\tau(i)}^n w_n$. If $\tau(i) = \tau(l)$, then $\sigma(v_{l+1} - v_{l+1}) \in A(\Gamma_{D_n})(1)$; this implies that $i = l$. Thus $\tau$ is one-to-one, and so is in $S_n$. We want to show that $\tau$ is in $D_n$ and further restrict the coefficients.

A second consequence of (1) is that $c_i^i - c_i^{i+1}$ is zero if and only if $l \neq \tau(i), \tau(i) + 1$.

Let $z = \sum_{k \neq \tau(i), \tau(i)+1} c_k^i w_k = \sum_{k \neq \tau(i), \tau(i)+1} c_k^{i+1} w_k$. Then $(a_{\tau(i)}^i v_{\tau(i)\tau(i)+1} + b_{\tau(i)}^1 w_1 + \ldots + b_{\tau(i)}^n w_n)((c_i^i - c_i^{i+1})w_1 + \ldots + (c_i^n - c_i^{i+1})w_n) = (c_i^i - c_i^{i+1})w_{\tau(i)} + (c_i^{i+1} - c_i^{i+1})w_{\tau(i)+1} + z^2 - (c_i^{i+1} - c_i^{i+1})w_{\tau(i)} + (c_i^{i+1} - c_i^{i+1})w_{\tau(i)+1} + z^2 = (c_i^i - c_i^{i+1})w_{\tau(i)} + (c_i^{i+1} - c_i^{i+1})w_{\tau(i)+1} + 2z(c_i^i - c_i^{i+1})w_{\tau(i)} + (c_i^{i+1} - c_i^{i+1})w_{\tau(i)+1}^2 + (c_i^i - c_i^{i+1})w_{\tau(i)} + (c_i^{i+1} - c_i^{i+1})w_{\tau(i)+1}^2 - z(c_i^i - c_i^{i+1})w_{\tau(i)} + (c_i^{i+1} - c_i^{i+1})w_{\tau(i)+1}^2).

On the left-hand side, $w_k, k \neq \tau(i), \tau(i)+1$ is never the second term of the product of two $w_k$'s. Hence, $(c_i^i - c_i^{i+1})w_{\tau(i)} + (c_i^{i+1} - c_i^{i+1})w_{\tau(i)+1} + (c_i^{i+1} - c_i^{i+1})w_{\tau(i)+1} + z = 0$. This implies that either $c_i^i - c_i^{i+1}$ and $c_i^{i+1} = c_i^{i+1}$, which is a contradiction since $\sigma(w_i) = \sigma(w_{i+1})$, or $z = 0$. Thus $z = 0$ and so $c_k^i = c_k^{i+1} = 0$ for all $k \neq \tau(i), \tau(i)+1$. Furthermore, we now have that the left-hand side is in the subspace generated by $w_{\tau(i)}$, $w_{\tau(i)+1}$, so only $b_{\tau(i)}^1, b_{\tau(i)+1}^1$ can be nonzero.

Let us write down what we now know so far. For any $i$, $1 \leq i \leq n$, we have:

1. $\sigma(v_{i+1}) = a_{\tau(i)}^i v_{\tau(i)\tau(i)+1} + b_{\tau(i)}^1 w_1 + b_{\tau(i)}^i w_{\tau(i)+1}$
2. $\sigma(w_i) = c_i^i w_{\tau(i)} + c_i^{i+1} w_{\tau(i)+1}$
3. $\sigma(w_{i+1}) = c_i^{i+1} w_{\tau(i)} + c_i^{i+1} w_{\tau(i)+1}$

and

4. $(a_{\tau(i)}^i v_{\tau(i)\tau(i)+1} + b_{\tau(i)}^i w_{\tau(i)} + b_{\tau(i)+1}^i w_{\tau(i)+1})(c_i^i - c_i^{i+1})(w_{\tau(i)} - w_{\tau(i)+1}) = \alpha(w_j - w_{j+1})$
\[ a^i_{\tau(i)}(c^i_{\tau(i)} - c^{i+1}_{\tau(i)})(w^2_{\tau(i)} - w^2_{\tau(i)+1}) + (b^i_{\tau(i)}w_{\tau(i)} + b^{i+1}_{\tau(i)+1}w_{\tau(i)+1})(c^i_{\tau(i)} - c^{i+1}_{\tau(i)})(w_{\tau(i)} - w_{\tau(i)+1}) \mod R =
((c^i_{\tau(i)})^2 - (c^{i+1}_{\tau(i)})^2)w^2_{\tau(i)} + (c^i_{\tau(i)+1})^2(c^i_{\tau(i)} + c^{i+1}_{\tau(i)+1})(w_{\tau(i)}w_{\tau(i)+1} + w_{\tau(i)+1}w_{\tau(i)} + ((c^i_{\tau(i)+1})^2 - (c^{i+1}_{\tau(i)+1})^2)w^2_{\tau(i)+1}
\]

a) Assume now that \( n \geq 3 \).

Applying (1) and (3) above to \( i - 1 \) we find that \( \sigma(w_i) = c^i_{\tau(i)-1}w_{\tau(i-1)} + c^{i+1}_{\tau(i)-1}w_{\tau(i-1)+1} \). Because \( \sigma(w_i) \neq 0 \), \( c^i_{\tau(i)} \) and \( c^{i+1}_{\tau(i)+1} \) cannot both be zero. If \( c^i_{\tau(i)} \neq 0 \), then because \( \tau(i-1) \) cannot equal \( \tau(i) \), \( \tau(i-1) + 1 = \tau(i) \). But then \( \tau(i) + 1 \neq \tau(i-1) \) or \( \tau(i-1) + 1 \); so, \( c^{i+1}_{\tau(i)+1} = c^{i+1}_{\tau(i)-1} = 0 \). Thus, \( w_i \mapsto c^i_{\tau(i)}w_{\tau(i)} \). Furthermore, since \( \tau(i-1) + 1 = \tau(i) \), \( \tau = k \) some \( k \), \( 1 \leq k \leq n \); and so \( \tau \in D_n \).

If \( c^i_{\tau(i)} = 0 \), then \( c^{i+1}_{\tau(i)+1} \neq 0 \) and so \( \tau(i) + 1 = \tau(i-1) \) and \( c^{i+1}_{\tau(i)-1} = 0 \). Thus \( \tau = r^k \), for some \( k \), \( 1 \leq k \leq n \); again \( \tau \in D_n \).

Because we have \( c^i_{\tau(i)} - c^{i+1}_{\tau(i)+1} \neq -(c^{i+1}_{\tau(i)+1} - c^{i+1}_{\tau(i)+1}) \) and either \( c^{i+1}_{\tau(i)+1} = 0 \) or \( c^{i+1}_{\tau(i)+1} = c^{i+1}_{\tau(i)+1} \) we obtain that the coefficients of the images of \( w_i \) and \( w_{i+1} \) are the same; call this coefficient \( c \).

Thus,
\[
\begin{align*}
\tau_{\tau(i)}(c^i(w^2_{\tau(i)} - w^2_{\tau(i)+1}) + (b^i_{\tau(i)}w_{\tau(i)} + b^{i+1}_{\tau(i)+1}w_{\tau(i)+1})c(w_{\tau(i)} - w_{\tau(i)+1}) = c^2(w^2_{\tau(i)} - w^2_{\tau(i)+1})
\Rightarrow b^i_{\tau(i)} = b^{i+1}_{\tau(i)+1} = 0 \text{ and } a^i_{\tau(i)} = c \text{ for all } i.
\end{align*}
\]

What \( \sigma \) does on level one forces what happens on the levels above. We have determined thus far that there exists an automorphism \( \hat{\sigma} \) in \( k^* \times D_n \) that agrees with \( \sigma \) on the span of the \( v \)'s and \( w \)'s. Hence, by composing \( \sigma \) with the inverse of \( \hat{\sigma} \), we may assume that all \( v \)'s and \( w \)'s are fixed. Thus, using relation (2) above, we see that \( u \) must also be fixed (\( \sigma(u) = cu \)).

Therefore, \( \text{Aut}(A(\Gamma_{D_n})) = k^* \times D_n \).

b) Since in this case \( \tau(i+1) = \tau(i) + 1 = \tau(i-1) + 2 = \tau(i-1) \) in \( \mathbb{Z}/(2) \), the considerations of part (a) do not apply; \( c^i_{\tau(i)+1} = c^{i+1}_{\tau(i)-1} \) can be nonzero. Thus, \( w_i \) can go to a sum of multiples of \( w_1 \) and \( w_2 \); \( \sigma(w_i) = c^1_{\tau(i)}w_1 + c^2_{\tau(i)}w_2 \).

Because we only have one vertex in level two, \( v_1, v_2 \), it can only go to a multiple of itself plus multiples of \( w_1 \) and \( w_2 \). Thus we can drop the \( a \) and the superscripts on \( a \) and the superscripts on \( b_1 \); \( \sigma(v_1) = av_1 + bv_1 + bv_2 \).

We can rewrite (4) as \( a(c_1^1 - c_2^1)(w_2^1 - w_2^2) + (b_1w_1 + b_2w_2)(c_1^1 - c_2^1)(w_1 - w_2) = ((c_1^1)^2 - (c_2^1)^2)w_1^2 + (c_1^1c_2^2 - c_2^1c_2^2)(w_1w_2 + w_2w_1) + ((c_1^2)^2 - (c_2^2)^2)w_2^2 \). We can conclude from this that \( a + b_1 = c_1^1 + c_2^1, a + b_2 = c_1^2 + c_2^2, a + b_2 = c_1^2 + c_2^2, \) and \( -b_1(c_1^1 - c_2^1) = c_1^1c_2^2 - c_2^1c_2^2 = b_2(c_1^1 - c_2^1) \) \( \Rightarrow -b_1 = b_2 \) (else \( c_1^1c_2^2 = c_2^1c_2^2 \Rightarrow c_1^2 = c_2^2 \) and \( c_2^2 = c_2^2 \), which is not possible). These imply that \( 2a = c_1^1 + c_2^1 + c_2^1 + c_2^2 \Rightarrow a = c_1^1 + c_2^1 \Rightarrow b_1 = c_1^2 - c_2^2 = -b_2 \).

Write the element \( rv_1 + sw \) as the vector \( [ r \ s \ t ] \). Then a way to visualize what this automorphism group looks like is to consider the invertible transformation matrix \( M \) that sends \( [ r \ s \ t ] \rightarrow [ r \ s \ t ] \star M \)

\[
M = \begin{bmatrix}
  c_1^1 + c_2^1 & c_2^2 - c_1^1 & c_2^2 - c_1^2 \\
  0 & c_1^1 & c_2^2 \\
  0 & c_1^2 & c_1^1 + c_2^2 - c_2^1
\end{bmatrix}
\]

This matrix is conjugate to a triangular matrix and thus stabilizes a flag. The spaces \( M \) stabilizes can be found by solving \( [ r \ s \ t ] \star M = \alpha[ r \ s \ t ] \). \( M \) stabilizes the one-dimensional spaces \( k \begin{bmatrix} 1 & -1 & -1 \end{bmatrix} \) and \( k \begin{bmatrix} 0 & 1 & -1 \end{bmatrix} \).

Hilbert Series of \( A(\Gamma_{D_n}) \)

We will give here two bases for \( A(\Gamma_{D_n}) \), one for each of the two definitions of \( A(\Gamma) \) given in section 0. First we will give a basis in terms of the vertices (Proposition 0.2).

\[ \square \]
Proposition 1.2. A basis $\mathcal{B}$ of $A(\Gamma_{D_n})$ consists of the set of all words in $u, v_{i+1},$ and $w_i$ such that the following conditions on the words hold: the subword $v_{i+1}w_j$ only occurs if $j \neq i+1,$ the subword $uv_{i+1}$ only if $i = 1,$ and $w_{i+1}w_j$ only if $i = j = 1.$

Proof. We can describe a basis of monomials for $A(\Gamma_{D_n})$ using Bergman’s Diamond Lemma [Berg]. Put a partial order on the generators such that $u > v_{i+1} > w_j \forall i,j$ and $v_{i+1} > v_{j+1}$ and $w_i > w_j$ if $i > j.$ Order monomials lexicographically. The reductions are $w_{i+1}w_i \equiv w_{i+1} - v_i^2 + v_i^2 + (v_i - v_{i+1})w_{i+1}, \ 1 \leq i \leq n$ and $v_{i+1}w_{i+1} \equiv v_{i+1}w_i - v_i^2 + w_i^2 + w_{i+1}, \ 1 \leq i \leq n.$

We need to find a complete list of reductions so that all ambiguities resolve. The only ambiguity will occur when we have a word that ends in $v$ overlapping with one beginning with $v$; i.e. $w_{i+1}w_i^2w_{i+2}.
(uv_{i+1}w_i^2)w_{i+2} \equiv (uv_{i+1}w_i^2 + v_i^2 + (v_i - v_{i+1})w_{i+1})w_{i+2} \equiv w_{i+1}w_i^2 + v_i^2 - v_i^2 + w_i^2 + w_{i+1}^2 + v_i^2 + w_i^2 - v_i^2 - w_i^2 + w_{i+1}^2 - w_i^2 + w_{i+1}^2.

Thus we need to add an additional reduction; namely, $uv_{i+1}w_i \equiv uv_{i+1}w_i + v_i^2 + v_i^2 + v_{i+1}w_i - v_i^2 + v_{i+1}w_i^2 + v_i^2 + w_{i+1}^2 + v_i^3 + v_i^3 + w_i^3 - v_i^3 - w_i^3 + w_{i+1}^3 + w_{i+1}^3.

This does not create additional ambiguities since this reduction ends in $w$ and we have no reductions which begin in $w.$ Also, no reductions end in $u.$ Thus, all ambiguities now resolve.

Therefore, by Bergman’s Diamond Lemma, $A(\Gamma_{D_n})$ may be identified with the $k$-module of monomials which are irreducible under these reductions. Hence, $\mathcal{B}$ is a basis for $A(\Gamma_{D_n}).$

Next follows a basis in terms of edges (Thm 0.1).

Proposition 1.3. $\mathcal{B}' = \{e(x_1, k_1) \cdots e(x_l, k_l) : l \geq 0, x_1, ..., x_l \in \{u, v_1, ..., v_{n+1}, w_1, ..., w_n\}, 1 \leq k_i \leq |x_i|, (x_i, k_i) \gg (x_{i+1}, k_{i+1})\}$ is a basis for $A(\Gamma_{D_n}).$

Proof. This follows directly from Theorem 0.1.

In the preliminaries we stated that the algebra is generated by distinguished edges and so we can identify the distinguished edges with the vertices which are their tails - $e_v$ is identified with $v$. Thus $e(v, k)$ can be expressed as a product of $k$ vertices (recall we are writing $e(v, k)$ in lieu of $\hat{e}(v, k)$), and so there is a correlation between the bases $\mathcal{B}, \mathcal{B}'$ as follows:

$e(u, 3)$ $\leftrightarrow$ $uv_{12}w_1$ $\quad e(u, 2)$ $\leftrightarrow$ $uv_{12}$ $\quad e(u, 1)$ $\leftrightarrow$ $u$
$e(v_{i+1}, 2)$ $\leftrightarrow$ $v_{i+1}w_i$ $\quad e(v_{i+1}, 1)$ $\leftrightarrow$ $v_{i+1}$ $\quad e(v_i, 1)$ $\leftrightarrow$ $w_i$

It is important to observe that in the associated graded algebra, $\sigma \in Aut(A(\Gamma))$ permutes the elements of $gr\mathcal{B}$ and $gr\mathcal{B}'$.

Recall that the Hilbert series gives the graded dimension of an algebra; the coefficient of $t^k$ is the $k$-th graded dimension (see section 0 for the grading on our algebras). We write this as: $H(t) = \sum dim(A(\Gamma)[k]) t^k.$
Proposition 1.4. The Hilbert series for $A(\Gamma_{D_n})$ is

$$H(t) = \frac{1}{1 - (2n + 1)t + (2n - 1)t^2 - t^3} = \frac{1 - t}{1 - (2n + 2)t + 4nt^2 - 2nt^3 + t^4}$$

Proof. We will give two proofs of this proposition. The first one uses Proposition 1.2 and induction to count basis elements. This will give us a recursion that can then be written as a generating function. The second method of proof is much shorter and uses a theorem from [RSW] that gives a formula for the Hilbert series of an algebra associated to a directed, layered graph.

Method 1: By Proposition 1.2, there are $n(n - 1)$ subwords of the form $v_{i+1}w_j$ which can occur in an element of $B$ and exactly one of the forms $w_{i+1}w_{j+1}$ and $w_{i+1}w_j$. This means, of course, that there are $n$ subwords of the form $v_{i+1}w_j$ which cannot occur, $n - 1$ of the form $w_{i+1}$, and $n^2 - 1$ of the form $w_{i+1}w_j$.

Let $d_k = \dim(A(\Gamma_{D_n})(k))$.

We will proceed by induction.

- $d_0 = 1$
- $d_1 = 2n + 1$: Every word of length one belongs to the basis since all reducible subwords are of length greater than one. A basis is: $\{u, v_{i+1}, w_i\}$
- $d_2 = 4n^2 + 2n + 2$: There are $(2n + 1)^2$ elements of length two and $2n - 1$ of them are reducible. Hence, the dimension is $(2n + 1)^2 - (2n - 1) = 4n^2 + 2n + 2$. A basis is: $\{w_iu, w_iw_j, uu, w_{1,2}, uu_i, v_{i+1}u, w_{i+1}v_{j+1}, v_{i+1}w_j\}$

Use induction to determine $d_k$:

- If $x \in B$ is a word of length $k - 1$, then $w_ix \in B$. Thus there are $nd_{k-1}$ words of length $k$ in $B$ starting with $w_i$.
- If $x \in B$ is a word of length $k - 1$, then $v_{i+1}x \in B$ if and only if $x$ does not begin with $w_{i+1}$. As determined in the previous bullet, there are $nd_{k-2}$ basis elements starting with $w_j, 1 \leq j \leq n$ in degree $k - 1$, and thus $d_{k-2}$ of them beginning with $w_{i+1}$. Hence, for each $i$, there are $d_{k-1} - d_{k-2}$ possibilities for $x$. Therefore, there are $n(d_{k-1} - d_{k-2})$ words of length $k$ of the form $v_{i+1}x$.
- We will treat the case of words beginning with $u$ in three cases. If $x \in B$ of length $k - 2$, $ux \in B$ if and only if $x$ does not begin with $v_{i+1}, 2 \leq i \leq n$. There are $d_{k-1} - nd_{k-2} = n(d_{k-2} - d_{k-3})$ words beginning with $u$ in degree $k - 1$ (from previous bullets). Thus, there are that many words of the form $ux \in B$. Next $w_{1,2}x \in B$ if and only if $x$ does not begin with $w_i, 2 \leq i \leq n$. Thus, there are $d_{k-2} - (n - 1)d_{k-3}$ words of the form $w_{1,2}x$. Finally, $uw_i \in B$ for all $x$. Thus, there are $nd_{k-2}$ words of this form. This gives us a total of $d_{k-1} - 2nd_{k-2} + nd_{k-3} + d_{k-2} - (n - 1)d_{k-3} + nd_{k-2} = d_{k-1} - (n - 1)d_{k-2} - d_{k-3}$ words beginning with $u$.

Thus, $d_k = nd_{k-1} + n(d_{k-1} - d_{k-2}) + d_{k-1} - (n - 1)d_{k-2} + d_{k-3} = (2n + 1)d_{k-1} - (2n - 1)d_{k-2} + d_{k-3}$.

We can write this recurrence formula as a generating function following the method described by Wilf in [Wilf, §1.2]. Let $H(t) = \sum_{i \geq 0} d_it^i$ denote the generating function that we are trying to find. Let $d_{-2} = d_{-1} = 0, d_0 = 1$. Multiply both sides of the recursion by $t^i$ and sum over $i \geq 0$. Then on the left-hand side we have $d_1 + d_2t + d_3t^2 + ... = \frac{H(t) - d_0}{t}$. And on the right hand side we have $(2n + 1)H(t) - (2n - 1)tH(t) + t^2H(t)$.

Solving for $H(t)$:

$$H(t) = \frac{1}{1 - (2n + 1)t + (2n - 1)t^2 - t^3}.$$
Method 2: [[RWS], Thm 2] gives the Hilbert series formula as:

\[
H(\Gamma, t) = \frac{1 - t}{1 + \sum_{v_{i1} > \ldots > v_{il} \geq \ast} (-1)^{|v_{il}| - |v_{il}| + 1}}.
\]

In this example, the possible sequences indexing the sum are: \(u, v_{i1} > u, v_{i1} > v_{i1} > w_{i1}, v_{i1} > u, v_{i1} > v_{i1} > w_{i1}, v_{i1} > u, v_{i1} > v_{i1} > w_{i1}, v_{i1} > u, v_{i1} > u, v_{i1} > v_{i1} > w_{i1}, v_{i1} > u, v_{i1} > v_{i1} > u, v_{i1} > \ast, u > v_{i1} > \ast, u > v_{i1} > \ast, u > v_{i1} > v_{i1} > w_{i1} > \ast, u > v_{i1} > \ast, u > v_{i1} > v_{i1} > w_{i1} > \ast, \) and \(u > v_{i1} > w_{i1} > \ast, u > v_{i1} > \ast, u > v_{i1} > v_{i1} > w_{i1} > \ast, \) and \(u > v_{i1} > w_{i1} > \ast. \) Thus, the coefficients of \(t, t^2, t^3, \) and \(t^4\) are \(-2n + 2, n + 2n + n = 4n, n + n - 2n - 2n = -2n, \) and \(1 - 2n + 2n = 1, \) respectively.

The coefficient of \(t^k\) for \(k \geq 5\) is zero.

Thus \(H(A(\Gamma_{D_n}), t) = \frac{1 - t}{1 - (2n + 2)t + 4nt^2 - 2nt^3 + t^4}. \)

Graded trace generating functions of \(D_n\) acting on \(A(\Gamma_{D_n})\)

Pass to the associated graded algebra, \(\text{gr}A(\Gamma_{D_n}).\) Let \(\phi_1, \ldots, \phi_l,\) where \(l\) is the number of conjugacy classes of \(D_n,\) denote the distinct irreducible representations of \(D_n\) and let \(\chi_j\) denote the character afforded by \(\phi_j.\) \(D_n\) acts on each \(A(\Gamma_{D_n})[i],\) and so the completely reducible \(D_n\)-module \(A(\Gamma_{D_n})[i]\) may be written as \(\phi_j^{m_{ij}} \phi_j \cdot \) The bases \(B\) and \(B'\) are invariant under the automorphism \(\sigma.\) Therefore, the trace of \(\sigma\) on 

\[grA(\Gamma_{D_n})\text{, } Tr_{\sigma}\mid_{A(\Gamma_{D_n})}\text{,}\]

is the number of fixed basis elements.

Remark: \(Tr_{\sigma}\) is the dimension of the subalgebra \(A(\Gamma_{D_n}^\sigma),\) which is not the same as the dimension of the fixed point space. The subalgebra \(A(\Gamma_{D_n}^\sigma)\) described in section 0 is the span of the set of fixed elements of the basis. On the other hand, the fixed point space is the span of the sums of orbits of \(\sigma.\) Averages over orbits are in the fixed point space, but not in the subalgebra.

Graded trace generating functions:

We will give two methods by which to find the graded trace generating functions for general \(A(\Gamma).\) The first will be to essentially count allowable and non-allowable words - a generating function that gives the number of allowable words in each grading in the subalgebra \(A(\Gamma^\sigma).\) The second will generalize equation 2 to use on the subgraph \(\Gamma^\sigma\) and subalgebra \(A(\Gamma^\sigma).\) These graded trace generating functions will be used to find the multiplicities of irreducible representations.

Method 1 - Counting allowable and non-allowable fixed words:

The \(Tr^\sigma|_{A(D_n)[i]}\) is the number of fixed basis elements of degree \(i.\) In other words, the number of sequences \((x_1, k_1), \ldots, (x_l, k_l)\) such that \(1 \leq k_j \leq |x_j|, k_1 + \ldots + k_l = i, e(x_i, k_1) \cdots e(x_i, k_l)\) is irreducible and \(\sigma x_j = x_j \forall j.\) Recall that \(e(x_i, k)e(x_j, l)\) is reducible if there is a path from \(x_i\) to \(x_j\) and the level of \(x_i\) equals the level of \(x_j\) plus \(k.\) We want to count the number of \(x_j\) fixed by \(\sigma\) in level \(3 (a_3),\) in level \(2 (a_2),\) and in level \(1 (a_1).\)

Consider

\[
\frac{1}{1 - (a_1 t + a_2 t^2 + a_3 t^3)} = \sum_{i=0}^{\infty} (a_1 t + a_2 t^2 + a_3 t^3)^i.
\]

The coefficient of \(t^i\) is the number of monomials of degree \(i\) in fixed \(x_j;\) i.e. the graded trace. The only problem with this is that it counts all possible products - both irreducible and reducible. Hence, this must be adjusted.
Theorem 1.5. Let $|W|$ denote the bi-graded dimension of $W$ (a power series in $s$, $t$ where the dimension of $W_{i,j}$ is the coefficient of $st^j$). Let $A(\Gamma^a)$ be the algebra associated with the graph $\Gamma^a$, $V_\sigma$ the vector space of vertices of $\Gamma^a$, and $R$ the set of relations/reducible words. Thus $A(\Gamma^a) = T(V_\sigma)/<R> = T(V_\sigma)/A(\Gamma^a)RT(V_\sigma)$. Then

$$|A(\Gamma^a)| = \frac{1}{1 - |V_\sigma| + |R| - |RV_\sigma \cap V_\sigma R| + |RV_\sigma^2 \cap V_\sigma RV_\sigma \cap V_\sigma^2 R| - \ldots}$$

Proof. The graded dimension of $T(V_\sigma)$ is $\frac{1}{1 - |V_\sigma|}$. For simplicity of notation, let us denote $A(\Gamma^a)$ by $A$ and $V_\sigma$ by $V$ in this proof. As vector spaces, we can identify $A$ with the subspace of $T(V)$ not containing a subword in $R$.

Before giving the general proof, the special case when $RV \cap VR = (0)$ will be used to illustrate the general technique. Consider a reducible monomial $m = arx$ where $a \in A$, $r$ is the left-most subword from $R$, and $x \in T(V)$. Since $RV \cap VR = (0)$, there does not exist a $y$ such that $a = a'y$ and $yr \in RV$. Thus, $A = T(V)/ART(V)$ and $|A| = |T(V)| - |A||R||T(V)| = \frac{1}{1 - |V|} - \frac{|A||R|}{1 - |V|}$. Solving for $|A|$ we obtain $|A|(1 + \frac{|R|}{1 - |V|}) = \frac{1}{1 - |V|}$ which implies $|A| = \frac{1}{1 - |V| + |R|}$.

For $i \geq 0$ define $R^{(i)} := \bigcap_{j=0}^{i} V^{j} RV^{i-2}$; note that $R^{(0)} = R$. It will also be convenient to define $R^{(-1)} := V$ and $R^{(-2)} := k$. For $i \geq -2$, let $T_i := T(V)/R^{(i)}/(T(V)RV^{i+2}T(V) \cap T(V)R^{(i)})$. Also, define $T_{-3} := T(V)/(T(V)VT(V))$. Consider the quotient that defines $T_i$: in the quotient the $V^{i+2}$ moves $R$ past $R^{(i)}$ in $T(V)R^{(i)}$. In other words, $T_i$ is such that $R^{(i)}$ contains the left-most subword in $R$, and so the part of the word in $T(V)$ contains no subwords in $R$. Thus, $T_i \cong AR^{(i)}$. In particular, $T_0 = T(V)R/T(V)RV^2T(V) \cap T(V)R \cong AR$, $T_{-1} = T(V)VT(V) \cap T(V)R \cong AV$, $T_{-2} = T(V)/T(V)RT(V) \cong A$, and $T_{-3} = T(V)/T(V)VT(V) \cong k$.

Now define a map $\phi : T_i \rightarrow T_{i-1}$ for $i \geq -1$ by, for $a \in T(V)R^{(i)}$, $a + T(V)RV^{i+2}T(V) \cap T(V)R^{(i)} \mapsto a + T(V)RV^{i+1}T(V) \cap T(V)R^{(i-1)}$. Also, define $\phi_2 : T_{-2} \rightarrow T_{-3}$ by $a + T(V)RT(V) \mapsto a + T(V)VT(V)$. Because $VR^{(i)} = V(VR^{(i-1)} \cap R^{(i-1)}V) \subseteq V^{2}R^{(i-1)}$, $T(V)R^{(i)} \subseteq T(V)R^{(i-1)}$. Also, $T(V)RV^{i+2}T(V) \subseteq T(V)RV^{i+1}T(V)$ since $V^{i+2}T(V) = V^{i+1}VT(V) \subseteq V^{i+1}T(V)$. Thus, $\phi$ is a well-defined map.

Next we will show that the sequence $T_j \rightarrow T_{j-1} \rightarrow \cdots \rightarrow T_{-2} \rightarrow T_{-3} \rightarrow 0$ is exact (the sequence will not be infinite because $R^{(j+1)}$ is $(0)$ for some $j$). Now for $i \geq 0$, $\phi_{i-1}(a + T(V)RV^{i+2}T(V) \cap T(V)R^{(i)}) = a + T(V)RV^{i+1}T(V) \cap T(V)R^{(i-2)}$, but $a \in T(V)R^{(i)}$ and $a \in T(V)R^{(i)} = T(V)(RV^{i+2} \cdots \cap V^{i-2}R)$ \Rightarrow $a \in T(V)RV^{i+2} \Rightarrow a \in T(V)RV^{i+1}T(V) \subseteq T(V)R^{(i-1)}$. Thus, the image is $0$ and $\text{im} \phi_2 \subseteq \text{ker} \phi_{-1}$.

To show the other inclusion, note that $\ker \phi_{i-1} = \{a + T(V)RV^{i+2}T(V) \cap T(V)R^{(i-1)} : a \in T(V)RV^{i+1}T(V) \cap T(V)R^{(i-2)} \}$, and $a \in T(V)R^{(i-2)}$ since $a \in T(V)R^{(i-1)} \subseteq T(V)R^{(i-2)}$, and $a \in T(V)R^{(i-1)}$ and $T(V)RV^{i+2}T(V)$ imply that $a \in T(V)R^{(i)}$.

Furthermore, because the maps $\phi_i$ are homogeneous in degree, for $i \geq -1$ nothing gets mapped down to scalars. Thus, the $\text{im}(AV \rightarrow A) = A/k = \ker(\phi_2)$. Finally, for $a \in k$, $a \in A$ (and $\phi_2(a) = a + T(V)VT(V)$). Thus, $\phi_{-2}$ is surjective. Therefore, the sequence is exact.

Using the isomorphisms given above, we can write our exact sequence as $AR^{(j)} \rightarrow AR^{(j-1)} \rightarrow \cdots \rightarrow AR \rightarrow AV \rightarrow A \rightarrow k \rightarrow 0$. Therefore, $1 = |k| = \sum_{i \geq 0} (-1)^{|i|} |T_i| = |A| - |A||V| + \sum_{i \geq 0} (-1)^{|i|} |A||R^{(i)}| = |A|(1 - |V| + \sum_{i \geq 0} (-1)^{|i|} |R^{(i)}|)$ implies that $|A| = \frac{1}{1 - |V| + \sum_{i \geq 0} (-1)^{|i|} |R^{(i)}|}$.
Let us calculate the generating functions for our algebra. First of all, the identity acting on the algebra gives the graded dimension of the algebra. We will derive it using the theorem above to show that we get the same result as the Hilbert series given earlier.

Example calculation: The generators for the algebra are: $e(u, 3), e(u, 2), e(u, 1), e(v_{i+1}, 2)$, $e(v_{i+1}, 1), e(w_i), 1 \leq i \leq n$.

Non-allowable words: $e(u, 2)e(w_1), e(u, 1)e(v_{i+1}, 2), e(u, 1)e(v_{i+1}, 1), e(v_{i+1}, 1)e(w_i), e(v_{i+1}, 1)e(w_{i+1})$

Overlaps of the non-allowable words: $e(u, 1)e(v_{i+1}, 1)e(w_i), e(u, 1)e(v_{i+1}, 1)e(w_{i+1})$

Final count: $a_3 = 1 - (n + n) + (n + n) = 1, a_2 = (1 + n) - (n + n + n) = 1 - 2n, a_1 = 1 + n + n = 2n + 1$

Thus, we have

$$\frac{1}{1 - ((2n + 1)t - (2n - 1)t^2 + t^3)}$$

which agrees with the earlier results.

Similarly, only $u$ is fixed by $r$; so, we have $a_3 = a_2 = a_1 = 1$. Hence, the graded trace generating function for $r$ acting on the algebra is

$$\frac{1}{1 - (t + t^2 + t^3)}.$$ 

Also, we have for $n$ even and $s$ acting that $e(u, 3), e(u, 2), e(u, 1), e(v_{i+1}, 2), e(v_{i+1}, 1), e(v_{n/2+1}n/2+2, 2), e(v_{n/2+1}n/2+2, 1)$ are fixed and $e(u, 1)e(v_{i+1}, 2), e(u, 1)e(v_{i+1}, 1), i = 1, n/2 + 1$ are not allowed. Thus,

$$\frac{1}{1 - (3t + t^2 - t^3)}$$

We get the same function with $rs$ acting as well as when $n$ is odd and $s$ is acting.

Method 2 - Generalizing the Hilbert Series formula (2):

We would like to apply the function $H(\Gamma, t)$ (see equation (2)) to the subalgebras created by our fixed points. Take the subgraph of $\Gamma_{D_3}$ consisting of the points fixed by an automorphism $\sigma$. This generates a subalgebra of $gr_A(\Gamma_{D_3})$ in the way described in section 0. Thus, we are using equation (2) with the additional condition that the vertices in the sum are fixed by $\sigma \in Aut(A(\Gamma))$; call this modified formula $Tr_\sigma(A(\Gamma), t)$.

Theorem 1.6. Let $\Gamma$ be a layered graph with unique minimal element * of level 0 and $\sigma$ an automorphism of the graph. Let $\Gamma^*$ be the subgraph of $\Gamma$ with vertices being those fixed by $\sigma$ (as described in section 0). Denote the Hilbert series of the subalgebra $A(\Gamma^*)$, which is the graded trace function of $\sigma$ acting on $A(\Gamma)$, by $Tr_\sigma(A(\Gamma), t)$ (or $Tr_\sigma(t)$ when $A(\Gamma)$ is clear). Then

$$Tr_\sigma(A(\Gamma), t) = \frac{1 - t}{1 + \sum_{v_1, \ldots, v_i \in V_\sigma} (-1)^{|l|} l\cdot v_{i+1} - |v_i| + 1}.$$ 

Proof: This proof is a modified version of those of [RSW, RW], Thm 2, Thm 4.1. Write $Tr(t)$ for $Tr_\sigma(A(\Gamma), t)$ in this proof. Let $v_1, \ldots, v_i, v \in V_\sigma$. Recall that the basis for $A(\Gamma^*)$ is $B_\sigma = \{e(v_1, k_1) \cdots e(v_i, k_i) : v_1, \ldots, v_i \in V_\sigma, 1 \leq k_i \leq |v_i|, e(v_i, k_i) \neq e(v_{i+1}, k_{i+1})\}$.

For $v \in (V_\sigma)_+$, define $C_v = \bigcup_{k=1}^{v} e(v, k)B_\sigma$, $B_v = C_v \cap B_\sigma$, $D_v = C_v \setminus B_v$. Then $B_\sigma = \{\ast\} \cup \bigcup_{v \in (V_\sigma)_+} B_v$. Let $Tr_v = Tr_\sigma(B_v, t)$, the graded dimension of the span of $B_v$. Then $Tr(t) = 1 + \sum_{v \in (V_\sigma)_+} Tr_v(t)$. We also have $Tr_\sigma(C_v, t) = (t + \ldots + t^{l(v)}) Tr(t) = t^{l(v)/2} Tr(t)$ and, because $D_v = \bigcup_{v > w > \ast} e(v, w) - |w| B_w$,
\[ Tr_\sigma(D_v, t) = \sum_{v > w > *} t^{|v| - |w|} Tr_w(t). \]

Thus,
\[ Tr_v(t) = t \left( \frac{t^{|v|} - 1}{t - 1} \right) Tr(t) - \sum_{v > w > *} t^{|v| - |w|} Tr_w(t). \]

This equation may be written in matrix form. Put an order on \( V_\sigma \), arrange the elements in decreasing order, and index the elements of vectors and matrices by this ordered set. Let \( \vec{T}r(t) \) denote the column vector with \( v \)-vertex on each of levels one and two (the coefficient of \( t^{l(|v| - 1)} \) in the \( v \)-position, \( \vec{I} \) denote the column vector having 1 as each entry, and let \( \xi(t) \) denote the matrix with the entry in the \( (v, w) \)-position being \( t^{|v| - |w|} \) if \( v \geq w \) and 0 otherwise. Then \( \xi(t) \vec{T}r(t) = \vec{s}T t(t). \)

Now \( \zeta(t) - I \) is a strictly upper triangular matrix, and so \( \zeta(t) \) is invertible; \( \zeta^{-1}(t) = I - (\zeta(t) - I) + (\zeta(t) - I)^2 - \cdots \). Thus the \((v_1, v_2)\)-entry of \( \zeta^{-1}(t) \) is \( (v_1, v_2) \). Solving for \( T r(t) \) we get
\[ T r(t) = \frac{1}{1 - \vec{I}T \zeta^{-1}(t) \vec{s}} = \frac{1 - t}{1 - \sum_{v_1 > \cdots > v_2 \geq *} (-1)^{|v_1'| - |v_2'| + 1}.} \]

In general, if \( \sigma \) and \( \tau \) are conjugate, \( \Gamma^\sigma \) and \( \Gamma^\tau \) are isomorphic (by the conjugation acting on the subscripts of vertices in levels one and two). Thus, it is enough to find the graded trace functions for any one representative of each conjugacy class.

Now, we can get our graded trace generating functions by using equation (3) on the subalgebras of \( A(\Gamma_{D_n}) \).

Consider \( T r(t) - 1 = \vec{I}T \vec{T}r(t) = \vec{I}T \zeta^{-1}(t) \vec{s}T r(t). \) Solving for \( T r(t) \) we get
\[ T r(t) = \frac{1 - t}{1 - 2t - t^4} = \frac{1 - t}{1 - t(2 - t^3)} = \frac{1}{1 - (t^2 + t^3)}. \]

**Figure 2.** \( \Gamma^r_{D_n} \)

The automorphism \( s \) acting on the algebra when \( n \) is even fixes the top, minimal, and two vertices on level two \((v_{12} \text{ and } v_{n/2+1} n/2+2)\) (see figure 3). Similarly, when \( n \) is odd \( s \) fixes the top, minimal, and one vertex on each of levels one and two \((v_{12} \text{ and } w_{(n+3)/2})\). And, \( rs \) fixes the top, minimal and two vertices on level one \((w_2 \text{ and } w_{n/2+2})\). Thus, there are 4 vertices, two edges of length one, and two of length two. For the coefficient of \( t^4 \), we have \( u > * \), \( u > \text{vertex} > * \), and \( u > \text{vertex} > * \). Thus,
\[ T r_s(t) = T r_{rs}(t) = \frac{1 - t}{1 - (4t - 2t^2 - 2t^3 + t^4)} = \frac{1 - t}{1 - t(2 - t)(2 - t^2)} = \frac{1}{1 - (3t + t^2 - t^3)}. \]

Note: we will normally apply this method; although, both methods can theoretically be applied in all layered graph algebras.

**Representations of \( D_n \) acting on \( A(\Gamma_{D_n}) \)**
Recall that the character table for $D_n$ where $n = 2m$ is even:

\[
\begin{array}{ccccccc}
\chi_{\text{triv}} & 1 & 1 & \ldots & 1 & \ldots & 1 & 1 \\
\chi_{1-1} & 1 & 1 & \ldots & 1 & \ldots & 1 & -1 -1 \\
\chi_{-11} & 1 & -1 & \ldots & (-1)^{j} & \ldots & (-1)^{m} & 1 -1 \\
\chi_{-1-1} & 1 & -1 & \ldots & (-1)^{j} & \ldots & (-1)^{m} & -1 1 \\
\chi_{k} & 2 & 2 \cos(2\pi k/n) & \ldots & 2 \cos(2\pi k/n) & \ldots & 2 \cos(2\pi km/n) & 0 0 \\
\end{array}
\]

where $(1 \leq k \leq m - 1)$, $r = (12\ldots n)$ and $s = (12)(3n)(4n - 1)\ldots(\frac{n}{2} + 1 \frac{n}{2} + 2)$. Then $rs = (13)(4n)\ldots(\frac{n}{2} + 1 \frac{n}{2} + 3)$.

and when $n = 2m + 1$ is odd:

\[
\begin{array}{ccccccc}
\chi_{\text{triv}} & 1 & 1 & \ldots & 1 & \ldots & 1 & 1 \\
\chi_{1-1} & 1 & 1 & \ldots & 1 & \ldots & 1 & -1 \\
\chi_{k} & 2 & 2 \cos(2\pi k/n) & \ldots & 2 \cos(2\pi k/n) & \ldots & 2 \cos(2\pi km/n) & 0 \\
\end{array}
\]

where $(1 \leq k \leq m)$, $r = (12\ldots n)$ and $s = (12)(3n)\ldots(\frac{n+1}{2} \frac{n+5}{2})$.

Now let us determine the multiplicities of the irreducible representations. Fix $n$. Let the graded trace generating function be denoted by $Tr_\sigma(t) = \sum_i Tr_{\sigma,i}t^i$ where $Tr_{\sigma,i} = tr_{\sigma_i}|_{\Lambda(D_n)|}\chi_i$. Let $\phi$ be an irreducible representation of $D_n$, $m_\phi(t) = \sum_i m_{\phi,i}t^i$ where $m_{\phi,i}$ is the multiplicity of $\phi$ in $A(D_n)\chi_i$, and $C = [\chi_{\sigma_\phi}]$ be the character table of $D_n$.

Then, if we fix the degree, $Tr_{\sigma,i} = \sum_\phi \chi_{\sigma\phi}m_{\phi,i}$ and so, we have $Tr_\sigma(t) = \sum_\phi \chi_{\sigma\phi}m_\phi(t)$. Write $\tilde{Tr}(t) = [Tr_{\sigma_1}(t)\ldots Tr_{\sigma_i}(t)]^T$ and $\tilde{m}(t) = [m_\phi(t)\ldots m_\phi(t)]^T$. Finally,

\[
\tilde{Tr}(t) = C^T\tilde{m}(t) \implies \tilde{m}(t) = (C^T)^{-1}\tilde{Tr}(t).
\]

**Proposition 1.7.** Let $\tilde{m}(t)$ be the vector of the graded multiplicities of the irreducible representations of $D_n$ as described above. Set the graded trace functions to

\[
a = \frac{1}{1-(2n+1)i-(2n-1)\iota^2+i^2)}, \quad b = \frac{1}{1-(i+\iota+i^2)}, \quad \text{and} \quad c = \frac{1}{1-(3i+\iota-i^2)}.
\]

a) Let $n$ be even. Then,

\[
\tilde{m}(t) = \begin{bmatrix}
\frac{1}{2n}a + \frac{n-1}{2n}b + \frac{1}{2}c \\
\frac{1}{2n}a + \frac{n-1}{2n}b - \frac{1}{2}c \\
\frac{1}{2n}(a-b) \\
\frac{1}{2n}(a-b) \\
\frac{1}{2}(a-b) \\
\frac{1}{2}(a-b) \\
\end{bmatrix}
\]
1) Let $n=2m$ is even

$$n(t) = \begin{bmatrix}
\frac{1}{2n}a + \frac{a-1}{2n}b + \frac{1}{2}c \\
\frac{1}{2n}a + \frac{a-1}{2n}b - \frac{1}{2}c \\
\frac{1}{n}(a-b) \\
\ldots \\
\frac{1}{n}(a-b)
\end{bmatrix}$$

This is obtained from deleting the third and fourth entries in the $n$ is even case.

**Proof.** Multiply the transpose of the character table of $D_n$ by $n(t)$. The result is

$$\bar{Tr}(t) = \begin{bmatrix}
c \\
a \\
\ldots \\
b \\
b
\end{bmatrix}$$

as desired.

Notice that all of the representations are realized; and, with large multiplicity. The multiplicities are given in terms of sums of generating functions, so there is not as nice of a way for writing them as with the graded trace. Closed forms can be found; however, they are not easy to work with.

In conclusion, we now know the multiplicities of the representations for each $n$ and graded dimension.

**Representations of $D_n$ acting on $A(\Gamma_{D_n})$**

We will use the same methodology as for $A(\Gamma_{D_n})$ to determine the irreducible representations that are realized in $A(\Gamma_{D_n})$. (See Section 0 for the definition of the dual.)

**Proposition 1.8.** A basis for the graded dual algebra $A(\Gamma_{D_n})^{[1]}$ is: $u^*, v_{i+1}^*, w_i^* 1 \leq i \leq n$, $u^*v_{i+1}^*$ $1 \leq i \leq n - 1$, $v_{i+1}^*w_i^* 1 \leq i \leq n$, and $u^*v_{i+1}^*w_i^*$

**Proof.** The generators of $A(\Gamma_{D_n})$ are $u^*, v_{i+1}^*, w_i^* 1 \leq i \leq n$. In the associated graded algebra the relations are $v_{i+1}(w_1 - w_{i+1})$ and $u(v_{i+1} - v_{i+1} + w_{i+1})$. Thus, the relations in the dual are $u^{*2}, u^*w_i^*, v_{i+1}^*u^*$, $w_i^*u^*, w_i^*w_j^*, v_{i+1}^*v_{j+1}^*, w_i^*v_{j+1}^*, u^*(v_{12}^* + \ldots + v_{n1}^*), v_{i+1}^*w_i^*$ if $j \neq i, i + 1, v_{i+1}^*(w_i^* + w_{i+1}^*)$. The elements in the graded dual follow.

Now let us determine the trace on the graded pieces by seeing how each conjugacy class acts on the elements in the dual. There are only three degrees in the dual, $A(\Gamma_{D_n})^{[1]} = A(\Gamma_{D_n})^{[0]} \oplus A(\Gamma_{D_n})^{[1]} \oplus A(\Gamma_{D_n})^{[2]} \oplus A(\Gamma_{D_n})^{[3]}$, so we can calculate each independently.

Case 1: $n=2m$ is even

The traces on the graded pieces are:

| $Tr_{\sigma,1}$ | 1 | $r$ | $r^j$ | $r^m$ | $s$ | $rs$ |
|----------------|---|-----|-------|-------|----|-----|
| 2m+1          | 1 | 1   | 1     | 1     | 3  | 3   |
| 2m-1          | -1| -1  | -1    | -1    | -1 | -1  |
|                | 1 | 1   | 1     | 1     | -1 | -1  |

Now that we have the graded traces we can find the multiplicities of the representations by solving the system of equations: $\sum_\phi n_{\phi,i} \chi_{\sigma\phi}(x) = Tr_{\sigma,i}(x), x \in D_n$. They are given in the following table:
|     | $\chi_{\text{triv}}$ | $\chi_{1-1}$ | $\chi_{-1}$ | $\chi_{-1-1}$ | $\chi_k$ |
|-----|---------------------|----------------|-------------|---------------|--------|
| $m_{\phi,1}$ | 3                    | 0             | 1           | 1             | 2      |
| $m_{\phi,2}$ | 0                    | 1             | 1           | 1             | 2      |
| $m_{\phi,3}$ | 0                    | -1            | 0           | 0             | 0      |

Case 2: $n=2m+1$ is odd

The traces on the graded pieces are:

|     | $r$ | $r_j$ | $r_m$ | $s$ |
|-----|-----|-------|-------|-----|
| $Tr_{\sigma,1}$ | $2n+1$ | 1     | ...   | 1   | 3   |
| $Tr_{\sigma,2}$ | $2n-1$ | -1    | ...   | -1  | -1  |
| $Tr_{\sigma,3}$ | 1     | 1     | ...   | 1   | -1  |

The multiplicities are given in the following table:

|     | $\chi_{\text{triv}}$ | $\chi_{1-1}$ | $\chi_k$ |
|-----|---------------------|----------------|--------|
| $m_{\phi,1}$ | 3                    | 0             | 2      |
| $m_{\phi,2}$ | 0                    | 1             | 2      |
| $m_{\phi,3}$ | 0                    | -1            | 0      |

Notice that the graded traces and multiplicities are the same in both the even and odd cases.

These values give graded trace functions (in both even and odd cases) of:

$Tr_{(1)}(A(\Gamma_{D_n})^1, t) = 1 + (2n + 1)t + (2n - 1)t^2 + t^3$

$Tr_r(A(\Gamma_{D_n})^1, t) = 1 + t - t^2 + t^3$

$Tr_s(A(\Gamma_{D_n})^1, t) = Tr_{rs}(A(\Gamma_{D_n})^1, t) = 1 + 3t - t^2 - t^3$

Remark: $Tr_{\sigma}(A(\Gamma)^{id}, t)$ is not a generating function of a graded dimension, unlike in the case of $A(\Gamma)$.

**Koszulity property of $A(\Gamma_{D_n})$**

One property of Koszul algebras is that $H(A, t) * H(A^1, -t) = 1$. You can easily check that this is also true for the graded trace functions that we found for $A(\Gamma_{D_n})$ and its dual $A(\Gamma_{D_n})^{(1)}$ (recall that $A(\Gamma)^{1} = A(\Gamma)^{id}$). Namely, $Tr_{\sigma}(A(\Gamma_{D_n}), t) * Tr_{\sigma}(A(\Gamma_{D_n})^1, -t) = 1$ where $\sigma$ is an element in the automorphism group of the algebra.

2. $Q_n$

The algebras $Q_n$ are the algebras associated with the lattice of subsets of $\{1, 2, ..., n\}$. Label the vertices in level $i$ by $\{v_A : A \subseteq \{1, ..., n\}, |A| = i\}$. $Q_4$ is shown in figure 4 below. Their history and some properties are discussed in [GRSW].

![Figure 4. $Q_4$](image-url)
In [RSW], Thm 3], Retakh, Serconek, and Wilson prove that

\[ H(Q_n, t) = \frac{1 - t}{1 - (2 - t)^n} \]

using equation 2. Our goal is to find the representations of the automorphism group of \( Q_n \) acting on the algebra. In order to do this, we first have to find the graded trace functions (as in section 1). This section gives the graded trace generating functions for \( \sigma \in \text{Aut}(Q_n) \) acting on the algebra and its dual as well as the multiplicities of their irreducible representations.

Following the definition given in Proposition 0.2, the generators of \( Q_n \) are the vertices \( \{v_A : A \subseteq \{1, ..., n\}\} \) and the relations are

\[(*)\{v_A(v_{A\setminus i} - v_{A\setminus j}) - v^2_{A\setminus i} + v^2_{A\setminus j} + (v_{A\setminus i} - v_{A\setminus j})v_{A\setminus \{i,j\}} : A \subseteq \{1, ..., n\}, i, j \in A\}.

Recall that \( \text{Aut}(\Gamma) \) refers to filtration-preserving automorphisms (§0).

**Theorem 2.1.** If \( n \geq 3 \), \( \text{Aut}(Q_n) = k^* \times S_n \) (The \( n = 2 \) case is proved in section 1.)

**Proof.** This proof models that of Theorem 1.1. In order to determine the filtration-preserving automorphism group of the algebra, first note that any automorphism \( \tilde{\sigma} \) of the graph induces an automorphism \( \sigma \) of the algebra in that \( \tilde{\sigma} \) will preserve the vertices and their levels and the edges. Because the vertices are the generators of the algebra and the relations come from the paths, the algebra structure will be preserved as well. Thus if \( \tilde{\sigma} \) is an automorphism of the graph sending vertex \( v \) to vertex \( w \), then \( \sigma \) will send \( v \) to \( w \) in the algebra and will be an automorphism. Let us denote both by \( \sigma \).

As mentioned, any automorphism of the graph must preserve the set of vertices at each level and so acts on the set \( \{v_1, ..., v_n\} \) of all \( n \) vertices in level 1; so, we may say \( \sigma(v_i) = v_{\sigma(i)} \) (again slightly abusing the use of \( \sigma \)). Thus we can think of an automorphism of the graph as being a permutation in \( S_n \) acting on the subscripts/labels of the vertices of level 1. This will uniquely determine what happens on higher levels; i.e. \( \sigma(v_A) = v_{\sigma(A)} \). Labeling the vertices in levels two and higher by the vertices to which there is a path to in level one ensures that as long as the set of vertices in each level is preserved, the edges will be as well. Since for each subset of \( \{1, ..., n\} \) of cardinality \( i \) level \( i \) has a vertex labeled by that subset, every element of \( S_n \) is an automorphism of the graph. In other words, for every \( \tau \in S_n \), \( \tau \) will permute the vertices on each level.

So far we know that the automorphism group of \( Q_n \) contains \( S_n \). Also, for any scalar \( \alpha \), multiplication by \( \alpha \) is an automorphism because the relations are homogeneous. Thus \( \text{Aut}(Q_n) \supseteq k^* \times S_n \). Any automorphism of the algebra must preserve the relations; thus, for all subsets \( A \subseteq \{1, ..., n\} \) such that \( |A| \geq 2 \) and \( i, j \in A \), the image of \( v_A(v_{A\setminus i} - v_{A\setminus j}) - v^2_{A\setminus i} + v^2_{A\setminus j} + (v_{A\setminus i} - v_{A\setminus j})v_{A\setminus \{i,j\}} \) must equal zero. Let \( \sigma \in \text{Aut}(Q_n) \), \( \sigma(v_{i,j}) = a^i_{i,j}v_{12} + ... + a^2_{n-1,n}v_{n-1,n} + b^j_{i,j}v_1 + ... + b^2_{n,n}v_n \), and \( \sigma(v_i) = c^i_1v_1 + ... + c^i_nv_n \) for all \( i, j \) and with coefficients in the base field \( k \).

Now \( \sigma(v_{i,j}(v_{i,j}) = \sigma(v^2_{i,j} - v^2_{i,j}) \) implies

\[(a^i_{i,j}v_{i,j} + ... + a^2_{n-1,n}v_{n-1,n} + b^j_{i,j}v_1 + ... + b^2_{n,n}v_n)((c^i_1 - c^i_{1})v_1 + ... + (c^i_n - c^i_{n})v_n) = (c^1_1v_1 + ... + c^i_nv_n)^2 - (c^i_1v_1 + ... + c^i_nv_n)^2. \]

There are no \( v_{A}'s \) with \( |A| = 2 \) on the right-hand side, and so we must use our relations to eliminate them from the left-hand side. Thus, every occurrence of \( v_{i,j} \) must be followed by \( v_k - v_l \); and hence, \( c^i_k - c^i_l = (c^i_j - c^i_l) \). Therefore, if \( a^i_{i,j} \neq 0 \),

\[
(4) \quad (c^i_1 - c^i_{j})v_1 + ... + (c^i_n - c^i_{n})v_n = \alpha(v_k - v_l)
\]

for some \( \alpha \in k \).
This has two consequences. First, at most one \(a_{ij}^{ij}\) can be nonzero. If all \(a_{ij}^{ij}\) were zero, the element \(v_{ij} \notin (Q_n)_{(1)}\) would be sent to an element in \((Q_n)_{(1)}\), which we cannot allow because then \(\sigma\) would not be invertible. Thus \(a_{kl}^{ij}\) must be nonzero for exactly one \(\{ik\}\). Let us denote this set by \(\tau(i)\). Then \(\sigma(v_{ij}) = a_{ij}^{ij}v_{\tau(i)\tau(j)} + b_{ij}^{ij}v_{i} + \ldots + b_{kl}^{ij}v_{k}\). If \(\tau(ij) = \tau(kl)\), then \(\sigma(v_{ij} - v_{k}) \in (Q_n)_{(1)}\); this implies that \(\{ij\} = \{kl\}\). Thus \(\tau\) is one-to-one, and so is in \(S_n\).

A second consequence of (4) is that \(c_{ij}^r - c_{ij}^s\) is zero if and only if \(r \neq \tau(i), \tau(j)\).

We now have: \((a_{ij}^{ij} v_{\tau(i)\tau(j)} + b_{ij}^{ij}v_{i} + \ldots + b_{kl}^{ij}v_{k})(c_{ij}^r - c_{ij}^s)(v_{\tau(i)} - v_{\tau(j)}) = (c_{ij}^r v_{i} + \ldots + c_{ij}^s v_{n})^2 - (c_{ij}^r v_{i} + \ldots + c_{ij}^s v_{n})^2\). (Recall \(c_{ij}^r - c_{ij}^s = -(c_{ij}^r - c_{ij}^s)\).)

Let \(\tau = \sum_{r \neq \tau(i), \tau(j)} c_{ij}^r v_r = \sum c_{ij}^r v_r\). Then \((a_{ij}^{ij} v_{\tau(i)\tau(j)} + b_{ij}^{ij}v_{i} + \ldots + b_{kl}^{ij}v_{k})(c_{ij}^r - c_{ij}^s)(v_{\tau(i)} - v_{\tau(j)}) = (c_{ij}^r v_{\tau(i)} + c_{ij}^s v_{\tau(j)} + z)^2 - (c_{ij}^r v_{\tau(i)} + c_{ij}^s v_{\tau(j)} + z)^2 = (c_{ij}^r v_{\tau(i)} + c_{ij}^s v_{\tau(j)})^2 - (c_{ij}^r v_{\tau(i)} + c_{ij}^s v_{\tau(j)})^2\) + \(c_{ij}^r v_{\tau(i)} + c_{ij}^s v_{\tau(j)}z + z(c_{ij}^r v_{\tau(i)} + c_{ij}^s v_{\tau(j)}) - (c_{ij}^r v_{\tau(i)} + c_{ij}^s v_{\tau(j)})z - z(c_{ij}^r v_{\tau(i)} + c_{ij}^s v_{\tau(j)})\).

On the left-hand side, \(v_r\), for \(r \neq \tau(i), \tau(j)\), is never the second term of the product of two \(v_r\)'s. Hence, \((c_{ij}^r v_{\tau(i)} + c_{ij}^s v_{\tau(j)} - c_{ij}^r v_{\tau(j)} - c_{ij}^s v_{\tau(j)})z = 0\). This implies that either \(c_{ij}^r = c_{ij}^s\) and \(c_{ij}^r = c_{ij}^s\), which is a contradiction since \(\sigma(v_i) \neq \sigma(v_j)\), or \(z = 0\). Thus \(z = 0\) and so \(c_{ij}^r = c_{ij}^s = 0\) for all \(r \neq \tau(i), \tau(j)\).

Furthermore, we now have that the left-hand side is in the subspace generated by \(v_{\tau(i)}, v_{\tau(j)}\), so only \(b_{ij}^{ij}, b_{kl}^{ij}\) can be nonzero.

Let us write down what we know so far. For any \(i, j, 1 \leq i, j \leq n\), we have:

1) \(\sigma(v_{ij}) = a_{ij}^{ij} v_{\tau(j)\tau(i)} + b_{ij}^{ij}v_{i} + \ldots + b_{kl}^{ij}v_{k}\)
2) \(\sigma(v_i) = c_{ij}^r v_{\tau(i)} + c_{ij}^s v_{\tau(j)}\)
3) \(\sigma(v_j) = c_{ij}^r v_{\tau(j)} + c_{ij}^s v_{\tau(i)}\)

and

4) \((a_{ij}^{ij} v_{\tau(i)\tau(j)} + b_{ij}^{ij}v_{i} + \ldots + b_{kl}^{ij}v_{k})(c_{ij}^r - c_{ij}^s)(v_{\tau(i)} - v_{\tau(j)}) =
\quad a_{ij}^{ij}(c_{ij}^r - c_{ij}^s)(v_{\tau(i)} - v_{\tau(j)}) + (b_{ij}^{ij}v_{i} + b_{ij}^{ij}v_{j})\)
\quad + \(b_{ij}^{ij}v_{i} + b_{ij}^{ij}v_{j})(c_{ij}^r - c_{ij}^s)(v_{\tau(i)} - v_{\tau(j)}) =
\quad ((c_{ij}^r)^2 - (c_{ij}^s)^2)^2(v_{\tau(i)} - v_{\tau(j)})^2 + (c_{ij}^r - c_{ij}^s)(v_{\tau(i)} - v_{\tau(j)}) + \(c_{ij}^r v_{\tau(i)} + v_{\tau(j)} v_{\tau(j)}\) + \((c_{ij}^r)^2 - (c_{ij}^s)^2)^2(v_{\tau(i)} - v_{\tau(j)})\)

Assume now that \(n \geq 3\):

Applying (1) and (3) above to \(\{ik\}\) we find that \(\sigma(v_i) = c_{ij}^r v_{\tau(i)} + c_{ij}^s v_{\tau(k)}\). Because \(\sigma(v_i) \neq 0\), \(c_{ij}^r\) and \(c_{ij}^s\) cannot both be zero. Furthermore, \(\tau(i) \neq \tau(j) \neq \tau(k)\), so \(c_{ij}^r \neq 0\) and \(c_{ij}^s = c_{ij}^s = 0\). Thus, \(v_i \mapsto c_{ij}^r v_{\tau(i)}\).

Because we have \(c_{ij}^r - c_{ij}^s = -(c_{ij}^r - c_{ij}^s)\) and \(c_{ij}^s = 0\) = \(c_{ij}^r\), we obtain that the coefficients of the images of \(v_i\) and \(v_j\) are the same; call this coefficient \(c\).

Thus,
\(a_{ij}^{ij} v_{\tau(i)}^2 - v_{\tau(j)}^2 + (b_{ij}^{ij} v_{\tau(i)} + b_{ij}^{ij} v_{\tau(j)}) c(v_{\tau(i)} - v_{\tau(j)}) = c^2 v_{\tau(i)}^2 - v_{\tau(j)}^2\)
\(\Rightarrow b_{ij}^{ij} v_{\tau(j)} = 0\) and \(a_{ij}^{ij} = c\) for all \(i\).

What \(\sigma\) does on level one forces what happens on the levels above. We have determined thus far that there exists an automorphism \(\hat{\sigma}\) in \(k^* \times S_n\) that agrees with \(\sigma\) on the span of the \(v_j\)'s and \(v_k\)'s. Hence, by composing \(\sigma\) with the inverse of \(\hat{\sigma}\), we may assume that all \(v_j\)'s and \(v_k\)'s are fixed. Thus, using relation (4) above, we may determine that \(v_A, |A| \geq 3\) must also be fixed (levels one and two force level 3, then two and three force four, etc). (Note that setting \(\sigma(v_A) = cv_A\) does satisfy the relation, and so there are no further restrictions on \(\sigma\).)

Therefore, \(\text{Aut}(Q_n) = k^* \times S_n\). \qed
Proposition 2.2. \( \mathcal{B} = \{e(v_{A_1}, k_1) \cdots e(x_{A_i}, k_i) : l \geq 0, A_1, \ldots, A_i \subseteq \{1, \ldots, n\}, 1 \leq k_i \leq |v_{A_i}| = |A_i|, (v_{A_1}, k_1) \not= (v_{A_1+1}, k_{i+1}) \} \) is a basis for \( \mathbb{Q} \).

Proof. This follows directly from Theorem 0.1.

As in section 1, we are interested in finding graded trace functions so that we can find the multiplicities of the irreducible representations of \( \sigma \in \text{Aut}(\mathbb{Q}_n) \) acting on \( \mathbb{Q}_n \). Observe that \( \sigma \in \text{Aut}(\mathbb{Q}_n) \) permutes the elements of \( \mathcal{B} \); so, the graded trace is the number of fixed elements of \( \mathcal{B} \). These graded trace functions are also the graded dimensions of the subalgebras \( \mathbb{Q}_n \).

Recall that \( V_\sigma \) denotes the set of vertices in \( \Gamma \) fixed by \( \sigma \).

Theorem 2.3. Let \( \sigma \in \mathbb{S}_n \), \( \sigma = \sigma_1 \cdots \sigma_m \) be its cycle decomposition, and denote the length of \( \sigma_j \) by \( i_j \) \((i_j \geq 1, i_1 + \cdots + i_m = n)\). Then

\[
    T_{\sigma}(\mathbb{Q}_n, t) = \frac{1 - t}{1 - t \prod_{j=1}^{m} (2 - t^{i_j})}
\]

First of all, notice that when \( \sigma = (1) \), this yields \( H(\mathbb{Q}_n, t) \) given above.

We will prove this using equation 3 from section 1:

\[
    T_{\sigma}(\mathbb{A}(\Gamma), t) = \frac{1 - t}{1 + \sum_{v_1 \supset v_2 \supset \cdots \supset v_l = \emptyset} (-1)^l |v_1| - |v_l| + 1}.
\]

Let \( \|w\| \) be the number of \( \sigma \)-orbits of \( w \). Also, let \( O_j \) denote the non-trivial orbit of \( \sigma_j \).

Lemma 2.4. [RSW, Lemma 2] Let \( w \subseteq \{1, \ldots, n\} \) be fixed by \( \sigma \). Then

\[
    \sum_{w \supset \cdots w = \emptyset} (-1)^l = (-1)^{\|w\|+1}
\]

where all \( w_i \) are \( \sigma \)-fixed.

Proof. This proof is taken from [RSW] with slight modifications. If \( \|w\| = 1 \), then \( w = O_j \) for some \( j \). Thus, there are no fixed nonempty subsets of \( w \), and we get that both sides are equal to 1. Assume the result holds for all sets with \( \| \cdot \| < \|w\| \). Then

\[
    \sum_{w \supset \cdots w \supset \emptyset} (-1)^l = \sum_{w \supset \cdots w \supset \emptyset} \sum_{w \supset \cdots w \supset \emptyset} (-1)^l = \sum_{w \supset \cdots w \supset \emptyset} (-1)^{\|w\|}
\]

by the induction assumption. Now

\[
    \sum_{w \supset \emptyset} (-1)^{\|w\|} = \left( \sum_{w \supset \emptyset} (-1)^{\|w\|} \right) - (-1)^{|w|}, \quad \text{Say } w = O_1 \cup O_2 \cup \cdots \cup O_r.
\]

Then the number of \( w_2 \) such that \( w_2 \subseteq w \) and \( \|w_2\| = i \) is \( \binom{r}{i} \). Hence, we have

\[
    \sum_{w \supset \emptyset} (-1)^{\|w\|} = \left( \sum_{w \supset \emptyset} (-1)^{\|w\|} \right) - (-1)^{|w|} = \sum_{i=0}^{r} \binom{r}{i} (-1)^i + (-1)^{|w|+1} = (-1)^{|w|+1}, \quad \text{as desired, since the alternating sum of the binomial coefficients is zero.}
\]

\[
    \sum_{v = v_1 \supset v_2 \supset \cdots \supset v_l = w} (-1)^l = (-1)^{|v|-\|w\|+1}
\]

where all \( v_i \) are fixed by \( \sigma \).

Proof. This proof is taken from [RSW] with slight modifications. Let \( w' \) denote the complement of \( w \) in \( v \). Sets \( u \) satisfying \( v \supset u \supset w \) are in one-to-one correspondence with subsets of \( w' \) via the map \( u \mapsto u \cap w' \). Thus

\[
    \sum_{v = v_1 \supset v_2 \supset \cdots \supset v_l = w} (-1)^l = \sum_{w' = v'_1 \supset \cdots \supset v'_l = \emptyset} (-1)^{\|w'\|+1} \quad \text{by the lemma. Because } \|w'\| = \|v\| - \|w\|,
\]

this gives us what we want.
Proof. (of theorem): Let \( i_j > 1 \) for \( 1 \leq j \leq p \) and \( i_j = 1 \) for \( p < j \leq m \). To simplify notation, let \( i := i_1 + \cdots + i_p \).

By Corollary 2.5, 
\[
\sum_{v_1 \supset \cdots \supset v_\ell \supset \emptyset} (-1)^{\ell} t_{|v_1|-|v_\ell|+1} = \sum_{\{1, \ldots, n\} \supset v_1 \supset \cdots \supset v_\ell \supset \emptyset} (-1)^{|v_1|-|v_\ell|+1} t_{|v_1|-|v_\ell|+1} = \sum_{w \supset v_1 \supset \cdots \supset v_\ell \supset \emptyset} (-1)^{|w|+1} t_{|w|+1} = 
\sum_{w} 2^{m-\|w\|-1} (-1)^{|w|+1} t_{|w|+1}.
\]

The sets \( w \) are unions of orbits of a subset of \( O_1, \ldots, O_m \). Represent \( O_j \) is contained in \( w \) by \( a_j = 1 \) and \( a_j = 0 \) if not. Then the m-tuple \( \{a_1, \ldots, a_m\} \) tells us which orbits are contained in \( w \). We can then write 
\[
\sum_{w} 2^{m-\|w\|-1} (-1)^{|w|+1} t_{|w|+1} \text{ as } \sum_{a_1, \ldots, a_m \in \{0,1\}} (-1)^{\sum a_j+1} 2^{m-\sum a_j} \Sigma (a_i, i)+1.
\]

Consider when \( \sum a_j i_j = k \); in other words we are looking for the coefficient of \( t^k \). We can achieve this sum by choosing \( k \) orbits of size one \( (\sum a_j = k) \), by choosing one orbit \( O_j \) of size \( i_j \) and \( k - i_j \) of size one \( (\sum a_j = k - i_j + 1) \), etcetera, all the way up to choosing all orbits of size greater than one.

Tacking this all together we have that the coefficient of \( t^k+1 \) is:
\[
(-1)^{k+1} \left( \frac{m-p}{k} \right) 2^{m-k} + \sum_{j_1=1}^{p} (-1)^{k-(i_{j_1}-1)+1} \left( \frac{m-p}{k-i_{j_1}} \right) 2^{m-(k-i_{j_1}+1)} + \sum_{j_1 \neq j_2=1}^{p} (-1)^{k-(i_{j_1}-1)-(i_{j_2}-1)+1} \left( \frac{m-p}{k-i_{j_1}-i_{j_2}} \right) 2^{m-(k-i_{j_1}-i_{j_2}+2)} + \cdots + (-1)^{k-i+p+1} \left( \frac{m-p}{k-i} \right) 2^{m-(k-i+p)}
\]

Then,
\[
\sum_{k=0}^{n} (-1)^{k+1} \left( \frac{m-p}{k} \right) 2^{m-k} + \sum_{j_1=1}^{p} (-1)^{k-i_{j_1}+2} \left( \frac{m-p}{k-i_{j_1}} \right) 2^{m-k+i_{j_1}-1} + \cdots + (-1)^{k-i+p+1} \left( \frac{m-p}{k-i} \right) 2^{m-k+i-p}
\]

\[
= -t2^p(2-t)^{m-p} + (-1)^{2} 2^{p-1}(2-t)^{m-p} \sum_{j_1=1}^{p} t^{i_{j_1}+1} + (-1)^{3} 2^{p-2}(2-t)^{m-p} \sum_{j_1 \neq j_2=1}^{p} t^{i_{j_1}+i_{j_2}+1} + \cdots + (-1)^{m+1} t^{i+1}(2-t)^{m-p}
\]

\[
= -t(2-t)^{m-p} \left[ 2^p - 2^{p-1} \sum t^{i_{j_1}} + 2^{p-2} \sum t^{i_{j_1}+i_{j_2}} + \cdots + (1)^{m} t^{i} \right]
\]

\[
= -t(2-t)^{m-p}(2-t \cdots (2-t^p)).
\]

Therefore, we have (5). \( \square \)

Remark: Using method 1 from section 1 would necessitate writing out all of the elements, which is not practical and does not lend itself to finding a general solution.

Example 2.1. Here are the graded trace functions for \( Q_4 \):

\[
\begin{align*}
Tr_{(1)}(Q_4, t) &= \frac{1-t}{1-t(2-t)^4} & \quad Tr_{(12)}(Q_4, t) &= \frac{1-t}{1-t(2-t)(2-t)^2} \\
Tr_{(123)}(Q_4, t) &= \frac{1-t}{1-t(2-t^3)(2-t)} & \quad Tr_{(12)(34)}(Q_4, t) &= \frac{1-t}{1-t(2-t^2)^2} \\
Tr_{(1234)}(Q_4, t) &= \frac{1-t}{1-t(2-t^4)}
\end{align*}
\]
Representations of $S_n$ acting on $Q_n$

Now let us determine the multiplicities of the irreducible representations of $S_n$ acting on $Q_n$. Fix $n$. Let the graded trace generating function be denoted by $Tr_\sigma(t) = \sum_i Tr_{\sigma,i} t^i$ where $Tr_{\sigma,i} = tr_{\sigma|[Q_n]}$. Let $\phi$ be an irreducible representation of $S_n$, $m_\phi(t) = \sum_i m_{\phi,i} t^i$ where $m_{\phi,i}$ is the multiplicity of $\phi$ in $(Q_n)_{[i]}$, and $S = [\chi_{\sigma}]$ be the character table of $S_n$.

Then, if we fix the degree, $Tr_{\sigma,i} = \sum_\phi \chi_{\sigma\phi} m_{\phi,i} \rightarrow Tr_\sigma(t) = \sum_\phi \chi_{\sigma\phi} m_\phi(t)$. Write $\vec{T}_r(t) = [Tr_{\sigma_1}(t) ... Tr_{\sigma_l}(t)]^T$ and $\vec{m}(t) = [m_{\phi_1}(t) ... m_{\phi_l}(t)]^T$. Finally,

$$\vec{T}_r(t) = S^T \vec{m}(t) \implies \vec{m}(t) = (S^T)^{-1} \vec{T}_r(t).$$

Thus, to get the multiplicities of the irreducible representations, we multiply the inverse of the transpose of the character table of $S_n$ by the column vector of the trace values (the trace values of course are listed in the same order as the conjugacy classes in the character table).

Example 2.2. Irreducible Representations for $Q_4$:

The character table for $S_4$ is:

|     | (1) | (12) | (132) | (1234) | (12)(34) |
|-----|-----|------|-------|--------|----------|
| $\chi_{\text{triv}}$ | 1   | 1    | 1     | 1      | 1        |
| $\chi_{\text{sgn}}$  | 1   | -1   | 1     | -1     | 1        |
| $\chi_3$            | 2   | 0    | -1    | 0      | 2        |
| $\chi_{\text{reg}}$ | 3   | 1    | 0     | -1     | -1       |
| $\chi_{\text{sgn} \otimes \text{reg}}$ | 3   | -1   | 0     | 1      | -1       |

Let $a = Tr_{(1)}(Q_4,t) = \frac{1-t^4}{1-t^2}$, $b = Tr_{(12)}(Q_4,t) = \frac{1-t^4}{1-t^2}$, $c = Tr_{(132)}(Q_4,t) = \frac{1-t^4}{1-t^2}$, $d = Tr_{(1234)}(Q_4,t) = \frac{1-t^4}{1-t^2}$, $e = Tr_{(12)(34)}(Q_4,t) = \frac{1-t^4}{1-t^2}$.

Then, the multiplicities of the irreducible representations of $S_4$ acting on $Q_4$ as sums of the graded trace generating functions are:

$m_{\text{triv}} = \frac{1}{12}a + \frac{1}{3}b + \frac{1}{3}c + \frac{1}{3}d + \frac{1}{3}e$

$m_{\text{sgn}} = \frac{1}{12}a - \frac{1}{3}b + \frac{1}{3}c - \frac{1}{3}d + \frac{1}{3}e$

$m_3 = \frac{1}{12}a - \frac{1}{3}b + \frac{1}{3}c + \frac{1}{3}d + \frac{1}{3}e$

$m_{\text{reg}} = \frac{1}{3}a + \frac{1}{3}b - \frac{1}{3}d - \frac{1}{3}e$

$m_{\text{sgn} \otimes \text{reg}} = \frac{1}{3}a - \frac{1}{3}b + \frac{1}{3}d - \frac{1}{3}e$

The numerical values for the first few degrees are given in the following table:

|     | $\chi_{\text{triv}}$ | $\chi_{\text{sgn}}$ | $\chi_3$ | $\chi_{\text{reg}}$ | $\chi_{\text{sgn} \otimes \text{reg}}$ |
|-----|-----------------------|----------------------|----------|----------------------|-------------------------------------|
| $m_{\phi,1}$ | 4                     | 0                    | 1        | 3                    | 0                                    |
| $m_{\phi,2}$ | 26                    | 1                    | 17       | 36                   | 13                                   |
| $m_{\phi,3}$ | 219                   | 54                   | 239      | 434                  | 273                                  |

Unlike in the $A(\Gamma_D)$ case, we cannot write down one table giving all of the values in terms of the graded trace functions. However, we can give them in terms of the Frobenius formula.

Recall that Frobenius’ formula says $\chi_\lambda(C_i) = |\Delta(x)| \prod_j P_j(x)^{i_j}$ where $C_i$ is a representative from the conjugacy class $i$, $\lambda$ is a partition of $n$ (representing an irreducible representation), $\Delta(x) = \prod_{i<j} (x_i - x_j)$, $P_j(x) = x_1^j + ... + x_k^j$ where $k$ is at least the number of rows in $\lambda$, $i_j$ is the number of $j$-cycles in the
conjugacy class \(i\), \(l_1 = \lambda_1 + k - 1\), \(l_2 = \lambda_2 + k - 2\), ..., \(l_k = \lambda_k\), and \((l_1, \ldots, l_k)\) means take the coefficient of \(x_1^{l_1}x_2^{l_2} \ldots x_k^{l_k}\).

**Proposition 2.6.** Let \(r\) denote the degree and \(\lambda\) the irreducible representation. Then

\[
m_{\lambda, r} = \left[ \frac{1}{n!} \sum_{j \text{ partition of } n} \chi_{\lambda_1}(C_j) |C(j)| Tr(j) \right]_{(l_1, \ldots, l_k, r)}
\]

**Proof.** Let \(S\) be the character table of \(S_n\). \(S\) is an orthogonal matrix, so \(S^T S = D\), a diagonal matrix. This implies that \(S^{-1} = D^{-1} S^T\). Write \(S\) in terms of the Frobenius formula, so

\[
S = \begin{bmatrix}
\chi_{\lambda_1}(C_1) & \cdots & \chi_{\lambda_1}(C_k) \\
\vdots & \ddots & \vdots \\
\chi_{\lambda_k}(C_1) & \cdots & \chi_{\lambda_k}(C_k)
\end{bmatrix} \Rightarrow D = \begin{bmatrix}
\sum_i \chi_{\lambda_i}(C_1)^2 & 0 & \cdots & 0 \\
0 & \ddots & \vdots \\
0 & \ddots & 0 \\
0 & \cdots & 0 & \sum_i \chi_{\lambda_i}(C_k)^2
\end{bmatrix}
\]

\[
S^{-1} = \begin{bmatrix}
\chi_{\lambda_1}(C_1)/\sum_i \chi_{\lambda_i}(C_1)^2 & \cdots & \chi_{\lambda_1}(C_1)/\sum_i \chi_{\lambda_i}(C_1)^2 \\
\vdots & \ddots & \vdots \\
\chi_{\lambda_k}(C_k)/\sum_i \chi_{\lambda_i}(C_k)^2 & \cdots & \chi_{\lambda_k}(C_k)/\sum_i \chi_{\lambda_i}(C_k)^2
\end{bmatrix}
\]

Because \(\tilde{m}(t)S = T_r(t)\), \(\tilde{m}(t) = T_r(t)S^{-1}\); and so,

\[
m_{\lambda_i}(t) = \frac{\chi_{\lambda_i}(C_1) Tr_{\sigma_1} + \cdots + \chi_{\lambda_i}(C_k) Tr_{\sigma_k}}{\sum_i \chi_{\lambda_i}(C_i)^2}
\]

However, \(\sum_i \chi_{\lambda_i}(C_j)^2 = [S_n : C(j)] = n!/|C(j)|\), where \(|C(j)|\) is the size of the conjugacy class of partition \(j\). Thus,

\[
m_{\lambda_i, r} = \left[ \frac{1}{n!} \sum_{j \text{ partition of } n} \chi_{\lambda_i}(C_j) |C(j)| Tr(j) \right]_{(l_1, \ldots, l_k, r)}
\]

\[
Tr_{\sigma}(Q_n^1, t)
\]

We will now find the graded traces and representations of \(Q_n^1\). In this case, these are not graded dimensions. This definition of dual corresponds to \(Q_n^{id}\) as described in section 0.

\(Q_n^1\) is shown in [GGRSW, §6] as having the following basis:

Let \(B = \{b_1, \ldots, b_k\} \subseteq A \subseteq \{1, \ldots, n\}\) with \(b_1 > \ldots > b_k\). Let \(S(A : B) = s(A)s(A\setminus b_1) \cdots s(A\setminus b_1 \setminus \ldots \setminus b_k)\). Then \(S = \{S(A : B) : \min A \not\in B\} \cup \{\emptyset\}\) is a basis for \(Q_n^1\). The relations in the associated graded dual are:

1. \(s(A) \sum_{\sigma \in A} s(A\setminus \alpha), |A| \geq 2\)
2. \(s(A)s(A\setminus i)s(A\setminus i\setminus j) = -s(A)s(A\setminus j)s(A\setminus i\setminus j)\)
3. \(s(A)s(B) = 0\) if \(B \not\subseteq A\) or \(|B| \neq |A| - 1\).

As opposed to the case of \(Q_n\), \(\sigma\) does not permute the basis elements of \(Q_n^1\). Thus, it is not enough to count fixed basis elements to determine the trace. For each \(S(A : B) \in S\), we must write \(\sigma S(A : B) = S(\sigma A : \sigma B)\) as a linear combination of elements in \(S\). Write this as \(\sigma S(A : B) = \sum_{D \subseteq C \subseteq \{1, \ldots, n\}} a_{ABCD} S(C : D)\). Then the trace of \(\sigma\) on \(S(A : B)\), denoted by \(Tr_{\sigma} S(A : B)\), is \(a_{ABAB}\).
We are going to get three possible values for a basis element’s contribution to the trace: $-1, 0,$ or $1$. If $\sigma$ permutes the elements of $B$, we put them back into decreasing order using relation (2); this multiplies the element by $1$ or $-1$. Also, if $\min A \in \sigma B,$ we also use relation (1) to write $\sigma S(A : B)$ as a sum of basis elements where $\min A$ is not removed (this will include $S(A : B)$). This multiplies the element by $-1$. Finally, it is possible that $\sigma S(A : B)$ does not involve $S(A : B)$, and so the trace is zero.

Let us introduce some notation. For $\sigma \in S_n$, write $\sigma = \sigma_1 \ldots \sigma_m$, a product of disjoint cycles. If $B$ is $\sigma$-invariant, then let $l_B(\sigma)$ be the length of $\sigma$ restricted to the set $B$; i.e. if we write $\sigma$ as a product of transpositions, the length is how many of these transpositions contain an element of $B$. If there exists $c \in B$ such that $\sigma(c) = \min A$, then define $\sigma' := (c \min A) \sigma$. Denote the orbits of $\sigma$ by $\{O_1, O_2, \ldots, O_m\}$ and put an ordering on the orbits given by $O_i < O_j$ if the minimal element of $O_i$ is less than that of $O_j$. Say $O_1 < O_2 < \ldots < O_m$. Let $i_j$ be the size of $O_j$ (equals the length of $\sigma_j$). Let $r_1 < \ldots < r_l$.

**Proposition 2.7.**

$$Tr_\sigma S(A : B) = \begin{cases} 0 & \text{if } \sigma A \neq A \text{ or } \min A \in \sigma B \text{ and } \sigma(B \setminus b) \neq B \setminus \min A \text{ for some } b \in B \\ (-1)^{l_B(\sigma)} & \text{or } \min A \notin \sigma B \text{ and } \sigma(B) \neq B \\ (-1)^{l_B(\sigma')} + 1 & \min A \in \sigma B \text{ and for some } b \in B, \sigma(B \setminus b) = B \setminus \min A \end{cases}$$

**Proof.** We have three cases for how $\sigma$ acts on $S(A : B)$. First of all, if $\sigma A \neq A$, $Tr_\sigma S(A : B) = 0$ since it can be reduced to a basis element with the first term being $\sigma A$ (and so not the same element; hence, the trace is zero). Thus, assume $\sigma A = A$. Then $\sigma S(A : B) = S(A : \sigma B)$.

Case 1: If $\min A \in \sigma B$ and $\sigma(B \setminus b) \neq B \setminus \min A$ for some $b \in B$ or $\min A \notin \sigma B$ and $\sigma(B) \neq B$, then $Tr_\sigma S(A : B) = 0$.

The reason for this is that there are no relations to switch an element by more than one number. If these conditions hold, $\sigma S(A : B)$ will be sent to a different basis element.

Case 2: $\min A \notin \sigma B$, $\sigma B = B$.

Using relations of type (2), you can see that we multiply by $-1$ every time that we switch the order in which we remove an element from $A$. Since $\sigma S(A : B) = S(A : \sigma B)$ (recall $B$ is an ordered set), we will introduce $-1$ for every time we have to switch two elements of $B$ to get $\sigma B$ back into decreasing order. This is the length of $\sigma$ with respect to $B$. Therefore, the trace is $(-1)^{l_B(\sigma)}$.

Case 3: $\min A \in \sigma B$ and for some $b \in B$, $\sigma(B \setminus b) = B \setminus \min A$.

Again we are going to multiply $S(A : B)$ by $-1$ for every switch we have to make to get $\sigma(B)$ back into decreasing order. Also, since after applying $\sigma$ we will have removed $\min A$, we need to use relation (1) so that $\min A$ will be in $\sigma S(A : B).$ This will introduce $-1$ times a sum of basis elements. Now, all but one of these will cancel out due to the type (3) relations. (Besides, for trace we are only concerned with the multiple of our original basis element.) Thus, we have $-1$ for switching $\min A$ to $b$ and $-1$ each time we switch two terms to put them back in order. Since $\min A \in \sigma B$, $\exists c \in B \text{ s.t. } \sigma(c) = \min A$. Let $\sigma' = (c \min A) \sigma$. Then $\sigma'(B) = B$, and thus we are in case 2. So, $\sigma S(A : B) = (c \min A) \sigma S(A : B) = (c \min A) S(A : \sigma'B) = (-1)^{l_B(\sigma')}(c \min A) S(A : B) \sim (-1)^{l_B(\sigma')} + 1 S(A : B)$. Therefore, the trace is $(-1)^{l_B(\sigma')} + 1$. 

Now that we know what the trace is on each basis element, we want to find $Tr_\sigma (Q^i_n, t)$. 

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Theorem 2.8.

\[ Tr_\sigma(Q_n^t, t) = \frac{1 + t \prod_{k=1}^{m}(2 - (-t)^k)}{1 + t} \]

Proof. Case 2: \( \min A \notin \sigma B, \sigma B = B \)

Given \( B = \mathcal{O}_{r_1} \cup \ldots \cup \mathcal{O}_{r_t}, r_1 \neq 1 \), there are \( 2^{m-r_1-l} - 1 \) choices for \( A \). The reasoning for this is as follows. Because \( B \subseteq A \), \( A \) must contain all of \( \mathcal{O}_{r_1}, \ldots, \mathcal{O}_{r_t} \). Also, since \( \min A \notin \sigma B \), \( A \) must contain at least one \( \mathcal{O}_{r_0} \) such that \( r_0 < r_1 \).

Let us deal with this in two parts; first picking which orbits are in \( A \) greater than \( \mathcal{O}_{r_1} \) and second picking those which are less. There are \( 2^{m-r_1-l} - 1 \) choices for the former; we are picking whether each orbit not already chosen from \( \mathcal{O}_{r_1+1}, \ldots, \mathcal{O}_m \) is or is not in \( A \).

For the second part, we must have at least one orbit less than \( \mathcal{O}_{r_1} \) (as stated above). Thus, it is the number of ways of choosing a nonempty subset of \( \{ \mathcal{O}_1, \ldots, \mathcal{O}_{r_1-1} \} \), which is \( 2^{r_1-1} - 1 \).

Thus, in all, we have \( 2^{m-r_1-l} - 1 \) choices for \( A \) because \( \sigma B_1 \) is or is not in \( A \).

The contribution for all \( S(A : B) \) given \( B \) towards \( Tr_\sigma \) is \( (-1)^{r_1 + \ldots + r_t} (2^{-l} - 2^{m-r_1-l+1})^{r_1 + \ldots + r_t + 1} \).

We will need to sum this over all \( B \) and multiply by \( 1 + t \). This gives us

\[
\sum_{2 \leq r_1 < \ldots < r_t \leq m} (-1)^{r_t + \ldots + r_1 - l} (2^{-l} - 2^{m-r_1-l+1})^{r_t + \ldots + r_1 + 1} + \sum_{2 \leq r_2 < \ldots < r_t \leq m} (-1)^{r_t + \ldots + r_2 - l} (2^{-l} - 2^{m-r_1-l+1})^{r_t + \ldots + r_2 + 2}.
\]

Let us label this by \( c_1 + c_2 \) for ease of referencing later.

Case 3: \( \min A \in \sigma B \) and for some \( b \in B, \sigma B \setminus b = B \setminus \min A \)

Fix \( B, \) say \( B \subseteq \mathcal{O}_{r_1} \cup \ldots \cup \mathcal{O}_{r_t} \). \( B \) must contain all elements in \( \{ \mathcal{O}_{r_1}, \ldots, \mathcal{O}_{r_t} \} \) since \( B \) and \( \sigma B \) can only differ by one element; and, that must occur in \( \mathcal{O}_{r_1} \) because \( \min A \) must be in \( \mathcal{O}_{r_1} \) and cannot be in \( B \). Say \( \sigma r_1 = (c_{i_1} \ldots c_1 \min A) \). Then \( B \) must also contain consecutive elements \( \{c_1, \ldots, c_j\}, 1 \leq j \leq r_1 - 1 \), in \( \mathcal{O}_{r_1} \). If this were not the case, \( B \) and \( \sigma B \) would differ by more than one element \( (c_j \notin \sigma B) \).

Consider \( \sigma' = (c_j \min A) (\min A c_{i_1-1} \ldots c_1) \sigma r_2 \ldots \sigma r_t = (c_j \min A) (\min A c_1) \ldots (\min A c_{i_1-1}) \sigma r_2 \ldots \sigma r_t \). Then \( l_B(\sigma') = \sum_{k=2}^{l} (i_k - 1) + j + 1 \). Thus, by proposition 2.7, the trace of \( \sigma \) acting on \( S(A : B) \) is

\[
(-1)^{j+i_2+\ldots+i_{r_t}}(l-1)+2 = (-1)^{j+i_2+\ldots+i_{r_t}-l+1}.
\]

Given \( B, \) \( A \) must contain \( \{ \mathcal{O}_{r_1}, \ldots, \mathcal{O}_{r_t} \} \). Also, it may contain orbits greater than \( \mathcal{O}_{r_1} \), but not smaller than \( \min A \in \mathcal{O}_{r_1} \). Thus, there are \( 2^{m-r_1-l} - 1 \) choices for \( A \). Now there are \( i_1 - 1 \) subsets \( B \subseteq \mathcal{O}_{r_1} \cup \ldots \cup \mathcal{O}_{r_t}; \) \( j \) can be between 1 and \( i_1 - 1 \). Putting this all together, given \( \{ \mathcal{O}_{r_1}, \ldots, \mathcal{O}_{r_t} \}, S(A : B) \) contributes a total of \( 2^{m-r_1-l-1} \sum_{j=1}^{i_1-1} (-1)^{j+i_2+\ldots+i_{r_t}-l+1} t^{j+i_2+\ldots+i_{r_t}+1} \) towards the graded trace function.

We will need to sum this over all \( \{ \mathcal{O}_{r_1}, \ldots, \mathcal{O}_{r_t} \} \) and multiply by \( 1 + t \). This gives us

\[
\sum_{1 \leq r_1 < \ldots < r_t \leq m} 2^{m-r_1-l+1} (-1)^{i_2+\ldots+i_{r_t}-l} t^{i_2+\ldots+i_{r_t}+2} + \sum_{1 \leq r_2 < \ldots < r_t \leq m} 2^{m-r_1-l+1} (-1)^{i_2+\ldots+i_{r_t}-l} t^{i_2+\ldots+i_{r_t}+1}
\]

(notice that the sum over \( j \) is telescoping.) Let us label this by \( c_3 + c_4 \) for ease of referencing later.

Case 4: \( B = \emptyset \).
Because $\sigma A = A$, $Tr_\sigma S(A : B) = 1$. Thus we have a contribution of $1 + (2^m - 1)t$ towards the graded trace.

Multiplying by $1 + t$ gives us $1 + 2^m t + (2^m - 1)$, which we will label by $1 + 2^m t + c_5$.

If we sum over all possibilities for the traces and multiply by $1 + t$, we have that $Tr_\sigma(Q_n^1, t) = 1 + 2^m t + c_5 + c_1 + c_2 + c_3 + c_4$

Consider the following pieces of the equation.

$c_5 + c_3|_{t=1}$:
\[
(2^m - 1)t^2 + \sum_{1 \leq r_1 \leq m} 2^{m-r_1}(-1)^{r_1}t^2 = t^2[(2^m - 1) - \sum_{r_1=1}^{m} 2^{m-r_1}] = t^2(2^m - 1) - (2^{m-1} + 1) = 0.
\]

$c_2|_{r_1=r_2} + c_3|_{t>1}$:
\[
(2^{m-l+1} - 2^{m-r_2-l+1} + 1) (-1)^{r_2 + r_3} t^{r_2 + r_3 + r_1 + 2} + \sum_{r_1=1}^{r_2-1} 2^{m-r_1-l+1}(-1)^{r_2 + r_3 - r_1-l} t^{r_2 + r_3 + r_1 + 2} = (-1)^{r_2 + r_3 - r_1 - l} t^{r_2 + r_3 + r_1 + 2} [2^{m-l+1} - 2^{m-r_2-l+2} - \sum_{r_1=0}^{r_2-2} 2^{m-r_1-l}] = (-1)^{r_2 + r_3 - r_1 - l} t^{r_2 + r_3 + r_1 + 2} [2^{m-l+1} - 2^{m-r_2-l+2} - (\sum_{r_1=0}^{m-l} 2^r_1 - \sum_{r_1=0}^{m-r_2-l+1} 2^r_1)] = (-1)^{r_2 + r_3 - r_1 - l} t^{r_2 + r_3 + r_1 + 2} [2^{m-l+1} - 2^{m-r_2-l+2} - (2^{m-l+1} - 1 + (2^{m-(r_2-1)-l+1} - 1)] = 0.
\]

$c_1 + c_4|_{r_1 \neq 1}$:
\[
\sum_{1 \leq r_1 \leq m, 1 \leq l \leq m-1} (-1)^{r_1 + r_2 - l} t^{r_1 + r_2 + r_3 + r_1 + 1} [2^{m-l} - 2^{m-r_1-l+1} - 2^{m-r_1-l+1}] = \sum_{1 \leq r_1 \leq m, 1 \leq l \leq m-1} (-1)^{r_1 + r_2 - l} t^{r_1 + r_2 + r_3 + r_1 + 1} [2^{m-l} - 2^{m-r_1-l+1} - 2^{m-r_1-l+1}].
\]

$c_4|_{r_1=1}$:
\[
\sum_{1 \leq l \leq m} (-1)^{r_1 + r_2 - l} t^{r_1 + r_2 + r_3 + r_1 + 1} [2^{m-l} - 2^{m-r_1-l+1} - 2^{m-r_1-l+1}].
\]

Putting it all together:
\[
1 + t[2^m + \sum_{1 \leq r_1 \leq m, 1 \leq l \leq m} (-1)^{l} t^{2^{m-l} - t^{r_1 + r_2}}] = 1 + t \prod_{k=1}^{m} (2 - (-t)^k)
\]

Therefore,
\[
Tr_\sigma(Q_n^1, t) = \frac{1 + t \prod_{k=1}^{m} (2 - (-t)^k)}{1 + t}
\]

as desired.

\[\square\]

**Example 2.3.** Here are the graded trace functions for $Q_n^1$:

\[
Tr_{(1)}(Q_n^1, t) = \frac{1 + t(2 + t)^4}{1 + t} = 1 + 15t + 17t^2 + 7t^3 + t^4
\]
\[
Tr_{(12)}(Q_n^1, t) = \frac{1 + t(2 - t^2)(2 + t)^2}{1 + t} = 1 + 7t + t^2 - 3t^3 - t^4
\]
\[
\begin{align*}
Tr_{(123)}(Q_4^i, t) &= \frac{1 + t(2 + t^3)(2 + t)}{1 + t} = 1 + 3t - t^2 + t^4 \\
Tr_{(12)(34)}(Q_4^i, t) &= \frac{1 + t(2 - t^2)^2}{1 + t} = 1 + 3t - 3t^2 - t^3 + t^4 \\
Tr_{(1234)}(Q_4^i, t) &= \frac{1 + t(2 - t^4)}{1 + t} = 1 + t - t^2 + t^3 - t^4
\end{align*}
\]

Representations of \(S_n\) acting on \(Q_n^i\)

We do the same thing here as in the case of the algebra. Everything above (\(\tilde{m}(t) = (S^T)^{-1}T^r(t)\) and Proposition 2.6) is still true if you replace \(Tr_\sigma(Q_n, t)\) with \(Tr_\sigma(Q_n^i, t)\).

Example 2.4. Irreducible Representations of \(S_4\) acting on \(Q_4^i\):

There are only four degrees in the dual, so we can give all of the multiplicities in the following table:

| \(m_{\phi,1}\) | \(m_{\phi,2}\) | \(m_{\phi,3}\) | \(m_{\phi,4}\) |
|---|---|---|---|
| 4 | 0 | 1 | 3 |
| 0 | 0 | 1 | 3 |
| 0 | 1 | 0 | 0 |
| 0 | 1 | 0 | 0 |

Notice that all of the representations are realized in at least one grading, but not in every. Also, each representation occurs with a much smaller multiplicity than in the algebra.

Koszulity property of \(Q_n\)

One property of Koszul algebras is that \(H(A, t) \ast H(A^1, -t) = 1\). You can easily check that this is also true for the graded trace functions that we found for \(Q_n\) and \(Q_n^i\). Namely, \(Tr_\sigma(Q_n, t) \ast Tr_\sigma(Q_n^i, -t) = 1\) where \(\sigma\) is an element in the automorphism group of the algebra.

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