PREScribing CAPACITARY CURVATURE MEASURES ON
PLANAR CONVEX DOMAINS

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Abstract. For \( p \in (1, 2] \) and a bounded, convex, nonempty, open set \( \Omega \subset \mathbb{R}^2 \) let \( \mu_p(\bar{\Omega}, \cdot) \) be the \( p \)-capacitary curvature measure (generated by the closure \( \bar{\Omega} \) of \( \Omega \)) on the unit circle \( \mathbb{S}^1 \). This paper shows that such a problem of prescribing \( \mu_p \) on a planar convex domain: “Given a finite, nonnegative, Borel measure \( \mu \) on \( \mathbb{S}^1 \), find a bounded, convex, nonempty, open set \( \Omega \subset \mathbb{R}^2 \) such that \( d\mu_p(\bar{\Omega}, \cdot) = d\mu(\cdot) \)” is solvable if and only if \( \mu \) has centroid at the origin and its support \( \text{supp}(\mu) \) does not comprise any pair of antipodal points. And, the solution is unique up to translation. Moreover, if \( d\mu_p(\bar{\Omega}, \cdot) = \psi(\cdot) d\ell(\cdot) \) with \( \psi \in C^{k, \alpha} \) and \( d\ell \) being the standard arc-length element on \( \mathbb{S}^1 \), then \( \partial\Omega \) is of \( C^{k+2, \alpha} \).

1. Statement of Theorem 1.1

Continuing from [34] and [22, 23, 14, 35], we prove

Theorem 1.1. Let \( (p, k, \alpha) \in (1, 2] \times \mathbb{N} \times (0, 1) \) and \( \mu \) be a finite nonnegative Borel measure on the unit circle \( \mathbb{S}^1 \) of \( \mathbb{R}^2 \).

(i) Existence - there is a bounded, convex, nonempty, open subset \( \Omega \) of \( \mathbb{R}^2 \) such that \( d\mu_p(\bar{\Omega}, \cdot) = d\mu(\cdot) \) if and only if \( \mu \) has centroid at the origin and its support \( \text{supp}(\mu) \) does not comprise any pair of antipodal points.

(ii) Uniqueness - the domain \( \Omega \) in (i) is unique up to translation.

(iii) Regularity - if \( d\mu_p(\bar{\Omega}, \cdot) = \psi(\cdot) d\ell(\cdot) \), \( d\ell \) is the standard arc-length element on \( \mathbb{S}^1 \), and \( 0 < \psi \in C^{k, \alpha}(\mathbb{S}^1) \), i.e., its \( k \)-th derivative \( \psi^{(k)} \) is \( \alpha \)-Hölder continuous on \( \mathbb{S}^1 \), then the boundary \( \partial\Omega \) of \( \Omega \) is of \( C^{k+2, \alpha} \).

In the above and below, \( \mu_p(\bar{\Omega}, \cdot) \) is the \( p \)-capacitary curvature measure on \( \mathbb{S}^1 \) - more precisely - if \( u \) is the \( p \)-equilibrium potential \( u_{\Omega} \) of \( \bar{\Omega} \) - the closure of \( \Omega \) (cf. [25, 15, 13]), i.e., the unique solution \( u = u_{\bar{\Omega}} \) to the boundary value problem (for a model partial differential equation in geometric potential

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theory over $\mathbb{R}^2$; see e.g. [1, 2, 3]:

\[
\begin{cases}
\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u) = 0 & \text{in } \mathbb{R}^2 \setminus \Omega; \\
u = 0 & \text{on } \partial \Omega; \\
limit_{|x| \to \infty} u(x) = 1,
\end{cases}
\]

or

\[
\begin{cases}
\Delta_{p=2} u = \text{div}(\nabla u) = 0 & \text{in } \mathbb{R}^2 \setminus \hat{\Omega}; \\
u = 0 & \text{on } \partial \Omega; \\
0 < \liminf_{|x| \to \infty} \left(\frac{u(x)}{\log |x|}\right) \leq \limsup_{|x| \to \infty} \left(\frac{u(x)}{\log |x|}\right) < \infty,
\end{cases}
\]

then

\[
\mu_p(\hat{\Omega}, E) = \int_{g^{-1}(E)} |\nabla u|^p \, dH^1 = \int_{g^{-1}(E)} |\nabla u_\Omega|^p \, dH^1 \quad \forall \text{ Borel } E \subset S^1,
\]

where $dH^1$ is the standard 1-dimensional Hausdorff measure on $\partial \Omega$, $g^{-1} : S^1 \to \partial \Omega$ is the inverse of the Gauss map $g : \partial \Omega \to S^1$ (which is defined as the outer unit normal vector at $\partial \Omega$), and the non-tangential limit of $\nabla v = \nabla u_\Omega$ at each point of $\partial \Omega$ exists $H^1$-almost everywhere with $|\nabla u| = |\nabla u_\Omega| \in L^p(\partial \Omega, dH^1)$ (cf. [26, 27, 16]), and hence

\[
d\mu_p(\hat{\Omega}, \cdot) = g_*(|\nabla u|^p \, dH^1)(\cdot) = g_*(|\nabla u_\Omega|^p \, dH^1)(\cdot) \quad \text{on } S^1.
\]

Here it should be pointed out that not only the if-part of Theorem 1.1(i) implies [14, Theorem 1.2] under $1 < p < 2 = n$ and [23, Corollary 6.6] under $p = 2 = n$ due to the fact that $\text{supp}(\mu)$ comprising no any pair of antipodal points amounts to $\mu$ being unsupported on any equator (the intersection of $S^1$ with any line passing through the origin) but also Theorems 1.1(ii)&(iii) under $p \in (1, 2)$ have been established in [14, Theorems 1.2&1.4]. Our essential contribution to this direction is an establishment of Theorem 1.1(i) and the case $p = 2$ of Theorems 1.1(ii)&(iii).

Needless to say, Theorem 1.1 is not unimportant in that it is nonlinear-potential-theoretic generalization of the classical Minkowski problem in $\mathbb{R}^2$ concerning the existence, uniqueness and regularity of a planar convex domain with the prescribed curve measure

\[
d\mu_{cm} = g_*(dH^1) \quad \text{on } S^1
\]

defined by

\[
\mu_{cm}(E) = \int_{g^{-1}(E)} dH^1 = H^1(g^{-1}(E)) \quad \forall \text{ Borel } E \subset S^1.
\]

See e.g. [12, 21, 33, 24] and their references for an extensive discussion on this subject.
2. Preparational Material

Two-fold preparation for validating Theorem [14] is presented through this intermediate section.

On the one hand, it is necessary to recall three fundamental properties on the variational 1 < p < 2 capacity \( \text{pcap}(\bar{\Omega}) \) and the logarithmic capacity (or conformal radius or transfinite diameter) \( 2\text{cap}(\bar{\Omega}) \) of a compact, convex, nonempty set \( \bar{\Omega} \subset \mathbb{R}^2 \) (cf. [25, 15, 13, 23, 32]) determined by:

\[
\text{pcap}(\bar{\Omega}) = \lim_{|x| \to \infty} \left\{ 2\pi \left( \frac{2-p}{p-1} \right)^{p-1} |x|^{2-p} (1 - u_\Omega(x))^{p-1} \text{ as } p \in (1, 2); \\
\exp \left( 2\pi \left( \frac{2}{p-1} \right)^{p-1} |x| - u_\Omega(x) \right) \text{ as } p = 2,
\]

where \( d'H^2 \) stands for the two-dimensional Hausdorff measure on \( \mathbb{R}^2 \) and \( u_\Omega \) is the solution of either (eq \( 1 \)) or (eq \( 2 \)) and the following isocapacitary/isodiametric inequalities (cf. [14, 4, 32] and their relevant references):

\[
\left( \frac{A(\bar{\Omega})}{\pi} \right)^{\frac{1}{3}} \leq \left( \frac{\text{pcap}(\bar{\Omega})}{2\pi \left( \frac{2}{p-1} \right)^{p-1}} \right)^{\frac{1}{p-1}} \leq 2^{-1} \text{diam}(\bar{\Omega}) \text{ as } p \in (1, 2);
\]

\[2^{-1} \text{diam}(\bar{\Omega}) \leq 2\text{pcap}(\bar{\Omega}) \leq \text{diam}(\bar{\Omega}) \text{ as } p = 2.
\]

Thirdly, if \( h_\Omega(x) = \sup_{y \in \Omega} x \cdot y \) stands for the support function of \( \bar{\Omega} \), then (\( \star \)) can be formulated in the following way (cf. [14, Theorem 1.1]) for \( p \in (1, 2) \) and [35, Theorem 3.1] for \( p = 2 \):

\[
\int_{\partial \Omega} |\nabla u_\Omega(x)|^p x \cdot g(x) \ d'H^1(x) = \left\{ \begin{array}{ll}
\left( \frac{2-p}{p-1} \right)^{p-1} \text{pcap}(\bar{\Omega}) & \text{as } p \in (1, 2); \\
2\pi & \text{as } p = 2.
\end{array} \right.
\]

On the other hand, three key lemmas and their arguments are needed.

**Lemma 2.1.** Let \( p \in (1, 2) \) and \( \Omega \subset \mathbb{R}^2 \) be a bounded, convex, open set with non-empty interior. If \( u_\Omega \) is the p-equilibrium potential of \( \Omega \) and there is an origin-centered open disk \( D(o, r) \) with radius \( r > 0 \) such that \( \Omega \subset D(o, r) \), then there exists a constant \( c > 0 \) depending only on \( r \) such that \( |\nabla u_\Omega| \geq c \) almost everywhere on \( \partial \Omega \) with respect to \( d'H^1 \).
Proof. This follows directly from the case \( n = 2 \) of both \cite[Lemma 2.18]{14} (for \( p \in (1, 2) \)) and \cite[Theorem 3.2]{35} (for \( p = 2 \)). \qed

**Lemma 2.2.** For \( p \in (1, 2] \) and integer \( m \geq 3 \), a family \( \{\zeta_j\}_{j=1}^m \subset S^1 \), and any point \( p \in \mathbb{R}^m \) with all nonnegative components \( p_1, \ldots, p_m \) let

\[
\Omega(p) = \{ x \in \mathbb{R}^2 : x \cdot \zeta_j \leq p_j \ \forall \ j = 1, \ldots, m \} ; \\
\mathcal{M} = \{ p = (p_1, \ldots, p_m) \in \mathbb{R}^m : \text{pcap}(\Omega(p)) \geq 1 & p_j \geq 0 \ \forall \ j = 1, \ldots, m \} .
\]

Given a sequence of \( m \) positive numbers \( \{c_j\}_{j=1}^m \), set \( \Sigma(p) = \sum_{j=1}^m c_j p_j \). If \( \{\zeta_j\}_{j=1}^m \) obeys the following three conditions:

(i) for any \( \theta \in S^1 \) there is \( j \in \{1, \ldots, m\} \) such that \( |\theta \cdot \zeta_j| > 0 \);
(ii) \( |\zeta_j + \zeta_k| > 0 \ \forall \ j, k \in \{1, \ldots, m\} \);
(iii) \( \sum_{j=1}^m c_j \zeta_j = 0 \).

Then there exists a point \( p^* \in \mathcal{M} \) such that:

(iv) \( \inf_{p \in \mathcal{M}} \Sigma(p) = \Sigma(p^*) > 0 \);
(v) \( \Omega(p^*) \) is a polygon with \( \{F_j\}_{j=1}^m \) and \( \{\zeta_j\}_{j=1}^m \) as the only edges and outer unit normal vectors respectively;
(vi) the \( p \)-equilibrium potential \( u_{\Omega(p^*)} \) of \( \Omega(p^*) \) obeys

\[
c_{1 \leq j \leq m} = \tau_p^{-1} \Sigma(p^*) \int_{F_j} |\nabla u_{\Omega(p^*)}|^p d\mathcal{H}^1
\]

with

\[
\tau_p = \begin{cases} 
(2 - p)(p - 1)^{-1} & \text{as } p \in (1, 2) ; \\
2\pi & \text{as } p = 2 .
\end{cases}
\]

**Proof.** First of all, the argument for \cite[Theorem 5.4]{22} is modified to reveal that \( \Omega(p) \) is closed and bounded thanks to (i) which derives

\[
|x| \leq \sup_{j \in \{1, \ldots, m\}, \theta \in S^1} p_j |\theta \cdot \zeta_j|^{-1} \ \forall \ x \in \Omega(p) .
\]

Next, since \( \{c_j\}_{j=1}^m \) is fixed and

\[
\begin{cases} 
\Sigma(p) \leq \left( \sum_{j=1}^m c_j^2 \right)^{\frac{1}{2}} \left( \sum_{j=1}^m p_j^2 \right)^{\frac{1}{2}} ; \\
\inf_{p \in \mathcal{M}} \Sigma(p) < \infty ,
\end{cases}
\]

each minimizing sequence for \( \inf_{p \in \mathcal{M}} \Sigma(p) \) is bounded, and consequently, we can select a subsequence from the minimizing sequence that converges to \( p^* \). Now from the continuity of \( \text{pcap}(-) \) under the Hausdorff distance \( d_H(\cdot, \cdot) \) it follows that \( p^* \in \mathcal{M} \) is a minimizer. Of course,

\[
\begin{cases} 
\text{pcap}(\Omega(p^*)) = 1 ; \\
\inf_{p \in \mathcal{M}} \Sigma(p) = \Sigma(p^*) \ \forall \ p \in (1, 2] .
\end{cases}
\]
If \( \Sigma(p^*) = 0 \), then \( p^* \) is the origin, and hence condition (i) implies that \( \Omega(p^*) \) consists only of the origin, thereby yielding a contradiction
\[
1 = \text{pcap}(\Omega(p^*)) = 0.
\]
So, (iv) holds.

Furthermore, if the interior \( (\Omega(p^*))^\circ \) of \( \Omega(p^*) \) is empty, then (ii) can be used to deduce that \( \Omega(p^*) \) is contained in a compact convex set \( K \) with the Hausdorff dimension \( \dim_H(K) \leq 1. \)

- If \( \dim_H(K) = 0 \) then \( \Omega(p^*) \) comprises one point and hence
\[
0 = \text{pcap}(\Omega(p^*)) = 1,
\]
a contradiction.
- If \( \dim_H(K) = 1 \) then \( \Omega(p^*) \) reduces to a segment and hence there exists \( \zeta_j + \zeta_k = 0 \) for some \( j, k \in \{1, \ldots, m\} \) which is against the hypothesis (ii).

Thus, \( \Omega(p^*) \) has a non-empty interior, and consequently (v) holds.

Finally, in order to check (vi), observe that \( p^* \) is not unique. Given \( x_0 \in \mathbb{R}^2 \), if \( p \in M \) then an application of (iii) implies that
\[
q = \{p_j + x_0 \cdot \zeta_j\}_{j=1}^m
\]

enjoys
\[
\begin{cases}
\Omega(q) = x_0 + \Omega(p); \\
\Sigma(q) = \Sigma(p).
\end{cases}
\]

Due to the fact that \( \Omega(p^*) \) has non-empty interior, the origin may be translated to the interior of \( \Omega(p^*) \) so that each component \( p_j \) is positive. Let \( P \) be the collection of those vectors \( p = (p_1, \ldots, p_m) \) with
\[
\begin{cases}
p_j \geq 0; \\
\Sigma(tp + (1-t)p^*) = \Sigma(p^*) \quad \forall \quad t \in [0, 1].
\end{cases}
\]

Then
\[
\begin{cases}
p \in P; \\
t\Omega(p) + (1-t)\Omega(p^*) \subseteq \Omega(tp + (1-t)p^*) \quad \forall \quad t \in [0, 1],
\end{cases}
\]

plus \([14]\) Theorem 5.2] (for \( p = (1, 2) \)) and \([23]\) \((6.4)'\) or \([35]\) Theorem 4.4] (for \( p = 2 \)), ensures a constant \( w_j > 0 \) such that
\[
\sum_{j=1}^m (p_j - p_j^*)w_j = \lim_{t \to 0} t^{-1} \left( \text{pcap}(t\Omega(p) + (1-t)\Omega(p^*)) - \text{pcap}(\Omega(p^*)) \right) \leq 0.
\]

Whenever \( p \) is close to \( p^* = (p_1^*, \ldots, p_m^*) \), the support function \( h_{\Omega(p)} \) of \( \Omega(p) \) enjoys
\[
h_{\Omega(p)}(\zeta_j) = p_j \quad \forall \quad j \in \{1, \ldots, m\}.
\]
Recall $p_j^* > 0$. So

$$\sum_{j=1}^{m} (p_j - p_j^*)w_j = 0.$$ 

This last equation gives

$$w_j = \tau(p)(\sum p^*)^{-1} c_j \quad \forall \quad j \in \{1, \ldots, m\},$$

thereby completing the proof. □

**Lemma 2.3.** Let $p \in (1, 2]$ and $\mu$ be a finite, nonnegative, Borel measure comprising a finite sum of point masses on $\mathbb{S}^1$ such that:

(i) $\mu$ is not supported on any equator of $\mathbb{S}^1$, i.e., $\inf_{\theta \in \mathbb{S}^1} \int_{\mathbb{S}^1} |\theta \cdot \xi| \, d\mu(\xi) > 0$;

(ii) $\text{supp}(\mu)$ contains no any pair of antipodal points, i.e., if $\mu(\{\eta\}) > 0$ then $\mu(\{-\eta\}) = 0$;

(iii) $\int_{\mathbb{S}^1} \theta \cdot \xi \, d\mu(\xi) = 0 \quad \forall \theta \in \mathbb{S}^1$.

Then there exists a bounded, convex, nonempty, open polygon $O \subset \mathbb{R}^2$ such that $d\mu_p(O, \cdot) = d\mu(\cdot)$.

**Proof.** As in demonstrating [22, Lemma 5.7], we put

$$d\mu = \sum_{j=1}^{m} c_j \delta_{\zeta_j}$$

where $c_1, \ldots, c_m > 0$ are constants. Note that conditions (i), (ii) and (iii) in Lemma 2.2 amount to (i), (ii) and (iii) in Lemma 2.3, respectively. So, an application of Lemma 2.2 yields a bounded, convex, closed polygon $P$ containing the origin and a constant $c > 0$ such that

$$g_*((\nabla u_p)^p \, d\mathcal{H}^1) = c d\mu.$$

Note that if $rP$ is the $r$-dilation of $P$ then

$$g_*((\nabla u_p)^p \, d\mathcal{H}^1) = r^{1-p} g_*((\nabla u_p)^p \, d\mathcal{H}^1).$$

Thus, the desired result follows from choosing $r = \sqrt{\frac{1}{c}}$ and $\bar{O} = rP$. □

3. **Proof of Theorem 1.1**

(i) **Existence.** This comprises two parts.

The if-part. Suppose that $\mu$ has centroid at the origin and $\text{supp}(\mu)$ does not comprise any pair of antipodal points. Of course, the first supposed condition is just

$$\int_{\mathbb{S}^1} \theta \cdot \eta \, d\mu(\eta) = 0 \quad \forall \quad \theta \in \mathbb{S}^1.$$
However, the second one implies that $\mu$ is not supported on any equator (the intersection of the unit circle $S^1$ with any line through the origin) \{\theta, -\theta\} of $S^1$ where $\theta \in S^1$ - otherwise

$$\text{supp}(\mu) = \{\theta_0, -\theta_0\} \text{ for some } \theta_0 \in S^1.$$  

Conversely, if $\mu$ is unsupported on any equator then supp$(\mu)$ does not consist of any pair of antipodal points in $S^1$ - otherwise there is $\theta_1 \in S^1$ such that

$$\text{supp}(\mu) = \{\theta_1, -\theta_1\}, \text{ i.e., } \mu \text{ is supported on an equator of } S^1.$$  

Consequently,

$$0 < \kappa \leq \inf_{\theta \in S^1} \int_{S^1} |\theta \cdot \xi| \, d\mu(\xi).$$

Using the above analysis, we may take a sequence $\{\mu_j\}_{j=1}^\infty$ of finite, non-negative, Borel measures that are finite sums of point masses, not only converging to $\mu$ in the weak sense, but also satisfying (i)-(ii)-(iii) of Lemma 2.3 According to Lemma 2.3, for each $j$ there is a bounded, convex, closed set (polygon) $\bar{\Omega}_j \subset \mathbb{R}^2$ containing the origin such that the pull-back measure

$$d\mu_j(\bar{\Omega}_j, \cdot) = g_p(|\nabla u|)^p \, dH^1(\cdot)$$

is equal to $d\mu_j(\cdot)$. On the one hand, by Lemma 2.1 and (**) there is a constant $\kappa_j > 0$ independent of $j$ such that

$$\kappa_j \leq \left\{ \begin{array}{ll}
\left( \frac{p-1}{2-p} \right) (2\pi)^{-1} \text{pcap}(\bar{\Omega}_j) \left( \frac{1}{p} \right) & \text{for } 1 < p < 2 \\
\text{pcap}(\bar{\Omega}_j) & \text{for } p = 2
\end{array} \right.$$  

On the other hand, $\bar{\Omega}_j$ contains a segment $S_j$ such that its length is equal to diam$(\bar{\Omega}_j)$. Due to the translation-invariance of pcap$(\bar{\Omega}_j)$ it may be assumed that $S_j$ is the segment connecting $-2^{-1}\text{diam}(\bar{\Omega}_j)\theta_j$ and $2^{-1}\text{diam}(\bar{\Omega}_j)\theta_j$ where $\theta_j \in S^1$. If $j$ is big enough, then

$$\int_{S^1} h_{\bar{\Omega}_j} \, d\mu_j \geq \int_{S^1} h_{S_j} \, d\mu_j$$

$$\geq 2^{-1}\text{diam}(\bar{\Omega}_j) \int_{S^1} |\theta_j \cdot \xi| \, d\mu_j(\xi)$$

$$\geq 2^{-1}\text{diam}(\bar{\Omega}_j) \kappa,$$

and hence there is another constant $\kappa_2 > 0$ independent of $j$ such that $\kappa_2 \geq \text{diam}(\bar{\Omega}_j)$. Hence, an application of the Blaschke selection principle (see e.g. [31 Theorem 1.8.6]) derives that $\{\bar{\Omega}_j\}_{j=1}^\infty$ has a subsequence, still denoted by $\{\bar{\Omega}_j\}_{j=1}^\infty$, which converges to a bounded, compact, convex set $\bar{\Omega}_\infty \subset \mathbb{R}^2$ with pcap$(\bar{\Omega}_\infty) > 0$. In the sequel, we verify that the interior $(\bar{\Omega}_\infty)^\circ$ of $\bar{\Omega}_\infty$ is not empty. For this, assume $(\bar{\Omega}_\infty)^\circ = \emptyset$. Then the Hausdorff dimension $\text{dim}_H(\bar{\Omega}_\infty)$ of $\bar{\Omega}_\infty$ is strictly less than 2. If $\text{dim}_H(\bar{\Omega}_\infty) = 0$ then the convexity of $\bar{\Omega}_\infty$ ensures that $\bar{\Omega}_\infty$ is a single point and hence pcap$(\bar{\Omega}_\infty) = 0$. 

contradicting \( \text{pcap}(\Omega) > 0 \). This illustrates \( \dim_\mu(\Omega) = 1 \). Consequently, there exists a constant \( \kappa_2 > 0 \) and a point \( \xi \in S^1 \) such that the pull-back measure \( g_*(dH^1|_{\partial \Omega}) \) of \( H^1|_{\partial \Omega} \) to \( S^1 \) via the Gauss map \( g \) is equal to \( \kappa_3(\delta_\xi + \delta_{-\xi}) \). Upon using Lemma 2.1 we obtain a positive constant \( \kappa_4 \) (independent of \( j \) but dependent on \( \rho \) and the radius of an appropriate \( o \)-centered ball containing all \( \Omega_j \)) such that \( |\nabla u_{\bar{\Omega}}| \) holds almost everywhere on \( \partial \Omega_j \). Suppose that \( f \in C(S^1) \) (the class of all continuous functions on \( S^1 \)) is positive and its support is contained in a small neighbourhood \( N(\xi) \subset S^1 \) of \( \xi \in S^1 \) only. Now, we use Fatou’s lemma to derive

\[
\int_{N(\xi)} f \, d\mu = \liminf_{j \to \infty} \int_{S^1} f \, d\mu_j \\
\geq \kappa_4 \liminf_{j \to \infty} \int_{S^1} f \, g_*(dH^1|_{\partial \Omega}) \\
\geq \kappa_4 \int_{N(\xi)} \liminf_{j \to \infty} g_*(dH^1|_{\partial \Omega}) \\
= \kappa_4 \int_{N(\xi)} f \, g_*(dH^1|_{\partial \Omega}) \\
= \kappa_4 f(\xi).
\]

Thus, Radon-Nikodym’s differentiation of \( \mu \) with respect to the Dirac measure concentrated at \( \xi \) (cf. [17, page 42, Theorem 3]) implies that \( \mu \) must have a positive mass at \( \xi \), and similarly, \( \mu(\{\xi\}) > 0 \). Thus

\[
\text{supp}(\mu) \supset \{\xi, -\xi\}.
\]

Meanwhile, if

\[
\xi_0 \in S^1 \setminus \{\xi, -\xi\},
\]

then an application of the fact that the polygon \( \bar{\Omega}_j \) (whose Gauss map is denoted by \( g_j : \partial \Omega_j \to S^1 \)) approaches \( \Omega \) (which has only two outer unit normal vectors \( \pm \xi \)) ensures that \( \xi_0 \) is not in the set of all outer unit normal vectors of \( \bar{\Omega}_j \), thereby yielding

\[
\mathcal{H}^1(g_j^{-1}(\xi_0)) = 0 \quad \text{as \quad} j > N
\]

for a sufficiently large \( N \). According to [26, Theorems 1&3], there is \( q > p \) such that \( |\nabla u_{\bar{\Omega}}|^q \) is integrable on \( g_j^{-1}(\xi_0) \) with respect to \( d\mathcal{H}^1|_{\partial \Omega_j} \). This existence, the Hölder inequality, the weak convergence of \( \mu_j \), and Fatou’s
lemma, imply

\[ 0 \leq \mu(\{\xi_0\}) \leq \liminf_{j \to \infty} \mu_j(\{\xi_0\}) = \liminf_{j \to \infty} \int_{g_j^{-1}(\{\xi_0\})} |\nabla u_{\tilde{\Omega}_j}|^p \, d\mathcal{H}^1_{|\partial \Omega_j} \leq \liminf_{j \to \infty} \left( \int_{g_j^{-1}(\{\xi_0\})} |\nabla u_{\tilde{\Omega}_j}|^p \, d\mathcal{H}^1_{|\partial \Omega_j} \right)^{\frac{p}{p'}} \left( \mathcal{H}^1(g_j^{-1}(\{\xi_0\})) \right)^{1-\frac{p}{p'}} = 0. \]

Consequently, \( \mu(\{\xi_0\}) = 0 \). So,

\[ \text{supp}(\mu) = \{\xi, -\xi\}, \]

which contradicts the second supposed condition. In other words, \((\tilde{\Omega}_\infty)^o \neq \emptyset\). This, along with

\[ d\mu_p(\tilde{\Omega}_j, \cdot) = d\mu_j(\cdot) \]

and the weak convergence of \( \mu_j \to \mu \), derives

\[ d\mu_p(\tilde{\Omega}_\infty, \cdot) = d\mu(\cdot), \]

as desired.

The only-if part. Suppose that \( d\mu_p(\tilde{\Omega}, \cdot) = d\mu(\cdot) \) holds for a bounded, convex, nonempty, open set \( \Omega \subset \mathbb{R}^2 \). Note first that pcap(\cdot) is translation invariant. So

\[ \text{pcap}(\tilde{\Omega} + \{x_0\}) = \text{pcap}(\tilde{\Omega}) \quad \forall \quad x_0 \in \mathbb{R}^2. \]

However, the translation \( \tilde{\Omega} \mapsto \tilde{\Omega} + \{x_0\} \) changes \( x \cdot g \) to \( x \cdot g + x_0 \cdot x \). Thus, an application of (⋆⋆⋆) yields

\[
\int_{\partial(\tilde{\Omega} + \{x_0\})} (x \cdot g(x))|\nabla u_{\tilde{\Omega}_j + \{x_0\}}(x)|^p \, d\mathcal{H}^1(x) = \int_{\partial\tilde{\Omega}} (x_0 \cdot g(x))|\nabla u_{\tilde{\Omega}}(x)|^p \, d\mathcal{H}^1(x) + \int_{\tilde{\Omega}} (x \cdot g(x))|\nabla u_{\tilde{\Omega}}(x)|^p \, d\mathcal{H}^1(x). \]

Consequently,

\[ \int_{\partial\tilde{\Omega}} (x_0 \cdot g(x))|\nabla u_{\tilde{\Omega}}(x)|^p \, d\mathcal{H}^1(x) = 0. \]

This in turn implies the following linear constraint on \( \mu \):

\[ \int_{\mathbb{S}^1} \theta \cdot \eta \, d\mu(\theta) = \int_{\mathbb{S}^1} \theta \cdot \eta \, d\mu_p(\tilde{\Omega}, \theta) = 0 \quad \forall \quad \eta \in \mathbb{S}^1. \]
Next, let us validate that \( \text{supp}(\mu) \) does not comprise any pair of antipodal points. If this is not true, then there is \( \theta_0 \in \mathbb{S}^1 \) such that
\[
\text{supp}(\mu) = \{ \theta_0, -\theta_0 \}.
\]
However, the following considerations (partially motivated by [6, Lemma 4.1] handling the necessary part of a planar \( L_p \)-Minkowski problem from [30]) will show that this last identification cannot be valid.

Case \( o \in \Omega \). This, together with Lemma 2.1, ensures
\[
\{ \theta_0, -\theta_0 \} = \text{supp}(\mu_{\tilde{\Omega}, \cdot}) = \text{supp}(g_{\ast}(dH^1|_{\partial \Omega})).
\]
However, \( \tilde{\Omega} \) is not degenerate, so \( \text{supp}(g_{\ast}(dH^1|_{\partial \Omega})) \) cannot be \( \{ \theta_0, -\theta_0 \} \) - a contradiction occurs.

Case \( o \in \partial \Omega \). Denote by \( \Lambda \) the exterior normal cone at \( o \) such that
\[
\Lambda \cap \mathbb{S}^1 = \{ \eta \in \mathbb{S}^1 : h_{\Omega}(\eta) = 0 \}.
\]
Since \( \text{supp}(\mu) \) coincides with \( \text{supp}(\mu_{\tilde{\Omega}, \cdot}) \), it follows that \( h_{\Omega}(\theta_0) \) and \( h_{\Omega}(\theta_0) \) are positive. This in turn implies that \( \pm \theta_0 \) are not in \( \Lambda \). Without loss of generality we may assume that \( \Lambda \cap \mathbb{S}^1 \) is a subset of the following semi-circle
\[
T(-\theta_0, o) = \{ \zeta \in \mathbb{S}^1 : \zeta \cdot \theta_0 < 0 \}.
\]
Accordingly, if
\[
\eta \in T(\theta_0, o) = \{ \zeta \in \mathbb{S}^1 : \zeta \cdot \theta_0 > 0 \},
\]
then \( h_{\tilde{\Omega}}(\eta) > 0 \). Also because of
\[
g_{\ast}(dH^1|_{\partial \Omega})(T(\theta_0, o)) > 0
\]
and Lemma 2.1 (with a positive constant \( c \) depending only on \( p \) and \( r \) - the radius of a suitable ball \( D(o, r) \supset \Omega \)), we utilize
\[
\text{supp}(\mu) = \{ \theta_0, -\theta_0 \}
\]
to obtain the following contradictory computation:
\[
0 = \mu(T(\theta_0, o))
= \mu_p(\tilde{\Omega}, T(\theta_0, o))
= \int g_{\ast}^{-1}(T(\theta_0, o)) |\nabla u_{\tilde{\Omega}|}^p \, dH^1
\geq c^n H^1(g^{-1}(T(\theta_0, o))
> 0.
\]

(ii) Uniqueness. Suppose that \( \Omega_0, \Omega_1 \) are two solutions of the equation \( d\mu_p(\tilde{\Omega}, \cdot) = d\mu(\cdot) \). Then
\[
g_{\ast}(|\nabla u_{\Omega_0}|^p \, dH^1) = g_{\ast}(|\nabla u_{\Omega_1}|^p \, dH^1).
\]
To reach the conclusion that $\Omega_0$ and $\Omega_1$ are the same up to a translate, we define

$$[0, 1] \ni t \mapsto f_p(t) = \begin{cases} \left(p\text{cap}(\overline{(1-t)\Omega_0 + t\Omega_1})\right)^{\frac{2-p}{p}} & \text{as } p \in (1, 2); \\ p\text{cap}(1-t\tilde{\Omega}_0 + t\tilde{\Omega}_1) & \text{as } p = 2, \end{cases}$$

and handle the following two cases.

**Case $p \in (1, 2)$.** In a manner (cf. [11]) slightly different from proving [14, Theorem 1.2] (under $n = 2 > p > 1$), we use the chain rule, [14, Theorem 1.1] (under $n = 2$) and ($\star \star \star$) to get

$$f_p'(0) = \frac{(f_p(0))^{p-1}}{(2-p)} \int_{\partial \overline{\Omega}_0} (h_{\Omega_1}(g) - h_{\Omega_0}(g))|\nabla u_{\Omega_0}|^p \, d\mathcal{H}^1$$

$$= \frac{(f_p(0))^{p-1}}{(2-p)} \left( \int_{\partial \overline{\Omega}_0} h_{\Omega_1}(g)|\nabla u_{\Omega_0}|^p \, d\mathcal{H}^1 - \int_{\partial \overline{\Omega}_0} h_{\Omega_0}(g)|\nabla u_{\Omega_0}|^p \, d\mathcal{H}^1 \right)$$

$$= \frac{(f_p(0))^{p-1}}{(2-p)} \left( \int_{S^1} h_{\Omega_1}(g^*)|\nabla u_{\Omega_0}|^p \, d\mathcal{H}^1 - \int_{S^1} h_{\Omega_0}(g^*)|\nabla u_{\Omega_0}|^p \, d\mathcal{H}^1 \right)$$

$$= \frac{(f_p(0))^{p-1}}{(2-p)} \left( (f_p(1))^{2-p} - (f_p(0))^{2-p} \right).$$

According to [15], Theorem 1, $f_p$ is concave, and so

$$f_p(1) - f_p(0) \leq f_p'(0) = (f_p(0))^{p-1} \left( (f_p(1))^{2-p} - (f_p(0))^{2-p} \right).$$

This, along with exchanging $\tilde{\Omega}_0$ and $\tilde{\Omega}_1$, implies

$$\text{pcap}(\tilde{\Omega}_1) = f_p(1) \leq f_p(0) = \text{pcap}(\tilde{\Omega}_0) \leq f_p(1) = \text{pcap}(\tilde{\Omega}_1),$$

thereby producing $f_p'(0) = 0$ and $f_p$ being a constant thanks to the concavity of $f_p$. Since $\tilde{\Omega}_0$ and $\tilde{\Omega}_1$ have the same $p$-capacity, an application of the equality in [15], Theorem 1] yields that $\Omega_0$ is a translate of $\Omega_1$. 
Case \( p = 2 \). Referring to the argument for [35, Theorem 5.1] under \( n = 2 \), we employ [35, Theorems 4.4 & 3.1] to deduce

\[
f_2'(0) = (2\pi)^{-1} f_2(0) \int_{\partial \Omega_0} (h_{\Omega_1}(g) - h_{\Omega_0}(g)) \|\nabla u_{\Omega_0}\|^2 \, d\mathcal{H}^1
\]

\[
= (2\pi)^{-1} f_2(0) \left( \int_{\partial \Omega_0} h_{\Omega_1}(g) \|\nabla u_{\Omega_0}\|^2 \, d\mathcal{H}^1 - 2\pi \right)
\]

\[
= (2\pi)^{-1} f_2(0) \left( \int_{\partial \Omega_1} h_{\Omega_1}(g) \|\nabla u_{\Omega_0}\|^2 \, d\mathcal{H}^1 - 2\pi \right)
\]

\[
= (2\pi)^{-1} f_2(0) \left( \int_{\partial \Omega_1} h_{\Omega_1}(g) \|\nabla u_{\Omega_0}\|^2 \, d\mathcal{H}^1 - 2\pi \right)
\]

\[
= (2\pi)^{-1} f_2(0) (2\pi - 2\pi) = 0.
\]

Note that \( t \mapsto f_2(t) \) is concave on \([0, 1] \) (cf. [5, 13]). So \( f_2 \) is a constant function on \([0, 1] \), in particular, we have

\[
2\text{cap}(\bar{\Omega}_1) = f_2(1) = f_2(t) = f_2(0) = 2\text{cap}(\bar{\Omega}_0).
\]

As a consequence, the equation

\[
f_2(t) = f_2(0) \quad \forall \ t \in [0, 1]
\]

and [13, Theorem 3.1] ensure that \( \Omega_0 \) and \( \Omega_1 \) are the same up to translation and dilation. But,

\[
2\text{cap}(\bar{\Omega}_0) = 2\text{cap}(\bar{\Omega}_1)
\]

forces that \( \Omega_1 \) is only a translate of \( \Omega_0 \).

(iii) Regularity. [14, Theorem 1.4] covers the case \( 1 < p < 2 = n \). The argument for [14, Theorem 1.4] or for the regularity part of [22, Theorem 0.7] (cf. [22, Theorem 7.1] and [20]) under \( n = 2 \) can be modified to verify the case \( p = 2 \). For reader’s convenience, an outline of this verification under \( p \in (1, 2] \) is presented below.

Firstly, we observe that Lemmas 7.2-7.3-7.4 in [14] are still valid for the \((1, 2] \ni p\)-equilibrium potential \( u_{\Omega} \).

Secondly, [22, Lemma 6.16] can be used to produce two constants \( c > 0 \) and \( \epsilon \in (0, 1) \) (depending on the Lipschitz constant of \( \Omega \)) such that (cf. [14, Lemma 7.5] for \( p \in (1, 2] \) and [22, Theorem 6.5] for \( p = 2 \))

\[
\int_{H \cap \Omega} (\delta(x, H \cap \partial \Omega))^{1-p} \|\nabla u_{\Omega}(x)\|^p \, d\mathcal{H}^1(x) \leq c \mathcal{H}^1(\Omega \cap \partial \Omega) \inf_{H \cap \Omega} \|\nabla u_{\Omega}\|^p
\]

holds for any half-plane \( H \subset \mathbb{R}^2 \) with \( H \cap \mathcal{D}(o, r_{\text{int}}) = \emptyset \), where \( r_{\text{int}} \) is the inner radius of \( \Omega \), and \( \delta(x, H \cap \partial \Omega) \) is a normalized distance from \( x \) to \( H \cap \partial \Omega \).
Thirdly, from [14, Lemma 7.7] it follows that if
\[ d\mu_p(\tilde{\Omega}, \cdot) = \psi(\cdot) \, d\ell(\cdot) \]
is valid for some integrable function \( \psi \) being greater than a positive constant \( c \) on \( \mathbb{S}^1 \), and if \( \phi \) stands for the convex and Lipschitz function defined on a bounded open interval \( O \subset \mathbb{R}^1 \) whose graph
\[ G = \{(s, \phi(s)) : s \in O\} \]
is a portion of the convex curve \( \partial \Omega \), then \( \phi \) enjoys the following (1, 2] \( \ni p \)-Monge-Ampère equation in Alexandrov’s sense (cf. [19, p.6]):
\[ \phi''(s) = \det(\nabla^2 \phi(s)) \]
\[ = \left(1 + (\phi'(s))^2\right)^{\frac{3}{2}} |(\nabla u_\Omega)(s, \phi(s))|^p \psi(\xi)^{-1} \]
\[ = \left(1 + (\phi'(s))^2\right)^{\frac{3}{2}} |(\nabla u_\Omega)(s, \phi(s))|^p \psi(\xi)^{-1} \]
\[ \equiv \Phi_p(\tilde{\Omega}, s), \]
where
- \[ \frac{d}{ds} u_\Omega(s, \phi(s)) = (1, \phi'(s)) \cdot (\nabla u_\Omega)(s, \phi(s)) \]
is utilized to explain the action of \( \nabla u_\Omega \) at \((s, \phi(s)) \in G\);
- \[ s \mapsto \phi''(s) \left(1 + (\phi'(s))^2\right)^{-\frac{3}{2}} |(\nabla u_\Omega)(s, \phi(s))|^{-p} \]
is regarded as the \( p \)-equilibrium-potential-curvature on \( G \subset \partial \Omega \);
- \[ \xi = (\phi'(s), -1) \left(1 + (\phi'(s))^2\right)^{-\frac{1}{2}} \]
is written for the outer unit normal vector at \((s, \phi(s)) \in G\).

Fourthly, an application of the secondly-part and the thirdly-part above and [22, Theorem 7.1] derives that if \( \psi \) is bounded above and below by two positive constants then Caffarelli’s methodology developed in [10] can be adapted to establish that \( \partial \Omega \) is of \( C^{1,\varepsilon} \) for the above-found \( \varepsilon \in (0, 1) \). Now, for \( \alpha \in (0, 1) \) let the positive function \( \psi \) in
\[ d\mu_p(\tilde{\Omega}, \cdot) = \psi(\cdot) d\ell(\cdot) \]
belong to \( C^{0,\alpha}(\mathbb{S}^1) \). Since \( \partial \Omega \) is of \( C^{1,\varepsilon} \), a barrier argument, plus [28], yields that \( |\nabla u_\Omega| \) is not only bounded above and below by two positive constants (and so is \( \phi'' \) on \( O \)), but also \( |\nabla u_\Omega| \) is of \( C^{0,\varepsilon} \) up to \( \partial \Omega \). From the thirdly-part above it follows that \( \Phi_p(\tilde{\Omega}, \cdot) \) is of \( C^{0,\varepsilon_1} \) for some \( \varepsilon_1 \in (0, 1) \). This, along with \( \phi''(\cdot) = \Phi_p(\tilde{\Omega}, \cdot), \) gives that \( \phi \) is of \( C^{2,\varepsilon_1} \). As a consequence, we see that \( |\nabla u_\Omega| \) is of \( C^{1,\varepsilon_2} \) up to \( \partial \Omega \) for some \( \varepsilon_2 \in (0, 1) \), and thereby finding that \( \Phi_p(\tilde{\Omega}, \cdot) \) is of \( C^{0,\varepsilon_2} \). Accordingly, \( \partial \Omega \) being of \( C^{2,\varepsilon_2} \) follows from Caffarelli’s
three papers [7, 8, 9]. Continuing this initial process, we can reach the desired higher order regularity.

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References

[1] T. Adamowicz, On p-harmonic mappings in the plane. Nonlinear Anal. 71(2009)502-511.
[2] T. Adamowicz, The geometry of planar p-harmonic mappings: convexity, level curves and the isoperimetric inequality. Ann. Sc. Norm. Super. Pisa Cl. Sci. 14(2015)263-292.
[3] G. Aronsson, Aspects of p-harmonic functions in the plane. Summer School in Potential Theory (Joensuu, 1990), 9-34, Joensuu Yliop. Luonnont. Julk., 26, Univ. Joensuu, Joensuu, 1992.
[4] R.W. Barnard, K. Pearce and A.Y. Solynin, An isoperimetric inequality for logarithmic capacity. Ann. Acad. Sci. Fenn. Math. 27(2002)419-436.
[5] C. Borell, Hitting probability of killed Brownian motion: A study on geometric regularity. Ann. Sci. Ecole Norm. Supér. Paris 17(1984)451-467.
[6] K.J. Böröczky and H. T. Trinh, The planar L_p-Minkowski problem for 0 < p < 1. Adv. in Appl. Math. 87(2017)58-81.
[7] L. Caffarelli, Interior a priori estimates for solutions of fully non-linear equations. Ann. Math. 131(1989)189-213.
[8] L. Caffarelli, A localization property of viscosity solutions to the Monge-Ampère equation and their strict convexity. Ann. of Math. 131(1990)129-134.
[9] L. Caffarelli, Interior W^{2,p} estimates for solutions of the Monge-Ampère equation. Ann. Math. 131(1990)135-150.
[10] L. Caffarelli, Some regularity properties of solutions to the Monge-Ampère equation. Comm. Pure Appl. Math. 44(1991)965-969.
[11] L. Caffarelli, D. Jerison and E. H. Lieb, On the case of equality in the Brunn-Minkowski inequality for capacity. Adv. Math. 117(1996)193-207.
[12] S.-Y. Cheng and S.-T. Yau, On the regularity of the solution of the n-dimensional Minkowski problem. Comm. Pure Appl. Math. 29(1976)495-561.
[13] A. Colesanti and P. Cuoghi, The Brunn-Minkowski inequality for the n-dimensional logarithmic capacity. Potential Anal. 22(2005)289-304.
[14] A. Colesanti, K. Nyström, P. Salani, J. Xiao, D. Yang and G. Zhang, The Hadamard variational formula and the Minkowski problem for p-capacity. Adv. Math. 285(2015)1511-1588.
[15] A. Colesanti and P. Salani, The Brunn-Minkowski inequality for p-capacity of convex bodies. Math. Ann. 327(2003)459-479.
[16] B.E.J. Dahlberg, Estimates of harmonic measure. Arch. Rational Mech. Anal. 65(1977)275-288.
[17] L.C. Evans and R.F. Gariepy, Measure Theory and Fine Properties of Functions. CRC Press, 1992.
[18] R.J. Gardner and D. Hartenstine, Capacities, surface area, and radial sums. Adv. Math. 221(2009)601-626.
[19] C.E. Gutiérrez, The Monge-Ampère Equation. Progress in Nonlinear Differential Equations and Their Applications, Vol. 44, Birkhäuser, 2001.
[20] C.E. Gutiérrez and D. Hartenstine, Regularity of weak solutions to the Monge-Ampère equation. Trans. Amer. Math. Soc. 355(2003)2477-2500.
[21] D. Jerison, Prescribing harmonic measure on convex domains. Invent. Math. 105(1991)375-400.
[22] D. Jerison, A Minkowski problem for electrostatic capacity. Acta Math. 176(1996)1-47.
[23] D. Jerison, The direct method in the calculus of variations for convex bodies. Adv. Math. 122(1996)262-279.
[24] D. A. Klain, The Minkowski problem for polytopes. Adv. Math. 185(2004)270-288.
[25] J. L. Lewis, Capacitary functions in convex rings. Arch. Rational Mech. Anal. 66(1977)201-224.
[26] J. L. Lewis and K. Nyström, Boundary behaviour for p-harmonic functions in Lipschitz and starlike Lipschitz ring domains. Ann. Sci. École Norm. Sup. 40(2007)765-813.
[27] J. L. Lewis and K. Nyström, Regularity and free boundary regularity for the p-Laplacian in Lipschitz and $C^1$-domains. Ann. Acad. Sci. Fenn. Math. 33(2008)523-548.
[28] G. M. Lieberman, Boundary regularity for solutions of degenerate elliptic equations. Nonlinear Anal. 12(1988)1203-1219.
[29] M. Ludwig, J. Xiao and G. Zhang, Sharp convex Lorentz-Sobolev inequalities. Math. Ann. 350(2011)169-197.
[30] E. Lutwak, The Brunn-Minkowski-Firey theory. I. mixed volumes and the Minkowski problem. J. Differential Geom. 38(1993)131-150.
[31] R. Schneider, Convex Bodies: The Brunn-Minkowski Theory. Cambridge Univ. Press, 1993.
[32] A. Y. Solynin and V. A. Zalgaller, An isoperimetric inequality for logarithmic capacity of polygons. Ann. Math. 159(2004)277-303.
[33] V. Umanskiy, On solvability of two-dimensional $L_p$-Minkowski problem. Adv. Math. 180(2003)176-186.
[34] J. Xiao, On the variational p-capacity problem in the plane. Commun. Pure Appl. Anal. 14(2015)959-968.
[35] J. Xiao, Exploiting log-capacity in convex geometry. Asian J. Math. 22(2018)955-980.

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