On the Γ-limit of singular perturbation problems with optimal profiles which are not one-dimensional. Part III: The energies with non local terms

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Abstract

1 Introduction

The asymptotic behavior, when \( \varepsilon \to 0 \) of the family \( \{I_\varepsilon\}_{\varepsilon>0} \) of the functionals \( I_\varepsilon(\phi) : T \to \mathbb{R}^+ \cup \{0\} \cup \{+\infty\} \), where \( T \) is a given metric space, is partially described by the De Giorgi’s Γ-limits, defined by:

\[
(\Gamma - \lim \varepsilon \to 0^+ I_\varepsilon)(\phi) := \inf \left\{ \lim_{\varepsilon \to 0^+} I_\varepsilon(\phi_\varepsilon) : \phi_\varepsilon \to \phi \text{ in } T \right\},
\]

\[
(\Gamma - \lim \varepsilon \to 0^+ I_\varepsilon)(\phi) := \inf \left\{ \lim_{\varepsilon \to 0^+} I_\varepsilon(\phi_\varepsilon) : \phi_\varepsilon \to \phi \text{ in } T \right\},
\]

\[
(\Gamma - \lim I_\varepsilon)(\phi) := (\Gamma - \lim I_\varepsilon)(\phi) = (\Gamma - \lim I_\varepsilon)(\phi)
\]

in the case they are equal. (1.3)

Usually, for finding the Γ-limit of \( I_\varepsilon(\phi) \), we need to find two bounds.

(*) Firstly, we wish to find a lower bound, i.e. the functional \( \underline{I}(\phi) \) such that for every family \( \{\phi_\varepsilon\}_{\varepsilon>0} \), satisfying \( \phi_\varepsilon \to \phi \) as \( \varepsilon \to 0^+ \), we have \( \lim_{\varepsilon \to 0^+} I_\varepsilon(\phi_\varepsilon) \geq \underline{I}(\phi) \).

(**) Secondly, we wish to find an upper bound, i.e. the functional \( \overline{I}(\phi) \) such that there exists the family \( \{\psi_\varepsilon\}_{\varepsilon>0} \), satisfying \( \psi_\varepsilon \to \phi \) as \( \varepsilon \to 0^+ \), and we have \( \lim_{\varepsilon \to 0^+} I_\varepsilon(\psi_\varepsilon) \leq \overline{I}(\phi) \).

(***) If we obtain \( I_\varepsilon(\phi) = \overline{I}(\phi) \), then \( \overline{I}(\phi) \) will be the Γ-limit of \( I_\varepsilon(\phi) \).

Let \( G \in C^1(\mathbb{R}^{m \times N} \times \mathbb{R}^{m \times N(n-1)} \times \ldots \times \mathbb{R}^{m \times N} \times \mathbb{R}^m, \mathbb{R}) \) and \( W \in C^1(\mathbb{R}^m, \mathbb{R}) \) be nonnegative functions such that \( G(0,0,\ldots,0,b) = 0 \) and let \( \Psi \in C^1(\mathbb{R}^m, \mathbb{R}^{l \times N}) \). Consider the energy functional with nonlocal term defined for every \( \varepsilon > 0 \) by

\[
I_\varepsilon(\phi) = \int_\Omega \frac{1}{\varepsilon} G(\varepsilon^n \nabla \phi^n, \ldots, \varepsilon \nabla \phi, \phi) \, dx + \int_\Omega \frac{1}{\varepsilon} W(\phi) \, dx + \frac{1}{\varepsilon} \int_{\mathbb{R}^N} |\nabla \bar{H}_{\Psi(\phi)}|^2 \, dx \quad \text{for } \phi : \Omega \to \mathcal{M} \subset \mathbb{R}^m. \quad (1.4)
\]

Here given \( u : \Omega \to \mathbb{R}^{l \times N} \), \( \bar{H}_u : \mathbb{R}^N \to \mathbb{R}^l \) is defined by

\[
\begin{cases}
\Delta \bar{H}_u = \text{div} \{\chi_{\Omega} u\} \quad \text{in the sense of distributions in } \mathbb{R}^N, \\
\nabla \bar{H}_u \in L^2(\mathbb{R}^N, \mathbb{R}^{l \times N}),
\end{cases}
\]

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where $\chi_\Omega$ is the characteristic function of $\Omega$. One of the fields where functionals of type (1.4) are relevant is Micromagnetics (see [1], [14], [35], [36] and other). The full 3-dimensional model of ferromagnetic materials deals with an energy functional, which, up to a rescaling, has the form

$$E_\varepsilon(m) := \varepsilon \int_G |\nabla m|^2 \, dx + \frac{1}{\varepsilon^2} \int_G W(m) \, dx + \frac{1}{\varepsilon} \int_{\mathbb{R}^3} |\nabla \bar{H}_m|^2 \, dx,$$

where $G \subset \mathbb{R}^3$ is a bounded domain, $m : \Omega \to S^2$ stands for the magnetization, $\delta_\varepsilon > 0$ is a material parameter and $\bar{H}_m : \mathbb{R}^3 \to \mathbb{R}$ is defined, as before, by

$$\begin{cases}
\Delta \bar{H}_m = \text{div} \{ \chi_\Omega m \} & \text{in } \mathbb{R}^3, \\
\nabla \bar{H}_m \in L^2(\mathbb{R}^3, \mathbb{R}^3),
\end{cases}$$

The first term in (1.6) is usually called the exchange energy while the second is called the anisotropy energy and the third is called the demagnetization energy. One can consider the infinite cylindrical domain $\Omega = G \times \mathbb{R}$ and configurations which don’t depend on the last coordinate. These reduce the original model to a 2-dimensional one, where the energy, up to a rescaling, has the form

$$E_\varepsilon(m) := \varepsilon \int_G |\nabla m|^2 \, dx + \frac{1}{\varepsilon^2} \int_G W(m) \, dx + \frac{1}{\varepsilon} \int_{\mathbb{R}^2} |\nabla \bar{H}_m'|^2 \, dx,$$

where $G \subset \mathbb{R}^2$ is a bounded domain, $m = (m_1, m_2, m_3) : G \to S^2$ stands for the magnetization, $m' := (m_1, m_2) \in \mathbb{R}^2$ denotes the first two components of $m$, $\delta_\varepsilon > 0$ and $\bar{H}_m' : \mathbb{R}^2 \to \mathbb{R}$ is defined, as before, by

$$\begin{cases}
\Delta \bar{H}_m' = \text{div} \{ \chi_G m' \} & \text{in } \mathbb{R}^2, \\
\nabla \bar{H}_m' \in L^2(\mathbb{R}^2, \mathbb{R}^2),
\end{cases}$$

Note that in the case $\delta_\varepsilon = \varepsilon$ (i.e. the anisotropy and the demagnetization energies have the same order as $\varepsilon \to 0$) the energy-functionals in (1.6) and (1.5) are special cases of the energy in (1.4).

In this work, using the technique developed in [32] and [33], we construct the upper and the lower bounds as $\varepsilon \downarrow 0$ for the general energy of the form (1.4) under certain conditions on $M$ for functions $\phi \in BV$. In particular our upper bound improves, in general, one obtained in [25].

In order to reduce the problem (1.4) to the local problems studied in [32] and [33], the following trivial observation was made for the problem (1.4). For $\phi : \Omega \to M$, such that $\Psi(\phi) \in L^2$, consider the variational problem

$$J_{\Psi, \phi}(L) := \inf \left\{ P_{\Psi, \phi}(L) : \int_{\mathbb{R}^N} \left| L(x) + \chi_\Omega(x)\Psi(\phi(x)) \right|^2 \, dx : L \in L^2(\mathbb{R}^N, \mathbb{R}^{1 \times N}), \text{div} \, L \equiv 0 \right\}.$$  

Then

$$J_{\Psi, \phi}(L) = \int_{\mathbb{R}^N} \left| \nabla \bar{H}_{\Psi(\phi)}(x) \right|^2 \, dx,$$

where given $u : \Omega \to \mathbb{R}^{1 \times N}$, $\bar{H}_u : \mathbb{R}^N \to \mathbb{R}^i$ is defined by (1.5). Moreover

$$L_0(x) := \nabla \bar{H}_{\Psi(\phi)}(x) - \chi_\Omega(x)\Psi(\phi(x)) \quad \text{is a minimizer to (1.10)}.$$

Therefore the $\Gamma$-limit of the family of functionals (1.4) as $\phi_\varepsilon \to \phi$ where

$$\begin{cases}
W(\phi) = 0 \\
\text{div} \Psi(\phi) = 0 & \text{in } \Omega \\
\Psi(\phi) \cdot n = 0 & \text{on } \partial \Omega,
\end{cases}$$

is the same as the $\Gamma$-limit of the family of functionals

$$\bar{J}_\varepsilon(\phi, L) := \int_{\Omega} \left( \frac{1}{\varepsilon} G \left( \varepsilon^2 \nabla \phi^n, \ldots, \varepsilon \nabla \phi, \phi \right) \right) \, dx + \int_{\Omega} \frac{1}{\varepsilon} W(\phi) \, dx + \frac{1}{\varepsilon} \int_{\mathbb{R}^N} \left| L(x) + \chi_\Omega(x)\Psi(\phi(x)) \right|^2 \, dx$$

where $\phi : \Omega \to M$, and $\text{div} \, L = 0$, as a generalization.
Proposition 1.1. Let $\Psi(\psi) \in C(\mathbb{R}^m, \mathbb{R}^{1 \times N})$ which satisfies
\[
|\Psi(\psi)| \leq C_0|\psi|^{p/2} \quad \forall \psi \in \mathbb{R}^m,
\] (1.15)
for some constant $C_0 > 0$ and $p \geq 1$. Furthermore, for every $\varepsilon > 0$ consider the functional $E_\varepsilon(\phi(x)) : L^p(\Omega, \mathbb{R}^m) \to [0, +\infty) \cup \{+\infty\}$ which (possibly) can attain the infinite values. Next for every $\varepsilon > 0$ consider the functional $P_\varepsilon(\phi(x)) : L^p(\Omega, \mathbb{R}^m) \to [0, +\infty) \cup \{+\infty\}$, defined by
\[
P_\varepsilon(\phi(x)) := E_\varepsilon(\phi(x)) + \frac{1}{\delta_\varepsilon} \int_{\mathbb{R}^N} \left| \nabla_x H_{\Psi(\phi)}(x) \right|^2 \, dx,
\] (1.16)
where $\delta_\varepsilon > 0$ satisfies $\lim_{\varepsilon \to +0^+} \delta_\varepsilon = 0$ and given $\phi(x) \in L^p(\Omega, \mathbb{R}^m)$, $H_{\Psi(\phi)} : \mathbb{R}^N \to \mathbb{R}^l$ is defined by
\[
\begin{cases}
\Delta_x H_{\Psi(\phi)}(x) = \text{div} \left\{ \chi_{\Omega}(x) \cdot \Psi(\psi(x)) \right\} \quad \text{in } \mathbb{R}^N, \\
\nabla_x H_{\Psi(\phi)}(x) \in L^2(\mathbb{R}^N, \mathbb{R}^{l \times N}),
\end{cases}
\] (1.17)
with $\chi_{\Omega}(x) := 1$ if $x \in \Omega$ and $\chi_{\Omega}(x) := 0$ if $x \in \mathbb{R}^N \setminus \Omega$. Furthermore, for every $\varepsilon > 0$ consider the functional $Q_\varepsilon(\phi(x), L(x)) : L^p(\Omega, \mathbb{R}^m) \times L^2(\mathbb{R}^N, \mathbb{R}^{l \times N}) \to [0, +\infty) \cup \{+\infty\}$ defined by
\[
Q_\varepsilon(\phi(x), L(x)) := \begin{cases}
E_\varepsilon(\phi(x)) + \frac{1}{\delta_\varepsilon} \int_{\mathbb{R}^N} \left| L(x) + \chi_{\Omega}(x) \cdot \Psi(\phi(x)) \right|^2 \, dx & \text{if } \text{div}_x L(x) \equiv 0, \\
+\infty & \text{otherwise}.\end{cases}
\] (1.18)

Next for every $\phi(x) \in L^p(\Omega, \mathbb{R}^m)$, such that $\text{div}_x \left\{ \chi_{\Omega}(x) \cdot \Psi(\phi(x)) \right\} = 0$ in $\mathbb{R}^N$ set
\[
P(\phi) := \inf \left\{ \lim_{\varepsilon \to +0^+} P_\varepsilon(\phi_\varepsilon(x)) : \phi_\varepsilon(x) \to \phi(x) \text{ in } L^p(\Omega, \mathbb{R}^m) \right\},
\] (1.19)
\[
\overline{P}(\phi) := \inf \left\{ \liminf_{\varepsilon \to +0^+} P_\varepsilon(\phi_\varepsilon(x)) : \phi_\varepsilon(x) \to \phi(x) \text{ in } L^p(\Omega, \mathbb{R}^m) \right\},
\]
and
\[
Q(\phi) := \inf \left\{ \lim_{\varepsilon \to +0^+} Q_\varepsilon(\phi_\varepsilon(x), L_\varepsilon(x)) : \phi_\varepsilon(x) \to \phi(x) \text{ in } L^p(\Omega, \mathbb{R}^m) \right\},
\] (1.20)
and $L_\varepsilon(x) \to \left( -\chi_{\Omega}(x) \cdot \Psi(\phi(x)) \right)$ in $L^2(\mathbb{R}^N, \mathbb{R}^{l \times N})$,
\[
\underline{Q}(\phi) := \inf \left\{ \liminf_{\varepsilon \to +0^+} Q_\varepsilon(\phi_\varepsilon(x), L_\varepsilon(x)) : \phi_\varepsilon(x) \to \phi(x) \text{ in } L^p(\Omega, \mathbb{R}^m) \right\},
\]
and $L_\varepsilon(x) \to \left( -\chi_{\Omega}(x) \cdot F(\phi(x)) \right)$ in $L^2(\mathbb{R}^N, \mathbb{R}^{l \times N})$.

Then we have the following equalities
\[
P(\phi) = Q(\phi) \quad \text{and} \quad \overline{P}(\phi) = \underline{Q}(\phi).
\] (1.21)

Next since the energy (1.14) with $\mathcal{M} \equiv \mathbb{R}^m$ is a particular case of the functionals studied in [32], where we get the upper its bound and in [33], where we get its lower bound, we can apply this results to problem (1.4). Then we get the following Theorems providing the upper and the lower bound (see Theorems 2.1 and 2.2 for the proof).

Theorem 1.1. Let $\Omega \subset \mathbb{R}^N$ be an open set with locally Lipschitz’s boundary, let $G \in C^1(\mathbb{R}^{m \times N} \times \mathbb{R}^{m \times N}) \times \cdots \times \mathbb{R}^{m \times N} \times \mathbb{R}^m, \mathbb{R})$ and $W \in C^1(\mathbb{R}^m, \mathbb{R})$ be nonnegative functions such that $G(0, 0, \ldots, 0, b) = 0$ and let $\Psi \in C^1(\mathbb{R}^m, \mathbb{R}^{l \times N})$. Furthermore, let $\varphi \in BV(\mathbb{R}^N, \mathbb{R}^m) \cap L^\infty$ be such that $\|D\varphi\|(|\partial\Omega) = 0$, $W(\varphi(x)) = 0$ for
a.e. \( x \in \Omega \), \( \nabla x \Psi(\varphi(x)) = 0 \) in \( \Omega \) and \( \Psi(\varphi(x)) \cdot n(x) = 0 \) on \( \partial \Omega \). Then there exists a sequence \( \{ \psi_{\varepsilon} \}_{\varepsilon > 0} \subset C^\infty(\mathbb{R}^N, \mathbb{R}^m) \) such that \( \int_\Omega \psi_{\varepsilon}(x)dx = \int_\Omega \varphi(x)dx \), for every \( q \geq 1 \) we have \( \lim_{\varepsilon \to 0^+} \psi_{\varepsilon} = \varphi \) in \( L^q \) and

\[
\lim_{\varepsilon \to 0^+} I_{\varepsilon}(\psi_{\varepsilon}) := \int_\Omega \frac{1}{\varepsilon} G \left( \varepsilon^n \nabla \psi_{\varepsilon}, \ldots, \varepsilon \nabla \psi_{\varepsilon} \right) dx + \int_\Omega \frac{1}{\varepsilon} W(\psi_{\varepsilon}) dx + \frac{1}{\varepsilon} \int_{\mathbb{R}^N} |\nabla \tilde{H}_{\Psi(\psi_{\varepsilon})}|^2 dx \\
\leq \int_{\Omega \cap J_\varepsilon} \tilde{E}_{\text{per}} \left( \varphi^+ (x), \varphi^- (x), \nu(x) \right) dH^{N-1}(x),
\]

where \( \tilde{H}_{\text{per}} \) is defined by (1.23),

\[
\tilde{E}_{\text{per}} \left( \varphi^+, \varphi^-, \nu \right) := \inf \left\{ \int_{I_{\nu}} \frac{1}{L} \left( G \left( L^n \nabla L \zeta(y), \ldots, \nabla \zeta(y), \zeta(y) \right) dy + W(\zeta(y)) + |\nabla H_{\Psi, \zeta, \nu}(y)|^2 \right) \right\},
\]

where \( H_{\Psi, \zeta, \nu} \in W^{2,2}_{\text{loc}}(\mathbb{R}^N, \mathbb{R}^k) \) satisfies

\[
\begin{cases}
\Delta_y H_{\Psi, \zeta, \nu}(y) = \text{div}_y \Psi(\zeta(y)) \quad \text{in } I_{\nu}, \\
H_{\Psi, \zeta, \nu}(y + \nu_j) = H_{\Psi, \zeta, \nu}(y) \quad \forall y \in \mathbb{R}^N \text{ such that } |y \cdot \nu| < 1/2, \\
\frac{\partial}{\partial \nu} H_{\Psi, \zeta, \nu}(y) = 0 \quad \forall y \in \mathbb{R}^N \text{ such that } |y \cdot \nu| = 1/2,
\end{cases}
\]

and

\[
\mathcal{S}(\varphi^+, \varphi^-, I_{\nu}) := \left\{ \zeta \in C^0(\mathbb{R}^N, \mathbb{R}^m) : \zeta(y) = \varphi^- \text{ if } y \cdot \nu \leq -1/2, \right. \\
\left. \zeta(y) = \varphi^+ \text{ if } y \cdot \nu \geq 1/2 \text{ and } \zeta(y + \nu_j) = \zeta(y) \quad \forall j = 2, 3, \ldots, N \right\},
\]

Here

\[
I_{\nu} := \left\{ y \in \mathbb{R}^N : |y \cdot \nu_j| < 1/2 \quad \forall j = 1, 2, \ldots, N \right\},
\]

where \( \{ \nu_1, \nu_2, \ldots, \nu_N \} \subset \mathbb{R}^N \) is an orthonormal base in \( \mathbb{R}^N \) such that \( \nu_1 := \nu \).

**Theorem 1.2.** Let \( \Omega \subset \mathbb{R}^N \) and \( G, W, \Psi \) and \( \varphi \) be the same as in Theorem 1.1. Moreover, assume that there exists a constant \( C > 0 \) and \( p \geq 1 \) such that \( |\Psi(b)| \leq C(|b|^p + 1) \) for every \( b \in \mathbb{R}^m \) and \( |a_n|^p / C \leq G(a_n, \ldots, a_2, a_1, b) + W(b) \leq C \left( \sum_{j=1}^N |a_j|^p + |b|^p + 1 \right) \) for every \( a_j \in \mathbb{R}^{m \times N} \) and \( b \in \mathbb{R}^m \). Then for every sequence \( \{ \varphi_{\varepsilon} \}_{\varepsilon > 0} \subset W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^m) \) such that \( \varphi_{\varepsilon} \to \varphi \) in \( L^p(\Omega, \mathbb{R}^m) \) as \( \varepsilon \to 0^+ \), we have

\[
\lim_{\varepsilon \to 0^+} I_{\varepsilon}(\varphi_{\varepsilon}) := \lim_{\varepsilon \to 0^+} \left\{ \frac{1}{\varepsilon} \left( \int_\Omega \left( \varepsilon^n \nabla \varphi_{\varepsilon}(x), \ldots, \varepsilon \nabla \varphi_{\varepsilon}(x), \varphi_{\varepsilon}(x) \right) + W(\varphi_{\varepsilon}(x)) \right) dx + \frac{1}{\varepsilon} \int_{\mathbb{R}^N} |\nabla \tilde{H}(\varphi_{\varepsilon})|^2 dx \right\} \\
\geq \int_{\Omega \cap J_\varepsilon} \tilde{E}_0 \left( \varphi^+ (x), \varphi^- (x), \nu(x) \right) dH^{N-1}(x),
\]

where \( \tilde{H}_{\text{per}} \) is defined by (1.23) and

\[
\tilde{E}_0 \left( \varphi^+, \varphi^-, \nu \right) := \inf \left\{ \lim_{\varepsilon \to 0^+} \int_{I_{\nu}} \frac{1}{\varepsilon} \left( G \left( \varepsilon^n \nabla \zeta(y), \ldots, \varepsilon \nabla \zeta(y), \zeta(y) \right) + W(\zeta(y)) + |\nabla H_{\Psi, \zeta, \nu}(y)|^2 \right) \right\},
\]

where \( H_{\Psi, \zeta, \nu}^0 \in W^{1,2}_{\text{loc}}(I_{\nu}, \mathbb{R}^k) \) satisfies

\[
\Delta_y H_{\Psi, \zeta, \nu}^0(y) = \text{div}_y \Psi(\zeta(y)) \quad \text{in } I_{\nu},
\]
\[ S(\varphi^+, \varphi^-, I_\nu) \text{ is defined by } (1.25) \text{ and} \]
\[ \chi(y, \varphi^+, \varphi^-, \nu) := \begin{cases} \varphi^+ & \text{if } y \cdot \nu > 0, \\ \varphi^- & \text{if } y \cdot \nu < 0. \end{cases} \]

Here \( I_\nu := \{ y \in \mathbb{R}^N : |y \cdot \nu_j| < 1/2 \ \forall j = 1, 2 \ldots N \} \) where \( \{ \nu_1, \nu_2, \ldots, \nu_N \} \subset \mathbb{R}^N \) is an orthonormal base in \( \mathbb{R}^N \) such that \( \nu_1 := \nu \).

As the boundary conditions for \( H_{\Psi, \zeta, \nu} \) in (1.24) are different from those for \( H^0_{\Psi, \zeta, \nu} \), there is a natural question either in general upper bound obtained in Theorem 1.1 coincides with the lower bound obtained in Theorem 1.2. The answer yes will mean that we will find the full \( \Gamma \)-limit of \( I_\nu \) in the case of \( BV \cap L^\infty \) limiting functions. The equivalent question is either
\[ \hat{E}_{\text{per}}(\varphi^+, \varphi^-, \nu) = \hat{E}_0(\varphi^+, \varphi^-, \nu), \tag{1.29} \]
where \( E_{\text{per}}(\cdot) \) is defined in (1.28) and \( E_0(\cdot) \) is defined by (1.24). As we showed in (2.7) this is indeed the case when \( m = 1 \). Moreover, in the later case the optimal profiles are one dimensional. It can be showed that the question in the general case is equivalent to the question of equality of upper and lower bound arisen in (3.3).

Section 3 is devoted to the variational formulation of Method of Vanishing Viscosity for systems of Conservation Laws.

**Definition 1.1.** Let \( F(u) = \{ F_{\nu_j}(u) \} \in C^1(\mathbb{R}^k, \mathbb{R}^{k \times N}) \). Set \( \bar{F}_{\nu_j}(u) := (F_{\nu_j}(u), \ldots, F_k(u)) : \mathbb{R}^k \to \mathbb{R}^k \forall j \in \{1, 2, \ldots, N\} \). Consider the system of Conservation Laws
\[ \partial_t u + \text{div}_\nu F(u) = 0 \ \forall (x,t) \in \mathbb{R}^N \times (0, +\infty). \tag{1.30} \]
We say that the function \( \eta(u) \in C^1(\mathbb{R}^k, \mathbb{R}) \) is an entropy for the system (3.30) and \( \Psi(u) := \{ \Psi_1(u), \ldots, \Psi_N(u) \} \in C^1(\mathbb{R}^k, \mathbb{R}^N) \) is an entropy flux associated with \( \eta \) if we have
\[ \nabla_{\nu} \Psi_j(u) = \nabla_{\nu} \eta(u) \cdot \nabla_{\nu} \bar{F}_{\nu_j}(u) \ \forall u \in \mathbb{R}^k, j \in \{1, 2, \ldots, N\}. \tag{1.31} \]

Let \( F(u) = \{ F_{\nu_j}(u) \} \in C^1(\mathbb{R}^k, \mathbb{R}^{k \times N}) \), satisfying \( F(0) = 0 \), \( \eta(u) \in C^2(\mathbb{R}^k, \mathbb{R}) \) be an entropy for the system (1.30), which satisfies \( \eta(u) \geq 0 \) and \( \eta(0) = 0 \), and \( \Psi(u) := \{ \Psi_1(u), \ldots, \Psi_N(u) \} \in C^1(\mathbb{R}^k, \mathbb{R}^N) \) be a corresponding entropy flux associated with \( \eta \). Considered the following family of energy functionals \( \{ I_{\varepsilon,F}(u) \} \), defined for \( u(x,t) : \mathbb{R}^N \times [0, T] \to \mathbb{R}^k \) by
\[ I_{\varepsilon,F}(u) := \int_0^T \int_{\mathbb{R}^N} \left( \varepsilon \left| \nabla_{\nu} \{ \nabla_{\nu} \eta(u(x,t)) \} \right|^2 + \frac{1}{\varepsilon} \left| \nabla_{\nu} H_{F,u}(x,t) \right|^2 \right) dx \, dt + \int_{\mathbb{R}^N} \eta(u(x,T)) \, dx, \tag{1.32} \]
where \( H_{F,u}(x,t) \) satisfies
\[ \begin{cases} \Delta_{\nu} H_{F,u}(x,t) = \partial_t u(x,t) + \text{div}_x F(u(x,t)), \\ \nabla_{\nu} H_{F,u}(x,t) \in L^2(0,T; L^2(\mathbb{R}^N, \mathbb{R}^{k \times N})), \end{cases} \tag{1.33} \]
and we assume that
\[ u(x,t) \in L^2(0,T; W^{1,2}_0(\mathbb{R}^N, \mathbb{R}^k)) \cap C(0,T; L^2(\mathbb{R}^N, \mathbb{R}^k)) \cap L^\infty \text{ and } \partial_t u(x,t) \in L^2(0,T; W^{-1,2}(\mathbb{R}^N, \mathbb{R}^k)), \tag{1.34} \]
Since
\[ -\int_0^T \int_{\mathbb{R}^N} \nabla_{\nu} \{ \nabla_{\nu} \eta(u(x,t)) \} : \nabla_{\nu} H_{F,u}(x,t) \, dx \, dt = \int_0^T \int_{\mathbb{R}^N} \nabla_{\nu} \eta(u(x,t)) : \Delta_{\nu} H_{F,u}(x,t) \, dx \, dt \]
\[ = \int_0^T \int_{\mathbb{R}^N} \nabla_{\nu} \eta(u(x,t)) \cdot \left( \partial_t u(x,t) + \text{div}_x F(u(x,t)) \right) \, dx \, dt \]
\[ = \int_{\mathbb{R}^N} \left( \int_0^T \left( \partial_t \{ \eta(u(x,t)) \} \right) \, dt \right) dx + \int_0^T \int_{\mathbb{R}^N} \sum_{j=1}^N \nabla_{\nu} \eta(u(x,t)) \cdot \nabla_{\nu} \bar{F}_{\nu_j}(u(x,t)) \cdot \frac{\partial u}{\partial x_j}(x,t) \, dx \, dt \]
\[ = \int_{\mathbb{R}^N} \left( \eta(u(x,T)) - \eta(u(x,0)) \right) \, dx + \int_0^T \int_{\mathbb{R}^N} \text{div}_{x} \Psi(u(x,t)) \, dx \, dt = \int_{\mathbb{R}^N} \left( \eta(u(x,T)) - \eta(u(x,0)) \right) \, dx, \]
we can rewrite the expression of $I_{\varepsilon,F}(u)$ as

$$I_{\varepsilon,F}(u) = \int_0^T \int_{\mathbb{R}^N} \frac{1}{\varepsilon} \left| \nabla_x \{ \nabla_u \eta(u(x,t)) \} - \nabla_x H_{F,u}(x,t) \right|^2 \, dx \, dt + \int_{\mathbb{R}^N} \eta(u(x,0)) \, dx,$$

(1.35)

Thus if there exists a solution to

$$\begin{cases} \varepsilon \Delta_x \{ \nabla_u \eta(u(x,t)) \} = \partial_t u(x,t) + \text{div}_x F(u(x,t)) & \forall (x,t) \in \mathbb{R}^N \times (0,T), \\ u(x,0) = v_0(x) & \forall x \in \mathbb{R}^N. \end{cases}$$

(1.36)

for some $v_0(x) \in L^2(\mathbb{R}^N,\mathbb{R}^k) \cap L^\infty$ then, by (1.35), $u(x,t)$ is also a minimizer to

$$\inf \left\{ I_{\varepsilon,F}(u) : u(x,0) = v_0(x) \right\}.$$  

(1.37)

Moreover, in this case,

$$\inf \left\{ I_{\varepsilon,F}(u) : u(x,0) = v_0(x) \right\} = \int_{\mathbb{R}^N} \eta(v_0(x)) \, dx,$$

(1.38)

and the function $u(x,t) : \mathbb{R}^N \times [0,T) \rightarrow \mathbb{R}^k$ is a minimizer to (1.37) if and only if $u(x,t)$ is a solution to (1.36).

On the other hand it is clear that if minimizers $u_\varepsilon$ of (1.38) strongly converges in $L^2$ to some $u$, then $u$ is a solution of

$$\begin{cases} \partial_t u + \text{div}_x F(u) = 0 & \forall (x,t) \in \mathbb{R}^N \times (0,T) \\ u(x,0) = v_0(x). \end{cases}$$

(1.39)

Moreover, from the theory of $\Gamma$-limits it is well known that $u$ is a minimizer of the $\Gamma - \lim_{\varepsilon \rightarrow 0^+} I_{\varepsilon,F}(u)$. Thus, it is a natural question in the Method of Vanishing Viscosity for Conservation Laws to know the $\Gamma$-limit of the functional

$$J_{\varepsilon,F,v_0}(u) = \begin{cases} I_{\varepsilon,F}(u) & \text{if } u(x,0) \equiv v_0(x), \\ +\infty & \text{otherwise}. \end{cases}$$

(1.40)

Indeed, as we have a lack of uniqueness of solution to (1.36), the solution obtained by the Method of Vanishing Viscosity will be that, which minimizes the $\Gamma$-limit energy among all possible solutions of (1.39). The question of $\Gamma$-limit for $I_{\varepsilon,F}$ was arisen in [31]. In [27] we found its upper bound, achieved by one-dimensional profiles. Moreover we showed that this bound coincides with the $\Gamma$-limit in the case $k = 1$ i.e. in the case of scalar Conservation Law. In this paper we improve the upper bound in the case of systems and we construct also the lower bound.

As before, we can reduce the problem (1.32) to local problems studied in [32] and [33]. Indeed assume that

$$h_0 : \mathbb{R}^N \rightarrow \mathbb{R}^{k \times N}$$

satisfies $\text{div}_x h_0(x) \equiv v_0(x)$. Then set $L_u : \mathbb{R}^N \times (0,T) \rightarrow \mathbb{R}^{k \times N}$ by

$$L_u(x,t) := h_0(x) + \int_0^t \left\{ \nabla_x H_{F,u}(x,s) - F(x,u(s)) \right\} \, ds.$$  

(1.41)

where $H_{F,u}(x,t)$ satisfies (1.33). So $L_u(x,0) = h_0(x)$ and $\partial_t L_u(x,t) = \nabla_x H_{F,u}(x,t) - F(u(x,t))$. Thus $\text{div}_x L_u(x,0) = v_0(x)$ and $\partial_t \text{div}_x L_u(x,t) = \Delta_x H_{F,u}(x,t) - \text{div}_x F(u(x,t)) = \partial_t u(x,t)$. Therefore we get

$$\text{div}_x L_u(x,t) = u(x,t), \quad \text{div}_x L_u(x,0) = v_0(x) \quad \text{and} \quad \nabla_x H_{F,u}(x,t) = \partial_t L_u(x,t) + F(\text{div}_x L_u(x,t)),$$

(1.42)

Then we can rewrite the energy in (1.32) as

$$I_{\varepsilon,F}(u) = \frac{1}{2} \int_0^T \int_{\mathbb{R}^N} \left( \varepsilon \left| \nabla_x \{ \nabla \eta(u) \} \right|^2 + \frac{1}{\varepsilon} \left| \nabla_x H_{F,u} \right|^2 \right) \, dx \, dt + \frac{1}{2} \int_{\mathbb{R}^N} \eta(u(x,T)) \, dx =$$

$$\frac{1}{2} \int_0^T \int_{\mathbb{R}^N} \left( \varepsilon \left| \nabla_x \{ \nabla \eta(\text{div}_x L_u) \} \right|^2 + \frac{1}{\varepsilon} \left| \partial_t L_u + F(\text{div}_x L_u) \right|^2 \right) \, dx \, dt + \frac{1}{2} \int_{\mathbb{R}^N} \eta(\text{div}_x L_u(x,T)) \, dx.$$  

(1.43)
From the other hand define

$$I_{ε,F}(L) := \frac{1}{2} \int_0^T \int_{R^N} \left( \varepsilon |\nabla_x \{\nabla \eta(\text{div}_x L)\}|^2 + \frac{1}{\varepsilon} |\nabla \eta + F(\text{div}_x L)|^2 \right) dx dt + \frac{1}{2} \int_{R^N} \eta(\text{div}_x L(x,T)) dx$$

if $\text{div}_x L(x,0) = v_0(x)$ . \hspace{1cm} (1.44)

Then if for every $L \in L^2(R^N \times (0,T), R^{k \times N})$ such that $\partial_t L, \text{div}_x L, \nabla_x \text{div}_x L \in L^2$, we set $u(x,t) := \text{div}_x L(x,t)$ then $u \in L^2(R^N \times (0,T), R^k) \cap L^2(0,T; W^{-1,2}(R^N))$, $u(x,0) = v_0(x)$ and $\nabla_x H_{F,u} = (R + \partial_t L + F(\text{div}_x L))$, where $\text{div}_x R = 0$. Thus in particular

$$\int_0^T \int_{R^N} |\nabla_x H_{F,u}|^2 dx dt = \int_0^T \int_{R^N} |\partial_t L_u + F(\text{div}_x L_u)|^2 dx dt \leq \int_0^T \int_{R^N} |\partial_t L + F(\text{div}_x L)|^2 dx dt.$$

Thus, as before, we obtain that the $\Gamma - \lim$ and $\Gamma - \text{lim}$ of $I_{ε,F}$ when $u_ε \to u$ are the same as the $\Gamma - \lim$ and $\Gamma - \text{lim}$ of $I_{ε,F}$ as $(\text{div}_x L, \partial_t L) \to (u, -F(u))$. Applying the results of \hspace{1cm} and \hspace{1cm} we obtain the following theorems about the upper and lower bounds (see Theorems \hspace{1cm} and \hspace{1cm} for the equivalent formulations).

**Theorem 1.3.** Let $F(u) \in C^1(R^k, R^{k \times N})$ and $η(u) \in C^3(R^k, R)$ be an entropy for the system \hspace{1cm}, which satisfies $η(u) \geq 0$ and $η(0) = 0$. Furthermore, let $u(x,t) \in BV(R^N \times (0,T), R^k) \cap L^∞(0,T; L^2(R^N, R^k)) \cap L^∞$ be such that $u(x,t)$ is continuous in $[0,T]$ as a function of $t$ with the values in $L^∞(R^N, R^k)$ and satisfy the following Conservation Law on the strip:

$$\partial_t v(x,t) + \text{div}_x F(v(x,t)) = 0 \hspace{1cm} \forall (x,t) \in R^N \times (0,T). \hspace{1cm} (1.45)$$

Then there exists a sequence of functions $\{\tilde{v}_ε(x,t)\}_{ε \to 0} \subset L^∞(R^N \times (0,T), R^{k \times N})$ such that $\tilde{v}_ε(x,t) := \text{div}_x \tilde{v}_ε(x,t) \in L^2(0,T; H^1_0(R^N, R^k)) \cap C([0,T]; L^2(R^N, R^k)) \cap L^∞$ and $\tilde{L}_ε(x,t) := -\partial_t \tilde{v}_ε(x,t) \in L^2(R^N \times (0,T), R^{k \times N})$; $\tilde{u}_ε \to u$ in $\bigcap_{q \geq 1} L^q(R^N \times (0,T); R^k)$; $\tilde{L}_ε \to F(u)$ in $L^2(R^N \times (0,T), R^{k \times N})$; $\partial_t \tilde{u}_ε + \text{div}_x \tilde{L}_ε \equiv 0$, $\tilde{u}_ε(x,0) = u(x,0)$ and

$$\lim_{ε \to 0^+} \left\{ \int_0^T \int_{R^N} \left( \varepsilon |\nabla_x \{\nabla u \eta(\tilde{u}_ε(x,t))\}|^2 + \frac{1}{\varepsilon} |\nabla_x H_{F,\tilde{u}_ε}(x,t)|^2 \right) dx dt + \int_{R^N} \eta(\tilde{u}_ε(x,T)) dx \right\} \leq \int_{R^N} \tilde{E}_0(u^+(x,t), u^-(x,t), \nu(x,t)) \partial \mathcal{H}^N(x,t) + \int_{R^N} \eta(u(x,T)) dx, \hspace{1cm} (1.46)$$

where $H_{F,u}(x,t)$ satisfies \hspace{1cm},

$$\tilde{E}_0(u^+, u^-, \nu) := \inf \left\{ \int_{J_u} \left( L \nabla_y \{\nabla u \eta(\zeta(y,s))\}|^2 + \frac{1}{L} |\gamma(y,s) - F(\zeta(y,s))|^2 \right) dy ds : \right. \hspace{1cm}$$

$$L \in (0, +\infty), \hspace{1cm} \zeta \in Z^{(2)}(u^+, u^-, \nu), \hspace{1cm} \gamma \in Z^{(3)}(F(u^+), F(u^-), \nu), \hspace{1cm} \partial_s \zeta(y,s) + div_y \gamma(y,s) \equiv 0 \left. \right\} = \hspace{1cm}$$

$$\hat{E}_1(u^+, u^-, \nu) := \inf \left\{ \int_{J_u} \left( L \nabla_y \{\nabla u \eta(\zeta(y,s))\}|^2 + \frac{1}{L} |\partial_s \zeta(y,s) + div_y \gamma(y,s)|^2 \right) dy ds : \hspace{1cm}$$

$$L \in (0, +\infty), \hspace{1cm} \zeta \in Z^{(1)}(u^+, u^-, \nu) \right\}, \hspace{1cm} (1.47)$$

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with

\[ Z^{(1)}(u^+, u^-, \nu) := \begin{cases} 
\xi(y, s) \in D'(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}^{k \times N}) : \text{div}_y \xi(y, s) \in C^1(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}^k), \partial_x \xi(y, s) \in C^1(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}^{k \times N}), \\
(\text{div}_y \xi, -\partial_x \xi)(y, s) = (u^-, F(u^-)) \text{ if } y \cdot \nu \leq -1/2, (\text{div}_y \xi, -\partial_x \xi)(y, s) = (u^+, F(u^+)) \text{ if } y \cdot \nu \geq 1/2 \\
\text{and } (\text{div}_y \xi, -\partial_x \xi)((y, s) + \nu_j) = (\text{div}_y \xi, -\partial_x \xi)(y, s) \forall j = 2, 3, \ldots, N \end{cases}, \tag{1.48} \]

\[ Z^{(2)}(u^+, u^-, \nu) := \begin{cases} 
\xi(y, s) \in C^1(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}^k) : \xi(y, s) = u^+ \text{ if } y \cdot \nu \leq -1/2, \\
\xi(y, s) = u^- \text{ if } y \cdot \nu \geq 1/2 \text{ and } \xi((y, s) + \nu_j) = \xi(y, s) \forall j = 2, 3, \ldots, N \end{cases}, \tag{1.49} \]

\[ Z^{(3)}(A, B, \nu) := \begin{cases} 
\gamma(y, s) \in C^1(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}^k \times \mathbb{R}^N) : \gamma(y, s) = B \text{ if } y \cdot \nu \leq -1/2, \\
\gamma(y, s) = A \text{ if } y \cdot \nu \geq 1/2 \text{ and } \gamma((y, s) + \nu_j) = \gamma(y, s) \forall j = 2, 3, \ldots, N \end{cases}. \tag{1.50} \]

Here \( \bar{I}_{\nu} := \{ y \in \mathbb{R}^{N+1} : |y \cdot \nu_j| < 1/2 \ \forall j = 1, 2, \ldots, N \} \) where \( \{ \nu_1, \nu_2, \ldots, \nu_N, \nu_{N+1} \} \subset \mathbb{R}^{N+1} \) is an orthonormal base in \( \mathbb{R}^{N+1} \) such that \( \nu_1 = \nu \).

**Theorem 1.4.** Let \( F(u) \in C^1(\mathbb{R}^k, \mathbb{R}^{k \times N}) \) and \( \eta(u) \in C^3(\mathbb{R}^k, \mathbb{R}) \) be a convex entropy for the corresponding system [3350], which satisfies \( \eta(u) \geq 0, \eta(0) = 0 \) and \( |F(u)| \leq C(|u| + 1) \ \forall u \in \mathbb{R}^k \), for some constant \( C > 0 \). Furthermore, let \( u(x, t) \in BV(\mathbb{R}^N \times (0, T), \mathbb{R}^k) \cap L^\infty(0, T; L^2(\mathbb{R}^N, \mathbb{R}^k)) \) and \( \partial_t u \in L^2(\mathbb{R}^N \times (0, T), \mathbb{R}^{k \times N}) \) such that \( u \rightarrow u \) is continuous in \([0, T]\) as a function of \( t \) with the values in \( L^\infty(\mathbb{R}^N, \mathbb{R}^k) \) and satisfy [1433]. Then for every sequence of functions \( u_\varepsilon(x, t) \in L^2(0, T; H^1_0(\mathbb{R}^N, \mathbb{R}^k)) \cap C([0, T]; L^2(\mathbb{R}^N, \mathbb{R}^k)) \cap L^\infty \) and \( L_\varepsilon(x, t) \in L^2(\mathbb{R}^N \times (0, T), \mathbb{R}^{k \times N}) \) such that \( \partial_t u_\varepsilon + \text{div}_x L_\varepsilon = 0 \) we have

\[
\lim_{\varepsilon \to 0^+} \left\{ \int_0^T \int_{\mathbb{R}^N} \left( \varepsilon \left| \nabla_x \left\{ \text{div}_u \eta(u(x, t)) \right\} \right|^2 + \frac{1}{\varepsilon} \left| L_\varepsilon(x, t) - F(u_\varepsilon(x, t)) \right|^2 \right) \, dx \right\} = \\
\lim_{\varepsilon \to 0^+} \left\{ \int_0^T \int_{\mathbb{R}^N} \left( \varepsilon \left| \nabla_x \left\{ \text{div}_u \eta(u(x, t)) \right\} \right|^2 + \frac{1}{\varepsilon} \left| \partial_t u_\varepsilon(x, t) + F'(\text{div}_x u_\varepsilon(x, t)) \right|^2 \right) \, dt \right\} + \\
\int_{\mathbb{R}^N} \eta(\text{div}_x u_\varepsilon(x, T)) \, dx \geq \\
\lim_{\varepsilon \to 0^+} \left\{ \int_0^T \int_{\mathbb{R}^N} \left( \varepsilon \left| \nabla_x \left\{ \text{div}_u \eta(u(x, t)) \right\} \right|^2 + \frac{1}{\varepsilon} \left| \nabla_x H_{F, u_\varepsilon}(x, t) \right|^2 \right) \, dt \right\} + \\
\int_{\mathbb{R}^N} \eta(u(x, T)) \, dx \geq \\
\int_{I_u} \bar{I}_u^0(u^+(x, t), u^-(x, t), \nu(x, t)) \partial \mathcal{H}^N(x, t) + \int_{\mathbb{R}^N} \eta(u(x, T)) \, dx, \tag{1.51} \]

where \( H_{F, u}(x, t) \) satisfies [1333], \( v_\varepsilon(x, t) \in L^2_{\text{loc}}(\mathbb{R}^N \times (0, T), \mathbb{R}^{k \times N}) \) is such that \( u_\varepsilon(x, t) = \text{div}_x v_\varepsilon(x, t) \) and
Proof. Clearly \( V \subset L^2(\Omega, \mathbb{R}^{k \times N}) \) is a closed subspace of the Hilbert space \( L^2(\Omega, \mathbb{R}^{k \times N}) \). Therefore, clearly there exists a closed subspace \( V \subset L^2(\Omega, \mathbb{R}^{k \times N}) \), such that \( V \) is orthogonally complement of \( L_{k,N}(\Omega) \) in \( L^2(\Omega, \mathbb{R}^{k \times N}) \), i.e.

\[
\int_{\Omega} u(x) : v(x) \, dx = 0, \quad \forall u(x) \in L_{k,N}(\Omega) \forall v(x) \in V,
\]

(2.5)
and for every $h(x) \in L^2(\Omega, \mathbb{R}^{k \times N})$ there exist uniquely defined $u_h(x) \in L_{k,N}(\Omega)$ and $v_h(x) \in V$ such that $h(x) \equiv u_h(x) + v_h(x)$. Thus, in particular, there exist uniquely defined $P(x) \in L_{k,N}(\Omega)$ and $Q(x) \in V$ such that $-M(x) \equiv P(x) + Q(x)$. On the other hand for arbitrary $L(x) \in L_{k,N}(\Omega)$, using (2.5) we have
\[
J_M(L(x)) := \int_{\Omega} \left| L(x) + M(x) \right|^2 dx = \int_{\Omega} \left| (P(x) - L(x)) + Q(x) \right|^2 dx = \int_{\Omega} \left| P(x) - L(x) \right|^2 dx + \int_{\Omega} \left| Q(x) \right|^2 dx + \int_{\Omega} \left( P(x) - L(x) \right) : Q(x) dx = \int_{\Omega} \left| P(x) - L(x) \right|^2 dx + \int_{\Omega} \left( P(x) + M(x) \right)^2 dx = J_M(P(x)) + \int_{\Omega} \left| P(x) - L(x) \right|^2 dx. \tag{2.6}
\]
Thus $L_0(x) \equiv P(x)$ is unique minimizer to (2.3). Moreover, since $P(x) \in L_{k,N}(\Omega)$ and $Q(x) \in V$ we have
\[
\int_{\Omega} Q(x) : L(x) dx = 0, \quad \forall L(x) \in L_{k,N}(\Omega). \tag{2.7}
\]
Thus since $L_0(x) + M(x) \equiv -Q(x)$ we obtain
\[
\int_{\Omega} (L_0(x) + M(x)) : L(x) dx = 0, \quad \forall L(x) \in L_{k,N}(\Omega). \tag{2.8}
\]
In particular,
\[
\int_{\Omega} (L_0(x) + M(x)) : \delta(x) dx = 0, \quad \forall \delta(x) \in C^1_c(\overline{\Omega}, \mathbb{R}^{k \times N}) \text{ such that } \text{div}(\delta(x)) \equiv 0. \tag{2.9}
\]
Thus clearly there exists a function $H_M(x) \in W^{1,2}(\Omega, \mathbb{R}^k)$ such that $H_M(x) \in W^{1,2}(G, \mathbb{R}^k)$ for every bounded open subset $G \subset \Omega$ and $\nabla_x H_M(x) \equiv L_0(x) + M(x)$ on $\Omega$. Thus in particular
\[
\Delta_x H_M(x) = \text{div}_x \left( \nabla_x H_M \right) \equiv \text{div}_x L_0(x) + \text{div}_x M(x) = \text{div}_x M(x).
\]
Moreover obviously $\nabla_x H_M(x) \in L^2(\Omega, \mathbb{R}^{k \times N})$. Finally by (2.9) we have
\[
\int_{\partial \Omega} H_M(x) \cdot \left\{ \delta(x) \cdot n(x) \right\} dH^{N-1}(x) = \int_{\Omega} \left\{ \nabla_x H_M(x) : \delta(x) + H_M(x) \cdot \text{div}_x \delta(x) \right\} dx = 0 \quad \forall \delta(x) \in C^1_c(\overline{\Omega}, \mathbb{R}^{k \times N}) \text{ such that } \text{div}_x \delta(x) \equiv 0. \tag{2.10}
\]
Thus $H_M(x) = 0$ on $\partial \Omega$ in the sense of trace. This completes the proof. \qed

**Lemma 2.2.** Let $\Omega \subset \mathbb{R}^N$ be an open set with locally Lipschitz’s boundary and $G(\psi, x) : \mathbb{R}^m \times \Omega \to \mathbb{R}^{k \times N}$ be a measurable function, continuous by the first argument $\psi$, which satisfies
\[
\left| G(\psi, x) \right| \leq C_0 |\psi|^{p/2} + h_0(\psi) \quad \forall \psi \in \mathbb{R}^m, \ x \in \Omega, \tag{2.11}
\]
for some constant $C_0 > 0$, $p \geq 1$ and $h_0 \in L^2(\Omega, \mathbb{R})$. Furthermore, for every $\varepsilon > 0$ consider the functional $E_\varepsilon(\psi(x)) : L^p(\Omega, \mathbb{R}^m) \to [0, +\infty) \cup \{+\infty\}$ which (possibly) can attain the infinite values. Next for every $\varepsilon > 0$ consider the functional $P_\varepsilon(\psi(x)) : L^p(\Omega, \mathbb{R}^m) \to [0, +\infty) \cup \{+\infty\}$, defined by
\[
P_\varepsilon(\psi(x)) := E_\varepsilon(\psi(x)) + \frac{1}{\varepsilon \alpha} \int_{\mathbb{R}^N} \left| \nabla_x G_{\psi, \psi}(x) \right|^2 dx, \tag{2.12}
\]
where $\delta_\varepsilon > 0$ satisfies $\lim_{\varepsilon \to 0^+} \delta_\varepsilon = 0$ and given $\psi(x) \in L^p(\Omega, \mathbb{R}^m)$, $G_{\psi, \psi}(x) : \mathbb{R}^N \to \mathbb{R}^k$ is defined by
\[
\left\{ \begin{array}{l}
\Delta_x G_{\psi, \psi}(x) = \text{div}_x \left\{ \chi_\Omega(x) \cdot G(\psi(x), x) \right\} \quad \text{in } \mathbb{R}^N, \\
\nabla_x G_{\psi, \psi}(x) \in L^2(\mathbb{R}^N, \mathbb{R}^{k \times N}),
\end{array} \right. \tag{2.13}
\]
with $\chi_\Omega(x) := 1$ if $x \in \Omega$ and $\chi_\Omega(x) := 0$ if $x \in \mathbb{R}^N \setminus \Omega$. Furthermore, for every $\varepsilon > 0$ consider the functional $Q_\varepsilon(\psi(x), L(x)) : L^p(\Omega, \mathbb{R}^m) \times L^2(\mathbb{R}^N, \mathbb{R}^{k \times N}) \to [0, +\infty) \cup \{+\infty\}$ defined by
\[
Q_\varepsilon(\psi(x), L(x)) := \left\{ \begin{array}{l}
E_\varepsilon(\psi(x)) + \frac{1}{\delta_\varepsilon} \int_{\mathbb{R}^N} \left| L(x) + (\chi_\Omega(x) \cdot G(\psi(x), x) \right|^2 dx \quad \text{if } \text{div}_x L(x) = 0, \\
+\infty \quad \text{otherwise}.
\end{array} \right. \tag{2.14}
\]
Next for every \( \varphi(x) \in L^p(\Omega, \mathbb{R}^m) \), such that \( \text{div}_x \{ \chi_\Omega(x) \cdot G(\varphi(x), x) \} = 0 \) in \( \mathbb{R}^N \) set

\[
\begin{align*}
\mathcal{P}(\varphi) & := \inf \left\{ \lim_{\varepsilon \to 0^+} P_\varepsilon(\psi_\varepsilon(x)) : \psi_\varepsilon(x) \to \varphi(x) \text{ in } L^p(\Omega, \mathbb{R}^m) \right\}, \\
\mathcal{Q}(\varphi) & := \inf \left\{ \lim_{\varepsilon \to 0^+} Q_\varepsilon(\psi_\varepsilon(x), L_\varepsilon(x)) : \psi_\varepsilon(x) \to \varphi(x) \text{ in } L^p(\Omega, \mathbb{R}^m) \right\},
\end{align*}
\]

(2.15)

and

\[
\begin{align*}
\mathcal{P}(\varphi) & = \mathcal{Q}(\varphi) \quad \text{and} \quad \overline{\mathcal{P}}(\varphi) = \overline{\mathcal{Q}}(\varphi).
\end{align*}
\]

(2.17)

Proof. Fix some \( \varphi(x) \in L^p(\Omega, \mathbb{R}^m) \) such that \( \text{div}_x \{ \chi_\Omega(x) \cdot G(\varphi(x), x) \} = 0 \) in \( \mathbb{R}^N \). Then by (2.11) we have \( G(\varphi(x), x) \in L^2(\Omega, \mathbb{R}^{k \times N}) \) and thus

\[
- \chi_\Omega(x) \cdot G(\varphi(x), x) \in L^2(\mathbb{R}^N, \mathbb{R}^{k \times N}).
\]

(2.18)

Next fix some sequence \( \{ \psi_\varepsilon(x) \} \subset L^p(\Omega, \mathbb{R}^m) \) such that \( \psi_\varepsilon(x) \to \varphi(x) \) in \( L^p(\Omega, \mathbb{R}^m) \) as \( \varepsilon \to 0^+ \). Then by (2.11) we have

\[
- \chi_\Omega(x) \cdot G(\varphi(x), x) \to - \chi_\Omega(x) \cdot G(\varphi(x), x) \quad \text{in } L^2(\mathbb{R}^N, \mathbb{R}^{k \times N}).
\]

(2.19)

On the other hand by Lemma 2.1 together with (2.12), (2.13) and (2.14) clearly we have

\[
Q_\varepsilon(\psi_\varepsilon(x), L(x)) \geq P_\varepsilon(\psi_\varepsilon(x)) \quad \forall L(x) \in L^2(\mathbb{R}^N, \mathbb{R}^{k \times N}).
\]

(2.20)

Moreover, if we set

\[
\tilde{L}_\varepsilon(x) := \nabla_x V_{G, \psi}(x) - \chi_\Omega(x) \cdot G(\psi_\varepsilon(x), x) \quad \forall x \in \mathbb{R}^N
\]

(2.21)

then \( \tilde{L}_\varepsilon(x) \in L^2(\mathbb{R}^N, \mathbb{R}^{k \times N}) \), \( \text{div} \tilde{L}_\varepsilon(x) \equiv 0 \) and

\[
Q_\varepsilon(\psi_\varepsilon(x), \tilde{L}_\varepsilon(x)) = P_\varepsilon(\psi_\varepsilon(x)).
\]

(2.22)

In particular since \( \delta_\varepsilon \to 0 \) by (2.22), (2.14) and (2.11), for arbitrary sequence \( \varepsilon_n \to 0^+ \) as \( n \to +\infty \), we must have

\[
\tilde{L}_{\varepsilon_n}(x) \to - \chi_\Omega(x) \cdot G(\varphi(x), x) \quad \text{in } L^2(\mathbb{R}^N, \mathbb{R}^{k \times N}) \quad \text{if } \lim_{n \to +\infty} P_{\varepsilon_n}(\psi_{\varepsilon_n}(x)) < +\infty.
\]

(2.23)

Moreover, by (2.22), in this case we have

\[
\lim_{n \to +\infty} Q_{\varepsilon_n}(\psi_{\varepsilon_n}(x), \tilde{L}_{\varepsilon_n}(x)) = \lim_{n \to +\infty} P_{\varepsilon_n}(\psi_{\varepsilon_n}(x)) < +\infty.
\]

(2.24)

On the other hand by (2.20) for every sequence \( \tilde{L}_{\varepsilon_n}(x) \to - \chi_\Omega(x) \cdot G(\varphi(x), x) \) in \( L^2(\mathbb{R}^N, \mathbb{R}^{k \times N}) \) we obviously have

\[
\lim_{n \to +\infty} Q_{\varepsilon_n}(\psi_{\varepsilon_n}(x), L_{\varepsilon_n}(x)) = +\infty \quad \text{if } \lim_{n \to +\infty} P_{\varepsilon_n}(\psi_{\varepsilon_n}(x)) = +\infty.
\]

(2.25)
Therefore, by (2.23), (2.24) and (2.25) in any case there exists a sequence \( \tilde{L}_{\varepsilon_n}(x) \rightarrow -\chi(\Omega(x)) \cdot G(\varphi(x), x) \) in \( L^2(\mathbb{R}^N, \mathbb{R}^{k \times N}) \) so that

\[
\lim_{n \to +\infty} Q_{\varepsilon_n}(\psi_{\varepsilon_n}(x), \tilde{L}_{\varepsilon_n}(x)) = \lim_{n \to +\infty} P_{\varepsilon_n}(\psi_{\varepsilon_n}(x)).
\]

(2.26)

Thus since \( \varepsilon_n \to 0^+ \), \( \psi \to \varphi \) were chosen arbitrary, by (2.26) we deduce

\[
P(\varphi) \geq Q(\varphi) \quad \text{and} \quad P(\varphi) \geq Q(\varphi) \quad \forall \varphi \in L^p(\Omega, \mathbb{R}^m).
\]

(2.27)

On the other hand, by (2.20), clearly

\[
P(\varphi) \leq Q(\varphi) \quad \text{and} \quad P(\varphi) \leq Q(\varphi) \quad \forall \varphi \in L^p(\Omega, \mathbb{R}^m).
\]

(2.28)

This completes the proof. \[\square\]

Next plugging Lemma 2.1 into Theorem 4.2 in [32] we deduce the following upper bound result for problem with a non-local term.

**Theorem 2.1.** Let \( \Omega \subset \mathbb{R}^N \) be an open set with locally Lipschitz’s boundary. Furthermore, let \( G \in C^1(\mathbb{R}^m \times \mathbb{R}^q, \mathbb{R}^{k \times N}) \) and \( F \in C^1(\mathbb{R}^{m \times N^m} \times \ldots \times \mathbb{R}^{m \times N} \times \mathbb{R}^m \times \mathbb{R}^q, \mathbb{R}) \), be such that \( F \geq 0 \). Next let \( \varphi \in BV(\mathbb{R}^N, \mathbb{R}^m) \cap L^\infty \) and \( f \in BV_{loc}(\mathbb{R}^N, \mathbb{R}) \cap L^\infty \) be such that \( \|D\varphi\|_{(\partial \Omega)} = 0 \), \( F(0, \ldots, 0, \varphi(x), f(x)) = 0 \) for a.e. \( x \in \Omega \), \( \text{div}_x G(\varphi(x), f(x)) = 0 \) in \( \Omega \) and \( G(\varphi(x), f(x)) \cdot n(x) = 0 \) on \( \partial \Omega \). Then for every \( \delta > 0 \) there exists a sequence \( \{\psi_{\varepsilon}\}_{\varepsilon > 0} \subset C^\infty(\mathbb{R}^N, \mathbb{R}^m) \) such that \( \int_{\Omega} \psi_{\varepsilon}(x) dx = \int_{\Omega} \varphi(x) dx \), \( \lim_{\varepsilon \to 0^+} \psi_{\varepsilon} = \varphi \) in \( L^p \) and \( \lim_{\varepsilon \to 0^+} \varepsilon \nabla^j \psi_{\varepsilon} = 0 \) in \( L^p \) for every \( p \geq 1 \) and every \( j \in \{1, \ldots, n\} \), \( \{\varepsilon^{n} \nabla^n \psi_{\varepsilon}\}_{\varepsilon > 0}, \ldots, \{\varepsilon^{n} \nabla^n \psi_{\varepsilon}\}_{\varepsilon > 0} \) and \( \{\psi_{\varepsilon}\}_{\varepsilon > 0} \) are a bounded in \( L^\infty \) sequences, and we have

\[
\lim_{\varepsilon \to 0^+} \left\{ \frac{1}{\varepsilon} \int_{\Omega} F(\varepsilon^n \nabla^n \psi_{\varepsilon}(x), \ldots, \nabla \psi_{\varepsilon}(x), \psi_{\varepsilon}(x), f(x)) dx + \frac{1}{\varepsilon} \int_{\mathbb{R}^N} |\nabla V_{G, \psi}|^2 dx \right\}
\]

\[
\leq \int_{\Omega \cap J_{\varphi}} \hat{E}_{per}(\varphi^+(x), \varphi^-(x), \nu(x), x) d\mathcal{H}^{N-1}(x) + \delta,
\]

(2.29)

where \( V_{G, \psi} : \mathbb{R}^N \to \mathbb{R}^k \) is defined by

\[
\begin{cases}
\Delta_{x} V_{G, \psi}(x) = \text{div}_x \left\{ \chi(\Omega(x)) G(\psi(x), f(x)) \right\} & \text{in } \mathbb{R}^N, \\
\nabla V_{G, \psi} \in L^2(\mathbb{R}^N, \mathbb{R}^{k \times N}),
\end{cases}
\]

(2.30)

\[
\hat{E}_{per}(\varphi^+, \varphi^-, \nu, x) := \inf \left\{ \int_{I_{L}} \frac{1}{L} F(L^n \nabla^n \zeta(y), \ldots, L \nabla \zeta(y), \zeta(y), f^+(x)) dy + \right.
\]

\[
\int_{I_{L}} \frac{1}{L} F(L^n \nabla^n \zeta(y), \ldots, L \nabla \zeta(y), \zeta(y), f^-(x)) dy + \left. \int_{I_{L}} \frac{1}{L} |\nabla H_{G, \zeta, x, \nu}(y)|^2 dy : \right.
\]

\[
L \in (0, +\infty), \ z \in S(\varphi^+, \varphi^-, I_{L}) \right\},
\]

(2.31)

where \( H_{G, \zeta, x, \nu} \in W_{loc}^{2,2}(\mathbb{R}^N, \mathbb{R}^k) \) satisfying

\[
\begin{cases}
\Delta_{y} H_{G, \zeta, x, \nu}(y) = \text{div}_y G(\zeta(y), \sigma_{f, x}(y)) & \text{in } I_{\nu}, \\
H_{G, \zeta, x, \nu}(y + \nu) = H_{G, \zeta, x, \nu}(y) & \forall y \in \mathbb{R}^N \text{ such that } |y \cdot \nu| < 1/2, \\
\left. \frac{\partial}{\partial y} H_{G, \zeta, x, \nu}(y) = 0 \right\} \forall y \in \mathbb{R}^N \text{ such that } |y \cdot \nu| = 1/2,
\end{cases}
\]

(2.32)

with

\[
\sigma_{f, x}(y) := \begin{cases}
f^+(x) & \text{if } y \cdot \nu > 0, \\
f^-(x) & \text{if } y \cdot \nu < 0,
\end{cases}
\]

(2.33)
and
\[
S(\varphi^+, \varphi^-, I_\nu) := \left\{ \zeta \in C^\infty(\mathbb{R}^N, \mathbb{R}^m) : \begin{array}{l}
\zeta(y) = \varphi^- \text{ if } y \cdot \nu \leq -1/2,
\zeta(y) = \varphi^+ \text{ if } y \cdot \nu \geq 1/2 \text{ and } \zeta(y + \nu_j) = \zeta(y) \quad \forall j = 2, 3, \ldots, N
\end{array} \right\}, \quad (2.34)
\]

Here
\[
I_\nu := \left\{ y \in \mathbb{R}^N : |y \cdot \nu_j| < 1/2 \quad \forall j = 1, 2, \ldots, N \right\},
\]
\[
I_\nu^+ := \left\{ y \in \mathbb{R}^N : y \cdot \nu \in (0, 1/2) \text{ and } |y \cdot \nu_j| < 1/2 \quad \forall j = 2, 3, \ldots, N \right\},
\]
\[
I_\nu^- := \left\{ y \in \mathbb{R}^N : y \cdot \nu \in (-1/2, 0) \text{ and } |y \cdot \nu_j| < 1/2 \quad \forall j = 2, 3, \ldots, N \right\},
\]
where \( \{\nu_1, \nu_2, \ldots, \nu_N\} \subset \mathbb{R}^N \) is an orthonormal base in \( \mathbb{R}^N \) such that \( \nu_1 := \nu \).

Proof. Since \( \text{div}_x G(\varphi(x), f(x)) = 0 \) in \( \Omega \) and \( G(\varphi(x), f(x)) \cdot n(x) = 0 \) on \( \partial \Omega \) we easily deduce that
\[
\text{div}_x \{ \chi_\Omega(x) G(\varphi(x), f(x)) \} = 0 \quad \text{in } \mathbb{R}^N
\]
in the sense of distribution. Then by Theorem 4.2 in [32] we deduce that for every \( \delta > 0 \) there exist sequences \( \{\psi_\varepsilon\}_{\varepsilon > 0} \subset C^\infty(\mathbb{R}^N, \mathbb{R}^m) \) and \( \{L_\varepsilon\}_{\varepsilon > 0} \subset C^\infty(\mathbb{R}^N, \mathbb{R}^{k \times N}) \) such that
\[
\int_\Omega \psi_\varepsilon(x) dx = \int_\Omega \varphi(x) dx, \lim_{\varepsilon \to 0^+} \psi_\varepsilon = \varphi \text{ in } L^p \text{ and } \lim_{\varepsilon \to 0^+} \varepsilon^j \nabla^j \psi_\varepsilon = 0 \text{ in } L^p \quad \text{for every } p \geq 1 \text{ and every } j \in \{1, \ldots, n\}, \{\varepsilon^n \nabla^n \psi_\varepsilon\}_{\varepsilon > 0}, \{\varepsilon \nabla \psi_\varepsilon\}_{\varepsilon > 0}, \{\psi_\varepsilon\}_{\varepsilon > 0} \text{ are bounded in } L^\infty \text{ sequences, div } L_\varepsilon \equiv 0 \text{ in } \mathbb{R}^N, L_\varepsilon \to \chi_\Omega G(\varphi, f) \text{ in } L^2 \text{ and we have}
\]
\[
\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_{\Omega} F \left( \varepsilon^n \nabla^n \psi_\varepsilon(x), \ldots, \varepsilon \nabla \psi_\varepsilon(x), \psi_\varepsilon(x), f(x) \right) \chi_\Omega(x) dx + \frac{1}{\varepsilon} \int_{\Omega} \left| L_\varepsilon - \chi_\Omega G(\psi_\varepsilon, f) \right|^2 dx \leq \int_{\Omega \setminus J_\varepsilon} \tilde{E}_{\text{per}} \left( \varphi^+(x), \varphi^-(x), \nu(x), x \right) d\mathcal{H}^{N-1}(x) + \delta, \quad (2.35)
\]
where
\[
\tilde{E}_{\text{per}} \left( \varphi^+, \varphi^-, \nu, x \right) := \inf \left\{ \int_{I_\nu^+} \frac{1}{L} F(L^n \nabla^n \zeta(y), \ldots, L \nabla \zeta(y), \zeta(y), f^+(x)) \, dy + \int_{I_\nu^-} \frac{1}{L} F(L^n \nabla^n \zeta(y), \ldots, L \nabla \zeta(y), \zeta(y), f^-(x)) \, dy + \int_{I_\nu} \frac{1}{L} \left| \xi(y) - G(\zeta(y), \sigma_{f, \varepsilon}(y)) \right|^2 dy : \right. \left. L \in (0, +\infty), \right.
\]
\[
\zeta \in S(\varphi^+, \varphi^-, I_\nu), \right. \left. \xi \in S_0(\varphi^+, f^+, \varphi^-, f^-, I_\nu) \right\}, \quad (2.36)
\]
with
\[
S(\varphi^+, \varphi^-, I_\nu) := \left\{ \zeta \in C^\infty(\mathbb{R}^N, \mathbb{R}^m) : \begin{array}{l}
\zeta(y) = \varphi^- \text{ if } y \cdot \nu \leq -1/2,
\zeta(y) = \varphi^+ \text{ if } y \cdot \nu \geq 1/2 \text{ and } \zeta(y + \nu_j) = \zeta(y) \quad \forall j = 2, 3, \ldots, N
\end{array} \right\},
\]
\[
S_0(\varphi^+, f^+, \varphi^-, f^-, I_\nu) := \left\{ \xi \in C^\infty(\mathbb{R}^N, \mathbb{R}^{k \times N}) : \text{div}_x \xi(y) = 0, \xi(y) = G(\varphi^-, f^-) \text{ if } y \cdot \nu \leq -1/2,
\xi(y) = G(\varphi^+, f^+) \text{ if } y \cdot \nu \geq 1/2 \text{ and } \xi(y + \nu_j) = \xi(y) \quad \forall j = 2, 3, \ldots, N \right\}. \quad (2.37)
\]

Thus using Lemma 2.1 by (2.35) we deduce
\[
\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_{\Omega} F \left( \varepsilon^n \nabla^n \psi_\varepsilon(x), \ldots, \varepsilon \nabla \psi_\varepsilon(x), \psi_\varepsilon(x), f(x) \right) dx + \frac{1}{\varepsilon} \int_{\Omega} |\nabla V_{G, \psi_\varepsilon}|^2 dx \leq \int_{\Omega \setminus J_\varepsilon} \tilde{E}_{\text{per}} \left( \varphi^+(x), \varphi^-(x), \nu(x), x \right) d\mathcal{H}^{N-1}(x) + \delta, \quad (2.38)
\]

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where $V_{G,\varphi} : \mathbb{R}^N \to \mathbb{R}^k$ is defined by (2.30). Therefore, in order to complete the proof of the Theorem it is sufficient to prove that we always have
\begin{equation}
\hat{E}_{\text{per}}(\varphi^+, \varphi^-, \nu, x) = \hat{E}_{\text{per}}(\varphi^+, \varphi^-, \nu, x),
\tag{2.39}
\end{equation}
(see the definitions of the corresponding quantities in (2.30) and (2.31)). So fix some $\zeta \in S(\varphi^+, \varphi^-, I_\nu)$ and $L > 0$. Then it is sufficient to prove that
\begin{equation}
\inf \left\{ \int_{I_\nu} \left| \xi(y) - G(\zeta(y), \sigma_{f,x}(y)) \right|^2 \, dy : \xi \in S_0(\varphi^+, f^+, \varphi^-, f^-, I_\nu) \right\} = \int_{I_\nu} \left| \nabla H_{G,\zeta,x,\nu}(y) \right|^2 \, dy,
\tag{2.40}
\end{equation}
where $H_{G,\zeta,x,\nu} \in W^{2,2}_{\text{loc}}(\mathbb{R}^N, \mathbb{R}^k)$ satisfies (2.32). Indeed set
\begin{equation}
T(\varphi^+, f^+, \varphi^-, f^-, I_\nu) := \left\{ \xi \in L^2_{\text{loc}}(\mathbb{R}^N, \mathbb{R}^{k\times N}) : \text{div}_y \xi(y) = 0, \xi(y) = G(\varphi^-, f^-) \text{ if } y \cdot \nu < -1/2, \right. \\
\left. \xi(y) = G(\varphi^+, f^+) \text{ if } y \cdot \nu > 1/2 \text{ and } \xi(y + \nu_j) = \xi(y) \forall j = 2, \ldots, N \right\} \supseteq S_0(\varphi^+, f^+, \varphi^-, f^-, I_\nu).
\tag{2.41}
\end{equation}
Then clearly by the density arguments we have
\begin{equation}
\inf \left\{ \int_{I_\nu} \left| \xi(y) - G(\zeta(y), \sigma_{f,x}(y)) \right|^2 \, dy : \xi \in S_0(\varphi^+, f^+, \varphi^-, f^-, I_\nu) \right\} \\
\quad = \inf \left\{ \int_{I_\nu} \left| \xi(y) - G(\zeta(y), \sigma_{f,x}(y)) \right|^2 \, dy : \xi \in T(\varphi^+, f^+, \varphi^-, f^-, I_\nu) \right\}.
\tag{2.42}
\end{equation}
On the other hand clearly exists a minimizer to the r.h.s. of (2.41) i.e. there exists such $\xi_0 \in T(\varphi^+, f^+, \varphi^-, f^-, I_\nu)$ that
\begin{equation}
\int_{I_\nu} \left| \xi_0(y) - G(\zeta(y), \sigma_{f,x}(y)) \right|^2 \, dy \\
\quad = \inf \left\{ \int_{I_\nu} \left| \xi(y) - G(\zeta(y), \sigma_{f,x}(y)) \right|^2 \, dy : \xi \in T(\varphi^+, f^+, \varphi^-, f^-, I_\nu) \right\}.
\tag{2.43}
\end{equation}
Moreover, $\xi_0$ clearly satisfies
\begin{equation}
\int_{I_\nu} \left( \xi_0(y) - G(\zeta(y), \sigma_{f,x}(y)) \right) : \theta(y) \, dy \quad \text{for every } \theta \in L^2_{\text{loc}}(\mathbb{R}^N, \mathbb{R}^{k\times N})
\end{equation}
such that $\text{div}_y \theta(y) = 0$, $\theta(y) = 0$ if $|y \cdot \nu| > 1/2$, and $\theta(y + \nu_j) = \theta(y) \forall j = 2, \ldots, N$. (2.44)
In particular there exists $H \in W^{1,2}_{\text{loc}}(\mathbb{R}^N, \mathbb{R}^k)$, such that $H(y + \nu_j) = H(y) \forall y \in \mathbb{R}^N, \forall j = 2, \ldots, N$ and $\nabla_y H(y) = G(\zeta(y), \sigma_{f,x}(y)) - \xi_0(y)$ on $N_\nu$ where $N_\nu := \{ y \in \mathbb{R}^N : |y \cdot \nu| < 1/2 \}$. On the other hand since $\xi_0 \in T(\varphi^+, f^+, \varphi^-, f^-, I_\nu)$ we clearly have $\nabla_y H(y) \chi_{N_\nu}(y) = G(\zeta(y), \sigma_{f,x}(y)) - \xi_0(y)$ for every $y \in \mathbb{R}^N$. Thus since $\text{div}_y \xi_0(y) \equiv 0$ we obtain $\text{div}_y \{ \nabla_y H(y) \chi_{N_\nu}(y) \} = \text{div}_y G(\zeta(y), \sigma_{f,x}(y))$ on $\mathbb{R}^N$. Therefore, $H(y) \equiv H_{G,\zeta,x,\nu}$ where $H_{G,\zeta,x,\nu}$ satisfies (2.32). Plugging it into (2.43) we deduce
\begin{equation}
\int_{I_\nu} \left| \nabla H_{G,\zeta,x,\nu}(y) \right|^2 \, dy = \inf \left\{ \int_{I_\nu} \left| \xi(y) - G(\zeta(y), \sigma_{f,x}(y)) \right|^2 \, dy : \xi \in T(\varphi^+, f^+, \varphi^-, f^-, I_\nu) \right\}.
\tag{2.45}
\end{equation}
Therefore, using (2.42) and (2.45) we infer (2.40). This completes the proof.

Similarly plugging Lemma 2.1 into Theorem 2.3 in [33] we deduce the following abstract lower bound result for problem with a non-local term.
Theorem 2.2. Let $\Omega \subset \mathbb{R}^N$ be an open set with locally Lipschitz’s boundary. Furthermore, let $p \geq 1$ and $F \in C^0(\mathbb{R}^{m\times N} \times \ldots \times \mathbb{R}^{m\times N} \times \mathbb{R}^m, \mathbb{R})$ and $G \in C^1(\mathbb{R}^m, \mathbb{R}^{k\times N})$, be such that $F \geq 0$ and there exists a constant $C > 0$ such that $|G(b)| \leq C(|b|^{p/2} + 1)$ for every $b \in \mathbb{R}^m$ and $|a_n|^p / C \leq F(a_n, \ldots, a_2, a_1, b) \leq C\left( \sum_{j=1}^n |a_j|^p + |b|^p \right)$ for every $a_j \in \mathbb{R}^{m\times N_j}$ and $b \in \mathbb{R}^m$. Next let $\varphi \in BV(\Omega, \mathbb{R}^m) \cap L^p(\Omega, \mathbb{R}^m)$ be such that $F(0, \ldots, 0, \varphi(x)) = 0$ for a.e. $x \in \Omega$, $\text{div}_x G(\varphi(x)) = 0$ in $\Omega$ and $G(\varphi(x)) \cdot n(x) = 0$ on $\partial \Omega$. Then for every sequence $\{\varphi_\varepsilon\}_{\varepsilon > 0} \subset W^{n,p}_{loc} (\Omega, \mathbb{R}^m)$ such that $\varphi_\varepsilon \to \varphi$ in $L^p_{loc}(\Omega, \mathbb{R}^m)$ as $\varepsilon \to 0^+$, we have

$$
\lim_{\varepsilon \to 0^+} \left\{ \frac{1}{\varepsilon} \int_{\Omega} F\left(\varepsilon^n \nabla \varphi_\varepsilon(x), \ldots, \varepsilon \nabla \varphi_\varepsilon(x), \varphi_\varepsilon(x) \right) dx + \frac{1}{\varepsilon} \int_{\mathbb{R}^N} |\nabla V_{G, \varphi_\varepsilon}|^2 dx \right\} \geq \int_{\Omega \cap J_\varepsilon} \hat{E}_0(\varphi^+(x), \varphi^-(x), \nu(x)) dH^{N-1}(x),
$$

where $V_{G, \varphi} : \mathbb{R}^N \to \mathbb{R}$ is defined by

$$
\begin{aligned}
\Delta_x V_{G, \varphi}(x) &= \text{div}_x \left\{ \chi_G G(\varphi(x)) \right\} \quad \text{in } \mathbb{R}^N, \\
\nabla V_{G, \varphi} &\in L^2(\mathbb{R}^N, \mathbb{R}^{k\times N}),
\end{aligned}
$$

where $H^0_{G, \zeta, \nu} \in W^{1,2}_{loc}(I_\nu, \mathbb{R}^k)$ satisfies

$$
\Delta_y H^0_{G, \zeta, \nu}(y) = \text{div}_y G(\zeta(y)) \quad \text{in } I_\nu,
$$

and

$$
S^{(n)}_2(\varphi^+, \varphi^-, I_\nu) := \left\{ \zeta \in C^0(\mathbb{R}^N, \mathbb{R}_m) : \zeta(y) = \varphi^- \text{ if } y \cdot \nu \leq -1/2, \right. \\
\left. \zeta(y) = \varphi^+ \text{ if } y \cdot \nu \geq 1/2 \text{ and } \zeta(y + \nu_j) = \zeta(y) \quad \forall j = 2, 3, \ldots, N \right\},
$$

where $I_\nu := \{ y \in \mathbb{R}^N : |y \cdot \nu_j| < 1/2 \quad \forall j = 1, 2, \ldots, N \}$ and $\{ \nu_1, \nu_2, \ldots, \nu_N \} \subset \mathbb{R}^N$ is an orthonormal basis in $\mathbb{R}^N$ such that $\nu_1 := \nu$.

Proof. Let $\{\varphi_\varepsilon\}_{\varepsilon > 0} \subset W^{n,p}_{loc}(\Omega, \mathbb{R}^m)$ be such that $\varphi_\varepsilon \to \varphi$ in $L^p_{loc}(\Omega, \mathbb{R}^m)$ as $\varepsilon \to 0^+$. Without loss of generality we may assume

$$
D := \lim_{\varepsilon \to 0^+} \left\{ \frac{1}{\varepsilon} \int_{\Omega} F\left(\varepsilon^n \nabla \varphi_\varepsilon(x), \ldots, \varepsilon \nabla \varphi_\varepsilon(x), \varphi_\varepsilon(x) \right) dx + \frac{1}{\varepsilon} \int_{\mathbb{R}^N} |\nabla V_{G, \varphi_\varepsilon}|^2 dx \right\} < +\infty.
$$

Next set $L_\varepsilon := \chi_G G(\varphi(x)) - \nabla V_{G, \varphi_\varepsilon}$. Then we have div $L_\varepsilon \equiv 0$ on $\Omega$ and we have

$$
D = \lim_{\varepsilon \to 0^+} \left\{ \frac{1}{\varepsilon} \int_{\Omega} F\left(\varepsilon^n \nabla \varphi_\varepsilon(x), \ldots, \varepsilon \nabla \varphi_\varepsilon(x), \varphi_\varepsilon(x) \right) dx + \frac{1}{\varepsilon} \int_{\mathbb{R}^N} |L_\varepsilon - G(\varphi_\varepsilon(x))|^2 dx \right\} \geq \lim_{\varepsilon \to 0^+} \left\{ \frac{1}{\varepsilon} \int_{\Omega} F\left(\varepsilon^n \nabla \varphi_\varepsilon(x), \ldots, \varepsilon \nabla \varphi_\varepsilon(x), \varphi_\varepsilon(x) \right) dx + \frac{1}{\varepsilon} \int_{\Omega} \left| L_\varepsilon - G(\varphi_\varepsilon(x)) \right|^2 dx \right\}.
$$
Thus applying Theorem 2.3 in [33] we deduce
\[
D \geq \lim_{\varepsilon \to 0^+} \left\{ \frac{1}{\varepsilon} \int_{\Omega} F\left( \varepsilon^n \nabla \varphi_\varepsilon(x), \ldots, \varepsilon \nabla \varphi_\varepsilon(x), \varphi_\varepsilon(x) \right) dx + \frac{1}{\varepsilon} \int_{\Omega} |L_\varepsilon - G(\varphi_\varepsilon(x))|^2 dx \right\} \geq \int_{\Omega \cap \mathcal{J}_\nu} \tilde{E}_0\left( \varphi^+(x), \varphi^-(x), \nu(x) \right) d\mathcal{H}^{N-1}(x), \quad (2.53)
\]
where
\[
\tilde{E}_0\left( \varphi^+, \varphi^-, \nu, x \right) := \inf \left\{ \lim_{\varepsilon \to 0^+} \int_{\Omega} \frac{1}{\varepsilon} \left( F\left( \varepsilon^n \nabla \varphi_\varepsilon(x), \ldots, \varepsilon \nabla \varphi_\varepsilon(x), \varphi_\varepsilon(x) \right) + |\varphi_\varepsilon(x) - G(\varphi_\varepsilon(x))|^2 \right) dy : \right.
\]
\[
\left. \varphi_\varepsilon(x) \in W^{n,p}(I_\nu, \mathbb{R}^m), \varphi_\varepsilon(x) \in W^{n,2}(I_\nu, \mathbb{R}^{k \times N}) \; \text{such that} \; \text{div}_y \varphi_\varepsilon(x) \equiv 0, \right.
\]
\[
\left. \varphi_\varepsilon(x) \to \chi(y, \varphi^+, \varphi^-, \nu) \; \text{in} \; L^p(I_\nu, \mathbb{R}^m) \; \text{and} \; \varphi_\varepsilon(x) \to \chi(y, G(\varphi^+), G(\varphi^-), \nu) \; \text{in} \; L^2(I_\nu, \mathbb{R}^{k \times N}) \right\}. \quad (2.54)
\]
On the other hand by Proposition 3.1 and Lemma 3.3 in [33] we obtain
\[
\tilde{E}_0\left( \varphi^+, \varphi^-, \nu, x \right) \geq \inf \left\{ \lim_{\varepsilon \to 0^+} \int_{\Omega} \frac{1}{\varepsilon} \left( F\left( \varepsilon^n \nabla \varphi_\varepsilon(x), \ldots, \varepsilon \nabla \varphi_\varepsilon(x), \varphi_\varepsilon(x) \right) + |\varphi_\varepsilon(x) - G(\varphi_\varepsilon(x))|^2 \right) dy : \right.
\]
\[
\left. \varphi_\varepsilon(x) \in S^{(n)}_2(\varphi^+, \varphi^-, I_\nu), \varphi_\varepsilon(x) \in W^{n,2}(I_\nu, \mathbb{R}^{k \times N}) \; \text{such that} \; \text{div}_y \varphi_\varepsilon(x) \equiv 0, \right.
\]
\[
\left. \varphi_\varepsilon(x) \to \chi(y, \varphi^+, \varphi^-, \nu) \; \text{in} \; L^p(I_\nu, \mathbb{R}^m) \; \text{and} \; \varphi_\varepsilon(x) \to \chi(y, G(\varphi^+), G(\varphi^-), \nu) \; \text{in} \; L^2(I_\nu, \mathbb{R}^{k \times N}) \right\}. \quad (2.55)
\]
Therefore, using Lemma [2.31] we obtain
\[
\tilde{E}_0(\varphi^+, \varphi^-, \nu, x) \geq \tilde{E}_0(\varphi^+, \varphi^-, \nu, x), \quad (2.56)
\]
where \( \tilde{E}_0(\varphi^+, \varphi^-, \nu, x) \) is defined by (2.48). Thus, plugging (2.56) into (2.53) we deduce
\[
D = \lim_{\varepsilon \to 0^+} \left\{ \frac{1}{\varepsilon} \int_{\Omega} F\left( \varepsilon^n \nabla \varphi_\varepsilon(x), \ldots, \varepsilon \nabla \varphi_\varepsilon(x), \varphi_\varepsilon(x) \right) dx + \frac{1}{\varepsilon} \int_{\mathbb{R}^N} |\nabla V_{\varphi_\varepsilon}|^2 dx \right\} \geq \int_{\mathcal{J}_\nu} \tilde{E}_0\left( \varphi^+(x), \varphi^-(x), \nu(x) \right) d\mathcal{H}^{N-1}(x), \quad (2.57)
\]
and the result follows.

\[\square\]

3 The problem related to the theory of Conservation Laws

3.1 Some definitions and preliminaries

Definition 3.1. For a given Banach space \( X \) with the associated norm \( \| \cdot \|_X \) and a real interval \((a, b)\) we denote by \( L^q(a, b; X) \) the linear space of (equivalence classes of) strongly measurable (i.e. equivalent to some strongly Borel mapping) functions \( f : (a, b) \to X \) such that the functional
\[
\| f \|_{L^q(a, b; X)} := \left\{ \int_a^b \| f(t) \|_X^q dt \right\}^{1/q} \quad \text{if} \quad 1 \leq q < \infty
\]
\[
\sup_{t \in (a, b)} \| f(t) \|_X \quad \text{if} \quad q = \infty
\]
is finite. It is known that this functional defines a norm with respect to which \( L^q(a, b; X) \) becomes a Banach space.

Definition 3.2. Let \( \Omega \subset \mathbb{R}^N \) be an open set. We denote by \( \tilde{H}^1_0(\Omega, \mathbb{R}^k) \) the closure of \( C_c^\infty(\Omega, \mathbb{R}^k) \) with respect to the norm \( ||| \cdot ||| := \left( \int_{\Omega} |\nabla \varphi|^2 dx \right)^{1/2} \) (this space differ from \( W^{1,2}_0(\Omega, \mathbb{R}^k) \) only in the case of unbounded domain \( \Omega \)) and denote by \( H^{-1}(\Omega, \mathbb{R}^k) \) the space dual to \( \tilde{H}^1_0(\Omega, \mathbb{R}^k) \).
Remark 3.1. It is obvious that \( u \in \mathcal{D}'(\Omega, \mathbb{R}^k) \) belongs to \( \dot{H}^{-1}(\Omega, \mathbb{R}^k) \) if and only if there exists \( w \in \dot{H}_{0}^{1}(\Omega, \mathbb{R}^k) \) such that

\[
\int_{\Omega} \nabla w : \nabla \delta \, dx = - \langle u, \delta \rangle \quad \forall \delta \in C_{c}^{\infty}(\Omega, \mathbb{R}^k),
\]

(3.1)

Note that (3.1) is equivalent to that \( \Delta w = u \) as distributions. Moreover,

\[
|||w||| = \sup_{\delta \in \dot{H}_0^1(\Omega, \mathbb{R}^k), \, ||||\delta||| \leq 1} \langle u, \delta \rangle = |||u|||_{-1}.
\]

Finally observe that \( w \) is uniquely defined by \( u \).

Remark 3.2. It is obvious that \( u \in \mathcal{D}'(\Omega \times (0, T), \mathbb{R}^k) \) belongs to \( L^2(0, T; \dot{H}^{-1}(\Omega, \mathbb{R}^k)) \) if and only if there exists \( w \in L^2(0, T; \dot{H}_{0}^{1}(\Omega, \mathbb{R}^k)) \) such that

\[
\int_{0}^{T} \int_{\Omega} \nabla_x w(x, t) : \nabla_x \delta(x, t) \, dx \, dt = - \langle u, \delta \rangle \quad \forall \delta \in C_{c}^{\infty}(\Omega \times (0, T), \mathbb{R}^k).
\]

(3.2)

Note that (3.2) is equivalent to that \( \Delta_x w = u \) as distributions. Moreover,

\[
|||w|||_{L^2(0, T; \dot{H}_{0}^{1}(\Omega, \mathbb{R}^k))} = |||u|||_{L^2(0, T; \dot{H}^{-1}(\Omega, \mathbb{R}^k))}.
\]

Finally observe that \( w \) is uniquely defined by \( u \).

Lemma 3.1. Let \( \Omega \subset \mathbb{R}^N \) be an open set (possibly unbounded) with locally Lipschitz’s boundary and let \( T > 0 \). Furthermore, let \( G(\psi, x, t) : \mathbb{R}^k \times \Omega \times (0, T) \to \mathbb{R}^{k \times N} \) be a measurable function, continuous by the first argument \( \psi \), which satisfies

\[
|G(\psi, x, t)| \leq C_0 |\psi|^{p/2} + h_0(x, t) \quad \forall \psi \in \mathbb{R}^k, \, x \in \Omega,
\]

(3.3)

for some constant \( C_0 > 0 \), \( p \geq 1 \) and \( h_0 \in L^2(\Omega \times (0, T), \mathbb{R}) \). Furthermore, for every \( \varepsilon > 0 \) consider the functional \( E_{\varepsilon}(\psi(x, t)) : L^p(\Omega \times (0, T), \mathbb{R}^k) \to [0, +\infty) \cup \{+\infty\} \) which (possibly) can attain the infinite values.

Next for every \( \varepsilon > 0 \) consider the functional \( P_{\varepsilon}(\psi(x, t)) : L^p(\Omega \times (0, T), \mathbb{R}^k) \to [0, +\infty) \cup \{+\infty\} \), defined by

\[
P_{\varepsilon}(\psi(x, t)) := \begin{cases} 
E_{\varepsilon}(\psi(x, t)) + \frac{1}{\varepsilon} \int_{0}^{T} \int_{\Omega} \left| \nabla_x V_{G,\psi}(x, t) \right|^2 \, dx \, dt & \text{if } \partial_t \psi(x, t) \in L^2(0, T; \dot{H}^{-1}(\Omega, \mathbb{R}^k)), \\
+\infty & \text{otherwise},
\end{cases}
\]

(3.4)

where \( \delta_{\varepsilon} > 0 \) satisfies \( \lim_{\varepsilon \to 0^+} \delta_{\varepsilon} = 0 \) and given \( \psi(x, t) \in L^p(\Omega \times (0, T), \mathbb{R}^k) \), \( V_{G,\psi}(x, t) : \mathbb{R}^N \to \mathbb{R}^k \) is defined by

\[
\Delta_x V_{G,\psi}(x, t) = \partial_t \psi(x, t) + \text{div}_x G(\psi(x, t), x, t) \quad \text{in } \Omega \times (0, T),
\]

\[
V_{G,\psi}(x, t) \in L^2(0, T; \dot{H}_{0}^{1}(\Omega, \mathbb{R}^k)) \, ,
\]

(3.5)

Furthermore, for every \( \varepsilon > 0 \) consider the functional \( Q_{\varepsilon}(\psi(x, t), L(x, t)) : L^p(\Omega \times (0, T), \mathbb{R}^k) \times L^2(\Omega \times (0, T), \mathbb{R}^{k \times N}) \to [0, +\infty) \cup \{+\infty\} \) defined by

\[
Q_{\varepsilon}(\psi(x, t), L(x, t)) := \begin{cases} 
E_{\varepsilon}(\psi(x, t)) + \frac{1}{\varepsilon} \int_{0}^{T} \int_{\Omega} \left| L(x, t) - G(\psi(x, t), x, t) \right|^2 \, dx \, dt & \text{if } \partial_t \psi(x, t) + \text{div}_x L(x, t) = 0, \\
+\infty & \text{otherwise}.
\end{cases}
\]

(3.6)

Finally for every \( \varepsilon > 0 \) consider the functional \( R_{\varepsilon}(u) : \mathcal{D}'(\Omega \times (0, T), \mathbb{R}^{k \times N}) \to [0, +\infty) \cup \{+\infty\} \) defined by

\[
R_{\varepsilon}(u) := \begin{cases} 
E_{\varepsilon}(\text{div}_x u(x, t)) + \frac{1}{\varepsilon} \int_{0}^{T} \int_{\Omega} \left| \partial_t u(x, t) + G(\text{div}_x u(x, t), x, t) \right|^2 \, dx \, dt & \text{if } \partial_t u \in L^2(\Omega \times (0, T), \mathbb{R}^{k \times N}) \text{ and } \text{div}_x u \in L^p(\Omega \times (0, T), \mathbb{R}^k), \\
+\infty & \text{otherwise}.
\end{cases}
\]

(3.7)
Next for every \( \varphi(x,t) \in L^p(\Omega \times (0,T), \mathbb{R}^k) \), such that \( \partial_t \varphi(x,t) + \text{div}_x G(\varphi(x,t), x, t) = 0 \) on \( \Omega \), set

\[
\mathcal{P}(\varphi) := \inf \left\{ \lim_{\varepsilon \to 0^+} P_\varepsilon(\psi_\varepsilon(x,t)) : \psi_\varepsilon(x,t) \to \varphi(x,t) \text{ in } L^p(\Omega \times (0,T), \mathbb{R}^k) \right\},
\]

\[
\mathcal{P}_0(\varphi) := \inf \left\{ \lim_{\varepsilon \to 0^+} P_\varepsilon(\psi_\varepsilon(x,t)) : \psi_\varepsilon(x,t) \to \varphi(x,t) \text{ in } L^p(\Omega \times (0,T), \mathbb{R}^k) \right\},
\]

\[
\mathcal{Q}(\varphi) := \inf \left\{ \lim_{\varepsilon \to 0^+} Q_\varepsilon(\psi_\varepsilon(x,t), L_\varepsilon(x,t)) : \psi_\varepsilon(x,t) \to \varphi(x,t) \text{ in } L^p(\Omega \times (0,T), \mathbb{R}^k) \right\},
\]

\[
\mathcal{Q}_0(\varphi) := \inf \left\{ \lim_{\varepsilon \to 0^+} Q_\varepsilon(\psi_\varepsilon(x,t), L_\varepsilon(x,t)) : \psi_\varepsilon(x,t) \to \varphi(x,t) \text{ in } L^p(\Omega \times (0,T), \mathbb{R}^k) \right\},
\]

\[
\mathcal{R}(\varphi) := \inf \left\{ \lim_{\varepsilon \to 0^+} R_\varepsilon(u_\varepsilon(x,t)) : \text{div}_x u_\varepsilon(x) \to \varphi(x) \text{ in } L^p(\Omega \times (0,T), \mathbb{R}^k) \right\},
\]

\[
\mathcal{R}_0(\varphi) := \inf \left\{ \lim_{\varepsilon \to 0^+} R_\varepsilon(u_\varepsilon(x,t)) : \text{div}_x u_\varepsilon(x) \to \varphi(x) \text{ in } L^p(\Omega \times (0,T), \mathbb{R}^k) \right\},
\]

and

\[
\mathcal{P}(\varphi) = \mathcal{Q}(\varphi) = \mathcal{R}(\varphi) \quad \text{and} \quad \mathcal{P}_0(\varphi) = \mathcal{Q}_0(\varphi) = \mathcal{R}_0(\varphi).
\]

Proof. Fix some \( \psi(x,t) \in L^p(\Omega \times (0,T), \mathbb{R}^k) \), such that \( \partial_t \psi(x,t) \in L^2(0,T; \tilde{H}^{-1}(\Omega, \mathbb{R}^k)) \). Then by (3.3), we clearly have \( G(\psi(x,t), x, t) \in L^2(\Omega \times (0,T), \mathbb{R}^k) \) and then by Remark 3.2 we have \( \text{div}_x G(\psi(x,t), x, t) \in L^2(0,T; \tilde{H}^{-1}(\Omega, \mathbb{R}^k)) \). Thus by Remark 3.2 there exist uniquely defined \( H_\psi(x,t) \in L^2(0,T; \tilde{H}^{-1}(\Omega, \mathbb{R}^k)) \), \( D_{G,\psi}(x,t) \in L^2(0,T; \tilde{H}^1_0(\Omega, \mathbb{R}^k)) \) and \( V_{G,\psi}(x,t) \in L^2(0,T; \tilde{H}^1_0(\Omega, \mathbb{R}^k)) \) such that

\[
\begin{align*}
\Delta_x H_\psi(x,t) &= \partial_t \psi(x,t) \quad \text{in } \Omega \times (0,T), \\
\Delta_x D_{G,\psi}(x,t) &= \text{div}_x G(\psi(x,t), x, t) \quad \text{in } \Omega \times (0,T), \\
\Delta_x V_{G,\psi}(x,t) &= \partial_t \psi(x,t) + \text{div}_x G(\psi(x,t), x, t) \quad \text{in } \Omega \times (0,T),
\end{align*}
\]

and clearly

\[
V_{G,\psi}(x,t) \equiv H_\psi(x,t) + D_{G,\psi}(x,t) \quad \text{in } \Omega \times (0,T).
\]

Moreover, by Lemma 2.1 for every \( U(x,t) \in L^2(\Omega \times (0,T), \mathbb{R}^k) \), such that \( \text{div}_x U(x,t) \equiv 0 \) we have

\[
\int_0^T \int_{\Omega} \left| U(x,t) - \nabla_x H_\psi(x,t) - G(\psi(x,t), x, t) \right|^2 dxdt \geq \int_0^T \int_{\Omega} \left| \nabla_x V_{\psi}(x,t) \right|^2 dxdt,
\]

and if we denote

\[
U_{G,\psi}(x,t) := G(\psi(x,t), x, t) - \nabla_x D_{G,\psi}(x,t) \quad \forall (x,t) \in \Omega \times (0,T),
\]
then \( U_{G, \psi}(x, t) \in L^2(\Omega \times (0, T), \mathbb{R}^{k \times N}) \), \( \text{div}_x U_{G, \psi}(x, t) = 0 \) and
\[
\int_0^T \int_{\Omega} \left| U_{G, \psi}(x, t) - \nabla_x H_{\psi}(x, t) - G(\psi(x, t), x, t) \right|^2 dx dt = \int_0^T \int_{\Omega} \left| \nabla_x V_{\psi}(x, t) \right|^2 dx dt. \tag{3.16}
\]
In particular, by (3.14) for every \( L(x, t) \in L^2(\Omega \times (0, T), \mathbb{R}^{k \times N}) \), such that \( \text{div}_x L(x, t) + \partial_t \psi(x, t) \equiv 0 \) we have
\[
\int_0^T \int_{\Omega} \left| L(x, t) - G(\psi(x, t), x, t) \right|^2 dx dt \geq \int_0^T \int_{\Omega} \left| \nabla_x V_{\psi}(x, t) \right|^2 dx dt, \tag{3.17}
\]
and if we denote
\[
L_{G, \psi}(x, t) := G(\psi(x, t), x, t) - \nabla_x V_{G, \psi}(x, t) \quad \forall (x, t) \in \Omega \times (0, T),
\]
then \( L_{G, \psi}(x, t) \in L^2(\Omega \times (0, T), \mathbb{R}^{k \times N}) \), \( \text{div}_x L_{G, \psi}(x, t) + \partial_t \psi(x, t) \equiv 0 \) and
\[
\int_0^T \int_{\Omega} \left| L_{G, \psi}(x, t) - G(\psi(x, t), x, t) \right|^2 dx dt = \int_0^T \int_{\Omega} \left| \nabla_x V_{\psi}(x, t) \right|^2 dx dt. \tag{3.19}
\]
Finally define
\[
u(x, t) := K_{\psi}(x) - \int_0^t L_{G, \psi}(x, s) ds \quad \forall (x, t) \in \Omega \times (0, T),
\]
where \( K_{\psi}(x) : \Omega \to \mathbb{R}^{k \times N} \) satisfies \( \text{div}_x K_{\psi}(x) \equiv \psi(x, 0) \). Then since \( \text{div}_x L_{G, \psi}(x, t) + \partial_t \psi(x, t) \equiv 0 \) we deduce that
\[
\partial_t \nu(x, t) := -L_{G, \psi}(x, t) \quad \text{and} \quad \text{div}_x \nu(x, t) = \psi(x, t) \quad \forall (x, t) \in \Omega \times (0, T).
\]
Therefore, by (3.4), (3.6) and (3.7), using (3.19) and (3.21) we deduce
\[
P_{\varepsilon}(\psi(x, t)) = Q_{\varepsilon}(\psi(x, t), L_{G, \psi}(x, t)) = R_{\varepsilon}(\nu) .
\]
Moreover, by (3.6) and (3.17) we have
\[
P_{\varepsilon}(\psi(x, t)) \leq Q_{\varepsilon}(\psi(x, t), L(x, t)) \quad \forall L(x, t) \in L^2(\Omega \times (0, T), \mathbb{R}^{k \times N}) .
\]
Next fix some \( \varphi(x, t) \in L^p(\Omega \times (0, T), \mathbb{R}^k) \), such that
\[
\partial_t \varphi(x, t) + \text{div}_x G(\varphi(x, t), x, t) = 0 \quad \text{in} \quad \Omega \times (0, T).
\]
Then, using the fact that given \( \psi(x, t) \in L^p(\Omega \times (0, T), \mathbb{R}^k) \) and \( L(x, t) \in L^2(\Omega \times (0, T), \mathbb{R}^{k \times N}) \), such that \( \text{div}_x L(x, t) + \partial_t \psi(x, t) \equiv 0 \) we always have \( \partial_t \psi(x, t) \in L^2(0, T; H^{-1}(\Omega, \mathbb{R}^k)) \), by (3.23) we obtain
\[
\mathcal{P}(\varphi) \leq Q(\varphi) \quad \text{and} \quad \overline{\mathcal{P}}(\varphi) \leq \overline{Q}(\varphi).
\]
Furthermore, fix some sequence \( \{\psi_n(x, t)\} \subseteq L^p(\Omega \times (0, T), \mathbb{R}^k) \) such that \( \psi_n(x, t) \to \varphi(x, t) \) in \( L^p(\Omega \times (0, T), \mathbb{R}^k) \) as \( \varepsilon \to 0^+ \) and for a subsequence \( \varepsilon_n \downarrow 0 \) we have \( \lim_{n \to +\infty} P_{\varepsilon_n}(\psi_n(x, t)) < +\infty \). Then since \( \delta_{\varepsilon_n} \to 0^+ \), by (3.21) we deduce
\[
\lim_{n \to +\infty} V_{G, \psi_n}(x, t) = 0 \quad \text{in} \quad L^2(\Omega \times (0, T), \mathbb{R}^{k \times N}).
\]
On the other hand, since \( \psi_n(x, t) \to \varphi(x, t) \) in \( L^p(\Omega \times (0, T), \mathbb{R}^k) \), by (3.3) we have
\[
\lim_{\varepsilon \to 0^+} G(\psi_n(x, t), x, t) = G(\varphi(x, t), x, t) \quad \text{in} \quad L^2(\Omega \times (0, T), \mathbb{R}^{k \times N}).
\]
Thus by (3.15), (3.20) and (3.27) we deduce that
\[
\lim_{n \to +\infty} L_{G, \psi_n}(x, t) = G(\varphi(x, t), x, t) \quad \text{in} \quad L^2(\Omega \times (0, T), \mathbb{R}^{k \times N}),
\]
and therefore, by (3.21) we have
\[
\lim_{n \to +\infty} \partial_t \nu_{\varepsilon_n}(x, t) = -G(\varphi(x, t), x, t) \quad \text{in} \quad L^2(\Omega \times (0, T), \mathbb{R}^{k \times N}) \quad \text{and}
\]
\[
\lim_{n \to +\infty} \text{div}_x \nu_{\varepsilon_n}(x, t) = \varphi(x, t) \quad \text{in} \quad L^p(\Omega \times (0, T), \mathbb{R}^k). \tag{3.29}
\]
Moreover, by (3.22) we have
\[ \lim_{n \to +\infty} P_{\varepsilon_n}(\psi_{\varepsilon_n}(x,t)) = \lim_{n \to +\infty} Q_{\varepsilon_n}(\psi_{\varepsilon_n}(x,t), L_G, \psi_{\varepsilon_n}(x,t)) = \lim_{n \to +\infty} R_{\varepsilon_n}(u_{\psi_{\varepsilon_n}}). \] (3.30)

Thus since the sequence \( \{\psi_{\varepsilon}\} \) was arbitrary, we get
\[ P(\varphi) \geq Q(\varphi) \quad \text{and} \quad \overline{P}(\varphi) \geq \overline{Q}(\varphi). \] (3.31)

and
\[ P(\varphi) \geq R(\varphi) \quad \text{and} \quad \overline{P}(\varphi) \geq \overline{R}(\varphi). \] (3.32)

Thus plugging (3.31) into (3.25) we obtain
\[ P(\varphi) = Q(\varphi) \quad \text{and} \quad \overline{P}(\varphi) = \overline{Q}(\varphi). \] (3.33)

Finally fix arbitrary sequence \( \{u_{\varepsilon}\}_{\varepsilon > 0} \subset D'(\Omega \times (0,T), \mathbb{R}^{k \times N}) \) such that
\[ \text{div}_x u_{\varepsilon}(x) \to \varphi(x) \text{ in } L^p(\Omega \times (0,T), \mathbb{R}^k) \quad \text{and} \quad \partial_t u_{\varepsilon}(x) \to \left( -G \left( \varphi(x,t), x, t \right) \right) \text{ in } L^2(\Omega \times (0,T), \mathbb{R}^{k \times N}). \]

Then if we set
\[ L_{\varepsilon}(x,t) := -\partial_t u_{\varepsilon}(x,t) \quad \text{and} \quad \psi_{\varepsilon}(x,t) := \text{div}_x u_{\varepsilon}(x,t) \quad \forall (x,t) \in \Omega \times (0,T), \] (3.34)

we obtain \( \text{div} \ L_{\varepsilon} + \partial_t \psi_{\varepsilon} = 0 \) and
\[ \psi_{\varepsilon}(x,t) \to \varphi(x) \text{ in } L^p(\Omega \times (0,T), \mathbb{R}^k) \quad \text{and} \quad L_{\varepsilon}(x,t) \to G(\varphi(x,t), x, t) \text{ in } L^2(\Omega \times (0,T), \mathbb{R}^{k \times N}). \]

Moreover, by (3.34) we have
\[ Q_{\varepsilon}(\psi_{\varepsilon}(x,t), L_{\varepsilon}(x,t)) = R_{\varepsilon}(u_{\varepsilon}). \]

Therefore, since the sequence \( \{u_{\varepsilon}\}_{\varepsilon > 0} \) was arbitrary, we deduce
\[ Q(\varphi) \leq R(\varphi) \quad \text{and} \quad \overline{Q}(\varphi) \leq \overline{R}(\varphi). \] (3.35)

Thus by plugging (3.33), (3.32) and (3.35) we finally deduce (3.11).

**Definition 3.3.** Let \( F(u) = \{F_{ij}(u)\} \in C^1(\mathbb{R}^k, \mathbb{R}^{k \times N}) \). Set \( F_j(u) := (F_{1j}(u), \ldots, F_{kj}(u)) : \mathbb{R}^k \to \mathbb{R}^k \forall j \in \{1, 2, \ldots, N\} \). Consider the system of Conservation Laws
\[ \partial_t u + \text{div}_x F(u) = 0 \quad \forall (x,t) \in \mathbb{R}^N \times (0, +\infty). \] (3.36)

We say that the function \( \eta(u) \in C^1(\mathbb{R}^k, \mathbb{R}) \) is an entropy for the system (3.36) and \( \Psi(u) := (\Psi_1(u), \ldots, \Psi_N(u)) \in C^1(\mathbb{R}^k, \mathbb{R}^N) \) is an entropy flux associated with \( \eta \) if we have
\[ \nabla_u \Psi_j(u) = \nabla_u \eta(u) \cdot \nabla_u F_j(u) \quad \forall u \in \mathbb{R}^k, j \in \{1, 2, \ldots, N\}. \] (3.37)

Let \( F(u) = \{F_{ij}(u)\} \in C^1(\mathbb{R}^k, \mathbb{R}^{k \times N}) \) and \( \eta(u) \in C^2(\mathbb{R}^k, \mathbb{R}) \) be an entropy for the system (3.36), which satisfies \( \eta(u) \geq 0 \) and \( \eta(0) = 0 \), and \( \Psi(u) := (\Psi_1(u), \ldots, \Psi_N(u)) \in C^1(\mathbb{R}^k, \mathbb{R}^N) \) be a corresponding entropy flux associated with \( \eta \). Considered the following family of energy functionals \( \{I_{\varepsilon, F}(u)\} \), defined for \( u(x,t) : \mathbb{R}^N \times [0, T] \to \mathbb{R}^k \) by
\[ I_{\varepsilon, F}(u) := \int_0^T \int_{\mathbb{R}^N} \left( \frac{1}{\varepsilon} \left| \nabla \left( \sqrt{\varepsilon \nabla \eta(u(x,t))} \right) \right|^2 + \frac{1}{\varepsilon^2} \left| \nabla \eta(\sqrt{\varepsilon \nabla u(x,t)}) \right|^2 \right) dxdt + \int_{\mathbb{R}^N} \eta(u(x,T)) dx, \] (3.38)

where \( H_{F,u}(x,t) \in L^2(0,T; \hat{H}_0^1(\mathbb{R}^N, \mathbb{R}^N)) \) satisfies
\[ \Delta_x H_{F,u}(x,t) = \partial_t u(x,t) + \text{div}_x F(u(x,t)), \] (3.39)
and we assume that
\[
  u(x, t) \in L^2(0, T; \dot{H}_0^1(\mathbb{R}^N, \mathbb{R}^k)) \cap C(0, T; L^2(\mathbb{R}^N, \mathbb{R}^k)) \cap L^\infty \text{ and } \partial_t u(x, t) \in L^2(0, T; \dot{H}^{-1}(\mathbb{R}^N, \mathbb{R}^k)), \tag{3.40}
\]
Since
\[
  -\int_0^T \int_{\mathbb{R}^N} \nabla_x \{\nabla_u \eta(u(x, t))\} : \nabla_x H_{F,u}(x, t) \, dx \, dt = \int_0^T \int_{\mathbb{R}^N} \nabla_u \eta(u(x, t)) \cdot \Delta_x H_{F,u}(x, t) \, dx \, dt
\]
\[
  = \int_0^T \int_{\mathbb{R}^N} \nabla_u \eta(u(x, t)) \cdot \left(\partial_t u(x, t) + \text{div}_x F(u(x, t))\right) \, dx \, dt
\]
\[
  = \int_{\mathbb{R}^N} \left(\int_0^T \partial_t \eta(u(x, t)) \, dt\right) \, dx + \int_0^T \int_{\mathbb{R}^N} \sum_{j=1}^N \nabla_u \eta(u(x, t)) \cdot \nabla_x F_j(u(x, t)) \cdot \frac{\partial u}{\partial x_j}(x, t) \, dx \, dt
\]
\[
  = \int_{\mathbb{R}^N} \left(\eta(u(x, T)) - \eta(u(x, 0))\right) \, dx + \int_0^T \int_{\mathbb{R}^N} \text{div}_x \Psi(u(x, t)) \, dx \, dt = \int_{\mathbb{R}^N} \left(\eta(u(x, T)) - \eta(u(x, 0))\right) \, dx,
\]
we can rewrite the expression of $I_{\varepsilon,F}(u)$ as
\[
  I_{\varepsilon,F}(u) = \int_0^T \int_{\mathbb{R}^N} \frac{1}{\varepsilon} \left| \varepsilon \Delta_x \{\nabla_u \eta(u(x, t))\} - \nabla_x H_{F,u}(x, t) \right|^2 \, dx \, dt + \int_{\mathbb{R}^N} \eta(u(x, 0)) \, dx, \tag{3.41}
\]
Thus if there exists a solution to
\[
  \begin{cases}
    \varepsilon \Delta_x \{\nabla_u \eta(u(x, t))\} = \partial_t u(x, t) + \text{div}_x F(u(x, t)) & \forall (x, t) \in \mathbb{R}^N \times (0, T), \\
    u(x, 0) = v_0(x) & \forall x \in \mathbb{R}^N.
  \end{cases}
\tag{3.42}
\]
for some $v_0(x) \in L^2(\mathbb{R}^N, \mathbb{R}^k) \cap L^\infty$ then, by (3.41), $u(x, t)$ is also a minimizer to
\[
  \inf \{I_{\varepsilon,F}(u) : u(x, 0) = v_0(x)\}. \tag{3.43}
\]
Moreover, in this case,
\[
  \inf \{I_{\varepsilon,F}(u) : u(x, 0) = v_0(x)\} = \int_{\mathbb{R}^N} \eta(v_0(x)) \, dx, \tag{3.44}
\]
and the function $u(x, t) : \mathbb{R}^N \times [0, T] \to \mathbb{R}^k$ is a minimizer to (3.44) if and only if $u(x, t)$ is a solution to (3.42).

Thus it is a natural question in the Method of Vanishing Viscosity for Conservation Laws to know the $\Gamma$-limit of the functional
\[
  J_{\varepsilon,F,v_0}(u) = \begin{cases}
    I_{\varepsilon,F}(u) & \text{if } u(x, 0) \equiv v_0(x), \\
    +\infty & \text{otherwise}.
  \end{cases}
\tag{3.45}
\]

**Definition 3.4.** Let $F \in C^1(\mathbb{R}^k, \mathbb{R}^{k \times N})$ satisfying $F(0) = 0$. Denote by $\mathcal{E}_F^F$ the class of all $u(x, t) \in BV(\mathbb{R}^N \times (0, T), \mathbb{R}^k) \cap L^\infty(0, T; L^2(\mathbb{R}^N, \mathbb{R}^k)) \cap L^\infty$ such that $u(x, t)$ is continuous in $[0, T]$ as a function of $t$ with the values in $L^\infty(\mathbb{R}^N, \mathbb{R}^k)$ with respect to $L^\infty$-weak* topology and satisfy the following Conservation Law on the strip:
\[
  \partial_t u(x, t) + \text{div}_x F(u(x, t)) = 0 \quad \forall (x, t) \in \mathbb{R}^N \times (0, T). \tag{3.46}
\]

**Remark 3.3.** If $u(x, t) \in \mathcal{E}_F^F$ then we can continue $u(x, t)$ to be defined on $\mathbb{R} \times \mathbb{R}$, so that $u(x, t) \in BV(\mathbb{R} \times \mathbb{R}, \mathbb{R}^k) \cap L^\infty(\mathbb{R}; L^2(\mathbb{R}^N, \mathbb{R}^k)) \cap L^\infty$, $u(x, -t) \equiv u(x, t)$ $\forall (x, t) \in \mathbb{R}^N \times \mathbb{R}$ and for every $1 \leq q < +\infty$ $u(x, t)$ is continuous in $\mathbb{R}$ as a function of $t$ with the values in $L^q(\mathbb{R}^N, \mathbb{R}^k)$ with respect to $L^q$-strong topology.

**Lemma 3.2.** Let $F(u) \in C^1(\mathbb{R}^k, \mathbb{R}^{k \times N})$ satisfying $F(0) = 0$ and $\eta(u) \in C^3(\mathbb{R}^k, \mathbb{R})$ be an entropy for the corresponding system, which satisfies $\eta(u) \geq 0$ and $\eta(0) = 0$. Furthermore, let $u \in \mathcal{E}_F^F$ (see Definition 3.4 and Remark 3.3) and $\kappa(h) \in C^\infty(\mathbb{R}^{N+1})$ be a radial function, such that $\int_{\mathbb{R}^{N+1}} \kappa(h) dh = 1$. Then for every $\delta > 0$ there exists a sequence of functions $\{v_\varepsilon(x, t)\}_{\varepsilon > 0} \subset C^\infty(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}^k \times N)$ such that $u_\varepsilon(x, t) := \text{div}_x v_\varepsilon(x, t) \in W^{1,2}(\mathbb{R}^N \times (0, T), \mathbb{R}^k) \cap L^\infty$ and $L_\varepsilon(x, t) := -\partial_t v_\varepsilon(x, t) \in W^{1,2}(\mathbb{R}^N \times (0, T), \mathbb{R}^k \times N) \cap L^\infty$; $\{v_\varepsilon\}$, $\{L_\varepsilon\}$ and
\{\varepsilon \nabla_x u_\varepsilon\} are bounded in $L^\infty$ sequences; $u_\varepsilon \to u$, $L_\varepsilon \to F(u)$ and $\varepsilon \nabla_x u_\varepsilon \to 0$ as $\varepsilon \to 0^+$ in $L^q(\mathbb{R}^N \times (0, T))$; 
$\partial_t u_\varepsilon + \operatorname{div}_x L_\varepsilon \equiv 0$ and

\[
\lim_{\varepsilon \to 0^+} \int_0^T \int_{\mathbb{R}^N} \left( \varepsilon \left| \nabla_x \left\{ \nabla_y \eta(u_\varepsilon(x,t)) \right\} \right|^2 + \frac{1}{\varepsilon} \left| \nabla_x H_{F,u_\varepsilon}(x,t) \right|^2 \right) \, dx \, dt \
\leq \int_{\tilde{I}_u} \hat{E}_0 \left( u^+(x,t), u^-(x,t), \nu(x,t) \right) \partial H^N(x,t) + \delta, \quad (3.47)
\]

where $H_{F,u_\varepsilon}(x,t) \in L^2(0,T; \tilde{H}^1_0(\mathbb{R}^N, \mathbb{R}^k))$ satisfies

\[
\Delta_x H_{F,u_\varepsilon}(x,t) = \partial_t u_\varepsilon(x,t) + \operatorname{div}_x F(u_\varepsilon(x,t)), \quad (3.48)
\]

\[
\hat{E}_0(u^+, u^-, \nu) := \inf \left\{ \int_{\tilde{I}_u} \left( L \left| \nabla_y \left\{ \nabla_y \eta(\zeta(y,s)) \right\} \right|^2 + \frac{1}{L} \left| \gamma(y,s) - F(\zeta(y,s)) \right|^2 \right) \, dy \, ds : \right. 
\]

\begin{align*}
L & \in (0, +\infty), \quad \zeta \in \mathcal{Z}^{(2)}(u^+, u^-, \nu), \quad \gamma \in \mathcal{Z}^{(3)}(F(u^+), F(u^-), \nu), \quad \partial_\nu \gamma(y,s) + \operatorname{div}_y \gamma(y,s) \equiv 0 \bigg\} = \\
\hat{E}_1(u^+, u^-, \nu) := \inf \left\{ \int_{\tilde{I}_u} \left( L \left| \nabla_y \left\{ \nabla_y \xi(y,s) \right\} \right|^2 + \frac{1}{L} \left| \partial_\xi \xi(y,s) + F(\operatorname{div}_y \xi(y,s)) \right|^2 \right) \, dy \, ds : \\
L & \in (0, +\infty), \quad \xi \in \mathcal{Z}^{(1)}(u^+, u^-, \nu) \bigg\}, \quad (3.49)
\end{align*}

with

\[
\mathcal{Z}^{(1)}(u^+, u^-, \nu) := \\
\left\{ \xi(y,s) \in \mathcal{D}'(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}^{k \times N}) : \operatorname{div}_y \xi(y,s) \in C^1(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}^k), \quad \partial_\nu \xi(y,s) \in C^1(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}^{k \times N}), \right. \\
\left. ( \operatorname{div}_y \xi, -\partial_\nu \xi)(y,s) = (u^- , F(u^-)) \quad \text{if} \ y \cdot \nu \leq -1/2, \quad ( \operatorname{div}_y \xi, -\partial_\nu \xi)(y,s) = (u^+ , F(u^+)) \quad \text{if} \ y \cdot \nu \geq 1/2 \right. \\
\left. \text{and} \quad ( \operatorname{div}_y \xi, -\partial_\nu \xi)((y,s) + \nu_j) = ( \operatorname{div}_y \xi, -\partial_\nu \xi)(y,s) \quad \forall j = 2, 3, \ldots, N \right\}, \quad (3.50)
\]

\[
\mathcal{Z}^{(2)}(u^+, u^-, \nu) := \left\{ \zeta(y,s) \in C^1(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}^k) : \zeta(y,s) = u^- \quad \text{if} \ y \cdot \nu \leq -1/2, \right. \\
\left. \zeta(y,s) = u^+ \quad \text{if} \ y \cdot \nu \geq 1/2 \quad \text{and} \quad \zeta((y,s) + \nu_j) = \zeta(y,s) \quad \forall j = 2, 3, \ldots, N \right\}, \quad (3.51)
\]

\[
\mathcal{Z}^{(3)}(A, B, \nu) := \left\{ \gamma(y,s) \in C^1(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}^{k \times N}) : \gamma(y,s) = B \quad \text{if} \ y \cdot \nu \leq -1/2, \right. \\
\left. \gamma(y,s) = A \quad \text{if} \ y \cdot \nu \geq 1/2 \quad \text{and} \quad \gamma((y,s) + \nu_j) = \gamma(y,s) \quad \forall j = 2, 3, \ldots, N \right\}. \quad (3.52)
\]

Here $\tilde{I}_u := \{ y \in \mathbb{R}^{N+1} : |y \cdot \nu_j| < 1/2 \quad \forall j = 1, 2, \ldots, N \}$ where $\{\nu_1, \nu_2, \ldots, \nu_N, \nu_{N+1}\} \subset \mathbb{R}^{N+1}$ is an orthonormal base in $\mathbb{R}^{N+1}$ such that $\nu_1 := \nu$. Moreover, there exist $\sigma > 0$ and $R > 0$ (depending on $\delta$),
such that for every $0 < \varepsilon < 1$ and for every $(x, t) \in (\mathbb{R}^N \times \mathbb{R}) \setminus \{(x \in \mathbb{R}^N : |x| < R) \times (\sigma, T - \sigma)\}$ we have $u_{\varepsilon}(x, t) = u_{\varepsilon}^{(0)}(x, t)$ and $L_{\varepsilon}(x, t) = L_{\varepsilon}^{(0)}(x, t)$ where

$$u_{\varepsilon}^{(0)}(x) = \frac{1}{\varepsilon^{N+1}} \int_{\mathbb{R}} \int_{\mathbb{R}^N} \kappa\left(\frac{y - x}{\varepsilon}, \frac{s - t}{\varepsilon}\right) u(y, s) dy ds,$$

$$L_{\varepsilon}^{(0)}(x) = \frac{1}{\varepsilon^{N+1}} \int_{\mathbb{R}} \int_{\mathbb{R}^N} \kappa\left(\frac{y - x}{\varepsilon}, \frac{s - t}{\varepsilon}\right) F(u(y, s)) dy ds,$$

(see Remark 3.3). Finally for $\mathcal{H}^N$-a.e. $(x, t) \in J_u$ we have

$$\hat{E}_0\left(u^+(x, t), u^-(x, t), \nu(x, t)\right) = \left|\nu_x(x, t)\right| \hat{E}_0\left(u^+(x, t), u^-(x, t), \nu(x, t)\right), \tag{3.53}$$

with

$$\hat{E}_0(u^+, u^-, \nu) := \inf \left\{ \int_{J_u} \left( L \left|\nabla \eta \left( \nabla_{y} \nu \left( \nabla_{y} \eta \left( \nabla_{y} \nu \right) \right) \right) \right|^2 + \frac{1}{L} \left|\eta(y, s) - \hat{F}_{\nu}(\nabla_{y} \eta \left( \nabla_{y} \nu \right)) \right|^2 \right\} dy ds :$$

$$L \in (0, +\infty), \quad \xi \in \mathbb{Z}^{(2)}(u^+, u^-, \nu'), \quad \gamma \in \mathbb{Z}^{(3)}(\hat{F}_{\nu}(u^+), \hat{F}_{\nu}(u^-), \nu'), \quad \partial_{\xi} \gamma(y, s) + \text{div}_{y} \gamma(y, s) \equiv 0 \right\} = \left\{ \int_{J_u} \left( L \left|\nabla \eta \left( \nabla_{y} \nu \left( \nabla_{y} \gamma \right) \right) \right|^2 + \frac{1}{L} \left|\partial_{\xi} \gamma(y, s) + \hat{F}_{\nu}(\nabla_{y} \gamma(y, s)) \right|^2 \right\} dy ds :$$

$$L \in (0, +\infty), \quad \xi \in \mathbb{Z}^{(1)}(u^+, u^-, \nu') \right\}, \tag{3.54}$$

where we denote

$$\hat{F}_{\nu}(u) := F(u) + (\nu_{x}/[\nu_{y}]) \{u \otimes \nu_{y}\} \quad \forall u \in \mathbb{R}^k,$$

$$\nu = (\nu_{x}, \nu_{y}) = (\nu_{x}, \nu_{t}) \in \mathbb{R}^N \times \mathbb{R}$$

and $\nu' := (\nu_{y})/|\nu_{y}|, 0$. Here $I_{\nu'} := \{y \in \mathbb{R}^{N+1} : |y \cdot \nu_{y}'| < 1/2 \ \forall j = 1, 2, \ldots N\}$ where $\nu_{y}', \nu_{y}^1, \ldots, \nu_{y}^N, \nu_{y}^N, \ldots, \nu_{y}^{N+1} \subset \mathbb{R}^{N+1}$ is an orthonormal base in $\mathbb{R}^{N+1}$ such that $\nu_{y} := \nu$ and $\nu_{y}^{N+1} := (0, 0, \ldots, 1)$.

**Proof.** Using Theorem 4.1 in [32] we deduce that for every $\delta > 0$ there exists a sequence of functions

$$\{u_{\varepsilon}(x, t)\}_{\varepsilon > 0} \subset C^{\infty}(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}^k) \cap W^{1,2}(\mathbb{R}^N \times (0, T), \mathbb{R}^k) \cap L^N$${

and $\{L_{\varepsilon}(x, t)\}_{\varepsilon > 0} \subset C^{\infty}(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}^k \times N) \cap W^{1,2}(\mathbb{R}^N \times (0, T), \mathbb{R}^k \times \mathbb{R}^N) \cap L^N$ such that $u_{\varepsilon}, \{L_{\varepsilon}(x, t)\}$ and $\varepsilon \nabla_{x} u_{\varepsilon}$ are bounded in $L^N; u_{\varepsilon} \to u, L_{\varepsilon} \to F(u)$ and $\varepsilon \nabla_{x} u_{\varepsilon} \to 0$ as $\varepsilon \to 0^+$ in $L^q(\mathbb{R}^N \times (0, T))$; $\partial_{\nu_{x}} u_{\varepsilon} + \text{div}_{y} \nu_{x} L_{\varepsilon} \equiv 0$ and

$$\lim_{\varepsilon \to 0^+} \int_0^T \int_{\mathbb{R}^N} \left[ \left( \varepsilon \nabla_{y} \left( \nabla_{y} \eta \left( u_{\varepsilon}(x, t) \right) \right) \right)^2 + \frac{1}{\varepsilon} \left| L_{\varepsilon}(x, t) - F(u_{\varepsilon}(x, t)) \right|^2 \right] dx dt$$

$$= \lim_{\varepsilon \to 0^+} \int_0^T \int_{\mathbb{R}^N} \frac{1}{\varepsilon} \left( \left( \nabla_{y} u_{\varepsilon}(x, t) \right)^2 \cdot \left( \varepsilon \nabla_{x} u_{\varepsilon}(x, t) \right)^2 + \left| L_{\varepsilon}(x, t) - F(u_{\varepsilon}(x, t)) \right|^2 \right) dx dt$$

$$\leq \int_{J_u} \hat{E}_0\left(u^+(x, t), u^-(x, t), \nu(x, t)\right) \partial_{\mathcal{H}^N}(x, t) + \delta, \tag{3.56}$$

where

$$\hat{E}_0(u^+, u^-, \nu) := \inf \left\{ \int_{J_u} \left( L \left|\nabla \eta \left( \nabla_{y} \nu \left( \nabla_{y} \eta \right) \right) \right|^2 + \frac{1}{L} \left|\eta(y, s) - \hat{F}_{\nu}(\nabla_{y} \eta \left( \nabla_{y} \nu \right)) \right|^2 \right\} dy ds :$$

$$L \in (0, +\infty), \quad \xi \in \mathbb{Z}^{(2)}(u^+, u^-, \nu), \quad \gamma \in \mathbb{Z}^{(3)}(F(u^+), F(u^-), \nu), \quad \partial_{\xi} \gamma(y, s) + \text{div}_{y} \gamma(y, s) \equiv 0 \right\}, \tag{3.57}$$

and there exist $\sigma > 0$ and $R > 0$, such that for every $0 < \varepsilon < 1$ and every $(x, t) \in (\mathbb{R}^N \times \mathbb{R}) \setminus \{(x \in \mathbb{R}^N : |x| < R) \times (\sigma, T - \sigma)\}$ we have $u_{\varepsilon}(x, t) = u_{\varepsilon}^{(0)}(x, t)$ and $L_{\varepsilon}(x, t) = L_{\varepsilon}^{(0)}(x, t)$. Moreover, by Lemma 2.11 or by 3.17,
we obtain
\[
\lim_{\varepsilon \to 0^+} \int_0^T \int_{\mathbb{R}^N} \left( \varepsilon \left| \nabla_x \{ \nabla u \eta(u(x,t)) \} \right|^2 + \frac{1}{\varepsilon} \left| \nabla_x H_F(u(x,t)) \right|^2 \right) \, dx \, dt \leq \\
\lim_{\varepsilon \to 0^+} \int_0^T \int_{\mathbb{R}^N} \left( \varepsilon \left| \nabla_x \{ \nabla u \eta(u(x,t)) \} \right|^2 + \frac{1}{\varepsilon} \left| L_x(x,t) - F(u(x,t)) \right|^2 \right) \, dx \, dt = \\
\leq \int_I \frac{\varepsilon}{I} \left( u^+(x,t), u^-(x,t), \nu(x,t) \right) \partial \mathcal{H}^N(x,t) + \delta, \quad (3.58)
\]

Next, as before, define
\[
v_\varepsilon(x,t) := K_\varepsilon(x) - \int_0^t L_x(x,s) \, ds \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}, \quad (3.59)
\]
where $K_\varepsilon(x) \in C^\infty(\mathbb{R}^N, \mathbb{R}^{k \times N})$ satisfies $\text{div} \, K_\varepsilon(x) \equiv u_\varepsilon(x,0)$. Then clearly $v_\varepsilon(x,t) \in C^\infty(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}^{k \times N})$. Moreover, since $\text{div} \, L_x(x,t) + \partial_t u_\varepsilon(x,t) \equiv 0$ we deduce that
\[
\partial_t v_\varepsilon(x,t) = -L_x(x,t) \quad \text{and} \quad \text{div} \, v_\varepsilon(x,t) = u_\varepsilon(x,t) \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}. \quad (3.60)
\]

Next we will prove that
\[
\hat{E}_1 (u^+, u^-, \nu) := \inf \left\{ \int_{I_\nu} \left( L \left| \nabla_y \{ \nabla_x \eta(\text{div} \, \xi(y,s)) \} \right|^2 + \frac{1}{L} \left| \partial_s \xi(y,s) + F(\text{div} \, \xi(y,s)) \right|^2 \right) \, dy \, ds : \right. \\
\left. L \in (0, +\infty), \xi \in Z^{(1)}(u^+, u^-, \nu) \right\} = \\
\hat{E}_0 (u^+, u^-, \nu) := \inf \left\{ \int_{I_\nu} \left( L \left| \nabla_y \{ \nabla_y \xi(y,s) \} \right|^2 + \frac{1}{L} \left| \gamma(y,s) - F(\xi(y,s)) \right|^2 \right) \, dy \, ds : \right. \\
\left. L \in (0, +\infty), \xi \in Z^{(2)}(u^+, u^-, \nu), \gamma \in Z^{(3)}(F(u^+), F(u^-), \nu), \partial_s \xi(y,s) + \text{div} \, \gamma(y,s) \equiv 0 \right\}. \quad (3.61)
\]

Indeed, since for every $\xi \in Z^{(1)}(u^+, u^-, \nu)$ we clearly have $\text{div} \, \xi \in Z^{(2)}(u^+, u^-, \nu)$ and
\[
-\partial_s \xi \in Z^{(3)}(F(u^+), F(u^-), \nu)
\]
and since $\partial_s (\text{div} \, \xi) + \text{div} \, (-\partial_s \xi)$ we clearly have
\[
\hat{E}_1 (u^+, u^-, \nu) \geq \hat{E}_0 (u^+, u^-, \nu). \quad (3.62)
\]

On the other hand fix $\xi \in Z^{(2)}(u^+, u^-, \nu)$ and $\gamma \in Z^{(3)}(F(u^+), F(u^-), \nu)$ such that $\partial_s \xi(y,s) + \text{div} \, \gamma(y,s) \equiv 0$. Then define
\[
\xi(y,s) := Q(y) - \int_0^s \gamma(y,\tau) \, d\tau \quad \forall (y,s) \in \mathbb{R}^N \times \mathbb{R}, \quad (3.63)
\]
where $Q(y) \in C^1(\mathbb{R}^N, \mathbb{R}^k)$ is an arbitrary function which satisfies $\text{div} \, Q(y) = \xi(y,0)$. Then clearly $\xi \in C^1(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}^{k \times N})$ and moreover, since $\partial_s \xi(y,s) + \text{div} \, \gamma(y,s) \equiv 0$, we deduce easily that
\[
\partial_s \xi(y,s) := -\gamma(y,s) \quad \text{and} \quad \text{div} \, \xi(y,s) = \xi(y,s) \quad \forall (y,s) \in \mathbb{R}^N \times \mathbb{R}. \quad (3.64)
\]

Thus clearly $\hat{E}_1 (u^+, u^-, \nu) \leq \hat{E}_0 (u^+, u^-, \nu)$ and plugging it into (3.62) we deduce (3.61). By the same way we
obtain

$$
\hat{E}_1(u^+, u^-, \nu) := \inf \left\{ \int_{I_\nu} \left( L \left| \nabla_y \left\{ \nabla_u \eta \left( \text{div}_y \xi(y, s) \right) \right\} \right|^2 + \frac{1}{L} \left| \partial_s \xi(y, s) + \hat{F}_\nu \left( \text{div}_y \xi(y, s) \right) \right|^2 \right) \, dyds : \right.
\left. L \in (0, +\infty), \xi \in Z^{(1)}(u^+, u^-, \nu) \right\}
$$

$$
\hat{E}_0(u^+, u^-, \nu) := \inf \left\{ \int_{I_\nu} \left( L \left| \nabla_y \left\{ \nabla_u \eta \left( \xi(y, s) \right) \right\} \right|^2 + \frac{1}{L} \left| \gamma(y, s) - \hat{F}_\nu \left( \xi(y, s) \right) \right|^2 \right) \, dyds : \right.
\left. L \in (0, +\infty), \xi \in Z^{(2)}(u^+, u^-, \nu), \gamma \in Z^{(3)}(\hat{F}_\nu(u^+), \hat{F}_\nu(u^-), \nu), \partial_s \xi(y, s) + \text{div}_y \gamma(y, s) = 0 \right\}, \quad (3.65)
$$

Moreover, by Proposition 2.1 in [52], for every system of linearly independent vectors \( \{\eta_1, \eta_2, \ldots, \eta_{N+1}\} \), such that \( \eta_1 = \nu \) and \( \eta_j \cdot \nu = 0 \) for every \( j \in \{2, 3, \ldots, (N+1)\} \), we have

$$
\hat{E}_0(u^+, u^-, \nu) = \inf \left\{ \int_{I_{N+1}} \left( L \left| \nabla_y \left\{ \nabla_u \eta \left( \xi \left( \sum_{j=1}^{N+1} z_j \nu_j, s \right) \right) \right\} \right|^2 + \frac{1}{L} \left| \partial_s \xi \left( \sum_{j=1}^{N+1} z_j \nu_j, s \right) + \hat{F}_\nu \left( \text{div}_y \xi \left( \sum_{j=1}^{N+1} z_j \nu_j, s \right) \right) \right|^2 \right) \, dzds : \right.
\left. L \in (0, +\infty), \xi \in Z^{(1)}(u^+, u^-, \nu) \right\}
$$

$$
= \inf \left\{ \int_{I_{N+1}} \left( L \left| \nabla_y \left\{ \nabla_u \eta \left( \xi \left( \sum_{j=1}^{N+1} z_j \eta_j, s \right) \right) \right\} \right|^2 + \frac{1}{L} \left| \partial_s \xi \left( \sum_{j=1}^{N+1} z_j \eta_j, s \right) + \hat{F}_\nu \left( \text{div}_y \xi \left( \sum_{j=1}^{N+1} z_j \eta_j, s \right) \right) \right|^2 \right) \, dzds : \right.
\left. L \in (0, +\infty), \xi \in Z^{(1)}(u^+, u^-, \eta_1, \ldots, \eta_{N+1}) \right\}, \quad (3.66)
$$

where

$$
I_{N+1} := \left\{ s \in \mathbb{R}^N : -1/2 < s_j < 1/2 \quad \forall j = 1, 2, \ldots, N + 1 \right\} \quad (3.67)
$$

and

$$
Z^{(1)}(u^+, u^-, \eta_1, \ldots, \eta_{N+1}) := \left\{ \xi(y, s) \in C^1(\mathbb{R}^N \times \mathbb{R}^k \times \mathbb{R}^k) : \text{div}_y \xi(y, s) \in C^1(\mathbb{R}^N \times \mathbb{R}^k, \text{div}_y \xi(y, s)) \right\}
$$

(3.68)

$$
\tilde{Z}^{(2)}(u^+, u^-, \eta_1, \ldots, \eta_{N+1}) := \left\{ \xi(y, s) \in C^1(\mathbb{R}^N \times \mathbb{R}^k) : \text{div}_y \xi(y, s) = u^- \text{ if } y \cdot \eta_1 \leq -1/2, \right. \quad \sum_{j=1}^{N+1} z_j \eta_j \right\}
$$

(3.69)

$$
\tilde{Z}^{(3)}(A, B, \eta_1, \ldots, \eta_{N+1}) := \left\{ \gamma(y, s) \in C^1(\mathbb{R}^N \times \mathbb{R}^k) : \gamma(y, s) = \gamma \text{ if } y \cdot \eta_1 \leq -1/2, \right. \quad \text{if } y \cdot \eta_1 \geq 1/2 \quad \forall j = 1, 2, \ldots, N \right\} \quad (3.70)
$$
On the other hand, by the same way as before,

\[
\hat{E}_0(u^+, u^-, \nu) = \inf \left\{ \int_{I_{n+1}} \left( L \left| \nabla_y \left\{ \nabla_u \eta \left( \text{div}_y \left( \sum_{j=1}^{N+1} z_j \eta_j, s \right) \right) \right\} \right|^2 + \frac{1}{L} \left| \partial_x \left( \sum_{j=1}^{N+1} z_j \eta_j, s \right) \right|^2 \right\} dz ds : L \in (0, +\infty), \xi \in \mathcal{Z}^{(1)}(u^+, u^-, \eta_1, \ldots, \eta_{N+1}) \}
\]

\[
= \inf \left\{ \int_{I_{n+1}} \left( L \left| \nabla_y \left\{ \nabla_u \eta \left( \sum_{j=1}^{N+1} z_j \eta_j, s \right) \right\} \right|^2 + \frac{1}{L} \left| \sum_{j=1}^{N+1} z_j \eta_j, s \right|^2 \right\} dz ds : L \in (0, +\infty), \xi \in \mathcal{Z}^{(2)}(u^+, u^-, \eta_1, \ldots, \eta_{N+1}), \gamma \in \mathcal{Z}^{(3)}(\hat{F}_\nu(u^+), \hat{F}_\nu(u^-), \eta_1, \ldots, \eta_{N+1}), \partial_x \gamma(y, s) + \text{div}_y \gamma(y, s) \equiv 0 \right\}, \tag{3.71}
\]

In order to complete the proof, we need to deduce that for $\mathcal{H}^{N}\text{-a.e. } (x, t) \in J_u$ we have

\[
\hat{E}_0(u^+(x, t), u^-(x, t), \nu(x, t)) = |\nu_x(x, t)| \hat{E}_0(u^+(x, t), u^-(x, t), \nu(x, t)).
\]

Indeed, observe that by \textbf{[5.43]}, for $\mathcal{H}^{N}\text{-a.e. } (x, t) \in J_u$ we have

\[
(u^+(x, t) - u^-(x, t))\nu_x(x, t) + \left( F(u^+(x, t)) - F(u^-(x, t)) \right) \cdot \nu_x(x, t) = 0 \quad \mathcal{H}^{N}\text{-a.e. in } J_u. \tag{3.72}
\]

In particular since for $\mathcal{H}^{N}\text{-a.e. } (x, t) \in J_u$ we have $u^+(x, t) \neq u^-(x, t)$ and since $\nu(x, t) \not\equiv 0$ we deduce

\[
\nu_x(x, t) \not\equiv 0 \quad \text{for } \mathcal{H}^{N}\text{-a.e. } (x, t) \text{ in } J_u. \tag{3.73}
\]

On the other hand if $\nu_y \not\equiv 0$, consider the transformation of independent variables $(y, s) \leftrightarrow (\bar{y}, \bar{s})$, where

\[
\begin{align*}
\bar{y} := & |\nu_y| y + (s\nu_y/|\nu_y|) \nu_y, \\
\bar{s} := & |\nu_y| s,
\end{align*}
\]

and

\[
\begin{align*}
y := & \bar{y}/|\nu_y| - (s\nu_y/|\nu_y|^3) \nu_y, \\
s := & \bar{s}/|\nu_y|.
\end{align*}
\]

Then for every $\zeta \in \mathcal{Z}^{(2)}(u^+, u^-, \nu)$ and $\gamma \in \mathcal{Z}^{(3)}(F(u^+), F(u^-), \nu)$ such that $\partial_x \zeta(y, s) + \text{div}_y \gamma(y, s) \equiv 0$ define the transformations $\zeta \leftrightarrow \bar{\zeta}$ by

\[
\bar{\zeta}(\bar{y}, \bar{s}) := \zeta \left( \bar{y}/|\nu_y| - (s\nu_y/|\nu_y|^3) \nu_y, \bar{s}/|\nu_y| \right) = \zeta(y, s), \tag{3.75}
\]

and $\gamma \leftrightarrow \bar{\gamma}$ by

\[
\bar{\gamma}(\bar{y}, \bar{s}) := \gamma \left( \bar{y}/|\nu_y| - (s\nu_y/|\nu_y|^3) \nu_y, \bar{s}/|\nu_y| \right) + \left( \nu_y/|\nu_y|^2 \right) \left( \zeta \left( \bar{y}/|\nu_y| - (s\nu_y/|\nu_y|^3) \nu_y, \bar{s}/|\nu_y| \right) \otimes \nu_y \right)
\]

\[
= \gamma(y, s) + (\nu_y/|\nu_y|^2) \left( \zeta(y, s) \otimes \nu_y \right). \tag{3.76}
\]

Finally, consider the system of linearly independent vectors $\{\eta_1, \eta_2, \ldots, \eta_{N+1}\}$, defined by the identities

\[
\begin{align*}
\eta_1 := & \nu' = (\nu_y/|\nu_y|, 0), \\
\eta_j := & \left( \eta_j, \bar{y}, \eta_j, \bar{s} \right) := (|\nu_y| \nu_{j,y} + (\nu_y \nu_s/|\nu_y|) \nu_y, |\nu_y| \nu_{j,s}) \quad \forall j = 2, 3, \ldots, (N+1),
\end{align*}
\]

where we denote $\nu_{j,y}, \nu_{j,s} := \nu_j$. Then clearly, $\eta_1 = \nu'$ and $\eta_j \cdot \nu' = 0$ for every $j \in \{2, 3, \ldots, (N+1)\}$. Furthermore, we have $\zeta \in \mathcal{Z}^{(2)}(u^+, u^-, \nu)$ and $\gamma \in \mathcal{Z}^{(3)}(F(u^+), F(u^-), \nu)$ if and only if $\bar{\zeta} \in \mathcal{Z}^{(2)}(u^+, u^-, \eta_1, \ldots, \eta_{N+1})$ and $\bar{\gamma} \in \mathcal{Z}^{(3)}(\hat{F}_\nu(u^+), \hat{F}_\nu(u^-), \eta_1, \ldots, \eta_{N+1})$. Moreover, $\partial_x \zeta(y, s) + \text{div}_y \gamma(y, s) \equiv 0$ if and only if $\partial_x \bar{\zeta}(\bar{y}, \bar{s}) + \bar{\gamma}(\bar{y}, \bar{s}) \equiv 0$.
\[ \int_{I_{N+1}} \left| L \nabla_y \left\{ \nabla_y \eta \left( \sum_{j=1}^{N+1} z_j \nu_j, s \right) \right\} \right|^2 + \frac{1}{L} \left| \gamma \left( \sum_{j=1}^{N+1} z_j \nu_j, s \right) - F \left( \sum_{j=1}^{N+1} z_j \nu_j, s \right) \right|^2 \right| dz ds = \\
|\nu_y| \int_{I_{N+1}} \left( L|\nu_y| \left| \nabla_y \left\{ \nabla_y \eta \left( \sum_{j=1}^{N+1} z_j \eta_j, s \right) \right\} \right|^2 \\
+ \frac{1}{L|\nu_y|} \left| \gamma \left( \sum_{j=1}^{N+1} z_j \eta_j, s \right) - \hat{F}_\nu \left( \sum_{j=1}^{N+1} z_j \eta_j, s \right) \right|^2 \right| dz ds. \quad (3.78) \]

Therefore, by all this facts and (3.60), (3.71), we clearly obtain \( \hat{E}_0(u^+, u^-, \nu) = |\nu_y| \hat{E}_0(u^+, u^-, \nu) \). This completes the proof. \( \square \)

**Theorem 3.1.** Let \( F(u) \in C^1(\mathbb{R}^k, \mathbb{R}^{k \times N^k}) \) and \( \eta(u) \in C^3(\mathbb{R}^k, \mathbb{R}) \) be an entropy for the corresponding system (3.30), which satisfies \( \eta(u) \geq 0 \) and \( \eta(0) = 0 \). Furthermore, let \( u \in \mathcal{E}_T^0 \) (see Definition 3.4 and Remark 3.3).

Then for every \( \delta > 0 \) there exists a sequence of functions \( \{ \bar{u}_\varepsilon(x, t) \}_{\varepsilon > 0} \subset L^2_{\text{loc}}((\mathbb{R}^N \times (0, T), \mathbb{R}^{k \times N}) \) such that \( u_\varepsilon(x, t) := \text{div}_x \bar{u}_\varepsilon(x, t) \in L^2(0, T; H_0^1(\mathbb{R}^N, \mathbb{R}^k)) \cap C([0, T]; L^2(\mathbb{R}^N, \mathbb{R}^k)) \cap L^\infty \) and \( \tilde{L}_\varepsilon(x, t) := -\partial_t \bar{u}_\varepsilon(x, t) \in L^2(\mathbb{R}^N \times (0, T), \mathbb{R}^{k \times N}); \bar{u}_\varepsilon \to u \text{ in } \bigcap_{q \geq 1} L^q(\mathbb{R}^N \times (0, T); \mathbb{R}^k); \tilde{L}_\varepsilon \to F(u) \text{ in } L^2(\mathbb{R}^N \times (0, T), \mathbb{R}^{k \times N}); \{ \bar{u}_\varepsilon \}_{\varepsilon > 0} \text{ is bounded in } L^\infty \) sequence; \( \bar{u}_\varepsilon(x, 0) = u(x, 0) \); \( \partial_t \bar{u}_\varepsilon + \text{div}_x \tilde{L}_\varepsilon \equiv 0 \) and

\[
\lim_{\varepsilon \to 0^+} \left\{ \int_0^T \int_{\mathbb{R}^N} \left( \varepsilon \left| \nabla_x \left\{ \nabla_y \eta \left( \bar{u}_\varepsilon(x, t) \right) \right\} \right|^2 + \frac{1}{\varepsilon} \left| \nabla_x \nabla_y \bar{u}_\varepsilon(x, t) \right|^2 \right) dx dt + \int_{\mathbb{R}^N} \eta(\bar{u}_\varepsilon(x, T)) dx \right\} \leq \\
\lim_{\varepsilon \to 0^+} \left\{ \int_0^T \int_{\mathbb{R}^N} \left( \varepsilon \left| \nabla_x \left\{ \nabla_y \eta \left( \bar{u}_\varepsilon(x, t) \right) \right\} \right|^2 + \frac{1}{\varepsilon} \left| \tilde{L}_\varepsilon(x, t) - F(\bar{u}_\varepsilon(x, t)) \right|^2 \right) dx dt + \int_{\mathbb{R}^N} \eta(\bar{u}_\varepsilon(x, T)) dx \right\} \leq \\
\lim_{\varepsilon \to 0^+} \left\{ \int_0^T \int_{\mathbb{R}^N} \left( \varepsilon \left| \nabla_x \left\{ \nabla_y \eta \left( \text{div}_x \bar{u}_\varepsilon(x, t) \right) \right\} \right|^2 + \frac{1}{\varepsilon} \left| \partial_t \bar{u}_\varepsilon(x, t) + F(\text{div}_x \bar{u}_\varepsilon(x, t)) \right|^2 \right) dx dt \\
+ \int_{\mathbb{R}^N} \eta \left( \text{div}_x \bar{u}_\varepsilon(x, T) \right) dx \right\} \leq \int_{I_{u^+}} \hat{E}_0(u^+(x, t), u^-(x, t), \nu(x, t)) \partial \mathcal{H}^N(x, t) + \int_{\mathbb{R}^N} \eta(u(x, T)) dx + 2\delta, \quad (3.79) \]

where \( H_{F, \bar{u}_\varepsilon}(x, t) \in L^2(0, T; H_0^1(\mathbb{R}^N, \mathbb{R}^k)) \) satisfies

\[
\Delta_x H_{F, \bar{u}_\varepsilon}(x, t) = \partial_t \bar{u}_\varepsilon(x, t) + \text{div}_x F(\bar{u}_\varepsilon(x, t)), \quad (3.80) \]

\[
\hat{E}_0(u^+, u^-, \nu) := \inf \left\{ \int_{I_{u^+}} \left( L \left| \nabla_y \left\{ \nabla_y \eta \left( \xi(y, s) \right) \right\} \right|^2 + \frac{1}{T} \left| \gamma(y, s) - F(\xi(y, s)) \right|^2 \right) dy ds : \right. \\
L \in (0, +\infty), \xi \in \mathcal{Z}^{(2)}(u^+, u^-, \nu), \gamma \in \mathcal{Z}^{(3)}(F(u^+), F(u^-), \nu), \partial_s \xi(y, s) + \text{div}_y \gamma(y, s) \equiv 0 \right\} = \\
\hat{E}_1(u^+, u^-, \nu) := \inf \left\{ \int_{I_{u^+}} \left( L \left| \nabla_y \left\{ \nabla_y \eta \left( \text{div}_y \xi(y, s) \right) \right\} \right|^2 + \frac{1}{T} \left| \partial_s \xi(y, s) + F(\text{div}_y \xi(y, s)) \right|^2 \right) dy ds : \right. \\
L \in (0, +\infty), \xi \in \mathcal{Z}^{(1)}(u^+, u^-, \nu) \right\}, \quad (3.81) \]

with

\[
\mathcal{Z}^{(1)}(u^+, u^-, \nu) := \left\{ \xi(y, s) \in D'((\mathbb{R}^N \times \mathbb{R}^k) \times \mathbb{R}^N) : \text{div}_y \xi(y, s) \in C^1((\mathbb{R}^N \times \mathbb{R}^k), \partial_s \xi(y, s) \in C^1((\mathbb{R}^N \times \mathbb{R}^k), \right. \\
(\text{div}_y \xi, -\partial_s \xi)(y, s) = (u^-, F(u^-)) \text{ if } y \cdot \nu \leq -1/2, (\text{div}_y \xi, -\partial_s \xi)(y, s) = (u^+, F(u^+)) \text{ if } y \cdot \nu \geq 1/2 \\
\text{and } (\text{div}_y \xi, -\partial_s \xi)((y, s) + \nu) = (\text{div}_y \xi, -\partial_s \xi)(y, s) \forall j = 2, 3, \ldots, N, \right. \right\}, \quad (3.82) \]
\[ Z^{(2)}(u^+, u^-, \nu) := \begin{cases} 
\zeta(y, s) \in C^1(\mathbb{R}^N \times \mathbb{R}^k) : \zeta(y, s) = u^- \text{ if } y \cdot \nu \leq -1/2, \\
\zeta(y, s) = u^+ \text{ if } y \cdot \nu \geq 1/2 \text{ and } \zeta((y, s) + \nu_j) = \zeta(y, s) \forall j = 2, 3, \ldots, N \end{cases}, \] (3.83)

\[ Z^{(3)}(A, B, \nu) := \begin{cases} 
\gamma(y, s) \in C^1(\mathbb{R}^N \times \mathbb{R}^{k \times N}) : \gamma(y, s) = B \text{ if } y \cdot \nu \leq -1/2, \\
\gamma(y, s) = A \text{ if } y \cdot \nu \geq 1/2 \text{ and } \gamma((y, s) + \nu_j) = \gamma(y, s) \forall j = 2, 3, \ldots, N \end{cases}. \] (3.84)

Here \( I_{\nu} := \{ y \in \mathbb{R}^{N+1} : |y \cdot \nu_j| < 1/2 \quad \forall j = 1, 2, \ldots, N \} \) where \( \{ \nu_1, \nu_2, \ldots, \nu_N, \nu_{N+1} \} \subset \mathbb{R}^{N+1} \) is an orthonormal base in \( \mathbb{R}^{N+1} \) such that \( \nu_1 := \nu \). Finally for \( \mathcal{H}^N \)-a.e. \( (x, t) \in J_u \) we have

\[ \hat{E}_0(u^+(x, t), u^-(x, t), \nu(x, t)) = |\nu_x(x, t)| \hat{E}_0(u^+(x, t), u^-(x, t), \nu(x, t)), \] (3.85)

with

\[ \hat{E}_0(u^+, u^-, \nu) := \inf \left\{ \int_{I_{\nu}} \left( L|\nabla y\left\{ \nabla u_{\eta}(\zeta(y, s))\right\}|^2 + \frac{1}{L} |\gamma(y, s) - \hat{F}_\nu(\zeta(y, s))|^2 \right) dyds : \right. \]
\[ L \in (0, +\infty), \zeta \in Z^{(2)}(u^+, u^-, \nu'), \gamma \in Z^{(3)}(\hat{F}_\nu(u^+), \hat{F}_\nu(u^-), \nu'), \partial_y \zeta(y, s) + \text{div}_y \gamma(y, s) \equiv 0 \left. \right\} = \]

\[ \hat{E}_1(u^+, u^-, \nu) := \inf \left\{ \int_{I_{\nu'}} \left( L|\nabla y\left\{ \nabla u_{\eta}(\text{div}_y \gamma(y, s))\right\}|^2 + \frac{1}{L} |\partial_y \gamma(y, s) + \hat{F}_\nu(\text{div}_y \gamma(y, s))|^2 \right) dyds : \right. \]
\[ L \in (0, +\infty), \zeta \in Z^{(1)}(u^+, u^-, \nu') \left. \right\}, \] (3.86)

where we denote

\[ \hat{F}_\nu(u) := F(u) + (\nu_{\eta}|\nu_{\eta}|^2) \{ u \otimes \nu_{\eta} \} \quad \forall u \in \mathbb{R}^k \], \] (3.87)

\[ \nu = (\nu_y, \nu_x) = (\nu_x, \nu_t) \in \mathbb{R}^N \times \mathbb{R} \text{ and } \nu' := (\nu_{\eta}/|\nu_{\eta}|, 0). \] Here \( I_{\nu'} := \{ y \in \mathbb{R}^{N+1} : |y \cdot \nu'_j| < 1/2 \quad \forall j = 1, 2, \ldots, N \} \) where \( \{ \nu'_1, \nu'_2, \ldots, \nu'_N, \nu'_{N+1} \} \subset \mathbb{R}^{N+1} \) is an orthonormal base in \( \mathbb{R}^{N+1} \) such that \( \nu'_1 := \nu' \) and \( \nu'_{N+1} := (0, 0, 1, \ldots, 1) \).

**Proof.** Let \( \delta > 0 \) and \( \kappa_1(r) \in C_\infty^1(\mathbb{R}^{N+1}) \) be a radial function, such that \( \int_{\mathbb{R}^{N+1}} \kappa_1(r) dr = 1 \) and \( \kappa_1 \geq 0 \). For every \( h > 0 \) set \( \kappa_h(r) := \kappa_1(r)/h \). Then \( \kappa_h(r) \in C_\infty^1(\mathbb{R}^{N+1}) \) is a radial function, such that \( \int_{\mathbb{R}^{N+1}} \kappa_h(r) dr = 1 \) and \( \kappa_h \geq 0 \). Therefore, by Lemma 3.2 there exists a sequence of functions \( \{v_{h, \varepsilon}(x, t)\}_{\varepsilon > 0} \subset C_\infty^1(\mathbb{R}^N \times \mathbb{R}^{k \times N}) \) such that \( u_{h, \varepsilon}(x, t) := \text{div}_x v_{h, \varepsilon}(x, t) \in W^{1,2}(\mathbb{R}^N \times (0, T), \mathbb{R}^k) \cap L^\infty \) and \( L_{h, \varepsilon}(x, t) := -\partial_t v_{h, \varepsilon}(x, t) \in W^{1,2}(\mathbb{R}^N \times (0, T), \mathbb{R}^{k \times N}) \cap L^\infty \); \( \{u_{h, \varepsilon}\}, \{L_{h, \varepsilon}\} \text{ and } \{\text{div}_x v_{h, \varepsilon}\} \) are bounded in \( L^\infty \) sequences; \( u_{h, \varepsilon} \rightarrow u, L_{h, \varepsilon} \rightarrow F(u) \) and \( \varepsilon \text{div}_x v_{h, \varepsilon} \rightarrow 0 \) as \( \varepsilon \rightarrow 0^+ \) in \( L^q(\mathbb{R}^N \times (0, T)) \); \( \partial_x u_{h, \varepsilon} \rightarrow F(u) \) in \( L^q(\mathbb{R}^N \times (0, T)) \); and \( \partial_x u_{h, \varepsilon} \rightarrow F(u) \) in \( L^q(\mathbb{R}^N \times (0, T)) \); and

\[ \lim_{\varepsilon \rightarrow 0^+} \int_0^T \int_{\mathbb{R}^N} \left( \varepsilon|\nabla_y \left\{ \nabla u_{\eta}(u_{h, \varepsilon}(x, t))\right\}|^2 + \frac{1}{\varepsilon} |\nabla_x H_{F, u_{h, \varepsilon}}(x, t)|^2 \right) dxdt \leq \]

\[ \lim_{\varepsilon \rightarrow 0^+} \int_0^T \int_{\mathbb{R}^N} \left( \varepsilon|\nabla_y \left\{ \nabla u_{\eta}(u_{h, \varepsilon}(x, t))\right\}|^2 + \frac{1}{\varepsilon} |L_{h, \varepsilon}(x, t) - F(u_{h, \varepsilon}(x, t))|^2 \right) dxdt = \]

\[ \lim_{\varepsilon \rightarrow 0^+} \int_0^T \int_{\mathbb{R}^N} \left( \varepsilon|\nabla_y \left\{ \nabla u_{\eta}(\text{div}_x v_{h, \varepsilon}(x, t))\right\}|^2 + \frac{1}{\varepsilon} |\partial_t v_{h, \varepsilon}(x, t) + F(\text{div}_x v_{h, \varepsilon}(x, t))|^2 \right) dxdt \]

\[ \leq \int_{J_u} \hat{E}_0 \left( u^+(x, t), u^-(x, t), \nu(x, t) \right) \partial \mathcal{H}^N(x, t) + \delta, \] (3.88)
where $H_{F,u_{h,\varepsilon}}(x,t) \in L^2\left(0, T; \tilde{H}^1_0(\mathbb{R}^N, \mathbb{R}^k)\right)$ satisfies

$$\Delta_x H_{F,u_{h,\varepsilon}}(x,t) = \partial_t u_{h,\varepsilon}(x,t) + \text{div}_x F(u_{h,\varepsilon}(x,t)), \tag{3.89}$$

Moreover, there exist $\sigma > 0$ such that for every $0 < \varepsilon < 1$ and every $(x,t) \in (\mathbb{R}^N \times \mathbb{R}) \setminus \{(\mathbb{R}^N) \times (\sigma, T - \sigma)\}$ we have $u_{h,\varepsilon}(x,t) = u_{h,\varepsilon}^{(0)}(x,t)$ and $L_{\varepsilon}(x,t) = L_{h,\varepsilon}^{(0)}(x,t)$ where

$$u_{h,\varepsilon}^{(0)}(x) = \frac{1}{\varepsilon^{N+1}} \int_{\mathbb{R}} \int_{\mathbb{R}^N} \kappa_h \left(\frac{y-x}{\varepsilon}, \frac{s-t}{\varepsilon}\right) u(y,s) \, dyds,$$

$$L_{h,\varepsilon}^{(0)}(x) = \frac{1}{\varepsilon^{N+1}} \int_{\mathbb{R}} \int_{\mathbb{R}^N} \kappa_h \left(\frac{y-x}{\varepsilon}, \frac{s-t}{\varepsilon}\right) F(u(y,s)) \, dyds.$$

We need just slightly modify $u_{h,\varepsilon}$ in such a way that it will satisfy the condition $u_{h,\varepsilon}(x,0) \equiv u(x,0)$. Let $\chi_{\varepsilon}(x,t) \in L^\infty\left(0, T; L^2(\mathbb{R}^N, \mathbb{R}^k)\right)$ be a cut-off function satisfying $\theta(t) = 0$ for every $t \geq 1$ and $\theta(t) = 1$ for every $t \leq 1/2$. For every small $\varepsilon > 0$ define $\bar{u}_{h,\varepsilon}(x,t) \in L^\infty\left(0, T; L^2(\mathbb{R}^N, \mathbb{R}^k)\right)$ be a solution of the Heat Equation

$$\begin{aligned}
\varepsilon \Delta_x \chi_{\varepsilon} &= \partial_t \chi_{\varepsilon}, \\
\chi_{\varepsilon}(x,0) &= u(x,0). \tag{3.90}
\end{aligned}$$

It is clear that we may assume that $\chi_{\varepsilon}$ is $L^2$-strongly continuous in $[0, T]$ as a function of $t$ and $\chi_{\varepsilon}(x,0) = u(x,0)$. Moreover for every $0 \leq \hat{t} \leq T$ we have

$$2 \int_0^{\hat{t}} \int_{\mathbb{R}^N} \varepsilon |\nabla_x \chi_{\varepsilon}(x,s)|^2 \, dxdy = \int_{\mathbb{R}^N} u^2(x,0) \, dx - \int_{\mathbb{R}^N} \chi_{\varepsilon}^2(x,\hat{t}) \, dx. \tag{3.91}$$

Finally, by the well known maximum principle for the Heat Equation we clearly have $\|\chi_{\varepsilon}(\cdot, t)\|_{L^\infty} \leq \|u(\cdot, 0)\|_{L^\infty}$. Next let $\theta(t) \in C^\infty(\mathbb{R}, [0, 1])$ be a cut-off function satisfying $\theta(t) = 0$ for every $t \geq 1$ and $\theta(t) = 1$ for every $t \leq 1/2$. For every small $\varepsilon > 0$ define $\bar{u}_{h,\varepsilon}(x,t) \in L^\infty\left(0, T; L^2(\mathbb{R}^N, \mathbb{R}^k)\right)$ be a solution of the Heat Equation

$$\begin{aligned}
\bar{u}_{h,\varepsilon}(x,t) &:= u_{h,\varepsilon}(x,t) + \theta(t(h\varepsilon)^{-1}) (\chi_{\varepsilon}(x,t) - u_{h,\varepsilon}(x,0)) \\
&= u_{h,\varepsilon}(x,t) + \theta(t(h\varepsilon)^{-1}) (\chi_{\varepsilon}(x,t) - u_{h,\varepsilon}^{(0)}(x,0)). \tag{3.92}
\end{aligned}$$

Then $\bar{u}_{h,\varepsilon}$ is $L^2$-strongly continuous in $[0, T]$ as a function of $t$ and $\bar{u}_{h,\varepsilon}(x,0) = u(x,0)$. Moreover, $\bar{u}_{h,\varepsilon}(x,t) = u_{h,\varepsilon}(x,t)$ whenever $t \geq h\varepsilon$. Finally

$$\|\bar{u}_{h,\varepsilon} - u_{h,\varepsilon}\|_{L^\infty} + \|u_{h,\varepsilon}^{(0)}\|_{L^\infty} \leq C_0, \tag{3.93}$$

where $C_0 > 0$ is a constant which does not depend on $h$ and $\varepsilon$.

Now we want to prove that

$$\lim_{\varepsilon \to 0^+} \oint_{\varepsilon}^{h\varepsilon} \int_{\mathbb{R}^N} \left\{ \varepsilon \left|\nabla_x (\bar{u}_{h,\varepsilon} - u_{h,\varepsilon})\right|^2 + \frac{1}{\varepsilon} \left|\nabla_x D_{h,\varepsilon}(x,t)\right|^2 \right\} \, dxdy \leq C,$$ 

where $D_{h,\varepsilon}(x,t) \in L^2\left(0, T; \tilde{H}^1_0(\mathbb{R}^N, \mathbb{R}^k)\right)$ satisfies

$$\Delta_x D_{h,\varepsilon}(x,t) = \left\{ \partial_t (\bar{u}_{h,\varepsilon} - u_{h,\varepsilon}) + \text{div}_x F(\bar{u}_{h,\varepsilon} - F(u_{h,\varepsilon})) \right\}, \tag{3.95}$$

and $C > 0$ is a constant which does not depend on $h$ and $\varepsilon$.

First of all by (3.92) and (3.91) we observe that

$$\begin{aligned}
\lim_{\varepsilon \to 0^+} \oint_{\varepsilon}^{h\varepsilon} \int_{\mathbb{R}^N} \varepsilon \left|\nabla_x (\bar{u}_{h,\varepsilon} - u_{h,\varepsilon})\right|^2 \, dxdy &\leq \lim_{\varepsilon \to 0^+} 2\varepsilon^2 h \int_{\mathbb{R}^N} \left|\nabla_x u_{h,\varepsilon}^{(0)}(x,0)\right|^2 \, dx + \\
\lim_{\varepsilon \to 0^+} \oint_{\varepsilon}^{h\varepsilon} \int_{\mathbb{R}^N} 2\varepsilon \left|\nabla_x \chi_{\varepsilon}(x,t)\right|^2 \, dxdy &\leq \lim_{\varepsilon \to 0^+} 2h \int_{\mathbb{R}^N} \left|\nabla_x \kappa_h(z,s) \otimes u(x + \varepsilon z, \varepsilon s)\right| \, dzds \, dx \\
&\quad + \lim_{\varepsilon \to 0^+} \left( \int_{\mathbb{R}^N} \chi_{\varepsilon}^2(x,0) \, dx - \int_{\mathbb{R}^N} \chi_{\varepsilon}^2(x,h\varepsilon) \, dx \right) = 0. \tag{3.96}
\end{aligned}$$
On the other hand by (3.93) we deduce
\[
\lim_{\varepsilon \to 0^+} \int_0^{h_{\varepsilon}} \int_{\mathbb{R}^N} \frac{1}{\varepsilon} |\nabla_{\varepsilon} P_{h,\varepsilon}(x,t)|^2 \, dx dt \leq
\]
\[
C_1 \lim_{\varepsilon \to 0^+} \int_0^{h_{\varepsilon}} \int_{\mathbb{R}^N} \frac{1}{\varepsilon} |\bar{u}_{h,\varepsilon} - u_{h,\varepsilon}|^2 \, dx dt \leq
\]
\[
C_1 \lim_{\varepsilon \to 0^+} \int_0^{h_{\varepsilon}} \int_{\mathbb{R}^N} \frac{2}{\varepsilon} \left( \left| \chi_{\varepsilon}(x,t) \right|^2 + |u_{h,\varepsilon}^{(0)}(0,0)|^2 \right) \, dx dt \leq 4C_2h \sup_{t \in \mathbb{R}^N} u^2(x,t) dx = O(h). \quad (3.97)
\]
where \( P_{h,\varepsilon}(x,t) \in L^2(0,T; \dot{H}^1_0(\mathbb{R}^N, \mathbb{R}^k)) \) satisfies
\[
\Delta_x P_{h,\varepsilon}(x,t) = \text{div}_x \left( F(\bar{u}_{h,\varepsilon}) - F(u_{h,\varepsilon}) \right). \quad (3.98)
\]

Next, using (3.99) we infer
\[
\partial_t(\bar{u}_{h,\varepsilon} - u_{h,\varepsilon}) = \partial_t \left\{ \theta \left( t(h_{\varepsilon})^{-1} \right) \chi_{\varepsilon}(x,t) - u_{h,\varepsilon}^{(0)}(0,0) \right\} =
\]
\[
\theta \left( t(h_{\varepsilon})^{-1} \right) \partial_t \chi_{\varepsilon}(x,t) + (h_{\varepsilon})^{-1} \theta' \left( t(h_{\varepsilon})^{-1} \right) \left( \chi_{\varepsilon}(x,t) - u_{h,\varepsilon}^{(0)}(0,0) \right)
\]
\[
= \theta \left( t(h_{\varepsilon})^{-1} \right) \varepsilon \Delta_x \chi_{\varepsilon}(x,t) + (h_{\varepsilon})^{-1} \theta' \left( t(h_{\varepsilon})^{-1} \right) \left( \chi_{\varepsilon}(x,t) - u_{h,\varepsilon}^{(0)}(0,0) \right). \quad (3.99)
\]

Then consider \( Q_{h,\varepsilon}(x,t) \in L^2(0,T; \dot{H}^1_0(\mathbb{R}^N, \mathbb{R}^k)) \) such that
\[
\Delta_x Q_{h,\varepsilon}(x,t) = \partial_t(\bar{u}_{h,\varepsilon} - u_{h,\varepsilon}), \quad (3.100)
\]

By (3.99) we obtain
\[
\Delta_x Q_{h,\varepsilon}(x,t) = \theta \left( t(h_{\varepsilon})^{-1} \right) \varepsilon \Delta_x \chi_{\varepsilon}(x,t) + (h_{\varepsilon})^{-1} \theta' \left( t(h_{\varepsilon})^{-1} \right) \left( \chi_{\varepsilon}(x,t) - u_{h,\varepsilon}^{(0)}(0,0) \right)
\]
\[
= \theta \left( t(h_{\varepsilon})^{-1} \right) \varepsilon \Delta_x \chi_{\varepsilon}(x,t) + (h_{\varepsilon})^{-1} \theta' \left( t(h_{\varepsilon})^{-1} \right) \left( \chi_{\varepsilon}(x,t) - u_{h,\varepsilon}^{(0)}(x,0) \right) - \left( u_{h,\varepsilon}^{(0)}(x,0) - u(x,0) \right). \quad (3.101)
\]

Next, for every \( 0 < \rho < \varepsilon \) we have
\[
u_{h,\varepsilon}^{(0)}(x,0) - u_{h,\rho}^{(0)}(x,0) = \int_{\rho}^{\varepsilon} \frac{\partial u_{h,\cdot}^{(0)}(x,0)}{\partial \tau} d\tau = \int_{\rho}^{\varepsilon} \frac{\partial}{\partial \tau} \left( \frac{1}{\tau^{N+1}} \int_{\mathbb{R}^N} \kappa_h \left( \frac{y-x}{\tau}, \frac{s}{\tau} \right) u(y,s) dy ds \right) d\tau
\]
\[
= - \int_{\rho}^{\varepsilon} \left( \frac{1}{\tau^{N+2}} \int_{\mathbb{R}^N} \left\{ \frac{y-x}{\tau} \cdot \nabla_1 \kappa_h \left( \frac{y-x}{\tau}, \frac{s}{\tau} \right) + \frac{s}{\tau} \partial_2 \kappa_h \left( \frac{y-x}{\tau}, \frac{s}{\tau} \right) \right\} u(y,s) dy ds \right) d\tau
\]
\[
= \int_{\rho}^{\varepsilon} \left( \frac{1}{\tau^{N+1}} \int_{\mathbb{R}^N} \text{div}_x \left\{ \frac{y-x}{\tau} \cdot \nabla_1 \kappa_h \left( \frac{y-x}{\tau}, \frac{s}{\tau} \right) \right\} u(y,s) - \partial_s \left\{ \frac{1}{\tau^{N+1}} \kappa_h \left( \frac{y-x}{\tau}, \frac{s}{\tau} \right) \right\} \right) u(y,s) dy ds \right) d\tau
\]
\[
= \text{div}_x \int_{\rho}^{\varepsilon} \left( \frac{1}{\tau^{N+1}} \int_{\mathbb{R}^N} u(y,s) \otimes \left\{ \frac{y-x}{\tau} \cdot \kappa_h \left( \frac{y-x}{\tau}, \frac{s}{\tau} \right) \right\} dy ds \right) d\tau
\]
\[
- \int_{\rho}^{\varepsilon} \left( \frac{1}{\tau^{N+1}} \int_{\mathbb{R}^N} \partial_s \left\{ \frac{1}{\tau^{N+1}} \kappa_h \left( \frac{y-x}{\tau}, \frac{s}{\tau} \right) \right\} u(y,s) dy ds \right) d\tau
\]
\[
= \text{div}_x \int_{\rho}^{\varepsilon} \left( \frac{1}{\tau^{N+1}} \int_{\mathbb{R}^N} u(y,s) \otimes \left\{ \frac{y-x}{\tau} \cdot \kappa_h \left( \frac{y-x}{\tau}, \frac{s}{\tau} \right) \right\} dy ds \right) d\tau
\]
\[
- 2 \int_{\rho}^{\varepsilon} \left( \frac{1}{\tau^{N+1}} \int_{0}^{\infty} \partial_s \left\{ \frac{1}{\tau^{N+1}} \kappa_h \left( \frac{y-x}{\tau}, \frac{s}{\tau} \right) \right\} u(y,s) dy ds \right) d\tau, \quad (3.102)
\]
where we deduce the last equality by the fact that \( u(x, t) \equiv u(x, -t) \) (see Remark 3.10). Thus, by (3.102), for every \( \varphi(x) \in C_c^\infty(\mathbb{R}^N, \mathbb{R}^k) \) we have

\[
\int_{\mathbb{R}^N} \left( u_{h, \varepsilon}^{(0)}(x, 0) - u_{h, \varepsilon}^{(0)}(x, 0) \right) \cdot \varphi(x) dx =
\]

\[
- \int_{\mathbb{R}^N} \int_0^\varepsilon \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} u(y, s) \otimes \left\{ \frac{y-x}{	au} \kappa_h \left( \frac{y-x}{	au}, \frac{s}{	au} \right) \right\} dy ds \right) : \nabla_x \varphi(x) \ d\sigma dx
\]

\[
- 2 \int_{\mathbb{R}^N} \int_0^\varepsilon \left( \frac{1}{\tau^{N+1}} \int_{\mathbb{R}^N} \int_0^\infty \partial_s \left( \frac{s}{	au} \kappa_h \left( \frac{y-x}{	au}, \frac{s}{	au} \right) \varphi(x) \right) \cdot u(y, s) dy ds \right) d\tau dx , \tag{3.103}
\]

On the other hand by (3.10) and (3.103), for sufficiently small \( \varepsilon > 0 \) we deduce,

\[
\int_{\mathbb{R}^N} \left( u_{h, \varepsilon}^{(0)}(x, 0) - u_{h, \varepsilon}^{(0)}(x, 0) \right) \cdot \varphi(x) dx =
\]

\[
- \int_{\mathbb{R}^N} \int_0^\varepsilon \left( \frac{1}{\tau^{N+1}} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} u(y, s) \otimes \left\{ \frac{y-x}{	au} \kappa_h \left( \frac{y-x}{	au}, \frac{s}{	au} \right) \right\} dy ds \right) : \nabla_x \varphi(x) \ d\sigma dx
\]

\[
+ 2 \int_{\mathbb{R}^N} \int_0^\varepsilon \left( \frac{1}{\tau^{N+1}} \int_{\mathbb{R}^N} \int_0^\infty \varphi(x) \otimes \nabla_x \left( \frac{s}{	au} \kappa_h \left( \frac{y-x}{	au}, \frac{s}{	au} \right) \right) : F(u(y, s)) dy ds \right) d\tau dx =
\]

\[
- \int_{\mathbb{R}^N} \int_0^\varepsilon \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} u(y, s) \otimes \left\{ \frac{y-x}{	au} \kappa_h \left( \frac{y-x}{	au}, \frac{s}{	au} \right) \right\} dy ds \right) : \nabla_x \varphi(x) \ d\sigma dx
\]

\[
- 2 \int_{\mathbb{R}^N} \int_0^\varepsilon \left( \frac{1}{\tau^{N+1}} \int_{\mathbb{R}^N} \int_0^\infty \varphi(x) \otimes \nabla_x \left( \frac{s}{	au} \kappa_h \left( \frac{y-x}{	au}, \frac{s}{	au} \right) \right) : F(u(y, s)) dy ds \right) d\tau dy =
\]

\[
- \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \kappa_h(z, s) u(x + \tau z, \tau s) \otimes z \ d\tau ds \right) : \nabla_x \varphi(x) \ d\sigma dx
\]

\[
+ 2 \int_{\mathbb{R}^N} \int_0^\varepsilon \left( \int_{\mathbb{R}^N} \int_0^\infty s \kappa_h(z, s) F(u(x + \tau z, \tau s)) d\tau ds \right) : \nabla_x \varphi(x) \ d\sigma dx , \tag{3.104}
\]

Thus letting \( \rho \to 0^+ \) in (3.104) we obtain

\[
\int_{\mathbb{R}^N} \left( u_{h, \varepsilon}^{(0)}(x, 0) - u(x, 0) \right) \cdot \varphi(x) dx = - \int_{\mathbb{R}^N} \int_0^\varepsilon \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \kappa_h(z, s) u(x + \tau z, \tau s) \otimes z \ d\tau ds \right) : \nabla_x \varphi(x) \ d\sigma dx
\]

\[
+ 2 \int_{\mathbb{R}^N} \int_0^\varepsilon \left( \int_0^\infty \int_{\mathbb{R}^N} s \kappa_h(z, s) F(u(x + \tau z, \tau s)) d\tau ds \right) : \nabla_x \varphi(x) \ d\sigma dx =
\]

\[
- \varepsilon \int_{\mathbb{R}^N} \int_0^1 \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \kappa_h(z, s) u(x + \varepsilon \tau z, \varepsilon \tau s) \otimes z \ d\tau ds \right) : \nabla_x \varphi(x) \ d\sigma dx
\]

\[
+ 2 \varepsilon \int_{\mathbb{R}^N} \int_0^1 \left( \int_0^\infty \int_{\mathbb{R}^N} s \kappa_h(z, s) F(u(x + \varepsilon \tau z, \varepsilon \tau s)) d\tau ds \right) : \nabla_x \varphi(x) \ d\sigma dx , \tag{3.105}
\]

Thus

\[
u^{(0)}_{h, \varepsilon}(x, 0) - u(x, 0) = \varepsilon \ \text{div}_x \left\{ \int_0^1 \int_{\mathbb{R}^N} \kappa_h(z, s) u(x + \varepsilon \tau z, \varepsilon \tau s) \otimes z \ d\tau ds \right\} =
\]

\[
- 2 \int_0^1 \int_0^\infty \int_{\mathbb{R}^N} s \kappa_h(z, s) F(u(x + \varepsilon \tau z, \varepsilon \tau s)) d\tau ds \tau \right\} , \tag{3.106}
\]
Therefore, by (3.106), (3.107) and (3.101) we obtain
\[
\Delta_x Q_{h,\varepsilon}(x, t) = \operatorname{div}_x \left\{ \theta(t(h\varepsilon)^{-1})\varepsilon\nabla_x \chi_{\varepsilon}(x, t) + h^{-1}\theta'(t(h\varepsilon)^{-1})\nabla_x \left( \int_0^t \chi_{\varepsilon}(x, s)ds \right) \\
- h^{-1}\theta'(t(h\varepsilon)^{-1}) \left( \int_0^1 \int_{\mathbb{R}^N} \kappa_h(z, s)u(x + \varepsilon\tau z, \varepsilon\tau s) \otimes z \, dz \, d\tau \right) \\
- 2h \int_0^1 \int_0^\infty \int_{\mathbb{R}^N} s \kappa_h(z, s)F(u(x + \varepsilon\tau z, \varepsilon\tau s)) \, dz \, d\tau \right\},
\]
and since \( \kappa_h(r) := \kappa_1(r/h)/h^{N+1} \) we obtain
\[
\Delta_x Q_{h,\varepsilon}(x, t) = \operatorname{div}_x \left\{ \theta(t(h\varepsilon)^{-1})\varepsilon\nabla_x \chi_{\varepsilon}(x, t) + h^{-1}\theta'(t(h\varepsilon)^{-1})\nabla_x \left( \int_0^t \chi_{\varepsilon}(x, s)ds \right) \\
- h^{-1}\theta'(t(h\varepsilon)^{-1}) \left( \int_0^1 \int_{\mathbb{R}^N} \kappa_1(z, s)u(x + \varepsilon\tau z, \varepsilon\tau s) \otimes z \, dz \, d\tau \right) \\
- 2h \int_0^1 \int_0^\infty \int_{\mathbb{R}^N} \kappa_1(z, s)F(u(x + \varepsilon\tau z, \varepsilon\tau s)) \, dz \, d\tau \right\}.
\]
Thus
\[
\lim_{\varepsilon \to 0^+} \int_0^{h\varepsilon} \int_{\mathbb{R}^N} \left| \nabla_x Q_{h,\varepsilon}(x, t) \right|^2 \, dx \, dt \leq C \lim_{\varepsilon \to 0^+} \int_0^{h\varepsilon} \int_{\mathbb{R}^N} \varepsilon^2 \left| \nabla_x \chi_{\varepsilon}(x, t) \right|^2 \, dx \, dt \\
+ C \lim_{\varepsilon \to 0^+} \int_0^{h\varepsilon} \int_{\mathbb{R}^N} \left| \nabla_x \left( \int_0^t \chi_{\varepsilon}(x, s)ds \right) \right|^2 \, dx \, dt \\
+ C \int_0^{h\varepsilon} \int_{\mathbb{R}^N} \left( \int_0^1 \int_{\mathbb{R}^N} \kappa_1(z, s)u(x + \varepsilon\tau z, \varepsilon\tau s) \otimes z \, dz \, d\tau \right) \\
\left. \left( \int_0^{h\varepsilon} \int_{\mathbb{R}^N} \kappa_1(z, s)F(u(x + \varepsilon\tau z, \varepsilon\tau s)) \, dz \, d\tau \right) \right|^2 \, dx.
\]
Therefore,
\[
\lim_{\varepsilon \to 0^+} \int_0^{h\varepsilon} \int_{\mathbb{R}^N} \left| \nabla_x Q_{h,\varepsilon}(x, t) \right|^2 \, dx \, dt \leq C \left( \lim_{\varepsilon \to 0^+} \frac{1}{2} \left( \int_{\mathbb{R}^N} \chi_{\varepsilon}^2(x, 0)dx - \int_{\mathbb{R}^N} \chi_{\varepsilon}^2(x, h\varepsilon)dx \right) \\
+ \lim_{\varepsilon \to 0^+} \frac{1}{h^2\varepsilon} \int_0^{h\varepsilon} \int_{\mathbb{R}^N} \left| \nabla_x \left( \int_0^t \chi_{\varepsilon}(x, s)ds \right) \right|^2 \, dx \, dt \\
+ \lim_{\varepsilon \to 0^+} \frac{1}{h} \int_{\mathbb{R}^N} \left( \int_0^1 \int_{\mathbb{R}^N} \kappa_1(z, s)u(x + \varepsilon\tau z, \varepsilon\tau s) \otimes z \, dz \, d\tau \right) \\
\left. \left( \int_0^{h\varepsilon} \int_{\mathbb{R}^N} \kappa_1(z, s)F(u(x + \varepsilon\tau z, \varepsilon\tau s)) \, dz \, d\tau \right) \right|^2 \, dx \right) \\
= 0 + C \left( \lim_{\varepsilon \to 0^+} \frac{1}{h^2\varepsilon} \int_0^{h\varepsilon} \int_{\mathbb{R}^N} \left| \nabla_x \left( \int_0^t \chi_{\varepsilon}(x, s)ds \right) \right|^2 \, dx \, dt \\
+ \lim_{\varepsilon \to 0^+} \frac{1}{h} \int_{\mathbb{R}^N} \left( \int_0^1 \int_{\mathbb{R}^N} \kappa_1(z, s)u(x + \varepsilon\tau z, \varepsilon\tau s) \otimes z \, dz \, d\tau \right) \\
\left. \left( \int_0^{h\varepsilon} \int_{\mathbb{R}^N} \kappa_1(z, s)F(u(x + \varepsilon\tau z, \varepsilon\tau s)) \, dz \, d\tau \right) \right|^2 \, dx \right).
\]
On the other hand

\[
\lim_{\varepsilon \to 0^+} \frac{1}{h \varepsilon} \int_0^{h \varepsilon} \int_{\mathbb{R}^N} \left| \nabla_x \left( \int_0^t \chi_{\varepsilon}(x, s) ds \right) \right|^2 dx dt = \lim_{\varepsilon \to 0^+} \frac{1}{h \varepsilon} \int_0^{h \varepsilon} \int_{\mathbb{R}^N} \left| \nabla_x \chi_{\varepsilon}(x, s) \right|^2 ds dx dt \leq \\
\lim_{\varepsilon \to 0^+} \frac{1}{h^2 \varepsilon} \int_0^{h \varepsilon} \int_{\mathbb{R}^N} \left| \nabla_x \chi_{\varepsilon}(x, s) \right|^2 ds dx dt \leq \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_{\mathbb{R}^N} \left| \nabla_x \chi_{\varepsilon} \right|^2 dx = 0. \quad (3.112)
\]

Then, by (3.111) and (3.112) we obtain

\[
\lim_{\varepsilon \to 0^+} \int_0^{h \varepsilon} \int_{\mathbb{R}^N} \frac{1}{\varepsilon} \left| \nabla_x Q_{h \varepsilon}(x, t) \right|^2 dx dt \leq C \lim_{\varepsilon \to 0^+} \frac{1}{h} \int_0^{h \varepsilon} \int_{\mathbb{R}^N} \kappa_1(z, s) z \otimes u(x, \varepsilon \mathbf{H} z, \varepsilon \mathbf{H} s) dz ds dt \left\langle dx \right. \\
+ C \lim_{\varepsilon \to 0^+} \frac{h}{\varepsilon} \int_{\mathbb{R}^N} \int_0^{h \varepsilon} \int_{\mathbb{R}^N} 2 \int_0^\infty \int_{\mathbb{R}^N} s \kappa_1(z, s) F(u(x, \varepsilon \mathbf{H} z, \varepsilon \mathbf{H} s)) ds ds dt \left\langle dx \right. \\
= 0 + O(h). \quad (3.113)
\]

Thus by (3.109), (3.107) and (3.113) we deduce (3.114). Therefore, by (3.88), (3.94), and (3.93) we obtain

\[
\lim_{\varepsilon \to 0^+} \int_0^T \int_{\mathbb{R}^N} \left\langle \left| \nabla_x \left\{ \nabla u(\tilde{u}_{h \varepsilon}(x, t)) \right\} \right|^2 + 1 \varepsilon \left| \nabla_x F_u(\tilde{u}_{h \varepsilon}(x, t)) \right|^2 \right\rangle dx dt \leq \\
\int_{\mathbb{R}^N} \hat{E}_0(u^+(x, t), u^-(x, t), \mathbf{V}(x, t)) \partial \mathcal{H}(x) \delta + O(h^{1/2}), \quad (3.114)
\]

Therefore, taking \( h \) sufficiently small and \( \tilde{u}_{h \varepsilon} := \tilde{u}_{h \varepsilon} \) and \( \bar{L}_{\varepsilon} := -\nabla_x H_{F, \bar{u}_{h \varepsilon}} + F(\bar{u}_{h \varepsilon}) \), by (3.114) and the fact that \( \bar{u}_{h \varepsilon}(x, T) = u_{h \varepsilon}(x, T) = u^{(0)}_{h \varepsilon}(x, T) \to u(x, T) \) in \( L^2(\mathbb{R}^N, \mathbb{R}^k) \) we finally deduce (3.79). (where as before we can define the corresponding function \( \bar{u}_{h \varepsilon} \)).

The following Theorem provides the lower bound for the functional, related to conservation laws.

**Theorem 3.2.** Let \( F(u) \in C^1(\mathbb{R}^k, \mathbb{R}^{k \times N}) \) and \( \eta(u) \in C^3(\mathbb{R}^k, \mathbb{R}) \) be a convex entropy for the corresponding system (3.10), which satisfies \( \eta(u) \geq 0 \), \( \eta(0) = 0 \) and \( |F(u)| \leq C(|u| + 1) \) \( \forall u \in \mathbb{R}^k \), for some constant \( C > 0 \). Furthermore, let \( u \in \mathcal{E}_F \) (see Definition 3.3). Then for every sequence of functions \( u_{h \varepsilon}(x, t) \in L^2(\mathbb{R}^N, H_0^1(\mathbb{R}^N, \mathbb{R}^k)) \cap C(0, T; L^2(\mathbb{R}^N, \mathbb{R}^k)) \cap L^\infty \) and \( L_{h \varepsilon}(x, t) \in L^2(\mathbb{R}^N \times (0, T), \mathbb{R}^{k \times N}) \) such that \( u_{h \varepsilon} \to u \) in \( L^2(\mathbb{R}^N \times (0, T); \mathbb{R}^k) \), \( L_{h \varepsilon} \to F(u) \) in \( L^2(\mathbb{R}^N \times (0, T); \mathbb{R}^{k \times N}) \) and \( \partial_t \tilde{u}_{h \varepsilon} + \nabla_x \bar{L}_{\varepsilon} = 0 \) we have

\[
\lim_{\varepsilon \to 0^+} \left\{ \int_0^T \int_{\mathbb{R}^N} \left( \varepsilon \left| \nabla_x \left\{ \nabla u(\tilde{u}_{h \varepsilon}(x, t)) \right\} \right|^2 + \frac{1}{\varepsilon} \left| L_{h \varepsilon}(x, t) - F(u_{h \varepsilon}(x, t)) \right|^2 \right) dx dt \right\} = \\
\lim_{\varepsilon \to 0^+} \left\{ \int_0^T \int_{\mathbb{R}^N} \left( \varepsilon \left| \nabla_x \left\{ \nabla u \eta(\nabla_x v_{h \varepsilon}(x, t)) \right\} \right|^2 + \frac{1}{\varepsilon} \left| \partial_t v_{h \varepsilon}(x, t) + F(\nabla_x v_{h \varepsilon}(x, t)) \right|^2 \right) dx dt \right\} \\
+ \int_{\mathbb{R}^N} \eta \left( \nabla_x v_{h \varepsilon}(x, T) \right) dx \geq \\
\lim_{\varepsilon \to 0^+} \left\{ \int_0^T \int_{\mathbb{R}^N} \left( \varepsilon \left| \nabla_x \left\{ \nabla u(\tilde{u}_{h \varepsilon}(x, t)) \right\} \right|^2 + \frac{1}{\varepsilon} \left| \nabla_x H_{F, u_{h \varepsilon}}(x, t) \right|^2 \right) dx dt \right\} \\
+ \int_{\mathbb{R}^N} \eta \left( \nabla_x u_{h \varepsilon}(x, T) \right) dx \geq \\
\int_{\mathbb{R}^N} \hat{I}_0(u^+(x, t), u^-(x, t), \mathbf{V}(x, t)) \partial \mathcal{H}(x) + \int_{\mathbb{R}^N} \eta(u(x, T)) dx, \quad (3.115)
\]

where \( H_{F, \bar{u}_{h \varepsilon}}(x, t) \in L^2(0, T; \hat{H}^1_0(\mathbb{R}^N, \mathbb{R}^k)) \) satisfies

\[
\Delta_x H_{F, u_{h \varepsilon}}(x, t) = \partial_t u_{h \varepsilon}(x, t) + \nabla_x F\left(u_{h \varepsilon}(x, t)\right), \quad (3.116)
\]

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Finally for a.e. 
\( \nu \) and 
\( BV \)

in the similar way as we proved the equality (3.53).

\[
\hat{I}_0(u^+, u^-, \nu) := \inf \left\{ \lim_{\epsilon \to 0^+} \int_{I_{\nu}} \left( \epsilon \left| \nabla_y \left( \nabla u \eta (\zeta(y, s)) \right) \right|^2 + \frac{1}{\epsilon} \left| \gamma(y, s) - F(\zeta(y, s)) \right|^2 \right) dyds : \right. \\
\left. \zeta \to \chi(y, u^+, u^-, \nu) \text{ in } L^2(I_{\nu}, \mathbb{R}^k), \quad \gamma \to \hat{\chi}(y, u^+, u^-, \nu, F) \text{ in } L^2(I_{\nu}, \mathbb{R}^{k \times N}) \\
\quad \text{ and } \partial_y \zeta(y, s) + \nabla_y \gamma(y, s) \equiv 0 \right\} \\
= \inf \left\{ \lim_{\epsilon \to 0^+} \int_{I_{\nu}} \left( \epsilon \left| \nabla_y \left( \nabla u \eta (\zeta(y, s)) \right) \right|^2 + \frac{1}{\epsilon} \left| \partial_y \zeta(y, s) + F(\nabla y \zeta(y, s)) \right|^2 \right) dyds : \right. \\
\left. \text{div}_x \zeta \to \chi(y, u^+, u^-, \nu) \text{ in } L^2(I_{\nu}, \mathbb{R}^k) - \partial_y \zeta \to \hat{\chi}(y, u^+, u^-, \nu, F) \text{ in } L^2(I_{\nu}, \mathbb{R}^{k \times N}) \right\}, \quad (3.117)
\]

\[
\chi(y, u^+, u^-, \nu) := \begin{cases} 
  u^+ & \text{if } y \cdot \nu > 0, \\
  u^- & \text{if } y \cdot \nu < 0,
\end{cases}
\]
and

\[
\hat{\chi}(y, u^+, u^-, \nu, F) := \begin{cases} 
  F(u^+) & \text{if } y \cdot \nu > 0, \\
  F(u^-) & \text{if } y \cdot \nu < 0.
\end{cases}
\]

Finally for a.e. \((x, t) \in J_{\nu}\) we have 
\( \hat{I}_0(u^+(x), u^-(x), \nu(x)) = [\nu_x(x, t)] \hat{I}_0(u^+(x), u^-(x), \nu(x)) \), where

\[
\bar{I}_0(u^+, u^-, \nu) := \inf \left\{ \lim_{\epsilon \to 0^+} \int_{I_{\nu}} \left( \epsilon \left| \nabla_y \left( \nabla u \eta (\zeta(y, s)) \right) \right|^2 + \frac{1}{\epsilon} \left| \gamma(y, s) - \hat{F}_\nu(\zeta(y, s)) \right|^2 \right) dyds : \right. \\
\left. \zeta \to \chi(y, u^+, u^-, \nu') \text{ in } L^2(I_{\nu'}, \mathbb{R}^k), \quad \gamma \to \hat{\chi}(y, u^+, u^-, \nu', \hat{F}_\nu) \text{ in } L^2(I_{\nu'}, \mathbb{R}^{k \times N}) \\
\quad \text{ and } \partial_y \zeta(y, s) + \nabla_y \gamma(y, s) \equiv 0 \right\} \\
= \inf \left\{ \lim_{\epsilon \to 0^+} \int_{I_{\nu'}} \left( \epsilon \left| \nabla_y \left( \nabla u \eta (\zeta(y, s)) \right) \right|^2 + \frac{1}{\epsilon} \left| \partial_y \zeta(y, s) + \hat{F}_\nu(\nabla y \zeta(y, s)) \right|^2 \right) dyds : \right. \\
\left. \text{div}_x \zeta \to \chi(y, u^+, u^-, \nu') \text{ in } L^2(I_{\nu'}, \mathbb{R}^k) - \partial_y \zeta \to \hat{\chi}(y, u^+, u^-, \nu', \hat{F}_\nu) \text{ in } L^2(I_{\nu'}, \mathbb{R}^{k \times N}) \right\}, \quad (3.118)
\]

where we denote

\[
\hat{F}_\nu(u) := F(u) + \left( \nu_x / |\nu_y|^2 \right) \{ u \otimes \nu_y \} \quad \forall u \in \mathbb{R}^k,
\]

\( \nu = (\nu_y, \nu_s) = (\nu_x, \nu_t) \in \mathbb{R}^N \times \mathbb{R} \) and \( \nu' := (\nu_y / |\nu_y|, 0) \). Here \( I_{\nu} := \{ y \in \mathbb{R}^{N+1} : |y \cdot \nu_j| < 1/2 \quad \forall j = 1, 2, \ldots, N \} \) where \( \{ \nu_1, \nu_2, \ldots, \nu_N, \nu_{N+1} \} \subset \mathbb{R}^{N+1} \) is an orthonormal base in \( \mathbb{R}^{N+1} \) such that \( \nu_1 := \nu \) and \( \hat{I}_{\nu'} := \{ y \in \mathbb{R}^{N+1} : |y \cdot \nu'_j| < 1/2 \quad \forall j = 1, 2, \ldots, N \} \) where \( \{ \nu'_1, \nu'_2, \ldots, \nu'_N, \nu'_{N+1} \} \subset \mathbb{R}^{N+1} \) is an orthonormal base in \( \mathbb{R}^{N+1} \) such that \( \nu'_1 := \nu' \) and \( \nu'_{N+1} := (0, 0, \ldots, 1) \).

Proof: The inequality (3.118) follows by Theorem 3.1 in [33] or by Theorem 3.3 in [33]. Finally we can prove equality

\[
\hat{I}_0(u^+(x), u^-(x), \nu(x)) = [\nu_x(x, t)] \hat{I}_0(u^+(x), u^-(x), \nu(x))
\]
in the similar way as we proved the equality (3.33).

### A Notations and basic results about BV-functions

- For given a real topological linear space \( X \) we denote by \( X^* \) the dual space (the space of continuous linear functionals from \( X \) to \( \mathbb{R} \)).
• For given $h \in X$ and $x^* \in X^*$ we denote by $\langle h, x^* \rangle_{X \times X^*}$ the value in $\mathbb{R}$ of the functional $x^*$ on the vector $h$.

• For given two normed linear spaces $X$ and $Y$ we denote by $\mathcal{L}(X; Y)$ the linear space of continuous (bounded) linear operators from $X$ to $Y$.

• For given $A \in \mathcal{L}(X; Y)$ and $h \in X$ we denote by $A \cdot h$ the value in $Y$ of the operator $A$ on the vector $h$.

• For given two reflexive Banach spaces $X, Y$ and $S \in \mathcal{L}(X; Y)$ we denote by $S^* \in \mathcal{L}(Y^*; X^*)$ the corresponding adjoint operator, which satisfy

$$\langle x, S^* \cdot y^* \rangle_{X \times X^*} := \langle S \cdot x, y^* \rangle_{Y \times Y^*} \quad \text{for every } y^* \in Y^* \text{ and } x \in X.$$ 

• Given open set $G \subset \mathbb{R}^N$ we denote by $\mathcal{D}(G, \mathbb{R}^d)$ the real topological linear space of compactly supported $\mathbb{R}^d$-valued test functions i.e. $C_c^\infty(G, \mathbb{R}^d)$ with the usual topology.

• We denote $\mathcal{D}'(G, \mathbb{R}^d) := \{\mathcal{D}(G, \mathbb{R}^d)\}^*$ (the space of $\mathbb{R}^d$ valued distributions in $G$).

• Given $h \in \mathcal{D}'(G, \mathbb{R}^d)$ and $\delta \in \mathcal{D}(G, \mathbb{R}^d)$ we denote $\langle \delta, h \rangle := \langle \delta, h \rangle_{\mathcal{D}(G, \mathbb{R}^d) \times \mathcal{D}'(G, \mathbb{R}^d)}$ i.e. the value in $\mathbb{R}$ of the distribution $h$ on the test function $\delta$.

• Given a linear operator $A \in \mathcal{L}(\mathbb{R}^d; \mathbb{R}^k)$ and a distribution $h \in \mathcal{D}'(G, \mathbb{R}^d)$ we denote by $A \cdot h$ the distribution in $\mathcal{D}'(G, \mathbb{R}^k)$ defined by

$$\langle \delta, A \cdot h \rangle := \langle A^* \cdot \delta, h \rangle \quad \forall \delta \in \mathcal{D}(G, \mathbb{R}^k).$$

• Given $h \in \mathcal{D}'(G, \mathbb{R}^d)$ and $\delta \in \mathcal{D}(G, \mathbb{R})$ by $\langle \delta, h \rangle$ we denote the vector in $\mathbb{R}^d$ which satisfy $\langle \delta, h \rangle \cdot e := \langle \delta e, h \rangle$ for every $e \in \mathbb{R}^d$.

• For a $p \times q$ matrix $A$ with $ij$-th entry $a_{ij}$ we denote by $|A| = \left(\sum_{i=1}^{p} \sum_{j=1}^{q} a_{ij}^2\right)^{1/2}$ the Frobenius norm of $A$.

• For two matrices $A, B \in \mathbb{R}^{p \times q}$ with $ij$-th entries $a_{ij}$ and $b_{ij}$ respectively, we write $A : B := \sum_{i=1}^{p} \sum_{j=1}^{q} a_{ij} b_{ij}$.

• For the $p \times q$ matrix $A$ with $ij$-th entry $a_{ij}$ and for the $q \times d$ matrix $B$ with $ij$-th entry $b_{ij}$ we denote by $AB := A \cdot B$ their product, i.e. the $p \times d$ matrix, with $ij$-th entry $\sum_{k=1}^{q} a_{ik} b_{kj}$.

• We identify the $\mathbf{u} = (u_1, \ldots, u_q) \in \mathbb{R}^q$ with the $q \times 1$ matrix $A$ with $i$-th entry $u_i$, so that for the $p \times q$ matrix $A$ with $ij$-th entry $a_{ij}$ and for $\mathbf{v} = (v_1, v_2, \ldots, v_q) \in \mathbb{R}^q$ we denote by $A \mathbf{v} := A \cdot \mathbf{v}$ the $p$-dimensional vector $\mathbf{u} = (u_1, \ldots, u_p) \in \mathbb{R}^p$, given by $\mathbf{u}_i = \sum_{k=1}^{q} a_{ik} v_k$ for every $1 \leq i \leq p$.

• As usual $A^T$ denotes the transpose of the matrix $A$.

• For $\mathbf{u} = (u_1, \ldots, u_p) \in \mathbb{R}^p$ and $\mathbf{v} = (v_1, \ldots, v_p) \in \mathbb{R}^p$ we denote by $\mathbf{u} \cdot \mathbf{v} := \sum_{k=1}^{p} u_k v_k$ the standard scalar product. We also note that $\mathbf{u} \cdot \mathbf{v} = \mathbf{v}^T \mathbf{u}$ as products of matrices.

• For $\mathbf{u} = (u_1, \ldots, u_p) \in \mathbb{R}^p$ and $\mathbf{v} = (v_1, \ldots, v_q) \in \mathbb{R}^q$ we denote by $\mathbf{u} \otimes \mathbf{v}$ the $p \times q$ matrix with $ij$-th entry $u_i v_j$ (i.e. $\mathbf{u} \otimes \mathbf{v} = \mathbf{u} \mathbf{v}^T$ as product of matrices).

• For any $p \times q$ matrix $A$ with $ij$-th entry $a_{ij}$ and $\mathbf{v} = (v_1, v_2, \ldots, v_d) \in \mathbb{R}^d$ we denote by $A \otimes \mathbf{v}$ the $p \times q \times d$ tensor with $ijk$-th entry $a_{ij} v_k$.

• Given a vector valued function $f(x) = (f_1(x), \ldots, f_k(x)) : \Omega \rightarrow \mathbb{R}^k$ ($\Omega \subset \mathbb{R}^N$) we denote by $Df$ or by $\nabla_x f$ the $k \times N$ matrix with $ij$-th entry $\frac{\partial f_j}{\partial x_i}$.
Given a matrix valued function $F(x) := \{F_{ij}(x)\} : \mathbb{R}^N \to \mathbb{R}^{k \times N}$ ($\Omega \subset \mathbb{R}^N$) we denote by $\text{div} F$ the $\mathbb{R}^k$-valued vector field defined by $\text{div} F := (l_1, \ldots, l_k)$ where $l_i = \sum_{j=1}^{N} \frac{\partial F_{ij}}{\partial x_j}$.

Given a matrix valued function $F(x) = \{ f_{ij}(x) \}(1 \leq i \leq p, 1 \leq j \leq q) : \Omega \to \mathbb{R}^{p \times q}$ ($\Omega \subset \mathbb{R}^N$) we denote by $DF$ or by $\nabla_x F$ the $p \times q \times N$ tensor with $ijk$-th entry $\frac{\partial f_{ij}}{\partial x_k}$.

For every dimension $d$ we denote by $I$ the unit $d \times d$-matrix and by $O$ the null $d \times d$-matrix.

Given a vector valued measure $\mu = (\mu_1, \ldots, \mu_k)$ (where for any $1 \leq j \leq k$, $\mu_j$ is a finite signed measure) we denote by $\|\mu\|(E)$ its total variation measure of the set $E$.

For any $\mu$-measurable function $f$, we define the product measure $f : \mu$ by: $f \cdot \mu(E) = \int_E f \, d\mu$, for every $\mu$-measurable set $E$.

Throughout this paper we assume that $\Omega \subset \mathbb{R}^N$ is an open set.

In what follows we present some known results on BV-spaces. We rely mainly on the book [3] by Ambrosio, Fusco and Pallara. Other sources are the books by Hudjaev and Volpert [38], Giusti [19] and Evans and Gariepy [17]. We begin by introducing some notation. For every $\nu \in S^{N-1}$ (the unit sphere in $\mathbb{R}^N$) and $R > 0$ we set

$$
B^+_R(x, \nu) = \{ y \in \mathbb{R}^N : |y - x| < R, (y - x) \cdot \nu > 0 \}, \quad (A.1)
$$

$$
B^-_R(x, \nu) = \{ y \in \mathbb{R}^N : |y - x| < R, (y - x) \cdot \nu < 0 \}, \quad (A.2)
$$

$$
H^+_\nu(x, \nu) = \{ y \in \mathbb{R}^N : (y - x) \cdot \nu > 0 \}, \quad (A.3)
$$

$$
H^-_\nu(x, \nu) = \{ y \in \mathbb{R}^N : (y - x) \cdot \nu < 0 \} \quad (A.4)
$$

and

$$
H^0_\nu = \{ y \in \mathbb{R}^N : y \cdot \nu = 0 \} \quad (A.5)
$$

Next we recall the definition of the space of functions with bounded variation. In what follows, $L^N$ denotes the Lebesgue measure in $\mathbb{R}^N$.

**Definition A.1.** Let $\Omega$ be a domain in $\mathbb{R}^N$ and let $f \in L^1(\Omega, \mathbb{R}^m)$. We say that $f \in BV(\Omega, \mathbb{R}^m)$ if

$$
\int_{\Omega} |Df| := \sup \left\{ \int_\Omega \sum_{k=1}^{m} f_k \text{div} \varphi_k \, dL^N : \varphi_k \in C^1_c(\Omega, \mathbb{R}^N) \forall k, \sum_{k=1}^{m} |\varphi_k(x)|^2 \leq 1 \forall x \in \Omega \right\}
$$

is finite. In this case we define the BV-norm of $f$ by $\|f\|_{BV} := \|f\|_{L^1} + \int_{\Omega} |Df|$.

We recall below some basic notions in Geometric Measure Theory (see [3]).

**Definition A.2.** Let $\Omega$ be a domain in $\mathbb{R}^N$. Consider a function $f \in L^1_{\text{loc}}(\Omega, \mathbb{R}^m)$ and a point $x \in \Omega$.

i) We say that $x$ is a point of *approximate continuity* of $f$ if there exists $z \in \mathbb{R}^m$ such that

$$
\lim_{\rho \to 0^+} \frac{\int_{B_\rho(x)} |f(y) - z| \, dy}{L^N(B_\rho(x))} = 0.
$$

In this case $z$ is called an *approximate limit* of $f$ at $x$ and we denote $z$ by $\hat{f}(x)$. The set of points of approximate continuity of $f$ is denoted by $G_f$.

ii) We say that $x$ is an *approximate jump point* of $f$ if there exist $a, b \in \mathbb{R}^m$ and $\nu \in S^{N-1}$ such that $a \neq b$ and

$$
\lim_{\rho \to 0^+} \frac{\int_{B^+_\rho(x, \nu)} |f(y) - a| \, dy}{L^N(B_\rho(x))} = 0, \quad \lim_{\rho \to 0^+} \frac{\int_{B^-_\rho(x, \nu)} |f(y) - b| \, dy}{L^N(B_\rho(x))} = 0. \quad (A.6)
$$

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The triple \((a, b, \nu)\), uniquely determined by (A.6) up to a permutation of \((a, b)\) and a change of sign of \(\nu\), is denoted by \((f^+(x), f^-(x), \nu_f(x))\). We shall call \(\nu_f(x)\) the approximate jump vector and we shall sometimes write simply \(\nu(x)\) if the reference to the function \(f\) is clear. The set of approximate jump points is denoted by \(J_f\). A choice of \(\nu(x)\) for every \(x \in J_f\) (which is unique up to sign) determines an orientation of \(J_f\). At a point of approximate continuity \(x\), we shall use the convention \(f^+(x) = f^-(x) = \hat{f}(x)\).

We recall the following results on BV-functions that we shall use in the sequel. They are all taken from [4]. In all of them \(\Omega\) is a domain in \(\mathbb{R}^N\) and \(f\) belongs to \(BV(\Omega, \mathbb{R}^m)\).

**Theorem A.1** (Theorems 3.69 and 3.78 from [4]).

i) \(\mathcal{H}^{N-1}\)-almost every point in \(\Omega \setminus J_f\) is a point of approximate continuity of \(f\).

ii) The set \(J_f\) is a countably \(\mathcal{H}^{N-1}\)-rectifiable Borel set, oriented by \(\nu(x)\). In other words, \(J_f\) is \(\sigma\)-finite with respect to \(\mathcal{H}^{N-1}\), there exist countably many \(C^1\) hypersurfaces \(\{S_k\}_{k=1}^\infty\) such that \(\mathcal{H}^{N-1}(\bigcup_{k=1}^\infty S_k) = 0\), and for \(\mathcal{H}^{N-1}\)-almost every \(x \in J_f \cap S_k\), the approximate jump vector \(\nu(x)\) is normal to \(S_k\) at the point \(x\).

iii) \(\lfloor (f^+ - f^-) \otimes \nu_f \rfloor(x) \in L^1(J_f, d\mathcal{H}^{N-1}).\)

**Theorem A.2** (Theorems 3.92 and 3.78 from [4]). The distributional gradient \(Df\) can be decomposed as a sum of three Borel regular finite matrix-valued measures on \(\Omega\),

\[
Df = D^a f + D^c f + D^j f
\]

with

\[
D^a f = (\nabla f) \mathcal{L}^N \quad \text{and} \quad D^j f = (f^+ - f^-) \otimes \nu_f \mathcal{H}^{N-1} \llcorner J_f.
\]

\(D^a\), \(D^c\) and \(D^j\) are called absolutely continuous part, Cantor and jump part of \(Df\), respectively, and \(\nabla f \in L^1(\Omega, \mathbb{R}^{m \times N})\) is the approximate differential of \(f\). The three parts are mutually singular to each other. Moreover we have the following properties:

i) The support of \(D^c f\) is concentrated on a set of \(\mathcal{L}^N\)-measure zero, but \((D^c f)(B) = 0\) for any Borel set \(B \subset \Omega\) which is \(\sigma\)-finite with respect to \(\mathcal{H}^{N-1}\);

ii) \(\lfloor D^a f \rfloor(f^{-1}(H)) = 0\) and \(\lfloor D^c f \rfloor(f^{-1}(H)) = 0\) for every \(H \subset \mathbb{R}^m\) satisfying \(\mathcal{H}^1(H) = 0\).

**Theorem A.3** (Volpert chain rule, Theorems 3.96 and 3.99 from [4]). Let \(\Phi \in C^1(\mathbb{R}^m, \mathbb{R}^q)\) be a Lipschitz function satisfying \(\Phi(0) = 0\) if \(|\Omega| = \infty\). Then, \(v(x) = (\Phi \circ f)(x)\) belongs to \(BV(\Omega, \mathbb{R}^q)\) and we have

\[
D^a v = \nabla \Phi(f) \nabla f \mathcal{L}^N, \quad D^c v = \nabla \Phi(\hat{f}) D^c f, \quad D^j v = [\Phi(f^+) - \Phi(f^-)] \otimes \nu_f \mathcal{H}^{N-1} \llcorner J_f.
\]

We also recall that the trace operator \(T\) is a continuous map from \(BV(\Omega)\), endowed with the strong topology (or more generally, the topology induced by strict convergence), to \(L^1(\partial \Omega, \mathcal{H}^{N-1} \llcorner \partial \Omega)\), provided that \(\Omega\) has a bounded Lipschitz boundary (see [4] Theorems 3.87 and 3.88)).

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