Sets of Completely Decomposed Primes
in Extensions of Number Fields

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Let $p$ be a prime number and let $k(p)$ be the maximal $p$-extension of a number
field $k$. If $T$ is a set of primes of $k$, then $k^T(p)$ denotes the maximal $p$-extension of
$k$ which is completely decomposed at $T$. Assuming that $T$ is finite, the canonical
homomorphism

$$\phi^T(p) : \prod_{\mathfrak{p} \in T(k^T(p))} G_{\mathfrak{p}}(k(p)|k) \longrightarrow G(k(p)|k^T(p))$$

of the free pro-$p$-product of the decomposition groups $G_{\mathfrak{p}}(k(p)|k)$ into the Galois
group $G(k(p)|k^T(p))$ is an isomorphism, see [3] theorem (10.5.8); here the prime $\mathfrak{p}$ is an arbitrary extension of $p$ to $k(p)$.

In the profinite case, i.e. considering the maximal Galois extension $k^T$ which
is completely decomposed at the finite set $T$, there exist suitable extensions $\mathfrak{P}|\mathfrak{p}$ such that $\phi^T$ is an isomorphism of profinite groups. If $T = S_\infty$ is the set
archimedean primes, this is a result of Fried-Haran-Völklein [1] and in general it
is proven by Pop [4].

The fact that $\phi^T(p)$ is an isomorphism if $T$ is finite implies very strong proper-
ties for the extension $k^T(p)|k$. In particular, the (strict) cohomological dimension of the Galois group $G(k(p)|k^T(p))$ is equal to 2 (if $p = 2$ one has to require that $k$
is totally imaginary). Furthermore, we get for the corresponding local extensions that

$$(k^T(p))_{\mathfrak{P}} = k_{\mathfrak{p}}(p) \quad \text{for all primes } \mathfrak{P}|\mathfrak{p}, \, \mathfrak{p} \notin T,$$

i.e. $k^T(p)$ realizes the maximal $p$-extension $k_{\mathfrak{p}}(p)$ of the local fields $k_{\mathfrak{p}}$ for all primes not in $T$. In particular, the set $T$ is equal to the set $D(k^T(p)|k)$ of all primes of $k$ which decomposed completely in the extension $k^T(p)|k$. We will say that $T$ is saturated if it has this property and we call the stronger property $(\ast)$ that $T$ is strongly saturated. If the Dirichlet density $\delta(T)$ is positive, then $T$ is saturated if and only if $T$ is the set of completely decomposed primes of a finite Galois extension of $k$.
If \( T \) is an arbitrary set of primes of \( k \), we call the set \( \hat{T} = D(k^T(p)|k) \) the saturation of \( T \). The most interesting case is that \( T \) is an infinite set of primes of density zero. If \( T \) is saturated, then the extension \( k^T(p)|k \) is infinite. We will see that there exist infinite sets \( T \) of primes such that \( \delta(\hat{T}) > \delta(T) = 0 \), and also sets \( T \) such that \( \delta(T) = \delta(\hat{T}) = 0 \). An important example is the following.

**Theorem 1:** Let \( p \) be an odd prime number and let \( k \) be a CM-field containing the group \( \mu_p \) of all \( p \)-th roots of unity, with maximal totally real subfield \( k^+ \), i.e. \( k = k^+(\mu_p) \) is totally imaginary and \( [k : k^+] = 2 \). Let \( S_p = \{p|p\} \) and \( T = \{p|p \cap k^+ \text{ is inert in } k|k^+\} \).

Then \( T \cup S_p \) is strongly saturated.

Moreover, in the example above we get that the Galois group

\[
G((k^T)_{nr}(p)|k^T(p))
\]

of the maximal unramified \( p \)-extension of \( (k^T)_{nr}(p) \) of \( k^T(p) \) is a free pro-\( p \)-group. This will follow from a more general theorem which deals with a generalization of the notion of saturated sets.

A set \( T = T(k) \) is called stably saturated if the sets \( T(k') \) are saturated for every finite Galois extension \( k'|k \) inside \( k^T(p) \). These sets are necessarily of density 0 if they are not equal to set \( \mathcal{P} \) of all primes. Obviously, strongly saturated sets are stably saturated. We have the following theorem.

**Theorem 2:** Let \( T \neq \mathcal{P} \) be a stably saturated set of primes of a number field \( k \). Then the canonical map

\[
\phi^T(p) : \bigstar_{\mathfrak{P} \in T(k^T(p))} (G_{\mathfrak{P}}(k(p)|k), T_{\mathfrak{P}}(k(p)|k)) \longrightarrow G(k(p)/k^T(p))
\]

is an isomorphism.

Here \( \bigstar_{\mathfrak{P} \in T(k^T(p))} (G_{\mathfrak{P}}(k(p)|k), T_{\mathfrak{P}}(k(p)|k)) \) denotes the free corestricted pro-\( p \)-product of the decomposition groups \( G_{\mathfrak{P}}(k(p)|k) \) with respect to the inertia groups \( T_{\mathfrak{P}}(k(p)|k) \), see [2].

**Corollary:** In the situation of theorem 1 let \( \tilde{T} = T \cup S_p \). Then

\[
\bigstar_{\mathfrak{P} \in \tilde{T}(k^T(p))} (G_{\mathfrak{P}}(k(p)|k), T_{\mathfrak{P}}(k(p)|k)) \longrightarrow G(k(p)/k^{\tilde{T}}(p)).
\]

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1 Saturated sets of primes of a number field

We start with some remarks on complete lattices. Let

\[ A \xrightarrow{\varphi} B \xleftarrow{\psi} \]

be maps between complete lattices \((A, \subseteq)\) and \((B, \subseteq)\) with the following properties:

I. \(\varphi\) and \(\psi\) are order-reversing,
II. \(A \subseteq \psi \varphi (A)\) and \(B \subseteq \varphi \psi (B)\) for all \(A \in A\) and \(B \in B\).

For \(A \in A\) and \(B \in B\) we define the saturation

\[ \hat{A} := \psi \varphi (A) \text{ and } \hat{B} := \varphi \psi (B), \]

and call \(A \in A\) resp. \(B \in B\) to be saturated if \(A = \hat{A}\) resp. \(B = \hat{B}\). We put

\[ A_{\text{sat}} = \{ A \in A | A \text{ is saturated} \}, \quad B_{\text{sat}} = \{ B \in B | B \text{ is saturated} \}, \]

and we have the following properties:

(i) \(A_1 \subseteq A_2\) implies \(\hat{A}_1 \subseteq \hat{A}_2\) and \(B_1 \subseteq B_2\) implies \(\hat{B}_1 \subseteq \hat{B}_2\).
(ii) \(\varphi \psi \varphi = \varphi, \varphi \varphi \psi = \psi, \hat{A} = A, \hat{B} = B\).
(iii) \(B_{\text{sat}}\) is the image of \(A\) under \(\varphi\) and \(A_{\text{sat}}\) is the image of \(B\) under \(\psi\), i.e. \(\varphi: A \rightarrow B_{\text{sat}}\) and \(\psi: B \rightarrow A_{\text{sat}}\), and \(\varphi\) and \(\varphi\) induce bijections

\[ A_{\text{sat}} \xrightarrow{\varphi} B_{\text{sat}}. \]

(iv)

\[ \varphi \left( \bigcup_{i} A_i \right) = \bigcap_{i} \varphi (A_i), \quad \varphi \left( \bigcap_{i} \psi (B_i) \right) = \bigcup_{i} B_i. \]

In particular, the infimum of saturated elements is again saturated. The verification of these statements is straightforward using the properties I and II of the maps \(\varphi\) and \(\psi\). We define the equivalence relations

\[ A_1 \sim A_2 :\Leftrightarrow \varphi (A_1) = \varphi (A_2) \quad \text{for } A_1, A_2 \in A \]

and

\[ B_1 \sim B_2 :\Leftrightarrow \psi (B_1) = \psi (B_2) \quad \text{for } B_1, B_2 \in B, \]

and denote the classes by \([X]\). Obviously, \(X \sim \hat{X}\) by (ii) and for \(Y \in [X]\) we have \(Y \subseteq \hat{X}\), and so \(\hat{X}\) is the unique maximal element of \([X]\). Furthermore, the surjections \(\psi \varphi: A \rightarrow A_{\text{sat}}\) and \(\varphi \psi: B \rightarrow B_{\text{sat}}\) induce bijections

\[ \psi \varphi: (A/\sim) \rightarrow A_{\text{sat}}, \quad [A] \mapsto \hat{A}, \quad \text{and} \quad \varphi \psi: (B/\sim) \rightarrow B_{\text{sat}}, \quad [B] \mapsto \hat{B}. \]
Now let $K$ be a number field. We use the following notation: $S_\infty$, $S_\mathbb{R}$ and $S_\mathbb{C}$ are the sets of archimedean, real and complex primes of $K$, respectively, $\mathcal{P}$ is the set of all primes of $K$, and if $p$ is a prime number, then $S_p$ is the set of all primes of $K$ above $p$.

If $p$ is a prime of $K$, then $K_p$ is the completion of $K$ with respect to $p$. If $L|K$ is a Galois extension, then we denote the decomposition group and inertia group of the Galois group $G(L|K)$ with respect to $p$ by $G_p(L|K)$ and $T_p(L|K)$, respectively.

For a set $S$ of primes of $K$, let $\delta(S) = \delta_K(S)$ be its Dirichlet density. If $S = S(K)$ is a set of primes and $K'|K$ an algebraic extension of $K$, then we denote the set of primes of $K'$ consisting of all prolongations of $S$ by $S(K')$.

If $\mathfrak{c}$ is a class of finite groups which is closed under taking subgroups, homomorphic images and group extensions, then $K(\mathfrak{c})$ is the maximal pro-$\mathfrak{c}$-extension of $K$, and in particular if $p$ is a prime number, $K(p)$ denotes the maximal $p$-extension of $K$. By abuse of notation, we denote the maximal pro-$\mathfrak{c}$ extension of $K$ which is completely decomposed at $T$ by $K^T(\mathfrak{c})$, and $K^T(p)$ is the maximal $p$-extension of $K$ inside $K^T$.

Let

$$\mathcal{E}_K = \{ L \mid L \text{ is a Galois extension of } K \} \xrightarrow{\varphi}{T \mid T \text{ is a set of primes of } K} = S_K$$

where

$$\varphi(L) = D(L|K) \text{ is the set of primes which are completely decomposed in } L|K,$$

$$\psi(T) = K^T \text{ is the maximal Galois extension of } K \text{ which is completely decomposed at } T.$$

Obviously, the maps $\varphi$ and $\psi$ are order-reversing and

$$L \subseteq \psi\varphi(L) = K^{D(L|K)} =: \hat{L} \quad \text{and} \quad T \subseteq \varphi\psi(T) = D(K^T|K) =: \hat{T}.$$

Furthermore, we define the equivalence relations

$$L_1 \sim L_2 :\iff D(L_1|K) = D(L_2|K) \quad \text{for } L_1, L_2 \in \mathcal{E}_K$$

and

$$T_1 \sim T_2 :\iff K^{T_1} = K^{T_2} \quad \text{for } T_1, T_2 \in S_K.$$

**Definition 1.1** The extension $L \in \mathcal{E}_K$ is called **saturated** if $\hat{L} = L$, i.e.

$$L = K^{D(L|K)},$$

and the set $T \in S_K$ is called **saturated** if $\hat{T} = T$, i.e.

$$T = D(K^T|K).$$
We strengthen the notion of saturated sets in the following way:

**Definition 1.2** Let $T$ be a set of primes of $K$.

(i) The set $T = T(K)$ is called **stably saturated** if the sets $T(K') \in \mathcal{S}_{K'}$ are saturated for every finite Galois extension $K'|K$ inside $K^T$.

(ii) We call $T$ to be **strongly saturated** if $(K^T)_\mathfrak{p} = \bar{K}_p$ for all primes $\mathfrak{p} | p \not\in T$, where $\bar{K}_p$ is the algebraic closure of $K_p$.

**Remark 1:** If we consider the set $E_K(c) = \{L | L$ is a pro-$c$-extension of $K\}$, we define a set $T \in \mathcal{S}_K$ to be $c$-saturated if $T = D(K^T(c)|K)$, and an extension $L \in \mathcal{E}_K(c)$ is called **$c$-saturated** if $L = K^{D(L|K)}(c)$. Analogously, we define stably $c$-saturated and strongly $c$-saturated, e.g. $T$ is strongly $c$-saturated if $(K^T(c))_\mathfrak{p} = \bar{K}_p(\mathfrak{p})$ for all primes $\mathfrak{p}|p, p \not\in T$.

We say, a prime of $K$ is **redundant** (or more precise, $c$-redundant) if is totally decomposed in every extension inside $K(c)$. Obviously, redundant primes are necessarily archimedean primes and the complex primes are always redundant. But also real primes might be redundant if we restrict to pro-$c$-extensions, e.g. if we consider $p$-extensions where $p$ is an odd prime number. Therefore we make the following

**Convention:** In the following all considered primes are not redundant and $\mathcal{S}_K$ consists only of sets of non-redundant primes for the extension $K(c)|K$.

Immediately from the definitions (and the convention) above we get

**Lemma 1.3** Let $T \in \mathcal{S}_K$.

(i) $T$ is saturated if and only if $(K^T)_p \neq \bar{K}_p$ for all $p \not\in T$.

(ii) $T$ is stably saturated if and only if $(K^T)_p|\bar{K}_p$ is an infinite extension for all $p \not\in T$.

From the general remarks on partial ordered sets we see that there are bijections

$$(\mathcal{E}_K)_{sat} \xrightarrow{\varphi} (\mathcal{S}_K)_{sat}$$

and we have the following
Lemma 1.4

(i) \( T_1 \subseteq T_2 \) implies \( \hat{T}_1 \subseteq \hat{T}_2 \) and \( L_1 \subseteq L_2 \) implies \( \hat{L}_1 \subseteq \hat{L}_2 \).

(ii) \( K^T = K^\hat{T} \) and \( D(L|K) = D(\hat{L}|K) \).

(iii) \( D(\prod_i L_i|K) = \bigcap_i D(L_i|K) \) and \( K \bigcup_i T_i = \bigcap_i K^{T_i} \).

(iv) \( D(\bigcap_i K^{T_i}|K) = \hat{\bigcup_i T_i} \) and \( K \bigcap_i D(L_i|K) = \hat{\prod_i L_i} \).

Theorem 1.5

(i) If \( T \) is a finite set of primes of \( K \), then \( T \) is strongly saturated.

(ii) If \( L \) is a finite Galois extension of \( K \), then \( L \) is saturated.

Proof: (i) Let \( p \) be a prime number and let \( L|K \) be a finite Galois extension inside \( K^T \). Let \( p_0 \notin T \), \( \mathfrak{P}_0 \) a fixed extension of \( p_0 \) to \( K^T \) and \( \mathfrak{P}_0 \) the restriction of \( \mathfrak{P}_0 \) to \( L \). By the theorem of Grunwald/Wang (see [3], theorem (9.2.2)) the canonical homomorphism

\[
H^1(L, \mathbb{Z}/p\mathbb{Z}) \longrightarrow H^1(L_{\mathfrak{P}_0}, \mathbb{Z}/p\mathbb{Z}) \bigoplus \bigoplus_{\mathfrak{P} \in T(L)} H^1(L_{\mathfrak{P}}, \mathbb{Z}/p\mathbb{Z})
\]

is surjective. In particular, for every \( \alpha_{\mathfrak{P}_0} \in H^1(L_{\mathfrak{P}_0}, \mathbb{Z}/p\mathbb{Z}) \) there exists an element \( \beta \in H^1(L, \mathbb{Z}/p\mathbb{Z}) \) which is mapped to \( (\alpha_{\mathfrak{P}_0}, 0, \ldots, 0) \). But \( \beta \) lies in the subgroup \( H^1(K^T|L, \mathbb{Z}/p\mathbb{Z}) \) of \( H^1(L, \mathbb{Z}/p\mathbb{Z}) \). Therefore

\[
H^1(K^T|L, \mathbb{Z}/p\mathbb{Z}) \longrightarrow H^1(L_{\mathfrak{P}_0}, \mathbb{Z}/p\mathbb{Z})
\]

is surjective. Varying \( L \) and \( p \), it follows that the completion of \( K^T \) with respect to the prime \( \mathfrak{P}_0 \), \( \mathfrak{P}_0|p_0 \) and \( p_0 \notin T \), is equal to the algebraic closure of \( K_{p_0} \) (since \( G(K_{p_0}|K_{p_0}) \) is pro-solvable).

(ii) Let \( L' \) be a finite Galois extension of \( K \) with \( L \subseteq L' \subseteq K^{D(L|K)} \). Since \( D(L|K) \subseteq D(L'|K) \), we obtain for the densities of these sets the inequality \( \delta(D(L|K)) \leq \delta(D(L'|K)) \), and so, by \( \check{\text{C}}e\text{b}o\text{t}a\text{r}e\text{v}'s \) density theorem,

\[
[L' : K] = \delta(D(L'|K))^{-1} \leq \delta(D(L|K))^{-1} = [L : K].
\]

This shows that \( L' = L \) and so \( L = K^{D(L|K)} \). \( \square \)

Now we consider sets of primes which are infinite and of density equal to 0.

Proposition 1.6 For a set \( T \) of primes we have

\[
\delta(\hat{T}) = 0 \quad \Leftrightarrow \quad K^T|K \text{ is an infinite extension}.
\]
Proof: Since $\hat{T} = D(K^T|K)$, it follows that $\delta(\hat{T}) > 0$ if $K^T|K$ is a finite extension. Conversely, assume that $K^T|K$ is infinite and let $L$ be a finite Galois extension of $K$ inside $K^T$. Then

$$\frac{1}{[L : K]} = \delta(D(L|K)) \geq \delta(\hat{T}),$$

and so $\delta(\hat{T}) = 0$. □

Remark 2: From the proposition above and lemma (1.3)(ii) it follows that a stably saturated set of primes $T \neq \mathcal{P}$ has necessarily density equal to 0.

Remark 3: Let $n \in \mathbb{N}$. Then there exist infinite sets $T$ of primes such that

$$\frac{1}{n} = \delta(\hat{T}) > \delta(T) = 0.$$

Example 1: Let $L|K$ be a Galois extension of the number field $K$ such that $[L : K] = n$. The set $\mathcal{E}_{L|K}^{\text{fin}}$ of proper finite extensions of $L$ being Galois over $K$ is countable, say $\mathcal{E}_{L|K}^{\text{fin}} = \{L_i, i \in \mathbb{N}\}$. We choose for every $i \in \mathbb{N}$ an element $\sigma_i \in G(L_i|L)$, $\sigma_i \neq 1$, and a prime $p_i$ of $K$ which is unramified in $L_i|K$ having a Frobenius $(L_i|K \mathfrak{p}_i) = \sigma_i$. Since there exist infinitely many such primes, there is a $p_i$ such that $N_{K|\mathbb{Q}} p_i \geq i^2$. Then

$$\sum_{i=1}^{\infty} \frac{1}{N_{K|\mathbb{Q}} p_i} \leq \sum_{i=1}^{\infty} \frac{1}{i^2}$$

converges, and so the set $T = \{p_i, i \in \mathbb{N}\}$ has density equal to 0. Furthermore we have

$$K^T = L.$$

Indeed, since every $p \in T$ is completely decomposed in $L$, we have $L \subseteq K^T$ and a finite Galois extension $E|K$, $L \subsetneq E \subseteq K^T$ would be a field $L_i$ for some $i \in \mathbb{N}$ and so $p_i \in T$ would not be completely decomposed in $E|K$. Therefore $\hat{T} = D(L|K)$ and $\delta(\hat{T}) = \frac{1}{n} > 0$.

Remark 4: There exist infinite sets $T$ of primes such that

$$\delta(\hat{T}) = \delta(T) = 0.$$

Example 2: Assume that $K$ is a number field such that

(i) $K$ is not totally real,
(ii) there exists a proper subfield $E$ of $K$ such that $K|E$ is a cyclic extension.
We denote the Galois group $G(K|E)$ by $\Delta$. Let

$$T_0 = \{ p \mid p \cap E \text{ is inert in } K|E \}.$$  

Then $T_0$ is an infinite set of primes of $K$ of density equal to 0. Let $p$ be a prime number not dividing $[K : E]$ whose extensions to $E$ are completely decomposed or totally ramified in $K|E$. Let $K_\infty$ resp. $E_\infty$ be the compositum of all $\Z_p$-extension of $K$ resp. $E$. Then

$$G(K_\infty|K) = \Z_p^{r_2(K)+1+\delta_K}, \quad G(E_\infty|E) = \Z_p^{r_2(E)+1+\delta_E},$$

where $r_2$ denotes the number of complex places and $\delta$ is the so-called Leopoldt defect. We have a decomposition of $\Z_p[\Delta]$-modules

$$G(K_\infty|K) \cong G(E_\infty|E) \oplus M,$$

where $M$ is a $\Delta$-module with $M^\Delta = 0$ and $r = \text{rang}_{\Z_p} M \geq r_2(K) - r_2(E) > 0$, since $\delta_K \geq \delta_E$. Let $L$ be the subfield of $K_\infty$ corresponding to $G(K_\infty|K)^\Delta \cong G(E_\infty|E)$, i.e.

$$G(L|K) \cong M$$

and so $G(L|K)^\Delta = 0$. Observe that $L$ is a Galois extension of $E$. Now let $p \in T_0$. For the decomposition group with respect to $p$ we have the split exact sequence

$$0 \longrightarrow G_p(L|K) \longrightarrow G_p(L|E) \longrightarrow \Delta_p \longrightarrow 0.$$  

If $G_p(L|K)$ would be non-trivial, then $G_p(L|E)$ is not abelian since $\Delta_p = \Delta \neq 0$ acts non-trivially on $G_p(L|K) \subseteq G(L|K)$. But $p$ lies not above $p$ and so it is unramified in $L|K$, thus unramified in $L|E$ and so $G_p(L|E)$ is cyclic. Therefore all primes in $T_0$ are completely decomposed in the infinite extension $L|K$. Let

$$T = D(L|K).$$

Then $T$ is an infinite saturated set (containing $T_0$), i.e. $T = \hat{T}$, and since $K^T$ is an infinite extension (containing $L$), it follows from proposition (1.6) that $\delta(\hat{T}) = 0$. Furthermore, if $S_\infty \subseteq T$, then

$$(K^T)_p|K_p \quad \text{is an infinite extension if } p \notin T,$$

hence $T$ is stably saturated. This follows from the fact, that for a prime $p \notin D(L|K)$ the non-trivial decomposition group $G_p(L|K)$ has to be a infinite subgroup of $G(L|K) \cong \Z_p^r$.

An example for the situation above is a CM-field $K$ with maximal totally real subfield $K^+$. If $T_0(K^+) = \{ p \mid p \text{ is inert in } K \}$, then $T_0 = T_0(K)$ is infinite of density equal to zero. Let $K_{ac}$ be the anti-cyclotomic $\Z_p$-extension of $K$, $p \neq 2$,
and $T = D(K_{ac}|K)$. Then $T_0 \subseteq T = \hat{T}$ and so $K^T|K$ is an infinite extension and therefore $\delta(T) = 0$.

Now we consider the cardinality of an equivalence class $[T]$ of a set of primes $T$ of $K$. By definition, the saturation $\hat{T}$ of $T$ is the unique maximal element of $[T]$ (with respect to the inclusion). But the example 1 of remark 3 shows that there might be infinitely many different (even pairwise disjoint) minimal elements in the class $[T]$.

**Proposition 1.7**

(i) If $L$ is a finite Galois extension of $K$, then
\[
\#[L] = 1 \quad \text{and} \quad \#[D(L|K)] = \infty.
\]

(ii) If $T$ is a finite set of primes, then
\[
\#[T] = 1 \quad \text{and} \quad \#[K^T] = \infty.
\]

**Proof:** Let $L' \in [L]$. Then $L' \subseteq \hat{L} = L$ by theorem (1.5)(ii). Thus $L'|K$ is finite and so
\[
L' = K^{D(L|K)} = K^{D(L|K)} = L.
\]

In order to prove the second assertion of (i), let $L$ be a finite Galois extension of $K$ and let $T \subseteq D(L|K)$ be a subset of density equal to zero. Then
\[
L = K^{D(L|K)} = K^{D(L|K)} \setminus T,
\]
hence every subset $D(L|K) \setminus T$ with $\delta(T) = 0$ is contained in $[D(L|K)]$.

By theorem (1.5)(i), every subset of $T$ is (strongly) saturated. Thus the first assertion of (ii) follows. For the second let $S = T \cup \{p\}$, where $p$ is some prime not in $T$. By [3](10.5.8) we have for the Galois group of the extension $K^T(p)|K^S(p)$ the isomorphism
\[
\ast_{p \in S \setminus T(K^S(p))} G(K_p(p)|K_p) \sim G(K^T(p)|K^S(p)),
\]
where $p$ is some prime number and $E(p)|K$ denotes the maximal $p$-extension inside a Galois extension $E|K$. In particular, the extension $K^T|K^S$ is infinite. Since every Galois extension $L|K$ with $K^S \subseteq L \subseteq K^T$ is contained in $[K^T]$, the second assertion of (ii) follows. \qed
2 The maximal $p$-extension $k^T(p)$ of $k$

In the following we will consider $p$-extensions and by a saturated set of primes we always mean a $p$-saturated set (see remark 1 of the first section).

Let $S, T$ be sets of primes of a (not necessarily finite) number field $K$ and let $p$ be a prime of $K$. Let $K(p)$ be the maximal $p$-extension of $K$. Mostly we will drop the notion $(p)$. So let

$K_{nr}$ is the maximal unramified $p$-extension of $K$,
$K_S$ is the maximal $p$-extension of $K$ which is unramified outside $S$,
$K^T$ is the maximal $p$-extension of $K$ which is completely decomposed at $T$,
$K^T_S$ is the maximal $p$-extension of $K$ which is unramified outside $S$ and completely decomposed at $T$,

$G_p(K) = G_p(K(p)|K) \cong G(K_p(p)|K_p)$ is the decomposition group,
$T_p(K) = T_p(K(p)|K) \cong T(K_p(p)|K_p)$ is the inertia group with respect to $p$.

In the following $k$ will always denote a finite number field. If $K|k$ is an infinite extension of number fields and $S$ a set of primes of $k$, then $S(K)$ denotes the profinite space

$$S(K) = \lim_{\leftarrow} S(k') \cup \{*_{k'}\}$$

where $k'$ runs through the finite subextensions of $K|k$ and $S(k') \cup \{*_{k'}\}$ is the one-point compactification of the discrete set $S(k')$ of primes of $k'$ lying above $S$. According to [3] (10.5.8) and (10.5.10) we have

**Theorem 2.1** If $R' \subseteq R \subseteq S \subseteq S'$ are sets of primes of $k$ such that $\delta(S) = 1$ and $R$ is finite, then the canonical homomorphism

$$\phi_{S',S}^{R':R} : \bigstar_{p \in S' \setminus S(k^R_S)} T(k_p(p)|k_p) \star \bigstar_{p \in R' \setminus R(k^R_S)} G(k_p(p)|k_p) \longrightarrow G(k^R_{S'}|k^R_S)$$

is an isomorphism. Furthermore we have the following assertions concerning the (strict) cohomological dimension: Assume that $k$ is totally imaginary if $p = 2$, then

$$cd_p G(k^R_{S'}|k) = \text{scd}_p G(k^R_S|k) = 2.$$
use the notation

\[ \bigoplus'_{p \in T(K)} H^i(G_p(k(p)|K)) := \lim_{\rightarrow} \bigoplus_{k' \in T(k')} H^i(G_p(k(p)|k')) \]

where \( k' \) runs through all finite subextensions of \( K|k \).

**Proposition 2.2** Let \( p \) be a prime number and let \( T \) be a set of primes of a number field \( k \) of density \( \delta(T) = 0 \). Then the following assertions are equivalent:

(i) \( T \) is stably saturated.
(ii) \( \text{cd}_p G_p(k(p)|k^T) \leq 1 \) for all \( p \not\in T \).
(iii) The canonical map

\[ H^2(k(p)|k^T) \xrightarrow{\sim} \bigoplus'_{p \in T(k^T)} H^2(G_p(k)) \]

is an isomorphism.
(iv) The group \( G((k^T)_S|k^T) \) is free for every \( S \) with \( \delta(S) = 1 \), \( S \cap T = \emptyset \).

**Proof:** (i)\(\Leftrightarrow\)(ii): By lemma (1.3) the set \( T \) is stably saturated if and only if the extensions \( (k^T)_p|k^p \) are infinite for all primes \( p \not\in T \), i.e. if and only if \( \text{cd}_p G_p(k(p)|k^T) \leq 1 \), see [3] (7.1.8)(i).

In order to prove (ii)\(\Leftrightarrow\)(iii), first observe that \( k^T|k \) is an infinite extension as \( \delta(T) = 0 \). Using the Poitou-Tate theorem, see [3] (8.6.10), (10.4.8), and the Hasse principle, loc. cit. (9.1.16), and passing to the limit over all finite extensions \( k' \) inside \( k^T|k \), we obtain the isomorphism

\[ H^2(k(p)|k^T) \xrightarrow{\sim} \bigoplus'_{p \in T(k^T)} H^2(G_p(k)) \oplus \bigoplus'_{p \not\in T(k^T)} H^2(G_p(k(p)|k^T)) , \]

since \( \lim_{\rightarrow} H^0(G(k(p)|k'), \mu_p)^\vee = 0 \). This shows that (iii) is equivalent to

\[ H^2(G_p(k(p)|k^T)) = 0 \] for all \( p \not\in T(k^T) \),

and so we get (ii)\(\Leftrightarrow\)(iii).

From the Poitou-Tate exact sequences and the Hasse principle for the extensions \( k(p)|k \) and \( k_S|k \) (using \( \delta(S) = 1 \)) we get the exact sequence

\[ 0 \rightarrow H^2(k_S|k) \rightarrow H^2(k(p)|k) \rightarrow \bigoplus_{p \not\in S} H^2(G_p(k)) \rightarrow 0. \]
Passing to the limit from $k$ to $k^T$, we obtain the exact sequence

$$0 \to H^2((k^T)|k^T) \to H^2(k(k)|k^T) \to \bigoplus_{p \notin S(k^T)} H^2(G_p(k^T)) \to 0.$$  

Assuming (ii), we have $H^2(G_p(k(p)|k^T)) = 0$ for all $p \notin T(k^T)$, and so

$$\bigoplus_{p \notin S(k^T)} H^2(G_p(k(p)|k^T)) = \bigoplus_{p \in T(k^T)} H^2(G_p(k(p)|k^T)) = \bigoplus_{p \in T(k^T)} H^2(G_p(k))$$

for every $S$ with $S \cap T = \emptyset$. It follows that $H^2((k^T)|k^T) = 0$, i.e. assertion (iv) holds.

Finally, if $H^2((k^T)_T|k^T) = H^2(k_T|k^T) = 0$ for $\bar{T} = \mathcal{P}\setminus T$, then, using again the exact sequence above, assertion (iii) holds. \hfill \qed

There is another situation in which the pro-$p$ group $G((k^T)_S|k^T)$ is free.

**Theorem 2.3** Let $p$ be a prime number and let $T$ and $S$ are sets of primes of a number field $k$, where $T \cup S \neq \mathcal{P}$ is strongly saturated and $T \cap S = \emptyset$. Then the following holds:

(i) $(k^T)_S = k_T$ and $\operatorname{cd}_p G((k^T)_S|k^T) \leq 1$.

(ii) If $\delta(S) = 0$, then $\operatorname{cd}_p G((k^T)_{nr}|k^T) \leq 1$, i.e. the Galois group of the maximal unramified $p$-extension $(k^T)_{nr}|k^T$ is a free pro-$p$-group.

In particular, if $T \neq \mathcal{P}$ is a strongly saturated set, then

$$\operatorname{cd}_p G((k^T)_{nr}|k^T) \leq 1.$$

**Proof:** Since $T \cup S \neq \mathcal{P}$ is stably saturated, we have $\delta(T) \leq \delta(T \cup S) = 0$, hence $\delta(\bar{T}) = 1$, where $\bar{T} = \mathcal{P}\setminus T$. It follows from proposition (2.2) that the group $G(k_T|k^T)$ is free. Since $T \cup S$ is strongly saturated, the extension $k^{T\cup S}$ realizes the local extensions for all primes in $\overline{T \cup S} = \overline{T} \setminus S$, and so the extension $k^T$ has this property. Therefore $k_T|k^T$ is completely decomposed by $\overline{T \cup S}$, hence $k_T = (k^T)_T = (k^T)_S$. This proves (i).

Now assume that $\delta(S) = 0$, hence $\delta(\bar{T} \setminus S) = 1$. Let $K|k$ be a finite extension inside $k^T$. Using the Poitou-Tate theorem and the Hasse principle, we see that the canonical map

$$H^2(K_{\bar{T}\setminus S}|K) \longrightarrow H^2(K_{T}|K)$$

is injective. Passing to the limit, it follows that

$$H^2((k^T)_{T\setminus S}|k^T) \longrightarrow H^2(k_T|k^T)$$
is injective. Since \((k^T)^{T\setminus S} = (k^T)_{nr}\), the desired result follows from (i). 

By theorem (1.5)(i) finite sets are strongly saturated. Now we will show that there are also infinite strongly saturated sets.

**Theorem 2.4** Let \(p\) be an odd prime number and let \(k\) be a CM-field containing the group \(\mu_p\) of all \(p\)-th roots of unity, with maximal totally real subfield \(k^+\), i.e. \(k = k^+(\mu_p)\) is totally imaginary and \([k : k^+] = 2\). Let

\[
T = \{p \mid p \cap k^+ \text{ is inert in } k|k^+\}.
\]

Then the set \(T_0 = T \cup S_p\) of primes of \(k\) is strongly saturated. Furthermore, the Galois group \(G((k^{T_0})_{nr}|k^{T_0})\) is a free pro-\(p\)-group.

**Proof:** Let

\[
S = \{p \mid p \cap k^+ \text{ is decomposed in } k|k^+\}
\]

and \(S_1 = \mathcal{P}\setminus T_0 = S \setminus S_p\). Let \(S_0\) be a subset of \(S_1\) invariant under the action of \(G(k|k^+)\) such that \(V = S_1 \setminus S_0\) is finite (observe that \(T(k_p(p)|k_p)\) is cyclic for \(p \in V\) as \(V \cap S_p = \emptyset\)). Let \(K|k\) be a finite extension inside \(k^{T_0}\) being Galois over \(k^+\) (observe that \(k^{T_0}|k^+\) is a Galois extension as \(T_0\) is invariant under \(G(k|k^+)\)). First we show the following

**Claim:** There exists an abelian (not necessarily finite) \(p\)-extension \(L|K\), which is Galois over \(k^+\), central over \(k\), unramified by \(T_0(K)\), completely decomposed at \(S_p\) and ramified at each prime of \(V(K)\):

\[
T(k_p(p)|k_p)G_p(k(p)|k) \subseteq G_p(L|K) \text{ for all } p \in V(k).
\]

**Proof:** Consider the group extension

\[
1 \rightarrow G(K_{S_1 \cup S_p}^{S_p}|K_{S_0 \cup S_p}) \rightarrow G(K_{S_1 \cup S_p}^{S_p}|K) \rightarrow G(K_{S_0 \cup S_p}^{S_p}|K) \rightarrow 1.
\]

Since \(\delta(S_0) = 1\), we have \(H^2(G(K_{S_0 \cup S_p}^{S_p}|K), \mathbb{Q}_p/\mathbb{Z}_p) = 0\), see (2.1). In the proof of this claim we write \(H^i(E|F)\) for \(H^i(G(E|F), \mathbb{Q}_p/\mathbb{Z}_p)\). We obtain an exact sequence

\[
0 \rightarrow H^1(K_{S_0 \cup S_p}^{S_p}|K) \rightarrow H^1(K_{S_1 \cup S_p}^{S_p}|K) \rightarrow H^1(K_{S_1 \cup S_p}^{S_p}|K_{S_0 \cup S_p})^G(K_{S_0 \cup S_p}^{S_p}|K) \rightarrow 0,
\]

and so an exact sequence

\[
0 \rightarrow H^1(K_{S_0 \cup S_p}^{S_p}|K)^{G(K|k)} \rightarrow H^1(K_{S_1 \cup S_p}^{S_p}|K)^{G(K|k)} \rightarrow H^1(K_{S_1 \cup S_p}^{S_p}|K_{S_0 \cup S_p})^{G(K_{S_0 \cup S_p}^{S_p}|k)} \rightarrow H^1(K|k, H^1(K_{S_0 \cup S_p}^{S_p}|K)) \rightarrow H^1(K|k, H^1(K_{S_1 \cup S_p}^{S_p}|K))
\]
Using again that $H^2(G(K^S_p|K), \mathbb{Q}_p/\mathbb{Z}_p) = 0$ where $\tilde{S} = S_0 \cup S_p$ resp. $\tilde{S} = S_1 \cup S_p$, the Hochschild-Serre spectral sequences

$$E^{i,j}_2 = H^i(K|k, H^j(K^S_p|K)) \Rightarrow H^{i+j}(K^S_p|k)$$

show that in the commutative diagram

$$\begin{array}{ccc}
H^1(K|k, H^1(K^S_p|S_0 \cup S_p|K)) & \longrightarrow & H^1(K|k, H^1(K^S_p|S_1 \cup S_p|K)) \\
\downarrow d^{2,1} & & \downarrow d^{2,1} \\
H^3(K|k, H^0(K^S_p|S_0 \cup S_p|K)) & \longrightarrow & H^3(K|k, H^0(K^S_p|S_1 \cup S_p|K))
\end{array}$$

the differentials $d^{2,1}$ are isomorphisms. Thus we obtain an exact sequence

$$0 \longrightarrow (G(K^S_{S_1 \cup S_p}|K^S_{S_0 \cup S_p})^{ab})_{G(K^S_p|S_0 \cup S_p|k)} \longrightarrow$$

$$\big(G(K^S_{S_1 \cup S_p}|K)^{ab}\big)_{G(K|k)} \longrightarrow \big(G(K^S_{S_0 \cup S_p}|K)^{ab}\big)_{G(K|k)} \longrightarrow 0.$$  

Using (2.1), we obtain the isomorphism

$$\prod_{p \in V(k)} T(k_p(p)|K)_{G_p(k(p)|k)} \cong G(K^S_p|S_{0 \cup S_p})^{ab}_{G(K^S_p|S_0 \cup S_p|k)} \subseteq G(K^S_p|S_{1 \cup S_p})^{ab}_{G(K^S_p|S_1 \cup S_p|k)}.$$  

Thus the abelian extension $L|K$ with $G(L|K) = G(K^S_p|S_1 \cup S_p)|K)^{ab}_{G(K^S_p|S_1 \cup S_p|k)}$ has the desired properties and we proved the claim.

As $L|K$ is central over $k$, $G(k|k^+) \cong \sigma >$ acts on $G(L|K)$. Thus $G(L|K) = G(L|K)^+ \oplus G(L|K)^-$, where $G(L|K)^\pm = G(L|K)^{\pm 1}$. Let $L^\pm$ be defined by $G(L|L^\pm) = G(L|K)^\pm$, i.e. $G(L^\pm|K) = G(L|K)/G(L|K)^\mp$.

For $q \in T_0\backslash S_p$ we have the exact sequence

$$1 \longrightarrow G_q(L^-|k) \longrightarrow G_q(L^-|k^+) \longrightarrow G_q(k|k^+) \longrightarrow 1,$$

where $G_q(L^-|k) = G_q(L^-|K)$. Suppose that $G_q(L^-|k) \neq 1$. Since $G(k|k^+) = G_q(k|k^+)$ acts non-trivially on $G(L^-|K)$, the group $G_q(L^-|k^+)$ is non-abelian. On the other hand the extension $L^-|k^+$ is unramified at $q$. This contradiction shows that all primes of $T_0\backslash S_p$ are completely decomposed in $L^-|K$, and so in $L^-|k$. Since $L|K$ is completely decomposed at $S_p$, we obtain $L^- \subseteq k^{T_0}$.

Let $p \in V$ and so $p \cap k^+$ splits in $k|k^+$. Let $\bar{p} = p^\sigma$ be the conjugated prime and let $\bar{q}$ be a prolongation of $p$ to $K$ and $\bar{q}^\sigma = q^\sigma$. By the claim it follows that

$$\left(T(k_p(p)|K_{\bar{q}})_{G_{\bar{q}}(k(p)|k)} \oplus T(k_{\bar{q}}(p)|K_{\bar{q}})_{G_{\bar{q}}(k(p)|k)}\right)^-$$

injects into $G(L|K)^- \Rightarrow G(L^-|K) = G(L|K)/G(L|K)^+$, and so $L^-|K$ is ramified by $\bar{q}$.
Varying the set $S_0$ and the extension $K|k$, it follows that the extension $k^{T_0}|k$ realizes the ramified part of $k_p(p)|k_p$ for all $p \in S_1 = P \setminus T_0$. Since $k^{T_0}|k$ is a Galois extension, also the unramified part must be realized, i.e.

$$(k^{T_0})_P = k_p(p) \quad \text{for all } P, p \not\in T_0$$

(if $k_p(p)|(k^{T_0})_P$ would have a non-trivial unramified part, then, as the subgroup generated by the Frobenius automorphism is not normal, this extension would also have a ramified part).

The last assertion of the theorem follows from theorem (2.3).

Let

$$\prod_{t \in T}(A_t, B_t) = \{(a_t)_{t \in T} \in \prod_{t \in T} A_t \mid a_t \in B_t \text{ for almost all } t \in T\}$$

be the restricted product over a discrete set $T$ of abelian locally compact groups $A_t$ with respect to closed subgroups $B_t$. The topology is given by the subgroups $V$ with

(i) $V \cap A_t$ is open in $A_t$ for all $t \in T$,
(ii) $V \supseteq B_t$ for almost all $t \in T$.

Then we call

$$\prod_{t \in T}^c(A_t, B_t) := \lim_{\leftarrow V} \left(\prod_{t \in T}(A_t, B_t)/V, \right)$$

the compactification of $\prod_{t \in T}(A_t, B_t)$, where $V$ runs through all open subgroups of finite index in $\prod_{t \in T}(A_t, B_t)$. The the canonical map $\prod_{t \in T}(A_t, B_t) \to \prod_{t \in T}^c(A_t, B_t)$ has dense image.

We define the discretization of $\prod_{t \in T}(A_t, B_t)$ by

$$\prod_{t \in T}^d(A_t, B_t) := \lim_{\to W} W$$

where $W$ runs through the finite subgroups of $\prod_{t \in T}(A_t, B_t)$. If the subgroups $B_t$ of $A_t$, $t \in T$, are open and compact, then $\prod_{t \in T}(A_t, B_t)$ is locally compact. Using the equality

$$(\prod_{t \in T}(A_t, B_t))^\vee = \prod_{t \in T}(A_t^\vee, (A_t/B_t)^\vee),$$

we obtain

$$\prod_{t \in T}^d(A_t, B_t) = (((\prod_{t \in T}(A_t, B_t))^\vee)^\vee)^\vee \quad \text{and} \quad \prod_{t \in T}^c(A_t, B_t) = (((\prod_{t \in T}(A_t, B_t))^\vee)^d)^\vee,$$

where $^\vee$ denotes the Pontryagin-dual.
Proposition 2.5 Let $T$ be a set of primes of a number field $k$ with $\delta(T) = 0$. Then

(i) $G(k(p)/k^T)/G(k(p)/k^T)^* = \lim_{\rightarrow} \prod_{\mathfrak{p} \in T(K)}^C (G_{\mathfrak{p}}(k)/G_{\mathfrak{p}}(k)^*, \tilde{T}_{\mathfrak{p}}(k))$,

(ii) $H^1(G(k(p)/k^T)) = \lim_{\rightarrow} \prod_{\mathfrak{p} \in T(K)}^d (H^1(G_{\mathfrak{p}}(k)), H^1_{nr}(G_{\mathfrak{p}}(k)))$,

where $K$ runs through the finite subextensions of $k^T|k$ and $\tilde{T}_{\mathfrak{p}}(k)$ is the group $T_{\mathfrak{p}}(k)G_{\mathfrak{p}}(k)^*/G_{\mathfrak{p}}(k)^*$.

Proof: Since the set $\mathcal{P}\setminus T$ has density equal to 1, we get from the Poitou-Tate exact sequence and the Hasse principle the commutative and exact diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & H^1(k/k) & \longrightarrow & \prod_{\mathfrak{p} \in T} H^1(k_{\mathfrak{p}}/k_{\mathfrak{p}}, \mu_{\mathfrak{p}}) & \longrightarrow & 0 \\
& | & | & | & | & | & |
0 & \longrightarrow & H^1(k/k) & \longrightarrow & \prod_{\mathfrak{p}} H^1(k_{\mathfrak{p}}/k_{\mathfrak{p}}, \mu_{\mathfrak{p}}) & \longrightarrow & H^1(k/k, \mathbb{Z}/p\mathbb{Z})^\vee & \longrightarrow & 0 \\
& & & & & & & & \\
\prod_{\mathfrak{p} \in T} H^1(k_{\mathfrak{p}}/k_{\mathfrak{p}}, \mu_{\mathfrak{p}}) & \sim & \prod_{\mathfrak{p} \in T} H^1(k_{\mathfrak{p}}/k_{\mathfrak{p}}, \mathbb{Z}/p\mathbb{Z})^\vee & & & & & & \\
\end{array}
\]

where we use the local duality theorem

$H^1(k_{\mathfrak{p}}/k_{\mathfrak{p}}, \mu_{\mathfrak{p}}) \sim \sim H^1(G_{\mathfrak{p}}(k), \mathbb{Z}/p\mathbb{Z})^\vee = G_{\mathfrak{p}}(k)/G_{\mathfrak{p}}(k)^*$

and for $\mathfrak{p}$ not above $p$ or $\infty$ the isomorphism

$H^1_{nr}(k_{\mathfrak{p}}/k_{\mathfrak{p}}, \mu_{\mathfrak{p}}) \sim \sim (H^1(T_{\mathfrak{p}}(k), \mathbb{Z}/p\mathbb{Z})G_{\mathfrak{p}}(k))^\vee \cong T_{\mathfrak{p}}(k)G_{\mathfrak{p}}(k)^*/G_{\mathfrak{p}}(k)^*$,

see [3] (8.6.10), (9.1.9), (9.1.10), (7.2.15). Thus we get a continuous injection

$\prod_{\mathfrak{p} \in T(K)} (G_{\mathfrak{p}}(k)/G_{\mathfrak{p}}(k)^*, \tilde{T}_{\mathfrak{p}}(k)) \hookrightarrow G(k(p)/k)/G(k(p)/k)^*$

and so an injection $\prod_{\mathfrak{p} \in T(K)} (G_{\mathfrak{p}}(k)/G_{\mathfrak{p}}(k)^*, \tilde{T}_{\mathfrak{p}}(k)) \hookrightarrow G(k(p)/k)/G(k(p)/k)^*$. Passing to the limit, we obtain the injection

$\lim_{\rightarrow} \prod_{\mathfrak{p} \in T(K)}^C (G_{\mathfrak{p}}(k)/G_{\mathfrak{p}}(k)^*, \tilde{T}_{\mathfrak{p}}(k)) \hookrightarrow G(k(p)/k^T)/G(k(p)/k^T)^*$

which, by definition of the field $k^T$, is also surjective. Therefore we obtain (i) and (ii) is just the dual assertion. □
Proposition 2.6 Let \( p \) be a prime number and let \( T \neq \mathcal{P} \) be a stably saturated set of primes of a number field \( k \). Assume that \( k \) is totally imaginary if \( p = 2 \). Then

\[
\text{cd}_p \ G(k^T|k) \leq 2.
\]

Proof: We consider the Hochschild-Serre spectral sequence

\[
E_2^{ij} = H^i(G/N, H^j(N)) \Rightarrow E^{i+j} = H^{i+j}(G)
\]

for an extension \( 1 \to N \to G \to G/N \to 1 \) of pro-\( p \) groups. If \( \text{cd}_p(G) \leq 2 \), \( E_2^{11} = 0 \) and the edge homomorphism \( E_2 \to E_3^{02} \) is surjective, then it follows that \( E_3^{30} = 0 \), i.e. \( \text{cd}_p(G/N) \leq 2 \).

Now let \( G = G(k(p)|k) \) and \( N = G(k(p)|k^T) \). Using (2.2) and the Poitou-Tate theorem, it follows that the canonical map

\[
H^2(k(p)|k) \to \bigoplus_{p \in T(k)} H^2(G_p(k)) \sim \to H^0(k^T|k, H^2(k(p)|k^T))
\]

is surjective. Furthermore, using (2.5)(ii), we see that

\[
H^1(k(p)|k^T) = \lim_{\to K} H^1(\prod_{p \in T(k)}^c (G_p(k)/G_p(k)^*, \tilde{T}_p(k)))
\]

\[
= \lim_{\to K} \text{Coind}^{G(k)} H^1(\prod_{p \in T(k)}^c (G_p(k)/G_p(k)^*, \tilde{T}_p(k)))
\]

\[
= \text{Coind}^{G(k^T|k)} H^1(\prod_{p \in T(k)} (G_p(k)/G_p(k)^*, \tilde{T}_p(k)))
\]

is a \( G(k^T|k) \)-coinduced module, hence \( H^1(k^T|k, H^1(k(p)|k^T)) = 0 \). Therefore we obtain the desired result. \( \square \)

If \( T \) is a set of primes of \( k \), then let \( T(k^T) = \lim_{\to K} T(k) \cup \{ \ast_K \} \), where \( K \) runs through the finite subextensions of \( k^T/k \). The following theorem asserts that the Galois group \( G(k(p)/k^T) \), where \( T \neq \mathcal{P} \) is stably saturated, is a corestricted free pro-\( p \)-product of the family \( (G_{\mathfrak{p}}(k))_{\mathfrak{p} \in T(k^T)} \) of decomposition groups with respect to the continuous family \( (T_{\mathfrak{p}}(k))_{\mathfrak{p} \in T(k^T)} \) of inertia groups, see [2] for the definition.

Theorem 2.7 Let \( T \neq \mathcal{P} \) be a stably saturated set of primes of a number field \( k \). Then the canonical map

\[
\varphi : \bigoplus_{\mathfrak{p} \in T(k^T)} (G_{\mathfrak{p}}(k), T_{\mathfrak{p}}(k)) \sim \to G(k(p)/k^T)
\]

is an isomorphism.
Proof: We can apply [2] prop. (4.3) and have to show that the induced maps
\[ \varphi_* : H^1(G(k(p)/k^T), \mathbb{Z}/p\mathbb{Z}) \rightarrow \lim_{K \rightarrow T(K)} \prod_{\mathcal{T}(K)}^d (H^1(G_{\mathfrak{p}^q}(k), \mathbb{Z}/p\mathbb{Z}), H^1_{nr}(G_{\mathfrak{p}^q}(k), \mathbb{Z}/p\mathbb{Z})) \]
\[ \varphi_* : H^2(G(k(p)/k^T), \mathbb{Z}/p\mathbb{Z}) \rightarrow \lim_{K \rightarrow T(K)} \prod_{\mathcal{T}(K)}^d (H^2(G_{\mathfrak{p}^q}(k), \mathbb{Z}/p\mathbb{Z}), H^2_{nr}(G_{\mathfrak{p}^q}(k), \mathbb{Z}/p\mathbb{Z})) \]
are bijective resp. injective. Since \( H^2_{nr}(G_{\mathfrak{p}^q}(k)) = 0 \), it follows that
\[ H^2(G) = \lim_{K \rightarrow T(K)} \prod_{\mathcal{T}(K)}^d (H^2(G_{\mathfrak{p}^q}(k), H^2_{nr}(G_{\mathfrak{p}^q}(k)))) = \bigoplus_{\mathfrak{p} \in T(k^T)}^\prime H^2(G_{\mathfrak{p}}(k)). \]
Now the result follows from (2.5)(ii) and (2.2). \( \square \)

In the situation of theorem (2.4) we obtain

**Corollary 2.8**. Let \( p \) be an odd prime number and let \( k \) be a CM-field containing the group \( \mu_p \) of all \( p \)-th roots of unity, with maximal totally real subfield \( k^+ \), i.e. \( k = k^+(\mu_p) \) is totally imaginary and \([k : k^+] = 2\). Let
\[ T = \{ p \mid p \cap k^+ \text{ is inert in } k|k^+ \} \cup S_p. \]
Then
\[ \bigotimes_{\mathfrak{p} \in T(k^T)}^\ast (G_{\mathfrak{p}^q}(k), T_{\mathfrak{p}^q}(k)) \rightarrow G(k(p)/k^T). \]

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