Braiding operator via quantum cluster algebra

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Abstract
We construct a braiding operator in terms of the quantum dilogarithm function based on the quantum cluster algebra. We show that it is a \(q\)-deformation of the \(R\)-operator for which hyperbolic octahedron is assigned. Also shown is that, by taking \(q\) to be a root of unity, our braiding operator reduces to the Kashaev \(R^K\)-matrix up to a simple gauge-transformation.

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(Some figures may appear in colour only in the online journal)

1. Introduction

It is conjectured [27] that the hyperbolic volume of knot complement is given by

\[
\lim_{N \to \infty} \frac{2\pi}{N} \log \left| \langle K \rangle_N \right| = \text{Vol}(S^3 \setminus K),
\]  

(1.1)

where \(\langle K \rangle_N\) is the Kashaev invariant of knot \(K\). Kashaev constructed the quantum invariant \(\langle K \rangle_N\) based on the finite-dimensional representation of the quantum dilogarithm [25]. It was later realized that the Kashaev invariant \(\langle K \rangle_N\) coincides with the \(N\)-colored Jones polynomial at the \(N\)th root of unity [32],

\[
\langle K \rangle_N = J_N(K; q = e^{2\pi i/N}),
\]  

(1.2)
where the $N$-colored Jones polynomial is normalized to be $J_q$ (unknot; $q$) = 1. More precisely it was shown that Kashaev’s braiding matrix $R^K$ as a finite-dimensional representation of the Artin braid group with $n$ strands,

$$R_n = \left\{ \begin{array}{l}
\sigma_1, \sigma_2, \ldots, \sigma_{n-1} \\
\sigma_i \sigma_{i+1} = \sigma_{i+1} \sigma_i, \\
\sigma_i = \sigma_j, \quad \text{for } |i - j| > 1
\end{array} \right\}$$

(1.3)
is gauge-equivalent to the $R^I$-matrix for the $N$-colored Jones polynomial at the root of unity. As the colored Jones polynomial is well-understood in the framework of the quantum group $\mathfrak{g}\mathfrak{l}_2(q^2)$ (see, e.g., [30]), the Kashaev invariant $\langle K \rangle_N$ is regarded as an invariant for $\mathfrak{g}\mathfrak{l}_2(q^2)$. In [2, 24, 26] (see also [9]) studied is a relationship between $\langle K \rangle_N$ and the 6j-symbol of $U_q(\mathfrak{sl}_2)$, but the mathematical background of the Kashaev $R^K$-matrix itself still remains unclear, at least, to us.

Meanwhile, studies on the geometrical content of $\langle K \rangle_N$ have been much developed. It is now recognized that an ideal hyperbolic octahedron is assigned to each $R^K$-matrix [37], and proposed [8, 41] was a method to construct from a set of such octahedra the Neumann–Zagier potential function [34] which give the complex volume of knot complement. This observation is based on a fact that the hyperbolic volume of ideal tetrahedron is given in terms of the dilogarithm function (see, e.g., [38]), and that the $R^K$-matrix asymptotically consists of four dilogarithm functions [37].

In our previous paper [23], we constructed the $R$-operator from the viewpoint of the cluster algebra. We showed that the $R$-operator is geometrically interpreted as a hyperbolic octahedron which is the same assigned to the Kashaev $R^K$-matrix. The cluster algebra was originally introduced by Fomin and Zelevinsky [19] to study the total positivity in semi-simple Lie groups, and it is promising to clarify a deep connection with geometry [18] (see also [22, 33]).

The purpose of this article is to quantize the $R$-operator in [23]. Basic tool is the quantum cluster algebra [6, 16, 17, 31]. We shall construct the $R$-operator in terms of the quantum dilogarithm function as a conjugation for the quantum $R^q$-operator, and clarify a relationship with the Kashaev $R^K$-matrix. We show explicitly that the $R$-operator reduces to the $R^K$-matrix up to a gauge-transformation when the quantized parameter $q^2$ tends to the $N$th root of unity.

This paper is organized as follows. In section 2, we briefly review our previous results [22, 23]. We discuss a relationship between the cluster algebra and the hyperbolic geometry, and we recall a definition of the $R$-operator (2.9) which is illustrated as a hyperbolic octahedron. In section 3 we study a $q$-deformation of the $R$-operator. We construct the braiding $R^q$-operator (3.9) as a conjugation of the $R$-operator (3.15), which is written in terms of the quantum dilogarithm function. In a limit that $q^2$ goes to a root of unity, the $R$-operator reduces to the Kashaev $R^K$-matrix.

2. Cluster algebra and hyperbolic geometry

2.1. Cluster algebra

We briefly collect a notion of the cluster algebra. See [19] for detail.

Fix a positive integer $N$. Let $(\mathbf{x}, \mathbf{B})$ be a cluster seed, where $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ is a cluster variable, and an $N \times N$ skew-symmetric integral matrix $\mathbf{B} = (b_{ij})$ is an exchange matrix. The exchange matrix is depicted as a quiver which has $N$ vertices, by regarding

$$b_{ij} = \# \{ \text{arrows from } i \text{ to } j \} - \# \{ \text{arrows from } j \text{ to } i \}.$$ 

(2.1)
What is important is an operation on cluster seeds, which is called the mutation. For \( k = 1, \ldots, N \), the mutation \( \mu_k \) of \((x, B)\) is defined by

\[
\mu_k (x, B) = (\tilde{x}, \tilde{B}),
\]

(2.2)

where a cluster variable \( \tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_N) \) and an exchange matrix \( \tilde{B} = (\tilde{b}_{ij}) \) are respectively given by

\[
\tilde{x}_i = \begin{cases} 
    x_i, & \text{for } i \neq k, \\
    \frac{1}{x_k} \left( \prod_{j : b_{ij} > 0} x_j^{b_{ij}} + \prod_{j : b_{ij} < 0} x_j^{-b_{ij}} \right), & \text{for } i = k,
\end{cases}
\]

(2.3)

\[
\tilde{b}_{ij} = \begin{cases} 
    -b_{ij}, & \text{for } i = k \text{ or } j = k, \\
    b_{ij} + \frac{|b_{ik}| b_{kj} + b_{ik} |b_{kj}|}{2}, & \text{otherwise}.
\end{cases}
\]

(2.4)

In this article, for each seed \((x, B)\) we define the \( y \)-variable \( y = (y_1, \ldots, y_N) \) as

\[
y_j = \prod_k x_k^{b_{ij}}.
\]

(2.5)

The mutation of the cluster seed induces the mutation of the \( y \)-variable,

\[
\mu_k (y, B) = (\tilde{y}, \tilde{B}),
\]

(2.6)

where the exchange matrix \( \tilde{B} \) is (2.4), and \( \tilde{y} = \prod_k \tilde{x}_k^{\tilde{b}_{ij}} \) is given by

\[
\tilde{y}_i = \begin{cases} 
    y_k^{-1}, & \text{for } i = k, \\
    y (1 + y_k^{-1})^{-b_{ki}}, & \text{for } i \neq k, \ b_{ki} \geq 0, \\
    y (1 + y_k)^{-b_{ki}}, & \text{for } i \neq k, \ b_{ki} \leq 0.
\end{cases}
\]

(2.7)

2.2. Braiding operator

To study the braid group \( B_n \) (1.3), we set the exchange matrix \( B \) to be a \((3n + 1) \times (3n + 1)\) skew-symmetric matrix [23]

\[
B = \begin{pmatrix}
0 & 1 & -1 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
-1 & 0 & 0 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\
1 & 0 & 0 & -1 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & -1 & 1 & 0 & 1 & \cdots & \cdots & \cdots & 0 \\
0 & 0 & 0 & -1 & 0 & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 0 & -1 \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & -1 & 1 & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & -1 & 1 & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & -1 & 1 & 0
\end{pmatrix}
\]

(2.8)
See figure 1 for the associated quiver. We remark that the quiver gives rise to a triangulation of a punctured disk, whose edges correspond to the vertices in the quiver. See figure 2. We define the $R$-operator acting on a cluster seed $(x, B)$ by

$$R^i = s_{3i, 3i+2} s_{3i-1, 3i+2} s_{3i, 3i+3} \mu_{3i+1} \mu_{3i-1} \mu_{3i+3} \mu_{3i+1},$$

for $i = 1, \ldots, n - 1$, where $s_{ij}$ is the permutation of subscripts, e.g.,

$$s_{ij}(..., x_i, ..., x_j, ...) = (...x_j, ..., x_i, ...).$$

As the exchange matrix $B$ is invariant under $R$, we write $R^i(x, B)$ for $R^i(x, B)$ with

$$R^i(x) = (x_1, ..., x_{3i-3}, R^i(x_{3i-2}, ..., x_{3i+4}), x_{3i+5}, ..., x_{3n+1}),$$

where we have from (2.3)

$$R^i(x_1, x_2, ..., x_7) = \left(\begin{array}{c} x_1, x_2, x_3 x_5 + x_3 x_4 x_5 + x_1 x_2 x_6, \\
 x_2 x_4, x_1 x_3 x_5 + x_3 x_4 x_5 + x_1 x_3 x_4 x_5 + x_3 x_4 x_5 x_7 + x_1 x_2 x_6 x_7, \\
 x_2 x_4 x_6, x_3 x_4 x_5 + x_3 x_4 x_7 + x_2 x_6 x_7, \\
 x_4 x_6, x_3, x_7 \end{array}\right).$$

See figure 1 for the associated quiver. We remark that the quiver gives rise to a triangulation of a punctured disk, whose edges correspond to the vertices in the quiver. See figure 2. We define the $R$-operator acting on a cluster seed $(x, B)$ by

$$R^i = s_{3i, 3i+2} s_{3i-1, 3i+2} s_{3i, 3i+3} \mu_{3i+1} \mu_{3i-1} \mu_{3i+3} \mu_{3i+1},$$

for $i = 1, \ldots, n - 1$, where $s_{ij}$ is the permutation of subscripts, e.g.,

$$s_{ij}(..., x_i, ..., x_j, ...) = (...x_j, ..., x_i, ...).$$

As the exchange matrix $B$ is invariant under $R$, we write $R^i(x, B)$ for $R^i(x, B)$ with

$$R^i(x) = (x_1, ..., x_{3i-3}, R^i(x_{3i-2}, ..., x_{3i+4}), x_{3i+5}, ..., x_{3n+1}),$$

where we have from (2.3)

$$R^i(x_1, x_2, ..., x_7) = \left(\begin{array}{c} x_1, x_2, x_3 x_5 + x_3 x_4 x_5 + x_1 x_2 x_6, \\
 x_2 x_4, x_1 x_3 x_5 + x_3 x_4 x_5 + x_1 x_3 x_4 x_5 + x_3 x_4 x_5 x_7 + x_1 x_2 x_6 x_7, \\
 x_2 x_4 x_6, x_3 x_4 x_5 + x_3 x_4 x_7 + x_2 x_6 x_7, \\
 x_4 x_6, x_3, x_7 \end{array}\right).$$
By definition (2.5), the action on the \( y \)-variable \( y \) is induced as
\[
R(y) = \left( y_1, \ldots, y_{3l-3}, R_y(y_{3l-2}, \ldots, y_{3l+4}), y_{3l+5}, \ldots, y_{3n+1} \right),
\]
where (2.6) gives
\[
R_y(y_1, y_2, \ldots, y_7) = \left( y_1 \left( 1 + y_2 + y_2 y_6 \right), \frac{y_2 y_3 y_5 y_6}{1 + y_2 + y_6 + y_2 y_4 y_6}, \frac{y_4}{1 + y_2 + y_6 + y_2 y_4 y_6}, \frac{y_2 y_3 y_4 y_6}{1 + y_2 + y_6 + y_2 y_4 y_6}, \frac{y_4 y_6}{1 + y_2 + y_6 + y_2 y_4 y_6}, \left( 1 + y_6 + y_2 y_6 \right) y_7 \right).
\]
(2.13)

We use \( R \) in both (2.10) and (2.12) without confusion.

In [23], it was shown that the \( R \)-operator represents the braid group \( B_n \), and that we have,
\[
R^i R^{j+1} R^i = R^i R^{j+1} R^i, \quad R^i R^j = R^j R^i, \quad \text{for } |i - j| > 1.
\]
(2.14)

This can be proved by direct computations using (2.11) and (2.13). We note that the birational Yang–Baxter map in [11] is intrinsically same with (2.11). The braid relation (2.14) could as well be checked from a dual picture as follows. We recall that the mutation is regarded as a ‘flip’ of triangulation of a punctured disk [18]. Here a flip is meant to remove a common edge of two adjacent triangles and to reproduce another different diagonal edge of quadrilateral (see, for example, figure 3). This interpretation explains the action of the \( R \)-operator on a punctured disk as illustrated in figure 4. We find that the \( R \)-operator on the punctured disk is nothing but a half Dehn twist exchanging two punctures counter-clockwise. This clarification of the braid group is well-known (see, e.g., [7]), and the braid relation (2.14) follows immediately.

A geometrical interpretation of the \( R \)-operator is given from the three-dimensional picture of the flip [22]. The mutation in figure 3 acts on the \( y \)-variable as

![Figure 3](image_url)
The flip in figure 3 is interpreted as a gluing of a hyperbolic ideal tetrahedron to a triangulation of punctured disk as in figure 5. Here the ideal hyperbolic tetrahedron has a dihedral angle of edge. Consistency condition gives \( z \tilde{z} = 1 \), and we have \( \tilde{z}_1 = z_1 \tilde{z} \) and so on. This transformation is identified with the mutation of \( y \)-variable in figure 3 [22].

\[
\begin{align*}
\tilde{y}_1 &= y_1 (1 + y_3), \\
\tilde{y}_2 &= y_2 (1 + y_3^{-1})^{-1}, \\
\tilde{y}_3 &= y_3^{-1}, \\
\tilde{y}_4 &= y_4 (1 + y_3^{-1})^{-1}, \\
\tilde{y}_5 &= y_5 (1 + y_3).
\end{align*}
\]

The flip in figure 3 is interpreted as a gluing of a hyperbolic ideal tetrahedron to a triangulation of punctured disk as in figure 5. Here the ideal hyperbolic tetrahedron has a dihedral angle of edge. Consistency condition gives \( z \tilde{z} = 1 \), and we have \( \tilde{z}_1 = z_1 \tilde{z} \) and so on. This transformation is identified with the mutation of \( y \)-variable in figure 3 [22].

\[
\begin{align*}
\tilde{y}_1 &= y_1 (1 + y_3), \\
\tilde{y}_2 &= y_2 (1 + y_3^{-1})^{-1}, \\
\tilde{y}_3 &= y_3^{-1}, \\
\tilde{y}_4 &= y_4 (1 + y_3^{-1})^{-1}, \\
\tilde{y}_5 &= y_5 (1 + y_3).
\end{align*}
\]
shape parameter \( z \), and a dihedral angle of each edge is parameterized by \( z', z'' = (1-z)^{-1} \) as in figure 6 (see, e.g., [38]). Then each dihedral angle on triangulated surface after the gluing is read as

\[
\begin{align*}
\hat{z}_1 &= z_1 z', \\
\hat{z}_2 &= z_2 z'', \\
\hat{z}_3 &= z, \\
\hat{z}_4 &= z_4 z', \\
\hat{z}_5 &= z_5 z',
\end{align*}
\]

with a consistency condition \( z_1 z = 1 \). These two sets of equations indicate a correspondence between the \( y \)-variables and the dihedral angles of triangulated surface, \( z_k = -y_k \), and we conclude that the mutation is regarded as a gluing of an ideal tetrahedron with shape parameter \( z = -y_k \) to punctured surface.

As a consequence, the cluster \( R \)-operator (2.9), which consists of four mutations, can be regarded as an ideal octahedron in figure 7. See that every dihedral angle is written in terms of the \( y \)-variable. Accordingly, the hyperbolic volume of the octahedron for \( \tilde{y} = R(y) \) is given by

\[
D\left(-1/y_{3i+1}\right) + D\left(y_{3i+2}/y_{3i-2}\right) + D\left(-y_{3i+1}\right) + D\left(y_{3i+4}/y_{3i+4}\right),
\]

where \( D(z) \) is the Bloch–Wigner function (see, e.g., [42]),

\[
D(z) = \text{Li}_2(z) + \arg(1 - z) \log |z|.
\]

It should be noted that this type of the hyperbolic octahedron is used not only in studies of the Kashaev \( R^K \)-matrix [37] but in SnapPea algorithm [39]. Note also that the cluster variable \( x \) is identified with Zickert’s edge parameter [43]. See [23] for detail.
3. Quantization

3.1. Quantum cluster algebra

We recall a quantization of the cluster algebra based on [16, 17].

Fix a parameter $q$. The $y$-variable is quantized to be a $q$-commuting generator $Y = (Y_1, \ldots, Y_n)$ satisfying

$$Y_i Y_j = q^{2b_{ij}} Y_j Y_i, \quad (3.1)$$

where $B = (b_{ij})$ is the skew-symmetric exchange matrix used in the classical cluster algebra.

The $q$-commuting relation (3.1) is realized by

$$Y_i = e^{2\pi i \hat{b}_{ii}}, \quad (3.2)$$

where $q = e^{2\pi i}$ and

$$\left[ \hat{Y}_i, \hat{Y}_j \right] = \frac{i}{2\pi} b_{ij}. \quad (3.3)$$

The $q$-deformation of the mutation (2.6) on the $y$-variable is defined by

$$\mu_Y^q (Y, B) = (\hat{Y}, \hat{B}), \quad (3.4)$$

where the exchange matrix $\hat{B}$ is (2.4), and

$$\hat{Y}_i = \begin{cases} Y_i^{-1}, & \text{for } i = k, \\ Y_i \prod_{m=1}^{b_{ki}} \left( 1 + q^{2m-1} Y_k^{-1} \right)^{-1}, & \text{for } i \neq k, b_{ki} > 0, \\ Y_i \prod_{m=1}^{b_{ki}} \left( 1 + q^{2m-1} Y_k \right), & \text{for } i \neq k, b_{ki} \leq 0. \end{cases} \quad (3.5)$$

One sees that this reduces to the classical mutation (2.6) in $q \to 1$. 

Figure 7. Octahedron assigned to crossing (center). Four oriented tetrahedra (left) are glued together. A dihedral angle of each edge is given in terms of the $y$-variable, and we give dihedral angles around central axis (right). Here we assume $\hat{y} = \hat{R}(y)$. 

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It is known that the quantum mutation $\mu^q_k$ (3.4) is decomposed into

$$\mu^q_k = \mu^z_k \circ \mu^i_k.$$  \hspace{1cm} (3.6)

Here $\mu^i_k$ is given by

$$Y_i \mapsto \begin{cases} 
Y_i^{-1}, & \text{for } i = k, \\
q^{b_{kwi}} Y_i Y_k^{b_{ki}}, & \text{for } i \neq k, \ b_{ki} \geq 0, \\
Y_i, & \text{for } i \neq k, \ b_{ki} \leq 0,
\end{cases} \hspace{1cm} (3.7)$$

and $\mu^z_k$ is a conjugation by the quantum dilogarithm function

$$\mu^z_k \equiv \text{Ad} \left( \Phi \left( \frac{\psi_k}{\psi_k} \right) \right) ; \ Y_i \mapsto \Phi \left( \frac{\psi_k}{\psi_k} \right) Y_i \Phi \left( \frac{\psi_k}{\psi_k} \right)^{-1}. \hspace{1cm} (3.8)$$

See appendix A for definition and properties of the quantum dilogarithm function $\Phi(y)$. Note that the quantum mutation $\mu^q_k$ is not a conjugation in general.

### 3.2. Braiding operator

We now consider the quantum cluster algebra for the exchange matrix $B$ (2.8), whose quiver and dual picture as a triangulation are respectively given in figures 1 and 2.

As a natural quantization of the R-operator (2.9), we define a quantum braiding operator acting on $(Y, B)$ by

$$R^q = s_{3,3+2} s_{3-1,3+2} s_{3,3+3} \mu^q_{3+1} \mu^q_{3+1} \mu^q_{3+3} \mu^q_{3+1} \mu^q_{3+1} \mu^q_{3+3} \mu^q_{3+1}. \hspace{1cm} (3.9)$$

Along with the classical case the exchange matrix $B$ (2.8) is invariant under the operator $R^q$, and (3.5) gives an action on $Y$ as

$$R^q(Y) = (Y_1, \ldots, Y_{3-3}, R^q_3(Y_{3-2}, \ldots, Y_{3+4}), Y_{3+5}, \ldots, Y_{3n+1}). \hspace{1cm} (3.10)$$

where

$$R^q_3(Y_1, \ldots, Y_3) = \begin{pmatrix} 
Y_1 \left( 1 + q Y_2^* \right) \\
Y_3 \left( 1 + q Y_4^{-1} \right)^{-1} \left( 1 + q Y_5^{-1} \right)^{-1} \left( 1 + q Y_6^{-1} \right)^{-1} \left( 1 + q Y_4^{-1} \right)^{-1} \\
Y_2^{-1} \left( 1 + q Y_4^* \right) \\
Y_4^{-1} \\
Y_6^{-1} \left( 1 + q Y_4^* \right) \\
Y_3 \left( 1 + q Y_4^{-1} \right)^{-1} \left( 1 + q Y_5^{-1} \right)^{-1} \left( 1 + q Y_6^{-1} \right)^{-1} \left( 1 + q Y_4^{-1} \right)^{-1} \\
Y_1 \left( 1 + q Y_2^* \right)
\end{pmatrix}. \hspace{1cm} (3.11)$$

Here we have used

$$Y_4^* = Y_2 \left( 1 + q Y_4 \right), \quad Y_6^* = Y_6 \left( 1 + q Y_4 \right), \quad Y_4^* = Y_4^{-1} \left( 1 + q Y_2^* \right) \left( 1 + q Y_6^* \right). \hspace{1cm} (3.12)$$

Clearly (3.11) reduces to (2.13) when $q \to 1$.

In our noncommutative algebra (3.1) with (2.8), there exist central elements, $Y_{3i} Y_{3i}$ ($i = 1, \ldots, n$) and $Y_1 Y_4 \cdots Y_{3n+1}$. For simplicity we consider a subspace defined by
where $c \in \mathbb{R}$.

In this setting, we find that, in contrast to that $\mu^q_k$ (3.4) is not an adjoint operator, the $R^q$-operator (3.9) is written as a conjugation

$$R^q(Y) = Ad(R)(Y) = R Y R^{-1},$$

where

$$R = \Phi(\hat{\gamma}_{3i+1}^c) \Phi(\hat{\gamma}_{3i-1}^c) \Phi(\hat{\gamma}_{3i+3}^c)^{-1} \theta(c + \hat{\gamma}_{3i+1}).$$

See (A.10) for the definition of $\theta(z)$. We can check (3.14) by a direct computation using (A.5).

See appendix B. Furthermore we find that the $R$-operator (3.15) fulfills the braid relation

$$R^i R^j R^k R^l = R^l R^k R^j R^i,$$

for $|i - j| > 1$.

See appendix C for the proof. Note that the essentially same solution of the braid relation was studied in [13, 28].

As we have seen in the previous section that the classical $R$-operator (2.9) on the $y$-variable is interpreted as an ideal hyperbolic octahedron, the $R$-operator (3.15) introduced as an adjoint operator for the quantum $R^q$-operator (3.14) should be regarded as a quantum content of the octahedron. It is convincing since the function $\Phi(z)$ reduces, in a classical limit $b \to 0$, to the dilogarithm function (A.6), which is related to the hyperbolic volume of ideal tetrahedron (2.2).

3.3. Braiding matrix at generic $q$

We shall give an infinite-dimensional representation of the quantum $R$-operator (3.15). For this purpose, we set $c = c' + c''$ and

$$\hat{\gamma}_{3i-2} = \hat{\gamma}_{i-1} - \hat{\gamma}_i,$$
$$\hat{\gamma}_{3i-1} = \hat{\rho}_i + c',$$
$$\hat{\gamma}_{3i} = -\hat{\rho}_i + c''$$

where we mean $\hat{\gamma}_0 = \hat{\gamma}_{n+1} = 0$, and $\hat{\gamma}_i$ and $\hat{\rho}_i$ are generators of the Heisenberg algebra,

$$\left[ \hat{\gamma}_i, \hat{\gamma}_j \right] = \left[ \hat{\rho}_i, \hat{\rho}_j \right] = 0, \quad \left[ \hat{\gamma}_i, \hat{\rho}_j \right] = \frac{i}{2\pi} \delta_{ij}.$$

We define bases in coordinate and momentum spaces, $|x\rangle$ and $|p\rangle$, by

$$\hat{\gamma}_i |x\rangle = x_i |x\rangle, \quad \hat{\rho}_i |p\rangle = p_i |p\rangle.$$

These are orthonormal bases satisfying

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3 We thank Rinat Kashaev for kindly informing us.
\[
\langle x|\mathbf{r}\rangle = \prod_{i=1}^{n} \delta(x_i - x'_i), \quad \langle p|\mathbf{p}'\rangle = \prod_{i=1}^{n} \delta(p_i - p'_i),
\]
\[
\langle x|p\rangle = \langle p|x\rangle = e^{2\pi i \sum_{i=1}^{n} x_i p_i},
\]
\[
\int_{\mathbb{R}^n} \langle x|d\mathbf{r}\rangle = \int_{\mathbb{R}^n} \langle p|dp\rangle = 1.
\]

A matrix element \(\langle \mathbf{r}|\mathbf{R}|\mathbf{r}'\rangle\) of the \(\mathbf{R}\)-operator (3.15) is computed as follows. Using (3.17) we get
\[
\langle x_1, x_2 | \mathbf{R} | x_1', x_2' \rangle = \int_{\mathbb{R}^2} dp \Phi(x_1 - x_2) \Phi(p_i + c') \langle p|\mathbf{R}(-p_2 + c') \rangle \Phi(x_1' - x_2') \langle p|\mathbf{R}(-p_2 + c') \rangle \times \int_{\mathbb{R}^2} dp \Phi(p_1 + c') e^{2\pi i (x_1 - x_1')} \cdot \int_{\mathbb{R}^2} dp \Phi(-p_2 + c') e^{2\pi i (x_2 - x_2')}.
\]

Applying (A.9) and (A.10), we obtain
\[
\langle x_1, x_2 | \mathbf{R} | x_1', x_2' \rangle = \frac{\Phi(x_1 - x_2) \Phi(x_1' - x_1) \Phi(x_2' - x_1 + c_0) \Phi(x_2' - x_2 + c_0)}{\Phi(x_1 - x_1' + c_0) \Phi(x_1' - x_2 + c_0) \Phi(x_1' + c_0)} \times e^{2\pi i \left( c_1 (x_1' - x_1', c_2 (x_2' - x_1') + c_2' (x_2' - x_2') + c_2' (x_1 - x_2) + \frac{1}{2} (1 - 4 c_2^2) - c_2') \right)}.
\]

### 3.4. Braiding matrix at root of unity \(q^{2N} = 1\)

In our preceding construction, we have used the quantum dilogarithm function \(\Phi(z)\) (A.1) introduced by Faddeev [12]. It is well known that, due to that \(e^{2\pi i j} \) commute with \(e^{2\pi i k}\) for arbitrary \(j\) and \(k\), we can replace \(\Phi(z)\) by \(\Phi(\epsilon) e^{2\pi i c} q^{2N}\), i.e., we can drop a \(q\)-dependence in (A.3). For simplicity, we set \(c = 0\) further, and we pay attention to a finite-dimensional representation of the \(\mathbf{R}\)-operator
\[
\mathbf{R} = \frac{1}{\left( -q Y_4; q^2 \right) \infty} \cdot \frac{1}{\left( -q Y_2; q^3 \right) \infty} \cdot \frac{1}{\left( -q Y_2; q^3 \right) \infty} \cdot \frac{1}{\left( -q Y_4^{-1}; q^3 \right) \infty},
\]
where we have used the \(q\)-Pochhammer symbol (A.4).

We rely on a method of [5] to construct explicitly a finite-dimensional representation of the \(\mathbf{R}\)-operator (3.20). We set
\[
q = e^{-\frac{2\pi i}{N} \zeta},
\]
and study a limit \(q^2 \to \zeta\) by \(\epsilon \to 0\). Here \(\zeta\) is the \(N\)th root of unity,
\[
\zeta = e^{-\frac{2\pi i}{N}}, \quad \zeta^\frac{3}{2} = e^{-\frac{2\pi i}{N}}.
\]
In a limit $\epsilon \to 0$, an asymptotics of the $q$-infinite product is given by [5]

$$
(\chi; q^2)_\infty = e^{-\frac{\ln(q)}{1 - x^N}} \prod_{k=1}^{N-1} (1 - \zeta x^k)^{-\frac{1}{q}} + O(\epsilon),
$$

(3.23)

where $|x| < 1$, and we have used the Euler–Maclaurin formula. We then obtain

$$
(-qY; q^2)_\infty = e^{-\frac{\ln(q)}{1 - x^N}} d(\zeta^Y) + O(\epsilon),
$$

(3.24)

where $d(x)$ is defined by

$$
d(x) = (1 - x^N)^{-\frac{\ln(q)}{1 - x^N}} \prod_{k=1}^{N-1} (1 - \zeta x^k)^{-\frac{1}{q}}.
$$

(3.25)

We recall that for $\hat{u} \hat{v} = \zeta \hat{v} \hat{u}$ we have [5]

$$
e^{-\ln\hat{u}(Y)} \ast \hat{v} = \hat{v} (1 - \hat{u}^N)^{-1/N},
$$

$$
e^{-\ln\hat{u}(Y)} \ast \hat{u} = \hat{u} (1 - \hat{v}^N)^{-1/N},
$$

(3.26)

where we mean

$$
e^{\hat{a} \ast \hat{b}} = \lim_{\epsilon \to 0} \epsilon^{\hat{a} / \epsilon} \hat{b} e^{-\hat{a} / \epsilon}.
$$

(3.27)

Substituting (3.24) for (3.20), we have an asymptotic behavior in $\epsilon \to 0$

$$
\frac{1}{\mathbf{R}} \approx e^{-\frac{\ln(q)}{1 - x^N}} \prod_{k=1}^{N-1} (1 - \zeta x^k)^{-\frac{1}{q}}
\times \left( e^{-\text{Li}_2(-Y_4^N)} e^{-\text{Li}_2(-Y_6^N)} e^{-\text{Li}_2(-Y_4^N)} \ast \frac{1}{d(\zeta^2 Y_4)} \right)
\times \left( e^{-\text{Li}_2(-Y_4^N)} \ast \frac{1}{d(\zeta^2 Y_4)} \right) \cdot \frac{1}{d(\zeta^2 Y_6)} \cdot \frac{1}{d(\zeta^2 Y_4)}. \quad (3.28)
$$

With a help of (3.26), we find

$$
\frac{1}{\mathbf{R}} \approx e^{-\frac{\ln(q)}{1 - x^N}} \prod_{k=1}^{N-1} (1 - \zeta x^k)^{-\frac{1}{q}} \mathbf{R}.
$$

(3.29)

Here we obtain the dilogarithm factors in the right-hand side as a dominating term in a limit $\epsilon \to 0$, and $\mathbf{R}$ denotes the second dominating term given by

$$
\mathbf{R} = \frac{1}{d(\zeta^2 Y_4(1 + Y_4^N(1 + Y_6^N(1 + Y_4^N)))^2)} \cdot \frac{1}{d(\zeta^2 Y_4(1 + Y_6^N)))^2} \cdot \frac{1}{d(\zeta^2 Y_4)}.
$$

(3.30)

We note that we have $q^2 \to \zeta$ as $\epsilon \to 0$, and that the quantum $Y$-operator in the above $\mathbf{R}$-matrix fulfills
\[ Y_i \ Y_j = \zeta^{b_i j} Y_j \ Y_i, \tag{3.31} \]

with the root of unity \( \zeta \) (3.22).

We study a finite-dimensional matrix representation of the second dominating term \( R \) in (3.29). We use
\[ \omega = \zeta^{-1} = e^{2\pi i/N}, \tag{3.32} \]
and we define \( w(x, y|n) \) for \( n \in \mathbb{Z}_{\geq 0} \) and \( x, y \in \mathbb{C} \) satisfying \( x^N + y^N = 1 \) by
\[ w(x, y|0) = y^{\frac{1}{N}} \prod_{j=1}^{N-1} \frac{1 - \omega^{-j} x^j}{1 - \omega^{-j} x}. \]
\[ \frac{w(x, y|n)}{w(x, y|0)} = \prod_{j=1}^{n} \frac{y}{1 - \omega^{j} x}. \tag{3.33} \]

Following a convention [4] we also use a multi-valued function of \( x \) by
\[ w(x, n) = w(x, \Delta(x)|n), \tag{3.34} \]
where \( \Delta(x) \) is defined by
\[ \Delta(x) = (1 - x^N)^{1/N}. \tag{3.35} \]

Note that
\[ w(x, y|n + N) = w(x, y|n), \tag{3.36} \]
and that the function \( w(x, n) \) is related to the function \( d(x) \) defined in (3.25)
\[ w(x, 0) = \frac{1}{d(x)}. \tag{3.37} \]

The function \( w(x, y|n) \) is often used in studies of integrable models in statistical mechanics, and known are the following identities,
\[ w(x, y|n) = w(\omega^n x, y|0), \quad \prod_{k=0}^{N-1} w(x, y|k) = 1. \tag{3.38} \]

Furthermore we have
\[ \sum_{k=0}^{N-1} w(x, y|k) \omega^{nk} = N \frac{(x/y)^{\frac{N-1}{N}}}{\Delta(y, x)} \frac{1}{w(y, x|n - 1)}. \tag{3.39} \]

See appendix \( D \) for a definition of \( \Delta(y, x) \) (see also [3–5, 36]).

We introduce an \( N^2 \times N^2 \) matrix representation of (3.31),
\[ Y_2 = \omega^\frac{1}{2} \kappa_2 \ X \otimes \mathbf{1}, \]
\[ Y_4 = \omega^\frac{1}{2} \kappa_4 \ Z \otimes Z^{-1}, \]
\[ Y_6 = \omega^\frac{1}{2} \kappa_6 \mathbf{1} \otimes X^{-1}. \tag{3.40} \]

Here \( \kappa_i \in \mathbb{C}, |\kappa_i| < 1 \), and \( N \times N \) matrices \( Z \) and \( X \), satisfying \( Z \ X = \omega^{-1} \ X \ Z \), are defined by
\[ (Z)_{j,k} = \omega^j \delta_{j,k}, \quad (X)_{j,k} = \delta_{j,k-1}. \tag{3.41} \]
where the Kronecker delta has a period $N$. By substituting (3.40) for (3.30), we have

$$
R_{ij,kl} = \left[ \frac{1}{d(k_i^\prime \Delta(k_i^\prime) Z \otimes Z^{-1})} \right]_{ij,kl} \times \left[ \frac{1}{d(k_i^\prime 1 \otimes X^{-1})} \right]_{kl,ij},
$$

(3.42)

where we have used $\kappa_i^\prime = \kappa_1 \Delta(k_1^\prime)$ and $\kappa_i = \kappa_0 \Delta(k_0^\prime)$. By use of (3.37) and (3.39), we get

$$
R_{ij,kl} = \frac{(\kappa_i^\prime / \Delta(k_i^\prime))^{n_i}}{\lambda (\Delta(k_i^\prime), \kappa_i^\prime)} \frac{(\kappa_i^\prime / \Delta(k_i^\prime))^{n_i}}{\lambda (\Delta(k_i^\prime), \kappa_i^\prime)} \times \frac{w(\kappa_i^\prime)}{w(\Delta(k_i^\prime), i-j)} \frac{w(\omega_1^{-1} k_i^\prime, \ell-k)}{11 k_i^\prime},
$$

(3.43)

We set $\kappa_1 = 1 - \delta N$, $0 < \delta \ll 1$, and $\kappa_2, \kappa_0 > 0$. In a limit $\delta \to 0$, we find with a help of (3.33) that a dominating term behaves as

$$
R_{ij,kl} \propto \frac{(\omega)_{i-j} (\omega)_{k-l}}{\omega_1^{-1} k_i^\prime} \omega^{N-1+|i-j|+|k-l|-|i-j|-|k-l|}.
$$

(3.44)

Here $[n] = n \bmod N$ satisfying $0 \leq [n] < N$, and we mean $(\omega)_{[n]} = (\omega; \omega)_{[n]}$. The origin of $\omega^{[l-k]}$ is subtle. It is due to that the function $w(x, n) = w(x, \Delta(x)n)$ is a multi-valued function, $w(x, n) \propto w^n$, and that $w(\omega_1^{-1} k_i^\prime, \ell-k)$ which originates from $d(\omega_1^{-1} k_i^\prime 1 \otimes Z)$ in (3.42) crosses a branch-cut in getting $\Delta(\omega_1^{-1} k_i^\prime)$. In (3.44), we see that

$$
\theta_{i,j}^{\omega,\theta,\omega,\omega} \to \begin{cases} 1, & \text{when } [i-j] + [j-\ell] + [\ell-k-l] + [k-i] = N-1, \\ 0, & \text{otherwise}. \end{cases}
$$

As a result, we get

$$
R_{ij,kl} \delta \to \rho \frac{(\omega)_{i-j} (\omega)_{k-l}}{\omega_{i-j} (\omega)_{k-l}} \frac{\omega^{N-1+|i-j|+|k-l|-|i-j|-|k-l|}}{(\omega)_{[i-j]}},
$$

(3.45)

where $\rho$ is an irrelevant complex number. Here we mean $\rho = (\omega_{[n]} = (\omega; \omega)_{[n]}$, and we have used an identity

$$
(\omega)_{[n]} (\omega)_{[n-1]} = N.
$$

By construction, the $R$-matrix fulfills the braid relation

$$
(R \otimes 1) (1 \otimes R) (R \otimes 1) = (1 \otimes R) (R \otimes 1) (1 \otimes R). \quad (3.46)
$$
One notices that this is gauge-equivalent with the Kashaev $R^K$-matrix \[25, 32\]

\[
(R^K)_{i,j}^{k,l} = \frac{N \omega^{-1+i-k} \delta_{i,j}^{k,l}}{(\omega)^{i-l} (\omega)^{j-l} (\omega)^{j-k}}.
\]

To conclude, the Kashaev $R^K$-matrix corresponds to a finite-dimensional representation of the $R$-operator (3.15) which is constructed based on the quantum cluster algebra. As we have seen that the classical $R$-operator (2.9) is regarded as the hyperbolic octahedron in figure 7 and that a conjugation of the $R$-operator (3.15) is the quantum $R^q$-operator which reduces to the $R$-operator in a limit $q \to 1$, it is natural that both $R$- and $R^K$-matrices are realized as the octahedron in a limit $N \to \infty$. Correspondingly a matrix element (3.19) is an infinite-dimensional analogue of the Kashaev $R^K$-matrix.

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**Appendix A. Quantum dilogarithm**

We use the Faddeev quantum dilogarithm $\Phi(z)$ \[12\] defined by

\[
\Phi(z) = \exp\left(-\frac{1}{4} \int_{b+0} e^{-i z w} \frac{e^{-2i z w}}{\sinh(b w) \sinh(b^{-1} w)} \frac{dw}{w}\right).
\]

Here we assume $b \in \mathbb{C}$ with $\Im b > 0$, and we use

\[
q = e^{xb^2}, \quad \bar{q} = e^{-xb^2}, \quad c_b = \frac{i}{2} (b + b^{-1}).
\]

It is well known that we have

\[
\Phi(z) = \left(\frac{-\bar{q} e^{2ibz}; q^2}{-q e^{2ibz}; q^2}\right)_\infty
\]

where we have used the $q$-Pochhammer symbol

\[
(x; q)_n = \prod_{k=1}^n (1 - x q^{k-1}).
\]

It is easy to see that

\[
\Phi(z \pm i b) \equiv (1 + e^{2ibz} q^2) z 1 \Phi(z).
\]

The classical dilogarithm function is given in a limit $b \to 0$

\[
\Phi\left(\frac{z}{2\pi b}\right) \sim \exp\left(i \frac{z}{2\pi b^2} \text{Li}_2(-e^z)\right).
\]
The most important property of $\Phi(z)$ is the pentagon identity [14] (also [42]),

$$\Phi(\tilde{x}) \Phi(\tilde{p}) = \Phi(\hat{p}) \Phi(\tilde{x} + \hat{p}) \Phi(\tilde{x}),$$

where

$$[\tilde{x}, \tilde{p}] = \frac{1}{2\pi}.$$  \hspace{1cm} (A.7)

See [29] for a recent development on identities of the quantum dilogarithm functions. Notice that the function $\Phi(z)$ is used to construct the quantum invariant in [21] (see also [1, 10]).

Also known is the Fourier transformation formula for $\Phi(z)$

$$\int_{\mathbb{R}} \Phi(z) e^{2\pi i wz} dz = \Phi(-w - c_{b}) e^{i\pi(1 - 4c_{b}^2)/12}$$

$$= \frac{1}{\Phi(w + c_{b})} e^{-2\pi i w x + i(1 - 4c_{b}^2)/12}. \hspace{1cm} (A.9)$$

See [15, 35, 40] for detail.

We define $\theta(z)$ by

$$\theta(z) = \Phi(z) \Phi(-z)$$

$$= e^{-\pi z^2 + i\pi(1 + 2c_{b}^2)/6}. \hspace{1cm} (A.10)$$

We see that we have for $\tilde{x}$ and $\tilde{p}$ satisfying (A.8)

$$\theta(\tilde{p}) e^{2\pi i \tilde{p}} = e^{2\pi i \tilde{p}} \theta(\tilde{p}), \hspace{1cm} (A.11)$$

$$\theta(\tilde{x}) e^{2\pi i \tilde{x}} = e^{2\pi i \tilde{x}} \theta(\tilde{x}). \hspace{1cm} (A.12)$$

Appendix B. Proof of (3.14)

We show that $\hat{R} R \hat{R}^{-1}$ results in (3.11). We only give cases for $i = 2, 3$ explicitly, and the others are obtained in a similar manner. For a sake of simplicity, we write $\Phi_{i} = \Phi(\pm \frac{i}{\sqrt{2}})$, $\Phi_{i \pm j} = \Phi(\pm \frac{j}{\sqrt{2}})$, $\Phi_{i + j} = \Phi(\sqrt{2} \frac{j}{\sqrt{2}})$, and so on. For $i = 2, 3$, we compute as follows:

$$\hat{R} R \hat{R}^{-1}$$

$$= \Phi_{2} \Phi_{3} \Phi_{2}^{-1} \frac{1}{\Phi_{2} \Phi_{3} \Phi_{2}^{-1} \Phi_{2} \Phi_{3} \Phi_{2}^{-1} \Phi_{2}^{-1} \Phi_{2}^{-1} \Phi_{4}^{-1} \Phi_{4}^{-1}}$$

$$= \frac{q \Phi_{2} \Phi_{3} \Phi_{2}^{-1} \Phi_{2} \Phi_{3} \Phi_{4} \Phi_{4}^{-1} \Phi_{2}^{-1} \Phi_{2}^{-1}}{q \Phi_{2} \Phi_{3} \Phi_{2}^{-1} \Phi_{2} \Phi_{3} \Phi_{4} \Phi_{4} \Phi_{4}^{-1} \Phi_{2}^{-1} \Phi_{2}^{-1}} \hspace{1cm} \text{by (A.12) and (3.13)}$$

$$= \frac{q \Phi_{2} \Phi_{3} \Phi_{2}^{-1} \Phi_{2} \Phi_{3} \Phi_{4} \Phi_{4} \Phi_{4}^{-1} \Phi_{2}^{-1} \Phi_{2}^{-1}}{q \Phi_{3} \Phi_{2} \Phi_{3} \Phi_{4} \Phi_{4} \Phi_{4} \Phi_{4}^{-1} \Phi_{2}^{-1} \Phi_{2}^{-1}} \hspace{1cm} \text{by (A.5)}$$

$$= \frac{q \Phi_{2} \Phi_{3} \Phi_{2}^{-1} \Phi_{2} \Phi_{3} \Phi_{4} \Phi_{4} \Phi_{4}^{-1} \Phi_{2}^{-1} \Phi_{2}^{-1}}{q \Phi_{2} \Phi_{3} \Phi_{2}^{-1} \Phi_{2} \Phi_{3} \Phi_{4} \Phi_{4} \Phi_{4}^{-1} \Phi_{2}^{-1} \Phi_{2}^{-1}}$$

$$= \frac{q \Phi_{2} \Phi_{3} \Phi_{2}^{-1} \Phi_{2} \Phi_{3} \Phi_{4} \Phi_{4} \Phi_{4}^{-1} \Phi_{2}^{-1} \Phi_{2}^{-1}}{q \Phi_{2} \Phi_{3} \Phi_{2}^{-1} \Phi_{2} \Phi_{3} \Phi_{4} \Phi_{4} \Phi_{4}^{-1} \Phi_{2}^{-1} \Phi_{2}^{-1}}$$
Here we have used $e^{2abc} = qY_5Y_6$ at the second equality, and $Y'_2$, $Y'_6$ and $Y''_4$ are given by (3.12).

A case of $i = 3$ is as follows.

\[
\begin{align*}
R \ Y_3 \ R^{-1} &= \Phi_4 \Phi_2 \Phi_6 \Phi_4^{-1} \theta_{+4} \theta_{+4} \theta_{-4} \Phi_4 \Phi_6 \Phi_4^{-1} \\
&= q \Phi_2 \Phi_6 \Phi_2^{-1} Y'_2 Y'_4 \Phi_6 \Phi_2^{-1} \Phi_2^{-1} \Phi_4^{-1} \quad \text{by (A.11) and (A.13)} \\
&= \Phi_2 \Phi_6 Y'_2^{-1} \left(1 + qY'_4^{-1}\right) \Phi_6^{-1} \Phi_2^{-1} \Phi_4^{-1} \quad \text{by (A.5)} \\
&= Y'_2^{-1} \left(1 + qY'_4\right) \Phi_4^{-1} \quad \text{by (A.5)},
\end{align*}
\]

where we have used $e^{2abc} = qY'_2^{-1}Y_3^{-1}$ at the second equality.

**Appendix C. Proof of Braid relation (3.16)**

We shall check (3.16) for $i = 1$. We employ the notations in appendix B.

The proof is straightforward but tedious by use of (A.7), (A.11), and (A.12). We compute as follows;

\[
\begin{align*}
2 R \ Y_1 \ R^{-1} &= \Phi_7 \Phi_5 \Phi_9 \Phi_7^{-1} \theta_{+7} \theta_{+7} \Phi_4 \Phi_5 \Phi_4^{-1} \theta_{+4} \\
&= \Phi_7 \Phi_5 \Phi_9 \Phi_7^{-1} \Phi_4 \Phi_5 \Phi_7^{-1} \theta_{+7} \theta_{+7} \Phi_4^{-1} \theta_{+4} \theta_{+4} \theta_{+7} \quad \text{by (A.11)} \\
&= \Phi_7 \Phi_5 \Phi_9 \Phi_7^{-1} \Phi_4 \Phi_5 \Phi_7^{-1} \theta_{+4} \Phi_7^{-1} \theta_{+4} \quad \text{by (A.7)} \\
&= \Phi_7 \Phi_5 \Phi_9 \Phi_7^{-1} \Phi_4 \Phi_5 \Phi_7^{-1} \theta_{+7} \theta_{+7} \Phi_4^{-1} \theta_{+4} \theta_{+4} \theta_{+7} \quad \text{by (A.7)} \\
&= \Phi_7 \Phi_5 \Phi_9 \Phi_7^{-1} \Phi_4 \Phi_5 \Phi_7^{-1} \theta_{+7} \theta_{+7} \Phi_4^{-1} \theta_{+4} \theta_{+4} \theta_{+7} \quad \text{by (A.12)} \\
&= \Phi_7 \Phi_5 \Phi_9 \Phi_7^{-1} \Phi_4 \Phi_5 \Phi_7^{-1} \theta_{+7} \theta_{+7} \Phi_4^{-1} \theta_{+4} \theta_{+4} \theta_{+7} \quad \text{by (A.7)} \\
&= \Phi_7 \Phi_5 \Phi_9 \Phi_7^{-1} \Phi_4 \Phi_5 \Phi_7^{-1} \theta_{+7} \theta_{+7} \Phi_4^{-1} \theta_{+4} \theta_{+4} \theta_{+7} \quad \text{by (A.12)} \\
&= \Phi_7 \Phi_5 \Phi_9 \Phi_7^{-1} \Phi_4 \Phi_5 \Phi_7^{-1} \theta_{+7} \theta_{+7} \Phi_4^{-1} \theta_{+4} \theta_{+4} \theta_{+7} \quad \text{by (A.7)} \\
&= \Phi_7 \Phi_5 \Phi_9 \Phi_7^{-1} \Phi_4 \Phi_5 \Phi_7^{-1} \theta_{+7} \theta_{+7} \Phi_4^{-1} \theta_{+4} \theta_{+4} \theta_{+7} \quad \text{by (A.12)} \\
&= \Phi_7 \Phi_5 \Phi_9 \Phi_7^{-1} \Phi_4 \Phi_5 \Phi_7^{-1} \theta_{+7} \theta_{+7} \Phi_4^{-1} \theta_{+4} \theta_{+4} \theta_{+7} \quad \text{by (A.7)} \\
&= \Phi_7 \Phi_5 \Phi_9 \Phi_7^{-1} \Phi_4 \Phi_5 \Phi_7^{-1} \theta_{+7} \theta_{+7} \Phi_4^{-1} \theta_{+4} \theta_{+4} \theta_{+7} \quad \text{by (A.12)} \\
&= \Phi_7 \Phi_5 \Phi_9 \Phi_7^{-1} \Phi_4 \Phi_5 \Phi_7^{-1} \theta_{+7} \theta_{+7} \Phi_4^{-1} \theta_{+4} \theta_{+4} \theta_{+7} \quad \text{by (A.7)} \\
&= \Phi_7 \Phi_5 \Phi_9 \Phi_7^{-1} \Phi_4 \Phi_5 \Phi_7^{-1} \theta_{+7} \theta_{+7} \Phi_4^{-1} \theta_{+4} \theta_{+4} \theta_{+7} \quad \text{by (A.12)} \\
&= \Phi_7 \Phi_5 \Phi_9 \Phi_7^{-1} \Phi_4 \Phi_5 \Phi_7^{-1} \theta_{+7} \theta_{+7} \Phi_4^{-1} \theta_{+4} \theta_{+4} \theta_{+7} \quad \text{by (A.7)} \\
&= \Phi_7 \Phi_5 \Phi_9 \Phi_7^{-1} \Phi_4 \Phi_5 \Phi_7^{-1} \theta_{+7} \theta_{+7} \Phi_4^{-1} \theta_{+4} \theta_{+4} \theta_{+7} \quad \text{by (A.12)} \\
&= \Phi_7 \Phi_5 \Phi_9 \Phi_7^{-1} \Phi_4 \Phi_5 \Phi_7^{-1} \theta_{+7} \theta_{+7} \Phi_4^{-1} \theta_{+4} \theta_{+4} \theta_{+7} \quad \text{by (A.7)} \\
&= \Phi_7 \Phi_5 \Phi_9 \Phi_7^{-1} \Phi_4 \Phi_5 \Phi_7^{-1} \theta_{+7} \theta_{+7} \Phi_4^{-1} \theta_{+4} \theta_{+4} \theta_{+7} \quad \text{by (A.12)} \\
&= \Phi_7 \Phi_5 \Phi_9 \Phi_7^{-1} \Phi_4 \Phi_5 \Phi_7^{-1} \theta_{+7} \theta_{+7} \Phi_4^{-1} \theta_{+4} \theta_{+4} \theta_{+7} \quad \text{by (A.7)} \\
&= \Phi_7 \Phi_5 \Phi_9 \Phi_7^{-1} \Phi_4 \Phi_5 \Phi_7^{-1} \theta_{+7} \theta_{+7} \Phi_4^{-1} \theta_{+4} \theta_{+4} \theta_{+7} \quad \text{by (A.12)} \
\end{align*}
\]
Thus we get
\[
\sum_{k=0}^{n} (q^{-n}; q)_k (b; q)_k z^k = \frac{(c; q)_n}{(c; q)_n} \left( \frac{b z}{q} \right)^n \sum_{k=0}^{n} \frac{(q^{-n}; q)_k (q/z; q)_k (q^{1-n}/c; q)_k}{(b q^{1-n}/c; q)_k (q; q)_k} q^k,
\]
which reduces, in \( c \to 0 \), to
\[
\sum_{k=0}^{n} \frac{(q^{-n}; q)_k (b; q)_k}{(q; q)_k} z^k = \left( \frac{b z}{q} \right)^n \sum_{k=0}^{n} \frac{(q^{-n}; q)_k (q/z; q)_k}{(q; q)_k} \left( \frac{q}{b} \right)^k.
\]  
(D.1)

By setting \( n = N - 1 \) and \( q = \omega \) where \( \omega^{-N} = 1 \), we get
\[
\sum_{k=0}^{N-1} (b; \omega)_k z^k = \left( \frac{b z}{\omega} \right)^{N-1} \sum_{k=0}^{N-1} (\omega/z; \omega)_k \left( \frac{\omega}{b} \right)^k.
\]  
(D.2)
Using this identity, we compute as follows.

\[
\sum_{k=0}^{N-1} \omega^{-nk} w(x, y|k) = \frac{1}{w(x, y|0)} \sum_{k=0}^{N-1} (\omega x; \omega) \left( \frac{1}{y \omega^k} \right)^k \\
= \frac{1}{w(x, y|0)} \left( \frac{x}{y} \right)^{N-1} \omega^{n} \sum_{k=0}^{N-1} (y \omega^{n+k}; \omega) \frac{1}{x^k} \quad \text{by (D.2)} \\
= \frac{w(y \omega^n, x|0)}{w(x, y|0)} \left( \frac{x}{y} \right)^{N-1} \omega^n \sum_{k=0}^{N-1} \frac{1}{w(y \omega^k, x|k)}.
\]

Here we note that \( x^N + y^N = 1 \). As a result, we get from (3.36) and (3.38)

\[
\sum_{k=0}^{N-1} \omega^{-nk} w(x, y|k) = \omega^n w(y \omega^n, x|0) \left( \frac{x}{y} \right)^{N-1} \lambda(x, y), \quad \text{(D.3)}
\]

where

\[
\lambda(x, y) = \left( \frac{x}{y} \right)^{(N-1)/2} \sum_{k=0}^{N-1} \frac{1}{w(y, x|k)}.
\]

See that

\[
\lambda(x, \omega y) = \lambda(x, y), \quad \lambda(x, y) \xrightarrow{y \to 1} N \prod_{j=1}^{N-1} (1 - \omega^{-j/N}). \quad \text{(D.5)}
\]

The Fourier transform of (D.3) reduces to

\[
\sum_{k=0}^{N-1} w(y, x|k) \omega^{kn} = N \left( \frac{y/x}{(N-1)/2} \right) \frac{1}{\lambda(x, y)} \frac{1}{w(x, y|0)}.
\]

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