A SHARP ERROR ANALYSIS FOR THE DISCONTINUOUS GALERKIN METHOD OF OPTIMAL CONTROL PROBLEMS

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Abstract. In this paper, we are concerned with a nonlinear optimal control problem of ordinary differential equations. We consider discretization of the problem with the discontinuous Galerkin method with arbitrary order \( r \in \mathbb{N} \). Under suitable regularity assumptions on the cost functional and solutions of the state equations, we provide sharp estimates for the error of the approximate solutions. Numerical experiments are presented supporting the theoretical results.

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1. Introduction

In the present work, we discuss discontinuous Galerkin (DG) approximations to a nonlinear optimal control problem (OCP) of ordinary differential equations (ODEs). More

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precisely, we consider the following optimal control problem:

\[
\text{Minimize } J(u, x) := \int_0^T g(t, x(t), u(t)) \, dt \quad (1.1)
\]

subject to

\[
\begin{cases}
x'(t) = f(t, x(t), u(t)), & \text{a.e. on } [0, T], \\
x(0) = x_0, \\
u(t) \in U_{ad}, & \text{a.e. on } [0, T].
\end{cases}
\quad (1.2)
\]

Here \( u(t) \in \mathbb{R}^d \) is the control, and \( x(t) \in \mathbb{R}^d \) is the state of the system at time \( t \in [0, T] \).

Further, \( g : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) and \( f : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \) are given, and the set of admissible controls \( U_{ad} \subset U := L^\infty(0, T) \) is given by

\[
U_{ad} = \{ u \in \mathbb{R}^d : u_\ell \leq u \leq u_u \}
\]

for some \( u_\ell, u_u \in (\mathbb{R} \cup \{-\infty, \infty\})^d \).

There have been a lot of study on the numerical computation for the above problem. The numerical schemes need a discretization of the ODEs, for example, the Euler discretization for the OCPs of ODEs are well studied for sufficiently smooth optimal controls based on strong second-order optimality conditions \[1, 5, 6\]. For optimal control problems with control appearing linearly, the optimal control may be discontinuous, for an instance, bang-bang controller, and such conditions are not satisfied. In that respect, there have been many studies to develop new second-order optimality conditions for the optimal control problems with control appearing linearly \[2, 9, 12, 13\].

The Pseudo-spectral method is also popularly used for the discretization due to its capability of high-order accuracy for smooth solutions to the OCPs \([7, 14]\). However, the high-order accuracy of the Pseudo-spectral method is known to be often lost for bang-bang OCPs, where the solutions may not be smooth enough. To handle this issue, Henriques et al. \[10\] proposed a mesh refinement method based on a high-order DG method for the OCPs of ODEs. The DG method discretizes the time interval in small time subintervals, in which the weak formulation is employed. The test functions are usually taken as piecewise polynomials which can be discontinuous at boundaries of the time interval, see Section 2 for more detailed discussion. We refer to \[3, 8, 15\] and references therein for DG methods for ODEs.

In this paper, we provide a rigorous analysis for the DG discretization applied to the nonlinear OCP \((1.1)-(1.2)\) with arbitrary order \( r \in \mathbb{N} \cup \{0\} \) for general functions \( f \) and \( g \) with suitable smoothness. It is worth noticing that the control is not required to be linear in the state equations \((1.2)\), and the control space \( U_{ad} \) allows to take into account discontinuous controls. The constraints for controls are defined by lower and upper bounds. Moreover, the cost functional is also given in a general form, it may not be quadratic. Motivated from a recent work by Neitzel and Vexler \[11\], we consider a second-order sufficient condition \((2.5)\) and prove the existence of a sequence of locally optimal controls to discretized problem converging strongly in \( L^2((0, T); \mathbb{R}^d) \) to the OCP \((1.1)-(1.2)\). We also
establish a sharp convergence rate of that error estimate which depends on the regularity of optimal solutions and the degree of piecewise polynomials mentioned above, see Section 2 for details.

For notational simplicity, we denote by $I := (0, T)$, $X := L^2(I; \mathbb{R}^d)$, and $(v, w)_I = (v, w)_{L^2(I; \mathbb{R}^d)}$. We also use simplified notations:

$$
\| \cdot \|_{L^p(I)} := \| \cdot \|_{L^p(I; \mathbb{R}^d)} \quad \text{and} \quad \| \cdot \|_{W^{p, \infty}(I; \mathbb{R}^d)} := \| \cdot \|_{W^{p, \infty}(I; \mathbb{R}^d)}
$$

for $1 \leq p \leq \infty$. Throughout this paper, we assume that $f, g \in C([0, T]; W^{3, \infty}(\mathbb{R}^d \times \mathbb{R}^d))$ satisfy

$$
\sup_{0 \leq t \leq T} (\| f(t, \cdot, \cdot) \|_{W^{3, \infty}} + \| g(t, \cdot, \cdot) \|_{W^{3, \infty}}) \leq K \quad (1.3)
$$

for some $K > 0$.

We next introduce the control-to-state mapping $G : \mathcal{U} \to X \cap L^\infty(I; \mathbb{R}^d)$, $G(u) = x$, with $x$ solving (1.2). It induces the objective function $j : \mathcal{U} \to \mathbb{R}_+$, $u \mapsto J(u, G(u))$. This makes the optimal control problem (1.1)-(1.2) equivalent to

$$
\text{Minimize } j(u) \text{ subject to } u \in \mathcal{U}_{ad}. \quad (1.4)
$$

**Definition 1.1.** A control $\bar{u} \in \mathcal{U}_{ad}$ is a local solution of (1.4) if there exists a constant $\epsilon > 0$ such that $j(u) \geq j(\bar{u})$ holds for all $u \in \mathcal{U}_{ad}$ with $\| \bar{u} - u \|_{L^2(I)} \leq \epsilon$.

Numerical methods for optimal control problems can broadly be classified as either direct or indirect one. The direct method first approximates the OCPs in finite dimensional space and then applies an optimization method. The indirect method first uses Pontryagin’s maximum principle to deduce a system of ODEs for the state and the adjoint state, which are then solved by a suitable discretization of the ODEs.

To implement the numerical experiment in the current work, we shall use the indirect method. In order to solve the system of ODEs from the maximum principle, we apply the forward-backward method. This method first solves the state equation forward in time, and then solves the adjoint equation backward, and then updates the control. Iterating this procedure gives a fixed point, which solves the system of ODEs.

In Section 2, we explain the discretization of the ODEs and the OCP. Then we present the main results of the paper and provide some preliminary results. In Section 3 the adjoint problems are studied. Section 4 is devoted to study the second order analysis of the optimal solutions and the approximate optimal solutions. In Section 5 we prove the existence of the local solution to the approximate OCP, and establish the proof of the main results. Finally, in Section 6 we perform several numerical experiments for linear and nonlinear OCPs. In Appendix A we prove Lemma 3.2 and Lemma 3.3, which reformulate the first derivatives of the objective functionals in terms of the adjoint states. In Appendix B we derive the formulars on the second order derivatives of the objective functionals. Appendix C is devoted to prove a Lipshitz stability of the discretized version of the ODEs (1.2) with respect to the control variable. In Appendix D we prove a technical result used for computing the second order derivatives in Appendix B.
2. DG formulation

In this section, we describe the approximation of the OCP \((1.1)-(1.2)\) with the DG method, and then we state the main results of the current work.

First we consider the discretization of the following ODEs:

\[
\begin{aligned}
x'(t) &= F(t, x(t)), \quad t \in (0, T), \\
x(0) &= x_0,
\end{aligned}
\tag{2.1}
\]

where \(x : [0, T] \to \mathbb{R}^d, F : (0, T) \times \mathbb{R}^d \to \mathbb{R}^d\) is uniformly Lipschitz continuous with respect to \(u\), i.e.,

\[
\|F(t, u) - F(t, v)\| \leq L\|u - v\|, \quad u, v \in \mathbb{R}^d, \quad t \in (0, T)
\]

with a constant \(L > 0\).

Let \(M\) be a partition of \(I\) into \(N\) time intervals \(\{I_n\}_{n=1}^N\) given by \(I_n = (t_{n-1}, t_n)\) with nodes \(0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T\). Let \(h_n\) be the length of \(I_n\), i.e., \(h_n = t_n - t_{n-1}\), and we set \(h := \max_{1 \leq n \leq N} h_n\). We also define

\[
\varphi^+_n = \lim_{t \downarrow t_n} \varphi(t_n + t), \quad 0 \leq n \leq N - 1, \quad \varphi^-_n = \lim_{t \downarrow t_n} \varphi(t_n - t), \quad 1 \leq n \leq N.
\]

We also denote by the jumps across the nodes \([\varphi]_n := \varphi^+_n - \varphi^-_n\). We define

\[
X^r_h := \{ \varphi_h \in X : \varphi|_{I_n} \in P^r(I_n), \quad 1 \leq n \leq N\},
\]

where \(P^r(I_n)\) represents the set of all polynomials of \(t\) up to order \(r\) defined on \(I_n\) with coefficients in \(\mathbb{R}^d\). Then the approximate solution \(x_h\) of \((2.1)\) is given as

\[
\sum_{n=1}^N (x'(t) - F(t, x(t)), \varphi(t))_{I_n} + \sum_{n=2}^N ([x]_{n-1}, \varphi^+_n - \varphi^-_{n-1}) + (x^+_0, \varphi^+_0) = (x_0, \varphi^+_0)
\tag{2.2}
\]

for all \(\varphi \in X^r_h\). Here \((\cdot, \cdot)\) denotes inner product in \(\mathbb{R}^d\), and

\[
(A(t), B(t))_{I_n} = \int_{I_n} (A(t), B(t)) \, dt
\]

for \(A, B : I_n \to \mathbb{R}^d\).

We recall the error estimate for the DG approximation of \((2.1)\) from [15, Corollary 3.15].

**Theorem 2.1.** Let \(x(t)\) be the solution of \((2.1)\) such that \(x \in W^{k, \infty}(I; \mathbb{R}^d)\). Assume that the solution \(x\) belongs to \(C^k([0, T]; \mathbb{R}^d)\) for some \(k \in \mathbb{N}\). Let \(x_h \in X^r_h\) be the DG approximate solution to \((2.2)\) of order \(r \in \mathbb{N} \cup \{0\}\). Then we have

\[
\sup_{0 \leq t \leq T} |x_h(t) - x(t)| \leq C h^{\min(r+1, k)} \|x\|_{W^{k, \infty}(I; \mathbb{R}^d)},
\]

where \(C > 0\) is determined by \(L\), \(T\), and \(r\).

Now, for given \(u \in \mathcal{U}\), we consider the approximate solution \(x \in X^r_h\) of the control problem \((1.2)\) satisfying

\[
\sum_{n=1}^N (x'(t) - f(t, x(t), u(t)), \varphi(t))_{I_n} + \sum_{n=2}^N ([x]_{n-1}, \varphi^+_n - \varphi^-_{n-1}) + (x^+_0, \varphi^+_0) = (x_0, \varphi^+_0)
\tag{2.3}
\]
for all $\varphi \in X_r^\prime$.

We consider a discrete control-to-state mapping $G_h : \mathcal{U} \to X_r^h, u \mapsto G_h(u)$, where $G_h(u)$ is the solution of (2.3). We also introduce the discrete objective function $j_h : \mathcal{U} \to \mathbb{R}_+, u \mapsto J(u, G_h(u))$. This leads to the following discretized version of (1.1):

$$\min_{u_h \in \mathcal{U}_{ad}} j_h(u_h).$$

(2.4)

We now define a discrete local solution to (2.4).

**Definition 2.2.** A control $\bar{u}_h \in \mathcal{U}_{ad}$ is called a discrete local solution of (2.4) if there exists an $\varepsilon > 0$ such that $j_h(u) \geq j_h(\bar{u}_h)$ holds for all $u \in \mathcal{U}_{ad}$ with $\|u - \bar{u}_h\|_{L^2(I)} \leq \varepsilon$.

Throughout the paper, we will consider local solutions $\bar{u}$ to (1.4) satisfying the following non-degeneracy condition.

**Assumption 1.** Let $\bar{u} \in \mathcal{U}_{ad}$ be the local solution of (1.1). We assume that it satisfies

$$j''(\bar{u})(v,v) \geq \gamma \|v\|^2_{L^2(I)} \quad \text{for all } v \in \mathcal{U}$$

(2.5)

for some $\gamma > 0$.

In the first main result, we prove the existence of the local solution to the approximate problem (2.4).

**Theorem 2.3.** Suppose that $\bar{u} \in \mathcal{U}_{ad}$ is a local solution of (1.1) satisfying Assumption 1. Then, there exists $\varepsilon > 0$ and $h_0 > 0$ such that for $h \in (0, h_0)$ the approximate problem (2.4) has a local solution in $B_\varepsilon(\bar{u}) = \{u \in \mathcal{U}_{ad} : \|u - \bar{u}\|_{L^2(I)} < \varepsilon\}$.

The second main result is the following convergence estimate of the approximate solutions.

**Theorem 2.4.** Let $\bar{u} \in \mathcal{U}_{ad}$ be a local solution of (1.4) satisfying Assumption 1, let $\bar{u}_h$ be the approximate solution found in Theorem 2.3 and let $\lambda(\bar{u})$ be the adjoint state defined in Definition 3.1. Assume that the state $\bar{x} = G(\bar{u})$ belongs to $W^{k_1,\infty}(I; \mathbb{R}^d)$ and the adjoint state $\lambda(\bar{u})$ belongs to $W^{k_2,\infty}(I; \mathbb{R}^d)$. Then we have

$$\|\bar{u} - \bar{u}_h\|_{L^2(I)} = O(h^{\min\{r+1,k_1,k_2\}}).$$

The above result establishes the error estimate concerning the discretization of the ODEs in the OCPs. On the other hand, to implement a numerical computation to the OCP (1.4), one need also consider an approximation of the control space with a finite dimensional one. In Section 5, we will see that the proof of Theorem 2.4 can be extended to obtain the error analysis incorporating the discretization of the control space.

### 3. Adjoint solutions

This section is devoted to study the adjoint solutions to the OCP (1.1) and its discretized version (2.4).

We introduce a bilinear form $b(\cdot, \cdot)$ for $x \in C^1([0,T])$ and $\varphi \in X$ by

$$b(x, \varphi) := \int_0^T x'(t) \cdot \varphi(t) \, dt.$$  

(3.1)
Then, for a fixed control $u \in U$ and initial data $x_0$, a weak formulation of (1.2) can be written as
\[ b(x, \varphi) = \int_0^T f(t, x(t), u(t)) \cdot \varphi(t) \, dt \tag{3.2} \]
for all $\varphi \in X$.

**Definition 3.1.** For a control $u \in U$, we define the adjoint state $\lambda = \lambda(u) \in X$ as the solution to
\[ \lambda'(t) = -\nabla_x f(t, x(t), u(t)) \lambda(t) + \nabla_x g(t, x(t), u(t)) \]
with $\lambda(T) = 0$ for all $\varphi \in X$. It satisfies the weak formulation
\[ b(\phi, \lambda) = (\phi, \nabla_x f(\cdot, x, u) \lambda - \nabla_x g(\cdot, x, u))_{L^2(I)} \tag{3.3} \]
for all $\phi \in X$.

For $u, v \in U$, the derivative of $j$ at $u$ in the direction $v$ defined by
\[ j'(u)v := \lim_{t \to 0^+} \frac{j(u + tv) - j(u)}{t}. \tag{3.4} \]

It is well-known that the derivative of the cost functional can be calculated with the adjoint state, as described below.

**Lemma 3.2.** We have
\[ j'(u)(v) = (\partial_u g(\cdot, x, u) - \partial_u f(\cdot, x, u) \lambda(u), v)_I \tag{3.4} \]
for all $v \in U_{ad}$, where $x = G(u)$.

**Proof.** For the completeness of the paper, we give the proof in Appendix B.

Next we describe the adjoint problem for the approximate problem. For $x_h, \varphi \in X_h^r$, we define
\[ B(x, \varphi) := \sum_{n=1}^N (x', \varphi)_I + \sum_{n=2}^N ([x]_{n-1}, \varphi_{n-1}^+) + (x^0, \varphi_0^+). \tag{3.5} \]

For approximate solution $x_h = G_h(u) \in X_h^r$, the equation (2.3) with control $u \in U$ can be written as
\[ B(x_h, \varphi) = (f(\cdot, x_h, u), \varphi)_I + (x_0, \varphi_0^+) \quad \text{for all} \quad \varphi \in X_h^r. \tag{3.6} \]

Now we define the adjoint equation for the approximate problem (2.4).

**Definition 3.3.** The adjoint state $\lambda_h = \lambda_h(u) \in X_h^r$ is defined as the solution of the following discrete adjoint equation:
\[ B(\varphi, \lambda_h) = (\varphi, \partial_\varphi (\cdot, x_h, u) \lambda_h - \partial_\varphi g(\cdot, x_h, u))_I \quad \text{for all} \quad \varphi \in X_h^r. \tag{3.7} \]

In Appendix A we briefly explain how the adjoint equation (3.7) can be derived from the Lagrangian related to (2.4).

We also have an analogous result to Lemma 3.2.

**Lemma 3.4.** We have
\[ j_h'(u)(v) = (\partial_u g(\cdot, x_h, u) - \partial_u f(\cdot, x_h, u) \lambda_h, v)_I \quad \text{for all} \quad v \in U_{ad}, \tag{3.8} \]
where $x_h = G_h(u)$. 

Proof. The proof is given in Appendix B. □

In order to prove the main results in Section 2, we shall use the following lemma.

**Lemma 3.5.** Let $u \in U$. Assume that $x = G(u) \in W^{k_1, \infty}(I; \mathbb{R}^d)$ and $\lambda = \lambda(u) \in W^{k_2, \infty}(I; \mathbb{R}^d)$. Then we have
\[
\|\lambda(u) - \lambda_h(u)\|_{L^2(I)} = O(h^{\min\{k_1, k_2, r+1\}}).
\] (3.9)

Proof. We recall from (3.3) and (3.7) that $\lambda = \lambda(u)$ solves
\[
b(\varphi, \lambda) - (\varphi, \partial_x f(\cdot, x, u)\lambda)_{L^2(I)} = - (\varphi, \partial_x g(\cdot, x, u))_{I},
\]
and $\lambda_h = \lambda_h(u)$ solves
\[
B(\varphi, \lambda) - (\varphi, \partial_x f(\cdot, x, u)\lambda)_{L^2(I)}
\]
\[
= -(\varphi, \partial_x g(\cdot, x_h, u))_{L^2(I)} + (\varphi, (\partial_x f(\cdot, x_h, u) - \partial_x f(\cdot, x, u))\lambda)_{L^2(I)}.
\]
Here $x \in G(u) \in X$ and $x_h = G_h(u) \in X_h$. The estimate of $x - x_h$ is induced from Theorem 2.1 as follows:
\[
\|x - x_h\|_{L^\infty(I)} = O(h^{\min\{k_1, r+1\}}\|x\|_{W^{k_1, \infty}(I)}).
\] (3.10)

We consider $\zeta_h \in X_h$ solving
\[
B(\phi, \zeta_h) - (\phi, \partial_x f(\cdot, x, u)\zeta_h)_{I} = -(\phi, \partial_x g(\cdot, x, u))_{I}.
\]
Then, by Theorem 2.1, we have
\[
\|\zeta_h - \lambda\|_{L^\infty(I)} = O(h^{\min\{k_2, r+1\}}\|\lambda\|_{W^{k_2, \infty}(I)}).
\] (3.11)

By (3.10), we obtain
\[
\partial_x g(\cdot, x, u) - \partial_x g(\cdot, x_h, u) = O(h^{\min\{k_1, r+1\}})
\]
and
\[
(\partial_x f(\cdot, x_h, u) - \partial_x f(\cdot, x, u))\lambda_h(u) = O(h^{\min\{k_1, r+1\}}).
\]
This, together with Lemma C.4, yields
\[
\|\lambda_h - \zeta_h\|_{L^\infty(I)} = O(h^{\min\{k_1, r+1\}}).
\]
Combining this estimate with (3.11), we find that
\[
\|\lambda_h(u) - \lambda(u)\|_{L^\infty(I)} = O(h^{\min\{k_1, k_2, r+1\}}),
\]
which completes the proof. □

4. Second order analysis

In this section, we analyze the second order condition of the functions $j$ and $j_h$, which are essential in the convergence estimates in the next section.
4.1. Second order condition for $j$. We defined the solution mapping $G : \mathcal{U} \to X \cap L^\infty(I; \mathbb{R}^d)$ in the previous section. Here we present Lipschitz estimates for the solution mapping $G$, its derivative $G'$, and the solution to the adjoint equation.

**Lemma 4.1.** Let $u, \hat{u} \in \mathcal{U}_{ad}$ and $v \in \mathcal{U}$ be given. Then, there exists $C > 0$ such that

$$\|G(u) - G(\hat{u})\|_{L^\infty(I)} \leq C\|u - \hat{u}\|_{L^2(I)}, \quad \|G'(u)v - G'(\hat{u})v\|_{L^1(I)} \leq C\|u - \hat{u}\|_{L^2(I)}\|v\|_{L^2(I)},$$

and

$$\|\lambda(u) - \lambda(\hat{u})\|_{L^\infty(I)} \leq C\|u - \hat{u}\|_{L^2(I)}.$$

**Proof.** Let us denote by $x = G(u)$ and $\hat{x} = G(\hat{u})$. Then it follows from (3.2) that

$$(x - \hat{x})'(t) = f(t, x(t), u(t)) - f(t, \hat{x}(t), \hat{u}(t)).$$

Note that

$$|f(t, x(t), u(t)) - f(t, \hat{x}(t), \hat{u}(t))| \leq C|x(t) - x(t)| + C|\hat{x}(t) - u(t)|.$$ 

By applying Gronwall’s lemma, we get the inequality

$$\|x - \hat{x}\|_{L^\infty(I)} \leq C\|u - \hat{u}\|_{L^1(I)} \leq C\|u - \hat{u}\|_{L^2(I)}.$$

This gives the first inequality. For the second one, if we set $y = G'(u)v$ and $\hat{y} = G'(\hat{u})v$, then we find

$$(y - \hat{y})'(t) = \partial_x f(t, x(t), u(t))(y - \hat{y})(t) + (\partial_x f(t, x, u) - \partial_x f(t, \hat{x}, \hat{u}))\hat{y}(t) + (\partial_u f(t, x, u) - \partial_u f(t, \hat{x}, \hat{u}))v(t).$$

This together with the first assertion above yields

$$\|y - \hat{y}\|_{L^\infty(I)} \leq C\|(\partial_x f(\cdot, x, u) - \partial_x f(\cdot, \hat{x}, \hat{u}))\hat{y}\|_{L^1(I)} + C\|\partial_u f(\cdot, x, u) - \partial_u f(\cdot, \hat{x}, \hat{u})v\|_{L^1(I)} \leq C\|\|x - \hat{x}\|_{L^2(I)} + \|u - \hat{u}\|_{L^2(I)}\|\hat{y}\|_{L^2(I)} \leq C\|u - \hat{u}\|_{L^2(I)}\|y\|_{L^2(I)}.$$

For notational simplicity, we denote by $\lambda = \lambda(u)$ and $\hat{\lambda} = \lambda(\hat{u})$. Then, we get

$$-(\lambda - \hat{\lambda})'(t) = \partial_x f(\cdot, x, u)(\lambda - \hat{\lambda})(t) + (\partial_x f(\cdot, x, u) - \partial_x f(\cdot, \hat{x}, \hat{u}))(t) - (\partial_x g(\cdot, x, u) - \partial_x g(\cdot, \hat{x}, \hat{u}))(t).$$

Thus, we have

$$\|\lambda - \hat{\lambda}\|_{L^\infty(I)} \leq C\|(\partial_x f(\cdot, x, u) - \partial_x f(\cdot, \hat{x}, \hat{u})){\hat{\lambda}}\|_{L^1(I)} + C\|\partial_x g(\cdot, x, u) - \partial_x g(\cdot, \hat{x}, \hat{u})\|_{L^1(I)} \leq C(\|\hat{\lambda}\|_{L^\infty(I)} + 1)\|x - \hat{x}\|_{L^\infty(I)} + \|u - \hat{u}\|_{L^2(I)} \leq C\|u - \hat{u}\|_{L^2(I)},$$

where we used

$$\|\hat{\lambda}\|_{L^\infty(I)} \leq C\|\partial_x g\|_{L^\infty(I)}$$

due to $\hat{\lambda}(T) = 0$. This completes the proof. \qed
We now show that the second order condition of \( j \) holds near the optimal local solution \( \tilde{u} \in U_{ad} \).

**Lemma 4.2.** There exists \( \epsilon > 0 \) such that

\[
j''(u)(v, v) \geq \frac{\gamma}{2} \|v\|^2_{L^2(I)}
\]

holds for all \( v \in U \) and all \( u \in U_{ad} \) with \( \|u - \tilde{u}\|_{L^2(I)} \leq \epsilon \). Here \( \gamma > 0 \) is appeared in (2.5).

**Proof.** Let \( y(t) = G'(\bar{u})v \) and \( y(\bar{u})(t) = G'(\bar{u})v \). By using Lemma 3.1, we find

\[
j''(u)(v, v) - j''(\bar{u})(v, v)
\]

\[
= -\int_0^T \lambda(t) \left( \frac{\partial^2 f}{\partial x^2} (t, x, u) y^2(t) + 2 \frac{\partial^2 f}{\partial x \partial u} (t, x, u) y(t) v(t) + \frac{\partial^2 f}{\partial u^2} (t, x, u) v^2(t) \right) dt
\]

\[+ \int_0^T \frac{\partial^2 g}{\partial x^2} (t, x, u) y^2(t) + 2 \frac{\partial^2 g}{\partial x \partial u} (t, x, u) y(t) v(t) + \frac{\partial^2 g}{\partial u^2} (t, x, u) v^2(t) dt
\]

\[+ \int_0^T \lambda(t) \left( \frac{\partial^2 f}{\partial x^2} (t, \bar{x}, \bar{u}) \bar{y}^2(t) + 2 \frac{\partial^2 f}{\partial x \partial u} (t, \bar{x}, \bar{u}) \bar{y}(t) v(t) + \frac{\partial^2 f}{\partial u^2} (t, \bar{x}, \bar{u}) v^2(t) \right) dt
\]

\[- \int_0^T \frac{\partial^2 g}{\partial x^2} (t, \bar{x}, \bar{u}) \bar{y}^2(t) + 2 \frac{\partial^2 g}{\partial x \partial u} (t, \bar{x}, \bar{u}) \bar{y}(t) v(t) + \frac{\partial^2 g}{\partial u^2} (t, \bar{x}, \bar{u}) v^2(t) dt,
\]

where we denoted by \( \lambda(t) := \lambda(u)(t), x(t) := G(u)(t), \bar{\lambda}(t) := \lambda(\bar{u})(t), \) and \( \bar{x}(t) := G(\bar{u})(t) \).

On the other hand, it follows from Lemma 4.1 that

\[
\|x - \bar{x}\|_{L^\infty(I)} \leq C \|u - \bar{u}\|_{L^2(I)}, \quad \|y - \bar{y}\|_{L^2(I)} \leq C \|u - \bar{u}\|_{L^2(I)} \|v\|_{L^2(I)}
\]

\[
\|y\|_{L^2(I)} \leq C \|v\|_{L^2(I)}, \quad \|\lambda\|_{L^\infty(I)} + \|\bar{\lambda}\|_{L^\infty(I)} \leq C \|\partial_x g\|_{L^\infty(I)}, \quad \text{and}
\]

\[
\|\lambda - \bar{\lambda}\|_{L^\infty(I)} \leq C \|u - \bar{u}\|_{L^2(I)}
\]

This together with the following estimate

\[
\int_0^T |y^2(t) - \bar{y}^2(t)| dt \leq \int_0^T |y(t) + \bar{y}(t)||y(t) - \bar{y}(t)| dt
\]

\[\leq \|y - \bar{y}\|_{L^2(I)} (\|y\|_{L^2(I)} + \|\bar{y}\|_{L^2(I)})
\]

\[\leq C \|u - \bar{u}\|_{L^2(I)} \|v\|_{L^2(I)}^2
\]

yields

\[
|j''(u)(v, v) - j''(\bar{u})(v, v)| \leq C \|u - \bar{u}\|_{L^2(I)} \|v\|_{L^2(I)}^2
\]

Hence, we have

\[
j''(u)(v, v) = j''(\bar{u})(v, v) + (j''(u)(v, v) - j''(\bar{u})(v, v))
\]

\[\geq \gamma \|v\|_{L^2(I)}^2 - C \|u - \bar{u}\|_{L^2(I)} \|v\|_{L^2(I)}^2.
\]

By choosing \( \epsilon > 0 \) small enough, we conclude the desired result.

\[\square\]

As a consequence of this lemma, we have the following result.
Theorem 4.3. Let \( \bar{u} \in U_{ad} \) satisfy the first optimality condition and Assumption 1. Then, there exist constants \( \epsilon, \delta > 0 \) such that

\[
j(u) \geq j(\bar{u}) + \delta \|u - \bar{u}\|_{L^2(I)}^2
\]

for any \( u \in U_{ad} \) with \( \|u - \bar{u}\|_{L^2(I)} \leq \epsilon \).

Proof. By Taylor expansion, we get

\[
j(u) = j(\bar{u}) + j'(\bar{u})(u - \bar{u}) + \frac{1}{2} j''(\bar{u}_s)(u - \bar{u}, u - \bar{u}),
\]

where \( \bar{u}_s = \bar{u} + s(u - \bar{u}) \) for some \( s \in [0, 1] \). On the other hand, by the first optimality condition, we have

\[
j'(\bar{u})(u - \bar{u}) \geq 0 \quad \text{for all} \quad u \in U_{ad}.
\]

Moreover, we also find

\[
\|\bar{u} - \bar{u}_s\|_{L^2(I)} \leq s\|u - \bar{u}\|_{L^2(I)} \leq \epsilon.
\]

Using these observations and Lemma 4.2, we conclude

\[
j(u) \geq j(\bar{u}) + \frac{\gamma}{2} \|u - \bar{u}\|_{L^2(I)}^2.
\]

\[\square\]

4.2. Second order condition for \( j_h \). In this part, we investigate the second order condition for the discrete objective function \( j_h \). In a similar fashion as previous subsection, we first provide the Lipschitz estimates for \( G_h \) and the discrete adjoint state. Since the proof is almost same as Lemma 4.1, we omit it here.

Lemma 4.4. Let \( u, \bar{u} \in U_{ad} \) and \( v \in U \) be given. Then, there exists \( C > 0 \), independent of \( h \in (0,1) \), such that

\[
\|G_h(u) - G_h(\bar{u})\|_{L^\infty(I)} \leq C\|u - \bar{u}\|_{L^2(I)},
\]

\[
\|G'_h(u)v - G'_h(\bar{u})v\|_{L^2(I)} \leq C\|u - \bar{u}\|_{L^2(I)}\|v\|_{L^2(I)},
\]

and

\[
\|\lambda_h(u) - \lambda_h(\bar{u})\|_{L^\infty(I)} \leq C\|u - \bar{u}\|_{L^2(I)}.
\]

Proof. The proof of the first estimate is proved in Lemma C.4. The other estimates then can be obtained as in Lemma 4.1 with using the estimate of Lemma C.4. \[\square\]

Lemma 4.5. For \( u \in U_{ad} \), let \( x = G(u) \) be given by the solution of the state equation (3.12), and let \( y = G'(u)v \) for \( v \in U \). Let \( x_h = G_h(u) \) be the solution of the discrete state equation (4.3), and let \( y_h = G'_h(u)v \). Then we have

\[
\|y_h - y\|_{L^2(I)} \leq Ch\|v\|_{L^2(I)}.
\]

Proof. Define \( \tilde{y} \) by

\[
b(\tilde{y}, \varphi) = (\partial_x f(\cdot, x_h, u)\tilde{y} + \partial_u f(\cdot, x_h, u)v, \varphi)_I
\]

for \( \varphi \in X \), with the initial data \( \tilde{y}(0) = 0 \). Note that \( y \) satisfies

\[
b(y, \varphi) = (\partial_x f(\cdot, x, u)y + \partial_u f(\cdot, x, u)v, \varphi)_I
\]
with the initial data \( y(0) = 0 \). Combining these two equations, we get
\[
(\partial_t f(\cdot, x_h, u)(\bar{y} - y), \varphi) + ((\partial_x f(\cdot, x_h, u) \partial_x f(\cdot, x, u)) y, \varphi) + ((\partial_u f(\cdot, x_h, u) - \partial_u f(\cdot, x, u)) v, \varphi).
\]
This yields
\[
\|\bar{y} - y\|_{L^2(I)} \leq C\|x_h - x\|_{L^\infty(I)} \left(\|y\|_{L^2(I)} + \|v\|_{L^2(I)}\right)
\leq C\|x_h - x\|_{L^\infty(I)}\|v\|_{L^2(I)}
\leq C h\|v\|_{L^2(I)},
\]
where we used Theorem 2.1 and (4.1). On the other hand, \( y_h \) satisfies
\[
B(y_h, \varphi) = (\partial_x f(\cdot, x_h, u) y_h + \partial_u f(\cdot, x_h, u) v, \varphi).
\]
Comparing this with (4.3), we can use Theorem 2.1 to obtain the following error estimate:
\[
\|\bar{y} - y_h\|_{L^2(I)} \leq C h\|v\|_{L^2(I)}.
\]
Hence, we have
\[
\|y_h - y\|_{L^2(I)} \leq \|\bar{y} - y\|_{L^2(I)} + \|\bar{y} - y_h\|_{L^2(I)} \leq C h\|v\|_{L^2(I)}.
\]
This completes the proof. \(\square\)

**Lemma 4.6.** There exists \( \epsilon > 0 \) such that for all \( u \in U_{ad} \) with \( \|u - \bar{u}\|_{L^2(I)} \leq \epsilon \) and \( v \in U \), we have
\[
j_t^n(u)(v, v) \geq \frac{\gamma}{4}\|v\|_{L^2(I)}^2
\]
for \( h > 0 \) small enough.

**Proof.** We first claim that
\[
j_t^n(u)(v, v) - j_t^n(u)(v, v) \leq C h\|v\|_{L^2(I)}^2
\]
for \( h > 0 \) small enough, where \( C > 0 \) is independent of \( h \). Let \( x(t) = G(u)(t), \lambda(t) = \lambda(u)(t), x_h(t) = G_h(u)(t), \) and \( \lambda_h(t) = \lambda_h(u)(t) \). Also we let \( y = G'(u)v \) and \( y_h = G_h'(u)v \).

It follows from Lemmas 4.1 and 4.2 that
\[
j_t^n(u)(v, v) - j_t^n(u)(v, v)
\]
\[
= -\int_0^T \left(\partial_t f(\cdot, x, u)y^2(t) + 2\frac{\partial^2 f}{\partial x \partial u}(t, x, u)y(t)v(t) + \frac{\partial^2 f}{\partial u^2}(t, x, u)v^2(t)\right) dt
\]
\[
+ \int_0^T \left(\partial_t g(\cdot, x, u)y^2(t) + 2\frac{\partial^2 g}{\partial x \partial u}(t, x, u)y(t)v(t) + \frac{\partial^2 g}{\partial u^2}(t, x, u)v^2(t)\right) dt
\]
\[
+ \int_0^T \lambda_h(t)\left(\partial_t f(\cdot, x, u)y_h^2(t) + 2\frac{\partial^2 f}{\partial x \partial u}(t, x, u)y_h(t)v(t) + \frac{\partial^2 f}{\partial u^2}(t, x, u)v^2(t)\right) dt
\]
\[
- \int_0^T \lambda_h(t)\left(\partial_t g(\cdot, x, u)y_h^2(t) + 2\frac{\partial^2 g}{\partial x \partial u}(t, x, u)y_h(t)v(t) + \frac{\partial^2 g}{\partial u^2}(t, x, u)v^2(t)\right) dt dt.
\]

In order to show (4.4), by using a similar argument as in the proof of Lemma 4.2 it suffices to show that there exists \( C > 0 \), independent of \( h \), such that
\[
\|x - x_h\|_{L^\infty(I)} \leq C h, \quad \|y - y_h\|_{L^2(I)} \leq C h\|v\|_{L^2(I)}, \quad \|y_h\|_{L^2(I)} \leq C\|v\|_{L^2(I)}.
\]
\[ \| \lambda_h \|_{L^{\infty}(I)} \leq C, \quad \| \lambda - \lambda_h \|_{L^{\infty}(I)} \leq Ch, \tag{4.6} \]

and
\[ \int_0^T |y^2(t) - y_h^2(t)| \, dt \leq Ch\|v\|_{L^2(I)}^2. \]

The first and second inequality in (4.5) hold due to Theorem 2.1 and Lemma 4.5. For the third one in (4.5), we estimate
\[ \| y_h \|_{L^2(I)} \leq \| y - y_h \|_{L^2(I)} + \| y \|_{L^2(I)} \leq Ch\|v\|_{L^2(I)} + C\|v\|_{L^2(I)} \leq C\|v\|_{L^2(I)}, \]

for \( h \) small enough. By Lemma 3.5, the second inequality in (4.6) holds. We also find
\[ \| \lambda_h \|_{L^{\infty}(I)} \leq \| \lambda - \lambda_h \|_{L^{\infty}(I)} + \| \lambda \|_{L^{\infty}(I)} \leq Ch + C \leq C \]

for \( h \) small enough, and this asserts the first inequality in (4.6). Finally, we obtain
\[ \int_0^T |y^2(t) - y_h^2(t)| \, dt \leq \int_0^T |y(t) + y_h(t)||y(t) - y_h(t)| \, dt \]
\[ \leq \| y(t) - y_h(t) \|_{L^2(I)} \left( \| y \|_{L^2(I)} + \| y_h \|_{L^2(I)} \right) \]
\[ \leq Ch\|v\|_{L^2(I)}^2, \]

due to (4.5). All of the above estimates enable us to prove the claim (4.4). This together with Lemma 4.2 yields
\[ j''_h(u)(v,v) \geq j''(u)(v,v) - |j''_h(u)(v,v) - j''(u)(v,v)| \]
\[ \geq \frac{\gamma}{2} \| v \|^2_{L^2(I)} - Ch\|v\|_{L^2(I)}^2 \]
\[ \geq \frac{\gamma}{4} \| v \|^2_{L^2(I)} \]

for \( h > 0 \) small enough. \( \square \)

5. Convergence estimates

5.1. Semidiscrete case. We first prove the existence of the local solution to the approximate problem (2.4).

Proof of Theorem 2.3. Choose \( \epsilon > 0 \) as in Theorem 4.3. By compactness and continuity, \( j_h \) has a minimizer in
\[ \overline{B}_\epsilon(\bar{u}) = \{ u \in \mathcal{U}_{ad} : \| u - \bar{u} \|_{L^2(I)} \leq \epsilon \}. \]

It remains to show that the minimizer is achieved in the interior of \( \overline{B}_\epsilon(\bar{u}) \). To show this, we observe that
\[ \lim_{h \to 0} j_h(u) = j(u) \]
uniformly on \( \overline{B}_\epsilon(\bar{u}) \) by Theorem 2.1. Moreover we deduce from Theorem 4.3 that
\[ j(u) \geq j(\bar{u}) + \frac{\gamma}{2} \epsilon^2, \quad \text{if} \quad \| u - \bar{u} \|_{L^2(I)} = \epsilon. \]

Therefore, there exists \( h_0 > 0 \) such that for \( h \in (0, h_0) \) we have
\[ j_h(u) \geq j_h(\bar{u}) + \frac{\gamma}{4} \epsilon^2 \quad \text{if} \quad \| u - \bar{u} \|_{L^2(I)} = \epsilon, \]
since \( j_h(u) \) converges to \( j(u) \) as \( h \to 0 \), uniformly for \( u \in \mathcal{U} \). Thus, the minimizer is achieved in \( B_r(\bar{u}) \).

We now provide the details of the convergence estimate of the approximate solutions.

**Proof of Theorem 2.4.** Analogous to (4.2), the discrete first order necessary optimality condition for \( \bar{u}_h \in \mathcal{U}_{ad} \) reads

\[
j'_h(\bar{u}_h)(u - \bar{u}_h) \geq 0 \quad \text{for all} \quad u \in \mathcal{U}_{ad}.
\]

Summing up this and (4.2), we get

\[
0 \leq (j'(\bar{u}) - j'_h(\bar{u}_h))(\bar{u}_h - \bar{u})
= (j'(\bar{u}) - j'_h(\bar{u}_h))(\bar{u}_h - \bar{u}) + (j'_h(\bar{u}) - j'_h(\bar{u}_h))(\bar{u}_h - \bar{u}).
\]

Now, by applying the mean value theorem with a value \( t \in (0, 1) \), one has

\[
C\|\bar{u}_h - \bar{u}\|_{L^2(I)}^2 \leq j''(\bar{u}_h - t(\bar{u} - \bar{u}_h))(\bar{u}_h - \bar{u})
= (j'_h(\bar{u}_h) - j'_h(\bar{u}_h))(\bar{u}_h - \bar{u})
\leq (j'(\bar{u}) - j'_h(\bar{u}_h))(\bar{u}_h - \bar{u}),
\]

where we used Lemma 4.6 in the first inequality and (5.1) in the second inequality. For our aim, it only remains to estimate the right hand side. Let us express it using the adjoint solutions. From (3.4), we have

\[
j'(\bar{u})(\bar{u}_h - \bar{u}) = (\partial_u g(\cdot, \bar{x}, \bar{u}) - \partial_u f(\cdot, \bar{x}, \bar{u})\lambda(\bar{u}), \bar{u}_h - \bar{u})_I,
\]

and it follows from (3.8) that

\[
j'_h(\bar{u})(\bar{u}_h - \bar{u}) = (\partial_u g(\cdot, \bar{x}, \bar{u}) - \partial_u f(\cdot, \bar{x}, \bar{u})\lambda(\bar{u}), \bar{u}_h - \bar{u})_I.
\]

Here we remind that \( \bar{x}_h \in X^r_h \) denotes the solution to (2.3) with control \( \bar{u} \) and initial data \( x_0 \). Thus, we find

\[
j'(\bar{u}) - j'_h(\bar{u}_h)(\bar{u}_h - \bar{u})
= \left( \partial_u g(\cdot, \bar{x}, \bar{u}) - \partial_u g(\cdot, \bar{x}_h, \bar{u}), \bar{u}_h - \bar{u} \right)_I
- \left( \partial_u f(\cdot, \bar{x}, \bar{u})\lambda(\bar{u}) - \partial_u f(\cdot, \bar{x}_h, \bar{u})\lambda(\bar{u}), \bar{u}_h - \bar{u} \right)_I.
\]

Using Hölder’s inequality to the above, we deduce

\[
(j'(\bar{u}) - j'_h(\bar{u}_h))(\bar{u}_h - \bar{u})
\leq \|\partial_u g\|_{L^\infty} \|\bar{x} - \bar{x}_h\|_{L^2(I)} \|\bar{u}_h - \bar{u}\|_{L^2(I)}
+ \|\lambda(\bar{u})\|_{L^\infty(I)} \|\partial_u f(\cdot, \bar{x}, \bar{u}) - \partial_u f(\cdot, \bar{x}_h, \bar{u})\|_{L^2(I)} \|\bar{u}_h - \bar{u}\|_{L^2(I)}
+ \|\partial_u f(\cdot, \bar{x}_h, \bar{u})\|_{L^\infty} \|\lambda(\bar{u}) - \lambda_k(\bar{u})\|_{L^2(I)} \|\bar{u}_h - \bar{u}\|_{L^2(I)}
\leq ((C + K)\|\bar{x} - \bar{x}_h\|_{L^2(I)} + K\|\lambda(\bar{u}) - \lambda_k(\bar{u})\|_{L^2(I)}) \|\bar{u}_h - \bar{u}\|_{L^2(I)}.
\]

Now we apply (3.9) and (3.10) to get

\[
(j'(\bar{u}) - j'_h(\bar{u}_h))(\bar{u}_h - \bar{u}) \leq Ch^{\min\{k_1, k_2, r + 1\}} \|\bar{u}_h - \bar{u}\|_{L^2(I)}.
\]

(5.3)
Combining this with (5.3), we finally obtain
\[ \| \bar{u}_h - \bar{u} \|_{L^2(I)} \leq Ch^{\min\{k_1,k_2,r+1\}}. \]
This completes the proof. \(\square\)

5.2. Fully discrete case. Here we consider discretization \(U_h\) of the control space \(U\). For example, the space of step functions or the high-order DG space \(U_h = X_h^r\).

**Theorem 5.1.** Assume the same statements in Theorem 2.4. In addition, suppose that there exists a projection operator \(P_h : U \to U_h\) and a value \(a > 0\) such that \(\|P_h \bar{u} - \bar{u}\|_{L^2(I)} = O(h^a)\) for \(h \in (0,1)\). Let \(\bar{u}_h\) be the optimal solution to
\[ \text{Minimize } j_h(u) \text{ subject to } u \in U_h. \]
Then the following estimate holds:
\[ \| \bar{u}_h - \bar{u} \|_{L^2(I)} = O(h^{\min\{r+1,k_1,k_2,a/2\}}). \]
If we further assume that \(j'(\bar{u}) = 0\), then the above estimate can be improved to
\[ \| \bar{u}_h - \bar{u} \|_{L^2(I)} = O(h^{\min\{r+1,k_1,k_2,a\}}). \]

**Proof.** In this case, by the first optimality conditions on \(\bar{u}\) and \(\bar{u}_h\), we have
\[ j'(\bar{u})(\bar{u}_h - \bar{u}) \geq 0 \quad \text{and} \quad j_h'(\bar{u}_h)(P_h \bar{u} - \bar{u}_h) \geq 0. \]
The latter condition can be written as
\[ 0 \leq j_h'(\bar{u}_h)(\bar{u} - \bar{u}_h) + j_h'(\bar{u}_h)(P_h \bar{u} - \bar{u}) = j_h'(\bar{u}_h)(\bar{u} - \bar{u}_h) + R_h, \]
where \(R_h := j_h'(\bar{u}_h)(P_h \bar{u} - \bar{u})\). Summing up these two inequalities, we get
\[ 0 \leq (j'(\bar{u}) - j_h'(\bar{u}_h))\bar{u}_h - \bar{u} + R_h \]
\[ = (j'(\bar{u}) - j_h'(\bar{u}_h))\bar{u}_h - \bar{u} + (j_h'(\bar{u}) - j_h'(\bar{u}_h))(\bar{u}_h - \bar{u}) + R_h, \]
i.e.,
\[ (j_h' - j_h')\bar{u}_h - \bar{u}_h - \bar{u} \leq (j'(\bar{u}) - j_h'(\bar{u}_h))(\bar{u}_h - \bar{u}) + R_h. \] (5.4)
By the assumption of the theorem, we have
\[ \| R_h \|_{L^2(I)} = O(h^a). \] (5.5)
On the other hand, by applying the mean value theorem and Lemma 4.6 we obtain
\[ (j_h' - j_h')(\bar{u}_h - \bar{u}) = j_h''(\bar{u} + t(\bar{u} - \bar{u}_h))(\bar{u}_h - \bar{u}, \bar{u}_h - \bar{u}) \geq C\| \bar{u}_h - \bar{u}\|_{L^2(I)}. \]
Combining this with (5.4) yields
\[ \| \bar{u}_h - \bar{u} \|_{L^2(I)}^2 \leq C(j'(\bar{u}) - j_h'(\bar{u}_h))(\bar{u}_h - \bar{u}) + CR_h \]
We now use the estimate (5.3) in the previous proof, we have
\[ \| \bar{u}_h - \bar{u} \|_{L^2(I)}^2 \leq Ch^{\min\{k_1,k_2,r+1\}}\| \bar{u}_h - \bar{u}\|_{L^2(I)} + CR_h, \] (5.6)
which together with (5.5) gives the desired estimate
\[ \|\bar{u}_h - \bar{u}\|_{L^2(I)} = O(h^{\min\{r+1,k_1,k_2,a/2\}}). \]

When we further assume \( j'(\bar{u}) = 0 \), we have
\[ j_h'(\bar{u}_h) = (j_h'(\bar{u}_h) - j_h'(\bar{u})) + (j_h'(\bar{u}) - j'(\bar{u})). \]

Using this and (5.3), we find that
\[ |R_h| = |j_h'(\bar{u}_h)(P_h\bar{u} - \bar{u})| \leq C \left( \|\bar{u}_h - \bar{u}\|_{L^2(I)} + h^{\min\{k_1,k_2,r+1\}} \right) \|P_h\bar{u} - \bar{u}\|_{L^2(I)} \]
\[ \leq Ch^a \left( \|\bar{u}_h - \bar{u}\|_{L^2(I)} + h^{\min\{k_1,k_2,r+1\}} \right). \]

Inserting this into (5.6) we obtain
\[ \|\bar{u}_h - \bar{u}\|_{L^2(I)}^2 \leq Ch^{\min\{k_1,k_2,r+1\}} \|\bar{u}_h - \bar{u}\|_{L^2(I)} \]
\[ + Ch^a \left( \|\bar{u}_h - \bar{u}\|_{L^2(I)} + h^{\min\{k_1,k_2,r+1\}} \right). \]

It gives the desired estimate
\[ \|\bar{u}_h - \bar{u}\|_{L^2(I)} = O(h^{\min\{r+1,k_1,k_2,a\}}). \]

The proof is done. \( \square \)

6. Numerical experiments

In this section, we present several numerical experiments which validate our theoretical results. We employed the forward-backward DG methods [4] to solve the examples of the OCPs.

6.1. Linear problem. Let us consider the following simple one dimensional OCP, which has been used as an example [16], that consists of maximizing the functional
\[ J = \frac{1}{2} \int_0^1 x^2(t) + u^2(t) \, dt \]
subject to the state equation
\[ x'(t) = -x(t) + u(t), \quad x(0) = 1, \quad (6.1) \]
and \( \mathcal{U} = L^2([0,1]) \). Using a similar idea as in Section 3 based on the maximum principle, we can derive the adjoint equation to the above optimal control problem:
\[ \lambda'(t) = \lambda(t) - x(t), \quad \lambda(1) = 0. \]

Furthermore, we also find that the optimal solutions \( \bar{u} = -\lambda \) and \( \bar{x} \) satisfies (6.2). Thus we have the solution
\[ \bar{x}(t) = \frac{\sqrt{2} \cosh(\sqrt{2}(t - 1)) - \sinh(\sqrt{2}(t - 1))}{\sqrt{2} \cosh(\sqrt{2}) + \sinh(\sqrt{2})} \]
and
\[ \bar{u}(t) = \frac{\sinh(\sqrt{2}(t - 1))}{\sqrt{2} \cosh(\sqrt{2}) + \sinh(\sqrt{2})}. \]
For fixed \( r \in \mathbb{N} \), we use \( X_r^n \) for the approximate space of \( \mathcal{U} \). In Table 1 we report the discrete \( L^2 \) error between optimal solutions and its approximations for the above optimal control problem. Here \( r + 1 \) is the number of grid points on each time interval \( I_n \), and we used the equidistant points for our numerical computations. The numerical result confirms that the error is of order \( h^{r+1} \) as proved in Theorem 2.4.

### Table 1. Discrete \( L^2 \) error: \( \| \bar{x} - x_h \|_{L^2(I)} \) and \( \| \bar{u} - u_h \|_{L^2(I)} \)

| \( h \) | \( r = 1 \) | \( r = 2 \) | \( r = 3 \) |
|---|---|---|---|
| \( (0.1) \times 2^0 \) | \( 1.9455e-03 \) | \( 2.6708e-05 \) | \( 2.8964e-07 \) |
| \( (0.1) \times 2^{-1} \) | \( 4.8861e-04 \) | \( 3.3523e-06 \) | \( 1.8172e-08 \) |
| \( (0.1) \times 2^{-2} \) | \( 1.2240e-04 \) | \( 4.0780e-05 \) | \( 1.9521e-08 \) |
| \( (0.1) \times 2^{-3} \) | \( 3.0629e-05 \) | \( 2.5748e-06 \) | \( 6.5673e-10 \) |
| \( (0.1) \times 2^{-4} \) | \( 7.6607e-06 \) | \( 2.5748e-06 \) | \( 2.5890e-11 \) |
| \( (0.1) \times 2^{-5} \) | \( 1.9156e-06 \) | \( 6.4477e-07 \) | \( 2.7555e-11 \) |

6.2. **Nonlinear problem.** In this part, we consider the following nonlinear optimal control problem:

\[
J = \frac{1}{2} \int_0^{1/5} x^2(t) + u^2(t) \, dt
\]

subject to the state equation

\[
x'(t) = x^2(t) + u(t), \quad x(0) = 2.
\]

In this case, the corresponding adjoint equation and optimal control are given as follows.

\[
\lambda'(t) = -x(t)(1 + \lambda(t)) \quad \text{and} \quad \bar{u}(t) = -\lambda(t),
\]

and thus the optimal solution \( \bar{x} \) solves

\[
x'(t) = x^2(t) - \lambda(t), \quad x(0) = 2.
\]
In this case, since we have no explicit form of the actual solutions, we take the reference solutions $\bar{x}_h$ (resp., $\bar{u}_h$) with $h = (0.1) \times 2^{-9}$ instead of $\bar{x}$ (resp., $\bar{u}$). In Table 2, we arrange the discrete $L^2$ error between reference solutions and its approximations.

**Table 2.** Discrete $L^2$ error: $\|\bar{x} - \bar{x}_h\|_{L^2(I)}$ and $\|\bar{u} - \bar{u}_h\|_{L^2(I)}$

| $r$  | $h$              | $\|\bar{x} - \bar{x}_h\|_{L^2(I)}$ | $\|\bar{u} - \bar{u}_h\|_{L^2(I)}$ | $\log_2 \frac{\|\bar{x} - \bar{x}_h\|_{L^2(I)}}{\|\bar{x}_h\|_{L^2(I)}}$ | $\log_2 \frac{\|\bar{u} - \bar{u}_h\|_{L^2(I)}}{\|\bar{u}_h\|_{L^2(I)}}$ |
|------|------------------|-----------------------------------|-----------------------------------|------------------------------------------|------------------------------------------|
| $1$  | 0.1              | 1.3006e-02                        | 2.6587e-03                        | 1.51                                     | 1.95                                     |
|      | $(0.1) \times 2^{-1}$ | 4.5715e-03                         | 6.8872e-04                        | 1.78                                     | 2.02                                     |
|      | $(0.1) \times 2^{-2}$ | 1.3286e-03                         | 1.7024e-04                        | 1.90                                     | 2.01                                     |
|      | $(0.1) \times 2^{-3}$ | 3.5677e-04                         | 4.2187e-05                        | 1.95                                     | 2.01                                     |
|      | $(0.1) \times 2^{-4}$ | 9.2305e-05                         | 1.0492e-05                        | 1.95                                     | 2.01                                     |
|      | $(0.1) \times 2^{-5}$ | 2.3420e-05                         | 2.6101e-06                        | 1.98                                     | 2.01                                     |
| $2$  | 0.1              | 7.9288e-04                        | 7.1751e-05                        | 2.23                                     | 3.40                                     |
|      | $(0.1) \times 2^{-1}$ | 1.6928e-04                         | 6.8412e-06                        | 2.62                                     | 3.25                                     |
|      | $(0.1) \times 2^{-2}$ | 2.7566e-05                         | 7.2059e-07                        | 2.81                                     | 3.10                                     |
|      | $(0.1) \times 2^{-3}$ | 3.9391e-06                         | 8.4373e-08                        | 2.90                                     | 3.03                                     |
|      | $(0.1) \times 2^{-4}$ | 5.2676e-07                         | 1.0332e-08                        | 2.95                                     | 3.01                                     |
|      | $(0.1) \times 2^{-5}$ | 6.8107e-08                         | 1.2833e-09                        | 2.98                                     | 3.01                                     |
| $3$  | 0.1              | 4.8978e-05                        | 2.3326e-06                        | 3.07                                     | 5.53                                     |
|      | $(0.1) \times 2^{-1}$ | 5.8217e-06                         | 2.0158e-07                        | 3.07                                     | 3.53                                     |
|      | $(0.1) \times 2^{-2}$ | 5.0236e-07                         | 1.3655e-08                        | 3.53                                     | 3.88                                     |
|      | $(0.1) \times 2^{-3}$ | 3.6929e-08                         | 8.7619e-10                        | 3.77                                     | 3.96                                     |
|      | $(0.1) \times 2^{-4}$ | 2.5037e-09                         | 5.5551e-11                        | 3.88                                     | 3.98                                     |
|      | $(0.1) \times 2^{-5}$ | 1.6329e-10                         | 3.6858e-12                        | 3.94                                     | 3.91                                     |

**Appendix A. Proofs of Lemma 3.2 and Lemma 3.4**

In this part, we give the proofs of Lemma 3.2 and Lemma 3.4. Before presenting it, we shall explain how to derive the discrete adjoint equation (3.7) from the Lagrangian associated to (2.4).

Let us first write the Lagrangian of the problem (1.1) and (3.6) as follows:

$$L_h(x, u, \lambda) := \int_0^T g(t, x_h(t), u(t)) \, dt + B(x_h, \lambda_h) - (f(\cdot, x_h, u), \lambda_h)_I - (x_0, \lambda_h^0),$$

(A.1)

where the bilinear operator $B(\cdot, \cdot)$ is given by (3.6). If we compute the functional derivatives of the above Lagrangian (A.1) with respect to the adjoint state $\lambda_h$, then $\frac{\delta L_h}{\delta \lambda_h} = 0$ leads (3.6). We now derive the equation of discrete adjoint state. Using the integration by parts, we find

$$B(x_h, \lambda_h) = - \sum_{n=1}^N (x_h, \lambda'_{h,n})_n - \sum_{n=1}^{N-1} (x_{h,n}, [\lambda_h]_n) + (x_{h,N}, \lambda^{-}_{h,N}).$$
This enables us to rewrite the Lagrangian (A.1) as

$$L_h(x, u, \lambda) = \int_0^T g(t, x_h(t), u(t)) \, dt - \sum_{n=1}^N (x_h, \lambda')_{I_n} - (f(\cdot, x_h, u), \lambda)_{I} - \sum_{n=1}^{N-1} (x_{h,n}, [\lambda_h]_n) + (x_{h,N}, \lambda_{h,N}) - (x_0, \lambda_{h,0})^T,$$

and this further implies

$$0 = \frac{\delta L_h(x, u, \lambda)}{\delta x_h}(\psi_h)$$

for all \(\psi_h \in X_h^T\), where we applied the integration by parts for \((\psi_h, \lambda')_{I_n}\) to derive the second equality. The above equality corresponds to the adjoint equation (3.7).

**Proof of Lemma 3.2.** In order to compute the functional derivative of \(j\) with respect to \(u\), we consider \(j(u + sv) = J(u + sv, G(u + sv))\) with \(v \in U\) and \(s \in \mathbb{R}_+\). If we set \(x^s(t) := G(u(t) + sv(t)) = x(t) + sy(t) + o(s)\) with \(y \in X\), we can easily find \(y = G'(u)v\) satisfies

$$y'(t) = \frac{\partial f}{\partial x}(t, x, u)y(t) + \frac{\partial f}{\partial u}(t, x, u)v(t),$$

with the initial condition \(y(0) = 0\). Recall from (3.3) that the adjoint state \(\lambda(t) = \lambda(u)(t)\) satisfies

$$\lambda'(t) = \frac{\partial g}{\partial x}(t, x, u) - \lambda(t)\frac{\partial f}{\partial x}(t, x, u).$$

Then, we have

$$j'(u)v = \frac{d}{ds}j(u + sv)\bigg|_{s=0} = \int_0^T \frac{\partial g}{\partial u}(t, x(t), u(t))v(t) \, dt + \int_0^T \frac{\partial g}{\partial x}(t, x(t), u(t))y(t) \, dt$$

$$= \int_0^T \left( \frac{\partial g}{\partial u}(t, x(t), u(t)) - \lambda(t)\frac{\partial f}{\partial u}(t, x(t), u(t)) \right) v(t) \, dt,$$

where we used

$$\int_0^T \frac{\partial g}{\partial x}(t, x(t), u(t))y(t) \, dt = \int_0^T \left( \lambda'(t) + \lambda(t)\frac{\partial f}{\partial x}(t, x(t), u(t)) \right) y(t) \, dt$$

$$= -\int_0^T \lambda(t)\frac{\partial f}{\partial u}(t, x(t), u(t))v(t) \, dt,$$

due to (A.3), (A.4), \(y(0) = 0\), and \(\lambda(T) = 0\). \(\square\)
Proof of Lemma 3.4. The proof is very similar to Lemma 3.2. We consider \( j_h(u + sv) = J(u + sv, G_h(u + sv)) \) with \( v \in \mathcal{U} \) and \( s \in \mathbb{R}_+ \). It is not difficult to see that \( x_h^s := G_h(u + sv) = x_h + sy_h + o(s) \), where \( y_h = G_h'(u)v \in X_h^t \) satisfies the following equation:

\[
B(y_h, \varphi) = \left( \frac{\partial f}{\partial x}(\cdot, x_h, u)y_h + \frac{\partial f}{\partial u}(\cdot, x_h, u)v, \varphi \right)_I \quad \text{for all } \varphi \in X_h^t. \quad (A.5)
\]

Note that

\[
\frac{d}{ds} G_h(u + sv) \bigg|_{s=0} = \frac{d}{ds} x_h^s \bigg|_{s=0} = y_h,
\]

and so we obtain

\[
j_h'(u)v = \left. \frac{d}{ds} j_h(u + sv) \right|_{s=0} = \int_0^T \frac{\partial g}{\partial u}(t, x_h(t), u(t))y_h(t) dt + \int_0^T \frac{\partial g}{\partial x}(t, x_h(t), u(t))y_h(t) dt.
\]

We then take \( \psi_h = y_h \) in (A.2) to get

\[
\int_0^T \frac{\partial g}{\partial x}(t, x_h(t), u(t))y_h(t) dt = \sum_{n=1}^N (y_h, \lambda_h^n)_{I_n} + \left( \frac{\partial f}{\partial x}(\cdot, x_h, u)y_h, \lambda_h \right)_I + \sum_{n=1}^{N-1} (y_h, [\lambda_h]_{n-1}, [\lambda_h]_n) - (y_{h,N}^-, \lambda_{h,N}^-).
\]

On the other hand, by using the integration by parts, we find

\[
\sum_{n=1}^N (y_h, \lambda_h^n)_{I_n} + \sum_{n=1}^{N-1} (y_h, [\lambda_h]_{n-1}, [\lambda_h]_n) - (y_{h,N}^-, \lambda_{h,N}^-) = -\sum_{n=1}^{N-1} (y_h, \lambda_h^n)_{I_n} - \sum_{n=2}^{N} (y_h, \lambda_h^n)_{I_n-1} - (y_{h,0}^+, \lambda_{h,0}^+)
\]

\[
= -B(w_h, \lambda_h),
\]

where \( B(\cdot, \cdot) \) is appeared in (3.5). This yields

\[
\int_0^T \frac{\partial g}{\partial x}(t, x_h(t), u(t))y_h(t) dt = -B(y_h, \lambda_h) + \left( \frac{\partial f}{\partial x}(\cdot, x_h, u)y_h, \lambda_h \right)_I
\]

\[
= - \left( \frac{\partial f}{\partial u}(\cdot, x_h, u)v, \lambda_h \right)_I,
\]

due to (A.5). This, together with (A.6), concludes

\[
j_h'(u)v = \int_0^T \left( \frac{\partial g}{\partial u}(t, x_h(t), u(t)) - \frac{\partial f}{\partial u}(t, x_h(t), u(t))\lambda_h(t) \right) v(t) dt,
\]

where \( v \in \mathcal{U} \). \( \square \)
Appendix B. Derivations of the Second Order Derivative of Objective Functions

In this appendix, we provide details of the derivation of the second order derivative of objective function $j$ and the discrete one $j_h$.

Lemma B.1. Let $j$ be the objective function for the optimal control problem (1.1)-(1.2). Then, for $u \in \mathcal{U}_{ad}$ and $v \in \mathcal{U}$, we have

$$j''(u, v) = - \int_0^T \lambda(t) \left( \frac{\partial^2 f}{(\partial x)^2}(t, x(t), u(t)) y^2(t) + 2 \frac{\partial^2 f}{\partial x \partial u}(t, x(t), u(t)) y(t)v(t) + \frac{\partial^2 g}{(\partial u)^2}(t, x(t), u(t)) y^2(t) \right) dt$$

$$- \int_0^T \lambda(t) \frac{\partial^2 f}{(\partial u)^2}(t, x(t), u(t)) v^2(t) dt + \int_0^T \frac{\partial^2 g}{(\partial x)^2}(t, x(t), u(t)) y^2(t) dt$$

$$+ \int_0^T 2 \frac{\partial^2 g}{\partial x \partial u}(t, x(t), u(t)) y(t)v(t) dt + \int_0^T \frac{\partial^2 g}{(\partial u)^2}(t, x(t), u(t)) v^2(t) dt.$$

Proof. Similarly as in Appendix A we consider $j(u + sv) = J(u + sv, G(u + sv))$ with $v \in \mathcal{U}$ and $s \in \mathbb{R}_+$ and set $x^s(t) := G(u(t) + sv(t))$. It is not difficult to see that $x^s(t) = x(t) + sy(t) + (s^2/2)z(t) + o(s^2)$, where $y \in X$ is given as in (A.3) and $z \in X$ is the solution to

$$z'(t) = \frac{\partial^2 f}{(\partial x)^2}(t, x(t), u(t)) y^2(t) + 2 \frac{\partial^2 f}{\partial x \partial u}(t, x(t), u(t)) y(t)v(t) + \frac{\partial^2 f}{(\partial u)^2}(t, x(t), u(t)) v^2(t)$$

$$+ \frac{\partial f}{\partial x}(t, x(t), u(t)) z(t),$$

with the initial condition $z(0) = 0$. Then we obtain

$$j''(u, v) = \frac{d^2}{ds^2} j(u + sv)\bigg|_{s=0}$$

$$= \frac{d^2}{ds^2} \int_0^T g(t, x^s(t), u(t) + sv(t)) dt \bigg|_{s=0}$$

$$= \int_0^T \frac{\partial g}{\partial x}(t, x(t), u(t)) z(t) dt + \int_0^T \frac{\partial^2 g}{(\partial x)^2}(t, x(t), u(t)) y^2(t) dt$$

$$+ \int_0^T 2 \frac{\partial^2 g}{\partial x \partial u}(t, x(t), u(t)) y(t)v(t) dt + \int_0^T \frac{\partial^2 g}{(\partial u)^2}(t, x(t), u(t)) v^2(t) dt. \quad \text{(B.1)}$$
On the other hand, we use (A.1) to get
\[
\int_0^T \frac{\partial g}{\partial x}(t, x(t), u(t)) z(t) \, dt \\
= \int_0^T \lambda(t) z(t) \, dt + \int_0^T \frac{\partial f}{\partial x}(t, x(t), u(t)) \lambda(t) z(t) \, dt \\
= - \int_0^T \lambda(t) z'(t) \, dt + \int_0^T \frac{\partial f}{\partial x}(t, x(t), u(t)) \lambda(t) z(t) \, dt \\
= - \int_0^T \lambda(t) \left( \frac{\partial^2 f}{(\partial x)^2}(t, x(t), u(t)) y^2(t) + 2 \frac{\partial^2 f}{\partial x \partial u}(t, x(t), u(t)) y(t)v(t) \right) \, dt \\
- \int_0^T \lambda(t) \frac{\partial^2 f}{(\partial u)^2}(t, x(t), u(t)) v^2(t) \, dt,
\]
where we used \( \lambda(T) = 0 \) and \( z(0) = 0 \). By combining the above with (B.1), we have
\[
j''(u)(v, v) = - \int_0^T \lambda(t) \left( \frac{\partial^2 f}{(\partial x)^2}(t, x(t), u(t)) y^2(t) + 2 \frac{\partial^2 f}{\partial x \partial u}(t, x(t), u(t)) y(t)v(t) \right) \, dt \\
- \int_0^T \lambda(t) \frac{\partial^2 f}{(\partial u)^2}(t, x(t), u(t)) v^2(t) \, dt + \int_0^T \frac{\partial^2 g}{(\partial x)^2}(t, x(t), u(t)) y^2(t) \, dt \\
+ \int_0^T 2 \frac{\partial^2 g}{\partial x \partial u}(t, x, u) y(t)v(t) \, dt + \int_0^T \frac{\partial^2 g}{(\partial u)^2}(t, x(t), u(t)) v^2(t) \, dt.
\]

Next we proceed the similar calculation for the approximate solution.

**Lemma B.2.** Let \( j_h \) be the discrete objective function for the optimal control problem (1.1)-(1.2). Then, for \( u \in \mathcal{U}_{ad} \) and \( v \in \mathcal{U} \), we have
\[
j_h''(u)(v, v) = - \int_0^T \left( \frac{\partial^2 f}{(\partial x)^2}(t, x_h, u) y_h^2(t) + 2 \frac{\partial^2 f}{\partial x \partial u}(t, x_h, u) y_h(t)v(t) + \frac{\partial^2 f}{(\partial u)^2}(t, x_h, u) v^2(t) \right) \lambda_h(t) \, dt \\
+ \int_0^T \left( \frac{\partial^2 g}{(\partial x)^2}(t, x_h, u) y_h^2(t) + 2 \frac{\partial^2 g}{\partial x \partial u}(t, x_h, u) y_h(t)v(t) + \frac{\partial^2 g}{(\partial u)^2}(t, x_h, u) v^2(t) \right) \, dt.
\]

**Proof.** Similarly as in the proof of Lemma 3.4, we consider \( j_h(u+sv) = J(u+sv, G_h(u+sv)) \) with \( v \in \mathcal{U} \) and \( s \in \mathbb{R}_+ \) and set \( x_h^s := G_h(u+sv) \). We have \( x_h^s = x_h + sy_h + (s^2/2)z_h + o(s^2) \), where \( y_h = G_h'(u)v \in X_h^s \) and \( z_h \in X_h^s \) satisfies
\[
B(z_h, \phi) \\
= \int_0^T \left( \frac{\partial^2 f}{(\partial x)^2}(t, x_h, u) y_h^2(t) + 2 \frac{\partial^2 f}{\partial x \partial u}(t, x_h, u) y_h(t)v(t) + \frac{\partial^2 f}{(\partial u)^2}(t, x_h, u) v^2(t) \right) \phi(t) \, dt \\
+ \int_0^T \frac{\partial f}{\partial x}(t, x_h, u) z_h(t) \phi(t) \, dt.
\]
Now a straightforward computation gives
\[ j''_h(u)(v, v) = \frac{d^2}{ds^2} \int_0^T g(t, x_h^s(t), u(t) + sv(t)) dt \bigg|_{s=0} \]
\[ = \int_0^T \frac{\partial g}{\partial x}(t, x_h(t), u(t)) z_h(t) dt + \int_0^T \frac{\partial^2 g}{(\partial x)^2}(t, x_h(t), u(t)) y_h^2(t) dt \]
\[ + \int_0^T 2 \frac{\partial^2 g}{\partial x \partial u}(t, x_h(t), u(t)) y_h(t)v(t) dt + \int_0^T \frac{\partial^2 g}{(\partial u)^2}(t, x_h(t), u(t)) v^2(t) dt. \]

Note that the discrete adjoint state \( \lambda_h(t) = \lambda_h(u)(t) \) satisfies
\[-B(\psi, \lambda_h) + \left( \frac{\partial f}{\partial x}(t, x_h, u) \lambda_h, \psi \right)_I = \left( \frac{\partial f}{\partial x}(t, x_h, u), \psi \right)_I \]
for all \( X_h^r \). Thus by considering \( \psi = z_h \in X_h^r \), we find
\[ \left( \frac{\partial g}{\partial x}(t, x_h, u), z_h \right)_I \]
\[ = -B(z_h, \lambda_h) + \left( \frac{\partial f}{\partial x}(t, x_h, u) \lambda_h, z_h \right)_I \]
\[ = -\int_0^T \left( \frac{\partial^2 f}{(\partial x)^2}(t, x_h, u) y_h^2(t) + 2 \frac{\partial^2 f}{\partial x \partial u}(t, x_h, u) y_h(t)v(t) + \frac{\partial^2 f}{(\partial u)^2}(t, x_h, u)v^2(t) \right) \lambda_h(t) dt. \]

Combining the above equalities, we find that
\[ j''_h(u)(v, v) \]
\[ = -\int_0^T \left( \frac{\partial^2 f}{(\partial x)^2}(t, x_h, u) y_h^2(t) + 2 \frac{\partial^2 f}{\partial x \partial u}(t, x_h, u) y_h(t)v(t) + \frac{\partial^2 f}{(\partial u)^2}(t, x_h, u)v^2(t) \right) \lambda_h(t) dt \]
\[ + \int_0^T \left( \frac{\partial^2 g}{(\partial x)^2}(t, x_h, u) y_h^2(t) + 2 \frac{\partial^2 g}{\partial x \partial u}(t, x_h, u) y_h(t)v(t) + \frac{\partial^2 g}{(\partial u)^2}(t, x_h, u)v^2(t) \right) dt. \]

This completes the proof. \( \square \)

**Appendix C. Lipschitz estimates**

We recall from [15, Lemma 2.4] the following lemma.

**Lemma C.1.** Let \( I = (a, b) \) and \( k = b - a > 0 \). Then
\[ \int_a^b |\phi(t)|^2 dt \leq \frac{1}{k} \sum_{i=1}^d \left( \int_a^b \phi_i(t) dt \right)^2 + \frac{1}{2} \int_a^b (b - t)(t - a)|\phi'(t)|^2 dt \]
for all \( \phi(t) = (\phi_1(t), \cdots, \phi_d(t)) \in P^r((a, b); \mathbb{R}^d) \), \( r \in \mathbb{N}_0 \).

The next result is from [15, Lemma 3.1].

**Lemma C.2.** For \( I = (a, b) \) and \( r \in \mathbb{N}_0 \), we have
\[ \|\phi\|^2_{L^2(I)} \leq C \log(r + 1) \int_a^b |\phi'(t)|^2(t - a) dt + C|\phi(b)|^2 \]
for all \( \phi(t) = (\phi_1(t), \cdots, \phi_d(t)) \in P^r((a, b); \mathbb{R}^d) \). Here \( C > 0 \) is independent of \( r, a, b, \) and \( d \).
We shall use the following Gronwall inequality.

**Lemma C.3.** Let \( \{a_n\}_{n=1}^N \) and \( \{b_n\}_{n=1}^N \) be sequences of non-negative numbers with \( b_1 \leq b_2 \leq \cdots \leq b_N \). Assume that for a value \( h \in (0,1/2) \) we have

\[
(1-h)b_{n+1} \leq b_n + a_n
\]

for \( n \in \mathbb{N} \). Fix a value \( M \in \mathbb{N} \). Then there exists a constant \( C_\infty > 0 \) independent of \( h \in (0,1/2) \) such that

\[
b_n \leq e^{Cnh} \sum_{k=1}^n a_k
\]

for any \( n \in \mathbb{N} \) with \( n \leq N/h \).

**Proof.** The proof is obtained by applying an induction. \( \square \)

**Lemma C.4.** There exists a constant \( C > 0 \) independent of \( h > 0 \) such that

\[
\|G_h(u_1) - G_h(u_2)\|_{L^\infty(I)} \leq C\|u_1 - u_2\|_{L^2(I)}
\]

for all \( u_1, u_2 \in \mathcal{U}_{ad} \) and \( h > 0 \) small enough.

**Proof.** We note that

\[
\sum_{n=1}^N ((x_1 - x_2)'(t), \phi(t))_{I_n} - \sum_{n=1}^N (f(t, x_1(t), u_1(t)) - f(t, x_2(t), u_2(t)), \phi(t))_{I_n}
\]

\[
+ \sum_{n=2}^N ([x_1 - x_2]_{n-1}, \phi_{n-1}^+)_{I_n} + ([x_{10} - x_{20}]^+, \phi_0^+)_{I_1} = 0.
\]

To obtain the desired estimates, for each \( n \in \{1, \cdots, N\} \) we shall take the following test functions \( \phi \in X_f^N \) supported on \( I_n \) given as

\[
\phi(t) = (x_1 - x_2)(t)1_{I_n}(t),
\]

\[
\phi(t) = (t-t_{n-1})(x_1 - x_2)'(t)1_{I_n}(t), \quad \text{and}
\]

\[
\phi(t) = (t-t_{n-1})1_{I_n}(t),
\]

where \( 1_{I_n} : I \to \{0,1\} \) denotes the indicator function, that is, \( 1_{I_n}(t) = 1 \) for \( t \in I_n \) and \( 1_{I_n}(t) = 0 \) for \( t \in I \setminus I_n \). First we take \( \phi(t) = (x_1 - x_2)(t)1_{I_n}(t) \) for \( n = 1, 2, \cdots, N \). Then,

\[
((x_1 - x_2)'(t), (x_1 - x_2)(t))_{I_n} - (f(t, x_1(t), u_1(t)) - f(t, x_2(t), u_2(t)), (x_1 - x_2)(t))_{I_n}
\]

\[
+ ([x_1 - x_2]_{n-1}, (x_1 - x_2)^+_{n-1}) = 0.
\]

(C.1)

Notice that

\[
([x_1 - x_2]_{n-1}, (x_1 - x_2)^+_{n-1}) = ((x_1 - x_2)^+_{n-1})^2 - (x_1 - x_2)^+_{n-1}, (x_1 - x_2)^+_{n-1})
\]

Using this in (C.1), we find

\[
\frac{1}{2}|(x_1 - x_2)_{n-1}|^2 - \frac{1}{2}|(x_1 - x_2)^+_{n-1}|^2 + |(x_1 - x_2)^+_{n-1}|^2
\]

\[
= ((x_1 - x_2)_{n-1}, (x_1 - x_2)^+_{n-1}) + (f(t, x_1(t), u_1(t)) - f(t, x_2(t), u_2(t)), (x_1 - x_2)(t))_{I_n}.
\]
By applying Cauchy-Schwarz inequality, we obtain
\[
\frac{1}{2}|(x_1 - x_2)^n| \leq \frac{1}{2}|(x_1 - x_2)^{n-1}|^2 + \langle f(t, x_1(t), u_1(t)) - f(t, x_2(t), u_2(t)), (x_1 - x_2)(t)\rangle_{I_n}.
\]

By using the assumption on \( f \) in (1.3), we further estimate
\[
\frac{1}{2}|(x_1 - x_2)^n| \leq \frac{1}{2}|(x_1 - x_2)^{n-1}|^2 + C\|(x_1 - x_2)\|_{L^2(I_n)}^2 + C\|(u_1 - u_2)\|_{L^2(I_n)}^2. \tag{C.2}
\]

Secondly, we take \( \phi(t) = (t - t_{n-1})(x_1 - x_2)'(t)1_{I_n}(t) \) to have
\[
\langle (x_1 - x_2)'(t), (t - t_{n-1})(x_1 - x_2)'(t)\rangle_{I_n} = \langle f(t, x_1(t), u_1(t)) - f(t, x_2(t), u_2(t)), (t - t_{n-1})(x_1 - x_2)'(t)\rangle_{I_n}.
\]

By using Hölder’s inequality, we get
\[
\int_{I_n} (t - t_{n-1})|(x_1 - x_2)'(t)|^2 \, dt \leq \int_{I_n} |t - t_{n-1}|\|(x_1 - x_2)(t)\|^2 + \|(u_1 - u_2)(t)\|^2 \, dt. \tag{C.3}
\]

Notice that
\[
\langle (x_1 - x_2)'(t), (t - t_{n-1})\rangle_{I_n} = -\int_{I_n} (x_1 - x_2)(t) \, dt + (x_1 - x_2)(t_n)(t_n - t_{n-1}).
\]

Thus, choosing \( \phi(t) = (t - t_{n-1})1_{I_n}(t) \) gives
\[
\int_{I_n} (x_1 - x_2)(t) \, dt + (x_1 - x_2)(t_n)(t_n - t_{n-1})
= -\int_{I_n} (f(t, x_1(t), u_1(t)) - f(t, x_2(t), u_2(t))) (t - t_{n-1}) \, dt,
\]
and subsequently, this yields
\[
\left| \int_{I_n} (x_1 - x_2)(t) \, dt \right|^2
\leq 2h_n^2 \left[(x_1 - x_2)^n\right]^2 + 2\int_{I_n} \|(f_1 - f_2)(t)\|^2 \, dt \int_{I_n} (t_n - t)^2 \, dt
\leq 2h_n^2 \left[(x_1 - x_2)^n\right]^2 + Ch_n^3 \int_{I_n} \left(\|(x_1 - x_2)(t)\|^2 + \|(u_1 - u_2)(t)\|^2\right) \, dt,
\]
where \( h_n = t_n - t_{n-1} \) and \( f_i(t) = f(t, x_i(t), u_i(t)) \) for \( i = 1, 2 \). This together with Lemma [C.1] asserts
\[
\left| \int_{I_n} (x_1 - x_2)(t) \, dt \right|^2
\leq 2h_n^2 \left[(x_1 - x_2)^n\right]^2 + Ch_n^2 \int_{I_n} (t - t_{n-1})\|(x_1 - x_2)'(t)\|^2 \, dt
+ Ch_n^3 \int_{I_n} \|(u_1 - u_2)(t)\|^2 \, dt \tag{C.4}
\]
for $h > 0$ small enough. Combining (C.2) and (C.3), we find
\[
\int_{I_n} (t-t_{n-1})|(x_1-x_2)'(t)|^2 \, dt + |(x_1-x_2)_{n-1}|^2 \\
\leq C\|x_1-x_2\|^2_{L^2(I_n)} + C \int_{I_n} |(u_1-u_2)(t)|^2 \, dt + |(x_1-x_2)_{n-1}|^2 \\
\leq \frac{C}{h_n} \int_{I_n} (x_1-x_2)(t) \, dt |^2 + Ch_n \int_{I_n} (t-t_{n-1})|(x_1-x_2)'(t)|^2 \, dt + |(x_1-x_2)_{n-1}|^2 + C \int_{I_n} |(u_1-u_2)(t)|^2 \, dt,
\]
where we applied Lemma C.1 in the second inequality. This, together with (C.4), we obtain
\[
\int_{I_n} (t-t_{n-1})|(x_1-x_2)'(t)|^2 \, dt + |(x_1-x_2)_{n-1}|^2 \\
\leq 2h_n|(x_1-x_2)_{n-2}|^2 + |(x_1-x_2)_{n-1}|^2 + C \int_{I_n} |(u_1-u_2)(t)|^2 \, dt
\]
for $h > 0$ small enough. Now we apply Lemma C.3 to find
\[
\int_{I_n} (t-t_{n-1})|(x_1-x_2)'(t)|^2 \, dt + |(x_1-x_2)_{n-1}|^2 \leq C \int_{0}^{T} |(u_1-u_2)(t)|^2 \, dt.
\]
Finally, by applying Lemma C.2 to the above, we obtain the desired estimate. \qed

**Appendix D. Taylor Expansion of the Solutions to ODEs**

Here we give some rigorous proofs for the Taylor expansion of the solutions to ODEs and its discretized version, which are used in Lemma B.1 and Lemma B.2.

**Lemma D.1.** Let $x^s = G(u + sv)$.

1. Then $x^s$ can be written as
   \[
   x^s(t) = x(t) + sy(t) + r(t),
   \]
   where $r(t)$ satisfies $|r(t)| \leq Cs^2$ for $t \in [0, T]$ with a constant $C > 0$ independent of $s$.

2. Then $x^s$ can be written as
   \[
   x^s(t) = x(t) + sy(t) + \frac{s^2}{2}z(t) + r(t),
   \]
   where $r(t)$ satisfies $|r(t)| \leq Cs^3$ for $t \in [0, T]$ with a constant $C > 0$ independent of $s$.

**Proof.** Recall that $x^s$, $x$, and $y$ satisfy
\[
(x^s)'(t) = f(t, x^s(t), u(t) + sv(t)),
\]
\[
x'(t) = f(t, x(t), u(t)),
\]
and
\[
y'(t) = \frac{\partial f}{\partial x}(t, x(t), u(t)) + \frac{\partial f}{\partial u}(t, x(t), u(t))v(t),
\]
\[
y'(t) = \frac{\partial f}{\partial x}(t, x(t), u(t)) + \frac{\partial f}{\partial u}(t, x(t), u(t))v(t),
\]
\[
y'(t) = \frac{\partial f}{\partial x}(t, x(t), u(t)) + \frac{\partial f}{\partial u}(t, x(t), u(t))v(t),
\]
respectively. Then, we find that
\[
\begin{align*}
r'(t) &= f(t, R(t) + x(t) + sy(t), u(t) + sv(t)) - f(t, x(t), u(t)) \\
&= f(t, r(t) + x(t) + sy(t), u(t) + sv(t)) - f(t, x(t) + sy(t), u(t) + sv(t)) \\
&
+ f(t, x(t) + sy(t), u(t) + sv(t)) - s \left( \frac{\partial f}{\partial x}(t, x(t), u(t))y(t) + \frac{\partial f}{\partial u}(t, x(t), u(t))v(t) \right) \\
&\leq C|r(t)| + O(s^3).
\end{align*}
\]

Now, by applying the Gronwall inequality, we deduce \( r(t) = O(s^2) \) for \( t \in [0, T] \). This proves the estimate (D.1). Similar argument can be applied to prove the estimate (D.2).

Next we consider the discretized version.

**Lemma D.2.** Let \( x_h^s = G_h(u + sv) \).

1. Then \( x^s \) can be written as
\[
\begin{align*}
x^s_h(t) &= x_h(t) + sy_h(t) + r(t), \\
&\quad \text{(D.3) where } r(t) \text{ satisfies } |r(t)| \leq Cs^2 \text{ for } t \in [0, T] \text{ with a constant } C > 0 \text{ independent of } s.
\end{align*}
\]

2. Then \( x^s \) can be written as
\[
\begin{align*}
x^s_h(t) &= x_h(t) + sy_h(t) + \frac{s^2}{2} z(t) + r(t), \\
&\quad \text{(D.4) where } r(t) \text{ satisfies } |r(t)| \leq Cs^3 \text{ for } t \in [0, T] \text{ with a constant } C > 0 \text{ independent of } s.
\end{align*}
\]

**Proof.** We recall that
\[
\begin{align*}
B(x^s_h, \phi) &= (f(t, x^s_h(t), u + sv), \phi)_I, \\
B(x_h, \phi) &= (f(t, x_h(t), u), \phi)_I, \quad \text{and} \\
B(y_h, \phi) &= \left( \frac{\partial f}{\partial x}(t, x_h(t), u)y_h + \frac{\partial f}{\partial u}(t, x_h(t), u(t))v(t), \phi \right)_I.
\end{align*}
\]

Now we set \( z_h = x^s_h - x_h - sy_h \). Then,
\[
\begin{align*}
B(z_h, \phi) &= (f(z_h + x_h + sy_h, u + sv, t) - f(x_h + sy_h, u + sv, t), \phi)_I \\
&\quad - \left( f(x_h + sy_h, u + sv, t) - f(x_h, u, t) \\
&\quad - s \left( \frac{\partial f}{\partial x}(t, x_h(t), u(t))y_h + \frac{\partial f}{\partial u}(t, x_h(t), u(t)) \right), \phi \right)_I \\
&= (O(z_h), \phi)_I + (O(s^2), \phi)_I.
\end{align*}
\]
At this stage, arguing as in the proof of Lemma 4.3 we obtain \( z_h(t) = O(s^2) \) for \( t \in [0, T] \). This gives the desired estimate (D.3). The estimate (D.4) can be obtained in a similar way. □

References

[1] W. Alt, On the approximation of infinite optimization problems with an application to optimal control problems, Appl. Math. Optim., 12, (1984), 15–27.
[2] W. Alt, U. Felgenhauer, and M. Seydenschwanz, Euler discretization for a class of nonlinear optimal control problems with control appearing linearly, Comput. Optim. Appl., 69, (2018), 825–856.
[3] M. Baccouch, Analysis of a posteriori error estimates of the discontinuous Galerkin method for nonlinear ordinary differential equations, Appl. Numer. Math., 106, (2016), 129–153.
[4] M. Delfour, W. Hager, and F. Trochu, Discontinuous Galerkin methods for ordinary differential equations, Math. Comp., 36, (1981), 455–473.
[5] A. L. Dontchev and W. W. Hager, Lipschitzian stability in nonlinear control and optimization, SIAM J. Control Optim., 31, (1993), 569–603.
[6] A. L. Dontchev and W. W. Hager, The Euler approximation in state constrained optimal control, Math. Comp., 70, (2000), 173–203.
[7] G. Elmagar, M. A. Kazemi, M. Razzaghi, The pseudospectral Legendre method for discretizing optimal control problems, IEEE T. Automat. Contr., 40, (1995), 1793–1796.
[8] D. Estep, A posteriori error bounds and global error control for approximation of ordinary differential equations, SIAM J. Numer. Anal., 32, (1995), 1–48.
[9] U. Felgenhauer, On stability of bang-bang type controls, SIAM J. Control Optim., 41, (2003), 1843–1867.
[10] J. Henriques, J. Lemos, J.Eca, L. Gato, A. Falcão, A high-order discontinuous Galerkin method with mesh refinement for optimal control, Automatica, 85, (2017), 70–82.
[11] I. Neitzel and B. Vexler, A priori error estimates for space-time finite element discretization of semilinear parabolic optimal control problems, Numer. Math., 120, (2012), 345–386.
[12] N. P. Osmolovskii and H. Maurer, Equivalence of second order optimality conditions for bang-bang control problems. Par 1: main results, Control Cybern., 34, (2005), 927–950.
[13] N. P. Osmolovskii and H. Maurer, Equivalence of second order optimality conditions for bang-bang control problems. Par 2: proofs, variational derivatives and representations, Control Cybern., 36, (2007), 5–45.
[14] I. M. Ross, M. Karpenko, A review of pseudospectral optimal control: From theory to flight. Annual Reviews in Control, 36, (2012), 182–197.
[15] D. Schötzau and C. Schwab, An hp a priori error analysis of the DG time-stepping method for initial value problems, Calcolo, 37, (2000), 207–232.
[16] J. Vlassenbroeck and R. Van Dooren, A Chebyshev technique for solving nonlinear optimal control problems, IEEE T. Automat. Contr., 33, (1988), 333–349.

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