The Tractability of SHAP-scores over Deterministic and Decomposable Boolean Circuits

Marcelo Arenas¹,³, Pablo Barceló²,³, Leopoldo Bertossi³,⁴, and Mikaël Monet³

¹Department of Computer Science, Universidad Católica de Chile
²Institute for Mathematical and Computational Engineering, Universidad Católica de Chile
³IMFD Chile
⁴Universidad Adolfo Ibáñez & Data Observatory Foundation

July 29, 2020

Abstract. Scores based on Shapley values are currently widely used for providing explanations to classification results over machine learning models. A prime example of this corresponds to the influential SHAP-score, a version of the Shapley value in which the contribution of a set $S$ of features from a given entity $e$ over a model $M$ is defined as the expected value in $M$ of the set of entities $e'$ that coincide with $e$ over all features in $S$. While in general computing Shapley values is a computationally intractable problem, it has recently been claimed that the SHAP-score can be computed in polynomial time over the class of decision trees. In this paper, we provide a proof of a stronger result over Boolean models: the SHAP-score can be computed in polynomial time over deterministic and decomposable Boolean circuits, also known as tractable probabilistic circuits. Such circuits encompass a wide range of Boolean circuits and binary decision diagrams classes, including binary decision trees and Ordered Binary Decision Diagrams (OBDDs). Moreover, we establish the computational limits of the notion of SHAP-score by showing that computing it over a class of Boolean models is always (polynomially) as hard as the model counting problem for this class (under some mild condition). This implies, for instance, that computing the SHAP-score for DNF propositional formulae is a $\#P$-hard problem, and, thus, that determinism is essential for the circuits that we consider.

1 Introduction

The Shapley value is a game theory notion first introduced by Shapley in 1953, and whose goal is to quantify the contribution of a player to a coalition game [16, 15]. Since then, it has been applied in many disciplines such as economy, politics, social choices theory, and computer science. In computer science, for instance, the Shapley value has been used in knowledge representation to measure the degree of inconsistency of a propositional knowledge base [8]; in data management to measure the contribution of a tuple to a query answer [9]; in network analysis to quantify the degree of centrality of nodes [12]; in bioinformatics to identify the most relevant genes in a co-expression network [3], and so on.

More recently, the Shapley value has been used in machine learning to provide explanations for the outcomes of classification models, on the basis of numerical scores assigned to the participating feature values [11]. It is this setting that we will be interested in. As an example, let us imagine a journal editor that uses a learned classifier $M$ (e.g., a decision tree) to help determine if papers should be accepted or not (because there are too many submissions); given a paper, denoted $e$, the classifier $M$ labels $e$ with $M(e) = 1$ if the paper should be accepted, and $M(e) = 0$ if it should be rejected. Now, some researchers submit their work $e$, and the classifier returns $M(e) = 0$. Naturally, the authors request an explanation. A common approach in this kind of scenario consists in giving scores to the feature values in $e$, to quantify their relevance to the classification outcome. The higher the score of a feature value,
the more explanatory that value is. In this example for instance, the fact that the paper has value 4 for feature \textit{Average\_number\_typos\_per\_line} could be given the highest score.

The SHAP-score, a scoring function whose definition is based on the Shapley-value, was proposed for this task and has become particularly influential [10, 11]. For a given classifier $M$, entity $e$ and feature $x$, the SHAP-score $\text{SHAP}(M, e, x)$ intuitively represents the importance of the feature value $e(x)$ to the classification result $M(e)$. In its general formulation, $\text{SHAP}(M, e, x)$ is a weighted average of differences of expected values of the labels (c.f. Section 2 for the formal definitions). Unfortunately, computing quantities that are based on the notion of Shapley-value is in general intractable. Indeed, in many scenarios the computation turns out to be \#P-hard [7, 6, 9], which makes the notion difficult to use (if not impossible) for practical purposes. Therefore, a natural question is whether these intractability results also apply to the SHAP-score, and also determine in which cases the computation can be done efficiently. This is what we do in this paper.

\textbf{Hypotheses.} We focus on classifiers working with binary feature values only (i.e., the value of each feature can be 0 or 1), and that return 1 (accept) or 0 (reject) for each entity. We will call these \textit{Boolean classifiers}. The second assumption that we make is to consider that the probability distribution on the population of entities is the \textit{uniform probability space}. Since we consider only binary feature values, this is the uniform distribution on an outcome space consisting of fixed-length binary sequences. We note here that these two restrictions (considering Boolean classifiers, and the uniform probability space over the inputs) are relevant in many practical scenarios.

\textbf{Boolean classifiers studied.} We study Boolean classifiers defined as \textit{deterministic and decomposable} Boolean circuits, a model that has been widely studied in the field of knowledge compilation [4]. Such circuits encompass a wide range of Boolean circuits and binary decision diagrams classes that are considered in knowledge compilation and in AI more generally. For instance, they generalize binary decision trees, Ordered Binary Decision Diagrams (OBDDs), Free Binary Decision Diagrams (FBDDs), Deterministic and Decomposable Negation Normal Forms (d-DNNFs), etc [4, 1, 5]. These circuits are also known under the name \textit{tractable probabilistic circuits}, which is used in recent literature [20, 17, 18, 19, 21, 13]. For this paper, the reader not familiar with knowledge compilation can simply think about deterministic and decomposable circuits as a powerful tool for showing the tractability of computing \textit{SHAP}-score on several Boolean classifier classes at the same time.

\textbf{Contributions.} In this setting, our main contribution is to show that for classifiers given as deterministic and decomposable circuits, the computation of the \textit{SHAP}-score is tractable. Formally, we prove:

\textbf{Theorem 1.1.} The following problem can be solved in polynomial time. Given as input a deterministic and decomposable circuit $C$ over a set of features $X$, an entity $e : X \rightarrow \{0, 1\}$, and a feature $x \in X$, compute the value $\text{SHAP}(C, e, x)$.

In particular, since binary decision trees, OBDDs, FBDDs and d-DNNFs are all restricted kinds of deterministic and decomposable circuits, we obtain as a consequence of Theorem 1.1 that this problem is also in polynomial time for these classes. For instance, for binary decision trees we obtain:

\textbf{Corollary 1.2.} The following problem can be solved in polynomial time. Given as input a binary decision tree $T$ over a set of features $X$, an entity $e : X \rightarrow \{0, 1\}$, and a feature $x \in X$, compute the value $\text{SHAP}(T, e, x)$.

We note that this corollary recaptures and proves a recent result of [10], where the authors claimed without proof that this task is tractable for decision trees. In addition, we point out that Theorem 1.1 is a nontrivial extension of the result for decision trees, as it is known that deterministic and decomposable circuits can be exponentially more succinct than binary decision trees (in fact, than FBDDs) at representing Boolean classifiers [4, 1].

An important observation is that the determinism assumption in Theorem 1.1 is necessary for obtaining tractability. Indeed, as stated next, the problem of computing \textit{SHAP}(\alpha, e, x), when $\alpha$ is a propositional formula in Disjunctive Normal Form (DNF), is \#P-hard. Such formulae are in fact restricted kinds of decomposable circuits.
Proposition 1.3. The following problem is \#P-hard. Given as input a DNF formula $\alpha$ over a set of features $X$, an entity $e : X \to \{0, 1\}$, and a feature $x \in X$, compute the value $\text{SHAP}(\alpha, e, x)$.

In fact, we prove Proposition 1.3 by showing a more general result: computing the SHAP-score over a class of Boolean models is always polynomially as hard as the model counting problem for that class, under some mild condition; see Lemma 4.1 for the details. Since counting the number of models of DNF formulae is a \#P-hard problem [14], this establishes Proposition 1.3.

Paper structure. We give preliminaries in Section 2, where we formally define the notions of Boolean classifier and of SHAP-score, as well as the main Boolean classifier classes that we consider. In Section 3, we prove that the SHAP-score can be computed in polynomial time for deterministic and decomposable Boolean circuits. In Section 4, we show that computing the SHAP-score is as hard as the model counting problem.

2 Preliminaries

2.1 Entities and Boolean classifiers

Let $X$ be a finite set of features, also called variables. An entity over $X$ is a function $e : X \to \{0, 1\}$. We denote by $\text{ent}(X)$ the set of all entities over $X$.

Example 2.1. Let $X = \{\text{Is\_well\_formatted}, \text{Is\_well\_motivated}, \text{Contains\_typos}\}$ be a set of features. A particular paper could be represented by an entity $e$ such that $e(\text{Is\_well\_formatted}) = 0$, $e(\text{Is\_well\_motivated}) = 1$, $e(\text{Contains\_typos}) = 0$. Then $\text{ent}(X)$ is the set containing all $2^3 = 8$ different entities over $X$.

A Boolean classifier $M$ over $X$ is simply a function $M : \text{ent}(X) \to \{0, 1\}$ that maps every entity over $X$ to 0 or 1. We say that $M$ accepts an entity $e$ when $M(e) = 1$, and that it rejects it if $M(e) = 0$.

2.2 The Shapley value of a Boolean classifier

Let $M$ be a Boolean classifier over a set $X$ of features. Consider the uniform probability distribution over the set $\text{ent}(X)$ of entities, and notice that $M : \text{ent}(X) \to \{0, 1\}$ can be considered as a random variable. Moreover, given an entity $e$ over $X$ and a subset $S \subseteq X$ of features, define $\text{cons}(e, S)$ as the set of all entities that coincide with $e$ over each feature in $S$, that is, $\text{cons}(e, S) := \{e' \in \text{ent}(X) \mid e'(x) = e(x)\}$ for each $x \in S$.\footnote{"cons" stands for consistent.} Then, given an entity $e \in \text{ent}(X)$ and a subset $S \subseteq X$ of features, define the expected value of $S$ in $e$ with respect to $M$ as

$$
\phi(M, e, S) := \mathbb{E}[M \mid \text{cons}(e, S)] = \sum_{e' \in \text{cons}(e, S)} \frac{1}{2^{|S|}} M(e').
$$

Intuitively, $\phi(M, e, S)$ corresponds to the probability that $M(e') = 1$ holds, conditioned on the inputs $e' \in \text{ent}(X)$ to coincide with $e$ over each feature in $S$.

Definition 2.2. Given a Boolean classifier $M$ over a set of features $X$, an entity $e$ over $X$, and a feature $x \in X$, the Shapley value of feature $x$ in $e$ with respect to $M$ is defined as

$$
\text{SHAP}(M, e, x) := \sum_{S \subseteq X \setminus \{x\}} \frac{|S|! \cdot (|X| - |S| - 1)!}{|X|!} \left( \phi(M, e, S \cup \{x\}) - \phi(M, e, S) \right).
$$

(1)

Observe that, from the definitions, a high positive value of $\text{SHAP}(M, e, x)$ intuitively means that setting $x$ to $e(x)$ strongly leans the classifier towards acceptance, while a high negative value of $\text{SHAP}(M, e, x)$ means that setting $x$ to $e(x)$ strongly leans the classifier towards rejection.
2.3 Representation of Boolean classifiers

**Binary Decision Diagrams.** A binary decision diagram (BDD) over a set of variables \( X \) is a rooted directed acyclic graph \( D \) such that: (i) each internal node is labeled with a variable from \( X \), and has exactly two outgoing edges: one labeled 0, the other one labeled 1; and (ii) each leaf is labeled either 0 or 1. Such a BDD represents a Boolean classifier in the following way. Let \( e \) be an entity over \( X \), and let \( \pi_e = u_1, \ldots, u_m \) be the unique path in \( D \) satisfying the following conditions: (a) \( u_1 \) is the root of \( D \); (b) \( u_m \) is a leaf of \( D \); and (c) for every \( i \in \{1, \ldots, m-1\} \), if the label of \( u_i \) is \( x \in X \), then the label of the edge \((u_i, u_{i+1})\) is equal to \( \ell(x) \). Then the value of \( e \) in \( D \), denoted by \( D(e) \), is defined as the label of the leaf \( u_m \). Moreover, a binary decision diagram \( D \) is free (FBDD) if for every path from the root to a leaf, no two nodes on that path have the same label, and a binary decision tree is an FBDD whose underlying graph is a tree.

**Boolean circuits.** Boolean classifiers can also be represented as Boolean circuits. More precisely, a Boolean circuit over a set of variables \( X \) is a directed acyclic graph \( C \) such that

(i) Every node without incoming edges is either a variable gate or a constant gate. A variable gate is labeled with a variable from \( X \), and a constant gate is labeled with 0 or 1.

(ii) Every node with incoming edges is called a logic gate, and is labeled with a symbol \( \land, \lor \) or \( \neg \). If such a node is labeled with the symbol \( \neg \), then it has exactly one incoming edge;\(^2\)

(iii) Exactly one node does not have any outgoing edges, and this node is called the output gate of \( C \).

Such a Boolean circuit \( C \) represents a Boolean classifier in the following way. Let \( e \) be an entity over the set of variables \( X \). Then, for every node \( g \) of \( C \), we define its value \( \text{val}(g, e) \) as follows. If \( g \) is a variable gate with label \( x \in X \), then \( \text{val}(g, e) = e(x) \), and if \( g \) is a constant gate with label \( \ell \in \{0, 1\} \), then \( \text{val}(g, e) = \ell \). Otherwise, \( g \) is a logic gate with incoming edges \( (g_1, g), \ldots, (g_m, g) \), with \( m \geq 1 \) and \( g_1, \ldots, g_m \) called input gates of \( g \); and \( \text{val}(g, e) \) is then defined as follows. If the label of \( g \) is \( \neg \), then \( m = 1 \) and \( \text{val}(g, e) = 1 - \text{val}(g_1, e) \). If the label of \( g \) is \( \land \), then \( \text{val}(g, e) = \min\{\text{val}(g_1, e), \ldots, \text{val}(g_m, e)\} \). If the label of \( g \) is \( \lor \), then \( \text{val}(g, e) = \max\{\text{val}(g_1, e), \ldots, \text{val}(g_m, e)\} \). Finally, the value of \( e \) in \( C \), denoted by \( C(e) \), is defined as \( \text{val}(g_{\text{output}}, e) \), where \( g_{\text{output}} \) is the output gate of \( C \).

Several restriction of Boolean circuits have been studied. Let \( C \) be a Boolean circuit over a set of variables \( X \) and \( g \) a gate of \( C \). The Boolean circuit \( C_g \) over \( X \) is defined by considering the subgraph of \( C \) induced by the set of gates \( g' \) of \( C \) for which there exists a path from \( g' \) to \( g \) in \( C \). Notice that \( g \) is the output gate of \( C_g \). The set \( \text{var}(g) \) is defined as the set of variables \( x \in X \) such that there exists a path from a variable gate with label \( x \) to \( g \) in \( C \). Then, an \( \lor \)-gate \( g \) of \( C \) is said to be deterministic if for every pair \( g_1, g_2 \) of distinct input gates of \( g \), the Boolean circuits \( C_{g_1} \) and \( C_{g_2} \) are disjoint in the sense that there is no entity \( e \) that is accepted by both \( C_{g_1} \) and \( C_{g_2} \) (that is, there is no entity \( e \in \text{ent}(X) \) such that \( C_{g_1}(e) = C_{g_2}(e) = 1 \)). The circuit \( C \) is called deterministic if every \( \lor \)-gate of \( C \) is deterministic. An \( \land \)-gate \( g \) of \( C \) is said to be decomposable if for every pair \( g_1, g_2 \) of distinct input gates of \( g \), we have that \( \text{var}(g_1) \cap \text{var}(g_2) = \emptyset \). Then, \( C \) is called decomposable if every \( \land \)-gate of \( C \) is decomposable.

We point out here that deterministic and decomposable Boolean circuits encompass a wide range of binary decision diagrams and Boolean circuits that have been considered in knowledge compilation; in particular, they include binary decision trees, ordered binary decision diagrams (OBDDs), free binary decision diagrams, and structured versions of them. We refer the reader to [4, 1] for detailed studies of knowledge compilation classes and of their precise relationships.

**Example 2.3 (Folklore).** Given an FBDD \( D \) over a set of variables \( X \), we explain how \( D \) can be encoded as a deterministic and decomposable Boolean circuit \( C \) over \( X \). Notice that the technique used in this example also apply to binary decision trees, as they are a particular case of FBDDs. The construction of \( C \) is done by traversing the structure of \( D \) in a bottom-up manner. In particular, for every node \( u \) of \( D \), we construct a deterministic and decomposable circuit \( C(u) \) that is equivalent to the FBDD represented by the subgraph of \( D \) rooted at \( u \). More precisely, for a leaf \( u \) of \( D \) that is labeled with \( \ell \in \{0, 1\} \), we define \( C(u) \) to be the Boolean circuit consisting of only one constant gate with label \( \ell \). For an internal

---

\(^2\)Recall that the fan-in of a gate is the number of its input gates. In our definition of Boolean circuits, we allow unbounded fan-in \( \land \) and \( \lor \)-gates.
node $u$ of $D$ labeled with variable $x \in X$, let $u_0$ and $u_1$ be the nodes that we reach from $u$ by following the 0- and 1-labeled edge, respectively. Then $\alpha(u)$ is the Boolean circuit depicted in the following figure:

For the $\lor$-gate shown in the figure, if an entity $e$ is accepted by the Boolean circuit in its left-hand size, then $e(x) = 0$, while if an entity $e$ is accepted by the Boolean circuit in its right-hand size, then $e(x) = 1$. Hence, we have that this $\lor$-gate is deterministic, from which we conclude that $\alpha(u)$ is deterministic, as $\alpha(u_0)$ and $\alpha(u_1)$ are also deterministic by construction. Moreover, the $\land$-gates shown in the figure are decomposable as variable $x$ is mentioned neither in $\alpha(u_0)$ nor in $\alpha(u_1)$: this is because $D$ is a free BDD. Thus, we conclude that $\alpha(u)$ is decomposable, as $\alpha(u_0)$ and $\alpha(u_1)$ are decomposable by construction. Finally, if $u_{\text{root}}$ is the root of $D$, then by construction we have that $\alpha(u_{\text{root}})$ is a deterministic and decomposable Boolean circuit equivalent to $D$. Note that this encoding can trivially be done in linear time. Thus, we often say, by abuse of terminology, that “FBDDs (or binary decision trees) are restricted kinds of deterministic and decomposable circuits”.

**Proviso.** By slightly abusing notation, if a BDD $D$ (resp., a Boolean Circuit $C$) represents a Boolean classifier $M$, then we use term $\text{SHAP}(D, e, x)$ (resp., $\text{SHAP}(C, e, x)$) to represent $\text{SHAP}(M, e, x)$.

## 3 Computing SHAP-score for deterministic and decomposable circuits

In this section, we prove our main tractability result, namely, that computing SHAP-scores for Boolean classifiers given as deterministic and decomposable Boolean circuits can be done in polynomial time; see Theorem 1.1 for the formal statement. First, we need to introduce some notation. Let $M$ be a Boolean classifier over a set of features $X$. We write $\text{SAT}(M) \subseteq \text{ent}(X)$ for the set of entities that are accepted by $M$, and $\#\text{SAT}(M)$ for the cardinality of this set. Let $e, e' \in \text{ent}(X)$ be a pair of entities over $X$. We define $\text{sim}(e, e')$ to be the set of features on which $e$ and $e'$ coincide, that is, $\text{sim}(e, e') := \{x \in X \mid e(x) = e'(x)\}$. Given a Boolean classifier $M$ over $X$, an entity $e \in \text{ent}(X)$ and a natural number $k \leq |X|$, we define the set $\text{SAT}(M, e, k) := \text{SAT}(M) \cap \{e' \in \text{ent}(X) \mid |\text{sim}(e, e')| = k\}$, in other words, the set of entities $e'$ that are accepted by $M$ and which coincide with $e$ in exactly $k$ features. Naturally, we write $\#\text{SAT}(M, e, k)$ for the size of $\text{SAT}(M, e, k)$.

**Example 3.1.** Continuing Example 2.1, let us consider the Boolean classifier $M$ that accepts an entity $e$ (representing a paper) if and only if it is well motivated, that is, if and only if $e(\text{Is\_well\_motivated}) = 1$. Then $\text{SAT}(M)$ is the set containing all the papers that are well motivated, so $\#\text{SAT}(M) = 4$. Now, consider the entity $e$ from Example 2.1. One should check that $\#\text{SAT}(M, e, 0) = 0$, $\#\text{SAT}(M, e, 1) = 1$, $\#\text{SAT}(M, e, 2) = 2$ and $\#\text{SAT}(M, e, 3) = 1$.

Our proof of Theorem 1.1 is technical and is divided into two modular parts. The first part, which is developed in Section 3.1, consists in showing that the problem of computing $\text{SHAP}(-, -, -)$ can be reduced in polynomial time to that of computing $\#\text{SAT}(-, -, -)$. This part of the proof is a sequence of formula manipulations, and it only uses the fact that deterministic and decomposable circuits can be efficiently conditioned on a variable value (to be defined in Section 3.1). In the second part of the proof, which is developed at Section 3.2, we show that computing $\#\text{SAT}(-, -, -)$ can be done in polynomial time for deterministic and decomposable Boolean circuits. It is in this part that the magic of deterministic and decomposable circuits really operates.
3.1 Reducing in polynomial-time from \( \text{SHAP}(\cdot, \cdot, \cdot) \) to \( \#\text{SAT}(\cdot, \cdot, \cdot) \)

In this section, we show that for deterministic and decomposable Boolean circuits, the computation of the SHAP-score can be reduced in polynomial time to the computation of \( \#\text{SAT}(\cdot, \cdot, \cdot) \). To achieve this, we will need two more definitions. Let \( M \) be a Boolean classifier over a set of features \( X \) and \( x \in X \), and let Boolean classifiers \( M_{+x} : \text{ent}(X \setminus \{x\}) \to \{0, 1\} \) and \( M_{-x} : \text{ent}(X \setminus \{x\}) \to \{0, 1\} \) be defined as follows. For \( e \in \text{ent}(X \setminus \{x\}) \), let us write \( e_{+x} \) and \( e_{-x} \) the entities over \( X \) such that \( e_{+x}(x) = 1 \), \( e_{-x}(x) = 0 \) and \( e_{+x}(y) = e_{-x}(y) = e(y) \) for every \( y \in X \setminus \{x\} \). Then define \( M_{+x}(e) := M(e_{+x}) \) and \( M_{-x}(e) := M(e_{-x}) \).

In the literature, \( M_{+x} \) (resp., \( M_{-x} \)) is called the conditioning by \( x \) (resp., by \( \neg x \)) of \( M \). It is clear that conditioning can be done in linear time for a Boolean circuit \( C \) by replacing every gate with label \( x \) by a constant gate with label 1 (resp., 0). We write \( C_{+x} \) (resp., \( C_{-x} \)) the Boolean circuit obtained via this transformation. It is easy to check that, if \( C \) is a deterministic and decomposable Boolean circuit, then \( C_{+x} \) and \( C_{-x} \) are deterministic and decomposable as well.

We now introduce the second definition that we will need. For a Boolean classifier \( M \) over a set of variables \( X \), an entity \( e \in \text{ent}(X) \) and a natural number \( k \leq |X| \), define

\[
H(M, e, k) := \sum_{S \subseteq X \setminus \{x\}} \sum_{|S| = k} M(e').
\]

We first explain in Section 3.1.1 how computing \( \text{SHAP}(\cdot, \cdot, \cdot) \) can be reduced in polynomial time to the problem of computing \( H(\cdot, \cdot, \cdot) \), and then show in Section 3.1.2 that computing \( H(\cdot, \cdot, \cdot) \) can be reduced in polynomial time to computing \( \#\text{SAT}(\cdot, \cdot, \cdot) \).

3.1.1 Reducing from \( \text{SHAP}(\cdot, \cdot, \cdot) \) to \( H(\cdot, \cdot, \cdot) \)

We wish to compute \( \text{SHAP}(C, e, x) \), for a given deterministic and decomposable circuit \( C \) over a set of variables \( X \), entity \( e \in \text{ent}(X) \) and feature \( x \in X \). Define

\[
\text{Diff}_k(C, e, x) := \sum_{S \subseteq X \setminus \{x\}} (\phi(C, e, S \cup \{x\}) - \phi(C, e, S)),
\]

and let \( n = |X| \). We then have:

\[
\text{SHAP}(C, e, x) = \sum_{S \subseteq X \setminus \{x\}} \frac{|S|!(n - |S| - 1)!}{n!} (\phi(C, e, S \cup \{x\}) - \phi(C, e, S))
\]

\[
= \sum_{k=0}^{n-1} \frac{k!(n-k-1)!}{n!} \text{Diff}_k(C, e, x).
\]

Therefore, it is enough to show how to compute in polynomial time the quantities \( \text{Diff}_k(C, e, x) \) for each \( k \in \{0, \ldots, n-1\} \). By definition of \( \phi(\cdot, \cdot, \cdot) \), we have that

\[
\text{Diff}_k(C, e, x) = \left[ \sum_{S \subseteq X \setminus \{x\} \atop |S| = k} \frac{1}{2^n - (k+1)} \sum_{e' \in \text{cons}(e, S \cup \{x\})} C(e') \right] - \left[ \sum_{S \subseteq X \setminus \{x\} \atop |S| = k} \frac{1}{2^n - k} \sum_{e' \in \text{cons}(e, S)} C(e') \right].
\]

In this expression, let \( \alpha \) and \( \beta \) be the left- and right-hand side terms in the subtraction. Moreover, for an entity \( e \in \text{ent}(X) \) and \( S \subseteq X \), let \( e|_S \) be the entity over \( S \) that is obtained by restricting \( e \) to the domain \( S \) (that is, \( e|_S \in \text{ent}(S) \) and \( e|_S(y) := e(y) \) for every \( y \in S \)). Then, looking closer at \( \beta \), we conclude that

\[
\beta = \sum_{S \subseteq X \setminus \{x\} \atop |S| = k} \frac{1}{2^n - k} \sum_{e' \in \text{cons}(e, S)} C(e')
\]
\[
\begin{align*}
&= \left[ \sum_{\substack{S \subseteq X \setminus \{x\} \atop |S| = k}} \frac{1}{2^{n-k}} \sum_{e' \in \mathsf{cons}(e,S)} C(e') \right] \\
&\quad + \left[ \sum_{\substack{S \subseteq X \setminus \{x\} \atop |S| = k}} \frac{1}{2^{n-k}} \sum_{e' \in \mathsf{cons}(e,S)} C(e') \right] \\
&= \left[ \frac{1}{2^{n-k}} \sum_{\substack{S \subseteq X \setminus \{x\} \atop |S| = k}} \sum_{e'' \in \mathsf{cons}(e'|X \setminus \{x\},S)} C_{+x}(e'') \right] \\
&\quad + \left[ \frac{1}{2^{n-k}} \sum_{\substack{S \subseteq X \setminus \{x\} \atop |S| = k}} \sum_{e'' \in \mathsf{cons}(e'|X \setminus \{x\},S)} C_{-x}(e'') \right] \\
&= \frac{1}{2^{n-k}} \left( H(C_{+x}, e_{|X \setminus \{x\}|}, k) + H(C_{-x}, e_{|X \setminus \{x\}|}, k) \right).
\end{align*}
\]

The last equality is obtained simply by using the definition of \( H(\cdot, \cdot, \cdot) \). Hence, if we can compute in polynomial time \( H(\cdot, \cdot, \cdot) \) for deterministic and decomposable Boolean circuits, then we can compute \( \beta \) in polynomial time as \( C_{+x} \) and \( C_{-x} \) can be computed in linear time from \( C \), and they are deterministic and decomposable Boolean circuits as well. We now inspect the term \( \alpha \).

\[
\begin{align*}
\alpha &= \sum_{\substack{S \subseteq X \setminus \{x\} \atop |S| = k}} \frac{1}{2^{n-(k+1)}} \sum_{e' \in \mathsf{cons}(e,S \cup \{x\})} C(e') \\
&= \frac{1}{2^{n-(k+1)}} \sum_{\substack{S \subseteq X \setminus \{x\} \atop |S| = k}} \sum_{e' \in \mathsf{cons}(e,S \cup \{x\})} C(e').
\end{align*}
\]

But then observe that, for \( S \subseteq X \setminus \{x\} \) and \( e' \in \mathsf{cons}(e,S \cup \{x\}) \), it holds that

\[
C(e') = \begin{cases} 
C_{+x}(e'|X \setminus \{x\}) & \text{if } e(x) = 1, \\
C_{-x}(e'|X \setminus \{x\}) & \text{if } e(x) = 0.
\end{cases}
\]

Therefore, if \( e(x) = 1 \), we have that

\[
\begin{align*}
\alpha &= \frac{1}{2^{n-(k+1)}} \sum_{\substack{S \subseteq X \setminus \{x\} \atop |S| = k}} \sum_{e'' \in \mathsf{cons}(e'|X \setminus \{x\},S)} C_{+x}(e'') \\
&= \frac{1}{2^{n-(k+1)}} \sum_{\substack{S \subseteq X \setminus \{x\} \atop |S| = k}} \sum_{e'' \in \mathsf{cons}(e'|X \setminus \{x\},S)} C_{+x}(e'') \\
&= \frac{1}{2^{n-(k+1)}} \sum_{\substack{S \subseteq X \setminus \{x\} \atop |S| = k}} H(C_{+x}, e_{|X \setminus \{x\}|}, k).
\end{align*}
\]

whereas if \( e(x) = 0 \), we have that

\[
\begin{align*}
\alpha &= \frac{1}{2^{n-(k+1)}} \sum_{\substack{S \subseteq X \setminus \{x\} \atop |S| = k}} H(C_{-x}, e_{|X \setminus \{x\}|}, k).
\end{align*}
\]

Hence, again, if we were able to compute in polynomial time \( H(\cdot, \cdot, \cdot) \) for deterministic and decomposable Boolean circuits, then we could compute \( \alpha \) in polynomial time (as deterministic and decomposable Boolean circuits \( C_{+x} \) and \( C_{-x} \) can be computed in linear time from \( C \)). But then we deduce from (\text{\textsection}) that \( \mathsf{Diff}_k(C, e, x) \) could be computed in polynomial time for each \( k \in \{0, \ldots, n - 1\} \), from which we have that \( \mathsf{SHAP}(C, e, x) \) could be computed in polynomial time, therefore concluding the existence of the reduction claimed in this section.
3.1.2 Reducing from $H(\cdot, \cdot, \cdot)$ to $\#\text{SAT}(\cdot, \cdot, \cdot)$

We now show that computing $H(\cdot, \cdot, \cdot)$ can be reduced in polynomial time to computing $\#\text{SAT}(\cdot, \cdot, \cdot)$. Given as input a deterministic and decomposable circuit $C$ over a set of variables $X$, an entity $e \in \text{ent}(X)$ and a natural number $k \leq |X|$, we have to compute

$$H(C, e, k) = \sum_{S \subseteq X} \sum_{|S| = k} C(e').$$

Let us now consider an entity $e'' \in \text{ent}(X)$ and reason about how many times $e''$ will occur as a summand in the above expression. First of all, it is clear that if $|\sim(e, e'')| < k$, then $e''$ will not appear in the sum; this is because if $e' \in \text{cons}(e, S)$ for some $S \subseteq X$ such that $|S| = k$, then $S \subseteq \sim(e, e')$ and, thus, $k \leq |\sim(e, e')|$. Now, how many times does an entity $e'' \in \text{ent}(X)$ such that $|\sim(e, e'')| \geq k$ occur as a summand in the expression? The answer is simple: once per $S \subseteq \sim(e, e'')$ of size $k$. Since there are $(\lceil |\sim(e, e'')| \rceil)$ such sets $S$, we obtain that

$$H(C, e, k) = \sum_{e'' \in \text{ent}(X)} \sum_{|\sim(e, e'')| \geq k} C(e'') \cdot \binom{|\sim(e, e'')|}{k} = \sum_{e'' \in \text{SAT}(C)} \sum_{|\sim(e, e'')| \geq k} \binom{|\sim(e, e'')|}{k} = \sum_{\ell = k}^{n} \binom{\ell}{k} \sum_{e'' \in \text{SAT}(C), |\sim(e, e'')| = \ell} 1 = \sum_{\ell = k}^{n} \binom{\ell}{k} \cdot \#\text{SAT}(C, e, \ell),$$

with the last equality being obtained by using the definition of $\#\text{SAT}(\cdot, \cdot, \cdot)$. This concludes the reduction of this section and, hence, the first part of the proof.

3.2 Computing $\#\text{SAT}(\cdot, \cdot, \cdot)$ in polynomial time

We now take care of the second part of the proof of Theorem 1.1, i.e., proving that computing $\#\text{SAT}(\cdot, \cdot, \cdot)$ for deterministic and decomposable Boolean circuits can be done in polynomial time. Formally:

**Lemma 3.2.** The following problem can be solved in polynomial time. Given as input a deterministic and decomposable Boolean circuit $C$ over a set of variables $X$, an entity $e \in \text{ent}(X)$ and a natural number $\ell \leq |X|$, compute the quantity $\#\text{SAT}(C, e, \ell)$.

We first perform two preprocessing steps on $C$, which will simplify the proof.

Rewriting to fan-in at most 2. First, we modify the circuit $C$ so that the fan-in of every $\lor$- and $\land$-gate is at most 2. This can simply be done in linear time by rewriting every $\land$-gate (resp., and $\lor$-gate) of fan-in $m > 2$ with a chain of $m-1$ $\land$-gates (resp., $\lor$-gates) of fan-in 2. It is clear that the resulting Boolean circuit is deterministic and decomposable. Hence, from now on we assume that the fan-in of every $\lor$- and $\land$-gate of $C$ is at most 2.
Smoothing the circuit. We say that a deterministic and decomposable circuit $C$ is smooth \cite{4, 21} if for every $\lor$-gate $g$ and input gates $g_1, g_2$ of $g$, we have that $\text{var}(g_1) = \text{var}(g_2)$, and we call such an $\lor$-gate smooth. We modify as follows the circuit $C$ so that it becomes smooth. Recall that by the previous paragraph, we assume that the fan-in of every $\lor$-gate is at most 2. For an $\lor$-gate $g$ of $C$ having two input gates $g_1, g_2$ violating the smoothness condition, define $S_1 := \text{var}(g_1) \setminus \text{var}(g_2)$ and $S_2 := \text{var}(g_2) \setminus \text{var}(g_1)$, and let $d_{S_1}, d_{S_2}$ be Boolean circuits defined as follows. If $S_1 = \emptyset$, then $d_{S_1}$ consist of the single constant gate 1. Otherwise, $d_{S_1}$ encodes the propositional formula $\land_{x \in S_1}(x \lor \neg x)$, but it is constructed in such a way that every $\land$- and $\lor$-gate has fan-in at most 2.

Boolean circuit $d_{S_2}$ is constructed exactly as $d_{S_1}$ but considering the set of variables $S_2$ instead of $S_1$. Observe that $\text{var}(d_{S_1}) = S_1$, $\text{var}(d_{S_2}) = S_2$ and $d_{S_1}, d_{S_2}$ always evaluate to 1. Then, we transform $g$ into a smooth $\lor$-gate by replacing gate $g_1$ by a decomposable $\land$-gate $(g_1 \land d_{S_1})$, and gate $g_2$ by a decomposable $\land$-gate $(g_2 \land d_{S_2})$. This does not change the Boolean classifier computed. Moreover, since $\text{var}(g_1 \land d_{S_1}) = \text{var}(g_2 \land d_{S_2}) = \text{var}(g_1) \cup \text{var}(g_2)$, we have that $g$ is now smooth. Finally, the resulting Boolean circuit is deterministic and decomposable. Hence, by repeating the previous procedure for each non-smooth $\lor$-gate, we conclude that $C$ can be transformed into an equivalent smooth Boolean circuit in polynomial time, which is deterministic and decomposable, and where each gate has fan-in at most 2. Thus, from now on we also assume that $C$ is smooth.

\textbf{Proof of Lemma 3.2.} Let $C$ be a deterministic and decomposable Boolean circuit $C$ over a set of variables $X$, $e \in \text{ent}(X)$, $\ell$ a natural number such that $\ell \leq |X|$ and $n = |X|$. For a gate $g$ of $C$, let $R_g$ be the Boolean circuit over $\text{var}(g)$ that is defined by considering the subgraph of $C$ induced by the set of gates $g'$ in $C$ for which there exists a path from $g'$ to $g$ in $C$. Notice that $R_g$ is a deterministic and decomposable Boolean circuit with output gate $g$. Moreover, for a gate $g$ and natural number $k \leq |\text{var}(g)|$, define $\alpha^k_{g} := \#\text{SAT}(R_g, e_{\text{var}(g)}, k)$, which we recall is the number of entities $e' \in \text{ent}(\text{var}(g))$ such that $e'$ satisfies $R_g$ and $|\text{sim}(e_{\text{var}(g)}, e')| = k$. We will show how to compute all the values $\alpha^k_{g}$ for every gate $g$ of $C$ and $k \in \{0, \ldots, |\text{var}(g)|\}$ in polynomial time. This will conclude the proof since, for the output gate $g_{\text{output}}$ of $C$, we have that $\alpha^{\text{sim}}_{g_{\text{output}}} = \#\text{SAT}(C, e, \ell)$. Next we explain how to compute these values by bottom-up induction on $C$.

\textbf{Variable gate.} $g$ is a variable gate with label $y \in X$, so that $\text{var}(g) = \{y\}$. Then we have that $\alpha^0_{g} = 1 - e(y)$ and $\alpha^{1}_{g} = e(y)$.

\textbf{Constant gate.} $g$ is a constant gate with label $a \in \{0, 1\}$. Then $\alpha^0_{g} = a$.

\textbf{$\neg$-gate.} $g$ is a $\neg$-gate with input gate $g'$. Notice that $\text{var}(g) = \text{var}(g')$. Then, since $|\text{var}(g)|$ is equal to the number of entities $e' \in \text{ent}(\text{var}(g))$ such that $|\text{sim}(e_{\text{var}(g)}, e')| = k$, we have that $\alpha^k_{g} = \binom{|\text{var}(g)|}{k} - \alpha^k_{g'}$ for every $k \in \{0, \ldots, |\text{var}(g)|\}$. By induction, the values $\alpha^k_{g}$ for $k \in \{0, \ldots, |\text{var}(g)|\}$ have already been computed. Moreover, the values $|\text{var}(g)|$ for $k \in \{0, \ldots, |\text{var}(g)|\}$ can be computed in polynomial time since $n$ and $|\text{var}(g)|$ are all given in unary when computing the SHAP-score. Thus, we can compute all the values $\alpha^k_{g}$ for $k \in \{0, \ldots, |\text{var}(g)|\}$ in polynomial time.

\textbf{$\lor$-gate.} $g$ is an $\lor$-gate. By assumption, recall that $g$ is deterministic, smooth and has fan-in at most 2. If $g$ has only one input $g'$, then clearly $\text{var}(g) = \text{var}(g')$ and $\alpha^k_{g} = \alpha^k_{g'}$ for every $k \in \{0, \ldots, |\text{var}(g)|\}$. Thus, assume that $g$ has exactly two input gates $g_1$ and $g_2$, and recall that $\text{var}(g_1) = \text{var}(g_2) = \text{var}(g)$, because $g$ is smooth. Moreover, given that $g$ is deterministic and smooth, we have that

$$\text{SAT}(R_g) = \text{SAT}(R_{g_1}) \cup \text{SAT}(R_{g_2}),$$

where $\text{SAT}(R_{g_1}) \cap \text{SAT}(R_{g_2}) = \emptyset$. By intersecting these three sets with the set $\{e' \in \text{var}(g) \mid |\text{sim}(e_{\text{var}(g)}, e')| = k\}$, we obtain by definition of $\text{SAT}(\cdot, \cdot, \cdot)$ that $\text{SAT}(R_g, e_{\text{var}(g)}, k) = \text{SAT}(R_{g_1}, e_{\text{var}(g)}, k) \cup \text{SAT}(R_{g_2}, e_{\text{var}(g)}, k)$.

\footnote{Indeed, we recall the mathematical convention that there is a unique entity with the empty domain and, hence, a unique function over $\emptyset$.}
where again $\text{SAT}(R_{g_1}, e_{\var(g)}, k) \cap \text{SAT}(R_{g_2}, e_{\var(g)}, k) = \emptyset$. We conclude that:

$$\#\text{SAT}(R_g, e_{\var(g)}, k) = \#\text{SAT}(R_{g_1}, e_{\var(g)}, k) + \#\text{SAT}(R_{g_2}, e_{\var(g)}, k)$$

Thus, we have that $\alpha^k_g = \alpha^k_{g_1} + \alpha^k_{g_2}$ for every $k \in \{0, \ldots, |\var(g)|\}$. By induction, the values $\alpha^k_{g_1}$ and $\alpha^k_{g_2}$, for each $k \in \{0, \ldots, |\var(g)|\}$, have already been computed. Therefore, we can compute all the values $\alpha^k_g$ for $k \in \{0, \ldots, |\var(g)|\}$ in polynomial time.

$\wedge$-gate. $g$ is an $\wedge$-gate. By assumption, recall that $g$ is decomposable and has fan-in at most 2. If $g$ has only one input $g'$, then clearly $\var(g) = \var(g')$ and $\alpha^k_g = \alpha^k_{g'}$ for every $k \in \{0, \ldots, |\var(g)|\}$. Thus, assume that $g$ has exactly two input gates $g_1$ and $g_2$. We will need the following notation.

For two disjoint sets of variables $X_1$, $X_2$ and entities $e_1 \in \text{ent}(X_1)$, $e_2 \in \text{ent}(X_2)$, we denote by $e_1 \cup e_2$ the entity over $X_1 \cup X_2$ that coincides with $e_1$ over $X_1$ and with $e_2$ over $X_2$ (that is, $e_1 \cup e_2 \in \text{ent}(X_1 \cup X_2)$, $(e_1 \cup e_2)(x_1) = e_1(x_1)$ for every $x_1 \in X_1$, and $(e_1 \cup e_2)(x_2) = e_2(x_2)$ for every $x_2 \in X_2$). Moreover, for two sets $S_1 \subseteq \text{ent}(X_1)$, $S_2 \subseteq \text{ent}(X_2)$, we denote by $S_1 \otimes S_2$ the set of entities over $X_1 \cup X_2$ defined as

$$S_1 \otimes S_2 := \{ e_1 \cup e_2 \mid e_1 \in S_1 \text{ and } e_2 \in S_2 \}.$$ 

Given that $g$ is a decomposable $\wedge$-gate, we have that:

$$\text{SAT}(R_g) = \text{SAT}(R_{g_1}) \otimes \text{SAT}(R_{g_2}).$$

Moreover, we have that $\text{SAT}(R_g, e_{\var(g)}, k) = \text{SAT}(R_g) \cap \{ e' \in \var(g) \mid |\sim(e_{\var(g)}, e')| = k \}$ and

$$\text{SAT}(R_{g_1}) \otimes \text{SAT}(R_{g_2}) \cap \{ e' \in \var(g) \mid |\sim(e_{\var(g)}, e')| = k \} = \{ e_1 \cup e_2 \mid e_1 \in \text{SAT}(R_{g_1}) \text{ and } e_2 \in \text{SAT}(R_{g_2}) \} \cap \{ e' \in \var(g) \mid |\sim(e_{\var(g)}, e')| = k \} = \{ e_1 \cup e_2 \mid e_1 \in \text{SAT}(R_{g_1}), e_2 \in \text{SAT}(R_{g_2}) \text{ and } |\sim(e_{\var(g)}, e_1, e_2)| = k \} = \{ e_1 \cup e_2 \mid e_1 \in \text{SAT}(R_{g_1}), e_2 \in \text{SAT}(R_{g_2}), \text{ and there exist } i \in \{0, \ldots, |\var(g_1)|\}, j \in \{0, \ldots, |\var(g_2)|\} \text{ such that } |\sim(e_{\var(g_1)}, e_1)| = i, |\sim(e_{\var(g_2)}, e_2)| = j, \text{ and } i + j = k \} = \bigcup_{i \in \{0, \ldots, |\var(g_1)|\}} \{ e_1 \mid e_1 \in \text{SAT}(R_{g_1}), |\sim(e_{\var(g_1)}, e_1)| = i \} \otimes \{ e_2 \mid e_2 \in \text{SAT}(R_{g_2}), |\sim(e_{\var(g_2)}, e_2)| = j \} = \bigcup_{i \in \{0, \ldots, |\var(g_1)|\}, j \in \{0, \ldots, |\var(g_2)|\}, i + j = k} \text{SAT}(R_{g_1}, e_{\var(g_1)}, i) \otimes \text{SAT}(R_{g_2}, e_{\var(g_2)}, j).$$

Combining the previous results, we obtain that

$$\text{SAT}(R_g, e_{\var(g)}, k) = \bigcup_{i \in \{0, \ldots, |\var(g_1)|\}, j \in \{0, \ldots, |\var(g_2)|\}, i + j = k} \text{SAT}(R_{g_1}, e_{\var(g_1)}, i) \otimes \text{SAT}(R_{g_2}, e_{\var(g_2)}, j).$$

Thus, given that for every pair $i_1, i_2 \in \{0, \ldots, |\var(g_1)|\}$ such that $i_1 \neq i_2$, it holds that

$$\text{SAT}(R_{g_1}, e_{\var(g_1)}, i_1) \cap \text{SAT}(R_{g_1}, e_{\var(g_1)}, i_2) = \emptyset$$

(and similarly for $R_{g_2}$), we conclude by the definitions of $\alpha^k_g$, $\alpha^k_{g_1}$, $\alpha^k_{g_2}$ that

$$\alpha^k_g = \sum_{i \in \{0, \ldots, |\var(g_1)|\}} \alpha^i_{g_1} \cdot \alpha^j_{g_2},$$

By induction, the values $\alpha^i_{g_1}$ and $\alpha^j_{g_2}$, for each $i \in \{0, \ldots, |\var(g_1)|\}$ and $j \in \{0, \ldots, |\var(g_2)|\}$, have already been computed. Therefore, we can compute all the values $\alpha^k_g$ for $k \in \{0, \ldots, |\var(g)|\}$ in polynomial time.

This concludes the proof of Lemma 3.2 and, hence, the proof of Theorem 1.1. \qed
4 Limits for exact computation of SHAP-score

In this section, we establish the computational limits of the notion of SHAP-score. To this end, for a class $\mathcal{C}$ of Boolean classifiers, we define the evaluation problem for $\mathcal{C}$, denoted by $\text{EVAL}(\mathcal{C})$, as the problem of accepting the language $\{(M,e) \mid M \in \mathcal{C}, M e \}$. Notice that the evaluation problem can be solved in polynomial time for all the Boolean circuits and binary decision diagrams classes considered in this paper. In what follows, we show that if $\text{EVAL}(\mathcal{C})$ can be solved in polynomial time for a class of Boolean classifiers $\mathcal{C}$, then the problem of computing the number of entities accepted by a Boolean classifier $M$ in $\mathcal{C}$ can be reduced in polynomial time to the problem of computing the SHAP-score for $M$. More precisely,

**Lemma 4.1** (Generalization of [2, Theorem 5.1]). Let $M$ be a Boolean classifier over a set of features $X$. Then, letting $n = |X|$, for every $e \in \text{ent}(X)$ we have:

$$\# \text{SAT}(M) = 2^n \left( M(e) - \sum_{x \in X} \text{SHAP}(M, e, x) \right).$$

(2)

Therefore, if $\mathcal{C}$ is a class of Boolean classifiers such that $\text{EVAL}(\mathcal{C})$ can be solved in polynomial time, then the computation of $\# \text{SAT}(\cdot)$ can be reduced in polynomial time to the computation of $\text{SHAP}(\cdot, \cdot, \cdot)$ for the class $\mathcal{C}$.

It is important to notice that Proposition 1.3 is a corollary of Lemma 4.1, as $\# \text{SAT}(\cdot)$ is $\#P$-hard for the class of propositional formulae in DNF [14], and it is clear that the evaluation problem can be solved in polynomial time for this class. Also, we point out that Theorem 5.1 in [2] was stated for Boolean classifiers given as propositional formulae in Conjunctive Normal Form (CNF), and for the probability space on the inputs that is the product space. However, a close inspection of their proof reveals that it actually works for the uniform space, and for any class of models whose evaluation problem is in polynomial time.

**Proof of Lemma 4.1.** The validity of Equation (2) will be consequence of the following property: for every Boolean classifier $M$ over $X$ and entity $e \in \text{ent}(X)$, it holds that

$$\sum_{x \in X} \text{SHAP}(M, e, x) = \phi(M, e, X) - \phi(M, e, \emptyset).$$

(3)

This property is often called the efficiency property of the Shapley value. Although this is folklore, we prove Equation (3) here for the reader’s convenience. For a permutation $\pi : X \to \{1, \ldots, n\}$ and $x \in X$, let $S^x_\pi$ denote the set of features that appear before $x$ in $\pi$. Formally, $S^x_\pi := \{y \in X \mid \pi(y) < \pi(x)\}$. Then, letting $\Pi(X)$ be the set of all permutations $\pi : X \to \{1, \ldots, n\}$, observe that Equation (1) can be rewritten as

$$\text{SHAP}(M, e, x) = \frac{1}{n!} \sum_{\pi \in \Pi(X)} \left( \phi(M, e, S^x_\pi \cup \{x\}) - \phi(M, e, S^x_\pi) \right).$$

Hence, we have that

$$\sum_{x \in X} \text{SHAP}(M, e, x) = \frac{1}{n!} \sum_{x \in X} \sum_{\pi \in \Pi(X)} \left( \phi(M, e, S^x_\pi \cup \{x\}) - \phi(M, e, S^x_\pi) \right)$$

$$= \frac{1}{n!} \sum_{\pi \in \Pi(X)} \sum_{x \in X} \left( \phi(M, e, S^x_\pi \cup \{x\}) - \phi(M, e, S^x_\pi) \right)$$

$$= \frac{1}{n!} \sum_{\pi \in \Pi(X)} \left( \phi(M, e, X) - \phi(M, e, \emptyset) \right),$$

And, in fact, for any “reasonable” class of models that could be considered in machine learning; indeed, this is the problem of determining the prediction of a learned model $M$ on an entity $e$. 

11
where the last equality is obtained by noticing that the inner sum is a telescoping sum. This establishes Equation (3). Now, we simply use the definition of $\phi(\cdot,\cdot,\cdot)$ in this equation to obtain

$$\sum_{x \in X} \text{SHAP}(M,e,x) = M(e) - \frac{1}{2^n} \sum_{e' \in \text{ent}(X)} M(e')$$

$$= M(e) - \frac{\#\text{SAT}(M)}{2^n},$$

thus proving Equation (2) and concluding the proof. \qed

Acknowledgements. Bertossi is a member of the Academic Network of RelationalAI Inc., where his interest in this topic started. Barceló is funded by Fondecyt grant 1200967. All authors are funded by the Millennium Institute for Foundational Research on Data (IMFD Chile).

References

[1] A. Amarilli, F. Capelli, M. Monet, and P. Senellart. Connecting knowledge compilation classes and width parameters. Theory Comput. Syst., 64(5):861–914, 2020.

[2] L. Bertossi, J. Li, M. Schleich, D. Suciu, and Z. Vagena. Causality-based explanation of classification outcomes. In Proceedings of the Fourth Workshop on Data Management for End-To-End Machine Learning, DEEM@SIGMOD 2020, pages 6:1–6:10, 2020.

[3] G. Cesari, E. Algaba, S. Moretti, and J. A. Nepomuceno. An application of the Shapley value to the analysis of co-expression networks. Applied network science, 3(1):35, 2018.

[4] A. Darwiche. On the tractable counting of theory models and its application to truth maintenance and belief revision. J. Applied Non-Classical Logics, 11(1-2), 2001.

[5] A. Darwiche and A. Hirth. On the reasons behind decisions. arXiv preprint arXiv:2002.09284, 2020.

[6] X. Deng and C. H. Papadimitriou. On the complexity of cooperative solution concepts. Mathematics of operations research, 19(2):257–266, 1994.

[7] U. Faigle and W. Kern. The Shapley value for cooperative games under precedence constraints. International Journal of Game Theory, 21(3):249–266, 1992.

[8] A. Hunter and S. Konieczny. On the measure of conflicts: Shapley inconsistency values. Artificial Intelligence, 174(14):1007–1026, 2010.

[9] E. Livshits, L. E. Bertossi, B. Kimelfeld, and M. Sebag. The Shapley value of tuples in query answering. In 23rd International Conference on Database Theory, ICDT 2020, March 30-April 2, 2020, Copenhagen, Denmark, volume 155, pages 20:1–20:19, 2020.

[10] S. M. Lundberg, G. Erion, H. Chen, A. DeGrave, J. M. Prutkin, B. Nair, R. Katz, J. Himmelfarb, N. Bansal, and S.-I. Lee. From local explanations to global understanding with explainable AI for trees. Nature machine intelligence, 2(1):2522–5839, 2020.

[11] S. M. Lundberg and S.-I. Lee. A unified approach to interpreting model predictions. In Advances in neural information processing systems, pages 4765–4774, 2017.

[12] T. P. Michalak, K. V. Aadithya, P. L. Szczepanski, B. Ravindran, and N. R. Jennings. Efficient computation of the Shapley value for game-theoretic network centrality. Journal of Artificial Intelligence Research, 46:607–650, 2013.

[13] R. Peharz, S. Lang, A. Vergari, K. Stedzner, A. Molina, M. Trapp, G. V. d. Broeck, K. Kersting, and Z. Ghahramani. Einsum networks: Fast and scalable learning of tractable probabilistic circuits. arXiv preprint arXiv:2004.06231, 2020.
[14] J. S. Provan and M. O. Ball. The complexity of counting cuts and of computing the probability that a graph is connected. *SIAM Journal on Computing*, 12(4):777–788, 1983.

[15] A. E. Roth. *The Shapley value: essays in honor of Lloyd S. Shapley*. Cambridge University Press, 1988.

[16] L. S. Shapley. A value for n-person games. *Contributions to the Theory of Games*, 2(28):307–317, 1953.

[17] W. Shi, A. Shih, A. Darwiche, and A. Choi. On tractable representations of binary neural networks. *arXiv preprint arXiv:2004.02082*, 2020.

[18] A. Shih, A. Choi, and A. Darwiche. Formal verification of Bayesian network classifiers. In *International Conference on Probabilistic Graphical Models*, pages 427–438, 2018.

[19] A. Shih, A. Choi, and A. Darwiche. A symbolic approach to explaining Bayesian network classifiers. *arXiv preprint arXiv:1805.03364*, 2018.

[20] A. Shih, A. Darwiche, and A. Choi. Verifying binarized neural networks by Angluin-style learning. In *International Conference on Theory and Applications of Satisfiability Testing*, pages 354–370. Springer, 2019.

[21] A. Shih, G. Van den Broeck, P. Beame, and A. Amarilli. Smoothing structured decomposable circuits. In *Advances in Neural Information Processing Systems*, pages 11416–11426, 2019.