Metamaterials from modified CPT-odd standard model extension and minimum length

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Here we discuss the standard model extension in the presence of CPT-odd Lorentz violation (LV) sector and of a deformed Heisenberg algebra that leads to a non-commutative theory with minimum length (ML). We derive the set of Maxwell equations emerging from this theory and considered the consequences with respect to the usual effects of electromagnetic waves and material media. We then considered the set of modified equations in material media and investigate the metamaterial behaviour as a consequence of LV and ML. We show that a negative index refraction can be derived from the presence of a non-commutativity suitably tuned by the $\beta$ parameter, while in the presence of LV, we obtained the set of modified Maxwell equation in terms of the corresponding material fields with terms depending explicitly from the terms of interaction between the material fields depending on non-commutativity with the background field due to CPT-odd LV. We conclude that a new set of metamaterials can be derived as a consequence of CPT-odd LV and non-commutativity with minimum length.

\section{I. INTRODUCTION}

The induction of Spontaneous Lorentz Violation (SLV) symmetry by the presence of a tensorial background is one of the possible scenarios in the non-homogeneous spacetime that could to explain Planck scale physics and dark physics scenarios. The residues of quantum gravity and metric fluctuations where the standard model (SM) has achieved its
validity limit, brought a more general picture represented by Standard Model Extension. The Standard Model Extended (SME) is an effective field theory obtained from SM (the model that uniquely describes the interactions that govern elementary particles, namely electromagnetic, weak nuclear and strong nuclear interactions, but does not incorporate the Gravitational) by the addition of terms that incorporate the violations of the Lorentz and CPT symmetry. In this scenario, Lorentz symmetry violation (LV) is a natural phenomena in high energy scales, inducing both spontaneous Lorentz symmetry violation (SLV) caused by a tensorial background in one side, and the breaking made by generalization of uncertainty principle, that is associated to a non-commutative geometry. Kostelecký and Samuel [1] proposed the idea of SLV when they initiated investigations of a possible physics beyond the Standard Model (SM). They have suggested that SLV could occur in a scenario of string field theory by means of non-scalar fields (vacuum of fields that have a tensor nature) by the presence of a non-scalar background [2, 3]. As a consequence, the prediction involves the assumption that in a more fundamental theory signals could be emitted from more fundamental fields by SLV symmetry. SME keeps the gauge invariance, conservation of energy and momentum and the covariance under observer rotations and boosts [4]. In this context, it is well-known that the presence of terms of LV symmetry imposes at least one privileged direction in the spacetime. In recent years, studies of the LV symmetry have been made in several different contexts [5–28].

On the other hand, non-commutative geometry (NCG) was developed some decades ago by A. Connes [29] and it was realized that the NCG would be a scheme to extend the SM in several forms [30]. In particular, it appears naturally in string theory scenarios [31, 32], that should lead to effective theories describing scenarios beyond SM and that recover at low energy limits known physical results from SM.

One possible way to explore the implementation of NCG theories is by the deformation of the Heisenberg algebra. In a modified Heisenberg algebra, by adding certain small corrections to the canonical commutation relations, it leads, as shown by A. Kempf and contributors [33–37], to the minimum uncertainty in the position measurement, $\Delta x_0$, called minimum length. The existence of this minimum length was also suggested by quantum gravity and string theory [38, 39].

Quesne and Tkachuk have introduced recently a Lorentz covariant deformed algebra in a
quantized $D+1$-dimension $[41,42]$, whose generalized commutation relations can be written:

\[ \frac{1}{i\hbar} [X^\mu, P^\nu] = (-1 + \beta P^2)\eta^{\mu\nu} + \beta' P^\mu P^\nu \]  \hspace{1cm} (1)

\[ \frac{1}{i\hbar} [P^\mu, P^\nu] = 0, \] \hspace{1cm} (2)

\[ \frac{1}{i\hbar} [X^\mu, X^\nu] = \frac{[(2\beta - \beta') - (2\beta^2 + \beta\beta')P^2](P^\mu X^\nu - P^\nu X^\mu)}{(1 - \beta P^2)}, \] \hspace{1cm} (3)

where $P^2 = P_\rho P^\rho$, $\mu, \nu, \rho = 0, 1, \cdots, D$, $\eta_{\mu\nu} = \eta^{\mu\nu} = \text{diag}(1, -1, -1, \cdots, -1)$, and $\beta, \beta' > 0$ are deformation parameters. From uncertainty relation, a minimum length can be achieved for these commutation relations, that is given by

\[ (\delta X_i)_0 = \hbar \sqrt{(D\beta + \beta') [1 - \beta \langle (P^0)^2 \rangle]}, \quad \forall i \in \{1, \cdots, D\}. \]

The representation algebra $[43]$, satisfying the above commutation relations at first order in $\beta, \beta'$ are given by:

\[ X^\mu = x^\mu - \frac{(2\beta - \beta')}{4}(x^\mu p^2 + p^2 x^\mu), \] \hspace{1cm} (4)

\[ P^\mu = \left(1 - \frac{\beta'}{2} p^2\right) p^\mu, \] \hspace{1cm} (5)

where $p^2 = p_\rho p^\rho$. The corresponding the position and momentum operators $x^\mu$ and $P^\mu = i\hbar \partial^\mu$ will lead to the equivalent relations

\[ X^\mu = x^\mu + \frac{(2\beta - \beta')\hbar^2}{4}(x^\mu \Box + \Box x^\mu), \] \hspace{1cm} (6)

\[ P^\mu = \left(1 + \frac{\beta'}{2} \hbar^2 \Box\right) i\hbar \partial^\mu, \] \hspace{1cm} (7)

where $\Box = \partial_\mu \partial^\mu$.

At second order in $\hbar$, this reduces to

\[ X^\mu = x^\mu \left(1 + \frac{(2\beta - \beta')\hbar^2}{4} \Box\right), \] \hspace{1cm} (8)

\[ P^\mu = i\hbar \partial^\mu \] \hspace{1cm} (9)

and up to third order in $\hbar$,

\[ X^\mu = x^\mu \left(1 + \frac{(2\beta - \beta')\hbar^2}{4} \Box\right), \] \hspace{1cm} (10)

\[ P^\mu = (1 + \beta \hbar^2 \Box) i\hbar \partial^\mu, \] \hspace{1cm} (11)
We also can consider the case $\beta' = 2\beta$, where

\begin{align*}
X^\mu &= x^\mu, \\
P^\mu &= (1 + \beta\hbar^2 \Box) i\hbar \dot{x}^\mu,
\end{align*}

As a consequence, taking $\beta' = 2\beta$ and second order in $\hbar$ we recover an usual canonical transformation. The hydrogen atom is one of the simplest quantum systems that allows theoretical predictions of high accuracy, and is well-studied experimentally offering the most precise amount of measures \[44\]. There are many papers where the energy spectrum of the hydrogen atom in the presence minimum length is calculated \[45, 48, 49\], some of which have divergences in levels $s$ ($n = 1$) \[45\] and recently it was studied the Stark effect with minimum length \[46\].

We investigate the scenario of anisotropy in polarized electromagnetic waves \[50\] generated by the presence of a Lorentz symmetry breaking term defined by a term $\epsilon_{\mu\nu\alpha\beta} V^\mu A^\nu F^{\alpha\beta}$ \[47\]. The Lorentz symmetry violation is located in the fourvector $V^\mu$ behavior. The effects of these anisotropies in the nature of vacuum polarized electromagnetic waves is then discussed.

The structure of this paper is the following: Section II we consider the CPT-odd sector of SME in the presence of a minimum length; section III we consider the transformation of polarized electromagnetic wave under the effect of a second order ML. Secion IV, we consider the non-commutativity in the approximation to reduce the effect of transformation restricted to the momentum sector, what leads to consequences in both electromagnetic waves and in the presence of source terms. In section V, we discuss the metamaterial behaviour with negative refractive index and in section VI we derive the set of modified Maxwell equations in material media, depending on the CPT-odd terms and ML. Finally, in section VII we leave our concluding remarks.

II. THE ODD GAUGE SECTOR OF THE STANDARD MODEL EXTENDED WITH MINIMUM LENGTH

We start with the following CPT-odd gauge sector from SME \[2, 3\]

\begin{equation}
\mathcal{L}_{2N+1} = -\frac{1}{4\mu_0} \mathcal{F}_{\mu\nu} F^{\mu\nu} - \frac{\chi}{4\mu_0} \epsilon_{\mu\nu\alpha\beta} V^\mu A^\nu F^{\alpha\beta} - A^\mu J^\mu,
\end{equation}
where $F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the tensor of the electromagnetic field constructed from the gauge field $A_\mu = (A^0, \mathbf{A})$. The term $\epsilon_{\mu \nu \alpha \beta} V^\mu A^\nu F^{\alpha \beta}$ leads to a possible scenario of anisotropy generated by LV [47], where Lorentz symmetry violation is located in the vector $V^\mu$ behavior.

We will write this Lagrangian in the presence of a NCG given by (6) and (13), corresponding to the transformations

$$x^\mu \to X^\mu = x^\mu,$$

$$\partial^\mu \to \nabla^\mu = (1 + \beta \hbar^2 \Box) \partial^\mu.$$

Neglecting terms of non-linear order in $\beta$, we arrive at

$$\mathcal{L}_{2N+1, \text{NCG}} = \mathcal{L}_{2N+1} - \frac{1}{2 \mu_0} \beta \hbar^2 F_{\mu \nu} \Box F^{\mu \nu} - \frac{\chi}{4 \mu_0} \beta \hbar^2 \epsilon_{\mu \nu \alpha \beta} V^\mu A^\nu \Box F^{\alpha \beta}. \tag{16}$$

The corresponding modified Maxwell equations of motion are given by

$$(1 + 2 \beta \hbar^2 \Box) \partial_\nu F^\nu \mu + \frac{\chi}{2} \epsilon^{\mu \lambda \alpha \beta} V_\lambda (1 + \beta \hbar^2 \Box) F_{\alpha \beta} = \mu_0 J^\mu. \tag{17}$$

The usual Bianchi identity is left invariant

$$\partial_\nu F_{\alpha \beta} + \partial_\alpha F_{\beta \nu} + \partial_\beta F_{\nu \alpha} = 0. \tag{18}$$

III. TRANSFORMATION UP TO SECOND ORDER IN $\hbar$ AND POLARIZED PLANE WAVES

Let us consider a set of polarized plane wave solutions $E(x^\mu) = E_0 \exp (k_\mu x^\mu)$ and $B(x^\mu) = B_0 \exp (k_\mu x^\mu)$. Under the transformations (8) and (9), these fields are changed to following

$$E(x^\mu + \frac{(2 \beta - \beta') \hbar^2}{4} (x^\mu \Box + \Box x^\mu)) = E_0 \exp \left(k_\mu \left[x^\mu + \frac{(2 \beta - \beta') \hbar^2}{4} (x^\mu \Box + \Box x^\mu)\right]\right), \tag{19}$$

$$B(x^\mu + \frac{(2 \beta - \beta') \hbar^2}{4} (x^\mu \Box + \Box x^\mu)) = B_0 \exp \left(k_\mu \left[x^\mu + \frac{(2 \beta - \beta') \hbar^2}{4} (x^\mu \Box + \Box x^\mu)\right]\right), \tag{20}$$

and consequently the fields turn to behave as operators. We rewrite in the form

$$E(x^\mu + \frac{(2 \beta - \beta') \hbar^2}{4} (x^\mu \Box + \Box x^\mu)) = E(x^\mu) \exp \left(k_\mu \left[\frac{(2 \beta - \beta') \hbar^2}{4} (x^\mu \Box + \Box x^\mu)\right]\right), \tag{21}$$

$$B(x^\mu + \frac{(2 \beta - \beta') \hbar^2}{4} (x^\mu \Box + \Box x^\mu)) = B(x^\mu) \exp \left(k_\mu \left[\frac{(2 \beta - \beta') \hbar^2}{4} (x^\mu \Box + \Box x^\mu)\right]\right). \tag{22}$$
Taking this up to the first order, we can rewrite
\[
E(x^\mu + \frac{(2\beta - \beta')\hbar^2}{4}(x^\mu \Box + \Box x^\mu)) = E(x^\mu)[1 + \left(\frac{(2\beta - \beta')\hbar^2}{4}(x^\mu \Box + \Box x^\mu)\right)],
\]
(23)
\[
B(x^\mu + \frac{(2\beta - \beta')\hbar^2}{4}(x^\mu \Box + \Box x^\mu)) = B(x^\mu)[1 + \left(\frac{(2\beta - \beta')\hbar^2}{4}(x^\mu \Box + \Box x^\mu)\right)],
\]
(24)
Restrict to a one dimensional propagation we have
\[
E(x', t') = E(x, t)[1 + i\frac{(2\beta - \beta')\hbar^2}{4}(kx - \omega t)\Box + i\frac{(2\beta - \beta')\hbar^2}{4}(kx - \omega t)\Box],
\]
(25)
\[
B(x', t') = B(x, t)[1 + i\frac{(2\beta - \beta')\hbar^2}{4}(kx - \omega t)\Box + i\frac{(2\beta - \beta')\hbar^2}{4}(kx - \omega t)\Box],
\]
(26)
As a consequence, the presence of a second order transformation in \(\hbar\) implies a contribution resulting from a quantization, where the fields acts as quantum observable.

IV. NCG TRANSFORMATION WITH \(\beta' = 2\beta\) AND MODIFIED MAXWELL EQUATIONS BY MINIMUM LENGTH

A. Electromagnetic waves

Considering electromagnetic waves in the presence of a minimum length, we have the simplified form for these equations in the vacuum
\[
(1 + \beta \hbar^2 \Box) \nabla \cdot E = 0
\]
(27)
\[
(1 + \beta \hbar^2 \Box) \nabla \cdot B = 0,
\]
(28)
\[
(1 + \beta \hbar^2 \Box) \nabla \times E = -(1 + \beta \hbar^2 \Box) \frac{\partial B}{\partial t},
\]
(29)
\[
(1 + \beta \hbar^2 \Box) \nabla \times B = \mu_0 \epsilon_0 (1 + \beta \hbar^2 \Box) \frac{\partial E}{\partial t},
\]
(30)
Let us consider the defined fields
\[
\tilde{E} = (1 + \beta \hbar^2 \Box)E,
\]
(31)
\[
\tilde{B} = (1 + \beta \hbar^2 \Box)B,
\]
(32)
We then have that for the linearly polarized solution for the electromagnetic waves
\[
E(x, t) = E_0 \cos(kx - \omega t),
\]
(33)
\[
B(x, t) = B_0 \cos(kx - \omega t),
\]
(34)
the modified fields will have the form

$$\tilde{E}(x,t) = E_0 \cos(kx - \omega t) + \beta \hbar^2 (-k^2 + \frac{\omega^2}{c^2}) E_0 \cos(kx - \omega t),$$  \hspace{1cm} (36)

$$\tilde{B}(x,t) = E_0 \cos(kx - \omega t) + \beta \hbar^2 (-k^2 + \frac{\omega^2}{c^2}) E_0 \cos(kx - \omega t),$$  \hspace{1cm} (37)

$$\tilde{E}(x,t) = E_0 \cos(kx - \omega t) + \beta \hbar^2 (-k^2 + \frac{\omega^2}{c^2}) E_0 \cos(kx - \omega t),$$  \hspace{1cm} (38)

As a consequence, the electromagnetic waves are left invariant in this case, since the left-hand side of these equation vanishes when the electromagnetic fields are wave solutions. This result, asserts that the non-commutativity displayed does not affect electromagnetic waves.

### B. Source terms

The presence of a minimum length transformation due to non-commutativity (17) induces a changing in the Maxwell equations, particularly in the Gauss and Ampère-Maxwell laws. The set of modified Maxwell equations in the presence of source terms is given by

$$\begin{align*}
(1 + \beta \hbar^2 \Box) \nabla \cdot E &= \frac{\rho}{\varepsilon_0}, \\
(1 + \beta \hbar^2 \Box) \nabla \cdot B &= 0, \\
(1 + \beta \hbar^2 \Box) \nabla \times E &= -(1 + \beta \hbar^2 \Box) \frac{\partial B}{\partial t}, \\
(1 + \beta \hbar^2 \Box) \nabla \times B &= \mu_0 J + \mu_0 \varepsilon_0 (1 + \beta \hbar^2 \Box) \frac{\partial E}{\partial t}.
\end{align*}$$

As usual, let us split the current density in terms of free, polarization and magnetization contributions $J = J_f + J_P + J_M$, and the charge density in terms of free and polarization terms $\rho = \rho_f + \rho_P$. The material media contribution is associated to the presence of an electric polarization $P$ and a magnetization $M$. These allows to write

$$J = J_f + \frac{\partial P}{\partial t} + \nabla \times M$$  \hspace{1cm} (43)

$$\rho = \rho_f - \nabla \cdot P$$  \hspace{1cm} (44)
We can then rewrite the previous equations as

\[ \nabla \cdot \left[ (1 + \beta \hbar^2 \Box) \mathbf{E} + \frac{\mathbf{P}}{\varepsilon_0} \right] = \frac{\rho_f}{\varepsilon_0}, \]  
(45)
\[ \nabla \cdot [(1 + \beta h^2 \Box) \mathbf{B}] = 0, \]  
(46)
\[ \nabla \times [(1 + \beta h^2 \Box) \mathbf{E}] = -\frac{\partial [(1 + \beta h^2 \Box) \mathbf{B}]}{\partial t}, \]  
(47)
\[ \nabla \times [(1 + \beta h^2 \Box) \mathbf{B} - \mu_0 \mathbf{M}] = \mu_0 \mathbf{J}_f + \frac{\partial [\mu_0 \varepsilon_0 (1 + \beta h^2 \Box) \mathbf{E} + \mu_0 \mathbf{P}]}{\partial t}, \]  
(48)

The corresponding fields in material media can be defined with \( \beta \) dependence

\[ \mathbf{D}_\beta = \varepsilon_0 (1 + \beta h^2 \Box) \mathbf{E} + \mathbf{P} \]
\[ = \mathbf{D} + \beta h^2 \Box \mathbf{E} \]  
(49)

where we also have a generalized response \( \mathbf{H} \) to the material media

\[ \mathbf{H}_\beta = \frac{(1 + \beta h^2 \Box)}{\mu_0} \mathbf{B} - \mathbf{M} \]
\[ = \mathbf{H} + \frac{\beta h^2 \Box}{\mu_0} \mathbf{B}. \]  
(50)

We then have the set of modified equations rewritten as

\[ \nabla \cdot \mathbf{D}_\beta = \rho_f, \]  
(51)
\[ \mu_0 \nabla \cdot [\mathbf{H}_\beta + \mathbf{M}] = 0, \]  
(52)
\[ \frac{1}{\varepsilon_0} \nabla \times [\mathbf{D}_\beta - \mathbf{P}] = -\mu_0 \frac{\partial [\mathbf{H}_\beta + \mathbf{M}]}{\partial t}, \]  
(53)
\[ \nabla \times \mathbf{H}_\beta = \mathbf{J}_f + \frac{\partial \mathbf{D}_\beta}{\partial t}, \]  
(54)

Using the constitutive relation for polarization \( \mathbf{P} = \varepsilon_0 \chi_e \mathbf{E} \) and the generalized one for magnetization

\[ \mathbf{M} = \chi_{m,\beta} \mathbf{H}_\beta, \]

where \( \chi_{m,\beta} \) is a \( \beta \)-dependent parameter. We can then write generalized relations for the fields in material media

\[ \mathbf{D}_\beta = \varepsilon_0 \left( 1 + \beta h^2 \Box + \chi_e \right) \mathbf{E}, \]  
(55)
\[ \mathbf{H}_\beta = \frac{1}{(1 + \chi_{m,\beta}) \mu_0} (1 + \beta h^2 \Box) \mathbf{B}. \]  
(56)
V. METAMATERIALS FROM MODIFIED MAXWELL EQUATIONS WITH MINIMUM LENGTH

In the Fourier transformed space in the previous result, we have

\[
D_\beta(p, p_0 = \omega) = \varepsilon_0 \left(1 + \beta \hbar^2 p_\mu p^\mu + \chi_e\right) E(p, p_0 = \omega),
\]

\[
H_\beta(p, p_0 = \omega) = \frac{1}{(1 + \chi_{m,\beta})\mu_0} \left(1 + \beta \hbar^2 p_\mu p^\mu\right) B(p, p_0 = \omega).
\]

(57)\quad (58)

As a consequence, the material media will be identified by mean of material permittivity and permeabilities

\[
\varepsilon_\beta = \varepsilon_0 \left(1 + \beta \hbar^2 p_\mu p^\mu + \chi_e\right)
\]

\[
\mu_\beta = \frac{(1 + \chi_{m,\beta})\mu_0}{(1 + \beta \hbar^2 p_\mu p^\mu)}
\]

(59)\quad (60)

Taking case where \(|\chi_{m,\beta}| \gg 1\) and \(\chi_e << 1\), we have the refraction index given by a dependence in the generalized NCG parameter

\[
n_\beta = \sqrt{\chi_{m,\beta}}
\]

(61)

Considering \(\chi_{m,\beta}\) given by a complex term

\[
\chi_{m,\beta} = \mathcal{T}_\beta e^{i\vartheta_\beta}
\]

(62)

we then have an negative index refraction metamaterial for \(\vartheta_\beta = 2\pi\), where \(e^{i\vartheta_\beta/2} = -1\), given by

\[
n_\beta = -\sqrt{\mathcal{T}_\beta}.
\]

(63)

This result shows that a metamaterial like behaviour can be achieved in the presence of a non-commutative geometry with artificial control of the non-commutative parameters in material media.
VI. MODIFIED MAXWELL EQUATIONS FROM CPT-ODD STANDARD MODEL EXTENSION AND MINIMUM LENGTH

The set of modified Maxwell equations in the CPT-odd standard model extension and minimum length described above is given by

\[ (1 + 2\beta\hbar^2\Box) \nabla \cdot E - c\chi V \cdot (1 + 2\beta\hbar^2\Box) B = \frac{\rho}{\varepsilon_0} \quad (64) \]
\[ \nabla \cdot B = 0, \quad (65) \]
\[ \nabla \times E = -\frac{\partial B}{\partial t}, \quad (66) \]
\[ (1 + 2\beta\hbar^2\Box) \left( \nabla \times B - \frac{1}{c^2} \frac{\partial E}{\partial t} \right) - \chi V_0 (1 + \beta \hbar^2\Box) B + \chi V \times (1 + \beta \hbar^2\Box) \frac{E}{c} = \mu_0 J \quad (67) \]

We can define field dependent generalized charge density and charge current, given by

\[ \mathcal{\tilde{\rho}}(B, V) = \frac{\rho}{\varepsilon_0} + c\chi V \cdot (1 + 2\beta\hbar^2\Box) B \]
and

\[ \mu_0 \mathcal{\tilde{J}}(E, B, V, V_0) = \chi V_0 (1 + \beta \hbar^2\Box) B - \chi V \times (1 + \beta \hbar^2\Box) \frac{E}{c} + \mu_0 J \]

The equations can then be rewritten as

\[ (1 + 2\beta\hbar^2\Box) \nabla \cdot E = \frac{\mathcal{\tilde{\rho}}(B, V)}{\varepsilon} \quad (68) \]
\[ \nabla \cdot B = 0, \quad (69) \]
\[ \nabla \times E = -\frac{\partial B}{\partial t}, \quad (70) \]
\[ (1 + 2\beta\hbar^2\Box) \left( \nabla \times B - \frac{1}{c^2} \frac{\partial E}{\partial t} \right) = \mu_0 \mathcal{\tilde{J}}(E, B, V, V_0) \quad (71) \]

We now split the current density in terms of free, polarization and magnetization and LV contributions

\[ J = J_f + J_P + J_M + J_W, \quad (72) \]
\[ \rho = \rho_f + \rho_P + \rho_W \quad (73) \]
where we have generalized current and charge density responses to the LV background given by

\[ J_W = -\chi V_0 M - \frac{\chi}{\mu_0 \varepsilon_0} V \times \frac{P}{c} \quad (74) \]

\[ \rho_W = -\varepsilon_0 \mu_0 \chi V \cdot M. \quad (75) \]

The generalized charge density can be written

\[ \frac{\tilde{\rho}(B, V)}{\varepsilon_0} = \frac{\rho_f - \nabla \cdot P}{\varepsilon_0} - \mu_0 \chi V \cdot M + \mu_0 \chi V \cdot (1+2\beta h^2 \Box) B \]

We can then write

\[ \frac{\tilde{\rho}(B, V)}{\varepsilon_0} = \frac{\rho_f - \nabla \cdot P}{\varepsilon_0} + \mu_0 \chi V \cdot H. \]

and the generalized current density is given by

\[ \mu_0 \tilde{J}(E, B, V, V_0) = \chi V_0 (1+\beta h^2 \Box) B - \chi V \times (1+\beta h^2 \Box) \frac{E}{c} \]

\[ + \mu_0 \chi V_0 M - \frac{\chi}{\varepsilon_0} V \times \frac{P}{c} + \mu_0 \left( J_f + \frac{\partial P}{\partial t} + \nabla \times M \right) \quad (76) \]

we can write it

\[ \mu_0 \tilde{J}(E, B, V, V_0) = \mu_0 \chi V_0 H - \frac{\chi}{\varepsilon_0 c} V \times D - \mu_0 \left( J_f + \frac{\partial P}{\partial t} + \nabla \times M \right) \]

The Modified Maxwell equations in material media are then written in the form

\[ \nabla \cdot D_\beta = \rho_f + \frac{\chi}{c} V \cdot H, \quad (77) \]

\[ \nabla \cdot [H_\beta + M] = 0, \quad (78) \]

\[ \nabla \times [D_\beta - P] = -\frac{1}{c^2} \frac{\partial [H_\beta + M]}{\partial t}, \quad (79) \]

\[ \nabla \times H_\beta = J_f + \frac{\partial D_\beta}{\partial t} + \chi V_0 H_\beta - c \chi V \times D_\beta, \quad (80) \]

We note that these set of equations include the presence of the vectorial background due to Lorentz violation interacting with the material fields. Additionally, the presence of non-commutativity is encapsulated in the fields and can be used in the way to build a suitable metamaterial.
VII. CONCLUDING REMARKS

We have considered an standard model extension involving a CPT-odd sector in the presence of a non-commutative geometry with minimum length. We considered the cases of electromagnetic waves and material media. In particular, we considered the case of metamaterial behaviour leading to the presence of a negative index of refraction in dependent of the modified dielectric contributions. We also derived a set of modified Maxwell equations in material media, where the presence of the tensor background, as a result of Lorentz violation, explicitly appears in the interaction terms with the material fields.

As the standard model extension can be used in this context to derive suitable metamaterials with novel properties, we can consider this result as an important step to determine the relationship between metamaterial behaviour, in particular with negative index refraction, with the presence of non-commutative geometry and Lorentz violation.

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