THE BOLTZMANN EQUATION WITH FRICTIONAL FORCE
FOR VERY SOFT POTENTIALS IN THE WHOLE SPACE

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Abstract. We develop a general energy method for proving the optimal time decay rates of the higher-order spatial derivatives of solutions to the Boltzmann-type and Landau-type systems in the whole space, for both hard potentials and soft potentials. With the help of this method, we establish the global existence and temporal convergence rates of solution near a given global Maxwellian to the Cauchy problem on the Boltzmann equation with frictional force for very soft potentials i.e. \(-3 < \gamma < -2\).

1. Introduction and main results.

1.1. The problem. This paper is concerned with the following Boltzmann equation with external force proportional to the macroscopic velocity \(u(t, x)\) in the whole space \(\mathbb{R}^3\):

\[
\begin{align*}
\partial_t F + v \cdot \nabla_x F - \zeta u \cdot \nabla_v F &= Q(F, F), \\
u &= \frac{\int_{\mathbb{R}^3} v Fdv}{\int_{\mathbb{R}^3} Fdv}
\end{align*}
\]

with initial data

\[
F(0, x, v) = F_0(x, v).
\]

Here \(F = F(t, x, v) \geq 0\) stands for the velocity distribution functions for the particles with position \(x = (x_1, x_2, x_3) \in \mathbb{R}^3\) and velocity \(v = (v_1, v_2, v_3) \in \mathbb{R}^3\) at time \(t \geq 0\) and the term \(\zeta u\) represents the frictional force which is proportional to the macroscopic velocity \(u(t, x)\) and we can normalize the positive constant \(\zeta\) to be 1.

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without loss of generality. The bilinear collision operator $Q(F, G)$ acting only on the velocity variable is defined by

$$Q(F, G)(v) = \int_{\mathbb{R}^3 \times S^2} |v - v_*|^\gamma q_0(\theta) \{ F(v_*)G(v') - F(v)G(v) \} d\omega dv_*,$$  \hspace{1cm} (3)$$

where in terms of velocities $v_*$ and $v$ before the collision, velocities $v'$ and $v'_*$ after the collision are defined by

$$v' = v - [(v - v_*) \cdot \omega] \omega, \quad v'_* = v_* + [(v - v_*) \cdot \omega] \omega, \quad \omega \in S^2$$

which follow from the conservation of momentum and kinetic energy during the collision process

$$v + v_* = v' + v'_*, \quad |v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2$$

The function $|v - v_*|^\gamma q_0(\theta)$ in (3) is the cross-section depending only on $\cos \theta = (v - v_*) \cdot \omega/|v - v_*|$ and $|v - v_*|$, and it satisfies $-3 < \gamma \leq 1$ and it is assumed to satisfy the Grad’s angular cutoff assumption

$$0 \leq q_0(\theta) \leq C |\cos \theta|. \hspace{1cm} (4)$$

The exponent $\gamma$ is determined by potential of intermolecular forces, which is called the hard potential when $0 \leq \gamma \leq 1$ including the Maxwell model $\gamma = 0$ and the hard sphere model $\gamma = 1, q_0(\theta) = C |\cos \theta|$, and the soft potential when $-3 < \gamma < 0$, especially, the case $-2 \leq \gamma < 0$ is called the moderately soft potential and $-3 < \gamma < -2$ very soft potential, cf.[1, 2, 3, 16].

The Boltzmann equation with frictional force describes the motion of the rarefied gas with friction force on the microscopic aspect. The corresponding compressible Euler system with frictional damping describes the macroscopic motion of the fluid. As is well known, there is a close relationship between the two equations. For example, the first-order approximation of the Boltzmann equation with frictional force by Hilbert expansion is the compressible Euler system with frictional damping formally when the number of Knudsen is small. Also for the compressible Euler equation with frictional damping, Hsiao-Liu [22] has shown that large time behavior of its global solutions can be described by the compressible flow through porous media. Since there are many open problems in Euler system with frictional damping, We can use the properties of the solution of the microscopic equation i.e. Boltzmann equation with frictional force to study the properties of the macroscopic equation, especially for the large time of the solution. So the well-posed of the solution to the Boltzmann equation with frictional force becomes very meaningful.

Motived by the nonlinear energy method developed by Guo [19] and Liu-Yang-Yu [25] to deal with the Boltzmann equation, Vong [28], Wang-Jiang [29] and Lei-Wan [23] have obtained the desired global solvability result near Maxwellian of such a global solution towards equilibrium to the Boltzmann equation with frictional force for $-2 \leq \gamma \leq 1$. A natural question is that how to treat the very soft potentials i.e. $-3 < \gamma < -2$. For other kinetic models, such as Vlasov-Poisson(or Maxwell)-Boltzmann systems, Landau-type systems, interested readers can refer to the references [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 18, 20, 24, 27, 31] for more details.

In the following text, we will solve this problem, and in the proving procedure, we will provide a general method for proving the time decay rates of the higher-order spatial derivatives of solutions to the Boltzmann-type and Landau-type systems in the whole space, which conclude both hard potentials and soft potentials.
To this end, if we set
\[ F(t, x, v) = \mu + \mu^\frac{1}{2} f(t, x, v), \quad \mu = \mu(v) = (2\pi)^{-3/2} e^{-|v|^2/2} \]
then the Cauchy problem (1)-(2) can be reformulated as
\[
\partial_t f + v \cdot \nabla_x f - u \cdot \nabla_v f + \frac{1}{2} u \cdot v f + u \cdot v f + Lf = \Gamma(f, f),
\]
\[
\left(1 + \int_{\mathbb{R}^3} \mu^\frac{1}{2} f dv\right) u = \int_{\mathbb{R}^3} \mu^\frac{1}{2} vz f dv
\]
with given initial data
\[ f(0, x, v) = f_0(x, v). \]
Here \( Lf = -\mu^{-\frac{1}{2}} \left[ Q \left( \mu, \mu^\frac{1}{2} f \right) + Q \left( \mu^\frac{1}{2} f, \mu \right) \right] \) and \( \Gamma(f, g) = \mu^{-\frac{1}{2}} Q \left( \mu^\frac{1}{2} f, \mu^\frac{1}{2} g \right) \) denote the linearized and nonlinear collision operators, respectively. It is well-known, cf. [17], that the null space of \( L \) is
\[ \ker L = \text{span}\left\{ \mu^\frac{1}{2}, \ v_1 \mu^\frac{1}{2}, \ v_2 \mu^\frac{1}{2}, \ v_3 \mu^\frac{1}{2}, \ |v|^2 \mu^\frac{1}{2} \right\}. \]
For given \( f(t, x, v) \), one can decompose \( f(t, x, v) \) uniquely as
\[ f = Pf + (I - P)f \]
with \( P \) being the orthogonal projection from \( L^2_v(\mathbb{R}^3) \) to \( \ker L \) defined by
\[ Pf = \left\{ \begin{array}{l}
\mu^\frac{1}{2} f = \left\{ a(t, x) + v \cdot b(t, x) + (|v|^2 - 3)c(t, x) \right\} \mu^\frac{1}{2}, \\
a = a = \int_{\mathbb{R}^3} \mu^\frac{1}{2} f dv, \\
b = b = \int_{\mathbb{R}^3} v \mu^\frac{1}{2} f dv, \\
c = c = a = \frac{1}{2} \int_{\mathbb{R}^3} (|v|^2 - 3) \mu^\frac{1}{2} f dv.
\end{array} \right. \]
\( Pf \) and \((I - P)f\) are called the macroscopic component and the microscopic component of \( f(t, x, v) \), respectively.

Under the Grad’s angular cutoff assumption (4), \( L \) is non-negative and coercive in the sense that there is a constant \( \kappa_0 > 0 \) such that
\[
\int_{\mathbb{R}^3} f L f dv \geq \kappa_0 \int_{\mathbb{R}^3} \nu(v) |(I - P)f|^2 dv, \quad \nu = \nu(v) \sim (1 + |v|)^7
\]
holds for \( f = f(t, x, v) \). Furthermore, \( L \) can be written as \( L = \nu - K \), where \( \nu \) denotes the collision frequency and \( K \) is a velocity integral operator with a real symmetric integral kernel \( K(v, v_*) \). The explicit representation of \( \nu \) and \( K \) will be given in the next section.

**Notations.**

- \( C \) denotes some positive constant (generally large) and \( \lambda \) denotes some positive constant (generally small), where both \( C \) and \( \lambda \) may take different values in different places.
- \( A \lesssim B \) means that there is a generic constant \( C > 0 \) such that \( A \leq CB \).
- \( A \sim B \) means \( A \lesssim B \) and \( B \lesssim A \).
- For an integer \( m \geq 0 \), we use \( H^m_v \), \( H^m_x \), \( H^m_v \) to denote the usual Hilbert spaces \( H^m_v(\mathbb{R}^3_\times \mathbb{R}^3_v) \), \( H^m(\mathbb{R}^3_\times \mathbb{R}^3_v) \), \( H^m_v(\mathbb{R}^3_v) \), respectively, and \( L^2_v, L^2_x, L^2_v \) are used for the case when \( m = 0 \). \( \langle \cdot, \cdot \rangle \) denotes the \( L^2 \) inner product in \( \mathbb{R}^3_v \), with the \( L^2 \) norm \( | \cdot | \) or \( | \cdot |_2 \). And \( \langle \cdot, \cdot \rangle \) denotes the \( L^2 \) inner product in \( \mathbb{R}^3_\times \mathbb{R}^3_v \) or \( \mathbb{R}^4_v \) with the \( L^2 \) norm \( \| \cdot \| \).
• For $q \geq 1$, we also define the mixed velocity-space Lebesgue space $Z_q = L^2_x(L^2_v) = L^2(\mathbb{R}^3, L^2(\mathbb{R}^3_v))$ with the norm

$$\|f\|_{Z_q} = \left( \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3_v} |f(x,v)|^{q/2} dv \right)^{2/q} dx \right)^{1/q}$$

for $f = f(x,v) \in Z_q$.

• We introduce the norms $\| \cdot \|_{H^m}$ with $m \geq 0$ given by

$$\|f\|^2_{H^m} \equiv \|f\|^2_{L^2(H^m)}, \quad \|f\|^2_{H^m} \equiv \|f\|^2_{L^2(H^m)}, \quad \|f\|^2_{H^m} \equiv \|f\|^2_{H^m}.$$  

Here $H^m = H^m(\mathbb{R}^3)$ is the standard homogeneous $L^2$ based Sobolev space equipped with norm $\|f\|^2_{H^m} = \int_{\mathbb{R}^3} \|\partial_k f\|_{L^2}^2 dk$. $B_C \subset \mathbb{R}^3$ denotes the ball of radius $C$ centered at the origin, and $L^2(B_C)$ stands for the space $L^2$ over $B_C$ and likewise for other spaces.

• For multi-indices $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)$, we denote $\partial_\beta = \partial_{\beta_1} \partial_{\beta_2} \partial_{\beta_3}$, $\partial_\alpha = \partial_{\alpha_1} \partial_{\alpha_2} \partial_{\alpha_3}$. And for simplicity, we employ $\epsilon_i$ to denote $\alpha_i = 1$ and $\beta_i = 0$ for $j \neq i$. As usual, the length of $\alpha$ is $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$, and $\beta \leq \alpha$ means that $\beta_j \leq \alpha_j$ for each $j = 1, 2, 3$, and $\alpha' < \alpha$ means that $\alpha' \leq \alpha$ and $|\alpha'| < |\alpha|$.

### 1.2. Main results.

Before giving our main results, we introduce the following mixed time-velocity weight function

$$\tilde{w}_l_{-|\beta|,-\gamma}(t,v) = (v)^{-\gamma} \tilde{w}_{l-|\beta|,-\gamma}(t,v),$$  

where $0 < \gamma < 1, |\beta| \leq l, 0 < \gamma < \frac{1}{4}$, which had been applied in [12, 23] to get the extra dissipative terms by the energy estimates such as:

$$(1 + t)^{-1-\gamma} \left\| \langle v \rangle \tilde{w}_{l-|\beta|,-\gamma} \partial_\beta \{ I - P \} f \right\|^2$$

which can control the following terms:

$$\sum_{|\alpha_1| \geq 1} \left( \tilde{w}_{l-|\beta|,-\gamma} \partial_\beta \{ I - P \} f \tilde{w}_{l-|\beta|,-\gamma} \partial_\alpha \{ I - P \} f \right)$$

$$= \sum_{|\alpha_1| \geq 1} \left( \tilde{w}_{l-|\beta|,-\gamma} \partial_\alpha u \tilde{w}_{l-|\beta|,-\gamma} \partial_\beta \{ I - P \} f \tilde{w}_{l-|\beta|,-\gamma} \partial_\alpha \{ I - P \} f \right)$$

$$\lesssim \sum_{|\alpha_1| \geq 1} \int_{\mathbb{R}^3_x \times \mathbb{R}^3_v} |\tilde{w}_{l-|\beta|,-\gamma} \partial_\alpha u \langle v \rangle^{-\gamma} \tilde{w}_{l-|\beta|,-\gamma} \partial_\beta \{ I - P \} f \tilde{w}_{l-|\beta|,-\gamma} \partial_\alpha \{ I - P \} f| \langle v \rangle^{-\gamma} |\tilde{w}_{l-|\beta|,-\gamma} \partial_\beta \{ I - P \} f| dv dx$$

if and only if $-2 \leq \gamma < 0$ and $u$ has a better temporal decay estimate. The reference [23] gives a detailed proof. However, when $-3 < \gamma < -2$, the above strategy can not hold any more!

To overcome the above difficulties, we introduce the following general weight function $w_{l-|\beta|,\kappa}(t,v)$

$$w_{l-|\beta|,\kappa} \equiv w_{l-|\beta|,\kappa}(t,v) = (v)^{\kappa} \langle \tilde{w}_{l-|\beta|,-\gamma} \rangle \tilde{w}_{l-|\beta|,-\gamma} \partial_\beta \{ I - P \} f,$$

$$\kappa \geq 0, \quad 0 < q \ll 1,$$  

where $\tilde{w}_{l-|\beta|,-\gamma}$ is the time-velocity weight function given in (9).
where the precise range of the parameter $\vartheta$ will be specified later, also define several temporal energy functionals which stands for the energy of the gas molecules:

$$E^{(j)}_{t,N,\kappa}(t) = \sum_{|\alpha|+|\beta| \leq N, |\alpha| \geq j} \left\| w_{t-|\beta|,\kappa} \partial^\beta f \right\|^2$$  \hspace{1cm} (13)

where $N \geq 0$ is an integer, and $\ell \geq N$ is a constant. Meanwhile the energy dissipation rate functional $D^{(j)}_{t,N,\kappa}(t)$ corresponding to $E^{(j)}_{t,N,\kappa}(t)$ is defined as

$$D^{(j)}_{t,N,\kappa}(t) = \sum_{|\alpha| \leq N-1} \| \nabla \partial^\alpha (a, b, c) \|^2 + \| \nabla b \|^2 + \sum_{|\alpha|+|\beta| \leq N, |\alpha| \geq j} \left\| \mu^\frac{1}{2} w_{t-|\beta|,\kappa} \partial^\beta \{I-P\} f \right\|^2 + (1+t)^{-(1+\vartheta)} \sum_{|\alpha|+|\beta| \leq N, |\alpha| \geq j} \left\| (v) \frac{1}{2} w_{t-|\beta|,\kappa} \partial^\beta \{I-P\} f \right\|^2. \hspace{1cm} (14)$$

**Theorem 1.1.** Assume $-3 < \gamma < -2$, $0 < \vartheta \leq \min\{\frac{1}{2}, \frac{3}{1-2\vartheta}\}$, $0 < q \ll 1$, $\langle v \rangle = \sqrt{1+v^2}$, $F_0(x, v) = \mu(v) + \mu^\frac{1}{2}(v) f_0(x, v) \geq 0$ and take $l^* \geq -\gamma l_0 - \frac{3}{2} \gamma - \frac{3}{4} \gamma^* \sigma^*$ with $\sigma^* = \max_{|\alpha|+|\beta| \leq N} \{\sigma_{|\alpha|+|\beta|,|\beta|}\}$, $l_j \geq N+4$ for $0 \leq j \leq N-1$ and $l_{j-1} = l_j + 2$ for $1 \leq j \leq N-1$, then if

$$Y_0 = \sum_{|\alpha|+|\beta| \leq N} \left\| \langle v \rangle^{|\alpha|-|\beta|} e^{q(v) \partial^\beta} f_0 \right\| + \| f_0 \|_{L_1} \hspace{1cm} (15)$$

is sufficiently small, there exists a unique global solution $f(t, x, v)$ to the Cauchy problem (5) – (6) such that $F(t, x, v) = \mu(v) + \mu^\frac{1}{2}(v) f(t, x, v) \geq 0$, and also have

$$E^{(j)}_{t_j,N,\kappa,\gamma}(t) \lesssim (1+t)^{-\frac{3}{2}+\gamma^*} Y_0^2. \hspace{1cm} (16)$$

where the energy functional $E^{(j)}_{t_j,N,\kappa,\gamma}(t)$ is defined in (13), $\sigma_{n,j}$ is defined as:

$$\sigma_{n,j} - \sigma_{n,j-1} = \frac{2(1+\gamma)}{\gamma-1} (1+\vartheta), \hspace{0.5cm} \sigma_{n,0} = 0, \hspace{0.5cm} 1 \leq j \leq n.$$

**Remark 1.**

- It is worth pointing that we use the weighted functional $e^{\frac{q(v)}{1+\gamma^*}}$ in (15) instead of $e^{\frac{q(v)}{1+\vartheta}}$ in [12], the advantage is that the requirement of perturbation $f(0, x, v) = f_0(x, v)$ is more weaker in the term $\|w_{t-|\beta|,\kappa} \partial^\beta f_0\|$ of $Y_0$ in (15).
- In our proof, we provide a general method for proving the optimal time decay rates of the higher-order spatial derivatives of solutions to the Boltzmann-type and Landau-type systems in the whole space for both hard potentials and soft potentials. This method is different from that in [21].

Now we present the main ideas in the proof as the following three folds:
First of all, take \( w_{\ell - |\beta|, \kappa} \) with \( \kappa = -\gamma \), as the above discussion, when \(-3 < \gamma < -2\), unlike (11), we estimates the terms as follows:

\[
\sum_{|\alpha_1| \geq 1} \left( \partial^{\alpha_1} u \cdot \nabla \cdot \partial_\beta^{-\alpha_1} \{ I - P \} f, w_{\ell - |\beta|, -\gamma}^2 \partial_\beta \{ I - P \} f \right)
\]

\[
= \sum_{|\alpha_1| \geq 1} \left( \partial^{\alpha_1} u (v)^{-\gamma} \cdot w_{\ell - |\beta + e_i|, -\gamma} \partial_\beta^{-\alpha_1} \{ I - P \} f, w_{\ell - |\beta|, -\gamma} \partial_\beta \{ I - P \} f \right)
\]

\[
= \sum_{|\alpha_1| \geq 1} \left( \partial^{\alpha_1} u (v)^{-\gamma} \cdot w_{\ell - |\beta + e_i|, -\gamma} \partial_\beta^{-\alpha_1} \{ I - P \} f, \langle v \rangle \frac{3}{2} w_{\ell - |\beta|, -\gamma} \partial_\beta \{ I - P \} f \right)
\]

\[
\lesssim \mathcal{E}_{\ell + \frac{3}{2}, N, -\gamma}^{(0)}(t) \left( \sum_{|\alpha| \leq N-1} \left\| \nabla \partial_\alpha b \right\|^2 + \left\| \langle v \rangle \frac{3}{2} w_{\ell - |\beta|, -\gamma} \partial_\beta \{ I - P \} f \right\|^2 \right)
\]

Thus, under the smallness of \( \mathcal{E}_{\ell + \frac{3}{2}, N, -\gamma}^{(0)}(t) \), one has by the energy estimates:

\[
\frac{d}{dt} \mathcal{E}_{\ell, N, -\gamma}^{(0)}(t) + \mathcal{D}_{\ell, N, -\gamma}^{(0)}(t) \leq 0
\]

With the help of linear decay analysis and Duhamel principle, if we suppose the energy function \( \mathcal{E}_{\ell + \frac{3}{2}, N, -\gamma}^{(0)}(t) \) with \( \ell = l_0 \) is sufficiently small, then

\[
(1 + t)^{\frac{3}{2}} \mathcal{E}_{l_0, N, -\gamma}^{(0)}(t) \lesssim \varepsilon
\]

where \( \varepsilon \) is a some small positive constant.

For \( 1 \leq j \leq N - 1 \), similar to the energy estimates of \( \mathcal{E}_{\ell, N, -\gamma}^{(0)}(t) \), one also has

\[
\frac{d}{dt} \mathcal{E}_{\ell, N, -\gamma}^{(j)}(t) + \mathcal{D}_{\ell, N, -\gamma}^{(j)}(t) \leq \chi_j \sum_{j_1 \leq j - 1} \mathcal{E}_{\ell, N, -\gamma}^{(j_1)}(t) \mathcal{E}_{\ell + \frac{3}{2}, N, -\gamma}^{(j - j_1)}(t)
\]

by mathematical induction, where the \( \chi \) is the characteristic function. If we take \( l_{j - 1} = l_j + 2 \) and suppose the energy function \( \mathcal{E}_{l_0 + \frac{3}{2}, N, -\gamma}^{(0)}(t) \) is sufficiently small, then

\[
(1 + t)^{\frac{3}{2} + j} \mathcal{E}_{l_j, N, -\gamma}^{(j)}(t) \lesssim \varepsilon
\]

which implies that

\[
(1 + t)^{\frac{3}{2} + j} \left\| \nabla^j u \right\|^2 \lesssim \varepsilon
\]

(ii). Secondly, unlike (11) and (17), we take \( w_{\ell - |\beta|, \kappa} \) with \( \kappa = 1, \ell = l^* \) where \( l^* \) is an undetermined constant which is strictly greater than \( -\gamma \left( l_0 + \frac{5}{2} \right) \), since

\[
w_{l^* - |\beta|, 1}(t, v) = w_{l^* - |\beta|, 1}(t, v) \times w_{l^* - |\beta|, 1, 1}(t, v) \times \langle v \rangle,
\]

then

\[
\sum_{|\alpha_1| \geq 1} \left( \partial^{\alpha_1} u \cdot \nabla \cdot \partial_\beta^{-\alpha_1} \{ I - P \} f, w_{l^* - |\beta|, 1}^2 \partial_\beta \{ I - P \} f \right)
\]

\[
\lesssim \sum_{|\alpha_1| \geq 1} \int_{\mathbb{R}^3_+ \times \mathbb{R}^1_+} \left| \partial^{\alpha_1} u \right| \left| \langle v \rangle \right|^\frac{3}{2} w_{l^* - |\beta + e_i|, 1} \partial_\beta^{-\alpha_1} \{ I - P \} f \left| dv \right| dx
\]

\[
\times \left| \langle v \rangle \right|^\frac{3}{2} w_{l^* - |\beta|, 1} \partial_\beta \{ I - P \} f \left| d\mu \right|
\]
which can be controlled by the extra dissipative terms
\[(1 + t)^{-1-\vartheta} \left\| \langle v \rangle^{\frac{1}{2}} w_{t-|\beta|,1} \partial_\beta^\alpha I - P \right\|^2 \]
and
\[(1 + t)^{-1-\vartheta} \left\| \langle v \rangle^{\frac{1}{2}} w_{t-|\beta+\epsilon_1|,1} \partial_\beta^{\alpha+\epsilon_1} I - P \right\|^2 \]
where we use the time decay property (18). However, for the transport term $v \cdot \nabla x f$,
\[
(\partial_\beta^\alpha (v \cdot \nabla_x (I - P) f), w_{t-|\beta|,1} \partial_\beta^\alpha (I - P) f)
\]
which leads to how to control
\[
\left\| \langle v \rangle^{\frac{1}{2}} w_{t-|\beta-\epsilon_1|,1} \partial_\beta^{\alpha+\epsilon_1} (I - P) f \right\|^2.
\]
Since $\frac{1}{2} < -\frac{1}{3} < 1 < \frac{1}{2}$ holds for all $-3 < \gamma < -2$, it does not lead to the increase of the weight if we neglect the factor $(1 + t)^{-1-\vartheta}$ in the extra dissipative term (10). Therefore, if we set different time increase rate $\sigma_{n,j}$ for
\[
\sum_{|\alpha|+|\beta|=n,|\beta|=j} \left\| w_{t-\beta} \partial_\beta^\alpha (I - P) f \right\|^2
\]
with
\[
\sigma_{n,j} - \sigma_{n,j-1} = \frac{2(1 + \gamma)}{\gamma - 1} (1 + \vartheta), \sigma_{n,0} = 0, 1 \leq j \leq n
\]
then
\[
\sum_{|\alpha|+|\beta|=n, |\beta|=j, 1 \leq k \leq n} (1 + t)^{-\sigma_{n,k}} \left\| \langle v \rangle^{\frac{1}{2}} w_{t-|\beta|+1,1} \partial_\beta^{\alpha+\epsilon_1} (I - P) f \right\|^2
\]
\[
\lesssim \sum_{|\alpha|+|\beta|=n, |\beta|=j, 1 \leq k \leq n} (1 + t)^{-\sigma_{n,j-1} - \frac{2(1 + \gamma)}{\gamma - 1} (1 + \vartheta)} \left\| \langle v \rangle^{\frac{1}{2}} w_{t-|\beta|+1,1} \partial_\beta^{\alpha+\epsilon_1} (I - P) f \right\|^{\frac{2(1 + \gamma)}{\gamma - 1}}
\]
\[
\times \left\| \langle v \rangle^{\frac{1}{2}} w_{t-\beta+|\alpha|,1} \partial_\beta^{\alpha+\epsilon_1} (I - P) f \right\|^{-\frac{2\gamma-6}{\gamma-1}}
\]
\[
\lesssim \sum_{|\alpha|+|\beta|=n, |\beta|=j, 1 \leq k \leq n} \left\{ \left(1 + t\right)^{-\sigma_{n,j-1} - 1 - \vartheta} \left\| \langle v \rangle^{\frac{1}{2}} w_{t-\beta+|\alpha|,1} \partial_\beta^{\alpha+\epsilon_1} (I - P) f \right\|^2 \right.\)

\[
\left. + (1 + t)^{-\sigma_{n,j-1}} \left\| \langle v \rangle^{\frac{1}{2}} w_{t-\beta+|\alpha|,1} \partial_\beta^{\alpha+\epsilon_1} (I - P) f \right\|^2 \right\}.
\]

Based on the above argument, if we assume that $\mathcal{E}^{(0)}_{t_0 + \frac{1}{2}, N, -\gamma}(t)$ is sufficiently small, then we have
\[
(1 + t)^{-\sigma_{n,j}} ||w_{t+|\beta|+1,1} \partial_\beta^\alpha I - P||^2 \lesssim \varepsilon.
\]

(iii). Finally, in order to guarantee the smallness of $\mathcal{E}^{(0)}_{t_0 + \frac{1}{2}, N, -\gamma}(t)$, we split it into macroscopic part and microscopic part, i.e.
\[
\sum_{|\alpha|+|\beta| \leq N} ||w_{t_0 + \frac{1}{2} - |\alpha|, -\gamma} \partial_\beta^\alpha P f||^2
\]
We list in the following lemma velocity weighted estimates on the collision frequency

\[ \nu \]

\[ L \]

estimates on the linearized collision operator

this section, we cite some fundamental results concerning the weighted energy type

proofs can be found in \([12, 30]\). The first lemma concerns the linearized operator

L \[ L \]

Preliminaries. Our manuscript is organized as follows. In Section 2, we will give some key

Recall that

\[ L \]

\[ K \]

L \[ L \]

2.

2. Preliminaries. Recall that \( \nu = \nu - K \) is defined by

\[ \nu(v) = \int_{\mathbb{R}^3 \times S^2} |v - v_*|^7 q_0(\theta) \mu(v_\ast) d\omega dv_\ast \sim (1 + |v|)^7, \]

and

\[ K f(v) = \int_{\mathbb{R}^3 \times S^2} |v - v_*|^7 q_0(\theta) \mu(v_\ast) \frac{1}{2} \mu(v') \frac{1}{2} f(v') d\omega dv_\ast + \int_{\mathbb{R}^3 \times S^2} |v - v_*|^7 q_0(\theta) \mu(v_\ast) \frac{1}{2} \mu(v') \frac{1}{2} f(v') d\omega dv_\ast - \int_{\mathbb{R}^3 \times S^2} |v - v_*|^7 q_0(\theta) \mu(v_\ast) \frac{1}{2} \mu(v) \frac{1}{2} f(v_\ast) d\omega dv_\ast = \int_{\mathbb{R}^3} K(v, v_\ast) f(v_\ast) dv_\ast. \]

We list in the following lemma velocity weighted estimates on the collision frequency

\[ \nu(v) \]

and integral operator \( K \) with respect to the velocity weighted function \((12)\). In this section, we cite some fundamental results concerning the weighted energy type

estimates on the linearized collision operator \( L \) and the nonlinear term \( \Gamma \), whose

proofs can be found in \([12, 30]\). The first lemma concerns the linearized operator \( L \).
Lemma 2.1. (cf. [12]) Let $0 < \kappa < 3$, $\ell \in \mathbb{R}$, and $0 \leq q \leq 1$, If $|\beta| > 0$, and for any $\eta > 0$, there is $C_\eta > 0$ such that

$$\int_{\mathbb{R}^3} w_{\ell,\kappa}^2(v)\partial_\beta (\nu(v)f)\partial_\beta f dv \geq \int_{\mathbb{R}^3} \nu(v)w_{\ell,\kappa}^2(v)|\partial_\beta f|^2 dv$$

$$- \eta \sum_{|\beta_1| \leq |\beta|} \int_{\mathbb{R}^3} \nu(v)w_{\ell,\kappa}^2(v)|\partial_{\beta_1} f|^2 dv$$

$$- C_\eta \int_{\mathbb{R}^3 \times \mathbb{R}^3} \chi_{|v| \leq 2C_\alpha(v)^{-2\gamma\ell}} |f|^2 dv.$$  \hfill (25)

Furthermore, if $|\beta| \geq 0$, then for any $\eta > 0$, there is $C_\eta > 0$ such that

$$\left| \int_{\mathbb{R}^3} w_{\ell,\kappa}^2(v)\partial_\beta (Kf) dv \right| \leq \left\{ \eta \sum_{|\beta_1| \leq |\beta|} \left( \int_{\mathbb{R}^3} \nu(v)w_{\ell,\kappa}^2(v)|\partial_{\beta_1} f|^2 dv \right)^{\frac{1}{2}} \right.$$  

$$\left. + C_\eta \left( \int_{\mathbb{R}^3} \chi_{|v| \leq 2C_\alpha(v)^{-2\gamma\ell}} |f|^2 dv \right)^{\frac{1}{2}} \right\} \left( \int_{\mathbb{R}^3} \nu(v)w_{\ell,\kappa}^2|g|^2 dv \right)^{\frac{1}{2}}. \hfill (26)$$

The following two lemmas concern the estimates on nonlinear term $\Gamma$. The second lemma is concerned with the corresponding weighted estimates on the nonlinear term $\Gamma$. For this purpose, similar to that of [26], we can get that

$$\partial_\beta^\alpha \Gamma(g_1, g_2) = \sum \mathcal{C}^\beta_\alpha \mathcal{C}^\alpha_\delta \Gamma_0 \left( \partial_1^\alpha g_1, \partial_2^\alpha g_2 \right) \hfill (27)$$

$$\equiv \sum \mathcal{C}^\beta_\alpha \mathcal{C}^\alpha_\delta \int_{\mathbb{R}^3 \times S^2} \left| v - v' \right|^\gamma q_0(\theta) \partial_{\beta_\delta}[\mu(v_\cdot)^\frac{1}{2}]$$

$$\times \left\{ \partial_1^\alpha g_1(v') \partial_2^\alpha g_2(v') + \partial_1^\alpha g_1(v) \partial_2^\alpha g_2(v') \right. \hfill (25)$$

$$\left. - \partial_1^\alpha g_1(v) \partial_2^\alpha g_2(v_s) - \partial_1^\alpha g_1(v) \partial_2^\alpha g_2(v_s) \right\} d\omega dv_s,$$

where $g_i = g_i(t, x, v)$ ($i = 1, 2$) and the summations are taken for all $\beta_0 + \beta_1 + \beta_2 = \beta, \alpha_1 + \alpha_2 = \alpha$. From Lemma 3 in [26], one can deduce that

**Lemma 2.2.** (cf.[26]) Assume $\kappa \ell \geq 0$. Let $-3 < \gamma < 0$, $N \geq 4$, $g_i = g_i(t, x, v)$, $\beta_0 + \beta_1 + \beta_2 = \beta$ and $\alpha_1 + \alpha_2 = \alpha$, we have the following results:

When $|\alpha_1| + |\beta_1| \leq N$, we have

$$\left< w_{\ell,\kappa}^2 \Gamma^0 \left( \partial_1^\alpha g_1, \partial_2^\alpha g_2 \right), \partial_3^\alpha g_3 \right> \lesssim \sum_{m \leq 2} \left\{ \left| \nabla_v \left\{ \mu^\delta \partial_1^\alpha g_1 \right\} \right| + \left| w_{\ell,\kappa} \partial_1^\alpha g_1 \right| \right\} \hfill (28)$$

$$\times \left| w_{\ell,\kappa} \partial_2^\alpha g_2 \right|_{L^2_v} \left| w_{\ell,\kappa} \partial_3^\alpha g_3 \right|_{L^2_v}$$

or

$$\left< w_{\ell,\kappa}^2 \Gamma^0 \left( \partial_1^\alpha g_1, \partial_2^\alpha g_2 \right), \partial_3^\alpha g_3 \right> \lesssim \sum_{m \leq 2} \left\{ \left| \nabla_v \left\{ \mu^\delta \partial_2^\alpha g_2 \right\} \right| + \left| w_{\ell,\kappa} \partial_2^\alpha g_2 \right| \right\} \hfill (29)$$

$$\times \left| w_{\ell,\kappa} \partial_1^\alpha g_1 \right|_{L^2_v} \left| w_{\ell,\kappa} \partial_3^\alpha g_3 \right|_{L^2_v}.$$
Furthermore, it holds that

\[
\| (v)^{-\gamma} T(g_1, g_2) \|_{L_x^2}^2 \lesssim \max \left\{ \sum_{|\beta| \leq 2} \| (v)^{-\gamma(|\beta|)} \partial_\beta g_1 \|_{L_x^2}^2, \sum_{|\beta| \leq 2} \| (v)^{-\gamma} g_1 \|_{L_x^2} \right\},
\]

(30)

Lemma 2.3. (cf.[23]) Recall the fluid-type system, there is a temporal interactive functional \( E_{\text{int}}^{(j)}(t) \) such that

\[
| E_{\text{int}}^{(j)}(t) | \lesssim \sum_{j \leq |\alpha| \leq N} \| \partial^\alpha f \|^2
\]

and

\[
\frac{d}{dt} E_N^{(j)}(t) + \sum_{j \leq |\alpha| \leq N-1} \| \partial^\alpha \nabla_x (a, b, c) \|^2 \
\lesssim \sum_{j \leq |\alpha| \leq N} \| \partial^\alpha (I - P) f \|^2 + \sum_{j_1 \leq j} E_{l,n,-\gamma}^{(j_1)}(t) D_{l,n,-\gamma}^{(j-j_1)}(t)
\]

(32)

hold for any \( 0 < t \leq T \).

Lemma 2.4. Let \(-3 < \gamma < 0, t \geq 0, t' > \frac{3}{4}, \alpha \geq 0, k = |\alpha|, \) and assume that

\[
\| (v)^{-\gamma} (t+t') f_0 \|_{Z_1} + \| (v)^{-\gamma (t+t')} f_0 \| < \infty.
\]

(33)

Then the evolution operator satisfies

\[
\| (v)^{-\gamma} e^{tB} f_0 \| \lesssim (1 + t) - \sigma_k (\| (v)^{-\gamma (t+t')} f_0 \|_{Z_1} + \| (v)^{-\gamma (t+t')} \partial^\alpha f_0 \|)
\]

(34)

for any \( t > 0 \).

Proof. The proof is similar with Theorem 3.1 of [11], here we omit it for brevity.

The following lemma concerns the nonlinear estimates.

Lemma 2.5. For \( j \leq N - 2, -3 < \gamma < -2, \) it holds that

\[
(\partial_\beta^\alpha (u \cdot \nabla_v f), w_\ell^{-|\beta|,-\gamma} \partial_\beta^\alpha f) \lesssim \chi_j = 0 E_{l+\frac{3}{4},N,-\gamma}^{(0)}(t) D_{l,n,-\gamma}^{(0)}(t)
\]

(35)

\[
+ \chi_{j \geq 1} \sum_{j_1 \leq j-1} E_{l,n,-\gamma}^{(j)}(t) E_{l+\frac{3}{4},N,-\gamma}^{(j-j_1)}(t) + \epsilon D_{l,n,-\gamma}^{(j)}(t)
\]

with \( |\alpha| + |\beta| \leq N, |\alpha| \geq j \).

Proof. For example, for \( j \geq 1, \)

\[
\sum_{|\alpha| \geq j} (\partial^\alpha u_i \partial^\alpha_{\beta+\epsilon_1} (I - P) f, w_\ell^{-|\beta|,-\gamma} \partial^\alpha_\beta (I - P) f)
\]

\[
\lesssim \| \partial^\alpha u_i \|_{L_x^2} \| (v)^{-\frac{3}{2}} w_\ell^{-|\beta|+\epsilon_1,-\gamma} \partial^\alpha_{\beta+\epsilon_1} (I - P) f \| \| (v)^{\frac{3}{2}} w_\ell^{-|\beta|,-\gamma} \partial^\alpha_\beta (I - P) f \|
\]

\[
\lesssim \| \partial^\alpha u_i \|_{L_x^2} \| (v)^{-\frac{3}{2}} w_\ell^{-|\beta|+\epsilon_1,-\gamma} \partial^\alpha_{\beta+\epsilon_1} (I - P) f \| \| (v)^{\frac{3}{2}} w_\ell^{-|\beta|,-\gamma} \partial^\alpha_\beta (I - P) f \|^2 + \epsilon \| w_\ell^{-|\beta|,-\gamma} \partial^\alpha_\beta (I - P) f \|^2
\]

\[
\lesssim E_{l,n,-\gamma}^{(j-1)}(t) E_{l+\frac{3}{4},N,-\gamma}^{(j)}(t) + \epsilon D_{l,n,-\gamma}^{(j)}(t)
\]

(35) follows by the similar way. Thus we have completed the proof of this lemma.

By the same virtue, we have the following two lemmas.
Lemma 2.6. Assume $-3 < \gamma < -2$, $j \leq N - 1$, and $|\alpha| + |\beta| \leq N, |\alpha| \geq j$, one has
\[
\sum_{j_1 \leq j-1} \mathcal{E}_{t,\gamma}(t) + \mathcal{E}_{t,\gamma}(t) + \mathcal{E}_{t,\gamma}(t) + \mathcal{E}_{t,\gamma}(t) + \mathcal{E}_{t,\gamma}(t),
\]
and
\[
\sum_{j_1 \leq j-1} \mathcal{E}_{t,\gamma}(t) + \mathcal{E}_{t,\gamma}(t) + \mathcal{E}_{t,\gamma}(t) + \mathcal{E}_{t,\gamma}(t) + \mathcal{E}_{t,\gamma}(t).
\]

3. The proofs of main results. This section is devoted to proving our main results based on the continuation argument. Local existence for the Cauchy problem (5)-(6) in certain weighted Sobolev space is now well-understood, cf. [20], thus we will get the global existence with the standard continuity argument if we can close a global priori estimate in the same weighted Sobolev space in which the local solution is constructed.

To make the presentation clear, we divide the rest of this section into three subsections. The first one focuses on deducing the temporal time decay of the energy functional on $f(t,x,v)$. The second is to control the energy functional $\mathcal{E}_{t,N,1}(s)$ with the weight function $w_{\ell-|\beta|,1}(t,v)$ which includes the time increment rate. The third is to prove the boundedness of the energy functional $\mathcal{E}_{t,\gamma}(t)$ with the weight function $w_{\ell+\frac{3}{4}-|\beta|,\gamma}(t,v)$. The last is to close the a priori estimate and complete the proof of Theorem 1.1.

3.1. The temporal time decay rates of solution on $f(t,x,v)$. This subsection concentrates on obtaining the Lyapunov inequality for $\mathcal{E}_{t,N,\gamma}(t)$.

Lemma 3.1. Let $\ell \geq N$, then
\[
\frac{d}{dt} \mathcal{E}_{t,N,\gamma}(t) + \mathcal{D}_{t,N,\gamma}(t) + \mathcal{E}_{t,N,\gamma}(t) + \mathcal{E}_{t,N,\gamma}(t) + \mathcal{E}_{t,N,\gamma}(t) + \mathcal{E}_{t,N,\gamma}(t) + \mathcal{E}_{t,N,\gamma}(t) + \mathcal{E}_{t,N,\gamma}(t).
\]

Proof. To this end, we rewrite (5) as
\[
\begin{cases}
\partial_t f + v \cdot \nabla_x f + b \cdot v \mu \frac{\partial}{\partial v} f - u \cdot \nabla_x f + \frac{1}{2} u \cdot v f + (u - b) \cdot v \mu \frac{\partial}{\partial v} f + Lf = \Gamma(f,f), \\
(1 + a)u = b.
\end{cases}
\]
We divide the proof into three steps:

**Step 1.** Take summation over $j \leq |\alpha| \leq N$, with the help of Lemma 2.5, 2.6, it holds that

$$
\frac{d}{dt} \sum_{j \leq |\alpha| \leq N} \| \partial^\alpha f \|^2 + \sum_{1 \leq |\alpha| \leq N} \left( \| \partial^\alpha b \|^2 + \| \nu^{\frac{1}{2}} \partial^\alpha (I - P) f \|^2 \right)
\leq \chi_j = 0 \mathcal{E}_{\ell + \frac{3}{2}, N, -\gamma}(t) D_{\ell, N, -\gamma}(t) + \chi_j \sum_{j_1 \leq j-1} \mathcal{E}_{\ell, N, -\gamma}(t) \mathcal{E}_{\ell + \frac{3}{2}, N, -\gamma}(t).
\tag{40}
$$

**Step 2.** Energy estimates with the weight function $w_{\ell - |\beta|, -\gamma}(t, v)$: Let us write down the time evolution equation of $\{I - P\} f$:

$$
\partial_t \{I - P\} f + v \cdot \nabla_x \{I - P\} f + \nu(v) \{I - P\} f
= K \{I - P\} f + \Gamma(f, f) + u \cdot \nabla_x \{I - P\} f - \frac{1}{2} u \cdot v \{I - P\} f
+ u \cdot \nabla_x P f - \frac{1}{2} u \cdot v P f - v \cdot \nabla_x P f + P (v \cdot \nabla_x f + \frac{1}{2} u \cdot v f - u \cdot \nabla_v f).
\tag{41}
$$

Firstly, multiplying (41) by $w_{\ell - |\beta|, -\gamma}(t, v) \partial^\alpha \{I - P\} f$ with $|\alpha| = j$, and integrating it over $\mathbb{R}^3 \times \mathbb{R}^3$, and applying Lemma 2.5, 2.6, we have

$$
\frac{1}{2} \frac{d}{dt} \| w_{\ell - |\beta|, -\gamma} \partial^\alpha \{I - P\} f \|^2 + \| w_{\ell - |\beta|, -\gamma} \partial^\alpha \{I - P\} f \|^2_v
+ \frac{1}{(1 + t)^{1+\sigma}} \| \nu^{\frac{1}{2}} w_{\ell - |\beta|, -\gamma} \partial^\alpha \{I - P\} f \|^2
\leq \chi_j = 0 \mathcal{E}_{\ell + \frac{3}{2}, N, -\gamma}(t) D_{\ell, N, -\gamma}(t) + \chi_j \sum_{j_1 \leq j-1} \mathcal{E}_{\ell, N, -\gamma}(t) \mathcal{E}_{\ell + \frac{3}{2}, N, -\gamma}(t)
+ \| \nabla^j \{I - P\} f \|^2_{L^2_{\xi} L^2_v} + \| \nabla^{j+1} (a, b, c) \|^2 + \varepsilon \| w_{\ell - |\beta|, -\gamma} \partial^\alpha \{I - P\} f \|^2_v.
\tag{42}
$$

Secondly, for the weighted estimates on the terms containing only $x$ derivatives, we take $\partial_x^\beta$ to (39) with $j + 1 \leq |\alpha| \leq N$, then multiply it by $w_{\ell - |\beta|, -\gamma}(t, v) \partial^\alpha f$ and integrate it over $\mathbb{R}^3 \times \mathbb{R}^3$, and we have

$$
\frac{d}{dt} \sum_{j+1 \leq |\alpha| \leq N} \| w_{\ell, -\gamma} \partial^\alpha f \|^2 + \sum_{j+1 \leq |\alpha| \leq N} \| w_{\ell, -\gamma} \partial^\alpha f \|^2_v
+ \sum_{j+1 \leq |\alpha| \leq N} \frac{1}{(1 + t)^{1+\sigma}} \| \nu^{\frac{1}{2}} w_{\ell, -\gamma} \partial^\alpha f \|^2
\leq \chi_j = 0 \mathcal{E}_{\ell + \frac{3}{2}, N, -\gamma}(t) D_{\ell, N, -\gamma}(t) + \chi_j \sum_{j_1 \leq j-1} \mathcal{E}_{\ell, N, -\gamma}(t) \mathcal{E}_{\ell + \frac{3}{2}, N, -\gamma}(t)
+ \sum_{j+1 \leq |\alpha| \leq N} \| \partial^\alpha f \|^2_v + \varepsilon \sum_{j+1 \leq |\alpha| \leq N} \| w_{\ell, -\gamma} \partial^\alpha f \|^2_v.
\tag{43}
$$

Thirdly, for the weighted estimates on the mixed $x-v$ derivatives: applying $\partial_x^\beta$ with $1 \leq |\beta| \leq N$ and $|\alpha| + |\beta| \leq N$ to (41), multiplying it by $w_{\ell - |\beta|, \ell}(t, v) \partial_x^\beta \{I - P\} f$ and integrating over $\mathbb{R}^3 \times \mathbb{R}^3$, applying Lemma 2.5, 2.6, taking summation over $\{|\beta| = m, |\alpha| + |\beta| \leq N \}$ for each given $1 \leq m \leq N$ and then taking the proper linear combination of those $N - 1$ estimates with properly chosen constants $C_m > 0$
and \( \eta > 0 \) small enough, we have

\[
\frac{1}{2} \frac{d}{dt} \sum_{m=1}^{N} C_m \sum_{|\alpha|+|\beta| \leq N, |\alpha| \geq j, |\beta| = m} \| w_{t-|\beta|, -\gamma} \partial_\beta^\alpha (I - P) f \|^2 \\
+ \sum_{|\alpha|+|\beta| \leq N, |\beta| \geq j+1} \| w_{t-|\beta|, -\gamma} \partial_\beta^\alpha (I - P) f \|^2 \\
+ \frac{1}{(1+t)^{1+\theta}} \sum_{|\alpha|+|\beta| \leq N, |\beta| \geq j+1} \| \langle v \rangle^{3/2} w_{t-|\beta|, -\gamma} \partial_\beta^\alpha (I - P) f \|^2 
\]  

(44)

\[
\leq \chi_{j=0} \mathcal{E}_{\ell, N, -\gamma}^{(0)} (t) \mathcal{D}_{\ell, N, -\gamma}^{(0)} (t) + \chi_{j \geq 1} \sum_{j_1 \leq j-1} \mathcal{E}_{\ell, N, -\gamma}^{(j_1)} (t) \mathcal{E}_{\ell, N, -\gamma}^{(j-1-j_1)} (t) \\
+ \sum_{j \leq |\alpha| \leq N} \| \partial^\alpha (I - P) f \|^2 + \| \nabla^{j+1} (a, b, c) \|_{H^j} 
\]

Step 3. In this step, we take a proper linear combination of the estimates (40),(64) (44) and (32). We can obtain that

\[
\frac{d}{dt} \mathcal{E}_{\ell, N, -\gamma}^{(j)} (t) + \mathcal{D}_{\ell, N, -\gamma}^{(j)} (t) \leq \chi_{j=0} \mathcal{E}_{\ell, N, -\gamma}^{(0)} (t) \mathcal{D}_{\ell, N, -\gamma}^{(0)} (t) \\
+ \chi_{j \geq 1} \sum_{j_1 \leq j-1} \mathcal{E}_{\ell, N, -\gamma}^{(j_1)} (t) \mathcal{E}_{\ell, N, -\gamma}^{(j-1-j_1)} (t) 
\]

where \( \mathcal{E}_{\ell, N, -\gamma}^{(j)} (t) \) is given by

\[
\mathcal{E}_{\ell, N, -\gamma}^{(j)} (t) = C_2 \left\{ C_1 \left[ \sum_{j \leq |\alpha| \leq N} \| \partial^\alpha f \|^2 + \kappa \mathcal{E}_{\ell, N, -\gamma}^{(j)} (t) \right] + \sum_{|\alpha| = j} \| w_{t-\gamma, v} \partial^\alpha (I - P) f \|^2 \\
+ \sum_{j+1 \leq |\alpha| \leq N} \| w_{t-\gamma, \partial^\alpha f} \|^2 \right\} + \sum_{m=1}^{N} C_m \sum_{|\alpha|+|\beta| \leq N, |\alpha| \geq j, |\beta| = m} \| w_{t-|\beta|, -\gamma} \partial_\beta^\alpha (I - P) f \|^2 
\]

where \( \kappa > 0 \) is small enough and \( C_1 \) and \( C_2 \) are chosen large sufficiently. It is easy to see

\[
\mathcal{E}_{\ell, N, -\gamma}^{(j)} (t) \sim \sum_{j \leq |\alpha| \leq N} \| \partial^\alpha P f \|^2 + \sum_{|\alpha|+|\beta| \leq N, |\alpha| \geq j} \| w_{t-|\beta|, -\gamma} \partial_\beta^\alpha (I - P) f \|^2. 
\]

thus we have completed the proof of the Lemma.

From the process of proof, in fact, we have not used the the exponential function \( e^{(1+m)\nu} \) of the weighted function (12).

The following lemma concerns the decay of the macro term of \( f(t) \).

Lemma 3.2. For \( j \leq N - 1 \), one has

\[
\| \nabla^j P f \| \leq (1+t)^{-\frac{3}{2}-\frac{j}{2}} \left( \| f_0 \|_{H^1 \cap Z_1} + \sup_{0 \leq s \leq t} \left\{ (1+s)^{3/2} \mathcal{E}_{N, N, -\gamma}^{(0)} (s) \right\} \right). \quad (45)
\]
Lemma 3.3. 

Proof.

\[ \| \nabla^2 P f \| \lesssim \| w^{-m} \nabla^2 f \| \lesssim \| w^{-m} \nabla^2 f(t) \| \]
\[ \lesssim (1 + t)^{-\frac{3}{4} - \frac{s}{2}} \| w^{-m + \ell} f_0 \|_{H^1 \cap Z_i} \]
\[ + \int_0^t (1 + t - s)^{-\frac{3}{4} - \frac{s}{2}} \| w^{-m + \ell} g(s) \|_{H^1 \cap Z_i} ds, \quad (46) \]

where we have used the fact that

\[ g(t, x, v) = \Gamma(f, f) + u \cdot \nabla_v f - \frac{1}{2} u \cdot v f + (b - u) \cdot v \mu^\frac{1}{2}, \]

and \( m \) can be chosen an enough large positive constant. By (30), one has

\[ \| w^{-m + \ell} \Gamma(f, f) \|_{H^1 \cap Z_i} \lesssim \mathcal{E}^{(0)}_{N, N, -\gamma}(s) \]

and by Sobolev inequality, one also has

\[ \left\| w^{-m + \ell} \left\{ u \cdot \nabla_v f - \frac{1}{2} u \cdot v f + (b - u) \cdot v \mu^\frac{1}{2} \right\} \right\|_{H^1 \cap Z_i} \]
\[ \lesssim \| w^{-m + \ell} u \cdot \nabla_v f \|_{H^1 \cap Z_i} \]
\[ + \| w^{-m + \ell} u \cdot \nabla_v f \|_{H^1 \cap Z_i} + \| w^{-m + \ell} (b - u) \cdot v \mu^\frac{1}{2} \|_{H^1 \cap Z_i} \]
\[ \lesssim \mathcal{E}^{(0)}_{N, N, -\gamma}(s), \]

then we arrive at

\[ \| w^{-m + \ell} g(s) \|_{H^1 \cap Z_i} \lesssim \mathcal{E}^{(0)}_{N, N, -\gamma}(s) \lesssim (1 + s)^{-\frac{3}{4} \sup_{0 \leq s \leq t} \left\{ (1 + s)^{\frac{3}{2}} \mathcal{E}^{(0)}_{N, N, -\gamma}(s) \right\}. (47) \]

So we can obtain that

\[ \| \nabla^2 P f \| \lesssim (1 + t)^{-\frac{3}{4} - \frac{s}{2}} \| w^{-m + \ell} f_0 \|_{H^1 \cap Z_i} + \sup_{0 \leq s \leq t} \left\{ (1 + s)^{\frac{3}{2}} \mathcal{E}^{(0)}_{N, N, -\gamma}(s) \right\} \]
\[ \times \int_0^t (1 + t - s)^{-\frac{3}{4} - \frac{s}{2} (1 + s)^{-\frac{3}{2}}} ds \]
\[ \lesssim (1 + t)^{-\frac{3}{4} - \frac{s}{2}} \left( \| f_0 \|_{H^1 \cap Z_i} + \sup_{0 \leq s \leq t} \left\{ (1 + s)^{\frac{3}{2}} \mathcal{E}^{(0)}_{N, N, -\gamma}(s) \right\} \right), \quad (48) \]

where we have used the fact that

\[ \int_0^t (1 + t - s)^{-\frac{3}{4} - \frac{s}{2} (1 + s)^{-\frac{3}{2}}} ds \lesssim (1 + t)^{-\frac{3}{4} - \frac{s}{2}}. \]

Thus the proof of this lemma is complete. \( \square \)

Now we are ready to deduce the temporal time decay rates of \( \mathcal{E}^{(j)}_{\ell, N, -\gamma}(t) \).

Lemma 3.3. For \( l_j \geq N \), take \( l_{j-1} = l_j + 2 \), suppose the energy function \( \mathcal{E}^{(0)}_{\ell_0 + \frac{3}{2}, -\gamma, N}(t) \) is sufficiently small, then we have \( \mathcal{E}^{(j)}_{l_j, N, -\gamma}(t) \) satisfies

\[ (1 + t)^{\frac{3}{2} + \frac{3}{4} \mathcal{E}^{(j)}_{l_j, N, -\gamma}(t)} \]
\[ \lesssim \mathcal{E}^{(j)}_{l_j, N, -\gamma}(0) + \left( \| f_0 \|_{H^\infty \cap Z_i} + \sup_{0 \leq s \leq t} \left\{ (1 + s)^{\frac{3}{2}} \mathcal{E}^{(0)}_{N, N, -\gamma}(s) \right\} \right)^2. \quad (49) \]
Proof. For $j = 0$, if $\mathcal{E}_{t, N, -\gamma}^{(0)}(t)$ is sufficiently small, then
\[
\frac{d}{dt} \mathcal{E}_{t, N, -\gamma}^{(0)}(t) + D_{t, N, -\gamma}^{(0)}(t) \leq 0,
\] (50)
taking $\ell = l_0$ and multiplying the above inequality by $(1 + t)^{-1 + \varepsilon}$ give
\[
\frac{d}{dt} \left( (1 + t)^{1/2 + \varepsilon} \mathcal{E}_{t, N, -\gamma}^{(0)}(t) \right) + (1 + t)^{1/2 + \varepsilon} D_{t, N, -\gamma}^{(0)}(t) \lesssim (1 + t)^{1/2 + \varepsilon} \mathcal{E}_{t, N, -\gamma}^{(0)}(t)
\] (51)
where we have used $\langle v \rangle^{-\gamma(l_0 + |\beta|)} = \langle v \rangle^{2} \langle v \rangle^{-\gamma(l_0 + |\beta|)}$ and $\varepsilon$ is a small positive number. In a similar way, taking $\ell = l_0 + 1/2$ in (50) and multiplying (50) by $(1 + t)^{1/2 + \varepsilon}$, one has
\[
\frac{d}{dt} \left( (1 + t)^{1/2 + \varepsilon} \mathcal{E}_{t, N, -\gamma}^{(0)}(t) \right) + (1 + t)^{1/2 + \varepsilon} D_{t, N, -\gamma}^{(0)}(t) \lesssim (1 + t)^{1/2 + \varepsilon} \mathcal{E}_{t, N, -\gamma}^{(0)}(t)
\] (52)
A proper linear combination of (51), (52) and (50) with $\ell = l_0 + 1$ imply that
\[
\frac{d}{dt} \left( (1 + t)^{1/2 + \varepsilon} \mathcal{E}_{t, N, -\gamma}^{(0)}(t) \right) + (1 + t)^{1/2 + \varepsilon} \mathcal{E}_{t, N, -\gamma}^{(0)}(t) + \mathcal{E}_{t, N, -\gamma}^{(0)}(t) \lesssim (1 + t)^{1/2 + \varepsilon} \mathcal{E}_{t, N, -\gamma}^{(0)}(t)
\] (53)
Integrating the above inequality, one has
\[
(1 + t)^{1/2 + \varepsilon} \mathcal{E}_{t, N, -\gamma}^{(0)}(t) \lesssim \int_0^t (1 + \tau)^{-1 + \varepsilon} d\tau \left( \|w_m + \gamma f_0\|_{H_{1}} \cap Z_{1} + \sup_{0 \leq s \leq t} \left\{ (1 + s)^{1/2} \mathcal{E}_{t, N, -\gamma}^{(0)}(s) \right\} \right)^2
\] (54)
which gives (49) with $j = 0$.

For $1 \leq j \leq N - 1$, similar to the energy estimates of $\mathcal{E}_{t, N, -\gamma}^{(j)}(t)$, one also has
\[
\frac{d}{dt} \mathcal{E}_{t, N, -\gamma}^{(j)}(t) + D_{t, N, -\gamma}^{(j)}(t) \leq \chi_{j \geq 1} \sum_{j_1 \leq j-1} \mathcal{E}_{t, N, -\gamma}^{(j_1)}(t) \mathcal{E}_{t, N, -\gamma}^{(j-1-j_1)}(t)
\]
by mathematical induction. If we take $l_{j-1} = l_j + 2$ and suppose the energy function $\mathcal{E}_{t, N, -\gamma}^{(0)}(t)$ is sufficiently small, then (49) holds for $1 \leq j \leq N - 1$, thus the proof of Lemma 3.3 is complete. \qed
Remark 2. \(\bullet\) From this lemma, we know that to establish the time decay rate of \(E^{(j)}_{l_i,N,-\gamma}(t)\), we should assume the functional \(E^{(0)}_{l_0+\frac{1}{3},N,-\gamma}(t)\) is sufficiently small.
\(\bullet\) Indeed, combining Lemma 3.1, Lemma 3.2 with Lemma 3.3 gives a general method of proving the optimal time decay rates of the higher-order spatial derivatives of solutions to the Boltzmann-type and Landau-type systems in the whole space, for both hard potentials and soft potentials.

3.2. The energy functional \(\tilde{E}_{r,N,1}(t)\) with the weight function \(w_{l,r-|\beta|,1}(t,v)\).

This subsection concentrates on obtaining the estimate of the energy functional \(\tilde{E}_{r,N,1}(t)\) defined in (56) with the weight function \(w_{l,r-|\beta|,1}(t,v)\).

Lemma 3.4. \(\text{Let } N \geq 4, l^* > l_0 + \frac{5}{2} \text{ and } \theta = \min\{\frac{1}{2}, \frac{3}{1-2\gamma}\}, \text{ and suppose that}\)

\[
\sup_{0 \leq s \leq t} E^{(0)}_{l_0+\frac{1}{3},N,-\gamma}(s) + \sup_{0 \leq s \leq t} \tilde{E}_{r,N,1}(s)
+ \left( \|f_0\|_{H^N \cap Z_1} + \sup_{0 \leq s \leq t} (1 + s)^{\frac{3}{2}} E^{(0)}_{r,N,-\gamma}(s) \right)^2
\]

is small enough, one holds

\[
\tilde{E}_{r,N,1}(t) \leq \sum_{|\alpha|+|\beta| \leq N} \|w_{l,r-|\beta|,1}(0,v)\partial_{\beta}^2 f(0,x,v)\|^2
\]

where

\[
\tilde{E}_{r,N,1}(t) \sim \sum_{|\alpha|+|\beta| = n, 1 \leq n \leq N} (1 + t)^{-\sigma_{n,j}} \|w_{l,r-|\beta|,1}\partial_{\beta}^3 \{I - P\} f\|^2
+ \sum_{|\alpha| = n, 1 \leq n \leq N} \|w_{l,r,1}\partial^\alpha f\|^2 + \|w_{l,r,1}\{I - P\} f\|^2.
\]

Proof. Firstly, multiplying (41) by \(w_{l,r,1}(t,v)\{I - P\} f\) and integrating it over \(\mathbb{R}^3 \times \mathbb{R}^3_v\), we have

\[
\frac{1}{2} \frac{d}{dt} \|w_{l,r,1}\{I - P\} f\|^2 + (\nu(v), w_{l,r,1}^2\{I - P\} f)^2
+ \frac{q(1 + t)^{1+\theta}}{(1 + t)^{1+\theta}} \|w_{l,r,1}\{I - P\} f(v)^2\|^2
\leq \left( \langle K\{I - P\} f, w_{l,r,1}^2\{I - P\} f \rangle \right)_{l_1}
+ \left( \langle u \cdot \nabla f - \frac{1}{2} u \cdot v f - v \cdot \nabla_x P f, w_{l,r,1}^2\{I - P\} f \rangle \right)_{l_2}
+ \left( \langle P(u \cdot \nabla_x f + \frac{1}{2} u \cdot v f - u \cdot \nabla_x P f), w_{l,r,1}^2\{I - P\} f \rangle \right)_{l_3}
+ \left( \langle u \cdot \nabla \{I - P\} f, w_{l,r,1}^2\{I - P\} f \rangle \right)_{l_4}
\]
From Lemma 2.1, Hölder’s inequality in $x$ integration and Cauchy-Schwarz’s inequality, $I_1$ can be bounded by

$$
\varepsilon \| w_{l_1,1} \{ I - P \} f \|^2 + C_N \| \{ I - P \} f \|^2.
$$

(58)

It is straightforward to estimate $I_2$ and $I_3$, we have

$$
I_2 + I_3 \leq \varepsilon \| w_{l_1,1} \{ I - P \} f \|^2 + C_N \left\{ \| \{ I - P \} f \|^2 + \| \nabla x \{ I - P \} f \|^2 + \| \nabla x (a, b, c) \|^2 \right\}.
$$

(59)

If we employed the inequality (45) and the fact that $-\frac{3}{2} \leq -1 - \vartheta$ for $0 < \vartheta \leq \frac{1}{2}$, then we can get

$$
I_4 + I_5 \lesssim u \| u \|_{L^\infty} \left\{ \| (v)^{\frac{1}{2}} w_{l_1,1} \{ I - P \} f \|^2 + \| (v)^{\frac{1}{2}} w_{l_1,1} \{ I - P \} f \|^2 \right\}
$$

$$
\lesssim (1 + t)^{-\frac{3}{2}} \left[ \| f_0 \|_{H^N \cap Z_{l_1}} + \sup_{0 \leq s \leq t} \left\{ (1 + s)^{\frac{1}{2}} \mathcal{E}^{(0)}_{N,N,-\gamma}(s) \right\} \right].
$$

(60)

For $I_6$, by (28) and (29), one has

$$
I_6 \lesssim \int_{\mathbb{R}^2} |\mu^\delta \nabla_v f|^2 |f|^2 dx + \varepsilon \| w_{l_1,1} \{ I - P \} f \|^2
$$

$$
\lesssim \| \mu^\delta \nabla_v f \|^2 \| f \|_{L^2_x L^2_t}^2 + \varepsilon \| w_{l_1,1} \{ I - P \} f \|^2
$$

(61)

Plugging (58), (59), (60) and (61) into (57) implies that

$$
\frac{d}{dt} \| w_{l_1,1} \{ I - P \} f \|^2 + \| w_{l_1,1} \{ I - P \} f \|^2 + \frac{1}{(1 + t)^{1+\vartheta}} \| (v)^{\frac{1}{2}} w_{l_1,1} \{ I - P \} f \|^2
$$

$$
\lesssim \| \{ I - P \} f \|^2_{L^2_x H^N_{l_1}} + \| \nabla x (a, b, c) \|^2 + \| \mu^\delta \nabla_v f \|^2 \| f \|_{L^2_x L^2_t}
$$

(62)

where we used the fact that

$$
\sup_{0 \leq s \leq t} \mathcal{E}^{(0)}_{l_0 + \frac{1}{2}, N, -\gamma}(s) + \left\{ \| f_0 \|_{H^N \cap Z_{l_1}} + \sup_{0 \leq s \leq t} \left\{ (1 + s)^{\frac{1}{2}} \mathcal{E}^{(0)}_{N,N,-\gamma}(s) \right\} \right\}^2
$$

is sufficiently small.

Secondly, for the weighted estimates on the terms containing only $x$ derivatives, we take $\partial_x^\alpha$ to (39)$_1$ with $1 \leq |\alpha| \leq N$, then multiply it by $w_{l_1,1}^2 (t, v) \partial_x^\alpha f$ and integrate.
Here we have used
\[ Lf = L(I - P)f = \nu(I - P)f + K(I - P)f. \]
For the second term on the left hand of the above equation,
\[
(v(v)\partial^a(I - P)f, w_{l,1}^2 \partial^a f) \\
= \|\partial^a_{l,1}(t, v)\partial^a(I - P)f\|^2 + (v\partial^a(I - P)f, w_{l,1}^2 \partial^a P f) \\
\geq \frac{1}{2} \|w_{l,1} \partial^a(I - P)f\|_v^2 - C\|\partial^a(a, b, c)\|^2.
\]
For \( I_7 \) on the right hand of (63), we have from Lemma 2.1
\[ I_7 \leq \varepsilon \|w_{l,1} \partial^a(I - P)f\|_v^2 + \|\partial^a(I - P)f\|_v^2 + \|\partial^a(a, b, c)\|^2. \]
Obviously, \( I_8 \) can be bounded by \( \eta \|v\|\partial^a f\|^2 + C\eta\|\partial^a u\|^2, I_9 \) can be bounded by:
\[
I_9 \leq C(0)_{t, l} \|\nabla f\|_{H_{L_{l,1}}}^2 + \varepsilon \|w_{l,1} \partial^a f\|_v^2 + \|u\|_{L_v^2} \|v\|_{L_v^3} \|w_{l,1} \partial^a f\|_v^2 \\
+ \sum_{1 \leq |\alpha|_1 \leq |\alpha|} \int_{\mathbb{R}^3} |\partial^{\alpha_1} u| \left( |v|^{\frac{1}{2}} w_{l,1} \partial^{\alpha - \alpha_1}(I - P)f \right)^2 \\
+ \left( |v|^{\frac{1}{2}} w_{l,1} \partial^{\alpha - \alpha_1}(I - P)f \right)^2 \right) dx + \varepsilon (1 + t)^{-1-\theta} (v)^{\frac{1}{2}} w_{l,1} \partial^a f\|^2.
\]
For \( I_{10} \), Lemma 2.2 tells us that:
\[
I_{10} \leq \sum_{|\alpha| \leq N - 4, m \leq 2} \int_{\mathbb{R}^3} \left| \nabla_m \left( \mu^\delta \partial^{\alpha_1} f \right) \right| \left| w_{l,1} \partial^{\alpha - \alpha_1} f \right|_{v} \left| w_{l,1} \partial^a f \right|_{v} dx \\
\leq \sum_{|\alpha| = N - 3 \text{ or } N - 2, m \leq 2} \int_{\mathbb{R}^3} \left| \nabla_m \left( \mu^\delta \partial^{\alpha_1} f \right) \right| \left| w_{l,1} \partial^{\alpha - \alpha_1} f \right|_{v} \left| w_{l,1} \partial^a f \right|_{v} dx \\
\leq \sum_{|\alpha| \geq N - 1, m \leq 2} \int_{\mathbb{R}^3} \left| \nabla_m \left( \mu^\delta \partial^{\alpha - \alpha_1} f \right) \right| \left| w_{l,1} \partial^{\alpha_1} f \right|_{v} \left| w_{l,1} \partial^a f \right|_{v} dx \\
\leq \sum_{|\alpha| \leq N} \int_{\mathbb{R}^3} \left| w_{l,1} \partial^{\alpha_1} f \right| \left| w_{l,1} \partial^{\alpha - \alpha_1} f \right|_{v} \left| w_{l,1} \partial^a f \right|_{v} dx.
\]
By using $L^2 - L^\infty - L^2$, $L^\infty - L^2 - L^2$, $L^6 - L^3 - L^2$ or $L^3 - L^6 - L^2$ type Hölder inequalities with respect to space derivative $x$, one has

$$\sum_{i=1}^{3} I_{10,i} \lesssim \mathcal{E}^{(0)}_{0,N,-\gamma}(t) \sum_{1 \leq |\alpha| \leq N} \|w_{1,-1} \partial^\alpha f\|^2 + \varepsilon \|w_{1,-1} \partial^\alpha f\|^2.$$

and

$$I_{10,4} \lesssim \sum_{1 \leq |\alpha| \leq N} \|w_{1,-1} \partial^\alpha f\|^2 \times \sum_{1 \leq |\alpha| \leq N} \|w_{1,-1} \partial^\alpha f\|^2 + \varepsilon \|w_{1,-1} \partial^\alpha f\|^2.$$

Consequently

$$I_{10} \lesssim \left\{ \mathcal{E}^{(0)}_{0,N,-\gamma}(t) + \sum_{1 \leq |\alpha| \leq N} \|w_{1,-1} \partial^\alpha f\|^2 \right\} \times \sum_{1 \leq |\alpha| \leq N} \|w_{1,-1} \partial^\alpha f\|^2 + \eta \|w_{1,-1} \partial^\alpha f\|^2.$$

Collecting the above estimates, for $1 \leq |\alpha| \leq N$, one has

$$\frac{d}{dt} \|w_{1,-1} \partial^\alpha f\|^2 + \|w_{1,-1} \partial^\alpha \{I - P\}f\|^2 + \frac{1}{(1 + t)^{1+\theta}} \|\langle v \rangle^{\frac{1}{2}} w_{1,-1} \partial^\alpha f\|^2$$

$$\lesssim \|\partial^\alpha \{I - P\}f\|^2 + \|\nabla_x \{a, b, c\}\|_{H^{N-1}}$$

$$+ \left\{ \mathcal{E}^{(0)}_{0,N,-\gamma}(t) + \sum_{1 \leq |\alpha| \leq N} \|w_{1,-1} \partial^\alpha f\|^2 \right\} \times \sum_{1 \leq |\alpha| \leq N} \|w_{1,-1} \partial^\alpha f\|^2$$

$$+ \sum_{1 \leq |\alpha| \leq N} \int_{\mathbb{R}^3} |\partial^\alpha u| \left( \|\langle v \rangle^{\frac{1}{2}} w_{1,-1} \partial^\alpha \{I - P\}f\|^2$$

$$+ \|\langle v \rangle^{\frac{1}{2}} w_{1,-1,1} \partial_{v_i}^\alpha \{I - P\}f\|^2 \right) dx$$

\textbf{Thirdly}, for the weighted estimates on the mixed $x - v$ derivatives: applying $\partial^\beta_\gamma$ with $|\beta| = j, 1 \leq j \leq N$ and $|\alpha| + |\beta| = n, 1 \leq n \leq N$ to (57), multiplying it by $w_{1,-|\beta|,1}(t,v) \partial^\beta_\gamma \{I - P\}f$ and integrating over $\mathbb{R}^3 \times \mathbb{R}^3$, we have

$$\frac{d}{dt} \|w_{1,-|\beta|,1} \partial^\beta_\gamma \{I - P\}f\|^2 + \frac{1}{(1 + t)^{1+\theta}} \|\langle v \rangle^{\frac{1}{2}} w_{1,-|\beta|,1} \partial^\beta_\gamma \{I - P\}f\|^2$$

$$+ \left( \partial_\beta \nu(v) \partial^\alpha \{I - P\}f , w_{1,-|\beta|,1} \partial^\beta_\gamma \{I - P\}f \right)$$

$$\lesssim \left\{ \partial_\beta \left( K \partial^\alpha \{I - P\}f , w_{1,-|\beta|,1} \partial^\beta_\gamma \{I - P\}f \right) \right\}_{I_{11}}$$

$$+ \left( \partial_\beta \left( v \cdot \nabla_x \partial^\alpha \{I - P\}f , w_{1,-|\beta|,1} \partial^\beta_\gamma \{I - P\}f \right) \right\}_{I_{12}}$$

$$+ \left( \partial^\beta \left[ u \cdot \nabla_v Pf - \frac{1}{2} u \cdot v Pf - v \cdot \nabla_x Pf \right] , w_{1,-|\beta|,1} \partial^\beta_\gamma \{I - P\}f \right) \right\}_{I_{13}}$$
\[
\begin{align*}
&+ \left( \partial_\beta^\alpha \left[ P(v \cdot \nabla_x f + \frac{1}{2} u \cdot v f - u \cdot \nabla_x f) \right], w_{t,v}^2(t,v) \partial_\beta^\alpha \{I - P\} f \right)_{14} \\
&+ \left( \partial_\beta^\alpha \left[u \cdot \nabla_v \{I - P\} f - \frac{1}{2} u \cdot v \{I - P\} f \right], w_{t,v}^2(t,v) \partial_\beta^\alpha \{I - P\} f \right)_{15} \\
&+ \left( \partial_\beta^\alpha \Gamma(f,f), w_{t,v}^2(t,v) \partial_\beta^\alpha \{I - P\} f \right)_{16}
\end{align*}
\]

Now we estimate term by term as follows. For the third term on the left hand of (65), we have
\[
\begin{align*}
&\left( \partial_\beta^\alpha \{v(v)\{I - P\} f\}, w_{t,v}^2 \partial_\beta^\alpha \{I - P\} f \right) \\
&\geq \|w_{t,v} \partial_\beta^\alpha \{I - P\} f\|_v^2 - \eta \sum_{|\beta_1| \leq |\beta|} \|w_{t,v} \partial_\beta^\alpha \{I - P\} f\|_v^2 \\
&- C_\eta \|X\| \leq C_\eta \langle v \rangle^{-\gamma(t-|\beta|)} \partial_\beta^\alpha \{I - P\} f\|_v^2.
\end{align*}
\]

For the first term \(I_{11}\) on the right hand of (65), we have from Lemma 2.1
\[
I_{11} \lesssim \eta \sum_{|\beta_1| \leq |\beta|} \|w_{t,v} \partial_\beta^\alpha \{I - P\} f\|_v^2 + \|\partial_\beta^\alpha \{I - P\} f\|_v^2.
\]

For the second term \(I_{12}\) on the right hand of (65), we have
\[
\begin{align*}
I_{12} &= \left| (\partial_{\beta - e_1}^\alpha \{I - P\} f, w_{t,v}^2 \partial_{\beta - e_1}^\alpha \{I - P\} f) \right| \\
&= \left| \langle v \rangle^{-\frac{1}{2}} w_{t,v} \partial_{\beta - e_1}^\alpha \{I - P\} f, \langle v \rangle^{-\frac{1}{2}} \langle v \rangle^{-\frac{1}{2}} w_{t,v} \partial_{\beta - e_1}^\alpha \{I - P\} f \right| \\
&\lesssim \eta \|w_{t,v} \partial_{\beta - e_1}^\alpha \{I - P\} f\|_v^2 + \|\langle v \rangle^{-\frac{1}{2}} w_{t,v} \partial_{\beta - e_1}^\alpha \{I - P\} f\|_v^2.
\end{align*}
\]

It is straightforward to estimate \(I_{13}\) and \(I_{14}\) on the right hand of (65), the two terms are bounded by
\[
\eta \|w_{t,v} \partial_{\beta - e_1}^\alpha \{I - P\} f\|_v^2 + C_\eta \left\{ \sum_{|\alpha| \leq N} \|\partial_\alpha \{I - P\} f\|_v^2 + \|\nabla_x (a,b,c)\|_{H^{-1}}^2 \right\}.
\]

\(I_{15}\) can be bounded by:
\[
\begin{align*}
I_{15} &\lesssim \|v(0)\|_{H^{-1}_{L^2}}^2 + (1 + t)^{1+\theta} \|u\|_{L^2}^2 \|\langle v \rangle^{-\frac{1}{2}} w_{t,v} \partial_{\beta - e_1}^\alpha \{I - P\} f\|_v^2 \\
&+ (1 + t)^{1+\theta} \sum_{1 \leq |\alpha_1| \leq |\alpha|} \int_{\mathbb{R}^2} |\partial_{\alpha_1} u|^2 \left( \langle v \rangle^{-\frac{1}{2}} w_{t,v} \partial_{\beta - \alpha_1}^{-\alpha_1} \{I - P\} f \right)_v^2 dx \\
&+ \langle v \rangle^{-\frac{1}{2}} w_{t,v} \partial_{\beta - e_1}^{-\alpha_1} \{I - P\} f \|_{L^2}^2 dx \\
&+ \varepsilon (1 + t)^{-\alpha} \|\langle v \rangle^{-\frac{1}{2}} w_{t,v} \partial_{\beta - e_1}^\alpha \{I - P\} f\|_v^2 + \varepsilon \|w_{t,v} \partial_{\beta - e_1}^\alpha \{I - P\} f\|_v^2.
\end{align*}
\]
Similar with $I_{10}$, by applying Sobolev and Hölder inequalities, one has:

$$I_{16} \lesssim \mathcal{E}^{(0)}_{0, N, -\gamma}(t) \left\{ D_{0, N, -\gamma}^{(0)}(t) + \sum_{\alpha_1 \leq n, \beta_1 \leq \beta} \| w_{\nu, -|\beta_1|, 1} \partial_{\beta_1}^{\alpha_1} (I - P) f \|^2 \right\}$$

$$+ \sum_{|\alpha_1| \leq |\alpha|, |\beta_1| \leq |\beta|} \| w_{\nu, -|\beta_1|, 1} \partial_{\beta_1}^{\alpha_1} (I - P) f \|^2 \| w_{\nu, -|\beta_1|, 1} \partial_{\beta_1}^{\alpha_1} (I - P) f \|^2$$

$$+ \epsilon \| w_{\nu, -|\beta_1|, 1} \partial_{\beta_1}^{\alpha_1} (I - P) f \|^2 \nu \right\}$$

Thus plugging the estimates on $I_{11} \sim I_{16}$ into (65) yields

$$\frac{d}{dt} \sum_{|\alpha_1| + |\beta_1| = n, |\beta_1| = j} \| w_{\nu, -|\beta_1|, 1} \partial_{\beta_1}^{\alpha_1} (I - P) f \|^2$$

$$+ \sum_{|\alpha_1| + |\beta_1| = n, |\beta_1| = j} \| w_{\nu, -|\beta_1|, 1} \partial_{\beta_1}^{\alpha_1} (I - P) f \|^2 \nu \right\}$$

$$+ (1 + t)^{-1-\vartheta} \sum_{|\alpha_1| + |\beta_1| = n, |\beta_1| = j} \| \left( \langle v \rangle \frac{1}{2} w_{\nu, -|\beta_1|, 1} \partial_{\beta_1}^{\alpha_1} (I - P) f \right) \|^2$$

$$\lesssim \eta \sum_{|\alpha_1| + |\beta_1| = n, |\beta_1| = j} \| w_{\nu, -|\beta_1|, 1} \partial_{\beta_1}^{\alpha_1} (I - P) f \|^2 \nu \right\}$$

$$+ \sum_{|\alpha_1| + |\beta_1| = n, |\beta_1| = j} \| \nabla_x (a, b, c) \|^2_{H^{\gamma-1}} + \sum_{|\alpha| \leq n} \| \partial_{\alpha} (I - P) f \|^2 \nu \right\}$$

$$+ \mathcal{E}^{(0)}_{0, N, -\gamma}(t) \left\{ D_{0, N, -\gamma}^{(0)}(t) + \sum_{|\alpha_1| + |\beta_1| = n, |\beta_1| = j} \| w_{\nu, -|\beta_1|, 1} \partial_{\beta_1}^{\alpha_1} (I - P) f \|^2 \right\}$$

$$+ (1 + t)^{1+\vartheta} \sum_{|\alpha_1| + |\beta_1| = n, |\beta_1| = j} \int_{\mathbb{R}^3} |\partial_{\alpha_1} u|^2 \left( \left\langle \langle v \rangle \frac{1}{2} w_{\nu, -|\beta_1|, 1} \partial_{\beta_1}^{\alpha_1} (I - P) f \right\rangle \right) \nu \right\}$$

To control the last term on the right hand side of the above inequality, we introduce $(1 + t)^{-\sigma_{n,j}}$ in which $\sigma_{n,j} - \sigma_{n,j-1} = \frac{2(1+\gamma)}{\gamma-2} (1 + \vartheta), \; 1 \leq j \leq n, \; \sigma_{n,0} = 0$, then one can deduce that

$$\sum_{|\alpha| + |\beta| = n, |\beta| = j} (1 + t)^{-\sigma_{n,j}} \left\langle \langle v \rangle \frac{1}{2} w_{\nu, -|\beta|, 1} \partial_{\beta}^{\alpha+e_i} (I - P) f \right\rangle \right\}$$

$$\lesssim \sum_{|\alpha| + |\beta| = n, |\beta| = j} (1 + t)^{-\sigma_{n,j}-1} \left\langle \langle v \rangle \frac{1}{2} w_{\nu, -|\beta|, 1} \partial_{\beta}^{\alpha+e_i} (I - P) f \right\rangle \frac{4(1+\gamma)}{\gamma-1}$$
Based on (67), multiplying (64) and (66) by \((1 + t)^{-\sigma_{n,j}}\), meanwhile taking summation over \(0 \leq j \leq n\), one deduces

\[
\frac{d}{dt} \left\{ \sum_{|\alpha| + |\beta| = n, |\beta| = j, 1 \leq j \leq n} (1 + t)^{-\sigma_{n,j}} \left\| w_{t^-, j+1,1} \partial^\alpha \partial^\beta \{I - P\} f \right\|^2 + \sum_{|\alpha| = n} \left\| w_{t^-, 1,1} \partial^\alpha f \right\|^2 \right\}
\]

\[
+ \sum_{|\alpha| + |\beta| = n, |\beta| = j, 1 \leq j \leq n} (1 + t)^{-\sigma_{n,j}} \left\| w_{t^-, j,1} \partial^\alpha \partial^\beta \{I - P\} f \right\|^2 + \sum_{|\alpha| = n} \left\| w_{t^-, 1,1} \partial^\alpha f \right\|^2
\]

\[
+ \sum_{|\alpha| = n} (1 + t)^{-\sigma_{n,j}} \left\| \langle v \rangle^{2} w_{t^-, j,1} \partial^\alpha \{I - P\} f \right\|^2
\]

\[
\leq \sum_{|\alpha| + |\beta| = n, |\beta| = j, |\beta_1| < j, 1 \leq j \leq n} \left\| \partial^\alpha \{I - P\} f \right\|^2 + \left\| \nabla_x (a, b, c) \right\|^2_{H^{N-1}}
\]

\[
+ \sum_{|\alpha| \leq N} \left\| \partial^\alpha \{I - P\} f \right\|^2 + \left\| \nabla_x (a, b, c) \right\|^2_{H^{N-1}}
\]

\[
+ \sum_{|\alpha| + |\beta| = n, |\beta| = j, 1 \leq j \leq n} (1 + t)^{1 + \vartheta - \sigma_{n,j}} \int_{R^2} \left( \left\| \partial^\alpha u \right\|^2 \right) \left( \left\| \langle v \rangle^{2} w_{t^-, j,1} \partial^\alpha - \partial^\alpha \{I - P\} f \right\|^2_{L^2_x} \right)
\]

\[
+ \left( \left\| \langle v \rangle^{2} w_{t^-, j,1} \partial^\alpha - \partial^\alpha \{I - P\} f \right\|^2_{L^2_x} \right) dx
\]

\[
+ \sum_{1 \leq j \leq |\alpha| \leq n} \int_{R^2} \left( \left\| \partial^\alpha u \right\|^2 \right) \left( \left\| \langle v \rangle^{2} w_{t^-, 1,1} \partial^\alpha - \partial^\alpha \{I - P\} f \right\|^2_{L^2_x} \right)
\]

\[
+ \left( \left\| \langle v \rangle^{2} w_{t^-, 1,1} \partial^\alpha - \partial^\alpha \{I - P\} f \right\|^2_{L^2_x} \right) dx
\]

\[
\times \left\{ D_{0,N,-\gamma}(t) + \sum_{|\alpha| + |\beta| = n, |\beta| = j, 1 \leq j \leq n} \left\| w_{t^-, j,1} \partial^\alpha \partial^\beta \{I - P\} f \right\|^2 \right\}
\]

\[
+ \sum_{1 \leq |\alpha| \leq n} \int_{R^2} \left( \left\| \partial^\alpha u \right\|^2 \right) \left( \left\| \langle v \rangle^{2} w_{t^-, 1,1} \partial^\alpha - \partial^\alpha \{I - P\} f \right\|^2_{L^2_x} \right)
\]

\[
+ \left( \left\| \langle v \rangle^{2} w_{t^-, 1,1} \partial^\alpha - \partial^\alpha \{I - P\} f \right\|^2_{L^2_x} \right) dx
\]
difficulty term is

Among “Three-terms” on the right hand side of the above inequality, the most
difficulty term is

which can be bounded by

where we have used (49) of Lemma 3.3 and the fact that \( \sigma_{n,j} - \sigma_{n,j-1} = \frac{2(1+\vartheta)}{\gamma-2} (1+\vartheta) \) for \( 1 \leq j \leq n \) such that

for \( 0 < \vartheta \leq \frac{3}{1-2\vartheta} \). Therefore, if

is sufficiently small, taking summation over \( 1 \leq n \leq N \), then one has

\[
\frac{d}{dt} \left\{ \sum_{|\alpha|+|\beta|=n,\beta_i=\varsigma} (1+t)^{-\sigma_{n,j}} \|w_{l^*_{-j,1}} \partial_{\beta_i}^n (I-P) f\|_{H^l} \right\} \\
+ \sum_{|\alpha|=n,1 \leq n \leq N} \|w_{l^*_{-j,1}} \partial_{\alpha} f\|_{H^l} \\
+ \sum_{|\alpha|+|\beta|=n,1 \leq n \leq N} (1+t)^{-\sigma_{n,j}} \|w_{l^*_{-j,1}} \partial_{\beta}^n (I-P) f\|_{H^l} \\
\]
Lemma 3.5. We have the following lemma:

\begin{align*}
3.3. \quad & \text{Energy functional } E^{(0)}_{l_0+\frac{2}{3},N,-\gamma}(t) \text{ with weight function } w_{l_0+\frac{2}{3},-\beta,-\gamma}(t,v).

\text{Recall Remark 2, we ask that } E^{(0)}_{l_0+\frac{2}{3},N,-\gamma}(t) \text{ is sufficiently small. To this end, we have the following lemma:}

\textbf{Lemma 3.5.} \text{Under the assumptions on the above lemma, furthermore, let } l^* \geq -\gamma l_0 - \frac{5}{2} \gamma - \frac{5\gamma}{3} \sigma^* \text{ with } \sigma^* = \max_{|\alpha|+|\beta| \leq N} \{\sigma_{|\alpha|+|\beta|,|\beta|}\}, \text{ then } E^{(0)}_{l_0+\frac{2}{3},N,-\gamma}(t) \text{ satisfies}

\begin{align*}
E^{(0)}_{l_0+\frac{2}{3},N,-\gamma}(t) & \lesssim \sum_{|\alpha|+|\beta| \leq N} \|w_{l^*,1}(0,v)\partial^\alpha \partial^\beta f(0,x,v)\|^2.
\end{align*}

(71)

where we have used the fact that

\begin{align*}
(1+t)^{-\sigma_n} & \leq (1+t)^{-\sigma_n,|\beta_1|, |\beta_1| < j}.
\end{align*}

A proper linear combination of (62), (69) and (50) implies that

\begin{align*}
\frac{d}{dt} \left\{ \sum_{|\alpha|+|\beta| \leq N \text{ and } |\beta| = j} \|w_{l^*,1}(I - P)f\|^2 + \|w_{l^*,1}(I - P)f\| \right\}
\end{align*}

Integrating (70) from 0 to \( t \) implies (55) by the definition (56), thus the proof of this lemma is complete. \( \Box \)

3.3. Energy functional \( E^{(0)}_{l_0+\frac{2}{3},N,-\gamma}(t) \) with weight function \( w_{l_0+\frac{2}{3},-\beta,-\gamma}(t,v) \).

Recall Remark 2, we ask that \( E^{(0)}_{l_0+\frac{2}{3},N,-\gamma}(t) \) is sufficiently small. To this end, we have the following lemma:

\textbf{Lemma 3.5.} \text{Under the assumptions on the above lemma, furthermore, let } l^* \geq -\gamma l_0 - \frac{5}{2} \gamma - \frac{5\gamma}{3} \sigma^* \text{ with } \sigma^* = \max_{|\alpha|+|\beta| \leq N} \{\sigma_{|\alpha|+|\beta|,|\beta|}\}, \text{ then } E^{(0)}_{l_0+\frac{2}{3},N,-\gamma}(t) \text{ satisfies}

\begin{align*}
E^{(0)}_{l_0+\frac{2}{3},N,-\gamma}(t) & \lesssim \sum_{|\alpha|+|\beta| \leq N} \|w_{l^*,1}(0,v)\partial^\alpha \partial^\beta f(0,x,v)\|^2.
\end{align*}

(71)
Proof. We split \( \varepsilon_{l_0 + \frac{1}{2}, N, -\gamma}(t) \) into macroscopic part and microscopic part, i.e.

\[
\sum_{|\alpha| + |\beta| \leq N} \| w_{l_0 + \frac{1}{2} - |\alpha| - |\beta|, -\gamma} \partial_\beta^n P f \|^2
\]

and

\[
\sum_{|\alpha| + |\beta| \leq N} \| w_{l_0 + \frac{1}{2} - |\alpha| - |\beta|, -\gamma} \partial_\beta^n (I - P) f \|^2.
\]

If we take \( l^* \geq -\gamma l_0 - \frac{5}{2} \gamma - \frac{5\gamma}{3} \sigma^* \) with \( \sigma^* = \max \{ \sigma_{|\alpha| + |\beta|, |\beta|} \} \) such that

\[
\| w_{l_0 + \frac{1}{2} - |\alpha| - |\beta|, -\gamma} \partial_\beta^n (I - P) f \|^2 
\lesssim \left\{ (1 + t)^{\frac{3}{2}} \| w_{l_0 - |\alpha| - |\beta|, -\gamma} \partial_\beta^n (I - P) f \|^2 \right\} \left( 1 + t \right)^{-\frac{\sigma_{|\alpha| + |\beta|, |\beta|}}{\gamma}} 
\times \left\{ (1 + t)^{-\sigma_{|\alpha| + |\beta|, |\beta|}} \| w_{l_0 - |\alpha| - |\beta|, 1} \partial_\beta^n (I - P) f \|^2 \right\} \left( 1 + t \right)^{-\frac{\sigma_{|\alpha| + |\beta|, |\beta|}}{\gamma}}
\lesssim (1 + t)^{\frac{3}{2}} \varepsilon_{l_0, N, -\gamma}(t) + (1 + t)^{-\sigma_{|\alpha| + |\beta|, |\beta|}} \| w_{l_0 - |\alpha| - |\beta|, 1} \partial_\beta^n (I - P) f \|^2
\lesssim \sum_{|\alpha| + |\beta| \leq N} \| w_{l_0 - |\alpha| - |\beta|, 1} \partial_\beta^n f(0, x, v) \|^2.
\]

This completes the proof of Lemma 3.5. \( \square \)

Recall the definition (56) of \( \tilde{\varepsilon}_{l^*, N, 1}(t) \), based on the above lemmas, we are ready to construct the a priori estimates

\[
X(t) = \sup_{0 \leq s \leq t} \left\{ (1 + s)^{\frac{3}{2}} \varepsilon_{l_0, N, -\gamma}(s) + \varepsilon_{l_0 + \frac{1}{2}, N, -\gamma}(s) + \tilde{\varepsilon}_{l^*, N, 1}(s) \right\} \leq M, \quad 0 < t \leq T,
\]

where \( M \) is a sufficiently small positive constant and the parameters \( l_0, l^* \) are given in the above lemma.

3.4. The proof of Theorem 1.1. Combining (49), (55) and (71), one has by (73)

\[
X(t) \lesssim Y_0^2 + X^2(t)
\]

It is immediate to follow from the a priori estimate that

\[
X(t) \lesssim Y_0^2
\]

holds true for any \( 0 \leq t \leq T \), as long as \( Y_0 \) is sufficiently small.

With the above preparation in hand, we now turn to prove Theorem 1.1. The local existence and uniqueness of the solution \( f(t, x, v) \) to the Cauchy problem (5) and (6) can be proved, the details are omitted for simplicity; see [17] with a little modification. Now we have obtained the uniform-in-time estimate (74) over \( 0 < t \leq T \) with \( 0 < T \leq \infty \). By the standard continuity argument, the global existence follows provided \( Y_0 \) defined by (15) is sufficiently small. This completes the proof of Theorem 1.1.

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