The correlations of the QCD Dirac eigenvalues are studied with use of an extended chiral random matrix model. The inclusion of spatial dependence which the original model lacks enables us to investigate the effects of diffusion modes. We get analytical expressions of level correlation functions with non-universal behavior caused by diffusion modes which is characterized by Thouless energy. Pion mode is shown to be responsible for these diffusion effects when QCD vacuum is considered a disordered medium.

1 Introduction

It is now widely recognized that a chiral random matrix theory (ChRMT) is a useful theoretical tool to describe non-perturbative aspects of QCD. Like other complicated quantum systems, statistical properties of QCD Dirac operator show high universality described by this model. A possible relation between chiral phase transition and Anderson localization transition (a metal-insulator transition induced by a random impurity potential) which was originally suggested in the context of an instanton liquid model is currently under intensive study.

In order to discuss such relations, however, the original model is not appropriate because the model lacks space-time coordinate dependence and can only describe a `metallic' region. In this region, energy eigenfunctions extend over the whole system. The correlations of energy levels are very large and can be correctly described by RMT. But, if impurity potential becomes strong, the wavefunction tends to localize. The correlations become small and the diffusion effects become important.

Recently, some authors have discussed the limitation of the original model. Comparing with numerical simulations, they found that the deviations from the results of the original model could be seen beyond a certain energy scale. Further there were some analytical arguments supporting the above results.

There are some arguments that Anderson transition can only happen in quenched theories (see e.g. Ref. 5).
This energy scale is termed Thouless energy after a quantity with a similar role in Anderson problem. In Anderson problem, Thouless energy $E_c$ is defined as $D/L^2$ where $D$ is a diffusion constant and $L$ is a system size. If energy scale we consider is much smaller than $E_c$, the spatial structure of a system can be ignored and we can use a random matrix model with a proper symmetry as a good approximation to describe statistical properties of a system. But once energy scale becomes comparable to or larger than $E_c$, we can no longer ignore the spatial dependence of the system.

The intuitive explanation of above results is as follows; Since a typical distance of a diffusing particle with a diffusion constant $D$ during a time period $t$ is $\sqrt{Dt}$, a time scale in which a particle is diffusing throughout a sample with linear dimension $L$ is $\tau_c = L^2/D$. The corresponding energy scale, $1/\tau_c$, is Thouless energy. Hence, when we consider times much greater than $\tau_c$ (or energies much smaller than $E_c$), a particle has enough time to wander everywhere in the sample and would finally forget from where it starts. Then every point in the sample becomes equivalent and the spatial structure of the original system can be ignored.

Besides the above intuitive reasoning, Thouless energy also appears in expressions of various statistical quantities in Anderson problem. In the context of impurity perturbation technique, Thouless energy is defined as a first excitation energy of fundamental diffusion modes, diffuson. If all excitation energies of the modes with finite wave numbers are much greater than the energy scale we consider, we can ignore contributions from these modes. There remains the lowest mode with 0 wave number which is constant throughout the sample. Keeping only the lowest mode amounts to neglecting the spatial dependence.

Turning to the statistical properties of QCD Dirac operator, the authors of Ref.\[6, 7\] followed the above intuitive argument and suggested that Thouless energy in QCD is the right hand side of Eq.(7). Hence if relevant energy scale exceeds this quantity, we should expect deviations from ChRMT due to the effects of diffusion modes. At present, however, we do not know concrete expressions of these effects, which is necessary to make the arguments quantitative.

In this paper, we aim to go beyond the qualitative argument and to calculate these effects of diffusion analytically. Guided by a model used to describe disordered electrons in metals[11], we extend the original chiral random matrix model; a single random matrix is replaced by a set of mutually coupled random matrices each of which is localized in a finite space-time region. This model is defined in section 2. Partition function is calculated and comparison with the nonlinear sigma model is made.

To calculate the level density in RMT, one usually use the orthogonal polynomial method[12] or the supersymmetry method[13, 14]. In the present paper, we use the supersymmetry method because this method is easy to handle the extended model. This is formulated in section 3.

The averaged level density (section 4) and the two-point level correlation function (section 5) are calculated perturbatively. In comparison with the non-chiral RMT, we confirm that the Thouless energy in QCD is related to the pion decay constant as originally suggested.

In section 6, we also calculate the two-point correlation functions which correspond to meson propagators. We find that a pion propagator derived in the present model corresponds to a diffuson propagator in Anderson model. Namely, pion mode plays the
role of diffusion mode in disordered QCD vacuum.

2 Chiral random matrix model

2.1 Original chiral random matrix model

Original chiral random matrix theory is defined as follows \cite{1}. Generally, Euclidean Dirac operator matrix in gauge field can be written as

\[
H = \begin{pmatrix}
0 & W \\
W^\dagger & 0
\end{pmatrix}
= \begin{pmatrix}
0 & \omega_1 - i\omega_2 \\
\omega_1 + i\omega_2 & 0
\end{pmatrix}
\tag{1}
\]

where \( W \) is an \( N \times N \) complex matrix and \( \omega_i \) is a real matrix. That is, we consider the case of unitary ensemble which has the symmetry for fundamental fermions with three colors. \( N \) goes to infinity in the thermodynamic limit. Matrix \( W \) depends on the gauge field and treated as random matrix. Gauge field path integral is replaced by Gaussian ensemble average \cite{1} as

\[
Z_{RMT} = \int dW \prod_{f=1}^{N_f} \det \left( \frac{m_f}{iW^\dagger} \right) \exp \left( -\frac{N}{2} \Sigma^2 \text{Tr} WW^\dagger \right)
\tag{2}
\]

where \( m_f \) are quark masses and \( \Sigma \) denotes chiral condensate. This function is expressed by the integral of auxiliary field \( U \) which is an \( N_f \times N_f \) unitary matrix \cite{1} as

\[
Z_{RMT} = \int dU \exp \left( N \Sigma \text{Re} \text{Tr} (m_q U) \right).
\tag{3}
\]

This function is the same as the partition function of nonlinear sigma model without space-time dependence,

\[
Z_{NLS} = \int [dU] \exp \left( -\int d^4x \mathcal{L} \right)
\tag{4}
\]

\[
\to \int dU \exp \left( V \Sigma \text{Re} \text{Tr} (m_q U) \right)
\tag{5}
\]

\[
\mathcal{L} = \frac{f_\pi^2}{4} \text{Tr} \left( \partial_\mu U^\dagger(x) \partial_\mu U(x) \right) - \Sigma \text{Re} \left( \text{Tr} m_q U(x) \right) + \cdots
\tag{6}
\]

where \( f_\pi \) is a pion decay constant. In the R.H.S. of the arrow, we ignore the space time dependence of \( U(x) \). That is, ChRMT is a zero dimensional theory. In ChRMT, we focus our attention on quasi-zero mode levels of Dirac operator. As long as we consider these quasi-zero modes, we can use the partition function of ChRMT without spatial dependence. It is known that the microscopic spectral density which is the level density of quasi-zero modes does not depend on the assumption of Gaussian ensemble average \cite{15}.

In nonlinear sigma model, the condition for the zero-momentum sector to dominate is \( 1/m_\pi \gg L \) \cite{10} where \( m_\pi \) is the pion mass and \( L \) is a system size. This condition
means Pion compton wavelength is greater than the system size. When $1/m_π \sim L$, contributions from nonzero modes are important and we cannot neglect the kinetic term. Using Gell-mann–Oakes–Renner relation, this condition is written as follows

$$m_q \sim \frac{f_q^2/2\Sigma}{L^2}.$$  \hspace{1cm} (7)

The authors of Ref.[6, 7] suggested that Thouless energy in QCD is the right hand side quantity of Eq.(7). If relevant energy scale exceeds Thouless energy, we should expect that the deviations from ChRMT appear, just like the case of Anderson model. This diffusion effects are discussed in Anderson model[11]. We try to do the same thing in QCD.

### 2.2 Extended chiral random matrix model

We must introduce the parameter corresponding to the pion decay constant. To consider spatial dependence, the random matrix $W$ is replaced by a set of mutually coupled random matrices as

$$W \rightarrow \begin{pmatrix} W_1 & v_1 & \cdots & 0 \\ v_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & v_1 \\ 0 & v_1 & \cdots & W_n \end{pmatrix} \hspace{1cm} (8)$$

where $W_i$ is an $l \times l$ random matrix and $v$ is a coupling constant. Label $i$ means space-time coordinate. Coupling matrix $v_1$ is chosen diagonal for simplicity. $1_l$ denotes an $l$ dimensional unit matrix. Eq.(8) is the expression for the one dimensional system. But we can consider any dimensions. The size of one block matrix, $l$, represents ‘internal’ degrees of freedom. In Anderson model, this size corresponds to an elastic scattering length.

Janik et al. suggested that this size corresponds to the constituent quark mass[8]. In instanton liquid model, particularly in finite temperatures, the instantons and anti-instantons forms molecules. A random matrix model motivated by this molecule formation is discussed in Ref.[18]. The size of these molecules can be considered $l$ in the present model.

A probability distribution of the random matrices is taken to be Gaussian

$$P(H) \propto \exp \left( -\Sigma^2 \sum_{i=1}^n \text{Tr} W_i W_i^\dagger \right) \hspace{1cm} (9)$$

where $\Sigma$ is a parameter which corresponds to the chiral condensate. When $n = 1$, this model reduces to the original one.

Partition function is calculated using auxiliary field as

$$Z_{RMT} = \int \prod_i^n dW_i \prod_{f=1}^{N_f} \det \left( \begin{pmatrix} m_f & iW_i^\dagger \\ iW_i & m_f \end{pmatrix} \right) \exp \left( -\Sigma^2 \sum_{i=1}^n \text{Tr} W_i W_i^\dagger \right)$$

$$= \int \prod_{i=1}^n dA_i \exp \left[ -\Sigma^2 \sum_{i=1}^n \text{Tr} A_i A_i^\dagger \right] \det \left( \begin{pmatrix} A_i^\dagger + m_f & i\bar{v} \\ i\bar{v} & A + m_f \end{pmatrix} \right) \hspace{1cm} (10)$$

A similar chiral random hopping model was used in Ref.[17] in order to describe superconductor/normal-metal systems.
where

\[ A + m_f = \begin{pmatrix} (A_1 + m_f) \cdot 1_l & 0 \\ 0 & (A_n + m_f) \cdot 1_l \end{pmatrix} \] (11)

\[ \tilde{v} = v \begin{pmatrix} 0 & 1_{N_f l} & \cdots & 1_{N_f l} \\ 1_{N_f l} & 0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & 1_{N_f l} \\ 0 & 1_{N_f l} & 0 \end{pmatrix} \] (12)

and \( A_i \) is an \( N_f \times N_f \) matrix. This integral is evaluated with use of the saddle-point approximation. The saddle-point manifold for \( v = 0 \) is \( A_i = U_i / \Sigma \) where \( U_i \) is an \( N_f \times N_f \) unitary matrix. We obtain the partition function as follows

\[ Z_{RMT} = \int \prod_{i=1}^{n} dU_i \exp \left[ \Sigma l \sum_{i=1}^{n} \text{Tr} m_q(U_i + U_i^\dagger) \\
+ \Sigma^2 v^2 l \sum_{i=1}^{n-1} \text{Tr} (U_{i+1}^\dagger U_i + U_i^\dagger U_{i+1}) + \cdots \right]. \] (13)

The coupling parameter \( v \) is assumed to be a small quantity and expanded up to the second power. This must correspond to the partition function of the nonlinear sigma model,

\[ Z_{NLS} = \int \prod_{i} dU_i \exp \left[ \frac{\Sigma}{2} l \sum_{i=1}^{n} \text{Tr} m_q(U_i + U_i^\dagger) \\
+ f_\pi^2 \frac{2\pi}{4l} \sum_{i=1}^{n-1} \text{Tr} (U_{i+1}^\dagger U_i + U_i^\dagger U_{i+1}) + \cdots \right]. \] (14)

Here, we set the lattice spacing \( a = 1 \). Comparing Eq.(13) with Eq.(14), we find the coupling parameter \( v \) is related with the pion decay constant as

\[ v^2 = \frac{f_\pi^2}{2\Sigma^2 l^2}. \] (15)

In the original ChRMT, the pion decay constant is treated as an infinite quantity. We note that \( v \) is small but \( f_\pi \) is a large number. This means \( l \) is a large number. In the original model, the matrix size must be taken infinity in order to get the universal results. In the present case, the size of one block matrix \( l \) must be taken infinity. On the other hand, it is not necessary to take an infinite limit of the number of the block \( n \).

### 3 Supersymmetry method

To calculate the level density, we use supersymmetry method[13, 14]. We use the notations and conventions in Ref.[14]. By this method, calculation of ensemble average can be performed easily. However at present this method cannot be applied for unquenched
calculation due to the presence of the fermion determinant. We use quenched approximation for this reason. In original ChRMT, calculation using this method is performed by the authors of Ref.\[19, 20, 21, 22, 23\].

Level density can be expressed as
\[
\rho(E) = -\frac{1}{\pi} \text{Im} \text{tr} G^{(R)}(E),
\]
where the (retarded) Green function is defined as
\[
G^{(R)}(E) = \frac{1}{E^+ - H}.
\]
\(H\) is a \(2nl \times 2nl\) matrix. This is derived from the generating function:
\[
Z(E, J) = \frac{\det(E^+ - H + J)}{\det(E^+ - H - J)} = \int d\psi \exp \left[ i\psi^\dagger (E^+ - H - Jk)\psi \right]
\]
\(k = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\)

where \(\psi\) is the 4\(nl\)-component graded vector,
\[
\psi^T = (\phi^T(+)^T, \phi^T(-)^T, \chi^T(+)^T, \chi^T(-)^T).
\]

In the above expression, \(\phi(\pm)\) is the \(nl\)-component bosonic vector and \(\chi(\pm)\) is the \(nl\)-component fermionic vector,
\[
\phi^T(\pm) = (\phi_1^T(\pm), \ldots, \phi_n^T(\pm))
= (\phi_{11}(\pm), \ldots, \phi_{1l}(\pm), \ldots, \phi_{n1}(\pm), \ldots, \phi_{nl}(\pm))
\]
\[
\chi^T(\pm) = (\chi_1^T(\pm), \ldots, \chi_n^T(\pm))
= (\chi_{11}(\pm), \ldots, \chi_{1l}(\pm), \ldots, \chi_{n1}(\pm), \ldots, \chi_{nl}(\pm))
\]
where the symbol \(\pm\) expresses the chiral structure of the matrix.

Taking derivatives with respect to the external field \(J\), we can get the Green function. Generating function is normalized to unity when \(J = 0\) and it is easy to calculate the ensemble average.

In Eq.(16), chiral basis is used. We perform a unitary rotation as a matter of convenience and get
\[
H = \begin{pmatrix} \omega_1 & -i\omega_2 \\ i\omega_2 & -\omega_1 \end{pmatrix}.
\]
This basis is used in Ref.[13]. We follow their way and notations.
In this basis, Dirac operator matrix is written as

\[
H = \begin{pmatrix}
\omega_1 + V & -i\omega_2 \\
-i\omega_2 & -\omega_1 - V
\end{pmatrix}
\]  

(24)

\[
\omega_1 = \begin{pmatrix}
\omega_{11} & 0 \\
0 & \omega_{1n}
\end{pmatrix}, \quad \omega_2 = \begin{pmatrix}
\omega_{21} & 0 \\
0 & \omega_{2n}
\end{pmatrix}
\]

\[
V = \begin{pmatrix}
0 & v_{1l} & 0 \\
v_{1l} & 0 & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
0 & v_{1l} & 0
\end{pmatrix},
\]

where \(\omega_{1i}, \omega_{2i}\) is an \(l \times l\) matrix and \(W_i = \omega_{1i} + i\omega_{2i}\).

Performing the ensemble average over the random matrix, Eq.(18) leads to the following expression\[19\],

\[
\langle Z(E,J) \rangle = \int dW Z(E,J) \exp \left(-l\Sigma^2 \sum_{i=1}^n \text{Tr} W_i W_i^\dagger \right)
\]

\[
= \int d\psi \exp \left[-\frac{1}{4l\Sigma^2} \sum_{i=1}^n \text{Tr} \tilde{A}^2(i) + i\psi^\dagger (E^+ - V - Jk)\psi \right].
\]  

(25)

Here, \(\tilde{A}(i)\) is a 4 \(\times\) 4 graded matrix,

\[
\tilde{A}(i) = \frac{1}{\sqrt{2}} (A_z(i) - \Sigma_y A_z(i) \Sigma_y)
\]

(26)

\[
A_z(i) = \Sigma_z^{1/2} \sum_{\mu=1}^l \psi_{\mu i}^\dagger \psi_{\mu i} \Sigma_z^{1/2}
\]

(27)

where \(\psi_{\mu i}\) is a 4 dimensional graded vector

\[
\psi_{\mu i}^T = (\phi_{\mu i}(+), \chi_{\mu i}(+), \phi_{\mu i}(-), \chi_{\mu i}(-))
\]

(28)

and \(\Sigma_z(\Sigma_y)\) is the 4 \(\times\) 4 Pauli matrix \(\sigma_z \otimes 1_2\) \((\sigma_y \otimes 1_2)\).

Next, we carry out the Hubbard-Stratonovitch transformation. Eq.(25) is expressed with use of the auxiliary variable \(Q\) as

\[
\langle Z(E,J) \rangle = \int dQ d\psi \exp \left[-\frac{l}{2} \sum_{i=1}^n \text{Tr} Q^2(i) - \frac{i}{\sqrt{2}\Sigma} \sum_{i=1}^n \text{Tr} Q(i) \tilde{A}(i) + i\psi^\dagger (E^+ - V - Jk)\psi \right].
\]  

(29)

A 4 \(\times\) 4 graded matrix \(Q\) is required to have the same symmetry as \(\tilde{A}\). Since \(\tilde{A}\) anticommutes with \(\Sigma_y\), we must impose the constraint

\[
\{Q(i), \Sigma_y\} = 0.
\]

(30)

As a consequence, \(\text{Tr} Q(i) \tilde{A}(i) = \sqrt{2} \text{Tr} Q(i) A_z(i)\).
Changing the integration variable to $\psi \rightarrow \Sigma z/2\psi$ and performing the integral over $\psi$, the generating function is expressed as follows

$$\langle Z(E, J) \rangle = \int dQ \exp \left[ -\frac{1}{2} \sum_{i=1}^{n} \text{trg} Q^2(i) - l \sum_{i=1}^{n} \text{trg} \ln \left( E + \Sigma z - \frac{1}{\Sigma} Q(i) \right) \right]$$

$$+ vl^2 \sum_{i=1}^{n} \text{trg} \ln \left( E + \Sigma z - \frac{1}{\Sigma} Q(i) \right)^{-1} \left( E + \Sigma z - \frac{1}{\Sigma} Q(i + 1) \right)^{-1} + \cdots$$

$$+ \sum_{i=1}^{n} \text{trg} \ln \left( E + \Sigma z - \frac{1}{\Sigma} Q(i) \right)^{-1} Jk + \cdots \right].$$

where $v$ is treated as a small parameter and expanded in a power series. The external field $J$ is a diagonal matrix.

4 Microscopic spectral density

4.1 Saddle-point

We apply a saddle-point approximation to Eq.(31). Since parameter $v$ is a small quantity, we consider the saddle-point for $v = 0$. The saddle-point equation is

$$- Q(i) + \frac{1}{E + \Sigma z - Q(i)} = 0. \quad (32)$$

For $E \sim \mathcal{O}(l^0)$, the saddle-point manifold consists only of the single point

$$Q(i) = \Sigma z \left( \frac{E \Sigma}{2} - i \sqrt{1 - \left( \frac{E \Sigma}{2} \right)^2} \right), \quad (33)$$

where minus sign in front of the square root is chosen so that it is consistent with the sign of the infinitesimal imaginary part in $E^+$. Level density is expressed as follows

$$\langle \rho(E) \rangle = - \frac{1}{2\pi} \text{Im} \text{tr} \frac{\partial}{\partial J} \langle Z(E, J) \rangle \bigg|_{J=0}$$

$$= - \frac{\Sigma l}{2\pi} \int dQ \sum_{i=1}^{n} \text{trg} k \Sigma z Q(i) \exp \left[ -\frac{l}{2} \sum_{i=1}^{n} \text{trg} Q^2(i) + \cdots \right]. \quad (34)$$

Using the saddle-point approximation, we can get the familiar semicircle law

$$\langle \rho(E) \rangle = \frac{1}{\Delta} \sqrt{1 - \left( \frac{\pi E}{4nl\Delta} \right)^2}$$

where $\Delta = \pi/2nl\Sigma$.

For $E \sim \mathcal{O}(l^{-1})$, saddle-point equation becomes

$$Q^2(i) = -1. \quad (36)$$

This is not a single point and we must perform the integration over the saddle-point manifold.
Level density is expressed as follows
\[
\langle \rho(E) \rangle = \frac{\Sigma l}{2\pi} \text{Im} \int_{Q^2 = -1} dQ \sum_{i=1}^{n} \text{trg} k \Sigma_z Q(i) \times \exp \left[ -lE \Sigma \sum_{i=1}^{n} \text{trg} Q(i) \Sigma_z + l \Sigma^2 v^2 \sum_{i=1}^{n} \text{trg} Q(i) Q(i+1) + \cdots \right].
\]
(37)

To proceed further, we must parametrize the saddle-point manifold. In Ref.[19], the \(4 \times 4\) graded matrix \(Q\) is parametrized as
\[
Q = T_0^{-1}(-i\Sigma_z)T_0
\]
(38)
\[
T_0 = \begin{pmatrix} 1 + t^2 & \frac{it}{2} \\ -it & 1 + t^2 \end{pmatrix},
\]
(39)
where \(2 \times 2\) graded matrix \(t\) is given by
\[
t = \begin{pmatrix} a & \rho^* \\ \rho & ib \end{pmatrix}.
\]
(40)

We omit the indices \(i\) in \(Q(i)\) for the sake of simplicity of expressions. Here, \(a\) and \(b\) are real bosonic variables and \(\rho\) and \(\rho^*\) are fermionic variables. Ref.[19] uses a diagonal parametrization of \(t\),
\[
t = ut_0u^{-1}
\]
(41)
\[
t_0 = \begin{pmatrix} \mu_1 & 0 \\ 0 & i\mu \end{pmatrix}, \quad u = \exp \left( \begin{pmatrix} 0 & \xi \\ \xi^* & 0 \end{pmatrix} \right).
\]
(42)

In this parametrization, measure \(dQ\) is calculated as
\[
dQ = d\theta_1 d\theta \frac{d\xi d\xi^*}{2\pi} \frac{1}{2\cosh \theta_1 \cos \theta - i \sinh \theta_1 \sin \theta - 1}
\]
(43)
\[
= d\rho d\rho^* \frac{d\rho d\rho^*}{2\pi} \left[ 1 - \frac{1}{2} (\text{trg} t)^2 + \cdots \right]
\]
(44)
where
\[
\mu_1 = \sinh \frac{\theta_1}{2},
\]
(45)
\[
\mu = \sin \frac{\theta}{2}.
\]
(46)

In diagonal parametrization, the measure \(dQ\) is expressed compactly. In the exact calculation for the original ChRMT, this parametrization is used. But we cannot do exact calculation in the present case because we must treat all the nonzero modes. Calculation becomes very difficult and we must rely on perturbation. The expression of \(dQ\) in the form of Eq.(44) is suitable for perturbative calculation.
4.2 Perturbative calculation

We take the continuum limit of Eq.(57). The continuum limit for a $d$ dimensional system is done by the following replacement

$$n \rightarrow n^d, \quad l \rightarrow l^d, \quad \sum_{i=1}^{n} \rightarrow \frac{1}{l^d} \int d^d x, \quad Q(i + 1) - Q(i) \rightarrow l \nabla Q(x).$$

(47)

The level density is expressed as

$$\langle \rho(E) \rangle = -\frac{\Sigma}{2\pi} \text{Im} \int [dQ] \left( \int d^d x \text{trg} (k\Sigma_x Q(x)) \right)$$

$$\times \exp \left[ -\int d^d x \left( \Sigma \text{trg} Q(x) \Sigma_x - \frac{f_\pi^2}{4} \text{trg} Q(x) \nabla^2 Q(x) \right) \right].$$

(48)

Here, we used the relation $v^2 = f_\pi^2/2 \Sigma L^2$.

Eq.(48) is expressed by $2 \times 2$ graded matrices $t(x)$ as

$$\langle \rho(E) \rangle = \frac{2\Sigma}{\pi} \text{Re} \int [dt] \left\{ \int d^d x (1 + \text{trg} kt^2(x)) \right\}$$

$$\times \exp \left[ -\int d^d x \left( -4i \Sigma \text{trg} t^2(x) - 2f_\pi^2 \text{trg} t(x) \nabla^2 t(x) + \cdots \right) \right]$$

$$\times \left\{ 1 - \frac{1}{2l^d} \int d^d x (\text{trg} t(x))^2 + \frac{1}{8l^d} \int d^d x d^d y (\text{trg} t(x))^2 (\text{trg} t(y))^2$$

$$+ \frac{1}{4l^d} \int d^d x t(x) \text{trg} t^3(x) + \cdots \right\}. \quad (49)$$

We perform the perturbative calculation. Keeping the bilinear terms in the exponential, the other terms are expanded in a power series of $t$ and we get

$$\langle \rho(E) \rangle = \frac{2\Sigma(n l)^d}{\pi} \text{Re} \left\langle 1 + \frac{1}{(n l)^d} \int d^d x \text{trg} kt^2(x) - \frac{1}{2l^d} \int d^d x (\text{trg} t(x))^2$$

$$- \frac{1}{2(n l)^d l^d} \int d^d x d^d y \text{trg} kt^2(x) (\text{trg} t(y))^2 + \frac{1}{8l^d} \int d^d x d^d y (\text{trg} t(x))^2 (\text{trg} t(y))^2$$

$$+ \frac{1}{4l^d} \int d^d x t(x) \text{trg} t^3(x) + 2f_\pi^2 \int d^d x (t(x) \nabla t(x))^2 + \cdots \right\} \right\rangle \quad (50)$$

$$\langle \langle \cdots \rangle \rangle = \int [dt] (\cdots) \exp \left[ -\int d^d x \left( -2f_\pi^2 \text{trg} t(x) \nabla^2 t(x) - 4i \Sigma \text{trg} t^2(x) \right) \right]. \quad (51)$$

The calculation is just a Gaussian integral and can be performed easily with help of the following identity for the arbitrary graded matrix $A,B$:

$$\langle \langle \text{trg} At(x) Bt(x) \rangle \rangle = \Pi(E; x, y) \text{trg} A \text{trg} B \quad (52)$$

$$\langle \langle \text{trg} At(x) \text{trg} Bt(x) \rangle \rangle = \Pi(E; x, y) \text{trg} AB, \quad (53)$$

where $\Pi(E; x, y)$ is the fundamental propagator of this model,

$$\Pi(E; x, y) = \sum_k \Pi(E, k) e^{i k(x-y)} \quad (54)$$

$$\Pi(E, k) = \frac{1}{4\pi \left( \frac{f_\pi^2}{2} - i \frac{\Delta}{2} \right)} \quad (55)$$

$$k_i = \frac{\pi n_i}{nl} (n_i = 0, \pm 1, \ldots), \quad \Delta = \frac{\pi}{2\Sigma(n l)^d}, \quad D = \frac{f_\pi^2}{2\Sigma}. \quad (56)$$
The perturbative calculation is carried out and we obtain the following expression

\[ \langle \rho(E) \rangle = \frac{1}{\Delta} \left\{ 1 - \frac{n^d}{8\pi^2} \sum_k \text{Re} \left( \frac{1}{\left( k^2 - i\frac{E}{\Delta} \right)^2} \right) + \cdots \right\}. \quad (56) \]

In this calculation, we treat the propagator as a small quantity because the pion decay constant is a large quantity. This is correct for the nonzero modes but not correct for the zero mode because the term \( Dk^2/\Delta \) vanishes at \( k = 0 \). As a result, oscillations due to level repulsion are not found. This is because we use the perturbation for all modes. Hence, for small energy regions, the above expression is no longer valid due to the divergence of \( k = 0 \) mode at \( E = 0 \). In these regions there are general arguments that the microscopic spectral density should take the universal form \(^{10}\).

5 Two-point level correlation function

Two-point level correlation function is calculated in the same way. This function can be derived from the ensemble average of the product of the generating functions as

\[
W(E_1, E_2) = \langle \text{tr} G^{(R)}(E_1) \text{tr} G^{(R)}(E_2) \rangle
\]

\[ = \frac{1}{4} \text{tr} \left( \frac{\partial}{\partial J^{(1)}} \right) \text{tr} \left( \frac{\partial}{\partial J^{(2)}} \right) \langle Z(E_1, J^{(1)}) Z(E_2, J^{(2)}) \rangle. \quad (57)\]

In ChRMT, we have the following identity

\[
\text{tr} G^{(A)}(E) = -\text{tr} G^{(R)}(-E). \quad (58)
\]

Here, \( G^{(A)}(E) \) is the advanced Green function. With use of this identity, the two-point level correlation function is expressed as follows

\[ \langle \rho(E_1) \rho(E_2) \rangle = -\frac{1}{(2\pi)^2} \left[ W(E_1, E_2) + W(-E_1, E_2) \right. \]

\[ + W(E_1, -E_2) + W(-E_1, -E_2) \left]. \quad (59) \]

Now let us turn to the calculation of \( W(E_1, E_2) \). The generating function is expressed with use of graded matrices and vectors as

\[
Z(E_1, J^{(1)}) Z(E_2, J^{(2)}) = \int d\hat{\psi} \exp \left[ i\hat{\psi}^\dagger (\hat{E} - H - \hat{J}k)\hat{\psi} \right] \quad (60)
\]

where

\[
\hat{\psi} = \begin{pmatrix} \psi^{(1)} \\ \psi^{(2)} \end{pmatrix}, \quad \hat{E} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}, \quad \hat{J} = \begin{pmatrix} J^{(1)} & 0 \\ 0 & J^{(2)} \end{pmatrix}. \quad (61)
\]

As in the previous section, the ensemble average and Hubbard-Stratonovitch transformation is performed and we obtain

\[
\langle Z(E_1, J^{(1)}) Z(E_2, J^{(2)}) \rangle = \int dQ \exp \left[ -\frac{l}{2} \sum_{i=1}^n \text{trg}^2 Q(i) \right. \]

\[ -l \sum_{i=1}^n \text{trg} \ln(\hat{E}\Sigma_z - \frac{1}{\Sigma} Q(i)) + lv^2 \sum_{i=1}^n \text{trg} (\hat{E}\Sigma_z - \frac{1}{\Sigma} Q(i))^{-1}(\hat{E}\Sigma_z - \frac{1}{\Sigma} Q(i + 1))^{-1} \]

\[ + \sum_{i=1}^n \text{trg} (\hat{E}\Sigma_z - \frac{1}{\Sigma} Q(i))^{-1} \hat{J}_{ii} k + \cdots \]. \quad (62) \]
Here, $Q$ is an $8 \times 8$ graded matrix. For $E \sim \mathcal{O}(l^{-1})$, the saddle point is $Q(i)^2 = -1$. Parametrization is as follows:

$$Q = T_0^{-1}(-i\Sigma_z)T_0$$

$$T_0 = T_uT_{ch}.$$  

An $8 \times 8$ graded matrix $T_0$ is separated by the chiral rotation part $T_{ch}$ and the unitary rotation part $T_u$ as

$$T_{ch} = \begin{pmatrix}
(1 + t_1^2)^{1/2} & 0 & it_1 & 0 \\
0 & (1 + t_2^2)^{1/2} & 0 & it_2 \\
-it_1 & 0 & (1 + t_1^2)^{1/2} & 0 \\
0 & -it_2 & 0 & (1 + t_2^2)^{1/2}
\end{pmatrix}$$

$$T_u = \begin{pmatrix}
(1 + t_{12}t_{21})^{1/2} & 0 & 0 & it_{12} \\
0 & (1 + t_{21}t_{12})^{1/2} & it_{21} & 0 \\
0 & it_{12} & (1 + t_{12}t_{21})^{1/2} & 0 \\
-it_{21} & 0 & 0 & (1 + t_{21}t_{12})^{1/2}
\end{pmatrix}.$$  

Here, $2 \times 2$ graded matrix $t$ is defined as

$$t_1 = \begin{pmatrix}
a_1 & \rho_1^* \\
\rho_1 & ib_1
\end{pmatrix},
\quad t_2 = \begin{pmatrix}
a_2 & \rho_2^* \\
\rho_2 & ib_2
\end{pmatrix}$$

$$t_{12} = \begin{pmatrix}
c & i\eta \\
\sigma & id
\end{pmatrix},
\quad t_{21} = \begin{pmatrix}
c^* & \sigma^* \\
-i\eta^* & id^*
\end{pmatrix}.$$  

The chiral rotation is due to the chiral structure of the matrix and the unitary rotation is due to the calculation of the two-point level density. The former is the same as the calculation of the one-point level density in section 4 and the latter is the same as the calculation of the two-point level density in the standard (non-chiral) RMT.

Taking derivative with respect to $J$, we can obtain $W(E_1, E_2)$ as

$$W(E_1, E_2) = \frac{l^2\Sigma^2}{4} \int_{Q^2=-1} dQ \sum_{i,j} \text{trg} \left( k\Sigma_z \frac{1+\Lambda}{2} Q(i) \right) \text{trg} \left( k\Sigma_z \frac{1-\Lambda}{2} Q(i) \right) \times \exp \left[ -l\Sigma \sum_{i=1}^n \text{trg} \hat{E}^+ \Sigma_z Q(i) + lv^2\Sigma^2 \sum_{i=1}^n \text{trg} Q(i)Q(i+1) \right]$$

where

$$\Lambda = \text{diag}(1, -1, 1, -1).$$

In this representation, $8 \times 8$ graded matrix is aligned as the following block structure: chiral, two-point, boson-fermion,

$$M = \begin{pmatrix}
M(++) & M(+-) \\
M(-+) & M(-)
\end{pmatrix}$$

$$M(++) = \begin{pmatrix}
M^{(11)}(++) & M^{(12)}(++) \\
M^{(21)}(++) & M^{(22)}(++)
\end{pmatrix}$$

$$M^{(11)}(++) = \begin{pmatrix}
M_{BB}^{(11)}(++) & M_{BF}^{(11)}(++) \\
M_{FB}^{(11)}(++) & M_{FF}^{(11)}(++)
\end{pmatrix}. $$

$$M^{(12)}(++) = \begin{pmatrix}
M_{BB}^{(12)}(++) & M_{BF}^{(12)}(++) \\
M_{FB}^{(12)}(++) & M_{FF}^{(12)}(++)
\end{pmatrix}.$$
We calculate the integral of Eq. (68) perturbatively. The measure $dQ$ is separated into the chiral part and unitary (two-point) part: $dQ = d\mu(T_{ch})d\mu(T_u)$. The chiral part is the same as the previous section. Jacobian of the unitary part is equal to unity if we take the matrix elements of $t_{12}, t_{21}$ as the integral variables. Then we get

$$dQ = dt_1 dt_2 \left( 1 - \frac{1}{2} \text{tr} t^2_1 - \frac{1}{2} \text{tr} t^2_2 + \cdots \right) dt_{12} dt_{21}.$$  \hfill (73)

The calculation is carried out in the same way as the previous section and yields

$$W(E_1, E_2) = -4(nl)^2 \Sigma^2 + 8 \Sigma^2 n^d(nl)^2 \sum_k \left( \Pi^2(E_1, k) + \Pi^2(E_2, k) \right) - 16 \Sigma^2 (nl)^2 \sum_k \Pi^2 \left( \frac{E_1 + E_2}{2}, k \right) + \cdots.$$  \hfill (74)

Two-point level density is expressed as

$$\langle \rho(E_1) \rho(E_2) \rangle - \langle \rho(E_1) \rangle \langle \rho(E_2) \rangle = \frac{1}{\Delta^2} \sum_k \left[ \Pi^2(E, k) + \Pi^2(-E, k) + \Pi^2(\omega/2, k) + \Pi^2(-\omega/2, k) \right] + \cdots,$$  \hfill (75)

where $E = (E_1 + E_2)/2$ and $\omega = E_1 - E_2$. $E$-dependent terms are characteristic in ChRMT. Comparing this expression with the usual (non-chiral) RMT, we confirm that Thouless energy in QCD is

$$E^c = \frac{\pi^2}{2} \Sigma L^2.$$  \hfill (76)

We can calculate number variance from the two-point level density. This is defined as

$$\Sigma^2(z) = \int_{-z/\Delta}^{z/\Delta} \int_{-z/\Delta}^{z/\Delta} dx dy \Delta^2 \left( \langle \rho(E_1) \rho(E_2) \rangle - \langle \rho(E_1) \rangle \langle \rho(E_2) \rangle \right),$$  \hfill (77)

where $x = E_1/\Delta$ and $y = E_2/\Delta$. This is the measure of fluctuations of the level number in the finite level interval $z$.

Integration is performed and we obtain

$$\Sigma^2(z) = \frac{1}{\pi^2} \sum_k \ln \left( 1 + \frac{z^2}{4k^2} \right).$$  \hfill (78)

We note that the $E$-dependent terms and $\omega$-dependent terms give the same contributions to the number variance. Eq. (75) does not treat the zero mode exactly. We expect, however, that this approximation does not change the behavior of $\Sigma^2(z)$ drastically. This is because it is known from the results of the usual RMT that what is missing in the perturbative result of the two-point level density is the oscillating behavior of the exact result and the integration in Eq. (77) cancels these oscillations. Taking the continuum limit, we obtain

$$\Sigma^2(z) \propto \frac{1}{\pi^2} \left( \frac{\Delta z}{2D} \right)^{d/2} \frac{V}{\pi^d} \int d^d k' \ln \left( 1 + \frac{1}{k'^2} \right) \frac{\Delta^2}{2}.$$  \hfill (79)
We find that the number variance depends on the dimensionality of the system. This is very different from the original ChRMT where $\Sigma^2(z) \sim \log z$ for large $z$. The similar behavior is found in Ref.[24] for usual (non-chiral) RMT and interpreted as change from level correlation of RMT to those of non-correlated energy levels (Poisson statistics: $\Sigma^2(z) \sim z$). This is consistent with intuitive discussion.

6 Two-point correlation function

The spatial dependence, which we introduced into the original ChRMT in this paper, enables us to calculate the correlation between two spatial points. By formal replacement of the energy $E^+$ to $im$, the Green function $G^{(R)}(E)$ becomes a quark propagator. The ensemble average of a matrix element

$$G^{(R)}(im; x, y, \mu, \nu) = \langle x, \mu | \frac{1}{im-H} | y, \nu \rangle$$

(80)

can be calculated from the generating function as

$$\langle G(im; x, y, \mu, \nu) \rangle = \frac{1}{2} \frac{\partial}{\partial J_{xy,\mu\nu}} \langle Z(im, J) \rangle.$$ (81)

The external field $J$ is not diagonal in this case. Label $x, y$ represents the position of block matrices. Label $\mu, \nu$ represents the 'internal' degrees of freedom which are described and by random matrices. We get $\langle G(im; x, y, \mu, \nu) \rangle = 0$ if $|x - y| > l$. This behavior means the size $l$ is a reciprocal of the constituent quark mass. This is consistent with the discussion of the Ref.[6].

We can also consider 'meson' propagator in this model. This is the product of the Green functions and can be calculated from the generating function as

$$\langle \text{tr} \sum_{\mu,\nu} G(im; x, y, \mu, \nu)G(im; y, x, \nu, \mu) \rangle =$$

$$\frac{1}{4} \sum_{\mu,\nu} \left( \frac{\partial}{\partial J_{x\mu y\nu}(+)} \frac{\partial}{\partial J_{x\mu y\nu}(+)} + \frac{\partial}{\partial J_{x\mu y\nu}(+)} \frac{\partial}{\partial J_{y\nu x\mu}(-)} \right) \langle Z(E, J) \rangle$$

(82)

and

$$\langle \text{tr} \sum_{\mu,\nu} G(im; x, y, \mu, \nu)\gamma_5 G(im; y, x, \nu, \mu)\gamma_5 \rangle =$$

$$\frac{1}{4} \sum_{\mu,\nu} \left( \frac{\partial}{\partial J_{x\mu y\nu}(+)} \frac{\partial}{\partial J_{x\mu y\nu}(-)} + \frac{\partial}{\partial J_{x\mu y\nu}(+)} \frac{\partial}{\partial J_{y\nu x\mu}(+)} \right) \langle Z(E, J) \rangle.$$ (83)

Here, trace is taken over the chiral degrees of freedom. Summation over $\mu,\nu$ is taken because this degrees of freedom is described by random matrices. Eq.(82) corresponds to
scalar (sigma) propagator and Eq.(83) corresponds to pseudo scalar (pion) propagator \[3\]. We consider the long range correlation between the point \(x \neq y\). In the present basis, matrix \(\gamma_5\) is expressed as

\[
\gamma_5 = \begin{pmatrix}
0 & -1 \\
-1 & 0
\end{pmatrix}.
\] (84)

Eq.(82) and (83) are calculated perturbatively and we obtain

\[
\left\langle \text{tr} \sum_{\mu,\nu} G(im; x, y, \mu, \nu) G(im; y, x, \nu, \mu) \right\rangle = 64 \Sigma^2 l^d \int d^d z \Pi(im; x, y) \Pi(im; y, z) \Pi(im; z, x) + \cdots
\] (85)

\[
\left\langle \text{tr} \sum_{\mu,\nu} G(im; x, y, \mu, \nu) \gamma_5 G(im; y, x, \nu, \mu) \gamma_5 \right\rangle = -16 l^2 \Sigma^2 \Pi(im; x, y) + \cdots
\] (86)

where

\[
\Pi(im; x, y) = \sum_k \Pi(im, k) \exp(ik(x - y))
\] (87)

\[
\Pi(im, k) = \frac{1}{4\pi \left(\frac{D}{2} k^2 + m^2\right)}
\]

\[
= \frac{1}{4\pi \frac{D}{2} \left(k^2 + m^2\right)}.
\] (88)

Here \(\Pi(im; k)\) is obtained from \(\Pi(E; k)\) by the replacement of \(E\) by \(im\). The above equation shows that \(\Pi(im; x, y)\) corresponds to the pion propagator. We find that the scalar propagator can be neglected by comparing with the pseudo scalar propagator. Pseudo-scalar mode is important. This is consistent with the NG theorem. As was shown in Section 2.2, the present model is equivalent to the nonlinear sigma model. Hence it is reasonable that the pion propagator appears as a fundamental ingredient in the calculation of various correlation functions. The calculation of the two-point correlation function was also done in Ref.[9] with use of partially quenched chiral perturbation theory.

\(\Pi(E; x, y)\) is a propagator of the fundamental mode in the perturbative calculation. In Anderson problem, this mode is called diffuson. Eq.(86) shows the pion mode in QCD plays the same role as the diffuson mode in Anderson model. The fundamental mode appealing in the chiral perturbation theory, that is, pion, corresponds to the fundamental mode appealing in the Anderson model, diffuson.

### 7 Conclusions

In this paper, we examined the level correlations of QCD Dirac operator. We introduced the extended chiral random matrix model including the spatial dependence.

\[3\]We must insert the isospin matrix but this is irrelevant in this calculation.
First, we showed the partition function of this random matrix model is equivalent to that of a nonlinear sigma model. This equivalence has been previously shown only for the 0-dimensional case without spatial dependence. As far as we know, this is a first explicit derivation of nonlinear sigma model (with derivative terms) from the kind of chiral random matrix models.

To calculate level statistics, we used supersymmetry method developed by Efetov. Using the perturbative expansion, we got the expression of the level density and two-point level correlation function.

Comparing the expression of this function with the usual RMT, we confirm that the Thouless energy in QCD is $E_c = f^2_\pi / 2\Sigma L^2$. Number variance was calculated from the two-point level correlation function and found to depend on the dimensionality of the system. This is very different from the standard universal result and interpreted as change from level correlation of RMT to those of non-correlated energy levels.

For the first time, the correlation functions of pseudo-scalar type as well as scalar type between two spatial points could be calculated for the chiral random matrix model. The previous work with use of partially quenched chiral perturbation theory only gives the correlation function of pseudo-scalar type. The reason why scalar type correlation function can be calculated in this work is that quark propagators can be defined for the chiral random matrix model.

These functions demonstrate that the pion propagator is nothing but the diffusion propagator which is a fundamental mode responsible for diffusion effects in Anderson model. Namely, pion mode plays a role of diffusion when we consider QCD vacuum as a disordered medium. On the other hand, sigma-like mode does not appear as a fundamental mode in this model.

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