Heavy Flavor Contributions to QCD Sum Rules and the Running Coupling Constant\(^1\)

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Abstract. We have calculated first and second order corrections to several sum rules measured in deep inelastic lepton-hadron scattering. These corrections, which are due to heavy flavors only, are compared with the existing perturbation series which is computed for massless quarks up to third order in the strong coupling constant \(\alpha_s\). A study of the perturbation series reveals that the large logarithms of the type \(\ln^i Q^2/m^2\) dominate the perturbation series at much larger values than those given by the usual matching conditions imposed on the \(\alpha_s(\mu)\). Therefore these matching conditions cannot be used to extrapolate the running coupling constant from small \(\mu\) to very large scales like \(\mu = M_Z\). An alternative description of the running coupling constant in the MOM-scheme is proposed.

INTRODUCTION

Structure functions, measured in deep inelastic lepton-hadron scattering (see Fig. 1)

\[ l_1(k_1) + H(p) \rightarrow l_2(k_2) + 'X', \tag{1} \]

where ‘\(X\)’ denotes any inclusive final hadronic state, provide us with an excellent test of perturbative QCD. The in and outgoing leptons are represented by \(l_1\) and \(l_2\) respectively and the hadron is denoted by \(H\). On the Born level the reaction proceeds via the exchange of one of the vector bosons \(V\) of the standard model which are given by \(\gamma\), \(Z\) and \(W^\pm\). The virtuality of the vector boson \(V\) and the C.M. energy are given by \(q^2 = -Q^2 < 0\) and \(S = (p + k_1)^2\) respectively. Further the scaling variables are defined by \(x = Q^2/2p.q\) and \(y = p.q/p.k_1\). The structure functions show up in the polarized and unpolarized cross sections. Starting with the spin-averaged cross section for \(V = \gamma\) we obtain

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\[ \frac{d^2\sigma}{dxdy} = \frac{4\pi\alpha^2}{(Q^2)^2} S \left[ (1 - y)F_2(x, Q^2) + xy^2F_1(x, Q^2) \right]. \] (2)

For \( V = W^\pm \) and \( Q^2 \ll M_H^2 \) the unpolarized cross section for the charged current process becomes

\[ \frac{d^2\sigma^\pm}{dxdy} = \frac{G_F^2}{2\pi} S \left[ (1 - y)F_2(x, Q^2) + xy^2F_1(x, Q^2) \mp xy(1 - \frac{y}{2})F_3(x, Q^2) \right]. \] (3)

Finally we are also interested in the spin structure functions which are measured in polarized scattering. In the case of \( V = \gamma \) the cross section becomes

\[ \frac{d^2\sigma^{H(\rightarrow)}}{dxdy} - \frac{d^2\sigma^{H(\leftarrow)}}{dxdy} = \frac{8\pi\alpha^2}{Q^2} \left[ (2 - y)g_1(x, Q^2) \right], \] (4)

from which one can extract the longitudinal spin structure function \( g_1 \). The transverse spin structure function \( g_2 \) is obtained from the cross section

\[ \frac{d^3\sigma^{H(\rightarrow)}}{dxdy\,d\phi} - \frac{d^3\sigma^{H(\leftarrow)}}{dxdy\,d\phi} = \cos \phi \frac{4\alpha^2}{Q^2} \left( \frac{4M^2x(1 - y)}{yS} \right)^{1/2} \left[ yg_1(x, Q^2) + 2g_2(x, Q^2) \right], \] (5)

where \( \phi \) is the angle between the spin of the hadron and the plane spanned by the lepton momenta \( \vec{k}_i \). Further we have indicated between the brackets the spin of the incoming lepton \( l_1 \) and we have neglected in the expressions above all power corrections of the type \( M^2/S \) where \( M \) denotes the mass of the hadron \( H \) (see Fig. 1).

One of the most important predictions of QCD is the \( Q^2 \)-evolution of the structure functions. However the \( x \)-dependence cannot be predicted yet. The latter

![FIGURE 1. Kinematics of deep inelastic lepton-hadron scattering.](image-url)
follows from non-perturbative QCD which is at such a premature stage that the \( x \)-dependence cannot be determined. This problem can be avoided when one integrates over the latter variable which leads to the sum rules discussed below. Notice that this integration requires a full knowledge about the range \( 0 < x < 1 \). Since there are no data available for \( x < 10^{-4} \) and \( x > 0.8 \) one has to make extrapolations into this region. This will introduce an error which is very hard to estimate. In the past various techniques have been used to compute the sum rules. The most known among them are

1. Infinite momentum frame techniques used in current algebra.
2. Dispersion relations which are derived from the Regge behavior of the structure functions \( F(x, Q^2) \) at small \( x \).
3. Operator product expansion (OPE) techniques which in leading twist are equivalent to the parton model.

Using the last technique the derivation of the sum rules proceeds as follows. First let us define the hadronic structure tensor which appears in the derivation of the cross sections presented above. It is given by

\[
W^{\mu\nu}(p,q,s) = \frac{1}{4\pi} \int d^4z \ e^{iq.z} \langle p,s| [J^\mu(z), J^\nu(0)] |p,s\rangle
= W_S^{\mu\nu}(p,q) + i W_A^{\mu\nu}(p,q,s). \tag{6}
\]

This tensor can be split up into a symmetric and a antisymmetric part i.e.

\[
W_S^{\mu\nu}(p,q) = \left( g^{\mu\nu} + \frac{q^{\mu} q^{\nu}}{q^2} \right) F_1(x, Q^2)
+ \left( p^\nu - \frac{p q}{q^2} q^\nu \right) \left( p^\mu - \frac{p q}{q^2} q^\mu \right) \frac{F_1(x, Q^2)}{p q}, \tag{7}
\]

\[
W_A^{\mu\nu}(p,q,s) = -\frac{M}{p q} \epsilon^{\mu\nu\alpha\beta} q_\alpha \left[ s_\beta g_1(x, Q^2) + (s_\beta - \frac{s q}{p q} p_\beta) g_2(x, Q^2) \right]. \tag{8}
\]

In the Bjorken limit (\( Q^2 \to \infty \) and \( x \) is fixed) the integrand in Eq. (6) is dominated by the lightcone \( z^2 = 0 \). This allows us to make a lightcone expansion of the current commutator

\[
[J(z), J(0)]_{z^2=0} = \sum_{\tau} \sum_{N} C^{N,\tau}(z^2 \mu^2) O^{N,\tau}(\mu^2, 0), \tag{9}
\]

where for convenience we have dropped the Lorentz indices of the currents. In the expression above \( \tau \) and \( N \) denote the twist and spin of the operator \( O^{N,\tau} \) respectively. The latter and the singular coefficient function \( C^{N,\tau} \) are understood
to be renormalized where $\mu$ represents the renormalization scale. After insertion of expression (11) into Eq. (6) one obtains the $N$th moment of the structure function which equals

$$\int_0^1 dx x^{N-1} F(x, Q^2) = \sum_{\tau} \left( \frac{M^2}{Q^2} \right)^{\frac{\tau}{2} - 1} A^{(N),\tau}(\mu^2) C^{(N),\tau} \left( \frac{Q^2}{\mu^2} \right).$$

(10)

In momentum space the operator matrix element (OME) and the coefficient function are defined by

$$A^{(N),\tau}(\mu^2) = \langle p | O^{N,\tau}(\mu^2, 0) | p \rangle,$$

(11)

and

$$C^{(N),\tau} \left( \frac{Q^2}{\mu^2} \right) = \int d^4z e^{i\mathbf{q} \cdot \mathbf{z}} C^{N,\tau}(z^2 \mu^2),$$

(12)

respectively. Limiting ourselves to twist two contributions only the non-singlet part (w.r.t flavor) of the structure functions is determined by the quark operator

$$O_k^N \equiv O_k^{\mu_1 \mu_2 \ldots \mu_N}(x) = \bar{\psi}(x)\gamma^{\mu_1}D^{\mu_2} \ldots D^{\mu_N} \lambda_k \psi(x),$$

(13)

where $D^\mu$ denotes the covariant derivative and $\lambda_k$ are the generators of the flavor group $SU_F(n_f)$. In the case of $N = 1$ the following operators are conserved

$$O_k^\mu(x) = \bar{\psi}(x)\gamma^{\mu} \lambda_k \psi(x) \quad \text{and} \quad O_k^{\mu,5}(x) = \bar{\psi}(x)\gamma^{\mu} \gamma^5 \lambda_k \psi(x),$$

(14)

which means that they do not have to renormalized. Hence the corresponding OME’s and coefficient functions are independent of the scale $\mu$ i.e.

$$\int_0^1 dx F(x, Q^2) = A_k^{(1)} C_k^{(1)}(Q^2), \quad \text{with} \quad \langle p | O_k^\mu(0) | p \rangle = A_k^{(1)} p^\mu,$$

(15)

where $A_k^{(1)}$ can be determined from $SU_F(n_f)$ and a low energy theorem. In general the coefficient function $C^{(1)}$ can be expanded in a power series in $\alpha_s(\mu^2)$. However in two cases this function becomes trivial. The first example is the Adler sum rule [1]

$$\int_0^1 \frac{dx}{x} \left[ F_2^{np}(x, Q^2) - F_2^{np}(x, Q^2) \right] = K(n_f),$$

$$K(3) = 2 + 2 \sin^2 \theta_c \ (SU_F(3)), \quad K(4) = 2 \ (SU_F(4)).$$

(16)

Here $\theta_c$ denotes the Cabibbo angle and for the constant $K(n_f)$ we have quoted the values given by the flavor group $SU_F(n_f)$ for $n_f = 3, 4$. The second example is the Burkhardt-Cottingham sum rule [2] given by

$$\int_0^1 dx g_2(x, Q^2) = 0.$$
1. The values on the right-hand side are independent of the method used for the derivation of the sum rules.
2. No power corrections of the type \((\Lambda^2/Q^2)^i\) (higher twist).
3. No mass corrections (e.g. due to heavy flavors).
4. No QCD corrections.

The coefficient function has a nontrivial form in the following cases provided the results follow from OPE. The first one is the polarized Bjorken sum rule\([3]\)
\[
\Delta g_1(Q^2) \equiv \int_0^1 dx \left[ g_{1p}^{ep}(x, Q^2) - g_{1n}^{en}(x, Q^2) \right] = \frac{1}{6} \left| \frac{G_A}{G_V} \right| C^{g_1}(n_f, Q^2). \tag{18}
\]
The second one is represented by the unpolarized Bjorken sum rule\([3]\)
\[
\Delta F_1(Q^2) \equiv \int_0^1 dx \left[ F_{1p}^{ep}(x, Q^2) - F_{1n}^{en}(x, Q^2) \right] = K(n_f) C^{F_1}(n_f, Q^2),
\]
\[
K(3) = 1 + \sin^2 \theta_c \ (SU_F(3)), \quad K(4) = 1 \ (SU_F(4)), \tag{19}
\]
The third one is the Gross-Llewellyn Smith sum rule\([5]\)
\[
\Delta F_3(Q^2) \equiv \int_0^1 dx \left[ F_{3p}^{op}(x, Q^2) + F_{3n}^{op}(x, Q^2) \right] = K(n_f) C^{F_3}(n_f, Q^2),
\]
\[
K(3) = 6 - 2 \sin^2 \theta_c \ (SU_F(3)), \quad K(4) = 6 \ (SU_F(4)). \tag{20}
\]
The coefficient function can be expanded in \(\alpha_s\) as
\[
C^r(n_f, Q^2) = \sum_{i=0}^\infty \left( \frac{\alpha_s(n_f, Q^2)}{\pi} \right)^i c_i^r(n_f), \quad r = g_1, F_1, F_3. \tag{21}
\]
The sum rules in Eqs. (18)-(20) have the following properties
1. For \(m_q = 0\) the \(c_i^r\) are known up to order \(\alpha_s^3\) for \(i \leq 3\) (see\([6], [7]\)).
2. Higher twist corrections are estimated in\([8]\)
3. Heavy flavor corrections are computed up to order \(\alpha_s^2\) in\([9]\)

In leading twist the coefficient functions can be computed in the QCD improved parton model. In this model the structure function can be written as
\[
F(x, Q^2) = \sum_a c_a^2 \int_0^1 dz_1 \int_0^1 dz_2 \, \delta(x - z_1 z_2) f_a^H(z_1) \hat{F}_a(z_2, \frac{Q^2}{m_q^2})
\]
\[
\equiv \sum_a c_a^2 f_a^H \otimes \mathcal{F}_a(\frac{Q^2}{m_q^2}), \tag{22}
\]
FIGURE 2. heavy flavor loop contribution to quark-quark scattering.

where $\hat{f}_a^H$ denotes the bare parton density and the parton structure function $F_a$ represents the QCD corrections including the Born approximation. Notice that $F_a$ contains mass singularities. They are regulated by giving the light quark a mass which is sufficient in the non-singlet case. Subsequently these singularities are removed via mass factorization which reads as follows

$$F_a(Q^2, m_q^2) = A_{aa}(\frac{\mu^2}{m_q^2}) \otimes C_a(Q^2, \mu^2).$$

(23)

Hence the hadronic structure function can be written as

$$F(Q^2) = \sum_a e^2_a f_a^H(\mu^2) \otimes C_a(Q^2, \mu^2), \quad \text{with} \quad f_a^H(\mu^2) = A_{aa}(\frac{\mu^2}{m_q^2}) \otimes \hat{f}_a^H,$$

(24)

where $f_a^H$ denotes the renormalized parton density which like the coefficient function depends on the factorization scale $\mu$. Performing the Mellin transformation one obtains

$$\int_0^1 dx x^{N-1} F(x, Q^2) = A^{(N)}(\mu^2) C^{(N)}(\frac{Q^2}{\mu^2}),$$

(25)

so that one can make the following identifications

$$A^{(N)}(\mu^2) = \int_0^1 dz z^{N-1} \sum_a e^2_a f_a^H(z, \mu^2), \quad C^{(N)}(\frac{Q^2}{\mu^2}) = \int_0^1 dy y^{N-1} C(y, \frac{Q^2}{\mu^2}),$$

(26)

which yields the coefficient function $C^{(1)}$

**HEAVY FLAVOR THRESHOLDS IN THE STRONG COUPLING CONSTANT**

Coupling constant renormalization in the case of light flavors (massless quarks) conventionally proceeds in the $\overline{\text{MS}}$-scheme. Using n-dimensional regularization the
coupling constant gets a dimension i.e. $g_s^u M^{4-n}$ where $g_s^u$ is the bare (unrenormalized) coupling constant. The renormalization of the perturbation series is then carried out by the substitution

$$
g_s^u \rightarrow g_s(\mu^2) \left[ 1 + \frac{\alpha_s(\mu^2)}{8\pi} \beta_0(n_f) \left\{ \frac{2}{\varepsilon} + \gamma_E - \ln 4\pi + \ln \frac{\mu^2}{M^2} \right\} + \cdots \right],$$

(27)

with $\alpha_s = g_s^2/4\pi$ and $\beta_0(n_f) = 11 - 2/3 n_f$. The renormalized coupling constant $\alpha_s$ satisfies the differential equation

$$\mu^2 \frac{d \alpha_s(n_f, \mu^2)}{d \mu^2} = \beta(\alpha_s(n_f, \mu^2)), \quad \rightarrow \quad \alpha_s(n_f, \mu^2) = \frac{4\pi}{\beta_0(n_f) \ln \frac{\mu^2}{\Lambda^2}} + \cdots.$$  

(28)

The solution obtained from the equation above is quite appropriate since the scale $\mu$ is much larger than the light quark mass. However in the case of heavy quarks we encounter a problem which will be illustrated by the following example. Let us consider (light) quark-quark scattering as shown in Fig. 2. Including the heavy quark loop the cross section behaves like

$$\sigma \sim \alpha_s^2(n_f, \mu^2) \frac{1}{t} + 2 \alpha_s^3(n_f, \mu^2) \frac{1}{t} \Pi_1\left(\frac{t}{m^2}\right) + \cdots,$$  

(29)

where $\Pi_1(x)$ denotes the one-loop vacuum polarization function which behaves asymptotically as

$$\Pi_1\left(\frac{t}{m^2}\right) \sim -\beta_{0,Q} \ln \left(\frac{-t}{m^2}\right), \quad \beta_{0,Q} = -\frac{2}{3}.$$  

(30)

This behavior leads to large corrections in the perturbation series which have to be resummed as follows. First we have to make the substitution

$$\alpha_s(n_f, \mu^2) \rightarrow \alpha_s(n_f + 1, \mu^2)[1 + \alpha_s(n_f + 1, \mu^2) \beta_{0,Q} \ln \frac{\mu^2}{m^2} \cdots],$$  

(31)

in Eq. (29) so that for $-t \gg m^2$ the cross section behaves like

$$\sigma \sim \alpha_s^2(n_f + 1, \mu^2) \frac{1}{t} + 2 \alpha_s^3(n_f + 1, \mu^2) \frac{1}{t} \beta_{0,Q} \ln \frac{\mu^2}{m^2}.$$  

(32)

Next $\alpha_s(n_f + 1, \mu^2)$ satisfies the massless renormalization group equation in (28) but now for $n_f + 1$ light flavors with the boundary condition [10]

$$\alpha_s(n_f, \Lambda_n, \mu^2) = \alpha_s(n_f + 1, \Lambda_{n+1}, \mu^2),$$  

(33)

where the matching value is given by $\mu = m$. In this way one obtains a better behavior of the perturbation series than shown by Eq. (29) in particular for $\mu^2 = -t$. However the representation of the cross section in (32) at $n_f + 1$ light flavors is only correct if $-t \gg m^2$. This follows from the vacuum polarization function $\Pi_1(-t/m^2)$ which gets close to the logarithm in Eq. (30) for $-t > 21 m^2$ within 10%. For $-t \approx m^2$ one does not reach the asymptotic form of $\Pi_1(-t/m^2)$ so that in this case it is much better to use the representation for the cross section at $n_f$ flavors in Eq. (29). It also shows that the matching scale $\mu$ in Eq. (31) has to be chosen at a much larger value e.g. $\mu^2 = 21 m^2$ instead of $\mu^2 = m^2$. 
HEAVY FLAVOR CONTRIBUTIONS TO THE POLARIZED BJORKEN SUM RULE

The sum rule in Eq. (18) receives contributions from the following sub processes (see Fig. 2 in [9])

\[ q + \gamma^* \rightarrow q + Q + \bar{Q}, \]
\[ q + \gamma^* \rightarrow q + g \quad \text{one-loop gluon self energy correction,} \]
\[ q + \gamma^* \rightarrow q \quad \text{two-loop vertex correction with one-loop gluon self energy.} \] (34)

The correction to the sum rule up to order $\alpha_s^2$ can be expressed as

\[
C_{g1}^{n_f}(n_f, Q^2) = 1 + \frac{\alpha_s(n_f, \mu^2)}{\pi} a_1 + \left( \frac{\alpha_s(n_f, \mu^2)}{\pi} \right)^2 \left[ -a_1 \beta_0(n_f) \ln \frac{Q^2}{\mu^2} + a_2 + n_f b_2 \right] + H_Q \left( \frac{Q^2}{m^2} \right),
\] (35)

where $H_Q$ denotes the heavy quark contribution. The two regions of interest are

\[
Q^2 \ll m^2 \rightarrow H_Q \left( \frac{Q^2}{m^2} \right) \sim \frac{Q^2}{m^2} \ln \frac{Q^2}{m^2},
\] (36)
\[
Q^2 \gg m^2 \rightarrow H_Q \left( \frac{Q^2}{m^2} \right) = -a_1 \beta_{0,Q} \ln \frac{Q^2}{m^2} + b_2.
\] (37)

The first limit shows that heavy quarks of mass $m$ decouple from the perturbation series when their mass is much larger than the virtuality of the photon. In the region $Q^2 \gg m^2$ we observe the same logarithmic behavior as for $\Pi_1$ in Eq. (30). However here $C_{g1}^{n_f}(n_f, Q^2)$ becomes asymptotic for $Q^2 > 40 m^2$ within 15%. Hence the value of $Q^2$ is about twice as large as the one found for $-t$ in quark-quark scattering. This we have checked for charm production. Here we choose $n_f = 3$ and $m = m_c = 1.5 \text{ GeV}/c^2$ in Eq. (35) and the results are shown in table 1 (see [9]). Further we observe that the heavy flavor corrections to the sum rule (21)

| $Q^2$ (GeV/c$^2$) | light flavors | charm contribution$^a$ | charm contribution$^b$ | charm contribution$^c$ |
|------------------|--------------|------------------------|------------------------|------------------------|
| 3                | 0.795        | -0.688.10^{-3}         | 0.451.10^{-2}          | -0.265.10^{-1}         |
| 10               | 0.883        | -0.788.10^{-3}         | 0.569.10^{-3}          | -0.586.10^{-2}         |
| 100              | 0.931        | -0.107.10^{-2}         | -0.0912.10^{-2}        | -0.121.10^{-2}         |

$^a$ Exact formula Eq. (35).
$^b$ Asymptotic formula according to Eq. (37).
$^c$ Estimated order $\alpha_s^2$ correction according to [11].
are very small. They are even smaller than the estimated order \( \alpha_s^4 \) corrections in [11] in spite of the fact that the heavy flavor contribution \( H_Q \) starts in order \( \alpha_s^2 \). The reason that the logarithmic behavior in process (36) emerges at a much larger scale than the reaction in Fig. 2 can be explained as follows. In Fig. 2 the gluon momentum \( k \) entering the self energy is an external variable since \( t = k^2 \). However in the case of process (36) one has to integrate over \( k \) so that the latter is not an external variable anymore. Therefore the role of \( t \) is taken over by \( Q^2 \). At very large \( Q^2 \) the perturbation series in Eq. (35) can be improved by following the same procedure as outlined below Eq. (30). The result is that instead of Eq. (35) one obtains the improved perturbation series

\[
C^{g1}(n_f + 1, Q^2) = 1 + \frac{\alpha_s(n_f + 1, \mu^2)}{\pi} a_1 + \left( \frac{\alpha_s(n_f + 1, \mu^2)}{\pi} \right)^2 \left[ -a_1 \beta_0(n_f + 1) \ln \frac{Q^2}{\mu^2} \right. \\
\left. + a_2 + (n_f + 1)b_2 \right],
\]

which behaves much better for the choice \( \mu^2 = Q^2 \) provided \( Q^2 \gg m^2 \). However the expression above is not a correct representation when \( Q^2 \approx m^2 \) as we have already seen below Eq. (33). Hence \( \mu^2 = m^2 \) is not a good matching scale where the running coupling constant jumps from \( n_f \) to \( n_f + 1 \) flavors. Therefore this scale has to be chosen at a much larger value than what is usually done in the literature. This also means that extrapolations from a measurement of \( \alpha_s(n_f, \mu^2) \) at small \( \mu \) to large values like \( \mu = M_Z \) have to be distrusted. A scheme (hereafter called MOM) which incorporates the effect of the heavy flavor thresholds in the running coupling constant in a much better way is proposed in [12]. Here one simply resums the vacuum polarization function as follows

\[
\alpha_s^{\text{MOM}}(\mu^2) = \frac{\alpha_s(3, \mu^2)}{1 + \alpha_s(3, \mu^2) U_1 + \alpha_s(3, \mu^2) \left( U_2 / U_1 \right) \ln \left( 1 + \alpha_s(3, \mu^2) U_1 \right)},
\]

with

\[
U_i = \sum_{f=c,b,t} \left[ \Pi_i \left( \frac{\mu^2}{m_f^2} \right) - \Pi_i \left( \frac{\mu_0^2}{m_f^2} \right) \right], \quad i = 1, 2 \cdots,
\]

where the coupling constant \( \alpha_s(3, \mu^2) \) is represented in the \( \overline{\text{MS}} \)-scheme. Expression (39) agrees rather well with the numerical solution of the renormalization group equation in Eq. (28) where the \( \beta \)-function depends on \( m \). This in particular holds when the two-loop self energy contributions \( \Pi_2 \) are included. From the results discussed above we have seen that the scales where both \( \Pi_1 \) and \( H_Q \) behave logarithmically are relatively much closer to each other than to \( \mu = m \). Hence it is better to put \( n_f = 3 \) in Eq. (35) and to make the replacement

\[
\alpha_s(3, \mu^2) = \alpha_s^{\text{MOM}}(\mu^2) \left[ 1 + \frac{\alpha_s^{\text{MOM}}(\mu^2)}{\pi} U_1 + \cdots \right],
\]
than to follow the procedure outlined below Eq. (30). In this way one gets a continuous transition through the flavor thresholds, also with respect to derivatives, of the perturbation series of $C^{a_1}$ in Eq. (35). Similar results were also found for the sum rules in Eqs. (19) and (20) (see [9]). Finally we conclude

1. Heavy flavor corrections to the sum rules are very small and they do not exceed the estimated order $\alpha_s^4$ contributions due to light quarks.

2. The matching scale in the literature (here $\mu = m$) where the running coupling constant jumps from $n_f$ to $n_f + 1$ flavors is chosen to be too small.

3. The MOM scheme as chosen in Eq. (39) gives a better representation for the perturbation series than the usual $\overline{\text{MS}}$-scheme with the matching condition in Eq. (33).

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