Using a New Projection Approach to Find the Optimal Solution for Nonlinear Systems of Monotone Equation

N. k. Dreeb, L. H. Hashim, K. H. Hashim and Mushtak A. K. Shiker

1,3,4 Department of Mathematics, College of Education for Pure Sciences, University of Babylon, Hilla, Iraq.
2Department of Planning and Studies, University of Babylon, Hilla, Iraq.

E-mail: 1 nabiha250222@gmail.com, 2 luainfo@gmail.com, 3 teacher89karrar@gmail.com.
* Corresponding author: 4 mmttmmhh@yahoo.com.

Abstract. There are many algorithms are used to solve systems of nonlinear monotone equations with various advantages and disadvantages, including the line search algorithm, trust region algorithm, projection algorithm and others. In this paper we used a new projection algorithm to solve these systems. The projection methods are considered one of the effective free derivative methods to solve systems of nonlinear monotone equations. The framework of this method is that the current iterate is separated strictly from the solution set of the problem in each iteration by a suitable hyperplane which constructs by the new algorithm. Then, in order to determine the new approximation, the current iteration is projected on this hyperplane. The global convergence of the proposed algorithm is proven under standard assumptions. The numerical results showed that the suggested algorithm is very efficiency and promising.

Keywords: Nonlinear system of equations. Projection method. Monotone strategy. Global convergence.

1. Introduction

In sciences with different fields and aspects, nonlinear systems and their solutions are of great importance in them, as they are an important part of mathematics and physics (because most physical systems are nonlinear), as well as their importance in engineering, especially mechanical engineering, electricity, management, economics, population growth, weather and other natural phenomena. While it is possible to convert nonlinear problems into linear problems with multiple variables, it can be said that the study of nonlinear problems was side by side with the linear system.

Consider the following nonlinear system of equations

\[ F(x) = 0, \] (1)
Where $F$ is a continuous and monotone function from $R^n$ to $R^n$, condition of monotony mean:

$$(F(x) - F(y))^T (x - y) \geq 0, \forall x, y \in R^n. \quad (2)$$

Because of the difficulty of finding a solution to nonlinear system equations in many mathematical and engineering applications, we can only rely on iterative methods that use the iterative procedure to obtain approximate solutions. Newton's method may be one of the best numerical methods that use the iterative method to solve these systems, and it is considered a simple way to find approximate values of equations.

In recent years, many modifications have been made to the Newton method, these suggested methods may be equivalent to (or better than) the Newton method to solve the nonlinear system of equations. The line search method and the trust region method are the most important two methods to solve these systems.

The basic idea of the line search technique is how to find the step length within a specific direction, while the trust region technique is how to find an neighborhood to the current step $x_k$, provided that the new frequency falls within the area that is the trust region determined by its radius, also, this technique is used to solve unconstrained Optimization [1].

Some methods proved ineffective for solving large-scale nonlinear system of equations as Newton method and quasi-Newton methods [2-5] because they need to solve the Jacobian matrix or an approximation of it in each iterative.

In our work, we used a new projection technique to solve large-scale systems of nonlinear equations. It can be said that the simplest idea of the projection method is interested in separating the present approximation from the result set of the problem (1) by appropriate hyperplane that is built in each iterate and then projecting this approximation on the same hyperplane to obtain the new approximation. Several authors use conjugate gradient approaches combining with projection techniques for solving (1) as well as optimization issues [6, 7].

Solodov and Svaiter suggested the first projection approach in 1998 and it showed the totally convergent of solving nonlinear problems [8].

The authors worked in vary fields such as optimization, operation research and nonlinear systems, but in this work, see [9-19], we used a new algorithm to solve the nonlinear systems, we proved its global convergence. Then we compare with two famous methods, SBM method in [20] and DFPB1 in [1], the new algorithm will be more efficient.

2. The Framework

The projection technique is one of the ways that proved to be active in solving nonlinear problems and it is a suitable and applicable way to solve large-scale difficulties, these methods use a series of repetitions to arrive to the next iterate

$$x_{k+1} = x_k + \alpha_k d_k, \quad (3)$$

where $\alpha_k$ is a step length and $d_k$ is the step direction, these processes are called an iterative procedures, so the projection techniques are called iterative methods. The projection approaches are family of derivative free. To define these effective methods we use the projection operator $\Theta_\Omega[.]$.

Let $\Theta_\Omega[.]$ be a mapping from $R^n$ to $\Omega$, where $\Omega$ is non-empty closed convex set [21]:

$$\Theta_\Omega[x] = \text{argmin} \{\|x - z\|, z \in \Omega\}, \forall x \in R^n. \quad (4)$$
The projection operator has interesting features [22] is non-expansive property:
\[ \| \Phi_\Omega (x) - \Phi_\Omega (y) \| \leq \| x - y \|, \quad \forall \ x, y \in \mathbb{R}^n. \] (5)
As a result produces,
\[ \| \Phi_\Omega (x) - y \| \leq \| x - y \|, \quad \forall \ x, y \in \Omega . \] (6)
After a series of iterations, in every iteration, the present approximation \( x_k \) is isolated from the result set of the problem by the hyperplane \( H_k \) that is construction by using a line search technique.
\[ H_k = \{ x \in \mathbb{R}^n / F (z_k)^T (x - z_k) = 0 \} , \] (7)
where
\[ z_k = x_k + \alpha_k d_k . \] (8)
By Solodov and Svaiter’s suggestion [8], the following iterate \( x_{k+1} \) can be resolute by projection \( z_K \) onto \( H_k \), where
\[ C_K = \{ x \in \mathbb{R}^n / F (z_k)^T (x - z_k) \leq 0 \} . \] (9)
The approximation that is best among all result of system (1) can be determined by projection \( x_k \) onto \( C_k \), but \( x_k \notin C_k \). Then the following approximation, \( x_{k+1} \), can be determined by
\[ x_{k+1} = P_C (x_k) = \arg \min \{ \| x - x_k \| \mid x \in C_k \} \]
So,
\[ x_{k+1} = x_k - \frac{F(z_k)^T (x_k - z_k)}{\| F(z_k) \|^2} F(z_k). \] (10)
The suggested method, built on the projection free-derivatives method for the system of nonlinear equations, determine a direction \( d_k \),
a new direction has foreword as
\[ d_k = \begin{cases} -F(x_k) & \text{if } k = 0, \\ -\mu_k F(x_k) + \tau_k & \text{otherwise}, \end{cases} \] (11)
Where \( \mu_k = \frac{s_k^T s_k}{y_k^2} \), \( s_k = x_{k+1} - x_k \), \( y_k = F(x_{k+1}) - F(x_k) \)
With \( \tau_k = \frac{F(x_{k+1}) y_k}{\| F(x_k) \|^2} \)
Generally used the direction \( d_k \) which satisfies.
\[ F_k^T d_k \leq -C \| F_k \|^2 , \] (12)
\[ F (z_K)^T (x_k - z_k) > 0 , \] (13)
where \( C \) is appositive constant.
In 2018, Mushtak and Amini [2] introduced a new line search strategy for separating hyperplane in projection technique, encourage us to take advantages of this line search, which needs \( \alpha_k = \{ \beta \theta^i : i = 0,1,2,... \} \) satisfies the condition
\[-F (x_k + \alpha_k d_k)^T d_k \geq \theta \lambda_k \alpha_k \|F(z_k)\|, \tag{14}\]

Where \(\lambda_k = \frac{\lambda}{1 + \|d_k\|^2}\), and \(\theta, \lambda\) are parameters.

Our new algorithm will be stated as below.

**Algorithm 1 (NBM)**

**Input:** An initial point \(x_0 \in \mathbb{R}^n\), and the parameters \(\theta, \lambda, \varepsilon \in (0.2)\) and \(\beta \in (0.1)\).

**Start**

Set \(k = 0\);
\[F_0 = F(x_0)\]
\[d_0 = -F_0\]

While \(\|F_K\| > \varepsilon\).

**Step 1:** compute \(\|F_K\|\). If \(\|F_K\| \leq \varepsilon\) stop.

Set \(\alpha_k = \beta\):

Find the minimum index \(i_k \in \{1, 2, 3, \ldots\}\) such that
\[-F (x_k + \alpha_k d_k)^T d_k \geq \theta \lambda_k \alpha_k \|F(z_k)\|,\]

where \(\lambda_k = \frac{\lambda}{1 + \|d_k\|^2}\).

While \(\alpha_k = \theta^{i_k} \alpha_k\):

Set \(z_k \leftarrow x_k + \alpha_k d_k\).

End while

**Step 2:** If \(\|F(z_k)\| \leq \varepsilon\), stop. Otherwise compute \(x_{k+1}\) by (10):

**Step 3:** Compute \(d_k\) by (11).
\[F_{K+1} \leftarrow F(x_{k+1})\; ;\]

If \(F_k^T d_k > -\varepsilon \|F_k\|^2\)
\[d_k = -F_k\]

End If

\(k \leftarrow k + 1\).

End while

**End**

**Remark (R1)** [8]: from stage 3 of Algorithm 1, it is easy to note that the introduced direction satisfy the sufficient descent condition, and for any \(k\), \(F_k^T d_k \leq -\varepsilon \|F_k\|^2\).
3. Convergence possessions:

In this part, We need some interesting lemmas and assumptions in showing the global convergence of algorithm 1.

Assumption (B1): The result set of (1) is nonempty.

Assumption (B2): The mapping $F(x)$ is Lipschitz continuous on $R^n$ such that there exists a positive constant $M$, i.e.
\[
\|F(x) - F(y)\| \leq M \|x - y\|, \forall x, y \in R^n.
\]

Assumption (B3): The mapping $F(x)$ is monotone on $R^n$ such that
\[
(F(x) - F(y))^T (x - y) \geq 0, \forall x, y \in R^n.
\]

Lemma (L1) [22]

Let the set $\Omega \subseteq R^n$ be nonempty closed convex set and The projection operator $\rho_\Omega(x)$ be the projection of $x$ onto closed convex set $\Omega$. For any $x, y \in R^n$, The next statements hold:

i) \hspace{1cm} $\forall \ i \in \Omega, \langle \rho_\Omega(x) - x, z - \rho_\Omega(x) \rangle \geq 0$

ii) \hspace{1cm} $\langle \rho_\Omega(x) - \rho_\Omega(y), x - y \rangle \geq 0$, and the inequality is strict when $\rho_\Omega(x) \neq \rho_\Omega(y)$

iii) \hspace{1cm} $\|\rho_\Omega(x) - \rho_\Omega(y)\| \leq \|x - y\|

Lemma (L2) [8] Assume the assumption $B_1, B_2$ and $B_3$ hold and the sequence $\{x_k\}$ is generated via algorithm 1. For any $x^*$ such that $F(x^*) = 0$, then
\[
\|x_{k+1} - x^*\|^2 \leq \|x_k - x^*\|^2 - \|x_{k+1} - x_k\|^2. \tag{15}
\]

And the sequence $\{x_k\}$ is bounded. Moreover, either the sequence $\{x_k\}$ is finite although the last iterate is a solution of (1) or the sequence $\{x_k\}$ is infinite and
\[
\lim_{k \to \infty} \|x_{k+1} - x_k\| = 0. \tag{16}
\]

Proof Let $x^* \in R^n$ be any point such that $F(x^*) = 0$ by monotonicity of $F$
\[
\langle F(y), x^* - y \rangle \leq 0.
\]
The hyperplane $H = \{s \in R^n / \langle F(y), s - y \rangle = 0\}$, separates $x_k$ from $x^*$, it is easy to satisfy that $x_{k+1}$ is the projection of $x_k$ onto the hyperplane $H$. Sine $x^*$ belongs to this hyperplane, from properties of the projection operator [22] we get
\[
\|x_k - x^*\|^2 = \|x_k - x_{k+1}\|^2 + \|x_{k+1} - x^*\|^2 + 2\langle x_k - x_{k+1}, x_{k+1} - x^* \rangle \geq \left(\frac{\langle F(y), x_k - y \rangle}{\|F(y)\|}\right)^2 + \|x_{k+1} - x^*\|^2. \quad \Box
\]

Lemma (L3) [8] Assume that the assumption $B_1, B_2$ and $B_3$ holds and the sequences $\{x_k\}$ and $\{z_k\}$ are generated by Algorithm 1, then
\[
\alpha_k \geq \min\{\beta, \frac{\theta^2\|F_k\|^2}{(M\|\alpha\|^2 + \|F(z_k)\|)}\}. \tag{17}
\]
Proof By the line search rule (14), if \( \alpha_k \neq \beta \) then \( \tilde{\alpha}_k = \theta^{-1} \alpha_k \) does not satisfy (14) this mean that
\[
-F(x_k + \theta^{-1} \alpha_k d_k)^T d_k < \theta \lambda \theta^{-1} \alpha_k \gamma_k \|F(x_k)\| \leq \lambda \alpha_k \|F(z_k)\|
\]
Where \( \gamma_k = \frac{1}{1+\|d_k\|^2} \).

By the Lipchitz continuity of \( F \) and (12) we get:
\[
C \|F_k\|^2 \leq - F_k^T d_k = (F(z_k) - F(x_k))^T d_k - F(z_k)^T d_k
\leq \|F(z_k) - F(x_k)\| \|d_k\| + \lambda \alpha_k \|F(z_k)\|
= \alpha_k (M \|d_k\|^2 + \lambda \|F(z_k)\|).
\]
So,
\[
\alpha_k \geq \frac{\theta c \|F_k\|^2}{(M \|d_k\|^2 + \lambda \|F(z_k)\|)}.
\]
the proof is complete and the equation (17) is correct. \( \square \)

The results of Lemma L3 found that the line search of algorithm 1 is well defined.

Theorem (T1) Assume that \((B_2)\) and \((B_3)\) hold and the sequence \( \{x_k\} \) is generated by Algorithm 1, then:
\[
\lim_{k \to \infty} \|F_k\| = 0 \tag{18}
\]

Proof From (10) and (14) we get
\[
\|x_{k+1} - x_k\|^2 \geq \frac{|F(z_k)^T (x_k - z_k)|}{\|F(z_k)\|} = \frac{-\alpha_k F(z_k)^T d_k}{\|F(z_k)\|} \geq \frac{2 \alpha_k^2 \|F(z_k)\|}{(1+\|d_k\|^2)\|F(z_k)\|} = \frac{2 \alpha_k^2}{1+\|d_k\|^2} \tag{19}
\]
By lemma 3 from [1], the sequence of direction \( \{d_k\} \) that generated by algorithm 1 are bounded.
there is a constant \( N > 0 \) such that
\[
\|F(x_k)\| \leq N. \tag{20}
\]
And result that for all \( k \), there exists a constant \( L > 0 \) such that
\[
\|d_k\| \leq L. \tag{21}
\]
By the Lipchitz continuity of \( F \), it can be concluded that:
\[
\|F(z_k)\| \leq \|F(z_k) - F(x_k)\| + \|F(x_k)\| \leq M(z_k - x_k) + N = M \alpha_k \|d_k\| + N \tag{22}
\]
From (19) together with (21) gives
\[
\|x_{k+1} - x_k\|^2 \geq \frac{2 \alpha_k^2}{1+L^2}.
\]
So,
\[
\lim_{k \to \infty} \|x_{k+1} - x_k\|^2 \geq \lim_{k \to \infty} \left( \frac{2 \alpha_k^2}{1+L^2} \right).
\]
\[
\lim_{k \to \infty} \alpha_k \|d_k\| = 0 \quad (23)
\]

Now, by using Cauchy Schwartz inequality along with (12), we get

\[
C \|F_k\|^2 \leq -F_k^T d_k \leq \|F_k\| \|d_k\|
\]

So,
\[
\|d_k\| \geq C \|F_k\|. \quad (24)
\]

For all \(k\). Giving to this condition and (23), it follows that
\[
\lim_{k \to \infty} \alpha_k = 0 \quad (25)
\]

On the other hand, Multiplying (17) by \(\|d_k\|^2\) result that
\[
\alpha_k \|d_k\|^2 \geq \min\left\{ \beta \|d_k\|^2, \frac{\theta C L^2 \|F_k\|^2}{(M L^2 + \lambda (M \alpha_k \|d_k\| + N))} \right\} \quad (26)
\]

From (24) and (26) we have
\[
\lambda_0 \|F_k\|^2 \leq \alpha_k \|d_k\|^2, \quad (27)
\]

Where
\[
\lambda_0 = \min\left\{ \beta c^2, \frac{\theta C L^2}{(M L^2 + \lambda (M \alpha_k \|d_k\| + N))} \right\}.
\]

The relation (23) and (27) conclude that
\[
\lim_{k \to \infty} \|F_k\| = 0.
\]

4. **Numerical Experiment**

In this section, we compare the performance of the new algorithm (NBM) with two famous algorithms:

**SBM:** This technique is taken from Yan, Q.R., et al [20] and it uses two modified HS approaches with the projection technique in Solodov and Svaiter (1998) [8].

**DFPB1:** This technique is taken from Mushtak A.K. and Zahra Sahib [1].

We compared the numerical results of the three approaches with reference to \(N_i\) (number of iterations), \(N_f\) (number of function evaluations) and CPU time (time which required for each algorithm to reach the solution). For this purpose we used the test problems in [20] with 5000-50000 as dimensions for the taken primary points.

The PC that used to run these algorithm was with 4 GB RAM and CPU 2.70 GHz. MATLAB R2014a programming environment is used to write all the codes. The terminate value is \(\|F_k\| \leq 10^{-8}\) or \(\|F(z_k)\| \leq 10^{-4}\), or the whole number of iterates surpasses 500000 for all algorithms to be terminated. The parameters are stated as: \(\theta = 0.4\), \(\beta = 0.9\), \(\lambda = 0.1\), \(\epsilon = 10^{-8}\). The numerical results are registered in tables 1 and 2. Where table 1 contains \(N_i\) and \(N_f\), while table 2 contains the CPU time results.
Table 4.1: Numerical results ( $N_i$, $N_f$ )

| P   | Dim. | S.P | New $N_i$ | MHS $N_i$ | DFPB $N_i$ | New $N_f$ | MHS $N_f$ | DFPB $N_f$ |
|-----|------|-----|-----------|-----------|------------|-----------|-----------|------------|
| $P_1$ | 50000 | $x_0$ | 11 | 91 | 1421 | 13288 | 1421 | 13288 |
| 50000 | $x_1$ | 19 | 200 | 1421 | 13288 | 1421 | 13288 |
| 50000 | $x_2$ | 13 | 110 | 142 | 880 | 142 | 880 |
| 50000 | $x_3$ | 16 | 162 | 142 | 880 | 142 | 880 |
| 50000 | $x_4$ | 18 | 281 | 5404 | 53012 | 9 | 21 |
| 50000 | $x_5$ | 9 | 96 | 17 | 65 | 17 | 65 |
| 50000 | $x_6$ | 25 | 188 | 4439 | 40363 | 88 | 504 |
| 50000 | $x_7$ | 25 | 198 | 2252 | 19238 | 88 | 504 |
| $P_2$ | 50000 | $x_0$ | 11 | 91 | 1421 | 13288 | 1421 | 13288 |
| 50000 | $x_1$ | 18 | 228 | 1371 | 12933 | 1421 | 13288 |
| 50000 | $x_2$ | 13 | 110 | 142 | 880 | 142 | 880 |
| 50000 | $x_3$ | 19 | 256 | 155 | 1086 | 142 | 880 |
| 50000 | $x_4$ | 80 | 1120 | 3859 | 36651 | 9 | 21 |
| 50000 | $x_5$ | 9 | 96 | 17 | 65 | 17 | 65 |
| 50000 | $x_6$ | 56 | 666 | 7274 | 72430 | 88 | 504 |
| 50000 | $x_7$ | 49 | 498 | 581 | 3673 | 88 | 504 |
| $P_3$ | 10000 | $x_0$ | 16975 | 213353 | 418081 | 3099340 | 178397 | 721899 |
| 10000 | $x_1$ | 67640 | 981540 | 415836 | 3005275 | 187559 | 759768 |
| 10000 | $x_2$ | 14647 | 189321 | 390297 | 2977500 | 162500 | 656715 |
| 10000 | $x_3$ | 55909 | 839601 | 402816 | 2960816 | 178197 | 720729 |
| 10000 | $x_4$ | 47471 | 729546 | 371496 | 2767595 | 165523 | 668916 |
| 10000 | $x_5$ | 54499 | 855259 | 368053 | 2766114 | 162292 | 655726 |
| 10000 | $x_6$ | 22901 | 321922 | 163175 | 1263922 | 68435 | 276645 |
| 10000 | $x_7$ | 22313 | 312113 | 153214 | 1138115 | 68436 | 276649 |
| $P_4$ | 10000 | $x_0$ | 24 | 255 | 20751 | 230201 | 469 | 4001 |
| 10000 | $x_1$ | 370 | 5289 | 7709 | 73104 | 1494 | 14896 |
| 10000 | $x_2$ | 18 | 183 | 2231 | 20120 | 155 | 1029 |
| 10000 | $x_3$ | 365 | 5294 | 12747 | 135225 | 244 | 1713 |
| 10000 | $x_4$ | 198 | 2896 | 27023 | 302729 | 85 | 395 |
| 10000 | $x_5$ | 144 | 2092 | 11484 | 121948 | 76 | 314 |
| 10000 | $x_6$ | 45 | 568 | 17308 | 191471 | 113 | 620 |
| 10000 | $x_7$ | 27 | 291 | 2215 | 18879 | 113 | 620 |
| $P_5$ | 5000 | $x_0$ | 494 | 5834 | 154735 | 2156312 | 75926 | 1027957 |
| 5000 | $x_1$ | 379 | 4285 | 147988 | 2085617 | 75401 | 1019862 |
| 5000 | $x_2$ | 278 | 3260 | 153015 | 2148172 | 75883 | 1027291 |
| 5000 | $x_3$ | 341 | 3944 | 149814 | 2109510 | 341 | 3944 |
| 5000 | $x_4$ | 346 | 3797 | 158114 | 2198306 | 75839 | 1026614 |
| 5000 | $x_5$ | 333 | 3718 | 156045 | 2174637 | 75843 | 1026679 |
| 5000 | $x_6$ | 384 | 4216 | 146366 | 2075659 | 75872 | 1027109 |
| 5000 | $x_7$ | 301 | 3459 | 152515 | 2142193 | 75850 | 1026796 |
Table 4.1: Numerical results ($N_i, N_f$) – continued

| P   | Dim. | S.P | New  | MHS  | DFPB |
|-----|------|-----|------|------|------|
|     |      |     | $N_i$| $N_f$| $N_i$| $N_f$| $N_i$| $N_f$| $N_i$| $N_f$| $N_i$| $N_f$|
| $P_6$ | 50000 | $x_0$ | 11  | 82  | 3353 | 28949 | 1009 | 8383 |
|      | 50000 | $x_1$ | 18  | 210 | 8993 | 88244 | 1914 | 17404|
|      | 50000 | $x_2$ | 19  | 245 | 3932 | 35528 | 262  | 1813 |
|      | 50000 | $x_3$ | 23  | 300 | 142  | 880   | 554  | 4288 |
|      | 50000 | $x_4$ | 17  | 210 | 4450 | 40138 | 386  | 2805 |
|      | 50000 | $x_5$ | 15  | 179 | 17   | 65    | 376  | 2725 |
|      | 50000 | $x_6$ | 19  | 250 | 1615 | 12611 | 333  | 2381 |
|      | 50000 | $x_7$ | 21  | 284 | 2057 | 16704 | 333  | 2381 |
| $P_7$ | 50000 | $x_0$ | 5   | 12  | 18560| 45738 | 1421 | 13288|
|      | 50000 | $x_1$ | 22  | 280 | 1675 | 15191 | 1421 | 13288|
|      | 50000 | $x_2$ | 13  | 110 | 17325| 34836 | 142  | 880 |
|      | 50000 | $x_3$ | 19  | 256 | 158  | 1099  | 142  | 880 |
|      | 50000 | $x_4$ | 12  | 138 | 993  | 1988  | 9    | 21  |
|      | 50000 | $x_5$ | 4   | 16  | 16926| 33854 | 17   | 65 |
|      | 50000 | $x_6$ | 151 | 1728| 17089| 34332 | 88   | 504 |
|      | 50000 | $x_7$ | 45  | 430 | 17089| 34332 | 88   | 504 |

Table 4.2: Numerical results (CPU time)

| P   | Dim. | S.P | New     | MHS     | DFPB     |
|-----|------|-----|---------|---------|----------|
|     |      |     | $N_i$   | $N_f$   | $N_i$    |
| $P_1$ | 50000 | $x_0$ | 0.15625 | 21.21875| 20.62500|
|      | 50000 | $x_1$ | 0.26562 | 22.26562| 20.90625|
|      | 50000 | $x_2$ | 0.14062 | 1.15625 | 1.06250 |
|      | 50000 | $x_3$ | 0.15625 | 1.14062 | 1.00000 |
|      | 50000 | $x_4$ | 0.32812 | 55.01562| 0.01562 |
|      | 50000 | $x_5$ | 0.14062 | 0.18750 | 0.10937 |
|      | 50000 | $x_6$ | 0.26562 | 43.78125| 0.06937 |
|      | 50000 | $x_7$ | 0.25000 | 20.68750| 0.51562 |
| $P_2$ | 50000 | $x_0$ | 0.12500 | 21.50000| 20.79687|
|      | 50000 | $x_1$ | 0.28125 | 21.09375| 20.93750|
|      | 50000 | $x_2$ | 0.10937 | 1.12500 | 1.00000 |
|      | 50000 | $x_3$ | 0.31250 | 1.32812 | 1.04687 |
|      | 50000 | $x_4$ | 1.10937 | 39.98437| 0.10937 |
|      | 50000 | $x_5$ | 0.14062 | 0.10937 | 0.12500 |
|      | 50000 | $x_6$ | 0.62500 | 80.98437| 0.53125 |
|      | 50000 | $x_7$ | 0.54687 | 4.65625 | 0.68750 |
From table 1, we can see that the new approach (NBM) is better than the other two methods (SBM) and (DFPB1), that it has a number of iterations and number of evaluation functions less than in the other methods in most of the problems with most of initial points. As well as the results in table 2, we

| P_3 | Dim. | S.P | New | MHS | DFPB |
|-----|------|-----|-----|-----|------|
| 10000 | x_0  | 0.42221 | 6.36439 | 1.47290 |
| 10000 | x_1  | 1.95898 | 6.28664 | 1.57473 |
| 10000 | x_2  | 0.37418 | 6.14095 | 1.33835 |
| 10000 | x_3  | 1.66826 | 6.05081 | 1.48701 |
| 10000 | x_4  | 1.47228 | 5.68717 | 1.36782 |
| 10000 | x_5  | 1.68854 | 5.71673 | 1.35753 |
| 10000 | x_6  | 0.63462 | 2.60884 | 0.57164 |
| 10000 | x_7  | 0.61637 | 2.35051 | 0.57096 |

| P_4 | Dim. | S.P | New | MHS | DFPB |
|-----|------|-----|-----|-----|------|
| 10000 | x_0  | 0.09375 | 70.75000 | 1.20312 |
| 10000 | x_1  | 1.34375 | 22.57812 | 4.18750 |
| 10000 | x_2  | 0.03125 | 6.34375 | 0.31250 |
| 10000 | x_3  | 1.37500 | 41.62500 | 0.50000 |
| 10000 | x_4  | 0.81250 | 92.67187 | 0.12500 |
| 10000 | x_5  | 0.53125 | 37.10937 | 0.09375 |
| 10000 | x_6  | 0.12500 | 58.79687 | 0.15625 |
| 10000 | x_7  | 0.06250 | 5.85937 | 0.18750 |

| P_5 | Dim. | S.P | New | MHS | DFPB |
|-----|------|-----|-----|-----|------|
| 5000 | x_0  | 0.98437 | 3.34671 | 1.55750 |
| 5000 | x_1  | 0.68750 | 3.17734 | 1.54468 |
| 5000 | x_2  | 0.51562 | 3.32609 | 1.56765 |
| 5000 | x_3  | 0.53125 | 3.32687 | 0.00593 |
| 5000 | x_4  | 0.59375 | 3.40265 | 1.56156 |
| 5000 | x_5  | 0.60937 | 3.35593 | 1.56953 |
| 5000 | x_6  | 0.71875 | 3.16562 | 1.56218 |
| 5000 | x_7  | 0.57812 | 3.28640 | 1.57656 |

| P_6 | Dim. | S.P | New | MHS | DFPB |
|-----|------|-----|-----|-----|------|
| 50000 | x_0  | 0.21875 | 0.58468 | 16.76562 |
| 50000 | x_1  | 0.43750 | 1.73859 | 32.28125 |
| 50000 | x_2  | 0.53125 | 0.71421 | 3.25000 |
| 50000 | x_3  | 0.60937 | 0.01062 | 8.23437 |
| 50000 | x_4  | 0.48437 | 0.79390 | 5.18750 |
| 50000 | x_5  | 0.35937 | 0.00109 | 5.43750 |
| 50000 | x_6  | 0.56250 | 0.25359 | 4.46875 |
| 50000 | x_7  | 0.67187 | 0.34156 | 4.34375 |

| P_7 | Dim. | S.P | New | MHS | DFPB |
|-----|------|-----|-----|-----|------|
| 50000 | x_0  | 0.01562 | 92.42187 | 20.92187 |
| 50000 | x_1  | 0.34375 | 22.23437 | 21.37500 |
| 50000 | x_2  | 0.09375 | 74.35937 | 0.92187 |
| 50000 | x_3  | 0.20312 | 1.15625 | 1.09375 |
| 50000 | x_4  | 0.09375 | 4.14062 | 0.03125 |
| 50000 | x_5  | 0.03125 | 73.79687 | 0.12500 |
| 50000 | x_6  | 1.65625 | 74.93750 | 0.59375 |
| 50000 | x_7  | 0.40625 | 73.34375 | 0.60937 |
can see that the CPU time spent by the new technique (NBM) is lower than in the other two methods in most of problem, that indicated the efficiency and quality of our new method.

5. Conclusions

The current work suggests a new projection technique for solving a system of large-scale nonlinear monotone equations. The projection-based algorithms does not use any feature function or derivatives. It considered as a part of the class of derivative-free function-value based approaches. Likewise. For the new method we established the global convergence which is proved under classic assumptions. The numerical results showed that the suggested method is so efficient.

6. References

[1] Shiker M A K and Sahib Z 2018 A modified technique for solving unconstrained optimization, J. Eng. Applied Sci., 13, 9667-9671.
[2] Shiker M A K and Amini K 2018 A new projection-based algorithm for solving a large scale nonlinear system of monotone equations, Croatian operational research review, crorr, 9, 63-73.
[3] Mahdi M M and Shiker M A K 2020 Solving systems of nonlinear monotone equations by using a new projection approach, “in press”, accepted paper for publication in Journal of Physics: Conference Series, International Conference of Modern Applications on Information and Communication Technology (ICMAICT) - Iraq.
[4] Mahdi M M and Shiker M A K 2020 Three-term of new conjugate gradient projection approach under Wolfe condition to solve unconstrained optimization problems, Journal of Advanced Research in Dynamical and Control Systems, 12, 788-795.
[5] Mahdi M M and Shiker M A K 2020 A new projection technique for developing a Liu-Storey method to solve nonlinear systems of monotone equations, J. Phys.: Conf. Ser. 1591 012030.
[6] Mahdi M M and Shiker M A K 2020 Three terms of derivative free projection technique for solving nonlinear monotone equations, J. Phys.: Conf. Ser. 1591 012031.
[7] Mahdi M M and Shiker M A K, 2020 A New Class of Three-Term Dou- ble Projection Approach for Solving Nonlinear Monotone Equations, J. Phys.: Conf. Ser. 1664 012147
[8] Solodov M V and Svaiter B F 1998 A globally convergent inexact Newton method for systems of monotone equations, in: M. Fukushima L Qi (Eds.), Reformulation: Nonsmooth, Piecewise Smooth, Semismooth and Smoothing Methods, Kluwer Academic Publishers, 355-369.
[9] Hashim K H and Shiker M A K 2020 Using a new line search method with gradient direction to solve nonlinear systems of equations, “in press”, accepted paper for publication in Journal of Physics: Conference Series, International Conference of Modern Applications on Information and Communication Technology (ICMAICT) - Iraq.
[10] Dwail H H Mahdi M M Wasi H A Hashim K H Dreeb N K Hussein A H and Shiker M A K 2020 A new modified TR algorithm with adaptive radius to solve a nonlinear systems of equations, “in press”, accepted paper for publication in Journal of Physics: Conference Series, International Conference of Modern Applications on Information and Communication Technology (ICMAICT) - Iraq.
[11] Hussein H A and Shiker M A K 2020 A modification to Vogel's approximation method to Solve transportation problems, J. Phys.: Conf. Ser. 1591 012029.
[12] Hussein H A and Shiker M A K 2020 Two New Effective Methods to Find the Optimal Solution for the Assignment Problems, Journal of Advanced Research in Dynamical and Control Systems, 12, 49-54.
[13] Wasi H A and Shiker M A K 2020 A modified of FR method to solve unconstrained optimization, “in press”, accepted paper for publication in *Journal of Physics: Conference Series*, International Conference of Modern Applications on Information and Communication Technology (ICMAICT) - Iraq.

[14] Hussein H A and Shiker M A K and Zabiba M S M 2020 A new revised efficient of VAM to find the initial solution for the transportation problem, *J. Phys.: Conf. Ser.* 1591 012032.

[15] Wasi, H A and Shiker, M A K, 2020 Proposed CG method to solve unconstrained optimization problems”, “in press, accepted paper for publication in *Journal of Physics: Conference Series*, International Conference of Modern Applications on Information and Communication Technology (ICMAICT) - Iraq.

[16] Dwail H H and Shiker M A K, 2020 Using a trust region method with nonmonotone technique to solve unrestricted optimization problem, *J. Phys.: Conf. Ser.* 1664 012128.

[17] Dwail H H and Shiker M A K, 2020 Reducing the time that TRM requires to solve systems of nonlinear equations *IOP Conf. Ser.: Mater. Sci. Eng.* 928 042043.

[18] Wasi H A and Shiker M A K 2020 A new hybrid CGM for unconstrained optimization problems, *J. Phys.: Conf. Ser.* 1664 012077.

[19] Hassan Z A H H and Shiker M A K 2018 Using of generalized baye’s theorem to evaluate the reliability of aircraft systems, *Journal of Engineering and Applied Sciences*, (Special Issue13), 10797-10801.

[20] Yan Q R and Peng X Z and Li D H 2010 A Globally convergent derivative-free method for solving large-scale nonlinear monotone equations, *Journal of Computational and Applied Mathematics*, 234, 649-657.

[21] Wang Y J and Xiu N H and Zhang J Z 2003 Modified extra gradient method for variational inequalities and verification of solution existence, *Journal of optimization theory and applications*, 119, 167 – 183.

[22] Zarantonello E H 1971 Projections on convex sets in Hilbert space and spectral theory, In Zarantonello E. H. (Ed), Contributions to nonlinear Functional analysis, *Academic press, New York*. 

12