Further investigations of Rényi entropy power inequalities and an entropic characterization of s-concave densities

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Abstract

We investigate the role of convexity in Rényi entropy power inequalities. After proving that a general Rényi entropy power inequality in the style of Bobkov-Chistyakov (2015) fails when the order \( r \in (0, 1) \), we show that random vectors with s-concave densities do satisfy such a Rényi entropy power inequality. Along the way, we establish convergence in the Central Limit Theorem for Rényi entropy of order \( r \in (0, 1) \) for a large class of random vectors. This generalizes a celebrated result of Barron (1986). Additionally, we give an entropic characterization of the class of s-concave densities, which extends a classical result of Cover and Zhang (1994).

1 Introduction

Let \( X \) be a random vector in \( \mathbb{R}^d \). Suppose that \( X \) has the density \( f \) with respect to the Lebesgue measure. For \( r \in (0, 1) \cup (1, \infty) \), the Rényi entropy of order \( r \) (or simply, \( r \)-Rényi entropy) is defined as

\[
h_r(X) = \frac{1}{1 - r} \log \int_{\mathbb{R}^d} f(x)^r \, dx. \tag{1}
\]

For \( r \in \{0, 1, \infty\} \), the \( r \)-Rényi entropy can be extended continuously such that the RHS of (1) is \( \log |\text{supp}(f)| \) for \( r = 0 \); \( -\int_{\mathbb{R}^d} f(x) \log f(x) \, dx \) for \( r = 1 \); and \( -\log \|f\|_\infty \) for \( r = \infty \). The case \( r = 1 \) corresponds to the classical Shannon differential entropy. Here, we denote by \( |\text{supp}(f)| \) the Lebesgue measure of the support of \( f \), and \( \|f\|_\infty \) represents the essential supremum of \( f \). The \( r \)-Rényi entropy power is defined by

\[
N_r(X) = e^{2h_r(X)/d}. \tag{2}
\]

In the following, we drop the subscript when \( r = 1 \).

The classical Entropy Power Inequality (henceforth, EPI) of Shannon [36] and Stam [38], states that the entropy power \( N(X) \) is super-additive on the sum of independent random vectors. There has been recent success on extensions of the EPI from the Shannon differential entropy to \( r \)-Rényi entropy. In [8, 9], Bobkov and Chistyakov showed that, at the expense of an absolute constant \( c > 0 \), the following Rényi EPI of order \( r \in [1, \infty] \) holds

\[
N_r(X_1 + \cdots + X_n) \geq c \sum_{i=1}^n N_r(X_i). \tag{3}
\]

Ram and Sason soon after gave a sharpened summation dependent constant [39]. Savaré and Toscani [33] showed that a modified Rényi entropy power was concave along the solution of some nonlinear heat equation, which generalizes Costa’s concavity of entropy power [18]. Bobkov and Marsiglietti [10] proved the following variant of Rényi EPI

\[
N_r(X + Y)^\alpha \geq N_r(X)^\alpha + N_r(Y)^\alpha \tag{4}
\]

for \( r > 1 \) and some exponent \( \alpha \) only depending on \( r \). It is clear that (4) holds for more than two summands. Improvement of the exponent \( \alpha \) was given by Li [26].

One of our goals is to establish analogues of (3) and (4) when the Rényi parameter \( r \in (0, 1) \). Both (3) and (4) can be derived from Young’s convolution inequality conjugated by the entropic comparison inequality \( h_{r_1}(X) \geq h_{r_2}(X) \) for any \( 0 \leq r_1 \leq r_2 \). The latter is an immediate consequence of Jensen’s
inequality. When the Rényi parameter \( r \in (0,1) \), analogues of (3) and (4) require a reverse entropic comparison inequality aforementioned. This technical issue prevents a general Rényi EPI of order \( r \in (0,1) \) for generic random vectors. Our first result shows that a general Rényi EPI of the form (3) indeed fails for all \( r \in (0,1) \).

**Theorem 1.** For any \( r \in (0,1) \) and \( \varepsilon > 0 \), there exist independent random vectors \( X_1, \cdots, X_n \) in \( \mathbb{R}^d \), for some \( d \geq 1 \) and \( n \geq 2 \), such that

\[
N_r(X_1 + \cdots + X_n) < \varepsilon \sum_{i=1}^n N_r(X_i).
\]

(5)

We have explicit construction of such random vectors. They are essentially truncations of some spherically symmetric random vectors with finite covariance matrices and infinite Rényi entropies of order \( r \in (0,1) \). The key point is a \( r \)-Rényi entropic Central Limit Theorem (henceforth, CLT); that is, the \( r \)-Rényi entropy of their normalized sum converges to the \( r \)-Rényi entropy of a Gaussian. This implies that after appropriate normalization the LHS of (3) is finite, but the RHS of (3) can be as large as possible. The entropic CLT has been studied for a long time. A celebrated result of Barron [3] shows the convergence in Shannon entropy in the CLT (see [25] for a multidimensional setting). The recent work of Bobkov and Marsiglietti [11] studies the convergence in Rényi entropy of order \( r > 1 \) in the CLT for real-valued random variables. In Section 2, we consider the analogue of [11, Theorem 1.1] in higher dimensions and prove the Rényi entropic CLT of order \( r \in (0,1) \) for a large class of random vectors.

As mentioned above, the reverse entropic comparison inequality prevents Rényi EPIs of order \( r \in (0,1) \) for generic random variables. However, a large class of random variables with the so-called \( s \)-concave densities do satisfy such a reverse entropic comparison inequality. Our next results show that Rényi EPI of order \( r \in (0,1) \) holds for such densities. This extends the earlier work of Marsiglietti and Melbourne [30, 31] for log-concave densities (which corresponds to the \( s = 0 \) case).

Let \( s \in [-\infty, \infty] \). A function \( f: \mathbb{R}^d \to [0,\infty) \) is called \( s \)-concave if the inequality

\[
f((1-\lambda)x + \lambda y) \geq ((1-\lambda)f(x)^s + \lambda f(y)^s)^{1/s}
\]

holds for all \( x, y \in \mathbb{R}^d \) such that \( f(x)f(y) > 0 \) and \( \lambda \in (0,1) \). For \( s \in \{-\infty,0,\infty\} \), the RHS of (6) is understood in the limiting sense; that is \( \min\{f(x),f(y)\} \) for \( s = -\infty \), \( f(x)^{1-\lambda}f(y)^{\lambda} \) for \( s = 0 \), and \( \max\{f(x),f(y)\} \) for \( s = \infty \). The case \( s = 0 \) corresponds to log-concave functions. The study of measures with \( s \)-concave densities was initiated by Borell in the seminal work [12, 13]. One can think of \( s \)-concave densities, in particular log-concave densities, as functional versions of convex sets. There has been a recent stream of research on a formal parallel relation between functional inequalities of \( s \)-concave densities and geometric inequalities of convex sets.

**Theorem 2.** Given \( s \in (-1/d,0) \) and \( r \in (-sd,1) \), there exists \( c = c(s,r,d,n) \) such that for all independent random vectors \( X_1, \cdots, X_n \) with \( s \)-concave densities in \( \mathbb{R}^d \),

\[
N_r(X_1 + \cdots + X_n) \geq c \sum_{i=1}^n N_r(X_i).
\]

(7)

In particular, one can take

\[
c = r^\frac{1}{1-r'} \left(1 + \frac{1}{n|r'|}\right)^{1+n|r'|} \left( \prod_{k=1}^d \frac{(1 + ks)^{r'(n-1)}(1 + \frac{ks}{r'})^{1+|r'|}}{(1 + ks(1 + \frac{1}{n|r'|}))^{1+n|r'|}} \right)^\frac{1}{d},
\]

(8)

where \( r' = r/(r-1) \) is the Hölder conjugate of \( r \).

**Theorem 3.** Given \( s \in (-1/d,0) \), there exist \( 0 < r_0 < 1 \) and \( \alpha = \alpha(s,r,d) \) such that for \( r \in (r_0,1) \) and independent random vectors \( X \) and \( Y \) in \( \mathbb{R}^d \) with \( s \)-concave densities,

\[
N_r(X+Y)^\alpha \geq N_r(X)^\alpha + N_r(Y)^\alpha.
\]

(9)
In particular, one can take
\[ r_0 = \left(1 - \frac{2}{1 + \sqrt{3}} \left(1 + \frac{1}{sd}\right)\right)^{-1}, \]  
and
\[ \alpha = \left(1 + \frac{\log r + (r + 1) \log \frac{r+1}{r} + C(s)}{(1 - r) \log 2}\right)^{-1}, \]
where
\[ C(s) = \frac{2}{d} \sum_{k=1}^{d} \left( \log \left(1 + \frac{ks}{r}\right) + r \log(1 + ks) - (r + 1) \log \left(1 + \frac{ks(r+1)}{2r}\right) \right). \]

Owing to the convexity, random vectors with s-concave densities also satisfy a reverse EPI, which was first proved by Bobkov and Madiman [7]. This can be seen as the functional lifting of Milman’s well known reverse Brunn-Minkowski inequality [32]. Motivated by Busemann’s theorem [16] in convex geometry, Ball, Nayar, and Tkocz [2] conjectured that the following reverse EPI
\[ N(X + Y)^{1/2} \leq N(X)^{1/2} + N(Y)^{1/2} \]  
holds for any symmetric log-concave random vector \((X, Y) \in \mathbb{R}^2\). The \(r\)-Rényi entropy analogue was asked in [29], and the \(r = 2\) case was soon verified in [26]. It was also observed in [26] that the \(r\)-Rényi entropy analogue is equivalent to the convexity of \(p\)-cross-section body in convex geometry introduced by Gardner and Giannopoulos [22]. The equivalent linearization of \((13)\) reads as follows.

Let \((X, Y) \in \mathbb{R}^2\) be a symmetric log-concave random vector such that \(h(X) = h(Y)\). Then for any \(\lambda \in [0, 1]\) we have
\[ h((1 - \lambda)X + \lambda Y) \leq h(X). \]  
A classical result of Cover and Zhang [19] showed that the above inequality holds under a stronger assumption that \(X\) and \(Y\) have the same log-concave distribution. The following theorem extends Cover and Zhang’s result from log-concave densities to general \(s\)-concave densities. This gives an entropic characterization of \(s\)-concave densities and implies a reverse Rényi EPI for random vectors with the same \(s\)-concave density.

**Theorem 4.** Let \(f\) be a probability density function on \(\mathbb{R}^d\). Then
\[ \sup_{X_i \sim f} h_r \left( \sum_{i=1}^{n} \lambda_i X_i \right) = h_r(X_1) \]  
holds for all \(\lambda_i \geq 0\) such that \(\sum_{i=1}^{n} \lambda_i = 1\) if and only if the density \(f\) is \((r - 1)\)-concave.

The paper is organized as follows. Section 2 explores convergence in the CLT in the entropic sense. For \(r > 1\) convergence is fully characterized for random vectors in \(\mathbb{R}^d\), while for \(r \in (0, 1)\) sufficient conditions with application to Rényi EPIs are explored. Precisely, convergence is proven for log-concave random vectors and random vectors with radially symmetric unimodal densities and compact support. As an application of the Rényi entropic CLT, we prove in Section 3 that a general \(r\)-Rényi EPI fails when \(r \in (0, 1)\), thus establishing Theorem 1. We also complement this result by proving Theorems 2 and 3. In the last section, we provide an entropic characterization of the class of \(s\)-concave densities, and discuss applications to a reverse Rényi EPI.

## 2 Rényi entropic CLT

Let \(\{X_n\}_{n \in \mathbb{N}}\) be a sequence of independent identically distributed (henceforth, i.i.d.) centered random vectors in \(\mathbb{R}^d\) with finite covariance matrix. We denote by \(Z_n\) the normalized sum
\[ Z_n = \frac{X_1 + \cdots + X_n}{\sqrt{n}}. \]
An important tool used to prove various forms of CLT is the characteristic function. Recall that the characteristic function of a random vector $X$ is defined by
\[ \varphi_X(t) = \mathbb{E}[e^{i(t,X)}], \quad t \in \mathbb{R}^d. \] (17)

Before providing sufficient conditions for convergence in CLT in Rényi entropy of order $r \in (0,1)$, we first extend [11, Theorem 1.1] to higher dimensions.

**Theorem 5.** Let $r > 1$. Let $X_1, \cdots, X_n$ be i.i.d. centered random vectors in $\mathbb{R}^d$. We denote by $\rho_n$ the density of $Z_n$ defined in (16). The following statements are equivalent.

1. $h_r(Z_n) \to h_r(Z)$ as $n \to +\infty$, where $Z$ is a Gaussian random vector with mean 0 and the same covariance matrix as $X_1$.

2. $h_r(Z_n)$ is finite for some integer $n_0$.

3. $\int_{\mathbb{R}^d} |\varphi_{X_1}(t)|^\nu \, dt < +\infty$ for some $\nu \geq 1$.

4. $Z_{n_0}$ has a bounded density $\rho_{n_0}$ for some integer $n_0$.

**Proof.**

1 $\implies$ 2: Assume that $h_r(Z_n) \to h_r(Z)$ as $n \to +\infty$. Then, there exists an integer $n_0$ such that
\[ h_r(Z) - 1 < h_r(Z_{n_0}) < h_r(Z) + 1. \] (18)

Since $h_r(Z)$ is finite, we deduce that $h_r(Z_{n_0})$ is finite as well.

2 $\implies$ 3: Assume that $h_r(Z_{n_0})$ is finite for some integer $n_0$. Then, $Z_{n_0}$ has a density $\rho_{n_0}$ which is in $L^r(\mathbb{R}^d)$, and thus $Z_n$ has a density $\rho_n \in L^r$ for any $n \geq n_0$ by the convolution structure of $Z_n$. If $r \geq 2$, then $\rho_n \in L^2(\mathbb{R}^d)$. Hence by Plancherel identity, $\varphi_{Z_n} \in L^2(\mathbb{R}^d)$. It follows that
\[ \int_{\mathbb{R}^d} |\varphi_{Z_n}(t)|^2 \, dt = \int_{\mathbb{R}^d} |\varphi_{X_1}(t/\sqrt{n})|^2 \, dt < +\infty. \] (19)

We deduce that for $\nu = 2n_0$,
\[ \int_{\mathbb{R}^d} |\varphi_{X_1}(t)|^\nu \, dt < +\infty. \] (20)

If $r \in (1,2)$, then by the Hausdorff-Young inequality,
\[ \|\varphi_{Z_n}\|_{L^r'} \leq \frac{1}{(2\pi)^{d/2}} \|ho_n\|_{L^r}, \] (21)

where $r'$ is the conjugate of $r$. Hence, for $\nu = r'n_0$,
\[ \int_{\mathbb{R}^d} |\varphi_{X_1}(t)|^\nu \, dt < +\infty. \] (22)

3 $\implies$ 4: Since $\int_{\mathbb{R}^d} |\varphi_{X_1}(t)|^\nu \, dt < +\infty$ for some $\nu \geq 1$, one may apply Gnedenko’s local limit theorems (see [23]), which is valid in arbitrary dimension (see [5]). In particular,
\[ \lim_{\nu \to +\infty} \sup_{x \in \mathbb{R}^d} |\rho_n(x) - \phi_\Sigma(x)| = 0, \] (23)

where $\phi_\Sigma$ denotes the density of a Gaussian random vector with mean 0 and same covariance matrix as $X_1$. We deduce that there exists an integer $n_0$ and a constant $M > 0$ such that $\rho_n \leq M$ for all $n \geq n_0$.

4 $\implies$ 1: Since $\rho_{n_0}$ is bounded, then $\rho_{n_0} \in L^2$, and we deduce by Plancherel identity that
\[ \int_{\mathbb{R}^d} |\varphi_{X_1}(t)|^\nu \, dt < +\infty \text{ for } \nu = 2n_0. \] Hence [23] holds and there exists $M > 0$ such that $\rho_n \leq M$ for all $n \geq n_0$. Let us show that $\int \rho_n \to \int \phi_\Sigma$ as $n \to +\infty$, where $\phi_\Sigma$ denotes the density of a Gaussian random vector with mean 0 and same covariance matrix as $X_1$. From the central limit theorem, there exists $T > 0$ such that for all $n$ large enough,
\[ \int_{|x|>T} \rho_n(x) \, dx < \varepsilon, \quad \int_{|x|>T} \phi_\Sigma(x) \, dx < \varepsilon. \] (24)
Hence,
\[
\int_{|x|>T} \rho_n(x)^r \, dx \leq M^{r-1} \int_{|x|>T} \rho_n(x) \, dx < M^{r-1} \varepsilon, \tag{25}
\]
and similarly for \( \int_{|x|>T} \phi^r_X \). Hence, for any \( \delta > 0 \), there exists \( T > 0 \) such that for all \( n \) large enough,
\[
\left| \int_{|x|>T} \rho_n(x)^r \, dx - \int_{|x|>T} \phi^r_X(x)^r \, dx \right| < \delta. \tag{26}
\]
On the other hand, by (23), for all \( T > 0 \), the function \( \rho_n^r(x)1_{\{|x|\leq T\}} \) converges everywhere to \( \phi^r_{\Sigma}(x)1_{\{|x|\leq T\}} \) as \( n \to +\infty \). Since \( \rho_n^r(x)1_{\{|x|\leq T\}} \) is dominated by the integrable function \( M^{r}1_{\{|x|\leq T\}} \), one may use the Lebesgue dominated theorem to conclude that
\[
\lim_{n \to +\infty} \left| \int_{|x| \leq T} \rho_n(x)^r \, dx - \int_{|x| \leq T} \phi^r_X(x)^r \, dx \right| = 0. \tag{27}
\]

Remark 6. Theorem 3 fails when \( r \in (0, 1) \). For example, one can consider i.i.d. random variables with a bounded density \( \rho(x) \) such that \( \int_{\mathbb{R}^d} \rho(x)^r \, dx = +\infty \) (e.g., Cauchy-type distributions). The implication \( 4 \Rightarrow 2 \) (and thus \( 4 \Rightarrow 1 \)) will not hold since by Jensen inequality \( h_r(Z_n) \geq h_r(X_1/\sqrt{n}) = \infty \) for all \( n \geq 1 \). As observed by Barron [3], the implication \( 1 \Rightarrow 4 \) does not necessarily hold in the Shannon entropy case \( r = 1 \).

Our following results yield sufficient conditions for a CLT to hold for Rényi entropies of order \( r \in (0, 1) \) for a large classes of random vectors in \( \mathbb{R}^d \).

Theorem 7. Let \( r \in (0, 1) \). Let \( X_1, \ldots, X_n \) be i.i.d. centered log-concave random vectors in \( \mathbb{R}^d \). Then we have \( h_r(Z_n) < +\infty \) for all \( n \geq 1 \), and

\[
\lim_{n \to \infty} h_r(Z_n) = h_r(Z),
\]
where \( Z_n \) is the normalized sum in (16) and \( Z \) is a Gaussian random vector with mean 0 and the same covariance matrix as \( X_1 \).

Proof. Since log-concavity is preserved under independent sum, \( Z_n \) is log-concave for all \( n \geq 1 \). Hence, for all \( n \geq 1 \), \( Z_n \) has a bounded log-concave density \( \rho_n \), which satisfies
\[
\rho_n(x) \leq e^{-a_n|x|^r + b_n}, \tag{28}
\]
for all \( x \in \mathbb{R}^d \), for some constants \( a_n > 0 \), \( b_n \in \mathbb{R} \) possibly depending on the dimension (see, e.g., [14]). Hence, for all \( n \geq 1 \),
\[
\int_{\mathbb{R}^d} \rho_n(x)^r \, dx \leq \int_{\mathbb{R}^d} e^{-r(a_n|x|^r + b_n)} \, dx < +\infty. \tag{29}
\]
We deduce that \( h_r(Z_n) < +\infty \) for all \( n \geq 1 \).

The boundedness of \( \rho_n \) implies that (23) holds, and thus there exists an integer \( n_0 \) such that for all \( n \geq n_0 \),
\[
\rho_n(0) > \frac{1}{2} \phi^r_{\Sigma}(0), \tag{30}
\]
where \( \Sigma \) is the covariance matrix of \( X_1 \) (and thus does not depend on \( n \)). Moreover, since \( \rho_n \) is log-concave, one has for all \( x \in \mathbb{R}^d \),
\[
\rho_n(rx) = \rho_n((1-r)0 + rx) \geq \rho_n(0)^{1-r} \rho_n(x)^r \geq \frac{1}{2^{1-r}} \phi^r_{\Sigma}(0)^{1-r} \rho_n(x)^r. \tag{31}
\]
Hence, for all $T > 0$,
\[
\int_{|x| > T} \rho_n(x)^r \, dx \leq \frac{2^{1-r}}{\phi_\Sigma(0)^{1-r}} \int_{|x| > T} \rho_n(rx) \, dx
\]
\[
= \frac{2^{1-r}}{r^d \phi_\Sigma(0)^{1-r}} \mathbb{P} (|Z_n| > rT)
\]
\[
\leq \frac{1}{T^{d+2} \phi_\Sigma(0)^{1-r} r},
\]
where the last inequality follows from Markov’s inequality and the fact that
\[
\mathbb{E} [|Z_n|^2] = \frac{\mathbb{E} [|X_1|^2] + \cdots + \mathbb{E} [|X_n|^2]}{n} = \mathbb{E} [|X_1|^2].
\]  
Hence, for every $\varepsilon > 0$, one may choose a positive number $T$ such that for all $n$ large enough,
\[
\int_{|x| > T} \rho_n(x)^r \, dx < \varepsilon, \quad \int_{|x| > T} \phi_\Sigma(x)^r \, dx < \varepsilon,
\]
and hence
\[
\left| \int_{|x| > T} \rho_n(x)^r \, dx - \int_{|x| > T} \phi_\Sigma(x)^r \, dx \right| < \varepsilon.
\]
On the other hand, from (23), we conclude as in the proof of Theorem 5 that for all $T > 0$,
\[
\lim_{n \to +\infty} \left( \int_{|x| \leq T} \rho_n(x)^r \, dx - \int_{|x| \leq T} \phi_\Sigma(x)^r \, dx \right) = 0.
\]

A function $f : \mathbb{R}^d \to \mathbb{R}$ is called unimodal if the super-level sets $\{x \in \mathbb{R}^d : f(x) > t\}$ are convex for all $t \in \mathbb{R}$. Next, we provide a convergence result for random vectors in $\mathbb{R}^d$ with unimodal densities under additional symmetry assumptions. First, we need the following stability result.

**Proposition 8.** The class of spherically symmetric unimodal random variables is stable under convolution.

**Proof.** Suppose that $f_i$ are densities such that $f_i(Tx) = f_i(x)$ for an orthogonal map $T$ and $|x| \leq |y|$ implies $f_i(x) \geq f_i(y)$. By the layer cake decomposition, we write
\[
f_i(x) = \int_0^\infty 1_{\{\langle u, v \rangle : f_i(u) > v\}}(x, \lambda) \, d\lambda.
\]
After applying Fubini-Tonelli,
\[
f_1 * f_2(x) = \int_{\mathbb{R}^d} f_1(x - y) f_2(y) \, dy
\]
\[
= \int_0^\infty \int_0^\infty \left( \int_{\mathbb{R}^d} 1_{\{\langle u, v \rangle : f_1(u) > v\}}(x - y, \lambda_1) 1_{\{\langle u, v \rangle : f_2(u) > v\}}(y, \lambda_2) \, dy \right) \, d\lambda_1 \, d\lambda_2.
\]
Notice that by the spherical symmetry and decreasingness of $f_i$,
\[
B_{\lambda_i} = \{u : f_i(u) > \lambda_i\}
\]
is an origin symmetric ball. Thus we can write the integrand in (40) as
\[
\int_{\mathbb{R}^d} 1_{B_{\lambda_1}}(x - y) 1_{B_{\lambda_2}}(y) \, dy = 1_{B_{\lambda_1}} * 1_{B_{\lambda_2}}(x).
\]
This quantity is clearly dependent only on $|x|$, giving spherical symmetry. Additionally as the convolution of two log-concave functions, $1_{B_{\lambda_1}} * 1_{B_{\lambda_2}}$ is log-concave as well. It follows that for every $\lambda_1, \lambda_2$, and $|x| \leq |y|$ we have
\[
1_{B_{\lambda_1}} * 1_{B_{\lambda_2}}(x) \geq 1_{B_{\lambda_1}} * 1_{B_{\lambda_2}}(y).
\]
Integrating this inequality completes the proof. 
\]
Let us establish large deviation and pointwise inequalities for radially symmetric unimodal densities with compact support.

**Theorem 9** (Hoeffding [21]). Let $X_1, \cdots, X_n$ be independent random variables with mean 0 and bounded in $(a_i, b_i)$, respectively. One has for all $T > 0$,

$$
\mathbb{P}\left( \sum_{i=1}^{n} X_i > T \right) \leq \exp\left( -\frac{2T^2}{\sum_{i=1}^{n}(b_i - a_i)^2} \right). \tag{44}
$$

The following result is Hoeffding’s inequality in higher dimensions.

**Lemma 10.** Let $X_1, \cdots, X_n$ be centered independent random vectors in $\mathbb{R}^d$ satisfying $\mathbb{P}(|X_i| > R) = 0$ for some $R > 0$. One has for all $T > 0$,

$$
\mathbb{P}\left( \left| \frac{X_1 + \cdots + X_n}{\sqrt{n}} \right| > T \right) \leq 2d \exp\left( -\frac{T^2}{d^2 R^2} \right). \tag{45}
$$

**Proof.** Let $X_{i,j}$ be the $j$-th coordinate of the random vector $X_i$. Then we have

$$
\mathbb{P}\left( \left| \frac{X_1 + \cdots + X_n}{\sqrt{n}} \right| > T \right) \leq \mathbb{P}\left( \bigcup_{j=1}^{d} \left\{ \left| X_{1,j} + \cdots + X_{n,j} \right| > \frac{T\sqrt{n}}{d} \right\} \right) \tag{46}
$$

$$
\leq \sum_{j=1}^{d} \mathbb{P}\left( \left| X_{1,j} + \cdots + X_{n,j} \right| > \frac{T\sqrt{n}}{d} \right) \tag{47}
$$

$$
\leq 2d \exp\left( -\frac{T^2}{d^2 R^2} \right). \tag{48}
$$

where inequality (46) follows from the pigeon-hole principle, (47) from a union bound, and (48) follows from applying Theorem 9 to $X_{1,j} + \cdots + X_{n,j}$ and $(-X_{1,j}) + \cdots + (-X_{n,j})$. \hfill \Box

We deduce the following pointwise estimate for unimodal radially symmetric and bounded random variables.

**Corollary 11.** Let $X_1, \cdots, X_n$ be i.i.d. random vectors with radially symmetric unimodal density supported on the origin-centered Euclidean ball of radius $R > 0$. Let $\rho_n$ denote the density of the normalized sum $Z_n$. Then there exists $c_d > 0$ such that for $|x| > 2$,

$$
\rho_n(x) \leq c_d \exp\left( -\frac{(|x| - 1)^2}{d^2 R^2} \right). \tag{49}
$$

**Proof.** Stating Lemma 10 in terms of $\rho_n$, we have

$$
\int_{|w| > T} \rho_n(w)dw \leq 2d \exp\left( -\frac{T^2}{d^2 R^2} \right). \tag{50}
$$

Since the class of radially symmetric unimodal random variables is stable under independent summation by Proposition 5, $\rho_n$ is radially symmetric and unimodal, so that

$$
\rho_n(x) \leq \frac{\int_{B_{|x|}} \rho_n(w)dw}{\text{Vol}(B_{|x|})} \tag{51}
$$

$$
\leq \frac{\int_{|w| \geq |x| - 1} \rho_n(w)dw}{(2d - 1)\omega_d} \tag{52}
$$

where $B_{|x|}$ represents the Euclidean ball of radius $|x|$ centered at the origin and $\omega_d$ is the volume of the unit ball. Note that

$$
\text{Vol}(B_{|x|}) - |x|^d \geq (2d - 1)\omega_d, \tag{53}
$$
since \( t \mapsto t^d - (t - 1)^d \) is increasing, so that (52) follows. Now applying (50) we have

\[
\rho_n(x) \leq \frac{\int_{|w| \geq |x| - 1} \rho_n(w)dw}{(2^d - 1)\omega_d} \leq \frac{2d}{(2^d - 1)\omega_d} \exp\left(-\frac{(|x| - 1)^2}{d^2 R^2}\right)
\]

and our result holds with

\[
c_d = \frac{2d}{(2^d - 1)\omega_d}.
\]

We are now ready to establish a convergence result for bounded radially symmetric unimodal random vectors.

**Theorem 12.** Let \( r \in (0, 1) \). Let \( X_1, \ldots, X_n \) be i.i.d. random vectors in \( \mathbb{R}^d \) with a radially symmetric unimodal density with compact support. Then we have

\[
\lim_{n \to \infty} h_r(Z_n) = h_r(Z),
\]

where \( Z_n \) is the normalized sum in (16) and \( Z \) is a Gaussian random vector with mean 0 and the same covariance matrix as \( X_1 \).

**Proof.** Let us denote by \( \rho_n \) the density of \((X_1 + \cdots + X_n)/\sqrt{n}\). Since \( \rho_1 \in L^1 \), it follows that \( \rho_n, n \geq 2 \), are continuous, and since \( \rho_n \) are, in addition, radially symmetric unimodal densities by Proposition 8, then \( \rho_n, n \geq 2 \), are bounded. One may thus apply (23) together with Lebesgue dominated convergence to conclude that for all \( T > 0 \),

\[
\lim_{n \to +\infty} \left| \int_{|x| \leq T} \rho_n(x)^r dx - \int_{|x| \leq T} \phi_{\Sigma}(x)^r dx \right| = 0.
\]

On the other hand, by Corollary 11 one may choose \( T > 0 \) such that for all \( n \geq 1 \),

\[
\int_{|x| > T} \rho_n(x)^r dx < \varepsilon, \quad \int_{|x| > T} \phi_{\Sigma}(x)^r dx < \varepsilon,
\]

and hence

\[
\left| \int_{|x| > T} \rho_n(x)^r dx - \int_{|x| > T} \phi_{\Sigma}(x)^r dx \right| < \varepsilon.
\]

\[\square\]

### 3 Rényi EPIs of order \( r \in (0, 1) \)

A striking difference between Rényi EPIs of order \( r \in (0, 1) \) and \( r \geq 1 \) is the lack of an absolute constant. Indeed, it was shown in [9] that for \( r \geq 1 \) Rényi EPI of the form (3) holds for generic independent random vectors with an absolute constant \( c_r \geq \frac{1}{e} r^{\frac{1}{1-r}} \). The following subsection shows that such a Rényi EPI does not exist for \( r \in (0, 1) \).

#### 3.1 Failure of a generic Rényi EPI

**Definition 13.** For \( r \in [0, \infty] \), we define \( c_r \) as the largest number such that the following inequality holds for any independent random vectors \( X_1, \ldots, X_n \) in \( \mathbb{R}^d \),

\[
N_r(X_1 + \cdots + X_n) \geq c_r \sum_{i=1}^n N_r(X_i).
\]
Then we can rephrase Theorem 14 as follows.

**Theorem 14.** For \( r \in (0,1) \), the constant \( c_r \) defined in (61) satisfies \( c_r = 0 \).

The motivating observation for this line of argument is the fact that for \( r \in (0,1) \), there exist random variables with finite variances and infinite \( r \)-Rényi entropies. One might anticipate that this could contradict the existence of an \( r \)-Rényi EPI, as the CLT forces the normalized sum of such i.i.d. random variables to become ‘more Gaussian’. Heuristically, one anticipates for large \( n \), and \( X_1, \cdots, X_n \) independently drawn from such a distribution, that \( N_r(X_1 + \cdots + X_n)/n = N_r(Z_1/n) \) should approach \( N_r(Z) \), where \( Z_n = (X_1 + \cdots + X_n)/\sqrt{n} \) is the normalized sum and \( Z \) is a Gaussian vector with the same variance as \( X_1 \), while \( \sum_{i=1}^n N_r(X_i)/n = N_r(X_1) \) is infinite.

**Proof of Theorem 14.** Let us consider the following density

\[
f(x) = f_{R,p,d}(x) = C_R(1 + |x|)^{-p}1_{B_R}(x), \quad x \in \mathbb{R}^d, \tag{62}
\]

with \( p, R > 0 \) and \( C_R \) implicitly determined to make \( f \) a density. Note that \( f \) is bounded, unimodal, and radially symmetric. Thus its covariance matrix is a multiple of the identity, i.e., \( \sigma^2_R I \) for some \( \sigma_R > 0 \). Computing in spherical coordinates one can easily see that \( \lim_{R \to \infty} C_R \) is finite for \( p > d \), and we can thus define a density \( f_{\infty,p,d} \). What is more, when \( p > d + 2 \), the limiting density \( f_{\infty,p,d} \) has a finite covariance matrix, and has finite \( r \)-Rényi entropy if and only if \( p > \frac{d}{r} \).

Now fix \( r \in (0,1) \) and take the dimension to be \( d^* = \min\{d \in \mathbb{N} : d > \frac{2}{1-r} \} \), and \( p \in (d^* + 2, \frac{2}{1-r}] \). In this case, the limit density \( f_{\infty,p,d^*} \) is well defined and it has finite covariance matrix \( \sigma^2_{\infty} I \), but the corresponding \( r \)-Rényi entropy is infinite. Now we select independent random vectors \( X_1, \cdots, X_n \) from the distribution \( f_{R,p,d^*} \). Since \( f_{R,p,d^*} \) is a radially symmetric unimodal density with compact support, one may apply Theorem 12 to conclude that

\[
\lim_{n \to \infty} N_r(Z_1/n) = \sigma^2_R N_r(Z_{1d}),
\]

where \( Z_{1d} \) is the standard \( d \)-dimensional Gaussian. Notice that

\[
\lim_{R \to \infty} N_r(X_1) = \infty,
\]

while

\[
\lim_{R \to \infty} \sigma_R = \sigma_{\infty} < \infty.
\]

Given \( M > 0 \), we can take \( R \) large enough such that \( N_r(X_1) \geq M \), and \( |\sigma^2_R - \sigma^2_{\infty}| \leq 1 \). Then we can take \( n \) large enough such that

\[
N_r(Z_1/n) \leq (\sigma^2_{\infty} + 2)N_r(Z_{1d}).
\]

We conclude that for the inequality (61) to hold we must have

\[
c_r \leq \frac{(\sigma^2_{\infty} + 2)N_r(Z_{1d})}{M}
\]

for all \( M > 0 \). Taking \( M \to \infty \) this can only hold if \( c_r = 0 \).

**Remark 15.** Random vectors in our proof of Theorem 14 has identical \( s \)-concave density with \( s < 0 \) and \( |s| \geq r/d \). In the following section, we will prove a complement result by showing that \( r \)-Rényi EPI of order \( r \in (0,1) \) does hold for \( s \)-concave densities when \( s < 0 \) and \( |s| < r/d \).

### 3.2 Rényi EPIs for \( s \)-concave densities

As showed above, a generic Rényi EPI of the form (3) fails for \( r \in (0,1) \). In this part, we establish Rényi EPIs of the form (3) and (4) for an important class of random vectors with \( s \)-concave densities (see (6)). The constants \( c \) and \( \alpha \) are no longer dimension-free as in the case of log-concave densities [30], this is owing to the absence of exponential tails.

Following Lieb [28], we prove Theorems 2 and 3 by showing their equivalent linearizations. The following linearization of (3) and (4) is due to Rioul [34]. The \( c = 1 \) case has been used in [26].
Theorem 16 ([34]). Let $X_1, \ldots, X_n$ be independent random vectors in $\mathbb{R}^d$. The following statements are equivalent.

- There exist a constant $c > 0$ and an exponent $\alpha > 0$ such that
  \[ N_{\alpha}^\ast \left( \sum_{i=1}^n X_i \right) \geq c \sum_{i=1}^n N_{\alpha}^\ast (X_i). \]  
  \[ (68) \]

- For any $\lambda_1, \ldots, \lambda_n \geq 0$ such that $\sum_{i=1}^n \lambda_i = 1$, one has
  \[ h_r \left( \sum_{i=1}^n \sqrt{\lambda_i} X_i \right) - \sum_{i=1}^n \lambda_i h_{r_i} (X_i) \geq \frac{d}{2} \left( \frac{\log c}{\alpha} + \left( \frac{1}{\alpha} - 1 \right) H(\lambda) \right), \]  
  where $H(\lambda) \triangleq H(\lambda_1, \ldots, \lambda_n)$ is the discrete entropy defined as
  \[ H(\lambda) = - \sum_{i=1}^n \lambda_i \log \lambda_i. \]  
  \[ (69) \]

One of the ingredients used to establish [69] is Young’s sharp convolution inequality [4, 14]. Its information-theoretic formulation was given in [20], which we recall below. We denote by $r'$ the H"{o}lder conjugate of $r$, i.e.,
  \[ \frac{1}{r} + \frac{1}{r'} = 1. \]  
  \[ (71) \]

Theorem 17 ([14, 20]). Let $r > 0$. Let $\lambda_1, \ldots, \lambda_n \geq 0$ such that $\sum_{i=1}^n \lambda_i = 1$, and let $r_1, \ldots, r_n$ be positive reals such that $\lambda_i = r'/r_i$. Then, for all independent random vectors $X_1, \ldots, X_n$ in $\mathbb{R}^d$, one has
  \[ h_r \left( \sum_{i=1}^n \sqrt{\lambda_i} X_i \right) - \sum_{i=1}^n \lambda_i h_{r_i} (X_i) \geq \frac{d}{2} \left( \frac{\log r}{r} - \sum_{i=1}^n \log r_i \right). \]  
  \[ (72) \]

The second ingredient is a comparison between Rényi entropies $h_r$ and $h_{r_i}$. When $r > 1$, we have $1 < r_i < r$, and Jensen’s inequality implies that $h_r \leq h_{r_i}$. In this case, one can deduce [69] from (72) with $h_{r_i}$ replaced by $h_r$. However, when $r \in (0, 1)$, the order of $r$ and $r_i$ are reversed, i.e., $0 < r < r_i < 1$, and we need a reverse entropy comparison inequality. The so-called $s$-concave densities do satisfy such a reverse entropy comparison inequality. The following result of Fradelizi, Li, and Madiman [21] serves this purpose.

Theorem 18 ([21]). Let $s \in \mathbb{R}$. Let $f : \mathbb{R}^d \to [0, +\infty)$ be an integrable $s$-concave function. Then, the function
  \[ G(r) = C(r) \int_{\mathbb{R}^d} f(x)^r \, dx \]  
  \[ (73) \]
  is log-concave for $r > \max \{ 0, -sd \}$, where
  \[ C(r) = (r + s) \cdots (r + sd). \]  
  \[ (74) \]

We deduce the following Rényi entropic comparison for random vectors with $s$-concave densities.

Corollary 19. Let $X$ be a random vector in $\mathbb{R}^d$ with an $s$-concave density. For $-sd < r < q < 1$, we have
  \[ h_q (X) \geq h_r (X) + \log \frac{C(r)^{q/r} C(1)^{-q}}{C(q)^{1/r}}. \]  
  \[ (75) \]

Proof. Write $q = (1 - \lambda) \cdot r + \lambda \cdot 1$. By Theorem 18 we have
  \[ G(q) \geq G(r)^{1 - \lambda} G(1)^{\lambda} = G(r)^{1 - \frac{q}{r}} G(1)^{\frac{q}{r}}. \]  
  \[ (76) \]

Rewrite the above inequality in terms of entropy power
  \[ C(q)^{\frac{q}{r}} \geq C(r)^{\frac{q}{r}} \frac{C(q)^{\frac{q}{r}}}{C(r)^{\frac{q}{r}}} N_r (X) C(1)^{\frac{q}{r}}. \]  
  \[ (77) \]

The desired result follows from taking the logarithm of the above inequality. \(\square\)
Put together Theorem 17 and Corollary 19. We obtain the following Rényi EPI valid for a single Rényi parameter \( r \in (0, 1) \) in the class of \( s \)-concave random variables.

**Theorem 20.** Let \( s \in (-1/d, 0) \) and \( r \in (-sd, 1) \). Let \( X_1, \ldots, X_n \) be independent random vectors in \( \mathbb{R}^d \) with \( s \)-concave densities. Then, for all \( \lambda = (\lambda_1, \ldots, \lambda_n) \in [0, 1]^n \) such that \( \sum_{i=1}^n \lambda_i = 1 \), we have

\[
h_r \left( \sum_{i=1}^n \sqrt[\lambda_i]{X_i} \right) - \sum_{i=1}^n \lambda_i h_r(X_i) \geq \frac{d}{2} A(\lambda) + \sum_{k=1}^d g_k(\lambda),
\]

where

\[
A(\lambda) = r' \left( \left( 1 - \frac{1}{r' \lambda} \right) \log \left( 1 - \frac{1}{r'} \right) - \sum_{i=1}^n \left( 1 - \frac{\lambda_i}{r'} \right) \log \left( 1 - \frac{\lambda_i}{r'} \right) \right),
\]

\[
g_k(\lambda) = (1 - n) r' \log (1 + k s) + (1 - r') \log \left( 1 + \frac{k s}{r} \right) + r' \sum_{i=1}^n \left( 1 - \frac{\lambda_i}{r'} \right) \log \left( 1 + k s \left( 1 - \frac{\lambda_i}{r'} \right) \right).
\]

**Proof.** Let \( r_i \) be defined by \( \lambda_i = r' / r_i' \), where \( r' \) and \( r_i' \) are Hölder conjugates of \( r \) and \( r_i \), respectively. Combining Theorem 17 with Corollary 19 we have

\[
h_r \left( \sum_{i=1}^n \sqrt[\lambda_i]{X_i} \right) - \sum_{i=1}^n \lambda_i h_r(X_i) \geq \frac{d}{2} r' \left( \frac{\log r}{r} - \sum_{i=1}^n \log \frac{r_i}{r} \right) + \sum_{i=1}^n \lambda_i \log \frac{C(r) \frac{1}{1-r} C(1) \frac{1}{1-r'} (1-1/r)}{C(r_i) \frac{1}{1-r_i}}.
\]

Notice that \( C(r) = r^d D(r) \), where \( D(r) = (1 + s/r) \cdots (1 + sd/r) \). Thus,

\[
\sum_{i=1}^n \lambda_i \log \frac{C(r) \frac{1}{1-r} C(1) \frac{1}{1-r'} (1-1/r)}{C(r_i) \frac{1}{1-r_i}} = \sum_{i=1}^n \lambda_i \left( \log D(r) \frac{1}{1-r'} + \left( \frac{1}{1-r_i} - \frac{1}{1-r} \right) \log D(1) - \frac{\log D(r_i)}{1-r_i} \right) + d \left( \frac{\log r}{1-r} - \sum_{i=1}^n \lambda_i \log \frac{r_i}{1-r_i} \right).
\]

Using the identities \( 1/(1-r) = 1-r' \) and \( \lambda_i/(1-r_i) = \lambda_i - r' \), we have

\[
\sum_{i=1}^n \lambda_i \left( \log D(r) \frac{1}{1-r'} + \left( \frac{1}{1-r_i} - \frac{1}{1-r} \right) \log D(1) - \frac{\log D(r_i)}{1-r_i} \right)
\]

\[
= (1 - r') \log D(r) + (1 - n) r' \log D(1) + \sum_{k=1}^d \sum_{i=1}^n (r_i - \lambda_i) \log \left( 1 + \frac{k s}{r_i} \right)
\]

\[
= \sum_{k=1}^d \left( (1 - r') \log \left( 1 + \frac{k s}{r} \right) + (1 - n) r' \log (1 + k s) + \sum_{i=1}^n (r_i - \lambda_i) \log \left( 1 + \frac{k s}{r_i} \right) \right)
\]

\[
= \sum_{k=1}^d g_k(\lambda),
\]

the last identity follows from \( 1/r_i = 1 - \lambda_i/r' \). Hence, the RHS of (79) can be written as

\[
\frac{d}{2} r' \left( \frac{\log r}{r} - \sum_{i=1}^n \frac{\log r_i}{r_i} \right) + d \left( \frac{\log r}{1-r} - \sum_{i=1}^n \lambda_i \frac{\log r_i}{1-r_i} \right) + \sum_{k=1}^d g_k(\lambda) = \frac{d}{2} A(\lambda) + \sum_{k=1}^d g_k(\lambda).
\]

Having Theorem 16 and Theorem 20 at hand, we are ready to prove the main results.
### 3.2.1 Proof of Theorem 2

Combine Theorems 16 and 20. Then it suffices to find \( c \) such that for all \( \lambda = (\lambda_1, \cdots, \lambda_n) \in [0, 1]^n \) satisfying \( \sum_{i=1}^{n} \lambda_i = 1 \),

\[
\frac{d}{2} A(\lambda) + \sum_{k=1}^{d} g_k(\lambda) \geq \frac{d}{2} \log c. \tag{86}
\]

Hence, we set

\[
c = \inf_{\lambda} \exp \left( A(\lambda) + \frac{2}{d} \sum_{k=1}^{d} g_k(\lambda) \right), \tag{87}
\]

where the infimum runs over all \( \lambda = (\lambda_1, \cdots, \lambda_n) \in [0, 1]^n \) such that \( \sum_{i=1}^{n} \lambda_i = 1 \). For fixed \( r \), both \( A(\lambda) \) and \( g_k(\lambda) \) are sum of one-dimensional convex functions of the form \((1 + x) \log (1 + x)\). Furthermore, both \( A(\lambda) \) and \( g_k(\lambda) \) are permutation invariant. Hence, the minimum is achieved at \( \lambda = (1/n, \cdots, 1/n) \). This yields the value of \( c \) in Theorem 2.

### 3.2.2 Proof of Theorem 3

The following lemma in [30] serves us in the proof of Theorem 3.

**Lemma 21** ([30]). Let \( c > 0 \). Let \( L, F : [0, c] \to [0, \infty) \) be twice differentiable on \( (0, c) \), continuous on \([0, c]\), such that \( L(0) = F(0) = 0 \) and \( L'(c) = F'(c) = 0 \). Let us also assume that \( F(x) > 0 \) for \( x > 0 \), that \( F \) is strictly increasing, and that \( F' \) is strictly decreasing. Then \( \frac{L'}{F} \) increasing on \((0, c)\) implies that \( \frac{L}{F} \) is increasing on \((0, c)\) as well. In particular,

\[
\max_{x \in [0, c]} \frac{L(x)}{F(x)} = \frac{L(c)}{F(c)}.
\]

**Proof of Theorem 3** Using Theorems 16 and 20 with \( n = 2 \), it suffices to find \( \alpha \) such that for all \( \lambda \in [0, 1] \),

\[
\frac{d}{2} A(\lambda) + \sum_{k=1}^{d} g_k(\lambda) \geq \frac{d}{2} \left( \frac{1}{\alpha} - 1 \right) H(\lambda), \tag{88}
\]

where,

\[
A(\lambda) = \lambda' \left( \left( 1 - \frac{1}{r'} \right) \log \left( 1 - \frac{1}{r'} \right) - \left( 1 - \frac{\lambda}{r'} \right) \log \left( 1 - \frac{\lambda}{r'} \right) - \left( \frac{1 - \frac{1-\lambda}{r'}}{r'} \right) \log \left( \frac{1 - \frac{1-\lambda}{r'}}{r'} \right) \right), \tag{89}
\]

\[
g_k(\lambda) = \left( 1 - r' \right) \log \left( 1 + \frac{ks}{r} \right) - r' \log (1 + ks) \quad + \quad \lambda' \left( \left( 1 - \frac{\lambda}{r'} \right) \log \left( 1 + ks \left( 1 - \frac{\lambda}{r'} \right) \right) + \left( \frac{1 - \lambda}{r'} \right) \log \left( 1 + ks \left( 1 - \frac{\lambda}{r'} \right) \right) \right). \tag{90}
\]

We can set

\[
\alpha = \left( 1 - \sup_{0 \leq \lambda \leq 1} \left( \frac{A(\lambda)}{H(\lambda)} - \frac{2}{d} \sum_{k=1}^{d} \frac{g_k(\lambda)}{H(\lambda)} \right) \right)^{-1}. \tag{91}
\]

We will show that the optimal value is achieved at \( \lambda = 1/2 \). Since the function is symmetric about \( \lambda = 1/2 \), it suffices to show that

\[
- \frac{A(\lambda)}{H(\lambda)} - \frac{2}{d} \sum_{k=1}^{n} \frac{g_k(\lambda)}{H(\lambda)} \tag{92}
\]

is increasing on \([0, 1/2]\). It has been shown in [29] that \(-A(\lambda)/H(\lambda)\) is increasing on \([0, 1/2]\). We will show that every \(-g_k(\lambda)/H(\lambda)\) is also increasing on \([0, 1/2]\), by applying Lemma 21. Note that \(-g_k(\lambda), H(\lambda) \geq 0 \). Also, one can check that \( g_k(0) = g_k(1) = 0 \) and \( g_k'(1/2) = 0 \). Elementary calculation yields

\[
H''(\lambda) = -\frac{1}{(1-\lambda)} \tag{93}
\]
Let us define $x = \frac{1}{|r'|}$ and $y = \frac{1}{|r'|}$. Then one can check that

$$-g_k''(\lambda) = \frac{ks}{|r'|} \left( \frac{1}{1 + ks(1 + x)} + \frac{1}{1 + ks(1 + y)} + \frac{1}{(1 + ks(1 + x))^2} + \frac{1}{(1 + ks(1 + y))^2} \right).$$

(94)

Hence, we have

$$-\frac{g_k''(\lambda)}{H''(\lambda)} = ksr'W(x),$$

where

$$W(x) = xy \left( \frac{1}{1 + ks(1 + x)} + \frac{1}{1 + ks(1 + y)} + \frac{1}{(1 + ks(1 + x))^2} + \frac{1}{(1 + ks(1 + y))^2} \right)$$

(96)

with $y = \frac{1}{|r'|} - x$. Since $s, r' < 0$, it suffices to show that $W(x)$ is increasing over $[0, \frac{1}{2|r'|}]$. We rewrite $W$ in the following way

$$W(x) = W_1(x) + W_2(x),$$

where

$$W_1(x) = xy \left( \frac{1}{1 + ks(1 + x)} + \frac{1}{1 + ks(1 + y)} \right),$$

and

$$W_2(x) = xy \left( \frac{1}{(1 + ks(1 + x))^2} + \frac{1}{(1 + ks(1 + y))^2} \right).$$

(98)

(99)

We will show that both $W_1(x)$ and $W_2(x)$ are increasing on $[0, \frac{1}{2|r'|}]$. Now let us focus on $W_1$. Since $y = \frac{1}{|r'|} - x$, it is easy to see that

$$W_1'(x) = \left( \frac{1}{|r'|} - 2x \right) \left( \frac{1}{1 + ks(1 + x)} + \frac{1}{1 + ks(1 + y)} \right) - kxy \left( \frac{1}{(1 + ks(1 + x))^2} - \frac{1}{(1 + ks(1 + y))^2} \right).$$

(100)

Let us denote

$$a \triangleq a(x) = 1 + ks(1 + x)$$

(101)

$$b \triangleq b(x) = 1 + ks(1 + y) = 1 + ks \left( \frac{1}{|r'|} - x + 1 \right).$$

(102)

The condition $r > -sd$ implies that $a, b \geq 0$. With these notations, we have

$$W_1'(x) = \left( \frac{1}{a} + \frac{1}{b} \right) \left( \frac{1}{|r'|} - 2x \right) - kxy \left( \frac{1}{a} - \frac{1}{b} \right)$$

(103)

$$= \left( \frac{1}{a} + \frac{1}{b} \right) \left( \frac{1}{|r'|} - 2x \right) \left( 1 - (ks)^2 xy \right),$$

(104)

where the last identity follows from

$$\frac{1}{a} - \frac{1}{b} = ks \frac{1}{ab} \left( \frac{1}{|r'|} - 2x \right).$$

(105)

Since $a, b \geq 0$ and $x \in [0, \frac{1}{2|r'|}]$, it suffices to show that

$$ab - (ks)^2 xy \geq 0.$$  

(106)

Using (101) and (102), we have

$$ab - (ks)^2 xy = (1 + ks) \left( 1 + \frac{ks}{r} \right).$$

(107)

Then the desired statement follows from that $s > -1/d$ and $r > -sd$. We conclude that $W_1$ is increasing on $[0, \frac{1}{2|r'|}].$
It remains to show that \(W_2(x)\) is increasing on \([0, \frac{1}{2|r'|}]\). Recall the definition of \(W_2(x)\) in (139), it is easy to check that
\[
W_2'(x) = \left(\frac{1}{|r'|} - 2x\right) \left(\frac{1}{a^2} + \frac{1}{b^2}\right) - 2ksxy \left(\frac{1}{a^2} - \frac{1}{b^2}\right)
\]
(108)
\[
= \frac{b - a}{ks} \left(\frac{1}{a^2} + \frac{1}{b^2}\right) - 2ksxy \left(\frac{1}{a^2} - \frac{1}{b^2}\right)
\]
(109)
\[
= \frac{b - a}{ksa^3b^3}T(x),
\]
(110)
where
\[
T(x) = ab(a^2 + b^2) - 2k^2s^2xy(a^2 + ab + b^2).
\]
(111)
Since
\[
\frac{b - a}{ks} = \frac{1}{|r'|} - 2x \geq 0, \quad x \in \left[0, \frac{1}{2|r'|}\right],
\]
(112)
it suffices to show that \(T(x) \geq 0\) for \(x \in \left[0, \frac{1}{2|r'|}\right]\). Using the identities
\[
a'(x)b(x) + a(x)b'(x) = ks(b - a) = -a(x)a'(x) - b(x)b'(x),
\]
(113)
one can check that
\[
T'(x) = ks(a-b)U(x),
\]
(114)
where
\[
U(x) = a^2 + b^2 + 4ab - 2k^2s^2xy.
\]
(115)
Notice that \(U'(x) \equiv 0\), which implies that \(U(x)\) is a constant. Since \(a, b \geq 0\), we have
\[
U(0) = a^2 + b^2 + 4ab > 0.
\]
(116)
Hence, \(T'(x) \leq 0\), i.e., \(T(x)\) is decreasing. Therefore, since \(a = b\) when \(x = \frac{1}{2|r'|}\), we have
\[
T(x) \geq T\left(\frac{1}{2|r'|}\right) = 2a^2(a^2 - 3k^2s^2x^2) \quad \text{at } x = \frac{1}{2|r'|},
\]
(117)
It suffices to have
\[
a^2 \geq 3k^2s^2x^2, \quad x = \frac{1}{2|r'|},
\]
(118)
which is equivalent to
\[
\frac{1}{|r'|} \leq \frac{2}{1 + \sqrt{3}} \left(\frac{1}{ks} - 1\right).
\]
(119)
This finishes the proof that every \(-g_k(\lambda)/H(\lambda)\) is also increasing on \([0, 1/2]\). Then the numerical value of \(\alpha\) in theorem 3 follows from setting \(\lambda = 1/2\) in (91).

**Remark 22.** Our optimization argument heavily relies on that \(-A(\lambda)/H(\lambda)\) and \(-g_k(\lambda)/H(\lambda)\) are monotonically increasing for \(\lambda \in [0, 1/2]\). As observed in [23], the monotonicity of \(-A(\lambda)/H(\lambda)\) does not depend on the value of \(r\). Numerical examples show that \(-g_k(\lambda)/H(\lambda)\) is not monotone when \(r\) is small. This is one of the reasons for the restriction \(r > r_0\).

**Remark 23.** Note that the condition \(r > -sd\) of Theorem 18 can be rewritten as
\[
\frac{1}{|r'|} < \left(\frac{1}{d|s|} - 1\right).
\]
(120)
We do not know whether Theorem 4 holds when
\[
\frac{2}{1 + \sqrt{3}} \left(\frac{1}{d|s|} - 1\right) < \frac{1}{|r'|} < \left(\frac{1}{d|s|} - 1\right).
\]
(121)
4 An entropic characterization of s-concave densities

Let \( X \) and \( Y \) be real-valued random variables (possibly dependent) with the fixed identical density \( f \). Cover and Zhang \([19]\) proved that

\[
h(X + Y) \leq h(2X)
\]  
(122)

holds for every coupling of \( X \) and \( Y \) if and only if \( f \) is log-concave. This gives an entropic characterization of one-dimensional log-concave densities. We will extend Cover and Zhang’s result to Rényi entropies of random vectors with \( s \)-concave densities (defined in \([9]\)), which particularly include log-concave densities as a special case. This was previously proved in \([27]\) under the stronger assumption that \( f \) is continuous.

Firstly, we introduce some classical variations of convexity and concavity which will be used later in our proof.

**Definition 24.** For a fixed \( \lambda \in (0, 1) \), a function \( f : \mathbb{R}^d \to \mathbb{R} \) with convex support is called almost \( \lambda \)-convex if

\[
f((1 - \lambda)x + \lambda y) \leq (1 - \lambda)f(x) + \lambda f(y)
\]  
(123)

holds for almost every pair \( x, y \) in the domain of \( f \). We say that \( f \) is \( \lambda \)-convex if inequality \((123)\) holds for any pair \( x, y \) in the domain of \( f \). We say that \( f \) is convex if \( f \) is \( \lambda \)-convex for any \( \lambda \in (0, 1) \).

The above definition is equivalent to that \( f \) satisfies the following inequality

\[
f(\lambda x + (1 - \lambda)y) + f((1 - \lambda)x + \lambda y) \leq f(x) + f(y).
\]  
(124)

One can define almost \( \lambda \)-concavity, \( \lambda \)-concavity, and concavity by reversing the inequalities \((123)\) or \((124)\). Theorem 1 of \([1]\) implies that an almost \( \lambda \)-convex function is identical to a \( \lambda \)-convex function except on a set of Lebesgue measure 0. (For the theorem there, one can take the ideals \( I_1 \) and \( I_2 \) as the family of sets with Lebesgue measure 0 in \( \mathbb{R}^d \) and \( \mathbb{R}^{2d} \), respectively). In general, \( \lambda \)-convexity is not equivalent to convexity, as it is not a strong enough notion to imply continuity, at least not in a logical framework that accepts the axiom of choice. Indeed, counterexamples can be constructed using a Hamel basis for \( \mathbb{R} \) as a vector space over \( \mathbb{Q} \). However, in the case that \( f \) is Lebesgue measurable, a classical result of Blumberg \([6]\) and Sierpinski \([37]\) (see also \([17]\) in more general setting) shows that \( \lambda \)-convexity implies continuity thus convexity.

**Theorem 25.** Let \( s > -1/d \). A random vector \( X \) in \( \mathbb{R}^d \) has density \( f \) being \( s \)-concave if and only if

\[
h_r(\lambda X + (1 - \lambda)Y) \leq h_r(X)
\]  
(125)

holds for any \( \lambda \in (0, 1) \) and \( r = 1 + s \). Equality is achieved when \( X \equiv Y \).

**Proof.** We only prove the statement for \( s > 0 \), equivalently \( r > 1 \). The proof for \(-1/d < s < 0\), equivalently \( 1 - 1/d < r < 1 \), is similar and sketched below.

**Sufficiency:** Let \( g \) be the density of \( \lambda X + (1 - \lambda)Y \). Then we have

\[
h_r(X) = \frac{1}{1-r} \log \mathbb{E}f^{-1}(X)
\]  
(126)

\[
= \frac{1}{1-r} \log(\lambda \mathbb{E}f^{-1}(X) + (1 - \lambda)\mathbb{E}f^{-1}(Y))
\]  
(127)

\[
\geq \frac{1}{1-r} \log \mathbb{E}f^{-1}(\lambda X + (1 - \lambda)Y)
\]  
(128)

\[
= \frac{1}{1-r} \log \int_{\mathbb{R}^d} f(x)^{-1}g(x)dx
\]  
(129)

\[
\geq \frac{1}{1-r} \log \left(\int_{\mathbb{R}^d} f(x)^r dx\right)^{1-\frac{1}{r}} \left(\int_{\mathbb{R}^d} g(x)^r dx\right)^{\frac{1}{r}}
\]  
(130)

\[
= \frac{r-1}{r} h_r(g(X)) + \frac{1}{r} h_r(\lambda X + (1 - \lambda)Y).
\]  
(131)
This is equivalent to the desired result. The second identity follows from the assumption that $X$ and $Y$ have the same distribution. In the first inequality, we use the concavity of $f^{r-1}$ and that $\frac{1}{1-r} \log x$ is decreasing when $r > 1$. The second inequality follows from Hölder’s inequality and that $\frac{1}{1-r} \log x$ is decreasing when $r > 1$. Note that the argument also works for $1 - 1/d < r < 1$ in conjunction with the convexity of $f^{r-1}$, the reverse Hölder inequality and that $\frac{1}{1-r} \log x$ is increasing, instead of decreasing when $0 < r < 1$.

**Necessity:** We will prove the statement by contradiction. The following example is borrowed from Cover and Zhang [19]. It might be helpful for the readers to better view the ‘mass transferring’ argument used in the proof. Consider the density $f(x) = 3/2$ in the intervals $(0, 1/3)$ and $(2/3, 1)$. It is clear that $f$ is not $(r - 1)$-concave. The joint distribution of $(X, Y)$ with $Y \equiv X$ concentrates on the diagonal line $y = x$. Its Radon-Nikodym derivative $g$ with respect to the Lebesgue measure on the line $y = x$ exists and is shown in Fig. 1. We move a part of the values of this Radon-Nikodym derivative to the lines $y = x - 2/3$ and $y = x + 2/3$. The new Radon-Nikodym derivative $\hat{g}$ is shown in Fig. 2. Let $(\hat{X}, \hat{Y})$ be a pair of random variables whose joint distribution possesses this new Radon-Nikodym derivative. It is easy to see that $\hat{X}$ and $\hat{Y}$ still have the same density $f$. But $\hat{X} + \hat{Y}$ is uniformly distributed on $(0, 2)$, and $h_r(\hat{X} + \hat{Y}) = \log 2$. One can check that $h_r(2X) = \log(4/3)$.

Now we turn to the general case. Suppose that $f$ is not $(r - 1)$-concave, i.e., $f^{r-1}$ is not concave (for $r > 1$). Invoke the equivalence of $\lambda$-concavity and concavity for Lebesgue measurable probability density functions. For any $\lambda \in (0, 1)$ there exists $(x, y)$ such that the following inequality holds

$$f^{r-1}(x) + f^{r-1}(y) > f^{r-1}(\lambda x + (1 - \lambda)y) + f^{r-1}((1 - \lambda)x + \lambda y). \quad (132)$$

We denote by $A_\lambda$ the collection of such pairs $(x, y)$. If $m(A_\lambda) = 0$, i.e., the reverse of (132) holds almost everywhere, then $f^{r-1}$ is an almost $\lambda$-concave function, and $f^{r-1}$ is identical to a $\lambda$-concave function except on a set of the Lebesgue measure 0. Without changing the distribution of $X$, we can modify $f$ such that $f^{r-1}$ is $\lambda$-concave. Using the equivalence of $\lambda$-concavity and concavity, after modification $f^{r-1}$ is concave, i.e., $f$ is $(r - 1)$-concave. Therefore, if $f$ is not $(r - 1)$-concave, for any $\lambda \in (0, 1)$ we must have $m(A_\lambda) > 0$. Then for any $\lambda \in (0, 1)$ there exists $(x, y)$ such that (132) holds for a set of $x$ with positive measure. We rephrase the argument in a form suitable for our purpose. For any $\lambda \in (0, 1)$, there is $x_0 \neq 0$ such that the set

$$\Lambda = \{ x \in \mathbb{R}^d : f(x + x_0)^{r-1} + f(x - x_0)^{r-1} > f(x + (2\lambda - 1)x_0)^{r-1} + f(x - (2\lambda - 1)x_0)^{r-1} \} \quad (133)$$

has positive measure. For $\epsilon > 0$, we denote by $\Lambda(\epsilon)$ a ball of radius $\epsilon$ whose intersection with $\Lambda$ has positive measure. Let $g(x, y)$ be the Radon-Nikodym derivative (with respect to the Lebesgue measure on $\mathbb{R}^d$) of the joint distribution of $(X, Y)$ such that $X \equiv Y$. (So $g$ is supported on the ‘diagonal line’ $y = x$). Now we build a new density $\hat{g}$ by translating a small amount of ‘mass’ from ‘diagonal’ points $(x - x_0, x - x_0)$ and $(x + x_0, x + x_0)$ to ‘off-diagonal’ points $(x - x_0, x + x_0), (x + x_0, x - x_0)$. To be
more precise, we define the new joint density \( \hat{g} \) as

\[
\hat{g}(x, y) = g(x, y) + \sqrt{d(\lambda^2 + (1 - \lambda)^2)} \delta \left( 1_{\{x-x_0,x+x_0\} \in \Lambda(\epsilon)} \right)
\]

(134)

\[
+ 1_{\{x+x_0,x-x_0\} \in \Lambda(\epsilon)} - 1_{\{x-x_0,x-x_0\} \in \Lambda(\epsilon)}
\]

(135)

\[
- 1_{\{x+x_0,x+x_0\} \in \Lambda(\epsilon)}
\]

(136)

where \( \delta > 0 \) and \( 1_A \) is the indicator function of the set \( A \). When \( \delta > 0 \) is small enough, \( \hat{g}(x, y) \) is non-negative everywhere. Furthermore, our construction preserves the ‘total mass’. Hence, the function \( \hat{g}(x, y) \) is indeed a probability density. Let \((\hat{X}, \hat{Y})\) be a pair with joint density \( \hat{g}(x, y) \). The marginals \( X \) and \( Y \) have the same distribution as that of \( X \), since the ‘positive mass’ on ‘off-diagonal’ points cancels the ‘negative mass’ on ‘diagonal’ points when we project in the \( x \) and \( y \) direction. We claim that the pair \((\hat{X}, \hat{Y})\) has larger entropy \( h_r(\lambda X + (1 - \lambda) Y) \). The density of \( \lambda X + (1 - \lambda) Y \) is

\[
\hat{f}(x) = f(x) + \delta \left( 1_{\Lambda(\epsilon)} + (2\lambda - 1)x_0 + 1_{\Lambda(\epsilon)} - (2\lambda - 1)x_0 - 1_{\Lambda(\epsilon)} + x_0 - 1_{\Lambda(\epsilon)} - x_0 \right).
\]

(137)

Let \( \Omega \) denote the union of \( \Lambda(\epsilon) + (2\lambda - 1)x_0, \Lambda(\epsilon) - (2\lambda - 1)x_0, \Lambda(\epsilon) + x_0 \) and \( \Lambda(\epsilon) - x_0 \). Then we have

\[
h_r(\lambda \hat{X} + (1 - \lambda) \hat{Y}) = \frac{1}{1 - r} \log \left( \int_{\Omega} \hat{f}(x)^r dx + \int_{\Omega^c} f(x)^r dx \right).
\]

(138)

For \( \epsilon > 0 \) small, since \( x_0 \neq 0 \), \( \Omega \) is the union of disjoint translates of \( \Lambda(\epsilon) \). When \( \delta > 0 \) is sufficiently small, we have

\[
\int_{\Omega} \hat{f}(x)^r dx = \int_{\Lambda(\epsilon)} \left[ (f(x - (2\lambda - 1)x_0) + \delta)^r + (f(x + (2\lambda - 1)x_0) + \delta)^r \right] dx
\]

(139)

\[
+ (f(x - x_0) - \delta)^r + (f(x - x_0) - \delta)^r dx
\]

(140)

\[
< \int_{\Lambda(\epsilon)} [f(x - (2\lambda - 1)x_0)^r + f(x + (2\lambda - 1)x_0)^r + f(x + x_0)^r + f(x - x_0)^r] dx
\]

(141)

\[
= \int_{\Omega} f(x)^r dx,
\]

(142)

where inequality (141) follows from the observation that for \( x \in \Lambda(\epsilon) \) the derivative of the integrand at \( \delta = 0 \) is

\[
r \left[ (f(x - (2\lambda - 1)x_0))^{r-1} + (f(x + (2\lambda - 1)x_0))^{r-1} - f(x - x_0)^{r-1} - f(x + x_0)^{r-1} \right] < 0.
\]

(143)

Since \( r > 1 \), we have

\[
h_r(\lambda \hat{X} + (1 - \lambda) \hat{Y}) > \frac{1}{1 - r} \log \left( \int_{\Omega} f(x)^r dx + \int_{\Omega^c} f(x)^r dx \right) = h(X).
\]

(144)

In the case \( 0 < r < 1 \), we define the set \( \Lambda \) by revering the inequality, and inequality (141) will be also reversed. We will arrive at the same result.

\[\square\]

Remark 26. We mention that the proof of sufficiency is an immediate consequence of Theorem 3.36 in [29]. The theorem there draws heavily on the ideas of [39], where a related study, deriving the Schur convexity of Rényi entropies under the assumption of exchangeability and \( s \)-concavity of the random variables, generalizing Yu’s results in [40] on the entropies of sums of i.i.d. log-concave random variables.

Taking \( \lambda = 1/2 \) in Theorem 25, we have the following reverse Rényi EPI for dependent random variables with the same distribution.

Corollary 27. Let \( s > -1/d \). Let \( X \) and \( Y \) be (possibly dependent) random vectors in \( \mathbb{R}^d \) with the same density \( f \) being \( s \)-concave. Then we have

\[
N_r(X + Y) \leq 4N_r(X), \tag{145}
\]

where \( r = 1 + s \).
Theorem 25 also implies the following seemingly stronger form.

**Theorem 28.** Let $f$ be a probability density function on $\mathbb{R}^d$. Then

$$\sup_{X_i \sim f} h_r \left( \sum_{i=1}^{n} \lambda_i X_i \right) = h_r(X_1)$$

(146)

holds for all $\lambda_i \geq 0$ such that $\sum_{i=1}^{n} \lambda_i = 1$ if and only if the density $f$ is $(r - 1)$-concave.

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