Massless spin 2 interacting with massive higher spins in $d = 3$

Yu. M. Zinoviev

Institute for High Energy Physics of National Research Center “Kurchatov Institute”, Protvino, Moscow Region, 142281, Russia

E-mail: yuri.zinoviev@ihep.ru

ABSTRACT: In this paper we consider massless spin 2 interacting with the massive arbitrary spin fermions in $d = 3$. First of all, we study all possible deformations for the massive fermion unfolded equations in presence of a massless spin-2 field. We find three linearly independent solutions one of which corresponds to the standard gravitational interactions. Then for all three cases we reconstruct appropriate Lagrangian formulation.

KEYWORDS: Field Theories in Lower Dimensions, Gauge Symmetry, Higher Spin Gravity, Higher Spin Symmetry

ArXiv ePrint: 2211.09405
1 Introduction

For a long time three dimensional space serves as a nice playground for the investigation of higher spin interactions. The most well known example is a Blencowe theory [1], describing interactions for infinite set of massless bosonic and fermionic fields. In the metric-like formalism a complete classification of the cubic interaction vertices for the bosonic fields was elaborated in [2, 3]. For the spins $s \geq 2$ it was shown that for any three spins $s_1, s_2, s_3$ satisfying a so-called triangular inequality $s_i + s_{i+1} > s_{i+2}$ there exist just one parity even and one parity odd vertex. Using a frame-like formalism we confirmed these results and extended them including massless fermions with spins $s \geq 3/2$ as well [4]. In the frame-like formalism it is evident that it is not possible to construct any quartic or higher order vertices.\(^\text{1}\) In spite of this fact, there exists a large number of closed models with a finite number of higher spin fields (see e.g. [6–9]).

Recently, a complete classification for the cubic vertices for massive fields was developed in the light-cone formalism [10] (see also [11]). The main result is that for any three spins there exists at least one (sometimes two) cubic vertex. At the same time, in our analyse based on the frame-like formalism [4] the triangular inequality also appeared. The Lorentz covariant formulation of the complete set of cubic vertices for the massive fields and their relation with the classification of [10] is still an open question.

\[^\text{1}\text{For the general discussion on the relation between frame-like and metric-like formalism see [5].}\]
One more interesting and important case — interactions of massless and massive higher spins. Recall that the massless fields in $d = 3$ do not have any physical degrees of freedom, so they simply do not exist in the light-cone formalism. That is why the classification of [10] deals with the massive fields only. Till now the most important result is a Prokushkin-Vasiliev theory [12, 13] which contain a complete set of non-linear unfolded equations describing interaction of the infinite set of massless bosonic and fermionic fields with massive scalars and spinors. In [4] working both with Lagrangian and unfolded formulations we consider possible generalization to the massive higher spins. Once again we have found that the triangular inequality is necessary.

Our analysis was based on the frame-like gauge invariant formalism for the massive bosonic and fermionic fields [14, 15], where gauge invariance is achieved due to the introduction of appropriate number of Stueckelberg fields. In such approach there appears a huge ambiguity related with the (higher derivative) field redefinitions [16–18] when one and the same theory may appear in its non-abelian, abelian or trivially gauge invariant incarnations. In particular, we faced this problem trying to construct a Lagrangian formulation for massive higher spin supermultiplet. The solution found was based on the usage of the unfolded formulation for the massive higher spins [19, 20]. Recall, that such formulation contains three sets of unfolded equations: for the one-forms, for the Stueckelberg zero-forms and for the gauge invariant zero-forms. In this approach to find supertransformations one has to consider a deformation of the unfolded equations in presence of a massless spin-3/2 field. In the sector of the gauge invariant zero-forms such procedure appears to be non-ambiguous and consistency allows one to promote these results on two other sectors [21]. After that we managed to reconstruct the Lagrangian formulation for these supermultiplets [22, 23].

In this work we follow the same strategy for the case of massless spin 2 field interacting with the massive higher spins. Namely, we begin with the deformation of the unfolded equations in presence of a massless spin-2 in the sector of the gauge invariant zero-forms. We found three solutions one of which (naturally) corresponds to the standard gravitational interaction. Then by consistency we promote these solution to the two other sectors. At last we show that in all three cases it is possible to reconstruct a Lagrangian formulation for the appropriate cubic vertices. The gravitational interactions for massive higher spin bosons were considered previously [24, 25], so in this work we restrict ourselves with the fermionic fields only.

The layout of the paper is simple. In section 2 we provide all necessary information on the Lagrangian and unfolded formulation for the massive higher spin fermions. Section 3 devoted to the deformation of the unfolded equations, while in section 4 Lagrangian formulation is reconstructed.

Notation and conventions. We work in the frame-like multispinor formalism where all objects are forms having a number of completely symmetric spinor indices (see [23] for details). A coordinate free description of flat three dimensional space is given by the back-
ground frame one-form $e^{k(2)}$ and a background Lorentz covariant derivative $D$ such that

$$D \wedge D = 0, \quad D \wedge e^{k(2)} = 0.$$ 

Also we use two and three forms defined as

$$e^{\alpha(2)} \wedge e^{\beta(2)} = \varepsilon^{\alpha\beta} E^{\alpha\beta}, \quad E^{\alpha(2)} \wedge e^{\beta(2)} = \varepsilon^{\alpha\beta} e^{\alpha\beta} E.$$ 

The massless spin 2 is described by a physical one-form $H^{\alpha(2)}$ and an auxiliary one-form $\Omega^{\alpha(2)}$ with the free Lagrangian

$$L_0 = \Omega_{\alpha\beta} e^{\beta(2)} \Omega^{\alpha(2)} + \Omega_{\alpha(2)} DH^{\alpha(2)},$$

which is invariant under the following gauge transformations

$$\delta \Omega^{\alpha(2)} = D\eta^{\alpha(2)}, \quad \delta H^{\alpha(2)} = D\xi^{\alpha(2)} + \varepsilon^{\alpha\beta} \eta^{\alpha\beta}.$$ 

Equations of motion coincide with the zero gauge invariant curvature equations

$$R^{\alpha(2)} = D\Omega^{\alpha(2)} = 0, \quad T^{\alpha(2)} = DH^{\alpha(2)} + \varepsilon^{\alpha\beta} \Omega^{\alpha\beta} = 0.$$ 

2 Massive spin-(s+1/2) fermion

Gauge invariant formulation of the massive spin-(s + 1/2) fermions $s \geq 1$ in $d = 3$ [15, 22] requires a set of one-forms $\Phi_{\alpha(2k+1)}$, $0 \leq k \leq s - 1$ and zero-form $\phi^{\alpha}$. The free Lagrangian:

$$\begin{align*}
\frac{1}{4} L_0 &= \sum_{k=0}^{s-1} \left( \frac{2s+1}{2} \right)^{k+1} \Phi_{\alpha(2k+1)} D\Phi_{\alpha(2k+1)} + \frac{1}{2} \varepsilon_{\alpha(2)} E^{\alpha(2)} D\phi^{\alpha} \\
&\quad + \sum_{k=1}^{s-1} (-1)^{k+1} \phi_{\alpha(2k)} e^{(2)}(2) \Phi_{\alpha(2k+1)} + a_0 \Phi_{\alpha} E^{\alpha} \phi^{\alpha} \\
&\quad + \sum_{k=0}^{s-1} (-1)^{k+1} \frac{b_k}{2} \phi_{\alpha(2k)} e^{(2)}(2) \Phi_{\alpha(2k+1)} - \frac{3b_0}{2} E\phi_{\alpha} \phi^{\alpha}.
\end{align*}$$

(2.1)

Here

$$b_k = \frac{(2s + 1)}{(2k + 3)}, \quad a_k^2 = \frac{(s + k + 1)(s - k)}{2(k + 1)(2k + 1)} M^2, \quad a_0^2 = 2s(s + 1)M^2. \quad (2.2)$$

This Lagrangian is invariant under the following gauge transformations with the fermionic parameters $\zeta^{\alpha(2k+1)}$, $0 \leq k \leq s - 2$:

$$\begin{align*}
\delta_0 \Phi^{\alpha(2s-1)} &= D\zeta^{\alpha(2s-1)} + \frac{b_{s-1}}{(2s - 1)} e^{\alpha} \zeta^{\alpha(2s-2)} + \frac{a_{s-1}}{(s - 1)(2s - 1)} \zeta^{\alpha(2s-1)} \\
\delta_0 \Phi^{\alpha(2k+1)} &= D\zeta^{\alpha(2k+1)} + \frac{b_k}{(2k + 1)} e^{\alpha} \zeta^{\alpha(2k+1)} \\
&\quad + \frac{a_k}{k(2k + 1)} e^{\alpha(2)}(2) \zeta^{\alpha(2k+1)} + a_{k+1} e^{\beta(2)}(2) \zeta^{\alpha(2k+1)}(2), \\
\delta_0 \Phi^{\alpha} &= D\zeta^{\alpha} + b_0 e^{\alpha} \zeta^{\alpha} + a_1 e^{\beta(2)}(2) \zeta^{\alpha(2)}, \\
\delta_0 \phi^{\alpha} &= a_0 \zeta^{\alpha}.
\end{align*}$$

(2.3)

Here and in what follows all fermionic objects are assumed to be anti-commuting quantities.
Note that such formalism admits an infinite spin limit \( s \to \infty, M \to 0, Ms = \mu = \text{const} \) [26] (see [27] for review):\)
\[
b_k = \frac{2\mu}{(2k + 3)}, \quad a_k^2 = \frac{\mu^2}{2(k + 1)(2k + 1)}, \quad a_0^2 = 2\mu^2. \quad (2.4)
\]
Note also that in some cases it appears to be convenient to work with partially gauge fixed version then one sets \( \phi^\alpha = 0 \). Namely, the invariance of the Lagrangian under the \( \zeta^\alpha \) transformations implies the following relation
\[
D \frac{\delta L_0}{\delta \Phi_\alpha'} + b_0 e^\alpha_\beta \frac{\delta L_0}{\delta \Phi_\beta} - a_1 e^\beta_\gamma \frac{\delta L_0}{\delta \Phi_\alpha(2\beta)} = a_0 \frac{\delta L_0}{\delta \phi_\alpha}. \quad (2.5)
\]
Thus the equation for the field \( \phi_\alpha \) follows from the equations of other fields. In turn, this means that we can consistently set \( \phi^\alpha = 0 \) (see e.g. discussion in [29]). In this case the Lagrangian is still invariant under the transformations with the parameters \( \zeta^{\alpha(2k+1)} \), \( 1 \leq k \leq s - 1 \), while instead of \( \zeta^\alpha \)-invariance we have a constraint:\)
\[
0 \approx D \frac{\delta L_0}{\delta \Phi_\alpha'} + b_0 e^\alpha_\beta \frac{\delta L_0}{\delta \Phi_\beta} - a_1 e^\beta_\gamma \frac{\delta L_0}{\delta \Phi_\alpha(2\beta)} = -a_0^2 E^\alpha_\beta \phi^\beta. \quad (2.6)
\]
In [28] we proved that such formalism does describe one physical degree of freedom with mass \( M \) and helicity \( s + 1/2 \).

For each one-form \( \Phi^{\alpha(2k+1)} \) there exists a gauge invariant two-form \( \mathcal{F}^{\alpha(2k+1)} \):
\[
\mathcal{F}^{\alpha(2k+1)} = D\Phi^{\alpha(2k+1)} + \frac{b_k}{2(k + 1)} e^\alpha_\beta \Phi^{(2k)\beta} + \frac{a_k}{k(2k + 1)} e^{(2)\alpha} \Phi^{(2k-1)} + a_{k+1} e^{(2)\beta} \Phi^{(2k+1)\beta}, \quad (2.7)
\]
But to construct a gauge invariant one-form for the Stueckelberg zero-form \( \phi^\alpha \) one has introduce an extra zero-form \( \phi^{(3)} \) playing the role of Stueckelberg field for the \( \zeta^{\alpha(3)} \) transformation
\[
\delta \phi^{(3)} = a_0 \zeta^{(3)}. \quad (2.8)
\]
Then, to construct a gauge invariant one-form for \( \phi^{(3)} \) one has introduce an extra zero-form \( \phi^{(5)} \) and so on. The procedure stops at \( \phi^{(2s-1)} \) so that the complete set of gauge invariant one-forms looks like:
\[
\begin{align*}
\mathcal{C}^\alpha &= D\phi^\alpha - a_0 \Phi^\alpha + b_0 e^\alpha_\beta \phi^\beta + c_1 e^\beta_\gamma \phi^{(2)\gamma}, \\
\mathcal{C}^{(2k+1)} &= D\phi^{(2k+1)} - a_0 \Phi^{(2k+1)} + \frac{b_k}{2(k + 1)} e^\alpha_\beta \phi^{(2k+1)\beta} \\
&\quad + \frac{a_k}{k(2k + 1)} e^{(2)\alpha} \phi^{(2k-1)} + a_{k+1} e^{(2)\gamma} \phi^{(2k+1)\gamma}(2.8), \\
\mathcal{C}^{(2s-1)} &= D\phi^{(2s-1)} - a_0 \Phi^{(2s-1)} + \frac{b_{s-1}}{(2s - 1)} e^\alpha_\beta \phi^{(2s-1)\beta} \\
&\quad + \frac{a_{s-1}}{(s - 1)(2s - 1)} e^{(2)\alpha} \phi^{(2s-3)},
\end{align*}
\]
\[\text{In the spin-tensor formalism this constraint is equivalent to the } \gamma^\mu \Phi_\mu = 0.\]
where
\[ \delta \phi^{(2k+1)} = a_0 \phi^{(2k+1)}. \] (2.9)

Unfolded equations for the massive fermions [21] (see [19, 20] for the bosonic fields) look as follows. On-shell all gauge invariant two-forms \( F \approx 0 \) and all gauge invariant one-forms \( C \approx 0 \) except the highest one:
\[ 0 \approx C^{(2s-1)} + e^{(2)} \phi^{(2s-1)\beta(2)} \] (2.10)

Here the zero-form \( \phi^{(2s+1)} \) is the first representative of an infinite chain of the gauge invariant zero-forms \( \phi^{(2k+1)} \), \( k \geq s \), with the corresponding unfolded equations
\[ 0 = D\phi^{(2k+1)} + e_{(2)}^{\alpha} \phi^{(2k+1)\beta(2)} + c_k e^{\alpha} \beta \phi^{(2k)\beta} + d_k e^{(2)} \phi^{(2k-1)}, \] (2.11)

where
\[ c_k = \frac{(2s+1)M}{(2k+1)(2k+3)}, \quad d_k = -\frac{(k-s)(k+s+1)}{2k(2k+1)^2} M^2. \] (2.12)

Thus the complete set of the unfolded equations contains three sectors: for the one-forms, for the Stueckelberg zero-forms and for the gauge invariant zero-forms.

Note that for \( s = 0 \) eq. (2.11) provides a full set of the unfolded equations for massive spin-1/2 field (no gauge one-forms or Stueckelberg zero-forms in this case). The free Lagrangian for this field in our formalism looks like:
\[ \frac{1}{4} \mathcal{L}_0 = \frac{1}{2} \mathcal{E}^{\alpha} \beta D\phi^{\beta} - \frac{M}{2} \mathcal{E} \phi^{\alpha}. \] (2.13)

3 Deformation of the unfolded equations

Recall that the Prokushkin-Vasiliev theory [12, 13] contains a complete system of the non-linear unfolded equations describing an interaction of massive scalars and spinors with the infinite set of the massless higher spin fields. In this section we consider a deformation of the unfolded equations for the massive higher spin fermions in the presence of massless spin-2 field as a very first step towards a possible generalization of this theory. We begin with the sector of the gauge invariant zero-forms and then try to promote the results into the sectors of the Stueckelberg zero-forms and the one-forms.

3.1 Sector of the gauge invariant zero-forms

Let us consider two fermions: \( \phi \) with mass \( M_1 \) and spin \( s_1 \) and \( \psi \) with mass \( M_2 \) and spin \( s_2 \). The most general ansatz for the deformed unfolded equations of the first fermion has the form (see [4] for general discussion):
\[ 0 = D\phi^{(2k+1)} + e_{(2)}^{\alpha} \phi^{(2k+1)\beta(2)} + c_k e^{\alpha} \beta \phi^{(2k)\beta} + d_k e^{(2)} \phi^{(2k-1)} + f_{1,k} H^{\alpha(2k)^\beta(2)} + f_{2,k} H^{(2)} e^{\alpha(2k)\beta} + f_{3,k} H^{(2)\alpha(2k-1)} + g_{1,k} \Omega^{\alpha(2k)^\beta(2)} + g_{2,k} \Omega^{(2)\alpha(2k)\beta} + g_{3,k} \Omega^{(2)\alpha(2k-1)}. \] (3.1)

Similarly for the second fermion with \( \tilde{c}, \tilde{d}, \tilde{g} \) and \( \tilde{f} \). First of all, consistency requires that masses must be equal \( M_1 = M_2 = M \) while spins may be equal \( s_1 = s_2 \) or differ by one unit \( s_1 = s_2 \pm 1 \). We have found three linearly independent solutions.
Case I — gravity. The two spins are equal so we can restrict ourselves with just one fermion. The solution can be written as

\[ 0 = \hat{D}\phi^{(2k+1)} + e_{(2)}\phi^{(2k+1)\beta(2)} + c_k e_{\beta}\phi^{(2k+2)} + d_k \epsilon^{(2)} \phi^{(2k-1)}, \]

where

\[ \hat{D}\phi^{(2k+1)} = D\phi^{(2k+1)} + g_0\Omega^{\beta}\phi^{(2k+2)}, \quad \epsilon^{(2)} = \epsilon^{(2)} + g_0H^{(2)} \]

This solution (which works for the case \( s = 0 \) as well) corresponds to the standard minimal gravitational interaction (hence the name). Indeed, in the next approximation we need

\[ 0 = D\Omega^{(2)} + \frac{g_0}{2} \Omega^{\alpha}\Omega^{\beta}, \]

\[ 0 = DH^{(2)} + (e^{(2)} + g_0H^{(2)})\Omega^{\alpha}\]

Case II — non-gravity 1. Here two spins are also equal and we consider one fermion only. Then the solution has the form \((k \geq s, s \geq 0)\)

\[ 0 = D\phi^{(2k+1)} + e_{(2)}\phi^{(2k+1)\beta(2)} + c_k e_{\beta}\phi^{(2k+2)} + d_k \epsilon^{(2)} \phi^{(2k-1)}, \]

where

\[ \epsilon^{(2)} = \epsilon^{(2)} + g_0\Omega^{(2)}. \]

No deformations for spin 2 equations in this case.

Case III — non-gravity 2. Now spins \( s_1 = s + 1/2 \) and \( s_2 = s - 1/2 \). General solution (which works for the special case \( s = 1 \) when \( s_1 = 3/2 \) and \( s_2 = 1/2 \) has the form:

\[ 0 = D\phi^{(2k+1)} + e_{(2)}\phi^{(2k+1)\beta(2)} + c_k e_{\beta}\phi^{(2k+2)} + d_k \epsilon^{(2)} \phi^{(2k-1)} \]

\[ + g_0\Omega^{(2)}\psi^{(2k+1)\beta(2)} + g_{2,k} \Omega^{\beta}\psi^{(2k+2)} + g_{3,k} \Omega^{(2)}\psi^{(2k-1)}, \]

\[ 0 = D\psi^{(2k+1)} + e_{(2)}\psi^{(2k+1)\beta(2)} + c_k e_{\beta}\phi^{(2k+2)} + d_k \epsilon^{(2)} \phi^{(2k-1)} \]

\[ + g_0\Omega^{(2)}\phi^{(2k+1)\beta(2)} + g_{2,k} \Omega^{\beta}\phi^{(2k+2)} + g_{3,k} \Omega^{(2)}\phi^{(2k-1)}, \]

where

\[ g_{2,k} = \frac{2(k + s + 1)M}{(2k + 1)(2k + 3)}g_0, \quad g_{3,k} = \frac{(k + s)(k + s + 1)M^2}{2k(2k + 1)(2k + 2)}g_0, \]

\[ \tilde{g}_{2,k} = -\frac{2(k - s + 1)M}{(2k + 1)(2k + 3)}g_0, \quad \tilde{g}_{3,k} = \frac{(k - s)(k - s + 1)M^2}{2k(2k + 1)(2k + 2)}g_0. \]

Let us stress that in this case the \( \eta^{(2)} \)-transformations have nothing to do with the usual Lorentz transformations because they transform two fields with different spins into each other.

The structure of quadratic terms corresponds to non-gravity 1, so to construct closed theory one can try to consider two massless spin-2 fields (say, \( \omega^{(2)} \) for non-gravity 1 and \( \Omega^{(2)} \) for non-gravity 2) simultaneously. We will not proceed along this line, simply note that in this case we will need something like

\[ 0 = D\omega^{(2)} + \frac{g_0}{2} \Omega^{\alpha}\Omega^{\beta}, \quad 0 = D\Omega^{(2)}. \]
3.2 Gravity

In this case the deformed equations can be straightforwardly written with the standard minimal substitution rules. For the one-forms we obtain (1 \leq k \leq s - 1)

\begin{align*}
0 &= \hat{D} \Phi^a(2k+1) + \frac{b_k}{(2k + 1)} \hat{e}^a_{\beta} \Phi^{(2k)\beta} \\
&\quad + \frac{a_k}{k(2k + 1)} \hat{e}^{\alpha(2)} \Phi^{(2k-1)} + a_{k+1} \hat{e}_{\beta(2)} \Phi^{(2k+1)\beta(2)}, \\
0 &= \hat{D} \Phi^a + b_0 \hat{e}^a_{\beta} \Phi^\beta + a_1 \hat{e}_{\beta(2)} \Phi^{\beta(2)} - \frac{a_0 g_0}{4} e^\alpha_{\beta} H^\beta_\gamma + H^\alpha_{\beta} e^\beta_\gamma \phi^\gamma, \tag{3.12}
\end{align*}

while the deformed equations for the Stueckelberg zero-forms look like (1 \leq k \leq s - 2):

\begin{align*}
0 &= \hat{D} \Phi^a - a_0 \Phi^a + b_0 \hat{e}^a_{\beta} \Phi^\beta + a_1 \hat{e}_{\beta(2)} \Phi^{\beta(2)}, \\
0 &= \hat{D} \Phi^a(2k+1) - a_0 \Phi^a(2k+1) + \frac{b_k}{(2k + 1)} \hat{e}^a_{\beta} \Phi^{(2k)\beta} \\
&\quad + \frac{a_k}{k(2k + 1)} \hat{e}^{\alpha(2)} \Phi^{(2k-1)} + a_{k+1} \hat{e}_{\beta(2)} \Phi^{(2k+1)\beta(2)}, \\
0 &= \hat{D} \Phi^a + b_0 \hat{e}^a_{\beta} \Phi^\beta + a_1 \hat{e}_{\beta(2)} \Phi^{\beta(2)} - \frac{a_0 g_0}{4} e^\alpha_{\beta} \Omega^\beta_\gamma + \Omega^\alpha_{\beta} e^\beta_\gamma \phi^\gamma, \tag{3.13}
\end{align*}

Here \( \hat{D} \) and \( \hat{e} \) are the same as in (3.3).

3.3 Non-gravity 1

Similarly, for the deformed equations of the one-forms we obtain (1 \leq k \leq s - 1)

\begin{align*}
0 &= D \Phi^a(2k+1) + \frac{b_k}{(2k + 1)} \hat{e}^a_{\beta} \Phi^{(2k)\beta} \\
&\quad + \frac{a_k}{k(2k + 1)} \hat{e}^{\alpha(2)} \Phi^{(2k-1)} + a_{k+1} \hat{e}_{\beta(2)} \Phi^{(2k+1)\beta(2)}, \\
0 &= D \Phi^a + b_0 \hat{e}^a_{\beta} \Phi^\beta + a_1 \hat{e}_{\beta(2)} \Phi^{\beta(2)} - \frac{a_0 g_0}{4} e^\alpha_{\beta} \Omega^\beta_\gamma + \Omega^\alpha_{\beta} e^\beta_\gamma \phi^\gamma, \tag{3.14}
\end{align*}

while the deformed equations for the Stueckelberg zero-forms look like (1 \leq k \leq s - 2):

\begin{align*}
0 &= D \Phi^a - a_0 \Phi^a + b_0 \hat{e}^a_{\beta} \Phi^\beta + c_1 \hat{e}_{\beta(2)} \Phi^{\beta(2)}, \\
0 &= D \Phi^a(2k+1) - a_0 \Phi^a(2k+1) + \frac{b_k}{(2k + 1)} \hat{e}^a_{\beta} \Phi^{(2k)\beta} \\
&\quad + \frac{a_k}{k(2k + 1)} \hat{e}^{\alpha(2)} \Phi^{(2k-1)} + a_{k+1} \hat{e}_{\beta(2)} \Phi^{(2k+1)\beta(2)}, \\
0 &= D \Phi^a + b_0 \hat{e}^a_{\beta} \Phi^\beta + a_1 \hat{e}_{\beta(2)} \Phi^{\beta(2)} - \frac{a_0 g_0}{4} e^\alpha_{\beta} \Omega^\beta_\gamma + \Omega^\alpha_{\beta} e^\beta_\gamma \phi^\gamma, \tag{3.15}
\end{align*}

Here \( \hat{e} \) is the same as in (3.6).
3.4 Non-gravity 2

In this case we have to consider two fields with different spins $s_1 = s + 1/2$ and $s_2 = s - 1/2$. Let us begin with the general case $s > 1$ (special case $s = 1$ see below) and consider the following ansatz for the Stueckelberg zero-forms of the first field $(1 \leq k \leq s - 2)$:

\begin{align*}
0 &= D\phi^a - a_0\Phi^a + b_0 e^a_{\beta\phi^\beta} + a_1 e_{\beta(2)}\phi^{a\beta(2)} + h_{1,0}\Omega^a_{\beta}\psi^\beta + h_{3,0}\Omega_{\beta(2)}\psi^{a\beta(2)}, \\
0 &= D\phi^{a(2k+1)} - a_0\Phi^{a(2k+1)} + \frac{b_k}{(2k+1)} e^a_{\beta\phi^{a(2k)\beta}} \\
&+ \frac{a_k}{k(2k+1)} e^{a(2)} e^a_{\beta\phi^{a(2k-1)\beta}} + a_{k+1} e_{\beta(2)}\phi^{a(2k+1)\beta(2)} \\
&+ h_{1,k}\Omega^a_{\beta}\psi^{a(2k)\beta} + h_{2,k}\Omega^{a(2)}\psi^{a(2k-1)\beta} + h_{3,k}\Omega_{\beta(2)}\psi^{a(2k+1)\beta(2)},
\end{align*}

(3.16)

Note that the terms with the coefficients $h_{3,s-2}$, $h_{1,s-1}$ and $h_{3,s-1}$ contain gauge invariant zero-forms of the second fields. Consistency requirement gives:

\begin{align*}
&h_{1,k} = 2\sqrt{2(s + k + 1)(s - k - 1)} M^2 g_0, \quad h_{1,s-1} = \frac{4M s g_0}{(2s-1)(2s+1)}, \\
&h_{2,k} = \frac{1}{k(2k+1)} \sqrt{\frac{(s + k + 1)(s + k)}{(k + k)(k + 1)}} M^2 g_0, \\
&h_{3,k} = \sqrt{\frac{(s - k - 2)(s - k - 1)}{(k + 2)(k + 3)}} M^2 g_0, \quad h_{3,s-2} = a_{s-1} g_0, \quad h_{3,s-1} = g_0.
\end{align*}

(3.17)

This in turn requires the following deformations for the one-forms $(1 \leq k \leq s - 2)$:

\begin{align*}
0 &= D\Phi^{a(2s-1)} + \frac{b_{s-1}}{(2s-1)} e^a_{\beta\Phi^{a(2s-2)\beta}} + \frac{a_{s-1}}{(s-1)(2s-1)} e^{a(2)} e^a_{\beta\Phi^{a(2s-3)\beta}}, \\
&+ \frac{a_{s-1}}{h_{2,s-2}\Omega^{(2)}\Phi^{a(2s-3)\beta}}, \\
0 &= D\Phi^{a(2k+1)} + \frac{b_k}{(2k+1)} e^a_{\beta\Phi^{a(2k)\beta}} \\
&+ \frac{a_k}{k(2k+1)} e^{a(2)} e^a_{\beta\Phi^{a(2k-1)\beta}} + a_{k+1} e_{\beta(2)}\Phi^{a(2k+1)\beta(2)} \\
&+ \frac{a_{k+1}}{h_{1,k}\Omega^a_{\beta}\Phi^{a(2k)\beta} + h_{2,k}\Omega^{a(2)}\Phi^{a(2k-1)\beta} + h_{3,k}\Omega_{\beta(2)}\Phi^{a(2k+1)\beta(2)}]},
\end{align*}

(3.18)

\begin{align*}
0 &= D\Phi^a + b_0 e^a_{\beta\Phi^\beta} + a_1 e_{\beta(2)}\Phi^{a\beta(2)} - a_0 E^a_{\beta\phi^\beta} \\
&+ \frac{a_0}{h_{1,0}\Omega^a_{\beta}\Phi^\beta + h_{3,0}\Omega_{\beta(2)}\Phi^{a\beta(2)} - \frac{a_0 M g_0}{4} (e^a_{\beta\Omega^\gamma \gamma} + e_{\beta\Omega^\gamma\gamma})\phi^\gamma},
\end{align*}
Similarly, the ansatz for the Stueckelberg zero-forms of the second field (1 ≤ k ≤ s − 3):

\[
0 = D\psi^\alpha - \tilde{a}_0 \psi^\alpha + \tilde{b}_0 e^\alpha_\beta \psi^\beta + \tilde{a}_1 e_\beta(2)\psi_\beta(2) + \tilde{h}_{1,0} \Omega^\alpha_\beta \phi^\beta + \tilde{h}_{3,0} \Omega_\beta(2)\phi^\beta(2),
\]

\[
0 = D\psi^\alpha(2k+1) - \tilde{a}_0 \psi^\alpha(2k+1) + \frac{\tilde{b}_k}{(2k + 1)} e^\alpha_\beta \psi^\alpha(2k+1)_\beta
\]

\[
+ \tilde{a}_k + \frac{\tilde{a}_{k+1} e_\beta(2)}{(2k + 1)} \psi^\alpha(2k+1)_\beta(2)
\]

\[
+ \tilde{h}_{1,k} \Omega^\alpha_\beta \phi^\beta(2k+1)_\beta + \tilde{h}_{2,k} \Omega_\beta(2)\phi^\beta(2k+1)_\beta(2) + \tilde{h}_{3,k} \Omega_\beta(2)\phi_\beta(2k+1)_\beta(2),
\]

(3.19)

\[
0 = D\psi^\alpha(2s-3) = \tilde{a}_0 \psi^\alpha(2s-3) + \frac{\tilde{b}_s}{(2s - 3)} e^\alpha_\beta \psi^\alpha(2s-4)_\beta
\]

\[
+ \tilde{a}_{s-2} + \frac{\tilde{a}_{s-1} e_\beta(2)}{(2s - 3)} \psi^\alpha(2s-3)_\beta(2)
\]

\[
+ \tilde{h}_{1,s-2} \Omega^\alpha_\beta \phi^\beta(2s-4)_\beta + \tilde{h}_{2,s-2} \Omega_\beta(2)\phi^\beta(2s-5)_\beta(2) + \tilde{h}_{3,s-2} \Omega_\beta(2)\phi^\beta(2s-3)_\beta(2).
\]

Note that all terms with the coefficients \( \tilde{h} \) contain Stueckelberg zero forms of the first field.

We obtain:

\[
\tilde{h}_{1,k} = \frac{2\sqrt{(s + k + 1)(s - k - 1)}}{(2k + 1)(2k + 3)} \tilde{g}_0.
\]

\[
\tilde{h}_{2,k} = \frac{1}{k(2k + 1)} \sqrt{\frac{(s - k - 1)(s - k)}{2(2k + 1)(2k + 1)}} \tilde{g}_0, \quad k > 0,
\]

(3.20)

\[
\tilde{h}_{3,k} = \sqrt{\frac{(s + k + 2)(s + k + 1)}{2(2k + 2)(2k + 3)}} \tilde{g}_0.
\]

This in turn requires the following deformations for the one-forms (1 ≤ k ≤ s − 2):

\[
0 = D\psi^\alpha(2k+1) + \frac{\tilde{b}_k}{(2k + 1)} e^\alpha_\beta \psi^\alpha(2k+1)_\beta
\]

\[
+ \tilde{a}_k + \frac{\tilde{a}_{k+1} e_\beta(2)}{(2k + 1)} \psi^\alpha(2k+1)_\beta(2)
\]

\[
+ \tilde{a}_0 \tilde{h}_{1,k} \Omega^\alpha_\beta \phi^\beta(2k+1)_\beta + \tilde{h}_{2,k} \Omega_\beta(2)\phi^\beta(2k+1)_\beta(2) + \tilde{h}_{3,k} \Omega_\beta(2)\phi_\beta(2k+1)_\beta(2),
\]

(3.21)

\[
0 = D\psi^\alpha + \tilde{b}_0 e^\alpha_\beta \phi^\beta + \tilde{a}_1 e_\beta(2)\phi^\beta(2) - \tilde{a}_0 \tilde{h}_{1,0} \Omega^\alpha_\beta \phi^\beta(2) + \tilde{h}_{3,0} \Omega_\beta(2)\phi^\beta(2)
\]

\[
+ \frac{\tilde{a}_0}{3M} \tilde{g}_0 \Omega^\alpha_\beta \gamma + \Omega_\beta e^\beta_\gamma \phi^\gamma.
\]

Now let us turn to the special case with spin-3/2 and spin-1/2 fields. Deformed unfolded equations for the gauge invariant zero-forms were already given above, so to complete the solution we need corresponding equations for the one-form \( \Phi^\alpha \) and Stueckelberg
zero-form $\phi^\alpha$. We obtain:

$$0 = D\Phi^\alpha + Me^\alpha_\beta \Phi^\beta - 2ME^\alpha_\beta \phi^\beta + \frac{4M^2 g_0}{3} [e^\alpha_\beta \Omega^\beta_\gamma + \Omega^\alpha_\beta \epsilon^\beta_\gamma] \psi^\gamma,$$

$$0 = D\phi^\alpha - 2M\Phi^\alpha + Me^\alpha_\beta \phi^\beta + e^\beta_\beta \phi^\alpha(2)$$

$$+ g_0 \Omega_\beta(2) \psi^\alpha(2) + \frac{4Mg_0}{3} \Omega^\alpha_\beta \psi^\beta.$$

### 4 Cubic vertices

In this section we reconstruct the Lagrangian formulations for all three cases above. Recall that the free Lagrangian contains the one-forms $\Phi^\alpha(2k+1), 0 \leq k \leq s - 1$ and one zero-form $\phi^\alpha$ only, while all other zero-forms are interpreted as the higher derivatives of the Lagrangian fields. Analyzing the unfolded equations obtained above, we see that for the gravity case it is enough to consider terms with no more than one derivative (taking into account that the auxiliary field $\Omega^\alpha(2)$ is equivalent to the first derivative of the physical field $H^\alpha(2)$), while in the two non-gravity cases we have to consider terms with up to two derivatives.

#### 4.1 Gravity

In this case the cubic vertex (we set $g_0 = 1$ for simplicity) can be written as

$$\frac{1}{4} L_1 = \sum_{k=0}^{s-1} (-1)^k \frac{1}{2} (2k+1) \Phi^\alpha(2k+1) \Omega^\beta_\gamma \Phi^\alpha(2k+1) + \frac{b_k}{2} \Phi^\alpha(2k+1) H^\beta_\gamma \Phi^\alpha(2k+1)$$

$$+ \sum_{k=1}^{s-1} (-1)^k a_k \Phi^\alpha(2k-1) H^\beta(2) \Phi^\alpha(2k-1) + \frac{1}{4} \epsilon^\alpha_\beta H^\beta_\gamma + H^\alpha_\beta \epsilon^\beta_\gamma D\phi^\gamma$$

$$+ \frac{a_0}{4} \Phi^\alpha \epsilon^\alpha_\beta H^\beta_\gamma + H^\alpha_\beta \epsilon^\beta_\gamma \phi^\gamma - \frac{3b_0}{4} (EH) \phi^\alpha.$$

(4.1)

All the gauge transformations (for the massless spin two as well as for massive fermion) requires non-trivial corrections.

**$\eta^\alpha(2)$-transformations.** The gauge invariance requires:

$$\delta \Phi^\alpha(2k+1) = -\eta^\alpha_\beta \Phi^\alpha(2k+1), \quad \delta \phi^\alpha = -\eta^\alpha_\beta \phi^\beta.$$

(4.2)

**$\xi^\alpha(2)$-transformations.** Here we obtain:

$$\delta \Phi^\alpha(2k+1) = -\frac{b_k}{(2k+1)} \xi^\alpha_\beta \Phi^\alpha(2k+1) - \frac{a_k}{k(2k+1)} \xi^\alpha(2) \Phi^\alpha(2k-1) - a_{k+1} \xi_\beta(2) \Phi^\alpha(2k+1),$$

$$\delta \Phi^\alpha = -b_0 \xi^\alpha_\beta \Phi^\beta + a_1 \xi_\beta(2) \Phi^\alpha(2) - \frac{a_0}{4} (\epsilon^\alpha_\beta \xi^\beta_\gamma - \xi^\alpha_\beta \epsilon^\beta_\gamma) \phi^\gamma,$$

(4.3)

$$\delta \phi^\alpha = -b_0 \xi^\alpha_\beta \phi^\beta + a_1 \xi_\beta(2) \phi^\alpha(2).$$
\(\zeta^{(2k+1)}\)-transformations. The corrections for the fermion look like:

\[
\delta \Phi^{\alpha (2k+1)} = \frac{b_k}{(2k+1)} H^{\beta} \alpha \zeta^{(2k+1)\beta} + \frac{a_k}{k(2k+1)} H^{(2)} \alpha \zeta^{(2k-1)} + a_{k+1} H^{\beta(2)} \zeta^{(2k+1)\beta(2)},
\]

\[
\delta \Phi^{\alpha} = \Omega^{\alpha} \beta \zeta^{\beta} + b_0 H^{\alpha} \beta \zeta^{\beta} + a_1 H^{\beta(2)} \zeta^{\alpha \beta(2)}, \tag{4.4}
\]

while for the massless spin-2 they have the form:

\[
\delta \Omega^{(2)} = \sum_{k=1}^{n-1} (-1)^{k+1} \left[ b_k \Phi^{\alpha (2k+1) \beta (2k)} + a_k \Phi^{(2k) (2k-1) \beta (2k)} - a_k H^{(2k-1) \beta (2k)} \right]
- b_0 \Phi^{\alpha \beta} + \frac{a_0}{8} \left( \phi^a \xi^{a \beta} \zeta^{\beta} + \phi^{\beta \alpha} \zeta^{\alpha} \right), \tag{4.5}
\]

\[
\delta H^{(2)} = \sum_{k=0}^{n-1} (-1)^{k+1} (2k+1) \Phi^{(2k) \beta (2k)} \zeta^{\alpha \beta (2k)}.
\]

Note that the last formulas are completely consistent with corrections to the massless spin-2 curvatures:

\[
\Delta R^{(2)} = \sum_{k=0}^{n-1} (-1)^{k+1} \frac{b_k}{2} \Phi^{(2k) \beta (2k)} + \sum_{k=1}^{n-1} (-1)^{k+1} a_k \Phi^{(2k) (2k-1) \beta (2k-1)}
+ \frac{1}{16} \left( \phi^{\beta \alpha} \zeta^{\alpha \beta} \Phi^{\beta (2k)} - \phi^a \xi^{a \beta} D \phi^\beta - \frac{a_0}{8} \left( \phi^{\beta \alpha} \zeta^{\alpha \beta} + \phi^{\alpha \beta} \zeta^a \right) \right) - \frac{3b_0}{4} E^{(2) \beta \Phi^{\beta}}, \tag{4.6}
\]

\[
\Delta T^{(2)} = \sum_{k=0}^{n-1} (-1)^{k+1} \frac{(2k+1)}{2} \Phi^{(2k) \beta (2k)} - \frac{1}{4} E^{(2) \beta \Phi^{\beta}},
\]

Note also that on partial gauge fixing \(\phi^a = 0\) the invariance under the \(\eta^{a (2)}\) and \(\zeta^{a (2k+1)}\), \(k > 0\) transformations still remain, while the Lagrangian ceases to be invariant under \(\xi^{a (2)}\) transformations:

\[
\delta \xi^{a} \mathcal{L} = - \frac{a_0^2}{4} \Phi^{\alpha \beta \gamma} \Phi^{\alpha \beta \gamma}.
\]

The reason is that our field \(\phi^a\) has non-trivial \(\zeta^{a (2)}\)-transformation and after gauge fixing we loose its contribution. However, using once again the fact that the \(\phi^a\) equation follows from the equations of other fields, we may try to compensate this non-invariance. Technical problem is that to write the corresponding corrections in the frame-like formalism one has to introduce inverse frame \(\hat{\xi}^{a (2)}\), so that additional contribution to variations will be (schematically):

\[
\delta \xi^{a} \mathcal{L} \sim \Phi^{a \beta \gamma} \Phi^{\alpha \beta \gamma} \sim \Phi^{a \beta \gamma} \xi^{\beta \gamma} \Phi^{\alpha \beta \gamma}.
\]

Another possibility is to use a metric-like multispinor formalism as we used in [17]. As for the \(\zeta^{a}\)-transformations, similarly to the free case instead of \(\zeta^{a}\)-invariance we obtain a constraint

\[
0 = E^{\alpha \beta} \Phi^{\beta} + \frac{1}{4} (e^{\alpha \beta} H^{\beta \gamma} + H^{\alpha \beta} e^{\beta \gamma}) \Phi^{\gamma}, \tag{4.7}
\]

which is just a covariantization of the free constraint (2.6).

For completeness, let us give here the cubic vertex for the massive spin-1/2 field:

\[
\mathcal{L}_1 = - \frac{1}{4} (E \Omega) \phi^a \phi^a + \frac{1}{8} \phi^a [e^{\alpha \beta} H^{\beta \gamma} + H^{\alpha \beta} e^{\beta \gamma}] D \phi^\gamma - \frac{M}{4} (EH) \phi^a \phi^a. \tag{4.8}
\]
4.2 Non-gravity 1

In this case the cubic vertex (we also set $g_0 = 1$) can be written as follows:

$$
\frac{1}{i}L_1 = \sum_{k=0}^{s-1} (-1)^{k+1} \frac{b_k}{2} \Phi_{\alpha(2k)\beta} \Omega^{\beta}_{\gamma} \Phi^{(2k)\gamma} + \sum_{k=1}^{s-1} (-1)^{k+1} a_k \Phi_{\alpha(2k-1)\beta(2)\gamma} \Phi^{(2k-1)\gamma} \label{4.9}
$$

$$
+ \frac{1}{8} \phi_\alpha [e^\alpha_\beta \Omega^{\beta}_{\gamma} + \Omega^{\beta}_{\beta} e^\beta_{\gamma}] D\phi^\gamma + \frac{a_0}{4} \Phi_\alpha [e^\alpha_\beta \Omega^{\beta}_{\gamma} + \Omega^{\beta}_{\beta} e^\beta_{\gamma}] \phi^\gamma - \frac{3b_0}{4} (E\Omega) \phi_\alpha \phi^\alpha.
$$

The field $H^{(2)}$ does not enter the Lagrangian so the vertex is trivially invariant under the $\zeta^{(2)}$-transformations.

$\eta^{(2)}$-transformations. Here we need the following corrections:

$$
\delta \Phi^{(2k+1)} = - \frac{b_k}{(2k+1)} \eta_{\alpha\beta} \Phi^{(2k)\beta} - \frac{a_k}{k(2k+1)} \eta^{(2)} \Phi^{(2k-1)} - a_{k+1} \eta_{\beta(2)} \Phi^{(2k+1)\beta},
$$

$$
\delta \Phi^\alpha = -b_0 \eta^{\alpha}_{\beta} \Phi^\beta - a_1 \eta_{\beta(2)} \Phi^{\alpha(2)} - \frac{a_0}{4} (e^{\alpha\beta} \eta^{\gamma}_{\beta} - \eta^{\alpha}_{\beta} e^\beta_{\gamma}) \phi^\gamma,
$$

$$
\delta \phi^\alpha = -b_0 \eta^{\alpha}_{\beta} \phi^\beta - a_1 \eta_{\beta(2)} \phi^{\alpha(2)}.
$$

$\zeta^{(2k+1)}$-transformations. For the fermions the corrections look like:

$$
\delta \Phi^{(2k+1)} = \frac{b_k}{(2k+1)} \Omega^{\beta}_{\gamma} \zeta^{(2k)\beta} + \frac{a_k}{k(2k+1)} \Omega^{(2)} \zeta^{(2k-1)} + a_{k+1} \Omega^{(2)} \zeta^{(2k+1)\beta},
$$

$$
\delta \Phi^\alpha = b_0 \Omega^{\beta}_{\gamma} \phi^\beta + a_1 \Omega^{(2)} \zeta^{\alpha(2)},
$$

while for the spin-2 field we obtain:

$$\delta H^{(2)} = \sum_{k=1}^{s-1} (-1)^{k+1} [b_k \Phi^{(2k)\beta} \zeta^{\beta(2)} + a_k \Phi^{(2)\beta(2k-1)} \zeta^{\beta(2k-1)} + a_k \Phi^{(2)\beta(2k-1)} \zeta^{\beta(2k-1)}]$$

$$
- b_0 \Phi^\alpha \zeta^\alpha + \frac{a_0}{8} (\phi^\alpha e^{\alpha\beta} \zeta^\beta + \phi^\beta e^{\beta\alpha} \zeta^\alpha).
$$

Note that in this case the last formula is consistent with the corrections to the spin 2 curvature:

$$
\Delta T^{(2)} = \sum_{k=0}^{s-1} (-1)^{k+1} \frac{b_k}{2} \Phi^{(2k)\beta} \Phi^{\alpha(2k)\beta} + \sum_{k=1}^{s-1} (-1)^{k+1} a_k \Phi^{(2k-1)\beta(2)\gamma} \Phi^{(2k-1)\gamma} \label{4.13}
$$

$$
+ \frac{1}{16} (\phi^\beta e^{\beta\alpha} D\phi^\alpha - \phi^\alpha e^{\alpha\beta} D\phi^\beta) - \frac{a_0}{8} (\Phi^{\beta\gamma} \psi^\alpha - \Phi^\alpha \psi^{\beta\alpha} \phi^\beta) - \frac{3b_0}{4} E^{(2)} \phi^\alpha \phi^\beta.
$$

Note that for the massive spin-1/2 case the vertex takes a very simple form:

$$
L_1 = \frac{1}{8} \phi_\alpha [e^\alpha_\beta \Omega^{\beta}_{\gamma} + \Omega^{\beta}_{\beta} e^\beta_{\gamma}] D\phi^\gamma - \frac{M}{4} (E\Omega) \phi_\alpha \phi^\alpha.
$$

(4.14)
4.3 Non-gravity 2

In this case we have two independent sets of the unfolded equations with independent coupling constants \( g_0 \) and \( \tilde{g}_0 \). To see if they can be consistent with the Lagrangian formulation, consider the following ansatz for the cubic vertex:

\[
\mathcal{L}_1 = \sum_{k=0}^{k=0} (-1)^{k+1}[\kappa_1 k \Phi_{\alpha(2k+1)\beta(2k+2)} \Omega^{(2k+1)\gamma} \Psi^{(2k+2)\gamma} - \kappa_2 k \Phi_{\alpha(2k+2)\beta(2k+1)} \Omega^{(2k+2)\gamma} \Psi^{(2k+1)\gamma}]
\]

\[
+\kappa_3 k \Phi_{\alpha(2k+1)\beta(2k+2)} \Omega^{(2k+1)\gamma} \Psi^{(2k+2)\gamma} + \rho_1 \Phi_{\alpha(2k+1)\beta(2k+2)} \Omega^{(2k+1)\gamma} \Psi^{(2k+2)\gamma} + \rho_2 \Phi_{\alpha(2k+2)\beta(2k+1)} \Omega^{(2k+2)\gamma} \Psi^{(2k+1)\gamma} + \rho_3 \Phi_{\alpha(2k+1)\beta(2k+2)} \Omega^{(2k+1)\gamma} \Psi^{(2k+2)\gamma} + \phi_4 (\rho_4 \Phi_{\alpha(2k+1)\beta(2k+2)} \Omega^{(2k+1)\gamma} \Psi^{(2k+2)\gamma} + \rho_6 \Phi_{\alpha(2k+2)\beta(2k+1)} \Omega^{(2k+2)\gamma} \Psi^{(2k+1)\gamma} \psi_\gamma).
\]

(4.15)

Consistency of such ansatz with the unfolded equations requires:

\[
\kappa_{1,k} = \frac{\tilde{a}_0}{a_0} (2k+1)h_{1,k} = \frac{a_0}{a_0} (2k+1)\tilde{h}_{1,k},
\]

\[
\kappa_{2,k} = \frac{\tilde{a}_0}{a_0} (k+1)(2k+3)h_{2,k+1} = \frac{a_0}{a_0} \tilde{h}_{3,k},
\]

\[
\kappa_{3,k} = \frac{\tilde{a}_0}{a_0} h_{3,k} = \frac{a_0}{a_0} (k+1)(2k+3)\tilde{h}_{2,k+1}.
\]

(4.16)

Besides, gauge invariance requires:

\[
\rho_1 = \frac{\tilde{a}_0 M}{4} g_0, \quad \rho_2 = \frac{a_0 M}{4} \tilde{g}_0,
\]

\[
\rho_3 = \rho_4 = \frac{\rho_1}{2a_0} = \frac{\rho_2}{2a_0},
\]

\[
\rho_5 = \frac{a_0}{\tilde{a}_0} (2s+1)\tilde{g}_0, \quad \rho_6 = \frac{a_0}{\tilde{a}_0} (2s-1)\tilde{g}_0.
\]

(4.17)

All these relations are satisfied provided

\[
\tilde{a}_0^2 M^2 g_0 = a_0^2 \tilde{g}_0.
\]

(4.18)

As in the previous case the Lagrangian is trivially invariant under \( \xi^{(2)} \)-transformations.

**\( \eta^{(2)} \)-transformations.** Here we need the following corrections:

\[
\delta \Phi^{\alpha(2k+1)} = -\frac{\tilde{a}_0}{a_0} \left[ h_{1,k} \eta^{(2k+1)\beta} \psi^{(2k+2)\beta} + h_{2,k} \eta^{(2k+2)\beta} \psi^{(2k+1)\beta} + h_{3,k} \eta^{(2k+1)\beta(2k+2)} \right],
\]

\[
\delta \Phi^{\alpha} = -\frac{\tilde{a}_0}{a_0} \left[ h_{1,0} \eta^{(2k+1)\beta} \psi^{(2k+2)\beta} + h_{3,0} \eta^{(2k+2)\beta(2k+1)} \right] + \rho_1 (e^\alpha_{\beta} \eta^{(2k+1)\beta} \psi^{(2k+2)\beta} + e^\alpha_{\beta} \psi^{(2k+1)\beta} \psi^{(2k+2)\beta}) \psi_\gamma,
\]

(4.19)

\[
\delta \psi^{\alpha} = -h_{1,0} \eta^{(2k+1)\beta} \psi^{(2k+2)\beta} - h_{3,0} \eta^{(2k+2)\beta(2k+1)} \psi^{(2k+1)\beta}.
\]

and similarly for \( \Psi \) with \( \Psi \leftrightarrow \Phi, a_0 \leftrightarrow \tilde{a}_0 \) and \( h \rightarrow \tilde{h} \).
\( \tilde{\zeta}^{(2k+1)} \)-transformations. The corrections for the fermions have the form:

\[
\delta \Phi_\alpha^{(2k+1)} = \tilde{a}_0 \tilde{a}_0 \left[ h_{1,k} \Omega^\alpha_\beta \tilde{\zeta}_\beta^{(2k+1)} \right] + h_{2,k} \Omega^{(2)}_\beta \tilde{\zeta}_\beta^{(2k-1)} + h_{3,k} \Omega_\beta^{(2)} \tilde{\zeta}_\beta^{(2k+1)},
\]

\[
\delta \Phi_\alpha = \tilde{a}_0 \tilde{a}_0 \left[ h_{1,0} \Omega^\alpha_\beta \tilde{\zeta}_\beta^{(2)} \right] + h_{3,0} \Omega_\beta^{(2)} \tilde{\zeta}_\beta^{(2)},
\]

(4.20)

and similarly for \( \Psi \). While the \( \zeta \)-transformations for the \( H^{(2)} \) field can be extracted from the corrections to the torsion:

\[
\Delta T^{(2)} = \sum_{k=0}^{s-1} (-1)^{k+1} \left[ \kappa_{1,k} \Phi^{\alpha \beta (2k)} \Psi^{\alpha \beta} + \kappa_{2,k} \Phi^{(2) (2k+1)} \Psi^{(2) (2k+1)} \right]
\]

\[
+ \kappa_{3,k} \Phi^{(2) (2k+1)} \Psi^{(2) (2k+1)} \]

\[
- \frac{\rho_4}{2} (\Phi_{\beta} e^{\beta \alpha} \psi^\alpha - \Phi^\alpha e^{\alpha \beta} \psi^\beta) - \frac{\rho_4}{2} (\Phi_{\beta} e^{\beta \alpha} \phi^\alpha - \Phi^\alpha e^{\alpha \beta} \phi^\beta)
\]

\[
- \frac{\rho_5}{2} (\Phi_{\beta} e^{\beta \alpha} D^\alpha \psi^\beta - \phi^\alpha e^{\alpha \beta} D^\beta \psi^\alpha) - \frac{\rho_5}{2} (\Phi_{\beta} e^{\beta \alpha} D^\alpha \phi^\beta - \phi^\alpha e^{\alpha \beta} D^\beta \phi^\alpha)
\]

\[
- \frac{\rho_6}{2} \phi^\beta E^{\beta \alpha} \psi^\alpha + \frac{\rho_6}{2} \phi^\alpha E^{\alpha \beta} \psi^\beta.
\]

For completeness, let us give here the cubic vertex for the special case with spin-3/2 and spin-1/2 fields:

\[
\mathcal{L}_1 = g_0 \Phi_\alpha (e^\alpha_\beta \Omega^\beta_\gamma + \Omega^\alpha_\beta e^\beta_\gamma) D\psi^\gamma + g_0 \psi_\alpha (e^\alpha_\beta \Omega^\beta_\gamma + \Omega^\alpha_\beta e^\beta_\gamma) D\phi^\gamma
\]

\[
+ 4 M g_0 \phi_\alpha (e^\alpha_\beta \Omega^\beta_\gamma + \Omega^\alpha_\beta e^\beta_\gamma) \psi^\gamma + 2 M g_0 \phi_\alpha (3 E^\alpha_\beta \Omega^\beta_\gamma + \Omega^\alpha_\beta E^\beta_\gamma) \psi^\gamma.
\]

5 Conclusion

In this work we considered an interaction of the massless spin 2 field with the massive higher spin fermions. The procedure begins with the deformation of the unfolded equations in the sector of the gauge invariant zero-forms where we have found three linearly independent solutions. Then for all three cases we promoted this solutions to the two other sectors and finally reconstructed corresponding cubic vertices. Physically the most important case is of course gravitational interaction but we think that it is important that the procedure works for all three cases. It may seems as not the shortest way to the Lagrangian cubic vertices but such procedure allows us to determine the minimum possible number of derivatives and in some sense to resolve the ambiguity related with the field redefinitions. It would be interesting to try to extend these approach to higher spins, first of all to spins 5/2 and 3.

Open Access. This article is distributed under the terms of the Creative Commons Attribution License (CC-BY 4.0), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited. SCOAP3 supports the goals of the International Year of Basic Sciences for Sustainable Development.
References

[1] M.P. Blencowe, A consistent interacting massless higher spin field theory in $D = (2 + 1)$, *Class. Quant. Grav.* **6** (1989) 443 [SPIRE].

[2] K. Mkrtchyan, Cubic interactions of massless bosonic fields in three dimensions, *Phys. Rev. Lett.* **120** (2018) 221601 [arXiv:1712.10003] [SPIRE].

[3] P. Kessel and K. Mkrtchyan, Cubic interactions of massless bosonic fields in three dimensions II: parity-odd and Chern-Simons vertices, *Phys. Rev. D* **97** (2018) 106021 [arXiv:1803.02737] [SPIRE].

[4] Y.M. Zinoviev, On higher spin cubic interactions in $d = 3$, *JHEP* **11** (2021) 022 [arXiv:2109.08480] [SPIRE].

[5] M. Grigoriev, K. Mkrtchyan and E. Skvortsov, Matter-free higher spin gravities in $3D$: partially-massless fields and general structure, *Phys. Rev. D* **102** (2020) 066003 [arXiv:2005.05931] [SPIRE].

[6] A. Campoleoni, S. Fredenhagen, S. Pfenninger and S. Theisen, Asymptotic symmetries of three-dimensional gravity coupled to higher-spin fields, *JHEP* **11** (2010) 007 [arXiv:1008.4744] [SPIRE].

[7] B. Chen, J. Long and Y.-N. Wang, Black holes in truncated higher spin $AdS_3$ gravity, *JHEP* **12** (2012) 052 [arXiv:1209.6185] [SPIRE].

[8] H.S. Tan, Exploring three-dimensional higher-spin supergravity based on $sl(N|N − 1)$ Chern-Simons theories, *JHEP* **11** (2012) 063 [arXiv:1208.2277] [SPIRE].

[9] Y.M. Zinoviev, Hypergravity in $AdS_3$, *Phys. Lett. B* **739** (2014) 106 [arXiv:1408.2912] [SPIRE].

[10] R.R. Metsaev, Cubic interactions of arbitrary spin fields in 3d flat space, *J. Phys. A* **53** (2020) 445401 [arXiv:2005.12224] [SPIRE].

[11] E. Skvortsov, T. Tran and M. Tsulaia, A stringy theory in three dimensions and massive higher spins, *Phys. Rev. D* **102** (2020) 126010 [arXiv:2006.05809] [SPIRE].

[12] S.F. Prokushkin and M.A. Vasiliev, Higher spin gauge interactions for massive matter fields in 3D $AdS$ space-time, *Nucl. Phys. B* **545** (1999) 385 [hep-th/9806236] [SPIRE].

[13] S.F. Prokushkin, A.Y. Segal and M.A. Vasiliev, Coordinate free action for $AdS_3$ higher spin matter systems, *Phys. Lett. B* **478** (2000) 333 [hep-th/9912280] [SPIRE].

[14] I.L. Buchbinder, T.V. Snegirev and Y.M. Zinoviev, Gauge invariant Lagrangian formulation of massive higher spin fields in $(A)dS_3$ space, *Phys. Lett. B* **716** (2012) 243 [arXiv:1207.1215] [SPIRE].

[15] I.L. Buchbinder, T.V. Snegirev and Y.M. Zinoviev, Frame-like gauge invariant Lagrangian formulation of massive fermionic higher spin fields in $AdS_3$ space, *Phys. Lett. B* **738** (2014) 258 [arXiv:1407.3918] [SPIRE].

[16] N. Boulanger, C. Deffayet, S. Garcia-Saenz and L. Traina, Consistent deformations of free massive field theories in the Stueckelberg formulation, *JHEP* **07** (2018) 021 [arXiv:1806.04695] [SPIRE].

[17] M.V. Khabarov and Y.M. Zinoviev, On massive spin-3/2 in the Fradkin-Vasiliev formalism, *Class. Quant. Grav.* **38** (2021) 195012 [arXiv:2105.01325] [SPIRE].
[18] M.V. Khabarov and Y.M. Zinoviev, *On massive spin-2 in the Fradkin-Vasiliev formalism. II. General massive case*, Nucl. Phys. B 973 (2021) 115591 [arXiv:2107.05900 [nSPIRE]].

[19] N. Boulanger, D. Ponomarev, E. Sezgin and P. Sundell, *New unfolded higher spin systems in AdS$_3$*, Class. Quant. Grav. 32 (2015) 155002 [arXiv:1412.8209 [nSPIRE]].

[20] Y.M. Zinoviev, *Massive higher spins in d = 3 unfolded*, J. Phys. A 49 (2016) 095401 [arXiv:1509.00968 [nSPIRE]].

[21] I.L. Buchbinder, T.V. Snegirev and Y.M. Zinoviev, *Unfolded equations for massive higher spin supermultiplets in AdS$_3$*, JHEP 08 (2016) 075 [arXiv:1606.02475 [nSPIRE]].

[22] I.L. Buchbinder, T.V. Snegirev and Y.M. Zinoviev, *Lagrangian description of massive higher spin supermultiplets in AdS$_3$ space*, JHEP 08 (2017) 021 [arXiv:1705.06163 [nSPIRE]].

[23] I.L. Buchbinder, T.V. Snegirev and Y.M. Zinoviev, *Supersymmetric higher spin models in three dimensional spaces*, Symmetry 10 (2017) 9 [arXiv:1711.11450 [nSPIRE]].

[24] Y.M. Zinoviev, *On massive gravity and bigravity in three dimensions*, Class. Quant. Grav. 30 (2013) 055005 [arXiv:1205.6892 [nSPIRE]].

[25] I.L. Buchbinder, T.V. Snegirev and Y.M. Zinoviev, *On gravitational interactions for massive higher spins in AdS$_3$*, J. Phys. A 46 (2013) 214015 [arXiv:1208.0183 [nSPIRE]].

[26] Y.M. Zinoviev, *Infinite spin fields in d = 3 and beyond*, Universe 3 (2017) 63 [arXiv:1707.08832 [nSPIRE]].

[27] X. Bekaert and E.D. Skvortsov, *Elementary particles with continuous spin*, Int. J. Mod. Phys. A 32 (2017) 1730019 [arXiv:1708.01030 [nSPIRE]].

[28] M.V. Khabarov and Y.M. Zinoviev, *On massive higher spins in d = 3*, JHEP 04 (2022) 055 [arXiv:2201.09491 [nSPIRE]].

[29] I. Bandos et al., *The Goldstino brane, the constrained superfields and matter in N = 1 supergravity*, JHEP 11 (2016) 109 [arXiv:1608.05908 [nSPIRE]].