Conditioning of Gaussian processes and a zero area Brownian bridge

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Abstract

We generalize the notion of Gaussian bridges by conditioning Gaussian processes given that certain linear functionals of the sample paths vanish. We show the equivalence of the laws of the unconditioned and the conditioned process and by an application of Girsanov’s theorem, we show that the conditioned process follows a stochastic differential equation (SDE) whenever the unconditioned process does. In the Markovian case, we are able to determine the coefficients in the SDE of the conditioned process explicitly. Our main example is Brownian motion on [0, 1] pinned down in 0 at time 1 and conditioned to have vanishing area spanned by the sample paths.

Keywords: Gaussian processes, Conditioning, Brownian bridge, Series expansions

1. Introduction

Let $X = \{X_s\}_{s \in [0,T]}$ be a Gaussian process with values in the space of continuous functions $C([0,T])$ and assume $E X_s = 0$ for all $s \in [0,T]$. Let $A$ be a given finite set of linear functionals on $C([0,T])$. In this work we consider the conditioned process of $X$ given that the linear functionals in $A$ acting on $X$ vanish. More formally, the conditioned process of $X$ with respect to $A$ is a continuous Gaussian process $X^{(A)} = \{X^{(A)}_s\}_{s \in [0,T]}$ such that the law of $X^{(A)}$ under $P$ is the same as the law of $X$ under $P^{(A)}$, where $P^{(A)}$ is defined as

$$P^{(A)}(\cdot) = P(\cdot | a(X) = 0 \text{ for all } a \in A).$$ (1)

The random variables $X_s$, $0 \leq s \leq T$, and $a(X)$, $a \in A$, are centered Gaussian random variables and hence conditioning becomes orthogonal projection in the Gaussian Hilbert space spanned by the random variables $X_s$, $0 \leq s \leq T$ (see for example Chapter 9 in [6]).

A well studied example is that of Gaussian bridges (see for example [5] and [1]). In this case the set $A$ only consists of the element $\delta_T$, where $\delta_T$...
denotes point evaluation of a function at point $T$. For example the standard linear Brownian motion on $[0,1]$ conditioned to have $W_1 = 0$ (i.e., $A = \{\delta_1\}$) yields the Brownian bridge $B$ on $[0,1]$. An anticipative representation of $B$ is

$$B_s = W_s - sW_1, \quad 0 \leq s \leq 1, \quad (2)$$

and a non-anticipative representation (i.e., adapted to the natural filtration of $W$) of $B$ is

$$dB_s = dW_s - \frac{B_s}{1-s} ds, \quad B_0 = 0, \quad 0 \leq s < 1. \quad (3)$$

The present work generalizes the setting of Gaussian bridges by allowing several and more general conditions in (1). Our main example (studied in Section 6.1) is the Brownian motion $W$ conditioned to have $W_1 = 0$ and $I_1 = \int_0^1 W_x \, dx = 0$ (i.e., $A = \{\delta_1, a_0\}$ with $a_0(f) = \int_0^1 f(x) \, dx$, $f \in C([0,1])$). We call the conditioned process the zero area Brownian bridge and denote it by $M$. Figure 1 shows a typical path of $M$. An anticipative representation of $M$ (corresponding to (2) for $B$) is

$$M_s = W_s - s(3s - 2)W_1 - 6s(1-s)I_1, \quad 0 \leq s \leq 1,$$

and a non-anticipative representation of $M$ (corresponding to (3) for $B$) is

$$dM_s = dW_s - \frac{4M_s}{1-s} ds - \frac{6J_s}{(1-s)^2} ds, \quad M_0 = 0, \quad 0 \leq s < 1,$$

where $J_s = \int_0^s M_x \, dx$. In particular, the two dimensional process $(M_s, J_s)_{s \in [0,1]}$ is a time-inhomogeneous Markov process.

We fix some notations and introduce the conditioned process properly. Then we state the main results of the paper.
1.1. Notations and definition of the conditioned process

Let $C([0,T])$ be the space of continuous functions on $[0,T]$ equipped with the supremum norm

$$
\|f\|_{\infty} = \sup_{0 \leq s \leq T} |f(s)|, \quad f \in C([0,T]).
$$

Then $(C([0,T]), \| \cdot \|_{\infty})$ becomes a separable Banach space. For a continuous function $f \in C([0,T])$ and an element $a \in C([0,T])^*$ we write $a(f)$ for the evaluation map. Let $\mathcal{C}$ denote the Borel $\sigma$-algebra on $C([0,T])$. The dual space $C([0,T])^*$ of $C([0,T])$ can be identified with the space of signed finite Borel measure on $[0,T]$ (see Appendix C in [1]). We use the notation $a(f)$ and $\int f(s) a(ds)$ interchangeably. In particular, we use the second form if the integration only runs over a subset of $[0,T]$. By $\delta_s$, $s \in [0,T]$, we denote the point evaluation at point $s$, i.e., $\delta_s(f) = f(s)$, $f \in C([0,T])$. For $0 \leq s \leq T$, let $\mathfrak{F}_s \subset \mathcal{C}$ be the smallest $\sigma$-algebra on $C([0,T])$ such that all $\delta_r$, $0 \leq r \leq s$, are $\mathfrak{F}_s \cdot \mathcal{B}(\mathbb{R})$ measurable, where $\mathcal{B}(\mathbb{R})$ is the Borel $\sigma$-algebra on $\mathbb{R}$.

Let $X = (X_s)_{s \in [0,T]}$ be a continuous Gaussian process defined on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$. Assume $\mathbb{E}X_s = 0$ for all $s \in [0,T]$ and let $R_X : [0,T] \times [0,T] \rightarrow \mathbb{R}$ be the covariance function of $X$, $R_X(s,t) = \mathbb{E}X_sX_t$. A condition for $X$ is an element $a \in C([0,T])^*$ and $X$ fulfills the condition $a$ if $a(X) = 0$, almost surely. Let $A \subset C([0,T])^*$ be a finite set of conditions. We define a probability measure $\mathbb{P}^{(A)}$ on $(\Omega, \mathfrak{F})$ by

$$
\mathbb{P}^{(A)}(F) = \mathbb{P}(F \mid a(X) = 0 \text{ for all } a \in A), \quad F \in \mathfrak{F},
$$

and let $P_X^{(A)}$ be the induced measure on $(C([0,T]), \mathcal{C})$ of $X$ under $\mathbb{P}^{(A)}$. Though we condition on an event of probability zero in (1), the measure $\mathbb{P}^{(A)}$ is well defined since $a(X)$ is Gaussian and we condition on $a(X) = 0$ for all $a \in A$ (see also Section 9.3 in [2]).

A continuous Gaussian process $X^{(A)} = (X^{(A)}_s)_{s \in [0,T]}$ defined on a probability space $(\Omega', \mathfrak{F}', \mathbb{P}')$ is a conditioned process of $X$ with respect to the set of conditions $A$ if its induced measure $\mathbb{P}_{X^{(A)}}$ on $(C([0,T]), \mathcal{C})$ coincides with $P_X^{(A)}$. The conditioned process is thus only defined in law. The process $X$ defined on $(\Omega, \mathfrak{F}, \mathbb{P}^{(A)})$ is a version of the conditioned Gaussian process of $X$ (defined on $(\Omega, \mathfrak{F}, \mathbb{P})$) with respect to the conditions $A$.

1.2. Main results

Let $N$ be the number of conditions in $A$. In Section 2 we will introduce a separable Hilbert space $H$ and a linear and bounded operator $u : H \rightarrow C([0,T])$ such that

$$
X = \sum_{i=1}^{N} \omega_i(u e_i) + \sum_{j=1}^{\infty} \omega'_j(n f_j) \quad \text{and} \quad X^{(A)} = \sum_{j=1}^{\infty} \omega'_j(n f_j)
$$

in law for sequences $(e_i)_{i=1}^{N} \subset H$ and $(f_j)_{j=1}^{\infty} \subset H$ such that $\{e_1, \ldots, e_N, f_1, f_2, \ldots\}$ forms an orthonormal basis in $H$, and sequences of independent standard normal random variables $(\omega_i)_{i=0}^{N}$ and $(\omega'_j)_{i=0}^{\infty}$. Based on these series expansions
we find basic properties of the conditioned process. In particular its covariance structure (Proposition 1) and an anticipative representation (Theorem 3).

Let \((e_i)_{i=1}^\infty \subset H\) and \((f_j)_{j=1}^\infty \subset H\) be as in (5) and let \(H^A\) be the closed linear span of \(\{f_j : j \geq 1\}\). In Section 3 we show that \(\mathbb{P}_X\) and \(\mathbb{P}_{X(A)}\) are equivalent on \(\mathfrak{F}_s\) if and only if for every \(e_i\) there is an \(e'_i \in H^A\) with \((ue'_i)(x) = (ue_i)(x)\), for all \(0 \leq x \leq s\).

In Section 4 we show that, under some assumptions on \(X\) and \(A\), the process \(X^A\) solves a stochastic differential equation of the form

\[
dX^A_s = adW_s + \delta(s, X^A)ds, \quad X^A_0 = 0, \quad 0 \leq s < T, \tag{6}
\]

where \(W\) is a standard linear Brownian motion and \(\delta\) is a progressively measurable functional on \(C([0, T])\).

In Section 3 we assume that \(X\) is a Markov process. Defining \(I^{A,1}, \ldots, I^{A,N}\) by

\[
I^{A,1}_s = \int_0^s X^A_t a_1(dx), \quad \text{where } A = \{a_1, \ldots, a_N\},
\]

it is shown in Theorem 7 that \((X^A_s, I^{A,1}_s, \ldots, I^{A,N}_s)_{s \in [0, T]}\) is a Markov process as well. Based on this result we find a formula for \(\mathbb{E}[X^A_t | \mathfrak{F}^X_s]\), \(s \leq t\), where \(\mathfrak{F}^X_s\) denotes the natural filtration of \(X^A\) (Theorem 8), which enables us to find the \(\delta\) in (6) explicitly.

2. A series expansion and basic properties of \(X^A\)

The aim of this section is to find a series expansion of \(X^A\) analogous to that in (5). As a preliminary we start with a subsection on processes generated by an operator. Throughout, let \((\omega_i)_{i=0}^\infty\) and \((\omega'_i)_{i=0}^\infty\) be sequences of independent standard normal random variables defined on a probability space \((\Omega, \mathfrak{F}, \mathbb{P})\).

2.1. Gaussian processes defined by an operator

Let \(v : H \to C([0, T])\) be a linear and bounded operator from a separable Hilbert space \(H\) into \(C([0, T])\) and let \(v^* : C([0, T])^* \to H\) be the adjoint operator of \(v\), i.e., \((v^*a, h) = a(vh)\) for all \(h \in H\) and \(a \in C([0, T])^*\). Let \((\cdot, \cdot)\) denote the scalar product on \(H\) and \(\|\cdot\|\) its induced norm.

For an orthonormal basis \((h_i)_{i=1}^\infty \subset H\) define

\[
Z_s = \sum_{i=1}^\infty \omega_i(vh_i)(s) = \sum_{i=1}^\infty \omega_i(v^*\delta_s, h_i). \tag{7}
\]

The series on the right hand side of (7) converges almost surely for each \(s \in [0, T]\) because of

\[
\sum_{i=1}^\infty |(v^*\delta_s, h_i)|^2 = \|v^*\delta_s\|^2 < \infty.
\]

The exceptional null set in (7) in general depends on \(s \in [0, T]\). So (7) defines a not necessarily continuous Gaussian process \(Z = (Z_s)_{s \in [0, T]}\). If the series

\[
Z = \sum_{i=1}^\infty \omega_i(vh_i)
\]
converges almost surely in $C([0, T])$ we say that $v$ generates the continuous Gaussian process $Z$.

For the covariance function $R_Z(s, t) = \mathbb{E}Z_sZ_t$ of $Z$ it holds

$$R_Z(s, t) = \sum_{i=1}^{\infty} (vh_i)(s)(vh_i)(t) = \langle v^* \delta_s, v^* \delta_t \rangle. \quad (8)$$

Hence, a change of the orthonormal basis in (7) gives another process $Z'$, in general different from $Z$. But, by (8), $Z$ and $Z'$ have the same finite-dimensional distributions, i.e., $Z'$ is a version of $Z$.

2.2. A series expansion of the process $X^{(A)}$

The following result will be crucial for our work.

**Theorem 1** (Theorem 3.5.1 in [2]). For the continuous Gaussian process $X = (X_s)_{s \in [0, T]}$ there is a separable Hilbert space $H$ and a linear and bounded operator $u : H \to C([0, T])$ such that for every orthonormal basis $(h_i)_{i=1}^{\infty} \subset H$ the series

$$\sum_{i=1}^{\infty} \omega_i(uh_i) \quad (9)$$

converges almost surely in $C([0, T])$ and

$$X_s = \sum_{i=1}^{\infty} \omega_i(uh_i)(s)$$

holds in the sense of finite-dimensional distributions.

We define the closed linear subspace

$$H^{(A)} = \{h \in H : a(uh) = 0 \text{ for all } a \in A \} \subset H$$

and call it the reduced Hilbert space with respect to $A$. Let $H^{(A)} \subset H$ be the orthogonal complement of $H^{(A)}$ (we write $H^{(A)} = H \ominus H^{(A)}$). We call $H^{(A)}$ the detached subspace of $H$ with respect to $A$. By definition of $u^*$,

$$H^{(A)} = \{h \in H : \langle u^* a, h \rangle = 0 \text{ for all } a \in A \}$$

$$= \{h \in H : h \text{ is orthogonal to } u^* a \text{ for all } a \in A \} \subset H,$$

and thus $H^{(A)}$ is spanned the elements $u^* a$,

$$H^{(A)} = \text{span}\{u^* a : a \in A\}, \quad (10)$$

implying that $H^{(A)}$ is (at most) of dimension $N$.

Define

$$X^{(A)} = \sum_{i=1}^{\infty} \omega_i(uh_i), \quad (11)$$

5
where \((f_i)_{i=1}^\infty \subset H(A)\) is an orthonormal basis in \(H(A)\). By [8], the law of \(X(A)\) differs from \([9]\) only by a finite number of terms (given that we assume that \(\{f_1, f_2, \ldots\}\) is a subset of \(\{h_1, h_2, \ldots\}\) the series in \([11]\) converges in \(C([0, T])\) almost surely.

**Theorem 2.** The process \(X(A)\) defined in [11] is a conditioned process of \(X\) with respect to \(A\).

*Proof.* We have to show \(\mathbb{P}^{(A)}(X \in F) = \mathbb{P}(X^{(A)} \in F)\) for all \(F \in \mathcal{C}\) with \(\mathbb{P}^{(A)}\) defined in [11]. Let \((e_i)^N_{i=1} \subset H(A)\) be an orthonormal basis in \(H(A)\). Then the processes \(X\) and

\[
X^{(A)} + \sum_{i=1}^{N} \omega'_i(ue_i)
\]

coincide in law. We thus have

\[
\mathbb{P}^{(A)}(X \in F) = \mathbb{P}(X \in F \mid a(X) = 0 \ \forall a \in A)
\]

\[
= \mathbb{P}
\left(X^{(A)} + \sum_{i=1}^{N} \omega'_i(ue_i) \in F \mid a(X^{(A)}) + \sum_{i=1}^{N} \omega'_i a(ue_i) = 0 \ \forall a \in A
\right).
\]

By definition of \(X^{(A)}\) we have \(a(X^{(A)}) = 0\). The requirement \(\sum_{i=0}^{N} \omega'_i a(ue_i) = 0\) for all \(a \in A\) implies \(\omega'_i = 0\) for all \(1 \leq i \leq N\) assume \(\omega'_{i_0} \neq 0\) for some \(i_0\). Then, for all \(a \in A\), we have \(a(ue) = 0\) for the non-zero element \(e = \sum_{i=1}^{N} \omega'_i e_i\) implying that \(e \in H(A)\). This is a contradiction to the fact that the spaces \(H(A)\) and \(H(A)\) are orthogonal. Hence,

\[
\mathbb{P}^{(A)}(X \in F) = \mathbb{P}
\left(X^{(A)} + \sum_{i=1}^{N} \omega'_i(ue_i) \in F \mid \omega'_i = 0 \ 1 \leq i \leq N
\right)
\]

\[
= \mathbb{P}(X^{(A)} \in F).
\]

Let \(R_{X^{(A)}}\) be the covariance function of the conditioned process \(X^{(A)}\) of \(X\) with respect to \(A \subset C([0, T])^*\).

**Proposition 1.** Let \((e_i)^N_{i=1} \subset H(A)\) be an orthonormal basis in the detached subspace \(H(A)\). Then

\[
R_{X^{(A)}}(s, t) = R_X(s, t) - \sum_{i=1}^{N} (ue_i)(s)(ue_i)(t).
\]

*Proof.* Let \((f_j)^\infty_{j=1} \subset H(A)\) be an orthonormal basis in \(H(A)\). Then an orthonormal basis in \(H = H(A) \oplus H(A)\) is \(\{e_1, \ldots, e_N, f_1, f_2, \ldots\}\) and thus, by [8],

\[
R_X(s, t) = \sum_{i=1}^{N} (ue_i)(s)(ue_i)(t) + \sum_{j=1}^{\infty} (uf_j)(s)(uf_j)(t).
\]
Hence,

\[ R_X(A)(s, t) = \sum_{j=1}^{\infty} (uf_j)(s)(uf_j)(t) \]

\[ = R_X(s, t) - \sum_{i=1}^{N} (ue_i)(s)(ue_i)(t). \]

\[ \square \]

2.3. Anticipative representation

Define Gaussian processes \( I^1, \ldots, I^N \) by

\[ I^i_s = \int_0^s X_x a_i(dx), \quad 0 \leq s \leq T, \quad 1 \leq i \leq N. \quad (12) \]

**Proposition 2.** The conditions \( a_1, \ldots, a_N \) can be chosen such that the random variables \( I^1, \ldots, I^N \) are independent and standard normal and the set \( \{ u^*a_i : 1 \leq i \leq N \} \) is an orthonormal basis in \( H(A) \).

**Proof.** Let the conditions \( a_1, \ldots, a_N \) be arbitrary. Then the Gram-Schmidt orthonormalization \( \hat{I}^i_T = I^i_T / \mathbb{E}[I^i_T]^2 \),

\[ \hat{I}^i_T = \hat{I}^i_T / \mathbb{E}[\hat{I}^i_T]^2, \quad \text{where} \quad \hat{I}^i_T = I^i_T - \sum_{j=1}^{i-1} \mathbb{E}[I^j_T \hat{I}^i_T] \hat{I}^j_T, \quad i = 2, \ldots, N, \quad (13) \]

yields independent standard normal random variables \( \hat{I}^1_T, \ldots, \hat{I}^N_T \) and conditioning on \( \hat{I}^1_T = 0 \) is equivalent to conditioning on \( \hat{I}^1_T = 0 \) almost surely for \( 1 \leq i \leq N \). Now define measures \( \hat{a}_1, \ldots, \hat{a}_N \) by \( \hat{a}_1 = a_1 / \mathbb{E}[I^1_T]^2 \) and

\[ \hat{a}_i = \hat{a}_i / \mathbb{E}[\hat{I}^i_T]^2, \quad \text{where} \quad \hat{a}_i = a_i - \sum_{j=1}^{i-1} \mathbb{E}[I^j_T \hat{I}^i_T] \hat{a}_j. \]

Then we have \( \hat{I}^i_T = \int_0^T X_x \hat{a}_i(dx) \), i.e., we obtain independent standard normal random variables and conditioning with respect to \( \{ a_1, \ldots, a_N \} \) is equivalent to conditioning with respect to \( \{ \hat{a}_1, \ldots, \hat{a}_N \} \).

Moreover, we show that \( \{ u^*\hat{a}_1, \ldots, u^*\hat{a}_N \} \) is an orthonormal set in \( H \) and thus, by (10), an orthonormal basis in \( H(A) \); let \( \{ h_n \}_{n=1}^{\infty} \subset H \) be an orthonormal basis in \( H \). Then, for \( 1 \leq i, j \leq N \),

\[ \langle u^*\hat{a}_i, u^*\hat{a}_j \rangle = \sum_{n=1}^{\infty} \langle h_n, u^*\hat{a}_i \rangle \langle h_n, u^*\hat{a}_j \rangle = \sum_{n=1}^{\infty} \hat{a}_i(uh_n)\hat{a}_j(uh_n) \]

\[ = \sum_{n=1}^{\infty} \int_0^T (uh_n)(x) \hat{a}_i(dx) \int_0^T (uh_n)(y) \hat{a}_j(dy) \]

\[ = \int_0^T \int_0^T \sum_{n=1}^{\infty} (uh_n)(x)(uh_n)(y) \hat{a}_i(dx) \hat{a}_j(dy). \]
By (8), we have
\[ \sum_{n=1}^{\infty} (uh_n)(x)(uh_n)(y) = EX_x X_y \]
and thus
\[ \langle u^* \hat{a}_i, u^* \hat{a}_j \rangle = E \int_0^T X_x \hat{a}_i(dx) \hat{a}_j(dy) \]
\[ = E \left[ \int_0^T X_x \hat{a}_i(dx) \int_0^T X_y \hat{a}_j(dy) \right] \]
\[ = E [\hat{I}^i_T \hat{I}^j_T] = \delta_{i,j}, \]
where \( \delta_{i,j} \) denotes the Kronecker delta.

The following result follows directly from the general theory of conditioning of Gaussian random variables (see Chapter 9 in [6]).

**Proposition 3.** Let \( a_1, \ldots, a_N \) be such that the random variables \( I^1_T, \ldots, I^N_T \) are independent and standard normal random variables. Then an anticipative representation for \( X^{(A)} \) is
\[ X^{(A)}_x = X_x - \sum_{i=1}^{N} E \left[ X_x I^i_T \right] I^i_T. \]

We drop the requirement that \( I^1_T, \ldots, I^N_T \) are orthonormal but we still assume that the set \( \{ u^* a_i : 1 \leq i \leq N \} \subset H(A) \) is linearly independent in \( H \). Let \( (e_i)_{i=1}^{N} \subset H(A) \) be an orthonormal basis \( H(A) \) and define a matrix \( B \) and a vector \( b(X) \) by
\[ B = \begin{pmatrix}
  a_1(ue_1) & a_1(ue_2) & \cdots & a_1(ue_N) \\
  a_2(ue_1) & a_2(ue_2) & \cdots & a_2(ue_N) \\
  \vdots & \vdots & \ddots & \vdots \\
  a_N(ue_1) & a_N(ue_2) & \cdots & a_N(ue_N)
\end{pmatrix} \quad \text{and} \quad b(X) = \begin{pmatrix}
  a_1(X) \\
  a_2(X) \\
  \vdots \\
  a_N(X)
\end{pmatrix}. \]

**Theorem 3.** The matrix \( B \) is invertible and an anticipative representation of the conditioned process \( X^{(A)} \) is
\[ X^{(A)} = X - \sum_{i=1}^{N} \xi_i(X)(ue_i), \]
where \( \xi(X) = (\xi_1(X), \ldots, \xi_N(X))^T \) is given by \( \xi(X) = B^{-1} b(X) \).

**Proof.** In order to show that the matrix \( B \) is invertible, we show that the rank of \( B \) is \( N \). Since the \( e_i \)'s form an orthonormal basis in the Hilbert space spanned by \( \{ u^* a_1, \ldots, u^* a_N \} \), the rank of \( B \) is equal to the rank of \( B' \) with
\[ B' = \begin{pmatrix}
  a_1(uu^* a_1) & a_1(uu^* a_2) & \cdots & a_1(uu^* a_N) \\
  a_2(uu^* a_1) & a_2(uu^* a_2) & \cdots & a_2(uu^* a_N) \\
  \vdots & \vdots & \ddots & \vdots \\
  a_N(uu^* a_1) & a_N(uu^* a_2) & \cdots & a_N(uu^* a_N)
\end{pmatrix}. \]
Hence, it is enough to show that the columns of $B'$ are linearly independent. We assume

$$0 = \left( \sum_{i=1}^{N} \lambda_i a_1(u u^* a_i), \ldots, \sum_{i=1}^{N} \lambda_i a_N(u u^* a_i) \right).$$

Then,

$$0 = \sum_{j=1}^{N} \lambda_j \sum_{i=1}^{N} \lambda_i a_j(u u^* a_i) = \sum_{i,j=1}^{N} \lambda_i \lambda_j \langle u^* a_i, u^* a_j \rangle = \left\| \sum_{i=1}^{N} \lambda_i u^* a_i \right\|^2$$

which yields the requirement $\sum_{i=1}^{N} \lambda_i u^* a_i = 0$ and thus $\lambda_i = 0$, $1 \leq i \leq N$, since $\{u^* a_1, \ldots, u^* a_N\}$ is assumed to be linearly independent in $H$. Hence, the rank of $B'$ and $B$ is $N$ and the matrix $B$ is invertible.

Formula (14) follows from

$$X = X^{(A)} + \sum_{i=1}^{N} \omega_i (u e_i),$$

where $\omega_1, \ldots, \omega_N$ are independent standard normal random variables. Once we see a realization $X(\omega)$ of $X$ we do not know a priori, which values the $\omega_i$'s attained. But we can calculate them from the fact that

$$0 = a_j(X^{(A)}) = a_j(X) - \sum_{i=1}^{N} \omega_i a_j(u e_i)$$

for all $1 \leq j \leq N$, which leads to the system of linear equations $B \xi = b(X)$, its solution $\xi(X)$ and the claimed representation for $X^{(A)}$.

3. Equivalence of measures

Let $X$ be a continuous Gaussian process and let $X^{(A)}$ be the conditioned process of $X$ with respect to a finite set of conditions $A = \{a_1, \ldots, a_N\}$. Moreover, let $P_X$ and $P_{X^{(A)}}$ be the induced measures of $X$ and $X^{(A)}$ on $(C([0, T]), C)$.

We can not expect that $P_X$ and $P_{X^{(A)}}$ are equivalent on $C$ since

$$P_X(\{f \in C([0, T]) : a(f) = 0 \ \forall a \in A\}) = 0$$

in case that $X$ does not fulfill all conditions in $A$, while

$$P_{X^{(A)}}(\{f \in C([0, T]) : a(f) = 0 \ \forall a \in A\}) = 1.$$ 

In this section, we show that $P_X$ and $P_{X^{(A)}}$ are equivalent on a suitable sub-$\sigma$-algebra of $C$.

Let $X$ be generated by the operator $u : H \to C([0, T])$ and let $\{e_1, \ldots, e_N\}$ be an orthonormal basis in the detached Hilbert space $H^{(A)} \subset H$ (w.l.o.g. we assume $\dim(H^{(A)}) = N$; otherwise let some of the $e_i$'s be 0). Recall that $\mathcal{F}_t \subset C$ is the smallest $\sigma$-algebra on $C([0, T])$ such that all $\delta_r$, $0 \leq r \leq s$, are $\mathcal{F}_s$-$\mathcal{B}(\mathbb{R})$-measurable.
Theorem 4. The probability measures $\mathbb{P}_X$ and $\mathbb{P}_{X(A)}$ are equivalent on $\mathcal{F}_s$ if and only if there exist $e_i' \in H^{(A)}$, $1 \leq i \leq N$, such that
\[
(ue_i')(x) = (ue_i)(x), \quad 0 \leq x \leq s.
\]
(15)
Otherwise $\mathbb{P}_X$ and $\mathbb{P}_{X(A)}$ are orthogonal on $\mathcal{F}_s$.

We prepare for the proof of Theorem 4 by introducing some additional notation and proving an auxiliary result. For $d \geq 1$, let $\mathbb{P}_d$ be the standard Gaussian law on $(\mathbb{R}, \mathcal{B}(\mathbb{R}^d))$, i.e., $\mathbb{P}_d = \bigotimes_{i=1}^d \mathbb{P}_1$, where $\mathbb{P}_1$ is the standard normal law on $\mathbb{R}$, and consider the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with $\Omega = \bigotimes_{i=1}^\infty \mathbb{R}$, $\mathcal{A} = \bigotimes_{i=1}^\infty \mathcal{B}(\mathbb{R})$, and $\mathbb{P} = \bigotimes_{i=1}^\infty \mathbb{P}_1$. Let $(\xi_{ij})_{i,j=1}^\infty$ be such that $(\xi_{ij})_{i,j=1}^\infty \in l_2$, i.e.,
\[
\sum_{j=1}^\infty \xi_{ij}^2 < \infty \quad \text{for all } i \geq 1.
\]
(16)
We introduce the mapping $M : \Omega \to \Omega$ by
\[
(\omega_1, \omega_2, \ldots) \mapsto (\omega'_1, \omega'_2, \ldots),
\]
\[
\omega'_j = \sum_{i=1}^N \omega_i \xi_{ij} + \omega_{i+N}.
\]
(17)

Proposition 4. For $F \in \mathcal{A}$ with $\mathbb{P}(F) > 0$ it holds $\mathbb{P}(M(F)) > 0$.

Proof. Let $F \in \mathcal{A}$ with $\mathbb{P}(F) > 0$. For an element $x \in \mathbb{R}^N$ define
\[
F_x = \{y \in \Omega : (x, y) \in F\} \subset \Omega,
\]
\[
F'_x = \{(x, y) \in \Omega : y \in F_x\} \subset \Omega.
\]
Then
\[
0 < \mathbb{P}(F) = \int_{\mathbb{R}^N} \mathbb{P}(F_x) \mathbb{P}_N(dx)
\]
which implies the existence of a $z = (z_1, \ldots, z_N) \in \mathbb{R}^N$ with $\mathbb{P}(F_z) > 0$. Define the element $l = (l_1, l_2, \ldots) \in \Omega$ by
\[
l_j = \sum_{i=1}^N \xi_{ij} z_i.
\]
By Jensen’s inequality,
\[
\sum_{j=1}^\infty l_j^2 = \sum_{j=1}^\infty \left( \sum_{i=1}^N \xi_{ij} z_i \right)^2 \leq N \sum_{j=1}^\infty \sum_{i=1}^N \xi_{ij}^2 z_i^2 = N \sum_{i=1}^N z_i^2 \sum_{j=1}^\infty \xi_{ij}^2 < \infty
\]
and thus $l \in l_2$. Define $\tau_l : \Omega \to \Omega$ by $\tau_l(\omega) = \omega - l$. Then, for the subset $F'_z \subset F$ it holds $M(F'_z) = l + F_z = \tau_l^{-1}(F_z)$ and thus
\[
\mathbb{P}(M(F)) \geq \mathbb{P}(M(F'_z)) = \mathbb{P}(\tau_l^{-1}(F_z)) = \mathbb{P} \circ \tau_l^{-1}(F_z).
\]
(18)
The probability space $(\Omega, \mathcal{A}, \mathbb{P})$ is the canonical model for the Gaussian process $Z = (Z_n)_{n \in \mathbb{N}}$ with covariance $\mathbb{E}Z_m Z_n = \delta_{m,n}$, $m, n \in \mathbb{N}$. The Cameron-Martin space associated with $Z$ is $l_2$ and thus, since $l \in l_2$, the probability measures $\mathbb{P}$ and $\mathbb{P} \circ \tau_l^{-1}$ are equivalent (Theorem 14.17 in [6]). Hence, since $\mathbb{P}(F_z) > 0$ we have, by (18),
\[
\mathbb{P}(M(F)) \geq \mathbb{P} \circ \tau_l^{-1}(F_z) > 0.
\]
We are only interested in the laws of \(X\) and \(X^{(A)}\) and might thus, without loss of generality, assume that they are defined on the probability space \((\Omega, \mathfrak{F}, P)\). Let \(\{f_1, f_2, \ldots\}\) be an orthonormal basis in the reduced Hilbert space \(H^{(A)} \subset H\).

We may write \(X : \Omega \to C([0, T])\) as

\[
X(\omega) = X(\omega_1, \omega_2, \ldots) = \sum_{i=1}^{N} \omega_i (ue_i) + \sum_{j=1}^{\infty} \omega_{j+N} (uf_j),
\]

and \(X^{(A)} : \Omega \to C([0, T])\) as

\[
X^{(A)}(\omega) = X^{(A)}(\omega_1, \omega_2, \ldots) = \sum_{j=1}^{\infty} \omega_j (uf_j).
\]

**Proof of sufficiency in Theorem 4.** Let \(e_i' \in H^{(A)}\) be such that

\[
(ue_i')(x) = (ue_j)(x), \quad 0 \leq x \leq s, \quad 1 \leq i \leq N,
\]

and define \(\xi_{ij} = \langle e_i', f_j \rangle\), \(1 \leq i \leq N, j \geq 1\). Then \((\xi_{ij})_{i,j=1}^{\infty}\) fulfills (16) and we have

\[
(ue_i) = (ue_i') = \sum_{j=1}^{\infty} \xi_{ij}(uf_j)
\]
on \([0, s]\). Plugging this into (19), we obtain by (17) and (20),

\[
X(\omega) = \sum_{j=1}^{\infty} \left( \sum_{i=1}^{N} \omega_i \xi_{ij} + \omega_{j+N} \right) (uf_j) = X^{(A)}(M(\omega))
\]
on \([0, s]\).

Let \(F \in \mathfrak{F}_s\) with \(P_X(F) = P(X^{-1}(F)) > 0\). Since \(M\) is surjective,

\[
P_{X^{(A)}}(F) = P(\{\omega' : X^{(A)}(\omega') \in F\})
= P(\{M(\omega) : \omega \in \Omega \quad \text{and} \quad X^{(A)}(M(\omega)) \in F\}).
\]

Because of \(F \in \mathfrak{F}_s\) and \(X^{(A)}(M(\omega)) = X(\omega)\) on \([0, s]\),

\[
P_{X^{(A)}}(F) = P(\{M(\omega) : \omega \in \Omega \quad \text{and} \quad X(\omega) \in F\})
= P(M(X^{-1}(F))).
\]

Since \(P(X^{-1}(F)) > 0\), Proposition 4 yields \(P(M(X^{-1}(F))) > 0\) and thus \(P_{X^{(A)}}(F) > 0\).

To show the converse, let \(F \in \mathfrak{F}_s\) with \(P_{X^{(A)}}(F) > 0\). Since \(M\) is surjective we have

\[
P_{X^{(A)}}(F) = P(\omega' \in \Omega : X^{(A)}(\omega') \in F)
\leq P(\omega \in \Omega : X^{(A)}(M(\omega)) \in F)
\]

and thus, because of \(F \in \mathfrak{F}_s\) and \(X^{(A)}(M(\omega)) = X(\omega)\) on \([0, s]\),

\[
0 < P(\{\omega \in \Omega : X(\omega) \in F\}) = P_X(F).
\]

\[\square\]
**Proof of necessity in Theorem 4.** Fix \(0 \leq s \leq T\), assume without loss of generality that \(N = 1\), and set \(e = e_1\). Define \(u_s : H \to C([0, s])\) by \((u_s h)(x) = (uh)(x)\), \(h \in H\), \(x \in [0, s]\), and assume that for all \(e' \in H^{(A)}\) there is an \(x \in [0, s]\) such that \((ue)(x) \neq (ue')(x)\). Then \(e\) is orthogonal to \(\ker(u_s) \subset H\), the kernel of \(u_s\). Define \(H_s = H \ominus \ker(u_s)\) and let \(\{e, h_1, h_2, \ldots\}\) be an orthonormal basis in \(H_s\). Introduce a new scalar product \(\langle \cdot, \cdot \rangle_s\) on the set \(\{e, h_1, h_2, \ldots\}\) by

\[
\langle e, e \rangle_s = 1, \quad \langle e, h_i \rangle_s = 0, \quad \text{and} \quad \langle h_i, h_j \rangle_s = \delta_{i,j}/i^2,
\]

and let \(H'_s\) be the closed linear span of \(\{e, h_1, h_2, \ldots\}\) under \(\langle \cdot, \cdot \rangle_s\). Then \(H'_s\) is a Hilbert space and we have \(H_s \subset H'_s\). Let \(u'_s\) be the extension of \(u_s\) from \(H_s\) to \(H'_s\) and let \(b\) be a linear functional on \(H'_s\) such that \(b(e) \neq 0\) and \(b(h_i) = 0\). The operator \(u'_s\) is a linear bijection from \(H'_s\) to \(u'_s(H'_s)\) and we have \(\sum_{i=1}^{\infty} \omega_i h_i \in H'_s\) almost surely. Moreover,

\[
X^{(A)} = \sum_{i=1}^{\infty} \omega_i (u'_s h_i) \quad \text{and} \quad X = X^{(A)} + \omega_0 (u'_s e)
\]

in distribution on \(\mathcal{F}_s\). Applying the \(\mathcal{F}_s\)-measurable linear mapping \(b \circ u'^{-1}_s : u'_s(H'_s) \to \mathbb{R}\) yields

\[
(b \circ u'^{-1}_s)(X^{(A)}) = \sum_{i=1}^{\infty} (b \circ u'^{-1}_s \circ u'_s)(\omega_i h_i) = \sum_{i=1}^{\infty} \omega_i b(h_i) = 0,
\]

but

\[
(b \circ u'^{-1}_s)(X) = (b \circ u'^{-1}_s)(X^{(A)}) + (b \circ u'^{-1}_s)(\omega_0 (u'_s e)) = \omega_0 b(e) \neq 0
\]

almost surely. Hence, the measures \(\mathbb{P}_X\) and \(\mathbb{P}_{X^{(A)}}\) are orthogonal on \(\mathcal{F}_s\). \(\Box\)

4. **Non-anticipative representations**

Now, we consider alternative, non-anticipative representations for \(X^{(A)}\) in the same setting as in the previous section. We assume that the supremum over all \(0 \leq s \leq T\) for which \(\{15\}\) holds is \(T\). If this is not the case, the following calculations can only be performed on an interval \([0, T_s) \subset [0, T]\).

Recall that a progressively measurable functional on \(C([0, T])\) is a mapping \(\beta : [0, T] \times C([0, T]) \to \mathbb{R}\) such that for each \(0 \leq s \leq T\), the restriction of \(\beta\) to \([0, s] \times C([0, T])\) is \(\mathcal{B}([0, s]) \otimes \mathcal{B}(\mathbb{R})\)-measurable.

Let \(W = (W_s)_{s \in [0, T]}\) be a standard linear Brownian motion defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and assume that there is a \(0 \neq \alpha \in \mathbb{R}\) and a progressively measurable functional \(\beta\) on \(C([0, T])\) with

\[
\int_0^S |\beta(x, X)| \, dx < \infty \quad (21)
\]

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\( \mathbb{P} \)-almost surely for all \( S < T \), such that \( X \) is a (strong) solution of the stochastic differential equation

\[
dX_s = \alpha dW_s + \beta(s, X) ds, \quad X_0 = 0, \quad 0 \leq s < T. \tag{22}
\]

In order to apply the results from the previous section, it proves to be useful to assume without loss of generality \( \Omega = C([0, T]) \) (recall that we do not distinguish between Gaussian processes with the same law): by (22) and since \( \alpha \neq 0 \),

\[
W_s = \alpha^{-1} X_s - \alpha^{-1} \int_0^s \beta(x, X) \, dx, \quad 0 \leq s < T.
\]

Let \( \mathbb{P}_X \) be the induced measure of \( X \) on the space \( (C([0, T]), \mathcal{C}) \). Define the processes \( \hat{X} : (C([0, T]), \mathcal{C}) \to (C([0, T]), \mathcal{C}) \) and \( \hat{W} : (C([0, T]), \mathcal{C}) \to (C([0, T]), \mathcal{C}) \) by \((\hat{X} f)(s) = f(s) \) and

\[
(\hat{W} f)(s) = \alpha^{-1} (\hat{X} f)(s) - \alpha^{-1} \int_0^s \beta(x, \hat{X} f) \, dx
\]

for \( 0 \leq s < T \) and \( f \in C([0, T]) \). Then, on \( (C([0, T]), \mathcal{C}, \mathbb{P}_X) \), \( \hat{W} \) is a standard Brownian motion, \( \hat{X} = X \) in distribution, and we have

\[
d\hat{X}_s = \alpha d\hat{W}_s + \beta(s, \hat{X}) ds, \quad \hat{X}_0 = 0, \quad 0 \leq s < T
\]

with

\[
\int_0^S |\beta(x, \hat{X})| \, dx < \infty
\]

\( \mathbb{P}_X \)-almost surely for all \( S < T \). From the construction follows that the natural filtration of \( \hat{W} \) and \( \hat{X} \) is \( \mathcal{F} \).

4.1. Existence of a describing SDE

Let \( \mathbb{P}_{X^{(A)}} \) be the induced measure of \( X^{(A)} \) on \( (C([0, T]), \mathcal{C}) \).

**Theorem 5.** There is a Brownian motion \( W' = (W'_s)_{s \in [0, T]} \) defined on the probability space \( (C([0, T]), \mathcal{C}, \mathbb{P}_{X^{(A)}}) \) and a progressively measurable functional \( \delta \) on \( C([0, T]) \) with

\[
\int_0^S |\delta(x, X^{(A)})| \, dx < \infty \tag{23}
\]

\( \mathbb{P}_{X^{(A)}} \)-almost surely for all \( S < T \) such that the conditioned process \( X^{(A)} \) is a (strong) solution of the stochastic differential equation

\[
dX^{(A)}_s = \alpha dW'_s + \delta(s, X^{(A)}) ds, \quad X^{(A)}_0 = 0, \quad 0 \leq s < T. \tag{24}
\]

**Proof.** We consider the mapping \( Y : (C([0, T]), \mathcal{C}) \to (C([0, T]), \mathcal{C}) \) defined by \( Y(f) = f \) for \( f \in C([0, T]) \). Then, under the measure \( \mathbb{P}_X \), \( Y \) is a version of \( X \) and under the measure \( \mathbb{P}_{X^{(A)}} \), \( Y \) is a version of \( X^{(A)} \). Under \( \mathbb{P}_X \), the
semimartingale $Y = (Y_s)_{s \in [0,T]}$ has the decomposition $Y = M + A$, where $M$ is a continuous martingale and $A$ a finite variation process,

$$M_s = \alpha W_s, \quad A_s = \int_0^s \beta(x,Y) \, dx.$$  

By Theorem 4 the measures $\mathbb{P}_X$ and $\mathbb{P}_{X(A)}$ are equivalent on $\mathcal{F}_s$ for all $0 \leq s < T$. Hence,

$$Z_s = \mathbb{E}_{\mathbb{P}_X} \left[ \frac{d\mathbb{P}_{X(A)}}{d\mathbb{P}_X} \right]_{\mathcal{F}_s}, \quad 0 \leq s < T,$$

is an almost sure non-negative continuous $(\mathbb{P}_X, \mathcal{F})$-martingale. By Girsanov’s Theorem (see e.g. Theorem III.35 in [9]), $Y$ is a semimartingale under $\mathbb{P}_{X(A)}$ with decomposition $Y = L + C$ with

$$L_s = M_s - \int_0^s Z_x^{-1} d[Z,M]_x,$$

being a local martingale under $\mathbb{P}_{X(A)}$, where $[Z,M]$ denotes the quadratic co-variation process of $M$ and $Z$, and $C = Y - L$ is a $\mathbb{P}_{X(A)}$-finite variation process. By the martingale representation theorem (see e.g. Theorem 4.3.4 in [8]) there is an adapted stochastic process $\gamma$ such that

$$Z_s = \int_0^s \gamma(x) \, dW_x \quad \text{and} \quad \mathbb{E}_{\mathbb{P}_X} \left[ \int_0^s \gamma^2(x) \, dx \right] < \infty.$$  

Since $M = \alpha W$ it follows $d[Z,M]_x = \alpha \gamma(x) \, dx$ under $\mathbb{P}_X$ and thus under $\mathbb{P}_{X(A)}$. Hence, by (25),

$$Y_s = L_s + (Y_s - L_s)$$

$$= L_s + (M_s + A_s - L_s)$$

$$= \alpha \left( W_s - \int_0^s Z_x^{-1} \gamma(x) \, dx \right) + \left( \int_0^s \left( \alpha Z_x^{-1} \gamma(x) + \beta(x,Y) \right) \, dx \right).$$

The quadratic variation process of the first bracket is $s$ under $\mathbb{P}_X$ and thus under $\mathbb{P}_{X(A)}$. By Lévy’s characterization of Brownian motion,

$$W'_s = W_s - \int_0^s Z_x^{-1} \gamma(x) \, dx$$  

(26)

is a Brownian motion under $\mathbb{P}_{X(A)}$. That is,

$$Y_s = \alpha W'_s + \int_0^s \left( \alpha Z_x^{-1} \gamma(x) + \beta(x,Y) \right) \, dx, \quad 0 \leq s < T.$$  

Since the natural filtration of $Y$ is $\mathcal{F}$ and the process $(\alpha Z_x^{-1} \gamma(x) + \beta(x,Y))_{0 \leq s < T}$ is adapted to this filtration we have

$$\alpha Z_x^{-1} \gamma(x) + \beta(x,Y) = \delta(x,Y)$$
for some progressively measurable functional \( \delta \) on \( C([0, T]) \). Moreover, from (21) and (26) it follows
\[
\int_0^S |\delta(x, Y)| \, dx < \infty
\]
\( P_{X(A)} \)-almost surely for all \( S < T \). \( \square \)

4.2. Determination of the drift

Theorem 5 provides us with a progressively measurable functional \( \delta \) on \( C([0, T]) \) for which
\[
\int_0^S |\delta(x, X^{(A)})| \, dx < \infty
\]
almost surely for all \( S < T \). But in the following we need more than this.

**Proposition 5.** The progressively measurable functional \( \delta \) in Theorem 5 satisfies
\[
E \int_0^S |\delta(x, X^{(A)})| \, dx < \infty, \quad S < T.
\]

**Proof.** From (23) we know \(|\delta(s, X^{(A)})| < \infty \) almost surely for almost all \( 0 \leq s \leq S \) and thus the limit in
\[
\delta(s, X^{(A)}) = \lim_{\varepsilon \searrow 0} \varepsilon^{-1} \int_s^{s+\varepsilon} \delta(x, X^{(A)}) \, dx
\]
\[
= \lim_{\varepsilon \searrow 0} \varepsilon^{-1} \left( X^{(A)}_{s+\varepsilon} - X^{(A)}_s - \alpha W_{s+\varepsilon} + \alpha W_s \right)
\]
exists and is, as the limit of Gaussian random variables, a Gaussian random variable.

Let \( \sigma^2(x) = E|\delta(x, X^{(A)})|^2 \) be the variance of \( \delta(x, X^{(A)}) \) and for \( n \in \mathbb{N} \) set \( \delta_n(x) = \min\{1, n/\sigma(x)\} \delta(x, X^{(A)}) \). Then
\[
\sigma_n^2(x) = E|\delta_n(x)|^2 = \min\{\sigma^2(x), n^2\} \leq n^2 \quad \text{and} \quad \sigma_n^2(x) \nearrow \sigma^2(x) \text{ for all } x \text{ as } n \to \infty.
\]
Since \( \delta_n(x) \) is Gaussian we have \( E[\delta_n(x)] = \sqrt{2/\pi} \sigma_n(x) \) and by the Cauchy-Schwartz inequality
\[
E|\delta_n(x)\delta_n(y)| \leq \sqrt{E[\delta_n(x)]^2 E[\delta_n(y)]^2} = \sigma_n(x)\sigma_n(y).
\]
Define
\[
Z = \int_0^S |\delta(x, X^{(A)})| \, dx \quad \text{and} \quad Z_n = \int_0^S |\delta_n(x)| \, dx.
\]
Then we have \( Z_n \leq Z \) and
\[
E[Z_n] = \int_0^S E|\delta_n(x)| \, dx = \frac{\sqrt{2}}{\pi} \int_0^S \sigma_n(x) \, dx \nearrow \frac{\sqrt{2}}{\pi} \int_0^S \sigma(x) \, dx
\]
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as \( n \to \infty \). Moreover,

\[
\mathbb{E}Z_n^2 = \mathbb{E} \int_0^S \int_0^S \delta_n(x)\delta_n(y) \, dx \, dy \leq \int_0^S \int_0^S |\delta_n(x)\delta_n(y)| \, dx \, dy \\
\leq \int_0^S \int_0^S \sigma_n(x)\sigma_n(y) \, dx \, dy = \left( \int_0^S \sigma_n(x) \, dx \right)^2.
\]

Thus, for the variance \( \text{Var} Z_n = \mathbb{E}Z_n^2 - (\mathbb{E}Z_n)^2 \),

\[
\text{Var} Z_n = \left( \int_0^S \sigma_n(x) \, dx \right)^2 - \left( \sqrt{\frac{2}{\pi}} \int_0^S \sigma_n(x) \, dx \right)^2 \\
= (1 - 2/\pi) \left( \int_0^S \sigma_n(x) \, dx \right)^2.
\]

Since \( Z_n \leq Z \) it follows for \( \varepsilon > 0 \)

\[
\mathbb{P} \left( Z \leq \varepsilon \int_0^S \sigma_n(x) \, dx \right) \leq \mathbb{P} \left( Z_n \leq \varepsilon \int_0^S \sigma_n(x) \, dx \right) \\
= \mathbb{P} \left( \mathbb{E}Z_n - Z_n \geq \mathbb{E}Z_n - \varepsilon \int_0^S \sigma_n(x) \, dx \right) \\
\leq \mathbb{P} \left( |\mathbb{E}Z_n - Z_n| \geq (\sqrt{2/\pi} - \varepsilon) \int_0^S \sigma_n(x) \, dx \right).
\]

By Chebyshev’s inequality,

\[
\mathbb{P} \left( Z \leq \varepsilon \int_0^S \sigma_n(x) \, dx \right) \leq \frac{\text{Var} Z_n}{(\sqrt{2/\pi} - \varepsilon)^2 \left( \int_0^S \sigma_n(x) \, dx \right)^2} \\
= \frac{(1 - 2/\pi) \left( \int_0^S \sigma_n(x) \, dx \right)^2}{(\sqrt{2/\pi} - \varepsilon)^2 \left( \int_0^S \sigma_n(x) \, dx \right)^2} \\
= \frac{1 - 2/\pi}{(\sqrt{2/\pi} - \varepsilon)^2}.
\]

Thus, for \( \varepsilon > 0 \) small enough,

\[
\mathbb{P} \left( Z \leq \varepsilon \int_0^S \sigma_n(x) \, dx \right) \leq c < 1.
\]

Note that the constant \( c \) depends only on \( \varepsilon \) but not on \( n \in \mathbb{N} \). Hence, taking
the limit $n \to \infty$, we obtain by the monotone convergence theorem

$$0 < \mathbb{P} \left( Z > \varepsilon \int_0^S \sigma(x) \, dx \right) = \mathbb{P} \left( \varepsilon^{-1} \int_0^S |\delta(x, X^{(A)})| \, dx > \int_0^S \sigma(x) \, dx \right).$$

Since $\int_0^S |\delta(x, X^{(A)})| \, dx < \infty$ almost surely it follows $\int_0^S \sigma(x) \, dx < \infty$ and finally

$$\mathbb{E} \int_0^S |\delta(x, X^{(A)})| \, dx = \sqrt{\frac{2}{\pi}} \int_0^S \sigma(x) \, dx < \infty.$$ 

**Theorem 6.** Almost surely, for almost all $0 \leq s < T$, the drift term $\delta(s, X^{(A)})$ in Theorem 5 is

$$\delta(s, X^{(A)}) = \lim_{r \searrow 0} \frac{\mathbb{E} [X^{(A)}_{s+r} | \mathcal{F}_s] - X^{(A)}_s}{r}.$$ 

**Proof.** Let $s \geq 0$ be fixed. By (24), for $r > 0$,

$$X^{(A)}_{s+r} = X^{(A)}_s + \alpha W'_{s+r} - \alpha W'_s + \int_s^{s+r} \delta(x, X^{(A)}) \, dx.$$

Hence, since $X^{(A)}_s$ is $\mathcal{F}_s$-measurable,

$$\mathbb{E} [X^{(A)}_{s+r} | \mathcal{F}_s] = X^{(A)}_s + \mathbb{E} [\alpha W'_{s+r} - \alpha W'_s | \mathcal{F}_s] + \mathbb{E} \left[ \int_s^{s+r} \delta(x, X^{(A)}) \, dx \bigg| \mathcal{F}_s \right].$$

Since $W'$ has independent increments with mean 0, the second term vanishes. By Proposition 5 we can apply Fubini’s theorem to the third term and get

$$\mathbb{E} [X^{(A)}_{s+r} | \mathcal{F}_s] = X^{(A)}_s + \int_s^{s+r} \mathbb{E} [\delta(x, X^{(A)}) | \mathcal{F}_s] \, dx.$$

Finally (see e.g. Corollary 2.14 in [7]),

$$\lim_{r \searrow 0} \frac{\mathbb{E} [X^{(A)}_{s+r} | \mathcal{F}_s] - X^{(A)}_s}{r} = \lim_{r \searrow 0} \frac{1}{r} \int_s^{s+r} \mathbb{E} [\delta(x, X^{(A)}) | \mathcal{F}_s] \, dx$$

$$= \mathbb{E} [\delta(s, X^{(A)}) | \mathcal{F}_s]$$

$$= \delta(s, X^{(A)})$$

for almost all $s \geq 0$. 

**5. The Markov property and the expected future**

In this section we assume that the Gaussian process $X = (X_s)_{s \in [0,T]}$ is a Markov process. Let $X^{(A)} = (X^{(A)}_s)_{s \in [0,T]}$ be the conditioned process of $X$ with respect to $A = \{a_1, \ldots, a_N\}$ and let $\mathcal{F}^{X^{(A)}} = (\mathcal{F}^{X^{(A)}}_s)_{s \in [0,T]}$ be the natural filtration of $X^{(A)}$. The process $X^{(A)}$ is in general not a Markov process as well.
5.1. Retrieving the Markov property

Define Gaussian processes $I^{(A)}_s$ by

$$I^{(A)}_s = \int_0^s X^{(A)}_x a_i(dx), \quad 0 \leq s \leq T, \quad 0 \leq i \leq N.$$  

**Theorem 7.** The Gaussian process $(X^{(A)}, I^{(A)}_1, \ldots, I^{(A)}_N)$ is an $(N + 1)$-dimensional (in general time-inhomogeneous) Markov process.

First, we show the result for the case that $X$ is Brownian motion and then the general case.

**Proof of Theorem 7 for $X$ Brownian motion.** We assume that $a_1(X), \ldots, a_n(X)$ are independent standard normal random variables. Without loss of generality we can do so by Proposition 2. For every $0 \leq s \leq t \leq T$ we define the Gaussian random variable $Z_{s,t}$ by

$$Z_{s,t} = X^{(A)}_t - \mathbb{E}[X^{(A)}_t | \mathcal{F}^{X^{(A)}}_s].$$

Then $Z_{s,t}$ is independent from $\mathcal{F}^{X^{(A)}}_s$. We show that

$$Z_{s,t} = X^{(A)}_t - \mathbb{E}[X^{(A)}_t | \{X^{(A)}_s, I^{(A)}_1, \ldots, I^{(A)}_N\}],$$  

which implies that $\mathbb{E}[X^{(A)}_t | \mathcal{F}^{X^{(A)}}_s] = \mathbb{E}[X^{(A)}_t | \{X^{(A)}_s, I^{(A)}_1, \ldots, I^{(A)}_N\}]$. Since the natural filtration of $X^{(A)}$ and $(X^{(A)}, I^{(A)}_1, \ldots, I^{(A)}_N)$ coincide, this will prove the theorem.

Set $\psi_i(y, s) = a_i(I_{[y, s]})$ and rewrite the Gaussian processes $I^i$ in (12) as

$$I^i_s = \int_0^s X_x a_i(dx) = \int_0^s \int_0^x dX_x a_i(dx) = \int_0^s a_i(dx) dX_x$$

$$= \int_0^s \psi_i(y, s) dX_x, \quad 0 \leq s \leq T, \quad 0 \leq i \leq N.$$  

We condition the process $X$ on $a_i(X) = I^i_T = 0$ almost surely for $1 \leq i \leq N$. Since we assume $I^1_T, \ldots, I^N_T$ to be independent random variables with $\mathbb{E}[I^i_T]^2 = 1$, the conditioned process $X^{(A)}$ and the processes $I^{(A)}_s$ are (as in Proposition 3) given by

$$X^{(A)}_s = X_s - \sum_{j=1}^N I^j_T \mathbb{E}[X_s I^j_T] \quad \text{and} \quad I^{(A)}_s = I^i_s - \sum_{j=1}^N I^j_T \mathbb{E}[I^s I^j_T],$$

$$0 \leq s \leq T, \quad 0 \leq i \leq N.$$  

Now, define Gaussian processes $J^i$ and $J^{(A)}_s$ by

$$J^i_s = \psi_i(s, T) X_s + I^i_s = \int_0^s (\psi_i(s, T) + \psi_i(y, s)) dX_y$$

$$= \int_0^s \psi_i(y, T) dX_y,$$

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and
\[
J_s^{(A),i} = \psi_i(s, T) X_s^{(A)} + I_s^{(A),i} \quad (31)
\]
\[
= \psi_i(s, T) \left( X_s - \sum_{j=1}^{N} I_t^{j} E \left[ X_s I_{t}^{j} \right] \right) + I_s^{i} - \sum_{j=1}^{N} I_t^{j} E \left[ I_s^{i} I_{t}^{j} \right] \\
= \psi_i(s, T) X_s + I_s^{i} - \sum_{j=1}^{N} I_t^{j} E \left[ (\psi_i(s, T) X_s + I_s^{i}) I_{t}^{j} \right] \\
= J_s^{i} - \sum_{j=1}^{N} I_t^{j} E \left[ J_s^{i} I_{t}^{j} \right], \quad 0 \leq s \leq T, 0 \leq i \leq N. \quad (32)
\]

By (31), it is enough to show
\[
Z_{s,t} = X_t^{(A)} - E[X_t^{(A)} | \{ X_s^{(A)}, J_s^{(A),1}, \ldots, J_s^{(A),N} \}]
\]
in order to show (27). Define
\[
Z_{s,t}^{*} = X_t^{(A)} - X_s^{(A)} - \sum_{i=1}^{N} b_i(s, t) J_s^{(A),i}, \quad (33)
\]
where the \(b_i\)’s are chosen such that \(Z_{s,t}^{*}\) is independent from \(J_s^{(A),i}, 1 \leq i \leq N\), i.e., we require
\[
0 = E \left[ Z_{s,t}^{*} J_s^{(A),i} \right] = E \left[ Z_{s,t}^{*} \left( J_s^{i} - \sum_{j=1}^{N} I_t^{j} E \left[ J_s^{i} I_{t}^{j} \right] \right) \right] \quad (34)
\]
\[
= E \left[ Z_{s,t}^{*} J_s^{i} \right] - \sum_{j=1}^{N} E \left[ Z_{s,t}^{*} I_{t}^{j} \right] E \left[ J_s^{i} I_{t}^{j} \right].
\]

By (29) and (32),
\[
Z_{s,t}^{*} = X_t - \sum_{i=1}^{N} I_t^{i} E \left[ X_s I_{t}^{i} \right] - X_s + \sum_{i=1}^{N} I_t^{i} E \left[ X_s I_{t}^{i} \right] \\
- \sum_{i=1}^{N} b_i(s, t) \left( J_s^{i} - \sum_{j=1}^{N} I_t^{j} E \left[ J_s^{i} I_{t}^{j} \right] \right) \\
= X_t - X_s - \sum_{i=1}^{N} b_i(s, t) J_s^{i} - \sum_{i=1}^{N} I_t^{i} E \left[ \left( X_t - X_s - \sum_{j=1}^{N} b_j(s, t) J_s^{j} \right) I_{t}^{i} \right],
\]
and thus

\[ \mathbb{E}[Z_{s,t}^* I_T^j] = \mathbb{E} \left[ (X_t - X_s - \sum_{i=1}^{N} b_i(s,t) J_s^i) I_T^j \right] \quad (36) \]

\[ - \sum_{i=1}^{N} \mathbb{E} [I_T^j I_T^i] \mathbb{E} \left[ (X_t - X_s - \sum_{k=1}^{N} b_k(s,t) J_s^k) I_T^i \right] = 0, \]

since we assumed \( \mathbb{E} I_T^j I_T^i = \delta_{i,j} \). Moreover,

\[ \mathbb{E}[Z_{s,t}^* J_s^j] = \mathbb{E} \left[ (X_t - X_s) J_s^j - \sum_{j=1}^{N} b_j(s,t) \mathbb{E} [J_s^j] \right] \]

\[ - \sum_{j=1}^{N} \mathbb{E} \left[ (X_t - X_s - \sum_{k=1}^{N} b_k(s,t) J_s^k) I_T^j \right] \mathbb{E} [I_T^j J_s^j], \]

where \( \mathbb{E} [(X_t - X_s) J_s^j] = 0 \) and \( \mathbb{E} [I_T^j J_s^j] = \mathbb{E} [J_s^j] \) by (28) and (30). Hence, (34) reduces to

\[ 0 = - \sum_{j=1}^{N} b_j(s,t) - \sum_{j=1}^{N} \mathbb{E} \left[ (X_t - X_s - \sum_{k=1}^{N} b_k(s,t) J_s^k) I_T^j \right]. \quad (37) \]

By (29) and (36), for all \( 0 \leq u \leq s, \)

\[ \mathbb{E}[Z_{s,t}^* X_u^{(A)}] = \mathbb{E}[Z_{s,t}^* X_u] - \sum_{j=1}^{N} \mathbb{E} [X_u I_T^j] \mathbb{E} [Z_{s,t}^* I_T^j] \]

\[ = \mathbb{E}[Z_{s,t}^* X_u]. \]

We replace \( Z_{s,t}^* \) by (35) and obtain

\[ \mathbb{E}[Z_{s,t}^* X_u^{(A)}] = \mathbb{E} [(X_t - X_s) X_u] - \sum_{i=1}^{N} b_i(s,t) \mathbb{E} [J_s^i X_u] \]

\[ - \sum_{i=1}^{N} \mathbb{E} \left[ (X_t - X_s - \sum_{j=1}^{N} b_j(s,t) J_s^j) I_T^i \right] \mathbb{E} [I_T^i X_u]. \]

Since, \( \mathbb{E} [(X_t - X_s) X_u] = 0 \) and \( \mathbb{E} [I_T^i X_u] = \mathbb{E} [J_s^i X_u] \) by (28) and (30) it follows

\[ \mathbb{E}[Z_{s,t}^* X_u^{(A)}] \]

\[ = \mathbb{E} [J_s^i X_u] \left( - \sum_{i=1}^{N} b_i(s,t) - \sum_{i=1}^{N} \mathbb{E} \left[ (X_t - X_s - \sum_{j=1}^{N} b_j(s,t) J_s^j) I_T^i \right] \right) \]

and thus \( \mathbb{E}[Z_{s,t}^* X_u^{(A)}] = 0 \) by (37). This implies \( Z_{s,t}^* = Z_{s,t} \). Hence, the theorem is proven for the case that \( X \) is standard linear Brownian motion. □
We now turn to the general case. For simplicity we assume that there are constants \(0 \leq c_1 \leq c_2 \leq T\) such that \(R_X(s,t) \neq 0\) for \(c_1 < s, t < c_2\) and \(R_X(s,s) = 0\) for \(0 \leq s < c_1\) and \(c_2 < s \leq T\). In [3] it was shown that there are (up to a constant) uniquely defined functions \(f : [0,T] \to \mathbb{R}\) and \(g : [0,T] \to \mathbb{R}\) such that \(h = f/g\) (with the convention \(0/0 = 0\)) is a non-negative, non-decreasing function on \([0,T]\) and

\[
R_X(s,t) = f(s \land t)g(s \lor t), \quad 0 \leq s, t \leq T.
\]

This implies

\[
X_s = g(s)W_{h(s)}
\]

in finite-dimensional distributions, where \(W = (W_s)_{s \geq 0}\) is a standard linear Brownian motion:

\[
\mathbb{E}g(s)W_{h(s)}g(t)W_{h(t)} = g(s)g(t)(h(s) \land h(t)) = g(s)g(t)h(s \land t)
\]

\[
= g(s)\frac{f(s \land t)}{g(s \land t)} = f(s \land t)g(s \lor t).
\]

Proof of Theorem 7 in the general case. We proceed in two steps: (i) we show Theorem 7 for \((\tilde{g}(s)W_s)_{s \in [0,T]}\) for every positive function \(\tilde{g}\); (ii) we prove the theorem for the process \((\tilde{X}_{h(s)})_{s \in [0,T]}\), where we assume the correctness of the theorem for the process \(\tilde{X}\).

Let \(h^{-1}\) be the inverse function of \(h\) (which exists since \(h\) is a non-decreasing function), i.e., we have \(h^{-1}(h(s)) = s\) for all \(0 \leq s \leq T\), and define \(\tilde{g} = g \circ h^{-1}\). Then, by (i), Theorem 7 holds true for \(\tilde{X} = \tilde{g}W\) and thus, by (ii), Theorem 7 holds true for \(X = \tilde{X} \circ h\), i.e.,

\[
X_s = \tilde{X}_{h(s)} = \tilde{g}(h(s))W_{h(s)} = (g \circ h^{-1} \circ h)(s)W_{h(s)} = g(s)W_{h(s)}.
\]

We prove (i): the Brownian motion \(W\) and the process \(\tilde{X} = \tilde{g}W\) on \([0,T]\) are generated by \(u : L_2([0,T]) \to C([0,T])\) and \(u_{\tilde{g}} : L_2([0,T]) \to C([0,T])\) with

\[
(ue)(s) = \int_0^se(x)\,dx, \quad (u_{\tilde{g}}e)(s) = \tilde{g}(s)\int_0^se(x)\,dx,
\]

\(e \in L_2([0,T]), \ 0 \leq s \leq T\). Define measures \(a_i^\tilde{g}\) by \(a_i^\tilde{g}(B) = \int_B \tilde{g}(x)a_i(dx),\ B \in \mathcal{B}([0,T]), \ 1 \leq i \leq N\). Then, \(a_i(\tilde{X}) = a_i^\tilde{g}(W)\) and \((u^*a_i^\tilde{g})(x) = (u_{\tilde{g}}^*a_i)(x) = a_i^\tilde{g}(\mathbb{I}_{[x,T]}).\) Hence,

\[
(u_{\tilde{g}}u_{\tilde{g}}^*a_i)(s) = \tilde{g}(s)\int_0^sa_i^\tilde{g}(\mathbb{I}_{[x,T]})\,dx = \tilde{g}(s)(u^*a_i^\tilde{g})(s).
\]

By Proposition 2 we may assume that the random variables \(a_1(\tilde{X}), \ldots, a_N(\tilde{X})\) are independent standard normal and thus, for \(W(A)\) being the conditioned
process of $W$ by $A^\delta = \{a_1^\delta, \ldots, a_N^\delta\}$ and $\hat{X}^{(A)}$ being the conditioned process of $X$ by $A$, $0 \leq s, t \leq T$,

$$
\mathbb{E}\left[\hat{X}_s^{(A)}\hat{X}_t^{(A)}\right] = \bar{g}(s)\bar{g}(t)(s \wedge t) - \sum_{i=1}^N (u_2 u^*_2 a_i)(s)(u_2 u^*_2 a_i)(t)
$$

$$
= \bar{g}(s)\bar{g}(t) \left( s \wedge t - \sum_{i=1}^N (uu^* a^\delta_i)(s)(uu^* a^\delta_i)(t) \right)
$$

$$
= \mathbb{E}\left[\bar{g}(s)W_s^{(A^\delta)}\bar{g}(t)W_t^{(A^\delta)}\right],
$$

i.e., the processes $\hat{X}^{(A)}$ and $\bar{g}W^{(A^\delta)}$ coincide in law. Consider the integrated processes $I^{(A),i}$ and $L^{(A^\delta),i}$ given by

$$
I^{(A),i}_s = \int_0^s \hat{X}^{(A)}(x) a_i(dx), \quad L^{(A^\delta),i}_s = \int_0^s W^{(A^\delta)}(x) a^\delta_i(dx), \quad 1 \leq i \leq N.
$$

From the proof of Theorem 7 for the case that $X$ is Brownian motion we know that $(W^{(A^\delta)}, L^{(A^\delta),1}, \ldots, L^{(A^\delta),N})$ is a Markov process. Since $\hat{X}^{(A)}/\bar{g}$ and $W^{(A^\delta)}$ coincide in law this implies that $(\hat{X}^{(A)}/\bar{g}, I^{(A),1}, \ldots, I^{(A),N})$ is a Markov process, where we used

$$
\int_0^s \hat{X}^{(A)}/\bar{g}(x) a^\delta_i(dx) = \int_0^s \hat{X}^{(A)}(x) a_i(dx) = I^{(A),i}_s, \quad 1 \leq i \leq N.
$$

Finally, this implies that $(\hat{X}^{(a),1}, \ldots, I^{(A),N})$ is a Markov process. This proves (i).

We prove (ii): Assume that Theorem 7 holds true for $\hat{X} = (\hat{X}_s)_{s \in [0,T]}$ and let $\tilde{X}$ be generated by $u : H \to C([0,T])$. Moreover, let $h$ be a non-negative, increasing function on $[0,T]$ with $h(T) = T'$. Define $X = (X_s)_{s \in [0,T]} = (\hat{X}_s h(s))_{s \in [0,T]}$. Then $X$ is generated by $uh : H \to C([0,T])$ with $(uh_e)(s) = (ue)(h(s)), e \in H$. Define measures $a^h_i$ by $a^h_i(B) = (a_i \circ h^{-1})(B), B \in \mathcal{B}([0,T]), 1 \leq i \leq N$. Then,

$$
a_i(X) = \int_0^T X_a a_i(dx) = \int_0^T \hat{X}_a h(x) a_i(dx) = \int_{h(0)}^{h(T)} \hat{X}_a (a \circ h^{-1})(dx).
$$

(38)

If $h(0) > 0$ then, since $h$ is increasing, $h^{-1}([0,h(0)]) = \emptyset$, and thus

$$
a_i(X) = \int_0^{T'} \hat{X}_a (a \circ h^{-1})(dx) = a^h_i(X).
$$

(39)

In the same way we get for all $e \in H$,

$$
\langle u^*a_i, e \rangle = \int_0^T (ue)(h(x)) a_i(dx) = \int_0^{T'} (ue)(x) a^h_i(dx) = \langle u^*a^h_i, e \rangle
$$
and thus \( u_i^* a_i = u^* a_i^h \), \( 1 \leq i \leq N \). By Proposition 2 we may assume that the random variables \( a_1(X), \ldots, a_N(X) \) are independent standard normal and thus, for \( X^{(A)} \) being the conditioned process of \( X \) with respect to \( A \) and \( \tilde{X}^{(A^b)} \) being the conditioned process of \( \tilde{X} \) with respect to \( A^b = \{a_1^b, \ldots, a_N^b\} \), \( 0 \leq s, t \leq T \),

\[
\mathbb{E} \left[ X_s^{(A)} X_t^{(A)} \right] = \mathbb{E} [X_s X_t] - \sum_{i=1}^{N} (u_i^* a_i)(s)(u_i^* a_i)(t) = \mathbb{E} \left[ \tilde{X}_{h(s)} \tilde{X}_{h(t)} \right] - \sum_{i=1}^{N} (u_i^* a_i^b)(h(s))(u_i^* a_i^b)(h(t)) = \mathbb{E} \left[ \tilde{X}^{(A^b)} \tilde{X}^{(A^b)} \right],
\]

i.e., the processes \( X^{(A)} \) and \( \tilde{X}^{(A^b)} \) coincide in law. Consider the integrated processes \( L^{(A^b),i} \) given by \( L^{(A^b),i}_{s} = \int_{0}^{s} \tilde{X}^{(A^b)} a_i^b (dx) \), \( 1 \leq i \leq N \). Then, as in (38) and (39), for \( 0 \leq s \leq T \),

\[
I^{(A)}_{s} = \int_{0}^{s} X^{(A)}_{u} a_i (dx) = \int_{0}^{s} \tilde{X}^{(A^b)}_{h(u)} a_i (dx) = \int_{0}^{h(s)} \tilde{X}^{(A^b)}_{h(u)} a_i^b (dx) = L^{(A^b)}_{h(s)}
\]

in finite-dimensional distributions. By the assumption on \( \tilde{X} \) the process \( (\tilde{X}^{(A^b)}_{s}, L^{(A^b),1}_{s}, \ldots, L^{(A^b),N}_{s})_{s \in [0,T]} \) is a Markov process implying that \( (\tilde{X}^{(A^b)}_{h(s)}, L^{(A^b),1}_{h(s)}, \ldots, L^{(A^b),N}_{h(s)})_{s \in [0,T]} \) is a Markov process as well. Since \( (X^{(A)}, I^{(A),1}, \ldots, I^{(A),N})_{s \in [0,T]} \) and \( (\tilde{X}^{(A^b)}_{h(s)}, L^{(A^b),1}_{h(s)}, \ldots, L^{(A^b),N}_{h(s)})_{s \in [0,T]} \) coincide in law we conclude that \( (X^{(A)}, I^{(A),1}, \ldots, I^{(A),N})_{s \in [0,T]} \) is a Markov process. \( \square \)

5.2. The expected future

Now, we can give an explicit formula for \( \mathbb{E} [X_t^{(A)} | \tilde{S}^{X^{(A)}}_s] \), \( s < t \leq T \). This together with Theorem 5 enables us to calculate the drift term in Theorem 6 in the case that \( X \) is Markovian. Define a matrix \( D_s \) by

\[
D_s = \begin{pmatrix}
g(s) & (ue_1)(s) & \cdots & (ue_N)(s) \\
\int_{a_1} g(x) a_1 (dx) & \int_{a_1} (ue_1)(x) a_1 (dx) & \cdots & \int_{a_1} (ue_N)(x) a_1 (dx) \\
\vdots & \vdots & \ddots & \vdots \\
\int_{a_N} g(x) a_N (dx) & \int_{a_N} (ue_1)(x) a_N (dx) & \cdots & \int_{a_N} (ue_N)(x) a_N (dx)
\end{pmatrix}
\]

and a vector \( d_s \) by

\[
d_s = \left( \begin{array}{c}
X_s^{(A)} \\
-I^{(A),1} s \\
\vdots \\
-I^{(A),N} s
\end{array} \right)^T.
\]

**Theorem 8.** For every \( s < t \) there are \( \tilde{S}^{X^{(A)}}_s \)-measurable random variables \( \xi_0, \ldots, \xi_N \) such that

\[
\mathbb{E} [X_t^{(A)} | \tilde{S}^{X^{(A)}}_s] = \xi_0 g(t) + \sum_{i=1}^{N} \xi_i (ue_i)(t).
\]

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Assume that the matrix \( D_s \) is invertible. Then \( \xi = (\xi_0, \ldots, \xi_N)^T \) is given by \( \xi = D_s^{-1}d_s \).

Proof. For \( s < t \) and \( 1 \leq i \leq N \), we have

\[
E \left[ I_s^{(A),i} X_t^{(A)} \right] = E \left[ \int_0^s X_x^{(A)} a_i(dx) X_t^{(A)} \right] = \int_0^s E \left[ X_x^{(A)} X_t^{(A)} \right] a_i(dx)
\]

\[
= \int_0^s R_{X^{(A)}}(x,t) a_i(dx)
\]

\[
= \int_0^s \left( f(x)g(t) - \sum_{i=1}^N (ue_i)(x)(ue_i)(t) \right) a_i(dx)
\]

\[
= g(t) \int_0^s f(x) a_i(dx) - \sum_{i=1}^N (ue_i)(t) \int_0^s (ue_i)(x) a_i(dx).
\]

In particular, \( E \left[ I_s^{(A),i} X_t^{(A)} \right] \) is a deterministic linear combination of \( g(t) \) and \( (ue_i)(t), 1 \leq i \leq N \).

By Theorem 7

\[
E[X_t^{(A)} | \mathcal{F}_s^{X^{(A)}}] = E[X_t^{(A)} | \{ X_s^{(A), i}, I_s^{(A),1}, \ldots, I_s^{(A),N} \}].
\]

Assume without loss of generality that \( \{ X_s^{(A), i}, I_s^{(A),1}, \ldots, I_s^{(A),N} \} \) are orthonormal random variables (otherwise orthonormalize them similar to (13)). Then, by the general theory of conditioning of Gaussian random variables,

\[
E[X_t^{(A)} | \mathcal{F}_s^{X^{(A)}}] = X_s^{(A)}E \left[ X_t^{(A)} X_s^{(A)} \right] + \sum_{i=1}^N I_s^{(A),i}E \left[ X_t^{(A)} I_s^{(A),i} \right].
\]

Since \( E \left[ X_s^{(A)} X_t^{(A)} \right] \) and \( E \left[ I_s^{(A),i} X_t^{(A)} \right] \) are deterministic linear combinations of \( g(t) \) and \( (ue_i)(t), 1 \leq i \leq N \), there are \( \mathcal{F}_s^{X^{(A)}} \)-measurable random variables \( \xi_0, \ldots, \xi_N \) such that

\[
E[X_t^{(A)} | \mathcal{F}_s^{X^{(A)}}] = \xi_0 g(t) + \sum_{i=1}^N \xi_i(ue_i)(t).
\]

In order to determine \( \xi_0, \ldots, \xi_N \), consider the process \( Z = (Z_t)_{t \in [0,T]} \) defined by

\[
Z_t = \begin{cases} X_t^{(A)}, & \text{for } t \leq s, \\ E[X_t^{(A)} | \mathcal{F}_s^{X^{(A)}}], & \text{for } t > s. \end{cases}
\]

\( Z \) is continuous and fulfills the conditions \( a_1, \ldots, a_N \), i.e.,

\[
Z_s = X_s^{(A)} = \lim_{t \searrow s} Z_t = \lim_{t \searrow s} E[X_t^{(A)} | \mathcal{F}_s^{X^{(A)}}] = \xi_0 g(s) + \sum_{i=1}^N \xi_i(ue_i)(s),
\]

\[
\text{for } t \searrow s.
\]
and

\[ 0 = \int_0^T Z_x a_j(dx) = \int_0^s X_x^{(A)} a_j(dx) + \int_s^T \mathbb{E}[X_x^{(A)} | \mathcal{F}_s] a_j(dx) = I_s^{(A), j} + \xi_0 \int_{s+}^T g(x) a_j(dx) + \sum_{i=1}^N \xi_i \int_{s+}^T (ue_i)(x) a_j(dx), \]

i.e.,

\[ -I_s^{(A), j} = \xi_0 \int_{s+}^T g(x) a_j(dx) + \sum_{i=1}^N \xi_i \int_{s+}^T (ue_i)(x) a_j(dx), \quad 1 \leq j \leq N. \]

This leads to the system of linear equations \( D_s \xi = d_s \) and its solution \( \xi = D_s^{-1} d_s. \)

6. Examples

6.1. The zero area Brownian bridge

The standard linear Brownian motion \( W = (W_s)_{s \in [0, 1]} \) on \([0, 1]\) is generated by the operator \( u : L^2([0, 1]) \rightarrow C([0, 1]) \) with

\[ (uh)(s) = \int_0^s h(x) \, dx \]

for \( h \in L^2([0, 1]). \) For example, the trigonometric basis in \( L^2([0, 1]), \)

\[ \{ e_n : n \geq 0 \} = \{ 1 \} \cup \{ \sqrt{2} \cos(\pi n x) : n \geq 1 \}, \]

for which \( (ue_0)(s) = s \) and

\[ (ue_n)(s) = \int_0^s \sqrt{2} \cos(\pi n x) \, dx = \sqrt{2} \frac{\sin(\pi n s)}{\pi n}, \]

yields the well known representation

\[ W_s = \omega_0 s + \sqrt{2} \sum_{n=1}^\infty \omega_n \frac{\sin(\pi n s)}{\pi n}. \]

Let \( M = (M_s)_{s \in [0, 1]} \) be the Brownian motion conditioned to be zero at time 1 and with integral zero, i.e., \( M = W^{(A)} \) for \( A = \{ \delta_1, a_0 \} \subset C([0, 1])^* \) with

\[ \delta_1(f) = f(1) \quad \text{and} \quad a_0(f) = \int_0^1 f(s) \, ds, \quad f \in C([0, 1]). \]

It holds

\[ (u^* \delta_1)(x) = 1 \quad \text{and} \quad (u^*a_0)(x) = 1 - x. \]
The detached subspace $H_A$ of $L_2([0,1])$ with respect to the set of conditions $A = \{\delta_1, a_0\} \subset C([0,1])^*$ is thus $H_A = \text{span}\{1, 1-x\}$. An orthonormal basis in $H_A$ is $\{e_1, e_2\} = \{1, \sqrt{3}(2x-1)\}$. Hence, according to Proposition 1 the covariance of the zero area Brownian bridge $M = W(A)$ is given by $(0 \leq s, t \leq 1)$

$$R_M(s,t) = R_W(s,t) - (ue_1)(s)(ue_1)(t) - (ue_2)(s)(ue_2)(s)$$

$$= \min\{s, t\} - \int_0^s dx \int_0^t dy - \int_0^s \sqrt{3}(1-2x) dx \int_0^t \sqrt{3}(1-2y) dy$$

$$= \min\{s, t\} - st - 3(s - s^2)(t - t^2).$$

Using the notation from Theorem 3 the matrix $B$ and the vector $b$ become

$$B = \begin{pmatrix} \delta_1(u e_1) & \delta_1(u e_2) \\ a_0(u e_1) & a_0(u e_2) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1/2 & 1/(2\sqrt{3}) \end{pmatrix}$$

and $b = \begin{pmatrix} \delta_1(W) \\ a_0(W) \end{pmatrix} = \begin{pmatrix} W_1 \\ I_1 \end{pmatrix},$

where $I_s = \int_0^s W_t dx$. Solving the linear equation system $B\xi = b$ yields

$$\xi_1 = W_1 \quad \text{and} \quad \xi_2 = \sqrt{3}(2I_1 - W_1).$$

Then, by Theorem 3 an anticipative representation for $M$ is

$$M_s = W_s - W_1s - \sqrt{3}(2I_1 - W_1)\sqrt{3}(s - s^2)$$

$$= W_s - s(3s - 2)W_1 - 6s(1-s)I_1.$$

Let $\mathbb{P}_W$ and $\mathbb{P}_M$ be the induced measures of $W$ and $M$ on $(C([0,1]),C)$. For every $s < 1$ the condition in (15) is fulfilled. Hence, by Theorem 4 the measures $\mathbb{P}_W$ and $\mathbb{P}_M$ are equivalent on $\mathfrak{F}_s$, for every $s < 1$, where, as in Theorem 4, $\mathfrak{F}_s \subset C$ is the smallest $\sigma$-algebra on $C([0,T])$ such that all point evaluation functionals $\delta_x, 0 \leq x \leq s$, are $\mathfrak{F}_s - \mathfrak{B}(\mathbb{R})$-measurable.

By Theorem 5, $M$ is a solution of the stochastic differential equation

$$dM_s = dW_s + \delta(s,M)ds, \quad M_0 = 0, \quad 0 \leq s < 1,$$

where $\delta$ is a progressively measurable functional on $C([0,1])$. By Theorem 6

$$\delta(s,M) = \lim_{r \to 0} \mathbb{E}[M_{s+r} | \mathfrak{F}_s^M] - M_s, \quad 0 \leq s < 1,$$

where $\mathfrak{F}_s^M$ is the natural filtration of $M$ at time $s$. Define $J_s = \int_0^s M_t dx$, $0 \leq s \leq 1$. Since $(W_s)_{s \in [0,1]}$ is a Markov process, $(M_s, J_s)_{s \in [0,1]}$ is a Markov process as well by Theorem 7. By Theorem 8, for $0 \leq s \leq t < 1$, we have

$$\mathbb{E}[M_t | \mathfrak{F}_s^M] = \xi_0 + \xi_1s + \xi_2\sqrt{3}(t - t^2),$$

where $\xi = (\xi_0, \xi_1, \xi_2)$ is the solution of the system of linear equations $D_s\xi = d_s$ with $d_s = (M_s, 0, -J_s)$ and

$$D_s = \begin{pmatrix} 1 & s & \sqrt{3}(s - s^2) \\ 1 & 1/2 & \sqrt{3}(1 - s^2)/2 \\ 1-s & (1-s^2)/2 & \sqrt{3}(1 - s^2)/2 - (1-s^2)/\sqrt{3} \end{pmatrix}. $$

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Solving this system of linear equations yields

\[ \begin{align*}
\xi_0 &= \frac{M_s(2s^2 - s - 1) - 6Js}{(s-1)^3} \\
\xi_1 &= -\frac{M_s(2s^2 - s - 1) - 6Js}{(s-1)^3} \\
\xi_2 &= -\sqrt{3}\frac{M_s(s-1) - 2Js}{(s-1)^3},
\end{align*} \]

and thus

\[ \mathbb{E}[M_t \mid \mathcal{F}_s^M] = \frac{M_s(2s^2 - s - 1) - 6Js}{(s-1)^3} - \frac{3(t-t^2)}{(s-1)^3} \frac{M_s(s-1) - 2Js}{(s-1)^3}. \]

We have

\[ \lim_{r \searrow 0} \frac{\mathbb{E}[M_{s+r} \mid \mathcal{F}_s^M] - M_s}{r} = -\frac{4M_s}{1-s} - \frac{6Js}{(1-s)^2}. \]

Hence, \( M \) has the stochastic differential

\[ dM_s = dW_s - \frac{4M_s}{1-s} ds - \frac{6Js}{(1-s)^2} ds, \quad M_0 = 0, \quad 0 \leq s < 1. \]

### 6.2. Gaussian bridges

The conditioning of a Gaussian process on \([0, T]\) to be zero at time \( T \) is a well-studied but important example (see for example [5]). This leads to Gaussian bridges: let \( X_s = (X_s)_{s \in [0, T]} \) be a continuous Gaussian process and let \( \delta_T \in C([0, T])^* \) be the evaluation functional at point \( T \). Then \( X^{(\delta_T)} \) is called the bridge process of \( X \).

**Proposition 6.** The covariance \( R_{X^{(\delta_T)}}(s, t) = \mathbb{E}X_s^{(\delta_T)}X_t^{(\delta_T)} \) is

\[ R_{X^{(\delta_T)}}(s, t) = R_X(s, t) - \frac{R_X(s, T) R_X(t, T)}{R_X(T, T)}, \quad 0 \leq s, t \leq T, \]

where \( R_X(s, t) = \mathbb{E}X_sX_t \) is the covariance function of \( X \), and a anticipative representation for \( X^{(\delta_T)} \) is

\[ X_s^{(\delta_T)} = X_s - \frac{R_X(s, T)}{R_X(T, T)} X_T, \quad 0 \leq s \leq T. \]

**Proof.** Let \( X \) be generated by the linear and bounded operator \( u : H \to C([0, T]) \) and let \( (e_i)_{i=1}^\infty \) be an orthonormal basis in the separable Hilbert space \( H \). By [10], the detached Hilbert space \( H^{(\delta_T)} \) with respect to the condition \( \delta_T \) is spanned by

\[ u^* \delta_T = \sum_{i=1}^\infty (u^* \delta_T, e_i) e_i = \sum_{i=1}^\infty (ue_i)(T)e_i. \]
By Parseval’s identity and (8),

\[
\|u^*\delta_T\|^2 = \sum_{i=1}^{\infty} |(u^*\delta_T, e_i)|^2 = \sum_{i=1}^{\infty} (ue_i)(T)(ue_i)(T) = R_X(T, T).
\]

Hence, by Proposition 1 in the first line and (8) in the last line

\[
R_X(\delta_T, s, t) = R_X(s, t) - \frac{\sum_{i=1}^{\infty} (ue_i)(T)(ue_i)(s) \sum_{i=1}^{\infty} (ue_i)(T)(ue_i)(t)}{R_X(T, T)}
\]

\[
= R_X(s, t) - \frac{R_X(s, T)R_X(t, T)}{R_X(T, T)}.
\]

The anticipative representation of \(X(\delta_T)\) follows by Theorem 3.

\[\square\]

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