Kolmogorov scaling from random force fields

M. H. Jensen\textsuperscript{1(a)}, K. Sneppen\textsuperscript{1(b)} and L. Angheluta\textsuperscript{2(c)}

\textsuperscript{1} Niels Bohr Institute - Blegdamsvej 17, Dk 2100, Copenhagen, Denmark, EU\textsuperscript{(d)}
\textsuperscript{2} Center for Physics of Geological Processes, University of Oslo - Oslo, Norway

received 11 April 2008; accepted in final form 25 August 2008
published online 24 September 2008

PACS 05.20.Jj – Statistical mechanics of classical fluids
PACS 05.40.-a – Fluctuation phenomena, random processes, noise, and Brownian motion
PACS 47.27.-i – Turbulent flows

Abstract – We show that the classical Kolmogorov and Richardson scaling laws in fully developed turbulence are consistent with a random Gaussian force field. Numerical simulations of a shell model for turbulence suggest that the fluctuations in the force (acceleration) field are scale independent throughout the inertial regime. We find that Lagrangian statistics of the relative velocity in a turbulent flow is determined by the typical force field, whereas the multiscaling is associated to extreme events in the force field fluctuations.

Copyright \textcopyright{} EPLA, 2008

In studies of fully developed turbulence, two discoveries are highly noticeable as fundamental and seminal. One regards Richardson’s study of the enhanced dispersion of particles advected by a turbulent flow [1]. The other is the Kolmogorov’s fundamental derivation, essentially based on dimensional arguments, of the energy spectrum in fully developed turbulence [2]. Both theories employ the energy cascade, from the integral scale down to the dissipation scale, as the paradigmatic physical picture of the energy dissipation flow. Indeed, pair-particles passively advected by turbulence exhibit a superdiffusive behavior with their relative distance given by Richardson’s scaling as observed both experimentally [3], analytically [4], and in direct numerical simulations [5,6]. In the velocity space no superdiffusive behavior is needed to substantiate superdiffusion in real space.

In this letter, we show that the velocity increments generated by a white-noise force field are sufficient to generate the superdiffusive behavior, as well as the Kolmogorov energy spectrum. To put it in very simple terms: integrating “up” from the random acceleration field to the velocity field and subsequently to the displacement is enough to reproduce the well-known scaling laws.

To clarify the underlying physical picture, we consider a simple stochastic model of relative dispersion in a white-noise acceleration field given by

$$\frac{d\Delta v}{dt} = \Delta F(t),$$

(1)

$$\langle \Delta F(t')\Delta F(t'') \rangle = 6\epsilon^* \delta(t' - t''),$$

(2)

where $\Delta v(t) = v_1(t) - v_2(t)$ is the velocity difference between the two particles moving along the two trajectories $r_1(t)$ and $r_2(t)$, and $\Delta F(t) = F_1(t) - F_2(t)$ is the relative force. The prefactor $6$ instead of the usual factor $2$ appearing in the force correlation is due to the parametrization of the relative dispersion in terms of the diffusion constant $\epsilon^*$ for a single particle in the velocity space. The $\delta$-function may have, in principle, a finite width given by the time correlation of the relative random field along the two trajectories. This width will be determined both by the time it takes to pass a correlation length for a given force realization, and the time it takes to change the force in a certain point of the system.

In this set up, the relative velocity field, $\Delta v(t) = \int_0^t \Delta F(s)ds$, is a Wiener process with a Gaussian distribution, which leads to

$$\langle \Delta v(t_1)\Delta v(t_2) \rangle = \int_0^{t_1} \int_0^{t_2} \langle \Delta F(t')\Delta F(t'') \rangle dt' dt''$$

$$= 6\epsilon^* \int_0^{t_1} \int_0^{t_2} \delta(t' - t'') dt' dt''$$

$$= 6\epsilon^* \int_0^{\min(t_1,t_2)} dt'$$

$$= 6\epsilon^* \min(t_1,t_2),$$

(3)
implying that the mean square velocity difference is
\begin{equation}
\langle \Delta v^2(t) \rangle = 6 \epsilon^* t.
\end{equation}
The relative separation is described by a non-Gaussian distribution with the second moment satisfying Richardson’s scaling, that is
\[ \langle \Delta r^2(t) \rangle = \int_0^t \int_0^t \langle \Delta v(t_1) \Delta v(t_2) \rangle dt_1 dt_2 = 6 \epsilon^* t^3 \] 
By eliminating the time dependence of the relative velocity and distance, we obtain the exact Kolmogorov scaling,
\begin{equation}
\langle \Delta v^2(t) \rangle = 2^{2/3} 3 \epsilon^{*2/3} \langle \Delta r^2(t) \rangle^{1/3},
\end{equation}
in the Lagrangian framework (for the higher moments see footnote 1). Thus, Kolmogorov scaling is consistent with the assumption that the dispersion is driven by sufficiently random and uncorrelated acceleration fields.

In deriving eq. (6) we assumed that the relative velocity is obtained by following the Lagrangian trajectories, which in a real turbulent flow may differ from the typical velocity increments separated by the distance \( r \) (Eulerian measurement of the velocity differences) [7]. Many other stochastic models for relative dispersion have been proposed in the literature. Several of these models were based on Markov processes with some functional assumptions of the velocity profiles [8–12]. Other models were based on Langevin dynamics to various approximations of the Navier-Stokes equations [13].

Equation (4) implies that \( 2 \epsilon^* \) is the diffusion constant for the relative velocity. For a Lagrangian stochastic flow generated by the white-noise acceleration field, \( \epsilon^* \) can be estimated as
\[ \langle \Delta \vec{v}(t) \Delta \vec{F}(t) \rangle = \int_0^t ds \langle \Delta \vec{F}(s) \cdot \Delta \vec{F}(t) \rangle = \int_0^t ds \langle \Delta \vec{F}(s) \cdot \Delta \vec{F}(t) \rangle = 6 \epsilon^* t^3. \] 
From dimensional considerations, \( \epsilon^* \) has the same units [length \(^2\)/time \(^3\)] as the standard energy dissipation rate \( \epsilon \) characterizing the turbulence cascade.

To examine how the Lagrangian white-noise acceleration relates to the anomalous scaling laws in a more realistic turbulent field, we consider the kinematics of pair particles advected by the homogeneous turbulent flow obtained by a real-space transformation of the GOY shell model [14,15]. This model proposed originally by Gledzer, Yamada and Ohkitani [16,17] provides a description of the turbulent motion embodied in the Navier-Stokes equations. The GOY model is formulated on a \( N \)-discrete set of wave numbers, \( k_n = 2^n \), with the associated Fourier modes \( u_n \) evolving according to
\begin{equation}
\frac{d}{dt} + \nu k_n^2 u_n = i k_n \left( a_n u_n u_{n+1} + u_n^* u_{n+1}^* + \frac{c_n}{4} u_{n-1}^* u_{n-2}^* \right) + f \delta_{n,1},
\end{equation}
for \( n = 1 \ldots N \). The coefficients of the non-linear terms are constrained by two conservation laws, namely the total energy, \( E = \sum_n |u_n|^2 \), and the helicity (for 3-D), \( H = \sum_n (-1)^n k_n |u_n|^2 \), or the enstrophy (for 2-D), \( Z = \sum_n k_n^4 |u_n|^2 \), in the inviscid limit, i.e. \( f = \nu = 0 \) [18].

Therefore, they may be expressed in terms of a free parameter only \( \delta \in [0,2] \), \( a_n = 1 \), \( b_n+1 = -\delta \), \( c_n+2 = -(1-\delta) \). As observed by Kadanoff [19], one obtains the canonical value \( \delta = 1/2 \), when the 3d-helicity is conserved. The set (8) of \( N \)-coupled ordinary differential equations can be numerically integrated by standard techniques [20]. We have used standard parameters in this paper \( N = 19 \), \( \nu = 10^{-6} \), \( k_0 = 2 \cdot 10^{-4} \), \( f = 5 \cdot 10^{-3} \).

The GOY model is defined in \( k \)-space but we study particle dispersion in direct space obtained by an inverse Fourier transform [14] of the form
\[ \vec{v}(\vec{r},t) = \sum_{n=1}^N \vec{e}_n [u_n(t) e^{i k_n \cdot \vec{r}} + c.c.]. \] 
Here the wave vectors are \( \vec{k}_n = k_n \vec{e}_n \) where \( \vec{e}_n \) is a unit vector in a random direction, for each shell \( n \) and \( \vec{e}_n \) are unit vectors in random directions. We ensure that the velocity field is incompressible, \( \nabla \cdot \vec{v} = 0 \), by constraining \( \vec{e}_n \cdot \vec{e}_n = 0, \forall n \). In our numerical computations we consider the vectors \( \vec{e}_n \) and \( \vec{e}_n \) quenched in time but averaged over many different realizations of these.

As an example of the motion in this field, fig. 1 shows the trajectories of two passively advected particles. As the relative distance diverges in time, the two particles experience different force fields, which in turn typically increase the difference in the relative velocities of the two particles. The figure shows the individual particles as they are advected, first together and later diverging away from each other when they are encased in different eddies.

Figure 2 examines the noise in the effective force field \( \langle \Delta F \Delta F' \rangle \) for the relative motion of the two advected particles. In fig. 2 we use viscosity \( \nu = 10^{-6} \), with a Kolmogorov scale \( \Delta r \sim 1.0 \cdot 10^{-4} \). The noise amplitude is plotted vs. the average square distance between the particles \( \langle \Delta r^2 \rangle = (r_1(t) - r_2(t))^2 \), with the time as parametrization of the curves, as in eq. (6). The average is over independent trials of the two advected particles. One observes that both the
Kolmogorov scaling from random force fields

\[ \Delta \mathbf{r}(t) = \mathbf{r}_1(t) - \mathbf{r}_2(t) \]

\[ \frac{d\Delta \mathbf{r}}{dt} = \mathbf{v}(r(t)) \]

\[ \frac{d\Delta \mathbf{F}}{dt} = F(r(t)) \]

**Fig. 1:** Two particles being advected in a random force field, generated by the GOY shell model.

**Fig. 2:** The squared relative acceleration \( \Delta \mathbf{F} \) and its infinite moment versus \( \Delta \mathbf{r}^2 \). The thin lines are for the Lagrangian trajectories where distances and velocities are parameterized by the time of advection. The squares represent the corresponding Eulerian measures. The straight line represents standard Kolmogorov scaling \( \langle \Delta \mathbf{v}^2 \rangle \propto \langle \Delta \mathbf{r}^2 \rangle^{1/3} \).

The typical value of the squared noise and its infinite moment \( \langle \max \{ \Delta \mathbf{v}^2(t) \} \rangle \) at any distance is constant throughout the inertial range, \textit{i.e.} above the Kolmogorov scale.

We conclude that the force field is equivalent to Gaussian white noise, and therefore the structure function of this turbulent field should be close to the one predicted by eq. (6). This is confirmed in fig. 3 where we show the deviations in velocity as a function of the square distance between the particles, that is the plot is parameterized through the time \( t \) as indicated in eq. (6). One indeed sees that \( \langle \Delta \mathbf{v}^2(t) \rangle \) vs. \( \langle \Delta \mathbf{r}^2(t) \rangle \) scales with an exponent close to 1/3 in agreement with our expectations. Concerning the Richardson scaling law eq. (6), we observe in the GOY simulations a long Batchelor regime \( \langle \Delta \mathbf{v}^2(t) \rangle \sim t^2 \) before it reaches the Richardson law in agreement with recent experimental observations [21]. To study pair particle dispersion, we thus advocate to perform the scaling plot eq. (6) with time as parameter and believe this is why we observe similar behavior for both Lagrangian and Eulerian measurements. For completeness, we in fig. 3 also show the infinite moment of the velocity, and remark that this higher moment scales with an exponent close to 0.23. This signals multidiffusion [22] where extreme velocity differences sometimes, but rarely, are reached after short separations. In the current context, we see these extreme deviations as a measure of very unlikely and intermittent events which only add little to the typical behavior of the flow. Indeed also the Eulerian statistics shows clear multiscaling as expected [14].

While our intuition has been based on the Lagrangian picture of advected particles, it is interesting that the corresponding Eulerian quantities behave similarly. This is demonstrated in simulations where we now fix the distance between two points, and then calculate, respectively, the difference in velocity and acceleration. The squares in figs. 2 and 3 show how \( \langle \Delta \mathbf{F}^2(r) \rangle \) and \( \langle \Delta \mathbf{v}^2(r) \rangle \) vary with the square relative distance between the investigated points. From fig. 2 we see that the value of the plateau for the random force field is a direct consequence of its random expectation at any large distance. Therefore, there is nothing special about the selection of advected points in the Lagrangian case. In fact the onset of the plateau is slightly sharper in the Eulerian case, presumably reflecting averaging associated to the underlying time parameter in the Lagrangian advection. Similarly, there is no significant difference for the structure functions shown in fig. 3. That the Lagrangian and Eulerian quantities behave similarly...
for relative velocity differences have also been observed in previous studies, see for instance [23,24].

Using eq. (7) we estimate $\langle \Delta v \Delta F \rangle \sim 0.1$ throughout the inertial range in the GOY model simulations, see fig. 4. This value of the effective velocity diffusion constant is larger than the average energy dissipation at the Kolmogorov scale, estimated from the energy input $R e \langle u \cdot f \rangle = 0.001$ in the GOY model. This discrepancy in the effective diffusion terms we attribute to the huge contributions from the maybe unrealistically huge spikes in the acceleration of the GOY model. Spikes which of course are absent in the simple white noise calculation of eq. (7). These spikes also gives rise to multifidiffusion, as discussed above.

We believe that the acceleration field as shown in fig. 2 should be experimentally accessible either by particle tracking in a 3-D flow [25,26] or from probe measurements in channel flows employing the Taylor hypothesis. In the first case the acceleration is easily estimated from the temporal variations in the velocity field of the 3-D advected particles. In fact [25] investigated fluctuations in single path accelerations, which were found to be larger than Gaussian expectations. More interestingly, in relation to our work, [25] also reports accelerations that are independent of the Reynolds number, resembling our findings of an acceleration difference that is independent of scale.

Overall we have seen that the variance of the force field reaches an average value that is independent of the distance between the advected points in the turbulent fluid. Already at distances slightly above the Kolmogorov scale the two particles often receive random “kicks” which are as large at small scales as they are at the integral scale. Thus, huge accelerations are associated to the very small scales, presumably to the core of eddies at the verge of their destruction by dissipation. The acceleration between two particles is primarily dependent on how close each of them are to the center of an eddy. Since accelerations are largest at the core of eddies, the relative acceleration will be dominated by the one particle that circles fastest around its eddy [27].

When examining the distribution of the accelerations at a fixed distance, the GOY shell model simulations predict a broad power law like behavior with a cutoff which is independent of the distance (as demonstrated by the constant max norm). The size of the cutoff is determined by the size of the forcing and the scale at which this forcing is acting (in our simulations, the scale is $\Delta r = 1$). In experiments the single particle acceleration has broad tails, characterized by a stretched exponential [25]. Thus already the fat tails are narrower than a power law and thus also much narrower than the GOY model results. Thus a Gaussian assumption is fairly reasonable, with limitations only imposed by time correlations. The extreme events in the tails of the single particle acceleration are presumably correlated [25], reflecting fast circulation around the core of a vortex as seen in DNS simulation [27]. In that case the repeated circulation of large accelerations contribute less to the velocity drift than a non-correlated acceleration of same magnitude would do. Also the effective $\epsilon^*$ would be lower than the one deduced directly from the width of the single-particle acceleration distribution measured in [25]. In real turbulence, the trapping of particles in vortex cores becomes less important with increasing Reynolds number [28], in which case $\epsilon^*$ estimated from eq. (7) would be close to the effective diffusion constant for the velocity field.

In conclusion, the motion associated with the relatively slow turn-over dynamics of the large eddies is not needed for obtaining Richardson or Kolmogorov statistics. These two seminal laws can be deduced from the simple assumption of a random force field that fluctuates with an amplitude set by the system size and with a correlation time set by the Kolmogorov scale. Obviously these assumptions are not realistic, but to some extent the fat tails and time correlations are coupled in a way that reduces the effect of both on the velocity dispersion. However, correlations can definitely have other effects on the space time structure of the overall flow as demonstrated by the multifractal models [29]. Our objectives have been to demonstrate that the motion of two particles in a turbulent motion can be captured by simple assumptions of an effectively random force field.

***

We are grateful to E. BODENSCHATZ, H. NAKANISHI, S. PIGOLOTTI and Y. POMEAU for valuable discussions. We thank the Danish National Research Foundation for support through the Center for Models of Life.
REFERENCES

[1] Richardson L. F., Proc. R. Soc. London, Ser. A, 110 (1926) 709.
[2] Kolmogorov A. N., C. R. Acad. Sci. USSR, 30 (1941) 301; 32 (1941) 16.
[3] Berg J., Lüthi B., Mann J. and Ott S., Phys. Rev. E, 74 (2006) 016304.
[4] Falkovich G., Gawedzki K. and Vergassola M., Rev. Mod. Phys., 73 (2001) 1.
[5] Boffetta G. and Sokolov I. M., Phys. Fluids, 14 (2002) 9.
[6] Boffetta G. and Sokolov I. M., Phys. Rev. Lett., 88 (2002) 9.
[7] Mordant N., Delour J., Lévéque E., Michel O., Arnéodo A. and Pinton J.-F., J. Stat. Phys., 113 (2003) 5/6.
[8] Durbin P. A., J. Fluid Mech., 100 (1980) 279.
[9] Thomson D. J., J. Fluid Mech., 210 (1990) 113.
[10] Borgas M. S. and Sawford B. L., J. Fluid Mech., 279 (1994) 69.
[11] Pedrizzetti G. and Novikov E. A., J. Fluid Mech., 280 (1994) 69.
[12] Borgas M. S. and Yeung P. K., J. Fluid Mech., 503 (2004) 125.
[13] Heppe B. M. O., J. Fluid Mech., 357 (1998) 167.
[14] Jensen M. H., Phys. Rev. Lett., 83 (1999) 76.
[15] Bohr T., Jensen M. H., Paladin G. and Vulpiani A., Dynamical Systems Approach to Turbulence (Cambridge University Press, Cambridge) 1998.
[16] Gledzer E. B., Sov. Phys. Dokl., 18 (1973) 216.
[17] Yamada M. and Ohkitani K., J. Phys. Soc. Jpn., 56 (1987) 4210; Prog. Theor. Phys., 79 (1988) 1265.
[18] Biferale L. and Kerr R. M., Phys. Rev. E, 52 (1995) 6.
[19] Kadanoff L., Lohse D., Wang J. and Benzi R., Phys. Fluids, 7 (1995) 617.
[20] Pisarenko D., Biferale L., Courvoisier D., Frisch U. and Vergassola M., Phys. Fluids A, 5 (1993) 10.
[21] Bourgoin M., Ouellette N. T., Xu H., Berg J. and Bodenschatz E., Science, 311 (2006) 835.
[22] Sneppen K. and Jensen M. H., Phys. Rev. E, 49 (1994) 919.
[23] Nicolleau F. and Vassilicos J. C., Phys. Rev. Lett., 90 (2003) 024503.
[24] Nicolleau F. and Yu G., Phys. Fluids, 16 (2004) 2309.
[25] La Porta A., Voth G. A., Crawford A. M., Alexander J. and Bodenschatz E., Nature, 409 (2001) 1017.
[26] Lüthi B., Berg J., Ott S. and Mann J., Phys. Fluids, 19 (2007) 045110.
[27] Toschi F., Biferale L., Boffetta G., Celani A., Devenish B. J. and Lanotte A., J. Turb., 6 (2005) N15; Biferale L., Boffetta G., Celani A., Lanotte A. and Toschi F., Phys. Fluids, 17 (2005) 021701.
[28] Yeung P. K., Pope S. B., Kurth E. A. and Lamorgese A. G., J. Fluid Mech., 582 (2007) 399.
[29] Arnéodo A. et al., Phys. Rev. Lett., 100 (2008) 254504.