On maximal agreement couplings

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August 5, 2016

We call a coupling of two stochastic processes which maximizes the time until the first disagreement a maximal agreement coupling. We show that such a coupling always exists. Furthermore, it is possible to construct a lower bound on the disagreement time which is independent of one of the two processes.

1 Introduction and Results

Let \((E, \mathcal{E})\) be a Polish space equipped with the Borel \(\sigma\)-algebra. Let \((Z^1_t)_{t \in \mathbb{N}}, (Z^2_t)_{t \in \mathbb{N}}\) be two \(E\)-valued stochastic processes on the canonical path space \((E^\mathbb{N}, \mathcal{E}^\mathbb{N})\) with laws \(\mu^1, \mu^2\). We simply write \(Z = (Z_t)_{t \in \mathbb{N}}\) for a generic element of \(E^\mathbb{N}\).

A coupling of the measures \(\mu^1\) and \(\mu^2\) is a measure \(\hat{\mu}\) on the product space \(E^\mathbb{N} \times E^\mathbb{N}\) where the marginals are given by \(\mu^1\) and \(\mu^2\).

For a sub-\(\sigma\)-algebra \(\mathcal{F} \subset E^\mathbb{N}\), denote the total variation distance with respect to \(\mathcal{F}\) by

\[
\| \mu^1 - \mu^2 \|_{\mathcal{F} - \text{TV}} := \sup_{A \in \mathcal{F}} (\mu^1(A) - \mu^2(A)).
\]

A classical question is how quickly \(Z^1\) and \(Z^2\) can be coupled, that is finding a coupling under which the last time \(Z^1\) and \(Z^2\) disagree is as small as possible. More formally, let

\[
\sigma_0 := \inf \{ t \geq 0 : Z^1_s = Z^2_s \forall s \geq t \}
\]

and \(\mathcal{G}_t := \sigma(Z_s : s \geq t)\). For any possible coupling \(\hat{\mu}\) the coupling inequality

\[
\hat{\mu}(\sigma_0 \geq t) \geq \| \mu^1 - \mu^2 \|_{\mathcal{G}_t - \text{TV}},
\]

provides a universal lower bound. A maximal coupling is a coupling for which (1) is an equality for all \(t \in \mathbb{N}\), and it is well-known that such a coupling always exists [2, 1, 3].

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We are interested in the opposite question, namely we want to find a coupling so that the first disagreement time or decoupling time
\[ \sigma := \inf \{ t \geq 0 : Z^1_t \neq Z^2_t \}\]
is as big as possible. There is a corresponding coupling inequality for this question as well. Let \( F_t := \sigma(Z_s : 0 \leq s \leq t) \).

**Lemma 1.1.** For any coupling \( \hat{\mu} \) of \( \mu^1 \) and \( \mu^2 \),
\[ \hat{\mu}(\sigma > t) \leq 1 - \| \mu^1 - \mu^2 \|_{\text{TV}} \quad \forall \ t \in \mathbb{N}. \] (2)

We call a coupling for which (2) is sharp for all \( t \in \mathbb{N} \) a maximal agreement coupling.

**Theorem 1.2.** There exists a maximal agreement coupling of \( \mu^1 \) and \( \mu^2 \).

In general such a coupling is not unique, since there are no conditions on the joint distribution of \( Z^1 \) and \( Z^2 \) after the decoupling time \( \sigma \). In fact, any coupling of the marginals after the decoupling time can be used to construct a maximal agreement coupling. Of course for this to be of use we need to describe the marginals first.

To this end we use the language of regular conditional probabilities. Fix \( t \in \mathbb{N} \), \( i \in \{1, 2\} \). Since \( E \) is a Polish space regular conditional probabilities of \( Z^i \) given the first \( t+1 \) steps exist. For \( z \in E^{t+1} \) we write \( \hat{\mu}^i(\cdot|Z^i = z) \) for the regular conditional law of \( Z^i \) given \( Z^i = z \). We adopt similar notation for the regular conditional probabilities of other probability measures, in particular for couplings.

**Theorem 1.3.** Let \( \hat{\mu} \) be a maximal agreement coupling.

a) For \( t \in \mathbb{N} \), \( s \geq t \), \( z \in E^s \), \( i = 1, 2 \), the marginals after the decoupling time are given by
\[ \hat{\mu}(Z^i \in \cdot | Z^i = z, \sigma \geq t) = \mu^i(\cdot|z) \quad \hat{\mu} \text{- a.s.} \]

b) For \( t \in \mathbb{N} \) and \( z, z' \in E^{t+1} \) with \( z_0, \ldots, t-1 = z_0', \ldots, t-1 \) let \( \hat{\mu}^i_z z' \) be a coupling of \( \mu^i(\cdot|z) \) and \( \mu^2(\cdot|z') \). Assume that the map \((Z^1, Z^2, \sigma) \mapsto \hat{\mu}^i_z z' \) is measurable. Then
\[ \hat{\mu}' := \int \hat{\mu}^i_z z' \ d\hat{\mu} \]
is a maximal agreement coupling.

It is clear that the event \( \{ \sigma = t \} \) contains information about \( Z^1 \) and \( Z^2 \). This is unavoidable, but also undesirable. In particular properties of the first disagreement time \( \sigma \) cannot assumed to be stable under conditioning: \( \hat{\mu}(\sigma = \infty) = \inf_{t \in \mathbb{N}} \hat{\mu}(\sigma \geq t) \) might be positive, but \( \hat{\mu}(\sigma \geq t|Z^1 \in A_t) \to 0 \) for a decreasing sequence of events \( A_t \), \( A_t \in F_t \).

The second main result of this article is a remedy to this problem. There exists a lower bound \( \tau \) on \( \sigma \) which is independent of \( Z^1 \). With this independence there is no problem in the above example when using \( \tau \) instead of \( \sigma \).
There is also a r.v. $\tau$. It is possible to construct a (non-degenerate) time $\tilde{\tau}$.

Theorem 1.4. Clearly, in the theorem the roles of $Z$.

In particular, if

$\tilde{\tau}$.

Lemma 1.5. Assume $E$ is countable. Define for $i = 1, 2$

Then $\kappa_t \leq 1 - \delta_t^{(2)}$ and

\begin{align*}
\kappa_t &\leq (\delta_t^{(1)})^{-1} \sup \{ \mu^2(Z_t = e|Z = z) - \mu^1(Z_t = e|Z = z) : z \in E^{t-1}, e \in E \}.
\end{align*}

We finish with two remarks. First, we address an (impossible) generalization of Theorem 1.4. Clearly, in the theorem the roles of $Z^1$ and $Z^2$ can be reversed, so that there is also a r.v. $\tau'$ with $\tau' \leq \sigma$ and $\tau'$ independent of $Z^2$. One might wonder if it is possible to construct a (non-degenerate) time $\tilde{\tau}$ which satisfies $\tilde{\tau} \leq \sigma$ and which is independent of $Z^1$ and independent of $Z^2$ (clearly it cannot be independent of both simultaneously). However, this is not possible, as the following argument shows: Let $f : E \to \mathbb{R}$, $t \geq 0$. Then

\begin{align*}
\mu^1(f(Z_t)) - \mu^2(f(Z_t)) &= \tilde{\nu} (f(Z_t^1) - f(Z_t^2)) \\
&= \tilde{\nu} (f(Z_t^1 1_{\tilde{\tau} \leq t}) - f(Z_t^2 1_{\tilde{\tau} \leq t})) \\
&= \tilde{\nu} (f(Z_t^1 1_{\tilde{\tau} \leq t}) - \tilde{\nu} (f(Z_t^2 1_{\tilde{\tau} \leq t}))
\end{align*}

By the assumed individual independence, this equals

\begin{align*}
\tilde{\nu}(\tilde{\tau} \leq t) (\mu^1(f(Z_t)) - \mu^2(f(Z_t)))
\end{align*}

which implies $\tilde{\tau} = 0$ a.s.

For the final remark we consider applying the results to Markov chains. Let $X^1$ and $X^2$ be two Markov chains with the same transition kernel but possibly different starting points $x_1$ and $x_2$ on a Polish space $F$. Clearly a maximal coupling of $X^1$ and $X^2$ is trivial, $\sigma = \infty$ if $x_1 = x_2$ and $\sigma = 0$ otherwise. However, let $\phi : F \to E$ and consider $Z_t^i = \phi(X_t^i)$, $t \in \mathbb{N}$, $i = 1, 2$. For these induced processes a maximal agreement coupling is both meaningful and interesting. For example $\phi$ could be a course-graining map or a projection on a lower-dimensional state space.
2 Preliminaries and the proof of Lemma 1.1

Before going into the proofs we need some more notation and concepts. We say \( \nu \) is a (sub-)probability measure when the total mass \( |\nu| \) is less or equal to 1. For two sub-probability measures \( \nu^1 \) and \( \nu^2 \), we say \( \nu^1 \leq \nu^2 \) if \( \nu^1(A) \leq \nu^2(A) \) for any event \( A \), or equivalently \( \nu^1 \ll \nu^2 \) and \( \frac{d\nu^1}{d\nu^2} \leq 1 \). The minimum \( \nu^1 \wedge \nu^2 \) is the largest sub-probability measure \( \nu \) which satisfies \( \nu \leq \nu^1 \) and \( \nu \leq \nu^2 \). With \( \nu|\mathcal{F}_i \) we denote the restriction of the measure \( \nu \) to the \( \sigma \)-algebra \( \mathcal{F}_i \). For \( t \in \mathbb{N} \), \( z \in E^{t+1} \), the regular conditional probability \( \nu(\cdot|Z = z) \) of a sub-probability measure \( \nu \) is the regular conditional probability of the probability measure \( \nu/|\nu| \). A consequence of this convention is that for an event \( A \) with \( \nu(A) > 0 \) we have \( \nu(\cdot|Z = z, A) = \nu(\cdot|A|Z = z) \).

The proof of the coupling inequality in Lemma 1.1 is a simple computation using the minimum of two measures.

Proof of Lemma 1.1 By the maximality of \( \nu_i := \mu^1|\mathcal{F}_i \wedge \mu^2|\mathcal{F}_i \) we have that the measures \( \mu^1|\mathcal{F}_i - \nu_i \) and \( \mu^2|\mathcal{F}_i - \nu_i \) are mutually singular, and hence
\[
\| \mu^1 - \mu^2 \|_{\mathcal{F}_i - TV} = 1 - |\nu_i |.
\]
Furthermore, for \( i = 1, 2 \) and any coupling \( \widehat{\mu} \) and \( A \in \mathcal{F}_i \),
\[
\widehat{\mu}(Z^i_{0,...,t} \in A, \sigma > t) = \widehat{\mu}(Z^2_{0,...,t} \in A, \sigma > t) \leq \mu^1(A),
\]
which by the maximality of \( \nu_i \) implies \( \widehat{\mu}(Z^i_{0,...,t} \in A, \sigma > t) \leq \nu_i, i = 1, 2 \). Therefore
\[
\widehat{\mu}(\sigma > t) \leq |\nu_i | = 1 - \| \mu^1 - \mu^2 \|_{\mathcal{F}_i - TV}.
\]

3 Proofs of Theorems 1.2 and 1.3

The proof of Theorem 1.2 is an explicity construction. It uses the same strategy as the proof for the existence of a maximal coupling found in [4](Theorem 4.6.1). The key difference is that we work with the increasing sequence of \( \sigma \)-algebras \( (\mathcal{F}_i) \). In contrast the construction of the maximal coupling makes use of the decreasing sequence \( (\mathcal{G}_i) \). This difference means an inductive argument from the largest \( \sigma \)-algebra downwards is not possible.

Proof of Theorem 1.2 We will iteratively define a sequence of sub-probability measures which will allow us to construct the coupling. We start by setting \( \pi_0 := \mu^1 \), \( i = 1, 2 \) and \( \pi_0 := \pi_0|\mathcal{F}_0 \wedge \pi_0|\mathcal{F}_0 \), the largest common component of the two measures on the \( \sigma \)-algebra \( \mathcal{F}_0 \). Note that we can interpret \( \pi_0 \) as a sub-probability measure on \( E \). Next we set \( \pi_1(i) := \int_E \pi_0(\cdot|z)\pi_0(dz) \), which is the extension of \( \pi_0 \) to a sub-probability measure on \( E^N \) which satisfies \( \pi_1 \leq \pi_0 \), finally we set \( \mu^1_0 := \pi_0 - \pi_1, i = 1, 2 \). Iterating,
we define
\[\pi_t := \underline{\mu}_t^i|_{\mathcal{F}_t} \land \underline{\mu}_t^i|_{\mathcal{F}_t},\]
\[\overline{\mu}_{t+1}(\cdot) := \int_{E^{t+1}} \overline{\mu}_t^i(\cdot | z) \pi_t(dz),\]
\[\mu_t^i := \underline{\mu}_t - \overline{\mu}_{t+1}.\]  
From the construction we immediately obtain that
\[\mu^1 = \underline{\mu}_0 \geq \mu_1 \geq \ldots, \quad \sum_{s=0}^t \mu_s^i = \mu^i - \overline{\mu}_{t+1}^i \leq \mu^i,\]
\[\mu_t^i|_{\mathcal{F}_t} = \pi_{t-1}|_{\mathcal{F}_t} - \pi_t|_{\mathcal{F}_t}, \quad \| \mu_t^i - \mu_t^2 \|_{TV} = 0, \quad 0 \leq s < t.\]
As a consequence, we can define \(\mu^\infty_i := \mu^i - \sum_{s=0}^\infty \mu_s^i \geq 0\). Furthermore,
\[\mu^\infty_i|_{\mathcal{F}_t} = \left[ \mu^i - \sum_{s=0}^t \mu_s^i \right]|_{\mathcal{F}_t} + \sum_{s=t+1}^\infty \pi_s|_{\mathcal{F}_t} = \pi_t + \sum_{s=t+1}^\infty \pi_s|_{\mathcal{F}_t},\]
which shows that \(\mu^1 = \mu^2\).

To obtain a coupling, let \(\widehat{\mu}_0 := \mu_0 \otimes \mu_0^2\), and
\[\widehat{\mu}_t := \int_{E^t} \mu_t^1(\cdot | z) \otimes \mu_t^2(\cdot | z) \mu_t^i|_{\mathcal{F}_{t-1}}(dz), \quad 1 \leq t \leq \infty,\]
where for \(t = \infty\) we have the degenerate case with \(z \in E^\infty\) and \(\mu^\infty_i(\cdot | z) = \delta_z\).

Define \(\widehat{\mu} = \widehat{\mu}_0 + \widehat{\mu}_1 + \ldots + \widehat{\mu}_\infty\), for which a direct computation shows that the marginals are \(\mu^1\) and \(\mu^2\), hence \(\widehat{\mu}\) is a coupling. What remains to show is that this is indeed a maximal agreement coupling.

First we will show that for all \(t \in \mathbb{N} \cup \{\infty\}\),
\[\widehat{\mu}(\cdot, \sigma = t) = \widehat{\mu}_t(\cdot),\]
which is equivalent to \(\widehat{\mu}_t(\sigma \neq t) = 0\) for all \(t \in \mathbb{N} \cup \{\infty\}\). By construction \(\widehat{\mu}_t(\sigma < t) = 0\), and
\[\mu_t^1|_{\mathcal{F}_t} \land \mu_t^2|_{\mathcal{F}_t} = (\underline{\mu}_t^1|_{\mathcal{F}_t} - \overline{\mu}_{t+1}^1|_{\mathcal{F}_t}) \land (\underline{\mu}_t^2|_{\mathcal{F}_t} - \overline{\mu}_{t+1}^2|_{\mathcal{F}_t}) = \underline{\mu}_t^2|_{\mathcal{F}_t} \land \underline{\mu}_t^1|_{\mathcal{F}_t} - \pi_t = 0.\]
Therefore \(\widehat{\mu}_t(\sigma \leq t) = |\widehat{\mu}_t|\), the total mass of \(\widehat{\mu}_t\), and hence \(\widehat{\mu}_t(\sigma > t) = 0\).

With (3) we can now verify that \(\widehat{\mu}\) is indeed a maximal agreement coupling:
\[\| \mu^1 - \mu^2 \|_{TV} = 1 - \| \mu^1|_{\mathcal{F}_t} \land \mu^2|_{\mathcal{F}_t} = 1 - \| \underline{\mu}_{t+1}^1 \|\]
and by (2) and (7)
\[\| \mu^1 - \mu^2 \|_{TV} \leq \widehat{\mu}(\sigma \leq t) = \sum_{s=0}^t |\widehat{\mu}_s| = 1 - |\underline{\mu}_{t+1}^1| = \| \mu^1 - \mu^2 \|_{TV},\]
which shows that (2) is an equality for all \(t\) and hence \(\widehat{\mu}\) is indeed a maximal agreement coupling. \(\square\)
The proof of Theorem 1.3 is mostly a refinement of the construction of the maximal agreement coupling above. We first show that various regular conditional probabilities of the building blocks of \( \hat{\mu} \) can be expressed via \( \mu^1 \) and \( \mu^2 \).

**Lemma 3.1.** In the construction of the maximal agreement coupling of Theorem 1.3, it holds that \( \overline{\mu}_t^i(\cdot|z) = \mu^i(\cdot|z) \) for all \( s \geq t, \overline{\mu}_t^i|_{\mathcal{F}_{s-1}} \)-a.e. \( z \in E^s \), and \( \mu^i_1(\cdot|z) = \mu^i(\cdot|z) \) for all \( s \geq t, \mu^i_1|_{\mathcal{F}_{s}} \)-a.e. \( z \in E^{s+1} \).

**Proof.** First we show that \( \overline{\mu}_t^i(\cdot|z) = \mu^i(\cdot|z) \) for \( \overline{\mu}_t^i|_{\mathcal{F}_{s}} \)-a.e. \( z \in E^s \), and the proof is done by induction. The claim is clearly true for \( t = 0 \), since \( \overline{\mu}_0^i = \mu^i \). Assume now the claim is true for \( t \in \mathbb{N} \). Let \( s \geq t + 1 \), \( z \in E^{t+1} \) and \( z' \in E^{s+1} \) with \( z'_0 \cdots z'_t = z \). Since \( \overline{\mu}_{t+1}^i(\cdot|z) = \int_{E^{t+1}} \overline{\mu}_{t+1}^i(\cdot|\gamma)\pi_t(d\gamma) \) and \( z \in E^{t+1} \), we have \( \overline{\mu}_{t+1}^i(\cdot|z) = \overline{\mu}_t^i(\cdot|z) \) for \( \pi_t \)-a.e. \( z \in E^{t+1} \). Since \( \overline{\mu}_{t+1}^i|_{\mathcal{F}_t} = \pi_t \leq \overline{\mu}_t^i|_{\mathcal{F}_t} \), the induction hypothesis implies \( \overline{\mu}_{t+1}^i(\cdot|z) = \mu^i(\cdot|z) \) for \( \overline{\mu}_{t+1}^i|_{\mathcal{F}_t} \)-a.e. \( z \in E^{t+1} \). To obtain the statement for \( z' \) we use the fact that \( \mu^i(\cdot|z') \) is a version of the regular conditional probability \( \nu_z(\cdot|z') \), where \( \nu_z = \mu^i(\cdot|z) \).

For the second claim, let \( s \geq t, A \in \mathcal{G}_{s+1} \) and \( B \in \mathcal{F}_s \). Then, using the definition of \( \mu^i_1 \) and the first claim,

\[
\int_B \mu^i_1(A|z)\mu^i_1|_{\mathcal{F}_s}(dz) = \mu^i_1(A \cap B) = \overline{\mu}_t^i(A \cap B) - \overline{\mu}_{t+1}^i(A \cap B)
\]

\[
= \int_B \mu(A|z)(\overline{\mu}_t^i - \overline{\mu}_{t+1}^i)|_{\mathcal{F}_s}(dz) = \int_B \mu^i(A|z)\mu^i_1|_{\mathcal{F}_s}(dz).
\]

**Proof of Theorem 1.3** Part a): First assume that \( \hat{\mu} \) is the maximal agreement coupling constructed in Theorem 1.2. By (i), (ii) and Lemma 3.1,

\( \hat{\mu}(\cdot|Z^i = z, \sigma \geq t) \leq \hat{\mu}(\cdot, \sigma \geq t|Z^i = z) = \overline{\mu}_t^i(\cdot|Z^i = z) = \mu^i_1(\cdot|z) = \mu^i(\cdot|z) \),

which shows the claim for this maximal agreement coupling. Assume now that \( \hat{\mu}' \) is some other maximal agreement coupling. Define the sub-probability measure \( \pi^i_t(A) := \hat{\mu}'(Z^i \in A, \sigma > t), A \in \mathcal{F}_t, i = 1, 2 \). The definition of \( \pi^i_t \) does not depend on the choice of \( i \) since \( \sigma > t \) and \( A \in \mathcal{F}_t \). Therefore \( \pi^i_t \leq \mu^i_t \) for \( i = 1, 2 \). By the maximal agreement property of \( \hat{\mu}'_t \), \( \pi^i_t = \pi_t \), which implies \( \pi^i_t = \pi_t \). Defining \( \overline{\mu}^i_t(\cdot|z) := \hat{\mu}'(Z^i \in \cdot, \sigma > t) \) and \( \mu^i_t = \overline{\mu}^i_t - \overline{\mu}^i_{t+1} \), the proof of Lemma 3.1 and the above argument for \( \hat{\mu} \) are true for \( \hat{\mu}' \) as well, using only \( \pi^i_t = \pi_t \).

For part b), in (ii) we replace \( \mu^i_1(\cdot|z) \odot \mu^i_2(\cdot|z) \) by

\[
\int_{E \times E} \hat{\mu}^i_t(\cdot, \gamma_z)(\cdot, \gamma_z)(Z^1 \in \cdot, Z^2 \in \cdot) \mu^i_1(Z_t \in \cdot|z) \odot \mu^i_2(Z_t \in \cdot|z) \ (d\gamma_1, \gamma_2).
\]

By Lemma 3.1 the marginals stay the same, so we obtain a valid coupling of \( \mu^1 \) and \( \mu^2 \). And since the change affects only the evolution after the decoupling time, the maximal agreement property remains unaffected.
4 Proof of Theorem 1.4

This proof relies on a refinement of the construction of the maximal agreement coupling in the previous section. The next lemma is the key ingredient. Basically, it is the analogous statement of Theorem 1.4 for a single time point \( t \).

**Lemma 4.1.** Fix \( t \in \mathbb{N} \). A maximal agreement coupling \( \tilde{\mu} \) of \( \mu^1 \) and \( \mu^2 \) can be extended to a coupling \( \tilde{\mu}^1 \) on \( E^N \times E^N \times \{0,1\} \) containing an additional random variable \( Y_t \in \{0,1\} \) with the following properties:

a) \( \tilde{\mu}^1(Y_t = 1) = \kappa_t \), where \( \kappa_t \) is as in Theorem 1.4;

b) \( Y_t \) is independent of \( Z^1 \) and \( \{ \sigma > t - 1 \} \);

c) \( \{ \sigma = t \} \subseteq \{ \sigma > t - 1, Y_t = 1 \} \).

**Proof.** Assume that \( \kappa_t \in (0,1) \), otherwise the statement is trivial. Furthermore assume for now that \( \tilde{\mu} \) is the maximal agreement coupling constructed in the proof of Theorem 1.4. For \( A \subset E^t \) and \( B \subset E \), we write

\[
\kappa_t(A, B) := \tilde{\mu}(\sigma = t | Z^1_t \in B, Z^1_{0,...,t-1} \in A, \sigma > t).
\]

Since \( \tilde{\mu}(Z^1 \in \cdot, \sigma = s) = \mu^s_1(\cdot) \) and \( \mu^s_1 = \tilde{\mu}^1_t - \tilde{\mu}^1_{t+1} \), we have

\[
\kappa_t(A, B) = \frac{\tilde{\mu}(\sigma = t, Z^1_t \in B, Z^1_{0,...,t-1} \in A)}{\tilde{\mu}(\sigma \geq t, Z^1_t \in B, Z^1_{0,...,t-1} \in A)} = 1 - \tilde{\mu}^1_{t+1}(Z_t \in B, Z_{0,...,t-1} \in A) = \pi_t(A, B).
\]

We want to show that \( \kappa_t(A, B) \leq \kappa_t \). To this end, by (8) and Lemma 3.1

\[
\pi_t(A, B, Z_{0,...,t-1} \in A) = \int_A \int_B \int_{\mathbb{R}} \mu^1(Z_t \in dy|z) \mu^2(Z_t \in dy|z) \pi_{t-1}(dz)
\]

where we used in the last line that \( \mu^1(Z_t \in \cdot|z) \ll \mu^2(Z_t \in \cdot|z) \) (for a.e. \( z \)) since \( \kappa_t < 1 \). By using the fact that for any \( a \in \mathbb{R} \), \( a = (a \wedge 1)(a \vee 1) \), we can upper bound the above by

\[
\text{ess sup}_{\mu \in \mathcal{A}, \nu \in \mathcal{B}} \left( \frac{d\mu^1(Z_t \in \cdot|z)}{d\mu^2(Z_t \in \cdot|z)}(y) \right) \int_A \int_B \mu^1(Z_t \in \cdot|z) \wedge 1 \mu^2(Z_t \in dy|z) \pi_{t-1}(dz)
\]

\[
= \text{ess sup}_{\mu \in \mathcal{A}, \nu \in \mathcal{B}} \left( \frac{d\mu^1(Z_t \in \cdot|z)}{d\mu^2(Z_t \in \cdot|z)}(y) \right) \int_A \int_B \left[ \mu^1(Z_t \in \cdot|z) \wedge \mu^2(Z_t \in \cdot|z) \right] (dy) \pi_{t-1}(dz)
\]

\[
\leq (1 - \kappa_t)^{-1} \pi_t(Z_t \in B, Z_{0,...,t-1} \in A) = (1 - \kappa_t)^{-1} \mu^1_{t+1}(Z_t \in B, Z_{0,...,t-1} \in A),
\]
where in the last line we used \(3\) and \(4\). It follows that \(5\) is indeed less or equal to \(\kappa_t\). Define now for \(z \in E^{t+1}\): 
\[\kappa_t(z) := \tilde{\mu}(\sigma = t|Z^1 = z, \sigma \geq t).\]
Since \(\kappa_t(A, B) \leq \kappa_t\) for all \(A, B\) we have also that \(\kappa_t(z) \leq \kappa_t\) for \(\tilde{\mu}_{t+1}\)-a.e. \(z \in E^{t+1}\).

We can define the extended coupling \(\tilde{\mu}_t^Y\) on \(E^N \times E^N \times \{0, 1\}\) via \(\tilde{\mu}_s^Y = \tilde{\mu}_s \otimes (\kappa_t \delta_1 + (1 - \kappa_t) \delta_0)\), \(s \leq t\), \(\tilde{\nu}_t^Y = \tilde{\mu}_t \otimes \delta_1\) and

\[\tilde{\nu}_t^Y = \int_{E^N \times E^N} \tilde{\mu}_s(|Z^1 = Z^2 = z) \otimes \left(\left(1 - \frac{1 - \kappa_t}{1 - \kappa_t(z_0, \ldots, t)}\right) \delta_1 + \frac{1 - \kappa_t}{1 - \kappa_t(z_0, \ldots, t)} \delta_0\right) \mu_1^Z|_{\mathcal{F}_s} (dz)\]

for \(s > t\), and we set \(\tilde{\mu}_t^Y = \tilde{\mu}_0^Y + \cdots + \tilde{\mu}_t^Y\).

What remains is to verify that properties a), b) and c) hold. Property c) follows from \(\tilde{\mu}_t^Y(\cdot, \sigma = t) = \tilde{\mu}_t^Y\) and the definition of \(\tilde{\mu}_t^Y\). For a) and b), let \(A \in \mathcal{F}_t\). By the construction of \(\tilde{\mu}_t^Y\),

\[\tilde{\mu}_t^Y(Y_t = 1, Z^1 \in A, \sigma \geq t) = \tilde{\mu}_t^Y(Y_t = 1, Z^1 \in A) + \cdots + \tilde{\mu}_t^Y(Y_t = 1, Z^1 \in A)\]

\[= \mu_t^1(Z^1 \in A) + \tilde{\mu}_t^Y(1, Z^1 \in A) = \mu_t^1(1, Z^1 \in A) + \int_{A} \left(1 - \frac{1 - \kappa_t}{1 - \kappa_t(z)}\right) \tilde{\mu}_t^1|_{\mathcal{F}_s} (dz).\]

By \(3\) and \(\kappa_t < 1\),

\[(1 - \kappa_t(A, B))^{-1} = \frac{\tilde{\mu}_t^1(Z_t \in B, Z_{0, \ldots, t-1} \in A)}{\tilde{\mu}_{t+1}^1(Z_t \in B, Z_{0, \ldots, t-1} \in A)} < \infty,

from which follows that \(\tilde{\mu}_t^1|_{\mathcal{F}_s} \prec \mu_t^1|_{\mathcal{F}_s}\) and

\[\frac{d\mu_t^1|_{\mathcal{F}_s}}{d\tilde{\mu}_t^1|_{\mathcal{F}_s}}(z) = (1 - \kappa_t(z))^{-1}.\]

Together with \(5\) we obtain

\[\tilde{\mu}_s^Y(Y_t = 1, Z^1 \in A, \sigma \geq t) = \mu_t^1(A) + \tilde{\mu}_t^1(A) - (1 - \kappa_t) \mu_t^1(A) = \kappa_t \tilde{\mu}_t^1(A) = \kappa_t \tilde{\mu}_t^Y(Z^1 \in A, \sigma \geq t).\]

This shows both that \(\tilde{\mu}_t^Y(Y_t = 1) = \kappa_t\) and independence of \(Z_0, \ldots, t\) and \(\{\sigma \geq t\}\). To obtain the full independence of \(Z^1\), let \(B \in \sigma(Z_{t+1}^1, \ldots, Z_t^1)\) for \(s > t\) arbitrary. Then, by \(4\) and Lemma 3.3 \(6\) changes to

\[\tilde{\mu}_s^Y(Y_t = 1, Z^1 \in A \cap B, \sigma \geq t) \]

\[= \mu_t^1(A \cap B) + \int_{A} \left(1 - \frac{1 - \kappa_t}{1 - \kappa_t(z)}\right) \mu_t^1(B|z) \tilde{\mu}_t^1|_{\mathcal{F}_s} (dz).\]

With the same computation as in \(10\) we get

\[\tilde{\mu}_s^Y(Y_t = 1, Z^1 \in A \cap B, \sigma \geq t) = \kappa_t \tilde{\mu}_t^Y(Z^1 \in A \cap B, \sigma \geq t),\]
which completes the proof for $\mu$.

To show the statement for a general maximal agreement coupling $\tilde{\mu}$ we use the same strategy as in the proof of Theorem 4.3 We define $\pi_t$, $\mu_t$ and $\nu_t$ in terms of $\tilde{\mu}$:

$$
\pi_t := \tilde{\mu}(Z^1 \in \cdot, \sigma > t | \mathcal{F}_t), \\
\mu_t := \tilde{\mu}(\cdot, \sigma = t), \\
\nu_t := \tilde{\mu}(Z^1 \in \cdot), \\
\mu_t^i := \tilde{\mu}_t(Z^i \in \cdot), \\
\nu_t^i := \tilde{\mu}_t(Z^i \in \cdot, \sigma \geq t).
$$

We restate that $\pi_t$ is universal in maximal agreement couplings, as was shown in the proof of Theorem 4.3 Using this the above construction of $\hat{\mu}^{Y^i_t}$ follows through the same.

Theorem 4.4 is a generalization of Lemma 4.4 and the proof reflects this.

**Proof of Theorem 4.4** We will introduce random variables $(Y_t)_{t \in \mathbb{N}}$ in such a way that the law of $(Z^1, Z^2, Y_t)$ is given by the coupling $\tilde{\mu}^{Y^i_t}$ constructed in Lemma 4.4 We do this by using the way $\mu_s$ is extended to $\tilde{\mu}_s^{Y^i_t}$ simultaneously for all $Y_t$. For $s, t \in \mathbb{N}$ and $z \in E^{s+1}$ let

$$
\nu_{s,t}(z) := \begin{cases} \kappa_t \delta_1 + (1 - \kappa_t) \delta_0, & s < t; \\
\delta_1, & s = t; \\
(1 - \frac{1 - \kappa_t}{1 - \kappa_s(z_0, \ldots, z_t)}) \delta_1 + \frac{1 - \kappa_s}{1 - \kappa_s(z_0, \ldots, z_t)} \delta_0, & s > t. \end{cases}
$$

Note that $\nu_{s,t}(z)$ is the distribution of $Y_t$ given $\{\sigma = s\}$ and $Z^1 = z$. By simply taking the product measures we obtain a coupling $\hat{\nu} = \hat{\nu}_0 + \ldots + \hat{\nu}_\infty$,

$$
\hat{\nu}_s = \int_{E^{s+1}} \tilde{\mu}_s(z) \otimes \prod_{t=0}^\infty \nu_{s,t}(z) \mu_t^i | \mathcal{F}_s(dz),
$$

where $\mu_t^i$ and $\tilde{\mu}_s$ are given by (12). This construction indeed extends the maximal agreement coupling $\tilde{\mu}$ by a sequence $(Y_t)_{t \in \mathbb{N}}$ and the marginal of $(Z^1, Z^2, Y_t)$ is given by $\hat{\mu}^{Y^i_t}$.

Let $\tau := \inf\{t \geq 0 : Y_t = 1\}$. By construction, $\hat{\nu}_s(Y_t = 1) = 1$. This implies $Y_\sigma = 1$ and hence $\tau \leq \sigma$ a.s. Furthermore we get $\hat{\nu}_s(\tau = t) = 0$ for all $t > s$.

Let $A \subset E^\mathbb{N}$ be an arbitrary event. We have

$$
\hat{\nu}(Z^1 \in A, \tau > t) = \hat{\nu}(Z^1 \in A, Y_t = 0, \tau > t - 1) \\
= (\hat{\nu}_{t+1} + \ldots + \hat{\nu}_\infty)(Z^1 \in A, Y_t = \ldots = Y_0 = 0).
$$

For $r > t$,

$$
\hat{\nu}_s(Z^1 \in A, Y_t = \ldots = Y_0 = 0) \\
= \int_{E^{r+1}} \tilde{\mu}_s(Z^1 \in A | Z^1 = z) \otimes \prod_{s=0}^r \nu_{r,s}(z) (Y_s = 0) \mu_t^i | \mathcal{F}_s(dz) \\
= \int_{E^{r+1}} \mu_t^i(A | z) \prod_{s=0}^r \frac{1 - \kappa_s}{1 - \kappa_s(z_0, \ldots, z_s)} \mu_t^i | \mathcal{F}_s(dz).
$$
By Lemma 3.1, $\mu^1_r(A|z) = \mu^1(A|z)$. Summing over $r > t$, we get
\[
\hat{\nu}(Z^1 \in A, \tau > t) = \left( \prod_{s=0}^{t} (1 - \kappa_s) \right) \int_{E_{t+1}} \mu^1_t(A|z) \prod_{s=0}^{t} \frac{1}{1 - \kappa_{s}(z_0, \ldots, s)} \mu^1_{t+1}|x_s(dz).
\]

By (10), \((1 - \kappa_t(z_0, \ldots, t))^{-1} = \frac{\mu^1_t|x_t}{\mu^1_{t+1}|x_t} \). Together with Lemma 3.1 this allows us to simplify the integral to
\[
\int_{E_t} \mu^1_t(A|z) \prod_{s=0}^{t-1} \frac{1}{1 - \kappa_{s}(z_0, \ldots, s)} \mu^1_{t+1}|x_{t-1}(dz).
\]
Repeating the argument shows that it in fact equals $\int_{E} \mu^1_t(A|z) \mu^1_0|F_0(dz) = \mu^1(A)$, which shows that
\[
\hat{\nu}(Z^1 \in A, \tau > t) = \left( \prod_{s=0}^{t} (1 - \kappa_s) \right) \mu^1(A) = \hat{\nu}(\tau > t) \hat{\nu}(Z^1 \in A).
\]

\[\blacksquare\]

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