Desingularization of arithmetic surfaces: algorithmic aspects

Anne Frühbis-Krüger and Stefan Wewers

Abstract The quest for regular models of arithmetic surfaces allows different viewpoints and approaches: using valuations or a covering by charts. In this article, we sketch both approaches and then show in a concrete example, how surprisingly beneficial it can be to exploit properties and techniques from both worlds simultaneously.

1 Introduction

Resolution of singularities in dimension 2 was first proved by Jung in 1908 [16], but it was not until Hironaka’s work in 1964 [15] that this could also be mastered in dimensions beyond 3. However, Hironaka’s result only applies to characteristic zero, but not to positive or mixed characteristic. There the general question is still wide open with partial results for low dimensions. In particular, Lipman gave a construction for 2-dimensional schemes in full generality in [17].

Lipman’s result includes the case of an arithmetic surface, i.e. integral models of curves over number fields. In fact, the existence of (minimal) regular models of curves over number fields is a cornerstone of modern arithmetic geometry. Important early results are for instance the existence of a minimal regular model of an elliptic curve by Néron ([23]) and Tate’s algorithm ([30]) for computing it explicitly.

In this paper we study a particular series of examples of surface singularities which is a special case of a construction due to Lorenzini ([19], [20]). The singularity in question is a wild quotient singularity. More precisely, the singular point lies on an arithmetic surface of mixed characteristic \((0, p)\) which is the quotient of a regular surface by a cyclic group of prime order \(p\), such that the group action has
isolated fixed points. We prove that in our example one obtains a series of rational deter-
minantal singularities of multiplicity \( p \), and we are able to write down explicit
equations for these (see Proposition 3.4).

Determinantal rings (of expected codimension) are well-studied objects in comm-
mutative algebra: the free resolution is the Eagon-Northcott complex and hence
many invariants of the ring such as projective dimension, depth, Castelnuovo-
Mumford regularity, etc. are known (see e.g. [8], [3]). Beyond that, such singular-
ities (in the geometric case) are an active area of current research in singularity
theory studying e.g. classification questions, invariants, notions of equivalence and
topological properties, see e.g. [10], [24], [32]. We show, by a direct computation, 
that the resolution in our arithmetic setting is completely analogous to the geometric
case.

Both for deriving the equations of our singularities and for resolving them, we
employ and mix two rather different approaches to represent and to compute with
arithmetic surfaces. The first approach is more standard and consists in representing
a surfaces as a finite union of affine charts, and the coordinate ring of each affine
chart as a finitely generated algebra over the ground ring. From this point of view,
computations with arithmetic surfaces can be performed with standard tools from
computer algebra, like standard bases (e.g. in SINGULAR [6]). However, these tech-
niques are not yet as mature in the arithmetic case as they are in the geometric case.

The second approach uses valuations as its main tool. We work over a discrete
valuation ring \( R \). An arithmetic surface \( X \) over \( \text{Spec} \, R \) is considered as an \( R \)-model of
its generic fiber \( X_K \) (a smooth curve over \( K = \text{Frac}(R) \)). Then any (normal) \( R \)-model
\( X \) of \( X_K \) is determined by a finite set \( V(X) \) of discrete valuations on the function
field of \( X_K \) corresponding to the irreducible components of the special fiber of \( X \).
A priori, it is not clear how to extract useful information about the model \( X \) from
the set \( V(X) \). Nevertheless, in joint work with J. Rüth the second named author has
used this technique successfully for computing semistable reduction of curves (see
e.g. [28]).

The paper is structured as follows. In Section 2 we give some general definitions
concerning arithmetic surfaces, and we present our two approaches for representing
them explicitly. Section 3 then presents our series of wild quotient singularities. In
the final section, we compute, in one concrete example of our wild quotient singular-
ities, an explicit desingularization.

2 Arithmetic surfaces and models of curves

2.1 General definitions

Definition 2.1. By a surface we mean an integral and noetherian scheme \( X \) of di-

dimension 2. An arithmetic surface is a surface \( X \) together with a faithfully flat mor-

phism \( f : X \to S = \text{Spec}(R) \) of finite type, where \( R \) is a Dedekind domain. To avoid
technicalities, we always assume that $R$ (and hence $X$) is excellent. Moreover, we will assume in addition that $X$ is normal, unless we explicitly say otherwise.

A common situation where arithmetic surfaces occur is the following. Let $R$ be a Dedekind domain, $K = \text{Frac}(R)$ and $X_K$ a smooth and projective curve over $K$. An $R$-model of $X_K$ is an arithmetic surface $X \to \text{Spec}(R)$, together with an identification of $X_K$ with the generic fiber of $X$, i.e. $X_K = X \otimes_R K$.

For the following discussion we fix an arithmetic surface $X \to \text{Spec}(R)$. We write $X^{\text{sing}}$ for the subset of points whose local ring is not regular. Since we assume that $X$ is normal, $X^{\text{sing}}$ is closed of codimension 2 and hence consists of a finite set of closed points of $X$. A point $\xi \in X^{\text{sing}}$ is called a singularity of $X$. (If we drop the normality condition, then $X^{\text{sing}}$ may also have components of codimension 1.)

By a modification of $X$ we mean a proper birational map $f : X' \to X$. A modification is an isomorphism outside a finite set of closed points. If $f$ is an isomorphism away from a single point $\xi \in X$, then $\xi$ is called the center of the modification and $E := f^{-1}(\xi) \subset X'$ the exceptional fiber or exceptional locus (we endow $E$ with the reduced subscheme structure). Note that $E$ is a connected scheme of dimension one.

We will use the notation

$$E = \bigcup_{i=1}^{n} C_i,$$

where the $C_i$ are the irreducible components. Each of them is a projective curve over the residue field $k = k(\xi)$. If the modification changes more than a single point, we will still denote the exceptional locus by $E$, but $E$ obviously does not need to be connected any more.

**Definition 2.2.** Let $p : X \to S$ be an arithmetic surface and $\xi \in X^{\text{sing}}$ a singularity. A desingularization of $\xi \in X$ is a modification $f : X' \to X$ with center $\xi$ and exceptional fiber $E = f^{-1}(\xi) \subset X'$ such that every point $\xi' \in E$ is a regular point of $X'$. A desingularization of $X$ is a modification consisting of desingularizations at all points of $X^{\text{sing}}$.

By a theorem Lipman ([17]), a desingularization of $X$ always exists by means of a sequence of normalizations and blow-ups. Depending on the situation we often want $f$ to satisfy further conditions. We list some of them:

(a) The exceptional divisor $E$ is a normal crossing divisor of $X'$.
(b) Let $s := p(x)$. Then the fiber $X'_s$ of $X'$ over $s$ is a normal crossing divisor on $X'$ (when endowed with the reduced subscheme structure).
(c) The desingularization $f : X' \to X$ is minimal (among all desingularizations of $\xi \in X$).
(d) $f : X' \to X$ is minimal among all desingularizations satisfying (a) (resp. (b)).

Choosing a different approach than Lipman and avoiding normalizations completely, Cossart, Janssen and Saito proved a desingularization algorithm relying only on blow-ups at regular centers in [4], see also [5]. The approach allows to additionally satisfy yet another rather common condition:
(e) If $X \subset W$ for some regular scheme $W$, then desingularization of $X$ can be achieved by modifications of $W$ which are isomorphisms outside $X^\text{sing}$.

### 2.2 Presentation by affine charts

We are interested in the problem of computing a desingularization $f : X' \to X$ of a given singularity $\xi \in X$ on an arithmetic surface explicitly. Before we can even state this problem precisely, we have to say something about the way in which the surface $X$ is represented.

The most obvious way to present $X$ is to write it as a union of affine charts, 

$$X = \bigcup_{j=1}^r U_j, \quad U_j = \text{Spec} A_j.$$  

Here each $A_j$ is a finitely generated $R$-algebra whose fraction field is the function field $F(X)$ of $X$. After choosing a set of generators of $A_j/R$, we can obtain a presentation 'by generators and relations'. This means that

$$A_j = R[\underline{x}] / I_j,$$

where $\underline{x} = (x_1, \ldots, x_{n_j})$ is a set of indeterminates and $I_j \subset R[\underline{x}]$ is an ideal. Choosing a list of generators of $I_j$, we obtain a presentation

$$R[\underline{x}]^{m_j} \to R[\underline{x}] \to A_j \to 0.$$

Taking into account the relations among the generators of the ideal $I_j$ this presentation extends to

$$R[\underline{x}]^{m_j} \to R[\underline{x}]^{m_j} \to R[\underline{x}] \to A_j \to 0,$$

where the matrix describing the left-most map is usually referred to as the first syzygy matrix of $I_j$ or $A_j$ respectively. Iteratively forming higher syzygies, this leads to free resolutions, i.e. exact sequences of free $R[\underline{x}]$-modules. As $R[\underline{x}]$ is a polynomial ring over a Dedekind domain, it has global dimension $n_j + 1$ and hence $A_j$ possesses a free resolution of length at most $n_j + 1$. Working locally at a maximal ideal $m \subset R[\underline{x}]$, this allows e.g. the calculation of the $m$-depth of $A_j$ by the Auslander-Buchsbaum formula.

In the subsequent sections, we shall encounter examples placing us in a particular situation, for which free resolutions are well understood: determinantal varieties corresponding to maximal minors. For these, $I_j$ is generated by the maximal minors of an $m \times n$ matrix defining a variety of codimension $(m - t + 1)(n - t + 1)$, where $t = \min \{m, n\}$. Most prominently, the Hilbert-Burch theorem (see for instance [8]) relates Cohen-Macaulay codimension 2 varieties to the $t$-minors of their first syzygy

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1 as before $W$ should be excellent, noetherian, integral  
2 thanks to Grothendieck
matrix, which is of size $t \times (t + 1)$, and ensures the map given by this matrix to be injective.

### 2.3 Presentation using valuations

An alternative way to present an arithmetic surface is the following. To describe it it is convenient to assume that $R$ is a local ring. Then $R$ is actually the valuation ring of a discrete valuation $v_K : K^\times \to \mathbb{Q}$ of its fraction field $K = \text{Frac}(R)$. We choose a uniformizer $\pi$ of $v_K$ (i.e. a generator of the maximal ideal $p \triangleleft R$) and normalize $v_K$ such that $v_K(\pi) = 1$. We denote the residue field of $v_K$ by $k$. In addition we make the following assumption:

**Assumption 2.3.** The valuation $v_K$ is either henselian, or its residue field $k$ is algebraic over a finite field.

We fix a smooth projective curve $X_K$ over $K$. Note that $X_K$ is uniquely determined by its function field $F_X$, and conversely every finitely generated field extension $F/K$ of transcendence degree 1 is the function field of a smooth projective curve $X_K$.

Let $X$ be an $R$-model of $X_K$, $X_s$ its special fiber and

$$X_s = \bigcup_i \bar{X}_i$$

its decomposition into irreducible components. Then each component $\bar{X}_i$ is a prime divisor on the surface $X$. Because $X$ is normal, $\bar{X}_i$ gives rise to a discrete valuation $v_i$ on $F_X$ such that $v_i(\pi) > 0$. We normalize $v_i$ such that $v_i(\pi) = 1$, i.e. such that $v_i|_K = v_K$. By definition, the residue field $k(v_i)$ of $v_i$ is the function field of the component $\bar{X}_i$. In particular, $k(v_i)$ is function field over $k$ of transcendence degree 1.

A discrete valuation $v$ on the function field $F_X$ is called geometric if $v|_K = v_K$ and the residue field $k(v)$ is a finitely generated extension of $k$ of transcendence degree 1. Let $V(F_X)$ denote the set of geometric valuations. Given a model $X$ of $X_K$, we write

$$V(X) := \{v_1, \ldots, v_r\} \subset V(F_X)$$

for the set of geometric valuations corresponding to the components of the special fiber of $X$.

**Theorem 2.4.** The map

$$X \mapsto V(X)$$

is a bijection between the set of isomorphism classes of $R$-models of $X_K$ and the set of finite nonempty subsets of $V(F_X)$.  

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3 Historically, this was actually the first method, pioneered by Deuring [7] more than 10 years before the invention of schemes.

4 More generally, we could have assumed that $(K, v_K)$ satisfies the local Skolem property, see [13]
Furthermore, given two models $X, X'$ of $X_K$, there exists a map $X' \to X$ which is the identity on $X_K$ (and which is then automatically a modification) if and only if $V(X) \subset V(X')$.

Proof. See [12] or [27].

By the above theorem models of a given smooth projective curve $X_K$ over a valued field $(K, v_K)$ can be defined simply by specifying a finite list of valuations. An obvious drawback of this approach is that it is not obvious how to extract detailed information on the model $X$ from the set $V(X)$. A priori, $V(X)$ only gives ‘birational’ information on the special fiber $X_s$. For instance, it is not immediate to see whether the model $X$ is regular.

So far, the above approach based on valuations has proved to be very useful for the computation of semistable models (see [28]). We intend to extend it to other problems in the future. In §4.2 we will see a first attempt to use it for desingularization.

### 2.4 Computational tools

In this section we report on some ongoing work to implement computational tools for dealing with arithmetic surfaces and their desingularization.

**Valuation based approach**

As we have explained in §2.3 it is in principle possible to describe arithmetic surfaces over a local field purely in terms of valuations. In order to use this approach for explicit computations, one needs a way to write down, manipulate and compute with geometric valuations. Fortunately, such methods are available (but maybe not as widely known as they should). Our approach goes back to work of MacLane ([21], [22]). In the present context (i.e. for describing models of curves over local fields) it has been developed systematically in Julian Rüth’s PhD thesis ([27]).

We will not go into details, but for later use we need to introduce the notion of an *inductive valuation*. Let $K$ be a field with a discrete valuation $v_K$ and valuation ring $R$ as before. Let $v$ be an extension of $v_K$ to a geometric valuation on the rational function field $K(x)$. We assume in addition that $v(x) \geq 0$ (i.e. that $R[x]$ is contained in the valuation ring of $v$). Let $\phi \in R[x]$ be a monic integral polynomial, and let $\lambda \in \mathbb{Q}$ be a rational number satisfying $\lambda > v(\phi)$. If $\phi$ is a key polynomial for $v$ (see [27], Definition 4.7) then we can define a new geometric valuation $v'$ (called an augmentation of $v$) with the property that

$$v'(\phi) = \lambda, \quad v'(f) = v(f) \text{ for } f \in K[x] \text{ with } \deg(f) < \deg(\phi).$$

See [27], Definition 4.9. We write
\[ v' = [v, v'(\phi) = \lambda]. \]

The process of augmenting a given geometric valuation can be iterated. A geometric valuation \( v \) on \( K(x) \) which is obtained by a sequence of augmentations, starting from the Gauss valuation with respect to \( x \), is called an \textit{inductive valuation}. It can be written as

\[ v = v_n = [v_0, v_1(\phi_1) = \lambda_1, \ldots, v_n(\phi_n) = \lambda_n]. \quad (1) \]

Here \( v_0 \) is the Gauss valuation, \( \lambda_i \in \mathbb{Q} \) and \( \phi_i \in R[x] \) is monic. Furthermore, \( \phi_i \) is a key polynomial for \( v_{i-1} \) and \( \lambda_i > v_{i-1}(\phi_i) \). By [27], Theorem 4.31, every geometric valuation \( v \) on \( K(x) \) with \( v(x) \geq 0 \) can be written as an inductive valuation.

The notion of inductive valuation can be extended in several ways. Firstly, by replacing \( x \) with \( x^{-1} \) if necessary, we can drop the condition \( v(x) \geq 0 \). Hence we can write every geometric valuation on \( K(x) \) as an inductive valuation. Secondly, for the last augmentation step in (1) we can allow the value \( \lambda_n = \infty \). The resulting \( v_n \) is then only a \textit{pseudo-valuation} and induces a true valuation on the quotient ring \( L := K[x]/(\phi_n) \) (which is a field because key polynomials are irreducible). Thirdly, given an arbitrary finite extension \( L/K \), we can compute the (finite) set of extensions \( w \) of \( v_K \) to \( L \) as follows. We write \( L = K[x]/(f) \) for an irreducible polynomial \( f \in K[x] \). If \( f \) is irreducible over the completion \( \hat{K} \) of \( K \) with respect to \( v_K \), then there exists a unique extension \( w \) of \( v \) to \( L \) which can be written as an inductive pseudo-valuation on \( K[x] \) (with \( \phi_n = f \)). In general, let \( f = \prod f_i \) be the factorization into irreducibles over \( \hat{K} \). Then each factor \( f_i \) gives rise to an extension \( w_i \) of \( v \) to \( L \).

Considering \( w_i \) as a pseudo-valuation on \( K[x] \), MacLane shows that \( w_i \) can be written as a \textit{limit valuation} of a chain of inductive valuations \( v_n \). By this we mean that \( v_n \) is an augmentation of \( v_{n-1} \), and for every \( \alpha = (g(x) \mod (f)) \in L \) there exists \( n \geq 0 \) such that \( w_i(\alpha) = v_n(g) = v_{n+1}(g) = \ldots \).

MacLane’s theory is constructive and can be used to implement algorithms for dealing with discrete valuations on a fairly large class of fields. A Sage package written by Julian Rüth called mac_lane (26) is available under \url{github.com/saraedum/mac_lane}

It can be used to define and compute with discrete valuations of the following kind:

- \( p \)-adic valuations on number fields.
- Geometric valuations \( v \) on function fields \( F/K \) (of dimension 1) whose restriction to \( K \) is either trivial, or can be defined by this package.

Given a valuation \( v \) on a field \( K \) of the above kind and a finite separable extension \( L/K \), it is possible to compute the set of all extension of \( v \) to \( K \).

\textbf{Chart based approach}

On the other hand, a description by affine charts as in [22] not only emphasizes the similarity to the geometric setting, it also allows the use of computational techniques such as standard bases (whenever a suitably powerful arithmetic for computations in \( R \) is available). This, in turn, opens up a whole portfolio of algorithms ranging from
basic functionality like elimination or ideal quotients to more sophisticated algorithms such as blowing up and normalization, which eventually permit to practically implement the above mentioned algorithms of Lipman and of Cossart-Janssen-Saito for desingularization of 2-dimensional schemes. Note at this point that neither of the two algorithms imposes the condition of normality on the surfaces to be resolved.

In a nutshell, the desingularization problem for 2-dimensional schemes is the problem of finding suitable centers which improve the singularity without introducing new complications. In this context, 0-dimensional centers for blow-ups usually do not pose any major problems: such blow-ups at different centers may be interchanged, as they are isomorphisms outside their respective centers and hence do not interact. However, even resolving a 0-dimensional singular point in the geometric case may already require the use of 1-dimensional centers to achieve a regular model and normal crossing divisors. These curves can exhibit significantly more structure than sets of points, e.g. they can possess intersecting components or non-regular branches. So the central problems in resolving the singularities of 2-dimensional schemes are ensuring improvement in each step and treating 1-dimensional loci which need to be improved. In particular for the latter, the two aforementioned approaches differ significantly.

The key idea behind Lipman’s algorithm \[17\] is that normal varieties are regular in codimension 1, i.e. that their singular locus is 0-dimensional. Thus a normalization step can always ensure that only sets of points will be required for subsequent blowing up:

**Theorem 2.5** (\[17\]). Let \( X \) be an excellent, noetherian, reduced scheme of dimension 2, then \( X \) posses a desingularization by a finite sequence of birational morphisms of the form

\[
X_r \overset{\pi_r \circ n_r}{\longrightarrow} \cdots \overset{\pi_2 \circ n_2}{\longrightarrow} X_1 \overset{\pi_1 \circ n_1}{\longrightarrow} X_0 = X,
\]

where \( \pi_i \) denotes a blow up at a finite number of points, \( n_i \) a normalization and \( X_r \) is regular.

While blowing up is algorithmically straightforward e.g. using an elimination (see e.g. \[9\]), the hard step is the normalization. Although there has been significant improvement in the efficiency of Grauert-Remmert style normalization algorithms in the last decade (see e.g. \[14\], \[1\]), this is still a bottleneck when working over a Dedekind domain \( \mathcal{R} \) instead of a field. The crucial step here is the choice of a suitable test ideal, i.e. a radical ideal contained in the ideal of the non-normal locus and containing a non-zerodivisor. In the geometric case, the ideal of the singular locus – generated by the original set of generators and the appropriate minors of the Jacobian matrix – is well-suited for this task, but in the current setting it also sees fibre singularities which do not contribute to the non-regular locus. Hence the approximation of the non-normal locus by this test ideal is rather coarse and significantly impedes efficiency. In practice, a better approximation of the non-normal locus is achieved by constructing a test ideal following an idea of Hironaka’s termination
criterion: we use the locus where Hironaka’s invariant $\nu^*$, i.e. the tuple of orders (in the sense of orders of power series) of the elements of a local standard basis, sorted by increasing order, is lexicographically greater than a tuple of ones.

The approach of Cossart-Janssen-Saito [4] (CJS for short) on the other hand, avoids normalization completely and allows well-chosen 1-dimensional centers, whenever necessary; when choosing centers, it takes into account the full history of blowing ups leading to the current situation. In constrast to Lipman’s approach, this algorithm yields an embedded desingularization. Nevertheless, a key step is again the use of the locus where $\nu^*$ lexicographically exceeds a tuple of ones. But then, no normalization follows, instead the singularities of this locus are first resolved before it is itself used as a 1-dimensional center. Each arising exceptional curve in this process remembers when it was created and whether its center was of dimension 0 or 1, because this information is crucial in the choice of center for ensuring improvement as well as normal crossing of exceptional curves.

A beta version of the first algorithm is available as SINGULAR-library resliman.lib and is planned to become part of the distribution in the near future. A prototype implementation of the CJS-algorithm has been implemented and is closely related to an ongoing PhD-project on a parallel approach to resolution of singularities using the gpi-space parallelization environment (for recent progress along this train of thought see [2], [25]).

3 Explicit construction of wild quotient singularities

In this section we describe a series of examples for arithmetic surfaces with interesting singularities. The general construction is due to Lorenzini (see [19] and [20]). Our contribution is to explicitly describe the (local) ring of the singularity by generators and relations. In the next section we also describe the desingularization in an equally explicit way.

Let $R$ be a discrete valuation ring, with maximal ideal $p$, residue field $k = R/p$ and fraction field $K$. Let $v_K$ denote the corresponding discrete valuation on $K$. We assume that $k$ has positive characteristic $p$ and that $v_K$ is henselian (in particular, Assumption 2.3 holds).

Let $X_K$ be a smooth, projective and absolutely irreducible curve over $K$, of genus $g$. We assume that $X_K$ has potentially good reduction with respect to $v_K$. This means that there exists a finite extension $L/K$ and a smooth model $Y$ of $X_L := X_K \otimes_K L$ over the integral closure $R_L$ of $R$ in $L$. Note that $R_L$ is a discrete valuation ring corresponding to the unique extension $v_L$ of $v_K$ to $L$. We assume in addition that $L/K$ is a Galois extension, and that the natural action of $G := \text{Gal}(L/K)$ on $X_L$ extends to an action on $Y$. Under this assumption, we can form the quotient scheme $X_Y = X_L/G$. It is an $R$-model of $X_K$.

The model $Y$ is regular because $Y \to \text{Spec}(R)$ is smooth by assumption. However, the quotient scheme $X = Y/G$ may have singularities. In fact, let $\xi \in X_l$ be a closed
point on the special fiber of $X$, and let $\eta \in Y$, be a point above $\xi$. Let $I_\eta \subset G$ denote the inertia subgroup of $\eta$ in $G$. If $I_\eta = 1$ then the map $Y \to X$ is étale in $\eta$. It follows that $X$ is regular in $\xi$ because $Y$ is regular in $\eta$.

In general, the locus of points with $I_\eta \neq 1$ may consists of the entire closed fiber $Y$, and hence be a subset of codimension 1 on $Y$. To obtain isolated quotient singularities we impose the following condition:

**Assumption 3.1.** The action of $G$ on the special fiber $Y_s$ is generically free.

Under this assumption, there are at most a finite number of points $\eta \in Y_s$ with nontrivial inertia $I_\eta \neq 1$. Let $\xi_1, \ldots, \xi_r \in X_s$ be the images of the points $\eta \in Y_s$ with $I_\eta \neq 1$. Then $\xi_1, \ldots, \xi_r$ are precisely the singularities of the model $X$.

**Remark 3.2.** In Lorenzini’s original setting, Assumption 3.1 holds automatically because the curve $Y$ has genus $g(Y) \geq 2$. In our series of examples we have $g(Y) = 0$, but the assumption holds nevertheless.

### 3.1 An explicit example

Let $p$ be a prime number, $K$ a number field and $p \mid p$ a prime ideal of $\mathcal{O}_K$ over $p$. Let $v_K$ denote the discrete valuation on $K$ corresponding to $p$ and $R$ the valuation ring of $v_K$. Let $L/K$ be a Galois extension of degree $p$ which is totally ramified at $p$. This means that $v_K$ has a unique extension $v_L$ to $L$. Let $\sigma$ be a generator of the cyclic group $G = \text{Gal}(L/K)$. Let $\pi_L$ be a uniformizer for $v_L$. We normalize $v_L$ such that $v_L(\pi_L) = 1$. Set

$$m := v_L(\sigma(\pi_L) - \pi_L).$$

Then $m \geq 2$ is the first and only break in the filtration of $G$ by higher ramification groups. We let $a \in k^\times$ denote the image of the element $(\sigma(\pi_L) - \pi_L)/\pi_L^m \in R^\times$.

Let $X_K := \mathbb{P}^1_L$ be the projective line over $K$. We identify the function field $F_X$ with the rational function field $K(x)$ in the indeterminate $x$. Then $L(x)$ is the function field of $X_L := \mathbb{P}^1_K$. We define an element

$$y := \frac{x - \pi_L}{\pi_L^m} \in L(x).$$

Clearly, $L(x) = L(y)$, and so $y$, considered as a rational function on $X_L$, gives rise to an isomorphism $X_L \cong \mathbb{P}^1_L$. We let $Y$ denote the smooth $R_L$-model of $X_L$ such that $y$ extends to an isomorphism $Y \cong \mathbb{P}^1_{R_L}$. By an easy calculation we see that $\sigma(y) = ay + b$, with $a \in R_L^\times$ and $b \in R_L$. Furthermore,

$$\sigma(y) \equiv y + u \pmod{\pi_L}.$$

In geometric terms this means that the action of $G$ on $X_L$ extends to the smooth model $Y$, and that the restriction of this action to the special fiber $Y_s \cong \mathbb{P}^1_k$ is generically free (and hence Assumption 3.1 holds). In fact, the action of $G$ is fix point free.
on the affine line \( \text{Spec} k[y] \), and if \( \eta \in Y \) denote the point corresponding to \( y = \infty \) then \( I_\eta = G \).

Let \( \xi \in X \) denote the image of \( \eta \). By construction, \( \xi \) is a wild quotient singularity, and it is the only singular point on \( X \). Our goal is to write down explicitly an affine chart \( U = \text{Spec} A \subset X \) containing \( \xi \).

To state our result we need some more notation. Let \( \phi \in K[x] \) denote the minimal polynomial of \( \pi \) over \( K \).

\[
\phi = x^p + \sum_{i=0}^{p-1} a_i x^i = p \prod_{k=0}^{p-1} (x - \sigma^k(\pi_L)),
\]

where \( a_0, \ldots, a_{p-1} \in p \). The constant coefficient

\[
\pi_K := a_0 = N_{L/K}(\pi_L)
\]

is actually a prime element of \( R \), i.e. \( \phi \) is an Eisenstein polynomial.

The following lemma gives a characterization of the model \( X \) in terms of the set \( V(X) \) of valuations corresponding to the irreducible components of the special fiber (as in Theorem 2.4).

**Lemma 3.3.** We have

\[
V(X) = \{ v \}
\]

where \( v \) is the inductive valuation on \( K(x) \) extending \( v_K \) given by

\[
v : [v(x) = 1/p, v(\phi) = m].
\]

(See \[27\], §4.4 for the relevant notation.)

**Proof.** It is clear that \( V(Y) = \{ w \} \), where \( w \) is the Gauss valuation on \( F(X_L) = L(y) \) with respect to the parameter \( y \) and the valuation \( v_L \). Since \( Y \to X = y / G \) is a finite morphism between (normal) models of their respective generic fibers, we have \( V(X) = \{ v \} \), where \( v \) is the restriction of \( w \) to the subfield \( F(X_K) = K(x) \subset F(X_L) = L(y) \). It remains to identify \( v \) with the inductive valuation given in the statement of the lemma.

We will use the characterization of an inductive valuation which is implicit in \[27\], §4.4. Let \( v' \) be a valuation on \( K(x) \) which extends \( v_K \) and satisfies

\[
v'(x) \geq 0, \quad v'(\phi) \geq m.
\]

Then we claim that \( v(f) \leq v'(f) \) for all \( f \in K[x] \). By \[27\], Theorem 4.56, the claim implies that

\[
v = [v(x) = 1/p, v(\phi) = m].
\]

To prove the claim, we choose an extension \( w' \) of \( v' \) to the overfield \( L(y) \). Then

\[
m \leq v'(\phi) = \sum_{i=0}^{p-1} w'(x - \sigma^i(\pi_L)) = \sum_{i=0}^{p-1} w'(\pi_L^{i}y + \pi_L - \sigma^i(\pi_L)). \tag{2}
\]
By definition we have

\[ w'(\pi_L) = v_L(\pi_L) = 1/p, \quad w'(\pi_L - \sigma'(\pi_L)) = v_L(\pi_L - \sigma'(\pi_L)) \geq m/p. \] (3)

Combining (2), (3) and the strong triangle inequality we conclude that

\[ w'(y) \geq 0. \]

The valuation \( w \) being the Gauss valuation with respect to \( y \) and \( v_L \) this implies \( w(f) \leq v'(f) \) for all \( f \in K[x] \). But \( K[x] \subset K[y] \), and therefore \( v(f) \leq v'(f) \) for all \( f \in K[y] \). This proves the claim and also the lemma.

Let \( D_K \subset X_K \) be the divisor of zeroes of \( \phi \), and let \( D \subset X \) be the closure of \( D_K \). Let \( U := X - D \) denote the complement.

**Proposition 3.4.** 1. We have \( U = \text{Spec} A \), where \( A \subset F_X = K(x) \) is the sub-\( R \)-algebra generated by the elements \( x_0, \ldots, x_{p-1} \), where

\[ x_i := \pi_K x_i^{-1}, \quad i = 0, \ldots, p-1. \]

The point \( \xi \) lies on \( U \) and corresponds to the maximal ideal

\[ m := (\pi_K, x_0, \ldots, x_{p-1}) \triangleleft A. \]

2. The ideal of relations between the generators \( x_0, \ldots, x_{p-1} \) is generated by the \( 2 \times 2 \) minors of the matrix

\[ M := \begin{pmatrix} x_0 & x_1 \\ x_1 & x_2 \\ \vdots & \vdots \\ x_{p-2} & x_{p-1} \\ x_{p-1} & z \end{pmatrix}, \quad \text{with} \quad z := \pi_K^m - \sum_{i=0}^{p-1} a_i x_i. \]

**Proof.** It follows from [18], Corollary 5.3.24, that the divisor \( D \subset X \) is ample, and hence \( U := X - D = \text{Spec}(A) \) is affine. Since \( X \) is normal, the ring \( A \) consists precisely of all rational functions \( f \in K(x) \) with \( \text{ord}_Z(f) \geq 0 \), for any prime divisor \( Z \subset X \) distinct from \( D \).

A prime divisor \( Z \subset X \) is either horizontal (i.e. the closure of a closed point on \( X_K \)) or equal to \( X_s \). By Lemma 3.3, \( X_s \) is a prime divisor with corresponding valuation \( v \) on \( K(x) \). It follows that

\[ A = \{ f \in A_K \mid v(f) \geq 0 \}, \]

where

\[ A_K = K[\phi^{-1}, x\phi^{-1}, \ldots, x^{p-1}\phi^{-1}]. \]

In order to make the condition \( v(f) \geq 0 \) more explicit, we write \( f \in A_K \) in the form

\[ f = c_0 + \sum_{i=0}^{r-1} \sum_{j=0}^{p-1} c_{i,j} x^i \phi^{j-r}, \]
with $c_0, c_{i,j} \in K$. Then Lemma 3.3 shows that

$$v(f) = \min \{ v_K(c_0), v_K(c_{i,j}) + j/p - m(r-i) \}.$$ 

So the condition $v(f) \geq 0$ is equivalent to

$$v_K(c_{i,j}) + j/p \geq m(r-i),$$

for $i = 0, \ldots, r-1$ and $j = 0, \ldots, p-1$. It follows that

$$A = R[x_0, \ldots, x_{p-1}], \quad \text{where } x_j := \pi^n_K x^j \phi^{-1}.$$ 

This is the first part of Statement (i); the second part is obvious.

To prove Statement (ii) we let $I$ be the ideal in the polynomial ring $R[x] = \mathbb{R}[x_0, \ldots, x_{p-1}]$ generated by the $2 \times 2$-minors of the matrix $M$. It is easy to check that the generators of $A$ satisfy these relations. Therefore, we have a surjective map $A' := R[x_0, \ldots, x_{p-1}]/I \to A$. We want to prove that $A' = A$.

Let $A'' := A'[x_0^{-1}]$ and consider the matrix $M$ with entries in $A''$. By definition we have $\text{rk}M \leq 1$, and the upper left entry $x_0$ is a unit. An elementary argument shows that there exists $t \in A''$ such that

$$x_0 \phi(t) = \pi^n_K, \quad x_i = t'x_0, \quad i = 1, \ldots, p-1.$$ 

It follows that

$$A'' = R[x_0, x_0^{-1}, t \mid x_0 \phi(t) = \pi^n_K].$$

In particular $A''/R[x_0, x_0^{-1}]$ is a finite flat and generically étale extension of degree $p$.

We deduce that $A''$ is an integral domain of dimension 2. Looking at the equations defining $A'$, it is easy to see that

$$(x_0)_{\text{rad}} = (x_0, \ldots, x_{p-1})$$

and that $A'/ (x_0)_{\text{rad}} \cong R$ has dimension 1. Together with $\dim A'' = 2$ this implies that $\dim A' = 2$. Therefore, $A'$ is a determinantal ring of the 'expected' codimension $(p-2+1)(2-2+1) = p-1$. Now a theorem of Eagon and Hochster shows that $A'$ is Cohen-Macaulay (see [8], Theorem 18.18 for a textbook reference). Every associated prime of a Cohen-Macaulay ring is minimal ([8], Corollary 18.10). Since $A'' = A'[x_0^{-1}]$ is an integral domain, it follows that $A'$ is an integral domain as well.

The analysis of $A''$ from above also shows that

$$A''_K = A'_K[x_0^{-1}] = A_K[x_0^{-1}] = K[x, \phi^{-1}].$$

It follows that $J = \ker(A' \to A)$ is an ideal of codimension $\geq 1$. But $A, A'$ have the same dimension, so $J$ consists of zero divisor. On the other hand, we have shown above that $A'$ is an integral domain. Hence $J = 0$. This completes the proof of Proposition 3.3
Example 3.5. The simplest special case of Proposition 3.4 where the resulting singularity is not a complete intersection is for $p = 3$. To make this even more explicit, we set $K := \mathbb{Q}$ and let $v_K$ denote the $3$-adic valuation on $K$ and $R := \mathbb{Z}_{(3)}$ the valuation ring (the localization of $\mathbb{Z}$ at $3$). Moreover, we set

$$\phi := x^3 - 3x^2 + 3.$$ 

The splitting field $L/K$ of $\phi$ is a Galois extension of degree 3 which is totally ramified at $p = 3$. Indeed, we can factor $\phi$ as

$$\phi = (x - \pi)(x - \sigma(\pi))(x - \sigma^2(\pi)) = (x - \pi)(x - \pi - \pi^2 + 3\pi)(x - \pi + \pi^2 - 3),$$

where $\pi$ is prime elements for the unique extension $v_L$ of $v_K$ to $L$. We see that

$$m := v_L(\pi - \sigma(\pi)) = 2.$$ 

The resulting singularity $\xi$ of the model $X$ of $X_K = \mathbb{P}^1_K$ constructed above is a rational triple point.

Remark 3.6. The generic fiber $X_K$ of our model $X$ is a curve of genus zero and so is not, strictly speaking, an example of the situation studied by Lorenzini. But we can easily modify our construction to get examples with arbitrary high genus. For instance, choose $m > 1$, $p \nmid m$ and consider the Kummer cover $Y_K \to X_K$ of smooth projective curves with generic equation

$$Y_K : y^m = \phi(x).$$

Then $g(Y_K) \geq 2$ (except for $p = 3$ and $m = 2$ when $g(Y_K) = 1$). Let $Y$ denote the normalization of the $R$-model $X$ inside the function field of $Y_K$. Then $Y$ is a (normal) $R$-model of $Y_K$. It can easily be shown that $Y$ has a unique singular point $\eta$ (which is the unique point in the inverse image of $\xi \in X$), and that $\eta \in Y$ is a wild quotient singularity in the sense of [20]. We intend to study this situation in a subsequent paper.

4 An explicit resolution

To keep the construction of a desingularization in an explicit example as concise as possible we now focus on the specific Example 3.5. This case already illustrates the general situation quite well, but is still sufficiently small to avoid lengthy explicit computations.

Set $K := \mathbb{Q}$ and let $v_K$ denote the 3-adic valuation on $K$ and $R := \mathbb{Z}_{(3)}$ the valuation ring (the localization of $\mathbb{Z}$ at $3$). Let $v_0$ denote the Gauss valuation on $K(x)$ with respect to $x$. We define an inductive valuation $v$ on $K(x)$ as follows:

$$v := [v_0, v(x) = 1/3, v(x^3 - 3x^2 + 3) = 2].$$
Let $X$ be the model of $X_K := \mathbb{P}_k^1$ with $V(X) = \{v\}$. We have shown in the preceding section that $X$ has a unique singularity $\xi$ with an affine open neighborhood $U = \text{Spec} A$, where

$$A = R[x, y, z]/I,$$

and where $I$ is the ideal generated by the 2-minors of the matrix

$$M = \begin{pmatrix} x & y \\ y & z \\ z & 3x - 3z - 9 \end{pmatrix}.$$ 

The singular point $\xi$ corresponds to the maximal ideal $m = (3, x, y, z) \triangleleft A$.

### 4.1 Explicit blowups and Tjurina modifications

Our goal is to construct explicitly a desingularization $f : X' \to X$ of $\xi$. For ease of notation we replace the projective scheme $X$ by the affine open subset $U = \text{Spec} A$.

We not only know that $A$ is Cohen-Macaulay of codimension 2, we are in an even better setting, the situation of the Hilbert-Burch theorem, which then implies that a free resolution of $A$ is of the form

$$0 \to R[x, y, z]^2 \xrightarrow{M} R[x, y, z]^3 \to R[x, y, z] \to A \to 0,$$

i.e. the Eagon-Northcott complex of $M$.

At first glance this seems to be unrelated to our task of desingularizing $A$. However, these structural observations point us to well known results in the complex geometric case: In the late 1960s, Gergana Tjurina classified the rational triple point singularities over the complex numbers in [31] and constructed minimal desingularizations thereof in a direct way. Our given matrix $M$ structurally corresponds to a singularity of type $H_5$ in Tjurina’s article, which we will refer to as $Y$ here and for which a presentation matrix (over $\mathbb{C}[x, y, z, w]$) is of the form

$$N = \begin{pmatrix} x & y \\ y & z \\ z & wx - w^2 \end{pmatrix}.$$ 

The last entry can be replaced by $wx - wz - w^2$ without changing the analytic type of the singularity as is shown in the classification of simple Cohen-Macaulay codimension 2 singularities in [10]. This similarity suggests to try and mimic the philosophy of Tjurina’s choice of centers for the desingularization of $X$.

Tjurina’s first step towards a resolution of singularities is nowadays called a Tjurina modification and is based on the observation that at each point of $Y$ except the
origin the row space of the presentation matrix defines a unique direction in $\mathbb{C}^2$ and hence a point in the Grassmanian of lines in 2-space. Resolving indeterminacies of this rational map into the Grassmanian then yields the Tjurina transform which can then be described by the equations

$$ N \cdot \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. $$

(For a more detailed treatment of Tjurina modifications see the first section of [11].)

Three further blow-ups, each at the (0-dimensional) singular locus, which happens to be the non-normal crossing locus of the exceptional curves in the second and third blow-up, then lead Tjurina to a desingularization. The exceptional locus of this sequence of blow-ups consists of 6 curves of genus zero, where the one originating from the Tjurina modification is the only one with self-intersection $-3$; all others have self-intersection $-2$. The dual graph of the resolution is of the form:

```
-2 -2 -2 -2 -2

-3
```

**Fig. 1** Tjurina’s intersection graph $H_5$

Returning to our setting, we can mimic these steps, obtaining the following as ideal of the Tjurina transform:

$$ I_{X_1} = \langle sx - ty, sy - tz, sz - t(3z - 3x - 9) \rangle $$

By direct computation, it is easy to see that $X_1$ is regular except above 3 and that above 3 the non-regular locus is contained in the chart $t \neq 0$. The exceptional curve $C_0$ which arose in this blow-up is a $\mathbb{P}^1$ and corresponds to the ideal $\langle x, y, z, 3 \rangle$. Passing to the chart $t \neq 0$, we can harmlessly eliminate the variables $y$ and $z$ according to the first two generators. This essentially leaves a hypersurface described by the ideal

$$ I_{X_1, \text{new}} = \langle s^3x - 3s^2x + 3x + 9 \rangle \subset R[x, s] $$

and an exceptional curve $I_{C_0} = \langle x, 3 \rangle$. The non-regular locus of this hypersurface corresponds to $\langle x, s, 3 \rangle$ as a direct computation shows; this is the center of the upcoming blow-up, which leads to 3 charts, two of which only contain regular points.
and only see normal crossing divisors. In the remaining chart \((y_1 \neq 0)\), the strict transform is given by

\[ I_{X_2} = \langle 3 - y_2 s, s^2 y_0 - s^2 y_0 y_2 + y_0 y_2 + y_2^2 \rangle, \]

the strict transform of the exceptional curve \(C_0\) by \(\langle 3, y_0, y_2 \rangle\) and the two components \(C_1\) and \(C_2\) of the new exceptional curve \(E_2\) by \(\langle 3, s, y_2 (y_0 + y_2) \rangle\). As the non-regular locus is given by \(\langle 3, s, y_0, y_2 \rangle\) and the non-normal crossing locus of the exceptional curves is the same point, analogous to Tjurina’s setting, this point has to be chosen as upcoming center. After blowing up this point of \(X_2\), we see in one chart that each of the two components \(C_1\) and \(C_2\) of the preceding exceptional curve \(E_2\) meets one component of the new exceptional curve \(E_3\); more precisely, \(C_1\) meets \(C_3\) and \(C_2\) meets \(C_4\). In another chart, we see that the transform of \(C_0\) meets both \(C_3\) and \(C_4\) at the origin, which is also the only singular point. Blowing up this point then introduces yet another exceptional curve \(C_5\) meeting \(C_0, C_3\) and \(C_4\); at this stage, the strict transform is regular and the exceptional divisor is normal crossing. All exceptional curves are \(-2\)-curves except the \(-3\)-curve \(C_0\). Hence we obtained the dual graph:

\[ \begin{array}{cccccc}
C_1 & C_3 & C_5 & C_4 & C_2 \\
\circ & \circ & \circ & \circ & \circ \\
& & & & & \\
& & & & & C_0 \\
\end{array} \]

Fig. 2 The intersection graph of the desingularization of \(X\)

An explicit comparison of the computations of Tjurina and of the one presented in our setting shows that that all computational steps as well as the final result are analogous in both cases. This certainly raises the question whether other singularities from Tjurina’s list also have an analogue arising from the construction of Section 3 and what geometric properties the singularities corresponding to the matrices of the previous section might exhibit.

Remark 4.1. 1. In the above calculation, we saw that we could safely replace the matrix \(N\), which is the normal form in the classification of simple Cohen-Macaulay codimension 2 singularities [10], by a matrix say \(N'\) which directly corresponds to the original matrix \(M\), differing only by using a variable \(w\) instead of \(\pi_k = 3\). The isomorphism of the local rings of the singularities represented by \(N\) and \(N'\) does not involve any change of \(w\), whence we could hope
for an equivalent isomorphism for $M$. This, however, does not exist, as the isomorphism over $\mathbb{C}$ involves the multiplicative inverse of 3.

2. As in the explicit example here, all the determinantal singularities from Proposition 3.4 allow a Tjurina modification at the origin of the respective chart at the beginning of the desingularization; this provides an exceptional curve $C_0$. After this step, we see only one singular point, an $A_{pm-1}$ singularity. This latter singularity is well known to have a dual graph of resolution which is a chain with $pm - 1$ vertices and $pm - 2$ edges, where the middle vertex corresponds to the youngest exceptional curve. This middle vertex is the position, where the edge connecting the vertex corresponding to $C_0$ to the chain.

### 4.2 A posteriori description via valuations

We return to our original notation, i.e. $X$ denotes the $R$-model of $X_K = \mathbb{P}^1_K$ with $V(X) = \{v\}$ (and not its affine subset $\text{Spec}A$). Also, $x$ again denotes the original coordinate function on $X_K$.

The computation of the previous section show that there exist a desingularization $f : X' \to X$ of $\xi$ such that the exceptional fiber $E := f^{-1}(\xi)$ is a normal crossing divisor and consists of 6 smooth rational curves, with an intersection graph given in Fig. 2. The arithmetic surface $X'$ is itself an $R$-model of $X_K$ and is hence completely determined by the set $V(X')$ of geometric valuations of $K(x)$ corresponding to the irreducible components of the special fiber $X'_s$. But $X'_s$ consists precisely of the strict transform $C_6$ of $X_s$ (which corresponds to the valuation $v_6 := v$) and the 6 components $C_0, \ldots, C_5$ of the exceptional divisor.

The obvious question is: what are the valuations corresponding to the components $C_i, i = 0, \ldots, 5$?

**Proposition 4.2.** Let $v_i$ denote the valuation on $K(x)$ corresponding to the component $C_i$, for $i = 0, \ldots, 5$. We normalize $v_i$ such that $v_i(3) = 1$ (i.e. such that $v_i|K = v_K$). Then $v_0$ is the Gauss valuation with respect to the coordinate $x$. For $i = 1, 3, 5$,

$$v_i = [v_0, v_i(x) = r_i], \quad r_i = \begin{cases} 1/3, & i = 5, \\ 1/2, & i = 3, \\ 1, & i = 1. \end{cases}$$

For $i = 2, 4$ we have

$$v_i = [v_0, v_i(x) = 1/3, v_i(\phi) = s_i], \quad s_i = \begin{cases} 4/3, & i = 4, \\ 5/3, & i = 2. \end{cases}$$

**Proof.** This can be checked by a direct (but somewhat involved) computation, using the explicit description of the desingularization by affine charts in §4.1. As an illus-
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It suffices to consider the first step of the desingularization, the Tyurina modification $X_1 \to X$. We use the notation from p.16. The affine chart of $X_1$ defined by $t \neq 0$ has the form

$$\text{Spec} R[x_0, s \mid s^3x_0 - 3sx_0^2 + 3x_0 + 9 = 0]$$

and the exceptional divisor $E_1 \subset X_1$ is given on this chart by $I_{E_1} = (x_0^3)$. So $\text{Spec} F_3[s]$ is an affine open of $E_1$, and hence $E_1$ is a projective line. We claim that $E_1$, as a prime divisor on $X$, gives rise to the valuation $v_0$ (the Gauss valuation with respect to $x$).

We write $x_0, s$ as rational functions in $x$:

$$x_0 = 9\phi^{-1}, \quad s = \frac{x_1}{x_0} = x.$$

Now we see that the generators of the ideal $I_{E_1}$ have positive valuation ($v_0(x_0) = 2$) and $s$ is a $v_0$-unit and is a generator of its residue field. This shows that the prime divisor $E_1 \subset X_1$ corresponds to the valuation $v_0$. As the component $C_0$ of the desingularization $X' \to X$ is simply the strict transform of $E_1$ under the map $X' \to X_1$, we have proved the proposition for $i = 0$. For $i = 1, \ldots, 5$ one can proceed in a similar way. \qed

**Remark 4.3.**

1. We have found the set $V(X') = \{v_0, \ldots, v_6\}$ after computing the desingularization $X' \to X$. By Theorem 2.4 $X'$ is determined by $V(X')$. Could we have found $V(X')$ by some other method, and would this give an alternative way to compute desingularization? In this simple case it is indeed possible to check the regularity of $X'$ (and the fact that $X'_i$ is a normal crossing divisor) purely in terms of the set of valuations $\{v_0, \ldots, v_6\}$. More details will be given elsewhere.

2. If we accept that $X'$ is regular and $X'_i$ is a normal crossing divisor, it is easy to compute the self intersection numbers of the irreducible components $C_i$, as follows. Let

$$\tilde{E} := (3) = \sum_{i=0}^{6} m_iC_i \in \text{Div}(X)$$

be the principal divisor of the prime 3. For each $i$ the integer $m_i$ (the multiplicity of the component $C_i$) is equal to the ramification index of the extension $K(x)/K$ with respect to $v_i$. It is easy to read off $m_i$ from the explicit description of the $v_i$ in Proposition 4.2.

$$m_0 = 1, \ m_1 = 1, \ m_2 = 3, \ m_3 = 2, \ m_4 = 3, \ m_5 = 3, \ m_6 = 3.$$

Since $\tilde{E}$ is a principal divisor, we have
\[ 0 = (C_i, \mathcal{E}) = \sum_{j=0}^{6} m_j(C_i, C_j) , \]

for \( i = 0, \ldots, 6 \), see e.g. \[29\], §IV.7. The component graph from Fig. 2 tells us what \((C_i, C_j)\) is for \( i \neq j \) (either 1 or 0). Now the self intersection numbers \((C_i, C_i)\) can be computed easily. We find that

\[
(C_i, C_i) = \begin{cases} 
-3, & i = 0, \\
-2, & i = 1, \ldots, 5, \\
-1, & i = 6.
\end{cases}
\]

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