GRAPH HOMOLOGY: KOSZUL AND VERDIER DUALITY

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Abstract. We show that Verdier duality for certain sheaves on the moduli spaces of graphs associated to Koszul operads corresponds to Koszul duality of operads. This in particular gives a conceptual explanation of the appearance of graph cohomology of both the commutative and Lie types in computations of the cohomology of the outer automorphism group of a free group. Another consequence is an explicit computation of dualizing sheaves on spaces of metric graphs, thus characterizing to which extent these spaces are different from oriented orbifolds. We also provide a relation between the cohomology of the space of metric ribbon graphs, known to be homotopy equivalent to the moduli space of Riemann surfaces, and the cohomology of a certain sheaf on the space of usual metric graphs.

Introduction

The popularity of graph homology owes largely to the fact that the cohomology of two important spaces in mathematics, the classifying space $Y_n$ of the outer automorphism group of the free group on $n$ generators and the (decorated) moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces of genus $g$ with $n$ punctures, even though generally intractable, may be computed via a deceptively simple combinatorial construction, called graph homology, see M. Culler and K. Vogtmann [2] and R. C. Penner [14]. These results, combined with further study of graph homology by M. Kontsevich [10], rendered the following identifications:

\[ H_\bullet(Y_n, k) \cong H^{3n-4-\bullet}_\text{Lie}(n), \]
\[ H^\bullet_c(Y_n, k) \cong \check{H}^\bullet_{\text{Comm}}(n), \]
\[ H_\bullet(\mathcal{M}_{g,n}, k) \cong H^{6g+3n-7-\bullet}_{\text{Ass}}(g, n), \]
\[ H^\bullet_c(\mathcal{M}_{g,n}, k) \cong \check{H}^\bullet_{\text{Ass}}(g, n), \]

where $k$ is a coefficient field of characteristic zero, $H^\bullet_c$ denotes cohomology with compact supports, and in the right-hand side, we have graph cohomology of various flavors, Lie, commutative, and associative, with trivial or twisted coefficients.

The appearance of Koszul-dual operads in the right-hand side as corresponding to the homology vs. cohomology with compact supports in the left-hand side is quite suggestive: it hints on a relationship between some kind of Poincaré duality for spaces and Koszul duality for operads.

In this paper we show that this relationship indeed takes place and in fact prove more general results, Theorems 3.9 and 4.4, which show that up to an orientation twist, Verdier duality on the moduli space of graphs transfers a certain constructible sheaf corresponding to an operad $\mathcal{O}$ to the sheaf corresponding to the dg-dual operad $\mathcal{D}\mathcal{O}$, which is quasi-isomorphic to the Koszul-dual operad $\mathcal{O}^!$, if $\mathcal{O}$ happens to be Koszul. The idea of a relationship between the two dualities originates from the paper [6] by Ginzburg and Kapranov, who noticed that Verdier duality for sheaves on buildings (spaces of metric trees) provided a sheaf-theoretic interpretation of Koszul duality for operads. Koszulity was thus interpreted in terms of the vanishing of higher cohomology for corresponding sheaves, while in our paper it translates into a duality statement between highly nontrivial cohomology groups of spaces of metric graphs.

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The moduli spaces we consider are non-compact. It seems likely that similar results hold for certain compactifications of our moduli spaces and we intend to return to this in the future.

**Notation.** Throughout this paper we work with vector spaces, graded vector spaces, and dg-vector spaces or complexes — all finite-dimensional in each degree and bounded, over a ground field $k$, which is assumed to be of characteristic zero with the exception of Section 1. We consider chain complexes $V_\bullet = \bigoplus_{i \in \mathbb{Z}} V_i$ with a differential $d : V_i \to V_{i-1}$ and cochain complexes $V^\bullet = \bigoplus_{i \in \mathbb{Z}} V^i$ with a differential $d : V^i \to V^{i+1}$.

The (degree) shift $V[1]$ of a complex $V$ has components $(V[1])_i = V_{i+1}$ in the category of chain complexes and $(V[1])^i = V^{i+1}$ in the category of cochain complexes. For chain complexes the degree shift is also known as desuspension.

The functor $V \mapsto V^*$ of taking the linear dual acts within each of the two categories:

$$V^*_i = (V_{-i})^*, \quad d^* : V^*_i \to V^*_{i-1},$$

while another functor, $V \mapsto V^\vee$, takes the category of chain complexes to that of cochain ones:

$$(V^\vee)^j = (V^{-j})^*, \quad d^* : (V^\vee)^j \to (V^\vee)^{j+1},$$

while another functor, $V \mapsto V^\vee$, takes the category of chain complexes to that of cochain ones:

$$(V^\vee)^j = (V_j)^*, \quad d^\vee : (V^\vee)^j \to (V^\vee)^{j+1}.$$}

Note that $(V[1])^* \cong V^*[-1]$ and $(V[1])^\vee \cong V^\vee[1]$. The double dual $V^{**}$ of a chain complex $V$ is naturally isomorphic to $V$, while $V^*\vee \cong V^\vee*$ and the functor $V \mapsto V^\vee*$ is an equivalence of categories of chain and cochain complexes. Clearly $(V^\vee)^* \cong V_{-1}$.

An ungraded vector space $V$ could be assumed to lie in degree 0, and it will be clear from the context whether this (trivial) grading is considered homological or cohomological. If dim $V = n$ we will call the determinant of $V$ the one-dimensional graded vector space $\text{Det}(V) = S^n(V[-1]) = \Lambda^n(V)[-n]$, concentrated in degree $n$. Note that $\text{Det}(V)^* [-2n] \cong \text{Det}(V^*)$. We will use negative powers of one-dimensional graded vector spaces for the corresponding positive tensor powers of their $*$-duals, so that

$$\text{Det}^{-p}(V) = ((\text{Det} V)^*)^\otimes p.$$}

For a finite collection $\{V_\alpha \mid \alpha \in I\}$ of finite-dimensional vector spaces, we have a natural identification

$$\bigotimes_{\alpha \in I} V_\alpha[-1] \cong \text{Det}(I) \otimes \bigotimes_{\alpha \in I} V_\alpha.$$}

If $S$ is a finite set, let $\text{Det}(S) := \text{Det}(k^S)$. Since there is a canonical isomorphism $(k^S)^* \cong k^S$, we have $\text{Det}(S)^* [-2|S|] \cong \text{Det}(S)$. Note also that $\text{Det}^2(S) \cong k[-2|S|]$.

For a simplex $\sigma$, the symbol $\text{Det}(\sigma)$ will denote the determinant of the set of vertices of $\sigma$. When the ground field $k = \mathbb{R}$, a choice of a nonzero element in $\text{Det}(\sigma)$ up to a positive real factor is equivalent to providing $\sigma$ with an orientation in the usual sense.

**1. Verdier duality for simplicial complexes**

In this section we formulate and prove certain results on Verdier duality for sheaves on simplicial complexes. These results, in a slightly different situation of spaces stratified into cells, were stated in [3].

**Definition 1.1.** Let $X$ be a finite simplicial complex. A sheaf of dg-vector spaces over a ground field $k$ on $X$ is called constructible, if its restriction to each face of $X$ is a constant sheaf whose stalk is a dg-vector space.

**Remark 1.2.** Ginzburg and Kapranov use the term “combinatorial sheaf.” We follow the more conventional terminology adopted in, e.g. [9].

Any simplicial complex $X$ admits an open covering $U_\sigma$ where $\sigma$ runs through the faces of $X$; namely $U_\sigma$ is the open star of $\sigma$, the union of the interiors of those faces of $X$ which contain $\sigma$. Any sheaf determines a contravariant functor from the poset $\{U_\sigma\}$ into the category of dg-vector spaces. Conversely, let $\mathcal{F}$ be a constructible sheaf on $X$. Let $x \in X$ and consider the face $\sigma$
of smallest dimension containing $x$. Then the set of sections of $\mathscr{F}$ over any sufficiently small neighborhood of $X$ will coincide with $\Gamma(X, U_\sigma)$. Therefore, $\mathscr{F}$ is completely determined by the corresponding functor.

Consider the category whose objects are the simplices of $X$ and the morphisms are inclusions of faces. We will call a coefficient system on $X$ any covariant functor from this category to the category of dg-vector spaces.

**Proposition 1.3.** There is a one-to-one correspondence between constructible sheaves and coefficient systems on a simplicial complex $X$.

*Proof.* Indeed, it suffices to note that the category of faces of $X$ is opposite to the category of open stars of $X$. □

**Remark 1.4.** The cohomology of a constructible sheaf could be computed using the Čech complex of the covering $\{U_\sigma\}$, as follows from Kashiwara-Schapira [9], Proposition 8.1.4. Cohomology in this paper will always mean hypercohomology.

**Proposition 1.5.** Let $\mathscr{F}$ be a constructible sheaf and $\{\mathscr{F}_\sigma\}$ be the corresponding coefficient system on $X$. Then the cohomology of a constructible sheaf on $X$ coincides with the cohomology of the cochain complex

\[
\bigoplus_{\tau} \mathscr{F}_\tau \otimes \text{Det}(\tau)[1]
\]

on which the differential acts as the sum of the internal differential on $\mathscr{F}$ and a map

\[
\mathscr{F}_\sigma \otimes \text{Det}(\sigma)[1] \mapsto \sum_{\tau \supset \sigma, \dim \tau = \dim \sigma + 1} \mathscr{F}_\tau \otimes \text{Det}(\tau)[1],
\]

where the last map is induced by inclusions $\sigma \hookrightarrow \tau$.

*Proof.* According to Remark 1.4, the cohomology of $\mathscr{F}$ could be computed using the Čech (bi)complex of the covering of $\{U_\sigma\}$ of $X$. A simple inspection shows that this complex is isomorphic to the complex (1.1). □

We will now discuss Verdier duality in the simplicial context. Recall that for a sheaf $\mathscr{F}$, considered as an object of the derived category of sheaves on $X$, its Verdier dual $D\mathscr{F}$ is characterized by the property $R\Gamma(U, D\mathscr{F}) = [R\Gamma_c(U, \mathscr{F})]^*$. It is easy to see that for a constructible sheaf $\mathscr{F}$, its Verdier dual complex $D\mathscr{F}$ will have constructible cohomology and therefore, by [9], Theorem 8.1.10, can be represented by a complex of constructible sheaves.

**Proposition 1.6.** Let $\mathscr{F}$ be a constructible sheaf on $X$ and $D\mathscr{F}$ is its Verdier dual. Then $D\mathscr{F}$ is represented by the constructible complex $\sigma \mapsto D\mathscr{F}_\sigma$, where $D\mathscr{F}_\sigma$ is the following cochain complex:

\[
\bigoplus_{\tau \supset \sigma} (\mathscr{F}_\tau \otimes \text{Det}(\tau)[1])^*
\]

whose differential is the dual to that in (1.1).

*Proof.* Consider the open star $\text{st}(\sigma)$ of the simplex $\sigma$. We will denote by $i : \text{st}(\sigma) \to \overline{\text{st}(\sigma)}$ the inclusion of $\text{st}(\sigma)$ into its closure. Then $i_!(\mathscr{F}|_{\text{st}(\sigma)})$ is a constructible sheaf on the simplicial complex $\overline{\text{st}(\sigma)}$. It follows that

\[
R\Gamma_c(\mathscr{F}, \text{st}(\sigma)) = R\Gamma_c i_!(\mathscr{F}|_{\text{st}(\sigma)}) = R\Gamma i_!(\mathscr{F}|_{\text{st}(\sigma)}).
\]

Note that $\overline{\text{st}(\sigma)}$ is the union of all simplices containing $\sigma$. The sheaf $i_!(\mathscr{F}|_{\text{st}(\sigma)})$ corresponds to the coefficients system on $\overline{\text{st}(\sigma)}$ so that $(i_!(\mathscr{F}|_{\text{st}(\sigma)}))_\tau = \begin{cases} \mathscr{F}_\tau & \text{if } \tau \supset \sigma, \\ 0 & \text{otherwise.} \end{cases}$ Now Corollary 1.5.

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implies that \( R\Gamma(U, F) \) is represented by the complex
\[
\bigoplus_{\tau \supset \sigma} F_\tau \otimes \text{Det}(\tau)[1],
\]
and the desired statement follows.

□

**Remark 1.7.** This result was formulated in the stratified setting in [6], Proposition 3.5.12 (b).

## 2. Equivariant Verdier Duality

In this section we generalize our theory to the case of orbi-simplicial complexes. We will not discuss orbi-simplicial complexes in full generality, restricting ourselves to the case when there exists a global group action. For the rest of the paper, the ground field \( k \) will have characteristic 0. Let \( X \) be a topological space and \( G \) be a group acting properly discontinuously on \( X \). That means that the stabilizer \( G_x \) of every point \( x \in X \) is finite and every point \( x \in X \) has a neighborhood \( U_x \) such that \( gU_x \cap U_x = \emptyset \) if \( g \notin G_x \). Let \( Y \) denote the space of orbits \( X/G \) and by \( f : X \to Y \) the projection map. We now recall some standard definitions and facts about equivariant sheaves, cf. [7] or a more modern reference [3].

**Definition 2.1.** A \( G \)-equivariant sheaf \( \mathcal{F} \) on \( X \) is a sheaf of \( k \)-vector spaces with a \( G \)-action. More precisely, for any \( g \in G \) and any open set \( U \subset X \) there is an isomorphism \( g_U : \Gamma(U, \mathcal{F}) \to \Gamma(gU, \mathcal{F}) \) which is compatible with the restriction maps in the sense that for any open subsets \( V \subset U \) in \( X \) the following diagram is commutative:

\[
\begin{array}{ccc}
\Gamma(U, \mathcal{F}) & \xrightarrow{g_U} & \Gamma(gU, \mathcal{F}) \\
\downarrow & & \downarrow \\
\Gamma(V, \mathcal{F}) & \xrightarrow{g_V} & \Gamma(gV, \mathcal{F})
\end{array}
\]

where the downward arrows are the restriction maps. In addition we require the following cocycle conditions:

- \( 1_U \) is the identity isomorphism for any open set \( U \).
- \( h g_U \circ g_U = (h \circ g)_U \) for any \( h, g \in G \) and any open set \( U \subset X \).

Note that \( \Gamma(X, \mathcal{F}) \) has a \( G \)-action. We will denote by \( \Gamma^G(X, \mathcal{F}) \) the space of \( G \)-invariants:
\[
\Gamma^G(X, \mathcal{F}) = [\Gamma(X, \mathcal{F})]^G.
\]
A morphism \( \mathcal{F}_1 \to \mathcal{F}_2 \) between two equivariant sheaves is an element in \( \Gamma^G(X, \text{Hom}(\mathcal{F}_1, \mathcal{F}_2)) \). \( G \)-equivariant sheaves on \( X \) form an abelian category. For any sheaf \( \mathcal{F} \) on \( Y \) the sheaf \( f^{-1} \mathcal{F} \) is naturally a \( G \)-equivariant sheaf on \( X \). The direct image sheaf \( f_* \mathcal{F} \) is a \( G \)-equivariant sheaf on \( Y \) where \( G \) is assumed to act trivially on \( Y \).

**Definition 2.2.** The \( G \)-equivariant direct image \( f_*^G \mathcal{F} \) is the sheaf of \( G \)-invariants of \( f_* \mathcal{F} \) so that for \( V \in Y \) we have \( \Gamma(V, f_*^G \mathcal{F}) = \Gamma(V, f_* \mathcal{F})^G \).

The functor \( f^{-1} \) embeds the category of sheaves on \( Y \) as a full subcategory into the category of \( G \)-equivariant sheaves on \( X \). Moreover, \( f_*^G \circ f^{-1} \) is isomorphic to the identity functor on the category of sheaves on \( Y \). Since the functor \( f_*^G \) is exact these statements continue to hold on the level of derived categories, cf. [3], Theorem 8.6.1.

Now assume that \( X \) is a finite-dimensional simplicial complex and that \( G \) acts simplicially, i.e. for any simplex \( \sigma \in X \) and \( g \in G \) the image \( g(\sigma) \) is another simplex of \( X \) and \( g : \sigma \to g(\sigma) \) is an affine map. Our standing assumptions on the action imply that the stabilizer of each simplex is finite. As a topological space \( Y \) is glued from \textit{orbi-simplices}, i.e. quotients of simplices by actions of finite groups. One has one \( n \)-dimensional orbi-simplex of \( Y \) for each orbit of the action of \( G \) on the set of \( n \)-simplices of \( X \).

**Definition 2.3.** A sheaf \( \mathcal{F} \) on \( Y \) is called constructible if \( f^{-1} \mathcal{F} \) is constructible on \( X \).
In other words a constructible sheaf is constant when restricted onto each orbi-simplex. Just as in the non-equivariant situation, a constructible sheaf $\mathcal{F}$ on $Y$ is equivalent to a coefficient system on $Y$, i.e. a functor $\sigma \mapsto \mathcal{F}_\sigma$ from the poset of orbi-simplices of $Y$ into $k$-vector spaces.

Then we have the following (almost verbatim) analogue of Corollary 1.5.

Proposition 2.4. Let $\mathcal{F}$ be a constructible sheaf on $Y$ and $\{\mathcal{F}_\sigma\}$ be the corresponding coefficient system on $Y$. Then the cohomology of a constructible sheaf on $X$ coincides with the cohomology of the complex

$$\bigoplus_{\tau} \mathcal{F}_\tau \otimes \text{Det}(\tau)[1].$$

Here the direct sum is over the orbi-simplices of $Y$, and the differential acts as in the non-equivariant situation.

Proof. According to the correspondence between equivariant sheaves on $X$ and non-equivariant sheaves on $Y$ we have an isomorphism $R\Gamma^G(X, f^{-1}\mathcal{F}) \cong R\Gamma(Y, \mathcal{F})$. Since the complex (2.1) is just the complex of $G$-invariants of the Čech complex of $f^{-1}\mathcal{F}$ and the latter does compute $R\Gamma(X, \mathcal{F})$ by Corollary 1.5 the statement of our proposition follows. \hfill $\square$

Similar arguments can be used to prove the following analogue of Proposition 1.6.

Proposition 2.5. Let $\mathcal{F}$ be a constructible sheaf on $Y$ and $D\mathcal{F}$ be its Verdier dual. Then $D\mathcal{F}$ is represented by the constructible complex $\sigma \mapsto D\mathcal{F}_\sigma$ where $D\mathcal{F}_\sigma$ is the following complex:

$$\bigoplus_{\tau \supset \sigma} (\mathcal{F}_\tau \otimes \text{Det}(\tau)[1])^*.$$ 

Here $\tau$ runs over the orbi-simplices of $Y$ having $\sigma$ as a face, the grading convention and the formula for the differential are the same as in the non-equivariant situation. \hfill $\square$

Remark 2.6. We need a mild generalization of the above results. Suppose that $X'$ is a $G$-subcomplex of $X$ and $Y' := X'/G \subset Y = X/G$. We require that $G$ act properly discontinuously on $X \setminus X'$, but not necessarily on the whole $X$. If $\sigma$ is a simplex in $X \setminus X'$ we will still refer to its image in $Y$ as an orbi-simplex.

Furthermore, let $\mathcal{F}$ be a sheaf on $Y$ for which $i^{-1}\mathcal{F}$ is a constructible sheaf on $X$ and $\mathcal{F}|_{Y'} = 0$.

Then Propositions 2.4 and 2.5 continue to hold. In other words formula (2.1) still computes the cohomology of $\mathcal{F}$ and (2.2) represents the sheaf $D\mathcal{F}$. To see that one only has to note that

- the functor $i^{-1}$ restricted to the category of sheaves on $Y \setminus Y'$ still embeds the derived category of sheaves on $Y \setminus Y'$ into the derived category of sheaves on $X \setminus X'$ and
- for the inclusion $j : X \setminus X' \hookrightarrow X$ the functor $j_!$, the direct image with compact support is exact.

3. Graph complexes and spaces of metric graphs

3.1. Graph complexes. A graph is specified by a set of vertices, a set of half-edges, and (rather obvious) combinatorial relations between them, cf. for example, [5] for precise definitions. One may also think of a graph as an isomorphism class of a 1-dimensional CW complex. We will only consider connected, finite graphs whose vertices have valence three or higher, i.e., for each vertex the number of incident half-edges must be at least three. The sets of vertices and edges of a graph $\Gamma$ will be denoted by $V(\Gamma)$ and $E(\Gamma)$, respectively. The set of half-edges incident to a vertex $v \in V(\Gamma)$ will be denoted by $H(v)$.

Let $O$ be a cyclic operad in the category of chain complexes of $k$-vector spaces. For simplicity, we will assume that $O(1) = k$ and $O(n)$ is a finite-dimensional dg-vector space for each $n \geq 2$, as this is the case for the standard examples of $O = \text{Comm}$, $\text{Ass}$, and $\text{Lie}$. The more general case of an admissible operad, see [6][3.1.5], can also be treated by taking tensor products over the associative algebra $K = O(1)$, rather than the ground field $k$. 

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If $S$ is a set of $n + 1$ elements, $n > 0$, one can define $\mathcal{O}(\{S\})$ by using the coinvariants trick:

$$\mathcal{O}(\{S\}) := (\mathcal{O}(n) \times \text{Iso}(S, [n]))S_{n+1},$$

where $\text{Iso}(S, [n])$ is the set of bijections between $S$ and $[n] := \{0, 1, \ldots, n\}$ and the symmetric group $S_{n+1}$ acts diagonally. Recall the notion of an $\mathcal{O}$-graph complex $\mathbb{H}$.

**Definition 3.1.** An $\mathcal{O}$-decorated graph or simply an $\mathcal{O}$-graph is a graph $\Gamma$ together with a decoration which associates to any vertex $v$ of $\Gamma$ an element in $\mathcal{O}((H(v)))$.

The space of $\mathcal{O}$-decorations on $\Gamma$ is the chain complex

$$\Gamma^\mathcal{O} = \bigotimes_{v \in V(\Gamma)} \mathcal{O}((H(v))).$$

The orientation space of a graph $\Gamma$ is the one-dimensional graded vector space $\text{Or}(\Gamma) := \text{Det}(E(\Gamma)) \otimes \text{Det}^{-1} H_1(\Gamma)[\chi]$, concentrated in degree $e(\Gamma) - 1$, where $e(\Gamma) = |E(\Gamma)|$ and $\chi = \chi(\Gamma)$ is the Euler characteristic of the graph $\Gamma$ as a CW complex. A twisted orientation space of a graph $\Gamma$ is the vector space $\text{Det}(E(\Gamma))[1]$. A (twisted) orientation on a graph $\Gamma$ is a choice of a nonzero element or in $\text{Or}(\Gamma) (\text{Det}(E(\Gamma))[1])$, respectively.

**Remark 3.2.** When $k = \mathbb{R}$, an orientation on a graph (up to a positive real factor) is equivalent to an ordering of its vertices and directing its edges (up to even permutation), cf. [5, 15, 1].

The following cyclic operads are of particular importance:

1. The commutative operad $\text{Comm}(n) = k$ for $n > 0$;
2. The associative operad $\text{Ass}(n) = k[S_n]$ for $n > 0$;
3. The Lie operad, whose $n$th space $\mathcal{L}$ie$(n)$ is the $k$-vector space spanned by all Lie monomials in $n$ variables containing each variable exactly once.

The corresponding $\mathcal{O}$-graphs are called commutative, ribbon and Lie graphs, respectively.

**Definition 3.3.** The $\mathcal{O}$-graph complex is the following complex of $k$-vector spaces:

$$C^\mathcal{O} = \bigoplus_\Gamma \Gamma^\mathcal{O} \otimes \text{Or}(\Gamma),$$

where the summation runs over the isomorphism classes $\Gamma$ of graphs. The grading comes from the internal grading on $\mathcal{O}$ and the grading on $\text{Or}(\Gamma)$, which sits in degree $e(\Gamma) - 1$, so that, provided that $\mathcal{O}$ is non-negatively graded, the graph complex in general would end in degree $-\chi(\Gamma)$, corresponding to graphs with one vertex. The differential is the sum of the internal differential coming from the operad $\mathcal{O}$ and the graph differential $d : C_n^\mathcal{O} \rightarrow C_{n-1}^\mathcal{O}$ which acts as follows:

$$(3.1) \quad d(\Gamma \otimes \text{or}) = \sum_e \Gamma_e \otimes \text{or}_e,$$

where $\Gamma$ is an $\mathcal{O}$-graph and $\text{or} \in \text{Or}(\Gamma) \setminus \{0\}$ is an orientation on $\Gamma$. Here $\Gamma_e$ is the graph obtained from $\Gamma$ by contracting an edge $e$ and the summation is taken over all edges of $\Gamma$ which are not loops.

The orientation $\text{or}_e$ of $\Gamma_e$ is induced from the orientation of $\Gamma$ in such a way that $\text{or} = e \wedge \text{or}_e$. The $\mathcal{O}$-decoration on $\Gamma$ is defined as follows. Suppose that the two endpoints $v_1$ and $v_2$ of the edge $e$ have valences $n_1$ and $n_2$, respectively. Then the vertex obtained by coalescing $v_1$ and $v_2$ is decorated by the element $v_1(\mathcal{O}) \circ v_2(\mathcal{O}) \in \mathcal{O}(n_1 + n_2 - 3)$, where $\mathcal{O}(n_1 - 1) \circ \mathcal{O}(n_2 - 1) \rightarrow \mathcal{O}(n_1 + n_2 - 3)$ is a structure map of the operad $\mathcal{O}$.

Similarly, the twisted graph complex $\widetilde{C}^\mathcal{O}$ is formed by the isomorphism classes of $\mathcal{O}$-graphs with twisted orientation; the grading and the differential are defined like in the untwisted case, so that, for example, terms corresponding to graphs $\Gamma$ with a single vertex decorated by an element of $\mathcal{O}$ of degree zero would sit in degree $-\chi(\Gamma)$.

The homology of the complexes $C^\mathcal{O}$ and $\widetilde{C}^\mathcal{O}$ are denoted by $H^\mathcal{O}$ and $\widetilde{H}^\mathcal{O}$ respectively and called $\mathcal{O}$-graph homology. The cohomology of the $k$-dual cochain complexes $C^\mathcal{O}_* = [C^\mathcal{O}]^\vee$ are
\[\bigoplus \Gamma (\Gamma^{\bullet})^{\vee} \otimes \text{Det}(E(\Gamma)) \otimes \text{Det}^{-1}(H_1(\Gamma)^*)[\chi] \text{ and } \widetilde{C}_n^{\bullet} = \left[\widetilde{C}_n^{\Gamma^{\bullet}}\right]^{\vee} = \bigoplus \Gamma (\Gamma^{\bullet})^{\vee} \otimes \text{Det}(E(\Gamma))[1] \text{ are called } \Theta\text{-graph cohomology and twisted } \Theta\text{-graph cohomology, respectively.}

**Remark 3.4.** The graph complexes associated to two quasi-isomorphic operads are themselves quasi-isomorphic as a standard spectral sequence argument makes clear.

For an integer \(n > 1\), we will consider graphs \(\Gamma\) with \(H_1(\Gamma, \mathbb{Z})\) being a free abelian group of rank \(n\). These graphs form a subcomplex \(C_{\Gamma^{\bullet}}(n)\); clearly \(C_{\Gamma^{\bullet}} = \bigoplus_n C_{\Gamma^{\bullet}}(n)\). In what follows the number \(n\) will be understood but not explicitly mentioned. Note that \(C_{\Gamma^{\bullet}}\) are complexes of finite-dimensional vector spaces and have finite lengths.

Furthermore in the case \(\Theta = \text{ss}\) an \(\Theta\)-graph – a ribbon graph – has an additional invariant, the number \(b > 0\) of boundary components, cf. for example, the survey [8]. It is convenient to introduce the *genus* \(g \geq 0\) of a ribbon graph by the formula \(g = 1/2(n + 1 - b)\); then graphs with fixed \(g\) and \(b\) form a subcomplex \(C_{\Gamma^{\text{ss}}}(g, b)\) inside \(C_{\Gamma^{\text{ss}}} \) and \(C_{\Gamma^{\text{ss}}}(n) \cong \bigoplus_{g \geq 0, b \geq 1} \bigoplus_{2g - b = 1 - n < 0} C_{\Gamma^{\text{ss}}}(g, b)\).

### 3.2. Metric graphs

We now introduce the moduli space of metric graphs, cf. [2]. A **metric graph** is a graph \(\Gamma\) together with a map \(l : E(\Gamma) \to \mathbb{R}_+\); the positive number \(l(e)\) is called the **length** of the edge \(e\).

A **marking** on a metric graph is a homotopy equivalence \(\sqcup^n S^1 \to \Gamma\) from an \(n\)-fold wedge of circles to \(\Gamma\). The set of markings on \(\Gamma\) is acted on by the group of isometries of \(\Gamma\) and marked graphs belonging to the same orbit are called equivalent.

We associate to any metric graph \(\Gamma\) with \(\sum_{e \in E(\Gamma)} l(e) = 1\) a point in the simplex \(\Delta^{e(\Gamma) - 1}\) whose barycentric coordinates are given by the lengths of the edges of \(\Gamma\). Collapsing edges corresponds to passing to the faces of the corresponding simplex and after a suitable identification we obtain a topological space \(X_n\), the so-called **Outer Space**. A point in \(X_n\) is a marked graph and the group of outer automorphisms of the free group on \(n\) generators, denoted by \(\text{Out}(F_n)\), acts on \(X_n\) by changing markings. The space \(X_n\) is not a simplicial complex owing to the existence of loops but adding formally the missing faces (ideal simplices), one obtains a simplicial complex \(\overline{X}_n\), on which the group \(\text{Out}(F_n)\) continues to act, together with an inclusion \(i : Y_n \to \overline{X}_n\).

The space \(X_n\) is contractible and \(\text{Out}(F_n)\) acts on it properly discontinuously, so \(Y_n := X_n/\text{Out}(F_n)\) is a rationally classifying space of \(\text{Out}(F_n)\). Each isomorphism class of graphs contributes an orbi-simplex in \(\overline{Y}_n\). The remaining simplices will be referred to as **ideal orbi-simplices**.

Note that \(\overline{Y}_n := Y_n/\text{Out}(F_n)\) is not a simplicial orbi-complex since \(\text{Out}(F_n)\) does not act on the ideal simplices properly discontinuously. However this will not affect our results since the sheaves we are interested in will vanish on those bad simplices, cf. Remark 2.6. We will now introduce certain constructible sheaves on \(\mathcal{H}\) on \(\overline{Y}_n\).

**Definition 3.5.**

1. For an orbi-simplex \(\sigma\) corresponding to a graph \(\Gamma\), we set \(\overline{\mathcal{H}}_{\sigma} = \text{Det}^{-1}(H_1(\Gamma))[\sim n]\). If \(\sigma\) is an ideal orbi-simplex we set \(\overline{\mathcal{H}}_{\sigma} = 0\). The complex \(i^{-1}\overline{\mathcal{H}}\) on \(Y_n\) will be denoted by \(\mathcal{H}\).

2. Associated to a cyclic operad \(\Theta\) is a constructible complex \(\overline{\mathcal{F}}_{\Theta}\) on \(\overline{Y}_n\) defined as follows. For an oriented simplex \(\sigma \in \overline{Y}_n\) corresponding to a graph \(\Gamma\), we set \(\mathcal{F}_{\sigma}^{\Theta}\) to be the cochain complex \(k\)-dual to the chain complex \(\Gamma^{\Theta}\) of \(\Theta\)-decorations on \(\Gamma\):

\[\mathcal{F}_{\sigma}^{\Theta} := (\Gamma^{\Theta})^{\vee} .\]

If \(\sigma\) is an ideal simplex we set \(\mathcal{F}_{\sigma}^{\Theta} = 0\). If \(\sigma \subset \tau\) is a face of \(\tau\), the corresponding morphism \(\mathcal{F}_{\sigma}^{\Theta} \to \mathcal{F}_{\tau}^{\Theta}\) is defined as the dual to the one obtained by the operad composition of decorations along the edges in the graph \(\Gamma_{\tau}\) being contracted to obtain \(\Gamma_{\sigma}\), where \(\Gamma_{\sigma}\) and \(\Gamma_{\tau}\) are the graphs corresponding to the simplices \(\sigma\) and \(\tau\), respectively. The complex of sheaves \(i^{-1}\overline{\mathcal{F}}_{\Theta}\) on \(Y_n\) will be denoted by \(\mathcal{F}_{\Theta}\).
Remark 3.6.

- Clearly \(i_! \mathcal{F}^\theta \cong \mathcal{F}^\theta \); similarly \(i_* \mathcal{H} \cong \mathcal{H} \). Furthermore \(\mathcal{H} \) is a locally free sheaf on \(Y_n\).
- For two quasi-isomorphic operads \(\mathcal{O} \) and \(\mathcal{O}'\), the complexes \(\mathcal{F}_\mathcal{O} \) and \(\mathcal{F}_{\mathcal{O}'} \) are quasi-isomorphic.

Theorem 3.7. There are canonical isomorphisms of graded \(k\)-vector spaces:

1. \(\check{H}^*_{\mathcal{O}}(n) \cong H^*(Y_n, \mathcal{F}^\theta) \) for the twisted \(\mathcal{O}\)-graph cohomology;
2. \(H^*_{\mathcal{O}}(n) \cong H^*(Y_n, \mathcal{F}^\theta \otimes \mathcal{H}) \) for the \(\mathcal{O}\)-graph cohomology.

Proof. We shall only prove part (1), the argument for (2) being virtually identical, as the standard orientation on a graph \(\Gamma\) differs from the twisted one by \(\text{Det}^{-1}(H_1(\Gamma))[-n]\). It suffices to prove the isomorphism \(\check{H}^*_{\mathcal{O}}(n) \cong H^*(Y_n, \mathcal{F}^\theta)\). Using Proposition 2.4 (or, rather, its modification as in Remark 2.6), we see that the complex computing \(H^*(Y_n, \mathcal{F}^\theta)\) coincides with the complex \(\check{C}^*_{\mathcal{O}}(n)\), so the statement of the theorem follows. \(\Box\)

Given a cyclic operad \(\mathcal{O}\) of chain complexes, let us recall the construction of its \(dg\)-dual operad \(D\mathcal{O}\) from \([3,4,5]\). The \(n\)th component \(D\mathcal{O}(n)\) of the \(dg\)-dual cyclic operad \(D\mathcal{O}\) for each \(n > 0\) is defined as the linear dual of the space of oriented, unrooted trees decorated by a certain degree shift \(s\mathcal{O}[-1]\) with leaves labeled by \(0, 1, \ldots, n\). More precisely,

\[
(3.2) \quad D\mathcal{O}(n) := \bigoplus_{\text{unrooted } n+1\text{-trees } T} (T^{s\mathcal{O}[-1]})^*,
\]

where the summation runs over the isomorphism classes \(T\) of (unrooted) trees with vertices of valence at least three and \(n + 1\) leaves thought of as half-edges with free ends and labeled by numbers \(0, 1, \ldots, n\). Here \(s\mathcal{O}\) is the cyclic-operad suspension, \([4]\):

\[
(s\mathcal{O}(n)) := \text{Det}^{-1}(k^n[-2]) \otimes \mathcal{O}(n),
\]

which results in

\[
(s\mathcal{O}((S))) = \text{Det}^{-1}(k^2[-2]) \otimes \mathcal{O}((S))
\]

for a finite set \(S\). Also, \(T^{s\mathcal{O}[-1]}\) is the space of \(s\mathcal{O}[-1]\)-decorations on \(T\). Thus, if \(\mathcal{O}\) happens to be concentrated in degree zero, \(D\mathcal{O}(n)\) will be a chain complex spanning degrees \(n - 2\) through \(0\). The differential on \(D\mathcal{O}(n)\) is the sum of the internal differential coming from the complex of \(\mathcal{O}\)-decorations on a tree and the differential linear dual to the differential \((3.1)\) restricted from graphs to trees.

Remark 3.8. If \(S\) is a set of \(n + 1\) elements, one can make precise sense out of \(D\mathcal{O}(S)\) by considering trees whose leaves are labeled by the elements of \(S\).

Theorem 3.9. There is a canonical isomorphism in the derived category of sheaves on \(Y_n\):

\[
D\mathcal{F}^\theta \cong F^{D\mathcal{O}} \otimes \mathcal{H}[4 - 3n],
\]

where \(D\mathcal{F}\) is the Verdier dual sheaf and \(D\mathcal{O}\) the \(dg\)-dual operad.

Proof. It suffices to provide a canonical isomorphism

\[
(3.3) \quad D\mathcal{F}^\theta \cong F^{D\mathcal{O}} \otimes \mathcal{H}[4 - 3n].
\]

To prove it, we will evaluate \((3.3)\) on an orbi-simplex \(\sigma\) and establish an isomorphism, natural with respect to isomorphisms of the corresponding graphs \(\Gamma_{\sigma}\).

By definition,

\[
\mathcal{F}^{D\mathcal{O}}_{\Gamma_{\sigma}} = (\Gamma_{\sigma}^{D\mathcal{O}})^{\vee} = \bigotimes_{v \in V(\Gamma_{\sigma})} D\mathcal{O}((H(v)))^{\vee} = \bigoplus_{v \in V(\Gamma_{\sigma})} \bigoplus_{\text{unrooted } H(v)\text{-trees } T_v} (T^{s\mathcal{O}[-1]}_v)^{\vee}.
\]

Note that a graph \(\Gamma_{\sigma}\) with each vertex \(v\) decorated by a tree \(T_v\) whose leaves are labeled by the set \(H(v)\) of half-edges emanating from \(v\) is literally the same as a graph \(\Gamma_{\tau}\) with a collection of subtrees, such that contracting each of these subtrees returns the graph \(\Gamma_{\sigma}\). We will call such
graph $\Gamma_\tau$ a vertex expansion of $\Gamma_\sigma$. Moreover, $s\mathcal{O}[-1] \text{-decorations on the trees } T_v \text{ will obviously result in } s\mathcal{O}[-1] \text{-decorations on the graphs } \Gamma_\tau$. Thus, we see that

$$B_\sigma^{D\mathcal{O}} = \bigoplus_{\text{vertex expansions } \Gamma_\tau \text{ of } \Gamma_\sigma} \left(\Gamma_\tau^{s\mathcal{O}[-1]}\right)^{^\text{sv}}.$$ 

Now let us identify $\left(\Gamma_\sigma^{s\mathcal{O}[-1]}\right)^{^\text{sv}}$:

$$\left(\Gamma_\sigma^{s\mathcal{O}[-1]}\right)^{^\text{sv}} = \bigotimes_{v \in V(\Gamma)} \text{Det}(H(v)) \otimes \mathcal{O}((H(v))^{\text{sv}}[3] \cong \left(\Gamma_\sigma^{\mathcal{O}}\right)^{^\text{sv}} \otimes \bigotimes_{v \in V(\Gamma)} \text{Det}(H(v))[3].$$

Leaving the factor $\left(\Gamma_\sigma^{\mathcal{O}}\right)^{^\text{sv}}$ out for the time being, let us deal with orientations. We have

$$\bigotimes_{v \in V(\Gamma)} \text{Det}(H(v))[3] \cong \text{Det}^{-3}(V(\Gamma)) \otimes \bigotimes_{v \in V(\Gamma)} \text{Det}(H(v))$$

$$\cong \text{Det}^{-1}(V(\Gamma))[2v(\Gamma)] \otimes \bigotimes_{v \in V(\Gamma)} \text{Det}(H(v)),$$

where $v(\Gamma) = |V(\Gamma)|$. Note that the set $\prod_{v \in V(\Gamma)} H(v)$ is naturally isomorphic to the set $\prod_{e \in E(\Gamma)} H(e)$, where $H(e)$ is the set of (two) half-edges making up an edge $e$, as both sets count the set of half-edges $H(\Gamma)$ of the graph, the former by grouping the set of half-edges by vertices, the latter by edges. By passing to determinants, we obtain

$$\bigotimes_{v \in V(\Gamma)} \text{Det}(H(v))[3] \cong \text{Det}^{-1}(V(\Gamma))[2v(\Gamma)] \otimes \bigotimes_{e \in E(\Gamma)} \text{Det}(H(e)).$$

Note that the exact sequence

$$0 \to H_1(\Gamma) \to C_1(\Gamma) \to C_0(\Gamma) \to H_0(\Gamma) \to 0$$

yields a canonical isomorphism

$$\text{Det}(H_0(\Gamma)) \otimes \text{Det}^{-1} H_1(\Gamma) \cong C_0(\Gamma) \otimes \text{Det}^{-1} C_1(\Gamma).$$

Further, we have the following natural isomorphisms:

$$\text{Det} C_0(\Gamma) \cong \text{Det} V(\Gamma),$$

$$\text{Det} C_1(\Gamma) \cong \bigotimes_{e \in E(\Gamma)} \text{Det} H(e)[1]$$

$$\cong \text{Det}^{-1} E(\Gamma) \otimes \bigotimes_{e \in E(\Gamma)} \text{Det} H(e),$$

$$\text{Det}(H_0(\Gamma)) \cong k[-1].$$

We conclude that the last expression in (3.4) is isomorphic to

$$\text{Det}(E(\Gamma)) \otimes \text{Det}(H_1(\Gamma)) \otimes \text{Det}^{-1}(H_0(\Gamma))[2v(\Gamma)]$$

$$\cong \text{Det}^{-1}(E(\Gamma)) \otimes \text{Det}(H_1(\Gamma)) \otimes \text{Det}^{-1}(H_0(\Gamma))[2(v(\Gamma) - e(\Gamma))]$$

$$\cong \text{Or}^{-1}(\Gamma)[4 - 3n],$$

which implies

$$B_\sigma^{D\mathcal{O}} \cong \bigoplus_{\text{vertex expansions } \Gamma_\tau \text{ of } \Gamma_\sigma} \left(\Gamma_\tau^{\mathcal{O}}\right)^{^\text{sv}} \otimes \text{Or}^{-1}(\Gamma_\tau)[4 - 3n].$$

The differential on the complex $B_\sigma^{D\mathcal{O}}$ is the sum of the internal differential on $\mathcal{O}$ and the summation over all contractions of a given graph $\Gamma_\tau$ along the edges arising in the vertex expansions of $\Gamma_\sigma$ of the corresponding operad compositions. This is similar to the differential in the graph complex $C_\tau^{\mathcal{O}}$, with the same effect on the orientation factor, except that the resulting grading is now cohomological.
Now let us turn to the left-hand side of (3.3). According to Proposition 2.5 and Remark 2.6, the object \( D\mathcal{F}^\sigma \) is represented by the complex

\[
D\mathcal{F}^\sigma \cong \bigoplus_{\tau \subset \sigma} (\mathcal{F}^\sigma \otimes \text{Det}(\tau)[1])^* = \bigoplus_{\text{vertex expansions } \Gamma^\sigma \text{ of } \Gamma_{\sigma}} \left( \Gamma^\sigma \right)^{\star \star} \otimes \text{Det}^{-1}(E(\Gamma))[1],
\]

with the same differential as for the twisted graph complex \( \tilde{C}^\sigma \). This immediately implies (3.3).

Remark 3.10. Note that the identification of \( \mathcal{F}^{D\mathcal{O}} \) in the beginning of the proof of Theorem 3.9 shows that the complex computing the cohomology of \( \mathcal{F}^{D\mathcal{O}} \) is a graph complex decorated by a decorated tree complex, all complexes being cochain complexes. That complex can obviously be identified with a decorated (cochain) graph complex, and the rest of the proof of Theorem 3.9 expresses the resulting decoration through the \( \mathcal{O} \)-decoration.

The following corollary describes the dualizing sheaf on \( Y_n \); note that it is concentrated in a single degree as a direct consequence of the fact that the operad \( \text{Comm} \) is Koszul.

Corollary 3.11. The dualizing sheaf on \( Y_n \) is isomorphic to \( \mathcal{F}^{L\mathcal{E}} \otimes \mathcal{H}[4 - 3n] \).

Let \( \tilde{k} \) denote the one-dimensional \( \text{Out}(F_n) \)-module corresponding to the local system \( \mathcal{H} \), concentrated in degree zero; an element in \( \text{Out}(F_n) \) acts on \( \tilde{k} \) as multiplication by 1 or \(-1\), equal to the determinant of the linear map induced on \( H_1(\Gamma) \). Then we have the following result which was formulated by Kontsevich in [10] and given a different proof in [1].

Corollary 3.12. There are the following isomorphisms of graded \( k \)-vector spaces:

1. \( H_\bullet(\text{Out}(F_n), k) \cong H_{\tilde{k}^{3n-4}}^{L\mathcal{E}}(n) \);
2. \( H_\bullet(\text{Out}(F_n), \tilde{k}) \cong \tilde{H}_{\tilde{k}^{3n-4}}^{L\mathcal{E}}(n) \).

Proof. As usual, we limit ourselves with proving the first statement. We have

\[
H_\bullet(\text{Out}(F_n), k) \cong H_\bullet(Y_n, k)
\cong [H^\bullet(Y_n, k)]^{\vee}
\cong [H^\bullet(Y_n, \mathcal{F}^{comm})]^{\vee}
\cong H^\bullet(Y_n, D\mathcal{F}^{comm})^{\star \vee}
\cong H^\bullet(Y_n, \mathcal{F}^{D\mathcal{O}} \otimes \mathcal{H}[4 - 3n])^{\star \vee}
\cong H^{\bullet + 4 - 3n}(Y_n, \mathcal{F}^{comm} \otimes \mathcal{H})^{\star \vee}
\cong H^{3n - 4}_{\mathcal{F}^{L\mathcal{E}}}(Y_n, \mathcal{F}^{L\mathcal{E}} \otimes \mathcal{H})
\cong H_{\mathcal{F}^{L\mathcal{E}}}^{3n - 4}(\mathcal{F}^{L\mathcal{E}} \otimes \mathcal{H}),
\]

as required. Note that we used the fact that \( D = D^{-1} \) where \( D \) is the functor of taking the Verdier dual. \( \square \)

Remark 3.13. Compare this to a more straightforward computation to get a relation between the cohomology of \( Y_n \) with compact supports and commutative graph cohomology:

\[
H^\bullet_c(Y_n, k) = H^\bullet_c(Y_n, \mathcal{F}^{comm}) = \tilde{H}^\bullet_{comm}(n).
\]

4. Ribbon Graphs

The theory developed in the previous section has an analogue for non-\( \Sigma \) operads and ribbon graph complexes. Recall that a ribbon graph is an \( \mathcal{O} \)-decorated graph; this is equivalent to having a cyclic ordering on the set of half-edges around each vertex. Given a ribbon graph \( \Gamma \), there is a canonical way of producing a compact, oriented surface with boundary \( S(\Gamma) \) of which the graph \( \Gamma \) is a deformation retract. In this way one attaches to a ribbon graph two invariants:
the genus $g \geq 0$ and the number $n \geq 1$ of boundary components of the corresponding surface, $2 - 2g - n < 0$. An isomorphism between two ribbon graphs is an isomorphism preserving the cyclic ordering around each vertex. We will not specify whether the boundary components should be fixed (not necessarily point-wise) under an isomorphism or allowed to be permuted; both versions admit completely parallel treatments.

The mapping class group $\Gamma_{g,n}$ is the group of isotopy classes of orientation-preserving diffeomorphisms of an oriented surface of genus $g$ with $n$ boundary components. Again, we shall be ambiguous whether or not $\Gamma_{g,n}$ permutes the boundary components of a surface.

Now let $\mathcal{O}$ be a cyclic (chain) $k$-operad with $\mathcal{O}(1) = k$ without the action of the symmetric group, a so-called non-$\Sigma$ operad. We introduce the notions of a ribbon $\mathcal{O}$-graph complex $C^{\text{Rib}}_{\mathcal{O}}$, its (cochain) dual $C^{\text{c}
olimits\text{Rib}}_{\mathcal{O}}$, as well as the twisted versions $\tilde{C}^{\text{Rib}}_{\mathcal{O}}$ and $\tilde{C}^{\text{c}
olimits\text{Rib}}_{\mathcal{O}}$ in precisely the same way as in the previous section. For two quasi-isomorphic operads, the corresponding ribbon graph complexes will be quasi-isomorphic. The subcomplex in $C^{\text{Rib}}_{\mathcal{O}}$ consisting of ribbon graphs with fixed $g$ and $n$ will be denoted by $C^{\text{Rib}}_{\mathcal{O}}(g,n)$ ($C^{\text{c}
olimits\text{Rib}}_{\mathcal{O}}(g,n)$ for the cohomological version). It is easy to see that

$$C^{\text{Rib}}_{\mathcal{O}} \cong \bigoplus_{g \geq 0, n \geq 1 \atop 2 - 2g - n < 0} C^{\text{Rib}}_{\mathcal{O}}(g,n).$$

The most important example of a ribbon $\mathcal{O}$-graph complex corresponds to the associative non-$\Sigma$ operad $\mathcal{T}(m) = k$ in all degrees $m \geq 1$. In this case we have an isomorphism $C^{\text{Rib}}_{\mathcal{T}}(g,n) \cong C^{\ss\text{Rib}}_{\mathcal{T}}(g,n)$ and similarly for the twisted versions, cf. the discussion at the end of Section 3.1.

We will now introduce the space of metric ribbon graphs $\mathbb{M}_{g,n}$. A point $\Gamma$ in $\mathbb{M}_{g,n}$ is an isomorphism class of ribbon graphs of genus $g$ with $n$ boundary components such that each edge $e \in E(\Gamma)$ is supplied with length $l(e) > 0$; we require that $\sum_{e \in E(\Gamma)} l(e) = 1$. The space $\mathbb{M}_{g,n}$ naturally compactifies to a simplicial orbi-complex $\mathbb{M}_{g,n}$. Namely, one introduces a contractible space of marked ribbon graphs $\mathcal{A}_{g,n}$, also known as the arc complex, on which $\Gamma_{g,n}$ acts properly discontinuously. The space $\mathcal{A}_{g,n}$ admits a simplicial closure $\overline{\mathcal{A}}_{g,n}$ (also known as the arc complex) and we denote by $\overline{\mathbb{M}}_{g,n}$ the quotient of $\overline{\mathcal{A}}_{g,n}$ by $\Gamma_{g,n}$.

**Remark 4.1.** The space $\mathbb{M}_{g,n}$ is known to be homeomorphic to the Cartesian product of the open standard simplex $\Delta^{n-1}$ and the moduli space of Riemann surfaces of genus $g$ with $n$ labeled punctures or its quotient by the diagonal action of the symmetric group $S_n$, depending on whether we allow graph isomorphisms permuting the boundary components. A version of the simplicial compactification of $\mathbb{M}_{g,n}$ was constructed by Kontsevich in [11] in connection with his proof of the Witten conjecture. The reader is referred to the papers by Looijenga [12], Zvonkine [16], and Mondello [13] for details and comparison of $\overline{\mathbb{M}}_{g,n}$ to the Deligne-Mumford compactification.

Definitions of the sheaves $\mathcal{F}^\mathcal{O}$, $\mathcal{F}^\mathcal{O}$, and $\mathcal{M}$ on $\mathbb{M}_{g,n}$ and $\overline{\mathbb{M}}_{g,n}$ transfer verbatim from the corresponding definitions in the previous section. We can now formulate analogues of the main results from Section 3. We are now proved in precisely the same way as Theorems 3.7 and 3.9.

**Theorem 4.2.** There are canonical isomorphisms of $k$-vector spaces:

1. $\tilde{H}^\bullet_{\text{Rib}}(g,n) \cong H^\bullet_c(\mathbb{M}_{g,n}, \mathcal{F}^\mathcal{O})$;
2. $\tilde{H}^\bullet_{\text{Rib}}(g,n) \cong H^\bullet_c(\mathbb{M}_{g,n}, \mathcal{F}^{\mathcal{O}} \otimes \mathcal{M})$.

**Corollary 4.3.**

$$H^\cdot_c(\mathcal{Y}_r, \mathcal{A}^{\ss\text{ss}}) \cong \bigoplus_{g \geq 0, n \geq 1 \atop 2 - 2g - n = 1 - r} H^\cdot_c(\mathbb{M}_{g,n}, k) \quad \text{for } r > 1.$$
Theorem 4.4. There is a canonical isomorphism in the derived category of sheaves on \(\mathcal{M}_{g,n}\):
\[
D\mathcal{F}^\Theta \cong \mathcal{F}^{D\Theta} \otimes \mathcal{H}[7 \cdot 6g - 3n].
\]
Here \(D\mathcal{F}\) is the dg-dual non-\(\Sigma\) operad to \(\mathcal{F}\), defined for non-\(\Sigma\) operads the same way as in (3.2), except that the trees must be planar.

The analogue of Corollary 3.12 reads as follows.

Corollary 4.5. There are the following isomorphisms of graded \(k\)-vector spaces:
\[
\begin{align*}
(1) \quad H^\bullet(\mathcal{M}_{g,n}, k) & \cong H_{\text{Rib}^F}^{6g+3n-7-\bullet}(g,n) \cong H_{\Gamma_{\text{Ass}}}^{6g+3n-7-\bullet}(g,n); \\
(2) \quad H^\bullet(\mathcal{M}_{g,n}, \tilde{k}) & \cong H_{\text{Rib}^F}^{6g+3n-7-\bullet}(g,n) \cong H_{\Gamma_{\text{Ass}}}^{6g+3n-7-\bullet}(g,n).
\end{align*}
\]

The following corollary describes the dualizing sheaf on \(\mathcal{M}_{g,n}\).

Corollary 4.6. The dualizing sheaf on \(\mathcal{M}_{g,n}\) is isomorphic to \(H[7 \cdot 6g - 3n]\).

Remark 4.7. Note that the dualizing sheaf is locally constant in agreement with the well-known fact that \(\mathcal{M}_{g,n}\) is an orbifold. Therefore, Verdier duality on \(\mathcal{M}_{g,n}\) turns into Poincaré-Lefschetz duality:
\[
H^\bullet(\mathcal{M}_{g,n}, k) = H_{c}^{6g+3n-7-\bullet}(\mathcal{M}_{g,n}, \tilde{k}).
\]
When \(\mathcal{M}_{g,n}\) stands for the moduli space of ribbon graphs with labeled boundary components, it will be an orientable orbifold, in which case \(\tilde{k} \cong k\), but taking its quotient by the symmetric group permuting the boundary components to get the other version of \(\mathcal{M}_{g,n}\) will destroy orientability, and the dualizing sheaf will no longer be constant.

References

[1] J. Conant, K. Vogtmann. On a theorem of Kontsevich. Algebr. Geom. Topol. 3 (2003), 1167-1224.
[2] M. Culler, K. Vogtmann. Moduli of graphs and automorphisms of free groups. Invent. Math. 84 (1986), no. 1, 91-119.
[3] J. Bernstein, V. Lunts. Equivariant sheaves and functors. Lecture Notes in Mathematics, 1578. Springer-Verlag, Berlin, 1994.
[4] E. Getzler, M. M. Kapranov. Cyclic operads and cyclic homology. Geometry, topology, & physics, 167–201, Conf. Proc. Lecture Notes Geom. Topology, IV, Int. Press, Cambridge, MA, 1995.
[5] E. Getzler, M. M. Kapranov. Modular operads. Compositio Math. 110 (1998), no. 1, 65–126.
[6] V. Ginzburg, M. Kapranov. Koszul duality for operads. Duke Math. J. 76 (1994), no. 1, 203–272.
[7] A. Grothendieck. Sur quelques points d’algèbre homologique. Tôhoku Math. J. (2) 9 (1957), 119-221.
[8] R. Hain, E. Loosjenga. Mapping class groups and moduli spaces of curves. Algebraic geometry: Santa Cruz 1995, 97–142, Proc. Sympos. Pure Math., 62, Part 2, Amer. Math. Soc., Providence, RI, 1997.
[9] M. Kashiwara, P. Schapira. Sheaves on manifolds. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 292, Springer-Verlag, Berlin, 1990.
[10] M. Kontsevich. Formal (non)-commutative symplectic geometry, in ‘The Gelfand Mathematics seminars, 1990-1992, Birhäuser, Boston, 1993, 173–187.
[11] M. Kontsevich. Intersection theory on the moduli space of curves and the matrix Airy function. Comm. Math. Phys. 147 (1992), no. 1, 1–23.
[12] E. Looijenga. Cellular decompositions of compactified moduli spaces of pointed curves. The moduli space of curves (Texel Island, 1994), 369–400, Progr. Math., 129, Birhäuser Boston, Boston, MA, 1995.
[13] G. Mondello. Combinatorial classes on moduli space of curves are tautological, 2004, Issue 44, Pages 2329-2390.
[14] R. C. Penner. The moduli space of a punctured surface and perturbative series. Bull. Amer. Math. Soc. (N.S.) 15 (1986), no. 1, 73–77.
[15] D. Thurston. Integral Expressions for the Vassiliev Knot Invariants. arXiv:math.QA/9901110
[16] D. Zvonkine. Strebel differentials on stable curves and Kontsevich’s proof of Witten’s conjecture. arXiv:math.AG/0209071
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