REALIZATIONS OF AFFINE LIE ALGEBRA $A_1^{(1)}$ AT NEGATIVE LEVELS

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Abstract. A realization of the affine Lie algebra $A_1^{(1)}$ and the relevant $\mathbb{Z}$-algebra at negative level $-k$ is given in terms of parafermions. This generalizes the recent work on realization of the affine Lie algebra at the critical level.

1. Introduction

Since the sixties the theory of affine Lie algebras has been one of the popular subjects in mathematical physics. Its physical applications and mathematical properties usually depend on whether one can use the matrix method to give a concrete realization or representation. This approach has been used in dual resonance models, infinitesimal Bäcklund transformations in soliton theory etc. The first concrete realization of the affine Lie algebra $\widehat{sl}(2)$ was Lepowsky-Wilson’s vertex operator representation at level one [18] and was then generalized to arbitrary types by Kac-Kazhdan-Lepowsky-Wilson [15]. Later the homogeneous realization of simply laced affine Lie algebras at level one was given by I. Frenkel-Kac [12] and Segal [19]. Fermionic realizations were also constructed by I. Frenkel [10] and Kac-Peterson [16], and were generalized to arbitrary types by Feingold and I. Frenkel [9].

Representations of the affine Lie algebras at other levels also have attracted a lot of attention [17, 14, 7, 5, 6]. Wakimoto [20] derived a general scheme to realize the affine Lie algebra of type $A_1^{(1)}$ and this was generalized to higher rank by Feigin and E. Frenkel [8]. Generally speaking, highest weight representations of the affine Lie algebras with integral levels are built from the theory of vertex operators in terms of bosonic or fermionic operators. Recently Adamovic [1] used vertex superalgebras to study critical modules for $\widehat{sl}_2(\mathbb{C})$. Dunbar et al [7] also gave a new representation of $\widehat{sl}_2(\mathbb{C})$ at the critical level using some technique similar to semi-infinite wedge products. This paper is a generalization of their work to arbitrary negative integral level using the theory of parafermions. Parafermions are introduced in statistical mechanics and conformal field theory, they are also related to Majorana fermions, fractional superstring, mirror symmetry, and have close connections with exclusion statistics, quantum computations, Bose-Einstein condensates etc [2, 3, 4]. In particular, the parafermions, sometimes regarded as $\mathbb{Z}$-algebras proposed in [22] contribute to various extensions of the Ising model and 3-state Potts model, all of which are basically relevant to $A_1^{(1)}$. These works

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show that the $Z$-algebra at a positive integral level is identical with that of $A_1^{(1)}$-parafermions.

In the monograph [5], Dong and Lepowsky constructed canonical generalized vertex operator algebras for $\hat{\mathfrak{g}}$ (of simply laced types $\hat{A}$, $\hat{D}$ or $\hat{E}$) and pointed out that the corresponding quotient space for the vacuum space of any positive integer level $k$ standard $\mathfrak{g}$-module is a module of the generalized vertex operator algebra. Furthermore, as an illustration, they showed in details the construction for $A_1^{(1)}$. They used the vacuum space of $L(k,0)$ ($k \in \mathbb{N}$) in terms of a natural Heisenberg subalgebra of $A_1^{(1)}$ to define a quotient spaces of this vacuum space by the action of an infinite cyclic group, and then realized the parafermion algebra as the canonically modified $Z$-algebra acting on certain quotient spaces.

In the recent work of [7] the affine Lie algebra $\hat{\mathfrak{sl}}_2(\mathbb{C})$ at the critical level $-2$ was realized using the generalized Clifford algebra. This shows that the case of negative level can be treated by parafermions as well. In this paper we generalize this result and realize the affine Lie algebra $\hat{\mathfrak{sl}}_2$ at negative levels by parafermions. Although many results at negative levels are quite similar to the positive integral levels, we still give a complete treatment of the realization with the hope that this may be useful to understand Lusztig’s theory of the relationship between quantum groups and affine Lie algebras. For completeness we include all necessary computation of operator product expansions of parafermions and also provide the detailed verifications of the $Z$-algebra relations.

The paper is organized as follows. In section two we first recall the basic definitions. The later part of section two reviews some basic results of parafermions based on [7], and briefly explains the physicists’ approach to parafermion fields with respect to each form of current algebras, operator product expansions and so on, and also studies in detail the generalized commutation relations and in particular modifications needed in the paper. In section three, a parafermionic representation of $A_1^{(1)}$ at level $-k$ ($k \in \mathbb{N}$) is constructed and corresponding results for the associated $Z$-algebra are given.

2. Basic definitions

2.1. The affine Lie algebra $\hat{\mathfrak{sl}}_2$. Let $\hat{\mathfrak{sl}}_2(\mathbb{C})$ be the affine Lie algebra of type $A_1^{(1)}$, which is generated by a 1-dimensional central $c$, a degree derivation $d = 1 \otimes t \partial t$ and elements $a(m) = a \otimes t^m \in \mathfrak{sl}_2(\mathbb{C}) \otimes \mathbb{C}[t, t^{-1}]$, where $\mathbb{C}[t, t^{-1}]$ is the algebra of Laurent polynomials in the indeterminate $t$. The Lie bracket operation is defined by

\begin{align}
[c, \hat{\mathfrak{sl}}_2(\mathbb{C})], [d, a(m)] &= ma(m) \\
[a(m), b(n)] &= [a, b](m + n) + Tr(ab)mc\delta_{m+n,0}
\end{align}

for all $m, n \in \mathbb{Z}, a, b \in \mathfrak{sl}_2(\mathbb{C})$. The Chevalley basis of $\mathfrak{sl}_2$ consists of $X, Y, H$:

\begin{align*}
H &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
\end{align*}

with brackets

\begin{align}
[H, X] &= 2X, [H, Y] = -2Y, [X, Y] = H
\end{align}
Besides the presentation of $A_1^{(1)}$ with the basis \{\(H(p), X(m), Y(n), c \mid p, m, n \in \mathbb{Z}\}\), satisfying the commutation relations (1), there is also the Kac-Moody definition by the Chevalley generators \{\(h_i, e_j, f_k| i, j, k \in \{0, 1\}\}\), subject to the conditions

\[
[h_i, e_j] = A_{ij}e_j, [h_i, f_j] = -A_{ij}f_j, [e_i, f_j] = \delta_{ij}h_j,
\]

where \(A = (A_{ij}) = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}\) is the generalized Cartan matrix.

The two equivalent descriptions of $A_1^{(1)}$ are related under the following correspondence

\[
e_0 \leftrightarrow Y(1), f_0 \leftrightarrow X(-1), h_0 = -H(0) + c,
\]

\[
e_1 \leftrightarrow X(0), f_1 \leftrightarrow Y(0), h_1 = H(0)
\]

Recall that the weight space \(V_\mu = \{v \in V \mid h \cdot v = \mu(h)v, \forall h \in \mathfrak{h}\}\), where \(\mathfrak{h} = (\oplus_{n \in \mathbb{Z}} H(n)) \oplus \mathbb{C}c \oplus \mathbb{C}d\) is the Cartan subalgebra of \(\mathfrak{sl}_2(\mathbb{C})\). A highest weight module \(V(\lambda) = \oplus_{\mu \leq \lambda} V_\mu\), or the highest weight representation, is the space generated by a highest weight vector \(v_\lambda\) of weight \(\lambda\) such that \(e_i v_\lambda = 0, h_i v_\lambda = \lambda(h_i)v_\lambda\). The central element \(c\) acts on \(V(\lambda)\) as a scalar \(k\), which will be called the level of the module.

The elements of Heisenberg subalgebra \(\mathfrak{h}' = (\oplus_{n \neq 0} H(n)) \oplus \mathbb{C}c\) of $A_1^{(1)}$ obey the following relations which are special cases of Eq. (1):

\[
[H(m), H(n)] = 2mc\delta_{m+n, 0}
\]

Given level \(-k\), any negative integer. Let \(S(\mathfrak{h}'^-)\) the space of symmetric polynomials generated by elements in \(\mathfrak{h}'^- = \oplus_{n < 0} H(n)\). Then there is a canonical representation of the Heisenberg algebra \(\mathfrak{h}'\) on \(S(\mathfrak{h}'^-)\) via the actions in accordance with Eq. (1):

\[
c \cdot 1 = k, H(0) \cdot v = 0,
\]

\[
H(m) \cdot v = H(m)v, m < 0
\]

\[
H(m) \cdot v = -2mk\delta_{H(m)}(v), m > 0
\]

In fact, this can be verified pretty straightforward, one needs only to observe that \([H(m), H(n)] \cdot v = -2mk\delta_{m+n, 0}v\) is valid under bracket relations.

We denote by \(a(z) = \sum_{m \in \mathbb{Z}} a(m)z^{-m}\) the power formal series. Here and later, \(z, w\) mean any formal variables. In this form, \(A_1^{(1)}\) is usually called a current algebra.

To write commutation relations in formal series, we need to introduce the formal \(\delta\)-function \(\delta(\frac{w}{z}) = \sum_{m \in \mathbb{Z}} \frac{w^m}{z^m}\), which possesses the fundamental property: for any \(f(w, z) \in \text{End}(V)\) such that

\[
\lim_{w \to z} f(w, z) = f(z, z), \quad f(w, z)\delta(\frac{w}{z}) = f(z, z)\delta(\frac{w}{z})
\]

exists. More information on delta functions can be found in [11].

The commutation relations of the affine Lie algebra can now be given as follows.

\[
[H(z), X(w)] = 2X(w)\delta(\frac{w}{z})
\]

\[
[H(z), Y(w)] = -2X(w)\delta(\frac{w}{z})
\]

\[
[X(z), Y(w)] = H(w)\delta(\frac{w}{z}) - kw\partial_w \delta(\frac{w}{z})
\]
2.2. Parafermions. We now discuss the parafermion theory [21]. Let $\Phi$ be the root system of the simple Lie algebra $\mathfrak{g}$ and let $M$ (or $M \mod kM_L$) denote the root lattice spanned by $\Phi$, where $-k$ is identified with the level in the corresponding affine Lie algebra $\hat{\mathfrak{g}}$ and $M_L$ is the long root sublattice. Let $E_\alpha$ be the root vector of $\mathfrak{g}$, and we normalize the Chevalley basis of $\mathfrak{g}$ via $[E_\alpha, E_\beta] = \epsilon_{\alpha, \beta} E_{\alpha + \beta}$ if $\alpha + \beta \in \Phi$.

It is well-known that $\epsilon_{\alpha, \beta} \in \mathbb{Z}$.

General parafermion is defined for elements of $M$, but we will focus on parafermionic fields $\psi_\alpha(z)$, $\psi_\beta(w)$ for roots $\alpha, \beta \in \Phi$ [13]. For two such parafermions the radial ordered product is defined as a multivalued function owning to the mutually semilocal property between them (cf. [21]). Instead of (anti-)commutativity the key relation is

$$R(\psi_\alpha(z)\psi_\beta(w)) = (-1)^{\frac{\alpha \beta}{2\pi}} R(\psi_\beta(w)\psi_\alpha(z)).$$

For simplicity we will drop the symbol $R$. For $\alpha, \beta \in \Phi$ the operator product expansion for two parafermions can be formulated as (cf. [2])

$$\psi_\alpha(z)\psi_\beta(w)(z-w)^{-\frac{\alpha \beta}{2\pi}} = N(\psi_\alpha(z)\psi_\beta(w)) + \frac{\epsilon_{\alpha, \beta} E_{\alpha + \beta}(w)}{z-w} \psi_{\alpha + \beta}(w) + \frac{\delta_{\alpha + \beta, 0} I_{\psi_\alpha(z)\psi_\beta(w)}}{(z-w)^2}$$

in which $N(\psi_\alpha(z)\psi_\beta(w))$ can be seen as infinitesimal of higher order in $z-w$, namely inside $O(z-w)$. Note that the regular part of the expression in parentheses satisfy

$$N(\psi_\alpha(z)\psi_\beta(w)) = N(\psi_\beta(w)\psi_\alpha(z)),$$

and

$$\epsilon_{\alpha, \beta} = \begin{cases} \epsilon_{\alpha, \beta}/\sqrt{-k}, & \text{if } \alpha + \beta \in \Phi \\ 0, & \text{otherwise} \end{cases},$$

where $I_{\psi_\alpha(z)\psi_\beta(w)}$ are some constants to be fixed later.

According to the parafermion theory, the conformal dimension of $\psi_l(z)$, $l \in \Phi$ is defined by $\Delta_l = \frac{l^2}{2k} + n(l)$ [13], where $n(l)$ is the minimal number of roots $\alpha_i$ in $\Phi$ by which $l$ can be composed, $\alpha = \sum_{i=1}^{n(l)} \alpha_i$. Note that Eq. (8) can be equivalently written as

$$\psi_\alpha(z)\psi_\beta(w) = (z-w)^{\Delta_{\alpha + \beta} - \Delta_\alpha - \Delta_\beta}[\delta_{\alpha + \beta, 0} I_{\psi_\alpha(z)\psi_\beta(w)} + \epsilon_{\alpha, \beta} \psi_{\alpha + \beta}(w) + \ldots].$$

In this case $\Delta_\alpha = \Delta_{-\alpha} = 1$ and $\Delta_0 = 0$, therefore Eq. (9) is simply

$$\psi_\alpha(z)\psi_\beta(w)(z-w)^{-\frac{\alpha \beta}{2\pi}} = (z-w)^{n(\alpha + \beta) - 2}[\delta_{\alpha + \beta, 0} I_{\psi_\alpha(z)\psi_\beta(w)} + \epsilon_{\alpha, \beta} \psi_{\alpha + \beta}(w) + O(z-w)].$$

For $\psi_{\pm \alpha}(z)$ associated to Lie algebra $\mathfrak{sl}_2(\mathbb{C})$, we find that $I_{\psi_{\pm \alpha}(z)\psi_{\pm \alpha}(w)} = -kzw$ for $\alpha = -\beta$, and it is 1 otherwise. We define the normal ordered product

$$\psi_\alpha(z)\psi_\beta(w) : (z-w)^{-\frac{\alpha \beta}{2\pi}} = N(\psi_\alpha(z)\psi_\beta(w)).$$

We define the contraction function by

$$\psi_\alpha(z)\psi_\beta(w)(z-w)^{-\frac{\alpha \beta}{2\pi}} = \psi_\alpha(z)\psi_\beta(w)(z-w)^{-\frac{\alpha \beta}{2\pi}} : \psi_\alpha(z)\psi_\beta(w) : (z-w)^{-\frac{\alpha \beta}{2\pi}}.$$
Then Eq. (8) can be written by
\[
(12) \quad \psi_{\alpha}(z)\psi_{\beta}(w)(z - w) = \frac{\alpha_\beta}{z - w},
\]
\[
: \psi_{\alpha}(z)\psi_{\alpha}(w): (z - w)^{-\frac{\alpha_\beta}{z - w}}, \quad i\alpha = \beta
\]
\[
: \psi_{\alpha}(z)\psi_{-\alpha}(w): (z - w)^{\frac{\alpha_\beta}{z - w}} + \frac{kw}{(z - w)^2}, \quad i\alpha = -\beta
\]

Now for the affine Lie algebra \( \hat{sl}_2 \) we define the \( Z \)-algebra operators \( A_\alpha(z) \), \( A^*_{-\alpha}(z) \) for \( \psi_\alpha(z), \psi_{-\alpha}(z) \). Using Eq. (12), we get the following equations:
\[
A_\alpha(z)A^*_\alpha(w)(z - w)^{\frac{\alpha_\beta}{z - w}} = A^*_\alpha(z)A_\alpha(w)(z - w)^{\frac{\alpha_\beta}{z - w}} = 0,
\]
\[
A_\alpha(z)A^*_\alpha(w)(z - w)^{\frac{\alpha_\beta}{z - w}} = \frac{-kzw}{(z - w)^2},
\]
\[
A^*_\alpha(w)A_\alpha(z)(w - z)^{\frac{\alpha_\beta}{w - z}} = \frac{-kzw}{(w - z)^2}.
\]

The operator \( A_\alpha \) (or \( A^*_{-\alpha} \)) acts on the field operator \( \Phi_{\lambda,\bar{\lambda}}(w, \bar{w}) \) with charge \( (\lambda, \bar{\lambda}) \) (cf. [2] [3] [22]) as follows.
\[
(14) \quad A_\alpha(z)\Phi_{\lambda,\bar{\lambda}}(w, \bar{w}) = \sum_{m=\infty}^{\infty} (z - w)^{-m - \frac{\alpha_\lambda}{2\pi i}} A^\alpha_{m, \lambda} \Phi_{\lambda,\bar{\lambda}}(w, \bar{w}),
\]
then the component operator \( A^\alpha_{m, \lambda} \) acts on \( \Phi_{\lambda,\bar{\lambda}} \) via
\[
A^\alpha_{m, \lambda} \Phi_{\lambda,\bar{\lambda}}(w, \bar{w}) = \int_\mathbb{C} \frac{dz}{2\pi i} (z - w)^{-m - \frac{\alpha_\lambda}{2\pi i}} A_\alpha(z)\Phi_{\lambda,\bar{\lambda}}(w, \bar{w}).
\]

We are interested only in parafermions \( \psi_{\alpha} \), carrying charge \( (\alpha, 0) \) with \( \Phi_{\alpha,0}(w, \bar{w}) \)
\[
(15) \quad A_\alpha(z)\Phi_{\alpha,0}(w, \bar{w}) = \sum_{m=\infty}^{\infty} (z - w)^{-m - \frac{\alpha_\lambda}{2\pi i}} A^\alpha_{m, \alpha} \Phi_{\alpha,0}(w, \bar{w}).
\]

### 2.3. Action of the group algebra.

The group algebra \( \mathbb{C}(\mathbb{Z}_\alpha) \) is the associative algebra generated by \( e^{n\alpha} \) \((n \in \mathbb{Z})\) under the multiplication
\[
(16) \quad e^0 = 1, \quad e^{m\alpha} \cdot e^{n\alpha} = e^{(m+n)\alpha}.
\]
where \( m, n \in \mathbb{Z} \). The group algebra \( \mathbb{C}(\mathbb{Z}_\alpha) \) acts on itself via multiplication, and we also introduce the operator \( h(0) \) \((h \in \mathfrak{h})\) which acts on \( \mathbb{C}(\mathbb{Z}_\alpha) \) by
\[
h(0) : \mathbb{C}(\mathbb{Z}_\alpha) \rightarrow \mathbb{C}(\mathbb{Z}_\alpha)
\]
\[
e^{\alpha} \mapsto (h, \alpha)e^{\alpha}
\]
so we get \([h(0), e^{\alpha}] = (h, \alpha)e^{\alpha}\). Using the operator \( h(0) \) we naturally define the operator \( z^h \) \(\in \text{End} \mathbb{C}(\mathbb{Z}_\alpha) \) \((z)\) (can be seen as \( z^{h(0)}) \) for \( h \in \mathbb{Z}_\alpha \) by
\[
(17) \quad z^h \cdot e^{\alpha} = z^{(h, \alpha)}e^{\alpha}.
\]

Then we get
\[
(18) \quad [\alpha(0), z^\beta] = 0,
\]
\[
z^\alpha e^\beta = z^{(\alpha, \beta)}e^\beta z^\alpha = e^\beta z^{\alpha + (\alpha, \beta)}.
\]
3. Construction of the Parafermion Representations of $A^{(1)}_1$ and $Z$-algebra

3.1. Action of Heisenberg subalgebra. We define the following exponential operators on the space $S(h^-)$ and their properties are given in Proposition 1

\[
E_\pm^+(z) = e^{\sum_{n>0} \frac{H(-n)}{kn} z^n}
\]
\[
E_\pm^-(z) = e^{\sum_{n>0} \frac{H(n)}{kn} z^n}
\]

Proposition 1. On the space $S(h^-)$ we have

\[
E_\pm^+(z)E_\pm^-(z) = E_\pm^-(z)E_\pm^+(z) = 1
\]

(19)

\[
E_\pm^+(z)E_\pm^-(w) = E_\pm^-(w)E_\pm^+(z)
\]

(20)

\[
E_\pm^+(z)E_\pm^-(w) = E_\pm^-(w)E_\pm^+(z)
\]

(21)

Proof. These identities are proved by the Campbell-Hausdorf-Witt theorem. The commutativity relations are easy consequence of the fact that $H(m)$ and $H(n)$ commute if $m \neq -n$. For the other identities we compute that

\[
\partial_z (E_\pm^-(z)E_\pm^-(z)) = \partial_z (\exp (- \sum_{n \neq 0} \frac{H(n)}{kn} z^{-n})) = E_\pm^+(z)E_\pm^-(z) \sum_{n \neq 0} \frac{H(n)}{kn} z^{-n-1}.
\]

For the last two relations in Eq. (21), we use the identity $e^{x_1}e^{x_2} = e^{x_2}e^{x_1}$ if $x_1, x_2$ commute with $[x_1, x_2]$:}

\[
E_\pm^+(z)E_\pm^-(w) = E_\pm^-(w)E_\pm^+(z)\exp (\sum_{m>0} \frac{-H(-m)}{km} z^{-m} - \sum_{n>0} \frac{H(n)}{kn} w^{-n})
\]

\[
= E_\pm^-(w)E_\pm^+(z)\exp (- \sum_{m, n>0} \frac{2mc\delta_{m-n, 0}}{k^2 mn} z^m w^{-n})
\]

\[
E_\pm^-(w)E_\pm^+(z)(1 - \frac{z}{w})^{-2}.
\]

3.2. The realization. Let $V = S(h^-) \otimes <\Phi_{\alpha,0}(\omega, \varphi) > \otimes \mathbb{C}(\mathbb{Z} \alpha)$, we define the map $\pi : \widehat{sl}_2(\mathbb{C}) \to \text{End}(V) \{z\}$ as follows:

\[
X(z) \mapsto E_\pm^+(z)E_\pm^+(z) \otimes A_{\alpha}(z)e^{\alpha} z^{-\frac{c}{d}}
\]

\[
Y(z) \mapsto E_\pm^-(z)E_\pm^-(z) \otimes A^*_{-\alpha}(z)e^{-\alpha} z^{\frac{c}{d}}
\]

\[
H(z) \mapsto H(z) \otimes 1
\]

\[
c \mapsto -k
\]

\[
d \mapsto \text{deg}.
\]
Theorem 2. \((\pi, V)\) defines a representation of \(A_1^{(1)}\).

Proof. For convenience, we just check the relations in Eq. (6).

\[
X(z)X(w) \rightarrow E_+^+(z)E_+^-(z)E_+^+(w)E_+^-(w) \otimes A_\alpha(z)e^\alpha z^{-\frac{\alpha}{2}} A_\alpha(w)e^\alpha w^{-\frac{\alpha}{2}} = E_+^+(z)E_+^-(w)E_+^+(z)E_+^-(w)(1 - \frac{w}{z})z^{-\frac{\alpha}{2}} \otimes A_\alpha(z)A_\alpha(w)e^{2\alpha z^{-\frac{\alpha}{2}} + \frac{\alpha}{2} w^{-\frac{\alpha}{2}}}
\]

\[
= E_+^+(z)E_+^-(w)E_+^+(z)E_+^-(w)(1 - \frac{w}{z})z^{-\frac{\alpha}{2}} \otimes (\underbrace{A_\alpha(z)A_\alpha(w)} + (w) \delta_{\alpha}(\frac{w}{z}) + \frac{\alpha}{2} e^{2\alpha z^{-\frac{\alpha}{2}} + \frac{\alpha}{2} w^{-\frac{\alpha}{2}}})
\]

Thus we get the computation as expected

\[
[X(z), X(w)] = X(z)X(w) - X(w)X(z) = E_+^+(z)E_+^-(w) \otimes \underbrace{A_\alpha(z)A_\alpha(w)} + (w) \delta_{\alpha}(\frac{w}{z}) + \frac{\alpha}{2} e^{2\alpha z^{-\frac{\alpha}{2}} + \frac{\alpha}{2} w^{-\frac{\alpha}{2}}}
\]

Hence,

\[
[X(z), X(w)] = X(z)X(w) - X(w)X(z) = E_+^+(z)E_+^-(w) \otimes (\underbrace{A_\alpha(z)A_\alpha(w)} + (w) \delta_{\alpha}(\frac{w}{z}) + \frac{\alpha}{2} e^{2\alpha z^{-\frac{\alpha}{2}} + \frac{\alpha}{2} w^{-\frac{\alpha}{2}}}) = 0
\]

By similar method we get \([Y(z)Y(w)] = 0\). Next notice that

\[
X(z)Y(w) \rightarrow E_+^+(z)E_+^-(z)E_+^+(w)E_+^-(w) \otimes A_\alpha(z)e^\alpha z^{-\frac{\alpha}{2}} A_\alpha^\ast(w)e^{-\alpha w\frac{\alpha}{2}}
\]

\[
= E_+^+(z)E_+^-(w)E_+^+(z)E_+^-(w)(1 - \frac{w}{z})z^{-\frac{\alpha}{2}} \otimes (\underbrace{A_\alpha(z)A_\alpha^\ast(w)} + (w) \delta_{\alpha}(\frac{w}{z}) + \frac{\alpha}{2} e^{-2\alpha z^{-\frac{\alpha}{2}} + \frac{\alpha}{2} w^{-\frac{\alpha}{2}}})
\]

Hence,

\[
[X(z), Y(w)] = X(z)Y(w) - Y(w)X(z) = E_+^+(z)E_+^-(w) \otimes (\underbrace{A_\alpha(z)A_\alpha^\ast(w)} + (w) \delta_{\alpha}(\frac{w}{z}) + \frac{\alpha}{2} e^{-2\alpha z^{-\frac{\alpha}{2}} + \frac{\alpha}{2} w^{-\frac{\alpha}{2}}}) = 0
\]
It is easy to compute that

\[ [H(z), E^+_m(w)] = \sum_{m < \mathbb{Z}} [H(m), e^\omega \frac{H(m-n)}{m} w^m] z^{-m} E^+_m(w) \sum_{n > 0} |H(m), H(-n)| z^{-m} w^n \]

\[ = E^+_m(w) \sum_{n > 0} \frac{2m\delta_{m-n,0}}{kn} z^{-m} w^n = E^+_m(w) \sum_{n > 0} -2z^{-n} w^n. \]

A similar calculation for \([H(z), E^+_m(w)],[H(z), E^-_m(w)],[H(z), E^+_m(w)]\) yields

\[ [H(z), X(w)] = [H(z), E^+_m(w)]E^+_m(w) \otimes A_\alpha(w)e^\alpha w^{-\frac{\bar{w}}{z}} \]

\[ + E^+_m(w)[H(z), E^+_m(w)] \otimes A_\alpha(w)e^\alpha w^{-\frac{\bar{w}}{z}} + E^+_m(w)E^+_m(w) \otimes A_\alpha(w)[H(0), e^\alpha] w^{-\frac{\bar{w}}{z}} \]

\[ = 2E^+_m(w)E^+_m(w) \left( \sum_{n > 0} z^{-n} w^n + \sum_{n < 0} z^{-n} w^n + 1 \right) \otimes A_\alpha(w)e^\alpha w^{-\frac{\bar{w}}{z}} \]

\[ = 2E^+_m(w)E^+_m(w) \otimes A_\alpha(w)e^\alpha w^{-\frac{\bar{w}}{z}} = 2X(w)\delta^\alpha(y \bar{z}). \]

It is immediate that \([H(z), Y(w)] = -2Y(w)\delta^\alpha(y \bar{z}). \]

The action of \(c\) shows that the representation of \(A_1^{(1)}\) just obtained has the level \(-k\).

3.3. The \(Z\)-algebra. Furthermore we can get the representation of \(Z\)-algebra as in [7]. We remark that this \(Z\)-algebra is fundamentally different from the \(Z\)-algebra in [21]. Taken the same definition of formal power series \(Z^\pm(z), x(\phi_1, z), x(\phi_2, z)\):

\(Z^+(z) = Z(\alpha, z) = E^+_m(z)X(z)E^-_m(z), Z^-(z) = Z(-\alpha, z) = E^+_m(z)Y(z)E^-_m(z), \)

and the generalized commutator brackets

\[ \{x(\phi_1, z), x(\phi_2, z)\} = x(\phi_1, z)x(\phi_2, z)(1 - \frac{\bar{w}}{z})^{(\phi_1, \phi_2)}_e - x(\phi_2, z)x(\phi_1, z)(1 - \frac{\bar{w}}{z})^{(\phi_1, \phi_2)}_e, \]

for \(\phi_1, \phi_2 = \pm \alpha\), we can check that the lemmas given in paper [7] still hold. We state them here in Lemma 3 and Lemma 4.

**Lemma 3.** Let \(Z\)-operators \(Z(z) = Z^+(z), Z^-(z)\), we have that

\[ [E^+_m(z), Z(w)] = 0, \quad [E^-_m(z), Z(w)] = 0 \]

**Proof.** For \(n \neq 0\), simple calculation yields

\[ [H(n), X(w)] = \sum_{m \in \mathbb{Z}} [H(n), X(m)] w^{-m} = \sum_{m \in \mathbb{Z}} 2X(m + n)w^{-m} = 2X(w)w^n \]

and write \(x_1 = \frac{\partial}{\partial s}(e^{sx})|_{s=0}\), the following equations follow from \([x_1, e^{sx}] = e^{sx}[x_1, x_2]\)

\[ [H(n), E^-_m(w)] = E^-_m(w)[H(n), \sum_{m > 0} \frac{H(m)}{km} w^{-m}] \]

\[ = E^-_m(w) \sum_{m > 0} \frac{2m\delta_{m-n,0}}{kn} w^{-m} = -2E^-_m(w)w^n \delta_{m, n < 0}, \]

\[ [H(n), E^+_m(w)] = -2E^+_m(w)w^n \delta_{m, n > 0}. \]
Note that

\[
[H(n), Z^+(w)] = [H(n), E^+_+(w)X(w)E^+_-(w)]
\]
\[
= [H(n), E^+_-(w)]X(w)E^+_-(w) + E^+_-(w)[H(n), X(w)]E^+_-(w) + E^+_-(w)X(w)[H(n), E^+_-(w)]
\]
\[
= (-2w^n \delta_{m,n>0} + 2w^n + 2w^n \delta_{m,n<0})E^+_-(w)X(w)E^+_-(w) = 0.
\]

Similarly, \([H(n), Z^-(w)] = [H(-n), Z^+(w)] = 0\), so

\[
[E^+_+(z), Z^+(w)] = E^+_+(z) \left[ \sum_{n>0} \frac{H(n)}{kn} z^{-n}, Z^+(w) \right] = 0
\]

The calculation for the other brackets is similar.

**Lemma 4.** One has

\[
\begin{align*}
[[Z^+(z), Z^+(w)] &= 0 \\
[[Z^+(z), Z^-(w)] &= H(0) \delta(\frac{m}{z}) - kw\partial_{w}\delta(\frac{m}{z})
\end{align*}
\]

**Proof.** We calculate the product using Proposition 1 together with Lemma 3

\[
Z^-(z)Z^-(w) = Z^-(z)E^+_+(w)Y(w)E^+_+(w)E^+_+(w)Z^-(z)Y(w)E^+_+(w)
\]
\[
= E^+_+(w)E^+_+(z)Y(z)(E^+_-(w)E^+_+(w))E^+_+(z)Y(w)E^+_+(w)
\]
\[
= E^+_+(w)E^+_+(z)Y(z)E^+_+(w)E^+_+(z)(1 - \frac{w}{z})^{\frac{2}{2}}Y(w)E^+_+(w)
\]
\[
= E^+_+(w)E^+_+(z)Y(z)E^+_+(w)(E^+_+(z)Z^-(w))(1 - \frac{w}{z})^{\frac{2}{2}}
\]
\[
= E^+_+(w)E^+_+(z)Y(z)Y(w)E^+_+(w)E^+_+(w)(1 - \frac{w}{z})^{\frac{2}{2}}
\]
\[
= E^+_+(w)E^+_+(z)Y(z)Y(w)E^+_+(w)E^+_+(w)(1 - \frac{w}{z})^{\frac{2}{2}}
\]

Consequently,

\[
[[Z^-(z)Z^-(w)] = Z^-(z)Z^-(w)(1 - \frac{w}{z})^{\frac{2}{2}} - Z^-(w)Z^-(z)(1 - \frac{z}{w})^{\frac{2}{2}}
\]
\[
= E^+_+(z)E^+_+(w)[Y(z), Y(w)]E^+_+(z)E^+_+(w) = 0
\]

The calculation for \([[Z^+(z), Z^+(w)]]\) is similar.

Also

\[
Z^+(z)Z^+(w) = E^+_+(w)E^+_+(z)X(z)Y(w)E^+_+(w)E^+_+(z)(1 - \frac{w}{z})^{\frac{2}{2}}
\]
\[
Z^-(w)Z^+(z) = E^+_+(z)E^+_+(w)Y(w)X(z)E^+_+(z)E^+_+(w)(1 - \frac{z}{w})^{\frac{2}{2}}
\]
\[ [Z^+(z)Z^-(w)] = Z^+(z)Z^-(w)(1 - \frac{w}{z})^\frac{\beta}{2} - Z^-(w)Z^+(z)(1 - \frac{z}{w})^\frac{\beta}{2} \]
\[ = E^+_{\pm}(w)E^\pm_{\pm}(z)[X(z)Y(w)]E^+_{\pm}(w)E^-_{\pm}(z) \]
\[ = E^+_{\pm}(z)E^\pm_{\pm}(w)H(w)\delta(w)E^+_{\pm}(z)E^-_{\pm}(w) - E^+_{\pm}(z)E^\pm_{\pm}(w)kw\partial_w\delta(w)E^+_{\pm}(z)E^-_{\pm}(w) \]
\[ = H(w)\delta(w) - kw\left(\partial_w (E^+_{\pm}(z)E^\pm_{\pm}(w)E^-_{\pm}(z)E^+_{\pm}(w)\delta(w))\right) \]
\[ - \partial_w (E^+_{\pm}(z)E^\pm_{\pm}(w)E^-_{\pm}(z)E^+_{\pm}(w))\delta(w) \]
\[ = \sum_{m \in \mathbb{Z}} H(m)w^{-m}\delta(w) - kw\sum_{m \neq 0} H(m)w^{-m-1}\delta(w) \]
\[ = H(0)\delta(w) - kw\partial_w\delta(w) \]

\[ \square \]

For \( A_1^{(1)} \)-module \( V = S(\mathfrak{h}^-) \otimes \langle \Phi_{\alpha,0}(\omega, \varpi) \rangle \otimes \mathbb{C}(\mathbb{Z}\alpha) \) in Theorem 2, we define the vacuum space \( \Omega(V) \) of \( V \) by

\[ \Omega(V) = \{ v \in V, \eta^+=\oplus_{n>0}H(n) | \eta^+ \cdot v = 0 \}. \]

Observe that we can decompose \( V \) by \( V = S(\mathfrak{h}^-) \otimes \Omega(V) \), then we get \( \Omega(V) = \Phi_{\alpha,0}(\omega, \varpi) \otimes \mathbb{C}(\mathbb{Z}\alpha) \) and furthermore.

**Theorem 5.** The map \( \pi_\Omega : Z \rightarrow gl(\Omega(V)) \) gives a representation of \( Z \)-algebra on the vacuum space \( \Omega(V) \) at level \(-k\) via the action:

\[ Z^+(z) \mapsto A_\alpha(z)e^\alpha z^{-\frac{\beta}{2}} \]
\[ Z^-(z) \mapsto A^*_{-\alpha}(z)e^{-\alpha} z^{\frac{\beta}{2}} \]

**Proof.** Under the map \( \pi \) we have

\[ Z^+(z)Z^+(w) \mapsto A_\alpha(z)e^\alpha z^{-\frac{\beta}{2}}A_\alpha(w)e^\alpha w^{-\frac{\beta}{2}} \]
\[ = A_\alpha(z)A_\alpha(w)e^{2\alpha}(zw)^{-\frac{\beta}{2}} \]

Therefore,

\[ [[Z^+(z), Z^+(w)]] = Z^+(z)Z^+(w)(1 - \frac{w}{z})^{-\frac{\beta}{2}} - Z^+(w)Z^+(z)(1 - \frac{z}{w})^{-\frac{\beta}{2}} \]
\[ \mapsto (A_\alpha(z)A_\alpha(w)(z-w)^{-\frac{\beta}{2}} - A_\alpha(w)A_\alpha(z)(w-z)^{-\frac{\beta}{2}})e^{2\alpha}(zw)^{-\frac{\beta}{2}} \]
\[ + (A_\alpha(z)A_\alpha(w)(z-w)^{-\frac{\beta}{2}} - A_\alpha(w)A_\alpha(z)(w-z)^{-\frac{\beta}{2}})e^{2\alpha}(zw)^{-\frac{\beta}{2}} = 0 \]

Similar calculations produce \([Z^-(z), Z^-(w)] = 0\]. Note that

\[ Z^+(z)Z^-(w) \mapsto A_\alpha(z)e^\alpha z^{-\frac{\beta}{2}}A^*_{-\alpha}(w)e^{-\alpha} w^{\frac{\beta}{2}} \]
\[ = A_\alpha(z)A^*_{-\alpha}(w)z^{\frac{\beta}{2}} z^{-\frac{\beta}{2}} w^{\frac{\beta}{2}} \]
\[ Z^-(w)Z^+(z) \mapsto A^*_{-\alpha}(w)A_\alpha(z)w^{\frac{\beta}{2}} z^{-\frac{\beta}{2}} w^{\frac{\beta}{2}} \]
Then
\[
\llbracket Z^+(z), Z^-(w) \rrbracket = Z^+(z) Z^-(w) (1 - \frac{w}{z}) \frac{\hat{\alpha}}{\hat{\alpha}} - Z^-(w) Z^+(z) (1 - \frac{z}{w}) \frac{\hat{\alpha}}{\hat{\alpha}}
\]
\[
\rightarrow (A_\alpha(z) A^-_\alpha(w)(z - w)\frac{\hat{\alpha}}{\hat{\alpha}} - A^-_\alpha(w) A_\alpha(z)(w - z)\frac{\hat{\alpha}}{\hat{\alpha}}) z^{-\frac{\hat{\alpha}}{\hat{\alpha}}} w^{\frac{\hat{\alpha}}{\hat{\alpha}}}
\]
\[
= (A_\alpha(z) A^-_\alpha(w)(z - w)\frac{\hat{\alpha}}{\hat{\alpha}} - A^-_\alpha(w) A_\alpha(z)(w - z)\frac{\hat{\alpha}}{\hat{\alpha}}) z^{-\frac{\hat{\alpha}}{\hat{\alpha}}} w^{\frac{\hat{\alpha}}{\hat{\alpha}}}
\]
\[
+ (A_\alpha(z) A^-_\alpha(w)(z - w)\frac{\hat{\alpha}}{\hat{\alpha}} - A^-_\alpha(w) A_\alpha(z)(w - z)\frac{\hat{\alpha}}{\hat{\alpha}}) z^{-\frac{\hat{\alpha}}{\hat{\alpha}}} w^{\frac{\hat{\alpha}}{\hat{\alpha}}}
\]
\[
= \left( -\frac{kzw}{(z - w)^2} - \frac{-kzw}{(w - z)^2} \right) z^{-\frac{\hat{\alpha}}{\hat{\alpha}}} w^{\frac{\hat{\alpha}}{\hat{\alpha}}} = -kw \partial_w \delta \left( \frac{w}{z} \right) z^{-\frac{\hat{\alpha}}{\hat{\alpha}}} w^{\frac{\hat{\alpha}}{\hat{\alpha}}}
\]
\[
= -kw \partial_w \left( \delta \left( \frac{w}{z} \right) z^{-\frac{\hat{\alpha}}{\hat{\alpha}}} w^{\frac{\hat{\alpha}}{\hat{\alpha}}} \right) + kw \delta \left( \frac{w}{z} \right) \partial_w (z^{-\frac{\hat{\alpha}}{\hat{\alpha}}} w^{\frac{\hat{\alpha}}{\hat{\alpha}}})
\]
\[
= -kw \partial_w \delta \left( \frac{w}{z} \right) + az \frac{\hat{\alpha}}{\hat{\alpha}} w^{-\frac{\hat{\alpha}}{\hat{\alpha}}} \delta \left( \frac{w}{z} \right) = H(0) \delta \left( \frac{w}{z} \right) - kw \partial_w \delta \left( \frac{w}{z} \right),
\]
from which the theorem follows. $\square$

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