On data recovery with restraints on the spectrum range and the process range

Nikolai Dokuchaev

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Abstract

The paper considers recovery of signals from incomplete observations and a problem of determination of the allowed quantity of missed observations, i.e. the problem of determination of the size of the uniqueness sets for a given data recovery procedures. The paper suggests a way to bypass solution of this uniqueness problem via imposing restrictions investigates possibility of data recovery for classes of finite sequences under a special discretization of the process range. It is shown that these sequences can be dense in the space of all sequences and that the uniqueness sets for them can be singletons. Some robustness with respect to rounding of input data can be achieved via including additional observations.

Key words: data recovery, data compression, discrete Fourier transform, spectrum range discretization, process range discretization, Diophantine equations.

1 Introduction

The paper investigates possibility of recovery of finite sequences from partial observations in the setting with insufficient statistics where the probability distributions are unknown. In other words, this setting is oriented on the sequences being considered as sole sequences rather than members of an ensemble; this feature is typical in signal processing for speech, images, human activity analysis, traffic control, etc.

The data recovery problem was studied intensively in different settings exploring different restrictions on classes of underlying processes classes. Usually, recoverability is associated with
sparsity or certain restrictions on the spectrum support such as bandlimitiness or the presence of spectrum gaps; see e.g. [4, 5, 9, 21] and references therein. The classical Nyquist-Shannon sampling theorem establishes that a band-limited continuous time function can be recovered without error from a discrete sample taken with a sampling rate that is at least twice the maximum frequency present in the signal (the Nyquist critical rate). This principle defines the choice of the sampling rate in almost all signal processing protocols.

It is known that, for signals with certain structure, the Nyquist rate could be excessive for signal recovery; see e.g. [3, 19]. For example, it is known that a sparse enough subsequence or an one-sided semi-infinite subsequence can be removed from an oversampling sequence [15, 25].

Some paradigm changing results were obtained in [4, 5, 6, 11] and consequent papers in the so-called "compressive sensing" setting for finite sequences. Methods for these sequences can be immediately applied to digital processes and computer algorithms since they would not require adaptation to inevitable data truncation, unlike results obtained for continuous processes or for infinite discrete time processes.

The compressive sensing explores sparsity of signals, i.e. restrictions on the number of non-zero members of the underlying finite sequences; the location of these non-zero members is not specified and is assumed to be unknown. The main result of [4, 5, 6, 10, 11, 12] was a new method of signal recovery from a relatively small number of measurement in frequency domain given certain sparsity in time domain. (Equivalently, this result can be reformulated for measurements on time domain given the sparsity in frequency domain). The recovery algorithm suggested was based on $\ell_1$-minimization / Basis Pursuit Denoising method.

Quantification of recoverability criterions was presented in the form of an asymptotic estimate of the required number $|U|$ of observed Fourier coefficients versus $S$, where $S$ is the number of nonzero members of the underlying finite sequences with $N$ elements. It was shown [5] that recovery with overwhelming probability can be ensured given that $|U| \gtrsim CS \log(N)$ for some constant $C > 0$. There were some other modifications such as $S \ll |U|/\log(N/|U|)$ [10]; see also [11, 12, 16, 24]. As was mentioned in [12], these estimates are not sharp and can be improved. Furthermore, asymptotically lossless linear recoverability for $|D| \sim S + o(N)$ was proved in [26] using Shannon-theoretic setting under probabilistic assumptions for i.i.d. components of the underlying sequences with known distributions (which was essential). It was shown therein that recoverability can be achieved with $|U| \sim N\rho + o(N)$, where $\rho$ is the (upper) Rényi information dimension of the distribution. Since $\rho \leq S/N$ for sparse signals, it is a significant improvement. Moreover, it was also shown also that this estimate for $|U|$ cannot be improved in this probabilistic
setting [26]. An impact of the noise contamination on compressed sampling was studied in [2, 7]; various alternative setting were considered in [13, 14, 17, 18]. This illustrates how challenging is the problem of determination of the allowed quantity of missed observations and the related problem of uniqueness of recovery result.

The present paper considers data recovery problem for finite sequences and suggests a way to bypass solution of this uniqueness problem given that a process matching available observations is found somehow. This is achieved via imposing some restrictions on the process range described are defined by a special discretization of the spectrum range or the process range. It appears that this approach allows to construct classes of sequences that are ε-dense in the space of all sequences and, at the same time, have singleton uniqueness sets (Theorems 1, 3-4 below). The implied recovery procedure is neither numerically feasible nor stable since it would require to solve a Diophantine-type equation (equation (1), (2), (3) below). However, any solution of the data recovery problem obtained by any method will be automatically a correct error-free solution. For example, the solution obtained via ℓ1-minimization in the compressing sensing approach is guaranteed to be a correct one if all restrictions on matching the available observations are satisfied.

To address robustness of recoverability, we considered an alternative setting where the data recovery is robust with respect to rounding of the input processes. with rounded underlying processes where observation of first $S$ Fourier coefficients is sufficient to recover sparse sequences with no more than $S$ nonzero terms (Theorem 5).

It can be noted that the original ArXiv version of this paper included Theorem 1 only; Theorem 3 was added on 4 December 2017, Theorem 5 was added on June 22 2018, Theorems 2 and 4 was added on June 22 2018.

2 Some definitions and background

For a integer $N > 0$, let $X$ be the set of mappings $x : D \rightarrow \mathbb{C}^N$, where $D \triangleq \{0, 1, ..., N - 1\}$. This set can be associated with the space $\mathbb{C}^N$ as well as with the space of $N$-periodic sequences in $\mathbb{C}$. We consider $X$ as a linear normed space with the standard norm from $\mathbb{C}^N$.

Let us consider the discrete Fourier transform as a mapping $F : X \rightarrow X$ such that $F(x) = Qx$, where $Q = \left\{ \frac{1}{\sqrt{N}} e^{-ikt/N} \right\}_{k,t=0}^{N-1}$ is the DFT matrix, $i = \sqrt{-1}$.

Let $\nu$, $\nu_1$, $\mu$, and $\mu_1$, be positive integers.

For $a \in \mathbb{R}$, $a \geq 0$, let $\lfloor a \rfloor = \{k \in \mathbb{Z} : a \in [k, k + 1)\}$. For $a \in \mathbb{R}$, $a < 0$, let $\lfloor a \rfloor = \{k \in \mathbb{Z} : x \in (k - 1, k]\}$. For $a \in \mathbb{R}$, let $\rho_{\nu, \mu}(a) = \nu^{-\mu} [\nu^\mu a]$. We extend this function on complex numbers
such that
\[
\rho_{\nu,\mu}(z) = \rho_{\nu,\mu}(\Re z) + i \rho_{\nu,\mu}(\Im z), \quad z \in \mathbb{C}.
\]

Similarly, we define rounding function \( \rho_{\nu,\mu} : \mathbb{C}^N \to \mathbb{C}^N \), meaning the corresponding component-wise rounding.

Let \( X_{\nu,\mu} = \rho_{\nu,\mu}(X) \); this is the set of sequences from \( X \) with rounded components.

3 Some cases where uniqueness sets are singletons

3.1 The case where components of underlying process are observable

In this section, we consider a problem of recovery of \( y \in X \) from available observations of some of its components.

**Definition 1.** Let a subset \( U \) of \( D \) and a subset \( Y \) of \( X \) be given. If any \( y \in Y \) is uniquely defined by its trace \( y|_U \), then we say that \( U \) is an uniqueness set with respect to \( Y \).

**Theorem 1.** For any \( \varepsilon > 0 \) and any \( d \in \{1, \ldots, N - 1\} \), there exists a set \( Y_{\varepsilon} \) such that the following holds.

(i) The set \( Y_{\varepsilon} \) is closed in \( X \) and is such that if \( \hat{y} \in Y_{\varepsilon} \) and \( \tilde{y} \in Y_{\varepsilon} \) then \( \hat{y} - \tilde{y} \in Y_{\varepsilon} \).

(ii) The set \( Y_{\varepsilon} \) is \( \varepsilon \)-dense in \( X \).

(iii) The singleton \( U = \{d\} \) is a uniqueness set with respect to \( Y_{\varepsilon} \).

**Proof.** Let \( d \in \{1, \ldots, N - 1\} \) be fixed. To prove the theorem, we construct required sets \( Y_{\varepsilon} \) as sets of sequences with restrictions on their spectrum range.

Let \( \xi = (\xi_0, \ldots, \xi_{N-1}) \in X \) be defined such that
\[
\xi_k = e^{i2(\nu_1 \cdot \mu_1 (\pi - \pi))dk/N}.
\]

Let \( Y_{d,\nu_1,\mu_1} \) be the set of all \( y \in X \) such that there exists \( X = (X_0, \ldots, X_{N-1}) \in X_{\nu,\mu} \) such that \( Y_k = \xi_k X_k \) for \( k = 0, 1, \ldots, N - 1 \), where \( Y = (Y_0, \ldots, Y_{N-1}) = \mathcal{F}y \).

It can be noted that a set \( Y_{d,\nu_1,\mu_1} \) is defined by restrictions on the range of the spectrum \( Y = \mathcal{F}y \) of its members.

Clearly, for any \( \varepsilon > 0 \), there exist large enough \( \nu, \nu_1, \mu, \) and \( \mu_1 \), such that the set \( Y_{d,\nu_1,\mu_1} \) is \( \varepsilon \)-dense in \( X \). In addition, condition (i) in Theorem 1 is satisfied for \( Y_{d,\nu_1,\mu_1} \) for all \( \nu, \nu_1, \mu, \) and \( \mu_1 \).
Let \( y \in \mathcal{Y}_{d,\nu,\mu,\nu_1,\mu_1} \), and let \( Y = (Y_0, ..., Y_{N-1}) = \mathcal{F}y, X = (X_0, ..., X_{N-1}) \in \mathcal{X}_{\nu,\mu} \), \( Y_k = \xi_k X_k \). We have that

\[
y_d = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{i2\pi dk/N} Y_k = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{i2\pi dk/N} \xi_k X_k.
\]

Let

\[
\omega_k = 2\rho_{\nu_1,\mu_1}(\pi) dk/N.
\]

By the choice of \( \xi_k \) and by the definitions, we have that \( Y_0 = X_0 \) and

\[
y_d = \frac{1}{\sqrt{N}} \left[ X_0 + \sum_{k=1}^{N-1} e^{i\omega_k} X_k \right]. \tag{1}
\]

To prove the theorem, it suffices to show that, for any \( \nu, \nu_1, \mu, \mu_1 \), and any \( y \in \mathcal{Y}_{d,\nu,\mu,\nu_1,\mu_1} \), there exists an unique \( X = (X_0, ..., X_{N-1}) \in \mathcal{X}_{\nu,\mu} \) satisfying (1).

For this, it suffices to show that, for any \( \nu, \nu_1, \mu, \) and \( \mu_1 \), we have that if \( \hat{y}_d = \tilde{y}_d \) for some \( \hat{y}, \tilde{y} \in \mathcal{Y}_{d,\nu,\mu,\nu_1,\mu_1} \), then \( \hat{y} = \tilde{y} \).

We are now in the position to complete the proof. By condition (i) in Theorem 1, it suffices to show that if \( y_d = 0 \) for \( y \in \mathcal{Y}_{d,\nu,\mu,\nu_1,\mu_1} \), then equation (1) has only zero solution \( X \) in \( \mathcal{X}_{\nu,\mu} \). Let us show this.

Since \( y \in \mathcal{Y}_{d,\nu,\mu,\nu_1,\mu_1} \), it follows from the definitions that \( X_k \) are rational numbers for \( k = 0, 1, ..., N - 1 \). In addition, \( \omega_k \) are rational numbers as well. By the Lindemann–Weierstrass Theorem, it follows that \( X_k = 0 \) for all \( k \) (see [1], Chapter 1, Theorem 1.4). This completes the proof. \( \square \).

Remark 1. Theorem 1 allows the following obvious modification: for any \( d \in D, \varepsilon > 0, \) and any set \( G \subset \mathcal{X} \), there exists a \( \varepsilon \)-dense in \( \mathcal{X} \cap G \) set \( \mathcal{Y}_\varepsilon \subset \mathcal{X} \cap G \) such that its uniqueness set is a singleton.

3.2 The case where Fourier coefficients are observable

In this section, we consider a setting where, for a given \( y \in \mathcal{X} \), we observe some components of \( Y = Qy \).

Definition 2. Let a subset \( U \) of \( D \) and a subset \( \mathcal{Y} \) of \( \mathcal{X} \) be given. If any \( y \in \mathcal{Y} \) is uniquely defined by the trace \( Y|_U \), where \( Y = Qy \), then we say that \( U \) is a uniqueness set in the frequency domain with respect to \( \mathcal{Y} \).
Let $X_S$ be the set of all $y \in X$ such that $\sum_{k \in D} I_{\{y_k \neq 0\}} \leq S$. Let $S \in \{1, ..., N - 1\}$ be given.

**Theorem 2.** (i) If $N$ is a prime number, then any set $U \subset D$ such that $|U| = 2S$ is a uniqueness set in the frequency domain with respect to $Y_S$. ([5], Theorem 1.1).

(ii) Let $U \subset D$ be such that $|U| = 2S$ and that there exists $u \in D$ such that $U = \{u, u+1, ..., u + 2S\}$. Then $U$ is a uniqueness set in the frequency domain with respect to $Y_S$.

**Proof of Theorem 2 (ii).** Let $M = |U| = 2S$. Let $T \subset D$ be such that $|T| = M$, $T = \{t_1, ..., t_M\}$.

Consider the matrix

$$Q_{U,T} = \frac{1}{\sqrt{N}} \begin{pmatrix}
  e^{-iut_1 \pi/N} & e^{-iut_2 \pi/N} & \cdots & e^{-iut_M \pi/N} \\
  e^{-i(u+1)t_1 \pi/N} & e^{-i(u+1)t_2 \pi/N} & \cdots & e^{-i(u+1)t_M \pi/N} \\
  \vdots & \vdots & \ddots & \vdots \\
  e^{-i(u+M)t_1 \pi/N} & e^{-i(u+M)t_2 \pi/N} & \cdots & e^{-i(u+M)t_M \pi/N}
\end{pmatrix} \in \mathbb{C}^{M \times M}.$$

If $u = 0$ then this is a non-degenerate Vandermonde matrix with a nonzero determinant that we denote $V$. If $u \neq 0$ then $|\det Q_{U,T}| = |V \prod_{t \in T} e^{-iut \pi/N}| \neq 0$. The remaining part of the proof repeats the proof of Theorem 1.1 from [5]. □

The following theorem shows that the recovery uniqueness can be ensured for much smaller sets given additional restrictions on the processes range.

**Theorem 3.** For any $\varepsilon > 0$ and any $d \in \{1, ..., N - 1\}$, there exists a set $\hat{Y}_\varepsilon$ such that the following holds.

(i) The set $\hat{Y}_\varepsilon$ is closed in $X$ and is such that if $\hat{y} \in \hat{Y}_\varepsilon$ and $\tilde{y} \in \hat{Y}_\varepsilon$ then $\hat{y} - \tilde{y} \in \hat{Y}_\varepsilon$.

(ii) The set $\hat{Y}_\varepsilon$ is $\varepsilon$-dense in $X$.

(iii) The singleton $U = \{d\}$ is a uniqueness set in the frequency domain with respect to $\hat{Y}_\varepsilon$.

**Proof.** The proof is similar to the proof of Theorem 1 however, we provide it for the sake of completeness.

Let $d \in \{1, ..., N - 1\}$ be fixed. To prove the theorem, we construct required sets $\hat{Y}_\varepsilon$ as sets of sequences with restrictions on their range.

Let $\zeta = (\zeta_0, ..., \zeta_{N-1}) \in X$ be defined such that

$$\zeta_k = e^{-i2(\rho_{\nu_1, \nu_1}(\pi) - \pi)dk/N}.$$

Let $\hat{Y}_{d,\nu,\nu_1,\mu_1}$ be the set of all $y \in X$ such that there exists $x = (x_0, x_1, ..., x_{N-1}) \in X_{\nu,\mu}$ such that $y_k = \zeta_k x_k$ for $k = 0, 1, ..., N - 1$. 
It can be noted that a set \( \hat{Y}_{d,\nu,\mu,\nu,1,\mu,1} \) is defined by restrictions on the range of its members \( y \).
Clearly, for any \( \varepsilon > 0 \), there exist large enough \( \nu, \nu_1, \mu, \) and \( \mu_1 \), such that the set \( \hat{Y}_{d,\nu,\mu,\nu,1,\mu,1} \) is \( \varepsilon \)-dense in \( X \). In addition, condition (i) in Theorem 3 is satisfied for \( \hat{Y}_{d,\nu,\mu,\nu,1,\mu,1} \) for all \( \nu, \nu_1, \mu, \) and \( \mu_1 \).

Let \( y = (y_0, \ldots, y_{N-1}) \in \hat{Y}_{d,\nu,\mu,\nu,1,\mu,1} \), let \( Y = (Y_0, \ldots, Y_{N-1}) = \mathcal{F}y \), and let \( x = (x_0, \ldots, x_{N-1}) \in \mathcal{X}_{\nu,\mu} \) be such that \( y_k = \zeta_k x_k \). We have that

\[
Y_d = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{-i2\pi dk/N} y_k = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{-i2\pi dk/N} \zeta_k x_k.
\]

By the choice of \( \zeta_k \) and by the definitions, we have that \( y_0 = x_0 \) and

\[
Y_d = \frac{1}{\sqrt{N}} \left[ x_0 + \sum_{k=1}^{N-1} e^{-i\omega_k} x_k \right],
\]

where

\[
\omega_k = 2\rho_{\nu_1,\mu_1}(\pi) dk/N.
\]

To prove the theorem, it suffices to show that, for any \( \nu, \nu_1, \mu, \) and \( \mu_1 \), and any \( y \in \hat{Y}_{d,\nu,\mu,\nu,1,\mu,1} \), there exists an unique \( x = (x_0, \ldots, x_{N-1}) \in \mathcal{X}_{\nu,\mu} \) satisfying (2).

For this, it suffices to show that, for any \( \nu \) and \( \mu \), we have that if \( \hat{Y}_d = \tilde{Y}_d \) for some \( \tilde{y}, \tilde{y} \in \mathcal{X}_{\nu,\mu} \), \( \tilde{Y} = \mathcal{F}(\tilde{y}) \), and \( \tilde{y} = \mathcal{F}(\tilde{y}) \), then \( \tilde{y} = \tilde{y} \).

We are now in the position to complete the proof. By condition (i) in Theorem 3 it suffices to show that if \( Y_d = 0 \) for \( y \in \mathcal{X}_{\nu,\mu} \) and \( Y = \mathcal{F}y \), then equation (2) has only zero solution \( x \) in \( \mathcal{X}_{\nu,\mu} \). Let us show this.

Since \( y \in \hat{X}_{\nu,\mu} \), it follows from the definitions that \( x_k \) are rational numbers for \( k = 0, 1, \ldots, N-1 \). In addition, \( \omega_k \) are rational numbers as well. By the Lindemann–Weierstrass Theorem again, it follows that \( x_k = 0 \) for all \( k \) (see [1], Chapter 1, Theorem 1.4). This completes the proof. □.

**Remark 2.** Similarly to Theorem 1, Theorem 3 allows the following obvious modification: for any \( d \in S, \varepsilon > 0 \), and any set \( \hat{G} \subset \mathcal{X} \), there exists a \( \varepsilon \)-dense in \( \mathcal{X} \cap \hat{G} \) set \( \hat{Y}_\varepsilon \subset \mathcal{X} \cap \hat{G} \) such that its uniqueness set in the frequency domain is a singleton.

### 3.3 The case where Z-transform is observable

In this section, we consider a setting where, for a given \( y \in \mathcal{X} \), we observe some values of its Z-transform \( Y = Zy \) defined as

\[
Y(z) = \sum_{k=0}^{N} z^{-k} y_k, \quad z \in \mathbb{C}.
\]
Let $T \triangleq \{ z \in \mathbb{C} : |z| = 1 \}$.

**Definition 3.** Let a subset $U$ of $T$ and a subset $Y$ of $\mathcal{X}$ be given. If any $y \in Y$ is uniquely defined by the trace $Y|_U$, where $Y = Zy$, then we say that $U$ is a uniqueness set in the frequency domain with respect to $Y$.

**Theorem 4.** For any $\varepsilon > 0$ and any algebraic number $\omega \in (-\pi, \pi] \setminus \{0\}$, there exists a set $\tilde{Y}_\varepsilon$ such that the following holds.

(i) The set $\tilde{Y}_\varepsilon$ is closed in $\mathcal{X}$ and is such that if $\tilde{y} \in \tilde{Y}_\varepsilon$ and $\bar{y} \in \tilde{Y}_\varepsilon$ then $\tilde{y} - \bar{y} \in \tilde{Y}_\varepsilon$.

(ii) The set $\tilde{Y}_\varepsilon$ is $\varepsilon$-dense in $\mathcal{X}$.

(iii) The singleton $U = \{ e^{i\omega} \}$ is a uniqueness set in the frequency domain with respect to $\tilde{Y}_\varepsilon$.

**Proof.** The proof is similar to the proof of Theorems 1-3; we provide it for the sake of completeness.

Let an algebraic number $\omega \in (-\pi, \pi] \setminus \{0\}$ be fixed. To prove the theorem, we construct required sets $\tilde{Y}_\varepsilon$.

Clearly, condition (i) in Theorem 4 is satisfied for the sets $\mathcal{X}_{\nu,\mu}$ for all $\nu$ and $\mu$. In addition, for any $\varepsilon > 0$, there exist large enough $\nu$ and $\mu$, such that the set $\mathcal{X}_{\nu,\mu}$ is $\varepsilon$-dense in $\mathcal{X}$. Let $\tilde{Y}_\varepsilon$ be selected as the corresponding set $\mathcal{X}_{\nu,\mu}$ selected for this $\varepsilon$.

Let $y = (y_0, ..., y_{N-1}) \in \mathcal{X}_{\mu,\nu}$, and let $Y = Zy$. We have that

$$Y(e^{i\omega}) = \sum_{k=0}^{N-1} e^{-i\omega k} y_k. \tag{3}$$

To prove the theorem, it suffices to show that, for any $\nu$ and $\mu$, and any $y \in \mathcal{X}_{\nu,\mu}$, there exists at most one $X = (X_0, ..., X_{N-1}) \in \mathcal{X}_{\nu,\mu}$ satisfying (3).

For this, it suffices to show that, for any $\nu$ and $\mu$, we have that if $\tilde{Y}(e^{i\omega}) = \tilde{Y}(e^{i\omega})$ for some $\tilde{y}, \bar{y} \in \mathcal{X}_{\nu,\mu}$, $\tilde{Y} = Z\tilde{y}$, and $\bar{Y} = Z\bar{y}$, then $\tilde{y} = \bar{y}$.

Furthermore, by condition (i) in Theorem 4 it suffices to show that if $Y(e^{i\omega}) = 0$ for $y \in \mathcal{X}_{\nu,\mu}$ and $Y = Zy$, then equation (3) has only zero solution $x$ in $\mathcal{X}_{\nu,\mu}$. Let us show this.

Since $y \in \mathcal{X}_{\nu,\mu}$, it follows that the components of $y$ are rational numbers. In addition, $i\omega k$ are algebraic numbers. By the Lindemann–Weierstrass Theorem again, it follows that $x_k = 0$ for all $k$ (see [1], Chapter 1, Theorem 1.4). This completes the proof. □.

**Remark 3.** Similarly to Theorems 1-3, Theorem 4 allows the following obvious modification: for any algebraic number $\omega \in (-\pi, \pi] \setminus \{0\}$, any $\varepsilon > 0$, and any set $\tilde{G} \subset \mathcal{X}$, there exists a $\varepsilon$-dense in $\mathcal{X} \cap \tilde{G}$ set $\tilde{Y}_\varepsilon \subset \mathcal{X} \cap \tilde{G}$ such that its uniqueness set in the frequency domain is a singleton.
4 The case of multiple observations

4.1 Extended systems with additional observations

Assume that, in the setting of Theorem 1, there are available observations of \( y_t \) for \( y = (y_0, ..., y_{N-1}) \in Y_{d,\nu,\mu,\nu_1,\mu_1} \) at \( t \in D_1 \cup \{d\} \), where \( D_1 \) is a subset of \( D \). In this case, equation (1) can be supplemented with equations

\[
y_t = \frac{1}{\sqrt{N}} \left[ y_0 + \sum_{k=1}^{N-1} e^{i2\pi tk/N} \xi_k X_k \right], \quad t \in D_1.
\]

This system has a unique solution, since even a single equation with \( t = d \) has a unique solution. Therefore, any solution of this system (for example, obtained via minimization of \( \|Y\|_{\ell_1} \) as in the compressive sensing approach) ensures an error-free recovery of the underlying process.

Similar reasoning can be applied in the setting of Theorem 3 that, in the setting of the proof of this theorem, there are available observations of \( Y_\omega \) for \( y \in \hat{Y}_{d,\nu,\mu,\nu_1,\mu_1} \) and \( Y = (Y_0, ..., Y_{N-1}) = F y \) at \( \omega \in D \), where \( D \) is a subset of \( S \) such that \( d \in D \). In this case, equation (2) can be replaced by a system of equations

\[
Y_\omega = \frac{1}{\sqrt{N}} \left[ y_0 + \sum_{k=1}^{N-1} e^{-i2\pi \omega k/N} y_k \right], \quad \omega \in D
\]

for an unknown vector \( y = \{y_k\} \in X \) such that \( \mu_k = \zeta_k x_k \). We know that this system has a unique solution, since even a single equation with \( \omega = d \) has a unique solution. Therefore, any solution of this system (for example, obtained via minimization of \( \|y\|_{\ell_1} \) as in the compressive sensing approach) ensures an error-free recovery of the underlying process.

If the sets \( \hat{G} \) in Remarks 1 and 2 are bounded, then the sets \( \hat{Y}_{d,\nu,\mu,\nu_1,\mu_1} \cap \hat{G} \) and \( \hat{Y}_{d,\nu,\mu,\nu_1,\mu_1} \cap \hat{G} \) are finite. In this case, for certain range of \( N, G, \) and \( \hat{G} \), the solution of equations and (4) can be obtained with a brute-force search. Ever-growing available computational power will allow larger and larger \( N, G, \) and \( \hat{G} \).

Similar reasoning can be applied in the setting of Theorem 4.

4.2 Robustness with respect to rounding for sparse signals with additional observations

In this section, we consider a setting where, for a given \( x \in X \), we observe some components of \( X = Qx \).
Definition 4. Let a subset \( U \) of \( D \) of cardinality \(|U|\) and a subset \( Y \) of \( X \) be given such that \( U \) is a uniqueness set in the frequency domain with respect to \( Y \) in the sense of Definition 2. Let \( A : C^{|U|} \to C^N \) be a mapping such that \( A(X|_U) = x \), where \( X = Qx \), i.e. this mapping represents a recovery algorithm of \( x \) from \( X|_U \). We say that this algorithm is robust with respect to data rounding if, for any \( \delta > 0 \), there exists \( \bar{\mu} = \bar{\mu}(\delta, N, U) > 0 \) such that \( |\hat{x}_{\nu,\mu} - x| \leq \delta \) for any \( \mu \geq \bar{\mu} \) and any \( x \in Y \) such that \(|x| \leq 1\). Here \( \hat{x}_{\nu,\mu} = A(X_{\nu,\mu}|_U) \), where \( X_{\nu,\mu} = Q(R_{\nu,\mu}(x)) \).

In the definition above, the estimate \( \hat{x}_{\nu,\mu} \) of \( x \) is obtained as the output of the corresponding algorithm with the rounded input process.

For an integer \( S \in \{1, ..., N\} \), let \( X_S \) be set of all \( x \in X \) with no more than \( S \) non-zero components.

Theorem 5. For any \( \varepsilon > 0 \), there exists a set \( Y_\varepsilon \) such that the following holds.

(i) The set \( Y_\varepsilon \) is \( \varepsilon \)-dense in \( X \).

(ii) The set \( U = \{0, 1, ..., S - 1\} \) is a uniqueness set with respect to \( Y_\varepsilon \cap X_S \).

(iii) There exists an algorithm of recovery \( x \in Y_\varepsilon \cap X_S \) from \( X|_U \) that is robust with respect to data rounding in the sense of Definition 4.

Proof of Theorem 5. For \( a \in \mathbb{R} \), \( k \in \mathbb{Z} \), let function \( p_{\nu,k}(a) : \mathbb{R} \to \{0, 1, ..., \nu\} \) be defined as the corresponding term in the representation

\[
a = \sum_{k=-\infty}^{\infty} p_{\nu,k}(a) \nu^{-k}.
\]

We extend this function on complex numbers such that

\[
p_{\nu,k}(z) = p_{\nu,k}(\text{Re } z) + ip_{\nu,k}(\text{Im } z), \quad z \in \mathbb{C}.
\]

For \( z \in \mathbb{C} \) and integers \( k \in \{0, 1, ..., N - 1\} \), \( M > 0 \), and \( \nu > 0 \), let \( \zeta_{\nu,M,k}(z) \in \mathbb{C} \) be defined such that

\[
\zeta_{\nu,M,k}(z) = \rho_{\nu,M}(z) + \nu^{-M-k} \mathbb{I}_{z \neq 0}.
\] (4)

By the definition,

\[
R_{\nu,M}(\zeta_{\nu,M,k}(z)) = R_{\nu,M}(z), \quad p_{\nu,M+k}(\zeta_{\nu,M,k}(z)) = \mathbb{I}_{z \neq 0},
\]

\[
p_{\nu,M+m}(\zeta_{\nu,M,k}(z)) = 0, \quad m \in \mathbb{Z}, \quad m > 0, \quad m \neq k.
\]
Let $\mathcal{Y}_{\nu, M}$ be the set of all $x = (x_0, x_1, ..., x_{N-1}) \in \mathcal{X}$ such that $x_k = \zeta_{\nu, M, k}(x_k)$ for $k = 0, 1, ..., N-1$.

It can be noted that the class $\mathcal{Y}_{\nu, M}$ is defined by restrictions of the rounding type on the range of its members.

Let us show that, for any integer $\nu \geq 2$, the sets $\mathcal{Y}_{\nu, M}$ are such as required in the theorem statement.

Clearly, for any $\varepsilon > 0$ and $\nu \geq 2$, there exist large enough $M$, such that the set $\mathcal{Y}_{\nu, M}$ is $\varepsilon$-dense in $\mathcal{X}$. Hence the required property (i) holds.

Let $\hat{\mathcal{Y}}_{\nu, M, S}$ we the set of all vectors $(X_0, ..., X_S)$ such that there exists $x = (x_0, ..., x_{N-1}) \in \mathcal{Y}_{\nu, M} \cap \mathcal{X}_S$ such that

$$X_\omega = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{-i2\pi\omega k/N} x_k, \quad \omega \in U, \quad \text{subject to} \quad x \in \mathcal{Y}_{\nu, M} \cap \mathcal{X}_S. \quad (5)$$

In particular,

$$X_0 = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x_k. \quad (6)$$

By the definitions, it follows that

$$p_{\nu, M+k}(X_0) = I_{\{x_k \neq 0\}}. \quad (7)$$

Let us show that, for any $\nu$ and $M$, and any $(X_0, ..., X_S) \in \hat{\mathcal{Y}}_{\nu, M, S}$, system (5) has a unique solution $x \in \mathcal{Y}_{\nu, M, S} \cap \mathcal{X}_S$.

Let $K(x) = \{k_1, ..., k_S\} \subset D$ be a set such that supp $x \subset K(x)$. For certainty, we presume that this set is formed from minimal possible numbers. Since $p_{\nu, M+k}(X_0) = 0$ for $k \notin K(x)$, it follows that the sets of solution for system (5) is the same as for the system

$$X_\omega = \frac{1}{\sqrt{N}} \sum_{k \in K(x)} e^{-i2\pi\omega k/N} x_k, \quad \omega \in U, \quad \text{subject to} \quad x \in \mathcal{Y}_{\nu, M} \cap \mathcal{X}_S. \quad (8)$$

Let $Q_{K(x)} \in \mathcal{C}^{S \times S}$ be the matrix of this system. Similarly to the proof of Theorem (ii), where we use now $M = S$ and $u = 0$, we obtain that this is a Vandermonde matrix. Hence the system has a unique solution which is also an exact solution of the problem of recovery of $x$ from observations $(X_0, X_1, ..., X_{S-1})$. Hence the required property (ii) holds.

Furthermore, let us prove that the property (iii) holds for the recovery algorithm consisting of calculation of $K(x)$ and consequent solution of system (8) as described in the proof above.

Let us observe first that $K(x) = K(R_{\nu, \bar{\mu}}(x))$ for $\bar{\mu} = M + 2^N$ for any $M > 0$ and $x \in \mathcal{X}_{\nu, M} \cap \mathcal{X}_S$. 

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Furthermore, \( \sup_{x \in \mathcal{X}} \| Q_{K(x)}^{-1} \| < +\infty \) since there exists a finite number of possible choices for \( K(\cdot) \). Here \( \| \cdot \| \) is the Frobenius matrix norm.

Clearly, one can select large enough integer \( \mu > 0 \) such that

\[
| R_{\nu,\mu}(x) - x | \leq \delta \sup_{x \in \mathcal{X}} \| Q_{K(x)}^{-1} \| \forall x \in \mathcal{X} : |x| \leq 1.
\]

If, in addition, \( \mu \geq \bar{\mu} \), then \( K(R_{\nu,\mu}(x)) = K(x) \) and

\[
| Q_{K(x)}^{-1} R_{\nu,\mu}(x) - Q_{K(x)}^{-1} x | \leq \delta.
\]

Hence the required property (iii) holds. This completes the proof of Theorem 5. \( \square \).

**Remark 4.** If \( N \) is a prime number, then, by Chebotarev Lemma (see, e.g. [23]), the matrix \( Q_{K(x)} \) of system (3) is non-degenerate for any choice of \( U = \{\omega_1, \ldots, \omega_S\} \subset D \). In this case, Theorem 5 can be extended on the case of \( U = \{\omega_1, \ldots, \omega_S\} \subset D \) such that \( \omega_1 = 0 \) and with arbitrarily selected \( \{\omega_j\}_{j>1} \).

**Remark 5.** As can be seen from the proof, recovery of the set \( K(x) \) requires quite precise representation of \( X_0 = \sum_{k=0}^{N-1} x_k \), which is numerically challenging for large \( N \). The robustness of recovery established in the theorem takes effect for quite large \( \mu \) only.

## 5 Discussion and future research

Traditionally, possibilities of data recovery and extrapolation are associated with spectrum degeneracy such as bandlimitiness, the presence of spectrum gaps, and data sparsity. Theorems 1, 3-4 suggest to explore restraints on the process range or process spectrum. These theorems establish that there are \( \varepsilon \)-dense sets of sequences that are uniquely defined by a single measurement. The corresponding ranges are defined by a special type of discretization that involves adjustment using \( \xi_k \) or \( \zeta_k \). Sparsity, bandlimiteness, or presence spectrum gaps, are not required for this.

Theorems 1-4 do not lead to an efficient numerical algorithm. Formally, these theorems and their proof imply a data compression and consequent recovery procedure. For example, Theorem 3 implies the following procedure: (i) a sequence \( x \in \mathcal{X} \) can be approximated by some close enough \( y \in \hat{\mathcal{Y}}_{d,\nu,\mu,\nu_1,\mu_1} \); (ii) this \( y \) can be recovered via rational solutions \( \{x_k\} \) of equations (2) respectively which are versions of Diophantine equation. Currently, it is unclear what kind of computational power would be sufficient to solve these equations what quantity of information is required to code a ”rounded” version \( x_d \) for a given class \( \mathcal{Y}_\nu \). This problem is beyond the scope of this paper; review of some related methods and some references can be found, e.g., in [8, 22].
It appears that some robustness with respect to rounding can be achieved in a setting with
sequences with rounded components under additional restrictions on their sparsity and with addi-
tional observations of Fourier coefficients (Theorem 5). In this setting, a different kind of rounding
was used, comparing with Theorems 1-4. However, the recovery would require precise summation
that could be computationally expensive for large N.

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