Design and Simulation of Meshing Performance of Modified Straight Bevel Gears

Mingzhang Chen 1, Xiaoshuang Xiong 2,* and Wuhao Zhuang 1

1 Hubei Key Laboratory of Advanced Technology for Automotive Components, Wuhan University of Technology, Wuhan 430070, China; chenmingzhang@whut.edu.cn (M.C.); zhuangwuhao@whut.edu.cn (W.Z.)
2 Hubei Key Laboratory of Digital Textile Equipment, Wuhan Textile University, Wuhan 430200, China
* Correspondence: xuxiong@wtu.edu.cn; Tel.: +86-15927687697

Abstract: As key components to transmit power and motion between intersecting shafts, it is necessary to design feasible tooth axial modification to improve the meshing performance and bearing capacity of straight bevel gears. The main purpose of this paper is to propose an effective axial modification method of straight bevel gears considering alignment errors. In this paper, the meshing performance of two kinds of tooth axial modification method (tooth end relief and symmetric crowned modification) for straight bevel gears is investigated by the finite element analysis (FEA). The results show that the tooth end relief is an optimal method to enhance the meshing performance of gears in different installations for decreasing transmission errors, reducing maximum contact stress and bending stress and improving the distribution of contact stress and bending stress. This research provides a suitable tooth end relief method of straight bevel gear with alignment errors.

Keywords: alignment error; contact stress and bending stress; straight bevel gear; symmetric crowned modification; tooth end relief; transmission error

1. Introduction

As key components to transmit power and motion between intersecting shafts, straight bevel gears are extensively applied in automobiles and mechanisms for the advantages such as smooth transmission, high load-carrying capacity and low noise. In practical straight bevel gear drives, the non-uniform deformation of tooth and alignment errors of gear pairs always exist. However, non-uniform deformation of tooth and alignment errors of gear pairs directly affect the meshing performance and transmission quality of straight bevel gear drives, which may decrease the transmission stability and load uniformity of gear drives greatly and increase the vibration and noise of gear drives dramatically. Therefore, it is of great importance to put forward an optimal design of straight bevel gears to enhance the stability and safety of the gear transmission in different alignment errors.

In recent years, much research on gear tooth meshing performance and gear tooth modification have been carried out. In gear tooth meshing performance aspects, Xie et al. [1] combined the theoretical calculation and finite element analysis to investigate spur gears meshing performance. Ouyang et al. [2] established a new tribo-dynamic model of the spur gear pair to predict the meshing performance. Wang and Hua [3] obtained meshing performance of non-modification spiral bevel gear pairs with alignment errors by finite element analysis (FEA). Lin [4] studied effect of machining errors and alignment errors on the meshing performance by FEA. Litvin et al. [5] investigated the meshing performance of loaded spiral bevel gears with alignment errors by FEA. Zhu et al. [6] studied the influence of shaft alignment errors on loaded tooth contact in crossed gears. The paper...
published by Chen and Tang [7] presented the contact path and transmission errors of spur gears with single alignment error and coupling alignment errors.

In gear tooth modification aspects, Zhang et al. [8] provided an optimal isometric modification of involute straight bevel gears through dynamic contact FEA. Liu et al. [9] studied the relation between mesh characteristics and tooth modifications in gear pairs. Motahar et al. [10] proposed a profile modification method of bevel gears for optimizing meshing performance. Cao [11] applied the TCA (tooth contact analysis) to analyze the contact path of modification straight bevel gears in different alignment errors. Using TCA, the influence of tooth modification on tooth contact in spiral bevel gears is investigated by Simon [12] and Zhang [13], who proposed the tooth end relief of double circular-arc helical gears for reducing the transmission errors. Samani et al. [14] put forward a novel tooth surface modification method to decrease the transmission errors. Chen [15] discussed two axial modification methods of loaded straight bevel gears with the deformation of the bearing axle.

However, the approach for the optimal axial modification of straight bevel gears in alignment errors has not been provided. Therefore, it is necessary to find an effective axial modification method of straight bevel gears in alignment errors. In this paper, an optimal tooth end relief of straight bevel gears is proposed and the accurate tooth end relief tooth surface equations are built. Furthermore, the meshing performance of modification straight bevel gears such as contacting area, contact stress, bending stress and transmission errors are investigated.

2. Theoretical Background

As shown in Figure 1, two bevel gear shape planing tools are applied for the generation of straight bevel gears. The planing tool 2 is applied for the generation of left-hand profile of the straight bevel gear. The planing tool 2’ is applied for the generation of right-hand profile of the straight bevel gear. The blade of the two bevel gear shape cutters is of a straight line profile, which is perpendicular to line OO₀. The angle between two bevel gear planing tools 2 and 2’ is defined as θ. And the angle θ is described by the Equation (1).

\[ \theta = \frac{360}{z} \]  

(1)

where z is the number of teeth.

Fixed coordinate systems S₀, Sᵣ, S₂ are rigidly connected to the cutting machine. Movable coordinate systems Sₜ and S₁ are rigidly connected to the planing tool 2 and gear, respectively. They are rotated about the Z₀-axis and Z₂-axis, respectively.

![Figure 1. Concept of straight bevel gear planning.](image)
The angular velocity of gear rotated about the Z-axis is defined as \( \omega_1 \) and the angular velocity of planing tool 2 rotated about the Z-axis is defined as \( \omega_2 \). The ratio of instantaneous angular velocities of the gear and planing tool 2 is defined as \( i = \omega_1/\omega_2 = 1/\sin \alpha \). In the cutting process, the mesh between point C at the base circle of planing tool 2 with the gear generates the spherical involute profile of the gear. The other point, except point C, of the straight line profile of planing tool 2 does not generate the gear profile, although it takes part in the cutting. The mathematical generation of gear teeth by the planing tool 2 is described by the following Equation (2) [16].

\[
\begin{align*}
\overrightarrow{r}_i &= M_u \cdot M_w \cdot M'_w \cdot M_g \cdot \overrightarrow{r}_g \\
&= \begin{bmatrix}
\cos \varphi_i & -\sin \varphi_i & 0 & 0 & 0 & 1 \\
\sin \varphi_i & \cos \varphi_i & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\cos \alpha & 0 & -\sin \alpha & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
\sin \alpha & 0 & \cos \alpha & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\cos \varphi_g & \sin \varphi_g & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
\sin \varphi_g & -\cos \varphi_g & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
r \cos \varphi_g \sin \alpha + \sin \varphi_g \sin \varphi_i \\
r \cos \varphi_g \sin \alpha \sin \varphi_i - \sin \varphi_g \cos \varphi_i \\
r \cos \varphi_g \cos \alpha
\end{bmatrix}
\end{align*}
\]

(2)

where \( \overrightarrow{r}_i \) is the radius vector of gear tooth surface point, \( \overrightarrow{r}_g \) is the radius vector of point C at the base circle of gear planing tool 2. Matrices \( M_i \) provide the coordinate transformations from the movable coordinate system \( S_i \) attached to the planing tool 2 to the movable coordinate system \( S_i \) attached to the gear. According to the ratio of instantaneous angular velocities of the gear and planing tool 2, the relation between \( \varphi_g \) and \( \varphi_i \) can be described by the Equation (3).

\[
\varphi_g = \varphi_i \sin \alpha
\]

(3)

In the practical cutting of the gear, the angle \( \alpha \) is equal to the angle of base cone. The rotational angle \( \varphi_g \) must be larger than the critical angle \( \varphi_c \). The critical angle \( \varphi_c \) is represented by the Equation (4).

\[
\varphi_c = \cos^{-1}\left(\cos \theta_a / \cos \theta_b \right) - \cos^{-1}\left(\cos \theta_c / \cos \theta_b \right)
\]

(4)

where \( \theta_a \) and \( \theta_b \) are the angle of tip cone and base cone, respectively, and \( \theta_c \) is the angle of cutting beginning cone.

Combined the Equations (1)–(4), the assembling models which are illustrated in Figure 2 are accomplished in 3D modeling software [17]. The basic geometrical data of the sample straight bevel gear pair are shown in Table 1.

**Table 1.** Basic geometric data of the example straight bevel gear pair.

| Parameter                        | Jiangsu Pacific Gear Transmission Co. LTD, Taizhou, China | Driving Pinion | Driven Gear |
|----------------------------------|---------------------------------------------------------|----------------|-------------|
| Module (mm)                      | \( m_u \)                                               | 3.7792         | 3.7792      |
| Shaft angle (deg)                | \( \Sigma \)                                            | 90             | 90          |
| Number of teeth                  | \( z \)                                                 | 10             | 14          |
| Pressure angle (deg)             | \( \alpha \)                                            | 22.5           | 22.5        |
| Modification coefficient of height | \( x \)                                              | -0.1812        | 0.1812      |
| Shearing modification coefficient | \( x_0 \)                                              | 0.05           | -0.05       |
3. Establishment of Finite Element Models

Finite Element Analysis (FEA) simulates the real physical system (geometry and loading condition) by using a mathematical approximation method. In addition, simple and interacting elements are applied to approximate the real system of infinite unknowns. FEA considers the solution domain to be composed of many small interconnected subdomains called finite elements, assumes an appropriate (simpler) approximate solution for each element, and then deduces the solution to the problem by solving the general conditions (such as the equilibrium conditions of the structure) of this domain. This solution is not an exact solution, but an approximate solution, because the actual problem is replaced by a simpler one. Because it is difficult to get an accurate solution for most practical problems, the finite element method not only has high precision, but also can adapt to various complex shapes, therefore FEA becomes an effective means of engineering analysis.

The specific processes to create the finite element model of straight bevel gear pair are shown as follows:

(1) Based material property and load parameter. The finite element model of a straight bevel gear pair is performed with Solid185 (8-node isoparametric hexahedron element). The material is 40Cr steel with the properties of Young’s modulus \( E = 2.1 \times 10^5 \) MPa and Poisson’s ratio \( \mu = 0.3 \). The friction coefficient between two gears is 0.1 and the applied torque \( T = 50 \text{ Nm} \); and

(2) Apply load and boundary conditions. Nodes on the bottom rim of the driven gear are fixed. Moreover, the radial and axial degrees of freedom of nodes on the bottom rim of the driving pinion are fixed and only the rational degree of freedom is set as free and coupled. Furthermore, the torque \( (T = 50 \text{ N-m}) \) is applied on the nodes of the bottom rim of the driving pinion. The established finite element model of straight bevel gear pair established by software ANSYS is shown in Figure 3.
4. Analysis of the Circumferential Displacements of Tooth

The tooth load and the circumferential displacement of tooth change greatly when the tooth pairs enter in mesh or exit out mesh. This can cause great vibrations and noise. Four critical transferring positions $P_1$, $P_2$, $P_3$ and $P_4$ (Figure 4) in the meshing cycle are chosen for investigating the tooth circumferential displacement differences in the longitudinal direction.

![Figure 4. Illustration of gear mesh.](image)

The circumferential displacement difference can be calculated by Equation (5) [13].

\[
\Delta \delta_i = |\delta_{im} - \delta_{iu}|
\]  

(5)

where subscript $i$ represents the critical position $P_1$, $P_2$, $P_3$ and $P_4$; $\Delta \delta_i$ is the circumferential displacement difference at the critical position; $\delta_{im}$ is the circumferential displacement of meshing tooth along the contact line at the critical position. $\delta_{iu}$ is the circumferential displacement of the tooth without mesh along the contact line at the corresponding position.

When $T = 50$ Nm, the circumferential displacement difference curves at critical positions of the driving pinion are shown in Figure 5.

As shown in Figure 5, the circumferential displacement differences increase gradually from the toe of the tooth to the heel of the tooth and reach the maximum at the heel of the tooth. It can be also seen that the circumferential displacement differences reach the maximum at the critical position $P_3$ which is the final point of single pair contact. In other words, the comprehensive deformation of the tooth is biggest at $P_3$ because one tooth bears all load and the bending torque is great. These circumferential displacement difference curves provide a significant basis for modification.

![Figure 5. The circumferential displacement difference curves at positions $P_1$, $P_2$, $P_3$ and $P_4$.](image)
5. Determination of the Axial Modification of the Tooth.

As mentioned above, the circumferential displacement difference at the heel of the tooth is larger than the other parts of the tooth. The contact stress is both higher at the toe and heel of the tooth. So the tooth ends are damaged easily. Therefore, in order to improve the meshing performance of gear drives, the teeth of the driving pinion need axial modification to lower the stress concentration at the tooth end. In this paper, two kinds of modification methods of symmetric crowned modification and tooth end relief are used to design teeth. Moreover, an optimal tooth modification method is proposed by comprehensively analyzing the meshing performance of gear drives in different alignment errors.

The key symmetric crowned modification parameter is the modification dimension $\Delta T$. The modification parameters of tooth end relief contain the modification dimension $\Delta T$ and the relief length $\Delta L$ (Figure 6). As shown in Figure 5, it can be seen that the maximal circumferential displacement difference of the toe and the heel of the tooth is 10.2 and 14.12 $\mu$m, respectively. Figure 7 presents the distribution of contact stress along the direction of the tooth width. It shows that the contact stress increases sharply at the tooth end and the length of the contact stress increasing sharply area at the tooth end is about one tenth of the tooth width.

![Figure 6](image)

**Figure 6.** (a) Symmetric crowned modification; (b) tooth end relief.

![Figure 7](image)

**Figure 7.** The distribution of contact stress along the direction of tooth width.

In the paper, the relief lengths selected are equal to the one tenth of the tooth width, namely, $\Delta L_1 = \Delta L_2 = 0.1b = 0.975$ mm. Modification parameters are presented in Table 2. After the modification parameters are determined, the modification gear tooth surface
equation can be derived based on the modification parameters and the geometric relationship.

Table 2. Modification method and parameter.

| Jiangsu Pacific Gear Transmission Co. LTD, Taizhou, China | Modification Method | Modification Parameter |
|----------------------------------------------------------|---------------------|-----------------------|
| Case1 | Symmetric crowned modification | $\Delta T = 11 \mu m$ |
| Case2 | Symmetric crowned modification | $\Delta T = 15 \mu m$ |
| Case3 | Tooth end relief | $\Delta T_1 = \Delta T_2 = \Delta 11 \mu m$ |
| Case4 | Tooth end relief | $\Delta T_1 = 11 \mu m$, $\Delta T_2 = \Delta 15 \mu m$ |

5.1. Derivation of Symmetric Crowned Modification Gear Tooth Surface

The arc radius of symmetric crowned modification can be calculated by the following Equation (6) [13]:

$$R_c = \frac{b^2}{8\Delta T}$$

The modification arc is represented by the following Equation (7) [13]:

$$\begin{align*}
y(r, R_c) &= R_c - \sqrt{R_c^2 - [r - (R - \frac{1}{2}b)]^2} \\
x(r, R_c) &= r \\
R - b &\leq r \leq R
\end{align*}$$

where $R_c$ is the arc radius, $b$ is the tooth width, $\Delta T$ is the modification dimension and $R$ is the outer cone distance.

The schematic drawing of the symmetric crowned modification arc curve is presented in Figure 8. As shown in Figure 8, arc $M'M$ is the symmetric crowned modification arc curve and line $P'P$ is the unmodified curve. The line $P'M'$ is perpendicular to the line $P'O_c$ and the line $PM$ is perpendicular to line $PO$. The unit vector $\vec{n}_p$ which is parallel to the line $PM$ can be derived by following Equation (8):

$$\vec{n}_p = \begin{bmatrix}
\frac{(\sin(\beta \sin \alpha) \cos \beta - \cos(\beta \sin \alpha) \sin \alpha \cos \beta)}{\sqrt{\cos^2(\beta \sin \alpha) \sin^2 \alpha + \sin^2(\beta \sin \alpha)}} \\
\frac{(\cos(\beta \sin \alpha) \sin \alpha \cos \beta + \sin(\beta \sin \alpha) \sin \beta)}{\sqrt{\cos^2(\beta \sin \alpha) \sin^2 \alpha + \sin^2(\beta \sin \alpha)}} \\
0
\end{bmatrix}$$

So, according to the relationship between the point $P$ on the unmodified gear tooth surface and the point $M$ on the modification gear tooth surface, the modification gear tooth surface $\Sigma_1$ (Figure 6a) can be generated by the following Equation (9):

$$\begin{align*}
x &= r \cos(\beta \sin \alpha) \sin \alpha \cos \beta + r \sin(\beta \sin \alpha) \sin \beta + y(r, R_c) x_{r'p} \\
y &= r \cos(\beta \sin \alpha) \sin \alpha \sin \beta - r \sin(\beta \sin \alpha) \cos \beta + y(r, R_c) y_{r'p} \\
z &= r \cos(\beta \sin \alpha) \cos \alpha
\end{align*}$$
5.2. Derivation of Tooth End Relief Gear Tooth Surface

The schematic drawing of the tooth end relief is shown as Figure 9. As shown in Figure 9b, the arc radius of modification curves $G'N'$ and $GN$ can be calculated by the following Equation (10) [13]:

$$R_i = \frac{\Delta L_i^2}{2\Delta T_i}$$  \hspace{1cm} (10)

The modification arc $G'N'$ is represented by the following Equation (11) [13]:

$$\begin{align*}
y_i(r, R_i) &= R_i - \frac{\sqrt{R_i^2 - [r - (R - b + \Delta L_i)]^2}}{r - (R - b + \Delta L_i)} \\
x_i(r, R_i) &= r \\
R - b \leq r \leq R - b + \Delta L_i
\end{align*}$$  \hspace{1cm} (11)

The modification arc $GN$ is represented by the following Equation (12) [13]:

$$\begin{align*}
y_2(r, R_{i2}) &= R_{i2} - \frac{\sqrt{R_{i2}^2 - [r - (R - \Delta L_{i2})]^2}}{r - (R - \Delta L_{i2})} \\
x_2(r, R_{i2}) &= r \\
R - \Delta L_{i2} \leq r \leq R
\end{align*}$$  \hspace{1cm} (12)

where subscript $i$ represents the arc curve $G'N'$ and $GN$, $R_i$ is the arc radius, $b$ is the tooth width, $\Delta T_i$ is the modification dimension, $\Delta L_i$ is the relief length and $R$ is the outer cone distance.

Like the symmetric crowned modification, the unit vector $\hat{n}_p$ which is parallel to the line $PG$ can be also calculated by the Equation (9). The modification gear tooth surface $\Sigma_1$ and $\Sigma_2$ (Figure 9b) can be generated by the following Equation (13):

$$\begin{align*}
x &= r \cos(\beta \sin \alpha) \sin \alpha \cos \beta + r \sin(\beta \sin \alpha) \sin \beta + y_i(r, R_i) \frac{y_{i-}}{y_{i-}} \\
y &= r \cos(\beta \sin \alpha) \sin \alpha \sin \beta - r \sin(\beta \sin \alpha) \cos \beta + y_i(r, R_i) \frac{y_{i-}}{y_{i-}} \\
z &= r \cos(\beta \sin \alpha) \cos \alpha
\end{align*}$$  \hspace{1cm} (13)
where subscript $i = 1, 2$.

![Diagram](image)

**Figure 9.** Schematic illustration of the tooth end relief (a) modification principle (b) modification curves $G'N'$ and $GN$.

### 6. Results and Discussions

In practical straight bevel gear drives, alignment errors always exist and directly affect the meshing performance and transmission quality of straight bevel gear drives, which may decrease the transmission stability and load uniformity of gear drives greatly and increase the vibration and noise of gear drives dramatically. In this paper, the meshing performance of modified gear drives both in standard installment and with alignment errors has been studied to evaluate the tooth modification effect. We choose the alignment errors in Table 3 [17].

**Table 3.** Values of alignment errors.

| Names of Parameters                              | Value  |
|-------------------------------------------------|--------|
| Axial displacement of pinion, $\Delta P$ (mm)   | 0.1    |
| Axial displacement of gear, $\Delta G$ (mm)     | 0.1    |
| Change of center distance, $\Delta E$ (mm)      | $-0.02, 0.02$ |
| Change of crossing angle, $\Delta \gamma$ (deg) | 1      |

#### 6.1. Influence of Modification on the Tooth Contact Area

As usually known, the modification of the tooth can directly affect the contact area, and then affect the value and distribution of contact stress. For this reason, it is necessary to analyze the contact area and contact stress to evaluate the modification of the tooth. In this paper, the contact area of meshing tooth pair at the pitch cone has been investigated. Figure 10 presents the distribution of contact stress on the tooth surface of no modification, Case4 (tooth end relief) and Case1 (symmetric crowned modification) in standard installation, respectively. As shown in Figure 10a, the contact area of no modification gear tooth without misalignment along the tooth width direction looks like a line. Furthermore, the contact stress is higher at the heel of the tooth, which is called the edge effect. It can be seen from Figure 10b that the contact area of the tooth end relief gear tooth without misalignment is linear and is at the unmodified area of the tooth. Furthermore, the contact stress reaches a maximum at the modification start point near the heel of the tooth. As shown in Figure 10c, the contact ellipse area of the symmetric crowned modification gear tooth without misalignment is at the middle of the tooth.
Figure 10. Distribution of contact stress on tooth surface for pinion (left) and the gear (right) in standard installment. (a) No modification, (b) Case4 and (c) Case1.

Figures 11 and 12 show the distribution of contact stress on the tooth surface of no modification, Case4 and Case1, when $\Delta P = 0.1$ and when $\Delta G = 0.1$, respectively. As shown in Figures 11a and 12a, the gear pair of no modification appears to end contact when the local stress concentration at the heel of the tooth when $\Delta P = 0.1$ or $\Delta G = 0.1$. It can be seen from Figures 11b and 12b that the contact area of tooth end relief gear tooth, when $\Delta P = 0.1$ or $\Delta G = 0.1$, is at the modification start point near the heel of the tooth. As presented in Figures 11c and 12c, the contact ellipse area of the symmetric crowned modification gear tooth, when $\Delta P = 0.1$ or $\Delta G = 0.1$, moves a little to the heel of the tooth from the middle of the tooth.
Figure 11. Distribution of contact stress on tooth surface for pinion (left) and the gear (right) when $\Delta P = 0.1$. (a) No modification, (b) Case4 and (c) Case1.
Figure 12. Distribution of contact stress on the tooth surface for pinion (left) and the gear (right) when $\Delta G = 0.1$. (a) No modification, (b) Case4 and (c) Case1.

Figures 13 and 14 show the distribution of contact stress on the tooth surface of no modification, Case4 and Case1, when $\Delta E = 0.02$ and $\Delta E = -0.02$, respectively. As shown in Figures 13a and 14a, the gear pair of no modification appears to end contact and local stress at the toe and the heel of the tooth, when $\Delta E = 0.02$ or $\Delta E = -0.02$, respectively. It can be seen from Figure 13b that the contact area of the tooth end relief gear tooth when $\Delta E = 0.02$ is at the modification start point near the toe of the tooth. However, the contact area of the tooth end relief gear tooth when $\Delta E = -0.02$ is at the modification start point near the heel of the tooth, as shown in Figure 14b. As shown in Figure 13c, the contact ellipse area of the symmetric crowned modification gear tooth moves a little to the toe of the tooth from the middle of the tooth, when $\Delta E = 0.02$. However, as presented in Figure 14c, the contact ellipse area of the symmetric crowned modification gear tooth moves a little to the heel of the tooth from the middle of the tooth, when $\Delta E = -0.02$. 
Figure 13. Distribution of contact stress on the tooth surface for pinion (left) and the gear (right) when $\Delta E = 0.02$. (a) No modification, (b) Case4 and (c) Case1.
Figure 14. Distribution of contact stress on the tooth surface for pinion (left) and the gear (right) when $\Delta E = -0.02$. (a) No modification, (b) Case 4 and (c) Case 1.

Figure 15 presents the distribution of contact stress on the tooth surface of no modification, Case 4 and Case 1, when $\Delta y = 1$. As shown in Figure 15, the contact areas of the tooth pair of no modification, tooth end relief and symmetric crowned modification are basically the same as that in the standard installation.
As mentioned above, the two modification methods can all avoid edge effect caused by shearing action. By comparison of the contact area of no modification, tooth end relief and symmetric crowned modification, it can be found that both the two modification methods can improve the contact area and increase the bearing capacity of the gear pair.

6.2. Influence of Modification on the Tooth Contact Stress

According to the study by Deng [18], the fatigue failure of the driving pinion including contact and bending fatigue is the main fatigue failure for the straight bevel gear pair in engineering machinery. So, in this paper, it is necessary to pay more attention to the contact stress and bending stress for the modification driving opinion. In Table 4, we list the average values of the contact stress of no modification, tooth end relief and symmetric crowned modification in the single pair contact zone in different installment conditions. Figure 16a illustrates the variation of contact stress in a whole cycle for the unmodified gears in different installment conditions. It can be seen from Figure 16a that the alignment
errors $\Delta P$, $\Delta G$ and $\Delta E$ have great effect on the contact stress for the unmodified gears and the alignment error $\Delta \gamma$ has little effect on the contact stress compared with standard installment. Furthermore, the alignment errors increase the maximal contact stress greatly. This is because the alignment errors $\Delta P$, $\Delta G$ and $\Delta E$ result in the end contact of gear pairs.

Figure 16b shows the variation of contact stress in a whole cycle for the tooth end relief gears in different installment conditions. As shown in Figure 16b, the maximal contact stress in all kinds of installment condition reduces greatly by the application of tooth end relief. Moreover, the contact stress in the single pair contact zone for the tooth end relief gears with alignment errors also reduces drastically which can be also seen from Table 4.

Table 4. Average values of the contact stress for the driving in single contact zone.

| Jiangsu Pacific Gear Transmission Co. LTD, Taizhou, China | Without Misalignment/MPa | $\Delta P = 0.1$ /MPa | $\Delta G = 0.1$ /MPa | $\Delta E = -0.02$ /MPa | $\Delta E = 0.02$ /MPa | $\Delta \gamma = 1$ /MPa |
|----------------------------------------------------------|--------------------------|-----------------------|------------------------|------------------------|------------------------|------------------------|
| Unmodified gear                                          | 594.4                    | 698.3                 | 849.5                  | 705.4                  | 661.4                  | 602.4                  |
| Case1                                                    | 465.9                    | 483.9                 | 476.2                  | 471.1                  | 471.0                  | 475.3                  |
| Case2                                                    | 491.5                    | 508.0                 | 496.0                  | 491.1                  | 496.4                  | 493.8                  |
| Case3                                                    | 443.7                    | 504.4                 | 522.7                  | 489.1                  | 431.1                  | 449.3                  |
| Case4                                                    | 441.3                    | 494.9                 | 503.8                  | 482.6                  | 419.0                  | 442.0                  |

Figure 16c shows the variation of contact stress in a whole cycle for the symmetric crowned modification gears with alignment errors. As presented in Figure 16c, the contact stress in the whole cycle for the symmetric crowned modification gears with alignment errors decreases dramatically. Furthermore, it can be also found that the alignment errors have little effect on the contact stress for the symmetric crowned modification gears. By the application of symmetric crowned modification, the fluctuation of contact stress in the single pair contact zone reduces greatly and the contact stress decreases drastically too. It can be found from Table 4 that the contact stress of tooth end relief without misalignment is smaller than that of symmetric crowned modification. This mainly because the contact area of tooth end relief is larger and spread uniformly along the tooth width unmodified zone (Figure 10b).
From the above results, it can be concluded that both the tooth end relief and the symmetric crowned modification can decrease the contact stress greatly and improve the bearing capacity of gear pair. Moreover, the contact stress of tooth end relief (Case4) is smaller than the other three.

6.3. Influence of Modification on the Distribution of Tooth Contact Stress

In this paper, the contact stress distribution of meshing tooth pair at the pitch cone has been investigated. Figure 17 presents the contact stress distribution for the unmodified gear along the tooth width in different installment conditions. As shown in Figure 17, the contact stress without misalignment is larger at the tooth end, especially the heel of the tooth. The contact stress is concentrated in the heel of the tooth, when $\Delta G = 0.1$, $\Delta E = -0.02$. On the contrary, the contact stress is concentrated in the toe of tooth, when $\Delta E = 0.02$. Furthermore, the contact stress is concentrated in the position close to the heel of the tooth, when $\Delta P = 0.1$, although the contact stress declines a little at the heel of the tooth. This is because the alignment error $\Delta P = 0.1$ causes a little excursion for the contact point from the heel to the toe along the pinion tooth width. The conclusion can be obtained from Figure 17 that the alignment errors cause the serious edge contact of the unmodified gear again.

Figure 17. Contact stress distribution for the unmodified driving opinion along tooth width in different installment conditions.

Figure 18 shows the contact stress distribution for the tooth end relief gear along the tooth width in different installment conditions. As shown in Figure 18, the contact stress of the tooth end relief gear tooth in different installment conditions is mainly concentrated
in the unmodified zone along the tooth width and is larger at the two beginning modification points. The contact stress of two beginning modification points is lower than the end contact stress of the unmodified gear. It can be also seen from Figure 18 that the contact stress of the tooth end decreases greatly through the tooth end relief.

![Figure 18](image1)

**Figure 18.** Contact stress distribution for the tooth end relief driving pinion (Case4) along the tooth width in different.

The contact stress distribution for the symmetric crowned modification gear along the tooth width in different installment conditions is presented in Figure 19. As shown in Figure 19, the contact stress in different installment conditions is mainly concentrated in the middle of the tooth width and the contact stress of the tooth end decreases greatly, through symmetric crowned modification. Furthermore, the maximum contact stress in the middle of the tooth width is lower than the end contact stress of the unmodified gear.

![Figure 19](image2)

**Figure 19.** Contact stress distribution for the symmetric crowned modification driving pinion (Case1) along the tooth width in different installment conditions.

As mentioned above, both tooth end relief and symmetric crowned modification can eliminate the edge contact of the straight bevel gear tooth caused by alignment errors.

6.4. Influence of Modification on the Distribution of Tooth Bending Stress

In this paper, the bending stress distribution of the driving pinion at the pitch cone has been investigated. Figure 20 presents the bending stress distribution for the driving pinion of no modification, tooth end relief and symmetric crowned modification in different installment conditions, respectively. It can be also seen from Figure 20a that the point of maximum bending stress is near the toe of the tooth, when \( \Delta E = 0.02, \Delta P = 1 \) or without misalignment. However, the point of maximum bending stress is near the heel of the tooth, when \( \Delta E = -0.02, \Delta P = 1 \) or \( \Delta G = 0.1 \). From the analysis above, it is concluded that
the alignment errors have a great effect on the bending stress distribution for the unmodified driving pinion. Compared to Figure 20a–c, it can be found that the point of maximum bending stress for the both three kinds tooth is near the toe of tooth, when $\Delta E = 0.02$, $\Delta \gamma = 1$ or without misalignment. However, the point of maximum bending stress of tooth end relief and symmetric crowned modification is in the middle of the tooth, when $\Delta E = -0.02$, $\Delta P = 1$ or $\Delta G = 0.1$. Moreover, the distribution of bending stress for tooth end relief and symmetric crown modification is uniform, and the bending stress for tooth end relief is smaller.

![Bending stress distribution](image)

**Figure 20.** Bending stress distribution for the driving pinion in different installment conditions—(a) no modification, (b) tooth end relief (Case 4) and (c) symmetric crowned modification (Case 1).

As mentioned above, it can be obtained that the improvement of the distribution of bending stress of tooth end relief is better than symmetric crowned modification.

6.5. Influence of Modification on the Transmission Error

One important purpose of modification is to reduce the noise and vibration of gear drives. And the noise and vibration are closely related to transmission error. So it is necessary to analyze the influence of modification on the transmission error and then evaluate the modification effect. The transmission error is the difference between the angle displacement of gears in reality and in theory. In this paper, the angle displacement of the bottom rim of the pinion is used to represent the transmission error. The angle displacement of the bottom rim of the pinion can be derived by Equation (14) [19]:

$$\Delta \phi_1 = \frac{\delta}{r_z}$$  \hspace{1cm} (14)

where: $\delta$ is the circumferential displacement of the nodes on the bottom rim of the pinion under loaded, $r_z$ is the radius of the bottom rim of the pinion. $\Delta \phi_1$ is the rotation angle of the driving pinion.

Figure 21 presents the curves of static transmission errors of the driving pinion of no modification, tooth end relief and symmetric crown modification in the whole meshing cycle in different installment conditions, respectively. The results of investigation are as follows:
(1) As shown in Figure 21a, the transmission errors of unmodified gear pairs increase greatly, when \( \Delta P = 0.1, \Delta G = 0.1, \Delta E = -0.02 \) or \( \Delta E = 0.02 \). However, the increase of transmission errors is not significant, when \( \Delta y = 1 \). From the analysis above, it can be concluded that the transmission errors of straight bevel gear pairs are sensitive to the alignment error \( \Delta P, \Delta G \) and \( \Delta E \), but not sensitive to the alignment error \( \Delta y \).

(2) Compared Figure 21a,b, it can be found that the variation of transmission errors of tooth end relief is basically the same as that of no modification in different installment conditions. Compared to Figure 21a,c, it is obtained that the variation of transmission errors of symmetric crown modification caused by alignment errors is not great; and

(3) The transmission error amplitudes of unmodified gear and different modifications under different installment conditions are presented in Figure 22. From Figure 22, it is found that the tooth end relief (Case4) is relatively optimal which can greatly reduce the vibration and noise of gear drives, and Case1 and Case3 are moderate, but Case2 is not well. From Case2, it can be found that excessive modification value will decrease the meshing stiffness, and then increase the transmission error amplitude.

![Figure 21](image1)

**Figure 21.** Variation of transmission errors of driving pinion in different installment conditions—(a) no modification, (b) tooth end relief (Case4) and (c) symmetric crowned modification (Case1).

![Figure 22](image2)

**Figure 22.** Variation of transmission error amplitudes of driving pinion under different modification in different installment conditions.
7. Conclusions

From the analysis in this article of the simulation of meshing performance of modified straight bevel gears, it can be concluded that:

(1) Both the tooth end relief and the symmetric crowned modification can avoid the edge effect caused by shearing action and alignment errors, and improve the contact area which increases the bearing capacity of gear pair. Moreover, Case4 (tooth end relief) is the best modification method for decreasing the contacting stress greatly.

(2) Both the tooth end relief and the symmetric crowned modification can reduce the sensitivity of bending stress to alignment errors. Furthermore, Case4 (tooth end relief) is the best modification method for decreasing the bending stress greatly and reducing the sensitivity of bending stress to alignment errors.

(3) Case4 (tooth end relief) is the most optimal method to decrease the transmission error amplitude and reduce the meshing impact. However, excessive modification value will decrease the meshing stiffness, and then increase the transmission error amplitude such as Case2.

(4) Results show that Case4 (tooth end relief) is the better method to avoid the edge contact, improve the contacting and bending fatigue life and reduce the vibration and noise for straight bevel gear driving.

This research contributes to improving meshing performance of straight bevel gears. Based on the investigation of this paper, the more effective modification method to improve meshing performance of straight bevel gears will be investigated in the subsequent works.

Author Contributions: M.C.: investigation, methodology, writing—original draft preparation. XX.: supervision, writing—review and editing. W.Z.: funding acquisition, resources, writing—review and editing. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the National Natural Science Foundation of China (No. 51575416), Natural Science Foundation of Hubei Province (No. 2019CFA041), Natural Science Foundation of Hubei Province (No. 2020CFB389), Independent Innovation Foundation of Wuhan University of Technology (2019IVA102), China Postdoctoral Science Foundation (2020M672429).

Institutional Review Board Statement: “Not applicable” for studies not involving humans or animals.

Informed Consent Statement: “Not applicable” for studies not involving humans or animals.

Data Availability Statement: Data sharing not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

Nomenclature

\( \theta \) Angle between two bevel gear planing tools (deg)
\( \theta_a \) Angle of tip cone (deg)
\( \theta_b \) Angle of base cone (deg)
\( \theta_c \) Angle of cutting beginning cone (deg)
\( \Delta \theta_i \) Rotation angle of driving pinion (deg)
\( z \) Number of teeth
\( r_i \) Radius vector of gear tooth surface point
\( r_r \) Radius vector of point at base circle of gear planing tool
\( r_e \) Radius of the bottom rim of the pinion (mm)
\( M_1 \) Matrix (N.m)
\( \alpha \) Pressure angle (deg)
\( \Delta \delta_i \) Circumferential displacement difference at critical position (mm)
\( \delta_m \) Circumferential displacement of meshing tooth along contact line at critical position (mm)
\( \delta_o \) Circumferential displacement without mesh along contact line (mm)
\( \delta \) Circumferential displacement of nodes on bottom rim of pinion under loaded (mm)
References

1. Xie, C. Improved analytical models for mesh stiffness and load sharing ratio of spur gears considering structure coupling effect. Mech. Syst. Signal Process. 2018, 111, 331–347.

2. Ouyang, T.; Huang, H.; Zhang, N.; Mo, C.; Chen, N. A model to predict tribo-dynamic performance of a spur gear pair. Tribol. Inter. 2017, 116, 449–459.

3. Wang, B.; Hua, L. Computerized design and FE simulation of meshing of involute spiral bevel gears with alignment errors. Adv. Mater. Res. 2011, 199–200, 386–391.

4. Lin, T.; He, Z. Analytical method for coupled transmission error of helical gear system with machining errors, assembly errors and tooth modifications. Mech. Syst. Signal Process. 2017, 91, 167–182.

5. Litvin, F.L.; Vecchiato, D.; Yukishima, K.; Azar, A.F. Reduction of noise of loaded and unloaded misaligned gear drives. Comput. Method. Appl. Mech. Eng. 2006, 195, 5523–5536.

6. Zhu, C.; Song, C.; Lim, T.C.; Peng, T. Pitch cone design and influence of misalignments on tooth contact behaviors of crossed beveloid gears. Mech. Mach. Theory 2013, 59, 48–64.

7. Chen, X.; Tang, J. Tooth contact analysis of spur face gear drives with alignment errors. In Proceedings of the 2011 Second International Conference on Digital Manufacturing and Automation, Hunan, China, 5–7 August 2011; pp. 1368–1371.

8. Zhang, F.; Tian, X.; Cui, H. The modification design of involute straight bevel gear. IERI Procedia 2012, 3, 52–59.

9. Liu, S.; Song, C.; Zhu, C.; Ni, G. Effects of tooth modifications on mesh characteristics of crossed beveloid gear pair with small shaft angle. Mech. Mach. Theory 2018, 119, 142–160.

10. Motahar, H.; Samani, F.S.; Molaei, M. Nonlinear vibration of the bevel gear with teeth profile modification. Nonlinear Dyn. 2015, 83, 1875–1884.

11. Cao, X.; Lou, J.; Ma, Z. Sensitivity analysis of installation errors of the straight bevel gear modification tooth surface. J. Mech. Trans. 2014, 38, 40–43. (In Chinese)

12. Simon, V.V. Influence of tooth modifications on tooth contact in face-hobbed spiral bevel gears. Mech. Mach. Theory 2011, 46, 1980–1998.

13. Zhang, H.; Lin, H.; Han, X. Computerized design and simulation of meshing of modified double circular-arc helical gears by tooth end relief with helix. Mech. Mach. Theory 2010, 45, 46–64.

14. Samani, F.S.; Molaei, M.; Pellicano, F. Nonlinear vibration of the spiral bevel gear with a novel tooth surface modification method. Meccanica 2019, 54, 1071–1081.

15. Chen, X. The Modification Methods of Straight Bevel Gear. Ph.D. Thesis, Hua Zhong University of Science and Technology, Wuhan, China, 2006; pp. 73–90. (In Chinese)

16. Shunmugam, M.S.; Narayana, S.V.R.S.; Jayarakash, V. Establishing gear tooth surface geometry and normal deviation. Mech. Mach. Theory 1998, 33, 525–534.

17. Han, X.; Hua, L.; Deng, S.; Luo, Q. Influence of alignment errors on contact pressure during straight bevel gear meshing process. Chin. J. Mech. Eng. 2015, 28, 1089–1099.

18. Deng, S.; Hua, L.; Han, X.; Hunag, S. Finite element analysis of contact fatigue and bending fatigue of a theoretical assembling straight bevel gear pair. J. Cent. South. Univ. 2013, 20, 279–292.

19. Shutting, L. Effect of addendum on contact strength bending strength and basic performance parameters of a pair of spur gears. Mech. Mach. Theory 2008, 13, 1543–1556.