Angular ordering is an important feature of perturbative QCD with a deep theoretical origin and many phenomenological consequences. It is the result of destructive interference: outside angular ordered regions amplitudes involving soft gluons cancel. This property is quite general, and it is present in both time-like processes, such as $e^+e^-$ annihilation, and in space-like processes, such as deep inelastic scattering (DIS).

In DIS, angular ordering is essential for describing the structure of the final state, but not for the gluon density at small $x$. This is because in the resummation of singular terms of the gluon density, there is a cancellation between the real and virtual contributions. As a result, to leading order the small-$x$ gluon density is obtained by resumming $\ln x$ powers coming only from IR singularities, and angular ordering contributes only to subleading corrections.

In this talk, as a first step of a systematic study of multi-parton emission in DIS, the effect of angular ordering on the small-$x$ evolution of the gluon structure function is studied with both analytical and numerical techniques.

The detailed analysis of angular ordering in multi-parton emission at small $x$ and in the related virtual corrections shows that to leading order the initial-state gluon emission can be formulated as a branching process (Fig. 1) in which angular ordering is taken into account both in real emissions and virtual corrections.

The emission process takes place in the angular ordered region given by $\theta_i > \theta_{i-1}$ with $\theta_i$ the angle of the emitted gluon $q_i$ with respect to the incoming gluon $k_0$. In terms of the emitted transverse momenta $q_i$, this region becomes $q_i > z_{i-1}q_{i-1}$ and the branching distribution for the emission of gluon $i$ reads

$$dP_i = \frac{d^2q_i}{\pi q_i^2} \frac{\bar{\alpha}_S}{z_i} \Delta(z_i, q_i, k_i) \theta(q_i - z_{i-1}q_{i-1}),$$

(1)

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where

\[ \ln \Delta(z_i, q_i, k_i) = - \int_{z_i}^{1} \frac{dz'}{z'} \frac{\bar{\alpha}_S}{z'} \int \frac{dq'^2}{q'^2} \theta(k_i - q') \theta(q' - z' q_i) \]  

(2)

is the form factor which resums important virtual corrections for small \( z_i \).

Angular ordering provides a lower bound on transverse momenta, so that no collinear cutoff is needed other than a small virtuality for the first incoming gluon. On the other hand, in order to deduce a recurrence relation for the inclusive distribution one has to introduce an additional dependence on a momentum variable \( p \). That variable corresponds to the transverse momentum associated with the maximum available angle \( \bar{\theta} \) for the last emission, which in DIS is settled by the angle of the quarks produced in the boson-gluon fusion. The dependence on \( p \) is through

\[ \theta_n < \bar{\theta} \Rightarrow z_n q_n < p, \]  

(3)

where \( p \simeq x E \bar{\theta} \) and \( x E \) is the energy of the \( n \)-th gluon, which undergoes the hard collision at the scale \( Q \).

The distribution \( A(x, k, p) \) for emitting \( n \) initial state gluons satisfies the equation (CCFM equation 4)

\[ A(x, k, p) = A(0)(x, k, p) + \int \frac{d^2q}{\pi q^2} \frac{dz}{z} \bar{\alpha}_S \Delta(z, q, k) \theta(p - z q) A \left( \frac{x}{z}, |k + q|, q \right), \]  

(4)

where the inhomogeneous term \( A(0)(x, k, p) \) is the distribution for no gluon emission.

The neglecting of the \( p \)-dependence in \( A(x, k, p) \) corresponds to neglecting angular ordering. In this case the transverse momenta have no lower bound, and we need to introduce a collinear cutoff \( \mu \) to avoid singularities. The gluon density \( F(x, k) \) thus obtained satisfies the recurrence relation:

\[ F(x, k) = F^{(0)}(x, k) + \int \frac{d^2q}{\pi q^2} \frac{dz}{z} \bar{\alpha}_S \Delta^{(0)}(z, q, k) \theta(q - \mu) F \left( \frac{x}{z}, |k + q| \right), \]  

(5)

with the form factor

\[ \ln \Delta^{(0)}(z, k) = - \int_{z}^{1} \frac{dz'}{z'} \frac{\bar{\alpha}_S}{z'} \int \frac{dq'^2}{q'^2} \theta(k - q') \theta(q' - \mu). \]  

(6)

\(^a\)The usual Sudakov form factor is not included in the single-branching kernel, since it is cancelled by soft emissions.

\(^b\)It can be proved\(^b\) that the gluon density \( A(x, k, p) \) becomes independent of \( p \) for \( p \to \infty \).
In the limit $\mu \to 0$ — which can be safely performed — (5) prove to be equivalent to the BFKL equation.

The analytic treatment of the CCFM equation is more complicated than that of the BFKL equation because the gluon density contains one extra parameter, $p$. We take the eigensolutions of (4) in the form

$$xA(x,k,p) = x^{-\omega} \frac{k^2}{k_0^2} \tilde{\gamma} G\left(\frac{p}{k}\right),$$

where $\tilde{\gamma}$ and $\omega$ are related through the CCFM characteristic function

$$1 = \frac{\bar{\alpha}_S}{\omega} \tilde{\chi}(\tilde{\gamma}, \alpha_S),$$

and the function $G\left(\frac{p}{k}\right)$ takes into account angular ordering, parameterising the unknown dependence on $p$.

For $0 < \tilde{\gamma} < 1$ fixed, one obtains a coupled pair of equations for $G$ and $\tilde{\chi}$:

$$p\partial_p G\left(\frac{p}{k}\right) = \tilde{\alpha}_S \int \frac{d^2q}{\pi q^2} \left(\frac{p}{q}\right)^{\tilde{\alpha}_S \tilde{\chi}} \Delta\left(\frac{p}{q}, q, k\right) G\left(\frac{q}{|k+q|} k^2\right) \left(\frac{|k+q|^2}{k^2}\right)^{\tilde{\gamma}-1},$$

$$\tilde{\chi} = \int \frac{d^2q}{\pi q^2} \left[\frac{(k+q)^2}{k^2}\right]^{\tilde{\gamma}-1} G\left(\frac{q}{|k+q|}\right) - \theta(k-q) G\left(\frac{q}{k}\right),$$

with the initial condition $G(\infty) = 1$.

By putting $G = 1$ in this last equation, one notes that $\tilde{\chi}$ becomes just the well-known BFKL characteristic function. Since $1 - G\left(\frac{p}{k}\right)$ is formally of order $\alpha_S$, this demonstrates that angular ordering has a next-to-leading effect on structure function evolution.

Though a number of asymptotic properties of the function $G\left(\frac{p}{k}\right)$ have been determined, it has not so far been possible to obtain its full analytic form.

In order to gain further insight into the effects of angular ordering on the structure function, a numerical analysis is needed, which we have carried out both for BFKL and CCFM equations.

Fig. 2 shows the results for $\tilde{\chi}$ compared to the BFKL characteristic functions as a function of $\tilde{\gamma}$ for various $\alpha_S$. The difference $\delta\chi = \chi - \tilde{\chi}$ is positive, increases with $\tilde{\gamma}$, and increases with $\alpha_S$. Moreover we find $\delta\chi \sim \tilde{\gamma}$ for $\tilde{\gamma} \to 0$ ($\tilde{\alpha}_S$ small and fixed) and $\delta\chi \sim \tilde{\alpha}_S$ for $\tilde{\alpha}_S \to 0$ ($\tilde{\gamma}$ small and fixed). This implies that the next-to-leading correction to the gluon anomalous dimension coming from angular ordering is of order $\tilde{\alpha}_S^\frac{1}{2}$.
With respect to the BFKL case, the position of the minimum of the characteristic function $\tilde{\chi}$ gets shifted to the right, the value of the minimum is lowered and — in contrast to the BFKL case — there is no longer even a divergence at $\gamma = 1$. This behaviour of $\tilde{\chi}$ reduces the exponent $\omega_c$ of the small-$x$ growth of the structure function, in accordance with the fact that angular ordering reduces the phase space for evolution.

In Fig. 3a and Fig. 3b we plot as a function of $\alpha_S$ the values $\tilde{\chi}_c$ and $\tilde{\gamma}_c$ with $\tilde{\chi}_c$ the minimum of $\tilde{\chi}$ and $\tilde{\gamma}_c$ its position. As expected the differences compared to the BFKL values $\chi_c = 4 \ln 2$ and $\gamma_c = \frac{1}{2}$ are of order $\tilde{\alpha}_S$.

Fig. 3c shows the second derivative, $\tilde{\chi}''_c$, of the characteristic function at its minimum; this quantity is important phenomenologically because the dif-
fusion in $\ln k$ is inversely proportional to $\sqrt{\tilde{\chi}'}$. From this result, one can therefore conclude that the inclusion of angular ordering significantly reduces the diffusion compared to the BFKL case.

The loss of symmetry under $\gamma \to 1 - \gamma$ relates to the loss of symmetry between small and large scales: while in BFKL regions of small and large momenta are equally important, in the CCFM case angular ordering favours instead the region of larger $k$. However, at each intermediate branching, the region of vanishing momentum is still reachable for $x \to 0$, so that the evolution still contains non-perturbative components.

The neglecting of angular ordering has no effect on the structure functions at leading order. This is no longer true for exclusive quantities. Indeed, the cancellation of collinear singularities between real emissions and virtual corrections is no longer guaranteed for the modified kernel which enter the evolution equations for associated distribution.

The inclusion of angular ordering is therefore expected to have relevant effects when simple exclusive quantities, associated with one-gluon inclusive distributions, are considered.

Although the analysis of this subject is far from being completed, preliminary calculations confirm that both the shapes and the normalisations of final state quantities are sensitive to the phase spaces reduction associated with angular ordering.

Fig. 4a shows the distribution of the number of initial state gluons emitted. As expected from the different behaviour in the collinear region, BFKL branching has more emissions and a broader tail with respect to the CCFM case. Fig. 4b shows the $p_t$-distribution in rapidity. As expected, angular ordering suppress the radiation in the central and high rapidity regions.

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\footnote{This is strictly true only for the solution in the saddle-point approximation; nevertheless this quantity remains a good indicator, due to the mild asymptotic behaviour of the $G$ function.}
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