VARIATIONS ON A THEME BY HIGMAN

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ABSTRACT. We propose elementary and explicit presentations of groups that have no amenable quotients and yet are SQ-universal. Examples include groups with a finite \( K(\pi, 1) \), no Kazhdan subgroups and no Haagerup quotients.

1. INTRODUCTION

In 1951, G. Higman defined the group

\[
\text{Hig}_n = \langle a_i (i \in \mathbb{Z}/n\mathbb{Z}) : [a_{i-1}, a_i] = a_i \rangle
\]

and proved that for \( n \geq 4 \) it is infinite without non-trivial finite quotient [8]. Since the presentation (1) is explicit and simple, A. Thom suggested that \( \text{Hig}_n \) is a good candidate to contradict approximation properties for groups and proved such a result in [20]. Perhaps the most elusive approximation property is still soficity [6, 21]; but a non-sofic group would in particular not be residually amenable, a statement we do not know for the Higman groups (cf. also [7]). The purpose of this note is to propound variations of Higman’s construction with no non-trivial amenable quotients at all.

There are several known sources of groups without amenable quotients since it suffices to take a (non-amenable) simple group to avoid all possible quotients. However, as Thelonius Sphere Monk observed, simple ain’t easy. To wit, one had to wait until the breakthrough of Burger–Mozes [2, 3] for simple groups of type \( F \), i.e. admitting a finite \( K(\pi, 1) \). Before this, no torsion-free finitely presented simple groups were known.

The examples below are of a completely opposite nature because they admit a wealth of quotients: indeed, like \( \text{Hig}_n \), they are SQ-universal, i.e. contain any countable group in a suitable quotient. It follows that they have uncountably many quotients [13 §III], despite having no amenable quotients.

We shall start with the easiest examples, whose cyclic structure is directly inspired by (1). Below that, we propose a cleaner construction, starting from copies of \( \mathbb{Z} \) only, which might be a better candidate to contradict approximation properties; the price to pay is to replace the cycle by a more complicated graph.

Disclaimer. No claim is made to produce the first examples of groups with a hodgepodge of sundry properties (for instance, if \( G \) is a Burger–Mozes group, then \( G \ast G \) satisfies many properties of \( G_n \) in Theorem 2 below, though with “amenable” instead of “Haagerup”). Our goal is to suggest transparent presentations for which the stated properties are explicit and their proofs effective.

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1.A. Starting from large groups. Given a group $K$, an element $x \in K$ and a positive integer $n$, we define the group

$$K^{(n,x)} = \langle K_i \mid (i \in \mathbb{Z}/n\mathbb{Z}) : [x_{i-1}, x_i] = x_i \rangle,$$

where $K_i, x_i$ denote $n$ independent copies of $K, x$. Thus, $K^{(n,e)} = K^{*n}$ and $\text{Hig}_n = \mathbb{Z}^{(n,1)}$.

We recall that a group is normally generated by a subset if no proper normal subgroup contains that subset. Following the ideas of Higman and Schupp, we obtain:

**Proposition 1.** Let $K$ be a group normally generated by an element $x$ of infinite order and let $n \geq 4$.

(i) If $K$ has no infinite amenable quotient (e.g. if $K$ is Kazhdan), then $K^{(n,x)}$ has no non-trivial amenable quotient.

(ii) If $K$ is finitely presented, torsion-free, type $F_\infty$, or type $F$, then $K^{(n,x)}$ has the corresponding property.

(iii) Every countable group embeds into some quotient of $K^{(n,x)}$.

**Remark.** Suppose that $\mathcal{C}$ is any class of groups closed under taking subgroups. The proof of (i) shows: if every quotient of $K$ in $\mathcal{C}$ is finite, then $K^{(n,x)}$ has no non-trivial quotient in $\mathcal{C}$. For instance, if $K$ is Kazhdan, then $K^{(n,x)}$ has no non-trivial quotient with the Haagerup property.

**Example.** The group $K = \text{SL}_d(\mathbb{Z})$ is an infinite, finitely presented (even type $F_\infty$) Kazhdan group for all $d \geq 3$ and the Steinberg relations show that it is normally generated by any elementary matrix (with coefficient 1). Alternatively, the Steinberg group itself $K = \text{St}_d(\mathbb{Z})$ has the same properties (it is Kazhdan because it is a finite extension of $\text{SL}_d(\mathbb{Z})$, see e.g. [12, 10.1]). This gives us the following presentations of SQ-universal type $F_\infty$ groups without Haagerup quotients:

$$S_{d,n} = \left\langle E_i^{p,q} \mid (i \in \mathbb{Z}/n\mathbb{Z}, 1 \leq p \neq q \leq d) : [E_i^{p,q}, E_i^{q,r}] = E_i^{p,r} (p \neq r \neq q), [E_i^{p,q}, E_i^{r,s}] = e (q \neq r, p \neq s \neq r)\right\rangle.$$

The choice of the pair $(1,2)$ is arbitrary and any other elementary matrix for $x$ gives an isomorphic group. If we use the Magnus–Nielsen presentation [9, 19] of $\text{SL}_d(\mathbb{Z})$ instead of the Steinberg group, we have to add the relations $(E_i^{1,2}(E_i^{2,1})^{-1}E_i^{1,2})^4 = e$.

These groups are not, however, torsion-free. Although congruence subgroups of $\text{SL}_d(\mathbb{Z})$ are torsion-free (and even type F by [16]), the latter are never normally generated by a single element because they have large abelianizations.

This construction can be transposed to other Chevalley groups.

Notice that if in addition $K$ is just infinite, like for instance $K = \text{SL}_d(\mathbb{Z})$ for $d$ odd [11], then this construction shows that $K$ embeds into all non-trivial quotients of $K^{(n,x)}$, such as for instance the simple quotients obtained from maximal normal subgroups.

1.B. An example built from $\mathbb{Z}$. Consider the semi-direct product

$$L = (\mathbb{Z}[1/2])^2 \rtimes (\mathbb{Z} \times F_2)$$
where the generator $h$ of $\mathbb{Z}$ acts on $(\mathbb{Z}[1/2])^2$ by multiplication by 2, and the generators $u, v$ of the free group $F_2$ act by multiplication by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

respectively. In particular the group $L$ is torsion-free, linear and finitely presented. It is generated by $\{x, y, h, u, v\}$ where $(x, y)$ is the standard basis of $\mathbb{Z}^2$.

We define a group $G_n$ by fusing together $n$ copies $L_i$ of $L$ in a circular fashion along the corresponding generators as follows:

$$G_n = \langle L_i : (h_i, u_i, v_i) = (y_{i-2}, x_{i-2}, y_{i-1}), i \in \mathbb{Z}/n\mathbb{Z} \rangle. \tag{2}$$

It is easy to write down an explicit presentation of $G_n$. Observe first that $L_i$ with our choice of generators, has a presentation with the following set $R$ of relations

$$R(x, y, h, u, v) : \quad e = [x, y] = [x, u] = [y, v] = [h, u] = [h, v],$$

$$[h, x] = [u, y] = x, \quad [h, y] = [v, x] = y.$$ 

Now (2) is equivalent to the finite presentation

$$G_n = \langle x_i, y_i : R(x_i, y_i, y_{i-2}, x_{i-2}, y_{i-1}), i \in \mathbb{Z}/n\mathbb{Z} \rangle. \tag{3}$$

We find these groups more elementary than $K^{(n, x)}$ (with Kazhdan $K$) and hope that they will be easier to use in applications. In return, we have to work more than before to deduce some of the following properties.

**Theorem 2.** Let $n \geq 8$.

(i) The group $G_n$ has no non-trivial Haagerup quotient.

(ii) Any quotient with a $\frac{1}{30}$-Følner set for the generators $x_i, y_i$ is trivial.

(iii) The only Kazhdan subgroup of $G_n$ is the trivial group.

(iv) The group $G_n$ admits a finite $K(\pi, 1)$.

(v) The group $G_n$ can be constructed starting from copies of $\mathbb{Z}$, using amalgamated free products, semi-direct products and HNN-extensions.

(vi) Every countable group embeds into some quotient of $G_n$ if $n \geq 9$.

(vii) The groups $G_m$ are trivial for $m \leq 4$ and $m = 6$.

The restriction $n \geq 9$ is probably not needed in (vi) but makes it very easy to check Schupp’s criterion for SQ-universality. We have not elucidated $G_5$ and $G_7$.

**Scholium.** We should like to point out a general type of presentations subsuming the examples above. Consider a group $L$ and two finite sets $A, P \subseteq L$. We think of elements in $A$ as “active”, whilst those in $P$ are “passive”. Consider furthermore a transitive labelled oriented graph $g$ whose edges are labelled by $P \times A$. To every vertex $i$ of $g$ we associate an independent copy $L_i$ of $L$. We then form the group

$$G = \langle L_i, i \in g : p_j = a_k \text{ if } \exists (p, a)\text{-labelled edge from } j \text{ to } k \rangle.$$ 

In order to get a manageable group from this presentation, we would like to ensure at the very least that each $L_i$ embeds. A favourable case is when $A$ is a basis for a free subgroup in $L$ and the edges spread the passive elements of $P_i$ incoming to a vertex $k$ over copies $L_j$ for suitably distinct $j$s. (In our case, we allowed a commutation in $A_k$ because it was going to hold also among the corresponding $P_j$.)
The trade-off is that this spreading should remain limited compared to the girth of the cycles in \( g \) along which we can cut the amalgamation scheme. Higman’s groups and the groups \( K^{(n,x)} \) use a simple \( n \)-cycle for \( g \); as for \( G_n \), we depict its graph in the figure below for \( n = 8 \); the orientation is implicit from the labelling.

Notation. Our convention for commutators is \([\alpha, \beta] = \alpha \beta \alpha^{-1} \beta^{-1}\); Higman used a different convention for (1) but this does not affect the group \( \text{Hig}_n \). Given a subset \( E \) of a group \( H \), we denote the subgroup it generates by \( \langle E \rangle \) or by \( \langle E \rangle_H \) when \( H \) needs to be clarified.

2. Proof of Proposition 1

This proposition really is just a variation on the work of Higman and Schupp. For (i) we start by recalling the following.

Lemma 3 (Higman’s circular argument). Let \( f \) be a homomorphism from \( \text{Hig}_n \) to another group. If \( f(a_i) \) has finite order for some \( i \), then \( f \) is trivial.

Proof (see also [14, p. 547]). The relations in (1) imply inductively that \( f(a_i) \) has finite order \( r_i \geq 1 \) for all \( i \). Suppose for a contradiction that \( r_i > 1 \) for some \( i \), hence for all \( i \) by the relations (1). Let \( p \) be the smallest prime dividing any \( r_j \). The relation \( a_{j-1}^{r_j-1}a_j^{-r_j-1} = a_j^{2^{r_j-1}} \) implies that \( 2^{r_j-1} - 1 \) is a multiple of \( r_j \) and hence of \( p \). In particular, \( p \neq 2 \) and the order \( s > 1 \) of 2 in \( (\mathbb{Z}/p\mathbb{Z})^\times \) divides \( r_j-1 \). This contradicts the choice of \( p \) because \( s \leq p - 1 \).

Suppose now that \( f \) is a homomorphism from \( K^{(n,x)} \) to an amenable group. The image of \( K_i \) in \( K^{(n,x)} \) is mapped by \( f \) to a finite group, so that in particular \( f(x_i) \) has finite order for all \( i \). Since we have a homomorphism \( \text{Hig}_n \to K^{(n,x)} \) sending \( a_i \) to \( x_i \), we deduce from Lemma 3 that \( f(x_i) \) is in fact trivial. Since \( K \) is normally generated by \( x \), it follows that \( f(K) \) is trivial. We conclude that \( f \) is trivial because the various \( K_i \) generate \( K^{(n,x)} \).
The two other points follow once we re-construct $K^{(n,x)}$ as a suitable amalgam. Recall that $x$ has infinite order; thus

\[ L = \langle K, h : [h, x] = x \rangle \]

is an HNN-extension; we define $L_i, h_i$ similarly. Now

\[ H = \langle L_0, L_1 : x_0 = h_1 \rangle \]

is a free product with amalgamation (because $x_0$ has infinite order) and therefore, using also the HNN-structure of (4), it follows that $\langle h_0, x_1 \rangle$ is a free group on $h_0, x_1$. Likewise, since $n \geq 4$, we deduce that

\[ H' = \langle L_2, \ldots, L_{n-1} : x_2 = h_3, \ldots, x_{n-2} = h_{n-1} \rangle \]

is a (successive) free product with amalgamation and that $h_2, x_{n-1}$ are a basis of a free group in $H'$. Therefore, we obtain $K^{(n,x)}$ by amalgamating $H$ and $H'$ over the groups $\langle h_0, x_1 \rangle$ and $\langle x_{n-1}, h_2 \rangle$ by identifying the free generators in the order given here.

Now the finiteness properties of (4) all follow since $K^{(n,x)}$ was obtained from copies of $K$ by finitely many HNN-extensions and amalgamated free products (see e.g. [5 §7]). Regarding SQ-universality, P. Schupp proved that it suffices to find a blocking pair for $\langle h_0, x_1 \rangle$ in $H$, see Thm. II in [18]. A blocking pair is provided for instance by any distinct non-trivial powers of an element $t \in H$ such that $t, h_0, x_1$ form a basis of a free group (see the comment after Thm. II in [18]). Just as in Lemma 4.3 of [18], the element $t = x_0^{-1}x_1h_0x_1^{-1}x_0$ will do.

3. Proof of Theorem [2]

We now turn to the groups $L$ and $G_n$ defined in part 1.B of the Introduction and fix some more notation. Denote by Heis($a, \beta, \zeta$) the (discrete) Heisenberg group with generators $a, \beta$ and central generator $\zeta$. More precisely, it is defined by the relations $[a, \beta] = \zeta$ and $[\zeta, a] = [\zeta, \beta] = e$. For instance, $\{v, x, y\}$ (or just $\{v, x\}$) generate a copy of Heis($v, x, y$) in $L$.

We shall use repeatedly, but tacitly, the following fundamental property of a free product with amalgamation $A \ast_C B$. If $A' < A$ and $B' < B$ are subgroups whose intersections with $C$ yield the same subgroup $C' < C$, then the canonical map $A' \ast_{C'} B' \to A \ast_C B$ is an embedding [15 8.11].

We embed $L$ into a larger group $J$ generated by $L$ together with an additional generator $z$ by defining the following free product with amalgamation:

\[ J = L \ast_{(h,v)} \text{Heis}(h, z, v). \]

Although $h$ and $v$ already occur in our definition of $L$, there is no ambiguity since they form a basis of a copy of $\mathbb{Z}^2$ both in $L$ and in $\text{Heis}(h, z, v)$. In particular, $L$ is indeed canonically embedded in $J$.

When we want to consider normal forms for this amalgamation (cf. [17 §1] or Thm. 4.4 in [10]), it is convenient that there are very nice coset representatives of $\langle h, v \rangle$ in each factor. Indeed, in $\text{Heis}(h, z, v)$, we can simply take the group $\langle z \rangle$. In $L$ written as

\[ L = (\mathbb{Z}[1/2])^2 \rtimes (\langle h \rangle \times \langle u, v \rangle), \]

we can take as set of representatives the group $(\mathbb{Z}[1/2])^2 \rtimes K$, where $K \lhd \langle u, v \rangle$ is the kernel of the morphism killing $v$. 
As before, we shall denote by $J_i$ a family of independent copies of $J$. We further denote by $z_i$ the corresponding additional generator. Then we have an equivalent presentation of $G_n$ given by

$$\left\langle J_i : \begin{align*}
v_{i-1} &= h_i \\
x_{i-1} &= z_i \\
y_{i-1} &= v_i \\
z_{i-1} &= u_i
\end{align*} \quad \forall i \in \mathbb{Z}/n\mathbb{Z} \right\rangle.$$ 

The advantage is that each relation involves only successive indices $i - 1$ and $i$.

We define inductively the groups $D_r$ for $r \in \mathbb{N}$, starting with $D_0 = J_0$, by the presentation

$$D_r = \left\langle D_{r-1}, J_r : \begin{align*}
v_{r-1} &= h_r \\
x_{r-1} &= z_r \\
y_{r-1} &= v_r \\
z_{r-1} &= u_r
\end{align*} \right\rangle.$$ 

We claim that this is in fact a free product with amalgamation of $D_{r-1}$ and $J_r$. More precisely, we claim that the subgroups of $J$ given respectively by

$$Q = \langle v, x, y, z \rangle_J \quad \text{and} \quad T = \langle h, z, v, u \rangle_J$$

are isomorphic under matching their generators in the order listed in (7). This claim, transported to the various $J_i$, implies in particular by induction that $D_r$ is indeed a free product with amalgamation $D_r \cong D_{r-1} *_{Q_{r-1}=T_r} J_r$, where $Q_i, T_i$ denote the corresponding subgroups of $J_i$.

To prove the claim, we note first that the structure of $Q$ is revealed by observing which subgroups are generated by $\{v, x, y\}$ and by $\{v, z\}$ in the amalgamation (5) defining $J$. Both intersect $\langle h, v \rangle$ exactly in $\langle v \rangle$ and thus $Q$ is itself a free product with amalgamation $Q = \text{Heis}(v, x, y) *_{\langle v \rangle} \langle v, z \rangle_J$ with $\langle v, z \rangle_J \cong \mathbb{Z}^2$.

As for $T$, given its relations, we have an epimorphism $Q \to T$ given by the above matching of generators; we need to show that it is in fact injective. To this end, consider that $T$ is generated by its subgroups $\text{Heis}(h, z, v)$ and $\langle h, v, u \rangle_J$. Since $L$ is a factor of $J_i$, the latter is $\langle h, v, u \rangle_L \cong \mathbb{Z} \times F_2$. Thus $T$ is an amalgamated free product $\text{Heis}(h, z, v) *_{\langle h, v \rangle} \langle h, v, u \rangle_L$. The injectivity now follows. In conclusion, $D_r$ is the following iterated free product with amalgamations:

$$D_r \cong J_0 *_{Q_0=T_1} J_1 *_{Q_1=T_2} \cdots *_{Q_{r-1}=T_r} J_r.$$ 

We also need to understand the intersection $Q \cap T$, which contains at least the group $\langle z, v \rangle_J \cong \mathbb{Z}^2$. In fact, this intersection is exactly $\langle z, v \rangle_J$. This follows by examining the normal form for the particularly simple choice of coset representatives made above.

As a consequence, we deduce that when $r \geq 3$, the subgroups $T_0$ and $Q_r$ of

$$D_r \cong (J_0 *_{Q_0=T_1} J_1) *_{Q_1=T_2} \cdots *_{Q_{r-2}=T_{r-3}} (J_{r-1} *_{Q_{r-1}=T_r} J_r)$$

intersect trivially and hence generate a free product $T_0 * Q_r$.

Finally, to close the circle, we will use the assumption $n \geq 8$ and glue $D_{n-3}$ with a copy $D'_3$ of $D_3$ as follows. We shift indices in the $D_3$ factor to obtain the isomorphic group

$$D'_3 = J_{n-4} *_{Q_{n-4}=T_{n-3}} \cdots *_{Q_{n-2}=T_{n-1}} J_{n-1}.$$
In $D'_3$, the subgroups $T_{n-4}$ and $Q_{n-1}$ generate $T_{n-4} \ast Q_{n-1}$. Since we have constructed isomorphisms $T_0 \cong Q_{n-1}$ and $Q_{n-5} \cong T_{n-4}$, we have a corresponding isomorphism

$$\varphi: T_0 \ast Q_{n-5} \rightarrow Q_{n-1} \ast T_{n-4}$$

and therefore we have a free product with amalgamation

$$(9) \quad D_{n-5} \ast_\varphi D'_3.$$ 

Since this is a rewriting of the presentation (6), we have indeed constructed $G_n$ as an amalgam whenever $n \geq 8$. In particular, $L_i$ is embedded in $G_n$.

At this point, we have established point (v) of Theorem 2, observing that (8) applied to $Q$ whenever $n = \sqrt{2^k - 2}$.

Finally reach $Z$, which has no non-trivial Kazhdan subgroup. This implies that any Kazhdan subgroup of $G_n$ can be recursively constrained into the factors of any amalgam. By (v) we finally reach $Z$, which has no non-trivial Kazhdan subgroup.

For (vi), we indulge in the expedience of $n \geq 9$. This allows us to see from the decomposition (8) applied to $r = n - 4$ that we have a free product

$$(T_0, u_2x_2, Q_r)_{D_r} = T_0 \ast \langle u_2x_2 \rangle \ast Q_r.$$ 

Indeed, reasoning within $J$, we see that $\langle ux \rangle$ intersects both $Q$ and $T$ trivially (and is infinite). This implies that any two distinct non-trivial powers of $u_2x_2$ constitute a blocking pair for $T_0 \ast Q_r$ in $D_r$, see again [18]. We conclude that $G_n$ is SQ-universal.

Turning to (i) we first observe that every generator in the presentation (3) functions as a self-destruct button for the group $G_n$, i.e. normally generates $G_n$.

**Lemma 4.** Let $f$ be a homomorphism from $G_n$ to another group. If $f$ sends some $x_i$ or some $y_j$ to the identity, then $f$ is trivial.

**Proof.** The element $u_jv_j^{-1}u_i$ conjugates $x_i$ to $y_j^{-1}$ and therefore we can assume that $f(y_i)$ is trivial. Since $y_i = v_{i+1}$, the relation $[v_{i+1}, x_{i+1}] = y_{i+1}$ implies inductively that $f(y_j)$ vanishes for all $j$. Conjugating by $u_jv_j^{-1}u_j$, we find that all generators in (3) are trivialized by $f$. \hfill \Box

Let now $f$ be a homomorphism from $G_n$ to some Haagerup group. The subgroup $\langle x, y \rangle$ of $\langle x, y \rangle \times \langle u, v \rangle$ has the relative property (T). Indeed, the proof of the corresponding statement for $Z^2 \times SL_2(Z)$ only depends on the image of $SL_2(Z)$ in the automorphism group of $Z^2$, see e.g. [1]. Therefore, $f(\langle x_i, y_j \rangle)$ is finite for all $i$.

On the other hand, the presentation (2) shows that we have a morphism $Hig_{\mathbb{Q}^n} \rightarrow G_n$ defined by $a_i \mapsto y_2$. By Higman’s argument (Lemma 3), it follows that $f(y_2)$ is trivial for all $i$. We conclude from Lemma 4 that $f$ is trivial.

For (ii) we use the explicit relative Kazhdan pair $(S_0, \varepsilon_0)$ provided by M. Burger, Example 2 p. 40 in [1]. Here $S_0$ is a certain generating set of $Z^2 \times SL_2(Z)$ and $\varepsilon_0 = \sqrt{2 - \sqrt{3}}$. Being a relative Kazhdan pair admits that any unitary representation of $Z^2 \times SL_2(Z)$ with $(S_0, \varepsilon_0)$-invariant vectors admits $Z^2$-invariant vectors, see [4]. We denote by $S = \{x, y, u, v\}$ our usual generators of $Z^2 \times F_2$ and write $\mathcal{S} = S \cup S^{-1} \cup \{\varepsilon\}$; then $(S, \varepsilon)$-invariance is equivalent
to \((\overline{S}, \varepsilon)\)-invariance. The set \(S_0\) from [11, Ex. 2] is contained in \(\overline{S}^3\) under the map \(F_2 \rightarrow \text{SL}_2(\mathbb{Z})\) and therefore every \((S, \varepsilon_0/3)\)-invariant vector is \((S_0, \varepsilon_0)\)-invariant. Now (ii) follows because \(\varepsilon_0/3 > 1/6\) and because any \((S, \varepsilon)\)-Følner set gives a \((S, \sqrt{\varepsilon})\)-invariant vector.

**Remark.** The corresponding argument provides also a lower bound on Følner constants for quotients of \(K^{(n,x)}\) when \(K\) is Kazhdan.

It only remains to prove (vii). Consider again the homomorphism \(\text{Hig}_n \rightarrow G_n\) above. When \(n\) is even, this factors through a morphism \(\text{Hig}_{n/2} \rightarrow G_n\). Since \(\text{Hig}_r\) is trivial for \(r \leq 3\) (see [8]), it follows that \(y_0\) is trivial when \(n = 4, 6\); now Lemma 4 shows that \(G_n\) is trivial. The same argument applied to the original map \(\text{Hig}_n \rightarrow G_n\) takes care of \(n \leq 3\).

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