Wall-crossing between stable and co-stable ADHM data

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Abstract

We prove a formula between Nekrasov partition functions defined from stable and co-stable ADHM data for the plane following a method by Nakajima-Yoshioka [NY2] based on the theory of wall-crossing formula developed by Mochizuki [Mo]. This formula is similar to conjectures by Ito-Maruyoshi-Okuda [IMO] (4.1), (4.2) for $A_1$ singularity.

1 Introduction

Nekrasov partition functions are introduced in [Ne]. They are defined by integrations

$$Z = \sum_{n=0}^{\infty} q^n \int_{M(r,n)} \psi$$

on moduli spaces $M(r,n)$ of framed sheaves on the plane $\mathbb{P}^2$ with the rank $r$ and the second Chern class $n$. Here $\psi$ are various equivariant cohomology classes on $M(r,n)$ corresponding to physical theories. These integrations are defined by localization for torus actions on moduli spaces (cf. §2.3). In particular we consider $T^2 \times T^r \times T^{N_f}$-actions on $M(r,n)$ with $0 \leq N_f \leq 2r$ and $T = \mathbb{C}^*$, where $T^2$-actions are induced by the diagonal action on $\mathbb{C}^2 \subset \mathbb{P}^2$, $T^r$-actions are induced by scale change of framings, and trivial $T^{N_f}$ actions on $M(r,n)$.

Nekrasov’s conjecture states that these partition functions give deformations of the Seiberg-Witten prepotentials for $N = 2$ SUSY Yang-Mills theory. This conjecture is proven in [BE], [NO] and [NY1] independently. In [NY1] they study the case for $\psi = 1$. Furthermore in [GNY] they study the case where $\psi$ is defined by the equivariant Euler class

$$\psi = e \left( \bigoplus_{j=1}^{N_f} \mathcal{V} \otimes \left( \frac{e^{m_j}}{\sqrt{t_1 t_2}} \right) \right), \quad (1)$$

for $(t_1, t_2) \in T^2, (e^{m_1}, \ldots, e^{m_{N_f}}) \in T^{N_f}$, in particular with $N_f = 1$ and $r = 2$, where $\mathcal{V}$ are tautological bundles on $M(r,n)$. They extend arguments in [NY1] using the theory of perverse coherent sheaves [NY2]. Combining with Mochizuki’s formula [Mo] they proved the Witten’s
conjecture \([W]\) relating Donaldson invariants with Seiberg-Witten invariants for complex projective surfaces.

On the other hand in \([IMO]\), similar functions are considered on ALE spaces of type \(A_{p-1}\) and quotient stacks \([\mathbb{C}^2/\mathbb{Z}_p]\) for \(p > 1\). They conjecture formulas \([IMO\ (4.1), (4.2)]\) among these functions. Physical background of Nekrasov partition functions is to compute integrations on moduli spaces of instantons on ALE spaces. Since moduli of instantons are singular, they use moduli of framed sheaves as resolutions.

In the setting of \([IMO]\) we can consider two resolutions, moduli spaces of framed sheaves on ALE spaces of type \(A_{p-1}\) and quotient stacks \([\mathbb{C}^2/\mathbb{Z}_p]\) for \(p > 1\). They conjecture formulas \([IMO, (4.1), (4.2)]\) among these functions. Physical background of Nekrasov partition functions is to compute integrations on moduli spaces of instantons on ALE spaces. Since moduli of instantons are singular, they use moduli of framed sheaves as resolutions.

In this paper we treat the case where \(p = 1\), and hence both the ALE space and the quotient stack coincide with \(\mathbb{C}^2\). But we still have two stability conditions and corresponding two resolutions from the viewpoint of the ADHM description. These are moduli of stable ADHM data and co-stable ADHM data, which are isomorphic as manifolds, but having different torus actions.

We consider Nekrasov partition functions defined from \(\psi\) in \([1]\) with \(N_f = 2r\). We compare Nekrasov partition functions defined from stable and co-stable ADHM data, and prove a formula in Theorem 2.6 similar to the above conjecture by \([IMO]\).

If we take \(\varepsilon = (\varepsilon_1, \varepsilon_2)\), \(a = (a_1, \ldots, a_r)\) and \(m^{N_f} = (m_1, \ldots, m_{N_f})\) corresponding to characters of \(T^2, T^r\) and \(T^{N_f}\), they form a polynomial ring \(\mathbb{Z}[\varepsilon, a, m^{N_f}]\) isomorphic to the \(T^2 \times T^r \times T^{N_f}\)-equivariant Chow ring of a point. Nekrasov partition functions \(Z = Z(\varepsilon, a, m^{N_f}, q)\) take values in the quotient field \(\mathbb{Q}(\varepsilon, a, m^{N_f})\), and ones defined from integrations on moduli of co-stable ADHM data are equal to \(Z(-\varepsilon, a, m^{N_f}, q)\).

As an application of Theorem 2.6 we determine the odd degree parts of \(\varepsilon_1 \varepsilon_2 \log Z(\varepsilon, a, m^{N_f})\) for \(0 \leq N_f \leq 2r - 2\) in \(\mathbb{Q}[\varepsilon_1, \varepsilon_2]\). These are equal to zero for \(N_f \leq 2r - 2\), and have non-zero coefficients only in \(\varepsilon_1 + \varepsilon_2\) for \(N_f = 2r - 1, 2r\). It was known that only coefficients for \(\varepsilon_1 + \varepsilon_2\) are determined in the case where \(N_f \leq 2r - 1\) (cf. \([NY1\ Lemma 7.1]\)), and this method can not be applied to the case where \(N_f = 2r\).

Our proof follows the method in \([NY2]\) based on the theory of wall-crossing formula developed in \([MO]\). We apply the method to the case where skyscraper sheaves destabilize framed sheaves on the wall. This case is not treated before. We will show in \([O]\) that similar wall-crossing formulas solve the above conjecture by \([IMO]\).

The paper is organized as follows. In §2 we recall ADHM description of framed sheaves, give a statement of main result in Theorem 2.6 and an application for Nekrasov partition functions. In §3 we recall quiver description of moduli of framed sheaves, and reduce a proof of Theorem 2.6 to wall-crossing formulas. In §4 we introduce enhanced master spaces used in
Mochizuki method [Mo], [NY2]. In §5 we study obstruction theories for moduli stacks. In §6 we compute wall-crossing formulas, and complete a proof of Theorem 2.6. In Appendix A we compute integrations on Hilbert schemes following [Na3].

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2 Framed sheaves on \( \mathbb{P}^2 \)

We recall ADHM descriptions of framed sheaves on \( \mathbb{P}^2 \) from [NY1], and introduce partition functions and our main result Theorem 2.6.

2.1 Framed sheaves and ADHM data

We consider the projective plane \( \mathbb{P}^2 \) over \( \mathbb{C} \) and the line \( \ell_\infty = \{x_0 = 0\} \), where \( [x_0, x_1, x_2] \) is the homogeneous coordinate of \( \mathbb{P}^2 \).

**Definition 2.1.** A framed sheaf on \( \mathbb{P}^2 \) is a pair \((E, \Phi)\) of
- a torsion free sheaf \( E \) on \( \mathbb{P}^2 \), and
- an isomorphism \( \Phi: E|_{\ell_\infty} \cong \mathcal{O}_{\ell_\infty} \).

We remark that framed sheaves are automatically slope semistable. Moduli spaces of framed sheaves were constructed in [HL] in more general framework. In this paper we construct the moduli spaces via ADHM description. To introduce ADHM description of framed sheaves we take finite dimensional vector spaces \( Q = \mathbb{C}^2, W = \mathbb{C}^r \) and \( V = \mathbb{C}^n \).

**Definition 2.2.** ADHM data on vector spaces \( W, V \) are collections of linear maps \((B, z, w)\) such that
\[
\wedge B + zw = 0 \in \text{Hom}_\mathbb{C}(\wedge Q^\vee \otimes V, V),
\]
where \( \wedge B \) is the restriction of \( B^2: Q^\vee \otimes Q^\vee \otimes V \to V \) to the subspace \( \wedge Q^\vee \otimes V \) of \( Q^\vee \otimes Q^\vee \otimes V \).

If we take the canonical basis \( e_1, e_2 \) of \( Q = \mathbb{C}^2 \) and put \( B_i = B \circ e_i^*: V \to V \), then the equation \( \wedge B + zw = 0 \) is equivalent to \([B_1, B_2] + zw = 0\).

**Definition 2.3.** \((B, z, w)\) is said to be stable if there exists no subspace \( S \subset V \) other than \( S = V \) such that \( B_i(S) \subset S \) for \( i = 1, 2 \) and \( \text{im} \ i \subset S \).

We put
\[
M(r, n) = \{(B, z, w) \mid \text{stable ADHM data on } W = \mathbb{C}^r, V = \mathbb{C}^n\}/\text{GL}(V),
\]
where \( \text{GL}(V) \) acts by base change of \( V \), that is, \( g(B_1, B_2, z, w) = (gB_1g^{-1}, gB_2g^{-1}, gz, wg^{-1}) \).

We have natural \( \text{GL}(Q) \times \text{GL}(W) \)-action on \( M(r, n) \). In particular \( M(r, n) \) has \( T^2 \times T^r \)-equivariant structure, where \( T^2 \) and \( T^r \) are diagonal tori of \( \text{GL}(Q) \) and \( \text{GL}(W) \) respectively.
**Theorem 2.4 ([13]).** We have an isomorphism from $M(r, n)$ to the moduli of isomorphism classes of framed sheaves $(E, \Phi)$ with $\operatorname{rk}(E) = r = \dim W, c_2(E) = n = \dim V$.

A framed sheaf $(E, \Phi)$ corresponding to $(B, z, w) \in M(r, n)$ via the above isomorphism is defined by the complex

$$C^* = \left( \mathcal{O}_{\mathbb{P}^2}(-1) \otimes \mathcal{O}_{\mathbb{P}^2} \overset{\Phi}{\to} \mathcal{O}_{\mathbb{P}^2} \otimes V \overset{\tau}{\to} \mathcal{O}_{\mathbb{P}^2} \otimes V \oplus \mathcal{O}_{\mathbb{P}^2} \otimes W \overset{\sigma}{\to} \mathcal{O}_{\mathbb{P}^2}(1) \otimes V \right),$$

where $\sigma = \begin{bmatrix} B_1 & 0 \\ B_2 & 0 \end{bmatrix}$, $\tau = \begin{bmatrix} -B_2x_0 + 1 & 0 \\ B_1x_0 & 0 \end{bmatrix}$, and the condition $[B_1, B_2] + zw = 0$ implies $\tau \sigma = 0$. We define $E = \ker \tau / \ker \sigma$. Then substituting $z_0 = 0$ we have a natural isomorphism $\Phi: E|_{\ell_\infty} \cong W \otimes \mathcal{O}_{\ell_\infty}$. This gives an isomorphism (see [Na1] Theorem 2.1 for the proof). Hereafter via this isomorphism we identify $M(r, n)$ and the moduli of isomorphism classes of framed sheaves $(E, \Phi)$ with $\operatorname{rk}(E) = r, c_2(E) = n$.

### 2.2 Torus action on $M(r, n)$

To describe torus fixed points of $M(r, n)$ we give a sheaf description of the torus action on $M(r, n)$ introduced in the previous subsection. For $T = \mathbb{C}^*$, we put $\tilde{T} = T^2 \times T^r \times T^{2r}$ and

$$t = (t_1, t_2) \in T^2, e^a = (e^{a_1}, \ldots, e^{a_r}) \in T^r, e^m = (e^{m_1}, \ldots, e^{m_{2r}}) \in T^{2r},$$

where $a = (a_1, \ldots, a_r), m = (m_1, \ldots, m_{2r})$. For $t = (t_1, t_2) \in T^2$ we consider the morphism $F_t: \mathbb{P}^2 \to \mathbb{P}^2$ defined by $[x_0, x_1, x_2] \mapsto [x_0, t_1 x_1, t_1 x_2]$. We identify $e^a = (e^{a_1}, \ldots, e^{a_r}) \in T^r$ with the diagonal matrix $\text{diag}(e^{a_1}, \ldots, e^{a_r})$. We define $(t, e^a)(E, \Phi) = (E', \Phi')$ by $E' = (F_t^{-1})^* E$, and $\Phi'$ is defined by the following commutative diagram

$$
\begin{array}{ccc}
E'|_{\ell_\infty} & \xrightarrow{\Phi'} & \mathcal{O}_{\ell_\infty}^{\mathbb{P}^r} \\
\downarrow{(F_t^{-1})^*} & & \downarrow{(F_t^{-1})^*} \\
(E|_{\ell_\infty})^{\mathbb{P}^r} & \xrightarrow{(F_t^{-1})^* \Phi} & \mathcal{O}_{\ell_\infty}^{\mathbb{P}^r}
\end{array}
$$

Let $T^{2r}$ act on $M(r, n)$ trivially. These actions on $M(r, n)$ are compatible with ones defined by ADHM data in the previous section via the isomorphism in Theorem 2.4.

**Proposition 2.5 ([NY1 Proposition 2.9]).** The $\tilde{T}$-fixed points of $M(r, n)$ are given by

$$I_{\bar{Y}} = (I_{Y_1} \oplus \cdots \oplus I_{Y_r}, \Phi),$$

where $\bar{Y} = (Y_1, \ldots, Y_r)$ is $r$-tuple of Young diagrams with $\sum_{i=1}^r |Y_i| = n$, $I_{Y_1}, \ldots, I_{Y_r}$ are corresponding monomial ideals supported at $[1, 0, 0] \in \mathbb{P}^2$, and $\Phi$ is a direct sum of natural isomorphisms $I_{Y_i}|_{\ell_\infty} \cong \mathcal{O}_{\ell_\infty}$ induced by the inclusion $I_{Y_i} \subset \mathcal{O}_{\mathbb{P}^2}$.
By Theorem 2.5 we have a universal framed sheaf \((E, \tilde{\Phi})\) on \(\mathbb{P}^2 \times M(r, n)\), that is, for each point \((E, \Phi) \in M(r, n)\) we have a unique isomorphism

\[ E|_{\mathbb{P}^2 \times ((E, \Phi))} \cong E \]

such that \(\tilde{\Phi}\) and \(\Phi\) commute on \(\ell_{\infty}\). This unique isomorphism gives a \(\tilde{T}\)-equivariant structure of \(E\).

We consider the tautological bundle \(\mathcal{V} = V_{r,n} = (\mathbb{R}^1 p_2)_*, (E \otimes p_1^* \mathcal{O}_{\mathbb{P}^2}(-1))\), where \(p_1: \mathbb{P}^2 \times M(r, n) \to \mathbb{P}^2, p_2: \mathbb{P}^2 \times M(r, n) \to M(r, n)\) are projections. Then \(\mathcal{V}\) is a vector bundle with the fiber \(H^1(\mathbb{P}^2, E(-1))\) over \((E, \Phi)\). The \(\tilde{T}\)-action on \(E\) induces a \(\tilde{T}\)-equivariant structure of \(\mathcal{V}\).

We also consider the tangent bundle \(T M(r, n)\) with the natural \(\tilde{T}\)-action.

### 2.3 Main result and Nekrasov Partition functions

Let \(A^T_2(X)\) be the \(\tilde{T}\)-equivariant Chow group of a \(\tilde{T}\)-space \(X\) with rational coefficients. They are modules over the \(\tilde{T}\)-equivariant Chow ring \(A^*_T(\text{pt})\) of a point, isomorphic to \(S(\tilde{T}) = \mathbb{Q}[e, a, m]\), where \(e = (e_1, e_2), a = (a_1, \ldots, a_r)\) and \(m = (m_1, \ldots, m_{2r})\) correspond to characters of \(\tilde{T}\) with eigen-values \(t, e^a, e^m\). The quotient field of \(S(\tilde{T})\) is denoted by \(S\).

We have a projective morphisms \(\pi: M(r, n) \to M_0(r, n)\), where \(M_0(r, n)\) is the Uhlenbeck (partial) compactification of the moduli space \(M^\text{reg}_0(r, n)\) of framed locally free sheaves \((E, \Phi)\). This morphism induces a homomorphism \(\pi_*: A^T_2(M(r, n)) \to A^T_2(M_0(r, n))\). In the next section we will explain the construction in [Na1] via the quiver description. Set-theoretically we have a bijection

\[ M_0(r, n) \to \bigsqcup_{n' = 0}^n M^\text{reg}_0(r, n') \times S^{n-n'}(\mathbb{C}^2), \]

and the \(\tilde{T}\)-fixed points set \(M_0(r, n)^{\tilde{T}}\) consists of only one point \(n[0]\).

By the localisation theorem [EG] Theorem 1 we have an isomorphism \((i_0)_*: S \cong A^T_2(M_0(r, n)) \otimes S\), where \(i_0: M_0(r, n)^{\tilde{T}} = \{n[0]\} \to M_0(r, n)\) is the inclusion. For \(\psi \in A^T_2(M(r, n))\), we put

\[ \int_{M(r, n)} \psi = (i_0)_*^{-1} \pi_* (\alpha \cap [M(r, n)]) \in S.\]

By Proposition 2.5 we have a commutative diagram

\[
\begin{array}{ccc}
A^T_2(M(r, n)) \otimes S & \overset{\cong}{\cong} & \bigoplus_{(\alpha)} S \\
\pi_* \downarrow & & \downarrow \Sigma \\
A^T_2(M_0(r, n)) \otimes S & \overset{\cong}{\cong} & S
\end{array}
\]
where \( \iota_{\mathcal{P}} : \{I_{\mathcal{P}}\} \to M(r,n) \) is the inclusion, and \((\iota_{\mathcal{P}})^{-1}(\psi) = \frac{\iota_{\mathcal{P}}^{*}\psi}{\iota_{\mathcal{P}}(TM(r,n))}\). Hence we have

\[
\int_{M(r,n)} \psi = \sum_{\psi} \frac{\iota_{\mathcal{P}}^{*}\psi}{\iota_{\mathcal{P}}(TM(r,n))} \in \mathcal{S}.
\]

We consider the equivariant Euler class \( e(\mathcal{F}_{r}(\mathcal{V})) \) in \( \mathcal{S} \) of a \( \tilde{T} \)-equivariant vector bundle

\[
\mathcal{F}_{r}(\mathcal{V}) = \left( V \otimes \frac{e^{m_{1}}}{\sqrt{t_{1}t_{2}}} \right) \oplus \cdots \oplus \left( V \otimes \frac{e^{m_{2r}}}{\sqrt{t_{1}t_{2}}} \right).
\]

Here we consider a homomorphism \( \tilde{T}' = \tilde{T} \to \tilde{T} \) defined by

\[
(t_{1}',t_{2}',e^{a'},e^{m'}) \mapsto (t_{1},t_{2},e^{a},e^{m}) = ((t_{1}')^{2},(t_{2}')^{2},e^{a'},e^{m'}),
\]

and use identification \( t_{1}' = \sqrt{t_{1}}, t_{2}' = \sqrt{t_{2}} \) and \( A_{\tilde{T}}^{*}(pt) \otimes S \cong \mathcal{S} \). For fixed \( r > 0 \), we put

\[
\alpha_{n} = \alpha_{n}(\varepsilon,a,m) = \int_{M(r,n)} e(\mathcal{F}_{r}(\mathcal{V})) = \sum_{\psi} \frac{\iota_{\mathcal{P}}^{*}e(\mathcal{F}_{r}(\mathcal{V}))}{\iota_{\mathcal{P}}^{*}e(TM(r,n))} \in \mathcal{S},
\]

where we omit \( r \) in notation \( \alpha_{n} \), since in this paper we always fix \( r \) and no confusion does not occur.

We put \( \varepsilon_{+} = \varepsilon_{1} + \varepsilon_{2}, \varepsilon_{+} = (\varepsilon_{+}, \ldots, \varepsilon_{+}) \), and consider \( \beta_{n} = \beta_{n}(\varepsilon,a,m) = \alpha_{n}(\varepsilon,-a,-m) \). We have our main theorem similar to the conjectured relations \( [IMO] \ (4.1), \ (4.2) \).

**Theorem 2.6.** We have

\[
\beta_{n} - \alpha_{n} = \sum_{k=1}^{n} (-1)^{k(r+1)} u_{r}(u_{r} - 1) \cdots (u_{r} - k + 1) \alpha_{n-k},
\]

where \( u_{r} = \frac{\varepsilon_{1}(2 \sum_{|a_{i}|=1} a_{i}^{2} + \sum_{|a_{i}|=1} m_{i})}{\varepsilon_{1} \varepsilon_{2}} \).

We consider the Nekrasov partition function \( Z(\varepsilon,a,m,q) = \sum_{n=0}^{\infty} \alpha_{n}(\varepsilon,a,m)q^{n} \) as in the introduction. Then this theorem says that we have

\[
Z(\varepsilon,-a,-m,q) = (1 - (-1)^{r} q)^{u_{r}} Z(\varepsilon,a,m,q).
\]

Since \( Z(\varepsilon,a,m,q) = Z(-\varepsilon,-a,-m,q) \) as in \( [NY1] \ Lemma \ 6.3 \ (3) \), we have

\[
Z(-\varepsilon,a,m,q) = (1 - (-1)^{r} q)^{u_{r}} Z(\varepsilon,a,m,q).
\]  \( (2) \)

### 2.4 Application

Over each fixed point in \( M(r,n) \) corresponding to \( r\)-tuple \( \tilde{Y} = (Y_{1}, \ldots, Y_{r}) \) of Young diagrams, the fibre of the tautological bundle \( V \) is isomorphic to

\[
\bigoplus_{\alpha=1}^{r} \bigoplus_{(i,j) \in Y_{\alpha}} e^{a_{\alpha}} l_{1}^{i} l_{2}^{j+1}
\]
We note that we substitute as \( \tilde{m} \) where

\[ \lambda_{\alpha,i} = 0 \quad \text{when} \quad i \quad \text{is larger than the width of the diagram} \, Y_\alpha. \]

Let \( Y_\alpha = \{ \lambda_\alpha, 1, \lambda_\alpha, 2, \ldots \} \) be a Young diagram where \( \lambda_{\alpha,i} \) is the height of the \( i \)-th column. We set \( \lambda_{\alpha,i} = 0 \) when \( i \) is larger than the width of the diagram \( Y_\alpha \). Let \( Y_\alpha^T = \{ \lambda_\alpha', 1, \lambda_\alpha', 2, \ldots \} \) be its transpose. For a box \( s = (i, j) \) in the \( i \)-th column and the \( j \)-th row, we define its arm-length \( a_{Y_\alpha}(s) \) and leg-length \( l_{Y_\alpha}(s) \) with respect to the diagram \( Y_\alpha \) by \( a_{Y_\alpha}(s) = \lambda_{\alpha,i} - j \) and \( l_{Y_\alpha}(s) = \lambda_{\alpha,j} - i \). The fibre of \( TM(r,n) \) over a fixed point corresponding to a datum \( \vec{Y} = (Y_1, \ldots, Y_r) \) is isomorphic to \( \bigoplus_{\alpha,i=1}^{\alpha,j} N_{\alpha,\beta}(t_1, t_2) \) as \( \tilde{T} \)-modules, where

\[
N_{\alpha,\beta}(t_1, t_2) = e_\beta e_\alpha^{-1} \times \left\{ \bigoplus_{s \in Y_\alpha} \left( -l_{Y_\alpha}(s) t_2 \right)^{a_{Y_\alpha}(s) + 1} \bigoplus_{t \in Y_{\beta}} \left( l_{Y_\alpha}(t) + 1 \right)^{a_{Y_\alpha}(t) + 1} \right\}.
\]

Thus we have

\[
Z(\varepsilon, \alpha, m, q) = \sum_{n=0}^{\infty} \sum_{|Y|=n} \frac{\prod_{f=1}^{2r} t_\varepsilon^f e(V \otimes \frac{e_m}{\sqrt{t_1 t_2}})}{t_\varepsilon^f TM(r,n)} q^n,
\]

where we substitute

\[
i_\varepsilon^f \left( V \otimes \frac{e_m}{\sqrt{t_1 t_2}} \right) = \prod_{\alpha=1}^{r} \prod_{(i,j) \in Y_\alpha} \left( a_\alpha + \left( -i + \frac{1}{2} \right) \varepsilon_1 + \left( -j + \frac{1}{2} \right) \varepsilon_2 + m_f \right)
\]

\[
i_\varepsilon^f TM(r,n) = \prod_{\alpha,\beta=1}^{r} \left( \prod_{s \in Y_\alpha} (a_\beta - a_\alpha - l_{Y_\alpha}(s) \varepsilon_1 + (a_{Y_\alpha}(s) + 1) \varepsilon_2) \times \prod_{t \in Y_{\beta}} (a_\beta - a_\alpha + (l_{Y_\alpha}(t) + 1) \varepsilon_1 - a_{Y_\alpha}(t) \varepsilon_2) \right).
\]

For application we introduce partition functions for \( 0 \leq N_f \leq 2r \)

\[
Z^{\psi_{N_f}}(\varepsilon, \alpha, m^{N_f}, q) = \sum_{n=0}^{\infty} q^n \int_{M(r,n)} \psi_{N_f}.
\]

where \( m^{N_f} = (m_1, \ldots, m_{N_f}) \) and

\[
\psi_{N_f} = e \left( \bigoplus_{j=1}^{N_f} V \otimes \frac{e_m}{\sqrt{t_1 t_2}} \right).
\]

We note that \( m = m^{2r} \) and \( Z(\varepsilon, \alpha, m, q) = Z^{\psi_{2r}}(\varepsilon, \alpha, m, q) \). We have

\[
Z^{\psi_{N_f - 1}}(\varepsilon, \alpha, m^{N_f - 1}, q) = \left. Z^{\psi_{N_f}}(\varepsilon, \alpha, m^{N_f - 1}, q, q) \right|_{q=0}.
\]

(3)
By (2) we have
\[ \log \frac{Z_{\psi^2 r}(-\varepsilon, a, m, q)}{Z_{\psi^2 r}(\varepsilon, a, m, q)} = u_r \log(1 - (-1)^r q). \]
Hence by (3) we have
\[ \log \frac{Z_{\psi^2 r-1}(-\varepsilon, a, m^{2r-1}, q)}{Z_{\psi^2 r-1}(\varepsilon, a, m^{2r-1}, q)} = (-1)^{r+1} \frac{\varepsilon_+}{\varepsilon_1 \varepsilon_2} q. \]
Similarly, for \(0 \leq N_f \leq 2r - 2\) we have
\[ \log \frac{Z_{N_f}(-\varepsilon, a, m^{N_f}, q)}{Z_{N_f}(\varepsilon, a, m^{N_f}, q)} = 0. \] (4)
Hence we determined the odd degree part of \(\varepsilon_1 \varepsilon_2 \log Z_{N_f}(\varepsilon, a, m^{N_f}, q)\) with respect to \(\varepsilon_1, \varepsilon_2\).

In particular it is equal to zero unless \(N_f \geq 2r - 1\).

3 Reduction to wall-crossing

In this section we reduce a proof of Theorem 2.6 to analysis of wall-crossing phenomena between stable and co-stable ADHM data.

3.1 Quiver description of moduli spaces of framed sheaves

For later purpose we modify the definition of ADHM data following [C]. We introduce a quiver \(\Gamma\) with relations, consisting of two vertices \(0, \infty\), and two arrows \(A_1, A_2\) from \(0\) to \(0\), \(r\) arrows \(\{\gamma_k\}_{k=1}^r\) from \(\infty\) to \(0\) and another \(r\) arrows \(\{\delta_k\}_{k=1}^r\) from \(0\) to \(\infty\). Relations are defined by
\[ [A_1, A_2] + \sum_{k=1}^r \gamma_k \delta_k = 0. \]
\(\Gamma\)-representations \(X\) consist of vector spaces \(X_0, X_\infty\) and linear maps among \(X_0\) and \(X_\infty\) corresponding to \(A_1, A_2, \gamma_k\) and \(\delta_k\) for \(k = 1, \ldots, n\). Then ADHM data on \(V, W\) are equivalent to \(\Gamma\)-representations \(X\) with
\[ X_0 = V, X_\infty = \begin{cases} \mathbb{C} & \text{if } W = \mathbb{C}^r, \\ 0 & \text{if } W = 0, \end{cases} \]
via \(B_1 = A_1, B_2 = A_2, z = \sum \gamma_k w_k^*, w = \sum \delta_k w_k\), where \(w_1, \ldots, w_r\) is the canonical basis of \(W = \mathbb{C}^r\). Hereafter we identify ADHM data \((B, z, w)\) and \(\Gamma\)-representations \(X\). We also write the vector space \(X_0 \oplus X_\infty\) by \(X\).

ADHM data \((B_1, B_2, z, w)\) are called co-stable if \((^tB_2, ^tB_1, ^tw, ^tz)\) are stable. Via the above correspondence between ADHM data and \(\Gamma\)-representations, the stability (resp. costability) is equivalent to \(\zeta(1, -n)\)-stability with \(\zeta < 0\) (resp. \(\zeta > 0\)). We put
\[ M_c^c(r, n) = \{(B, z, w) : \text{co-stable ADHM data on } W = \mathbb{C}^r, V = \mathbb{C}^n\}/GL(V). \]
To construct moduli spaces we introduce two affine spaces

\[ M = \mathbb{M}(r, n) = \mathbb{M}(W, V) = \text{Hom}_{\mathbb{C}}(Q^r \otimes V, V) \oplus \text{Hom}_{\mathbb{C}}(W, V) \oplus \text{Hom}_{\mathbb{C}}(\wedge Q^r \otimes V, W) \]

and \( L = L(n) = L(V) = \text{Hom}_{\mathbb{C}}(\wedge Q^r \otimes V, V) \), and consider a map

\[ \mu : M \to L_n(B, z, w) \mapsto \mu(B, z, w) = \wedge B + zw. \]

For \( \zeta \in \mathbb{R} \), we consider an open locus \( \mu^{-1}(0)^\zeta = \{ X \in \mu^{-1}(0) \mid X \text{ is } \zeta(-1, n) \text{-stable} \} \). If we take \( \zeta > 0 \), then we have \( M(r, n) = [\mu^{-1}(0)^{-\zeta}/G] \) and \( M^c(r, n) = [\mu^{-1}(0)^{\zeta}/G] \), where \( G = \text{GL}(V) \).

In this description the tautological vector bundle \( V \) on \( M(r, n) \) defined in \( \mathbb{22} \) is isomorphic to \( [\mu^{-1}(0)^{-\zeta} \times V/G] \), where \( G \) acts on \( V \) naturally. We also write by \( V \) the similar vector bundle \( [\mu^{-1}(0)^{\zeta} \times V/G] \) on \( M^c(r, n) \) and call tautological bundle. We define the Uhlenbeck compactification by \( M_0(r, n) = \text{Spec} \mathbb{C}[\mu^{-1}(0)]^G \). Then via the above construction we have a \( \tilde{T} \)-equivariant projective morphism \( \pi : M(r, n) \to M_0(r, n) \).

### 3.2 Relations between stable and co-stable ADHM data

We define \( \tilde{T} \)-equivariant morphisms \( D : M(r, n) \to M^c(r, n) \) by a \( G \times \tilde{T} \)-equivariant morphism

\[ \mu^{-1}(0)^{-\zeta} \to \mu^{-1}(0)^{\zeta}, (B_1, B_2, z, w) \mapsto (B_2, B_1, w, z), \]

via a group homomorphism

\[ G \times \tilde{T} \to G \times \tilde{T}, (g, t_1, t_2, e^a, e^m) \mapsto (t_1 t_2 g^{-1}, t_2, t_1, e^{-a}, e^m). \]

For each \( \tilde{T} \)-fixed point \( I_{\tilde{\psi}} \) in \( M(r, n) \) as in Proposition \( \mathbb{22} \), we consider the following embedding of the co-stable point

\[ \iota_{\tilde{\psi}} : \{ D(I_{\tilde{\psi}}) \} \to M^c(r, n). \]

Here we put \( \tilde{Y}^t = (Y_1^t, \ldots, Y_d^t) \) and \( Y_1^t \) are transposes of Young diagrams \( Y_1 \). If we have a decomposition \( \iota_{\tilde{\psi}}^* e(V) = \prod_{k=1}^n F_k(e, a) \) in \( S \) by linear polynomials \( \{ F_k \}_{k=1}^n \), then at \( D(I_{\tilde{\psi}}) \) in \( M^c(r, n) \) we have

\[ \iota_{\tilde{\psi}}^* e(V \otimes e^{m_f}) = \prod_{k=1}^n \left( -F_k(e, -a) + \frac{e_a}{2} + m_f \right) \]

\[ = (-1)^n \prod_{k=1}^n \left( F_k(e, -a) - \frac{e_a}{2} - m_f \right) \]

\[ = (-1)^n \alpha_a(e, a, m_f) \mid_{a=-a, m_f=-m_f}. \]

Similarly we have \( \iota_{\tilde{\psi}}^* e(TM^c(r, n)) = \iota_{\tilde{\psi}}^* e(TM(r, n)) \mid_{a=-a} \). Thus we get

\[ \int_{M^c(r, n)} e(F_\alpha(V)) = \alpha_a(e, -a, -m) = \beta_a(e, a, m). \]
4 Enhanced master spaces

We apply Mochizuki method [Mo] to the quiver description in the previous section following [NY2]. The argument in this section is totally similar to [NY2] except that we do not give a sheaf description. Hence we often omit proofs, but for some statements we give proofs for understandings. See [NY2] for complete proofs.

4.1 ADHM data with full flags

For vector spaces $V = \mathbb{C}^n, W = \mathbb{C}^r$ we consider ADHM data $(B, z, w)$ on $V, W$ with full flags $F^\bullet$ of $V$.

**Definition 4.1.** For $0 \leq \ell \leq n$, $(B, z, w, F^\bullet)$ is $\ell$-stable if the following two conditions hold:

1. For a non-zero subspace $S \subset V$, if $B_1(S), B_2(S) \subset S$ and $S \subset \ker w$, then we have $S \cap F^\ell = 0$.
2. For a proper subspace $T \subset V$, if $B_1(T), B_2(T) \subset T$ and $\im z \subset T$, then we have $F^\ell \not\subset T$.

We write by $\tilde{M}^\ell(r, n)$ moduli of $\ell$-stable ADHM data with full flags of $V$ constructed in the next subsection. We remark that when $\ell = 0$ (resp. $\ell = n$), an object $(B, z, w, F^\bullet)$ is $\ell$-stable if and only if $(B, z, w)$ is stable (resp. co-stable). Hence we see that $\tilde{M}^0(r, n)$ and $\tilde{M}^n(r, n)$ is the full flag bundle of tautological bundles on $M(r, n)$ and $M^c(r, n)$ respectively.

For later analysis we need stability parameters. For $\zeta \in \mathbb{R}$ and $\eta = (\eta_1, \ldots, \eta_n) \in \mathbb{Q}_r^n > 0$, we introduce $(\zeta, \eta)$-stability for ADHM data with full flags as follows. We take a $\Gamma$-representation $X = X_0 \oplus X_\infty$ with a full flag $F^\bullet$ of $X_0$. For a non-zero graded subspace $P = P_0 \oplus P_\infty$ of $X = X_0 \oplus X_\infty$, we define

$$\mu_{(\zeta, \eta)}(P) = \zeta((\dim P_0 - n \dim P_\infty) + \sum \eta_i \dim(P_0 \cap F^i)) \div \dim P_0 + \dim P_\infty.$$  

We say that $(X, F^\bullet)$ is $(\zeta, \eta)$-semistable if for any non-zero proper subrepresentation $P$ of $X$ we have

$$\mu_{(\zeta, \eta)}(P) \leq \mu_{(\zeta, \eta)}(X).$$

If the inequality is always strict unless $P = X$, we say that $(X, F^\bullet)$ is $(\zeta, \eta)$-stable.

Consider the following condition

$$\eta_k > (n + 1) \sum_{i=\ell+1}^{n} i \eta_i,$$  

(6)  

$$\quad (n + 1) \sum_{i=\ell+1}^{n} i \eta_i < \min \left( \sum_{i=1}^{\ell} i \eta_i - (n + 1)\zeta, \sum_{i=1}^{\ell} i \eta_i + (n + 1)\zeta + \eta_\ell \right).$$  

(7)
For example, first we take \( \eta_1, \ldots, \eta_\ell \in \mathbb{Q}_{>0} \) so that it satisfies (6) when we put \( \eta_{\ell+1} = \cdots = \eta_n = 0 \). Then we take \( \zeta > 0 \) so that it satisfies
\[
\sum_{i=1}^\ell i \eta_i - \eta_\ell < (n+1)\zeta < \sum_{i=1}^\ell i \eta_i.
\]
Finally we take \( \eta_{\ell+1}, \ldots, \eta_n \) satisfying (6) and (7).

Here we only use (7). The condition (6) will be used in §4.4.

**Proposition 4.2** (Mo, Proposition 4.2.4, NY2, Lemma 5.6). Assume condition (7). Then the \( \ell \)-stability is equivalent to the \( (\zeta, \eta) \)-stability. Furthermore the \( (\zeta, \eta) \)-semistability automatically implies the \( (\zeta, \eta) \)-stability.

**Proof.** Suppose that we have a non-zero sub-representation \( P \) of \( X \) with \( \dim P_0 = p > 0, P_\infty = 0 \). Then \( \mu_{(\zeta, \eta)}(P) \leq \mu_{(\zeta, \eta)}(X) \) means
\[
\zeta + \frac{\sum_{i=1}^n \eta_i \dim(P_0 \cap F^i)}{p} \leq \frac{\sum_{i=1}^n i \eta_i}{n+1}.
\]
By (7) this holds if and only if \( P_0 \cap F^\ell = 0 \). Moreover, the equality never holds.

Next suppose that we have a non-zero proper sub-representation \( P \) of \( X \) with \( \dim X_0/P_0 = p > 0, P_\infty = \mathbb{C} \). Then \( \mu_{(\zeta, \eta)}(P) \leq \mu_{(\zeta, \eta)}(X) \) means \( \mu_{\zeta, \eta}(X/P) \geq \mu_{\zeta, \eta}(X) \)
\[
\zeta + \frac{\sum_{i=1}^n \eta_i \dim(F^i/P_0 \cap F^i)}{p} \geq \frac{\sum_{i=1}^n i \eta_i}{n+1}.
\]
By (7) this holds if and only if \( F^\ell \not\subset P_0 \). Moreover, the equality never holds. \(\square\)

We also consider the following condition on \( \eta \):
\[
\sum_{i=1}^n k_i \eta_i \neq 0 \text{ for any } (k_1, \ldots, k_n) \in \mathbb{Z}^n \setminus \{0\} \text{ with } |k_i| \leq n^2.
\] (8)

From NY2 we have the following lemma.

**Lemma 4.3** (Mo, Lemma 4.3.9, NY2, Lemma 5.16). Assume that \( \eta \) satisfies (5). If \( (X, F^*) \) is \( (\zeta, \eta) \)-semistable, then its stabilizer is either trivial or \( \mathbb{C}^* \). In the latter case \( (X, F^*) \) has a unique decomposition \( (X_3, F_3^*) \oplus (X_4, F_4^*) \) such that both \( (X_3, F_3^*) \) and \( (X_4, F_4^*) \) are \( (\zeta, \eta) \)-stable, and \( \mu_{(\zeta, \eta)}(X_3) = \mu_{(\zeta, \eta)}(X_2) \). The stabilizer comes from that of the factor \( (X_4, F_4^*) \) with \( (X_4)_\infty = 0 \).
4.2 Moduli stacks and $C_h^*$-action

We consider ADHM data on $V = \mathbb{C}^n, W = \mathbb{C}^r$. Let $\mathbb{n}$ denote the set $\{1, \ldots, n\}$ of integers from 1 to $n$ and $F = F(V, \mathbb{n})$ denote the full flag variety of $V$. We consider natural projections $\rho_i : F \to G_i = Gr(V, i)$ to Grassmanian manifolds $G_i$ of $i$-dimensional subspace of $V$ and pull-backs $\rho_i^* \mathcal{O}_{G_i}(1)$ by Plucker embeddings.

In the following, for $\ell = 1, \ldots, n$ we take $\zeta^- < 0, \zeta > 0$ and $\eta \in \mathbb{Q}_{>0}^n$ such that $(\zeta, \eta)$ satisfies (6), (7), and $\eta$ satisfies (8), and $|\zeta|, |\eta|$ are sufficiently smaller than $|\zeta^-|$. We take a positive integer $k$ enough divisible such that $k\zeta, k\zeta^-$ and $k\eta$ are all integer valued, and consider ample $G$-linearizations

$$
L^+ = (\hat{M} \times (\det V)^{\otimes k\zeta}) \boxtimes \bigotimes_{i=1}^n \rho_i^* \mathcal{O}_{G_i}(k\eta_i),
$$

$$
L^- = (\hat{M} \times (\det V)^{\otimes k\zeta^-}) \boxtimes \bigotimes_{i=1}^n \rho_i^* \mathcal{O}_{G_i}(k\eta_i)
$$
on $\hat{M} = \hat{M}(r, n) = M \times F$, where $M = \hat{M}(r, n)$ is used in (5) to define $\mu : \hat{M} \to L$. We consider the composition $\hat{\mu} : \hat{M} \to L$ of the projection $\hat{M} \to M$ and $\mu : M \to L$, and semistable loci $\hat{\mu}^{-1}(0)^+$ and $\hat{\mu}^{-1}(0)^-$ with respect $L^+$ and $L^-$ respectively. Then by our choice of $\zeta^-, \zeta, \eta$ we have $\hat{M}^+(r, n) = [\hat{\mu}^{-1}(0)^+/G]$ and $M(r, n) \times F = [\mu^{-1}(0)^-/G]$.

We put $\hat{\mathcal{M}} = \hat{\mathcal{M}}(r, n) = \mathcal{P}(L^- \oplus L^+)$ and consider a composition $\hat{\mu} : \hat{\mathcal{M}} \to L$ of the projection $\hat{\mathcal{M}} \to M$ and $\mu : M \to L$. Then we have a natural $G = GL(V)$-action on $\hat{\mathcal{M}}$ keeping $\hat{\mu}^{-1}(0)$, and the $G$-equivariant tautological line bundle $\mathcal{O}(1)$ on $\hat{\mu}^{-1}(0)$ defining semistable locus $\hat{\mu}^{-1}(0)$.

We define enhanced master space by $\mathcal{M} = [\hat{\mu}^{-1}(0)^+/G]$. The projection $\hat{\mu}^{-1}(0) \to \mu^{-1}(0)$ induces a proper morphism $\Pi : \mathcal{M} \to M_0(r, n)$.

We have a $C_h^*$-action on $\mathcal{M}$ defined by

$$
(X, F^*, [z_-, z_+]) \mapsto (X, F^*, [e^h z_-, z_+]).
$$

Since $\mathcal{M} \subset [\hat{\mathcal{M}}(r, n)/G]$ is a separated Deligne-Mumford stack, we have an open substack $\mathcal{N}$ of $[\hat{\mathcal{M}}(r, n)/G]$ containing $\mathcal{M}$ such that $\mathcal{N}$ is a $\hat{T} \times C_h^*$-invariant smooth Deligne-Mumford stack. $C_h^*$-fixed points set $\mathcal{N}^{C_h^*}$ of $\mathcal{N}$ is defined by the zero locus of vector fields generated by the $C_h^*$-action. We define $C_h^*$-fixed points set $\mathcal{M}^{C_h^*}$ of $\mathcal{M}$ by $\mathcal{M}^{C_h^*} = \mathcal{M} \times_{\mathcal{N}} \mathcal{N}^{C_h^*}$ (cf. [GP, Appendix C]).

4.3 Hilbert schemes parametrizing destabilizing objects

We introduce another stability condition.

**Definition 4.4.** $(X, F^*)$ is $+$-stable if $X_\infty = 0$ and for any $B$-invariant proper subspace $S \subset X_0$ we have $S \cap F^1 = 0$. 

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We consider a moduli stack $M_p^+ = \left[ \left( \mu^{-1}(0) \times \mathbb{C}_{p^*} \right)^+ / \text{GL}(V) \right]$ of $+$-stable objects. Here $\left( \mu^{-1}(0) \times \mathbb{C}_{p^*} \right)^+$ is the $+$-stable locus, and $\text{GL}(V)$ acts on $\mathbb{C}_{p^*}$ by $g \cdot p_1 = (\det g)^{-D} p_2$ for $g \in \text{GL}(V)$. This space $M_p^+$ parametrizes $+$-stable ADHM data on $V_2 = \mathbb{C}^p$ with full flags of $V_2$ and orientations $\rho$: $\det V_2^\otimes D \cong \mathbb{C}$ for $D = k(\zeta - \zeta^-) \in \mathbb{Z}$.

On the other hand, we consider the moduli $M(1, p)$ of ADHM data on $V_2 = \mathbb{C}^p, W_2 = \mathbb{C}$, and a line bundle $(\det V_2)^D$ on $M(1, p)$, where $V_2$ is the tautological bundle on $M(1, p)$ corresponding to $V_2$.

**Proposition 4.5** ([NY2 Proposition 5.9]). We have the following.

(1) $M_p^+$ is a full flag bundle $\text{Flag}(\mathcal{V}_2/\mathcal{O}_{M(1,p)}(p-1))$ over the quotient stack $\left[ \left( (\det \mathcal{V}_2)^D \right)^\times / \mathbb{C}_u^* \right]$, where $\mathbb{C}_u^*$ acts on $(\det \mathcal{V}_2)^\times$ by fiber-wise multiplication of $u^{-pD}$.

(2) The homomorphism $\mathbb{C}_u^* \rightarrow \mathbb{C}_u^*$ given by $s = u^{-pD}$ induces an étale and finite morphism
\[
\left[ \left( (\det \mathcal{V}_2)^D \right)^\times / \mathbb{C}_u^* \right] \rightarrow M(1, p).
\]

**Proof.** (1) For any element $(B_1, B_2, F^*, \rho) \in M_p^+$, the $+$-stability implies that $\mathbb{C}[B_1, B_2]F^1 = V_2$. This means that for any non-zero element $v \in F^1$, $(B_1, B_2, v, 0)$ is a stable ADHM data on $V_2, W_2 = \mathbb{C}$. Hence by forgetting $F^i$ for $i > 1$, we get elements in
\[
\{(B, v, 0, \rho) \in M(W_2, V_2) \times \mathbb{C}^* \mid (B, v, 0) \text{ is stable ADHM data }\} / \text{GL}(V_2) \times \mathbb{C}_u^*],
\]
where $(g, u) \in \text{GL}(V_2) \times \mathbb{C}_u^*$ acts by $(g B_1 g^{-1}, g B_2 g^{-1}, u g v, 0, (\det g)^D \rho)$. Then this space is isomorphic to $\left[ \left( (\det \mathcal{V}_2)^D \right)^\times / \mathbb{C}_u^* \right]$ by a homomorphism $\text{GL}(V_2) \times \mathbb{C}_u^* \rightarrow \text{GL}(V_2) \times \mathbb{C}_u^*$ defined by $(g, u) \mapsto (ug, u)$.

(2) It follows from (1) since $\left[ \left( (\det \mathcal{V}_2)^D \right)^\times / \mathbb{C}_u^* \right] = M(1, p)$.

We write by $h: M_p^+ \rightarrow M(1, p)$ the composition of the morphisms $M_p^+ \rightarrow \left[ \left( (\det \mathcal{V}_2)^D \right)^\times / \mathbb{C}_u^* \right]$ and $\left[ \left( (\det \mathcal{V}_2)^D \right)^\times / \mathbb{C}_u^* \right] \rightarrow M(1, p)$ in the above proposition. The pull-back $h^*\mathcal{V}_2$ are also denoted by $\mathcal{V}_2$.

### 4.4 Decompositions of fixed points sets of $\mathbb{C}_h^*$-action

We take $x \in P(L^- \oplus L^+) \setminus (P(L^-) \sqcup P(L^+))$ over $(X, F^*)$. 


Lemma 4.6 (cf. [T1 Sections 3, 4]). A point \( x \in \mathbb{P}(L_- \oplus L^+) \setminus (\mathbb{P}(L_-) \sqcup \mathbb{P}(L_+)) \) is semistable if and only if \((X, F^\bullet) = (\zeta', \eta)\)-semistable for some \( \zeta' \) on the segment connecting \( \zeta^- \) and \( \zeta \).

We assume that \( x \) represents a \( C^*_h \)-fixed point in \( \mathcal{M} \), then \( x \) has a non-trivial stabilizer group of \( G \). By Lemma 3.3 \((X, F^\bullet)\) is strictly \( (\zeta', \eta)\)-semistable and we have a direct sum decomposition \((X, F^\bullet) = (X_2, F_2^\bullet) \oplus (X_\ell, F_\ell^\bullet)\) with \((X_\ell)_\infty = 0\), and \( \mu(\zeta', \eta)(X_2) = \mu(\zeta', \eta)(X). \)

We put \( I_\alpha = \{ i \in \mathbb{N} | F_\alpha^i/F_\alpha^{i-1} \neq 0 \} \) for \( \alpha = \gamma, \sharp \) so that \( \mathbb{N} = I_\gamma \sqcup I_\sharp \). The datum \((I_\gamma, I_\sharp)\) is called the decomposition type of the \( C^*_h \)-fixed point represented by \( x \).

Lemma 4.7 ([Mo Lemma 4.4.3], [NY2 Lemma 5.25]). We have \( \min(I_\sharp) \leq \ell \).

Conversely suppose that an object \((X, F^\bullet) = (X_\gamma, F_\gamma^\bullet) \oplus (X_\sharp, F_\sharp^\bullet)\) with the decomposition type \((I_\gamma, I_\sharp)\) with \( \min(I_\sharp) \leq \ell \) is given. Then by \( [T1] \) we have \( \mu(\zeta', \eta)(X_\gamma) > \mu(\zeta', \eta)(X) \). On the other hand since \(|I_\sharp|\) is enough smaller than \(|\zeta^-|\), we have \( \mu(\zeta', \eta)(X_\gamma) > \mu(\zeta', \eta)(X) \). Hence we can find \( \zeta' \) on the segment connecting \( \zeta \) and \( \zeta^- \) such that \( \mu(\zeta', \eta)(X_\gamma) = \mu(\zeta', \eta)(X) \).

Lemma 4.8 ([Mo Proposition 4.4.4], [NY2 Lemma 5.26]). We have the following.

(1) \((X_\gamma, F_\gamma)\) is \((\zeta', \eta)\)-stable if and only if it is \((\min(I_\gamma) - 1)\)-stable.

(2) \((X_\sharp, F_\sharp)\) is \((\zeta', \eta)\)-stable if and only if it is \((\sharp)\)-stable.

Proof. Let \( S \subset X_\gamma \) be a submodule. We first suppose \( S_\infty = 0 \). Then the inequality \( \mu(\zeta', \eta)(S) < \mu(\zeta', \eta)(X_\gamma) = \mu(\zeta', \eta)(X_\gamma) \) is equivalent to

\[
\sum_i \eta_i \dim(S_0 \cap F_\gamma^i) < \sum_i \eta_i \dim(F_\gamma^i) \frac{\dim S_0}{p}.
\]

Since \( \eta_i \) for \( i \geq \min(I_\gamma) \) is much smaller than \( \eta_{\min(I_\gamma) - 1} \) by \( \Box \), if the inequality holds, then we must have \( S_0 \cap F_{\gamma}^{\min(I_\gamma) - 1} = 0 \). Conversely if we have \( S_0 \cap F_{\gamma}^{\min(I_\gamma) - 1} = 0 \), then since \( F_{\gamma}^{\min(I_\gamma) - 1} = F_{\gamma}^{\min(I_\gamma)} \) again by \( \Box \) the above inequality holds.

Next, suppose \( S_\infty = \mathbb{C} \). Then the inequality \( \mu(\zeta', \eta)(S) < \mu(\zeta', \eta)(X_\gamma) = \mu(\zeta', \eta)(X_\gamma) \) is equivalent to

\[
\sum_i \eta_i \dim(F_\gamma^i/S_0 \cap F_\gamma^i) \frac{\dim(X_\gamma/S_0)}{p} > \sum_i \eta_i \dim(F_\gamma^i) \frac{\dim(X_\gamma)}{p}.
\]

This is equivalent to \( F_{\gamma}^{\min(I_\gamma) - 1} \subset S_0 \) by the same argument as the above. Thus \((X_\gamma, F_\gamma^\bullet)\) is \((\zeta', \eta)\)-stable if and only if it is \((\min(I_\gamma) - 1)\)-stable. \( \Box \)

We assume that \((X, F^\bullet) = (X_\gamma, F_\gamma^\bullet) \oplus (X_\sharp, F_\sharp^\bullet)\) satisfies conditions in the above lemma. Let \( V = V_\gamma \oplus V_\sharp \) be a corresponding direct sum decomposition of \( V \). Then we have

\[
(id_{V_\gamma} \oplus e^{p\pi i})(X, F^\bullet, [e^b z_-, z_+]) = (X, F^\bullet, [z_-, z_+]).
\]

Hence \( x \) represents a \( C^*_h \)-fixed point in \( \mathcal{M} \). These observations lead to the following theorem.
Theorem 4.9 ([NY2 Theorem 5.18]). For the above $\mathbb{C}_h^*$-action on $\mathcal{M}$ defined by $[z_-, z_+] \mapsto [e^h z_-, z_+]$ we have

$$\mathcal{M}^{\mathbb{C}_h^*} = \mathcal{M}_+ \sqcup \mathcal{M}_- \sqcup \bigsqcup_{\mathcal{J} \in S^\ell} \mathcal{M}_3,$$

where $\mathcal{M}_\pm = \{z_\pm = 0\}$, $S^\ell = \{\mathcal{J} = (I_1, I_2) \mid \mathbb{N} = I_3 \sqcup I_2, I_2 \neq 0, \min(I_1) \leq \ell\}$, and for $\mathcal{J} \in S^\ell$

$$\mathcal{M}_3 = \{(X_3, \tilde{X}_3, \rho) \mid \tilde{X}_3 = (X_3, F^*_3) \text{ is $\ell$-stable}, X_3 = (X_3, F^*_3) \text{ is $+$-stable}, \rho : \det^{\otimes D} \cong \mathbb{C}\}.$$

We have finite étale morphisms $F : S_3 \to \mathcal{M}_3$, $G : S_3 \to \tilde{M}^\ell(r, n - p) \times M^+_p$ of degree $\frac{1}{p \mathbb{T}}$, where $p = |I_1|, D = k(\zeta - \zeta^-)$. Furthermore there is a line bundle $L_{S_3}$ on $S_3$ such that $L_{S_3}^{\otimes D} \cong G^*(\det V^\vee)$, and we have $F^* \mathcal{V} = G^* V_3 \oplus \left(G^*(V_3 \otimes e^{\mathbb{T} F}) \otimes L_{S_3}\right)$ as a $\mathbb{C}_h^*$-equivariant vector bundles on $S_3$, where $V_3, V_4$ are tautological bundles on $M^{\min(I_1)-1}(r, n - p), M^+_p$ corresponding to $V_3, V_4$ respectively.

Proof. We have to only show assertions about finite étale morphisms $F, G$ and a line bundle $L_S$. First we study stack theoretic fixed points sets $\mathcal{M}_3$ for $\mathcal{J} = (I_1, I_2) \in S^\ell$.

We fix a decomposition $V = V_3 \oplus V_4$ such that $V_3 = \mathbb{C}^{n-p}, V_4 = \mathbb{C}^p$, where $p = |I_1|$. Then the decomposition type $\mathcal{J}$ gives a closed embedding $F_3 : F(V_3, p) \times F(V_4, p) \subset F(V, p)$ such that $G$-orbits of points in $F(V_3, p) \times F(V_4, p)$ cover $F(V, p)$, and $GL(V_3) \times GL(V_4)$ is the subgroup of elements in $G$ keeping $F(V_3, p) \times F(V_4, p)$. Hence if we put $X = [\tilde{M}(r, n) / G]$, we have

$$X \cong \left[\left(M(r, n) \times F(V_3, p) \times F(V_4, p)\right) \times \tilde{M}\right] / (GL(V_3) \times GL(V_4)).$$

We consider a group action $\mathbb{C}_h^* \times \mathcal{M} \to \mathcal{M}$ defined by

$$(X, F^*, [z_-, z_+]) \mapsto \left(\text{id}_{V_3} \oplus e^{\mathbb{T} F} \text{id}_{V_4}\right) \left(X, F^*, [e^h z_-, z_+]\right)$$

(10)

so that data $(X_3 \oplus X_4, F^*_3 \oplus F^*_4, [z_-, z_+])$ representing $\mathbb{C}_h^*$-fixed points in $\mathcal{M}$ of decomposition type $\mathcal{J}$ are fixed, where $(X_3, F^*_3)$ and $(X_4, F^*_4)$ are ADHM data with full flags on $V_3, W$ and $V_4, 0$ respectively. This action is equal to the original $\mathbb{C}_h^*$-action (9), since the difference is absorbed in $G$-action. Using this we can see that we have an open subset $X_3 \subset X$ such that

$$X_3^{\mathbb{C}_h^*} = \left[\left(M(W, V_3) \times M(0, V_4)\right) \times \tilde{M}\right] / (GL(V_3) \times GL(V_4)).$$

(11)

We put $\tilde{M} = \tilde{M}(W, V_3) / GL(V_3)$, where $\tilde{M} : \tilde{M}(W, V_3) \to \mathbb{L}$, and $g_\delta \in GL(V_3)$ acts by $g_\delta \cdot \rho_\delta = (\det g_\delta)^D \rho_\delta$. Then by Lemma 4.8 and (11) we have an isomorphism

$$\mathcal{M}_3 = \left(\mathcal{M} \times M^+_p\right) / \mathbb{C}_h^* \cong \mathcal{M} \times_{X} X_3^{\mathbb{C}_h^*}.$$
In this paper all moduli stacks are constructed by the following way. Let $X$, $G$, $V$, $\rho$, $\rho'$ be $H$-equivariant vector bundles on $Y$ and $\rho$ acts trivially on $V$. Then we have an induced $H$-equivariant vector bundle $\rho' \otimes V$. Hence we have $S_3 \cong \bar{M}/C_\ast^\prime \times M^+_p$ induced by a group homomorphism.

By $C_\ast^\prime$-action on $X$ we have $\rho(V|_{M_3}) = V_\circ \otimes \left(V_\circ \otimes e^{\frac{2\pi i}{\hbar}} \right)$. We put $S_3 = (\bar{M} \times M^+_p)/C_\ast$, where $s = u^{\hbar}$. Then $C_\ast \to C_\ast^\prime$, $s \mapsto u = s^pD$ induces a finite morphism $F: S_3 \to M_3$ of degree $\frac{1}{p!}D$, and we have $S_3 \cong \bar{M}/C_\ast^\prime \times M^+_p$ since $(X_\circ, s^{-pD} \rho_\circ) = s^{-1}V_\circ(X_\circ, \rho_\circ)$ and $C_\ast$ acts on $M^+_p$ trivially. Hence we have $F^*(V|_{M_3})$ corresponds to $V_\circ \otimes \left(V_\circ \otimes e^{\frac{2\pi i}{\hbar}} \otimes s\right)$ via the isomorphism $\rho(V|_{M_3})$.

On the other hand we have $\bar{M}/C_\ast^\prime \cong \bar{M}^{\min(\mathbb{Z})}_{\min(\mathbb{Z})} (r,n-p)$ and we have a finite morphism $G: S_3 \to \bar{M}^{\min(\mathbb{Z})}_{\min(\mathbb{Z})} (r,n-p)$ induced by $C_\ast^\prime \to C_\ast$, $s \mapsto u = s^pD$. The determinant line bundle $\det V_\circ$ on $\bar{M}/C_\ast^\prime \cong \bar{M}^{\min(\mathbb{Z})}_{\min(\mathbb{Z})} (r,n-p)$ corresponds to $\mathbb{C} \otimes u^{-1}$ as a $C_\ast^\prime$-equivariant line bundle, hence $G^*(\det V_\circ^\vee)$ corresponds to $\mathbb{C}$-equivariant line bundle $\mathbb{C} \otimes s^pD$ on $\bar{M}/C_\ast^\prime \times M^+_p$. If we put $L_S = \mathbb{C} \otimes s$ on $S_3$, we have $L^\otimes_{S} = G^*(\det V_\circ^\vee)$ and $F^*V = G^*V_\circ \otimes \left(G^* \left(V_\circ \otimes e^{\frac{2\pi i}{\hbar}} \right) \otimes L_S \right)$ on $S_3$.

5 Obstruction theories

In this section we compute and compare obstruction theories among moduli stacks. We recall that an obstruction theory for a Deligne-Mumford stack $\mathcal{Z}$ is a homomorphism $\text{ob}_\mathcal{Z}: Ob_{\mathcal{Z}} \to L_{\mathcal{Z}}$ in the derived category $D(\mathcal{Z})$ of quasi-coherent sheaves on $\mathcal{Z}$ such that the cohomology $H^i(Ob_{\mathcal{Z}})$ of the complex $Ob_{\mathcal{Z}}$ is coherent for $i = -1, 0, 1$, $H^i(\text{ob}_{\mathcal{Z}})$ are isomorphisms for $i \geq 0$, and $H^{-1}(\text{ob}_{\mathcal{Z}})$ is surjective. It is called perfect, if it is quasi-isomorphic to a complex of locally free sheaves $F^{-1} \to F^0 \to F^1$ in the derived category $D(\mathcal{Z})$. For more details, see [Mo] §2.4.

5.1 Setting

In this paper all moduli stacks are constructed by the following way. Let $Y = \mathbb{A}^1$ be an affine space, $X$ a smooth scheme with a group $H$-action $\rho$, and $H$-equivariant morphism $\varphi: X \to Y$ with respect to $\rho$ and the trivial $H$-action on $Y$. Then we have an induced $H$-action on $\varphi^{-1}(0)$. We take semistable locus $\varphi^{-1}(0)^{ss}$ of $\varphi^{-1}(0)$ with respect to a stability condition for the $H$-action and define the quotient stack $\mathcal{Z} = [\varphi^{-1}(0)^{ss}/H]$. 

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To construct an obstruction theory \( ob_Z : Ob_Z \to L_Z \), we identify coherent sheaves on \( Z \) and \( H \)-equivariant coherent sheaves on \( \varphi^{-1}(0)^{ss} \). Let \( I \) denote the ideal of the definition of \( \varphi^{-1}(0) \) in \( X \) generated by the pull-back of the affine coordinate \( y_1, \ldots, y_l \) of \( Y \). We consider a \( H \)-equivariant complex

\[
0 \to I/I^2|_{\varphi^{-1}(0)^{ss}} \to \Omega_X|_{\varphi^{-1}(0)^{ss}} \nu^* \to h^* \otimes \mathcal{O}_{\varphi^{-1}(0)^{ss}} \to 0,
\]

where \( \Omega_X|_{\varphi^{-1}(0)^{ss}} \) stands on degree 0, and \( h \) is the Lie algebra of \( H \). The last map \( \nu \) is the restriction to \( \{1_H\} \times \varphi^{-1}(0)^{ss} \) of the composition of the pull-back \( \rho^*: \Omega_X \to \Omega_{H \times X} \) by the group action \( \rho \) and the projection \( \Omega_{H \times X} \to h^* \otimes \mathcal{O}_{H \times X} \). We have a homomorphism from this complex \([12]\) to \( L_Z \), induced by the transitivity triangle. This is an isomorphism after truncation at degree \(-1\) by [11] Corollaire 3.1.3 in chapter III.

By composing surjections \( \varphi^* \Omega_Y|_{\varphi^{-1}(0)} \to I/I^2, dy_i \mapsto \varphi^*(y_i) \mod I^2 \) in \([12]\), we define a complex \( Ob_Z \) by the following \( H \)-equivariant complex:

\[
0 \to \varphi^* \Omega_Y|_{\varphi^{-1}(0)} \nu^* \to \Omega_X|_{\varphi^{-1}(0)^{ss}} \nu^* \to h^* \otimes \mathcal{O}_{\varphi^{-1}(0)^{ss}} \to 0,
\]

and we have a homomorphism from \( Ob_Z \) to the complex \([12]\). This gives a perfect obstruction theory \( ob_Z : Ob_Z \to L_Z \) by composing with the above homomorphism from the complex \([12]\) to \( L_Z \). When \( \varphi^{-1}(0) \) is a complete intersection, then by [11] Corollaire 3.2.7 in chapter III and [MG] Proposition 2.3.3, the obstruction theory \( ob_Z \) is an isomorphism. This complex is the dual of the tangent complex in [NY2] §4.2 in the case where we consider moduli of framed sheaves on the blow-up of the plane.

We suppose that \( X, Y \) admit \( \tilde{T} \times C_H^a \)-actions compatible with \( H \)-actions such that \( \varphi \) is \( \tilde{T} \times C_H^a \)-equivariant. Then \( Ob_Z \) and \( L_Z \) have \( \tilde{T} \times C_H^a \)-equivariant structure and \( ob_Z \) is also \( \tilde{T} \times C_H^a \)-equivariant. By [BP] we have a virtual fundamental cycle \([Z]^{vir} \) in \( \lim_{\leftarrow n} A_{C_H}^C (Z \times_{\tilde{T}} E_n) \), where \( E_n \to E_n/\tilde{T} \) is a finite dimensional approximation of the classifying space \( E\tilde{T} \to B\tilde{T} \). Furthermore if \( Z \) is a separated Deligne-Mumford stack, then we have an open substack \( X \) of \( [X/\tilde{H}] \) containing \( Z \) such that \( X \) is a smooth Deligne-Mumford stack kept by \( \tilde{T} \times C_H^a \)-action. \( C_H^a \)-fixed points set \( X^{C_H^a} \) of \( X \) is defined by the zero locus of vector fields generated by the \( C_H^a \)-action. We define \( C_H^a \)-fixed points set \( Z^{C_H^a} \) of \( Z \) by \( Z^{C_H^a} = Z \times_X X^{C_H^a} \).

For components \( Z_i \) of \( Z^{C_H^a} \), restrictions \( (Ob_Z)|_{Z_i} \) has a fibrewise \( \tilde{T} \times C_H^a \)-equivariant structures, hence we have direct sum decompositions of \( (Ob_Z)|_{Z_i} \) into invariant parts \( (Ob_Z)|_{Z_i}^{in} \) and moving parts \( (Ob_Z)|_{Z_i}^{mov} \). By [GP] invariant parts \( (Ob_Z)|_{Z_i}^{in} \) define perfect obstruction theories of \( Z_i \), hence virtual fundamental cycles \([Z_i]^{vir} \) in \( \lim_{\leftarrow n} A_{C_H}^C (Z_i \times_{\tilde{T}} E_n) \). We define virtual normal bundles \( \mathcal{N}(Z_i) \) of \( Z_i \) in \( Z \) by \( \mathcal{N}(Z_i) = (Ob_Z)|_{Z_i}^{mov} \) in the \( K \)-group of \( Z \). By [GP] we have

\[
[Z]^{vir} = \sum_{i} t_i \cdot [Z_i]^{vir} \in \lim_{\leftarrow n} A_{C_H}^C (Z \times_{\tilde{T}} E_n),
\]

(14)
where \( \iota_i : Z_i \to Z \) is the inclusion.

For example if we put \( Y = \mathbb{L}, X = \hat{M}, \varphi = \mu, H = G \) and take the stability condition with respect to \( O(1) \) as in the previous section, then we get \( Z = \mathcal{M} \) the enhanced master space. Since \( \hat{\mu} \) is \( \hat{T} \times \mathbb{C}^*_g \)-equivariant, and \( \hat{\mu}^{-1}(0) \) is a complete intersection by [9], Theorem 1.2], we have a \( \hat{T} \times \mathbb{C}^*_g \)-equivariant perfect obstruction theory \( \text{ob}_{\mathcal{M}} = \text{id}_{\mathcal{M}} : \mathcal{O}_{\mathcal{M}} = L_{\mathcal{M}} \) described by [13]. Similarly for \( \mathcal{M}_{\pm} = \{ z_{\pm} = 0 \} \), we have \( \hat{T} \times \mathbb{C}^*_g \)-equivariant perfect obstruction theories \( \text{ob}_{\mathcal{M}_{\pm}} = \text{id}_{\mathcal{M}_{\pm}} : \mathcal{O}_{\mathcal{M}_{\pm}} = L_{\mathcal{M}_{\pm}} \) described by [13]. It is easy to see that the following lemma holds.

**Lemma 5.1 ([Mo] Proposition 5.9.2).** We have \( (\iota_{\pm}^* \text{Ob}_{\mathcal{M}})^{inv} = \text{Ob}_{\mathcal{M}_{\pm}} \) and \( \mathfrak{N}(\mathcal{M}_{\pm}) = e^{\pm h} \).

In the following, for vector bundles \( \mathcal{E}, \mathcal{F} \) on a stack \( Z \), we write by \( \mathcal{H}om(\mathcal{E}, \mathcal{F}) \) the vector bundle \( \mathcal{E}^\vee \otimes \mathcal{F} \) on \( Z \).

### 5.2 Obstruction theories for decomposition

We consider a decomposition type \( \mathfrak{d} = (I_5, I_4) \in S^f \) and fix a decomposition \( V = V_5 \oplus V_2 \) with \( \dim V_2 = |I_4| = p \). We consider \( \tilde{M}^{min(I_4)-1}(r, n - p) \) in Theorem 4.9 as a moduli spaces of ADHM data on \( W, V_5 \) with full flags of \( V_5 \). We write by \( W \) the tautological bundle on \( \tilde{M}^{min(I_4)-1}(r, n - p) \) corresponding to \( W \). We consider \( \mathbb{C}^*_g \)-action on \( \mathcal{M} \) as in (10), and put \( V'_2 = V_2 \otimes e^{\frac{2\pi i}{p}} \otimes (\det V_5)^{-\frac{1}{2}} \). We consider universal flags \( \mathcal{F}_y \) and \( \mathcal{F}_2 \) on \( \tilde{M}^{min(I_4)-1}(r, n - p) \) and \( M^+_p \).

An obstruction theory \( \text{ob}_{\mathcal{M}_3} \) is given as in the previous subsection taking

\[
\varphi = \mu_5 \times \mu_2 : \hat{M}(W, V_5) \times \mathbb{C}^*_p \times \mathcal{M}_p \to \mathbb{L}(V_5) \times \mathbb{L}(V_2),
\]

\[
H = \text{GL}(V_5) \times \mathbb{C}^*_p \times \text{GL}(V_2) \times \mathbb{C}^*_p, \text{ and the product of } (\min(I_4) - 1)\text{-stability and } +\text{-stability. Here } \mu_5 : \hat{M}(W, V_5) \times \mathbb{C}^*_p \to \mathbb{L}(V_5) \text{ is the composition of the projection } \hat{M}(W, V_5) \times \mathbb{C}^*_p \to \mathbb{M}(W, V_5) \text{ and } \mu : \mathbb{M}(W, V_5) \to \mathbb{L}(V_5).
\]

**Proposition 5.2 ([Mo] Proposition 5.9.3).** We have the following.

1. We have \( (\iota_5^* \text{Ob}_{\mathcal{M}})^{inv} = \text{Ob}_{\mathcal{M}_3} \).
2. We have \( F^* \mathfrak{N}(\mathcal{M}_3) = G^*(N_0 + \mathfrak{N}(V_5, V'_2, W)) \) in the equivariant \( K \)-group of \( S_3 \), where

\[
N_0 = \bigoplus_{i > j} \left( \text{Hom} \left( \mathcal{F}_5^i / \mathcal{F}_5^{i-1}, \mathcal{F}_5^j / \mathcal{F}_5^{j-1} \right) \oplus \text{Hom} \left( \mathcal{F}_5^i / \mathcal{F}_5^{i-1}, \mathcal{F}_5^j / \mathcal{F}_5^{j-1} \right) \right)
\]

\[
\mathfrak{N}(V_5, V'_2, W) = \text{Hom}(Q^\vee \otimes V_5, V'_2) + \text{Hom}(Q^\vee \otimes V'_2, V_5) + \text{Hom}(V_5, V'_2) + \text{Hom}(\wedge^2 Q^\vee \otimes V_5, V'_2) + \text{Hom}(\wedge^2 Q^\vee \otimes V'_2, V_5) - \text{Hom}(V_5, V'_2) - \text{Hom}(V'_2, V_5).
\]
Proof. Using $\mathbb{C}^*_p$-action by (10) we compute $(i^*_\mathcal{L}\Omega_{\mathcal{M}})^{inv}$ and $(i^*_\mathcal{L}\Omega_{\mathcal{M}})^{inv} = \mathcal{N}(\mathcal{M}_3)$. By Theorem 1.2 we have the assertion.

We describe the cotangent complex $L_{\mathcal{M}_p^+}$ by (13) taking $\varphi: X \to Y$ and $H$ in (5.1) as follows. We consider an open subset

$$U = \{(B, z, w) \in \mathcal{M}(W_\mathcal{V}, V_\mathcal{V}) \mid z \neq 0\}$$

of $\mathcal{M}(1, p)$, where $W_\mathcal{V} = \mathbb{C}, V_\mathcal{V} = \mathbb{C}^p$. Then $z \in \text{Hom}(W_\mathcal{V}, V_\mathcal{V})$ gives an inclusion $\mathcal{O}_U \to V_\mathcal{V} = U \times V_\mathcal{V}$ of vector bundles on $U$. We consider the direct sum decomposition

$$\mathcal{M}_p^+ = \mathcal{O}(\mathcal{V}_\mathcal{W}/\mathcal{O}_U, p-1) \times \mathbb{C}^*_p.$$

As $\varphi$ in (5.1) we take the composition $\mu_\mathcal{L}^+: \mathcal{M}_p^+ \to \mathcal{L}(V_\mathcal{V})$ of the projection $\mathcal{M}_p^+ \to U$ and $\mu|_U: U \to \mathcal{L}(V_\mathcal{V})$, and $H = \text{GL}(V_\mathcal{V}) \times \mathbb{C}^*_p$. We take a stability condition corresponding to $+\text{-stability}$ and write the stable locus by $\mu^{-1}_\mathcal{L}(0)^{ss}$. Then by Proposition 5.1 we have

$$M_{\mathcal{L}_p}^+ \equiv [\mu^{-1}_\mathcal{L}(0)^{ss}/\text{GL}(V_\mathcal{V}) \times \mathbb{C}^*_p]$$

and the cotangent complex $L_{\mathcal{M}_p^+}$ is described as in (13).

To define an obstruction theory $\text{ob}_{\mathcal{L}_p^+}$ we first introduce an obstruction theory $\text{ob}_\mathcal{L}_2$ for the moduli space $M(1, p)$ of stable ADHM data on $W_\mathcal{V} = \mathbb{C}, V_\mathcal{V} = \mathbb{C}^p$. The cotangent complex $L_{M(1, p)}$ is defined by (13) taking $\varphi = \mu: M(1, p) \to \mathcal{L}(V_\mathcal{V})$ and $H = \text{GL}(V_\mathcal{V})$, since $\mu^{-1}(0)$ is a complete intersection by (4.2). We consider the direct sum decomposition

$$\Omega_{M(1, p)} = \Omega'_{M(1, p)} \oplus \Omega''_{M(1, p)}$$

where $\Omega'_{M(1, p)} = M(1, p) \times (\text{Hom}_\mathbb{C}(Q^\vee \otimes V_\mathcal{V}, V_\mathcal{V}) \otimes \text{Hom}_\mathbb{C}(W_\mathcal{V}, V_\mathcal{V}))^\vee$ and $\Omega''_{M(1, p)} = M(1, p) \times \text{Hom}_\mathbb{C}(\wedge Q^\vee \otimes V_\mathcal{V}, W_\mathcal{V})^\vee$. We replace the middle term in $L_{M(1, p)}$ with $\Omega'_{M(1, p)}$ to get

$$\text{Ob}_2 = \left(0 \to \mu^* \Omega_{\mathcal{L}(V_\mathcal{V})}|_{\mu^{-1}(0)^{ss}} \to \Omega'_{M(1, p)}|_{\mu^{-1}(0)^{ss}} \to \mathfrak{g}(V_\mathcal{W})^\vee \otimes \mathcal{O}_{\mu^{-1}(0)^{ss}} \right),$$

where morphisms in the complex are induced from ones in $L_{M(1, p)}$ via the natural projection and the injection between $\Omega_{M(1, p)}$ and $\Omega'_{M(1, p)}$. The injection $\Omega'_{M(1, p)} \to \Omega_{M(1, p)}$ induce a morphism $\text{ob}_2: \text{Ob}_2 \to L_{M(1, p)}$ of complexes. This gives an obstruction theory.

On $M_{\mathcal{L}_p}^+$ we have a distinguished triangle

$$L_{M_{\mathcal{L}_p}^+}/M(1, p)[{-1}] \to h^*L_{M(1, p)} \to L_{M_{\mathcal{L}_p}^+} \to L_{M_{\mathcal{L}_p}^+}/M(1, p).$$

Via the projection $L_{M(1, p)} \to \text{Ob}_2$ we have $L_{M_{\mathcal{L}_p}^+}/M(1, p)[{-1}] \to h^*\text{Ob}_2$. We define

$$\text{Ob}_{M_{\mathcal{L}_p}^+} = \text{Cone}(L_{M_{\mathcal{L}_p}^+}/M(1, p)[{-1}] \to h^*\text{Ob}_2)$$

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and \( ob_{M^+_p} : Ob_{M^+_p} \to L_{M^+_p} \) to be the induced morphism. This defines a virtual fundamental cycle \([M^+_p]\). By the construction we have \( h^*[M(1,p)]^\text{vir} = [M^+_p] \).

**Proposition 5.3** (\[Mo\] Proposition 5.8.1). We have

\[
F^* ob_{M_3} = G^* p_1^* Ob_{\tilde M^{\min(I_2)}(r,n-p)} \oplus G^* p_2^* Ob_{M^+_p},
\]

where \( p_1, p_2 \) are projections from \( \tilde M^{\min(I_2)}(r,n) \times M^+_p \) to \( \tilde M^{\min(I_2)}(r,n-p), M^+_p \).

**Proof.** Via the isomorphism \( S_3 \cong \tilde M/C^* \times M^+_p \) as in the proof of Theorem 4.9 we can check the assertion. \( \square \)

By this proposition we have \( F^*[M_3] = G^* \left( [\tilde M^{\min(I_2)}(r,n-p)] \times [M^+_p] \right) \).

### 5.3 Relative tangent bundles for flags

We consider the pull-back \( \Theta^{rel} \) of the relative tangent bundle of \([\tilde \mu^{-1}(0)/G] \) over \([\mu^{-1}(0)/G] \). Over \( M_3 \), we have a decomposition \( \Theta^{rel} = \Theta^{rel}_b \oplus \Theta^{rel}_s \oplus N_0 \), where

\[
\Theta^{rel}_\alpha = \bigoplus_{i>j} \text{Hom} \left( F^j_/F^i_/^{-1}, F^j_/F^i_/^{-1} \right)
\]

for \( \alpha = b, s \), and \( N_0 \) defined in §5.2.

We also consider the relative tangent bundle \( \Theta' \) of \( h : M^+_p \to M(1,p) \). Then we have an exact sequence

\[
0 \to \Theta' \to \Theta_s \to h^* \left( \mathcal{V}_s/\mathcal{O}_{M(1,p)} \right) \to 0,
\]

where \( \mathcal{V}_s/\mathcal{O}_{M(1,p)} \) is the quotient by the tautological homomorphism \( \mathcal{O}_{M(1,p)} \to \mathcal{V}_s \). We see that \( L_{M^+_p/M(1,p)} \cong (\Theta')^\vee \) and by the definition we have

\[
Ob_{M^+_p} = h^* Ob_{M(1,p)} + (\Theta')^\vee \tag{16}
\]

in the \( K \)-group of \( M^+_p \).

### 6 Wall-crossing formulas

In this section we derive wall-crossing formulas following \[Mo\] Chapter 7 and \[NY2\] §6, and give a proof of Theorem 2.6.
6.1 Localization

Let us consider \( \psi = \psi(V) \in A^*_T \times \mathbb{C}_n^* (M(r,n)) \) to be the Euler class of \( \tilde{T} \times \mathbb{C}_n^* \)-equivariant \( K \)-theory class defined by a linear combination of tensor products of \( V, V^* \), and \( \tilde{T} \)-modules. For example, we take \( \psi = 1, \psi = e(F_c(V)) \), or \( \psi = e(TM(r,n) \otimes e^{m_1}) \). We also write by \( \psi \) the class defined by the same formula on \( M^c(r,n), \tilde{M}^f(r,n), \mathcal{M} \), and so on.

Let us consider the pull-back \( \Theta^{rel} \) to \( \tilde{M}^f(r,n) \) of the relative tangent bundle of \([\tilde{\mu}^{-1}(0)/G]\) over \([\mu^{-1}(0)/G]\). We also write by \( \Theta^{rel} \) the pull-back to the enhanced master space \( \mathcal{M} \). We introduce \( \tilde{\psi} = \frac{1}{n} \psi \cup e(\Theta^{rel}) \) on \( \tilde{M}^f(r,n) \) and \( \mathcal{M} \) so that

\[
\int_{\tilde{M}^f(r,n)} \tilde{\psi} = \int_{M^c(r,n)} \psi, \quad \int_{\tilde{M}^f(r,n)} \tilde{\psi} = \int_{M^c(r,n)} \psi.
\]

We consider integrations \( \int_{\mathcal{M}} \tilde{\psi} \cup e(\mathcal{O}_M \otimes e^h) \) over enhanced master spaces. This integration is defined by

\[
\int_{\mathcal{M}} \tilde{\psi} \cup e(\mathcal{O}_M \otimes e^h) = (\alpha_0)^{-1} \Pi_* \left( \tilde{\psi} \cup e(\mathcal{O}_M \otimes e^h) \right) \cap [\mathcal{M}]^{vir} \in S[h, h^{-1}],
\]

where \( \Pi: \mathcal{M} \to M_0(r,n) \) as in \([12,2] \) \( \iota_0: \{n[0]\} \to M_0(r,n) \) as in \([2,3]\) and \( [\mathcal{M}]^{vir} = [\mathcal{M}] \) is the (virtual) fundamental cycle defined by the obstruction theory \( ob_M = id_{L_M} \). We also use similar push-forward homomorphisms from homology groups of various moduli stacks to define integrals, for example \( \int_{\mathcal{M}_2^+}, \int_{\mathcal{M}_2^-} \).

By \([6,4]\), we have the following commutative diagram:

\[
\begin{array}{ccc}
\lim_{\varphi \to \eta} A^\mathbb{C}_c^* (M \times \bar{T} E_n) & \xrightarrow{\varphi} & \lim_{\varphi \to \eta} A^\mathbb{C}_c^* (\mathcal{M} \times \bar{T} E_n) \\
{f_M} & & {f_M + f_{M^+} + \Sigma_3 f_{M_3}} \\
\lim_{\varphi \to \eta} A^\mathbb{C}_c^* (M_0(r,n) \times \bar{T} E_n) & \xrightarrow{\varphi} & \lim_{\varphi \to \eta} A^\mathbb{C}_c^* (M_0(r,n) \times \bar{T} E_n)
\end{array}
\]

where the upper horizontal arrow is given by

\[
\frac{\ell_+}{e(\mathfrak{H}(M_-))} + \frac{\ell_-}{e(\mathfrak{H}(M_+))} + \sum_{3 \in \mathbb{S}} \frac{\ell_3}{e(\mathfrak{H}(M_3))}.
\]

Hence we have

\[
\int_{\mathcal{M}} \tilde{\psi} \cup e(\mathcal{O}_M \otimes e^h) = \int_{M^c} \tilde{\psi} \cup e(\mathcal{O}_M \otimes e^h) + \int_{M^c} \tilde{\psi} \cup e(\mathcal{O}_M \otimes e^h) + \sum_{3 \in \mathbb{S}} \int_{M_3} \tilde{\psi} \cup e(\mathcal{O}_M \otimes e^h).
\]
If we substitute $\hbar = 0$, then the left hand side is equal to zero. Hence by Lemma 5.1 and $\mathcal{M}_+ = \tilde{M}^\ell(r, n)$, $\mathcal{M}_- = \tilde{M}^0(r, n)$, we have

$$\int_{\tilde{M}^\ell(r, n)} \tilde{\psi} - \int_{M(r, n)} \psi = -\frac{1}{n!} \sum_{J \in S^\ell} \text{Res}_{\hbar = 0} \int_{\mathcal{M}_3} \frac{\partial J \tilde{\psi}}{e(\mathfrak{H}(\mathcal{M}_3))}. \quad (17)$$

By Theorem 4.9, Proposition 5.2 and Proposition 5.3 we have

$$\int_{\mathcal{M}_3} e(\mathfrak{H}(\mathcal{M}_3)) \frac{\partial J \tilde{\psi}}{J} = \int_{\tilde{M}^{\text{min}(I)^{\ell-1}(r, n-p)} \times M_p^{+ \text{vir}}} \psi(V_2 \boxtimes V_2') \cup e(\mathfrak{H}(V_2, V_2', W)) \cup e(\Theta^\text{rel}_\ell), \quad (18)$$

where $V_2' = V_2 \otimes \frac{\hbar}{pD} \otimes (\det V_2)^{-\frac{1}{2}}$ as in Proposition 5.2 (2). Here $\int_{[M_p^+]^{\text{vir}}} \cdot$ is the push-forward by the projection $p_1: \tilde{M}^{\text{min}(I)^{\ell-1}(r, n-p)} \times M_p^{+ \text{vir}} \rightarrow \tilde{M}^{\text{min}(I)^{\ell-1}(r, n-p)}$, and

$$[\tilde{M}^{\text{min}(I)^{\ell-1}(r, n-p)} \times M_p^{+ \text{vir}}]$$

is the virtual fundamental cycle defined by the obstruction theory $p_1^* \phi_{\tilde{M}^{\text{min}(I)^{\ell-1}(r, n-p)} \times M_p^{+ \text{vir}}}$ as in Proposition 5.3.

### 6.2 Computations of residues

We simplify integrations as in [Mo, Proof of Theorem 7.2.4]. Since $e(V_2') = \sum_{i=1}^{p} c_i(V_2') \left( \frac{h - c_1(V_2)}{pD} \right)^{p-i}$, the integrand of $\int_{[M_p^+]^{\text{vir}}}$ in the above equation (18) is of the form $\sum_{j=-\infty}^{\infty} A_j \left( \frac{h - c_1(V_2)}{pD} \right)^j$, where $A_j$ does not depend on $\hbar$. We have

$$\text{Res}_{\hbar = 0} \sum_{j=-\infty}^{\infty} A_j \left( \frac{h - c_1(V_2)}{pD} \right)^j = pDA_{-1} = \text{Res}_{\hbar = 0} \sum_{j=-\infty}^{\infty} A_j \left( \frac{h}{pD} \right)^j.$$

Hence we have

$$\text{Res}_{\hbar = 0} \int_{[M_p^+]^{\text{vir}}} \psi(V_2 \boxtimes V_2') \cup e(\mathfrak{H}(V_2, V_2', W)) \cup e(\Theta^\text{rel}_\ell)$$

$$= \text{Res}_{\hbar = 0} \int_{[M_p^+]^{\text{vir}}} \psi(V_2 \boxtimes V_2' \otimes \frac{\hbar}{pD}) \cup e(\mathfrak{H}(V_2 \otimes e(\hbar), W)) \cup e(\Theta^\text{rel}_\ell).$$
As in [NY2] §6.3 using $\text{Res}_{h=0} f(h) = pD \text{Res}_{h=0} f(pDh)$ for $h = \frac{h}{p^n}$, this is equal to

$$pD \text{Res}_{h=0} \int_{[M^p]_{\text{vir}}} \psi(V_0 \boxtimes V_2 \otimes \mathcal{O}) + e(\mathfrak{M(V_0, V_2 \otimes e^h, \mathcal{W}))} \cup e(\Theta^{rel})$$

$$= (p - 1)! \text{Res}_{h=0} \int_{[M(1,p)]_{\text{vir}}} \psi(V_0 \boxtimes V_2 \otimes e^h) + e(\mathfrak{M(V_0, V_2 \otimes e^h, \mathcal{W}))} \cup e(\mathcal{V}_Y).$$

The last equality follows from Proposition 4.55 and (16). By localization theorem this is

$$(p - 1)! \text{Res}_{h=0} \sum \iota_{\psi^p}(V_0 \boxtimes V_2 \otimes e^h) \cup e(\mathfrak{M(V_0, V_2 \otimes e^h, \mathcal{W})) \times \frac{\iota^*_{\psi}(V_0/\mathcal{O}_{M(1,p)})}{\iota^*_{\psi}(\mathcal{O}_Y)},$$

where the sum is taken over the set of Young diagrams $Y$ with the weight $|Y| = p$, and $\iota_{\psi^p}$ is the inclusion $\tilde{M}^{\text{min}(p-1)}(r, n - p) \times \{Y\} \rightarrow \tilde{M}^{\text{min}(p-1)}(r, n - p) \times M(1, p)$.

In the following we consider the case where $\rho = (F_\theta(V_0)).$ Then we have

$$\psi(V_0 \boxtimes V_2) = e(F_\theta(V_0)) \cup e(F_\theta(V_2)).$$

Since the degree of $\iota^*_{\psi^p}(F_\theta(V_2 \otimes e^h)) \times \iota^*_{\psi}(\mathfrak{M(V_0, V_2 \otimes e^h, \mathcal{W}))}$ with respect to $h$ is equal to 0, we have

$$\text{Res}_{h=0} \left(\iota_{\theta^p}(F_\theta(V_2 \otimes e^h)) \times \iota_{\psi}(\mathfrak{M(V_0, V_2 \otimes e^h, \mathcal{W}))}\right)$$

$$= \text{Res}_{h=0} \iota_{\theta^p}(F_\theta(V_2 \otimes e^h)) + \iota_{\psi}(\mathfrak{M(V_0, V_2 \otimes e^h, \mathcal{W}))} + \left(-1\right)^{rp} \left(2 \sum_{n=1}^{r} a_n + 2r \sum_{f=1}^{m_f} m_f\right).$$

Thus we have

$$\text{Res}_{h=0} \int_{M^p} \frac{e(F_\theta(V_0))e(\Theta^{rel})}{\mathfrak{M(M^p))}}$$

$$= \int_{\tilde{M}^{\text{min}(p-1)}(r, n - p)} e(F_\theta(V_0))e(\Theta^{rel}) \times$$

$$\left(-1\right)^{rp} \left(2 \sum_{n=1}^{r} a_n + 2r \sum_{f=1}^{m_f} m_f\right) \times \sum_{Y,|Y| = p} \iota_{\psi}^*(V_0/\mathcal{O}_{M(1,p)}) \times \frac{\iota_{\psi}^*(\mathcal{O}_Y)}{\iota_{\psi}^*(\mathcal{O}_Y)}.$$  \hspace{1cm} (19)

Putting the following proposition together with (17), (19), we have

$$\frac{1}{n!} \int_{\tilde{M}^{\theta}(r, n)} e(F_\theta(V_0) \oplus \Theta^{rel}) - \int_{M(r, n)} e(F_\theta(V_0)) =$$

$$\left(-1\right)^{rp+1} \frac{(p - 1)!}{n!} \sum_{Y \in S,} \int_{\tilde{M}^{\text{min}(p-1)}(r, n - p)} e(F_\theta(V_0))e(\Theta^{rel}), \hspace{1cm} (20)$$

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where \( u_r = \varepsilon_r (2 \sum_{a=1}^ra_k + \sum_{f=1}^m f_j) \).

**Proposition 6.1.** We have

\[
\sum_{|Y|=p} \frac{i_Y^* e(V_2/O_{M(1,p)})}{i_Y^* e(\mathcal{O}_{\mathcal{L}})} = \frac{\varepsilon_+}{p \varepsilon_2}.
\]

**Proof.** By definition (15) of \( \mathcal{O}_{\mathcal{L}} \), we have

\[
e(\mathcal{O}_{\mathcal{L}}) = \frac{e(TM(1,p))}{e(\text{Hom}(\wedge Q^r \otimes V_2, W_2))}.
\]

where \( W_2 = \mathcal{O}_{M(1,p)} \) is the tautological bundle corresponding to \( W_2 \). Since \( \wedge Q = t_1t_2 \) as a \( \tilde{T} \)-module, we have

\[
\sum_{|Y|=p} \frac{i_Y^* e(V_2/O_{M(1,p)})}{i_Y^* e(\mathcal{O}_{\mathcal{L}})} = \sum_{|Y|=p} \frac{i_Y^* e(V_2/O_{M(1,p)}) i_Y^* e(V_2^r \otimes t_1t_2)}{i_Y^* e(TM(1,p))} = \int_{M(1,p)} e\left((V_2/O_{M(1,p)}) \otimes V_2^r \otimes t_1t_2\right)
\]

\[
= \left( \frac{1}{m_1} \int_{M(1,p)} e(V_2 \otimes e^{m_1} \otimes V_2^r \otimes e^{m_2}) \right)_{m_1=0, m_2=\varepsilon_+} = \varepsilon_+.\]

The last equality follows from Proposition A.1 in Appendix A. \( \square \)

### 6.3 Proof of Theorem 2.6

Here we complete wall-crossing formula by the equation (20), and prove Theorem 2.6. For \( 0 \leq q \leq n \), if we put

\[
\gamma_{q,n} = \frac{1}{n!} \int_{M^q(n)} e(F_r(V) \otimes \Theta^{rel}),
\]

then we have \( \gamma_{0,n} = \alpha_n, \gamma_{n,n} = \beta_n \). We consider the set \( \text{Dec}^i(n) \) of \( i \) tuple \( J^i = (I^i_1, \ldots, I^i_i) \) of non-empty subsets of \( n \) such that \( \min(I^i_1) > \cdots > \min(I^i_i) \). We put \( |J^i| = |I^i_1| \cdots + |I^i_i| \) for each \( J^i \in \text{Dec}^i(n) \).
Lemma 6.2 ([Mo, Lemma 7.6.5], [NY2, Lemma 6.6]). We have

\[ \beta_n - \alpha_n = \sum_{1 \leq i < j} \sum_{3^j \in \text{Dec}^i(n)} (-1)^{r(3^j)} \left( \frac{(|I^j_0| - 1)! \cdots (|I^j_r| - 1)! (n - |3^j|)!}{n!} \right) u^j_i \alpha_{n - |3^j|}. \]

Proof. We use induction on \( j > 0 \). When \( j = 1 \), this equation is nothing but (20) for \( \ell = n \), since we have \( \text{Dec}^1(n) \cong S^n \). Then applying (20) repeatedly we get the assertion for any \( j > 0 \).

Since \( \text{Dec}^i(n) = \emptyset \) for \( j > n \), by the above lemma we have

\[ \beta_n - \alpha_n = \sum_{1 \leq i \leq n} \sum_{3^j \in \text{Dec}^i(n)} (-1)^{r(3^j)} \left( \frac{(|I^j_0| - 1)! \cdots (|I^j_r| - 1)! (n - |3^j|)!}{n!} \right) u^j_i \alpha_{n - |3^j|}. \]

We consider a map \( \rho_i : \text{Dec}^i(n) \to \mathbb{N}^i \) sending \( 3^j \) to \( (|I^j_0|, \ldots, |I^j_r|) \). By [Mo, Lemma 7.6.6] or [NY2, Lemma 6.8], for any \( \vec{p}(i) = (p_1, \ldots, p_i) \in \mathbb{N}^i \) such that \( |\vec{p}(i)| = p_1 + \cdots + p_i \leq n \) we have

\[ |\rho_i^{-1}(\vec{p}(i))| \frac{(p_1 - 1)! \cdots (p_i - 1)! (n - |\vec{p}(i)|)!}{n!} = \frac{1}{\prod_{j=1}^i \sum_{1 \leq h \leq j} p_h}, \]

and hence

\[ \beta_n - \alpha_n = \sum_{1 \leq i \leq n} \sum_{|\vec{p}(i)| \leq n} \frac{(-1)^{r(\vec{p}(i)) + u^i} \alpha_{n - |\vec{p}(i)|}}{\prod_{j=1}^i \sum_{1 \leq h \leq j} p_h}. \]

To prove Theorem (20) for each \( k = 1, \ldots, n \) we must show

\[ \sum_{i=1}^{k} \sum_{p_1 + \cdots + p_i = k} (-1)^i \frac{u^i}{\prod_{j=1}^{i} \sum_{1 \leq h \leq j} p_h} = (-1)^k \frac{u_r (u_r - 1) \cdots (u_r - k + 1)}{k!}. \] (21)

The right hand side of (21) is equal to

\[ \sum_{i=1}^{k} \sum_{r_1 + \cdots + r_i = k} (-1)^i \frac{u^i}{\prod_{j=1}^{i} r_j}. \]

Hence (21) follows from the bijection

\( (p_1, \ldots, p_i) \mapsto (r_1, \ldots, r_i) = (p_1, p_1 + p_2, \ldots, p_1 + \cdots + p_i). \)
6.4 Partition functions defined from other classes

We consider Nekrasov partition functions $Z_{\psi}(\varepsilon, a, m^N_f, q) = \sum_{n=0}^{\infty} q^n \int_{\mathcal{M}(r,n)} \psi$ defined from $\psi \in A_\psi^*(M(r,n))$ other than $\psi = e(\mathcal{F}_r(V))$.

Here we consider the case where $N_f = 0$ and $\psi = 1$, or $N_f = 1$ and $\psi = e(TM(r,n) \otimes e^{\sqrt{t_1 t_2} m_1})$.

Then in both cases, residues in (17) with respect to $\hbar$ are equal to zero. Hence the right hand side of (17) is equal to zero. As a result we have

$$Z_{\psi}(-\varepsilon, a, m^N_f, q) = Z_{\psi}(\varepsilon, a, m^N_f, q).$$

When $N_f = 0$ and $\psi = 1$, this formula also implies (4). When $N_f = r = 1$ and $\psi = e(TM(1,n) \otimes e^{\sqrt{t_1 t_2} m_1})$, by [CO, Corollary 1] we have

$$Z_{\psi}(\varepsilon, a, m_1, q) = \prod_n (1 - q^n)^{\frac{\varepsilon_2}{2} - m_1 - 1}$$

and we can also check this formula.

A Integrations on Hilbert schemes

In this appendix we compute integrations on Hilbert schemes $M(1,n)$. Here we substitute $a = 0$ in $\mathcal{S}$, and hence integrations take values in $\mathbb{Q}(\varepsilon_1, \varepsilon_2, m_1, m_2)$.

Proposition A.1. We have

$$\sum_{n=0}^{\infty} q^n \int_{\mathcal{M}(1,n)} e(\mathcal{V} \otimes e^{m_1} \otimes \mathcal{V}' \otimes e^{m_2}) = (1 - q)^{-m_1 m_2 \varepsilon_1 \varepsilon_2 / 4 \varepsilon_1 \varepsilon_2}$$

that is, $\int_{\mathcal{M}(1,n)} e(\mathcal{V} \otimes e^{m_1} \otimes \mathcal{V}' \otimes e^{m_2}) = \prod_{n=1}^{m_2} \frac{(\varepsilon_1 \varepsilon_2 / n) + 1}{n}.$

Proof. We have

$$\int_{\mathcal{M}(1,n)} e(\mathcal{V} \otimes e^{m_1} \otimes \mathcal{V}' \otimes e^{m_2}) = \sum_{|Y|=n} \chi_{\mathcal{V}}\chi^{\mathcal{V}'} \frac{e(\mathcal{V} \otimes e^{m_1}) e(\mathcal{V}' \otimes e^{m_2})}{e(TM(1,n))} \chi_{\mathcal{V}}$$

$$= \lim_{t \to 0} \sum_{|Y|=n} \chi_{\mathcal{V}}^{\mathcal{V}^\vee \wedge_{-1} (TM(1,n))} \chi_{\mathcal{V}}^{\mathcal{V}^\vee \wedge_{-1} (TM(1,n))} |_{t_1 = e^{\varepsilon_1}, t_2 = e^{\varepsilon_2}},$$
where $\text{ch}$ denotes the Hilbert series (cf. [NY1, §4]), and $\wedge_u(E) = \sum u^i(\wedge^i E)$ in $K_T(M(1,p))[u]$. To compute the sum we follow the notation and the method in [Na3], where we substitute $t = t_1^{-1}, q = t_2^{-1}$. We use plethystic substitution (cf. [Na3, 1 (i)]) by symmetric functions

$$\Omega = \prod_{i=1}^{\infty} \frac{1}{1-x_i} = \exp \left( \sum_{n=1}^{\infty} \frac{p_n}{n} \right) = \sum_{n=0}^{\infty} h_n,$$

where $p_n$ and $h_n$ are the $n^{th}$ power and complete symmetric functions respectively.

From [Ma, VI (6.11')] or [H, (3.5.20)] we have $\tilde{H}_\mu[1-u; t_1, t_2] = \text{ch}_\star Y(\wedge_u V \vee \#)$, where $\tilde{H}_\mu(x; t_1, t_2)$ is the modified Macdonald polynomials (cf. [Na3, 1 (ii)]), and $\mu$ is the partition corresponding to the Young diagram $Y$. Hence by the Cauchy formula [Na3, (1.7)] we have

$$\sum_{n=0}^{\infty} \sum_{|Y| = n} \frac{\text{ch}_\star Y \wedge_{-u_1} (V') \text{ch}_\star Y \wedge_{-u_2} (V)}{\text{ch}_\star Y \wedge_1 (T^* M(1,n))} q^n = \Omega \left[ \frac{(1-u_1)(1-u_2)}{(1-t_1^{-1})(1-t_2^{-1})} \right] = \exp \left( \sum_{n=1}^{\infty} \frac{(1-u_1^n)(1-u_2^n)}{(1-t_1^{-n})(1-t_2^{-n})} \frac{q^n}{n} \right).$$

Hence we have

$$\sum_{n=0}^{\infty} q^n \int_{M(1,n)} e(V \otimes e^{m_1} \oplus V^\vee \otimes e^{m_2}) = \lim_{t \to 0} \exp \left( \sum_{n=1}^{\infty} \frac{(1-e^{nm_1}t)(1-e^{nm_2}t)}{(1-e^{-nx_1}t)(1-e^{-nx_2}t)} \frac{q^n}{n} \right) = \exp \left( \frac{m_1 m_2}{\epsilon_1 \epsilon_2} \sum_{n=1}^{\infty} \frac{q^n}{n} \right) = (1-q)^{-\frac{m_1 m_2}{\epsilon_1 \epsilon_2}}.$$

We consider the Nekrasov partition function $Z(\varepsilon, a, m) = \sum_{n=0}^{\infty} q^n \int_{M(1,n)} e(F_1(V))$ for rank $r = 1$. By substituting $m_1 = a - \frac{\varepsilon_1}{2} + m_1, m_2 = -a + \frac{\varepsilon_1}{2} - m_2$ in Proposition [A.1] we have

$$Z(\varepsilon, a, m) = (1+q)^{\alpha_1},$$

where $\alpha_1 = \frac{(a - \frac{\varepsilon_1}{2} + m_1)(a - \frac{\varepsilon_1}{2} + m_2)}{\varepsilon_1 \varepsilon_2}$.

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