ON 3-UNIFORM HYPERGRAPHS AVOIDING A CYCLE OF LENGTH FOUR

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Abstract. In this note we show that the maximum number of edges in a 3-uniform hypergraph without a Berge cycle of length four is at most $(1 + o(1)) \frac{n^{3/2}}{\sqrt{10}}$. This improves earlier estimates by Györi and Lemons and by Füredi and Özkahya.

1. Introduction

Given a hypergraph $H$, let $V(H)$ and $E(H)$ denote the set of vertices and edges of $H$. A hypergraph is called $r$-uniform if all of its edges have size $r$. Berge introduced the following definitions of a path and cycle in a hypergraph.

Definition 1. A Berge cycle of length $\ell$ in a hypergraph is a set of $\ell$ distinct vertices $\{v_1, \ldots, v_\ell\}$ and $\ell$ distinct edges $\{e_1, \ldots, e_\ell\}$ such that $\{v_i, v_{i+1}\} \subseteq e_i$ with indices taken modulo $\ell$. A Berge path of length $\ell$ is a set of $\ell + 1$ distinct vertices $\{v_1, \ldots, v_{\ell+1}\}$ and $\ell$ distinct edges $\{e_1, \ldots, e_\ell\}$ such that for $1 \leq i \leq \ell$ we have $\{v_i, v_{i+1}\} \subseteq e_i$.

Let $\text{ex}_r(n, BC_\ell)$ denote the maximum number of edges in a $r$-uniform hypergraph without a Berge cycle of length $\ell$. In the case $r = 2$ we write simply $\text{ex}(n, C_\ell)$.

A well-known result of Bondy and Simonovits [2] asserts that for all $\ell \geq 2$ we have $\text{ex}(n, C_\ell) = O(n^{1+1/\ell})$, however the order of magnitude is only known to be sharp in the cases $\ell = 2, 3, 5$. Erdős, Rényi and Sós [3] proved the asymptotic result $\text{ex}(n, C_4) = \frac{n^{3/2}}{2} + o(n^{3/2})$. For hypergraphs of higher uniformity Györi and Lemons [6] extended the Bondy Simonovits theorem and showed in particular that $\text{ex}_3(n, BC_4) = O(n^{3/2})$. It follows from the results of Füredi and Özkahya [4] that $\text{ex}_3(n, BC_4) \leq (1 + o(1))\frac{2}{3}n^{3/2}$ (see Theorem 2 in [4]). In this note we significantly improve this bound as follows.

Theorem 1.

$(1 - o(1)) \frac{n^{3/2}}{3\sqrt{3}} \leq \text{ex}_3(n, C_4) \leq (1 + o(1)) \frac{n^{3/2}}{\sqrt{10}}$.

The lower bound comes from a construction originating in Bollobás and Györi [11] (stated more generally in [5]). We take a $C_4$-free bipartite graph with color classes of size $n/3$ and $\frac{(2n/3)^{3/2}}{2\sqrt{2}} = \frac{n^{3/2}}{3\sqrt{3}}$ edges asymptotically. For every vertex $v$ in one of the color classes, we take an additional vertex $v'$ and add it to every edge in the graph incident to $v$. This results in

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a 3-uniform hypergraph on \( n \) vertices with \( \frac{3}{\sqrt{3}} \) edges asymptotically, and it is easy to verify this hypergraph contains no Berge \( C_4 \).

2. Proof of the upper bound in Theorem 1

Now, we prove the upper bound. Let \( \mathcal{H} \) be a 3-uniform hypergraph with no Berge \( C_4 \) and no isolated vertices. A block \( B \) of a hypergraph \( \mathcal{H} \) is defined to be a maximal subhypergraph of \( \mathcal{H} \) with the property that for any two edges \( e, f \in E(B) \), there is a sequence of edges of \( \mathcal{H} \), \( e = e_1, e_2, \ldots, e_t = f \), such that \( |e_i \cap e_{i+1}| = 2 \) for all \( 1 \leq i \leq t - 1 \) and \( V(B) = \cup_{h \in E(B)} h \). It is easy to see that the blocks of \( \mathcal{H} \) define a unique partition of \( E(\mathcal{H}) \).

For a block \( B \) and an edge \( h \in E(B) \), we say \( h \) is a leaf if there exists \( x \in h \) such that the only edge of \( B \) incident to \( x \) is \( h \). It is simple to observe that the set of non-leaf edges of a block \( B \) is either the empty set, a single edge or the edges of a complete hypergraph on 4-vertices minus an edge, \( K_4^{(3)} \). Even more if the set of non-leaf edges of \( B \) is \( E(K_4^{(3)}) \), then \( B = K_4^{(3)} \). This implies that the set \( B(\mathcal{H}) = \{ B | B \text{ is a block in } \mathcal{H} \} \) of all blocks of \( \mathcal{H} \), can be partitioned into the following types of blocks:

1. We say \( B \in B(\mathcal{H}) \) is type 1 if there exists an edge \( e \in E(\mathcal{B}) \) such that for all distinct \( f_1, f_2 \in E(\mathcal{B}) \), \( f_1 \neq e \), we have \( |e \cap f_i| = 2 \), for \( i = 1, 2 \) and \( f_1 \cap f_2 \subseteq e \).
2. We say \( B \in B(\mathcal{H}) \) is type 2 if \( B = K_4^{(3)} \).

Define the 2-shadow of a hypergraph to be the graph on the same set of vertices whose edges are all pairs of vertices \( \{x, y\} \) for which there exists an edge \( e \in E(\mathcal{H}) \) such that \( \{x, y\} \subseteq e \). We denote the 2-shadow of a hypergraph \( \mathcal{H} \) by \( \partial \mathcal{H} \). The proof of Theorem 1 will proceed by estimating the number of 3-paths (3-vertex paths) in the 2-shadow of a Berge \( C_4 \)-free hypergraph in two different ways. To this end, we introduce several notions of the degree of a vertex. Given a vertex \( v \) in a hypergraph \( \mathcal{H} \), \( d(v) \) denotes the classical hypergraph degree of \( v \), in particular \( d(v) = |\{ h \in E(\mathcal{H}) : v \in h \}| \). Let \( d_s(v) \) be the (graph) degree of \( v \) in the 2-shadow of the hypergraph, in particular \( d_s(v) = |\{ e \in E(\partial \mathcal{H}) : v \in e \}| \). Then, we define the excess degree of the vertex \( v \) to be \( d_{ex}(v) = d_s(v) - d(v) \). Finally, we define the block degree \( d_b(v) \) to be the total number of blocks containing an edge which contains \( v \).

Notice that for every 4-cycle \( x_1, x_2, x_3, x_4, x_1 \) of \( \partial \mathcal{H} \), there exists three distinct integers \( 1 \leq i < j < k \leq 4 \) such that \( \{x_i, x_j, x_k\} \in E(\mathcal{H}) \), otherwise \( \mathcal{H} \) contains a copy of Berge \( C_4 \). We call this edge a representative edge of this 4-cycle. Note that each 4-cycle of \( \partial \mathcal{H} \) has either 1, 2 or 3 representative edges. Two edges of \( \mathcal{H} \) sharing two vertices yield a \( C_4 \) in \( \partial \mathcal{H} \). However these are not only types of \( C_4 \)'s in \( \partial \mathcal{H} \). We call a 4-cycle of \( \partial \mathcal{H} \) rare if the induced subhypergraph of \( \mathcal{H} \) on the vertices of cycle does not contain two edges sharing a diagonal pair of vertices of the 4-cycle. In the following claim, we show that the number of such cycles is small.

We define a particular type of 3-path of \( \partial \mathcal{H} \). A 3-path, \( x_1, x_2, x_3 \), is called good if \( \{x_1, x_2, x_3\} \notin E(\mathcal{H}) \) and there is no \( x \in V(\mathcal{H}) \) such that \( x, x_1, x_2, x_3, x \) is a rare cycle of \( \partial \mathcal{H} \).

**Claim 1.** For any \( a, b \in V(\mathcal{H}) \), there are at most two good 3-paths in \( \partial \mathcal{H} \) with end points \( a \) and \( b \).

**Proof.** Suppose, by contradiction, that there are three distinct vertices \( v_1, v_2, v_3 \) different from \( a \) and \( b \) such that \( a, v_i, b \) forms a good 3-path of \( \partial \mathcal{H} \) for all integer \( 1 \leq i \leq 3 \). It follows
that there are three Berge paths \( a, e, v, f, b \), for all integer \( 1 \leq i \leq 3 \) in \( \mathcal{H} \). Note that those edges are not necessarily distinct. But we have \( e_i \neq f_i \) and \( e_i \neq f_j, i \neq j \), since \( \{a, v_i\} \subset e_i \) and \( \{b, v_j\} \subset f_j \) and \( \mathcal{H} \) is 3-uniform. Note that if \( e_2 = e_3 \), then \( e_2 = \{a, v_2, v_3\} \), hence \( e_1 \neq e_2 \). Similarly we have either \( f_1 \neq f_2 \) or \( f_1 \neq f_3 \). We may assume, without loss of generality, that \( e_1 \neq e_2, e_3 \). It follows that either \( a, e_1, v_1, f_1, b, f_2, v_2, e_2, a \) or \( a, e_1, v_1, f_1, b, f_3, v_3, e_3 \), \( a \) is a Berge \( C_4 \), a contradiction. □

Claim 2. There are at most \( 6|E(\mathcal{H})| \) rare 4-cycles in \( \partial \mathcal{H} \).

Proof. We fix an edge \( \{a, b, c\} \in E(\mathcal{H}) \). It suffices to show that the edge \( \{a, b, c\} \) is representative of at most 6 rare 4-cycles (that is, \( \{a, b, c\} \) is contained in the vertex set of at most 6 rare 4-cycles). Suppose by contradiction that this is not true. Observe that there are three possible positions for a fixed vertex \( v \) among the vertices of a four cycle in \( \partial \mathcal{H} \) with \( \{a, b, c\} \).

By the pigeonhole principle there are 3 distinct vertices \( v_1, v_2, v_3 \) different from \( a, b \) or \( c \) with the same position in the 4-cycle. Without loss of generality, we may assume they form a 4-cycle in the order \( v_1, a, c, b, v_1 \). Therefore from the definition of a rare 4-cycle, there are at least three good 3-paths in \( \partial \mathcal{H} \) from \( a \) to \( b \), a contradiction to Claim 1. □

Using Claim 2, it is easy to see that the number of 3-paths in \( \partial \mathcal{H} \) which are not good is at most \( 3|E(\mathcal{H})| + 3 \cdot 6|E(\mathcal{H})| = 21|E(\mathcal{H})| \). Here we use the fact that each rare 4-cycle induces an edge of \( \mathcal{H} \).

By conditioning on the middle vertex of the 3-path, we have the following estimate on the number of 3-paths in \( \partial \mathcal{H} \):

\[
\#(3\text{-paths in } \partial \mathcal{H}) = \sum_{v \in V(\mathcal{H})} \binom{d_s(v)}{2} = \sum_{v \in V(\mathcal{H})} \binom{d(v) + d_{ex}(v)}{2}. 
\]

The following claim provides an upper bound on the number of good 3-paths in \( \partial \mathcal{H} \).

Claim 3.

\[
\#(\text{good 3-paths in } \partial \mathcal{H}) \leq 2\binom{n}{2} - 4\sum_{v \in V(\mathcal{H})} \binom{d_b(v)}{2}. 
\]

Proof. Fix a vertex \( v \) and consider two adjacent edges \( \{v, x_1, x_2\} \) and \( \{v, y_1, y_2\} \) such that they belong to the different blocks; clearly the vertices \( v, x_1, x_2, y_1, y_2 \) are all distinct. We claim that there is at most one good 3-path, namely \( x_i, v, y_j, \) between \( x_i \) and \( y_j \), for each \( i, j \in \{1, 2\} \). Suppose this is not the case, then without loss of generality, there exists \( u \neq v \) such that \( x_1, u, y_1 \) is a good 3-path. By the definition of a good 3-path, there are two distinct edges \( h_x, h_y \in \mathcal{H} \) such that \( x_1, u \in h_x \) and \( y_1, u \in h_y \). If \( \{v, x_1, x_2\}, \{v, y_1, y_2\}, h_x \) and \( h_y \) are all different edges, then clearly there is a Berge 4-cycle. Therefore either \( \{v, x_1, x_2\} = h_x \) or \( \{v, y_1, y_2\} = h_y \). Hence we have \( u \in \{x_2, y_2\} \), without loss of generality we may assume \( u = x_2 \). Observe that the 4-cycle \( x_1, x_2, y_1, v \) of \( \partial \mathcal{H} \) contains a good 3-path and so by definition the 4-cycle \( x_1, x_2, y_1, v \) is not a rare 4-cycle. Hence we have a contradiction to the statement that edges \( \{v, x_1, x_2\} \) and \( \{v, y_1, y_2\} \) belong to the different blocks. Concluding that there is at most one good path between \( x_i \) and \( y_j \). So there are at least \( 4\sum_{v \in V(\mathcal{H})} \binom{d_b(v)}{2} \) pairs of vertices which have at most one good 3-path between them. From Claim 1 for each pair of vertices there are at most two of good 3-paths in \( \partial \mathcal{H} \). These observations complete the proof of Claim 3. □
Thus, since the number of 3-paths which are not good is at most 21 \(|E(\mathcal{H})|\), we have
\[
\sum_{v \in V(\mathcal{H})} \left( \frac{d(v) + d_{ex}(v)}{2} \right) = \#(3\text{-paths in } \partial \mathcal{H}) \leq 2 \left( \frac{n}{2} \right) - 4 \sum_{v \in V(\mathcal{H})} \left( \frac{d_b(v)}{2} \right) + 21 |E(\mathcal{H})|.
\] (1)

Now, we will obtain estimates for \(\sum_{v \in V(\mathcal{H})} d_{ex}(v)\) and \(\sum_{v \in V(\mathcal{H})} d_b(v)\). For each block \(B\) and \(v \in V(\mathcal{B})\), let \(d_{ex}^B(v)\) denote an excess degree of \(v\) inside the hypergraph \(\mathcal{B}\). If \(B\) is type 1, then every vertex \(v \in V(\mathcal{B})\) has \(d_{ex}^B(v) \geq 1\), so for type 1 blocks, \(\sum_{v \in V(\mathcal{B})} d_{ex}^B(v) \geq |V(\mathcal{B})|\). It is easy to see that for every block \(B\) we have \(|V(\mathcal{B})| > |E(\mathcal{B})|\), so \(\sum_{v \in V(\mathcal{B})} d_{ex}^B(v) > |E(\mathcal{B})|\), for every type 1 block \(B\). If \(B\) is a type 2 block, then \(\sum_{v \in V(\mathcal{B})} d_{ex}^B(v) = 3 = |E(\mathcal{B})|\). Therefore,
\[
\sum_{v \in V(\mathcal{B})} d_{ex}^B(v) \geq |E(\mathcal{B})|
\]
for every block \(B\) in \(B(\mathcal{H})\). This together with the fact that the blocks define a partition of the edges \(E(\mathcal{H})\) implies
\[
\sum_{v \in V(\mathcal{H})} d_{ex}(v) = \sum_{B \in B(\mathcal{H})} \sum_{v \in V(\mathcal{B})} d_{ex}^B(v) \geq \sum_{B \in B(\mathcal{H})} |E(\mathcal{B})| = |E(\mathcal{H})|.
\] (2)

On the other hand, a simple double counting argument yields
\[
\sum_{v \in V(\mathcal{H})} d_b(v) = \sum_{B \in B(\mathcal{H})} |V(\mathcal{B})|.
\]

Therefore,
\[
\sum_{v \in V(\mathcal{H})} d_b(v) = \sum_{B \in B(\mathcal{H})} |V(\mathcal{B})| \geq \sum_{B \in B(\mathcal{H})} |\mathcal{B}| = |E(\mathcal{H})|.
\] (3)

Now we will use the inequalities derived so far to get desired upper bound on \(|E(\mathcal{H})|\).

By [2],
\[
4 |E(\mathcal{H})| = 3 |E(\mathcal{H})| + |E(\mathcal{H})| \leq \sum_{v \in V(\mathcal{H})} (d(v) + d_{ex}(v)).
\]

Since \(\left(\frac{z}{2}\right)\) is a convex function, by Jensen’s inequality we have
\[
\left( \frac{1}{n} \sum_{v \in V(\mathcal{H})} (d(v) + d_{ex}(v)) \right) \leq \frac{1}{n} \sum_{v \in V(\mathcal{H})} \left( \frac{d(v) + d_{ex}(v)}{2} \right).
\]

Combining the above two inequalities we get
\[
n \left( \frac{4|E(\mathcal{H})|}{2n} \right) \leq \sum_{v \in V(\mathcal{H})} \left( \frac{d(v) + d_{ex}(v)}{2} \right).
\] (4)

Similarly, by [3] and Jensen’s inequality, we have
\[
n \left( \frac{|E(\mathcal{H})|}{2n} \right) \leq \sum_{v \in V(\mathcal{H})} \left( \frac{d_b(v)}{2} \right).
\] (5)
Combining (1), (4) and (5) we obtain
\[ n \left( \frac{|E(H)|}{n} \right) + 4n \left( \frac{|E(H)|}{n} \right) \leq 2 \binom{n}{2} + 21 |E(H)|. \] (6)

Rearranging (6) yields the desired bound,
\[ |E(H)| \leq (1 + o(1)) \frac{n^{3/2}}{\sqrt{10}}. \]

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