Improved Bethe-Heitler formula

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Abstract

The bremsstrahlung cross section of electron in the atomic electric field is re-derived using the time ordered perturbative theory. The results are compared with the Bethe-Heitler formula. We indicate that both the TOPT-description and a soft version for the bremsstrahlung process predict a strong screening parameter-dependent cross section, which is missed by previous bremsstrahlung theory.

keywords: Bremsstrahlung; QED; Screening effect

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1 Introduction

When electrons scatter off electric field of proton or nucleus, they can emit real photons. This is bremsstrahlung (braking radiation). Bremsstrahlung appears in nearly all branches of physics. Bethe and Heitler first gave a quantum-mechanical description of the bremsstrahlung emission at the Coulomb potential of an infinite heavy atom [1]. The Bethe-Heitler formula is an elementary and important equation in quantum electromagnetic dynamics (QED) and astrophysics.

The bremsstrahlung process of electron in the Coulomb field is a second-order process, which involves the photon emission of electron and the Coulomb scattering. However, these two sub-processes have divergences. The former is infrared divergence, while the later origins from the long-range $1/r$ potential.

For a neutral atom, the nuclear Coulomb potential is completely screened by the electron cloud, which reduces a significant contribution from scattering distance larger than the atomic radius. Therefore, the Coulomb potential in the S-matrix element should be replaced by a phenomenological screening potential. However, the complex correlations between the above mentioned two sub-processes hinder us to obtained an exact analytical solution for the integrated cross section. Bethe and Heitler take an extra model to introduce the screening radius $R$ in the solution and the result predicts a slower $\ln R$-dependent cross section. The Bethe-Heitler formula was broadly applied in astrophysics.

Recently, a puzzled difference of the energy spectra of electron/positron at GeV-TeV energy band in cosmic-ray [2-5] raises our doubts to the validity of the Bethe-Heitler formula [6]. We found that the bremsstrahlung cross section at the soft photon limit (the photon energy $\omega = 0$) contradicts with the predictions of the Bethe-Heitler formula. It means that the Bethe-Heitler formula should be improved at a general case of $\omega \neq 0$. For this sake, we re-derive the cross section formula of the bremsstrahlung emission with the
screening potential in this work, where the time ordered perturbative theory (TOPT) [7] will be used to separate the scattering and radiation processes at the equivalent photon (or Weizsäcker-Williams) approximation. This method was successfully used to decompose a complex Feynman diagram to several simpler sub-processes in our previous works [8].

We consider the scattering of electron on a light nucleus, where the recoil effect is not negligible, since most targets in the interstellar medium are light nuclei. Besides, we will show that the contributions of interference terms between scattering and radiation sub-processes can be neglected due to the recoil effect, thus we can further decompose the process. The new bremsstrahlung formula (34) predicts a strong $R^2$-dependent cross section. The result will inspire us to review the traditional electromagnetic shower theory at the extreme conditions.

The paper is organized as follows. In Sec. 2 we detail the derivation of the bremsstrahlung formula using the TOPT. Then we compare the results with the Bethe-Heitler formula in Sec. 3. A short summary is given in Sec. 4.
2 The bremsstrahlung cross section with screening potential

The Bethe-Heitler formula assumes that the target atom is infinitely heavy. For using the TOPT in the following derivation, we consider a more general case: electron scattering off a finite heavy atom. The differential cross section of the bremsstrahlung emission (Fig. 1) in the covariant perturbation theory at the leading order approximation is [9]

\[
d\sigma = \frac{m_e M_0}{\sqrt{(p_i P_i)^2 - m_e^2 M_0^2}} |M_{p_i P_i \rightarrow p_f P_f k}|^2 (2\pi)^4 \delta^4(p_i + P_i - p_f - P_f - k) \]

where the screening photon propagator in the matrix takes

\[
\frac{i g_{\mu \nu}}{q^2 - \mu^2 + i\epsilon}.
\]

The screening parameter \(\mu\) has the dimension of mass and \(1/\mu \sim R\), \(R\) is the atom radius for a neutral atom.

According to the TOPT, a covariant Feynman propagator in \(M_{p_i P_i \rightarrow p_f P_f k}\)

\[
S = \int d^4l \frac{i\gamma \cdot l + m_e}{l^2 - m_e^2 + i\epsilon},
\]

may decompose to a forward and a backward components:

\[
S_F = \frac{i}{2E_i \omega + E_f - E_i} \gamma \cdot \hat{l} + m_e, \quad \text{forward}
\]

and

\[
S_B = \frac{i}{2E_i \omega - E_f - E_i} \gamma \cdot \hat{l} + m_e, \quad \text{backward}
\]
Figure 1: Two elemental bremsstrahlung amplitudes.

Figure 2: The TOPT decomposition of Fig. 1. Dashed lines indicate the time ordered of the process.

Note that \( l(E_l, \vec{l}_T, l_L) \) is off-mass shell \( l^2 \neq m_e^2 \), while \( \hat{l} = (E_l, \vec{l}_T, l_L) \) or \( \hat{l} = (-E_l, \vec{l}_T, l_L) \) are on-mass shell, i.e., \( \hat{l}^2 = m_e^2 \).

It seems that the TOPT decomposition complicates the calculation with increasing the propagators (Fig. 2). However, the backward component will be suppressed at higher energy and small emitted angle. For example, we take \( \vec{l} \) along the z-direction, and define \( v \) as the momentum fraction of \( \hat{l} \) carried by the longitudinal momentum \( k_L \) of photon,

\[
v = \frac{k_L}{l} \approx \frac{\omega}{E_l}, \tag{6}
\]

where we neglect the electron mass \( m_e \) at high energy and note that \( \hat{l} \) is on-mass shell.
At high energy and small emitted angle we denote

\[ \hat{l} = (E_l, \vec{l}_T, \vec{l}_L) = (E_l, \vec{0}, E_l), \]  

(7)

\[ k = (\omega, \vec{k}_T, k_L) = \left( vE_l + \frac{\vec{k}_T^2}{2vE_l}, \vec{k}_T, vE_l \right), \]  

(8)

and

\[ p_f = (E_f, \vec{p}_{f,T}, p_{f,L}) = \left( (1 - v)E_l + \frac{\vec{k}_T^2}{2(1 - v)E_l}, -\vec{k}_T, (1 - v)E_l \right). \]  

(9)

If \( v \neq 0 \) and 1 one can find that

\[ S_F \approx \frac{1}{2E_l} \frac{1}{\omega + E_f - E_l} \approx \frac{v(1 - v)}{k_T^2}, \]  

(10)

which is much larger than

\[ S_B \approx \frac{1}{2E_l} \frac{1}{\omega - E_f + E_l} \approx \frac{1}{4vE_f^2}. \]  

(11)

Therefore, the contributions of the backward propagator are negligible. This not only reduces the number of diagrams, but also allows us to factorize the complex Feynman graph due to the on-mass shell of the forward propagator. This is the theoretical basic of the equivalent photon approximation. We use the processes in Fig. 3 to show this approximation.

We take the laboratory frame, where the target atom is at rest, but the incident electron has a high energy. This is an infinite momentum frame for the electron. Note that the physical picture of the same process has different appearances in the different coordinate frames, even in the different infinite momentum frames. In the above mentioned frame, both the longitudinal and vertical momenta of the virtual photon generally does
Figure 3: Four TOPT diagrams after neglecting the contributions of the backward components at high energy and small scattering angle.

not disappear. Thus, the contributions of Figs. 3b and 3c are not negligible due to the coherence between Fig. 2a and 2c.

Now we consider the recoil effect. The electron-atom interaction time is

$$\tau \sim \frac{1}{\nu},$$

(12)

$\nu$ is the energy loss of the incident electron. The radiation time is

$$T_1 \simeq T_2 \sim \frac{1}{E_f + \omega - E_i} = \frac{E_i \nu (1 - \nu)}{k_T^2},$$

(13)

We set $\nu = \eta E_i$. One can find that at high energy $E_i$ and small scattering angle ($E_i \gg k_T$) a small energy loss ($\eta \ll 1$) may sufficiently lead to

$$\tau < T_{1,2},$$

(14)

i.e., the scattering time $\tau$ can not cover two time periods $T_1$ and $T_2$ in a same bremsstrahlung event. In this case, the contributions of the interferant processes in Figs. 3b and 3c are
inhibited. After removing these coherent diagrams, using the on-mass shell of the momentum $\hat{l}$, the process can further decompose to two sub-processes (Fig. 4).

We discuss the process involving (a) in Fig. 4, Eq. (1) becomes

$$d\sigma_a = \frac{m_e M_0}{\sqrt{(p_i P_i)^2 - m_e^2 M_0^2}} |M_{p_i \rightarrow lP_f}|^2 \frac{M_0 d^3 \tilde{P}_f}{(2\pi)^3 E_f} \frac{m_e d^3 \tilde{p}_f}{(2\pi)^3 E_f} (2\pi)^4 \delta^4(p_i + P_i - p_f - P_f - k)$$

$$\left(\frac{1}{2E_i}\right)^2 \left(\frac{1}{E_f + \omega - E_i}\right)^2 |M_{\hat{l} \rightarrow p_f k}|^2 \frac{2\pi d^3 k}{(2\pi)^3 \omega}$$

$$\equiv d\tilde{\sigma}_a dP_a,$$

(15)

where

$$d\tilde{\sigma}_a = \frac{m_e M_0}{\sqrt{(p_i P_i)^2 - m_e^2 M_0^2}} |M_{p_i \rightarrow lP_f}|^2 \frac{M_0 d^3 \tilde{P}_f}{(2\pi)^3 E_f} \frac{m_e d^3 \tilde{p}_f}{(2\pi)^3 E_f} (2\pi)^4 \delta^4(p_i + P_i - p_f - P_f - k),$$

(16)

and

$$dP_a = \frac{1}{4\pi^2} \left(\frac{1}{2E_i}\right)^2 \left(\frac{1}{E_f + \omega - E_i}\right)^2 |M_{\hat{l} \rightarrow p_f k}|^2 \frac{q^3 k^3}{\omega}$$

(17)

We calculate $d\tilde{\sigma}_a$ using

$$|M_{p_i \rightarrow lP_f}|^2 = \frac{e^2 (Ze)^2 (4\pi)^2}{4m_e^2 M_0^2 (q^2 - \mu^2)^2} \left[ p_i^\mu p_i^\nu + p_f^\mu p_f^\nu - g^{\mu\nu} (l \cdot p_i - m_e^2) \right]$$

$$\left[ P_f^\mu P_i^\nu + P_i^\mu P_f^\nu - g^{\mu\nu} (P_f \cdot P_i - M_0^2) \right],$$

(18)
where we use $l$ to replace $\hat{l}$ in the matrix since $E_l \simeq E_\hat{l}$ for the small emitted angle. The result is

$$d\sigma_a = \frac{Z^2\alpha^2}{4E_i^2} \frac{1}{(\sin^2 \frac{\theta}{2} + \mu^2/4E_i(E_f + \omega))^2} \frac{\cos^2 \frac{\theta}{2} - \frac{q^2}{2M_0^2} \sin^2 \frac{\theta}{2}}{1 + \frac{2E_i}{M_0} \sin^2 \frac{\theta}{2}} d\Omega$$

$$= \frac{Z^2\alpha^2}{4E_i^2} \frac{1}{((1 + \frac{\mu^2}{2E_iM_0}) \sin^2 \frac{\theta}{2} + \mu^2/4E_i^2)^2} \frac{\cos^2 \frac{\theta}{2} - \frac{q^2}{2M_0^2} \sin^2 \frac{\theta}{2}}{1 + \frac{2E_i}{M_0} \sin^2 \frac{\theta}{2}} d\Omega$$

$$\simeq \frac{Z^2\alpha^2}{4E_i^2} \frac{1}{(\sin^2 \frac{\theta}{2} + \mu^2/4E_i^2)^2} \frac{\cos^2 \frac{\theta}{2} - \frac{q^2}{2M_0^2} \sin^2 \frac{\theta}{2}}{1 + \frac{2E_i}{M_0} \sin^2 \frac{\theta}{2}} d\Omega.$$  \hspace{1cm} (19)

Where we used the 4-transfer momentum

$$q^2 = -4E_i(E_f + \omega) \sin^2 \frac{\theta}{2},$$  \hspace{1cm} (20)

the energy momentum conservation

$$\nu = E_i - E_f - \omega = \frac{2E_i(E_f + \omega)}{M_0} \sin^2 \frac{\theta}{2},$$  \hspace{1cm} (21)

and

$$E_f + \omega = E_i \left(1 + \frac{2E_i}{M_0} \sin^2 \frac{\theta}{2}\right)^{-1}.$$  \hspace{1cm} (22)

Note that a $q^2$-dependent term in Eq. (19) is absent when the target is a spin-0 particle, however, it is does not change the following results.

On the other hand, through a simple calculation, we obtain

$$dP_a \equiv \frac{\alpha}{2\pi} P_A dvdk_\perp^2 = \frac{\alpha}{2\pi} \frac{(1 + (1 - v)^2)(1 - v)}{v} dvdk_\perp^2.$$  \hspace{1cm} (23),

where $v = \omega/(E_f + \omega) = \omega/(E_i - \nu)$. In the calculation, we turn the $z$-axis direction from $\vec{p}_i$ to $\vec{l}$.

Combining Eqs. (19) and (23), we have
\[
\begin{align*}
\frac{d\sigma_a}{4E_i^2} & = \frac{Z^2\alpha^2}{\sin^2 \frac{\theta}{2} + \mu^2/4E_i^2)^2} \frac{\cos^2 \frac{\theta}{2} - \frac{q^2}{2M_0^2} \sin^2 \frac{\theta}{2}}{1 + \frac{2E_i}{M_0} \sin^2 \frac{\theta}{2}} \frac{\alpha}{2\pi} \frac{(1 + (1 - v)^2)(1 - v)}{v} d\Omega dv d\ln k_T^2. \\
\end{align*}
\] (24)

Using

\[
\int \frac{dv}{v} = \ln \frac{\omega_{\text{max}}}{\omega_{\text{min}}} = \ln \frac{E_i}{\omega_{\text{min}}},
\] (25)

and

\[
k_T^2 \equiv \beta(-q^2)
\] (26)

with \(\beta \leq 1\) since \(k_T^2\) origins from \(q^2\), we have

\[
\int d\ln k_T^2 = \ln \frac{k_{T,max}^2}{k_{T,min}^2} = \ln \frac{-q^2}{\Lambda^2},
\] (27)

where we introduce a cut-off \(\Lambda^2\) to regularize the collinear divergence. Usually, the value \(\Lambda\) relates to the measurement resolution. Thus, we have the bremsstrahlung cross section in the differential form

\[
\begin{align*}
\frac{d\sigma_a}{4E_i^2} & = \frac{Z^2\alpha^2}{\sin^2 \frac{\theta}{2} + \mu^2/4E_i(E_f + \omega))^2} \frac{\cos^2 \frac{\theta}{2} - \frac{q^2}{2M_0^2} \sin^2 \frac{\theta}{2}}{1 + \frac{2E_i}{M_0} \sin^2 \frac{\theta}{2}} \frac{\alpha}{2\pi} \frac{(1 + (1 - v)^2)(1 - v)}{v} d\Omega dv d\Omega \\
\end{align*}
\] (28)

where we reorganize

\[
\begin{align*}
\frac{d\sigma_a}{4E_i^2} & = \frac{Z^2\alpha^2}{\sin^2 \frac{\theta}{2} + \mu^2/4E_i(E_f + \omega))^2} \frac{\cos^2 \frac{\theta}{2} - \frac{q^2}{2M_0^2} \sin^2 \frac{\theta}{2}}{1 + \frac{2E_i}{M_0} \sin^2 \frac{\theta}{2}} \frac{\alpha}{2\pi} \frac{(1 + (1 - v)^2)(1 - v)}{v} d\Omega dv d\Omega,
\end{align*}
\] (29)

and

\[
\frac{\alpha}{2\pi} P_a = \frac{\alpha}{2\pi} \frac{(1 + (1 - v)^2)(1 - v)}{v}.
\] (30)
We calculate the integrated bremsstrahlung cross section at a given initial energy through the angle-integral. Note that $-q^2 \geq \Lambda^2$ implies

$$\sin^2(\theta_{\text{min}}/2) = \Lambda^2/(4E_i^2) \equiv \epsilon. \quad (31)$$

We decompose

$$\ln \frac{-q^2}{\Lambda^2} = \ln \frac{4E_i^2}{\Lambda^2} - \ln \left(1 + \frac{2E_i}{M_0} \sin^2 \theta/2 \right) + \ln(\sin^2 \theta/2). \quad (32)$$

The contribution of the second term on the right side is negligible comparing with that of the first term. The first term can be integrated and it contributes $2Z^2\alpha^2/\mu^2 + \Lambda^2 \ln(4E_i^2/\Lambda^2)$ at the leading order approximation. However, the third integral is not so lucky. Through the numeric computations we find that the contribution of the third term is almost $\sim 2\beta Z^2\alpha^2/(\mu^2 + \Lambda^2) \ln(4E_i^2/\Lambda^2)$ with $\beta \sim -0.7$. Thus we have

$$d\sigma_a \simeq 0.6Z^2\alpha^3 \frac{(1 + (1 - v)^2)(1 - v)}{v} dv \times \begin{cases} \frac{1}{\mu} \ln \frac{4E_i^2}{\Lambda^2}, & \Lambda > \mu \\ \frac{1}{\mu^2} \ln \frac{4E_i^2}{\mu^2}, & \Lambda \leq \mu \end{cases} \quad (33)$$

If keeping a leading term $1/v$ in $P_A$, we obtain the total cross section

$$\sigma_a \simeq 1.2Z^2\alpha^3 \ln \frac{E_i}{\omega_{\text{min}}} \times \begin{cases} \frac{1}{\mu} \ln \frac{4E_i^2}{\Lambda^2}, & \Lambda > \mu \\ \frac{1}{\mu^2} \ln \frac{4E_i^2}{\mu^2}, & \Lambda \leq \mu \end{cases} \quad (34)$$

One can understand the physical sense of $\Lambda$ as follows. The parameters $\mu$ and $\Lambda$ appear in a same factor $1/(\mu^2 + \Lambda^2)$ of the cross section. Since $1/\mu$ is a space scale of the screening Coulomb potential, $1/\Lambda$ should also be related to a space character about the process. In fact, $\theta_{\text{min}}$ in Eq. (31) relates to a maximum impact parameter $b_{\text{max}}$ for the scattering of electron in a central Coulomb potential. Therefore, $1/\Lambda \sim b_{\text{max}}$ and Eq. (34) has two space scales.

The results (33) and (34) show that as long as $\Lambda < \mu$, the bremsstrahlung cross section
is almost proportional to the geometric area of the atomic Coulomb field $\sim R^2$, rather than a weaker $\ln R$-dependence that the Bethe-Heitler formula predicted.

Now we calculate the process involving (b) in Fig. 4. Its difference from $d\sigma_\alpha$ is not only the interpretations of $p_i \leftrightarrow \hat{l}$ and $\hat{l} \leftrightarrow p_f$, but also they have the different phase spaces. Corresponding to Eq. (15) we have

$$d\sigma_b = \frac{m_eM_0}{\sqrt{(p_iP_i)^2 - m_e^2M_0^2}} |M_{\hat{l}P_i\rightarrow p_iP_f}|^2 \frac{M_0^2p_f^4}{(2\pi)^3E_f^2} \frac{m_e^2p_f^4}{(2\pi)^3E_f^2} (2\pi)^4 \delta^4(p_i + P_i - p_f - P_f - k)$$

$$= \left(\frac{1}{2E_\nu}\right)^2 \left(\frac{1}{E_\nu + \omega - E_i}\right)^2 |M_{p_i\rightarrow \hat{l}k}|^2 \frac{2\pi d^3k}{(2\pi)^3\omega}$$

$$= d\hat{\sigma}_b \frac{\alpha^2}{2\pi} P d\nu d\Omega,$$ (37)

where

$$d\hat{\sigma}_b = \frac{Z^2}{{\lambda^2}} \ln -\frac{q^2}{{\lambda^2}} \frac{1}{{\left(\sin^2 \frac{\theta}{2} + \mu^2/4E_f(E_i - \omega)\right)^2}} \frac{\cos^2 \frac{\theta}{2} - \frac{q^2}{2M_0^2} \sin^2 \frac{\theta}{2}}{1 + \frac{2(E_i - \omega)}{M_0} \sin^2 \frac{\theta}{2}} d\Omega$$

$$= \frac{Z^2}{{\lambda^2}} \ln -\frac{q^2}{{\lambda^2}} \frac{1}{{\left(1 - \frac{\mu^2}{2E_fM_0}\right)^2 \sin^2 \frac{\theta}{2} + \mu^2/4E_f^2)^2}} \frac{\cos^2 \frac{\theta}{2} - \frac{q^2}{2M_0^2} \sin^2 \frac{\theta}{2}}{1 + \frac{2(E_i - \omega)}{M_0} \sin^2 \frac{\theta}{2}} d\Omega$$

$$\approx \frac{Z^2}{{\lambda^2}} \ln -\frac{q^2}{{\lambda^2}} \frac{1}{{\left(\sin^2 \frac{\theta}{2} + \frac{\mu^2}{4E_f^2}\right)^2}} \frac{\cos^2 \frac{\theta}{2} - \frac{q^2}{2M_0^2} \sin^2 \frac{\theta}{2}}{1 + \frac{2(E_i - \omega)}{M_0} \sin^2 \frac{\theta}{2}} d\Omega.$$ (38)

Note that

$$E_i - \omega = E_f \left(1 - \frac{2E_f}{M_0} \sin^2 \frac{\theta}{2}\right)^{-1},$$ (39)

$$q^2 = -4E_f(E_i - \omega) \sin^2 \frac{\theta}{2},$$ (40)

and the energy momentum conservation

$$\nu = E_i - E_f - \omega = \frac{2E_f(E_i - \omega)}{M_0} \sin^2 \frac{\theta}{2}.$$ (41)
are used.

After integral, we have

\[ \hat{\sigma}_b = \hat{\sigma}_a \left( \frac{E_f}{E_i} \right)^2 < \hat{\sigma}_a. \]  \hspace{1cm} (42)
3 Comparing with the Bethe-Heitler formula

The differential cross section of the Bethe-Heitler formula for the bremsstrahlung [1] is

\[
d\sigma_{B-H} = \frac{Z^2 \alpha^3}{2\pi^2} \frac{|\vec{p}_f|}{|\vec{p}_i|} d\omega d\Omega_c d\Omega_k \left[ \frac{p_f^2 \sin^2 \theta_f}{(E_f - |\vec{p}_f| \cos \theta_f)^2 (4E_i^2 - q^2)} + \frac{p_i^2 \sin^2 \theta_i}{(E_i - |\vec{p}_i| \cos \theta_i)^2 (4E_f^2 - q^2)} \right] + 2\omega^2 \frac{\vec{p}_i^2 \sin \theta_i + \vec{p}_f^2 \sin \theta_f}{(E_i - |\vec{p}_i| \cos \theta_i)(E_f - |\vec{p}_f| \cos \theta_f)} - 2 \frac{|\vec{p}_i||\vec{p}_f| \sin \theta_i \sin \theta_f \cos \phi}{(E_i - |\vec{p}_i| \cos \theta_i)(E_f - |\vec{p}_f| \cos \theta_f)} (2E_i^2 + 2E_f^2 - q^2), \tag{43}
\]

where \( \theta_i, \theta_f \) are the angles between \( \vec{k} \) and \( \vec{p}_i, \vec{p}_f \) respectively; \( \phi \) the angle between \( (\vec{p}_i \vec{k}) \) plane and \( (\vec{p}_f \vec{k}) \) plane.

The Bethe-Heitler formula does not consider the recoil effect, therefore the scattering time of the electron with an infinite heavy target is \( \tau = \infty \). Thus, we can not distinguish whether the photon is emitted from the incoming or outgoing electrons. In opposite to the Bethe-Heitler formula, Eq. (28) presents two factorized sub-processes, because of the recoil effect suppresses the contributions of Figs. 3b and 3c.

After integral over angles, the differential cross section (43) at high energy \( E_i, E_f \gg m_e \) can be simplified as [7]

\[
d\sigma_{B-H} \simeq \frac{Z^2 \alpha^3}{m_e^2} \frac{d\omega}{\omega} \frac{4}{E_i^2 + E_f^2 - \frac{2}{3} E_i E_f} \left( \log \left( \frac{2E_i E_f}{m_e \omega} \right) - \frac{1}{2} \right). \tag{44}
\]

Unfortunately, Eq. (44) does not present the screening effect since it used a pure Coulomb potential. Bethe and Heitler define a term \( 2E_i E_f / m_e^2 \omega \) in Eq. (44) as a screening radius \( R \), i.e.,

\[
d\sigma_{B-H} = \frac{Z^2 \alpha^3}{m_e^2} \frac{d\omega}{\omega} \frac{4}{E_i^2 + E_f^2 - \frac{2}{3} E_i E_f} \left( \log(m_e R) - \frac{1}{2} \right), \tag{45}
\]

or
\[ d\sigma_{B-H} = \frac{Z^2\alpha^3}{m_e^2} \frac{d\omega}{\omega E_i^2} \left( E_i^2 + E_f^2 - \frac{2}{3} E_f E_i \right) \left( \log(137 Z^{-1/3}) - \frac{1}{2} \right), \] (46)

where the Thomas-Fermi model is used. However, this method "freezes" unreasonably the variables \( E_i E_f / \omega \) and has following uncertainty: if setting any factor \( 1 = \left( \frac{(2E_i E_f)/(m_e \omega)}{((m_e \omega) / (2E_i E_f))^n} \right) \) into Eq. (44), one can get the different \( R \)-dependent results.

For further comparison, we refer Eq. (33) to rewrite Eq. (45) as

\[ d\sigma_{B-H} = \tilde{\sigma}_{B-H} \frac{\alpha}{2\pi} P_{B-H} dv, \] (47)

where

\[ \tilde{\sigma}_{B-H} = \frac{8\pi Z^2 \alpha^2}{m_e^2} \left( \log(m_e R) - \frac{1}{2} \right), \] (48)

and

\[ \frac{\alpha}{2\pi} P_{B-H}dv = \frac{\alpha}{2\pi} \frac{d\omega}{\omega E_i^2} \left( E_i^2 + E_f^2 - \frac{2}{3} E_f E_i \right). \] (49)

Using \( v = \omega/E_i \) (note that \( \nu = 0 \) in the Bethe-Heitler formula) and \( d\omega/\omega = dv/v \), we have

\[ \frac{\alpha}{2\pi} P_{B-H}(v) = \frac{\alpha}{2\pi} \left[ \frac{1 + (1 - v)^2}{v} - \frac{2(1 - v)}{3v} \right]. \] (50)

Thus, the Bethe-Heitler formula (47) becomes

\[ d\sigma_{B-H} \simeq \frac{4Z^2 \alpha^3}{m_e^2} \left( \ln(137 Z^{-1/3}) - \frac{1}{2} \right) \left[ \frac{1 + (1 - v)^2}{v} - \frac{2(1 - v)}{3v} \right] dv. \] (51)
Figure 5: The bremsstrahlung process on a Coulomb potential at $\omega \to 0$. The double lines are the eikonal form of the electron propagators.

Figure 6: The factorized bremsstrahlung process for a soft photon version.
This result has a similar leading behavior $\sim 1/v$ as Eq. (30). However, there is a completely different $R$-dependence in $\tilde{\sigma}_{B-H}$.

It is well known that at limit $\omega \rightarrow 0$ any process leading to photon emission can be factorized [10]. The bremsstrahlung cross section has its soft version. Two electron propagators in Fig. 5, which are indicated by the double lines, are the eikonal form at the soft photon limit. A corresponding factorized differential cross section (see Fig. 6) is [9]

$$
\frac{d\sigma_{\text{soft}}}{d\Omega} = \frac{d\sigma_{\text{Ruth}}}{d\Omega} \frac{2\alpha}{\pi} \ln \frac{E_i}{\omega_{\text{min}}} \left\{ \begin{array}{ll}
\frac{4}{3}v^2 \sin^2 \frac{1}{2}\theta, & NR \\
\ln \frac{-q^2}{m_e^2} - 1, & ER
\end{array} \right.
$$

Using the screening potential, we have [6]

$$
\sigma_{\text{NR soft}} \simeq \frac{8Z^2\alpha^3}{E_i^2} \ln \frac{E_i}{\omega_{\text{min}}} \left[ \ln \frac{\mu^2 + 2E_i^2}{\mu^2} - 2 \right],
$$

at the nonrelativistic (NR) limit, which has the $\ln R$-dependent cross section similar to Eq. (48) but with a $1/E_i^2$-suppletion. We emphasize that Eq. (52) at the NR-limit is valid only at a very narrow kinematics range near $E_i \sim m_e$, where the contributions of order $O((v/c)^4)$ are almost vanished. Any term without scattering angle $\theta$ in Eq. (52) will appear a strongly $R^2$-dependent bremsstrahlung cross section, if they can not be completely canceled at $E_i \gg m_e$.

We consider the integral of Eq. (52) at the ER limit. Because of $m_e^2$ in $\ln(-q^2/m_e^2)$ is not introduced as a cut-off parameter [9], the theory itself does not have any restrictions on the value of $q^2$. Therefore, $-q^2 < m_e^2$ is allowed. It implies a negative cross section. Taking a step back, if we regard this $m_e$ as a cut-off parameter, the result shows that the scattering will be restricted inside a small range $r \sim 1/|q| < 1/m_e \sim 10^3 \text{ fm}$, which is smaller much than atomic radius $\sim 10^5 \text{ fm}$, and the cross section becomes irrelevant to the screening parameter $\mu$. According to Eq. (27), $m_e^2$ in $\ln(-q^2/m_e^2)$ should be replaced by a general cut-off parameter $\Lambda^2$ using the following substitution
\[ \ln\left(-\frac{q^2}{m_e^2}\right) \to \ln\left[\frac{-\frac{q^2}{m_e^2}}{-\frac{q_{\text{min}}^2}{m_e^2}}\right] \equiv \ln \frac{-q^2}{\Lambda^2}, \] (54)

\(\Lambda\) is the process-dependent and we request \(-q^2 > \Lambda^2\). Thus, the cross section at the ER limit reads

\[ \sigma_{\text{soft}}^{ER} = \frac{Z^2\alpha^3\pi}{2E_i^4} \int_{\theta_{\text{min}}}^{\pi} d\theta \frac{\sin \theta}{(\sin^2 \frac{\theta}{2} + \frac{\mu^2}{4E_i^2})^2} \frac{2\alpha}{\pi} \left[ \ln \left(\frac{-q^2}{\Lambda^2}\right) - 1 \right] \ln \frac{E_i}{\omega_{\text{min}}}, \] (55)

where \(\sin^2(\theta_{\text{min}}/2) \equiv \Lambda^2/(4E_i^2)\). We get [6]

\[ \sigma_{\text{soft}}^{ER} \simeq \begin{cases} 
\frac{4Z^2\alpha^3}{\Lambda^4} \ln \frac{4E_i^2}{\Lambda^2} \ln \frac{E_i}{\omega_{\text{min}}}, & \Lambda > \mu \\
\frac{4Z^2\alpha^3}{\mu^4} \ln \frac{4E_i^2}{\mu^2} \ln \frac{E_i}{\omega_{\text{min}}}, & \Lambda \leq \mu 
\end{cases} \] (56)

it is compatible with Eq. (34).

The coefficients of Eq. (34) are smaller than that of Eq. (56). The reason is that the contributions of \(d\sigma_b\) are neglected in Eq. (34). Besides, the contribution of the third term in Eq. (32) is more negative due to the recoil effect. Therefore, our formula is consistent with the soft version of bremsstrahlung, but both contradict the Bethe-Heitler formula. According to the QED-results either Eq. (34) or Eq. (56), we conclude that the screening parameter \(\mu\) or \(1/R\) is defined in a wrong location in the Bethe-Heitler formula (45) (or (47)) for high energy bremsstrahlung.

Now we try to answer why a strong screening scale-dependence of the bremsstrahlung cross section has not been discovered in a long time? If \(\Lambda \gg \mu\), for example, \(\Lambda \simeq m_e\), Eq. (34) has structure

\[ \sigma \sim \frac{1}{m_e} \ln \frac{4E_i^2}{m_e^2} \ln \frac{E_i}{\omega_{\text{min}}}, \] (57)

which is similar to Eq. (56) with \(\Lambda = m_e\). The result is irrelevant to the screening parameter \(\mu\) and without a strong \(R^2\)-dependent effect. We think that the measurements of the
differential bremsstrahlung cross sections belong to this example. They detect the angular or energy distributions for the photons. The separation of the detected photon from the electron is restricted by the instrument resolution, which has a larger parameter Λ. On the other hand, the $R^2$-dependent effect may obviously appear in the total bremsstrahlung cross section, where the angle and energy of the projected particles are integrated over all possible phase space and they have a minimum value of Λ. The measurement of the radiation length $\lambda_A \sim 1/\sigma_A$ is such an example. In practical applications, there are several uncertainties: a quantitative relationship $R \sim \mu$, the value of $\omega_{\min}$ and the corrections of the approximations. We have suggested to measure the high energy electron spectra when they pass through the completely ionized and extremely thin atmosphere [6]. Where the atomic Coulomb potential may expand to a macroscopic spatial scale $\sim 10^{-3}$ cm, which is much larger than an atomic radius $\sim 10^{-8}$ cm and can provide a big $R^2$-dependent effect. Besides, the above mentioned uncertainties can be effectively canceled though the comparison with a normal radiation length. This application can be simplified as

$$\sigma_A(E_i, \mu, \omega_{\min}) = \sigma_0(E_i, \mu_0, \omega_{\min}) \left(\frac{\mu_0}{\mu}\right)^2,$$  \hspace{1cm} (58)$$
or for the radiation length

$$\lambda_A(E_i, \mu, \omega_{\min}) = \lambda_0(E_i, \mu_0, \omega_{\min}) \left(\frac{\mu}{\mu_0}\right)^2,$$  \hspace{1cm} (59)$$
where $1/\mu_0$ is the radius of a referring neutral atom and $\sigma_0$ or $\lambda_0$ are fixed by the corresponding data.

We should mention the classical bremsstrahlung theory. It is the radiation of accelerated charged particle during its collisions with atomic electric field. Analogy to the soft photon limit of quantum theory, under low frequency limit the intensity of radiation is written as a factorized form [11].
\[
\frac{d\chi}{d\omega dQ} \equiv \frac{dI}{d\omega} \frac{d\sigma_{\text{Ruth}}}{dQ},
\]

(60)

where \(\omega\) is radiation frequency and \(\sigma_{\text{Ruth}}\) the classical Rutherford cross section. Using

\[
\lim_{\omega \to 0} \frac{dI}{d\omega} = \frac{2}{3\pi} \frac{\alpha}{m_e^2} Q^2
\]

(61)

and

\[
\frac{d\sigma_{\text{Ruth}}}{dQ} = 8\pi \left(\frac{Z\alpha}{\beta}\right)^2 \frac{Q}{(Q^2 + \mu^2)^2},
\]

(62)

we have

\[
\frac{d\chi}{d\omega} = 16 \frac{Z^2 \alpha^3}{3 m_e^2 \beta^2} \int_{Q_{\text{min}}}^{Q_{\text{max}}} \frac{Q^3 dQ}{(Q^2 + \mu^2)^2} = 16 \frac{Z^2 \alpha^3}{3 m_e^2 \beta^2} \left[ \ln \left( \frac{Q_{\text{max}}}{\mu} \right) - \frac{1}{2} \right],
\]

(63)

where \(Q_{\text{min}} \ll \mu\) is used. The result is \(\chi \sim \ln R\). This is not surprising, since Eq. (61) corresponds really to the nonrelativistic limit in Eq. (52). Therefore, the description of bremsstrahlung in the classical electrodynamics using (63) at high energy is not sufficient.
4 Summary

The Bethe-Heitler formula describes bremsstrahlung of high energy electrons in a pure Coulomb potential, which may lead to an infinite total cross section since the Coulomb scattering is a long-range interaction. A natural method is to use a screening potential to replace the Coulomb potential. However, the complex interference effect between scattering and radiation sub-processes makes a difficulty for us to get an analytical solution if considering the screening potential. It brings the uncertainty in the Bethe-Heitler formula.

For this sake, we re-derive the formula for the bremsstrahlung cross section of electron in the atomic field using the TOPT framework. We prove that the recoil corrections of a finite mass atom at high energy may further decompose the bremsstrahlung cross section to two sub-processes at the equivalent photon approximation. The improved bremsstrahlung formula contains the screening potential and predicts a strong $R^2$-dependent bremsstrahlung cross section. The results remind us to review the traditional electromagnetic shower theory at the extreme conditions.

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