Non-Traditional Intervals and Their Use. Which Ones Really Make Sense?

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Received February 8, 2022; in final form, November 16, 2022; accepted January 30, 2023

Abstract—The paper discusses the question of why intervals, which are the main object of Interval Analysis, have exactly the form that we know well and habitually use, and not some other. In particular, we investigate why traditional intervals are closed, i.e., contain their endpoints, and also what is wrong with an empty interval. A second question considered in the work is how expedient it is to expand the set of traditional intervals by some other objects. We show that improper (“reversed”) intervals and the arithmetic of such intervals (the Kaucher complete interval arithmetic) are very useful from many different points of view.

DOI: 10.1134/S1995423923020088

Keywords: interval analysis, interval, non-traditional intervals, classical interval arithmetic, Kaucher interval arithmetic.

1. INTRODUCTION

The article discusses one of the basic objects of Interval Analysis, namely, the concept of interval. Recall that classical intervals are closed, connected, and bounded subsets of the real line $\mathbb{R}$, i.e., sets of the form

\[ [a, b] = \{ x \in \mathbb{R} | a \leq x \leq b \} \] (1)

(see [1, 3, 14, 28, 30, 31] and other books on Interval Analysis). The set of all intervals (usually with arithmetic operations on it) is denoted by $\mathbb{I}$. Multidimensional intervals are their generalizations, in one sense or another.

The issues that will be covered below were raised in an online discussion on \texttt{reliable_computing-mailing list} (see [40]) that took place in the spring of 2018. Its starting point was the question of whether it is advisable to introduce and further use open and half-open intervals, such as $[a, b[, ]a, b[, ]a, b]$, in addition to the existing closed intervals of the form (1) (we denote various types of intervals in the style of N. Bourbaki). The author had experience of working with similar objects and therefore took an active part in the ensuing discussion. A summary of the views on these issues constitutes the core of this article. Previously, some ideas of the following text have been published in book [1, Sec. 1.11].

Hereafter, by “traditional intervals” we mean usual intervals of the form (1) that constitute the classical interval arithmetic $\mathbb{I}$, and the “non-traditional” intervals in Section 3 are open and half-open intervals. Further, in Section 4, we consider improper (“reversed”) intervals from Kaucher interval arithmetic $\mathbb{K}$ (algebraic and order completion of $\mathbb{I}$) as non-traditional intervals. The short Section 5 discusses the usefulness of a special “empty interval.” Our notation follows the informal international standard [23].

In addition to the traditional “endpoint form” of intervals (1), there are also other equivalent representations. For example, the midpoint–radius form of intervals,

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\( (m, r) = m + r [-1, 1] \), where \( m = \frac{1}{2}(a + b), \ r = \frac{1}{2}(b - a) \),

is very popular in various applications. But in our work we will not consider interval representation forms other than the canonical (1).

Infinite and semi-infinite intervals of the form \([-\infty, p], [q, \infty]\), and \([-\infty, p] \cap [q, \infty]\) can also be classified as non-traditional. They have been first considered by W. Kahan [17] and constitute the so-called Kahan interval arithmetic (see details in [1, 22, 25]). Over the years such intervals have received numerous applications in Interval Analysis, and hence they do not need additional justification from our side. As a consequence, we do not consider these non-traditional intervals in our work.

2. WHY INTERVALS ARE CLOSED IN INTERVAL ANALYSIS?

Intervals, as they were proposed in the seminal works of the founders of Interval Analysis—T. Sunaga [48], R.E. Moore [29], L.V. Kantorovich [18], and others—are closed segments of the numerical axis. But one of the popular myths about Interval Analysis and interval arithmetic \( IR \) (widespread among beginners and veterans alike) is that it has the disadvantage of supporting only intervals which contain their endpoints. This allegedly makes it impossible to perform some basic set operations (like complementation), bounds its expressive power, limits arithmetic operations, etc.

Well, is it really a drawback that intervals are closed and that they have no open endpoints? Or, on the contrary, do we not understand the underlying reasons behind \( IR \)?

This kind of questions was asked by W. Kahan more than half a century ago (see [17]), and he was the first to propose non-closed intervals (although he did not realize his idea to the end). Then the same issues came up in 1998 and then in 2018.

Of course, in practical applications of Interval Analysis these questions are not so relevant, because all numerical values of continuously varying quantities encountered in practice are approximate, all measurements are performed with non-zero error, etc. Therefore, it is almost impossible to trace the difference between a point and other values that differ infinitesimally little from it. Therefore, in engineering practice the question of whether intervals are open or closed seems rather abstract and unimportant. But this question is important in theory and in calculations with intervals.

There are arguments in favor of admitting non-traditional open and half-open intervals in computation and in our reasoning in general, in favor of giving them “citizenship rights” in Interval Analysis. Sometimes such intervals seem to significantly expand our capabilities. For example, let \([0, 1]\) be a traditional closed interval and let \([0, 1)\) be an interval half-open at zero, then

\[
\begin{align*}
[0, 1] &= [\infty, \infty], & \text{but} & \quad \left[ \frac{0, 1}{0, 1} \right] = [0, \infty],
\end{align*}
\]

since the first fraction must contain the result of the division \(0/0\), and this is not the case for the second fraction.

In other words, the benefit from the fact that, in (2), the denominator is open at one of its endpoints is half the real line! This is very helpful in the interval Newton method for enclosing zeros of functions. Its iterations are defined by the formula

\[
X^{(k+1)} \leftarrow X^{(k)} \cap N(X^{(k)}, \tilde{x}^{(k)}), \quad \tilde{x}^{(k)} \in X^{(k)}, \quad k = 0, 1, 2, \ldots,
\]

where

\[
N(X, \tilde{x}) := \tilde{x} - \frac{f(\tilde{x})}{f'(X)}
\]

is the interval Newton operator (see, e.g., [1, 14, 28,30, 31]). If, in the fraction of (4), the numerator and denominator coincided with those of (2), we would get very large improvement of the result. Then the
interval Newton method finds solutions to equations much faster, areas that obviously do not contain solutions are better eliminated, etc.

A similar situation can arise when implementing the interval Gauss–Seidel method for interval linear equations (see, e.g., [1, 14, 22, 30, 31]). One also needs to divide by an interval there and then intersect the result with another interval, and in this case we can again get the same great improvement of the final result.

More than 20 years ago I worked for a Novosibirsk software company called UniPro, which then carried out orders for Sun Microsystems, Inc. Our team was implementing Sun’s interval Fortran-95 (see [50]), a programming language with built-in interval data type and operations with it. Just at that time our project manager, Bill Walster from Sun Microsystems, was writing a book [14] with E. Hansen, which was devoted to interval methods for global optimization and equation solving. He was greatly impressed by the above observation on the interval Newton method. Hence, it was decided to implement something like open or half-open intervals, at least partially, insofar as the sign of the zero, which was inherent to floating-point computer arithmetics according to IEEE 754/854 standards, made it possible to easily implement open and closed endpoints at zero.

In the computer, an interval \([a, b]\) is naturally represented by a pair of real numbers, \((a, b)\), and, for zero endpoints, we can take that

\[
\begin{align*}
(a, +0) & \text{ means } [a, 0], \\
(a, -0) & \text{ means } [a, 0[ \\
(-0, b) & \text{ means } ]0, b], \\
(+0, b) & \text{ means } ]0, b]
\end{align*}
\]

for \(a < 0\), and \(b > 0\).

Approximately the same was proposed for the implementation of intervals not closed at zero by W. Kahan in his work [17]. The possibility of such a natural implementation was very tempting, and then the modified division in (2) seemed to be available. Therefore, Bill Walster immediately jumped on this idea.

In our team, I was responsible for semantic testing and general mathematical consulting, so I had to think deeply about the consequences of introducing the new construction into the language. As I delved into the question, I realized that to implement open intervals, even partially, was hardly possible and did not make much sense. It turned out that there are very important and even fundamental mathematical reasons why the intervals should be closed. The fact is that, with their mathematical properties, bounded closed intervals essentially differ from non-closed, open and half-open, intervals. That was exactly what I reported to Bill Walster, and the implementation of non-closed intervals in Fortran-95 was canceled.

Then the discussion in our team acquired a broader context, and the participants began to discuss questions about whether any other intervals, besides the classical ones, are needed at all. Is it necessary to introduce the empty set into our algebraic systems of intervals? These topics will be discussed in Sections 4 and 5 of this work, but first we will consider what is so good about closed intervals.

3. CLOSED INTERVALS ARE COMPACT SETS, COMPLETE LATTICES, AND COMPLETE METRIC SPACES

Let us recall that a subset \(S\) of a topological space \(X\) is called compact, if for every open cover of \(S\) there exists a finite subcover of \(S\) [9, 41, 47]. In fact, the concept of compactness formalizes the idea that it is possible to exhaust a set by a finite number of arbitrarily small open sets. Bounded closed intervals are compact sets in \(\mathbb{R}\), while open and half-open intervals are non-compact, with all the ensuing consequences.

Considered as metric spaces, i.e., sets with an abstract distance (metric), non-closed intervals are not complete spaces (see [9, 41]): fundamental sequences (also called Cauchy sequences) of elements from such non-closed intervals do not necessarily converge within such intervals. For example, the sequence of numbers \(1/k, k = 1, 2, \ldots\), has no limit within the half-open interval \([0, 1)\).

A partially ordered set in which we can freely take supremum and infimum for each pair of elements is called lattice [6]. A partially ordered set is called complete lattice if all of its subsets have a supremum and an infimum. Non-closed intervals are not complete lattices with respect to the standard order “\(\leq\)” on the real line, i.e., one cannot take infima and suprema, with respect to the order “\(\leq\)” for every subset of
a non-closed interval. An example is the same sequence of numbers $1/k$, $k = 1, 2, \ldots$, in the half-open interval $]0, 1]$.

The above facts have many unpleasant consequences for practice, which will be listed below.

**Non-compactness.** On compact closed intervals, continuous functions reach their extrema, i.e., their minimal and maximal values (the Weierstrass extreme value theorem). But on non-compact, open and half-open intervals continuous functions may not reach their extreme values.

**Brouwer fixed point theorem and Banach fixed-point theorem.** The Brouwer’s fixed point theorem (see, e.g., [11, 34, 51]) states that for any continuous function $\phi$ mapping a compact convex set from $\mathbb{R}^n$ to itself there is a point $\tilde{x}$ such that $\tilde{x} = \phi(\tilde{x})$ (a “fixed point” that remains unchanged). Obtained in 1909–1912, this result has become one of the cornerstones of computational Interval Analysis, since intervals in $\mathbb{R}$ and their multidimensional analogs are convex compact sets. Given an equation $f(x) = 0$, we can always reduce it to a fixed-point form $x = \phi(x)$ and then proceed as follows. If, using interval methods for enclosing the ranges of functions, we verify the fulfillment of the Brouwer fixed-point theorem on an interval $X$ for the mapping $\phi$, i.e., that the inclusion $\phi(X) \subseteq X$ is valid, then we rigorously prove the existence of a solution to the fixed-point equation $x = \phi(x)$ within $X$.

The above constructive proof method for solutions to equations is an integral part of important interval methods, in particular, the Krawczyk method, the interval Newton method, and the Hansen–Sengupta method (see, e.g., [1, 14, 22, 28, 30, 31]).

The Banach fixed-point theorem (also known as the contraction mapping theorem; see, e.g., [11, 34, 51]) states that a complete metric space $X$ with a contraction mapping $\phi : X \to X$ has a unique fixed-point $\tilde{x}$, i.e., such that $\tilde{x} = \phi(\tilde{x})$. It is also an important tool of Computational Interval Analysis, because traditional intervals are complete metric spaces and, hence, the Banach fixed-point theorem allows one to prove the existence and even uniqueness of solutions to equations. Often in Computational Interval Analysis, the Schröder fixed point theorem is used for the same purposes [8, 31].

For non-closed intervals and their multidimensional analogs the Brouwer fixed point theorem, the Banach fixed-point theorem, as well as the Schröder fixed point theorem are not valid. Therefore, these interval tests for the existence of solutions to equations and systems of equations, that is, the interval Newton method, the Krawczyk method, and the Hansen–Sengupta method do not work in full with non-closed intervals.

Thus, with non-closed intervals, Computational Interval Analysis is deprived of its most powerful tools, which are widely used in solving equations, systems of linear and nonlinear equations, as well as in global optimization.

**Nested intervals principle.** It is known to be one of the popular interval tools for both theory and verified computing. Let us recall its formulation (see [1, 3, 30, 31]): Every nested interval sequence \[ \{X_k\}_{k=1}^{\infty}, \text{i.e., such that } X_{k+1} \subseteq X_k \text{ for all } k, \text{ converges and has a limit } \bigcap_{k=1}^{\infty} X_k. \]

In the above statement, $X_k$ can be either one-dimensional intervals or interval boxes in $\mathbb{R}^n$.

It turns out to be incorrect for non-traditional intervals. A sequence of half-open intervals does not necessarily converge to anything and can have an empty intersection. For example,

\[ \bigcap_{k=1}^{\infty} ]0, \frac{1}{k} [ = \emptyset. \]

This is a great loss. Let us recall that the theory of interval integrals and the interval estimates of the integrals of a real function are based on this principle (see [7, 38]).

The most practical and efficient interval methods for solving operator equations (integral and differential) are based on the nested intervals principle and some fixed point theorems, in particular, the Banach fixed-point theorem and the Schröder fixed-point theorem [8, 31]. They also become invalid, since non-closed intervals are not complete topological spaces.
Birkhoff–Tarski theorem and Kantorovich lemma. These are popular fixed-point theorems for partially ordered sets, analogs of topological fixed-point theorems that we have formulated earlier. The Birkhoff–Tarski principle (also called the Knaster–Tarski principle [6, 11, 49]) states that if $X$ is a complete lattice and $\phi : X \to X$ is an isotone (order preserving) function, then $\phi$ has a fixed point $\bar{x} \in X$, i.e., such that $\bar{x} = \phi(\bar{x})$. A feature of this result is the absence of special requirements for the continuity of the function $\phi$, the form of the set $X$, and its topological properties.

The Birkhoff–Tarski theorem is incorrect for non-closed intervals that are not complete lattices with respect to the standard order $\leq$ on $\mathbb{R}$ and with respect to inclusion ordering between intervals. The Kantorovich lemma [34, Sec. 13.2] is a similar useful result, which is a fixed-point theorem for isotone mappings. It also becomes invalid for non-closed intervals.

Distance between various types of intervals. How should we calculate the distance between $[\alpha, \beta]$ and $[\alpha, \beta]$, two intervals that differ in only one endpoint?

In mathematics, distance (deviation) is usually formalized by a metric, a function that gives a distance between each pair of elements of a set. The metric is defined axiomatically, as a nonnegative function that satisfies three axioms: identity of indiscernibles, symmetry, and triangle inequality (for details see, e.g., [8–10]).

The distance between intervals is known to be defined as follows [1, 3, 28, 30, 31]:

$$\text{dist} (a, b) = \max\{|a - b|, |\alpha - \beta|\}.$$ 

It should be equal to zero for $a = [\alpha, \beta]$ and $b = [\alpha, \beta]$. Thus, one of the main purposes of distance, which is to distinguish between elements of a set that do not coincide with each other, is not fulfilled. Moreover, it turns out that metric (distance) cannot be introduced in any way on the set of all closed and non-closed intervals, that is, this set is essentially non-metrizable as a topological space.

The Arkhangel’skii metrization theorem (see, e.g. [10]) asserts that the topology of a space can be determined by a metric if and only if this space satisfies the first axiom of separation (so-called T1-axiom) and has a countable fundamental family of open neighborhoods. Axiom T1 is the weakest axiom of separability, it requires that any one of two points of the space has a neighborhood not containing the other point. It is easy to see that the space of all closed and non-closed intervals does not satisfy even this weakest axiom: a half-open interval $[a, b]$ and its closure $[a, b]$ cannot be surrounded by such neighborhoods.

The failure of axiom T1 is a very serious evidence of the fact that the topological space under consideration is very exotic, even pathologic. For us, in Interval Analysis, it implies that a meaningful calculus on the set of closed and non-closed intervals will most likely never be constructed. Of course, this does not exclude individual episodic applications of non-closed intervals in certain particular situations. But in general, alas . . .

4. CAN OTHER NON-TRADITIONAL INTERVALS BE USEFUL IN INTERVAL ANALYSIS?

Next, let us turn to other kinds of non-traditional intervals, such as improper (“reversed”) intervals, like $[2.3, 1.4]$, $[1.05, -2.5]$, etc.

Some time ago, A. Neumaier reviewed in [32] the properties and applications of some non-traditional interval arithmetics (he called them “non-standard arithmetics,” an unfortunate term, as if someone issued a standard for the various types of intervals). But since A. Neumaier himself is a pessimist and does not believe much in the usefulness of these arithmetics, his review of applications turned out to be also pessimistic, almost like an obituary. Below, we will try to give another overview of the capabilities of improper intervals from a more general point of view and show that, sooner or later, they will take their rightful place among the mathematical tools of natural and social sciences.
In the previous section, we considered intervals from the viewpoint of the science of Topology. Let us now consider sets of intervals and, more precisely, various interval arithmetics, from the viewpoint of another great science—Algebra. It is often called “the science of algebraic systems,” i.e., a science that studies sets with certain operations and relations defined on them. Let us look at the operations existing on the set of intervals.

In terms of Algebra, operations can be different. If an associative binary operation is defined on a set of some elements, this set is called a semigroup, a monoid, or a group, depending on what properties of this operation are. Strictly speaking, interval arithmetic is an algebraic system on which more than one operation is defined, but for our analysis it is sufficient to consider these operations one at a time.

A semigroup is the weakest formation, where almost nothing is required of the binary associative operation between the elements.

A monoid is a semigroup with a neutral element. A reminder: the neutral element or identity element is a special kind of element with respect to the binary operation on that set, which leaves other elements unchanged when combined with them.

A group is an algebraic system where the operation in question is reversible, that is, for any element there is an inverse element with respect to this operation. Performing the operation between an element and its inverse one results in the identity element.

In general, it is much more comfortable to work with a group than with a monoid or a semigroup. Implicit awareness of this fact has been one of the driving forces behind the expansion of popular and well-known algebraic systems over the past millennia. Recall that this is why the simplest natural numbers were once expanded to integers, then to rational numbers, and then to real and complex numbers, and so on (although this process was not linear).

Why? The fact is, the operation in a group is “predictable” and “invertible” by its results. We can restore the operands from the result of the operation. We can perform algebraic manipulations in a group more easily and with less restrictions. In other words, our mathematical tools in a group are richer than in a semigroup or monoid.

Specifically, in a group with an operation “∗,” if we have an equality

\[ a \ast c = b \ast c, \]

we can conclude that

\[ a = b. \]

And if

\[ a \ast b = c, \]

then

\[ a = b^{-1} \ast c, \]

where \( b^{-1} \) is the inverse of \( b \) with respect to the operation “∗.” Additionally, we can solve equations in the group, which is not possible in semigroups and monoids in the general case.

Do we really need such capabilities in Interval Analysis? My answer is definitely “yes.” The author, for example, needs it, and he certainly knows that many others also need such things. The above is especially important when we solve so-called “inverse problems,” when it is necessary to restore the preimage of a function by its value. A special case of “inverse problems” is the well-known problem of solving equations and systems of equations.

If we cannot restore preimages in elementary interval operations, we do not have adequate tools to solve “inverse problems” in general.

Yet another obvious example where the above algebraic properties prove to be indispensable is in metrology and measurement theory. This is the fundamental concept of measurement error. Recall
that by definition the error is the difference of an approximate value of a quantity and its exact ideal value. In the natural sciences and engineering, this latter is understood as the true value of a physical quantity, that is, the value that ideally reflects the considered quantity or phenomenon within the framework of the model (theory) we have adopted to describe it. Anyway, the difference in the above formulation means the algebraic difference, i.e., addition with the opposite element with respect to addition. If a measurement result and/or the true value of a quantity are of the interval type, it is not possible to correctly find the error in the classical interval arithmetic, since there is neither algebraic subtraction nor elements that are opposite to proper intervals with respect to addition.

In a similar situation in Geometry, when the Minkowski sum of sets is considered, and it is required to “inverse” it, the so-called Hukuhara difference is introduced \[15\] according to the following rule:

\[ A \ominus B = C \iff A = B + C. \]

With respect to the classical interval arithmetic \( \mathbb{R} \), it is better not to limit ourselves to partial reversal of operations, but to correct the situation fundamentally by performing algebraic completion of \( \mathbb{R} \).

### 4.2. Algebra II

Let us consider further facts from Algebra. Even if an algebraic system with an associated binary operation is not a group, we can judge how good or bad this operation is in terms of its “invertibility.” The condition

\[ a \ast c = b \ast c \Rightarrow a = b \quad (5) \]

is called cancellation law. If it holds in a semigroup or monoid, this is a sign that the operation “\( \ast \)” has good “invertibility properties,” and it is almost as that in a group. Moreover, such a semigroup can often be enlarged to a group, or, in other words, this semigroup can be isomorphically embedded in a group.

The corresponding result from Algebra is as follows:

**Theorem.** Every commutative semigroup that satisfies the cancellation law can be isomorphically embedded in a commutative group.

The reader can see details, for example, in book [24, Chap. 2, Sec. 5].

Well, what about our interval arithmetic \( \mathbb{R} \)? It is an Abelian (commutative) semigroup with respect to both addition and multiplication. For addition, the cancellation law is evidently satisfied, but for multiplication the cancellation law is fulfilled only for intervals that do not contain zero. In the general case, the cancellation law is not valid, as one can see from the following example:

\[ [-1, 2] \cdot [2, 3] = [-3, 6] = [-1, 2] \cdot [1, 3]. \]

As a consequence, we can embed the interval arithmetic in a broader algebraic system in which every element has an additive inverse (opposite) element, and any interval that does not contain zero has a multiplicative inverse.

This is the well-known Kaucher interval arithmetic \( \mathbb{K} \), developed in the PhD thesis of Edgar Kaucher [19], which was defended in Karlsruhe, Germany, in 1973, under the supervision of Prof. Ulrich Kulisch. The main results of this dissertation were included in articles [20, 21]. Earlier an idea of algebraic extension and completion of the classical interval arithmetic was also implemented in a preprint of H.-J. Ortolf [35], although it was not elaborated in detail.

The operation opposite to addition in \( \mathbb{K} \), the so-called “algebraic subtraction,” is an analog of the Hukuhara difference [15] and is usually denoted by the same symbol “\( \ominus \).” In the example of interval measurement error from the preceding subsection, we can, therefore, define it as algebraic difference \((\tilde{x} \ominus x^*)\) of a measured value \(\tilde{x}\) and a true value \(x^*\) of a physical quantity.

In the Kaucher arithmetic \( \mathbb{K} \), “inclusion ordering,” which plays such an important role in Interval Analysis, naturally extends the usual inclusion order in the set-theoretical sense, namely...
\[ a \subseteq b \iff a \geq b \text{ and } a \leq b, \]

and \( \mathbb{KR} \) is a lattice with respect to the above inclusion. Specifically, in \( \mathbb{KR} \), the minimum “\( \land \)" and maximum “\( \lor \)" of \( a \) and \( b \) with respect to inclusion, which are defined as

\[
\begin{align*}
a \land b &= [\max\{a, b\}, \min\{\pi, \overline{b}\}], \\
a \lor b &= [\min\{a, b\}, \max\{\pi, \overline{b}\}],
\end{align*}
\]

can always be executed. In particular, if intervals \( a \) and \( b \) do not intersect, their minimum is an “improper interval.”

A lot is said about the Kaucher interval arithmetic in the standard IEEE 1788-2015 for implementation of interval arithmetic on digital computers [16], although this arithmetic itself has yet to become a daily working tool for people using interval computation.

It is necessary to say that E. Kaucher’s work [19] is nontrivial, since he had to extend interval multiplication with the help of not only algebraic considerations, but also with an inclusion order relation, which was due to the lack of a multiplicative cancellation law. This also results in the fact that the Kaucher interval arithmetic has some “strange” features, such as, e.g., nontrivial zero divisors:

\[ [-1, 1] \cdot [2, -3] = 0, \]

which can be easily explained and interpreted from a more advanced standpoint. Namely,

\[
[-1, 1] \cdot [2, -3] = \bigvee_{x \in [-1, 1]} \bigwedge_{y \in [-3, 2]} (x \cdot y) = 0,
\]

according to the min-max definition of arithmetic operations in the Kaucher interval arithmetic [1, 19, 45].

Moreover, as is often the case in mathematics, progress in an area immediately leads to advances in other areas related to that one. For the intervals in the new algebraically completed interval arithmetic, new logical interpretations of the arithmetic operations are possible. They were developed in works of Spanish researchers in the 70-90s of the XX century and summarized in book [42]. An alternative presentation of this theory can be found in [11, 13].

Anyway, it makes sense to conclude this section by stressing the crucial role of the cancellation law in semigroups. As we have already said, this is a sign of partial “invertibility” of the operation under study, and this fact greatly simplifies the solution of various inverse problems in specific semigroups.

4.3. Algebra and beyond

We can consider the arguments of the previous subsections in a slightly different context and show a different standpoint.

For several thousand years, there exists a very general and very powerful method for solving various mathematical problems, which is called the “method of equations.” Its essence is

- to designate the sought-for value by a special symbol
- (usually a letter called “unknown variable”)
- and then
- to write out an equality (or several equalities, i.e., a system)
- that the solution to the problem of interest must satisfy.
An equality with an unknown variable whose value we have to find is called equation. Further, to solve the original problem, it is necessary to solve the equation, i.e., find, in one way or another, the value of the unknown variable (which can be a number, a function, etc.) that satisfies the constructed equation or system of equations.

The convenience and generality of this method is that the equation can be “very implicit” with respect to the unknown quantity. Moreover, the ways in which we search for its solution do not necessarily have meaningful practical sense with respect to the unknown variable. Instead, they can be very formal manipulations that are only mathematical in nature. It is only important that the resulting solution of the equation has a practical meaning. With what kind of mathematics we get it is not so important.

A nontrivial fact that some of readers (and even experts in Interval Analysis) may not realize: it is also useful in Interval Analysis to solve equations, Interval Equations. It is useful to find solutions of interval equations in the general mathematical sense described above. We call them “formal solutions,” since the nature of operations involved in the equations may be not necessarily algebraic. Anyway, doing this, of course, is best in an algebraically completed interval arithmetic, that is, in $\mathbb{KR}$.

“Formal solutions” to interval equations were first considered in 1969 by the Romanian mathematician S. Berti [5], where they were not named in any way. Berti studied an interval quadratic equation and simply drew attention to the fact that the concept of solving an interval equation can also be given such a meaning. Then H. Ratschek and W. Sauer [39] studied such solutions for a single interval linear equation, and they used the term “algebraic solution.” In [33], K. Nickel considered formal solutions to interval linear systems of equations in complex interval arithmetics, but did not name them in any specific way. The author and other researchers previously used the term “algebraic solutions” [27, 37, 43, 44, 52], but now we strongly recommend the term “formal solutions” (see [42, 45, 46] and many others).

For example, the interval $[0, 1]$ is a formal solution to the interval quadratic equation

$$[1, 2] x^2 + [-1, 1] x = [-1, 3].$$

The interval function $x(t) = 10.5 \cdot [e^t, e^{2t}]$ of a real argument $t$ is a formal solution to the interval differential equation

$$\frac{dx(t)}{dt} = [1, 2] x(t).$$

The interval function $x(t) = [0, 2t]$ on $[0, 1]$ is a formal solution to the Fredholm interval integral equation of the second kind

$$x(t) + \int_0^1 (1.5s + t) x(s) \, ds = [0, 3t + 1].$$

The last two (purely illustrative) examples show the main drawback of the term “algebraic solution”: it emphasizes the algebraic nature of the operations that form the interval equation in question, therefore, talking about “algebraic” solutions of interval differential, integral and like equations is at least incorrect.

Let us recall some results obtained in the 60–90s of the last century that show the usefulness of “formal solutions.”

**Enclosing the united solution set.** Let us consider an interval system of linear algebraic equations $Ax = b$, with an interval $m \times n$-matrix $A$ and an interval right-hand side $m$-vector $b$. Its united solution set is known to be the set

$$\Xi_{uni}(A, b) = \{ x \in \mathbb{R}^n \mid Ax = b \text{ for some } A \in A \text{ and } b \in b \},$$

i.e., the set of solutions to all point systems $Ax = b$ with $A \in A$ and $b \in b$. Interval estimation of the united solution set is an important practical problem, which is also one of the classic problems of Interval Analysis. Hundreds of articles have been devoted to it from the 60s of the last century up to the present time.
It is easy to show that the united solution set of the original system of equations coincides with the united solution set of the fixed-point system

\[ x = (I - A)x + b. \]

Next, a formal solution to the above fixed-point interval system gives an enclosure (outer interval box) of the united solution set if the spectral radius of the matrix \(|I - A|\) composed of the moduli of elements of \((I - A)\) is less than 1. This is a well-known result of Apostolatos and Kulisch [4], obtained in 1968, which we reformulate in new terms convenient to our purposes. The reader can also find this result in [3, beginning of Chap. 12].

**Inner estimation of the united solution set.** If an interval system of linear equations \(Ax = b\) is given, a proper formal solution to the interval system

\[(\text{dual } A)x = b,\]

where “dual” is dualization in the Kaucher arithmetic (interchanging the endpoints of an interval), provides an inner box of the united solution set. This inner box is almost always inclusion maximal (that is, it touches the boundaries of the united solution set, see [1]).

**Inner estimation of the tolerable solution set.** The tolerable solution set for an interval linear system \(Ax = b\) is known to be

\[ \Xi_{\text{tol}}(A, b) = \{ x \in \mathbb{R}^n \mid Ax \in b \text{ for every } A \in A \}, \]

i.e., the set of all such vectors \(x\) that the product \(Ax\) falls within the right-hand side vector \(b\) for every \(A \in A\). This is the second, in importance, among solution sets for interval systems of equations.

Any proper formal solution to the interval system

\[ Ax = b \]

(with the same form as the initial interval system) gives an inner box of the tolerable solution set. It is also inclusion maximal in most cases.

**Enclosing the tolerable solution set.** Any proper formal solution to the interval system

\[ x = (I - (\text{dual } A))x + b \]

gives an enclosure of the tolerable solution set if the spectral radius of \(|I - A|\) is less than 1 (see [1]).

And so on. This list is indeed very extensive, and we could continue it, but the above is enough for our short note. Naturally, there exist generalizations of the above results to nonlinear systems (see, e.g., [42]).

In conclusion, it is worth noting that, historically, the short term “solution” as applied to interval equations has taken on a slightly different meaning. Since the early 60s of the last century, when speaking about a solution of an interval equation, one has been referring to a solution of some extended problem related to this equation. For example, “to find an interval enclosure for the united solution set of an interval equation” (a typical example of such terminology is [36]). Or, “to find an inner interval box within a tolerable solution set” (this is the so-called interval tolerance problem). And so on. In other words, the situation at this point is similar to what we have in the theory of differential equations, where we do not consider solutions to individual differential equations, per se. Usually some problem related to the differential equation in question is formulated (initial value problem, boundary value problem, etc.) to impose additional constraints on the desired solution without which the statement would be incomplete and meaningless. Then solutions to this extended problem, not to the single equation itself, are considered. The same is true for Interval Analysis.
5. EMPTY INTERVALS

Empty intervals are useful in some cases, although these are mostly used in the classical interval arithmetic equipped with set-theoretic operations. It usually results from intersection, e.g.,

\[ [1, 2] \cap [3, 4] = \emptyset. \]

In the Kaucher interval arithmetic, it is sometimes advisable to use minimum with respect to inclusion instead of this. Taking minimum by inclusion is an operation similar in purpose to intersection, but “more friendly.” For example,

\[ [1, 2] \land [3, 4] = [3, 2], \]

and we thus get a non-empty result which can lead to nontrivial conclusions in the course of further reasoning.

What happens when we add the empty set (“empty interval”) to an interval arithmetic? We have then

\[ a + \emptyset = \emptyset + a = \emptyset, \]
\[ a - \emptyset = \emptyset - a = \emptyset, \]
\[ a \cdot \emptyset = \emptyset \cdot a = \emptyset, \]
\[ a/\emptyset = \emptyset/a = \emptyset, \]

and the cancellation law (5) is ruined for both addition and multiplication in the interval arithmetic.

In the 2018 discussion, one of the participants, John Gustafson, compared the empty set to zero, i.e., to 0. That is the wrong metaphor: unlike noble identity elements like 0 and 1, the empty set is a kind of “vampire” in the algebraic sense, judging by equalities (6).

No invertibility of operations. No embedding in a larger and more complete algebraic system. The interval arithmetics with the empty set are suitable to solve mostly “direct problems” and perform chains of calculations in the forward direction.

In particular, the Kaucher interval arithmetic is incompatible with the empty set, as we can see from the above discussion.

6. CONCLUSIONS

The purpose of our article was to show that some types of intervals have unfavorable mathematical properties, while other types of intervals, on the contrary, are very useful. In particular, we tried to show that adding open and semi-open intervals to the traditional closed intervals is a complex and controversial undertaking. It will invalidate most of the numerical methods that make interval computing successful, and, thus, despite its apparent advantages, it will not help to improve interval analysis and its applications. Nevertheless, the conclusions drawn in the work should not be considered a final verdict which puts an end to the use of open and semi-open intervals in calculations in general.

Various applications of open and half-open intervals, as well as empty intervals, can and should be developed in those situations where they can essentially help us in modeling reality and solving various practical problems. For example, the use of half-open intervals turns out to be fruitful in formalizing, by methods of fuzzy set theory, some aspects of human reasoning, which one of the reviewers kindly pointed out to the author. This technique is described, e.g., in [2, Sec. 8].

ACKNOWLEDGEMENTS

The author thanks R. Baker Kearfott for his suggestion to rework the results of the online discussion into the present text, as well as the reviewers for useful comments that helped to improve the article.
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