Complex Tori, Theta Groups and Their Jordan Properties

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Received March 18, 2019; revised July 21, 2019; accepted October 15, 2019

Abstract—We prove that an analog of Jordan’s theorem on finite subgroups of general linear groups does not hold for the group of bimeromorphic automorphisms of a product of the complex projective line and a complex torus of positive algebraic dimension.

DOI: 10.1134/S0081543819060026

1. INTRODUCTION

1.1. As usual, $\mathbb{CP}^1$ denotes the complex projective line. Recall that a group $G$ is called Jordan (V. L. Popov [8]) if there exists a positive integer $J$ such that the following condition holds. If $B$ is a finite subgroup of $G$, then there exists an abelian normal subgroup $A$ of $B$ such that the index $[B:A]$ does not exceed $J$. If this is the case, then the smallest $J$ for which this condition holds is called the Jordan constant of $G$ and denoted by $J_G$; otherwise, we say that $G$ is not Jordan and its Jordan constant is $\infty$. Popov [9] proved that every complex or real Lie group with finitely many connected components is Jordan. (His result also covers the case when the group of connected components is bounded [9].)

If $Z$ is a connected complex manifold, then we write $\text{Bim}(Z)$ for its group of bimeromorphic automorphisms and $\text{Aut}(Z)$ for its subgroup of all biholomorphic automorphisms of $Z$.

The Jordan properties of $\text{Aut}(Z)$ and $\text{Bim}(Z)$ when $Z$ is a compact complex manifold have been studied recently by Sh. Meng and D.-Q. Zhang [6] and Yu. Prokhorov and C. Shramov [10, 11]. In particular, Prokhorov and Shramov have classified all the surfaces with non-Jordan $\text{Bim}$ (the case of projective surfaces was done earlier by Popov and the author in [8, 14]. See also [9], where Jordan properties of the groups of biholomorphic automorphisms for certain compact and non-compact complex manifolds were studied.)

The aim of this paper is to study the Jordan properties of $\text{Aut}(Y)$ and $\text{Bim}(Y)$ where $Y$ are certain $\mathbb{CP}^1$-bundles over complex tori. Recall [7] that a complex compact manifold $X$ is a complex torus if it is (biholomorphic to) a connected compact complex Lie group (such a group is always commutative [7]). It is known [2, Ch. 2, Sect. 6] that the algebraic dimension $\text{dim}_a(X)$ of $X$ is positive if and only if $X$ admits as a quotient torus a positive-dimensional complex abelian variety. If $x \in X$, then we write $T_x \in \text{Aut}(X)$ for the translation map

$$T_x: X \to X, \quad z \mapsto z + x \quad \forall z \in X. \quad (1.1)$$

Clearly, all $T_x$ constitute a commutative subgroup in $\text{Aut}(X)$, because

$$T_x \circ T_y = T_{x+y} \quad \forall x, y \in X.$$ 

If $\mathcal{V}$ a holomorphic vector bundle over $X$, then for every $\lambda \in \mathbb{C}^*$ we write

$$\text{mult}(\lambda) = \text{mult}_{\mathcal{V}}(\lambda) \in \text{Aut}(\mathcal{V})$$

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for the holomorphic automorphism of the total space of $\mathcal{V}$ that acts as multiplication by $\lambda$ in every fiber. The map
\begin{equation}
mult = \mult_{\mathcal{V}}: \mathbb{C}^* \to \text{Aut}(\mathcal{V}), \quad \lambda \mapsto \mult_{\mathcal{V}}(\lambda)
\end{equation}
is an injective group homomorphism. We write $\text{Aut}_0(\mathcal{V})$ for the centralizer of $\mult_{\mathcal{V}}(\mathbb{C}^*)$ in $\text{Aut}(\mathcal{V})$. Clearly, $\text{Aut}_0(\mathcal{V})$ is a subgroup of $\text{Aut}(\mathcal{V})$ that contains $\mult_{\mathcal{V}}(\mathbb{C}^*)$.

We write $1_X$ for the trivial holomorphic line bundle $X \times \mathbb{C}$ on $X$. If $\mathcal{L}$ is a holomorphic line bundle over $X$, then we write $\mathcal{L}_x$ for its fiber over $x \in X$ and $Y_\mathcal{L}$ for the $\mathbb{CP}^1$-bundle over $X$ that is the projectivization $\mathbb{P}(\mathcal{L} \oplus 1_X)$ of the rank 2 vector bundle $\mathcal{L} \oplus 1_X$ over $X$.

**Example 1.1.** If $\mathcal{L} = 1_X$, then $Y_\mathcal{L} = \mathbb{P}(1_X \oplus 1_X) = X \times \mathbb{CP}^1$.

1.2. Let $\mathcal{L}$ be a holomorphic line bundle over $X$. We write $K(\mathcal{L})$ for the set of all $x \in X$ such that $\mathcal{L}$ is isomorphic to the induced holomorphic line bundle $T^*_x \mathcal{L}$ on $X$. It is known [5, §1.5] that $K(\mathcal{L})$ is a subgroup of $X$ that coincides with the kernel of a certain holomorphic Lie group homomorphism from $X$ to the dual torus of $X$. This implies that $K(\mathcal{L})$ is a closed (hence compact) complex commutative Lie subgroup in $X$ and therefore has finitely many connected components. We write $K(\mathcal{L})^0$ for the identity component of $K(\mathcal{L})$; by definition, $K(\mathcal{L})^0$ is a complex subtorus in $X$,

$$K(\mathcal{L})^0 \subset K(\mathcal{L}) \subset X;$$

the compactness of $K(\mathcal{L})$ implies that the quotient $K(\mathcal{L})/K(\mathcal{L})^0$ is a finite commutative group.

Let us consider the subgroup $S(\mathcal{L}) \subset \text{Aut}(\mathcal{L})$ of all holomorphic automorphisms $u$ of the total space of $\mathcal{L}$ that satisfy the following conditions:

(i) there exists $x \in X$ such that $u: \mathcal{L} \to \mathcal{L}$ is a lifting of $T_x: X \to X$, i.e., the diagram

\begin{equation}
\begin{array}{ccc}
\mathcal{L} & \xrightarrow{u} & \mathcal{L} \\
\downarrow & & \downarrow \\
X & \xrightarrow{T_x} & X
\end{array}
\end{equation}

is commutative;

(ii) for every $z \in X$ the map between the fibers of $\mathcal{L}$ over $z$ and $z + x$ induced by $u$ is a linear isomorphism of one-dimensional $\mathbb{C}$-vector spaces.

By definition,

$$\mult_{\mathcal{L}}(\mathbb{C}^*) \subset S(\mathcal{L}) \subset \text{Aut}_0(\mathcal{L}).$$

There is a natural group homomorphism

$$\rho = \rho_{\mathcal{L}}: S(\mathcal{L}) \to X$$

that sends $u$ to $x$ if $u$ is a lifting of $T_x$. Clearly, $\ker(\rho_{\mathcal{L}}) = \mult_{\mathcal{L}}(\mathbb{C}^*) \cong \mathbb{C}^*$. This means that $S(\mathcal{L})$ is included in an exact sequence of groups

$$1 \to \mathbb{C}^* \xrightarrow{\mult_{\mathcal{L}}} S(\mathcal{L}) \xrightarrow{\rho_{\mathcal{L}}} X.$$  

**Remark 1.2.** Let $\psi: \mathcal{L}_1 \cong \mathcal{L}_2$ be an isomorphism of holomorphic line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ over $X$. Then $K(\mathcal{L}_1) = K(\mathcal{L}_2)$, and all the isomorphisms between $\mathcal{L}_1$ and $\mathcal{L}_2$ are of the form $\mult_{\mathcal{L}_2}(c) = \psi \mult_{\mathcal{L}_1}(c)$ where $c$ runs through $\mathbb{C}^*$. This implies that the group isomorphism

$$\psi_\mathcal{L}: S(\mathcal{L}_1) \cong S(\mathcal{L}_2), \quad u \mapsto \psi u \psi^{-1}$$
induced by \( \psi \) does not depend on the choice of \( \psi \). In addition,
\[
\psi_S(\text{mult}_{L_1}(c)) = \text{mult}_{L_2}(c) \quad \forall c \in \mathbb{C}^*
\] (1.7)
and \( \psi_S \) may be extended to the commutative diagram
\[
\begin{array}{ccc}
S(L_1) & \xrightarrow{\psi_S} & S(L_2) \\
\downarrow{\rho_{L_1}} & & \downarrow{\rho_{L_2}} \\
X & & X
\end{array}
\] (1.8)

In what follows we write \( #(C) \) for the number of elements of a finite set \( C \). By a short exact sequence of complex (respectively, real) Lie groups we mean a short exact sequence of groups such that each of these groups is a complex (respectively, real) Lie group and all the homomorphisms involved are homomorphisms of the corresponding complex (respectively, real) Lie groups. We do not assume these groups to be connected or to have finitely many connected components. The following assertion was inspired by results of D. Mumford [7, Sect. 23], who dealt with abelian varieties.

**Theorem 1.3.** If \( L \) is any holomorphic line bundle over \( X \), then the group \( S(L) \) carries the natural structure of a complex Lie group such that the following assertions hold:

(i) the action map \( S(L) \times L \rightarrow L \), \((u,l) \mapsto u(l) \ \forall u \in S(L), \ l \in L \) of \( S(L) \) on the total space of \( L \) is holomorphic;

(ii) \( \rho_L(S(L)) = K(L) \) and the short exact sequence of groups induced by (1.5),
\[
1 \rightarrow \mathbb{C}^* \xrightarrow{\text{mult}_L} S(L) \xrightarrow{\rho_L} K(L) \rightarrow 0,
\] (1.9)
is a short exact sequence of complex Lie groups;

(iii) consider the preimage \( S(L)^0 := \rho_L^{-1}(K(L)^0) \subset S(L) \), which is a normal clopen complex Lie subgroup of finite index \( #(K(L)/K(L)^0) \) in \( S(L) \). Then \( S(L)^0 \) is the identity component and the center of \( S(L) \). In particular, \( S(L) \) is commutative if and only if \( K(L) \) is connected;

(iv) if \( \psi: L \rightarrow L' \) is an isomorphism of holomorphic vector bundles over \( X \), then the map \( \psi_S: S(L) \cong S(L') \) defined in (1.6) is an isomorphism of complex Lie groups.

**Corollary 1.4.** If \( L \in \text{Pic}^0(X) \), then \( S(L) \) is commutative.

**Proof.** It is known [5, Corollary 1.9] that if \( L \in \text{Pic}^0(X) \), then \( K(L) = X \) and therefore \( K(L) \) is connected. Now the desired result follows from Theorem 1.3(iii). \( \square \)

The following assertion was actually proved in [15] in the case when \( \dim(X) = 1 \) (see also [1, Corollary 3.6]).

**Theorem 1.5.** Let \( L \) be a holomorphic line bundle over \( X \). Then there is a group embedding
\[
\Upsilon_L: S(L) \hookrightarrow \text{Aut}(Y_L)
\]
of \( S(L) \) into the group \( \text{Aut}(Y_L) \) of holomorphic automorphisms of \( Y_L = \mathbb{P}(L \oplus 1_X) \) such that the action map
\[
S(L) \times Y_L \rightarrow Y_L, \quad (u, \eta) \mapsto \Upsilon_L(u)(\eta) \quad \forall u \in S(L), \ \eta \in Y_L
\]
is holomorphic. In addition, the action of every \( u \in S(L) \) on \( Y_L \) is a lifting of \( T_x: X \rightarrow X \) where \( x = \rho_L(u) \). In other words, the following diagram is commutative:
\[
\begin{array}{ccc}
Y_L & \xrightarrow{T_x} & Y_L \\
\downarrow{\Upsilon_L(u)} & & \downarrow{\Upsilon_L(u)} \\
X & \xrightarrow{T_x} & X
\end{array}
\] (1.10)
The following assertion was actually proved in [14] in the case when $X$ is an abelian variety and $\mathcal{L}$ is ample.

**Theorem 1.6.** The Jordan constant of $S(\mathcal{L})$ is $\sqrt{\#(K(\mathcal{L})/K(\mathcal{L})^0)}$.

We use Theorem 1.5 and ideas related to Theorem 1.6 in the proof of the following main result of this paper.

**Theorem 1.7.** Let $X$ be a complex torus of positive algebraic dimension. Then the group $\text{Bin}(X \times \mathbb{CP}^1)$ is not Jordan.

The special case of Theorem 1.7 when $X$ is a complex abelian variety was done in [14]. We also prove the following generalizations of Theorem 1.7.

**Theorem 1.8.** Let $\psi: X \to A$ be a surjective holomorphic group homomorphism from a complex torus $X$ onto a complex abelian variety $A$ of positive dimension. Let $\mathcal{F}$ be a holomorphic line bundle on $X$ that enjoys the following property:

- There exists a holomorphic line bundle $\mathcal{M}$ on $A$ and a holomorphic line bundle $\mathcal{F}_0 \in \text{Pic}^0(X)$ such that $\mathcal{F}$ is isomorphic to the tensor product $\psi^*\mathcal{M} \otimes \mathcal{F}_0$.

Then the group $\text{Bin}(Y_{\mathcal{F}})$ is not Jordan.

**Example 1.9.** Taking $X = A$ and $\psi$ the identity map, we infer from Theorem 1.8 that if $X$ is a positive-dimensional complex abelian variety, then the group $\text{Bin}(Y_{\mathcal{F}})$ is not Jordan for any holomorphic line bundle over $X$. (Actually, this assertion follows from the results of [14].)

**Theorem 1.10.** Let $X$ be a complex torus and $\mathcal{F}$ a holomorphic line bundle on $X$. Let $X_0$ be a complex subtorus in $X$ that enjoys the following properties:

(i) $X_0$ has positive dimension;
(ii) the quotient $A := X/X_0$ is a complex abelian variety of positive dimension;
(iii) the restriction of $\mathcal{F}$ to $X_0$ lies in $\text{Pic}^0(X_0)$;
(iv) $\text{Hom}(X_0, A) = \{0\}$.

Then the group $\text{Bin}(Y_{\mathcal{F}})$ is not Jordan.

**Example 1.11.** Suppose that $X$ is a two-dimensional complex torus that contains a one-dimensional subtorus $X_0$. Then $X_0$ is a one-dimensional abelian variety (elliptic curve) and the quotient $X_1 = X/X_0$ is also a one-dimensional torus and therefore is also an elliptic curve. Now the condition $\text{Hom}(X_0, X_1) = \{0\}$ means that $X_0$ and $X_1$ are not isogenous. It follows from Theorem 1.10 that if $X_0$ and $X_1$ are not isogenous and $\mathcal{F}$ is a holomorphic line bundle on $X$ whose restriction to $X_0$ lies in $\text{Pic}^0(X_0)$ (i.e., has degree 0), then $\text{Bin}(Y_{\mathcal{F}})$ is not Jordan.

The paper is organized as follows. In Section 2 we discuss certain natural nonlinear transformation groups that act in complex vector spaces. Section 3 deals mostly with linear algebra (Hermitian forms, lattices, discriminant groups) related to holomorphic bundles on complex tori via the Appell–Humbert theorem. In Section 4 we discuss in detail theta groups, which are pretty well known in the case of abelian varieties [7, 5]. Theorems 1.3 and 1.5 are proved in Section 5. Jordan properties of theta groups are discussed in Section 6; they are used in the proof of Theorem 1.6 in Section 7. Theorem 1.7 is proved in Section 8. Section 9 deals with pencils (one-dimensional families) of Hermitian forms; its results are used in Section 10 in the proofs of Theorems 1.10 and 1.8. In Section 11 we discuss theta groups corresponding to line bundles in $\text{Pic}^0$ and identify them with the complement of the zero section in the total space of the line bundle. (In particular, we give another proof of their commutativity.)
2. PRELIMINARIES

2.1. Throughout the paper we will freely use the well-known commutator pairing

$$e: C \times C \to A$$

(2.1)

that arises from a short exact sequence of groups (central extension of $C$ by $A$)

$$1 \to A \to B \xrightarrow{\pi} C \to 1$$

(2.2)

where $A$ is a central subgroup of $B$ and $C$ is a commutative group. Recall that in order to find $e(c_1, c_2) \in A$ for $c_1, c_2 \in C$, one has to choose preimages $b_1, b_2 \in B$ with respect to the surjection $q: B \to C$, i.e.,

$$q(b_1) = c_1, \quad q(b_2) = c_2,$$

and put

$$e(c_1, c_2) := b_1 b_2 b_1^{-1} b_2^{-1} \in A;$$

(2.3)

e$(c_1, c_2)$ does not depend on the choice of $b_1, b_2$. It is well known that $e$ is bimultiplicative and alternating. It follows from the very definition of $e$ that a subgroup $K \subset B$ is commutative if and only if its image $q(K)$ is an isotropic subgroup of $C$ with respect to $e$.

2.2. Let $V \cong \mathbb{C}^g$ be a finite-dimensional complex vector space of finite positive dimension $g$. Let $L \cong \mathbb{C}$ be a one-dimensional complex vector space and $V_L := V \times L$. We regard $V_L$ as a complex manifold and write $\text{Aut}(V_L)$ for the group of holomorphic automorphisms of $V_L$. For every $\lambda \in \mathbb{C}^*$ we write $\text{mult}(\lambda)$ for the holomorphic automorphism of $V_L$ defined by

$$\text{mult}(\lambda): (v, c) \mapsto (v, \lambda c) \quad \forall v \in V, \quad c \in L.$$  

The map

$$\text{mult}: \mathbb{C}^* \to \text{Aut}(V_L), \quad \lambda \mapsto \text{mult}(\lambda)$$

is an injective group homomorphism with image $\text{mult}(\mathbb{C}^*)$. We write $\text{Aut}_0(V_L)$ for the centralizer of $\text{mult}(\mathbb{C}^*)$ in $\text{Aut}(V_L)$. Clearly, $\text{Aut}_0(V_L)$ is a subgroup of $\text{Aut}(V_L)$ containing $\text{mult}(\mathbb{C}^*)$. In what follows we discuss certain subgroups of $\text{Aut}_0(V_L)$ related to line bundles on complex tori of the form $V/L$ where $L$ is a lattice of maximal rank in $V$, i.e., a discrete subgroup of rank $2 \dim_{\mathbb{C}}(V) = 2g$.

2.3. Let $H: V \times V \to \mathbb{C}$ be a Hermitian form on $V$ and

$$E: V \times V \to \mathbb{R}, \quad u, v \mapsto \text{Im}(H(u, v))$$

its imaginary part. Then $E$ is an alternating $\mathbb{R}$-bilinear form such that

$$E(iu, iv) = E(u, v), \quad H(u, v) = E(u, iv) + iE(u, v) \quad \forall u, v \in V.$$  

(2.4)

As usual, consider the kernel of $H$

$$\ker(H) := \{ u \in V \mid H(u, v) = 0 \quad \forall v \in V \},$$  

(2.5)

which is a $\mathbb{C}$-vector subspace in $V$.

For every $u \in V$ let us consider $B_{H,u} \in \text{Aut}_0(V_L)$ defined as follows:

$$B_{H,u}(v, c) = (v + u, e^{\pi H(v, u)} c) \quad \forall v \in V, \quad c \in L.$$
Clearly, $B_{H,0}$ is the identity automorphism of $V_L$. If $u_1, u_2 \in V$, then
\[
B_{H,u_2} \circ B_{H,u_1}((v, c)) = (v + u_1 + u_2, e^{\pi H(v+u_1,u_2)}e^{\pi H(v,u_1)c})
\]
\[
= (v + u_1 + u_2, e^{\pi H(u_1,u_2)}e^{\pi H(v,u_1+u_2)c}) = \text{mult}(e^{\pi H(u_1,u_2)}) \circ B_{H,u_1+u_2}((v, c)).
\] (2.6)
This implies that in $\text{Aut}_0(V_L)$ we have
\[
B_{H,u_1}B_{H,u_2} = \text{mult}(e^{\pi H(u_1,u_2)}) \circ B_{H,u_1+u_2}
\]
and therefore
\[
B_{H,u_2}B_{H,u_1}B_{H,u_2}^{-1} \circ B_{H,u_1}^{-1} = \text{mult}(e^{\pi H(u_1,u_2)})(\text{mult}(e^{\pi H(u_2,u_1)}))^{-1} = \text{mult}(e^{2\pi i E(u_1,u_2)}).
\]
In particular, $B_{H,u_1}$ and $B_{H,u_2}$ commute if and only if $E(u_1, u_2) \in \mathbb{Z}$. In addition, it follows from (2.6) applied to $u_1 = u$ and $u_2 = -u$ that
\[
B_{H,u} = \text{mult}(e^{-\pi H(u,u)}) \circ B_{H,-u}, \quad (\text{mult}(\lambda) \circ B_{H,u})^{-1} = \text{mult}(\frac{e^{-\pi H(u,u)}}{\lambda}) \circ B_{H,-u}.
\] (2.7)
We write $\tilde{\mathcal{G}}(H, V)$ for the subset
\[
\{ \text{mult}(\lambda) \circ B_{H,u} \mid \lambda \in \mathbb{C}^*, \ u \in V \} \subset \text{Aut}_0(V_L).
\]
It follows from (2.6) that $\tilde{\mathcal{G}}(H, V)$ is the subgroup of $\text{Aut}_0(V_L)$ generated by $\text{mult}(\mathbb{C}^*)$ and all $B_{H,u}$, $u \in V$. Clearly, $\tilde{\mathcal{G}}(H, V)$ is included in the short exact sequence
\[
1 \to \mathbb{C}^* \xrightarrow{\text{mult}} \tilde{\mathcal{G}}(H, V) \xrightarrow{\kappa} V \to 0
\] (2.8)
where $\text{mult}(\mathbb{C}^*)$ is a central subgroup of $\tilde{\mathcal{G}}(H, V)$ and $\kappa : \tilde{\mathcal{G}}(H, V) \to V$ is a surjective group homomorphism that “kills” $\text{mult}(\mathbb{C}^*)$ while
\[
\kappa(B_{H,u}) = u \quad \forall u \in V.
\]
In other words, each $u \in V$ lifts to $B_{H,u} \in \tilde{\mathcal{G}}(H, V)$. This implies that the commutator pairing $V \times V \to \text{mult}(\mathbb{C}^*)$ associated with the central extension (2.8) coincides with
\[
u_1, u_2 \mapsto \text{mult}(e^{2\pi i E(u_1,u_2)}) \quad \forall u_1, u_2 \in V.
\]
In particular, if $H = 0$, then $E = 0$ and $\tilde{\mathcal{G}}(H, V)$ is a commutative group. More generally, if $\Pi \subset V$ is an additive subgroup in $V$, then we may define $\tilde{\mathcal{G}}(H, \Pi)$ as the subgroup of $\text{Aut}_0(V_L)$ generated by $\text{mult}(\mathbb{C}^*)$ and all $B_{H,u}$, $u \in \Pi$. Clearly, $\tilde{\mathcal{G}}(H, \Pi) = \kappa^{-1}(\Pi)$ is included in the short exact sequence
\[
1 \to \mathbb{C}^* \xrightarrow{\text{mult}} \tilde{\mathcal{G}}(H, \Pi) \xrightarrow{\kappa} \Pi \to 0
\] (2.9)
where $\text{mult}(\mathbb{C}^*)$ is a central subgroup of $\tilde{\mathcal{G}}(H, \Pi)$. Each $u \in \Pi$ lifts to $B_{H,u} \in \tilde{\mathcal{G}}(H, \Pi)$, and the commutator pairing $\Pi \times \Pi \to \text{mult}(\mathbb{C}^*)$ associated with the central extension (2.9) coincides with
\[
u_1, u_2 \mapsto \text{mult}(e^{2\pi i E(u_1,u_2)}) \quad \forall u_1, u_2 \in \Pi.
\]
In particular, if the restriction of $H$ to $\Pi$ is identically zero, then $\tilde{\mathcal{G}}(H, \Pi)$ is a commutative group. It follows from (2.6) that $\tilde{\mathcal{G}}(H, \ker(H))$ is a central subgroup in $\tilde{\mathcal{G}}(H, V)$. 
2.4. The aim of this subsection and Subsection 2.5 is to endow $\mathbf{G}(H, V)$ with the natural structure of a connected real Lie group and certain of its subgroups $\mathbf{G}(H, \Pi)$ (including the one with $\Pi = \ker(H)$) with the natural structure of a complex Lie group. Let us consider the complex manifold $V \times^H \mathbb{C}^*: = V \times \mathbb{C}^*$ endowed with the composition law

$$(V \times^H \mathbb{C}^*) \times (V \times^H \mathbb{C}^*) \rightarrow V \times^H \mathbb{C}^*, \quad (u, \lambda), (v, \mu) \mapsto (u, \lambda) \circ (v, \mu) := (u + v, \lambda e^{\pi H(u, v)}).$$

The bijectivity of the map

$$\Psi_H: V \times^H \mathbb{C}^* \rightarrow \mathbf{G}(H, V), \quad (u, \lambda) \mapsto \text{mult}(\lambda) \circ B_u$$

combined with (2.6) and (2.7) proves that the composition law (2.10) makes $V \times^H \mathbb{C}^*$ into a group with identity element $(0, 1)$ and the operation of taking the inverse defined by the map

$$V \times^H \mathbb{C}^* \rightarrow V \times^H \mathbb{C}^*, \quad (v, \lambda) \mapsto (v, \lambda)^{-1} := \left(-u, \frac{e^{-\pi H(u, v)}}{\lambda}\right).$$

In addition, $\Psi_H$ is a group isomorphism. Formulas (2.6) and (2.7) show that the group $V \times^H \mathbb{C}^*$ is actually a real Lie group with real structure induced by the natural complex structure on $V \times^H \mathbb{C}^*$. (However, if $H = 0$, then the real Lie group $V \times^H \mathbb{C}^*$ is actually a commutative complex Lie group.) Clearly, $V \times^H \mathbb{C}^*$ is included in the short exact sequence of real Lie groups

$$1 \rightarrow \mathbb{C}^* \xrightarrow{\lambda \mapsto (0, \lambda)} V \times^H \mathbb{C}^* \xrightarrow{(v, \lambda) \mapsto v} V \rightarrow 0.$$  

(2.12)

Applying $\Psi_H$ to (2.12), we find that (2.8) is actually a short exact sequence of real Lie groups.

Let $\Pi$ be a closed additive subgroup of $V$. The third theorem of Cartan implies that $\Pi$ is a real Lie subgroup of $V$. Clearly,

$$\Pi \times^H \mathbb{C}^*: = \Pi \times \mathbb{C}^* \subset V \times \mathbb{C}^* = V \times^H \mathbb{C}^*$$

is a closed subgroup of $V \times^H \mathbb{C}^*$ and therefore is a real Lie subgroup of $V \times^H \mathbb{C}^*$. Notice that

$$\Psi_H(\Pi \times^H \mathbb{C}^*) = \mathbf{G}(H, \Pi),$$

which implies that $\Psi_H: \Pi \times^H \mathbb{C}^* \rightarrow \mathbf{G}(H, \Pi)$ is a group isomorphism that provides $\mathbf{G}(H, \Pi)$ with the structure of a real Lie group. Clearly, $\Pi \times^H \mathbb{C}^*$ is included in the short exact sequence of real Lie groups

$$1 \rightarrow \mathbb{C}^* \xrightarrow{\lambda \mapsto (0, \lambda)} \Pi \times^H \mathbb{C}^* \xrightarrow{(v, \lambda) \mapsto v} \Pi \rightarrow 0.$$  

(2.13)

Applying $\Psi_H$ to (2.13), we find that (2.9) is actually a short exact sequence of real Lie groups.

Remark 2.1. Let $\Pi^0$ be the identity component of $\Pi$, which is a connected real Lie subgroup of $V$, i.e., is an $\mathbb{R}$-vector subspace of $V$. Then $\Pi^0 \times^H \mathbb{C}^*$ is the connected component of $\Pi \times^H \mathbb{C}^*$ that contains the identity element $(0, 1)$ of the group law, i.e., the identity component of $\Pi \times^H \mathbb{C}^*$. It follows that $\mathbf{G}(H, \Pi^0)$ is the identity component of $\mathbf{G}(H, \Pi)$. This implies that $\Pi/\Pi^0$ is canonically isomorphic to the group $\mathbf{G}(H, \Pi)/\mathbf{G}(H, \Pi^0)$ of connected components of $\mathbf{G}(H, \Pi)$.

Example 2.2. If $\Pi = \ker(H)$, then the group law on $\ker(H) \times^H \mathbb{C}^*$ is

$$(u, \lambda), (v, \mu) \mapsto (u, \lambda) \circ (v, \mu) := (u + v, \lambda e^{\pi H(u, v)}) = (u + v, \lambda e^{\pi 0}) = (u + v, \lambda \mu),$$

since $H(u, v) = 0$ for all $u, v \in \ker(H)$. This means that $\ker(H) \times^H \mathbb{C}^*$ is actually the direct product $\ker(H) \times \mathbb{C}^*$ of complex Lie groups $\ker(H)$ and $\mathbb{C}^*$; in particular, it is connected and commutative.
2.5. Now and till the end of this section let us assume that \( \Pi^0 = \ker(H) \). Since \( \ker(H) \) is a complex vector subspace in \( V \), it is a complex Lie subgroup in \( V \) and therefore \( \Pi \) is also a closed complex Lie subgroup in \( V \). This implies that \( \Pi \times^H \mathbb{C}^* \) is a closed complex submanifold of \( V \times^H \mathbb{C}^* \). Recall that \( \Pi \times^H \mathbb{C}^* \) is a closed real Lie subgroup of the real Lie group \( V \times^H \mathbb{C}^* \). We claim that \( \Pi \times^H \mathbb{C}^* \) is actually a complex Lie group.

**Lemma 2.3.** If \( \Pi^0 = \ker(H) \), then the real Lie group \( \Pi \times^H \mathbb{C}^* \) is the complex Lie group with respect to the structure of the complex manifold on \( \Pi \times^H \mathbb{C}^* \) described above. In addition, (2.13) is a short exact sequence of complex Lie groups.

**Proof.** We need to check that the maps (2.6) and (2.7) restricted to \( (\Pi \times^H \mathbb{C}^*) \times (\Pi \times^H \mathbb{C}^*) \) and \( \Pi \times^H \mathbb{C}^* \), respectively, are holomorphic (not just real analytic). The complex analyticity condition could be checked locally, in the open neighborhoods

\[
(\Pi \times^H \mathbb{C}^*) \times (\Pi \times^H \mathbb{C}^*) \subseteq (\Pi \times^H \mathbb{C}^*) \times (\Pi \times^H \mathbb{C}^*)
\]

of points \((u_0, \lambda_0), (v_0, \mu_0) \in (\Pi \times^H \mathbb{C}^*) \times (\Pi \times^H \mathbb{C}^*)\). Then (2.6) gives us the composition law

\[
(u_0 + u, \lambda) \circ (v_0 + v, \mu) = (u_0 + v_0 + u + v, e^{\pi H(u_0 + u, v_0 + v)} \lambda \mu) = (u_0 + v_0 + u + v, e^{\pi H(u_0, v_0)} \lambda \mu),
\]

which is obviously holomorphic in \( u, v \in \ker(H) \) and \( \lambda, \mu \in \mathbb{C}^* \). Similarly, (2.7) gives us the operation of taking the inverse

\[
(u_0 + u, \lambda) \mapsto (u_0 + u, \lambda)^{-1} = \left(-u_0 - u, \frac{e^{-\pi H(u_0 + u, u_0 + u)}}{\lambda}\right) = \left(-u_0 - u, \frac{e^{-\pi H(u_0, 0)}}{\lambda}\right),
\]

which is obviously holomorphic in \( u \in \ker(H) \) and \( \lambda \in \mathbb{C}^* \). The second assertion of the lemma is also obvious. \( \square \)

**Remark 2.4.** Let \( \Pi^0 = \ker(H) \).

1. Lemma 2.3 and the bijectivity of \( \Psi_H \) allow us to endow the real Lie group \( \tilde{\mathbf{G}}(H, \Pi) = \Psi_H(\Pi \times^H \mathbb{C}^*) \) with the compatible natural structure of a complex Lie group, whose identity component

\[
\tilde{\mathbf{G}}(H, \Pi^0) = \tilde{\mathbf{G}}(H, \ker(H)) = \Psi_H(\ker(H) \times^H \mathbb{C}^*)
\]

is a central subgroup of \( \tilde{\mathbf{G}}(H, \Pi) \) (and even of \( \tilde{\mathbf{G}}(H, V) \)).

2. The action map

\[
\tilde{\mathbf{G}}(H, \Pi) \times V_L \rightarrow V_L, \quad \Psi_H(u, \lambda), (v, c) \mapsto \text{mult}(\lambda) \circ B_u((v, c)) = (v + u, \lambda e^{\pi H(v, u)} c)
\]

is holomorphic. Indeed, it suffices to check that it is holomorphic at all \( \Psi_H(u, c) \) from the open subgroup \( \tilde{\mathbf{G}}(H, \ker(H)) \). In this case \( H(v, u) = 0 \) and we get the map

\[
(\ker(H) \times \mathbb{C}^*) \times V_L \rightarrow V_L, \quad (u, \lambda), (v, c) \mapsto (v + u, \lambda c),
\]

which is obviously holomorphic. Clearly, (2.9) is a short exact sequence of complex Lie groups.

3. HERMITIAN FORMS, LATTICES AND LINE BUNDLES

In what follows, a lattice is an additive discrete subgroup in a finite-dimensional complex (or real) vector space.

Let \( X \) be a positive-dimensional complex torus, i.e., \( X = V/L \) where \( V \cong \mathbb{C}^g \) a finite-dimensional complex vector space of positive dimension \( g \) and \( L \subset V \) a lattice of maximal possible rank \( 2g \).
We view $X$ as a connected complex commutative compact Lie group. By the Appell–Humbert theorem [7, 5], holomorphic line bundles $L$ on $X$ are classified (up to an isomorphism) by Appell–Humbert data $(H, \alpha)$ where $H : V \times V \to \mathbb{C}$ is a Hermitian form on $V$ and $\alpha : L \to U(1)$ is a map from $L$ to the unit circle $U(1)$ that enjoy the following properties:

$$E(l_1, l_2) := \text{Im}(H(l_1, l_2)) \in \mathbb{Z}, \quad \alpha(l_1 + l_2) = (-1)^{E(l_1, l_2)} \alpha(l_1) \alpha(l_2) \quad \forall l_1, l_2 \in L. \quad (3.1)$$

We denote by $\mathcal{L}(H, \alpha)$ the corresponding line bundle on $X$, whose definition is recalled in Subsection 3.2.

### 3.1. Consider the following discrete action of the group $L$ on $V_L$ by holomorphic automorphisms. An element $l \in L$ acts as

$$\mathcal{A}_{H,l} : V_L \to V_L, \quad (v, c) \mapsto (v + l, \alpha(l)e^{\pi H(v, l)} + \pi H(l, l)/2 \cdot c) \quad \forall v \in V, \quad c \in \mathbb{C}.$$ 

In other words,

$$\mathcal{A}_{H,l} = \text{mult}(\alpha(l)e^{\pi H(l, l)/2}) \mathcal{B}_{H,l} \in \text{Auto}(V_L).$$

In particular,

$$\mathcal{A}_{H,l} \in \bar{\mathcal{G}}(H, L) \quad \forall l \in L.$$ 

It is well known (and could be easily checked by direct computations) that

$$\mathcal{A}_{H,l_1} \circ \mathcal{A}_{H,l_2} = \mathcal{A}_{H,l_1 + l_2} \quad \forall l_1, l_2 \in L.$$ 

In particular, the map

$$\mathcal{A}^L : L \to \text{Auto}(V_L), \quad l \mapsto \mathcal{A}_{H,l} \quad (3.2)$$

is an injective group homomorphism, whose image we denote by

$$\bar{L} = \bar{L}(H, \alpha) := \mathcal{A}^L(L) \subset \bar{\mathcal{G}}(H, V) \subset \text{Auto}(V_L). \quad (3.3)$$

Clearly, $\bar{L}$ is a subgroup of $\text{Auto}(V_L)$ that meets $\text{mult}(\mathbb{C}^\times)$ only at the identity element. In addition,

$$\bar{L} = \bar{L}(H, \alpha) \subset \bar{\mathcal{G}}(H, L) \subset \text{Auto}(V_L). \quad (3.4)$$

It is also clear that for each additive subgroup $\Pi \subset V$ we have

$$\bar{L} \cap \bar{\mathcal{G}}(\Pi) = \mathcal{A}^L(\Pi \cap L) = \{\mathcal{A}_{H,l} \mid l \in \Pi \cap L\}. \quad (3.5)$$

### 3.2. The holomorphic line bundle $\mathcal{L}(H, \alpha) \to X$ is defined [7, Sect. 2] as the quotient

$$V_L/\bar{L} = V_L/L \to V/L = X.$$ 

In particular, $V_L/\bar{L}$ is the total space of the holomorphic line bundle $\mathcal{L}(H, \alpha)$.

In the obvious notation,

$$\mathcal{L}(H, \alpha) \otimes \mathcal{L}(H', \alpha') \cong \mathcal{L}(H + H', \alpha \alpha'). \quad (3.6)$$

In particular, $1_X$ is isomorphic to $\mathcal{L}(0, \alpha_0)$ where

$$\alpha_0 : L \to \{1\} \subset U(1) \quad (3.7)$$

is the trivial character of $L$.

**Definition 3.1.** One says [7, 5, 2] that a holomorphic line bundle on $X$ lies in $\text{Pic}^0(X)$ if it is isomorphic to $\mathcal{L}(0, \alpha)$ for some $\alpha$, i.e., the corresponding Hermitian form is zero.
3.3. Since $L$ spans the $\mathbb{R}$-vector space $V$, we have
\[
\ker(H) = \{ x \in V \mid H(x, l) = 0 \ \forall l \in L \}.
\]

Let
\[
L_E^\perp := \{ x \in V \mid E(x, l) \in \mathbb{Z} \ \forall l \in L \}.
\]

Clearly, $L_E^\perp$ is a closed (not necessarily connected) real Lie subgroup of $V$ that contains $L$ as a discrete subgroup. In particular, the identity component $(L_E^\perp)^0$ of $L_E^\perp$ is an $\mathbb{R}$-vector subspace of $V$.

**Lemma 3.2.** (i) $(L_E^\perp)^0 = \ker(H)$. In particular, $(L_E^\perp)^0$ is a $\mathbb{C}$-vector subspace of $V$ and $L_E^\perp$ is a closed complex Lie subgroup of $V$.

(ii) $\tilde{G}(H, L_E^\perp)$ is a complex Lie group that is included in the short exact sequence of complex Lie groups
\[
1 \to \mathbb{C}^* \xrightarrow{\text{mult}} \tilde{G}(H, L_E^\perp) \xrightarrow{\kappa} L_E^\perp \to 0
\]
defined in (2.9) for $\Pi = L_E^\perp$.

(iii) $\tilde{G}(H, \ker(H)) = \kappa^{-1}(\ker(H))$ is the identity component of $\tilde{G}(H, L_E^\perp)$, which is a central clopen subgroup of $\tilde{G}(H, L_E^\perp)$ containing $\text{mult}(\mathbb{C}^*)$ and included in the short exact sequence of complex Lie groups
\[
1 \to \mathbb{C}^* \xrightarrow{\text{mult}} \tilde{G}(H, \ker(H)) \xrightarrow{\kappa} \ker(H) \to 0
\]
defined in (2.9) for $\Pi = \ker(H)$; it is also induced from (3.9) by $\ker(H) \subset L_E^\perp$.

(iv) The action map $\tilde{G}(H, L_E^\perp) \times V_\mathbb{L} \to V_\mathbb{L}$ is holomorphic.

**Proof.** Clearly,
\[
(L_E^\perp)^0 \subset \{ v \in V \mid E(l, v) = 0 \ \forall l \in L \} =: V_0.
\]

Since $E$ is $\mathbb{R}$-bilinear, $V_0$ is a real vector subspace of $V$. In view of the first formula in (2.4), $V_0 = \mathbb{i}V_0$, i.e., $V_0$ is a complex vector subspace of $V$. Since $L$ spans $V$ over $\mathbb{R}$ and $E$ is $\mathbb{R}$-bilinear, we have
\[
V_0 = \{ v \in V \mid E(u, v) = 0 \ \forall u \in V \}.
\]

Since $V_0$ is a $\mathbb{C}$-vector subspace, (3.12) and the second formula in (2.4) imply that
\[
V_0 = \{ v \in V \mid H(u, v) = 0 \ \forall u \in V \} = \ker(H).
\]

Now (3.12) and (3.13) imply that
\[
V_0 = \ker(H) \subset (L_E^\perp)^0,
\]

because $\ker(H)$ is connected. In view of (3.11),
\[
\ker(H) = V_0 \supset (L_E^\perp)^0.
\]

This implies that $\ker(H) = (L_E^\perp)^0$, which proves (i). Now assertions (ii)–(iv) follow from Remark 2.4. □

3.4. Let us put
\[
L_0 := L \cap \ker(H) = \{ l \in L \mid E(l, v) = 0 \ \forall v \in V \} = \{ l \in L \mid E(l, m) = 0 \ \forall m \in L \}.
\]

Then $L_0$ is a free saturated $\mathbb{Z}$-submodule of $L$ and $E$ induces a nondegenerate alternating bilinear form on $L/L_0$. In particular, the rank of the free $\mathbb{Z}$-module $L/L_0$ is even. Since the rank of $L$ is
even, the rank of \( L_0 \) is also even. Let \( 2g_0 \) be the rank of \( L_0 \). Then the rank of \( L/L_0 \) is \( 2(g - g_0) \). Notice also that since \( L_0 \) is a lattice in \( \ker(H) \),

\[
2g_0 \leq \dim_{\mathbb{R}}(\ker(H)).
\]

Since \( L_0 \) is saturated in \( L \), there exists a saturated free \( \mathbb{Z} \)-submodule \( L_1 \subset L \) of rank \( 2g - 2g_0 \) such that

\[
L = L_0 \oplus L_1.
\]

This implies that

\[
V = L \otimes \mathbb{R} = (L_0 \otimes \mathbb{R}) \oplus (L_1 \otimes \mathbb{R}).
\]

Clearly, the restriction

\[
E|_{L_1} : L_1 \times L_1 \to \mathbb{Z}
\]

of \( E \) to \( L_1 \) is a nondegenerate alternating bilinear form. This implies that the restriction

\[
E|_{L_1 \otimes \mathbb{R}} : (L_1 \otimes \mathbb{R}) \times (L_1 \otimes \mathbb{R}) \to \mathbb{R}
\]

is a nondegenerate alternating \( \mathbb{R} \)-bilinear form. It follows that

\[
(L_1 \otimes \mathbb{R}) \cap \ker(H) = \{0\}
\]

and therefore

\[
2g = \dim_{\mathbb{R}}(V) \geq \dim_{\mathbb{R}}(L_1 \otimes \mathbb{R}) + \dim_{\mathbb{R}}(\ker(H))
\]

\[
= (2g - 2g_0) + \dim_{\mathbb{R}}(\ker(H)) \geq (2g - 2g_0) + 2g_0 = 2g.
\]

This implies that \( \dim_{\mathbb{R}}(\ker(H)) = 2g_0 \), i.e., \( L_0 \) is a lattice of maximal rank in the real vector space \( V \) and

\[
\ker(H) = L_0 \otimes \mathbb{R}.
\]

**Remark 3.3.** It follows from (3.1) that the restriction of \( \alpha \) to \( L_0 \) is a group homomorphism, i.e.,

\[
\alpha(l_1 + l_2) = \alpha(l_1)\alpha(l_2) \quad \forall l_1, l_2 \in L_0.
\]

(3.14)

### 4. Theta Groups

We keep the notation and assumptions of Section 3.

**4.1.** Suppose that \( L \neq L_0 \), i.e., \( E \neq 0 \) and so \( H \neq 0 \). This means that \( L_1 \) is a free \( \mathbb{Z} \)-module of **positive** rank \( 2(g - g_0) \). Let us choose once and for all a basis \( \{l_1, \ldots, l_{2g - 2g_0}\} \) of \( L_1 \) and consider the **skew-symmetric nondegenerate** square matrix \( \tilde{E} \) of \( E|_{L_1} \) associated with this basis, whose order is \( 2g - 2g_0 \) and entries are

\[
\tilde{E}_{ij} := E|_{L_1}(l_i, l_j) = E(l_i, l_j) \in \mathbb{Z}. \quad \text{(4.1)}
\]

The determinant \( \det(E|_{L_1}) \) of \( \tilde{E} \) is a nonzero integer that does not depend on the choice of the basis. Since \( \tilde{E} \) is skew-symmetric, \( \det(E|_{L_1}) \) is the square of the Pfaffian of \( \tilde{E} \). Since all the entries of \( \tilde{E} \) are integers, its Pfaffian is also an integer and therefore \( \det(E|_{L_1}) \) is a square in \( \mathbb{Z} \); in particular, it is a **positive integer**. Its square root \( \sqrt{\det(E|_{L_1})} \) will play a prominent role in Section 6. On the other hand, \( \det(E|_{L_1}) \) admits the following well-known interpretation. Let

\[
L_{1,E}^+ = \{ \tilde{x} \in L_1 \otimes \mathbb{R} \mid E(\tilde{x}, l) \in \mathbb{Z} \ \forall l \in L \}.
\]
The nondegeneracy of $E|_{L_1}$ implies that $L_{1,E}^+$ is a free $\mathbb{Z}$-module of rank $2g - 2g_0$ that is contained in $L_1 \otimes \mathbb{Q}$ and contains $L_1$ as a subgroup of finite index. It is well known that

$$[L_{1,E}^+:L_1] = \#(L_{1,E}^+/L_1) = \det(E|_{L_1}).$$ (4.2)

Let us also point out the following obvious but useful equality:

$$L_{E}^+ = \ker(H) \oplus L_{1,E}^+ = (L_0 \otimes \mathbb{R}) \oplus L_{1,E}^+. $$ (4.3)

In order to get an explicit description of the finite discriminant group $L_{1,E}^+/L_1$, notice that the structure theorem for alternating nondegenerate bilinear forms over $\mathbb{Z}$ implies the existence of a basis

$$e_1, f_1, \ldots, e_{g-g_0}, f_{g-g_0} \in L_1$$

of the free $\mathbb{Z}$-module $L_1$ and positive integers $d_1(E), \ldots, d_{g-g_0}(E)$ that enjoy the following properties. Each $d_i(E)$ divides $d_{i+1}(E)$ (if $1 \leq i < g_0$), and

$$L_1 = \bigoplus_{i=1}^{g-g_0} \left( \mathbb{Z} \cdot e_i \oplus \mathbb{Z} \cdot f_i \right), \quad E(e_i, f_j) = 0 \quad \text{if} \quad i \neq j, \quad E(e_i, f_i) = d_i(E) \quad \forall i$$

(see also [2, pp. 7–8]). It follows that

$$\det(\bar{E}) = \left( \prod_{i=1}^{g-g_0} d_i(E) \right)^2, \quad L_{1,E}^+ = \bigoplus_{i=1}^{g-g_0} \frac{1}{d_i(E)} \left( \mathbb{Z} \cdot e_i \oplus \mathbb{Z} \cdot f_i \right).$$

If we define free $\mathbb{Z}$-submodules

$$U_i := \mathbb{Z} \cdot e_i \oplus \mathbb{Z} \cdot f_i \subset L_1$$

of rank 2, then we get a direct orthogonal (with respect to $E$) splitting

$$L = L_0 \oplus L_1, \quad L_1 = \bigoplus_{i=1}^{g-g_0} U_i, \quad L_{1,E}^+ = \bigoplus_{i=1}^{g-g_0} \frac{1}{d_i(E)} U_i. $$ (4.4)

In addition,

$$E\left( \frac{1}{d_i(E)} c_i, \frac{1}{d_i(E)} f_i \right) = \frac{1}{d_i(E)} \quad \forall i, \quad E\left( \frac{1}{d_i(E)} U_i, \frac{1}{d_j(E)} U_j \right) = \{0\} \quad \forall i \neq j. $$ (4.5)

It follows from (4.4) that

$$L_{1,E}^+/L_1 \cong \bigoplus_{i=1}^{g-g_0} \left( \frac{1}{d_i(E)} U_i \right) / U_i \cong \bigoplus_{i=1}^{g-g_0} \left( \mathbb{Z} / d_i(E) \mathbb{Z} \right)^2,$$

$$\#(L_{1,E}^+/L_1) = \left( \prod_{i=1}^{g-g_0} d_i(E) \right)^2 = \det(\bar{E}).$$ (4.6)

(4.7)

It also follows from (4.4) that

$$X = V/L \supseteq L_{E}^+/L = \left( \ker(H)/L_0 \right) \oplus (L_{1,E}^+/L_1) = \left( \ker(H)/L_0 \right) \oplus \left( \bigoplus_{i=1}^{g-g_0} \left( \frac{1}{d_i(E)} U_i \right) / U_i \right)$$

(see also [2, pp. 7–8]).
Remark 4.1. Suppose that $H \neq 0$. It follows from [5, § 1.5] that

$$K(\mathcal{L}(H, \alpha)) = L_1^E/L \subset X.$$  \hfill{(4.8)}

Now (4.7) implies that $\ker(H)/L_0$ is the identity component $K(\mathcal{L}(H, \alpha))^0$ of the complex Lie group $K(\mathcal{L}(H, \alpha))$, while the group $K(\mathcal{L}(H, \alpha))/K(\mathcal{L}(H, \alpha))^0$ is isomorphic to $L_{1,E}^1/L_1$ and therefore

$$\#(K(\mathcal{L}(H, \alpha))/K(\mathcal{L}(H, \alpha))^0) = \#(L_{1,E}^1/L_1) = \det(E|_{L_1}) = \left(\prod_{i=1}^{g-g_0} d_i(E)\right)^2. \hfill{(4.9)}$$

4.2. Let us consider the alternating bilinear pairing

$$e_E: L_{1,E}^1/L_1 \times L_{1,E}^1/L_1 \to \mathbb{C}^*, \hfill{(4.10)}

(v_1 + L_1, v_2 + L_1) \mapsto e^{2\pi i E(v_1, v_2)} \quad \forall v_1 + L_1, v_2 + L_1 \in L_{1,E}^1/L_1.$$

It follows from (4.5) and (4.4) that $e_E$ is a nondegenerate pairing.

Lemma 4.2. Suppose that $H \neq 0$. Let $n$ be a positive integer such that

$$E(l_1, l_2) \in n\mathbb{Z} \quad \forall l_1, l_2 \in L. \hfill{(4.11)}$$

Then $\#(L_{1,E}^1/L_1)$ is divisible by $n^2$.

Proof. Since $H \neq 0$, we have $g_0 < g$, i.e., $g - g_0 \geq 1$. It follows from (4.1) that all the entries of the order $2(g - g_0)$ square matrix $E$ are divisible by $n$ in $\mathbb{Z}$ and therefore $\det(E)$ is divisible by $n^{2(g-g_0)}$ in $\mathbb{Z}$. This implies that $\#(L_{1,E}^1/L_1) = \det(E)$ is divisible by $n^{2(g-g_0)}$ and therefore is divisible by $n^2$. \hfill{□}

Theorem 4.3. (i) $\widetilde{L} = \widetilde{L}(H, \alpha)$ is a central discrete subgroup of $\widetilde{\mathfrak{G}}(H, L_{E}^1)$ that meets $\text{mult}(\mathbb{C}^*)$ only at the identity.

(ii) The intersection

$$L_0 := \widetilde{L} \cap \widetilde{\mathfrak{G}}(H, \ker(H)) = A^L(L_0) = \{A_{H,l} \mid l \in L_0\}$$

is a discrete subgroup in the commutative connected complex Lie group $\widetilde{\mathfrak{G}}(H, \ker(H))$.

(iii) The commutative connected complex Lie group $\widetilde{\mathfrak{G}}(H, \ker(H))/L_0$ is isomorphic to the quotient $\ker(H) \times \mathbb{C}^*)/L_0$ where $L_0 := \{(l, \alpha(l)) \mid l \in L_0\}$ is a discrete subgroup in $\ker(H) \times \mathbb{C}^*$.

Proof. We have already seen that $\widetilde{L}$ meets $\text{mult}(\mathbb{C}^*)$ only at the identity and $\widetilde{L} \subset \widetilde{\mathfrak{G}}(H, L)$. Since $L_{E}^1$ contains $L$, we have

$$\widetilde{\mathfrak{G}}(H, L) = \kappa^{-1}(L) \subset \kappa^{-1}(L_{E}^1) = \widetilde{\mathfrak{G}}(H, L_{E}^1)$$

and therefore $\widetilde{L} \subset \widetilde{\mathfrak{G}}(H, L_{E}^1)$. Recall that $E(L_{E}^1, L) \subset \mathbb{Z}$. So, if $\widetilde{l} \in \widetilde{L}$ and $\phi \in \widetilde{\mathfrak{G}}(H, L_{E}^1)$, then $E(\kappa(\widetilde{l}), \kappa(\phi)) \in \mathbb{Z}$ and therefore $\widetilde{l}$ and $\phi$ commute (see the very end of Subsection 2.3). This implies that $\widetilde{L}$ is a central subgroup of $\widetilde{\mathfrak{G}}(H, L_{E}^1)$. In order to check the discreteness of $\widetilde{L}$, recall that $\kappa(\widetilde{L}) = L$ is a discrete subgroup in $L_{E}^1$. Hence, if $\widetilde{L}$ is not discrete, the intersection $\widetilde{L} \cap \ker(\kappa)$ is infinite. However, $\ker(\kappa) = \text{mult}(\mathbb{C}^*)$ and we know that $\widetilde{L}$ meets $\text{mult}(\mathbb{C}^*)$ only at a single element. The obtained contradiction proves that $\widetilde{L}$ is discrete. This proves (i). Since $L_0 = L \cap \ker(H)$, assertion (ii) follows from (i) combined with (3.5) applied to $\Pi = \ker(H)$. Now assertion (iii) follows from (ii) combined with Example 2.2. \hfill{□}

Remark 4.4. The same arguments prove that $\widetilde{\mathfrak{G}}(H, L)$ is a central subgroup of $\widetilde{\mathfrak{G}}(H, L_{E}^1)$. In fact, the natural “multiplication map”

$$\text{mult}(\mathbb{C}^*) \times \widetilde{L}(H, \alpha) \to \widetilde{\mathfrak{G}}(H, L)$$

is a group isomorphism.
4.3. Recall that applying the short exact sequence (2.9) to \( \Pi = L_E^\perp \), we get a short exact sequence (3.9) of complex Lie groups, where the image \( \text{mult}(\mathbb{C}^*) \) is a central subgroup of \( \tilde{\mathfrak{G}}(H, L_E^\perp) \). Each \( u \in L_E^\perp \) lifts to \( \mathcal{B}_{H,u} \in \tilde{\mathfrak{G}}(H, L_E^\perp) \), and the commutator pairing

\[
L_E^\perp \times L_E^\perp \to \text{mult}(\mathbb{C}^*)
\]

associated with (3.9) coincides with

\[
u_1, u_2 \mapsto \text{mult}(e^{2\pi i E(u_1, u_2)}) \quad \forall u_1, u_2 \in L_E^\perp.
\]

Recall that

\[
\tilde{L} \subset \tilde{\mathfrak{G}}(H, L_E^\perp) \subset \tilde{\mathfrak{G}}(H, V) \subset \text{Aut}(V_L)
\]

is a central discrete subgroup \( \tilde{L} \) of \( \tilde{\mathfrak{G}}(H, L_E^\perp) \) that acts discretely on \( V_L \) and

\[
\mathcal{L}(H, \alpha) = V_L/\tilde{L} = V_L/(H, \alpha).
\]

This gives us the natural embedding of the complex Lie quotient group

\[
\mathfrak{G}(H, \alpha) := \tilde{\mathfrak{G}}(H, L_E^\perp)/\tilde{L} = \tilde{\mathfrak{G}}(H, L_E^\perp)/(H, \alpha)
\]

into the group \( \text{Aut}(\mathcal{L}(H, \alpha)) \) of holomorphic automorphisms of the total space of \( \mathcal{L}(H, \alpha) \). Further, we will identify \( \mathfrak{G}(H, \alpha) = \tilde{\mathfrak{G}}(H, L_E^\perp)/\tilde{L} \) with its (isomorphic) image in \( \text{Aut}(\mathcal{L}(H, \alpha)) \). It follows from Lemma 3.2(iii) that the action map

\[
\mathfrak{G}(H, \alpha) \times \mathcal{L}(H, \alpha) \to \mathcal{L}(H, \alpha)
\]

is holomorphic.

4.4. It follows from (3.9) and (4.8) that \( \mathfrak{G}(H, \alpha) = \tilde{\mathfrak{G}}(H, L_E^\perp)/\tilde{L} \) is included in a short exact sequence of complex Lie groups

\[
1 \to \mathbb{C}^* \to \mathfrak{G}(H, \alpha) \xrightarrow{\pi} L_E^\perp/L = K(\mathcal{L}(H, \alpha)) \to 0. \tag{4.12}
\]

Here the image of \( \mathbb{C}^* \) is a central subgroup in \( \mathfrak{G}(H, \alpha) \): each

\[
\lambda \in \mathbb{C}^* \to \mathfrak{G}(H, \alpha) \subset \text{Aut}(\mathcal{L}(H, \alpha))
\]

acts on the total space of \( \mathcal{L}(H, \alpha) \) as the multiplication by \( \lambda \) at every fiber of \( \mathcal{L}(H, \alpha) \to X \), i.e., \( \lambda \) is mapped to \( \text{mult}_{\mathcal{L}(H, \alpha)}(\lambda) \); the surjective complex Lie group homomorphism

\[
\pi: \mathfrak{G}(H, \alpha) = \tilde{\mathfrak{G}}(H, L_E^\perp)/\tilde{L} \to L_E^\perp/L
\]

“kills” the image of \( \mathbb{C}^* \) and sends a coset \( \mathcal{B}_{H,u} \tilde{L} \) to \( u + L \) for every \( u \in L_E^\perp \).

Clearly, the commutator pairing associated with the central extension (4.12) is

\[
e_{H,\alpha}: L_E^\perp/L \times L_E^\perp/L \to \mathbb{C}^*, \quad u_1 + L, u_2 + L \mapsto e^{2\pi i E(u_1, u_2)} \quad \forall u_1 + L, u_2 + L \in L_E^\perp/L. \tag{4.13}
\]

Remark 4.5.

1. Clearly, the restriction of \( e_{H,\alpha} \) to \( L_{1,E}^\perp/L_1 \times L_{1,E}^\perp/L_1 \) coincides with the nondegenerate pairing (4.10).

2. Clearly, \( \ker(H)/L_0 \) lies in the kernel of \( e_{H,\alpha} \). Combining this with assertion 1, we find that \( \ker(H)/L_0 \) coincides with the kernel of \( e_{H,\alpha} \), since

\[
L_E^\perp/L = (\ker(H)/L_0) \oplus (L_{1,E}^\perp/L_1).
\]
Theorem 4.6. The identity component $\mathfrak{G}(H, \alpha)^0$ of $\mathfrak{G}(H, \alpha)$ coincides with the preimage $\kappa^{-1}(\ker(H)/L_0)$ of
\[ \ker(H)/L_0 \subset L_E^\perp/L \subset V/L = X \]
and is canonically isomorphic as a complex Lie group to the quotient $\widetilde{\mathfrak{G}}(H, \ker(H))/\tilde{L}_0$. In particular, $\mathfrak{G}(H, \alpha)^0$ is a central subgroup of $\mathfrak{G}(H, \alpha)$ that is isomorphic as a complex Lie group to $(\ker(H) \times \mathbb{C}^*)/\tilde{L}_0$.

Proof. It follows from Theorem 4.3 that the subgroup $\mathfrak{G}(H, \alpha)^0$ is the image of $\tilde{\mathfrak{G}}(H, \ker(H))$ in $\tilde{\mathfrak{G}}(H, L_E^\perp)/\tilde{L} = \mathfrak{G}(H, \alpha)$ and this image coincides with
\[ \tilde{\mathfrak{G}}(H, \ker(H))/\tilde{L} \cap \tilde{\mathfrak{G}}(H, \ker(H))) = \tilde{\mathfrak{G}}(H, \ker(H))/\tilde{L}_0 \subset \tilde{\mathfrak{G}}(H, L_E^\perp)/\tilde{L} = \mathfrak{G}(H, \alpha). \]
Since $\ker(H)/L_0$ is the identity component of $L_E^\perp/L$, (4.12) implies that $\mathfrak{G}(H, \alpha)^0 \subset \kappa^{-1}(\ker(H)/L_0)$. The connectedness of $\mathbb{C}^*$ implies that its image in $\mathfrak{G}(H, \alpha)$ (see (4.12)) lies in $\mathfrak{G}(H, \alpha)^0$. The exactness of (4.12) implies that in order to prove the desired equality, it suffices to check that for every $u + L_0 \in \ker(H)/L_0$ (with $u \in \ker(H)$) there is $u \in \mathfrak{G}(H, \alpha)^0$ with $\tilde{\pi}(u) = u + L_0$. Due to (2.9) and (3.9),
\[ u := B_{H,A} L_0 \in \tilde{\mathfrak{G}}(H, \ker(H))/\tilde{L}_0 = \mathfrak{G}(H, \alpha)^0 \]
does the trick. Now the last assertion of the theorem follows from Theorem 4.3(iii). \qed

Theorem 4.7. Let $\tilde{B}$ be a subgroup of $\mathfrak{G}(H, \alpha)$ and
\[ B := \tilde{\pi}(\tilde{B}) \subset L^\perp/L \]
be its image, which is a subgroup in $L^\perp/L$.

(i) If $N$ is a subgroup of $B$, then
\[ \tilde{N} := \tilde{\pi}^{-1}(N) \cap \tilde{B} = \{ u \in \tilde{B} \subset \mathfrak{G}(H, \alpha) \mid \tilde{\pi}(u) \in B \} \]
is a normal subgroup in $\tilde{B}$. In addition, if $\tilde{B}$ is finite, then $[B : N]$ and $[\tilde{B} : \tilde{N}]$ coincide.

(ii) The subgroup $\tilde{B}$ is commutative if and only if $B$ is isotropic with respect to $e_{H, \alpha}$.

(iii) Suppose that $H \neq 0$ and
\[ B \subset L^\perp_{1,E}/L_1 \subset L^\perp/L. \]
Then $\tilde{B}$ is commutative if and only if $B$ is isotropic with respect to $e_E$. If this is the case, then the index $[(L^\perp_{1,E}/L_1) : B]$ is divisible by $\sqrt{\#(L^\perp_{1,E}/L_1)}$.

(iv) Suppose that $H \neq 0$ and $n$ is a positive integer such that
\[ E(L, L) \subset n\mathbb{Z}. \]
Let $\tilde{A}$ be a commutative subgroup of $\mathfrak{G}(H, \alpha)$ and let
\[ A := \tilde{\pi}(\tilde{A}) \subset L^\perp/L \]
be its image, which is a subgroup in $L^\perp/L$. If $A \subset L^\perp_{1,E}/L_1$, then the index $[(L^\perp_{1,E}/L_1) : B]$ is divisible by $n$.

(v) Suppose that $H \neq 0$ and
\[ \text{pr}_2: L^\perp/L = (\ker(H)/L_0) \oplus (L^\perp_{1,E}/L) \to (L^\perp_{1,E}/L_1) \quad (4.14) \]
is the projection map. Let
\[ B_1 := \text{pr}_2^1(B) = \text{pr}_2^2(\tilde{B}) \subset L^\perp_{1,E}/L_1. \]
Then $\tilde{B}$ is commutative if and only if $B_1$ is isotropic with respect to $e_E$. If this is the case, then the index $[(L^\perp_{1,E}/L_1) : B]$ is divisible by $\sqrt{\#(L^\perp_{1,E}/L_1)}$. 

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**Proof.** Let us prove (i). Since $B \subset L_{1,E}^\perp/L_1$ is commutative, its every subgroup, including $N$, is normal in $B$. Let us consider the surjective homomorphism

$$\pi: \tilde{B} \to B$$

and denote its kernel by $\tilde{B}_0$, which is a normal subgroup in $\tilde{B}$. The surjectivity of (4.15) implies that the preimage $\tilde{N} \subset \tilde{B}$ of $N$ is also normal in $\tilde{B}$; in addition, $\tilde{N}$ contains $\tilde{B}_0$, which is normal in $\tilde{N}$. The surjection (4.15) induces group isomorphisms

$$\tilde{B}/\tilde{B}_0 \cong B, \quad \tilde{N}/\tilde{B}_0 \cong N.$$

If $\tilde{B}$ is finite, then all the other groups involved are also finite and

$$\#(\tilde{B}) = \#(\tilde{B}_0) \cdot \#(B), \quad \#(\tilde{N}) = \#(\tilde{B}_0) \cdot \#(N),$$

which implies that

$$[\tilde{B} : \tilde{N}] = \frac{\#(\tilde{B})}{\#(N)} = \frac{\#(\tilde{B}_0) \cdot \#(B)}{\#(B_0) \cdot \#(N)} = \frac{\#(B)}{\#(N)} = [B : N].$$

It follows that $[\tilde{B} : \tilde{N}] = [B : N]$. This completes the proof of assertion (i).

Assertion (ii) follows from the description (4.13) of $e_{H,\alpha}$ as the commutator pairing associated with the central extension (4.12).

The first assertion of (iii) follows from (ii) and Remark 4.5. The second assertion of (iii) follows from the first one and the nondegeneracy of $e_E$.

Assertion (iv) follows from (iii) combined with Lemma 4.2.

Let us prove (v). By (ii), $\tilde{B}$ is commutative if and only if $B$ is isotropic with respect to $e_{H,\alpha}$. Let

$$x_1, x_2 \in B \subset L^\perp/L = \ker(H)/L_0 \oplus L_{1,E}^\perp/L_1 \subset X.$$

We have

$$x_1 = h_1 + l_1, \quad x_2 = h_2 + l_2, \quad h_j \in \ker(H)/L_0, \quad l_j \in L_{1,E}^\perp/L_1.$$

By Remark 4.5, each $h_j$ lies in the kernel of $e_{H,\alpha}$. This implies that

$$e_{H,\alpha}(x_1, x_2) = e_{H,\alpha}(l_1, l_2) = e_E(l_1, l_2).$$

Therefore, $B$ is isotropic with respect to $e_{H,\alpha}$ if and only if $B_1$ is isotropic with respect to $e_E$. The remaining assertion about the index follows from the nondegeneracy of $e_E$. \[\square\]

**4.5.** We call $\mathfrak{G}(H, \alpha)$ the theta group of $\mathcal{L}(H, \alpha)$. Recall (Remark 4.1) that

$$K(\mathcal{L}(H, \alpha)) = L^\perp/L \subset V/L = X.$$  (4.16)

Clearly,

$$\mathfrak{G}(H, \alpha) \subset S(\mathcal{L}(H, \alpha)).$$

More precisely, all elements of $\text{mult}_{\mathcal{L}(H, \alpha)}(\mathbb{C}^*)$ are liftings of the identity automorphism $T_e$ of $X$ (where $e$ is the zero of group law on $X$), while $B_{H,\alpha}\tilde{L}$ is a lifting of $T_x$ where

$$x = u + L \in L_{E}^\perp/L = K(\mathcal{L}(H, \alpha)) \subset V/L = X.$$

It follows that

$$\tilde{\pi}: \mathfrak{G}(H, \alpha) \to L_{E}^\perp/L = K(\mathcal{L}(H, \alpha)) \subset X.$$
coincides with the restriction of
\[ \rho = \rho_{\mathcal{L}(H,\alpha)}: \mathcal{S}(\mathcal{L}(H,\alpha)) \to X \]
(defined in Subsection 1.2) to \( \mathfrak{g}(H,\alpha) \subset \mathcal{S}(\mathcal{L}(H,\alpha)) \).

**Theorem 4.8.** The identity component \( \mathfrak{g}(H,\alpha)^0 = \pi^{-1}(\text{ker}(H)/L_0) \) of the complex Lie group \( \mathfrak{g}(H,\alpha) \) is the center of \( \mathfrak{g}(H,\alpha) \), which is included in the short exact sequence of complex Lie groups
\[ 1 \to \mathbb{C}^* \to \mathfrak{g}(H,\alpha)^0 \to \ker(H)/L_0 \to 0. \]
In particular, \( \mathfrak{g}(H,\alpha) \) is commutative if \( H = 0 \).

**Proof.** It follows from the results of Subsection 4.4 that \( u \in \mathfrak{g}(H,\alpha) \) lies in the center of \( \mathfrak{g}(H,\alpha) \) if and only if
\[ x := \pi(u) \in L_E^1/L \]
satisfies
\[ x = v + L \quad \text{with} \quad v \in L_E^1, \quad e^{2\pi i E(v,w)} = 1 \quad \forall w \in L_E^1, \]
i.e.,
\[ E(v, w) = \text{Im}(H(v, w)) \in \mathbb{Z} \quad \forall w \in L_E^1. \]  \hfill (4.17)
Clearly, each \( v \in \ker(H) \) satisfies (4.17), and therefore the center of \( \mathfrak{g}(H,\alpha) \) contains
\[ \pi^{-1}(\ker(H)/(\ker(H) \cap L)) = \pi^{-1}(\ker(H)/L_0) = \mathfrak{g}(H,\alpha)^0. \]
In particular, if \( H = 0 \), then
\[ \ker(H) = V, \quad L_0 = L, \quad L_E^1 = V, \quad \ker(H)/L_0 = L_E^1/L; \]
hence \( \mathfrak{g}(H,\alpha) \) coincides with its central subgroup \( \mathfrak{g}(H,\alpha)^0 \) and is therefore commutative.

Now suppose that (in the notation of Subsection 3.4)
\[ v \notin \ker(H) \oplus \bigoplus_{i=1}^{g-g_0} U_i. \]
This implies that \( H \neq 0 \) and there exist \( u_0 \in \ker(H) \) and \( u_i \in (1/d_i)U_i \) (for all \( i \) with \( 1 \leq i \leq g - g_0 \)) such that not all \( u_i \) are in \( U_i \) and
\[ u = u_0 + \sum_{i=1}^{g-g_0} u_i. \]
Suppose that \( u_j \notin U_j \) for a certain \( j \in \{1, \ldots, g - g_0\} \). Then \( u_j = a_j e_j + b_j f_j \) where \( a_j, b_j \in (1/d_j)\mathbb{Z} \) and at least one of the coefficients \( a_j \) and \( b_j \) is not an integer. Recall that
\[ \frac{1}{d_j} e_j, \frac{1}{d_j} f_j \in L_E^1. \]
However,
\[ E\left(u, \frac{1}{d_j} e_j\right) = b_j, \quad E\left(u, \frac{1}{d_j} f_j\right) = a_j \]
and therefore (4.17) does not hold. This implies that if \( u \) is a central element of \( \mathfrak{g}(H,\alpha) \), then \( v \in \ker(H) \oplus \bigoplus_{i=1}^{g-g_0} U_i \), i.e.,
\[ \pi(u) \in \ker(H)/L_0, \]
which means that \( u \in \mathfrak{g}(H,\alpha)^0 \). This completes the proof. \( \square \)
Theorem 4.9. If \( \mathcal{L} = \mathcal{L}(H, \alpha) \), then \( \mathfrak{G}(H, \alpha) = \mathfrak{S}(\mathcal{L}(H, \alpha)) \). In particular,
\[
\rho_{\mathcal{L}(H, \alpha)} = \pi, \quad K(\mathcal{L}(H, \alpha)) = \pi(\mathfrak{G}(H, \alpha)) = \rho_{\mathcal{L}(H, \alpha)}(\mathfrak{S}(\mathcal{L}(H, \alpha))). \tag{4.18}
\]

Remark 4.10. Recall (Remark 4.1) that \( \ker(H)/L_0 \) is the identity component of \( K(\mathcal{L}(H, \alpha)) \). Now the results of Subsection 4.5 combined with Theorem 4.9 imply that
\[
\mathfrak{G}(H, \alpha)^0 = \pi^{-1}(\ker(H)/L_0) = \rho^{-1}(K(\mathcal{L}(H, \alpha))^0) = \mathfrak{S}(\mathcal{L}(H, \alpha))^0. \tag{4.19}
\]

Proof of Theorem 4.9. We write \( \rho: \mathcal{L} \to X \) for the structure morphism from the total space of the holomorphic line bundle to its base.

Let \( u \in \mathfrak{S}(\mathcal{L}) \subset \text{Aut}(\mathcal{L}) \). By the definition of \( \mathfrak{S}(\mathcal{L}) \), there is \( x \in X \) such that \( u: \mathcal{L} \to \mathcal{L} \) is a lifting of \( T_x: X \to X \). In particular, the restriction of \( u \) to the fibers of \( \mathcal{L} \) induces linear isomorphisms
\[
u_z: \mathcal{L}_z \cong \mathcal{L}_{z+x}
\]
between the fibers of \( \mathcal{L} \) over \( z \) and \( x+z \) for all \( z \in X \).

It follows from [12, Ch. I, Proposition 2.14] (applied to \( f = T_x \)) that there exists an induced holomorphic line bundle
\[T^*_x \mathcal{L} = \{(l, z) \in \mathcal{L} \times X \mid p(l) = z+x\}\]
over \( X \) with the structure morphism
\[T^*_x \mathcal{L} \to X, \quad (l, z) \mapsto z\]
and a holomorphic map of total spaces of holomorphic line bundles over \( X \),
\[(T_x)_*: T^*_x \mathcal{L} \to \mathcal{L}, \quad (l, z) \mapsto l,\]
that lifts \( T_x \) and induces \( \mathbb{C} \)-linear isomorphisms between the corresponding fibers \((T^*_x \mathcal{L})_z \) and \( \mathcal{L}_{z+x} \) for all \( z \in X \). Clearly, \((T_x)_* \) is a biholomorphic isomorphism: indeed, its inverse is defined by
\[l \mapsto (l, z) = (l, p(l) - x)\]
It follows that the composition
\[(T_x)_* \circ u^{-1}: \mathcal{L} \to \mathcal{L} \to T^*_x \mathcal{L}\]
is an isomorphism of holomorphic line bundles \( \mathcal{L} \) and \( T^*_x \mathcal{L} \) over \( X \). Therefore, the holomorphic line bundles \( \mathcal{L}(H, \alpha) = \mathcal{L} \) and \( T^*_x \mathcal{L}(H, \alpha) = T^*_x \mathcal{L} \) over \( X \) are isomorphic. Hence
\[x \in K(\mathcal{L}(H, \alpha)) = L^1_E/L \subset V/L = X.\]

Pick \( u \in L^1_E \) with \( u + L = x \). Then \( uB^{-1}_{H,u} \) is a holomorphic automorphism of \( \mathcal{L}(H, \alpha) \) that leaves every fiber \( \mathcal{L}(H, \alpha)_z \) invariant and acts on it as a \( \mathbb{C} \)-linear automorphism. By the compactness and connectedness of \( X \), there is a nonzero scalar \( \lambda \in \mathbb{C}^* \) such that \( uB^{-1}_{H,u} \) acts as multiplication by \( \lambda \) on every fiber. It follows that \( uB^{-1}_{H,u} \) lies in \( \mathfrak{G}(H, \alpha) \). Since \( B_{H,u} \) lies in \( \mathfrak{G}(H, \alpha) \) as well, we conclude that \( u \in \mathfrak{G}(H, \alpha) \). \( \Box \)

Remark 4.11. It follows from Theorem 4.9 combined with (4.12) and the results of Subsection 4.5 that \( \mathfrak{S}(\mathcal{L}(H, \alpha)) = \mathfrak{G}(H, \alpha) \) is included in a short exact sequence of complex Lie groups
\[1 \to \mathbb{C}^* \to \mathfrak{S}(\mathcal{L}(H, \alpha)) = \mathfrak{G}(H, \alpha) \xrightarrow{\rho_{\mathcal{L}(H, \alpha)}} K(\mathcal{L}(H, \alpha)) \to 0. \tag{4.20}\]
5. PROOFS OF THEOREMS 1.3 AND 1.5

**Definition 5.1.** Let $\mathcal{L}$ be a holomorphic line bundle on $X$. Let us choose an isomorphism of holomorphic line bundles $\psi: \mathcal{L} \cong \mathcal{L}(H, \alpha)$ for suitable AH-data $(H, \alpha)$ where $(H, \alpha)$ is uniquely determined by the isomorphism class of $\mathcal{L}$. By Remark 1.2 combined with Theorem 4.9, there is a certain group isomorphism $\psi_S: \mathbf{S}(\mathcal{L}) \cong \mathcal{G}(H, \alpha)$ that does not depend on the choice of $\psi$. Then there is a canonical structure of a complex Lie group on $\mathbf{S}(\mathcal{L})$ such that the group isomorphism $\psi_S: \mathbf{S}(\mathcal{L}) \cong \mathcal{G}(H, \alpha)$ is an isomorphism of complex Lie groups.

**Corollary 5.2.** The action map $\mathbf{S}(\mathcal{L}) \times \mathcal{L} \to \mathcal{L}$ and the group homomorphism $\rho_{\mathcal{L}}: \mathbf{S}(\mathcal{L}) \to X$ are holomorphic.

**Proof.** We may assume that $\mathcal{L} = \mathcal{L}(H, \alpha)$ and therefore $\mathbf{S}(\mathcal{L}) = \mathcal{G}(H, \alpha)$. Then our assertion follows from the results of Subsection 4.3 and Theorem 4.9. □

**Proof of Theorem 1.3.** Assertions (i) and (ii) are contained in Corollary 5.2 and Theorem 4.9. Assertion (iv) follows from the very Definition 5.1. In order to check (iii), let us assume that $\mathcal{L} = \mathcal{L}(H, \alpha)$ and therefore

$$\mathbf{S}(\mathcal{L}) = \mathcal{G}(H, \alpha), \quad \rho_{\mathcal{L}} = \pi, \quad \mathbf{S}(\mathcal{L})^0 = \mathcal{G}(H, \alpha)^0, \quad K(\mathcal{L})^0 = \ker(H)/L_0.$$ 

Then all the assertions of (iii) follow from Theorems 4.6 and 4.8. □

**Proof of Theorem 1.5.** Denote by $\mathcal{V}$ the rank 2 vector bundle $\mathcal{V} = \mathcal{L} \oplus 1_X$ over $X$. By definition, $Y_{\mathcal{L}}$ is the projectivization of $\mathcal{V}$.

First, let us define an embedding

$$\mathbf{S}(\mathcal{L}) \hookrightarrow \text{Aut}_0(\mathcal{V}) = \text{Aut}_0(\mathcal{L} \oplus 1_X).$$

In order to do that, recall that each $u \in \mathbf{S}(\mathcal{L}) \subset \text{Aut}_0(\mathcal{L})$ is a lifting of $T_x: X \to X$ where $x = \rho_{\mathcal{L}}(u) \in X$. This allows us to define the action of $u$ on $1_X = X \times \mathbb{C}$ as

$$\pi_1(u): X \times \mathbb{C} \to X \times \mathbb{C}, \quad (z, \lambda) \mapsto (z + \rho_{\mathcal{L}}(u), \lambda) \quad \forall z \in X, \lambda \in \mathbb{C}.$$ 

By Corollary 5.2, $\rho_{\mathcal{L}}$ is a homomorphism of complex Lie groups and, hence, is holomorphic; therefore, the corresponding action map

$$\mathbf{S}(\mathcal{L}) \times 1_X \to 1_X, \quad u, (z, \lambda) \mapsto (z + \rho_{\mathcal{L}}(u), \lambda)$$

is holomorphic. This gives us a (non-injective) group homomorphism

$$\pi_1: \mathbf{S}(\mathcal{L}) \to \text{Aut}_0(1_X),$$

whose image meets the “scalar automorphisms” $\text{mult}_{1_X}(\mathbb{C}^*)$ only at the identity automorphism $1_X$. Taking the “direct sum” of the embedding $\mathbf{S}(\mathcal{L}) \subset \text{Aut}_0(\mathcal{L})$ with $\pi_1$, we get a group embedding

$$\pi_2: \mathbf{S}(\mathcal{L}) \hookrightarrow \text{Aut}_0(\mathcal{L} \oplus 1_X) = \text{Aut}_0(\mathcal{V}),$$

whose image also meets precisely one element of $\text{mult}_{\mathcal{V}}(\mathbb{C}^*)$, namely, the identity automorphism of $\mathcal{V}$. Clearly, the corresponding action map

$$\mathbf{S}(\mathcal{L}) \times \mathcal{V} \to \mathcal{V},$$

$$u, (I_z; (z, \lambda)) \mapsto (u(I_z); z + \rho_{\mathcal{L}}(u, \lambda)) \quad \forall z \in X, \quad I_z \in \mathcal{L}_z, \quad \lambda \in \mathbb{C}, \quad u \in \mathbf{S}(\mathcal{L})$$

is holomorphic as well, since the action map $\mathbf{S}(\mathcal{L}) \times \mathcal{L} \to \mathcal{L}$ is holomorphic by Corollary 5.2. It is also clear that $\pi_2(u)$ is a lifting of $T_x$ with $x = \rho_{\mathcal{L}}(u)$. It follows that the group homomorphism

$$\Upsilon_{\mathcal{L}}: \mathbf{S}(\mathcal{L}) \to \text{Aut}(\mathbb{P}(\mathcal{V})) = \text{Aut}(Y_{\mathcal{L}})$$

induced by $\pi_2$ is an embedding, the corresponding action map $\mathbf{S}(\mathcal{L}) \times Y_{\mathcal{L}} \to Y_{\mathcal{L}}$ is holomorphic, and $\Upsilon_{\mathcal{L}}(u)$ is a lifting of $T_x$ with $x = \rho_{\mathcal{L}}(u)$. □
6. JORDAN PROPERTIES OF THETA GROUPS

We keep the notation and assumptions of Section 3.

**Theorem 6.1.** Suppose that $H \neq 0$. Then $\mathfrak{G}(H, \alpha)$ is Jordan and its Jordan constant is

$$\sqrt{\#(L_{1,E}^1/L_1)} = \prod_{i=1}^{g-9_0} d_i(E).$$

**Corollary 6.2.** Let $H \neq 0$ and $n$ be a positive integer such that

$$E(L, L) \subset n\mathbb{Z}.$$ 

Then the Jordan constant $J_{\mathfrak{G}(H, \alpha)}$ of $\mathfrak{G}(H, \alpha)$ is divisible by $n$. In particular, $J_{\mathfrak{G}(H, \alpha)} \geq n$.

**Proof.** By Lemma 4.2, $\#(L_{1,E}^1/L_1)$ is divisible by $n^2$. Now the desired result follows readily from Theorem 6.1. □

We will need the following lemma, which will be proved at the end of this section.

**Lemma 6.3.** Let $\Delta$ be a finite subgroup in $K(\mathcal{L}(H, \alpha))$. Then there exists a finite subgroup $\tilde{\Delta}$ in $\mathfrak{G}(H, \alpha)$ such that $\pi(\tilde{\Delta}) = \Delta$.

**Proof of Theorem 6.1.** Let $\tilde{B}$ be a finite subgroup in $\mathfrak{G}(H, \alpha)$. Consider its images

$$B = \pi(\tilde{B}) \subset K(\mathcal{L}(H, \alpha)) = (\ker(H)/L_0) \oplus (L_{1,E}^1/L_1), \quad B_1 = \text{pr}_2(B) = \text{pr}_2(\pi(\tilde{B})) \subset L_{1,E}^1/L_1.$$ 

Let $A_1$ be a maximal isotropic subgroup in $B_1$ with respect to $e_E$. The nondegenerate pairing $e_E$ gives rise to an embedding

$$B_1/A_1 \hookrightarrow \text{Hom}(A_1, \mathbb{C}^*), \quad b + A_1 \mapsto \{a \mapsto e_E(a, b) \forall a \in A_1\}.$$ 

Since the orders of the finite commutative groups $A_1$ and $\text{Hom}(A_1, \mathbb{C}^*)$ coincide, $\#(A_1)$ divides $\#(B_1/A_1)$ and therefore $\#(B_1/A_1)^2$ divides $\#(B_1)$, which in turn divides $\#(L_{1,E}^1/L_1)$. It follows that the index

$$[B_1 : A_1] = \#(B_1/A_1)$$

does not exceed (actually divides) $\sqrt{\#(L_{1,E}^1/L_1)}$. Let us consider the subgroup

$$\tilde{A} := (\text{pr}_2 \pi)^{-1}(A_1) \cap \tilde{B}.$$ 

Since $A_1$ is isotropic, it follows from Theorem 4.7(v) that $\tilde{A}$ is a commutative subgroup. Since $A_1$ is obviously normal in the (commutative) group $B_1$, the preimage $\tilde{A}$ of $A_1$ with respect to the surjective homomorphism

$$\tilde{B} \xrightarrow{\text{pr}_2 \pi} B_2$$

is a normal subgroup of $\tilde{B}$ whose index does not exceed (actually equals) $[B_1 : A_1]$, which, in turn, does not exceed $\sqrt{\#(L_{1,E}^1/L_1)}$. It follows that $\mathfrak{G}(H, \alpha)$ is Jordan and its Jordan constant does not exceed $\sqrt{\#(L_{1,E}^1/L_1)}$. We need to prove that the Jordan constant is at least $\sqrt{\#(L_{1,E}^1/L_1)}$.

In order to do that, let us consider the finite subgroup

$$\Delta := L_{1,E}^1/L_1 = \{0\} \oplus (L_{1,E}^1/L_1) \subset (\ker(H)/L_0) \oplus (L_{1,E}^1/L_1) = K(\mathcal{L}(H, \alpha)).$$

By Lemma 6.3, there is a finite subgroup $\tilde{\Delta} \subset \mathfrak{G}(H, \alpha)$ such that

$$\pi(\tilde{\Delta}) = \Delta.$$
Let $A' \subset \tilde{\Delta}$ be a commutative normal subgroup of $\tilde{\Delta}$. By Theorem 4.7(iii), the subgroup

$$A = \pi(A') \subset \Delta = L_{1,E}^+ / L_1$$

is an isotropic subgroup with respect to the pairing $e_E$ and the index $[(L_{1,E}^+ / L_1) : A]$ is divisible by $\sqrt{\#(L_{1,E}^+ / L_1)}$. Let us define

$$\tilde{A} := \pi^{-1}(A) \cap \tilde{\Delta} \subset \tilde{\Delta}.$$ 

By Theorem 4.7(i), $\tilde{A}$ is a normal subgroup of $\tilde{\Delta}$ and

$$[\tilde{\Delta} : \tilde{A}] = [L_{1,E}^+ / L_1 : A].$$

This implies that $[\tilde{\Delta} : \tilde{A}]$ is divisible by $\sqrt{\#(L_{1,E}^+ / L_1)}$.

Clearly, $\tilde{A}$ contains $A'$. This implies that the index $[\tilde{\Delta} : A']$ is divisible by $[\tilde{\Delta} : \tilde{A}]$ and is therefore divisible by $\sqrt{\#(L_{1,E}^+ / L_1)}$. However, if $U$ is a maximal isotropic subgroup in $L_{1,E}^+ / L_1$, then

$$\#(U) = \sqrt{\#(L_{1,E}^+ / L_1)} = [L_{1,E}^+ / L_1 : U].$$

Let

$$\tilde{U} := \pi^{-1}(U) \cap \tilde{\Delta} \subset \tilde{\Delta}.$$ 

By assertions (i) and (iii) of Theorem 4.7, $\tilde{U}$ is a commutative normal subgroup in $L_{1,E}^+ / L_1$ of index $\sqrt{\#(L_{1,E}^+ / L_1)}$. It follows that the Jordan constant of $G(H, \alpha)^0$ is at least $\sqrt{\#(L_{1,E}^+ / L_1)}$. This completes the proof. \[\Box\]

**Proof of Lemma 6.3.** In what follows we identify $\mathbb{C}^*$ with its image in $G(H, \alpha)$ and view it as a certain central subgroup of $G(H, \alpha)$. Let $d$ be the exponent of $\Delta$. Let us consider the finite multiplicative group $\mu_d$ of all $d$th roots of unity and the finite multiplicative group $\mu_{d^2}$ of all $d^2$th roots of unity in $\mathbb{C}$. We have

$$\mu_d \subset \mu_{d^2} \subset \mathbb{C}^* \subset G(H, \alpha).$$

For every $x \in \Delta$ choose its lifting $\bar{x} \in G(H, \alpha)$ with the same order as $x$ and such that the lifting $(\bar{-}x)$ of $-x$ coincides with $\bar{x}^{-1}$. (This is possible, since $\mathbb{C}^*$ is a central divisible subgroup in $G(H, \alpha)$.) Let us consider the finite set

$$\tilde{\Delta} = \{ \gamma \bar{x} \mid \gamma \in \mu_{d^2}, \ x \in \Delta \} \subset G(H, \alpha).$$

Clearly, $\pi(\gamma \bar{x}) = x$ and therefore $\pi(\tilde{\Delta}) = \Delta$. It remains to check that $\tilde{\Delta}$ is a subgroup in $G(H, \alpha)$. Let $x_1, x_2 \in \Delta$ and $x_3 = x_1 + x_2 \in \Delta$. We need to compare $\bar{x}_1 \bar{x}_2$ and $\bar{x}_3$ in $G(H, \alpha)$. Notice that there is $\gamma \in \mathbb{C}^*$ such that

$$\bar{x}_3 = \gamma \bar{x}_1 \bar{x}_2.$$ 

In addition,

$$\bar{x}_1^d = \bar{x}_2^d = \bar{x}_3^d = 1 \in \mathbb{C}^* \subset G(H, \alpha).$$

On the other hand, we have

$$\gamma_0 := \bar{x}_1 \bar{x}_2 \bar{x}_1^{-1} \bar{x}_2^{-1} \in \mu_d \in \mathbb{C}^* \subset G(H, \alpha).$$
since the orders of both $\tilde{x}_1$ and $\tilde{x}_2$ divide $d$. It follows that the images of $\tilde{x}_1$ and $\tilde{x}_2$ in the quotient $\mathcal{G}(H, \alpha)/\mu_d$ do commute, and therefore the image of $\tilde{x}_1 \tilde{x}_2$ in $\mathcal{G}(H, \alpha)/\mu_d$ has order dividing $d$. This means that

$$(\tilde{x}_1 \tilde{x}_2)^d \in \mu_d$$

and therefore

$$(\tilde{x}_1 \tilde{x}_2)^{d^2} = 1.$$

It follows that

$$1 = \tilde{x}_3^{d^2} = (\gamma \tilde{x}_1 \tilde{x}_2)^{d^2} = \gamma^{d^2} (\tilde{x}_1 \tilde{x}_2)^{d^2} = \gamma^{d^2} \cdot 1 = 1.$$

This implies that $\gamma^{d^2} = 1$ and therefore

$$\tilde{x}_1 \tilde{x}_2 = \gamma^{-1} \tilde{x}_3 \in \tilde{\Delta}.$$

It follows that $\tilde{\Delta}$ is a subgroup (see also [4, Ch. I, Exercise 3 for § 4]). □

7. PROOF OF THEOREM 1.6

We may and will assume that $\mathcal{L} = L(H, \alpha)$. We keep the notation and assumptions of Section 6.

Proof of Theorem 1.6. By Theorem 4.9, $\mathcal{S}(\mathcal{L}) = \mathcal{G}(H, \alpha)$. By Theorem 4.8,

$$\mathcal{G}(H, \alpha)^0 := \sqrt{\#(\ker(H)/L_0)}$$

is the center of $\mathcal{G}(H, \alpha)$.

Suppose that $H = 0$. Then $K(\mathcal{L}(H, \alpha)) = X$ (see [5, Corollary 1.9]); in particular, it is connected, i.e., the number of its connected components is one. On the other hand, by Theorem 4.8, $\mathcal{G}(H, \alpha)$ is commutative and therefore its Jordan constant is 1. This yields the desired result when $H = 0$.

Suppose that $H \neq 0$. By Theorem 6.1, the Jordan constant of $\mathcal{G}(H, \alpha)$ is $\sqrt{\#(L_{1,E}^+/L_1)}$. By Remark 4.1, $L_{1,E}^+/L_1$ is isomorphic to the group $K(\mathcal{L}(H, \alpha))/K(\mathcal{L}(H, \alpha))^0$ of connected components of $K(\mathcal{L}(H, \alpha))$. This implies that the Jordan constant of $\mathcal{G}(H, \alpha)$ is $\sqrt{\#(K(\mathcal{L}(H, \alpha))/K(\mathcal{L}(H, \alpha))^0)}$, which completes the proof. □

8. $\mathbb{CP}^1$-BUNDLES OVER COMPLEX TORI

We start with the following elementary but useful observations that allow us to handle the groups of bimeromorphic automorphisms of $\mathbb{CP}^1$-bundles by using information about the groups of biholomorphic automorphisms.

Remark 8.1. Let $\mathcal{L}$ and $\mathcal{N}$ be holomorphic line bundles over $X$. Assume that $\mathcal{L}$ admits a nonzero holomorphic section, say $s$. Let $n$ be a positive integer.

1. Clearly, $\mathcal{L}^n$ also admits a nonzero holomorphic section $s^\otimes n$.

2. The holomorphic $\mathbb{C}$-linear map of rank 2 holomorphic vector bundles on $X$

$$\mathcal{N} \oplus 1_X \to (\mathcal{N} \otimes \mathcal{L}) \oplus 1_X, \quad (t_x; x, \lambda) \mapsto (t_x \otimes s(x); x, \lambda) \quad \forall x \in X, \quad t_x \in \mathcal{N}_x, \quad \lambda \in \mathbb{C}$$

induces a bimeromorphic isomorphism of the corresponding $\mathbb{CP}^1$-bundles $\mathbb{P}(\mathcal{N} \oplus 1_X)$ = $Y_{\mathcal{N}}$ and $\mathbb{P}((\mathcal{N} \otimes \mathcal{L}) \oplus 1_X) = Y_{\mathcal{N} \otimes \mathcal{L}}$ over $X$. Therefore, the groups Bim($Y_{\mathcal{N}}$) and Bim($Y_{\mathcal{N} \otimes \mathcal{L}}$) are isomorphic.

Taking into account that $\mathcal{L}^n$ also admits a nonzero holomorphic section, we find that the groups Bim($Y_{\mathcal{N}}$) and Bim($Y_{\mathcal{N} \otimes \mathcal{L}^n}$) are isomorphic for all positive integers $n$. 

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3. It follows from assertion 2 applied to \( N = 1_X \) combined with Example 1.1 that for all positive integers \( n \) the \( \mathbb{CP}^1 \)-bundles \( X \times \mathbb{CP}^1 \) and \( Y_{\mathbb{C}^n} \) are bimeromorphic, and therefore the groups \( \text{Bim}(X \times \mathbb{CP}^1) \) and \( \text{Bim}(Y_{\mathbb{C}^n}) \) are isomorphic.

4. It follows from assertions 2 and 3 that for all positive integers \( n \) the group \( \text{Bim}(Y_N) \) contains a subgroup isomorphic to \( \mathbb{S}(N \otimes \mathcal{L}^n) \) and the group \( \text{Bim}(X \times \mathbb{CP}^1) \) contains a subgroup isomorphic to \( \mathbb{S}(\mathcal{L}^n) \). We will use this observation (together with Theorem 1.3) in the proof of Theorems 1.7 and 1.8.

**Proof of Theorem 1.7.** Since \( X \) has positive algebraic dimension, it follows from the results of [2, Ch. 2, Sect. 6] that there exists a surjective holomorphic homomorphism \( \psi: X \to A \) to a positive-dimensional complex abelian variety \( A \). There exists a very ample holomorphic line bundle \( \mathcal{M} \) on \( A \) such that the group \( H^0(A, \mathcal{M}) \) of global sections of \( \mathcal{M} \) has \( \mathbb{C} \)-dimension at least 2. Let us consider the induced holomorphic line bundle \( \psi_* \mathcal{M} \) on \( X \). Since \( \psi \) is surjective, the group \( H^0(X, \psi_* \mathcal{M}) \) of global sections of \( \psi_* \mathcal{M} \) also has \( \mathbb{C} \)-dimension at least 2, because \( H^0(A, \mathcal{M}) \) embeds into \( H^0(X, \psi_* \mathcal{M}) \). There exist \( AH \) data \( (H, \alpha) \) on \( X \) such \( \psi_* \mathcal{M} \) is isomorphic to \( \mathcal{L}(H, \alpha) \). This implies that \( \mathcal{L}(H, \alpha) \) has at least two linearly independent nonzero holomorphic sections. Now if \( H = 0 \), then \( \mathcal{L}(H, \alpha) = \mathcal{L}(0, \alpha) \) and one of the following two conditions holds:

(i) \( \alpha = \alpha_0 \), i.e., \( \mathcal{L}(H, \alpha) = \mathcal{L}(0, \alpha_0) \) is isomorphic to \( 1_X \) and

\[
H^0(X, \mathcal{L}(H, \alpha)) = H^0(X, \mathcal{L}(0, \alpha_0)) = H^0(X, 1_X) = \mathbb{C};
\]

(ii) \( \alpha \neq \alpha_0 \), in which case it follows from [5, Theorem 2.1] that

\[
H^0(X, \mathcal{L}(H, \alpha)) = H^0(X, \mathcal{L}(0, \alpha)) = \{0\}.
\]

Since the \( \mathbb{C} \)-dimension of \( H^0(X, \mathcal{L}(H, \alpha)) \) is at least 2, neither (i) nor (ii) holds. This implies that \( H \neq 0 \).

Let \( n \) be a positive integer. Then \( nH \neq 0 \) and the holomorphic line bundle

\[
\mathcal{L}(nH, \alpha^n) \cong \mathcal{L}(H, \alpha)^{\otimes n}
\]

over \( X \) also admits a nonzero holomorphic section. Notice that

\[
E_n := \text{Im}(nH) = nE \quad \text{where} \quad E = \text{Im}(H).
\]

In particular, \( E_n(L, L) \subset n\mathbb{Z} \). It follows from Corollary 6.2 applied to \( \mathcal{L}(nH, \alpha^n) \) that the Jordan constant of \( \mathfrak{G}(nH, \alpha^n) \) is at least \( n \). By Theorem 1.5, there is a group embedding

\[
\mathfrak{G}(nH, \alpha^n) \hookrightarrow \text{Aut}(\mathbb{P}(\mathcal{L}(nH, \alpha^n) \oplus 1_X)).
\]

By Remark 8.1, \( \text{Bim}(X \times \mathbb{CP}^1) \) and \( \text{Bim}(\mathbb{P}(\mathcal{L}(nH, \alpha^n) \oplus 1_X)) \) are isomorphic. This implies that for all positive integers \( n \) the group \( \text{Bim}(X \times \mathbb{CP}^1) \) contains a subgroup whose Jordan constant is at least \( n \). It follows that \( \text{Bim}(X \times \mathbb{CP}^1) \) is not Jordan.

\[\square\]

9. PENCILS OF HERMITIAN FORMS

In order to prove Theorem 1.8, we need to construct families of Hermitian forms and corresponding alternating forms. We keep the notation and assumptions of Section 3.

Let \( H: V \times V \to \mathbb{C} \) be a Hermitian form. Consider its imaginary part

\[
E: V \times V \to \mathbb{R}, \quad (u, v) \mapsto \text{Im}(H(u, v)),
\]

which is an alternating \( \mathbb{R} \)-bilinear form on \( V \). Assume that

\[
E(L, L) \subset \mathbb{Z}.
\]
**Definition 9.1.** We say that $H$ is dominated by $H$ if

$$\ker(H) \subset \ker(H).$$

For every positive integer $n$ let us consider the Hermitian form

$$H_n := H + nH: V \times V \to \mathbb{C},$$

whose imaginary part

$$E_n := \text{Im}(H_n) = E + nE: V \times V \to \mathbb{R}$$

is an alternating $\mathbb{R}$-bilinear form on $V$. Clearly, for all $n$

$$E_n(L, L) \subset \mathbb{Z}.$$  

If $H$ is dominated by $H$, then every $H_n$ is also dominated by $H$.

**Theorem 9.2.** Suppose that $H \neq 0$ (i.e., $g > g_0$) and that $H$ is dominated by $H$. Then for all but finitely many $n$

$$\ker(H) = \ker(H)$$  \hspace{1cm} (9.1)

and the restriction

$$E_n|_{L_1}: L_1 \times L_1 \to \mathbb{Z}$$  \hspace{1cm} (9.2)

of $E_n$ to $L_1$ is a nondegenerate alternating bilinear form.

**Proof.** Let $\widetilde{E}$ be the square matrix of $E|_{L_1}$ of order $2g - 2g_0$ with respect to the basis $\{\tilde{t}_1, \ldots, \tilde{t}_{2g-2g_0}\}$ of $L_1$. (Recall that $\widetilde{E}$ is the matrix of $E|_{L_1}$ with respect to the same basis.) Then for all $n$ the matrix $\widetilde{E} + n\widetilde{E}$ coincides with the matrix $\widetilde{E}_n$ of $E_n|_{L_1}$ with respect to $\{\tilde{t}_1, \ldots, \tilde{t}_{2g-2g_0}\}$. Recall that the matrix $\widetilde{E}$ is nondegenerate and consider the polynomial

$$f_{H,H}(T) := \det(\widetilde{E}) \det(\widetilde{E}^{-1}\widetilde{E} + T) \in \mathbb{Q}[T].$$  \hspace{1cm} (9.3)

Clearly, $f_{H,H}(T)$ is a degree $2g - 2g_0$ polynomial with (positive) leading coefficient $\det(\widetilde{E})$. We have

$$\det(\widetilde{E}_n) = \det(\widetilde{E} + n\widetilde{E}) = \det\left(\widetilde{E}(\widetilde{E}^{-1}\widetilde{E} + n)\right) = \det(\widetilde{E}) \det(\widetilde{E}^{-1}\widetilde{E} + n) = f_{H,H}(n).$$

In other words,

$$\det(\widetilde{E}_n) = f_{H,H}(n).$$  \hspace{1cm} (9.4)

Since $f_{H,H}(T)$ is not a constant, $\det(\widetilde{E}_n) \neq 0$ for all but finitely many $n$.

Let us assume that $\det(\widetilde{E}_n) \neq 0$, which is true for all but finitely many positive integers $n$. Then $E_n|_{L_1}$ is nondegenerate. It follows that the restriction of $E_n$ to $L_1 \otimes \mathbb{R}$ is nondegenerate as well. On the other hand, the restriction of $E_n$ to $\ker(H)$ is identically zero. This implies that $\ker(E_n) = \ker(H)$ and therefore

$$\ker(H_n) = \ker(E_n) = \ker(H).$$  \hspace{1cm} $\square$

**Definition 9.3.** Suppose that $H \neq 0$ and $n$ is a positive integer such that $\ker(H) = \ker(H)$ and $E_n|_{L_1}$ is a nondegenerate (by Theorem 9.2, these properties hold for all but finitely many positive integers $n$). Let us define $L_{1,E_n}^+$ as

$$L_{1,E_n}^+ = \{ \bar{x} \in L_1 \otimes \mathbb{R} \mid E_n(\bar{x}, l) \in \mathbb{Z} \forall l \in L \}.$$
Remark 9.4. Applying the arguments of Subsection 4.1 to the nondegenerate $E_n \mid L_1$ (instead of $E \mid L_1$), we find that $L_1^+_{1,E_n}$ is a free $\mathbb{Z}$-module of rank $2g - 2g_0$ that lies in $L_1 \otimes \mathbb{Q}$ and contains $L_1$ as a subgroup of finite index, i.e., the quotient $L_1^+_{1,E_n}/L_1$ is a finite commutative group.

It follows from (9.4) and the arguments of Subsection 4.1 applied to $E_n$ (instead of $E$) that

$$\#(L_1^+_{1,E_n}/L_1) = \det(\tilde{E}_n) = f_{H,H}(n).$$

(9.5)

Since $f_{H,H}(T)$ is a polynomial of positive degree, $\#(L_1^+_{1,E_n}/L_1)$ tends to infinity as $n$ tends to infinity, i.e., $\sqrt{\#(L_1^+_{1,E_n}/L_1)}$ tends to infinity as well.

Theorem 9.5. Let $H \neq 0$ be a positive semidefinite Hermitian form, $H$ a Hermitian form that is dominated by $H$, $(H, \alpha)$ AH data, $\mathcal{L}(H, \alpha)$ the corresponding holomorphic line bundle on $X$, and $Y_{\mathcal{L}(H, \alpha)}$ the corresponding $\mathbb{CP}^1$-bundle on $X$. Then the group $\text{Bim}(Y_{\mathcal{L}(H, \alpha)})$ is not Jordan.

Proof. Replacing $H$ by $2H$, we may and will assume that its imaginary part $E$ satisfies the condition $E(L, L) \subset 2\mathbb{Z}$. Then $(H, \alpha_0)$ is AH data. Since $H$ is positive semidefinite, it follows from [5, Theorem 2.1] that the holomorphic line bundle $\mathcal{L}(H, \alpha_0)$ admits a nonzero holomorphic section. Since

$$H_n = H + nH, \quad \alpha = \alpha \cdot \alpha_0^n$$

for all positive integers $n$, we find that

$$\mathcal{L}(H_n, \alpha) \cong \mathcal{L}(H, \alpha) \otimes \mathcal{L}(H, \alpha_0)^n.$$ 

It follows from Remark 8.1 that the groups $\text{Bim}(Y_{\mathcal{L}(H, \alpha)})$ and $\text{Bim}(Y_{\mathcal{L}(H_n, \alpha)})$ are isomorphic. On the other hand, by Theorem 1.5, $\text{Aut}(Y_{\mathcal{L}(H_n, \alpha)})$ contains a subgroup isomorphic to $\mathfrak{G}(H_n, \alpha)$. In view of Theorems 9.2 and 6.1 (applied to $(H_n, \alpha)$), for all sufficiently large $n$ the Jordan constant of $\mathfrak{G}(H_n, \alpha)$ is $\sqrt{\#(L_1^+_{1,E_n}/L_1)}$. It follows from Remark 9.4 that the Jordan constant of $\mathfrak{G}(H_n, \alpha)$ tends to infinity as $n$ tends to infinity. Since each

$$\mathfrak{G}(H_n, \alpha) \hookrightarrow \text{Aut}(Y_{\mathcal{L}(H_n, \alpha)}) \subset \text{Bim}(Y_{\mathcal{L}(H_n, \alpha)})$$

is isomorphic to a certain subgroup of $\text{Bim}(Y_{\mathcal{L}(H, \alpha)})$, we conclude that the Jordan constant of $\text{Bim}(Y_{\mathcal{L}(H, \alpha)})$ is $\infty$, i.e., $\text{Bim}(Y_{\mathcal{L}(H, \alpha)})$ is not Jordan. \qed

10. COMPLEX TORI AND ABELIAN VARIETIES

A complex abelian variety $A$ of positive dimension is a complex torus $W/\Gamma$ where $W$ is a $\mathbb{C}$-vector space of finite positive dimension and $\Gamma \subset W$ is a discrete additive group of maximal rank $2\dim_{\mathbb{C}}(W)$. In addition, there exists a polarization, i.e., a positive definite Hermitian form

$$H_A: W \times W \to \mathbb{C}$$

such that

$$\text{Im}(H_A(\gamma_1, \gamma_2)) \in \mathbb{Z} \quad \forall \gamma_1, \gamma_2 \in \Gamma.$$

Proof of Theorem 1.8. Every surjective holomorphic homomorphism $\psi: X \to A$ is induced by a certain surjective $\mathbb{C}$-linear map $\overline{\psi}: V \to W$ such that $\psi(L) \subset \Gamma$ in the sense that

$$\psi(v + L) = \overline{\psi}(v) + \Gamma \in W/\Gamma = A \quad \forall v + L \in V/L = X.$$ 

Every holomorphic line bundle $\mathcal{M}$ on $A$ is isomorphic to $\mathcal{L}(H_A, \beta)$ for certain AH data $(H_A, \beta)$ where the Hermitian form

$$H_A: W \times W \to \mathbb{C}$$
satisfies the condition
\[ E_A(\gamma_1, \gamma_2) := \text{Im}(H_A(\gamma_1, \gamma_2)) \in \mathbb{Z} \quad \forall \gamma_1, \gamma_2 \in \Gamma \]
and the map \( \beta: \Gamma \to U(1) \) satisfies the condition
\[ \beta(\gamma_1 + \gamma_2) = (-1)^{E_A(\gamma_1, \gamma_2)} \beta(\gamma_2) \beta(\gamma_1) \quad \forall \gamma_1, \gamma_2 \in \Gamma. \]
In addition, it follows from [3, Lemma 2.3.4] that the induced holomorphic line bundle \( \psi^*\mathcal{M} \) on \( X \) is isomorphic to \( \mathcal{L}(H, \alpha_1) \) where
\[ H: V \times V \to \mathbb{C}, \quad H(v_1, v_2) = H_A(\overline{\psi}(v_1), \overline{\psi}(v_2)), \quad \alpha_1: L \to U(1), \quad \alpha_1(l) = \beta(\overline{\psi}(l)). \]
Clearly,
\[ \ker(\overline{\psi}) \subset \ker(H) \subset V. \]
Let us choose a polarization \( H_A \) on \( A \) and consider the Hermitian form
\[ H: V \times V \to \mathbb{C}, \quad H(v_1, v_2) = H_A(\overline{\psi}(v_1), \overline{\psi}(v_2)). \]
Clearly, \( H \neq 0 \), it is positive semidefinite, and for all \( l_1, l_2 \in L \)
\[ \text{Im}(H(l_1, l_2)) = \text{Im}(H_A(\overline{\psi}(l_1), \overline{\psi}(l_2))) \in \mathbb{Z}, \]
because \( \overline{\psi}(l_1), \overline{\psi}(l_2) \in \Gamma \). On the other hand, since \( H_A \) is positive and therefore nondegenerate, \( \ker(H) = \ker(\overline{\psi}) \). This implies that \( \ker(H) \subset \ker(H) \), i.e., \( H \) is dominated by \( H \). It follows from Theorem 9.5 that the group \( \text{Bim}(Y_{\mathcal{L}(H, \alpha)}) \) is not Jordan for any holomorphic line bundle \( \mathcal{L}(H, \alpha) \) where \( \alpha: L \to U(1) \) is any map such that \( (H, \alpha) \) is AH data. On the other hand, every holomorphic line bundle on \( X \) that is isomorphic to \( \mathcal{L}(H, \alpha_1) \otimes \mathcal{F}_0 \) with \( \mathcal{F}_0 \in \text{Pic}^0(X) \) is isomorphic to \( \mathcal{L}(H, \alpha) \) for suitable \( \alpha \). In order to finish the proof, one has only to recall that \( \mathcal{L}(H, \alpha_1) \) is isomorphic to \( \psi^*\mathcal{M} \). \( \square \)

**Proof of Theorem 1.10.** A nonzero complex subtorus \( X_0 \subset X \) and the quotient \( A = X/X_0 \) admit the following description. There exists a nonzero \( \mathbb{C} \)-vector subspace \( U \subset V \) such that \( L_U = L \cap U \) is a lattice of rank \( 2\dim_{\mathbb{C}}(W) \) in \( U \), the quotient \( L/L_U \) is a lattice of rank \( 2\dim_{\mathbb{C}}(V/U) \) in the nonzero \( \mathbb{C} \)-vector space \( W := V/U \), and
\[ X_0 = U/L_U \subset V/L = X, \quad A = (V/U)/(L/L_U) = W/\Gamma \]
where
\[ \Gamma := L/L_U \subset V/U = W. \]
We may assume that \( \mathcal{F} = \mathcal{L}(H, \alpha) \) for certain AH data \( (H, \alpha) \) on \( X \) where \( H \) is a Hermitian form
\[ H: V \times V \to \mathbb{C}, \]
whose imaginary part
\[ E := \text{Im}(H): V \times V \to \mathbb{R} \]
is integer-valued on \( L \times L \). The restriction of \( \mathcal{F} \) to \( X_0 \) lies in \( \text{Pic}^0(X_0) \). It follows from [3, Lemma 2.3.4] that
\[ H(U, U) = \{0\}. \]
This implies that \( H \) induces a biadditive form
\[
S: U \times W (= V/U) \rightarrow \mathbb{C}, \quad S(u, v + U) = H(u, v)
\]
such that
\[
S(\lambda u, w) = \lambda S(u, w), \quad S(u, \lambda w) = \overline{\lambda} S(u, w)
\]
for all \( u \in U \), \( w \in W \) and \( \lambda \in \mathbb{C} \). In addition,
\[
\text{Im}(S(l, \gamma)) \in \mathbb{Z} \quad \forall l \in L_U \subset U, \quad \gamma \in \Gamma \subset W.
\]
(10.1)
Clearly,
\[
S = 0 \quad \text{if and only if} \quad U \subset \ker(H).
\]
(10.2)
Consider the \( \dim_{\mathbb{C}}(W) \)-dimensional \( \mathbb{C} \)-vector space \( \Hom_{\text{antilin}}(W, \mathbb{C}) \) of \( \mathbb{C} \)-antilinear maps
\[
h: W \rightarrow \mathbb{C}, \quad h(w_1 + w_2) = h(w_1) + h(w_2), \quad h(\lambda w) = \overline{\lambda} h(w) \quad \forall w_1, w_2, w \in W, \quad \lambda \in \mathbb{C},
\]
and the lattice
\[
\Gamma_{\text{antilin}} := \{ h \in \Hom_{\text{antilin}}(W, \mathbb{C}) \mid \text{Im}(h(\gamma)) \in \mathbb{Z} \quad \forall \gamma \in \mathbb{Z} \} \subset \Hom_{\text{antilin}}(W, \mathbb{C})
\]
of rank \( 2 \dim_{\mathbb{C}}(W) \). The form \( S \) defines the \( \mathbb{C} \)-linear homomorphism of vector spaces
\[
a_S: U \rightarrow \Hom_{\text{antilin}}(W, \mathbb{C}), \quad u \mapsto \{ w \mapsto S(u, w) \}.
\]
(10.3)
Clearly,
\[
a_S = 0 \quad \text{if and only if} \quad U \subset \ker(H).
\]
(10.4)
In view of (10.1), \( a_U(L_U) \subset \Gamma_{\text{antilin}} \). This implies that \( a_S \) induces a holomorphic homomorphism of complex tori
\[
b_S: U/L_U \rightarrow \Hom_{\text{antilin}}(W, \mathbb{C})/\Gamma_{\text{antilin}}, \quad u + L_U \mapsto a_S(u) + \Gamma_{\text{antilin}}.
\]
Recall that \( U/L_U = X_0 \) and \( W/\Gamma \) is our complex abelian variety \( A \). It is proved in [7, Sect. 3] that \( \Hom_{\text{antilin}}(W, \mathbb{C})/\Gamma_{\text{antilin}} \) is the dual abelian variety \( \hat{A} \) of \( A \). We know that \( \text{Hom}(X_0, A) = \{0\} \).
Since every abelian variety and its dual are isogenous, \( \text{Hom}(X_0, \hat{A}) = \{0\} \) as well. It follows that \( b_S = 0 \). This means that the \( \mathbb{C} \)-vector subspace \( a_S(U) \) lies in the lattice \( \Gamma_{\text{antilin}} \) and therefore \( a_S = 0 \).
By (10.3), \( S = 0 \). Now it follows from (10.2) that \( U \subset \ker(H) \). This implies that there is a Hermitian form \( H_A: W \times W \rightarrow \mathbb{C} \) on \( W = V/U \) such that
\[
H_A(v_1 + U, v_2 + U) = H(v_1, v_2) \quad \forall v_1, v_2 \in V, \quad v_1 + U, v_2 + U \in V/U = W.
\]
(10.5)
We have
\[
\text{Im}(H_A(l_1 + L_U, l_2 + L_U)) = \text{Im}(H(l_1, l_2)) \in \mathbb{Z} \quad \forall l_1, l_2 \in L, \quad l_1 + L_U, l_2 + L_U \in L/L_U = \Gamma.
\]
By [5, Lemma 1.6], there exists a map \( \beta: \Gamma \rightarrow U(1) \) such that \( (H_A, \beta) \) is AH data on \( A \). Let \( \mathcal{L}(H_A, \beta) \) be the corresponding holomorphic line bundle on \( A \). The inverse image \( \psi^* \mathcal{L}(H_A, \beta) \) on \( X \) is a holomorphic line bundle on \( X \) that is isomorphic to some \( \mathcal{L}(H', \alpha') \). It follows from [3, Lemma 2.3.4] that the Hermitian form \( H' \) on \( V \) and the map \( \alpha': L \rightarrow U(1) \) are as follows:
\[
H'(v_1, v_2) = H_A(v_1 + U, v_2 + U) \quad \forall v_1, v_2 \in U, \quad \alpha'(l) = \beta(l + L_U) \quad \forall l \in L.
\]
(10.6)
It follows from (10.5) and (10.4) that \( H' = H \). This means that \( \psi^* \mathcal{L}(H_A, \beta) \) is isomorphic to \( \mathcal{L}(H, \alpha_1) \). Since \( \mathcal{F} = \mathcal{L}(H, \alpha) \), this line bundle is isomorphic to \( \psi^* \mathcal{L}(H_A, \beta) \otimes \mathcal{F}_0 \) where \( \mathcal{F}_0 = \mathcal{L}(0, \alpha/\alpha_1) \in \text{Pic}^0(X) \). Now the desired result follows from Theorem 1.10. \qed
11. Pic\(^0\) AND THETA GROUPS

In this section we revisit theta groups that correspond to the case \(H = 0\). The main idea is to identify the theta group of a line bundle from \(\text{Pic}^0\) and the total space of the bundle with zero section removed (see [13, 15], where the case of abelian varieties was discussed).

Recall that a holomorphic line bundle \(\mathcal{L}\) over \(X\) lies in \(\text{Pic}^0(X)\) if the corresponding Hermitian form \(H\) is zero, i.e., \(\mathcal{L} \cong L(0, \alpha)\). If this is the case, then

\[
\alpha : L \to U(1) \subset \mathbb{C}^*
\]

is a group homomorphism and \(L(0, \alpha)\) is the quotient of the direct product \(V \times \mathbb{C}\) modulo the following action of \(L\):

\[
(v, c) \mapsto (v + l, \alpha(l)c) \quad \forall l \in L, \quad v \in V, \quad c \in \mathbb{C}.
\]

On the other hand, the \(\mathbb{C}^*\)-bundle \(\mathcal{L}(0, \alpha)^*\) over \(X\) obtained from \(\mathcal{L}(0, \alpha)\) by removing the zero section may be viewed as the quotient \((V \times \mathbb{C}^*)/\mathcal{L}\) of the commutative complex Lie group \(V \times \mathbb{C}^*\) by its discrete subgroup

\[
\mathcal{L} := \{(l, \alpha(l)) \mid l \in L\} \subset L \times \mathbb{C}^*.
\]

In particular, \(\mathcal{L}(0, \alpha)^*\) carries the natural structure of a commutative complex Lie group. It is included in the short exact sequence of commutative complex Lie groups

\[
1 \to \mathbb{C}^* \to \mathcal{L}(0, \alpha)^* \to (V/L =) X \to 0.
\]

Notice that the natural faithful action of \(V \times \mathbb{C}^*\) on \(V \times \mathbb{C}\) descends to the faithful action of \(\mathcal{L}(0, \alpha)^*\) on \(\mathcal{L}(0, \alpha)\), so one may view \(\mathcal{L}(0, \alpha)^*\) as a subgroup of \(\text{Aut}(\mathcal{L}(0, \alpha))\).

**Remark 11.1.** Clearly, \(\mathcal{L}(0, \alpha)^* \subset \text{S}(\mathcal{L}(0, \alpha)) \subset \text{Aut}(\mathcal{L}(0, \alpha))\) and for every \(c \in \mathbb{C}^*\)

\[
(0, c)\mathcal{L} \in \mathcal{L}(0, \alpha)^* \subset \text{S}(\mathcal{L}(0, \alpha)) \subset \text{Aut}(\mathcal{L}(0, \alpha))
\]

acts as multiplication by \(c\) in all fibers of \(\mathcal{L}(0, \alpha) \to X\).

**Theorem 11.2.** \(\mathcal{L}(\alpha, 0)^* = \text{S}(\mathcal{L}(0, \alpha))\). In particular, \(\text{S}(\mathcal{L}(0, \alpha))\) is commutative.

**Proof.** Let \(u \in \text{S}(\mathcal{L}(0, \alpha))\). Then there is \(y \in X\) such that \(u\) is a lifting of \(T_y\). Choose \(\tilde{y} \in \mathcal{L}(0, \alpha)^*\) that lifts \(T_y\) as well. For example, take \(v \in V\) such that \(y = V + L\) and put

\[
\tilde{y} = (v, 1)\tilde{L} \in (V \times \mathbb{C}^*)/\tilde{L}.
\]

Then \(u\tilde{y}^{-1}\) is an automorphism of \(\mathcal{L}(0, \alpha)\) that sends every fiber of \(\mathcal{L}(0, \alpha) \to X\) into itself and acts on every such fiber as a \(\mathbb{C}\)-linear automorphism. This means that there is a nowhere vanishing holomorphic function \(f\) on \(X\) such that \(u\tilde{y}^{-1}\) acts on the fiber \(\mathcal{L}(0, \alpha)_z\) as the multiplication by \(f(z) \in \mathbb{C}^*\) for all \(z \in X\). Since \(X\) is compact and connected, there is \(c \in \mathbb{C}^*\) such that \(f(z) = c\) for all \(z \in X\). It follows from Remark 11.1 that \(u\tilde{y}^{-1} \in \mathcal{L}(0, \alpha)^*\). Since \(\tilde{y} \in \mathcal{L}(0, \alpha)^*\), we have \(u = (u\tilde{y}^{-1})\tilde{y} \in \mathcal{L}(0, \alpha)^*\). This completes the proof.  \(\square\)

**ACKNOWLEDGMENTS**

This paper is a result of an attempt to answer questions of Constantin Shramov. I am grateful to him for interesting stimulating questions and discussions. My special thanks go to Vladimir Popov, whose thoughtful comments helped to improve the exposition.

Part of this work was done in May–July 2018 when I visited the Max Planck Institut für Mathematik (Bonn, Germany), whose hospitality and support are gratefully acknowledged.
FUNDING

The work was supported in part by the Simons Foundation Collaboration Grant no. 585711.

REFERENCES

1. T. Bandman and Yu. G. Zarhin, “Jordan groups, conic bundles and abelian varieties,” Algebr. Geom. 4 (2), 229–246 (2017).
2. Ch. Birkenhake and H. Lange, Complex Tori (Birkhäuser, Boston, 1999), Prog. Math. 177.
3. Ch. Birkenhake and H. Lange, Complex Abelian Varieties, 2nd ed. (Springer, Berlin, 2004), Grundl. Math. Wiss. 302.
4. N. Bourbaki, Algebra I. Chapters 1–3 (Springer, Berlin, 1989), Elements of Mathematics.
5. G. R. Kempf, Complex Abelian Varieties and Theta Functions (Berlin, Springer, 1991).
6. Sh. Meng and D.-Q. Zhang, “Jordan property for non-linear algebraic groups and projective varieties,” Am. J. Math. 140 (4), 1133–1145 (2018).
7. D. Mumford, Abelian Varieties, 2nd ed. (Oxford Univ. Press, London, 1974).
8. V. L. Popov, “On the Makar-Limanov, Derksen invariants, and finite automorphism groups of algebraic varieties,” in Affine Algebraic Geometry: The Russell Festschrift (Am. Math. Soc., Providence, RI, 2011), CRM Proc. Lect. Notes 54, pp. 289–311.
9. V. L. Popov, “The Jordan property for Lie groups and automorphism groups of complex spaces,” Math. Notes 103 (5), 811–819 (2018).
10. Yu. Prokhorov and C. Shramov, “Automorphism groups of compact complex surfaces,” arXiv:1708.03566 [math.AG].
11. Yu. Prokhorov and C. Shramov, “Automorphism groups of Inoue and Kodaira surfaces,” arXiv:1812.02400 [math.AG].
12. R. O. Wells Jr., Differential Analysis on Complex Manifolds, 3rd ed. (Springer, New York, 2008), Grundl. Math. Wiss. 65.
13. Yu. G. Zarhin, “Local heights and Néron pairings,” Proc. Steklov Inst. Math. 208, 100–114 (1995) [transl. from Tr. Mat. Inst. Steklova 208, 111–127 (1995)].
14. Yu. G. Zarhin, “Theta groups and products of abelian and rational varieties,” Proc. Edinb. Math. Soc., Ser. 2, 57 (1), 299–304 (2014).
15. Yu. G. Zarhin, “Jordan groups and elliptic ruled surfaces,” Transform. Groups 20 (2), 557–572 (2015).

This article was submitted by the author simultaneously in Russian and English.