Abstract: We introduce the concept of the primitivity of independent set in vertex-transitive graphs, and investigate the relationship between the primitivity and the structure of maximum independent sets in direct products of vertex-transitive graphs. As a consequence of our main results, we positively solve an open problem related to the structure of independent sets in powers of vertex-transitive graphs. © 2010 Wiley Periodicals, Inc. J Graph Theory 67: 218–225, 2011

Keywords: direct product; independent set; primitivity
1. INTRODUCTION

The direct product $G \times H$ of two graphs $G$ and $H$ is defined by

$$V(G \times H) = V(G) \times V(H)$$

and

$$E(G \times H) = \{((u_1, u_2), (v_1, v_2)) : [u_1, v_1] \in E(G) \text{ and } [u_2, v_2] \in E(H)\}.$$  

For a graph $G$, let $G^n = G \times \cdots \times G$ denote the $n$th power of $G$.

It is clear that if $I$ is an independent set of $G$ (or $H$), then $I \times H$ (or $G \times I$) is an independent set of $G \times H$. We say that $G \times H$ is MIS-normal (maximum-independent-set-normal) if each of its maximum independent sets is of this form. Then the independence number

$$\alpha(G \times H) = \max\{\alpha(G)|H|, \alpha(H)|G|\}$$  \hspace{1cm} (1)

if $G \times H$ is MIS-normal. A product $G_1 \times G_2 \times \cdots \times G_n$ is said to be MIS-normal if all of its maximum independent sets are preimages of projections of maximum independent sets of one of its factors.

This poses two immediate problems: whether (1) holds for all graphs $G$ and $H$, and whether $G \times H$ is MIS-normal when (1) holds. In general, however, (1) does not hold for some non-vertex-transitive graphs (see [5]). So, Tardif [8] asked whether (1) holds for all vertex-transitive graphs $G$ and $H$. Larose and Tardif [6] investigated the relationship between the projectivity and the structure of maximal independent sets in powers of a circular graph, Kneser graph, or truncated simplex. Recently, Mario and Vera [7] proved that (1) holds for some special vertex-transitive graphs, e.g., circular graphs and Kneser graphs. In fact, Frankl [4] proved in 1996, 1 year before Tardif’s question was posed, that (1) holds for Kneser graphs. Subsequently, Ahlswede et al. [1] generalized Frankl’s result.

In the context of vertex-transitive graphs, the “No-Homomorphism” lemma of Albertson and Collins [2] is useful to get bounds on the size of independent sets.

**Lemma 1.1** (Albertson and Collins [2]). Let $G$ and $H$ be two graphs such that $G$ is vertex-transitive and there exists a homomorphism $\phi : H \rightarrow G$. Then $\alpha(G)/|V(G)| \leq \alpha(H)/|V(H)|$, and the equality holds if and only if for any independent set $I$ of cardinality $\alpha(G)$ in $G$, $\phi^{-1}(I)$ is an independent set of cardinality $\alpha(H)$ in $H$.

By this lemma, it is easy to deduce that $\alpha(G^n) = \alpha(G)|V(G)|^{n-1}$ for any vertex-transitive graph $G$ and positive integer $n$ (see [6]). So it is natural to ask whether $G^n$ is MIS-normal. Evidently, if $G^n$ is MIS-normal for some $n > 2$, so is $G^2$. Conversely, Larose and Tardif [6] posed the following problem.

**Problem 1.2** (see Larose and Tardif [6]). Let $G$ be a non-bipartite vertex-transitive graph. If $G^2$ is MIS-normal, is the same for all powers of $G$?

This article is organized as follows. In the next section, we introduce a concept of the primitivity of independent sets in a vertex-transitive graph, and prove that the primitivity can be preserved in direct products under certain conditions. Based on these
results, we establish in Section 3 a direct product theorem on the MIS-normality. As a consequence, Problem 1.2 is solved.

2. PRIMITIVITY OF INDEPENDENT SETS

In the sequel of this article, let $G$ and $H$ be vertex-transitive graphs. By $I(G)$ we denote the set of all maximum independent sets of $G$. For any subset $A$ of $V(G)$, let $\alpha(A)$ denote the independence number of the induced subgraph of $G$ by $A$, and we define

$$N_G(A) = \{b \in G : (a, b) \in E(G) \text{ for some } a \in A\},$$

$$N_G[A] = N_G(A) \cup A \text{ and } \overline{N}_G[A] = G - N_G[A].$$

In Lemma 1.1, by taking $H$ as an induced subgraph of $G$ and $\phi$ as the embedding mapping, we obtain the following lemma (cf. [3]).

Lemma 2.1. $\alpha(G) / |V(G)| \leq \alpha(B) / |B|$ holds for all $B \subseteq V(G)$. Equality implies that $|S \cap B| = \alpha(B)$ for every $S \in I(G)$.

A graph $G$ is said to be non-empty if $E(G) \neq \emptyset$. Lemma 2.1 implies that $\alpha(G) \leq |V(G)|/2$ for all non-empty vertex-transitive graphs. Equality holds if and only if $G$ is bipartite, which we state as a corollary for reference.

Corollary 2.2. Let $G$ be a non-empty vertex-transitive graph. Then $\alpha(G) / |G| \leq \frac{1}{2}$, and equality holds if and only if $G$ is bipartite.

Proposition 2.3. Let $A$ be an independent set of $G$. Then $|A|/|N_G[A]| \leq \alpha(G)/|V(G)|$. Equality implies that $|S \cap N_G[A]| = |A|$ for every $S \in I(G)$, and in particularly $A \subseteq S$ for some $S \in I(G)$.

Proof. Since $A$ is an independent set, clearly

$$\frac{|A| + \alpha(N_G[A])}{|N_G[A]| + |\overline{N}_G[A]|} \leq \frac{\alpha(G)}{|V(G)|}.$$

By Lemma 2.1, we see that $\alpha(N_G[A])/|\overline{N}_G[A]| \geq \alpha(G)/|V(G)|$, so $|A|/|N_G[A]| \leq \alpha(G)/|V(G)|$. Equality in the latter implies equality in the former. In this case any $S \in I(G)$ must be the union of a maximum independent set in $\overline{N}_G[A]$ and an independent set of size $|A|$ in $N_G[A]$, and thus $|S \cap N_G[A]| = |A|$.

An independent set $A$ in $G$ is said to be imprimitive if $|A| < \alpha(G)$ and $|A|/|N_G[A]| = \alpha(G)/|V(G)|$. We say that $G$ is IS-imprimitive if $G$ has an imprimitive independent set. In the other case, $G$ is IS-primitive.

Proposition 2.4. Let $A$ be a maximum imprimitive independent set of $G$. Set $B = \overline{N}_G[A]$. Then $\alpha(B)/|B| = \alpha(G)/|V(G)|$ and $\{\sigma(B) : \sigma \in \text{Aut}(G)\}$ forms a non-trivial partition of $V(G)$, i.e., $\sigma(B) \cap B = \emptyset$ or $B$ for each $\sigma \in \text{Aut}(G)$.

Proof. Clearly $(|A| + \alpha(B))/(|N_G[A]| + |B|) \leq \alpha(G)/|V(G)|$. Combining the condition of $A$ and Lemma 2.1, we have $\alpha(B)/|B| = \alpha(G)/|V(G)|$. By definition, $N_G[\sigma(A)] = \sigma(N_G[A])$ and $\alpha(N_G[A])/|\overline{N}_G[A]| \geq \alpha(G)/|V(G)|$, so $|A|/|N_G[A]| \leq \alpha(G)/|V(G)|$. Equality in the latter implies equality in the former. In this case any $S \in I(G)$ must be the union of a maximum independent set in $\overline{N}_G[A]$ and an independent set of size $|A|$. We say that $G$ is IS-imprimitive if $G$ has an imprimitive independent set. In the other case, $G$ is IS-primitive.
By definition we see that

\[ |V(G)| > |N_G[A] \cup \sigma(N_G[A])| > |N_G[A]|. \tag{2} \]

Let \( C = \sigma(A) \cup (A - N_G[\sigma(A)]) \). Then \( C \) is also an independent set and

\[ N_G[C] \subseteq N_G[A] \cup \sigma(N_G[A]). \]

By Proposition 2.3, \(|S \cap N_G[A]| = |A|\) for all \( S \in I(G)\), which implies that \((S - N_G[A]) \cup A \in I(G)\) for all \( S \in I(G)\). Similarly,

\[
((S - N_G[A]) \cup A) - N_G[\sigma(A)] \cup \sigma(A) = (S - N_G[A] \cup N_G[\sigma(A)]) \cup (A - N_G[\sigma(A)]) \cup \sigma(A) \\
= (S - N_G[A] \cup N_G[\sigma(A)]) \cup C
\]

is also a maximum independent set of \( G \), which implies \(|S \cap (N_G[A] \cup N_G[\sigma(A)])| = |C|\) for all \( S \in I(G)\).

Given a \( u \in V(G) \), suppose that there are \( r \) \( S \)'s in \( I(G) \) such that \( u \in S \). Since \( G \) is vertex-transitive, the number \( r \) is independent of the choice of \( u \). Thus \( r|V(G)| = \vartheta(G)|I(G)| \). On the other hand, since \(|S \cap (N_G[A] \cup N_G[\sigma(A)])| = |C|\) for all \( S \in I(G)\), \(|C|/|I(G)| = r|N_G[A] \cup N_G[\sigma(A)]|\). Combining the above two equalities, we have \(|C|/(|N_G[A] \cup N_G[\sigma(A)])| = \vartheta(G)/|V(G)|\). Thus, by Proposition 2.3 we have

\[
\frac{\vartheta(G)}{|V(G)|} \geq \frac{|C|}{N_G[C]} \geq \frac{|C|}{|N_G[A] \cup N_G[\sigma(A)]|} = \frac{\vartheta(G)}{|V(G)|},
\]

which implies \( N_G[C] = N_G[A] \cup N_G[\sigma(A)] \) and \(|C|/|N_G[C]| = \vartheta(G)/|V(G)|\). By (2), we have \(|A| < |C| < \vartheta(G)|\), contradicting the maximality of \(|A|\). This completes the proof.

The concept of primitivity comes from permutation groups: A permutation group \( \Gamma \) acting on a set \( X \) is called primitive if \( \Gamma \) preserves no non-trivial partition of \( X \). In the other case, \( \Gamma \) is imprimitive. As usual (see e.g. [6]), a vertex-transitive graph \( G \) is called primitive if its automorphism group, as a permutation group on \( V(G) \), is primitive. By Proposition 2.4 we see that if \( G \) is primitive, then \( G \) is IS-primitive. But the converse is not true.

For any \( S \subseteq V(G) \times V(H) \), \( a \in G \) and \( u \in H \), define

\[
\hat{c}_G(u,S) = \{ b \in G : (b,u) \in S \}, \quad \hat{c}_H(a,S) = \{ v \in H : (a,v) \in S \},
\]

and

\[
\hat{c}_G(S) = \{ b \in G : \hat{c}_H(b,S) \neq \emptyset \}, \quad \hat{c}_H(S) = \{ v \in H : \hat{c}_G(v,S) \neq \emptyset \}.
\]

By definition we see that \( \hat{c}_G(S) \) and \( \hat{c}_H(S) \) are in fact the projections of \( S \) on \( G \) and \( H \), respectively.

**Lemma 2.5.** Let \( G \) and \( H \) be two vertex-transitive graphs. If \( G \times H \) is MIS-normal and IS-imprimitive, then one of the following two possible cases holds: (i) \( \vartheta(H)/|H| = \vartheta(G)/|G| \), and one of them is IS-imprimitive or both \( G \) and \( H \) are bipartite; (ii) \( \vartheta(H)/|H| < \vartheta(G)/|G| \), and \( G \) is IS-imprimitive.

*Journal of Graph Theory* DOI 10.1002/jgt
Proof. Throughout this proof, we denote \( N_{G \times H}[A] \) by \( N[A] \) for brevity. Suppose that \( G \times H \) is IS-imprimitive and let \( A \) be a maximum primitive independent set of \( G \times H \). Without loss of generality, we assume that \( \alpha(H)/|V(H)| \leq \alpha(G)/|V(G)| \), then \( \alpha(G \times H) = \alpha(G)|V(H)| \). And thus \( |A|/|N[A]| = \alpha(G \times H)/|V(G \times H)| = \alpha(G)/|V(G)| \). If \( E(G) = \emptyset \), the result is trivial; so we suppose \( E(G) \neq \emptyset \). Then Corollary 2.2 implies that \( \alpha(H)/|V(H)| \leq \alpha(G)/|V(G)| \leq 1/2 \). By Proposition 2.3, there exists some \( S \in I(G \times H) \) such that \( A = S \cap N[A] \). Since \( G \times H \) is MIS-normal, there exists some \( S' \in I(G) \) such that \( A = S' \times H \). And thus \( A = (S' \times H) \cap N[A] \). Set \( B = \overline{N}_G[A] \). Then, by Proposition 2.4, \( \sigma(B) \cap B = \emptyset \) or \( B \) for every \( \sigma \in \text{Aut}(G \times H) \).

Set \( C = \hat{\varepsilon}_G(B) \). For every pair \( a \) and \( b \) of \( C \), select \( u \in \hat{\varepsilon}_H(a,B) \) and \( v \in \hat{\varepsilon}_H(b,B) \). Since \( G \) and \( H \) are vertex-transitive, there exist \( \gamma \in \text{Aut}(G) \) and \( \tau \in \text{Aut}(H) \) such that \( a = \gamma(b) \) and \( u = \tau(v) \). It is clear that \( \sigma = (\gamma, \tau) \in \text{Aut}(G \times H) \) and \( (a,u) = (b,v) \in \sigma(B) \cap B \).

By Proposition 2.4, we conclude that \( \sigma(B) = B \). Thus, we have \( \hat{\varepsilon}_H(a,B) = \tau(\hat{\varepsilon}_H(b,B)) \).

Therefore, \( |\hat{\varepsilon}_H(a,B)| = |\hat{\varepsilon}_H(b,B)| \) for any \( a,b \in C \). In the following, we will complete the proof by two cases.

Case 1. \( C \neq V(G) \). Set \( \overline{C} = (V(G) - C) \). Then \( (\overline{C} \times H) \cap B = \emptyset \), and thus \( \overline{C} \times H \subseteq N[A] \). For every \( S'' \in I(G) \), it is clear that \( (S'' \times H) \) is a maximum independent set of \( G \times H \). Since \( \alpha(H)/|V(H)| = \alpha(G)/|V(G)| \) and \( |\hat{\varepsilon}_H(a,B)| \) for all \( a,b \in \hat{\varepsilon}_G(B) \), from Lemma 2.1 and the MIS-normality of \( G \times H \) it follows that

\[
\frac{|S'' \times H| \cap B}{|B|} = \frac{|S'' \cap C|}{|C|} = \frac{\alpha(G)}{|V(G)|}.
\]

Thus for every \( S'' \in I(G) \),

\[
\frac{\alpha(G)}{|V(G)|} \geq \frac{|S''|}{|V(G)|} = \frac{|S'' \cap C| + |S'' \cap \overline{C}|}{|C| + |\overline{C}|} = \frac{|S'' \cap C|}{|C|}.
\]

Recall that \( \overline{C} \times H \subseteq N[A] \) and \( A \subseteq S' \times H \), it is easy to see that \( A = N[A] \cap (S' \times H) \) and \( \hat{\varepsilon}_G(A \cap (\overline{C} \times H)) = S' \cap \overline{C} \). Setting \( F = S' \cap \overline{C} \), we have that \( a \times H \subseteq A \) for every \( a \in F \). If \( N_G[F] \cap C \neq \emptyset \), then there exist \( a \in F \) and \( b \in C \) such that \( (a,b) \in E(G) \). Since \( B = \overline{N}_G[A] \) and \( a \times H \subseteq A \), by definition, \( (b,u) \in \overline{N}[a \times H] \) for every \( u \in \hat{\varepsilon}_H(b,B) \). Hence \( N_H[H] \neq \emptyset \) and \( E(H) = \emptyset \), which contradicts that \( \alpha(H)/|H| \leq 1/2 \). Thus \( N_G[F] \cap C = \emptyset \), i.e., \( N_G[F] \subseteq \overline{C} \).

By Proposition 2.3 and (3),

\[
\frac{\alpha(G)}{|V(G)|} \geq \frac{|F|}{|N_G[F]|} = \frac{|S'' \cap \overline{C}|}{|\overline{C}|} = \frac{\alpha(G)}{|V(G)|}.
\]

Therefore \( |F|/|N_G[F]| = \alpha(G)/|V(G)| \); so \( G \) is IS-imprimitive. In this case, it is clear that \( G \times H \) is not MIS-normal if \( \alpha(G)/|V(G)| = \alpha(H)/|V(H)| \), so the MIS-normality of \( G \times H \) implies that \( \alpha(G)/|V(G)| > \alpha(H)/|V(H)| \), and hence (ii) holds.

Case 2. \( C = V(G) \). Since \( |\hat{\varepsilon}_H(a,B)| = |\hat{\varepsilon}_H(b,B)| \) for all \( a,b \in V(G) \), we have \( \hat{\varepsilon}_G(N[A]) = V(G) \) and \( |\hat{\varepsilon}_H(a,N[A])| = |\hat{\varepsilon}_H(b,N[A])| < |H| \) for all \( a,b \in V(G) \). Since \( A = (S' \times H) \cap N[A] \), \( \hat{\varepsilon}_H(a,N[A]) \subseteq \hat{\varepsilon}_H(a,S' \times H) \) for all \( a \in \hat{\varepsilon}_G(A) \). Thus \( \hat{\varepsilon}_H(a,A) = \hat{\varepsilon}_H(a,N[A]) \) for all \( a \in \hat{\varepsilon}_G(A) \). Select two vertices \( a \) and \( b \) of \( V(G) \) such that \( a \in \hat{\varepsilon}_G(A) \) and \( (a,b) \in E(G) \). Then, for every \( u \in [V(H) - \hat{\varepsilon}_H(b,N[A])] \) and \( v \in \hat{\varepsilon}_H(a,N[A]) \), it is
clear that \([b, u], (a, v) \notin E(G \times H)\); so \((u, v) \notin E(H)\). This means \(u \not\in N_H(\hat{\varphi}_H(a, N[A]))\), that is,

\[ V(H) - \hat{\varphi}_H(b, N[A]) \subseteq V(H) - N_H(\hat{\varphi}_H(a, N[A])). \quad (4) \]

If \(\hat{\varphi}_H(b, N[A]) = \hat{\varphi}_H(a, N[A])\), it follows from (4) that \(H\) is disconnected, and then \(G \times H\) is not MIS-normal since \(\varpi(G) / |V(G)| \geq \varpi(H) / |V(H)|\), contradicting that \(G \times H\) is MIS-normal. Therefore, \(\hat{\varphi}_H(b, N[A]) \neq \hat{\varphi}_H(a, N[A])\). Set \(D = \hat{\varphi}_H(a, N[A]) - \hat{\varphi}_H(b, N[A])\).

It is easy to check that

\[ 2|D| = |\hat{\varphi}_H(a, N[A]) \cup \hat{\varphi}_H(b, N[A]) - \hat{\varphi}_H(a, N[A]) \cap \hat{\varphi}_H(b, N[A])|. \]

Since \(D \subseteq H - \hat{\varphi}_H(b, N[A])\) and \(D \subseteq \hat{\varphi}_G(a, N[A])\), by (4), we have

\[ D \subseteq V(H) - \hat{\varphi}_H(b, N[A]) \subseteq V(H) - N_H(\hat{\varphi}_H(a, N[A])) \subseteq V(H) - N_H(D). \]

So \(D\) is an independent set of \(H\) and

\[ N_H[D] \subseteq D \cup [\hat{\varphi}_H(b, N[A]) - \hat{\varphi}_H(a, N[A])] \]

\[ = \hat{\varphi}_H(a, N[A]) \cup \hat{\varphi}_H(b, N[A]) - \hat{\varphi}_H(a, N[A]) \cap \hat{\varphi}_H(b, N[A]), \]

which implies that \(\frac{1}{2} \geq \frac{\varpi(H)}{|V(H)|} \geq \frac{|D|}{|N_H[D]|} \geq \frac{1}{2}\). Thus \(\frac{\varpi(G)}{|V(G)|} = \frac{\varpi(H)}{|V(H)|} = \frac{1}{2}\). By Corollary 2.2, \(G\) and \(H\) are both bipartite; so (i) holds and the proof is completed. \(\square\)

**Theorem 2.6.** Let \(G\) and \(H\) be two non-bipartite vertex-transitive graph such that \(\varpi(H) / |V(H)| = \varpi(G) / |V(G)|\). If \(G \times H\) is MIS-normal, then \(G\), \(H\) and \(G \times H\) are all IS-primitive.

**Proof.** First, suppose that \(G\) is IS-imprimitive and let \(A\) be an imprimitive independent set in \(G\). For any \(S \in I(H)\), let \(S' = (\overline{N_G}[A] \times S) \cup (A \times H)\). It is clear that \(S'\) is an independent set of \(G \times H\) and

\[ |S'| = |\overline{N_G}[A]| \varpi(H) + |A||V(H)| = (|\overline{N_G}[A]| + |N_G[A]|) \varpi(H) \]

\[ = |V(G)| \varpi(H) = \varpi(G \times H), \]

i.e., \(S'\) is a maximum independent set of \(G \times H\), contradicting the MIS-normality of \(G\). Therefore, \(G\) is IS-primitive. Similarly, \(H\) is also IS-primitive. By Lemma 2.5, \(G \times H\) is IS-primitive. \(\square\)

### 3. MIS-NORMALITY OF THE PRODUCTS OF GRAPHS

The following theorem is the main result on the MIS-normality of products of vertex-transitive graphs in this article.

**Theorem 3.1.** Let \(G\) and \(H\) be two vertex-transitive graphs. Suppose that there exists an induced subgraph \(G'\) of \(G\) such that \(G' \times H\) is MIS-normal and \(\varpi(G') / |V(G')| = \varpi(G) / |V(G)|\). Then either: (i) \(G \times H\) is MIS-normal, or (ii) \(\varpi(G) / |V(G)| = \varpi(H) / |V(H)|\) and \(G\) is IS-imprimitive, or (iii) \(\varpi(G) / |V(G)| < \varpi(H) / |V(H)|\) and \(G\) is disconnected.
Let $a$ and $G$ be an independent set of $C$. Since $|E| = |V(C)|$, the MIS-normality of $G' \times H$, we have the following inequality:

$$\frac{\alpha(G \times H)}{|V(G)||V(H)|} \leq \frac{\alpha(G') \times H)}{|V(G')||V(H)|} = \max \left\{ \frac{\alpha(G)}{|V(G)|}, \frac{\alpha(H)}{|V(H)|} \right\} \leq \frac{\alpha(G \times H)}{|V(G)||V(H)|},$$

yielding

$$\frac{\alpha(G \times H)}{|V(G)||V(H)|} = \frac{\alpha(G ) \times H)}{|V(G')||V(H)|} = \max \left\{ \frac{\alpha(G)}{|V(G)|}, \frac{\alpha(H)}{|V(H)|} \right\}. \quad (5)$$

For every $\sigma \in \text{Aut}(G)$, it is clear that $\sigma(G') \times H$ is MIS-normal. Let $S$ be a maximum independent set of $G \times H$. By Lemma 2.1 and (5), $S \cap (\sigma(G') \times H)$ is a maximum independent set of $\sigma(G') \times H$. Clearly, for each $a \in \hat{c}(S)$, there is a $\sigma \in \text{Aut}(G)$ such that $a \in \sigma(G')$. We therefore have that $|\hat{c}(a, S)| = |H|$ or $\alpha(H)$ for each $a \in \hat{c}(S)$. In the following, we distinguish three cases to complete the proof.

Case 1. $|\hat{c}(a, S)| = |V(H)|$ for every $a \in \hat{c}(S)$. By (5), we obtain that $|\hat{c}(S)| = \alpha(G)$. Since $E(H) \neq \emptyset$, $\hat{c}(S)$ is an independent set of $G$. This implies that $S = \hat{c}(S) \times H$.

Case 2. $|\hat{c}(a, S)| = \alpha(H)$ for every $a \in \hat{c}(S)$. By (5), we have that $\hat{c}(S) = G$, $\alpha(H) / |V(H)| \geq \alpha(G) / |V(G)|$ and $\hat{c}(a, S)$ is a maximum independent set of $H$ for every $a \in G$. Let $a$ be a fixed vertex of $G$, and set

$$C = \{c \in G : \hat{c}(c, S) = \hat{c}(a, S)\}.$$

If $C = G$, then $S = G \times \hat{c}(a, S)$. If $C \neq G$, then choose $d \in G - C$ and $c \in C$. Since $\hat{c}(c, S) \neq \hat{c}(d, S)$, there exists $u \in \hat{c}(c, S)$ and $v \in \hat{c}(d, S)$ such that $(u, v) \in E(H)$ and $[(c, u), (d, v)] \not\in E(G \times H)$. This implies that $c, d \in E(G)$ and thus $S$ is disconnected.

Case 3: $|\hat{c}(a, S)| = |V(H)|$ and $|\hat{c}(b, S)| = \alpha(H)$ for some $a, b \in \hat{c}(S)$. By (5), $\alpha(H) / |V(H)| = \alpha(G) / |V(G)|$ and $\alpha(G \times H) = \alpha(G) / |V(H)| = \alpha(H) / |V(G)|$. Set

$$C = \{c \in G : |\hat{c}(c, S)| = |V(H)|\} \quad \text{and} \quad D = \{d \in G : |\hat{c}(d, S)| = \alpha(H)\}.$$

Since $E(H) \neq \emptyset$, it is clear that $C$ is an independent set of $G$ and $(c, d) \not\in E(G)$ for every $c \in C$ and $d \in D$. So $N_G(C) \subseteq V(G) - D$. Moreover,

$$|S| = \alpha(H) / |V(G)| = |C||V(H)| + |D|\alpha(H).$$

Thus $|C| / |N_G(C)| \geq |C| / (|V(G)| - |D|) = \alpha(H) / |V(H)| = \alpha(G) / |V(G)|$. By Proposition 2.3, $|C| / |N_G(C)| = \alpha(G) / |V(G)|$, that is, $G$ is IS-imprimitive.

This completes the proof.

The following Corollary solves Problem 1.2 in a bit more general setting.

**Corollary 3.2.** Let $G$ be a vertex-transitive, non-bipartite graph. If $G^2$ is MIS-normal, then $G^n$ is also MIS-normal and IS-primitive for all $n \geq 3$.

**Proof.** We prove by induction on $n$. Since $G^2$ is MIS-normal, by Theorem 2.6, $G$ and $G^2$ are both IS-primitive. Assume that $G^d$ is MIS-normal and IS-primitive for all $d = 2, \ldots, n - 1$. We now prove that $G^n$ is MIS-normal and IS-primitive. Note that

Journal of Graph Theory DOI 10.1002/jgt
Let $G^n = G^2 \times G^{n-2}$. Let $G'$ be some subgraph of $G^2$ that is isomorphic to $G$, for instance, the subgraph induced by the set of vertices $\{(u, u) : u \in V(G)\}$. It is clear that

$$\frac{\chi(G')}{|V(G')|} = \frac{\chi(G)}{|V(G)|} = \frac{\chi(G^2)}{|V(G^2)|},$$

and $G' \times G^{n-2}$ is isomorphic to $G^{n-1}$. Thus by assumption, $G' \times G^{n-2}$ is MIS-normal. By Theorem 3.1 and Theorem 2.6, it is easy to see that $G^n$ is MIS-normal and IS-primitive. This completes the proof.

ACKNOWLEDGMENTS

The author is greatly indebted to the anonymous referees for giving useful comments and suggestions that have considerably improved the manuscript. He is grateful also for many valuable discussions with Professor J. Wang and Professor C. J. Zhou. This article is supported by the National Natural Science Foundation of China (No. 10826084) and Zhejiang Innovation Project (Grant No. T200905).

REFERENCES

[1] R. Ahlswede, H. Aydinian, and L. H. Khachatrian, The intersection theorem for direct products, European J Combin 19 (1998), 649–661.
[2] M. O. Albertson and K. L. Collins, Homomorphisms of 3-chromatic graphs, Discrete Math 54 (1985), 127–132.
[3] P. J. Cameron and C. Y. Ku, Intersecting families of permutations, European J Comb 24 (2003), 881–890.
[4] P. Frankl, An Erdős-Ko-Rado Theorem for direct products, European J Combin 17 (1996), 727–730.
[5] P. K. Jha and S. Klavžar, Independence in direct-product graphs, Ars Combin 50 (1998), 53–60.
[6] B. Larose and C. Tardif, Projectivity and independent sets in powers of graphs, J Graph Theory 40 (2002), 162–171.
[7] V. P. Mario and J. Vera, Independent and coloring properties of direct products of some vertex-transitive graphs, Discrete Math 306 (2006), 2275–2281.
[8] C. Tardif, Graph products and the chromatic difference sequence of vertex-transitive graphs, Discrete Math 185 (1998), 193–200.