The Unharmonic dc SQUID Energy Level Splitting

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Abstract. A DC SQUID with Josephson junctions characterized by nonsinusoidal current-phase relation is being considered as a basis for a phase qubit. It has been shown that the second and third harmonic components in the current-phase relation are able to provide a double-well potential and the energy level splitting. The threshold condition for the double-well formation has been determined taking into account the impact of both harmonics. The splitting of the ground state energy level has been calculated as a function of the harmonic amplitudes for different ratio $s$ of characteristic Josephson energy $E_J$ to the Coulomb energy $E_{Q0}$. It has been shown that the gap value comes to about $7E_{Q0}$ with increase of the ratio $s$. No external field needed, no bias current required and no circular currents are the major advantages of such a qubit.

1. Introduction

One of the promising types of phase qubits [1] is the so-called “silent” qubit based on nonsinusoidal current-phase relation, which is observed in particular for submicron high-$T_c$ grain boundary Josephson junctions [2]. The “silent qubit” considered recently in [3,4] is a dc superconducting quantum interference device (SQUID) with D/D grain boundary junctions (consisting of 2 d-wave superconducting films separated by a grain boundary), which are characterized by a second harmonic in the current-phase relation (CPR). The main advantages of the device are (i) protection of the operating point from the fluctuations of the external fields, already on the classical level, (ii) the absence of a state-dependent spontaneous current in the loop of the qubit, and (iii) the absence of bias magnetic flux [3,4].

At the same time there is experimental evidence that CPR of some D/D junctions contains both the second and third harmonics [5]. Therefore this paper is aimed at the analysis of the role of the third harmonic itself and together with second harmonic.

In general case the considered CPR can be written as follows:

$$I_j(\varphi) = A_j \sin \varphi_j - B_j \sin 2\varphi_j + C_j \sin 3\varphi_j, \ldots, j = 1, 2$$  \hspace{1cm} (1)

According to experimental data for conventional Josephson junctions [5] the amplitudes of the first and second harmonics should have opposite signs, while no restriction for C is known. In the case of so-called $\pi$-junctions amplitude $A$ is negative.

2. Double-well potential

The qubit inductance $L$ must be as small as possible to eliminate any crosstalk between neighboring qubits. In the case of negligibly small normalized inductance $\tilde{I} = 2\pi LL_0/\Phi_0 \ll 1$, the total magnetic flux $\Phi$ through the interferometer loop approaches the applied magnetic flux $\Phi_0$, and Josephson energy $E_J$ of the system is given by the following expression (constant term is omitted):
where $M = M_1 - M_2$, $\mathbf{\bar{s}} = (M_1 + M_2)/2$, and $M = M_e \frac{\sum}{\sqrt{2}}$, and $I_0$ is a normalizing current. The profile of the energy versus phase $\mathbf{\bar{s}}$ is presented in Fig. 1a.

According to (2), when no external flux is applied, the energy potential is always a symmetric function of the phase $\mathbf{\bar{s}}$ regardless of any asymmetry in the junction parameters. $E_J$ can show double-well form, and the local minimum positions $\mathbf{\bar{s}}_r$ and $\mathbf{\bar{s}}_l$ may be found from extremum condition

$$\left( \frac{\partial E_J(\varphi, \theta)}{\partial \theta} \right)_{\theta=0} = 0,$$

(3)

If the second harmonic amplitudes are high enough to fulfill threshold relation (see domain in Fig. 1b)

$$|2(B_1 + B_2)| > |(A_1 + A_2)|, \quad (A_1 + A_2)/2(B_1 + B_2) > 0,$$

(4)

the double-well form takes place with the symmetrically located minima at

$$\theta^{*}_{r,l} = \pm \arccos\left(\frac{A_1 + A_2}{2(B_1 + B_2)}\right) + 2\pi n, n = 0, 1, 2...$$

(5)

In the case of identical CPR of the junctions (junction areas may be different) the energy potential remains symmetric regardless of an applied magnetic field. However, if the CPRs do not coincide, the applied magnetic field always breaks the symmetry of the potential. In the extreme case of the junction area difference ($a_1 \gg a_2$) corresponding to proceeding to the one-junction interferometer, the energy potential becomes always symmetric.

Coexistence of the second and third harmonics results in a new threshold condition for the double-well formation instead of condition (4):

$$-4(C_1 + C_2) > (B_1 + B_2) + D^{ij} > 4(C_1 + C_2), \quad \text{or} \quad -4(C_1 + C_2) > (B_1 + B_2) - D^{ij} > 4(C_1 + C_2),$$

(6b)

where $D = (B_1 + B_2)^2 + 4(C_1 + C_2)^2 - 4(B_1 + B_2)(A_1 + A_2), D > 0$.

One should emphasize that only third harmonic existence is also able to provide the double-well form, if the following condition is fulfilled (see domain in Fig. 1b)

$$-3 < (A_1 + A_2) / (C_1 + C_2) < 1.$$

(7)

3. Energy level splitting

If the characteristic Josephson energy $E_C = \Phi_0 I_0 / 2\pi$ is much more than the characteristic Coulomb energy $E_Q = e^2/(2C)$ (C is the junction capacitances), Josephson-junction phases $\phi_1$ and $\phi_2$ (or phases $\theta$ and $\varphi$) are well-defined generalized coordinates of the system under consideration. This system is characterized by the following Hamiltonian:

$$H = -E_Q \frac{\partial^2}{\partial \phi_1^2} - E_Q \frac{\partial^2}{\partial \phi_2^2} + E_J(\phi_1, \phi_2),$$

(8)

where $E_J(\phi_1, \phi_2)$ is the Josephson energy (2). If no external magnetic flux is applied ($\varphi_e = 0$), the Schrödinger equation can be read in terms of $\theta$ and $\varphi$ as follows:

![Fig. 1. (a) Equivalent scheme of the “silent” qubit proposed and its potential profiles at $A_1 = -C_1 = I_0, B_1 = B_2 = 0, A_2 = -C_2 = -0.8 I_0$. (b) Domains of the double-well potential existence for the unharmonic SQUID at the presence of the only second harmonic or of the only third harmonic in the current phase relation of the junctions.](image)
where $\Psi$ is the wave function, $\varepsilon = E/E_0$ normalized eigenvalue of Hamiltonian (8) and $\omega_j = E_j/E_0$.

There are two possible approaches to solve the quantum mechanical problem: (i) the use an oscillator model and (ii) the analysis by Mathieu equation.

### 3.2. Oscillator Model.

In the frame of this approach the potential (2) should be approximated by parabolic wells near points of local minima by retaining only $-\text{quadratic terms}$. This allows us to find a discrete set of steady state energy levels $E_i$. Existence of tunneling trough the barrier between two wells leads to splitting of the energy levels. Specifically the ground state level $E_0$ splits into two levels $E^\pm = E_0 \pm \Delta$.

The splitting amplitude $\Delta$ is defined by both the height $V$ and the half-width $a$ of the potential barrier between the wells:

$$\Delta = V \times \exp \left(-2a(E_0)\sqrt{(V-E_0)/E_0}\right).$$

At $C_j = C_2 = 0$, the ground level $E_0$, barrier height $V$ and barrier half-width $a$ are as follows:

$$E_0 = 2E_0\sqrt{s(B_1 + B_2)[1 - (A_1 + A_2)^2/4(B_1 + B_2)^2]} I_0,$$

$$V = \Phi_2(B_1 + B_2) \left[ \frac{A_1 + A_2}{4(B_1 + B_2)^2} - \left( \frac{A_1 + A_2}{B_1 + B_2} \right)^2 + 1 \right], \quad \theta = |\theta^*| = |\theta^*|,$$

where $s = E_0/E_0$ is the ratio of the characteristic Josephson energy to the Coulomb energy. Maximum value of the splitting peaks at the total amplitude value $(B_1 + B_2) = (1,2 \ldots 1,6)I_0$ and becomes $(3,5,7)E_0$. The maximum position depends on ratio $s$, as it is shown in Fig. 2.

If CPR contains only third harmonic $(B_j = 0)$ the barrier height $V$ and the ground level $E_0$ are described by the following formulas:

$$E_0 = \frac{1}{2} E_{0} \sqrt{\frac{2s}{I_0} \left[ (2A_1 + 2A_2 - 18C_1 - 18C_2) \frac{1 - (A_1 + A_2)^2}{4(C_1 + C_2)^2} + 6(C_1 + C_2 - A_1 - A_2) \right]^{1/2}},$$

$$V = \Phi_2 \left[ (A_1 + A_2 - C_1 - C_2) \left( \frac{1}{4} - \frac{A_1 + A_2}{4(C_1 + C_2)} - 1 \right)^{1/2} \frac{4}{3} (C_1 + C_2) \left( \frac{1}{4} - \frac{A_1 + A_2}{4(C_1 + C_2)} \right)^{1/2} \right]^{1/2}, \quad \alpha = \arccos \left( \frac{1}{4} \frac{A_1 + A_2}{4(C_1 + C_2)} \right)^{1/2}.$$

Fig. 2 presents the splitting gap behavior for the two cases when CPR contains either only second harmonic $(C_j = 0)$ or only third harmonic $(B_j = 0)$.
3.2. Mathieu Equation Method

When no external flux is applied ($\phi_\varepsilon = 0$) and the CPR does not contain third harmonic, the Schrödinger equation (8) can be read as follows:

$$\frac{\partial^2 \Psi}{\partial \theta^2} + \left( \frac{s}{2} (A_1 + A_2) \cos \theta - \frac{s}{4} (B_1 + B_2) \cos 2\theta + \frac{\varepsilon}{2} \right) \Psi = 0,$$

(15)

This equation can be reduced to the Mathieu equation in two extreme cases corresponding to negligibly small values of either B-determined term or A-determined term. In the latter case equation (15) reduces to the following Mathieu equation:

$$\frac{\partial^2 \Psi}{\partial \theta^2} + \left( \frac{\varepsilon}{2} - 2q \cos 2\theta \right) \Psi = 0,$$

(16)

where $2q = s(B_1 + B_2)/4$. Using the specified recurrent formulas [6] for this equation, the eigenenergies $\varepsilon_0^\pm$ (splitted ground level) and $\varepsilon_1^\pm$ have been calculated numerically and are presented in Fig. 3. The obtained ground energy level splitting corresponds to the one calculated in the oscillator model at $|B_1 + B_2| >> |A_1 + A_2|$

Fig. 3. Eigenvalues for Mathieu equation (16) versus parameter $2q = s(B_1 + B_2)/4$. The eigenvalues $\varepsilon_0^\pm$, $\varepsilon_1^\pm$ correspond to the split energy levels: the ground level $E_0$ and the first level $E_1$. Energy $\varepsilon$ is normalized by characteristic Coulomb energy $E_{Q0}$ and counted off from maximum value of the potential barrier between wells.

4. Conclusions

Existence of MQT of the superconductive phase $\varphi$ in high-T$_C$ Josephson junctions has been recently demonstrated [2]. Furthermore, it has been reported that the ratio $\alpha$ of the second harmonic amplitude to the first harmonic amplitude in CPR of YBCO junctions ranges from $\alpha \approx 0.1$ at $T = 900$ mK to a saturated value $\alpha \approx 0.7$ below $T = 100$ mK [2]. Therefore there are all reasons to believe that the “silent” phase qubit can be built on the base of such high-T$_C$ superconductors.

There are two possible ways to measure states of the “silent” qubit. Firstly, circular currents in the qubit can appear only in the case of Josephson junctions with different CPR. Thus, applying a small external flux $\phi_\varepsilon$, one can induce small state-dependent currents, and the qubit states can be measured by means of impedance measurement technique [7]. At the same time the qubit remains “silent”, because the noise caused by the flux is of higher order of vanishing than the currents [4]. Secondly, the possible measurement technique to distinguish the no-current states of the qubit is described in [7].

This work was supported in part by CRDF RUP1-1493-MO-05, ISTC Grant 2369 and Russian Grant for Scientific School (contract 02.445.11.7169)

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