Topological Rényi and Entanglement Entropy for a 2d q-deformed $U(N)$ Yang-Mills theory and its Chern-Simons dual

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Abstract:
Rényi and entanglement entropies are constructed for 2d q-deformed topological Yang-Mills theories with gauge group $U(N)$, as well as the dual 3d Chern-Simons (CS) theory on Seifert manifolds. When $q = \exp[2\pi i/(N + K)]$, and $K$ is odd, the topological Rényi entropy and Wilson line observables of the CS theory can be expressed in terms of the modular transformation matrices of the WZW theory, $\hat{U}(N)_{K,N(K+N)}$. If both $K$ and $N$ are odd, there is a level-rank duality of the 2d qYM theory and of the associated CS theory, as well as that of the Rényi and entanglement entropies, and Wilson line observables.
1. Introduction

Entanglement entropy (EE) provides a useful tool for the study of various aspects of quantum field theory, gravitation, AdS/CFT, conformal field theory (CFT) as well as applications to condensed matter physics. The computation of EE in the presence of gauge fields poses certain difficulties, as in general it is not possible to separate the Hilbert space into the product of two Hilbert spaces without violating Gauss’s law on the boundary of a bi-partition. Typical proposals to deal with this issue have been formulated in terms of boundary conditions of a lattice gauge theory [1–23].

However, there are special cases where the EE can be computed in the presence of continuum gauge fields. Among these are

1) The WZW $\hat{SU}(N)_1 = \hat{U}(N)_1/U(1)$ theory as described by $N$ free Dirac fermions coupled to a $U(1)$ constraint gauge field [24–26],

2) Left-right entanglement entropy (LREE) for WZW models on a circle [27–30],

3) 2d non-Abelian pure gauge theory [31], and

4) 2d q-deformed Yang-Mills theory and its Chern-Simons (CS) dual [32–34].

It is the last theory which is the subject of this paper. Hopefully, these examples will give insights for a better understanding of the computation of EE in the presence of gauge fields.
The 2d q-deformed YM theory on a genus g Riemann surface is a topological theory \[32–34\], \[35–40\], \[41–43\] which has a close connection to CS theory on a Seifert manifold. In this paper we consider the U(N) WZW theory defined by

\[
\hat{U}(N)_{K,N(K+N)} = [\hat{SU}(N)_K \times \hat{U}(1)_{N(K+N)}]/\mathbb{Z}_N
\]

which is only well defined for \(K\) odd. The partition function of the 2dq U(N) YM theory on the Riemann surface \(\Sigma_g\) with genus \(g\), is given up to an overall normalization by \[32–34\]

\[
Z_{qYM}[U(N), q, p, \theta] \sim \sum_{\mathcal{R}} (\dim_q \mathcal{R})^{2-2g} q^{-\frac{1}{2}pC_2(\mathcal{R})} e^{i\theta C_1(\mathcal{R})}
\]

where the sum is over all representations \(\mathcal{R}\) of U(N), \(\dim_q \mathcal{R}\) is the q-dimension of \(\mathcal{R}\), and \(C_2(\mathcal{R})\) and \(C_1(\mathcal{R})\) are quadratic and linear Casimir operators appropriate to \(\mathcal{R}\).

Consider the subsystem \(A\) to be the union of \(l\) disjoint intervals, referred to as \(l\) cuts, and the \(n-\) sheeted ramified cover \(\Sigma_n\) of \(\Sigma_g\), with Euler characteristic

\[
\chi(\Sigma_n) = n\chi(\Sigma_g) - 2l(n-1)
\]

where \(\chi(\Sigma_g) = 2 - 2g\) and \(l \geq 1\). Equations \([1.2]\) and \([1.3]\) provide the basic information for computing the Rényi entropy with the replica trick for the case we are considering.

When \(q = \exp[2\pi i/(N + K)]\), and \(K\) and \(p\) both integers, the set of \(U(N)\) representations are described by Young tableaux with no more than \(K\) columns. Moreover, the quantities in \([1.2]\) may then be expressed in terms of the modular transformation matrices of the WZW theory. Finally, when \(\theta = 0\), \(K\) odd and \(p\) an integer, the partition function \([1.2]\) may be expressed as that of a CS theory on the Seifert manifold \(M(g,p)\), where \(p\) is the first Chern class of the circle bundle over \(\Sigma_q\). This last relation gives a dual description of the Rényi and entanglement entropies. These considerations can also be extended to certain Wilson line observables \[32–34\].

Further, when \(N\) and \(K\) are both odd, the qYM theory exhibits \(N \leftrightarrow K\) duality \[32–34\], \[44–49\] provided that \(q = \exp[2\pi i/(N + K)]\) and \(\theta = 0 \mod 2\pi/(K + N)\). Another set of examples with similar themes is the left-right entanglement entropy (LREE) for WZW models on a circle \[27–30\], which is also related to the entropy of a (2+1) topological field theory (TQFT) \[41–43\].

2. Rényi and entanglement entropies

The partition function of the 2d q-deformed YM theory on a genus g Riemann surface \(\Sigma_g\) is given by \([1.2]\)

\[\text{Representations } \mathcal{R} \text{ of } \hat{U}(N)_{K,N(K+N)} \text{ are defined by the}
\]

\[\text{An extensive exposition of the necessary representation theory for this paper is contained in ref. } [32]\]
representation

\[(R, Q) \text{ of } \hat{SU}(N)_K \times \hat{U}(N)_{N(K+N)} \quad (2.1)\]

where

\[Q = r \mod N \quad (2.2)\]

with \(r\) the number of boxes of the Young tableau associated to \(R\). The quadratic Casimir operator

\[C_2(R) = C_2(R, Q) = NQ + T(R) \quad (2.3)\]

and

\[T(R) = \sum_{i=1}^{N} \bar{l}_i (\bar{l}_i - 2i + 1) \quad (2.4)\]

for an extended Young tableau \(R\) with row lengths \(\bar{l}_i\), so that

\[C_1(R) = \sum_{i=1}^{N} \bar{l}_i \quad (2.5)\]

In order to compute the Rényi entropy by means of the replica trick, one replaces the base manifold \(\Sigma_g\) by the \(n\)-sheeted ramified cover \(\Sigma_n\) of \(\Sigma_g\) with Euler characteristic \((1.3)\). The Rényi entropy for integer \(n\) is

\[S_n = \frac{1}{1 - n} \ln \text{tr} \rho_A^n \quad (2.6)\]

where the reduced density matrix \(\rho_A = \text{tr}_{\bar{A}} \rho\), with the trace taken over the complement \(\bar{A}\). Then

\[\text{tr} \rho_A^n = \frac{Z_n}{Z^n} \quad (2.7)\]

and the EE is

\[S_{EE} = - \lim_{n \to 1} \frac{\partial}{\partial n} \frac{Z_n}{Z^n} \quad (2.8)\]

\[= \ln Z - \frac{1}{Z} \lim_{n \to 1} \frac{\partial}{\partial n} Z_n \]

Equations \((1.2), (1.3)\) and \((2.7)\) give

\[\text{tr} \rho_A^n = \frac{\sum_{R} \left[ \left( \text{dim}_q \chi_R \right)^n \chi(\Sigma_g) - 2l(n-1) q^{-\frac{1}{2} \rho C_2(R)} e^{in\theta C_1(R)} \right] \chi(\Sigma_g) \sum_{R} \left( \text{dim}_q \chi_R \right)^n \chi(\Sigma_g) q^{-\frac{1}{2} \rho C_2(R)} e^{in\theta C_1(R)} \right]}{\left( \sum_{R} \chi(\Sigma_g) \sum_{R} \left( \text{dim}_q \chi_R \right)^n \chi(\Sigma_g) q^{-\frac{1}{2} \rho C_2(R)} e^{in\theta C_1(R)} \right)} \quad (2.9)\]
so that
\[ S_{EE} = \ln \left[ \sum_{\mathcal{R}} (\dim_q \mathcal{R})^{\chi(\Sigma_g)} q^{-\frac{1}{2} p C_2(\mathcal{R})} e^{i \theta C_1(\mathcal{R})} \right] \]
\[ - \sum_{\mathcal{R}} \left[ (\dim_q \mathcal{R})^{\chi(\Sigma_g)} q^{-\frac{1}{2} p C_2(\mathcal{R})} e^{i \theta C_1(\mathcal{R})} \ln [(\dim_q \mathcal{R})^{\chi(\Sigma_g)}-2l q^{-\frac{1}{2} p C_2(\mathcal{R})} e^{i \theta C_1(\mathcal{R})}] \right] \]
\[ \sum_{\mathcal{R}} (\dim_q \mathcal{R})^{\chi(\Sigma_g)} q^{-\frac{1}{2} p C_2(\mathcal{R})} e^{i \theta C_1(\mathcal{R})} \]

Equations (2.9) and (2.10) are the central results of this paper. Various specializations of the 2d qYM theory are obtained from (2.9) and (2.10) as we now detail.

When \( q = \exp[2\pi i/(N + K)] \), \( p \in \mathbb{Z} \), \( \theta = \frac{2\pi p}{N+K} \mod \frac{2\pi}{N+K} \), \( K \) odd, the sum (1.2) is now restricted to Young tableaux with no more than \( K \) columns [32–34]. If further \( \theta = 2\pi t/(N + K) \), with \( t \in \mathbb{Z} \), the quantities that appear in (1.2) may be expressed in terms of the modular transformation matrices of \( \hat{U}(N)_{K,N(N+K)} \), namely [32–34]
\[
(\dim_q \mathcal{R}) = (\dim_q R) = \left( \frac{S_{0R}}{S_{00}} \right) \hat{S}_{U(N)_{K}} = \left( \frac{S_{0R}}{S_{00}} \right) \hat{U}(N)_{K,N(N+K)}
\]  
and
\[
q^{\frac{1}{2} C_2(\mathcal{R})} = T_{RR}/T_{00},
\]

while \( C_1(\mathcal{R}) \) is given by (2.3). Then (2.11) and (2.12) are substituted into (2.9) and (2.10) to obtain the corresponding Rényi and entanglement entropies.

Finally, for \( q = \exp[2\pi i/(N + K)] \), and \( \theta = 0 \), the partition function may be expressed in terms of the level-K \( U(N) \) CS theory on the Seifert manifold \( \mathcal{M}_{(g,p)} \) [with Seifert framing]. Then (1.2) reduces to [32–34]
\[
Z_{qYM}[U(N), e^{2\pi i/(N+K)}, p, \theta = 0] = T_{00}^{-p} S_{00}^{2g-2} Z_{CS}[\mathcal{M}_{(g,p)}, U(N), K] \quad (K \ \text{odd})
\]  

where \( \mathcal{M}(g, p) \) is the circle bundle over \( \Sigma_g \) with first Chern class \( p \). One then proceeds to the Rényi and EE by the substitution [31]
\[
2g - 2 = \chi(\Sigma_g) \rightarrow \chi(\Sigma_n) = n\chi(\Sigma_g) - 2l(n - 1)
\]

with \( l \geq 1 \), as in (1.3). The appropriate restrictions of (2.7) – (2.10) provide the Rényi and EE in terms of modular transformation matrices, and the Seifert manifold, as per (2.14). The computations with these substitutions are straightforward, so that we omit the details.

\footnote{In [31] the EE is discussed for 2d non-Abelian pure gauge theory, an area preserving theory. Here we are considering topological gauge theories. We have greatly benefited from the considerations of [31].}
3. The Chern-Simons dual

Since CS theory is a topological field theory, its observables are topological invariants of the three manifold \( \mathcal{M} \) on which the theory is defined. The partition function for the theory depends only on \( \mathcal{M} \), the gauge group \( G \), the CS coupling \( K \), and the framing of the manifold. Among other gauge invariant observables are Wilson lines defined by \( \text{Tr}_R P \exp(\oint A) \) around a closed path in \( \mathcal{M} \), with the trace taken in some representation \( R \) of the \( G \).

Consider CS theory on a Seifert manifold \( \mathcal{M}(g, p) \), which is a circle bundle over \( \Sigma_g \) with first Chern class \( p \), together with the link consisting of \( n-1 \) circles carrying representations \( R_2, \cdots, R_n \), all linked to a single circle carrying representation \( R_1 \). This has the expectation value in Seifert framing [32–34]

\[
W_{R_1 \cdots R_n}[\mathcal{M}(g, p), G, K] = \sum_R (T_{RR})^{-p} S_{0R}^{2-g-2n} \prod_{i=1}^{n} S_{RR_i} \tag{3.1}
\]

where \( T_{RR} \) and \( S_{RR_i} \) are the modular transformations matrices of the \( \hat{g}_K \) WZW model.

It has been shown [32–34] that a certain class of Wilson line observables in 2d qYM theory with \( q = \exp[2\pi i/(N + K)] \), are proportional to the links of CS theory on \( \mathcal{M}(g, p) \). For \( K \) odd these observables of \( U(N) \) qYM theory may be expressed in terms of modular transformation matrices \( \hat{U}(N)_{K,N(N+K)} \). That is [32–34]

\[
W_{R_1 \cdots R_n}[U(N), e^{2\pi i/(N+K)}, p, 0] \sim \sum_R S_{0R}^{2-g-2n} (T_{RR})^{-p} \prod_{i=1}^{n} S_{RR_i} \quad \text{for \( K \) odd} \tag{3.2}
\]

\[
= W_{R_1 \cdots R_n}[\mathcal{M}(g, p), U(N), K] \tag{3.3}
\]

with Seifert framing.

In (3.2) the sum is restricted to \( R \) integrable, i.e. to a Young Tableaux with no more than \( N \) rows and \( K \) columns. Then the link described by (3.1) for the CS theory provides the identity (3.3). The relation of \( R \) to \( R \) is in (2.1–2.5).

Equations (3.2–3.3), combined with (2.14) allows us to compute the additional EE for the link observables described by (3.1).

We adopt the notation of [31] to define the expectation value of an operator \( X \) that is diagonal in \( R \), where

\[
X_R = \langle R | X | R \rangle \tag{3.4}
\]

then

\[
\langle X \rangle = \frac{\sum_R [((\dim_q R)_{\chi(\Sigma_g)} q^{-\frac{1}{2} p C_2(\mathcal{R})} e^{i \theta C_1(\mathcal{R})} X_R)]}{\sum_R (\dim_q R)_{\chi(\Sigma_g)} q^{-\frac{1}{2} p C_2(\mathcal{R})} e^{i \theta C_1(\mathcal{R})}} \tag{3.5}
\]
Specialize this to \((3.2-3.3)\), with \(\theta = 0\), and \(\langle W_{R_1 \cdots R_n} \rangle\) on the 2d qYM side and \(\langle W_{R_1 \cdots R_n} \rangle\) on the CS side. Define, together with \((2.14),(3.2)\) and \((3.3)\)
\[
f(R) = (\dim_q \mathcal{R})^{\chi(\Sigma_g)} q^{-\frac{1}{2}pC_2(\mathcal{R})}
\]
then
\[
\ln \langle W_{R_1 \cdots R_n} \rangle = \ln \left[ \sum_{R} f(R) W_{R_1 \cdots R_n} \right] - \ln \left[ \sum_{R} f(R) \right] \tag{3.7}
\]
\[
\ln \langle W_{R_1 \cdots R_n} \rangle = \ln \left[ \sum_{R} f(R) W_{R_1 \cdots R_n} \right] - \ln \left[ \sum_{R} f(R) \right] \tag{3.8}
\]
where \((3.7)\) and \((3.8)\) are for the 2d qYM side and CS side respectively. Therefore the additional EE dual to the Wilson line observables described by \((3.2-3.3)\) is given by \((3.7-3.8)\)

4. Level-rank duality

Restrict \(N\) and \(K\) both to odd values and \(p\) even, so that both \(\hat{U}(N)_{K,N(N+K)}\) and \(\hat{U}(K)_{N,N(N+K)}\) are well defined. Then there is a level-rank dual map \([32–34]\) of the following: \(\text{Tr} \rho_A, S_{EE}, Z_{qYM}, W_{R_1 \cdots R_n}\) and \(\Delta S_{EE}\). For example,
\[
W_{R_1 \cdots R_n} [\mathcal{M}(g, p), U(N), K] = e^{i\pi p(KN+1)/12} W^{*}_{\tilde{R}_1 \cdots \tilde{R}_n} [\mathcal{M}(g, p), U(N), K] \tag{4.1}
\]
Analogous relations for \((2.9), (2.10), (2.13)\) and \((3.7-3.8)\) are valid for the \(\tilde{Sp}(n)_k\) WZW model, where \(n = \text{rank}Sp(n)\). That is, for \(p\) even
\[
W_{R_1 \cdots R_n} [\mathcal{M}(g, p), Sp(n), k] = e^{i\pi nk/6} W^{*}_{\tilde{R}_1 \cdots \tilde{R}_n} [\mathcal{M}(g, p), Sp(n), k] \tag{4.2}
\]
In equations \((4.1)\) and \((4.2)\) the representations \(\tilde{R}\) is that obtained from \(R\) with the rows and columns of the Young Tableaux interchanged.

5. Summary

In this paper we examined a particular case where the topological Rényi and entanglement entropies can be computed in the presence of continuum gauge fields. In this context we considered the 2d q-deformed \(U(N)\) Yang-Mills theory and its Chern-Simons dual, and particular Wilson loop link configurations.

It remains to be seen whether the example of this paper, and others, will be sufficient to provide insights for the treatment of Rényi and EE in theories with gauge fields without recourse to lattice \([1–23]\) or analogous regulators.

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