Commutators on $\ell_\infty$

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Abstract

The operators on $\ell_\infty$ which are commutators are those not of the form $\lambda I + S$ with $\lambda \neq 0$ and $S$ strictly singular.

1 Introduction

The commutator of two elements $A$ and $B$ in a Banach algebra is given by

$$[A, B] = AB - BA.$$  

A natural problem that arises in the study of derivations on a Banach algebra $\mathcal{A}$ is to classify the commutators in the algebra. Using a result of Wintner([18]), who proved that the identity in a unital Banach algebra is not a commutator, with no effort one can also show that no operator of the form $\lambda I + K$, where $K$ belongs to a norm closed ideal $I(\mathcal{A})$ of $\mathcal{L}(\mathcal{A})$ and $\lambda \neq 0$, is a commutator in the Banach algebra $\mathcal{L}(\mathcal{A})$ of all bounded linear operators on the Banach space $\mathcal{A}$. The latter fact can be easily seen just by observing that the quotient algebra $\mathcal{L}(\mathcal{A})/I(\mathcal{A})$ also satisfies the conditions of Wintner’s theorem.

In 1965 Brown and Pearcy ([5]) made a breakthrough by proving that the only operators on $\ell_2$ that are not commutators are the ones of the form $\lambda I + K$, where $K$ is compact and $\lambda \neq 0$. Their result suggests what the classification on the other classical sequence spaces might be, and, in 1972, Apostol ([8]) proved that every non-commutator on the space $\ell_p$ for $1 < p < \infty$ is of the form $\lambda I + K$, where $K$ is compact and $\lambda \neq 0$. One year later he proved that the same classification holds in the case of $\mathcal{A} = c_0$ ([4]). Apostol proved some partial results on $\ell_1$, but only 30 year later was the same classification proved for $\mathcal{A} = \ell_1$ by the first author ([6]). Note that if $\mathcal{A} = \ell_p$ ($1 \leq p < \infty$) or $\mathcal{A} = c_0$, the ideal of compact operators $K(\mathcal{A})$ is the largest proper ideal in $\mathcal{L}(\mathcal{A})$ ([8], see also [17, Theorem 6.2]). The classification of the commutators on $\ell_p$, $1 \leq p < \infty$, and partial results on other spaces suggest the following

**Conjecture.** Let $\mathcal{A}$ be a Banach space such that $\mathcal{A} \simeq (\sum \mathcal{A})_p$, $1 \leq p \leq \infty$ or $p = 0$ (we say that such a space admits a Pelczynski decomposition). Assume that $\mathcal{L}(\mathcal{A})$ has a largest ideal $\mathcal{M}$. Then every non-commutator on $\mathcal{A}$ has the form $\lambda I + K$, where $K \in \mathcal{M}$ and $\lambda \neq 0$.

In [8] Apostol obtained a partial result regarding the commutators on $\ell_\infty$. He proved that if $T \in \mathcal{L}(\ell_\infty)$ and there exists a sequence of projections $(P_n)_{n=1}^\infty$ on $\ell_\infty$ such that $P_n(\ell_\infty) \simeq \ell_\infty$ for $n = 1, 2, \ldots$ and $\|P_nT\| \to 0$ as $n \to \infty$, then $T$ is a commutator. This condition is clearly satisfied if $T$ is a compact operator, but, as the first author showed in [6], it is also satisfied if $T$ is strictly singular, which is an essential step for proving the conjecture for $\ell_\infty$.

In order to give a positive answer to the conjecture one has to prove

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Every operator \( T \in \mathcal{M} \) is a commutator

If \( T \in \mathcal{L}(X) \) is not of the form \( \lambda I + K \), where \( K \in \mathcal{M} \) and \( \lambda \neq 0 \), then \( T \) is a commutator.

In this paper we will give positive answer to this conjecture for the space \( \ell_\infty \).

## 2 Notation and basic results

For a Banach space \( X \) denote by the \( \mathcal{L}(X) \), \( K(X) \), \( C(X) \) and \( S_X \) the space of all bounded linear operators, the ideal of compact operators, the set of all finite co-dimensional subspaces of \( X \) and the unit sphere of \( X \). By \textit{ideal} we always mean closed, non-zero, proper ideal. A map from a Banach space \( X \) to a Banach space \( Y \) is said to be strictly singular if whenever the restriction of \( T \) to a subspace \( M \) of \( X \) has a continuous inverse, \( M \) is finite dimensional, in the case where \( X = Y \), the set of strictly singular operators forms an ideal which we will denote by \( S(X) \). Recall that for \( X = \ell_p \), \( 1 \leq p < \infty \), \( S(X) = K(X) \) \( [3] \) and on \( \ell_\infty \) the ideals of strictly singular and weakly compact operators coincide \( [1] \) \textit{Theorem 5.5.1} \). A Banach space \( X \) is called \textit{prime} if each infinite-dimensional complemented subspace of \( X \) is isomorphic to \( X \). The spaces \( \ell_p \), \( 1 \leq p \leq \infty \), are all prime (cf. \( [13] \) \textit{Theorem 2.a.3} and \textit{Theorem 2.a.7}). For any two subspaces (possibly not closed) \( X \) and \( Y \) of a Banach space \( Z \) let

\[
\text{d}(X, Y) = \inf \{ \| x - y \| : x \in S_X, y \in Y \}.
\]

A well known consequence of the open mapping theorem is that for any two closed subspaces \( X \) and \( Y \) of \( Z \), if \( X \cap Y = \{ 0 \} \) then \( X + Y \) is a closed subspace of \( Z \) if and only if \( \text{d}(X, Y) > 0 \). Note also that \( \text{d}(X, Y) = 0 \) if and only if \( \text{d}(Y, X) = 0 \). First we prove a proposition that will later allow us to consider translations of an operator \( T \) by a multiple of the identity instead of the operator \( T \) itself.

**Proposition 2.1.** Let \( X \) be a Banach space and \( T \in \mathcal{L}(X) \) be such that there exists a subspace \( Y \subset X \) for which \( T \) is an isomorphism on \( Y \) and \( d(Y, TY) > 0 \). Then for every \( \lambda \in \mathbb{C} \), \( (T - \lambda I)|_Y \) is an isomorphism and \( d(Y, (T - \lambda I)Y) > 0 \).

**Proof.** First, note that the two hypotheses on \( Y \) (that \( T \) is an isomorphism on \( Y \) and \( d(Y, TY) > 0 \)) are together equivalent to the existence of a constant \( c > 0 \) s.t. for all \( y \in S_Y \), \( d(Ty, Y) > c \). To see this, let us first assume that the hypotheses of the theorem are satisfied. Then there exists a constant \( C \) such that \( \| Ty \| \geq C \) for every \( y \in S_Y \). For an arbitrary \( y \in S_Y \), let \( z_y = \frac{Ty}{\| Ty \|} \) and then we clearly have

\[
d(Ty, Y) = \| Ty \| d(z_y, Y) \geq Cd(Ty, Y) =: c > 0.
\]

To show the other direction note that for \( y \in S_Y \), \( 0 < c < d(Ty, Y) = \| Ty \| d(z_y, Y) \leq \| Ty \| d(z_y, Y) \). Taking the infimum over all \( z_y \in S_Y \) in the last inequality, we obtain that \( d(Ty, Y) > 0 \) and hence \( d(Y, TY) > 0 \). On the other hand, for all \( y \in S_Y \) we have

\[
0 < c < d(Ty, Y) \leq \| Ty - \frac{c}{2}y \| \leq \| Ty \| + \frac{c}{2},
\]

hence \( \| Ty \| \geq \frac{c}{2} \), which in turn implies that \( T \) is an isomorphism on \( Y \).

Now it is easy to finish the proof. The condition \( d(Ty, Y) > c \) for all \( y \in S_Y \) is clearly satisfied if we substitute \( T \) with \( T - \lambda I \) since for a fixed \( y \in S_Y \),

\[
d((T - \lambda I)y, Y) = \inf_{z \in Y} \|(T - \lambda I)y - z\| = \inf_{z \in Y} \|Ty - z\| = d(Ty, Y),
\]

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hence \((T - \lambda I)|_Y\) is an isomorphism and \(d(Y, (T - \lambda I)Y) > 0\).

Note the following two simple facts:

- If \(T: \mathcal{X} \to \mathcal{X}\) is a commutator on \(\mathcal{X}\) and \(S: \mathcal{X} \to \mathcal{Y}\) is an onto isomorphism, then \(STS^{-1}\) is a commutator on \(\mathcal{Y}\).
- Let \(T: \mathcal{X} \to \mathcal{X}\) be such that there exists \(X_1 \subset \mathcal{X}\) for which \(T|_{X_1}\) is an isomorphism and \(d(X_1, TX_1) > 0\). If \(S: \mathcal{X} \to \mathcal{Y}\) is an onto isomorphism, then there exists \(Y_1 \subset \mathcal{Y}, Y_1 \simeq X_1\), such that \(STS^{-1}|_{Y_1}\) is an isomorphism and \(d(Y_1, STS^{-1}Y_1) > 0\) (in fact \(Y_1 = SX_1\)). Note also that if \(X_1\) is complemented in \(\mathcal{X}\), then \(Y_1\) is complemented in \(\mathcal{Y}\).

Using the two facts above, sometimes we will replace an operator \(T\) by an operator \(T_1\) which is similar to \(T\) and possibly acts on another Banach space.

If \(\{Y_i\}_{i=0}^\infty\) is a sequence of arbitrary Banach spaces, by \(\left\{\sum_{i=0}^\infty Y_i\right\}_p\) we denote the space of all sequences \(\{y_i\}_{i=0}^\infty\) where \(y_i \in Y_i, i = 0, 1, \ldots\), such that \(\|y_i\|_{Y_i} \in \ell_p\) with the norm \(\|(y_i)\| = \|\|y_i\|_{Y_i}\|_p\) (if \(Y_i \equiv Y\) for every \(i = 0, 1, \ldots\) we will use the notation \(\left\{\sum Y\right\}_p\)). We will only consider the case where all the spaces \(Y_i, i = 0, 1, \ldots\), are uniformly isomorphic to a Banach space \(Y\), that is, there exists a constant \(\lambda > 0\) and sequence of isomorphisms \(\{T_i: Y_i \to Y\}_{i=0}^\infty\) such that \(\|T^{-1}\| = 1\) and \(\|T\| \leq \lambda\). In this case we define an isomorphism \(U: \left\{\sum_{i=0}^\infty Y_i\right\}_p \to \left\{\sum Y\right\}_p\) via \(T_i\) by

\[
U(y_0, y_1, \ldots) = (T_0(y_0), T_1(y_1), \ldots),
\]

and it is easy to see that \(\|U\| \leq \lambda\) and \(\|U^{-1}\| = 1\). Sometimes we will identify the space \(\left\{\sum_{i=0}^\infty Y_i\right\}_p\) with \(\left\{\sum Y\right\}_p\) via the isomorphism \(U\) when there is no ambiguity how the properties of an operator \(T\) on \(\left\{\sum_{i=0}^\infty Y_i\right\}_p\) translate to the properties of the operator \(UTU^{-1}\) on \(\left\{\sum Y\right\}_p\).

For \(y = (y_i) \in \left\{\sum Y\right\}_p, y_i \in Y\), define the following two operators:

\[
R(y) = (0, y_0, y_1, \ldots), \quad L(y) = (y_1, y_2, \ldots).
\]

The operators \(L\) and \(R\) are, respectively, the left and the right shift on the space \(\left\{\sum Y\right\}_p\). Denote by \(P_i, i = 0, 1, \ldots\), the natural, norm one, projection from \(\left\{\sum Y\right\}_p\) onto the \(i\)-th component of \(\left\{\sum Y\right\}_p\), which we denote by \(Y^i\). We should note that if \(Y \simeq \left\{\sum Y\right\}_p\), then some of the results in this paper are similar to results in [6], but initially we do not require this condition, and, in particular, some of the results we prove here have applications to spaces like \(\left\{\sum \ell_q\right\}_p\) for arbitrary \(1 \leq p, q \leq \infty\). Our first proposition shows some basic properties of the left and the right shift as well as the fact that all the powers of \(L\) and \(R\) are uniformly bounded, which will play an important role in the sequel. Since the proof follows immediately from the definitions we will omit it.

Proposition 2.2. Consider the Banach space \(\left\{\sum Y\right\}_p\). We have the following identities

\[
\|L^n\| = 1, \|R^n\| = 1 \text{ for every } n = 1, 2, \ldots
\]

\[
LP_0 = P_0R = 0, \quad LR = I, \quad RL = I - P_0, \quad RP_i = P_{i+1}R, \quad P_iL = LP_{i+1} \text{ for } i \geq 0.
\]

Note that we can define a left and right shift on \(\left\{\sum_{i=0}^\infty Y_i\right\}_p\) by \(\bar{L} = U^{-1}LU\) and \(\bar{R} = U^{-1}RU\), and, using the above proposition, we immediately have \(\|\bar{R}^n\| \leq \lambda\) and \(\|\bar{L}^n\| \leq \lambda\). If there is no ambiguity, we will denote the left and the right shift on \(\left\{\sum_{i=0}^\infty Y_i\right\}_p\) simply by \(L\) and \(R\).
Following the ideas in [3], for $1 \leq p < \infty$ and $p = 0$ define the set

$$\mathcal{A} = \{ T \in (\sum Y)^p : \sum_{n=0}^{\infty} R^nTL^n \text{ is strongly convergent}\},$$

and for $T \in \mathcal{A}$ define

$$T_A = \sum_{n=0}^{\infty} R^nTL^n.$$

Now using the fact that an operator $T$ is a commutator if and only if $T$ is in the range of $D_S$ for some $S$, where $D_S$ is the inner derivation determined by $S$, defined by $D_S(T) = ST - TS$, it is easy to see ([6, Lemma 3]) that if $T \in \mathcal{A}$ then

$$T = D_L(RT_A) = -D_R(T_AL),$$

hence $T$ is a commutator.

### 3 Commutators on $(\sum Y)^p$

The ideas in this section are similar to the ideas in [3], but here we present them from a different point of view, in a more general setting and we also include the case $p = \infty$. The following lemma is a generalization of [3, Lemma 2.8] in the case $p = \infty$ and [6, Corollary 7] in the case $1 \leq p < \infty$ and $p = 0$. The proof presented here follows the ideas of the proof in [6]. Of course, some of the ideas can be traced back to the classic paper of Brown and Pearcy ([5]) and to Apostol's papers [3], [4], and the references therein.

**Lemma 3.1.** Let $T \in \mathcal{L}((\sum Y)^p)$. Then the operators $P_0T$ and $TP_0$ are commutators.

*Proof.* The proof shows that $P_0T$ is in the range of $D_L$ and $TP_0$ is in the range of $D_R$. We will consider two cases depending on $p$.

**Case I : $p = \infty$**

In this case we first observe that the series

$$S_0 = \sum_{n=0}^{\infty} R^nP_0TL^n$$

is pointwise convergent coordinatewise. Indeed, let $x \in (\sum Y)^\infty$ and define $y_n = R^nP_0TL^n x$ for $n = 0, 1, \ldots$. Note that from the definition we immediately have $y_n \in Y^n$ so the sum $\sum_{n=0}^{\infty} y_n$ converges in the product topology on $(\sum Y)^\infty$ to a point in $(\sum Y)^\infty$ since $\|y_n\| \leq \|R^n\| \|P_0\| \|T\| \|L^n\| \|x\| \leq \|T\| \|x\|$.

Secondly, we observe that $S_0$ and $L$ commute. Because $L$ and $R$ are continuous operators on $(\sum Y)^\infty$ with the product topology and $LR = I$, we have

$$S_0Lx = \sum_{n=0}^{\infty} R^nP_0TL^{n+1}x = L\left(\sum_{n=1}^{\infty} R^nP_0TL^n x\right) = L\left(\sum_{n=0}^{\infty} R^nP_0TL^n x\right) - LP_0Tx$$

$$= LS_0x - 0$$

since $LP_0 = 0$. That is, $D_LS_0 = 0$, as desired.
Let us consider the case

\[ P \]

In this case the proof is similar to the proof of [6, Lemma 6 and Corollary 7] and we include it for completeness.

Since

\[ TP_0 = \sum_{n=0}^{\infty} (I - RL)P_0TL^n x \]

1. **Case II:**

The proof of the statement that \( TP_0 \) is a commutator involves a similar modification of the proof of [3, Lemma 2.8]. Again, consider the series

\[ S = \sum_{n=0}^{\infty} P^n TP_0 L^n. \]

This is pointwise convergent coordinatewise and \( SL = LS \) (from the above reasoning applied to the operator \( TP_0 \)), and

\[ D_R(-SL) = -D_R(LS) = -RLS + LSR = -(I - P_0)S + LSR \]

\[ = -S + P_0S + S = P_0TP_0. \]

Now it is easy to see that

\[ D_R(LTP_0 - SL) = RLTP_0 - LTP_0R + P_0TP_0 = (I - P_0)TP_0 + P_0TP_0 = TP_0. \]

**Case II:** \( 1 \leq p < \infty \) or \( p = 0 \)

In this case the proof is similar to the proof of [5, Lemma 6 and Corollary 7] and we include it for completeness. Let us consider the case \( p \geq 1 \) first. For any \( y \in \mathcal{A} \) we have

\[ \left\| \sum_{n=m}^{m+r} R^n P^n TP_j L^n y \right\|_p = \left\| \sum_{n=m}^{m+r} R^n P^n TP_j L^n P_j y \right\|_p = \sum_{n=m}^{m+r} \left\| R^n P^n TP_j L^n P_j y \right\|_p \]

\[ \leq \left\| P^n TP_j \right\|_p \sum_{n=m}^{m+r} \left\| P_j y \right\|_p \leq \left\| P^n TP_j \right\|_p \sum_{n=m}^{\infty} \left\| P_j y \right\|_p. \]

Since \( \sum_{n=m}^{\infty} \left\| P_j y \right\|_p \to 0 \) as \( m \to \infty \) we have that \( \sum_{n=0}^{\infty} R^n P^n TP_j L^n \) is strongly convergent and \( P^n TP_j \in \mathcal{A} \).

For \( p = 0 \) a similar calculation shows

\[ \left\| \sum_{n=m}^{m+r} R^n P^n TP_j L^n y \right\| = \left\| \sum_{n=m}^{m+r} R^n P^n TP_j L^n P_j y \right\| = \max_{m \leq n \leq m+r} \left\| R^n P^n TP_j L^n P_j y \right\| \]

\[ \leq \left\| P^n TP_j \right\| \max_{m \leq n \leq m+r} \left\| P_j y \right\| \]

and since \( \max_{m \leq n \leq m+r} \left\| P_j y \right\| \to 0 \) as \( m \to \infty \) we apply the same argument as in the case \( p \geq 1 \) to obtain \( P^n TP_j \in \mathcal{A} \).

Using \( P^n TP_j \in \mathcal{A} \) for \( i = j = 0 \) and [5] we have

\[ P_0 TP_0 = D_L(R(P_0 TP_0)_{\mathcal{A}L}) = -D_R((P_0 TP_0)_{\mathcal{A}L}). \]

Again, as in [3, Corollary 7], via direct computation we obtain

\[ TP_0 = D_R(LTP_0 - (P_0 TP_0)_{\mathcal{A}L}) \]

\[ P_0 T = D_L(-P_0 TR + R(P_0 TP_0)_{\mathcal{A}L}). \]

\[ \square \]
Now we switch our attention to Banach spaces which in addition satisfy $\mathcal{X} \simeq \left( \sum \mathcal{A} \right)_p$ for some $1 \leq p \leq \infty$ or $p = 0$. Note that the Banach space $\left( \sum Y \right)_p$ satisfies this condition regardless of the space $Y$, hence we will be able to use the results we proved so far in this section. We begin with a definition.

**Definition 3.2.** Let $\mathcal{X}$ be a Banach space such that $\mathcal{X} \simeq \left( \sum \mathcal{A} \right)_p$, $1 \leq p \leq \infty$ or $p = 0$. We say that $\mathcal{D} = \{X_i\}_{i=0}^{\infty}$ is a decomposition of $\mathcal{X}$ if it forms an $\ell_p$ or $c_0$ decomposition of $\mathcal{X}$ into subspaces which are uniformly isomorphic to $\mathcal{X}$; that is, if the following three conditions are satisfied:

- There are uniformly bounded projections $P_i$ on $\mathcal{X}$ with $P_i \mathcal{X} = X_i$ and $P_i P_j = 0$ for $i \neq j$.
- There exists a collection of isomorphisms $\psi_i : X_i \to \mathcal{X}$, $i \in \mathbb{N}$, such that $\|\psi_i^{-1}\| = 1$ and $\lambda = \sup_{i \in \mathbb{N}} \|\psi_i\| < \infty$.
- The formula $Sx = (\psi_i P_i x)$ defines a surjective isomorphism from $\mathcal{X}$ onto $\left( \sum \mathcal{A} \right)_p$.

If $\mathcal{D} = \{X_i\}_{i=0}^{\infty}$ is a decomposition of $\mathcal{X}$ we have $\mathcal{X} \simeq \left( \sum \mathcal{A} \right)_p \simeq \left( \sum_{i=0}^{\infty} X_i \right)_p$, where the second isomorphic relation is via the isomorphism $U$ defined in (1). Using this simple observation we will often identify $\mathcal{X}$ with $\left( \sum_{i=0}^{\infty} X_i \right)_p$. Our next theorem is similar to [6, Theorem 16] and [3, Theorem 4.6], but we state it and prove it in a more general setting and also include the case $p = \infty$.

**Theorem 3.3.** Let $\mathcal{X}$ be a Banach space such that $\mathcal{X} \simeq \left( \sum \mathcal{A} \right)_p$, $1 \leq p \leq \infty$ or $p = 0$. Let $T \in \mathcal{L}(\mathcal{X})$ be such that there exists a subspace $X \subset \mathcal{X}$ such that $\mathcal{X} \simeq \mathcal{X}$, $T|_X$ is an isomorphism, $X + T(X)$ is complemented in $\mathcal{X}$ and $d(X, T(X)) > 0$. Then there exists a decomposition $\mathcal{D}$ of $\mathcal{X}$ such that $T$ is similar to a matrix operator of the form

$$\begin{pmatrix} * & L \\ * & * \end{pmatrix}$$

on $\mathcal{X} \oplus \mathcal{X}$, where $L$ is the left shift associated with $\mathcal{D}$.

**Proof.** Clearly $\mathcal{X} = X \oplus T(X) \oplus Z$ where $Z$ is complemented in $\mathcal{X}$. Note that without loss of generality we can assume that $Z$ is isomorphic to $\mathcal{X}$. Indeed, if this is not the case, let $X = X_1 \oplus X_2$, $X \simeq X_1 \simeq X_2$ and $X_1, X_2$ complemented in $\mathcal{X}$ (hence also complemented in $\mathcal{X}$). Then $d(X_1, T(X_1)) > 0$ and $\mathcal{X} = X_1 \oplus T(X_1) \oplus Z_1$ where $Z_1$ is a complemented subspace of $\mathcal{X}$, which contains the subspace $X_2 \subset \mathcal{X}$, such that $X_2$ is isomorphic to $\mathcal{X}$ and complemented in $Z$. Applying the Pelczyński decomposition technique ([4, Proposition 4]), we conclude that $Z_1$ is isomorphic to $X$. This observation plays an important role and will allow us to construct the decompositions we need during the rest of the proof.

Denote by $I - P$ the projection onto $T(X)$ with kernel $X + Z$. Consider two decompositions $\mathcal{D}_1 = \{X_i\}_{i=0}^{\infty}$, $\mathcal{D}_2 = \{Y_i\}_{i=0}^{\infty}$ of $\mathcal{X}$ such that $T(X) = Y_0 = X_1 \oplus X_2 \oplus \cdots$, $X_0 = Y_1 \oplus Y_2 \oplus \cdots$, $\mathcal{Y} = X$, and $Z = Y_2 \oplus Y_3 \oplus \cdots$. Define a map $S$

$$S \varphi = L_{\mathcal{D}_1} \varphi \oplus L_{\mathcal{D}_2} \varphi, \quad \varphi \in \mathcal{X}$$

from $\mathcal{X}$ to $\mathcal{X} \oplus \mathcal{X}$. The map $S$ is invertible ($S^{-1}(a, b) = R_{\mathcal{D}_1} a + R_{\mathcal{D}_2} b$). Just using the definition of $S$ and the formula for $S^{-1}$ we see that

$$STS^{-1}(a, b) = ST(R_{\mathcal{D}_1} a + R_{\mathcal{D}_2} b) = S(TR_{\mathcal{D}_1} a + TR_{\mathcal{D}_2} b) = (L_{\mathcal{D}_1} TR_{\mathcal{D}_1} a + L_{\mathcal{D}_1} TR_{\mathcal{D}_2} b) \oplus (L_{\mathcal{D}_2} TR_{\mathcal{D}_1} a + L_{\mathcal{D}_2} TR_{\mathcal{D}_2} b)),$$
Let
\[ A = P_{\gamma}TR_{D_2} = (I - P)TR_{D_2} \] (11)
and note that \( A_{\gamma_{\gamma^*}} \equiv A_{(I - P)X} : (I - P)X \to (I - P)X \) is onto and invertible since \( R_{D_2} \) is an isomorphism on \( P_{\gamma}X \) and \( R_{D_2}(P_{\gamma}X) = Y_1 = X \). Here we used the fact that \( P_{\gamma}T \) is an isomorphism on \( X (PX = X) \). Denote by \( T_0 : (I - P)X \to (I - P)X \) the inverse of \( A_{\gamma_{\gamma^*}} \) (note that \( T_0 \) is an automorphism on \( (I - P)X \)) and consider \( G : X \to X \) defined by

\[ G = T_0(I - P) - T_0A. \]

We will show that \( G^{-1} = A + P \). In fact, from the definitions of \( A \) and \( T_0 \) it is clear that

\[ AT_0(I - P) = T_0A(I - P) = I - P, \quad PT_0(I - P) = PA = 0, \quad (I - P)A = A \]

and since \( A \) maps onto \( (I - P)X \) and \( AT_0 = I_{(I - P)X} \) we also have

\[ A - AT_0A = 0. \]

Now using (12) and (13) we compute

\[
\begin{align*}
(A + P)G &= (A + P)(I + T_0(I - P) - T_0A) \\
&= A + AT_0(I - P) - AT_0A + P = I - P + P = I \\
G(A + P) &= (I + T_0(I - P) - T_0A)(A + P) \\
&= A + P + T_0(I - P)A + T_0(I - P)P - T_0AA - T_0AP \\
&= A + P + T_0A - T_0AA - T_0AP \\
&= P + (I - T_0A)A + T_0A(I - P) \\
&= P + (I - T_0A)(I - P)A + (I - P) \\
&= I + ((I - P) - T_0A(I - P))A \\
&= I + (I - P - (I - P))A = I.
\end{align*}
\]

Using a similarity we obtain

\[
\begin{pmatrix}
I & 0 \\
0 & G^{-1}
\end{pmatrix}
\begin{pmatrix}
* & L_{D_1}TR_{D_2} \\
* & *
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
0 & G
\end{pmatrix}
= \begin{pmatrix}
* & L_{D_1}TR_{D_2}G \\
* & *
\end{pmatrix}.
\]

It is clear that we will be done if we show that \( L_{D_1} = L_{D_1}TR_{D_2}G \). In order to do this consider the equation \((A + P)G = I \Leftrightarrow AG + PG = I\). Multiplying both sides of the last equation on the left by \( L_{D_1} \) gives us \( L_{D_1}AG + L_{D_1}PG = L_{D_1} \). Using \( L_{D_1}P = L_{D_1}P_{X_0} = 0 \) we obtain \( L_{D_1}AG = L_{D_1} \). Finally, substituting \( A \) from (11) in the last equation yields

\[ L_{D_1} = L_{D_1}AG = L_{D_1}P_{X_0}TR_{D_2}G = L_{D_1}(I - P_{X_0})TR_{D_2}G = L_{D_1}TR_{D_2}G, \]

which finishes the proof. \( \square \)

The following theorem was proved in \( \text{[3]} \) for \( X = \ell_p, \; 1 < p < \infty \), but inessential modifications give the result in these general settings.
Theorem 3.4. Let $X$ be a Banach space such that $X \simeq (\sum X)_p$. Let $D$ be a decomposition of $X$ and let $L$ be the left shift associated with it. Then the matrix operator

$$
\begin{pmatrix}
T_1 & L \\
T_2 & T_3
\end{pmatrix}
$$

acting on $X \oplus X$ is a commutator.

Proof. Let $D = \{X_i\}$ be the given decomposition. Consider a decomposition $D_1 = \{Y_i\}$ such that $Y_0 = \bigoplus_{i=1}^{\infty} X_i$ and $X_0 = \bigoplus_{i=1}^{\infty} Y_i$. Now there exists an operator $G$ such that $D_{L_D}G = R_{D_1}D_1(T_1 + T_3)$. This can be done using Lemma 3.1, since $R_{D_1}L_{D_1} = I - P_{Y_0} = P_{X_0}$. By making the similarity

$$
\tilde{T} := \begin{pmatrix} I & 0 \\ G & I \end{pmatrix} \begin{pmatrix} T_1 & L \\ T_2 & T_3 \end{pmatrix} \begin{pmatrix} I & 0 \\ -G & I \end{pmatrix} = \begin{pmatrix} T_1 - LG & L \\ * & T_3 + GL \end{pmatrix}
$$

we have $T_1 + T_3 - LG + GL = T_1 + T_3 - D_{L_D}G = T_1 + T_3 - R_{D_1}L_{D_1}(T_1 + T_3) = P_{Y_0}(T_1 + T_3)$. Using Corollary 3.1, again we deduce that $T_1 + T_3 - LG + GL$ is a commutator. Thus by replacing $T$ by $\tilde{T}$ we can assume that $T_1 + T_3$ is a commutator, say $T_1 + T_3 = AB - BA$ and $\|A\| < 1/2$ (this can be done by scaling). Denote by $M_T$ left multiplication by the operator $T$. Then $\|M_{R(D_A)}\| < 1$ where $R$ is the right shift associated with $D$. The operator $T_0 = (M_T - M_{R(D_A)})^{-1} M_R(T_3B - T_2)$ is well defined and it is easy to see that

$$
\begin{pmatrix} A & 0 \\ T_3 & A - L \end{pmatrix} \begin{pmatrix} B & I \\ T_0 & 0 \end{pmatrix} - \begin{pmatrix} B & I \\ T_0 & 0 \end{pmatrix} \begin{pmatrix} A & 0 \\ T_3 & A - L \end{pmatrix} = \begin{pmatrix} T_1 & L \\ T_2 & T_3 \end{pmatrix}.
$$

This finishes the proof. \hfill $\Box$

4 Operators on $\ell_\infty$

Definition 4.1. The left essential spectrum of $T \in \mathcal{L}(X)$ is the set \cite{2} Def 1.1

$$
\sigma_{l.e.}(T) = \{\lambda \in \mathbb{C} : \inf_{x \in S_Y} \|T - \lambda I\| = 0 \text{ for all } Y \subset X \text{ s.t. } \operatorname{codim}(Y) < \infty\}.
$$

Apostol \cite{2} Theorem 1.4] proved that for any $T \in \mathcal{L}(X)$, $\sigma_{l.e.}(T)$ is a closed non-void set. The following lemma is a characterization of the operators not of the form $\lambda I + K$ on the classical Banach sequence spaces. The proof presented here follows Apostol’s ideas \cite{3} Lemma 4.1], but it is presented in a more general way.

Lemma 4.2. Let $X$ be a Banach space isomorphic to $\ell_p$ for $1 \leq p < \infty$ or $c_0$ and let $T \in \mathcal{L}(X)$. Then the following are equivalent

1. $T - \lambda I$ is not a compact operator for any $\lambda \in \mathbb{C}$.
2. There exists an infinite dimensional complemented subspace $Y \subset X$ such that $Y \simeq X$, $T|_Y$ is an isomorphism and $d(Y, T(Y)) > 0$.

Proof. (2) $\implies$ (1)
Assume that $T = \lambda I + K$ for some $\lambda \in \mathbb{C}$ and some $K \in \mathcal{K}(X)$. Clearly $\lambda \neq 0$ since $T|_Y$ is an isomorphism.
Now there exists a sequence \( \{x_i\}_{i=1}^{\infty} \subset S_Y \) such that \( \|Kx_n\| \to 0 \) as \( n \to \infty \). Let \( y_n = T \left( \frac{x_n}{\lambda} \right) \) and note that
\[
\|x_n - y_n\| = \left\| x_n - (\lambda I + K) \left( \frac{x_n}{\lambda} \right) \right\| = \left\| x_n - x_n - K \left( \frac{x_n}{\lambda} \right) \right\| = \frac{\|Kx_n\|}{\lambda} \to 0
\]
as \( n \to \infty \) which contradicts the assumption \( d(Y, T(Y)) > 0 \). Thus \( T - \lambda I \) is not a compact operator for any \( \lambda \in \mathbb{C} \).

(1) \( \implies \) (2)

The proof in this direction follows the ideas of the proof of Lemma 4.1 from [3]. Let \( \lambda \in \sigma_{l.e.}(T) \). Then \( T_1 = T - \lambda I \) is not a compact operator and \( 0 \in \sigma_{l.e.}(T_1) \). Using just the definition of the left essential spectrum, we find a normalized block basis sequence \( \{x_i\}_{i=1}^{\infty} \) of the standard unit vector basis of \( X \) such that \( \|T_1 x_n\| < \frac{1}{2^n} \) for \( n = 1, 2, \ldots \). Thus if we denote \( Z = \text{span} \{x_i : i = 1, 2, \ldots\} \) we have \( Z \simeq X' \) and \( T_1|_Z \) is a compact operator. Let \( I - P \) be a bounded projection from \( X \) onto \( Z \) ([13, Lemma 1]) so that \( T_1(I - P) \) is compact. Now consider the operator \( T_2 = (I - P)T_1P \). We have two possibilities:

**Case I.** Assume that \( T_2 = (I - P)T_1P \) is not a compact operator. Then there exists an infinite dimensional subspace \( Y_1 \subset PX \) on which \( T_2 \) is an isomorphism and hence using ([14] Lemma 2) if necessary, we find a complemented subspace \( Y \subset PX \) such that \( T_2 \) is an isomorphism on \( Y \). By the construction of the operator \( T_2 \) we immediately have \( d(Y, (I - P)T_1P(Y)) > 0 \) and hence \( d(Y, T_1(Y)) > 0 \). Note that since \( X' \) is prime and \( Y \) is complemented in \( X' \), \( Y \simeq X' \) is automatic. Now we are in position to use Proposition [21] to conclude that \( d(Y, T(Y)) > 0 \).

**Case II.** Now we can assume that the operator \( (I - P)T_1P \) is compact. Since \( T_1(I - P) \) is compact and using
\[
T_1 = T_1(I - P) + (I - P)T_1P + PT_1P
\]
we conclude that the operator \( PT_1P \) is not compact. Using \( X' \equiv PX \oplus (I - P)X' \), we identify \( PX \oplus (I - P)X' \) with \( X' \oplus X' \) via an isomorphism \( U \), such that \( U \) maps \( PX \) onto the first copy of \( X \) in the sum \( X' \oplus X' \). Without loss of generality we assume that \( T_1 = \left( \begin{array}{cc} T_{11} & T_{12} \\ T_{21} & T_{22} \end{array} \right) \) is acting on \( X' \oplus X' \). Denote by \( P = \left( \begin{array}{cc} I & 0 \\ 0 & 0 \end{array} \right) \) the projection from \( X' \oplus X' \) onto the first copy of \( X' \). In the new settings, we have that \( T_{11} \) is not compact and \( T_{21}, T_{22} \) and \( T_{12} \) are compact operators. Define the operator \( S \) on \( X' \oplus X' \) in the following way:
\[
\sqrt{2}S = \left( \begin{array}{cc} I & I \\ I & -I \end{array} \right).
\]
Clearly \( S^2 = I \) hence \( S = S^{-1} \). Now consider the operator \( 2(I - P)S^{-1}T_1SP \). A simple calculation shows that
\[
2(I - P)S^{-1}T_1SP = \left( \begin{array}{cc} 0 & 0 \\ T_{11} + T_{12} - T_{21} - T_{22} & 0 \end{array} \right)
\]
hence \( (I - P)S^{-1}T_1SP \) is not compact. Now we can continue as in the previous case to conclude that there exists a complemented subspace \( Y \subset X' \) in the first copy of \( X' \oplus X' \) for which \( d(Y, S^{-1}T_1S(Y)) > 0 \) and hence \( d(SY, T_1(SY)) > 0 \). Again using Proposition [21] we conclude that \( d(Y, T(Y)) > 0 \).}

**Remark 4.3.** We should note that the two conditions in the preceding lemma are equivalent to a third one, which is the same as (2) plus the additional condition that \( Y \oplus T(Y) \) is complemented in \( X' \). This is essentially what was used for proving the complete classification of the commutators on \( \ell_1 \) in [6], and \( \ell_p \), \( 1 < p < \infty \), and \( c_0 \) in [2] and [3]. The last mentioned condition will also play an important role in the proof of the complete classification of the commutators on \( \ell_\infty \), but we should point out that once we have an infinite dimensional subspace \( Y \subset \ell_\infty \) such that \( Y \simeq \ell_\infty \), \( T|_Y \) is an isomorphism and \( d(Y, T(Y)) > 0 \), then \( Y' \) and \( Y \oplus T(Y) \) will be automatically complemented in \( \ell_\infty \).

**Lemma 4.4.** Let \( T \in \mathcal{L}(\ell_\infty) \) and denote by \( I \) the identity operator on \( \ell_\infty \). Then the following are equivalent
(a) For each subspace \( X \subset \ell_\infty \), \( X \simeq c_0 \), there exists a constant \( \lambda_X \) and a compact operator \( K_X : X \to \ell_\infty \) depending on \( X \) such that \( T|_X = \lambda_X I_X + K_X \).
(b) There exists a constant \( \lambda \) such that \( T = \lambda I + S \), where \( S \in \mathcal{S}(\ell_\infty) \).
Proof. Clearly (b) implies (a), since every strictly singular operator from $c_0$ to any Banach space is compact \((\text{[1]} \text{ Theorem 2.4.10)}\). For proving the other direction we will first show that for every two subspaces $X, Y$ such that $X \cong Y \cong c_0$ we have $\lambda_X = \lambda_Y$. We have several cases.

**Case I.** $X \cap Y = \{0\}$, $d(X, Y) > 0$. 
Let $\{x_i\}_{i=1}^\infty$ and $\{y_i\}_{i=1}^\infty$ be bases for $X$ and $Y$, respectively, which are equivalent to the usual unit vector basis of $c_0$. Consider the sequence $\{z_i\}_{i=1}^\infty$ such that $z_{2i} = x_i$, $z_{2i-1} = y_i$ for $i = 1, 2, \ldots$. If we denote $Z = \text{span}\{z_i : i = 1, 2, \ldots\}$, then clearly $Z \cong c_0$, and, using the assumption of the lemma, we have that $T|_Z = \lambda_Z I|_Z + K_Z$. Now using $X \subset Z$ we have that $\lambda_X I|_X + K_X = (\lambda_Z I|_Z + K_Z)|_X$, hence

$$(\lambda_X - \lambda_Z) I|_X = (K_Z)|_X - K_X.$$ 

The last equation is only possible if $\lambda_X = \lambda_Z$ since the identity is never a compact operator on a infinite dimensional subspace. Similarly $\lambda_Y = \lambda_Z$ and hence $\lambda_X = \lambda_Y$.

**Case II.** $X \cap Y = \{0\}$, $d(X, Y) = 0$. 
Again let $\{x_i\}_{i=1}^\infty$ and $\{y_i\}_{i=1}^\infty$ be bases of $X$ and $Y$, respectively, which are equivalent to the usual unit vector basis of $c_0$ and assume also that $\lambda_X \neq \lambda_Y$. There exists a normalized block basis $\{u_i\}_{i=1}^\infty$ of $\{x_i\}_{i=1}^\infty$ and a normalized block basis $\{v_i\}_{i=1}^\infty$ of $\{y_i\}_{i=1}^\infty$ such that $\|u_i - v_i\| < \frac{1}{i}$. Then $\|u_i - v_i\| \to 0 \Rightarrow \|Tu_i - Tv_i\| \to 0 \Rightarrow \|\lambda_X u_i + K_X u_i - \lambda_Y v_i - K_Y v_i\| \to 0$. Since $u_i \to 0$ weakly (as a bounded block basis of the standard unit vector basis of $c_0$) we have $\|K_X u_i\| \to 0$ and using $\|u_i - v_i\| \to 0$ we conclude that

$$\|(\lambda_X - \lambda_Y) v_i - K_Y v_i\| \to 0.$$ 

Then there exists $N \in \mathbb{N}$ such that $\|K_Y v_i\| > \frac{|\lambda_X - \lambda_Y|}{2} \|v_i\|$ for $i > N$, which is impossible because $K_Y$ is a compact operator. Thus, in this case we also have $\lambda_X = \lambda_Y$.

**Case III.** $X \cap Y = Z \neq \{0\}$, $\dim(Z) = \infty$. 
In this case we have $(\lambda_X I|_X + K_X)|_Z = (\lambda_Y I|_Y + K_Y)|_Z$ and, as in the first case, we rewrite the preceding equation in the form

$$(\lambda_X I|_X - \lambda_Y I|_Y)|_Z = (K_Y - K_X)|_Z.$$ 

Again, as in **Case I**, the last equation is only possible if $\lambda_X = \lambda_Y$ since the identity is never a compact operator on an infinite dimensional subspace.

**Case IV.** $X \cap Y = Z \neq \{0\}$, $\dim(Z) < \infty$. 
Let $X = Z \bigoplus X_1$ and $Y = Z \bigoplus Y_1$. Then $X_1 \cap Y_1 = \{0\}$, $X_1 \cong Y_1 \cong c_0$ and we can reduce to one of the previous cases.

Let us denote $S = T - \lambda I$ where $\lambda = \lambda_X$ for arbitrary $X \subset \ell_\infty$, $X \cong c_0$. If $S$ is not a strictly singular operator, then there is a subspace $Z \subset \ell_\infty$, $Z \cong \ell_\infty$ such that $S|_Z$ is an isomorphism \((\text{[10]} \text{ Corollary 1.4)}\), hence we can find $Z_1 \subset Z \subset \ell_\infty$, $Z_1 \cong c_0$, such that $S|_{Z_1}$ is an isomorphism. This contradicts the assumption that $S|_{Z_1}$ is a compact operator.

The following corollary is an immediate consequence of Lemma 4.4.

**Corollary 4.5.** Suppose $T \in \mathcal{L}(\ell_\infty)$ is such that $T - \lambda I \notin \mathcal{S}(\ell_\infty)$ for any $\lambda \in \mathbb{C}$. Then there exist a subspace $X \subset \ell_\infty$, $X \cong c_0$ such that $(T - \lambda I)|_X$ is not a compact operator for any $\lambda \in \mathbb{C}$.

**Theorem 4.6.** Let $T \in \mathcal{L}(\ell_\infty)$ be such that $T - \lambda I \notin \mathcal{S}(\ell_\infty)$ for any $\lambda$. Then there exists a subspace $X \subset \ell_\infty$ such that $X \cong c_0$, $T|_X$ is an isomorphism and $d(X, T(X)) > 0$. 

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Case II. The operator $T_1 = (I - P)TP$ is not compact. Since $T_1$ is a non-compact operator from $X \simeq c_0$ into a Banach space we have that $T_1$ is an isomorphism on some subspace $Y \subset X$, $Y \simeq c_0$ ([11, Theorem 2.4.10]). Clearly, from the form of the operator $T_1$ we have $d(Y, T_1(Y)) = d(Y, (I - P)TP(Y)) > 0$ and hence $d(Y, T(Y)) > 0$.

Case II. If $(I - P)TP$ is compact and $\lambda \in \mathbb{C}$, then $(I - P)TP - \lambda I_{\|} Z = TP - \lambda I_{\|} Z$ is not compact. Now for $T_2 := TP : X \to X$ we apply Lemma [4.2] to conclude that there exists a subspace $Y \subset X$, $Y \simeq c_0$ such that $d(Y, PT(Y)) = d(Y, PT(Y)) > 0$ and hence $d(Y, T(Y)) > 0$. 

The following theorem is an analog of Lemma 4.2 for the space $\ell_\infty$.

**Theorem 4.7.** Let $T \in \mathcal{L}(\ell_\infty)$ be such that $T - \lambda I \notin S(\ell_\infty)$ for any $\lambda \in \mathbb{C}$. Then there exists a subspace $X \subset \ell_\infty$ such that $X \simeq \ell_\infty$, $T|_X$ is an isomorphism and $d(X, T(X)) > 0$.

**Proof.** From Theorem [1.0] we have a subspace $Y \subset \ell_\infty$, $Y \simeq c_0$ such that $T|_Y$ is an isomorphism and $d(Y, T(Y)) > 0$. Let $N_1 = \{3i + k : i = 0, 1, \ldots\}$ for $k = 1, 2, 3$.

There exists an isomorphism $\overline{S} : Y \oplus TY \to c_0(N_1) \oplus c_0(N_2)$ such that $\overline{S}(Y) = c_0(N_1)$ and $\overline{S}(TY) = c_0(N_2)$. Note that the space $Y \oplus TY$ is indeed a closed subspace of $\ell_\infty$ due to the fact that $d(Y, T(Y)) > 0$. Now we use [12, Theorem 3] to extend $\overline{S}$ to an automorphism $S$ on $\ell_\infty$. Let $T_1 = STS^{-1}$ and consider the operator $(P_{N_2}T_1)|_{\ell_\infty(N_1)} : \ell_\infty(N_1) \to \ell_\infty(N_2)$, where $P_{N_2}$ is the natural projection onto $\ell_\infty(N_2)$. Since $T_1(c_0(N_1)) = c_0(N_2)$, by [10, Proposition 1.2] there exists an infinite set $M \subset N_1$ such that $(P_{N_2}T_1)|_{\ell_\infty(M)}$ is an isomorphism. This immediately yields

$$d(\ell_\infty(M), P_{N_2}T_1(\ell_\infty(M))) > 0$$

and hence

$$d(\ell_\infty(M), T_1(\ell_\infty(M))) > 0. \tag{14}$$

Finally, recall that $T_1 = STS^{-1}$, thus

$$d(\ell_\infty(M), STS^{-1}(\ell_\infty(M))) > 0$$

and hence $d(S^{-1}(\ell_\infty(M)), TS^{-1}(\ell_\infty(M))) > 0$. 

Finally, we can prove our main result.

**Theorem 4.8.** An operator $T \in \mathcal{L}(\ell_\infty)$ is a commutator if and only if $T - \lambda I \notin S(\ell_\infty)$ for any $\lambda \neq 0$.

**Proof.** Note first that if $T$ is a commutator, from the remarks we made in the introduction it follows that $T - \lambda I$ cannot be strictly singular for any $\lambda \neq 0$. For proving the other direction we have to consider two cases:

**Case I.** If $T \in S(\ell_\infty)$ ($\lambda = 0$), the statement of the theorem follows from [3, Theorem 23].

**Case II.** If $T - \lambda I \notin S(\ell_\infty)$ for any $\lambda \in \mathbb{C}$, then we apply Theorem 4.7 to get $X \subset \ell_\infty$ such that $X \simeq \ell_\infty$, $T|_X$ an isomorphism and $d(X, TX) > 0$. The subspace $X + TX$ is isomorphic to $\ell_\infty$ and thus is complemented in $\ell_\infty$. Theorem [3,3] now yields that $T$ is similar to an operator of the form \[
\begin{pmatrix}
* & L \\
* & *
\end{pmatrix}
\]. Finally, we apply Theorem [3,4] to complete the proof. 

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5 Remarks and problems

We end this note with some comments and questions that arise from our work. First consider the set

$$\mathcal{M}_X = \{ T \in \mathcal{L}(\mathcal{X}) : I_X \text{ does not factor through } T \}.$$ 

This set comes naturally from our investigation of the commutators on $\ell_p$ for $1 \leq p \leq \infty$. We know ([6, Theorem 18], [3, Theorem 4.8], [4, Theorem 2.6]) that the non-commutators on $\ell_p$, $1 \leq p < \infty$ and $c_0$ have the form $\lambda I + K$ where $K \in \mathcal{M}_X$ and $\lambda \neq 0$, where $\mathcal{M}_X = \mathcal{K}(\ell_p)$ is actually the largest ideal in $\mathcal{L}(\ell_p)$ ([8]), and, in this paper we showed (Theorem 4.8) that the non-commutators on $\ell_\infty$ have the form $\lambda I + S$ where $S \in \mathcal{M}_X$ and $\lambda \neq 0$, where $\mathcal{M}_X = S(\ell_\infty)$. Thus, it is natural to ask the question for which Banach spaces $\mathcal{X}$ is the set $\mathcal{M}_X$ the largest ideal in $\mathcal{L}(\mathcal{X})$? Let us also mention that in addition to the already mentioned spaces, if $\mathcal{X} = L_p(0,1)$, $1 \leq p < \infty$, then $\mathcal{M}_X$ is again the largest ideal in $\mathcal{L}(\mathcal{X})$ (cf. [7] for the case $p = 1$ and [9, Proposition 9.11] for $p > 1$).

First note that the set $\mathcal{M}_X$ is closed under left and right multiplication with operators from $\mathcal{L}(\mathcal{X})$, so the question whether $\mathcal{M}_X$ is an ideal is equivalent to the question whether $\mathcal{M}_X$ is closed under addition. Note also that if $\mathcal{M}_X$ is an ideal then it is automatically the largest ideal in $\mathcal{L}(\mathcal{X})$ and hence closed, so the question we will consider is under what conditions we have

$$\mathcal{M}_X + \mathcal{M}_X \subseteq \mathcal{M}_X. \quad (15)$$

The following proposition gives a sufficient condition for (15) to hold.

**Proposition 5.1.** Let $\mathcal{X}$ be a Banach space such that for every $T \in \mathcal{L}(\mathcal{X})$ we have $T \notin \mathcal{M}_X$ or $I - T \notin \mathcal{M}_X$. Then $\mathcal{M}_X$ is the largest (hence closed) ideal in $\mathcal{L}(\mathcal{X})$.

**Proof.** Let $S, T \in \mathcal{M}_X$ and assume that $S + T \notin \mathcal{M}_X$. By our assumption, there exist two operators $U: \mathcal{X} \to \mathcal{X}$ and $V: \mathcal{X} \to \mathcal{X}$ which make the following diagram commute:

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{S+T} & \mathcal{X} \\
U \downarrow & & \downarrow V \\
\mathcal{X} & \xrightarrow{I} & \mathcal{X}
\end{array}
\]

Denote $W = (S + T)U(\mathcal{X})$ and let $P: \mathcal{X} \to W$ be a projection onto $W$ (we can take $P = (S + T)UV$). Clearly $VP(S+T)U = I$. Now $S, T \in \mathcal{M}_X$ implies $VPSU, VPST \in \mathcal{M}_X$ which is a contradiction since $VPSU + VPTU = I$. 

Let us just mention that the conditions of the proposition above are satisfied for $\mathcal{X} = C([0,1])$ ([11, Proposition 2.1]) hence $\mathcal{M}_X$ is the largest ideal in $\mathcal{L}(C([0,1]))$ as well.

We should point out that there are Banach spaces for which $\mathcal{M}_X$ is not an ideal in $\mathcal{L}(\mathcal{X})$. In the space $\ell_p \oplus \ell_q$, $1 \leq p < q < \infty$, there are exactly two maximal ideals ([15]), namely, the closure of the ideal of the operators that factor through $\ell_p$, which we will denote by $\alpha_p$, and the closure of the ideal of the operators that factor through $\ell_q$, which we will denote by $\alpha_q$. In this particular space, the first author proved a necessary and sufficient condition for an operator to be a commutator:
Theorem 5.2. ([6, Theorem 20]) Let \( P_{\ell_p} \) and \( P_{\ell_q} \) be the natural projections from \( \ell_p \oplus \ell_q \) onto \( \ell_p \) and \( \ell_q \), respectively. Then \( T \) is a commutator if and only of \( P_{\ell_p}TP_{\ell_p} \) and \( P_{\ell_q}TP_{\ell_q} \) are commutators as operators acting on \( \ell_p \) and \( \ell_q \) respectively.

If we denote \( T = (T_{11}, T_{12}, T_{21}, T_{22}) \), the last theorem implies that \( T \) is not a commutator if and only if \( T_{11} \) or \( T_{22} \) is not a commutator as an operator acting on \( \ell_p \) or \( \ell_q \) respectively. Now using the classification of the commutators on \( \ell_p \) for \( 1 \leq p < \infty \) and the results in [13], it is easy to deduce that an operator on \( \ell_p \oplus \ell_q \) is not a commutator if and only if it has the form \( \lambda I + K \) where \( \lambda \neq 0 \) and \( K \in \alpha_p \cup \alpha_q \). We can generalize this fact, but first we need a definition and a Lemma that follows easily from [6, Corollary 21].

**Property P.** We say that a Banach space \( X \) has property \( P \) if \( T \in \mathcal{L}(X) \) is not a commutator if and only if \( T = \lambda I + S \), where \( \lambda \neq 0 \) and \( S \) belongs to some proper ideal of \( \mathcal{L}(X) \).

All the Banach spaces we have considered so far have property \( P \) and our goal now is to show that property \( P \) is closed under taking finite sums under certain conditions imposed on the elements of the sum.

**Lemma 5.3.** Let \( \{X_i\}_{i=1}^n \) be a finite sequence of Banach spaces that have property \( P \). Assume also that all operators \( A : X_i \to X_j \) that factor through \( X_k \) are in the intersection of all maximal ideals in \( \mathcal{L}(X_i) \) for each \( i, j, k = 1, 2, \ldots, n \), \( i \neq j \). Let \( X = X_1 \oplus X_2 \oplus \cdots \oplus X_n \) and let \( P_i \) be the natural projections from \( X \) onto \( X_i \) for \( i = 1, 2, \ldots, n \). Then \( T \in \mathcal{L}(X) \) is a commutator if and only if for each \( 1 \leq i \leq n \), \( P_iTP_i \) is a commutator as an operator acting on \( X_i \).

**Proof.** The proof is by induction and it mimics the proof of [6, Corollary 21]. First consider the case \( n = 2 \). Let \( T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) where \( A : X_1 \to X_1, D : X_2 \to X_2, B : X_2 \to X_1, C : X_1 \to X_2 \). If \( T \) is a commutator, then \( T = [T_1, T_2] \) for some \( T_1, T_2 \in \mathcal{L}(X) \). Write \( T_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} \) for \( i = 1, 2 \). A simple computation shows that

\[
T = \begin{pmatrix} [A_1, A_2] + B_1C_2 - B_2C_1 & A_1B_2 + B_1D_2 - A_2B_1 - B_2D_1 \\ C_1A_2 + D_1C_2 - C_2A_1 - D_2C_1 & [D_1, D_2] + C_1B_2 - C_2B_1 \end{pmatrix}.
\]

From the fact that \( X_1 \) and \( X_2 \) have property \( P \), and the fact that the \( B_1C_2, B_2C_1 \) lie in the intersection of all maximal ideals in \( \mathcal{L}(X_1) \) and \( C_1B_2, C_2B_1 \) lie in the intersection of all maximal ideals in \( \mathcal{L}(X_2) \) we immediately deduce that the diagonal entries in the last representation of \( T \) are commutators. In the preceding argument we used the fact that a perturbation of a commutator on a Banach space \( X \) having property \( P \) by an operator that lies in the intersection of all maximal ideals in \( \mathcal{L}(X) \) is still a commutator. To show this fact assume that \( A \in \mathcal{L}(X) \) is a commutator, \( B \in \mathcal{L}(X) \) lies in the intersection of all maximal ideals in \( \mathcal{L}(X) \) and \( A + B = \lambda I + S \) where \( S \) is an element of some ideal \( M \) in \( \mathcal{L}(X) \). Now using the simple observation that every ideal is contained in some maximal ideal, we conclude that \( S - B \) is contained in a maximal ideal, say \( M \) containing \( M \) hence \( A - \lambda I \in M \), which is a contradiction with the assumption that \( X \) has property \( P \).

For the other direction we apply [6, Lemma 19] which concludes the proof in the case \( n = 2 \). The general case follows from the same considerations as in the case \( n = 2 \) in a obvious way.

Our last corollary shows that property \( P \) is preserved under taking finite sums of Banach spaces having property \( P \) and some additional assumptions as in Lemma 5.3.
Corollary 5.4. Let \( \{X_i\}_{i=1}^n \) be a finite sequence of Banach spaces that have property \( P \). Assume also that all operators \( A: X_i \to X_j \) that factor through \( X_j \) are in the intersection of all maximal ideals in \( \mathcal{L}(X_i) \) for each \( i, j = 1, 2, \ldots, n, \ i \neq j \). Then \( X = X_1 \oplus X_2 \oplus \cdots \oplus X_n \) has property \( P \).

Proof. Assume that \( T \in \mathcal{L}(\mathcal{X}) \) is not a commutator. Using Lemma 5.3, this can happen if and only if \( P_iTP_i \) is not commutator on \( X_i \) for some \( i \in \{1, 2, \ldots, n\} \) and without loss of generality assume that \( i = 1 \). Since \( P_iTP_i \) is not a commutator and \( X_1 \) has property \( P \) then \( P_iTP_i = \lambda I_{X_1} + S \) where \( S \) belongs to some maximal ideal \( J \) of \( \mathcal{L}(X_1) \). Consider

\[
M = \{ T \in \mathcal{L}(\mathcal{X}) : P_iTP_i \in J \}. \tag{16}
\]

Clearly, if \( T \in M \) and \( A \in \mathcal{L}(\mathcal{X}) \), then \( AT, TA \in M \) because of the assumption on the operators from \( X_1 \) to \( X_1 \) that factor through \( X_j \). It is also obvious that \( M \) is closed under addition, hence \( M \) is an ideal. Now it is easy to see that \( T - \lambda I \in M \) which shows that all non-commutators have the form \( \lambda I + S \), where \( \lambda \neq 0 \) and \( S \) belongs to some proper ideal of \( \mathcal{L}(\mathcal{X}) \).

The other direction follows from our comment in the beginning of the introduction that no operator of the form \( \lambda I + S \) can be a commutator for any \( \lambda \neq 0 \) and any operator \( S \) which lies in a proper ideal of \( \mathcal{L}(\mathcal{X}) \).

\[\square\]

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