Longitudinal electric conductivity and dielectric permeability in quantum plasma with constant collision frequency in Mermin’ approach

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Abstract

Detailed deducing of formulas for longitudinal electric conductivity and dielectric permeability in the quantum degenerate collisional plasma with constant collision frequency in Mermin’ approach is given. The kinetic Schrödinger–Boltzmann equation in momentum space in relaxation approximation is used.

It is shown that when collision frequency of plasma particles tends to zero (plasma passes to collisionless one), the deduced formula for dielectric function passes to the known Lindhard’ formula for collisionless plasmas.

It is shown that the deduced formula for dielectric permeability coincides with known Mermin’s formula.

Graphic research of the real and imaginary parts of dielectric function is made.

Graphic comparison of the real and imaginary parts of dielectric function for quantum and classical plasma also is made.

The module of derivative dielectric function also has been investigated graphically.

\textbf{Key words:} Lindhard, Mermin, quantum collisional plasma, conductance, rate equation, density matrix, commutator, degenerate plasma.

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Introduction

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In the known work of Mermin [1] on the basis of the analysis of nonequilibrium density matrix in \( \tau \)-approximation has been obtained expression for longitudinal dielectric permeability of quantum collisional plasmas for case of constant collision frequency of plasmas particles.

Earlier in the work of Klimontovich and Silin [2] and after that in the work of Lindhard [3] has been obtained expression for longitudinal and transverse dielectric permeability of quantum collisionless plasmas.

By Kliewer and Fuchs [4] it has been shown, that direct generalisation of formulas of Lindhard on the case of collisional plasmas is incorrectly. This lack for the longitudinal dielectric permeability has been eliminated in the work of Mermin [1]. Next in the works [5] and [6] has been given attempt to deduce Mermin’s formula.

For collisional plasmas correct formulas longitudinal and transverse electric conductivity and dielectric permeability are received accordingly in works [7] and [8]. In these works Wigner—Vlasov—Boltzmann kinetic equation in relaxation approximation in coordinate space is used. In work [9] the formula for transverse electric conductivity has been deduced for quantum collisional plasmas with use of the kinetic equation by Mermin’ approach (in momentum space).

In the present article formulas for longitudinal electric conductivity and dielectric permeability in the quantum degenerate collisional plasma with constant collisional frequency by Mermin’ approach are received. The kinetic Schrödinger—Boltzmann equation in momentum space in relaxation approximation is used.

It is shown also, that when frequency of collisions of particles of plasma tends to zero (plasma passes to collisionless one), the deduced formula for dielectric function passes to the known Lindhard’ formula for collisionless plasmas. It is shown, that when frequency of collisions is a constant, the deduced formula for dielectric function passes in known Mermin’ formula.

It is shown that the deduced formula for dielectric permeability coincides with known Mermin’s formula.
Last years there is a steady interest to research properties of quantum plasma [10]–[27].

1. Kinetic Schrödinger—Boltzmann equation for density matrix

Let the vector potential of an electromagnetic field is harmonious, i.e. changes as \( \varphi(\mathbf{r}, t) = \varphi(\mathbf{r}) \exp(-i\omega t) \). Relation between scalar potential and intensity of the electric field it is given by expression \( \mathbf{E}(\mathbf{q}) = -\nabla \varphi(\mathbf{q}) \). The equilibrium matrix of density looks like

\[
\tilde{\rho} = \frac{1}{\exp \frac{H - \mu}{k_B T} + 1}, \quad \mu = \mu_0 + \delta \mu.
\]

Here \( T \) is the temperature, \( k_B \) is the Boltzmann constant, \( \mu_0 \) is the chemical potential of plasma in an equilibrium condition, \( \delta \mu \) is the correction to the chemical potential, caused presence of variable electric field, \( H \) is the Hamiltonian.

Hamiltonian looks like here \( H = H_0 + H_1 \), where \( H_0 = \frac{\mathbf{p}^2}{2m} \), \( H_1 = e\varphi \). Here \( m, e \) are mass and charge of electron, \( \mathbf{p} = \hbar \mathbf{k} \) is the electron momentum.

Let’s designate an equilibrium matrix of density in absence of an external field through \( \tilde{\rho}_0 \):

\[
\tilde{\rho}_0 = \frac{1}{\exp \frac{H_0 - \mu_0}{k_B T} + 1}.
\]

Density matrix it is possible to present an equilibrium matrix of density in the form

\[
\tilde{\rho} = \tilde{\rho}_0 + \tilde{\rho}_1.
\]

Here \( \tilde{\rho}_1 \) is the correction to the equilibrium matrix of density, caused by presence of an electromagnetic field.

In linear approximation we receive

\[
[H, \tilde{\rho}] = [H_0, \tilde{\rho}_1] + [H_1, \tilde{\rho}_0],
\]
and

\[ [H, \hat{\rho}] = 0. \]

Here \([H, \hat{\rho}] = H \hat{\rho} - \hat{\rho} H\) is the commutator.

Let’s notice that the vector \(|k\rangle\) is the eigen vector of operators \(H\) and \(p\). Thus

\[ H|k\rangle = \mathcal{E}_k |k\rangle, \quad \langle k|H = \mathcal{E}_k \langle k|, \quad p|k\rangle = \hbar k|k\rangle, \quad \langle k|p = \hbar k \langle k|. \]

Let’s notice that for the operator \(L\) the relationship is carried out

\[ \langle k_1 |L|k_2 \rangle = \frac{1}{(2\pi)^3} \int \exp(-i\mathbf{k}_1 \cdot \mathbf{r}) \exp(i\mathbf{k}_2 \cdot \mathbf{r}) d\mathbf{r}. \]

By means of this relation it is received

\[ \langle k_1 |[[H_0, \tilde{\rho}]], k_2 \rangle = - \langle k_1 |[H_1, \tilde{\rho}_0]|k_2 \rangle. \]

Let’s write down this equality in the explicit form

\[ \langle k_1 |H_0 \tilde{\rho}_1 |k_2 \rangle - \langle k_1 |\tilde{\rho}_1 H_0 |k_2 \rangle = - \langle k_1 |H_1 \tilde{\rho}_0 |k_2 \rangle + \langle k_1 |\tilde{\rho}_0 H_1 |k_2 \rangle. \]

From here we receive that

\[ (\mathcal{E}_{k_1} - \mathcal{E}_{k_2}) \langle k_1 |\tilde{\rho}_1 |k_2 \rangle = (f_{k_1} - f_{k_2}) \langle k_1 |H_1 |k_2 \rangle = e(f_{k_1} - f_{k_2}) \langle k_1 |\varphi |k_2 \rangle. \]

Here

\[ \mathcal{E}_k = \frac{\hbar^2 k^2}{2m}, \quad f_k = \frac{1}{\exp \left( \frac{E_k - \mu_0}{k_B T} \right) + 1}. \]

Considering that

\[ \langle k_1 |\varphi |k_2 \rangle = \frac{1}{(2\pi)^3} \int \exp(-i(k_1 - k_2) \cdot \mathbf{r}) \varphi(\mathbf{r}) d\mathbf{r} = \varphi(k_1 - k_2), \]

we receive

\[ (\mathcal{E}_{k_1} - \mathcal{E}_{k_2}) \tilde{\rho}_1(k_1, k_2) = e(f_{k_1} - f_{k_2}) \varphi(k_1 - k_2). \]

The kinetic Schrödinger–Boltzmann equation for the density matrix in \(\tau\)-approximation with constant frequency of collisions looks like

\[ i\hbar \frac{\partial \rho}{\partial t} = [H, \rho] + \frac{i\hbar}{\tau} (\tilde{\rho} - \rho), \]
or
\[
\frac{\partial \rho}{\partial t} = -\frac{i}{\hbar} [H, \rho] + \nu (\tilde{\rho} - \rho). \tag{1.1}
\]

Here \(\tau\) is the average time of free electrons path, \(\nu = 1/\tau\) is the frequency of collisions.

In linear approximation the density matrix we will search in the form
\[
\rho = \tilde{\rho}_0 + \rho_1.
\]

Then in linear approach the equation (1.2) looks like
\[
i\hbar \frac{\partial \rho_1}{\partial t} = [H_0, \rho_1] + [H_1, \rho_0] + i\nu \hbar (\tilde{\rho}_1 - \rho_1). \tag{1.2}
\]

Let’s consider, that \(\rho_1 \sim \exp(-i\omega t)\). From here for \(\rho_1\) we receive the relation
\[
\hbar \omega \langle k_1 | \rho_1 | k_2 \rangle = (\mathcal{E}_{k_1} - \mathcal{E}_{k_2}) \langle k_1 | \rho_1 | k_2 \rangle - e(f_{k_1} - f_{k_2}) \varphi(k_1 - k_2) +
+i\nu \hbar \langle k_1 | \tilde{\rho}_1 - \rho_1 | k_2 \rangle. \tag{1.2}
\]

We transform this equation to the form
\[
(\mathcal{E}_{k_2} - \mathcal{E}_{k_1} + \hbar \omega + i\nu \hbar) \langle k_1 | \rho_1 | k_2 \rangle = -e(f_{k_1} - f_{k_2}) \langle k_1 | \varphi | k_2 \rangle +
+i\nu \hbar \langle k_1 | \tilde{\rho}_1 | k_2 \rangle.
\]

From this equation we obtain
\[
\langle k_1 | \rho_1 | k_2 \rangle = \rho_1(k_1, k_2) = -e \frac{f_{k_1} - f_{k_2}}{\mathcal{E}_{k_2} - \mathcal{E}_{k_1} + \hbar \omega + i\nu \hbar} \varphi(k_1 - k_2) +
+i\nu \hbar \tilde{\rho}_1(k_1, k_2)
\tag{1.3}
\]

The relation (1.3) represents the solution of linear Schrödinger—Boltzmann equation, expressed through perturbation to equilibrium matrix of density \(\tilde{\rho}_1(k_1 - k_2) = \langle k_1 | \tilde{\rho}_1 | k_2 \rangle\). Let’s find this perturbation.

Let’s take advantage of the obvious relation
\[
[H - \mu, \tilde{\rho}] = 0.
\]

In linear approximation from here it is received
\[
[H_0 - \mu_0, \tilde{\rho}_1] + [H_1 - \delta \mu, \tilde{\rho}_0] = 0.
\]
Transforming the first commutator from here we receive

\[ [H_0, \tilde{\rho}_1] = \left[ \delta \mu - H_1, \tilde{\rho}_0 \right], \]

or

\[ [H_0, \tilde{\rho}_1] = \left[ \delta \mu - e\varphi, \tilde{\rho}_0 \right]. \]

Let’s designate now

\[ \delta \mu_* = \delta \mu - e\varphi. \]

Then the previous equality write down in the form

\[ [H_0, \tilde{\rho}_1] = \left[ \delta \mu_*, \tilde{\rho}_0 \right]. \]

From here we receive that

\[
\langle k_1 | \tilde{\rho}_1 | k_2 \rangle = -(f_{k_1} - f_{k_2}) \langle k_1 | \delta \mu_* | k_2 \rangle,
\]

from which

\[
\langle k_1 | \tilde{\rho}_1 | k_2 \rangle = -\frac{f_{k_1} - f_{k_2}}{\varepsilon_{k_1} - \varepsilon_{k_2}} \langle k_1 | \delta \mu_* | k_2 \rangle,
\]

or, that all the same,

\[ \tilde{\rho}_1(k_1, k_2) = -\frac{f_{k_1} - f_{k_2}}{\varepsilon_{k_1} - \varepsilon_{k_2}} \delta \mu_*(k_1 - k_2). \]

We have received perturbation to the equilibrium matrix of the density, expressed through perturbation of chemical potential. The last we will find from the preservation law of numerical density.

We put next

\[ k_1 = k + \frac{q}{2}, \quad q_2 = k - \frac{q}{2} \]

and rewrite in this terms equations (1.4), (1.5) и (1.6). We receive following equalities

\[
\hbar \omega \langle k + \frac{q}{2} | \rho_1 | k - \frac{q}{2} \rangle = \left( \varepsilon_{k+q/2} - \varepsilon_{k-q/2} \right) \langle k + \frac{q}{2} | \rho_1 | k - \frac{q}{2} \rangle -
\]

\[
- e\varphi(q)(f_{k+q/2} - f_{k-q/2}) + i\hbar \langle k + \frac{q}{2} | \tilde{\rho}_1 - \rho_1 | k - \frac{q}{2} \rangle,
\]

(1.2)
\[
\langle k + \frac{q}{2} | \rho_1 | k - \frac{q}{2} \rangle \equiv \rho_1(q) = -\frac{e\varphi(q)(f_{k+q/2} - f_{k-q/2})}{\mathcal{E}_{k-q/2} - \mathcal{E}_{k+q/2} + \hbar \omega + i\nu \hbar} + \\
+ \frac{i\nu \hbar \langle k + \frac{q}{2} | \tilde{\rho}_1 | k - \frac{q}{2} \rangle}{\mathcal{E}_{k-q/2} - \mathcal{E}_{k+q/2} + \hbar \omega + i\nu \hbar},
\]

(1.3')

\[
\langle k + \frac{q}{2} | \tilde{\rho}_1 | k - \frac{q}{2} \rangle = -\frac{f_{k+q/2} - f_{k-q/2}}{\mathcal{E}_{k+q/2} - \mathcal{E}_{k-q/2}} \langle k + \frac{q}{2} | \delta \mu_\ast | k - \frac{q}{2} \rangle.
\]

(1.4')

2. Perturbation of chemical potential

The quantity \(\delta \mu\) (or \(\delta \mu_\ast\)) is responsible for the local preservation of number of particles (electrons). This local law preservation looks like [1]

\[
\omega \delta n(q, \omega, \nu) = q \delta j(q, \omega, \nu).
\]

(2.1)

In equation (2.1) \(\delta n(q, \omega, \nu), \delta j(q, \omega, \nu) = j(q, \omega, \nu)\) are change of concentration and stream density of electrons under action electric field, and

\[
\delta n(q, \omega, \nu) = \int \frac{dk}{4\pi^3} \langle k + \frac{q}{2} | \rho_1 | k - \frac{q}{2} \rangle,
\]

\[
\delta j(q, \omega, \nu) = \int \frac{\hbar kd\mathbf{k}}{4\pi^2m} \langle k + \frac{q}{2} | \rho_1 | k - \frac{q}{2} \rangle.
\]

Let’s return to the equation (1.2) which have been written down concerning perturbation to the matrix density. From this equation follows, that

\[
\int \frac{\hbar \omega d\mathbf{k}}{4\pi^3} \langle k + \frac{q}{2} | \rho_1 | k - \frac{q}{2} \rangle =
\]

\[
= \int \frac{dk}{4\pi^3} (\mathcal{E}_{k+q/2} - \mathcal{E}_{k-q/2}) \langle k + \frac{q}{2} | \rho_1 | k - \frac{q}{2} \rangle - \\
- e\varphi(q) \int \frac{dk}{4\pi^3} (f_{k+q/2} - f_{k-q/2}) + 
\]
\[ +i\hbar \nu \int \frac{dk}{4\pi^3} \langle k + \frac{q}{2} | \hat{\rho}_1 - \rho_1 | k - \frac{q}{2} \rangle. \]

Let's transform this equality, using obvious parities
\[ \mathcal{E}_{k+q/2} - \mathcal{E}_{k-q/2} = \frac{\hbar^2 kq}{m}, \] (2.2)
and
\[ \int \frac{dk}{4\pi^3} \left( f_{k-q/2} - f_{k+q/2} \right) = 0. \]

As result last equality can be rewrite in the form
\[ \hbar \int \frac{dk}{4\pi^3} \left( \omega \langle k + \frac{q}{2} | \rho_1 | k - \frac{q}{2} \rangle - \frac{\hbar kq}{m} \langle k + \frac{q}{2} | \rho_1 | k - \frac{q}{2} \rangle \right) = \]
\[ = i\hbar \nu \int \frac{dk}{4\pi^3} \langle k + \frac{q}{2} | \hat{\rho}_1 - \rho_1 | k - \frac{q}{2} \rangle. \]

Expression in the left part of this relation according to (2.1) is equal to zero
\[ \hbar \omega \delta n(q, \omega, \nu) - \hbar q \delta j(q, \omega, \nu) = 0. \]

The last equality is true owing to the law of local preservation of number particles. From here follows, that
\[ \int \frac{dk}{4\pi^3} \langle k + \frac{q}{2} | \hat{\rho}_1 - \rho_1 | k - \frac{q}{2} \rangle = 0. \]

This equality is equivalent to the following
\[ \int \frac{dk}{4\pi^3} \langle k + \frac{q}{2} | \hat{\rho}_1 | k - \frac{q}{2} \rangle = \]
\[ = \int \frac{dk}{4\pi^3} \langle k + \frac{q}{2} | \rho_1 | k - \frac{q}{2} \rangle. \]

Considering earlier received expression (1.4) for \( \langle k_1 | \hat{\rho}_1 | k_2 \rangle \) (or relation (1.4’) for \( \langle k + q/2 | \hat{\rho}_1 | k - q/2 \rangle \), from here we have
\[ \delta \mu_+(q) \int \frac{dk}{4\pi^3} \frac{f_{k+q/2} - f_{k-q/2}}{\mathcal{E}_{k-q/2} - \mathcal{E}_{k+q/2}} = \]
\[ = \int \frac{dk}{4\pi^3} \langle k + \frac{q}{2} | \rho_1 | k - \frac{q}{2} \rangle. \]

Thus for perturbation quantity \( \delta \mu_+(q) \) it is received
\[ \delta \mu_+(q) = \frac{1}{B(q, 0)} \int \frac{dk}{4\pi^3} \langle k + \frac{q}{2} | \rho_1 | k - \frac{q}{2} \rangle. \] (2.3)
Here the following designation is accepted

\[ B(q, 0) = \int \frac{d\mathbf{k}}{4\pi^3} \frac{f_{k+q/2} - f_{k-q/2}}{\varepsilon_{k+q/2} - \varepsilon_{k+q/2}}. \]

From equation (1.3) we obtain

\[ [\varepsilon_{k_2} - \varepsilon_{k_1} + \hbar(\omega + i\nu)] \langle k_1 | \rho_1 | k_2 \rangle = \]

\[ = -e(f_{k_1} - f_{k_2})\varphi(k_1 - k_2) + i\hbar \langle k_1 | \hat{\rho}_1 | k_2 \rangle. \]

Last component in this equality we will replace according to (1.4). We receive that

\[ \left[ \varepsilon_{k_2} - \varepsilon_{k_1} + \hbar(\omega + i\nu) \right] \langle k_1 | \rho_1 | k_2 \rangle = \]

\[ = -e(f_{k_1} - f_{k_2})\varphi(k_1 - k_2) - i\hbar \frac{f_{k_1} - f_{k_2}}{\varepsilon_{k_1} - \varepsilon_{k_2}} \delta \mu_s(k_1 - k_2). \]

From this equation we obtain expression for \( \langle k_1 | \rho_1 | k_2 \rangle \):

\[ \langle k_1 | \rho_1 | k_2 \rangle = -\frac{e(f_{k_1} - f_{k_2})\varphi(k_1 - k_2)}{\varepsilon_{k_2} - \varepsilon_{k_1} + \hbar \omega + i\hbar \nu} - \]

\[ -i\hbar \frac{f_{k_1} - f_{k_2}}{\varepsilon_{k_1} - \varepsilon_{k_2}} \frac{\delta \mu_s(k_1 - k_2)}{\hbar \omega + i\hbar \nu}, \] \hspace{1cm} (2.4)

or, after decomposition on partial fractions,

\[ \langle k_1 | \rho_1 | k_2 \rangle = -\frac{e(f_{k_1} - f_{k_2})\varphi(k_1 - k_2)}{\varepsilon_{k_2} - \varepsilon_{k_1} + \hbar \omega + i\hbar \nu} + \]

\[ + \frac{i\nu}{\omega + i\nu} \cdot \frac{f_{k_1} - f_{k_2}}{\varepsilon_{k_2} - \varepsilon_{k_1}} \delta \mu_s(k_1 - k_2) - \]

\[ - \frac{i\nu}{\omega + i\nu} \cdot \frac{f_{k_1} - f_{k_2}}{\varepsilon_{k_1} - \varepsilon_{k_2} + \hbar \omega + i\hbar \nu} \delta \mu_s(k_1 - k_2), \] \hspace{1cm} (2.4')

Passing to variables \( k \) and \( q \), from here we receive

\[ \langle k + \frac{q}{2} | \rho_1 | k - \frac{q}{2} \rangle = -\frac{e(f_{k+q/2} - f_{k-q/2})\varphi(q)}{\varepsilon_{k-q/2} - \varepsilon_{k+q/2} + \hbar(\omega + i\nu)} + \]

\[ + \frac{i\nu}{\omega + i\nu} \cdot \frac{f_{k+q/2} - f_{k-q/2}}{\varepsilon_{k-q/2} - \varepsilon_{k+q/2}} \delta \mu_s(q, \omega, \nu) - \]

\[ - \frac{i\nu}{\omega + i\nu} \cdot \frac{(f_{k+q/2} - f_{k-q/2})\delta \mu_s(q, \omega, \nu)}{\varepsilon_{k+q/2} + \hbar(\omega + i\nu)} \] \hspace{1cm} (2.4'')
Let’s substitute expression (2.4") in the formula for perturbation of chemical potential (2.3). On this way for perturbation it is received the following expression

$$\delta \mu_*(q) = -\frac{(\omega + i\nu)B(q, \omega + i\nu)}{\omega B(q, 0) + i\nu B(q, \omega + i\nu)} e\varphi(q).$$ \hspace{1cm} (2.5)$$

Here

$$B(q, \omega + i\nu) = \int \frac{dk}{4\pi^3} \frac{f_{k+q/2} - f_{k-q/2}}{\varepsilon_{k-q/2} - \varepsilon_{k+q/2} + \hbar(\omega + i\nu)}.$$ 

**3. Electric conductivity and dielectric permeability**

Let’s substitute (2.5) in (2.4") and in the received expression we will result similar members. As result we receive the following expression

$$\langle k + \frac{q}{2} | \rho_1 | k - \frac{q}{2} \rangle = -e\varphi(q) \frac{f_{k+q/2} - f_{k-q/2}}{\omega B(q, 0) + i\nu B(q, \omega + i\nu)} \times$$

$$\times \left[ \frac{\omega B(q, 0)}{\varepsilon_{k-q/2} - \varepsilon_{k+q/2} + \hbar(\omega + i\nu)} + \frac{i\nu B(q, \omega + i\nu)}{\varepsilon_{k-q/2} - \varepsilon_{k+q/2}} \right]. \hspace{1cm} (3.1)$$

The current density $j_e(q, \omega, \nu)$ is calculated by $\langle k_1 | \rho_1 | k_2 \rangle$

$$j_e(q, \omega, \nu) = e j(q, \omega, \nu) = \frac{eh}{m} \int \frac{kdk}{4\pi^3} \langle k + \frac{q}{2} | \rho_1 | k - \frac{q}{2} \rangle.$$ 

Thus intensity of electric field is connected with potential of this field by relation $E(q, \omega) = -i q \varphi(q, \omega)$, From here the field potential is equal

$$\varphi(q, \omega) = \frac{iqE(q, \omega)}{q^2}.$$ 

Hence, expression for current density $j_e(q, \omega, \nu)$ by means of relation (3.1) it is possible to rewrite in the form

$$j_e(q, \omega, \nu) \equiv \sigma_i E(q, \omega) =$$

$$= -\frac{e^2h}{mq^2} E(q, \omega) \int \frac{kdk}{4\pi^3} \frac{f_{k+q/2} - f_{k-q/2}}{\omega B(q, 0) + i\nu B(q, \omega + i\nu)} \times$$
\[
\times \left[ \frac{\omega B(q, 0)}{\varepsilon_{k-q/2} - \varepsilon_{k+q/2} + \hbar(\omega + i\nu)} + \frac{i\nu B(q, \omega + i\nu)}{\varepsilon_{k-q/2} - \varepsilon_{k+q/2}} \right].
\]

From here we receive
\[
\sigma_l(q, \omega, \nu) = -ie^2 \frac{\hbar}{mq^2} \int \frac{k q dk}{4\pi^3} \frac{f_{k+q/2} - f_{k-q/2}}{\omega B(q, 0) + i\nu B(q, \omega + i\nu)} \times
\]
\[
\times \left[ \frac{\omega B(q, 0)}{\varepsilon_{k-q/2} - \varepsilon_{k+q/2} + \hbar(\omega + i\nu)} + \frac{i\nu B(q, \omega + i\nu)}{\varepsilon_{k-q/2} - \varepsilon_{k+q/2}} \right].
\]

The scalar production \(k q\) we find from relation (2.2):
\[
k q = \frac{m}{\hbar^2} \left( \varepsilon_{k+q/2} - \varepsilon_{k-q/2} \right).
\]
By means of this relation we will write expression for dielectric permeability (3.2) to the following form
\[
\sigma_l(q, \omega, \nu) = -i e^2 \frac{\omega(\omega + i\nu)B(q, 0)B(q, \omega + i\nu)}{q^2 \omega B(q, 0) + i\nu B(q, \omega + i\nu)}. \tag{3.3}
\]

This formula expresses longitudinal electric conductivity into quantum collisional plasma. On the basis of (3.3) we will write expression for longitudinal dielectric permeability into the quantum collisional plasma
\[
\varepsilon_l(q, \omega, \nu) = 1 + 4\pi e^2 \frac{\omega(\omega + i\nu)B(q, 0)B(q, \omega + i\nu)}{q^2 \omega B(q, 0) + i\nu B(q, \omega + i\nu)}. \tag{3.4}
\]
From the formula (3.4) it is visible that at \(\omega = 0\) we receive
\[
\varepsilon_l(q, \omega, \nu) = 1 + \frac{4\pi e^2}{q^2} B(q, 0).
\]
Thus, at \(\omega = 0\) dielectric function does not depend from frequency of particles collisions of plasma.

At \(\nu = 0\) from (3.4) we receive
\[
\varepsilon_l(q, \omega, \nu) = 1 + \frac{4\pi e^2}{q^2} B(q, \omega).
\]
Thus, at \(\nu = 0\) dielectric function passes in the known formula received by Klimontovich and Silin in 1952 and after that by Lindhard in 1954.
4. Comparison with Mermin’ formula

We will write out Mermin’ formula for dielectric function [1]

\[
\varepsilon_{\text{Mermin}}^l(q, \omega, \nu) = 1 + \frac{(1 + i/\omega\tau)(\varepsilon^0(q, \omega + i/\tau) - 1)}{1 + (i/\omega\tau)(\varepsilon^0(q, \omega + i/\tau) - 1)/(\varepsilon^0(q, 0) - 1)}. \tag{4.1}
\]

In expression (4.1) \( \varepsilon^0(q, 0) \) is the Lindhard’ dielectric function for collisionless plasma,

\[
\varepsilon^0(q, \omega) = 1 + \frac{4\pi e^2}{q^2} B(q, \omega),
\]

\[
B(q, \omega) = \int \frac{dp}{4\pi^3} \frac{f_{p+q/2} - f_{p-q/2}}{\varepsilon_{p-q/2} - \varepsilon_{p+q/2} + \omega},
\]

\[
B(q, 0) = \int \frac{dp}{4\pi^3} \frac{f_{p+q/2} - f_{p-q/2}}{\varepsilon_{p-q/2} - \varepsilon_{p+q/2}}.
\]

Let’s transform the formula (4.1), noticing, that \( 1 + i/\omega\tau = (\omega + i\nu)/\omega \), to the form

\[
\varepsilon_{\text{Mermin}}^l(q, \omega, \nu) = 1 + \frac{(\omega + i\nu)[\varepsilon^0(q, \omega + i\nu) - 1][\varepsilon^0(q, 0) - 1]}{\omega[\varepsilon^0(q, 0) - 1] + i\nu[\varepsilon^0(q, \omega + i\nu) - 1]}. \tag{4.2}
\]

We rewrite Mermin’s formula (4.2) in terms of integrals \( B(q, \omega) \) and \( B(q, 0) \)

\[
\varepsilon_{\text{Mermin}}^l(q, \omega, \nu) = 1 + \frac{4\pi e^2(\omega + i\nu)B(q, \omega + i\nu)B(q, 0)}{q^2 \omega B(q, 0) + i\nu B(q, \omega + i\nu)}. \tag{4.3}
\]

Comparison of Mermin’ formula (4.3) with the formula (3.4) shows their full coincidence.

5. Quantum degenerate plasma

Let’s calculate integrals

\[
B(q, \omega + i\nu) = \int \frac{dk}{4\pi^3} \frac{f_{k+q/2} - f_{k-q/2}}{\varepsilon_{k-q/2} - \varepsilon_{k+q/2} + \hbar(\omega + i\nu)}. \tag{5.1}
\]
For degenerate plasma in formula (5.1) there are designations
\[ f_k = \Theta(E_F - E_k), \quad \varepsilon_k = \frac{\hbar^2 k^2}{2m}, \quad \varepsilon_F = \frac{p_{F}^2}{2m}, \]
\[ \Theta(x) \] is the Heaviside function,
\[ \Theta(x) = \begin{cases} 
1, & x > 0, \\
0, & x < 0. 
\end{cases} \]

Let’s present integral (5.1) in the form of a difference of two integrals. We will enter in these integrals obvious replacement of variables. It is as a result received
\[ B(q, \omega + i\nu) = \int \frac{dk}{4\pi^3} \frac{f_k[2\varepsilon_k - \varepsilon_{k+q} - \varepsilon_{k-q}]}{[\varepsilon_{k-q} - \varepsilon_k + \hbar(\omega + i\nu)][\varepsilon_k - \varepsilon_{k+q} + \hbar(\omega + i\omega)]}. \] (5.2)

Let’s pass to integration on the dimensionless wave vector \( \mathbf{P} = \frac{k}{k_F} \), vectors \( \mathbf{P} \) and \( \mathbf{q} \) let’s direct lengthways axes \( x \), believing \( \mathbf{P} = P_x(1, 0, 0) \) and \( \mathbf{q} = q(1, 0, 0) \). We introduce one more wave vector \( \mathbf{k} = \frac{q}{k_F} \). Then
\[ \varepsilon_{k-q} - \varepsilon_k + \hbar(\omega + i\nu) = -\frac{\hbar^2(2k\mathbf{q} + q^2)}{2m} + \hbar(\omega + i\nu) = \]
\[ = -\frac{2\hbar^2 q}{2m}(k_x - \frac{q}{2}) + \hbar(\omega + i\nu) = -2k\varepsilon_F(P_x - \frac{k}{2}) + \hbar(\omega + i\nu) = \]
\[ = -2k\varepsilon_F\left(P_x - \frac{k}{2} - \frac{\hbar(\omega + i\nu)}{2k\varepsilon_F}\right) = -2k\varepsilon_F\left(P_x - \frac{k}{2} - \frac{z}{k}\right). \]
Here
\[ z = \frac{\hbar(\omega + i\nu)}{2E_F} = \frac{\omega + i\nu}{k_Fv_F}, \quad x = \frac{\omega}{k_Fv_F}, \quad y = \frac{\nu}{k_Fv_F}. \]

Similarly,
\[ \varepsilon_k - \varepsilon_{k+q} + \hbar(\omega + i\nu) = -2k\varepsilon_F\left(P_x + \frac{k}{2} - \frac{z}{k}\right), \]
the kernel of integral (5.2) is equal:

\[
\frac{2\mathcal{E}_k - \mathcal{E}_{k+q} - \mathcal{E}_{k-q}}{[\mathcal{E}_{k-q} - \mathcal{E}_k + \hbar(\omega + i\nu)][\mathcal{E}_k - \mathcal{E}_{k+q} + \hbar(\omega + i\omega)]} = \\
= -\frac{1}{2\mathcal{E}_F}\left[\left(P_x - \frac{z}{k}\right)^2 - \left(\frac{k}{2}\right)^2\right].
\]

In the integral (5.2)

\[
f_k = \Theta(\mathcal{E}_F - \mathcal{E}_F P^2) = \Theta(1 - P^2), \quad \mathcal{E}_k = \mathcal{E}_F P^2.
\]

Now the integral (5.2) is equal

\[
B(k, z) = -\frac{k_F^3}{8\pi^3\mathcal{E}_F} \int \frac{\Theta(1 - P^2)}{(P_x - z/k)^2 - (k/2)^2} = \\
= -\frac{3N}{4\pi mv_F^2} \int_{P^2 < 1} d^3P \left(\frac{P_x}{(P_x - z/k)^2 - (k/2)^2}\right).
\]

It is easy to see, that

\[
B(k, z) = -\frac{3N}{4mv_F^2} \int_{-1}^{1} \frac{(1 - P_x^2)dP_x}{(P_x - z/k)^2 - (k/2)^2} = \\
= -\frac{3N}{4mv_F^2} \left[-2 + \frac{1}{k} \left[1 - \left(\frac{z}{k}\right)^2 - \left(\frac{k}{2}\right)^2\right] \ln \left(\frac{z/k}{k} - 1 - k - (k/2)^2\right) - \\
\frac{z}{k} \ln \left(\frac{(1 - z/k)^2 - (k/2)^2}{(1 + z/k)^2 - (k/2)^2}\right)\right].
\]

From this equality follows, that

\[
B(k, 0) = -\frac{3N}{4mv_F^2} \int_{-1}^{1} \frac{(1 - P_x^2)dP_x}{P_x^2 - (k/2)^2} = \\
= -\frac{3N}{4mv_F^2} \left[-2 + \frac{(2 - k)(2 + k)}{2k} \ln \frac{k - 2}{k + 2}\right].
\]

By means of these integrals it is found electric conductivity

\[
\sigma_I(x, y, k) = i\frac{3e^2N\omega}{4mv_Fk^2k^2} \frac{(x + iy)b(k, z)b(k, 0)}{xb(k, 0) + iyb(k, z)}, \quad (5.3)
\]
where
\[
    b(k, z) = \int_{-1}^{1} \frac{(1 - \tau^2)d\tau}{\tau^2 - (k/2)^2} = 
\]
\[
    = -2 + \frac{1}{k} \left[ 1 - (z/k)^2 - (k/2)^2 \right] \ln \frac{(z/k)^2 - 1 + k - (k/2)^2}{(z/k)^2 - 1 - k - (k/2)^2} - \frac{z}{k} \ln \frac{(1 - z/k)^2 - (k/2)^2}{(1 + z/k)^2 - (k/2)^2},
\]
\[
b(k, 0) = \int_{-1}^{1} \frac{(1 - \tau^2)d\tau}{\tau^2 - (k/2)^2} = -2 + \frac{(2 - k)(2 + k)}{2k} \ln \frac{k - 2}{k + 2}.
\]

In the formula (5.3) we will allocate static conductivity \(\sigma_0 = \frac{e^2N}{mv}\).

Then
\[
    \frac{\sigma_l(x, y, k)}{\sigma_0} = \frac{3}{4} \cdot \frac{\hbar \omega}{mv_F^2} \cdot \frac{\hbar \nu}{mv_F^2} \cdot \frac{1}{k^2} \cdot \frac{(x + iy)b(k, 0)b(k, z)}{xb(k, 0) + iyb(k, z)}. \tag{5.4}
\]

By means of (5.4) we will write the formula for dielectric permeability
\[
    \varepsilon_l(x, y, k) = 1 - \frac{3\omega_p^2}{4\omega^2} \cdot \left( \frac{\hbar \omega}{mv_F^2} \right)^2 \cdot \frac{1}{k^2} \cdot \frac{(x + iy)b(k, z)b(k, 0)}{xb(k, 0) + iyb(k, z)}, \tag{5.5}
\]
or, in equivalent form
\[
    \varepsilon_l(x, y, k) = 1 - \frac{3}{4k^2} \cdot \left( \frac{\omega_p \hbar}{mv_F^2} \right)^2 \cdot \frac{(x + iy)b(k, z)b(k, 0)}{xb(k, 0) + iyb(k, z)}. \tag{5.5'}
\]

Having entered dimensionless plasma (Langmuir) frequency
\[
    x_p = \frac{\omega_p \hbar}{mv_F^2} = \frac{\omega_p \hbar}{p_F v_F} = \frac{\omega_p}{k_F v_F},
\]
let’s copy the formula (5.5’) in the form
\[
    \varepsilon_l(x, y, k) = 1 - \frac{3x_p^2}{4k^2} \cdot \frac{(x + iy)b(k, z)b(k, 0)}{xb(k, 0) + iyb(k, z)}. \tag{5.5''}
\]

6. Degenerate classical (Fermi) plasma
We take expression of dielectric function for classical degenerate plasmas

\[ \varepsilon_{\text{classic}}(\omega, \nu, q) = 1 + \frac{3\omega_p^2}{q^2v_F^2} \cdot \frac{1 - (1 - i\omega\tau)T_0(\omega, \nu, q)}{1 - T_0(\omega, \nu, q)}, \]

(6.1)

where \( q \) is the wave number,

\[ T_0(\omega, \nu, q) = \frac{1}{2} \int_{-1}^{1} \frac{d\mu}{1 - i\omega\tau + iqv_F\tau\mu}. \]

Let’s result the formula (6.1) to the obvious form

\[ \varepsilon_{\text{classic}}(\omega, \nu, q) = 1 + \frac{3\omega_p^2}{q^2v_F^2} \cdot \frac{1 + \frac{\omega + i\nu}{2qv_F} \ln \frac{(\omega + i\nu)/v_F - q}{(\omega + i\nu)/v_F + q}}{1 + \frac{i\nu}{2qv_F} \ln \frac{(\omega + i\nu)/v_F - q}{(\omega + i\nu)/v_F + q}}. \]

(6.2)

Let’s copy the formula (6.2) in dimensionless variables

\[ x = \frac{\omega}{k_Fv_F}, \quad y = \frac{\nu}{k_Fv_F}, \quad k = \frac{q}{k_F}, \quad x_p = \frac{\omega_p}{k_Fv_F}. \]

We obtain that

\[ \varepsilon_{\text{classic}}(x, y, k) = 1 + \frac{3x_p^2}{k^2} \cdot \frac{1 + \frac{z}{2k} \ln \frac{z - k}{z + k}}{1 + \frac{i\nu}{2k} \ln \frac{z - k}{z + k}}, \]

(6.3)

where \( z = x + iy \).

In some questions instead of the formula (6.3) it is more convenient to use following expression of dielectric function

\[ \varepsilon_{\text{classic}}(x, y, k) = 1 + \frac{3x_p^2}{k^2} \cdot \frac{1 + \frac{z}{2k} b_0(z, k)}{1 + i\nu b_0(z, k)}, \]

(6.4)

where

\[ b_0(z, k) = \frac{1}{2k} \int_{-1}^{1} \frac{d\mu}{\mu - z/k} = \frac{1}{2k} \ln \frac{z - k}{z + k}. \]

In all drawings dimensionless plasma frequency is taken to equal unit: \( x_p = 1 \).
On figs. 1–8 graphics of the real and imaginary parts of dielectric function of quantum collisional degenerate plasmas are given. On figs. 1 and 2 curves 1, 2, 3 correspond to functions $\varepsilon_l(k) = \varepsilon_l(0.1, 0.001, k)$, $\varepsilon_l(k) = \varepsilon_l(0.15, 0.001, k)$, $\varepsilon_l(k) = \varepsilon_l(0.2, 0.001, k)$.

On figs. 3 and 4 curves 1, 2, 3 correspond to functions $\varepsilon_l(k) = \varepsilon_l(0.9, 0.001, k)$, $\varepsilon_l(k) = \varepsilon_l(1.0, 0.001, k)$, $\varepsilon_l(k) = \varepsilon_l(1.1, 0.001, k)$.

On figs. 5 and 6 curves 1, 2, 3 correspond to functions $\varepsilon_l(x) = \varepsilon_l(x, 0.001, 0.1)$, $\varepsilon_l(x) = \varepsilon_l(x, 0.001, 0.12)$, $\varepsilon_l(x) = \varepsilon_l(x, 0.001, 0.15)$.

On figs. 7 and 8 curves 1, 2, 3 correspond to functions $\varepsilon_l(y) = \varepsilon_l(0.11, 0.001, y)$, $\varepsilon_l(y) = \varepsilon_l(0.12, 0.001, y)$, $\varepsilon_l(y) = \varepsilon_l(0.13, 0.001, y)$.

On figs. 9–12 comparison of the real and imaginary parts of dielectric function quantum and classical collisional degenerate plasmas at $y = 0.001$ is given. On figs. 9 and 10 the case $x = 0.3$ is represented, on figs. 11 and 12 the case $x = 1$ is represented. Curves 1 and 2 correspond to quantum $\varepsilon_l$ and classical $\varepsilon_l^{\text{classic}}$ plasmas.

On figs. 13 and 14 the module of derivative $|d\varepsilon_l/dk|$ of dielectric function of quantum (curves 1) and classical (curves 2) plasmas for the case $x = 0.5$ at $y = 0.0001$ (fig. 13) and $y = 0.001$ (fig. 14) is represented.

7. Conclusion

In the present work the detailed conclusion of formulas for the longitudinal electric conductivity and dielectric permeability of quantum degenerate collisional plasmas is resulted.

For this purpose the kinetic the equation with integral of collisions in relaxation form in momentum space is used.

Graphic research of properties the real and imaginary parts of dielectric function of quantum collisional plasmas and graphic comparison of the real and imaginary parts of dielectric function between quantum and classical plasmas is shown.
Fig. 1. Real parts of dielectric function, $x_p = 1, y = 0.001$. Curves 1, 2, 3 correspond to values of dimensionless frequency of electric field $x = 0.1, 0.15, 0.2$.

Fig. 2. Imaginary parts of dielectric function, $x_p = 1, y = 0.001$. Curves 1, 2, 3 correspond to values of dimensionless frequency of electric field $x = 0.1, 0.15, 0.2$. 
Fig. 3. Real parts of dielectric function, \( x_p = 1, y = 0.001 \). Curves 1, 2, 3 correspond to values of dimensionless frequency of electric field \( x = 0.9, 1.0, 1.1 \).

Fig. 4. Imaginary parts of dielectric function, \( x_p = 1, y = 0.001 \). Curves 1, 2, 3 correspond to values of dimensionless frequency of electric field \( x = 0.9, 1.0, 1.1 \).
Fig. 5. Real parts of dielectric function, \( x_p = 1, y = 0.001 \). Curves 1, 2, 3 correspond to values of dimensionless wave number \( k = 0.1, 0.12, 0.15 \).

Fig. 6. Imaginary parts of dielectric function, \( x_p = 1, y = 0.001 \). Curves 1, 2, 3 correspond to values of dimensionless wave number \( k = 0.1, 0.12, 0.15 \).
Fig. 7. Real parts of dielectric function, $x_p = 1, k = 0.1$. Curves 1, 2, 3 correspond to values of dimensionless collision frequency $x = 0.11, 0.12, 0.13$.

Fig. 8. Imaginary parts of dielectric function, $x_p = 1, k = 0.1$. Curves 1, 2, 3 correspond to values of dimensionless collision frequency $x = 0.11, 0.12, 0.13$. 
Fig. 9. Real parts of dielectric function, $x_p = 1$, $x = 0.3$, $y = 0.001$. Curves 1 and 2 correspond to quantum and classical plasmas.

Fig. 10. Imaginary parts of dielectric function, $x_p = 1$, $x = 0.3$, $y = 0.001$. Curves 1 and 2 correspond to quantum and classical plasmas.
Fig. 11. Real parts of dielectric function, \( x_p = 1, x = 1, y = 0.001 \). Curves 1 and 2 correspond to quantum and classical plasmas.

Fig. 12. Imaginary parts of dielectric function, \( x_p = 1, x = 1, y = 0.001 \). Curves 1 and 2 correspond to quantum and classical plasmas.
Fig. 13. Module of derivative $|d\varepsilon_l/dk|$ of dielectric function, $x_p = 1$, $x = 0.05$, $y = 0.0001$. Curves 1 and 2 correspond to quantum and classical plasmas.

Fig. 14. Module of derivative $|d\varepsilon_l/dk|$ of dielectric function, $x_p = 1$, $x = 0.05$, $y = 0.001$. Curves 1 and 2 correspond to quantum and classical plasmas.
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