1. Introduction and statement of the main results

In this paper we consider the sharp pointwise estimates for the gradients of real–valued bounded harmonic functions. We will first recall the known estimates of this type in the plane and in the space.

For every fixed \( z = (x, y) \in \mathbb{R}^2_+ \) there holds the optimal gradient estimate

\[
|\nabla V(z)| \leq \frac{2}{\pi} \frac{1}{y} |V|_\infty,
\]

where \( V \) is an arbitrary bounded harmonic functions in the upper half–plane \( H^2 = \mathbb{R}^2_+ \), and \( |V|_\infty = \sup_{z \in \mathbb{R}^2_+} |V(z)| \). Using the conformal transformation of the unit disk \( B^2 \) onto \( \mathbb{R}^2_+ \) one easily derives

\[
|\nabla U(z)| \leq \frac{4}{\pi} \frac{1}{1 - |z|^2} |U|_\infty,
\]

for \( x \in B^2 \), where \( U \) is harmonic and bounded in the unit disc \( B^2 \). This classical result is improved in the recent paper of D. Kalaj and M. Vuorinen [3]. Their form of the above inequality says that

\[
|\nabla U(z)| \leq \frac{4}{\pi} \frac{1 - U(z)^2}{1 - |z|^2}.
\]

This relation requires that \( U \) is bounded by 1 in the disc \( B^2 \). The proof of the inequality \((1.3)\) lies on the classical Schwarz lemma for bounded analytic functions.

Recently G. Kresin and V. Maz’ya [8] proved the following generalization od \((1.1)\):

\[
|\nabla V(x)| \leq \frac{4}{\sqrt{\pi}} \frac{(n-1)(n-1)/2}{\Gamma((n-1)/2)} \frac{1}{\Gamma((n-1)/2) x_n} |V|_\infty.
\]

Here, \( V \) is a bounded harmonic function in the half–space \( \mathbb{R}^n_+ \), \( |V|_\infty = \sup_{y \in \mathbb{R}^n_+} |V(y)| \), and \( x = (x', x_n) \in \mathbb{R}^n_+ \) is fixed. These optimal pointwise estimates arise while proving Khavinson conjecture in the halfspace setting. In order to formulate the conjecture we introduce the notation we need.

For every fixed \( x \) let \( C(x) \) denote the optimal number for the gradient estimate

\[
|\nabla U(x)| \leq C(x)|U|_\infty,
\]
where $U$ is harmonic and bounded in $B^n$ or $\mathbb{R}_n^+$. Similarly, for every $v \in \mathbb{R}^n$, $|v| = 1$ denote by $C(x, v)$ the optimal number for the gradient estimate in the direction $v$, i.e., the smallest number such that the following relation holds

$$ |\langle \nabla U(x), v \rangle| \leq C(x, v)|U|_\infty $$

for every bounded and harmonic $U$. Since

$$ |\nabla U(x)| = \sup_{v \in \partial B^n} |\langle \nabla U(x), v \rangle|, $$

it follows that

(1.6) \quad C(x) = \sup_v C(x, v).

It turned out that the variational problem (1.6) is a hard problem, especially in the unit ball setting. The generalized Khavinson conjecture states that

**Conjecture 1.1.** For $x \in B^n$ we have

(1.7) \quad C(x) = C(x, n_x),

where $n_x = x/|x|$ is the vector normal to the boundary at $x$.

In 1992, Khavinson [6] obtained the optimal estimate in the normal direction of the gradient of bounded harmonic functions in the unit ball in $\mathbb{R}^3$. In a private conversation with K. Gresin and V. Maz’ya he believed that the same estimate hold for the norm of the gradient, i.e., that the above conjecture is true for the unit ball $B^3$.

In their recent paper [8] and in their book [9], G. Kresin and V. Maz’ya considered the Khavinson problem from a more general aspect including harmonic functions with $L^p$-boundary values $(1 \leq p \lesssim \infty)$. They formulated the generalized Khavinson conjecture and proved it for bounded harmonic functions in $\mathbb{R}_n^+$. In this context we have $n_x = e_n$ for all $x \in \mathbb{R}_n^+$. After replacing $C(x)$ with $C(x, e_n)$ in (1.5), they obtained (1.4).

M. Marković in a recent paper [11] proved the conjecture when $x$ is near the boundary, i.e., if $1 - \epsilon \leq |x| < 1$. Therefore, in (1.5) one can replace $C(x)$ with $C(x, n_x)$, if $|x|$ is near 1. In this paper we prove the conjecture for $n = 4$, i.e. we prove the following theorem

**Theorem 1.2.** For $x \in B^4$ we have

(1.8) \quad C(x) = C(x, n_x),

where $n_x = x/|x|$ is the vector normal to the boundary at $x$.

A reformulated version of Theorem 1.2 is the following theorem, whose proof follows directly from Theorem 2.9 and relation (2.1) below.

**Theorem 1.3.** Let $n = 4$. Then we have the sharp inequality for every $x \in B^4$, $r = |x|:

$$ |\nabla u(x)| \leq \frac{r\sqrt{4 - r^2} (2 + r^2) + 4 (1 - r^2) \tan^{-1} \left[ \frac{r\sqrt{4 - r^2}}{r^2 - 2} \right]}{\pi (1 - r^2)r^3} |u|_\infty, \quad u \in h^\infty(B^4). $$

Here and in the sequel, $h^\infty(B^4)$ is the Hardy space of bounded harmonic functions on the unit ball $B^4$ (cf. [1]).
Corollary 1.4. For the decreasing diffeomorphism $\mathcal{C} : [0, 1] \to \left[\frac{3\sqrt{3}}{2\pi}, \frac{16}{3\pi}\right]$, defined by
\[
\mathcal{C}(r) = \frac{r \sqrt{4 - r^2} (2 + r^2) + 4 (1 - r^2) \tan^{-1} \left[\frac{r \sqrt{4 - r^2}}{-2 + r^2}\right]}{\pi r^3},
\]
we have the sharp inequality for every $x \in B^4$, $r = |x|$:
\[
|\nabla u(x)| \leq \frac{\mathcal{C}(r)}{1 - r^2} |u|_{\infty} \quad u \in h^\infty(B^4).
\]

Remark 1.5. Observe that for $R^4_+$, Kresin - Maz'ya inequality (1.4) reads as
\[
(1.9) \quad |\nabla V(x)| \leq 3\sqrt{3} \frac{1}{2\pi x_4} |V|_{\infty}.
\]

Proof of corollary. We need to prove that $\mathcal{C}(r)$ is a decreasing function. We have that
\[
\mathcal{C}'(r) = -\frac{\left((-2 + r)(2 + r) (-6 + r^2) - 4 \sqrt{4 - r^2} (-3 + r^2) \tan^{-1} \left[\frac{r \sqrt{4 - r^2}}{-2 + r^2}\right]\right)}{\pi r^4 \sqrt{4 - r^2}}.
\]
So $\mathcal{C}'(r) \leq 0$ if and only if
\[
v(r) = \frac{r (4 - r^2) (6 - r^2)}{4 (3 - r^2) \sqrt{4 - r^2}} + \tan^{-1} \left[\frac{r \sqrt{4 - r^2}}{-2 + r^2}\right] \geq 0.
\]
Since
\[
v'(r) = \frac{r^4 \sqrt{4 - r^2}}{2 (3 - r^2)^2} \geq 0,
\]
and $v(0) = 0$ and the claim follows. \qed

2. The technical lemmas

Let $r = |x|$. For $n \geq 3$, let $\omega_n$ be the area of $S^{n-1}$. Marković in [11] proved that
\[
(C) \quad C(x) = \frac{1}{1 - r} \sup_{z > 0} C(z, r),
\]
where
\[
(C) \quad C(z, r) = \frac{4\omega_{n-2}}{\omega_n} \frac{2^{n-1}}{(1 + r)^{n-1}} \frac{1}{\sqrt{1 + z^2}} \int_0^1 \frac{\Psi_r(zt) + \Psi_r(-zt)}{\sqrt{(1 - t^2)^{1-n}}} dt.
\]
Here
\[
\Psi_r(z) = \int_0^{\sqrt{z^2 + 1 - \alpha_r^2}} n - \beta_r + nw - \beta_r w^2 \frac{1}{(1 + w^2)^{n/2 + 1/(1 + k_r^2 w^2)^{n/2 - 1}}} w^{n-2} dw,
\]
and
\[
k_r = \frac{1 - r}{1 + r}, \quad \alpha_r = \frac{r(n - 2)}{n}, \quad \beta_r = \frac{(n - (n - 2)r)}{2}.
\]
Further, in the same paper he has showed that the conjectured equality (1.7) is equivalent to the equality
\[
(2.4) \quad \sup_{z > 0} C(z, r) = C(0, r).
\]
Our goal is to prove (2.4) for $n = 4$. 

2.1. Explicit representation of $\Psi$ for $n = 4$. Let us recalculate the integrand in (2.3):

\[ Q(w) = \frac{w^2 \left( 4 + \frac{1}{3}(-4 + 2r) - \frac{1}{2}(4 - 2r)w^2 + 4wz \right)}{(1 + w^2)^3 \left( 1 + \frac{(1-r)^2w^2}{(1+r)^2} \right)} \]

\[ = \frac{(1+r)^2w^2 \left( 2 + r - 2w^2 + rw^2 + 4wz \right)}{(1 + w^2)^3 \left( (1+r)^2 + (-1+r)^2w^2 \right)} \]

\[ = \frac{(1+r)^2}{r (1 + w^2)^3} - \frac{(1+r)^2(1 + 4r)}{4r^2 (1 + w^2)^2} \]

\[ + \frac{(1+r)^2zw}{16r^3 (1 + w^2)} - \frac{(-1 + r)^2(1 + r)^4 zw}{16r^3 ((1 + r)^2 + (-1 + r)^2w^2)} \]

By elementary integration and since

\[ \int \frac{1}{(1 + w^2)^3} \, dw = \frac{1}{8} \left( \frac{w (5 + 3w^2)}{(1 + w^2)^2} + 3 \tan^{-1}[w] \right) \]

while

\[ \int \frac{1}{(1 + w^2)^2} \, dw = \frac{1}{2} \left( \frac{w}{1 + w^2} + \tan^{-1}[w] \right) \]

we obtain

\[ R(w) = \frac{32r^3}{(1+r)^2} \int Q(w) \, dw \]

\[ = \frac{4rw \left( 1 + w^2 + r \left( -1 + w^2 \right) \right) - 4r \left( 1 + r^2 + (1 + r)^2w^2 \right) z}{(1 + w^2)^2} \]

\[ + 2 \left( -1 + r^2 \right) \tan^{-1}[w] + 2 \left( -1 + r^2 \right) \tan^{-1} \left[ \frac{(-1 + r)w}{1 + r} \right] \]

\[ + (-1 + r^2)^2 z \log \left[ \frac{(1 + r)^2 + (-1 + r)^2w^2}{1 + w^2} \right]. \]

Thus we have
Lemma 2.1. For $r \in (0, 1)$ and $z > 0$ we have

$$
\Psi_r(z) = \frac{(1-r)(1+r)^3}{64r^3} \times \left( r \left( 4 + r^2(4 + r) \right) z + 4 \left( 1 + r^2 \right) z^3 + (2 + r^2 + 2 \left( 1 + r^2 \right) z^2 \right) \sqrt{4 - r^2 + 4z^2} \right)

\frac{(1 + z^2)(1 - r^2)}{(1 + r)^2} \left( -2\right)

+ 4 \tan^{-1} \left[ r \left( \frac{-2rz + (r^2 - 2)\sqrt{4 - r^2 + 4z^2}}{(-2 + r^2)^2 - 4(-1 + r^2)z^2} \right) \right]

+ 2 \left( 1 - r^2 \right) z \log \left[ 1 + z \left( z + r^2z - r\sqrt{4 - r^2 + 4z^2} \right) \right].

Proof. In view of (2.5) and (2.3) we obtain

$$
\frac{32r^3}{(1 + r)^2} \Psi_r(z) = \frac{32r^3}{(1 + r)^2} \int_0^{\frac{z + \sqrt{z^2 + 1 - \alpha_r}}{1 - \alpha_r}} Q(w)dw = R \left( w + \sqrt{2} \frac{z + 1 - \alpha_r}{1 - \alpha_r} \right) - R(0),
$$

which after some elementary transformations implies the lemma. □

2.2. Explicit representation of $C$ for $n = 4$. From Lemma 2.1 we have

Lemma 2.2. For $r \in (0, 1)$ and $z > 0$ we have

$$
\Psi_r(zt) + \Psi_r(-zt) = \frac{(1-r)(1+r)^3}{16r^3} \left( r \sqrt{4 - r^2 + 4t^2z^2} \left( 2 + r^2 + 2 \left( 1 + r^2 \right) t^2z^2 \right) \right)

\frac{2 \left( 1 - r^2 \right) (1 + t^2z^2)}{(1 + r)^2} \left( -2 \right)

- \tan^{-1} \left[ r \left( \frac{2rzt + (2 - r^2)\sqrt{4 - r^2 + 4t^2z^2}}{(-2 + r^2)^2 - 4(-1 + r^2) t^2z^2} \right) \right]

+ \tan^{-1} \left[ r \left( \frac{2rzt - (2 - r^2)\sqrt{4 - r^2 + 4t^2z^2}}{(-2 + r^2)^2 - 4(-1 + r^2) t^2z^2} \right) \right]

- \left( 1 - r^2 \right) t z \tanh^{-1} \left[ \frac{rtz\sqrt{4 - r^2 + 4t^2z^2}}{1 + (1 + r^2)t^2z^2} \right].
$$

Using integration by parts for $V = t$ and

$$
U = - \tan^{-1} \left[ r \left( \frac{2rzt + (2 - r^2)\sqrt{4 - r^2 + 4t^2z^2}}{(-2 + r^2)^2 - 4(-1 + r^2) t^2z^2} \right) \right]

+ \tan^{-1} \left[ r \left( \frac{2rzt - (2 - r^2)\sqrt{4 - r^2 + 4t^2z^2}}{(-2 + r^2)^2 - 4(-1 + r^2) t^2z^2} \right) \right].
$$
and in view of the formula

\[ U' = \frac{r t z^2 \left( (-2 + r^2)^2 + 2 \left( 2 - 3r^2 + r^4 \right) t^2 z^2 \right)}{(1 + t^2 z^2) \sqrt{4 - r^2 + 4t^2 z^2} \left( 1 + (-1 + r^2)^2 t^2 z^2 \right)}, \]

which can be proved by a direct computation, we obtain

\[ \int U(t) dV = tU(t) - \int t \frac{r t z^2 \left( (-2 + r^2)^2 + 2 \left( 2 - 3r^2 + r^4 \right) t^2 z^2 \right)}{(1 + t^2 z^2) \sqrt{4 - r^2 + 4t^2 z^2} \left( 1 + (-1 + r^2)^2 t^2 z^2 \right)} dt. \]

Similarly, for \( V_1 = t^2/2 \) and

\[ U_1 = -z(1 - r^2) \tanh^{-1} \left[ \frac{r t z \sqrt{4 - r^2 + 4t^2 z^2}}{1 + (1 - r^2) t^2 z^2} \right] \]

we obtain

\[ U'_1 = \frac{r z^2 \left( -4 + 5r^2 - r^4 + (-4 + 7r^2 - 4r^4 + r^6) t^2 z^2 \right)}{(1 + t^2 z^2) \sqrt{4 - r^2 + 4t^2 z^2} \left( 1 + (-1 + r^2)^2 t^2 z^2 \right)}, \]

and then

\[ \int U_1(t) dV_1 = U_1 V_1 - \int \frac{2 t^2 r z^2 \left( -4 + 5r^2 - r^4 + (-4 + 7r^2 - 4r^4 + r^6) t^2 z^2 \right)}{(1 + t^2 z^2) \sqrt{4 - r^2 + 4t^2 z^2} \left( 1 + (-1 + r^2)^2 t^2 z^2 \right)} dt. \]

Furthermore we have

\[ VU' + V_1 U'_1 = -\frac{r t^2 z^2 \left( -4 + 3r^2 - r^4 - (4 - 5r^2 + r^6) t^2 z^2 \right)}{2 (1 + t^2 z^2) \sqrt{4 - r^2 + 4t^2 z^2} \left( 1 + (-1 + r^2)^2 t^2 z^2 \right)}; \]

Let

\[ Y = \frac{r \sqrt{4 - r^2 + 4t^2 z^2} \left( 2 + r^2 + 2 \left( 1 + r^2 \right) t^2 z^2 \right)}{2 (1 - r^2) (1 + t^2 z^2)} \]

and

\[ X = -VU' - V_1 U'_1 + Y. \]

By using the formula

\[ \int \frac{a}{\sqrt{b + ct^2}} \left( 1 + \frac{(c-a^2) t^2}{b} \right) dt = \tanh^{-1} \left[ \frac{at}{\sqrt{b + ct^2}} \right] \]

and the representation

\[ 2 \left( 1 - r^2 \right) zX = \frac{4r \left( 1 + r^2 \right) t^2 z^3}{\sqrt{4 - r^2 + 4t^2 z^2}} + r \left( 1 + r^2 \right) z \sqrt{4 - r^2 + 4t^2 z^2} \]

\[ + \frac{r z}{\sqrt{4 - r^2 + 4t^2 z^2} (1 + t^2 z^2)} - \frac{r (3 + r^2) z}{\sqrt{4 - r^2 + 4t^2 z^2} \left( 1 + (-1 + r^2)^2 t^2 z^2 \right)}, \]
Lemma 2.3. For \( t \in [0, 1] \) we have

\[
\int_0^t (\Psi_r(zs) + \Psi_r(-zs))ds = \frac{1}{16r^3} (1 + r)^2 \left( \frac{1}{2} r (1 + r^2) tz\sqrt{4 - r^2 + 4t^2z^2} + \frac{1}{2} \tanh^{-1} \left[ \frac{rtz}{\sqrt{4 - r^2 + 4t^2z^2}} \right] - \tanh^{-1} \left[ \frac{r(-3 + r^2)tz}{\sqrt{4 - r^2 + 4t^2z^2}} \right] \right)
\]

\[
+ \frac{1}{2} \tanh^{-1} \left[ \frac{rtz}{\sqrt{4 - r^2 + 4t^2z^2}} \right] - \frac{1}{2} \tanh^{-1} \left[ \frac{r(-3 + r^2)tz}{\sqrt{4 - r^2 + 4t^2z^2}} \right]
\]

\[
+ (-1 + r^2) tz \tan^{-1} \left[ \frac{r(2rtz + (2 - r^2)\sqrt{4 - r^2 + 4t^2z^2})}{(-2 + r^2)^2 - 4(-1 + r^2)t^2z^2} \right]
\]

\[
- (-1 + r^2) tz \tan^{-1} \left[ \frac{r(2rtz - (2 - r^2)\sqrt{4 - r^2 + 4t^2z^2})}{(-2 + r^2)^2 - 4(-1 + r^2)t^2z^2} \right]
\]

\[
- \frac{1}{2} (-1 + r^2)^2 t^2z^2 \tanh^{-1} \left[ \frac{rtz\sqrt{4 - r^2 + 4t^2z^2}}{1 + (1 + r^2)t^2z^2} \right]\).
\]

By putting \( t = 1 \) in Lemma 2.3 and using (2.2), in view of

\[
\frac{4\omega_{1-2}}{\omega_4} \frac{2^{4-1}}{(1 + r)^{4-1}} = \frac{4 \cdot 2\pi \cdot 8}{2\pi^2 (1 + r)^3} = \frac{32}{\pi (1 + r)^3},
\]

we obtain

Lemma 2.4. For \( r \in (0, 1) \) and \( z > 0 \) we have

\[
C(r, z) = \frac{2(1 - r)}{\pi r^3 \sqrt{1 + z^2}} (h_1(z) + h_2(z) + h_3(z)),
\]

where

\[
h_1(z) = \frac{r(1 + r^2)\sqrt{4 - r^2 + 4z^2}}{2(1 - r^2)} + \frac{\tanh^{-1} \left[ \frac{rz}{\sqrt{4 - r^2 + 4z^2}} \right] - \tanh^{-1} \left[ \frac{r(-3 + r^2)z}{\sqrt{4 - r^2 + 4z^2}} \right]}{2(1 - r^2) z},
\]
and
\[ h_2(z) = \tan^{-1} \left[ \frac{r \left( 2rz - (2 - r^2)\sqrt{4 - r^2 + 4z^2} \right)}{(-2 + r^2)^2 - 4 (-1 + r^2) z^2} \right] - \tan^{-1} \left[ \frac{r \left( 2rz + (2 - r^2)\sqrt{4 - r^2 + 4z^2} \right)}{(-2 + r^2)^2 - 4 (-1 + r^2) z^2} \right] \]

and
\[ h_3(z) = \frac{1}{2} (1 + r^2) z \tanh^{-1} \left[ \frac{rz\sqrt{4 - r^2 + 4z^2}}{1 + (1 + r^2) z^2} \right]. \]

Finally we need the following lemmata

**Lemma 2.5.** Let
\[ L(z) = \tanh^{-1} \left[ \frac{r z}{\sqrt{4 - r^2 + 4z^2}} \right] - \tanh^{-1} \left[ \frac{r (-3 + r^2) z}{\sqrt{4 - r^2 + 4z^2}} \right]. \]

Then, \( L(z) \leq rz\sqrt{4 - r^2} \).

**Proof.** By differentiating \( L \) we obtain
\[ L'(z) = \frac{r \left( 4 - r^2 + (4 - 3r^2 + r^4) z^2 \right)}{(1 + z^2) \sqrt{4 - r^2 + 4z^2} \left( 1 + (-1 + r^2)^2 z^2 \right)}. \]

Since
\[ \frac{\partial_z \left( 4 - r^2 + (4 - 3r^2 + r^4) z^2 \right)}{(1 + z^2) \left( 1 + (-1 + r^2)^2 z^2 \right)} = -\frac{2z}{\left( 1 + z^2 \right)^2} + \frac{2 \left( -3 + r^2 \right) \left( -1 + r^2 \right) z}{(1 + (-1 + r^2)^2 z^2)^2} \leq 0, \]

it follows that \( (L(z) - rz\sqrt{4 - r^2})' \leq L'(0) - r\sqrt{4 - r^2} = 0 \). So \( L(z) \leq rz\sqrt{4 - r^2} \). \( \square \)

**Lemma 2.6.** Let
\[ g_1(z) = \frac{r\sqrt{4 - r^2} + r \left( 1 + r^2 \right) \sqrt{4 - r^2 + 4z^2}}{\sqrt{1 + z^2}}. \]

Then \( \sup_{z > 0} g_1(z) = g_1(0) = r\sqrt{4 - r^2} + r \left( 1 + r^2 \right) \sqrt{4 - r^2} \).

**Proof.** Since
\[ g_1'(z) = \frac{r z \left( r^2 + r^4 - \sqrt{(-4 + r^2) \left( r^2 - 4 \left( 1 + z^2 \right) \right)} \right)}{\left( 1 + z^2 \right) \sqrt{4 - r^2 + 4z^2}} \]

and
\[ (r^2 + r^4)^2 - (-4 + r^2)(r^2 - 4(1 + z^2)) = 2r^6 + r^8 - 16(1 + z^2) + 4r^2(2 + z^2) \leq -5 - 12z^2 \leq 0, \]

it follows that \( g_1'(z) \leq 0 \) for \( z \geq 0 \). Thus \( g_1(z) \leq g_1(0) \) for \( z \geq 0 \) what we needed to prove. \( \square \)
Lemma 2.7. Let
\[ h_2(z) = \tan^{-1} \left[ \frac{r \left(2rz + (2-r^2)\sqrt{4-r^2 + 4z^2}\right)}{(-2 + r^2)^2 - 4(-1 + r^2) z^2} \right] - \tan^{-1} \left[ \frac{r \left(2rz - (2-r^2)\sqrt{4-r^2 + 4z^2}\right)}{(-2 + r^2)^2 - 4(-1 + r^2) z^2} \right]. \]
Then \( \sup_{z>0} h_2(z) = h_2(0) = 2 \tan^{-1} \left[ r \frac{\sqrt{4-r^2}}{2-r^2} \right]. \)

Proof. We have
\[ h_2'(z) = -2r (2-r^2) z \left( -2 + r^2 + 2(-1 + r^2) z^2 \right) \frac{(1+z^2)\sqrt{4-r^2 + 4z^2}}{1 + (-1 + r^2)^2 z^2}, \]
so \( h_2'(z) \leq 0 \), and \( h_2(z) \leq h_2(0) \) for every \( z \).

By using the formula \( \frac{1}{2} \log \frac{1+a}{1-a} = \tanh^{-1}(a) \), for \( a = r \sqrt{4-r^2 + 4z^2} \frac{1}{1+(-1+r^2)z^2} < 1 \), we obtain

Lemma 2.8. For \( z > 0 \) and \( 0 < r < 1 \) we have
\[ h_3(z) := \frac{1}{2} (-1 + r^2) z \tanh^{-1} \left[ \frac{r \sqrt{4-r^2 + 4z^2}}{1 + (1+r^2) z^2} \right] \leq h_3(0) = 0. \]

2.3. Proof of the main result.

Theorem 2.9. For \( r \in (0, 1) \) we have
\[ \sup_{z>0} C(z, r) = C(0, r) = \left( r \sqrt{4-r^2} (2+r^2) + 4 (1-r^2) \tan^{-1} \left[ r \frac{\sqrt{4-r^2}}{2-r^2} \right] \right) \frac{1}{\pi (1+r)^3}. \]

Proof. From Lemmas 2.4, 2.5, 2.6, 2.7 and 2.8 we obtain
\[ C(z, r) = \frac{2(1-r)}{\pi r^3 \sqrt{1+z^2}} (h_1(z) + h_2(z) + h_3(z)) \]
\[ \leq \frac{2(1-r)}{\pi r^3 \sqrt{1+z^2}} \left( \frac{r (1+r^2) \sqrt{4-r^2 + 4z^2}}{2 (1-r^2)} + \frac{L(z)}{2 (1-r^2) z} + h_2(0) + h_3(0) \right) \]
\[ \leq \frac{2(1-r)}{\pi r^3 \sqrt{1+z^2}} \left( \frac{r (1+r^2) \sqrt{4-r^2 + 4z^2}}{2 (1-r^2)} + \frac{r z \sqrt{4-r^2}}{2 (1-r^2) z} + h_2(0) + h_3(0) \right) \]
\[ = \frac{1-r}{\pi r^3 (1-r^2)} g_1(z) + \frac{2(1-r)}{\pi r^3 \sqrt{1+z^2}} (h_2(0) + h_3(0)) \]
\[ \leq \frac{1-r}{\pi r^3 (1-r^2)} g_1(0) + \frac{2(1-r)}{\pi r^3} (h_2(0) + h_3(0)) \]
\[ = \frac{2(1-r)}{\pi r^3} (h_1(0) + h_2(0) + h_3(0)) = C(0, r) \]
\[ = \left( r \sqrt{4-r^2} (2+r^2) + 4 (1-r^2) \tan^{-1} \left[ r \frac{\sqrt{4-r^2}}{2-r^2} \right] \right) \frac{1}{\pi (1+r)^3}. \]

So \( \sup_{z>0} C(z, r) = C(0, r) \) what we needed to prove. \( \square \)
Remark 2.10. It seems that the same strategy for \( n \neq 4 \) does not work, because the integrand that appear in definition of the function \( C \) is much more complicated.

References

[1] Sh. Axler, P. Bourdon and W. Ramey, *Harmonic function theory*, Springer, New York (1992).

[2] F. Colonna, *The Bloch constant of bounded harmonic mappings*, Indiana University Math. J. **38** (1989), 829–840.

[3] D. Kalaj and M. Vuorinen, *On harmonic functions and the Schwarz lemma*, Proc. Amer. Math. Soc. **140** (2012), 161–165.

[4] D. Kalaj and M. Marković, *Optimal estimates for harmonic functions in the unit ball*, Positivity **16** (2012), 771–782.

[5] D. Kalaj and M. Marković, *Optimal Estimates for the Gradient of Harmonic Functions in the Unit Disk*, Complex Analysis and Operator Theory **7** (2013), 1167–1183.

[6] D. Khavinson, *An extremal problem for harmonic functions in the ball*, Canadian Math. Bulletin **35** (1992), 218–220.

[7] G. Kresin and V. Maz’ya, *Sharp pointwise estimates for directional derivatives of harmonic functions in a multidimensional ball*, J. Math. Sci. **169** (2010), 167–187.

[8] G. Kresin and V. Maz’ya, *Optimal estimates for the gradient of harmonic functions in the multidimensional half-space*, Discrete Contin. Dyn. Syst. **28** (2010), 425–440.

[9] G. Kresin and V. Maz’ya, *Sharp real-part theorems*. Springer, Berlin, Jan. 1, 2007. (Lecture Notes in Mathematics, 1903). ISBN: 978-3-540-69573-8.

[10] M. Marković, *On harmonic functions and the hyperbolic metric*, Indagationes Mathematicae **26** (2015), 19–23.

[11] M. Marković, *Proof of the Khavinson conjecture near the boundary of the unit ball*, arXiv:1508.00125v1.

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