Berezin quantization
and unitary representations of Lie groups

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Abstract

In 1974, Berezin proposed a quantum theory for dynamical systems having a Kähler manifold as their phase space. The system states were represented by holomorphic functions on the manifold. For any homogeneous Kähler manifold, the Lie algebra of its group of motions may be represented either by holomorphic differential operators ("quantum theory"), or by functions on the manifold with Poisson brackets, generated by the Kähler structure ("classical theory"). The Kähler potentials and the corresponding Lie algebras are constructed now explicitly for all unitary representations of any compact simple Lie group. The quantum dynamics can be represented in terms of a phase-space path integral, and the action principle appears in the semi-classical approximation.
1 Introduction

1.1 Historical background

In a series of papers [1, 2, 3, 4], written in the beginning of the 70-ties, Berezin developed a new approach to description of quantum dynamical systems on non-Euclidean phase-space manifolds. The approach was essentially based upon two basic implements: functional formalism and complex analysis.

From its very beginning, quantum mechanics was expressed in terms of the Hilbert space of state vectors (represented, as a rule, by wave functions in the coordinate representation) and observables given by operators acting in that space. The fundamental observables, like energy, momentum or angular momentum, were represented by differential operators, so differential equations played the dominant role in the early quantum theory. Fock was probably the first[5, 6] who understood, still in the early days of quantum theory, that functional methods may be extremely powerful, especially, in applications of the method of second quantization to quantum field theory. Schwinger's quantum action principle and Feynman’s path integrals were introduced into quantum electrodynamics and did lead to its triumph in the late 40-ties. Berezin was exposed to that brilliant development when he was a student in the 50-ties and took part in the Landau seminars at the Institute of Physical Problems in Moscow. At that time, there was evidently a missing element in the functional representation of quantum electrodynamics: a proper description of fermion fields. Schwinger employed second-kind variables[7] in his action principle, in order to describe fermion fields, electrons in particular, but the formalism of anti-commuting numbers was not yet developed systematically. Berezin invented the integral in anti-commuting (Grassmann) variables, and the unified functional approach to quantal systems with Bose and Fermi degrees of freedom was presented in a complete form in his thesis. His famous book[8] was based upon the thesis, and it has been the mathematical foundation for the future development of methods of supersymmetry in physics[9]. Later, spinning string models prompted an elaboration of the functional approach to particle spin dynamics and the Poisson brackets on super-phase spaces were defined[10, 11].

The second quantization in its functional form is naturally related to complex analysis. Fock[12], and later Bargmann[14], used the holomorphic representation of functionals in quantum theory. (For reviews, see e.g. Refs.[12]
Investigation of properties of normal and anti-normal symbols of the evolution operator enabled Berezin to derive sophisticated theorems about spectra of Hamiltonian operators for the Euclidean phase space. Those ideas were further developed later, e.g. in Ref. On the other hand, Berezin discovered that the formalism can be extended to non-Euclidean dynamical systems, as soon as the corresponding phase-space manifold has a complex Kählerian structure.

1.2 Quantization in Euclidean phase spaces

Berezin’s line of reasoning was fairly straightforward. It was known that in the Euclidean case, the states of any quantal system can be represented by holomorphic functions \( \psi(z) \), where \( z \equiv \{z_1, \ldots, z_m\} \in \mathbb{C}^m \) and \( m \) is the number of degrees of freedom. The structure of the Hilbert space was introduced in the space of the state vectors by means of the inner product

\[
(\psi_2, \psi_1) = \int_{\mathbb{C}^m} \overline{\psi_2}(z)\psi_1(z)e^{-(z \cdot \bar{z})}d\mu(z, \bar{z}). \tag{1}
\]

Here \( (z \cdot \bar{z}) \equiv \sum_{\alpha=1}^{m} z_{\alpha}\bar{z}_{\alpha} \) is the Euclidean scalar product, and the integral is taken with the standard Liouville measure

\[
d\mu(z, \bar{z}) \equiv \prod_{\alpha=1}^{m} \frac{dz_{\alpha} \wedge d\bar{z}_{\alpha}}{2\pi i} = \prod_{\alpha=1}^{m} \frac{dq_{\alpha}dp_{\alpha}}{2\pi \hbar}, \quad z = (q + iP)/\sqrt{2\hbar}, \tag{2}
\]

where \( q \) and \( p \) are the usual coordinate and momentum vectors. The inner product is invariant under translations in the phase space, if the wave functions have appropriate transformation properties,

\[
z \to z' = \epsilon z + a \sim \psi(z) \to \psi'(z) = e^{i(\epsilon z \cdot \bar{a}) + \frac{1}{2}(a \cdot \bar{a})}\psi(z'), \tag{3}
\]

where \( a \) is any complex \( m \)-vector and \( |\epsilon| = 1 \). After three consecutive translations by \( a, b \) and \( -a - b \), the phase space remains invariable, while each wave function gets a constant phase shift

\[
\psi(z) \to \psi'(z) = e^{i\varphi}\psi(z), \quad \varphi = [(b \cdot \bar{a}) - (a \cdot \bar{b})]/2i. \tag{4}
\]

Evidently, \( \frac{1}{2}\kappa \) is just the Euclidean area of the triangle built upon the vectors \( a \) and \( b \). In general, any continuous translation in the phase space, given by
a closed line in $\mathbb{C}^m$, results in a phase shift, which is the Euclidean area of the minimal surface stretched upon the line. Thus the standard symplectic structure appears from Eq. (1), and one gets a geometric interpretation for the canonical quantization: the space of state vectors is a line bundle based upon the phase space. The phase space translations and the phase shifts together constitute a Lie group $W_m$ (Schwinger’s special canonical group $[17]$). The corresponding Lie algebra (the Heisenberg–Weyl algebra) is given by the canonical commutation relations for its basis elements, which are the coordinate and momentum operators (or the creation and annihilation operators) and the unit operator which generates the phase shifts (the gauge group $U(1) \subset W_m$).

The Berezin quantization stems from the extension of the holomorphic quantization to phase spaces which are Kähler manifolds having non-flat geometries. The inner product is defined by means of the integral with an appropriate invariant measure, and the Hermitean scalar product in the exponent is substituted for the Kähler potential, see Eq. (5) below.

1.3 Contents of this paper

The next Section presents a short review of Berezin’s quantization principle. A special attention is paid to the properties of cocycle functions which define the structure of the line bundle representing the space of state vectors. Two examples are also given: the two-dimensional sphere and the Lobachevsky plane. Together with the usual phase plane, given in the previous subsection, they represent all possible quantizations for one degree of freedom. The quantization is given by a non-negative integer $l$, which specifies the quantized curvature of the phase space. In the compact case, i.e. for the sphere, the total number of states of the system is finite, $N = l + 1$. Section 3 shows a correspondence between the quantization on a homogeneous Kähler manifold and the representation of the Lie algebra for its group of motion. The Lie algebra is represented in two alternative ways: i) by holomorphic differential operators, ii) by their symbols, i.e. functions on the phase space and the appropriate Poisson brackets. This gives the actual meaning to the word “quantization”. In Section 4 the Kähler potentials are constructed explicitly for all compact simple Lie groups. Complex coordinates correspond to positive roots of the Lie algebra, and fundamental Kähler potentials are expressed in terms of generalized determinants (considered in Appendix). A
description of dynamics is discussed in Section 5. An analogue of the path integral is derived, which leads to an action functional in the semi-classical approximation, and the classical Hamilton equations of motion result from the corresponding variational principle.

The paper is addressed mainly to physicists, and mathematical arguments are more intuitive than rigorous.

2 Quantization on homogeneous Kähler manifolds

2.1 The Hilbert space of state vectors

We shall consider a m-dimensional Kähler manifold $\mathcal{M}$ with local complex coordinates $z_\alpha (\alpha = 1, \cdots, m)$, where a group $\mathcal{G}$ is acting holomorphically, i.e. $z \rightarrow gz, \forall g \in \mathcal{G}$ (notations follow mainly Ref.[18]). State vectors of the quantal system are defined as holomorphic sections of the holomorphic line bundle $\mathcal{L}$ over $\mathcal{M}$. The state vectors are represented by locally holomorphic wave functions $\psi(z)$. The Hilbert space structure is assigned to $\mathcal{L}$ by means of the following $\mathcal{G}$-invariant inner product,

$$\langle \psi_2, \psi_1 \rangle = \int_{\mathcal{M}} \overline{\psi_2(z)} \psi_1(z) \exp[-K(z, \bar{z})]d\mu(z, \bar{z}).$$

(5)

Here $K(z, \bar{z})$ is the Kähler potential, associated with the line bundle $\mathcal{L}$ and defined in any open coordinate neighbourhood of the manifold $\mathcal{M}$. The meaning of the integral includes, of course, the sum over the neighbourhoods covering $\mathcal{M}$. The invariance is imposed by the transformation law for the wave functions, which is consistent with that for the Kähler potential, namely for any $g \in \mathcal{G}$

$$K(z, \bar{z}) \rightarrow K(gz, \bar{gz}) = K(z, \bar{z}) + \Phi(z; g) + \overline{\Phi(z; g)},$$

(6)

$$\psi(z) \rightarrow \psi(gz) = \exp[\Phi(z; g)](\hat{U}(g^{-1})\psi)(z),$$

(7)

where $\hat{U}(g)$ is a unitary operator representing the group element $g$ in the Hilbert space $\mathcal{L}$, $\hat{U}(g_1)\hat{U}(g_2) = \hat{U}(g_1g_2)$, and the cocycle function $\Phi(z; g)$ is (locally) holomorphic.
The (invariant) integration measure is expressed, as usual, in terms of the $m$-th power of the corresponding Kähler (1,1)-form $\omega$,

$$d\mu(z, \bar{z}) \equiv C \frac{\omega}{2\pi i} \wedge \cdots \wedge \frac{\omega}{2\pi i} \ (m \text{ times}),$$

where $C$ is a normalization constant (see Eq. (12) below) and the factor $2\pi i$ is introduced for future convenience. (We assume, of course, that the form $\omega$ is non-degenerate, so the integrals do not vanish identically.) The (1,1)-form $\omega$ has the following local representation

$$\omega \equiv \omega_{\alpha\bar{\beta}}(z, \bar{z}) dz^\alpha \wedge d\bar{z}^\beta; \quad \omega_{\alpha\bar{\beta}} = \partial_{\alpha} \partial_{\bar{\beta}} K,$$

where $\partial_{\alpha} \equiv \partial/\partial z^\alpha$, $\partial_{\bar{\beta}} \equiv \partial/\partial \bar{z}^\beta$. The form $\omega$ is closed, $\partial \omega = 0$, $\bar{\partial} \omega = 0$, and invariant under the group transformations, as follows from Eq. (6).

Let us make two more assumptions.

(A) Group $G$ acts transitively in $M$, i.e. for any two points there is a group element transforming one of them into the other.

(B) Excluding from $M$ a manifold $X$ of a lower dimensionality, one gets a domain with a simple topology, $M \setminus X \equiv \mathbb{C}^m$.

It follows that any holomorphic function invariant under $G$ is a constant. This property is very important for the following. Various manifolds $X$, to be excluded in order to reduce $M$ to $\mathbb{C}^m$, are transformed into each other under the group transformations. Because of (B), the inner product in Eq. (5) can be considered as the integral over $\mathbb{C}^m$ and is independent of the choice of $X$.

### 2.2 Symbols of linear operators

For the manifolds of our concern here, the Kähler potential can be considered as a boundary value of a function $K(\zeta, \bar{z})$, holomorphic in the first variable and anti-holomorphic in the second one. Berezin introduced a “supercomplete set” of state vectors

$$\Psi_v(z) \equiv \exp[K(z, \bar{v})],$$

This set is an important class of generalized coherent states (an exposition and an abundant bibliography can be found in Refs. [19, 20]).

One can prove that, under a proper normalization of the integration measure, any element of the Hilbert space is reproduced by the integral on the
manifold,

\[ \psi(\zeta) \equiv \int_\mathcal{M} \psi(z) \exp \left[ K(\zeta, \bar{z}) - K(z, \bar{z}) \right] d\mu(z, \bar{z}), \quad \forall \psi \in \mathcal{L}. \]  

(11)

The proof stems from the fact that, because of (6)-(7), the ratio of the integral in the r.h.s. to \( \psi(\zeta) \) is invariant under the group transformations. This number is independent on \( \psi \), and setting a proper value of \( C \) in Eq. (8), one can make it equal to 1. In order to prove that, one can calculate \( (\psi, \psi) \), applying (10) first to a coherent state and then to \( \psi \), getting an identity. Moreover, for the manifolds we are dealing with, \( K(0, \bar{z}) \equiv 0 \), and the following integral exists

\[ \int_\mathcal{M} \exp\left[-K(z, \bar{z})\right]d\mu(z, \bar{z}) = 1. \]  

(12)

In the other words, all constant sections of \( \mathcal{L} \) belong to the Hilbert space, and one can assume that \( \psi_0(z) \equiv 1 \) has the unit norm. (In typical physical problems a constant \( \psi \) corresponds to the “vacuum”, i.e. the system ground state for a proper Hamiltonian.) Thus the constant \( C \) in Eq. (8) is expressed simply in terms of an integral over the manifold.

The reproducing kernel given in terms of the Kähler potential in Eq. (11), has an expansion in terms of any orthonormal basis \( \{\phi_\nu(z)\} \) in \( \mathcal{L} \),

\[ \exp[K(\zeta, \bar{z})] = \sum_\nu \phi_\nu(\zeta)\bar{\phi}_\nu(z). \]  

(13)

This equality gives also an expansion of the coherent state (10) in the basis of the orthonormal states.

Linear operators in the Hilbert space, in particular describing observables of the quantal system, are represented with their symbols in the following way: \( \hat{A} \to A(z, \bar{z}) \) means

\[ (\hat{A}\psi)(\zeta) = \int_\mathcal{M} A(\zeta, \bar{z})\psi(z) \exp[K(\zeta, \bar{z}) - K(z, \bar{z})] d\mu(z, \bar{z}). \]  

(14)

The symbol representation has the following nice properties.

1. The symbol of the unit operator is just 1, \( \hat{I} \to I(\zeta, \bar{z}) \equiv 1 \).

2. The trace of any operator is given by the integral of its symbol,

\[ \text{tr}(\hat{A}) = \int_\mathcal{M} A(z, \bar{z})d\mu(z, \bar{z}). \]  

(15)
For compact manifolds, the trace of the unity operator $\hat{I}$ exists, the volume is finite and equals the total number of states $N$,

$$\text{tr}(\hat{I}) \equiv N = \int_{\mathcal{M}} d\mu(z, \bar{z}). \quad (16)$$

3. The Hermitean conjugation in the Hilbert space is represented by the complex conjugation of the symbol and transposition of its arguments,

$$\hat{A}^\dagger \rightarrow A^\ast(\zeta, \bar{z}) = \overline{A(z, \bar{\zeta})}. \quad (17)$$

4. The symbol for the product of operators is given by an integral of the product of their symbols (the $*$-product)

$$\hat{A}\hat{B} \rightarrow (A \ast B)(\zeta, \bar{\eta}) \equiv \int_{\mathcal{M}} A(\zeta, \bar{z})B(z, \bar{\eta}) \times \exp[K(\zeta, \bar{z}) - K(z, \bar{z}) + K(z, \bar{\eta}) - K(\zeta, \bar{\eta})] d\mu(z, \bar{z}). \quad (18)$$

In particular, one has an analogue of the Gaussian integral,

$$\exp[K(\zeta, \bar{\zeta})] = \int_{\mathcal{M}} \exp[K(\zeta, \bar{z}) - K(z, \bar{z}) + K(z, \bar{\zeta})] d\mu(z, \bar{z}). \quad (19)$$

The scalar products of the coherent states (10) are obtained from (19).

In general, any system state is given by a (positive semi-definite) density operator $\hat{\rho}$, which can be also represented with its symbol $\rho(z, \bar{z})$. For any operator $\hat{A}$, its expectation value in the given state is

$$<A>_{\hat{\rho}} \equiv \text{tr}(\hat{A}\hat{\rho}) = \int_{\mathcal{M}} A(\zeta, \bar{z})\rho(z, \bar{\zeta}) \times \exp[K(\zeta, \bar{z}) - K(z, \bar{z}) + K(z, \bar{\zeta})] d\mu(z, \bar{z}) d\mu(z, \bar{\zeta}).$$

Thus the associative algebra of observables for the quantal system is constructed completely in terms of the operator symbols.

### 2.3 Cocycle functions

The cocycle functions, defined in Eq. (6), have the following properties,

$$\Phi(z, e) = 0, \quad e - \text{unity in } \mathcal{G}, \quad (21)$$

$$\Phi(gz; g^{-1}) = -\Phi(z; g), \quad \forall g \in \mathcal{G}, \quad (22)$$

$$\Phi(z; g_2g_1) = \Phi(g_2z; g_1) + \Phi(z; g_2), \quad \forall g_1, g_2 \in \mathcal{G}. \quad (23)$$
Writing the latter condition in the infinitesimal form, one gets differential equations for the cocycle functions. In order to write them down, we need some notations.

Let $g$ be the Lie Algebra of the group $G$. Introducing a basis $\tau_a$ in $g$, with $a = 1, \cdots, n \equiv \dim g$, one has Cartesian coordinates for the group elements $g = \exp(-\xi^a \tau_a)$, and the corresponding (left) Lie derivatives $D_a$ on the group manifold. The action of the group $G$ on the manifold $M$ determines the holomorphic Killing fields

$$\nabla_a = \kappa^a(z) \partial_a, \quad \kappa^a(z) \equiv D_a(gz)^a |_{g=e}.$$  

The conjugate Killing field and the differential operator $\nabla_a$ are defined similarly. The Lie derivative and the Killing derivative satisfy the commutation relations of the Lie algebra $g$,

$$[\tau_a, \tau_b] = f^c_{ab} \tau_c \quad \Rightarrow \quad [D_a, D_b] = f^c_{ab} D_c, \quad [\nabla_a, \nabla_b] = f^c_{ab} \nabla_c.$$  

A holomorphic vector field is associated with the cocycle functions,

$$\varphi_a(z) = D_a \Phi(z; g) |_{g=e}.$$  

Now the differential equations for the cocycle functions are written down as follows,

$$D_a \Phi = \nabla_a \Phi + \varphi_a.$$  

The equations are consistent, as soon as $\varphi_a(z)$ satisfy the linear differential equations

$$\nabla_a \varphi_b - \nabla_b \varphi_a = f^c_{ab} \varphi_c.$$  

In Eq. (6) the cocycle functions are defined up to an arbitrary imaginary term, but this term determines the phase shift of the wave functions in Eq. (7). As soon as the wave functions must be one-valued functions on $M$, the additional term must be a multiple of $2\pi i$, and one gets a boundary condition on $\Phi$. Namely, let us consider a compact one-parameter subgroup of $G$, $u(t) \in U(1) \subset G$, where $0 \leq t < 2\pi$. For any such a subgroup, one must have

$$\Phi(z; u(2\pi)) = 2\pi il,$$  

where $l$ is an integer, depending, in principle, on the equivalence class of the subgroups $U(1)$. This is a quantization condition for the Kähler manifolds, which is necessary for consistency of the quantum theory, based upon
the line bundle structure, presented above. The quantization condition is of exactly the same nature as the Dirac quantization for magnetic charge \([21]\), as explained by Wu and Yang \([22]\). If this condition holds, the corresponding Kähler potential is called integral. One can see that integral of \(\omega\) over any two-dimensional cycle stretched upon a line homotopical to the group trajectory is given by Eq. \((29)\).

### 2.4 Sphere and pseudosphere

The simplest examples presented by Berezin \([4]\) for \(m = 1\) are the two-dimensional sphere \(S^2\) and the pseudosphere \(H^2\) (the Lobachevsky plane). The following table is a summary for these two manifolds,

| \(\mathcal{M}\)   | \(S^2\)         | \(H^2\)   |
|-------------------|-----------------|-----------|
| \(G\)            | \(SU(2)\)       | \(SU(1,1)\) |
| \(K(z, \bar{z})\) | \(l\ \log(1 + z\bar{z})\) | \(-l(1 + 1)\ \log(1 - z\bar{z})\) |
| \(d\mu(z, \bar{z})\) | \((l + 1)(1 + z\bar{z})^{-2}dz \wedge d\bar{z}/2\pi i\) | \(l(1 - z\bar{z})^{-2}dz \wedge d\bar{z}/2\pi i\) |
| \(gz\)           | \((\alpha z - \beta)/(\bar{\beta}z + \alpha)\) | \((\alpha z + \beta)/(\bar{\beta}z + \alpha)\) |
| \(\Phi(z; g)\)   | \(-l\ \log(\bar{\beta}z + \alpha)\) | \((l + 1)\ \log(\bar{\beta}z + \alpha)\) |

In both the cases, \(l\) is any positive integer, \(\alpha\) and \(\beta\) are complex group parameters. An orthonormal basis in \(L\) is given by functions \(\phi_\nu = c_\nu z^\nu\), where \(\nu\) is a nonnegative integer and \(c_\nu\) is a normalization constant. For the compact case, \(S^2\), the manifold has a finite volume, which equals the total number of states, \(V = N = l + 1\). (The norm of \(\phi_\nu\), Eq. \((5)\), does not exist if \(\nu > l\).) For the pseudosphere, the Hilbert space is infinite, and the domain in \(\mathbb{C}\) is noncompact, \(z\bar{z} < 1\). The monomials given above are always eigen-functions of the Hermitean operator \(\hat{H} = zd/dz\), which has the integer spectrum, like the one-dimensional harmonic oscillator in the Euclidean case. Thus we have got all the unitary representations of \(SU(2)\) (corresponding to the angular momentum values \(\frac{1}{2}l\)), and the representations of the discrete series for \(SU(1,1)\).

The usual quantum mechanics in one degree of freedom is the boundary case between \(S^2\) and \(H^2\). Actually, if the system motion is confined to a restricted domain \(|z| \ll 1\), one can introduce the usual phase space and write \(z = (q + ip)/\sqrt{2R}\), so that for \(R \to \infty\) one gets the Hilbert space of Eq. \((1)\) for \(m = 1\), \(\hbar = R^2/l\) and \(l \gg 1\). Berezin noted that \(\hbar^{-1}\) must have a discrete.
spectrum for $S^2$; he remarked that this condition “seems extravagant” [4]. In fact, the “quantization” of the Kähler structures is typical; it results from Eq. (29), if one has a nontrivial representation of a compact subgroup in the group of motions. Here the subgroup is $U(1)$: $\beta = 0, \alpha = e^{it}$.

The quantization on sphere was proposed by Souriau [23]. However, a decade before, Klauder [24] was probably the first who considered the quantization on sphere (actually, on the direct product of infinitely many spheres) for the spinor representation (the particular case of $l = 1$), introduced the coherent states, and used the method for quantum field theory. Berezin [3] considered classical symmetric complex domains, and the Lobachevsky plane was the simplest particular case.

3 The Lie algebra and the Poisson brackets

The group action in $\mathcal{L}$ is represented by the unitary operator

$$\hat{U}(g)\psi(z) = \exp[-\Phi(z; g^{-1})]\psi(g^{-1}z), \quad (31)$$

which leads to a representation of the Lie algebra $\mathfrak{g}$ in terms of the holomorphic first-order differential operators,

$$\tau_a \rightarrow \hat{T}_a = \nabla_a - \varphi_a(z), \quad [\hat{T}_a, \hat{T}_b] = f_{ab}^c \hat{T}_c, \quad (32)$$

cf. Eqs. (25) and (28).

The symbol representation for the operators is obtained from Eq. (11),

$$\langle \hat{U}(g)\psi(\zeta) \rangle = \int_\mathcal{M} U_g(\zeta, \bar{z})\psi(z) \exp\left[K(\zeta, \bar{z}) - K(z, \bar{z})\right]d\mu(z, \bar{z}), \quad (33)$$

$$U_g(\zeta, \bar{z}) = \exp\left[K(g^{-1}\zeta, \bar{z}) - K(\zeta, \bar{z}) - \Phi(\zeta; g^{-1})\right], \quad (34)$$

$$\equiv \exp\left[K(\zeta, g\bar{z}) - K(\zeta, \bar{z}) + \Phi(g\bar{z}; g)\right].$$

$$T_a(\zeta, \bar{z}) = \nabla_a K(\zeta, \bar{z}) - \varphi_a(\zeta) \equiv -\bar{T}_a(z, \zeta) = -\nabla_a K(\zeta, \bar{z}) + \varphi_a(z). \quad (35)$$

(The second equality is true for the real basis, where $\hat{T}_a = -\hat{T}_a^\dagger$.) Thus the symbols of the Lie algebra are expressed in terms of the Killing fields and the symplectic one-form generated by the Kähler structure,

$$T_a(\zeta, \bar{z}) = \kappa(\zeta)_{\alpha}^a \Lambda_\alpha(\zeta, \bar{z}) - \varphi_a(\zeta), \quad dK(\zeta, \bar{z}) = \Lambda_\alpha d\zeta^\alpha + \Lambda_\beta d\bar{z}^\beta. \quad (36)$$
Assuming that the (1,1)-form $\omega$ is non-degenerate, one can define the Poisson brackets in $\mathcal{M}$ by means of a field $\varpi$ dual to $\omega$. Namely, for any two symbols $A(z,\bar{z})$ and $B(z,\bar{z})$ one has

$$\{A,B\}_{P.B.} \equiv \varpi^{\alpha\bar{\beta}} (\partial_\alpha A \partial_{\bar{\beta}} B - \partial_\alpha B \partial_{\bar{\beta}} A) = -\{B,A\}_{P.B.}$$

(37)

$$\omega_{\alpha\bar{\beta}} \varpi^{\alpha\gamma} = \delta^{\gamma}_{\bar{\beta}}, \quad \omega_{\alpha\bar{\beta}} \varpi^{\gamma\bar{\beta}} = \delta^{\gamma}_{\alpha}.$$  

The Jacobi identity results from the fact that $\omega$ in (9) is a closed form. Using (35), one can show that the Poisson brackets provide with a representation for the Lie algebra $\mathfrak{g}$,

$$\{T_a, T_b\}_{P.B.} = \nabla_a T_b - \nabla_b T_a = f^c_{ab} T_c.$$  

(38)

The comparison between Eqs. (32) and (38) shows the meaning of the “quantization” performed. The Lie multiplication in $\mathfrak{g}$, Eqs. (25) and (32), plays the same role as the canonical commutation relations in the standard quantum theory: there is a one-to-one correspondence between commutators and the Poisson brackets for the canonical variables $T_a$. Of course, the correspondence is maintained necessarily only for the canonical variables, like it holds for coordinates and momenta in the usual theory.

The coherent states (10) are solutions to the following differential equations

$$\hat{T}_a \psi_v(z) = T_a(z,\bar{v}) \psi_v(z).$$  

(39)

As in the usual quantum mechanics, they minimize the uncertainty relations (cf. Ref.[20]).

4 Kähler structures for compact Lie groups

4.1 Borel coordinates

The Kähler potentials have been constructed explicitly for all compact simple Lie groups[26, 27, 28, 29]. The complex parameters are introduced following the Borel method[30].

The following notations will be used. Let $\mathcal{G}$ be a compact simple Lie group, and $\mathcal{T}$ be its maximal Abelian subgroup (the maximal torus); the coset space $\mathcal{F} = \mathcal{G}/\mathcal{T}$ is the flag manifold. The local complex parametrization in $\mathcal{F}$
is introduced by means of the canonical diffeomorphism $G/T \cong G^c/P$. Here $G^c$ is the complex extension of $G$; the parabolic subgroup $P$ is a semi-direct product of $T$ and the Borel subgroup $B \subset G^c$. We shall employ the canonical basis $\{\tau_a\} = \{h_j, e_{\pm\alpha}\}$ in the Lie algebra $g$, where $j = 1, \ldots, r \equiv \text{rank}(g)$ and $\{\alpha\} \in \Delta^+_g$ are the positive roots of $g$ (see e.g. in Ref. [31]). The number of the positive roots is $n_+ = \frac{1}{2}[\dim(g) - \text{rank}(g)]$. The Lie products of the basis elements, important for following, are

$$[h_j, h_k] = 0, \quad [h_j, e_{\alpha}] = (\alpha \cdot w_j)e_{\alpha}, \quad [e_{\alpha}, e_{\beta}] = \chi(\alpha, \beta)e_{\alpha+\beta}. \quad (40)$$

Here $\chi(\alpha, \beta)$ is a function on the root lattice, which vanishes if $\alpha + \beta \notin \Delta^+_g$, and $w_j$ are the fundamental weight vectors. For any unitary irreducible group representation, its dominant weight $l$ is given by a sum of the fundamental weights with nonnegative integer coefficients,

$$l = \sum_{j=1}^{r} \mu^j w_j. \quad (41)$$

Given the canonical basis, the Lie algebra $g^c$ is splitted into three subalgebras, $g^c = g^- \oplus t^c \oplus g^+$, corresponding to three subsets of the basis elements, $\{e_{-\alpha}\}$, $\{h_j\}$, $\{e_{\alpha}\}$. Respectively, the Lie algebra $g^+$ generates a nilpotent subgroup $G^+ \subset G^c$, and the Lie algebra $p = g^- \oplus t^c$ generates $P$. The complex parameters which are introduced in $F$ correspond to the positive roots of $g$, and (complex) $\dim(F) = n_+$,

$$f(z) = \exp(\sum_{\alpha \in \Delta^+_g} z^\alpha e_{\alpha}) \in G^+, \quad z^\alpha \in \mathbb{C}. \quad (42)$$

Similarly, one can write elements of the parabolic subgroup. Any element of the complex group has a unique Mackey decomposition,

$$\forall g \in G^c, \quad g = fp, \quad f \in G^+, \quad p \in P, \quad (43)$$

$$p = \exp(\sum_{j=1}^{r} x_j h_j + \sum_{\alpha \in \Delta^+_g} y^\alpha e_{-\alpha}).$$

(The decomposition is valid for all $g$, except for a subset of a lower dimensionality). As soon as $f(z)$ is an element of a nilpotent group, its matrix representations are polynomials of $z^\alpha$. The local form (42) for $f$ is valid
in a neighbourhood of the point \( z^\alpha = 0 \), i.e. the origin of the coordinate system in \( \mathcal{F} \). Transition to other domains of \( \mathcal{F} \), covering the manifold completely, can be performed by means of the group transformations. In order to obtain the decomposition (43) for any given \( g \), one has first to calculate \( f(z)^{-1}g = p \), using the fact that \( f(z)^{-1} \in \mathcal{G}^+ \) is a polynomial of \( z_\alpha \) in any group representation. Setting \( p = \exp(\eta) \), \( \eta \in \mathbf{g}^c \), one requires that \( \eta \) has no \( e_\alpha \)-components. Thus one gets \( n_+ \) algebraic equations for \( n_+ \) variables \( z_\alpha \).

As soon as the equations are solved and \( z \)'s are found as functions of \( g \), one gets \( f(z) \) and \( p = f^{-1}g \).

The group \( \mathcal{G}^c \) acts on \( \mathcal{F} \) by left multiplications. Actually, for any \( g \) and \( z \) one has a unique decomposition,

\[
g f(z) = f(gz) p(z; g), \quad p(z; g) \in \mathcal{P},
\]

where \( gz \) is a rational function of \( z \). For any element \( g \) which does not drive the point with coordinates \( z^\alpha \) outside the coordinate neighbourhood containing the origin where (42) is valid, \( gz \) and \( p(z; g) \) can be obtained from (44) by means of algebraic operations. Performing two consecutive transformations, like in Eq. (23), one gets

\[
p(z; g_2 g_1) = p(g_2 z; g_1) p(z, g_2), \quad \forall g_1, g_2 \in \mathcal{G}^c.
\]

The decomposition in Eq. (44) shows the way to the desired Kähler structure.

### 4.2 Kähler potentials

In a work by Bando, Kuramoto, Maskawa and Uehara\[26\] the Kähler potentials were expressed in terms of the fundamental unitary representation of \( \mathcal{G} \). The representation \( g \rightarrow \hat{g} \) (hat stands for the matrix in this section) is also a representation of \( \mathcal{G}^c \), but the elements of \( \mathcal{G}^+ \) and \( \mathcal{P} \) are not unitary, if they do not belong to \( \mathcal{G} \). The partial solution of Eqs. (6), (7) is given in terms of the generalized determinant (see Appendix),

\[
\begin{align*}
K^j(\zeta, \bar{z}) &= \log \det' \left( \hat{f}(\zeta) \theta_j \hat{f}(z)^\dagger \right), \\
\Phi^j(z; g) &= - \log \det' \left( \hat{g} \hat{f}(z) \theta_j \hat{f}(gz)^{-1} \right).
\end{align*}
\]

Here \( \theta_j \) is a projection matrix which satisfies the following conditions

\[
\begin{align*}
\theta_j &= \theta_j^\dagger, \quad \theta_j^2 = \theta_j, \quad \det' \theta_j = 1, \\
\theta_j \hat{p} \theta_j &= \hat{p} \theta_j, \quad \forall p \in \mathcal{P}.
\end{align*}
\]
There are \( r \) independent matrices of this kind \((j = 1, \cdots , r)\), having different ranks. It is easy to see, using Eqs. (44) and (A4), that for each \( \theta_j \)

\[
\det'[\hat{f}(g\zeta)\theta\hat{f}(gz)] = \frac{\det'[\hat{g}\hat{f}(\zeta)\hat{p}^{-1}\theta(\hat{p}^{-1})^t\hat{f}(z)^t\hat{g}^t]}{\det'[\hat{p}^{-1}\theta(\hat{p}^{-1})^t]} \frac{\det'[\theta]}{\det'[\theta]}.
\]

(49)

The second factor is a holomorphic function of \( \zeta \) times an anti-holomorphic function of \( z \), owing to Eq. (A3). One can see (cf. Eq. (50) below) that \( \text{rank}(\theta_j) \) equals the number of weights \( w \) in the fundamental representation, which satisfy the condition \( (w \cdot w_j) = (w_1 \cdot w_j) \). (Here \( w_1 \) is the dominant weight of the fundamental representation, i.e. the lowest fundamental weight.) In Appendix B, the ranks of \( \theta_j \) are given for all the classical groups and for the exceptional group \( G_2 \).

A matrix having the properties described above can be constructed for any subgroup \( U(1) \subset \mathcal{G} \), and the basis matrices \( \theta_j \) correspond to the components of the maximal torus \( \mathcal{T} \). In order to see that, let us take a (noncompact) subgroup \( q(t) = \exp(-ith_j) \in \mathcal{G}^* \), and consider the transformation \( p \rightarrow p_t = q(t)pq_j(t)^{-1} \). (Note that \( h_j \) is anti-self-adjoint for unitary representations.) It is easy to see that \( p_t \in \mathcal{P} \) and its parameters are \( y_t^\alpha = y^\alpha \exp[-t(\alpha \cdot w_j)] \).

As soon as \( (\alpha \cdot w_j) \geq 0 \) for all positive roots \( \alpha \), only those parameters \( y^\beta \) survive in the limit \( t \rightarrow \infty \), for which \( (\beta \cdot w_j) = 0 \). In the other words, \( p_\infty \) is reduced to a subgroup \( \mathcal{P}_j \), generated by \( e^{-\beta} \) for the roots \( \beta \) satisfying the above condition. This observation suggests the following construction of the projection matrices in terms of elements of the Cartan subalgebra. For any unitary representation, where \( \hat{h}_j \) is diagonal, \( \theta_j \) is also diagonal and has 1 at the sites where \( -i\hat{h}_j \) has its maximum eigen-value, which is \( v_j \equiv (w_j \cdot w_1) \). All the other elements of \( \theta_j \) are 0. Thus the fundamental Kähler potentials can be also written in the following invariant form

\[
K^j(\zeta, z) = \lim_{\lambda \rightarrow 0} \lim_{t \rightarrow \infty} \frac{\det[\hat{f}(\zeta)\hat{q}_j(t)\hat{f}(z)^t] - \lambda \exp(v_j t)]}{\det[\hat{q}_j(t) - \lambda \exp(v_j t)]}.
\]

(50)

The order of the limits cannot be changed, of course.

Evidently, the sum of two Kähler potentials has also the desired transformation property (6), so the fundamental potentials generate a lattice.
Ultimately, the *general* solution is given for any group representation with a dominant weight \( l \),

\[ K^{(l)}(\zeta, \bar{z}) = \sum_{j=1}^{r} l_j K_j^{(l)}(\zeta, \bar{z}), \tag{51} \]

\[ \Phi^{(l)}(z; g) = \sum_{j=1}^{r} l_j \Phi_j^{(l)}(z; g). \tag{52} \]

The fundamental potentials (46) are logarithms of polynomials in \( \zeta \) and \( z \), so the function \( \exp[K^{(l)}] \), which appears in the integrals in Section 2, is a polynomial, and its degree is determined by \( l \).

If any unitary representation of \( G \) would be employed in Eq. (46), the result can be reduced to a sum of the fundamental potentials \( K_j^{(l)} \), as soon as that representation can be extracted from a direct product of the fundamental ones.

Having the expression for the Kähler potential, one gets immediately symbols for elements of the group \( G \) and its Lie algebra \( g \),

\[ U^{(l)}_g(\zeta, \bar{z}) = \prod_{j=1}^{r} \left[ \frac{\det'(\hat{g} \hat{f}(\zeta) \theta_j \hat{f}(z)^\dagger)}{\det'\left(\hat{f}(\zeta) \theta_j \hat{f}(z)^\dagger\right)} \right]^{l_j}, \tag{53} \]

\[ T^{(l)}_a(\zeta, \bar{z}) = \text{tr}\left(\Theta^{(l)}(\zeta, \bar{z}) \tau_a\right). \tag{54} \]

Here the trace is taken in the fundamental representation, and the matrix \( \Theta \), called sometimes *momentum map*, has been described in Ref.\[29\]. It is a sum of the fundamental components,

\[ \Theta^{(l)}(\zeta, \bar{z}) = \sum_{j=1}^{r} l_j \Theta_j(\zeta, \bar{z}). \tag{55} \]

The fundamental momentum maps \( \Theta_j(z, \bar{z}) \) may be considered as the projections \( \theta_j \) transported from the origin of \( \mathcal{M} \) to its arbitrary point. In the other words, there exists an element \( v(\zeta, \bar{z}) \in G \), such that

\[ \Theta_j(\zeta, \bar{z}) = \hat{v} \theta_j \hat{v}^{-1}, \quad \hat{v}^{-1} = \hat{v}^\dagger. \tag{56} \]

It was shown\[29\] that \( \Theta_j(\zeta, \bar{z}) \exp[K_j^{(l)}(\zeta, \bar{z})] \) is a polynomial in \( \zeta \) and \( z \). The Killing fields and the Lie algebra cocycles \( \varphi_a \), which are present in the general expression for \( T_a \) in Eq. (36), have been also given in Ref.\[29\].
It is easy to see that \( K^j \) satisfy the following differential equations
\[
\nabla_\beta(\zeta)K^j(\zeta, \bar{z}) - \nabla_\beta(z)K^j(\zeta, \bar{z}) = 0, \quad \forall \beta : (\beta \cdot w_j) = 0.
\]
Therefore the \((1,1)\)-form \( \omega \), defined in Eq. (9), is degenerate if there is a non-empty set of roots \( \sigma \), for which \( K(l) \) satisfies Eqs. (57),
\[
(1 \cdot \sigma) = 0, \quad \sigma \in \Delta^+_s \subset \Delta^+_g.
\]
Actually, the set \( \Delta^+_s \) is generated by the primitive roots \( \gamma_j \), for which \( l_j = 0 \). Now the Kähler structure is introduced in a subset of \( \mathcal{F} \), given by the constraints \( z_\sigma = 0 \). The flag manifold \( \mathcal{F} \) can be considered as a fiber bundle, \( \mathcal{M} \) being its base, where the unitary group representation \( R_l \) generates a non-degenerate Kähler structure. The local coordinates on \( \mathcal{M} \) are \( z_\alpha \), where \( \alpha \in \Delta^+_g \setminus \Delta^+_s \). Respectively, the little group of \( \mathcal{M} \) is larger than the maximal torus,
\[
\mathcal{M} = \mathcal{G}/\mathcal{H}, \quad \mathcal{H} = S \otimes T',
\]
where \( S \) is the semi-simple Lie group having \( s \) as its Lie algebra, and \( T' \subset T \) is a torus generated by those basis elements \( h_j \), for which \( l_j \neq 0 \). This construction has a clear interpretation in terms of Dynkin graphs\[28\]. Given a group representation \( R_l \), one has to eliminate from the Dynkin graph the nodes for which \( l_j \neq 0 \). The remaining nodes indicate a semi-simple Lie algebra \( s \subset h \subset g \). The parabolic subgroup \( \mathcal{P} \) is extended respectively; its Lie algebra is \( p = g^- + t^c + s^c \).

### 4.3 Unitary groups

Fot \( \mathcal{G} = SU(r+1) \), the fundamental representation is \((r+1)\)-dimensional. The primitive roots and the fundamental weights are given in terms of an orthonormal basis \( \{ \epsilon_i \} \) in the Euclidean space \( \mathbb{R}^{r+1} \) (see e.g. in Ref.\[31\])
\[
\gamma_j = \epsilon_j - \epsilon_{j+1}, \quad w_j = \sum_{i=1}^{j} \epsilon_i - \frac{j}{r+1} \sum_{i=1}^{r+1} \epsilon_i.
\]
Now \( v_j = 1 - j/(r+1) \), and \( \text{rank}(\theta_j) = j \). The local coordinates corresponding to the positive roots \( \alpha = \epsilon_j - \epsilon_k \) are elements of a triangular matrix \( \hat{z} \), i.e. \( z_{jk}, 1 \leq j < k \leq (r+1) \) (other elements of the matrix are zero), and the complex dimensionality of \( \mathcal{F} \) is \( \frac{1}{2} r(r+1) \). The matrix \( \hat{f}(z) \) is triangular; its
diagonal elements are 1, and polynomials in \( z^\alpha \) stand above the diagonal. (The general form of the polynomials can be written down easily.) The manifold \( \mathcal{F} \) has an additional symmetry under a reflection of the root space, under which \( j \to r - j \), and \( \theta_{r-j} \to v(I - \theta_j)v^{-1} \), where \( v \) is a matrix reversing the order of components in the representation space, corresponding to the automorphism of the root system, specific for the unitary groups.

If the number of nonzero components \( l_j \) is \( k < r \), the phase space is a section of the flag manifold, \( \mathcal{M} \subset \mathcal{F} \). Now the gauge group is larger than \( T_r \), namely \( H = T_k \otimes \prod SU(r_i + 1) \), where \( \sum r_i = r - k \). The corresponding Dynkin graph is obtained by elimination of \( k \) links from the chain describing \( SU(r + 1) \). If only one component is nonzero, say, \( l_q = l \), one gets the Grassmann manifold \( \mathcal{M} = Gr(p, q) \equiv U(p + q)/U(p) \otimes U(q) \) (where \( 1 \leq q \leq p < r + 1 \equiv p + q \)) which is a rank-one section of \( \mathcal{F} \). The complex dimensionality of the manifold is \( pq \), and the local coordinates are elements of a \( p \times q \) matrix \( \hat{z} \), so that the elements of \( \hat{f} \) are \( f_{jn} = \delta_{jn} + z_{j,n-p} \), where \( 1 \leq j \leq p \), and \( p + 1 \leq n \leq p + q \). The resulting Kähler potential is

\[
K(z, \bar{z}) = l \log \det(I_q + \hat{z}^\dagger \hat{z}').
\]

For \( q = 1 \), \( \hat{z} \) is a complex vector, \( \mathcal{M} \equiv \mathbb{C}P^r \) is the complex projective space, and one gets the familiar Fubini – Study metric[18]. The metrics is “quantized”, since \( l \) is integer.

Extending the arguments given in Section 2.4, one can claim that if \( l \) is large it evaluates the area of two-dimensional cross sections of the phase space \( \mathcal{M} \) in the \( \hbar \) units, i.e. \( \hbar \sim l^{-1} \). If the rank of \( \mathcal{M} \) is \( k > 1 \), one has a number of nonzero numbers \( l_j \). Introducing a local system of coordinates and momenta, one gets an apparent anisotropy in \( \hbar \), which cannot be eliminated because of the boundary conditions.

5 Dynamics

Until this section, we have been discussing quantum kinematics (as defined in Schwinger’s book[17]). In order to introduce dynamics, one needs a Hamiltonian \( \hat{H} \), to be given as a function of the canonical variables \( \hat{T}_a \), like Hamiltonians of the standard theory are expressed in terms of coordinates and momenta. Thus \( \hat{H} \) belongs to the universal envelopping algebra of \( \mathfrak{g} \). The problem is to get the evolution operator \( \hat{G}(t) = \exp(-it\hat{H}) \), where \( t \) stands
for time, and to find solutions to the Heisenberg and/or Schrödinger equations of motion,
\[
d\hat{T}_a/dt = -i[\hat{T}_a, \hat{H}], \quad \hat{T}_a(t) = \hat{G}(t)^{-1}\hat{T}_a(0)\hat{G}(t); \quad (62)
\]
\[
d\hat{\rho}/dt = i[\hat{\rho}, \hat{H}], \quad \hat{\rho}(t) = \hat{G}(t)\hat{\rho}(0)\hat{G}(t)^{-1}, \quad (63)
\]
where \(\hat{\rho}\) is the system density operator. In the functional approach, one has to calculate the partition function \(Z(t)\) and the generating functional \(F_\eta\), which shows the system response to external (time-dependent) source terms in the Hamiltonian,
\[
i\partial \hat{G}_\eta/\partial t = [\hat{H} + i\eta^a(t)\hat{T}_a] \hat{G}_\eta, \quad \hat{G}_\eta(t_0, t_0) = \hat{I}, \quad (64)
\]
\[
F_\eta = \lim_{t_0 \rightarrow \pm \infty} \text{Tr} \hat{G}_\eta(t_+, t_-), \quad Z(t) = \text{Tr} \hat{G}_0(t_0 + t, t_0). \quad (65)
\]
If the symbols are used, the equations of motion are linear integro-differential equations, and the trace is an integral in \(\mathcal{M}\). In constructing possible Hamiltonians as functions of \(\hat{T}_a\), one should have in mind that for compact spaces the group representations are finite-dimensional, and there are some identities for any given representation. In particular, each Casimir operator \(\hat{C}_n\) is the unit operator times a number, which depends on \(\mathbf{l}\).

The solution is obtained immediately for Hamiltonians which are elements of \(\mathfrak{g}\) combined with the Casimir operators, i.e. \(\hat{H} = \sum \mu_n \hat{C}_n + i\xi^a \hat{T}_a\), where \(\mu_n\) and \(\xi^a\) are real, as soon as \(\hat{H}\) is self-adjoint. In this case, the evolution operator is just an element of \(\mathcal{G}\), its symbol is given by Eq. (53), and \(Z(t)\) is the representation character, times \(\exp(-it\sum \mu_n \lambda_n)\), where \(\lambda_n\) are eigenvalues of \(\hat{C}_n\).

Let us consider, for instance, the Hamiltonian \(\hat{H} = \mu \hat{C}_2 + i\xi^a \hat{T}_a\) for sphere \(S^2 \equiv SU(2)/U(1)\). Its symbol is
\[
H(z, \bar{z}) = \frac{\mu}{4}l(l+2) + l(1 + z\bar{z})^{-1}[\xi^1(z + \bar{z}) - i\xi^2(z - \bar{z}) + \xi^3(1 - z\bar{z})]. \quad (66)
\]
By means of a change of coordinates (rotation) one can set \(\xi^1 = 0 = \xi^2\), and the spectrum is obtained immediately, \(\varepsilon_\nu = \text{const} + \xi\nu\), where \(\xi = |\xi^3|\) and \(\nu = 0, 1, \cdots, l\). The constant \(\mu\) may be chosen to set \(\varepsilon_0 = 0\). The partition function is
\[
Z(t) = \exp(-il\xi t)\frac{\sin(l+1)\xi t}{\sin\xi t}. \quad (67)
\]
The classical trajectories are the parallels of latitude on the sphere, namely, 
\[ z(t) = z(0)e^{-\imath \xi t}. \]
In the limit, discussed in section 2.4, where \( l \to \infty \) and 
\[ z = (q + \imath p)/\sqrt{2l\hbar}, \]
we get the standard Hamiltonian of the one-dimensional oscillator, retaining terms of the order of \( zz \), i.e. \( \hbar H(p, q) = \xi(p^2 + q^2) + \text{const.} \)

The semi-classical approximation works usually in the situations where 
large parameters are present in exponential integrands, and the integrals are 
evaluated by means of the steepest descent method. This is true as well for 
dynamics on homogeneous Kähler manifolds. The large parameter appears 
if some components of \( l \) are large. By the way, the total number of states, 
which is proportional to the volume of \( M \), is also large in this case. The large 
parameter must be present also in the Hamiltonian \( H \), otherwise the classical 
dynamics would be trivial. In order to get the semi-classical approximation, 
let us construct an analogue of the path integral. The standard method is 
to use the identity
\[ \hat{U}_t \equiv e^{-\imath \hat{H}t} = \left(e^{-\imath \hat{H}t/N}\right)^N, \quad \forall N, \] (68)
and write an approximate expression for the symbol of the evolution operator 
at small times, \( \tau = t/N \), where \( N \to \infty \),
\[ e^{-\imath \tau \hat{H}} \to e^{-\imath \tau H(z, \bar{z})} + O(\tau^2). \] (69)

In principle, this is not correct for our manifolds, because each symbol must 
be a rational function, i.e. a polynomial divided by \( \exp(K) \), and this form 
does not survive the exponentiation. In the limit of large \( l \), however, the 
polynomials are of a very high degree, and this condition is not too restrictive.
Multiplying \( N \) operators (69) and calculating the trace (cf. eq. (20)), one 
gets the partition function
\[ Z(t) = \lim_{N \to \infty} \int_M \cdots \int d\mu(z_n, \bar{z}_n) \]
\[ \exp \left\{ \sum_{n=1}^{N} [K(z_n, \bar{z}_{n+1}) - K(z_n, \bar{z}_n) - i\tau H(z_n, \bar{z}_{n+1})] \right\}. \] (70)
The boundary condition is \( \bar{z}_{N+1} = \bar{z}_1 \), so the sequence of points on \( M \) can 
be considered as closed. The exponent has an extremum under the following 
conditions, for \( n = 1, \ldots, N \),
\[ \Lambda_\alpha(z_n, \bar{z}_{n+1}) - \Lambda_\alpha(z_n, \bar{z}_n) - i\tau \partial_\alpha H(z_n, \bar{z}_{n+1}) = 0, \]
\[ \bar{\Lambda}_\alpha(z_{n-1}, \bar{z}_n) - \bar{\Lambda}_\alpha(z_n, \bar{z}_n) - i\tau \partial_{\bar{\alpha}} H(z_{n-1}, \bar{z}_n) = 0, \] (71)

20
where $\Lambda$ and $\bar{\Lambda}$ are the partial derivatives of $K$, as given in Eq. (36). Under reasonable conditions on the Hamiltonian, in the limit $\tau \to 0$, the region $z_{n+1} \to z_n$ contributes predominantly to the integral, so that

$$\Lambda^\alpha(z_n, \bar{z}_{n+1}) - \Lambda^\alpha(z_n, \bar{z}_n) \approx \omega^{\alpha\beta}(z_n, \bar{z}_n)(\bar{z}_{n+1} - \bar{z}_n),$$

and equations (71) become the usual Hamilton equations of motion, which can be set to the familiar form with the Poisson brackets defined in Eq. (37),

$$\frac{dz^\alpha}{dt} = \{H, z^\alpha\}_{P.B.}, \quad \frac{d\bar{z}^\alpha}{dt} = \{H, \bar{z}^\alpha\}_{P.B.}.$$ \hspace{1cm} (73)

These equations of motion can be also derived from the variational principle applied to the action which results from the sum in Eq. (70) if the difference of two Kähler potentials is replaced by the differential,

$$A_t = \int_0^t \left\{ \frac{1}{2} i [\Lambda^\alpha(z, \bar{z})dz^\alpha - \bar{\Lambda}^\alpha(z, \bar{z})d\bar{z}^\alpha] - H(z, \bar{z})d\tau \right\}.$$ \hspace{1cm} (74)

(We added the total derivative $\frac{1}{2}dK$ to the integrand to make the action real. Note that $K$ is not defined globally on $M$, but the addition is possible because of the cocycle condition.) The action functional is calculated on closed trajectories, $z(0) = z(t)$.

The arguments presented above support the meaning of the construction as a method of quantization: the quantum theory has its classical limit described by the action functional with the Kählerian symplectic form. The classical theory is local and does not require integer coefficients at the fundamental Kähler potentials.

6 Concluding remarks

Berezin’s method results in the following construction. For any unitary representation $R_l$ of a compact simple group $G$, one has a compact homogeneous Kähler manifold $M \equiv G/H$ and a Hilbert space $L$ of (locally) holomorphic functions which can be considered as a line bundle upon $M$. According to the Borel – Weil – Bott theorem, the Hilbert-space representation is unitary. The Lie algebra $g$ is realized by means of linear differential operators in $L$, or by means of Poisson brackets for functions on $M$. This is the essence of
quantization. The Kähler potentials have been constructed explicitly for all compact Lie groups.

Until now, the Berezin quantization had no real physical applications, except for some “model building”. An extension to a new theory of quantized fields will probably be the next step.

We conclude with a few remarks on subjects which have been beyond the scope of this paper.

1. Non-compact groups. The method can be extended easily to non-compact (locally-compact) groups which are subgroups of $G^c$. Let $\eta$ be a non-degenerate matrix in the fundamental representation space of $G$. A subgroup $\tilde{G} \subset G^c$ can be specified with a pseudo-unitarity condition $\hat{g} \eta \hat{g}^\dagger = \eta$. The group $\tilde{G}$ is non-compact if $\eta$ is not positive-definite. The arguments of Section 4 are valid, if instead of Eq. (46) the fundamental potentials are defined by

$$K^j(\zeta, \bar{z}) = \log \det' \left( \hat{f}(\zeta) \theta_j \hat{f}(z)^\dagger \eta \right)$$  (75)

(it is assumed that $\det'(\theta_j \eta) = 1$). The pseudosphere in section 2.4 is obtained in this way from $SU(1,1) \subset SL(2,\mathbb{C})$ with $\eta = \text{diag}(1,-1)$. More information on non-compact homogeneous Kähler manifolds can be found elsewhere\cite{32}. The manifold is non-compact, as it is an image of a bounded domain in $\mathbb{C}^m$ where $\det'$ in Eq. (75) is positive. The boundary of the domain is an analogue of the absolute for the Lobachevsky plane. The Kähler manifolds appear for discrete series of unitary (infinite-dimensional) representations of non-compact groups. In general, homogeneous manifolds of Lie groups cannot be provided with a Kähler structure, their geometry is too complicated (see e.g. in Ref.\cite{34}). In particular, it is impossible for $\tilde{G}/\tilde{H}$, if $\mathcal{H}$ is non-compact.

2. Infinite-dimensional limit. The main building blocks of the method admit an extension to the limit of infinite-dimensional groups. For example, infinite-dimensional Hilbert – Schmidt Grassmannians have been considered\cite{32}. It was found that the classical Hamiltonians generate a Lie subalgebra of the total Poisson-bracket Lie algebra, isomorphic to the central extension of $g$ which include the (quantum) anomaly. The geometric quantization in the $N \to \infty$ limit has been also considered in a recent paper\cite{33}.

3. Super-manifolds. For the compact case, representations of the Lie groups can be constructed also in super-manifolds, i.e. in the Grassmann algebra with anti-commuting generators (the exterior algebra\cite{17}). An example of $SO(3)$ was considered and applied to the description of spinning
Different group representations are obtained on super-manifolds with different numbers of Fermi-type degrees of freedom. Two approaches, based upon the Kähler manifolds and the super-manifolds, are equivalent but quite different technically. In field theory the equivalence is known as “bosonization of fermion models”.

4. Integrals on Kähler manifolds. Starting from the known transformation properties, one can calculate a class of integrals on $M$, like Eq. (19). A particular result is Weyl’s formula for the representation dimensionality and a similar formula for volumes of the manifolds. The Weyl group can be also realized in $M$. (The result will be published elsewhere.)

5. Integrable systems. Investigation of Lie subalgebras of the universal enveloping algebra of $g$ may provide with a new insight into the theory of integrable systems (cf. the example in Section 5).

6. Additional references. Only a part of a great number of works dealing with the Berezin approach to quantization has been mentioned here. Besides those mentioned, see, for instance, Refs. [35, 36, 37, 38], and references therein.

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Appendix

A. Generalized determinant

For any self-adjoint positive semi-definite linear operator $A$, its generalized determinant, $\det' A$, is defined as the product of all its non-zero eigen-values. Given a pair of semi-definite operators of the same rank, $A$ and $B$, one has

$$\frac{\det' A}{\det' B} = \lim_{\lambda \to 0} \frac{\det(A - \lambda)}{\det(B - \lambda)}.$$  (76)
Applying the identity
\[
\log \frac{\alpha}{\beta} = -\int_0^\infty \left( e^{-t\alpha} - e^{-t\beta} \right) \frac{dt}{t}, \quad \forall \alpha, \beta > 0,
\]  
(77)
to the eigen-values of \( A \) and \( B \), and summing up, one gets
\[
\log \frac{\det' A}{\det' B} = -\int_0^\infty \left[ \text{Tr}(e^{-tA}) - \text{Tr}(e^{-tB}) \right] \frac{dt}{t}.
\]  
(78)
This equality enables one to define the generalized determinants for elliptic operators having definite partition functions; it has been used in quantum field theory\[39\].

We declare that two operators belong to the same class, and write \( A \sim B \), if there exists a non-singular operator \( P \), relating them, \( B = PAP^\dagger \). One can show that for any non-singular operator \( F \),
\[
\frac{\det' FAF^\dagger}{\det' A} = \frac{\det' FBF^\dagger}{\det' B}, \quad \text{if } A \sim B.
\]  
(79)
Thus the ratio depends only on the class containing \( A \) and \( B \). One can also show that
\[
\frac{\det' F_2F_1AF_1^\dagger F_2^\dagger}{\det' A} = \frac{\det' F_1AF_1^\dagger \det' F_2AF_2^\dagger}{\det' A}.
\]  
(80)
This equality is an extension of the usual one, \( \det(F_1F_2) = \det F_1 \det F_2 \).

**B. Projection matrices for classical groups and \( G_2 \)**

We shall use the notations of Ref.[31]. Roots and weights for the simple Lie algebras of rank \( r \) are given in terms of the orthonormal basis \( \{ \epsilon_j \} \) in the root space, except for \( A_r \) and \( G_2 \), where the root space is described as a \( r \)-dimensional hyperplane in the \((r + 1)\)-dimensional Euclidean space (the root space is normal to the vector \( \varepsilon = \sum_{j=1}^{r+1} \epsilon_j \)). The primitive roots \( \gamma_j \) are
\[
A_r : \quad \epsilon_j - \epsilon_{j+1} \quad \quad (1 \leq j \leq r);
B_r : \quad \epsilon_j - \epsilon_{j+1}, \epsilon_r \quad \quad (1 \leq j < r);
C_r : \quad \epsilon_j - \epsilon_{j+1}, 2\epsilon_r \quad \quad (1 \leq j < r);
D_r : \quad \epsilon_j - \epsilon_{j+1}, \epsilon_{r-1} + \epsilon_r \quad \quad (1 \leq j < r);
G_2 : \quad \epsilon_1 - 2\epsilon_2 + \epsilon_3, \epsilon_2 - \epsilon_3.
\]  
(81)
The corresponding fundamental weights \( w_j \) are (the lowest vector \( w_1 \) is the last in the line)

\[
\begin{align*}
A_r &: \quad \sum_{i=1}^{j} \epsilon_i - \frac{1}{r+1} \epsilon_j \quad (1 \leq j \leq r); \\
B_r &: \quad \sum_{i=1}^{j} \epsilon_i + \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_r) \quad (1 \leq j \leq r-1); \\
C_r &: \quad \sum_{i=1}^{j} \epsilon_i \quad (1 \leq j \leq r); \\
D_r &: \quad \sum_{i=1}^{j} \epsilon_i, \quad \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_r), \quad \frac{1}{2}(\epsilon_1 + \cdots - \epsilon_r) \quad (1 \leq j \leq r-2); \\
G_2 &: \quad 2\epsilon_1 - \epsilon_2 - \epsilon_3, \quad \epsilon_1 - \epsilon_3.
\end{align*}
\]

The weights of the fundamental representation, having \( w_1 \) as its dominant weight, and its dimensionality \( d_f \) are given by

\[
\begin{align*}
A_r &: \quad \epsilon_j - \frac{1}{r+1} \epsilon, \quad d_f = r + 1; \\
B_r &: \quad \frac{1}{2}(\pm \epsilon_1 \pm \epsilon_2 \cdots \pm \epsilon_r), \quad d_f = 2^r; \\
C_r &: \quad (\pm \epsilon_1 \pm \epsilon_2 \cdots \pm \epsilon_r), \quad d_f = 2^r; \\
D_r &: \quad \frac{1}{2}(\pm \epsilon_1 \pm \epsilon_2 \cdots \pm \epsilon_r), \quad d_f = 2^{r-1}; \\
G_2 &: \quad \pm(\epsilon_1 - \epsilon_3), \pm(\epsilon_1 - \epsilon_2), \pm(\epsilon_2 - \epsilon_3), \quad 0, \quad d_f = 7.
\end{align*}
\]

(For \( D_r \) the number of minuses in each weight vector is odd.) Ultimately, the ranks of the projection matrices \( \theta_j \), corresponding to the fundamental weights \( w_j \), are given in the following Dynkin graphs. The rank of \( \theta_1 \), corresponding to \( w_1 \), is 1.

\[
\begin{align*}
SU(r+1) &\sim A_r : \quad \includegraphics[width=0.5\textwidth]{suDynkin} \\
SO(2r+1) &\sim B_r : \quad \includegraphics[width=0.5\textwidth]{soDynkin} \\
Sp(2r) &\sim C_r : \quad \includegraphics[width=0.5\textwidth]{spDynkin} \\
SO(2r) &\sim D_r : \quad \includegraphics[width=0.5\textwidth]{soDynkin} \\
G_2 &\sim G_2 : \quad \includegraphics[width=0.5\textwidth]{g2Dynkin}
\end{align*}
\]

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Similarly, one can calculate the ranks of \( \theta_j \) for the other 4 exceptional groups. It is remarkable that all the projection matrices can be constructed recursively.

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