How far apart are classical
and quantum systems? *

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Abstract
As is well known, classical systems approximate quantum ones –
but how well? We introduce a definition of a “distance” on classical
and quantum phase spaces that offers a measure of their separation.
Such a distance scale provides a means to measure the quality of ap-
proximate solutions to various problems. A few simple applications
are discussed.

Introduction
The purpose of this paper is to introduce a “distance function” which mea-
sures the “distance” between classical and quantum systems. Such a distance
measure may be used, for example, to compare two different approximation
schemes applied to a single quantum system. Thus, one can decide which
scheme is “better” in the sense measured by the proposed distance function.

Let us start our analysis with a brief review of what constitutes a solution
set for each regime: quantum and classical. We confine our attention to pure
states. For quantum mechanics, a pure state is determined by a vector \(|\psi\rangle\)

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in a Hilbert space $H$. In fact, the physical content of a pure state is uniquely specified by a unit vector up to an overall phase factor. Equivalently, a pure state is defined by a one-dimensional projection operator which we denote by $|\psi\rangle\langle\psi|$, leaving implicit the fact that $|\psi\rangle$ is a unit vector, $\langle\psi|\psi\rangle = \|\psi\|^2 = 1$.

The equation of motion for quantum mechanics, i.e., Schrödinger’s equation (with $\hbar = 1$),

$$i(d/dt)|\psi(t)\rangle = \mathcal{H}|\psi(t)\rangle,$$

where we have assumed for simplicity a time independent Hamiltonian $\mathcal{H}$, has a solution set

$$S_Q \equiv \{|\psi(t)\rangle : |\psi(t)\rangle = e^{-i\mathcal{H}t}|\psi(0)\rangle, \|\psi(0)\| = 1, \ 0 \leq t \leq T\}$$

that depends on the initial state $|\psi(0)\rangle$ and the particular self-adjoint Hamiltonian $\mathcal{H}$.

For classical mechanics, the equations of motion are Hamilton’s equations

$$\dot{q} = \partial H/\partial p, \quad \dot{p} = -\partial H/\partial q,$$

which, for a suitable classical Hamiltonian $H = H(p, q)$, have a solution set

$$S_C \equiv \{(p(t), q(t)) : \dot{q} = \partial H/\partial p, \ \dot{p} = -\partial H/\partial q, \ (p(0), q(0)), \ 0 \leq t \leq T\}.$$

The state space of quantum mechanics is a complex Hilbert space (modulo phase and normalization), while, for a single degree of freedom, the state space of classical mechanics is a real two-dimensional symplectic manifold, the phase space. It is evident that given this version of quantum and classical theories, the two systems are so different from each other that any meaningful distance seems hopeless to define.

As a step in the right direction, let us recall the action principles of the two disciplines. First, we observe that the quantum action functional may be taken to be

$$I_Q = \int [i\langle\psi(t)|(d/dt)|\psi(t)\rangle - \langle\psi(t)|\mathcal{H}|\psi(t)\rangle] \, dt,$$

and that stationary variation of this functional with respect to $\langle\psi(t)|$ gives rise to Schrödinger’s equation, Eq. (1). Second, we note that the classical action functional is given by

$$I_C = \int [p(t)(d/dt)q(t) - H(p(t), q(t))] \, dt,$$
and that stationary variations of this functional with respect to $p(t)$ and $q(t)$ give rise to Hamilton’s equations, Eq. (3). It seems that the two theories are still difficult to relate to each other.

Coherent states provide the bridge that connects the quantum and classical action functionals [1]. Let $|\eta\rangle$ be a fairly general unit vector in Hilbert space subject to the modest requirement that $\langle\eta|Q|\eta\rangle = \langle\eta|P|\eta\rangle = 0$, where $Q$ and $P$ are the usual self-adjoint Heisenberg kinematical variables. Next, we introduce the two-parameter family of coherent states, defined by

$$|p,q\rangle \equiv e^{-iqP}e^{ipQ}|\eta\rangle,$$

(7)

for all $(p, q) \in \mathbb{R}^2$. Since $[Q, P] = i\mathbb{1}$, it readily follows that

$$\langle p, q|Q|p, q\rangle = q, \quad \langle p, q|P|p, q\rangle = p.$$

(8)

The set of coherent states spans the Hilbert space, a fact that is implicit in the standard resolution of unity which holds for any unit fiducial vector $|\eta\rangle$ [1, 2].

Let us next analyze the following question: What is the consequence of varying the quantum action functional over a limited set of states, say, over just the coherent states? Thus, let us consider

$$I_Q = \int [i\langle p(t), q(t)|(d/dt)|p(t), q(t)\rangle - \langle p(t), q(t)|H|p(t), q(t)\rangle] dt,$$

(9)

which is readily evaluated as

$$I_Q = \int [p(t)(d/dt)q(t) - H(p(t), q(t))] dt,$$

(10)

where in this expression

$$H(p, q) \equiv \langle p, q|H|p, q\rangle.$$

(11)

Evidently, the equations of motion that follow from the quantum action principle subject to variation only within the set of coherent states is entirely equivalent to the classical equations of motion for a classical Hamiltonian defined by (11).

The connection of the quantum and classical action functionals – and hence their respective equations of motion – that arises by using the coherent states has, at last, put the two theories within a common framework. However, this fact does not fully resolve our situation since the action functional $I$ does not represent a proper distance functional.
Distance Choices

Let us recall the three fundamental properties of a “distance function”, $D(1, 2)$, between two elements “1” and “2”. These properties are:

1) $D(1, 2) \geq 0$; $D(1, 2) = 0 \iff \text{“1”=“2”}$ ,
2) $D(2, 1) = D(1, 2)$ ,
3) $D(1, 3) \leq D(1, 2) + D(2, 3)$.

Frequently, one of our systems (say “2”) will have zero distance, and so $D(1, 2)$ will be a function of just one argument, say “1”. In this case we shall simply write $D(1)$ representing the distance of the element “1” from the “zero” element. For the most part we shall deal with this simpler version, while at the end we shall consider the more general situation of $D(1, 2)$.

There are some additional conditions that we would like to impose on any distance function that we adopt. Expressed in a rather casual manner, the additional conditions we choose are:

1) $D(\text{true quantum solution}) = 0$ ,
2) $D(S; 0 \leq t \leq T) + D(S; T \leq t \leq T + U) = D(S; 0 \leq t \leq T + U)$ ,
3) $\lim_{\hbar \to 0} D(\text{classical solution}) = 0$ .

In item 2, $S$ denotes the solution set in question (cf., Eqs. (2) and (4)), apart from the time intervals involved.

Since we have found common ground for the quantum and classical formulations in Hilbert space, let us next recall some standard “distance expressions” used in Hilbert space. Restricting attention to unit vectors, we consider the distance function induced by:

1) Vector norm
   $$D_V(|1\rangle, |2\rangle) \equiv \| |1\rangle - |2\rangle \| = \sqrt{2 - \langle 1|2 \rangle - \langle 2|1 \rangle} ,$$

2) Ray vector norm
   $$D_R(|1\rangle, |2\rangle) \equiv \inf_{\alpha} \| |1\rangle - e^{-i\alpha}|2\rangle \| = \sqrt{2 - |\langle 1|2 \rangle|} ,$$

3) Operator norm
   $$D_O(|1\rangle, |2\rangle) \equiv \| |1\rangle \langle 1| - |2\rangle \langle 2| \| = \sqrt{1 - |\langle 1|2 \rangle|^2} .$$

Here the norm of a bounded operator $B$ is defined as $\|B\| = \sup_{|\psi\rangle} \|B|\psi\rangle\|$ over all normalized states, $\|\psi\| = 1$.

It follows that $0 \leq D_O \leq D_R \leq D_O + D_O^2$ as well as $0 \leq D_R \leq D_V$. From
this we learn that the norms $D_R$ and $D_O$ are equivalent (induce identical topologies), while for infinitesimal distances (i.e., $D_R \ll 1$), the norms $D_R$ and $D_O$ are equal. Finally, we note that $D_V$ is inequivalent to both $D_R$ and $D_O$. Based on these several properties, as well as reasons of simplicity, we choose $D_R$ to define our desired “distance function”.

**Choice of Distance**

We construct our distance $D$ as a continuous limit of piecewise segments. In particular, we adopt

\[
D \equiv \inf_{\alpha} \lim_{\epsilon \to 0} \sum_{l=1}^{N} \| (\psi_{l+1}) - e^{-i(\alpha_{l+1}-\alpha_l)}e^{-i\mathcal{H}\epsilon}|\psi_l\rangle \|
\]

\[
= \inf_{\alpha} \lim_{\epsilon \to 0} \sum_{l=1}^{N} \| i(|\psi_{l+1}) - |\psi_l\rangle) - i(e^{-i(\alpha_{l+1}-\alpha_l)-i\mathcal{H}\epsilon - 1})|\psi_l\rangle \| , \tag{12}
\]

where $\epsilon = T/N$. More specifically, we define

\[
D \equiv \inf_{\alpha} \int_{0}^{T} \| i(d/dt)|\psi(t)\rangle - [\dot{\alpha} + \mathcal{H}]|\psi(t)\rangle \| dt . \tag{13}
\]

Here $\alpha(t)$, $0 \leq t \leq T$, represents a function over which the infimum takes place. Equation (13) represents the basic definition introduced in this paper.

We immediately see that $D$(true quantum solution) = 0, and $D(S; 0 \leq t \leq T) + D(S; T \leq t \leq T + U) = D(S; 0 \leq t \leq T + U)$.

**Canonical examples**

Let us discuss next the “distance” appropriate to several classical systems. The general expression becomes

\[
D = \inf_{\alpha} \int_{0}^{T} \| i(d/dt)|p,q\rangle - [\dot{\alpha} + \mathcal{H}(P,Q)]|p,q\rangle \| dt
\]

\[
= \inf_{\alpha} \int_{0}^{T} \| \{\dot{\eta}(P + p) - \dot{p}Q - \dot{\alpha} - \mathcal{H}(P + p, Q + q)\} |\eta\rangle \| dt . \tag{14}
\]
As a first example, consider the harmonic oscillator with unit mass and
unit angular frequency, i.e., let us choose $H = \frac{1}{2}(P^2 + Q^2)$. Thus we deal with

$$D = \inf_{\alpha} \int_0^T \| \{ \dot{q}(P + p) - \dot{p}Q - \dot{\alpha} - \frac{1}{2}(p^2 + q^2)
-pP - qQ - \frac{1}{2}(P^2 + Q^2) \} |\eta\rangle \| \, dt .$$ (15)

The least value of $D$ arises if we choose

$$\dot{q} = p, \quad \dot{p} = -q, \quad \alpha = \frac{1}{2}(pq - t), \quad (P^2 + Q^2 - 1)|\eta\rangle = 0 ,$$ (16)

which leads to $D$(classical harmonic oscillator) $= 0$, as may have been anticipated.

More interesting is an anharmonic oscillator such as $H = \frac{1}{2}P^2 + \frac{1}{4}Q^4$. Let us initially choose

$$\dot{q} = p, \quad \dot{p} = -q^3, \quad \dot{\alpha} = pq - c - \frac{1}{2}p^2 - \frac{1}{4}q^4 + u, \quad (H - c)|\eta\rangle = 0 ,$$ (17)

where $u$ remains to be determined. This choice leads to

$$D = \inf_u \int_0^T \| \{ u + qQ^3 + \frac{3}{2}q^2Q^2 \} |\eta\rangle \| \, dt$$

$$= \inf_u \int_0^T \left[ u^2 + q^2\langle Q^6 \rangle + \frac{9}{4}q^4\langle Q^4 \rangle + 3uq^2\langle Q^2 \rangle \right]^{1/2} \, dt$$

$$= \inf_u \int_0^T \left[ (u + \frac{3}{2}q^2\langle Q^2 \rangle)^2 + q^2\langle Q^6 \rangle + \frac{9}{4}q^4\langle Q^4 \rangle - \langle Q^2 \rangle^2 \right]^{1/2} \, dt$$

$$= \int_0^T \left[ q^2\langle Q^6 \rangle + \frac{9}{4}\langle Q^4 \rangle - \langle Q^2 \rangle^2 \right]^{1/2} \, dt ,$$ (18)

where $\langle Q^r \rangle \equiv \langle \eta|Q^r|\eta\rangle$ vanishes for the present example if $r$ is odd. To
determine $D$ we have chosen $|\eta\rangle$ to be the ground state of $H$, a condition
that fixes $c$, in principle. Observe further for the present example that for
all even and positive $r$, $\lim_{h \to 0} \langle Q^r \rangle = 0$. Consequently, we further learn that
$\lim_{h \to 0} D$(classical anharmonic oscillator) $= 0$. 

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Spin example

It is noteworthy that the present definition for distance may be extended to other systems such as spin systems. A general class of coherent states for spin systems may be chosen as

$$|\theta,\phi\rangle \equiv e^{-i\phi S_3} e^{-i\theta S_2} |s, m\rangle,$$

(19)

where $S_3 |s, m\rangle = m |s, m\rangle$ and $\Sigma_j S_j^2 |s, m\rangle = s(s+1) |s, m\rangle$ [3]. Normally, one considers fiducial vectors $|s, s\rangle$ (or $|s, -s\rangle$), but let us focus on a rather uncommon choice, namely $|s, m\rangle = |1, 0\rangle$ [3]. In a standard representation in which $S_3$ is diagonal, these states are given by

$$|\theta, \phi\rangle = \begin{pmatrix} e^{-i\phi} \sin(\theta)/\sqrt{2} \\
\cos(\theta) \\
e^{i\phi} \sin(\theta)/\sqrt{2} \end{pmatrix}. \tag{20}$$

We choose for our example the Hamiltonian

$$\mathcal{H} = \lambda S_3^2 = \lambda \begin{pmatrix} 1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1 \end{pmatrix}. \tag{21}$$

We are then led to the distance

$$D = \inf_\alpha \int_0^T \|i(d/dt)|\theta, \phi\rangle - [\dot{\alpha} + \mathcal{H}] |\theta, \phi\rangle\| \, dt$$

$$= \inf_\alpha \int_0^T [\dot{\theta}^2 + \sin^2(\theta) \dot{\phi}^2 + \dot{\alpha}^2 + 2\dot{\alpha}\lambda \sin^2(\theta) + \lambda^2 \sin^2(\theta)]^{1/2} \, dt$$

$$= \int_0^T [\dot{\theta}^2 + \sin^2(\theta) \dot{\phi}^2 + \lambda^2 \sin^2(\theta) \cos^2(\theta)]^{1/2} \, dt. \tag{22}$$

It may be thought that this example corresponds to the distance for the corresponding classical spin system, but that would be incorrect. This fact follows since $|\eta\rangle = |s, m\rangle$ and in that case

$$i \langle \theta, \phi | d |\theta, \phi\rangle = m \cos(\theta) \, d\phi; \tag{23}$$

therefore this kinematical factor actually vanishes when $m = 0$. Thus there is no classical set of equations of motion in the present case, and this example illustrates how it is possible to define and use the distance function even when the restricted set of Hilbert states which is chosen fails to correspond to a conventional classical system.
**Extended coherent states**

In another direction, one may also consider so-called extended coherent states which are defined, for a single degree of freedom, by expressions of the form

\[ |p, q, r, s, \ldots \rangle \equiv \cdots e^{isY} e^{irX} e^{-iqP} e^{ipQ} |\eta\rangle , \tag{24} \]

where \( P \) and \( Q \), as usual, form an irreducible Heisenberg pair, and \( X = X(P, Q), Y = Y(P, Q), \ldots \). These states already form a conventional set of coherent states for \( p \) and \( q \); the extra variables, \( r, s, \ldots \), are not needed but may (in compact form) also be used in forming a resolution of unity. In any case, the distance definition in the present case becomes

\[ D = \inf_{\alpha} \int_{0}^{T} \| i(d/dt)|p, q, r, s, \ldots \rangle - [\dot{\alpha} + \mathcal{H}]/p, q, r, s, \ldots \| dt . \tag{25} \]

In general, the use of extended coherent states leads to a smaller value for \( D \) than if one had restricted oneself only to canonical coherent states.

**Pair Distance**

As our final topic we return to the issue of the distance between two systems. Let

\[ W_j(\alpha_j, t) \equiv i(d/dt)|\psi_j(t)\rangle - [\dot{\alpha}_j + \mathcal{H}_j]/\psi_j(t)\rangle , \quad j = 1, 2 , \tag{26} \]

denote the essential combination frequently used above. Consider the expression

\[ \| W_1(\alpha_1, t) - e^{-i\gamma(t)} W_2(\alpha_2, t) \| . \tag{27} \]

We will define the distance between system “1” and system “2” as

\[ D(1, 2) \equiv D_{12} \equiv \inf_{\alpha_1, \alpha_2, \gamma} \int_{0}^{T} \| W_1(\alpha_1, t) - e^{-i\gamma(t)} W_2(\alpha_2, t) \| dt . \tag{28} \]

We can bound this distance by first noting that

\[ \| W_1(\alpha_1, t) \| + \| W_2(\alpha_2, t) \| \geq \| W_1(\alpha_1, t) - e^{-i\gamma(t)} W_2(\alpha_2, t) \| \]

\[ \geq \left\| \| W_1(\alpha_1, t) \| - \| W_2(\alpha_2, t) \| \right\| \]

\[ \geq \left\{ \begin{array}{l} \| W_1(\alpha_1, t) \| - \| W_2(\alpha_2, t) \| \\ \| W_2(\alpha_2, t) \| - \| W_1(\alpha_1, t) \| \end{array} \right. . \tag{29} \]
As stated, this equation holds for all $t$ and for any choice of $\alpha_1$ and $\alpha_2$. Therefore, by integrating over $t$ and taking $\inf_{\alpha_1,\alpha_2,\gamma}$, we learn that

$$D_1 + D_2 \geq D_{12} \geq |D_1 - D_2| .$$  \hspace{1cm} (30)

Consequently, if (say) $D_2 = 0$, as would be the case if $|\psi_2(t)\rangle = \exp(-itH_2) |\psi_2(0)\rangle$, then

$$D_{12} = D_1 .$$  \hspace{1cm} (31)

Thus the distance of system “1” to any true quantum system is just what we have called $D (= D_1)$ previously. On the other hand, if one enquires about the distance between two general classical systems, then it would be necessary to use Eq. (28). Of course, for two classical systems, there are no doubt a number of plausible distance functions that would suggest themselves as well, and it may be useful to use the specific context to help decide which expression to choose.

**Conclusion**

In this article we have introduced a “distance function”, Eq. (13), that measures the distance of any Hilbert space temporal path from a true quantum solution. In particular, such a function can be used to measure the distance of a classical solution from the corresponding quantum solution. In Eq. (28) we have introduced a compatible definition for the distance between any two Hilbert space temporal paths.

In this article we have confined our attention to pure states. An interesting extension of the present work would be to study mixed states and their dynamical evolution. It is conjectured that the use of density matrices and their equation of motion in the sense of von Neumann, along with the distance function based on the operator norm, may be suitable to define an associated distance function in that case.

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