LOVÁSZ’S HOM-COUNTING THEOREM BY INCLUSION-EXCLUSION PRINCIPLE

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ABSTRACT. Let $\mathcal{C}$ be the category of finite graphs. Lovász (1967) shows that if $|\text{Hom}(X, A)| = |\text{Hom}(X, B)|$ holds for any $X$, then $A$ is isomorphic to $B$. Pultr (1973) gives a categorical generalization using a similar argument. Both proofs assume that each object has a finite number of isomorphism classes of subobjects. Generalizations without this assumption are given by Dawar, Jakl, and Reggio (2021) and Reggio (2021). Here another generalization without this assumption is given, with a shorter proof. Examples of categories are given, for which our theorem is applicable, but the existing theorems are not.

1. Introduction

In a category, it clearly holds that

$$A \cong B \Rightarrow |\text{Hom}(X, A)| = |\text{Hom}(X, B)|$$

for all objects $X$. A category where the converse holds is said to be combinatorial (Definition 2.6). This notion is introduced by Pultr [11] after the Lovász’s memorial work [6], and a considerable amount of studies exists: a direct generalization of Lovász’s proof for more general categories is given by Pultr [11], a different approach by Isbell [5], and some new types of proofs are given by Dawar, Jakl, and Reggio [3] and Reggio [12]. This property and its generalization in the category of graphs is widely studied, sometimes from computational aspects, see Cai-Govorov [2] and its references. The aim of this paper is to give yet another simple sufficient condition for a category to be combinatorial (Main Theorem 2.11). In the last section, we show some example categories, to separate the scope of the existing theorems and ours. There is a category to which our theorem is applicable, but the other theorems are not.

2. Preliminary and Main Theorem

The notions of mono, epi, pullback, pushout, and subobject are standard, see MacLane [9]. The term “quotient object” depends on the literature (in Mitchell [10, p.7] as a dual of subobject, and in Pultr [11, Section 1.1] in a different sense), so here we use a less common word:

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2020 Mathematics Subject Classification. 05C60 Isomorphism problems in graph theory (reconstruction conjecture, etc.) and homomorphisms (subgraph embedding, etc.) 18A20 Epimorphisms, monomorphisms, special classes of morphisms, null morphisms 68R10 Graph theory (including graph drawing) in computer science.

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Definition 2.1. A subobject of $A$ is a mono $m : B → A$. A supobject of $A$ is an epi $e : A → C$.

Definition 2.2. Let $C$ be a category. For supobjects $q : X → Q$ and $q' : X → Q'$, we say $q \geq q'$ if there is an $h$ with $q' = h \circ q$. This gives a partial order on the isomorphism classes of the supobjects of $X$. The largest supobject is the isomorphism class given by $id_X$. If $q$ is proper (i.e., nonisomorphic, see Definition 2.1) then the supobject $q : X → Q$ is said to be proper. (This is equivalent to $q < id_X$.)

A maximal supobject is a supobject that is maximal among the proper subobjects.

Dual notions are similarly defined for subobjects of $X$. To make clear, for subobjects $m : M → X$ and $m' : M' → X$, $m ≤ m'$ if $m = m' \circ h$ for some $h$. The largest subobjects are isomorphic to $id_X$.

Definition 2.3. ([11] Definition 4.3.2)

An epimorphism $e : A → B$ is an extremal epimorphism if $e = m \circ g$ where $m$ is mono, then $m$ is an isomorphism. An epimorphism is a proper epimorphism, if it is not an isomorphism.

Definition 2.4. ([10] p.12)

Let $f : X → Y$ be a morphism. An image of $f$ is a subobject $m : im f → Y$ such that there is a $g : X → im f$ with $f = m \circ g$, and if $f = m' \circ g'$ with another subobject $m' : Z → Y$, then $m = m' \circ h$ for some $h : Z → im f$ (i.e. $m ≤ m'$).

Since $m$ is mono, $g' = h \circ g$ follows.

Dually, a coimage of $f$ is a supobject $e : X → coim f$, such that there is an $g : coim f → Y$ with $f = g \circ e$, and if $f = g' \circ e'$ with another supobject $e' : X → Z$, then $e = h \circ e'$ for some $h : Z → coim f$ (i.e. $e ≤ e'$).

Definition 2.5. A category $C$ is locally finite, if for any objects $A, B$, Hom($A, B$) is a finite set.

This terminology seems now common [8] [12], but a different term “quasifinite” is used in Pultr [11]. The following notion combinatorial is the theme of this paper.

Definition 2.6. ([11] 1.7 Definition)

A locally finite category $C$ is said to be combinatorial, if for all objects $X$

$$|\text{Hom}(X, A)| = |\text{Hom}(X, B)|$$

hold then $A$ is isomorphic to $B$.

Lovász [8] proved that the categories of operations with finite structures (including the category of finite graphs) are combinatorial. Pultr gives a categorical generalization, using a similar argument. See Theorem 2.11. We start from some preliminary.

Lemma 2.7. (Lovász [8] Lemma 1)

Let $C$ be a locally finite category. If there are monomorphisms $m : A → B$ and $n : B → A$, then $m$ and $n$ are isomorphisms. Dually, if there are epimorphisms $e : A → B$ and $f : B → A$, then $e$ and $f$ are isomorphisms.

Proof. We prove only the dual. Since $|\text{Hom}(B, B)|$ is finite, the compositions $(ef)^n$ for $n \in \mathbb{N}$ must coincide for different $n$, say, for $n$ and $n + m$ with $m ≥ 1$. Since $(ef)^n$ is epi, $(ef)^n = (ef)^m(ef)^n$ implies $(ef)^m = id_B$, and hence $f$ is a splitting monomorphism: putting $g := (ef)^m e$, $gf = id_B$. Thus $fgf = f$, and since $f$ is epi, $fg = id_A$. \qed
Lemma 2.8. Let \( \mathcal{C} \) be a category. Let \( q_i : X \to Q_i, \ i = 1, 2, \) be supobjects (Definition 2.1). Suppose that \( q_1, q_2 \) has a pushout \( q_3 : X \to Q_3 \). Let \( Y \) be an object. Then, inside \( \text{Hom}(X, Y) \), we have

\[
q_3^* \text{Hom}(Q_3, Y) = q_1^* \text{Hom}(Q_1, Y) \cap q_2^* \text{Hom}(Q_2, Y)
\]

holds, where

\[
q_i^* \text{Hom}(Q_i, X) = \{ f \circ q_i \in \text{Hom}(X, Y) \mid f \in \text{Hom}(Q_i, X) \}.
\]

Proof. This follows from the definition of the pushout: by Yoneda functor \( \text{Hom}(\cdot, Y) \), a pushout is mapped to a pullback. Since \( q_i (i = 1, 2) \) are epi, the morphisms between Hom’s are injective, and the pullback is isomorphic to the intersection. \( \square \)

Let us denote by \( \text{Mono}(A, B) \) the set of monomorphisms between \( A \) and \( B \).

Definition 2.9. Let \( \mathcal{C} \) be a category. Let \( \mathcal{I} \) denote a subclass of monomorphisms, including the identities. The set of \( \mathcal{I} \)-monomorphism from \( A \) to \( B \) is denoted by

\[
\text{Mono}_\mathcal{I}(A, B) \subset \text{Mono}(A, B).
\]

Definition 2.10. For each \( X \), we specify a subclass \( \mathcal{M} \)-supobject of the class of the supobjects \( q : X \to Q \).

For most applications considered, \( \mathcal{M} \) equals to the class of maximal epimorphisms (see Definition 2.2), and \( \mathcal{I} \) equals to the class of monomorphisms. We introduced these notions, to make the condition of Main Theorem 2.11 as weak as possible.

We state the main theorem of this paper.

Theorem 2.11. (Main Theorem) Let \( \mathcal{M} \) and \( \mathcal{I} \) be as in Definitions 2.10, 2.9. Let \( \mathcal{C} \) be a locally finite category satisfying the following conditions.

1. For any finite number of \( \mathcal{M} \)-supobjects \( e_i : X \to Q_i, \ i = 1, 2, \ldots, m \), there exists a pushout.
2. For any object, the set of isomorphism classes of its \( \mathcal{M} \)-supobjects is finite.
3. If \( f : X \to Y \) is not \( \mathcal{I} \)-mono, it factors through an \( \mathcal{M} \)-supobject \( X \to Q \).
4. If \( f : X \to Y \) factors through an \( \mathcal{M} \)-supobject \( X \to Q \), then \( f \) is not \( \mathcal{I} \)-mono.

Then, \( \mathcal{C} \) is combinatorial (Definition 2.6).

Proof. Suppose that

\[
| \text{Hom}(X, A) | = | \text{Hom}(X, B) |
\]

holds for any \( X \). Let \( Z \) be an arbitrary object. Let \( q_i : Z \to Q_i (i = 1, 2, \ldots, m) \) be the representatives of the \( \mathcal{M} \)-supobjects of \( Z \) (they are finite by Condition 2).

Take \( f \in \text{Hom}(Z, A) \). By Conditions 3 and 4, \( f \) is not \( \mathcal{I} \)-mono if and only if \( f \) factors through one of \( Q_i \). Thus, we have

\[
\text{Mono}_\mathcal{I}(Z, A) = \text{Hom}(Z, A) \setminus \bigcup_{i=1}^{m} q_i^* \text{Hom}(Q_i, A).
\]
Now we use Lemma 2.8 and the inclusion-exclusion principle to obtain
\[
\text{monoI}(Z, A) = \text{Hom}(Z, A) - \sum_{1 \leq i \leq m} |q_i^* \text{Hom}(Q_i, A)| \\
+ \sum_{1 \leq i < j \leq m} |q_i^* \text{Hom}(Q_i, A) \cap q_j^* \text{Hom}(Q_j, A)| \\
- \sum_{1 \leq i < j < k \leq m} |q_i^* \text{Hom}(Q_i, A) \cap q_j^* \text{Hom}(Q_j, A) \cap q_k^* \text{Hom}(Q_k, A)| \\
\]
+ \cdots
\]
\[
= \text{Hom}(Z, A) - \sum_{1 \leq i \leq m} |q_i^* \text{Hom}(Q_i, A)| \\
+ \sum_{1 \leq i < j \leq m} |q_{ij}^* \text{Hom}(Q_i, \prod_{j \neq k} Q_j, A)| \\
- \sum_{1 \leq i < j < k \leq m} |q^*_{ijk} \text{Hom}(Q_i, \prod_{j \neq k} Q_j, \prod_{k \neq i} Q_k, A)| + \cdots, \tag{2.1}
\]
where
\[
q_{ij} : Z \to Q_i \prod_{j \neq i} Q_j
\]
denotes the pushout of \(q_i\) and \(q_j\),
\[
q_{ijk} : Z \to Q_i \prod_{j \neq i} Q_j \prod_{k \neq i} Q_k
\]
denotes the pushout of \(q_i, q_j, q_k\), and so on. Since the expression (2.1) is given by a combination of \(|\text{Hom}(\cdot, A)|\), we have the same value when \(A\) is replaced with \(B\). Namely,
\[
\text{monoI}(Z, A) = \text{monoI}(Z, B).
\]
If we put \(Z = A\), the left-hand side contains \(\text{id}_A\), hence there is a monomorphism \(A \to B\). The symmetric argument gives a monomorphism \(B \to A\), and Lovász’s Lemma 2.7 completes the proof. \(\square\)

Often, the following conditions are (stronger but) easier to check.

**Theorem 2.12.**

Let \(\mathcal{C}\) be a locally finite category satisfying the following conditions.

1. For any epimorphisms \(e_i : X \to Q_i, i = 1, 2\), there exists a pushout.
2. For any object, the set of isomorphism classes of its maximal supobjects is finite.
3. For any proper supobject \(q : X \to Q\), there is a maximal supobject \(q' : X \to Q'\) such that \(q' \geq q\) (see Definition 2.2 for the terminology).
4. If \(f : X \to Y\) is not mono, it factors through a proper supobject \(X \to Q\).
5. If \(f : X \to Y\) factors through a proper supobject \(X \to Q\), then \(f\) is not mono.

Then, \(\mathcal{C}\) is combinatorial.

**Proof.** This is obtained from the above Theorem 2.11 by considering the case where \(M\)-supobjects are the proper supobjects (Definition 2.2) and \(I\) is the class of monomorphisms. Note that the last three conditions imply the last two conditions.
Indeed, if $f$ is not-mono, it factors through a proper supobject, and then through a maximal supobject. Conversely, if $f$ factors through a maximal supobject (one of the proper supobjects), then $f$ is not mono. □

An even weaker form is the following.

**Corollary 2.13.** Among the five conditions in Theorem 2.12, we replace Conditions 4 and 5 with

4') Any morphism has a coimage.

5') $f: X \to Y$ is mono if and only if $X \to \text{coim } f$ is an isomorphism.

Under these five conditions, $C$ is combinatorial.

**Proof.** Suppose these conditions. If $f: X \to Y$ is not mono, then $X \to \text{coim } f$ is a proper supobject by (5'), which implies (4). If $f: X \to Y$ factors through a proper subobject $X \to Q$, then $X \to Q \to \text{coim } (f)$ given by the universality of the coimage is not an isomorphism (since if isomorphic, then $X \to Q$ is a splitting monomorphism and epimorphism, thus an isomorphism, contradicting the assumption). Hence $f$ is not mono by (5'), which implies (5). □

We remark the following, related to (5').

**Lemma 2.14.** Let $C$ be a category with coimages. The followings are equivalent.

1) For any $f: X \to Y$, if $X \to \text{coim } f$ is an isomorphism, then $f$ is mono.

2) For any $f: X \to Y$, $\text{coim } f \to Y$ is mono.

**Proof.** The second condition implies the first. For the converse, let $g: \text{coim } f \to Y$. Take $\text{coim } f \to \text{coim } g$. This is a morphism of supobjects of $X$, and $\text{coim } f \geq \text{coim } g$. By the universality of $\text{coim } f$, we have a morphism $\text{coim } g \to \text{coim } f$ of supobjects of $X$. Hence the converse inequality holds. Consequently, these are isomorphisms, and by the first condition, $g$ is mono. □

**Remark 2.15.** One can show that the categories of finite undirected graphs, finite directed graphs, the functor-categories $\text{FinSets}^C$ from a finite category $C$ to the category of finite sets and a category of finite groups satisfy the conditions of Corollary 2.13 (hence those of the main Theorem 2.11), hence are combinatorial. (Pultr’s theorem is also applicable for these examples.)

To see the strongness of such a statement, consider the category of finite groups. For finite groups $A, B, C$, suppose that $A \times B \cong A \times C$ holds. Then, for any $Z$,

$$|\text{Hom}(Z, A)||\text{Hom}(Z, B)| = |\text{Hom}(Z, A \times B)| = |\text{Hom}(Z, A \times C)| = |\text{Hom}(Z, A)||\text{Hom}(Z, C)|,$$

and since $|\text{Hom}(Z, A)| \geq 1$, we have

$$|\text{Hom}(Z, B)| = |\text{Hom}(Z, C)|,$$

and combinatoriality implies that $B \cong C$. This is non-trivial, see for example [4]. This property is well-studied as a cancellation law, see Lovász [7].

Note also that Dawar-Jakl-Reggio’s generalization [3, Theorem 5] (see Theorem 3.4 below) can not be applied to finite groups, since finite groups have pushouts for epimorphisms, but the pushout of $\mathbb{Z}/2$ and $\mathbb{Z}/3$ under the trivial group does not exist (it is known that the pushout is $PSL(2, \mathbb{Z})$ in the category of groups). Reggio’s generalization (Theorem 3.5 below) works, see [12, Example 4.6].
3. A variant of Pultr’s result for a comparison

The following theorem is a slight generalization of Pultr’s theorem [11 2.2 Theorem] which is based on an argument given by Lovász [6 (5), p.326]. We include a proof of this, mainly to show the difference from the proof of Main Theorem 2.11 and partly because the statement is slightly stronger, and to give a variant (Theorem 3.3 below).

**Theorem 3.1.** Let $\mathcal{C}$ be a locally finite category satisfying the following conditions.

1. For each object, the isomorphism classes of its subobjects is finite.
2. Every morphism $f$ has its image.
3. If $h = g \circ f$ and $g$ is an image of $h$, then $f$ is an extremal epimorphism.

Then, $\mathcal{C}$ is combinatorial.

We shall give a proof soon. The above theorem is slightly stronger than the following original Pultr’s theorem. Note that a quasifinite category in Pultr’s terminology is a locally finite category in our terminology.

**Theorem 3.2.** (Pultr [11 2.2 Theorem]) Let $\mathcal{C}$ be a locally finite category satisfying the conditions (1), (2) in the above Theorem 3.1, and

4. Every quotient is an epimorphism.

Then, $\mathcal{C}$ is combinatorial.

We shall define the term quotient now, but use only in the rest of this section to avoid confusions. A morphism $e : A \to B$ is a quotient, if in $e = \text{id}_B \circ e$, $\text{id}_B$ is an image of $e$ [11 1.1 Definition]. Theorem 3.1 implies Theorem 3.2 as follows.

**Proof.** It suffices to show that if every quotient is an epimorphism, then the third condition of Theorem 3.1 follows. Suppose that $f = g \circ h$ with $g$ being an image of $f$. Then $h$ is a quotient [11 1.4 1)]. By the third condition, $h$ is a quotient and epi, which implies that $h$ is an extremal epimorphism [11 1.2 Remark]. □

For a comparison to our main result Theorem 2.11, we would like to give a proof of Theorem 3.1 which is very similar to those by Lovász and Pultr.

**Proof.** (of Theorem 3.1). Let $T$ be a system of objects of $\mathcal{C}$ containing exactly one representative from each isomorphism class. We construct a mapping

$$\text{Hom}(A, B) \to T$$

by mapping $h$ to the class of $\text{im} \ h$. Thus if $h = g \circ f$ with $g$ being the image of $h$ with domain $T$ (one can choose a unique $T$ and $g$ by the uniqueness of the image and by the definition of $T$), then $h$ is mapped to $T$. This gives a disjoint decomposition

$$\text{Hom}(A, B) = \coprod_{T \in T} \text{Hom}(A, B)_T,$$

where $\text{Hom}(A, B)_T$ denotes the inverse image of $T$. Since $\mathcal{C}$ is locally finite, this is a finite sum. We consider a mapping for $T$ appeared in the sum (3.2)

$$\text{Extr}(A, T) \times \text{Mono}(T, B) \to \text{Hom}(A, B)_T, \quad (f, g) \mapsto g \circ f,$$

where $\text{Extr}(A, T)$ means the set of extremal epimorphisms. Take an $f \in \text{Extr}(A, T)$, and a $g \in \text{Mono}(T, B)$. Let $g'$ be an image of $g \circ f$. Thus we have $g \circ f = g' \circ f'$. The universality of the image implies that there is a mono $m$ with $g \circ m = g'$ and $f = m \circ f'$ (Definition 2.4). Since $f$ is an extremal epi, $m$ is isomorphic. This
implies that $g$ is an image of $g \circ f$, hence (3.3) is well-defined. It is surjective since if $h \in \text{Hom}(A, B)_T$, then there are $f, g$ with $h = g \circ f$ such that $g$ is an image of $h$ with domain $T$ by the comment after (3.1). Then $f$ is an extremal epimorphism by Condition (3) which means the surjectivity of (3.3). The fiber, i.e., the inverse image of one point in (3.3) has the same cardinality as $\text{Isom}(T, T)$, since the group acts faithfully and transitively on the fiber, as follows. Fix an $(f, g)$. Then $(f', g')$ maps to the same element in $\text{Hom}(A, B)_T$ if and only if $g' = g \circ f'$. Let $h$ be this composition. By the above argument, $g$ and $g'$ are images of $h$, and it follows that $g' = g \circ m$ for an isomorphism $m \in \text{Isom}(T, T)$. Thus $\text{Isom}(T, T)$ transitively acts on the fiber by $(f, g) \mapsto (m^{-1} f, gm)$. Since $g$ is mono, such an $m$ is unique, which shows the faithfulness. Thus (3.3) implies

$$|\text{Extr}(A, T)| \times |\text{Mono}(T, B)| = |\text{Isom}(T, T)| \times |\text{Hom}(A, B)_T|$$

(3.4)

and hence

$$|\text{Hom}(A, B)| = \sum_{T \in \mathcal{T}} |\text{Isom}(T, T)|^{-1} |\text{Extr}(A, T)| \times |\text{Mono}(T, B)|.$$ 

(3.5)

We claim that

$$|\text{Hom}(T, B)| = |\text{Hom}(T, C)| \text{ for all } T \in \mathcal{T}$$

implies that

$$|\text{Mono}(T, B)| = |\text{Mono}(T, C)| \text{ for all } T \in \mathcal{T}.$$ 

Then, by putting $T = B$ there is a monomorphism $B \to C$, and a symmetric argument gives a monomorphism $C \to B$, and Lemma (2.7) completes the proof. Let us prove the claim. From $|\text{Hom}(T, B)| = |\text{Hom}(T, C)|$, (3.5) implies

$$0 = \sum_{T' \neq T} \frac{|\text{Extr}(T, T')|}{|\text{Isom}(T', T')|} (|\text{Mono}(T', B)| - |\text{Mono}(T', C)|) + (|\text{Mono}(T, B)| - |\text{Mono}(T, C)|).$$

(Since $\text{Extr}(T, T) = \text{Isom}(T, T)$, by Lemma (2.7) for epi.) This implies that if

$$|\text{Mono}(T, B)| \neq |\text{Mono}(T, C)|$$

then

$$|\text{Mono}(T_1, B)| \neq |\text{Mono}(T_1, C)|$$

(3.6)

for some $T_1$ with a non-isomorphic extremal epimorphism $e : T \to T_1$. By the assumption,

$$|\text{Hom}(T_1, B)| = |\text{Hom}(T_1, C)|$$

holds, and we may iterate the same argument for $T_1$, to have $T_2$ with a proper extremal epimorphism $e_1 : T_1 \to T_2$ (For proper, see Definition (2.7)). In this way, we have an infinite sequence of objects $e_i : T_i \to T_{i+1}$. They are mutually non-isomorphic. (Assume any two are isomorphic, say $T_i$ and $T_j$, $i < j$. Then there is an isomorphism, hence an epimorphism $T_j \to T_i$, and Lemma (2.7) implies that the epimorphism $T_i \to T_j$ is an isomorphism, which implies $e_i$ is a monomorphism, and an extremal epimorphism, and thus $e_i$ is an isomorphism, leading to a contradiction.) These $T_i$’s are subobjects of $B$ or $C$, since in (3.6) one of the two is not empty, so one of $B$ and $C$ has infinitely many non-isomorphic subobjects $T_i$. This contradicts the finiteness in Condition (1). □

Recall that a preordered set is well-founded, if every non-empty subset has a minimal element. From the proof above, the following variant holds, which is our second main result.
Theorem 3.3. Theorem 3.1 also holds if the condition (1) is replaced with:

(1') For each object, the preordered class of its supobjects is well-founded.

Proof. If $C$ is not combinatorial, then in the proof, $T_i$ gives an infinite sequence of strictly decreasing supobjects of $T$, contradicting (1'). □

We remark that Dawar, Jakl, Reggio [3, Theorem 5] gives a different sufficient condition, as follows.

Theorem 3.4. (Dawar-Jakl-Reggio) Let $C$ be a locally finite category. If $C$ has pushouts and a proper factorization system, then it is combinatorial.

Reggio gives another sufficient condition [12, Theorem 4.3]

Theorem 3.5. (Reggio) Let $C$ be a locally finite category. If $C$ has a proper factorization system $(Q, M)$ such that $C$ is $Q$-well-founded. Then it is combinatorial.

We don’t describe the notion of proper factorization systems here, see [12, Appendix A].

4. Examples of categories to separate the scope of theorems

Here we construct a category, for which a weaker form of Main Theorem 2.12 (and consequently Main Theorem 2.11 and Reggio’s Theorem 3.5) are applicable, but Pultr’s Theorem 3.1 (or the method by Lovász), its variant Theorem 3.3 and Dawar-Jakl-Reggio’s Theorem 3.4 are not applicable.

Definition 4.1. We define a category $C$ as follows.

1. Objects are $P_i$ for all integer $i$.
2. $\text{Hom}(P_i, P_i) = \{\text{id}_{P_i}\}$.
3. $\text{Hom}(P_i, P_j) = \{a, b\}$ for $i > j$, $a \neq b$.
4. $\text{Hom}(P_i, P_j) = \emptyset$ for $i < j$.

Composition lows are $aa = a, ba = a, ab = b, bb = b$.

It is easy to check that this forms a category.

Lemma 4.2. Every non-identity morphism in $C$ is mono but not epi.

Proof. The equality $aa = ba$ implies $a$ is not epi, and so is $b$ since $ab = bb$. On the other hand, $aa \neq ab$ implies $a$ is mono, as well as $b$ by $ba \neq bb$. □

Lemma 4.3. The above category $C$ satisfies the five conditions in the weak form of Main Theorem 2.12 and hence is combinatorial.

Proof. Clearly, $C$ is locally finite. In Theorem 2.12 (1) follows since there is no epimorphism but the identities. Thus there are no proper supobjects, no maximal supobjects. Thus (2) and (3) are satisfied. Since every morphism is mono, (4) follows. Since there is no proper supobject, (5) holds. □

Lemma 4.4. In the above $C$, every object has infinitely many isomorphism classes of subobjects. Thus, Pultr’s Theorem 3.1 (i.e. Lovász’s argument [6]) will not work. The supobjects of each object are not well-founded, and thus the variant Theorem 3.3 is neither applicable.

Dawar-Jakl-Reggio’s Theorem 3.4 can not be applied to $C$, because they have no pushout for $a, b$ (no commutative squares for $a, b$ with the same domain).

Reggio’s Theorem 3.5 can be applied to $C$, with $(Q, M)$ being $\{(\text{id}), \{a, b, \text{id}\}\}$. 

Proposition 4.5. We shall see the ordered set $\mathbb{Z}$ as a category, with objects integers, and $\text{Hom}(i,j)$ is a singleton (an empty set respectively) if $i \geq j$ ($i < j$ respectively). Then, Main Theorem 2.11 is applicable, but Pultr’s Theorem 3.1, its variant Theorem 3.3, Dawar-Jakl-Reggio’s Theorem 3.4, Reggio’s Theorem 3.5 are not applicable.

Proof. Every morphisms are mono and epi. For Main Theorem 2.11 we take $\mathcal{M}$-supobjects to be empty, and $\mathcal{I}$ to be the set of all morphisms. Then all the conditions in Theorem 2.11 are satisfied.

There are infinitely many non-isomorphic subobjects for each object, and hence Pultr’s theorem cannot be applied. Since the supobjects have no minimal elements, it variant Theorem 3.3 cannot be applied. Since there is a morphism which is mono and epi but not isomorphic, [12] Lemma A.2 (b) shows that this category has no proper factorization system, and hence Dawar-Jakl-Reggio’s Theorem and Reggio’s Theorem are not applicable. □

Proposition 4.6. See the ordered set $\mathbb{N}$ as a category as above. Then, Theorem 3.1 is not applicable, but its variant Theorem 3.3 is applicable.

This is because $\mathbb{N}$ has infinitely many subobjects, but finitely many supobjects. A converse statement holds for the converse ordered set $\mathbb{N}$. We summarize the applicabilities in Table 1. For the ordered sets, Main Theorem 2.11 is applied with $\mathcal{M}$ empty and $\mathcal{I}$ all the morphisms.

|       | Main | Weak | Pultr | Its variant | Dawar-et.al. | Reggio |
|-------|------|------|-------|-------------|--------------|--------|
| digraphs       | yes  | yes  | yes   | yes         | yes          | yes    |
| finite groups   | yes  | yes  | yes   | yes         | no           | yes    |
| Definition      | yes  | yes  | no    | no          | no           | yes    |
| $\mathbb{Z}$     | yes  | no   | no    | no          | no           | no     |
| $\mathbb{N}$     | yes  | no   | no    | yes         | no           | no     |
| $\mathbb{N}^-$  | yes  | no   | no    | yes         | no           | no     |

Table 1. Scope of the theorems for combinatoriality

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