Closed form expression of the multivariate standard Normal distribution under a weighted sum constraint

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Abstract

In this letter we derive the \((n−1)\)-dimensional distribution corresponding to a \(n\)-dimensional i.i.d. Normal standard vector \(Z = (Z_1, Z_2, \ldots, Z_n)\) subjected to the weighted sum constraint \(\sum_{i=1}^n w_i Z_i = c\), \(w_i \neq 0\). We first address the \(n = 2\) case before proceeding with the general \(n \geq 2\) case. The resulting distribution is a Normal distribution whose mean vector \(\mu\) and covariance matrix \(\Sigma\) are explicitly derived as a function of \(w_1, \ldots, w_n, c\). The derivation of the density relies on a very specific positive definite matrix for which the determinant and inverse can be computed analytically.

1 Introduction

Factor models are extensively used in statistical modeling. In banking and finance for instance, it is a standard procedure to introduce a dependence structure among loans in credit risk modeling, see e.g., Li’s model [2016] but also Hull and White [2004], Andersen and Sidenius [2004], Vrins [2009], Laurent and Sestier [2016], just to name a few. In such models, the credit worthiness of the \(i\)-th entity is typically modeled as a random variable \(X_i\) defined as a weighted sum of common factors \((Y_1, \ldots, Y_J)\) accounting for the state of the global economy, the sector, the region, etc, and an idiosyncratic variable \(e_i\). In the popular case of a Gaussian copula model, all these factors are Normally distributed. The \(Y_j\) factors do not need to be independent, but can be decomposed (via a Cholesky transform) as a weighted sum of independent Normal risk factors \(\tilde{Z} := (\tilde{Z}_1, \tilde{Z}_2, \ldots, \tilde{Z}_n)\). The knowledge of a default event of the \(i\)-th reference entity reveals that the credit worthiness variable \(X_i\) reached the (assumed to be known) default threshold \(c_i\). The event \(X_i = c_i\) carries some information about the distribution of the underlying factors in that specific state. In particular, the vector \(\tilde{Z}\) is no longer standard Normal being told that \(X_i = c_i\); risk measures (e.g., value-at-risk) of the portfolio built from the outstanding loans might be strongly impacted by this information. This raises the following question: given a value \(y\) for the weighted sum \(w'Z\), what is the distribution of \(Z\)? Even if the analytical form of the conditional distribution is unknown, it is of course straightforward to sample a vector \(\hat{Z}\) of \(n\) Normal variables such that the weighted sum is \(y\). One possibility is to sample \(Z_j \sim N(0, 1)\) for \(j \in \{1, 2, \ldots, n-1\}\) and then set \(Z_n = (y - \sum_{i=1}^{n-1} w_i Z_i)/w_n\). Another possibility would be to sample a vector of \(n\) i.i.d. standard Normal variables \(\hat{Z} = (\hat{Z}_1, \hat{Z}_2, \ldots, \hat{Z}_n)\), compute \(\hat{y} := w'\hat{Z}\) and rescale the \(\hat{Z}\) to set \(Z = \frac{w}{\sqrt{w_n}}\hat{Z}\). Alternatively, one could take \(Z_i = \hat{Z}_i + (y - \hat{y})/(nw_i)\). However, none of these approaches yield the correct answer. The later requires the knowledge of the conditional distribution.

In this letter, we derive the conditional distribution associated to the \((w'Z = c)\)-slice of the \(n\)-dimensional standard Normal density when \(w_i \neq 0\) for all \(i \in \{1, 2, \ldots, n\}\). Interestingly, it is a \((n−1)\)-Normal whose mean vector and covariance matrix can be computed in closed form, respectively given by

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\[ \mu(c, w) := \frac{c}{w_2^2 \|w\|_2^2} \text{diag}(ww') \quad \Sigma_{i,j}(w) := \frac{w_i^2}{\|w\|_2^2} \left( \delta_{ij} \left( \|w\|_2^2 - w_i^2 \right) + (\delta_{ij} - 1)w_j^2 \right). \]

The distribution of \( Z \) can be obtained by simple rescaling of that of \( X \) as \( X = DZ \) where \( D = \text{diag}(w) \) is an invertible diagonal matrix. We first address the \((n = 2)\)-case before moving to the general case \( n \geq 2 \). The result derives from the analytical properties of a square positive definite matrix having a very specific form.

2 Bivariate case

We are looking for the distribution of \((X_1, X_2)\) given that \( X_1 + X_2 = c \). If \( Z_i \sim \mathcal{N}(0, 1) \text{ iid}, \) then \( X_1 \sim \mathcal{N}(1, w_1) \) are independent normal variables with standard deviation \( w_1 \). We note \( \phi(x; \mu, \sigma) \) the density associated to \( \mathcal{N}(\mu, \sigma) \).

We first compute the conditional density using Bayes

\[ f_X(x|x_1 + x_2 = c) = \frac{f_{X_1, X_2}(x_1, x_2|x_1 + x_2 = c)}{f_{x_1 + x_2}(c)} = \frac{f_{X_1, X_2}(x_1, x_2; x_1 + x_2 = c)}{f_{x_1 + x_2}(c)} \]

where the denominator is the centered Normal density with standard deviation \( \sqrt{w_1^2 + w_2^2} \):

\[ k_1(c, w) := f_{x_1 + x_2}(c) = \phi \left( c; \sqrt{w_1^2 + w_2^2} \right). \]

The numerator reads

\[ \frac{1}{\sqrt{2\pi w_1}} e^{-\frac{x_1^2}{2w_1^2}} \frac{1}{\sqrt{2\pi w_2}} e^{-\frac{x_2^2}{2w_2^2}} = \frac{1}{2\pi w_1 w_2} e^{-\frac{(x_1/w_1)^2 + (x_2/(w_2))^2}{2}}. \]

One can thus develop and complete the square to get

\[ \frac{1}{2\pi w_1 w_2} e^{-\frac{x_1^2}{2w_1^2}} e^{-\frac{c^2 - 2x_1c + x_1^2}{2w_1^2}} = \frac{e^{-\frac{x_1^2}{2w_1^2}} e^{-\frac{c^2 - 2x_1c + x_1^2}{2w_1^2}}}{2\pi w_1 w_2} \left( \frac{1}{w_1^2} + \frac{1}{w_2^2} \right)^{\frac{3}{2}} \]

and

\[ e^{-\frac{x_1^2}{2w_1^2}} e^{-\frac{c^2}{2w_2^2}} \frac{1}{2w_1 w_2} e^{-\frac{x_1^2}{2w_1^2}} \left( \frac{1}{w_1^2} + \frac{1}{w_2^2} \right)^{\frac{3}{2}} \left( x_1 - \frac{c}{w_1^2} \left( \frac{1}{w_1^2} + \frac{1}{w_2^2} \right) \right)^2 = k_1(c, w) \phi \left( x, \frac{c}{w_1^2 \left( \frac{1}{w_1^2} + \frac{1}{w_2^2} \right)} \left( \frac{1}{w_1^2} + \frac{1}{w_2^2} \right)^{-1} \right). \]

Hence, the conditional density \( f(x_1, x_2|c) \) of \((X_1, X_2)\) at \((x_1, c - x_1)\) is given by \( f(x_1) \) where

\[ f(x) := \phi \left( x; \mu(x, w), \sigma(w) \right) \quad \sigma(w) := \left( \frac{1}{w_1^2} + \frac{1}{w_2^2} \right)^{-1} \quad \mu(c, w) := \frac{c}{w_2 \left( \frac{1}{w_1^2} + \frac{1}{w_2^2} \right)} = \frac{c}{w_2} \sigma^2(w). \]

\[ 2 \]
3 Extension to higher dimensions

As before we compute the conditional density starting from Bayes' theorem,

\[ f_X \left( x \Bigg| \sum_{i=1}^{n} x_i = c \right) := f_{X_1, \ldots, X_n} \left( x_1, \ldots, x_n \Bigg| \sum_{i=1}^{n} x_i = c \right) = \frac{f_{X_1, \ldots, X_n} \left( x_1, \ldots, x_n; \sum_{i=1}^{n} x_i = c \right)}{f_{\sum_{i=1}^{n} x_i} \left( c \right)}. \]

The denominator collapses to the one-dimensional centered Normal density with variance \( w'w \):

\[ k_1(c, w) := f_{\sum_{i=1}^{n} x_i} \left( c \right) = \phi \left( c; 0, \sqrt{\sum_{i=1}^{n} w_i^2} \right). \]

The numerator can be written as

\[
\left( \prod_{i=1}^{n-1} e^{-\frac{x_i^2}{2w_i^2}} \right) e^{-\frac{\left( \sum_{i=1}^{n-1} x_i \right)^2}{2w_n^2}} = \left( \prod_{i=1}^{n-1} e^{-\frac{x_i^2}{2w_i^2}} \right) e^{-\frac{c^2 - 2c \sum_{i=1}^{n-1} x_i + \sum_{i=1}^{n-1} x_i^2 + \sum_{i=1}^{n-1} \sum_{j \neq i} x_i x_j}{2w_n^2}} \sqrt{2\pi w_n}.
\]

Hence, the conditional density looks like that of a \((n-1)\)-th dimensional Normal pdf:

\[
f_X \left( x \Bigg| \sum_{i=1}^{n} x_i = c \right) = k_2 \exp \left\{ -\frac{1}{2} \sum_{i=1}^{n-1} \left( \frac{1}{w_i^2} + \frac{1}{w_n^2} \right) x_i^2 + \frac{x_i}{w_n^2} \sum_{j=1,j \neq i}^{n-1} x_j - 2c \frac{x_i}{w_n^2} - \frac{c^2}{2w_n^2} \right\}
\]

where

\[
k_2 := k(c, w) e^{\frac{c^2}{2\sum_{i=1}^{n} w_i^2}} \quad \text{and} \quad k(c, w) := \frac{1}{(2\pi)^{n-1} \prod_{i=1}^{n} w_i}.
\]

In order for this density to belong to the Normal family, it needs to take the form of \( \phi(x; \mu, \Sigma) \) where \( \Sigma \) is a valid (positive definite) covariance matrix. In the sequel, we prove that \( f_X \left( x \Bigg| \sum_{i=1}^{n} x_i = c \right) \) does indeed have such a form and confirm that the corresponding matrix \( \Sigma \) is positive definite by determining the entries \( \alpha_{i,j} \) of \( \Sigma^{-1} \), the inverse of the \((n-1)\)-dimensional covariance matrix \( \Sigma \), and showing that \( \Sigma^{-1} \) is invertible and positive definite. Moreover, we compute analytically \( \Sigma \) and its determinant \( |\Sigma| \) as well as the corresponding mean vector \( \mu = (\mu_1, \mu_2, \ldots, \mu_{n-1}) \).

We start with the development of the Normal density of dimension \( n-1 \):

\[
\phi(x; \mu, \Sigma) = K \exp \left\{ -\frac{1}{2} \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \alpha_{i,j} x_i x_j - \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \alpha_{i,j} \mu_i x_j - \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \alpha_{i,j} \mu_j x_i + \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \alpha_{i,j} \mu_i \mu_j \right\}
\]

\[
= K \exp \left\{ -\frac{1}{2} \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \alpha_{i,j} x_i x_j - \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \alpha_{i,j} \mu_i x_j - \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \alpha_{i,j} \mu_j x_i + \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \alpha_{i,j} \mu_i \mu_j \right\}
\]

\[
= K \exp \left\{ -\frac{1}{2} \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \alpha_{i,j} x_i^2 + x_i \sum_{j=1,j \neq i}^{n-1} \alpha_{i,j} x_j - x_i \sum_{j=1}^{n-1} \left( \alpha_{j,i} + \alpha_{i,j} \right) \mu_j + \mu_i \sum_{j=1}^{n-1} \alpha_{i,j} \mu_j \right\}
\]

where \( K := 1/\sqrt{(2\pi)^{n-1} |\Sigma|} \). To determine the expression of the covariance matrix and mean vector of the conditional density (1) (assuming it is indeed Normal), it remains to determine the entries of
\( \mu, \Sigma^{-1} \) by inspection, comparing the expression of conditional density in (1) with that of the multivariate Normal (2).

Leaving only \( k(c, w) \) as a factor in front of the exponential in (1), the independent term (i.e. the term that does not appear as a factor of any \( x_i \)) reads without any loss of generality as

\[
\frac{c^2}{2 \sum_{i=1}^{n} w_i^2} - \frac{c^2}{2w_n^2} = \frac{c^2}{2w_n^2} \sum_{i=1}^{n-1} w_i^2 = -\frac{c^2}{2w_n^2} \sum_{j=1}^{n} w_j^2 \sum_{i=1}^{n-1} \gamma_i w_i^2
\]

for any \((\gamma_1, \gamma_2, \ldots, \gamma_{n-1})\) satisfying \(\sum_{i=1}^{n-1} \gamma_i w_i^2 = \sum_{i=1}^{n-1} w_i^2\).\(^1\) Comparing (1) and (2), it comes that the expression

\[
\left( \sum_{j=1}^{n} \frac{x_j^2}{w_j^2} \right) x_j + \sum_{j=1}^{n-1} \alpha_{ij} x_j - \gamma_i w_i^2 \sum_{j=1}^{n} \alpha_{ij} x_j = \frac{c^2}{w_n^2} \sum_{j=1}^{n} \alpha_{ij} x_j + \frac{c^2}{w_n^2} \sum_{j=1}^{n} \gamma_i w_i^2
\]

must agree with

\[
\alpha_{ij} x_j^2 + x_j \sum_{j=1, j \neq i}^{n-1} \alpha_{ij} x_j - x_j \sum_{j=1}^{n-1} (\alpha_{ij} + \alpha_{ji}) \mu_j + \mu_i \sum_{j=1}^{n-1} \alpha_{ij} \mu_j
\]

for all \(x_1, x_2, \ldots, x_{n-1}\). Equating the \(x_i x_j\) terms in (3) and (4) uniquely determines the components of \(\Sigma^{-1}\), \(\alpha_{ij} := (\Sigma^{-1})_{i,j} = \frac{1}{w_i} + \frac{1}{w_j} \) and \(\alpha_{ij,j} := (\Sigma^{-1})_{i,j,j} = \frac{1}{w_i^2}\). It remains to show that \(k(c, w) = K\), to find the expressions of the \(\mu_i\)'s from the \(x_i\) terms, provide the expression of \(\Sigma\) by inverting \(\Sigma^{-1}\) and finally, to check that the independent terms in (3) and (4) agree and that the implied \(\gamma_i\)'s comply with \(\sum_{i=1}^{n-1} \gamma_i w_i^2 = \sum_{i=1}^{n-1} w_i^2\). To that end, we rely on the following lemma (proven in the end of the paper).

**Lemma 1.** Let \(\delta_{ij}\) be the Kronecker delta and \(A(m)\) denote a matrix with \((i, j)\) elements \(A_{ij}(m) = a_i \delta_{ij} + a_o, a_k > 0\) for all \(k \in \{0, 1, \ldots, m\}\). Define \(\pi(m) := \prod_{k=0}^{m} a_k\) and \(s(m) := \sum_{k=0}^{m} 1/a_k\). Then:

(i) \(A(m)\) is positive definite;

(ii) its determinant is given by

\[
|A(m)| = \prod_{k=0}^{m} a_k = \pi(m)s(m)\;
\]

(iii) the elements of the inverse \(B(m) := (A(m))^{-1}\) are given by

\[
B_{i,j}(m) = \frac{1}{a_i s(m)} \left( \delta_{ij} a_i s(m) - 1 \right) + \delta_{ij} - 1 \right)
\]

As \(\Sigma^{-1}\) takes the form \(A(n-1)\) with \(a_0 \leftarrow 1/w_n^2\) and \(a_i \leftarrow 1/w_i^2\) for \(i \in \{1, 2, \ldots, n-1\}\) we can call Lemma 1 (i) to show that \(\Sigma^{-1}\) is symmetric and positive definite, proving that \(\Sigma\) is a valid covariance matrix satisfying \(|\Sigma| > 0\). From Lemma 1 (ii), \(k(c, w) = K\) as\(^2\)

\[
|\Sigma^{-1}| = \left( \sum_{i,j=1}^{n} \frac{1}{w_i w_j} \right) \left( \sum_{i=1}^{n} w_i^2 \right) = \frac{\sum_{i=1}^{n} w_i^2}{\prod_{i=1}^{n} w_i^2} = 1/\sqrt{|\Sigma|} = \sqrt{|\Sigma^{-1}|} = \sqrt{\frac{\sum_{k=1}^{n} w_k^2}{\prod_{k=1}^{n} w_k^2}}.
\]

We can then use Lemma 1 (iii) to determine \(B(n-1)\), the elements \(\beta_{i,j}\) of \(\Sigma\). Setting \(\|w\|_2 := \sqrt{\sum_{k=1}^{n} w_k^2}\)

\[
\beta_{i,j} = \frac{w_i^2}{\|w\|_2^2} (\delta_{ij} (\|w\|_2^2 - w_i^2) + (\delta_{ij} - 1) w_i^2) \;
\]

\(^1\)The constant case \(\gamma_i = 1\) might be a solution but it is not guaranteed at this stage.

\(^2\)Observe that in \(A(n-1)\) the summation and product indices agree with that of the \(a_i\), i.e. range from 0 to \(n - 1\), but the index of \(w_1\) ranges from 1 to \(n\).
Finally, the mean vector is obtained by equating the $x_i$ terms in (3) and (4). Using that $\Sigma^{-1}$ is symmetric, we observe that for all $i \in \{1, 2, \ldots, n-1\}$:

$$\frac{2c}{w_n^2} = 2 \sum_{j=1}^{n-1} \alpha_{i,j} \mu_j \Rightarrow \sum_{j=1}^{n-1} \alpha_{i,j} \mu_j = \frac{c}{w_n^2}. \quad (5)$$

Hence, $\Sigma^{-1} \mu = \frac{c}{w_n^2} 1_{n-1}$ where $1_{m}$ is the $m$-dimensional column vector with $m$ entries all set to $1$ so that

$$\mu_i = \frac{c}{w_n^2} \sum_{j=1}^{n-1} \beta_{i,j} = \frac{cw_i^2}{\|w\|_2^2}.$$

It remains to check that these expressions for $\mu$ and $\Sigma$ also comply with the independent term. Equating the independent terms of (3) and (4) and calling (5) yields

$$\frac{c^2 \gamma_i w_i^2}{w_n^2 \|w\|_2^2} = \mu_i \frac{c}{w_n^2} \Rightarrow \mu_i = \frac{c \gamma_i w_i^2}{\|w\|_2^2}$$

which holds true provided that we take $\gamma_i = 1$. This concludes the derivation of the conditional law as these $\gamma_i$’s comply with the constraint $\sum_{i=1}^{n-1} \gamma_i w_i^2 = \sum_{i=1}^{n-1} w_i^2 = \|w\|_2^2 - w_n^2$.

### Appendix: proof of Lemma 1

The matrix $A(m)$ is the sum of two positive definite matrices: a diagonal matrix with strictly positive entries $a_1, \ldots, a_m$ and a constant matrix with entries all set to $a_0 > 0$. Hence, $A(m)$ is positive definite, showing (i).

Let us now compute the determinant of $A(m)$. We proceed by recursion, showing that it is true for $m+1$ whenever it holds for $m \geq 2$. It is obvious to check that it is true for $m = 2$. The key point is to notice that it is enough to establish the following recursion rule:

$$|A(m+1)| = \pi(m+1)s(m+1) = \sum_{k=0}^{m} \pi(m+1-a_i) a_i + \pi(m+1-a_{m+1}) a_{m+1} = a_{m+1} |A(m)| + \pi(m). \quad (6)$$

We now apply the standard procedure for computing determinants, taking the product of each element $A(m)_{m+1,j}$ of the last row of $A(m)$ with the corresponding cofactor matrix $A(m)_{m+1,j}$ and computing the sum. Recall that the cofactor matrix associated to $A(m)_{i,j}$ is the submatrix $A(m)_{i,j}$ obtained by deleting the $i$-th row and $j$-th column of $A(m)$ Gentle [2007]. This yields

$$|A(m+1)| = a_0 \sum_{i=1}^{m} (-1)^{m+1+i} |A(m+1)_{m+1,j}| + (a_{m+1} + a_0) |A(m+1)_{m+1,m+1}|$$

where $|A(m+1)_{i,j}|$ is the minor associated to the $(i,j)$ element of $A(m)$, i.e. the determinant of the cofactor matrix $A(m+1)_{i,j}$. Interestingly, the cofactor matrices $A(m+1)_{i,j}$ take a form that is similar to $A(m)$. For instance $A(m+1)_{m+1,m+1} = A(m)$ and $A(m+1)_{m+1,m}$ is just $A(m)$ with $a_m \leftarrow 0$. Similarly, $A(m+1)_{m+1,m} = A(m)$ with $a_i \leftarrow 0$ provided that we shift all columns to the left, and put the last column back in first place (potentially changing the sign of the corresponding determinant), etc. More generally, for $i \in \{1, 2, \ldots, m\}$, the determinant of the $(i,j)$ cofactor matrix of $A(m)$, $|A(m+1)_{i,j}|$ is exactly that of $A(m)$ with $a_i \leftarrow a_{m+1}$ if $i = j$ or that of $A(m)$ with $a_i \leftarrow 0$ and $a_j \leftarrow a_{m+1}$ when $j \neq i$, up to some permutations of rows and columns. In fact:

$$|A(m+1)_{i,j}| = \sum_{k=0, k \neq i}^{m+1} \prod_{p=0, p \neq j}^{m} \frac{a_p}{a_k} a_{m+1} |A(m)| + \prod_{k=0, k \neq i}^{m} a_k = \frac{\pi(m+1)}{a_i} \sum_{k=0, k \neq i}^{m+1} \frac{1}{a_k} \quad (6)$$

$$|A(m+1)_{i,j}| = -(-1)^{i+j} \left( \sum_{k=0, k \neq (i,j)}^{m+1} \frac{\pi(m+1)}{a_k} a_i + \frac{\pi(m)}{a_i} a_{m+1} \right) = -(-1)^{i+j} \frac{\pi(m+1)}{a_i a_j}. \quad (7)$$

The minor $|A(m+1)_{m+1,i}|$ when $i \neq m+1$ can be obtained from the expression of $|A(m)|$ provided that we adjust the sign and replace $a_i$ by 0:
|A(m + 1)^{m+1,i}| = -(-1)^{i+m+1} \frac{\pi(m)}{a_i}, i \in \{1, 2, \ldots, m\}

(recall that A(m) is symmetric so that A(m)^{m+1,i} = A(m+1)^{i,m+1}). Therefore,

\[ |A(m + 1)| = (a_{m+1} + a_0)|A(m)| + a_0 \sum_{i=1}^{m} (-1)^{m+1+i} |A(m + 1)^{m+1,i}| \]

\[ = a_{m+1}|A(m)| + a_0|A(m)| + a_0 \sum_{i=1}^{m} (-1)^{2(m+1+i)} \frac{\pi(m)}{a_i} \]

\[ = a_{m+1}|A(m)| + a_0 \left( \frac{\pi(m)}{a_0} + \sum_{i=1}^{m} \frac{\pi(m)}{a_i} \right) - a_0 \sum_{i=1}^{m} \frac{\pi(m)}{a_i} \]

\[ = a_{m+1}|A(m)| + \pi(m) \]

and this recursion is equivalent to (ii).

Finally, the expression of \( B_{ij}(m) \) of \( B(m) := (A(m))^{-1} \) are given by \( 1/|A(m)| \) times the adjunct matrix of \( A(m) \), which is the (symmetric) cofactor matrix \( C(m) \). Observe that the elements \( C_{i,j}(m) \) are given by \((-1)^{i+j} M(m)_{i,j}\) where \( M(m)_{i,j} \) is the minor associated to \( A(m)_{i,j} \), i.e. \( |A(m)|^{i,j} \). Using the minors expressions (6) and (7) derived above replacing \( m \) by \( m - 1 \) yields:

\[ B(m)_{i,i} = \frac{|A(m)|^{i,i}}{|A(m)|} = \frac{\sum_{k=0}^{m} a_k a_{i-k} \frac{1}{a_k}}{\sum_{k=0}^{m} a_k} = \frac{s(m) - 1/a_i}{a_i s(m)} = \frac{a_i s(m) - 1}{a_i^2 s(m)} \]

\[ B(m)_{i,j \neq i} = (-1)^{i+j} \frac{|A(m)|^{i,j \neq i}}{|A(m)|} = -\frac{\pi(m)}{a_ia_j|A(m)|} = \frac{-1}{a_ia_j \sum_{k=0}^{m} a_k} = \frac{-1}{a_ia_j s(m)}. \]

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