ON A CONJECTURE OF SERRE ON ABELIAN THREEFOLDS

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En genre 3, le théorème de Torelli s’applique de façon moins satisfaisante :
on doit extraire une mystérieuse racine carrée (J.-P.S., Collected Papers, n° 129)

Abstract. In this article, we give a reformulation of a result from Howe, Leprevost and Poonen on a three dimensional family of abelian threefolds. We also link their result to a conjecture of Serre on a precise form of Torelli theorem for genus 3 curves.

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1. Introduction

1.1. Geometric Torelli’s theorem. Let $K$ be an algebraically closed field. If $X$ is a (smooth algebraic projective) curve of genus $g$ over $K$, the Jacobian $\text{Jac} X$ of $X$ is an abelian variety of dimension $g$, and $\text{Jac} X$ has a canonical principal polarization $\lambda$. We obtain in this way a morphism

$$\text{Jac} : M_g \longrightarrow A_g$$

from the space $M_g$ of ($K$-isomorphism classes of) curves of genus $g$ to the space $A_g$ of ($K$-isomorphism classes of) $g$-dimensional principally polarized abelian varieties (p.p.a.v.).

According to Torelli’s Theorem, proved one century ago, the map $X \mapsto (\text{Jac} X, \Theta)$ is injective. An algebraic proof was provided by Weil [18] half a century ago, and it is a long time studied question to characterize the image of this map.

If $g = 3$, these spaces are both of dimension $3g - 3 = g(g + 1)/2 = 6$. According to Hoyt [7] and Oort and Ueno [15], the image of $M_3$ is exactly the space of indecomposable principally polarized threefolds. Recall that $(A, \lambda)$ is decomposable if there is an abelian subvariety $B$ of $A$ neither equal to 0 nor to $A$, such that the restriction of $\lambda$ to $B$ is a principal polarization, and indecomposable otherwise. This was a problem left unsolved by Weil in [18].

Given a principally polarized abelian threefold $(A, \lambda)$ over $K$, two natural questions arise:

(i) How can we decide if the polarization is indecomposable?

(ii) How can we decide if $A$ is the Jacobian of a hyperelliptic curve?

Actually, both questions were answered by Igusa in 1967 [9] when $K = \mathbb{C}$, making use of a particular modular form $\chi_{18}$ on the Siegel upper half-space (see Th. 3.5.2 below).

1.2. Arithmetic Torelli’s theorem. Assume now that $K$ is an arbitrary field. Then, as Serre noticed in [12], the above correspondence is no longer true. Let $(A, \lambda)$ be a p.p.a.v. of dimension $g$ over $K$, and assume that $(A, \lambda)$ is isomorphic over $K$ to the Jacobian of a curve $X$ of genus $g$.

Theorem 1.2.1 (Serre). The following alternative holds:

(i) If $X$ is hyperelliptic, there exists a model $X/K$ of $X$ and a $K$-isomorphism between the p.p.a.v. $(\text{Jac} X, \Theta)$ and $(A, \lambda)$.

(ii) If $X$ is non hyperelliptic, there exists a model $X/K$ of $X$ and a quadratic character $\varepsilon : \text{Gal}(K_s/K) \rightarrow \{\pm 1\}$ such that $(\text{Jac} X, \Theta)$ is isomorphic to the twist $(A, \lambda)_\varepsilon$ of $(A, \lambda)$ by $\varepsilon$.

In particular, if $\varepsilon$ is not trivial, this implies that $\text{Jac} X$ is not isomorphic to $A$ over $K$, but only over a quadratic extension, and $(A, \lambda)$ is not isomorphic over $K$ to the Jacobian of a curve.

1.3. Serre’s conjecture. Let us come back to the case $g = 3$. Let there be given an indecomposable principally polarized abelian threefold $(A, \lambda)$ defined over $K$.

In a letter to Top [17] in 2003, J.-P. Serre asked two questions:

(i) How to decide, knowing only $(A, \lambda)$, that $X$ is hyperelliptic?

(ii) If $X$ is not hyperelliptic, how to find the quadratic character $\varepsilon$?

He proposed, in the case $K \subset \mathbb{C}$, the following conjecture:

Conjecture 1.3.1. Let $(A, \lambda)$ be an indecomposable principally polarized abelian threefold over $K$ isomorphic over $\overline{K}$ to the Jacobian of a curve $X$ of genus 3. Then there is an invariant $\chi_{18}(A, \lambda)$ such that

(i) $\chi_{18}(A, \lambda) = 0$ is and only if $X$ is hyperelliptic;
We use here the notation $\chi_{18}$ to emphasize that this conjecture was inspired by the results obtained by Igusa, and also much earlier by Klein [11] (see the remark after Cor. 4.3.3). In this article Klein relates (up to an undetermined constant) the modular form $\chi_{18}$ and the square of the discriminant of the quartic $X$ (when $\chi_{18}(\tau) \neq 0$). This invariant seemed to Serre a good choice to find this “mysterious square root”.

We plan to answer in the affirmative this conjecture for a family of abelian threefolds which are isogenous to the product of three elliptic curves (see Cor. 4.3.2). This will rely on the work of Howe, Leprevost and Poonen [8] for which we propose a natural rephrasing. For any field $K$ of characteristic different from 2, they consider abelian threefolds $(A, \lambda)$ defined as a quotient of three elliptic curves (with the trivial polarization) by a certain subgroup of 2-torsion points. For this threedimensional family, they make explicit the equation of the related curve and express the character $\varepsilon$ by a invariant $T$ involving the coefficients of the elliptic curves. In the first part, we show that $T$ can be naturally interpreted as a determinant. In a second phase, we take $K \subset \mathbb{C}$ and by uniformization, we express $T$ in terms of certain Thetanullwerte of the elliptic curves. Then using the duplication and transformation formula we express the modular form $\chi_{18}(A, \lambda)$ in terms of the same Thetanullwerte and compare the two expressions. We also obtain a proof of Klein’s result in this particular case and give the constant involved (see Cor. 4.3.3).

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2. Ciani Quartics

In this section we reformulate a result of [8] on a three-dimensional family of nonhyperelliptic genus 3 curves. In particular, this gives a more natural point of view on Prop. 15 of [8].

2.1. Definition of Ciani quartics. Edgardo Ciani gave in 1899 [3] a classification of nonsingular complex plane quartics curves based on the number of involutions in their automorphism group. We describe below the family of quartics admitting (at least) two commuting involutions (different from identity).

Let $K$ be a field with char $K \neq 2$, and $\text{Sym}_3(K)$ the vector space of symmetric matrices of size 3 with coefficients in $K$. Let

$$Q_m(x, y, z) = t.v.m.v, \quad v = (x^2, y^2, z^2), \quad m = \begin{bmatrix} a_1 & b_3 & b_2 \\ b_3 & a_2 & b_1 \\ b_2 & b_1 & a_3 \end{bmatrix} \in \text{Sym}_3(K).$$

Then

$$Q_m(x, y, z) = a_1x^4 + a_2y^4 + a_3z^4 + 2(b_1y^2z^2 + b_2x^2z^2 + b_3x^2y^2)$$

is a ternary quartic, and the map $m \mapsto Q_m$ is an isomorphism of $\text{Sym}_3(K)$ to the vector space of ternary quartic forms invariant under the three involutions

$$\sigma_1(x, y, z) = (-x, y, z), \quad \sigma_2(x, y, z) = (x, -y, z), \quad \sigma_3(x, y, z) = (x, y, -z).$$
The form \(Q_m\) is the zero locus of a plane quartic curve \(X_m\), whose automorphism group contains the Vierergruppe \(V_4 = (\mathbb{Z}/2\mathbb{Z})^2\).

If \(X_m\) is a nonsingular curve, we say that \(X_m\) is a Ciani quartic and that \(Q_m\) is a Ciani form. Now, E. Ciani (loc. cit.) proved that a plane quartic admitting two commuting involutions is geometrically isomorphic to a Ciani quartic (a more recent reference is [1]).

**Proposition 2.1.1.** If \(X\) is a plane quartic curve defined over \(K\), admitting at least two commuting involutions, also defined over \(K\), then there is \(m \in \text{Sym}_3(K)\) such that \(X\) is isomorphic to \(X_m\) over \(K\).

**Proof.** Let \(M_1\) and \(M_2\) in \(\text{PGL}_3(K)\) inducing two commuting involutions of \(X\). Then \(M_1^2 = \alpha_1 I\) with \(\alpha_1 \in K\) and \(\det(M_1)^2 = \alpha_1^3\), hence \(\alpha_1\) is a square and we can assume, by dividing \(M_1\) by \(\sqrt{\alpha_1}\), that \(M_1^2 = I\). The two matrices \(M_i\) commute so we can diagonalize them in the same basis: after a change of coordinates, we can suppose that \(M_i\) are (projectively) equal to

\[
I_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad I_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]

This implies that a quartic equation \(Q(x, y, z) = 0\) of \(X\) in these new coordinates must be invariant by the involutions \(\sigma_1\) and \(\sigma_2\) above, hence, \(Q\) is a Ciani form. \(\square\)

### 2.2. Discriminant of a ternary form

Our definition of a Ciani form includes that its zero locus must be a nonsingular curve. This condition is fulfilled if and only if the discriminant of the form is not 0. In order to obtain a criterion for this condition, we develop an algorithm for the discriminant of a general ternary form.

The **multivariate resultant** \(\text{Res}(f_1, \ldots, f_n)\) of \(n\) forms \(f_1, \ldots, f_n\) in \(n\) variables with coefficients in a field \(K\) is an irreducible polynomial in the coefficients of \(f_1, \ldots, f_n\) which vanishes whenever \(f_1, \ldots, f_n\) have a common non-zero root. One requires that the resultant is irreducible over \(\mathbb{Z}\), i.e. it has integral coefficients with greatest divisor equal to 1, and moreover

\[
\text{Res}(x_1^{d_1}, \ldots, x_n^{d_n}) = 1
\]

for any \((d_1, \ldots, d_n) \in \mathbb{N}^n\). The resultant exists and is unique. There is a remarkable determinantal formula for the resultant of 3 ternary forms of the same degree \(d\), due to Sylvester; see [6] for a modern exposition and a proof. We give this formula in the case \(d = 3\). Then \(\text{Res}(f_1, f_2, f_3)\) is a form of degree 27 in 30 unknowns. We shall express \(\text{Res}(f_1, f_2, f_3)\) as the determinant of a square matrix of size 15.

Let \(I_d\) be the set of sequences \(\nu = (\nu_1, \nu_2, \nu_3)\) with \(\nu_1 + \nu_2 + \nu_3 = d\). The monomials \(x^\nu = x_1^{\nu_1} x_2^{\nu_2} x_3^{\nu_3}\), where \(\nu \in I_d\), form the canonical basis of the space \(V_d\) of ternary forms of degree \(d\), which is of dimension \((d + 1)(d + 2)/2\).

For any monomial \(x^\nu \in V_2\) with \(\nu_1 + \nu_2 + \nu_3 = 2\), we choose arbitrary representations \(f_i = x_1^{\nu_{i,1}} f_{i,1} + x_2^{\nu_{i,2}} f_{i,2} + x_3^{\nu_{i,3}} f_{i,3}\) (\(1 \leq i \leq 3\)),

where \(f_{i,j}\) are forms of degree \(2 - \nu_j\), for \(1 \leq i, j \leq 3\). Such a representation is always possible, although not unique. Now we define

\[
S(x^\nu) = \det \begin{bmatrix} f_{1,1} & f_{1,2} & f_{1,3} \\ f_{2,1} & f_{2,2} & f_{2,3} \\ f_{3,1} & f_{3,2} & f_{3,3} \end{bmatrix}.
\]

Note that this determinant is indeed a ternary form of degree

\[
(2 - \nu_1) + (2 - \nu_2) + (2 - \nu_3) = 6 - 2 = 4.
\]
Since the monomials $x^\nu$ with $\nu \in I_2$ make up a basis of $V_2$, we have thus defined a linear map $S : V_2 \rightarrow V_4$. We consider the linear map

$$T : V_1 \times V_1 \times V_1 \times V_2 \rightarrow V_4$$

given by

$$T(l_1, l_2, l_3, g) = l_1f_1 + l_2f_2 + l_3f_3 + S(g).$$

Now one proves that the determinant of $T$ is independent of the choices made in the definition of $S$, and Sylvester’s formula holds:

$$\text{Res}(f_1, f_2, f_3) = \det T.$$

Generally speaking, the matrix of $T$ involves 864 monomials. Now, let $Q$ be a ternary form of degree $d$, and $X$ be the plane projective curve which is the zero locus of $Q$. Call $q_1, q_2, q_3$ the partial derivatives of $Q$. The discriminant of $Q$ is $\text{Disc}(Q) = \text{Res}(q_1, q_2, q_3)$. It is a form of degree $(3(d - 1))^2$ in the coefficients of $Q$, and $X$ is non-singular if and only if $\text{Disc}(Q) \neq 0$. The discriminant is an invariant of ternary forms: if $g \in \text{GL}_3(K)$, then

$$\text{Disc}(gQ) = (\det g)^w \text{Disc}Q,$$

where $w = d(d - 1)^2$.

If $Q$ is a quartic form $\text{Disc}Q$ is a form of degree 27 in the coefficients, with $w = 36$ in (1). Applying Sylvester’s formula, we get:

**Proposition 2.2.1.** Let $m \in \text{Sym}_3(K)$ and $c_i = a_i a_k - b_i^2$ is the cofactor of $a_i$ for $1 \leq i \leq 3$. If $(q_1, q_2, q_3)$ are the partial derivatives of the ternary quartic $Q_m$, then $\text{Disc}Q_m = 2^{36}\text{D}(m)$, where

$$\text{D}(m) = a_1 a_2 a_3 (c_1 c_2 c_3)^2 \det(m)^4.$$ 

Note that this result was obtained by Edge [4], in a more intricate way.

We denote by $S$ the set of $m \in \text{Sym}_3(K)$ such that

$$a_1 a_2 a_3 \neq 0, \quad c_1 c_2 c_3 \neq 0.$$ 

Now, Prop. 2.2.1 implies that the curve $X_m$ is nonsingular if and only if $m$ belongs to the set $S^\times = S \cap \text{GL}_3(K)$.

**Lemma 2.2.2.** The map $m \mapsto Q_m$ from $S^\times$ to the set $Q$ of Ciani forms is a bijection. \hfill \Box

The automorphisms of $X_m$ induce a simple description of its Jacobian. In order to make it explicit, we need to introduce a certain product of elliptic curves.

### 2.3. Product of elliptic curves

We introduce the following notations: let

$$E_i : y^2 = x(x^2 - 4b_i x - 4c_i), \quad (b_i \in K, c_i \in K^\times) \quad (i = 1, 2, 3),$$

three elliptic curves with $(0, 0)$ as a rational 2-torsion point. The discriminant of $E_i$ is $\Delta_i = 2^{12}c_i^2 \delta_i$, where $\delta_i = b_i^2 + c_i \in K^\times$. We assume that there exists a square root $\rho \in K^\times$ of $\delta(A)$, that is, $\Delta_1 \Delta_2 \Delta_3$ is a square in $K$. We denote by $A$ the set of products $E_1 \times E_2 \times E_3$ of such curves and we define

$$\bar{A} = \{ \bar{A} = (A, \rho) \in A \times K^\times \mid \rho^2 = \delta(A) \}.$$ 

If $\bar{A} \in \bar{A}$, we put $a_i = \rho/\delta_i$ and

$$\text{Mat}(\bar{A}) = \begin{bmatrix} a_1 & b_3 & b_2 \\ b_1 & a_2 & b_1 \\ b_2 & b_1 & a_3 \end{bmatrix} \in S.$$ 

Conversely, a matrix $m \in S$ defines an abelian threefold $A(m) \in A$, which is the product of the curves

$$E_i : y^2 = x(x^2 - 4b_i x - 4c_i) \quad (i = 1, 2, 3).$$
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Then
\[ \delta_i = b_i^2 + c_i = a_j a_k \in K^\times, \quad \Delta_1 \Delta_2 \Delta_3 = (2^{18} a_1 a_2 a_3 c_1 c_2 c_3)^2, \quad \delta(A) = (a_1 a_2 a_3)^2. \]

We define \( \rho(m) = a_1 a_2 a_3, \) and \( \mathbf{Ab}(m) = (A(m), \rho(m)) \in \tilde{\mathbb{A}}. \)

**Lemma 2.3.1.** The maps

\[ \text{Mat}: \tilde{\mathbb{A}} \longrightarrow S, \quad \text{Ab}: S \longrightarrow \tilde{\mathbb{A}}, \]

are mutually inverse bijections. \( \square \)

The two lemmas 2.2.2 and 2.3.1 provide a natural map from the set \( Q \) of Ciani quartic forms to \( \tilde{\mathbb{A}}. \) This map has actually a geometric meaning, and in order to explain it, we introduce the following notation. If \( m \in \text{GL}_3(K) \), we denote by \( \text{Cof}(m) \) the cofactor matrix of \( m \), satisfying

\[ m^t \text{Cof} m = \det m, \quad \det \text{Cof} m = (\det m)^2, \quad \text{Cof} \text{Cof} m = \left( \det m \right)^2 m. \]

Let \( Q_m \) be a Ciani form associated to \( m \in S^\times \) and \( X_m \) be the corresponding Ciani quartic

\[ X_m : Q_m(x, y, z) = F_m(x^2, y^2, z^2) = 0. \]

By quotient, we get three genus one curves

\[ C_1 := X_m / < 1, \sigma_1 > : F(yz, x^2, y^2) = 0, \]
\[ C_2 := X_m / < 1, \sigma_2 > : F(xz, y^2, z^2) = 0, \]
\[ C_3 := X_m / < 1, \sigma_3 > : F(xy, z^2, x^2) = 0, \]

where \( \sigma_i \) \((i = 1, 2, 3)\) are the involutions of \( X_m. \) Another change of variables maps the genus 1 quartics \( C_i \) to the elliptic curves

\[ F_i : y^2 = x(x^2 - 4d_i x - 4a_i \det(m)), \quad (i = 1, 2, 3). \]

In this way, we get a map

\[ \varphi : X_m \longrightarrow B_m = F_1 \times F_2 \times F_3. \]

Let us now look more closely at \( B_m. \) The identity

\[ \text{Cof} \text{Cof} m = (\det m).m \]

implies that the cofactor of \( c_i \) is \( a_i \det m. \) Hence,

\[ \mathbf{Ab}(\text{Cof} m) = (B_m, c_1 c_2 c_3). \]

Since the Jacobian is the Albanese variety of \( X_m, \) we get a factorization

\[ X_m \]
\[ \downarrow \varphi \]
\[ \text{Jac} X_m \longrightarrow B_m \]

where \( \iota \) is a canonical embedding. Since the images of the regular differential forms on \( F_i \) make a basis of those on \( X_m, \) we obtain:

**Proposition 2.3.2.** The map

\[ \Phi : \text{Jac} X_m \longrightarrow \mathbf{A}(\text{Cof} m) \]

is a \((2, 2, 2)\)-isogeny defined over \( K. \) \( \square \)
The correspondences

\[ \begin{array}{ccc}
m & \longrightarrow & \text{Cof } m \\
\downarrow & & \downarrow \\
Q_m & \longrightarrow & A(\text{Cof } m) \xrightarrow{\text{isg}} \text{Jac } X_m
\end{array} \]

lead to a commutative diagram, where \( Q \) is the space of Ciani quartics over \( K \):

\[ \begin{array}{ccc}
\mathbb{S}^4 & \xrightarrow{\text{Cof}} & \mathbb{S}^4 \\
\downarrow & = & \downarrow \\
Q & \xrightarrow{\text{"Jac"}} & \tilde{\mathbb{A}}
\end{array} \]

In the next section we describe the kernel of the isogeny \( \Phi \).

2.4. The theory of Howe, Leprevost and Poonen revisited. The previous isogeny can be made more precise as we recall from [8]. There are some differences between their notation and ours, see the remark at the end of Sec 2.5 for a comparison. Let us introduce couples \((A, W)\), where:

(i) \( A \in \mathcal{A} \) as defined in \( \S 2.3 \)

The Weil pairings on the factors combine to give a non degenerate alternating pairing \( e_2 \) on the finite group scheme \( A[2] \) over \( K \).

(ii) \( W \) is a totally isotropic indecomposable subspace of \( A[2] \) defined over \( K \).

Choose a basis \((P_i, Q_i) \in E_i[2]\), that is, a level 2 structure on \( E_i \). This defines a level 2 structure on \( A \). In [8, Lem.13] it is proved that after a labeling of the 2-torsion points we can write

\[ W = \left\{ (O, O, O), (O, Q_2, Q_3), (Q_1, O, Q_3), \right. \]

\[ \left. (Q_1, Q_2, O), (P_1, P_2, P_3), (P_1, R_2, R_3), (R_1, P_2, R_3), (R_1, R_2, P_3) \right\} \]

with

\[ Q_i = (0, 0), \ P_i = (0, 2b_i + \rho_i), \ R_i = (0, 2b_i - \rho_i), \ \rho_1\rho_2\rho_3 = \rho_W, \]

and the four possible choices of \( \rho_1, \rho_2, \rho_3 \) leading to the same value of \( \rho_1\rho_2\rho_3 \) give the same subgroup \( W \). Conversely, if \( (A, \rho) \in \tilde{\mathbb{A}} \) is given, if we choose \( \rho_1, \rho_2, \rho_3 \) in such a way that \( \rho_1\rho_2\rho_3 = \rho \), and if we define \( P_i \) and \( R_i \) as above, then we can define a subgroup \( W_\rho \) by (2).

Lemma 2.4.1. The map \( (A, \rho) \mapsto (A, W_\rho) \) from \( \tilde{\mathbb{A}} \) to the set of couples \((A, W)\) as defined above is a bijection. \( \square \)

We take on \( A = A(m) \) the principal polarization \( \lambda \) which is the product of the canonical polarizations on each factor. Then we have a commutative diagram

\[ \begin{array}{ccc}
A^{2\lambda} & \longrightarrow & A^\vee \\
\downarrow \pi & & \downarrow \pi \\
A' \xrightarrow{\lambda'} & \longrightarrow & (A')^\vee
\end{array} \]

with a unique principal polarization \( \lambda' \) on \( A' = A'(m) = A(m)/W_{\rho(m)} \). From [8, Prop.15] we get:

Theorem 2.4.2. The composition of the isogeny \( \Phi \) and of the projection \( \pi \) leads to an isomorphism of p.p.a.v.:

\[ \text{Jac } X_m \longrightarrow A'(\text{Cof } m). \]
As a corollary, for any \( m \), the \( \text{p.p.a.v.} \ (A', \lambda') \) is indecomposable.
The reverse direction is more interesting and will give an algebraic answer to Serre’s
conjecture.

2.5. Relation with Serre’s conjecture. We need the following elementary lemma
from linear algebra.

**Lemma 2.5.1.** The map \( m \mapsto \text{Cof} \ m \) induces an exact sequence
\[
1 \longrightarrow \{ \pm 1 \} \longrightarrow \text{GL}_3(K) \xrightarrow{\text{Cof}} G^{x^2}(K) \longrightarrow 1
\]
with
\[
G^{x^2}(K) = \{ m \in \text{GL}_3(K) \mid \det m \in K^{x^2} \}.
\]
Let \( (A, \rho) = \tilde{A} \in A \) and \( m = \text{Mat}(\tilde{A}) \).

**Theorem 2.5.2.** The following results hold.

(i) if \( \text{T}(\tilde{A}) = 0 \), that is, \( m \in S \setminus S^x \), there is a hyperelliptic curve \( X \) of genus 3
such that \( A'(m) \) is isomorphic to the Jacobian of \( X \).

(ii) if \( \text{T}(\tilde{A}) \neq 0 \), that is, \( m \in S^x \), then there exists a non hyperelliptic curve
of genus 3 defined over \( K \) whose Jacobian is isomorphic to \( A'(m) \) if and
only if \( \text{T}(\tilde{A}) \) is a square in \( K \).

**Proof.** The first part is [8, Prop.14] where the hyperelliptic curve is constructed
explicitly. For the second part, if \( \det(m) \) is a square, then using Lem. 2.5.1, we see
that there exists a matrix \( m' \in S^x \) such that \( m = \text{Cof}(m') \) and we apply Th. 2.4.2.

If \( d = \det(m) \) is not a square, let \( m_d = dm \) and \( \tilde{A}_d = \text{Ab}(m_d) \). \( A(m_d) \) is defined by
\[
E_i : y^2 = x(x^2 - 4b_i dx - 4c_i d^2) \quad (i = 1, 2, 3),
\]
Thus \( A(m_d) \) is a quadratic twist of \( A(m) \). Now, \( \det(m_d) \) is a square, so there exists
\( m' \) such that \( \text{Jac}(X_{m'}) \) is isomorphic to \( A'(m_d) \). Since \( A'(m_d) \) is a quadratic twist
of \( A'(m) \) and is the Jacobian of a non hyperelliptic curve, Th. 2.5.1 shows that
\( A'(m) \) cannot be a Jacobian.

**Corollary 2.5.3.** With the same notation as above:

(i) if \( \text{T}(\tilde{A}) \in K^{x^2} \), there is an isogeny defined over \( K \)
\[
\text{Jac} X_{m'} \longrightarrow A(m), \quad \text{Cof} m' = m,
\]

(ii) if \( \text{T}(\tilde{A}) \notin K^{x^2} \), there is an isogeny defined over \( K \)
\[
\text{Jac} X_{m'} \longrightarrow A(m_d) \quad \text{Cof} m' = dm. \quad \square
\]

We hope to give in a near future a geometric interpretation of the connection of
Serre’s problem with the determinant of certain quadratic forms in the general case.

**Remark.** In [8], Howe, Leprevost and Poonen write the elliptic curves
\[
y^2 = x(x^2 + A_i x + B_i) \quad \text{avec} \ A_i, B_i \in K \quad (i = 1, 2, 3).
\]
So
\[
A_i = -4b_i, \quad B_i = -4c_i,
\]
\[
\Delta_i = A_i^2 - 4B_i = 16(b_i^2 + c_i) = 16\delta_i,
\]
\[
d_i = -(A_i + 2x(P_i)) = 4b_i - 4(b_i + \rho_i) = -4\rho_i, \quad d_i^2 = \Delta_i.
\]
And the factor
\[
T_0(\tilde{A}) = d_1d_2d_3(A_1^2 + A_2^2 + A_3^2 - 1) - 2A_1A_2A_3,
\]
which is
\[ T_0(\tilde{A}) = 64[\rho(\frac{b_1^2}{\delta_1}) + \frac{b_2^2}{\delta_2} + \frac{b_3^2}{\delta_3} - 1) + 2b_1b_2b_3]. \]

We then have \( T_0(\tilde{A}) = 64T(\tilde{A}). \)

3. Complex abelian varieties

We recall in this section some well known propositions on abelian varieties over \( \mathbb{C} \) and fix the notation.

3.1. The symplectic group. If \( V \) is a module of rank \( 2g \) over a commutative ring \( R \) and if \( E \) is a non-degenerate alternating bilinear form on \( V \), a basis \( (a_i)_{1 \leq i \leq 2g} \) of \( V \) is said symplectic if the matrix \( (E(a_i, a_j)) = J \), where
\[ J = \begin{bmatrix} 0 & 1_g \\ -1_g & 0 \end{bmatrix}. \]

The group of matrices
\[ M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \text{SL}_{2g}(R) \]
such that \( M.J.M^t = J \) is the symplectic group \( \text{Sp}_{2g}(R) \). It acts simply transitively on the set of symplectic bases of \( V \).

Lemma 3.1.1 ([2, Lem.8.2.1]). If \( M \in \text{Sp}_{2g}(R) \) the following conditions are equivalent.

(i) \( M \in \text{Sp}_{2g}(R) \).
(ii) \( ^tA.C \) and \( ^tB.D \) are symmetric, and \( ^tA.D - ^tC.B = 1_g \).
(iii) \( A^tB \) and \( C^tD \) are symmetric and \( A^tD - B^tC = 1_g \). \( \Box \)

The group \( \text{Sp}_{2g}(R) \) is the group of \( R \)-rational points of a Chevalley group scheme \( \text{Sp}_{2g} \), which contains certain remarkable subgroups defined as follows. The reductive subgroup \( M \) of \( \text{Sp}_{2g} \) is the subgroup which respects the canonical decomposition \( Z^{2g} = Z^g \oplus Z^g \). Elements of \( M(\mathbb{Z}) \) are
\[ M = \begin{bmatrix} A & 0 \\ 0 & ^tA^{-1} \end{bmatrix}, \quad A \in \text{GL}_g(\mathbb{Z}). \]

The unipotent subgroup \( U \) is the stability group leaving pointwise fixed the canonical totally isotropic subspace \( V_0 \), which is the first direct summand in the standard decomposition. Elements of \( U(\mathbb{Z}) \) are
\[ U = \begin{bmatrix} 1_g & B \\ 0 & 1_g \end{bmatrix}, \quad ^tB = B, \quad B \in \text{M}_g(\mathbb{Z}). \]

The unipotent subgroup \( V \) opposite to \( U \) is the stability group of the second direct summand in the standard decomposition. One has \( V = ^tU = J.U.J^{-1} \). Elements of \( V(\mathbb{Z}) \) are
\[ V = \begin{bmatrix} 1_g & 0 \\ C & 1_g \end{bmatrix}, \quad ^tC = C, \quad C \in \text{M}_g(\mathbb{Z}). \]

The subgroup \( P = M \ltimes U \) is the parabolic subgroup of \( \text{Sp}_{2g} \) normalizing \( V_0 \), and \( P \) is actually a maximal parabolic subgroup. Elements of \( P(\mathbb{Z}) \) are
\[ P = \begin{bmatrix} A & B \\ 0 & ^tA^{-1} \end{bmatrix}, \quad A^tB = B^tA, \quad A \in \text{GL}_g(\mathbb{Z}), \quad B \in \text{M}_g(\mathbb{Z}). \]
3.2. Abelian varieties. Let $\Omega = [w_1 \ldots w_{2g}] \in M_{g,2g}(\mathbb{C})$, where $w_1, \ldots, w_{2g}$ are column vectors giving a basis of $\mathbb{C}^g$ on $\mathbb{R}$. It generates a lattice

$$\Lambda = \Omega \mathbb{Z}^{2g} \subset \mathbb{C}^g.$$ 

Let $\mathcal{R}_g$ be the set of matrices $\Omega \in M_{g,2g}(\mathbb{C})$ satisfying the Riemann conditions

$$\Omega . J \cdot \Omega = 0, \quad 2i(\overline{\Omega} . J^{-1} . \Omega)^{-1} > 0$$

(> 0 means positive definite). We call such a matrix $\Omega$ a period matrix. If $\Omega \in \mathcal{R}_g$, the torus $A_\Omega = \mathbb{C}^g/\Lambda$ is an abelian variety of dimension $g$ with a principal polarization $\lambda$ represented by the hermitian form $H = 2i(\overline{\Omega} . J^{-1} . \Omega)^{-1}$ (see [2] Lem.4.2.3). The group $GL_g(\mathbb{C})$ acts on the right on $\mathcal{R}_g$. If we write

$$\Omega = [(w_1 \ldots w_g) (w_{g+1} \ldots w_{2g})] = [\Omega_1 \ \Omega_2], \quad \text{where } \Omega_i \in M_g(\mathbb{C}),$$

we get $W. [\Omega_1 \ \Omega_2] = [W.\Omega_1 \ W.\Omega_2]$ for any $W \in GL_g(\mathbb{C})$. This action induces an isomorphism of p.p.a.v. In particular if we choose $W = \Omega_2^{-1}$, we see that $A_\Omega$ is isomorphic to the p.p.a.v.

$$A_\tau = A_{\Omega(\tau)}, \quad \Omega(\tau) = [\tau \ 1_g], \quad \tau = \tau(\Omega) = \Omega_2^{-1} \Omega_1,$$

and $\Omega \in \mathcal{R}_g$ if and only if $\tau(\Omega)$ belongs to the Siegel upper half plane

$$\mathbb{H}_g = \{ \tau \in M_g(\mathbb{C}) \mid \tau > 0, \text{Im} \tau > 0 \}.$$ 

We call a matrix $\tau \in \mathbb{H}_g$ a Riemann matrix. The Siegel modular group $\Gamma_g = Sp_{2g}(\mathbb{Z})$ acts on the right on $\mathcal{R}_g$: if $\Omega \in \mathcal{R}_g$ and if $M \in \Gamma_g$,

$$\Omega . M = [\Omega_1 \ \Omega_2] \begin{bmatrix} A & B \\ C & D \end{bmatrix} = [\Omega_1 A + \Omega_2 C \ \Omega_1 B + \Omega_2 D].$$

This action corresponds to a change of symplectic basis. The group $\Gamma_g$ also acts on the left on the Siegel upper half plane: if $\tau \in \mathbb{H}_g$, we denote

$$M . \tau = (A\tau + B)(C\tau + D)^{-1}.$$ 

Both actions are linked by

$$M . \tau(\Omega) = \tau(\Omega^t M) .$$

3.3. Isotropy and quotients. For any maximal isotropic subgroup $V \subset F_2^{2g}$, we have the transporter

$$\text{Trans}(V) = \{ M \in Sp_{2g}(F_2) \mid MV_0 = V \},$$

$V_0$ being the canonical maximal isotropic subgroup generated by the vectors $e_1, \ldots, e_g$ of the canonical basis. Since $Sp_{2g}(F_2)$ permutes transitively the maximal isotropic subgroups of $F_2^{2g}$, the transporter is a left coset: $\text{Trans}(V) = M_0 P(F_2)$, for any $M_0 \in \text{Trans}(V)$. Hence, the set of maximal isotropic subgroups is the quotient set $Sp_{2g}(F_2)/P(F_2)$, a set with 135 elements if $g = 3$.

Let now $\Omega \in \mathcal{R}_g$, $\Lambda = \Omega \mathbb{Z}^{2g}$ and $(A, \lambda) = (A_\Omega, H)$ be the corresponding p.p.a.v. of dimension $g$. The linear map $\alpha : \mathbb{Z}^{2g} \to 1/4 \Lambda$ such that

$$\alpha(x) = \frac{1}{2} \Omega . x$$

defines a level 2 symplectic structure on $A[2]$, that is, an isomorphism

$$\tilde{\alpha} : F_2^{2g} \xrightarrow{\sim} A[2]$$
We define \( 0 \) we deduce that \( \Gamma \).

We introduce now the congruence subgroup

\[ \Gamma_{0,\eta}(2) = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma_{\eta} \mid C \equiv 0 \pmod{2} \right\}. \]

This is the transposed subgroup of the group \( \Gamma_{\eta}(2) \), see §[A.11]. From Prop. [A.11] we deduce that \( \Gamma_{0,\eta}(2) = P(\mathbb{Z}).\Gamma_{\eta}(2) \), hence

\[ \text{Trans}(W) = M\Gamma_{0,\eta}(2) \]

for any \( M \in \text{Trans}(W) \).

**Proposition 3.3.1.** With the previous notation, if \( \tau = \tau(\Omega) \) then \( \frac{1}{2}M.\tau \) is a Riemann matrix of the p.p.a.v. \( A/W \) for all \( M \in \text{Trans}(W) \).

**Proof.** If \( M \in \text{Trans}(W) \), \( W \pmod{\Lambda} \) is generated by the vectors

\[ \frac{1}{2} \Omega. M e_1, \ldots, \frac{1}{2} \Omega. M e_g, \]

and the matrix

\[ \Omega' = \Omega. M H, \quad H = \begin{bmatrix} \frac{1}{2} I_g & 0 \\ 0 & I_g \end{bmatrix}, \]

generates \( \Lambda_W \).

Using (3), we get

\[ \tau(\Omega') = \langle (M H).\tau \rangle = \frac{1}{2} (M \tau). \]

By [14] Prop. 16.8, the polarization \( 2\lambda \) of \( A \) reduces to a principal polarization \( \lambda' \) on \( A' = A/W \). This last corresponds canonically to \( \Omega' \) since

\[ 2i(\Omega'/J^{-1}.\Omega')^{-1} = 2i(\Omega M H J^{-1} H' M' \Omega)^{-1} = 2 \cdot 2i(\Omega J^{-1} \Omega)^{-1}. \]

\( \square \)

### 3.4. Theta functions.

We recall the definition of theta functions with (entire) characteristics \( [\varepsilon] = [\varepsilon_2] \) where \( \varepsilon_1, \varepsilon_2 \in \mathbb{Z}^g \), following [2]. The *classical* theta function is

\[ \vartheta_{[\varepsilon]}(z, \tau) = \sum_{n \in \mathbb{Z}^g} q^{(n+\varepsilon/2)(n+\varepsilon/2)+2(n+\varepsilon/2)(z+\varepsilon/2)} \quad (\tau \in \mathbb{H}_g, z \in \mathbb{C}^g). \]

The Thetanullwerte are the values at \( z = 0 \) of these functions, and we denote

\[ \vartheta_{[\varepsilon]}(\tau) = \vartheta_{[\varepsilon]}(0, \tau). \]

We now state two formulas.

**Proposition 3.4.1** (duplication formula, see [16] Cor.IIA2.1 and [10] IV.th.2). Let \( [\varepsilon_1] \) and \( [\varepsilon_2] \) be two characteristics and \( \tau \in \mathbb{H}_g \). Then

\[ \vartheta_{[\varepsilon_2]}(\tau/2) \vartheta_{[\varepsilon_1]}(\tau/2) = \sum_{\mu \in (\mathbb{Z}/2\mathbb{Z})^g} (-1)^{\mu} \cdot \vartheta_{[\varepsilon_1-\mu]}(\tau) \cdot \vartheta_{[\varepsilon_2-\mu]}(\tau). \]
The second is called *transformation formula*. Let \( M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}_{2g}(\mathbb{Z}) \). We let \( M \) acts on the characteristics in the following way

\[
[M, \varepsilon] = M. [\varepsilon_1 \varepsilon_2] = \begin{bmatrix} D_{\varepsilon_1} - C_{\varepsilon_2} + (C'D)_{10} \\ -B_{\varepsilon_1} + A_{\varepsilon_2} + (A'B)_{10} \end{bmatrix}
\]

where \( P_0 \) denotes the diagonal of the matrix \( P \).

**Proposition 3.4.2** ([103, V.5.2]).

\[
\vartheta[M, \varepsilon](M, \tau) = \kappa(M) \cdot w^{\phi(\varepsilon_1, \varepsilon_2)(M)} \cdot j(M, \tau)^{1/2} \cdot \vartheta[\varepsilon](\tau)
\]

where \( \kappa(M)^2 \) is a root of 1 depending only on \( M \), \( \omega = e^{i\pi/4} \),

\[
j(M, \tau) = \det(C\tau + D)
\]

and

\[
\phi(\varepsilon_1, \varepsilon_2)(M) = \varepsilon_1^tDB\varepsilon_1 - 2\varepsilon_1^tBC\varepsilon_2 + \varepsilon_2^tCA\varepsilon_2 - 2(D\varepsilon_1 - C\varepsilon_2) \cdot (A'B)_0.
\]

We will need a slightly modified version of the previous result.

**Corollary 3.4.3.** For any characteristic \([\varepsilon_1 \varepsilon_2]\) and for any \( M \in \text{Sp}_{2g}(\mathbb{Z}) \) we have

(5) \[
\vartheta[\varepsilon_1 \varepsilon_2](M, \tau) = c(M, \tau) \cdot \omega^{-\phi(\varepsilon_1, \varepsilon_2)(M^{-1})} \cdot \vartheta[\varepsilon_1^t(C'(D' B)' - (C'B)' + (A'B)_{10}) + \varepsilon_2^t(C'A - (A'B)_{10}) + \varepsilon_1^t(C'B + (A'B)_{10}) + \varepsilon_2^t(CA + (A'B)_{10}) + \varepsilon_2^t(DB + (A'B)_{10})] \cdot (\tau)
\]

where

\[
c(M, \tau) = \kappa(M^{-1})^{-1} \cdot j(M, \tau)
\]

**Proof.** To inverse the action on the characteristics, we let \( \tau' = M^{-1} \cdot \tau \) in the transformation formula. Note that

\[
M^{-1} = \begin{pmatrix} t' & -t \\ -tC & t'A \end{pmatrix}
\]

and that \([M, \varepsilon] = [t'M^{-1}(\varepsilon_1)] + [t'(C'D)_{10} + (A'B)_{10}]\). Thus we get the action on the characteristics. For the factor \( j(M, \tau) \) note that

\[
j(M, \tau) = \det(C\tau + D) = \det(CM^{-1} \cdot \tau' + D)
\]

\[
= \det(C(\tau' - t'B)^{-1} - t'A)^{-1} + D
\]

\[
= \det(C(\tau' - t'B)^{-1} + D(-C\tau' + t'A)) \cdot \det(-C\tau' + t'A)^{-1}
\]

\[
= \det(-C\tau' + t'A)^{-1} = j(M^{-1}, \tau')^{-1}
\]

using Lem. 3.1.1. □

**Corollary 3.4.4.** Let \( \Omega = [\Omega_1 \Omega_2] \) be a period matrix and \( \tau = \tau(\Omega) = \Omega^{-1}_2 \Omega_1 \in \mathbb{H}_g \). Let \( \Omega' = \Omega M = [\Omega_1' \Omega_2'] \). Then

\[
j(M, \tau(\Omega)) = \det(\Omega_2)^{-1} \cdot \det(\Omega_1').
\]

**Proof.** We compute

\[
\det(C\tau + D) = \det(\tau'C + t'D)
\]

\[
= \det(\Omega_2)^{-1} \det(\Omega_1'C + \Omega_2'D)
\]

\[
= \det(\Omega_2)^{-1} \cdot \det(\Omega_1'),
\]

the last expression coming from (3). □
3.5. The modular function \( \chi_k \). Recall that a characteristic \([\varepsilon_1, \varepsilon_2]\) is \textit{even} (resp. \textit{odd}) if \( \varepsilon_1, \varepsilon_2 \equiv 0 \pmod{2} \) (resp. \( \varepsilon_1, \varepsilon_2 \equiv 1 \pmod{2} \)). Let \( S_g \) (resp. \( U_g \)) be the set of even (resp. odd) characteristics \([\varepsilon_1, \varepsilon_2]\) with coefficients in \( \{0,1\} \). It is well known that

\[
\#S_g = 2^{g-1}(2^g + 1), \quad \#U_g = 2^{g-1}(2^g - 1).
\]

Let \( \Omega = [\Omega_1 \Omega_2] \in \mathbb{R}_g \) and \( \tau = \Omega_2^{-1}\Omega_1 \in \mathbb{H}_g \) be a Riemann matrix. For \( g \geq 2 \), we denote \( k = \#S_g/2 \) and we are interested in the following expressions:

\[
\chi_k(\tau) = \prod_{\varepsilon \in S_g} \vartheta[\varepsilon](\tau).
\]

Recall that a function \( f \) is a \textit{modular form} of weight \( w \) for the congruence subgroup \( \Gamma \subset \Gamma_g \) if for all \( \tau \in \mathbb{H}_g \) and \( M \in \Gamma \) one has

\[
f(M.\tau) = j(M, \tau)^w f(\tau).
\]

Using Cor. 3.4.4, we get

**Corollary 3.5.1.** Let \( f \) be a modular form of weight \( k \) for \( \Gamma \) on \( \mathbb{H}_g \). For \( \Omega = [\Omega_1 \Omega_2] \in \mathbb{R}_g \), we define \( \tau = \Omega_2^{-1}\Omega_1 \in \mathbb{H}_g \) a Riemann matrix and

\[
f(\Omega) := \det(\Omega_2)^{-k} f(\tau).
\]

Then for all \( M \in \Gamma \)

\[
f(\Omega.M) = f(\Omega).
\]

In his beautiful paper [9], Igusa proves the following result [loc. cit., Lem. 10 & 11]. Denote by \( \Sigma_{140} \) the thirty-fifth elementary symmetric function of the eighth power of the even Thetanullwerte.

**Theorem 3.5.2.** For \( g \geq 3 \), the product \( \chi_k(\tau) \) is a modular form of weight \( k \) for the group \( \Gamma_g \). Moreover, If \( g = 3 \) and \( \tau \in \mathbb{H}_3 \), then:

(i) \( A_\tau \) is decomposable if \( \chi_{18}(\tau) = \Sigma_{140}(\tau) = 0 \).

(ii) \( A_\tau \) is a hyperelliptic Jacobian if \( \chi_{18}(\tau) = 0 \) and \( \Sigma_{140}(\tau) \neq 0 \).

(iii) \( A_\tau \) is a non hyperelliptic Jacobian if \( \chi_{18}(\tau) \neq 0 \). □

This theorem gives an answer to the two questions raised in Sec. 1.1 over \( \mathbb{C} \).

In the sequel, we will need the following result to prove the independence of our results from the choices we will make. The proof is the case \( g = 3 \) of Th. A.1.2.

**Proposition 3.5.3.** The product \( \tau \mapsto \chi_{18}(\frac{1}{2}\tau) \) is a modular form on \( \mathbb{H}_3 \) of weight 18 for \( \Gamma_3^0(2) \).

4. Comparison of analytic and algebraic discriminants

In this part, we make the link between the algebraic result Th 2.5.2 and Serre’s conjecture on the modular function \( \chi_{18} \). To do so, we first compute a quantity related easily to \( T(A) \) in terms of the Thetanullwerte on the elliptic curves. Then, after a good choice of a symplectic matrix \( N \) (related to the subgroup \( W \) we use for the quotient), we compute \( \chi_{18}((N.\tau)/2) \) in terms of the same Thetanullwerte. Thus, we express \( \chi_{18} \) on the quotient \( A/W \). Finally we compare the expressions to get Serre’s conjecture.
4.1. **Expression of the algebraic discriminant.** We come back to the hypotheses of §2.3 and specialize to the case $K \subset \mathbb{C}$. Let $A = E_1 \times E_2 \times E_3 \in A$, where

$$E_i : y^2 = x(x^2 - 4b_i x - 4c_i), \quad (b_i \in K, c_i \in K^\times) \quad (i = 1, 2, 3).$$

We choose a root $\rho$ of $\delta(A)$, and put $m = \text{Mat}(\tilde{A})$ with $\tilde{A} = (A, \rho)$. Let

$$X(m) := (a_1 a_2 a_3)^4 (c_1 c_2 c_3)^2 \det m.$$

Since $T(\tilde{A}) = \det m$, $T(\tilde{A})$ is a square in $K$ if and only if $X(m)$ is a square in $K$. The function $X$ appears naturally in our problem since it is related to the function $D(m)$ (Prop. 2.2.1) by

$$X(Cof m) = D(m)^2,$$

and this reflects Serre’s conjecture according to Th. 2.4.2. In order to determine the expression of $X(m)$ is terms of the Thetanullwerte, we use the following uniformization. The curves $E_i$ can be written as

$$E(\omega_{1i}, \omega_{2i}) : y^2 = x(x + \frac{\pi^2}{\omega^2_2} \theta_{\omega}^{i} (\tau_i)^4)(X + \frac{\pi^2}{\omega^2_2} \theta_{\omega}^{i} (\tau_i)^4),$$

with $[\omega_{1i} \omega_{2i}] \in \mathbb{R}_1$ and $\tau_i = \frac{\omega_{1i}}{\omega_{2i}}$. We identify $\mathbb{R}_1^3$ with the set of matrices

$$\Omega = [\Omega_1 \Omega_2] = \begin{bmatrix} \omega_{11} & 0 & 0 \\ 0 & \omega_{12} & 0 \\ 0 & 0 & \omega_{13} \end{bmatrix}, \begin{bmatrix} \omega_{21} & 0 & 0 \\ 0 & \omega_{22} & 0 \\ 0 & 0 & \omega_{23} \end{bmatrix}$$

such that

$$\tau = \tau(\Omega) = \Omega_2^{-1} \Omega_1 = \begin{bmatrix} \tau_1 & 0 & 0 \\ 0 & \tau_2 & 0 \\ 0 & 0 & \tau_3 \end{bmatrix} \in \mathbb{H}_3.$$ We define

$$A(\Omega) := E(\omega_{11}, \omega_{21}) \times E(\omega_{12}, \omega_{22}) \times E(\omega_{13}, \omega_{23}),$$

$$\rho(\Omega) := \frac{\pi^6}{64(\det \Omega_2)^2} \theta_{\omega}^{i} \left[ \begin{array}{c} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{array} \right] (\tau).$$

This defines an element $\tilde{A}(\Omega) := (A(\Omega), \rho(\Omega)) \in \tilde{A}$, and a matrix $m(\Omega) := \text{Mat}(\tilde{A}(\Omega))$. For $1 \leq i \leq 3$, denote

$$\vartheta_{0i} = \theta_{\omega}^{i} (\tau_i), \quad \vartheta_{11} = \theta_{\omega}^{[1]} (\tau_i), \quad \vartheta_{21} = \theta_{\omega}^{[1]} (\tau_i).$$

The coefficients of $A(\Omega)$ and $m(\Omega)$ are

$$a_i = \frac{\pi^2}{4} \frac{\omega_{2i}^2}{(\omega_{2i} \omega_{2k})^2} \vartheta_{0k}^4 \vartheta_{1k}^4, \quad b_i = -\frac{\pi^2}{4 \omega_{2i}^2} (\vartheta_{01}^4 + \vartheta_{21}^4), \quad c_i = -\frac{\pi^4}{4 \omega_{2i}^2} \vartheta_{01}^4 \vartheta_{21}^4,$$

where $(i, j, k)$ is a cyclic permutation. The determinant of $m(\Omega)$ is expressed as follows. Let

$$a = \vartheta_{01}^2 \vartheta_{02}^2 \vartheta_{03}^2, \quad b = \vartheta_{01}^2 \vartheta_{22}^2 \vartheta_{03}^2, \quad c = \vartheta_{21}^2 \vartheta_{02}^2 \vartheta_{03}^2, \quad d = \vartheta_{21}^2 \vartheta_{22}^2 \vartheta_{23}^2,$$

$$R_1 = (a + b + c + d)(a + b - c - d)(a - b + c + d)(a - b - c - d),$$

Then

$$\det m(\Omega) = \frac{\pi^6}{2^4 \cdot \prod_{i=1}^{3} (\omega_{2i}^2 (\vartheta_{0i}^4 - \vartheta_{2i}^4))} \cdot R_1.$$
Thus we get

\[
X(m(Ω)) = \left(\frac{\pi^{12}}{2^6} \prod_{i=1}^{3} \omega^4_{2i} \cdot \prod_{i=1}^{3} (\vartheta_{0i}^4 - \vartheta_{2i}^4)^2 \cdot \left(\frac{\pi^{6}}{2^6} \prod_{i=1}^{3} \omega^2_{2i} \cdot \prod_{i=1}^{3} (\vartheta_{0i}^4 - \vartheta_{2i}^4)^2 \right)\right) \cdot R_1
\]

\[
= \frac{\pi^{54}}{2^{40}} \cdot \det(Ω)^{-18} \cdot \left(\prod_{i=1}^{3} \vartheta_{0i}^6 \vartheta_{2i}^8 (\vartheta_{0i}^4 - \vartheta_{2i}^4)^{3}\right) \cdot R_1.
\]

4.2. The **subgroup** \(W\). With the notation of Sec. 2.4 we can always assume the following correspondences

\[
P_i \leftrightarrow \frac{\omega_{1i}}{2}, \quad Q_i \leftrightarrow \frac{\omega_{2i}}{2}, \quad R_i = P_i + Q_i \leftrightarrow \frac{\omega_{1i} + \omega_{2i}}{2}
\]

for the points of \(E_i\). The characteristics associated to the points of \(W\) (see § 2.3) are

\[
\begin{bmatrix}
on000 \
000 \
000 
\end{bmatrix}, \begin{bmatrix}
on00 \
001 \
000 
\end{bmatrix}, \begin{bmatrix}
on00 \
001 \
000 
\end{bmatrix}, \begin{bmatrix}
000 \
000 \
000 
\end{bmatrix}, \begin{bmatrix}
111 \
010 \
000 
\end{bmatrix}, \begin{bmatrix}
111 \
010 \
000 
\end{bmatrix}, \begin{bmatrix}
111 \
010 \
000 
\end{bmatrix}, \begin{bmatrix}
111 \
010 \
000 
\end{bmatrix}.
\]

It defines a maximal isotropic subgroup \(V\) of \(F_2^6\). A basis of \(V\) over \(F_2\) is given by the three vectors

\[
\alpha_1 = \begin{bmatrix}000 \\ 011\end{bmatrix}, \quad \alpha_2 = \begin{bmatrix}000 \\ 010\end{bmatrix}, \quad \alpha_3 = \begin{bmatrix}111 \\ 000\end{bmatrix}.
\]

The matrix

\[
N = \begin{bmatrix}0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\]

belongs to \(Γ_3\) and satisfies \(N.ε_i \equiv \alpha_i \pmod{2}\) if \(1 ≤ i ≤ 3\), thus \(N \in \text{Trans}(W)\).

The set

\[
Γ_g(1, 2) = \left\{ \begin{bmatrix}A & B \\ C & D\end{bmatrix} \in Γ_g \ | \ \begin{bmatrix}A & B \\ C & D\end{bmatrix}_0 \equiv \begin{bmatrix}C & D\end{bmatrix}_0 \equiv 0 \pmod{2} \right\},
\]

is a subgroup of \(Γ\), and \(κ^2\) is a character of \(Γ_g(1, 2)\) [10, p. 181].

**Lemma 4.2.1.** The matrices \(N\) and \(^TN\) are in \(Γ_3(1, 2)\), and \(κ(N)^2 = κ(^TN)^2 = ±1\).

**Proof.** We have \(N = LQ\), where

\[
L = \begin{bmatrix}A & 0 \\ 0 & A^{-1}\end{bmatrix}, \quad \text{with} \quad A = \begin{bmatrix}0 & -1 & 1 \\ 0 & 0 & 1 \\ -1 & 0 & 1\end{bmatrix},
\]

and

\[
Q = \begin{bmatrix}0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}.
\]

One checks easily that \(L, L, Q, ^TQ\) belong to \(Γ_3(1, 2)\), hence, \(N\) and \(^TN\) are in \(Γ_3(1, 2)\) as well. If

\[
M = \begin{bmatrix}A & B \\ 0 & D\end{bmatrix} \in \mathbb{P}(Z),
\]

then...
then $\kappa(M)^2 = \det D$, see [9, Lem. 7, p. 181]. Now

$$Q^2 = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix}, \quad \text{with} \quad S = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$ 

From this we deduce that

$$\kappa(L)^2 = \det A = 1, \quad \kappa(Q)^4 = \kappa(Q^2)^2 = \det S = 1,$$

hence, $\kappa(N)^2 = \kappa(tN)^2 = \pm 1$. \hfill $\Box$

**Proposition 4.2.2.** Let $\Omega = \Omega NH$. Then

$$\tau(\Omega') = \frac{1}{2}tN.\tau$$

is a Riemann matrix for $A'(m)$. Moreover, the value $\chi_{18}(\Omega')$ is independent on the choice of $N \in \text{Trans}(W)$.

**Proof.** The first assertion comes from Prop. 3.3.1, the second from Prop. 3.5.3 \hfill $\Box$

4.3. **Expression of $\chi_{18}(\Omega')$ as a discriminant.** Our main result in this section is the following

**Theorem 4.3.1.** Let $\Omega \in \mathcal{R}_4^3$ and $A(\Omega)$ be the corresponding abelian threefold, let $m = m(\Omega) \in S$ be the associated matrix, and $\Omega' \in \mathcal{R}_3$ be a Riemann matrix of $A(\Omega)/W$. Then

$$(\frac{\pi}{2})^{54} \cdot \chi_{18}(\Omega') = X(m).$$

**Proof.** The strategy is the following. Let $N$ be the matrix defined in § 4.2 and define $\tau' = tN.\tau = 2\tau(\Omega')$.

(i) Pair the Thetanullwerte in $\tau'/2$ such that one can apply the duplication formula (4). We then obtain expressions in terms of Thetanullwerte in $\tau'$. Such a pairing is not unique and one makes here a choice which allows an easy comparison of the final formulas.

(ii) For each of the Thetanullwerte in $\tau'$, apply the transformation formula (5) to obtain an expression in $\tau$.

(iii) Finally, since $\tau = \text{diag}(\tau_1, \tau_2, \tau_3)$, we get

$$\vartheta_{[a_1 b_1 c_1]} \tau) = \prod_{i=1}^{3} \vartheta_{[a_i]} (\tau_i).$$

Let

$$c(N) = \kappa(tN)^{-2} \det(\Omega_2)^{-1} \det(\Omega_2') = \pm \det(\Omega_2)^{-1} \det(\Omega_2')$$

by Lem. 4.2.1.

Applying steps (i) to (iii) with the software MAGMA (see http://iml.univ-mrs.fr/~ritzen/programme/check), we get the following 18 identities, where we write

$$\vartheta_{[000]} \vartheta_{[000]} = \vartheta_{[000]} \vartheta_{[000]} (\tau'/2) \vartheta_{[000]} \vartheta_{[000]} (\tau'/2), \quad c = c(N).$$

We make the pairing in such a way that the expressions of $\vartheta_{[000]} \vartheta_{[000]}$ do not contain $\vartheta_{14}$ terms. The first four are, with the preceding notation,

$$\vartheta_{[000]} \vartheta_{[000]} = c(a + b + c + d) \quad \vartheta_{[000]} \vartheta_{[000]} = c(a + b - c - d) \quad \vartheta_{[000]} \vartheta_{[000]} = -c(a - b - c + d) \quad \vartheta_{[000]} \vartheta_{[000]} = -c(a - b + c - d)$$

and the remaining 14 are

$$\vartheta_{[010]} \vartheta_{[000]} = 2c(\vartheta_{01} \vartheta_{21} \vartheta_{02} \vartheta_{22} \vartheta_{03}^2 + \vartheta_{01} \vartheta_{21} \vartheta_{02} \vartheta_{22} \vartheta_{23}^2)$$
In other words, Serre’s conjecture is true for our three dimensional family \( \Omega \) of abelian threefolds.

**Corollary 4.3.2.** Let \( K \subset \mathbb{C} \) and \( m \in \mathbb{S}^\times \) with coefficients in \( K \). Let \( A'(m) \) be the associated abelian threefold and \( \Omega' \) be one of its period matrix. Then

\[
\left( \frac{\pi}{2} \right)^{54} \cdot \chi_{18}(\Omega') \in \mathbb{K}^2
\]

if and only if \( A'(m) \) is the Jacobian of a non hyperelliptic genus 3 curve.

In other words, Serre’s conjecture is true for our three dimensional family \( \Omega \) of abelian threefolds.

**Corollary 4.3.3.** If \( m \in \mathbb{S}^\times \) and \( \Omega_m \) is a period matrix associated to the non hyperelliptic genus 3 curve \( X_m \) with Ciani form \( Q_m \) then

\[
\chi_{18}(\Omega_m) = \left( \frac{1}{2\pi} \right)^{54} \cdot \text{Disc}(Q_m)^2. 
\]

**Proof.** Using Th 2.4.2 and \( \text{(6)} \) we get

\[
\left( \frac{\pi}{2} \right)^{54} \cdot \chi_{18}(\Omega_m) = X(\text{Cof} \ m) = D(m)^2 = (2^{-54} \cdot \text{Disc} \ Q_m)^2.
\]
Remark. When $m \in S \setminus S^\times$, the abelian variety $A'(m)$ comes from a hyperelliptic curve and the above formula degenerates. However in [13] and [5] we find a beautiful formula for the hyperelliptic case in every genus. Let

$$C : Y_2 = a_{2g+2}X^{2g+2} + \ldots + a_0 = a_{2g+2}(X - \alpha_1) \cdots (X - \alpha_{2g+2})$$

and

$$\Delta_{\text{alg}}(C) = a_{2g+2}^{2g+2} \prod_{j<k}(\alpha_j - \alpha_k)^2.$$ 

They define also a modular form on $H_g$

$$\delta(\tau) = \prod_{\varepsilon \in T} \vartheta[\varepsilon](\tau)^8$$

where $T$ is a certain subset of even theta characteristic. One has

$$\Delta_{\text{alg}}(C)^{2n} = (2\pi)^{4r} \det(\Omega_1)^{-4r} \delta(\tau)^2$$

where

$$r = \binom{2g+2}{g+1}, \quad n = \binom{2g}{g+1},$$

and $\tau = \tau(\Omega)$ for a certain period matrix $\Omega = [\Omega_1, \Omega_2]$ of $\text{Jac}(C)$.

Remark. Denote by $V_3^4$ the 15-dimensional affine open set of ternary quartics. Felix Klein proved in 1889 that there is a map

$$\Omega : V_3^4 \rightarrow \mathbb{R}_g$$

such that if $\Omega(Q) = [\Omega_1, \Omega_2]$ and $X : Q = 0$, then $\text{Jac} X = A_{\Omega(Q)}$ and

$$\chi_{18}(\Omega) = c \text{Disc}(Q)^2,$$

with some unspecified constant $c \in \mathbb{C}$. We prove here that $c = (1/2\pi)^{54}$. Using this precise version of Klein’s formula, it is almost obvious to extend our theorem to the general case. However, we did not include it, for we think that a good presentation should include a modern proof of Klein’s result. We plan to do this in a forthcoming article.

Appendix A.

A.1. Modularity of $\chi_k$. Let

$$\Gamma_g(2) = \{M \in \Gamma_g \mid M \equiv 1_{2g}(\text{mod } 2)\}$$

and recall that the sequence

$$1 \rightarrow \Gamma_g(2) \rightarrow \Gamma_g \rightarrow \text{Sp}_{2g}(\mathbb{F}_2) \rightarrow 1$$

is exact. We introduce the congruence subgroup

$$\Gamma^0_g(2) = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \text{Sp}_g(\mathbb{Z}) \mid B \equiv 0(\text{mod } 2) \right\}. $$

We need a set of generators for this subgroup. For any integer $n \geq 1$, define

$$M(n) = M(\mathbb{Z}) \cap \Gamma_g(n), \quad U(n) = U(\mathbb{Z}) \cap \Gamma_g(n), \quad V(n) = V(\mathbb{Z}) \cap \Gamma_g(n).$$

with

$$\Gamma_g(n) = \{M \in \Gamma_g \mid M \equiv 1_{2g}(\text{mod } n)\}.$$ 

Proposition A.1.1. The subgroups $M(1)$, $U(2)$ and $V(1)$ generate $\Gamma^0_g(2)$, and $\Gamma^0_g(2) = \Gamma_g(2).M(\mathbb{Z}).V(\mathbb{Z})$. 

Proof. First, the subgroups \( M(2), U(2) \) and \( V(2) \) generate \( \Gamma_g(2) \), see [10] p. 179. Let
\[
\Gamma_g^1(2) = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \text{Sp}_g(\mathbb{Z}) \mid A \equiv D \equiv 1 \pmod{2} \text{ and } B \equiv 0 \pmod{2} \right\}.
\]
There is the following diagram, where the vertical arrow is the transpose of the reduction modulo 2:
\[
\begin{array}{cccc}
\Gamma_g(2) & \to & \Gamma_g^1(2) & \to & \Gamma_g^0(2) & \to & \Gamma_g(1) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & U(\mathbb{F}_2) & \to & \text{P}(\mathbb{F}_2) & \to & \text{Sp}_g(\mathbb{F}_2)
\end{array}
\]
Then, if \( M \in \Gamma_g^1(2) \) is written as usual
\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\begin{bmatrix}
1_g & 0 \\
0 & 1_g
\end{bmatrix}
= \begin{bmatrix}
A + BC & B \\
C + DC & D
\end{bmatrix} \in \Gamma_g(2),
\]
and if \( M \in \Gamma_g^0(2) \), then
\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\begin{bmatrix}
A^{-1} & 0 \\
0 & 1_A
\end{bmatrix}
= \begin{bmatrix}
1_g & B^tA \\
CA^{-1} & D^tA
\end{bmatrix} \in \Gamma_g^1(2).
\]
\( \square \)

Theorem A.1.2. Assume \( g \geq 3 \). The function \( \chi_k(\frac{j}{2} \tau) \) is a modular form on \( \mathbb{H}^g \) of weight \( k \) for \( \Gamma_g^0(2) \).

Proof. J.-I. Igusa proved [9] p. 850] that \( \chi_k(\tau) \) is a modular form of weight \( k \) for \( \Gamma_g(1) \) if \( g \geq 3 \). Let
\[
H = \begin{bmatrix} \frac{j}{2}1_g & 0 \\ 0 & 1_g \end{bmatrix}, \quad H.\tau = \frac{j}{2}\tau.
\]
Let \( f(\tau) = \chi_k(\frac{j}{2} \tau) = \chi_k(\tau) \). It is sufficient to check that
\[
f(M.\tau) = j(M, \tau)^k f(\tau)
\]
if \( M \) belongs to one of the generating subgroups described in Prop. A.1.1. First, if \( M \in M(1) \), then \( H.M = M.H \), hence,
\[
f(M.\tau) = \chi_k(H.M.\tau) = \chi_k(M.H.\tau) = j(M, H.\tau)^k \chi_k(\tau) = j(M, \tau)^k f(\tau),
\]
since \( j(M, \tau) = \pm 1 \) for every \( M \in \text{M}(\mathbb{Z}) \) does not depend on \( \tau \in \mathbb{H}_g \). Now, if \( U \in U(2) \), then
\[
U = U'' = \begin{bmatrix} I_3 & 2B \\ 0 & I_3 \end{bmatrix}, \quad \text{where } U' = \begin{bmatrix} I_3 & B \\ 0 & I_3 \end{bmatrix} \in \text{U}(\mathbb{Z}),
\]
and \( H.U = H.U'' = U'.H \). This implies
\[
f(U.\tau) = \chi_k(H.U'' \tau) = \chi_k(U'.H.\tau) = j(U', H.\tau)^k \chi_k(\tau) = j(U, \tau)^k f(\tau),
\]
since \( j(U^n, \tau) = 1 \) for every \( U \in \text{U}(\mathbb{Z}) \). If \( V \in V(1) \), then \( H.V = V^2.H \). Hence
\[
f(V.\tau) = \chi_k(H.V.\tau) = \chi_k(V^2.H.\tau) = j(V^2, H.\tau)^k \chi_k(\tau) = j(V, \tau)^k \chi_k(\tau) = j(V, \tau)^k f(\tau),
\]
since \( j(V^2, \tau) = j(V, 2\tau) \) for every \( V \in V(1) \). \( \square \)
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