A modulation technique for the blow-up profile of the vector-valued semilinear wave equation

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Abstract

We consider a vector-valued blow-up solution with values in $\mathbb{R}^m$ for the semilinear wave equation with power nonlinearity in one space dimension (this is a system of PDEs). We first characterize all the solutions of the associated stationary problem as an $m$-parameter family. Then, we show that the solution in self-similar variables approaches some particular stationary one in the energy norm, in the non-characteristic cases. Our analysis is not just a simple adaptation of the already handled real or complex case. In particular, there is a new structure of the set a stationary solutions.

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1 Introduction

We consider the vector-valued semilinear wave equation

$$\begin{cases} \partial_t^2 u = \partial_x^2 u + |u|^{p-1}u, \\ u(0) = u_0 \text{ and } u_t(0) = u_1, \end{cases}$$

where here and all over the paper $|\cdot|$ is the euclidian norm in $\mathbb{R}^m$, $u(t) : x \in \mathbb{R} \rightarrow u(x,t) \in \mathbb{R}^m$, $m \geq 2$, $p > 1$, $u_0 \in H^1_{loc,u}$ and $u_1 \in L^2_{loc,u}$ with $||v||^2_{H^1_{loc,u}} = \sup_{a \in \mathbb{R}} \int_{|x-a|<1} |v(x)|^2 dx$ and $||v||^2_{H^1_{loc,u}} = ||v||^2_{L^2_{loc,u}} + ||\nabla v||^2_{L^2_{loc,u}}$.

The Cauchy problem for equation (1) in the space $H^1_{loc,u} \times L^2_{loc,u}$ follows from the finite speed of propagation and the wellposedness in $H^1 \times L^2$. See for instance Ginibre, Soffer and Velo [9], Ginibre and Velo [10], Lindblad and Sogge [14] (for the local in time wellposedness in $H^1 \times L^2$). Existence of blow-up solutions follows from ODE techniques or the energy-based

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blow-up criterion of [13]. More blow-up results can be found in Caffarelli and Friedman [6], Alinhac [1] and [2], Kichenassamy and Littman [12], [11] Shatah and Struwe [25]).

The real case (in one space dimension) has been understood completely, in a series of papers by Merle and Zaag [18], [19], [21] and [22] and in Côte and Zaag [7] (see also the note [20]). Recently, the authors give an extension to higher dimensions in [24] and [23], where the blow-up behavior is given, together with some stability results.

For other types of nonlinearities, we mention the recent contribution of Azaiez, Masmoudi and Zaag in [3], where we study the semilinear wave equation with exponential nonlinearity, in particular we give the blow-up rate with some estimations.

In [4], we consider the complex-valued solution of (1) (or \(\mathbb{R}^2\)-valued solution), characterize all stationary solutions and give a trapping result. The main obstruction in extending those results to the vector case \(m \geq 3\) was the question of classification of all self similar solutions of (1) in the energy space. In this paper we solve that problem and show that the real valued and complex valued classification also hold in the vector-valued case \(m \geq 3\) (see Proposition 2 below), with an adequate choice in \(S^{m-1}\). This is in fact our main contribution in this paper, and it allows us to generalize the results of the complex case to the vector valued case \(m \geq 3\).

In this paper, we aim at proving similar results for the general case \(u(x, t) \in \mathbb{R}^m\), for \(m \geq 3\).

Let us first introduce some notations before stating our results.

If \(u\) is a blow-up solution of (1), we define (see for example Alinhac [1]) a continuous curve \(\Gamma\) as the graph of a function \(x \to T(x)\) such that the domain of definition of \(u\) (or the maximal influence domain of \(\Gamma\)) is

\[
D_u = \{(x, t) | t < T(x)\}.
\]

From the finite speed of propagation, \(T\) is a \(1\)-Lipschitz function. The time \(\bar{T} = \inf_{x \in \mathbb{R}} T(x)\) and the graph \(\Gamma\) are called (respectively) the blow-up time and the blow-up graph of \(u\).

Let us introduce the following non-degeneracy condition for \(\Gamma\). If we introduce for all \(x \in \mathbb{R}\), \(t \leq T(x)\) and \(\delta > 0\), the cone

\[
C_{x, t, \delta} = \{(\xi, \tau) \neq (x, t) | 0 \leq \tau \leq t - \delta|\xi - x|\},
\]

then our non-degeneracy condition is the following: \(x_0\) is a non-characteristic point if

\[
\exists \delta = \delta(x_0) \in (0, 1) \text{ such that } u \text{ is defined on } C_{x_0, T(x_0), \delta_0}.
\]

If condition (2) is not true, then we call \(x_0\) a characteristic point. Already when \(u\) is real-valued, we know from [21] and [7] that there exist blow-up solutions with characteristic points.

Given some \(x_0 \in \mathbb{R}\), we introduce the following self-similar change of variables:

\[
w_{x_0}(y, s) = (T(x_0) - t)^{\frac{2}{p-1}}u(x, t), \quad y = \frac{x - x_0}{T(x_0) - t}, \quad s = -\log(T(x_0) - t).
\]

This change of variables transforms the backward light cone with vertex \((x_0, T(x_0))\) into the infinite cylinder \((y, s) \in (-1, 1) \times [-\log T(x_0), +\infty)\). The function \(w_{x_0}\) (we write \(w\) for simplicity) satisfies the following equation for all \(|y| < 1\) and \(s \geq -\log T(x_0)\):

\[
\partial_s^2 w = Lw - \frac{2(p + 1)}{(p - 1)^2}w + |w|^{p-1}w - \frac{p + 3}{p - 1}\partial_s w - 2y\partial_y w
\]

where \(Lw = \frac{1}{\rho} \partial_y (\rho(1 - y^2)\partial_y w)\) and \(\rho(y) = (1 - y^2)^{\frac{2}{p-1}}\).
This equation will be studied in the space
\[ \mathcal{H} = \{ q \in H^{1}_{loc} \times L^{2}_{loc}((-1,1), \mathbb{R}^{m}) \mid \| q \|_{\mathcal{H}}^{2} = \int_{-1}^{1} (|q_{1}|^{2} + |q_{1}'|^{2}(1-y^{2}) + |q_{2}|^{2}) \rho \, dy < +\infty \}, \]
which is the energy space for \( w \). Note that \( \mathcal{H} = \mathcal{H}_{0} \times L^{2}_{\rho} \) where
\[ \mathcal{H}_{0} = \{ r \in H^{1}_{loc}((-1,1), \mathbb{R}^{m}) \mid \| r \|_{\mathcal{H}_{0}}^{2} = \int_{-1}^{1} (|r'|^{2}(1-y^{2}) + |r|^{2}) \rho \, dy < +\infty \}. \]

In some places in our proof and when this is natural, the notation \( \mathcal{H}, \mathcal{H}_{0} \) and \( L^{2}_{\rho} \) may stand for real-valued spaces. Let us define
\[ E(w, \partial_{s}w) = \int_{-1}^{1} \left( \frac{1}{2} |\partial_{s}w|^{2} + \frac{1}{2} |\partial_{y}w|^{2}(1-y^{2}) + \frac{p+1}{(p-1)^{2}} |w|^{\frac{p+1}{p-1}} - \frac{1}{p+1} |w|^{p+1} \right) \rho \, dy. \]

By the argument of Antonini and Merle \[3\], which works straightforwardly in the vector-valued case, we see that \( E \) is a Lyapunov functional for equation (1).

### 1.1 Blow-up rate

Only in this subsection, the space dimension will be extended to any \( N \geq 1 \). We assume in addition that \( p \) is conformal or sub-conformal:
\[ 1 < p \leq p_{c} = 1 + \frac{4}{N-1}. \]

We recall that for the real case of equation (11), Merle and Zaag determined in \[15\] and \[16\] the blow-up rate for (11) in the region \( \{(x,t) \mid t < T\} \) in a first step. Then in \[17\], they extended their result to the whole domain of definition \( \{(x,t) \mid t < T(x)\} \). In fact, the proof of \[15\], \[16\] and \[17\] is valid for vector-valued solutions, since the energy structure (see (8)), which is the main ingredient of the proof, is preserved. This is the growth estimate near the blow-up surface for solutions of equation (11).

**Proposition 1.** (Growth estimate near the blow-up surface for solutions of equation (11)) If \( u \) is a solution of (11) with blow-up surface \( \Gamma : \{ x \rightarrow T(x) \} \), and if \( x_{0} \in \mathbb{R}^{N} \) is non-characteristic (in the sense (2)) then,

(i) **(Uniform bounds on \( w \))** For all \( s \geq - \log \frac{T(x_{0})}{4} \):
\[ ||w_{x_{0}}(s)||_{H^{1}(B)} + ||\partial_{s}w_{x_{0}}(s)||_{L^{2}(B)} \leq K. \]

(ii) **(Uniform bounds on \( u \))** For all \( t \in [\frac{4}{T(x_{0})}, T(x_{0})] \):
\[ (T(x_{0}) - t)^{\frac{2}{p-1}} \frac{||u(t)||_{L^{2}(B(x_{0},T(x_{0})-t))}}{T(x_{0}) - t} \]
\[ + (T(x_{0}) - t)^{\frac{2}{p-1} + 1} \left( \frac{||\partial_{t}u(t)||_{L^{2}(B(x_{0},T(x_{0})-t))}}{T(x_{0}) - t} \frac{1}{N/2} + \frac{||\nabla u(t)||_{L^{2}(B(x_{0},T(x_{0})-t))}}{T(x_{0}) - t} \right)^{N/2} \leq K, \]
where the constant \( K \) depends only on \( N, p \), and on an upper bound on \( T(x_{0}), 1/T(x_{0}), \delta_{0}(x_{0}) \) and the initial data in \( H^{1}_{loc,u} \times L^{2}_{loc,u} \).
1.2 Blow-up profile

This result is our main novelty. In the following, we characterize the set of stationary solutions for vector-valued solutions.

Proposition 2. (Characterization of all stationary solutions of equation (4) in $\mathcal{H}_0$). (i) Consider $w \in \mathcal{H}_0$ a stationary solution of (4). Then, either $w \equiv 0$ or there exist $d \in (-1, 1)$ and $\Omega \in S^{m-1}$ such that $w(y) = \Omega \kappa(d, y)$ where

$$
\forall (d, y) \in (-1, 1)^2, \kappa(d, y) = \kappa_0 (1 - d^2)^{\frac{1}{p-1}} \text{ and } \kappa_0 = \left(\frac{2(p+1)}{(p-1)^2}\right)^{\frac{1}{p-1}}.
$$

(ii) It holds that

$$
E(0, 0) = 0 \text{ and } \forall d \in (-1, 1), \forall \Omega \in S^{m-1}, E(\kappa(d, \cdot)\Omega, 0) = E(\kappa_0, 0) > 0
$$

where $E$ is given by (8).

Thanks to the existence of the Lyapunov functional $E(w, \partial_s w)$ defined in (8), we show that when $x_0$ is non-characteristic, then $w_{x_0}$ approaches the set of non-zero stationary solutions:

Proposition 3. (Approaching the set of non-zero stationary solutions near a non-characteristic point) Consider $u$ a solution of (4) with blow-up curve $\Gamma : \{x \to T(x)\}$. If $x_0 \in \mathbb{R}$ is non-characteristic, then:

(A.i) $\inf_{\Omega \in S^{m-1}, |d| < 1} \|w_{x_0}(\cdot, s) - \kappa(d, \cdot)\Omega\|_{H^1(-1, 1)} + \|\partial_s w_{x_0}\|_{L^2(-1, 1)} \to 0$ as $s \to \infty$.

(A.ii) $E(w_{x_0}(s), \partial_s w_{x_0}(s)) \to E(\kappa_0, 0)$ as $s \to \infty$.

We write the fundamental theorem of our paper:

Theorem 4. (Trapping near the set of non-zero stationary solutions of (4)) There exist positive $\epsilon_0$, $\mu_0$ and $C_0$ such that if $w \in C([s^*, \infty), \mathcal{H})$ for some $s^* \in \mathbb{R}$ is a solution of equation (4) such that

$$
\forall s \geq s^*, E(w(s), \partial_s w(s)) \geq E(\kappa_0, 0),
$$

and

$$
\left\| \begin{pmatrix} w(s^*) \\ \partial_s w(s^*) \end{pmatrix} - \begin{pmatrix} \kappa(d^*, \cdot)\Omega^* \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \leq \epsilon^*
$$

for some $d^* \in (-1, 1), \Omega^* \in S^{m-1}$ and $\epsilon^* \in (0, \epsilon_0]$, then there exists $d_\infty \in (-1, 1)$ and $\Omega_\infty \in S^{m-1}$ such that

$$
|\arg\tanh d_\infty - \arg\tanh d^*| + |\Omega_\infty - \Omega^*| \leq C_0 \epsilon^*
$$

and for all $s \geq s^*$:

$$
\left\| \begin{pmatrix} w(s) \\ \partial_s w(s) \end{pmatrix} - \begin{pmatrix} \kappa(d_\infty, \cdot)\Omega_\infty \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \leq C_0 \epsilon^* e^{-\mu_0(s-s^*)}.
$$

Combining Proposition and Theorem we derive the existence of a blow-up profile near non-characteristic points in the following:
Theorem 5. (Blow-up profile near a non-characteristic point) If \( u \) a solution of (1) with blow-up curve \( \Gamma : \{x \to T(x)\} \) and \( x_0 \in \mathbb{R} \) is non-characteristic (in the sense (2)), then there exist \( d_\infty(x_0) \in (-1, 1) \), \( \Omega_\infty(x_0) \in S^{m-1} \) and \( s^*(x_0) \geq -\log T(x_0) \) such that for all \( s \geq s^*(x_0) \), (13) holds with \( \epsilon^* = \epsilon_0 \), where \( C_0 \) and \( \epsilon_0 \) are given in Theorem 4. Moreover,

\[
||w_{x_0}(s) - \kappa(d_\infty(x_0))\Omega_\infty(x_0)||_{H^1(-1,1)} + ||\partial_s w_{x_0}(s)||_{L^2(-1,1)} \to 0 \quad \text{as} \quad s \to \infty.
\]

Remark: From the Sobolev embedding, we know that the convergence takes place also in \( L^\infty \), in the sense that

\[
||w_{x_0}(s) - \kappa(d_\infty(x_0))\Omega_\infty(x_0)||_{L^\infty(-1,1)} \to 0 \quad \text{as} \quad s \to \infty.
\]

In this paper, we give the proofs of Proposition 2 and Theorem 4, which present the novelties of this work comparing with the handled real and complex cases, since Propositions 2, 3 and Theorem 5 can be generalized from the real case treated in [18] without any difficulty.

Let us remark that our paper is not a simple adaptation of the complex case. In fact, the vector-valued structure of our solution implies a new characterization of the set of stationary solutions in \( \mathbb{R}^m \) (see Proposition 2 above). In addition, in order to apply the modulation theory, we need more parameters, and for that, a suitable \( m \times m \) rotation matrix will be defined (see (60) and (61) below; see the beginning of the proof of Proposition 3.3 page 19 below), and we have to treat delicately the terms coming from the rotation matrix.

This paper is organized as follows:
- In Section 2, we give the proof of Proposition 2
- In Section 3, we give the proof of Theorem 4

2 Characterization of the set of stationary solutions

In this section, we prove Proposition 2 which characterizes all \( \mathcal{H}_0 \) solutions of

\[
\frac{1}{\rho}(\rho(1 - y^2)w')' - \frac{2(p + 1)}{(p - 1)^2}w + |w|^{p-1}w = 0, \tag{14}
\]

the stationary version of (4). Note that since 0 and \( \kappa_0 \Omega \) are trivial solutions to equation (14) for any \( \Omega \in S^{m-1} \), we see from a Lorentz transformation (see Lemma 2.6 page 54 in [18]) that \( \mathcal{T}_d e^{i\theta} \kappa_0 = \kappa(d, y) \) is also a stationary solution to (4). Let us introduce the set

\[
S \equiv \{0, \kappa(d, \cdot)\Omega, |d| < 1, \Omega \in S^{m-1}\}. \tag{15}
\]

Now, we prove Proposition 2 which states that there are no more solutions of (14) in \( \mathcal{H}_0 \) outside the set \( S \).

We first prove (ii), since its proof is short.

(ii) Since we clearly have from the definition (8) that \( E(0,0) = 0 \), we will compute \( E(\Omega \kappa(d, \cdot), 0) \).

From (8) and the proof of the real case treated in page 59 in [18], we see that

\[
E(\kappa(d, \cdot)\Omega, 0) = E(\kappa(d, \cdot), 0) = E(\kappa_0, 0) > 0.
\]

Thus, (10) follows.

(i) Consider \( w \in \mathcal{H}_0 \) an \( \mathbb{R}^m \) non-zero solution of (14). Let us prove that there are some \( d \in (-1, 1) \) and \( \Omega \in S^{m-1} \) such that \( w = \kappa(d, \cdot)\Omega \). For this purpose, define

\[
\xi = \frac{1}{2} \log \left( \frac{1 + y}{1 - y} \right) \quad \text{(that is} \quad y = \tanh \xi) \quad \text{and} \quad \bar{w}(\xi) = w(y)(1 - y^2)^{-\frac{1}{p-1}}. \tag{16}
\]
As in the real case, we see from straightforward calculations that \( \bar{w} \neq 0 \) is a \( H^1(\mathbb{R}) \) solution to
\[
\partial^2_t \bar{w} + |\bar{w}|^{p-1} \bar{w} - \frac{4}{(p-1)^2} \bar{w} = 0, \quad \forall \xi \in \mathbb{R}.
\]

Our aim is to prove the existence of \( \Omega \in \mathbb{R} \) and \( \xi_0 \in \mathbb{R} \) such that \( \bar{w}(\xi) = \Omega \bar{k}(\xi + \xi_0) \) where
\[
\bar{k}(\xi) = \frac{\kappa_0}{\cosh^{p-1}(\xi)}.
\]

Since \( \bar{w} \in H^1(\mathbb{R}) \subset C^2(\mathbb{R}) \), we see that \( \bar{w} \) is a strong \( C^2 \) solution of equation (17). Since \( \bar{w} \neq 0 \), there exists \( \xi_0 \in \mathbb{R} \) such that \( \bar{w}(\xi_0) \neq 0 \). By invariance of (17) by translation, we may suppose that \( \xi_0 = 0 \). Let
\[
G^* = \{ \xi \in \mathbb{R} | \bar{w}(\xi) \neq 0 \},
\]
a nonempty open set by continuity. Note that \( G^* \) contains some non empty interval \( I \) containing 0.

We introduce \( \rho \) and \( \Omega \) by
\[
\rho = |\bar{w}|; \quad \Omega = \frac{\bar{w}}{|\bar{w}|}, \quad \text{whenever } \xi \in G^*.
\]

From equation (17), we see that
\[
\rho'' \Omega + 2 \rho' \Omega' + \rho \Omega'' + \rho^2 \Omega - \frac{4}{(p-1)^2} \rho \Omega = 0.
\]

Now, since \( |\Omega| = 1 \), we immediately see that \( \Omega', \Omega = 0 \) and \( \Omega'', \Omega + |\Omega'|^2 = 0 \).

Let \( H(\xi) = |\Omega|^2 \). Projecting equation (19) according to \( \Omega \) and \( \Omega' \) we see that
\[
\forall \xi \in G^*, \begin{cases} 
\rho''(\xi) - \rho(\xi)H(\xi) - c_0 \rho(\xi) + \rho(\xi)^p = 0, \\
4\rho'(\xi)H(\xi) + \rho(\xi)H'(\xi) = 0
\end{cases}
\]

Integrating the second equation on the interval \( I \subset G^* \), we see that for all \( \xi \in I \),
\[
H(\xi) = \frac{H(0)(\rho(0))^4}{(\rho(\xi))^4}. \quad \text{Plugging this in the first equation, we get}
\]
\[
\forall \xi \in I, \rho''(\xi) - \frac{\mu}{(\rho(\xi))^3} - c_0 \rho(\xi) + \rho^p(\xi) = 0 \quad \text{where } \mu = H(0)(\rho(0))^4.
\]

Now let
\[
G = \left\{ \xi \in G^*, \forall \xi' \in I_\xi, \quad H(\xi') = \frac{H(0)(\rho(0))^4}{(\rho(\xi'))^4} \right\},
\]

where \( I_\xi = [0, \xi] \) if \( \xi \geq 0 \) or \( I_\xi = (\xi, 0] \) if \( \xi \leq 0 \). Note that \( I \subset G \). Now, we give the following:

**Lemma 2.1.** There exists \( \epsilon_0 > 0 \) such that
\[
\forall \xi \in G, \forall \xi' \in I_\xi, \quad 0 < \epsilon_0 \leq |\bar{w}(\xi')| \leq \frac{1}{\epsilon_0}.
\]

**Proof.** The proof is the same as in the complex-case, see page 5898 in [4]. But for the reader’s convenience and for the sake of self-containedness, we recall it here. Take \( \xi \in G \). By definition (22) of \( G \), we see that equation (21) is satisfied for all \( \xi' \in I_\xi \). Multiplying \( \rho''(\xi) - \frac{\mu}{(\rho(\xi))^3} - c_0 \rho(\xi) + \rho^p(\xi) = 0 \) by \( \rho' \) and integrating between 0 and \( \xi \), we get:
\[
\forall \xi \in I_\xi, \quad \mathcal{E}(\xi') = \mathcal{E}(0), \quad \text{where } \mathcal{E}(\xi') = \frac{1}{2} (\rho'(\xi'))^2 + \frac{\mu}{2((\rho(\xi'))^2} - \frac{c_0}{2} \rho^2(\xi') + \frac{\rho^{p+1}(\xi')}{p+1},
\]
or equivalently,
\[ \forall \xi' \in I_\xi, \ F(\rho(\xi')) = \frac{1}{2} \rho'(\xi')^2 \geq 0 \text{ where } F(r) = \frac{\mu}{2r^2} + \frac{c_0}{2}r^2 - \frac{r^{p+1}}{p+1} + \mathcal{E}(0). \]
Since \( F(r) \to -\infty \) as \( r \to 0 \) or \( r \to \infty \), there exists \( \epsilon_0 = \epsilon_0(\mu, E(0)) > 0 \) such that \( \epsilon_0 \leq \rho(\xi') \leq \frac{1}{\epsilon_0} \), which yields to the conclusion of the Claim \((\text{2.1})\).

We claim the following:

**Lemma 2.2.** It holds that \( G = \mathbb{R} \).

**Proof.** Note first that by construction, \( G \) is a nonempty interval (note that \( 0 \in I \subset G \) where \( I \) is defined right before \((\text{18})\)). We have only to prove that \( \sup G = +\infty \), since the fact that \( \inf G = -\infty \) can be deduced by replacing \( \bar{w}(\xi) \) by \( \bar{w}(-\xi) \).
By contradiction, suppose that \( \sup G = a < +\infty \).

First of all, by Lemma \((\text{2.1})\) we have for all \( \xi' \in [0, a), 0 < \epsilon_0 \leq |\bar{w}(\xi')| \leq \frac{1}{\epsilon_0} \). By continuity, this holds also for \( \xi' = a \), hence, \( \bar{w}(a) \neq 0 \), and \( a \in G^* \). Furthermore, by definition of \( G \) and continuity, we see that
\[ \forall \xi \in [0, a], H(\xi) = \frac{H(0)|\rho(0)|^4}{\rho'(\xi)^4}. \]
(23)
Therefore, we see that \( a \in G \). By continuity, we can write for all \( \xi \in (a - \delta, a + \delta) \), where \( \delta > 0 \) is small enough,
\[
\begin{cases}
\rho''(\xi) - \rho(\xi)H(\xi) - c_0\rho(\xi) + \rho(\xi)^p = 0, \\
4\rho'(\xi)H(\xi) + \rho(\xi)H'(\xi) = 0.
\end{cases}
\]
From the second equation and \((\text{23})\) applied with \( \xi = a \), we see that \( H(\xi) = \frac{H(0)|\rho(0)|^4}{(\rho(\xi))^4} \). Therefore, it follows that \((a, a + \delta) \in G \), which contradicts the fact that \( a = \sup G \). \( \blacksquare \)

Note from Lemma \((\text{2.2})\) that \((\text{20})\) and \((\text{21})\) holds for all \( \xi \in \mathbb{R} \). We claim that \( H(0) = 0 \). Indeed, if not, then by \((\text{21})\), we have \( \mu \neq 0 \), and since \( G = \mathbb{R} \), we see from Lemma \((\text{2.1})\) that for all \( \xi \in \mathbb{R}, |\bar{w}(\xi)| \geq \epsilon_0 \), therefore \( w \notin L^2(\mathbb{R}) \), which contradicts the fact that \( \bar{w} \in H^1(\mathbb{R}) \). Thus, \( H(0) = 0 \), and \( \mu = 0 \). By uniqueness of solutions to the second equation of \((\text{20})\), we see that \( H(\xi) = 0 \) for all \( \xi \in \mathbb{R} \), so \( \Omega(\xi) = \Omega(0) \), and
\[
\begin{align*}
\bar{w}(0) &= \rho(0)|\Omega(0)|, \\
\bar{w}'(0) &= \rho'(0)|\Omega(0)|.
\end{align*}
\]
Let \( W \) be the maximal real-valued solution of
\[
\begin{cases}
W'' - c_0W + |W|^{p-1}W = 0 \\
W(0) = \rho(0) \\
W'(0) = \rho'(0).
\end{cases}
\]
By uniqueness of the Cauchy problem of equation \((\text{17})\), we have for all \( \xi \in \mathbb{R}, \bar{w}(\xi) = W(\xi)|\Omega(0)| \), and as \( \bar{w} \in H^1(\mathbb{R}) \), \( W \) is also in \( H^1(\mathbb{R}) \). It is then classical that there exists \( \xi_0 \) such that for all \( \xi \in \mathbb{R}, W(\xi) = \bar{k}(\xi + \xi_0) \) (remember that \( \rho(0) > 0 \), hence we only select positive solutions here). In addition, for \( \Omega_0 = \Omega(0), \bar{w}(\xi) = \bar{k}(\xi + \xi_0)|\Omega_0|. \) Thus, for \( d = \tanh\xi_0 \in (-1, 1) \) and \( y = \tanh\xi \), we get
\[
\bar{w}(\xi) = \kappa_0 \left[ 1 - \tanh(\xi + \xi_0)^2 \right]^{\frac{1}{p-1}} \Omega_0 = \kappa_0 \left[ 1 - \left( \tanh \xi + \tanh \xi_0 \right) \frac{2}{1 + \tanh \xi \tanh \xi_0} \right]^{\frac{1}{p-1}} \Omega_0
\]
\[
= \kappa_0 \left[ 1 - \left( \frac{y + d}{1 + dy} \right)^2 \right]^{\frac{1}{p-1}} \Omega_0 = \kappa_0 \left[ \frac{(1 - d^2)(1 - y^2)^2}{(1 + dy)^2} \right]^{\frac{1}{p-1}} \Omega_0 = \kappa(d, y)(1 - y^2)^{\frac{p}{p-1}} \Omega_0.
\]
By (16), we see that $w(y) = \kappa(d, y)\Omega_0$. This concludes the proof of Proposition 2.

3 Outline of the proof of Theorem 4

The proof of Theorem 4 is not a simple adaptation of the complex-case to the vector-valued case, in fact, it involves a delicate modulation. In this section, we will outline the proof, insisting on the novelties, and only recalling the features which are the same as in the real-valued complex-valued cases.

This section is organized as follows:
- In Subsection 3.1, we linearize equation (4) around $\kappa(d, y)e_1$ where $e_1 = (1, 0, ..., 0)$ and figure-out that, with respect to the complex-valued case, our linear operator is just a superposition of one copy of the real part operator, with $(m-1)$ copies of the imaginary part operator.
- In Subsection 3.2, we recall from [18] the spectral properties of the real-part operator.
- In Subsection 3.3, we recall from [4] the spectral properties of the imaginary-part operator.
- In Subsection 3.4, assuming that $\Omega^* = e_1$ (possible thanks to rotation invariance of (4)), we introduce a modulation technique adapted to the vector-valued case. This part makes the originality of our work with respect to the complex-valued case.
- In Subsection 3.5, we write down the equations satisfied by the modulation parameters along with the PDE satisfied by $q(y, s)$ and its components.
- In Subsection 3.6, we conclude the proof of Theorem 4.

3.1 The linearized operator around a non-zero stationary solution

We study the properties of the linearized operator of equation (4) around the stationary solution $\kappa(d, y)$ (9).

Let us introduce $q = (q_1, q_2) \in \mathbb{R}^m \times \mathbb{R}^m$ for all $s \in [s_0, \infty)$, for a given $s_0 \in \mathbb{R}$, by

$$
\begin{pmatrix}
    w(y, s) \\
    \partial_s w(y, s)
\end{pmatrix} = \begin{pmatrix}
    \kappa(d, y)e_1 \\
    0
\end{pmatrix} + \begin{pmatrix}
    q_1(y, s) \\
    q_2(y, s)
\end{pmatrix},
$$

(24)

Let us introduce the coordinates of $q_1$ and $q_2$ by

$q_1 = (q_{1,1}, q_{1,2}, ..., q_{1,m}), \quad q_2 = (q_{2,1}, q_{2,2}, ..., q_{2,m}).$

We see from equation (4), that $q$ satisfies the following equation for all $s \geq s_0$:

$$
\frac{\partial}{\partial s} \begin{pmatrix}
    q_1 \\
    q_2
\end{pmatrix} = L_d \begin{pmatrix}
    q_1 \\
    q_2
\end{pmatrix} + \begin{pmatrix}
    0 \\
    f_d(q_1)
\end{pmatrix},
$$

(25)

where

$L_d \begin{pmatrix}
    q_1 \\
    q_2
\end{pmatrix} = \begin{pmatrix}
    \mathcal{L}q_1 + \tilde{\psi}(d, y)q_{1,1}e_1 + \sum_{j=2}^m q_j \tilde{\psi}(d, y)q_{1,j}e_j - \frac{p+3}{p-1}q_2 - 2y\partial_y q_2
\end{pmatrix},$

(26)

$$
\tilde{\psi}(d, y) = p\kappa(d, y)^{p-1} - \frac{2(p+1)}{(p-1)^2}
$$

(27)

$f_d(q_1) = f_{d,1}(q_1)e_1 + \sum_{j=2}^m f_{d,j}(q_1)e_j,$

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where
\[ f_{d,1}(q_1) = |\kappa(d, y)e_1 + q_1|^{p-1}(\kappa(d, y) + q_{1,1}) - \kappa(d, y)^{p} - p\kappa^{p-1}(d, y)q_{1,1}. \]  
(28)

\[ f_{d,j}(q_1) = |\kappa(d, y)e_1 + q_1|^{p-1}q_{1,j} - \kappa^{p-1}(d, y)q_{1,j}. \]  
(29)

Projecting (25) on the first coordinate, we get for all \( s \geq s_0 \):
\[ \frac{\partial}{\partial s} \begin{pmatrix} q_{1,1} \\ q_{2,1} \end{pmatrix} = \bar{L}_d \begin{pmatrix} q_{1,1} \\ q_{2,1} \end{pmatrix} + \begin{pmatrix} 0 \\ f_{d,1}(q_1) \end{pmatrix}, \]  
(30)
where \( \bar{L}_d \) is given by:
\[ \bar{L}_d \begin{pmatrix} q_{1,1} \\ q_{2,1} \end{pmatrix} = \left( \mathcal{L}_{q_{1,1}} + \bar{\psi}(d, y)q_{1,1} - \frac{p+3}{p-1}q_{2,1} - 2y\partial_y q_{2,1} \right). \]  
(31)

Now, projecting equation (25) on the \( j \)-th coordinate with \( j = 2, .., m \), we see that
\[ \frac{\partial}{\partial s} \begin{pmatrix} q_{1,j} \\ q_{2,j} \end{pmatrix} = \tilde{L}_d \begin{pmatrix} q_{1,j} \\ q_{2,j} \end{pmatrix} + \begin{pmatrix} 0 \\ f_{d,j}(q_1) \end{pmatrix}, \]  
(32)
where
\[ \tilde{L}_d \begin{pmatrix} q_{1,j} \\ q_{2,j} \end{pmatrix} = \left( \mathcal{L}_{q_{1,j}} + \bar{\psi}(d, y)q_{1,j} - \frac{p+3}{p-1}q_{2,j} - 2y\partial_y q_{2,j} \right). \]  
(33)

Remark: Our linearized operator \( L_d \) is in fact diagonal in the sense that
\[ L_d \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \bar{L}_d \begin{pmatrix} q_{1,1} \\ q_{2,1} \end{pmatrix} e_1 + \sum_{j=2}^{m} \tilde{L}_d \begin{pmatrix} q_{1,j} \\ q_{2,j} \end{pmatrix} e_j. \]

We mention that for \( j = 1 \), equation (30) is the same as the equation satisfied by the real part of the solution in the complex case (see Section 3 page 5899 in [1]), whereas for \( j = 2, .., m \), equation (32) is the same as the equation satisfied by the imaginary part of the solution operator in the complex case. Thus, the reader will have no difficulty in adapting the remaining part of the proof to the vector-valued case. Thus, the dynamical system formulation we performed when \( m = 2 \) can be adapted straightforwardly to the case \( m \geq 3 \).

Note from (6) that we have
\[ \|q\|_H = [\phi(q, q)]^{\frac{1}{2}} < +\infty, \]
where the inner product \( \phi \) is defined by
\[ \phi(q, r) = \phi \left( \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \right) = \int_{-1}^{1} (q_1 r_1 + q_2 r_2 + q_{1,1} r_{1,1} (1 - y^2)) \rho dy. \]
where \( q_1, r_1 = \sum_{j=1}^{m} q_{1,1} r_{1,1} \) is the standard inner product in \( \mathbb{R}^m \), with similar expressions for \( q_1, r_1 \) and \( q_2, r_2 \).

Using integration by parts and the definition of \( \mathcal{L} \), we have the following:
\[ \phi(q, r) = \int_{-1}^{1} (q_1 \cdot (-\mathcal{L} r_1 + r_1) + q_2 \cdot r_2) \rho dy. \]  
(34)

In the following two sections, we recall from [15] and [4] the spectral properties of \( \bar{L}_d \) and \( \tilde{L}_d \).
### 3.2 Spectral theory of the operator $\tilde{L}_d$

From Section 4 in [18], we know that $\tilde{L}_d$ has two nonnegative eigenvalues $\lambda = 1$ and $\lambda = 0$ with eigenfunctions

$$\tilde{F}_1^d(y) = (1 - d^2)^{\frac{p}{p-1}} \left( (1 + dy)^{\frac{p+1}{p-1}} \right) \text{ and } \tilde{F}_0^d(y) = (1 - d^2)^{\frac{1}{p-1}} \left( \frac{y+d}{(1+dy)^{\frac{p+1}{p-1}}} \right). \quad (35)$$

Note that for some $C_0 > 0$ and any $\lambda \in \{0, 1\}$, we have

$$\forall |d| < 1, \quad \frac{1}{C_0} \leq ||\tilde{F}_d^x||_H \leq C_0 \text{ and } ||\partial_d \tilde{F}_d^x||_H \leq \frac{C_0}{1 - d^2}. \quad (36)$$

Also, we know that $\tilde{L}_d^*$ the conjugate operator of $\tilde{L}_d$ with respect to $\phi$ is given by

$$\tilde{L}_d^*(r_1, r_2) = \left( -\mathcal{L}r_1 + r_1 + \frac{p+3}{p-1}r_2 + 2yr'_2 - \frac{8}{(p-1)(1-y)r} \right) \quad r = R_d(r_2)$$

for any $(r_1, r_2) \in (\mathcal{D}(\mathcal{L}))^2$, where $r = R_d(r_2)$ is the unique solution of $-\mathcal{L}r + r = \mathcal{L}r_2 + \tilde{\psi}(d, y)r_2$.

Here, the domain $\mathcal{D}(\mathcal{L})$ of $\mathcal{L}$ defined in [5] is the set of all $r \in L^2_{\nu}$ such that $\mathcal{L}r \in L^2_{\nu}$.

Furthermore, $\tilde{L}_d^*$ has two nonnegative eigenvalues $\lambda = 0$ and $\lambda = 1$ with eigenfunctions $\tilde{W}_d^x$ such that

$$\tilde{W}_{1,2}^d(y) = \tilde{c}_1 \left( (1-y^2)^{\frac{1}{p-1}} (1-d)^{\frac{1}{p-1}} \right) \quad \tilde{W}_{0,2}^d(y) = \tilde{c}_0 \left( (y+d)^{\frac{1}{p-1}} (1+dy)^{\frac{1}{p-1}} \right),$$

with

$$\tilde{c}_1 = 2(\frac{2}{p-1} + \lambda) \left( \frac{y}{1-y^2} \right)^{1-\lambda} \rho(y) dy,$$

and $\tilde{W}_{1,1}^d$ is the unique solution of the equation

$$-\mathcal{L}r + r = \left( \lambda - \frac{p+3}{p-1} \right) r_2 - 2yr'_2 + \frac{8}{p-1} \frac{r_2}{1-y^2} \quad r_2 = \tilde{W}_{1,2}^d.$$

We also have for $\lambda = 0, 1$

$$||\tilde{W}_d^x||_H + (1 - d^2)||\partial_d \tilde{W}_d^x||_H \leq C, \forall |d| < 1. \quad (37)$$

Note that we have the following relations for $\lambda = 0$ or $\lambda = 1$

$$\phi(\tilde{W}_{1,2}^d, \tilde{F}_{1,2}^d) = 1 \text{ and } \phi(\tilde{W}_{1,1}^d, \tilde{F}_{1-1,1}^d) = 0. \quad (38)$$

Let us introduce for $\lambda \in \{0, 1\}$ the projectors $\tilde{\pi}_\lambda(r)$, and $\tilde{\pi}_d^d(r)$ for any $r \in \mathcal{H}$ by

$$\tilde{\pi}_\lambda^d(r) = \phi(\tilde{W}_{\lambda}^d, r), \quad (39)$$

$$r = \tilde{\pi}_0^d(r) \tilde{F}_0^d(y) + \tilde{\pi}_1^d(r) \tilde{F}_1^d(y) + \tilde{\pi}_d^d(r), \quad (40)$$

---

1 In section 4 of [18], we had non explicit normalizing constants $\tilde{c}_\lambda = \tilde{c}_\lambda(d)$. In Lemma 2.4 in [24], the authors compute the explicit dependence of $\tilde{c}_\lambda(d)$. 

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and the space
\[ \tilde{\mathcal{H}}^d \equiv \{ r \in \mathcal{H} \mid \tilde{\pi}^d_1(r) = \tilde{\pi}^d_0(r) = 0 \}. \]

Introducing the bilinear form
\[ \tilde{\varphi}_d(q, r) = \int_{-1}^{1} (-\tilde{\psi}(d, y)q_1 r_1 + q_1 r_1'(1 - y^2) + q_2 r_2)\rho dy, \quad (41) \]
where \( \tilde{\psi}(d, y) \) is defined in \[20\], we recall from Proposition 4.7 page 90 in \[18\] that there exists \( C_0 > 0 \) such that for all \( |d| < 1 \), for all \( r \in \tilde{\mathcal{H}}^d \),
\[ \frac{1}{C_0} ||r||^2_{\mathcal{H}} \leq \tilde{\varphi}_d(r, r) \leq C_0 ||r||^2_{\mathcal{H}}. \quad (42) \]
Furthermore, if \( r \in \mathcal{H} \), then
\[ \frac{1}{C_0} ||r||_{\mathcal{H}} \leq \tilde{\varphi}_d(r, r) \leq C_0 ||r||_{\mathcal{H}} \text{ where } r_+ = \tilde{\pi}^d_1(r). \quad (43) \]

In the following section we recall from \[4\] the spectral properties of \( \tilde{L}_d \).

### 3.3 Spectral theory of the operator \( \tilde{L}_d \)

From Section 3 in \[4\], we know that \( \tilde{L}_d \) has one nonnegative eigenvalue \( \lambda = 0 \) with eigenfunction
\[ \tilde{F}^d_0(y) = \begin{pmatrix} \kappa(d, y) \\ 0 \end{pmatrix}. \quad (44) \]
Note that for some \( C_0 > 0 \) we have
\[ \forall |d| < 1, \quad \frac{1}{C_0} \leq ||\tilde{F}^d||_{\mathcal{H}} \leq C_0 \text{ and } ||\partial_d \tilde{F}^d||_{\mathcal{H}} \leq \frac{C_0}{1 - d^2}. \quad (45) \]

We know also that the operator \( \tilde{L}_d^* \) conjugate of \( \tilde{L}_d \) with respect to \( \phi \) is given by
\[ \tilde{L}_d^* \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} -\mathcal{L}r_1 + r_1 + \frac{p+3}{p-1}r_2 + 2yr_2' - \frac{8}{(p-1)(1-y^2)}r_2^2 \\ \cdot \end{pmatrix} \quad (46) \]
for any \((r_1, r_2) \in (\mathcal{D}(\mathcal{L}))^2\), where \( r = \tilde{R}_d(r_2) \) is the unique solution of
\[ -\mathcal{L}r + r = \mathcal{L}r_2 + \tilde{\psi}(d, y)r_2. \quad (47) \]

Furthermore, \( \tilde{L}_d^* \) have one nonnegative eigenvalue \( \lambda = 0 \) with eigenfunction \( \tilde{W}^d_0 \) such that
\[ \tilde{W}^d_{0,2}(y) = \tilde{c}0\kappa(d, y) \quad \text{and} \quad \frac{1}{\tilde{c}_0} = \frac{4\kappa^2_0}{p-1} \int_{-1}^{1} \frac{\rho(y)}{1-y^2}dy \quad (48) \]
and \( \tilde{W}^d_{0,1} \) is the unique solution of the equation
\[ -\mathcal{L}r + r = -\frac{p+3}{p-1}r_2 - 2yr_2' + \frac{8}{(p-1)(1-y^2)}r_2^2 \quad (49) \]
with \( r_2 = \tilde{W}^d_{0,2} \).
We also have for \( \lambda = 0, 1 \)
\[ ||\tilde{W}^d_0||_{\mathcal{H}} + (1 - d^2)||\partial_d \tilde{W}^d_0||_{\mathcal{H}} \leq C, \forall |d| < 1. \quad (50) \]
Moreover, we have
\[ \phi(\tilde{W}_0^d, \tilde{F}_0^d) = 1. \] (51)

Let us introduce the projectors \( \tilde{\pi}_d^0(r) \) and \( \tilde{\pi}_d^-(r) \) for any \( r \in \mathcal{H} \) by
\[ \tilde{\pi}_d^0(r) = \phi(\tilde{W}_0^d, r), \] (52)
\[ r = \tilde{\pi}_d^0(r) \tilde{F}_0^d(y) + \tilde{\pi}_d^-(r). \] (53)

and the space
\[ \tilde{\mathcal{H}}_d^d \equiv \{ r \in \mathcal{H} | \tilde{\pi}_d^0(r) = 0 \}. \] (54)

Introducing the bilinear form
\[ \tilde{\varphi}_d(q, r) = \int_{-1}^{1} (-\tilde{\psi}(d, y) q_1 r_1 + q'_1 r'_1 (1 - y^2) + q_2 r_2) \rho dy, \] (55)
where \( \tilde{\psi}(d, y) \) is defined in (27), we recall from Proposition 3.7 page 5906 in [4] that there exists \( C_0 > 0 \) such that for all \( |d| < 1 \), for all \( r \in \tilde{\mathcal{H}}_d^d \),
\[ \frac{1}{C_0} ||r||^2_{\mathcal{H}} \leq \tilde{\varphi}_d(r, r) \leq C_0 ||r||^2_{\mathcal{H}}. \] (56)

### 3.4 A modulation technique

We start the proof of Theorem 4 here.

Let us consider \( w \in C([s^*, \infty), \mathcal{H}) \) for some \( s^* \in \mathbb{R} \) a solution of equation (4) such that
\[ \forall s \geq s^*, E(w(s), \partial_s w(s)) \geq E(\kappa_0, 0) \]
and
\[ \left\| \left( \frac{w(s^*)}{\partial_s w(s^*)} \right) - \left( \kappa(d^*, \Omega^*) \right) \right\|_{\mathcal{H}} \leq \epsilon^* \] (57)
for some \( d^* \in (-1, 1) \), \( \Omega^* \in \mathbb{S}^{m-1} \) and \( \epsilon^* > 0 \) to be chosen small enough.

Our aim is to show the convergence of \((w(s), \partial_s w(s))\) as \( s \to \infty \) to some \((\kappa(d_\infty, 0)\Omega_\infty, 0)\), for some \((d_\infty, \Omega_\infty)\) close to \((d^*, \Omega^*)\).

As one can see from (57), \((w, \partial_s w)\) is close to a one representative of the family of the non-zero stationary solution
\[ S^* \equiv \{ (\kappa(d, y), 0)\Omega, |d| < 1, \Omega \in \mathbb{S}^{m-1} \}. \]

From the continuity of \((w, \partial_s w)\) from \([s^*, \infty)\) to \( \mathcal{H} \), \((w(s), \partial_s w(s))\) will stay close to a soliton from \( S^* \), at least for a short time after \( s^* \). In fact, we can do better, and impose some orthogonality conditions, killing the zero directions of the linearized operator of equation (4) (see the operator \( L_d \) defined in (25)).

From the invariance of equation (4) under rotations in \( \mathbb{R}^m \), we may assume that
\[ \Omega^* = e_1. \] (58)
We recall that at this level of the study in the complex case (i.e. for \( m = 2 \)), we were able to modulate \((w, \partial_s w)\) as follows

\[
\begin{pmatrix}
  w(y, s) \\
  \partial_s w(y, s)
\end{pmatrix} = e^{i \theta(s)} \begin{pmatrix}
  \kappa(d(s), y) \\
  0
\end{pmatrix} + \begin{pmatrix}
  q_1(y, s) \\
  q_2(y, s)
\end{pmatrix},
\]

for some well chosen \( d(s) \in (-1, 1) \) and \( \theta(s) \in \mathbb{R} \), such that

\[
\pi_0 d(s) \begin{pmatrix}
  q_{1,1}(s) \\
  q_{2,1}(s)
\end{pmatrix} = \pi_0(s) \begin{pmatrix}
  q_{1,2}(s) \\
  q_{2,2}(s)
\end{pmatrix} = 0
\]

where \( \pi_0^d \) and \( \pi_0^d \) are defined in (39) and (52) and \( q = (q_1, q_2) \) is small in \( \mathcal{H} \).

From (59), we see that we have a rotation in the complex plane, which has to be generalized to the vector-valued case. In order to do so, we introduce for \( i = 2, \ldots, m \)

\[
R_i = \begin{pmatrix}
  \cos \theta_i & 0 & \cdots & -\sin \theta_i & \cdots & 0 \\
  0 & 1 & \cdots & 0 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
  \sin \theta_i & 0 & \cdots & \cos \theta_i & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & 0 & \cdots & 1
\end{pmatrix}.
\]

(60)

Note that \( R_i \) is an \( m \times m \) orthonormal matrix which rotates the \((e_1, e_i)\)-plane by an angle \( \theta_i \) and leaves all other directions invariant. We introduce \( R_\theta \) by

\[
R_\theta \equiv R_2 R_3 \cdots R_m,
\]

(61)

where \( \theta = (\theta_2, \theta_3, \ldots, \theta_m) \). Clearly, \( R_\theta \) is an \( m \times m \) orthonormal matrix. We also define \( A_j \) by

\[
A_j = R_\theta^{-1} \frac{\partial R_\theta}{\partial \theta_j}.
\]

(62)

In the appendix, we show a different expression for \( A_j \):

\[
A_j = \frac{\partial R_\theta^{-1}}{\partial \theta_j} R_\theta.
\]

(63)

In fact, this formalism is borrowed from Filippas and Merle [8] who introduced the modulation technique for the vector-valued heat equation

\[
\partial_t u = \Delta u + |u|^{p-1} u.
\]

We are ready to give our modulation technique result well adapted to the vector-valued case:

**Proposition 3.1. (Modulation of \( w \) with respect to \( \kappa(d, \cdot) \Omega \), where \( \Omega \in \mathbb{R}^{m-1} \))** There exists \( \epsilon_0 > 0 \) and \( K_1 > 0 \) such that for all \( \epsilon \leq \epsilon_0 \) if \( v \in \mathcal{H} \), \( d \in (-1, 1) \) and \( \hat{\theta} = (\hat{\theta}_2, \ldots, \hat{\theta}_m) \in \mathbb{R}^{m-1} \) are such that

\[
\forall i = 2, \ldots, m, \cos \hat{\theta}_i \geq \frac{3}{4} \text{ and } ||\hat{q}||_H \leq \epsilon \text{ where } v = R_{\hat{\theta}} \begin{pmatrix}
  \kappa(d, \cdot) e_1 \\
  0
\end{pmatrix} + \hat{q},
\]

(63)
then, there exist \( d \in (-1, 1) \), \( \hat{\theta} = (\hat{\theta}_2, \ldots, \hat{\theta}_m) \in \mathbb{R}^{m-1} \) such that

\[
\bar{\pi}^d_{0} \begin{pmatrix} q_{1,1} \\ q_{2,1} \end{pmatrix} = 0, \quad \text{and} \quad \hat{\pi}^d_{0} \begin{pmatrix} q_{1,j} \\ q_{2,j} \end{pmatrix} = 0, \quad \forall j = 2, \ldots, m,
\]

(64)

where \( q = (q_1, q_2) \) is defined by:

\[
\left| \log \left( 1 + \frac{1 + d}{1 - d} \right) - \log \left( 1 + \frac{1 + \hat{d}}{1 - \hat{d}} \right) \right| + |\theta - \hat{\theta}| \leq C_0 ||\hat{q}||_H \leq K_1 \epsilon,
\]

\( \forall i = 2, \ldots, m \), \( \cos \theta_i \geq \frac{1}{2} \) and \( ||q||_H \leq K_1 \epsilon \).

In order to prove this proposition, we need the following estimates on the matrix \( A_j \) given in (62) and (63):

**Lemma 3.2 (Orthogonality and continuity results related to the matrix \( A_i \) (62)).**

i) For any \( i \in \{2, \ldots, m\} \),

\[
A_i e_1 = \left( \prod_{j=i+1}^{m} \cos \theta_j \right) e_i
\]

ii) For any \( i \in \{2, \ldots, m\}, z \in \mathbb{R}^m \), we have

\[
|A_i(z)| \leq |z|.
\]

*Proof.* The proof is straightforward though a bit technical. For that reason, we give it in Appendix A. \( \blacksquare \)

Now, we are ready to prove Proposition 3.1.

**Proof of Proposition 3.1** The proof is similar to the complex-valued case. However, since our notations are somehow complicated, we give details for the reader’s convenience.

First, we recall that \( \theta = (\theta_2, \theta_3, \ldots, \theta_m) \in \mathbb{R}^{m-1} \).

From (39) and (52), we see that the condition (64) becomes \( \Phi(v, d, \theta) = 0 \) where \( \Phi \in C(H \times (\mathbb{R} \times \mathbb{R}^{-1} \times \mathbb{R}^{m-1}, \mathbb{R}^m) \) is defined by

\[
\Phi(v, d, \theta) = \begin{pmatrix} \tilde{\Phi}(v, d, \theta) \\ \tilde{\Phi}_2(v, d, \theta) \\ \vdots \\ \tilde{\Phi}_m(v, d, \theta) \end{pmatrix} = \begin{pmatrix} \phi \left( \begin{pmatrix} V_{1,1} \\ V_{2,1} \end{pmatrix} - \begin{pmatrix} \kappa(d,.) \\ 0 \end{pmatrix}, \hat{W}_0^d \right) \\ \phi \left( \begin{pmatrix} V_{1,j} \\ V_{2,j} \end{pmatrix}, \hat{W}_0^d \right)_{j=2 \ldots m} \end{pmatrix}
\]

(65)

where \( V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \in \mathbb{R}^m \times \mathbb{R}^m \) is given by \( V = R_\theta^{-1} v \).

We claim that we can apply the implicit function theorem to \( \Phi \) near the point \((\hat{v}, \hat{d}, \hat{\theta})\) with \( \hat{v} = R_\theta(\kappa(d,.)e_1, 0) \). Three facts have to be checked:

1-First, note that \( \hat{v} = R_\theta^{-1}(\hat{v}) \), hence

\[
\Phi(R_\theta(\kappa(d,.)e_1, 0), \hat{d}, \hat{\theta}) = 0.
\]

2-Then, we compute from (65), for all \( u \in H \),

\[
D_v \Phi(v, d, \theta)(u) = \phi \left( \begin{pmatrix} U_{1,1} \\ U_{2,1} \end{pmatrix}, \hat{W}_0^d \right),
\]
and for all $j = 2 \ldots m$, we have

$$D_v \Phi_j(v, d, \theta)(u) = \phi\left(\begin{pmatrix} U_{1,j} \\ U_{2,j} \end{pmatrix}, \tilde{W}_0^d\right),$$

so we have from (67) and (50)

$$||D_v \Phi(v, d, \theta)|| \leq C_0 \text{ and } ||D_v \Phi_j(v, d, \theta)|| \leq C_0. \quad (66)$$

3.- Let $J(\Phi, \Phi_{j=2..m})$ the jacobian matrix of $\Phi$ with respect to $(d, \theta)$, and $D$ its determinant so

$$J \equiv \begin{pmatrix} \partial_d \bar{\phi} & \partial_{\theta_2} \bar{\phi} & \ldots & \partial_{\theta_m} \bar{\phi} \\ \partial_d \tilde{\phi}_2 & \partial_{\theta_2} \tilde{\phi}_2 & \ldots & \partial_{\theta_m} \tilde{\phi}_2 \\ \vdots & \vdots & \vdots & \vdots \\ \partial_d \tilde{\phi}_m & \partial_{\theta_2} \tilde{\phi}_m & \ldots & \partial_{\theta_m} \tilde{\phi}_m \end{pmatrix}. \quad (67)$$

Then, we compute from (65):

$$\partial_d \bar{\Phi} = -\phi((\partial_d \kappa(d, .), 0), \tilde{W}_0^d) + \phi\left(V_{1,1} V_{2,1}, \left(\kappa(d, .) - \left(\begin{array}{c} 0 \\ 0 \end{array}\right)\right), \tilde{W}_0^d\right), \quad (68)$$

and for $i, j = 2 \ldots m$

$$\begin{align*}
\partial_d \bar{\Phi}_j &= \phi\left(V_{1,j} V_{2,j}, \partial_d \tilde{W}_0^d\right), \quad (69) \\
\partial_{\theta_i} \bar{\Phi} &= \phi(\partial_{\theta_i} \left(V_{1,1} V_{2,1}\right), \tilde{W}_0^d) = \phi\left(\left< e_1, \frac{\partial \bar{\kappa}(d, \theta)}{\partial \theta_1} v_1 \right>, \tilde{W}_0^d\right), \quad (70) \\
\partial_{\theta_j} \bar{\Phi}_j &= \phi\left(\left< e_j, \frac{\partial \bar{\kappa}(d, \theta)}{\partial \theta_j} \right>, \tilde{W}_0^d\right). \quad (71)
\end{align*}$$

Now, we assume that

$$|\theta| + |\log \frac{1 + d}{1 - d} - \log \frac{1 + d}{1 - d}| + \|v - R_\theta \left(\kappa(d, \cdot) e_1\right)\|_\mathcal{H} \leq \epsilon_1 \quad (73)$$

for some small $\epsilon_1 > 0$.

In the following, we estimate each of the derivatives whose expressions were given above.

- Since

$$\begin{pmatrix} \partial_d \kappa(d, y) \\ 0 \end{pmatrix} = \frac{-2\kappa_0}{(p - 1)(1 - d^2)} \tilde{F}_d,$$

by definition (35) and (9), it follows from the orthogonality condition (38) that

$$\phi((\partial_d \kappa(d, .), 0), \tilde{W}_0^d) = \frac{-2\kappa_0}{(p - 1)(1 - d^2)}.$$

Therefore, from (68), we write

$$\begin{align*}
\partial_d \bar{\Phi} &= \frac{2\kappa_0}{(p - 1)(1 - d^2)} + \phi\left(V_{1,1} V_{2,1}, \left(\kappa(d, .) - \left(\begin{array}{c} 0 \\ 0 \end{array}\right)\right), \partial_d \tilde{W}_0^d\right). \quad (74)
\end{align*}$$
Since \( \begin{pmatrix} V_{1,1} \\ V_{2,1} \end{pmatrix} = \left( \begin{array}{c} \langle e_1, R_{\theta}^{-1} v_1 \rangle \\ \langle e_1, R_{\theta}^{-1} v_2 \rangle \end{array} \right) \), we write

\[
\| \begin{pmatrix} V_{1,1} \\ V_{2,1} \end{pmatrix} - \left( \begin{array}{c} \kappa(d, .) \\ 0 \end{array} \right) \|_H \leq \| R_{\theta}^{-1} v - \left( \begin{array}{c} \kappa(d, .) e_1 \\ 0 \end{array} \right) \|_H \quad (75)
\]

\[
\leq \| (R_{\theta}^{-1} - R_{\theta'}^{-1}) v \|_H + \| R_{\theta}^{-1} v - \left( \begin{array}{c} \kappa(d, .) e_1 \\ 0 \end{array} \right) \|_H + \| \kappa(d, .) - \kappa(d, .) \|_H.
\]

Since,

\[
\forall \theta, \theta' \in \mathbb{R}, |R_{\theta} - R_{\theta'}| + |R_{\theta}^{-1} - R_{\theta'}^{-1}| \leq C|\theta - \theta'|,
\]

(see \(67\) below for \( R_{\theta} \), and use an adhoc change of variables for \( R_{\theta}^{-1} \)), recalling the following continuity result from estimate (174) page 101 in \([18]\):

\[
\| \kappa(d_1, .) - \kappa(d_2, .) \|_{H_0} \leq C \left| \frac{1 + d_1}{1 - d_1} - \frac{1 + d_2}{1 - d_2} \right|,
\]

(76)

we see from the Cauchy-Schwartz inequality, (75), (37) and (73) that

\[
\| \begin{pmatrix} V_{1,1} \\ V_{2,1} \end{pmatrix} - \left( \begin{array}{c} \kappa(d, .) \\ 0 \end{array} \right) \|_H \leq \| V - \left( \begin{array}{c} \kappa(d, .) e_1 \\ 0 \end{array} \right) \|_H \leq C \epsilon_1.
\]

\[
|\partial_\theta \bar{\Phi} - \frac{2 \kappa_0}{(p - 1)(1 - d^2)}| \leq C \epsilon_1.
\]

(78)

- Since

\[
\begin{pmatrix} V_{1,j} \\ V_{2,j} \end{pmatrix} = \left( \begin{array}{c} \langle e_j, R_{\theta}^{-1} v_1 \rangle \\ \langle e_j, R_{\theta}^{-1} v_2 \rangle \end{array} \right),
\]

we write

\[
\| \begin{pmatrix} V_{1,j} \\ V_{2,j} \end{pmatrix} \|_H \leq \| R_{\theta}^{-1} v - \left( \begin{array}{c} \kappa(d, .) e_1 \\ 0 \end{array} \right) \|_H \leq C \epsilon_1
\]

(79)

by the same argument as for (75). Using the Cauchy-Schwarz inequality together with (50), we see from (69) that

\[
|\partial_\theta \bar{\Phi}_j| \leq \frac{C \epsilon_1}{1 - d^2}.
\]

(80)

From (63), we see that \( \frac{\partial R_{\theta}^{-1}}{\partial \theta_i} = A_i R_{\theta}^{-1} \). Therefore using ii) of Lemma 3.2 and the fact that the rotation \( R_{\theta} \) does not change the norm in \( H \), we write

\[
\| \frac{\partial R_{\theta}^{-1}}{\partial \theta_i} (v) - \left( \begin{array}{c} \kappa(d, .) A_i(e_1) \\ 0 \end{array} \right) \|_H \leq \| v - R_{\theta} \left( \begin{array}{c} \kappa(d, .) e_1 \\ 0 \end{array} \right) \|_H \leq C \epsilon_1,
\]

(81)

by the same argument as for (75). Using the Cauchy-Schwarz identity together with (37), we see from (70) that

\[
|\partial_\theta \bar{\Phi}_j| \leq C \epsilon_1.
\]

(82)

- By the same argument as for (82), we obtain from (71)

\[
|\partial_\theta \bar{\Phi}_j| \leq C \epsilon_1 \text{ if } i \neq j.
\]

(83)
Now, if \( i = j \), noting from (63) that \( \frac{\partial R_{\tau_1}^{-1}}{\partial y_i}v = A_iR_{\tau_1}^{-1}(v) = A_iV \), applying the operator \( A_i \) to (77), then taking the scalar product with \( e_i \), we see from Lemma 3.2 that

\[
\left\| \begin{pmatrix} \langle e_i, \frac{\partial R_{\tau_1}^{-1}}{\partial y_i}v_1 \rangle \\ \langle e_i, \frac{\partial R_{\tau_1}^{-1}}{\partial y_i}v_2 \rangle \end{pmatrix} - \left( \kappa(d_i,.) \prod_{k=i+1}^{m} \cos \theta_k \right) \right\|_H \leq C \epsilon_1.
\]

Since we know from (44) and (51) that

\[
\phi(\kappa(d,.)\dot{W}_0^d) = 1,
\]

it follows from (71) that

\[
\left| \partial \phi_i - \prod_{k=i+1}^{m} \cos \theta_k \right| \leq C \epsilon_1. 
\]

Collecting (78), (80), (82), (83) and (84) we see that

\[
|D - \frac{2\kappa_0}{(p-1)(1-d^2)} - \cos \theta_3(\cos \theta_4)^2...(\cos \theta_m)^{m-2}| \leq \frac{C \epsilon_1}{1-d^2}.
\]

Since

\[
\cos \theta_i \geq \frac{3}{4}
\]

by hypothesis, we have the non-degeneracy of \( \Phi \) (voir (65)) near the point \((\dot{v}, \hat{d}, \hat{\theta})\) with \( \dot{v} = R_{\eta}(\kappa(\hat{d},.)e_1,0) \). Applying the implicit function theorem, we conclude the proof of Proposition 3.1.

### 3.5 Dynamics of \( q, d \) and \( \theta \)

Let us apply Proposition 3.1 with \( v = (w, \partial_s w)(s^*) \), \( \hat{d} = d^* \) and \( \hat{\theta} = 0 \). Clearly, from (67) and (58), we have \( ||\dot{q}||_H \leq \epsilon^* \). Assuming that

\[
\epsilon^* \leq \epsilon_0
\]

defined in Proposition 3.1, we see that the proposition applies, and from the continuity of \( (w, \partial_s w) \) from \([s^*, \infty)\) to \( \mathcal{H} \), we have a maximal \( \bar{s} > s^* \), such that \( (w(s), \partial_s w(s)) \) can be modulated in the sense that

\[
\begin{pmatrix}
  w(y,s) \\
  \partial_s w(y,s)
\end{pmatrix}
= R_{\theta(s)} \left[ \begin{pmatrix}
  \kappa(d(s),y)e_1 \\
  0
\end{pmatrix} + \begin{pmatrix}
  q_1(y,s) \\
  q_2(y,s)
\end{pmatrix} \right],
\]

where the parameters \( d(s) \in (-1,1) \) and \( \theta(s) = (\theta_2(s),...,\theta_m(s)) \) are such that for all \( s \in [s^*, \bar{s}] \)

\[
\pi_0^{-d(s)} \begin{pmatrix}
  q_{1,1}(s) \\
  q_{2,1}(s)
\end{pmatrix} = 0, \text{ and } \pi_0^{-d(s)} \begin{pmatrix}
  q_{1,j}(s) \\
  q_{2,j}(s)
\end{pmatrix} = 0, \forall j = 2,..m,
\]

and

\[
\forall i = 2,..,m \cos \theta_i(s) \geq \frac{1}{2} \text{ and } ||q(s)||_H \leq \epsilon \equiv 2K_0K_1\epsilon^*,
\]

where \( K_1 > 0 \) is defined in Proposition 3.1 and \( K_1 > 1 \) is a constant that will be fixed below in (141).
Two cases then arise:
- Case 1: \( \bar{s} = +\infty \);
- Case 2: \( \bar{s} < +\infty \); in this case, we have an equality case in (87), i.e. \( \cos \theta_i(\bar{s}) = \frac{1}{2} \) for some \( i = 2, \ldots, m \), or \( ||q(\bar{s})||_H = 2K_0K_1 \varepsilon^4 \).

At this stage, we see that controlling the solution \((w(s), \partial_s w(s)) \in \mathcal{H}\) is equivalent to controlling \(q \in \mathcal{H}, d \in (-1, 1)\) and \(\theta(s) \in \mathbb{R}^{m-1}\).

Before giving the dynamics of this parameters, we need to introduce some notations.

From (86), we will expand \( \bar{q} \) and \( \bar{\bar{q}} \) respectively according to the spectrum of the linear operators \( \bar{L}_d \) and \( \bar{L}_d \) as in (40) and (53):

\[
\begin{align*}
\forall j \in \{1, \ldots, m\}, \quad \begin{pmatrix} q_{1,j}(y, s) \\ q_{2,j}(y, s) \end{pmatrix} &= \begin{pmatrix} q_{-1,1}(y, s) \\ q_{-2,1}(y, s) \end{pmatrix} + \begin{pmatrix} q_{-1,j}(y, s) \\ q_{-2,j}(y, s) \end{pmatrix} \\ \alpha_{1,j} &= \pi_{d_1}^{\bar{q}}(q_{1,1}) - \bar{q}^{\bar{d}}(q_{1,2}) \\ \alpha_{0,j} &= \bar{q}^{\bar{d}}(q_{2,1}) - \bar{q}^{\bar{d}}(q_{2,2}) \\ \alpha_{1,j} &= \pi_{d_1}^{-}\bar{q}(q_{1,1}) - \bar{q}^{-d}(q_{1,2}) \\ \alpha_{0,j} &= \bar{q}^{-d}(q_{2,1}) - \bar{q}^{-d}(q_{2,2}) \\ \forall j \in \{1, \ldots, m\}, \quad \alpha_{1,j}(s) &= \sqrt{\tilde{\varphi}_d\left(\begin{pmatrix} q_{1,1} \\ q_{2,1} \end{pmatrix}, \begin{pmatrix} q_{1,1} \\ q_{2,1} \end{pmatrix}\right)} \quad \alpha_{0,j}(s) = \sqrt{\tilde{\varphi}_d\left(\begin{pmatrix} q_{1,1} \\ q_{2,1} \end{pmatrix}, \begin{pmatrix} q_{1,1} \\ q_{2,1} \end{pmatrix}\right)} \\ \alpha_{1,j}(s) &= \sqrt{\tilde{\varphi}_d\left(\begin{pmatrix} q_{1,1} \\ q_{2,1} \end{pmatrix}, \begin{pmatrix} q_{1,1} \\ q_{2,1} \end{pmatrix}\right)} \quad \alpha_{0,j}(s) = \sqrt{\tilde{\varphi}_d\left(\begin{pmatrix} q_{1,1} \\ q_{2,1} \end{pmatrix}, \begin{pmatrix} q_{1,1} \\ q_{2,1} \end{pmatrix}\right)}
\end{align*}
\]

From (88), (89), (12), (38) and (56), we see that for all \( s \geq s_0 \),

\[
\begin{align*}
\frac{1}{C_0} \alpha_{1,1}(s) &\leq \| \begin{pmatrix} q_{1,1} \\ q_{2,1} \end{pmatrix} \|_\mathcal{H} \leq C_0 \alpha_{1,1}(s) \\
\frac{1}{C_0} \| \alpha_{1,1}(s) + \alpha_{1,1}(s) \| &\leq \| \begin{pmatrix} q_{1,1} \\ q_{2,1} \end{pmatrix} \|_\mathcal{H} \leq C_0 (|\alpha_{1,1}(s)| + \alpha_{1,1}(s)) \\
\frac{1}{C_0} \alpha_{1,j}(s) &\leq \| \begin{pmatrix} q_{1,j} \\ q_{2,j} \end{pmatrix} \|_\mathcal{H} \leq C_0 \alpha_{1,j}(s) \\
\frac{1}{C_0} \alpha_{1,j}(s) &\leq \| \begin{pmatrix} q_{1,j} \\ q_{2,j} \end{pmatrix} \|_\mathcal{H} \leq C_0 \alpha_{1,j}(s)
\end{align*}
\]

for some \( C_0 > 0 \). In the following proposition, we derive from (103) and (104) differential inequalities satisfied by \( \alpha_{1,1}(s), \alpha_{1,j}(s), \alpha_{1,j}(s), \theta_i(s) \) and \( d(s) \). Introducing

\[
R_- (s) = - \int_{-1}^{1} \mathcal{F}_d(q_1) \rho dy,
\]

where

\[
\mathcal{F}_d(q_1(y, s)) = \frac{\kappa(d, \cdot)v_1 + q_1^{p+1}}{p+1} - \frac{\kappa(d, \cdot)v_1 + q_1^{p+1}}{p+1} - \frac{\kappa(d, \cdot)v_1 + q_1^{p+1}}{p+1} - \frac{\kappa(d, \cdot)v_1 + q_1^{p+1}}{p+1} + \frac{\kappa(d, \cdot)v_1^{p-1}}{2} \sum_{j=2}^{m} q_{1,j}^{2} + \frac{m}{2} q_{1,j}^{2},
\]

we claim the following
Proposition 3.3. (Dynamics of the parameters) For $\epsilon^*$ small enough and for all $s \in [s^*, \bar{s})$, we have:

(i) (Control of the modulation parameter)
\[
\sum_{i=2}^{m} |\theta_i'| + \frac{|d'|}{1 - d'^2} \leq C_0 ||q||^2_{L^2}. \tag{95}
\]

(ii) (Projection of equation (103) on the different eigenspaces of $\tilde{L}_d$ and $\tilde{L}_d$)
\[
|\alpha_1(s) - \alpha_1(s)| \leq C_0 ||q||^2_{L^2}.
\]
\[
(R_+ + \frac{1}{2}(\alpha_1 + \alpha_{2,j}))' \leq -\frac{4}{p-1} \int_{-1}^{1} (q_{-2,1} + q_{-2,j}) \frac{\rho}{1 - y^2} dy + C_0 ||q||^3_{L^2}, \tag{97}
\]
for $j \in \{2, ..., m\}$ and $R_-$ defined in (93), satisfying
\[
|R_-(s)| \leq C_0 ||q||^{1+p}_{L^2} \text{ where } p = \min(p, 2) > 1. \tag{98}
\]

(iii) (An additional relation)
\[
\frac{d}{ds} \int_{-1}^{1} q_{1,1} q_{2,1} \rho \leq -\frac{4}{5} \alpha_{-1}^2 + C_0 \int_{-1}^{1} q_{-2,1} \frac{\rho}{1 - y^2} + C_0 ||q||^2_{L^2}. \tag{99}
\]
For $j \in \{2, ..., m\}$, we have:
\[
\frac{d}{ds} \int_{-1}^{1} q_{1,j} q_{2,1} \rho \leq -\frac{4}{5} \alpha_{-j}^2 + C_0 \int_{-1}^{1} q_{-2,j} \frac{\rho}{1 - y^2} + C_0 ||q||^2_{L^2}. \tag{100}
\]

(iv) (Energy barrier)
\[
\alpha_1(s) \leq C_0 \alpha_{-1}(s) + C_1 \sum_{j=2}^{m} \alpha_{-j}(s). \tag{101}
\]

Proof. The proof follows the general framework developed by Merle and Zaag in the real case (see Proposition 5.2 in [18]), then adapted to the complex-valued case in [4] (see Proposition 4.2 page 5915 in [4]). However, new ideas are needed, mainly because we have $(m - 1)$ rotation parameters in the modulation technique (see Proposition 3.1 above), rather than only one in the complex-valued case. For that reason, in the following, we give details only for the "new" terms, referring the reader to the earlier literature for the "old" terms.

Let us first write an equation satisfied by $q$ defined in (85). We put the equation (4) satisfied by $w$ in vectorial form:
\[
\partial_s w_1 = w_2, \quad \partial_s w_2 = \mathcal{L} w_1 - \frac{2(p+1)}{p-1} w_1 + |w_1|^{p-1} w_1 - \frac{p+3}{p-1} w_2 - 2y \partial_y w_2. \tag{102}
\]
We replace all the terms of (102) by their expressions from (85). Precisely, for the terms of the right hand side of (102) we have:
\[
\partial_s w_1 = R_{\theta} (d \partial_d k e_1 + \partial_s q_1) + \sum_{i=1}^{m} \theta_i \frac{\partial R_{\theta}}{\partial \theta_i} (\kappa_d e_1 + q_1),
\]
\[
\partial_s w_2 = R_{\theta} (\partial_s q_1) + \sum_{i=1}^{m} \theta_i \frac{\partial R_{\theta}}{\partial \theta_i} (q_2).
\]
For the terms on the left hand side of (102) we have:
\[
w_2 = R_{\theta}(q_2), \quad \mathcal{L} w_1 = R_{\theta}(\mathcal{L}(\kappa_d e_1) + \mathcal{L} q_1), \quad |w_1| = |\kappa_d e_1 + q_1|, \quad \partial_y w_2 = R_{\theta} \partial_y q_2.
\]
Then, multiplying by \( R_g^{-1} \), using the fact that \((\kappa(d, \cdot), 0)\) is a stationary solution and dissociating the first and \(j\)th component of these equations, we get for all \(s \in [s^*, \bar{s})\), for all \(j \in \{2, ..., m\}:

\[
\frac{\partial}{\partial s} \begin{pmatrix} q_{1,1} \\ q_{2,1} \end{pmatrix} = \bar{L}_{d(s)} \begin{pmatrix} q_{1,1} \\ q_{2,1} \end{pmatrix} + \begin{pmatrix} 0 \\ f_{d(s),1}(q_1) \end{pmatrix} - d'(s) \begin{pmatrix} \partial_d \kappa(d, y) \\ 0 \end{pmatrix} - \sum_{i=2}^m \theta_i'(s) \begin{pmatrix} a_{i,1,1} \\ a_{i,2,1} \end{pmatrix},
\]

\[
\frac{\partial}{\partial s} \begin{pmatrix} q_{1,j} \\ q_{2,j} \end{pmatrix} = \bar{L}_{d(s)} \begin{pmatrix} q_{1,j} \\ q_{2,j} \end{pmatrix} + \begin{pmatrix} 0 \\ f_{d(s),j}(q_1) \end{pmatrix} - \sum_{i=2}^m \theta_i'(s) \begin{pmatrix} a_{i,1,j} \\ a_{i,2,j} \end{pmatrix},
\]

where \(\bar{L}_{d(s)}, f_{d(s),1}\) and \(f_{d(s),j}\) are defined in (31), (33), (28) and (29), and \(a_i\) by

\[
\begin{pmatrix} a_{i,1} \\ a_{i,2} \end{pmatrix} = \frac{A_i(\kappa d e_1 + q_1)}{A_i(q_2)},
\]

(105)

with \(a_{i,1} = (a_{i,1,1}, a_{i,1,2}, ..., a_{i,1,m}) \in \mathbb{R}^m\) and \(a_{i,2} = (a_{i,2,1}, a_{i,2,2}, ..., a_{i,2,m}) \in \mathbb{R}^m\).

Let \(i \in \{2, ..., m\}\), Projecting equation (103) with the projector \(\bar{\pi}_\lambda^d (39)\) for \(\lambda = 0\) and \(\lambda = 1\), we write

\[
\bar{\pi}_\lambda^d (\partial_s \begin{pmatrix} q_{1,1} \\ q_{2,1} \end{pmatrix}) = \bar{\pi}_\lambda^d (\bar{L}_{d(s)} \begin{pmatrix} q_{1,1} \\ q_{2,1} \end{pmatrix}) + \bar{\pi}_\lambda^d \begin{pmatrix} 0 \\ f_{d(s),1}(q_1) \end{pmatrix} - d'(s) \bar{\pi}_\lambda^d \begin{pmatrix} \partial_d \kappa(d, y) \\ 0 \end{pmatrix} - \sum_{i=2}^m \theta_i'(s) \bar{\pi}_\lambda^d \begin{pmatrix} a_{i,1,1} \\ a_{i,2,1} \end{pmatrix}.
\]

(106)

Note that, expect the last term, all the terms of (106) can be controled exactly like the real case using (87) (for details see page 105 in [18]). So, we recall that we have:

\[
|\bar{\pi}_\lambda^d (\partial_s \begin{pmatrix} q_{1,1} \\ q_{2,1} \end{pmatrix}) - \alpha'_{\lambda,1}| \leq \frac{C_0}{1 - d^2} ||d'|| ||q||_\mathcal{H},
\]

(107)

\[
\bar{\pi}_\lambda^d (\bar{L}_{d} \begin{pmatrix} q_{1,1} \\ q_{2,1} \end{pmatrix}) = \lambda \alpha_{\lambda,1},
\]

(108)

\[
|\bar{\pi}_\lambda^d \begin{pmatrix} 0 \\ f_{d(s),1}(q_1) \end{pmatrix}| \leq C_0 ||q||_\mathcal{H}^2,
\]

(109)

\[
|\bar{\pi}_\lambda^d \begin{pmatrix} \partial_d \kappa(d, y) \\ 0 \end{pmatrix}| = -\frac{2\kappa_0}{(p - 1)(1 - d^2)} \bar{\pi}_\lambda^d (F_0^d) = -\frac{2\kappa_0}{(p - 1)(1 - d^2)} \delta_{\lambda,0}.
\]

(110)

Now, we focus on the study of the last term of (106). From the definition of \(a_i\) (105) and \(i\) of Lemma 3.2 we have:

\[
\begin{pmatrix} a_{i,1,1} \\ a_{i,2,1} \end{pmatrix} = \kappa_d \begin{pmatrix} < e_1, A_i(e_1) > \\ 0 \end{pmatrix} + \begin{pmatrix} < e_1, A_i(q_1) > \\ < e_1, A_i(q_2) > \end{pmatrix} = \begin{pmatrix} < e_1, A_i(q_1) > \\ < e_1, A_i(q_2) > \end{pmatrix}.
\]

Applying the projector \(\bar{\pi}_\lambda^d (39)\), we get

\[
|\bar{\pi}_\lambda^d \begin{pmatrix} a_{i,1,1} \\ a_{i,2,1} \end{pmatrix}| = |\bar{\pi}_\lambda^d \begin{pmatrix} < e_1, A_i(q_1) > \\ < e_1, A_i(q_2) > \end{pmatrix}| \leq C ||< e_1, A_i(q_1) > ||_\mathcal{H} + ||< e_1, A_i(q_2) > ||_{L_0^2}.
\]

(111)
Using \( \text{ii)} \) of Lemma 3.2, we have:

\[
\| < e_1, A_1(q_2) > \|_{L^2_{\nu}}^2 = \int_{-1}^{1} < e_1, A_1(q_2) >^2 \, \rho dy \leq \int_{-1}^{1} |A_1(q_2)|^2 \, \rho dy \leq \int_{-1}^{1} |q_2|^2 \, \rho dy,
\]

(112)

and by the same way, using \( \text{ii)} \) of Lemma 3.2 and the definition of \( \mathcal{H}_0 \) (7), we have

\[
\| < e_1, A_1(q_1) > \|_{\mathcal{H}_0}^2 = \int_{-1}^{1} < e_1, A_1(q_1) >^2 \, \rho dy + \int_{-1}^{1} (|e_1, A_1(\partial_y q_1)|^2 (1 - y^2) \, \rho dy
\]

\[
\leq \int_{-1}^{1} (|q_1|^2 + (1 - y^2)|\partial_y q_1|^2) \, \rho dy.
\]

(113)

From (111), (112) and (113), we have

\[
|\tilde{\alpha}^d \left( \frac{a_{i,1,1}}{a_{i,2,1}} \right)| \leq C_0 \|q\|_{\mathcal{H}}.
\]

(114)

Using (107), (107), (108), (109), (110), (111), and the fact that \( \alpha_{0,1} \equiv \alpha'_{0,1} \equiv 0 \) (see (90)), we get for \( \lambda = 0, 1 \):

\[
\frac{2\kappa_0}{(p - 1)(1 - d^2)} |d'||q||_{\mathcal{H}} + C_0 \|q\|^2_{\mathcal{H}} + C_0 \|q\|_{\mathcal{H}} \sum_{j=2}^{m} |\theta'_j|.
\]

(115)

\[
|\tilde{\alpha}_1^d(s) - \tilde{\alpha}_1(s)| \leq \frac{C_0}{1 - d^2} |d'||q||_{\mathcal{H}} + C_0 \|q\|^2_{\mathcal{H}} + C_0 \|q\|_{\mathcal{H}} \sum_{j=2}^{m} |\theta'_j|.
\]

(116)

Now, projecting equation (104) with the projector \( \tilde{\pi}_0^d \) (52), where \( j \in \{2, \cdots, m\} \), we get:

\[
\tilde{\pi}_0^d(\partial_s \left( \frac{q_{1,j}}{q_{2,j}} \right)) = \tilde{\pi}_0^d(\tilde{L}_d(s) \left( \frac{q_{1,j}}{q_{2,j}} \right)) + \tilde{\pi}_0^d \left( \frac{0}{f_{d(s),j}(q_1)} \right) - \sum_{i=2}^{m} \theta'_i(s) \tilde{\pi}_0^d \left( \frac{a_{i,1,j}}{a_{i,2,j}} \right).
\]

(117)

From the complex-valued case we recall that we have (for details see page 5917 in [4], together with Lemma 3.2):

\[
|\tilde{\pi}_0^d(\partial_s \left( \frac{q_{1,j}}{q_{2,j}} \right))| \leq \frac{C_0}{1 - d^2} |d'||q||_{\mathcal{H}},
\]

(118)

\[
\tilde{\pi}_0^d(\tilde{L}_d \left( \frac{q_{1,j}}{q_{2,j}} \right)) = 0,
\]

(119)

\[
|\tilde{\pi}_0^d \left( \frac{0}{f_{d(s),j}(q_1)} \right)| \leq C_0 |q||_{\mathcal{H}},
\]

(120)

\[
\tilde{\pi}_0^d \left( \frac{\kappa_d}{0} \right) = 1.
\]

(121)

Thus, only the last term in (117) remains to be treated in the following.

From the definition of \( a_i \) (105), we recall that

\[
\left( \frac{a_{i,1,j}}{a_{i,2,j}} \right) = \left( \begin{array}{c} < e_j, A_i(e_1) > \\ 0 \end{array} \right) + \left( \begin{array}{c} < e_j, A_i(q_1) > \\ < e_j, A_i(q_2) > \end{array} \right).
\]

(122)

By \( \text{i)} \) of Lemma 3.2

\[
\sum_{i=2}^{m} \theta'_i(s) \left( \frac{a_{i,1,j}}{a_{i,2,j}} \right) \theta'_j(s) \left( \prod_{l=j+1}^{m} \cos \theta_l \kappa_d \right) + \sum_{i=2}^{m} \theta'_i(s) \left( < e_j, A_i(q_1) > \\ < e_j, A_i(q_2) > \right),
\]

(123)
where by convention $\prod_{l=m+1}^{m} \cos \theta_l = 1$ if $j = m$.

Applying the projection $\tilde{\pi}_0^d$ to (123) and using (121), we see that

$$\left| \sum_{i=2}^{m} \theta_j(s) \tilde{\pi}_0^d \left( \frac{a_{i,1,j}}{\alpha_{i,2,j}} \right) - \theta_j(s) \prod_{l=j+1}^{m} \cos \theta_l \right| \leq \sum_{i=2}^{m} \left| \theta_j(s) \right| \left| \tilde{\pi}_0^d \left( \langle e_j, A_i(q_1) \rangle \right) \right|$$

$$\leq C_0 ||q||_H \sum_{i=2}^{m} |\theta_j(s)|,$$

where, we use the fact that

$$\left| \tilde{\pi}_0^d \left( \langle e_j, A_i(q_1) \rangle \right) \right| \leq C_0 ||q||_H,$$

which follows by the same techniques as in (111) (112) and (113).

Using (117), (118), (119), (120) and (124), and recalling from (87) that

$$\prod_{l=j+1}^{m} \cos \theta_l \geq (\frac{1}{2})^{m-j},$$

we get for any $j \in \{2, \ldots, m\}$:

$$\left| \theta_j(s) \right| \leq \frac{C_0}{1 - d^2} |d'||q||_H + C_0 ||q||_H^2 + C_0 ||q||_H \sum_{i=2}^{m} |\theta_i|,$$

Using (126) together with (115), we see that

$$\sum_{j=2}^{m} \theta_j(s) + \frac{|d'|}{1 - d^2} \leq C_0 \frac{|d'|}{1 - d^2} ||q||_H + C_0 ||q||_H^2 + C_0 ||q||_H \sum_{i=2}^{m} |\theta_i|,$$

Thus, using again (87) and taking $\epsilon$ small enough, we get

$$\sum_{j=2}^{m} \theta_j(s) + \frac{|d'|}{1 - d^2} \leq C_0 ||q||_H^2,$$

which yields (95). Then, using (116) together with (95) gives (96).

For estimations (98) (99) (100) (101), the study in the complex case (Subsection 4.3 page 5914 in [4]) can be adapted without any difficulty to the vector-valued case. For the reader convenience, we detail for example the energy barrier (101):

Using the definition of $q(y, s)$ (52), we can make an expansion of $E(w(s), \partial_s w(s))$ (8) for $q \to 0$ in $H$ and get after from straightforward computations:

$$E(w(s), \partial_s w(s)) = E(\kappa_0, 0) + \frac{1}{2} \left[ \tilde{\varphi}_d \left( \left( q_{1,1}^{1,1}, q_{1,1}^{2,1} \right) \right) + \sum_{i=2}^{m} \tilde{\varphi}_d \left( \left( q_{1,j}^{1,j}, q_{2,j}^{2,j} \right) \right) \right] - \int_{-1}^{1} F_d(q_1) \rho dy$$

where $\tilde{\varphi}_d, \varphi_d$ and $F_d(q_1)$ are defined in (11), (55) and (94).
Using the argument in the real case (see page 113 in [18]) we see that for some $C_0, C_1 > 0$ we have:

$$\varphi_d\left(\frac{q_{1,1}}{q_{2,1}}, \frac{q_{1,1}}{q_{2,1}}\right) \leq C_0\alpha_{1,1}^2 - C_1\alpha_{-1}^2. \quad (128)$$

From (89), (91) and (50), we see by definition that

$$0 \leq \varphi_d\left(\frac{q_{1,j}}{q_{2,j}}, \frac{q_{1,j}}{q_{2,j}}\right) = \alpha_{-j}^2. \quad (129)$$

Since we have from (93), (98), (87), (129) and (92):

$$\left|\int_{-1}^{1} F_d(q_1)p\,dy\right| \leq C\|q(s)\|_{\mathcal{H}}^{\beta+1} \leq C\epsilon^{\beta-1}(\alpha_{1,1}^2 + \alpha_{-1}^2 + \sum_{i=2}^{m} \alpha_{-j}^2), \quad (130)$$

Using (11), (127), (128) and (130), we see that taking $\epsilon$ small enough so that $C\epsilon^{\beta-1} \leq \frac{C_1}{4}$, we get

$$0 \leq E(w(s), \partial_s w(s)) - E(\kappa_0, 0) \leq \left(\frac{C_0}{2} + \frac{C_1}{4}\right)\alpha_{-1}^2 - \frac{C_1}{4}\alpha_{1,1}^2 + \left(\frac{1}{2} + \frac{C_1}{4}\right)\sum_{i=2}^{m} \alpha_{-j}^2,$$

which yields (101).

### 3.6 Exponential decay of the different components

Our aim is to show that $\|q(s)\|_{\mathcal{H}} \to 0$ and that both $\theta$ and $d$ converge as $s \to \infty$. An important issue will be to show that the unstable mode $\alpha_{1,1}$, which satisfies equation (92) never dominates. This is true thanks to item (iv) in Proposition 3.3.

If we introduce

$$\lambda(s) = \frac{1}{2}\log\left(\frac{1 + d(s)}{1 - d(s)}\right), a(s) = \alpha_{1,1}(s)^2 \text{ and } b(s) = \alpha_{-1}(s)^2 + \sum_{j=2}^{m} \alpha_{-j}(s)^2 + R_-(s) \quad (131)$$

(note that $d(s) = \tanh(\lambda(s))$), then we see from (88), (82) and (77) that for all $s \in [s^*, \bar{s}]$

$$|R_-(s)| = |b(s) - (\alpha_{-1}(s)^2 + \sum_{j=2}^{m} \alpha_{-j}(s)^2)| \leq C_0\epsilon^{\beta-1}(\alpha_{1,1}(s)^2 + \alpha_{-1}(s)^2 + \sum_{j=2}^{m} \alpha_{-j}(s)^2),$$

hence

$$\frac{99}{100}\alpha_{-1}(s)^2 + \frac{99}{100}\sum_{j=2}^{m} \alpha_{-j}(s)^2 - \frac{1}{100}a \leq b \leq \frac{101}{100}\alpha_{-1}(s)^2 + \sum_{j=2}^{m} \alpha_{-j}(s)^2 + \frac{1}{100}a \quad (132)$$

for $\epsilon$ small enough. Therefore, using Proposition 3.3, estimate (77), (82) and the fact that $\lambda'(s) = \frac{d'(s)}{1 - d(s)^2}$, we derive the following:

**Claim 3.4. (Relations between $a$, $b$, $\lambda$, $\theta$, $\int_{-1}^{1} q_{1,1}q_{2,1}\rho$ and $\int_{-1}^{1} q_{1,j}q_{2,j}\rho$)** There exist positive $\epsilon_4$, $K_4$ and $K_5$ such that if $\epsilon^* \leq \epsilon_4$, then we have for all $s \in [s^*, \bar{s}]$ and $j = 2, \ldots, m$:

(i) (Size of the solution)

$$\frac{1}{K_4}(a(s) + b(s)) \leq \|q(s)\|_{\mathcal{H}}^2 \leq K_4(a(s) + b(s)) \leq K_4^2\epsilon^2, \quad (133)$$

$$|\theta'(s)| + |\lambda'(s)| \leq K_4(a(s) + b(s)) \leq K_4^2\|q(s)\|_{\mathcal{H}}^2, \quad (134)$$

$$\left|\int_{-1}^{1} q_{1,1}q_{1,1}\rho\right| \leq K_4(a(s) + b(s)), \quad (135)$$

$$\left|\int_{-1}^{1} q_{1,j}q_{1,j}\rho\right| \leq K_4 b(s), \quad (136)$$

and (132) holds.
(ii) (Equations)
\[ \frac{3}{2} a - K_4 \epsilon b \leq a' \leq \frac{5}{2} a - K_4 \epsilon b, \]

\[ y' \leq -\frac{8}{p-1} \int_{-1}^{1} (q_{-2,1}^2 + q_{-2,j}^2) \frac{\rho}{1 - y^2} dy + K_4 \epsilon (a + b), \]

\[ \frac{d}{ds} \int_{-1}^{1} (q_{1,1} q_{2,1} + q_{1,j} q_{2,j}) \rho \leq -\frac{3}{5} b + K_4 \int_{-1}^{1} (q_{-2,1}^2 + q_{-2,j}^2) \frac{\rho}{1 - y^2} + K_4 a. \]

(iii) (Energy barrier) we have
\[ a(s) \leq K_5 b(s). \]

End of the Proof of Theorem 4. Now, we are ready to finish the proof of Theorem 4 just started at the beginning of Section 3.3. Let us define \( s_2 \in [s^*, \bar{s}] \) as the first \( s \in [s^*, \bar{s}] \) such that
\[ a(s) \geq \frac{b(s)}{5K_4}, \]
where \( K_4 \) is introduced in Corollary 3.4, or \( s_2^* = \bar{s} \) if (139) is never satisfied on \([s^*, \bar{s}]\). We claim the following:

Claim 3.5. There exist positive \( \epsilon_6, \mu_6, K_6 \) and \( f \in C^1([s^*, s_2^*]) \) such that if \( \epsilon \leq \epsilon_6 \), then for all \( s \in [s^*, s_2^*] \):

(i) \[ \frac{1}{2} f(s) \leq b(s) \leq 2f(s) \text{ and } f'(s) \leq -2\mu_6 f(s), \]

(ii) \[ ||q(s)||_{\mathcal{H}} \leq K_6 ||q(s^*)||_{\mathcal{H}} e^{-\mu_6(s-s^*)} \leq K_6 K_1 e^{-\mu_6(s-s^*}). \]

Proof. The proof of Claim 5.6 page 115 in [18] remains valid where \( f(s) \) is given by
\[ f(s) = b(s) + \eta_6 \int_{-1}^{1} (q_{1,1} q_{2,1} + \sum_{j=2}^{m} q_{1,j} q_{2,j}) \rho, \]
where \( \eta_6 > 0 \) is fixed small independent of \( \epsilon \).

Claim 3.6. (i) There exists \( \epsilon_7 > 0 \) such that for all \( \sigma > 0 \), there exists \( K_7(\sigma) > 0 \) such that if \( \epsilon \leq \epsilon_7 \), then
\[ \forall s \in [s_2^*, \min(s_2^* + \sigma, \bar{s})], ||q(s)||_{\mathcal{H}} \leq K_7 ||q(s^*)||_{\mathcal{H}} e^{-\mu_6(s-s^*)} \leq K_7 K_1 e^{-\mu_6(s-s^*)} \]
and
\[ |\theta_i(s)| \leq C \frac{(K_7 K_1 \epsilon^s)^2}{2\mu_6} \]
where \( \mu_6 \) has been introduced in Claim 3.5.

(ii) There exists \( \epsilon_8 > 0 \) such that if \( \epsilon \leq \epsilon_8 \), then
\[ \forall s \in (s_2^*, \bar{s}), b(s) \leq a(s) \left( 5K_4 e^{-\frac{(s-s^*)^2}{2}} + \frac{1}{4K_5} \right) \]
where \( K_4 \) and \( K_5 \) have been introduced in Corollary 3.4.

Proof. The proof is the same as the proof of Claim 5.7 page 117 in [18].
Now, in order to conclude the proof of Theorem 4, we fix \( \sigma_0 > 0 \) such that
\[
5K_4^{2\alpha} + \frac{1}{4K_5} \leq \frac{1}{2K_5},
\]
where \( K_4 \) and \( K_5 \) are introduced in Claim 3.4. Then, we fix the value of
\[
K_0 = \max(2, K_6, K_7(\sigma_0)),
\]
and the constants are defined in Claims 3.5 and 3.6. Then, we fix
\[
\epsilon_0 = \min \left( 1, \epsilon_1, \frac{\epsilon_1}{2K_0K_1} \right) \quad \text{for } i \in \{4, 6, 7, 8\}
\]
and the constants are defined in Claims 3.4, 3.5 and 3.6. Now, if \( \epsilon^* \leq \epsilon_0 \), then Claim 3.4, Claim 3.5 and Claim 3.6 apply. We claim that for all \( s \in [s^*, \bar{s}] \),
\[
\|q(s)\|_{\mathcal{H}} \leq K_0\|q(s^*)\|_{\mathcal{H}}e^{\mu_\epsilon(s-s^*)} \leq K_0K_1e^{\mu_\epsilon(s-s^*)} = \frac{\epsilon}{2}e^{-\mu_\epsilon(s-s^*)}.
\]
Indeed, if \( s \in [s^*, \min(s_2^* + \sigma_0, \bar{s})] \), then this comes from (ii) of Claim 3.5 or (i) of Claim 3.6 and the definition of \( K_0 \).

Now, if \( s_2^* + \sigma_0 < \bar{s} \) and \( s \in [s_2^* + \sigma_0, \bar{s}] \), then we have from (140) and the definition of \( \sigma_0 \),
\[
b(s) \leq \frac{a(s)}{2K_5} \quad \text{on the one hand.}
\]
On the other hand, from (iii) in Claim 3.4 we have \( a(s) \leq K_3b(s) \), hence, \( a(s) = b(s) = 0 \) and from (133), \( q(y, s) \equiv 0 \), hence (142) is satisfied trivially.

In particular, we have for all \( s \in [s^*, \bar{s}] \), \( \|q\|_{\mathcal{H}} \leq \frac{\epsilon}{2} \) and \( \cos \theta_i \geq 1 - C_4\frac{\mu_\epsilon}{\mu_0} \geq \frac{3}{4} \), hence, by definition of \( \bar{s} \) given right before (85), this means that \( \bar{s} = \infty \).

From (i) of Claim 3.6 and (133), we have
\[
\forall s \geq s^*, \|q(s)\|_{\mathcal{H}} + |\theta(s)| + |\lambda(s)| \leq K_0^2 e^{2\mu_\epsilon(s-s^*)},
\]
where \( \theta(s) = (\theta_2(s), ..., \theta_m(s)) \).

defintion of \( \lambda_\infty \) and \( \lambda_\infty - \lambda(s) \).

Hence, there is \( \theta_\infty \in \mathbb{R}^{m-1} \), \( \lambda_\infty \) in \( \mathbb{R} \) such that \( \theta(s) \to \theta_\infty \), \( \lambda(s) \to \lambda_\infty \) as \( s \to \infty \) and
\[
\forall s \geq s^*, |\lambda_\infty - \lambda(s)| \leq C_1\epsilon^2 e^{-2\mu_\epsilon(s-s^*)} = C_2\epsilon^2 e^{-2\mu_\epsilon(s-s^*)}
\]
for some positive \( C_1 \) and \( C_2 \). Taking \( s = s^* \) here, we see that
\[
|\lambda_\infty - \lambda^*| + |\theta_\infty| \leq C_0\epsilon^*,
\]
where \( \Omega = R_{\theta_\infty}(e_1) \). If \( d_\infty = \tanh \lambda_\infty \), then we see that \( |d_\infty - d^*| \leq C_3(1 - d^2)\epsilon^* \).

We use the definition of \( q \) (85), (143), (144) and (145) we write
\[
\left\| \begin{pmatrix} w(s) \\ \partial_s w(s) \end{pmatrix} - \begin{pmatrix} \kappa(d_\infty, \cdot)\Omega_\infty \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \leq \left\| \begin{pmatrix} w(s) \\ \partial_s w(s) \end{pmatrix} - \begin{pmatrix} \kappa(d(s), \cdot)\Omega_\infty \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} + \left\| (\kappa(d(s), \cdot) - \kappa(d_\infty, \cdot))\Omega_\infty \right\|_{\mathcal{H}}
\]
where, we used the fact that \( \theta \in \mathbb{R}^{m-1} \mapsto \mathcal{O}^m \) is a Lipschitz function (see (67) to be convinced) and \( \lambda \in \mathbb{R} \to \kappa(d, \cdot) \in \mathcal{H}_0 \) is also Lipschitz, where \( d = \tanh \lambda \) (see (76)). This concludes the proof of Theorem 4 in the case where \( \Omega^* = e_1 \) (see (55)). From rotation invariance of equation (4), this yields the conclusion of Theorem 4 in the general case.
A A some technical estimates

In this section, we give the proof of estimate (63) and Lemma 3.2.

Proof of estimate (63):
Using (61), we see that
\[
\frac{\partial R_\theta}{\partial \theta_j} = R_{2\ldots R_{j-1}} \frac{\partial R_j}{\partial \theta_j} R_j \ldots R_m. \tag{147}
\]
From (60), we see that
\[
\frac{\partial R_\theta}{\partial \theta_j} = \Pi_j \circ R_j(\theta_j + \frac{\pi}{2}) = R_j(\theta_j + \frac{\pi}{2}) \circ \Pi_j, \tag{148}
\]
where \(\Pi_j\) is the orthogonal projection on the plane spanned by \(e_1\) and \(e_j\), and the rotation \(R_j(\alpha)\) is given by considering the matrix of \(R_j\) defined in (60), and changing \(\theta_j\) into \(\alpha\).
Since
\[
\frac{\partial R_j^{-1}}{\partial \theta_j} R_j(\theta_j + \frac{\pi}{2}) = \partial R_j(\frac{\pi}{2}),
\]
it follows from (61) and (148) that
\[
R_\theta^{-1} \frac{\partial R_\theta}{\partial \theta_j} = R_m^{-1} \ldots R_{j+1}^{-1} R_j(\frac{\pi}{2}) \Pi_j R_{j+1} \ldots R_m. \tag{149}
\]
By the same argument, we drive that \(\frac{\partial R_j^{-1}}{\partial \theta_j} \partial R_\theta\) has the same expression, thus, (63) holds from (62) and (149).

Now, we give the proof of Lemma 3.2.

Proof of Lemma 3.2:
i) We first give the expression of the \(m \times m\) matrix \(R_\theta\) defined (61). Indeed, using (60) and (61), we have:
\[
R_\theta \equiv \begin{pmatrix}
\varphi_{2,m} & -\sin \theta_2 & \cdots & -\sin \theta_j \varphi_{2,l-1} & \cdots & -\sin \theta_m \varphi_{2,m-1} \\
\sin \theta_2 \varphi_{3,m} & \cos \theta_2 & & & & \\
\vdots & \ddots & \ddots & \ddots & \ddots & \\
\sin \theta_{k-1} \varphi_{k+1,m} & 0 & \cdots & \cdots & 0 & \cos \theta_j \\
\sin \theta_k \varphi_{m+1,m} & 0 & \cdots & \cdots & 0 & \cos \theta_m
\end{pmatrix} \tag{150}
\]
where for \(k \geq 1, l \geq 2:\n\]
\[
R_{\theta, k, l} = \begin{cases}
-\sin \theta_{l} \varphi_{2,l-1} & \text{if } k = 1 \\
-\sin \theta_{k} \sin \theta_{l} \varphi_{k+1,l-1} & \text{if } 2 \leq k \leq l - 1 \\
\cos \theta_{k} & \text{if } k = l \\
0 & \text{if } k \geq l + 1
\end{cases} \tag{151}
\]
Therefore, we will prove the following identities, which imply item \( i \):

(A) For all \( i, j \in \{2, ..., m\} \), such that \( i \neq j \), we have

\[
< e_j, A_i(e_1) > = 0.
\]

(B) For all \( i \in \{2, ..., m\} \)

\[
< e_1, A_ie_1 > = 0.
\]

(C) For all \( i \in \{2, ..., m\} \), we have

\[
< e_i, A_i e_1 > = \varphi_{i+1,m},
\]

where \( A_i \) and \( \varphi_{i+1,m} \) are given in (152) and (152).

**Proof of (A).** Let \( i, j \in \{2, ..., m\} \), such that \( i \neq j \). The idea is to compute \( < R_\theta e_j, \frac{\partial R_\theta}{\partial \theta_i} e_1 > \) instead of \( < e_j, A_i e_1 > \). In fact, using the conservation of the inner product after a rotation and the fact that \( A_i = R_\theta^{-1} \frac{\partial R_\theta}{\partial \theta_i} \) (by (62)), we have:

\[
< e_j, A_i e_1 > = < R_\theta e_j, R_\theta A_i e_1 > = < R_\theta e_j, \frac{\partial R_\theta}{\partial \theta_i} e_1 >.
\]

In the following, we distinguish two cases:

- Case 1: \( i \leq j - 1 \),
- Case 2: \( i \geq j + 1 \).

We first handle Case 1.

**Case 1:** \( i \leq j - 1 \). Using (152) and its derivative with respect to \( \theta_i \), we write:

\[
R_\theta e_j = (R_\theta e_j)_{k=1,\ldots,m} = \begin{cases}
-\sin \theta_j \varphi_{2,j-1}, & \text{if } k = 1 \\
-\sin \theta_k \sin \theta_j \varphi_{k+1,j-1} & \text{if } 2 \leq k \leq j - 1 \\
\cos \theta_j & \text{if } k = j \\
0 & \text{if } k \geq j + 1
\end{cases}
\]

(154)

and

\[
\frac{\partial R_\theta}{\partial \theta_i} e_1 = (\frac{\partial R_\theta}{\partial \theta_i} e_1)_{k=1,\ldots,m} = \begin{cases}
-\sin \theta_i \varphi_{2,m,k} \cos \theta_i & \text{if } k = 1 \\
-\sin \theta_i \sin \theta_k \varphi_{k+1,m} \cos \theta_i & \text{if } 2 \leq k \leq i - 1 \\
\cos \theta_i \varphi_{i+1,m} & \text{if } k = i \\
0 & \text{if } k \geq i + 1
\end{cases}
\]

(155)

Therefore,

\[
< R_\theta e_j, \frac{\partial R_\theta}{\partial \theta_i} e_1 > = \sum_{k=1}^{m} R_{\theta,k,j} \frac{\partial R_\theta}{\partial \theta_i} e_{k,1}
\]

\[
= \sin \theta_i \varphi_{2,m,k} \cos \theta_i \left( \varphi_{2,j-1} + \sum_{k=2}^{i-1} \sin \theta_k \sin \theta_i \varphi_{k+1,m,k} \cos \theta_i \times \sin \theta_k \sin \theta_j \varphi_{k+1,j-1} \right)
\]

\[
- \cos \theta_i \varphi_{i+1,m,k} \sin \theta_i \sin \theta_j \varphi_{i+1,j-1}
\]

\[
= \sin \theta_i \sin \theta_j \left( \varphi_{2,m,k} \cos \theta_i \varphi_{2,j-1} + \sum_{k=2}^{i-1} \sin \theta_k \varphi_{k+1,m,k} \cos \theta_i \times \varphi_{k+1,j-1} - \cos \theta_i \varphi_{i+1,m,k} \varphi_{i+1,j-1} \right)
\]

(156)
In order to transform the sum term in the previous identity, we make in the following a finite induction where the parameter \( q \) decreases from \( i - 1 \) to 1:

**Lemma A.1.** We have:

\[
\forall q \in \{1, \ldots, i - 1\}, \quad \sum_{k=2}^{i-1} \sin \frac{\theta_k^2 \varphi_{k+1,m}}{\cos \theta_i} \times \varphi_{k+1,j-1} - \cos \theta_i \varphi_{i+1,m} \varphi_{i+1,j-1} = (157)
\]

\[
\sum_{k=2}^{q} \sin \frac{\theta_k^2 \varphi_{k+1,m}}{\cos \theta_i} \times \varphi_{k+1,j-1} - \cos \theta_i \frac{\varphi_{q+1,m} \varphi_{q+1,j-1}}{\cos \theta_i}.
\]

**Remark:** If \( q = 1 \), the sum in the right hand side is naturally zero.

**Proof.** See below. ■

Applying this Lemma, we conclude the proof of (A) in Case 1 (i.e. when \( i \leq j - 1 \)). Indeed, from (156) and Lemma A.1 with \( q = 1 \) we write

\[
< R_\theta e_j, \frac{\partial R_\theta}{\partial \theta_i} e_1 >= \sin \theta_i \sin \theta_j \left( \frac{\varphi_{2,m} \varphi_{2,j-1}}{\cos \theta_i} - \frac{\varphi_{2,m} \varphi_{2,j-1}}{\cos \theta_i} \right) = 0.
\]

It remains now to prove Lemma A.1.

**Proof of Lemma A.1.** First, we give the following:

**Claim A.2.** We have

\[
\varphi_{i,j-1} = \cos \theta_i \varphi_{i+1,j-1}.
\]

**Proof.** Since \( i \leq j - 1 \), we have two cases:
- If \( i \leq j - 2 \): trivial.
- If \( i = j - 1 \): \( \varphi_{i,j-1} = \varphi_{i,i} = \cos \theta_i \) and \( \varphi_{i+1,j-1} = \varphi_{i+1,i} = 1 \), and the result follows. ■

Now, we are ready to start the proof of Lemma A.1. Let us prove the result using an induction with a decreasing index. For \( q = i - 1 \), (157) is satisfied using Claim A.2. Assume now that (157) is true for \( q = i - 1, \ldots, 2 \) and let us prove it for \( q = 1 \). Using (157) with \( q \), we write

\[
\sum_{k=2}^{i-1} \sin \frac{\theta_k^2 \varphi_{k+1,m}}{\cos \theta_i} \times \varphi_{k+1,j-1} - \cos \theta_i \varphi_{i+1,m} \varphi_{i+1,j-1} =
\]

\[
\sum_{k=2}^{q-1} \sin \frac{\theta_k^2 \varphi_{k+1,m}}{\cos \theta_i} \times \varphi_{k+1,j-1} + \sin \theta_q^2 \frac{\varphi_{q+1,m}}{\cos \theta_i} \times (\varphi_{q+1,j-1} - \frac{\varphi_{q+1,m} \varphi_{q+1,j-1}}{\cos \theta_i} =
\]

\[
\sum_{k=2}^{q-1} \sin \frac{\theta_k^2 \varphi_{k+1,m}}{\cos \theta_i} \times \varphi_{k+1,j-1} - \cos \theta_q^2 \frac{\varphi_{q+1,m}}{\cos \theta_i} \times \varphi_{q+1,j-1} =
\]

\[
\sum_{k=2}^{q-1} \sin \frac{\theta_k^2 \varphi_{k+1,m}}{\cos \theta_i} \times \varphi_{k+1,j-1} - \frac{\varphi_{q+1,m} \varphi_{q+1,j-1}}{\cos \theta_i}.
\]

Thus, (157) is satisfied for \( q = 1 \). This concludes the proof of Lemma A.1 and identity (A) when \( i \leq j - 1 \). ■
Now, we handle Case 2.

- **Case 2**: $i \geq j + 1$.

Using (154) and (155), we write:

\[
< R_\theta e_j, \frac{\partial R_\theta}{\partial \theta_i} e_1 > = \sum_{k=1}^m R_{\theta, k, j} \frac{\partial R_\theta}{\partial \theta_i, k, 1} = \sin \theta_i \sin \theta_j \left( \frac{\varphi_{2, m}}{\cos \theta_i} \varphi_{2, j-1} + \sum_{k=2}^{j-1} (\sin^2 \theta_k \frac{\varphi_{k+1, m}}{\cos \theta_i} \varphi_{k+1, j-1} - \cos \theta_j \frac{\varphi_{j+1, m}}{\cos \theta_i}) \right). \tag{158}
\]

In order to transform the sum term in the previous identity, we make in the following a finite induction where the parameter $q$ decreases from $j - 1$ to 1:

**Lemma A.3.** We have:

\[
\forall q \in \{1, \ldots, j - 1\}, \quad \sum_{k=2}^{j-1} \sin^2 \theta_k \frac{\varphi_{k+1, m}}{\cos \theta_i} \varphi_{k+1, j-1} = \cos \theta_j \frac{\varphi_{j+1, m}}{\cos \theta_i} = \left( \sum_{k=2}^q \sin^2 \theta_k \frac{\varphi_{k+1, m}}{\cos \theta_i} \varphi_{k+1, j-1} - \varphi_{q+1, j-1} \frac{\varphi_{q+1, m}}{\cos \theta_i} \right). \tag{159}
\]

**Remark:** If $q = 1$, the sum in the right hand side is naturally zero.

**Proof.** See below.

Applying this Lemma, we conclude the proof of (A) in Case 2 (i. e. when $i \geq j + 1$). Indeed, from (158) and Lemma A.3 with $q = 1$ we write

\[
< R_\theta e_j, \frac{\partial R_\theta}{\partial \theta_i} e_1 > = \sin \theta_i \sin \theta_j \left( \frac{\varphi_{2, m}}{\cos \theta_i} \varphi_{2, j-1} - \cos \theta_j \frac{\varphi_{j+1, m}}{\cos \theta_i} \varphi_{2, j-1} \right) = 0.
\]

It remains now to prove Lemma A.3.

**Proof of Lemma A.3.** We prove the result using an induction with a decreasing index. For $q = j - 1$, (159) is satisfied. Assume now that (159) is true for $q = j - 1, \ldots, 2$ and let us prove it for $q - 1$. Using (159) with $q$, we write

\[
\sum_{k=2}^{j-1} \sin^2 \theta_k \frac{\varphi_{k+1, m}}{\cos \theta_i} \varphi_{k+1, j-1} = \cos \theta_j \frac{\varphi_{j+1, m}}{\cos \theta_i} = \left( \sum_{k=2}^q \sin^2 \theta_k \frac{\varphi_{k+1, m}}{\cos \theta_i} \varphi_{k+1, j-1} - \varphi_{q+1, j-1} \frac{\varphi_{q+1, m}}{\cos \theta_i} \right).
\]

Thus, (159) is satisfied for $q - 1$. This concludes the proof of Lemma A.3.

\[
\square
\]

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Therefore, using (155), we have:

\[ < e_1, A_i e_1 > = < R_\theta e_1, \frac{\partial R_\theta}{\partial \theta_i} e_1 > . \]

From (150), we have:

\[ R_\theta e_1 = (\varphi_{2, m}, \sin \theta_2 \varphi_{3, m}, \ldots, \sin \theta_i \varphi_{i+1, m}, \ldots, \sin \theta_m) . \]

Therefore, using (155), we have:

\[ < R_\theta e_1, \frac{\partial R_\theta}{\partial \theta_i} e_1 > = - \cos \theta_i \sin \theta_i \left( \frac{\varphi_{2, m}}{\cos \theta_i} \right)^2 + \sum_{k=2}^{i-1} \sin^2 \theta_k \left( \frac{\varphi_{k+1, m}}{\cos \theta_i} \right)^2 + \cos \theta_i \sin \theta_i (\varphi_{i+1, m})^2. \]

In order to transform the sum term in the previous identity, we make in the following a finite induction:

**Lemma A.4.** We have:

\[ \forall q \in \{2, \ldots, i-1\}, \left( \frac{\varphi_{2, m}}{\cos \theta_i} \right)^2 + \sum_{l=2}^{i-1} \sin^2 \theta_l \left( \frac{\varphi_{l+1, m}}{\cos \theta_i} \right)^2 = \left( \frac{\varphi_{q, m}}{\cos \theta_i} \right)^2 + \sum_{l=q}^{i-1} \sin^2 \theta_l \left( \frac{\varphi_{l+1, m}}{\cos \theta_i} \right)^2. \tag{160} \]

**Remark:** If \( q = i \), the sum in the right hand side is naturally zero.

Using Lemma A.4, with \( q = i \) we get

\[ < R_\theta e_1, \frac{\partial R_\theta}{\partial \theta_i} e_1 > = - \cos \theta_i \sin \theta_i (\varphi_{i+1, m})^2 + \cos \theta_i \sin \theta_i (\varphi_{i+1, m})^2 = 0, \]

which yields the result. In order to conclude (B) we give the proof of Lemma A.4.

**Proof of Lemma A.4.** We proceed by induction for \( q \in \{2, \ldots, i-1\} \). For \( q = 2 \), (160) is satisfied. Assume that (160) is true for \( q = 2, \ldots, i-1 \) and prove it for \( q + 1 \). Using (160) with \( q \), we write

\[ \left( \frac{\varphi_{2, m}}{\cos \theta_i} \right)^2 + \sum_{l=2}^{i-1} \sin^2 \theta_l \left( \frac{\varphi_{l+1, m}}{\cos \theta_i} \right)^2 = \left( \frac{\varphi_{q, m}}{\cos \theta_i} \right)^2 + \sum_{l=q}^{i-1} \sin^2 \theta_l \left( \frac{\varphi_{l+1, m}}{\cos \theta_i} \right)^2 \]

\[ = \cos^2 \theta_q \left( \frac{\varphi_{q+1, m}}{\cos \theta_i} \right)^2 + \sin^2 \theta_q \left( \frac{\varphi_{q+1, m}}{\cos \theta_i} \right)^2 + \sum_{l=q+1}^{i-1} \sin^2 \theta_l \left( \frac{\varphi_{l+1, m}}{\cos \theta_i} \right)^2 \]

\[ = \left( \frac{\varphi_{q+1, m}}{\cos \theta_i} \right)^2 + \sum_{l=q+1}^{i-1} \sin^2 \theta_l \left( \frac{\varphi_{l+1, m}}{\cos \theta_i} \right)^2. \]

Thus (160) is satisfied for \( q + 1 \). This concludes the proof of Lemma A.4 and identity (B). \( \blacksquare \)

**Proof of (C):** Consider \( i \in \{2, \ldots, m\} \). As for (153) we have:

\[ < e_i, A_i e_1 > = < R_\theta e_i, \frac{\partial R_\theta}{\partial \theta_i} e_1 > . \]

Using (151) and (155),

\[ < e_i, A_i e_1 > = \sin^2 \theta_i \varphi_{i+1, m} \left( \varphi_{2, i-1}^2 + \sum_{k=2}^{i-1} \sin^2 \theta_k \varphi_{k+1, i-1}^2 \right) + \cos^2 \theta_i \varphi_{i+1, m}. \tag{161} \]

In order to transform the sum term in the previous identity, we make in the following a finite induction:
Lemma A.5. We have: \( \forall q \in \{2, ..., i\}, \)
\[
\varphi_{2,i-1}^2 + \sum_{l=2}^{i-1} \sin^2 \theta_l \varphi_{l+1,i-1}^2 = \varphi_{q,i-1}^2 + \sum_{l=q}^{i-1} \sin^2 \theta_l \varphi_{l+1,i-1}^2. \tag{162}
\]

Remark: If \( q = i \), the sum in the right hand side is naturally zero.

From (161) and (162) with \( q = i \) we get
\[
< R_{\theta} e_i, \frac{\partial R_{\theta}}{\partial \theta_i} > = \sin^2 \theta_i \varphi_{i+1,m} + \cos^2 \theta_i \varphi_{i+1,m} = \varphi_{i+1,m}.
\]
which yields the result. In order to conclude \((C)\) we give the proof of Lemma A.5

Proof of Lemma A.5. We proceed by induction for \( q \in \{2, ..., i\} \). For \( q = 2 \), (162) is satisfied. Assume now that (162) is true for \( q = 2, ..., i - 1 \) and prove it for \( q + 1 \). Using (162) with \( q \), we write
\[
\varphi_{2,i-1}^2 + \sum_{l=2}^{i-1} \sin^2 \theta_l \varphi_{l+1,i-1}^2 = \varphi_{q,i-1}^2 + \sum_{l=q}^{i-1} \sin^2 \theta_l \varphi_{l+1,i-1}^2
\]
\[
= \cos^2 \theta_q \varphi_{q+1,i-1} + \sin^2 \theta_q \varphi_{q+1,i-1} + \sum_{l=q+1}^{i-1} \sin^2 \theta_l \varphi_{l+1,i-1}
\]
\[
= \varphi_{q+1,i-1} + \sum_{l=q+1}^{i-1} \sin^2 \theta_l \varphi_{l+1,i-1}.
\]

Thus (162) is satisfied for \( q + 1 \). This concludes the proof of Lemma A.5. \( \blacksquare \)

ii) We recall from (61) that we have
\[
\frac{\partial R_{\theta}}{\partial \theta_j} = R_2 \cdots R_{j-1} \frac{\partial R_j}{\partial \theta_j} R_{j+1} \cdots R_m,
\]
so by (62), \( A_j \) is given explicitly by
\[
A_j = R_m^{-1} R_{m-1}^{-1} \cdots R_{j+1}^{-1} \frac{\partial R_j}{\partial \theta_j} R_{j+1} \cdots R_m.
\]
From a straightforward geometrical observation, we can see that the rotation conserves the euclidian norm in \( \mathbb{R}^m \). For \( \frac{\partial R_j}{\partial \theta_j} \), it can be seen as a composition of a projection on the plane \( (e_1, e_j) \) and a rotation with angle \( \theta_j + \frac{\pi}{2} \), which decreases the norm. This concludes the proof of Lemma 3.2.

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