THE DIFFUSIVE LIMITS OF TWO SPECIES VLASOV-MAXWELL-BOLTZMANN EQUATIONS

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Abstract. In this work, we mainly concern the limiting behavior of the electromagnetic field of two species Vlasov-Maxwell-Boltzmann system in diffusive limits. As knudsen numbers go to zero, the electric magnetic and magnetic field may preserve or vanish. We verify rigorously Navier-Stokes, Navier-Stokes-Poisson and Navier-Stokes Maxwell limit of the two species Vlasov-Maxwell-Boltzmann system on the torus in three dimension. The justification is based on the unified and uniform estimates of solutions to the dimensionless Vlasov-Maxwell-Boltzmann. The uniform estimates of solutions are obtained by employing the hypocoercivity of the linear Boltzmann operator and constructing an equation containing damping term of electric field.

1. Introduction and Motivation

The evolution of the dilute gas consisting of charged particles is described by the Vlasov-Maxwell-Boltzmann (VMB) system which models the dynamics of charged particle under auto-induced electromagnetic field. There exists an alternative description for the gas: the fluid dynamics. These two ways are deeply connected. From the point of view of physics, the fluid regime is those with small Knudsen numbers which is defined as the ratio of the mean free path of the molecular to the physical length scale. As the Knudsen number goes to zero, the gas under consideration satisfies continuum mechanics. Due to its electromagnetic field, the hydrodynamic limits in formal level are quite involved (see [5, 16] for instance). While cations and anions are with the same mass (see [16] with different mass), D. Arsénio and L. Saint-Raymond performed a very systematic formal analysis in [1]. From [1, Sec.2.4], the dimensionless two species Vlasov-Maxwell-Boltzmann equations are

\[
\begin{align*}
\begin{cases}
\partial_t f_e^+ + v \cdot \nabla_x f_e^+ + (\alpha_v E_e + \beta_v v \times B_e) \cdot \nabla_v f_e^+ = \frac{1}{\epsilon} Q(f_e^+, f_e^+) + \frac{1}{\epsilon} Q(f_e^+, f_e^-), \\
\partial_t f_e^- + v \cdot \nabla_x f_e^- - (\alpha_v E_e + \beta_v v \times B_e) \cdot \nabla_v f_e^- = \frac{1}{\epsilon} Q(f_e^-, f_e^+) + \frac{1}{\epsilon} Q(f_e^-, f_e^-), \\
\gamma_e \partial_t E_e - \nabla \times B_e = -\frac{\beta_e}{\epsilon} \int_{\mathbb{R}^3} (f_e^+ - f_e^-) \, dv, \\
\gamma_e \partial_t B_e + \nabla \times E_e = 0, \\
\n\text{div} B_e = 0, \quad \epsilon \cdot \text{div} E_e = \int_{\mathbb{R}^3} (f_e^+ - f_e^-) \, Mdv.
\end{cases}
\end{align*}
\]

where

- \( \epsilon \): the knudsen number;
- \( \alpha_v \): strength of the electric induction;
- \( \beta_v \): strength of the magnetic induction;
- \( \gamma_e \): ratio of the bulk velocity to the speed of the light;
- \( E_e \): electric field;
- \( B_e \): magnetic field;
- \( f_e^\pm \): number density of anion or cation.

Specially, \( \alpha_e, \beta_e \) and \( \gamma_e \) satisfy the following relations:

\[
\alpha_e \gamma_e = \epsilon \cdot \beta_e. 
\]

\( f_e^\pm = f_e^\pm(t, x, v) \) denotes the density of the charges with velocity \( v \) (\( v \in \mathbb{R}^3 \)) on position \( x \) (\( x \in \mathbb{T}^3 \)) at time \( t \) (\( t > 0 \)). The charged particles are moving under the Lorentz force (represented by the third term on the right hand of the first equation in (1)) and the inter-particle collisions (modeled by \( Q(f, f) \)) along their trajectories. The self-generated electromagnetic field is described by the Maxwell’s equation (the equations from the second to the last in (1)). In sequence, they are the Ampère’s equation (the second, \( E \) is the electric field), Faraday’s equation (the third, \( B \) denotes the magnetic field) and the Gauss’s law (the last). \( \text{div} B = 0 \)
means that there is no magnetic monopole. For the two species equations, the notations can be understood in the similar way. The differences are that directions of the Lorentz force acting on ions and electrons are opposite.

The collision operator $Q$ is defined as follows:

$$Q(f, f) = \int_{\mathbb{R}^3 \times S^2} (f'f' - f'f) \sigma(v - v_\omega, \omega) dv_\omega d\omega,$$

where $f' = f(v')$, $f' = f(v')$, $f = f(v_\omega)$. Here, $v$ and $v_\omega$ denote the velocities of two particle before the collision and $v'$ and $v'_\omega$ are their velocities after collision. The collision between particles is elastic, i.e., satisfies the conservation law of momentum and energy:

$$\begin{cases} v + v_\omega = v' + v'_\omega, \\ |v|^2 + |v_\omega|^2 = |v'|^2 + |v'_\omega|^2, \end{cases}$$

and

$$\begin{cases} v' = \frac{v + v_\omega}{2} + \frac{|v - v_\omega|}{2} \omega, \\ v'_\omega = \frac{v + v_\omega}{2} - \frac{|v - v_\omega|}{2} \omega, \end{cases} \text{ with } |v - v_\omega| \in S^2 \text{ (the unit sphere in } \mathbb{R}^3).$$

The nonnegative $\sigma(v - v_\omega, \omega)$, called cross-section, is a function of $|v - v_\omega|$ and cosine of the derivation angle $(\frac{v - v_\omega}{|v - v_\omega|}, \omega)$.

For two species cases, it is more convenient to consider sum and difference of $f^+_e$ and $f^-_e$. Denoting

$$F_e = f^+_e + f^-_e, \quad G_e = f^+_e - f^-_e,$$

then we have

$$\begin{cases} \epsilon \partial_t F_e + \bar{v} \cdot \nabla x F_e + (\alpha_e \epsilon E_e + \beta_e \epsilon v \times B_e) \cdot \nabla \epsilon G_e = \frac{1}{\epsilon} Q(F_e, F_e) \\ \epsilon \partial_t G_e + \bar{v} \cdot \nabla x G_e + (\alpha_e \epsilon E_e + \beta_e \epsilon v \times B_e) \cdot \nabla \epsilon F_e = \frac{1}{\epsilon} Q(G_e, F_e) \\ \gamma_e \partial_t E_e - \nabla \times B_e = -\frac{\epsilon}{\gamma_e} \int_{\mathbb{R}^3} G_e v dx, \\ \gamma_e \partial_t B_e + \nabla \times E_e = 0, \\ \text{div} B_e = 0, \text{ div} E_e = \frac{\epsilon}{\gamma_e} \int_{\mathbb{R}^3} G_e dv. \end{cases} \tag{4}$$

Plugging the following ansatz into (4)

$$F_e = M(1 + \epsilon f_e), \quad G_e = \epsilon M g_e,$$

it follows that

$$\begin{cases} \partial_t f_e + \frac{1}{\epsilon} \bar{v} \cdot \nabla x f_e + \frac{1}{\epsilon} \mathcal{L}(f_e) = N_f, \\ \partial_t g_e + \frac{1}{\epsilon} \bar{v} \cdot \nabla x g_e + \frac{\alpha_e}{\epsilon} E_e \cdot v + \frac{1}{\epsilon} \mathcal{L}(g_e) = N_g, \\ \gamma_e \partial_t E_e - \nabla \times B_e = -\frac{1}{\gamma_e} \bar{v} f_e, \\ \gamma_e \partial_t B_e + \nabla \times E_e = 0, \\ \text{div} B_e = 0, \text{ div} E_e = \frac{\epsilon}{\gamma_e} n_e, \end{cases} \tag{5}$$

with

$$j_e = \int_{\mathbb{R}^3} g_e \bar{v} Mdv, \quad n_e = \int_{\mathbb{R}^3} g_e M dv,$$

$$N_f = \frac{\epsilon}{\gamma_e} E_e \cdot v g_e - \left(\frac{\alpha_e}{\epsilon} E_e + \frac{\beta_e}{\epsilon} v \times B_e\right) \cdot \nabla v g_e + \frac{1}{\gamma_e} \Gamma(f_e, f_e),$$

$$N_g = \frac{\alpha_e}{\epsilon} E_e \cdot v f_e - \left(\frac{\alpha_e}{\epsilon} E_e + \frac{\beta_e}{\epsilon} v \times B_e\right) \cdot \nabla v f_e + \frac{1}{\gamma_e} \Gamma(g_e, f_e),$$

$$-\mathcal{L}(w) = (M)^{-1} Q(M w, M), \quad -\mathcal{L}(w) = (M)^{-1} \left(Q(M w, M) + Q(M, M w)\right).$$

In this work, we concern the hydrodynamics limit, i.e., the transition from kinetic system to macroscopic fluid equations. The mathematical justification of the hydrodynamics limit for the Boltzmann type equations can be divided into two sorts: the renormalized solution framework or classic solution framework. For the work on the existence of renormalized solutions and fluid limit in renormalized solutions work, one can check [3, 8, 10, 21, 23, 24, 25]. In this work, we only concern the classic solution framework. The existence of classic solution to VMB system can be found in [9, 13, 29] and the references therein. The rigorous verification
work started from the Boltzmann equation and then was generalized to Vlasov-Poisson-Boltzmann or Vlasov-Maxwell-Boltzmann system. For the Boltzmann equation, the limiting system is just Navier-Stokes system under diffusive limit. The justification work could be found in [4, 7, 14, 19] and the references therein. For the VPB system, according to [1], the limiting fluids equation could be Navier-Stokes equation or Navier-Stokes-Poisson equation. For the diffusive limit of the Vlasov-Poisson-Boltzmann equation, we refer to [12, 20, 22, 30] and the references therein for the rigorous justification work.

Due to the presence of the electromagnetic field, the structure of limiting fluid equations of VMB system is richer. The electric or magnetic field may vanish as the Knudsen number tends to zero. The Navier-Stokes-Poisson limit of VMB system was verified in [15] by Hilbert expansion method. The Navier-Stokes-Maxwell limit can be found in [18] by Hilbert expansion method and in [17] based on the uniform estimates with respect to the knudsen number. The authors in [15, 17, 18] all consider the diffusive limit of the VMB system, but the electromagnetic field in the limiting equation are different. There only exists electric field in [15]. Both electric field and magnetic field preserve in [17, 18]. The limit behavior of the electromagnetic field in VMB system is determined by the scalings of the strength of electric induction \( \alpha \) and magnetic induction \( \beta \). For details, denoting

\[
\alpha = \lim_{\varepsilon \to 0} \alpha\varepsilon, \quad \beta = \lim_{\varepsilon \to 0} \beta\varepsilon,
\]

then the type of limiting fluid equation are as follows

1. \( \alpha > 0, \beta > 0 \), \( \rightarrow \) Electric field and magnetic field (see (10));
2. \( \alpha > 0, \beta = 0 \), \( \rightarrow \) only electric field (see (11));
3. \( \alpha = 0, \beta = 0 \), \( \rightarrow \) No electromagnetic field (see (12)).

The goal of this work is to verify rigorously (6). We try to justify the role of \( \alpha \) and \( \beta \) in determining the limiting system. In other word, we shall verify the diffusive limit of two species Vlasov-Maxwell-Boltzmann for the three cases in (6). Since there exist lots of cases for each one in (6), for simplicity, we choose the following three main scalings for example:

1. \( \alpha = \varepsilon, \beta = 1 \), \( \rightarrow \) Navier-Stokes-Maxwell system;
2. \( \alpha = \varepsilon, \beta = \varepsilon \), \( \rightarrow \) Navier-Stokes-Poisson system;
3. \( \alpha = \varepsilon^2, \beta = \varepsilon^2 \), \( \rightarrow \) Navier-Stokes system.

Of course, our method can be generalized to more scalings (see Remark 3.6). The justification of this transition phenomenon is based on the uniform estimates of solutions \((f, g, B, E)\) with respect to the knudsen number. Since the local coercivity properties of the linear Boltzmann operator, the difficulty of obtaining the uniform estimates is how to get the dissipative energy of solutions, special for the macroscopic parts.

The approach of this work is motivated by the method used in [7, 28] where the authors added a “mixed term” up to the instant energy to deduce the hypocoercivity properties of the Boltzmann equations. By employing the similar strategy, we can get the dissipative energy(see (9) for the meanings) of \( f_\varepsilon \) and \( g_\varepsilon \). Since the equations of \( B_\varepsilon \) and \( E_\varepsilon \) are hyperbolic, the key difficulty is how to obtain the dissipative estimates of the electromagnetic field for all the three cases in (7) at the same time. The strategy of dealing with this difficulty is to construct a equation with damping term of \( E_\varepsilon \). In details, multiplying the equation of \( g_\varepsilon \) by \( \tilde{v} \) (see (24)) and then integrating over \( \mathbb{R}^3 \), we can obtain that

\[
-\partial_t \tilde{g}_\varepsilon + \cdots + \sigma \frac{\varepsilon^2}{\varepsilon^2} E_\varepsilon = \cdots .
\]

There exists kind of damping term \( E_\varepsilon \) in the above equation. Based on “mixture norm”, we can obtain the dissipative estimates of the curl part of \( E_\varepsilon \) first. After that, we can obtain the dissipative estimates of the curl part of \( B_\varepsilon \) by employing the Ampère and Faraday’s equation. Then we can recover the dissipative estimates of electromagnetic field by the Helmholtz decomposition. As for the three difference cases in (7), the first priority is to obtain the dissipative estimates of electromagnetic field related to \( \alpha \) and \( \beta \) (see the coefficient \( \varepsilon^2 \) in (30)). After determining the coefficient \( \frac{\varepsilon^2}{\varepsilon^2} \), the inequality can be closed with the help of (2) which is very natural and play a essential role in obtaining the uniform estimates.

The novelty of this work is twofold. One is that we verify the diffusive limits for the three main scalings (7) at the same time. We also give a rigorous proof for the formal derivation in [1] (only for two species with strong interaction). On the other hand, the method of this work is based on the uniform estimates obtained
by “mixed norm” introduced in [28, 7]. Specially, as mentioned before, we employ a new way of obtaining the dissipative estimates of electromagnetic field. Indeed, for instance([17]), noticing that the magnetic field is divergence free, by employing the third and forth equation in (5), one can obtain the dissipative energy estimates by employing the wave equation of \( B_s \). Besides, based on the micro-macro decomposition, the norm used in [13] must contain the temporal derivative to obtain the dissipative estimate of electromagnetic field( see [13, Lemma 9] for more details). We also give an alternative proof for the existence of classic solutions to the VMB system and the three limiting fluid systems.

The rest of this paper is made up of three sections. We shall introduce the preliminaries in the Section 2, such as notations, assumption on kernel and initial data. The main results can be found in Sec 3 where we also explain our strategy and difficulties. The Section 4 consists in deriving the uniform a prior estimates and is the most important part of this work. Based on the uniform estimates for all the three cases in (7), the fluid limits will be verified in Sec. 5.

2. Preliminaries

2.1. Notations and Terms. For estimates like this

\[
\frac{d}{dt} E(t) + D(t) \leq \cdots ,
\]

where \( E(t) \) and \( D(t) \) are positive functions of \( t \). We call \( E(t) \) by the instant energy and \( D(t) \) the dissipative energy (estimates). \( \nabla_x f = \nabla_{i_1} \nabla_{i_2} \nabla_{i_3} f \) is the \( i \)-th derivative of \( f \) with \( i_1 + i_2 + i_3 = i \). Specially, \( \nabla_x f \) is the gradient of scalar function \( f \). In the similar way, we can define \( \nabla_v^i f \) and \( \nabla_v f \). The norms of \( f \) are defined as follows:

\[
\|f\|_{L^2_v}^2 = \int_{\mathbb{R}^3} f^2 Mdv, \quad \|f\|_{L^2_x}^2 = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f^2 Mdvdx,
\]

\[
\|f\|_{H^s_v}^2 = \sum_{k=0}^s \|\nabla_k f\|_{L^2_v}^2, \quad \|f\|_{H^s_x}^2 = \sum_{k=0}^s \|\nabla_k f\|_{L^2_x}^2.
\]

Denoting \( \hat{v}^2 = 1 + |v| \), the norms with weight on \( v \) are defined as follows:

\[
\|f\|_{L^2_{\hat{v}}}^2 = \|f\hat{v}\|_{L^2_v}^2, \quad \|f\|_{H^s_{\hat{v}}}^2 = \sum_{k=0}^s \|\nabla_k f\hat{v}\|_{L^2_v}^2,
\]

\[
\|f\|_{L^2_{\hat{v}, x}}^2 = \int_{\mathbb{R}^3} f^2 \hat{v}^2 Mdv, \quad \|f\|_{H^s_{\hat{v}, x}}^2 = \sum_{k=0}^s \sum_{i+j=k} \|\nabla_i \nabla_j f\|_{L^2_{\hat{v}, x}}^2.
\]

In this work, the \( C \) denotes some constant independent of \( \epsilon \) and is different from lines to lines. The \( a \lesssim b \) means that there exists some constant \( C \) independent of \( \epsilon \) such that \( a \leq Cb \).

2.2. Limiting equations. For the convenience of stating our main results, we list three fluid equations. The first one is the Navier-Stokes-Maxwell system where exists Lorentz force(both electric and magnetic field):

\[
\begin{cases}
\partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla P = n \cdot E + j \times B, \\
\partial_t \theta + u \cdot \nabla \theta - \kappa \Delta \theta = 0, \\
\text{div} u = 0, \quad \rho + \theta = 0, \\
\partial_t E - \nabla \times B = -j, \\
\partial_t B + \nabla \times E = 0, \\
\text{div} B = 0, \quad \text{div} E = n, \\
j = \sigma(E + u \times B) - \sigma \nabla x n + nu.
\end{cases}
\]
While there is only Coulomb force, we obtain the Navier-Stokes-Poisson system:

\[
\begin{aligned}
\partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla P &= n \cdot E, \\
\partial_t \theta + u \cdot \nabla \theta - \kappa \Delta \theta &= 0, \\
\partial_t n + \text{div}(nu) - \sigma \Delta n + \sigma n &= 0, \\
\text{div} u &= 0, \quad \rho + \theta = 0, \\
\nabla \times E &= 0, \quad \text{div} E = n.
\end{aligned}
\tag{11}
\]

The last case is two fluid Navier-Stokes system where the electromagnetic field vanishes:

\[
\begin{aligned}
\partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla P &= 0, \\
\partial_t \theta + u \cdot \nabla \theta - \kappa \Delta \theta &= 0, \\
\partial_t n + u \cdot \nabla n - \sigma \Delta n &= 0, \\
\text{div} u &= 0, \quad \rho + \theta = 0.
\end{aligned}
\tag{12}
\]

In the above system, the \(\nu, \kappa, \sigma\) are strictly positive constants and will be clearly defined in (25).

2.3. Assumption on the linear operators. This section consists in stating the assumption on the linear Boltzmann operators: \(L\) and \(L\). According to our notations, by simple computation, we can infer

\[
L(h) = (M)^{-1} Q(M, h, M) = \int_{\mathbb{R}^3 \times \mathbb{S}^2} (h' - h) M(v, \omega) b(v - v_s, \omega) dv_s d\omega
\]

\[
= -\int_{\mathbb{R}^3 \times \mathbb{S}^2} h' M(v, \omega) b(v - v_s, \omega) dv_s d\omega + \int_{\mathbb{R}^3 \times \mathbb{S}^2} M(v, \omega) b(v - v_s, \omega) dv_s d\omega h
\]

\[
:= -\Phi(h) + \Lambda(v) \cdot h,
\]

and

\[
L(h) = (M)^{-1} Q(M, h, M) + (M)^{-1} Q(M, M) = \Phi(h) - \Lambda(v) \cdot h + (M)^{-1} Q(M, M)
\]

\[
:= -K(h) + \Lambda(v) \cdot h.
\]

In summary, the operators satisfy

\[
L = -K + \Lambda, \quad L = -\Phi + \Lambda.
\tag{13}
\]

From [11], the operator \(\Lambda(v)\) is coercive. Even though the compact parts of \(L\) and \(L\) are different, according to [5, 13, 26, 27], the similar assumption can be established. To avoid using too many notations of constants and without loss of generality, we set the constant to be one.

**H1.** This assumption is on the coercive operator \(\Lambda\) in (13).

- For any \(h, g \in L^2_v\),

\[
\|h\|_{L^2_{\nu, \Lambda}}^2 \leq \int_{\mathbb{R}^3} L(h) \cdot h \text{Md}v \leq C \|h\|_{L^2_{\nu, \Lambda}}^2,
\]

and

\[
| \int_{\mathbb{R}^3} \Lambda(h) \cdot g \text{Md}v | \leq C \|h\|_{L^2_{\nu, \Lambda}} \|g\|_{L^2_{\nu, \Lambda}}.
\tag{14}
\]

- With respect to the derivative of \(v\), the operator \(\Lambda\) admits “a defect of coercivity”, i.e., there exist some strictly positive constant \(\delta\) and \(C_\delta\) such that

\[
\int_{\mathbb{R}^3} \nabla_v \Lambda(h) \cdot \nabla_v h \text{Md}v \geq (1 - \delta) \|\nabla_v h\|_{L^2_{\nu, \Lambda}}^2 - C_\delta \|h\|_{L^2_{\nu, \Lambda}}^2, \quad 0 < \delta < 1.
\tag{15}
\]

and for the higher order derivative,

\[
\int_{\mathbb{R}^3} \int_{\mathbb{T}^3} \nabla_v \cdot \nabla^j \Lambda(h) \cdot \nabla^j_v h \text{Md}v dx \geq (1 - \delta) \|\nabla_v \cdot \nabla^j_v h\|_{L^2_{\nu, \Lambda}}^2 - C_\delta \|h\|_{H^{j+1}}^2.
\tag{16}
\]
**H2.** This assumption is about the compact operators in (13). For any \( \delta > 0 \), there exists some positive constant \( C_3 \) such that

\[
| \int_{\mathbb{R}^3} \nabla_v K(h) \cdot \nabla_v h \text{Mdv} | + | \int_{\mathbb{R}^3} \nabla_v \Phi(h) \cdot \nabla_v h \text{Mdv} | \leq C_3 \| h \|_{L^2_{v, \Lambda}}^2 + \delta \| \nabla_v h \|_{L^2_{v, \Lambda}}^2 ,
\]

and for higher order derivative,

\[
| \int_{T^3} \int_{\mathbb{R}^3} \nabla^i_v \nabla^j_v K(h) \cdot \nabla^i_v \nabla^j_v h \text{Mdv} | + | \int_{T^3} \int_{\mathbb{R}^3} \nabla^i_v \nabla^j_v \Phi(h) \cdot \nabla^i_v \nabla^j_v h \text{Mdv} | \leq C_3 \| h \|_{L^2_{v, \Lambda} + 1}^2 + \delta \| \nabla^i_v \nabla^j_v h \|_{L^2_{v, \Lambda}}^2 .
\]

**H3.** (Relaxation to the local equilibrium.) This assumption is on the linear Boltzmann operators and their kernel spaces. The linear Boltzmann operators \( L \) and \( L \) are closed and self-adjoint operators from \( L^2_v \) to \( L^2_v \).

The kernel space of \( L \) is spanned by \( 1, v_1, v_2, v_3, |v|^2 \). The kernel space of \( L \) is spanned by \( 1 \). Furthermore, \( L \) and \( L \) satisfy “local coercivity assumption”: there exists \( c_2 > 0 \) such that

\[
\int_{\mathbb{R}^3} L(g) \cdot g \text{Mdv} \geq \| g - P g \|_{L^2_{v, \Lambda}}^2 ,
\]

\[
\int_{\mathbb{R}^3} L(h) \cdot h \text{Mdv} \geq \| h - P h \|_{L^2_{v, \Lambda}}^2 ,
\]

where \( P \) is the projection operator of \( L \) and \( L \) onto their kernel space respectively, the macroscopic part. In this work, while the projection \( P \) is related to \( L \), i.e., the first one in (19),

\[
P g = \int_{\mathbb{R}^3} g \text{Mdv} + v \cdot \int_{\mathbb{R}^3} g v \text{Mdv} + \frac{|v|^2 - 3}{2} \int_{\mathbb{R}^3} g \frac{|v|^2 - 3}{3} \text{Mdv}.
\]

If the projection \( P \) is related to \( L \), i.e., the second one in (19),

\[
P h = \int_{\mathbb{R}^3} h \text{Mdv}.
\]

In this work, we use the same \( P \) for the projection operator of \( L \) and \( L \) onto their kernel space.

Furthermore, we also assume that

\[
| \int_{\mathbb{R}^3} f \cdot L(g) \text{Mdv} | \leq C \| f \|_{L^2_{v, \Lambda}} \| g \|_{L^2_{v, \Lambda} \Lambda}, \quad | \int_{\mathbb{R}^3} f \cdot L(g) \text{Mdv} | \leq C \| f \|_{L^2_{v, \Lambda}} \| g \|_{L^2_{v, \Lambda} \Lambda}, \quad \forall f, g \in L^2_{v, \Lambda}.
\]

**H4.** This assumption is on \( \Gamma(g, h) \) and \( \Gamma(g, h) \).

- For any \( g, h \in L^2, \Gamma(g, h) \in \text{Ker}(L)^\perp, \Gamma(g, h) \in \text{Ker}(L)^\perp. \)
- For the non-linear operator and \( s \geq 3 \)

\[
| \int_{T^3} \int_{\mathbb{R}^3} \nabla^i_v \Gamma(g, h) \cdot f \text{Mdvdx} | \leq \| (g, h) \|_{H^s_{v}} \| (g, h) \|_{H^s_{v}} \| f \|_{L^2_{v, \Lambda}}, \quad i \geq 1, \quad s = i + j.
\]

Besides, we introduce matrices \( A(v) \) and \( B(v) \), vector \( \hat{v} \) as follows

\[
A(v) = v \otimes v - \frac{|v|^2}{3} I, \quad B(v) = v \left( \frac{|v|^2}{2} - \frac{5}{2} \right), \quad \mathcal{L} \hat{A}(v) = A(v), \quad \mathcal{L} \hat{B}(v) = B(v), \quad \mathcal{L} \hat{v} = v.
\]

and define

\[
\nu = \frac{1}{15} \sum_{1 \leq i, j \leq 3} \int_{\mathbb{R}^3} A_{ij} \hat{A}_{ij} \text{Mdv}, \quad \kappa = \frac{2}{15} \sum_{1 \leq i, j \leq 3} \int_{\mathbb{R}^3} B_{ij} \hat{B}_{ij} \text{Mdv}, \quad \sigma = \frac{1}{3} \int_{\mathbb{R}^3} \hat{v} \cdot v \text{Mdv}.
\]

The matrix (24) and constants (25) will be very useful during the justification of fluid limits.
2.4. Assumption on the initial data. Denoting
\[ \rho_e = \int_{\mathbb{R}^3} f_e Mdv, \quad u_e = \int_{\mathbb{R}^3} f_e v Mdv, \quad \theta_e = \int_{\mathbb{R}^3} \left( \frac{|v|^2}{2} - 1 \right) f_e Mdv, \quad n_e = \int_{\mathbb{R}^3} g_e Mdv, \]

similar to [7, 13], by assuming the initial data have the same mass, velocity and total energy as the steady case, we can assume the initial data
\begin{align*}
\int_{\mathbb{T}^3} & (u_e + \gamma_e E_e \times B_e)(0)dx = 0, \\
\int_{\mathbb{T}^3} & \left( \theta_e + \epsilon \cdot \frac{|E_e|^2 + |B_e|^2}{4} \right)(0)dx = 0, \\
\int_{\mathbb{T}^3} & \rho_e(0)dx = \int_{\mathbb{T}^3} n_e(0)dx = 0, \quad \int_{\mathbb{T}^3} B_e(0)dx = 0.
\end{align*}

3. Main results

We recall the three main scalings in (7).
\[ \alpha_e = \epsilon, \quad \gamma_e = \beta_e = 1; \quad \alpha_e = \gamma_e = \beta_e = \epsilon; \quad \alpha_e = \epsilon^2, \quad \gamma_e = \epsilon, \quad \beta_e = \epsilon^2. \]

Define
\[ H^s_e(t) := \| (f_e, g_e, B_e, E_e) \|^2_{H^s_{L^2}} + \epsilon^2 \| (\nabla v f_e, \nabla v g_e) \|^2_{H^{s-1}}. \]

**Theorem 3.1.** Under the assumption in the section 2.3 and the assumption (26) on the initial data, for all the three scalings (27–29) on \( \alpha_e, \beta_e \) and \( \gamma_e \), there exists some small enough constant \( c_0 \) such that for any \( 0 < \epsilon \leq 1 \) if the initial data \( (f_e(0), g_e(0), E_e(0), B_e(0)) \) satisfy
\[ H^s_e(0) \leq c_0, \quad s \geq 3, \]
then system (5) admit a unique global classical solution \( (f_e, g_e, B_e, E_e) \) satisfying for any \( t > 0 \)
\[ \sup_{0 \leq s \leq t} H^s_e(t) + \frac{1}{4} \int_0^t \left( \| (f_e, g_e) \|^2_{H^s_{L^2}} + \frac{\alpha_e^2}{\epsilon^2} \| (E_e, B_e) \|^2_{H^{s-1}_2} + \frac{1}{\epsilon^2 \alpha_e} \| (f_e^1, g_e^1) \|^2_{H^{s-1}_x} \right) (s) ds \leq \frac{c_1}{\epsilon^2} H^s_e(0), \]
where \( c_1 \) and \( c_2 \) are positive constants only dependent of the Sobolev embedding constant.

**Remark 3.2.** We use an equivalent norm \( \tilde{H}^s_e \) (122) instead of \( H^s_e \) to obtain the prior estimate (30). The constants \( c_1 \) and \( c_2 \) come from the equivalent relation of \( H^s_e \) and \( \tilde{H}^s_e \); see (124).

**Remark 3.3.** We comment here on the dissipative estimates of \( B_e \) and \( E_e \). On one hand, while initial data belongs to \( H^s \) space, we only obtain the order \( s-1 \) dissipative estimates of \( E_e \) and \( B_e \). This can be understood like this: from (8) and (35),
\[ -\partial_j \tilde{J}_e + \frac{\alpha_e}{\epsilon^2} E_e - \frac{1}{\epsilon} \text{div} \int_{\mathbb{R}^3} \tilde{v} \otimes v g_e Mdv = \cdots, \]
there already exists one derivative for the third term on the left hand of the above equation.

Furthermore, by the Helmholtz decomposition, we can split \( E_e \) and \( B_e \) to two parts: curl-free part and divergence-free part. The loss derivative only occurs in the process of obtaining \( \nabla \times E_e \) and \( \nabla \times B_e \). Indeed,
\[ \text{div} E_e = n_{e_i} \text{div} B_e = 0. \]

For the Vlasov-Poisson-Boltzmann system, since the electromagnetic field is gradient flow, there is no loss of derivative.

**Remark 3.4.** There exists a coefficient \( \frac{\alpha_e^2}{\epsilon^2} \) before the dissipative estimates of the electromagnetic field. Under the scalings (29), i.e., \( \alpha_e = o(\epsilon) \), the dissipative estimates of \( E_e \) is very weak. Due to the loss of derivative and the coefficient \( \frac{\alpha_e^2}{\epsilon^2} \), we need to adjust the coefficient and derivative carefully to close the inequality.
Before stating the limiting, let \( u_0, \theta_0, n_0, E_0, B_0 \in H^s_x \) and satisfy (up to a subsequence)

\[
P_{u_\epsilon}(0) \to u_0, \ \frac{3}{5}\theta_\epsilon(0) - \frac{2}{5}p_\epsilon(0) \to \theta_0, n_\epsilon(0) \to 0, \ \ E_\epsilon(0) \to E_0, \ \ B_\epsilon(0) \to B_0 \text{ in } H^{s-1}_x.
\]

where \( P \) is the Leray projector.

**Theorem 3.5 (Fluid limit).** Under the assumption in the section 2.3 and the assumption (26) on the initial data, for the solutions \( f_\epsilon, g_\epsilon, E_\epsilon, B_\epsilon \) constructed in Theorem 3.1, it follows that

\[
f_\epsilon \to \rho + u \cdot v + \frac{|v|^2 - 3}{2}\theta, \ g_\epsilon \to n(t, x), \ \text{in } L^2(0, +\infty; H^{s-1}_x),
\]

\[
E_\epsilon \to E, \ B_\epsilon \to B, \ \text{in } L^2((0, T); H^{s-1}_x))(\text{for any } T > 0),
\]

with \( \rho, u, \theta, E, B \) satisfying

\[
\rho, u, \theta, n, E, B \in L^\infty((0, \infty); H^s_x).
\]

and

- for case (27), \( \rho, n, u, \theta, E \) and \( B \) are global classic solutions to Navier-Stokes-Maxwell system with initial data \((u_0, \theta_0, B_0, E_0)\);
- for case (28), \( \rho, u, \theta, n \) and \( E \) are global classic solutions to Naiver-Stokes-Poisson system with initial data \((u_0, \theta_0, n_0)\);
- for case (29), \( \rho, u, \theta \) and \( n \) are global classic solutions to two fluids Navier-Stokes-Fourier system with initial data \((u_0, \theta_0, n_0)\).

Furthermore, for any \( \tau > 0 \), we can infer

\[
P_{u_\epsilon} \to u, \ \frac{3}{5}\theta_\epsilon - \frac{2}{5}p_\epsilon \to \theta, \ \text{in } C([\tau, +\infty); H^{s-1}_x).
\]

**Remark 3.6.** The main result of Theorem 3.1 and Theorem 3.5 also work for more general scalings. (28)

can be generalized to

\[
\alpha_\epsilon = \epsilon, \ \beta_\epsilon = o(1).
\]

(27) can be generalized to

\[
\alpha_\epsilon = o(\epsilon), \ \beta_\epsilon = o(1), \ \gamma_\epsilon = o(1), \ \gamma_\epsilon \lesssim \frac{\alpha_\epsilon}{\epsilon},
\]

where

\[
o(1) = \{z_\epsilon | z_\epsilon \to 0, \ \epsilon \to 0\}, \ o(\epsilon) = \{z_\epsilon | z_\epsilon \to 0, \ \epsilon \to 0\}.
\]

**Remark 3.7.** During the justification of the fluid limit, the key point is to verify the Ohm’s law

\[
\frac{1}{\epsilon} j_\epsilon \to j = \sigma(\alpha E + \beta u \times B) - \sigma \nabla_x n + nu, \ \text{in the distributional sense.}
\]

**Remark 3.8.** Let \( \tilde{u}_0, \tilde{\rho}_0 \) and \( \tilde{\theta}_0 \) be the limits of \( u_\epsilon(0), \ \rho_\epsilon(0) \) and \( \theta_\epsilon(0) \) in the distributional sense. If the initial data are well-prepared, i.e.,

\[
\text{div} \tilde{u}_0 = 0, \ \tilde{\rho}_0 = \tilde{\theta}_0 = 0,
\]

then (33) can be improved to

\[
P_{u_\epsilon} \to u, \ \frac{3}{5}\theta_\epsilon - \frac{2}{5}p_\epsilon \to \theta, \ \text{in } C([0, +\infty); H^{s-1}_x).
\]

**Remark 3.9.** The main results of this work can be generalized to more collisional collisional kernels such as Fokker–Planck and Landau kernel, see [7, Apendix A] for details.

**3.1. The strategy and difficulty of the proof.** We only sketch the proof of Theorem 3.1. The proof of Theorem 3.5 is based on the local conservation laws of VMB system. For the uniform estimates, the goal is to obtain inequality like this

\[
\frac{d}{dt}E(t) + D(t) \leq E(t)D(t),
\]

where

\[
E(t) \approx \|(f_\epsilon, g_\epsilon, E_\epsilon, B_\epsilon)\|_{H^s_x}^2 + \epsilon^2\|\nabla v f_\epsilon, \nabla g_\epsilon\|_{H^{s-1}_x}^2,
\]

\[
D(t) \approx \|(f_\epsilon, g_\epsilon, \frac{1}{\tau}f_\epsilon + \frac{1}{\tau}g_\epsilon)\|_{H^s_x}^2 + \frac{\alpha^2}{\epsilon^2}\|(B_\epsilon, E_\epsilon)\|_{H^{s-1}_x}^2.
\]

In what follows, we take the first equation of (5)

\[
\partial_t f_\epsilon + \frac{1}{\tau}v \cdot \nabla_x f_\epsilon + \frac{1}{\tau^2}L(f_\epsilon) = \cdots
\]
to explain the “mixture norm” skills used in [28] and [7]. Because of the local coercivity properties of $\mathcal{L}$ and $L$, we can obtain

$$z_1 \cdot \frac{d}{dt} \| f_\epsilon \|_{H^1_3}^2 + z_1 \cdot \frac{1}{\tau^2} \| f_\epsilon^{\perp} \|_{H^1_{L^2}}^2 \leq \cdots.$$  

For the mixture term $\int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \nabla_x f_\epsilon \cdot \nabla_v f_\epsilon M_\epsilon \, dv \, dx$,

$$z_2 \cdot \frac{d}{dt} \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \nabla_x f_\epsilon \cdot \nabla_v f_\epsilon M_\epsilon \, dv \, dx + z_2 \cdot \| \nabla_x f_\epsilon \|_{L^2}^2 \leq \cdots. \tag{34}$$

In the above inequality, there exists the dissipative estimate of $f_\epsilon$. The $L^2$ estimate of $f_\epsilon$ can be obtained by Poincare’s inequality and the local conservation laws. Furthermore, we also need to estimate $\nabla_v f_\epsilon$

$$z_3 \cdot \epsilon^2 \frac{d}{dt} \| \nabla_v f_\epsilon \|_{L^2}^2 + z_3 \cdot \| \nabla_v f_\epsilon \|_{L^2}^2 \leq \cdots.$$  

Choosing proper constants $z_1$, $z_2$ and $z_3$ to let the “mixture norm” satisfy

$$z_1 \| f_\epsilon \|_{H^1_3}^2 + z_2 \cdot \epsilon \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \nabla_x f_\epsilon \cdot \nabla_v f_\epsilon M_\epsilon \, dv \, dx + z_3 \cdot \epsilon^2 \| \nabla_v f_\epsilon \|_{L^2}^2 \approx \| f_\epsilon \|_{H^1_3}^2,$$

one can obtain that

$$\frac{d}{dt} \| f_\epsilon \|_{H^1_3}^2 + \| f_\epsilon \|_{H^1_3}^2 \leq \cdots.$$  

The Lemma 4.2, Lemma 4.5 and Lemma 4.4 consist in establishing the similar above inequalities for $f_\epsilon$ and $g_\epsilon$. From the previous analysis, the key point of [28] and [7] is to obtain (34) and then employ the Poincare’s inequality to recover zero-order dissipative estimates solutions. But there exist new difficulty for the dimensionless system (5). Indeed, for the Boltzmann system, if the mean value of the initial datum is zero, then the solution preserves this properties. But for (5), from the global conservation laws (see Lemma 4.1), the mean value of $f_\epsilon$ and $g_\epsilon$ may be not zero. The Poincare’s inequality can not be directly used. The idea is to employ the global conservation law to overcome this difficulty. The second difficulty comes from the singular term with coefficient $\frac{\beta}{\tau}$ in (5). Indeed, this term is hard to bound, while $\beta = O(1)$. This difficulty can be dealt with by splitting $f_\epsilon$ and $g_\epsilon$ into macroscopic parts and microscopic parts (see Remark 4.3).  

The estimates of electromagnetic parts are new in this work. For the electromagnetic field, as mentioned before, multiplying the second equation by $\tilde{v} M$ and integrating the resulting equation over $\mathbb{R}^3$, it follows that

$$-\frac{\partial \tilde{j}_e}{\partial t} + \frac{\epsilon}{\tau} E_\epsilon - \frac{1}{\tau} \text{div} \int_{\mathbb{R}^3} \tilde{v} \otimes v g_\epsilon M_\epsilon - \frac{1}{\tau} j_\epsilon = \cdots. \tag{35}$$

The key point is to multiply $\tilde{v}$ other than $v$. One of the advantage is that the very singular term $\frac{1}{\tau} j_\epsilon$ can be canceled by employ the equation of $E_\epsilon$ (see (101)). Based on the above equation, one can obtain the dissipative estimates of $E_\epsilon$ (see Lemma 4.6). For the dissipative estimates of $B_\epsilon$, by the third and forth equations in (5), one can obtain that

$$-\gamma \frac{d}{dt} \int_{\mathbb{T}^3} E_\epsilon \cdot \nabla B_\epsilon \, dx + \| \nabla \times B_\epsilon \|_{L^2}^2 - \| \nabla \times E_\epsilon \|_{L^2}^2 \leq \cdots.$$  

Since we have obtain the dissipative estimates of $E_\epsilon$, one can finally obtain the dissipative of $B_\epsilon$ by the Helmholtz decomposition and Poincare’s inequality.  

With the uniform estimate at our disposal, the limiting equations can be deduced from the local conservation of laws of Vlasov-Maxwell-Boltzmann by employing the properties of the linear Boltzmann operator and the structure of Vlasov-Maxwell-Boltzmann system.

4. A priori estimates

This section is devoted to proving the existence of solutions to (5), i.e., Theorem 3.1. The key ingredient is the uniform priori estimate of solutions. The proof is quite involved. We split the whole proof into four lemmas.
4.1. The analysis of global conservation laws. The mean value of the macroscopic part of \( f_\varepsilon \) and \( g_\varepsilon \) is necessary to obtain the dissipative estimates of \( g_\varepsilon \) and \( f_\varepsilon \).

**Lemma 4.1.** The classic solution of system (5) enjoy the following conservation law:

\[
\frac{d}{dt} \int_{\mathbb{T}^3} (u_\varepsilon + \gamma_\varepsilon E_\varepsilon \times B_\varepsilon) \, dx = 0,
\]

\[
\frac{d}{dt} \int_{\mathbb{T}^3} \left( \theta_\varepsilon + \varepsilon \cdot \frac{(E_\varepsilon)^2 + |B_\varepsilon|^2}{3} \right) \, dx = 0,
\]

\[
\frac{d}{dt} \int_{\mathbb{T}^3} \rho_\varepsilon(t) \, dx = \frac{d}{dt} \int_{\mathbb{T}^3} n_\varepsilon(t) \, dx = \frac{d}{dt} \int_{\mathbb{T}^3} B_\varepsilon(t) \, dx = 0.
\]

*Proof.* Recalling that

\[
v \cdot (v \times B_\varepsilon) = 0,
\]

then we can rewrite system (5) as

\[
\begin{aligned}
\partial_t f_\varepsilon + \frac{v}{\varepsilon} \cdot \nabla_z f_\varepsilon - \frac{1}{\varepsilon} \mathcal{L}(f_\varepsilon) &= -\frac{\alpha_\varepsilon E_\varepsilon + \beta_\varepsilon v \times B_\varepsilon}{\varepsilon} \cdot \nabla_v (M g_\varepsilon) + \frac{1}{\varepsilon} \Gamma(f_\varepsilon, f_\varepsilon), \\
\partial_t g_\varepsilon + \frac{v}{\varepsilon} \cdot \nabla_z g_\varepsilon - \frac{\alpha_\varepsilon E_\varepsilon}{\varepsilon^2} v - \frac{1}{\varepsilon} \mathcal{L}(g_\varepsilon) &= -\frac{\alpha_\varepsilon E_\varepsilon + \beta_\varepsilon v \times B_\varepsilon}{\varepsilon^2} \cdot \nabla_v (M f_\varepsilon) + \frac{1}{2\varepsilon} \Gamma(g_\varepsilon, f_\varepsilon), \\
\gamma_\varepsilon \partial_t E_\varepsilon + \nabla \times B_\varepsilon &= -\frac{\beta_\varepsilon}{\varepsilon} J_\varepsilon, \\
\gamma_\varepsilon \partial_t B_\varepsilon + \nabla \times E_\varepsilon &= 0, \\
\text{div} B_\varepsilon &= 0, \quad \text{div} E_\varepsilon = \frac{\alpha_\varepsilon}{\varepsilon} \int_{\mathbb{R}^3} g_\varepsilon M \, dv.
\end{aligned}
\]

Then from (37), it follows that

\[
\frac{d}{dt} \int_{\mathbb{T}^3} \rho_\varepsilon \, dx = 0, \quad \frac{d}{dt} \int_{\mathbb{T}^3} n_\varepsilon(t) \, dx = 0.
\]

The local conservation law of velocity is

\[
\begin{aligned}
\frac{d}{dt} \int_{\mathbb{T}^3} u_\varepsilon(t) \, dx &= -\int_{\mathbb{T}^3} v \cdot \left( \frac{\alpha_\varepsilon E_\varepsilon + \beta_\varepsilon v \times B_\varepsilon}{\varepsilon} \cdot \nabla_v (M g_\varepsilon) \right) \, dv \, dx \\
&= \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \left( \frac{\alpha_\varepsilon E_\varepsilon + \beta_\varepsilon v \times B_\varepsilon}{\varepsilon} \cdot (M g_\varepsilon) \right) \, dv \, dx \\
&= \int_{\mathbb{T}^3} \text{div} E_\varepsilon \cdot E_\varepsilon \, dx + \frac{\beta_\varepsilon}{\varepsilon} \int_{\mathbb{T}^3} j_\varepsilon \times B_\varepsilon \, dx.
\end{aligned}
\]

By simple computation,

\[
\text{div}(E_\varepsilon \otimes E_\varepsilon) = \text{div} E_\varepsilon \cdot E_\varepsilon + (E_\varepsilon \cdot \nabla) E_\varepsilon,
\]

\[
(\nabla \times E_\varepsilon) \times E_\varepsilon = (E_\varepsilon \cdot \nabla) E_\varepsilon - \frac{1}{2} \nabla |E_\varepsilon|^2,
\]

plugging (40) into (39), we can infer that

\[
\frac{d}{dt} \int_{\mathbb{T}^3} u_\varepsilon(t) \, dx = \frac{\beta_\varepsilon}{\varepsilon} \int_{\mathbb{T}^3} j_\varepsilon \times B_\varepsilon \, dx - \int_{\mathbb{T}^3} (E_\varepsilon \cdot \nabla) E_\varepsilon \, dx.
\]

Multiplying the third equation of (37) by \( \times B_\varepsilon \) and the forth equation by \( \times E_\varepsilon \) respectively, then it follows that

\[
\gamma_\varepsilon \frac{d}{dt} \int_{\mathbb{T}^3} E_\varepsilon \times B_\varepsilon \, dx - \int_{\mathbb{T}^3} (\nabla \times B_\varepsilon) \times B_\varepsilon + (\nabla \times E_\varepsilon) \times E_\varepsilon \, dx = -\frac{\beta_\varepsilon}{\varepsilon} \int_{\mathbb{T}^3} j_\varepsilon \times B_\varepsilon \, dx.
\]

With the help of (40) and \( \text{div} B_\varepsilon = 0 \), we can infer that

\[
(\nabla \times B_\varepsilon) \times B_\varepsilon + (\nabla \times E_\varepsilon) \times E_\varepsilon = (E_\varepsilon \cdot \nabla) E_\varepsilon + (B_\varepsilon \cdot \nabla) B_\varepsilon - \frac{1}{2} \nabla \left( |E_\varepsilon|^2 + |B_\varepsilon|^2 \right)
\]

\[
= (E_\varepsilon \cdot \nabla) E_\varepsilon + (B_\varepsilon \cdot \nabla) B_\varepsilon - \frac{1}{2} \nabla \left( |E_\varepsilon|^2 + |B_\varepsilon|^2 \right)
\]

\[
= (E_\varepsilon \cdot \nabla) E_\varepsilon - \text{div}(B_\varepsilon \otimes B_\varepsilon) - \frac{1}{2 \varepsilon} \nabla \left( |E_\varepsilon|^2 + |B_\varepsilon|^2 \right).
\]

Combining (43), (41) and (42), we can infer that

\[
\frac{d}{dt} \int_{\mathbb{T}^3} (u_\varepsilon + \gamma_\varepsilon E_\varepsilon \times B_\varepsilon) \, dx = 0.
\]
Multiplying the first equation of (37) by $\frac{|v|^2-3}{3}$ and then integrating over the phase space, we can infer that
\[
\frac{d}{dt} \int_{T^3} \theta_\epsilon(t)dx = - \int_{T^3} \int_{\mathbb{R}^3} \left( \frac{v}{\epsilon} \frac{r}{3} \left( \frac{\alpha_x E_x + \beta_x v \times B_x}{\epsilon} \right) \cdot \nabla v (Mg_\epsilon) \right) dv dx \\
= \frac{2}{3} \int_{T^3} \int_{\mathbb{R}^3} \left( \frac{v}{\epsilon} \frac{r}{3} \left( \frac{\alpha_x E_x + \beta_x v \times B_x}{\epsilon} \right) (Mg_\epsilon) \right) dv dx \\
= \frac{2\gamma}{3\gamma} \int_{T^3} E_\epsilon \cdot j_\epsilon dx.
\]  
By the similar trick of deducing (42), we can infer that
\[
\epsilon \frac{d}{dt} \int_{T^3} |E_\epsilon|^2 + |B_\epsilon|^2 dx = \frac{\beta}{\gamma} \int_{T^3} E_\epsilon \cdot j_\epsilon dx.
\]  
Combining (45) and (46), we can infer that
\[
\frac{d}{dt} \int_{T^3} \left( \theta_\epsilon + \epsilon \frac{2}{3} |E_\epsilon|^2 + |B_\epsilon|^2 \right) (t) dx = 0. \tag{47}
\]
\[
\square
\]

Since the proof of uniform estimates is very involved, we split it into several parts.

4.2. The dissipative estimates of the microscopic part.

**Lemma 4.2** (only related to $\nabla^k_x$). Under the assumption in the section 2.3 and the assumption (26) on the initial data, if $(f_\epsilon, g_\epsilon, B_\epsilon, E_\epsilon)$ are solutions to (5), then
\[
\frac{d}{dt} \| (f_\epsilon, g_\epsilon, E_\epsilon, B_\epsilon) \|_{H^2_x}^2 + \frac{\beta_\epsilon}{\gamma^2} \left\| (f_\epsilon^\perp, g_\epsilon^\perp) \right\|_{H^2_x}^2 \\
\leq C \| (E_\epsilon, B_\epsilon, f_\epsilon, g_\epsilon) \|_{H^2_x}^2 \left\| (f_\epsilon, g_\epsilon) \right\|_{H^2_x}^2. \tag{48}
\]

**Remark 4.3.** The term $\frac{\beta_\epsilon}{\gamma^2} v \times B_\epsilon \cdot \nabla v g_\epsilon$ brings new difficulties. Indeed, There already exists derivative with respect to $v$ in the Lorentz force. While $\beta_\epsilon = O(1)$, $\frac{\beta_\epsilon}{\gamma^2} = \frac{1}{\epsilon}$. The idea to deal with this difficulty is to decompose $f_\epsilon$ and $g_\epsilon$ into fluid part and microscopic part (see (54)).

**Proof.** Applying $\nabla^k_x$ to the first four equations of (37) and then multiplying the resulting equations by $\nabla^k_x f_\epsilon, M, \nabla^k_x g_\epsilon, M, \nabla^k_x E_\epsilon M$ and $\nabla^k_x B_\epsilon M$, the integration over the phase space leads to
\[
\frac{d}{dt} \| \nabla^k_x (f_\epsilon, g_\epsilon, E_\epsilon, B_\epsilon) \|_{L^2_x}^2 + \frac{\beta_\epsilon}{\gamma^2} \int_{T^3} E_\epsilon \cdot j_\epsilon dx + \frac{\beta_\epsilon}{\gamma^2} \int_{T^3} E_\epsilon \cdot j_\epsilon dx \\
= \frac{1}{\epsilon^2} \int_{T^3} \int_{\mathbb{R}^3} \left( \nabla^k_x f_\epsilon \cdot \nabla^k_x f_\epsilon + L \left( \nabla^k_x g_\epsilon \cdot \nabla^k_x g_\epsilon \right) M v dx \\
= - \frac{1}{\epsilon^2} \int_{T^3} \int_{\mathbb{R}^3} \left( \nabla^k_x \left( \frac{\alpha_x E_x + \beta_x v \times B_x}{Me} \cdot \nabla v (Mg_\epsilon) \right) \nabla^k_x f_\epsilon + \nabla^k_x \left( \frac{\alpha_x E_x + \beta_x v \times B_x}{Me} \cdot \nabla v (Mf_\epsilon) \right) \nabla^k_x g_\epsilon \right) M v dx \\
+ \frac{1}{\epsilon^2} \int_{T^3} \int_{\mathbb{R}^3} \left( \nabla^k_x \Gamma (f_\epsilon, f_\epsilon) \cdot \nabla^k_x f_\epsilon + \nabla^k_x \Gamma (g_\epsilon, f_\epsilon) \cdot \nabla^k_x g_\epsilon \right) M v dx \\
:= D_1 + D_2.
\]
Recalling that $\alpha, \gamma_\epsilon = \epsilon \beta_\epsilon$ and by the assumptions on $L$ and $L$, we can infer that
\[
\frac{d}{dt} \| \nabla^k_x (f_\epsilon, g_\epsilon, E_\epsilon, B_\epsilon) \|_{L^2_x}^2 + \frac{1}{\epsilon^2} \left( \| \nabla^k_x f_\epsilon^\perp \|_{L^2_x}^2 + \| \nabla^k_x g_\epsilon^\perp \|_{L^2_x}^2 \right) \leq D_1 + D_2. \tag{50}
\]

Now, we try to estimate $D_1$ first. For $\beta = 1$, $\frac{\beta_\epsilon}{\gamma^2} = \epsilon^{-1}$ is unbounded while $\epsilon \to 0$. Since $\alpha \leq \epsilon$, $\frac{\beta_\epsilon}{\gamma^2} \leq 1$. Thus we only need to pay attention to the term with coefficient $\frac{\beta_\epsilon}{\gamma^2}$. Secondly, while $k = s$ and all the derivative acts on $g_\epsilon$ and $f_\epsilon$, that is to say,
\[
\int_{T^3} \int_{\mathbb{R}^3} \left( \left( \frac{\alpha_x E_x + \beta_x v \times B_x}{Me} \right) (M \nabla^k_x \nabla v g_\epsilon) \right) \nabla^k_x f_\epsilon + \left( \frac{\alpha_x E_x + \beta_x v \times B_x}{Me} \right) (M \nabla^k_x \nabla v f_\epsilon) \nabla^k_x g_\epsilon \right) M v dx,
\]
the above term can not be directly controlled. To overcome these difficulties, we first split

\[ D_1 = \int_{T^3} \int_{\mathbb{R}^3} \left( \left( \frac{\alpha_x E_x + \beta_x v \times B_x}{\varepsilon} \cdot \nabla_x \nabla_x^k (M g_x) \right) \nabla_x^k \psi \right) M \, dv \, dx \\
+ \int_{T^3} \int_{\mathbb{R}^3} \left( \nabla_x^k \left( \frac{\alpha_x E_x + \beta_x v \times B_x}{\varepsilon} \cdot \nabla_x \nabla_x^k (M f_x) \right) \nabla_x^k \phi \right) M \, dv \, dx + D_r \\
:= D_2 + D_r. \tag{51} \]

\( D_2 \) is simple and can be bounded by integrating by parts over the phase space. Indeed, by simple computation, we can conclude that

\[ D_2 = \int_{T^3} \int_{\mathbb{R}^3} \left( \frac{\alpha_x E_x + \beta_x v \times B_x}{\varepsilon} \cdot \nabla_x \nabla_x^k (M (f_x + g_x)) \right) \nabla_x^k (f_x + g_x) \, dv \, dx \\
- \int_{T^3} \int_{\mathbb{R}^3} \left( \frac{\alpha_x E_x + \beta_x v \times B_x}{\varepsilon} \cdot \nabla_x \nabla_x^k (M f_x) \right) \nabla_x^k (f_x) \, dv \, dx \\
- \int_{T^3} \int_{\mathbb{R}^3} \left( \frac{\alpha_x E_x + \beta_x v \times B_x}{\varepsilon} \cdot \nabla_x \nabla_x^k (M g_x) \right) \nabla_x^k (g_x) \, dv \, dx \\
= -\frac{1}{2} \int_{T^3} \int_{\mathbb{R}^3} \left( \frac{\alpha_x E_x + \beta_x v \times B_x}{\varepsilon} \cdot \nabla_x \nabla_x^k (f_x + g_x) \right) \nabla_x^k (f_x + g_x) \, M \, dv \, dx \\
+ \frac{1}{2} \int_{T^3} \int_{\mathbb{R}^3} \left( \frac{\alpha_x E_x + \beta_x v \times B_x}{\varepsilon} \cdot \nabla_x \nabla_x^k (f_x) \right) \nabla_x^k (f_x) \, M \, dv \, dx \\
+ \frac{1}{2} \int_{T^3} \int_{\mathbb{R}^3} \left( \frac{\alpha_x E_x + \beta_x v \times B_x}{\varepsilon} \cdot \nabla_x \nabla_x^k (g_x) \right) \nabla_x^k (g_x) \, M \, dv \, dx \\
= -\frac{\alpha_x}{\varepsilon} \int_{T^3} \int_{\mathbb{R}^3} v \cdot E_x \nabla_x^k f_x \nabla_x^k (g_x) \, dv \, dx \\
\lesssim \frac{\alpha_x}{\varepsilon} \|E_x\|_{H^1} \|f_x\|_{H^k} \|g_x\|_{H^k_x} \] 

To deal with \( D_r \),

\[ D_r = \sum_{i \geq 1} \int_{T^3} \int_{\mathbb{R}^3} \nabla_x^k \left( \frac{\alpha_x E_x + \beta_x v \times B_x}{\varepsilon} \cdot \nabla_x \nabla_x^k (M g_x) \right) \nabla_x^k f_x \, M \, dv \, dx \\
+ \sum_{i \geq 1} \int_{T^3} \int_{\mathbb{R}^3} \nabla_x^k \left( \frac{\alpha_x E_x + \beta_x v \times B_x}{\varepsilon} \cdot \nabla_x \nabla_x^k (M f_x) \right) \nabla_x^k g_x \, M \, dv \, dx \\
:= D_{r1} + D_{r2}. \tag{53} \]

We need to use the structure of \( g_x \) to control \( D_{r2} \). Indeed, noticing that

\[ g_x(t, x, v) = n_x(t, x) + g_x^0(t, x, v), \]
thus, we can infer that $\nabla_v \mathcal{P} g_\epsilon = 0$. With the help of this fact, we can infer that

$$D_{r_2} = \sum_{i+j=k} \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \nabla^i \left( \frac{\alpha E_\epsilon + \beta v \times B_\epsilon}{\epsilon} \right) \cdot \nabla_v \nabla^j_x (M f_\epsilon) \nabla^k_x g_\epsilon \, dv \, dx$$

$$= \sum_{i+j=k} \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \nabla^i \left( \frac{\alpha E_\epsilon + \beta v \times B_\epsilon}{\epsilon} \right) \cdot \nabla_v \nabla^j_x (M f_\epsilon) \nabla^k_x g^\bot_\epsilon \, dv \, dx$$

$$+ \sum_{i+j=k} \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \nabla^i \left( \frac{\alpha E_\epsilon + \beta v \times B_\epsilon}{\epsilon} \right) \cdot \nabla_v \nabla^j_x (M f_\epsilon) \nabla^k_x \mathcal{P} g_\epsilon \, dv \, dx$$

$$= \sum_{i+j=k} \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \nabla^i \left( \frac{\alpha E_\epsilon + \beta v \times B_\epsilon}{\epsilon} \right) \cdot \nabla_v \nabla^j_x (M f_\epsilon) \nabla^k_x g^\bot_\epsilon \, dv \, dx$$

$$= \frac{\alpha}{\epsilon} \sum_{i+j=k} \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} v \cdot \nabla^i_x E_\epsilon \cdot \nabla_v \nabla^j_x f_\epsilon \cdot \nabla^k_x g^\bot_\epsilon \, Md \, dx$$

$$+ \frac{\alpha}{\epsilon} \sum_{i+j=k} \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \nabla^i_x E_\epsilon \cdot \nabla_v \nabla^j_x f_\epsilon \cdot \nabla^k_x g^\bot_\epsilon \, Md \, dx$$

$$+ \frac{\beta}{\epsilon} \sum_{i+j=k} \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} v \times \nabla^i_x B_\epsilon \cdot \nabla_v \nabla^j_x f_\epsilon \cdot \nabla^k_x g^\bot_\epsilon \, Md \, dx$$

$$= D_{r_{21}} + D_{r_{22}} + D_{r_{23}}.$$ 

The way of controlling $D_{r_{21}}, D_{r_{22}}$ and $D_{r_{23}}$ are similar. Here, we only take $D_{r_{23}}$ for example.

$$D_{r_{23}} \leq \frac{\beta}{\epsilon} \sum_{i+j=k} \int_{\mathbb{T}^3} \left| \nabla^i_x B_\epsilon \right| \int_{\mathbb{R}^3} |v| |\nabla_v \nabla^j_x f_\epsilon| \cdot |\nabla^k_x g^\bot_\epsilon| \, Md \, dx$$

$$\leq \frac{\beta}{\epsilon} \sum_{i+j=k} \int_{\mathbb{T}^3} \left| \nabla^i_x B_\epsilon(t, x) \right| \left| \nabla_v \nabla^j_x f_\epsilon(t, x) \right| L^2_{H^2_v} \left| \nabla^k_x g^\bot_\epsilon(t, x) \right| L^2_{H^2_v} \, dx$$

$$\leq \frac{\beta}{\epsilon} \sum_{i+j=k} \int_{\mathbb{T}^3} \left| \nabla^i_x B_\epsilon(t, x) \right| \left| \nabla_v \nabla^j_x f_\epsilon(t, x) \right| L^\infty_{L^2_{H^2_v}} \left| \nabla^k_x g^\bot_\epsilon(t, x) \right| L^2_{L^2_{H^2_v}} \, dx$$

$$+ \frac{\beta}{\epsilon} \sum_{i+j=k} \int_{\mathbb{T}^3} \left| \nabla^i_x B_\epsilon(t, x) \right| \left| \nabla_v \nabla^j_x f_\epsilon(t, x) \right| L^\infty_{L^2_{H^2_v}} \left| \nabla^k_x g^\bot_\epsilon(t, x) \right| L^2_{L^2_{H^2_v}} \, dx$$

$$\leq \frac{\beta}{\epsilon} \left| \nabla_v \nabla^j_x f_\epsilon(t, x) \right| L^\infty_{L^2_{H^2_v}} \sum_{i+j=k} \int_{\mathbb{T}^3} \left| \nabla^i_x B_\epsilon(t, x) \right| \left| \nabla^k_x g^\bot_\epsilon(t, x) \right| L^2_{L^2_{H^2_v}} \, dx$$

$$+ \frac{\beta}{\epsilon} \left| \nabla^i_x B_\epsilon(t, x) \right| L^\infty_{L^2_{H^2_v}} \sum_{i+j=k} \int_{\mathbb{T}^3} \left| \nabla_v \nabla^j_x f_\epsilon(t, x) \right| \left| \nabla^k_x g^\bot_\epsilon(t, x) \right| L^2_{L^2_{H^2_v}} \, dx.$$

Then by the Sobolev embedding inequalities (for $x$) and Hölder inequality, we can infer

$$D_{r_2} \leq C \| (E_\epsilon, B_\epsilon) \|_{H^2}^2 \| f_\epsilon \|_{H^2_v}^2 + \frac{1}{8\epsilon^2} \| \nabla^k_x g^\bot_\epsilon \|_{L^2_{H^2_v}}^2.$$

(55)
The trick of controlling $D_{r1}$ is very complicated. Except for employing the structure of $g_{\epsilon}$, we also need to split $f_{\epsilon}$ into macroscopic part and microscopic part. Indeed,

$$D_{r1} = \sum_{i \geq 1}^{i+j=k} \int_{T \in R^3} \nabla_x^i \left( \frac{\alpha E_e + \beta v \times B_e}{\epsilon} \right) \cdot \nabla_x^j (M_{g_{\epsilon}}) \nabla_x^k f_{\epsilon} \, dv \, dx$$

$$+ \sum_{i \geq 1}^{i+j=k} \int_{T \in R^3} \nabla_x^i \left( \frac{\alpha E_e + \beta v \times B_e}{\epsilon} \right) \cdot \nabla_x^j (M_{g_{\epsilon}}) \nabla_x^k \mathcal{P} f_{\epsilon} \, dv \, dx$$

$$+ \sum_{i \geq 1}^{i+j=k} \int_{T \in R^3} \nabla_x^i \left( \frac{\alpha E_e + \beta v \times B_e}{\epsilon} \right) \cdot \nabla_x^j (M_{g_{\epsilon}}) \nabla_x^k \mathcal{P} f_{\epsilon} \, dv \, dx.$$  \hspace{1cm} (56)

Except for the second term in the right hand of (56), the other two terms are easy to be bounded. Indeed, for the first term in (56), we can infer that

$$\sum_{i \geq 1}^{i+j=k} \int_{T \in R^3} \nabla_x^i \left( \frac{\alpha E_e + \beta v \times B_e}{\epsilon} \right) \cdot \nabla_x^j (M_{g_{\epsilon}}) \nabla_x^k f_{\epsilon} \, dv \, dx$$

$$= -\frac{A_\epsilon}{e} \sum_{i \geq 1}^{i+j=k} \int_{T \in R^3} v \cdot \nabla_x^i E_e \cdot \nabla_x^j g_{\epsilon} \nabla_x^k \mathcal{P} f_{\epsilon} \, dv \, dx$$

$$+ \frac{A_\epsilon}{\epsilon} \sum_{i \geq 1}^{i+j=k} \int_{T \in R^3} \nabla_x^i E_e \cdot \nabla_x^j g_{\epsilon} \nabla_x^k \mathcal{P} f_{\epsilon} \, dv \, dx$$

$$\leq C \|(E_e, B_e)\|_{H^2}^2 \|(f_{\epsilon}, g_{\epsilon})\|_{H^\Lambda}^2 + \frac{1}{10\epsilon x} \|\nabla_x^k f_{\epsilon} \|_{L^\Lambda}^2.$$ \hspace{1cm} (57)

Similarly, we can infer that

$$\sum_{i \geq 1}^{i+j=k} \int_{T \in R^3} \nabla_x^i \left( \frac{\alpha E_e + \beta v \times B_e}{\epsilon} \right) \cdot \nabla_x^j (M_{g_{\epsilon}}) \nabla_x^k \mathcal{P} f_{\epsilon} \, dv \, dx$$

$$\leq C \|(E_e, B_e)\|_{H^2}^2 \|(f_{\epsilon}, g_{\epsilon})\|_{H^\Lambda}^2 + \frac{1}{10\epsilon x} \|\nabla_x^k f_{\epsilon} \|_{L^\Lambda}^2.$$ \hspace{1cm} (58)

The second term in (56) is more direct. Recalling that

$$\mathcal{P} f_{\epsilon} = \rho_{\epsilon} + u_{\epsilon} \cdot v + \theta_{\epsilon} \|v\|^{2-3}_2, \quad \nabla_{\epsilon} n_{\epsilon} = 0,$$

then it follows that

$$\sum_{i \geq 1}^{i+j=k} \int_{T \in R^3} \nabla_x^i \left( \frac{\alpha E_e + \beta v \times B_e}{\epsilon} \right) \cdot \nabla_x^j (M_{g_{\epsilon}}) \nabla_x^k \mathcal{P} f_{\epsilon} \, dv \, dx$$

$$= -\frac{A_\epsilon}{e} \sum_{i \geq 1}^{i+j=k} \int_{T \in R^3} (v \cdot \nabla_x^i E_e) \cdot \nabla_x^j n_{\epsilon} \nabla_x^k \mathcal{P} f_{\epsilon} \, dv \, dx$$

$$\leq C \|(E_e, B_e, f_{\epsilon}, g_{\epsilon})\|_{H^2}^2 \|(f_{\epsilon}, g_{\epsilon})\|_{H^\Lambda}^2.$$ \hspace{1cm} (59)

With the help of (56), (57), (58), (59) and (55), we can finally conclude

$$D_{1r} \leq C \|(E_e, B_e, f_{\epsilon}, g_{\epsilon})\|_{H^2}^2 \|(f_{\epsilon}, g_{\epsilon})\|_{H^\Lambda}^2 + \frac{1}{10\epsilon x} \|\nabla_x^k f_{\epsilon} \|_{L^\Lambda}^2,$$ \hspace{1cm} (60)

and

$$D_r \leq C \|(E_e, B_e, f_{\epsilon}, g_{\epsilon})\|_{H^2}^2 \|(f_{\epsilon}, g_{\epsilon})\|_{H^\Lambda}^2 + \frac{1}{10\epsilon x} \|\nabla_x^k (f_{\epsilon}^L, g_{\epsilon}^L)\|_{L^\Lambda}^2.$$ \hspace{1cm} (61)

Finally, with the help of (52) and (61), for the $D_1$ in (50), we can obtain that

$$D_1 \leq C \|(E_e, B_e, f_{\epsilon}, g_{\epsilon})\|_{H^2}^2 \|(f_{\epsilon}, g_{\epsilon})\|_{H^\Lambda}^2 + \frac{1}{8\epsilon x} \|\nabla_x^k (f_{\epsilon}^L, g_{\epsilon}^L)\|_{L^\Lambda}^2.$$ \hspace{1cm} (62)
For the nonlinear collision operator \((D_n\text{ in (50)})\), by the assumption, we can infer that
\[
D_n = \frac{1}{2} \int_{\mathbb{T}^d} \left( \nabla_x^k \Gamma(f, f_\epsilon) \cdot \nabla_x^k f_\epsilon + \nabla_x^k \Gamma(g, f_\epsilon) \cdot \nabla_x^k g_\epsilon \right) \text{Md}v dx \\
\leq C \|(f, g_\epsilon)\|_{H_x^2} \|f_\epsilon, g_\epsilon\|_{H_{\alpha, 2}} \|\nabla_x^k (f, g, f_\epsilon)\|_{L_A^2} (63)
\]
Combing \((49), (50), (62)\) and \((63)\), we complete the proof of this lemma.

Lemma 4.2 is only related to the derivative of \(f_\epsilon\) and \(g_\epsilon\) with respect to \(x\). The following lemma is devoted to the estimates of \(\nabla_x^i \nabla_x^j g_\epsilon\) \((i \geq 1)\). Loosely speaking, the goal of the following lemma is to obtain estimates like this
\[
\epsilon^2 \frac{d}{dt} \|\nabla_v f, \nabla_v g_\epsilon\|_{H_{\alpha, 2}}^2 + \|\nabla_v f, \nabla_v g_\epsilon\|_{H_{\alpha, 2}}^2 \leq \|(f_\epsilon, g_\epsilon, \alpha E_\epsilon)\|_{H_{\alpha, 2}}^2 + \cdots, \ s = i + j, \ i \geq 1.
\]
Since there exists a coefficient \(\epsilon^2\) before \(\frac{1}{H_{\alpha, 2}} \|\nabla_x^i \nabla_x^j f_\epsilon, \nabla_x^i \nabla_x^j g_\epsilon\|_{L^2}^2\). Compared to Lemma 4.2, it is simpler. Indeed, the singular term (the one with coefficient \(\frac{1}{\epsilon^2}\)) is easily estimated. Before stating this lemma, we introduce the equivalent norm of \(\dot{H}^k\)
\[
\dot{H}^k(t) = 8 \sum_{i+j=k} \|\nabla_x^i \nabla_x^j f_\epsilon, \nabla_x^i \nabla_x^j g_\epsilon\|_{L^2}^2 + \|\nabla_x^k f_\epsilon, \nabla_x^k g_\epsilon\|_{L^2}^2
\]

Lemma 4.4. Under the assumption in the section 2.3 and the assumption (26) on the initial data, if \((f_\epsilon, g_\epsilon, B_\epsilon, E_\epsilon)\) are solutions to (5), then
\[
\epsilon^2 \frac{d}{dt} \left( \frac{1}{\epsilon^2} \sum_{m=1}^{s \epsilon} \hat{H}_{v, e}^m(t) + \hat{H}_{v, e}^s(t) \right) + \frac{1}{\epsilon} \|\nabla_v f, \nabla_v g_\epsilon\|_{H_{\alpha, 2}}^2 \\
\leq b_1 \|(f, g_\epsilon)\|_{H_{\alpha, 2}}^2 + b_1 \alpha^2 \|E_\epsilon\|_{H_{\alpha, 2}}^2 + C \|(f_\epsilon, g_\epsilon, E_\epsilon, B_\epsilon)\|_{H_{\alpha, 2}} \|\nabla_x^k (f_\epsilon, g_\epsilon)\|_{L_A^2}^2 (64)
\]
where \(c_1\) comes from the computation and is only dependent of the Sobolev embedding constants.

Proof. Applying \(\nabla_x^i \nabla_x^j\) to the first two equations of (5), then multiplying the resulting equations by \(\epsilon^2 \nabla_x^i \nabla_x^j f_\epsilon, M\) and \(\epsilon^2 \nabla_x^i \nabla_x^j g_\epsilon, M\) respectively, the integration over \(\mathbb{T}^d \times \mathbb{R}^3\) leads to
\[
\epsilon^2 \frac{d}{dt} \left( \int_{\mathbb{T}^d} \int_{\mathbb{R}^3} \left( \nabla_x^i \nabla_x^j \nabla_v^k f_\epsilon \cdot \nabla_x^i \nabla_x^j \nabla_v^k f_\epsilon + \nabla_x^i \nabla_x^j \nabla_v^k L(g, f_\epsilon) \cdot \nabla_x^i \nabla_x^j \nabla_v^k g_\epsilon \right) \text{Md}v dx \right) \\
= -\epsilon \int_{\mathbb{T}^d} \int_{\mathbb{R}^3} \left( \nabla_x^i \nabla_x^j \nabla_v^k (v \cdot \nabla x f_\epsilon) \cdot \nabla_x^i \nabla_x^j \nabla_v^k f_\epsilon + \nabla_x^i \nabla_x^j \nabla_v^k (v \cdot \nabla x g, f_\epsilon) \cdot \nabla_x^i \nabla_x^j \nabla_v^k g_\epsilon \right) \text{Md}v dx \\
+ \epsilon \int_{\mathbb{T}^d} \int_{\mathbb{R}^3} \left( \nabla_x^i \nabla_x^j \nabla_v^k N_f \cdot \nabla_x^i \nabla_x^j \nabla_v^k f_\epsilon + \nabla_x^i \nabla_x^j \nabla_v^k N_g \cdot \nabla_x^i \nabla_x^j \nabla_v^k g_\epsilon \right) \text{Md}v dx \\
+ \alpha E_\epsilon \int_{\mathbb{T}^d} \int_{\mathbb{R}^3} \left( \nabla_x^i \nabla_x^j \nabla_v^k (v \cdot E_\epsilon) \cdot \nabla_x^i \nabla_x^j \nabla_v^k g_\epsilon \right) \text{Md}v dx \\
= T_1 + T_2 + T_3.
\]
For the left hand of \((65)\), by the assumption on the linear Boltzmann operator, we can infer that
\[
\int_{\mathbb{T}^d} \int_{\mathbb{R}^3} \left( \nabla_x^i \nabla_x^j \nabla_v^k L(f, f_\epsilon) \cdot \nabla_x^i \nabla_x^j \nabla_v^k f_\epsilon + \nabla_x^i \nabla_x^j \nabla_v^k L(g, f) \cdot \nabla_x^i \nabla_x^j \nabla_v^k g_\epsilon \right) \text{Md}v dx.
\]
We first deal with the quadratic term \((T_1 \text{ and } T_3)\) in the right hand of \((65)\). For \(T_3\), by Hölder inequality, we can infer that
\[
T_3 \leq 4 \alpha^2 \|\nabla_x^i \nabla_x^j (v \cdot E_\epsilon)\|_{L^2}^2 + \frac{1}{16} \|\nabla_x^i \nabla_x^j g_\epsilon\|_{L^2}^2.
\]
For $T_1$, by integration by parts over the phase space, we can infer that
\begin{equation}
T_1 \leq \varepsilon \|\nabla v^{-1} \nabla f \|_{L^2} \|\nabla v \nabla f \|_{L^2} + \varepsilon \|\nabla v^{-1} \nabla g \|_{L^2} \|\nabla v \nabla g \|_{L^2} 
\end{equation}
\begin{equation}
\leq 4 \|\nabla v^{-1} \nabla f \|_{L^2}^2 + \frac{1}{4} \|\nabla v \nabla g \|_{L^2}^2.
\end{equation}

For the triple term $T_2$, the way of controlling this term is similar to that of $D_1$ and $D_n$ in Lemma 4.2. Recalling that
\begin{align*}
N_f &= \frac{\alpha}{\varepsilon} E_v \cdot v \cdot g \cdot \frac{\beta}{\varepsilon} \left( E_v + \frac{2}{\varepsilon} v \times B_v \right) \cdot \nabla g + \frac{1}{2} \Gamma(f, f), \\
N_g &= \frac{\alpha}{\varepsilon} E_v \cdot v \cdot f \cdot \frac{\beta}{\varepsilon} \left( E_v + \frac{2}{\varepsilon} v \times B_v \right) \cdot \nabla f + \frac{1}{2} \Gamma(f, f),
\end{align*}
we can split $T_2$ as follows
\begin{align*}
T_2 &= \varepsilon \iint_{B^3} \nabla v \nabla \left( \alpha E_v \cdot v \cdot g \cdot \left( \frac{\alpha}{\varepsilon} E_v + \frac{\beta}{\varepsilon} \left( E_v + \frac{2}{\varepsilon} v \times B_v \right) \cdot \nabla g + \frac{1}{2} \Gamma(f, f) \right) \cdot \nabla v \nabla f \cdot m \cdot d \nu \cdot d x \\
&\quad + \varepsilon \iint_{B^3} \nabla v \nabla \left( \alpha E_v \cdot v \cdot f \cdot \left( \frac{\alpha}{\varepsilon} E_v + \frac{\beta}{\varepsilon} \left( E_v + \frac{2}{\varepsilon} v \times B_v \right) \cdot \nabla f + \frac{1}{2} \Gamma(f, f) \right) \cdot \nabla v \nabla f \cdot m \cdot d \nu \cdot d x
\end{align*}
\begin{equation}
= T_{21} + T_{22} + T_{23}.
\end{equation}

By the same trick of dealing with $D_1$, we can infer that
\begin{equation}
T_{21} + T_{22} \leq \varepsilon \|\varepsilon (E_v, B_v)\|_{L^2} \|\varepsilon (f, g)\|_{H^1}^2.
\end{equation}

By the assumption on the quadratic collision operator, we can infer
\begin{equation}
T_{23} \leq \varepsilon \|\varepsilon (f, g)\|_{H^1} \|\varepsilon (f, g)\|_{H^1}.
\end{equation}

Combining (69), (70) and (71), we can infer that
\begin{equation}
T_2 \leq C \|\varepsilon (f, g, E_v, B_v)\|_{H^2} \|\varepsilon (f, g)\|_{L^2}^2 + C \varepsilon \|\varepsilon (f, g)\|_{H^1} \|\varepsilon (f, g)\|_{H^1}.
\end{equation}

Combining (66), (68), (67) and (72), we can infer that
\begin{align*}
\frac{c^2}{2} \frac{d}{d t} \left( \|\nabla v \nabla f \|_{L^2} \right)^2 + \frac{4}{3} \left( \|\nabla v \nabla g \|_{L^2} \right)^2 \\
\leq 4 \|\nabla v \nabla f \|_{L^2}^2 + 4 \|\nabla v \nabla g \|_{L^2}^2 + C \|\varepsilon (f, g)\|_{H^1}^2.
\end{align*}

For $i = 1, j = k - 1$, (73) becomes
\begin{align*}
\frac{c^2}{2} \frac{d}{d t} \left( \|\nabla^{-1} v \nabla f \|_{L^2} \right)^2 + \frac{4}{3} \left( \|\nabla v \nabla g \|_{L^2} \right)^2 \\
\leq 4 \|\nabla^{-1} v \nabla f \|_{L^2}^2 + 4 \|\nabla^{-1} v \nabla g \|_{L^2}^2 + C \|\varepsilon (f, g)\|_{H^1}^2.
\end{align*}

For $i = 2, j = k - 2$, (73) becomes
\begin{align*}
\frac{c^2}{2} \frac{d}{d t} \left( \|\nabla^{-2} v \nabla f \|_{L^2} \right)^2 + \frac{4}{3} \left( \|\nabla v \nabla g \|_{L^2} \right)^2 \\
\leq 4 \|\nabla^{-2} v \nabla f \|_{L^2}^2 + C \|\varepsilon (f, g)\|_{H^1}^2.
\end{align*}

For $i = k, j = 0$, (73) becomes
\begin{align*}
\frac{c^2}{2} \frac{d}{d t} \left( \|\nabla v \nabla f \|_{L^2} \right)^2 + \frac{4}{3} \left( \|\nabla v \nabla g \|_{L^2} \right)^2 \\
\leq 4 \|\nabla^{-1} v \nabla f \|_{L^2}^2 + C \|\varepsilon (f, g)\|_{H^1}^2.
\end{align*}
From $8 \times (74) + 8 \times (75) + \cdots + (76)$, it follows that there exists $d_1$ such that

$$
\epsilon^2 \frac{d}{2dt} \left( 8 \sum_{i \geq 1, j \geq 1} \|(\nabla_v \nabla_x^j f_e, \nabla_v \nabla_x^j g_e)\|_L^2 + \|(\nabla_v^k f_e, \nabla_v^k g_e)\|_L^2 \right) + \frac{4}{2d} \sum_{i \geq 1, j \geq 1} \|(\nabla_v \nabla_x^j f_e, \nabla_v \nabla_x^j g_e)\|_L^2

\leq 4\alpha^2 \|(E_e)\|_{H^{k-1}_v}^2 + C \|(f_e, g_e)\|_{H^{k-1}_v}^2 + C \|(f_e, g_e, E_e, B_e)\|_{H^2_v} \|(f_e, g_e)\|_{H^2_v}

+ C \epsilon \|(\nabla_v f_e, \nabla_v g_e)\|_{H^{k-1}} \|(f_e, g_e)\|_{H^{k-1}} + d_1 \|(\nabla_v f_e, \nabla_v g_e)\|_{H^{k-2}}.
$$

(77)

Denoting

$$
\dot{H}^k_{v,e}(t) = 8 \sum_{i \geq 1, j \geq 1} \|(\nabla_v \nabla_x^j f_e, \nabla_v \nabla_x^j g_e)\|_L^2 + \|(\nabla_v^k f_e, \nabla_v^k g_e)\|_L^2,
$$

and choosing $\frac{3d}{4} \geq d_1 + \frac{4}{2d} (d_2 \geq 1)$, then we can deduce from (77)

$$
\epsilon^2 \frac{d}{2dt} \left( \frac{8d}{3} \sum_{k=1}^s \int_{\mathbb{R}^3} \dot{H}^m_{v,e}(t) + \frac{2}{4} \|(\nabla_v f_e, \nabla_v g_e)\|_{H^{k-1}_v}^2 \right)

\leq C \|(f_e, g_e)\|_{H^{k-1}_v}^2 + C \alpha^2 \|E_e\|_{H^{k-1}_v}^2 + C \|(f_e, g_e, E_e, B_e)\|_{H^2_v} \|(f_e, g_e)\|_{H^2_v}

+ C \epsilon \|(\nabla_v f_e, \nabla_v g_e)\|_{H^{k-1}} \|(f_e, g_e)\|_{H^{k-1}}.$$

(78)

where $\|h\|_{H^a} = \|h\|_{L^2}$.

By the similar method of deducing (77), we can infer that

$$
\epsilon^2 \frac{d}{2dt} \left( \frac{8d}{3} \sum_{k=1}^s \int_{\mathbb{R}^3} \dot{H}^m_{v,e}(t) \right) + \frac{2}{4} \|(\nabla_v f_e, \nabla_v g_e)\|_{H^{k-1}_v}^2

\leq C \|(f_e, g_e, \alpha E_e)\|_{H^{k-1}_v}^2 + C \|(f_e, g_e, E_e, B_e)\|_{H^2_v} \|(f_e, g_e)\|_{H^2_v}

+ C \epsilon \|(\nabla_v f_e, \nabla_v g_e)\|_{H^{k-1}} \|(f_e, g_e)\|_{H^{k-1}}.
$$

(79)

4.3. The dissipative estimates of the macroscopic parts. From Lemma 4.4, if we want to close the whole inequalities, we still need to obtain the dissipative energy estimates of $f_e$, $g_e$ and $E_e$. This subsection is devoted to the dissipative energy of $f_e$ and $g_e$.

**Lemma 4.5.** Under the assumption in the section 2.3 and the assumption (26) on the initial data, if $(f_e, g_e, B_e, E_e)$ are solutions to (5), then

$$
\epsilon \frac{d}{dt} \sum_{k=1}^s \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (\nabla_x \nabla_x^{k-1} f_e, \nabla_v \nabla_x^{k-1} f_e + \nabla_x \nabla_x^{k-1} g_e, \nabla_v \nabla_x^{k-1} g_e) \cdot Mdvdx

+ \frac{4}{2d} \|(\nabla_x f_e, \nabla_v g_e)\|_{H^{k-1}_v}^2 + \|\text{div} E_e\|_{H^{k-1}_v}^2 - \delta_2 \|\nabla_v (f_e, g_e)\|_{H^{k-1}_v}^2 - \delta_1 \|E_e\|_{H^{k-1}_v}^2

\leq \left( \frac{\delta}{d} + \frac{1}{d} \right) \frac{4}{2d} \|(f_e^k, g_e^k)\|_{H^{k-1}_v}^2 + C \|(f_e, g_e, E_e, B_e)\|_{H^2_v} \|(f_e, g_e)\|_{H^2_v}^2.
$$

(80)
Proof. Applying \( \nabla_v \nabla_x^{k-1} \) to the first two equations of (5), multiplying them by \( \nabla_x \nabla_x^{k-1} f_\varepsilon \) and \( \nabla_x \nabla_x^{k-1} g_\varepsilon \) respectively and then integrating the resulting equation over the phase space, we can infer that

\[
\epsilon \frac{d}{dt} \int_{\mathbb{R}^3} \left( \nabla_x \nabla_x^{k-1} f_\varepsilon \cdot \nabla_x \nabla_x^{k-1} f_\varepsilon + \nabla_x \nabla_x^{k-1} g_\varepsilon \cdot \nabla_x \nabla_x^{k-1} g_\varepsilon \right) \text{Md} \text{d}v \text{d}x
\]

\[
+ \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left( \nabla_v \nabla_x^{k-1} (v \cdot \nabla_x f_\varepsilon) \cdot \nabla_x \nabla_x^{k-1} f_\varepsilon + \nabla_x \nabla_x^{k-1} (v \cdot \nabla_x f_\varepsilon) \cdot \nabla_v \nabla_x^{k-1} f_\varepsilon \right) \text{Md} \text{d}v \text{d}x
\]

\[
+ \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left( \nabla_v \nabla_x^{k-1} (v \cdot \nabla_x g_\varepsilon) \cdot \nabla_x \nabla_x^{k-1} g_\varepsilon + \nabla_x \nabla_x^{k-1} (v \cdot \nabla_x g_\varepsilon) \cdot \nabla_v \nabla_x^{k-1} g_\varepsilon \right) \text{Md} \text{d}v \text{d}x
\]

\[
- \frac{1}{\varepsilon} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left( \nabla_v \nabla_x^{k-1} \mathcal{L}(f_\varepsilon) \cdot \nabla_x \nabla_x^{k-1} f_\varepsilon + \nabla_x \nabla_x^{k-1} \mathcal{L}(f_\varepsilon) \cdot \nabla_v \nabla_x^{k-1} f_\varepsilon \right) \text{Md} \text{d}v \text{d}x
\]

\[
- \frac{1}{\varepsilon} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left( \nabla_v \nabla_x^{k-1} \mathcal{L}(g_\varepsilon) \cdot \nabla_x \nabla_x^{k-1} g_\varepsilon + \nabla_x \nabla_x^{k-1} \mathcal{L}(g_\varepsilon) \cdot \nabla_v \nabla_x^{k-1} g_\varepsilon \right) \text{Md} \text{d}v \text{d}x
\]

\[
= \mathcal{M}_3 + \mathcal{M}_4.
\]

We will control the left hand of (81) respectively. By integration by parts, we can infer

\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left( \nabla_v \nabla_x^{k-1} (v \cdot \nabla_x f_\varepsilon) \cdot \nabla_x \nabla_x^{k-1} f_\varepsilon + \nabla_x \nabla_x^{k-1} (v \cdot \nabla_x f_\varepsilon) \cdot \nabla_v \nabla_x^{k-1} f_\varepsilon \right) \text{Md} \text{d}v \text{d}x
\]

\[
= \| v \cdot \nabla_x \nabla_x^{k-1} f_\varepsilon \|_2^2 \| v \cdot \nabla_v \nabla_x^{k-1} f_\varepsilon \|_2^2 + 2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \nabla_x \nabla_x^{k-1} (v \cdot \nabla_x f_\varepsilon) \cdot \nabla_v \nabla_x^{k-1} f_\varepsilon \text{M} \text{d}v \text{d}x.
\]
For $M$ by the assumptions on the linear Boltzmann operator, we can deduce that for $M$ is easy to be controlled. Indeed,

$$|M_1| \leq \frac{1}{\varepsilon} \left| \int_{\mathbb{R}^3} \nabla \cdot (\nabla x \nabla x^{-1} L(g_\varepsilon) \cdot \nabla v \nabla x^{-1} g_\varepsilon) \, \mathrm{d}v \, \mathrm{d}x \right| \leq \frac{1}{\varepsilon} \left( \| \nabla f_\varepsilon^+ \|_{L^2}^2 + \frac{\delta}{\varepsilon} \| \nabla v \nabla x^{-1} f_\varepsilon \|_{L^2}^2 \right).$$

For $M_{12}$, by the decomposition of $\nabla x \nabla x^{-1} f_\varepsilon$, we can obtain that

$$|M_{12}| \leq \frac{1}{\varepsilon} \left| \int_{\mathbb{R}^3} \nabla v \nabla x^{-1} L(f_\varepsilon) \cdot \nabla x \nabla x^{-1} f_\varepsilon \, \mathrm{d}v \, \mathrm{d}x \right| \leq \frac{1}{\varepsilon} \left| \int_{\mathbb{R}^3} \nabla v \nabla x^{-1} L(f_\varepsilon) \cdot \nabla x \nabla x^{-1} Pf_\varepsilon \, \mathrm{d}v \, \mathrm{d}x \right| + \frac{1}{\varepsilon} \left| \int_{\mathbb{R}^3} \nabla v \nabla L(f_\varepsilon^+) \cdot \nabla x \nabla x^{-1} f_\varepsilon \, \mathrm{d}v \, \mathrm{d}x \right| = M_{121} + M_{122}.$$

For $M_{121}$, by integration by parts, it follows that

$$M_{121} = \frac{1}{\varepsilon} \left| \int_{\mathbb{R}^3} \left( \nabla \nabla x^{-1} f_\varepsilon \cdot \nabla x \nabla x^{-1} \nabla v (P f_\varepsilon M) \right) \, \mathrm{d}v \, \mathrm{d}x \right| \leq C \frac{1}{\varepsilon} \| \nabla x^{-1} f_\varepsilon \|_{L^2} \| \nabla x \nabla x^{-1} \nabla v \| \| \nabla x \nabla x^{-1} Pf_\varepsilon \|_{L^2} \leq \frac{C}{\varepsilon} \| \nabla x^{-1} f_\varepsilon \|_{H^1_{\alpha_x}}^2 + \frac{\delta}{\varepsilon} \| \nabla x \nabla x^{-1} f_\varepsilon \|_{H^1_{\alpha_x}}^2.$$

By the assumptions on the linear Boltzmann operator, we can deduce that for $M_{122}$

$$M_{122} \leq C \frac{1}{\varepsilon} \left( \| \nabla x^{-1} f_\varepsilon \|_{L^2} + \| \nabla v \nabla x^{-1} f_\varepsilon \|_{L^2} \right) \| \nabla x \nabla x^{-1} f_\varepsilon \|_{L^2} \leq \frac{C}{\varepsilon^2} \| \nabla x^{-1} f_\varepsilon \|_{H^1_{\alpha_x}}^2 + \frac{\delta}{\varepsilon} \| \nabla x \nabla x^{-1} f_\varepsilon \|_{H^1_{\alpha_x}}^2.$$

With help of the related estimates of $M_1$, we can finally have

$$|M_1| \leq \frac{C}{\varepsilon^2} \| \nabla x^{-1} f_\varepsilon \|_{H^1_{\alpha_x}}^2 + \delta \| \nabla v \nabla x^{-1} f_\varepsilon \|_{H^1_{\alpha_x}}^2 + \frac{1}{\varepsilon} \| \nabla x \nabla x^{-1} g_\varepsilon \|_{L^2}^2.$$

Similarly, for $M_2$, we can obtain that

$$|M_2| \leq \frac{C}{\varepsilon^2} \| \nabla x^{-1} g_\varepsilon \|_{H^1_{\alpha_x}}^2 + \delta \| \nabla v \nabla x^{-1} g_\varepsilon \|_{H^1_{\alpha_x}}^2 + \frac{1}{\varepsilon} \| \nabla x \nabla x^{-1} g_\varepsilon \|_{L^2}^2.$$

For the last term in the left hand of (81), by integration by parts, it follows that

$$\int_{\mathbb{R}^3} \left( \nabla \nabla x^{-1} (v \cdot E_\varepsilon) \cdot \nabla x \nabla x^{-1} g_\varepsilon + \nabla x \nabla x^{-1} (v \cdot E_\varepsilon) \cdot \nabla v \nabla x^{-1} g_\varepsilon \right) \, \mathrm{d}v \, \mathrm{d}s \leq -\| \text{div}(\nabla x^{-1} E_\varepsilon) \|_{L^2}^2 + \delta \frac{\alpha^2}{\varepsilon^2} \| \nabla x^{-1} E_\varepsilon \|_{L^2}^2 + \frac{C}{\varepsilon} \| \nabla x g_\varepsilon \|_{L^2}^2,$$

where we have used the fact that

$$\text{div} E_\varepsilon = \frac{\alpha}{\varepsilon} \int_{\mathbb{R}^3} g_\varepsilon \, \mathrm{d}v, \quad \nabla v g_\varepsilon = \nabla x g_\varepsilon^+.$$
In the light of (81), (86), (89), (90) and (91), there exists some $c_2$ such that

$$
\begin{align*}
\epsilon \frac{d}{dt} \int_{\mathbb{R}^3} (\nabla \times \nabla^{k-1} f_x \cdot \nabla_v \nabla^{k-1} f_x + \nabla_x \nabla^{k-1} g_x \cdot \nabla_v \nabla^{k-1} g_x) \, M \, dx \\
+ \frac{3}{4} \| (\nabla \times \nabla^{k-1} f_x, \nabla_x \nabla^{k-1} g_x) \|_{L^2}^2 - \frac{c_2}{\epsilon^2} \| \nabla^{k-1} (f_x, g_x) \|_{H^3_x}^2 \\
- c_2 \alpha^2 \| \nabla^k E_x \|_{L^2}^2 - \delta \| \nabla \nabla^{k-1} (f_x, g_x) \|_{L^2}^2 \\
\leq M_3 + M_4.
\end{align*}
$$

(92)

With the same trick to that of deducing (52) and (86) (integrating by parts three time), we can deduce that

$$
M_3 + M_4 \leq C\| (f_x, g_x, E_x, B_x) \|_{H^3_x} \| (f_x, g_x) \|_{H^3_x}.
$$

(93)

\[ \square \]

4.4. The dissipative estimates of the electromagnetic parts.

**Lemma 4.6.** Under the assumption in the section 2.3 and the assumption (26) on the initial data, if $(f_x, g_x, B_x, E_x)$ are solutions to (5), then for $k \leq s - 2$,

$$
\begin{align*}
\frac{d}{dt} \int_{T^3} (|\nabla \times \nabla^k E_x|^2 + |\nabla \times \nabla^k B_x|^2 - 2\alpha \cdot \nabla \times \nabla^k E_x \cdot \nabla \times \nabla^k E_x) \, dx \\
+ \frac{2\alpha^2}{\epsilon^2} \| \nabla \times \nabla^k E_x \|_{L^2}^2 - \frac{\alpha^2}{\epsilon^2} \| \nabla \times \nabla^k B_x \|_{L^2}^2 \\
\leq C \| (f_x, g_x, E_x, B_x) \|_{H^3_x}^2 \| (f_x, g_x) \|_{H^3_x}^2 + C(1 + \frac{2}{\epsilon^2}) \| \nabla^k g_x \|_{H^2_x}^2.
\end{align*}
$$

(94)

and

$$
\begin{align*}
\frac{d}{dt} \int_{T^3} (|E_x|^2 + |B_x|^2 - 2\alpha \cdot j_x \cdot E_x) \, dx \\
+ \frac{2\alpha^2}{\epsilon^2} \| E_x \|_{L^2}^2 - \frac{\alpha^2}{\epsilon^2} \| \nabla \times B_x \|_{L^2}^2 \\
\leq C \| (f_x, g_x, E_x, B_x) \|_{H^3_x}^2 \| (f_x, g_x) \|_{H^3_x}^2 + C(1 + \frac{2}{\epsilon^2}) \| g_x \|_{H^2_x}^2.
\end{align*}
$$

(95)

**Proof.** Multiplying the second equation of (37) by $\hat{v}M$ and then integrating over $\mathbb{R}^3$, we can infer that

$$
\partial_t \hat{j}_v \cdot \frac{1}{2} \text{div} \int_{\mathbb{R}^3} \hat{v} \otimes v, g_x M \, dx - \frac{\alpha}{\epsilon^2} \hat{E}_x - \frac{1}{\epsilon^2} \int_{\mathbb{R}^3} L(g_x) \hat{v} M \, dx = \int_{\mathbb{R}^3} \hat{v} N_3 M \, dx,
$$

(96)

with

$$
\sigma = \frac{1}{4} \int_{\mathbb{R}^3} \hat{v} \cdot v M \, dx.
$$

(97)

Applying the curl operator to the above equation, we finally obtain that

$$
\begin{align*}
\partial_t \nabla \times \nabla^{k} j_x \cdot \frac{1}{2} \nabla \times \left( \text{div} \int_{\mathbb{R}^3} \hat{v} \otimes v \nabla^k g_x M \, dx \right) - \frac{\alpha}{\epsilon^2} \nabla \times \nabla^k E_x + \frac{1}{\epsilon^2} \nabla \times \nabla^{k} j_x = \nabla \times \int_{\mathbb{R}^3} \hat{v} \nabla^k N_3 M \, dx,
\end{align*}
$$

(98)

where we have used the fact that

$$
\hat{j}_x = \int_{\mathbb{R}^3} g_x \cdot v M \, dx = \int_{\mathbb{R}^3} L(g_x) \cdot \hat{v} M \, dx.
$$

Multiplying (98) by $-\alpha \nabla \times \nabla^k E_x$, then integrating over torus, we finally deduce that

$$
\begin{align*}
-\alpha \int_{T^3} \partial_t \nabla \times \nabla^{k} j_x \cdot \nabla \times \nabla^k E_x \, dx - \frac{\alpha}{\epsilon^2} \int_{T^3} \nabla \times \left( \text{div} \int_{\mathbb{R}^3} \hat{v} \otimes v \nabla^k g_x M \, dx \right) \cdot \nabla \times \nabla^k E_x \, dx \\
+ \frac{2\alpha^2}{\epsilon^2} \| \nabla \times \nabla^k E_x \|_{L^2}^2 - \frac{\alpha^2}{\epsilon^2} \int_{T^3} \nabla \times \nabla^k j_x \cdot \nabla \times \nabla^k E_x \, dx = \alpha \int_{T^3} \nabla \times \int_{\mathbb{R}^3} \hat{v} \nabla^k N_3 M \cdot \nabla \times \nabla^k E_x \, dx.
\end{align*}
$$

(99)

Again, applying the curl operator to the third equation of (5)

$$
\gamma \partial_t \nabla \times E_x - \nabla \times \nabla \times B_x + \frac{\beta}{\epsilon} \nabla \times j_x = 0.
$$
From the above equation, we can infer that
\[
- \alpha \epsilon \int_{T^3} \nabla \times \nabla ^k j \cdot \partial_t \nabla \times \nabla ^k E \, dx + \frac{\alpha}{\gamma} \int_{T^3} \nabla \times \nabla \times \nabla ^k B \cdot \nabla \times \nabla ^k j \, dx \\
- \frac{\alpha \epsilon}{\gamma} \int_{T^3} \nabla \times \nabla ^k j \cdot \nabla \times \nabla ^k j \, dx = 0. 
\]

(100)

The last term in the right of (100) is hard to control. Applying the curl operator to the Ampere’s equation and Maxwell’s equation in (5), then we can infer that
\[
\frac{d}{dt} \| \nabla \times \nabla ^k E \cdot \nabla \times \nabla ^k B \|_{L^2}^2 = - \frac{\alpha}{\gamma} \int_{T^3} \nabla \times \nabla ^k j \cdot \nabla \times \nabla ^k E \, dx. 
\]

(101)

From (99), (100) and (101),
\[
\frac{d}{dt} \int_{T^3} \left( |\nabla \times \nabla ^k E|^2 + |\nabla \times \nabla ^k B|^2 - 2 \alpha \epsilon \cdot \nabla \times \nabla ^k j \cdot \nabla \times \nabla ^k E \right) \, dx + \sigma \frac{\alpha^2}{\gamma} \| \nabla \times \nabla ^k E \|_{L^2}^2 \\
= \frac{\alpha}{\gamma} \int_{T^3} \nabla \times \left( \text{div} \int_{R^3} \nabla \times \nabla ^k E \, dx \right) \cdot \nabla \times \nabla ^k E \, dx \\
+ \frac{\alpha^2}{\gamma} \int_{T^3} \nabla \times \nabla ^k j \cdot \nabla \times \nabla ^k j \, dx - \frac{\alpha}{\gamma} \int_{T^3} \nabla \times \nabla \times \nabla ^k B \cdot \nabla \times \nabla ^k j \, dx \\
- \alpha \epsilon \int_{T^3} \nabla \times \int_{R^3} \nabla \times \nabla ^k N g \cdot \nabla \times \nabla ^k E \, dx, 
\]

where we have used the following fact
\[
\alpha \epsilon \gamma = \epsilon \beta \epsilon. 
\]

For the first terms in the right hand of (102) and noticing that \( \nabla \times \nabla ^k E \in \text{Ker}^k \),
\[
\frac{\alpha}{\gamma} \int_{T^3} \left( \text{div} \int_{R^3} \nabla \times \nabla ^k E \, dx \right) \cdot \nabla \times \nabla ^k E \, dx \leq C \| \nabla \times \nabla ^k E \|_{L^2}^2 + \frac{\alpha^2}{16 \gamma} \| \nabla \times \nabla ^k E \|_{L^2}^2. 
\]

For the third term in (102), by integration by parts, it follows
\[
\frac{\alpha}{\gamma} \int_{T^3} \nabla \times \nabla \times \nabla ^k B \cdot \nabla \times \nabla ^k j \, dx = \int_{T^3} \nabla \times \nabla \times \nabla ^k j \cdot \nabla \times \nabla ^k B \, dx \\
\leq C \frac{\alpha^2}{\gamma} \| \nabla \times \nabla ^k E \|_{L^2}^2 + \frac{\alpha^2}{16 \gamma} \| \nabla \times \nabla ^k B \|_{L^2}^2, 
\]

where we have used the following fact that
\[
\| \nabla \times \nabla ^k j \|_{L^2}^2 \leq \| \nabla \times \int_{R^3} g \, dx \|_{L^2}^2. 
\]

(103)

Similarly, we can infer that
\[
\frac{\alpha^2}{\gamma} \int_{T^3} \nabla \times \nabla ^k j \cdot \nabla \times \nabla ^k j \, dx \leq C \frac{\alpha^2}{\gamma} \| \nabla \times \nabla ^k j \|_{L^2}^2. 
\]

(104)

For the last one in the right hand of (102), noticing that \( k \leq s - 2 \), it follows that
\[
\alpha \epsilon \int_{T^3} \nabla \times \int_{R^3} \nabla \times \nabla ^k N g \cdot \nabla \times \nabla ^k E \, dx \\
= \alpha \epsilon \cdot \epsilon \int_{T^3} \nabla \times \int_{R^3} \nabla \times \nabla ^k (E \cdot v \cdot f) \, dx \\
- \int_{T^3} \nabla \times \int_{R^3} \nabla \times \nabla ^k ((\alpha \epsilon + \beta \epsilon v \times B) \cdot \nabla f \cdot \Gamma(g, f) \cdot \nabla \times \nabla ^k E \, dx \\
\leq C \| \langle f, g \rangle \|_{R^3}^2 \| \langle f, g \rangle \|_{H^2}^2 + \frac{\alpha^2}{16 \gamma} \| \nabla \times \nabla ^k E \|_{L^2}^2. 
\]

(105)
By the similar way of deducing (102), we can infer that
\[
\frac{d}{dt} \int_{\mathbb{T}^3} \left( |E_\epsilon|^2 + |B_\epsilon|^2 - 2\alpha_\epsilon \cdot \tilde{j}_\epsilon \cdot E_\epsilon \right) dx + \sigma \frac{\alpha_\epsilon^2}{\tau^2} \|E_\epsilon\|_{L^2}^2 \\
= \frac{\alpha_\epsilon}{\tau} \int_{\mathbb{T}^3} \left( \text{div} \int_{\mathbb{R}^3} \tilde{v} \otimes v_g \cdot Mdv \right) \cdot E_\epsilon dx \\
+ \frac{\alpha_\epsilon^2}{\tau^2} \int_{\mathbb{T}^3} j_\epsilon \cdot \tilde{j}_\epsilon dx - \frac{\alpha_\epsilon}{\tau} \int_{\mathbb{T}^3} \nabla \times B_\epsilon \cdot \tilde{j}_\epsilon dx \\
- \alpha_\epsilon \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \tilde{v} N_g \cdot Mdv \cdot E_\epsilon dx,
\]
and
\[
\frac{d}{dt} \int_{\mathbb{T}^3} \left( |E_\epsilon|^2 + |B_\epsilon|^2 - 2\alpha_\epsilon \cdot \tilde{j}_\epsilon \cdot E_\epsilon \right) dx \\
+ \frac{3\alpha_\epsilon^2}{\tau^2} \|E_\epsilon\|_{L^2}^2 - \frac{\alpha_\epsilon^2}{16} \|\nabla \times B_\epsilon\|_{L^2}^2 \\
\leq C \|(f_\epsilon, g_\epsilon, E_\epsilon, B_\epsilon)\|_{H^\frac{1}{2}}^2 \|(f_\epsilon, g_\epsilon)\|_{H^1}^2 + C(1 + \frac{\sigma}{\tau^2}) \|g_\epsilon\|_{H^\frac{1}{2}}^2.
\]
In summary, we complete the proof. \(\Box\)

The next lemma gives the \(L^2\) dissipative estimates of \(\nabla \times B\).

**Lemma 4.7.** Under the assumption in the section 2.3 and the assumption (26) on the initial data, if \((f_\epsilon, g_\epsilon, E_\epsilon, B_\epsilon)\) are solutions to (5), then for \(k \leq s - 1\).
\[
-\frac{2\alpha_\epsilon^2}{\tau^2} \frac{d}{dt} \int_{\mathbb{T}^3} \nabla^k \nabla^k B_\epsilon \cdot \nabla \times \nabla^k B_\epsilon dx + \frac{3\alpha_\epsilon^2}{\tau^2} \|\nabla \times \nabla^k B_\epsilon\|_{L^2}^2 - \frac{\alpha_\epsilon^2}{\tau^2} \|\nabla \times \nabla^k E_\epsilon\|_{L^2}^2 \leq C \frac{\sigma^2}{\tau^2} \|\nabla^k g_\epsilon\|_{L^2}^2.
\]

**Proof.** Applying \(\nabla^k\) to the third and forth equations of (5), then multiplying the resulting equation by \(-\nabla \times \nabla^k B\) and \(-\nabla \times \nabla^k E\) respectively, then we can infer that
\[
-\frac{2\alpha_\epsilon^2}{\tau^2} \frac{d}{dt} \int_{\mathbb{T}^3} \nabla^k \nabla^k E_\epsilon \cdot \nabla \times \nabla^k B_\epsilon dx + \frac{\alpha_\epsilon^2}{\tau^2} \|\nabla \times \nabla^k B_\epsilon\|_{L^2}^2 - \frac{\alpha_\epsilon^2}{\tau^2} \|\nabla \times \nabla^k E_\epsilon\|_{L^2}^2 = -\frac{\beta_\epsilon^2}{\tau^2} \int_{\mathbb{T}^3} \nabla^k j_\epsilon \cdot \nabla \times \nabla^k B_\epsilon dx.
\]
Recalling that \(\frac{\alpha_\epsilon}{\tau} \leq 1\), then we can infer that
\[
-\frac{\beta_\epsilon^2}{\tau^2} \int_{\mathbb{T}^3} \nabla^k j_\epsilon \cdot \nabla \times \nabla^k B_\epsilon dx \leq C \frac{\sigma^2}{\tau^2} \|\nabla^k g_\epsilon\|_{L^2}^2 + \frac{\alpha_\epsilon^2}{\tau^2} \|\nabla \times \nabla^k B_\epsilon\|_{L^2}^2.
\]
and
\[
-\frac{2\alpha_\epsilon^2}{\tau^2} \frac{d}{dt} \int_{\mathbb{T}^3} \nabla^k \nabla^k E_\epsilon \cdot \nabla \times \nabla^k B_\epsilon dx + \frac{\alpha_\epsilon^2}{\tau^2} \|\nabla \times \nabla^k B_\epsilon\|_{L^2}^2 - \frac{\alpha_\epsilon^2}{\tau^2} \|\nabla \times \nabla^k E_\epsilon\|_{L^2}^2 \leq C \frac{\sigma^2}{\tau^2} \|\nabla^k g_\epsilon\|_{L^2}^2,
\]
where we have used (103). \(\Box\)

Combining Lemma 4.6 and Lemma 4.7, we can obtain the following lemma.

**Lemma 4.8.** Under the assumption in the section 2.3 and the assumption (26) on the initial data, if \((f_\epsilon, g_\epsilon, E_\epsilon, B_\epsilon)\) are solutions to (5), then for \(k \leq s - 2\),
\[
\frac{d}{dt} \sum_{k=0}^{s-2} \int_{\mathbb{T}^3} \left( |\nabla \times \nabla^k E_\epsilon|^2 + |\nabla \times \nabla^k B_\epsilon|^2 - 2\alpha_\epsilon \cdot \nabla \times \nabla^k \tilde{j}_\epsilon \cdot \nabla \times \nabla^k E_\epsilon - \frac{2\alpha_\epsilon^2}{\tau^2} \nabla^k E_\epsilon \cdot \nabla \times \nabla^k B_\epsilon \right) dx \\
+ \frac{d}{dt} \int_{\mathbb{T}^3} (|E_\epsilon|^2 + |B_\epsilon|^2 - 2\alpha_\epsilon \cdot \tilde{j}_\epsilon \cdot E_\epsilon) dx + \frac{\alpha_\epsilon^2}{\tau^2} \|E_\epsilon\|_{L^2}^2 + \frac{\alpha_\epsilon^2}{\tau^2} \|\nabla \times E_\epsilon \times \nabla \times B_\epsilon\|_{L^2}^2 \\
\leq C \|\left(f_\epsilon, g_\epsilon, E_\epsilon, B_\epsilon\right)\|_{H^\frac{1}{2}} \left(\left(f_\epsilon, g_\epsilon\right)\right)_{H^1} + C(1 + \frac{\sigma}{\tau^2}) \|g_\epsilon\|_{H^\frac{1}{2}}^2.
\]

**Remark 4.9.** In this lemma, we obtain the dissipative estimates of \(E\) and \(B\). Noticing that the Ampere equation and Faraday equation are hyperbolic equations, there is no obvious dissipative effect. To obtain the dissipation of \(E\) and \(B\), we need the estimates of \(\nabla^2 g\) to close the inequalities. This is why that the second order derivative of initial data belonging to \(L^2\) space is necessary.
4.5. The whole estimates. In this section, we shall close the whole estimates. Before that, we first analyze the electromagnatic parts.

**Lemma 4.10.** Under the assumption in the section 2.3 and the assumption (26) on the initial data, if \((f_e, g, B_e, E_e)\) are solutions to (5), then there exists some small enough constant \(c_0\) such that

\[
\sup_{0 \leq s \leq t} H^s(t) + \frac{1}{4} \int_0^t \left( \|(f_e, g, e)\|_{H^{s-1}}^2 + \frac{\alpha^2}{\tau^2} \|(E_e, B_e)\|_{H^{s-1}}^2 \right) ds \leq \frac{c_1}{c_2} H^s(0),
\]

(110)

where \(c_1\) and \(c_2\) are positive constants only dependent of the Sobolev embedding constant.

**Proof:** This Lemma can be proved by employing the Poincare’s inequality and choosing proper constants. We split the proof into three steps.

**Step one:** obtain the dissipative estimates of the electromagnetic field

Denoting

\[
H^s_{e,c}(t) = \sum_{k=0}^{s-2} \int_{\mathbb{R}^3} \left( |\nabla \times \nabla f_e|^2 + |\nabla \times \nabla g_e|^2 - 2\alpha c \cdot \nabla \times \nabla f_e \cdot \nabla \times \nabla g_e \right) dx
\]

(111)

and

\[
H^s_{d,c}(t) = \epsilon \sum_{k=0}^{s-2} \int_{\mathbb{R}^3} \left( |\nabla \times \nabla f_e|^2 + |\nabla \times \nabla g_e|^2 - 2\alpha \cdot \nabla \times \nabla f_e \cdot \nabla \times \nabla g_e \right) dx
\]

(112)

then combining (80) and (109) up, we can infer that

\[
\begin{align*}
\frac{d}{dt} \left( H^s_{e,c} + H^s_{d,c} \right)(t) &+ \frac{1}{4} \left( \|(\nabla \times f_e, \nabla \times g_e)\|_{H^{s-1}}^2 - \delta_2 \|\nabla v(f_e, g_e)\|_{H^{s-1}}^2 \right) \\
&+ \|\nabla \nabla E_e\|_{H^{s-1}}^2 - \delta_1 \frac{\alpha^2}{\tau^2} \|E_e\|_{H^{s-1}}^2 + \frac{3\alpha^2}{4} \|E_e\|_{L^2}^2 + \frac{\alpha^2}{16} \|\nabla \nabla E_e, \nabla \times B_e\|_{H^{s-1}}^2 \leq C \left( \|(f_e, g_e, E_e, B_e)\|_{H^{s-1}}^2 + \epsilon \left( 1 + \frac{\epsilon^2}{\beta^2} + \frac{1}{\beta^2} \right) \|f_e, g_e\|_{H^{s-1}}^2 \right)
\end{align*}
\]

(113)

Noticing that we have obtained the \(L^2\) estimates of the curl part and divergence part of \(E_e\), the estimate of \(\nabla E_e\) can be recovered by Hodge decomposition, that is to say,

\[
\|\nabla E_e\|_{L^2}^2 \leq C \left( \|\nabla E_e\|_{L^2}^2 + \|\nabla \times E_e\|_{L^2}^2 \right).
\]

For the magnetic field \(B_e\), since \(\text{div} B_e = 0\),

\[
\|\nabla B_e\|_{L^2}^2 \leq C \|\nabla \times B_e\|_{L^2}^2.
\]

Furthermore, the forth equation in (5), the mean value of \(B\) on the torus are zero. By Poincare’s inequality, the estimate of \(B_e\) is

\[
\|B_e\|_{L^2}^2 \leq C \|\nabla B_e\|_{L^2}^2.
\]

So for the electromagnetic field and some \(c_1 > 0\), we can obtain that

\[
c_4 \frac{\alpha^2}{\tau^2} \|(B_e, E_e)\|_{H^{s-1}}^2 \leq \|\nabla E_e\|_{H^{s-1}}^2 + \frac{3\alpha^2}{4} \|E_e\|_{L^2}^2 + \frac{\alpha^2}{16} \|\nabla \nabla E_e, \nabla \times B_e\|_{H^{s-1}}^2.
\]

Choosing \(\delta_1 = \frac{\Phi}{4}\), then we can infer that

\[
\begin{align*}
\frac{d}{dt} \left( H^s_{e,c} + H^s_{d,c} \right)(t) &+ \frac{1}{4} \left( \|(\nabla \times f_e, \nabla \times g_e)\|_{H^{s-1}}^2 - \delta_2 \|\nabla v(f_e, g_e)\|_{H^{s-1}}^2 \right) \\
&+ \|\nabla \nabla E_e\|_{H^{s-1}}^2 - \delta_1 \frac{\alpha^2}{\tau^2} \|E_e\|_{H^{s-1}}^2 + \frac{3\alpha^2}{4} \|E_e\|_{L^2}^2 + \frac{\alpha^2}{16} \|\nabla \nabla E_e, \nabla \times B_e\|_{H^{s-1}}^2 \leq C \left( \|(f_e, g_e, E_e, B_e)\|_{H^{s-1}}^2 + \epsilon \left( 1 + \frac{\epsilon^2}{\beta^2} + \frac{1}{\beta^2} \right) \|f_e, g_e\|_{H^{s-1}}^2 \right)
\end{align*}
\]

(114)

**Step two:** obtain the dissipative estimates of the macroscopic part of \(f_e\) and \(g_e\)

By Lemma 4.1, noticing that \(\int_{\mathbb{R}^3} P g_e dx = 0\) for any \(t > 0\), thus we can infer that

\[
\|g_e\|_{H^s}^2 \leq C \left( \|\nabla g_e\|_{H^{s-1}}^2 + \|g_e\|_{H^{s-1}}^2 \right).
\]

(115)
The macroscopic part of \( f_e \) is more complicated. From Lemma 4.1,
\[
\int_{\mathbb{R}^3} \mathcal{P} f_e(t) dx = -\gamma_e v \cdot \int_{\mathbb{R}^3} E_e \times B_e dx - \frac{c_6}{6} \| (E_e, B_e) \|_{L^2}^2. \tag{116}
\]
Thus, by Poincare’s inequality, we can infer that
\[
\| f_e \|_{H^s}^2 \leq C \left( \| \nabla f_e \|_{H^s_{x}}^2 + \| f_e \|_{H^s_{x}}^2 + \| \nabla (E_e, B_e) \|_{H^s_{x}}^2 (E_e, B_e) \|_{H^s_{x}}^2 \right). \tag{117}
\]
Together with (114), (117) and (115), there exists some \( c_6 > 0 \) such that
\[
\begin{align*}
\frac{d}{dt} \left( H_{v,e}^{s-1} + H_{d,e}^{s} \right) &+ c_6 \| (f_e, g_e) \|_{H^s_{x}}^2 \| f_e \|_{H^s_{x}}^2 + \mathcal{L}_d \| (f_e, g_e) \|_{H^s_{x}}^2 + \mathcal{L}_d \| (f_e, g_e) \|_{H^s_{x}}^2 \\
&\leq C \| (f_e, g_e, E_e, B_e) \|_{H^s_{x}}^2 \| f_e \|_{H^s_{x}}^2 + C \| (f_e, g_e, E_e, B_e) \|_{H^s_{x}}^2 (E_e, B_e) \|_{H^s_{x}}^2 \| (f_e, g_e) \|_{H^s_{x}}^2.
\end{align*}
\]
\[
\text{Step three: Closing the entire estimates}
\]
\[
e^2 \frac{d}{dt} \left( \frac{8}{3} \sum_{m=1}^{s-1} \hat{H}_{v,e}^m (t) + \hat{H}_{v,e}^s (t) \right) + \frac{c_6}{2} \| (\nabla f_e, \nabla g_e) \|_{H^s_{x}}^2 \tag{119}
\]
Denoting
\[
H_{v,e}^s = \left( \frac{8}{3} \sum_{m=1}^{s-1} \hat{H}_{v,e}^m (t) + \hat{H}_{v,e}^s (t) \right),
\]
and choosing \( b_4 \) and \( \delta_2 \) such that
\[
b_1 \cdot c_6 \geq b_1 + 1, \quad \frac{3c_4}{2} \cdot b_4 \geq \frac{1}{2}, \quad b_4 \cdot \delta_2 = \frac{1}{3},
\]
then we can infer that there exists some \( c_6 \) such that
\[
\begin{align*}
\frac{d}{dt} \left( b_4 \cdot H_{v,e}^{s-1} + b_4 \cdot H_{d,e}^{s} + H_{v,e}^{s} \right) &+ \frac{c_6}{2} \left( \| (f_e, g_e) \|_{H^s_{x}}^2 + \| (\nabla f_e, \nabla g_e) \|_{H^s_{x}}^2 + \| (E_e, B_e) \|_{H^s_{x}}^2 \right) \\
&\leq C \| (f_e, g_e, E_e, B_e) \|_{H^s_{x}}^2 \| (f_e, g_e, E_e, B_e) \|_{H^s_{x}}^2 + C \| (f_e, g_e, E_e, B_e) \|_{H^s_{x}}^2 (E_e, B_e) \|_{H^s_{x}}^2 \| (f_e, g_e) \|_{H^s_{x}}^2.
\end{align*}
\]
By the relation (7), we can infer that
\[
\gamma_e \leq \frac{c_6}{2}, \quad \epsilon \leq \frac{c_6}{2}, \quad \frac{c_6}{\gamma_e} \leq \frac{1}{3}.
\]
Denoting
\[
\hat{H}_e^s := b_5 \| (f_e, g_e, E_e, B_e) \|_{H^s_{x}}^2 + b_4 \cdot H_{v,e}^{s-1} + b_4 \cdot H_{d,e}^{s} + H_{v,e}^{s},
\]
and choosing \( b_5 \) such that
\[
\hat{H}_e^s \approx H_e^s, \quad \frac{b_5}{c_6} \geq c_6 \left( 1 + \frac{c_6}{2} \right) + \frac{1}{2c_6},
\]
then we have
\[
\frac{d}{dt} \hat{H}_e^s + \frac{1}{2} \left( \| (f_e, g_e) \|_{H^s_{x}}^2 + \frac{c_6}{2} \| (E_e, B_e) \|_{H^s_{x}}^2 \right) + \| (f_e, g_e) \|_{H^s_{x}}^2 \| (f_e, g_e) \|_{H^s_{x}}^2.
\]
Since \( \hat{H}_e^s \) is equivalent to \( H_e^s \), there exists some \( 0 < c_1 < 1 \) and \( c_4 > 0 \) such that
\[
c_1 \| H_e^s \| \leq \hat{H}_e^s \leq c_4 \| H_e^s \|. \tag{124}
\]
As long as the initial data satisfy that
\[
H_e^s (0) \leq c_0 := \frac{1}{4c_4c_5},
\]
\[
H_e^s \leq c_0 H_e^s.
\]

(125)
then it follow that for any $t > 0$

$$\sup_{0 \leq s \leq t} \tilde{H}_s^p(t) + \frac{1}{t} \int_0^t \left( \| (f_c, g_c, B_c, E_c) \|_{H_s^0}^2 + \frac{\alpha^2}{\epsilon^2} \| (E_c, B_c) \|_{H_s^{-1}}^2 + \frac{1}{\epsilon^2} \| (f_c^\perp, g_c^\perp) \|_{H_s^0}^2 \right) (s) \, ds \leq \tilde{H}_s^p(0).$$  \hspace{1cm} (126)

By (124), we can infer that

$$\sup_{0 \leq s \leq t} H_s^p(t) + \frac{1}{t} \int_0^t \left( \| (f_c, g_c, B_c, E_c) \|_{H_s^0}^2 + \frac{\alpha^2}{\epsilon^2} \| (E_c, B_c) \|_{H_s^{-1}}^2 + \frac{1}{\epsilon^2} \| (f_c^\perp, g_c^\perp) \|_{H_s^0}^2 \right) (s) \, ds \leq \frac{C}{\epsilon^2} H_s^p(0).$$

We complete the proof of this lemma. \hfill \square

In what follows, we sketch the idea of constructing approximate solutions and complete the proof of Theorem 3.1.

$$\begin{align*}
\begin{aligned}
\partial_t f^n_c + \frac{1}{\epsilon} v \cdot \nabla x f^n_c &= -\frac{1}{\epsilon} \mathcal{L}(f^n_c) + \frac{\alpha}{\epsilon^2} B^n_c \cdot \nabla v (M g^n_c) = \frac{1}{\epsilon^2} \Gamma(f^n_{\epsilon^{-1}}, f^n_{\epsilon^{-1}}), \\
\partial_t g^n_c + \frac{1}{\epsilon} v \cdot \nabla x g^n_c &= -\frac{1}{\epsilon} \mathcal{L}(g^n_c) + \frac{\alpha}{\epsilon^2} B^n_c \cdot \nabla v (M g^n_c) = \frac{1}{\epsilon^2} \Gamma(g^n_{\epsilon^{-1}}, f^n_{\epsilon^{-1}}), \\
\gamma_\epsilon \partial_n E^n_c - \nabla \times B^n_c &= -\frac{\beta}{\epsilon^2} J^n_c, \\
\gamma_\epsilon \partial_n B^n_c + \nabla \times E^n_c &= 0, \\
div B^n_c = 0, & \text{div } E^n_c = \frac{\alpha}{\epsilon} \int_{\mathbb{R}^3} g^n_c Mdv, \\
f^n_c = f^0, & g^n_c = g^0, \quad E^n_c = B^n_c = 0.
\end{aligned}
\end{align*} \hspace{1cm} (127)$$

The approximate solutions can be constructed by iteration method and induction method. Then based on the uniform estimates, the solutions can be obtained by employing Rellich-Kondrachov compactness theorem.

5. Verify the limit

In this section, we will complete the proof of Theorem 3.5 based on the uniform estimates (110).

The proof of Theorem 3.5. The proof is based on the local conservation laws and uniform estimates. In Sec. 4, we have obtain the following prior estimates: for any $t > 0$

$$\sup_{0 \leq s \leq t} \| (f_c, g_c, B_c, E_c) \|_{H_s^0} + \int_0^t \left( \| (f_c, g_c) \|_{H_s^0}^2 + \frac{1}{\epsilon^2} \| (f_c^\perp, g_c^\perp) \|_{H_s^0}^2 \right) (s) \, ds \leq C_0.$$  \hspace{1cm} (128)

We split the proof into three steps. Before that, denoting

$$\alpha = \lim_{\epsilon \to 0} \frac{\alpha_\epsilon}{\epsilon}, \quad \beta = \lim_{\epsilon \to 0} \beta_\epsilon, \quad \gamma = \lim_{\epsilon \to 0} \gamma_\epsilon.$$  \hspace{1cm} (129)

**Step 1: the strong convergence of $f_c$ and $g_c$.**

First, since $\int_0^\infty \| (f_c, g_c) \|_{H_s^0} \, ds \leq C_0$, then there exist some $f, g \in L^2((0, \infty); H^s)$ such that

$$f_c \to f, \quad g_c \to g, \quad \text{in } L^2((0, +\infty); H^s).$$  \hspace{1cm} (130)

On the other hand, noticing that there exists a coefficient $\frac{1}{\epsilon^2}$ before the microscopic part in (128), then we can infer that

$$f_c^\perp \to 0, \quad g_c^\perp \to 0, \quad \text{in } L^2((0, +\infty); H_s^1).$$  \hspace{1cm} (131)

All together, we can infer

$$f_c \to f, \quad g_c \to g, \quad \text{in } L^2((0, +\infty); H_s^0),$$

and

$$f \in \text{Ker} \mathcal{L}, \quad g \in \text{Ker} \mathcal{L}.$$  \hspace{1cm} (132)

By the properties of the Boltzmann operator, we can infer that there exists some $\rho(t), \ u(t), \ \theta(t) \in H_s^0$ and $n(t) \in H_s^1$ such that

$$f(t, x) = \rho(t, x) + u(t, x) \cdot v + \frac{|v|^2 - 3}{2} \theta(t, x), \quad g(t, x) = n(t, x).$$  \hspace{1cm} (133)

Recalling the macroscopic parts of $f_c$ and $g_c$, i.e.,

$$\rho_c = \int_{\mathbb{R}^3} f_c Mdv, \quad u_c = \int_{\mathbb{R}^3} f_c v Mdv, \quad \theta_c = \int_{\mathbb{R}^3} h_c \frac{|v|^2 - 3}{2} Mdv, \quad n_c = \int_{\mathbb{R}^3} d_c Mdv.$$
By (128), we can obtain that
\[
\rho_{\epsilon}(t, x), u_{\epsilon}(t, x), \theta_{\epsilon}(t, x), n_{\epsilon}(t, x) \in L^\infty([0, +\infty); H^s_x) \cap L^2([0, +\infty); H^s_x).
\] (132)

Based on the estimates (132), for any fixed \( t > 0 \), there exist \( \{\rho, u, \theta, n, B, E\} \subset H^s_x \) such that for any fixed \( T > 0 \)
\[
\rho_{\epsilon} \to \rho, u_{\epsilon} \to u, \theta_{\epsilon} \to \theta, n_{\epsilon} \to n, B_{\epsilon} \to B, E_{\epsilon} \to E, \text{ in, } L^2((0, T); H^{s-1}_x). \] (133)

The next step is to verify \( \rho, u, \theta \) and \( n \) satisfy the limiting fluid equations.

**Step 2: the local conservation laws.**

Copying the nonlinear equations (37) below,
\[
\begin{align*}
\partial_t \rho_{\epsilon} + \frac{1}{\epsilon} \nabla \cdot j_{\epsilon} &= 0, \\
\partial_t u_{\epsilon} + \frac{1}{\epsilon} \nabla \cdot (\rho_{\epsilon} \cdot u_{\epsilon}) &= \frac{\alpha}{\epsilon} \rho_{\epsilon} \cdot E_{\epsilon} + \frac{\beta}{\epsilon} j_{\epsilon} \cdot B_{\epsilon}, \\
\partial_t \theta_{\epsilon} + \frac{1}{\epsilon} \nabla \cdot (\rho_{\epsilon} \cdot \theta_{\epsilon}) &= \frac{\alpha}{\epsilon} \rho_{\epsilon} \cdot E_{\epsilon} \cdot v - \frac{1}{\epsilon^2} \nabla \cdot (\rho_{\epsilon} \cdot E_{\epsilon} \cdot v) + \frac{\Gamma}{\epsilon} (g_{\epsilon} \cdot f_{\epsilon}), \\
\partial_t n_{\epsilon} + \frac{1}{\epsilon} \nabla \cdot (n_{\epsilon} \cdot j_{\epsilon}) &= 0, \\
\gamma_{\epsilon} \partial_t E_{\epsilon} \cdot \nabla \times B_{\epsilon} &= - \frac{1}{\epsilon} \gamma_{\epsilon} j_{\epsilon}, \\
\gamma_{\epsilon} \partial_t j_{\epsilon} \cdot B_{\epsilon} + \nabla \times E_{\epsilon} &= 0, \\
\div B_{\epsilon} &= 0, \quad \div E_{\epsilon} = \frac{\alpha}{\epsilon} \int_{R^3} g_{\epsilon} M dv.
\end{align*}
\] (134)

from the above system, the local conservation laws, i.e., the equations of \( \rho_{\epsilon}, u_{\epsilon}, \theta_{\epsilon} \) and \( n_{\epsilon} \), are
\[
\begin{align*}
\partial_t \rho = \frac{1}{\epsilon} \nabla \cdot j_{\epsilon} &= 0, \\
\partial_t u - \frac{1}{\epsilon} \nabla \cdot (\rho \cdot u) &= \frac{\alpha}{\epsilon} \rho \cdot E + \frac{\beta}{\epsilon} j \cdot B, \\
\partial_t \theta + \frac{1}{\epsilon} \nabla \cdot (\rho \cdot \theta) &= \frac{\alpha}{\epsilon} \rho \cdot E \cdot v - \frac{1}{\epsilon^2} \nabla \cdot (\rho \cdot E \cdot v) + \frac{\Gamma}{\epsilon} (g \cdot f), \\
\partial_t n + \frac{1}{\epsilon} \nabla \cdot (n \cdot j) &= 0, \\
\gamma \partial_t E \cdot \nabla \times B &= - \frac{1}{\epsilon} \gamma j, \\
\gamma \partial_t j \cdot B + \nabla \times E &= 0, \\
\div B &= 0, \quad \div E = \frac{\alpha}{\epsilon} n.
\end{align*}
\] (135)

where
\[
A(v) = v \otimes v - \frac{|v|^2}{3} \mathbf{I}, \quad B(v) = v(\frac{|v|^2}{2} - \frac{3}{2}), \quad \mathcal{L} \hat{A}(v) = A(v), \quad \mathcal{L} \hat{B}(v) = B(v).
\] (136)

By the local conservation laws of mass, i.e., the first equation of (134), we can infer that in the distributional sense
\[
\div u_{\epsilon} \to \div u = 0.
\] (137)

Before verifying the the velocity and temperature equation, we need to understand the limiting behavior of \( j_{\epsilon} \).

**Step 3, the limiting behavior of \( \frac{1}{\epsilon} j_{\epsilon} \).**

From (96), we can infer that
\[
\frac{1}{\epsilon} j_{\epsilon} = -\epsilon \partial_t \tilde{j}_{\epsilon} - \div \int_{R^3} \tilde{v} \otimes v \epsilon g_{\epsilon} M dv + \sigma \frac{\alpha}{\epsilon} E \epsilon + \epsilon \int_{R^3} \tilde{v} n_{\epsilon} M dv.
\]

By the uniform estimates (128), for any \( t > 0 \),
\[
\int_0^t \| (j_{\epsilon}, \tilde{j}_{\epsilon})(s) \|_{L^2_x}^2 ds \leq C_0 \epsilon^2,
\] (137)
and
\[
\div \int_{R^3} \tilde{v} \otimes v g_{\epsilon} M dv = \div \int_{R^3} \tilde{v} \otimes v g_{\epsilon}^1 M dv + \sigma \nabla n_{\epsilon} = \sigma \nabla n_{\epsilon} + R_1(\epsilon).
\] (138)

where
\[
R_1(\epsilon) \to 0, \quad \text{in, } L^2((0, +\infty); H^{s-1}_x).
\]

For the last term in the right hand of \( \frac{1}{\epsilon} j_{\epsilon} \), by simple computation and decomposition, we can infer that
\[
\epsilon \int_{R^3} \tilde{v} N_{\epsilon} M dv = - \int_{R^3} \left( (\alpha \epsilon E + \beta \epsilon v \times B_{\epsilon}) \cdot \nabla (M_{\epsilon f}) \right) \tilde{v} dv + \int_{R^3} \Gamma (g_{\epsilon} f_{\epsilon}) \tilde{v} M dv
\]
\[
= -\beta \int_{R^3} (v \times B_{\epsilon} \cdot \nabla f_{\epsilon}) \tilde{v} M dv + n_{\epsilon} \int_{R^3} \Gamma (1, P f) \tilde{v} M dv + R_2(\epsilon),
\] (139)
where
\[ R_2(\epsilon) = \alpha \epsilon \int_{\mathbb{R}^3} (E_\epsilon \cdot \nabla_v(M_{f_\epsilon})) \hat{v} dv + \int_{\mathbb{R}^3} \Gamma(g_{f_\epsilon} f_\epsilon) \hat{v} M dv + \int_{\mathbb{R}^3} \Gamma(g_{f_\epsilon} f_\epsilon) \hat{v} M dv. \]

According to (129), (133) and (131), we can infer that in the distributional sense:
\[ \int_{\mathbb{R}^3} (v \times B_\epsilon \cdot \nabla_v f_\epsilon) \hat{v} M dv \rightarrow \sigma B \times u. \]  
(140)

Noticing the fact that \( P f_\epsilon = \rho_\epsilon + u_\epsilon \cdot v + \frac{|v|^2 - 3}{3} \theta_\epsilon \) and \( P f_\epsilon \in \text{Ker} L \), then we have
\[ L(u_\epsilon \cdot v + \frac{|v|^2 - 3}{3} \theta_\epsilon) = \Gamma(u_\epsilon \cdot v + \frac{|v|^2 - 3}{3} \theta_\epsilon, 1) + \Gamma(1, u_\epsilon \cdot v + \frac{|v|^2 - 3}{3} \theta_\epsilon) = 0. \]

Then, for the rest one in the second line of (139), we can deduce that
\[ \epsilon \int_{\mathbb{R}^3} \hat{v} N_3 M dv = n_\epsilon \int_{\mathbb{R}^3} \hat{v} \Gamma(1, P f_\epsilon) M dv \]
\[ = -n_\epsilon \int_{\mathbb{R}^3} \hat{v} \Gamma(u_\epsilon \cdot v + \frac{|v|^2 - 3}{3} \theta_\epsilon, 1) M dv \]
\[ = n_\epsilon \int_{\mathbb{R}^3} \hat{v} L(u_\epsilon \cdot v + \frac{|v|^2 - 3}{3} \theta_\epsilon) M dv \]
\[ = n_\epsilon \int_{\mathbb{R}^3} v(u_\epsilon \cdot v + \frac{|v|^2 - 3}{3} \theta_\epsilon) M dv \]
\[ = n_\epsilon \epsilon u_\epsilon. \]  
(141)

All together, we can infer that in the distributional sense:
\[ \epsilon \int_{\mathbb{R}^3} \hat{v} N_3 M dv \rightarrow \beta u \times B + nu. \]  
(142)

and
\[ \frac{1}{\epsilon} j_\epsilon \rightarrow j = \sigma(\alpha E + \beta u \times B) - \sigma \nabla_x n + nu. \]  
(143)

**Step 4. the convergence of conservation laws of momentum and temperature**

From (143), we can infer that in the distributional sense:
\[ j_\epsilon \rightarrow 0, \quad \frac{1}{\epsilon} j_\epsilon \rightarrow j = \frac{\sigma}{\epsilon}(\alpha E + \beta u \times B) - \frac{\sigma}{\epsilon} \nabla_x n + nu. \]  
(144)

Furthermore, for velocity and temperature equation of (134), we can infer that
\[ \partial_t u_\epsilon + \frac{1}{\epsilon} \text{div} \int_{\mathbb{R}^3} \hat{A} \mathcal{L}_\epsilon f \hat{f} dv + \frac{1}{\epsilon} \nabla_x (\rho_\epsilon + \theta_\epsilon) = \frac{\sigma}{\epsilon} n_\epsilon \cdot E_\epsilon + \frac{\beta}{\epsilon} j_\epsilon \times B_\epsilon, \]
\[ \partial_t \left( \frac{\sigma}{\epsilon} \theta_\epsilon - \frac{\sigma}{\epsilon} \rho_\epsilon \right) + \frac{2}{\epsilon} \text{div} \int_{\mathbb{R}^3} \hat{B} \mathcal{L}_\epsilon f \hat{f} dv = \frac{2}{\epsilon} j_\epsilon \cdot E_\epsilon. \]  
(145)

Based on (128) and (137), it follows that
\[ \nabla_x (\rho_\epsilon + \theta_\epsilon) \rightarrow 0, \quad \text{in the distributional sense}, \]
and
\[ \rho + \theta = 0. \]  
(146)

The (136) and (146) are Bousinessq equations. Now, we can deduce the velocity equations of the macroscopic equations. Denoting by \( \mathbf{P} \) the Leray projection operator on torus, from the local conservation laws (134), it follows that
\[ \partial_t \mathbf{P} u_\epsilon + \frac{1}{\epsilon} \mathbf{P} \left( \text{div} \int_{\mathbb{R}^3} \hat{A} \mathcal{L}_\epsilon f \hat{f} dv \right) = \mathbf{P} \left( \frac{\sigma}{\epsilon} n_\epsilon \cdot E_\epsilon + \frac{\beta}{\epsilon} j_\epsilon \times B_\epsilon \right). \]  
(147)

Based on the first equation of (37), it implies
\[ \frac{1}{\epsilon} \mathcal{L}(f_\epsilon) = -v \cdot \nabla_x f_\epsilon - \epsilon \partial_t f_\epsilon + \Gamma(f_\epsilon, f_\epsilon) + \alpha_\epsilon E_\epsilon \cdot v \cdot g_\epsilon - \alpha_\epsilon E_\epsilon + \beta_\epsilon v \times B_\epsilon \cdot \nabla_v g_\epsilon \]
\[ = \Gamma(f_\epsilon, f_\epsilon) - v \cdot \nabla_x f_\epsilon + R_\mathcal{L}(\epsilon). \]
By simple calculation (see [1, 3, 2]), we can infer that
\[
\int_{\mathbb{R}^3} \hat{A} \cdot \frac{1}{\varepsilon} \mathcal{L} h \varepsilon M dv = u_\varepsilon \otimes u_\varepsilon - \frac{|u_\varepsilon|^2}{3} I - \mu (\nabla u_\varepsilon + \nabla^T u_\varepsilon - \frac{2}{3} \text{div} u_\varepsilon) - R_f(\varepsilon)
\]
(148)
with
\[
R_f(\varepsilon) := \int_{\mathbb{R}^3} \hat{A} \cdot (R_L(\varepsilon) - v \cdot \nabla f_\varepsilon^\perp + \Gamma(f_\varepsilon^\perp, f_\varepsilon) + \Gamma(f_\varepsilon, f_\varepsilon^\perp)) M dv,
\]
and
\[
\mu = \frac{1}{15} \sum_{1 \leq i,j \leq 3} \int_{\mathbb{R}^3} A_{ij} \hat{A}_{ij} M dv.
\]
According to (128), by the weak convergence results (129) and the properties (131) of \( f \) and \( g \), it holds in the distributional sense that
\[
R_f(\varepsilon) \to 0.
\]
Based on (128), (143), (132) and (147), we can obtain that
\[
\partial_t \varepsilon \mathbf{u} \in H_{x}^{s-1}.
\]
Employing Aubin-Lions-Simon theorem (see [6]), we can infer
\[
\mathbf{u}_\varepsilon \in C((0, +\infty; H_{x}^{s-1}); \mathbf{u} \rightarrow \mathbf{u}, \text{ in } C((0, +\infty; H_{x}^{s-1}).
\]
Thus, according to (128), (132) and (148), we can infer in the distributional sense that
\[
\mathbf{u}_\varepsilon \rightarrow \mathbf{u}, \frac{1}{\varepsilon} \mathbf{P} \left( \text{div} \int_{\mathbb{R}^3} \hat{A} \mathcal{L} h \varepsilon M dv \right) \rightarrow \mathbf{u} \cdot \nabla u - \mu \Delta u.
\]
(151)
Combining the strong convergence properties (133), we can finally deduce that
\[
\partial_t u + u \cdot \nabla u - \mu \Delta u + \nabla P = \alpha n \cdot \mathbf{E} + \beta \cdot j \times \mathbf{B}.
\]
(152)
Similar to (148), we can infer that
\[
\frac{2}{\varepsilon} \int_{\mathbb{R}^3} \hat{B} \cdot \frac{1}{\varepsilon} \mathcal{L} g \varepsilon M dv = u_\varepsilon \cdot \theta_\varepsilon - \kappa \nabla \theta_\varepsilon - R_\theta(\varepsilon)
\]
(153)
with
\[
R_\theta(\varepsilon) := \frac{2}{\varepsilon} \int_{\mathbb{R}^3} \hat{B} \cdot (R_L(\varepsilon) - v \cdot \nabla f_\varepsilon^\perp + \Gamma(f_\varepsilon^\perp, f_\varepsilon) + \Gamma(f_\varepsilon, f_\varepsilon^\perp)) M dv, \kappa = \frac{2}{15} \sum_{1 \leq i \leq 3} \int_{\mathbb{R}^3} B_i \hat{B}_i M dv.
\]
Finally, the temperature equation becomes
\[
\partial_t \left( \theta_\varepsilon - \frac{2}{3} \rho \varepsilon \right) + \text{div}(u_\varepsilon \theta) - \kappa \Delta \theta_\varepsilon = \frac{2}{3} j \cdot \mathbf{E}_\varepsilon + \text{div} R_\theta(\varepsilon).
\]
(154)
According to (133), (146) and (144), by the similar analysis to obtaining (152), we can infer that
\[
\partial_t \theta + u \cdot \nabla \theta - \kappa \Delta \theta = 0.
\]
(155)
In the light of all the previous analysis in the section, it follows that we have verified that
\[
f_\varepsilon(t, x, v) \to \rho(t, x) + u(t, x) \cdot v + \frac{|v|^2 - 3}{2} \theta(t, x), g_\varepsilon(t, x, v) \to n(t, x), \text{ in } L^2((0, +\infty); H^s).
\]
(156)
with \( \rho, u, \theta \in L^\infty((0, +\infty); H^s) \) and satisfying
\[
\begin{aligned}
\partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla P = \alpha n \cdot \mathbf{E} + \beta \cdot j \times \mathbf{B}, \\
\partial_t \theta + u \cdot \nabla \theta - \kappa \Delta \theta = 0, \\
\text{div} u = 0, \rho + \theta = 0,
\end{aligned}
\]
(157)
\[
\begin{aligned}
j = \sigma (\alpha \mathbf{E} + \beta \mathbf{u} \times \mathbf{B}) - \sigma \nabla n + nu.
\end{aligned}
\]

**Step 5, the electromagnetic system**

For the Ampere’s equation and Faraday’s equation in (134), according to the uniform estimates (128) and (144), we can infer that in the distributional sense:
\[
\begin{aligned}
\gamma \partial_t E - \nabla \times \mathbf{B} &= -\beta j, \\
\gamma \partial_t B + \nabla \times E &= 0, \\
\text{div} B &= 0, \text{ div} E = \alpha n.
\end{aligned}
\]
(158)
Furthermore, from the fourth equation of (134), we can infer that
\[ \partial_t n + \text{div} j = 0. \]  
(159)

**Step 6, The final limiting fluid equation**

Combining (157) and (158), the limiting system is
\[
\begin{align*}
\partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla P &= \alpha n \cdot E + \beta \cdot j \times B, \\
\partial_t \theta + u \cdot \nabla \theta - \kappa \Delta \theta &= 0, \\
\text{div} u &= 0, \quad \rho + \theta = 0, \\
j &= \sigma(\alpha E + \beta u \times B) - \sigma \nabla \nabla n + nu, \\
\gamma \partial_t E - \nabla \times B &= -\beta j, \\
\gamma \partial_t B + \nabla \times E &= 0, \\
\text{div} B &= 0, \quad \text{div} E = \alpha n. 
\end{align*}
\]  
(160)

Then for the limiting fluid equations, we only need to plug the value of \(\alpha, \beta, \gamma\) into (160).

For (27), we have
\[ \alpha = \beta = \gamma = 1, \]  
(161)
and
\[
\begin{align*}
\partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla P &= n \cdot E + j \times B, \\
\partial_t \theta + u \cdot \nabla \theta - \kappa \Delta \theta &= 0, \\
\text{div} u &= 0, \quad \rho + \theta = 0, \\
\partial_t E - \nabla \times B &= -j, \\
\partial_t B + \nabla \times E &= 0, \\
\text{div} B &= 0, \quad \text{div} E = n, \\
j &= \sigma(E + u \times B) - \sigma \nabla \nabla n + nu 
\end{align*}
\]  
As for (28), it follows that
\[ \alpha = 1, \beta = \gamma = 0, \]  
(162)
and
\[
\begin{align*}
\partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla P &= n \cdot E, \\
\partial_t \theta + u \cdot \nabla \theta - \kappa \Delta \theta &= 0, \\
\partial_t n + \text{div}(nu) - \sigma \Delta n + \sigma n &= 0, \\
\text{div} u &= 0, \quad \rho + \theta = 0, \\
\nabla \times E &= 0, \quad \text{div} E = n, 
\end{align*}
\]  
where we have used (159).

With respect to (28), it follows that
\[ \alpha = \beta = \gamma = 0, \]  
(163)
and two fluids Navier-Stokes-Fourier system,
\[
\begin{align*}
\partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla P &= 0, \\
\partial_t \theta + u \cdot \nabla \theta - \kappa \Delta \theta &= 0, \\
\partial_t n + \text{div}(nu) - \sigma \Delta n &= 0, \\
\text{div} u &= 0, \quad \rho + \theta = 0. 
\end{align*}
\]
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