Critical hysteresis on dilute triangular lattice revisited

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Critical hysteresis in the zero-temperature random-field Ising model on a dilute triangular lattice was studied in Phys Rev E 91, 012131 (2015). The occupation probability \( c \) of a sublattice was decreased to transform a triangular lattice (\( c = 1 \)) to a honeycomb lattice (\( c = 0 \)). Numerical work suggested that critical hysteresis vanishes abruptly for \( c \leq 1/3 \). Subsequent developments in the field require us to revisit this result. An alternate approach presented here suggests absence of critical hysteresis over a larger range of dilution.

I. INTRODUCTION

The random-field Ising model \[1\] was introduced to study the effect of quenched disorder on a system’s ability to sustain long-range order in thermal equilibrium. After a rather prolonged debate it was resolved that the lower critical dimension of the Ising model \[2\] remains equal to two in the presence of quenched random fields \[3\]. Subsequently a zero-temperature version of the model (ZTRFIM) without thermal fluctuations but an on-site quenched random field distribution \( N[0,\sigma^2] \) was introduced \[4, 5\] as a model for disorder-driven hysteresis in ferromagnets and other similar systems \[6\]. Numerical simulations of ZTRFIM on a simple cubic lattice reveals a critical value of \( \sigma = \sigma_c \approx 2.16J \) (\( J \) being the nearest neighbor ferromagnetic exchange interaction). For \( \sigma < \sigma_c \) each half of the hysteresis loop shows a discontinuity in magnetization. The size of the discontinuity decreases to zero at a critical value of the applied field \( h_c \approx 1.435J \) as \( \sigma \) is increased to \( \sigma_c \). The behavior near \( \{h_c,\sigma_c\} \) shows scaling and universality quite similar to the one caused by critical thermal fluctuations at an equilibrium critical point. These aspects of the model are important in understanding hysteresis experiments and related theoretical issues. Initial numerical attempts to find a \( \sigma_c \) on the square lattice were inconclusive casting doubt on the lower critical dimension of the model. More extensive simulations \[7\] indicate \( \sigma_c \approx 0.54J \) and \( h_c \approx 1.275J \) on the 2d square lattice.

An exact solution \[8\] of ZTRFIM on a Bethe lattice of integer connectivity \( z \) shows that criticality occurs only if \( z \geq 4 \). Normally critical behavior on Bethe lattices is independent of \( z \) if \( z > 2 \) and is the same as in the mean field theory. Therefore the result for hysteresis is unusual and efforts have been made \[8\] to understand the physics behind it. A useful insight is obtained by extending the analysis to noninteger values of \( z \) \[10, 11\]. This is done by considering lattices where the connectivity of each node is distributed over a set of integers so that the average connectivity of a node has a noninteger value greater than two. Fortunately the problem can still be solved exactly and leads to the identification of a general criterion for the occurrence of critical hysteresis. The general criterion is that there should be a spanning path across the lattice and a fraction of sites on this path (even an arbitrarily small fraction) should have connectivity \( z \geq 4 \).

On periodic lattices, an exact solution of ZTRFIM is not available. Extant simulations indicate that the existence or absence of \( \sigma_c \) on a periodic lattice with uniform connectivity \( z \) is the same as on a Bethe lattice of connectivity \( z \). Criticality is absent on any lattice with \( z = 3 \) irrespective of the dimension \( d \) of the space in which the lattice is embedded \[12\]. Indeed critical hysteresis appears to be determined by a lower integer connectivity \( z_\ell = 4 \) rather than a lower critical dimension \( d_c = 2 \). As \( z \) increases above \( z_\ell \), the critical point becomes easier to observe in simulations. Compared with the intensive simulations on large square lattices, it takes a modest effort to observe criticality on a triangular lattice \[13\]. However the estimated value of \( \sigma_c \) appears to decrease slowly with increasing size of lattice. A study on lattices of size \( L \times L \) with \( L \leq 600 \) gives \( \sigma_c = 1.22 \) \[12\], while more extensive simulations on lattices of size \( L \leq 65536 \) yield \( \sigma_c = 0.86 \) \[14\]. We may remark that the critical exponents on the triangular lattice appear to be different from those on the square lattice \[14\]. This is puzzling in the context of the universality of critical phenomena and the broader implications of this result are not clear. At present \( L = 65536 \) is the largest linear size that has been studied thoroughly using available computers. One may ask if \( \sigma_c \) would decrease further in case much larger values of \( L \) were studied. Although extant numerical studies do not suggest \( \sigma_c \rightarrow 0 \) as \( L \rightarrow \infty \) but we are not aware of a rigorous argument for the same. Questions of this nature can not be resolved conclusively.

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by numerical studies. Criticality on a dilute lattice is even harder to settle numerically due to additional positional disorder. Keeping this in mind, our focus in the present paper is on systems of modest sizes and try to understand the qualitative trends in the basic data.

It has been argued that $\sigma_c = 0$ for an unsymmetric distribution of the random field in case $z = 3$ and $\sigma_c > 0$ for integer values $z \geq 4$ [12]. Our object here is to examine non-integer values of $z > 3$. A dilute (partially occupied) lattice of connectivity $z$ enables us to study a lattice of average connectivity $z_{av} < z$. We consider a triangular lattice $T = A + B + C$ with with one of its constituent sublattices, say $C$, having a reduced occupation probability $c$ [13]. The average connectivity on $T$ is then equal to $z_{av} = 6(1 + 2c)/(2 + c)$ and the average connectivity on $A$ or $B$ sublattice is equal to $z_{eff} = 3(1 + c)$. The connectivity of occupied sites on $C$ is equal to six. As $c$ is reduced from 1 to 0, we go from a triangular to a honeycomb lattice. Extant work indicates that $\sigma_c$ drops to zero at $c = 1/3$ within numerical errors. This appears reasonable at first sight because $z_{eff} = 4$ at $c = 1/3$ and $z_t = 4$ is the minimum integer connectivity for criticality. However the new criteria for critical hysteretic $T$ [14] does not require the average connectivity of a lattice to be equal to four. What is required is that some fraction of sites on a spanning path should have connectivity $z \geq 4$. This criterion is valid for Bethe lattices but it may apply to periodic lattices as well. If so, we may expect a non-zero $\sigma_c$ in the entire range $1 \geq c > 0$.

The reason for a discontinuity in the hysteresis loop on a Bethe lattice is that a fixed point corresponding to zero magnetization becomes unstable and splits into two stable fixed points for $\sigma < \sigma_c$. The size of the splitting is the size of the discontinuity. This is easily demonstrated by an analysis of the model on a Cayley tree [11]. We set the applied field equal to zero, and consider an initial configuration with all spins down except the spins on the surface of the tree. If the surface spins are equally likely to be up or down i.e. if the surface magnetization is zero it remains zero as spins are relaxed layer by layer towards the interior of the tree. Small perturbations to the surface magnetization behave differently depending on the connectivity $z$ of the lattice. If $z < 3$ the perturbations decrease and the magnetization in the deep interior remains zero. If $z \geq 4$, the perturbations diverge. A positive value of magnetization tends to increase, and a negative value tends to decrease as we move towards the interior. An important point is that this is not just a global property of a lattice of uniform connectivity $z$. On a lattice with mixed connectivity, each node depending on its connectivity $z$ increases or decreases the perturbation passing through it in a similar fashion. Larger the connectivity of the node, larger is the enhancement. Thus a small perturbation on the surface leads to a finite discontinuity in the deep interior of the tree if a fraction of nodes along the path have $z \geq 4$. Of course a spanning path is a prerequisite to reach the deep interior. However spanning paths are always there under our scheme of dilution. Even if $c = 0$, there are spanning paths on the honeycomb lattice; $c > 0$ introduces additional paths containing $C$ sites. The $C$ sites have connectivity equal to six. As long as there are some $C$ sites there are spanning paths punctuated by sites with connectivity equal to six. Remaining sites, the $A$ and $B$ sites have connectivity $3(1 + c)$ on the average. If $c \geq 1/3$ the average connectivity of each site on the spanning path is greater than four and we have a case for a relatively large discontinuity as observed in extant simulations. On the other hand, if $c < 1/3$, we should still expect a discontinuity albeit a much smaller one. The argument in favor of it is the enhancement effect of nodes with $z \geq 4$ on a Bethe lattice. It is not clear $a$ priori how loops on a periodic lattice may vacate this effect. This forms the motivation to review critical hysteretic on the dilute triangular lattice. However simulations presented below suggest that criticality on a dilute triangular lattice is qualitatively different from that on a dilute Bethe lattice.

II. NUMERICAL RESULTS

Our main interest is to understand the qualitative dependence of $\sigma_c$ on $L$ and $c$, and to check in particular if $\sigma_c$ drops to zero abruptly when $c$ drops below $c = 1/3$. The defining feature of $\sigma_c$ is that the discontinuity in the magnetization $m(h)$, say on the lower half of the hysteresis loop, reduces to zero as $h \to h_c$ and $\sigma \to \sigma_c$ from below. Exact solution on Bethe lattice and simulations on periodic lattices reveal that a discontinuity in magnetization is accompanied by a reversal of magnetization. Numerical determination of a discontinuity is rather problematic. For small $\sigma$, the graph $m(h)$ vs. $h$ near $m(h) = 0$ tends to be almost vertical anyway. A simulation based on a single realization of the random-field distribution necessarily shows a broken curve comprising a few irregularly placed discontinuities due to large fluctuations in the system. The number as well as positions of discontinuities vary from configuration to configuration and averaging over configurations results in a steep but smooth $m(h)$ curve. A genuine underlying discontinuity, if any, has to be inferred from the character of fluctuations. An added complication is that fluctuations at a discontinuity are different from those at the critical point where the discontinuity vanishes. Finally finite size scaling has to be employed to infer $\sigma_c$ in the thermodynamic limit. The estimate for $\sigma_c$ using finite size scaling should be independent of system sizes used in numerical simulations. However numerical uncertainties are large and diminish extremely slowly with increasing system size. As mentioned earlier, initial studies on triangular lattices of sizes $L \times L$ with $L \leq 600$ indicated $\sigma_c = 1.22$ [12] but more extensive simulations on lattices upto $L = 65536$ yield $\sigma_c = 0.86$ [14]. The procedure for determining $\sigma_c$ is rather indirect, tedious, cpu intensive, and various compromises have to be
made in order to draw reasonable conclusions. 

In this paper we adopt a different approach than used in previous studies. The basic idea is simple although the details have similar issues as in earlier studies. The new approach is useful in discerning important trends in the behavior of the model based on simulations of systems of modest sizes. For a fixed \( \sigma \) on an \( L \times L \) lattice, we count the total number of metastable states \( M \) (fixed points under zero temperature Glauber dynamics) comprising the lower half of hysteresis loop. As usual we increase the applied field \( h \) by a minimal amount to go from one fixed point to the next and keep \( h \) fixed during the relaxation process. We plot \( M \) as a function of \( \sigma \). It is a monotonically increasing function of \( \sigma \) without any discontinuity. The cpu time increases rapidly with increasing \( L \) and \( \sigma \). Fig.1 shows the result on a modest 33 \( \times \) 33 triangular lattice and \( 0 < \sigma \leq 50 \). The general features of Fig.1 are easy to understand. In the limit \( \sigma \rightarrow 0 \), the first spin to flip up initiates an infinite avalanche of flipped spins giving \( M=2 \). In the limit \( \sigma \rightarrow \infty \), spins flip up independently and \( M \) should approach \( L \times L \). We expect \( M \) to increase continuously from 2 to \( L \times L \) as \( \sigma \) increases from 0 to \( \infty \). This expectation is born out by Fig.1. If there is a critical value of \( \sigma \) separating discontinuous \( m(h) \) for \( \sigma < \sigma_c \) with continuous \( m(h) \) for \( \sigma > \sigma_c \) we ought to see its signature in Fig.1. A discontinuity in \( m(h) \) at \( \sigma_c \) would effectively reduce \( M \) in proportion to its size. This would result in some change in shape of \( M \) vs. \( \sigma \) graph at \( \sigma_c \). We find that this effect is present but too weak to be seen with naked eye in the main graph of Fig.1 or its magnified portion in the range \( 0 < \sigma < 4 \) shown in the left inset there. However, we do see an apparent inflexion point around \( \sigma_c \approx 1.8 \) if \( M \) is plotted on logscale scale as in the right inset. We now examine how \( \sigma_c \) shifts to lower values with increasing \( L \).

Fig.2 shows \( \log M \) vs. \( \sigma \) in the range \( 0 < \sigma < 2 \) for \( L=33,99,198,333,666 \), and 999. The results have been averaged over 10\(^4\) configurations of the random field distribution for \( L=33,99,198 \) and 10\(^3\) configurations for higher values of \( L \). As expected, the graphs start at \( M=2 \) and fan out towards \( \log L \) with increasing \( \sigma \). There is an apparent scaling with respect to \( L \). The scaling becomes clearer if we plot \( \log \frac{M}{L} \) vs. \( \sigma \) as shown in Fig.3. Let us call L33 the graph in Fig.3 corresponding to \( L=33 \) and similarly L99 etc. We find that L33 merges with L99 for \( \sigma \geq 1.8 \); L99 merges with L198 for \( \sigma \geq 1.5 \); L198 merges with L333 for \( \sigma \geq 1.4 \); L333 merges with L666 for \( \sigma \geq 1.3 \); L666 merges with L999 for \( \sigma \geq 1.2 \). It is reasonable to conclude that the range of \( \sigma \) where two graphs overlap is free of discontinuities in hysteresis. Thus we may interpret the results as indicating \( \sigma_c = 1.8, 1.5, 1.4, 1.3, 1.2 \) for systems of size 33\(^2\), 99\(^2\), 198\(^2\), 333\(^2\), 666\(^2\), and 999\(^2\) respectively. This sheds some light on previous results for \( \sigma_c \) in the literature. In reference [13] systems of size up to 600\(^2\) were studied and finite size analysis of data yielded \( \sigma_c = 1.22 \). Subsequently in reference [14] systems of size upto 65536\(^2\) were studied and finite size scaling indicated \( \sigma_c = 0.86 \). These results are consistent with Fig.3.

Simulations presented in Fig.3 demonstrate the existence of critical hysteresis on a triangular lattice. Of course, the result is not new [13,14], but it validates a new method. Our goal is to apply the new method to examine criticality on dilute triangular lattice and compare with previous results [13]. In preparation for this goal we apply the new method to the case \( c = 0 \) as well, i.e. on a honeycomb lattice. Fig.4 shows the results. Earlier studies have indicated the absence of critical hysteresis on a honeycomb lattice [12]. A comparison of Fig.3 with Fig.4 brings out characteristic differences between cases when critical hysteresis is present or absent. In both cases we see a threshold \( \sigma_{th} \) such that \( M = 2 \) for \( \sigma < \sigma_{th} \). Consequently plots of \( \log M/L^2 \) vs. \( \sigma \) for different \( L \) are widely separated in this region. For \( \sigma >> \sigma_{th} \), plots of \( \log M/L^2 \) vs. \( \sigma \) for different \( L \) merge with each other. This indicates the system to be in a phase of large disorder where spins flip up independently and associated avalanches are relatively small, localised, and uniformly distributed over \( L \times L \) lattice. If the area covered by an individual avalanche is much smaller than \( L \times L \), we may expect the number of avalanches or equivalently the number of metastable states per unit area of the lattice to be independent of \( L \) for a given \( \sigma \). This explains why the curves merge with each other for large \( \sigma \).

A closer examination of Fig.3 and Fig.4 reveals that the crossover from small disorder to large disorder behavior is qualitatively different in the two cases. On the triangular lattice, the curves for increasing \( L \) maintain their relative position over the entire range of \( \sigma \). A curve with larger \( L \) merges gently with the curve for previous largest \( L \) at a point which we identify as \( \sigma_c(L) \). The shape of the curve changes from concave up to convex up at \( \sigma_c(L) \). As \( L \) increases, \( \sigma_c(L) \) decreases. In contrast, the log \( M/L^2 \) vs. \( \sigma \) curves for different \( L \) on the honeycomb lattice reverse their order before merging with each other. This transformation takes place over a relatively narrow window \([0, \sigma]\) which shrinks further with increasing \( L \). The rise of \( M \) with \( \sigma \) becomes sharper with increasing \( L \) and shifts to smaller values of \( \sigma \). It indicates that the system in the thermodynamic limit is likely to be free of any discontinuities in magnetization for any \( \sigma > 0 \). In other words, \( \sigma_c \rightarrow 0 \) in the limit \( L \rightarrow \infty \). The absence of critical hysteresis on a honeycomb lattice has been proven for an asymmetric distribution of the random field. It was shown if on-site quenched random fields are positive with the half-width of their distribution going to zero, \( m(h) \) would increase smoothly from \(-1\) to \( 1 \) as \( h \) increases from \(-\infty\) to \( J \). A similar argument can be used to prove that more than half of the spins in the system would have turned up continuously at \( h = J \) for a Gaussian random field distribution. In other words, magnetization reversal would occur without a discontinuity as \( \sigma \rightarrow 0 \). Therefore critical hysteresis on the honeycomb lattice may be ruled out in the thermodynamic limit. Keeping in mind that finite size effects decrease logarithmically slowly, we take Fig.4 as characteristic of the absence of criticality in the finite systems studied here.
The above discussion provides us a reasonable signature of critical hysteresis which can be read off from log $M/L^2$ vs. $\sigma$ graphs. Fig.5, Fig.6, and Fig.7 show the results of simulations on dilute triangular lattice with $c = 0.3, 0.4,$ and 0.6 respectively. It is evident that the case $c = 0.3$ as well as $c = 0.4$ is similar to the case $c = 0$. Thus we conclude that critical hysteresis is absent in these cases. The graphs for $c = 0.6$ seem to have a mixed character. Results for $L = 33, 99, 198$ have the features of $c = 0$ while those for $L = 333, 666, 999$ appear closer in character to $c = 1$. To us it seems that lattice with $c = 0.6$ supports critical hysteresis albeit it is a borderline case as discussed below.

III. CONCLUDING REMARKS

Hysteresis in the zero-temperature random-field Ising model on honeycomb ($z = 3$) and triangular ($z = 6$) lattices is a difficult problem analytically as well as numerically. Extant work indicates that honeycomb lattice does not support critical hysteresis but the triangular lattice does so. However the critical point $\sigma_c(L)$ on the triangular lattice decreases extremely slowly with increasing system size $L$. Intensive numerical simulations on large systems ($N \approx 10^{10}$) have been used to estimate $\sigma_c$ in the limit $L \to \infty$. The problem on the dilute triangular lattice is even more challenging. Simulations on modest systems ($N \approx 10^6$) along with finite size scaling and percolation arguments predict $\sigma_c = 0$ for $c < 1/3$. At first sight this appears reasonable. It is similar to the behavior on Bethe lattices of uniform integer connectivity; $\sigma_c > 0$ if $z \geq 4$. A dilute triangular lattice with $c < 1/3$ corresponds to $z_{av} < 4$ and one may expect it to have $\sigma_c = 0$. However, later work on non-integer Bethe lattices showed that $\sigma_c > 0$ for $3 < z_{av} \leq 4$. If similarity between Bethe and periodic lattices were to hold in general, it would mean $\sigma_c > 0$ on dilute triangular lattice for $0 < c < 1/3$ as well. The motivation for the present work was to examine this point.

Somewhat unexpectedly the simulations presented here indicate $\sigma_c = 0$ for $0 < c < 0.6$ approximately. System size in the present simulations is of the same order ($N \approx 10^6$) as used in the previous study but processing of the data under finite size scaling hypothesis has been avoided. The reason is that the predictions of $\sigma_c$ under the finite size scaling are uncomfortably close to the smallest $\sigma$ where an infinite avalanche first occurs. This is particularly true in cases where we may expect $\sigma_c = 0$. Earlier results also showed a change of behavior at $c \approx 0.6$; $\sigma_c$ decreased almost linearly from its value at $c = 1$ to its value at $c \approx 0.6$, remained constant between $c = 0.6$ and $c = 1/3$ before dropping to zero. The reason for this was not clear but the new results suggest that $\sigma_c = 0$ for $c < 0.6$ approximately. The broad implication of this result is that dilution introduces a positional disorder in the system in addition to the random field. The positional disorder has a stronger effect on $\sigma_c$ on a periodic lattice than on a Bethe lattice.

We wish to conclude with two general remarks on hysteresis studies in ZTRFIM. Firstly, setting temperature and driving frequency equal to zero is an approximation. Hysteresis in physical systems is necessarily a finite temperature and finite time phenomena. A key feature of ZTRFIM is the occurrence of a fixed point under the zero-temperature dynamics. Scale invariance around the fixed point is directly related to experimental aspects of Barkhausen noise. The fixed point at $\sigma_c$ is lost if any of the two approximations are relaxed [10]. This is disconcerting but does not end the usefulness of ZTRFIM. The model has been applied to a variety of social phenomena including opinion dynamics and finite time phenomena. A key feature of ZTRFIM is the occurrence of a fixed point under the zero-temperature dynamics. Scale invariance around the fixed point is directly related to experimental aspects of Barkhausen noise.

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FIG. 1: Number of metastable states $M$ vs. $\sigma$ comprising the lower half of hysteresis loop on a triangular lattice of size $33 \times 33$ for Gaussian random field distributions $N[0, \sigma^2]$; for each $\sigma$ the value of $M$ is averaged over $10^4$ independent realizations of the distribution. The first inset on the left shows an enlarged view of the graph in the range $0 < \sigma < 4$. The second inset on the right shows the same data as in the first inset but on a logarithmic scale along the y-axis.
FIG. 2: $\log M$ vs. $\sigma$ in the range $0 < \sigma < 2$ on an $L \times L$ triangular lattice; $L = 33, 99, 198, 333, 666$, and $999$. $M$ is averaged over $10^4$ configurations for $L \leq 198$ and $10^3$ configurations for $L > 198$. As $L$ increases, the inflexion point $\sigma_c(L)$ in the corresponding graph shifts to lower $\sigma$. 
FIG. 3: The data in Fig. 2 for the triangular lattice is replotted to show $\log \frac{M}{L^2}$ vs. $\sigma$. $M \to 2$ as $\sigma \to 0$ irrespective of $L$. It is therefore an artifact of scaling that graphs fan out for smaller values of $\sigma$. The interesting feature is that they overlap for larger values of $\sigma$. This feature may be exploited to estimate $\sigma_c$.

FIG. 4: Log $\frac{M}{L^2}$ vs. $\sigma$ on an $L \times L$ honeycomb lattice for $0 < \sigma \leq 0.8$ and different values of $L$. A comparison with Fig. 3 indicates that critical hysteresis is absent on the honeycomb lattice (see text).
FIG. 5: Metastable states vs. $\sigma$ on a partially diluted triangular lattice with $c = 0.30$ in the range $0.1 \leq \sigma \leq 0.8$.

FIG. 6: Metastable states vs. $\sigma$ on a partially diluted triangular lattice with $c = 0.40$ in the range $0.1 \leq \sigma \leq 1.0$. 
FIG. 7: Metastable states vs. $\sigma$ on a partially diluted triangular lattice with $c = 0.60$ in the range $0.1 \leq \sigma \leq 2.0$. 

$log(M/L^2) - log((2+c)/3)$