TIGHT AND COVER-TO-JOIN REPRESENTATIONS OF SEMILATTICES AND INVERSE SEMIGROUPS

R. Exel

We discuss the relationship between tight and cover-to-join representations of semilattices and inverse semigroups, showing that a slight extension of the former, together with an appropriate selection of co-domains, makes the two notions equivalent. As a consequence, when constructing universal objects based on them, one is allowed to substitute cover-to-join for tight and vice-versa.

1. Introduction.

Exactly twelve years ago, to be precise on March 7, 2007, I posted a paper on the arXiv [3] describing the notion of tight representations of semilattices and inverse semigroups, which turned out to have many applications and in particular proved to be useful to give a unified perspective to a significant number of C*-algebras containing a preferred generating set of partial isometries ([1], [2], [4], [6], [8], [9], [14], [15]).

The notion of tight representations (described below for the convenience of the reader) is slightly involving as it depends on the analysis of certain pairs of finite sets X and Y, but it becomes much simplified when X is a singleton and Y is empty (see [4: Proposition 11.8]). In this simplified form it has been re-discovered and used in many subsequent works (e.g. [2], [10], [11], [12]) under the name of cover-to-join representations.

The notion of cover-to-join representations, requiring a smaller set of conditions, is consequently weaker and, as it turns out, strictly weaker, than the original notion of tightness. Nevertheless, besides being easier to formulate, the notion of cover-to-join representations has the advantage of being applicable to representations taking values in generalized Boolean algebras, that is, Boolean algebras without a unit. Explicitly mentioning the operation of complementation, tight representations only make sense for unital Boolean algebras.

The goal of this note is to describe an attempt to reconcile the notions of tight and cover-to-join representations: slightly extending the former, and adjusting for the appropriate co-domains, we show that, after all, the two notions coincide.

One of the main practical consequences of this fact is that the difference between the two notions becomes irrelevant for the purpose of constructing universal objects based on them, such as the completion of an inverse semigroup recently introduced in [12]. We are moreover able to fix a slight imprecision in the proof of [2: Theorem 2.2], at least as far as its consequence that the universal C*-algebras for tight vs. cover-to-join representations are isomorphic.

2. Generalized Boolean algebras.

We begin by recalling the well known notion of generalized Boolean algebras.

2.1. Definition. A generalized Boolean algebra [16: Definition 5] is a set B equipped with binary operations ∧ and ∨, and containing an element 0, such that for every a, b and c in B, one has that

(i) (commutativity) \( a \lor b = b \lor a \), and \( a \land b = b \land a \),
(ii) (associativity) \( (a \land b) \land c = a \land (b \land c) \),
(iii) (distributivity) \( a \land (b \lor c) = (a \land b) \lor (a \land c) \),
(iv) \( a \lor 0 = a \),
(v) (relative complement) if \( a = a \land b \), there is an element \( x \) in \( B \), such that \( x \lor a = b \), and \( x \land a = 0 \),
(vi) \( a \lor a = a = a \land a \).

* Universidade Federal de Santa Catarina and University of Nebraska at Lincoln.
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It follows that (ii) and (iii) also hold with $\lor$ and $\land$ interchanged, meaning that $\lor$ is associative [16: Theorems 55 & 14], and that $\lor$ distributes over $\land$ [16: Theorems 55 & 11].

When $a = a \land b$, as in (v), one writes $a \leq b$. It is then easy to see that $\leq$ is a partial order on $B$.

The element $x$ referred to in (v) is called the relative complement of $a$ in $b$, and it is usually denoted $b \setminus a$.

2.2. Definition. (cf. [16: Theorem 56]) A generalized Boolean algebra $B$ is called a Boolean algebra if there exists an element 1 in $B$, such that $a \land 1 = a$, for every $a$ in $B$.

For Boolean algebras, the complement of an element $a$ relative to 1 is often denoted $\neg a$.

Recall that an ideal of a generalized Boolean algebra $B$ is any nonempty subset $C$ of $B$ which is closed under $\lor$, and such that $a \leq b \in C \Rightarrow a \in C$.

Such an ideal is evidently also closed under $\land$ and under relative complements, so it is a generalized Boolean algebra in itself.

Given any nonempty subset $S$ of $B$, notice that the subset $C$ defined by

$$C = \{ a \in B : a \leq \bigvee_{z \in Z} z, \text{ for some finite subset } Z \subseteq S \},$$

is an ideal of $B$ and it is clearly the smallest ideal containing $S$, so we shall call it the ideal generated by $S$, and we shall denote it by $(S)$.

3. Tight and cover-to-join representations of semilattices.

From now on let us fix a semilattice $E$ (always assumed to have a zero element).

3.1. Definition. A representation of $E$ in a generalized Boolean algebra $B$ is any map $\pi : E \rightarrow B$, such that

(i) $\pi(0) = 0$, and
(ii) $\pi(x \land y) = \pi(x) \land \pi(y)$, for every $x$ and $y$ in $E$.

In order to spell out the definition of the notion of tight representations, introduced in [4], let $F$ be any subset of $E$. We then say that a given subset $Z \subseteq F$ is a cover for $F$, if for every nonzero $x$ in $F$, there exists some $z$ in $Z$, such that $z \land x \neq 0$.

Furthermore, if $X$ and $Y$ are finite subsets of $E$, we let

$$E^{X,Y} = \{ z \in E : z \leq x, \forall x \in X, \text{ and } z \perp y, \forall y \in Y \}.$$

3.2. Definition. (cf. [4: Definition 11.6]) A representation $\pi$ of $E$ in a Boolean algebra $B$ is said to be tight if, for any finite subsets $X$ and $Y$ of $E$, and for any finite cover $Z$ for $E^{X,Y}$, one has that

$$\bigvee_{z \in Z} \pi(z) = \bigwedge_{x \in X} \pi(x) \land \bigwedge_{y \in Y} \neg \pi(y).$$

(3.2.1)

Observe that if $Y$ is empty and $X$ is a singleton, say $X = \{x\}$, then

$$E^{X,Y} = E^{\{x\},\emptyset} = \{ z \in E : z \leq x \},$$

and if $Z$ is a cover for this set, then (3.2.1) reads

$$\bigvee_{z \in Z} \pi(z) = \pi(x).$$

(3.3)
To check that a given representation is tight, it is not enough to verify (3.3), as it is readily seen by considering the example in which \( E = \{0, 1\} \) and \( B \) is any Boolean algebra containing an element \( x \neq 1 \). Indeed, the map \( \pi : E \to B \) given by \( \pi(0) = 0 \), and \( \pi(1) = x \), satisfies all instances of (3.3) even thought it is not tight. The reader might wonder if the fact that \( \pi \) fails to preserve the unit is playing a part in this counter-example, but it is also easy to find examples of cover-to-join representations of non-unital semilattices which are not tight.

Representations \( \pi \) satisfying (3.3) whenever \( Z \) is a cover for \( E^{\{x\}} \) have been considered in [4: Proposition 11.8], and they have been called cover-to-join representations in [2].

It is a trivial matter to prove that a cover-to-join representation satisfies (3.2.1) whenever \( X \) is nonempty (see the proof of [4: Lemma 11.7]), so the question of whether a cover-to-join representation is indeed tight rests on verifying (3.2.1) when \( X \) is empty. In this case, and assuming that \( Z \) is a cover for \( E^{x,Y} \), it is easy to see that \( Z \cup Y \) is a cover for the whole of \( E \). Should we be dealing with a semilattice not admitting any finite cover, this situation will therefore never occur, that is, one will never be required to check (3.2.1) for an empty set \( X \), hence every cover-to-join representation is automatically tight.

This has in fact already been observed in [4: Proposition 11.8], which says that every cover-to-join representation is tight in case \( E \) does not admit any finite cover, as we have just discussed, but also if \( E \) contains a finite set \( X \) such that

\[
\bigvee_{x \in X} \pi(x) = 1. \tag{3.4}
\]

The latter condition is useful for dealing with characters, i.e. with representations of \( E \) in the Boolean algebra \( \{0, 1\} \), because the requirement that a character be nonzero immediately implies (3.4), so again cover-to-join suffices to prove tightness.

On the other hand, an advantage of the notion of cover-to-join representations is that it makes sense for representations in generalized Boolean algebras, while the reference to the unary operation \( \neg \) in (3.2.1) precludes it from being applied when the target algebra lacks a unit, that is, for a representation into a generalized Boolean algebra.

Again referring to the occurrence of \( \neg \) in (3.2.1), observe that if \( X \) is nonempty, then the right hand side of (3.2.1) lies in the ideal of \( B \) generated by the range of \( \pi \). This is because, even though \( \neg \pi(y) \) is not necessarily in \( \langle \pi(E) \rangle \), this term will appear besides \( \pi(x) \), for some \( x \) in \( X \), and hence

\[
\pi(x) \land \neg \pi(y) = \pi(x) \setminus (\pi(x) \land \pi(y)) \in \langle \pi(E) \rangle.
\]

This means that:

**3.5. Proposition.** If \( E \) is a semilattice not admitting any finite cover then, whenever \( X \) and \( Y \) are finite subsets of \( E \), and \( Z \) is a finite cover of \( E^{X,Y} \), the right hand side of (3.2.1) lies in \( \langle \pi(E) \rangle \).

As a consequence we see that definition (3.2) may be safely applied to a representation of \( E \) in a generalized Boolean algebra, as long as \( E \) does not admit a finite cover: despite the occurrence of \( \neg \) in (3.2.1), once its right hand side is expanded, it may always be expressed in terms of relative complements, hence avoiding the use of the missing unary operation \( \neg \).

We may therefore consider the following slight generalization of the notion of tight representations:

**3.6. Definition.** A representation \( \pi \) of \( E \) in a generalized Boolean algebra \( B \) is said to be tight if, either \( B \) is a Boolean algebra and \( \pi \) is tight in the sense of (3.2), or the following two conditions are verified:

- (i) \( E \) admits no finite cover, and
- (ii) (3.2.1) holds for any finite subsets \( X \) and \( Y \) of \( E \), and for any finite cover \( Z \) for \( E^{X,Y} \).

As already stressed, despite the occurrence of \( \neg \) in (3.2.1), condition (3.6.ii) will always make sense in a generalized Boolean algebra.

So here is a result that perhaps may be used to reconcile the notions of tightness and cover-to-join representations:
3.7. Theorem. Let $\pi$ be a representation of the semilattice $E$ in the generalized Boolean algebra $B$. Then

(i) if $\pi$ is tight then it is also cover-to-join,

(ii) if $\pi$ is cover-to-join then there exists an ideal $B'$ of $B$, containing the range of $\pi$, such that, once $\pi$ is seen as a representation of $E$ in $B'$, one has that $\pi$ is tight.

Proof. Point (i) being immediate, let us prove (ii). Under the assumption that $E$ does not admit any finite cover, we have that $\pi$ is tight as a representation into $B' = B$, by [4: Proposition 11.8], or rather by its obvious adaptation to generalized Boolean algebras.

It therefore remains to prove (ii) in case $E$ does admit a finite cover, say $Z$. Setting $e = \bigvee_{z \in Z} \pi(z)$, (3.7.1) we claim that

$$
\pi(x) \leq e, \quad \forall x \in E.
$$

To see this, pick $x$ in $E$ and notice that, since $Z$ is a cover for $E$, we have in particular that the set

$$\{ z \wedge x : z \in Z \}
$$

is a cover for $x$, so the cover-to-join property of $\pi$ implies that

$$
\pi(x) = \bigvee_{z \in Z} \pi(z \wedge x) \leq \bigvee_{z \in Z} \pi(z) = e,
$$

proving (3.7.2). We therefore let

$$
B' = \{ a \in B : a \leq e \},
$$

which is evidently an ideal of $B$ containing the range of $\pi$ by (3.7.2).

By (3.7.1) we then have that $\pi$ satisfies [4: Lemma 11.7.(i)], as long as we see $\pi$ as a representation of $E$ in $B'$, whose unit is clearly $e$. The result then follows from [4: Proposition 11.8]. □

4. Non-degenerate representations of semilattices.

The following is perhaps the most obvious adaptation of the notion of non-degenerate representations extensively used in the theory of operator algebras [17: Definition 9.3].

4.1. Definition. We shall say that a representation $\pi$ of a semilattice $E$ in a generalized Boolean algebra $B$ is non-degenerate if, for every $a$ in $B$, there is a finite subset $Z$ of $E$ such that $a \leq \bigvee_{z \in Z} \pi(z)$. In other words, $\pi$ is non-degenerate if and only if $B$ coincides with the ideal generated by the range of $\pi$.

Observe that, if both $E$ and $B$ have a unit, and if $\pi$ is a unital map, then $\pi$ is evidently non-degenerate. More generally, if $\pi$ satisfies (3.4), then the same is also clearly true.

The following result says that, by adjusting the co-domain of a representation, we can always make it non-degenerate.

4.2. Proposition. Let $\pi$ be a representation of $E$ in the generalized Boolean algebra $B$. Letting $C$ be the ideal of $B$ generated by the range of $\pi$, one has that $\pi$ is a non-degenerate representation of $E$ in $C$.

Proof. Obvious. □

For non-degenerate representations we have the following streamlined version of (3.7):

4.3. Corollary. Let $\pi$ be a non-degenerate representation of the semilattice $E$ in the generalized Boolean algebra $B$. Then $\pi$ is tight if and only if it is cover-to-join.

Proof. The “only if” direction being trivial, we concentrate on the “if” part, so let us assume that $\pi$ is cover-to-join. By (3.7) there exists an ideal $B'$ of $B$, containing the range of $\pi$, and such that $\pi$ is tight as a representation in $B'$. Such an ideal will therefore contain the ideal generated by $\pi(E)$, which coincides with $B$ by hypothesis. Therefore $B' = B$, and hence $\pi$ is tight as a representation into its default co-domain $B$. □
5. Representations of inverse semigroups.

By its very nature, the concept of a tight representation pertains to the realm of semilattices and Boolean algebras. However, given the relevance of the study of semilattices in the theory of inverse semigroups, tight representations have had a strong impact on the latter.

Recall that a Boolean inverse semigroup (see [5] but please observe that this notion is not equivalent to the homonym studied in [10] and [18]) is an inverse semigroup whose idempotent semilattice $E(S)$ is a Boolean algebra. In accordance with what we have been discussing up to now, it is sensible to give the following:

5.1. Definition.

(i) A generalized Boolean inverse semigroup is an inverse semigroup whose idempotent semilattice is a generalized Boolean algebra.

(ii) (cf. [4: Definition 13.1] and [5: Proposition 6.2]) If $S$ is any inverse semigroup and $T$ is a generalized Boolean inverse semigroup, we say that a homomorphism $\pi : S \rightarrow T$ (always assumed to preserve zero) is tight if the restriction of $\pi$ to $E(S)$ is a tight representation into $E(T)$, in the sense of (3.6).

(iii) If $\pi$ is as above, we say that $\pi$ is cover-to-join if the restriction of $\pi$ to $E(S)$ is cover-to-join.

We then have the following version of (3.7) and (4.2):

5.2. Corollary. Let $\pi$ be a representation of the inverse semigroup $S$ in the generalized Boolean inverse semigroup $T$. Then

(i) if $\pi$ is tight then it is also cover-to-join,

(ii) if $\pi$ is cover-to-join then there exists a generalized Boolean inverse sub-semigroup $T'$ of $T$, containing the range of $\pi$, such that, once $\pi$ is seen as a representation of $S$ in $T'$, one has that $\pi$ is tight.

(iii) if $\pi$ is cover-to-join, and if the restriction of $\pi$ to $E(S)$ is non-degenerate, then $\pi$ is tight.

Proof. The proof is essentially contained in the proofs of (3.7) and (4.2), except maybe for the proof of (ii) under the assumption that $E(S)$ admits a finite cover, say $Z$. In this case, let $e$ be as in (3.7.1) and put

$$T' = \{ t \in T : t^*t \leq e, tt^* \leq e \},$$

observing that $T'$ is clearly an inverse sub-semigroup of $T$, and that its idempotent semilattice is a Boolean algebra. Given any $s$ in $S$, observe that $s^*s$ lies in $E(S)$ and

$$\pi(s)^*\pi(s) = \pi((s^*s) \leq e,$$

where the last inequality above follows as in (3.7.2). By a similar reasoning one shows that also $\pi(s)\pi(s)^* \leq e$, so we see that $\pi(s)$ lies in $T'$, and we may then think of $\pi$ as a representation of $S$ in $T'$. As in (3.7), one may now easily prove that $\pi$ becomes a tight representation into $T'$. \(\square\)

6. Conclusion.

As a consequence of the above results, when defining universal objects (such as semigroups, algebras or C*-algebras) for a class of representations of inverse semigroups, one may safely substitute cover-to-join for tight and vice-versa. Given the widespread use of tight representations, there are many instances where the above principle applies. Below we spell out one such result to concretely illustrate our point, but similar results may be obtained as trivial reformulations of the following:

6.1. Theorem. Let $S$ be an inverse semigroup and let $C^*_\text{tight}(S)$ be the universal C*-algebra [4: Theorem 13.3] for tight Hilbert space representations of $S$ [4: Definition 13.1]. Also let $C^*_\text{cover-to-join}(S)$ be the universal C*-algebra for cover-to-join Hilbert space representations of $S$. Then

$$C^*_\text{tight}(S) \simeq C^*_\text{cover-to-join}(S).$$

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1 All inverse semigroups in this note are required to have a zero.
Proof. It suffices to prove that $C^*_\text{tight}(S)$ also has the universal property for cover-to-join representations. So let

$$
\pi : S \to B(H)
$$

be a cover-to-join representation of $S$ on some Hilbert space $H$. Should the idempotent semilattice of $S$ admit no finite covers, one has that $\pi$ is tight so there is nothing to do. On the other hand, assuming that $Z$ is a finite cover for $E(S)$, let $e$ be as in (3.7.1).

Writing $H_e$ for the range of $e$ and letting $K = H_e^\perp$, we then obviously have that $H = H_e \oplus K$. It then follows from (3.7.2) that each $\pi(s)$ decomposes as a direct sum of operators

$$
\pi(s) = \pi'(s) \oplus 0,
$$

thus defining a representation $\pi'$ of $S$ on $H_e$ which is clearly also cover-to-join. It is also clear that $\pi'$ is non-degenerate on $E(S)$, so we have by (5.2.iii) that $\pi'$ is tight. Therefore the universal property provides a $^*$-representation $\varphi'$ of $C^*_\text{tight}(S)$ on $B(H_e)$ coinciding with $\pi'$ on the canonical image of $S$ within $C^*_\text{tight}(S)$. It then follows that $\varphi := \varphi' \oplus 0$ coincides with $\pi$ on $S$, concluding the proof. □

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