On the blow-up of a normal singularity at maximal Cohen–Macaulay modules

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Abstract
In Raynaud and Gruson (Invent Math 13:1–89, 1971) and Raynaud (Compos Math 24:11–31, 1972) developed the theory of blowing-up an algebraic variety $X$ along a coherent sheaf $M$. However, not much is known about the singularities of the blow-up. In this article, we prove that if $X$ is a normal surface singularity and $M$ is a reflexive $O_X$-module, then such a blow-up arises naturally from the theory of McKay correspondence. We show that the normalization of the blow-up of Raynaud and Gruson is obtained by a resolution of $X$ such that the full sheaf $\mathcal{M}$ associated to $M$ (i.e., the reflexive hull of the pull-back of $M$) is globally generated and then contracting all the components of the exceptional divisor not intersecting the first Chern class of $\mathcal{M}$. Moreover, we prove that if $X$ is Gorenstein and $M$ is special in the sense of Wunram (Math Ann 279(4):583–598, 1988) and Riemenschneider (Compos Math 24:11–31, 1972) (generalized in Fernández de Bobadilla and Romano-Velázquez (Reflexive Modules on Normal Gorenstein Stein Surfaces, Their Deformations and Moduli, arXiv:1812.06543, 2018)), then the blow-up of Raynaud and Gruson is normal. Finally, we use the theory of matrix factorization developed by Eisenbud, to give concrete examples of such blow-ups.

Keywords Maximal Cohen–Macaulay module · Flatifying blowing-up · Gorenstein surface singularity · Matrix factorization

Mathematics Subject Classification Primary 13C14 · 13H10 · 14E16 · 32S25 · 32S05

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1 Introduction

Let $X$ be an algebraic variety and $M$ be a coherent $\mathcal{O}_X$-module. The ground field is always assumed to be the complex numbers. The Raynaud–Gruson flattening theorem [16, 31] states that under some hypothesis, there exists a finitely presented closed subscheme of $X$ (depending on $M$) such that the blow-up $f: X' \to X$ along the subscheme satisfies the properties: $f^*M/\text{tor}$ is flat over $X'$ and $f^*M/\text{tor}$ has the same rank as $M$. The construction is universal in the sense that for any morphism $\eta: Y \to X$ for which the pull-back of $M$ to $Y$ is flat modulo torsion, the morphism $\eta$ factors through $X'$. The existence of such blow-ups satisfying the universal minimality condition has numerous applications. Raynaud [16] uses this blow-up to prove the Chow’s lemma (i.e., given $X$ separated over $S$, we can find a blow-up $X'$ of $X$ such that $X'$ is quasi-projective over $S$). Hironaka [23] recovers this blow-up and the Chow’s lemma in the context of analytic geometry. Abramovich, Karu, Matsuki and Włodarczyk [1] use the Chow’s lemma to prove the weak factorization conjecture for birational maps (i.e., a birational map between complete non-singular varieties is a composition of blow-ups and blow-downs along smooth centers). Campana [5] uses the flattening theorem to study the geometry, arithmetic, and classification of compact Kähler manifolds. Hassett and Hyeon [22] use the flattening theorem to study the log canonical models for the moduli space of curves. Rossi [34] and later Villamayor [35] investigate the case of the blow-up at a coherent module $M$ over a (possible non-reduced) ring $R$ with applications to the flattening of projective morphisms and related problems in singularity theory like the Nash transformation (the closure of the graph of the Gauss map which assigns each regular point to its tangent space considered as an element in a Grassmanian manifold). Yasuda uses the blow-up at a coherent module to construct canonical birational modifications of a variety, called the higher Nash transformation [37] and the $F$-blowup [38].

In general, the Raynaud–Gruson blow-up may be very difficult to describe. Moreover, the blow-up is not regular in general. In some instances this blow-up has been studied in detail, for example: Hara, Sawada and Yasuda [21] prove that the $F$-blowup of a rational surface singularity has only rational singularities. In this article we prove that the Raynaud–Gruson blow-up has a very nice description in the case of $M$ a maximal Cohen–Macaulay module and $X$ a normal surface singularity. For this, we generalize some techniques and ideas given in [11] together with a careful study of the Raynaud–Gruson blow-up. In this setting, given $M$ a maximal Cohen–Macaulay $\mathcal{O}_X$-module, by [11] there exists an unique resolution $\pi: \tilde{X} \to X$ called the minimal adapted resolution associated to $M$ such that the associated full sheaf $\mathcal{M} := (\pi^*M)\vee$ is generated by global sections and $\tilde{X}$ is the minimal resolution of $X$ satisfying this property, where $(-)^\vee$ denotes the dual with respect to the structure sheaf. Recall, full sheaf was first defined by Esnault [9] for rational singularities and generalized by Kahn [24] for normal surface singularities. We prove (Theorem 4.5):

**Theorem 1** Let $(X, x)$ be the germ of a normal surface singularity. Let $M$ be a reflexive $\mathcal{O}_X$-module of rank $r$. Let $\pi: \tilde{X} \to X$ be the minimal adapted resolution associated to $M$ with exceptional divisor $E$. Let $E_1, \ldots, E_n$ be the irreducible components of $E$ and $\mathcal{M} := (\pi^*M)\vee$ be the full sheaf associated to $M$. Then, the normalization of the Raynaud–Gruson blow-up of $X$ at $M$ is obtained by contracting the irreducible components $E_i$ such that $c_1(\mathcal{M}) \cdot E_j = 0$.

In the case of normal, Gorenstein surface singularities and maximal Cohen–Macaulay modules we prove that the Raynaud–Gruson blow-up is related to the McKay correspondence. The McKay correspondence was constructed by McKay [29] and a conceptual geometric
understanding of the correspondence was achieved by a series of papers by Gonzalez-Springberg and Verdier [13], Artin and Verdier [2] and by Esnault and Knörrer [10]. This correspondence gives a bijection between the isomorphism classes of non-trivial indecomposable reflexive modules and the irreducible components of the exceptional divisor of the minimal resolution of a rational double point. Later Wunram [36] generalized the McKay correspondence to any rational singularity using the notion of *specialty*. If the singularity is Gorenstein we generalize in [11] the definition of specialty as follows: a maximal Cohen–Macaulay module $M$ of rank $r$ over a normal surface singularity $X$ is called *special* if the minimal adapted resolution $\pi : \tilde{X} \to X$ has the property that the dimension over $\mathbb{C}$ of the module $R^1\pi_* (\pi^* M)_\vee$ is equal to $rp_g$, where $p_g$ is the geometric genus of $X$. Using the notion of specialty we prove the following result (Theorem 4.4):

**Theorem 2** Let $(X, x)$ be the germ of a normal Gorenstein surface singularity and $M$ be a special module. Let $f : \text{Bl}_M(X) \to X$ the Raynaud–Gruson blow-up of $X$ at the module $M$. Then, $\text{Bl}_M(X)$ is normal.

Theorem 1 and Theorem 2 generalize the result of Hara, Sawada and Yasuda [21, Proposition 3.2] and the results of Gustavsen and Ile [17] to the case of normal surface singularities. Hara, Sawada and Yasuda prove that the $F$-blowup of a rational surface singularity of positive characteristic is dominated by the minimal resolution and has only rational singularities. Gustavsen and Ile prove the analogous statement in characteristic zero, i.e., they prove that the blow-up at any maximal Cohen–Macaulay module in a rational surface singularity of characteristic zero is a partial resolution dominated by the minimal resolution. The results of Gustavsen and Ile follow by the following two properties of rational singularities:

- the minimal resolution of a rational singularity is the minimal adapted resolution of every maximal Cohen–Macaulay module,
- the blow-up of a complete ideal (in the sense of Lipman [28]) is always normal.

For a general normal surface singularity both assertions fail. Indeed, by [11] we know that in general in a Gorenstein, normal, surface singularity, its minimal resolution is not the minimal adapted resolution of every maximal Cohen–Macaulay module. Moreover, in Example 5.6 we provide a normal, Gorenstein, surface singularity and a maximal Cohen–Macaulay module such that the Raynaud–Gruson blow-up is not normal. In order to prove both theorems we use the properties of the minimal adapted resolution and the techniques developed in [11].

As a consequence of Theorem 1 and Theorem 2 we generalize the McKay correspondence given by Artin and Verdier [2] as follows (Corollary 4.9):

**Corollary 3** Let $(X, x)$ be a normal Gorenstein surface singularity. Then, there exists a bijection between the following sets:

1. The set of special, indecomposable non-trivial $\mathcal{O}_X$-modules up to isomorphism.
2. The set of irreducible divisors $E$ over $x$, such at any resolution of $X$ where $E$ appears, the Gorenstein form has neither zeros nor poles along $E$.
3. The set of partial resolutions $\psi : Y \to X$ with irreducible exceptional divisor $E$ such that:

   (a) the Gorenstein form has neither zeros nor poles along $E \setminus \text{Sing } Y$.
   (b) the partial resolution is dominated by a resolution such that the Gorenstein form has neither zeros nor poles along its exceptional divisor.

In the last section we use the theory of matrix factorizations developed by Eisenbud [8] to compute some explicit examples of blow-up at reflexive modules. In particular we prove that
the fundamental module in the hypersurface singularity given by \( f = x^3 + z^3 + y^3 \) is not a special module. It is important to note that Hara, Sawada and Yasuda [21, Example 4.6] have an example of a simple elliptic singularity of positive characteristic in which the \( F \)-blowup is not normal. Hence, our example is the characteristic zero version of [21, Example 4.6].

The organization of this paper is as follows: In Sect. 2 we give preliminary results about full sheaves over normal, Gorenstein surface singularities, the Raynaud–Gruson blow-up in coherent sheaves and complete ideals. In Sect. 3 we define the notion of adapted resolutions over any normal singularity of arbitrary dimension. In Sect. 4 we prove our main results and generalize some results given in [11]. In Sect. 5 we recall some basics on matrix factorization and compute explicit examples of the Raynaud–Gruson blow-up of maximal Cohen–Macaulay modules.

2 Preliminaries

In this section we recall basics on full sheaves over Gorenstein singularities, the Raynaud–Gruson blow-up in coherent sheaves and complete ideals. We assume basic familiarity with dualizing sheaves, modules and normal surface singularities, see [4, 18, 20, 30] for more details.

2.1 Setting and notation

Throughout this article, we denote by \((X, x)\) either a complex analytic normal surface germ, or the spectrum of a normal complete \(\mathbb{C}\)-algebra of dimension 2. In few instances it will denote the spectrum of a normal complete \(\mathbb{C}\)-algebra of dimension \(n\).

In this situation \(X\) has a dualizing sheaf \(\omega_X\), and we also denote by \(\omega_X\) its stalk at \(x \in X\), which is called the dualizing module of the ring \(\mathcal{O}_{X,x}\) (see [20, Chapter 5 Sect. 3] for more details). If \((X, x)\) is a Gorenstein normal singularity, then the dualizing module coincides with \(\mathcal{O}_{X,x}\).

Let \(\pi: \tilde{X} \to X\), be a resolution of singularities, i.e., a proper holomorphic map from a smooth surface \(\tilde{X}\) to a given representative of \((X, x)\) such that \(\pi\) is biholomorphic in the complement of \(\pi^{-1}(x)\). The exceptional divisor is denoted by \(E := \pi^{-1}(x)\), with irreducible components \(E_1, \ldots, E_m\).

If \((X, x)\) is a Gorenstein surface singularity, there is a 2-form \(\Omega_{\tilde{X}}\) which is meromorphic in \(\tilde{X}\), and has neither zeros nor poles in \(\tilde{X} \setminus E\); this form is called the Gorenstein form. Let \(\text{div}(\Omega_{\tilde{X}}) = \sum q_i E_i\) be the divisor associated with the Gorenstein form. The coefficients \(q_i\) are independent of the choice of the form \(\Omega_{\tilde{X}}\) with these properties.

Remark 2.1 In the minimal model program the coefficients \(q_i\) are called the discrepancies.

Definition 2.2 Let \(\pi: \tilde{X} \to X\) be a resolution of a normal Gorenstein surface singularity. The canonical cycle is defined as \(Z_k := \sum_i -q_i E_i\), where the \(q_i\) are the coefficients defined above.

We say that \(\tilde{X}\) is small with respect to the Gorenstein form if \(Z_k\) is greater than or equal to 0.

Definition 2.3 The geometric genus of \(X\) is defined to be the dimension as a \(\mathbb{C}\)-vector space of \(H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})\) for any resolution. We denote the geometric genus by \(p_g\).
Following Reid [32] we have the following definition:

**Definition 2.4** Let \((X, x)\) be the germ of a normal surface singularity. A *partial resolution* is a proper birational morphism \(\pi: Y \to X\) with \(Y\) a normal variety such that \(\pi\) is biholomorphic in the complement of \(\pi^{-1}(x)\).

### 2.2 Cohen–Macaulay modules and reflexive modules

Let \(X\) be a normal variety. Let \(\text{Hom}_{\mathcal{O}_X}(\cdot, \cdot)\) and \(\text{Ext}^i_{\mathcal{O}_X}(\cdot, \cdot)\) be the sheaf theoretic Hom and Ext functors. The dual of an \(\mathcal{O}_X\)-module \(M\) is denoted by \(M^\vee := \text{Hom}_{\mathcal{O}_X}(M, \mathcal{O}_X)\) and its \(\omega_X\)-dual is \(\text{Hom}_{\mathcal{O}_X}(\text{Hom}_{\mathcal{O}_X}(M, \omega_X), \omega_X)\). An \(\mathcal{O}_X\)-module \(M\) is called *reflexive* (resp. \(\omega_X\)-reflexive) if the natural homomorphism from \(M\) to \(M^{\vee\vee}\) (resp. to \(\text{Hom}_{\mathcal{O}_X}(\text{Hom}_{\mathcal{O}_X}(M, \omega_X), \omega_X)\)) is an isomorphism.

An \(\mathcal{O}_X\)-module \(M\) is called *Cohen–Macaulay* if for every \(y \in X\) the depth of the stalk \(M_y\) is equal to the dimension of the support of the module \(M_y\). If the depth of \(M_y\) is equal to the dimension of \(\mathcal{O}_{X,y}\), then the module \(M_y\) is called *maximal Cohen–Macaulay*. A module is *indecomposable* if it cannot be written as a direct sum of two non-trivial submodules.

### 2.3 Full sheaves and the minimal adapted resolution

Let \((X, x)\) be the germ of a normal surface singularity and \(\pi: \tilde{X} \to X\), be a resolution. Recall, the following definition of full sheaves as in [24, Definition 1.1].

**Definition 2.5** An \(\mathcal{O}_{\tilde{X}}\)-module \(\mathcal{M}\) is called *full* if there is a reflexive \(\mathcal{O}_X\)-module \(M\) such that \(\mathcal{M} \cong (\pi_* M)^{\vee\vee}\). We call \(\mathcal{M}\) the full sheaf associated to \(M\).

Another important notion is the concept of specialty. For rational surface singularities, Wunram [36] and Riemenschneider [33] defined a special full sheaf as a full sheaf for which its dual has the first cohomology group equal to zero. In [11] the author and Bobadilla generalized this definition as follows:

**Definition 2.6** Let \(M\) be a reflexive \(\mathcal{O}_X\)-module of rank \(r\) and \(\mathcal{M}\) be the full sheaf associated to \(M\). The full sheaf \(\mathcal{M}\) is called *special* if \(\dim_{\mathbb{C}} \left(R^1 \pi_* \left(M^\vee\right)\right) = rp_g\) where \(p_g\) is the geometric genus.

We say that \(M\) is a *special module* if for any resolution, the full sheaf associated to \(M\) is special.

Let \(M\) be a reflexive \(\mathcal{O}_X\)-module. The minimal adapted resolution associated to \(M\) was defined in [11] and it plays a crucial role for the classification of special reflexive modules. Recall,

**Definition 2.7** Let \(M\) be a reflexive \(\mathcal{O}_X\)-module. The minimal resolution \(\pi: \tilde{X} \to X\) for which the associated full sheaf \((\pi_* M)^{\vee\vee}\) is generated by global sections is called the *minimal adapted resolution* associated to \(M\).

An example of a special module was computed in [11, Example 10.6]. A different example of a special module is as follows:
Example 2.8 Let \( f = x^{2n+1} + y^{2n+1} + z^{2n+1} \) with \( n \geq 1 \). The smooth line \( C = V(x, y + z) \) is contained in \( X = V(f) \). Therefore, by \cite[Proposition 5.13]{11} the dual of the module of relations of \( \mathcal{O}_C \) as an \( \mathcal{O}_X \)-module is special.

Remark 2.9 It is important to note that not every reflexive \( \mathcal{O}_X \)-module is special. For example, let \( X = V(x^3 + y^3 + z^3) \subset \mathbb{C}^3 \) and \( \pi : \tilde{X} \to X \) be the minimal resolution. Let \( M \) be the module of Zariski differentials. By \cite[p. 216]{3} the sheaf \( \pi^* M/\text{tor} \) is locally free, therefore
\[
(\pi^* M) \overset{\vee}{\cong} \pi^* M/\text{tor}.
\]
Since \( \pi^* M/\text{tor} \) is generated by global sections, so is \( (\pi^* M) \overset{\vee}{\cong} \). Therefore \( \pi : \tilde{X} \to X \) is the minimal adapted resolution associated to \( M \). Now, suppose that \( M \) is special. By \cite[Proposition 4.14]{11} the canonical cycle \( Z_K \) should be 0. This is impossible since \( X \) is not a rational double point singularity. Thus, \( M \) is a non-special reflexive module.

2.4 Blowing up at coherent sheaves

In this paper we use the description of the Raynaud–Gruson blow-up given by Villamor \cite{35}. Let \( R \) be a domain with quotient field \( K \). The fractional ideals are the finitely generated \( R \)-submodules of \( K \). Two fractional ideals \( J_1 \) and \( J_2 \) are isomorphic if and only if there exists some element \( k \) in \( K \) different from zero such that \( J_1 = k J_2 \). The norm of a module is defined in the class of all fractional ideals modulo isomorphism as follows:

**Definition 2.10** \cite[p. 123]{35} Let \( M \) be a finitely generated \( R \)-module of rank \( r \). The norm of \( M \) is the class
\[
\| M \|_R := \text{Im} \left( \bigwedge^r M \to \bigwedge^r M \otimes K \cong K \right) / \sim,
\]
where \( \sim \) denotes the isomorphism as fractional ideals.

The blow-up of \( R \) at the module \( M \) is described as follows:

**Theorem 2.11** \cite[Theorem 3.3]{35} Let \( M \) be a finitely generated \( R \)-module of rank \( r \). There exists a blow-up:
\[
f : \text{Bl}_M(R) \to \text{Spec}(R),
\]
with the following properties:

1. The sheaf \( f^* M/\text{tor} \) is a locally free sheaf of \( \mathcal{O}_{\text{Bl}_M(R)} \)-modules of rank \( r \).
2. (Universal property) For any morphism \( \sigma : Z \to \text{Spec}(R) \) such that \( \sigma^* M/\text{tor} \) is a locally free sheaf of \( \mathcal{O}_Z \)-modules of rank \( r \), there exists an unique morphism \( \beta : Z \to \text{Bl}_M(R) \) such that \( f \circ \beta = \sigma \).

As mentioned before, Villamayor constructed the above blow-up under more general assumptions, but in our setting Theorem 2.11 is sufficient. Furthermore, in our situation \( R \) is a domain, hence this blow-up is a proper birational morphism (see \cite{35} and \cite{34} for details).

**Remark 2.12** Recall that, the blow-up stated in Theorem 2.11 was constructed by Villamayor by taking the blow-up at the fractional ideal \( \| M \|_R \) (at any representative). This fact will be used later in the article.
2.5 Complete ideals

In this section we recall some basic notions of complete ideals. See [28] for more details. Let \( X \) be an integral scheme with sheaf of rational functions \( \mathcal{R}_X \). Let \( \mathcal{J} \) be a quasi-coherent \( \mathcal{O}_X \)-submodule of \( \mathcal{R}_X \). Denote by

\[
\mathcal{A} := \bigoplus_{n \geq 0} \mathcal{J}^n, \quad (\mathcal{J}^0 := \mathcal{O}_X), \quad \text{and} \quad \mathcal{R} := \bigoplus_{n \geq 0} \mathcal{R}_X^n, \quad (\mathcal{R}_X^0 := \mathcal{R}_X),
\]

where \( \mathcal{J}^n \) (resp. \( \mathcal{R}_X^n \)) is the product (as fractional ideal) of \( n \) copies of \( \mathcal{J} \) (resp. \( \mathcal{R}_X \)). Hence, the sheaves \( \mathcal{A} \) and \( \mathcal{R} \) are quasi-coherent graded \( \mathcal{O}_X \)-algebras. Let \( \mathcal{A}' \) be the integral closure of \( \mathcal{A} \) in \( \mathcal{R} \). Denote by \( \mathcal{J}_n \) the image of the following composition:

\[
\mathcal{A}' \subset \mathcal{R} \xrightarrow{pr_n} \mathcal{R}_X,
\]

where \( pr_n \) is the \( n \)-th projection.

**Definition 2.13** We say that \( \mathcal{J}_1 \) is the completion of \( \mathcal{J} \). Furthermore, \( \mathcal{J} \) is complete if \( \mathcal{J} = \mathcal{J}_1 \).

The following two lemmas will play a crucial role later.

**Lemma 2.14** [28, Lemma 5.2] Let \( X \) be an integral scheme and \( \mathcal{J} \) be a coherent \( \mathcal{O}_X \)-submodule of \( \mathcal{R}_X \) different from zero. If all the positive powers of \( \mathcal{J} \) are complete, then the scheme obtained by blowing-up \( \mathcal{J} \) is normal.

**Lemma 2.15** [28, Lemma 5.3] Let \( X, \mathcal{R}_X \) and \( \mathcal{J} \) be as in Lemma 2.14. Let \( f : X \to Y \),

be a quasi-compact, quasi-separated birational morphism. If \( \mathcal{J} \) is complete, then so is \( f_* \mathcal{J} \).

3 Adapted resolutions

In this section we introduce the notion of adapted resolutions of maximal Cohen–Macaulay modules over any dimension. We prove that if the singularity has dimension two, then the minimal adapted resolution is an adapted resolution (Proposition 3.3).

**Definition 3.1** Let \((X, x)\) be the complex analytic germ of a normal \( n \)-dimensional singularity. Let \( M \) be a maximal Cohen–Macaulay \( \mathcal{O}_X \)-module. A resolution

\[
\pi : \tilde{X} \to X,
\]

is called an adapted resolution associated to \( M \) if \( \pi^* M / \text{tor} \) is a locally free \( \mathcal{O}_{\tilde{X}} \)-module.

Note that, if \((X, x)\) is a normal surface singularity and \( M \) is a reflexive \( \mathcal{O}_X \)-module, then its minimal adapted resolution is an adapted resolution. The following proposition tell us that adapted resolutions exist in any dimension.

**Proposition 3.2** Let \((X, x)\) be the germ of a normal \( n \)-dimensional singularity. Let \( M \) be a maximal Cohen–Macaulay module and

\[
f : \text{Bl}_M(X) \to X,
\]
be the blow-up of $X$ at $M$. Then, any resolution

$$\sigma : Y \to \text{Bl}_M(X),$$

is an adapted resolution associated to $M$.

**Proof** The module $M$ is a maximal Cohen–Macaulay, therefore $M$ is free over the regular part of $X$. Hence, the morphism $f$ is an isomorphism over the regular part of $X$. Thus

$$f \circ \sigma : Y \to X,$$

is a resolution of $X$. Denote by

$$\tilde{M} := f^*M / \text{tor} \quad \text{and} \quad \tilde{\mathcal{M}} := \sigma^* \tilde{M},$$

where $\text{tor}$ denotes the torsion part of $f^*M$.

The sheaf $\tilde{M}$ is locally free and generated by global sections, therefore $\tilde{\mathcal{M}}$ is also locally free and generated by global sections. Now, consider the following two exact sequences:

\begin{align*}
0 & \to \text{tor} \to f^*M \to \tilde{M} \to 0, \quad (1) \\
0 & \to \text{tor} \to \sigma^* f^*M \to \sigma^* f^*M / \text{tor} \to 0. \quad (2)
\end{align*}

Applying the functor $\sigma^*$ to the exact sequence (1) and using (2), we get the following commutative diagram:

\begin{equation*}
\begin{array}{cccccc}
0 & \to & \ker & \xrightarrow{=} & \sigma^* f^*M & \xrightarrow{\alpha} & \tilde{\mathcal{M}} & \to & 0 \\
\uparrow & & \circ & & \circ & & \alpha & & \\
0 & \to & \text{tor} & \xrightarrow{=} & \sigma^* f^*M & \xrightarrow{\alpha} & \sigma^* f^*M / \text{tor} & \to & 0
\end{array}
\end{equation*}

Notice that the morphism $\alpha$ is an isomorphism. Indeed, it is clearly a surjection. Now, its kernel is torsion. But $\sigma^* f^*M / \text{tor}$ is torsion-free. Therefore, $\alpha$ is also injective. Since $\alpha$ is an isomorphism, the sheaf $\sigma^* f^*M / \text{tor}$ is locally free. This proves the proposition. \hfill \Box

As an application of Theorem 2.11 we can construct the minimal adapted resolution of a reflexive module as follows:

**Proposition 3.3** Let $(X, x)$ be the germ of a normal surface singularity. Let $M$ be a reflexive $\mathcal{O}_X$-module and

$$\pi : \tilde{X} \to X,$$

be the associated minimal adapted resolution. Let $f : \text{Bl}_M(X) \to X$ be the blow-up of $X$ at the module $M$ and

$$\rho : \tilde{\text{Bl}_M(X)}_{\text{min}} \to \text{Bl}_M(X),$$

be the minimal resolution of $\text{Bl}_M(X)$. Then, $\tilde{\text{Bl}_M(X)}_{\text{min}} \cong \tilde{X}$.

**Proof** Consider the following commutative diagram:

\begin{equation*}
\begin{array}{ccc}
\tilde{\text{Bl}_M(X)}_{\text{min}} & \xrightarrow{\phi} & \tilde{X} \\
\rho \downarrow & & \downarrow \pi \\
\text{Bl}_M(X) & \xrightarrow{f} & X
\end{array}
\end{equation*}
where $\phi$ is given by the universal property of the blow-up of $X$ at $M$ and $\varphi$ comes from the universal property of the minimal resolution of $\text{Bl}_M(X)$. By Definition 2.7, the resolution $\tilde{X}$ is the minimal resolution of $X$ such that the full sheaf $\mathcal{M} := (\pi^* M \otimes \varphi)$ is generated by global sections. By Proposition 3.2, the resolution $\text{Bl}_M(X)_{\text{min}}$ is an adapted resolution associated to $M$. By Definition 3.1, the sheaf $\rho^* f^* M / \text{tor}$ is locally free. Set $\tilde{M} := (\rho^* f^* M / \text{tor}) \otimes \varphi$. Since $\rho^* f^* M / \text{tor}$ is locally free, then $\tilde{M} \cong \rho^* f^* M / \text{tor}$. Since $\rho^* f^* M / \text{tor}$ is generated by global sections, so is $\tilde{M}$. Hence the morphism $\varphi$ is an isomorphism. This proves the proposition. □

4 Normality of the blow-up

In the previous section we used the blow-up of a reflexive module to recover its minimal adapted resolution, so it is natural to study the properties of this blow-up. In this section we prove that the Raynaud–Gruson blow-up of a special module over a Gorenstein, normal, surface singularity is a partial resolution (Theorem 4.4). Furthermore, we prove that this blow-up can be constructed via a resolution (depending on the reflexive module) and the first Chern class of the associated full sheaf (Theorem 4.5).

First, we observe that the pushforward via a resolution map commutes with tensor product modulo torsion of certain coherent modules.

Lemma 4.1 Let $(X, x)$ be the germ of a normal surface singularity. Let $\pi: \tilde{X} \to X$ be any resolution with $E$ the exceptional divisor. Let $\mathcal{M}$ be a $\mathcal{O}_{\tilde{X}}$-module generated by global sections and $A$ be an $\mathcal{O}_{\tilde{X}}$-module of dimension one such that its support intersects $E$ in a finite number of points. Then, the natural morphism

$$\alpha: \pi_* \mathcal{M} \otimes \pi_* A \to \pi_* (\mathcal{M} \otimes A),$$

(3)

is a surjection.

Proof Since $\mathcal{M}$ is generated by global sections, taking enough global sections $\phi_1, \ldots, \phi_s$ we can guarantee that the morphism given by the sections

$$\mathcal{O}_{\tilde{X}}^s(\phi_1, \ldots, \phi_s) \to \mathcal{M} \to 0,$$

(4)

and the induced map

$$\mathcal{O}_{\tilde{X}}^s \to \pi_* \mathcal{M} \to 0,$$

(5)

are surjective. Applying $- \otimes A$ to (4) we get

$$0 \to \ker \to \mathcal{O}_{\tilde{X}}^s \otimes A \to \mathcal{M} \otimes A \to 0.$$  

(6)

Applying the functor $\pi_*$ to the exact sequence (6) we get

$$0 \to \pi_* \ker \to \pi_* \left( \mathcal{O}_{\tilde{X}}^s \otimes A \right) \to \pi_* (\mathcal{M} \otimes A) \to R^1 \pi_* \ker \to \ldots$$

(7)

Since the support of $\ker$ is contained in the support of $A$ and the support of $\mathcal{M}$ has dimension one and it only intersects $E$ in a finite number of points, then $R^1 \pi_* \ker = 0$. Now, applying $- \otimes \pi_* A$ to (5) we get

$$\mathcal{O}_{\tilde{X}}^s \otimes \pi_* A \to \pi_* \mathcal{M} \otimes \pi_* A \to 0.$$  

(8)
By (7) and (8) we get the following commutative diagram
\[
\begin{align*}
\pi_* \left( \mathcal{O}_{\tilde{X}}^r \otimes \mathcal{A} \right) & \longrightarrow \pi_* (\mathcal{M} \otimes \mathcal{A}) \longrightarrow 0 \\
\pi_* \mathcal{M} & \longrightarrow \pi_* \mathcal{M} \otimes \pi_* \mathcal{A} \longrightarrow 0
\end{align*}
\]
where \( \alpha_1 \) is the natural map. Since,
\[
\pi_* \left( \mathcal{O}_{\tilde{X}}^r \otimes \mathcal{A} \right) \cong \pi_* \left( \mathcal{A}^r \right) \cong (\pi_* \mathcal{A})^r \cong \mathcal{O}_{\tilde{X}}^r \otimes \pi_* \mathcal{A},
\]
the morphism \( \alpha_1 \) is a surjection. By (9), we get that \( \alpha \) is also a surjection. \( \Box \)

From now on, let \((X, x)\) be the germ of a normal Gorenstein surface singularity and \(M\) be a special module of rank \(r\). Let \(\pi : \tilde{X} \rightarrow X\) be the minimal adapted resolution associated to \(M\) with \(E = \bigcup E_i\) the exceptional divisor with irreducible components \(E_i\). Let
\[
\omega := (\pi^* M)^{\vee} \quad \text{and} \quad \mathcal{L} := \text{det}(\mathcal{M}).
\]
By [11, Proposition 7.4], the full sheaf \(\omega\) is an extension of the determinant bundle \(\mathcal{L}\) by \(\mathcal{O}_{\tilde{X}}^{r-1}\).

Take \(r\) generic global sections \(\phi_1, \ldots, \phi_r\) of \(\omega\) and consider the following exact sequence given by the sections:
\[
0 \rightarrow \mathcal{O}_{\tilde{X}}^r(\phi_1, \ldots, \phi_r) \rightarrow \omega \rightarrow \mathcal{A}' \rightarrow 0.
\]
By [11, Lemma 5.4], the degeneracy module \(\mathcal{A}'\) is isomorphic to \(\mathcal{O}_D\), where \(D \subset \tilde{X}\) is a smooth curve meeting the exceptional divisor transversely at its smooth locus.

The following lemma tells us that the norm of \(M\) has a representative given by the global sections of \(\mathcal{L} := \text{det}(\mathcal{M})\).

**Lemma 4.2** Let \((X, x)\) be the germ of a normal Gorenstein surface singularity and \(M\) be a special \(\mathcal{O}_X\)-module. Let \(\pi : \tilde{X} \rightarrow X\) be the minimal adapted resolution associated to \(M\). Denote by \(\omega := (\pi^* M)^{\vee}\) and by \(\mathcal{L} := \text{det}(\mathcal{M})\). Then, \(\pi_* \mathcal{L}\) is a representative of \(\|M\|_{\mathcal{O}_X}\).

**Proof** Denote by \(r\) the rank of \(M\). By [11, Proposition 7.4], the full sheaf \(\omega\) is an extension of its determinant line bundle \(\mathcal{L}\) by \(\mathcal{O}_{\tilde{X}}^{r-1}\). Thus, we have the exact sequence
\[
0 \rightarrow \mathcal{O}_{\tilde{X}}^{r-1} \rightarrow \omega \rightarrow \mathcal{L} \rightarrow 0.
\]
By [11, Corollary 6.2] and [11, Lemma 7.5], we get \(\dim_\mathbb{C} (R^1 \pi_* \mathcal{M}) = rp_g\) and \(\dim_\mathbb{C} (R^1 \pi_* \mathcal{L}) = p_g\). Therefore, applying the functor \(\pi_* -\) to the exact sequence (10) we get,
\[
0 \rightarrow \mathcal{O}_{\tilde{X}}^{r-1} \rightarrow \omega \rightarrow \pi_* \mathcal{L} \rightarrow 0.
\]
Now the proof follows as the proof of [17, Lemma 4.1]. Indeed, once we have the exact sequences (10) and (11), the rest of the proof of [17, Lemma 4.1] is very general and it does not use the hypothesis of rationality. This finishes the proof. \( \Box \)

We want to prove that the blow-up of a special module is a partial resolution. By Lemma 4.2, Remark 2.12, Lemma 2.14 and Lemma 2.15, it is enough to prove that \((\pi_* \mathcal{L})^n\)
is complete for any positive integer \( n \) (where \((-)^n\) denotes the product of fractional ideals). This is going to be the strategy of the proof of Theorem 4.4.

The following remark tells us that the tensor product of locally free sheaves coincides with the product as a fractional ideal sheaves. We need this remark in order to prove that \((\pi_\ast \mathcal{L})^n\) is complete.

**Remark 4.3** Let \((X, x)\) be the germ of a normal singularity. Let \(\pi : \tilde{X} \to X\) be a resolution and let \(\mathcal{L}\) be a line bundle over \(\tilde{X}\). Notice that the natural morphism

\[
\mathcal{L}^\otimes n \to \mathcal{L}^n,
\]

is an isomorphism. Indeed, the morphism (12) is always a surjection. Now, the sheaf \(\mathcal{L}\) is locally free, hence it is flat. Therefore, the natural morphism is injective.

The following theorem tells us that the blow-up of a special module is a partial resolution.

**Theorem 4.4** Let \((X, x)\) be the germ of a normal Gorenstein surface singularity and \(M\) be a special module. Let \(f : \text{Bl}_M(X) \to X\) be the blow-up of \(X\) at the module \(M\). Then, \(\text{Bl}_M(X)\) is normal.

**Proof** Let

\[
\pi : \tilde{X} \to X,
\]

be the minimal adapted resolution associated to \(M\). Denote by \(\mathcal{M} = (\pi^\ast M) \vee\) and by \(\mathcal{L}\) its determinant. By [11, Remark 5.3], the resolution is small with respect to the Gorenstein form. Hence, the canonical cycle \(Z_K\) is non-negative. Take \(r\) generic global sections \(\phi_1, \ldots, \phi_r\) of \(M\). By [11, Lemma 5.4], we get the following exact sequence given by the sections:

\[
0 \to \mathcal{O}_{\tilde{X}}^r(\phi_1, \ldots, \phi_r) \to \mathcal{M} \to \mathcal{O}_D \to 0,
\]

where \(D \subset \tilde{X}\) is a smooth curve meeting the exceptional divisor transversely at its smooth locus. Moreover, \(D\) does not meet the support of \(Z_K\) (see [11, Proposition 5.14]). Therefore, for any positive integer \(n\) we have

\[
\text{Tor}_1^{\mathcal{O}_{\tilde{X}}}(\mathcal{O}_D, \mathcal{L}^\otimes n \otimes \mathcal{O}_{Z_K}) = 0 \quad \text{and} \quad \mathcal{O}_D \otimes (\mathcal{L}^\otimes n \otimes \mathcal{O}_{Z_K}) = 0.
\]

By these equalities, applying \(- \otimes (\mathcal{L}^\otimes n \otimes \mathcal{O}_{Z_K})\) to the exact sequence

\[
0 \to \mathcal{O}_{\tilde{X}} \to \mathcal{L} \to \mathcal{O}_D \to 0,
\]

we get

\[
\mathcal{L}^\otimes n \otimes \mathcal{O}_{Z_K} \cong \mathcal{L}^\otimes (n+1) \otimes \mathcal{O}_{Z_K}.
\]

Tensoring (13) with \(\mathcal{L}^\otimes n\) we get

\[
0 \to \mathcal{L}^\otimes n \to \mathcal{L}^\otimes (n+1) \to \mathcal{O}_D \otimes \mathcal{L}^\otimes n \to 0.
\]

Now, applying the functor \(\pi_\ast\) to the exact sequence (15) and using the isomorphism (14) we obtain the following commutative diagram:
\[
\cdots \rightarrow \pi_*(O_D \otimes \mathcal{L}^\otimes n) \rightarrow R^1\pi_*(\mathcal{L}^\otimes n) \rightarrow R^1\pi_*(O_D \otimes \mathcal{L}^\otimes(n+1)) \rightarrow 0
\]

\[
\begin{array}{cccccc}
0 & \rightarrow & R^1\pi_*(O_{ZK} \otimes \mathcal{L}^\otimes n) & \rightarrow & R^1\pi_*\left(\mathcal{L}^\otimes(n+1) \otimes \mathcal{O}_{ZK}\right) & \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & \\
0 & & 0 & & 0 &
\end{array}
\]

(16)

where the two columns are induced by the exact sequence

\[
0 \rightarrow \omega_{\tilde{X}} \rightarrow \mathcal{O}_{\tilde{X}} \rightarrow \mathcal{O}_{ZK} \rightarrow 0.
\]

(17)

Note that the existence of the exact sequence (17) follows from the fact that the resolution is small with respect to the Gorenstein form. We claim that for any positive integer we have

\[
\dim \mathbb{C}\left(R^1\pi_*(L^\otimes n)\right) = \dim \mathbb{C}\left(R^1\pi_*\left(O_{ZK} \otimes L^\otimes n\right)\right),
\]

\[
\dim \mathbb{C}\left(R^1\pi_*(L^\otimes n)\right) = \dim \mathbb{C}\left(R^1\pi_*\left(O_{ZK} \otimes L^\otimes n\right)\right) = p_g.
\]

(18)

Indeed, the proof of these equalities follows by induction on \(n\) as follows: The case \(n = 1\) of both equalities follows by [11, Lemma 7.5]. Suppose that the claim is true for some \(n = k\). We then prove the claim for \(n = k + 1\). By the diagram (16) taking \(n = k\) and the induction hypothesis we get that the first column is an isomorphism. Therefore,

\[
\dim \mathbb{C}\left(R^1\pi_*\left(L^\otimes(k+1)\right)\right) = \dim \mathbb{C}\left(R^1\pi_*\left(O_{ZK} \otimes L^\otimes(k+1)\right)\right).
\]

(19)

This proves the first equality. Now, by the first equality of (18) (which we have already proved), the diagram (16) and the induction hypothesis we get

\[
p_g = \dim \mathbb{C}\left(R^1\pi_*\left(L^\otimes k\right)\right) = \dim \mathbb{C}\left(R^1\pi_*\left(L^\otimes(k+1)\right)\right).
\]

This proves the second equality. Thus, the claim is true. Now, applying the functor \(\pi_*\) to the exact sequence (13) we get the exact sequence:

\[
0 \rightarrow \mathcal{O}_X \rightarrow \pi_*\mathcal{L} \rightarrow \pi_*O_D \rightarrow R^1\pi_*\mathcal{O}_{\tilde{X}} \rightarrow R^1\pi_*\mathcal{L} \rightarrow 0.
\]

(20)

By (18) and by the exact sequence (20) we get

\[
0 \rightarrow \mathcal{O}_X \rightarrow \pi_*\mathcal{L} \rightarrow \pi_*O_D \rightarrow 0.
\]

(21)

The exact sequences (13) and (21) tell us that \(\mathcal{L}\) is generated by global sections. Thus, the sheaf \(\mathcal{L}^\otimes n\) is also generated by global sections. Now, we prove by induction that the natural morphism

\[
(\pi_*\mathcal{L})^\otimes n \rightarrow \pi_*\left(\mathcal{L}^\otimes n\right),
\]

(22)

is a surjection for any positive integer \(n\). The case \(n = 1\) is clearly true. Assume that the assertion is true for some \(n = k\). We then prove the assertion for \(n = k + 1\). Consider the following commutative diagram obtained by tensoring (13) and (21) with \(\mathcal{L}^\otimes k\) and \((\pi_*\mathcal{L})^\otimes k\), respectively:

\[
\begin{array}{cccccc}
0 & \rightarrow & \pi_*\left(\mathcal{L}^\otimes k\right) & \rightarrow & \pi_*\left(\mathcal{L}^\otimes(k+1)\right) & \rightarrow & 0 \\
\alpha_1 & \uparrow & \circlearrowleft & \alpha_2 & \uparrow & \circlearrowleft & \alpha_3 & \uparrow \\
(\pi_*\mathcal{L})^\otimes k & \rightarrow & (\pi_*\mathcal{L})^\otimes(k+1) & \rightarrow & \pi_*O_D \otimes (\pi_*\mathcal{L})^\otimes k & \rightarrow & 0
\end{array}
\]
where the morphisms in the columns are the natural ones. The morphism $\sigma$ is a surjection by (18). Indeed, by (18) we get
\[
\dim_C \left( \text{rk}_* \left( \mathcal{L}^\otimes k \right) \right) = \dim_C \left( \text{rk}_* \left( \mathcal{L}^\otimes (k+1) \right) \right).
\]
Since $O_D$ and $O_D \otimes \mathcal{L}^\otimes k$ have the same support, we get $\text{rk}_* \left( O_D \otimes \mathcal{L}^\otimes k \right) = 0$. Therefore, $\sigma$ is a surjection. The morphism $\alpha_1$ is a surjection by the induction hypothesis. The morphism $\alpha_3$ is also a surjection by Lemma 4.1. This implies that $\alpha_2$ is a surjection. This proves the induction step. This proves the surjectivity of (22).

Now, we prove that $(\pi_* \mathcal{L})^n$ is complete for any positive integer $n$. By Remark 4.3, we have
\[
\pi_* \left( \mathcal{L}^\otimes n \right) = \pi_* \left( \mathcal{L}^n \right). \tag{23}
\]
Consider the composition
\[
(\pi_* \mathcal{L})^\otimes n \rightarrow \pi_* \left( \mathcal{L}^\otimes n \right) \xrightarrow{\sigma} \pi_* \left( \mathcal{L}^n \right). \tag{24}
\]
Since the natural map given in (22) is a surjection and by the equality (23), we get that the composition (24) is also a surjection. Therefore, $(\pi_* \mathcal{L})^n = \pi_* (\mathcal{L}^n)$. By (23), this implies $\pi_* \left( \mathcal{L}^\otimes n \right) = (\pi_* \mathcal{L})^n$. Since $\mathcal{L}^\otimes n$ is complete, then by Lemma 2.15 the ideal $(\pi_* \mathcal{L})^n$ is complete. Now, the theorem follows by Remark 2.12, Lemma 2.14 and Lemma 2.15. This proves the theorem. \hfill \square

The following theorem tells us how to recover the normalization of the blow-up of a reflexive module using the minimal adapted resolution.

**Theorem 4.5** Let $(X, x)$ be the germ of a normal surface singularity. Let $M$ be a reflexive $O_X$-module of rank $r$. Let $\pi: \tilde{X} \rightarrow X$ be the minimal adapted resolution associated to $M$ with exceptional divisor $E$. Let $E_1, \ldots, E_n$ be the irreducible components of $E$ and $\mathcal{M} := (\pi^* M)^\vee$ be the full sheaf associated to $M$. Then, the normalization of $\text{Bl}_M (X)$ is obtained by contracting the irreducible components $E_j$ such that $c_1 (\mathcal{M}) \cdot E_j = 0$.

**Proof** Let $E_1, \ldots, E_m$ be the irreducible components of $E$ such that $c_1 (\mathcal{M}) \cdot E_j \neq 0$. Let $h: \tilde{X} \rightarrow Y$, be the contraction of all the irreducible components of $E$ different from $E_1, \ldots, E_m$. Denote by $E' := \bigcup_{j=1}^m E_j$ and $S := h (E \setminus E')$. Notice that $S$ is a finite set of cardinality equal to the number of connected components of $E \setminus E'$. By Grauert’s contraction theorem [14] the variety $Y$ is normal. Let
\[
v: Y \rightarrow X,
\]
be the natural morphism such that $\pi = v \circ h$.

We prove that $h_* \mathcal{M}$ is a locally free $O_Y$-module generated by its global sections. Let $\phi_1, \ldots, \phi_r$ be generic global sections of $\mathcal{M}$. By [11, Lemma 5.4], the exact sequence given by the sections is
\[
0 \rightarrow \mathcal{O}_X^r (\phi_1, \ldots, \phi_r) \rightarrow \mathcal{M} \rightarrow O_D \rightarrow 0. \tag{25}
\]
Applying the functor $h_*$ to the exact sequence (25) we get
\[
0 \rightarrow \mathcal{O}_Y^r \rightarrow h_* \mathcal{M} \xrightarrow{\psi} h_* O_D. \tag{26}
\]
Notice that the morphism \( h \) is an isomorphism in the complement of \( E \setminus E' \). Moreover, since the Poincaré dual of the support of \( O_D \) is the first Chern class of \( M \), we get that \( D \) only intersects \( E' \). Hence, the support of \( h_*O_D \) and \( S \) are disjoint sets. Thus, the morphism \( \psi \) is a surjection. We claim that \( h_*M \) is a locally free sheaf generated by global sections. Indeed, first we prove that \( h_*M \) is locally free. Let \( p \) be any point in \( Y \). Since \( h \) is an isomorphism in the complement of \( E \setminus E' \), we only need to prove that for any point \( p \) in \( S \) the module \( h_*M_p \) is free. Let \( p \in S \). Since \( S \) and \( D \) are disjoint sets, then \( p \notin D \). Therefore, by (26) we get \( O_{Y,p}' \cong h_*M_p \). Thus, \( h_*M \) is a locally free sheaf. Now we prove that \( h_*M \) is generated by global sections. Note that \( v^*h_*M = \pi_*M = M \), hence \( h_*M \) and \( M \) have the same global sections. Recall that \( \phi_1, \ldots, \phi_r \) are generic global sections of \( M \). Since \( M \) is generated by global sections, we can complete such set of sections to a set of generators \( \phi_1, \ldots, \phi_r, \phi_{r+1}, \ldots, \phi_{r'} \). Hence, we have the following diagram,

\[
0 \longrightarrow O_{Y}' \longrightarrow h_*M \xrightarrow{\sigma} h_*O_D \longrightarrow 0
\]

where the map \( \phi : O_{Y}' \rightarrow h_*M \) is given by the sections \( \phi_1, \ldots, \phi_r, \phi_{r+1}, \ldots, \phi_{r'} \). Since \( h \) is an isomorphism in the complement of \( E \setminus E' \), we only need to prove that for any point \( p \) in \( S \) the morphism \( \phi_p : O_{Y,p}' \rightarrow h_*M_p \) is a surjection. Let \( p \in S \). Since \( S \) and \( D \) are disjoint sets, then \( p \notin D \). Thus, by the diagram (27) we get

\[
0 \longrightarrow O_{Y,p}' \longrightarrow h_*M_p \longrightarrow 0
\]

By (28), we get that the morphism \( \phi_p \) is a surjection. Therefore, the morphism \( \phi \) is a surjection, i.e., the sheaf \( h_*M \) is generated by global sections. Now, denote by \( \tilde{M} \) := \( (v^*M)' \) and consider the natural morphism

\[
\theta : h_*M \rightarrow \tilde{M}.
\]

Note that \( h_*M \) and \( \tilde{M} \) are both reflexive and \( \theta \) is an isomorphism outside the closed subset \( S \) of codimension two. Thus, by [19, Proposition 1.6] the map \( \theta \) is an isomorphism. Consequently, the sheaf \( \tilde{M} \) is locally free and it is generated by its global sections. Since \( \tilde{M} \) is generated by global sections, we get

\[
\tilde{M} \cong v^*M/\text{tor}.
\]

In particular, the sheaf \( v^*M/\text{tor} \) is locally free. Let

\[
n : \text{NB1}_M(X) \rightarrow \text{Bl}_M(X),
\]

be the normalization of \( \text{Bl}_M(X) \). Denote by \( \text{NB1}_M(X)_{\text{min}} \) the minimal resolution of \( \text{NB1}_M(X) \). By the universal property of the blow-up of \( X \) at \( M \) and the universal property of the normalization, there exist morphisms \( g : Y \rightarrow \text{Bl}_M(X) \) and \( \gamma : Y \rightarrow \text{NB1}_M(X) \).
such that the diagram commutes

\[
\begin{array}{ccc}
\text{NBl}_M(X)_{\text{min}} & \to & \text{NBl}_M(X) \\
\downarrow & & \downarrow n \\
\tilde{X} & \to & \text{Bl}_M(X) \\
\uparrow g & & \uparrow \gamma \\
\end{array}
\]

(29)

Now we prove that \( \gamma \) is an isomorphism. Suppose that there exists a point \( x_0 \) in the singular locus of \( \text{NBl}_M(X) \) such that \( E_j \subset h^{-1}(\gamma^{-1}(x_0)) \) where \( c_1(M) \cdot E_j \neq 0 \). Consider the following commutative triangle

\[
\begin{array}{ccc}
\tilde{X} & \to & \tilde{X}_j \\
\downarrow & & \downarrow f \\
\text{NBl}_M(X) & \to & \\
\gamma \circ h & & \\
\end{array}
\]

where \( v : \tilde{X} \to \tilde{X}_j \) is the contraction of \( E_j \). Since \( \tilde{X} \) is the minimal adapted resolution associated to \( M \), by [11, Sect. 5] (in particular the proof of [11, Proposition 5.1]) we get that \( \tilde{X}_j \) is a resolution of \( X \). Denote by \( \mathcal{M}_{\text{NBl}_M} \) the pullback (under the natural map) of \( M \) to \( \text{NBl}_M(X) \). Consider the following commutative diagram

\[
\begin{array}{ccc}
0 & \to & \ker \to \text{f}^* \mathcal{M}_{\text{NBl}_M} \to \text{f}^*(\mathcal{M}_{\text{NBl}_M}/\text{tor}) \to 0 \\
& & \uparrow \circ \uparrow \alpha \\
0 & \to & \text{tor} \to \text{f}^* \mathcal{M}_{\text{NBl}_M} \to (\text{f}^* \mathcal{M}_{\text{NBl}_M})/\text{tor} \to 0
\end{array}
\]

Since the kernel of \( \alpha \) is torsion and \((\mathcal{M}_{\text{NBl}_M})/\text{tor}\) is torsion-free, we get that \( \alpha \) is an isomorphism. Since \( \mathcal{M}_{\text{NBl}_M}/\text{tor} \) is locally free, so is \( \text{f}^* (\mathcal{M}_{\text{NBl}_M}/\text{tor}) \). Since \( \alpha \) is an isomorphism, the sheaf \((\text{f}^* \mathcal{M}_{\text{NBl}_M})/\text{tor}\) is locally free. Note that \( \text{f}^* \mathcal{M}_{\text{NBl}_M} \) is the pullback (under the natural map) of \( M \) to \( \tilde{X}_j \). Since \((\text{f}^* \mathcal{M}_{\text{NBl}_M})/\text{tor}\) is locally free, we get

\[
(\text{f}^* \mathcal{M}_{\text{NBl}_M})/\text{tor} \cong (\text{f}^* \mathcal{M}_{\text{NBl}_M})^{\vee}.
\]

Therefore, the full sheaf \((\text{f}^* \mathcal{M}_{\text{NBl}_M})^{\vee}\) associated to \( M \) is generated by global sections. This is impossible, since \( \tilde{X} \) is the minimal resolution where the full sheaf associated to \( M \) is generated by global sections. Therefore, for any point \( x_0 \) in the singular locus of \( \text{NBl}_M(X) \) the fiber \( \gamma^{-1}(x_0) \) is finite. By Proposition 3.3 we know that \( \tilde{X} \cong \text{NBl}_M(X)_{\text{min}} \). By (29) and the aforementioned isomorphism, we get that \( \gamma \) is a surjection. Now, suppose that there exists a point \( x_0 \) in the singular locus of \( \text{NBl}_M(X) \) such that the fiber \( \gamma^{-1}(x_0) \) has at least two points. This implies that \( h^{-1}(\gamma^{-1}(x_0)) \) is disconnected but this is impossible since the fibers should be connected. Therefore, \( \gamma \) is also injective. Since \( \gamma \) is a bijection and, \( Y \) and \( \text{NBl}_M(X) \) are normal, then \( \gamma \) is an isomorphism. This proves the theorem. \( \square \)

**Corollary 4.6** Let \((X, x)\) be the germ of a normal Gorenstein surface singularity. Let \( M \) be a special \( \mathcal{O}_X \)-module. Then, the blow-up of \( X \) at \( M \) has at worst normal Gorenstein singularities.

**Proof** Let \( \pi : \tilde{X} \to X \),
be the minimal adapted resolution associated to $M$ with exceptional divisor $E$ and $\mathcal{M} := (\pi^*M)^{\vee\vee}$ be the full sheaf associated to $M$. Let $E_1, \ldots, E_m$ be the irreducible components of $E$ such that $c_1(\mathcal{M}) \cdot E_j \neq 0$. If the singularity is rational, then the corollary follows by [6, Theorem 2.2]. Now assume that the singularity is not rational. Let $E_1, \ldots, E_m$ be the irreducible components of $E$ such that $c_1(M) \cdot E_j \neq 0$. If the singularity is rational, then the corollary follows by [6, Theorem 2.2]. Now assume that the singularity is not rational. Let $h : \tilde{X} \rightarrow Y$, be the contraction of all the irreducible components of $E$ different from $E_1, \ldots, E_m$. Denote by $E := \bigcup_{j=1}^m E_j$ and $S := h(E \setminus E')$. Recall that $S$ is a finite set.

By Theorem 4.4 and Theorem 4.5 we get $Y \cong \text{Bl}_M(X)$. By Grauert’s contraction theorem [14] the restriction

$$h|_{\tilde{X}\setminus(E\setminus E')} : \tilde{X} \setminus (E \setminus E') \rightarrow Y \setminus S,$$

is an isomorphism. Since $X$ is Gorenstein, the Gorenstein 2-form $\omega_{\tilde{X}}$ has neither zeros nor poles in $\tilde{X} \setminus E$. By [11, Corollary 6.2 and Theorem 6.1], we get $[c_1(\mathcal{M})] \cdot [Z_K] = 0$. Therefore, the Gorenstein 2-form has not either zeros or poles along $E'$ (see also [11, Corollary 7.12]). Hence, the Gorenstein 2-form $\omega_{\tilde{X}}$ does not have any zero or pole along $\tilde{X} \setminus (E \setminus E')$. Therefore, there exists a two form that does not vanishes over $Y \setminus S$, i.e., $Y$ has only Gorenstein singularities. This proves the corollary.

**Remark 4.7** Let $(X, x)$ be the germ of a normal Gorenstein surface singularity. Let $M$ be a special indecomposable $O_X$-module. By Corollary 4.6, the blow-up of $X$ at $M$ has at worst normal Gorenstein singularities. The only case when the blow-up of $X$ at $M$ is smooth is in the case of the $A_1$-singularity. Indeed, by Theorem 4.5 we get that $\text{Bl}_M(X)$ has only one irreducible component (here we use that $M$ is indecomposable). Therefore, $\text{Bl}_M(X)$ should be the minimal resolution of $X$ (any blow-up of the minimal resolution produces a new irreducible component but $\text{Bl}_M(X)$ only has one component). By Proposition 3.3, the resolution $\text{Bl}_M(X)$ is the minimal adapted resolution associated to $M$. By [11, Corollary 6.2 and Theorem 6.1], we get $Z_K = 0$. Since $X$ is Gorenstein and for the minimal resolution we have $Z_K = 0$, then $X$ should be a rational double point. Since $X$ and the $A_1$-singularity have the same minimal resolution, then $X$ is the $A_1$-singularity. In all other cases the blow-up of $X$ at $M$ always has normal Gorenstein singularities.

**Corollary 4.8** Let $(X, x)$ be the germ of a normal Gorenstein surface singularity. Then, two special modules $M$ and $M_0$ give isomorphic partial resolutions if and only if the same non-trivial indecomposable modules appear as summands of $M$ and as ones of $M_0$.

**Proof** The proof follows by Theorem 4.5, Proposition 3.3 and [11, Corollary 7.12].

Now, by Theorem 4.4 and Theorem 4.5 we generalize the McKay correspondence as follows:

**Corollary 4.9** Let $(X, x)$ be a normal Gorenstein surface singularity. Then, there exists a bijection between the following sets:

1. The set of special, indecomposable non-trivial $O_X$-modules up to isomorphism.
2. The set of irreducible divisors $E$ over $x$, such at any resolution of $X$ where $E$ appears, the Gorenstein form has neither zeros nor poles along $E$.
3. The set of partial resolutions $\psi : Y \rightarrow X$ with irreducible exceptional divisor $E$ such that:

```
(a) the partial resolution is dominated by a resolution which is small with respect to the Gorenstein form.
(b) the Gorenstein form does not have any zeros or poles along $E \setminus \text{Sing } Y$.

**Proof** Let $M$ be a special indecomposable non-trivial $\mathcal{O}_X$-module. By Corollary 4.6, the blow-up of $X$ at $M$ denoted by $f : \text{Bl}_M(X) \to X$, is a Gorenstein partial resolution with irreducible exceptional divisor. We associate to the module $M$ the partial resolution $f : \text{Bl}_M(X) \to X$. Hence, we have constructed a map from the set (1) to the set (3).

Let $\psi : Y \to X$ be a partial resolution satisfying the properties of (a) and (b). Therefore, there exists a resolution small with respect to the Gorenstein form such that the Gorenstein form does not have zeros or poles along $E'$, where $E'$ is the strict transform of $E$. By [11, Corollary 7.12] there exits $M$ a special indecomposable non-trivial $\mathcal{O}_X$-module. We associate to the partial resolution $\psi : Y \to X$ the module $M$. Hence, we have constructed a map from the set (3) to the set (1). The map from (1) to (3) and the map from (3) to (1) are inverses of each other. This proves the bijection between (1) and (3).

The bijection between (1) and (2) follows from [11, Corollary 7.12]. This proves the corollary. □

5 Applications via matrix factorizations

In this section we use the theory of matrix factorizations in order to compute examples of blow-ups at reflexive modules. We recall some preliminaries on matrix factorizations.

5.1 Matrix factorizations

From now on, let $(S, m)$ be a regular local ring and suppose that $R = S/I$ is Henselian, where $I$ is a principal ideal of $S$ generated by $f$.

Recall, the notion of matrix factorization given by Eisenbud [8] gives an equivalence of categories between maximal Cohen–Macaulay $R$-modules and pairs of matrices with entries in $S$ satisfying some conditions. We review this construction. See [8] or [39, Chapter 7] for more details.

**Definition 5.1** Let $R$ be the coordinate ring of the a hypersurface defined by $f$ in $(S, m)$. A matrix factorization of $f$ is an ordered pair of $n \times n$-matrices $(\Phi, \Psi)$ with entries in $S$ such that

$$\Phi \cdot \Psi = f \cdot \text{Id}_{S^n}, \quad \Psi \cdot \Phi = f \cdot \text{Id}_{S^n},$$

where $\text{Id}_{S^n}$ is the $n \times n$ identity matrix.

A morphism between matrix factorizations $(\Phi_1, \Psi_1)$ and $(\Phi_2, \Psi_2)$ is a pair of $n \times n$ matrices $(\alpha, \beta)$ with entries in $S$ such that

$$\alpha \cdot \Phi_1 = \Phi_2 \cdot \beta, \quad \beta \cdot \Psi_1 = \Psi_2 \cdot \alpha.$$

A matrix factorization is reduced if and only if

$$\text{Im } \Phi \subset mS^n \quad \text{and} \quad \text{Im } \Psi \subset mS^n.$$
Using matrix factorizations Eisenbud [8] proved the following:

**Theorem 5.2** ([8]) *There is a one-to-one correspondence between:

1. equivalence classes of reduced matrix factorizations of $f$.
2. isomorphism classes of non-trivial periodic minimal free resolutions of $R$-modules of periodicity two.
3. maximal Cohen–Macaulay $R$-modules without free summands.*

We sketch the idea of a part of the proof of Theorem 5.2 which will be used later in this section. Let $M$ be a maximal Cohen–Macaulay $R$-module without free summands. By the Auslander–Buchsbaum–Serre theorem we have

$$\text{proj dim}_S M = \dim S - \text{depth } M = 1.$$ 

Thus, there is a free resolution of $M$ as $S$-module of length 1:

$$0 \to S^n \xrightarrow{\Phi} S^n \to M \to 0.$$ 

Since $f$ annihilates the module $M$, we get $f \cdot S^n \subseteq \text{Im} \Phi$. Hence, there exists a matrix $\Psi$ with entries in $S$ such that

$$\Phi \cdot \Psi = f \cdot \text{Id}_S^n,$$

where the pair $(\Phi, \Psi)$ is an $n \times n$-matrix factorization of $f$ with $\text{coker} \Phi = M$.

Conversely, let $(\Phi, \Psi)$ be a matrix factorization of $f$. Denoting $\hat{\Phi}$ and $\hat{\Psi}$ the matrices $\Phi, \Psi$ modulo $(f)$ respectively, we have the following exact sequence of $R$-modules:

$$\cdots \to R^n \xrightarrow{\hat{\Psi}} R^n \xrightarrow{\hat{\Phi}} R^n \xrightarrow{\hat{\Psi}} \cdots$$

If $f$ is not a zero-divisor in $S$, then the complex

$$\cdots \to R^n \xrightarrow{\hat{\Psi}} R^n \xrightarrow{\hat{\Phi}} R^n \to \text{coker} \hat{\Phi} \to 0,$$

is exact. Hence it is a periodic free resolution of $\text{coker} \hat{\Phi}$ with periodicity two. Furthermore, the module $\text{coker} \hat{\Phi}$ is a maximal Cohen–Macaulay $R$-module.

**5.2 The blow-up at a matrix factorization.**

In this subsection we use the blow-up given by Villamayor [35] and the matrix factorizations in the case of $S = \mathbb{C}[x, y, z]$ and $R$ the coordinate ring of a normal hypersurface, i.e., $R = S/(f)$ with $f \in S$ and $R$ is a normal ring. Recall that any hypersurface singularity is a Gorenstein singularity.

Let $M$ be a reflexive $R$-module of rank $r$. By [35, Proposition 2.5] we can use the matrix factorization associated to $M$ to obtain a representative of $\|M\|_R$. Some computations can be done by hand, but in some cases we use the software SINGULAR 4-1-2 [7] and the libraries resolve.lib [12] and sing.lib [15].

**Example 5.3** Let $f = xy + zn + 1$, i.e., $R$ is the $A_n$-singularity. The matrix factorization of rational double points are well known, see for example [25]. In the case of the $A_n$-singularity the matrix factorizations are

$$\Phi_k = \begin{bmatrix} y & -zn+1-k \\ z^k & x \end{bmatrix}, \quad \Psi_k = \begin{bmatrix} x & zn+1-k \\ -z^k & y \end{bmatrix}, \quad (30)$$
where \( k \) is an integer such that \( 0 \leq k \leq n \). Using the matrix \( \Phi_k \) we get the following morphism:

\[
R^2 \xrightarrow{\Phi_k} R^2 \rightarrow M(\Phi_k) \rightarrow 0,
\]

where \( M(\Phi_k) := \text{coker} \Phi_k \). Let \( K \) be the kernel of the morphism \( \Phi_k \) and denote by

\[
K_1 := \left\{ \begin{bmatrix} y \\ z^k \end{bmatrix} \cdot g \in R^2 \mid g \in R \right\}.
\]

By [35, Proposition 2.5], the ideal \( I_k = (y, z^k) \) is a representative of \( \|M(\Phi_k)\|_R \). Therefore, the blow-up at \( M(\Phi_k) \) of \( R \) is the blow-up at the ideal \( I_k \). In this case \( \text{Bl}_{M(\Phi_k)}(R) \) has at most two singular points:

1. If \( \text{Bl}_{M(\Phi_k)}(R) \) has one singular point, then the singularity is \( xy + zn \).
2. If \( \text{Bl}_{M(\Phi_k)}(R) \) has two singular points, then one singularity is \( xy + zn - l \) and the other singularity is \( xy + zl + 1 \) with \( 1 \leq l \leq n - 2 \).

Both cases were exactly as predicted by Corollary 4.9 or [17].

We now consider a different singularity. From now on, let \( f = x^3 + y^3 + z^3 \) and \( R = \mathbb{C}\{x, y, z\}/(f) \). Hence \( R \) is a normal Gorenstein surface singularity. In this case all the reflexive modules were classified by Kahn [24] and all the special modules were classified in [11]. Several people have studied this singularity and the category of reflexive modules, for example [24, 26, 27]. First, we study the blow-up at the fundamental module of \( R \).

**Definition 5.4** ([39, Definition 11.5]) The fundamental exact sequence of \( R \) is the following exact sequence (unique, up to non-canonical isomorphism):

\[
0 \rightarrow R \rightarrow A \rightarrow R \rightarrow \mathbb{C} \rightarrow 0,
\]

corresponding to a non-zero element of \( \text{Ext}^2_R(R/m, R) \cong \mathbb{C} \). The module \( A \) is called the fundamental module of \( R \).

The fundamental module of \( R \) is an indecomposable reflexive module of rank 2 (see [39, Chapter 11] for more properties about the fundamental module). A natural question is the following: Is the fundamental module, a special module? We show:

**Proposition 5.5** Let \( f = x^3 + y^3 + z^3 \) and \( R = \mathbb{C}\{x, y, z\}/(f) \). Then, the fundamental module of \( R \) is not special.

**Proof** The idea is the same as in Example 5.3: we use the matrix factorization of \( A \) to compute the ideal that we need to blow-up. Then, we use Corollary 4.6 in order to check if the module is special. The matrix factorization associated to \( A \) was computed by Yoshino and Kawamoto in [40] and Laza, Pfister and Popescu in [26]. The periodic free resolution of \( A \) is the following:

\[
\cdots \rightarrow R^4 \xrightarrow{\Psi_A} R^4 \xrightarrow{\Phi_A} R^4 \rightarrow A \rightarrow 0,
\]

where

\[
\Phi_A = \begin{bmatrix}
  x^2 & -y & -z & 0 \\
  y^2 & x & 0 & -z \\
  z^2 & 0 & x & y \\
  0 & z^2 & -y^2 & x^2
\end{bmatrix}, \quad \Psi_A = \begin{bmatrix}
  x & y & z & 0 \\
  -y^2 & x^2 & 0 & z \\
  -z^2 & 0 & x^2 & -y \\
  0 & -z^2 & y^2 & x
\end{bmatrix}.
\]
By [35, Proposition 2.5], we can choose as a representative of \( \|A\|_R \) the ideal generated by all the \( 2 \times 2 \)-minors of the matrix given by

\[
D = \begin{bmatrix}
-z & 0 \\
0 & -z \\
x & y \\
y^2 & x^2
\end{bmatrix}.
\] (33)

By Theorem 2.11, the blow-up of \( R \) at \( A \) is the blow-up at the ideal

\[
I = (z^2, yz, xz, x^3 + y^3).
\] (34)

Using SINGULAR 4-1-2 [7] and the libraries resolve.lib [12] and sing.lib [15] one can check that the blow-up at the ideal \( I \) has 4 smooth charts, therefore the blow-up of \( R \) at \( A \) is a resolution. Then, by Corollary 4.6 and Remark 4.7 the module \( A \) is not special. \( \Box \)

We now give one example where the blow-up is not normal.

**Example 5.6** Consider the following matrix factorization of \( f \):

\[
\Phi = \begin{bmatrix}
x + y & -z^2 \\
z & x^2 - xy + y^2
\end{bmatrix}, \quad \Psi = \begin{bmatrix}
x^2 - xy + y^2 & z^2 \\
-z & x + y
\end{bmatrix}
\] (35)

Let \( M = \text{coker} \Phi \). Then, using SINGULAR 4-1-2 [7] and the libraries resolve.lib [12] and sing.lib [15] one can check that the blow-up of \( R \) at \( M \) does not have an isolated singularity, hence it is not normal.

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