Bispecial factors in circular non-pushy D0L languages

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Abstract

We study bispecial factors in fixed points of morphisms. In particular, we propose a simple method of how to find all bispecial words of non-pushy circular D0L-systems. This method can be formulated as an algorithm. Moreover, we prove that non-pushy circular D0L-systems are exactly those with finite critical exponent.

keywords: bispecial factors, circular D0L systems, non-pushy D0L systems, critical exponent

1 Introduction

Bispecial factors proved to be a powerful tool for better understanding of complexity of aperiodic sequences of symbols from a finite set. One of the most studied families of such sequences are fixed points of morphisms. In this paper we present a method of how to describe the structure of all bispecial factors in a given fixed point.

The method we describe here can be partially spotted in results of several authors: It is a sort of inverse of the algorithm by Cassaigne from paper [1] which is concerned by pattern avoidability. Very similar approach was used in [2] by Avgustinovich and Frid and in [3] by Frid to describe bispecial factors of biprefix circular morphisms and marked uniform morphisms, respectively. Actually, the fact that all bispecial factors in a fixed point can be generated as members of some easily constructed sequences was noticed in many papers where factor complexity was computed, see, e.g., [4, 5]. In this paper we formalize this approach and prove that it works for a very wide class of morphisms, namely non-pushy and circular morphisms. Moreover, it seems that the assumptions we need for proofs can be weakened or even omitted and the main theorems remain true.

The paper is organized as follows. In the next section we introduce necessary notation and notions and also explain the importance of bispecial factors. Since it is easier to explain the main result using examples than to formulate it as a theorem, we do so in Section 3. Section 4 contains proofs of the crucial theorems and in Section 5 we explain how to use our results to identify immediately all infinite special branches. In Section 6 we prove that non-pushy and circular morphisms are exactly those whose fixed points have finite critical exponent.

2 Preliminaries

Let \( \mathcal{A} = \{0, 1, \ldots, n - 1\}, n \geq 2 \), be a finite alphabet of \( n \) letters; if needed, we denote this particular \( n \)-letter alphabet as \( \mathcal{A}_n \). An infinite word over the alphabet \( \mathcal{A} \) is a

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sequence \( u = (u_i)_{i \geq 1} \) where \( u_i \in A \) for all \( i \geq 1 \). If \( v = u_ju_{j+1} \cdots u_{j+n-1}, \) \( j, n \geq 1 \), then \( v \) is said to be a factor of \( u \) of length \( n \), the empty word \( \epsilon \) is the factor of length 0. The set of all finite words over \( A \) is the free monoid \( A^* \), the set of nonempty finite words is denoted by \( A^+ = A^* \setminus \{ \epsilon \} \).

A map \( \varphi : A^* \to A^* \) is called a morphism if \( \varphi(wv) = \varphi(w)\varphi(v) \) for every \( w, v \in A^* \). Any morphism \( \varphi \) is uniquely given by the set of images of letters \( \varphi(a), a \in A \). If all these images are nonempty words, the morphism is called non-erasing. A famous example of a morphism is the Thue-Morse morphism \( \varphi_{TM} \) defined by

\[
\begin{align*}
\varphi_{TM}(0) &= 01, \\
\varphi_{TM}(1) &= 10.
\end{align*}
\]

This paper studies infinite fixed points of morphisms: an infinite word \( w \) is a fixed point of a morphism \( \varphi \) if \( \varphi(w) = w \). If \( \varphi^\ell(w) = w \) for some positive \( \ell \), \( w \) is a periodic point of \( \varphi \). The fixed point of \( \varphi_{TM} \) beginning in the letter 0 is the infinite word

\[
u_{TM} = \lim_{n \to \infty} \varphi^n_{TM}(0) = \varphi^n_{TM}(0) = 0110100110 \cdots ,
\]

which is called the Thue-Morse word.

An infinite word \( u \) is aperiodic if it is not eventually periodic, i.e., there are no finite words \( v \) and \( w \) such that \( u = vwvwv \cdots = vw^\omega \). If a word \( u = vw \), then \( v \) is a prefix of \( u \) and \( w \) is its suffix. In this case we put \( v^{-1}w = u \) and \( u^{-1}v = w \). Given a morphism \( \varphi \) on \( A \), if \( \varphi(a) \) is not a suffix of \( \varphi(b) \) for any distinct \( a, b \in A \), then \( \varphi \) is said to be suffix-free. Prefix-free morphisms are defined analogously.

The language of a fixed point \( u \) is the set of all its factors and is denoted by \( L(u) \).

When speaking about a morphism, we usually mean a morphism together with its particular infinite fixed point. But a morphism can have more than one fixed point and not all of them must have the same language (this is true if the morphism is primitive). For instance, consider the morphism \( 0 \mapsto 010, 1 \mapsto 11: \) it has two fixed points, one aperiodic starting in 0 and one periodic starting in 1. Therefore, instead of speaking only about a morphism we will always speak about a morphism and its particular infinite fixed point. A well-established way of how to do so is to treat a morphism and its fixed point as a D0L-system (see, e.g., [6] and [7]).

**Definition 1.** A triplet \( G = (A, \varphi, w) \) is called a D0L-system, where \( A \) is an alphabet, \( \varphi \) a morphism on \( A \), and \( w \in A^+ \) is an axiom. The language of \( G \) denoted by \( L(G) \) is the set of all factors of the words \( \varphi^n(w), n = 0, 1, \ldots \).

If \( \varphi \) is non-erasing, then the system is called PD0L-system.

In what follows, when referring to a D0L-system, we always mean a PD0L-system. In fact, for any D0L-system, it is possible to construct its elementary (not simplifiable) version which is a PD0L-system with an injective morphism [8] [9].

Clearly, if \( \varphi(a) = av \) for some \( a \in A, v \in A^+ \), and if \( \varphi \) is non-erasing, then the language of the D0L-system \( (A, \varphi, a) \) is the language of the infinite fixed point \( \varphi^n(a) \).

There are several tools which help us to study the structure of the language of D0L-systems. We mention here two basic ones: the factor complexity and critical exponent. The factor complexity of a language is the function \( C(n) \) which counts the number of factors of length \( n \). The factor complexity is usually obtained using the notion of special factors.

**Definition 2.** Let \( w \) be a factor of the language \( L(G) \) of a D0L-system \( G \) over \( A \).

The set of left extensions of \( w \) is defined as

\[
\text{Lext}(w) = \{ a \in A : aw \in L(G) \}.
\]
If \( \# \text{Lext}(w) \geq 2 \), then \( w \) is said to be a left special (LS) factor of \( \mathcal{L}(G) \).

In the analogous way we define the set of right extensions \( \text{Rext}(w) \) and a right special (RS) factor. If \( w \) is both left and right special, then it is called bispecial (BS).

The connection between special factors and the factor complexity is described in [5]; the complete knowledge of LS, RS, or BS factors enables to find the factor complexity.

The critical exponent is related to the repetitions in the language. Let \( w \) be a finite and nonempty word. Any finite prefix \( v \) of \( w^r = w\cdots w \) is a power of \( w \). We denote this by \( v = w^r \), in words \( v \) is \( r \)-power of \( w \), where \( r = |v|/|w| \). Further, we define the index of \( w \) in a language \( \mathcal{L}(G) \) of a D0L-system \( G \) as

\[
\text{ind}(w, G) = \sup \{ r \in \mathbb{Q} : w^r \in \mathcal{L}(G) \}.
\]

And finally, the critical exponent of the language \( \mathcal{L}(G) \) is the number

\[
\sup \{ \text{ind}(w) \mid w \in \mathcal{L}(G) \}.
\]

More details about critical exponent can be found, e.g., in [10]. Examples of how knowledge of BS factors can help to compute it for a fixed point of a morphism are in [11] and [12].

3 Explaining the main result

Since the main result of this paper is a tool rather than a theorem, we demonstrate it using example morphisms. The tool has two ingredients. First one is a mapping that maps a BS factor to another one and so, applied repetitively, it generates sequences of BS factors. This mapping is defined by two directed labeled graphs. The other ingredient is a finite set of BS factors such that the sequences generated from them by the mapping cover all BS factors in a given fixed point.

Let us consider the morphism \( \varphi_E \) defined by \( 0 \mapsto 012 \), \( 1 \mapsto 112 \), \( 2 \mapsto 102 \) and the corresponding D0L-system \( (\mathcal{A}_3, \varphi_E, 0) \) with the fixed point \( u_E \). The factor 212 is LS and has left extensions 0 and 1. If we apply the morphism \( \varphi_E \) on this structure – both on the factor and its two extensions –, the resulting factor \( \varphi_E(212) \) is no more LS since the respective extensions \( \varphi_E(0) = 012 \) and \( \varphi_E(1) = 112 \) end in the same letter 2. In order to obtain a LS factor, we have to cut off the longest common suffix of the new extensions, here it is 12, and append it to the beginning of the \( \varphi_E \)-image of the factor: the result is the LS factor \( 12\varphi_E(212) \) with left extensions 0 and 1. We can proceed in the same manner and obtain another LS factor \( 12\varphi_E(12)\varphi_E^2(212) \) again with the same extensions 0 and 1. Clearly, the same process works for RS factors and right extensions.

Let us formalize what we did in the previous paragraph. Instead of BS factors we will use a slightly different notion of BS triplets \( ((a, b), v, (c, d)) \), where \( v \) is a BS factor and \( (a, b) \) and \( (c, d) \) are unordered pairs of its left and right extensions, respectively. We assume that either \( avc \) and \( bvd \) or \( avd \) and \( bvc \) are factors. Thus, \( ((0, 1), 2121, (0, 1)) \) is a BS triplet in \( u_E \). In terms of the previous paragraph, we can get another BS triplet from this one, namely \( ((0, 1), 12\varphi_E(212), (0, 1)) \); we call this BS triplet the \( f \)-image of \( ((0, 1), 2121, (0, 1)) \). The fact that left extensions \( (0, 1) \) result again in extensions \( (0, 1) \) with appending of 12 can be represented as a directed edge from vertex \( (0, 1) \) to vertex \( (0, 1) \) with label 12. The edge corresponding to the right extensions starts
in \((0,1)\), ends again in \((0,1)\) and is labeled by the empty word. Applying this idea on all possible pairs of left and right extensions gives us two directed labeled graphs depicted in Figure 1. We call these graphs *graph of left and right prolongations*. With these graphs in hand, it is easy to generate infinitely many BS triplets from a given starting one.

![Graphs](image)

**Figure 1:** The graphs defining the \(f\)-image for the morphism \(\varphi_E\).

The other ingredient of our method bears on the fact that all BS triplets can be generated by taking repetitively \(f\)-image of only finitely many *initial* BS triplets. Initial BS triplets are those which are not \(f\)-images of another BS triplet. For instance, \(((0,1), 2112, (0,1))\) is not initial as it is the \(f\)-image of the BS triplet \(((0,1), 1, (0,2))\) which is initial. Later we will show how to find all the initial factors for a given fixed point. For the case of \(u_E\), we have eight initial BS triplets:

\[
\begin{align*}
((0,1), 121, (0,1)), & \quad ((0,1), 12, (0,1)), & \quad ((0,1), 21, (0,1)), & \quad ((0,1), 2, (0,1)), \\
((1,2), 1, (1,2)), & \quad ((0,2), 1, (1,2)), & \quad ((0,2), 1, (0,2)), & \quad ((1,2), 0, (1,2)).
\end{align*}
\]

The vertices of the graphs from Figure 1 are just all pairs of distinct letters, but the situation is not that simple for all morphisms. In fact, it happens for graph of left (right) prolongations only if the respective morphism is suffix-free (prefix-free). This case when the morphism is both prefix- and suffix-free has been already solved in [2], where not only describe the authors all BS factors, but they also give a formula for the factor complexity.

Let us consider the morphism \(\varphi_S\) defined by \(0 \mapsto 0012, 1 \mapsto 2, 2 \mapsto 012\) and the corresponding D0L-system \((A_3, \varphi_S, 0)\). Clearly, the morphism is not suffix-free. Let \(v\) be a LS factor with left extensions \((1,2)\). If we apply the morphism as above, we have a problem: the longest common suffix of factors \(\varphi_S(1) = 2\) and \(\varphi_S(2) = 012\) is 2 and so we do not know what are the left extensions of the LS factor \(2\varphi_S(v)\). A solution is to consider left extensions longer than one letter, in such a case we say left prolongation instead of left extension. Clearly, the factor 1 is always preceded by 0. Hence, let us consider left extensions \((01,2)\). Now, \(\varphi_S(01) = 00122\) is no more a suffix of \(\varphi_S(2) = 012\) and so we know the new left extensions: again \((01,2)\). In this way we can construct a complete graph defining the respective \(f\)-image, the result is in Figure 3 (the notation will be explained later).

To prove that a proper finite set of pairs of left and right extensions of arbitrary length always exists is not trivial. It is not simple even to describe the properties such sets should possess so that they define a correct \(f\)-image. We will call such sets *left* and *right forky sets*, see Definition 20. It may happen that a finite forky set does not exist and so our method fails. Therefore we will have to put some restriction on the D0L-systems considered: we will assume that the systems are *circular* and *non-pushy*. These notions are explained in the following section.
Finally, we can now state the main result of this paper: Given a circular non-pushy D0L-system with an aperiodic fixed point, there exist finite left and right forky sets defining two directed graphs and a finite set of initial BS triplets such that the corresponding \( f \)-image applied repetitively on the initial BS-triplets generates all BS factors.

# 4 Forky sets and initial factors

## 4.1 Circular and non-pushy D0L-systems

Any factor of a fixed point of a morphism \( \varphi \) can be decomposed into (possibly incomplete) \( \varphi \)-images of letters. For example, in the case of \( \varphi_E \) we have: 01210 is a factor of \( \varphi_E(0)\varphi_E(2) \), i.e., 01210 is composed by \( \varphi_E \)-images of 0 and 2. We denote this using bars, i.e., 012\textsuperscript{|}10. The decomposition may not be unique. For instance 210 is always decomposed as 2\textsuperscript{|}10 but we do not know whether 2 is a suffix of \( \varphi_E(0) \) or \( \varphi_E(1) \) (it cannot be a suffix of \( \varphi_E(2) \) since 22 is not a factor of \( u_E \)). In the case of the factor 1, we do not even know where to place the bar if not at all.

A factor can have more than one decomposition; however, if there is a common bar for all these decompositions, this bar is called a synchronizing point. Coming back to our example, 210 has a synchronizing point between 2 and 10, formally we say that \((2, 10)\) is a synchronizing point of 210.

**Definition 3** (Cassaigne [1]). Let \( \varphi \) be a morphism with a fixed point \( u \), \( \varphi \) injective on \( \mathcal{L}(u) \), and let \( w \) be a factor of \( u \). An ordered pair of factors \((w_1, w_2)\) is called a synchronizing point of \( w \) if \( w = w_1w_2 \) and \( \forall v_1, v_2 \in A^* \) \((v_1w_2 \in \varphi(\mathcal{L}(u)) \Rightarrow v_1w_1 \in \varphi(\mathcal{L}(u)) \) and \( v_2w_2 \in \varphi(\mathcal{L}(u))\)).

We denote this by \( w = w_1|_s w_2 \).

**Definition 4.** A D0L-system \( G = (A, \varphi, w) \) is circular if \( \varphi \) is injective on \( \mathcal{L}(G) \) and if there exists \( D \in \mathbb{N} \) such that any \( v \in \mathcal{L}(G) \) with \( |v| \geq D \) has at least one synchronizing point. This \( D \) is called a synchronizing delay.

Some examples of both circular and non-circular D0L-systems follow.

**Example 5.** The system \( G = (A_2, \varphi_{TM}, 0) \) is circular with a synchronizing delay 4. It is clear that any \( w \in \mathcal{L}(G) \) containing 00 or 11 has the synchronizing point \( w = \cdots 0, 0 \cdots \) or \( w = \cdots 1, 1 \cdots \). To see that, let us consider a word \( w \) of length 4 not containing these two factors. Without loss of generality, assume that \( w \) begins in 1, then \( w = 1010 \). This word can be decomposed into \( \varphi_{TM}(0) \) and \( \varphi_{TM}(1) \) in exactly two ways: \( 10|10 \) and \( 1|01 \). But the latter one is not admissible since it arises as the \( \varphi_{TM} \)-image of 000 which is not an element of \( \mathcal{L}(G) \).

**Example 6.** The system \( G = (A_2, \varphi, 0) \), where \( \varphi(0) = 01, \varphi(1) = 11, \) is not circular. Indeed, for all \( n \) the word \( 1^n \) has no synchronizing point since it can be decomposed as \( 11|11 \cdots \) and \( 1|11 \cdots \)

This example is very simple since the respective infinite fixed point 011111\cdots is eventually periodic. However, there are also aperiodic non-circular systems.
Example 7. The system $G = (A_3, \varphi, 0)$, where $\varphi(0) = 010, \varphi(1) = 22, \varphi(2) = 11$, is not circular. The argument is the same as in the previous example, since the words $1^n$ are for all $n \in \mathbb{N}$ elements of $L(G)$. However, the infinite word $\varphi^\omega(0) = 0102201011110102 \cdots$ is aperiodic.

One can notice that the languages in the both non-circular examples contain an arbitrary power of 1. It is not just a coincidence but a general rule.

**Theorem 8** (Mignosi and Séébold [13]). If a D0L-system is $k$-power-free (i.e., $L(G)$ does not contain the $k$-power of any word) for some $k \geq 1$, then it is circular.

Thus, non-circular fixed points must have infinite critical exponent, in fact, they must contain an unbounded power of some word.

**Theorem 9** (Ehrenfeucht and Rozenberg [14]). Given a D0L-system $G = (A, \varphi, w)$, if $L(G)$ contains a $k$-power for all $k \in \mathbb{N}$, then $G$ is strongly repetitive, i.e., there exists a nonempty $v \in L(G)$ such that $v^\ell \in L(G)$ for all $\ell \in \mathbb{N}$.

We see that non-circular systems have very special properties. Furthermore, the morphism of a non-circular system cannot be even primitive. A morphism $\varphi$ over $A$ is primitive if there is $k \in \mathbb{N}$ such that $\varphi^k(a)$ contains $b$ for all $a, b \in A$.

**Theorem 10** (Mossé [15]). Any D0L-system $G = (A, \varphi, a)$ with $\varphi$ injective on $G$ and primitive is circular.

No matter how non-circular systems seem to be bizarre, there is no known finite algorithm which would decide whether a given general D0L-system is circular or not. Of course, if the respective morphism is primitive, it is easy to prove it in finite steps. Later on we also prove that if the system is non-pushy, then the circularity is equivalent to repetitiveness which is decidable.

**Example 11.** An example of non-primitive but circular morphism is the one given by $0 \mapsto 0010, 1 \mapsto 1$. This is the Chacon morphism [16] and 5 is its synchronizing delay.

### 4.2 Non-pushy D0L-systems

The following two definitions and the lemma are taken from [13].

**Definition 12.** Let $G = (A, \varphi, w)$ be a D0L-system. A letter $b \in A$ has rank zero if $L(G_b)$, where $G_b = (A, \varphi, b)$, is finite.

**Definition 13.** A D0L-system $G = (A, \varphi, w)$ is pushy if for all $n \in \mathbb{N}$ there exists $v \in L(G)$ of length $n$ which is composed of letters that have rank zero; otherwise $G$ is non-pushy. If $G$ is non-pushy, then $q(G)$ denotes $q(G) = \max\{|v| \mid v \in L(G) \text{ is composed of letters that have rank zero}\}$.

**Lemma 14.**

1. It is decidable whether or not an arbitrary D0L-system is pushy.
2. If $G$ is pushy, then $L(G)$ is strongly repetitive (see Theorem 9).
3. If $G$ is non-pushy, then $q(G)$ is effectively computable.

\footnote{In the article [15] the circular systems are called “recognizable”.

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4. It is decidable whether or not an arbitrary D0L-system is strongly repetitive.

**Corollary 15** (Krieger [10]). Let \( G = (A, \varphi, a) \), \( a \in A \), be a non-pushy D0L-system and let \( u = \varphi^v(a) \) be an infinite fixed point of \( \varphi \). Then there exists a non-erasing morphism \( \varphi' \) and an effectively computable \( C \in \mathbb{N} \) such that \( u = (\varphi')^v(a) \) and for all \( v \in \mathcal{L}(G) \) with \( |\varphi'(v)| = |v| \) we have \( |v| < C \).

This means that for any word \( v \) of length at least \( C \) we have \( |\varphi(v)| \geq |v| + 1 \). More generally, if \( |v| \geq KC \) then \( |\varphi(v)| \geq |v| + K \). We will use this in the proof of the main theorem of the following subsection.

In the sequel, we always suppose that the \( G = (A, \varphi, a) \) is such that \( \varphi' \) can be taken equal to \( \varphi \) (in fact \( \varphi' \) is just a power of \( \varphi \), see the proof in [10]). This is without loss of generality since the language is the same.

### 4.3 Forky sets

Our aim is to define properly the notion of \( f \)-image introduced in Section 3. As explained, we need to have two directed labeled graphs defined on unordered pairs of left and right prolongations. Since left and right extensions usually refer to letters and the vertices of our graphs might be pairs of words, we give the following definition.

**Definition 16.** Let \( u \) be an infinite word and \( w \) its factor. The set of left prolongations of \( w \) is the set
\[
L_{\text{pro}}(w) = \{ v \in A^+ \mid vw \in \mathcal{L}(u) \}.
\]

In an analogous way we define the set of right prolongations \( R_{\text{pro}}(w) \).

The sets \( L_{\text{pro}}(w) \) and \( R_{\text{pro}}(w) \) are, in general, infinite.

Our aim is to specify a suitable finite sets \( B_L \) and \( B_R \) of (unordered) pairs of left and right prolongations such that it allows to define correctly an \( f \)-image of all triplets \( ((w_1, w_2), v, (w_3, w_4)) \), where \( v \) is a BS factor, \( (w_1, w_2) \) a pair of its left and \( (w_3, w_4) \) a pair of its right prolongations from \( B_L \) and \( B_R \), respectively. The \( f \)-image defined by the sets \( B_L \) and \( B_R \) are to be defined as a BS triplet \( ((w'_1, w'_2), v', (w'_3, w'_4)) \), where \( (w'_1, w'_2) \) and \( (w'_3, w'_4) \) are again in \( B_L \) and \( B_R \) and the factor \( v' = f_L(w'_1, w'_2)\varphi(v)f_R(w'_3, w'_4) \) is BS. The mappings \( f_L \) and \( f_R \) are defined as follows.

**Definition 17.** Let \( \varphi \) be a morphism over \( A \) and let \( (v_1, v_2) \) be an unordered pair of words from \( A^+ \). We define
\[
f_L(v_1, v_2) = \text{the longest common suffix of } \varphi(v_1) \text{ and } \varphi(v_2),
\]
\[
f_R(v_1, v_2) = \text{the longest common prefix of } \varphi(v_1) \text{ and } \varphi(v_2).
\]

The purpose of the following definitions is just to describe “good” choices of \( B_L \) and \( B_R \).

**Definition 18.** Let \( (w_1, w_2) \) and \( (v_1, v_2) \) be unordered pairs of words. We say that
(i) \( (w_1, w_2) \) is a prefix (suffix) of \( (v_1, v_2) \) if either \( w_1 \) is a prefix (suffix) of \( v_1 \) and \( w_2 \) of \( v_2 \), or \( w_1 \) is a prefix (suffix) of \( v_2 \) and \( w_2 \) of \( v_1 \);
(ii) \( (w_1, w_2) \) and \( (v_1, v_2) \) are \( L \)-aligned if
\[
(v_1 = uw_1 \text{ or } w_1 = uv_1) \text{ and } (v_2 = u'w_2 \text{ or } w_2 = u've_2)
\]
or
\[
(v_1 = u'w_2 \text{ or } w_2 = uv_1) \text{ and } (v_2 = u'w_1 \text{ or } w_1 = u've_2)
\]
for some words \( u, u' \).
Analogously, we define pairs which are R-aligned.

**Example 19.** The pairs \((01, 0)\) and \((001, 10)\) are L-aligned, while \((01, 0)\) and \((011, 10)\) are not L-aligned. Schematically, the notion of L-aligned pairs of words is depicted in Figure 2.

![Given two pairs of words: L-aligned and not L-aligned.](image)

Figure 2: L-aligned and not L-aligned pairs of words.

**Definition 20.** Let \(\varphi\) be a morphism with a fixed point \(u\). A finite set \(B_L\) of unordered pairs \((w_1, w_2)\) of nonempty factors of \(u\) is called L-forky if all the following conditions are satisfied:

(i) the last letters of \(w_1\) and \(w_2\) are different for all \((w_1, w_2)\) \(\in B_L\),

(ii) no distinct pairs \((w_1, w_2)\) and \((w'_1, w'_2)\) from \(B_L\) are L-aligned,

(iii) for any \(v_1, v_2 \in L(u) \setminus \{\epsilon\}\) with distinct last letters there exists \((w_1, w_2)\) \(\in B_L\) such that \((w_1, w_2)\) and \((v_1, v_2)\) are L-aligned,

(iv) for any \((w_1, w_2)\) \(\in B_L\) there exists \((w'_1, w'_2)\) \(\in B_L\) such that 

\[
(w'_1 f_L(w_1, w_2), w'_2 f_L(w_1, w_2))
\]

is a suffix of \((\varphi(w_1), \varphi(w_2))\).

Analogously we define an R-forky set.

Since the definition may look a bit intricate, we now comment on all the conditions. Condition (i) says that \(w_1\) and \(w_2\) are left prolongations of LS factors (note that all pairs of words \(w_1, w_2 \in L(u)\) are prolongations of the empty word, i.e., elements of \(\text{Lpro}(\epsilon)\)). Condition (ii) is required to avoid redundancy in \(B_L\). Condition (iii) ensures that any two left prolongations of any LS factor are included in \(B_L\) in the following sense: if we prolong or shorten them in a certain way we obtain a pair from \(B_L\). And, finally, Condition (iv) is there because of the definition of the \(f\)-image: we want to be able to apply it repetitively. Note that due to (ii) and (iii) the pair \((w'_1, w'_2)\) from (iv) is uniquely given. Note also that if (i) is satisfied and the words from all the pairs of \(B_L\) are of the same length, then (ii) and (iii) are satisfied automatically.
Example 21. Consider the morphism \( \varphi_S \) from Section 5 defined by \( 0 \mapsto 0012, 1 \mapsto 2, 2 \mapsto 012 \). This morphism is injective and primitive and so, by Theorem 10, the respective DOL-system is circular. One can easily prove that 3 is a synchronizing delay (note that all factors containing 2 has a synchronizing point \( \cdots 2 | \cdots \)).

Since \( \varphi_S \) is prefix-free, we get that the set
\[
B_R = \{(0, 1), (0, 2), (1, 2)\}
\]
is R-forky. However, this set is not L-forky: Condition (iv) is not satisfied for any pair since \( \varphi_S(1) \) is a suffix of \( \varphi_S(2) \) which is a suffix of \( \varphi_S(0) \). To remedy this, we consider left prolongations one letter longer which are ending in 1 and 2. Since the list of all factors of length 2 reads
\[
00, 01, 12, 20, 22
\]
the new pairs are
\[
(0, 01), (0, 12), (0, 22), (2, 01).
\]
For these pairs conditions (i) – (iii) are again satisfied. But (iv) is not satisfied for \( (0, 12) \) since \( f_L(0, 12) = 012 \) and \( \varphi_S(0)(012)^{-1}, \varphi_S(12)(012)^{-1} = (0, 2) \) has no suffix in list (2). Hence, we have to prolong 12 again. There is only one possibility, namely 012. The resulting set
\[
B_L = \{(0, 01), (0, 012), (0, 22), (2, 01)\}
\]
is then L-forky since now we get \( (\varphi_S(0)(012)^{-1}, \varphi_S(012)(012)^{-1}) = (0, 00122) \) with a suffix \( (0, 22) \in B_L \).

Theorem 22. Let \( \varphi \) be a morphism on \( A \) with a fixed point \( u = \varphi^*(a) \). If \( (A, \varphi, a) \) is circular non-pushy system, then it has L-forky and R-forky sets.

Proof. Set \( M = DC \), where \( D \) is a synchronizing delay and \( C \) is the constant from Corollary 13. Define
\[
B_L = \{(w_1, w_2) : w_1, w_2 \in L(u), |w_1| = |w_2| = M, \text{ and the last letters of } w_1 \text{ and } w_2 \text{ are different}\}. \tag{3}
\]
We claim that \( B_L \) is L-forky. Conditions (i) – (iii) from Definition 20 are trivially fulfilled. It remains to prove (iv).

It is clear that we must have \( |f_L(w_1, w_2)| \leq D \) for any \( (w_1, w_2) \) since \( f_L(w_1, w_2) \) without the last letter does not have any synchronizing point. Now, it suffices to realize that for any \( w \) of length \( M \) we have \( |\varphi(w)| \geq |w| + D = M + D \) and so \( (\varphi(w_1)(f_L(w_1, w_2))^{-1}, \varphi(w_2)(f_L(w_1, w_2))^{-1}) \) has a suffix in \( B_L \). The proof of existence of an R-forky set is perfectly the same. \( \square \)

This proof does not give us a general guideline how to construct L-forky and R-forky sets since, as we have recalled earlier, we do not have an effective algorithm computing a synchronizing delay for a general morphism. Moreover, the L-forky set constructed in the proof is usually too huge. As in the case of our example morphism \( \varphi_S \) from Example 21 (for this morphism \( C = 2 \) and \( D = 3 \)), there usually exists a smaller L-forky set.

Remark 23. The techniques of the proof of Theorem 22 are the same as those of the proof of Lemma 11 in [10]. This Lemma, however, is concerned with the notion of critical exponent.
Definition 24. Let \( \varphi \) be a morphism with a fixed point \( u \) and let \( \mathcal{B}_L \) be an L-forky set. We define the directed labeled graph of left prolongations \( GL_{\varphi}^{(\mathcal{B}_L)} \) as follows:

(i) the set of vertices is \( \mathcal{B}_L \),
(ii) there is an edge from \((w_1, w_2)\) to \((w_3, w_4)\) if \((w_3f_L(w_1, w_2), w_4f_L(w_1, w_2))\) is a suffix of \((\varphi(w_1), \varphi(w_2))\). The label of this edge is \( f_L(w_1, w_2) \).

In the same manner we define the graph of right prolongations \( GR_{\varphi}^{(\mathcal{B}_R)} \).

As a straightforward consequence of the definition of forky sets (especially of Condition (iv)) we have the following property of the graphs.

Lemma 25. Each vertex in a graph of left and right prolongations has its out-degree equal to one.

Consequently, any long enough path in the graph ends in a cycle and any component contains exactly one cycle.

Example 26. The graphs \( GL_{\varphi_S}^{(\mathcal{B}_L)} \) and \( GR_{\varphi_S}^{(\mathcal{B}_R)} \) for \( \varphi_S \) and for the sets \( \mathcal{B}_L \) and \( \mathcal{B}_R \) from Example 21 are in Figure 3.

\[ \text{Figure 3: The graphs } GL_{\varphi_S}^{(\mathcal{B}_L)} \text{ and } GR_{\varphi_S}^{(\mathcal{B}_R)} \text{ for the morphism } \varphi_S. \]

Definition 27. Let \( \varphi \) be a morphism on \( A \) with a fixed point \( u \) and let \( \mathcal{B}_L \) and \( \mathcal{B}_R \) be L-forky and R-forky sets, respectively. A triplet \(((w_1, w_2), v, (w_3, w_4))\) is called a bispecial (BS) triplet in \( u \) if \((w_1, w_2) \in \mathcal{B}_L\), \((w_3, w_4) \in \mathcal{B}_R\) and \(w_1vw_3, w_2vw_4 \in \mathcal{L}(u)\) or \(w_1vw_4, w_2vw_3 \in \mathcal{L}(u)\).

Lemma 28. Let \( \varphi \) be a morphism with a fixed point \( u \), let \( \mathcal{B}_L \) be an L-forky and \( \mathcal{B}_R \) an R-forky set and let \( T = ((w_1, w_2), v, (w_3, w_4)) \) be a bispecial triplet of \( u \). Let us denote by

(i) \( g_L(w_1, w_2) \) the end of the edge of \( GL_{\varphi}^{(\mathcal{B}_L)} \) starting in \((w_1, w_2)\),
(ii) \( g_R(w_3, w_4) \) the end of the edge of \( GR_{\varphi}^{(\mathcal{B}_R)} \) starting in \((w_3, w_4)\).
\[ T' = (gL(w_1, w_2), f_L(w_1, w_2)\varphi(v) f_R(w_3, w_4), g_R(w_3, w_4)) \]

is also a bispecial triplet of \( u \).

**Definition 29.** Denote \( B = (B_L, B_R) \). The bispecial triplet \( T' \) from the previous lemma is called the \( f_B \)-image of a bispecial triplet \( T = ((w_1, w_2), v, (w_3, w_4)) \).

**Example 30.** Consider again the morphism \( \varphi_S \). Let \( B = (B_L, B_R) \) where \( B_L \) and \( B_R \) are those from Example 24. Then \( ((0, 012), 0, (0, 1)) \) is a bispecial triplet since both 001 and 01200 are factors. Its \( f_B \)-image reads \( ((0, 22), 0120012, (0, 2)) \) for we have \( g_L(0, 012) = (0, 22), f_L(0, 012) = 012, g_R(0, 1) = (0, 2), \) and \( f_R(0, 1) = \epsilon \).

Condition (iv) from Definition 20 of forky sets allows us to get a compact formula for the \( (f_B)^n \)-image, i.e., \( f_B \)-image applied repetitively \( n \) times.

**Lemma 31.** Let \( \varphi \) be a morphism and let \( ((w_1, w_2), v, (w_3, w_4)) \) be a bispecial triplet for some forky sets \( B = (B_L, B_R) \). Then for all \( n \in \mathbb{N} \) it holds that its \( (f_B)^n \)-image equals

\[
(g_L^n(w_1, w_2), f_L(\varphi^{n-1}(w_1), \varphi^{n-1}(w_2))\varphi^n(v) f_R(\varphi^{n-1}(w_3), \varphi^{n-1}(w_4)), g_R^n(w_3, w_4)).
\]

### 4.4 Initial BS factors

From the previous subsection, we know how to get a sequence of BS factors from some starting one: we just apply \( f_B \)-image repetitively. The goal of the present subsection is to prove that for circular systems there exists a finite set of initial BS factors (triplets) such that any other BS factor is an \( (f_B)^n \)-image of one of them.

**Definition 32.** Let \( \varphi \) be a morphism injective on \( \mathcal{L}(u) \), where \( u \) is its fixed point, let \( B_L \) and \( B_R \) be L- and R-forky sets, and \( T = ((w_1, w_2), v, (w_3, w_4)) \) a bispecial triplet. Assume, without loss of generality, that \( w_1w_2w_3w_4w_1 \in \mathcal{L}(u) \). An ordered pair of factors \( (v_1, v_2) \) is called a BS-synchronizing point of \( T \) if \( v = v_1v_2 \) and

\[ \forall u_1, u_2, u_3, u_4 \in A^*, (u_1w_1w_2w_3u_3, u_2w_2w_4u_4) \in \mathcal{L}(u) \Rightarrow u_1w_1v_1, u_2w_2v_2, u_3v_3, u_4v_4 \in \mathcal{L}(u). \]

We denote this by \( v = v_1|w_1v_2 \).

The notion of BS-synchronizing point is weaker than the one of synchronizing point: it holds that if \( v = v_1|w_1v_2 \), then \( v = v_1|w_1v_2 \). The other direction is not true as it follows from this example:

**Example 33.** Given a morphism \( 0 \mapsto 010, 1 \mapsto 210, 2 \mapsto 220 \), the factor 0 has no synchronizing point. On the other hand, if we take it as the bispecial triplet \( ((1, 2), 0, (0, 2)) \) it has the BS-synchronizing point \( 0|\delta \).

**Definition 34.** Let \( \varphi \) be a morphism with a fixed point \( u \) which is injective on \( \mathcal{L}(u) \). A bispecial triplet \( T = ((w_1, w_2), v, (w_3, w_4)) \) is said to be initial if it does not have any BS-synchronizing point.

**Definition 35.** Let \( T = ((w_1, w_2), v, (w_3, w_4)) \) be a bispecial triplet which is not initial and let \( (v_1, v_2), (v_3, v_4), \ldots, (v_{2m-1}, v_{2m}) \) be all its BS-synchronizing points such that \( |v_1| < |v_3| < \cdots < |v_{2m-1}| \). The factor \( v_1 \) is said to be the non-synchronized prefix, \( v_{2m} \) the non-synchronized suffix and the factor \( (v_1)^{-1}v(v_{2m})^{-1} \) is called the synchronized factor of \( T \).
Now we state the main theorem of this section.

**Theorem 36.** Let \((A, \varphi, a)\) be a circular non-pushy D0L-system, \(B_L\) and \(B_R\) its \(L\)-forky and \(R\)-forky set, and \(u = \varphi^n(a)\) infinite. Then there exists a finite set \(I\) of bispecial triplets such that for any bispecial factor \(v\) there exist a bispecial triplet \(T \in I\) and \(n \in \mathbb{N}\) such that \(((w_1, w_2), v, (w_3, w_4)) = (f_B)^n(T)\) for some \((w_1, w_2) \in B_L\) and \((w_3, w_4) \in B_R\).

**Proof.** Let \(I\) be the set of initial bispecial triplets. The finiteness of \(I\) is a direct consequence of the definition of circularity: elements of \(I\) cannot be longer than the synchronizing delay. The rest of the statement follows from the fact that any non-initial triplet has at least one \(f_B\)-preimage.

To prove this, it suffices to realize that the synchronized factor of \(((w_1, w_2), v, (w_3, w_4))\) has unique \(\varphi\)-preimage \(v'\) (possibly the empty word) and that the non-synchronized prefix (resp. suffix) must be equal to \(f_L(w'_1, w'_2)\) for some \((w'_1, w'_2) \in B_L\) (resp. to \(f_R(w'_3, w'_4)\) for some \((w'_3, w'_4) \in B_R\)).

\[ \square \]

**Example 37.** We now find the set \(I\) for \(u_B\) with \(B = (B_L, B_R)\), where \(B_L\) and \(B_R\) are from Example [27] the graphs of prolongations are in Figure [3]. The only BS factors without synchronizing points are \(\epsilon\) and 0 since any other factor contains the letter 2 (and hence it has a synchronizing point) or is not BS. It remains to find all corresponding triplets:

\[((0, 01), \epsilon, (1, 2)), ((0, 01), \epsilon, (0, 2)), ((0, 012), \epsilon, (0, 1)), ((0, 012), \epsilon, (0, 2)), ((0, 012), \epsilon, (1, 2)), ((0, 22), \epsilon, (0, 1)), ((2, 01), \epsilon, (0, 2)), ((0, 012), 0, (0, 1)).\]

Since we are usually interested in nonempty BS factors, we can replace the bispecial triplets containing \(\epsilon\) with their \(f_B\)-images and get:

\[((2, 01), 2, (0, 2)), ((2, 01), 20, (0, 1)), ((0, 22), 012, (0, 2)), ((0, 012), 012, (0, 2)), ((0, 012), 1, (0, 1)).\]

There are only 6 bispecial triplets since \(((0, 012), \epsilon, (0, 1))\) and \(((0, 012), \epsilon, (1, 2))\) have the same \(f_B\)-image and so do \(((0, 01), \epsilon, (0, 2))\) and \(((2, 01), \epsilon, (0, 2))\).

## 5 Infinite special branches

In the preceding section we have described a tool allowing us to find all BS factors. It requires some effort to construct the graphs GL and GR and to find all initial bispecial triplets, but it can be done by an algorithm. However, even if we have all these necessities in hand, it may be still a long way to the complete knowledge of the structure of all BS factors. Nevertheless, there is a class of special factors which can be identified directly from the graphs GL and GR, namely, the prefixes (or suffixes) of the so-called infinite LS (or RS) branches.

**Definition 38.** An infinite word \(w\) is an infinite LS branch of an infinite word \(u\) if each prefix of \(w\) is a LS factor of \(u\). We put

\[ L \text{Ext}(w) = \bigcap_{v \text{prefix of } w} L \text{Ext}(v). \]

Infinite RS branches are defined in the same manner, only that they are infinite to the right.
Here are some (almost) obvious statements on infinite special branches in an infinite word:

**Proposition 39.**

(i) If $u$ is eventually periodic, then there is no infinite LS branch of $u$,

(ii) if $u$ is aperiodic, then there exists at least one infinite LS branch of $u$,

(iii) if $u$ is a fixed point of a primitive morphism, then the number of infinite LS branches is bounded.

**Proof.** Item (i) is obvious, (iii) is a direct consequence of the fact that the first difference of complexity is bounded [17]. The proof of item (ii) is due to the famous König’s infinity lemma [18] applied on sets $V_1, V_2, \ldots$, where the set $V_k$ comprises all LS factors of length $k$ and where $v_1 \in V_i$ is connected by an edge with $v_2 \in V_{i+1}$ if $v_1$ is prefix of $v_2$. 

Imagine now that we have an L-forky set $B_L$ and an infinite LS branch $w$. There must exist $(v_1, v_2) \in B_L$ such that $v_1w$ and $v_2w$ are factors for any prefix $w$ of $w$. Such a pair is called an infinite LS pair.

**Definition 40.** Let $(v_1, v_2)$ be an element of an L-forky set corresponding to a fixed point $u$ of a morphism $\varphi$. The ordered pair $((v_1, v_2), w)$ is called an infinite LS pair of for any prefix $w$ of $w$ the words $v_1w$ and $v_2w$ are factors of $u$.

Further, we define the $f_{B_L}$-image of an infinite LS pair $((v_1, v_2), w)$ as the infinite LS pair $((v'_1, v'_2), w')$, where $(v'_1, v'_2) = g_L(v_1, v_2)$ and $w' = f_L(v_1, v_2)\varphi(w)$.

Having the $f_{B_L}$-image of an infinite LS branch, we are again interested in its $f_{B_L}$-preimage.

**Lemma 41.** Let $(A, \varphi, a), a \in A$, be a circular D0L-system with an infinite fixed point $u = \varphi^n(a)$ and let $B_L$ be its L-forky set. Then any infinite LS pair is the $f_{B_L}$-image of a unique infinite LS pair.

**Proof.** Let $((v_1, v_2), w)$ be an infinite LS pair and let $D$ be a synchronizing delay of $\varphi$. Then any prefix of $w$ of length at least $D$ has the same left-most synchronizing point $(w_1, w_1^{-1}w)$. Since such $w$ is LS, then $w_1$ must be a label of an edge in $GL_{\varphi^n(a)}^{B_L}$ whose end-vertex is $(v_1, v_2)$ and starting one in $(v'_1, v'_2)$. The infinite word $(w_1)^{-1}w$ must have a unique $\varphi$-preimage $w'$. 

Since any infinite LS pair $((v_1, v_2), w)$ has an $f_{B_L}$-preimage, the in-degree of the vertex $(v_1, v_2)$ in the graph of left prolongations $GL_{\varphi^n(a)}^{B_L}$ must be at least one.

**Corollary 42.** Let $(A, \varphi, a)$ be a circular D0L-system with a fixed point $u$ and an L-forky set $B_L$. If $((v_1, v_2), w)$ is an infinite LS pair then $(v_1, v_2)$ is a vertex of a cycle in $GL_{\varphi^n(a)}^{B_L}$.

We know that the number of infinite LS pairs in a fixed point of a primitive morphism is finite (see Proposition [19]); the following proposition says that this is true even if we weaken the assumption from primitive to circular and non-pushy. The proof of the proposition will, moreover, give us a simple method of how to find all these infinite LS pairs.
Theorem 43. Let \((A, \varphi, a) \in A\) be a circular D0L-system with an L-forky set \(\mathcal{B}_L\) such that \(\varphi^n(a) = u\) is infinite. Then there exist only a finite number of infinite LS pairs.

Proof. Let \(((v_1, v_2), w)\) be an infinite LS pair and let \((v_1, v_2)\) be a vertex of a cycle in \(GL_{\mathcal{B}_L}^k\). Let us assume that the cycle is of length \(k\). Then it is labeled by words \(f_L(v_1, v_2), f_L(g_L(v_1, v_2)), \ldots, f_L(g_L^{-1}(v_1, v_2))\) where \(f_L(v_1, v_2)\) is the label of the edge starting in \((v_1, v_2)\) (see Figure 4). We distinguish two cases:

(a) At least one of the labels of the cycle is not the empty word. Applying \(k\) times Lemma 41, we can find the infinite LS pair \(((v_1, v_2), w')\) such that \(w\) is the \((f_L)^k\)-image of \(((v_1, v_2), w)\), i.e.,

\[ w = f_L(g_L^{-1}(v_1, v_2)) \cdots \varphi^{k-2}(f_L(g_L(v_1, v_2)))\varphi^{k-1}(f_L(v_1, v_2)) \varphi^k(w') = s\varphi^k(w'). \]

Since \(w'\) can be expressed again as \(w' = s\varphi^k(w'')\) for some infinite LS pair \(((v_1, v_2), w'')\), we have

\[ w = s\varphi^k(s)\varphi^{2k}(s)\varphi^{3k}(w''). \]

Continuing in this construction one can prove that \(s\varphi^k(s)\cdots\varphi^{nk}(s)\) is a prefix of \(w\) for all \(n \in \mathbb{N}\). Therefore, we get

\[ w = s\varphi^k(s)\varphi^{2k}(s)\varphi^{3k}(s)\cdots. \]

We have just shown that exactly one infinite LS pair corresponds to each vertex of the cycle.

(b) Now assume that all the labels of the cycle are empty words. In such a case the \((f_L)_{\mathcal{B}_L}\)-image coincides with \(\varphi\)-image, meaning that \((f_L)_{\mathcal{B}_L}^{-1}\)-image of \(((v_1, v_2), w)\) is \((g_L^{-1}(v_1, v_2), \varphi^j(w))\) for all \(j = 1, 2, \ldots\). We want to prove that \(w\) must be a periodic point of \(\varphi\). Consider the directed graph whose vertices are the first letters of \(\varphi(b), b \in A\), and there is an edge from \(b\) to \(c\) if \(c\) is the first letter of \(\varphi(b)\). Clearly, the first letter of \(w\), say \(b\), must be again a vertex of a cycle in this graph. Let \(\ell\) be the length of this cycle. For reasons analogous to those above the \((f_L)_{\mathcal{B}_L}^{-\ell}\)-image and \((f_L)_{\mathcal{B}_L}^{j\ell}\)-preimage of \(w\) must also begin in \(b\). Therefore, \(w\) contains the factor \(\varphi^j(b)\) as a prefix for all \(j = 1, 2, \ldots\) and this implies that \(w = (\varphi^j)^\omega(b)\), i.e., \(w\) is a periodic point of \(\varphi\).
Figure 5: The graph $GL_{\varphi_P}^{(B_1)}$ for morphism defined by (7), $B_1$ is the set of all unordered pairs of letters.

Since the number of vertices of $GL_{\varphi_P}^{(B_1)}$ and of periodic points is finite, the number of infinite LS pairs must be finite as well.

The previous proof is also a proof of the following corollary which gives us a method of how to find all infinite LS branches.

**Corollary 44.** Let $(A, \varphi, a), a \in A$, be a circular D0L-system, $u = \varphi^\omega(a)$ infinite with $B_L$ an $L$-forky set and let $((v_1, v_2), w)$ be an infinite LS pair. Then either $w$ is a periodic point of $\varphi$, i.e.,

$$w = \varphi^\ell(w) \quad \text{for some } \ell \geq 1,$$

and $(v_1, v_2)$ is a vertex of a cycle in $GL_{\varphi_P}^{(B_L)}$ labeled by $\epsilon$ only, or $w = s\varphi^\ell(s)\varphi^{2\ell}(s)\cdots$ is the unique solution of the equation

$$w = s\varphi^\ell(w),$$

where $(v_1, v_2)$ is a vertex of a cycle in $GL_{\varphi}$ containing at least one edge with a non-empty label, $\ell$ is the length of this cycle and

$$s = f_L(g_L^{-1}(v_1, v_2))\cdots \varphi^{\ell-2}(f_L(g_L(v_1, v_2)))\varphi^{\ell-1}(f_L(v_1, v_2)).$$

We demonstrate this method on an example morphism.

**Example 45.** We consider the morphism

$$\varphi_P : 1 \mapsto 1211, 2 \mapsto 311, 3 \mapsto 2412, 4 \mapsto 435, 5 \mapsto 534$$

with $u = \varphi_P^\omega(1)$. This morphism is suffix- and prefix-free and so the set of all unordered pairs of distinct letters is $L$-forky. The graph of left prolongations is in Figure 5. The morphism $\varphi_P$ has five periodic points

$$\varphi_P^\ast(1), \varphi_P^\ast(4), \varphi_P^\ast(5), (\varphi_P^\ast)^\omega(2), (\varphi_P^\ast)^\omega(3).$$

It is easy to show that

$$\text{Ext}(1) = \{1, 2, 3, 4, 5\}, \text{Ext}(2) = \{1, 4, 5\}, \text{Ext}(3) = \{1, 4, 5\}, \text{Ext}(4) = \{1, 2, 3\}, \text{Ext}(5) = \{1, 2, 3\}.$$
Looking at the graph of left prolongations depicted in Figure 5, we see that \( \varphi_2^4(4) \) and \( \varphi_2^5(5) \) are not infinite LS branches as none of the vertices \((1,2),(2,3)\) and \((1,3)\) is a vertex of a cycle labeled by \( \epsilon \) only. Hence, only \( \varphi_2^4(1), (\varphi_2^5)^2(2), (\varphi_2^5)^3(3) \) are infinite LS branches with left extensions 1, 4, 5.

As for infinite LS branches corresponding to Equation (8), in the case of our example, there is only one cycle which is not labeled by the empty word: the cycle between vertices \((1,2)\) and \((2,3)\). There are two (= the length of the cycle) equations corresponding to this cycle

\[ w = \varphi(11)\varphi_2^2(w) \quad \text{and} \quad w = 11\varphi_2^2(w). \]

They give us two infinite LS branches

\[ \varphi(11)\varphi_2^2(11)\varphi_2^5(11) \cdots, \]
\[ 11\varphi_2^2(11)\varphi_2^5(11) \cdots, \]

the former having left extensions 1 and 2 and the latter 2 and 3.

6 Assumptions and a connection with the critical exponent

The method of how to generate all BS factors of a given D0L-system we have described above bears on two facts: There exists L- and R-forky sets and the number of initial BS triplets is finite. The former was proved for circular and non-pushy systems and the later for circular ones only. Are these assumptions necessary or can they be weakened?

Let us take a morphism \( 0 \mapsto 001, 1 \mapsto 1 \) which is pushy and circular and its fixed point \( u \) starting in 0. If we try to construct an L-forky set as it has been defined in this paper, we will find out that it is not possible. A natural candidate for vertices of the graph of left prolongations are pairs \((0^n,1_m)\) with \( n \in \mathbb{N} \), but any finite set of such pairs does not satisfy the property (iv) of Definition 20. So it seems that to assume the morphism being non-pushy is inevitable for existence of forked sets. However, if we relax the definition and enable the pairs of factors to be infinitely long, we can find something like L-forky set even for this morphism: Define a directed graph of left prolongations such that it has only one vertex \((0^n,\cdots,111)\) and one loop on this vertex with label 1 and a directed graph of right prolongations with one vertex \((0,1)\) and a loop on it with empty label, then all BS factors in \( u \) are the \( f \)-images of BS-triplet \((0,\cdots,111,0,(0,1))\), namely 0, 1\( \varphi(0), 1\varphi(1)\varphi^2(0) \),

Now, consider a morphism \( 0 \mapsto 001, 1 \mapsto 11 \) and its fixed point \( u \) starting in 0. This morphism is non-pushy and non-circular. In this case, L- and R-forky sets exists, we can simply take \((0,1)\) and there is only one (nonempty) initial BS-triplet with no BS-synchronizing point \((0,1,0,(0,1))\). Its \( f \)-images again read 0, 1\( \varphi(0), 1\varphi(1)\varphi^2(0) \),

In fact, to prove that the set of initial BS-triplets is finite, we need to know only that there is not infinite number of BS-triplets without BS-synchronizing point and it seems to be true even for non-circular morphisms.

All considered morphisms for which our method does not work (or is not proved to work) have an infinite critical exponent. The following theorem says this is not a misleading observation but a general rule.

Theorem 46. Let \( G = (A, \varphi, w) \) be a D0L-system. Then the critical exponent of \( \mathcal{L}(G) \) is finite if and only if \( G \) is circular and non-pushy.
Proof. ($\Rightarrow$): Circularity follows from Theorem 8. $G$ being pushy is in contradiction with Lemma 14, thus, it is non-pushy.

($\Leftarrow$): Suppose the critical exponent of $L(G)$ is infinite and that $G$ is circular and non-pushy. According to Theorem 9, there exists a non-empty factor $v \in L(G)$ such that for all $n \in \mathbb{N}$, $v^n \in L(G)$. Take the shortest factor $v$ having such property. Since $G$ is circular, there exists a finite synchronizing delay $D$. Take $N \in \mathbb{N}$ such that $|v^N| \geq D$. Then $v^N$ contains a synchronizing point, i.e., $v^N = v_1 | v_2$. It is clear that $v^{N+1}$ contains at least two synchronizing points, i.e., $v_{N+1} = v_1 | v_2 v = v v_1 | v_2$. In general, $v^{N+k}$ contains $k + 1$ synchronizing points at fixed distances equal to $|v|$. Since $\varphi$ is injective, it implies that there exists a unique $z \in L(G)$ such that $v^{N+k} = p \varphi(z^k) s$ (for some factors $p$ and $s$) and $z^k \in L(G)$ for all $k \geq 0$. According to the choice of $v$, it is clear that $|\varphi(z)| = |z| = |v|$. Denote by $L_1(z)$ the set of letters occurring in $z$. It is clear that $\varphi(L_1(z)) = L_1(v)$ and $\forall a \in L_1(z)$ we have $|\varphi(a)| = 1$.

We can now repeat the process: take the factor $z$ to play the role of factor $v$. Thus, we can find an infinite sequence of factors $z_0 = z, z_1, z_2 \ldots$ such that $\varphi(L_1(z_{k+1})) = L_1(z_k)$ and $|z_k| = |z|$ for all $k \geq 0$. Since $A$ is finite, it is clear that there exists integers $m \neq \ell$ such that $L_1(z_m) = L_1(z_\ell)$. This implies that for all $k$ the factor $z_k$ is composed of letters of rank zero. This is a contradiction with $G$ being non-pushy.

\[\square\]

7 Conclusion

The tool we have introduced in this paper enables to construct an algorithm which can find all BS factors in a given circular non-pushy D0L-system so that it produces the graphs of prolongations and the set of initial BS factors – its slightly simplified version was implemented by Štěpán Starosta using SAGE. The scatch of the algorithm is as follows:

1. Decide whether the input D0L-system is strongly repetitive using the algorithm from [14]. If it is, then by the previous theorem and Theorem 9 the D0L-system is non-circular or pushy and our method does not work. If it is not, proceed with to steps.

2. Construct L- and R-forky sets: the details of the construction are a bit technical but the basic idea is the same we used in Example 21.

3. Find all initial BS triplets without any BS-synchronizing point. The fact that the system is circular ensures the algorithm stops after a finite number of steps.

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