Fuzzy Bag Models

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Abstract

We show how hadronic bag models can be generalized to implement effects of a smooth and extended boundary. Our approach is based on fuzzy set theory and can be straightforwardly applied to any type of bag model. We illustrate the underlying ideas by calculating static nucleon properties in a fuzzy chiral bag model.
Physical concepts and models are often based on idealizations. Usually, those arise either from insufficient knowledge of the underlying physics, or they are intended to make the theoretical description more transparent and more amenable to quantitative analysis. Bag models \[1,2\], which occupy a prominent place among hadron models and are widely used in areas ranging from hard scattering processes to dense nuclear matter, furnish a typical example for such idealizations. They impose the confinement of relativistic quarks inside hadrons, in a region of modified vacuum, by static boundary conditions at a bag radius \(R\). Whereas the real vacuum is expected to return to its normal phase outside of the hadron gradually, however, this simple prescription leads to an infinitely thin bag boundary and thus to an abrupt transition between the two phases.

Of course, such a rough and energetically unfavorable approximation must miss some relevant features of the physics of hadrons. Especially observables with an exceptional sensitivity to the characteristics of the boundary, such as for example some properties of excited and deformed hadrons or diffractive scattering cross sections (in particular at low energies), therefore require a more realistic description of the hadronic boundary. Previous attempts to go beyond the sharp bag-boundary approximation, however, were technically quite involved and limited to a specific model \[3\].

In the present letter we consider a novel implementation of extended boundaries, which is easy to apply to even the most complex bag models (including those with quantized surfaces). This approach can be rigorously formulated in terms of fuzzy set theory \[4,5\], in which ordinary sets are generalized by assigning partial memberships to their elements. By now, fuzzy sets have proven remarkably useful in quite diverse areas of model building, and it seems worthwhile and timely to explore their potential in physics. The application to the transition between the inside and outside regions of bag models suggests itself naturally since fuzzy sets were specifically designed to implement smooth transitions between unrealistically distinct domains in simplified models.

It is quite straightforward to see how such fuzzy boundaries arise. To start with, one considers the sharp surface of the standard bag model at a given radius as the sole element
of an ordinary set. By letting this set become fuzzy, an extended boundary – containing conventional bag surfaces of varying radii and weights as elements – emerges. In analogy with the boundary conditions of standard bag models, the underlying fuzzy set (the weight function) is prescribed according to general physical requirements. Possibly, it could be determined dynamically in a future, more advanced version of the model.

As just indicated, the central idea of our approach is to promote the bag radius from a real number $R$ to a fuzzy set $\rho$. In general, fuzzy sets \cite{4} consist of an ordinary reference set $\mathcal{X}$ and a real-valued membership function

$$\mu : \mathcal{X} \to [0, 1] \quad x \mapsto \mu(x),$$

which specifies the degree to which an element $x \in \mathcal{X}$ belongs to $\mu$. (Following common practice, we use the same symbol for both the fuzzy set and its membership function.) By definition, $\mu$ is an element of the fuzzy power set $\mathcal{F}(\mathcal{X})$ over $\mathcal{X}$. Taken as the truth value of a statement $x$, $\mu(x)$ defines a generalization of Boolean logic (called $L_1$ \cite{6}) in which the strict true-false alternative for $x$ is relaxed.

Accordingly, the fuzzy bag radius is represented by a membership function $\rho(R)$, which specifies the degree to which a sphere with radius $R$ belongs to the extended bag boundary. Therefore, its reference set $\mathcal{R} \subseteq [0, \infty]$ minimally contains the radii in the surface region. We denote the center (in radial direction) of the boundary by $R_0$ and its width by $\Delta$. Some of the potential of this description of the boundary originates from the fact \cite{5} that membership degrees in fuzzy sets are generally not additive\footnote{This can be seen directly from the membership degree of subsets $\mathcal{R}_1 \in \mathcal{R}$, which is given by $\rho(\mathcal{R}_1) = \sup \{\rho(R) | R \in \mathcal{R}_1\}$ \cite{4}. For the same reason, $\int_{\mathcal{X}} \mu(x) \, dx \neq 1$ in general. The probability $P(\mathcal{R}_1) = \int_{\mathcal{R}_1} \rho(R) \, dR / \int_{\mathcal{R}} \rho(R) \, dR$, on the other hand, is obviously additive. While many theorems of ordinary set theory continue to hold for fuzzy sets, there are further crucial exceptions, e.g. $\mu \cup \mu^c \neq \mathcal{X}$ and $\mu \cap \mu^c \neq \emptyset$ (if $\mu \neq \emptyset, \mathcal{X}$).} (in contrast, for example, to probabilities). This implies, e.g., that bag surfaces at different $R$ (i.e. their fuzzy weights) do not have...
to be independent. Instead, they can be coexisting and correlated in a common, extended boundary.

Since bag models do not provide any dynamics for the boundary, we have to rely on more general physical considerations to find the appropriate shape of $\rho$. First, we expect that the shell at radius $R_0$ belongs fully to the transition region, $\rho(R_0) = 1$, and that $\rho(R)$ rises (decreases) monotonically for $R < R_0$ ($R > R_0$). Thereby, $\rho$ becomes an element of $\mathcal{F}_I(\mathcal{R}) = \{\mu \in \mathcal{F}(\mathcal{R})| \exists R \in \mathcal{R} : \mu(R) = 1 \land \forall a, b, c \in \mathcal{R} : a \leq b \leq c \Rightarrow \mu(c) \geq \min\{\mu(a), \mu(b)\}\}$, the set of fuzzy intervals over $\mathcal{R}$. (Fuzzy intervals have particularly convenient calculational properties, see below.) Furthermore, in a two-phase model $\Delta$ should not be larger than $R_0$. In fact, smaller $\Delta \lesssim R_0/2$ are preferable since the inner region of the bag is more efficiently described in terms of quarks. Reasonable values for $R_0$ lie in the typical hadronic range of about $0.5 - 1.0$ fm.

Nontopological soliton models [7], which capture qualitative aspects of the transition between QCD vacuum phases in hadrons at the mean-field level, corroborate this picture. The typical surface shapes found in such models are very close to those considered above. In particular, they do not show significant asymmetries between the inner and outer parts of the surface. This suggests to use a Gaussian membership function

$$\rho^{(g)}(R) = \exp\left[\frac{-(R - R_0)^2}{2\Delta^2}\right]$$

for the fuzzy bag radius, which we will do below. In order to check the dependence of the results on the detailed shape of the membership function, we have also tested alternative choices such as the triangular form $\rho^{(t)}(R) = 1 - \left|\frac{R_0 - R}{2\Delta}\right|$ for $|R - R_0| \leq 2\Delta$, and $\rho^{(t)}(R) = 0$ otherwise. (Note that $\rho^{(t)} \subseteq \rho^{(g)}$.) In all cases, the standard bag model is recovered for $\Delta \to 0$.

The next step in the setup of the fuzzy bag model deals with the definition and calculation of observables. Starting from a conventional bag model with crisp bag radius, this is accomplished by employing the extension principle [8] of fuzzy set theory. Adapted to our context, it states that any map $A(R)$ from a (crisp) bag radius $R$ to an observable $A \in \mathcal{A}$ (as
calculated in conventional bag models) can be uniquely extended to a map from the fuzzy bag radius $\rho(R)$ to a fuzzy set

$$\nu: \mathcal{F}_I(\mathcal{R}) \to \mathcal{F}(\mathcal{A}), \quad \rho(\mathcal{R}) \mapsto \nu_{\rho}(\mathcal{A})$$

$$\nu_{\rho}(x) := \sup \{\rho(R) \mid R \in \mathcal{R} \land x = A(R)\}.$$  \hspace{1cm} (3)

Equation (3) quantifies how the fuzziness of the basic variable $R$ propagates into the observables. It follows directly from the rules which govern fuzzy sets \cite{8}. The nonlinearity of the supremum of the membership degrees $\rho(R_i)$ (where all $R_i$ are mapped to the same $A$) in Eq. (3) shows explicitly that the resulting membership degrees are not additive\cite{8}. As mentioned earlier, this implies that bag surfaces at different $R$ need not be independent and, therefore, do not mutually exclude each other from belonging to a common boundary.

In order to convert fuzzy-bag results, i.e. the fuzzy sets $\nu(A)$, into numerical predictions, they have to be mapped onto those real numbers $\tilde{A}$ which best represent their physical information content. To this end, we employ the standard centroid map \cite{9}

$$\tilde{A} = \frac{\int_A A \nu(A) \, dA}{\int \nu(A) \, dA}.$$ \hspace{1cm} (4)

(The integrals extend over $\mathcal{A}$.) In subsequent calculations, the fuzzy results $\nu(A)$ can also be used directly whenever the involved mathematical operations can be extended to fuzzy intervals.

\footnote{There is another important difference between fuzzy and linear measures like, for example, probability densities. Regarding $p(R) = \rho(R) / \int \rho(R') \, dR'$ as a probability density would imply that $dP = p(R) \, dR$ is the associated probability to find $R$ in an interval $[R, R + dR]$. Therefore, the induced probability density for $A$, $p(x) = \sum_{R_i \in \mathcal{R}} \rho(R_i) \left| \frac{dR_i}{dA} \right|_{A(R_i) = x}$ (where $R_i(A)$ is the local inverse of $A(R)$ in the $i$-th monotonicity interval), contains a Jacobian which relates the intervals $[R, R + dR]$ and $[A, A + dA]$. Since membership in fuzzy sets is defined "pointwise", such a Jacobian is absent in $\nu_{\rho}(x)$.}
The above steps complete the definition of the fuzzy bag model as the most direct and transparent fuzzy-set extension of the standard bag model. In principle, one could try to refine this model by employing more complex tools from fuzzy set theory (see, e.g., Ref. [10]). In view of the inherent limitations of the bag model itself, however, and of our fragmentary understanding of the physical mechanism which generates extended hadron boundaries, it seems likely that not much can be gained by such complications.

In order to illustrate the above concepts with a practical example, we now apply them to the nonlinear chiral bag model [2], which is based on the Lagrangian

\[ \mathcal{L}_{\chi BM} = (\bar{q} i\slashed{D} q - B) \Theta_V - \frac{1}{2} \bar{q} U_5 q \delta_V - \left( \frac{f_\pi^2}{4} \text{tr} [L_{\mu} L^\mu] - \frac{1}{32e^2} \text{tr} [L_{\mu}, L_\nu]^2 \right) \Theta_\bar{V}. \]  

Here, \( \Theta_V, \Theta_\bar{V} \) and \( \delta_V \) are the bag theta function, its complement and its derivative, \( q \) are the quark fields, and the pion fields \( \tilde{\phi} \) appear in the nonlinear realizations \( U = \exp(i\tilde{\phi}/f_\pi) \) and \( U_5 = \exp(i\tilde{\phi}_5/f_\pi) \) of the chiral group with \( L_{\mu} = U^\dagger \partial_\mu U \). Furthermore, \( B \simeq (150 \text{ MeV})^4 \) is the bag constant, \( f_\pi = 93 \text{ MeV} \) the pion decay constant, and \( e = 4.5 \). The mean-field solution has the hedgehog form \( \tilde{\phi} = \hat{r} F(r) \) for the pions and contains three valence quarks in the lowest-lying bag states. By slow rotation with angular velocity \( \Omega \) it can be projected onto nucleon quantum numbers.

The calculation of static nucleon observables in this model has recently been reviewed in Ref. [3]. For the following discussion, we select two results which illustrate characteristic properties of the fuzzy extension. The first is the total bag energy \( E \) in the hedgehog state. Its bag-radius dependence is indicated in Fig. 2 (as the dotted line, with \( R_0 = R \) for crisp bag radii). In the corresponding fuzzy bag model, the energy is uniquely extended to the fuzzy set

\[ \epsilon_\rho(x) = \sup \{ \rho(R) \mid R \in \mathcal{R}_\epsilon \land x = E(R) \}, \]  

which is plotted in Fig. 1a for \( R_0 = 0.7 \text{ fm}, \Delta = 0.3 \text{ fm}, \) and \( \mathcal{R}_\epsilon = [0, 1.5] \text{ fm} \). Note that \( E(R) \) cannot be inverted on \( \mathcal{R}_\epsilon \), so that the supremum in Eq. (6) plays an active role in shaping \( \epsilon_\rho(E) \).
As a second example, we consider the axial coupling $g_A$ of the nucleon, calculated to first order in the angular velocity $\Omega$ \cite{2}. It is plotted as a function of the bag radius in Fig. 3 (dotted line). In contrast to the hedgehog energy, $g_A(R)$ is monotonic. On the other hand, it shows a significantly stronger bag-radius dependence, varying by almost a factor of two for $0 \leq R \leq 1\text{fm}$. This is a well-known shortcoming of the chiral bag model since it implies a strong deviation from “Cheshire-Cat” behavior (see below). The corresponding fuzzy set

$$\gamma_{\rho}(x) = \sup \{ \rho(R) \mid R \in \mathcal{R}_\gamma \land x = g_A(R) \}. \quad (7)$$

is shown in Fig. 1b for $\mathcal{R}_\gamma = [0,1] \text{fm}$ with $R_0$ and $\Delta$ as above. The shapes of $\epsilon$ and $\gamma$ closely reflect the behavior of $E(R)$ and $g_A(R)$, and therefore depart significantly from the Gaussian (bag radius) set by which they are induced. Nevertheless, it can be shown that they remain fuzzy intervals for all $R_0$ and $\Delta$ \cite{11}. This is a generic property of fuzzy bag-model observables which is helpful in subsequent calculations involving these sets.

Next, we calculate the centroids of $\epsilon(E)$ and $\gamma(g_A)$ according to Eq. (4) and examine the dependence of the resulting fuzzy-bag observables $\tilde{E}$ and $\tilde{g}_A$ on location and extension of the boundary region. Figure 2 shows the hedgehog energy $\tilde{E}$ as a function of $R_0$ for different values of the “fuzziness” parameter $\Delta$. (In the following, we drop the tilde on fuzzy-bag results and identify them by their $R_0$-dependence.) The dotted line corresponds to $\Delta \to 0$, i.e. to the standard chiral bag model with $R = R_0$.

For increasing diffuseness of the boundary, the bag energy becomes less sensitive to $R_0$ until, beyond $\Delta \sim 0.4 \text{fm}$, it remains almost $R_0$-independent. The sensitivity of $g_A(R_0)$ to the position of the bag boundary (Fig. 3) decreases even more strongly for broader transition regions. With $\Delta = 0.4 \text{fm}$ and for $R_0$ in the range $0 \leq R_0 \leq 1\text{fm}$, $g_A$ deviates less than 10 % from its experimental value 1.26 \cite{12}. Furthermore, it is interesting to note that the extended boundary shifts the minimum of the fuzzy-bag energy towards smaller radii, from 0.85 to 0.5 fm. If interpreted variationally, this minimum might contain (after projection) some information on the nucleon’s size and structure (as long as the Cheshire-Cat principle is not perfectly re-
alized, see below). From this point of view, smaller radii are favored both by experiment (which finds, e.g., that even rather hard probes \(q^2 \lesssim 1 \text{ GeV}^2\) do not resolve the nucleon’s quark core) and by meson-exchange phenomenology [13].

The reduced sensitivity of fuzzy bag model results to the boundary position has its origin in the (generally) increasing support of fuzzy sets associated with stronger varying \(A(R)\). The ensuing, weaker \(R_0\)-dependence of the results is quite welcome since the unobservable bag radius lacks an unambiguous physical meaning, and since it reduces the parameter dependence of the model. Moreover, it better complies with the Cheshire-Cat principle [14], according to which bag-model results should become radius-independent to the extent to which the description of the physics in- and outside of the bag can be perfected (and thus made indistinguishable). The improved Cheshire-Cat behavior is an inherent feature of fuzzy bag models because there is no longer a strict distinction between inside and outside dynamics. Exact Cheshire-Cat models would, in fact, be identical to their fuzzy counterparts since fuzzification leaves bag-radius-independent results unaffected. (Such models are, in other words, fixed points under fuzzification.)

In order to get an idea of the model dependence associated with different boundary shapes, it is useful to adopt a fuzzy measure for the equality of two fuzzy sets \(\mu_1, \mu_2\) [9],

\[
\| \mu_1 = \mu_2 \| = \inf \left\{ 1 - |\mu_1(x) - \mu_2(x)| \mid x \in X \right\},
\]

which allows to compare the effects of, e.g., triangular and Gaussian boundaries quantitatively. With \(\| \rho^{(g)} = \rho^{(t)} \| \simeq 0.9\), we find \(\| \epsilon^{(g)} = \epsilon^{(t)} \| \simeq 0.85\), and \(\| \gamma^{(g)} = \gamma^{(t)} \| \simeq 0.9\), almost independently of \(\Delta\). The weak dependence of the induced fuzzy sets on the detailed shape of \(\rho\) implies an even weaker dependence of the numerical results and makes the predictions of the model rather robust³.

To summarize, fuzzy bag models as defined above extend standard bag models by incor-

³In general, fuzzy sets induce similar results if they are “locally monotonic”, i.e. as long as \(\mu_1(x_2) \leq \mu_1(x_1) \Leftrightarrow \mu_2(x_2) \leq \mu_2(x_1)\) holds for all \(x_1, x_2 \in X\) [5].
porting effects of a smooth phase boundary in terms of fuzzy set theory. Nevertheless, they maintain the appealing simplicity and the absolute confinement of conventional bag models. Moreover, the fuzzy boundary can mitigate artefacts caused by sharp bag surfaces, and it reduces the sensitivity of observables to the bag size.

The fuzzy bag model thus provides a convenient instrument for studying consequences of extended hadron surfaces in a simple and rather unbiased way (at least as long as the Cheshire-Cat principle is not exactly realized). It should be especially useful for the investigation of observables with an enhanced sensitivity to surface properties, such as those of excited and deformed hadronic states, and for studying interactions among hadrons, e.g. in low-energy diffractive scattering processes. On a more conceptual level, the study of surface effects could reveal new aspects of the underlying transition between two QCD vacuum phases (with and without valence quark sources).

The model has successfully passed its first confrontation with phenomenology at the level of static nucleon observables. In comparison with the corresponding crisp bag model the fuzzy bag energy and the axial coupling show, for example, less sensitivity to the bag size, and the prediction for $g_A$ is improved. The results depend little on details of the boundary shape and are almost uniquely determined by the parameters of the corresponding crisp bag model and the thickness of the surface, which is the only important new scale introduced by the extended boundary.

Despite these encouraging results, however, more extensive phenomenological applications of the fuzzy bag should, at the present stage, not take precedence over the further development of its conceptual basis. A step in this direction could be, e.g., to find a selfconsistent dynamical mechanism for the calculation of the fuzzy bag radius set (perhaps as a soliton). It should also be possible to find physical applications for fuzzy sets beyond the realms of the bag model and hadronic physics.
REFERENCES

[1] A. Chodos, R.L. Jaffe, K. Johnson, C.B. Thorn, V.F. Weisskopf, Phys. Rev. D 9, 3471 (1974).

[2] A. Hosaka and H. Toki, Phys. Rept. 277, 65 (1996), and references therein.

[3] Y. Nogami, A. Suzuki, and N. Yamanishi, Can. J. Phys. 62, 554 (1984).

[4] L.A. Zadeh, Information and Control 8, 338 (1965).

[5] D. Dubois and H. Prade, Fuzzy Sets, Theory and Applications, Academic Press, Orlando (1980);

[6] J. Lukasiewicz, Selected Works, Amsterdam and Warsaw (1970).

[7] see, for example, R. Friedberg and T.D. Lee, Phys. Rev. D 16, 1096 (1977); Phys. Rev. D 18, 2623 (1978); S. Kahana, G. Ripka, and V. Soni, Nucl. Phys A 415, 351 (1984); M.K. Banerjee, Prog. Part. Nucl. Phys. 31, 77 (1993).

[8] L.A. Zadeh, Information Sci. 8, 199, 301 (1975); 9, 43 (1975); R.R. Yager, Fuzzy Sets and Systems 18, 205 (1986).

[9] R. Kruse, J. Gebhardt, and F. Klawonn, Foundations of Fuzzy Systems, John Wiley, New York (1994).

[10] M. Mizumoto and K. Tanaka, Information and Control 48, 30 (1981).

[11] H. Forkel, in preparation.

[12] M. Aguilar-Benitez et al., Phys. Rev. D 54, 1 (1996).

[13] R. Machleidt, K. Holinde, and C. Elster, Phys. Rept. 149, 1 (1987).

[14] S. Nadkarni, H.B. Nielsen, and I. Zahed, Nucl. Phys. B 253, 308 (1984); M. Rho, Phys. Rept. 240, 1 (1994).
FIG. 1. The membership function of a) the bag energy and b) the axial coupling of the nucleon for $R_0 = 0.7$ fm and $\Delta = 0.3$ fm.

FIG. 2. The bag energy as a function of the central radius $R_0$ of the transition region, for $\Delta = 0$ fm (dotted line), 0.1 fm (dashed), 0.2 fm (dot-dashed), 0.3 fm (dot-dot-dashed), 0.4 fm (solid). The open circles correspond to the mean value of the energy, assuming a probabilistic interpretation of $\rho$ (with $R_0 = 0.3$ fm).

FIG. 3. The axial coupling of the nucleon as a function of $R_0$ for the same values of $\Delta$ as above.
Fig. 2
Fig. 3
Fig. 1b