Symbolic Hamburger-Noether expressions of plane curves and construction of AG codes

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Abstract
We present an algorithm to compute bases for the spaces \( \mathcal{L}(G) \) and \( \Omega(G) \), provided \( G \) is a rational divisor over a non-singular absolutely irreducible algebraic curve, and also another algorithm to compute the Weierstrass semigroup at \( P \) together with functions for each value in this semigroup, provided \( P \) is a rational branch of a singular plane model for the curve. The method is founded on the Brill-Noether algorithm by combining in a suitable way the theory of Hamburger-Noether expansions and the imposition of virtual passing conditions. Such algorithms are given in terms of symbolic computation by introducing the notion of symbolic Hamburger-Noether expressions. Everything can be applied to the effective construction of Algebraic Geometry codes and also in the decoding problem of such codes, including the case of the Feng and Rao scheme for one-point codes.

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Key words – algebraic curves, singular plane models, resolution of singularities, symbolic Hamburger-Noether expressions, virtual passing conditions, Brill-Noether theorem, the spaces \( \mathcal{L}(G) \) and \( \Omega(G) \), Weierstrass semigroups, Algebraic Geometry codes.

1 Introduction
Since the construction at the beginning of the 80’s by Goppa of linear codes using Algebraic Geometry (see [1]), the theory of algebraic-geometric codes has been extensively developed. Algebraic Geometry codes (AG codes in short) can be constructed from any smooth algebraic projective curve \( \bar{\chi} \) defined over a finite field \( F \) as images of linear maps involving either residues at certain rational points of differential forms in \( \Omega(G-D) \), or evaluations at such points of rational

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1
functions in $L(G)$, where $G$ is a rational divisor over $\mathbb{F}$ and $D = P_1 + \ldots + P_n$ is the formal sum of the rational points above considered, $G$ and $D$ having disjoint supports (details in section 6).

In order to construct such codes, the main difficulty in practice is the computation of vector bases for the spaces $L(G)$, since the construction with differential forms is derived by duality. When the curve is given by means of a plane singular birational model $\chi$, some general methods can be used for this task if one knows well enough the singularities, namely the Brill-Noether [14] and Coates methods [8].

On the other hand, nice codes need to have also good decoding algorithms. From the beginning of the 90’s several decoding methods have been developed (see [15] for a survey on this matter). In the case $G = mP$ for some extra rational point $P$ and $m > 0$, Feng and Rao gave in [9] a simple method based on a majority voting test, nowadays considered to be the most efficient decoding procedure. This method requires the previous knowledge of the Weierstrass semigroup of $\chi$ at the rational branch given by $P$, together with a rational function $f_l \in \mathbb{F}(\chi)$ regular outside $P$ and achieving a pole at $P$ of order $l$, for each $l$ in this semigroup, as we will show in section 6.

Again, the main difficulty turns out to be the computation of Weierstrass semigroups and such functions $f_l$. This difficulty is in fact the main obstacle for practical uses in Coding Theory, among others, of the construction by García and Stichtenoth in [11] of a sequence of curves achieving the Drinfeld-Vlăduţ bound given in [5].

Thus, the effective coding and decoding of AG codes depend on the resolution of two basic problems: computation of a vector basis for $L(G)$ and the computation of Weierstrass semigroups together with functions achieving its values. The objective of this paper is to give a complete symbolic-computation treatment of these basic problems from the knowledge of a singular plane birational model $\chi$ for the smooth curve $\tilde{\chi}$, what is actually the most usual way to give a curve. The algorithms of this paper are at present being implemented by the authors in the computer algebra system SINGULAR [13], created by Greuel, Pfister and Schoenemann.

Our approach is based on very classical ideas. First we consider Hamburger-Noether expansions from the symbolic viewpoint; more precisely, we introduce in the paper the so-called symbolic Hamburger-Noether expressions, which will provide us with both all the information on the singularities and (symbolic) parametrizations for all their rational branches. Hamburger-Noether expansions are developed in [8] for the case of irreducible curve singularities over algebraically closed fields. Here we will need not only the symbolic version but also the case of general plane curve singularities over perfect fields (finite fields in practice) as developed in [8].

In particular, from the knowledge of the singularities one can compute the adjunction divisor, and from it the imposing conditions test for being an adjoint (see sections 3 and 4). This becomes important for the approach to the first
basic problem, since via the Brill-Noether algorithm the computation of a vector basis for \( L(G) \) is reduced to computing vector bases for some concrete spaces of adjoints, which are obtained by imposing certain assigned conditions. We show how the computation of such adjoint bases can be done by using the so-called principle of discharge due to Enriques in \( 7 \) (see \( 4 \) and \( 17 \) for a modern treatment). Thus, our solution to the first basic problem is derived from the three classical theories of Hamburger-Noether (section 2), Brill-Noether (section 3) and Enriques (section 4).

The second basic problem is approached in similar terms. In fact, from a singular plane model the adjunction theory of plane curves can be applied to give an algorithm to compute the Weierstrass semigroup and the corresponding functions (see section 5). Again, this algorithm becomes effective using symbolic Hamburger-Noether expressions at the singularities of \( \chi \). Finally, we apply these methods to the construction of AG codes in section 6.

## 2 Symbolic Hamburger-Noether expressions of plane curve singularities

In this section, we will introduce the symbolic Hamburger-Noether expressions for a plane curve singularity. For this, we fix in the sequel an arbitrary perfect field \( \mathbb{F} \) and an absolutely irreducible projective algebraic plane curve \( \chi \) defined over \( \mathbb{F} \). For a closed point \( P \) of \( \chi \) with local ring \( R = \mathcal{O}_{\chi, P} \) we denote by a rational branch of \( \chi \) at \( P \) any maximal prime ideal of \( \mathfrak{m} \), where \( \mathfrak{m} \) denotes the semilocal ring given by the normalization of \( R \). The datum of such a maximal ideal is equivalent to give a minimal prime ideal of \( \hat{R} \), the completion with respect to the Jacobson radical of \( R \) (see \( 1 \) and \( 2 \) for details).

Assume that we have chosen an affine chart containing \( P \), and let \( A = \mathbb{F}[X, Y]/(f(X, Y)) \) be the affine ring of coordinates, \( f(X, Y) = 0 \) being the affine equation of the curve in this chart. Regarding \( P \) as a non-zero prime ideal of \( A \), one has \( k(P) \cong K \hookrightarrow \hat{A}_P \) and, for practical reasons, one can actually write \( K = \mathbb{F}[Z]/(Q(Z)) \) for an irreducible polynomial \( Q \in \mathbb{F}[Z] \). Thus, up to a translation in \( K[X, Y] \), we can assume that \( P \) is the origin, the defining ideal of \( P \) being then \( (X, Y) \).

With these notations, one has \( \hat{R} \cong K[[X, Y]]/(f(X, Y)) \), and hence there exists a natural morphism \( K[[X, Y]] \to \hat{R} \). This allows us to introduce the following definition.

**Definition 2.1** In the above conditions, a rational parametrization of \( \chi \) at \( P \) related to the coordinates \( X, Y \) is a \( K \)-algebra morphism

\[
    r : K[[X, Y]] \to K_1[[t]]
\]

continuous for the \( (X, Y) \)-adic and the \( t \)-adic topologies, such that \( \text{Im}(r) \subsetneq K_1 \) and \( f \in \ker(r) \), where \( K_1 \) is a finite extension of \( K \) and \( t \) is an indeterminate.
This is equivalent to give formal series \( x(t), y(t) \in K_1[[t]] \) with at least one non identically zero such that \( f(x(t), y(t)) \equiv 0 \).

We can associate to each rational parametrization \( r \) the rational branch given by the minimal prime ideal \( p = \ker(\hat{r}) \), where \( \hat{r} : \hat{R} \to K_1[[t]] \) is the natural morphism induced by \( r \). Thus, we say that \( r \) is a rational parametrization of the branch \( p \).

We say that another rational parametrization \( s : K[[X,Y]] \to K_2[[u]] \) is derived from \( r \), and it is denoted by \( s \triangleright r \), if there exists a formal series \( t(u) \in K_2[[u]] \) with positive order and a \( K \)-algebra morphism \( \sigma : K_1[[t]] \to K_2[[u]] \) with \( \sigma(t) = t(u) \), such that \( s = \sigma \circ r \). One has that \( \triangleright \) is a partial preorder, and we say that two rational parametrizations \( r \) and \( s \) are equivalent if \( s \triangleright r \) and \( r \triangleright s \). Thus, a rational parametrization \( r \) is called primitive if it is minimal (with respect to the partial preorder \( \triangleright \)) modulo equivalence, and moreover the extension \( K_1[I^\prime] \) is also minimal (that is, \( r(X) \) and \( r(Y) \) are not both in \( K'[[[t]] \) for some field \( K' \) with \( I^\prime \subseteq K' \subseteq K_1 \) and \( K' \neq K_1 \)). One actually has that rational branches at \( P \) are in bijection with equivalence classes of primitive rational parametrizations at \( P \) and, in particular, there always exist rational parametrizations (details again in [2]). By choosing a primitive rational parametrization for each rational branch one obtains a so-called standard set of rational parametrizations at \( P \), and our next aim is the effective computation of such a set by means of the so-called Hamburger-Noether expansions.

Although a general definition can be given for arbitrary singular curves, we will study only the case of plane curves in order to get effective computations. Thus, let \( \rho : I^\prime[[X,Y]] \to F[[u]] \) be a rational parametrization over \( I^\prime \) of the plane curve \( \chi \) defined at the point \( P \), \( F \) being a finite extension of the base field \( I^\prime \). Denote in short by \( O \) the local ring of \( \chi \) at \( P \). One can consider \( O \) in fact as a subring of \( I^\prime[[u]] \), the images of \( X \) and \( Y \) being a minimal system of generators of the maximal ideal of \( O \).

**Definition 2.2** We introduce the **Hamburger-Noether expansion** of \( \chi \) at \( P \) for the branch given by \( \rho \) to be a finite sequence \( D \) of expressions in the variables \( Z_{-1}, Z_0, Z_1, \ldots, Z_r \) of the form

\[
Z_{-1} = a_{0,1}Z_0^0 + a_{0,2}Z_0^2 + \ldots + a_{0,h_0}Z_0^{h_0} + Z_0^{h_0}Z_1
\]

\[
Z_0 = a_{1,2}Z_1^2 + a_{1,3}Z_1^3 + \ldots + a_{1,h_1}Z_1^{h_1} + Z_1^{h_1}Z_2
\]

\[
\vdots
\]

\[
Z_{r-2} = a_{r-1,2}Z_{r-1}^2 + a_{r-1,3}Z_{r-1}^3 + \ldots + a_{r-1,h_{r-1}}Z_{r-1}^{h_{r-1}} + Z_{r-1}^{h_{r-1}}Z_r
\]

\[
Z_{r-1} = \sum_{i \geq 1} a_{r,i}Z_r^i
\]
where \( r \) is a non-negative integer, \( a_{j,i} \in F \), \( a_{k,1} = 0 \) if \( k > 0 \), \( h_j \) are positive integers and moreover

\[
f(Z_0(Z_r), Z_{-1}(Z_r)) = 0 \quad \text{in} \quad F[[Z_r]]
\]

\( f \in F[[X, Y]] \) being a generator of the ideal \( \ker(\rho) \).

The existence of such expansions and the finiteness of the number of lines is referred to [1], [2] or [18]. In fact such an expansion \( \mathcal{D} \) always gives a primitive rational parametrization equivalent to \( \rho \) if we consider \( X \equiv Z_0 \) and \( Y \equiv Z_{-1} \) as a function of the local parameter \( s = Z_r \) by successive substitutions. Moreover, \( \mathcal{D} \) only depends on the branch given by \( \rho \) and the choice of the parameters \( x, y \) in \( \mathcal{O} \) given by the images of \( X, Y \) under \( \rho \). Thus, for \( X \) and \( Y \) fixed the (finite) set of all the possible non-equivalent Hamburger-Noether expansions form a standard set of rational parametrizations of \( \chi \) at \( P \) (see [2]).

**Remark 2.3** The role played by the Hamburger-Noether expansions in arbitrary characteristic is just the same as that classically played by the Puiseux expansions in characteristic 0, which are given by

\[
X(t) = \alpha t^\nu \\
Y(t) = \sum_{i \geq \nu} \lambda_i t^i
\]

where \( \alpha \in F^* \) and \( \lambda_i \in F \). The main problem of the Puiseux expansions is that they do not always exist in positive characteristic, and when such expansions exist they are rational but the problem of making them primitive is not at all trivial (see [1] or [6]). These are the reasons why we use Hamburger-Noether expansions.

Now we show how to compute the Hamburger-Noether expansions without having a priori any local parametrization of the branch, but only with the aid of the Newton polygon of the local equation of \( \chi \) at \( P \). We will do it for the case of only one rational branch at \( P \) for the sake of simplicity, but the method also works for several branches (in the reduced case) because of the fact that the Newton polygon would be the collection of those of each branch joined together with increasing slope (see [18] for further details).

More precisely, let \( F \) be a perfect field and let \( \chi \) be given in affine coordinates by the local equation \( f(X, Y) = \sum_{\alpha, \beta \geq 0} c_{\alpha\beta}X^\alpha Y^\beta = 0 \), \( f \) being an irreducible polynomial in \( F[X, Y] \). Assume that we want to study the point \( P = (0, 0) \) and that there is only one rational branch at the origin defined over \( F \). Then we consider the Newton diagram of \( f \)

\[
D(f) \doteq \{(\alpha, \beta) \mid c_{\alpha\beta} \neq 0\}
\]
and we call *Newton polygon* of \( f \) (at the origin) the set of all the bounded segments of the convex hull of \( D(f) + \mathbb{R}^2_+ \), and it will be denoted by \( P(f) \).

Excluding the trivial cases where the curve is one of the coordinate axes, let \( l \) (respectively \( n \)) be the minimum integer such that \((l, 0) \in D(f)\) (respectively \((0, n) \in D(f)\)). We can obviously assume that \( n \leq l \). In this case, the Newton polygon consists just of one segment with non-zero slope and extremes \((l, 0)\) and \((0, n)\).

If \( \Delta = P(f) \) is the Newton polygon we can define

\[
L(X, Y) = \sum_{(\alpha,\beta) \in \Delta} c_{\alpha\beta} X^\alpha Y^\beta
\]

One obviously has \( L(X, Y) = c D(X, Y) \) for some \( c \in \mathbb{F}^* \) and some \( D(X, Y) \) which is monic in \( Y \) and defined over \( \mathbb{F} \). Moreover, by using the Hensel lemma one has

\[
D(X, Y) = \prod_{j=1}^d (Y^{n'j} - \delta_j X^{l'j})^e
\]

for some \( \delta_j \in \mathbb{F}^* \), where \( ed = g \text{cd}(l, n) \). Then the *characteristic polynomial* of \( \Delta \) is given by

\[
\Phi_\Delta(\lambda) = \prod_{j=1}^d (\lambda - \delta_j)
\]

It is an irreducible polynomial over \( \mathbb{F} \) (that is, \( \delta_j \) are conjugate each other by the Galois group over \( \mathbb{F} \)). Moreover, one has \( l = l' ed \) and \( n = n' ed \), being \( g \text{cd}(l', n') = 1 \).

If we write \( l = qn + h \) with \( 0 \leq h < n \), we find one of the following two cases:

**Case 1:** \( h = 0 \), what implies \( ed = n, l' = q \) and \( n' = 1 \). Thus write

\[
a_{0,1} = \ldots = a_{0,l'-1} = 0, \quad \text{and} \quad a_{0,l'} = \delta
\]

\( \delta \) being a symbolic root of \( \Phi_\Delta(\lambda) \), we get that the first line of the Hamburger-Noether expansion starts with

\[
Z_{-1} = a_{0,l'} Z_0^{l'} + \ldots
\]

Then we transform \( f \) by

\[
T_1(f, \delta, l') = f(X, Y + \delta X^{l'}) = f_1(X, Y)
\]

getting \( f_1 \) with a segment of extremes \((l_1, 0)\) and \((0, n)\) as the Newton polygon, being \( l_1 > l \), and we iterate the process, taking into account that \( f_1 \) has the coefficients in the field \( \mathbb{F}_1 = \mathbb{F}[\lambda]/(\Phi_\Delta(\lambda)) \) and that it is irreducible over such field.

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1. We mean by a symbolic root of \( \Phi_\Delta(\lambda) \) that one substitutes \( \mathbb{F} \) by the field \( \mathbb{F}_1 = \mathbb{F}[\lambda]/(\Phi_\Delta(\lambda)) \) and one takes as \( \delta \) the residual class of \( \lambda \) in this field.

2. Notice that this process could terminate if there is no point of the form \((l_1, 0)\).
Case 2: $h > 0$; in this case, the first line of the Hamburger-Noether expansion is just

$$Z_{-1} = Z_0^h Z_1$$

Now, since the polynomial $U(f, l, n) = f(Y, X Y^q)$ is divisible by $Y^n q$, we can transform $f$ by

$$T(f, l, n) = \frac{f(Y, X Y^q)}{Y^n q} = f_1(X, Y)$$

Thus the obtained Newton polygon $\Delta_1$ has $(n, 0)$ and $(0, h)$ as extremes, being $h < n$, and its characteristic polynomial is $\Phi_{\Delta_1}(\lambda) = \lambda^e \Phi_\Delta(1/\lambda)$.

Then we repeat the process, looking for the next line of the Hamburger-Noether expansion, identifying in $T(f, l, n) X \equiv Z_1$ and $Y \equiv Z_0$.

If the case $h = 0$ is found some consecutive times during the computation of the line $k + 1$ of the Hamburger-Noether expansion, where $k \in \{0, 1, \ldots, r - 1\}$, we append the new obtained result to the previous part of that line until we get the case $h > 0$, where we append the term $Z_1^h Z_{k+1}$ and change to the next line.

Hence, by applying a finite number of transformations of type $T_1$ or $T$, we get a trivial Newton polygon where either $\min (l, n) = 1$ or there is no point in the vertical axis, and the procedure stops. In this case one has all the lines of the Hamburger-Noether expansion except the last one. But then the equation $f(X, Y)$ is transformed into a polynomial $g(Z_r, Z_{r-1})$, $g$ being defined over a field $\mathbb{F}'$ which is obtained by successive symbolic extensions of $\mathbb{F}$ and one has

$$\frac{\partial g}{\partial Z_{r-1}}(0, 0) \neq 0.$$  

This means that one can obtain as many terms as needed of the last line of the Hamburger-Noether expansion from the polynomial $g(Z_r, Z_{r-1})$ as an implicit function (that is, by indeterminate coefficients), since this line represents $Z_{r-1}$ as a formal series in the variable $Z_r$. Thus, we do not need in practice the data given by the (infinite) series of the last line of the Hamburger-Noether expansion, but only the (finite) data of the implicit equation $g$, which contains the same information. This data is what we call a symbolic Hamburger-Noether expression, and it can be computed in an effective way by the above method for every singular closed point of $\chi$ (initially written in a symbolic extension of the base field if such a point is not rational). Even more, we do all these computations in successive symbolic extensions of $\mathbb{F}$ instead of considering a sufficiently large extension of it, what in practice saves a lot of time.

In the case of several branches the characteristic polynomial is not irreducible and each branch corresponds to an irreducible factor of this polynomial and its corresponding symbolic root, proceeding as in the case of one branch with every factor in parallel. Hence, in the general case we have to add in each step of the previous algorithm a factorization procedure for the corresponding characteristic polynomial, what also has an effective solution. Each irreducible factor follows at least one of the rational branches, so that one has an algorithm in form of tree.
Thus, the branches of the tree given by this algorithm correspond one-to-one to the branches of the curve at the considered point, and for each tree branch one has associated (as a byproduct of the algorithm) the symbolic Hamburger-Noether expression corresponding to the curve branch. The computation of Hamburger-Noether expansions is a known method and it has been implemented with the computer algebra system SINGULAR [13].

Example 2.4 Let \( \chi \) be the projective plane curve over \( \mathbb{F}_2 \) given by
\[
F(X, Y, Z) = X^{10} + Y^8Z^2 + X^3Z^7 + YZ^9 = 0
\]
with the only singular point \( P = (0 : 1 : 0) \) which is rational over \( \mathbb{F}_2 \), being furthermore the unique point of \( \chi \) at infinity. Take the local equation
\[
f(X, Z) = X^{10} + X^3Z^7 + Z^9 + Z^2
\]
of \( \chi \) where \( P \) is the origin, and apply the Hamburger-Noether algorithm to this equation.

With the above notations, one has \( L(X, Z) = Z^2 + X^{10} = (Z + X^5)^2 \); thus \( l = 10, l' = 5, n = e = 2, n' = d = 1 \) and \( q = 5 \), being in the case \( h = 0 \).

The characteristic polynomial is \( \Phi(\lambda) = \lambda + 1 \) and thus the symbolic root is nothing but \( \delta = 1 \), that is, we do not need to enlarge the base field \( \mathbb{F}_2 \). Hence, one has
\[
a_{0,0} = \ldots = a_{0,4} = 0 \quad a_{0,5} = 1
\]
and we do the change
\[
f_1(X, Z) = f(X, Z + X^5) = Z^2 + X^{38} + \ldots
\]
being now \( L(X, Z) = (Z + X^{19})^2 \) and thus \( l = 38, l' = 19, n = e = 2, n' = d = 1, q = 19 \) and again \( h = 0 \); one also has \( \Phi(\lambda) = \lambda + 1 \) and \( \delta = 1 \). Thus
\[
a_{0,6} = \ldots = a_{0,18} = 0 \quad a_{0,19} = 1
\]
and we do the transform
\[
f_2(X, Z) = f_1(X, Z + X^{19}) = Z^2 + X^{45} + \ldots
\]
In this case, one has \( L(X, Z) = Z^2 + X^{45} \), obtaining \( l = l' = 45, n = n' = 2, d = e = 1 \) and \( q = 22 \), being now in the case \( h = 1 > 0 \) and we have to change the line in the Hamburger-Noether expansion without enlarging the base field.

Now the transform to do is
\[
f_3(X, Z) = \frac{f_2(Z, XZ^{22})}{Z^{44}} = Z + X^2 + \ldots
\]
being now the origin a non-singular point of the new equation and the procedure ends with \( r = 1 \). Thus, the symbolic Hamburger-Noether expressions at \( P \) are

\[
\left\{
\begin{array}{l}
Z_{-1} = Z_0^5 + Z_0^{19} + Z_0^{22}Z_1 \\
g(Z_1, Z_0) = Z_1^2Z_0^{154} + Z_1^2Z_0^{151} + Z_1^2Z_0^{137} + Z_1^2Z_0^{130} + Z_1^2Z_0^{27} + Z_1^2Z_0^{13} + \\
+ Z_1^2Z_0^{110} + Z_1^2Z_0^{113} + Z_1^2Z_0^{107} + Z_1^2Z_0^{104} + Z_1^2Z_0^{101} + Z_1^2Z_0^{96} + \\
+ Z_1^2Z_0^{98} + Z_1^2Z_0^{95} + Z_1^2Z_0^{90} + Z_0^{92} + Z_1^2Z_0^{83} + Z_1^2Z_0^{79} + \\
+ Z_1^2Z_0^{76} + Z_1^2Z_0^{78} + Z_1Z_0^{67} + Z_1^2Z_0^{52} + Z_0^{64} + Z_0^{50} + \\
+ Z_1^2Z_0^{54} + Z_1^2Z_0^{42} + Z_1^2Z_0^{40} + Z_0^{66} + Z_1^2Z_0^{28} + Z_0^{22} + \\
+ Z_1Z_0^{18} + Z_0^{15} + Z_1Z_0^{11} + Z_0^8 + Z_0^4 + Z_0
\end{array}
\right.
\]

3 Normalization, resolution and adjunction via symbolic Hamburger-Noether expressions

The purpose of this section is the revision of some classical concepts taking into account the symbolic Hamburger-Noether expressions which have been introduced in the previous section. Thus, for a given plane curve \( \chi \) one can consider its normalization, that is the proper birational morphism

\[
n : \tilde{\chi} \to \chi
\]

where \( \tilde{\chi} \) is the curve obtained by gluing together the affine charts given by the normalization of the affine graded \( \mathbb{F} \)-algebras \( A_U \) for all affine charts \( U \) of \( \chi \). The curve \( \tilde{\chi} \) can be obtained as the blowing-up of the conductor, that is the sheaf of ideals locally given by

\[
C_\chi(U) = \{ f \in \overline{O}_\chi(U) \mid f\overline{O}_\chi(U) \subseteq O_\chi(U) \}
\]

Nevertheless, it is better in practice to look at \( \tilde{\chi} \) as successive blowing-ups of all the closed points of \( \chi \) which are singular until we get a curve without singular points, since this approach can be explicitly described by equations. In each of those blowing-ups one has as result the corresponding strict transform \( \chi_i \) for \( i \geq 0 \) (starting from \( \chi_0 = \chi \)), defined as usual (see for example \([8]\) or \([14]\)). This process can be represented by a combinatorial object called the resolution forest \( T_\chi \), consisting of one weighted oriented tree for each singular closed point of \( \chi \), and which is constructed as follows:

1) The vertices represent the successive points which are obtained by blowing up singular points of the successive strict transforms \( \chi_i \) of \( \chi \) until one gets a non-singular point at the end of each branch of the process. Two such vertices \( p \) and \( q \) of one tree corresponding to the points \( P \) and \( Q \) are connected by an edge from \( p \) to \( q \) if \( Q \) is one of the points obtained by blowing-up \( P \).
2) On each edge $\vec{pq}$ of the forest we put a weight $\rho_{pq} = [k(Q) : k(P)]$, where $k(P)$ and $k(Q)$ are the corresponding residual fields of the local rings $\mathcal{O}_{\chi,P}$ and $\mathcal{O}_{\chi+1,Q}$.

3) If $p$ is the root of the tree corresponding to the singular point $P$ of $\chi$, then we put on $p$ an initial weight $[k(P) : \mathbb{F}]$. On all the other vertices of the forest we can assign two alternative weights which are equivalent if we know the weights on the edges. In both cases one assigns to $p$ a weight for each branch of the tree passing through $p$, where by a branch we denote any upper extremal point of the forest, and we say that such a branch $q$ passes through $p$ there is an oriented path from $p$ to $q$ in $T_3$. The two alternative weights on $p$ for each $q$ are the following:

(I) The multiplicity at $P$ of the rational branch $q$ corresponding to $q$ computed in the corresponding curve $\chi_P$ obtained by blowing-up $\chi$, that is the multiplicity $e_{p,q}$ of the noetherian ring $\mathcal{O}_{\chi_P,P}/q$ of dimension 1 (denoting here $q$ the corresponding minimal prime ideal $\hat{O}_{\chi_P,P}$).

(II) The order at $P$ of the rational branch $q$, that is the number $m_{p,q} = \min \{v_Q(f) \mid f \in \mathfrak{m}_{\chi_P,P} \}$, where $\mathfrak{m}_{\chi_P,P}$ is the maximal ideal of the local ring $\mathcal{O}_{\chi_P,P}$ and $v_Q$ denotes the normalized valuation (that is, with $\mathbb{Z}$ as group of values) corresponding to $Q$ regarded as a point of $\bar{\chi}$. The equivalence between both weights is given by the formula

$$m_{p,q} [k(Q) : k(P)] = e_{p,q}$$

Notice that the order is actually the multiplicity of each of the conjugate geometric branches lying over $P$, considering $\chi$ to be defined over the algebraic closure $\bar{\mathbb{F}}$ of $\mathbb{F}$. By substituting $\mathbb{F}$ by $\bar{\mathbb{F}}$ one obtains another combinatorial object which is much more complex than the one above described and that has all weights on the edges equal to 1 and hence $m_{p,q} = e_{p,q}$. This object can be reconstructed from the rational object $T_\chi$ (this is shown in [3]) and it does not show properly the structure of $\chi$ over $\mathbb{F}$, being thus $T_\chi$ a more precise invariant of the normalization.

We will show now that from the computation of symbolic Hamburger-Noether expressions one gets, as a byproduct, the desingularization of the curve (see [3] and [4] for more details). In fact, for simplicity consider again the case of only one rational branch. Let $f \in \mathbb{F}[X,Y]$ be a local equation of $\chi$ at $P$, supposed rational and $P = (0,0)$ in the affine coordinates $X,Y$ (otherwise we consider an initial symbolic extension $\mathbb{F}'$ instead of $\mathbb{F}$). If we write $l = qn + h$ as in the previous section, then the first $q$ infinitely near points $P = P_0, P_1, \ldots, P_{q-1}$

\footnote{Notice that such branches are in bijection with the rational branches at $P$ of the corresponding curve obtained by blowing-up $P$, and also with the closed points over $P$ of the normalization.}
are rational over $\mathbb{F}$, being $P_i = (0, 0)$, for $0 \leq i \leq q - 1$, in the local affine coordinates $\{X, \frac{Y}{X}\}$ at $P_i$.

If $h = 0$, then $P_q$ has the symbolic field $\mathbb{F}_1 = \mathbb{F}[\lambda]/(\Phi_\Delta(\lambda))$ as residual field, being $P_q = (0, 0)$ in the local affine coordinates related to $\mathbb{F}_1$ given by $\{X, \frac{Y}{X^\delta}\}$, $\delta$ being a symbolic root of the characteristic polynomial $\Phi_\Delta(\lambda)$.

If $h > 0$, then the new coordinates are $\{Z_1, Z_0\}$, $P_q$ is rational over $\mathbb{F}$ and $P_q = (0, 0)$ in these coordinates, $Z_1 = 0$ being now the exceptional divisor instead of $Z_0 = 0$. Anyway, by doing successively the above changes of variables one easily gets the corresponding total, strict or virtual transform of any divisor.

With this notation, the edges $\overrightarrow{p_{i-1}p_i}$ of the resolution forest $T_\chi$, $p_j$ corresponding to $P_j$, have weight 1 either if $i < q$ or if $i = q$ and $h > 0$, and weight $d$ if $i = q$ and $h = 0$. The value $e \cdot n'$ in each step is just the order of that branch at $P_0, \ldots, P_{q-1}$, and $n = d \cdot e \cdot n'$ is the multiplicity. The weights at $P_q$ appear in the next step of the algorithm, where $P_q$ plays the role of $P_0 = P$, and so on.

When one gets the trivial polygon by iterating this method, one obtains all the infinitely near points with all the weights of the combinatorial object $T_\chi$. When the procedure ends, one has the coordinates $\{Z_r, Z_{r-1}\}$ and the local equation $g(Z_r, Z_{r-1})$, satisfying $\frac{\partial g}{\partial Z_{r-1}}(0, 0) \neq 0$. Doing $s$ additional transformations of type $T_1$ one obtains the embedded resolution, being $Z^*_s$ the initial form of $g(Z_r, Z_{r-1})$.

In the case of several branches, the resolution can be obtained taking into account that there are as many irreducible factors of the characteristic polynomial as infinitely near points in the exceptional divisor, and the corresponding symbolic roots yield suitable local coordinates for such points, that is, everything can be done, branch by branch, with an algorithm in form of a tree.

**Example 3.1** In the example 2.4, one obtains the resolution tree of $\chi$ at $P$ as the sequence of points

$$P \equiv p_0 \rightarrow p_1 \rightarrow \ldots \rightarrow p_{21} \rightarrow p_{22} \equiv q$$

corresponding to rational points of multiplicity $e_{p_i,q} = 2$ if $i = 0, \ldots, 21$, and $e_{p_{22},q} = 1$, the weights of all the edges being 1 as the initial weight, since we have never enlarged the base field.

A useful information which one can derive from $T_\chi$ is the adjunction divisor $A$ of the singular plane curve $\chi$, and hence the so-called adjoint divisors. The adjunction divisor of $\chi$ is nothing but the effective divisor given by the conductor ideal $C_\chi$ on $\tilde{\chi}$ (notice that $\tilde{\chi}$ is the blowing-up of $C_\chi$). It can be computed from the resolution forest as follows.

Let $q_1, \ldots, q_l$ be the branches of $T_\chi$, and let $Q_1, \ldots, Q_l$ be the corresponding points of $\tilde{\chi}$, by identifying $\tilde{\chi}$ to $\chi_N$. 
For each vertex \( p \in \mathcal{T}_\chi \) set

\[
e_p = \sum_{j=1}^{l} e_{p,q_j}
\]

with the convention that \( e_{p,q_j} = 0 \) if the branch \( q_j \) does not pass through the vertex \( p \). Then, the adjunction divisor is given by

\[
\mathcal{A} = \sum_{j=1}^{l} \left( \sum_{p \in \mathcal{T}_\chi} m_{p,q_j}(e_p - 1) \right) Q_j
\]

In the sequel, we will denote \( d_Q = d_q = \sum_{p \in \mathcal{T}_\chi} m_{p,q_j}(e_p - 1) \). One has

\[
\deg \mathcal{A} = \sum_{p \in \mathcal{T}_\chi} e_p(e_p - 1) \deg P
\]

since \( \deg Q_j = \deg P \cdot [k(Q_j) : k(P)] \) and \( e_{p,q_j} = m_{p,q_j} \cdot [k(Q_j) : k(P)] \) for \( p \) in the branch \( q_j \).

Now if we want to give the definition of what an adjoint divisor is, we need first some notations. Let \( P \) be a closed point of the curve \( \chi \) embedded in \( S = \mathbb{P}^2 \) and consider the domains \( R = \mathcal{O}_{\chi_P} \) and \( \mathcal{O} = \mathcal{O}_S \). Thus, the conductor

\[
\mathcal{C}_P = \mathcal{C}_{\mathcal{R}/R} = \{ z \in \mathcal{R} | z \mathcal{R} \subseteq R \}
\]

is by definition an ideal in \( R \) and \( \mathcal{R} \) at the same time. As an ideal of \( R \), there exists another ideal \( \mathfrak{A}_P \) containing the kernel of the natural morphism \( \mathcal{O} \rightarrow R \) such that \( \mathfrak{A}_P \) is applied onto \( \mathcal{C}_P \) by this morphism. The ideal \( \mathfrak{A}_P \) is called the ideal of germs of adjoints of \( \chi \) at \( P \) over \( \mathbb{F} \). In a global situation, the ideal of adjoints \( \mathfrak{A} \) is defined as a sheaf of ideals of \( \mathcal{O}_S \) over \( S \) whose stalk at \( P \) is either \( \mathfrak{A}_P \) when \( P \in \chi \), or \( \mathcal{O}_{S,P} \) otherwise. In fact, for \( P \in \chi \) one has \( \mathfrak{A}_P = \mathcal{O}_{S,P} \) if and only if \( P \) is non-singular; hence \( \mathfrak{A} \) has a finite support and can be given by the finite set of data \( \{ \mathfrak{A}_P | P \in \text{Sing}(\chi) \} \).

On the other hand, with the above notations and following \(^4\), for \( P \in S \) and \( h \in \mathcal{O}_{S,P} \) with \( e_P(h) \geq e_p - 1 \) given, denote by \( H = \text{div}(h) \) the divisor defined by \( h \) on the surface \( S \), and consider \( \pi_P^* H = \text{div}(\pi_P^* h) = (e_p - 1) E_P + \hat{H} \), where \( \pi_P \) denotes the blowing-up at \( P \) and \( E_P \) the exceptional divisor of \( \pi_P \). Then \( \hat{H} \) is called the virtual transform of \( H \) (with respect to \( P \) and the weight \( e_p \)), and the multiplicity \( \mu_q(h) \triangleq e_q(\hat{H}) \) (for \( q \) proximate to \( p \), that is, the corresponding point \( Q \) is in the strict transform of the exceptional divisor created in the blowing-up of the point \( P \)) is called the virtual multiplicity of \( h \) at \( q \) related to \( e_p - 1 \). By induction, if one substitutes the surface \( S \) by the corresponding one
at the inductive step, and by taking the successive virtual transforms related to the values $e_r - 1$, one has in a similar way the concept of virtual multiplicity at any $q$ in $T_\chi$, where we take in successive steps the virtual multiplicity $\mu_r(h)$ instead of the value $e_p(h)$ taken in the first step. Then, one has

$$2p = \{ h \in OS, p \mid \mu_q(h) \geq e_q - 1 \ \forall q \geq p, \ q \in T_\chi \}$$

As a consequence, for a $\mathbb{F}$-rational divisor $D$ on the surface $S$ one has four equivalent ways to say that $D$ is an adjoint divisor, as follows:

(i) Adjoint by branches: if the intersection multiplicity of $D$ and $\chi$ at every rational branch $q$ of $\chi$ is at least the coefficient $d_q$ that appears in the adjunction divisor $A_\chi$.

(ii) Divisorial adjoint: if $N^*D \geq A$, where $N = i \circ n$, $n$ being the normalization of $\chi$ and $i$ the embedding of $\chi$ in $S$.

(iii) Arithmetic adjoint: if the local equation of $D$ at every point $P \in \chi$ is in $2p$.

(iv) Geometric adjoint: if the virtual multiplicity of $D$ at every infinitely near point corresponding to $T_\chi$ is greater or equal than the effective multiplicity of the strict transform of $\chi$ at this point minus one.

Adjoint are useful to describe the vector space of finite dimension

$$L(G) = \{ f \in \mathbb{F} (\tilde{\chi}) \mid (f) + G \geq 0 \} \cup \{ 0 \}$$

for an arbitrary $\mathbb{F}$-rational divisor $G$ on $\tilde{\chi}$, as derived from the classical Brill-Noether theorem. Assume that $\chi$ is given by the homogeneous polynomial $F \in \mathbb{F}[X_0, X_1, X_2]$. Take a divisor $G$ on $\tilde{\chi}$ that is rational over $\mathbb{F}$ and consider a form $H_0 \in \mathbb{F}[X_0, X_1, X_2]$ of degree $n$, with $n \in \mathbb{N} \setminus \{ 0 \}$, defined over $\mathbb{F}$, not divisible by $F$ and satisfying

$$N^*H_0 \geq G + A$$

Then, the Brill-Noether theorem states that

$$L(G) = \{ \frac{h}{h_0} \mid H \in \mathcal{F}_n, \ H \notin F \cdot \mathbb{F}[X_0, X_1, X_2] \text{ and } N^*H + G \geq N^*H_0 \} \cup \{ 0 \}$$

where $h, h_0 \in \mathbb{F}(\chi)$ denote respectively the rational functions $H, H_0$ restricted on $\chi$, and $\mathcal{F}_n \subset \mathbb{F}[X_0, X_1, X_2]$ denotes the set of forms of degree $n$.

This result allows us to compute a basis of $L(G)$ over $\mathbb{F}$ by means of the following algorithm, $G$ being an arbitrary rational divisor.
Algorithm 3.2 (Brill-Noether algorithm)

For a given $G$, define $J = G + A$ and $J_+ = \max \{ J, 0 \}$.

1. Take a large enough $n \in \mathbb{N}$ such that there exists $H \in \mathcal{F}_n$ not divisible by $F$ with $N^*H \geq J_+$, for instance $n > \max \left\{ m - 1, \frac{m}{2} + \frac{\deg J_+ - 3}{2} \right\}$, $m = \deg F$ being the degree of $\chi$ (see [14]).

2. Compute a basis over $\mathbb{F}$ of the vector space $V = \{ H \in \mathcal{F}_n : F | H \text{ or } N^*H \geq J_+ \} \cup \{ 0 \}$

3. Assumed $n \geq m$, compute a set of forms of $\mathcal{F}_n$ giving a basis over $\mathbb{F}$ of the vector space $V' = V/W$, where $W = \{ A \in \mathcal{F}_n : F | A \} \cup \{ 0 \}$.

4. Choose an element $H_0 \in V \setminus W$ and compute the divisor $N^*H_0$.

5. Compute a set of forms of $\mathcal{F}_n$ being linearly independent over $\mathbb{F}$ which generate (modulo $W$) the vector space of forms $H$ satisfying $N^*H \geq A + R$ (or $H = 0$), where $R = N^*H_0 - J$.

6. If $\{ H_1, \ldots, H_s \}$ is the basis obtained in (5) and for $i = 0, 1, \ldots, s$ we denote by $h_i \in \mathbb{F}(\chi)$ the functions $H_i$ restricted to $\chi$, then

$$B = \left\{ \frac{h_1}{h_0}, \ldots, \frac{h_s}{h_0} \right\}$$

is a basis of $\mathcal{L}(G)$ over $\mathbb{F}$.

This algorithm also allows us to determine a basis for the space

$$\Omega(G) = \{ \omega \in \Omega(\chi) | (\omega) \geq G \} \cup \{ 0 \}$$

In fact, for any non-zero differential form $\eta$ defined over $\mathbb{F}$ denote $K = (\eta)$ the corresponding canonical divisor, which is rational over $\mathbb{F}$; then one has the $\mathbb{F}$-isomorphism

$$\mathcal{L}(K - G) \rightarrow \Omega(G)$$

$$f \mapsto f\eta$$

for any rational divisor $G$. Hence, if $\{ f_1, \ldots, f_s \}$ is a $\mathbb{F}$-basis for $\mathcal{L}(K - G)$ then the set $\{ f_1\eta, \ldots, f_s\eta \}$ is a basis over $\mathbb{F}$ for $\Omega(G)$.

Notice that if we want this algorithm to be effective we must solve the following related problems:

(a) Compute the adjunction divisor $A$ for a plane curve $\chi$, what can be done from the resolution tree at every singular closed point of $\chi$. Notice that this was already done by means of symbolic Hamburger-Noether expressions.
(b) Compute the intersection divisor $N^*H$ of a homogeneous polynomial $H$ and the curve $\chi$, that is, the value $\nu_Q(H)$ at every rational branch $Q$ of $\chi$. This can be solved by means of the primitive rational parametrizations of such branches also given by their corresponding symbolic Hamburger-Noether expressions or, more precisely, by lazy evaluation of these parametrizations, (i.e. evaluation step by step whenever necessary for the own computation), since the searched values only depend on the first terms of such expansions.

(c) For a given rational divisor $J$ and a suitable $n \in \mathbb{N}$, compute a basis over $\mathbb{F}$ for the vector space

$$V(J, n) = \{ H \in \mathcal{F}_n : F|H \text{ or } N^*H \geq J \} \cup \{0\}$$

which is the aim of the next section. Note that it also can be done by means of the resolution trees and the rational parametrizations of $\chi$ computed again from the symbolic Hamburger-Noether expressions.

4 Computing bases for $\mathcal{L}(G)$

For a given plane curve $\chi$, the computation of a basis for $\mathcal{L}(G)$, $G$ being a rational divisor over $\bar{\chi}$, is reduced, by the Brill-Noether theorem, to compute bases for spaces of adjoints of a suitable degree $n$. We show in this section how to impose the required adjunction conditions from the symbolic Hamburger-Noether expressions at every rational branch of $\chi$, by using the classical ideas of Enriques testing passing conditions.

In practice we know the polynomial $F(X_0, X_1, X_2) \in \mathbb{F}[X_0, X_1, X_2]$ defining the absolutely irreducible curve $\chi$ in the projective plane, and we have the data of a divisor $G$ that is rational over $\mathbb{F}$, involving a finite number of rational branches of $\chi$ and their corresponding coefficients.

We first take a value of $n$ such that there exists an adjoint of degree $n$ satisfying $^5$

$$N^*H_0 \geq A + G$$

Now computing the residue $R = N^*H_0 - A - G$ one has to describe the space of homogeneous polynomials $H$ of degree $n$ such that $N^*H \geq A + R$, modulo the multiples of $F$.

The problem of finding $H_0$ consists just of imposing to $H_0$ the condition of being an adjoint together with having some extra zeros on the divisor $G$. On the other hand, in order to go on with the Brill-Noether algorithm to describe $\mathcal{L}(G)$ the problem is again the same but taking $R$ instead of $G$. Thus we have to study the conditions imposed by the inequality $N^*H \geq A + R$ on a homogeneous polynomial $H$ of degree $n$, $R$ being an arbitrary effective divisor.

$^5$We assume $G \geq 0$, but in general one can consider the divisor $J_+$ instead of $J = A + G$, according to the above notations.
There are two ways to proceed. For the first one, assume that from the symbolic Hamburger-Noether expressions we have computed by lazy evaluation the primitive rational parametrizations \((X(Z_r), Y(Z_r))\) given by the corresponding Hamburger-Noether expansions at every branch involved in the support of the adjunction divisor \(A\) and \(R\).

The Dedekind formula allows us to find the coefficient \(d_q\) of \(A\) at the rational branch \(q\), which is given by

\[
d_q = \text{ord}_t \left( \frac{f_Y(X(t), Y(t))}{X'(t)} \right) = \text{ord}_t \left( \frac{f_X(X(t), Y(t))}{Y'(t)} \right)
\]

\((X(t), Y(t))\) being a primitive rational parametrization of \(q\) (notice that either \(X'(t) \neq 0\) or \(Y'(t) \neq 0\)). The algorithm to compute the symbolic Hamburger-Noether expressions provides us with as many terms of such a parametrization as we need to obtain the above orders in \(t\), by successive substitution and lazy evaluation.

Now we consider the coefficient \(r_q\) of \(R\) at \(q\), and thus the local condition at \(q\) imposed to \(H\) by the inequality \(N^*H \geq A + R\) is given by

\[
\text{ord}_h h(X(t), Y(t)) \geq d_q + r_q
\]

\(h\) being the local affine equation of \(H\) in terms of the coordinates \(X, Y\) at the corresponding point \(P\). Again a suitable number of steps of the lazy evaluation suffices to describe the first \(d_q + r_q\) monomials of the Taylor expansion of \(h(X(t), Y(t))\) as a function of the indeterminate coefficients of \(H\), whose vanishing gives the required linear conditions, taking all the possible branches \(q\) in the support of \(A\) and \(R\).

The second way is just the imposition of virtual passing conditions through the infinitely near points of the configuration of resolution \(\mathcal{E}_\chi\) with virtual multiplicities \(e_p - 1\), what also yields linear conditions on \(H\). The resolution configuration \(\mathcal{E}_\chi\) stands here for the set of points \(P\) (at the successive blowing-ups) corresponding to the vertices \(p \in T_\chi\). Notice that from the symbolic Hamburger-Noether expressions one can derive not only the total information of \(\mathcal{E}_\chi\) but also the information on bigger configurations \(\mathcal{D}\) obtained by adding to \(\mathcal{E}_\chi\) finitely many points with multiplicity 1 at the end of every branch of \(T_\chi\). Furthermore, the algorithm to compute the symbolic Hamburger-Noether expressions gives us also the weights for the resolution tree and local coordinates at every infinitely near point, as we have seen in the previous section. On the other hand, we say that a divisor \(H\) passes (virtually) through a configuration \(\mathcal{D}\) of infinitely near points of \(\chi\) with virtual multiplicities \(\{\mu_P \mid P \in \mathcal{D}\}\) if the virtual multiplicity of \(H\) at every point \(P\) of \(\mathcal{D}\) (as defined in section 3) is greater or equal than \(\mu_P\), generalizing the concept of geometric adjoint given in the above section.

The total number of imposed linear conditions is

\[
\sum_{P \in \mathcal{E}_\chi} \frac{e_p(e_p - 1)}{2} \deg P = \frac{1}{2} \deg A
\]
since the condition $\mu_p(h) \geq e_p - 1$ is equivalent to the vanishing of $\frac{e_p - 1}{2} e_p$ coefficients, what yields this number of conditions over a field isomorphic to the residual field $k(P)$, and thus $\frac{1}{2} e_p (e_p - 1) \deg P$ conditions over the base field $\mathbb{F}$.

Moreover, such conditions are linear independent whenever $n \geq m - 3$, because of the Noether’s adjunction theorem, which is referred to the next section, and the virtual transform $H$ of $H$ can be computed from the symbolic Hamburger-Noether expressions. Note that the first $e_p - 1$ terms of the Taylor expansion of $\tilde{H}(X(t), Y(t))$ vanish.

Now we must add to $N^*H \geq A$ the conditions given by $R$. If $\text{supp } R$ does not contain any singular point (that is, the adjoint defined by $H_0$ passes through $C_\chi$ with actual multiplicities $e_p - 1$), then the condition $N^*H \geq A + R$ is equivalent to $N^*H \geq A$ and $N^*H \geq R$ at the same time, and thus the method is just the same as before. This situation can be assumed if $n$ is large enough, by a theorem of Serre about the vanishing of the cohomology, but in practice the estimate of such values of $n$ is very hard and we will give an alternative method to proceed.

Denote by $r_q$ the coefficient of $R$ at the rational branch $q$, being $r_q \geq 0$ by assumption. We will show that $N^*H \geq A + R$ can also be described with virtual passing conditions on $H$. In fact, consider the configuration $C^+_{\chi + R}$ given by adding to $C_\chi$ the first $r_q$ points of multiplicity 1 in the sequence of infinitely near points corresponding to the branch $q$, for all $q$ in the support of $R$.

Recall that the condition $N^*H \geq A + R$ can be written in terms of the local conditions $\text{ord}_h(X_q(t), Y_q(t)) \geq d_q + r_q$ for each rational branch $q$ in $C^+_{\chi + R}$, $(X_q(t), Y_q(t))$ being a primitive rational parametrization corresponding to $q$. From the inequalities ($\star$) one gets the following result.

**Proposition 4.1** Under the above conditions, the inequality $N^*H \geq A + R$ is equivalent to the condition that the hypersurface defined by $H$ passes through the points of $C^+_{\chi + R}$ with virtual multiplicities $e_p - 1$ if $p \in C_\chi$ and 1 if $p \in C^+_{\chi + R} \setminus C_\chi$.

**Proof:**

If $N^*H \geq A + R$ then $N^*H \geq A$, since $R \geq 0$. Thus, $H$ passes through the points $p \in C_\chi$ with virtual multiplicities $e_p - 1$. On the other hand, the formula ($\star$) shows that the virtual transform of $H$ at the first point of multiplicity 1 corresponding to the branch $q$ has intersection multiplicity at least $r_q$ with the strict transform of this branch; hence, $H$ passes through the last $r_q$ points of $C^+_{\chi + R} \setminus C_\chi$ corresponding to $q$ with virtual multiplicity 1.
Conversely, if $H$ passes through the points of $\mathcal{C}_\chi^{+,R}$ with the above virtual multiplicities, then $(\ast)$ is satisfied for any branch $q$ in $\mathcal{C}_\chi^{+,R}$.

\[ \square \]

**Remark 4.2** The above result is considered in [4] in the case $r_q = e_{N(q),q} - 1$ to study the behaviour of the polar curve of a plane curve in characteristic $0$. We have proved that in fact the result is also true in any characteristic and for arbitrary values of $r_q$ whenever $r_q \geq 0$. Notice that (in total) one considers a number of linear conditions equal to $\frac{1}{2}\deg A + \deg R$, but they may not be linearly independent.

**Remark 4.3** The theory of Enriques on plane curves with assigned singularities or, in more modern terms, the theory of Zariski-Lipman of complete ideals, allows us to substitute the weights $e_p - 1$ in $\mathcal{C}_\chi$ and $1$ in $\mathcal{C}_\chi^{+,R} \setminus \mathcal{C}_\chi$ by other weights $\overline{e_p}$ over $\mathcal{C}_\chi^{+,R}$ satisfying the so-called proximity inequalities, that is

\[ \overline{e_p} \geq \sum_{r \rightarrow p} e_r \quad \forall p \in \mathcal{C}_\chi^{+,R} \]

This substitution can be done by means of a combinatorial algorithm known as the principle of discharge (see for instance [4]). This algorithm is combinatorial in the sense that one can describe it just in terms of the embedded resolution forest associated to the configuration $\mathcal{C}_\chi^{+,R}$.

Also notice that the $r_q$ added points of multiplicity 1 in each branch $q$ can be deduced in practice from the symbolic Hamburger-Noether expressions computing the first $r_q$ terms of the Taylor expansion of the implicit function given by the polynomial $g(Z_r, Z_{r-1})$. As a consequence of all what we have exposed so far, we can state the following result.

**Theorem 4.4** For any absolutely irreducible plane curve $\chi$ defined over a finite field $\mathbb{F}$ and given by a polynomial $F(X_0, X_1, X_2) \in \mathbb{F}[X_0, X_1, X_2]$, and for any rational divisor $G$ on $\hat{\chi}$, there exists an algorithm which computes bases over $\mathbb{F}$ for $\mathcal{L}(G)$ consisting of the following steps:

1. **(1)** Compute the closed points of the projective plane which are singular for the curve $\chi$.

2. **(2)** Compute the symbolic Hamburger-Noether expressions at every singular closed point of $\chi$ by using successive symbolic extensions of the base field $\mathbb{F}$.
(3) Compute an adjoint $H_0$ for $\chi$ of degree $n \geq m - 3$ satisfying $N^*H_0 \geq A + G$, where $A$ is the adjunction divisor of the curve $\chi$ computed by means of the step (2), and then compute the residue $R = N^*H_0 - A - G$.

(4) Describe the linear conditions $N^*H \geq A + R$ in terms of the coefficients of a generic form $H$ of degree $n$ and the lazy parametrizations of the rational branches of $\chi$ computed also from (2), by using either the method given in this section or the principle of discharge.

(5) Apply the previous steps to get, by using the Brill-Noether method, a $\mathbb{F}$-basis for the vector space $L(G)$.

Remark 4.5

i) The computation of the needed symbolic extensions of $\mathbb{F}$ requires factorization of polynomials in one variable, what has an effective solution in computational algebra.

ii) In fact, we could apply the method to any computable perfect field $\mathbb{F}$, that is, when the operations in $\mathbb{F}$ can be done in an effective way (for instance, when $\mathbb{F}$ is any field of algebraic numbers).

5 Computing Weierstrass semigroups

As we will see later, the decoding procedure of Feng and Rao is just based on the computation of a basis for $L(lP)$, $P$ being a rational point of $\tilde{\chi}$, in the way that if $l \in \Gamma_P$, the Weierstrass semigroup $\Gamma_P$ consisting of the Weierstrass non-gaps at $P$, then such a basis is obtained by adding to a basis of $L((l - 1)P)$ a function $f_l$ with a unique pole at $P$ of order $l$. What we are going to do now is to show how one can compute the semigroup $\Gamma_P$ and the functions $f_l$ in a quite general situation by using the theory of adjoints. For this, we make use of the classical adjunction theorem.

Denote by $A_n$ the set of adjoints of degree $n$ of the curve $\chi$ embedded in $\mathbb{P}^2$ and denote $N = i \circ n$, $n$ being the normalization of $\chi$ and $i$ the embedding of $\chi$ in $\mathbb{P}^2$. For every $D \in A_n$ one can consider its pull-back, which is given by $N^*D = A + D'$ for certain $D'$. The **adjunction theorem**, due to Noether, says that if $n + 3 \geq \deg \chi$ the divisors $D' = N^*D - A$ for $D \in A_n$ are exactly those in the complete linear system $|K_{\tilde{\chi}} + (n - m + 3)L|$, $K_{\tilde{\chi}}$ being a canonical divisor on $\tilde{\chi}$, $L$ the hyperplane section divisor and $m = \deg \chi$ (see [12] for details).

This result means that local adjunction conditions are linearly independent if imposed on divisors of large enough degree, and this independence is in fact global, that is, when imposed on all the points of $\chi$ at the same time. In particular, if $n = m - 3$ one obtains the following result.
Proposition 5.1 For \( n = m - 3 \) one has an \( \mathbb{F} \)-isomorphism of complete linear systems
\[
A_n \to |K_{\tilde{\chi}}|
\]
\[
D \mapsto N^*D - A
\]
Notice that this map is injective since \( n < m \), and the dimension over \( \mathbb{F} \) of the vector space of forms of dimension \( m - 3 \) in three variables equals the arithmetic genus \( p_a(\chi) \). But now the total number of linearly independent adjunction conditions is \( \frac{1}{2} \deg A \), and thus the formula of the geometric genus \( g(\chi) = p_a(\chi) - \frac{1}{2} \deg A \) can be seen as a problem of virtual conditions through the configuration of resolution \( C_{\chi} \).

Under the same hypothesis as in the previous sections, assume that \( G = lP \), where \( l \) is a non-negative integer and \( P \) is a rational point of \( \tilde{\chi} \), that is, a rational branch defined over \( \mathbb{F} \) at a certain point of the curve \( \chi \). Then the Riemann-Roch formula can be applied to the divisors \( lP \) and \( (l - 1)P \), what yields the equality
\[
(\ell(lP) - \ell((l - 1)P)) - (i(lP) - i((l - 1)P)) = 1
\]
being \( 0 \leq \ell(lP) - \ell((l - 1)P) \leq 1 \) and \( -1 \leq i(lP) - i((l - 1)P) \leq 0 \). Therefore one has \( l \notin \Gamma_P \) if and only if \( l \geq 1 \) and there exists a differential form which is regular on \( \tilde{\chi} \) and with a zero at \( P \) of order \( l - 1 \).

Notice that \( l \in \Gamma_P \) if \( l \geq 2g \). From these remarks one can easily prove the following result by using the proposition 5.1.

Proposition 5.2 Let \( l \in \mathbb{Z} \) such that \( 1 \leq l \leq 2g - 2 \). Then:

(a) \( l \notin \Gamma_P \) if and only if there exists a homogeneous polynomial \( H_0 \) of degree \( m - 3 \) with \( N^*H_0 \geq A + (l - 1)P \) such that \( P \) is not in the support of the effective divisor \( N^*H_0 - A - (l - 1)P \).

(b) There exists \( l' \geq l \) with \( l' \notin \Gamma_P \) if and only if there exists a homogeneous polynomial \( H_0 \) of degree \( m - 3 \) such that \( N^*H_0 \geq A + (l - 1)P \).

As a consequence, the following result provides us with an effective method to do the preprocessing of one-point codes by using plane models for the used curve in a quite general situation.

Theorem 5.3 Under the same assumptions as above, there exists an algorithm founded in the theory of adjoints to compute the Weierstrass semigroup \( \Gamma_P \) together with functions \( f_l \) with a pole at \( P \) of order \( l \) and regular on \( \tilde{\chi} \setminus \{P\} \), for all \( l \in \Gamma_P \).

Proof:
Computing the Weierstrass semigroup:

Taking \( G = (l - 1)P \) instead of the divisor \( R \) in proposition 4.1 and using the configuration \( \mathcal{C}_\chi^+, G \) one can impose the linear conditions given by \( N^*H \geq A + (l - 1)P \) on forms \( H \) of degree \( m - 3 \), which are equivalent to virtual passing conditions through \( q \in \mathcal{C}_\chi \) with multiplicities \( e_q - 1 \) and through \( q \in \mathcal{C}_\chi^+ \) with multiplicity 1.

Then for \( l \) increasing from \( l = 0 \) (always in \( \Gamma_P \)) one imposes successively the linear conditions given by \( N^*H \geq A + lP \), adding one condition in each step. Thus, the added condition given by the new \( l \) is linearly independent of the previous conditions, by using the proposition 5.1, if and only if \( l \notin \Gamma_P \). All the \( g \) gaps of \( \Gamma_P \), and hence the semigroup itself, are computed in at most \( 2g \) steps.

Computing the functions \( f_l \):

There are two ways to proceed. One way is to compute the functions \( f_l \) for all \( l \leq \tilde{l} \), \( \tilde{l} \) being the largest non-gap that is needed in the computations with the considered one-point code. The other way is to compute first a generator system \( \mathcal{C}_\chi^+ \) for the Weierstrass semigroup and then give the functions only for all \( l \) in such a system, \( \tilde{l} \) being now the largest generator. Anyway, the method described below, which is a suitable application of the Brill-Noether algorithm, works in both cases.

(i) Compute a homogeneous polynomial \( H_0 \) not divisible by \( F \) of large enough degree \( n \) satisfying \( N^*H_0 \geq A + lP \), and take \( l \in \Gamma_P \) with \( l \leq \tilde{l} \).

(ii) Denoting \( N^*H_0 = A + lP + R_l \) one has \( R_{l-1} = R_l + P \), \( R_l \) being effective. Thus, for decreasing \( l \) we can impose the conditions \( N^*H_0 \geq A + R_l \) by means of the proposition 4.1 in order to find a homogeneous polynomial \( H_l \) of degree \( n \) not divisible by \( F \) such that \( N^*H_l \geq A + R_l \) but not satisfying the condition \( N^*H_0 \geq A + R_{l-1} \).

(iii) Thus, the function \( f_l = H_l/H_0 \) restricted to \( \chi \) is regular on \( \chi \setminus \{P\} \) and has a pole at \( P \) of order \( l \).

\( \square \)

Example 5.4 Let \( \chi \) be the Klein quartic over \( \mathbb{F}_2 \) given by the equation

\[
F(X, Y, Z) = X^3Y + Y^3Z + Z^3X = 0
\]

\( ^6 \) This generator system may be the set of all the primitive elements in the Weierstrass semigroup, which is contained in the set of the first \( g + e \) non-gaps, \( e \) being the minimum non-zero element in the semigroup. However, it is much better to take an Apéry system since then one can easily compute the Feng-Rao distance of the code (see \( ^6 \)).
whose adjunction divisor is $A = 0$, since $\chi$ is non-singular. We are going to compute the Weierstrass semigroup at $P = (0 : 0 : 1)$ with the above method.

Since $P$ is non-singular one easily obtains by lazy evaluation a local parametrization of $\chi$ at $P$ given by

$$
\begin{align*}
X(t) &= t^3 + t^{10} + \ldots \\
Y(t) &= t
\end{align*}
$$

In order to get the gaps of $\Gamma_P$ one uses adjoints of degree $m - 3 = 1$, whose generic equation is given by

$$
H(X, Y, Z) = aX + bY + cZ
$$

and substituting the first terms of the local parametrization at $P$ we get

$$
h(X(t), Y(t)) = c + bt + at^3 + at^{10} + \ldots
$$

and proceed as in the above theorem:

- $l = 1$ is obviously the first gap, since $g = p_a(\chi) = 3 > 0$, but anyway it can also be checked by the method, since $l = 0$ impose no condition whereas $l = 1$ impose the condition $\text{ord}_t h(X(t), Y(t)) \geq 1$, which is equivalent to $c = 0$.

- For $l = 2$, the inequality $\text{ord}_t h(X(t), Y(t)) \geq 2$ is equivalent to the conditions $c = b = 0$, which are linearly independent of those imposed by $l = 1$, and thus $l = 2$ is a new Weierstrass gap.

- If $l = 3$, then $\text{ord}_t h(X(t), Y(t)) \geq 3$ is again equivalent to $c = b = 0$. Therefore the new condition depends on the previous one and one has $3 \in \Gamma_P$.

- At last, when $l = 4$ the condition $\text{ord}_t h(X(t), Y(t)) \geq 4$ is equivalent to $c = b = a = 0$ and one obtains $l = 4$ as the third gap of $\Gamma_P$ and the procedure ends.

Thus the Weierstrass gaps are $l = 1, 2, 4$ and the minimal generator system is then $\{3, 5, 7\}$. We are going to compute a function $f_l$ for each of these three generators also with the method described above.

We apply first the Brill-Noether algorithm to $G = 7P$ to obtain a form $H_0$ of degree $n = 4$ not divisible by $F$ such that $\mathbf{N}^*H_0 \geq J_+ = J = G = 7P$. That is, taking $H_0$ as a generic form of degree 4 with coefficients as variables, the needed condition is equivalent to $\text{ord}_t H_0(X(t), Y(t), 1) \geq 7$, being $(X(t), Y(t))$

\footnote{Notice that every plane curve is adjoint to $\chi$, since $A = 0$.}

\footnote{Notice that this semigroup is not symmetric, since the conductor is $C = 5 < 6 = 2g$, and in fact this set of generators is the Apéry system related to $\ell = 3$.}
the above local parametrization. This can be easily tested with a computer and one gets for instance the form \( H_0 = X^2YZ \), which is not divisible by \( F \).

Now in order to compute \( \mathbf{N}^*H_0 \) we use the symmetry of \( F \) with respect to the three variables to get local parametrizations at the points \( Q_1 = (1 : 0 : 0) \) and \( Q_2 = (0 : 1 : 0) \). Thus, one easily obtains

\[
\mathbf{N}^*H_0 = 2\mathbf{N}^*(X) + \mathbf{N}^*(Y) + \mathbf{N}^*(Z) = 7P + 4Q_1 + 5Q_2
\]

Then, in order to get \( f_7 \) we compute \( R_7 = 4Q_1 + 5Q_2 \), and find with the above method a form \( H_7 \) of degree 4 not divisible by \( F \) such that \( \mathbf{N}^*H_7 \geq R_7 \) but not satisfying \( \mathbf{N}^*H_7 \geq R_6 = R_7 + P \). This is equivalent to the condition \( \mathbf{N}^*H_7 \geq R_7 \) together with the local condition at \( P \) given by

\[
\text{ord}_t H_7(X(t), Y(t), 1) = 0
\]

obtaining for instance \( H_7 = Z^4 \) and hence \( f_7 = \frac{Z^3}{X^2Y} \).

In a similar way one checks that \( H_5 = Y^2Z^2 \) satisfies \( \mathbf{N}^*H_5 \geq R_5 \) but not \( \mathbf{N}^*H_5 \geq R_4 \), obtaining \( f_5 = \frac{YZ}{X^2} \), and \( H_3 = XYZ^2 \) satisfies \( \mathbf{N}^*H_3 \geq R_3 \) but not \( \mathbf{N}^*H_3 \geq R_2 \), obtaining \( f_3 = \frac{Z}{X} \).

In particular, a basis of \( L(7P) \) over \( \mathbb{F}_2 \) is given by

\[
\{1, Z, Z^{\frac{1}{2}}, \frac{Z^2}{X^{\frac{1}{2}}}, \frac{Z^3}{X^2Y}\}
\]

There is an alternative way to get the functions \( f_i \) from the Brill-Noether algorithm. Assume that a basis \( \{h_1, \ldots, h_s\} \) of \( L(\tilde{l}P) \) over \( \mathbb{F} \) has been already computed and that \( \tilde{l} \) is not a gap. We propose a triangulation method which works by induction on the dimension \( s \) as follows:

1. By computing first the pole orders \( \{-v_P(h_i)\} \) at \( P \), assume that the functions \( \{h_i\} \) are ordered in such a way that these pole orders are increasing in \( i \).

2. At least the function \( h_s \) satisfies \( -v_P(h_s) = \tilde{l} \) and we set \( f_{\tilde{l}} \equiv h_s \). If any other \( h_j \) satisfies the same condition, there exists a non-zero constant \( \lambda_j \) in \( \mathbb{F} \) such that \( -v_P(h_j - \lambda_jh_s) < \tilde{l} \); then we change such functions \( h_j \) by \( g_j \equiv h_j - \lambda_jh_s \) and set \( g_k \equiv h_k \) for all the others. The result now is obviously another basis \( \{g_1, \ldots, g_s\} \) of \( L(\tilde{l}P) \) over \( \mathbb{F} \) but with only one function \( g_s = f_{\tilde{l}} \) whose pole at \( P \) has maximum order \( \tilde{l} \).

3. Since the functions \( g_i \) are linearly independent over \( \mathbb{F} \) and \( -v_P(g_i) < \tilde{l} \) for \( i < s \), one has obtained a basis \( \{g_1, \ldots, g_{s-1}\} \) of \( L(l'P) \) over \( \mathbb{F} \), where \( l' \) denotes the largest non-gap such that \( l' < \tilde{l} \). But now the dimension is \( s - 1 \) and we can continue by induction.
The above procedure also provides us with a function \( f \) for each non-gap \( l \leq \tilde{l} \). In fact, it can be used to compute the Weierstrass semigroup up to an integer \( \tilde{l} \), since the maximum non-gap \( l' \) such that \( l' \leq \tilde{l} \) is just \( \max \{-v_P(h_1), \ldots, -v_P(h_n)\} \), in the above notations, and so on by induction.

## 6 Effective construction of AG codes

Let \( \tilde{\chi} \) be a non-singular projective algebraic curve defined over a finite field \( \mathbb{F} \) such that \( \tilde{\chi} \) is irreducible over \( \mathbb{F} \). In order to define the Algebraic Geometry codes, take \( \mathbb{F} \)-rational points \( P_1, \ldots, P_n \) of the curve and a \( \mathbb{F} \)-rational divisor \( G \) (which can be assumed effective) having disjoint support with \( D = P_1 + \ldots + P_n \), and then consider the well-defined linear maps

\[
\begin{align*}
\text{ev}_D : \mathcal{L}(G) &\rightarrow \mathbb{F}^n \\
f &\mapsto (f(P_1), \ldots, f(P_n))
\end{align*}
\]

and

\[
\begin{align*}
\text{res}_D : \Omega(G - D) &\rightarrow \mathbb{F}^n \\
\omega &\mapsto (\text{res}_{P_1}(\omega), \ldots, \text{res}_{P_n}(\omega))
\end{align*}
\]

One defines the linear codes

\[
\begin{align*}
C_L &\equiv C_L(D, G) \doteq \text{Im}(\text{ev}_D) \\
C_\Omega &\equiv C_\Omega(D, G) \doteq \text{Im}(\text{res}_D)
\end{align*}
\]

The length of both codes is obviously \( n \), and one has \( (C_\Omega) = C_L^\perp \) by the residues theorem. On the other hand, given \( D \) and \( G \) as above there exists a differential form \( \omega \) such that \( C_L(D, G) = C_\Omega(D, D - G + (\omega)) \) and thus it suffices to deal with the codes of type \( C_\Omega \).

Denote by \( k(C) \) and \( d(C) \) the dimension over \( \mathbb{F} \) and the minimum distance of the linear code \( C \) respectively, \( d(C) \) being the minimum number of non-zero entries of a non-zero vector of \( C \). Goppa estimates for \( k(C) \) and \( d(C) \) are deduced from the Riemann-Roch formula as follows (see [19] for further details). If \( 2g - 2 < \deg G < n \); then

\[
\begin{align*}
(1) \quad &\begin{cases}
  k(C_L) = \deg G + 1 - g \\
  d(C_L) \geq n - \deg G
\end{cases} & (2) \quad &\begin{cases}
  k(C_\Omega) = n - \deg G + g - 1 \\
  d(C_\Omega) \geq \deg G + 2 - 2g
\end{cases}
\end{align*}
\]

The main problem to solve for the construction of such codes consists of computing bases for \( \mathcal{L}(G) \) (or \( \Omega(G - D) \)), finding points (rational or not) of the curve and evaluating functions of \( \mathcal{L}(G) \) at some rational points (or computing residues of differential forms in \( \Omega(G - D) \) at those points). Thus, with the assumption of having a (possibly singular) plane model \( \chi \) of the curve \( \tilde{\chi} \), and since the codes of type \( C_L \) and \( C_\Omega \) are not only dual each other but both classes of codes are essentially the same (see [18]), the computational algorithms that are involved in these problems will basically be reduced to the following ones:

1. Find all the closed singular points and all the \( \mathbb{F} \)-rational points of \( \chi \), what can be done by means of Gröbner bases computation (see [14]).
(2) Compute the order of a function at a rational point $P$ and evaluate the function at this point when possible, what can be done from lazy parametrizations at the rational branch corresponding to $P$. More precisely, if $\phi = G/H$ is a quotient of homogeneous polynomials of the same degree in three variables, and $(X(t), Y(t))$ is the rational parametrization obtained from the symbolic Hamburger-Noether expressions for the branch given by $P$, the order can be computed taking at $P$ the corresponding local affine equation $g/h$ of $\phi$ and doing the substitution

$$\frac{g(X(t), Y(t))}{h(X(t), Y(t))} = \frac{a_r t^r + \ldots}{b_s t^s + \ldots}$$

obtaining the order $r - s$ by lazy evaluation. Moreover, if $\phi$ is well-defined at $P$ (what always happens in the applications to Coding Theory), then the evaluation of $\phi$ at $P$ is $0$ if $r > s$, and $a_r/b_s$ if $r = s$, since we actually work at the point $t = 0$.

(3) Find a basis for $\mathcal{L}(G)$ using the Brill-Noether method, what has been presented in this paper in an original way by means of symbolic Hamburger-Noether expressions and testing virtual passing conditions following the Brill-Noether algorithm.

An interesting case is when $G = mP$, $P$ being an extra rational point of $\tilde{\chi}$. In this case the codes $C_m = C_\Omega(D, mP)$ can be decoded by the majority scheme of the Feng and Rao algorithm, which is so far the most efficient method for the considered codes (see [9]).

In order to apply this decoding method, one has to fix for every non-negative integer $i$ a function $f_i$ in $\mathbb{F}(\tilde{\chi})$ with only one pole at $P$ of order $i$ for those values of $i$ for which it is possible, i.e. for the integers in the Weierstrass semigroup $\Gamma = \Gamma_P$ of $\tilde{\chi}$ at $P$. For a received word $y = c + e$, where $c \in C_m$, one can consider the unidimensional and bidimensional syndromes given respectively by

$$s_i(y) = \sum_{k=1}^{n} e_k f_i(P_k) \quad \text{and} \quad s_{i,j}(y) = \sum_{k=1}^{n} e_k f_i(P_k) f_j(P_k)$$

Notice that the set $\{f_i \mid i \leq m, \ i \in \Gamma\}$ is actually a basis for $\mathcal{L}(mP)$ and hence one has

$$C_m = \{y \in \mathbb{F}^n \mid s_i(y) = 0 \ \text{for} \ i \leq m\}$$

thus we can calculate $s_i(y)$ from the received word $y$ as $s_i(y) = \sum_{k=1}^{n} y_k f_i(P_k)$ for $i \leq m$, and such syndromes are called known.

In fact, it is a known fact that if one has a sufficiently large number of unknown syndromes $s_{i,j}(y)$ for $i+j > m$ one could know the emitted word $c$, and all above syndromes can be computed by majority voting (see [3]). In practice, the main problem is computing $\Gamma$ and the functions $f_i$ achieving the values of
the semigroup $\Gamma$ in order to carry out this decoding algorithm. This is just the other problem which has been solved in this paper in a general situation, by using the symbolic Hamburger-Noether expressions and the theory of adjoints.

**Example 6.1** Let $\chi$ be again the curve given in the example 2.4 by the equation

$$F(X, Y, Z) = X^{10} + Y^8Z^2 + X^3Z^7 + YZ^9 = 0$$

defined over $\mathbb{F}_2$ and whose genus is $g = 14$. This curve has 64 affine rational points over $\mathbb{F}_8$ (namely $P_1, \ldots, P_{64}$) and only one point $P = (0 : 1 : 0)$ at infinity, which is the only singular point of $\chi$ and which was treated in the above example. Thus, if one takes an integer $m$ with $26 < m < 64$, one can construct a code $C_m = C_{\Omega}(D, mP)$, where $D = P_1 + \ldots + P_{64}$, whose parameters are $[64, 77 - m, \geq m - 26]$. For example, if $m = 51$ then the dimension is $k = 26$ and $C_{51}$ corrects any configuration of 12 errors. In order to construct such a code, one has first to compute the vector space $\mathcal{L}(51P)$ by the Brill-Noether method and then triangulate to obtain the Weierstrass semigroup and the functions, since this one-point code can be decoded by the Feng-Rao procedure. In order to illustrate the method, we consider a smaller space, namely $\mathcal{L}(15P)$.

By using the symbolic Hamburger-Noether expressions computed in the example 2.4 one gets, by means of the Brill-Noether algorithm, a basis for such a space, namely

$$\{h_1 = 1, h_2 = X + X^5 + Y^4, h_3 = Y + Y^4 + X^5, h_4 = X^5 + Y^4, h_5 = XY^4 + Y^16 + X^{20}\}$$

The values at $P$ of these functions are $0, 12, 12, 13$; thus $13$ is the largest non-gap in the range $[0, 15]$ and $f_{13} = h_5$ is a function achieving such value. We fix now $f_{12} = h_4$ and triangulate $g_2 = h_2 + h_4$ and $g_3 = h_3 + h_4$, according to the triangulation method described in the previous section, and one finally obtains the values $0, 8, 10, 12, 13$ with the corresponding functions. In fact, that is enough to construct the whole Weierstrass semigroup and all the possibly needed functions, since the sequence $\{8, 12, 13\}$ is telescopic and thus generates the semigroup (see [10]). Finally, by evaluating those functions at the points $P_1, \ldots, P_{64}$ one easily obtains a parity check matrix for the code $C_m$.

As a conclusion, our main contribution to the construction of AG codes is a new effective solution to the problems which are involved in such a construction by using symbolic Hamburger-Noether expressions of a plane model for the smooth curve and testing virtual passing conditions, on the basis of the Brill-Noether algorithm. This way is simpler than the usual method of blowing-ups and Puiseux expansions, in the sense that symbolic Hamburger-Noether expressions give at the same time the desingularization and the primitive rational parametrizations for the branches of the plane curve. On the other hand, we have given an effective solution to the general problem of computing the Weierstrass semigroup at a rational branch $P$ of a singular plane model by using
the theory of adjunction, together with functions achieving the pole orders in this semigroup, what is essential in the construction and decoding problem of one-point codes by means of the majority scheme of Feng and Rao.

References

[1] A. Campillo, *Algebroid curves in positive characteristic*, Lecture Notes in Math., vol. 813, Springer-Verlag (1980).

[2] A. Campillo and J. Castellanos, *Curve singularities*, Univ. Valladolid, preprint (1997).

[3] A. Campillo and J. I. Farrán, *Computing Weierstrass semigroups and the Feng-Rao distance from singular plane models*, to appear in “Finite Fields and their Applications” (1999).

[4] E. Casas, *Infinitely near imposed singularities and singularities of polar curves*, Math. Annalen 287, pp. 429-454 (1990).

[5] V. G. Drinfeld and S. G. Vlăduţ, *Number of points of an algebraic curve*, Funktsional’-nyi Analiz i Ego Prilozhenia 17, pp. 53-54 (1983).

[6] D. Duval, *Diverses questions relatives au calcul formel avec des nombres algébriques*, Ph.D. thesis, Univ. Grenoble (1987).

[7] F. Enriques and O. Chisini, *Teoria geometrica delle equazioni e delle funzioni algebriche*, Bologna (1918).

[8] J. I. Farrán, *Construcción y decodificación de códigos algebro-geométricos a partir de curvas planas: algoritmos y aplicaciones*, Ph.D. thesis, Univ. Valladolid (1997).

[9] G. L. Feng and T. R. N. Rao, *Decoding algebraic-geometric codes up to the designed minimum distance*, IEEE Trans. Inform. Theory 39, pp. 37-45 (1993).

[10] A. García and H. Stichtenoth, *A tower of Artin-Schreier extensions of function fields attaining the Drinfeld-Vlăduţ bound*, Inventiones Mathematicae 121, pp. 211-222 (1995).

[11] V. D. Goppa, *Geometry and codes*, Kluwer Academic Publishers (1988).

[12] D. Gorenstein, *An arithmetic theory of adjoint plane curves*, Trans. Amer. Math. Soc., vol. 72, pp. 414-436 (1952).
[13] G.-M. Greuel, G. Pfister and H. Schoenemann, *SINGULAR*, a computer algebra system for Commutative Algebra and Algebraic Geometry, Fachbereich Mathematik der Universität, D-67653 Kaiserslautern.

[14] G. Haché, *Construction effective des codes géométriques*, Ph.D. thesis, Univ. Paris 6 (1996).

[15] T. Høholdt and R. Pellikaan, *On the decoding of algebraic-geometric codes*, IEEE Trans. Inform. Theory 41, pp. 1589-1614 (1995).

[16] C. Kirfel and R. Pellikaan, *The minimum distance of codes in an array coming from telescopic semigroups*, IEEE Trans. Inform. Theory 41, pp. 1720-1732 (1995).

[17] J. Lipman, *On complete ideals in regular local rings*, Algebraic Geometry and Commutative Algebra in honor of M. Nagata, pp. 203-231 (1987).

[18] M. Rybowicz, *Sur le calcul des places et des anneaux d'entiers d'un corps de fonctions algébriques*, Ph.D. thesis, Limoges (1990).

[19] M. A. Tsfasman and S. G. Vlăduţ, *Algebraic-geometric codes*, Math. and its Appl., vol. 58, Kluwer Academic Pub., Amsterdam (1991).