External stability for Spherically Symmetric Solutions in Lorentz Breaking Massive Gravity

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(Dated: July 29, 2014)

We discuss spherically symmetric solutions for Stars and Black Holes in Lorentz-breaking massive gravity theories. This analysis is valid for St"uckelberg’s effective field theory formulation, for Lorentz Breaking Massive Bigravity and general extensions of gravity leading to an extra term – $S_{\gamma}$ added to the Newtonian potential. The approach consists in analyzing the stability of the geodesic equations out to the star radius, at the first order (deviation equation). The main result is a strong constrain in the the space of parameters of these theories. This strongly motivates an higher order geodetic analysis of perturbations, to understand if a class of spherically symmetric Lorentz-breaking massive gravity solutions for stars, black holes, and, in general, self-gravitating systems exists, stable and phenomenologically acceptable, in the no-trivial case $S \neq 0$.

PACS numbers: 98.80.-k, 95.35.+x, 95.35.+d, 04.50.+h

Keywords: Alternative gravity; Spherical Symmetry; Lorentz symmetry breaking.

1. INTRODUCTION

Can the graviton have a mass? This is one of the most intriguing questions of theoretical physics. In fact it is connected to other deep questions: i) Can General Relativity be an approximation of an extended theory of gravity? ii) What about the Dark-side of the universe? iii) Can we construct a modified theory of GR, breaking Lorentz symmetry in its own sector? It is crucial to understand and to clarify the connections between these aspects.

The first issue about the mass term was firstly considered by Fierz and Pauli (FP) \cite{FP}. Despite of this intuition, it was soon clarified that this theory is unphysical: the modification of the Newtonian potentials is discontinuous in the limit of $m \to 0$ (very small graviton mass) with a large deviation of 25\% to the light deflection from the Sun \cite{BM}. Alternatively, it was proposed that, in the full non-linear regime, the discontinuity can be avoided \cite{FP}. But FP theory is also problematic at quantum level. In fact the gauge symmetry is broken by the explicit mass $m$ term with cutoff $\sim (m^4M_P)^{1/5}$. This scale is lower than the expected $(mM_P)^{1/2}$\cite{Weinberg}. On the other hand, such shortcomings are avoided by considering a possible connection with the question (iii) cited above. In fact it is interesting to consider Lorentz-breaking massive terms \cite{LM}: this class of terms is free from ghosts at low and strong coupling scales. Mass terms, breaking the diffeomorphism invariance, have been considered by reintroducing the Goldstone field associated to the broken invariance \cite{Henning,Weinberg} and for a review \cite{review}. Considering LBMG as an effective field theory, the graviton mass is generated by the interaction with a suitable set of St"uckelberg fields. A set of four St"uckelberg fields $\phi^a$ $(a = 1, 2, 3, 4)$, is introduced by transformations under diffeomorphisms $\delta x^\mu = \zeta^\mu (x)$. Goldstone fields transform as scalars and they can realize a modified theory of gravity, at IR scales. The result is manifestly invariant under diffeomorphism. In particular, the Einstein-Hilbert action is extended with potential like $m^2\mathcal{V}(\mathcal{X},\mathcal{V},\mathcal{S})$, with $\mathcal{X} = -g^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^b$, $\mathcal{V} = -g^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^b$, $\mathcal{S} = -g^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^b$. $m$ is the graviton mass scale. Lorentz breaking is manifest in the action, but it was considered a rotationally invariant potential $\mathcal{V}$. Bigravity could be another interesting idea to realize a Lorentz Breaking Massive Gravity without ghosts and discontinuities \cite{Bigravity, Bigravity2}. Such a theory is based on two coupled Einstein field equations derived on a Riemannian manifold with two conjugated metrics and two geodesic structures \cite{Bigravity}. Bigravity is also a theory with an intriguing phenomenology in galactic physics (see \textsuperscript{12,13} and \textsuperscript{14}), strictly related to Mirror theory \textsuperscript{32}.

\cite{Lett} Other Lorentz breaking gravity models, generally known as Bumblebee gravity, are considered in \textsuperscript{15-22}. The strategy to break Lorentz symmetry is completely different in this case: the EH action is extended with an interaction term of the curvature tensor and a new vector field (see \textsuperscript{14}). Then the vector field takes an expectation value that spontaneously breaks Lorentz invariance. Another prospective comes from DGP model theories. As pointed out in \textsuperscript{22}, spontaneous Lorentz breaking could be induced by the expectation value of sources.
Assuming the spherical symmetry for the metric generated by a spherically symmetric source, it is possible to find out a set of coordinates in order to have metric

\[ ds^2 = -dt^2 J(r) + K(r) ds^2 + r^2 dΩ^2 \]  

(1)

where \( M \) and \( S \) are two integration constants, \( G = 1/16 \pi M P^2 \) is the Newton constant. \( \Lambda^2 \) is the "effective" cosmological constant; it is a particular combination of the parameters in the potential chosen in literature. The exponent \( \gamma \) is another combination of the parameters. We not enter in the details just discussed in [11].

The last power law in [2] is the new term coming from the gravity modification; \( 1/r \) and \( r^2 \), on the contrary, are also present in the Schwarzschild-de Sitter metric. The same solution of [2] was previously found in Lorentz Breaking Massive Bigravity in a certain class of interactions between the two metrics [15].

Recently the specific class of solutions discussed in [11], with a \( S \neq 0 \), are discovered to be unstable if \( S \neq 0 \) [17]. A complete no-perturbative analysis shows the presence of a hidden ghost. On the other hand, this strongly depends on the specific choice of the interaction potential \( V \). The scope of our paper is more general. In fact it could exist a class of ghost-free LBMG with a static-gravitational potential like [2]. For example kinetic mixing terms, between the metric and the Goldstones or the two metrics in bigravity, could enter in the Action, eliminating ghosts. These terms are not considered in literature. This could enlarge the space of the parameters with other peculiar effects to explore.

Alternatively, it could be possible to extend the EH kinetic term of LBMG models analyzed, as a local term \( f(R) \) (see [20] for a review of \( f(R) \) theories). \( f(R) \) gravity itself is an intriguing extension of GR with a lot of phenomenology in cosmology [27], galactic physics [28], gravitational waves [29] and neutron stars [30,37]. So, it could be interesting to consider \( f(R) \) extension with a Lorentz breaking potential.

Then, it is also possible to consider no-local kinetic terms in LBMG. Recently the possibility of a ghost-free massive bigravity with no-local terms was proposed in [31].

On the other hand, a more general and radical point of view about ghosts in extend theories of gravity is considered in [30]: it was proposed that ghosts in quantum gravity are not dangerous if we consider a different interpretation of the quantum mechanics. This could be also tested in gravitational waves.

So the purpose of this paper is to study the external stability of the trajectories in a LBMG metric [11] through the geodesic stability condition. The analysis would be "effective", without to specify the particular model and its problems. So the question of ghosts is beyond the interest of this paper. For "external stability", we want to study the geodesic stability out of the event horizon for black holes: but obviously this condition is automatically satisfied for stars. In other words, we are studying the geodesic structure for all classes of LBMG theories with a new term \( \sim r^\gamma \) in addition to the Newtonian gravitational potential. We could study this term for all the possible values of \( S, \gamma \) at priori. But a first constrain comes from the convergence of the Komar integral [32] (in [15] and [11] it is considered). In fact the gravitational energy is

\[ \mathcal{E} = -\frac{1}{4\pi G} \int_{∂S} d^2x \sqrt{\nu^\mu \nu^\nu} \mathcal{W}_{\nu,\mu} \]  

(3)

with \( S \) is the 3-surface, \( ∂S_\ell \) is boundary at fixed time \( \ell \). \( \nu^\mu \) is the normal versor of \( S \), \( u^\nu \) is the normal versor of the boundary. The integration leads to the result

\[ \mathcal{E} = m + S_\gamma R^{\gamma+1} \]  

(4)

In the limit of \( R \to \infty \), this integral converges to \( m \) only for \( \gamma < -1 \). So our analysis will be interesting for gravitational potential \( \gamma < -1 \). \( \gamma > -1 \) solutions are just excluded previously for the classical Komar bound.

The basic idea of the geodetic stability study is the following: let us assume to infinitesimally perturb a generic geodetic trajectory in the gravitational metric [11] as \( x^\mu \to x^\mu + \delta x^\mu \). If the 4-deviation \( \delta x^\mu(s) \) explodes exponentially as \( \delta x(s) \sim e^{ks} \) (\( k \) is a constant), we have to conclude that the trajectories around the solutions are unstable. A solution for a star (or a black hole), that cannot admit external stable circular (or quasi-circular) trajectories, is not phenomenologically acceptable. In the next section we show this analysis and the consequent constraints on the space of parameters for LBMG theories. In our analysis we will assume that \( \Lambda = 0 \), in asymptotically flat hypothesis.

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2 The integration can be performed into a boundary as a 2-sphere, \( W = \partial / \partial t \) the Killing vector associated with time direction, the integration is cutoff to the fixed radius \( R \).
2. THE GEODETIC STABILITY CONDITION

The trajectories, in the gravitational field background, are described by the geodesic equations
\[
\frac{d^2 x^\lambda}{ds^2} + \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0
\]
(5)
with \(\frac{dx^\mu}{ds}\), the 4-velocity, \(s\) the affine parameter along the geodesic. If we perturb the geodesic as \(x^\mu \rightarrow x^\mu + \delta x^\mu\), where \(\delta x^\mu\) is the 4-deviation, we obtain, as standard, the deviation equation
\[
\frac{d^2 \delta x^\lambda}{ds^2} + 2 \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{ds} \frac{d\delta x^\lambda}{ds} + \partial_\rho \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \delta x^\rho = 0.
\]
(6)
If we insert this into (1), fixing the constant \(k_0 = 1\) in (2), we have following geodetic equations:
\[
\frac{d^2 t}{ds^2} = 0, \quad \frac{1}{2} J'(r) \left( \frac{dt}{ds} \right)^2 - r \left( \frac{d\phi}{ds} \right)^2 = 0, \quad \frac{d^2 \theta}{ds^2} = 0, \quad \frac{d^2 \phi}{ds^2} = 0.
\]
(7)
The geodesic deviation, divided by components, is
\[
\frac{d^2 \delta x^0}{ds^2} + \frac{J'(r)}{J(r)} \frac{dt}{ds} \frac{d\delta x^1}{ds} = 0,
\]
(8)
\[
\frac{d^2 \delta x^1}{ds^2} + J(r)J'(r) \frac{dt}{ds} \frac{d\delta x^0}{ds} - 2r J(r) \frac{d\phi}{ds} \frac{d\delta x^3}{ds} + \left[ \frac{1}{2} \left( J^2(r) + J(r)J''(r) \right) \left( \frac{dt}{ds} \right)^2 - (J(r) + r J'(r)) \left( \frac{d\phi}{ds} \right)^2 \right] \delta x^1 = 0,
\]
(9)
\[
\frac{d^2 \delta x^2}{ds^2} + \left( \frac{d\phi}{ds} \right)^2 \delta x^2 = 0,
\]
(10)
\[
\frac{d^2 \delta x^3}{ds^2} + \frac{2 d\phi d\delta x^1}{r ds} = 0
\]
(11)
\((J'(r) = dJ(r)/dr)\) We consider the circular orbit in the plane \(\theta = \pi/2\), that in the \(ds^2\) gives
\[
J(r) \left( \frac{dt}{ds} \right)^2 - r^2 \left( \frac{d\phi}{ds} \right)^2 = 1
\]
(12)
and from this last and (7) we can obtain
\[
\left( \frac{d\phi}{ds} \right)^2 = \frac{J'(r)}{r [2 J(r) - r J'(r)]}, \quad \left( \frac{dt}{ds} \right)^2 = \frac{2}{2 J(r) - r J'(r)}
\]
(13)
From this we can eliminate the dependence on \(s\) in the deviation equations, obtaining
\[
\frac{d^2 \delta x^0}{d\phi^2} + \frac{J'(r)}{J(r)} \frac{dt}{d\phi} \frac{d\delta x^1}{d\phi} = 0
\]
(14)
\[
\frac{d^2 \delta x^1}{d\phi^2} + J(r) J'(r) \frac{dt}{d\phi} \frac{d\delta x^0}{d\phi} - 2r J(r) \frac{d\phi}{d\phi} \frac{d\delta x^3}{d\phi} + \left[ \frac{1}{2} \left( J^2(r) + J(r) J''(r) \right) \left( \frac{dt}{d\phi} \right)^2 - (J(r) + r J'(r)) \right] \delta x^1 = 0,
\]
(15)
\[
\frac{d^2 \delta x^2}{ds^2} + \delta x^2 = 0,
\]
(16)
\[
\frac{d^2 \delta x^3}{ds^2} + \frac{2 d\delta x^1}{r ds} = 0
\]
(17)
But the last two equations give just harmonic motions that means that the motion in the plane $\theta = \pi/2$ is stable. Remain to consider the other components. Now we insert the modified gravitational potential (2) into (14-15)

$$\frac{d^2 \delta x^0}{d\phi^2} + \frac{2GM + 2\gamma GM_r}{r - 2GM + 2GSr^{\gamma+1}} \frac{dt}{d\phi} \frac{d\delta x^1}{d\phi} = 0$$

(18)

$$0 = \frac{d^2 \delta x^1}{d\phi^2} + \left(1 - \frac{2GM}{r} + 2GSr^{\gamma} \right) \frac{d\delta x^0}{d\phi} \frac{dt}{d\phi} - 2(r - 2GM + 2GSr^{\gamma+1}) \frac{d\delta x^3}{d\phi}$$

(19)

$$+ \left\{ \frac{1}{2} \left[ 4G^2 \left( \frac{M}{r^2} + \gamma r^{\gamma-1} \right)^2 + \left(1 - \frac{2GM}{r} + 2GSr^{\gamma} \right) \left(2G\gamma S(\gamma - 1)r^{\gamma-2} - \frac{4GM}{r^3} \right) \right] \left(\frac{dt}{d\phi} \right)^2 - [1 + 2(1 + \gamma)GSr^{\gamma}] \right\} \delta x^1$$

We can insert in the equations the harmonic solutions

$$\delta x^0 = \delta x^0_0 e^{i\omega \phi}, \quad \delta x^1 = \delta x^1_0 e^{i\omega \phi}, \quad \delta x^3 = \delta x^3_0 e^{i\omega \phi}$$

(20)

where $\delta x^0_0, \delta x^1_0, \delta x^3_0$ are constants.

The conditions obtained are apparently no-trivial, but between these the following relevant relation with the gravitational potential (2) is

$$r^{3+\gamma} + \frac{2S}{M} > 0$$

(21)

The condition (21) is not so different from the classical condition for the Reissner-Nordström case. Let us imagine to put $\gamma = -2$ and $\frac{2S}{M} = -\frac{q^2}{m^2}$, with $q, m$ the charge and the mass of the particle. This is just the case of symmetric solution for a charged particle. Also in RN apparently difficult constraints can be reduced just to two simple and relevant ones [18]:

$$r - 6m > 0$$

(22)

$$r - \frac{q^2}{m} > 0$$

(23)

The condition (23) is exactly the corresponding to (21) with $\gamma = -2$ and $\frac{2S}{M} = -\frac{q^2}{m^2}$; $r + \frac{2S}{M} > 0$. So the bound (21) passes this simple check of consistence with classical results of GR.

Eq. (21) excludes a large region of parameters corresponding to $S < 0$ for $\gamma < -3$. In fact for $r \rightarrow \infty$, $r^{3+\gamma} |_{\gamma < -3} \rightarrow 0$ while $S$ remains constant: $0 > -\frac{2S}{M} = \text{const} > 0$ is clearly impossible. For $-3 < \gamma < -1$, solutions are possible because now for $r \rightarrow \infty$, $r^{3+\gamma} |_{\gamma > -3} \rightarrow \infty$ and consequently this is higher then the constant $\frac{2S}{M}$. This is expected considering the case of RN, that is stable under geodetic perturbation. But we have a bound from below that not all the solutions satisfy (and again this is true also for RN, with a bound depending on the charge and the mass). On the other hand, the class of solutions $S > 0$ are unconstrained from the condition (21).

3. CONCLUSIONS

We analyzed the geodesic stability for spherically-symmetric self-gravitating systems (stars and black holes) in Lorentz-Breaking-Massive-Gravity, valid for Stückelberg, Bigravity and other LBMG classes leading to a new term $-S\gamma r^\gamma$ in the static external potential. Clearly, not all the possible interaction potentials were studied in the precedent literature, in the various classes of theories, essentially for a formal difficulty in the calculations. The interaction potentials considered in literature, exclude very interesting kinetic mixing terms between the Goldstones and the metric, or between the two metrics in bigravity. It is also practically unexplored the possibility to extend EH kinetic terms of our metric, with $f(R)$-like or no-local ones, in LBMG theories. Finally, the interpretation of ghosts in
quantum gravity could lead to ambiguities in the probabilistic formulation of quantum mechanics, that not exists at all in the current interpretations of QM. On the other hand, the questions of consistence of a particular model are not the purposes of this paper.

In our analysis, we have assumed the case of asymptotically flat metrics, setting the effective cosmological term $\Lambda = 0$. It was noted, as cited previously, that $\gamma > -1$ leads to a divergent gravitational energy. As a consequence, our calculation assume the starting bound $\gamma < -1$.

Our goal is a strong restriction of the space of parameters for the static solutions. This simple calculation leads to exclude all the metrics with $S < 0$ for the non standard term $\sim -Sr^\gamma$. In other words, the new non standard term proportional to $\sim r^\gamma$ cannot be repulsive if $\gamma < -3$. For $-3 < \gamma < -1$ repulsive terms are possible, but with bounds from below depending on $S$ and $M$. For example, if we consider these conditions for all $\gamma < -1$, the space of the parameters in [11] is strongly reduced.

We note that our analysis is limited to the first order of the geodesic perturbation, a deeper analysis could constrain even more the space of parameters of the LBMG models. This strongly motivates an higher order geodetic analysis in order to understand if a class of spherically symmetric LBMG solutions for stars and black holes exists, stable and phenomenologically acceptable, in the no-trivial case $S \neq 0$. This problem could lead to important phenomenological implications. It is worth stressing the fact that spherically symmetric solutions with charges or axial symmetry could have a larger parameter space, allowing more "islands" of stable solutions in the space of parameters.

Finally, it is possible that spherically symmetric solutions for black holes develop inner blueshift instabilities under electromagnetic or gravitational perturbations but this is not relevant for ordinary stars. Furthermore, strong gravity regimes could give rise to additional pressure terms capable of stabilizing very peculiar massive objects [36, 37].

Acknowledgments

We would like to thank Denis Comelli, Luigi Pilo and Fabrizio Nesti for discussions and remarks on these subjects. SC is supported by INFN (iniziativa specifica TEONGRAV).

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