Work fluctuations and Jarzynski equality in stochastic resetting

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We consider the paradigm of an overdamped Brownian particle in a potential well, which is modulated through an external protocol, in the presence of stochastic resetting. Thus, in addition to the short range diffusive motion, the particle also experiences intermittent long jumps which reset the particle back at a preferred location. Due to the modulation of the trap, work is done on the system and we investigate the statistical properties of the work fluctuations. We find that the distribution function of the work typically, in asymptotic times, converges to a universal Gaussian form for any protocol as long as that is also renewed after each resetting event. When observed for a finite time, we show that the system does not generically obey the Jarzynsky equality which connects the finite time work fluctuations to the difference in free energy, albeit a restricted set of protocols which we identify herein. In stark contrast, the Jarzynsky equality is always fulfilled when the protocols continue to evolve without being reset. We present a set of exactly solvable models, demonstrate the validation of our theory and carry out numerical simulations to illustrate these findings.

Introduction.— Stochastic thermodynamics is a cornerstone in non-equilibrium statistical physics [1–5]. Microscopic systems satisfy stochastic laws of motion governed by force fields and thermal fluctuations which arise due to the surrounding. The subject then teaches us that thermodynamic observables such as work, heat, entropy production etc. measured along the stochastic trajectories taken from ensembles of such dynamics will fluctuate too. Understanding the distribution and the statistical properties of these fluctuations is of great interest since they hold a treasure trove of information about microscopic systems and how they respond to external perturbation. Indeed there has been a myriad of studies to understand e.g., non-equilibrium dynamics of biopolymers [6, 7], colloidal particles [8–13], efficiency of molecular bio-motors [14, 15] and microscopic engines [16], heat conduction [17, 18], electronic transport in quantum systems [19], trapped-ion systems [20] and many more [21]. Despite there exists a long catalogue of such diverse small systems with no apparent similarity, it is quite remarkable to find universal relations which are obeyed regardless. One of the most celebrated ones is perhaps the Jarzynski equality (JE) that relates the non-equilibrium fluctuations of the work to the equilibrium free energy difference [22–24]. Universalities of such kind have always been considered as a holy grail in physical sciences and in this paper we seek out for thermodynamic invariant principles in stochastic resetting systems [25].

Dynamics with stochastic reset has drawn a lot of attention recently because of its rich non-equilibrium properties [25–37] and its broad applicability in first passage processes [38–47]. Nevertheless, thermodynamical perspective of resetting systems has been largely overlooked so far. It was only recently when first and second laws of thermodynamics were interpreted by identifying the contributions to the total entropy production [48], and furthermore it was shown to satisfy a universal integral fluctuation relation [49]. While these first studies focused exclusively on the entropy production, efforts are yet to be made to understand other response functions. Moreover, not much is known about the distribution of these observables. In particular, one important observable is the work function which is produced due to external perturbations to the system. Work statistics encodes important features of an out-of-equilibrium thermodynamic process but its computation is usually quite daunting. Here, we set out to characterize work fluctuations in a stochastic system which is subjected to resetting. Our detailed analysis to this account then reveals emergence of robust universal pattern in work-fluctuations: firstly resetting renders work-fluctuations Gaussian independent of the nature of the external perturbation that produces it. Secondly, work fluctuations are found to obey the JE under certain conditions which we identify through this comprehensive study.

General theory.— For the sake of generality, we put forward our results in the paradigm framework of a one-dimensional overdamped Brownian particle in a potential $U(x, \lambda(t))$, which is modulated externally through the protocol $\lambda(t)$. Motion of such a particle is governed by the Langevin equation of the form

$$\dot{x}(t) = -\gamma^{-1} \partial_x U(x, \lambda(t)) + \sqrt{2D}\eta(t),$$

(1)

where $\gamma$ and $D$ are the friction and diffusion coefficients respectively that satisfy the fluctuation-dissipation relation, i.e., $D\gamma = k_BT$, with $k_B$ being the Boltzmann constant and $T$ is the temperature of the medium. We assume $\langle \eta(t) \rangle = 0$ and $\langle \eta(t)\eta(t') \rangle = \delta(t-t')$. Moreover, let us consider that the position of the particle at $t = 0$ is distributed according to the probability density function
At random times taken from an exponential distribution \( f(t) = e^{-rt} \), the particle in motion is stopped and teleported to the initial configuration.

The external modulation of the potential pertains to the work fluctuations. It is convenient to first define the moment generating function (MGF) namely

\[
H_r(k,t) = \int_{-\infty}^{\infty} dW \ e^{-kW} P_r(W,t) \tag{3}
\]

where \( P_r(W,t) \) is the PDF of the work at time \( t \), averaged over the initial distribution \( p_{ini}(x_0) \) and the underlying dynamics with stochastic resetting. One can then make use of the renewal structure of the resetting dynamics to construct a relation that connects the MGF for \( r > 0 \) to that of \( r = 0 \) for random initial and subsequent resetting positions

\[
\tilde{H}_r(k,s) = \frac{\tilde{H}_0(k,s + r)}{1 - r\tilde{H}_0(k,s + r)} \cdot \tag{4}
\]

where \( \tilde{H}_r(k,s) = \int_0^\infty dt \ e^{-st} H_r(k,t) \) and the subscript 0 indicates the observables with \( r = 0 \). A short proof of Eq. (4) is added to the end of the manuscript leaving the details in [50]; but it is imperative to stress the following points. Note that Eq. (4) holds for any initial condition and naturally adheres to a fixed initial condition which was derived in [51, 52], but in the absence of any protocol \( \lambda(t) \). In the presence of protocol, one needs to be meticulous since the structure of this equation relies on the fact that \( \lambda(t) \) is also renewed after each resetting. As we will see later, Eq. (4) does not hold when the protocol is unaffected under resetting [50].

The MGF, given by Eq. (4), can be inverted to obtain the full work statistics at a given time. Nonetheless, we will show that it suffices to know the first and second moment to predict the universal behavior of the work fluctuations in the large time limit. To this end, we first note that the \( n \)-th moment of \( W \) in Laplace space can be written as

\[
\tilde{W}_r^n(s) = \frac{s + r}{s} \left[ \tilde{W}_0^n(s + r) + r \sum_{i=1}^{n} \binom{n}{i} \tilde{W}_r^{n-i}(s) \tilde{W}_0^i(s + r) \right] \tag{5}
\]

where we have defined \( \tilde{W}_r^n(s) \equiv \int_0^\infty dt \ e^{-st} \langle W^n(t) \rangle_r \). Eq. (5) gives a simple recipe to compute all the moments of \( W \) recursively from the knowledge of the moments of the process without resetting.

**Universal work fluctuations.**—The infinite set of moments given by Eq. (5) contains the same information as that of the full distributions \( P_r(W,t) \). However, physical intuition tells us that not all the moments contribute significantly at long time. To see this, we consider a trajectory of time length \( t \) with multiple possible resetting events. The total work done along this long trajectory can then be decomposed into the sum of the partial works produced in each time interval between the resetting events. However, these intervals are statistically independent since the entire configuration of the system (comprising the particle and the trap) is renewed after each resetting event, and hence there are no correlations between the intervals. Therefore, for a long enough observation time \( t \) one would expect on an average \( \sim rt \) number of resetting events and the total work \( W(t) \) can then be written as \( W \approx W_1 + W_2 + W_3 + \cdots + W_{[rt]} \). Since the intervals are disjointed, the \( W_r \)-s are also independent and identically distributed. Moreover, if \( W_r \)-s are regular (with finite mean and variance), one would expect that the distribution of \( W \), according to the central limit theorem, would converge to a Gaussian irrespective of the nature of the potential and choice of the external protocol

\[
P_r(W,t) = \frac{1}{\sqrt{2\pi\sigma_t^2}} \exp \left[ -\frac{(W - \mu_t)^2}{2\sigma_t^2} \right] \tag{6}
\]

where the mean \( \mu_t \equiv \langle W \rangle \) and the variance \( \sigma_t^2 \equiv \langle W^2 \rangle - \langle W \rangle^2 \) are computed from Eq. (5). We demonstrate our results in the set up of a 1D Brownian particle confined in a harmonic trap \( U(x,\lambda(t)) = \kappa(t)[x - y(t)]^2/2 \), where
\(\kappa(t)\) and \(y(t)\) represent the stiffness and center of the trap respectively. Here we consider two different types of external modulation namely \(\lambda(t) = \{y(t), \kappa(t)\}\) with their independent dynamics: (i) \(y(t) = ut\ (u > 0)\) with fixed \(\kappa = \kappa_0\) (Fig. 1 panel A), and (ii) \(\kappa(t) = \kappa_0 + vt\ (\kappa_0, v > 0)\) with fixed \(y\) (Fig. 1 panel B). For each of the cases, we consider two different initial conditions (from where the initial position and the subsequent resetting positions are drawn) (a) fixed initial condition (FIC): \(p_{ini}(x_0) = \delta(x_0)\) i.e., the particle is reset to the origin, and (b) random initial condition (RIC) e.g., the equilibrium distribution \(p_{eq}(x_0) = p_{eq}(x_0) \propto \exp[-\beta U(x_0, 0(0))]\), where \(\beta = 1/k_BT\) is the inverse temperature. We compare the numerical distribution of the work for each one of these cases to the theoretical prediction given by Eq. (6).

We see an excellent Gaussian collapse \(P(z) = \frac{1}{\sqrt{2\pi}z}e^{-z^2/2}\) of all the work-PDFs in terms of the rescaled variable \(z = (W - \mu_L)/\sigma_L\). This is shown in Fig. 2.

Jarzynski equality—reset protocol. The JE relates the finite time work fluctuations to the equilibrium free energy and here we ask whether such relations hold generically in resetting systems. We consider the same set-up as before and assume that each resetting act refresh both the particle and the protocol. We further assume that the initial condition is taken from an equilibrium distribution \(p_{eq}(x_0) \propto \exp[-\beta U(x_0, 0(0))]\), which is an essential prerequisite for the JE. Employing Eq. (4) and substituting \(k = 0\) there, we find a renewal expression for the average of the exponentiated work which connects to the same with \(r = 0\) in the Laplace space [50]

\[\mathcal{L}_{t \to s} [\left\langle e^{-W} \right\rangle_r] = \frac{\mathcal{L}_{t \to s+r} [\left\langle e^{-W} \right\rangle_0]}{1 - r \mathcal{L}_{t \to s+r} [\left\langle e^{-W} \right\rangle_0]}, \tag{7}\]

where \(\mathcal{L}\) is the Laplace transform operator. Several comments are in order now. The exponential average on the RHS is along the trajectory without resetting and therefore must satisfy the JE i.e., \(\left\langle e^{-W[0, t]} \right\rangle_0 = e^{-\left\langle F_0(\lambda(t)) \right\rangle_0} - F_0(\lambda(t))\), where \(F_0(\lambda(t))\) is the free energy of the underlying system (i.e., when the dynamics is not interrupted by resetting) corresponding to the value of \(\lambda\) evaluated at time \(t\). However, it is evident that substituting this in Eq. (7) will not essentially lead to \(e^{-\Delta F_0(t)}\) along the entire trajectory of length \(t\) in the presence of resetting i.e., JE will not be obeyed generically for any arbitrary protocol. Nonetheless, we identify the protocols which will indeed satisfy this condition. This happens when the modulation of the protocol renders a linear change in the free energy i.e., \(\Delta F_0(t) = \alpha t\).

The trivial scenario i.e., \(\Delta F_0(t) = 0\) is true under any external perturbation which is of the following form: \(U(x, y(t)) = U(x - y(t))\). This could happen, e.g., when we move the center of the trap \(y(t)\) according to some specific schedule. On the other hand, the nontrivial linear change in \(\Delta F_0(\neq 0)\) occurs when e.g., the stiffness \(\kappa(t)\) is varied exponentially as a function of time. Utilizing this condition in Eq. (7), we obtain \(\left\langle e^{-W} \right\rangle_r = e^{-\Delta F_0(t)}\) which holds along the entire trajectory with multiple resetting events [50].

We now briefly summarize the numerical setups which are used to verify these findings. We have simulated an overdamped Brownian particle in a harmonic trap \(U(x, \lambda(t)) = \kappa(t)[x - y(t)]^2/2\) in the presence of resetting \((r = 0.5)\), and measured \(e^{-W}\) till time \(t = 5\). In Fig. 3a, we have shown the convergence of the statistical average \(\left\langle e^{-W} \right\rangle\) as a function of realizations \(N_F\) for the following protocol modulations (i) moving the center of the trap with \(y(t) = 0.2t\), (ii) changing stiffness with a power law \(\kappa(t) = \kappa_0(1 + 0.2t)^{-2}\), and (iii) an exponential law \(\kappa(t) = \kappa_0 e^{-0.2t}\). As before, we have regulated one protocol at a time keeping the others fixed. The horizontal lines shown in the panel correspond to the theoretical prediction of \(e^{-\Delta F_0(t)}\) which takes the values 1.0, 2.0 and \(\sim 1.65\) respectively for each of the modulations. The exact computation has been reserved to [50]. It is evident from Fig. 3a that the JE holds for modulations (i) and (iii), but not for modulation (ii).

Jarzynski equality is invariant under non-reset protocol.—The discussion so far focused on the case when we reset both the protocol and the particle. In the following, we relax this condition and assume that only the particle is reset while the protocol keeps evolving in time. Moreover, we assume that after each resetting event, position of the particle is drawn from the equilibrium distribution \(p_{eq}(x_0) \propto \exp[-\beta U(x_0, 0(t_i))]\) corresponding to \(\lambda\) measured at the times \(t_i\) of resetting. In this way, the particle is effectively equilibrated after each resetting event which is essential for the JE to hold.

![FIG. 2: Numerical computation of the PDF of the rescaled work on the harmonically confined Brownian particle for the linear modulation of the trap center i.e., \(y(t) = ut\) (panel a) and the stiffness i.e., \(\kappa(t) = \kappa_0 + vt\) (panel b) respectively. Simulations are performed for FIC (circle markers) and RIC (square markers) respectively for each of the above cases. Parameters for panel (a): \(\kappa_0 = 1.5, u = 0.2\) for FIC and \(\kappa_0 = 0.5, u = 0.5\) for RIC respectively where \(r = 0.5\) and \(t = 500\) for both FIC and RIC. Numerical simulations are corroborated with the theoretical prediction given by \(P(z) = e^{-z^2/2} \sqrt{2\pi}\) (solid line in both cases), and we see an excellent Gaussian collapse.](image-url)
rational to believe that such universality of the fluctuations become Gaussian as a fallout of the production. A detailed analysis of this problem, however, Gaussian fluctuations for other thermodynamic observables. Consequently, our approach also predicts emergence of fluctuations Gaussian for other thermodynamic observables as well in a similar manner.

This construction correlates the intervals between resetting events: since the initial configuration of a given interval depends on the time spent in the previous one and hence renewal structure of Eq. (4) is lost. However, notice that (i) after each resetting event, the particle is prepared at the equilibrium state as stated above, and (ii) consequently, the equality is satisfied in any interval between two resetting events. Taking these two facts into account, one can show that the equality holds along the entire trajectory independent of the nature of the protocol.

\[ \langle e^{-W} \rangle_r = e^{-\Delta F_0(t)} \]  

(8)

We numerically check Eq. (8) in Fig. 3b and show that indeed JE is invariant under non-reset protocol modifications.

Conclusions and outlook.— In summary, this letter discusses statistical properties of work fluctuations in a stochastic resetting system. We find that the introduction of resetting renders the work fluctuations Gaussian in the large time for the reset protocols. We infer that this is due to the renewal structure of the resetting process. Consequently, our approach also predicts emergence of Gaussian fluctuations for other thermodynamic observables such as dissipated heat, power flux or entropy production. A detailed analysis of this problem, however, remains to be seen. Furthermore, we note that only the typical fluctuations become Gaussian as a fallout of the central limit theorem. On the other hand, it is only rational to believe that such universality of the fluctuations will be lost while looking at the tail behavior (atypical fluctuations) of the work-distribution. An outstanding challenge would be to extend our approach to capture such scenarios using large deviation theory.

Our research is the first ever work to discuss JE in a resetting system. We present an extensive study to unravel different constraints on the temporal quantification of the protocols to preserve the JE. Naturally, our study opens up a new research avenue with a great appeal to the experimental demonstration of the work fluctuations in the resetting controlled single molecular systems using optical traps.

**Derivation.** To derive Eq. (4), we first note that the probability density for having \( n \) reset events along a stochastic trajectory of length \( t \) is given by

\[ \psi_n(t) = \prod_{j=1}^{n} dt_j f(t_j - t_{j-1}) \psi_0(t - t_n), \]  

(9)

where \( t_0 = 0 \) and \( \psi_0(t) = \int_0^\infty dt' f(t') \) is the probability of not having any reset event up to time \( t \). Thus \( \sum_{n=0}^{\infty} \psi_n(t) = 1 \) is the normalization condition. Any observable which is measured along a stochastic resetting trajectory has to be averaged over an ensemble of paths weighted with the probabilities given by Eq. (9). Therefore, we can write the moment-generating function \( H_r(k, t) = \langle e^{-kW} \rangle_r \) as a weighted average over the underlying non-resetting MGF taken with respect to Eq. (9)

\[ H_r(k, t) = \sum_{n=0}^{\infty} \prod_{j=1}^{n} \int_{t_{j-1}}^{t} dt_j f(t_j - t_{j-1}) H_0(k, t_j - t_{j-1}) \times \psi_0(t - t_n) H_0(k, t - t_n), \]  

(10)

where \( H_0(k, t) = \langle e^{-kW} \rangle_0 \) is the MGF corresponding to the work done along a stochastic trajectory with no-resetting events and properly averaged over the initial condition \( p_{ini}(x_0) \). We now use the fact that \( f(t) = re^{-rt} \) and take Laplace transform on the both sides of Eq. (10). Applying the convolution property of the Laplace transform, we arrive at Eq. (4) [50]. Derivation of Eq. (10) explicitly relies on the fact that the protocol \( \lambda(t) \) always renews from \( \lambda(0) \) after each resetting event and thus in this case, \( H_0 \) depends only on the time difference \( t_j - t_{j-1} \) at each interval. This is in sharp contrast to the situation when the protocol is not renewed, and thus \( H_0 \) not only depends on the duration of the evolution, i.e., the time difference \( t_j - t_{j-1} \), but also depends on the specific starting time \( t_{j-1} \) through the value of \( \lambda(t_{j-1}) \) (since the particle is prepared from the distribution \( p_{ini}(x_0) \propto \exp \left[ -U(x_0, \lambda(t_{j-1})) \right] \) after each resetting event). Therefore, it is not possible to get a closed form formula for the MGF like in Eq. (4) (See [50] for more details). This derivation sets the stage for exploring not only non-equilibrium work fluctuations but for other thermodynamic observables as well in a similar manner.
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Supplemental Material for “Work fluctuations and Jarzynski equality in stochastic resetting”

DERIVATION FOR EQ. (4) AND JARZYNSKI EQUALITY FOR RESET PROTOCOL

Derivation for Eq. (4)

In this section, we present a detailed derivation of Eq. (4) in the main text. To this end, let us first recall that $f(t)$ is the waiting time density between two reset events. As defined in the main text, $\psi_n(t)$ is the probability of having $n$ occurrences of resetting events at time $t$. Clearly, the survival probability $\psi_0(t)$ i.e., the probability of not having any reset event up to time $t$, is given by

$$\psi_0(t) = \int_t^{\infty} dt' f(t'). \quad (S1)$$

The subsequent $\psi_n(t)$ can be constructed recursively convolving the previous probabilities with $f(t)$ e.g., the probability that a single reset has occurred up to time $t$ is given by

$$\psi_1(t) = \int_0^t dt_1 f(t_1) \psi_0(t - t_1). \quad (S2)$$

Similarly, for $n = 2$, we can write

$$\psi_2(t) = \int_0^t dt_1 \int_0^{t_1} dt_2 f(t_2) f(t_2 - t_1) \psi_0(t - t_2). \quad (S3)$$

Thus, we have in general, for $n > 0$,

$$\psi_n(t) = \int_0^t dt_1 f(t_1) \psi_{n-1}(t - t_1) = \left[ \prod_{j=1}^{n} \int_{t_{j-1}}^{t_j} dt_j f(t_j - t_{j-1}) \right] \psi_0(t - t_n). \quad (S4)$$

where for brevity we have introduced $t_0 \equiv 0$ in our notation. Note that the integration variables $t_j$, are the times in which the $j$-th reset event occurs, see Fig. S1. Normalization condition for the probability $\psi_n(t)$ is given by

$$\sum_{n=0}^{\infty} \psi_n(t) = 1. \quad (S5)$$

When the waiting times between the reset events are drawn from an exponential distribution i.e., when $f(t) = re^{-rt}$, we obtain, using Eq. (S4), that the probability of having $n$ reset events follows a Poisson distribution

$$\psi_n(t) = \frac{(rt)^n}{n!} e^{-rt}. \quad (S6)$$

Any observable which is measured along a stochastic resetting trajectory has to be averaged over an ensemble of paths weighted with the probabilities given by Eq. (S4). We specifically focus on the moment generating function
The time evolution of \( t \) in a stochastic process in which \textit{both the particle and protocol are reset} at a rate \( r \). The MGF is formally defined as

\[
H_r(k, t) = \int_{-\infty}^{\infty} dW e^{-kW} P_r(W, t), \tag{S7}
\]

where the average is taken over the probability density function \( P_r(W, t) \) of having carried out a work equal to \( W \) after a time evolution of \( t \). We can split the process into the different intervals between successive reset events. Since work is an additive quantity along time, the function \( P_r(W, t) \) is a convolution of the non-resetting probabilities \( P_0(W, t) \). Therefore, we can write the moment generating function \( H_r \) in the resetting process as a product of the non-resetting moment generating functions \( H_0 \) in the following way

\[
H_r(k, t) = \sum_{n=0}^{\infty} \left[ \prod_{j=1}^{n} \int_{t_{j-1}}^{t} dt_j f(t_j - t_{j-1}) H_0(k, t_j - t_{j-1}) \right] \psi_0(t - t_n) H_0(k, t - t_n), \tag{S8}
\]

where we have taken into account the weight of every possible resetting path given by Eq. (S4). Note that \( H_0(k, t) \) is implicitly averaged over the same initial (or the subsequent resetting) condition for the position of the particle. This we denote as \( \rho_{ini}(x_0) \). Moreover we also condition on the fact that the protocol \( \lambda(t) \) renews from the value \( \lambda(0) \) after each resetting event (see Fig. S1). The elegant convolution structure in Eq. (S8) naturally compels one to take the Laplace transform. To this end, we define

\[
\hat{H}_r(k, s) \equiv \mathcal{L}_{t \to s} [H_r(k, t)] = \int_0^{\infty} dt \, e^{-st} H_r(k, t). \tag{S9}
\]

Doing the simple exercise for the first summand \((n = 1)\) in Eq. (S8), we find

\[
\int_0^{\infty} dt \, e^{-st} \int_0^{t} dt_1 \, [f(t_1)H_0(k, t_1)] \psi_0(t - t_1) H_0(k, t - t_1) = \mathcal{L}_{t \to s} [f(t)H_0(k, t)] \mathcal{L}_{t \to s} [\psi_0(t)H_0(k, t)] \tag{S10}
\]

and this extends to similar products of Laplace transforms when computing for any \( n \). This gives

\[
\hat{H}_r(k, s) = \mathcal{L}_{t \to s} [\psi_0(t)H_0(k, t)] \sum_{n=0}^{\infty} \{\mathcal{L}_{t \to s} [f(t)H_0(k, t)]\}^n. \tag{S11}
\]

We again consider \( f(t) = re^{-rt} \), substitute \( \psi_0(t) \) from Eq. (S1) and carry out the sum in the expression above. This results in Eq. (4) in the main text.

\textbf{Jarzynski equality}

We now turn our attention to discuss validation of the Jarzynski equality for the above said processes where both the particle and the protocol are set to their respective initial conditions after each resetting event. Fulfillment of the Jarzynski equality requires that the initial conditions are always drawn from an equilibrium distribution. This means that we need to consider that both the initial and subsequent resetting positions (which will work as the initial configuration for the consecutive intervals) of the particle are distributed accordingly to the equilibrium distribution \( p_{ini}(x_0) = p_{eq}(x_0) \propto \exp \left[-\beta U(x_0, \lambda(0))\right] \), corresponding to the initial value of the protocol \( \lambda(0) \), as depicted in Fig. S1. We can compute the exponential average involved in Jarzynski equation looking into \( H_r(k, t) \) in Eq. (S8) by substituting \( k = 1 \). Applying this in Eq. (4) and by noting that \( H_r(1, t) = \left\langle e^{-W} \right\rangle_r \), we immediately recover Eq. (7). Furthermore, we substitute \( k = 1 \) in Eq. (S8) to find

\[
\left\langle e^{-W} \right\rangle_r = H_r(1, t) = \sum_{n=0}^{\infty} \left\{ \prod_{j=1}^{n} \int_{t_{j-1}}^{t} dt_j f(t_j - t_{j-1}) \left\langle e^{-W[t_{j-1};t_j]} \right\rangle_0 \right\} \psi_0(t - t_n) \left\langle e^{-W[t_n;\infty]} \right\rangle_0, \tag{S12}
\]

where \( W[t_{j-1};t_j] \) is the work done in the interval that starts at time \( t_{j-1} \) and ends at \( t_j \) and the initial condition chosen at time \( t_{j-1} \) is simply \( p_{eq}(x_0) \), corresponding to \( \lambda(0) \). Under such conditions, the Jarzynski equality should
hold between the reset-intervals since the particle performs motion followed by Eq. (1) without resetting. Using this fact in Eq. (S12), we can write

$$\langle e^{-W} \rangle_r = \sum_{n=0}^{\infty} \prod_{j=1}^{n} \int_{t_{j-1}}^{t_j} dt_j f(t_j - t_{j-1}) e^{-\beta[F_0(t_j - t_{j-1}) - F_0(0)]} \psi_0(t - t_n) e^{-\beta[F_0(t - t_n) - F_0(0)]},$$  

(S13)

where $F_0(t)$ denotes the free energy of a non-resetting system that follows the protocol $\lambda(t)$. It is easy to see from Eq. (S13) that we cannot take the exponential terms out of the integration generically but it is possible to compute the sum as before in Laplace space and this gives

$$\mathcal{L}_{t \to s} [\langle e^{-W} \rangle_r] = \frac{\mathcal{L}_{t \to s + r} [e^{-\beta \Delta F_0(t)}]}{1 - r \mathcal{L}_{t \to s + r} [e^{-\beta \Delta F_0(t)}]},$$  

(S14)

where again we have made use of the fact that the waiting times between resets are taken from an exponential distribution with mean $1/r$. Note that Eq. (S14) is basically Eq. (7) after taking into account Jarzynski equality holds in processes without resetting. Neither the RHS of Eq. (S13) nor that of Eq. (S14) generically equates to $e^{-\beta \Delta F_0(t)}$ along the entire trajectory of length $t$ with multiple reset events. Thus, one cannot claim the validity of the Jarzynski equality for arbitrary protocols along the entire process. However, we will demonstrate in the following that there are indeed protocols for which this happens.

### Identification of protocols

A careful observation of Eq. (S13) hints that specific features of $\Delta F_0(t)$ may play a relevant role in the validity of an extended version of Jarzynski equality in resetting processes. Specifically, if the free energy increment is linear in time, $\Delta F_0(t) = \alpha t$, the exponential terms in Eq. (S13) can be combined giving a single exponential $e^{-\beta \Delta F_0(t)}$ that can be taken out of the integral. Then, using the normalization condition Eq. (S5), we get

$$\langle e^{-W} \rangle_r = e^{-\beta \Delta F_0(t)}.$$  

(S15)

Moreover, we can do an alternative computation using Eq. (S14),

$$\mathcal{L}_{t \to s} [\langle e^{-W} \rangle_r] = \frac{\mathcal{L}_{t \to s + r} [e^{-\beta \alpha t}]}{1 - r \mathcal{L}_{t \to s + r} [e^{-\beta \alpha t}]},$$  

$$= \frac{1}{s + \beta \alpha} = \mathcal{L}_{t \to s} [e^{-\beta \alpha t}].$$  

(S16)

In conclusion, we have proven that if we modulate the protocol such that free energy evolves linearly, one can trivially extend the Jarzynski equality in case of resetting process. The case of $\alpha = 0$ corresponds to a specific physical situation where the free energy remains unaffected. Any protocol which introduces a translation of the underlying potential without varying shape, i.e., $U(x, t) = U(x - y(t))$, will satisfy this criterion, e.g., moving the center of the trap with a constant velocity and others. In this case, one has $\langle e^{-W} \rangle_r = 1$, which is the Jarzynski equality for any such protocols.

### Necessity and sufficiency of linearity

In the preceding subsection, we have already shown that linear modulation of $\Delta F_0$ is a sufficient condition for having the extension of Jarzynski equality when we reset both the particle and the protocol. Herein, we prove that this condition is not only sufficient but also necessary in order to guarantee the validity of Jarzynski equality. The idea for the proof is imposing that the RHS of Eq. (S14) does not depend on $r$. Therefore, there will be no difference between $\langle e^{-W} \rangle_r$ and $\langle e^{-W} \rangle_0$ and the extension of Jarzynski equality to resetting systems automatically holds. Let us call $\phi(s + r) \equiv \mathcal{L}_{t \to s + r} [e^{-\beta \Delta F_0(t)}]$. We are interested in forcing

$$\frac{\partial}{\partial r} \left[ \frac{\phi(s + r)}{1 - r \phi(s + r)} \right] = 0,$$  

(S17)
for all \( r \). This requirement leads to a differential equation for \( \phi(s + r) \),
\[
\phi'(s + r) = -\phi^2(s + r),
\]
the solution of which is
\[
\phi(s + r) = \frac{1}{s + r + \beta \alpha},
\]
with \( \alpha \) being an arbitrary constant. We have introduced the integration constant as \( \beta \alpha \) for brevity. Inverting the Laplace transform in above we finally arrive at
\[
e^{-\beta \Delta F_0(t)} = \mathcal{L}^{-1}_{s \rightarrow r}[\phi(s + r)] = e^{-\beta \alpha t}.
\]
Thus, as brought forward above, the necessary and sufficient condition to assure validity of the extension of Jarzinski equality is the linear time dependence of \( \Delta F_0(t) = \alpha t \). For instance, as demonstrated in the main text and later in SM, the exponential modulation of the stiffness \( \kappa(t) \) as a function of time provides a linear \( \Delta F_0 \) leading to the fulfillment of (S15).

**DERIVATION OF JARZYNSKI EQUALITY FOR THE NON-RESET PROTOCOL**

![Diagram](https://via.placeholder.com/150)

**FIG. S2:** Reset process path with exactly \( n \) reset events with non-reset protocol.

We now discuss the ramifications of a resetting procedure where only the position is reset leaving the protocol non-intervened during the resetting events. Specifically, the protocol \( \lambda(t) \) continues to evolve according to a given schedule independent of the resetting phenomena (see Fig. S2). However, the initial (at \( t = t_0 = 0 \)) and subsequent reset positions of the particle at resetting time \( t = t_i \) (where \( t_i - t_{i-1} \) are distributed accordingly to \( f(t) \)) are extracted randomly from an equilibrium distribution corresponding to the appropriate value of the protocol at the start \( (t_0) \) and consecutive resetting times \( (t_i) \) respectively. In this resetting configuration, writing \( H_r \) in terms of the \( H_0 \), as in Eq. (S8), becomes more involved since each \( H_0 \) represents a different evolution with an extra dependence on time through the protocol \( \lambda(t) \). This is given by
\[
H_r(k, t) = \sum_{n=0}^{\infty} \left[ \prod_{j=1}^{n} \int_{t_{j-1}}^{t} dt_j f(t_j - t_{j-1})H_0(k, t_j - t_{j-1}; t_{j-1}) \right] \psi_0(t - t_n)H_0(k, t - t_n; t_n).
\]

As remarked above, in this case, \( H_0(k, t_j - t_{j-1}; t_{j-1}) \) not only depends on the duration of the evolution, i.e., the time difference \( t_j - t_{j-1} \), but also depends on the specific starting time \( t_{j-1} \) through the value of \( \lambda(t_{j-1}) \), unlike Eq. (S8) where \( H_0 \) depended on \( \lambda(0) \) (thus only on the time difference) at each interval. Therefore, \textit{it is not possible to get a closed form formula for the MGF like in Eq. (4)}. However, we now set aside the questions related to the generic properties of \( H_r(k, t) \) in this case and focus on the validity of the Jarzynski equality. To this end, we take a similar route as before and set \( k = 1 \) in Eq. (S21) which gives formally the same equation that in Eq. (S12). However, in this case, the work \( W_{[t_{j-1}; t_j]} \) done in the interval \( (t_{j-1} : t_j) \) has its initial condition chosen at time \( t_{j-1} \) from
\[p_{eq}(x_0) \propto \exp \left[ -\beta U(x_0, \lambda(t_{j-1})) \right] \]
corresponding to \( \lambda(t_{j-1}) \), as depicted in Fig. S2. Now we can exploit the fact that in each interval between resetting events Jarzynski equality holds such that we finally write
\[
\langle e^{-W} \rangle_r = \sum_{n=0}^{\infty} \left[ \prod_{j=1}^{n} \int_{t_{j-1}}^{t} dt_j f(t_j - t_{j-1})e^{-\beta[F_0(t_j) - F_0(t_{j-1})]} \right] \psi_0(t - t_n)e^{-\beta[F_0(t) - F_0(t_n)]} = e^{-\beta \Delta F_0(t)},
\]
where the cancellation of the exponentials naturally leads us to the Jarzynski equality given by Eq. (8). Remarkably, the validity of Jarzynski equality in the non-reset protocol is independent on the specifically considered $\lambda(t)$. This is in sharp contrast to the reset protocol, where Jarzynski equality was fulfilled only when the protocol modulation rendered to a linear evolution of $\Delta F_0$ in time.

**EXACT COMPUTATION OF FREE ENERGY IN THE CASE OF STIFFNESS MODULATION**

In this section, we compute the free energy change of a Brownian particle subjected to a time-dependent harmonic potential. In particular, we vary the stiffness $\kappa(t)$ to modulate the trap. The dynamics of the particle is given by the overdamped Langevin equation

$$\dot{x}(t) = -\gamma^{-1}\kappa(t)x + \sqrt{2D}\eta(t), \tag{S23}$$

where recall that $D$ and $\gamma$ are the diffusion and friction constant respectively. At $t = 0$, the particle is in equilibrium with the environment, thus its position density takes the shape of Boltzmann distribution

$$p_{ini}(x_0) = p_{eq}(x_0) = \sqrt{\frac{\kappa_0}{2\pi\gamma D}} \exp \left[ -\frac{\kappa_0 x_0^2}{2\gamma D} \right], \tag{S24}$$

where we have assumed $\kappa(0) = \kappa_0$. Variation of the protocol $\kappa(t)$ will induce a change in the free energy, and this is given by

$$\beta \Delta F_0(t) = \frac{1}{2} \ln \frac{\kappa(t)}{\kappa_0}, \tag{S25}$$

where we have used the usual definition of equilibrium free energy,

$$\beta F_0(t) = -\ln \left[ \int dx e^{-\beta U(x,\lambda(t))} \right]. \tag{S26}$$

Equation (S25) gives a simple working formula to compute the change in the free energy for any modulation of $\kappa(t)$. In the following, we compute two specific examples of such modulation namely (i) exponential variation, (ii) power law variation. Each of these cases is appended below.

**Exponential variation**

In this case, we assume that the stiffness varies exponentially with time so that $\kappa(t) = \kappa_0 e^{-m_e t}$, where $m_e$ is a constant. Applying Eq. (S25), we find

$$\beta \Delta F_0(t) = -\frac{m_e}{2} t \tag{S27}$$

which states that the change in free energy is linear in time. Therefore, as shown before, this protocol will guarantee the validity of Jarzynski equality, i.e., Eq. (S15). In particular, when $\beta = 1, m_e = 0.2$ and $t = 5$, we have $\Delta F_0 = -0.5$ so that $e^{-\Delta F_0} \sim 1.65$ as used in the main text.

**Power law variation**

We now consider the case when the stiffness has a power law variation with respect to time. Specifically, we consider $\kappa(t) = \kappa_0 (1 + m_p t)^{-2}$, with $m_p$ being constant. Using this in Eq. (S25), we find

$$\beta \Delta F_0(t) = -\ln(1 + m_p t), \tag{S28}$$

where the free energy changes logarithmically as a function of time. Also note here when $\beta = 1, m_p = 0.2$ and $t = 5$, we have $\Delta F_0 = -0.69315$ so that $e^{-\Delta F_0} = 2.0$ as mentioned in the main text.
Explicit computation of \( \langle e^{-W} \rangle_r \) can also be done by making use of Eq. (S14). We first write, in accordance with the Jarzynski relation,

\[
\langle e^{-W} \rangle_0 = e^{-\beta \Delta F_0(t)} = 1 + m_p t .
\]  \hfill (S29)

Substituting this expression into the RHS of Eq. (S14), we find

\[
\mathcal{L}_{t \rightarrow s} \left[ \langle e^{-W} \rangle_r \right] = \frac{m_p + r + s}{s(r + s) - m_p r}.
\]  \hfill (S30)

Finally inverting the Laplace transform in Eq. (S30) we obtain

\[
\langle e^{-W} \rangle_r = e^{-\frac{t}{2}} \left\{ \left( 2m_p + r \right) \sinh \left[ \frac{t}{2} \sqrt{r(4m_p + r)} \right] \right\} + \cosh \left[ \frac{t}{2} \sqrt{r(4m_p + r)} \right] .
\]  \hfill (S31)

Putting the values \( m_p = 0.2, r = 0.5, \) and \( t = 5 \) in the above expression, we get \( \langle e^{-W} \rangle_r = 2.273 \), which as expected, is non-identical to \( e^{-\Delta F_0} = 2.0 \) which was obtained using Eq. (S28) (see above). With this in hand we again confer that the Jarzynski equality does not hold since \( \Delta F_0 \) is not linear in time.

**DERIVATION OF EQ. (5)**

In this section we explicitly give a detailed derivation of Eq. (5). We start by rewriting Eq. (4) as

\[
\tilde{H}_0(k, s + r) = \tilde{H}_r(k, s) \left[ 1 - r \tilde{H}_0(k, s + r) \right] ,
\]  \hfill (S32)

We take the \( n \)-th order derivative respect to (\(-k\)) that yields

\[
\tilde{H}_0^{(n)}(k, s + r) = \sum_{l=0}^{n} \binom{n}{l} \tilde{H}_r^{(n-l)}(k, s) \left[ 1 - r \tilde{H}_0(k, s + r) \right]^{(l)} ,
\]  \hfill (S33)

where we have introduced the notation

\[
f^{(m)}(k, \cdot) = \frac{\partial^m f(k, \cdot)}{\partial (-k)^m} .
\]  \hfill (S34)

Solving for \( \tilde{H}_r^{(n)}(k, s) \) we get

\[
\tilde{H}_r^{(n)}(k, s) = \frac{\tilde{H}_0^{(n)}(k, s + r) - \sum_{l=1}^{n} \binom{n}{l} \tilde{H}_r^{(n-l)}(k, s) \left[ 1 - r \tilde{H}_0(k, s + r) \right]^{(l)}}{1 - r \tilde{H}_0(k, s + r)} .
\]  \hfill (S35)

Substituting

\[
\tilde{H}_r^{(n)}(k, s) \bigg|_{k=0} = \int_0^\infty dt \ e^{-st} \langle W^{n-l} \rangle_r = \mathcal{L}_s \left[ \langle W^{n-l} \rangle_r \right] ,
\]  \hfill (S36)

\[
\tilde{H}_0^{(n)}(k, s + r) \bigg|_{k=0} = \int_0^\infty dt \ e^{-(s+r)t} \langle W^n \rangle_0 = \mathcal{L}_{s+r} \langle W^n \rangle_0 ,
\]  \hfill (S37)

\[
1 - r \tilde{H}_0(0, s + r) = 1 - r \int_0^\infty dt \ e^{-(s+r)t} = \frac{s}{s + r} ,
\]  \hfill (S38)

\[
\left[ 1 - r \tilde{H}_0(k, s + r) \right]^{(n)} \bigg|_{k=0} = -r \int_0^\infty dt \ e^{-(s+r)t} \langle W^n \rangle_0 = -r \mathcal{L}_{s+r} \langle W^n \rangle_0 , \text{ for } n \geq 1,
\]  \hfill (S39)

into Eq. (S35), we finally reach Eq. (5).