Tomonaga’s Conjecture on Photon Self-Energy

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We investigate Tomonaga’s conjecture that the self-energy of photon should vanish to zero. In fact, the contribution of photon’s self-energy diagram violates the Lorentz invariance and therefore it is unphysical. In addition, there occurs no wave function renormalization of photon in the exact Lippmann-Schwinger equation for the vector potential and this confirms that the conjecture is correct. Further, it is shown that the gauge condition \( k_{\mu} \Pi^{\mu\nu}(k) = 0 \) of the vacuum polarization tensor does not hold and the relation is obtained simply due to a mathematical mistake in replacing the integration variables in the infinite integral.

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I. INTRODUCTION

A half century ago, Tomonaga stated that the self-energy of photon should vanish to zero even though he did not present any concrete proof at that time [1]. Since then, however, people have never considered his conjecture seriously, and up to now, the self-energy of photon has been included into the renormalization scheme by throwing away the quadratic divergence terms and by keeping only the terms which have the same shape as the Lagrangian density of photon. In his statement, he also claimed that the regularization scheme proposed by Pauli and Villars [2] should not be a proper method for the renormalization scheme, but unfortunately, however, there was no concrete proof to this statement either, as far as we can check his published works as research papers.

In this paper, we study his conjecture that the self-energy of photon should vanish to zero and prove that it is indeed a correct statement. The basic reason why the self-energy of photon should vanish is basically because the self-energy of photon arising from the vacuum polarization diagram violates the Lorentz invariance and therefore the process of the photon self-energy diagram is unphysical, and this is similar to the Feynman diagrams which violate the energy and momentum conservations. Further, the exact Lippmann-Schwinger equation for the vector potential \( A \) clearly shows that the vector field cannot be affected from the perturbation expansion [3]. Since this is the exact equation of motion for \( A \), there is no way that the vector potential can be influenced by any of the vacuum polarization effects.

The basic physics picture of the vacuum polarization was made up by Heisenberg in 1934, and in order to clarify the problem of the vacuum polarization effects, we are forced to examining the papers by Heisenberg and Euler [4, 5]. The evaluation of the self-energy of photon was carried out just in a similar way to the Feynman diagram calculations. They obtained the quadratic divergence and logarithmic divergence terms, in addition to the finite effects of the vacuum polarization, and they considered the corresponding Lagrangian density. The effective Lagrangian density Heisenberg and Euler proposed can take into account the effects of the vacuum polarization. Apart from the renormalization procedure, the vacuum polarization effect is almost the same as that of the modern version. However, if one looks into their calculation carefully, then one can easily realize that the effective Lagrangian density method is a dangerous attempt for physics. The Lagrangian density is the basic tool which can describe the physical processes, and if one changes its shape, then this means that the basic physical law itself is modified even if the matter fields are not present. This is indeed connected to the understanding of the concept of the vacuum in field theory. In fact, the fields which appear in the effective Lagrangian density should become operators in the next step calculations, and this should produce unphysical processes as a result.

On the other hand, the renormalization scheme of the self-energy of photon in the present day is to find and determine the same shape of the Lagrangian density after the renormalization of the vacuum polarization contributions. However, the procedure of the renormalization scheme of the self-energy of photon is entirely based on one simple relation which is called ”gauge condition” for the vacuum polarization tensor. In the renormalization scheme of QED, this gauge condition played the most important role as the excuse for throwing away the quadratic divergence contribution. Here, we show in a simple mathematical fashion that the gauge condition of \( \Pi^{\mu\nu}(k) \)

\[ k_{\mu} \Pi^{\mu\nu}(k) = 0 \]  (1.1)

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is invalid. As we show later, the proof of eq.(1.1) which has been done in most of the field theory textbooks is based on a mathematical mistake which is basically due to the wrong replacement of the integration variable in the infinite integral case. This problem of mathematics was, of course, realized and stated in the text book of Bjorken and Drell [6], but they made the same type of mistake in evaluating the vacuum polarization contributions in the regularization scheme by Pauli and Villars [2]. In this respect, one sees that the Pauli-Villars regularization does not change anything in physics, and it is a meaningless procedure. This can be well realized if one calculates any physical processes of loop diagrams such as $\pi^0 \rightarrow 2\gamma$ and the box diagram in photo-photon scattering, and they have no logarithmic divergences. Indeed, the evaluation of all of these physical processes shows that they are finite due to the kinematical cancellation of the divergent contributions, and in fact the Pauli and Villars regularization does not play any role in these calculations of the Feynman diagrams.

Before going to the discussion of the gauge condition of eq.(1.1), we should understand the physical meaning of the gauge invariance in connection with the QED renormalization scheme [7–11]. Any physical quantities calculated by fixing the gauge should not depend on the gauge choice, and in fact the gauge invariance of the electromagnetic interaction

$$H' = -e \int j_{\mu} A_{\mu} d^3r$$

(1.2)

can be guaranteed with the condition that the fermion current should be conserved, that is

$$\partial_{\mu} j_{\mu} = 0.$$  

(1.3)

The gauge invariance of $H'$ can be easily seen under the gauge transformation of $A_{\mu} = A'_{\mu} + \partial_{\mu} \chi$ as

$$H' = -e \int j_{\mu} (A_{\mu} - \partial_{\mu} \chi) d^3r = -e \int j_{\mu} A_{\mu} d^3r + e \int \partial_{\mu} (j^{\mu} \chi) d^3r = -e \int j_{\mu} A_{\mu} d^3r$$

(1.4)

where the surface integral vanishes and we made use of eq.(1.3). This is the most important condition of the gauge invariance in QED, and as long as we carry out the perturbation theory with free fermion basis states, then the calculated results are always gauge invariant. There is no additional constraint like eq.(1.1) in the QED renormalization scheme [12]. Since the calculations with eq.(1.1) give wrong answers to the evaluation of many basic Feynman diagrams, it is important to clarify the situation at the present stage.

II. TOMONAGA’S CONJECTURE

In this section, we prove that the self-energy of photon should vanish to zero. First, we start from the QED Lagrangian density which can describe the fermions interacting with the electromagnetic fields. It reads

$$\mathcal{L} = \bar{\psi}(i\partial_{\mu} \gamma^{\mu} - g A_{\mu} \gamma^{\mu} - m) \psi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

(2.1)

where $F^{\mu\nu}$ denotes the field strength and is given as

$$F^{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}.$$  

(2.2)

$A^{\mu}$ denotes the gauge field with $A^{\mu} = (A_0, A)$. In this case, we can obtain the Dirac equation with the electromagnetic interaction

$$(i\partial_{\mu} \gamma^{\mu} - g A_{\mu} \gamma^{\mu} - m) \psi = 0.$$  

(2.3)

Also, we find the Maxwell equation

$$\partial_{\mu} F^{\mu\nu} = g j^{\nu}$$

(2.4)

where the current density $j^{\nu}$ is defined as

$$j^{\nu} = \bar{\psi} \gamma^{\nu} \psi.$$  

(2.5)

As the gauge fixing condition, we can take the Coulomb gauge

$$\nabla \cdot A = 0.$$
A. Lippmann-Schwinger equations

If one wishes to carry out the S-matrix evaluation, then the Lippmann-Schwinger equations must be most suitable for figuring out what are physical observables out of the S-matrix contributions [3]. Therefore, we first write the Lippmann-Schwinger equations for fermions and vector field. The Lippmann-Schwinger equation for the fermion field $\psi$ becomes

$$\psi(x) = \psi_0(x) + g \int G_F(x, x') A_\mu(x') \gamma^\mu \psi(x') d^4x'$$  \hspace{1cm} (2.6)

where $G_F(x, x')$ denotes the Green function which satisfies the following equation

$$(i\partial_\mu \gamma^\mu - m)G_F(x, x') = \delta^4(x - x').$$  \hspace{1cm} (2.7)

$\psi_0(x)$ is the free fermion field solution. The Green function $G_F(x, x')$ can be explicitly written as

$$G_F(x, x') = \int \frac{1}{p_\mu \gamma^\mu - m + i\epsilon} e^{ip(x-x')} \frac{d^4p}{(2\pi)^4}.$$  \hspace{1cm} (2.8)

On the other hand, the Lippmann-Schwinger equation for the vector field $A$ becomes

$$A = A_0 + g \int G_0(x, x') j(x') d^4x'$$  \hspace{1cm} (2.9)

where $A_0$ denotes the free field solution of the vector field. Here, the Green function $G_0(x, x')$ satisfies

$$\left( \frac{\partial^2}{\partial t^2} - \nabla^2 \right) G_0(x, x') = \delta^4(x - x').$$  \hspace{1cm} (2.10)

This can be explicitly written as

$$G_0(x, x') = \int \frac{1}{-p_0^2 + p^2 + i\epsilon} e^{ip(x-x')} \frac{d^4p}{(2\pi)^4}.$$  \hspace{1cm} (2.11)

Since we employ the Coulomb gauge fixing, the equation of motion for the $A_0$ field becomes a constraint equation and thus can be solved exactly as

$$A_0(r) = \frac{g}{4\pi} \int \frac{j_0(r')}{|r - r'|} d^3r'.$$  \hspace{1cm} (2.12)

This is all that we need for the discussion of the renormalization procedure. Up to this point, we have not made any approximations for the equations of motion. Now we can quantize the fermion field $\psi$ as well as the vector field $A$, and the quantized fields should be denoted as $\hat{\psi}$, $\hat{A}$.

B. Wave Function Renormalization–Fermion Field

When we carry out the perturbation expansion, we can obtain the integral equations in powers of the coupling constant $g$ as

$$\hat{\psi}(x) = \hat{\psi}_0(x) + g \int G_F(x, x') \hat{A}_\mu(x') \gamma^\mu \hat{\psi}_0(x') d^4x' + g^2 \int G_F(x, x') \hat{A}_\mu(x') \gamma^\mu G_F(x', x'') \hat{A}_\nu(x'') \gamma^\nu \hat{\psi}_0(x'') d^4x' d^4x'' + \cdots.$$  \hspace{1cm} (2.13)

This equation clearly shows that the fermion field should be affected by the perturbation expansion, and if it diverges, then we have to renormalize the wave function so as to absorb the infinity. Indeed, the infinity is logarithmic divergence and can be well renormalized into the wave function $\psi_0$. 
C. Wave Function Renormalization—Vector Field

On the other hand, the situation for the vector field case is quite different from the fermion field. This can be easily seen from the equation of motion of eq.(2.9). In this case, there is no iteration possible for the $A$ field. The vector field $\hat{A}$ can be determined from eq.(2.9) only when the fermion numbers are conserved. The best example can be found when the annihilation of the fermion pair takes place. In this case, we can write eq. (2.9) as

$$\langle 0 | A | f \bar{f} \rangle = g \int G_0(x, x') \langle 0 | j(x') | f \bar{f} \rangle d^4x'$$  \hspace{1cm} (2.14)

where $| f \bar{f} \rangle$ denotes the fermion and anti-fermion state. Now, we can consider the following physical process of $e^+e^- \rightarrow e^+e^-$ which can be described in terms of the T-matrix as

$$T = -g(e\bar{e}) \int j \cdot A(e\bar{e}) = -g^2 \int \langle e\bar{e} | j(x) | 0 \rangle G_0(x, x') \langle 0 | j(x') | e\bar{e} \rangle d^4x' \hspace{1cm} (2.15)$$

where one can see that there appears no self-energy of photon term whatever one evaluates any physical processes in the Lippmann-Schwinger equation. From this equation, one finds that the vector field $\hat{A}$ cannot be affected by the renormalization procedure, and it always stays as a free state of photon. Since this is the exact equation of motion, there is no other possibility for the vector field. In this respect, it is just simple that the gauge field $\hat{A}$ always behaves as a free photon state in the evaluation of any Feynman diagrams.

D. Mass Renormalization

In the evaluation of the S-matrix in QED, there are some Feynman diagrams which are divergent, and therefore we should renormalize them into the mass term if they are consistent with fundamental symmetries.

1. Fermion Self-energy

The evaluation of the self-energy of fermions can be carried out in a straightforward way, and one can obtain the self-energy which has a logarithmic divergence of the momentum cut-off $\Lambda$. Since electron has a mass, one can renormalize this logarithmic divergence term into the new mass term. In this procedure, there is no conceptual difficulty and indeed one can relate this renormalized effect to the observed value of the Lamb shift in hydrogen atom, which is indeed a great success of the QED renormalization scheme.

Here, we should note that the evaluation of the Lamb shift energy has been done only when the non-relativistic wave function of the hydrogen atom is employed. This is made by Bethe [13], and he stated that the relativistic treatment should be in progress. However, since then, there has been no work presented which treats the relativistic wave function of hydrogen atom. This is, however, quite understandable since we cannot calculate the negative energy states of the hydrogen atom up to the present stage. On the other hand, people believe that they have calculated the Lamb shift energy relativistically, but this is only the relativistic correction on the Lamb shift energy, and the main term of the Lamb shift energy is still far from satisfactory since the Bethe’s treatment has the logarithmic divergence of the Lamb shift energy. But, in other words, since it is logarithmic divergence, it should be reliable for the Lamb shift energy up to the order of the magnitude evaluation.

2. Photon Self-energy

On the other hand, the evaluation of the self-energy of photon gives rise to the energy which has a quadratic divergence. There is no way to renormalize it into the renormalization scheme of QED since photon has no mass term. This clearly indicates that one should not consider the contributions of the photon self-energy diagrams into the renormalization scheme since they violate the Lorentz invariance. In this respect, one can now realize that the quadratic divergence term should be discarded because it is not consistent with the Lorentz invariance, and it has nothing to do with the gauge invariance.

For the energy of photon, we should be careful in which system we are calculating it. This is very important since, obviously, there is no system where photon is at rest. The system we are calculating should be the system in which
fermion is at rest. In this system, the energy of photon with its momentum $k$ must be described as $E_k = |k|$, and there is no other expression. Therefore, the Lagrangian density of the vector field $A_\mu$ should be always written as

$$\mathcal{L}_0 = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

and there should not be any modifications possible.

### III. VACUUM POLARIZATION- REVISIT

Up to this point, we have clarified that the self-energy of photon should vanish to zero, and therefore photon stays always massless and there is no effect from the renormalization procedure. However, there have been many works for the QED evaluation with the renormalization procedure which considered the renormalization of the vacuum polarization contribution. Therefore, we should clarify what should be the basic mistake in the treatment commonly used up to the present stage.

First, we should like to critically review the renormalization procedure of the self-energy of photon. The basic starting point of the renormalization procedure is connected to the quantity of the vacuum polarization tensor $\Pi_{\mu\nu}(k)$.

This quantity itself is not directly a physical observable, but somehow people have been making use of the vacuum polarization tensor.

#### A. Vacuum Polarization Tensor

If one carries out the self-energy diagram of photon, then one obtains the quadratic divergence contributions. This gives rise to some difficult problems, and here we show that there is no way to renormalize the contributions of the self-energy of photon into the Lagrangian density as the counter terms. First, we write the result of the standard calculation of the vacuum polarization

$$\Pi^{\mu\nu}(k) = ie^2 \int \frac{d^4 p}{(2\pi)^4} \text{Tr} \left[ \gamma^\mu \frac{1}{\slashed{p} - m + i\varepsilon} \gamma^\nu \frac{1}{\slashed{p} - \slashed{k} - m + i\varepsilon} \right] = \Pi^{\mu\nu}_{(1)}(k) + \Pi^{\mu\nu}_{(2)}(k)$$

where

$$\Pi^{\mu\nu}_{(1)}(k) = \frac{\alpha}{2\pi} \left( \Lambda^2 + m^2 - \frac{k^2}{6} \right) g^{\mu\nu}$$

and

$$\Pi^{\mu\nu}_{(2)}(k) = \frac{\alpha}{3\pi} (k^\mu k^\nu - k^2 g^{\mu\nu}) \left[ \ln \left( \frac{\Lambda^2}{m^2 e} \right) - 6 \int_0^1 \frac{dz}{z} (1 - z) \ln \left( 1 - \frac{k^2}{m^2 z(1 - z)} \right) \right]$$

where $\Lambda$ denotes the cutoff momentum. Here, the $\Pi^{\mu\nu}_{(1)}(k)$ corresponds to the quadratic divergence term and it has been claimed that this should be discarded since it violates the gauge invariance when one considers the counter term of the Lagrangian density. The $\Pi^{\mu\nu}_{(2)}(k)$ term can keep the gauge invariance, and therefore one can renormalize it into the new Lagrangian density.

#### B. Gauge Condition of $\Pi^{\mu\nu}(k)$

For a long time, people believe that the $\Pi^{\mu\nu}(k)$ should satisfy the relation of eq.(1.1)

$$k_\mu \Pi^{\mu\nu}(k) = 0$$

and this equation is called "gauge condition of $\Pi^{\mu\nu}(k)$". This is the basic reason why people discarded the first term of eq.(3.1). The proof of the above relation seems to be simple and straightforward as discussed in the text book of Bjorken and Drell [6]. Even though they noticed that the proof cannot be justified for the infinite integral, they accepted the relation as a result in the calculations. The serious problem is that, since then, most of the field theory textbooks took the gauge condition of $\Pi^{\mu\nu}(k)$ as granted.
Here, we show that the proof of eq.(1.1) is a simple mistake and the relation has no foundation at all. First, we present the usual method of the proof, and we rewrite the $k_{\mu} \Pi^{\mu\nu}(k)$ as

$$ k_{\mu} \Pi^{\mu\nu}(k) = i e^2 \int \frac{d^4p}{(2\pi)^4} \text{Tr} \left[ \left( \frac{1}{\not{p} - \not{k} - m + i\varepsilon} - \frac{1}{\not{p} - m + i\varepsilon} \right) \gamma^\nu \right]. \quad (3.3) $$

In the first term, the integration variable should be replaced as $q = p - k$ and thus one can prove that

$$ k_{\mu} \Pi^{\mu\nu}(k) = i e^2 \int \frac{d^4q}{(2\pi)^4} \text{Tr} \left[ \frac{1}{\not{q} - m + i\varepsilon} \gamma^\nu \right] - i e^2 \int \frac{d^4p}{(2\pi)^4} \text{Tr} \left[ \frac{1}{\not{p} - m + i\varepsilon} \gamma^\nu \right] = 0. \quad (3.4) $$

At a glance, this proof looks plausible. However, one can easily notice that the replacement of the integration variable is only meaningful when the integration is finite. In order to clarify the mathematical mistake in eq.(3.4), we present a good example which shows that one cannot make a replacement of the integral variable when the integral is infinity. Now, we evaluate the following integral

$$ Q = \int_{-\infty}^{\infty} \left( (x-a)^2 - x^2 \right) dx. \quad (3.5) $$

Here, if we replace the integral variable in the first term as $x' = x - a$, then we can rewrite eq.(9) as

$$ Q = \int_{-\infty}^{\infty} \left( x'^2 dx' - x^2 dx \right) = 0. \quad (3.6) $$

However, if we calculate it properly, then we find

$$ Q = \int_{-\infty}^{\infty} \left( (x-a)^2 - x^2 \right) dx = \int_{-\infty}^{\infty} (a^2 - 2ax) dx = a^2 \times \infty \quad (3.7) $$

which disagrees with eq.(3.6). If one wishes to carefully calculate eq.(3.5) by replacing the integration variable, then one should do as follows

$$ Q = \lim_{\Lambda \to \infty} \int_{-\Lambda}^{\Lambda} \left( (x-a)^2 - x^2 \right) dx = \lim_{\Lambda \to \infty} \left[ \int_{-\Lambda}^{-a} x'^2 dx' - \int_{-\Lambda}^{\Lambda} x^2 dx \right] = \lim_{\Lambda \to \infty} 2a^2\Lambda. \quad (3.8) $$

It is clear by now that the replacement of the integration variable in the infinite integral should not be made, and this is just the mistake which has been accepted as the gauge condition of the $\Pi^{\mu\nu}(k)$ in terms of eq.(1.1). Therefore, the requirement of the gauge condition of the vacuum polarization is unphysical. In this sense, the calculated result of the $\Pi^{\mu\nu}(k)$ should be the ones which are given in eqs.(3.2). Mathematically and physically, there is no other result of the $\Pi^{\mu\nu}(k)$ than eqs.(3.2).

Some of the readers of Bjorken and Drell’s text book [6] may have a question as to why they obtained the vanishing contribution of the $\Pi^{\mu\nu}_1(k)$ term. For this, one can easily check the defect of the scaling trick in which they ignored the scale dependence of the integral ranges, and one sees that the mistake in the scaling trick is just similar to the one which is shown in eq.(3.8). This clearly shows that one has to be careful for the variable change in the infinite integral.

**C. Number of Constraints**

If eq.(1.1) were correct, then the number of the constraints due to the gauge invariance condition would become four. However, this number is too large compared with the condition from the gauge invariance. The gauge fixing should be only one condition like Coulomb gauge fixing and so on. This clearly indicates that the condition of eq.(1.1) is obviously unphysical. The gauge fixing should be done before the field quantization, and there is no additional condition of the gauge invariance, apart from eq.(1.3) which is well satisfied in the perturbation theory of QED.
D. Dimensional Regularization

As is well known, the dimensional regularization \[14, 15\] gives the vanishing contribution to the \(\Pi^{\mu\nu}(k)\) term when the integration of four dimensions \(d^4p\) is replaced by \(d^Dp\) with \(D = 4 - \epsilon\). However, this regularization is quite odd since the result of \(\Pi^{\mu\nu}(1)(k)\) term cannot be recovered when one sets \(\epsilon \to 0\), in contrast to any other regularizations such as the \(\zeta\)–function regularization. For example, in the \(\zeta\)–function regularization, one can, of course, recover the original divergence when one sets \(\epsilon \to 0\). In this respect, the dimensional regularization must have some simple mathematical problems which have nothing to do with physics. On the other hand, the divergence of the \(\Pi^{\mu\nu}(2)(k)\) term can be recovered at the limit of \(\epsilon \to 0\), and this must be connected to the fact that the integration in this case is convergent in the complex plane as long as one takes the integration dimension of \(D = 4 - \epsilon\) where the contour integration at the infinite semi-circle of \(R\) vanishes as \(R^{-\epsilon}\) with \(R \to \infty\). Therefore, the dimensional regularization can only evaluate the \(\Pi^{\mu\nu}(2)(k)\) term properly.

On the other hand, the reason why the \(\Pi^{\mu\nu}(1)(k)\) term disappears in the dimensional regularization can be easily understood as follows. In terms of the contour integration at the infinite semi-circle of \(R\), it diverges like \(R^{2-\epsilon}\) with \(R \to \infty\), and therefore it cannot be recovered once it is thrown away. Thus, it is now obvious that there is no mathematical reason to justify the calculation of the dimensional regularization.

At this point it may be interesting to clarify the reason as to why the calculated result by the dimensional regularization can satisfy the gauge condition of eq.(1.1). From eq.(3.3), one can see that the infinite terms are set to zero in the calculation of the dimensional regularization while the finite terms can be properly evaluated by the replacement of the integral variable, and thus those finite terms of the first and the second terms in the right hand side cancel with each other. Therefore, the calculation of the dimensional regularization can indeed satisfy the gauge condition of eq.(1.1), but this is, of course, accidental.

E. Physical Processes Involving Vacuum Polarizations

In nature, there are a number of Feynman diagrams which involve the vacuum polarization. The best known physical process must be the \(\pi^0\) decay into two photons, \(\pi^0 \to \gamma + \gamma\). This process of the Feynman diagrams can be well calculated in terms of the nucleon and anti-nucleon pair creation where these fermions couple to photons \[16\]. In this calculation, one knows that the loop integral gives a finite result since the apparent logarithmic divergence vanishes to zero due to the kinematical cancellation. Also, the physical process of photon-photon scattering involves the box diagrams where electrons and positrons are created from the vacuum state. As is well known, the apparent logarithmic divergence of this box diagrams vanishes again due to the kinematical cancellation, and the evaluation of the Feynman diagrams gives a finite number. This is clear since all of the perturbative calculations employ the free fermion basis states which always satisfy the current conservation of \(\partial_{\mu}j^\mu = 0\). In these processes, one does not have any additional “gauge conditions” in the evaluation of the Feynman diagrams. In this respect, if the process is physical, then the corresponding Feynman diagram should become finite without any further constraints of the gauge invariance nor regularizations.

IV. EFFECTIVE LAGRANGIAN DENSITY OF HEISENBERG FOR VACUUM POLARIZATION

Here, we briefly review the calculations of the self-energy of photon by Heisenberg and Euler \[4, 5\]. The calculations are based on the vacuum polarization due to the interaction of photon with positive and negative energy fermions. In this calculation, they started from the assumption that the negative energy fermions can interact with the electromagnetic fields even if they are static fields. This is based on the misunderstanding that the negative energy fermions are present as if they were the same as the positive energy fermions. In reality, the negative energy fermions occupy the negative energy states which are specified by their momenta and there are no coordinate dependences in the negative energy fermions. The creation of the fermion and anti-fermion pairs can be only possible from the time dependent interactions.

A. Effective Lagrangian Density

As for the self-energy of photon, they obtain the results which contain the quadratic divergence and logarithmic divergence terms. In addition, they obtain some finite contributions. Up to this point, their results are practically
the same as the modern calculations. Now the problem arises when they construct the Lagrangian density from their calculated results. They write the effective Lagrangian density

\[ \mathcal{L} = \frac{1}{2}(E^2 - B^2) + \alpha \int_0^\infty \frac{d\eta}{\eta^2} e^{-\eta \eta^2 \left( \frac{1}{\mathcal{E}_0^{*}} \sqrt{E^2 - B^2} + \frac{2 \mathcal{C}_0}{\mathcal{E}_0^{*}} \mathcal{E}_0^{*} \mathcal{B} \mathbf{E} \cdot \mathbf{B} + c.c. \right) + \frac{\eta^2}{3} (E^2 - B^2) \right) } \]  

(4.1)

where \( \mathcal{E}_0 \) is given as \( \mathcal{E}_0 = \frac{m^2}{\alpha} \). This effective Lagrangian density formulation is a dangerous attempt since eventually those fields should be treated as field operators after the field quantization. We believe that any fields which are quantized should be fundamental fields, and these fields which appear in Lagrangian density of eq.(4.1) are not the fundamental fields any more. In this case, we may ask as to what it means by the effective Lagrangian density? The basic question is the physical meaning of the fields \( \mathbf{E} \) and \( \mathbf{B} \) since they interact with each other by themselves. In particular, one cannot define the free electromagnetic field from this Lagrangian density since it corresponds to the physical state with no matter fields. This is very serious, and in fact, it is not consistent with observations that photon is always in a free state.

In summary, the problem of the effective Lagrangian method is that it is inconsistent with the definition of the vacuum state and therefore it cannot agree with the renormalization scheme.

B. Proper Treatment and Renormalization Scheme

By now it becomes clear that the treatment of Heisenberg concerning the self-energy of photon is incorrect. From the physical quantity which is calculated in the second order perturbation theory of the vacuum polarization due to photon, one can obtain the energy of the process as the function of the four momentum \( k^\mu \) of photon

\[ \Delta E = f(k^\mu). \]  

(4.2)

What one should do or one can do is that one should find the same shape as the original Lagrangian density as given

\[ \mathcal{L}_0 = \frac{1}{2}(E^2 - B^2) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \]  

(4.3)

by redefining the fields \( \mathbf{E} \) and \( \mathbf{B} \). In this respect, one sees that the only possible shape of the vacuum polarization energy should be written as

\[ \Delta E = C_0 k^2 \]  

(4.4)

where \( C_0 \) denotes some constant which may well be an infinite quantity. This is clear since, by noting \( \epsilon_\mu k^\mu = 0 \) and \( \epsilon_\mu \epsilon^\mu = 1 \), one can rewrite the above energy as

\[ \Delta E = C_0 \epsilon^\mu (g_{\mu\nu} k^2 - k_\mu k_\nu) \epsilon^\nu \]  

(4.5)

which corresponds to the following Lagrangian density

\[ \mathcal{L}' = C_0 A^\mu (g_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu) A^\nu. \]  

(4.6)

This can be renormalized into the original Lagrangian density. However, in reality, the calculated result of the finite term is written by eq.(4.1), but in addition, there are other infinite terms present in the vacuum polarization contributions. The calculated result of the vacuum polarization energy can be given as

\[ \Delta E = \frac{\alpha}{2\pi} (\Lambda^2 + m^2) - \frac{\alpha}{3\pi} \left( \frac{1}{4} + C' \right) k^2 \]  

(4.7)

where \( C' \) is written as

\[ C' = \ln \left( \frac{\Lambda^2}{m^2 \epsilon} \right) - 6 \int_0^1 dz z (1 - z) \ln \left( 1 - \frac{k^2}{m^2 \epsilon} z (1 - z) \right). \]

Therefore, there is no way to renormalize the self-energy of photon into the original Lagrangian density, and the only physically correct way is to discard all of the self-energy of photon contributions. This conclusion is just the same as the one that is presented in section 2 in terms of Tomonaga’s conjecture. The renormalization procedure without the photon self-energy effect is most natural, and up to the present stage, there is no experimental evidence that shows any of the effects arising from the self-energy of photon.
C. Improper Application of Effective Lagrangian

For a long time, people have been applying the effective Lagrangian density of Heisenberg to physical processes, and some of the applications give rise to rather serious problems in the fundamental interactions.

1. Uehling Potential

The most important of all is the Uehling potential which is obtained by Uehling [17]. By now it is clear that there is no modification of the Coulomb potential, and there is no way to obtain the Uehling potential. This Uehling potential is obtained by making double mistakes. The first one is just related to the renormalization procedure, and the effective Lagrangian method gives rise to the unphysical effect as discussed above. The other important mistake is connected to the Coulomb potential. As can be easily seen from the field quantization procedure, one has to first fix the gauge, and by choosing the Coulomb gauge fixing, one can obtain the equation for the $A_0$ field which becomes a constraint equation. Therefore, one finds that the $A_0$ field becomes time independent and thus the $A_0$ field can be solved exactly as given in eq.(2.12). There is no modification of the Coulomb field, and this result is stronger than the renormalization procedure. This means that the $A_0$ field should not be quantized, and therefore it is not involved in the renormalization scheme from the beginning.

2. Photon-Photon Scattering

Further, there is some application of the effective Lagrangian density to the evaluation of the scattering cross section between two photons [18, 19]. The cross section which is derived by making use of the effective Lagrangian density has no foundation at all, and it is physically incorrect. The proper cross section between two photons has been calculated, and it is completely different from the old version of the photon-photon cross section [20]. In fact, the photon-photon cross section is rather large, and it should be well detectable if one can control the time scale of photon in order to make a head-on collisions. However, the beam focusing of photon at low energy may well be difficult, and it should be non-trivial to make the focusing of photon in a sufficiently high standard [21].

V. CONCLUSIONS

We have examined Tomonaga’s conjecture that the self-energy of photon should vanish to zero. Here, we prove that his conjecture is correct, and the self-energy of photon should indeed be taken to be zero. This proof is mainly based on the exact Lippmann-Schwinger equation for the vector potential which clearly shows that there is no wave function renormalization possible for any iteration procedures. In addition, the photon self-energy violates the Lorentz invariance, and it is unphysical. Therefore it is simply discarded so as to keep the basic symmetry property of the system.

In addition, we critically review the photon self-energy treatment which has been considered to be a standard procedure until now. It is known that the quadratic divergence of the photon self-energy diagram is the only defect in the renormalization scheme of QED, and people discarded this divergent parts by requiring the gauge condition of the vacuum polarization tensor $\Pi^{\mu\nu}(k)$ since it is believed that $k_\mu\Pi^{\mu\nu}(k) = 0$ should hold. This condition of the gauge invariance has been used in the calculations of many vacuum polarization diagrams. However, we prove that this relation does not hold since it is derived by the simple mathematical mistake. Even though some people noticed that the derivation is wrong in mathematics, they continue to employ this gauge condition of the vacuum polarization tensor in their calculations of Feynman diagrams. Therefore, there are quite a few examples of the Feynman diagram evaluations which are incorrect because of the wrong condition of the gauge invariance.

By now, it becomes clear that photon should always stay massless, and there is no renormalization of photon propagation. It is just simple. The only procedure one should consider is the self-energy of fermion, and there is neither conceptual nor technical difficulty of treating the renormalization of fermion self-energy.

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