A Fitting Lemma for \( \mathbb{Z}/2 \)-graded modules

by

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Abstract. We study the annihilator of the cokernel of a map of free \( \mathbb{Z}/2 \)-graded modules over a \( \mathbb{Z}/2 \)-graded skew-commutative algebra in characteristic 0 and define analogues of its Fitting ideals. We show that in the “generic” case the annihilator is given by a Fitting ideal, and explain relations between the Fitting ideal and the annihilator that hold in general. Our results generalize the classical Fitting Lemma, and extend the key result of Green [1999]. They depend on the Berele-Regev theory of representations of general linear Lie super-algebras.

Introduction.

The classical Fitting Lemma (Fitting [1936]) gives information about the annihilator of a module over a commutative ring in terms of a presentation of the module by generators and relations. More precisely, let

\[ \phi : R^m \to R^d \]

be a map of finitely generated free modules over a commutative ring \( R \), and for any integer \( t \geq 0 \) let \( I_t(\phi) \) denote the ideal in \( R \) generated by the \( t \times t \) minors of \( \phi \). Fitting’s result says that the module coker \( \phi \) is annihilated by \( I_d(\phi) \), and that if \( \phi \) is the generic map—represented by a matrix whose entries are distinct indeterminates—then the annihilator is equal to \( I_d(\phi) \). Thus \( I_d(\phi) \) is the best approximation to the annihilator that is compatible with base change. Moreover, \( I_d(\phi) \) is not too bad an approximation to \( \text{ann coker} \phi \) in the sense that \( I_d(\phi) \supset (\text{ann coker} \phi)^d \), or more precisely \( (\text{ann coker} \phi)I_t(\phi) \subset I_{t+1}(\phi) \) for all \( 0 \leq t < d \).

In this paper we will prove a corresponding result in the case of \( \mathbb{Z}/2 \)-graded modules over a skew-commutative \( \mathbb{Z}/2 \)-graded algebra containing a field \( K \) of characteristic 0. Let \( R \) be a \( \mathbb{Z}/2 \)-graded skew-commutative \( K \)-algebra: that is, \( R = R_0 \oplus R_1 \) as vector spaces, \( R_0 \) is a commutative central subalgebra, \( R_0R_j \subset R_{i+j} \text{ (mod } 2) \), and every element of \( R_1 \) squares to 0. Any homogeneous map \( \phi \) of \( \mathbb{Z}/2 \)-graded free \( R \)-modules may be written in the form

\[ \phi : R^m \oplus R^m(1) \xrightarrow{\begin{pmatrix} X & A \\ B & Y \end{pmatrix}} R^d \oplus R^c(1), \]

where \( X, Y \) are matrices of even elements of \( R \), and \( A, B \) are matrices of odd elements. We will define an ideal \( I_{\Lambda(d,e)} \) and show that it is contained in the annihilator of the cokernel of \( \phi \), with equality in the generic case where the entries of the matrices \( X, Y, A, B \) are indeterminates (that is, \( R \) is a polynomial ring on the entries of \( X \) and \( Y \) tensored with an exterior algebra on the entries of \( A \) and \( B \).) We give examples to show that the annihilator can be quite different in positive characteristic.

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Now let $K$ be a field, and let $U = U_0 \oplus U_1$ and $V = V_0 \oplus V_1$ be $\mathbb{Z}/2$-graded vector spaces of dimensions $(d,e)$ and $(m,n)$ respectively. We consider the generic ring

$$S = S(V \otimes U) := S(V_0 \otimes U_0) \otimes S(V_1 \otimes U_1) \otimes (V_0 \otimes U_1) \otimes \wedge (V_1 \otimes U_0),$$

where $S$ denotes the symmetric algebra and $\wedge$ denotes the exterior algebra, and the \textit{generic}, or \textit{tautological} map

$$\Phi : S \otimes V \longrightarrow S \otimes U^*.$$

This map $\Phi$ is defined by the condition that $\Phi|_V = 1 \otimes \eta : V \rightarrow V \otimes_K U \otimes_K U^* \subset R \otimes_K U^*$, where $\eta : K \rightarrow U \otimes_K U^*$ is the dual of the contraction $U^* \otimes_K U \rightarrow K$. We will make use of this notation throughout the paper.

We will compute the annihilator of the cokernel of $\Phi$. Of course if we specialize $\Phi$ to any map of free modules $\phi$ over a $\mathbb{Z}/2$-graded ring, preserving the grading, then we can derive elements in the annihilator of the cokernel of $\phi$ by specializing the annihilator of the cokernel of $\Phi$.

In the classical case, where $V$ and $U$ have only even parts ($e = n = 0$) the annihilator is an invariant ideal for the action of the product of general linear groups $GL(V) \times GL(U)$. Such invariant ideals have been studied by DeConcini, Eisenbud, and Procesi in [1980] and have a very simple arithmetic. In the general case Berele and Regev [1987] have developed a highly parallel theory, using the $\mathbb{Z}/2$-graded Lie algebra $\mathfrak{g} = \mathfrak{gl}(V) \times \mathfrak{gl}(U)$ in place of $GL(V) \times GL(U)$. They show that the generic ring $S$ is a semisimple representation of $\mathfrak{g}$ (even though not all the representations of $\mathfrak{g}$ are semisimple) and that the irreducible summands of $S$ of total degree $t$ are parametrized by certain partitions of the integer $t$, just as in the commutative case. The Berele-Regev theory is described in detail below, in Section 1 of this paper.

If $\Lambda$ is a partition, we write $I_\Lambda$ for the ideal of $S$ generated by the irreducible representation corresponding to $\Lambda$. If $\phi$ is a matrix representing any map of $\mathbb{Z}/2$-graded free modules over a $\mathbb{Z}/2$-graded skew-commutative $K$ algebra $R$, then there is a unique ring homomorphism $\alpha : S \rightarrow R$ such that $\phi = \alpha(\Phi)$, and we write $I_\Lambda(\phi) := \alpha(I_\Lambda(\Phi))R$ for the ideal generated by the image of $I_\Lambda = I_\Lambda(\Phi)$.

If $e = n = 0$ the classical Fitting Lemma shows that the annihilator of the module with generic presentation matrix as above is $I_d(\Phi)$. In representation-theoretic terms this is the ideal generated by the representation $\wedge^d V \otimes \wedge^d U$, the irreducible representation associated to the partition with one term $(d)$. In our notation, $I_d(\Phi) = I_{(d)} = I_{(d)}(\Phi)$. Here is the generalization which is the main result of this paper:

**Theorem 1.** Suppose that $K$ is a field of characteristic 0, and let

$$\phi : R^m \oplus R^n(1) \xrightarrow{\begin{pmatrix} X & A \\ B & Y \end{pmatrix}} R^d \oplus R^e(1),$$

be a $\mathbb{Z}/2$-graded map of free modules over a $\mathbb{Z}/2$-graded skew-commutative $K$-algebra $R$.

**a)** When $R = S$ and $\phi = \Phi$, the generic map defined above, the annihilator of the cokernel of $\Phi$ is $I_{\Lambda(d,e)}(\Phi)$, where $\Lambda(d,e)$ is the partition $(d + 1, d + 1, \ldots, d + 1, d)$ of $(d + 1)(e + 1) - 1$ into $e + 1$ parts. In general we have $I_{\Lambda(d,e)}(\phi) \subset \text{ann coker}(\phi)$.

**b)** If $x_1, \ldots, x_e \in \text{ann coker}(\phi)$, then $x_1 \ldots x_e \in I_{\Lambda(0,e)}(\phi)$. Moreover, if $0 \leq s \leq d - 1$, and $x_1, \ldots, x_{e+1} \in \text{ann coker}(\phi)$, then $x_1 \ldots x_{e+1} I_{\Lambda(s,e)}(\phi) \subset I_{\Lambda(s+1,e)}(\phi)$.

The proof is given in sections 2 and 3 below.
In the classical case \((e = n = 0)\) we can also describe the annihilator of coker \(\Phi\) by saying that it is nonzero only if \(m \geq d\), and then it is generated, as a \(\mathfrak{gl}(V) \times \mathfrak{gl}(U)\)-ideal, by an \(m \times m\) minor of \(\Phi\). To simplify the general statement we note that a shift of degree by 1 does not change the annihilator of the cokernel of \(\Phi\), but has the effect of interchanging \(m\) with \(n\) and \(d\) with \(e\).

**Corollary 2.** With notation as above, the annihilator of the cokernel of \(\Phi\) is nonzero only if

\[\begin{array}{ll}
a) & m > d \quad \text{(or symmetrically } n > e) \\
\end{array}\]

or

\[\begin{array}{ll}
b) & m = d \quad \text{and} \quad n = e.
\end{array}\]

In each of these cases the annihilator is generated as a \(g\)-ideal by one element \(Z\) of degree \(de + d + e\) defined as follows:

In case \(a)\) when \(m > d\)

\[Z = Z_1 \cdot X(1, \ldots, d|1, \ldots, d)\]

where \(X(1, \ldots, d|1, \ldots, d)\) is the \(d \times d\) minor of \(X\) corresponding to the first \(d\) columns and \(Z_1 = \prod_{j \leq e, k \leq d+1} b_{j,k}\) is the product of all the elements in the first \(d+1\) columns of \(B\) (and symmetrically if \(n > e\));

In case \(b)\)

\[Z = W_1 \cdots W_e \cdot det(X)\]

where \(W\) is the \((d + 1) \times (d + 1)\) minor of \(\Phi\) containing \(X\) and the entry \(y_{s,s}\), that is,

\[W_s = det(X) y_{s,s} + \sum_{1 \leq i, \leq d} \pm det(X(\hat{i}, \hat{j})) a_{i,s} b_{s,j}.\]

Corollary 2 follows from intermediate results in the proof of Theorem 1a. We next give some examples of Theorem 1a and Corollary 2.

**Example 1.** Suppose that \(d = n = 0\), so that the presentation matrix \(B\) has only odd degree entries. A central observation of Green [1999] is that the “exterior minors” of \(\Phi\) are in the annihilator of coker \(\Phi\). The element \(Z\) of Corollary 2 is the product of the elements in the first column of \(\Phi\). Quite generally, it is not hard to see that the product of all elements in a \(K\)-linear combination of the columns of \(B\) is an exterior minor in Green’s sense. The representation corresponding to the partition \((1, \ldots, 1)\) of \(e\) is generated by \(\binom{m+e-1}{e}\) such products, so \(\binom{m+e-1}{e}\) exterior minors generate the annihilator in the generic case.

For example, taking \(m = 2\), \(e = 2\) the annihilator of the cokernel of the generic matrix

\[
\begin{pmatrix}
b_{1,1} & b_{1,2} \\
b_{2,1} & b_{2,2}
\end{pmatrix},
\]

where the variables all have odd degree, is minimally generated by the three exterior minors

\[b_{1,1} b_{2,1}, \quad b_{1,2} b_{2,2}, \quad (b_{1,1} + b_{1,2})(b_{2,1} + b_{2,2}).\]

**Example 2.** Now suppose that our generic matrix has size \(2 \times 2\) with the first row even and the second row odd \((m = 2, \ n = 0, \ d = e = 1)\):

\[
\begin{pmatrix}
x_{1,1} & x_{1,2} \\
x_{1,1} & x_{1,2}
\end{pmatrix}.
\]
In this case our result shows that the cokernel has annihilator equal to the product

$$(x_{1,1}, x_{1,2})(b_{1,1}b_{1,2}, x_{1,1}b_{1,2} - x_{1,2}b_{1,1}),$$

which is minimally generated by 4 elements. The element $Z$ is $x_{1,1}b_{1,1}b_{1,2}$.

**Example 3.** As a final $2 \times 2$ example, consider the case $m = n = d = e = 1$ which for simplicity we write as

$$\begin{pmatrix} x & a \\ b & y \end{pmatrix}.$$ 

Here the annihilator of the cokernel is again minimally generated by 4 elements, namely

$$axy, bxy, (xy - ab)x, (xy + ab)y.$$ 

The element $Z$ is $(xy + ab)x$. In an Appendix we will explain the $g$ action on these elements.

**Positive characteristics.** Already with $m = n = d = e = 1$ as in Example 3 the annihilator is different in characteristic 2: in characteristic zero the annihilator is generated by forms of degree 3, but in characteristic 2 the algebra $R$ is commutative, so the determinant $xy - ab$ is in the annihilator as well.

The annihilator can differ in other characteristics as well. Macaulay2 computations show that the case $d = 1$, $e = p - 1$, $m = 2$, $n = 0$ is exceptional in characteristic $p$ for $p = 3, 5$ and 7. Perhaps the same holds for all primes $p$.

The cokernel of the generic matrix over the integers can also have $\mathbb{Z}$-torsion. For example Macaulay2 computation shows that if $d = 1$, $e = 2$, $m = 3$, $n = 1$ then the cokernel of $\Phi_{\mathbb{Z}}$ has 2-torsion.

Our interest in extending the Fitting Lemma was inspired by Mark Green’s paper [1999] where he shows that the exterior minors are in the annihilator. Green’s striking use of his result to prove one of the Eisenbud, Koh, Stillman conjectures on linear syzygies turns on the fact that if $N$ is a module over a polynomial ring $S = K[X_1, \ldots, X_m]$ then $T := \text{Tor}_{\ast}^S(K, M)$ is a module over the ring $R = \text{Ext}_{\ast}^S(K, K)$, which is an exterior algebra. Green in effect translated the hypothesis of the linear syzygy conjecture into a statement about the degree 1 part of the $R$-free presentation matrix of the submodule of $T$ representing the linear part of the resolution of $N$, and then showed that the exterior minors generated a certain power of the maximal ideal of the exterior algebra, which was sufficient to prove the Conjecture. Green’s result only gives information on the annihilator in the case where the elements of the presentation matrix are all odd. Elements of even degree in an exterior algebra can behave (if the number of variables is large) very much like variables in a polynomial ring, at least as far as expressions of bounded degree are concerned. Thus to extend Green’s work it seemed natural to deal with the case of $\mathbb{Z}/2$-graded algebras.

This work is part of a program to study modules and resolutions over exterior algebras; see Eisenbud-Floystad-Schreyer [2001], and Eisenbud-Popescu-Yuzvinsky [2000] for further information.

We would never have undertaken the project reported in this paper if we had not had the program Macaulay2 (www.math.uiuc.edu/Macaulay2) of Grayson and Stillman as a tool; its ability to compute in skew commutative algebras was invaluable in figuring out the pattern that the results should have and in assuring us that we were on the right track.
1. Berele-Regev Theory

For the proof of Theorem 1 we will use the beautiful results of Berele and Regev [1987] giving the structure of $R$ as a module over $\mathfrak{g}$. For the convenience of the reader we give a brief sketch of what is needed. We make use of the notation introduced above: $U = U_0 \oplus U_1$ and $V = V_0 \oplus V_1$ are $\mathbb{Z}/2\mathbb{Z}$ graded vector spaces over the field $K$ of characteristic 0 with $\dim U = (d,e)$ and $\dim V = (m,n)$.

The $\mathbb{Z}/2\mathbb{Z}$-graded Lie algebra $\mathfrak{gl}(V)$ is the vector space of $\mathbb{Z}/2\mathbb{Z}$-graded endomorphisms of $V = V_0 \oplus V_1$. Thus

$$\mathfrak{gl}(V) = \mathfrak{gl}(V)_0 \oplus \mathfrak{gl}(V)_1,$$

where $\mathfrak{gl}(V)_0$ is the set of endomorphisms preserving the grading of $V$ and $\mathfrak{gl}(V)_1$ is the set of endomorphisms of $V$ shifting the grading by 1. Additively

$$\mathfrak{gl}(V)_0 = \text{End}_K(V_0) \oplus \text{End}_K(V_1),$$

$$\mathfrak{gl}(V)_1 = \text{Hom}_K(V_0, V_1) \oplus \text{Hom}_K(V_1, V_0)$$

The commutator of the pair of homogeneous elements $x, y \in \mathfrak{gl}(V)$ is defined by the formula

$$[x, y] = xy - (-1)^{\deg(x)\deg(y)}yx.$$

By a $\mathfrak{gl}(V)$-module we mean a $\mathbb{Z}/2\mathbb{Z}$-graded vector space $M = M_0 \oplus M_1$ with a bilinear map of $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces $\circ : \mathfrak{gl}(V) \times M \to M$ satisfying the identity

$$[x, y] \circ m = x \circ (y \circ m) - (-1)^{\deg(x)\deg(y)}y \circ (x \circ m)$$

for homogeneous elements $x, y \in \mathfrak{gl}(V), m \in M$.

In contrast to the classical theory, not every representation of the $\mathbb{Z}/2\mathbb{Z}$-graded Lie algebra $\mathfrak{gl}(V)$ is semisimple. For example its natural action on mixed tensors $V^{\otimes k} \otimes V^{\ast \otimes l}$ is in general not completely reducible. However, its action on $V^{\otimes t}$ decomposes just as in the ungraded case:

**Proposition 1.1.** The action of $\mathfrak{gl}(V)$ on $V^{\otimes t}$ is completely reducible for each $t$. More precisely, the analogue of Schur’s double centralizer theorem holds and the irreducible $\mathfrak{gl}(V)$-modules occurring in the decomposition of $V^{\otimes t}$ are in 1-1 correspondence with irreducible representations of the symmetric group $\Sigma_t$ on $t$ letters. These irreducibles are the Schur functors

$$\mathcal{S}_\lambda(V) = e(\lambda)V^{\otimes t}$$

where $e(\lambda)$ is a Young idempotent corresponding to a partition $\lambda$ in the group ring of the symmetric group $\Sigma_t$.

This notation is consistent with the notation above in the sense that the $d$-th homogeneous component of the ring $\mathcal{S}(V)$ is $\mathcal{S}_d(V)$ where $d$ represents the partition $(d)$ with one part.

Here we use the symbol $\mathcal{S}_\lambda$ to denote the $\mathbb{Z}/2\mathbb{Z}$-graded version of the Schur functor $S_\lambda$; the latter acts on ungraded vector spaces. Recall that the functor $\wedge^\lambda$ is by definition the same as the functor $S_\lambda'$, where $\lambda'$ denotes the partition which is conjugate to $\lambda$. (For example, the conjugate partition to $(2)$ is $(1,1)$. ) We will extend this by writing $\wedge^\lambda := \mathcal{S}_\lambda$ for the $\mathbb{Z}/2\mathbb{Z}$-graded vesion. The partition $(d)$ with only one part will be denoted simply $d$, so for example $\mathcal{S}_2(V) = \wedge^{(1,1)}V = S_2(V_0) \oplus (V_0 \otimes V_1) \oplus \wedge^2 V_1$ and similarly $\wedge^2 V = \mathcal{S}_{(1,1)}V = \wedge^2 V_0 \oplus V_0 \otimes V_1 \oplus S_2(V_1)$. In each case the decomposition is as representations of the subalgebra $\mathfrak{gl}(V_0) \times \mathfrak{gl}(V_1) \subset \mathfrak{gl}(V)$. Similar decompositions hold for all $\mathcal{S}_d$ and $\wedge^d V$. (If we were not working in characteristic zero we would use divided powers in place of symmetric powers in the description of $\wedge V$.)
Proposition 1.1 implies that the parts of the representation theory of \( \mathfrak{gl}(V) \times \mathfrak{gl}(U) \) that involve only tensor products of \( V \) and \( U \) and their summands are parallel to the representation theory in the case \( V_1 = U_1 = 0 \), which is the classical representation theory of product of the two general linear Lie algebras \( \mathfrak{gl}(V_0) \times \mathfrak{gl}(U_0) \).

The Proposition also implies that the decompositions into irreducible representations of tensor products of the \( S_\lambda(V) \), as well as the decompositions of their symmetric and exterior powers, correspond to the decompositions in the even case: we just have to replace the ordinary Schur functors \( S, \wedge \) by their \( \mathbb{Z}/2 \)-graded analogues \( S, \wedge \).

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Corollary 1.2. The \( t \)th component \( S_t(V \otimes U) \) of \( S(V \otimes U) \) decomposes as a \( g \)-module as

\[
S_t(V \otimes U) = \oplus_{\lambda, |\lambda|=t} S_\lambda(V) \otimes S_\lambda(U).
\]

Another application of the same principle shows that to describe the annihilator of the cokernel of \( \Phi \), and what generates it, it suffices to describe which representations \( S_\lambda V \otimes S_\lambda U \) it contains:

Corollary 1.3. If \( I \subset S(V \otimes U) \) is a \( g \)-invariant ideal, then \( I \) is a sum of subrepresentations \( S_\lambda V \otimes S_\lambda U \). Moreover, the ideal generated by \( S_\lambda V \otimes S_\lambda U \) contains \( S_\mu V \otimes S_\mu U \) if and only if \( \mu \supset \lambda \).

Although it is not so simple to describe the vectors in \( S_t(V \otimes U) \) that lie in a given irreducible summand, we can, as in the commutative case, define a filtration that has these irreducible representations as successive factors. We start by defining a map \( \rho_t : \bigwedge^t V \otimes \bigwedge^t U \to S_t(V \otimes U) \) as the composite

\[
\bigwedge^t V \otimes \bigwedge^t U \longrightarrow \otimes^t V \otimes \otimes^t U \longrightarrow S_t(V \otimes U)
\]

where the first map is the tensor product of the two diagonal maps (here we use the sign conventions for \( \mathbb{Z}/2 \)-graded vector spaces) and the second map simply pairs corresponding factors. Thus

\[
\rho_t(v_1 \wedge \ldots \wedge v_t \otimes u_1 \wedge \ldots \wedge u_t) = \sum_{\sigma \in \Sigma_t} \pm(v_1 \otimes u_{\sigma(1)}) \ldots (v_t \otimes u_{\sigma(t)})
\]

where the sign \( \pm \) is the sign of the permutation \( \sigma \) adjusted by the rule that switching homogeneous elements \( x, y \) from either \( V \) or \( U \) means we multiply by \((-1)^{\deg(x)\deg(y)}\). For example, if \( V \) and \( U \) were both even, the image of this map would be the span of the \( t \times t \) minors of the generic matrix; when \( V \) is even and \( U \) is odd, the image is the span of the space of “exterior minors” as in Green [1999].

For any partition \( \lambda = (\lambda_1, \ldots, \lambda_s) \) we define \( F_\lambda \) to be the image of the composite map

\[
m \circ (\rho_{\lambda_1} \otimes \ldots \otimes \rho_{\lambda_s}) : \bigwedge^{\lambda_1} V \otimes \bigwedge^{\lambda_2} U \otimes \ldots \otimes \bigwedge^{\lambda_s} V \otimes \bigwedge^{\lambda_s} U \to S_{|\lambda|}(V \otimes U)
\]
where \( m \) denotes the multiplication map in \( S(V \otimes U) \).

As in the even case we order partitions of \( t \) by saying \( \lambda < \mu \) if and only if \( \lambda'_i > \mu'_i \) for the smallest number such that \( \lambda'_i \neq \mu'_i \). Finally, we define the subspaces

\[
F_{\leq \lambda} = \sum_{|\mu|=|\lambda|, \mu \leq \lambda} F_\mu \quad \subset \quad F_{< \lambda} = \sum_{|\mu|=|\lambda|, \mu < \lambda} F_\mu.
\]

In the classical case, \( F_{\leq \lambda} \) is spanned by certain products of minors of the generic matrix. The straightening law of Dubilet-Rota-Stein [1974] shows that we get a basis if we choose only “standard” products of these types, and the successive quotients in the filtration are the irreducible representations of \( GL(V) \times GL(U) \). The analogue in our \( \mathbb{Z}/2 \)-graded case is:

**Proposition 1.4.** The subspaces \( F_{\leq \lambda} \) define a \( \mathfrak{g} \)-invariant filtration on \( S_{\lambda}(V \otimes U) \). The quotient \( F_{\leq \lambda}/F_{< \lambda} \) is isomorphic to \( \wedge^\lambda V \otimes \wedge^\lambda U = S_{\lambda'} V \otimes S_{\lambda'} U \).

There is also one element of each irreducible representation which is easy to describe: the highest weight vector. To speak of highest weight vectors we must choose ordered bases \( \{ u_1, \ldots, u_d \} \) and \( \{ u'_1, \ldots, u'_d \} \) of \( U_0 \) and \( U_1 \), and ordered bases \( \{ v_1, \ldots, v_m \} \) and \( \{ v'_1, \ldots, v'_m \} \) of \( V_0 \) and \( V_1 \) respectively.

**Proposition 1.5.** Let \( \lambda = (\lambda_1, \ldots, \lambda_s) \) be a partition, and let \( w^1_i \in \wedge^{\lambda_i} (V) \) and \( w^2_i \in \wedge^{\lambda_i} (U) \) be the elements

\[
w^1_i = \begin{cases} v_1 \wedge \ldots \wedge v_{\lambda_i}, & \text{if } \lambda_i \leq m; \\ v_1 \wedge \ldots \wedge v_m \wedge v_i(\lambda_i-m), & \text{otherwise}. \end{cases}
\]

\[
w^2_i = \begin{cases} u_1 \wedge \ldots \wedge u_{\lambda_i}, & \text{if } \lambda_i \leq d; \\ u_1 \wedge \ldots \wedge u_d \wedge u_i(\lambda_i-d), & \text{otherwise}. \end{cases}
\]

The element

\[
c_{\lambda} = \prod_{i=1}^s \rho_{\lambda_i}(w^1_i \otimes w^2_i) \quad \in \quad S(V \otimes U)
\]

is the highest weight vector from the irreducible component \( \wedge^\lambda V \otimes \wedge^\lambda U = S_{\lambda'} V \otimes S_{\lambda'} U \), where \( \lambda' \) is the conjugate partition to \( \lambda \).

We end this section with a description of a set of generators for the representation \( \wedge^\lambda V \otimes \wedge^\lambda U \). We start with a double tableau, that is two sequences of tensors \( v_{i,1} \wedge \ldots \wedge v_{i,\lambda_i} \in \wedge^{\lambda_i} V \) and \( u_{i,1} \wedge \ldots \wedge u_{i,\lambda_i} \in \wedge^{\lambda_i} U \) \( (1 \leq i \leq s) \). We imagine that the elements \( v_{i,j} \in V \) correspond to the \( i \)-th row of the tableau \( S \) of shape \( \lambda \), and the elements \( u_{i,j} \in U \) correspond to the \( i \)-th row of another tableau \( T \) of shape \( \lambda \). We define

\[
\rho(S \otimes T) = \prod_{1 \leq i \leq s} \rho_{\lambda_i}(v_{i,1} \wedge \ldots \wedge v_{i,\lambda_i} \otimes u_{i,1} \wedge \ldots \wedge u_{i,\lambda_i}).
\]

We think of \( \lambda \) as a Ferrers diagram. If \( S \) is a tableau of shape \( \lambda \) and \( \sigma \) is a permutation of the boxes in \( \lambda \) then \( \sigma(S) \) is another tableau of shape \( \lambda \) (here we write \( \sigma \) as a product of transpositions, and introduce a minus sign whenever we interchange two elements of odd degree.) Let \( P(\lambda) \) be the group of permutations of the boxes in \( \lambda \) that preserve the columns of \( \lambda \).
Proposition 1.6. The representation $\bigwedge^\lambda V \otimes \bigwedge^\lambda U \subset R$ is generated by elements

$$\pi(S,T) = \sum_{\sigma \in P(\lambda)} \rho(\sigma S \otimes T),$$

or, equivalently, by

$$\pi'(S,T) = \sum_{\sigma \in P(\lambda)} \rho(S \otimes \sigma T)$$

where $S$ and $T$ range over all tableaux of shape $\lambda$.

Proof. We show that the $\pi(S,T)$ generate; the proof for $\pi'$ is similar. Since $\rho(S,T)$ is antisymmetric in the elements appearing in each row of $S$, the element $\sum_{\sigma \in P(\lambda)} \rho(\sigma S \otimes T)$ is the $\mathfrak{gl}(V)$-linear projection of $\rho(S,T)$ to the $\bigwedge^\lambda V$-isotypic component of $R$. By Corollary 1.2 we have $R = \oplus_\lambda \bigwedge^\lambda V \otimes \bigwedge^\lambda U$, so this isotypic component is $\bigwedge^\lambda V \otimes \bigwedge^\lambda U \subset R$.

2. Proof of Theorem 1a.

In this section $U$ and $V$ are $\mathbb{Z}/2$-graded vector spaces of dimensions $(d,e)$ and $(m,n)$ respectively, and $\Phi$ is the generic map, defined tautologically over $R = S(V \otimes U)$.

We write $\Lambda(d,e)$ for the partition with $e + 1$ parts $((d+1)^e, d) = (d+1, \ldots, d+1,d)$; that is, $\Lambda(d,e)$ corresponds to the Young diagram that is a $(d+1) \times (e+1)$ rectangle minus the box in the lower right hand corner. For example, $\Lambda(2,3)$ may be represented by the Young diagram

$$\Lambda(2,3) = \begin{array}{|c|c|c|c|}
\hline
\hline
\hline
\end{array}$$

For any partition $\lambda$ we denote by $I_\lambda$ the ideal in $R$ generated by the representation $\bigwedge^\lambda V \otimes \bigwedge^\lambda U$.

With this notation, Theorem 1 takes the form:

Theorem 1. The annihilator of the cokernel of $\Phi$ is equal to $I_{\Lambda(d,e)}$.

Theorem 1 implies that the representations appearing in the annihilator of coker $\Phi$ depend only on the dimension of $U$, not the dimension of $V$, as long as the dimension of $V_0$ is large (Corollary 2.2), and we begin by proving this. For the precise statement, we will use the following notation: Let $V'$ be another $\mathbb{Z}/2$-graded vector space, and let $\Phi'$ be the generic map $R' \otimes V' \rightarrow R' \otimes U^*$ where $R' = S(V' \otimes U)$. If $V$ is a summand of $V'$, so that $V' = V \oplus W$, then the ring $R = S(V \otimes U)$ can be identified with the subring of $R'$. We want to compare the annihilators of the modules coker $\Phi$ and coker $\Phi'$.

Proposition 2.1. If $V$ is a $\mathbb{Z}/2$-graded summand of $V'$ then $\text{ann coker } \Phi = R \cap \text{ann coker } \Phi'$. More precisely,

a) coker $\Phi$ is an $R$-submodule of coker $\Phi'$,

b) coker $\Phi'$ is a quotient of $(\text{coker } \Phi) \otimes R'$.

Proof. The first statement follows easily from a) and b).

For the proof of a) and b) we may write $V' = V \oplus W$ and we make use of the $\mathbb{N}$-grading of $R'$ for which $V \otimes U$ has degree 0 and $W \otimes U$ has degree 1 (this grading has nothing to do with the $\mathbb{Z}/2$-grading used elsewhere in this paper!) The map $\Phi'$ is homogeneous of degree 0 if we twist the summands of its source appropriately

$$\Phi' : R' \otimes W(-1) \oplus R' \otimes V \xrightarrow{(\Phi'_1, \Phi'_2)} R' \otimes U^*, \quad$$

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so we have an induced \( \mathbb{N} \) grading on \( \text{coker} \Phi' \). As \( \Phi'_0 = \Phi \), we see that \( (\text{coker} \Phi')_0 = \text{coker} \Phi \). Since the elements of \( R \) have degree 0, this is an \( R \)-submodule as required for a).

For b) it suffices to note that \( \text{coker} \Phi' \) is obtained from \( (\text{coker} \Phi) \otimes R' \) by factoring out the relations corresponding to \( W \otimes R' \). \( \blacksquare \)

**Corollary 2.2.** With notation \( U, V, V', \Phi, \Phi' \) as above, suppose that \( V \) is such that \( I_\lambda \neq 0 \) in \( S(V \otimes U) \). If \( I_\lambda \subset \text{ann} \text{coker} \Phi \), then \( I_\lambda \subset \text{ann} \text{coker} \Phi' \).

**Proof.** The inclusion \( R = S(V \otimes U) \subset R' = S(V' \otimes U) \) carries \( \wedge^\lambda V \otimes \wedge^\lambda U \) into \( \wedge^{\lambda'} V' \otimes \wedge^{\lambda'} U \). The conclusion now follows from Proposition 2.1 a) and b). \( \blacksquare \)

**Proof of Theorem 1.** We first show that \( I_{\Lambda(d,e)} \) is contained in the annihilator of \( M := \text{coker} \Phi \). By Corollary 2.2, it is enough, given \( U \), to produce one nonzero element from \( \wedge^{\Lambda(d,e)} V \otimes \wedge^{\Lambda(d,e)} U \subset S(V \otimes U) \) that annihilates the cokernel of \( \Phi \) for some space \( V \). By Corollary 2.2 it suffices to prove this result in the case \( m = d + 1, n = 0 \), that is, \( \text{dim} \ V = (d + 1, 0) \).

Let \( u_1, \ldots, u_d \) be a basis of \( U_0 \), let \( u'_1, \ldots, u'_d \) be a basis of \( U_1 \), and let \( v_1, \ldots, v_{d+1} \) be a basis of \( V = V_0 \). We denote the variables from the \( U_0 \otimes V_0 \) block by \( x_{i,k} \) (\( 1 \leq i \leq d, 1 \leq k \leq d+1 \)), and the variables from the \( U_1 \otimes V_0 \) block by \( b_{j,k} \) (\( 1 \leq j \leq e, 1 \leq k \leq d+1 \)) thus:

\[
\Phi = \begin{pmatrix} X \\ B \end{pmatrix}, \quad X = (x_{i,k}), \quad B = (b_{j,k}).
\]

Let

\[
Z = Z_1 \cdot X(1, \ldots, d \mid 1, \ldots, d)
\]

where \( Z_1 = \prod_{j,k} b_{j,k} \) is the product of all the entries of \( B \) and \( X(1, \ldots, d \mid 1, \ldots, d) \) is the \( d \times d \) minor of the matrix \( X \) corresponding to the first \( d \) columns.

We now show that \( Z \) annihilates \( M \). Indeed, by the classical Fitting Lemma we know that \( X(1, \ldots, d \mid 1, \ldots, d) \) annihilates the even generic module \( M/\mathfrak{S}(U_1^*) \). Thus every basis element \( u_k \) multiplied by \( X(1, \ldots, d \mid 1, \ldots, d) \) can be expressed modulo the image of \( \Phi \) as a linear combination of \( u'_1, \ldots, u'_e \) with coefficients of positive degree in the variables \( b_{j,k} \). Since these variables are odd, \( Z u_k = 0 \) in \( M \).

To see that \( Z u'_i \) is also 0 in \( M \), we use the classical Fitting Lemma again on the first \( d \) columns of the matrix of \( \Phi \) to see that for any \( 1 \leq i \leq d \) the element \( X(1, \ldots, d \mid 1, \ldots, d) u_i \) can be expressed, modulo the image of \( \Phi \), as a linear combination of the \( b_{n,t} u'_t \). On the other hand, if we multiply the last column of the matrix of \( \Phi \) by the product \( Z_1' \) of all the \( b_{j,k} \) except for the \( b_{l,d+1} \), we get an expression for \( Z_1 u'_t \), modulo the image of \( \Phi \), as a linear combination of \( Z_1' u_1, \ldots, Z_1' u_d \). Thus \( Z u'_i = X(1, \ldots, d \mid 1, \ldots, d) Z_1 u'_i = 0 \) in \( M \) as required.

Next we prove that the element \( Z \) is a weight vector (not generally a highest weight vector) and lies in \( \Lambda^{\Lambda(d,e)} V \otimes \Lambda^{\Lambda(d,e)} U \). Indeed, the element \( X(1, \ldots, d \mid 1, \ldots, d) \) is a weight vector in \( \Lambda^{d} V \otimes \Lambda^{d} U \). The element \( Z_1 \) is a weight vector in the representation \( \Lambda^{(d+1)^e} V \otimes \Lambda^{(d+1)^e} U \). The product is thus contained in the ideal \( \mathcal{F}_{\Lambda(d,e)} \). The element \( Z \) has degree \((e+1)(d+1) - 1 \) but it involves only \( d+1 \) elements from \( V = V_0 \). By Proposition 1.5 its weight can occur only in representations \( \Lambda^\lambda V \otimes \Lambda^\lambda U \subset S_\lambda(V \otimes U) \) with \( \lambda \) having all parts \( \leq d+1 \). Since \( \Lambda(d,e) \) is the only partition \( \lambda \) with \( \leq e+1 \) parts having \( |\lambda| = (d+1)(e+1) - 1 \) and each \( \lambda_i \leq d+1 \), we are done. This argument shows that \( I_{\Lambda(d,e)} \) is contained in the annihilator of the cokernel of \( \Phi \).

Now let \( \mu \) be a partition not containing \( \Lambda(d,e) \). To complete the proof of Theorem 1, we must show that the ideal \( I_\mu \) does not annihilate \( M = \text{coker} \Phi \) or, equivalently, that the highest weight vector \( c_\mu \) does not annihilate \( M \).

Since \( \mu \) does not contain \( \Lambda(d,e) \) it does not contain one of the extremal boxes of \( \Lambda(d,e) \). By shifting the gradings of \( V, U \) by 1 we do not alter the annihilator of the generic map, but we change
the notation so all partitions are changed to their conjugates. Thus we may assume that \( \mu_e \leq d \). By Corollary 2.2 we may further assume that \( n = 0 \), so that \( V = V_0 \), and that \( m \gg 0 \). To prove the Theorem, we will do induction on \( d \).

If \( d = 0 \) we must show that the annihilator of the cokernel of \( \Phi \) is contained in \( I_{(1^e)} \); or equivalently that it contains no \( I_\lambda \) where \( \lambda \) has fewer than \( e \) parts. Set

\[
Z_1 = \prod_{1 \leq j \leq e-1, 1 \leq k \leq m} b_{j,k};
\]

By Proposition 1.5, \( Z_1 \) is the highest weight vector in \( \bigwedge^{m_e-1} V \otimes \bigwedge^{m_e-1} U \). The element \( Z_1 u'_e \) is not in the image of \( \Phi \) because the coefficient of \( u'_e \) in any element from the image of \( \Phi \) is in the ideal generated by \( b_{e,1}, \ldots, b_{e,m} \), while \( Z_1 \) is not in this ideal. Since \( Z_1 \) does not annihilate \( M \), no \( I_\lambda \) such that \( \lambda \) has \( < e \) parts can annihilate \( M \).

In case \( d > 0 \) the matrix of \( \Phi \) will contain an even variable \( x_{1,1} \). To complete the induction we will invert this variable and use:

**Lemma 2.3.**

a) Over the ring \( R_1 = R[x_{1,1}^{-1}] \) the map \( \Phi \) can be reduced by row and column operations to the form

\[
\Phi' \oplus \text{id} : (V' \otimes_K R_1) \oplus R_1 \to U'^* \otimes_K \oplus R_1
\]

where \( V \) is a \( \mathbb{Z}/2 \)-graded vector space of dimension \((m-1,n)\) and \( U \) is a \( \mathbb{Z}/2 \)-graded vector space of dimension \((d-1,e)\). Moreover the ring \( R' \) generated over \( K \) by the entries of \( \Phi' \) is isomorphic to \( S(V' \otimes U') \) and \( R_1 \) is a flat extension of \( R' \).

b) The localization of the ideal \( I_\mu \) at \( x_{1,1} \) is isomorphic to the extension of the ideal \( J'_\nu \) from \( R' \) where \( \nu \) is the partition obtained from \( \mu \) by subtracting 1 from each nonzero part.

**Proof of Lemma 2.3.** Column and row reduction give the following formulas for the entries of \( \Phi' \):

\[
x'_{i,k} = x_{i,k} - \frac{x_{1,k}x_{i,1}}{x_{1,1}}, \quad a'_{i,l} = a_{i,l} - \frac{a_{1,l}x_{i,1}}{x_{1,1}},
\]

\[
b'_{j,k} = b_{j,k} - \frac{x_{1,k}b_{j,1}}{x_{1,1}}, \quad y'_{j,l} = y_{j,l} - \frac{a_{1,l}b_{j,1}}{x_{1,1}}.
\]

Consequently

\[
R_1 = R'[x_{1,1}, x_{1,1}^{-1}][x_{1,2}, \ldots, x_{1,m}, a_{1,1}, \ldots, a_{1,n}, x_{2,1}, \ldots, x_{d,1}, b_{1,1}, \ldots, b_{e,1}]
\]

in the sense of \( \mathbb{Z}/2 \)-graded algebras. This proves part a).

To prove part b) we first observe that the localization of the ideal \( I_{(t)}(\Phi) \) gives the ideal \( I_{(t-1)}(\Phi') \). Indeed, the ideal \( I_{(t)}(\Phi) \) is generated by \( \mathbb{Z}/2 \)-graded analogues of \( t \times t \) minors of \( \Phi \). After localization it becomes the ideal \( I_{(t)}(\Phi' \oplus \text{id}_{R_1}) \) generated by the \( \mathbb{Z}/2 \)-graded analogues of \( t \times t \) minors of \( \Phi' \oplus \text{id}_{R_1} \). Let us call the row and column of the matrix \( \Phi' \oplus \text{id}_{R_1} \) corresponding to the summand \( R_1 \) the distinguished row and column respectively. Every \( \mathbb{Z}/2 \)-graded analogue of a \( t \times t \) minor of \( \Phi' \oplus \text{id}_{R_1} \) is either a \( (t-1) \times (t-1) \) minor of \( \Phi' \) (in case it contains the distinguished row and column), zero (if it contains the distinguished row but not the distinguished column or vice versa), or a \( t \times t \) minor of \( \Phi' \) if it does not contain the distinguished row nor column.

To show that the result generalizes to an arbitrary partition \( \mu \) we order the bases so that the distinguished row and column come first. We saw in Proposition 6 that the highest weight vectors in \( \bigwedge^\mu V \otimes \bigwedge^\mu U \) are the products of minors of the matrix \( \Phi \) on some initial subsets of rows.
and column of $\Phi$, so after localization each factor will contain both the distinguished row and the distinguished column of $\Phi' \oplus \text{id}_{R_1}$.

**Completion of the Proof of Theorem 1.** Now suppose that $d > 0, n = 0$. We may of course assume that $m \neq 0$, so that the matrix of $\Phi$ contains the even variable $x_{1,1}$. It is enough to prove that $I_\nu M \neq 0$ after inverting $x_{1,1}$. The ideal $I_\nu$ will localize to the ideal $J'_\nu$ where $\nu$ is equal to $\mu$ with all parts decreased by 1. The graded vector space $U$ of dimension $(d, e)$ will change to the $\mathbb{Z}/2$-graded vector space $U'$ of dimension $(d - 1, e)$. The desired conclusion follows by induction on $d$.

**3. Proof of Theorem 1b.**

If $\phi: R^m \to R^d$ is a matrix representing a map of free modules over a commutative ring then, as we noted in the introduction, there are inclusions $\text{ann}(M) \cdot I_i(\phi) \subset I_{i+1}(\phi)$ for $0 \leq i < d$ and thus, by induction, $(\text{ann}(M))^d \subset I_d(\phi)$; see for example Eisenbud[1995]. To prove these inclusions one first notes that the cokernel of $\phi$ is the same as the cokernel of

$$\psi : V_0 \otimes R \oplus U_0^* \otimes R \to U_0^* \otimes R$$

where $\psi = (\phi, a \cdot \text{Id})$. Thus $I_1(\phi) = I_j(\psi)$ and $I_{j+1}(\phi) = I_{j+1}(\psi) \supset a \cdot I_j(\phi)$. We will carry out the same approach in $\mathbb{Z}/2$-graded case.

In this section we work with an arbitrary map

$$\phi : V \otimes R \to U^* \otimes R$$

of $\mathbb{Z}/2$-graded free modules over a $\mathbb{Z}/2$-graded commutative ring $R$. The first step is to show that, just as in the classical case, the ideals $I_\lambda(\phi)$ depends only on the cokernel of $\phi$ and on the number and degrees of the generators chosen.

**Lemma 3.1.** If $\alpha : V' \otimes R \to V \otimes R$ then

$$I_\lambda(\phi \alpha) \subset I_\lambda(\phi).$$

In particular, if $\psi : V' \otimes R \to U^* \otimes R$ has the same cokernel as $\phi$, then $I_\lambda(\phi) = I_\lambda(\psi)$.

**Proof.** The second statement follows from the first because each of the maps $\phi$ and $\psi$ factors through the other.

To prove the first statement, we use the notation of Proposition 1.6. For any map $W \otimes R \to U^* \otimes R$, and any tableaux $S$ and $T$ of elements in $W$ and $U$, both of shape $\lambda$, $\pi'_\psi(S, T)$ be the result of specializing the element $\pi'(S, T)$ defined for the generic map $\Phi$ when $\Phi$ is specialized to $\psi$. By Proposition 1.6 it is enough to show that when $W = V'$ the element $\pi'_\phi(S', T)$ is in $I_\lambda(\phi)$. We have

$$\rho_l(v'_1 \wedge \ldots v'_l \otimes u_1 \wedge \ldots u_l) = \sum_{i_1 < \ldots < i_l} \rho_l(v'_1 \wedge \ldots v'_{i_1} \otimes v_{i_1}^* \wedge \ldots v_{i_l}^* \otimes u_1 \wedge \ldots u_l)$$

where $v_1, \ldots, v_{m+n}$ and $v_1^*, \ldots, v_{m+n}^*$ are dual bases of $V$ and $V^*$. Using this identity to rewrite the formula for $\pi'_\psi(S', T)$, where $S'$ is a tableau of shape $\lambda$ with entries in $V'$ and $T$ is a tableau of shape $\lambda$ with entries in $U$, we see that $\pi'_\phi(S', T)$ is a linear combination of elements of the form $\pi'_\phi(S, T)$, where $S$ is a tableau of shape $\lambda$ with entries in $V$.

Lemma 3.1 implies in particular that two presentations of the same module with the same numbers of even and odd generators have the same ideals $I_\lambda(\phi)$. Similar arguments show that we can allow for presentations with different numbers of generators as long as we change the partitions suitably: if we add $d'$ even and $e'$ odd generators, then we have to expand $\lambda$ by adding $d'$ columns of length equal to the length of the first column and $e'$ rows of length equal to the length of the first row of the resulting partition (or vice versa). In this sense the ideals $I_\lambda(\phi)$ depend only on the cokernel of $\phi$.

The main result of this section is the following
**Theorem 3.2.** Let $R, U, V, \phi$ be as in the beginning of the introduction, and let $M = \text{coker} \phi$.

a) Let $s$ be an integer, $0 \leq s \leq d - 1$. If $x_1, \ldots, x_{e+1} \in \text{Ann}_R M$, then

$$x_1 \ldots x_{e+1} I_{\Lambda(s,e)}(\phi) \subset I_{\Lambda(s+1,e)}(\phi).$$

b) If $x_1, \ldots, x_e \in \text{Ann}_R M$, then $x_1 \ldots x_e \in I_{\Lambda(0,e)}(\phi)$.

As in the classical case we derive:

**Corollary 3.3.** Let $M$ be a $\mathbb{Z}/2$-graded module over a $\mathbb{Z}/2$-graded ring $R$, with the presentation $\phi : V \otimes R \to U^* \otimes R$. Assume that $\dim U = (d,e), \dim V = (m,n)$. Let $x_1, \ldots, x_{(d+1)(e+1)-1}$ be homogeneous elements from $\text{Ann}_R M$. Then $x_1 \ldots x_{(d+1)(e+1)-1} \in I_{\Lambda(d,e)}(\phi)$.

**Proof of Theorem 3.2.** We begin with part b). We work with a presentation $(\phi, \psi) : V \otimes R \oplus W \otimes R \to U^* \otimes R$ where $W$ is a $\mathbb{Z}/2$-graded vector space of dimension $e$ with the the $i$-th generator $u_i$ going to $x_i$ times the $i$-th generator $u_i$ of $U^*$. The parity of the generators of $W$ is adjusted so $\psi$ is of degree 0. Now taking a double tableau $(S, T)$ of the shape $(1^e)$ with $w_i$ and $u_i$ in the $i$-th row, and applying the definition above, we see that the generator $\pi(S, T)$ is just $x_1 \ldots x_e$.

To prove part a) we distinguish two cases. In the case $s < d - 1$ we use the presentation $(\phi, \psi) : V \otimes R \oplus W \otimes R \to U^* \otimes R$ where $W$ is a $\mathbb{Z}/2$-graded vector space of dimension $e + 1$ with the $i$-th generator $u_i$ going to $x_i$ times the $i$-th generator $u_i$ of $U^*$. The parity of the generators of $W$ is adjusted so $\psi$ is of degree 0. We can assume without a loss of generality that $\phi := \Phi$ is generic. Then it is enough to prove that $c_{\Lambda(s,e)} x_1 \ldots x_{e+1} \in I_{\Lambda(s+1,e)}$, where $c_{\Lambda(s,e)}$ is the highest weight vector defined as in Proposition 1.5.

We pick a tableau $(S, T)$ of shape $\Lambda(s+1, e)$ as follows. The entries $v_{i,j}$, $u_{i,j}$ in the $i$-th row are the same as in the canonical tableau, except the last ones. The last entry in the tableau $v$ in the $i$-th row is $w_i$, and the last entry in the $i$-th row is $u_{s+2}$. The element $\pi'(S, T)$ is easily seen to be $c_{\Lambda(s,e)} x_1 \ldots x_{e+1}$.

In the case $s = d - 1$ we use the presentation $(\phi, \psi) : V \otimes R \oplus W \otimes R \to U^* \otimes R$ where $W$ is a $\mathbb{Z}/2$-graded vector space of dimension $e + 1$ with the $i$-th generator $u_i$ going to $x_i$ times the $d + i$-th generator $u_i$ of $U^*$ for $1 \leq i \leq e$ and $w_{e+1}$ goes to $x_{e+1}$ times $u_d$. The parity of the generators of $W$ is adjusted so $\psi$ is of degree 0. We can assume without a loss of generality that $\phi := \Phi$ is generic. Then it is enough to prove that $c_{\Lambda(d-1,e)} x_1 \ldots x_{e+1} \in I_{\Lambda(d,e)}$, where $c_{\Lambda(d-1,e)}$ is the canonical tableau.

We pick a tableau $(S, T)$ of shape $\Lambda(s+1, e)$ as follows. The entries $v_{i,j}$, $u_{i,j}$ in the $i$-th row are the same as in the canonical tableau, except the last ones. The last entry in the tableau $v$ in the $i$-th row is $w_i$, and the last entry in the $i$-th row is $u_{d+i}$ for $1 \leq i \leq e$, and $u_d$ for $e + 1$-st row. The element $\pi'(S, T)$ is easily seen to be $c_{\Lambda(d-1,e)} x_1 \ldots x_{e+1}$.
4. The resolution of generic $\mathbb{Z}/2$-graded module.

In this section we work over the generic ring $R = S$ as in the introduction, and we conjecture the form of a minimal free resolution over $R$ of the cokernel $C$ of the generic map $\Phi$. This resolution is a natural generalization of the one constructed in [B-E] in the commutative case. We work over a field $K$ of characteristic 0. We define some $\mathbb{Z}/2$-graded free $R$-modules $F_i$ as follows:

$$F_0 = U^* \otimes R, \quad F_1 = V \otimes R$$

$$F_i = \bigoplus_{|\alpha| + |\beta| = i - 2} S \Theta(d,e,\alpha,\beta)V \otimes S \Lambda(d,e,\alpha,\beta)U \otimes R$$

where $\Lambda(d,e,\alpha,\beta) = (d + 1 + \beta_1, d + 1 + \beta_2, \ldots d + 1 + \beta_e, e, \alpha_1', \ldots, \alpha_s')$, $\Theta(d,e,\alpha,\beta) = (d + 1 + \alpha_1, d + 1 + \alpha_2, \ldots d + 1 + \alpha_e, e + 1, \beta_1', \ldots, \beta_s')$, and we sum over all pairs of partitions $\alpha, \beta$ with at most $e$ parts.

**Conjecture 4.1.** There exists an equivariant differential $d_i : F_i \to F_{i-1}$, linear for $i \geq 3, i = 1$ and of degree $|\Lambda(d,e)|$ for $i = 2$, that makes $F_\ast$ into a minimal $R$-free $\mathbb{Z}/2$-graded resolution of $C$.

In the even case ($U_1 = V_1 = 0$) the desired complex is the Buchsbaum-Rim resolution (see for example Buchsbaum and Eisenbud [1975]). We have checked the conjecture computationally, using Macaulay2, in a few more cases.

**Appendix. Comments on the action of $g$**

It may at first be surprising that the generators given in Examples 1-3 of the Introduction are permuted by the action of $g$, so we pause to make the action explicit in Example 3 (the other cases are similar and simpler).

When we think of $R = S(V \otimes U)$ as a $g$ module, we think of $g$ acting on the left. But we may identify $U \otimes V$ with $\text{Hom}(V, U^*)^* = \text{Hom}(U^*, V)$, and thus identify $R$ with the coordinate ring of the space $\text{Hom}(V, U^*)$. In this identification it is natural to think of the Lie algebra $g = gl(V) \times gl(U)$ as $gl(V) \times gl(U^*)$, with the $gl(U^*)$ acting on the right. To make this identification, we use the **supertranspose** which is the anti-isomorphism

$$gl(U) \to gl(U^*); \quad \begin{pmatrix} U_{0,0} & U_{0,1} \\ U_{1,0} & U_{1,1} \end{pmatrix} \mapsto \begin{pmatrix} U_{0,0}^t & U_{0,1}^t \\ -U_{1,0}^t & U_{1,1}^t \end{pmatrix}$$

Now consider the case presented in Example 3 of the introduction, whose notation we use. To see the action of $g$ let us act by two elements of the Lie algebra on the element $axy$. First we act with the element $v_{0,1}$ from $gl(V)$ changing an odd element to an even one. We get a sum of terms each of which is $axy$ with one changed factor; we can replace $a$ by $x$ or $y$ by $b$. Thus we get terms $xxy$ and $axb$. The first comes with positive sign ($v_{0,1}$ acts from the left, we replaced first factor, so there are no switches), the second term comes with negative sign (we replaced $y$ by $b$, so had to switch $v_{0,1}$ with $a$). Thus we get $x(xy - ab)$.

Let us also act on $axy$ by the Lie algebra element $u_{1,0}$ from $gl(U^*)$ exchanging an even element with an odd one. We get a sum of terms where each term is $axy$ with one factor changed; we can change $x$ to $b$ and $a$ to $y$. We get terms $yxy$ and $aby$. Both come without sign as $u_{1,0}$ acts from the right and $x, y$ have even degree. Thus we get $(xy + ab)y$.
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