Around the Carnot theorem

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Abstract

We study the Carnot theorem and the configuration of points and lines in connection with it. It is proven that certain significant points in the configuration lie on the same lines and same conics. The proof of an equivalent statement formulated by Bradley is given. An open conjecture, established by Bradley, is proved using the theorems of Carnot and Menelaus.

1 Introduction

Carnot’s theorem can be considered as a generalization of Ceva’s theorem. The theorem of Carnot gives a necessary and sufficient condition for two points on each side of a triangle to form a conic.

Theorem 1.1 (Carnot’s theorem). Let $\triangle ABC$ be a triangle and let $A_1, A_2$ be the points on the line $BC$, $B_1, B_2$ on the line $CA$ and $C_1$ and $C_2$ on the line $AB$. The points $A_1, A_2, B_1, B_2, C_1$ and $C_2$ lie on the same conic $C$ if and only if

$$
\frac{AC_1}{C_1B} \cdot \frac{AC_2}{C_2B} \cdot \frac{BA_1}{A_1C} \cdot \frac{BA_2}{A_2C} \cdot \frac{CB_1}{B_1A} \cdot \frac{CB_2}{B_2A} = 1.
$$

In Section 2 we give a classical proof of Carnot’s theorem, using the theorems of Menelaus and Pascal. This proof can be found in [2]. We also study some natural points and lines involved in the configuration and its relations to the side lines of triangle $\triangle ABC$. Theorems 2.2 and 2.4 summarize these results. These theorems are generalizations of classical Euclidean theorems for incircle of a triangle.

In Section 3 we give an synthetic proof of the following statement (see Figure 1) which was the first time formulated in [1]:

Theorem 1.2 (Bradley’s theorem). There is a conic $D$ such that the lines $AA_1, AA_2, BB_1, BB_2, CC_1$ and $CC_2$ are tangents of $D$ if and only if the points $A_1, A_2, B_1, B_2, C_1$ and $C_2$ lie on the same conic $C$.

Our goal is to prove an equivalent statement, Corollary 3.1 which together with the Poncelet Triangle theorem implies Bradley’s theorem.

In the paper [1], Bradley formulated the following conjecture (see Figure 2):

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Theorem 1.3 (Bradley’s theorem about quadrilaterals). Let $ABCD$ and $PQRS$ be quadrilateral which are in axial perspective, that is $T = AB \cap PQ$, $U = BC \cap QR$, $V = CD \cap S$, $W = DA \cap SP$ are collinear. The other twelve intersections of the sides of the quadrilaterals are marked with notation exemplified by $13 = AB \cap RS$, $42 = DA \cap QR$ etc, in such way that number 1 corresponds to the sides $AB$ and $PQ$, 2 to $BC$ and $QR$, 3 to $CD$ and $RS$ and 4 to $DA$ and $SP$. Then there exist four conics $C_1$, $C_2$, $C_3$ and $C_4$ such that the points $23$, $24$, $32$, $34$, $42$, $43$ lie on conic $C_1$, the points $13$, $14$, $31$, $34$, $42$, $43$ lie on conic $C_2$, the points $12$, $14$, $21$, $24$, $41$, $42$ lie on conic $C_3$ and $12$, $13$, $21$, $23$, $31$, $32$ lie on conic $C_4$.

This theorem is proved in Section 4.
Theorems of Ceva, Menelaos and Carnot are used in [6] as ‘prototheorems’ to build new theorems that involve lines and conics. It is shown in [6] and [5] that any oriented triangulated 2-manifold can be a frame. This procedure works for theorems studied in this paper as well. Deep relation among classical projective geometry and more advanced topics in mathematics and computer science is explained in *Perspectives on Projective Geometry*, an inspirative book by Jürgen Richter-Gerbert. [5]. Software ‘Cinderella’ developed by Ulrich Kortenkamp and Jürgen Richter-Gerbert is used as experimental tool for discovering new results about Carnot’s configuration.

2 Carnot’s theorem

We start this section with proof of the Carnot theorem.

![Figure 3: The Carnot theorem](image)

**Proof of Carnot’s theorem:** Let the points \( A_1, A_2, B_1, B_2, C_1 \) and \( C_2 \) lie on the same conic \( \mathcal{C} \) and let \( L \) be the intersection of the lines \( A_1C_1 \) and \( AC \), \( M \) the intersection of the lines \( B_1C_2 \) and \( BC \) and \( N \) the intersection of the lines \( A_2B_2 \) and \( AB \), Figure 3. By the Pascal theorem, the points \( L, M \) and \( N \) lie on the same line, and from the Menelaos theorem the following holds:

\[
\frac{AL}{LC} \cdot \frac{CM}{MB} \cdot \frac{BN}{NA} = -1. \quad (2)
\]

Applying the Menelaos theorem three times for the lines \( \triangle A_1C_1, B_1C_2 \) and \( A_2B_2 \) \( \triangle ABC \), we obtain:

\[
\frac{AL}{LC} \cdot \frac{CA_1}{A_1B} \cdot \frac{BC_1}{C_1A} = -1, \quad (3)
\]

\[
\frac{AB_1}{B_1C} \cdot \frac{CM}{MB} \cdot \frac{BC_2}{C_2A} = -1, \quad (4)
\]
Multiplying the relations (3), (4) and (5) and division by (2), yields the relation (1).

In the opposite direction, the proof is similar. By the Menelas theorem, the relations (3), (4) and (5) hold. From the relations (3), (4), (5) and (1) one can easily deduce the relation (2), so by the converse of the Menelas theorem, the points $L, M$ and $N$ lie on the same line. The converse of the Pascal theorem then implies that the points $A_3, A_4, D_3$ and $D_4$ lie on the line $BC$.

Theorem 2.1. The points $B_3, B_4, E_3$ and $E_4$ lie on the line $CA$, the points $C_3, C_4, F_3$ and $F_4$ lie on the line $AB$ and the points $A_3, A_4, D_3$ and $D_4$ lie on the line $BC$.

Proof: We shall prove that $B_3$ lie on the line $CA$.

Let $R$ be the intersection of $A_1C_2$ and $AC$ and $R'$ the intersection of the lines $F_1D_2$ and $AC$.

From the Menelaos theorem for the line $A_1C_2$, we obtain:
\[
\frac{CR}{RA} = -\frac{C_2B}{AC_2} \cdot \frac{A_1C}{BA_1}.
\]

Let $X$ be the intersection point of the lines $AA_2$ and $CC_1$. The Menelaos theorem for the line $F_1D_2$ and $\triangle AXC$ yields:
\[
\frac{CR'}{R' A} : \frac{AD_2}{D_2X} : \frac{XF_1}{F_1C} = -1.
\]
From the Carnot theorem for the conic $C$ and $\triangle AXC$ we obtain:
\[
\frac{\overrightarrow{AD_2}}{D_2X} \cdot \frac{\overrightarrow{AA_2}}{A_2X} \cdot \frac{\overrightarrow{XF_1}}{F_1C} \cdot \frac{\overrightarrow{XC_1}}{C_1A} \cdot \frac{\overrightarrow{CB_1}}{B_1A} \cdot \frac{\overrightarrow{CB_2}}{B_2A} = 1. \tag{8}
\]
By the Law of Sines we have:
\[
\overrightarrow{A_2X} \sin \angle A_2XC = \overrightarrow{A_2C} \sin \angle BCC_1
\]
and
\[
\overrightarrow{C_1C} \sin \angle BCC_1 = \overrightarrow{C_1B} \sin \beta.
\]
From these two equations one can deduce:
\[
\overrightarrow{A_2X} \cdot \overrightarrow{C_1C} \sin \angle A_2XC = \overrightarrow{A_2C} \cdot \overrightarrow{C_1B} \sin \beta. \tag{9}
\]
Similarly, the following equality holds:
\[
\frac{\overrightarrow{AA_2}}{A_2X} \cdot \frac{\overrightarrow{XC_1}}{C_1C} = \frac{\overrightarrow{BA_2}}{A_2C} \cdot \frac{\overrightarrow{AC_1}}{C_1B}. \tag{10}
\]
From (9) and (10) (using the equality $\angle C_1XA = \angle A_2XC$) we conclude that:
\[
\frac{\overrightarrow{AA_2}}{A_2X} \cdot \frac{\overrightarrow{XC_1}}{C_1C} = \frac{\overrightarrow{BA_2}}{A_2C} \cdot \frac{\overrightarrow{AC_1}}{C_1B}. \tag{11}
\]
Now, from the relations (7), (8) and (11) we have:
\[
\overrightarrow{CR} = \overrightarrow{BA_2} \cdot \overrightarrow{AC_1} \cdot \overrightarrow{CB_1} \cdot \overrightarrow{CB_2}.
\]
But, the Carnot relation [1] implies
\[
\frac{\overrightarrow{CR}}{R'A} = -\frac{\overrightarrow{C_2B}}{AC_2} \cdot \frac{\overrightarrow{A_1C}}{BA_1},
\]
and $R = R' \equiv B_3$.

The proof for the other points is analogous. $\Box$

From Pascal’s theorem the following theorem is true (see Figure 5):

**Theorem 2.2.** The following 8 triples of points $(A_3, B_3, C_3)$, $(D_3, E_3, C_4)$, $(A_3, E_4, F_3)$, $(D_3, B_3, F_4)$, $(A_4, E_3, F_4)$, $(D_4, E_3, C_3)$, $(D_4, B_4, F_3)$, and $(A_4, B_4, C_4)$ are collinear.

In the sequel, we encounter the relations of higher order. We use the theorem of Carnot to prove that certain points in the configuration lie on the same conic.

**Theorem 2.3.** The points $D_3$, $D_4$, $E_3$, $E_4$, $F_3$ and $F_4$ lie on the same conic $D$.

**Proof:** From the proof of Theorem 2.1 we also deduce that:
\[
\frac{\overrightarrow{CE_3}}{E_3A} = -\frac{\overrightarrow{C_1B}}{AC_1} \cdot \frac{\overrightarrow{A_1C}}{BA_1} \cdot \frac{\overrightarrow{AF_3}}{F_3B} - \frac{\overrightarrow{A_1C}}{BA_1} \cdot \frac{\overrightarrow{B_1C}}{CB_1} \cdot \frac{\overrightarrow{BD_3}}{D_3C} = -\frac{\overrightarrow{B_1A}}{CB_1} \cdot \frac{\overrightarrow{C_1B}}{AC_1}.
\]
Then the following holds by $[1]
\frac{CE_3}{E_3A} \cdot \frac{CE_4}{E_4A} \cdot \frac{AF_3}{F_3B} \cdot \frac{AF_4}{F_4B} \cdot \frac{BD_3}{D_3C} \cdot \frac{BD_4}{D_4C} = 1.$

By the converse of Carnot’s theorem, the points $D_3, D_4, E_3, E_4, F_3$ and $F_4$ lie on the same conic.

In the same fashion we prove that:

**Theorem 2.4.** The following 4 sextuples of the points $(D_3, D_4, E_3, E_4, F_3, F_4)$, $(A_3, A_4, B_3, B_4, F_3, F_4)$, $(A_3, A_4, E_3, E_4, C_3, C_4)$ and $(D_3, D_4, B_3, B_4, C_3, C_4)$ are the sextuples of the points lying on the same conic.
3 Bradley’s Theorem

In this section we give an elementary proof of the Bradley’s conjecture [1]. The first proof, given by Zoltán Szilasi in [7], used barycentric coordinates. We use different approach and prove several other interesting things about Carnot’s configuration.

Let $X_1$ be the intersection points of the lines $AA_1$ and $BB_1$, $X_2$ of $BB_1$ and $CC_1$ and $X_3$ of $CC_1$ and $AA_1$. Let $Y_1$ be the intersection points of the lines $AA_2$ and $BB_2$, $Y_2$ of $BB_2$ and $CC_2$ and $Y_3$ of $CC_2$ and $AA_2$.

Define $T_2$ as the intersection point of the lines $X_1Y_3$ and $X_3Y_1$. The points $T_3$ and $T_1$ are defined analogously.

**Theorem 3.1.** $T_2$ lies on the line $BC$, $T_3$ on $CA$ and $T_1$ on $AB$.

**Proof:** Let $T'$ be the intersection point of the lines $X_3Y_1$ and $BC$ and let $T''$ be the intersection point of the lines $X_1Y_3$ and $BC$.

By the Menelaos theorem applied at $\triangle ABA_1$ and the line $BC_2$ we obtain:

$$\frac{AX_3}{X_3A_1} = -\frac{CB}{A_1C} \cdot \frac{AC_1}{C_1B}.$$ \hfill (12)

The same reasoning for $\triangle ACA_2$ and the line $CC_1$ we obtain:

$$\frac{A_2Y_1}{Y_1A} = -\frac{B_2C}{A_2B} \cdot \frac{BA_2}{C_2B}.$$ \hfill (13)

Then from the Menelaos theorem for $\triangle AA_1A_2$ and the line $X_1Y_3$ we get:

$$\frac{A_1T}{T'A_2} = -\frac{AB_2}{AC_1} \cdot \frac{A_1C}{B_2C} \cdot \frac{C_1B}{BA_2}.$$ \hfill (12)
In the same fashion we prove that:

\[
\frac{\overrightarrow{A_1T'}}{T''A_2} = \frac{\overrightarrow{AC_2}}{AB_1} \cdot \frac{\overrightarrow{B_1C}}{A_2C} \cdot \frac{\overrightarrow{BA_1}}{C_2B}.
\] (13)

By the relation (1) we conclude that:

\[
\frac{\overrightarrow{A_1T'}}{T'A_2} = \frac{\overrightarrow{A_1T''}}{T''A_2},
\]

so \( T' \equiv T'' \equiv T_2 \).

For the points \( T_1 \) and \( T_3 \) the proof is analogous. \( \square \)

Since the points \( T_2, B \) and \( C \) are collinear, by the converse of Pascal’s theorem for the hexagon \( X_3Y_1Y_2Y_3X_1X_2 \) we get (see Figure 8):

![Figure 8: Corollary 3.1](image)

**Corollary 3.1.** The points \( X_1, X_2, X_3, Y_1, Y_2 \) and \( Y_3 \) lie on the same conic.

An immediate consequence of this fact is (see Figure 9):

**Corollary 3.2.** The points \( T_1, T_2 \) and \( T_3 \) lie on the same line.

Bradley’s theorem [1.2] directly follows from Corollary 3.1 and the Poncelet triangle theorem [1.4 Theorem 5, p.184-185], see Figure 10.

### 4 Proof of Theorem 1.3

In this section we give the proof of Theorem 1.3. The proof illustrates a nice application of the Menelaus and the Carnot theorems.

**Proof:** We prove that the points \( 23, 24, 32, 34, 42, 43 \) lie on conic \( C_1 \). The proof for other points is analogous.

Let \( X \) be the intersection point of the lines \( AD \) and \( BC \). We apply the Menelaus theorem for \( \triangle XDC \) and the lines \( SW, RU, SV \) and \( VW \) and get:
\[
\frac{\overrightarrow{XW}}{\overrightarrow{WD}} \cdot \frac{\overrightarrow{D(34)}}{\overrightarrow{(34)C}} \cdot \frac{\overrightarrow{C(24)}}{\overrightarrow{(24)X}} = -1,
\]
(14)

\[
\frac{\overrightarrow{X(41)}}{\overrightarrow{(41)D}} \cdot \frac{\overrightarrow{D(32)}}{\overrightarrow{(32)C}} \cdot \frac{\overrightarrow{CU}}{\overrightarrow{UX}} = -1,
\]
(15)

\[
\frac{\overrightarrow{X(43)}}{\overrightarrow{(43)D}} \cdot \frac{\overrightarrow{DV}}{\overrightarrow{VC}} \cdot \frac{\overrightarrow{C(23)}}{\overrightarrow{(23)X}} = -1,
\]
(16)

\[
\frac{\overrightarrow{DW}}{\overrightarrow{WX}} \cdot \frac{\overrightarrow{XU}}{\overrightarrow{UC}} \cdot \frac{\overrightarrow{CV}}{\overrightarrow{VD}} = -1.
\]
(17)
After multiplication of (14), (15), (16) and (17), we obtain:

\[
\frac{\overrightarrow{D(34)}}{(34)C} \cdot \frac{\overrightarrow{C(24)}}{(24)X} \cdot \frac{\overrightarrow{X(41)}}{(41)D} \cdot \frac{\overrightarrow{D(32)}}{(32)C} \cdot \frac{\overrightarrow{X(43)}}{(43)D} \cdot \frac{\overrightarrow{C(23)}}{(23)X} = 1.
\]

From the converse of Carnot’s theorem it follows that the points 23, 24, 32, 34, 42, 43 lie on the same conic. □

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