Higher-Spin Theories and $Sp(2M)$ Invariant Space–Time

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Abstract

Some methods of the “unfolded dynamics” machinery particularly useful for the analysis of higher spin gauge theories are summarized. A formulation of 4d conformal higher spin theories in $Sp(8)$ invariant space-time with matrix coordinates and its extension to $Sp(2M)$ invariant space-times are discussed. A new result on the global characterization of causality of physical events in the $Sp(2M)$ invariant space-time is announced.

1 Introduction

Higher spin (HS) gauge theories are gauge theories with tensorial gauge symmetry parameters. In the Fronsdal’s formulation \cite{Fronsdal}, for a spin $s$ totally symmetric gauge field $\phi_{n_1...n_s}$, the gauge parameter $\epsilon_{n_1...n_{s-1}}$ is totally symmetric in its indices and traceless $\epsilon_{m_1n_2...n_{s-1}} = 0$. The field $\phi_{n_1...n_s}$ is double traceless $\phi^{kl}_{n_1n_2...n_s} = 0$. (For other equivalent approaches to HS massless field see, e.g., \cite{Vasiliev99}). For every spin $s$ there exists \cite{Fronsdal} a unique action with at most two derivatives $S = \int d^d x \partial_{n_1...n_s}(x) \partial_{n_1...n_s}(x)$ invariant under the Abelian gauge transformations

$$
\delta \phi_{n_1...n_s}(x) = \partial_{\{n_1\} \epsilon_{n_2...n_s\}}(x), \quad \partial_n = \frac{\partial}{\partial x^n}.
$$

For $s = 1$ and 2 this reproduces the Maxwell theory and linearized Einstein theory. Fields of half-integer spins are described analogously. Since the lower spin gauge theories with spins $s = 1, 3/2$ and 2 play a key role in the theory of fundamental interactions it is interesting to understand whether there is some nontrivial theory behind spin $s > 2$ gauge fields, the HS gauge fields.
There is a number of motivations for studying HS gauge theories. From supergravity perspective, this is interesting because theories with HS fields may have more supersymmetries than the “maximal” supergravities with 32 supercharges like 11d SUGRA [3]. Recall that the limitation that the number of supercharges is ≤ 32 is a direct consequence of the requirement that s ≤ 2 for all fields in a supermultiplet (see, e.g., [4]). From superstring perspective, most obvious motivation is due to Stueckelberg symmetries in the string field theory [5], which have a form of some spontaneously broken HS gauge symmetries. Whatever a symmetric phase of the superstring theory is, Stueckelberg symmetries are expected to become unbroken HS symmetries in such a phase and, therefore, the superstring field theory has to become one or another version of the HS gauge theory. An important indication in the same direction is [6] that string amplitudes exhibit certain symmetries in the high-energy limit equivalent to the string mass parameter tending to zero. In all cases, the key issue is the HS gauge symmetry.

Unusual feature of the interacting HS gauge theory is that unbroken HS gauge symmetries do not allow flat space-time as a vacuum solution, requiring nonzero space-time curvature with anti-de Sitter (AdS) space-time as a most symmetric vacuum [7]. Recently, this unusual property received interpretation [8, 9, 10, 11, 12, 13] in the context of the AdS/CFT correspondence conjecture [14]. In particular, in [8, 9, 13] it was conjectured that HS gauge theories in AdS bulk are duals of some conformal models on the AdS boundary in the large $N$ limit with $g^2 N \to 0$ where $g^2$ is the boundary coupling constant, which is opposite to the limit $g^2 N \to \infty$ studied within the original AdS/CFT correspondence conjecture [14]. Again, this suggests that the HS gauge theory has a good chance to be related to a symmetric phase of superstring theory. On the other hand an explanation why the HS gauge theory has not been yet observed in the superstring theory may be just that no complete formulation of the latter is still known in the AdS background at the quantum level despite the progress achieved at the classical level [15].

Properties of the HS theories can to large extent be revealed from the structure of global HS symmetry algebras found originally in [16] for the simplest HS model in $AdS_4$. In [17] it was realized that $AdS_4$ HS algebras are certain star-product algebras with spinorial generating elements, while in [18] it was then conjectured that algebras of the same structure correspond to HS models in higher dimensions.\footnote{Recently, analogous realization of the string field theory star product in terms of infinite set of oscillators carrying vector space-time indices was proposed in [19].} The construction is as follows. Consider oscillators $a_A$
and $b^B$ satisfying the commutation relations

$$[a_A, b^B] = \delta^B_A.$$  

(1.2)

In HS applications, indices $A, B = 1, 2 \ldots M$ are interpreted as spinorial (then $M = 2^p$). The HS fields are described by the gauge potentials

$$dx^n \omega_n(a, b|x) = \sum_{p,q} dx^n \omega_{A_1 \ldots A_p}^{B_1 \ldots B_q}(x) b^{A_1} \ldots b^{A_p} a_{B_1} \ldots a_{B_q}$$  

(1.3)

taking values in the oscillator algebra. The field $dx^n \omega_n(a, b|x)$ is the HS generalization of the frame field and Lorentz connection in the Cartan formulation of gravity. The HS curvature and transformation law have the standard Yang–Mills form with the oscillator algebra in place of a Yang–Mills matrix algebra

$$R = d\omega + \omega \wedge \omega, \quad d = dx^n \partial_n,$$

$$\delta \omega = d\varepsilon + [\omega, \varepsilon].$$

(1.4)

(1.5)

The HS gauge symmetry parameter $\varepsilon(a, b|x)$ is an arbitrary function of the space-time coordinates $x^n$ and noncommutative auxiliary variables $a_A$ and $b^B$, which admits an expansion analogous to (1.3). Since this expansion contains infinitely many terms, the HS algebra of oscillators is infinite-dimensional. It contains however finite-dimensional subalgebras. Important example is given by the subalgebra $sp(2M)$ spanned by various bilinears in the oscillators

$$P_{AB} = a_Aa_B, \quad L_A^B = \frac{1}{2}\{a_A, b^B\}, \quad K^{AB} = b^Aa^B$$

(1.6)

and its superextension $osp(1, 2M)$ with supercharges $Q_A = a_A$ and $S^B = b^B$. Usual $AdS$ and conformal symmetry algebras belong to this $sp(2M)$. In particular, $AdS_3$ algebra $o(2, 2)$ is isomorphic to $sp(2) \oplus sp(2)$. (The doubling which extends to the whole $AdS_3$ HS system \cite{20} is introduced with the aid of additional involutive element $\psi (\psi^2 = 1)$ \cite{21}.) The 3d conformal (equivalently $AdS_4$) algebra $o(3, 2)$ is isomorphic to $sp(4|\mathbb{R})$. The 4d conformal (equivalently $AdS_5$) algebra $o(4, 2)$ is isomorphic to $su(2; 2) \subset sp(8)$. For higher dimensions usual space-time symmetry algebras are subalgebras of appropriate $sp(2^n)$. As was shown long ago in \cite{22} $osp(1, 2^n)$ provides a natural simple superextension of the higher dimensional space-time symmetries.

The following important properties of the HS theories follow directly from the structure of HS algebras.

Usual spin 2 gravitational fields take values in the algebra $sp(2M)$ spanned by bilinears of the oscillators. Higher spins are associated with the higher order
polynomials of the oscillators. Because higher-order polynomials do not close to a finite-dimensional algebra, it follows that an infinite tower of HS fields should be included once there is at least one HS field.

By virtue of the field equations, the quantum-mechanical nonlocality of the HS algebra is translated to the space-time nonlocality of HS interactions \[23\]. Since higher spins require higher derivatives \[24, 25\], the theory with infinite towers of higher spins requires infinitely many derivatives.

More recent observation \[11, 26\] we focus on in this report is that, starting from \(d = 4\), usual space-time symmetries in conformal HS models extend to larger symplectic symmetries acting on the conformal HS multiplets. We will see that this may affect considerably the geometric picture of the world leading to new geometries with the so called “central charge” coordinates included. This, however, does not affect visualization of our world as a four-dimensional space-time realized as a three-brane embedded into the ten-dimensional generalized space-time with matrix coordinates.

### 2 4d conformal free fields

Consider a Fock vacuum state \(|0\rangle\) satisfying

\[
a_A|0\rangle = 0
\]  

and a set of functions taking values in the Fock space (i.e., sections of the Fock fiber bundle over space-time)

\[
|\Phi(x)\rangle = C(b|x|0\rangle, \quad C(b|x) = \sum_{n=0}^{\infty} \frac{1}{n!} C_{A_1...A_n}(x) b^{A_1} \ldots b^{A_n}.
\]  

(2.2)

To make contact with the description of 4d relativistic fields in terms of two-component indices one replaces the index \(A = 1\ldots4\) by a pair of two-component indices \(A = (\alpha, \dot{\alpha}), (\alpha, \beta \ldots = 1, 2; \dot{\alpha}, \dot{\beta} \ldots = 1, 2)\) and uses the equivalent expansion

\[
C(b, \bar{b}|x) = \sum_{m,n=0}^{\infty} \frac{1}{m!n!} C_{\alpha_1...\alpha_m,\dot{\alpha}_1\ldots\dot{\alpha}_n}(x) b^{\alpha_1} \ldots b^{\alpha_m} \bar{b}^{\dot{\alpha}_1} \ldots \bar{b}^{\dot{\alpha}_n}.
\]  

(2.3)

The key observation is that the set of all massless conformal field equations in four dimensions can be formulated in the following compact form \[11\]

\[
D_0|\Phi(x)\rangle \equiv d|\Phi(x)\rangle + \omega_0|\Phi(x)\rangle = 0, \quad \omega_0 = -dx^{\alpha\dot{\alpha}} a_\alpha \bar{a}_{\dot{\alpha}}.
\]  

(2.4)
Here \( dx^{\alpha \beta} \) is the 4d Minkowski frame 1-form and \( \omega_0 \) is the gauge field associated with the generators of translations realized as bilinear combinations of mutually commuting oscillators. Note that analogous description of 3d conformal fields was given in [27] (see also [28]).

Equation (2.4) is equivalent to the equation
\[
\frac{\partial}{\partial x^{\alpha \beta}} C(b, \bar{b}|x) - \frac{\partial^2}{\partial b^\alpha \partial \bar{b}^\beta} C(b, \bar{b}|x) = 0,
\]
which decomposes into independent subsystems associated with different spins, singled out by the condition
\[
\left( b^\alpha \frac{\partial}{\partial b^\alpha} - \bar{b}^\beta \frac{\partial}{\partial \bar{b}^\beta} \right) C(b, \bar{b}|x) = \pm 2s C(b, \bar{b}|x),
\]
equivalent to
\[
N|\Phi(x)\rangle = \pm 2s|\Phi(x)\rangle, \quad N = a_\alpha b^\alpha - \bar{a}_\beta \bar{b}^\beta.
\]

The fact that 4d massless field equations can be reformulated in the form (2.5) was known long ago [29, 30]. The dynamical fields are those in the expansion (2.3), carrying either only dotted or only undotted indices. They are contained in the analytic (\( C(b, 0|x) \)) and antianalytic (\( C(0, \bar{b}|x) \)) parts, and describe scalar \( C \), spinor \( C_\alpha b^\alpha + \bar{C}_\beta \bar{b}^\beta \), spin 1 field strength \( C_{\alpha \beta} b^\alpha b^\beta + \bar{C}_{\alpha \beta} \bar{b}^\alpha \bar{b}^\beta \) and so on for spins 3/2 and higher. All components in \( C(b, \bar{b}|x) \) which depend both on \( b \) and on \( \bar{b} \) are auxiliary being expressed by (2.5) in terms of space-time derivatives of the dynamical fields. The nontrivial equations on the dynamical fields are
\[
\left( \frac{\partial^2}{\partial b^\alpha \partial x^{\alpha \beta}} - \frac{\partial^2}{\partial b^\alpha \partial x^{\gamma \beta}} \right) C(b, 0|x) = 0, \quad \left( \frac{\partial^2}{\partial b^\beta \partial x^{\alpha \beta}} - \frac{\partial^2}{\partial \bar{b}^\gamma \partial x^{\alpha \beta}} \right) C(0, \bar{b}|x) = 0
\]
for \( s > 0 \) and the Klein–Gordon equation for the spin zero scalar field \( C(0, 0|x) \). These are the usual massless equations formulated in terms of field strengths. The simple observation [11] that massless equations admit the described Fock space realization allows us, with the help of the unfolded dynamics machinery, to reveal their symmetries which turn out to be much richer than the usual conformal symmetry of massless fields and, in particular, contain \( sp(8) \) symmetry.
3 Unfolded dynamics

Let \( W^a(x) \) be some set of differential \( p \)-forms with \( p \geq 0 \) (0-forms are included), and the generalized curvatures \( R^a \) be defined by the relations

\[
R^a = dW^a + F^a(W), \quad (3.1)
\]

where \( F^a \) are some functions of \( W^b \) built with the aid of the exterior product of differential forms. Given \( F^a(W) \) with vanishing 0-form part \((\text{deg}(F^a) > 0)\) which satisfies the generalized Jacobi identity

\[
F^b \delta F^a \delta W^b \equiv 0, \quad (3.2)
\]

we say following [31] that it defines a free differential algebra. This property guarantees that the generalized Bianchi identity \( dR^a = R^b \delta F^a \delta W^b \) is satisfied, which implies that the differential equations on \( W^a \)

\[
R^a = 0 \quad (3.3)
\]

are consistent. These equations are invariant under the gauge transformations

\[
\delta W^a = de^a - \epsilon^b \frac{\delta F^a}{\delta W^b}, \quad (3.4)
\]

where \( e^a(x) \) is an arbitrary \((\text{deg}(W^a) - 1)\)–form (0-forms do not give rise to gauge parameters) because the generalized curvatures transform as \( \delta R^a = -R^c \frac{\delta}{\delta W^c} \left( \epsilon^b \frac{\delta F^a}{\delta W^b} \right) \). Also, since Eqs. (3.3) are formulated in terms of differential forms, they are manifestly general coordinate invariant. Elementary but crucial fact is that any dynamical system can be “unfolded” to the form (3.3) by introducing enough auxiliary fields. Note that in such approach, the co-frame field \( e \) from which the metric tensor is built is one of the 1-forms in the set \( W^a \).

It is supposed to be invertible, having a part of order 1 in any perturbative expansion. This provides a meaningful linearization of Eq. (3.3).

For the particular case when the set \( W^a \) consists of only 1-forms \( \omega^i \), the function \( F^i(\omega) \) is bilinear \( F^i = f^i_{jk} \omega^j \wedge \omega^k \) and the relation (3.2) amounts to the Jacobi identity for a Lie algebra \( g \) with the structure coefficients \( f^i_{jk} \) (or superalgebra if some of \( \omega^i \) carry an additional Grassmann grading). If the set \( W^a \) also contains some \( p \)-forms \( v^\alpha \) (e.g., 0-forms) and the functions \( F^\alpha \) are linear in \( v \), \( F^\alpha = t^\alpha_{i\beta} \omega^i \wedge v^\beta \), the relation (3.2) implies that the matrices \( t^\alpha_{i\beta} \) form some representation \( t \) of \( g \) while (3.3) contains zero-curvature equations of \( g 

\[
R^i = d\omega^i + f^i_{jk} \omega^j \wedge \omega^k = 0 \quad (3.5)
\]
along with the covariant constancy equation

\[ Dv^\alpha = dv^\alpha + t^\alpha_{\beta\gamma} \omega^\gamma \wedge v^\beta = 0 \quad (3.6) \]

for the representation \( t \). Thus the zero-curvature equation for some Lie algebra \( g \) along with the covariant constancy conditions for a set of differential forms taking values in some module over \( g \) give the simplest example of a free differential algebra. Note that this case is not necessarily dynamically trivial and can be used to describe some not empty free field dynamics. Here the zero-curvature equation for gauge fields of \( g \) describes some \( g \)-invariant background geometry while the covariant constancy conditions can describe free fields propagating in this geometry provided that the representation \( t \) is infinite-dimensional. For example, the free massless equations (2.4)

\[ D_{0}(\Phi(x)) = 0 \]

have unfolded form. The connection \( \omega_0 \) in (2.4) trivially satisfies the zero curvature equation \( R = d\omega_0 + \omega_0 \wedge \omega_0 = 0 \) because it is \( x \)-independent and involves only mutually commuting oscillators \( a \). In fact, instead of fixing \( \omega_0 \) in the particular form (2.4) it is enough to say that the frame part of \( \omega_0 \) is nondegenerate and the condition (3.5) is satisfied, that brings us to the formulation of the dynamics of massless fields in an arbitrary conformally flat background.

Equations (3.3) express exterior differentials of all fields \( W^a(x) \) via values of the fields themselves. As a result, to reconstruct \( W^a(x) \) everywhere by virtue of (3.3) modulo gauge ambiguity (3.4), it is enough to fix values of \( W^a(x_0) \) at any given point \( x_0 \). For this to be possible, the set of fields \( W^a(x) \) must be rich enough to describe all on-mass-shell nontrivial combinations of the space-time derivatives of dynamical fields under consideration. From the Poincare’ lemma it follows that nontrivial degrees of freedom are contained in the 0-forms because all degrees of freedom in \( p \)-forms with \( p > 0 \) are pure gauge, i.e., are fixed in terms of 0-forms modulo gauge ambiguity. Identification of field strengths with the 0-forms \( C \) occurs in terms of some deformation of the zero-curvature equation of the type (3.5) to

\[ R \equiv d\omega + \omega \wedge \omega + \omega \wedge \omega C + \ldots = 0 \quad (3.7) \]

which, in fact, is the starting point towards the full nonlinear dynamics in the form (3.3) [29]. In HS gauge theories, 0-forms are described by the generating functions \( C(a, b|x) \) analogous to those of gauge potentials (1.3) but taking values in a certain “twisted adjoint” representation of the HS algebra. In the conformal model discussed in this talk, the set of “dynamical fields” \( C(0, \bar{b}|x) \)

\(^2\text{Of course, this is only true in a topologically trivial situation with trivial cohomology of the de Rahm differential } d. \text{ Otherwise further restrictions on } W^a(x_0) \text{ can occur.}\)
and \( C(b, 0|x) \) also corresponds to all on-mass-shell nontrivial combinations of the gauge invariant fields strengths of massless fields.

Note that nontrivial deformations \((3.7)\) sometimes result from some zero-curvature and covariant constancy conditions in a larger system
\[
dW = W \wedge W, \quad DC = 0
\]
provided that the 0-forms \( C \) satisfy some nonlinear constraints
\[
\varphi(C) = 0, \quad (3.9)
\]
being invariant in the sense that \( D\varphi(C) = 0 \) as a consequence of \((3.8)\) (for more details and examples of HS gauge theories formulated this way see \([32, 23]\)). In this case the nontrivial dynamical content of the equations is hidden in the constraints \((3.9)\). The role of Eqs. \((3.8)\), which can be integrated in the pure gauge form (at least locally), is to map the content of the constraints \((3.9)\) to the geometric framework associated with the original coordinates \( x \). Such a formulation makes it trivial to extend the dynamical equations to larger (super)spaces without changing its dynamical content by simply adding extra coordinates \( y \) and extending the differential forms to the extra dimensions
\[
dx^n W_n(x) \rightarrow dx^n W_n(x, y) + dy^{n'} W_{n'}(x, y). \quad (3.10)
\]
The dynamical content of the system \((3.8), (3.9)\) is still encoded by the constraints \((3.9)\) on the zero forms \( C \) at any point, say \( C(0,0) \), of a larger space-time. This trick was suggested in \([11]\) where it was applied to the analysis of the HS dynamics in the \( sp(8) \) invariant space-time as we discuss below. More recently, it was applied in \([33]\) to derive the superspace form of the 4\( d \) HS equations of \([32]\). As discussed in \([11]\) there is a great variety of extensions of the HS dynamics to different space-times. The invariant content to be intact is the constraints \((3.9)\) \([32]\).

### 4 Symmetries

A nice feature of the unfolded formulation is that it makes symmetries \((3.4)\) manifest. Suppose that some vacuum solution \( W_0(x) \) of \((3.3)\) is fixed. This restricts the gauge symmetry parameters by the condition \( \delta W_0(x) = 0 \) equivalent to
\[
de^a - \epsilon^a_b \frac{\delta F^a}{\delta W^b|_{W=W_0}} = 0. \quad (4.1)
\]
If the 0-form part of the l.h.s. of (4.1), \( \epsilon^b \delta F^a \delta W^b \bigg|_{W=C_0} \), is nonzero for a chosen solution, this imposes some restrictions on the parameter \( \epsilon^a \). In many cases however \( F^a(W) \) is such that \( \delta F^a \delta W^b \bigg|_{W=0} = 0 \). Choosing a solution with \( C_0 = 0 \) one finds that for this case the left hand side of Eq. (4.1) is only nontrivial in the sector of \( p \)-forms with \( p > 0 \) which is a consistent differential condition because \( \mathcal{R}(W_0) = 0 \). Therefore, it can be integrated to fix \( \epsilon^a(x) \) in terms of \( \epsilon^a(x_0) \) at any given point \( x_0 \) modulo second rank “gauge symmetries” for gauge parameters \( \delta \epsilon^a = d\xi^a + \xi^b \delta F^a \delta W^b \). From the Poincaré lemma it follows that nontrivial leftover symmetries are parametrized by the 0-form parameters among \( \epsilon^a \). These are reconstructed uniquely in terms of their values \( \epsilon^a(x_0) \), which are arbitrary parameters of the global symmetries of the system. As a result, there are as many global symmetries as 0-forms among \( \epsilon^a(x) \) at any fixed \( x \), i.e., as 1-forms among \( W^a \).

Applying this machinery to the HS system in Sec. 2 we observe that the Fock space forms a module over the 4\( d \) conformal algebra \( su(2,2) \) realized by the generators

\[
P_{\alpha\bar{\beta}} = a_\alpha \bar{a}_{\bar{\beta}}, \quad K^{\alpha\bar{\beta}} = b_\alpha \bar{b}_{\bar{\beta}},
\]

\[
L_\alpha^\beta = a_\alpha b^\beta - \frac{1}{2} \delta_\alpha^\beta a_\gamma b^\gamma, \quad \bar{L}_{\bar{\alpha}}^{\bar{\beta}} = \bar{a}_{\bar{\alpha}} \bar{b}^{\bar{\beta}} - \frac{1}{2} \delta_{\bar{\alpha}}^{\bar{\beta}} \bar{a}_{\bar{\gamma}} \bar{b}^{\bar{\gamma}}, \quad D = \{a_\alpha, b^\alpha\} + \{\bar{a}_{\bar{\alpha}}, \bar{b}^{\bar{\alpha}}\}.
\]

The 1-form \( \omega_0 \) in (2.4) is the flat connection of the conformal algebra. According to the general analysis, this proves that Eq. (2.4) is conformally invariant. Since the conformal algebra \( su(2,2) \) belongs to the symplectic algebra \( sp(8) \) (1.6) which, in its turn, belongs to the infinite-dimensional HS algebra of all polynomials of the oscillators which acts on the Fock module, from our analysis it follows that the set of equations for massless fields of all spins contained in (2.4) has global symmetries which form these algebras. Thus the infinite-dimensional algebra of polynomials of oscillators of \( a_A \) and \( b_B \) is shown \[11\] to form a symmetry of the system of equations of all massless fields that extends the usual conformal symmetry. This confirms the conjecture of Fradkin and Linetsky \[34\] that (an appropriate reduction of) this algebra can be used as 4\( d \) conformal HS algebra (these authors studied a different 4\( d \) conformal model being the HS extension of the nonunitary \( C^2 \) conformal gravity.) Algebras of this type were used in \[10, 35\] as 5\( d \) HS algebras.

The unfolding machinery provides efficient tools for elucidating explicit form of the transformation laws. For example, for the conformal HS system in Sec. 2 the zero curvature connection admits the pure gauge representation

\[
\omega_0 = g^{-1} dg, \quad g = \exp -x^{\alpha\bar{\beta}} a_\alpha \bar{a}_{\bar{\beta}}.
\]
Then, the global symmetry parameter that leaves $\omega_0$ invariant is
\[ \varepsilon(a, \bar{a}, b, \bar{b}|x) = g^{-1}(a, \bar{a}, b, \bar{b}|x)\varepsilon_0(a, \bar{a}, b, \bar{b})g(a, \bar{a}, b, \bar{b}|x), \] (4.5)
where $\varepsilon_0(a, \bar{a}, b, \bar{b})$ is an arbitrary $x$-independent element of the HS algebra of oscillators which parametrizes the HS global symmetries. The transformation law for massless fields is
\[ \delta|\Phi \rangle = \varepsilon \delta|\Phi \rangle. \] (4.6)
The parameters $\varepsilon_0(a, \bar{a}, b, \bar{b})$ bilinear in oscillators describe the $sp(8)$ symmetry. Those of higher orders describe HS symmetries. Note that the parameters $\varepsilon_0(a, \bar{a}, b, \bar{b})$ being order-$n$ polynomials in the oscillators give rise to some local transformation laws with at most order $[\frac{1}{2}n]$ space-time derivatives. Formulas (4.5) and (4.6) allow one easily to derive explicit form of the respective transformation laws [11].

5 $\sigma_-$ cohomology

A useful tool of the unfolding machinery is the identification of the dynamical content of unfolded equations with certain cohomology groups. Consider a covariant constancy equation of the form
\[ (\mathcal{D} + \sigma_- + \sigma_+)C(X) = 0, \] (5.1)
where $\mathcal{D}$ and $\sigma_\pm$ satisfy the relations
\[ (\sigma_\pm)^2 = 0, \quad \mathcal{D}^2 + \{\sigma_-, \sigma_+\} = 0, \quad \{\mathcal{D}, \sigma_\pm\} = 0, \] (5.2)
which guarantee that the connection is flat. It is demanded that only the operator $\mathcal{D}$, that contains the de Rahm differential, acts nontrivially (differentiates) on the space-time coordinates while $\sigma_\pm$ act in the fiber $V$ in which $C(X)$ takes values. It is also assumed that there exists a grading operator $G$ diagonalizable in $V$ such that its spectrum in $V$ is bounded from below and
\[ [G, \mathcal{D}] = 0, \quad [G, \sigma_\pm] = \pm \sigma_\pm. \] (5.3)
For example, the conformal equations (2.5) have this form with $\mathcal{D} = d$, $\sigma_+ = 0$, $\sigma_- = -dx^{\alpha\beta} \frac{\partial^2}{\partial \sigma \partial b^\beta}$ and $G = \frac{1}{2}(h^\alpha \frac{\partial}{\partial \sigma^\alpha} + \bar{h}^\alpha \frac{\partial}{\partial \bar{\sigma}^\alpha})$. In fact, it is a typical situation with $V$ realized as a space of polynomials of some auxiliary variables, $\sigma_\pm$ being some differential operators acting on these auxiliary variables and $G$ counting
a (nonnegative) polynomial degree that guarantees that the spectrum of \( G \) is bounded from below.

The following useful facts are true. (i) Dynamical fields contained in the 0-forms \( C \) take values in the cohomology group \( H^0(\sigma_-) \). In other words, dynamical fields in \( X \)-space are those satisfying \( \sigma_-(C(X)) = 0 \). This is because all fields in \( V/H^0(\sigma_-) \) (i.e., such that \( \sigma_-(C(X)) \neq 0 \)) are auxiliary being expressed via the space-time derivatives \( (D) \) of the dynamical fields by virtue of Eqs. (5.1). Note that \( H^0(\sigma_-) \) is always nonzero because it at least contains the nontrivial subspace of \( V \) of minimal grade.

(ii) There are as many differential conditions on the dynamical 0-forms \( C \) in Eq. (5.1) as elements of the cohomology group \( H^1(\sigma_-) \). If grade \( k \) elements of \( H^1(\sigma_-) \) impose equations on grade \( l \) dynamical fields, an order of differential equations is \( k + 1 - l \). All other equations in (5.1) are either constraints, which express auxiliary fields via derivatives of the dynamical ones, or consequences of the dynamical equations. When \( H^1(\sigma_-) \) is zero (5.1) is equivalent to some (usually infinite) set of constraints which express all fields contained in \( C(X) \) via derivatives of the dynamical fields (which however have no dynamics in that case being restricted by no differential equations).

The proof is elementary [11]. Indeed, consider the decomposition of the space of 0-forms \( C \) into the direct sum of eigenspaces of \( G \). Let a field having definite eigenvalue \( s_k \) of \( G \) be denoted \( C_k \), \( k = 0, 1, 2, \ldots \). Suppose that the dynamical content of the equations (5.1) with the eigenvalues \( s_k \) with \( k \leq k_q \) is found. Applying the operator \( D \) to the left hand side of the equations (5.1) at \( k \leq k_q \) we obtain taking into account (5.2) that

\[
\sigma_-(D(C_{k_q+1})) = 0.
\]

Therefore \( D(C_{k_q+1}) \) is \( \sigma_- \) closed. If the group \( H^1(\sigma_-) \) is trivial in the grade \( k_q + 1 \) sector, any solution of (5.1) can be written in the form \( D(C_{k_q+1}) = \sigma_- \tilde{C}_{k_q+2} \) for some field \( \tilde{C}_{k_q+2} \). This, in turn, is equivalent to the statement that one can adjust \( C_{k_q+2} \) in such a way that \( \tilde{C}_{k_q+2} = 0 \) or, equivalently, that the part of the equation (5.1) of the grade \( k_q + 1 \) is some constraint that expresses \( C_{k_q+2} \) in terms of the derivatives of \( C_{k_q+1} \) (here the assumption is used that the operator \( \sigma_- \) is algebraic in the space-time sense, i.e. it does not contain space-time derivatives.) If \( H^1(\sigma_-) \) is nontrivial, this means that the equation (5.1) sends the corresponding cohomology class to zero and, therefore, not only expresses the field \( C_{k_q+2} \) in terms of derivatives of \( C_{k_q+1} \) but also imposes some additional differential conditions on \( C_{k_q+1} \). All other equations in (5.1) either imply constraints which express auxiliary fields via derivatives of the dynamical ones or are consequences of the dynamical equations.
For example, for the system of massless fields with \( \sigma = -\frac{\partial^2}{\partial b \partial \bar{b}} \), \( H^0(\sigma_-) \) consists of analytic and antianalytic functions giving rise to the dynamical massless fields \( C(b, 0|x) \) and \( C(0, \bar{b}|x) \). The dynamical equations along with the Klein–Gordon equation are associated with \( H^1(\sigma_-) \).

The following comments are now in order.

If \( C(X) \) are \( p \)-forms, dynamical fields and nontrivial dynamical equations are associated with \( H^p \) and \( H^{p+1}(\sigma_-) \), respectively.

Suppose that the symmetry algebra \( g \) admits a triple \( Z \)-graded structure \( g_0 \oplus g_- \oplus g_+ \) with Abelian subalgebras \( g_\pm \). Let \( \sigma_- = dX^AP_A \) where \( P_A \) is some basis in \( g_- \). The dynamical fields satisfying \( \sigma_-C = 0 \) then identify with the primary fields in \( C(X) \) satisfying \( P_AC(X) = 0 \). In other words, \( H^0(\sigma_-) \) consists of singular vectors (i.e., vacua) of various submodules over \( g \) in \( V \), which is consistent since any invariant submodule of \( V \) gives rise to a subsystem in \( \sigma_- \). \( H^0(\sigma_-) \) forms some representation of \( g_0 \). In most interesting physical applications \( H^0(\sigma_-) \) decomposes into (may be infinite) direct sum of finite-dimensional representations of \( g_0 \). This corresponds to a model with dynamical fields carrying finite dimensional representations of \( g_0 \). Note that for the case of usual conformal algebra, \( g_0 \) consists of the Lorentz algebra plus dilatations, i.e., dynamical fields are some Lorentz tensors carrying definite conformal weights.

Suppose that \( V_{1,2} = V_1 \otimes V_2 \). Let \( C_1(X) \in V_1 \) and \( C_2(X) \in V_2 \) solve \( (5.1) \) with some operators \( \sigma_1^\perp \) and \( \sigma_2^\perp \). Then

\[
C_{1,2}(X) = C_1(X) \otimes C_2(X)
\]

solves \( (5.1) \) with \( \sigma_1^\perp = \sigma_\perp \otimes Id + Id \otimes \sigma_\perp^2 \). If \( V_1 \) and \( V_2 \) form modules over some symmetry algebra \( g \), the same is true for \( V_{1,2} \). As a result, the unfolded formulation of the dynamical equations equips the variety of solutions of \( g \)-quasiinvariant partial differential equations with the natural associative product structure isomorphic to the tensor algebra of (semiinfinite) modules over \( g \). Note that this associative structure differs from that of the ring of solutions of first order differential equations. It maps solutions of some set of (not necessarily first order) \( g \)-quasiinvariant partial differential equations to solutions of some other \( g \)-quasiinvariant equations. In fact, this property is deeply related to the general AdS/CFT philosophy \[37\].

Note the roles of “translations” \( P_A \) and “special conformal” generators \( K^A \) of \( g_+ \) acting in the fiber is exchanged in the unfolded formulation compared to the standard induced representation approach \[38\] in which primaries are defined directly in the base manifold.
6 Field equations in \( Sp(2M) \) invariant space-time

According to the general argument in Sec. 4 the set of massless equations of all spins \( (2.4) \) admits \( sp(8) \) symmetry realized by local field transformations. Definite spin field equations are singled out by the condition \( (2.7) \). The operator \( N \) in \( (2.7) \) does not commute to the \( sp(8) \) generators, breaking \( sp(8) \) down to the conformal algebra \( su(2, 2) \) which is the centralizer of \( N \) in \( sp(8) \). The \( su(2, 2) \) acts on every spin individually. The generators in \( sp(8)/su(2, 2) \) mix different spins. There are two irreducible subspaces of \( sp(8) \) in the set of massless fields of all spins, those containing bosons or fermions (they form an irreducible representation of \( osp(1, 8) \)). Note that the Fock representation of \( sp(8) \) used in our construction is the nonunitary dual of the unitary Fock (i.e., singleton) representation of \( sp(8) \). As explained in \( [27, 11] \) this duality has a form of some Bogolyubov transform. That the unitary Fock representation of \( sp(8) \) reduces to the collection of massless representations of all spins of \( su(2, 2) \) was discussed, e.g., in \( [38, 39] \).

Our goal is to reformulate 4d dynamics of massless fields of all spins in an equivalent but manifestly \( sp(8) \) invariant form. To this end we have to use a \( sp(8) \) invariant space-time. As a minimum this will give us a technical device analogous to superspace in the context of supersymmetric theories. Very likely, however, this generalization may have much deeper effect on our understanding of the space-time geometry. Since it is designed for the description of just those sets of HS massless fields that appear in consistent nonlinear HS theories, the proposed formulation has a good chance to be related to a symmetric phase of a theory of fundamental interactions.

Recall that the compactified Minkowski space-time is the coset space \( SO(d, 2)/P \) where \( P \) is the parabolic subgroup of \( SO(d, 2) \) generated by special conformal, Lorentz and dilatation generators. Usual Minkowski space is the maximal cell of \( SO(d, 2)/P \). Let us proceed analogously with the symplectic group \( Sp(M) \) choosing the parabolic subgroup of \( Sp(M) \) generated by the “special conformal” generators \( K_{AB} \) and Lorentz + dilatation generators \( L_{AB} \) in \( (1.6) \). The dimension of the resulting compactified generalized space-time \( M = Sp(M)/P \) equals to the number of translation generators in \( (1.6) \), i.e., \( dim(M) = \frac{1}{2}M(M + 1) \). Local coordinates are real symmetric matrices \( X_{AB} = X^{BA} \). This space was discussed by many authors in different contexts \( [40, 38, 41, 42, 43] \). It allows the action of \( sp(2M) \) realized by the vector.
In this talk we will not distinguish between the compactified space $\mathcal{M}_M$ and its maximal cell having trivial topology of $R_+^{M(M+1)}$.

Let us now extend Eq. (2.4) to $\mathcal{M}_4$ with the help of the unfolded formalism machinery. This is achieved by means of the trick explained at the end of Sec. 3 via replacing (2.4) with

\[ D_0|\Phi(X)\rangle \equiv d|\Phi(X)\rangle + \sigma_-|\Phi(X)\rangle = 0, \quad \sigma_- = -dX^{AB} \frac{\partial^2}{\partial b^A \partial b^B}. \]  

(6.2)

\((d = dX^{AB} \frac{\partial}{\partial X^{AB}})\). Decomposing $X^{AB}$ as $x^{\alpha\dot{\alpha}}, x^{\alpha\beta}, \bar{x}^{\dot{\alpha}\dot{\beta}}$ with Hermitian coordinates $x^{\alpha\dot{\alpha}}$ of the usual Minkowski space-time and additional mutually conjugated six coordinates $x^{\alpha\beta}$ and $\bar{x}^{\dot{\alpha}\dot{\beta}}$ we find that this equation contains the original 4\textit{d} equation (2.3) along with the equations $\frac{\partial}{\partial x^{\alpha\beta}} C(b|X) = 0$ and their conjugates. Clearly these new equations just reconstruct the dependence on the additional six coordinates $x^{\alpha\beta}$ and $\bar{x}^{\dot{\alpha}\dot{\beta}}$ in terms of values of the original fields in the 4\textit{d} Minkowski space-time.

Now we are in a position to analyse Eq. (6.2) directly in the ten dimensional space-time $\mathcal{M}_4$. The analysis of its dynamical content is elementary in terms of $\sigma_-$ cohomology. Let us consider the case of arbitrary $M$. Obviously, $H^0(\sigma_-)$ is spanned by at most linear polynomials in $b$, i.e., dynamical fields are

\[ C^{\text{dyn}} = b(X) + b^A f_A(X). \]  

(6.3)

Here the scalar field $b(X)$ and spinor field $f_A(X)$ in $\mathcal{M}_M$ describe, respectively, all massless boson and fermion fields in the 4\textit{d} Minkowski space-time in the case of $M = 4$. It is not hard to see (explicit proof is given in [37]) that $H^1(\sigma_-)$ is spanned by the elements of the form $F_{CD,B} b^B dX^{CD}$, and $B_{CD,BA} b^B b^A dX^{CD}$, where $F_{CD,B}$ and $B_{CD,BA}$ are arbitrary tensors having symmetry properties of the Young tableaux \[ \begin{array}{c} \square \square \\ \square \square \end{array} \] and \[ \begin{array}{c} \square \square \\ \square \square \end{array} \], respectively. The nontrivial equations on the dynamical fields, which have these symmetry properties, are

\[ \left( \frac{\partial^2}{\partial X^{AB} \partial X^{CD}} - \frac{\partial^2}{\partial X^{AC} \partial X^{BD}} \right) b(X) = 0 \]  

(6.4)

for the boson field $b(X)$ and

\[ \frac{\partial}{\partial X^{AB}} f_C(X) - \frac{\partial}{\partial X^{AC}} f_B(X) = 0 \]  

(6.5)
for the fermion field $f_B(X)$. These are analogues of the Klein–Gordon and Dirac equations. (For $M = 2$ they coincide with the $3d$ massless equations.) Note that every solution of (6.5) satisfies (6.4).

As Eqs. (6.4) and (6.5) with $M = 4$ are equivalent to the original $4d$ equations they are expected to respect nice features of the original $4d$ system, such as unitarity, locality, microcausality etc. It is instructive to reanalyse these properties starting directly from Eqs. (6.4) and (6.5). Once these are linear equations one can analyze them via Fourier transform. For a particular harmonic

$$b(X) = b_0 \exp ik_{AB}X^{AB}$$

requires

$$k_{AB}k_{CD} = k_{AC}k_{BD}.$$  \hspace{1cm} (6.7)

This is solved by the twistor ansatz

$$k_{AB} = k_{\xi_A\xi_B}$$  \hspace{1cm} (6.8)

with an arbitrary commuting real “svector” $\xi_A$ and a factor $k$. For the proof it is enough to diagonalize the symmetric matrix $k_{AB}$ by a $SL_M$ transformation to see that the product of any two different eigenvalues is zero by (6.7) at $A \neq B$ and, therefore, any nonzero matrix $k_{AB}$ satisfying (6.7) has rank 1. Modulo rescalings of $k$ and $\xi_A$, there are two essentially different options in (6.8), with $k = 1$ or $k = -1$. These correspond to the positive and negative frequency solutions, respectively. The analysis of fermions is analogous. As a result the generic solutions of (6.4) and (6.5) have the form

$$b(X) = \frac{1}{\pi M} \int d^M \xi \left( b^+(\xi) \exp i\xi_A\xi_BX^{AB} + b^-(\xi) \exp -i\xi_A\xi_BX^{AB} \right),$$

$$f_C(X) = \frac{1}{\pi M} \int d^M \xi \xi_C \left( f^+(\xi) \exp i\xi_A\xi_BX^{AB} + f^-(\xi) \exp -i\xi_A\xi_BX^{AB} \right).$$

Both for the scalar $b(X)$ and svector $f_A(X)$, the space of solutions is parametrized by two functions of $M$ variables $\xi_A$. Because odd functions $b^\pm(\xi)$ and even functions $f^\pm(\xi)$ do not contribute to (6.9) and (6.10), respectively, we require

$$b^\pm(\xi) = b^\pm(-\xi), \quad f^\pm(\xi) = -f^\pm(-\xi).$$  \hspace{1cm} (6.11)

The integration in (6.9) and (6.10) is thus carried out over $R^M/Z_2$. The origin of coordinates $\xi_A = 0$ is invariant under the $Z_2$ reflection $\xi_A \rightarrow -\xi_A$, being a singular point of the conical orbifold $R^M/Z_2$. Note that the whole framework
is supersymmetric as is manifested by the fact that bosons and fermions have equal numbers of degrees of freedom for any \( M \).

As shown in [26], the form of the general solution (6.10), (6.9) allows manifestly \( sp(2M) \) covariant quantization. The commutation relations for quantized fields equivalent to those in 3\( d \) and 4\( d \) Minkowski space-time for \( M = 2 \) and \( M = 4 \), respectively, are as simple as

\[
[b^\pm(\xi_1), b^\pm(\xi_2)] = 0, \quad [b^-(\xi_1), b^+(\xi_2)] = \frac{1}{2}(\delta(\xi_1 - \xi_2) + \delta(\xi_1 + \xi_2)),
\]

\[
[f^\pm(\xi_1), f^\pm(\xi_2)] = 0, \quad [f^-(\xi_1), f^+(\xi_2)] = \frac{1}{2}(\delta(\xi_1 - \xi_2) - \delta(\xi_1 + \xi_2)),
\]

while the Green function is \( \det |X_1 - X_2|^{-\frac{1}{2}} \).

7 Time

Equations (6.4) and (6.5) describe propagation along the generalized light-like directions \( \Delta X^{AB} = \eta^A \eta^B \), where \( \eta^A \) is a contravariant “svector”. The sign choice here fixes a time arrow. Suppose that a light-like signal emitted from some point \( X_0^{AB} \) of the generalized space-time reaches some other point \( X_1^{AB} = X_0^{AB} + \eta^A \eta^B \) switching on a new process that emits a signal in a different light-like direction\(^4\). Provided this happens several times, any point

\[
\Delta X^{AB} = \sum_{i=0}^{M} \eta_i^A \eta_i^B,
\]

can be reached where \( \eta_i^A \) is a complete set of contravariant svectors. Formula (7.1) describes generic positive semi-definite symmetric matrix \( \Delta X^{AB} \).

We see that the relativistic geometry that follows from Eq. (6.4) identifies the future cone \( C^+_X \) of a point \( X_0 \) with the set of matrices \( X^{AB} \) such that \( \Delta X^{AB} = X^{AB} - X_0^{AB} \) is positive semi-definite. Time-like vectors are described by positive definite matrices

\[
\Delta X^{AB} \xi_A \xi_A > 0, \quad \forall \xi_A.
\]

\(^4\)The assumption that a process can be switched on locally may or may not be true for particular dynamical equations. In fact, as discussed below, Eqs. (6.4) and (6.5) do not admit full localization in \( \mathcal{M}_M \). This does not affect our conclusions on the causal structure of the generalized space-time, however.
Light-like vectors identify with degenerate positive semi-definite matrices. We will distinguish between rank-$k$ light-like directions described by matrices of rank $k$. The concepts of time-like and rank-$k$ light-like vectors are invariant under the generalized Lorentz group $SL_M$.

Equation (6.4) describes propagation along the most degenerate light-like directions of rank 1. To work out a form of the equations that describe propagation along less degenerate light-like directions one can use the tensor product construction described at the end of Sec. 5. As shown in [37], $n$–fold tensor product of the Fock representation in Eq. (6.2) gives rise to the field equations which describe propagation along rank $n$ light-like directions in $\mathcal{M}_M$.

The past cone $C_{X_0}^-$ is defined analogously as the set of negative semi-definite matrices

$$(X^{AB} - X_0^{AB})\xi_A\xi_B \leq 0, \quad \forall \xi_A.$$ (7.3)

If $Y \in C_X^+$ then $X \in C_Y^+$ and $2X - Y \in C_X^-$. $C_X^+$ is the convex cone: $\forall X_1, X_2 \in C_X^+, \lambda, \mu \in R^+, \lambda X_1 + \mu X_2 \in C_X^+$. Note that the analysis of future and past cones in $\mathcal{M}_M$ was given in [43] in a different context.

Let us now introduce a concept of space-like global Cauchy surface as such a submanifold $\Sigma$ of some (generalized) space-time manifold $\mathcal{M}$ that

(i) $\forall X_1, X_2 \in \Sigma$, $X_1 \notin C_{X_2}^+$ and $X_2 \notin C_{X_1}^+$ for $X_1 \neq X_2$.

(ii) any point of $\mathcal{M}$ belongs to either future or past cone of some point of $\Sigma$.

No point $Y \in \mathcal{M}$ can belong to the future cone of some point $X_1 \in \Sigma$ and past cone of some other point $X_2 \in \Sigma$.

In particular, the property (i) implies that no pair of observers on $\Sigma$ are allowed to exchange causal signals, i.e., a global Cauchy surface is space-like. Provided that $\mathcal{M}$ admits a fibration (or rather foliation) into a set of space-like global Cauchy surfaces, $\Sigma_t$ parametrized by some parameter(s) $t$, this defines the concept of time(s).

Let $T^{AB}$ be some fixed positive definite matrix. The axioms (i) and (ii) are satisfied with the space-like global Cauchy surfaces $\Sigma_t$ parametrized as

$$X^{AB} \in \Sigma_t : \quad X^{AB} = x^{AB} + tT^{AB},$$ (7.4)

where the space coordinates $x^{AB}$ are arbitrary $T-$ traceless matrices

$$x^{AB}T_{AB} = 0, \quad T_{AB}T^{BC} = \delta^C_A.$$ (7.5)

Indeed, the difference of any two matrices of the form (7.4) at fixed $t$ is traceless and therefore it is neither positive definite nor negative definite. As a result,
any two points of $\Sigma_t$ at some fixed $t$ are separated by a space-like interval. The rest of the axioms is a consequence of the trivial decomposition (7.4) of a matrix into the sum of its trace and traceless parts.

An important output of this analysis is that $\mathcal{M}_M$ has just one evolution parameter

$$t = \frac{1}{M} X^{AB} T_{AB}. \quad (7.6)$$

The ambiguity in the choice of a positive definite matrix $T^{AB}$ parameterizes the ambiguity in the choice of a particular coordinate frame like in Einstein special relativity: any two positive definite matrices $T_{1,2}^{AB}$ with equal determinants are related by some generalized $SL_M$ Lorentz transformation. The dilatation allows one to fix a scale of time in an arbitrary way. As shown in [26], $SL_M$ generalized Lorentz transformations exhibit a lot of similarity with the usual Lorentz transformations in Minkowski space-time including nontrivial limits on relative speeds of reference frames.

8 Localization and Clifford algebra

Using the ambiguity in $b^\pm(\xi)$ and $f^\pm(\xi)$ in the general solutions (6.12) and (6.13) it is possible to fix their dependence in at most $M$ coordinates that, generically, is much less than $\dim(\mathcal{M}_M) = \frac{1}{2}M(M+1)$. In particular, it is impossible to localize solutions in more than $M$ coordinates. This phenomenon is new compared to the experience with the usual Minkowski space-time: although fields live in $\frac{1}{2}M(M+1)$ dimensional space-time physical events occur in some $M$-dimensional subspace. This happens because the equations are overdetermined and contain constraints which do not allow to focus the waves in more than $M$ dimensions. (Analogously, the Gauss law does not allow electric field to be fixed arbitrarily on the initial Cauchy surface for a given charge distribution.)

It is useful to introduce [26] the concept of local Cauchy bundle $E$ as such $M$-dimensional surface in $\mathcal{M}_M$ (or a limiting surface with some of radii shrinking to zero) that the solutions (6.12) and (6.13) are fixed by their values on $E$. Note that while the concept of global Cauchy surface is a characteristics of $\mathcal{M}_M$ independent of particular dynamics, the concept of local Cauchy surface characterizes particular dynamical equations in $\mathcal{M}_M$. These two notions coincide in the Minkowski geometry but turn out to be different in a more general framework.

Now one can address a question whether there are some $d-1$ space-like
coordinates

\[ x^n = \sigma^n_{AB} X^{AB} \]  

such that, using the ambiguity in the functions \[ b^\pm(\xi) \], it is possible to build solutions of the field equations proportional to (derivatives of) the delta function \[ \delta^{d-1}(x-x_0) \] at any point \( x_0 \in R^{d-1} \). For this to be possible, the dual momenta \( k_n = \Gamma_n^{AB} \xi_A \xi_B \) have to provide a map of the orbifold \( R^M/Z_2 \) onto \( R^{d-1} \) with the only singularity at \( \xi_A = 0 \). Then, by changing integration variables from \( \xi_A \) to \( k_n \) plus, may be, some other variables in case \( d-1 < M \), one can get delta functions \( \delta^{d-1}(x-x_0) \) by integrating over \( k_n \).

Remarkably, this can be achieved if the matrices \( \gamma^n_{AB} = \sigma^n_{AC} T^{CB} \), where \( T^{CB} \) is the time axis matrix, satisfy Clifford relations

\[ \{ \gamma^n, \gamma^m \} = 2 \eta^{nm}. \]  

(This is possible when the index \( A \) describes a collection of some \( q \) spinors with \( M = q 2^p \).) Indeed, an immediate consequence of (8.2) is that the matrices \( \sigma^n_{AB} \) are traceless \( \sigma^n_{AB} T^{AB} = 0 \) whenever \( d \geq 3 \), thus belonging to the global Cauchy surface. Another important property is that the momenta

\[ k_n(\xi) = \sigma^n_{AB} \xi_A \xi_B \]  

map the cone \( R^M/Z_2 \) onto \( R^{d-1} \), i.e., varying real twistor parameters \( \xi_A \) it is possible to get arbitrary values of \( k_n(\xi) \). This results from the invariance of the construction under the space rotations \( SO(d-1) \) generated by \( M_{nm} = \frac{1}{4} \{ \gamma_n, \gamma_m \} \). By a space rotation one aligns a vector \( k_n(\xi) \) along any direction and then normalizes it arbitrarily by a rescaling of \( \xi_A \). That momenta \( k_n(\xi) \) span \( R^{d-1} \) allows for localization of the fields in \( d-1 \) space \( x^n \) coordinates dual to \( k_n \), i.e., by means of integration over \( k_n \), one can reach the delta-functional initial data \( \delta^{d-1}(x^n-x_0^n) \) localized at any point of the physical space \( R^{d-1} \).

Because the square of any linear combination of \( \gamma \) matrices is proportional to the unit matrix, for any vector \( a^n \) there exists such a basis in the space of \( \xi_A \) that

\[ T^{AB} = \delta^{AB}, \quad a^n \sigma^n_{AB} = \sqrt{a^2 Y^{AB}}, \quad a^2 = a^n a^m \eta_{nm}, \]  

where

\[ Y = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \]  

with all four blocks being \( M/2 \times M/2 \) matrices (\( M \) is assumed to be even). As a result, (nondegenerate) space-time matrix coordinates of the form

\[ \chi^{AB} = t T^{AB} + x^n \sigma^n_{AB} \]  

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can only have three values of the inertia index $I_\chi = M$ for $t > \sqrt{x^2}$, $I_\chi = -M$ for $t < -\sqrt{x^2}$ and $I_\chi = 0$ for $t^2 < x^2$, where $x^2 = x^nx^m\eta_{nm}$. This corresponds to the standard space-time picture with the future cone ($I_\chi > 0$), past cone ($I_\chi < 0$) and the space-like region ($I_\chi = 0$). This property of space Clifford coordinates implies microcausality in terms of the coordinates $x^n$ of the usual space-time [26]. (Note that the condition that $I_\chi$ can take in the linear space of matrices of the form (8.6) only maximal value $M$ (future), minimal value $-M$ (past) or some fixed intermediate value $I_\chi = I_{\text{space}} \neq \pm M$ for all values of $t$ and $x^n$ can be taken as an alternative definition leading to the $\gamma$-matrix relations [26].)

A new comment we would like to make here is that the Clifford realization of coordinates of local events provides a global characterization of causality. Consider two events characterized by local space-time coordinates (8.6) $\chi_1$ and $\chi_2$. A solution of the field equations localized at $\chi = \chi_{1,2}$ is characterized by some function $c(X)$ of

$$X^{AB} = \chi^{AB} + \Lambda^{AB}$$

(8.7)

with the dependence in $\chi$ localized at $\chi_{1,2}$ (i.e., being proportional to some derivatives of $\delta(\chi - \chi_{1,2})$) but not necessarily localized in the rest traceless coordinates $\Lambda^{AB}$ orthogonal to $\sigma_{nAB}$

$$\Lambda^{AB}\sigma_{nAB} = 0, \quad T^{AB}\sigma_{nAB} = 0.$$  \hspace{1cm} (8.8)

The question is whether it is possible that $\chi^{AB}$ is space-like while $X^{AB}$ is time-like for some $\Lambda^{AB}$. Remarkably, the answer is negative (otherwise, causal relationships beyond the usual local coordinate Minkowski picture would be possible). Namely, let $X_1, X_2 \in \mathcal{M}$, be some points where solutions corresponding to some two different events are nonzero. If $X_1 \in \mathcal{C}_X \pm X_2$ the Clifford local coordinates $\chi_{1,2}$ characterizing these solutions are in the same causal relationship (for any choice of Clifford coordinates). This fact guarantees that the concept of causality has global meaning and is fully characterized by the causality of local events associated with the Clifford coordinates.

The proof is elementary. Let $X^{AB} \in \mathcal{C}$, where $\mathcal{C} = \mathcal{C}_0^+$, i.e. $X^{AB}$ is positive definite. The dual cone $\mathcal{C}^*$ consists of all positive definite matrices $Y_{AB}$. The two spaces are identified once the matrix $T^{AB}$ is used to raise and lower indices. Positive definite matrices $\chi^{AB}$ of the form (8.6) also form a selfdual cone, which

\[ \text{Recall that a dual cone } \mathcal{C}^* \text{ of } \mathcal{C} \subset V, \text{ where } V \text{ is some vector space, consists of such elements } a \in V^* \text{ that the scalar product } (a, b) > 0, \forall b \in \mathcal{C}. \text{ As is easy to see, } Y_{AB}X^{AB} > 0 \text{ for any positive definite matrices } X^{AB} \text{ and } Y_{AB}, \text{ and, if one of the matrices, say } Y_{AB}, \text{ is not positive definite, then there exists such a positive definite matrix } X^{AB} \text{ that } Y_{AB}X^{AB} \leq 0. \]
is the usual future cone $t > 0$, $t^2 - x^2 > 0$ in the Minkowski space-time. It follows that $\chi_{AB}^* X^{AB} > 0$ for any positive definite $\chi_{AB}^*$. The property (8.8) implies that $\chi_{AB}^* \chi_{AB}^{AB} > 0$ for any $\chi_{AB}^*$. Since the Minkowski cone is selfdual in Minkowski space-time, $\chi^{AB}$ belongs to the Minkowski cone and therefore is positive definite.

These properties indicate that the Clifford algebra realization of space coordinates has deep relationship with the concept of locality and micro causality. In other words, the generalized space-time $\mathcal{M}_M$ is visualized via Clifford algebras. Let us stress that there is no metric tensor in the dynamical equations (6.4) and (6.5), i.e., the original $Sp(2M)$ invariant setup is insensitive not only to the choice of the conformal factor of the metric but does not contain the metric tensor at all. The space metric $\eta_{nm}$, which automatically turns out to be positive definite as a consequence of (8.2), appears in the theory upon identification of an appropriate Clifford algebra at the stage of defining the concept of local event. It is indeed physically meaningful to define a physical length measure (metric) simultaneously with the definition of a physical point. Thus the formulation of HS dynamics in the generalized space-time drives us to a point at which a role of the metric tensor has to be reconsidered as not being given ad hoc but resulting from the analysis of the concept of locality in a theory that has no metric tensor in its original formulation. Let us stress that this in no way means that the property of independence of a coordinate choice (i.e., equivalence principle) is lost. It is guaranteed by the unfolded form of the dynamical equations (5.3) formulated in terms of differential forms. The new message is that prior defining how to measure a distance between events one has to specify what is a local physical event.

For the lowest values of $M = 2, 4, 8$, the local Cauchy bundle $E$ has the structure $E = \sigma \times S$ with the base manifold (local Cauchy surface) $\sigma = R^2, R^3, R^5$ as the physical space and fiber compact manifolds $S = Z_2, S^1, SU(2)$, respectively. This corresponds to 3d, 4d and 6d Minkowski space-times, while the fibers $S^1$ and $SU(2)$ give rise to some spin degrees of freedom. In particular, for the case of $M = 4$, modes of $S^1$ give rise to the infinite tower of spins in 4d Minkowski space-time [26]. Once some local Cauchy bundle $E = R^{d-1} \times S$ is chosen to visualize $\mathcal{M}_M$, the interpretation of different subalgebras of the $Sp(2M)$ symmetry of the original equations becomes different. A transformation which maps the Minkowski space-time $\sigma \times R^1$ to itself leaving the fibers intact is some usual (Minkowski) conformal transformation (conformal symmetry is a subalgebra of $Sp(2M)$ as a consequence of the Clifford realization of space [26]). Another type of symmetry does not shift points of the Minkowski space-time acting only on the coordinates of the fiber. For the case
of $M = 4$ this is the $U(1)$ electro-magnetic duality transformation acting on all spins, which thus acquires a purely geometric interpretation in the generalized space-time framework. The $Sp(2M)$ transformations which shift $E$ in $\mathcal{M}_M$ look as nongeometric symmetries from the Minkowski space-time perspective, extending $su(2, 2) \oplus u(1)$ to $sp(8)$ which mixes fields of different spins.

9 Conclusions

The fundamental fact that infinite-dimensional HS symmetry algebras are oscillator star-product algebras has a consequence that they contain $sp(2M)$ as finite-dimensional subalgebras when there are $M$ pairs of oscillators. For lower dimensions these are isomorphic to the usual conformal or AdS space-time symmetry algebras. For higher dimensions, starting from the 4d conformal algebra (equivalently $AdS_5$ algebra), space-time symmetry algebras become proper subalgebras of the symplectic subalgebras of the HS algebras. In particular, the 4d conformal algebra $su(2, 2)$ is a subalgebra of the $sp(8)$ that acts on the infinite towers of massless fields. The idea that HS theories have to admit a description in a larger manifestly $sp(8)$ invariant space-time is as natural as the idea to describe supersymmetric theories in superspace. The relevant $Sp(2M)$ invariant space-time $\mathcal{M}_M$ turns out to be $\frac{1}{2}M(M + 1)$ dimensional having $M \times M$ symmetric matrices $X^{AB} = X^{BA}$ as local coordinates ($A, B = 1 \ldots M$). These coordinates contain usual space-time coordinates $x^n$ along with coordinates $y$ analogous to the central charge coordinates known to appear in the context of brane dynamics and $M$-theory algebras (see, e.g., [44, 39, 45, 43, 46] and references therein). The infinite towers of 4d massless fields of all spins are described by one scalar $b(X)$ for all bosons and one spinor $f_A(X)$ for all fermions which satisfy, respectively, the second-order (6.4) and first-order (6.5) partial differential equations in $\mathcal{M}_4$ equivalent to all massless equations in the 4d Minkowski space-time.

Remarkably, these equations do not contain any metric tensor in $\mathcal{M}_M$. This unusual feature is physically appropriate because the HS equations in $\mathcal{M}_M$ fix solutions by their values on some $M$-dimensional manifold called local Cauchy bundle $E$. One can say that $E$ is the space of physical events. On the top of that there is one time evolution parameter associated with a time-like direction along some positive definite matrix $T^{AB}$. For $M = 4$, $E = R^3 \times S^1$ [20]. Modes on $S^1$ give rise to the infinite towers of higher spins. The electro-magnetic duality symmetry gets geometric interpretation as a part of the $Sp(2M)$ symmetry which acts on the fiber $S^1$. Metric structure reappears in the theory at the
stage of identification of the space coordinates which describe local events (i.e., $R^3$ for the $M = 4$ case) in a quite remarkable way: matrix coordinates which admit localization form Clifford algebra. This introduces simultaneously the metric tensor and spinor structure in the physical space-time of local events with the indices $A,B$ interpreted as (a collection of) spinor indices in the physical space-time. The lesson is that having a mathematical space like $\mathcal{M}_M$ is not the same as to have a physical space-time where events described by one or another equations can happen. Different equations in the same space can visualize it differently. Apart from the simplest (rank 1) dynamical system in $\mathcal{M}_M$ discussed in this talk, supported by $M$-dimensional local Cauchy surfaces, there exist higher rank system supported by $rM$-dimensional local Cauchy surfaces $\mathcal{M}_M$. In striking similarity with the brane picture in superstring theory physical space-times visualized by different equations coexist as “branes” imbedded into $\mathcal{M}_M$.

Reconsideration of such fundamental notions of physics as local event and local distance (metric) suggested by the analysis of the HS gauge theory may significantly affect the present day understanding of space-time geometry and related physical theories including general relativity and quantum mechanics. The most exciting option is that this may help to unify these two fundamental physical theories on a new basis.

The main practical problem for the future is to develop a manifestly $Sp(2M)$ symmetric nonlinear HS theory. In particular, this will require to include gauge potentials into the formalism as the equations discussed so far are formulated in terms of field strengths.

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