Curvature actions on Spin($n$) bundles

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Abstract

We compute the number of linearly independent ways in which a tensor of Weyl type may act upon a given irreducible tensor-spinor bundle $V$ over a Riemannian manifold. Together with the analogous but easier problem involving actions of tensors of Einstein type, this enumerates the possible curvature actions on $V$.

1 Introduction

Let $V$ be an irreducible Spin($n$) vector bundle over an $n$-dimensional Riemannian spin manifold $(M,g)$. The main point of this paper is to give a simple formula for the number of ways in which the different parts of the Riemann curvature can act upon sections of $V$ in an equivariant way.

Irreducible Spin($n$) bundles are in one-to-one correspondence with irreducible finite dimensional representations of Spin($n$). These are identical with the irreducible finite dimensional representations of the Lie algebra $\mathfrak{so}(n)$. The correspondence of bundles with representations, or, in the general parlance, modules, is given by the associated bundle construction (see, e.g., [14]). Let $S$ be the Spin($n$)-principal bundle of spin frames over the base manifold $M$; given a Spin($n$)-module $(\varphi,V)$, we form the bundle $V = S \times_{\lambda} V$.

The Riemann curvature is a section of a bundle associated to a direct sum of irreducible bundles. These irreducible summands hold the various parts of the Riemann tensor – the Weyl conformal curvature tensor (or its self-dual and anti-self-dual parts in dimension 4), the Einstein (trace-free Ricci) tensor, and the scalar curvature. \textit{Weitzenböck formulas}, informally speaking, express the difference between two second-order differential operators on a bundle $V$ as a curvature action; that is, as a Spin($n$)-equivariant bundle homomorphism from $\mathcal{R} \otimes V$ to $V$, where $\mathcal{R}$ is the appropriate bundle of curvature tensors. Because of the importance of such formulas in geometric analysis, it is important to understand curvature actions. For example, the bundle of trace-free symmetric 2-tensors, of which the Einstein curvature is a section, acts on the bundle of one-forms by (in abstract index notation)

$$\varphi_a \mapsto \sigma^b a \varphi_b.$$ 

Algebraic Weyl tensors, i.e. sections of the bundle in which the Weyl curvature lives, are capable of acting on two-forms $\varphi_{ab}$ via

$$\varphi_{ab} \mapsto C_{ab}^{cd} \varphi_{cd}.$$
We find the number of actions of each part of the Riemann curvature tensor (or, more precisely, each part of a generic algebraic Riemann curvature tensor) on each irreducible bundle by computing the dimension of the space of equivariant homomorphisms as described above. By the associated bundle construction and the fact that the various parts of the curvature are sections of specific Spin\((n)\)-bundles, this is the same problem as that of computing the dimension of \(\text{Hom}_{\mathfrak{so}(n)}(U \otimes V, V)\) for certain irreducible \(\mathfrak{so}(n)\)-modules \(U\), and a general irreducible \(\mathfrak{so}(n)\)-module \(V\). The problem is the same since the associated bundle construction “promotes” each module homomorphism to the bundle setting; bundle homomorphisms “demote” to module homomorphisms just by evaluation at any point.

2 Some facts from representation theory

We refer to [13] and [12] for the following standard facts on \(\mathfrak{so}(n)\). Let \(n \geq 2\).

Integral weights for \(\mathfrak{so}(n)\) are \(\ell\)-tuples, \(\ell = [n/2]\), consisting entirely of integers, or entirely of half-integers:

\[
\Pi = \mathbb{Z}^{\ell} \cup \left(\frac{1}{2} + \mathbb{Z}\right)^{\ell}.
\]

For each finite-dimensional representation \((\varphi, V)\) of \(\mathfrak{so}(n)\), the vector space \(V\) is a direct sum of weight spaces \(V_\mu\), for various \(\mu\) in \(\Pi\). We put

\[
\Pi(\varphi) = \{\mu \in \Pi \mid V_\mu \neq 0\}.
\]

The direct sum decomposition

\[
V = \bigoplus_{\mu \in \Pi(\varphi)} V_\mu
\]

is of course not \(\mathfrak{so}(n)\)-invariant in general; each weight space is, however, invariant under a maximal abelian subalgebra of \(\mathfrak{so}(n)\). The multiplicity of a weight \(\mu\) in a finite dimensional representation \((\varphi, V)\) is the dimension of \(V_\mu\).

A dominant integral weight is an integral weight \(\lambda\) which satisfies the inequality condition

\[
\begin{align*}
\lambda_1 &\geq \lambda_2 \geq \cdots \geq \lambda_{\ell-1} \geq \lambda_\ell, & n \text{ even}, \\
\lambda_1 &\geq \lambda_2 \geq \cdots \geq \lambda_{\ell-1} \geq \lambda_\ell & n \text{ odd}.
\end{align*}
\]

(2.1)

The finite-dimensional irreducible representations of \(\mathfrak{so}(n)\) are parameterized by dominant integral weights, and in fact, the dominant weight parameter \(\lambda\) associated to a given irreducible representation is its (lexicographically) highest weight. In what follows, we fix a dominant integral weight \(\lambda\), and study the corresponding irreducible representation \(V(\lambda)\). The set of weights of \(V(\lambda)\) will be denoted \(\Pi(\lambda)\), and the multiplicity of the weight \(\mu\) in the representation \(V(\lambda)\) will be denoted by \(m_{\lambda}(\mu)\).

In writing dominant weights, we shall sometimes use the convention of Strichartz, in which terminal strings of zeros are omitted from the notation. Thus, for example, we write \((1, 1, 0, \ldots, 0)\) as \((1, 1)\).

Let \(e_a\) be the \(\ell\)-tuple with 1 in the \(a^{th}\) entry, and 0 in all other entries. Let

\[
\alpha_{\pm ab} = e_a \pm e_b.
\]
The roots of \( so(n) \) are the weights of the adjoint representation. With their multiplicities, these are

| root \( \mu \)   | multiplicity |
|------------------|--------------|
| 0                | \( \ell \)   |
| \( e_a - e_b \)  | 1            |
| \( e_a + e_b \)  | 1            |
| \( -e_a - e_b \) | 1            |
| \( e_a \)        | \( n - 2\ell \) |
| \( -e_a \)       | \( n - 2\ell \) |

The sum of the (lexicographically) positive roots in \( so(n) \) is

\[
2\rho = (n - 2, n - 4, \ldots, n - 2\ell) = \begin{cases} 
(n - 2, n - 4, \ldots, 2, 0), & \text{n even,} \\
(n - 2, n - 4, \ldots, 3, 1), & \text{n odd.}
\end{cases}
\]

Let \( \mu \in \Pi \), and let \( \Delta^+ \) be the set of positive roots. One knows that that a dominant weight \( \mu \) lies in \( \Pi(\lambda) \) if and only if

\[
\mu + \sum_{\alpha \in \Delta^+} k_\alpha \alpha = \lambda
\]

for some list of natural numbers \( k_\alpha \). If (2.2) holds, \( \mu \) is said to be subordinate to \( \lambda \). Freudenthal’s formula allows one to inductively compute the multiplicities \( m_\lambda(\mu) \), for \( \mu \in \Pi(\lambda) \), starting with the fact that \( m_\lambda(\lambda) = 1 \):

\[
(\|\tilde{\lambda}\|^2 - \|\tilde{\mu}\|^2)m_\lambda(\mu) = 2 \sum_{\alpha \in \Delta^+} \sum_{j=1}^{\infty} m_\lambda(\mu + j\alpha)(\mu + j\alpha, \alpha).
\]

Here if \( \mu \) is a weight, then \( \tilde{\mu} = \mu + \rho \). The inner product \( (\cdot, \cdot) \) on the right is the standard inner product in \( \mathbb{R}^\ell \). A very useful fact, which we shall often use implicitly, is that if \( \lambda \) is dominant, then \( \tilde{\lambda} \) is strictly dominant – it satisfies a version of (2.1) in which the \( \geq \) signs are replaced by \( > \) signs. Note that the sum on the right in (2.3) is finite because \( \Pi(\lambda) \) is finite. Freudenthal’s formula expresses the multiplicity \( m_\lambda(\mu) \) in terms of multiplicities \( m_\lambda(\nu) \) for \( \nu \) lexicographically higher than \( \mu \); thus the computation is truly inductive. To make the computation run, we simply have to find all positive root strings through dominant weights in \( \Pi(\lambda) \). Note that if \( \mu \) is dominant and subordinate to \( \lambda \), then \( \mu + j\alpha \) (for \( j \in \mathbb{N}, \alpha \in \Delta^+ \)) is dominant, and is either subordinate to \( \lambda \), or outside of \( \Pi(\lambda) \).

To get the multiplicities of non-dominant weights in \( V(\lambda) \), one uses the action of the Weyl group \( W \). In the case of \( so(n) \), the Weyl group acts on \( \Pi \) as follows. If \( n \) is odd, a given \( w \in W \) acts by permutation, together with any number of sign changes on entries of a weight. If \( n \) is even, a given \( w \in W \) acts by permutation and an even number of sign changes. An important point is that \( W \) acts on \( \Pi(\lambda) \) (not just on \( \Pi \)), and that

\[
m_\lambda(w \cdot \mu) = m_\lambda(\mu), \quad \forall \mu \in \Pi(\lambda), \ w \in W.
\]
Since each Weyl group orbit clearly contains a unique dominant weight, Freudenthal’s formula solves the problem of finding all weights, with their multiplicities, in $V(\lambda)$.

Given two dominant integral weights $\sigma$ and $\lambda$, an important and much-studied problem is that of computing the direct sum decomposition of $V(\sigma) \otimes V(\lambda)$. By Weyl’s Theorem [13], any finite-dimensional representation of $\mathfrak{so}(n)$ is completely reducible ($\mathfrak{so}(n)$ being semisimple):

$$V(\sigma) \otimes V(\lambda) \cong \bigoplus_{\kappa \in \Pi^{\text{DI}}} M_{\kappa}(V(\sigma) \otimes V(\lambda)),$$

where $\Pi^{\text{DI}}$ is the set of dominant integral weights. Here the multiplicities $M_{\kappa}(V(\sigma) \otimes V(\lambda))$, which give the number of isomorphic copies of $V(\kappa)$ in $V(\sigma) \otimes V(\lambda)$, are natural numbers; by finite-dimensionality, all but a finite number of these vanish. The Brauer-Kostant formula [8, 15] expresses the numbers $M_{\lambda}(V(\sigma) \otimes V(\tau))$ in terms of $\lambda$ and all the weights, with multiplicities, of $V(\sigma)$. This is ideal for the type of problem we have here, in which we would like to compute information about the tensor product of a fixed representation (that to which algebraic Weyl tensors are associated) with an arbitrary representation – we just need the weights and multiplicities of the fixed representation.

To state the Brauer-Kostant formula, we need the action of the Weyl group $W$ of $\mathfrak{so}(n)$ on $\Pi$. The sign of a Weyl group element, $\text{sgn} \, w$, is the sign of the permutation times the number of sign changes. The Brauer-Kostant formula says that

$$M_{\lambda}(V(\sigma) \otimes V(\tau)) = \sum_{\mu \in \Pi} m_{\sigma}(\mu) \sum_{w \in W} (\text{sgn} \, w) \delta^{\tau - \mu}_{w \cdot \lambda},$$

where $\delta$ is the Kronecker delta.

### 3 Computations

By [18], the bundle of algebraic Weyl tensors is associated to the representation $V(2, 2)$ for $n \geq 5$, and to $V(2, 2) \oplus V(2, -2)$ for $n = 4$. Thus the first task is to find the weights, with multiplicities, of these modules. As noted above, each Weyl group orbit contains a unique dominant weight, so we need only list the multiplicities of dominant weights subordinate to $(2, 2)$ (and to $(2, -2)$ when $n = 4$). In the following theorem, we get these weights and multiplicities, as well as the sizes of the Weyl group orbits associated to each. This latter bit of information is not strictly necessary to our calculations, but allows us to accomplish a reassuring check: that the weight multiplicities add up to the dimension of the module. In addition, we give the contribution of each Weyl group orbit to the expression on the right in the Brauer-Kostant formula for $M_{\lambda}(V(2, 2) \otimes V(\lambda))$ (and, when $n = 4$, the same expression with $(2, -2)$ in place of $(2, 2)$). Multiplying each contribution by the corresponding multiplicity and adding gives the number $M_{\lambda}(V(2, 2) \otimes V(\lambda))$ (and the same with $(2, -2)$ in place of $(2, 2)$ when $n = 4$); this information is collected in Theorems 2, 3, and 4 below.

To write the contributions to the right-hand side of the Brauer-Kostant formula, we need to define some parameters based on the dominant weight $\lambda$. If $1 \leq k \leq \ell,$
let
\[ \varepsilon_{\kappa_{\ell-k}, \ldots, \kappa_\ell} = \begin{cases} 1, & (\lambda_{\ell-k}, \ldots, \lambda_\ell) = (\kappa_{\ell-k}, \ldots, \kappa_\ell), \\ 0 &\text{otherwise}, \end{cases} \]

and let
\[ \lambda^{(k)} = (\lambda_1, \ldots, \lambda_{\ell-k}) \]

Furthermore, let
\begin{align*}
T &= \# \{ a \mid \lambda_a = \lambda_{a+1} = \lambda_{a+2} \}, \\
D &= \# \{ a \mid \lambda_a+1 = \lambda_a - 1 \}, \\
P &= \# \{ a \mid \lambda_a+1 = \lambda_a \}, \\
S &= \# \{ \{ a, b \} \mid \lambda_a+1 = \lambda_a, \lambda_b+1 = \lambda_b, \# \{ a, a+1, b, b+1 \} = 4 \}.
\end{align*}

To paraphrase, \( T \) is the number of “flat triples”, \( D \) is the number of “1-drops”, \( P \) the number of “flat pairs”, and \( S \) is the number of disjoint pairs of flat pairs.

**Theorem 1** The dominant weights of \( V(2, 2) \), with their multiplicities, orbit sizes, and orbit contributions to \( \mathcal{M}_A(\ell V(2, 2) \otimes V(\lambda)) \) (assuming, when \( n \) is even, that \( \lambda_\ell \geq 0 \)) are given by the following tables:

### \( n \geq 10 \) even:

| weight \( \mu \) | multiplicity | orbit size \#(W \cdot \mu) | contribution |
|------------------|--------------|-----------------------------|--------------|
| \( (2, 2) \)     | 1            | 2\ell(\ell - 1)             | \(-T - D - \varepsilon_{0,0,0} - \varepsilon_{1/2,1/2} \) |
| \( (2, 1, 1) \)  | 1            | 4\ell(\ell - 1)(\ell - 2)  | \(2T + 2\varepsilon_{0,0,0} - \varepsilon_{0,0}P(\lambda^{(2)}) \) |
| \( (2) \)        | \ell - 2     | 2\ell                      | \varepsilon_{0,0} |
| \( (1, 1, 1, 1) \)| 2            | \( \frac{2}{3}\ell(\ell - 1)(\ell - 2)(\ell - 3) \) | \( S + \varepsilon_{0,0}P(\lambda^{(2)}) \) |
| \( (1, 1) \)     | 2\ell - 3    | 2\ell(\ell - 1)            | \(-P - \varepsilon_{0,0} \) |
| \( 0 \)          | \ell(\ell - 1) | 1                          | 1            |

### \( n = 8 \):

| weight \( \mu \) | multiplicity | orbit size \#(W \cdot \mu) | contribution |
|------------------|--------------|-----------------------------|--------------|
| \( (2, 2) \)     | 1            | 2\ell(\ell - 1) = 24       | \(-T - D - \varepsilon_{0,0,0} - \varepsilon_{1/2,1/2} \) |
| \( (2, 1, 1) \)  | 1            | 4\ell(\ell - 1)(\ell - 2)  = 96 | \(2T + 2\varepsilon_{0,0,0} - \varepsilon_{0,0}P(\lambda^{(2)}) \) |
| \( (2) \)        | \ell - 2 = 2 | 2\ell = 8                   | \varepsilon_{0,0} |
| \( (1, 1, 1, 1) \)| 2            | 8                          | \( S \) |
| \( (1, 1, 1, -1) \)| 2            | 8                          | \varepsilon_{0,0}P(\lambda^{(2)}) |
| \( (1, 1) \)     | 2\ell - 3 = 5| 2\ell(\ell - 1) = 24       | \(-P - \varepsilon_{0,0} \) |
| \( 0 \)          | \ell(\ell - 1) = 12 | 1                          | 1            |
\[ n = 6: \]

| weight \( \mu \) | multiplicity | orbit size \#(W \cdot \mu) | contribution |
|-----------------|--------------|--------------------------|-------------|
| (2, 2)          | 1            | \( 2\ell(\ell - 1) = 12 \) | \(-T - D - \varepsilon_{0,0,0} - \varepsilon_{1/2,1/2}\) |
| (2, 1, 1)       | 1            | 12                       | \( T + \varepsilon_{0,0,0} \) |
| (2, 1, -1)      | 1            | 12                       | \( T + \varepsilon_{0,0,0} \) |
| (2)             | \( \ell - 2 = 1 \) | \( 2\ell = 6 \) | \( \varepsilon_{0,0} \) |
| (1, 1)          | \( 2\ell - 3 = 3 \) | \( 2\ell(\ell - 1) = 12 \) | \(-P - \varepsilon_{0,0} \) |
| 0               | \( \ell(\ell - 1) = 6 \) | 1                       | 1           |

\[ n \geq 9 \text{ odd:} \]

| weight \( \mu \) | multiplicity | orbit size \#(W \cdot \mu) | contribution |
|-----------------|--------------|--------------------------|-------------|
| (2, 2)          | 1            | \( 2\ell(\ell - 1) \) | \(-T - D - \varepsilon_{0,0,0} \) |
| (2, 1, 1)       | 1            | \( 4\ell(\ell - 1)(\ell - 2) \) | \( 2T + \varepsilon_{1/2}P(\lambda^{(1)}) \) |
| (2)             | \( \ell - 1 = 2 \) | \( 2\ell \) | \(-\varepsilon_{1/2} \) |
| (1, 1, 1, 1)    | 2            | \( \frac{2}{3}\ell(\ell - 1)(\ell - 2)(\ell - 3) \) | \( S \) |
| (1, 1, 1)       | 2            | \( \frac{4}{3}\ell(\ell - 1)(\ell - 2) \) | \( \varepsilon_{0}P(\lambda^{(1)}) \) |
| (1, 1)          | \( 2(\ell - 1) \) | \( 2\ell(\ell - 1) \) | \(-P \) |
| (1)             | \( 2(\ell - 1) \) | \( 2\ell \) | \(-\varepsilon_{0} \) |
| 0               | \( (\ell + 1)(\ell - 1) \) | 1                       | 1           |

\[ n = 7: \]

| weight \( \mu \) | multiplicity | orbit size \#(W \cdot \mu) | contribution |
|-----------------|--------------|--------------------------|-------------|
| (2, 2)          | 1            | \( 2\ell(\ell - 1) = 12 \) | \(-T - D - \varepsilon_{0,0,0} \) |
| (2, 1, 1)       | 1            | \( 4\ell(\ell - 1)(\ell - 2) = 24 \) | \( 2T + \varepsilon_{1/2}P(\lambda^{(1)}) \) |
| (2)             | \( \ell - 1 = 2 \) | \( 2\ell = 6 \) | \(-\varepsilon_{1/2} \) |
| (1, 1, 1)       | 2            | \( \frac{4}{3}\ell(\ell - 1)(\ell - 2) = 8 \) | \( \varepsilon_{0}P(\lambda^{(1)}) \) |
| (1, 1)          | \( 2(\ell - 1) = 4 \) | \( 2\ell(\ell - 1) = 12 \) | \(-P \) |
| (1)             | \( 2(\ell - 1) = 4 \) | \( 2\ell = 6 \) | \(-\varepsilon_{0} \) |
| 0               | \( (\ell + 1)(\ell - 1) = 8 \) | 1                       | 1           |

\[ n = 5: \]
Proof. The identification of the weights is a straightforward application of (2.2), and the orbit sizes are immediate from the above description of the Weyl group. The multiplicities may be computed inductively, as outlined above, once all positive root strings through dominant weights are identified.

If \( n \geq 10 \) is even, the positive root strings through dominant weights are described by

\[
(2,1,1) + \alpha_{23} = (2,2), \\
(1,1,1,1) + \alpha_{ab} \sim (2,1,1), \quad (4 \geq b > a), \\
(2) + \alpha_{ab}^{\pm} \sim (2,1,1) \quad (b > a > 1), \\
(1,1) + \alpha_{ab}^{\pm} \sim (1,1,1,1) \quad (b > a > 2), \\
(1,1) + \alpha_{1a}^{\pm} \sim (2,1,1) \quad (a > 2), \\
(1,1) + \alpha_{2a}^{\pm} \sim (2,1,1) \quad (a > 2), \\
(1,1) + \alpha_{12} = (2) \quad (a > 2), \\
(1,1) + \alpha_{12} = (2,2), \\
0 + \alpha_{ab}^{\pm} \sim (1,1) \text{ and } 0 + 2\alpha_{ab}^{\pm} \sim (2,2) \quad (b > a).
\]

(In the very last table, the “contribution” is to \( M_\lambda(V(2,-2) \otimes V(\lambda)) \) rather than \( M_\lambda(V(2,2) \otimes V(\lambda)) \).)
Here $\mu \sim \nu$ means that $\mu$ and $\nu$ are in the same Weyl group orbit.

If $n \geq 9$ is odd, we have the root strings (3.5) through weights, as well as the following:

\begin{align*}
(2, 1) + e_2 &= (2, 2), \\
(2, 1) + e_a &\sim (2, 1, 1) \quad (a > 2), \\
(1, 1, 1) + \alpha_{ab} &\sim (2, 1) \quad (3 \geq b > a), \\
(1, 1, 1) + e_a &\sim (2, 1, 1) \quad (3 \geq a), \\
(1, 1, 1) + e_a &\sim (1, 1, 1, 1) \quad (a > 3), \\
(2) + e_a &\sim (2, 1) \text{ and } (2) + 2e_a \sim (2, 2) \quad (a > 1), \\
(1, 1) + e_a &\sim (2, 1) \quad (2 \geq a), \\
(1, 1) + e_a &\sim (1, 1, 1) \text{ and } (1, 1) + 2e_a \sim (2, 1, 1) \quad (a > 2), \\
(1) + \alpha^+_{1a} &\sim (2, 1) \quad (a > 1), \\
(1) + \alpha^+_{ab} &\sim (1, 1, 1) \quad (b > a > 1), \\
(1) + e_1 &= (2), \\
(1) + e_a &\sim (1, 1) \text{ and } (1) + 2e_a \sim (2, 1) \quad (a > 1), \\
0 + e_a &\sim (1) \text{ and } 0 + 2e_a \sim (2) \quad (b > a). \\
\end{align*}

(3.6)

The contributions to $M_\lambda(V(2, 2) \otimes V(\lambda))$ for $n \geq 9$ arise as follows. In case of a flat triple $\lambda_a = \lambda_{a+1} = \lambda_{a+2}$,

\begin{align*}
(\ldots, \lambda_a, \lambda_{a+1}, \lambda_{a+2}, \ldots) - (\ldots, 2, 0, -2, \ldots) &= (\ldots, \lambda_a, \lambda_a - 1, \lambda_a - 2, \ldots) - (\ldots, 2, 0, -2, \ldots) \\
&= (\ldots, \lambda_a - 2, \lambda_a - 1, \lambda_a, \ldots) \\
&= (\text{transposition}) \cdot \lambda.
\end{align*}

(3.7)

In addition,

\begin{align*}
(\ldots, \tilde{\lambda}_a, \tilde{\lambda}_{a+1}, \tilde{\lambda}_{a+2}, \ldots) - (\ldots, 2, -1, -1, \ldots) &= (\ldots, \tilde{\lambda}_a - 2, \tilde{\lambda}_a - 1, \tilde{\lambda}_a, \ldots) = (3 - \text{cycle}) \cdot \tilde{\lambda}.
\end{align*}

(3.8)

and

\begin{align*}
(\ldots, \tilde{\lambda}_a, \tilde{\lambda}_{a+1}, \tilde{\lambda}_{a+2}, \ldots) - (\ldots, 1, 1, -2, \ldots) &= (\ldots, \tilde{\lambda}_a - 2, \tilde{\lambda}_a - 1, \tilde{\lambda}_a, \ldots) = (3 - \text{cycle}) \cdot \tilde{\lambda}.
\end{align*}

(3.8)

This accounts for the $T$ contributions in either dimension parity.
For a 1-drop $\lambda_{a+1} = \lambda_a - 1$,

$$
\lambda = (\ldots, \tilde{\lambda}_a, \tilde{\lambda}_{a+1}, \ldots) - (\ldots, 2, -2, \ldots) \\
= (\ldots, \tilde{\lambda}_a, \tilde{\lambda}_{a-1}, \ldots) - (\ldots, 2, -2, \ldots) \\
= (\ldots, \tilde{\lambda}_a - 2, \tilde{\lambda}_a, \ldots) \\
= (\text{transposition}) \cdot \tilde{\lambda}.
$$

(3.9)

This accounts for the $D$ contribution in either dimension parity.

In case of a pair of disjoint pairs, $\lambda_{a+1} = \lambda_a$, $\lambda_{b+1} = \lambda_b$, \#\{a, a+1, b, b+1\} = 4, we may assume $a + 1 < b$; then

$$
(\ldots, \tilde{\lambda}_a, \tilde{\lambda}_{a+1}, \ldots) - (\ldots, 1, -1, 1, -1) \\
= (\ldots, \tilde{\lambda}_a, \tilde{\lambda}_{a-1}, \ldots) - (\ldots, 1, -1, 1, -1) \\
= (\ldots, \tilde{\lambda}_a - 1, \tilde{\lambda}_a, \ldots) \\
= (\text{product of 2 transpositions}) \cdot \tilde{\lambda}.
$$

This accounts for the $S$ contribution in either dimension parity.

For each pair $\lambda_a = \lambda_{a+1}$,

$$
(\ldots, \tilde{\lambda}_a, \tilde{\lambda}_{a+1}, \ldots) - (\ldots, 1, -1, \ldots) \\
= (\ldots, \tilde{\lambda}_a, \tilde{\lambda}_{a-1}, \ldots) - (\ldots, 1, -1, \ldots) \\
= (\ldots, \tilde{\lambda}_a - 1, \tilde{\lambda}_a, \ldots) \\
= (\text{transposition}) \cdot \tilde{\lambda}.
$$

(3.10)

This accounts for the $P$ contribution in either dimension parity.

The contribution of 1, coming from the weight 0, is clear.

If $n \geq 6$ is even and $\lambda_{\ell-2} = \lambda_{\ell-1} = \lambda_{\ell} = 0$, then

$$
\tilde{\lambda} - (\ldots, 2, 0, -2) = (\ldots, 2, 1, 0) - (\ldots, 2, 0, -2) = (\ldots, 0, 1, 2) = (\text{transposition}) \cdot \tilde{\lambda}.
$$

Furthermore,

$$
\tilde{\lambda} - (\ldots, 2, -1, -1) = (\ldots, 0, 2, 1) = (3\text{ - cycle}) \cdot \tilde{\lambda}, \\
\tilde{\lambda} = (\ldots, 1, 1 - 2) = (\ldots, 1, 0, 2) = (3\text{ - cycle}) \cdot \tilde{\lambda},
$$

(3.11)

This accounts for the $\varepsilon_{0,0,0}$ contributions in the even case.

If $n$ is even and $\lambda_{\ell-1} = \lambda_{\ell} = 1/2$, then

$$
\tilde{\lambda} - (\ldots, 2, 2) = (\ldots, \frac{3}{2}, \frac{1}{2}) - (\ldots, 2, 2) \\
= (\ldots, -\frac{1}{2}, -\frac{1}{2}) \\
= (\text{transposition and 2 sign changes}) \cdot \tilde{\lambda}.
$$

(3.12)

This accounts for the $\varepsilon_{1/2,1/2}$ contribution in the even case.
If \( n \) is even and \( \lambda_{\ell - 1} = \lambda_{\ell} = 0 \), we have
\[
\tilde{\lambda} - (\ldots, 2, 0) = (\ldots, 1, 0) - (\ldots, 2, 0) = (\ldots, -1, 0) = (2 \text{ sign changes}) \cdot \tilde{\lambda}.
\]
Furthermore,
\[
\tilde{\lambda} - (\ldots, 1, 1) = (\ldots, 0, -1) = (\text{transposition and 2 sign changes}) \cdot \tilde{\lambda}. \quad (3.13)
\]
(Note that the contribution from \( \tilde{\lambda} - (\ldots, 1, -1) = (\ldots, 0, 1) = (\text{transposition}) \cdot \tilde{\lambda} \)
has already been counted, and is included in the \( P \) contribution.) This accounts for the \( \varepsilon_{0,0} \) contributions in the even case.

There are also some \( \varepsilon_{0,0} P(\lambda^{(2)}) \) contributions in the even case; these arise as follows. If \( \lambda_a = \lambda_{a+1}, \ a + 1 < \ell - 1, \) and \( \lambda_{\ell - 1} = \lambda_{\ell} = 0 \), we have
\[
\tilde{\lambda} - (\ldots, 1, -1, \ldots, 2, 0) \\
= (\ldots, \tilde{\lambda}_a, \tilde{\lambda}_a - 1, \ldots, 1, 0) - (\ldots, 1, -1, \ldots, 2, 0) \\
= (\ldots, \tilde{\lambda}_a - 1, \tilde{\lambda}_a, \ldots, -1, 0) \\
= (\text{transposition and 2 sign changes}) \cdot \tilde{\lambda}, \quad (3.14)
\]
and
\[
\tilde{\lambda} - (\ldots, 1, -1, \ldots, 1, 1) = (\ldots, \tilde{\lambda}_a, \tilde{\lambda}_a - 1, \ldots, 1, 0) - (\ldots, 1, -1, \ldots, 1, 1) \\
= (\ldots, \tilde{\lambda}_a - 1, \tilde{\lambda}_a, \ldots, 0, -1) \\
= (2 \text{ transpositions and 2 sign changes}) \cdot \tilde{\lambda}.
\]

If \( n \) is odd and \( \lambda_{\ell - 1} = \lambda_{\ell} = 0 \), then the calculation (3.12) applies. Furthermore,
\[
\tilde{\lambda} - (\ldots, 1, 2) = (\ldots, \frac{3}{2}, \frac{1}{2}) - (\ldots, 1, 2) \\
= (\ldots, \frac{1}{2}, -\frac{3}{2}) \\
= (\text{transposition and 1 sign change}) \cdot \tilde{\lambda},
\]
and
\[
\tilde{\lambda} - (\ldots, 1, 2) = (\ldots, \frac{3}{2}, \frac{1}{2}) - (\ldots, 2, -1) \\
= (\ldots, -\frac{1}{2}, \frac{3}{2}) \\
= (\text{transposition and 1 sign change}) \cdot \tilde{\lambda}.
\]

This accounts for the \( \varepsilon_{0,0} \) contributions in the odd case.

If \( \lambda_{\ell} = 0 \), then
\[
\tilde{\lambda} - (\ldots, 1) = (\ldots, \frac{1}{2}) - (\ldots, 1) \\
= (\ldots, -\frac{1}{2}) \\
= (1 \text{ sign change}) \cdot \tilde{\lambda}.
\]
This accounts for the $\varepsilon_0$ contribution in the odd case.

If $\lambda_\ell = 1/2$, then
\[
\tilde{\lambda} - (\ldots, 1, 1) = (\ldots, 1) - (\ldots, 1)
\]
\[
= (\ldots, -1)
\]
\[
= (\text{1 sign change}) \cdot \tilde{\lambda}.
\]

This accounts for the $\varepsilon_{1/2}$ contribution in the odd case.

There are also $\varepsilon_0 P^{(1)}(\lambda)$ and $\varepsilon_{1/2} P^{(1)}(\lambda)$ contributions in the odd case; these arise as follows. If $\lambda_\ell = 0$, $\lambda_a = \lambda_{a+1}$, and $a + 1 < \ell$, then
\[
\tilde{\lambda} - (1, -1, 1) = (\ldots, \lambda_a, \lambda_a - 1, \ldots, -\frac{1}{2}) - (1, -1, 1)
\]
\[
= (\ldots, \lambda_a - 1, \lambda_a, \ldots, -\frac{1}{2})
\]
\[
= (\text{transposition and 1 sign change}) \cdot \tilde{\lambda}.
\]

If $\lambda_\ell = 1/2$, $\lambda_a = \lambda_{a+1}$, and $a + 1 < \ell$, then
\[
\tilde{\lambda} - (1, -1, 1) = (\ldots, \lambda_a, \lambda_a - 1, \ldots, 1) - (1, -1, 2)
\]
\[
= (\ldots, \lambda_a - 1, \lambda_a, \ldots, -1)
\]
\[
= (\text{transposition and 1 sign change}) \cdot \tilde{\lambda}.
\]

For all other values of $\tilde{\lambda}' := \tilde{\lambda} - \mu$, where $\mu \in \Pi(\lambda)$, either $([\tilde{\lambda}_1], \ldots, [\tilde{\lambda}_\ell])$ is not a permutation of $(|\tilde{\lambda}_1|, \ldots, |\tilde{\lambda}_\ell|)$, or else $n$ is even and $\tilde{\lambda}_1 \ldots \tilde{\lambda}_\ell < 0$. (Recall that we have assumed $\lambda_\ell \geq 0$.) Thus the above exhausts all possible contributions.

If $n = 8$, we have the following changes to the case $n \geq 10$ even just considered. The weight $(1, 1, 1, 1, 1, 1, 1, 1)$ is replaced by the two weights $(1, 1, 1, 1, 1, \pm 1)$. The entries
\[
(1, 1, 1, 1, 1, -1) + \alpha_{\pm a} \sim (2, 1, 1) \quad (3 \geq b > a),
\]
\[
(1, 1, 1, -1) + \alpha_{a+1}^+ \sim (2, 1, 1) \quad (3 \geq a)
\]

need to be added to (3.3). A review of the weight multiplicity calculation shows that each weight $(1, 1, 1, 1, \pm 1)$ “still” has multiplicity 2. Each weight $(1, 1, 1, 1, \pm 1)$ has a Weyl group orbit of size 8. (If we continue the formula for the size of the Weyl orbit of $(1, 1, 1, 1)$ in large even dimension to dimension 8, we get 16; thus the orbit “splits equally” in the descent to dimension 8.) Reviewing the contributions to the Brauer-Kostant formula, the $S$ contribution is attributable to $(1, 1, 1, 1)$, while the $\varepsilon_{0, 0} P^{(2)}(\lambda)$ contribution is attributable to $(1, 1, 1, -1)$.

When we descend to $n = 6$, the weight $(1, 1, 1, 1, 1)$ disappears. The weight $(2, 1, 1)$ “splits” into the weights $(2, 1, 1, 1)$; each has an orbit size of 12, or half of that predicted by continuing the expression $4\ell(\ell - 1)(\ell - 2)$. The first entry of (3.3) is replaced by
\[
(2, 1, 1, 1, 1, 1, 1, 1) + \alpha_{23}^+ = (2, 2),
\]
and so the multiplicity of each weight $(2, 1, 1, 1, 1, 1, 1, 1)$ is 1. Note that the orbit size formula for this weight gives 0 when $n = 6$ is substituted, and that the quantities $S$ and
vanish. The $T$ contributions, as well as the $\varepsilon_{0,0,0}$ contributions, are evenly split, by (3.7, 3.8, 3.11).

When $n = 4$, there is only a single positive root string through dominant weights in each module $V(2, \pm 2)$, namely the $\alpha_{12}^\pm$ string starting at 0. The weights and multiplicities in the two tables result. Note that (2) is not a weight in either module, and that the formula for its multiplicity as a weight in the large even $n$ case, namely $\ell - 2$, evaluates to 0 when $n = 4$. The quantities $S$, $T$ and $\varepsilon_{0,0,0}$ vanish identically.

When $n = 7$, the large odd $n$ case changes as follows. The weight $(1, 1, 1, 1)$ disappears; the formula for its orbit size gives the value 0, and the quantity $S$ vanishes. When $n = 5$, the weights with 3 nonzero entries also disappear, and the expressions for their orbit sizes vanish. The quantities $S$, $T$, and $P(\lambda^{(1)})$ vanish identically.

This completes the proof of Theorem 1.

We can check some of our numbers by multiplying multiplicities by orbit sizes and adding; this should give the dimension of the module $V(2, 2)$ (and $V(2, -2)$ when $n = 4$). Doing this for $n \geq 6$ even, we get

$$\frac{\ell(\ell + 1)(2\ell + 1)(2\ell - 3)}{3}. \quad (3.15)$$

This checks against direct computation via Weyl’s dimension formula [13]. When $n = 4$ (3.15) is still an expression for $\dim C$; and its summands $V(2, \pm 2)$ are both 5-dimensional.

When $n \geq 5$ is odd, the sum of multiplicities times orbit sizes is

$$\frac{(\ell + 1)(\ell - 1)(2\ell + 1)(2\ell + 3)}{3}. \quad (3.16)$$

This checks against Weyl’s dimension formula. The quantities (3.15) and (3.16) have a unified expression in terms of the dimension $n$, namely

$$\frac{n(n + 1)(n + 2)(n - 3)}{12}.$$

It remains to add up the contributions to the Brauer-Kostant quantity.

**Theorem 2** Suppose $n \geq 4$ is even, and let

$$\mathcal{C} := \begin{cases} V(2, 2), & n \geq 6, \\ V(2, 2) \oplus V(2, -2), & n = 4. \end{cases}$$

If $\lambda_\ell \geq 0$, then

$$M_\lambda(\mathcal{C} \otimes V(\lambda)) = \ell(\ell - 1) + T - D + (\ell + 1)\varepsilon_{0,0,0}.$$

If $\lambda_\ell < 0$, then

$$M_\lambda(\mathcal{C} \otimes V(\lambda)) = M_{\bar{\lambda}}(\mathcal{C} \otimes V(\bar{\lambda})),\quad \text{where } \bar{\lambda} = (\lambda_1, \ldots, \lambda_{\ell - 1} - \lambda_\ell).$$

Moreover, $\bar{\lambda}$ is a weight in $V(2, 2)$.
Proof. For $n \geq 8$, this is just a matter of adding up. For $n = 6$, we add up and recall the fact, noted above, that $S$ and $\varepsilon_{0,0}P(\lambda(2))$ vanish identically. For $n = 4$, we add up and recall that $S$, $T$ and $\varepsilon_{0,0}$ vanish identically.

To handle the case in which $\lambda_{\ell} < 0$, we just need to note the symmetry of the problem with respect to the reflection $\mu \mapsto -\bar{\mu}$ on $\Pi$. ///

Theorem 3 Suppose $n = 4$. If $\lambda_{2} \geq 0$, then

$$\mathcal{M}_{\lambda}(V(2, 2) \otimes V(\lambda)) = 1 - \varepsilon_{1/2, 1/2} - \varepsilon_{0,0},$$
$$\mathcal{M}_{\lambda}(V(2, -2) \otimes V(\lambda)) = 1 - D - P.$$ 

If $\lambda_{2} < 0$, then

$$\mathcal{M}_{\lambda}(V(2, \pm 2) \otimes V(\lambda)) = \mathcal{M}_{\lambda}(V(2, \mp 2) \otimes V(\bar{\lambda})).$$

Theorem 3 is a refinement of 2 in the case $n = 4$, since $C \otimes V(\lambda) = V(2, 2) \otimes V(\lambda) \oplus V(2, -2) \otimes V(\lambda)$.

This is just a matter of adding up. Again, to handle the case in which $\lambda_{\ell} < 0$, we just need to note the symmetry of the problem with respect to the reflection $\mu \mapsto -\bar{\mu}$ on $\Pi$. ///

Theorem 4 If $n \geq 5$ is odd, then

$$\mathcal{M}_{\lambda}(V(2, 2) \otimes V(\lambda)) = (\ell + 1)(\ell - 1) + T - D + 2S - 2(\ell - 1)P$$
$$+ (P(\lambda^{(1)}) - \ell + 1)(2\varepsilon_{0} + \varepsilon_{1/2}) + \varepsilon_{0,0}. \quad (3.18)$$

Proof. This is just a matter of adding up. Recall that when $n = 7$, the quantity $S$ vanishes identically, and that when $n = 5$, the quantities $S$, $T$, and $P(\lambda^{(1)})$ vanish identically. ///

4 Some more compact expressions

Now suppose that $\lambda_{\ell} \geq 0$, and write

$$\lambda = (\alpha_{1}, \ldots, \alpha_{1}, \ldots, \alpha_{r}, \ldots, \alpha_{r}),$$ \hspace{1cm} (4.19)

where $\alpha_{1} > \ldots > \alpha_{r}$. In other words, group the strings of identical entries in $\lambda$, and denote the string lengths by $k_{1}, \ldots, k_{r}$. Let $\lambda^{\text{red}}$ be the result of eliminating singleton strings (strings with $k_{i} = 1$) from $\lambda$. The tuple $\lambda^{\text{red}}$ may have anywhere from 0 to $\ell$ entries. In analogy with (4.19), write

$$\lambda^{\text{red}} = (\beta_{1}, \ldots, \beta_{1}, \ldots, \beta_{s}, \ldots, \beta_{s}),$$
We claim that the terms in (3.17) and (3.18) have very simple expressions in terms of $r$ and $s$.

To see this, first note that if
\[ X := x_1 + \ldots + x_s, \]
then
\[ T = X - 2s, \]
\[ 2S = \sum_{i=1}^{s} (x_i - 2)(x_i - 3) + 2 \sum_{1 \leq i < j \leq s} (x_i - 1)(x_j - 1), \]
\[ = X^2 - (2s + 3)X + s(s + 5), \]
\[ P = X - s. \]

Since
\[ r - s = \#(\text{singleton strings}) = \ell - X, \]
this gives
\[ \ell(\ell - 1) + T + 2S - (2\ell - 3)P = (X - s - \ell + 1)^2 + s - \frac{1}{4} \]
\[ = \left( \frac{1}{2} - r \right)^2 + s - \frac{1}{4} \]
\[ = r^2 - r + s. \]

Since
\[ D = r - 1 + h, \]
where
\[ h := \#\{ a \mid \lambda_a \geq \lambda_{a+1} + 2 \}, \]
we have
\[ \ell(\ell - 1) + T - D + 2S - (2\ell - 3)P = (r - 1)^2 + h + s. \] (4.21)

($h$ may be described as the number of “steep drops” in $\lambda$.)

To simplify the block of terms
\[ \varepsilon_{0,0} + (P(\lambda^{(2)}) - \ell + 1)\varepsilon_{0,0}, \]
note that if $\varepsilon_{0,0} = 1$, then
\[ \varepsilon_{0,0} + (P(\lambda^{(2)}) - \ell + 1)\varepsilon_{0,0} = P(\lambda^{(2)}) - \ell + 2. \]
while if $\varepsilon_{0,0} = 0$ and $\varepsilon_{0,0} = 1$, then
\[ \varepsilon_{0,0} + (P(\lambda^{(2)}) - \ell + 1)\varepsilon_{0,0} = P(\lambda^{(2)}) - \ell + 1. \]

In each case, this is the quantity
\[ P - \ell = X - s - \ell = -r. \]

The conclusion is that
\[ \varepsilon_{0,0} + (P(\lambda^{(2)}) - \ell + 1)\varepsilon_{0,0} = -r\varepsilon_{0,0}. \] (4.22)

(4.21) and (4.22) give:
Theorem 5 Suppose $n \geq 4$ is even. If $\lambda_\ell \geq 0$, then
\[ M_\lambda(\mathcal{C} \otimes V(\lambda)) = (r - 1)^2 + s + h - r \varepsilon_{0,0} - \varepsilon_{1/2,1/2}. \]

If $\lambda_\ell < 0$, then $M_\lambda(\mathcal{C} \otimes V(\lambda)) = M_\lambda(\mathcal{C} \otimes V(\bar{\lambda})).$

If $n = 4$, then $(r, s)$ is either $(2, 0)$ or $(1, 1)$, so that
\[(r - 1)^2 + s = 1.\]

If $\varepsilon_{0,0} = 1$, then $r = 1$. As a result, by Theorem 5,
\[ M_\lambda(W \otimes V(\lambda)) = 1 + h - \varepsilon_{0,0} - \varepsilon_{1/2,1/2} \quad (n = 4, \lambda_2 \geq 0). \]

Together with the first equation of Theorem 3, this gives:

Theorem 6 Suppose $n = 4$. If $\lambda_2 \geq 0$, then
\[ M_\lambda(V(2,2) \otimes V(\lambda)) = 1 - \varepsilon_{1/2,1/2} - \varepsilon_{0,0}, \]
\[ M_\lambda(V(2,-2) \otimes V(\lambda)) = h. \]

If $\lambda_2 < 0$, then
\[ M_\lambda(V(2,\pm2) \otimes V(\lambda)) = M_\lambda(V(2,\mp2) \otimes V(\bar{\lambda})). \]

Now consider the odd-dimensional case. By (4.21) and (4.20),
\[(\ell + 1)(\ell - 1) + T - D + 2S - 2(\ell - 1)P = \ell - 1 - P + (r - 1)^2 + h + s \quad \text{(4.23)}.
\]

If $\varepsilon_{0,0} = 1$, then
\[ 2(P(\lambda^{(1)}) - \ell + 1)\varepsilon_0 + \varepsilon_{0,0} = 2(P - \ell) + 1. \]

If $\varepsilon_{0,0} = 0$ but $\varepsilon_0 = 1$, then
\[ 2(P(\lambda^{(1)}) - \ell + 1)\varepsilon_0 + \varepsilon_{0,0} = 2(P - \ell + 1). \]

Thus in all cases,
\[ 2(P(\lambda^{(1)}) - \ell + 1)\varepsilon_0 + \varepsilon_{0,0} = 2(P - \ell + 1)\varepsilon_0 - \varepsilon_{0,0} = -2(r - 1)\varepsilon_0 - \varepsilon_{0,0}. \quad \text{(4.24)} \]

If $\varepsilon_{1/2,1/2} = 1$, then
\[ (P(\lambda^{(1)}) - \ell + 1)\varepsilon_{1/2} = P - \ell. \]

If $\varepsilon_{1/2,1/2} = 0$ but $\varepsilon_{1/2} = 1$, then
\[ (P(\lambda^{(1)}) - \ell + 1)\varepsilon_{1/2} = P - \ell + 1. \]

Thus in all cases,
\[ (P(\lambda^{(1)}) - \ell + 1)\varepsilon_{1/2} = (P - \ell + 1)\varepsilon_{1/2} - \varepsilon_{1/2,1/2} = -(r - 1)\varepsilon_{1/2} - \varepsilon_{1/2,1/2}. \quad \text{(4.25)} \]

Putting together (4.23), (4.24), and (4.25), we have:
Theorem 7 If \( n \geq 5 \) is odd, then
\[
\mathcal{M}_\lambda(V(2, 2) \otimes V(\lambda)) = r(r - 1) + s + h - 2(r - 1)\epsilon_0 - \epsilon_{0,0} - (r - 1)\epsilon_{1/2} - \epsilon_{1/2,1/2}.
\]

Using the more compact formulas, it is possible to classify those \( \lambda \) for which \( \mathcal{M}_\lambda(V(2, 2) \otimes V(\lambda)) = 0 \). Suppose that \( n \geq 6 \) is even, and \( \lambda \geq 0 \). If \( r = 1 \), then \( s = 1 \) and \( h = 0 \), so
\[
\mathcal{M}_\lambda(V(2, 2) \otimes V(\lambda)) = 1 - \epsilon_{0,0} - \epsilon_{1/2,1/2}.
\]

If \( r = 2 \), then \( s \geq 1 \), and
\[
\epsilon_{1} + s \geq 2, \quad \text{with equality} \iff s = 1.
\]

If \( r \geq 3 \), then
\[
\mathcal{M}_\lambda(V(2, 2) \otimes V(\lambda)) \geq r^2 - 3r + 1 \geq 1.
\]
Thus \( \mathcal{M}_\lambda(V(2, 2) \otimes V(\lambda)) \) vanishes if and only if \( \lambda = 0 \), \( \left( \frac{1}{2}, \ldots, \frac{1}{2} \right) \), or \( (1) \). Letting \( \lambda \) take any sign, we just need to add \( \left( \frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2} \right) \) to this list.

If \( n = 4 \) and \( \lambda_2 \geq 0 \), then \( \mathcal{M}_\lambda(V(2, 2) \otimes V(\lambda)) \) vanishes if and only if \( \lambda = 0 \) or \( \left( \frac{1}{2}, \frac{1}{2} \right) \). \( \mathcal{M}_\lambda(V(2, -2) \otimes V(\lambda)) \) vanishes if and only if \( \lambda_1 - \lambda_2 = 0 \) or \( 1 \). If \( \lambda_2 < 0 \), we just reverse the roles of \( (2, 2) \) and \( (2, -2) \) in these statements. Note that in particular, \( V(1) \) is no longer on the list of modules which cannot be acted upon by \( V(2, 2) \).

For odd \( n \), it is sometimes useful to rewrite Theorem 7 as
\[
\mathcal{M}_\lambda(V(2, 2) \otimes V(\lambda)) = (r - 1)(r - 2\epsilon_0 - \epsilon_{1/2}) + s + h - \epsilon_{0,0} - \epsilon_{1/2,1/2}.
\]
(4.26)

Suppose \( n \geq 7 \) is odd. If \( r = 1 \), then \( s = 1 \) and \( h = 0 \), so that
\[
\mathcal{M}_\lambda(V(2, 2) \otimes V(\lambda)) = 1 - \epsilon_{0,0} - \epsilon_{1/2,1/2}.
\]

If \( r = 2 \), then \( s \geq 1 \), and (4.26) shows that
\[
\mathcal{M}_\lambda(V(2, 2) \otimes V(\lambda)) \geq 0, \quad \text{with equality} \iff (h = 0 \text{ and } \epsilon_{0,0} = 1).
\]

If \( r \geq 3 \), (4.26) shows that \( \mathcal{M}_\lambda(V(2, 2) \otimes V(\lambda)) \geq 1 \). Thus \( \mathcal{M}_\lambda(V(2, 2) \otimes V(\lambda)) \) vanishes if and only if \( \lambda = 0 \), \( \left( \frac{1}{2}, \ldots, \frac{1}{2} \right) \), or \( (1) \).

If \( n = 5 \), the discussion immediately above is altered as follows. When \( r = 2 \), one can no longer conclude that \( s \geq 1 \); in fact \( s \) is necessarily 0. (4.26) becomes
\[
\mathcal{M}_\lambda(V(2, 2) \otimes V(\lambda)) = 2 - 2\epsilon_0 - \epsilon_{1/2} + h.
\]
This can vanish only for \( \epsilon_0 = 1 \) and \( h = 0 \). But this is the situation only for \( \lambda = (1) \). Thus the answer for \( n = 5 \) is the same as for odd \( n \geq 7 \).

Summarizing, we have:

Theorem 8 If \( n \geq 6 \) is even, then \( \mathcal{M}_\lambda(V(2, 2) \otimes V(\lambda)) \) vanishes if and only if \( \lambda = 0 \), \( \left( \frac{1}{2}, \ldots, \frac{1}{2}, \pm \frac{1}{2} \right) \), or \( (1) \). If \( n = 4 \), then \( \mathcal{M}_\lambda(V(2, 2) \otimes V(\lambda)) \) vanishes if and only if \( \lambda_1 + \lambda_2 \in \{0, 1\} \), and \( \mathcal{M}_\lambda(V(2, -2) \otimes V(\lambda)) \) vanishes if and only if \( \lambda_1 - \lambda_2 \in \{0, 1\} \). If \( n \geq 5 \) is odd, then \( \mathcal{M}_\lambda(V(2, 2) \otimes V(\lambda)) \) vanishes if and only if \( \lambda = 0 \), \( \left( \frac{1}{2}, \ldots, \frac{1}{2} \right) \), or \( (1) \).
It is also of some interest to know when there is a unique action of the Weyl module; i.e., when $\mathcal{M}_\lambda(V(2,2) \otimes V(\lambda))$ (and $\mathcal{M}_\lambda(V(2,-2) \otimes V(\lambda))$ when $n = 4$) is 1. We shall chase this through case by case, omitting some of the (by now routine) details.

For $n \geq 6$ even, we have $\mathcal{M}_\lambda(V(2,2) \otimes V(\lambda)) = 1$ in the following cases:

- $r = 1$ : $(p, \ldots, p, \pm p)$, where $p \geq 1$;
- $r = 2$ : $(\frac{3}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, \pm \frac{1}{2})$,
  $(p, 0, \ldots, 0)$ \quad $p \in 2 + \mathbb{N}$,
  $(1, \ldots, 1, 0, \ldots, 0)$;
  at least 2 at least 2 \quad (4.27)
- $r \geq 3$ : none.

For $n = 4$, $\mathcal{M}_\lambda(V(2,2) \otimes V(\lambda))$ and $\mathcal{M}_\lambda(V(2,-2) \otimes V(\lambda))$ can only take on the values 0 and 1; thus the multiplicity 1 case is exactly the complement of the multiplicity 0 case studied above.

If $n \geq 5$ is odd, we have $\mathcal{M}_\lambda(V(2,2) \otimes V(\lambda)) = 1$ in the following cases:

- $r = 1$ : $(p, \ldots, p)$, where $p \geq 1$;
- $r = 2$ : $(\frac{3}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$,
  $(1, \ldots, 1, 0)$ \quad $(n \geq 7)$,
  $(p)$ \quad $p \in 2 + \mathbb{N}$,
  $(1, \ldots, 1, 0, \ldots, 0)$;
  at least 2 at least 2 \quad (4.28)
- $r \geq 3$ : none.

Summarizing, we have:

**Theorem 9** If $n \geq 6$ is even, then $\mathcal{M}_\lambda(V(2,2) \otimes V(\lambda)) = 1$ if and only if $\lambda$ is one of the dominant weights listed in (4.27). If $n = 4$, then $\mathcal{M}_\lambda(V(2,2) \otimes V(\lambda)) = 1$ if and only if $\lambda_1 + \lambda_2 \notin \{0, 1\}$, $\mathcal{M}_\lambda(V(2,-2) \otimes V(\lambda)) = 1$ vanishes if and only if $\lambda_1 - \lambda_2 \notin \{0, 1\}$. If $n \geq 5$ is odd, then $\mathcal{M}_\lambda(V(2,2) \otimes V(\lambda)) = 1$ if and only if $\lambda$ is one of the dominant weights listed in (4.28).

### 5 Other actions of the Riemann curvature

The Riemann curvature tensor $R$ is a section of the bundle

$$\mathcal{R} = \begin{cases} 
V(2,2) \oplus V(2) \oplus V(0), & n \geq 5, \\
V(2,2) \oplus V(2,-2) \oplus V(2) \oplus V(0), & n = 4, \\
V(2) \oplus V(0), & n = 3, \\
V(0), & n = 2.
\end{cases}$$
The Weyl tensor part(s), if any, lives in the bundle(s) $V(2, \pm 2)$. The $V(2)$ part carries the Einstein (trace-free Ricci) tensor, and the $V(0)$ part carries the scalar curvature. (See [18] for a detailed discussion.) To count the actions of the “rest” of the Riemann tensor, first note that $V(0) \otimes V(\lambda) \cong_{\text{ad}} V(\lambda)$, so there is always one action (namely multiplication) of the scalar curvature. (This computation was incidental to an investigation of conformally invariant operators in [3].) Let $n \geq 3$. By [9], the direct sum decomposition of $V(1) \otimes V(\lambda)$ into irreducibles is multiplicity-free, and contains the module $V(\sigma)$ if and only if $\sigma$ is dominant, and either $\sigma = \lambda \pm e_a$ for some $a$, or else $n$ is odd, $\lambda_\ell > 0$, and $\sigma = \lambda$. (This computation is accomplished quite easily using the Brauer-Kostant formula.) [3] shows that if $N(\lambda)$ is the number of irreducible summands in $V(1) \otimes V(\lambda)$, then

$$M_\lambda(V(2) \otimes V(\lambda)) = \left[ \frac{N(\lambda) - 1}{2} \right].$$

But the discussion immediately above shows that $N(\lambda)$ is given, in the notation of Sec. 3 above, by

$$N(\lambda) = \begin{cases} 
2\ell - 2P - \varepsilon_{0,0}, & n \geq 4 \text{ even}, \; \lambda_\ell \geq 0, \\
N(\bar{\lambda}), & n \geq 4 \text{ even}, \; \lambda_\ell < 0, \\
2\ell + 1 - 2P - 2\varepsilon_0 - \varepsilon_{1/2}, & n \geq 3 \text{ odd}.
\end{cases}$$

As a result, we have:

**Theorem 10**

$$M_\lambda(V(2) \otimes V(\lambda)) = \begin{cases} 
\ell - P - 1 = r - 1, & n \geq 4 \text{ even}, \; \lambda_\ell \geq 0, \\
M_\lambda(V(2) \otimes V(\bar{\lambda})), & n \geq 4 \text{ even}, \; \lambda_\ell < 0, \\
\ell - P - \varepsilon_0 - \varepsilon_{1/2} = r - \varepsilon_0 - \varepsilon_{1/2}, & n \geq 3 \text{ odd}.
\end{cases}$$

In particular, we see immediately that the Einstein tensor cannot act on the bundle $V(\lambda)$ for precisely the following dominant weights $\lambda$:

$$\begin{align*}
(p, \ldots, p, \pm p), \quad n \geq 4 \text{ even;} \\
0 \text{ and } \left(\frac{1}{2}, \ldots, \frac{1}{2}\right), \quad n \geq 3 \text{ odd.}
\end{align*}$$

(5.29)

We can also sum up the contributions of the Weyl, Einstein, and scalar curvatures to get:

**Theorem 11**

$$M_\lambda(\mathcal{R} \otimes V(\lambda)) = \begin{cases} 
r(r - 1) + s + h - r\varepsilon_{0,0} - \varepsilon_{1/2,1/2}, & n \geq 4 \text{ even}, \; \lambda_\ell \geq 0, \\
M_\lambda(\mathcal{R} \otimes V(\bar{\lambda})), & n \geq 4 \text{ even}, \; \lambda_\ell < 0, \\
r^2 + s + h + 1 - (2r - 1)\varepsilon_0 - \varepsilon_{0,0} - r\varepsilon_{1/2} - \varepsilon_{1/2,1/2}, & n \geq 3 \text{ odd}.
\end{cases}$$
Because of the scalar curvature action, $\mathcal{M}_\lambda(\mathcal{R} \otimes V(\lambda))$ is always at least 1. It is exactly 1 in precisely the following cases:

- $0$ or $(\frac{1}{2}, \ldots, \frac{1}{2}, \pm \frac{1}{2})$,
- $0$ or $(\frac{1}{2}, \ldots, \frac{1}{2})$,
- $n \geq 4$ even,
- $n \geq 3$ odd.

### 6 Some explicit tensor representations

The modules which figure prominently in the discussion immediately above all have more or less explicit tensor, or tensor-spinor, realizations. For properly half-integral $n$ are written, and summation convention is understood. If $V$ is a module which splits as $\Sigma_+ \oplus \Sigma_- \subseteq so(n)$ $V(\frac{1}{2}, \ldots, \frac{1}{2})$, $V(\frac{1}{2}, \ldots, \frac{1}{2})$, $V(\frac{1}{2}, \ldots, \frac{1}{2})$, $V(\frac{1}{2}, \ldots, \frac{1}{2})$. If $k < n/2$, the module

$$V(1, \ldots, 1, 0, \ldots, 0)$$

is realized by the alternating $k$-tensors $\Lambda^k$, and also by $\Lambda^{n-k}$. (The Hodge star operator provides an explicit equivariant isomorphism between these two realizations.)

If $n$ is odd, the modules $V(1, \ldots, 1, \pm 1)$ are realized in middle forms of the two different dualities (eigenvalues under the Hodge star), $\Lambda^{n/2}_+$. If $p$ is a natural number, the module $V(p)$ is realizable as that of trace-free symmetric $p$-tensors. The Weyl module $C$ may be realized as that of totally trace-free 4-tensors $C_{abcd}$ with

$$C_{abcd} = C_{cdab} = -C_{bacd} = -C_{acdb} - C_{adbc}.$$  

If $n$ is odd, the module $V(\frac{3}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ may be realized as the twistors; i.e. spinor-one-forms $\varphi_a$ with $\gamma^a \varphi_a = 0$. Here $\gamma^a$ are the Clifford matrices, only tensor indices are written, and summation convention is understood. If $n$ is even, the twistors split as $V(\frac{3}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) \oplus V(\frac{3}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2})$. More generally, modules of the form

$$V(\frac{3}{2}, \ldots, \frac{3}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, \pm \frac{1}{2})$$

are direct summands of modules of spinor-forms (see [3], [10]). A module of the form $V(p, \ldots, p, \pm p)$, for integral $p$, is a direct summand (in fact, the highest weight summand) of the $p$-fold symmetric tensor power $S^p_+$ of $V(1, \ldots, 1, \pm 1)$ (which in turn, by the above, is a differential form module). For properly half-integral $p$, we get the direct summand of $\Sigma_+ \otimes S^{p-\frac{1}{2}}_+$ containing the highest weight vector.

The fact that $V(2)$ cannot act on $V(0)$ nor on $V(\frac{1}{2}, \ldots, \frac{1}{2}, \pm \frac{1}{2})$ is actually immediate by weight considerations. First note that

$$\text{Hom}_{so(n)}(V(\sigma) \otimes V(\lambda), V(\tau)) \cong \text{Hom}_{so(n)}(V(\sigma), V(\lambda)^*V(\tau)),$$

where $V(\lambda)^*$ is the module dual to $V(\lambda)$. It is easily seen that $V(\lambda)^* \cong V(\lambda)$ unless $n$ has the form $4k + 2$ and $\lambda \neq 0$, in which case $V(\lambda)^* \cong V(\bar{\lambda})$. (Since the dual is irreducible, we just have to find the dominant weight in the Weyl group orbit of $-\lambda$. This is either $\lambda$ or $\bar{\lambda}$, as indicated.) Denote the highest weight of the dual module by $V(\lambda^*) := V(\lambda)^*$. Then

$$\lambda^* + \tau < \sigma \Rightarrow \mathcal{M}_\tau(V(\sigma) \otimes V(\lambda)) = 0,$$
where the “$<$” relation on the left is the lexicographical ordering. Since $0 + 0 < 2$ and $\frac{1}{2} + \frac{1}{2} < 2$, there is no action of $V(2)$ on $V(0)$, nor on $V(\frac{1}{2}, \ldots, \frac{1}{2}, \pm \frac{1}{2})$.

For the same reason, $V(2, 2)$ cannot act on the trivial module, nor on any spinor bundle, nor on $\Lambda^1$. This immediately gives the “if” half of Theorem 8 for $n > 4$.

These weight size considerations do not, however, give the complete lists of modules that cannot be acted upon (see the $n = 4$ case of Theorem 8 and the first line of (5.23)). And, of course, weight size considerations say nothing about the “only if” part of these statements.

An elementary point of contact of these results with some results in geometric analysis is visible when we look at the Weitzenböckian, a curvature action on differential forms which is the difference between the form Laplacian $\Delta$ and the Bochner Laplacian $\nabla^*\nabla$. By \cite{11}, p. 118, application of the Weitzenböckian to the $k$-form $\varphi_{a_1 \ldots a_k}$ for ($k \geq 1$) gives

$$
(B\varphi)_{a_1 \ldots a_k} = \frac{n - 2k}{n - 2} r_{b[1a} \varphi^{b} \varphi_{a_2 \ldots a_k]} + \frac{k - 1}{(n - 1)(n - 2)} K \varphi_{a_1 \ldots a_k} + \frac{k - 1}{2} C_{bc[a_1 a_2} \varphi^{bc} \varphi_{a_3 \ldots a_k]},
$$

(6.30)

where $K$, $r$, and $C$ are respectively the scalar, Ricci, and Weyl curvatures, and indices within square brackets are to be skewed (antisymmetrized) over. (The reference actually gives a formula for $(B\varphi, \varphi)$, where $(\cdot, \cdot)$ is the natural inner product. Since $B$ is symmetric, the formula for $B\varphi$ can be recovered.) If $k = n/2$, the $r$ coefficient vanishes, showing that the Einstein tensor is not involved in the formula. Our results show that it cannot be involved, since (1) the Weitzenböckian, like $\Delta$ and $\nabla^*\nabla$, carries each of the two middle-form bundles $\Lambda^{n/2}$ to itself; (2) our result (5.20) shows that there is no action of $V(2)$ on $V(1, \ldots, 1, \pm 1)$. The action of $V(2)$ on $\Lambda^k$ implicitly exhibited in (6.30), namely

$$
\varphi_{a_1 \ldots a_k} \mapsto \sigma b_{[a_1} \varphi^{b} \varphi_{a_2 \ldots a_k]}, \quad \sigma \in V(2),
$$

when used on middle-forms, thus has to interchange the modules $\Lambda_{\pm}^{n/2}$.

The absence of the Einstein tensor in the middle-form Weitzenböckian is an important point in the derivation of the Bourguignon Vanishing Theorem \cite{2}. Similarly, the Weyl tensor part of the Weitzenböckian must vanish for $k = 1$.

Among other things, the formula (6.30) exhibits the action of the Weyl module on $\Lambda^k$ for $k \neq 0, 1, n - 1, n$. (In the cases $k = n - 1, n$, the expression vanishes by an argument of “skewing on too many indices.”)

The results of \cite{3} show that each bundle $V(\lambda)$ admits either a 1— or a 0—dimensional space of second-order conformally covariant differential operators, modulo order 0 actions of the Weyl tensor. The results of this paper show how many of these actions of the Weyl tensor there are—in particular, they show that generically, there are many.

Let $A$ be a scalar-valued function on the set $\Pi^{DI}$ of dominant integral $\mathfrak{so}(n)$ weights. We say that $A$ vanishes generically if there is a finite set $F$ such that

$$
\lambda_1, \ldots, \lambda_\ell, \lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \ldots, \lambda_{\ell - 1} - |\lambda_\ell| \not\in F \Rightarrow a(\lambda) = 0.
$$

If $A - B$ vanishes generically, we say that $A \text{ gen} B$. For example,

$$
r \equiv \ell, \ s \equiv 0, h \equiv \ell - 1,
$$
and

$$\varepsilon_{\kappa_\ell - k \ldots \kappa_\ell} = 0$$

for \( k \geq 1 \). Our results say that \( \mathcal{M}_\lambda(V(2, 2) \otimes V(\lambda)) \overset{\text{gen}}{=} (\ell + 1)(\ell - 1) \) for \( n \geq 5 \) odd, and that \( \mathcal{M}_\lambda(C \otimes V(\lambda)) \overset{\text{gen}}{=} \ell(\ell - 1) \) for \( n \geq 4 \) even. When \( n = 4 \), \( \mathcal{M}_\lambda(V(2, 2) \otimes V(\lambda)) \overset{\text{gen}}{=} \mathcal{M}_\lambda(V(2, -2) \otimes V(\lambda)) \overset{\text{gen}}{=} 1 \).

Theorem \[ \text{[14]} \] indicates that there will be only one action of trace-free symmetric 2-tensors on trace-free symmetric \( p \)-tensors for \( p \geq 1 \); this is given by

$$\varphi_{a_1 \ldots a_p} \mapsto \sigma_b(a_1 \varphi_{a_2 \ldots a_p})_0,$$

where \((\cdots)_0\) is trace-free symmetrization on the enclosed indices. Theorem \[ \text{[14]} \] also says that a bundle of twistors, i.e. spinor-one-forms \( \psi_a \) with \( \gamma^a \psi_a = 0 \), admits only one action of \( V(2) \). (Recall that the twistor bundle is \( V(\frac{3}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) \) if \( n \geq 3 \) is odd, and \( V(\frac{3}{2}, \frac{3}{2}, \ldots, \frac{1}{2}) \otimes V(\frac{3}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2}) \) if \( n \geq 4 \) is even.) There are two linearly independent spinor one-forms that can be constructed from \( \sigma \otimes \psi \), namely

$$\sigma_a \gamma^b \psi_b \quad \text{and} \quad \sigma_b \gamma^b \gamma_a \psi_c.$$

But by the Clifford relation

$$\gamma^a \gamma^b + \gamma^b \gamma^a = -2g^{ab},$$

only one linear combination of these is a twistor, namely

$$\sigma_a \gamma^b \psi_b - \frac{1}{n - 2} \sigma_b \gamma^b \gamma_a \psi_c.$$

Similar statements can be made, for \( n \geq 6 \), about bundles of spinor 2-forms with \( \gamma^a \psi_{ab} = 0 \); here the action is

$$\sigma^c [a \psi^b]_c - \frac{1}{n - 4} \sigma_{cd} \gamma^c [a \psi^b]_d.$$

The highest weights of these bundles consist are \((\frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \ldots, \frac{1}{2})\), and, if \( n \) is even, \((\frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2})\). The generalization to spinor-\( k \)-forms \( \psi_{a_1 \ldots a_k} \) with \( \gamma^{a_1} \psi_{a_1 \ldots a_k} \) is clear.

We get an action of the Weyl bundle on trace-free symmetric \( p \)-tensors by taking

$$C^b_{(a_1} c a_2 \varphi_{a_3 \ldots a_p)bc},$$

as long as \( p \geq 2 \). Theorems \[ \text{[5]} \] and \[ \text{[6]} \] predict that there should be just one action of the Weyl tensor as long as \( n \geq 5 \). If \( n = 4 \), then by Theorem \[ \text{[6]} \], there should be one action by \( V(2, 2) \) (the self-dual Weyl tensors), and one by \( V(2, -2) \) (the anti-self-dual Weyl tensors). Both of these are also described by \( (6.31) \).

By the Clifford relations and the fact that Weyl tensors are trace-free,

$$\varphi_a \mapsto C_{bc} d a \gamma^b \gamma^c \psi_d$$

is an action of the Weyl bundle on twistors; by Theorems \[ \text{[5]} \] and \[ \text{[6]} \], this is the only action for \( n \geq 4 \) which, in the even-dimensional case, preserves each subbundle \((\frac{3}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, \pm \frac{1}{2})\).

When \( n = 4 \), \( V(2, \pm 2) \) is a submodule of \( V(1, \pm 1) \otimes V(1, \pm 1) \); thus \( \mathcal{C} \) is a submodule of \( \Lambda^2_+ \otimes \Lambda^2_+ \oplus \Lambda^2_- \otimes \Lambda^2_- \). In particular, the “mixed” summands \( \Lambda^2_+ \otimes \Lambda^2_- \) and \( \Lambda^2_- \otimes \Lambda^2_+ \) in \( \Lambda^2 \otimes \Lambda^2 \) do not contribute to \( \mathcal{C} \). As a result, \( V(2, \pm 2) \) should not be able to act on \( V(1, \mp 1) \); this is confirmed by Theorem \[ \text{[6]} \].
7 Epilogue

A question which is not addressed in this paper, but would be well worth the effort, is to compute the dimension of $\text{Hom}_{\text{so}(n)}(\mathcal{C} \otimes V(\lambda), V(\kappa))$ for $\kappa \neq \lambda$. The corresponding problem with $V(2)$ in place of $\mathcal{C}$ was addressed in [7], where it plays a part in classifying second-order conformally invariant operators between different Spin$(n)$-bundles.

Whenever $V(1) \otimes V(\lambda)$ has $V(\tau)$ as a summand, there is a first-order equivariant differential operator from $V(\lambda)$ to $V(\tau)$, the so-called generalized gradient. This comes from compressing the covariant derivative to act between irreducible summands:

$$\mathcal{V}(\lambda) \xrightarrow{\nabla} T^*M \otimes V(\lambda) = \mathcal{V}(\tau_1) \oplus \ldots \oplus \mathcal{V}(\tau_{N(\lambda)}) \xrightarrow{\text{Proj}_{\tau_u}} \mathcal{V}(\tau_u).$$

(Recall that the decomposition $\mathcal{V}(\tau_1) \oplus \ldots \oplus \mathcal{V}(\tau_{N(\lambda)})$ is multiplicity free.) The generalized gradient is the composition $G_{\tau_u, \lambda} = \text{Proj}_{\tau_u} \circ \nabla$. When we compose two gradients, say

$$V(\lambda) \xrightarrow{G_1} V(\tau) \xrightarrow{G_2} V(\kappa),$$

we either get an operator of order 2, or (by Weyl’s invariant theory), one of order 0. If $\text{Hom}_{\text{so}(n)}(V(2) \otimes V(\lambda), V(\kappa)) = 0$, then we must get an order 0 operator, since the leading symbol of an order 2 operator is canonically associated to an element of $\text{Hom}_{\text{so}(n)}(V(2) \otimes V(\lambda), V(\kappa))$. This order 0 operator is an action of the Riemann curvature to which the scalar curvature cannot contribute, since $\kappa \neq \lambda$. The Einstein tensor cannot contribute either, since it is a section of $V(2)$. Thus in the situation outlined here, the composition $G_2G_1$ is an action of the Weyl tensor. Knowing how many actions of the Weyl tensor there are from $V(\lambda)$ to $V(\kappa)$ might be valuable in the computation of this action. In particular, if there are no such actions, the composition must be vanish. This, in fact, is exactly what happens for the operators $dd$ of the de Rham complex.

It is not too hard to show (see [7]) that the possible compositions reaching $V(\kappa)$ from $V(\lambda)$, when $\kappa \neq \lambda$, are limited as follows. In (7.32), there are either 1 or 2 choices for $\tau$. When there is one choice, $G_2G_1$ might have order 2, or might be an action of the Weyl tensor. When there are two choices, we have a diagram of operators

$$\begin{array}{c}
\tilde{G}_2 \\
V(\tilde{\tau}) \xrightarrow{\tilde{G}_1} V(\tau) \xrightarrow{G_2} V(\kappa) \\
\uparrow \tilde{G}_1 \quad \uparrow G_2 \\
V(\lambda) \xrightarrow{G_1} V(\tau)
\end{array}$$

and there is a unique (up to constant multiples) linear combination of $G_2G_1$ and $\tilde{G}_2\tilde{G}_1$ which is an order 0 action of the Riemann curvature. This may contain
contributions from the Weyl tensor and the Einstein (trace-free Ricci) tensor, but not from the scalar curvature.

Such considerations are behind important theorems on overdetermined systems of differential equations on spinors \[1\], the solvability of which characterizes special manifolds. It is hoped that the present program of enumerating and classifying curvature actions might eventually be used to get analogous results based on twistors, higher spinor-forms, and sections of other geometric bundles.

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