CHAOS CONTROL IN A SPECIAL PENDULUM SYSTEM FOR ULTRA-SUBHARMONIC RESONANCE

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ABSTRACT. In this paper, we study the chaos control of pendulum system with vibration of suspension axis for ultra-subharmonic resonance by using Melnikov methods, and give a necessary condition for controlling heteroclinic chaos and homoclinic chaos, respectively. We give some bifurcation diagrams by numerical simulations, which indicate that the chaos behaviors for ultra-subharmonic resonance may be inhibited to periodic orbits by adjusting phase-difference of parametric excitation, and prove that results obtained are very effective in inhibiting chaos for ultra-subharmonic resonance.

1. Introduction. Pendulum system is a classical non-linear dynamic system, it has a very rich practical background and extensive application. There are plenty of practical issues which can be described by pendulum system, for examples, Josephson junctions [28,34,35], super-conducting derive [16], the synchronous electric motor models of a single machine infinite bus [32], shunted model of electrical rotator [31], and so on. Hence, many investigators are attracted by the nonlinear dynamic behavior of pendulum system. In recent decades, they have done a lot of research in this field, such as Lima and Pettini [30] studied the control chaos in a damped driven two-well Duffing oscillator by using Melnikov methods, Levi et al. [28] investigated the dynamics of Josephson junctions, D’Humieres et al. [16] studied the chaos and routes to chaos in the forced pendulum, Clifford et al. [2,12,13] studied the escape zone, rotating periodic orbits and chaotic behavior in the parametrically excited pendulum, Yagasaki et al. [38] considered the controlling chaos in a pendulum through feedforward and feedback control, Jing et al. [21,22] discussed the bifurcation and chaos in Josephson system, Chacón et al. [5] studied the natural symmetries and regularization of the forced pendulum by means of weak parametric modulations, and investigated the chaos control in a driven Josephson junction [6], Cao et al. [4] studied the suppressing chaos by weak resonant excitations in an externally-forced froude pendulum, Yang [40] investigated the inhibition

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of chaos in a pendulum equation and three-well duffing system by Melnikov methods, Ruslan et al. [33] controlled a nonlinear system to an invariant manifold in pendulum system by using quantized state feedback, Amer [1] studied the dynamical behavior of a rigid body suspended on an elastic spring as a pendulum model with three degrees of freedom, Costa et al. [14] investigated the dynamics of an SMA-pendulum system and possibility of thermal control of the system, they also studied the chaos control of a smart system composed of a pendulum coupled with shape memory alloy elements in paper [15]. Research on the periodic orbit, bifurcation, chaos and chaos suppression in the pendulum system can be seen in references [3, 7, 8, 20, 23, 24, 25, 26, 29, 36], and so on.

Although there are plenty of research on the pendulum system, chaos control in pendulum system is still an interesting and challenging problem. In this paper, we will use Melnikov methods to study the chaos control of the following pendulum system for ultra-subharmonic resonance.

\[
\dot{x} = y, \quad \dot{y} = -\alpha x - \delta y - [1 + f_0 \cos(\Omega t + \Psi)] \sin x + f_1 \cos(\omega t) \sin(x - \gamma),
\]

where \(\alpha\) is the spring constant, \(\delta\) denotes damping or friction, \(f_1 \cos(\omega t)\) represents the external excitations, \(f_0 \cos(\Omega t + \Psi)\) is parametric excitation (or chaos-suppressing excitation), \(\gamma\) denotes the deviation angle when suspension axis deviates from vertical direction under vibration. If \(\Omega/\omega = p/q, p \geq 1, q > 1\), ultra-subharmonic resonance occurs in the system (1), where \(p\) and \(q\) are relatively prime integers (which means \((p, q) = 1\)). The special cases of system (1) have been studied extensively. When \(\Psi = 0\) and \(f_1 = 0\), Landa [27], Bishop et al. [2] and Wiggins [37] studied the bifurcation and chaos of system (1), Clifford et al. [13] investigated the rotating periodic orbits of system (1). When the last item of System (1) is \(f_1 \sin(\omega t)\), Jing et al. [17, 18] investigated the bifurcation and chaos of system (1) by using Melnikov methods and second-order averaging method, the criteria of bifurcation and chaos existence of system (1) for periodic and quasi-periodic perturbation was gave, respectively, Yang et al. [39, 41] investigated the chaos control of system (1) by using Melnikov methods, Chen et al. [9, 10] investigated the bifurcation, chaos and chaos control of system (1) with a phase shift by using Melnikov methods and second-order averaging method. For \(\alpha = 0, \gamma = 0\) and \(\Psi = 0\), D’Humieres et al. [16] investigated the chaotic behaviors of system (1) through the experimental method, they found some interesting behaviors in chaotic region, for example, the intermittent behavior, symmetry breaking of periodic orbits and period-3 bifurcations. when \(\Psi = 0\), Fu et al. [19] investigated the bifurcation and chaos of system (1) by using Melnikov methods and second-order averaging method. Chen et al. [11] investigated the chaos control of system (1) for primary and subharmonic resonance.

From the conclusions obtained, we find that the chaos control of pendulum system mainly focuses on primary and subharmonic resonance, and very little work has been done for ultra-subharmonic resonances. In fact, ultra-subharmonic resonance can occur in system (1), there can be ultraharmonic and ultrasubharmonic solutions. So it is necessary to investigate the chaos control of (1) for ultra-subharmonic resonances. In this paper, by using Melnikov methods proposed in [6], we study the control of homoclinic and heteroclinic chaos for ultra-subharmonic resonances, respectively. The conclusion in paper [6, 11] is a sufficient condition, but holds only for subharmonic resonance. Our conclusion is a necessary condition, it holds for ultra-subharmonic resonance. Combining the existing results in [11] with the new results reported in this paper, a more complete understanding of the chaos control
of system (1) is given. At the same time, it provides guidance for the application of chaos control about system (1).

The organization of this paper is as follows. In Section 2, we provide the fixed points and phase portraits of the unperturbed system of (1) in order to show the existence of homoclinic orbit and heteroclinic orbit. In Section 3, by using Melnikov methods proposed in [6], we calculate the suitable parameter intervals in which the chaotic behavior can be inhibited to periodic orbits, and obtain the criteria for inhibition of the homoclinic and heteroclinic chaos, respectively. In Section 4, we give numerical simulation, and analyze the consistency and difference between numerical simulation and theoretical analysis. The conclusion is given in Section 5.

2. Fixed points and phase portraits for unperturbed system. Let \( \delta = f_0 = f_1 = 0 \), then the system (1) becomes the following unperturbed system

\[
\dot{x} = y, \quad \dot{y} = -\sin x - \alpha x
\]

with Hamiltonian function

\[
H(x, y) = \frac{1}{2} y^2 + 1 - \cos x + \frac{\alpha}{2} x^2.
\]

For \( \alpha = 0.1 \), system (2) has five equilibrium points: \( O(0, 0) \), \( C_2(5.67921, 0) \) and \( C_4(-5.67921, 0) \) are centers, \( C_1(3.49906, 0) \) and \( C_3(-3.49906, 0) \) are saddles. According to Hamiltonian function, the phase portraits of system (2) for \( \alpha = 0.1 \) is given in Fig. 1. Fig. 1 shows that \( C_1 \) is connected to \( C_3 \) by heteroclinic orbits \( \Gamma_{het}^+ \) and \( \Gamma_{het}^- \), \( C_1 \) and \( C_3 \) is connected to itself by homoclinic orbit \( \Gamma_{hom}^+ \) and \( \Gamma_{hom}^- \), respectively.

![Phase portrait of system (2) for \( \alpha = 0.1 \).](image)

For \( \alpha = 0.1 \), there are two heteroclinic orbits and two homoclinic orbits in Fig. 1. Therefore, we can use Melnikov methods [6] to study chaos control of the perturbed pendulum system (1) for ultra-subharmonic resonance. In the following sections, we use Melnikov methods [6] to study suppressing chaos of system (1) by adjusting parametric excitation.
3. Chaos inhibition conditions. The Melnikov function of system (1) can be expressed as

\[ M(t_0) = -\delta \int_{-\infty}^{\infty} g_0^2(t)dt + \int_{-\infty}^{\infty} y_0(t)\left[ f_1 \cos[\omega(t + t_0)] \sin[x_0(t) - \gamma] - f_0 \cos[\Omega(t + t_0)] + \Psi \sin x_0(t) \right]dt, \]

where \((x_0, y_0) = (x_0(t), y_0(t))\) is the unperturbed homoclinic or heteroclinic orbits.

We first compute the Melnikov function for the homoclinic orbits. Since \(x_0(t)\) is even and \(y_0(t)\) is odd in this case, the Melnikov function (3) can be simplified as

\[ M_1(t_0) = -2\delta \int_{0}^{\infty} y_0^2(t)dt - 2f_1 \sin(\omega t_0) \int_{0}^{\infty} y_0(t) \sin(\omega t) \sin[x_0(t) - \gamma]dt + 2f_0 \sin(\Omega t_0 + \Psi) \int_{0}^{\infty} y_0(t) \sin(\Omega t) \sin[x_0(t)]dt = -C_1 - A_1 \sin(\omega t_0) + B_1 \sin(\Omega t_0 + \Psi), \]

where \(C_1 = 2\delta \int_{0}^{\infty} y_0^2(t)dt\).

Next, for the heteroclinic orbits, \(x_0(t)\) is odd and \(y_0(t)\) is even. Thus the Melnikov function (3) can be simplified as

\[ M_2(t_0) = -C_2 - A_2 \sin(\omega t_0 + \theta) + B_2 \sin(\Omega t_0 + \Psi), \]

where

\[ A_2 = \sqrt{A^2 + B^2}, \quad B_2 = 2f_0 \int_{0}^{\infty} y_0(t) \sin(\Omega t) \sin[x_0(t)]dt, \quad C_2 = 2\delta \int_{0}^{\infty} y_0^2(t)dt, \]

\[ A = 2f_1 \sin(\gamma) \int_{0}^{\infty} y_0(t) \cos(\Omega t) x_0(t)dt, \quad B = 2f_1 \cos(\gamma) \int_{0}^{\infty} y_0(t) \sin(\omega t) \sin[x_0(t)]dt \]

and \(\theta = \arctan(A/B)\).

According to the Melnikov functions, we know that the system (1) without parametric excitation may be chaotic for \(f_0 = 0, A_1 - C_1 \geq 0\) or \(A_2 - C_2 \geq 0\), and the system (1) with parametric excitation may be chaotic for \(f_0 \neq 0, B_1 \leq A_1 - C_1\) or \(B_2 \leq A_2 - C_2\) (these relationships represent a sufficient condition for \(M_1(t_0)\) or \(M_2(t_0)\) to change sign at some \(t_0\)). Therefore, we obtain that a necessary condition for \(M_1(t_0)\) or \(M_2(t_0)\) always having the same sign is \(B_1 > A_1 - C_1\) for (4) or \(B_2 > A_2 - C_2\) for (5).

From the above analysis, we study chaos control with the following steps. First, we assume that the system (1) without parametric excitation exhibits chaotic dynamics such that the melnikov function (3) has a simple zero, that is, \(A_1 - C_1 \geq 0\) for (4) or \(A_2 - C_2 \geq 0\) for (5). Next, we consider the chaos control of system (1) with parametric excitation, by \(B_1 > A_1 - C_1\) for (4) or \(B_2 > A_2 - C_2\) for (5), we can obtain the optimal values of \(\Psi\) (denoted as \(\Psi_{\text{opt}}\)) and the suitable intervals of \(f_0\) for suppressing chaos. We can get the same results because of the symmetry of homoclinic or heteroclinic orbits.

We have studied the inhibition of homoclinic and heteroclinic chaos of the pendulum system (1) for primary and subharmonic resonance (\(\Omega/\omega = p/1, p \in N\)) in [11]. So, we only consider chaos control of the pendulum system (1) for ultra-subharmonic resonances (\(\Omega/\omega = p/q, p, q \in N, q > 1, (p, q) = 1\)) in this paper. Next, we should consider the inhibition of homoclinic and heteroclinic chaos, separately.

\[ A_1 = 2f_1 \int_{0}^{\infty} y_0(t) \sin(\omega t) \sin[x_0(t) - \gamma]dt, \quad B_1 = 2f_0 \int_{0}^{\infty} y_0(t) \sin(\Omega t) \sin[x_0(t)]dt \]
3.1 For homoclinic orbits. For the right homoclinic orbit, we have

\[ M^+(t_0) = -C_1 - A_1 \sin(\omega t_0) + B_1 \sin(\Omega t_0 + \Psi^+). \]  

If \( f_0 = 0 \), the corresponding Melnikov function

\[ M^+_1(t_0) = -C_1 - A_1 \sin(\omega t_0) \]  

changes sign at some \( t_0 \), i.e., \( C_1 < A_1 \). If we let the parametric excitation act on the system (1) \( (f_0 \neq 0) \) such that \( B_1 \leq A_1 - C_1 \), this inequality is a sufficient condition for \( M^+_1(t_1) \) to change sign at some \( t_0 \). Thus,

\[ B_1 > A_1 - C_1 \equiv B_{\text{min}} \]  

is a necessary condition for \( M^+_1(t_0) \) to always have the same sign, i.e. \( M_1(t_0) < 0 \).

In order to the inequality (8) be a sufficient for \( M_1(t_0) < 0 \) for all \( t_0 \), the inequality

\[ -A_1 \sin(\omega t_0) + B_1 \sin(\Omega t_0 + \Psi^+) \leq A_1 - B_1 \]  

must hold. A sufficient condition for inequality (9) to hold for \( t_0 \rightarrow \infty \) is

\[ \frac{p}{q} = \frac{4m + 3 - 2\Psi^+/\pi}{4n + 3}, \]  

where \( m \) and \( n \) are positive integers. The suitable initial phase satisfying Eq. (10) is denoted \( \Psi^{+\text{max}}_\text{min} \).

Although condition (10) is a necessary one for inequality (9) to hold for all \( t_1 \), it provides us a good situation in which \( M^+_1(t_0) \) is as near as possible to be tangency condition for \( B_1 = B_{\text{min}} \). This means that one has a good chance to eliminate chaotic attractors.

For obtaining an upper threshold of \( f_0 \), we add a condition which don’t enhance the initial chaos, i.e. \( B_1 < A_1 + C_1 \equiv B_{\text{max}} \), this inequality is a necessary condition for \( M^+_1(t_0) < 0 \). In order to this relationship to also be a sufficient condition for \( M^+_1(t_1) < 0, \forall t_1 \in R \), we must have

\[ -A_1 \sin(\omega t_0) + B_1 \sin(\Omega t_0 + \Psi^+) \leq B_1 - A_1 \]  

A necessary condition for inequality (11) holding is

\[ \frac{p}{q} = \frac{4m + 1 - 2\Psi^+/\pi}{4n + 1}, \]  

where \( m \) and \( n \) are positive integers. The suitable initial phase satisfying Eq. (12) is denoted \( \Psi^{+\text{max}}_\text{min} \).

In analogy with (10), equality (12) makes \( M^+_1(t_0) \) is as near as possible to be tangency condition for \( B_1 = B_{\text{max}} \), thus one has a good chance to eliminate chaotic attractors.

According to \( A_1 - C_1 < B_1 \) and \( B_1 < A_1 + C_1 \), we can get

\[ f_{0\text{min}} = (1 - \frac{C_1}{A_1})R \]  

(13)
and

\[ f_{0\text{max}} = (1 + \frac{C_{\text{hom}}}{A_{\text{hom}}})R, \]  

(14)

where \( R = f_1 \int_0^\infty y_0(t) \sin(\omega t) \sin[x_0(t) - \gamma] dt / \int_0^\infty y_0(t) \sin(\Omega t) \sin[x_0(t)] dt. \)

For the left homoclinic orbit, we have

\[ M^{-}(t_0) = -C_1 - A_1 \sin(\omega t_0) + B_1 \sin(\Omega t_0 + \Psi^{-}). \]  

(15)

By the same method, we can find the lower and upper thresholds of \( f_0 \) are \( f_{0\text{min}} = (1 - \frac{C_1}{A_1})R \) and \( f_{0\text{max}} = (1 + \frac{C_1}{A_1})R \), respectively, and the suitable initial phase \( \Psi_{\text{min,max}}^{-} \) satisfies

\[ \Psi_{\text{min,max}}^{+} \equiv \Psi_{\text{min,max}}^{-} \pmod{2\pi}. \]  

(16)

From the above analysis, we obtain the following result.

**Result 1.** Suppose the values of parameters \( \alpha, \Omega, \omega, \delta, \gamma \) and \( f_1 \) are given, and \( \frac{\Omega}{\omega} = \frac{p}{q} \) \((p, q \in \mathbb{N}^+, q > 1, (p, q) = 1)\). If \( f_0 \in ((1 - \frac{C_1}{A_1})R, (1 + \frac{C_1}{A_1})R) \), then there may be a good chance to eliminate the homoclinic chaos of system (1) for some values of \( \Psi_{\text{min,max}}^{+} \) and \( \Psi_{\text{min,max}}^{-} \), where \( \Psi_{\text{min}}^{+} \) and \( \Psi_{\text{min}}^{-} \) satisfy Eq. (10), \( \Psi_{\text{max}}^{+} \) and \( \Psi_{\text{max}}^{-} \) satisfy Eq. (12).

### 3.2. For heteroclinic orbits.

For the upper heteroclinic orbit, the Melnikov function is written as

\[ M_{2}^{+}(t_0) = -C_2 - A_2 \sin(\omega t_0 + \theta) + B_2 \sin(\Omega t_0 + \Phi^{+}). \]  

(17)

Let \( t_0 = t_0 - \theta/\omega \) and \( \Psi^{+} = \Phi^{+} + \Omega \theta/\omega \), the Melnikov function (17) can become

\[ M_{2}^{+}(t_0) = -C_2 - A_2 \sin(\omega t_0) + B_2 \sin(\Omega t_0 + \Phi^{+}). \]  

(18)

When \( f_0 = 0 \) (which means \( B_2 = 0 \)), we suppose the corresponding Melnikov function

\[ M_{2}^{+}(t_0) = -C_2 - A_2 \sin(\omega t_0) \]  

(19)

changes sign at some \( t_0 \), i.e. \( C_2 \leq A_2 \). If we let the parametric excitation act on the system (1) (i.e. \( f_0 \neq 0 \)) such that \( B_2 \leq A_2 - C_2 \), which is a sufficient condition for \( M_{2}^{+}(t_0) \) to change sign at some \( t_0 \). Thus, \( B_2 > A_2 - C_2 \equiv B_{\text{min}} \) is a necessary condition for \( M_{2}^{+}(t_0) < 0 \), \((t_1 \in R)\). For this inequality to also be a sufficient condition for \( M_{2}^{+}(t_0) < 0 \), \((t_1 \in R)\), there must be

\[ A_2 - B_2 \geq -A_2 \sin(\omega t_0) + B_2 \sin(\Omega t_0 + \Phi^{+}). \]  

(20)

A sufficient condition for the equality (20) to hold for \( t_1 \to \infty \) is

\[ \frac{p}{q} = \frac{4m + 3 - 2\Phi^{+}/\pi}{4n + 3}, \]  

(21)

where \( m, n \) are positive integers, the value of \( \Phi^{+} \) is denoted \( \Phi_{\text{min}}^{+} \), thus the suitable initial phase \( \Psi_{\text{min}}^{+} = \Phi_{\text{min}}^{+} + \theta \Omega/\omega \).

Although condition (21) is a necessary one for the inequality (19) to hold for all \( t_1 \), it gives good situations in which \( M_{2}^{+}(t_0) \) is as near as possible to be tangency
condition for $B_2 = B_{\text{min}}$. Therefore, one has a good chance to suppress chaos. For this case, the lower threshold of $f_0$ is written as

$$f_{0\text{min}} = (1 - \frac{C_2}{A_2})R_1,$$  \hspace{1cm} (22)

where $R_1 = A_2/\int_0^{+\infty} y_0 \sin(\Omega t) \sin[x_0(t)] dt$.

For getting an upper threshold of $f_0$, we add a condition which don’t enhance the initial chaos, i.e. $B_2 < A_2 + C_2 + \Omega_{\text{max}}$, this inequality is a necessary condition for $M_2^*(\tau_0) < 0, \forall t \in R$. For this relationship to be a sufficient condition for $M_2^*(\tau_0) < 0, \forall t \in R$, there must be

$$B_2 - A_2 \geq -A_2 \sin(\omega \tau_0) + B_2 \sin(\Omega \tau_0 + \Psi^+).$$ \hspace{1cm} (23)

A sufficient condition for inequality (23) holding for $\tau_0 \to \infty$ is

$$p = \frac{4m + 1 - 2\Phi^+ / \pi}{4n + 1}, \hspace{1cm} (24)$$

where $m, n$ are positive integers, the value of $\Phi^+$ is denoted $\Phi_{\text{max}}^+$, so the suitable initial phase $\Psi_{\text{max}}^+ = \Phi_{\text{max}}^+ + \Omega/\omega$.

In analogy with (21), equality (24) again makes $M_2^*(\tau_0)$ is as near as possible to be tangency condition for $B_2 = B_{\text{max}}$. For this case, we get the upper threshold of $f_0$, which is expressed to

$$f_{0\text{max}} = (1 + \frac{C_2}{A_2})R_1.$$ \hspace{1cm} (25)

For the lower heteroclinic orbit, we have

$$M_2^+(t_0) = -C_2 - A_2 \sin(\omega t_0 - \theta) + B_2 \sin(\Omega t_0 + \Psi^-).$$ \hspace{1cm} (26)

Let $t_0 = \tau_0 + \theta/\omega$ and $\Psi^- = \Phi^- - \Omega/\omega$, the Melnikov function (26) can become

$$M_2^-(t_0) = -C_2 - A_2 \sin(\omega \tau_0) + B_2 \sin(\Omega \tau_0 + \Phi^-).$$ \hspace{1cm} (27)

In analogy with the upper heteroclinic orbit, the lower and upper threshold of $f_0$ for the lower heteroclinic orbit are given in (22) and (25), respectively. The suitable initial phase $\Psi_{\text{min,max}}^- = \Phi_{\text{min,max}}^- - \Omega/\omega$ should satisfy

$$\Psi_{\text{min,max}}^+ - \Psi_{\text{min,max}}^- = \frac{2\Omega}{\omega} \theta \text{mod} 2\pi.$$ \hspace{1cm} (28)

By the above analysis, we can get the following result.

**Result 2.** Supposed the values of parameters $\alpha$, $\Omega$, $\omega$, $\delta$, $\gamma$ and $f_1$ are given, and $\Omega = \frac{p}{q}$ ($p, q \in N^+, q > 1, (p, q) = 1$). If $f_0 \in ((1 - \frac{C_2}{A_2})R_1, (1 + \frac{C_2}{A_2})R_1)$, then there may be a good chance to eliminate the homoclinic chaos of system (1) for some values of $\Psi_{\text{min,max}}^+$ and $\Psi_{\text{min,max}}^-$, where $\Psi_{\text{min,max}}^+ = \Phi_{\text{min,max}}^+ + \theta \Omega/\omega$, $\Psi_{\text{min,max}}^- = \Phi_{\text{min,max}}^- - \Omega/\omega$, $\Phi_{\text{min}}^+$ and $\Phi_{\text{min}}^-$ satisfy Eq. (21), $\Phi_{\text{max}}^+$ and $\Phi_{\text{max}}^-$ satisfy Eq. (24).

**4. Numerical simulations.** In this section, we give the numerical simulation to check up the theoretical results obtained in the previous sections. Our algorithms
for computing $A, B, A_1, B_1, C_1, A_2, B_2$ and $C_2$ basically include two steps: solving the differential equation (2) for $x_0(t)$ and $y_0(t)$ by using the fourth Runge-Kutta technique, and integrating the integrals $A, B, A_1, B_1, C_1, A_2, B_2$ and $C_2$. Fixing $\alpha = 0.1, \gamma = 0.01, \delta = 0.38, f_1 = 1.381$, initial value $(x, y) = (0.3499, 0)$ and other parameters are varied.

To check up our theoretical results, taking $\omega = 1.5$ and $f_0 = 0$. The inequalities $A_1 > C_1$ and $A_2 > C_2$ hold for $f_1 = 1.381$, which means that the system (1) may exhibit chaos. When $\omega = 1.5, \alpha = 0.1, \delta = 0.38, f_0 = 0$ and $f_1 = 1.381$, the corresponding chaotic attractor is given in Fig. 2. Next, we verify the suppressing chaos in system (1) for the right homoclinic orbit (corresponding $\Psi_{\min,\max}^+$) and upper heteroclinic orbit (corresponding $\Psi_{\min,\max}^+$), respectively.

![Fig. 2. The chaotic attractor of system (1) for $\alpha = 0.1, \omega = 1.5, \delta = 0.38, f_1 = 1.381, \gamma = 0.01$ and $f_0 = 0$.](image)

(1) For the suppressing of homoclinic chaos.

(i) $\frac{\Omega}{2} = \frac{1}{2}$.

Fixing $\Omega = 0.75$, the result 1 predicts $f_{0\min} = -1.146, f_{0\max} = 0.3915$ and $\Psi_{\min,\max}^+ = \{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\}$ for inhibiting homoclinic chaos. The bifurcation diagram in $(\Psi, x)$ plane is given in Fig. 3 for $\alpha = 0.1, f_1 = 1.381, f_0 = 0.2, \delta = 0.38, \Omega = 0.75$ and $\omega = 1.5$, which shows that the homoclinic chaos is controlled to periodic orbit at $\Psi = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$ and $\frac{7\pi}{4}$.
(ii) $\Omega = \frac{1}{3}$.

When $\Omega = 0.5$, we obtain $f_{0\text{min}} = -1.017$, $f_{0\text{max}} = 0.35$ and $\Psi_{\text{min, max}} = \left\{ \frac{\pi}{3}, \pi, \frac{5\pi}{3} \right\}$ for controlling homoclinic chaos by the result 1. The bifurcation diagram in $(\Psi, x)$ plane for $\alpha = 0.1$, $f_1 = 1.381$, $f_0 = 0.2$, $\delta = 0.38$, $\Omega = 0.5$ and $\omega = 1.5$ is given in Fig. 4, which indicates that the homoclinic chaos is inhibited to periodic orbit at $\Psi = \frac{\pi}{3}$, $\pi$ and $\frac{5\pi}{3}$.

(iii) $\Omega = \frac{2}{3}$.

When $\Omega = 1$, we obtain $f_{0\text{min}} = -1.9732$, $f_{0\text{max}} = 0.6741$ and $\Psi_{\text{min, max}} = \left\{ \frac{\pi}{6}, \frac{5\pi}{6}, \frac{9\pi}{6} \right\}$ for suppressing homoclinic chaos according to the result 1. For $\alpha = 0.1$, $f_1 = 1.381$, $f_0 = 0.4$, $\delta = 0.38$, $\Omega = 1$ and $\omega = 1.5$, the bifurcation diagram in $(\Psi, x)$ plane is given in Fig. 5, which indicates that the homoclinic chaos is suppressed to periodic orbit at $\Psi = \frac{\pi}{6}$, $\frac{5\pi}{6}$ and $\frac{9\pi}{6}$.

From cases (i), (ii) and (iii), we observe that the homoclinic chaos can be controlled to periodic orbit when the conditions of result 1 are satisfied. Although the
The conditions of result 1 are necessary, not sufficient, sometimes the effect of controlling homoclinic chaos is good.

Fig. 5. The bifurcation diagram of system (1) in \((\Psi, x)\) plane for \(\alpha = 0.1, f_1 = 1.381, f_0 = 0.4, \delta = 0.38, \Omega = 1\) and \(\omega = 1.5\).

(2) For the suppressing of heteroclinic chaos.

(i) \(\Omega / \omega = \frac{1}{2}\).

Fixing \(\Omega = 0.75\), the result 2 predicts \(f_{0,\text{min}} = 0.1769, f_{0,\text{max}} = 3.603\) and \(\Psi_{\text{min, max}}^+ = \{0.2492\pi, 0.7492\pi, 1.2492\pi, 1.7492\pi\}\) for inhibiting heteroclinic chaos. When \(\alpha = 0.1, f_1 = 1.381, f_0 = 2, \delta = 0.38, \Omega = 0.75\) and \(\omega = 1.5\), the bifurcation diagram in \((\Psi, x)\) plane is given in Fig. 6, which shows that the heteroclinic chaos is controlled to periodic orbit at \(\Psi = 0.2492\pi, 0.7492\pi, 1.2492\pi\) and \(1.7492\pi\).

Fig. 6. The bifurcation diagram of system (1) in \((\Psi, x)\) plane for \(\alpha = 0.1, f_1 = 1.381, f_0 = 2, \delta = 0.38, \Omega = 0.75\) and \(\omega = 1.5\).

(ii) \(\Omega / \omega = \frac{1}{3}\).

When \(\Omega = 0.5\), we get \(f_{0,\text{min}} = 0.261, f_{0,\text{max}} = 5.313\) and \(\Psi_{\text{min, max}}^+ = \{0.3327\pi, 0.9994\pi, 1.6661\pi\}\) for controlling heteroclinic chaos by the result 2. For \(\alpha = 0.1, f_1 = 1.381, \delta = 0.38, \Omega = 0.5\) and \(\omega = 1.5\), the bifurcation diagrams in \((\Psi, x)\) plane for \(f_0 = 1\) and \(f_0 = 4\) are given in Fig. 7(a) and (b), respectively. Fig. 7(b) indicates
that the heteroclinic chaos is inhibited to periodic orbit at $\Psi = 0.3327\pi$, $0.9994\pi$ and $1.6661\pi$. Fig. 7(a) indicates that the heteroclinic chaos can’t be inhibited to periodic orbit at $\Psi = 0.3327\pi$, $0.9994\pi$ and $1.6661\pi$, and demonstrates that the theoretical conditions for $q > 1$ are only necessary, not sufficient.

(iii) $\frac{\Omega}{\omega} = \frac{2}{3}$.

Fixing $\Omega = 1$, the result 2 predicts $f_{0\text{min}} = 0.1435$, $f_{0\text{max}} = 2.922$ and $\Psi_{\text{min,max}}^+ = \{0.1656\pi, 0.4989\pi, 0.8322\pi, 1.1656\pi, 1.4989\pi, 1.8322\pi\}$ for controlling heteroclinic chaos. When $\alpha = 0.1$, $f_1 = 1.381$, $\delta = 0.38$, $\Omega = 1$ and $\omega = 1.5$, the bifurcation diagram in $(\Psi,x)$ plane for $f_0 = 2$ and $f_0 = 2.5$ are given in Fig. 8(a) and (b), respectively. Fig. 8(b) indicates that the heteroclinic chaos is inhibited to periodic orbit at $\Psi = 0.1656\pi, 0.4989\pi, 0.8322\pi, 1.1656\pi, 1.4989\pi$ and $1.8322\pi$. Fig. 8(a) indicates that the heteroclinic chaos can’t be inhibited to periodic orbit at $\Psi = 0.1656\pi, 0.4989\pi, 0.8322\pi, 1.1656\pi, 1.4989\pi$ and $1.8322\pi$, and demonstrates that the theoretical conditions for $q > 1$ are only necessary, not sufficient.

Fig. 7. The bifurcation diagram of system (1) in $(\Psi,x)$ plane for $\alpha = 0.1$, $f_1 = 1.381$, $\delta = 0.38$, $\Omega = 0.5$ and $\omega = 1.5$: (a) $f_0 = 1$; (b) $f_0 = 4$. 
Fig. 8. The bifurcation diagram of system (1) in (Ψ,x) plane for α = 0.1, \( f_1 = 1.381 \), \( \delta = 0.38 \), \( \Omega = 1 \) and \( \omega = 1.5 \): (a) \( f_0 = 2 \); (b) \( f_0 = 2.5 \).

5. Conclusion. In this paper, we study the chaos control of pendulum system for ultra-subharmonic resonance by using Melnikov methods, and give the control conditions for heteroclinic chaos and homoclinic chaos, respectively. Numerical simulations indicate that the chaos behaviors for \( q > 1 \) can be inhibited to periodic orbits in most cases (see Figs. 3-6, 7(b) and 8(b)), but there are heteroclinic chaos or homoclinic chaos can’t be suppressed to periodic orbits for some values of \( f_0 \) (see 7(a) and 8(a)), which demonstrate that the theoretical conditions for ultra-subharmonic resonance are only necessary, not sufficient. Although our results are only necessary, the methods used in this paper is very effective in inhibiting chaos for ultra-subharmonic resonance and can be applied to verify numerical and experimental results.

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