Bulk and Boundary $S$ Matrices for the $SU(N)$ Chain

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Abstract

We consider both closed and open integrable antiferromagnetic chains constructed with the $SU(N)$-invariant $R$ matrix. For the closed chain, we extend the analyses of Sutherland and Kulish – Reshetikhin by considering also complex “string” solutions of the Bethe Ansatz equations. Such solutions are essential to describe general multiparticle excited states. We also explicitly determine the $SU(N)$ quantum numbers of the states. In particular, the model has particle-like excitations in the fundamental representations $[k]$ of $SU(N)$, with $k = 1, \ldots, N-1$. We directly compute the complete two-particle $S$ matrices for the cases $[1] \otimes [1]$ and $[1] \otimes [N-1]$. For the open chain with diagonal boundary fields, we show that the transfer matrix has the symmetry $SU(l) \times SU(N-l) \times U(1)$, as well as a new “duality” symmetry which maps $l \leftrightarrow N-l$. With the help of these symmetries, we compute by means of the Bethe Ansatz for particles of types $[1]$ and $[N-1]$ the corresponding boundary $S$ matrices.
1 Introduction

Integrable quantum spin chains are exactly solvable quantum mechanical models of $N$ quantum spins, of which the Heisenberg model solved by Bethe [1]-[3] is the prototype. In the antiferromagnetic regime, such spin chains can be regarded as lattice versions of corresponding integrable relativistic quantum field theories. For integrable spin chains, quantities of physical interest (spectrum, $S$ matrix, etc.) can be calculated exactly by direct means, starting from the microscopic Hamiltonian; while for the corresponding field theories, such exact information has been primarily obtained by indirect means, such as the “bootstrap” approach [4], [5] and semiclassical approximations.

Quantum spin chains have one spatial dimension and come in two topologies: “closed” (periodic boundary conditions) and “open”. The latter exhibit a rich variety of boundary phenomena, which – for integrable chains – can be investigated exactly.

There exists a systematic approach for constructing integrable quantum spin chains, called the Quantum Inverse Scattering Method. (For a recent review, see [6].) The basic building blocks for constructing closed chains are $R$ matrices, which are solutions of the Yang-Baxter equation

$$R_{12}(\lambda) \ R_{13}(\lambda + \lambda') \ R_{23}(\lambda') = R_{23}(\lambda') \ R_{13}(\lambda + \lambda') \ R_{12}(\lambda). \quad (1.1)$$

For constructing open chains, one needs also $K$ matrices, which are solutions of the boundary Yang-Baxter equation [7]-[10]

$$R_{12}(\lambda_1 - \lambda_2) K_1(\lambda_1) R_{21}(\lambda_1 + \lambda_2) K_2(\lambda_2) = K_2(\lambda_2) R_{12}(\lambda_1 + \lambda_2) K_1(\lambda_1) R_{21}(\lambda_1 - \lambda_2). \quad (1.2)$$

In this paper, we focus on integrable quantum spin chains constructed with the $R$ matrix [11]

$$R(\lambda) = \frac{1}{\lambda + i} \ (\lambda + i \mathcal{P}), \quad (1.3)$$

where $\mathcal{P}$ is the permutation matrix

$$\mathcal{P} x \otimes y = y \otimes x \quad (1.4)$$

for all vectors $x$ and $y$ in an $\mathcal{N}$-dimensional complex vector space $C_\mathcal{N}$. This $R$ matrix is $SU(\mathcal{N})$ invariant; i.e.,

$$[U \otimes U, R(\lambda)] = 0 \quad (1.5)$$

for all group elements $U$ in the defining representation of $SU(\mathcal{N})$. 

This paper has two main sections. In Section 2 we consider the closed integrable antiferromagnetic chain with \( N \) “spins” (vectors in \( \mathbb{C}^N \)), which has the Hamiltonian

\[
H_{\text{closed}} = \sum_{n=1}^{N-1} H_{nn+1} + H_{N1},
\]

(1.6)

where the two-site Hamiltonian \( H_{jk} \) is given by

\[
H_{jk} = \frac{i}{2} \frac{d}{d\lambda} P_{jk} R_{jk}(\lambda) \bigg|_{\lambda=0} = \frac{1}{2} (\mathcal{P} - 1)_{jk}.
\]

(1.7)

The model is \( SU(N) \) invariant, and the space of states is \( \mathbb{C}^{\otimes N} \). This a generalization of the antiferromagnetic Heisenberg model, which corresponds to the case \( N = 2 \).

This model was first investigated by Sutherland [12] and by Kulish and Reshetikhin [13]. These authors determined the energy eigenstates and eigenvalues in terms of \( N-1 \) types of roots of a system of Bethe Ansatz equations (BAE). Moreover, they found that the ground state corresponds to having \( N-1 \) “filled Fermi seas”; and they studied particle-like excited states corresponding to “holes” in these seas. These analyses considered only real roots of the BAE.

We extend these analyses by considering also complex “string” solutions of the BAE. Such solutions are essential to describe general multiparticle excited states. Moreover, we explicitly determine the \( SU(N) \) quantum numbers of the states. In particular, we show that the Bethe Ansatz state consisting of one hole in the \( k^{th} \) sea \( (k = 1, \ldots, N-1) \) and no complex strings is the highest weight of the fundamental representation \( [k] \), corresponding to a Young tableau with a single column of \( k \) boxes, as shown in Figure 1. That is, the

![Figure 1: Young tableau with a single column of \( k \) boxes, corresponding to the fundamental representation \( [k] \) of \( SU(N) \)](image_url)

\footnote{Such excitations have been called “kinks” or “spinons”. We refer to them here simply as “particles”.}

model has particle-like excitations in the fundamental representations \( [k] \) of \( SU(N) \), with \( k = 1, \ldots, N-1 \). Finally, we directly compute the complete two-particle \( S \) matrices for the cases \( [1] \otimes [1] \) and \( [1] \otimes [N-1] \). In the latter case, the singlet state is described by the Bethe Ansatz state consisting of two holes (one each in seas 1 and \( N-1 \)) as well as one string of
length 2 in each of the $\mathcal{N} - 1$ seas. Our results for the $S$ matrices coincide with those found by the bootstrap approach [3], with no additional CDD factors.

In Section 3, we consider the open integrable chain constructed with the $SU(\mathcal{N})$-invariant $R$ matrix (1.3), together with the $\mathcal{N} \times \mathcal{N}$ diagonal $K$ matrices given by [14]

$$K^\pm_{(l)}(\lambda, \xi_{\mp}) = \text{diag}(a^{\mp}_l, \ldots, a^{\mp}_{N-l}, b^{\mp}_l, \ldots, b^{\mp}_{N-l}), \quad (1.8)$$

where

$$a^- = i\xi_- - \lambda, \quad b^- = i\xi_- + \lambda, \quad a^+ = i\xi_+ + \lambda, \quad b^+ = i(\xi_+ - \mathcal{N}) - \lambda, \quad (1.9)$$

for any $l \in \{1, \ldots, \mathcal{N} - 1\}$. The Hamiltonian is given by

$$\mathcal{H}_{open} = \sum_{n=1}^{N-1} \mathcal{H}_{nn+1} + \frac{1}{4\xi_-} \frac{d}{d\lambda} K^-_{(1)}(\lambda, \xi_-)|_{\lambda=0} + \frac{\text{tr}_0 K^+_{(0)}(0, \xi_+)\mathcal{H}_{N0}}{\text{tr} K^+_{(0)}(0, \xi_+)}. \quad (1.10)$$

The parameters $\xi_{\mp}$, which may be regarded as certain boundary magnetic fields, break the $SU(\mathcal{N})$ symmetry down to $SU(l) \times SU(\mathcal{N} - l) \times U(1)$. Moreover, we find a new “duality” symmetry which maps $l \leftrightarrow \mathcal{N} - l$. With the help of these residual symmetries of the model, we compute for particles of types [1] and [N − 1] the corresponding boundary $S$ matrices, which describe scattering from the ends of the chain. This is the first direct calculation of boundary $S$ matrices for a model whose symmetry algebra has rank greater than one. For the case $\mathcal{N} = 2$, we recover the results of Refs. [15] and [16].

## 2 The closed chain

In this section, we consider the closed integrable chain constructed with the $SU(\mathcal{N})$-invariant $R$ matrix (1.3). The transfer matrix $t(\lambda)$ is given by

$$t(\lambda) = \text{tr}_0 T_0(\lambda), \quad (2.1)$$

where $T_0(\lambda)$ is the monodromy matrix

$$T_0(\lambda) = R_{0N}(\lambda) \cdots R_{01}(\lambda). \quad (2.2)$$

(As is customary, we suppress the quantum-space subscripts 1, \ldots, $N$ of $T_0(\lambda)$.) The Yang-Baxter equation (1.1) guarantees the commutativity of the transfer matrix

$$[t(\lambda), t(\lambda')] = 0. \quad (2.3)$$
The Hamiltonian (1.6) is proportional to the logarithmic derivative of the transfer matrix at \( \lambda = 0 \)

\[
\mathcal{H}_{\text{closed}} = \frac{i}{2} \frac{d}{d\lambda} \log t(\lambda) \bigg|_{\lambda=0}.
\] (2.4)

The “momentum” operator \( P \) is defined by

\[
P = -i \log t(0),
\] (2.5)

since \( t(0) \) is the one-site shift operator.

### 2.1 \( SU(N) \) generators

In the defining representation of \( SU(N) \), we identify the raising and lowering operators

\[
j^{+}(k) = e_{k,k+1}, \quad j^{-}(k) = e_{k+1,k}, \quad k = 1, \ldots, N - 1,
\] (2.6)

and the Cartan generators

\[
s^{(k)} = e_{k,k} - e_{k+1,k+1}, \quad k = 1, \ldots, N - 1,
\] (2.7)

where \( e_{k,l} \) are elementary \( N \times N \) matrices with matrix elements \( (e_{k,l})_{ab} = \delta_{k,a} \delta_{l,b} \). These generators obey the commutation relations

\[
\left[ j^{+}(k), j^{-}(l) \right] = \delta_{k,l}s^{(k)}, \quad \left[ s^{(k)}, j^{+}(l) \right] = (2\delta_{k,l} - \delta_{k,l+1} - \delta_{k+1,l}) j^{+}(l).
\] (2.8)

We denote by \( j^{\pm}(k), s^{(k)} \) the generators at site \( n \), e.g.,

\[
s^{(k)}_n = 1 \otimes \ldots \otimes 1 \otimes s^{(k)} \otimes 1 \otimes \ldots \otimes 1, \quad n = 1, \ldots, N,
\] (2.9)

and we denote by \( J^{\pm}(k), S^{(k)} \) the corresponding “total” generators acting on the full space of states

\[
J^{\pm}(k) = \sum_{n=1}^{N} j^{\pm}(k)_n, \quad S^{(k)} = \sum_{n=1}^{N} s^{(k)}_n, \quad k = 1, \ldots, N - 1.
\] (2.10)

The \( SU(N) \) invariance (1.5) of the \( R \) matrix implies that

\[
\left[ t(\lambda), J^{\pm}(k) \right] = \left[ t(\lambda), S^{(k)} \right] = 0, \quad k = 1, \ldots, N - 1.
\] (2.11)

For future reference, we now relate these generators to the standard Cartan-Weyl basis. We set

\[
E_{\alpha^i} = \frac{\sqrt{2}}{2} J^{+(i)}, \quad E_{-\alpha^i} = \frac{\sqrt{2}}{2} J^{-(i)},
\]

\[
H_i = \sum_{j=1}^{N-1} \mu_j^i S^{(j)}, \quad i = 1, \ldots, N - 1
\] (2.12)
(with the coefficients $\mu_i^j$ still to be determined), and we demand the commutation relations

\[ [H_i, E_\alpha] = \alpha_i E_\alpha, \quad [E_\alpha, E_-\alpha] = \sum_{i=1}^{N-1} \alpha_i H_i. \]  

(2.13)

The vectors $\alpha^j = (\alpha^i_1, \ldots, \alpha^i_{N-1})$ are the simple roots normalized to unity $\alpha^i \cdot \alpha^i = 1$. One finds

\[ \alpha^j = \left(0, \ldots, 0, -\sqrt{\frac{i - 1}{2i}}, \sqrt{\frac{i + 1}{2i}}, 0, \ldots, 0\right). \]  

(2.14)

From the second relation in Eq. (2.13), we obtain

\[ \alpha^j \cdot \mu^k = \frac{1}{2} \delta_{j,k}. \]  

(2.15)

This implies the important result that $\mu^k = (\mu^k_1, \ldots, \mu^k_{N-1})$, $k = 1, \ldots, N-1$ are the fundamental weights of $SU(N)$ (see, e.g., [17]).

### 2.2 Bethe Ansatz and string hypothesis

We now review the exact Bethe Ansatz solution of the closed $SU(N)$-invariant chain, and we use the string hypothesis to recast the BAE into a form which is particularly suitable for studying multiparticle excitations.

Since the operators $H_{\text{closed}}, P, S^{(k)}$ mutually commute, there exist simultaneous eigenstates $|E, P, S^{(k)}\rangle$. The so-called Bethe Ansatz states are the subset of these states which are highest weights of $SU(N)$, i.e.,

\[ J^{+(k)}| \rangle = 0, \quad k = 1, \ldots, N-1. \]  

(2.16)

(See, e.g., Refs. [3], [18], [19].) These states have been determined by both the coordinate [12] and algebraic [13], [20] Bethe Ansatz methods. In the latter approach, the Bethe Ansatz states are constructed using certain creation operators (elements of the monodromy matrix) depending on solutions $\{\lambda^{(k)}_\alpha\}$ of the BAE

\[ 1 = - \prod_{\beta=1}^{M^{(k-1)}} e_{-1}(\lambda^{(k)}_\alpha - \lambda^{(k-1)}_\beta) \prod_{\beta=1}^{M^{(k)}} e_2(\lambda^{(k)}_\alpha - \lambda^{(k)}_\beta) \prod_{\beta=1}^{M^{(k+1)}} e_{-1}(\lambda^{(k)}_\alpha - \lambda^{(k+1)}_\beta), \]

\[ \alpha = 1, \ldots, M^{(k)}; \quad k = 1, \ldots, N-1, \]  

(2.17)

\[ J^{-(k)} \]  

\[ \]
where

\[ e_n(\lambda) = \frac{\lambda + i n}{\lambda - \frac{i n}{2}}, \]  

and \( M^{(0)} = N, \quad M^{(N)} = 0, \quad \lambda_0^{(0)} = \lambda_{\alpha}^{(N)} = 0 \). The corresponding eigenvalues are given by

\[ E = - \frac{1}{2} \sum_{\alpha=1}^{M^{(1)}} \frac{1}{\lambda_\alpha^{(1)}^2 + \frac{1}{4}}, \]

\[ P = -i \sum_{\alpha=1}^{M^{(1)}} \log e_1(\lambda_\alpha^{(1)}), \]

\[ S^{(k)} = M^{(k-1)} + M^{(k+1)} - 2M^{(k)}. \]  

We adopt the “string hypothesis”, which states that in the thermodynamic \((N \to \infty)\) limit, all the solutions \(\{\lambda_1^{(k)}, \ldots, \lambda_{M^{(k)}}^{(k)}\}\) are collections of \(M^{(n,k)}\) strings of length \(n\) of the form (for \(M^{(n,k)} > 0\))

\[ \lambda^{(n,k,j)} = \lambda^{(n,k)} + i \left(\frac{n+1}{2} - j\right), \]  

where \(j = 1, \ldots, n; \quad \alpha = 1, \ldots, M^{(n,k)}; \quad k = 1, \ldots, N - 1; \quad n = 1, \ldots, \infty; \) and the “centers” \(\lambda^{(n,k)}\) are real. The total number of \(\lambda\)’s of type \(k\) is given by

\[ M^{(k)} = \sum_{n=1}^{\infty} nM^{(n,k)}, \quad k = 1, \ldots, N - 1. \]  

Implementing this hypothesis in the BAE and then forming the product \(\prod_{j=1}^{n}\) over the imaginary parts of the strings (see, e.g., \([3]\)), we obtain a set of equations for the centers \(\lambda^{(n,k)}\) given (up to an overall sign) by

\[
1 \quad = \quad \left\{ \prod_{m=1}^{\infty} \prod_{\beta=1}^{M^{(m,k-1)}} F_{nm}(\lambda_\alpha^{(n,k)} - \lambda_\beta^{(m,k-1)}) \right\} \left\{ \prod_{m=1}^{\infty} \prod_{\beta=1}^{M^{(m,k)}} E_{nm}(\lambda_\alpha^{(n,k)} - \lambda_\beta^{(m,k)}) \right\} \\
\quad \times \quad \left\{ \prod_{m=1}^{\infty} \prod_{\beta=1}^{M^{(m,k+1)}} F_{nm}(\lambda_\alpha^{(n,k)} - \lambda_\beta^{(m,k+1)}) \right\}, \quad \alpha = 1, \ldots, M^{(n,k)}, \quad k = 1, \ldots, N - 1.
\]

where

\[ E_{nm}(\lambda) = e_{|n-m|}(\lambda) e_{|n-m|+2}(\lambda) \cdots e_{n+m-2}(\lambda) e_{n+m}(\lambda), \]

\[ F_{nm}(\lambda) = e_{-(|n-m|+1)}(\lambda) e_{-(|n-m|+3)}(\lambda) \cdots e_{-(n+m-3)}(\lambda) e_{-(n+m-1)}(\lambda), \]

and \(M^{(n,0)} = N\delta_{n,1}, \quad M^{(n,N)} = 0, \quad \lambda_0^{(n,0)} = \lambda_{\alpha}^{(n,N)} = 0).
Since Eqs. (2.22) involve only products of phases, it is useful to take the logarithm. We obtain the following important equations for $\lambda^{(n,k)}$:

\[ h^{(n,k)}(\lambda^{(n,k)}) = J^{(n,k)}, \quad \alpha = 1, \ldots, M^{(n,k)}, \]
\[ k = 1, \ldots, N - 1, \quad n = 1, \ldots, \infty. \]  

(2.24)

The so-called counting function $h^{(n,k)}(\lambda)$ is defined by

\[
h^{(n,k)}(\lambda) = \frac{1}{2\pi} \left\{ -\sum_{m=1}^{\infty} \sum_{\beta=1}^{M^{(m,k)}} \Phi_{nm}(\lambda - \lambda^{(m,k-1)}_{\beta}) - \sum_{m=1}^{\infty} \sum_{\beta=1}^{M^{(m,k)}} \Xi_{nm}(\lambda - \lambda^{(m,k)}_{\beta}) \right\},
\]

where

\[
\Xi_{nm}(\lambda) = (1 - \delta_{n,m})q_{|n-m|}(\lambda) + 2q_{|n-m|+2}(\lambda) + \cdots + 2q_{n+m-2}(\lambda) + q_{n+m}(\lambda),
\]
\[
\Phi_{nm}(\lambda) = -\left[q_{|n-m|+1}(\lambda) + q_{|n-m|+3}(\lambda) + \cdots + q_{n+m-3}(\lambda) + q_{n+m-1}(\lambda)\right],
\]

and $q_n(\lambda)$ is the odd monotonic-increasing function defined (for $n > 0$) by

\[
q_n(\lambda) = \pi + i \log \epsilon_n(\lambda), \quad -\pi < q_n(\lambda) \leq \pi.
\]

(2.27)

Moreover, \{\(J^{(n,k)}_\alpha\)\} are integers or half-odd integers which satisfy

\[-J^{(n,k)}_{\text{max}} \leq J^{(n,k)}_\alpha \leq J^{(n,k)}_{\text{max}}, \]

(2.28)

where $J^{(n,k)}_{\text{max}}$ is given by

\[
J^{(n,k)}_{\text{max}} = \frac{1}{2} \left\{ M^{(n,k)} - 1 + \sum_{m=1}^{\infty} \min(m, n) \left[ M^{(m,k-1)} - M^{(m,k+1)} - 2M^{(m,k)} \right] \right\}.
\]

(2.29)

In deriving the last equation, we assume the prescription \(\beta\) that $J^{(n,k)}_\alpha \to J^{(n,k)}_{\text{max}} + n$ for $\lambda^{(n,k)}_\alpha \to \infty$. We further assume that \{\(J^{(n,k)}_\alpha\)\} can be regarded as “quantum numbers” of the model: for every set \{\(J^{(n,k)}_\alpha\)\} in the range (2.28) (no two of which are identical), there is a unique solution \{\(\lambda^{(n,k)}_\alpha\)\} (no two of which are identical) of (2.24).

Using the string hypothesis, the expressions (2.19) for the eigenvalues become

\[
E = -\pi \sum_{n=1}^{\infty} \sum_{\alpha=1}^{M^{(n,1)}} a_n(\lambda^{(n,1)}_\alpha),
\]
\[
P = -\pi \sum_{n=1}^{\infty} \sum_{\alpha=1}^{M^{(n,1)}} \left[ q_n(\lambda^{(n,1)}_\alpha) - \pi \right],
\]
\[
S^{(k)} = \sum_{n=1}^{\infty} n \left[ M^{(n,k-1)} + M^{(n,k+1)} - 2M^{(n,k)} \right].
\]

(2.30)
As already noted, $M^{(n,0)} = N \delta_{n,1}$, $M^{(n,N)} = 0$.

By invoking the string hypothesis, we have transformed the problem of finding complex solutions of the BAE (2.17) to the simpler problem of finding real solutions of the equations (2.24). We now proceed to discuss the ground state and excitations.

### 2.3 Ground state and excitations

One can argue\(^3\) that the ground state is described by only real roots (i.e., strings of length 1) and no holes. That is, $M^{(n,j)} = 0$ for $n > 1$ and $M^{(1,j)} = 2J_{\text{max}}^{(1,j)} + 1$. Hence, $M^{(1,j)} = N(N - j)/N$. Evidently, the ground state lies in the sector where $N/N$ is an integer. Moreover, this state has all $S^{(j)} = 0$, and therefore is a singlet of $SU(N)$, as expected for an antiferromagnet. Since there are no holes, this state corresponds to a set of $N - 1$ filled Fermi seas.

Excited states are described by root distributions with holes and (optionally) complex strings (i.e., strings of length greater than 1). We let $\nu^{(j)}$ denote the number of holes in the $j^{th}$ sea,

$$
\nu^{(j)} = \left(2J_{\text{max}}^{(1,j)} + 1\right) - M^{(1,j)}. 
$$

#### Case a: no complex strings

We first consider the simpler case of excited states with holes but no complex strings. We refer to this as case $\text{a}$. For this case, we obtain from Eqs. (2.29) and (2.30) the remarkably simple relation

$$
S^{(j)} = \nu^{(j)}. 
$$

It follows from Eq. (2.27) that the Cartan generators $H_i$ have the eigenvalues

$$
H_i = \sum_{j=1}^{N-1} \mu_i^j \nu^{(j)}. 
$$

We conclude that the Bethe Ansatz state with $\nu^{(j)}$ holes in the $j^{th}$ sea and no complex strings is a highest-weight state with highest weight $\mu$ given by

$$
\mu = \sum_{j=1}^{N-1} \mu^j \nu^{(j)}, 
$$

\(^3\)One argument is based on the observation that a system at temperature $T$ goes to its ground state as $T \to 0$. The thermodynamics for the case $N = 3$ was formulated, following [21] – [23], in Ref. [24].
where \( \mu_i \) are the fundamental weights of \( SU(N) \) (see Eq. (2.13)). In the corresponding Young tableau (see Figure 2), the number of boxes in the \( i \)th row is equal to \( \sum_{j=i}^{N-1} \nu(j) \).

Figure 2: Young tableau corresponding to a general irreducible representation of \( SU(N) \)

Equivalently, the representation can be denoted by the number of boxes in each column of the Young tableau:

\[
\begin{bmatrix}
\overbrace{N-1, \ldots, N-1}^\nu(N-1) & \overbrace{N-2, \ldots, N-2}^\nu(N-2) & \ldots & \overbrace{1, \ldots, 1}^\nu(1)
\end{bmatrix}.
\]

In particular, the state with a single hole in the \( k \)th sea (i.e., \( \nu(j) = \delta_{j,k} \)) is the highest weight of the fundamental representation \([k]\), corresponding to the Young tableau shown in Figure 4.

We label the holes in the range (2.28) by \( \{ \tilde{J}^{(1,j)}_\alpha \} \), \( \alpha = 1, \ldots, \nu(j) \). The corresponding hole rapidities \( \{ \tilde{\lambda}^{(j)}_\alpha \} \) are defined by

\[
h^{(1,j)}(\tilde{\lambda}^{(j)}_\alpha) = \tilde{J}^{(1,j)}_\alpha, \quad \alpha = 1, \ldots, \nu(j),
\]

where \( h^{(1,j)}(\lambda) \) is the counting function given in Eq. (2.25) with \( n = 1 \).

For \( N \to \infty \) and for each value of \( j \), the roots \( \{ \lambda^{(1,j)}_\alpha \} \) become dense on the real line, and are described by the corresponding densities \( \sigma^{(j)}(\lambda) \) given by

\[
\sigma^{(j)}(\lambda) = \frac{1}{N} \frac{d}{d\lambda} h^{(1,j)}(\lambda).
\]

Approximating the sums in \( h^{(1,j)}(\lambda) \) by integrals using \( ^\square \)

\[
\frac{1}{N} \sum_{\alpha=1}^{\nu(j)} g(\lambda^{(1,j)}_\alpha) \approx \int_{-\infty}^{\infty} g(\lambda') \sigma^{(j)}(\lambda') \, d\lambda' - \frac{1}{N} \sum_{\alpha=1}^{\nu(j)} g(\tilde{\lambda}^{(j)}_\alpha)
\]

leads to a system of linear integral equations

\[
\sum_{m=1}^{N-1} \left( (\delta + \mathcal{K})_{jm} \ast \sigma^{(m)} \right)(\lambda) = a_1(\lambda) \delta_{j,1} + \frac{1}{N} \sum_{m=1}^{N-1} \sum_{\alpha=1}^{\nu(m)} \mathcal{K}(\lambda - \tilde{\lambda}^{(m)}_\alpha)_{jm},
\]

\[
\quad j = 1, \ldots, N-1,
\]

\( ^4 \)Here \( g(\lambda) \) is an arbitrary function of \( \lambda \) which goes to 0 for \( \lambda \to \pm \infty \).
where

\[ \mathcal{K}(\lambda)_{jm} = a_2(\lambda)\delta_{m,j} - a_1(\lambda)(\delta_{m,j-1} + \delta_{m,j+1}), \]

\[ a_n(\lambda) = \frac{1}{2\pi} \frac{dq_n(\lambda)}{d\lambda} = \frac{1}{2\pi} \frac{n}{\lambda^2 + \frac{n^2}{4}}, \]  

(2.39)

as usual * denotes the convolution

\[ (f * g)(\lambda) = \int_{-\infty}^{\infty} f(\lambda - \lambda') g(\lambda') d\lambda', \]  

(2.40)

and \( \nu^{(0)} = \nu^{(N)} = 0. \)

This system of equations is solved by Fourier transforms, for which we use the following conventions:

\[ \hat{f}(\omega) \equiv \int_{-\infty}^{\infty} e^{i\omega \lambda} f(\lambda) d\lambda, \quad f(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega \lambda} \hat{f}(\omega) d\omega, \]  

(2.41)

and therefore

\[ \hat{a}_n(\omega) = e^{-n|\omega|/2}, \quad n > 0. \]  

(2.42)

The resolvent

\[ \mathcal{R}_{mm'}(\lambda) = (\delta(\lambda) + \mathcal{K}(\lambda))^{-1}_{mm'} \]  

(2.43)

has the Fourier transform \[12\]

\[ \hat{\mathcal{R}}_{mm'}(\omega) = \frac{e^{\omega/2} \sinh (m_\omega |\omega|/2) \sinh ((N - m_\omega)|\omega|/2)}{\sinh (N|\omega|/2) \sinh (|\omega|/2)}, \]  

(2.44)

where \( m_\omega = \max(m, m') \) and \( m_\omega = \min(m, m') \). The densities \( \sigma^{(j)}(\lambda) \) are therefore given by

\[ \sigma^{(j)}(\lambda) = s^{(j)}(\lambda) + \frac{1}{N} \sum_{m=1}^{N-1} \sum_{\alpha=1}^{\mu(m)} \left[ \delta(\lambda - \bar{\lambda}_\alpha^{(m)})\delta_{j,m} - \mathcal{R}_{jm}(\lambda - \bar{\lambda}_\alpha^{(m)}) \right], \]  

(2.45)

where the ground state densities \( s^{(j)}(\lambda) \) have the Fourier transforms

\[ s^{(j)}(\omega) = \frac{\sinh ((N - j)|\omega|/2)}{\sinh (N|\omega|/2)}. \]  

(2.46)

Heuristically, the \( 1/N \) terms in Eq. (2.45) describe the “polarization” of the Fermi seas due to the presence of holes.
We remark that Eq. (2.31) can be solved for the integers \( M^{(1,j)} \) in terms of the number of holes:

\[
M^{(1,j)} = \sum_{k=1}^{N-1} \hat{R}_{jk}(0) \left( -\nu^{(k)} + N\delta_{k,1} \right).
\] (2.47)

In particular, the state with a single hole in the \( k^{th} \) sea lies in the sector where \( (N - k)/N \) is an integer. This is a generalization of the fact \([2], [16]\) that for the Heisenberg chain \( (N = 2) \), the state with one hole lies in the sector \( N = \text{odd} \).

The energy and momentum are given by

\[
E = Ne_0 + \pi \sum_{j=1}^{N-1} \sum_{\alpha=1}^{N\nu^{(j)}} s^{(j)}(\tilde{\lambda}_\alpha^{(j)}),
\]

\[
P = Np_0 + \sum_{j=1}^{N-1} \sum_{\alpha=1}^{N\nu^{(j)}} p^{(j)}(\tilde{\lambda}_\alpha^{(j)}),
\] (2.48)

where the ground state energy and momentum per site are given by

\[
e_0 = -\frac{1}{N} \left[ \psi(1) - \psi\left(\frac{1}{N}\right) \right], \quad p_0 = \pi \left(\frac{N-1}{N^2}\right),
\] (2.49)

where \( \psi(z) = \frac{d}{dz} \log \Gamma(z) \), and \( p^{(j)}(\lambda) \) satisfies

\[
\frac{d}{d\lambda} p^{(j)}(\lambda) = 2\pi s^{(j)}(\lambda), \quad p^{(j)}(0) = -\pi \left(\frac{N-j}{N}\right).
\] (2.50)

In particular, a single hole in the \( k^{th} \) sea (which, as we have seen above, is in the fundamental representation \([k]\) of \( SU(N) \)) with rapidity \( \tilde{\lambda}^{(k)} \) is a particle-like excitation with energy \( \pi s^{(k)}(\tilde{\lambda}^{(k)}) \) and momentum \( p^{(k)}(\tilde{\lambda}^{(k)}) \).

**Case b: including strings of length 2**

In section 2.4, we compute the full two-particle scattering matrix for the cases \([1] \otimes [1]\) and \([1] \otimes [N-1]\). Each of these tensor products decomposes into a direct sum of two irreducible representations \([1] \otimes [1] = [1, 1] \oplus [2]\) and \([1] \otimes [N-1] = [N-1, 1] \oplus [N]\), in accordance

\[
\square \otimes \square = \square \oplus \square
\]

Figure 3: Young tableaux corresponding to \([1] \otimes [1] = [1, 1] \oplus [2] \)
Figure 4: Young tableaux corresponding to $[1] \otimes [N-1] = [N-1,1] \oplus [N]$

with the Young tableaux in Figures 3 and 4 respectively. The calculation of the $S$ matrix eigenvalues requires the densities of the two-particle states corresponding to the irreducible representations. The two-particle Bethe Ansatz states which are highest weights of $[1,1]$ and $[N-1,1]$ are of the form described above, with only real roots and two holes. Indeed, these states have $\nu^{(j)} = 2\delta_{j,1}$ and $\nu^{(j)} = \delta_{j,1} + \delta_{j,N-1}$, respectively. However, the two-particle states which are highest weights of $[2]$ and $[N]$ are not of the above form. In order to describe these states, we need in addition to two holes also strings of length 2.

Let us consider the more general case of Bethe Ansatz states with arbitrary values of $\nu^{(j)}$ and $M^{(2,j)}$, with $M^{(n,j)} = 0$ for $n > 2$. We refer to this as case b. For this case, we obtain the following generalization of our previous result (2.32) for the eigenvalues of the Cartan generators:

$$S^{(j)} = \nu^{(j)} + M^{(2,j-1)} + M^{(2,j+1)} - 2M^{(2,j)}, \quad (2.51)$$

Moreover, in the expression for $J^{(2,j)}_{\text{max}}$ we can eliminate the dependence on $\{M^{(1,k)}\}$:

$$J^{(2,j)}_{\text{max}} = \frac{1}{2} \left[ \nu^{(j)} + M^{(2,j-1)} + M^{(2,j+1)} - M^{(2,j)} - 1 \right]. \quad (2.52)$$

The densities are now given by

$$\sigma_{b}^{(j)}(\lambda) = \sigma_{a}^{(j)}(\lambda) - \frac{1}{N} \sum_{\alpha=1}^{M^{(2,j)}} a_1(\lambda - \lambda^{(2,j)}_{\alpha}), \quad j = 1, \ldots, N-1, \quad (2.53)$$

where $\sigma_{a}^{(j)}(\lambda)$ is the density given in Eq. (2.45) for the corresponding state with the same holes but no complex strings. In order to determine the centers $\lambda^{(2,j)}_{\alpha}$ of the 2-strings in terms of the hole positions, we recall that $h^{(2,j)}(\lambda^{(2,j)}_{\alpha}) = J^{(2,j)}_{\alpha}$. Passing from sums to integrals, we obtain the relations

$$2\pi J^{(2,j)}_{\alpha} = \sum_{m=1}^{N-1} \sum_{\beta=1}^{M^{(2,m)}} \left[ -q_2(\lambda^{(2,j)}_{\alpha} - \lambda^{(2,m)}_{\beta})\delta_{m,j} + q_1(\lambda^{(2,j)}_{\alpha} - \lambda^{(2,m)}_{\beta})(\delta_{m,j-1} + \delta_{m,j+1}) \right]$$

$$+ \sum_{\beta=1}^{\nu^{(j)}} q_1(\lambda^{(2,j)}_{\alpha} - \tilde{\lambda}^{(j)}_{\beta}), \quad \alpha = 1, \ldots, M^{(2,j)}, \quad j = 1, \ldots, N-1. \quad (2.54)$$
The energy and momentum are given by the same expressions in Eq. (2.48).

We now specialize to the cases of interest. We see from Eq. (2.51) that the two-particle Bethe Ansatz state which is the highest weight of [2] has two holes in sea 1 and one 2-string in sea 1 (i.e., $\nu^{(j)} = 2\delta_{j,1}$ and $M^{(2,j)} = \delta_{j,1}$). Eq. (2.52) implies that $J^{(2,1)}_{\text{max}} = 0$ and hence $J^{(2,1)}_{1} = 0$. From the first relation in Eq. (2.54) we conclude that the center $\lambda^{(2,1)}_{1}$ of the 2-string is midway between the two holes

$$\lambda^{(2,1)}_{1} = \frac{1}{2}\left(\tilde{\lambda}^{(1)}_{1} + \tilde{\lambda}^{(1)}_{2}\right),$$

independently of the value of $N$.

Finally, we consider the singlet [N] two-particle Bethe Ansatz state. This is the state with one hole in sea 1, one hole in sea $N - 1$, and one 2-string in each of the $N - 1$ seas (i.e., $\nu^{(j)} = \delta_{j,1} + \delta_{j,N-1}$ and $M^{(2,j)} = 1$ for $j = 1, \ldots, N - 1$). We observe that each $J^{(2,j)}_{1} = 0$. Remarkably, the relations (2.54) lead to a linear system of equations for $\lambda^{(2,j)}_{1}$ whose resolvent is $\tilde{R}_{jk}(0)$. We obtain for the centers of the 2-strings the result

$$\lambda^{(2,j)}_{1} = \tilde{\lambda}^{(1)}_{1} + \frac{j}{N}\left(\tilde{\lambda}^{(N-1)}_{1} - \tilde{\lambda}^{(1)}_{1}\right), \quad j = 1, \ldots, N - 1.$$  

(2.56)

This result is represented schematically (for $\tilde{\lambda}^{(1)}_{1} > \tilde{\lambda}^{(N-1)}_{1}$) in Figure 5.

Figure 5: The singlet [N] two-particle Bethe Ansatz state. Horizontal lines represent the $N - 1$ root distributions forming the Fermi seas, circles denote holes in these seas, and crosses mark the centers of the 2-strings.
2.4 Bulk $S$ matrix

Following Refs. [25], [26], we define the two-particle $S$ matrix $S^{[j] \otimes [k]}$ for particles of type $[j]$ and $[k]$ by the momentum quantization condition

$$\left(e^{ip^{(j)}(\lambda)}S^{[j] \otimes [k]} - 1\right)|\tilde{\lambda}^{(j)}, \lambda^{(k)}\rangle = 0,$$  \hspace{1cm} (2.57)

where the single-particle momentum $p^{(j)}(\lambda)$ is given by Eq. (2.50), and $\tilde{\lambda}^{(j)}, \lambda^{(k)}$ are the corresponding hole rapidities.

We focus our attention on the cases $[1] \otimes [1]$ and $[1] \otimes [N - 1]$, for which the $S$ matrices act in $C_N \otimes C_N$ and $C_N \otimes \bar{C_N}$, respectively. As already noted, for these cases, the tensor product $[j] \otimes [k]$ decomposes into a direct sum of precisely two irreducible representations. (See Figures 3 and 4.) The Bethe Ansatz states which are highest weights of these irreducible representations belong to the cases we denoted $a$ and $b$, with densities given by Eqs. (2.45) and (2.53), respectively. Specifically, the state corresponding to the completely antisymmetric Young tableau (i.e., with only 1 column) belongs to case $b$, and the other state belongs to case $a$.

We now compute the eigenvalues of $S^{[j] \otimes [k]}$. Let $S_a$ and $S_b$ be the eigenvalues of $S^{[j] \otimes [k]}$ corresponding to states belonging to cases $a$ and $b$, respectively. The identity

$$\frac{1}{2\pi} d\frac{d}{d\lambda} p^{(j)}(\lambda) + \sigma^{(j)}(\lambda) - s^{(j)}(\lambda) = \frac{1}{N} d\frac{d}{d\lambda} h^{(1,j)}(\lambda)$$ \hspace{1cm} (2.58)

can easily be obtained from Eqs. (2.30) and (2.50). Integrating from $-\infty$ to $\tilde{\lambda}^{(j)}$ and exponentiating, we obtain the relation

$$e^{ip^{(j)}(\lambda)N} e^{i2\pi N \int_{-\infty}^{\tilde{\lambda}^{(j)}} \left(\sigma^{(j)}(\lambda) - s^{(j)}(\lambda)\right) d\lambda} e^{i2\pi (h^{(1,j)}(-\infty) - \tilde{j}^{(1,j)})} e^{-iNp^{(j)}(-\infty)} = 1,$$ \hspace{1cm} (2.59)

where $h^{(1,j)}(\tilde{\lambda}^{(j)}) = \tilde{j}^{(1,j)}$. Comparing with Eq. (2.57), we see that (up to a rapidity-independent phase factor)

$$S_a \sim \exp \left\{i2\pi N \int_{-\infty}^{\tilde{\lambda}^{(j)}} \left(\sigma_a^{(j)}(\lambda) - s^{(j)}(\lambda)\right) d\lambda\right\},$$ \hspace{1cm} (2.60)

where $\sigma_a^{(j)}(\lambda)$ is given by Eq. (2.43) with $\nu^{(m)} = \delta_{m,j} + \delta_{m,k}$. The integral can be explicitly performed using the Fourier-space expression (2.44), as well as the identity  \hspace{1cm} (2.61)

$$\int_0^\infty \frac{1 - e^{-\beta x}}{1 - e^{-x}} \frac{(1 - e^{-\gamma x}) e^{-\mu x}}{x} dx = \log \frac{\Gamma(\mu)\Gamma(\mu + \beta + \gamma)}{\Gamma(\mu + \beta)\Gamma(\mu + \gamma)};$$

The case $[N - 1] \otimes [N - 1]$ is equivalent to the case $[1] \otimes [1]$. Other cases can presumably be treated along similar lines; however, they involve higher-dimensional representations of $SU(N)$, and the corresponding $S$ matrices have more than two distinct eigenvalues.
provided \( Re \mu > 0, Re \mu > -Re \beta, Re \mu > -Re \gamma, \) and \( Re \mu > -Re (\beta + \gamma) \). One finds

\[
S_a = \prod_{l=0}^{j-1} \frac{\Gamma \left( 1 + \frac{2l+j-k}{2N} - \frac{i\lambda}{N} \right) \Gamma \left( 1 + \frac{2l+j+k}{2N} + \frac{i\lambda}{N} \right)}{\Gamma \left( 1 + \frac{2l-j-k}{2N} + \frac{i\lambda}{N} \right) \Gamma \left( 1 + \frac{2l-j+k}{2N} - \frac{i\lambda}{N} \right)}, \quad \tilde{\lambda} \equiv \tilde{\lambda}^{(j)} - \tilde{\lambda}^{(k)}. \tag{2.62}
\]

In particular, for the states \([1,1]\) and \([N-1,1]\),

\[
S_{[1,1]} = \frac{\Gamma \left( 1 - \frac{1}{N}(1+i\tilde{\lambda}) \right) \Gamma \left( 1 + \frac{i\tilde{\lambda}}{N} \right)}{\Gamma \left( 1 + \frac{1}{N}(-1 + i\tilde{\lambda}) \right) \Gamma \left( 1 - \frac{i\tilde{\lambda}}{N} \right)}, \quad \tilde{\lambda} \equiv \tilde{\lambda}^{(1)} - \tilde{\lambda}^{(1)} \tag{2.63}
\]

\[
S_{[N-1,1]} = \frac{\Gamma \left( \frac{1}{2} - \frac{i\tilde{\lambda}}{N} \right) \Gamma \left( \frac{1}{2} + \frac{i\tilde{\lambda}}{N}(-1 + i\tilde{\lambda}) \right)}{\Gamma \left( \frac{1}{2} + \frac{i\tilde{\lambda}}{N} \right) \Gamma \left( \frac{1}{2} - \frac{i\tilde{\lambda}}{N}(1 + i\tilde{\lambda}) \right)}, \quad \tilde{\lambda} \equiv \tilde{\lambda}^{(1)} - \tilde{\lambda}^{(N-1)}. \tag{2.64}
\]

Although \( S_a \) has been determined only up to a rapidity-independent phase factor, the ratio \( S_b/S_a \) can be computed exactly:

\[
\frac{S_b}{S_a} = e^{i2\pi N \int_{-\infty}^\infty \frac{\sigma_b^{(j)}(\lambda) - \sigma_a^{(j)}(\lambda)}{\sigma_b^{(j)}(\lambda)} \, d\lambda} e^{i2\pi \left( h_b^{(1,j)}(\infty) - h_a^{(1,j)}(\infty) \right) \int_{-\infty}^\infty \lambda^{(1,j)}} e^{-i2\pi \left( j_b^{(1,j)} - j_a^{(1,j)} \right)} e^{i2\pi N \int_{-\infty}^\infty \frac{\sigma_a^{(j)}(\lambda)}{\sigma_a^{(j)}(\lambda)} \, d\lambda}
\]

\[
= \prod_{\alpha=1}^{M(2,j)} e_1(\tilde{\lambda}^{(j)} - \lambda_a^{(2,j)}), \tag{2.65}
\]

where \( \sigma_b^{(j)}(\lambda) \) is given by Eq. (2.53). In particular, for the states \([2] \) and \([N] \), we find

\[
\frac{S_{[2]}}{S_{[1,1]}} = e_1 \left( \frac{1}{2} (\tilde{\lambda}_1^{(1)} - \tilde{\lambda}_2^{(1)}) \right), \tag{2.66}
\]

\[
\frac{S_{[N]}}{S_{[N-1,1]}} = e_1 \left( \frac{1}{N} (\tilde{\lambda}_1^{(1)} - \tilde{\lambda}_1^{(N-1)}) \right), \tag{2.67}
\]

where we have used our results for the centers of the 2-strings (2.54) and (2.56), respectively.

Finally, we cast our results into matrix form. For the case \([1] \otimes [1] \), \( SU(N) \) symmetry implies that the complete two-particle \( S \) matrix is given by

\[
S^{[1] \otimes [1]} = S_{[1,1]}(\tilde{\lambda}) \left( \frac{1}{2} (1 + P) + S_{[2]}(\tilde{\lambda}) \frac{1}{2} (1 - P) \right)
\]

\[
= S_{[1,1]}(\tilde{\lambda}) \frac{i\tilde{\lambda} + P}{i\tilde{\lambda} + 1}, \quad \tilde{\lambda} = \tilde{\lambda}_1^{(1)} - \tilde{\lambda}_2^{(1)}, \tag{2.68}
\]

where \( S_{[1,1]}(\tilde{\lambda}) \) is given by Eq. (2.63). Moreover, for the case \([1] \otimes [N-1] \),

\[
S^{[1] \otimes [N-1]} = S_{[N-1,1]}(\tilde{\lambda}) \left( 1 - P_{[N]} \right) + S_{[N]}(\tilde{\lambda}) P_{[N]}
\]

\[
= S_{[N-1,1]}(\tilde{\lambda}) \left( 1 - \frac{N}{i\tilde{\lambda} + \frac{N}{2} P_{[N]} \right), \quad \tilde{\lambda} = \tilde{\lambda}_1^{(1)} - \tilde{\lambda}_1^{(N-1)}, \tag{2.69}
\]
where \( P_{[N]} \) is the projector onto the one-dimensional subspace \([N]\), namely, \( P_{[N]} = \frac{1}{N} \mathbf{P}^\dagger_1 \mathbf{P}_1 \); and \( S_{[N-1, 1]}(\tilde{\lambda}) \) is given by Eq. (2.64). These results agree with those found using the bootstrap approach by Ogievetsky et al. [5] without additional CDD factors.

3 The open chain

We turn now to the open integrable chain constructed with the \( SU(N) \)-invariant \( R \) matrix (1.3) and the diagonal \( K \) matrices (1.8). Although this \( R \) matrix does not have crossing symmetry for \( N > 2 \), it does have the property [28]

\[
\left( \left( \left( R_{12}(\lambda) \right)^{t_2} \right)^{-1} \right)^{t_2} \propto M_2 \ R_{12}(\lambda + 2\rho) \ M_2^{-1},
\]

with \( M = 1 \) and \( \rho = N/2 \). One can therefore prove [8], [29] the commutativity of the transfer matrix \( t_{(l)}(\lambda, \xi \pm) \) given by

\[
t_{(l)}(\lambda, \xi \pm) = \text{tr}_0 \ K_{(l)0}^+(\lambda, \xi) \ T_0(\lambda) \ K_{(l)0}^{-}(\lambda, \xi) \ \hat{T}_0(\lambda),
\]

where \( T_0(\lambda) \) is the monodromy matrix (2.2) and \( \hat{T}_0(\lambda) \) is given by

\[
\hat{T}_0(\lambda) = R_{10}(\lambda) \cdots R_{N0}(\lambda).
\]

The Hamiltonian (1.10) is related to the derivative of the transfer matrix at \( \lambda = 0 \)

\[
\mathcal{H}_{\text{open}} = \frac{1}{4\xi_-} \text{tr} \ K_{(l)0}^+(0, \xi) \frac{d}{d\lambda} \left. t_{(l)}(\lambda, \xi \pm) \right|_{\lambda=0} - \frac{i}{4} \frac{d}{d\lambda} \left. \log \text{tr} \ K_{(l)0}^+(\lambda, \xi \pm) \right|_{\lambda=0}.
\]

3.1 Symmetries of the transfer matrix

The \( SU(N) \) invariance (1.5) of the \( R \) matrix implies that

\[
U_2 \ R_{12}(\lambda) \ U_2^\dagger = U_1^\dagger \ R_{12}(\lambda) \ U_1
\]

for all \( U \in SU(N) \). The LHS can be regarded as a “quantum-space” transformation, while the RHS can be regarded as an “auxiliary-space” transformation. The quantum-space operator \( \mathcal{U} \) defined by

\[
\mathcal{U} = U_1 \ U_2 \cdots U_N
\]

\[\text{We take into account apparent typos in their Eqs. (2.19) and (2.23).}\]

\[\text{The more general transfer matrix } t_{(l_+, l_-)}(\lambda, \xi \pm) \text{ constructed with } K_{(l)0}^{\pm}(\lambda, \xi \pm) \text{ also forms a one-parameter commutative family. For simplicity, we consider here the special case } l_+ = l_- = l.\]
therefore has the following action on the monodromy matrices
\[ \mathcal{U} T_0(\lambda) \mathcal{U}^\dagger = U_0^\dagger T_0(\lambda) U_0, \]
\[ \mathcal{U} \hat{T}_0(\lambda) \mathcal{U}^\dagger = U_0^\dagger \hat{T}_0(\lambda) U_0, \] (3.7)
and the transfer matrix transforms as follows:
\[ \mathcal{U} t_\ell(\lambda, \xi_\mp) \mathcal{U}^\dagger = \text{tr}_0 \left\{ \left( U K_\ell(\lambda, \xi_\mp) U^\dagger \right)_0 T_0(\lambda) \left( U K_\ell(\lambda, \xi_\pm) U^\dagger \right)_0 \hat{T}_0(\lambda) \right\}. \] (3.8)
For \( U \in SU(l) \times SU(N-l) \times U(1), \) evidently
\[ U K_\ell(\lambda, \xi_\mp) U^\dagger = K_\ell(\lambda, \xi_\mp), \] (3.9)
and therefore
\[ \mathcal{U} t_\ell(\lambda, \xi_\mp) \mathcal{U}^\dagger = t_\ell(\lambda, \xi_\mp). \] (3.10)
That is, the transfer matrix has the invariance \( SU(l) \times SU(N-l) \times U(1). \) In particular, it commutes with all the \( SU(N) \) Cartan generators
\[ \left[ t_\ell(\lambda, \xi_\mp), S^{(k)} \right] = 0, \quad k = 1, \ldots, N-1. \] (3.11)
The transfer matrix also has a less evident — but very useful — “duality” symmetry which maps \( l \leftrightarrow N-l. \) This symmetry originates from the simple fact that under the transformations
\[ \xi_- \rightarrow -\xi_- , \]
\[ \xi_+ \rightarrow -\xi_+ + N , \] (3.12)
the elements of the \( K \) matrices transform into each other: \( a^\mp \leftrightarrow -b^\mp. \) Therefore,
\[ K_\ell(\lambda, \xi_\mp) \rightarrow -K_\ell(\lambda, \xi_\mp), \] (3.13)
where
\[ K_\ell(\lambda, \xi_\mp) = \text{diag} \left( b^\mp_1, \ldots, b^\mp_l, a^\mp_{l+1}, \ldots, a^\mp_{N-l} \right). \] (3.14)
Notice that \( K_\ell(\lambda, \xi_\mp) \) and \( K_{(N-l)}(\lambda, \xi_\mp) \) are related by a cyclic permutation. Thus, there exist matrices \( U_\ell \in SU(N) \) such that
\[ U_\ell^k K_\ell^k(\lambda, \xi_\mp) U_\ell^\dagger = K_{(N-l)}^k(\lambda, \xi_\mp) = -K_\ell^k(\lambda, \xi_\mp), \] (3.16)
\[ U_{(j,k)} \text{diag}(d_1, \ldots, d_j, \ldots, d_k, \ldots, d_N) U_{(j,k)}^\dagger = \text{diag}(d_1, \ldots, d_k, \ldots, d_j, \ldots, d_N). \] (3.15)
We choose the matrices \( U_\ell \) to be products of matrices of the type \( U_{(j,k)}. \) Note that Eq. (3.16) does not uniquely determine \( U_\ell. \)
where
\[
\begin{align*}
\xi_- &= -\xi_-, \\
\xi_+ &= -\xi_+ + \mathcal{N}, \\
l' &= \mathcal{N} - l.
\end{align*}
\] (3.17)

Correspondingly, we obtain the desired “duality” transformation property of the transfer matrix
\[
U(t) \ t(t)(\lambda, \xi_\mp) \ U(t) = t(t')(\lambda, \xi'_\mp),
\] (3.18)
where
\[
U(t) = U(t) \cdot U(t) \cdots U(t) \cdot \mathcal{N}.
\]

Notice that the square of this transformation is the identity. The transfer matrix is “self-dual” for \(\xi_- = \xi'_- = 0\), \(\xi_+ = \xi'_+ = \mathcal{N}/2\), and \(l = l' = \mathcal{N}/2\).

### 3.2 Dual pseudovacuum

The algebraic Bethe Ansatz is usually implemented with the pseudovacuum \(\omega(1)\), where
\[
\omega(k) = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \otimes^N \leftarrow k^{th}.
\] (3.19)

We shall also make use of Bethe Ansatz states constructed with the “dual” pseudovacuum,\(^9\)
\[
\omega' = U(t) \cdot \omega(1) = \omega(\mathcal{N})
\] (3.20)
in order to compute boundary \(S\) matrices. We can obtain the corresponding Bethe Ansatz equations (BAE) with the help of the following

**Lemma** Let \(g\) be a transformation on the parameters \(l\) and \(\xi_\mp\) of the boundary \(K\) matrices,
\[
g : \begin{cases} 
l \rightarrow l' \\
\xi_\mp \rightarrow \xi'_\mp
\end{cases}
\] (3.21)

\(^9\)We use here the matrix \(U(t)\) described in the previous footnote. A different choice of \(U(t)\) can lead to a different “dual” pseudovacuum. Nevertheless, the results given in Section 3.4 for the boundary \(S\) matrices do not depend on this choice.
which squares to the identity, i.e., \( g^2 = 1 \). Furthermore, let \( U(l) \) be a unitary transformation on \( C_N^\otimes N \) such that

\[
U(l) \ t(l)(\lambda , \xi_\pm) \ U(l)^\dagger = t(l')(\lambda , \xi'_\pm) \tag{3.22}
\]

\[
U(l) \ \omega = \omega'. \tag{3.23}
\]

Then the BAE for the transfer matrix \( t(l)(\lambda , \xi_\pm) \) with the pseudovacuum \( \omega' \) are the same as the BAE for the transfer matrix \( t(l')(\lambda , \xi'_\pm) \) with the pseudovacuum \( \omega \).

Before giving our general proof, it is instructive to examine a more explicit proof for the special case \( N = 2 \), which was first considered by Sklyanin \[8\]. For this case \( l = l' = 1 \), and we therefore suppress the label \( l \) and write the transfer matrix as \( t(\lambda , \xi_\pm) \). Moreover, we take \( U \) as in Eq. (3.6), with

\[
U = \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}. \tag{3.24}
\]

Then condition (3.22) is satisfied, with \( \xi'_\pm \) as in Eq. (3.17). We consider the pseudovacuum \( \omega = \omega(1) \), and thus \( \omega' = U \omega(1) = \omega(2) \). The algebraic Bethe Ansatz leads to the fundamental result

\[
t(\lambda , \xi_\pm) \ |\{\lambda_\alpha\}, \xi_-\rangle = \Lambda (\lambda , \{\lambda_\alpha\}, \xi_\pm) \ |\{\lambda_\alpha\}, \xi_-\rangle, \tag{3.25}
\]

where

\[
|\{\lambda_\alpha\}, \xi_-\rangle = B(\lambda_1 , \xi_-) \cdots B(\lambda_M , \xi_-) \ \omega, \tag{3.26}
\]

and \( \{\lambda_\alpha\} \) are solutions of the BAE with pseudovacuum \( \omega \). Multiplying both sides of Eq. (3.23) with \( U \), and using condition (3.22) as well as the fact \( U \ B(\lambda , \xi_-) \ U^\dagger = C(\lambda , \xi'_-), \) we see that

\[
t(\lambda , \xi'_\pm) \ |\{\lambda_\alpha\}, \xi'_-\rangle' = \Lambda (\lambda , \{\lambda_\alpha\}, \xi'_\pm) \ |\{\lambda_\alpha\}, \xi'_-\rangle', \tag{3.27}
\]

where

\[
|\{\lambda_\alpha\}, \xi'_-\rangle' = C(\lambda_1 , \xi_-) \cdots C(\lambda_M , \xi_-) \ \omega'. \tag{3.28}
\]

Changing \( \xi_\pm \to \xi'_\pm \) in Eq. (3.27), we obtain

\[
t(\lambda , \xi_\pm) \ |\{\lambda'_\alpha\}, \xi_-\rangle' = \Lambda \left(\lambda , \{\lambda'_\alpha\}, \xi'_\pm\right) \ |\{\lambda'_\alpha\}, \xi_-\rangle', \tag{3.29}
\]

where \( \{\lambda'_\alpha\} \) satisfy the same BAE as \( \{\lambda_\alpha\} \), except with \( \xi_\pm \to \xi'_\pm \).
We consider now the general case. The nested algebraic Bethe Ansatz leads to the result
\[ t_l(\lambda, \xi_\mp) \left| \right. = \Lambda_l(\lambda, \xi_\mp) \left| \right. , \] (3.30)
where the eigenstate \( \left| \right. \) is constructed with the pseudovacuum \( \omega \). Multiplying both sides by \( U_l \) and using condition (3.22) gives
\[ t_{l'}(\lambda, \xi_\mp') U_l \left| \right. = \Lambda_l(\lambda, \xi_\mp) U_l \left| \right. . \] (3.31)
Changing \( \xi_\mp \rightarrow \xi_\mp' \) and \( l \rightarrow l' \), we obtain
\[ t_l(\lambda, \xi_\mp) \left| \right. ' = \Lambda_l(\lambda, \xi_\mp') \left| \right. ', \] (3.32)
where \( \left| \right. ' \) is constructed with the pseudovacuum \( \omega' = U_l \omega \). Notice that the eigenvalue of the transfer matrix is \( \Lambda(l')(\lambda, \xi_\mp') \). Recalling (see, e.g., [6]) that the BAE are precisely the conditions that the eigenvalues have vanishing residues, we conclude that the BAE corresponding to the pseudovacuum \( \omega' \) are the same as the BAE corresponding to the pseudovacuum \( \omega \), except with \( \xi_\mp \rightarrow \xi_\mp' \) and \( l \rightarrow l' \). This concludes our proof of the Lemma.

### 3.3 Bethe Ansatz and multihole states

The eigenstates of the transfer matrix \( t_l(\lambda, \xi_\mp) \) with the pseudovacuum \( \omega(l) \) have been constructed in Refs. [14], [30]. The corresponding Bethe Ansatz equations (BAE) are given by \(^{10}\)
\[ 1 = \left[ e^{2\xi_+} e^{(l)}(\lambda_\alpha - \lambda_\alpha^{(k-1)}) e^{-2\xi_+} e^{(l)}(\lambda_\alpha + \lambda_\alpha^{(k-1)}) \prod_{\beta=1}^{M^{(k-1)}} e^{-1} e^{2\xi_+} e^{(l)}(\lambda_\alpha - \lambda_\alpha^{(k)}) e^{-2\xi_+} e^{(l)}(\lambda_\alpha + \lambda_\alpha^{(k)}) \right] \]
\[ \times \prod_{\beta=1}^{M^{(k+1)}} e^{-1} e^{2\xi_+} e^{(l)}(\lambda_\alpha - \lambda_\alpha^{(k+1)}) e^{-2\xi_+} e^{(l)}(\lambda_\alpha + \lambda_\alpha^{(k+1)}) \]
\[ \alpha = 1, \ldots , M^{(k)} , \quad k = 1, \ldots , N - 1 . \] (3.33)

As before, \( M^{(0)} = N \), \( M^{(N)} = 0 \), \( \lambda_\alpha^{(0)} = \lambda_\alpha^{(N)} = 0 \). The requirement that solutions of the BAE correspond to independent Bethe Ansatz states leads to the restriction \( \lambda_\alpha^{(k)} > 0 \). For later convenience, we restrict \( \xi_+ > \frac{1}{2}(N - 1) \), \( \xi_- > N - \frac{1}{2} \). (See Eqs. (3.42) and (3.43) below.)

\(^{10}\)Starting from Eq. (17) in [14], we make a shift of variables \( \mu_j^{(k)} \rightarrow \mu_j^{(k)} - k\eta_2 \); and we then take the isotropic limit by making the redefinitions \( \gamma = i\eta, \xi_\mp \rightarrow i\eta\xi_\mp, \mu_j^{(k)} = \eta \lambda_j^{(k)} \), and then letting \( \eta \rightarrow 0 \).
Since we need to consider only one-particle states in order to compute boundary $S$ matrices, we restrict our attention here to real solutions of the BAE, i.e., no complex strings. In terms of the counting functions $h_{(l)}^{(k)}(\lambda)$ defined by

$$
h_{(l)}^{(k)}(\lambda) = \frac{1}{2\pi} \{ q_{l}(\lambda) + \left[ -q_{2l-1}(\lambda) + q_{2l+2N+l}(\lambda) \right] \delta_{k,l} 
+ \sum_{\beta=1}^{M^{(k-1)}} \left[ q_{l}(\lambda - \lambda_{\beta}^{(k-1)}) + q_{l}(\lambda + \lambda_{\beta}^{(k-1)}) \right] 
- \sum_{\beta=1}^{M^{(k)}} \left[ q_{2l}(\lambda - \lambda_{\beta}^{(k)}) + q_{2l}(\lambda + \lambda_{\beta}^{(k)}) \right] 
+ \sum_{\beta=1}^{M^{(k+1)}} \left[ q_{l}(\lambda - \lambda_{\beta}^{(k+1)}) + q_{l}(\lambda + \lambda_{\beta}^{(k+1)}) \right] \},
$$

the BAE take the form

$$
h_{(l)}^{(k)}(\lambda_{\alpha}^{(k)}) = J_{\alpha}^{(k)}, \quad \alpha = 1, \ldots, M^{(k)}, \quad k = 1, \ldots, N - 1.
$$

(3.35)

Although ultimately we focus on one-hole states, it is convenient to first consider the more general case of multihole states. As in the previous section, we let $\nu^{(j)}$ denote the number of holes in the $j^{th}$ sea, and we define the hole rapidities $\{\tilde{\lambda}_{\alpha}^{(j)}\}$ by

$$
h_{(l)}^{(j)}(\tilde{\lambda}_{\alpha}^{(j)}) = J_{\alpha}^{(j)}, \quad \alpha = 1, \ldots, \nu^{(j)}.
$$

(3.36)

In the thermodynamic limit, the roots are described by densities $\sigma_{(l)}^{(j)}(\lambda)$ given by

$$
\sigma_{(l)}^{(j)}(\lambda) = \frac{1}{N} \frac{d}{d\lambda} h_{(l)}^{(j)}(\lambda).
$$

(3.37)

The sums in $h_{(l)}^{(j)}(\lambda)$ can be approximated by integrals using (see, e.g., [15])

$$
\frac{1}{N} \sum_{\alpha=1}^{M^{(j)}} g(\tilde{\lambda}_{\alpha}^{(j)}) \approx \int_{0}^{\infty} g(\lambda') \sigma_{(l)}^{(j)}(\lambda') d\lambda' - \frac{1}{N} \sum_{\alpha=1}^{\nu^{(j)}} g(\tilde{\lambda}_{\alpha}^{(j)}) - \frac{1}{2N} g(0).
$$

(3.38)

For the symmetric density $\sigma_{(l)}^{(j)}_{s}(\lambda)$ defined by

$$
\sigma_{(l)}^{(j)}_{s}(\lambda) = \left\{ \begin{array}{ll}
\sigma_{(l)}^{(j)}(\lambda) & \lambda > 0 \\
\sigma_{(l)}^{(j)}(-\lambda) & \lambda < 0
\end{array} \right.
$$

(3.39)

we obtain the system of linear integral equations

$$
\sum_{m=1}^{N-1} \left( (\delta + \mathcal{K})_{jm} * \sigma_{(l)}^{(m)}_{s} \right)(\lambda) = 2a_{1}(\lambda)\delta_{j,1} + \frac{1}{N} \left( a_{2}(\lambda) + a_{1}(\lambda)(-1 + \delta_{j,1} + \delta_{j,N-1}) \right) 
+ \left( -a_{2\xi_{-1}}(\lambda) + a_{2\xi_{-2N+1}}(\lambda) \right) \delta_{j,1} + \sum_{m=1}^{N-1} \sum_{a=1}^{N-1} \left( \mathcal{K}(\lambda - \tilde{\lambda}_{a}^{(m)})_{jm} + \mathcal{K}(\lambda + \tilde{\lambda}_{a}^{(m)})_{jm} \right),
$$

\begin{align*}
& j = 1, \ldots, N - 1,
\end{align*}

(3.40)
where $K(\lambda)_{jm}$ is defined in Eq. (2.39). The solution is given by

$$\sigma^{(j)}(\lambda) = 2s^{(j)}(\lambda) + \delta\sigma^{(j)}(\lambda) + \frac{1}{N} \left\{ \sum_{m=1}^{N-1} (R_{jm} \ast [a_2 + a_1 (-1 + \delta_{m,1} + \delta_{m,N-1})]) (\lambda) \right. \\
+ \sum_{m=1}^{N-1} \sum_{\nu^{(m)}} \left[ \delta(\lambda - \tilde{\lambda}^{(m)}_{\alpha}) \delta_{j,m} - R_{jm}(\lambda - \tilde{\lambda}^{(m)}_{\alpha}) + (\tilde{\lambda}^{(m)}_{\alpha} \rightarrow -\tilde{\lambda}^{(m)}_{\alpha}) \right] \right\}, \quad (3.41)$$

where the quantity $\delta\sigma^{(j)}(\lambda)$ defined by

$$\delta\sigma^{(j)}(\lambda) = \frac{1}{N} \left( R_{j,l} \ast \left( -a_2\xi_{-l} + a_2\xi_{-2N+l} \right) \right) (\lambda) \quad (3.42)$$

has the dependence on the boundary parameters $\xi_{\mp}$, and $R_{jm}(\lambda)$ is the resolvent, which has the Fourier transform (2.44).

We shall also need the densities $\sigma^{(j)}_{(l)}(\lambda)$ corresponding to the dual pseudovacuum $\omega'$ given by Eq. (3.20) in order to calculate boundary $S$ matrices. According to the Lemma, the BAE with the pseudovacuum $\omega'$ are given by Eq. (3.33), except with $\xi_{\pm} \rightarrow \xi'_{\mp}$ and $l \rightarrow l'$. It follows that the corresponding densities $\sigma^{(j)}_{(l)}(\lambda)$ are given by Eq. (3.41), except with

$$\delta\sigma^{(j)}_{(l)}(\lambda) = \frac{1}{N} \left( R_{j,N-l} \ast \left( a_2\xi_{-N+l} - a_2\xi_{-2N+l} \right) \right) (\lambda) . \quad (3.43)$$

As previously noted, the $K$ matrices in the transfer matrix $t_{(l)}(\lambda, \xi_{\pm})$ break the bulk $SU(N)$ symmetry. Hence, strictly speaking, states should be classified according to the unbroken symmetry $SU(l) \times SU(N-l) \times U(1)$. However, we expect that at points of the chain that are far from the boundary, the effects of the boundary should be “small”. In particular, in the bulk, multiparticle states should “approximately” form irreducible representations of $SU(N)$, as discussed for the closed chain in Section 2. Therefore, we shall continue to classify bulk multiparticle states by $SU(N)$ quantum numbers.

Consider now the special case of one-particle states of type [1], which form an $N$-dimensional representation. For the Bethe Ansatz state constructed with the pseudovacuum $\omega_{(k)}$ having one hole in sea 1, the Cartan generators have the eigenvalues $^{11}$

$$S^{(j)} = \delta_{j,k} - \delta_{j,k-1} . \quad (3.44)$$

This state is represented by the vector

$$\begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow k^{th} , \quad (3.45)$$

$^{11}$In particular, $S^{(j)} = \delta_{j,1}$ for $k = 1$, which is consistent with Eq. (2.32).
which is the eigenvector of the matrices \( \{ s^{(j)} \} \) given in Eq. (2.7) with the eigenvalues (3.44).

Similarly, the one-particle states of type \([ \mathcal{N} - 1 ] = [ \overline{1}] \) also form an \( \mathcal{N} \)-dimensional representation. The Bethe Ansatz state constructed with the pseudovacuum \( \omega_{(k)} \) having one hole in sea \( \mathcal{N} - 1 \) has the eigenvalues

\[
S^{(j)} = \delta_{j, \mathcal{N} - k} - \delta_{j, \mathcal{N} + 1 - k}.
\]

This state is represented by the vector

\[
\begin{pmatrix}
0 \\
\vdots \\
1 \\
\vdots \\
0
\end{pmatrix} \leftrightarrow (\mathcal{N} + 1 - k)^{th},
\]

keeping in mind that the Cartan generators are now represented by the matrices \( \{-s^{(j)*}\} \). The highest weight of \([ \overline{1}] \) is the negative of the lowest weight of \([1]\).

### 3.4 Boundary \( S \) matrices

We define the boundary \( S \) matrices \( S^{\pm}_{(l)} [j] \) for a particle of type \([j]\), in analogy with the bulk \( S \) matrix, by the quantization condition \( [15], [16] \)

\[
\left( e^{i2p^{(j)}(\tilde{\lambda})_{j}N} S^{+}_{(l)} [j] S^{-}_{(l)} [j] - 1 \right) |\tilde{\lambda}^{(j)}\rangle = 0,
\]

where \( p^{(j)}(\lambda) \) is defined by Eq. (2.50), and \( \tilde{\lambda}^{(j)} \) is the hole rapidity. There is an identity for the open chain which is analogous to the one given in Eq. (2.58) for the closed chain:

\[
\frac{1}{\pi} \frac{d}{d\lambda} p^{(j)}(\lambda) + \sigma^{(j)}_{(l)}(\lambda) - 2s^{(j)}(\lambda) = \frac{1}{\mathcal{N}} \frac{d}{d\lambda} \lambda^{(j)}(\lambda)
\]

For simplicity, we focus our attention on the cases \([1]\) and \([\mathcal{N} - 1] = [\overline{1}]\), for which the \( S \) matrices act in \( \mathbb{C}^{\mathcal{N}} \) and \( \mathbb{C}^{\mathcal{N}} \), respectively.

We first treat the case \([1]\). The \( SU(l) \times SU(\mathcal{N} - l) \times U(1) \) invariance of the transfer matrix implies that \( S^{\pm}_{(l)} [1] \) are diagonal \( \mathcal{N} \times \mathcal{N} \) matrices of the form

\[
S^{\pm}_{(l)} [1] = \text{diag} \left( \alpha^{\pm}_{(l)}, \ldots, \alpha^{\pm}_{(l)}, \beta^{\pm}_{(l)}, \ldots, \beta^{\pm}_{(l)} \right).
\]

Choosing the state \( |\tilde{\lambda}^{(1)}\rangle \) in Eq. (3.48) to be the Bethe Ansatz state constructed with the pseudovacuum \( \omega_{(1)} \) having one hole in sea 1, we see that (up to a rapidity-independent phase
and with the help of the identity (2.61) and the duplication formula for the gamma function

\[ \delta_{\sigma} \]

we find

\[ \int_{0}^{\hat{\lambda}(1)} \left[ R(\lambda - \hat{\lambda}(1)) + R(\lambda + \hat{\lambda}(1)) \right] d\lambda = \int_{0}^{\hat{\lambda}(1)} 2R(2\lambda) d\lambda, \quad (3.52) \]

and with the help of the identity (2.61) and the duplication formula for the gamma function

\[ 2^{2x-1} \Gamma(x) \Gamma \left( x + \frac{1}{2} \right) = \pi^{\frac{1}{2}} \Gamma(2x), \quad (3.53) \]

we find

\[ \alpha^{-}_{(l)} = S_{0}(\hat{\lambda}(1)) \frac{\Gamma \left( \frac{1}{N} \left( \xi - l - \frac{1}{2} + i\hat{\lambda}(1) \right) \right) \Gamma \left( \frac{1}{N} \left( \xi + N - \frac{1}{2} - i\hat{\lambda}(1) \right) \right)}{\Gamma \left( \frac{1}{N} \left( \xi - l - \frac{1}{2} - i\hat{\lambda}(1) \right) \right) \Gamma \left( \frac{1}{N} \left( \xi + N - \frac{1}{2} + i\hat{\lambda}(1) \right) \right)}, \]

\[ \alpha^{+}_{(l)} = S_{0}(\hat{\lambda}(1)) \frac{\Gamma \left( \frac{1}{N} \left( \xi + N + l - \frac{1}{2} - i\hat{\lambda}(1) \right) \right) \Gamma \left( \frac{1}{N} \left( \xi + \frac{1}{2} + i\hat{\lambda}(1) \right) \right)}{\Gamma \left( \frac{1}{N} \left( \xi + N + l - \frac{1}{2} + i\hat{\lambda}(1) \right) \right) \Gamma \left( \frac{1}{N} \left( \xi + \frac{1}{2} - i\hat{\lambda}(1) \right) \right)}, \quad (3.54) \]

where the prefactor \( S_{0}(\hat{\lambda}) \) is given by

\[ S_{0}(\hat{\lambda}) = \frac{\Gamma \left( \frac{1}{N} \left( \frac{1}{2}(N - 1) - i\hat{\lambda} \right) \right) \Gamma \left( \frac{1}{N} \left( N + i\hat{\lambda} \right) \right)}{\Gamma \left( \frac{1}{N} \left( \frac{1}{2}(N - 1) + i\hat{\lambda} \right) \right) \Gamma \left( \frac{1}{N} \left( N - i\hat{\lambda} \right) \right)}. \quad (3.55) \]

Moreover, choosing the state \( |\hat{\lambda}(1)\rangle \) in Eq. (3.48) to be the Bethe Ansatz state constructed with the dual pseudovacuum \( \omega' = \omega(\mathcal{N}) \) having one hole in sea 1, we obtain the relation

\[ \frac{\beta^{+}_{(l)} \beta^{-}_{(l)}}{\alpha^{+}_{(l)} \alpha^{-}_{(l)}} = \exp \left\{ i2\pi N \int_{0}^{\hat{\lambda}(1)} \left( \sigma^{(1)}_{(l)}(\lambda) - \sigma^{(1)}_{(l)}(\lambda) \right) d\lambda \right\}. \quad (3.56) \]

Note that

\[ \sigma^{(1)}_{(l)}(\lambda) - \sigma^{(1)}_{(l)}(\lambda) = \delta\sigma^{(1)}_{(l)}(\lambda) - \delta\sigma^{(1)}_{(l)}(\lambda) \]

\[ = \frac{1}{\mathcal{N}} \left( a_{2\xi_{-1}}(\lambda) - a_{2\xi_{-2N+2l-1}}(\lambda) \right), \quad (3.57) \]

where \( \delta\sigma^{(1)}_{(l)} \) and \( \delta\sigma^{(1)}_{(l)} \) are given by Eqs. (3.42) and (3.43), respectively. We conclude

\[ \frac{\alpha^{-}_{(l)}}{\beta^{+}_{(l)}} = -e_{2\xi_{-1}}(\hat{\lambda}(1)), \]

\[ \frac{\beta^{+}_{(l)}}{\alpha^{-}_{(l)}} = -e_{2\xi_{-2N+2l-1}}(\hat{\lambda}(1)), \quad (3.58) \]
where we have resolved the sign ambiguity by demanding that the $S$ matrix be proportional to the unit matrix for $\lambda^{(1)} = 0$.

Finally, we consider the case $[N-1]$. The boundary $S$ matrices $S^{\pm}_{(l)}[N-1]$ are diagonal $N \times N$ matrices of the form

$$S^{\pm}_{(l)}[N-1] = \text{diag}(\tilde{\alpha}^{\pm}_{(l)}, \ldots, \tilde{\beta}^{\pm}_{(l)}, \ldots, \tilde{\beta}^{\pm}_{(N-l)}).$$ (3.59)

For this case we must consider one hole in sea $N-1$. Noting that

$$\tilde{\beta}^{+}_{(l)} \tilde{\beta}^{-}_{(l)} \sim \exp \left\{ i2\pi N \int_{0}^{\tilde{\lambda}^{(N-1)}} \left( \sigma^{(N-1)}_{(l)}(\lambda) - 2s^{(N-1)}(\lambda) \right) d\lambda \right\},$$ (3.60)

(see Eq. (3.47)), we obtain

\[ \begin{align*}
\tilde{\beta}^{-}_{(l)} &= S_{0}(\tilde{\lambda}^{(N-1)}) \frac{1}{\Gamma\left(\frac{1}{N} \left( \xi_{-} + l + \frac{1}{2}(N-1) - i\tilde{\lambda}^{(N-1)} \right) \right)} \frac{1}{\Gamma\left(\frac{1}{N} \left( \xi_{-} + l + \frac{1}{2}(N-1) + i\tilde{\lambda}^{(N-1)} \right) \right)} \frac{1}{\Gamma\left(\frac{1}{N} \left( \xi_{-} + \frac{1}{2}(N-1) - i\tilde{\lambda}^{(N-1)} \right) \right)} \\
\tilde{\beta}^{+}_{(l)} &= S_{0}(\tilde{\lambda}^{(N-1)}) \frac{1}{\Gamma\left(\frac{1}{N} \left( \xi_{+} + l + \frac{1}{2}(N+1) + i\tilde{\lambda}^{(N-1)} \right) \right)} \frac{1}{\Gamma\left(\frac{1}{N} \left( \xi_{+} + l + \frac{1}{2}(N+1) - i\tilde{\lambda}^{(N-1)} \right) \right)} \frac{1}{\Gamma\left(\frac{1}{N} \left( \xi_{+} - \frac{1}{2}(N+1) - i\tilde{\lambda}^{(N-1)} \right) \right)}.
\end{align*} \] (3.61)

where $S_{0}(\tilde{\lambda})$ is given in Eq. (3.35). Moreover,

$$\frac{\tilde{\beta}^{-}_{(l)}}{\tilde{\alpha}^{-}_{(l)}} = -e^{2\xi_{-} + 2l - N-1}(\tilde{\lambda}^{(N-1)}),$$

$$\frac{\tilde{\alpha}^{+}_{(l)}}{\tilde{\beta}^{+}_{(l)}} = -e^{2\xi_{+} - N-1}(\tilde{\lambda}^{(N-1)}).$$ (3.62)

### 4 Discussion

We have shown how to describe general multiparticle states of the antiferromagnetic $SU(N)$ chain, in particular their $SU(N)$ quantum numbers, within the framework of the Bethe Ansatz/string hypothesis. The picture which emerges is a rich generalization of the $N=2$ case [2, 3]. The ubiquitous appearance of the kernel $(1 + \tilde{K}(\omega))_{jm}$, which is characterized (see, e.g., [20]) by the $SU(N)$ Dynkin diagram, is noteworthy. Moreover, we have computed both bulk and boundary scattering matrices for particles of types $[1]$ and $[N-1]$. It should be possible to extend this analysis to particles of any type $[k]$.

We have identified a “duality” symmetry of the open chain transfer matrix with diagonal boundary fields, which plays an important role in our computation of boundary $S$ matrices. It
may be interesting to investigate further the “self-dual” case. We expect that this symmetry is also present for the boundary $A^{(1)}_{N-1}$ Toda theories with diagonal boundary fields. Whether such symmetries persist for nondiagonal boundary fields is also an interesting question.

The “mixed” boundary condition case $l_+ \neq l_-$ merits further investigation. One expects that the boundary $S$ matrix for one end of the chain should be independent of the boundary conditions at the other end. However, for this case, the unbroken symmetry group of the transfer matrix is smaller, and therefore, the arguments presented here require further refinement.

Since the groups $SO(2N)$ and $E_N$, like $SU(N)$, have simply-laced Lie algebras, it should be possible to treat the corresponding integrable chains in a similar fashion. Finally, we note that the boundary $S$ matrix calculations presented here can be generalized \cite{31} to the trigonometric case, i.e., the open chain constructed with the $A^{(1)}_{N-1}$ $R$ and $K$ matrices.

5 Acknowledgments

We are grateful to O. Alvarez and L. Mezincescu for valuable discussions. This work was supported in part by the National Science Foundation under Grant PHY-9507829.

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