APPROXIMATION OF SHEAVES ON ALGEBRAIC STACKS

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Abstract. Raynaud–Gruson characterized flat and pure morphisms between affine schemes in terms of projective modules. We give a similar characterization for non-affine morphisms. As an application, we show that every quasi-coherent sheaf is the union of its finitely generated quasi-coherent subsheaves on any quasi-compact and quasi-separated algebraic stack.

Introduction

It is well-known that on a noetherian scheme every quasi-coherent sheaf is the union of its coherent subsheaves [EGA] Cor. 9.4.9]. This is also true for noetherian algebraic stacks [LMB] Prop. 15.4].

For a non-noetherian scheme or algebraic stack $X$, this question splits up into two questions.

(i) Is every quasi-coherent $\mathcal{O}_X$-module the union of its quasi-coherent submodules of finite type?

(ii) Is every quasi-coherent $\mathcal{O}_X$-module a directed colimit of finitely presented $\mathcal{O}_X$-modules?

When these questions have positive answers, we say that $X$ has the partial completeness property and completeness property respectively. The second property implies the first (take images).

It is known that quasi-compact and quasi-separated schemes have the completeness property [EGA, §6.9]. In [Ryd] Thm. A] it was shown that many stacks, including quasi-compact and quasi-separated algebraic spaces and Deligne–Mumford stacks, have the completeness property. With current technology, this result only applies to relatively few algebraic stacks with infinite stabilizer groups.

The main result of this paper settles the partial completeness property for every reasonable stack.

Theorem. Let $X$ be a quasi-compact and quasi-separated algebraic stack. Then every quasi-coherent $\mathcal{O}_X$-module is the union of its quasi-coherent submodules of finite type.

An important application of the theorem is that when $X$ in addition has affine stabilizer groups, then there exists a finitely presented filtration of $X$ with strata that are global quotient stacks [HR] Prop. 2.6 (i)].
This is used to obtain a criterion for an algebraic stack to have finite coho-
mological dimension [HR14a, Thm. 2.1] and to extend Tannaka duality to
non-noetherian stacks [HR14b, Thm. 1.3].

The key idea of the proof of the theorem is to use projective modules
instead of flat modules. The main lemma (5.2) on existence of minimal
modules goes back to Serre [SGA3, Exp. VIB, 11.8, 11.10.1] in the context
of coalgebras and comodules. Here projectivity cannot be replaced with
flatness. The bulk of the paper is used to extend this result to non-affine
pure morphisms (Theorem 5.3). For this we use a new characterization of
pure morphisms between stacks in terms of projectivity (Theorem 4.3). This
generalizes the characterization of affine pure morphisms due to Raynaud–
Gruson [RG71, Thm. I.3.3.5].

The main result naturally leads to the following conjectures.

**Conjecture A.** If \( X \) is a quasi-compact and quasi-separated algebraic stack,
then \( X \) has the completeness property.

**Conjecture B.** If \( X \) is a quasi-compact and quasi-separated algebraic stack,
then \( X \) has an approximation, that is, there exists a factorization \( X \to X_0 \to \text{Spec} \mathbb{Z} \)
where \( X \to X_0 \) is affine and \( X_0 \) is of finite presentation over \( \text{Spec} \mathbb{Z} \).

The second conjecture implies the first conjecture. The techniques of this
paper can be used to reduce the conjectures to a seemingly simpler situation
(see Remark 6.6).

In the first three sections, we recall and extend some notions from schemes
to algebraic stacks. This includes (1) locally free and locally projective mod-
ules, (2) assassins and schematically dominant morphisms, and (3) pure mor-
phisms. In the fourth section, we give a characterization of pure morphisms
in terms of projectivity (Theorem 4.3). In the fifth section, we prove the
existence of minimal subsheaves for pure morphisms. In the sixth section,
we prove the main theorem. In the last section, we give some applications
to the main theorem.

We follow the terminology of [SP] and do not impose any separation
conditions on a general algebraic stack. An algebraic stack is *quasi-separated*
if its diagonal is quasi-compact and quasi-separated, that is, if the diagonal
and the double diagonal are quasi-compact.

**Acknowledgments.** It is my pleasure to acknowledge useful discussions
with Jack Hall and useful comments from Martin Brandenburg.

1. **Locally free and locally projective modules**

In this section, we recall some standard results on infinitely generated
projective modules due to Kaplansky, Bass and Raynaud–Gruson.

**Definition (1.1).** Let \( X \) be an algebraic stack. We say that a quasi-
coherent sheaf \( \mathcal{F} \) is *locally free* (resp. *locally projective*) if there exists a
jointly surjective family of flat morphisms \( p_i: \text{Spec} A_i \to X \), locally of finite
presentation, such that \( p_i^* \mathcal{F} \) is free (resp. projective) for every \( i \).
We do not require that $p^*_x \mathcal{F}$ has finite rank in the definition of locally free. Note that the properties locally free and locally projective are stable under arbitrary pull-back and are local for the fppf-topology. We have the implications: locally free $\implies$ locally projective $\implies$ flat.

If $x \in |X|$ is a point, then we define the rank $\text{rk}_F(x)$ of $\mathcal{F}$ at $x$ as the cardinality of a basis of the $k$-vector space $\varphi^* \mathcal{F}$ for any representative $\varphi: \text{Spec} \ k \to X$ of $x$. Since flat morphisms that are locally of finite presentation are open, the rank of $\mathcal{F}$ is locally constant on $|X|$ if $\mathcal{F}$ is locally free.

The rank does not behave so well for flat modules that are not finitely generated. If $A = \mathbb{Z}$ and $M = \mathbb{Q}$, then the rank of $M$ is not upper semicontinuous. The rank of projective modules is more well-behaved.

**Lemma (1.2)** (Kaplansky [Kap58]). If $A$ is a local ring, then every projective $A$-module is free.

Thus, if $X$ is a (quasi-separated) algebraic stack and $\mathcal{F}$ is a locally projective $\mathcal{O}_X$-module, then

(i) the rank of $\mathcal{F}$ is constant on irreducible components of $X$; and

(ii) if $X$ has a finite number of irreducible components (e.g., $X$ noetherian), then the rank is locally constant.

Nevertheless, even if $M$ is projective and has finite rank, the rank need not be locally constant. Bass gives an example, due to Kaplansky, of a projective module of rank $\leq 1$ such that the locus where the module has rank 0 is closed but not open [Bas63, p. 31, (2)]. A flat module that has constant rank need not be so nice either as the following example shows.

**Example (1.3).** If $M \subseteq \mathbb{Q}$ is the $\mathbb{Z}$-submodule generated by $p^{-1}$ for every prime number $p$, then $M$ is flat of constant rank 1 but neither projective nor finitely generated.

**Proposition (1.4).** Let $X$ be an algebraic stack and let $\mathcal{F}$ be a quasi-coherent sheaf on $X$.

(i) If $X$ is an affine scheme, then $\mathcal{F}$ is locally projective if and only if $\mathcal{F}$ is projective.

(ii) If $X$ is a noetherian affine scheme and $\aleph \geq \aleph_0$ is an infinite cardinal, then $\mathcal{F}$ is projective with constant rank $\aleph$ if and only if $\mathcal{F}$ is free of rank $\aleph$.

(iii) If $X$ is noetherian, then $\mathcal{F}$ is locally projective of finite rank if and only if $\mathcal{F}$ is finitely generated and locally free.

(iv) If $X$ is noetherian, then $\mathcal{F}$ is locally projective if and only if $\mathcal{F}$ is locally free.

(v) If $X$ is a noetherian scheme, then $\mathcal{F}$ is locally free if and only if $\mathcal{F}$ is Zariski-locally free.

**Proof.** All conditions are clearly sufficient. The necessity of the first condition follows from [RG71 I.3.1.4] (countable rank) or [RG71 II.2.5.1 and II.3.1.3] (general case). That conditions (ii) and (iii) are necessary is [Bas63, Cor. 3.2 & Prop. 4.2] respectively. Since the rank of a locally projective sheaf
is locally constant on a noetherian stack, the necessity of conditions (iv) and (v) follow from (i), (ii) and (iii).

□

Remark (1.5). Without the noetherian assumptions, statements (iii) and (iv) are false. If statement (ii) holds without the noetherian assumption then so does (v). In particular, this would imply that on any stack $X$, a quasi-coherent sheaf $F$ is locally free if and only if $F$ is locally projective, has locally constant rank and is finitely generated over the open locus of finite rank.

2. RELATIVE ASSASSINS AND RELATIVE FAITHFULNESS

In this section we extend the notions of relative assassins [RG71, 3.2.2] and schematically dominant morphisms [EGAIV, 11.9–11.10] from schemes to algebraic stacks.

(2.1) Associated points — There is a unique notion of associated points of coherent sheaves on locally noetherian algebraic stacks such that

(i) it coincides with the usual one for schemes; and
(ii) if $f: X \to Y$ is a flat morphism between locally noetherian stacks and $F$ is a coherent $O_Y$-module, then $f(\text{Ass}_X(f^*F)) \subseteq \text{Ass}_Y(F)$ with equality if $f$ is surjective.

The usual assassin satisfies (ii) for morphisms between schemes and we may thus simply define $\text{Ass}_X(F)$ for a coherent sheaf $F$ on $X$ as $\text{Ass}_X(F) := p(\text{Ass}_U(p^*F))$ where $p: U \to X$ is a presentation. One can also give a more intrinsic definition, cf. [Lie07, 2.2.6.3–2.2.6.7]. We abbreviate $\text{Ass}(X) = \text{Ass}_X(O_X)$.

In particular, if $f: X \to Y$ is locally of finite type and $\xi \in |Y|$ is a point, then we may define $\text{Ass}_X(\xi) \subseteq |f|^{-1}(\xi)$ as the image of $\text{Ass}(X_y) \to |X|$ for any representative $y: \text{Spec} \, k \to Y$ of $\xi$.

Definition (2.2) ([RG71, Déf. 3.2.2]). Let $f: X \to Y$ be a morphism of algebraic stacks that is locally of finite type. The relative assassin $\text{Ass}(X/Y)$ is the subset $\bigcup_{y \in |Y|} \text{Ass}(X_y)$ of $|X|$.

Note that $X$ and $Y$ need not be noetherian in the definition above, but the finiteness condition ensures that the fibers are locally noetherian. If $p: X' \to X$ is a flat morphism, locally of finite type, then $p(\text{Ass}(X'/Y)) \subseteq \text{Ass}(X/Y)$ with equality if $p$ is surjective.

Definition (2.3). Let $f: X \to Y$ be a morphism of algebraic stacks. We say that $f$ is schematically dominant if $O_Y \to f_*O_X$ is injective as a morphism of lisse-étale sheaves.

This agrees with the usual definition for schemes [EGAIV, 11.10.2] since that notion is fppf-local on the target [EGAIV, 11.10.5 (ii) b)]. It follows that our notion is fppf-local on the target as well. If $p: X' \to X$ is another morphism and $f \circ p$ is schematically dominant, then so is $f$. If $f$ and $p$ are schematically dominant, then so is $f \circ p$. In particular, morphisms that are covering in the fppf topology are schematically dominant.
Definition (2.4). Let $S$ be an algebraic stack and let $f : X \to Y$ be a morphism of algebraic stacks over $S$. We say that $f$ is $S$-universally schematically dominant if $f' : X \times_S S' \to Y \times_S S'$ is schematically dominant for every morphism $S' \to S$.

Proposition (2.5). Let $S$, $X$ and $Y$ be algebraic stacks and let $f : X \to Y$ and $Y \to S$ be flat morphisms that are locally of finite presentation. The following are equivalent.

1. The morphism $f$ is $S$-universally schematically dominant.
2. The image $f(X)$ contains the relative assassin $\text{Ass}(Y/S)$.

Proof. Since $f$ is open and faithfully flat onto its image, we may assume that $f$ is an open immersion. As the question is fppf-local on $Y$ and $S$ we may assume that $Y$ and $S$ are affine schemes. The result is then by [EGAIV, Prop. 11.10.10] (or [RG71, Cor. 3.2.6]).

To abbreviate, we say that $f$ is $S$-faithfully flat and locally of finite presentation when the equivalent conditions of Proposition (2.5) hold. This terminology is explained by the following lemma.

Lemma (2.6). Let $f : X \to Y$ and $\pi : Y \to S$ be morphisms of algebraic stacks. Assume that $f$ is $S$-universally schematically dominant. Given $F \in \text{QCoh}(Y)$ and $G \in \text{QCoh}(S)$, we have that

1. the unit map $\eta_{\pi^{-1}G} : \pi^*G \to f_*f^*\pi^*G$ is injective; and
2. a morphism $\theta : F \to \pi^*G$ is zero if and only if $f^*\theta$ is zero.

Proof. Consider $S' = \text{Spec}(\mathcal{O}_S \oplus G)$, where $G$ is square-zero, and let $X' = X \times_S S'$ and $Y' = Y \times_S S'$. Then $f' : X' \to Y'$ is schematically dominant, that is, $\mathcal{O}_{Y'} \oplus \pi^*G \to f_* (\mathcal{O}_X \oplus f^*\pi^*G)$ is injective. It follows that $\eta_{\pi^{-1}G}$ is injective.

If $\theta$ is zero, then so is $f^*\theta$. Conversely, if $f^*\theta$ is zero, then so is $\eta_{\pi^{-1}G} \circ \theta = (f_*f^*\theta) \circ f_*\eta_{\mathcal{F}}$. It follows that $\theta$ is zero since $\eta_{\pi^{-1}G}$ is injective.

Lemma (2.7). Let $f : X \to Y$ and $\pi : Y \to S$ be flat morphisms, that are locally of finite presentation, between algebraic stacks. Let $\mathcal{F}_0 \subseteq \mathcal{F}$ be quasi-coherent $\mathcal{O}_S$-modules and let $\mathcal{G}_0 \subseteq \pi^*\mathcal{F}$ be a quasi-coherent $\mathcal{O}_Y$-submodule. Assume that $f$ is $S$-faithfully flat. Then $\mathcal{G}_0 \subseteq \pi^*\mathcal{F}_0$ if and only if $f^*\mathcal{G}_0 \subseteq f^*\pi^*\mathcal{F}_0$.

Proof. Let $\mathcal{F}' = \mathcal{F}/\mathcal{F}_0$. Consider the map $\theta : \mathcal{G}_0 \to \pi^*\mathcal{F} \to \pi^*\mathcal{F}'$. Then $\mathcal{G}_0 \subseteq \pi^*\mathcal{F}_0$ if and only if $\theta = 0$ and $f^*\mathcal{G}_0 \subseteq f^*\pi^*\mathcal{F}_0$ if and only if $f^*\theta = 0$. Thus, the result follows from the previous lemma.

3. Pure morphisms of algebraic stacks

We begin by recalling the definition of pure morphisms of schemes [RG71, Déf. 3.3.3].

Definition (3.1). Let $f : X \to S$ be a morphism of schemes, locally of finite type. Let $s \in S$ be a point and let $(\bar{S}, \bar{s}) \to (S, s)$ be the henselization and $\bar{X} = X \times_S \bar{S}$. We say that $f$ is pure along $X_s$ if for every point $s' \in \bar{S}$, every associated point $x' \in \text{Ass}(X_{s'})$ is the generization of a point in $X_s$. We say
that $f$ is \textit{universally pure}, if $f': X \times_S S' \to S'$ is pure for every morphism $S' \to S$.

(3.2) \textbf{Examples} — The two key examples of pure morphisms are proper morphisms and faithfully flat morphisms with fibers that are geometrically irreducible without embedded components [RG71, Ex. I.3.3.4].

(3.3) \textbf{Base change and descent} — If $S' \to S$ is \textit{faithfully flat} and $f'$ is pure, then $f$ is pure. This follows directly from [RG71, 3.2.3].

It is not clear whether every pure morphism is universally pure. It is known, however, that if $f$ is \textit{flat}, pure and of finite presentation, then $f$ is universally pure [RG71, 3.3.7].

(3.4) \textbf{Composition} — Let $f: X \to Y$ and $g: Y \to S$ be morphisms of schemes, locally of finite type. If $f$ and $g$ are pure, then $g \circ f$ need not be pure. For example, the composition $\text{Spec}(k[x, y]/xy - 1) \hookrightarrow \text{Spec} k[x, y] \to \text{Spec} k[x]$ is not pure. On the other hand, if $f$ is flat and pure and $g$ is pure, then $g \circ f$ is pure. Also, if $f$ is faithfully flat and $g \circ f$ is pure, then $g$ is pure. Indeed, for every point $s \in S$, the restriction to the fiber $f_s: X_s \to Y_s$ maps associated points to associated points.

To extend purity to morphisms of stacks, we give a slightly different definition.

\textbf{Definition (3.5).} Let $f: X \to S$ be a morphism between algebraic stacks that is quasi-separated and locally of finite type. We say that $f$ is \textit{weakly closed} if $f(Z)$ is closed for every closed irreducible subset $Z \subset |X|$, such that the generic point of $Z$ is associated in its fiber. We say that $f$ is \textit{universally weakly closed}, if $f': X \times_S S' \to S'$ is weakly closed for every morphism $S' \to S$.

The remarks in (3.2)–(3.4) hold for “pure” replaced by “weakly closed” except that flat weakly closed morphisms need not be universally weakly closed.

We have the following valuative criterion.

\textbf{Proposition (3.6).} Let $f: X \to S$ be a quasi-separated morphism of finite type between algebraic stacks. Then the following are equivalent:

(i) $f$ is universally weakly closed;
(ii) for every valuation ring $V$ and morphism $\text{Spec} V \to S$, the base change $X \times_S \text{Spec} V \to \text{Spec} V$ is weakly closed; and
(iii) for every valuation ring $V$, morphism $\text{Spec} V \to S$, and associated point $z$ in $X \times_S \text{Spec} K(V)$, the closure of $z$ is faithfully flat over $\text{Spec} V$.

\textbf{If $f$ is a morphism of schemes, then this is equivalent to:}

(i') $f$ is universally pure;

\textbf{Proof.} Clearly, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). If $f$ is a morphism of schemes, then trivially (i) $\Rightarrow$ (i') and we note that (i') $\Rightarrow$ (ii) since it is enough to verify (ii) for henselian valuation rings.

To see that (iii) $\Rightarrow$ (i) we can assume that $S$ is affine and it is enough to prove that $f$ is weakly closed. Let $z \in X$ be a point that is associated in its
fiber and let $Z = \{ z \}$. Since $f(Z)$ is pro-constructible, it is enough to prove that $f(Z)$ is closed under specializations. This can be verified after the base change $S' = \text{Spec} V \to S$ for every valuation ring $V$ and every dominant morphism $\text{Spec} V \to \{ f(z) \}$. Since $X' \to S'$ is faithfully flat by (iii), the result follows.

To abbreviate, we say that a flat morphism of finite presentation between algebraic stacks is pure if it is universally weakly closed. This coincides with the usual definition for schemes by (3.3). It also coincides with the definition of pure in [Rom11, B.1].

The following lemma, which is a direct transcription of an argument in [RG71, proof of Prop. 3.3.6], shows that a flat morphism $X \to S$ is weakly closed if and only if $\text{Ass}(X/S) \to S$ is closed.

Lemma (3.7). Let $S$ be a scheme and let $X$ be an algebraic stack that is flat and of finite presentation over $S$. Let $s, s' \in |S|$ and $x' \in \text{Ass}(X_{s'})$. If $|X_s| \cap \{ x' \} \neq \emptyset$, then $\text{Ass}(X_s) \cap \{ x' \} \neq \emptyset$.

Proof. We may assume that $S = \text{Spec} A$ is affine. Pick a smooth presentation $p: U = \text{Spec} B \to X$. If $|X_s| \cap \{ x' \} \neq \emptyset$, then there exists a point $u' \in U$ above $x'$ such that $|U_s| \cap \{ u' \} \neq \emptyset$. We may assume that $u'$ is maximal in $p^{-1}(x')$ and then $u' \in \text{Ass}(U_s)$. Since $p(\text{Ass}(U_s)) = \text{Ass}(X_s)$, it is enough to prove that $\text{Ass}(U_s) \cap \{ u' \} \neq \emptyset$.

Let $u \in |U_s| \cap \{ u' \}$ and let $\Sigma \subseteq O_{U,u}$ be the elements whose images in $O_{U,u} \otimes \kappa(s)$ are non-zero divisors. Then $O_{U,u} \to \Sigma^{-1}O_{U,u}$ is $A$-universally injective and $\Sigma^{-1}O_{U,u}$ is a semi-local ring whose maximal ideals are associated points of $U$. [RG71 3.2.5]. In particular, the morphism $O_{U,u} \otimes \kappa(s') \to (\Sigma^{-1}O_{U,u}) \otimes \kappa(s')$ is injective. Since $u'$ is associated in $\text{Spec}(O_{U,u} \otimes \kappa(s'))$, this means that $u' \in \text{Spec}(\Sigma^{-1}O_{U,u} \otimes \kappa(s'))$; hence $u'$ is a specialization of an associated point $u_1$ of $U_s$. □

4. Homological projectivity

The main theorem of [RG71 §I.3] is the following relation between purity and projectivity for affine morphisms.

Theorem (4.1) (Raynaud–Gruson). Let $f: X \to Y$ be an affine finitely presented morphism of schemes. The following are equivalent:

(i) $f$ is flat and pure.

(ii) $f_* O_X$ is locally projective.

(iii) $f_* O_X$ is locally free.

Proof. The equivalence between (i) and (ii) is [RG71 Thm. I.3.3.5]. The equivalence between (ii) and (iii) is [RG71 Cor. I.3.3.12]. Note that if $Y$ is noetherian, then the latter equivalence follows directly from Proposition (1.4) (iv). The non-noetherian case follows from the noetherian case using the equivalence between (i) and (ii) and using that pure morphisms behave well under approximation [RG71 Cor. I.3.3.10]. □

Local projectivity of $f_* O_X$ is not local on $X$. To obtain a non-affine analogue of the theorem above, we introduce the following definition.
**Definition (4.2).** Let \( f : X \to Y \) be a flat morphism of finite presentation between algebraic stacks. We say that \( f \) is **homologically projective** (resp. **strongly homologically projective**) if there exists

(i) an fppf-covering \( \{ \text{Spec}(A_i) \to Y \} \); and

(ii) morphisms \( q_i : \text{Spec}(B_i) \to X \times_Y \text{Spec}(A_i) \);

such that for every \( i \)

(a) the composition \( \text{Spec}(B_i) \to X \times_Y \text{Spec}(A_i) \to \text{Spec}(A_i) \) makes \( B_i \) into a projective \( A_i \)-module; and

(b) \( q_i \) is \( \text{Spec}(A_i) \)-faithfully flat and locally of finite presentation (resp. faithfully flat and locally of finite presentation).

Here “homological” is to indicate that projective is interpreted as in homological algebra and not as in algebraic geometry. It should not be confused with the notion of **cohomologically projective** morphisms in [Alp13, 3.18].

By definition, the notion of (strong) homological projectivity is stable under base change and fppf-local on the target. If \( p : X' \to X \) is faithfully flat and locally of finite presentation and \( f \circ p \) is (strongly) homologically projective, then \( f \) is (strongly) homologically projective but the converse does not hold. It is, a priori, not clear whether the composition of two (strongly) homologically projective morphisms is (strongly) homologically projective.

Recall that \( X \) has the resolution property if every quasi-coherent sheaf of finite type on \( X \) admits a surjection from a vector bundle.

**Theorem (4.3).** Let \( f : X \to Y \) be a morphism of algebraic stacks that is flat and of finite presentation. Consider the following conditions:

(i) \( f \) is affine and \( f_* \mathcal{O}_X \) is locally projective;

(ii) \( f \) is strongly homologically projective;

(iii) \( f \) is homologically projective; and

(iv) \( f \) is pure.

Then \( (i) \implies (ii) \implies (iii) \iff (iv) \). If \( f \) is affine, then all four conditions are equivalent. If \( X \) has the resolution property fppf-locally on \( Y \) (e.g., if \( f \) is quasi-affine), then \( (ii) \iff (iii) \).

**Proof.** From the definitions, it follows that \( (i) \implies (ii) \implies (iii) \). To prove that \( (iii) \implies (iv) \), we may assume that \( Y = \text{Spec} A \) and that there is a \( Y \)-faithfully flat and finitely presented morphism \( U = \text{Spec} B \to X \) such that \( B \) is a projective \( A \)-module. By Theorem (4.1), we have that \( U \to Y \) is pure. Since the image of \( U \) contains \( \text{Ass}(X/Y) \), it follows that \( X \to Y \) is pure. When \( f \) is affine, \( (iv) \implies (i) \) by Theorem (4.1).

For \( (iv) \implies (iii) \), suppose that \( f \) is pure. As before we may assume that \( Y \) is affine. Pick a smooth presentation \( U = \text{Spec} B \to X \). Let \( y \in Y \) be a point. Then, by [RG71, Prop. 3.3.2], there exists a commutative diagram

\[
\begin{array}{ccc}
U' & \longrightarrow & U \\
\downarrow & & \downarrow \\
Y' & \longrightarrow & Y
\end{array}
\]

and a point \( y' \) above \( y \) such that
• $U' \to U$ and $Y' \to Y$ are étale, and $\kappa(y) = \kappa(y')$;
• $U' = \text{Spec } B'$ and $Y' = \text{Spec } A'$ are affine and $B'$ is a projective $A'$-module; and
• the image of $U' \to U$ contains $\text{Ass}(U_y)$.

In particular, the image of $U' \to U \to X$ contains $\text{Ass}(X_y)$. After replacing $X$, $Y$ and $U$ by their pull-backs along the base change $Y' \to Y$, we may assume that $Y' = Y$.

We now claim that the image of $U' \to U \to X$ contains $\text{Ass}(X_{y'})$ for every generization $y'$ of $y$. To see this, let $x' \in \text{Ass}(X_{y'})$. By the definition of purity, there exists a point $x \in X_y \cap \{x'\}$. By Lemma [5.7], there exists a point $x_1 \in \text{Ass}(X_y) \cap \{x'\}$. Since $x_1$ is in the image of $U'$, so is its generization $x'$.

By [RG71, Lem. 3.3.9], there is then an open neighborhood $y \in V \subseteq Y$ such that the image of $U' \to U \to X$ contains $\text{Ass}(X_{y'})$ for every $y' \in V$. This means that $U' \to U \to X$ is $Y$-faithfully flat over $V$, that is, $X \to Y$ is homologically projective over $V$. As the question is local on $Y$, it follows that $X \to Y$ is homologically projective.

Under the additional assumption on $X$, we will prove that (iv) $\implies$ (ii). For this, we may work locally on $Y$ and assume that $Y = \text{Spec } A$ is affine and that $X$ has the resolution property. Then $X = [U/GL_n]$ for some quasi-affine scheme $U$ [Tot04, Gro13]. By Jouanolou’s trick, there is an affine vector bundle torsor $E \to U$ [Jou73, Lem. 1.5] (also see [Wei89, 4.3–4.4]). Since $E \to X$ is flat with geometrically irreducible fibers, hence flat and pure, it follows that $E \to X \to Y$ is pure [A3]. Since $E = \text{Spec } B$ is affine, we have that $B$ is projective; thus, $f$ is strongly homologically projective. \hfill $\square$

5. Existence of minimal subsheaves

Let $f : X \to Y$ be a faithfully flat morphism between quasi-compact algebraic stacks and let $\mathcal{F} \in \textbf{QCoh}(Y)$. Assume that $\mathcal{G}_0 \subseteq \mathcal{G} := f^* \mathcal{F}$ is a quasi-coherent subsheaf of finite type. If $\mathcal{F}$ is the union of its quasi-coherent subsheaves $\mathcal{F}_\lambda$ of finite type, then, for sufficiently large $\lambda$, we have that $\mathcal{G}_0 \subseteq f^* \mathcal{F}_\lambda$.

Conversely, if $\mathcal{G} = f^* \mathcal{F}$ is the union of its quasi-coherent subsheaves $\mathcal{G}_\lambda$ of finite type and for every $\mathcal{G}_\lambda$ there exists $\mathcal{F}_\lambda \subseteq \mathcal{F}$ of finite type such that $\mathcal{G}_\lambda \subseteq f^* \mathcal{F}_\lambda$, then $\mathcal{F}$ is the union of its subsheaves of finite type.

We will see that, under suitable hypotheses, for every $\mathcal{G}_\lambda$ of finite type as above there is a minimal $\mathcal{F}_\lambda$ as above and it is of finite type. This is, however, not always the case:

**Example (5.1).** Let $A$ be a discrete valuation ring with fraction field $K$ and uniformizing parameter $t$. Let $B = A \times K$, which is a faithfully flat $A$-algebra. Let $M = A$ and consider the submodule $N_0 = (0 \times K) \subseteq M \otimes_A B = B$. For every non-trivial ideal $M_n = (t^n) \subseteq A = M$, we then have that $N_0 \subseteq M_n \otimes_A B = (t^n) \times K$. But the intersection is $\bigcap M_n = 0$ and $N_0 \not\subseteq (\bigcap M_n) \otimes_A B = 0$. Hence, there is no minimal submodule $M'$ of $M$ such that $N_0 \subseteq M' \otimes_A B$.
The problem in this example is that infinite intersections does not commute with flat pull-back. This does not happen if we replace flatness with projectivity.

**Lemma (5.2) (Serre).** Let $A$ be a ring and let $B$ be an $A$-algebra which is projective as an $A$-module. Let $M$ be an $A$-module and let $N_0 \subseteq M \otimes_A B$ be a $B$-submodule. Then there is a unique minimal $A$-submodule $M_0 \subseteq M$ such that $N_0 \subseteq M_0 \otimes_A B$. Moreover,

(i) if $N_0$ is of finite type, then so is $M_0$, and

(ii) if $A'$ is an $A$-algebra and we let $B' = B \otimes_A A'$, $M' = M \otimes_A A'$, $M'_0 := \text{im}(M_0 \otimes_A A' \to M')$ and $N'_0 = \text{im}(N_0 \otimes_B B' \to M' \otimes_{A'} B')$, then $M'_0$ is the minimal $A'$-submodule of $M'$ such that $N'_0 \subseteq M'_0 \otimes_{A'} B'$.

**Proof.** Choose a free $A$-module $F$ such that $B$ is a direct summand of $F$ and pick a basis $\{e_i\}$ of $F$. Let $M_0 \subseteq M$ be an $A$-submodule. Then $M_0 \otimes_A B \subseteq M_0 \otimes_A F \subseteq M \otimes_A F$ and $M \otimes_A B \subseteq M \otimes_A F$. Let $x \in N_0$ be an element. Then $x = \sum_i x_i \otimes e_i$ in $M \otimes_A F$, and, using the retraction $F \to B$, we may also write $x = \sum_i x_i \otimes b_i$ in $M \otimes_A B$. Thus $x \in M_0 \otimes_A B$ if and only if $x_i \in M_0$ for every $i$. It follows that the minimal submodule $M_0$ is the submodule generated by the $x_i$’s when $x$ ranges over a set of generators of $N_0$. The remaining claims follows immediately from the construction of $M_0$. □

Using purity, we give the following global version.

**Theorem (5.3).** Let $f : X \to Y$ be a flat morphism of finite presentation between algebraic stacks. Assume that $f$ is pure. Let $\mathcal{F} \in \mathbf{QCoh}(Y)$ and let $\mathcal{G}_0 \subseteq \mathcal{G} := f^* \mathcal{F}$ be a quasi-coherent submodule. Then there is a unique minimal quasi-coherent submodule $\mathcal{F}_0 \subseteq \mathcal{F}$ such that $\mathcal{G}_0 \subseteq f^* \mathcal{F}_0$. Moreover,

(i) if $\mathcal{G}_0$ is of finite type, then so is $\mathcal{F}_0$, and

(ii) the submodule $\mathcal{F}_0$ remains minimal under arbitrary base change on $Y$.

**Proof.** By Theorem (1.3), $f$ is homologically projective. By fppf descent, it is enough to prove the statement after replacing $Y$ with an fppf cover. We may thus assume that $Y = \text{Spec} A$ and that there exists a $Y$-faithfully flat morphism $q : X' = \text{Spec} B \to X$ of finite presentation such that $B$ is a projective $A$-module. If $\mathcal{F}_0 \subseteq \mathcal{F}$ is a submodule, then $\mathcal{G}_0 \subseteq f^* \mathcal{F}_0$ if and only if $q^* \mathcal{G}_0 \subseteq q^* f^* \mathcal{F}_0$ (Lemma 2.7). We may thus replace $X$ with $X'$ and assume that $X$ and $Y$ are affine. The theorem is then Lemma (5.2). □

6. Approximation of quasi-coherent sheaves

Let $X$ be a quasi-compact and quasi-separated algebraic stack. We recall from [Ryd14, §4] that $X$ has the completeness property if every quasi-coherent $\mathcal{O}_X$-module is a directed colimit of finitely presented $\mathcal{O}_X$-modules. We say that $X$ has the partial completeness property if every quasi-coherent $\mathcal{O}_X$-module is the directed colimit of its finitely generated quasi-coherent submodules. In the terminology of [Ryd14, §4], these two conditions are the conditions (C1) and (C2) for the category $\mathbf{QCoh}(X)$ and they imply the corresponding facts for quasi-coherent $\mathcal{O}_X$-algebras.

We also make the following definition that extends [Ryd14, Def. 4.7].
Definition (6.1). An algebraic stack $X$ is semi-noetherian (resp. pseudo-noetherian) if it is quasi-compact, quasi-separated and $X'$ has the partial completeness property (resp. completeness property) for every finitely presented morphism $X' \to X$ of algebraic stacks.

Every pseudo-noetherian algebraic stack is semi-noetherian. Examples of pseudo-noetherian algebraic stacks are noetherian algebraic stacks, quasi-compact and quasi-separated schemes, algebraic spaces and Deligne–Mumford stacks [Ryd14, Thm. A].

Proposition (6.2). Let $f : X \to Y$ be a faithfully flat and pure morphism of finite presentation between quasi-compact and quasi-separated algebraic stacks. If $X$ has the partial completeness property, then so has $Y$. In particular, $X$ is semi-noetherian if and only if $Y$ is semi-noetherian.

Proof. Let $\mathcal{F} \in \text{QCoh}(Y)$ and let $\mathcal{G}_0 \subseteq f^* \mathcal{F}$ be a quasi-coherent submodule of finite type. By Theorem (6.3) there exists a minimal quasi-coherent subsheaf $\mathcal{F}_0 \subseteq \mathcal{F}$ such that $\mathcal{G}_0 \subseteq f^* \mathcal{F}_0$. Since $\mathcal{F}_0$ is of finite type it follows that $\mathcal{F}$ is the union of its quasi-coherent submodules of finite type and the first statement follows. If $Y$ is semi-noetherian, then it follows from the definition that $X$ is semi-noetherian since $f$ is of finite presentation. The converse follows from the first statement. □

Proposition (6.3). Let $X$ be an algebraic stack and let $p : X' \to X$ be étale, representable, surjective and of finite presentation. Then $X$ is semi-noetherian if and only if $X'$ is semi-noetherian.

Proof. This is proven exactly as [Ryd14, Prop. 4.11]: étale dévissage [Ryd11, Thm. D] is used to reduce the question to where $p$ is either finite, surjective and étale or an étale neighborhood. These cases follow from simplified versions of [Ryd14, Lem. 4.9 and 4.10] where “completeness property” is replaced with “partial completeness property”. □

Theorem (6.4). Let $f : X \to Y$ be a faithfully flat morphism of finite presentation with geometrically reduced fibers (e.g., $f$ smooth). If $X$ is semi-noetherian, then so is $Y$.

Proof. For simplicity, let us first describe the smooth case, which is the only case we need for the main theorem. Consider the connected fibration $X \to \pi_0(X/Y) \to Y$ [LMB00, 6.8] or [Rom11, Thm. 2.5.2]. Then $g : X \to \pi_0(X/Y)$ has smooth geometrically connected fibers and $h : \pi_0(X/Y) \to Y$ is étale, representable and of finite presentation, but not necessarily separated. In particular, $g$ is pure.

In the general case we instead use the irreducible component fibration of Romagny [Rom11, Thm. 2.5.2]. The unicomponent locus $U \subseteq X$ is the subset of points that belong to exactly one irreducible component of their fibers. It is open and quasi-compact and there is a factorization $U \to \text{Irr}(X/Y) \to Y$ where the first morphism has geometrically reduced and irreducible fibers and the second is surjective étale, representable and of finite presentation. Note that $U \to \text{Irr}(X/Y)$ is faithfully flat and pure and that $U$ is semi-noetherian.
It follows that $\text{Irr}(X/Y)$ and $Y$ are semi-noetherian by Propositions (6.2) and (6.3) respectively.

We now obtain the following equivalent form of the main theorem.

**Corollary (6.5).** Let $X$ be a quasi-compact and quasi-separated algebraic stack. Then $X$ is semi-noetherian.

**Proof.** Pick a smooth presentation $\text{Spec} B \to X$. Since $\text{Spec} B$ is semi-noetherian, the result follows from Theorem (6.4). \qed

**Remark (6.6).** To answer Conjectures A and B, we may argue as in the proof of Theorem (6.4) using [Ryd14, Prop. 4.11 and Lem. 7.9]. This reduces the situation to where $X$ has a smooth pure presentation $U \to X$. The author hopes that this will be useful to settle the conjectures.

### 7. Applications

We give two simple applications of the main theorem.

**Theorem (7.1) (Zariski’s main theorem).** Let $f: X \to Y$ be a morphism between quasi-compact and quasi-separated algebraic stacks. Then the following are equivalent:

- (i) $f$ is representable, separated and quasi-finite; and
- (ii) there is a factorization $f = j \circ f$ where $j$ is a quasi-compact immersion and $f$ is finite.

**Proof.** Argue exactly as in [Ryd14 Thm. 8.6 (ii)] (or [LMB00 Thm. 16.5]) but using the partial completeness property instead of the completeness property. \qed

**Proposition (7.2).** Let $X$ be a quasi-compact and quasi-separated algebraic stack and let $U \subseteq X$ be a quasi-compact open substack. Then there exists a finitely presented closed substack $\emptyset = X_0 \hookrightarrow X_1 \hookrightarrow \ldots \hookrightarrow X_n$ such that $|X_n| = |X|$ and the restriction of $W \to X$ to $X_k \setminus X_{k-1}$ is flat for every $k = 1, 2, \ldots, n$.

**Proof.** Let $I \subseteq O_X$ be the quasi-coherent sheaf of ideals defining $Z_{\text{red}} = (X \setminus U)_{\text{red}}$. Write $I = \bigcup I_\lambda$ as a union of quasi-coherent ideals of finite type. If $Z_\lambda$ denotes the finitely presented closed substack corresponding to $I_\lambda$, then $\cap Z_\lambda = Z_{\text{red}}$. Since $U$ is quasi-compact it follows that $|Z_\lambda| = |Z_{\text{red}}|$ for all sufficiently large $\lambda$. We may take $Z = Z_\lambda$ for any such $\lambda$. \qed

As a third application we have the existence of flattening stratifications for finitely presented morphisms.

**Theorem (7.3).** Let $X$ be a quasi-compact and quasi-separated algebraic stack and let $W \to X$ be a morphism of finite presentation. Then there exists a sequence of finitely presented closed substacks $\emptyset = X_0 \hookrightarrow X_1 \hookrightarrow \ldots \hookrightarrow X_n$ such that $|X_n| = |X|$ and the restriction of $W \to X$ to $X_k \setminus X_{k-1}$ is flat for every $k = 1, 2, \ldots, n$.

**Proof.** The result is well-known when $X$ is noetherian: let $X_n = X_{\text{red}}$; pick a smooth presentation $p: \text{Spec}(A) \to X_n$; choose a non-empty open subscheme $V \subseteq \text{Spec}(A)$ over which $W$ is flat (generic flatness); let $X_{n-1} = (X \setminus p(V))_{\text{red}}$. The result now follows by noetherian induction.
If $X$ is affine, the result follows by standard limit methods: there is a noetherian affine scheme $X_0$, a morphism $X \to X_0$ and a flat morphism $W_0 \to X_0$ of finite presentation that pull-backs to $W \to X$. The pull-back of a solution to the problem for $W_0 \to X_0$ gives a solution for $W \to X$.

In the general case, we pick a smooth presentation $p: X' = \text{Spec}(A) \to X$ and choose a filtration $X_0' \hookrightarrow X_1' \hookrightarrow \cdots \hookrightarrow X_n'$ that solves the problem over $X'$. We will prove that $X$ has a filtration of length $n$ that solves the problem. Set-theoretically, we will have $|X_k| = X \setminus p(X' \setminus X_k')$. If $n = 0$, the problem is trivial. By induction on $n$, we may assume that there exists a filtration of length $n - 1$ on every closed substack $Q \hookrightarrow X$ such that $|p^{-1}(Q)| \subseteq |X_{n-1}|$.

The subset $p(X' \setminus X_{n-1}')$ is open and quasi-compact, hence there is a finitely presented closed substack $Z \hookrightarrow X$ such that $X \setminus Z = p(X' \setminus X_{n-1}')$ (Proposition 7.2).

Since $p$ is smooth, we have that $p^{-1}(X_{\text{red}}) = X_{\text{red}}'$ and hence $p^{-1}(X_{\text{red}}') \hookrightarrow X'$ factors through $X_n'$. Writing the nilradical of $\mathcal{O}_X$ as a union of quasi-coherent ideals of finite type, we may write the nil-immersion $X_{\text{red}} \hookrightarrow X$ as an intersection of finitely presented nil-immersions $X_\lambda \hookrightarrow X$. For sufficiently large $\lambda$, we have that $p^{-1}(X_\lambda) \hookrightarrow X'$ factors through $X_n'$. Then $W \to X$ is flat over $X_\lambda \setminus Z$ for such $\lambda$ since $p^{-1}(X_\lambda) \setminus X_{n-1}' \to X_\lambda \setminus Z$ is smooth and surjective.

We let $X_n = X_\lambda$ and $Q = Z \cap X_\lambda$. Then, by induction there is a filtration $X_0 \hookrightarrow X_1 \hookrightarrow \cdots \hookrightarrow X_{n-1} \hookrightarrow Q$ with $|X_{n-1}| = |Q|$ such that $W \to X$ is flat over the strata. The result follows. \hfill \square

As a fourth application we have the existence of stratifications into gerbes for stacks with finitely presented inertia.

**Corollary (7.4).** Let $X$ be a quasi-compact and quasi-separated algebraic stack with inertia of finite presentation. Then there exists a sequence of finitely presented closed substacks $\emptyset = X_0 \hookrightarrow X_1 \hookrightarrow \ldots \hookrightarrow X_n$ such that $|X_n| = |X|$ and $X_k \setminus X_{k-1}$ is an fppf gerbe for every $k = 1, 2, \ldots, n$.

**Proof.** Apply Theorem (7.3) on $I_X \to X$. \hfill \square

For a general quasi-compact and quasi-separated algebraic stack, the inertia is only of finite type. In this case, it is not always possible to find finitely presented stratifications as in Corollary (7.4). In fact, sometimes even an infinite number of strata is required \cite{SP06RF}.

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