Neutrino oscillation in a space-time with torsion

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Abstract

Using the Einstein-Cartan-Dirac theory, we study the effect of torsion on neutrino oscillation. We see that torsion cannot induce neutrino oscillation, but affects it whenever oscillation exists for other reasons. We show that the torsion effect on neutrino oscillation is as important as the neutrino mass effect, whenever the ratio of neutrino number density to neutrino energy is $\sim 10^{69}$ cm$^{-3}$/eV, or the number density of the matter is $\sim 10^{69}$ cm$^{-3}$.

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1 Introduction

Results from several solar neutrino experiments, utilizing different detection techniques, consistently show a discrepancy between the measured $\nu_e$ flux from the sun and the $\nu_e$ flux predicted by various solar models. The origin of this solar neutrino deficit is not yet well known. A possible solution is neutrino flavour oscillations, which was first suggested by Pontecorvo [1] and then Maki et al. [2].

Several mechanisms for neutrino oscillations have been proposed (see for example [3]). One mechanism, for example, assumes that neutrinos have non-equal masses, and that the neutrino mass eigenstates are not the weak interaction eigenstates. The most famous version of this type of solution is the Mikheyev-Smirnov-Wolfenstein (MSW) mechanism [4, 5]. There have been also proposed some alternative mechanisms, relating neutrino oscillations to gravitational effects; e.g. to violation of equivalence principle [6], or to violation of the Lorentz group symmetry [7]. There have been also some efforts to study the effect of (possible) torsion of space-time on neutrino oscillations [8, 9]. In ref. [8], the authors have studied this effect, by assuming that the torsion eigenstates, i.e. the eigenstate of the interaction part of the hamiltonian $H_T$, are different from the weak interaction eigenstates. In the case that the two eigenstates are the same ($\theta_T = 0$, in their language), their results coincide with the ordinary vacuum oscillation and the effect of torsion does not appear. Here, we study this problem from a different point of view. First, we take the torsion eigenstates the same as the weak interaction ones. Second, our procedure is different from [8] and based on the technique that was introduced in [10]. We use this technique in the Einstein-Cartan-Dirac theory.

The structure of the paper is as follows. In section 2, we briefly discuss the Einstein-Cartan-Dirac theory, in which a Dirac field couples to the metric and torsion of the space–time. We see that in this theory, the torsion is completely determined by a pseudo–vector $A_\mu$, and the equation of motion fixes it to be proportional to the axial current of the Dirac particle. In section 3, we study the effect of torsion on neutrino oscillation, and show that although the torsion can not induce neutrino oscillation, but affects it whenever oscillation exists for other reasons. We find some lower bounds on physical quantities such that the oscillation induced by torsion be of the same order as the oscillation induced by neutrino mass.

2 Brief review of Einstein-Cartan-Dirac theory

The differential geometry of a four dimensional manifold $U^4$ is determined by two objects; the Riemannian metric $g_{\mu\nu}$, and the connection $\Gamma^\nu_{\alpha\beta}$. Conceptually these two objects are completely independent. The metric determines the inner product of vectors at each point, enabling us to define arclengths and thus distances. The connection determines the parallel
transport and thus the covariant derivative of tensor and spinor fields. The connection is said to be compatible with metric, if the parallel transport of vectors does not change both the length of vectors and the angle between them.

The most general connection, compatible with the Riemannian metric, is

\[ \Gamma^\alpha_{\mu
u} := \left\{ \alpha \right\}_{\mu
u} + K^\alpha_{\mu
u}, \]  

where

\[ \left\{ \alpha \right\}_{\mu
u} = \frac{1}{2} g^{\alpha\beta} \left( -g_{\mu\nu,\beta} + g_{\beta\mu,\nu} + g_{\nu\beta,\mu} \right), \]

is the usual Christoffel symbol, and \( K^\alpha_{\mu\nu} \) is a rank 3 tensor, called contorsion. The only restriction on \( K \) is that, when its upper index is lowered with the metric, it has the following symmetry property

\[ K_{\alpha\mu\beta} = -K_{\beta\mu\alpha}. \]

It therefore follows that, in a \( d \)-dimensional space-time, it has \( d^2(d - 1)/2 \) independent components.

Contorsion is related to the torsion tensor as follows

\[ K^\alpha_{\mu
u} := \frac{1}{2} g^{\alpha\beta} \left( T_{\beta\mu\nu} + T_{\mu\beta\nu} + T_{\nu\beta\mu} \right), \]

and the torsion \( T^\alpha_{\mu
u} \) itself is the antisymmetric part of the connection

\[ T^\alpha_{\mu
u} = \Gamma^\alpha_{\mu
u} - \Gamma^\alpha_{\nu\mu}. \]

It follows that the differential geometry of \( U^4 \) is determined by two tensor fields: 1) the metric tensor \( g_{\mu\nu} \) and, 2) the contorsion tensor \( K^\alpha_{\mu\nu} \), or equivalently the torsion tensor \( T^\alpha_{\mu\nu} \).

In general relativity, the space-time is considered to be torsion free, a priori, while in Eistein-Cartan theory this is not the case. Here, we quickly review this latter theory.

As usual, the curvature tensor is defined as

\[ R^\kappa_{\lambda\mu\nu} := \Gamma^\kappa_{\nu\lambda\mu} - \Gamma^\kappa_{\mu\lambda\nu} + \Gamma^\eta_{\mu\lambda} \Gamma^\kappa_{\nu\eta} - \Gamma^\eta_{\nu\lambda} \Gamma^\kappa_{\mu\eta}. \]

Let \( V^4 \) denote the same manifold with the same Riemannian metric, but with vanishing torsion. The Riemann curvature of \( V^4 \) is

\[ ^0R^\kappa_{\lambda\mu\nu} := \left\{ \kappa \right\}_{\nu\lambda\mu} - \left\{ \kappa \right\}_{\mu\lambda\nu} + \left\{ \eta \right\}_{\mu\lambda} \left\{ \kappa \right\}_{\nu\eta} - \left\{ \eta \right\}_{\nu\lambda} \left\{ \kappa \right\}_{\mu\eta}. \]

Defining

\[ \tau_{\mu} := g^{\alpha\beta} K_{\alpha\beta\mu}, \]

\[ A^\sigma := \frac{1}{3} \epsilon^{\sigma\alpha\mu\nu} K_{\alpha\mu\nu}, \]
the contorsion can be written as
\[ K_{\alpha\mu\nu} = \frac{1}{3}(g_{\alpha\mu}\tau_\nu - g_{\nu\mu}\tau_\alpha) + \frac{1}{2}A^\sigma\varepsilon_{\sigma\alpha\mu\nu} + U_{\alpha\mu\nu}. \] (10)

This expression is simply the decomposition of contorsion tensor. Here \( U_{\alpha\mu\nu} \) is in fact defined by the above equation, and has the following properties
\[ U_{\alpha\mu\nu} = -U_{\nu\mu\alpha}, \quad g^{\alpha\mu}U_{\alpha\mu\nu} = 0, \quad \varepsilon^{\sigma\alpha\mu\nu}U_{\alpha\mu\nu} = 0. \] (11)

\( \varepsilon_{\alpha\kappa\mu\nu} \) is the totally antisymmetric pseudo-tensor of rank 4. Now, it can be shown that the scalar curvature is
\[ R = \frac{1}{6}R - \frac{2}{\sqrt{g}}\partial_\kappa(\sqrt{g}\tau^\kappa) + \left( -\frac{1}{3}\tau^2 + \frac{3}{2}A^2 + U_{\alpha\mu\nu}U^{\mu\nu\alpha} \right), \] (12)

where \( \sqrt{g} = [-\det(g_{\mu\nu})]^{1/2} \). Einstein-Cartan theory is a theory of gravitation in which the space-time is a manifold with torsion (i.e., in our notation a \( U^4 \)). The action functional of the Einstein-Cartan theory is
\[ I_{EC} := -\frac{c^3}{16\pi G} \int d^4x \sqrt{g}R = \] (13)
\[ -\frac{c^3}{16\pi G} \int d^4x \left( \sqrt{g}R - 2\partial_\kappa(\sqrt{g}\tau^\kappa) + \sqrt{g} \left( -\frac{1}{3}\tau^\kappa\tau_\kappa + \frac{3}{2}A^\kappa A_\kappa + U_{\alpha\mu\nu}U^{\mu\nu\alpha} \right) \right). \] (14)

An important feature of this action is that the contorsion contribute in a total derivative plus an algebraic expression. Therefore, in Einstein-Cartan theory, the equation of motion of the torsion field is simply an algebraic equation
\[ A^\kappa = 0, \quad \tau^\kappa = 0, \quad U_{\alpha\mu\nu} = 0. \] (15)

In other words, in the absence of matter, the Einstein-Cartan action implies that there is no torsion.

Now let us couple this to a spin–1/2 field. The resulting theory is known as the Einstein-Cartan-Dirac (ECD) theory. The action of this theory is
\[ I_{ECD} := I_{EC} + I_{D}, \] (16)
where
\[ I_{D} := \int d^4x \sqrt{g}(-\hbar)\overline{\psi} \left( e^{\mu}_{a,\gamma} e^a(\partial_\mu + \Gamma_\mu) + \frac{mc}{\hbar} \right) \psi. \]

Here \( a \) is a tetrad index and \( \mu \) is a coordinate index. The coupling of metric and torsion to the Dirac field is through the covariant derivative \( D_\mu := \partial_\mu + \Gamma_\mu \), in which \( \Gamma_\mu \) is the spin connection,
\[ \Gamma_\mu := -\frac{i}{8}[\gamma^a, \gamma^b]e_a^\nu e_{b\nu\mu}. \] (18)
Here a vertical line $|$ means the covariant derivative on $U^4$

\[ e_{b|i|\nu} := e_{b|i|\nu} - \Gamma^\lambda_{\mu\nu} e_{b|\lambda} = e_{b|i|\nu} - \{\lambda_{\mu\nu}\} e_{b|\lambda} - K^\lambda_{\mu\nu} e_{b|\lambda}. \]  

(19)

We can write this as follows

\[ e_{b|i|\nu} := e_{b|i|\nu} - K^\lambda_{\mu\nu} e_{b|\lambda}, \]  

(20)

where

\[ e_{b|i|\nu} := e_{b|i|\nu} - \{\lambda_{\mu\nu}\} e_{b|\lambda}, \]  

(21)

is the covariant derivative on $V^4$.

It can be easily seen that the effect of torsion in (17) is to add to the usual Dirac action in a curved torsionless space-time ($V^4$), the following interaction term

\[ \int d^4x \sqrt{|g|} (6\bar{h} A_\mu J^\mu_5), \]  

(22)

where $J^\mu_5 = -i\bar{\psi}\gamma^\mu\gamma^5\psi$ is the axial current of the Dirac field. Now, taking variation of (16) with respect to $A_\mu$, $\tau^\mu$, and $U_{\alpha\mu\nu}$, leads to the following equations of motion

\[ \tau^\mu = 0, \quad U_{\alpha\mu\nu} = 0, \]  

(23)

\[ A_\mu = \frac{96\pi}{3} \frac{G\bar{h}}{c^3} J^\mu_5. \]  

(24)

In summary, in ECD theory the torsion pseudo-vector is proportional to the axial current of the fermion field, and this current is the source of torsion. The proportionality constant is the square of Planck length.

## 3 Effect of torsion on neutrino oscillation

Now everything is ready to study the effect of torsion on neutrino oscillation. We follow the same procedure that was introduced in [10], in which the neutrino oscillation in curved background was studied. If we ignore the background matter effect, the Dirac field equation of motion is

\[ \left[ \gamma^a e^a_\mu \left( \partial_\mu + \Gamma_\mu \right) + \frac{mc}{\hbar} \right] \psi = 0, \]  

(25)

where the torsion contribution comes from $\Gamma_\mu$ term, via eqs.(18) and (19). Using the identity

\[ \gamma^a [\gamma^b, \gamma^c] = 2\eta^{ab}\gamma^c - 2\eta^{ac}\gamma^b - 2i\varepsilon^{abc}\gamma_5\gamma_d, \]  

(26)

one can show that the only nonvanishing contribution from spin connection is

\[ \gamma^a e^a_\mu \Gamma_\mu = \gamma^a e^a_\mu \left\{ iA_\mu \left[ -\frac{1}{2\sqrt{g}} \gamma_5 \right] \right\}, \]  

(27)
where
\[ A^\mu = A^\mu_G + A^\mu_T. \] (28)

Here
\[ A^\mu_G = \frac{1}{4} \sqrt{g} e^\mu_{\alpha\beta\gamma\delta} (e_{\beta\nu,\sigma} - e_{\beta\sigma,\nu}) e^\nu_{e_d} e^\sigma_{e_d}, \] (29)
is the contribution of the Christoffel symbol, which is present in both \( V^4 \) and \( U^4 \) and was
drived in [10], and
\[ A^\mu_T = -6 \sqrt{g} A^\mu, \] (30)
is the contribution of the torsion field \( A^\mu \), which is present only in \( U^4 \). Now, a look at
eqs.(24), (25), (28), and (30) shows that in Einstein-Cartan-Dirac theory, there is a self
interaction among the fermion field, because of torsion. This interaction, however, is very
small, because of the coefficient in (24).2

As was argued in [10], we can add a term proportional to the identity to eq.(27), to group
it with term arising from matter effects. The result is
\[ \gamma^a e^\mu_a \Gamma_\mu = \gamma^a e^\mu_a (i A^\mu \mathcal{P}_L), \] (31)
where \( \mathcal{P}_L \) is the left–handed projection operator. Putting (31) in (25), we see that the
momentum operator \( P_\mu \), used in neutrino oscillation calculation, can be computed from
following mass shell condition
\[ (P_\mu + \hbar A'_\mu \mathcal{P}_L)(P^\mu + \hbar A'^\mu \mathcal{P}_L) = -M^2_f e^2; \] (32)
where now \( A'_\mu \) representing both spin connection \( A_\mu \) and matter (\( A_{f\mu} \)) contribution
\[ A'_\mu = A_\mu + A_{f\mu}, \] (33)
with [10]
\[ A_{f\mu} = \begin{pmatrix} -\sqrt{2} G_F e N^\mu_{e} & 0 \\ 0 & 0 \end{pmatrix}. \] (34)

\( M^2_f \) is the vacuum mass matrix in flavour basis
\[ M^2_f = U \begin{pmatrix} m^2_1 & 0 \\ 0 & m^2_2 \end{pmatrix} U^\dagger, \] (35)
\footnote{There is another point of view about the coupling constant that appear in spin–torsion interaction [9,11-13]. The torsionic contact interaction Lagrangian between two spin half particles is formally identical to the weak interaction Lagrangian and may be written in the \((V - A)\) form, if at least one of the two fermions is massles. This suggest that the spin torsion coupling constant \( G_T \), be also identified with the weak interaction Fermi constant. This suggest \( G_T / G \approx 10^{31} \).}
with

\[ U = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \]  

(36)

\( A_{f\mu} \) is the potential, driven in the context of MSW effect, \( N_e = n_e u^\mu \) is the number current of the electron fluid; \( n_e \) is the electron density in the fluid rest frame, and \( u^\mu \) is the fluid’s four velocity. \( G_F \) is Fermi constant.

Now for a general trajectory with affine parameter \( \lambda \) (\( x^\mu = x^\mu(\lambda) \)), and for relativistic neutrinos, ignoring terms of \( O(A'^2) \) and \( O(A'M_f) \), one finds that column vector of flavour amplitudes

\[ \chi(\lambda) = \begin{pmatrix} <\nu_e|\psi(\lambda)> \\ <\nu_\mu|\psi(\lambda)> \end{pmatrix} \]  

(37)

satisfies in the following differential equation [10]

\[ i\frac{d\chi}{d\lambda} = (\frac{M_f^2 c^2}{2} + \hbar p.A'P_L)\chi, \]  

(38)

where \( p^\mu = dx^\mu / d\lambda \) is the tangent vector to the null world line. In the above relation \( P^0 = p^0 \) and \( P^i = (1 - \epsilon)p^i \) (\( \epsilon << 1 \)). In this way one can calculate the effect of torsion, through eqs.(33) and (28), on neutrino oscillation. As in the case that was considered in [10], the total gravitational contribution \( A_\mu \) is proportional to identity matrix in flavour space, and can not induce neutrino oscillation on its own, but affects it when there are other off-diagonal terms.

To evaluate the order of magnitude of the effect of torsion on oscillation, let us consider a case with only the mass and torsion terms. We want to study the conditions under which the effect of torsion is of the same order as the mass effect, i.e.:

\[ \frac{1}{2} m_\nu^2 c^2 \sim 6 \sqrt{\frac{g}{3}} \frac{96\pi G\hbar}{c^3} p_\nu \cdot J_5. \]  

(39)

For a spin–1/2 particle with \( S_z = \hbar/2 \), momentum \( p_\nu^\mu = (E_\nu/c, 0, 0, p_\nu) \), and density \( \rho_\nu \) we have

\[ J_5^\mu = \rho_\nu \left( \frac{p_\nu c}{E_\nu}, 0, 0, 1 \right). \]  

(40)

Also, noting that \( P^2 = -m_\nu^2 c^2 \) and \( p_\nu^2 = 0 \), up to order \( \epsilon \) we have

\[ p_\nu^\mu = \left( \frac{E_\nu}{c}, 0, 0, \left( 1 + \frac{m_\nu^2 c^2}{2p_\nu^2} \right) P_\nu \right). \]  

(41)

If the source of torsion is also neutrino, \( p_\nu = P_\nu \) and \( E_\nu = E_\nu \), eq. (39) results

\[ \frac{\rho_\nu}{E_\nu} \sim 10^{69} \text{cm}^{-3} \text{eV}. \]  

(42)
This shows that torsion can affect the neutrino oscillation whenever its number density is very large or its energy is very low. If the source of torsion are some other spin–1/2 particles at rest, such as electrons or neutrons, eq. (39) restricts the matter number density as follows

\[ \rho_{\text{matter}} \sim 10^{69} \text{ cm}^{-3}. \] (43)

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