Free Products in R. Thompson’s Group V

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Abstract

We investigate free product structures in R. Thompson’s group V, primarily by studying the topological dynamics associated with V’s action on the Cantor Set. We show that the class of free products which can be embedded into V includes the free product of any two finite groups, the free product of any finite group with Q/Z, and the countable non-abelian free groups. We also show the somewhat surprising result that Z^2 * Z does not embed in V, even though V has many embedded copies of Z^2 and has many embedded copies of free products of pairs of its subgroups.

1 Introduction

We prove some results related to the subgroup structure of R. Thompson’s group V. In particular, we explore conditions on factor groups so that free products of non-trivial factors can embed into V. Our first result shows that V contains many free products of various of the isomorphism classes of its non-trivial subgroups. Our second theorem states that (although V contains many free products as above and many copies of Z^2) the group Z^2 * Z does not embed into V. Of particular interest in this exploration is that the non-embedding result seems very difficult to prove using algebraic methods (say, using a presentation of V). The present authors use topological dynamics (via the characterization of V as a group of homeomorphisms of the Cantor Set C) to attain these results.

Let FPV denote the class of groups which
1. admit decompositions as free products of pairs of non-trivial subgroups, and
2. embed into V.

Also, let A denote the smallest class of groups so that
1. A contains all finite groups,
2. Z ∈ A,
3. Q/Z ∈ A, and
4. A is closed under
   (a) isomorphism,
   (b) passing to subgroup, and
   (c) taking the direct product of any finite member with any member.

We now state our main results.

Theorem 1.1. If K_1, K_2 ∈ A are non-trivial groups, then the group K_1 * K_2 ∈ FPV.

Some previously observed consequences of the above theorem are firstly, that V contains embedded copies of all of the countable non-abelian free groups (an obvious fact), and secondly, that V contains embedded copies of PSL(2, Z) ≅ Z_2 * Z_3 (T ≤ V, and recall that T is C^0...
conjugate to the group of PSL(2,Z) homeomorphisms of $RP^1$ with rational breaks in slope (see [7, 13]).

On the other hand, the following theorem shows that while free products of groups in the isomorphism classes of $V$’s subgroups are often available in $V$, one cannot choose these subgroups indiscriminately.

**Theorem 1.2.** The group $Z^2 \ast Z$ does not embed in $FPV$.

The original motivation for the work in this paper sprang from the question “Does $Z^2 \ast Z$ embed in Thompson’s group $V$?”, which was asked of the first author by Mark Sapir. Some context is given below.

The team of Holt, Röver, Rees and Thomas introduce and analyze the class of groups which have context free co-word problem in [H]. They show this class of groups is nice in various ways. For instance, it is closed under direct products, standard restricted wreath products where the top group is in the subclass of groups with context-free word problem, passing to finitely generated subgroups, and passing to finite index overgroups. They further conjecture that the class is not closed under free product, and currently, one of the lead candidates for proving this conjecture is $(Z^3) \ast Z$.

In the paper [1], Lehnert and Schweitzer show that R. Thompson’s group $V$ is a CFC group. In particular, if $Z^2 \ast Z$ embeds in $V$, then it too would be a CFC group.

Thus, our chief result, in terms of this thread of the development of the theory of CFC groups, simply says that $Z^2 \ast Z$ remains a reasonable candidate for proving the conjecture that the class of CFC groups is not closed under free product.

There is another non-embedding result for $V$ which is of interest in this context. Higman in [9] uses his semi-normal forms to study the dynamics of automorphism groups $G_{n,r}$ acting on specific algebras (where $n \geq 2$ and $r$ are positive integers). Semi-normal forms can help detect infinite orbits under these actions, and other nice properties of elements of the groups $G_{n,r}$. In any case, $V = G_{2,1}$ in Higman’s notation, and Higman shows in [9], using semi-normal forms, that $GL(3, Z)$ does not embed into $G_{n,r}$ for any indices $n$ and $r$. (Brin’s revealing pair technology has many significant parallels with Higman’s semi-normal forms, though it was developed independently.)

There is a technical dynamical property $(\ast)$ for subgroups of $V$, and a set $D$ of subgroups of $V$ which satisfy $(\ast)$, which we call the demonstrative groups. These groups are easy to find as factor subgroups of free product subgroups in $V$; property $(\ast)$ is useful when attempting to build Ping-Pong constructions (we will briefly discuss Fricke and Klein’s classical “Ping-Pong Lemma” in the next section).

While we do not know that every group isomorphic to a group in $D$ can be found in $A$, it is not too hard to show that every group in $A$ can embed in $V$ as a group in $D$. Thus, Theorem 1.1 is actually a corollary of the following lemma.

**Lemma 1.3.** If $K_1, K_2 \in D$ are non-trivial groups then $K_1 \ast K_2 \in FPV$.

One reasonable place to try to extend the class $A$ is to replace property (4.c) with closure under the general extension of a finite member by any member, creating a class $B$. In this case $B$ would contain all of the virtually cyclic groups. It may be easy to prove that $D$ contains isomorphic copies of the virtually cyclic groups; the present authors have made no efforts in that direction.

We state the following questions.

**Question 1.** Is it true that if $G$ and $H$ are non-trivial subgroups of $V$, with $|G| \geq 3$ and $(G, H) \cong G \ast H$ in $V$, then there are distinct non-empty sets $P_G$ and $P_H$ in $\mathcal{C}$ so that for any non-trivial elements $g \in G$ and $h \in H$ we have $P_{Hg} \subseteq P_G$ and $P_{Gh} \subseteq P_H$?

Colloquially, must every free product of groups in $V$ arise from a Ping-Pong in $V$?

Let $CFPV$ be the smallest class of groups which contains $B$ and which is closed under

1. isomorphism,
2. passing to subgroups,
3. extending any finite member by any member, and
4. taking free products of any two of its members.

**Question 2.** Does $FPV = CFPV$?

On one final note, we should like to mention the Java software package which Roman Kogan of SUNY Stony Brook wrote at the NSF funded Cornell Summer 2008 Mathematics Research Experience for Undergraduates. His software provides a convenient interface for calculating and storing products, inverses, and conjugation in the higher dimensional Thompson groups $nV$ created by Brin in [4] (and thus, in the R. Thompson groups $F < T < V$ as well). While we did not use the software to prove anything in this paper, we often used it during exploration to verify that our initial constructions worked as intended.

The authors would like to thank Daniel Farley for his participation in early phases of this project. The first author would also like to thank Mark Sapir for interesting discussions with regards to the family of R. Thompson groups. This note arose as a consequence of one such discussion.

## 2 Basic tools

In this section, we define the terminology of the paper, and state some easy or known facts that will be useful for us in our arguments.

### 2.1 The Cantor Set

We define language and notation describing the Cantor Set, and various aspects and subsets thereof.

We let $T_2$ represent the infinite binary tree. We label the nodes of $T_2$ by finite binary strings in the usual fashion (thus, the node labelled 1101 corresponds to the node found by starting at the root of $T_2$, and travelling down to the right child twice, then to the left child, and finally to the right child, see the diagram below). We identify the Cantor Set $C$ with the infinite descending paths from the root of $T_2$, as is standard. We will say $x \in C$ *underlies a node $n$ of $T_2$* if $n$ is a node of $T_2$ and the path for $x$ passes through $n$. We will call the set of such points the *Cantor Set underlying* $n$, and denote this set by $C_n$.

By way of example, the set of points in the Cantor Set underlying “1101” is given as the set of elements of $\{0, 1\}^\infty$ which begin with the string “1101”. These correspond to all of the infinite descending paths on $T_2$ which start at the root and pass through the node “1101” labelled in the diagram below.

![Diagram of the Cantor Set](image)

Thus, for the example above, we have

$$C_{1101} = \{(s_n) \in C \mid (s_n) = 1101s_4s_5s_6\ldots\}$$
For a finite collection $S$ of points in $\mathcal{C}$, we will specify that a neighborhood $U$ of $S$ is precisely given as a finite union of the Cantor Sets underlying a finite set $N$ of nodes of $T_2$, where we further require that for each point $s \in S$, there is precisely one node $n_s \in N$ so that $s$ underlies the node $n_s$. In particular, a neighborhood of point $x$ in $\mathcal{C}$ will be thought of as the Cantor Set underlying some node $n$ of $T_2$, where $x$ underlies $n$. This specification of the usual notion of neighborhood should cause the reader no confusion in the body of this paper. Given a set $S \subset \mathcal{C}$, we will call any open subset $U \subset \mathcal{C}$ containing $S$ a general neighborhood of $S$. We shall require the use of a general neighborhood only one time in this note.

### 2.2 R. Thompson’s group $V$

The group $V$ is a specific collection of homeomorphisms of $\mathcal{C}$, under the operation of composition. Each such element $u$ of $V$ can be represented non-uniquely as a pair of finite binary subtrees of $T_2$, each with the same number $n$ of leaves, together with an element $\sigma \in \Sigma_n$, the permutation group on $n$ letters.

We will write such a representation as $u \sim (D, R, \sigma)$, where $D$ and $R$ are our finite binary trees, and $\sigma$ is the permutation, as mentioned above. We will often call this simply a tree-pair, and drop explicit mention of the permutation unless we need it in the course of events. We may write $P = (D, R)$ in this case, considering $P$ to be a tree pair (with permutation) which guides us as a rule defining the element $u$ of $V$ it is intended to represent. For the remainder of this section, we will assume $u$ and $P$ are fixed for the purpose of discussion, and to help us define terminology.

The rule which translates a tree-pair into a homeomorphism of the Cantor Set is as follows. Consider $D$ and $R$ as subtrees of $T_2$. For each leaf $i$ of $D$ (where $i \in \{1, 2, \ldots, n\}$), map the Cantor Set underlying $i$ in the unique orientation-preserving, bijective, affine fashion to the Cantor Set underlying the leaf $\sigma(i)$ of $R$. Thus $D$ represents the domain, and $R$ represents the range.

Thoughout this note, we will have elements of $V$ act on $\mathcal{C}$ on the right, so that if $x \in \mathcal{C}$, we will denote by $xv$ the image of $x$ under the action of $v$. If $X \subset \mathcal{C}$, and $v \in V$, then we will denote by $Xv$ the set $\{xv | x \in X\}$. Following these conventions, if $u$ is also an element of $V$, then the symbols $u^v$ and $[u, v]$ are defined by the equations $u^v = v^{-1}uv$ and $[u, v] = u^{-1}v^{-1}uv = (v^{-1})u \cdot v = u^{-1} \cdot u^v$.

If $v \in V$, we define $\text{Supp}(v) = \{x \in \mathcal{C} | xv \neq x\}$. We now have the following standard lemma from permutation group theory, which we use freely in the remainder.

**Lemma 2.1.** Let $u, v \in V$, then $\text{Supp}(u^v) = \text{Supp}(uv)$.

Finally, given $v \in V$ and $x \in \mathcal{C}$, we define the orbit of $x$ under $v$ to be the set

$$\mathcal{O}(x, v) = \{xv^k | k \in Z\}.$$

If this set does not have cardinality one, then we say that the orbit of $x$ under $v$ is non-trivial. Other language to this effect is to be interpreted in the obvious fashion and should cause the reader no confusion.

We note that given any $v \in V$, there is an induced "action" on "most" of the nodes of the infinite binary tree $T_2$, in the following sense. If we pass to any leaf $l$ of a representative tree pair for $v$, or any node $m$ below $l$ (lying further down some infinite descending path from the root which passes through $l$), then the node $l$ (or $m$) will be mapped to a node $n$ of $T_2$; that is, the Cantor set underlying $l$ (or $m$) will be carried affinely and bijectively to the Cantor Set underlying $n$. We thus say $lv = n$ or $mv = n$ when we are thinking of the induced action of $v$ on nodes of $T_2$. (Note, nodes above the leaves of the domain tree will very likely get "split" by $v$, so that there is always a finite subset of $T$ where this sort of induced action makes no sense.)

### 2.3 Revealing pairs

We will make use of Brin’s revealing pair technology (see [3]) in order to define interesting subsets of the Cantor Set $\mathcal{C}$ for specific elements of $V$. The general argument can be given
without using revealing pairs, but they provide a useful context for our discussion. We will define here everything which is required for this note. Lemmas and corollaries appearing in this subsection before Lemma 2.4 can all either be found in [4] or in [12], or are easy consequences of the results found therein.

For the tree pair \( P = (D, R) \), we can consider the common tree \( C = D \cap R \), which is the finite rooted subtree of \( T_2 \) with each node a node of both \( D \) and \( R \). Each leaf of \( C \) is either the root of a descending subtree of \( D \), or the root of a descending subtree of \( R \) (in either such case, we call these descending trees components of \( D \setminus R \) or of \( R \setminus D \), as the case may be), or a leaf of both \( D \) and \( R \) (in this case, we call the leaf a neutral leaf of \( P \)).

The tree pair \( P \) is called a revealing pair if it satisfies two conditions. The first condition is that for each complementary component \( X \) of \( D \setminus R \), \( X \) has a leaf \( r_X \) which, under iteration of the rule \( P \), travels through the neutral leaves of \( C \) until it is finally mapped to the root of \( X \) (\( r_X \) is unique for \( X \), and is called the repelling leaf of \( X \) or the repeller of \( X \)). The second condition is similar; if \( n_Y \) is the root of a component \( Y \) of \( R \setminus D \), then iteration of the rule \( P \) has \( n_Y \) travel through the neutral leaves of \( C \) until it finally maps beneath itself to a leaf \( l_Y \) of \( Y \) (the leaf \( l_y \) is called the attracting leaf of \( Y \) or the attractor of \( Y \)). By the discussion preceding Lemma 10.2 of [4], each element in \( V \) has a revealing pair representative (and as before, this is not unique).

Let \( v \in V \). An easy consequence of Proposition 10.1 of [4] is that there is a minimal non-negative power \( k \) so that \( v^k \) acts on \( C \) with no non-trivial finite orbits. Set \( u = v^k \), so that \( u \) admits no non-trivial finite orbits in its action on \( C \). Finally, let us assume \( P = (D, R) \) is actually a revealing pair representing \( u \).

We obtain a list of useful, obvious results, which we leave to the reader to verify. The phrase “maps to” in this following lemma is referring to the action of \( w \) on nodes of \( T_2 \).

**Lemma 2.2.** Suppose \( w \in V \) so that \( w \) admits no non-trivial finite orbits in its action on the Cantor Set \( C \). Suppose further that the revealing pair \( P_w = (D_w, R_w) \) represents \( w \).

1. Any repeller \( r_X \) of a component \( X \) of \( D_w \setminus R_w \) always maps to the root of \( X \) by the rule \( P_w \).
2. The root \( n_Y \) of any component \( Y \) of \( R_w \setminus D_w \) always maps to the attractor \( l_Y \) of \( Y \) by the rule \( P_w \).
3. The map \( w \) restricted to any Cantor Set underlying a node \( r_X \) or \( n_Y \) as above is affine with slope not equal to one.
4. Every point in \( C \) which is fixed by \( w \) and which does not underly a node \( r_X \) or \( n_Y \) as above lies under a neutral leaf \( n \) of the pair \( P_w \) upon which \( w \) must act as the identity.

We continue our discussion with the element \( u \) constructed previously.

By an application of the standard Contraction Lemma, we observe that if a leaf \( l \) of \( D \) is mapped above or below itself in \( T_2 \) by the rule \( P \), then there will be a unique fixed point in the Cantor Set underlying \( l \) (if \( l \) maps above itself, consider the inverse map \( u^{-1} \) in order to force a contraction). Fixed points underlying repellers of \( D \) will be called repelling fixed points of \( u \), and fixed points underlying attractors in components of \( D \setminus R \) will be called attracting fixed points of \( u \).

**Corollary 2.3.** Under the same hypotheses as in Lemma 2.2, for each repeller \( r_x \) there is a unique repelling fixed point \( p_x \) underlying it, and for each attractor \( l_y \) there is a unique attracting fixed point underlying it.

The diagram below illustrates a likely tree pair for such an element \( u \). This particular tree-pair indicates that the element \( u \) has one repelling fixed point (under 0010) and two attracting fixed points (under 10 and 11 respectively).
We call the repelling and attracting fixed points under the action of \( u \) the important points of \( u \), and denote the set of such as \( I(u) \).

Now, throughout the remainder of this paper, if we discuss the important points of an element \( w \) of \( V \), it is to be understood that \( w \) does not admit finite non-trivial orbits in its action on \( C \). In this case, given a revealing pair \( Q = (S, T) \) representing \( w \), the Cantor Set underlying each root of a component of \( S \setminus T \) represents a repelling basin for \( w \), and the Cantor Set underlying a root of a component of \( T \setminus S \) represents an attracting basin for \( w \).

Let \( \{E_i\}_{i=1}^n \) represent the components of \( D \setminus R \) and let \( \{F_j\}_{j=1}^m \) represent the components of \( R \setminus D \). For each \( i \), \( E_i \) contains a repelling leaf, as defined above, and all other leaves of \( E_i \) are called sources. Likewise, for each component \( F_j \), there is an attracting leaf, as defined above, and all other leaves are called sinks. For each source leaf \( s_0 \), there is a path \( s_0 = n_0, n_1, \ldots, n_t = s_k \) through neutral leaves \( n_1, \ldots, n_{t-1} \) of \( C \), and then visiting a sink \( s_k \), so that \( u^p \) will throw the Cantor Set underlying \( s_0 = n_0 \) onto the Cantor Set underlying \( n_p \) for all indices \( 0 \leq p \leq t \). We call this path the source-sink chain \( s_0 = s_k \) for \( P \). We can now make a bi-partite graph, whose vertices are labelled by the repelling and attracting basins, and whose edges correspond with and are labelled by source-sink chains connecting repelling basins to attracting basins. We call this graph the flow graph for \( P \). (In the presence of finite non-trivial orbits, the flow graph has other information appended to it as in \([1, 2]\). Here, we only define the portions of the standard flow graph required to support the discussion in the remainder of this note.)

For our example above, the flow graph is as diagrammed.

Let \( X \) now represent a connected component of the flow graph for \( P \). We can form a set, the Cantor Set underlying \( X \), by taking the union of the Cantor Sets underlying the attracting and repelling basins of \( X \), and underlying the neutral leaf nodes occurring in the source-sink chain labels of the edges of \( X \). This union is immediately independent of the revealing pair representing \( u \), so that we will also call it a component of support of \( u \). We will define the component support of \( u \), denoted \( \text{Supp}(u) \), as the union of the components of support of \( u \). We note that \( \text{Supp}(u) \) actually is the topological closure of the support of \( u \), since it consists of the support of \( u \) together with the important points of \( u \).

In particular, we have a useful lemma, which is easy to see if the above is understood.
Lemma 2.4. Let $p \in C$ and $g, h \in V$ so that they admit no finite non-trivial orbits and with $p \in I(g) \cap I(h)$. Then, there is a node $n$ in $T_2$ so that

1. $p$ underlies $n$,
2. $C_n \subseteq \text{Supp}(g) \cap \text{Supp}(h)$, and
3. the commutator $[g, h]$ acts as the identity over $C_n$.

We now give three background lemmas from [1], including their proofs for completeness.

Lemma 2.5. Let $g, h \in V$ so that they admit no finite non-trivial orbits, and so that $g$ and $h$ commute. We then have $I(g) \cap I(h) = I(g) \cap \text{Supp}(h) = I(h) \cap \text{Supp}(g)$.

Proof. We show that if $x \in I(g) \cap \text{Supp}(h)$, then $x \in I(h)$, from which the lemma immediately follows.

If $x$ is not an important point of $h$, then $g$ must have infinitely many important points, since the full orbit of $x$ under that action of $h$ consists of important points for the functions $g^{(n)} = g$.

Lemma 2.6. Let $g, h \in V$ so that they admit no finite non-trivial orbits, and so that $g$ and $h$ commute. Suppose $X$ is a component of support for $g$, and $Y$ is a component of support for $h$. If $X \cap Y \neq \emptyset$, then $X \subset \text{Supp}(h)$ and $Y \subset \text{Supp}(g)$.

Proof. Suppose $x \in X \cap Y$. By Lemma 2.4 we can choose $x$ so that $x$ is not an important point of $g$ or of $h$.

Let $x_- = \lim_{n \to -\infty} xg^{-n}$ and $x_+ = \lim_{n \to -\infty} xg^n$. It is immediate that these limits exist and that $x_-, x_+ \in I(g)$.

We further see that both $x_-$ and $x_+$ are important points for $h$ by applying Lemma 2.5. For example, if $x_+$ is not in $\text{Supp}(h)$ then there is $n \in N$ with $xg^n = xg^n \neq xhg^n$.

Now, as $X$ underlies a connected component $C$ of the flow graph of $g$, the set of important points of $g$ in $X$ is actually contained in the component support of $h$. Therefore, some neighborhood of these points in $X$ is actually contained in the component support of $h$. It now follows that all of $X$ must be contained in the component support of $h$, and by symmetry, all of $Y$ must be contained in the component support of $g$.

The following corollary follows from Lemma 2.6.

Corollary 2.7. Let $g, h \in V$ so that they admit no finite non-trivial orbits, and so that $g$ and $h$ commute. Let $C_g$ ($C_h$) be the components of support of $g$ ($h$) which intersect non-trivially the components of support of $h$ ($g$) non-trivially. Under these circumstances, we have $\text{Supp}(g) \cap \text{Supp}(h) = \bigcup_{X \in C_g} X = \bigcup_{Y \in C_h} Y$.

We are finally ready to prove the following lemma, which provides a fundamental tool in our later analysis.

Lemma 2.8. Let $g, h \in V$ so that they admit no finite non-trivial orbits, and so that $g$ and $h$ commute. Suppose $X$ is a component of support for $g$, and $Y$ is a component of support for $h$. If $X \cap Y \neq \emptyset$, then $X = Y$.

Proof. Suppose $g, h, X, Y$ as in the first two sentences of the statement.

Let $x \in X \cap Y \setminus I(g)$. Note that such an $x$ exists and is not an important point for either $g$ or $h$. Define $x_+ = \lim_{n \to -\infty} xg^n$ as before.

Suppose there is $m \in N$ so that $xh^m \not\in X$.

Since $x_+$ is an important point of $h$ as well as $g$, if $S$ is large enough, then $x_S = xg^S$ will be close enough to $x_+$ that $m$ applications of $h$ to $x_S$ will result in a point still in a basin of attraction of $g$ containing $x_+$ (recall such a basin is a Cantor Set underlying a root of a complimentary component of $R_g \setminus D_g$ for some representative revealing pair $P_g = (D_g, R_g)$ for $g$). In particular, we see that $xh^mg^S$ is not in $X$ while $xg^S h^m \in X$.  

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Our result now follows from the connectivity of the component of the flow graph of \( g \) over \( X \), the connectivity of the component of the flow graph of \( h \) over \( Y \), the fact that given any important point \( p \) in \( X \cap Y \), there is a neighborhood \( N_p \) of that point which is fully contained in both \( X \) and \( Y \) (so that all orbits in \( X \) under \( g \) which enter \( N_p \) must be fully contained in \( Y \) and all orbits in \( Y \) under \( h \) which enter \( N_p \) must stay in \( X \)), and the fact that every point in \( X \) or \( Y \) limits to the important points of \( g \) and \( h \) under repeated applications of \( g \) or \( h \).

The following lemma is reminiscent of a similar result by Brin and Squier for elements of Thompson’s group \( F \) (or actually, for \( \text{PL}_\mathbb{Z}(I) \)) in [5], and the proof is philosophically the same (although, the details here are slightly more complicated due to the presence of extra attractors and repellers). This lemma combines very powerfully with the Lemma 2.8.

**Lemma 2.9.** Suppose that \( g, h \in V \) so that they admit no finite non-trivial orbits. Suppose further that \( g \) and \( h \) have a common component of support \( X \), and that the actions of \( g \) and \( h \) commute over \( X \). Then there are non-trivial powers \( m \) and \( n \) so that \( g^m = h^n \) over \( X \).

**Proof.** Let us suppose \( g \) is represented by a revealing pair \( P_g = (D_g, R_g) \), so that given an important point \( q \) of \( g \), the phrase “the basin containing \( q \)” is well defined (ie., the Cantor Set underlying the root of the complimentary component of \( D_g \backslash R_g \) or of \( R_g \backslash D_g \) which has a repelling or attracting leaf as a node over \( q \)).

Fix \( p \), an important point of both \( g \) and \( h \) in \( X \). Since \( g \) and \( h \) are affine in a neighborhood \( N_p \) of \( p \), there are non-trivial powers \( m \) and \( n \) so that \( g^m = h^n \) on \( N_p \). Now the element \( g^m h^{-n} \) is trivial on \( N_p \). For each \( x \in I(g) \cap X \) where \( g^m h^{-n} \) is fixed on some neighborhood of \( x \), let \( N_x \) be such a neighborhood, and let \( \mathcal{N} \) be the union of these neighborhoods.

We now have that \( \mathcal{N} \) is actually a neighborhood of \( I(g) \cap X = I(h) \cap X \). Otherwise, there is a pair \( r, a \in I(g) \cap X \) with \( a \) an attracting fixed point of \( g \) and \( r \) a repelling fixed point of \( g \), where one of \( \{ a, r \} \) is in \( \mathcal{N} \) and the other is not in \( \mathcal{N} \), and where there is a source-sink chain from the basin \( B_r \) containing \( r \) to the basin \( B_a \) containing \( a \) (this follows from the connectivity of the component of the flow graph of \( g \) over \( X \)). In the case that \( a \in \mathcal{N} \), there is \( x \in \operatorname{Supp}(g^m h^{-n}) \cap B_r \) and a positive power \( k \) so that \( xg^k \in \mathcal{N} \cap B_a \). In the case that \( r \in \mathcal{N} \), there is \( x \in \operatorname{Supp}(g^m h^{-n}) \cap B_a \) so that a negative power \( k \) has \( xg^k \in \mathcal{N} \cap B_r \). In either of these cases, Lemma 2.1 now shows that \( (g^m h^{-n})^g \) has support where \( g^m h^{-n} \) acts as the identity, which is impossible since \( g \) commutes with \( g \) and with \( h \).

Now again by Lemma 2.1 \( g^m h^{-n} \) cannot have any support in \( X \). Otherwise, for any \( x \in X \cap \operatorname{Supp}(g^m h^{-n}) \), there is a positive power \( k \) of \( g \) so that \( xg^k \) is near to an attractor of \( g \) in \( X \), in particular, \( k \) can be taken large enough so that \( xy^k \in \mathcal{N} \). But then, \( (g^m h^{-n})^g \) has support in \( \mathcal{N} \).

### 3 Some free products which do occur in \( V \)

In this section we prove Theorem 1.1. Our main constructive tool is the standard Ping-Pong Lemma of Fricke and Klein [6].

#### 3.1 Free product recognition

We give the version of the Ping-Pong Lemma essentially as it appears in [8].

**Lemma 3.1.** *(Ping Pong Lemma)*

Let \( G \) be a group acting on a set \( X \) and let \( H_1, H_2 \) be two subgroups of \( G \) such that \( |H_1| \geq 3 \) and \( |H_2| \geq 2 \). Suppose there exist two non-empty subsets \( X_1 \) and \( X_2 \) of \( X \) such that the following hold:

- \( X_1 \) is not contained in \( X_2 \).
\begin{itemize}
  \item for every \( h_1 \in H_1 \), \( h_1 \neq 1 \) we have \( h_1(X_2) \subset X_1 \),
  \item for every \( h_2 \in H_2 \), \( h_2 \neq 1 \) we have \( h_2(X_1) \subset X_2 \),
\end{itemize}

Then the subgroup \( H = \langle H_1, H_2 \rangle \leq G \) of \( G \) generated by \( H_1 \) and \( H_2 \) is equal to the free product of \( H_1 \) and \( H_2 \):

\[ H = H_1 \ast H_2. \]

In order to make use the Ping-Pong Lemma, we first define a set of subgroups of \( V \) which are easy to use as factors in free product decompositions.

### 3.2 Demonstrative groups

It is now time to define the demonstrative groups mentioned in the introduction. We will say a subgroup \( G \leq V \) is \textit{demonstrative} if and only if there is a node \( n \in T_2 \) so that for every non-trivial \( g \in G \), \( g \) admits a revealing pair representation \( P_g = (D_g, R_g) \) so that \( n \) is a neutral leaf of \( P_g \) and so that \( n \) is moved to a different node of \( T_2 \) by the action of \( g \). We call any node \( p \in T_2 \) which satisfies the properties of \( n \) in the definition of a demonstrative group \( G \) a \textit{demonstration node for} \( G \). As in the introduction, we denote the set of demonstrative subgroups of \( V \) by the symbol \( D \).

Recall that given \( n \) a node of \( T_2 \), we denote by \( \mathcal{C}_n \) the Cantor Set underlying \( n \). The following is an easy dynamical fact pertaining to demonstrative groups.

**Lemma 3.2.** Let \( G \leq V \). If \( G \) is a demonstrative group then there is a node \( n \) of \( T_2 \) so that for all \( g \in G \),

\[ \mathcal{C}_ng \cap \mathcal{C}_n = \emptyset. \]

**Proof.** If \( G \) is a demonstrative group, then by taking a demonstration node for \( G \) as our \( n \) we will produce the desired result, since the Cantor Set underlying any non-fixed neutral leaf of a tree pair \( P \) is moved entirely off itself by the element of \( V \) represented by \( P \). \( \square \)

We now show that the set of demonstrative groups is closed under some nice operations.

**Lemma 3.3.** Suppose \( G \) and \( H \) are demonstrative groups, and \( G \) is finite. Then

1. every subgroup \( K \leq H \) is demonstrative, and
2. there is a demonstrative group \( L \) with \( K \cong G \times H \).

**Proof.** The first point is immediate from the definition of a demonstrative group.

The second point requires a bit more care. Let \( m \) be a demonstration node for \( G \). For each element \( g \in G \), let \( P_g = (D_g, R_g) \) be a revealing pair for \( g \) which has \( m \) as a neutral leaf. Since \( G \) is finite, the orbit of \( m \) in \( T_2 \) under the action of \( G \) is a finite collection \( O_m \) of nodes of \( T_2 \).

We note that Lemma 3.2 guarantees us that \( g_1 \neq g_2 \in G \) implies \( \mathcal{C}_mg_1 \cap \mathcal{C}_mg_2 = \emptyset \).

Let \( n \) be the demonstration node for \( H \). We will find \( H_m \), a copy of \( H \), which is demonstrative with demonstration node \( mn \) (concatenate the names of the nodes \( m \) and \( n \) in \( T_2 \)), where every element of \( H_m \) commutes with every element of \( G \), so that \( \langle G, H_m \rangle \cong G \times H \).

We build \( H_m \) as follows. For every element \( h \in H \), let \( P_h = (D_h, R_h) \) be a representative tree pair for \( h \) which has \( n \) as a neutral leaf, in accord with the definition of \( H \) being demonstrative with demonstration node \( n \). We now build the tree pair \( P_h = (D_h, R_h) \) for \( h \), the element of \( H_m \) which will be the image of \( h \) under the embedding sending \( H \) to \( H_m \).

Let \( T \) be a finite binary subtree of \( T_2 \) with root the root of \( T_2 \), which contains every node in \( O_m \) (there are infinitely many such if \( O_m \) does not form the set of leaves for a finite binary subtree of \( T_2 \) with root the root of \( T_2 \)). Define \( D_T \) to be the extension of the tree \( T \), where we append the tree \( D_h \) to each of the leaves of \( T \) in \( O_m \). Define \( R_T \) to be the extension of \( T \) we get when we append the tree \( R_h \) to each leaf of \( T \) in \( O_m \). Use the identity permutation on the leaves of \( T \) not in \( O_m \), and for a particular node \( l \in O_m \), use the corresponding permutation for \( h \) on the leaves under \( l \) in the domain and range trees \( D_T^h \) and \( R_T^h \).
It is immediate from construction that this produces a set of elements $H_m$ so that $\langle H_m \rangle \cong H$.

The node $mn$ is demonstrative for $H_m$. It is also demonstrative for $G$. Let $A$ be a tree which has $n$ as a node. Now, for each element $g$ of $G$, simply append $A$ to every leaf in the finite cycle of neutral leaves containing $m$ for the tree pair $P_g$, to make a new revealing tree pair for $g$ which now has the node $mn$ as a neutral leaf (extending the permutation in the obvious fashion).

Since $H_m$ acts the same way under every image of $m$ under the action of $G$, we see that the elements of $G$ and the elements of $H_m$ commute.

Remark 3.4. We note that from the proof above that if $G$ is a finite demonstrative group with demonstration node $m$ and $H$ is a demonstrative group with demonstration node $n$, then

1. when passing to a subgroup $K \leq G$, we can still retain $m$ as a demonstration node for $K$, and

2. one can use the node $mn$ as the demonstration node of the demonstrative direct product representative of $G \times H$.

Lemma 3.5. There are demonstrative embeddings of

1. $Q/Z$ in $V$,
2. every non-trivial cyclic group in $V$, and
3. every finite group in $V$.

In all of these cases, we can find demonstrative embeddings with demonstration node “0”.

Proof. The embedding of $Q/Z$ in $V$ as given in Proposition 5.6 of [3] is demonstrative. In Figure 2 of that article, either of the quarters “01” and “11” can serve as demonstration nodes for the embedded image of $Q/Z$ in $V$. More generally, the node for “$J_{2,1}$” given in the embedding described in that article (which node is chosen by the reader), can serve as the demonstration node for that embedding. We note that we can conjugate any choice of any such embedding of $Q/Z$ in $V$ by an element $\theta$ of $V$ which sends the node chosen for $J_{2,1}$ to the node “0”. Our new embedding will have the node “0” as a demonstration node.

It is quite easy to find demonstrative embeddings of non-trivial cyclic groups in $V$; the element $g$ given by the revealing pair in the diagram below generates an infinite cyclic group with demonstration node “0”.

![Diagram](image)

We now show that for any fixed positive natural number $n$, the symmetric group on $n$ letters $S_n$ has a demonstrative representation with demonstration node “0”.

Let $T$ be a binary tree with $n!$ leaves, which includes “0” as a leaf, with a secondary labeling on leaves using the elements of $S_n$ in some order (with the node “0” being labelled with the identity element). Represent each element $\alpha$ of $S_n$ by the tree pair $(T,T)$, and use the permutation which sends the node labelled $m$ to the node labelled with the group element $m \cdot \alpha$. It is immediate that this is a faithful representation of $S_n$, and that every node is moved by every
non-trivial element of \( S_n \). In particular, the node “0” labelled by the identity element in our initial labelling of \( T \) works as well as any other as our demonstration node.

Passing to a subgroup of a demonstrative group \( G \) with demonstration node \( m \) produces a demonstrative group with demonstration node \( m \) by Lemma 3.3 and Remark 3.4, so every finite group has a demonstrative representation with node “0” as a demonstration node.

In fact, by essentially the argument given above for the case of \( Q/Z \), if we have a demonstrative group \( G \), there is a conjugate version \( G' \) of \( G \) with demonstration node “0”, which therefore have the following lemma.

**Lemma 3.6.** Suppose \( G \) is a demonstrative subgroup of \( V \). There is an isomorphic copy \( G_0 \) of \( G \) with demonstration node “0”, and another isomorphic copy \( G_1 \) of \( G \) with demonstration node “1”.

The following is an immediate consequence of Lemma 3.3, Lemma 3.5, the definition of the demonstrative groups, and the definition of the class of groups \( A \).

**Corollary 3.7.** If \( G \) is a group in the class \( A \), then there is a demonstrative group \( K \) so that \( G \cong K \).

We are now ready to prove our first primary result.

**Proof.** (of Lemma 1.3 and therefore of Theorem 1.1): By Corollary 3.7, we need only show that given two demonstrative groups, we can find a copy of their free product in \( V \).

In general, this will follow easily from the Ping-Pong Lemma. We will need a separate argument for \( Z_2 \ast Z_2 \).

Let \( G \) and \( H \) be non-trivial demonstrative groups, not both isomorphic with \( Z_2 \). Let \( X_1 = C_1 \) and \( X_2 = C_0 \), that is, the right and left halves of the Cantor Set, respectively.

Let \( G_0 \) and \( H_1 \) be the copies of \( G \) and \( H \) with demonstration nodes “0” and “1”, respectively (as in Lemma 3.6).

We note that any non-trivial element of \( G_0 \) takes the whole of the left half of the Cantor Set \( C_0 \) into the right half \( C_1 \). Similarly, any non-trivial element of \( H_1 \) takes the whole of the right half of the Cantor Set into the left half of the Cantor Set. Therefore, by the Ping-Pong lemma, the group \( \langle G_0, H_1 \rangle \cong G_0 \ast H_1 \cong G \ast H \).

Now let \( G = \langle g \rangle \) and \( H = \langle h \rangle \), where \( g \) and \( h \) are represented by the tree pairs \( P_g = (D_g, R_g) \) and \( P_h = (D_h, R_h) \) below. Both \( g \) and \( h \) are order two, so \( G \cong Z_2 \cong H \). However, direct calculation shows that \( gh \) has infinite order. In particular, \( \langle G, H \rangle \cong Z_2 \ast Z_2 \).
4 Non-embedding results

We now begin to prove our second primary result, Theorem 12.

After we develop some algebraic processes with controlled topological dynamics, an algorithmic process will demonstrate the non-embedding of $Z^2 \ast Z$ into $V$.

4.1 Some commutators in $Z^2 \ast Z$

Let $X$ be a non-empty set. Let $X^{-1}$ be a set disjoint from $X$ in bijective correspondence with $X$. If $\tau : X \rightarrow X^{-1}$ is the bijection, for each $a \in X$, denote by $a^{-1}$ the element $\tau a$. If $z \in X^{-1}$, denote by $z^{-1}$ the element $\tau^{-1}(z)$. We will call $X$ the alphabet, and for any element $a \in (X \cup X^{-1})$, we will call $a$ a letter. We will call any finite string of letters a word in $X$. If $w = w_1 w_2 \cdots w_k$ is a word in $X$, then we will denote by $w^{-1}$ the word $w_k^{-1} w_{k-1}^{-1} \cdots w_1^{-1}$. For any integer $n$, define the expression $w^n$ as $n$ successive occurrences of the word $w$ if $n \geq 0$, and $-n$ successive occurrences of the word $w^{-1}$ if $n < 0$.

Now let $a$, $b$, and $c$ be words in $X$. We will say that a word $w$ in $X$ is an $(a, b, c)$-commutator if there are minimal integers $n > 0$, $x_1$, $y_1$, and $z_1$ with $|x_i| + |y_i| \neq 0$ and $z_i \neq 0$ for all indices $1 \leq i \leq n$, so that

$$w = [a^x b^y, [a^{x_2} b^{y_2}, \ldots [a^{x_{n-1}} b^{y_{n-1}}, [a^{x_n} b^{y_n}, c^{z_n}]^{z_{n-2}} \cdots z_1]].$$

Note that in this paper, the commutator bracket $[u, v]$ will always represent the expression $u^{-1} v^{-1} uw$, as before for general elements of $V$.

The following is immediate from the definition of an $(a, b, c)$-commutator.

Lemma 4.1. Suppose $X$ is an alphabet with $a$, $b$, and $c$ words in $X$, and let $t$ be an $(a, b, c)$-commutator. If $w$ is an $(a, b, t)$-commutator, then $w$ is an $(a, b, c)$-commutator.

In the next lemma, we abuse notation by treating words in the alphabet $\{a, b, c\}$ as elements of the group $Z^2 \ast Z$ given by the presentation $\langle a, b, c \mid [a, b] \rangle$ (recall that by our definition, words in an alphabet also include "inverse" letters). Given words $w_1$, $w_2$, $\ldots$, $w_j$ in the alphabet $\{a, b, c\}$, we denote by $\langle w_1, w_2, \ldots, w_j \rangle$ the subgroup of $Z^2 \ast Z$ generated by the elements represented by these words. Henceforth, we will freely confuse words in an alphabet with group elements when it seems unlikely to cause confusion.

Lemma 4.2. Let $i$, $j$, and $k$ be integers, and define $t = [a^i b^j, c^k]$. If $|i| + |j| \neq 0$ and $k \neq 0$, then $(a, b, d)$ factors as $\langle a, b \rangle \ast \langle d \rangle \cong Z^2 \ast Z$.

Proof. Let

$$w = A_0 T_0 A_1 T_1 \cdots A_n T_n$$

where $A_p \in \langle a, b \rangle$ and $T_p \in \langle t \rangle$ for all valid indices $p$, where $X_p \neq 1$ for $X \in \{A, T\}$ except possibly for $A_0$ or $T_n$, and where if $n = 0$, one of $A_0$, $T_0$ is non-trivial in $\langle a, b \rangle$ and $\langle t \rangle$ respectively. We say $w$ is given in form (*).

We will have our lemma if we can show that $w$ does not represent a trivial element in $Z^2 \ast Z$.

We proceed by induction on $n$. We will use the phrase resultant form for any expression written as

$$\prod_{p=0}^m a^{x_p} b^{y_p} c^{z_p}$$

where $m$ is an integer with $0 \leq m$, $|x_p| + |y_p| \neq 0$ if $p > 0$, and where $z_p \neq 0$ if $p < m$. We note that in all such forms, the resulting expression cannot represent the trivial element in $Z^2 \ast Z$ unless $m = x_0 = y_0 = z_0 = 0$.

We shall prove our result by showing that if $w$ is given as in form (*), where we further have that $T_n \neq 1$, then $w$ will have resultant expression ending with one of the two forms (**) below

$$c^{-k} a^{j} b^{i} c^{k}$$

$$c^{-k} a^{-j} b^{-i} c^{k} a^{j} b^{i}$$
(where in both forms \( f \) is a positive integer). In both cases \( w \) cannot be trivial in \( \mathbb{Z}^2 \times \mathbb{Z} \).

The more general result then follows: elements of the form \( a^x b^y \) are not trivial in \( \mathbb{Z}^2 \times \mathbb{Z} \) if \( |x| + |y| \neq 0 \), and any element of \( \mathbb{Z}^2 \times \mathbb{Z} \) with a resultant expression ending with either of the forms in (***) cannot be made trivial by a postmultiplication by a string \( a^x b^y \) for integer values of \( x \) and \( y \).

We now begin our induction.

1. Suppose \( w = A_0T_0 \).

In this base case, if \( A_0 \) is trivial, the resultant form is \( t^z \) for some \( z \neq 0 \). This expression ends with one of the two forms in the list (***) (the resultant form depends on the sign of \( z \)). In similar fashion, if both \( A_0 \) and \( T_0 \) are not trivial in \( \langle a, b \rangle \) and \( \langle t \rangle \) respectively, we then have

\[
w = A_0T_0 = a^x b^y t^z = a^x b^y (a^{-i} b^{-j} c^{-k} a^i b^j c^k)^z
\]

If \( z < 0 \), this word admits no simple cancellations in the group \( \mathbb{Z}^2 \times \mathbb{Z} \), and is thus effectively in the resultant non-trivial form in \( \mathbb{Z}^2 \times \mathbb{Z} \) already (formally, we will need to re-arrange the orders of some \( a \)'s and \( b \)'s after expanding the negative power \( z \)). In any case, the word will end in the form \( c^{-k} a^{-i} b^{-j} c^k a^i b^j \), which is in the bottom form of (***)

If \( z > 0 \), then \( w \) simplifies as

\[
w = a^{-i} b^{-j} c^{-k} a^i b^j c^k (a^{-i} b^{-j} c^{-k} a^i b^j c^k)^{z-1}
\]

which again is a non-trivial resultant form in \( \mathbb{Z}^2 \times \mathbb{Z} \), even if the leading \( a^{x-i} \) and \( b^{y-j} \) terms are trivial. In particular, this word ends in the top form of (***)

2. Suppose now that \( n \) is some positive integer and we know by induction that for any expression of the form

\[
\prod_{p=0}^m A_p t^{s_p}
\]

(where \( n > m \geq 0 \), \( A_p \neq 1 \) for all \( p > 0 \), and \( s_p \neq 0 \) for all \( p \)) the resultant form of our word ends in one of the two forms in (***)

We now show that \( w \) has resultant form in the list (***)

To begin, we note that \( w \) can be expressed as

\[
w = rA_n t^{s_n},
\]

where \( r \) is expressed in resultant form and ends in one of the two forms in (***) (by our induction hypothesis), where \( s_n \neq 0 \), and where \( A_n \) is not trivial in \( \langle a, b \rangle \)

There are now two cases in our analysis, each of which splits into two further subcases.

2.(a) Suppose \( r \) ends with the form \( c^{-k} a^{i} b^{j} c^{k} \) where \( f > 0 \), \( |i| + |j| \neq 0 \), and \( k \neq 0 \).

If \( s_n < 0 \) we obtain the following

\[
w = \ldots c^{-k} a^{i} b^{j} c^{k} \cdot A_n \cdot c^{-k} a^{-i} b^{-j} c^{k} a^{i} b^{j} \ldots \cdot c^{-k} a^{-i} b^{-j} c^{k} a^{i} b^{j}
\]

where there are \(- s_n\) copies of \( c^{-k} a^{-i} b^{-j} c^{k} a^{i} b^{j} \) at the end of this expression. In this case there is absolutely no internal cancelling, and the expression as written above is in resultant form (recall, \( A_n \) is not trivial in \( \langle a, b \rangle \)). This expression is in the bottom form of (***)

If \( s_n > 0 \) we obtain

\[
w = \ldots c^{-k} a^{i} b^{j} c^{k} A_n \cdot a^{-i} b^{-j} c^{-k} a^{i} b^{j} c^{k} \ldots \cdot a^{-i} b^{-j} c^{-k} a^{i} b^{j} c^{k}
\]

which either fails to cancel the \( A_n \) expression with \( a^{-i} b^{-j} \), in which case we obtain the top form of (***) or, which has \( A_n \) cancel with \( a^{-i} b^{-j} \), in which case the bold substring above cancels so that we obtain the form

\[
w = \ldots c^{-k} a^{i} b^{j} c^{k} a^{i} b^{j} c^{k} \ldots \cdot a^{-i} b^{-j} c^{-k} a^{i} b^{j} c^{k}
\]

\[= \ldots c^{-k} a^{i} b^{j} c^{k} \ldots \cdot a^{-i} b^{-j} c^{-k} a^{i} b^{j} c^{k}
\]

\[= \ldots c^{-k} a^{i} b^{j} c^{k} \ldots \cdot a^{-i} b^{-j} c^{-k} a^{i} b^{j} c^{k}
\]

\[= \ldots c^{-k} a^{i} b^{j} c^{k} \ldots \cdot a^{-i} b^{-j} c^{-k} a^{i} b^{j} c^{k}
\]

\[= \ldots c^{-k} a^{i} b^{j} c^{k} \ldots \cdot a^{-i} b^{-j} c^{-k} a^{i} b^{j} c^{k}
\]

\[= \ldots c^{-k} a^{i} b^{j} c^{k} \ldots \cdot a^{-i} b^{-j} c^{-k} a^{i} b^{j} c^{k}
\]

\[= \ldots c^{-k} a^{i} b^{j} c^{k} \ldots \cdot a^{-i} b^{-j} c^{-k} a^{i} b^{j} c^{k}
\]
Lemma 4.3. Let $\langle Z, W \rangle$. We now create an algorithm, whose net effect will be to show that $\langle Z, W \rangle$. We do this by taking a supposed embedding, and through a process of "improvements," we $\beta$ sets as follows. Let $\alpha$ which must still be isomorphic to $\beta$. Throughout the remainder of the paper, we will assume $\alpha, \beta, \gamma$ with well defined versions of $\alpha, \beta, \gamma$. Thus, in all cases, our expression for $w$ ends in one of the forms $\langle ** \rangle$.

The following follows from the previous two lemmas via a simple induction argument.

**Lemma 4.3.** Let $Z^2 \ast Z$ be presented as $\langle a, b, c | [a, b] \rangle$. If $t$ is an $(a, b, c)$-commutator, then $\langle a, b, c \rangle$ factors as $\langle a, b \rangle \ast \langle t \rangle \cong Z^2 \ast Z$.

### 4.2 $Z^2 \ast Z$ cannot embed in $V$

We now create an algorithm, whose net effect will be to show that $Z^2 \ast Z$ does not embed into $V$. We do this by taking a supposed embedding, and through a process of "improvements," we show that our embedded group actually admits torsion elements.

Throughout the remainder of the paper, we will assume $G = \langle a, b, c | [a, b] \rangle \cong Z^2 \ast Z$, and that $\phi : G \to V$ is an embedding. Define $\alpha = a\phi$, $\beta = b\phi$, and $\gamma = c\phi$.

Each subsubsection which follows carries out an algebraic selection of a subgroup of $\phi(G)$ which must still be isomorphic to $Z^2 \ast Z$ for various reasons. The reader should imagine we have improved our initial selection of $\phi$ at such times, so that we always enter the next subsubsection with well defined versions of $\alpha$, $\beta$, and $\gamma$. In each case, the selection will be motivated by "improving" the dynamics of the interactions of $\alpha$, $\beta$, and $\gamma$. The reader will need to note how these dynamics are improving as we progress through the process.

#### 4.2.1 Improve $\langle \alpha, \beta, \gamma \rangle$ so it acts without non-trivial finite periodic orbits

Our first step is use Proposition 10.1 of [4] to pass to powers of $\alpha$, $\beta$, and $\gamma$ so that none of the resulting elements admit non-trivial finite orbits. In this subsubsection we will also develop some notation for various sets and quantities which will be important in the remainder.

By Lemma 2.8 we see that $\alpha$ and $\beta$ have various components of support, some of which may be common to both elements. Each of the remaining components of support of $\alpha$ or $\beta$ will be disjoint from the support of $\beta$ or $\alpha$, respectively. In particular, we can give name to these sets as follows. Let $\{A_i\}_{i=1}^r$ represent the components of support of $\alpha$ which are disjoint from
the components of support of $\beta$. Let $\{B_i\}_{i=1}^s$ represent the components of support of $\beta$ which are disjoint from the components of support of $\alpha$. Finally, let $\{C_k\}_{k=1}^t$ represent the common components of support of $\alpha$ and $\beta$.

For each component of support $C_k$, fix non-trivial integers $m_k$ and $n_k$ so that $\alpha^{m_k} \beta^{n_k}$ has trivial action over $C_k$. These integers exist by Corollary 2.9.

### 4.2.2 Modify $\gamma$ so that $I(\gamma) \cap (I(\alpha) \cup I(\beta)) = \emptyset$

In this subsection we improve $\phi$ in stages. We repeatedly replace $\gamma$ with $(\alpha, \beta, \gamma)$-commutators. Our goal is to arrange the components of support of the final version of $\gamma$ so that the set of important points of $\gamma$ is disjoint from the sets of important points of both $\alpha$ and $\beta$. The resulting group $\langle \alpha, \beta, \gamma \rangle$ must still factor as $\langle \alpha, \beta \rangle \ast \langle \gamma \rangle \cong \mathbb{Z}^2 \ast \mathbb{Z}$ by Lemma 4.3.

Let $S = I(\gamma) \cap (I(\alpha) \cup I(\beta))$. If $S$ is non-empty, then define the $(\alpha, \beta, \gamma)$-commutator

$$\theta := [\alpha \beta, \gamma].$$

Otherwise, we already have the goal of the subsection and we can pass to the next subsection without modifying $\gamma$.

If $x \in S$, then either $x$ is an important point of $\alpha \beta$, or $\alpha \beta$ acts as the identity in a neighborhood of $x$ ($\alpha$ and $\beta$ may act as local inverses in a neighborhood of $x$). In either case, $\theta$ will act as the identity in a neighborhood of $x$ (either by invoking Lemma 2.4, or simply, if $\alpha \beta$ acts trivially near $x$, then the commutator resolves as $\gamma^{-1} \gamma$ in a neighborhood of $x$).

Improve $\gamma$ by replacing it with $\theta$.

We now should mention a useful lemma.

**Lemma 4.4.** If $y$ is an important point of $\alpha$ or $\beta$, and $\gamma$ acts as the identity in some neighborhood $M_y$ of $y$, then any $(\alpha, \beta, \gamma)$-commutator $\tau$ will act as the identity in some neighborhood $N_y$ of $y$.

**Proof.** We show that if $p, q$ are integers with $|p| + |q| > 0$, $z \neq 0$ is an integer, and $\tau = [\alpha^p \beta^q, \gamma^z]$, then $y$ has a neighborhood $N_y$ so that $\tau$ acts as the identity on $N_y$. The general lemma then follows by an easy induction.

Let $p, q, z$ and $\tau$ as in the previous paragraph. We have

$$\tau = (\gamma^{-1})^p \cdot \beta^q \cdot \gamma.$$

In particular, the support of $\tau$ is contained in the union $\text{Supp}(\gamma) \cup \text{Supp}(\tau) = \text{Supp}(\gamma) \cup \text{Supp}(\eta).$ Let $M_y$ be a neighborhood of $y$ disjoint from the action of $\gamma$, and let $m$ be the node of $T_2$ corresponding to $M_y$. Pass deeply into $T_2$ beneath the node $m$ to find a node $n$ which has $y$ underlying $n$, and so that $\alpha^p \beta^q$ acts affinely over the Cantor Set $N_n$ underlying $n$, and so that $N_n \alpha^{-p} \beta^{-q} \subset M_y$. It is immediate that such a node $n$ exists. As $\text{Supp}(\gamma)$ lies outside of $M_y$, the action of $\alpha^p \beta^q$ cannot throw the support of $\gamma$ into $N_y$. \qed

If our new $\gamma$ has new important points in common with the important points of $\alpha$ or the important points of $\beta$, then return to the beginning of this subsection, observing that by the lemma we have just proven, any further versions of $\gamma$ that we create will act as the identity in a neighborhood of the set $S$ which we defined in the beginning of this section.

This process must stop, since the important points of $\alpha$ and $\beta$ are finite in number. The final version $\gamma$ that we have created has no important points in common with the important points of $\alpha$ or the important points of $\beta$.

Proceed to the next subsection.
4.2.3 Improve $\gamma$ so that $\text{Supp}(\gamma) \cap (I(\alpha) \cup I(\beta)) = \emptyset$

This process requires a bit more care.

We may suppose immediately that $I(\gamma) \cap (I(\alpha) \cup I(\beta)) = \emptyset$, or we could not have entered this subsubsection in our algorithm.

If $\text{Supp}(\gamma)$ does not contain any of the important points of $\alpha$ or $\beta$, then proceed to the next subsubsection, otherwise continue in this subsection.

Suppose $x$ is an important point of $\alpha$ or $\beta$ which is in the support of $\gamma$. We must carefully consider two cases, depending on the dynamics of $\alpha$ near $y = x\gamma^{-1}$.

In the first case, $y$ is disjoint from the support of $\alpha$.

In this case, define $\theta = [\alpha, \gamma] = \alpha^{-1} \gamma^{-1} \alpha \gamma$. We observe the following.

$$x\theta = x\alpha \cdot \gamma^{-1} \alpha \gamma = x\gamma^{-1} \alpha \gamma = y\alpha \gamma = y\gamma = x$$

So $x$ is fixed by $\theta$.

If $x$ is an important point of $\alpha$, then unless $y$ is also an important point of $\alpha$, $x$ will be an important point of $\theta$ as well; the initial invocation of $\alpha^{-1}$ acts as an affine map fixing $x$ with slope not equal to one in a neighborhood of $x$, while the latter invocation of $\alpha$ acts as the identity near $y$. If $y$ is an important point of $\alpha$ as well, then $x$ will be an important point for $\theta$ only if the slope of $\alpha$ in small neighborhoods of $x$ is not the same as the slope of $\alpha$ in small neighborhoods of $y$ (otherwise, $\theta$ will act as the identity in some neighborhood of $x$).

If $x$ is not an important point of $\alpha$, then $x$ is an important point of $\beta$ which must be disjoint from the component support of $\alpha$. In this case, $\alpha$ acts as the identity in a neighborhood of $x$. Therefore, $\theta$ will either have $x$ as an important point (if $y$ is an important point of $\alpha$) or act as the identity in a neighborhood of $x$.

Suppose instead that $y$ is not disjoint from the support of $\alpha$. Then either $y$ is in a common component of support $C_k$ of $\alpha$ and $\beta$, or $y$ is in a component of support $A_i$ for $\alpha$ for some index $i$. In the first case we can use Corollary 2.9 to find $p$, $q$ non-trivial integers so that $\alpha^p \beta^q$ acts trivially over $C_k$, while in the second case, take $p = 0$ and $q = 1$, so that again $\alpha^p \beta^q$ is trivial over $A_i$. In either case, define $\theta = [\alpha^p \beta^q, \gamma]$, and $\theta$ will have $x$ as an important point if $x$ is an important point of $\alpha^p \beta^q$, as in the previous discussion, or $\theta$ will act trivially in a neighborhood of $x$.

Now replace $\gamma$ by $\theta$. If $\gamma$ has any important points in common with the important points of $\alpha$ or $\beta$, return to the previous subsubsection. Otherwise, if there are any further important points of $\alpha$ or $\beta$ in the support of $\gamma$, then return to the beginning of this subsubsection. Finally, if none of the important points of $\alpha$ or $\beta$ are in the component support of $\gamma$, we can proceed to the next subsubsection.

Note that if we had $\text{Supp}(\gamma) \cap (I(\alpha) \cup I(\beta)) \neq \emptyset$ at the beginning of this subsubsection, then the cardinality of that intersection is reduced by at least one by the process here (no new points of $I(\alpha) \cup I(\beta)$ enter the support of $\gamma$ under this process by Lemma 4.4). Further, again by Lemma 4.4, this subsubsection will only force the algorithm to return to the previous subsubsection a finite number of times, and the cardinality of $\text{Supp}(\gamma) \cap (I(\alpha) \cup I(\beta))$ does not increase when we apply the process of the previous section. Therefore, the algorithm given to this point eventually passes to the next subsubsection, and at that time none of the important points of $\alpha$ or of $\beta$ are in the component support of $\gamma$.

4.2.4 Finding torsion where none can exist

We can now assume that there is a neighborhood of the important points of $\alpha$ and $\beta$ so that $\gamma$ acts trivially over this neighborhood.

In this section, we find an element in $\langle \alpha, \beta, \gamma \rangle$, which is the $\phi$-image of a non-trivial element of $Z^2 \ast Z$, but where orbit dynamics show that the image element must actually either be trivial, or torsion with order two, three, or six. Thus, $\phi$ cannot actually embed $Z^2 \ast Z$ into $V$.

We first note the effects of conjugating $\gamma$ by powers of $\alpha$ and $\beta$, assuming the details of our current dynamical situation with respect to important points and supports.
Lemma 4.5. Suppose $\delta$, $\epsilon$, and $\theta$ are elements in $V$ with the properties that

1. $\delta$ and $\epsilon$ commute,
2. none of $\delta$, $\epsilon$ or $\theta$ admit non-trivial finite orbits in their action on the Cantor Set,
3. the support of $\theta$ is disjoint from a neighborhood of the important points of $\delta$ and $\epsilon$.

There are infinitely many pairs of non-zero integers $x$ and $y$ so that

$$\text{Supp}(\theta^{\delta^x \epsilon^y}) \cap \text{Supp}(\theta) \cap (\text{Supp}(\delta) \cup \text{Supp}(\epsilon)) = \emptyset.$$ 

Proof. The lemma follows immediately from the observation that there are non-zero integers $i$ and $j$ so that $\delta^i \epsilon^j$ is non-trivial over every component of common support of $\delta$ and $\epsilon$ (since there are only finitely many such components), and thus over every component of support of $\delta$ and $\epsilon$. Now for large enough integers $n$, setting $x = ni$ and $y = nj$ produces integers so that $\delta^x \epsilon^y$ throws the support of $\theta$ entirely off of itself within the supports of $\delta$ and $\epsilon$ (the resulting support within the support of $\delta$ and $\epsilon$ will be near to the important points of $\delta$ and $\epsilon$).

Thus we may use conjugation of $\gamma$ by powers of $\alpha$ and $\beta$ to find non-trivial elements in $\langle \alpha, \beta, \gamma \rangle$ whose actions within the supports of $\alpha$ and $\beta$ are disjoint from the action of $\gamma$.

Let $x$ and $y$ be the integers guaranteed by Lemma 4.5 where $\alpha$ plays the role of $\delta$, $\beta$ plays the role of $\beta$, and $\gamma$ plays the role of $\theta$. Define $\omega = [\gamma, \gamma^{\alpha^x \beta^y}]$, noting in passing that $[c, c^{\alpha^x \beta^y}]$ is not trivial in $Z^2 \ast Z$.

Note that the component support of $\omega$ is contained in the set $\Gamma = \text{Supp}(\gamma) \cup \text{Supp}(\gamma^{\alpha^x \beta^y})$. One can now demonstrate by direct calculation that every $x \in \Gamma$ must travel, under the action of $\langle \omega \rangle$, along an orbit of length either one, two, or three. This implies that the order of $\omega$ is one, two, three, or six, which contradicts the fact that $\omega$ is a non-trivial element in a torsion free group. In particular, there can be no embedding of $Z^2 \ast Z$ in $V$.

The remainder of this subsubsection verifies the calculation mentioned in the previous paragraph. The discussion is highly technical. We first carefully define fifteen disjoint subsets of the potential support of $\omega$. Then, we analyze the flow of points in these sets under the action of $\langle \omega \rangle$.

To assist the reader in tracking this process, the following schematic provides an informal guide to the arrangements of the fifteen sets in the context of the actions of $\gamma$, $\theta$, and $\alpha^x \beta^y$. 

![Diagram](https://via.placeholder.com/150)
Defining sets and notation:

To simplify notation, set \( \theta = \gamma^{\alpha^x \beta^y} \). Set

\[
\mathcal{I} = \mathcal{I}(\gamma) \cup \mathcal{I}(\theta).
\]

While the important points of \( \gamma \) and \( \theta \) cannot be important points of \( \omega \), the dynamical analysis of the action of \( \omega \) is greatly assisted by paying careful attention to a neighborhood of the points in \( \mathcal{I} \).

The set \( \mathcal{I} \) decomposes as a disjoint union of six sets, some of which might be empty.

\[
\mathcal{I} = \mathcal{R}_c \cup \mathcal{R}_g \cup \mathcal{R}_t \cup \mathcal{A}_c \cup \mathcal{A}_g \cup \mathcal{A}_t
\]

Here, \( \mathcal{R}_c \) is the set of repelling fixed points of \( \gamma \) and \( \theta \) which lie outside of the support of \( \alpha^x \beta^y \).

We note in passing that this is the same set for both \( \gamma \) and \( \theta \) (subscript \( c \) denotes the word “common”).

Similarly, \( \mathcal{A}_c \) is the set of attracting fixed points of \( \gamma \) and \( \theta \) which lie outside the support of \( \alpha^x \beta^y \).

The sets \( \mathcal{R}_g \) and \( \mathcal{A}_g \) are respectively the sets of repelling and attracting fixed points of \( \gamma \) which lie in the support of \( \alpha^x \beta^y \). One sees immediately that \( \mathcal{R}_c = \mathcal{R}_g \alpha^x \beta^y \) and \( \mathcal{A}_c = \mathcal{A}_g \alpha^x \beta^y \).

For each \( x \in \mathcal{R}_c \), let \( n_x \) denote a node of \( T_2 \) so that \( x \in \mathcal{C}_{n_x} \subseteq \text{Supp}(\gamma) \) and so that \( \mathcal{C}_{n_x} \cap \text{Supp}(\alpha^x \beta^y) = \emptyset \). Similarly, for each \( y \in \mathcal{A}_c \), let \( n_y \) denote a node of \( T_2 \) so that \( y \in \mathcal{C}_{n_y} \subseteq \text{Supp}(\gamma) \) and so that \( \mathcal{C}_{n_y} \cap \text{Supp}(\alpha^x \beta^y) = \emptyset \).

For each \( x \in \mathcal{R}_g \), let \( n_x \) denote a node of \( T_2 \) so that \( x \in \mathcal{C}_{n_x} \subseteq \overline{\text{Supp}(\gamma)} \) and so that \( \mathcal{C}_{n_x} \cap \text{Supp}(\alpha^x \beta^y) = \mathcal{C}_{n_x} \). Similarly, for each \( y \in \mathcal{A}_g \), let \( n_y \) denote a node of \( T_2 \) so that \( y \in \mathcal{C}_{n_y} \subseteq \overline{\text{Supp}(\gamma)} \) and so that \( \mathcal{C}_{n_y} \cap \text{Supp}(\alpha^x \beta^y) = \mathcal{C}_{n_y} \).

For each \( x \in \mathcal{R}_t \), let \( n_x \) denote a node of \( T_2 \) so that \( x \in \mathcal{C}_{n_x} \subseteq \text{Supp}(\gamma) \) and so that \( \mathcal{C}_{n_x} \cap \text{Supp}(\alpha^x \beta^y) = \mathcal{C}_{n_x} \). Similarly, for each \( y \in \mathcal{A}_t \), let \( n_y \) denote a node of \( T_2 \) so that \( y \in \mathcal{C}_{n_y} \subseteq \text{Supp}(\gamma) \) and so that \( \mathcal{C}_{n_y} \cap \text{Supp}(\alpha^x \beta^y) = \mathcal{C}_{n_y} \).

All of the nodes \( n_x \) and \( n_y \) chosen above can be chosen in such a fashion as to have all of the various properties mentioned above, as well as the further property that given any distinct pair of nodes, the underlying Cantor Sets of the nodes are disjoint. We assume that the nodes have been chosen in such a fashion. The reader may verify that such choices can be made.

Let \( R = \mathcal{R}_c \cup \mathcal{R}_g \cup \mathcal{R}_t \), and let \( A = \mathcal{A}_c \cup \mathcal{A}_g \cup \mathcal{A}_t \), and set

\[
\mathcal{N}_c = \cup_{x \in \mathcal{R}_c} \mathcal{C}_{n_x}
\]

\[
\mathcal{N}_a = \cup_{y \in \mathcal{A}_c} \mathcal{C}_{n_y}
\]

The set \( \mathcal{N}_c \) is a neighborhood of the repelling fixed points of \( \gamma \) and \( \theta \), which we will use often in our calculations below. A modified version of \( \mathcal{N}_a \) will play the role of a similar neighborhood around the attractors.

There is an integer \( K > 0 \) so that \( \mathcal{N}_c \gamma^K \cup \mathcal{N}_a \theta^K \cup \mathcal{N}_a = \text{Supp}(\gamma) \cup \text{Supp}(\theta) \). Replace \( \gamma \) by \( \gamma^K \), and redefine \( \theta \) as \( \gamma^{\alpha^x \beta^y} \) using the new \( \gamma \). We now have

\[
\mathcal{N}_c \gamma \cup \mathcal{N}_a \theta \cup \mathcal{N}_a = \text{Supp}(\gamma) \cup \text{Supp}(\theta).
\]

For each \( x \in R \), set \( U_x = \mathcal{C}_{n_x} \). For each \( y \in A \), set \( V_y = \mathcal{C}_{n_y} \). Given \( x \in R_c \), \( R_g \), or \( R_t \) respectively, we may denote \( U_x \) by \( U^c_x \), \( U^g_x \), or \( U^t_x \) for clarity. Similarly, given \( y \in A_c \), \( A_g \), or \( A_t \), we may denote \( V_y \) by \( V^c_y \), \( V^g_y \), or \( V^t_y \) respectively.

Set \( \mathcal{N}_c = \cup_{y \in A} \mathcal{C}_{n_y} \). This is a generalized neighborhood of the attracting fixed points of \( \gamma \) and \( \theta \) lying entirely in the supports of \( \gamma \) and \( \theta \). Denote by \( M \) the set \( (\text{Supp}(\gamma) \cup \text{Supp}(\theta)) \setminus (\mathcal{N}_c \cup \mathcal{N}_a) \). This is the set of potential support of \( \omega \) away from the neighborhoods of the attracting and repelling fixed points of \( \gamma \) and \( \theta \). (The set \( M \)}
is the “middle” of the potential support of \( \omega \). The set \( M \) decomposes as a disjoint union of sets
\[
M = \bigcup \text{Supp}(\alpha^2 \beta^y), \quad M_g = \bigcup \text{Supp}(\gamma) \cap \text{Supp}(\alpha^2 \beta^y), \quad \text{and } M_t = \bigcup \text{Supp}(\theta) \cap \text{Supp}(\alpha^2 \beta^y).
\]

We now use \( M \) to further decompose the sets \( U_x \) and \( V_y \) for \( x \in R \) and \( y \in A \).

Given \( x \in U_x \) set
\[
U_{x,1} = \{ x \in U_x \mid x \gamma \in M \text{ or } x \theta \in M \}
\]
and set \( U_{x,2} = U_x \setminus U_{x,1} \). (At least two applications of \( \gamma \) or \( \theta \) are required for a point in \( U_{x,2} \) to leave \( U_x \).) Extend this notation in the obvious fashion so that \( U_{x,i}^c \), \( U_{x,i}^g \), and \( U_{x,i}^t \) are well defined for indices \( i = 1 \) or \( i = 2 \). We will not need larger values of \( i \) (i loosely represents minimal “escape time” from \( U_x \) under the actions of \( \gamma \) or \( \theta \)) for the analysis to follow.

Finally, and in similar fashion, given \( y \in R \), define
\[
V_{y,1} = V_y \cap (M \theta \cup M \gamma)
\]
and set \( V_{y,2} = V_y \setminus V_{y,1} \). Extend this notation in like fashion so that the sets \( V_{y,i}^c \), \( V_{y,i}^g \), and \( V_{y,i}^t \) are well defined for indices \( i = 1 \) and \( i = 2 \).

We have now defined the fifteen types of (pairwise disjoint) subsets of the potential support of \( \omega \) which we will require in our analysis of the action of \( \omega \) on \( C \). In particular, these are \( U_{x,i}^c \), \( U_{x,i}^g \), \( V_{x,i}^c \), \( V_{x,i}^g \), \( V_{x,i}^t \), \( M_{ei} \), \( M_g \), and \( M_t \), where \( i = 1 \) or \( i = 2 \).

Analysis of dynamics:
We are now ready to prove the following lemma.

**Lemma 4.6.** The element \( \omega \) defined above has order 1, 2, 3 or 6.

**Proof.** The support of \( \omega \) is contained in the union of the support of \( \gamma \) and the support of \( \theta \). In particular, we need to trace the orbits of the points in each of the fifteen sets defined in the paragraph just before the “Analysis of dynamics” header.

That effort is simplified by the fact that the action of \( \langle \omega \rangle \) in many of those sets is trivial (as one might expect, given that \( \omega \) is a commutator).

We perform the orbit calculations for some of the sets before confirming the statement in the last paragraph, in order to acquaint the reader with a method of orbit calculation.

In the diagram below, if there are several arrows leaving a node, this represents the fact that a point may move to distinct locations depending on further subdivisions within the fifteen sets. It often occurs that a previous choice at a branch makes later choices invalid. In our first calculation, we will draw such invalid possibilities with a “dotted” arrow. Arrows are sometimes decorated with strings to help explain the dynamics.

If \( x \in R \), in the diagrams below, we will drop the occurrence of \( x \) from the names of the repulsive sets \( U_{x,1}^c \), \( U_{x,2}^c \), \( U_{x,1}^g \), \( U_{x,2}^g \), \( U_{x,1}^t \), and \( U_{x,2}^t \), writing instead names such as \( U_{1,1}^c \). We will treat the attractive \( V \) sets in similar fashion. This should lead to no confusion.

The reader may be assisted in following the calculations below by recalling that \( M_t = M_g \alpha^2 \beta^g \), \( U_{1}^c = U_{1}^g \alpha^2 \beta^g \), \( V_{1}^c = V_{1}^g \alpha^2 \beta^g \), and that these sets are “parallel” in some sense due to the relationship \( O(p, \theta) = O(p, \gamma) \alpha^2 \beta^g \) for \( p \in C \). The reader should also observe that \( \gamma|_{V_{x}} = \theta|_{V_{x}} \) and \( \gamma^{-1}|_{U_{x}} = \theta^{-1}|_{U_{x}} \) for \( * = 1 \) or \( * = 2 \).

We now begin to trace orbits.
Assume $x_0 \in M_c$.

\[
x_0 \in M_c \xrightarrow{\gamma^{-1}} x_1 \in U_1^c \xrightarrow{\theta^{-1}} x_2 \in U_2^c \xrightarrow{\gamma} x_1 \xrightarrow{\theta} x_0
\]

\[
x_3 \in U_1^g \xrightarrow{\theta^{-1}} x_3 \xrightarrow{\gamma} x_0 \xrightarrow{\theta} x_4 \in V_1^t \xrightarrow{\gamma} x_5 \in V_1^c
\]

\[
x_4 \xrightarrow{\gamma^{-1}} x_4 \xrightarrow{\theta^{-1}} x_0 \xrightarrow{\gamma} x_6 \in V_1^g \xrightarrow{\theta} x_6
\]

\[
x_5 \xrightarrow{\gamma^{-1}} x_0 \xrightarrow{\theta^{-1}} x_7 \in U_1^t \xrightarrow{\gamma} x_7 \xrightarrow{\theta} x_0
\]

\[
x_6 \xrightarrow{\gamma^{-1}} x_0 \xrightarrow{\theta^{-1}} x_8 \in U_1^t \xrightarrow{\gamma} x_8 \xrightarrow{\theta} x_0
\]

In this example, each diagram component represents the possibilities of one application of $\omega$. Under the action of $\langle \omega \rangle$, the point $x_0$ had potential orbits of length one ($\{x_0\}$), of length two ($\{x_0, x_5\}$), and of length three ($\{x_0, x_4, x_6\}$).

We now free the symbols $x_i$ used in the above calculation, so that we can enter into similar calculations below with different values for the $x_i$. Whenever we are about to trace the orbits of one of the fifteen sets, we will assume the variables $x_i$ are unbound and available.

In each of the calculations below, we will no longer draw explicit “dotted arrows” for potential branches which cannot actually occur given previous information within the calculation.

Suppose $x_0 \in M_t$.

\[
x_0 \in M_t \xrightarrow{\gamma^{-1}} x_0 \xrightarrow{\theta^{-1}} x_1 \in U_1^t \xrightarrow{\gamma} x_1 \xrightarrow{\theta} x_0
\]

\[
x_2 \in U_1^t \xrightarrow{\gamma} x_3 \in M_g \xrightarrow{\theta} x_3
\]

\[
x_3 \xrightarrow{\gamma^{-1}} x_2 \xrightarrow{\theta^{-1}} x_3 \in U_2^c \xrightarrow{\gamma} x_2 \xrightarrow{\theta} x_0
\]

We thus see that the orbits of points in $M_t$ are either trivial or of length two under the action of $\langle \omega \rangle$. 

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Suppose now that \( x_0 \in M_g \).

\[
\begin{align*}
x_0 \in M_g & \xrightarrow{\gamma^{-1}} x_1 \in U_1^c \xrightarrow{\theta^{-1}} x_1 \xrightarrow{\gamma} x_0 \xrightarrow{\theta} x_0 \\
x_2 \in U_1^c & \xrightarrow{\theta^{-1}} x_3 \in U_2^c \xrightarrow{\gamma} x_2 \xrightarrow{\theta} x_4 \in M_t \\
x_4 & \xrightarrow{\gamma^{-1}} x_4 \xrightarrow{\theta^{-1}} x_2 \xrightarrow{\gamma} x_0 \xrightarrow{\theta} x_0
\end{align*}
\]

So the points in \( M_g \) also have only trivial orbits or orbits of length two under the action of \( \langle \omega \rangle \).

Armed with these previous examples the reader should now be able to check that if an initial point \( p \) has \( p \in U_#^* \) then for any valid values of the symbols \# and *, \( \omega \) fixes \( p \). For \( p \in V_2^* \) with * not "c", it is also easy to see that \( p\omega = p \). Thus, we have seen that our lemma is supported over the sets \( M_c, M_g, M_t, U_1^c, U_2^c, U_3^c, U_4^c, U_5^c, U_6^c, V_2^c \) and \( V_2^c \). There remain four sets to check.

Suppose now that \( x_0 \in V_2^c \).

\[
\begin{align*}
x_0 \in V_2^c & \xrightarrow{\gamma^{-1}} x_1 \in V_1^c \xrightarrow{\theta^{-1}} x_2 \in M_c \xrightarrow{\gamma} x_1 \xrightarrow{\theta} x_0 \\
x_3 & \xrightarrow{\gamma^{-1}} x_3 \in M_t \xrightarrow{\gamma} x_3 \xrightarrow{\theta} x_1 \\
x_4 \in V_2^c & \xrightarrow{\theta^{-1}} x_5 \in V_1^c \xrightarrow{\gamma} x_4 \xrightarrow{\theta} x_0 \\
x_6 & \xrightarrow{\theta^{-1}} x_6 \in V_2^c \xrightarrow{\gamma} x_4 \xrightarrow{\theta} x_0 \\
x_1 & \xrightarrow{\gamma^{-1}} x_7 \in M_g \xrightarrow{\theta^{-1}} x_7 \xrightarrow{\gamma} x_1 \xrightarrow{\theta} x_0
\end{align*}
\]

In particular, the orbits in \( \mathcal{C} \) under the action of \( \langle \omega \rangle \) which actually intersect \( V_2^c \) are either trivial or of length two.

Suppose now \( x_0 \in V_1^c \).

\[
\begin{align*}
x_0 \in V_1^c & \xrightarrow{\gamma^{-1}} x_1 \in V_1^c \xrightarrow{\theta^{-1}} x_2 \in M_c \xrightarrow{\gamma} x_1 \xrightarrow{\theta} x_0 \\
x_3 & \xrightarrow{\gamma^{-1}} x_3 \in M_t \xrightarrow{\gamma} x_3 \xrightarrow{\theta} x_1 \\
x_4 \in V_2^c & \xrightarrow{\theta^{-1}} x_5 \in V_1^c \xrightarrow{\gamma} x_4 \xrightarrow{\theta} x_0 \\
x_6 & \xrightarrow{\theta^{-1}} x_6 \in V_2^c \xrightarrow{\gamma} x_4 \xrightarrow{\theta} x_0 \\
x_1 & \xrightarrow{\gamma^{-1}} x_7 \in M_g \xrightarrow{\theta^{-1}} x_7 \xrightarrow{\gamma} x_1 \xrightarrow{\theta} x_0
\end{align*}
\]

Thus, every orbit under the action of \( \langle \omega \rangle \) which intersects \( V_1^c \) is either trivial or of order two.
Suppose now $x_0 \in V^t_1$.

\[ x_0 \in V^t_1 \xrightarrow{\gamma^{-1}} x_0 \xrightarrow{\theta^{-1}} x_1 \in M_t \xrightarrow{\gamma} x_1 \xrightarrow{\theta} x_0 \]

\[ x_2 \in M_c \xrightarrow{\gamma} x_3 \in V^g_1 \xrightarrow{\theta} x_3 \]

\[ x_3 \xrightarrow{\gamma^{-1}} x_2 \xrightarrow{\theta^{-1}} x_4 \in U^t_1 \xrightarrow{\gamma} x_4 \xrightarrow{\theta} x_2 \]

\[ x_5 \in U^c_1 \xrightarrow{\gamma} x_2 \xrightarrow{\theta} x_0 \]

\[ x_2 \xrightarrow{\gamma^{-1}} x_6 \in U^g_1 \xrightarrow{\theta^{-1}} x_6 \xrightarrow{\gamma} x_2 \xrightarrow{\theta} x_0 \]

In this case, it is possible for points in $V^t_1$ to be in orbits of order one, two, or three under the action of $\langle \omega \rangle$.

Finally, suppose $x_0 \in V^g_1$.

\[ x_0 \in V^g_1 \xrightarrow{\gamma^{-1}} x_1 \in M_g \xrightarrow{\theta^{-1}} x_1 \xrightarrow{\gamma} x_0 \xrightarrow{\theta} x_0 \]

\[ x_2 \in M_c \xrightarrow{\theta^{-1}} x_3 \in U^t_1 \xrightarrow{\gamma} x_3 \xrightarrow{\theta} x_2 \]

\[ x_4 \in U^c_1 \xrightarrow{\gamma} x_2 \xrightarrow{\theta} x_5 = x_0 \alpha^x \beta^y \in V^t_1 \]

\[ x_2 \xrightarrow{\gamma^{-1}} x_6 \in U^g_1 \xrightarrow{\theta^{-1}} x_6 \xrightarrow{\gamma} x_2 \xrightarrow{\theta} x_5 = x_0 \alpha^x \beta^y \in V^t_1 \]

\[ x_5 \xrightarrow{\gamma^{-1}} x_5 \xrightarrow{\theta^{-1}} x_2 \xrightarrow{\gamma} x_0 \xrightarrow{\theta} x_0 \]

Hence, under the action of $\langle \omega \rangle$, we see $V^g_1$ is also a set where orbits in $C$ which intersect $V^g_1$ may be of order one, two, or three.

We have now shown that if $p \in C$, then the cardinality of $O(p, \omega)$ is one, two, or three. We conclude that the order of $\omega$ divides six.

We have therefore found a contradiction to the existence of our supposed embedding of $Z^2 \ast Z$ into $V$; the element $\omega$ is the image of a non-trivial element of $Z^2 \ast Z$ under an embedding, so $\omega$ must have infinite order. In particular, there are no injections from $Z^2 \ast Z$ into $V$. 

}\end{document}
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