On the re-interpretation of Wheeler-DeWitt equation

Avadhut V. Purohit

Chennai Mathematical Institute, H1 SIPCOT IT Park, Kelambakkam, Chennai, India - 603103

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I have shown that the field defined by the Wheeler-DeWitt equation for pure gravity is not a standard gravitational field. This field has some features that are common to the matter fields. The re-interpretation leads to the geometrization of quantum theory.

INTRODUCTION

The Wheeler-DeWitt theory quantizes the ADM formulation of gravity. The quantization scheme is similar to the quantization of non-relativistic particles. For free relativistic particle itself, the quantization by raising $E \rightarrow i\hbar \partial_t$ and $\vec{p} \rightarrow -i\hbar \nabla$ to operators on the Hilbert space faces several mathematical and conceptual issues. To resolve the issue of the existence of a virtual multi-particle quantum state, the concept of a ‘field’ was introduced by re-interpreting $E^2 = |\vec{p}|^2 + m^2_0$. Quantum field theories then describe the interaction of relativistic particles through coupling terms. In the general theory of relativity, a particular matter distribution has definite space-time geometry $g_{\mu\nu}(\phi)$. The third quantized theories re-interpreted the Wheeler-DeWitt equation to account for the virtual multi-geometric quantum state. Such quantum theories have gravitational field variables together with the matter fields forming a superspace. Quantum mechanically, $(q_{ab}, P^{ab})$ are first quantized as they are not obtained by re-interpretation. But matter fields are second quantized. Therefore, third quantized fields face severe criticism (refer to corresponding sections in [1] and [2]).

In this paper, I analyze the Wheeler-DeWitt equation for pure gravity in the light of standard quantum field theories. The field satisfies ADM constraints for pure gravity. Therefore, one would interpret that the field $\Phi$ is a pure gravitational field. But I observe, several other fields also obey ADM constraints for pure gravity. I also observe that these fields have non-trivial stress tensors. Whereas the stress tensor for pure gravitational field is $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0$. The field $\Phi(q_{ab})$ identified by coupling function $\mu^2$ always remains non-negative regardless of the signature of $R^{(3)}$. The linearized theory is even closer to the standard matter field theories. The other interpretation is that the field $\Phi$ describes the Universe. I show that this interpretation faces serious problems due to the existence of an interactive field theory.

I analyze the quantum theory to clarify these issues. The quantum theory happens to be the geometric (or gravitized) quantum theory rather than the quantum geometric theory. Refer [3] for the discussion on the gravitized quantum theory. The theory has the geometric quantum of the corresponding field. But there is no quantum of gravity.

The second section discusses the classical theory with a subsection having linearized theory. The third section has a super-gauge invariant field that also satisfies the ADM constraints for pure gravity. I study the quantum theory in the next section. The last section concludes the paper.

RE-INTERPRETATION AND SCALAR FIELD

The Wheeler-DeWitt equation written in the DeWitt’s coordinates

$$\left( \frac{1}{\sqrt{-G}} \partial_\mu \sqrt{-G} G^{\mu\nu} \partial_\nu + \mu^2 \right) \Phi = 0 \quad (1)$$

is interpreted as a classical field equation. The field $\Phi$ is defined over the superspace $\zeta^\mu := (\zeta, \zeta^A)$ with $A = 1, 2, 3, 4, 5$. The DeWitt super-metric borrowed from [1].

$$G_{\mu\nu} := \begin{pmatrix} 1 & 0 \\ 0 & -3\zeta^2 G_{AB} \end{pmatrix} \quad (2)$$

$$\bar{G}_{AB} := Tr \left( q^{-1} \frac{\partial q}{\partial \zeta} q^{-1} \frac{\partial q}{\partial \zeta^B} \right) \quad (3)$$

$$\sqrt{-G} := \sqrt{-\det G_{\mu\nu}} \quad (4)$$

$\bar{G}_{AB}$ is symmetric super-metric on 5D manifold $M$ identified with $SL(3, \mathbb{R})/SO(3, \mathbb{R})$ (refer [1] and [10] for more discussion on the geometry of superspace). The coupling parameter is defined as

$$\mu^2 (\zeta, \zeta^A) := -\frac{3\zeta^2}{32} R^{(3)} \quad (5)$$

The action functional for the geometric scalar field that gives field equation (1) is assumed to have the following form.

$$A_\Phi := \int \mathcal{D}\zeta \sqrt{-G} \left( \partial_\mu \Phi G^{\mu\nu} \partial_\nu \Phi - \mu^2 \Phi^2 \right) \quad (6)$$

$\mathcal{D}\zeta$ is suitable measure over 6D manifold. But for the classical geometric scalar field, different combinations of $P^{ab}$ and $q_{ab}$ gives non-equivalent field equations. I have taken the combination of field variables with a consistent self-adjoint extension. In other words, the combination allows the Hamiltonian operator to be self-adjoint.
On single spacetime-like interpretation $\Phi = e^{iq_{\mu\nu} P_{\mu\nu}}$, ADM Hamiltonian constraints for pure gravity in the DeWitt coordinates $(5.20, \text{[1]})$

$$P_0^2 - \frac{32}{3k^2} \bar{G}^{AB} P_A P_B + \frac{3\bar{c}}{32} R^{(3)} \approx 0 \quad (7)$$

get recovered. Invariance of an action under variation of $\zeta^\mu := (\zeta, \zeta^A)$ gives stress tensor.

$$T^\mu_\nu := \frac{\partial L}{\partial \left(\frac{\partial \Phi}{\partial \zeta^\mu}\right)} \frac{\partial \Phi}{\partial \zeta^\mu} - L \delta^\mu_\nu \quad (8)$$

For now, assume $\zeta^\mu$ as time and perform Legendre transformation to get the Hamiltonian

$$H_\Phi := \frac{1}{\sqrt{-G}} \sqrt{-G} \frac{\partial G_{\alpha\beta}}{\partial \zeta^\alpha} \quad (9)$$

$$H_\Phi = f \int \sqrt{-G} \left( \frac{\partial G_{\mu\nu}}{\partial \zeta^\mu} \frac{\partial \Phi}{\partial \zeta^\mu} + \frac{\partial G_{\mu\nu}}{\partial \zeta^\mu} \frac{\partial \Phi}{\partial \zeta^\mu} + \frac{\partial G_{\mu\nu}}{\partial \zeta^\mu} \frac{\partial \Phi}{\partial \zeta^\mu} \right) \quad (10)$$

For $\mu^2 < 0$, the field is self-coupled, i.e. it has quadratic $\Lambda \Phi^4$ with some $\Lambda > 0$. The Hamiltonian shows that the quadratic coupling will be non-negative regardless of the signature of the 3-Ricci scalar and justifies the use of $\zeta$ as the time for the geometric scalar field.

Note: The Ricci curvature scalar $R^{(3)}$ and cosmological constant $\Lambda$ are not on equal footing. The former one has quadratic coupling, whereas the latter one contributes to the vacuum.

### Linearized theory

I define $q_{ab} := \delta_{ab} + h_{ab}$ with $\delta_{ab} >> h_{ab}$. The five dimensional superspace $\zeta^A := (r_1, r_2, \theta_1, \theta_2, \theta_3)$ has three compact and two non-compact coordinates.

$$\left( \frac{\partial^2}{\partial \zeta^2} - \frac{\partial^2}{\partial \zeta^2} + \mu^2 \right) \Phi = 0 \quad (11)$$

The wave equation is asymptotic and it represents a wave passing through surface with intrinsic geometry defined by $[\text{1}]$. Asymptotically flat approximation does not change the intrinsic curvature $R^{(3)}$. It makes field $\Phi$ in a fixed background having intrinsic quantity $\mu^2$. Depending on the intrinsic geometry of the surface, the signal arrives at different times. In this sense, the role of intrinsic curvature is similar to the role of mass in the special relativistic QFTs. But $\mu^2$ is not the mass. Using Iwasawa decomposition $SL(3, \mathbb{R})/SO(3, \mathbb{R})$ can be written in terms of

$$\left( \begin{array}{ccc} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_1r_2 \end{array} \right) \left( \begin{array}{ccc} 1 & \theta_1 & \theta_2 \\ \theta_1 & 0 & \theta_1 \\ \theta_2 & \theta_1 & 0 \end{array} \right) \quad (12)$$

$r_1$ and $r_2$ produce deformations along axes by preserving volume. Whereas other three are deformations along respective plane. $\Phi(r_1)$ and $\Phi(r_2)$ represent $+$-polarized waves. Whereas $\Phi(\theta_1)$, $\Phi(\theta_2)$ and $\Phi(\theta_3)$ represent $\times$-polarized waves.

### Features

- The field $\Phi(q_{ab})$ is identified with an intrinsic property $\mu$ and defined over $q_{ab}$ which is the solution to intrinsic curvature $R^{(3)}$. The existence of dynamical background shows that the field is gravitational.
- Even when the 3-geometry have positive curvature, quadratic coupling remains positive. This shows resemblance of the field with the standard matter fields.
- The field $\Phi$ can also have charge.

$$A_{\text{complex}} := \int \mathcal{D} \zeta \sqrt{-G} \left( \partial_\mu \Phi^* G^{\mu\nu} \partial_\nu \Phi - \mu^2 \Phi^* \Phi \right) \quad (13)$$

The pure gravity on other hand, does not have opposite charges. This is the property that resembles charged matter.

- The existence of $\mu^2 = 0$ or $R^{(3)} = 0$ does not necessarily mean line element is zero.

$$ds^2 = \frac{1}{32} G_{ij} dx^i dx^j \quad (14)$$

For the 3-metric $q_{ab} := q(t) \text{ diag} (f(r), r^2, r^2 \sin^2 \theta)$, the line element is given as follows.

$$ds^2 = G_{abcd} dq_a dq_b dq_c dq_d \quad (15)$$

The distance in the superspace, in the large $q$ limit, $ds^2 = -4 \frac{a^2 dq df}{f^2}$ increases with $q$. It shows cosmological expansion. It decreases with increase in $f$, showing attraction between two objects. This distance is actually the distance between two 3-metrics (refer $[\text{1}]$).

- The energy of the field $\Phi$ is always well-defined. Gravitational field may not always have time-like Killing vector field. Therefore, defining energy in the general relativity is not straightforward.

### RE-INTERPRETATION AND GAUGE INvariance

The frame of reference is given by the set of lapse function $N$ and shift vector $N^a$. The field $\Phi$ contains information about reference frame which can be easily seen by single-geometric interpretation $\Phi \sim e^{\pm i n^a q_{ab}}$ leading to $[\text{7}]$. As discussed by Penrose $[\text{2}]$, the equivalence principle in quantum mechanics leads to the phase difference. The complex scalar field $\Phi$ is not invariant under transformation $\Phi \rightarrow e^{i \alpha(c^n)}$. The field $A_\mu$ makes $[\text{13}]$ gauge invariant and obey equivalence principle.

$$\mathcal{L}_{\text{complex}} = \frac{1}{2} \left( D_\mu \Phi^* \right) G^{\mu\nu} \left( D_\nu \Phi \right) - \frac{\alpha^2}{2} \Phi^* \Phi \quad (16)$$

$$D_\mu := \partial_\mu - i \alpha A_\mu \quad (17)$$
If the field \( \Phi \) is interpreted as a description of the universe then, the above Lagrangian that describes interaction between charged universe and vector-valued universe which violates the definition of universe itself. That would force us to interpret the field \( \Phi \) similar to the matter field. But the third quantized fields already contain matter fields forming a superspace.

The action functional for field \( A_\mu \) assumed to have the following form.

\[
A_{\text{vector}} := -\frac{1}{4} \int \mathcal{D} \zeta \sqrt{-G} F_{\mu \nu} F^{\mu \nu} \tag{18}
\]

\( F_{\mu \nu} \) is completely anti-symmetric tensor and therefore satisfies Bianchi identity.

\[
\partial_\mu F_{\nu \lambda} + \partial_\nu F_{\mu \lambda} + \partial_\lambda F_{\nu \mu} = 0 \tag{19}
\]

The momentum conjugate to \( A_\lambda \) is

\[
\Pi^{\lambda}_{\text{vec}} := -\frac{32\sqrt{-G}}{3\zeta^2} F_{\mu \nu} G^{\mu \nu} \tag{20}
\]

Hamiltonian for this field obtained using Legendre transformation

\[
\left\{ A_k (\zeta, \zeta^A), \Pi^{\lambda}_{\text{vec}} (\zeta, \zeta^A) \right\}_{\text{P.B.}} = \delta \left( \zeta^A, \zeta^A \right) \tag{21}
\]

\[
H_{\text{vector}} = \int \mathcal{D} \zeta \mathcal{H}_{\text{vector}} \tag{22}
\]

\[
\mathcal{H}_{\text{vector}} = \sqrt{-G} \left( -\frac{16}{3\pi \xi} F_{ij} G^{ij} F_{ij} - \frac{225}{16\pi^2} G^{ij} G^{jk} F_{kl} \right)
\]

\[
- A_0 \left( \partial_i \Pi^{i}_{\text{vec}} \right)
\]

The component \( A_0 \) is a Lagrange multiplier. Field equations in presence of source \( J^\mu \) and in the gauge selected above are obtained as

\[
\partial_\mu F^{\mu \nu} + \frac{F^{\mu \nu}}{\sqrt{-G}} \partial_\nu \left( \sqrt{-G} \right) = J^\nu \tag{23}
\]

In addition to other sources \( J^\nu \), the second geometric term also acts as source for the gauge field. In absence of source, fields \( B_A := \epsilon_{ABC} F_{BC} \) as well as \( E_A := F_{0A} \) satisfy Wheeler-DeWitt equation.

\[
\frac{1}{\sqrt{-G}} \partial_\mu \left( \sqrt{-G} G^{\mu \nu} \partial_\nu \right) \left( \frac{E_C}{B_C} \right) = 0 \tag{24}
\]

In asymptotic flat limit (i.e. \( \sqrt{-G} \approx 1 \)), the field equations become \( \partial_\mu F^{\mu \nu} = J^\nu \). Invariance of the action under \( \zeta^\mu \rightarrow \zeta^\mu + \delta \zeta^\mu \) gives stress-energy tensor

\[
T^\mu_\nu := -\left( F^{\mu \alpha} F_{\nu \alpha} - \frac{1}{4} \delta_\nu^\alpha F_{\alpha \beta} F^{\alpha \beta} \right) \tag{25}
\]

The superspace can be written in terms of 4-dimensional space-time \( \{ x^\mu \} \) and, the physical space-time is 4-dimensional. Therefore, the super-gauge field \( A_\mu \) has spin 1.

The field \( A_\mu \) also satisfies the ADM constraints for pure gravity and has a non-trivial stress tensor. If \( \Phi \) is interpreted as a pure gravitational field then, the field \( A_\mu \) would also have to interpret as a pure gravitational field. But the field \( A_\mu \) is vector-field and reacts differently to opposite charges. Thus it shows that fields \( \Phi \) and \( A_\mu \) are not pure gravitational fields.

**QUANTIZATION**

From the geometric perspective, I categorized the canonical quantum theories into three.

1. **Background dependent theories**: Fields defined over given space-time are quantized. But the space-time remains classical. The Hamiltonian \( \hat{H} \left( \hat{\phi}_i, \hat{\pi}^i \right) \) gives time evolution.

2. **Background independent theories**: These theories attempt to quantize the gravity. Since gravity is a dynamical theory of space-time itself, the Hamiltonian does not give physical time evolution. The Hamiltonian is a collection of first-class constraints \( H_{\text{grav}} + H_{\text{matter}} = 0 \). Such theories have relational dynamics where dynamical fields evolve relative to one another. These theories require a sharp change of interpretation of time in the flat space-time limit \( H_{\text{grav}} \approx 0 \). In other words, such theories do not naturally reduce to the quantum theory in the flat space-time.

3. **Geometrized quantum theory**: The quantum theory developed in this paper fall under this category. Here, the field \( \hat{q}_{\mu \nu} \) has intrinsic property \( \mu \) defined by \( \hat{5} \). But the background \( g_{\mu \nu} \) is a solution to \( R(3) \). The Hamiltonian \( H \neq 0 \) gives \( \zeta \)-time evolution. The flat space-time makes theory equivalent to the fixed background quantum theory without any change of interpretation. Refer \( \hat{9} \) for the discussion on the gravitization of quantum mechanics.

Mathematically also quantum theories fall into three categories.

1. **The first quantization**: Theories such as Non-relativistic quantum theory or Wheeler-DeWitt theory for pure gravity fall in this category. In the Wheeler-DeWitt theory with pure gravity, the gravitational field variables \( g_{\mu \nu} \) are not obtained by re-interpretation. \( g_{\mu \nu} \) obtained by the generalization of special relativity.

2. **The second quantization**: These theories are obtained by re-interpretation. \( x^\mu := (t, \vec{x}) \) are parameters and fields are observable.
3. The third quantization: From the discussion above it is clear that the theories with \((q_{ab}(t, \vec{x}), \Phi(t, \vec{x}))\) are mathematically dubious. Because gravitational field variables are first quantized and matter fields are second quantized. The theories with \(\Phi(\phi(t, \vec{x}))\) called as third quantized theories.

The theory developed in the paper is second quantized. Because the field \(\Phi\) is defined only over \(q_{ab}\).

Geometric quantum effects are assumed to take over the classical geometry near the singularity. Quantization comes with the cost of raising creation and annihilation operators to vectors on coordinate space. An advantage of coordinate space quantization is that, it is possible to relate \((\zeta, \zeta^A)\) to \((t, \vec{x})\) making interpretation easier. I define vector-valued annihilation and creation operators

\[
a_A := \frac{(-G)^{\frac{1}{2}}}{\sqrt{G} \sqrt{\nabla A}} \left( \frac{1}{(-G)^{\frac{1}{2}}} n_A + i \sqrt{\frac{32}{3 \xi^2} \partial n_A} + i \omega \Phi n_A \right) \tag{26}
\]

\[
a_A^\dagger := \frac{(-G)^{\frac{1}{2}}}{\sqrt{G} \sqrt{\nabla A}} \left( \frac{1}{(-G)^{\frac{1}{2}}} n_B - i \sqrt{\frac{32}{3 \xi^2} \partial n_B} - i \omega \Phi n_B \right)
\]

\[n^A := \frac{G^{A B} C^B}{\sqrt{G} A C} (-G)^{\frac{1}{2}} := (-\text{det} G) \omega \zeta^A\] is a solution to the Riccati equation \([23]\).

\[
a_A^\dagger G^{AB} a_B + a_A G^{AB} a_B^\dagger
\]

\[
= \frac{\sqrt{G}}{\sqrt{\nabla A}} \left( \frac{1}{(-G)^{\frac{1}{2}}} n_A G^{AC} \left( \frac{\partial \Phi}{\partial \zeta^C} \right) \Phi \right)
\]

\[+ \omega \Phi \left( \frac{1}{(-G)^{\frac{1}{2}}} n_A G^{AC} \left( \frac{\partial \Phi}{\partial \zeta^C} \right) \Phi \right)
\]

The last term can be written as

\[
- \frac{\partial \Phi}{\partial \zeta^C} \left( \omega \sqrt{-G} \sqrt{\frac{32}{3 \xi^2} n_A G^{AC}} \right) \omega \Phi^2
\]

\[= \frac{\partial \Phi}{\partial \zeta^C} \left( \omega \sqrt{-G} \sqrt{\frac{32}{3 \xi^2} n_A G^{AC}} \right) \omega \Phi^2
\]

Integral of the first term on the right hand side is surface Integral. Assume it to vanish on the surface. Then

\[
\omega \sqrt{-G} \frac{1}{\sqrt{\nabla A}} \sqrt{\frac{32}{3 \xi^2} n_A G^{AC}} \left( \frac{\partial \Phi}{\partial \zeta^C} \right) \Phi
\]

\[= - \frac{\partial \Phi}{\partial \zeta^C} \left( \omega \sqrt{-G} \sqrt{\frac{32}{3 \xi^2} n_A G^{AC}} \right) \omega \Phi^2
\]

Now \(\omega \in \mathbb{R}\) is chosen as the solution to following Riccati equation

\[
\sqrt{-G} \omega^2 - \frac{\partial}{\partial \zeta^C} \left( \omega \sqrt{-G} \sqrt{\frac{32}{3 \xi^2} n_A G^{AC}} \right) = -G \mu^2 \tag{30}
\]

The equation should be solved using ‘correct’ boundary conditions. Such a solution is unique. Examples of such boundary conditions are

- FLRW \(\kappa = 0\) model: \(\omega\) that makes the spectrum of the Hamiltonian operator continuous in the limit \(q(t) \to \infty\).
- Schwarzschild spacetime: \(\omega\) that makes the spectrum of the Hamiltonian operator continuous in the limit \(q_{ab} \to \eta_{ab}\) with \(\eta_{ab}\) being flat 3-metric.

Computing the non-trivial commutator,

\[
a_A G^{AB} a_B^\dagger - a^\dagger_A G^{AB} a_B
\]

\[= \frac{1}{2} (-G)^{\frac{1}{2}} \sqrt{\frac{32}{3 \xi^2} n_A G^{AB} \left( \frac{\partial \Phi}{\partial \zeta^C} \right) - \Pi \frac{\partial \Phi}{\partial \zeta^C}}
\]

\[+ \frac{1}{2} \omega (-G)^{\frac{1}{2}} \left( \Phi \Pi - \Pi \Phi \right)
\]

\[= \frac{1}{2} (-G)^{\frac{1}{2}} \sqrt{\frac{32}{3 \xi^2} n_A G^{AB} \left( \frac{\partial \Phi}{\partial \zeta^C} \right)} + \frac{1}{2} (-G)^{\frac{1}{2}} \omega \left( \Phi \Pi - \Pi \Phi \right)
\]

Using property of Dirac delta function \(f(x) \delta'(x) = -f'(x) \delta(x)\), we get

\[
[a, a^\dagger] = \frac{1}{2} \left( \frac{32}{3 \xi^2} \partial \zeta^C \left( (-G)^{\frac{1}{2}} n_C \right) - (-G)^{\frac{1}{2}} \omega \right) \delta(\tilde{\zeta} \tilde{C})\]

(32)

Since this is the quantization of geometry, the Planck energy \(\varepsilon_P\) appears in stead of \(h\). The above commutator was possible because the inverse metric \(\bar{G}^{AB}\) is symmetric. These identities allow us to write the Hamiltonian operator in the discrete space. (i.e. \(\int \mathcal{D} \sigma A \to \sum_{A}\))

\[
\mathbf{H}_\Phi = \sum_{A} a^\dagger_A G^{AB} a_B
\]

\[+ \frac{\delta(0)}{2} \varepsilon_P \sum_{\zeta A} \left( \frac{32}{3 \xi^2} \partial \zeta^C \left( (-G)^{\frac{1}{2}} n_C \right) - (-G)^{\frac{1}{2}} \omega \right) \tilde{n}_A
\]

(33)

The second term is the vacuum term. The quantum vacuum is a sea of constantly creating and annihilating geometries. Discarding this term and writing the Hamiltonian operator in terms of number operator \(\hat{n} = \sum_{A} \hat{n}_A\) with \(a^\dagger_A a_A := \left( \frac{32}{3 \xi^2} \partial \zeta^C \left( (-G)^{\frac{1}{2}} n_C \right) - (-G)^{\frac{1}{2}} \omega \right) \tilde{n}_A\)

\[
\mathbf{H}_\Phi = \varepsilon_P \sum_{\zeta A} \left( \frac{32}{3 \xi^2} \partial \zeta^C \left( (-G)^{\frac{1}{2}} n_C \right) - (-G)^{\frac{1}{2}} \omega \right) \tilde{n}_A
\]

(34)

The appearance of the differential equation \([34]\) is not surprising. It is a consequence of using coordinate space for quantization. The scalar quantum of the Klein-Gordon field satisfies \(\omega^2 = k^2 + m^2\). Similarly, the quantum of the geometric scalar field follows frequency \(\frac{\sqrt{32 \xi^2} \partial \zeta^C (-G)^{\frac{1}{2}} n_C - (-G)^{\frac{1}{2}} \omega }{2} \omega\) with \(\omega\) being solution to \([30]\). \(\Phi\) has a single degree of freedom. Therefore the quantum is scalar. \(\Pi\) is a collection of creation and destruction operators. But \(\Phi\) depends non-linearly on creation and annihilation operators.

**Momentum:** The momentum operator defined using stress tensor

\[
\hat{P}^C = - \frac{32}{3 \xi^2} \sum_{\zeta A} \left( G^{CB} \frac{\partial \Phi}{\partial \zeta^B} \right) \Pi
\]

(35)

does not share eigenstates with the Hamiltonian operator. This is because \(\Phi\) depends non-linearly on creation and annihilation operators. Therefore the quantum with
a particular 3-metric does not have well-defined momentum at the quantum level.

**Measurement:** What is measured is the 3-metric and not the position of the quantum. Assume that the quantum state is ‘prepared’ in the superposition of several 3-geometries. On the measurement, we can observe only one 3-geometry. Quantum mechanically, there is no limit on how fast the quantum can change its 3-geometry.

**Hilbert Space**

Canonical commutation relations in discrete superspace are given as

\[ [a^\dagger, a] = [a, a^\dagger] = 0 \]
\[ [a, a^\dagger] = \beta \delta_{\zeta, \zeta'} \text{ (say)} \]

Where \( \beta := \frac{2\pi}{\sqrt{\epsilon}} \partial_c (-G^{\dagger} \xi^c) - (-G^{\dagger} \xi^c) \) is set for simplicity. These commutation relations allow us to construct the Hilbert space following experience of Harmonic oscillator with a major difference. As we move from one geometry (i.e. a particular point in the superspace) to another geometry, \( \beta \) changes. The role of creation and annihilation operator get interchanged depending on the signature of \( \beta \).

Define vacuum \( |0\rangle \) which gets annihilated by annihilation operator. Assume \( \beta > 0 \),

\[ a_A |0\rangle_A = 0 \]

any \( k \)-th state can be obtained by using creation operator

\[ |k_{\zeta, \zeta'}\rangle^A = \frac{1}{\sqrt{k!}} \left( a^{A\dagger}(\zeta, \zeta') \right)^k |0\rangle^A \]

The state represents \( k \) number of quantum with \( (\zeta, \zeta^A) \). Since creation and annihilation operators are vector-valued in the superspace, \( a_{\zeta}^A G^{CA} = a^{A\dagger} \) is adjoint of \( a_A \). The self-adjointness of the Hamiltonian requires \( \omega \in \mathbb{R} \). The state vector can be raised or lowered by the metric \( G_{AB} \) or the inverse metric \( G^{AB} \).

\[ |k_{1\zeta, \zeta'}\rangle_A = \tilde{G}_{AB} |k_{1\zeta, \zeta'}\rangle_B \]

The inner product is defined as

\[ B(k_{\zeta, \zeta'}|k_{\zeta', \zeta'}\rangle_B = \delta_{k_1, k_2} \delta(\zeta, \zeta') \]

Note that \( B \) is not summed over in the left hand side of the equation. Stated \( \{ |k_{\zeta, \zeta'}\rangle^A \} \) with \( A = 1, 2, 3, 4, 5 \) together describe single-geometry and each of the state deals with a particular coordinate in the superspace.

**Time**

Although the geometrized quantum theory is not a quantization of geometry, field quantization results in the geometric quantum. Therefore I discuss the problem of time. Kucher ([17], chapter 2) classifies the problem of time in quantum gravity into three. Namely, the functional evoloution problem, the multiple-choice problem, and the Hilbert space problem.

1. In the ADM theory Hamiltonian constraints obey Poisson bracket \( \{ H(x), H(x') \} \approx 0 \). But After raising them to operators on the Hilbert space, we may get \( [H(x), H(x')] \neq 0 \). It is the functional evolution problem. The geometrized quantum theory does not have a problem with functional evolution. Because \( q_{ab} \) and \( P^{ab} \) are not observables in theory. Hamiltonian constraints re-interpreted as a classical field equation.

2. Depending on the choice of time, we may get a different quantum theory. In the case of relativistic quantization, the particle’s clock serves as the best clock. Similarly, the volume element of the Universe serves as the best clock for the geometrized quantum theory.

3. The third quantized theories face the Hilbert space problem ([17], chapter 11). If we follow Kucher’s analysis ([17], chapter 11), the problem is deeply rooted in the concept of particle. In the third quantized theory, the interpretation of a particle does not change. Since the space-time background is dynamic, it is not clear what is one-particle Hilbert space. At the classical level itself, the theory discussed in the paper deviates from standard matter fields and pure gravitational fields. In the geometrized quantum theory, everything is geometric, and single-geometry Hilbert space is known. It shows that the geometrized quantum theory does not have the Hilbert space problem.

**RESULTS**

- I have shown that the field satisfied by the Wheeler-DeWitt equation cannot be a pure gravitational field.
- I have also shown that the interpretation of the field \( \Phi \) as a description of a single universe faces serious problems. The correct interpretation is that the non-interactive fields describe isolated geometries.
- I observe that the gravitational field variables \( (q_{ab}, P^{cd}) \) and matter fields are not on equal footings. The gravitational field variables are first quantized. But matter fields are second quantized. Any theory that treats both fields on equal footing concerning quantum level is dubious.
• In addition to having geometric properties, fields satisfying the Wheeler-DeWitt equation have properties similar to the corresponding matter fields.

• The field $\Phi$ in the asymptotically flat space-time limit behaves similar to the matter field in the fixed space-time background.

• The quantum theory studied to clarify and further understand the re-interpretation shows that the procedure geometrizes the quantum theory itself.

• The theory is free from the problem of time.

• The theory does not attempt to quantize the geometry. It is a result of field quantization.

* Electronic address: avdhoot.purohit@gmail.com

[1] Bryce S. DeWitt. Quantum theory of gravity i: The canonical theory. 160, 1967. URL https://doi.org/10.1103/physrev.160.1113

[2] Richard Arnowitt, Stanley Deser, and Charles W. Misner. Republication of: The dynamics of general relativity. 40, 2008. URL https://doi.org/10.1016/0370-2693(88)90108-6

[3] Michael McGuigan. Third quantization and wheeler-de Witt equation. 38, 1988.

[4] S J Robles Pérez. Third quantization: modeling the universe as a 'particle' in a quantum field theory of the minisuperspace. 410, 2013. URL https://doi.org/10.1088/1742-6596/410/1/012133

[5] Rubakov VA. On third quantization and the cosmological constant. 214, 1988. URL https://doi.org/10.1016/0370-2693(88)90108-6

[6] Strominger A. Quantum Cosmology and Baby Universes. World Scientific, London, UK, 1990.

[7] Karel V. Kucher. Time and interpretations of quantum gravity. International Journal of Modern Physics D. 20(Suppl. 1), 2011. URL https://doi.org/10.1142/S0218271811019347

[8] Claus Kiefer. Oxford press, Oxford, UK, third edition, 2012.

[9] Roger Penrose. On the gravitization of quantum mechanics i: Quantum state reduction. 44, 2014. URL https://doi.org/10.1007/s10701-013-9770-0

[10] Domenico Giulini. What is the geometry of superspace ? 51, 1995. URL https://doi.org/10.1103/PhysRevD.51.5630