ON ERROR TERM ESTIMATES À LA WALFISZ FOR MEAN VALUES OF ARITHMETIC FUNCTIONS

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Abstract. Walfisz (1963) proved the asymptotic formula
\[ \sum_{n \leq x} \varphi(n) = \frac{3}{\pi^2} x^2 + O(x (\log x)^{\frac{3}{4}} (\log \log x)^{\frac{3}{4}}), \]
which improved the error term estimate of Mertens (1874) and had been the best possible estimate for more than 50 years. Recently, H.-Q. Liu (2016) improved Walfisz’s error term estimate to
\[ \sum_{n \leq x} \varphi(n) = \frac{3}{\pi^2} x^2 + O(x (\log x)^{\frac{3}{4}} (\log \log x)^{\frac{1}{3}}). \]
We generalize Liu’s result to a certain class of arithmetic functions and improve the result of Balakrishnan and Pētermāns (1996). To this end, we provide a refined version of Vinogradov’s combinatorial decomposition available for a wider class of multiplicative functions.

1. Introduction

Let \( \varphi(n) \) be the Euler totient function
\[ \varphi(n) = n \prod_{p | n} \left( 1 - \frac{1}{p} \right). \]
Then it is relatively elementary to prove the asymptotic formula
\[ \sum_{n \leq x} \varphi(n) = \frac{3}{\pi^2} x^2 + O(x \log x) \]
due to Mertens [11]. In 1963, Walfisz [18] improved this error term estimate to
\[ \sum_{n \leq x} \varphi(n) = \frac{3}{\pi^2} x^2 + O(x (\log x)^{\frac{3}{4}} (\log \log x)^{\frac{3}{4}}). \]
Walfisz’s improvement is based on two methods for exponential sums developed by Vinogradov, one of which is Vinogradov’s mean value theorem and the other is some combinatorial decomposition. Recently, H.-Q. Liu [10] obtained a further improvement in the \((\log \log x)\)-power:
\[ \sum_{n \leq x} \varphi(n) = \frac{3}{\pi^2} x^2 + O(x (\log x)^{\frac{3}{4}} (\log \log x)^{\frac{1}{3}}). \]

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The main ingredient of Liu’s improvement is to replace Vinogradov’s combinatorial decompositions by Vaughan’s identity [6, Proposition 13.5], which enables us to produce Type II sums with more efficient summation ranges.

Walfisz’s result (1) can be generalized to a certain class of arithmetic functions. Such a generalization was studied by Balakrishnan and Pétermann [1]. Their result can be summarized in the following two theorems.

**Theorem A** ([1, Theorem 1]). Let \( \alpha \) be a complex number and

\[
f(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}
\]

be a Dirichlet series absolutely convergent for \( \sigma > 1 - \lambda \) with some real number \( \lambda > 0 \). Define arithmetic functions \( a(n) \) and \( v(n) \) by

\[
\sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \zeta(s)\zeta(s+1)^\alpha f(s+1)
\]

and

\[
\sum_{n=1}^{\infty} \frac{v(n)}{n^s} = \zeta(s)^\alpha f(s)
\]

for \( \sigma > 1 \), where we take the branch of \( \zeta(s+1)^\alpha \) by \( \arg \zeta(s+1) = 0 \) on the positive real line. Then we have

\[
\sum_{n \leq x} a(n) = \zeta(2)^\alpha f(2)x + \sum_{r=0}^{\lfloor \text{Re} \alpha \rfloor} A_r (\log x)^{\alpha_r - r} - \sum_{n \leq y} \frac{v(n)}{n} \psi \left( \frac{x}{n} \right) + o(1)
\]

as \( x \to \infty \), where the coefficients \( (A_r) \) are computable from the Laurent expansion of \( \zeta(s)^\alpha f(s) \) at \( s = 1 \), \( y = x \exp(- (\log x)^\theta) \), \( \psi(x) = \{x\} - \frac{1}{2} \) and the Landau symbol \( o(1) \) depends on the assumptions of this theorem.

**Theorem B** ([14, Theorem 1]). Let \( v(n) \) be a real-valued multiplicative function. Assume that there exist real numbers \( \alpha_1, \beta \geq 0 \) and a sequence of real numbers \( (V_r)_{r=0}^\infty \) such that for any positive integer \( \lambda \) and real number \( x \geq 4 \), we have

\[
(h1) \sum_{n \leq x} |v(n)| = x \sum_{r=0}^{\lambda + [\alpha_1]} V_r (\log x)^{\alpha_1 - r} + O\left( x(\log x)^{-\lambda} \right),
\]

\[
(h2) \sum_{n \leq x} |v(n)|^2 \ll x(\log x)^{\beta},
\]

and

\[
(h3) \text{the function } v(p) \text{ is ultimately monotonic with respect to } p
\]

and the function \( v(p^n) \) is bounded for every prime \( p \) and \( n \geq 1 \). Then for \( x \geq 4 \) and \( \theta > 0 \), we have

\[
\sum_{n \leq y} \frac{v(n)}{n} \psi \left( \frac{x}{n} \right) \ll (\log x)^{2(\alpha_1 + 1)} (\log \log x)^{\frac{4(\alpha_1 + 1)}{3}},
\]

where \( y = x \exp(- (\log x)^\theta) \) and the implicit constant depends on \( \theta \) and the constants in the above hypotheses \( (h1), (h2), \) and \( (h3) \).
Remark 1. Balakrishnan and Pétermann [1, Errata] claimed an error in the proof of Lemma 3 of [1, p.52–53], which means, in turn, an error in the proof of Theorem 2 in the same paper. In particular, they remarked that the argument at the top of p.53 is not immediate with the condition (h1) in [1]. Therefore, we stated Theorem B, a corrected version of this theorem given by Pétermann [14]. However, we may use the condition (h2) of [1] instead of (h1) to recover Theorem 2 of [1]. We omit the details on this argument since Theorem 1 below also recovers this original theorem. For the related arguments, see the proof of Lemma 5 in Section 3.

Remark 2. Since the parameter $\alpha$ in [1] and the parameter $\alpha_1$ in [14] have slightly different meanings, we used the letter $\alpha_1$ to denote the parameter $\alpha$ in [14]. These two parameters are roughly connected by $\alpha = \alpha_1 + 1$.

Remark 3. In [1], the parameter $y$ is given by $y = x \exp(-(\log x)^b)$ with some fixed real number $b$ such that $0 < b < B$, where $B$ is the constant such that $0 < B \leq 1/2$ and we have $\zeta(s) \neq 0$ in the region

\[(6) \quad \sigma > 1 - (\log(|t| + 4))^{-(1-B)}, \quad |t|: \text{large}.\]

By using the Vinogradov-Korobov zero free region [5, Theorem 6.1], we can choose $B$ to be any positive real number $< 1/3$, where the case $B = 1/3$ is excluded. Thus we can take, for example, $b = 1/6$. We chose this specific value in Theorem A for the notational simplicity. Note that the Vinogradov-Korobov zero free region [5, Theorem 6.1] is not the same one as Vinogradov and Korobov originally claimed, which is regarded to be still unproven today. Furthermore, by using the Selberg–Delange method as in [16, Chapter II.5], we may take any $0 < b < 1/2$ just by using the above zero free region (6) with $B = 0$.

The main aim of this paper is to improve the above result of Balakrishnan and Pétermann. As we can see from Theorem B, their result is based on Walfisz’s result (1). So it is natural to ask some improvement up to the strength of Liu’s result (2). Our main result can be stated as follows.

**Theorem 1.** Let $v(n)$ be a complex-valued multiplicative function such that there exists a real number $C \geq 2$ satisfying the following three conditions:

(V1) $|v(p)| \leq C$ for every prime number $p$,

(V2) $\sum_{n \leq x} |v(n)|^2 \leq Cx(\log x)^C$ $(x \geq 4)$,

(V3) $\sum_{p_n \leq x} |v(p_{n+1}) - v(p_n)| \leq C(\log x)^C$ $(x \geq 4)$,

where $p_n$ is the $n$-th prime number. Assume that a real number $\kappa \geq 0$ satisfies

(V) $\sum_{n \leq x} \frac{|v(n)|}{n} \ll (\log x)^\kappa$

for $x \geq 4$. Then for $x \geq 4$ and $\theta > 0$, we have

(7) $\sum_{n \leq y} \frac{v(n)}{n} \psi \left(\frac{x}{n}\right) \ll (\log x)^\theta (\log \log x)^\theta$,

where $y \leq x \exp(-(\log x)^\theta)$ and the implicit constant depends only on $\theta$, $C$, and the implicit constant in the above condition (V).
Remark 4. The estimate of the type (V) automatically follows by the assumption (V2). Thus, the assumption (V) is given only for specifying the exponents in the resulting estimate (7).

Remark 5. Theorem A, Theorem B, and Theorem 1 are related to the sum

$$\sum_{n \leq x} \phi(n)$$

rather than the left-hand side of (1). However, it is possible to translate this type of sums to the sum of the type (1) by partial summation. See the arguments in [1, p.64–68]. The author is also planning to give the related details somewhere. In particular, for the sum of the type

$$\sum_{n \leq x} \phi(n)^it$$

with some real number \( t \neq 0 \), we need to employ partial summation more carefully than in [1, p.64–68].

The estimate (7) gives an error term estimate of the strength of Liu’s result (2). It is natural to apply Liu’s approach [10] to prove (7). However, the author of this paper could not succeed to use Liu’s approach straightforwardly. In order to prove Theorem 1, we first prepare an estimate for the exponential sum over primes

$$\sum_{P < p \leq P'} e \left( \frac{Q}{p} \right), \quad P < P' \leq 2P$$

similarly to Main Lemma of Pétermann [14]. Main Lemma of Pétermann can be improved easily by using Liu’s approach, which is just a translation of Liu’s proof for the Möbius function in terms of the von Mangoldt function. However, our final result deals with more general arithmetic functions \( v(n) \), and the author could not find a combinatorial identity for such general multiplicative functions. Thus we need to return to the original approach of Vinogradov. In order to achieve the improved error term estimate even by using the decomposition of the Vinogradov-style, we use such a decomposition finer than the decomposition used by Walfisz or by Pétermann. For the details, see Lemma 9.

Not only improving the final error term estimate itself, Theorem 1 also relaxes the necessary assumptions in Theorem B. We now compare some assumptions of Theorem A and Theorem 1:

(1) We first removed the assumption that \( v(n) \) is real-valued. This is done just by remove the monotonicity on \( v(p) \) as in (h3) and by introducing a weaker assumption (V1) and (V3). Note that we also removed the assumption on the values of \( v(n) \) at the higher prime powers.

(2) The condition (V2) just states the same assumption as (h2).

(3) We removed the strong assumption (h1) and replaced by (V), which is the same assumption as in Theorem 2 of [1]. So, in particular, our Theorem 1 recovers Theorem 2 of [1].

(4) As we mentioned in Remark 2, we roughly have \( \alpha_1 = \alpha - 1 \). In Theorem B, we assumed that \( \alpha_1 \geq 0 \) so that, in principle, we are restricted to the case \( \alpha \geq 1 \). In Theorem 1, we roughly have \( \kappa = \alpha \) and we assumed only \( \kappa \geq 0 \). Thus, Theorem 1 is applicable for a wider range of \( \alpha \) than Theorem B.
As we can see from the above comparison, Theorem 1 is available for a wider class of multiplicative function than Theorem B. Actually, Theorem 1 is relaxed enough to obtain the following theorem.

**Theorem 2.** Under the same hypothesis as in Theorem A, we have

$$\sum_{n \leq x} a(n) = \zeta(2)^{\alpha} f(2)x + \sum_{r=0}^{[\Re \alpha]} A_r (\log x)^{\alpha-r} + O((\log x)^{2|\alpha|} (\log \log x)^{|\alpha|})$$

for $x \geq 4$, where the implicit constant depends on the hypothesis in Theorem A.

This enables us to apply the method of Balakrishnan–Pétermann directly to the generating function (4) without checking any additional assumption besides the assumptions in Theorem A. We shall prove Theorem 2 in Section 6.

**Remark 6.** Recently, Drappeau and Topacogullari [2] gave a new combinatorial decomposition for general $\tau_\alpha(n)$ with any complex number $\alpha$ in the context of the generalized Titchmarsh divisor problems. As we will see in the proof of Theorem 2, especially in the argument after (83), the function $v(n)$ in Theorem A can be reduced to $\tau_\alpha(n)$. More precisely, $v(n)$ can be decomposed into the convolution of $\tau_\alpha(n)$ and an arithmetic function $b(n)$ for which the series (3) converges absolutely for $\sigma < 1$. (For the same principle, see also Lemma 2.2 of [2].) Thus, for the proof of Theorem 2, the method of Drappeau and Topacogullari is available. However, our Theorem 1 deals with a slightly wider class of arithmetic functions. For example, the multiplicative function $v(n)$ defined by

$$v(p) = 2 + \frac{1}{\log \log (p + 4)}, \quad v(p^\nu) = 0 \quad (\nu \geq 2)$$

satisfies the conditions (V1), (V2), and (V3), but cannot be decomposed into the convolution of $\tau_\alpha(n)$ and $b(n)$ for which the series (3) converges for $\sigma < 1$. Indeed, the only possible choice of $\alpha$ for this example is $\alpha = 2$, but for this case,

$$b(p) = v(p) + \tau_{-2}(p) = \frac{1}{\log \log (p + 4)}$$

so that

$$\sum_{n \leq x} |b(n)| \geq \sum_{p \leq x} \frac{|b(p)|}{p} = \sum_{p \leq x} \frac{1}{p \log \log (p + 4)} \rightarrow \infty$$

as $x \rightarrow \infty$. Thus, at least in Theorem 1, our combinatorial decomposition seems to be slightly more general than the decomposition of Drappeau and Topacogullari.

2. **Notation**

The letter $p$ denotes a prime number and $p_n$ denotes the $n$-th prime number. By $s = \sigma + it$, we denote a complex variable $s$.

For a positive integer $n$, we denote by $\omega(n)$ the number of distinct prime factors of $n$ and by $\Omega(n)$ the number of prime factors of $n$ counted with multiplicity. As usual, $\Lambda(n)$ is the von Mangoldt function, $\mu(n)$ is the Môbius function, $\varphi(n)$ is the Euler totient function, and $\sigma(n)$ is the divisor summatory function. The function $\tau(n)$ is the divisor function, i.e. it denotes the number of positive divisors of $n$. 
More generally, for a complex number $\alpha$, we define the divisor function $\tau_\alpha(n)$ by the generation function

$$
\zeta(s)^\alpha = \sum_{n=1}^\infty \frac{\tau_\alpha(n)}{n^s} \quad \sigma > 1,
$$

where the branch of $\zeta(s)^\alpha$ is taken by $\arg \zeta(s) = 0$ for $s > 1$. Note that $\tau_2(n) = \tau(n)$.

For a positive integer $n$, we define $p_{\text{max}}(n)$ and $p_{\text{min}}(n)$ be the largest and the smallest prime factor of $n > 1$, respectively, and as a convention, we define $p_{\text{max}}(1) = 1$ and $p_{\text{min}}(1) = +\infty$. By $\psi(x, y)$, we denote the number of $y$-smooth numbers $\leq x$, i.e.

$$
\psi(x, y) = |\{n \leq x \mid p_{\text{max}}(n) \leq y\}|.
$$

We use the conditions (h1), (h2), (h3), (V1), (V2), (V3), (V) on multiplicative functions. See Theorem A and Theorem 1. The letters $C$ and $\kappa$ always denote the constants in (V1), (V2), (V3), (V). By saying a multiplicative function, we exclude the constant function 0.

We denote the fractional part of a real number $x$ by $\{x\}$ and let

$$
\psi(x) = \{x\} - \frac{1}{2}.
$$

The function $e(x)$ is defined by $e(x) = e^{2\pi i x}$ as usual.

The letter $B$ denotes the constant used for describing admissible ranges of several parameters in each Theorem or Lemma, e.g. see the condition (11). Thus $B$ has the same meaning during a fixed Theorem or Lemma and their proof, but it may have different meanings in different context. In order to denote the constant $B$ used in some preceding context, we use letters $B_1, B_2, \ldots$ instead of $B$. We emphasize the dependance of $B$ on some letters $A, C, \ldots$ by writing $B = B(A, C, \ldots)$.

If Theorem or Lemma is stated with the phrase “where the implicit constant depends only on $a, b, c, \ldots$”, then every implicit constant in the corresponding proof may also depend on $a, b, c, \ldots$ even without special mentions.

### 3. Exponential sums over primes

In this section, we prepare an estimate for exponential sums over primes. As we have mentioned, this estimate corresponds to Main Lemma of Pétermann [14] or the result of Liu [10]. We follow the method of Liu in order to improve Main Lemma of Pétermann [14] and simplify the proof.

We first prepare an estimate for the exponential sum

$$
\sum_{P < n \leq P'} e \left( \frac{Q}{n} \right), \quad P < P' \leq 2P, \quad e(x) = e^{2\pi i x}
$$

based on Vinogradov’s mean value theorem. We use Vinogradov’s mean value theorem through an application of the following lemma due to Karatsuba [7].

**Lemma 1** (Karatsuba’s lemma [7, Theorem 1]). Let $k$ be a positive integer, $X, P$ be real numbers with $P \geq 1$ and $f(x)$ be a real-valued function defined and $(k+1)$-times continuously differentiable on $[X, X + P]$. Assume that there are four constants

$$
0 < c_0 < 1, \quad 0 < c_3 \leq c_2 < c_1 < 1
$$

(8)
and positive integers
\[(9) \quad c_0 k \leq r \leq k, \quad 1 \leq j_1 < j_2 < \cdots < j_r \leq k\]
satisfying the following conditions:

(A) we have
\[
\left| \frac{f^{(k+1)}(x)}{(k+1)!} \right| \leq P^{c_1} \quad \text{on } [X, X+P],
\]

(B) we have for every \( j \in \{j_1, \ldots, j_r\} \),
\[
P^{-c_2 j} \left| \frac{f^{(j)}(x)}{j!} \right| \leq P^{-c_3 j}
\]
on \([X, X+P]\).

Then there is a constant \( 0 < \gamma \leq 1 \) such that for any \( P_1 \leq P \),
\[
\left| \sum_{X < n \leq X+P_1} e(f(n)) \right| \ll P^{1-\frac{\gamma}{2}},
\]
where the constant \( \gamma \) and the implicit constant depends only on \( c_0, c_1, c_2, \) and \( c_3 \).

Remark 7. In [7], there is one more constant \( c_4 \). However, since this constant \( c_4 \) can be taken by \( c_4 = (c_1 - c_2)/2 \) in the above setting, we did not introduce the constant \( c_4 \) for the notational simplicity.

We also make use of the Kusmin–Landau inequality:

**Lemma 2** (Kusmin–Landau inequality [4, Theorem 2.1]). Let \( X, P, \lambda \) be real numbers with \( P \geq 1 \) and \( \lambda > 0 \), and \( f(x) \) be a real-valued function defined and continuously differentiable on \([X, X+P]\). Assume also that \( f'(x) \) is monotonic and \( \lambda < |f'(x)| < 1 - \lambda \) on \([X, X+P]\). Then, we have
\[
\sum_{X < n \leq X+P} e(f(n)) \ll \lambda^{-1},
\]
where the implicit constant is absolute.

**Lemma 3.** Let \( P, P', Q \geq 4 \) be real numbers with \( P < P' \leq 2P \). Then
\[
\sum_{P < n \leq P'} e \left( \frac{Q}{n} \right) \ll P \exp \left( -\gamma \frac{(\log P)^3}{(\log Q)^2} \right) + P^2 Q^{-1},
\]
where the implicit constant and the constant \( \gamma > 0 \) are absolute.

**Proof.** Let \( f(x) = Q/x \). We may assume \( P > 2^{12} \) since otherwise
\[
P \exp \left( -\gamma \frac{(\log P)^3}{(\log Q)^2} \right) \geq P \exp \left( -\frac{12 \log 2)^3}{(2 \log 2)^2} \right) \gg P
\]
so the assertion is trivial. We may also assume \( P \leq Q^{2}/2 \) since if \( P > Q^{2}/2 \), then we may apply Lemma 2. Indeed, in this case, \( f'(x) \) is increasing on \([P, 2P]\),
\[
|f'(x)| \leq Q P^{-2} \leq 2^{2} P^{-\frac{1}{2}} \leq \frac{1}{2}
\]
and $|f'(x)| \gg QP^{-2}$ on $[P, 2P]$. Thus, Lemma 2 gives

$$\sum_{P < n \leq P'} e\left(\frac{Q}{n}\right) \ll P^2 Q^{-1}.$$ 

Therefore, we shall consider the case $2^{12} < P \leq Q^2/2$.

We apply Lemma 1 with

$$X = P, \quad P_1 = P' - P, \quad k = \left[\frac{26 \log Q}{\log P}\right],$$

$$c_0 = \frac{1}{39}, \quad c_1 = \frac{25}{26}, \quad c_2 = \frac{23}{24}, \quad c_3 = \frac{1}{4},$$

and

$$j_1 < \cdots < j_r: \text{all integers in } \left(2 \log \frac{Q^2}{2}, \frac{4 \log Q^2}{\log P}\right)$$

so that

$$r = \left\lfloor \frac{4 \log Q^2}{\log P}\right\rfloor - \left\lfloor \frac{2 \log Q^2}{\log P}\right\rfloor.$$ 

With this choice, the condition (8) is obviously satisfied. For the condition (9), the size of $j_1, \ldots, j_r$ clearly satisfies the condition. Also, we have

$$r = \left\lfloor \frac{4 \log Q^2}{\log P}\right\rfloor - \left\lfloor \frac{2 \log Q^2}{\log P}\right\rfloor \geq 2 \log \frac{Q^2}{\log P} - 1 \geq \log \frac{Q^2}{\log P} \geq 1$$

so that $j_1, \ldots, j_r$ is not an empty sequence and

$$c_0 k \leq \frac{2 \log Q}{3 \log P} \leq r \leq \frac{4 \log Q^2}{\log P} \leq k.$$ 

This assures the condition (9). The remaining conditions for Lemma 1 are (A) and (B). The condition (A) can be checked as

$$\left|\frac{f^{(k+1)}(x)}{(k+1)!}\right| = \frac{Q}{x^{k+2}} \leq \frac{Q}{P^{k+2}} \leq \frac{Q}{P^{k+1}} = P^{-c_1(k+1)} \frac{Q}{P^{(1-c_1)(k+1)}} \leq P^{-c_1(k+1)},$$

where we used

$$(1 - c_1)(k + 1) = \frac{1}{26} \left(\left\lfloor \frac{26 \log Q}{\log P}\right\rfloor + 1\right) \geq \log \frac{Q}{\log P}.$$ 

We move on to the condition (B). For $j \in \{j_1, \ldots, j_r\}$, by definition, we have

$$2 \frac{\log Q^2}{\log P} < j \leq 4 \frac{\log Q^2}{\log P},$$

Therefore, for the upper bound of the condition (B), we find

$$\left|\frac{f^{(j)}(x)}{j!}\right| = \frac{Q}{x^{j+1}} \leq \frac{Q}{P^j} \leq P^{-c_3 j}.$$ 

For the lower bound, by using the assumption $P \leq Q^2/2$, we obtain

$$\left|\frac{f^{(j)}(x)}{j!}\right| \geq \frac{Q}{(2P)^{j+1}} \geq \frac{Q^2}{(2P)^j}.$$
Since we are also assuming $2^{12} < P$, 

$$\left| \frac{f^{(j)}(x)}{j!} \right| \geq \frac{Q^2}{P^{12}} = P^{-c_2j} \frac{Q^2}{P^{12}} \geq P^{-c_2j}$$

by using the inequality (10).

Therefore, we can apply Lemma 1 to obtain

$$\sum_{P < n \leq P'} e\left( \frac{Q}{n} \right) \ll P^{1 - \gamma}$$

where the implicit constant and $\gamma$ is now absolute. Since

$$k = \left\lfloor \frac{26 \log Q}{\log P} \right\rfloor \leq \frac{26 \log Q}{\log P},$$

we obtain

$$P^{1 - \gamma} \leq P \exp \left( -\frac{\gamma}{676} \left( \log Q \right)^2 \right).$$

This completes the proof. \hfill \Box

We next recall Vaughan’s identity, the main ingredient of Liu’s improvement.

**Lemma 4** (Vaughan’s identity). For a real number $z \geq 2$ and any integer $n > z$,

$$\Lambda(n) = a_1(n) - a_2(n) + a_3(n),$$

where

$$a_1(n) = \sum_{uv = n \atop u, v \leq z} \mu(u) \log v, \quad a_2(n) = \sum_{uv = n \atop u \leq z^2} \left( \sum_{d \mid m = u} \mu(d) \Lambda(m) \right),$$

$$a_3(n) = \sum_{uv = n \atop u, v > z} \left( \sum_{d \mid m = u} \mu(d) \right) \Lambda(v).$$

**Proof.** See [6, p. 344, Section 13.4]. \hfill \Box

After applying Vaughan’s identity, the exponential sum is decomposed into Type I and Type II sums as usual. Therefore, we next prove the Type II sum estimate with exponential sum estimate of the Vinogradov type.

**Lemma 5** (Type II sum estimate). For any real number $A \geq 1$, there exists a real number $B = B(A) \geq 1$ such that for any sequences of complex numbers

$$A = (\alpha_u)_{u=1}^\infty, \quad B = (\beta_v)_{v=1}^\infty$$

and any real numbers $P, P', U, U', V, V', Q \geq 4$ with

$$P < P' \leq 2P, \quad U < U' \leq 2U, \quad V < V' \leq 2V',$$

we have

$$\sum_{P < uv \leq P' \atop U < uv \leq U' \atop V < uv \leq V'} \alpha_u \beta_v e\left( \frac{Q}{uv} \right) \ll \left( P^{\frac{1}{2}} (\log Q)^{-A} + PQ^{-\frac{1}{2}} \right) \|A\| \|B\| (\log Q)^{\frac{1}{2}}$$

provided

$$U, V \geq \exp\left( B(\log Q)^{\frac{1}{2}} (\log \log Q)^{\frac{1}{2}} \right),$$

\hfill (11)
where
\[ ||\mathcal{A}||^2 = \sum_{U < u \leq U'} |\alpha_u|^2, \quad ||\mathcal{B}||^2 = \sum_{V < v \leq V'} |\beta_v|^2 \]
and the implicit constant depends only on \( \mathcal{A} \).

Proof. We let \( 0 < \gamma \leq 1 \) be the constant in Lemma 3 and take \( B = (2A/\gamma)^{\frac{1}{4}} \). We then employ some trivial reductions. By symmetry between \( u \) and \( v \), it suffices to consider the case \( U \geq V \). If the sum in the assertion is non-empty, then we have \( P < U'V' \) and \( UV \leq P' \). Therefore, we may assume
\[ P/4 \leq UV \leq 2P. \] (12)
We may assume \( Q \) is larger than some constant depending only on \( \mathcal{A} \) since otherwise the Cauchy–Schwarz inequality gives
\[ \sum_{P < uv \leq P'} \sum_{U < u \leq U'} \sum_{V < v \leq V'} \alpha_u \beta_v e \left( \frac{Q}{uv} \right) \ll ||\mathcal{A}|| ||\mathcal{B}|| \frac{P}{\gamma} \]
so that the assertion follows. Similarly, we may assume \( P \leq Q/32 \) since otherwise \( PQ^{-\frac{1}{2}} \gg P^{\frac{1}{2}} \) so that (13) again gives the assertion.

After the above reductions, we use the Cauchy–Schwarz inequality to obtain
\[ \left| \sum_{P < uv \leq P'} \sum_{U < u \leq U'} \sum_{V < v \leq V'} \alpha_u \beta_v e \left( \frac{Q}{uv} \right) \right|^2 \leq ||\mathcal{A}||^2 \left| \sum_{P < u < v \leq P'/u} \sum_{V < v \leq V'} \beta_v e \left( \frac{Q}{uv} \right) \right|^2. \] (14)
We then expand the square in the latter factor as
\[ \sum_{U < u \leq U'} \left| \sum_{P < u < v \leq P'/u} \sum_{V < v \leq V'} \beta_v e \left( \frac{Q}{uv} \right) \right|^2 \]
\[ \leq \sum_{V < v_1, v_2 \leq V'} |\beta_{v_1} \beta_{v_2}| \left| \sum_{U(v_1, v_2) < u \leq U'(v_1, v_2)} e \left( \frac{Q}{u} \left( \frac{1}{v_1} - \frac{1}{v_2} \right) \right) \right| \]
\[ = \sum_{v_1 = v_2} + 2 \sum_{v_1 < v_2} = S_1 + 2S_2, \quad \text{say}, \]
where \( U(v_1, v_2) = \max(U, P/v_1, P/v_2), \quad U'(v_1, v_2) = \min(U', P'/v_1, P'/v_2) \).
Note that obviously
\[ U \leq U(v_1, v_2), \quad U'(v_1, v_2) \leq U' \leq 2U \leq 2U(v_1, v_2). \]
for every \( v_1, v_2 \). Thus for \( S_1 \), we have
\[ S_1 \ll U ||\mathcal{B}||^2 \ll PV^{-1} ||\mathcal{B}||^2 \ll P(\log Q)^{-2} ||\mathcal{B}||^2 \]
by (11) and (12). We apply Lemma 3 for \( S_2 \). If \( V < v_1 < v_2 \leq V' \), then
\[ Q \left( \frac{1}{v_1} - \frac{1}{v_2} \right) = \frac{Q(v_2 - v_1)}{v_1 v_2} \geq \frac{Q}{4V^2} \geq \frac{Q}{4UV} \geq \frac{Q}{8P} \geq 4 \]
since $U \geq V$, $UV \leq P'$ and $P \leq Q/32$. Thus we may apply Lemma 3 to the sum
\[ \sum_{U(v_1, v_2) < u \leq U'(v_1, v_2)} e\left( \frac{Q}{u} \left( \frac{1}{v_1} - \frac{1}{v_2} \right) \right). \]

By using
\[ U \leq U(v_1, v_2) \leq 2PV^{-1}, \quad Q \left( \frac{1}{v_1} - \frac{1}{v_2} \right) \leq Q \]
and
\[ U(v_1, v_2)^2 \left( Q \left( \frac{1}{v_1} - \frac{1}{v_2} \right) \right)^{-1} \ll (PV^{-1})^2Q^{-1} \frac{v_1v_2}{|v_1 - v_2|} \ll \frac{P^2Q^{-1}}{|v_1 - v_2|} \]
we obtain
\[ \sum_{U(v_1, v_2) < u \leq U'(v_1, v_2)} e\left( \frac{Q}{u} \left( \frac{1}{v_1} - \frac{1}{v_2} \right) \right) \ll PV^{-1} \exp \left( -\gamma \frac{(\log U)^3}{(\log Q)^2} \right) + \frac{P^2Q^{-1}}{|v_1 - v_2|} \ll PV^{-1}(\log Q)^{-2A} + \frac{P^2Q^{-1}}{|v_1 - v_2|} \]
by (11) since we took the constant $B$ by $B = (2A/\gamma)^{\frac{1}{2}}$. By using
\[ \sum_{V < v_1, v_2 \leq V'} |\beta_{v_1}\beta_{v_2}| \leq V \sum_{V < v \leq V'} |\beta_v|^2 = V\|B\|^2 \]
and
\[ \sum_{V < v_1 \neq v_2 \leq V'} \frac{|\beta_{v_1}\beta_{v_2}|}{|v_1 - v_2|} \ll \sum_{V < v_1 \leq V'} |\beta_{v_1}|^2 \sum_{V < v_2 \leq V', v_1 \neq v_2} \frac{1}{|v_1 - v_2|} \ll (\log Q)\|B\|^2, \]
where we used the symmetry between $v_1$ and $v_2$, we arrive at
\[ S_2 \ll \left( P(\log Q)^{-2A} + P^2Q^{-1}(\log Q) \right) \|B\|^2 \ll \left( P(\log Q)^{-2A} + P^2Q^{-1} \right) \|B\|^2(\log Q). \]

By combining (16) and (17) and inserting into (15),
\[ \sum_{U < u \leq U'} \left| \sum_{P/u < u' < P/u' \atop V < v \leq V'} \beta_v e\left( \frac{Q}{uv} \right) \right|^2 \ll \left( P(\log Q)^{-2A} + P^2Q^{-1} \right) \|B\|^2(\log Q). \]

Substituting this estimate into (14), we obtain the lemma. \hfill \Box

We now consider the exponential sum over primes.

**Lemma 6.** For any real number $A \geq 1$, there exists a real number $B = B(A) \geq 1$ such that for any real numbers $P, P', Q \geq 4$ with $P < P' \leq 2P$,
\[ \sum_{P < n \leq P'} \Lambda(n)e\left( \frac{Q}{n} \right) \ll \left( P(\log Q)^{-A} + P^2Q^{-\frac{1}{2}} \right) (\log Q)^5 \]
provided
\[ P \geq \exp(B(\log Q)^{\frac{3}{2}}(\log \log Q)^{\frac{1}{2}}), \]
where the implicit constant depends only on $A$.\hfill \Box
Proof. We let $0 < \gamma \leq 1$ be the constant in Lemma 3 and $B_1(A)$ be the constant $B(A)$ in Lemma 5 and take $B = \max(3(A/\gamma)^3, 3B_1(A))$ for the current proof. We may assume that $Q$ is larger than some constant depending only on $A$ since otherwise the trivial estimate

$$\sum_{P < n \leq P'} \Lambda(n)e\left(\frac{Q}{n}\right) \ll P$$

is enough. Also, if $P > Q$, then we have $P^\frac{2}{3}Q^{-\frac{1}{2}} \gg P$ so the assertion again trivially follows. Hence we may further assume $P \leq Q$.

We use Lemma 4 with $z = P^\frac{4}{3}$. This gives a decomposition

$$\sum_{P < n \leq P'} \Lambda(n)e\left(\frac{Q}{n}\right) = S_1 - S_2 + S_3,$$

where

$$S_1 = \sum_{P < uv \leq P' \atop u \leq P^\frac{4}{3}} \mu(u)(\log v)e\left(\frac{Q}{uv}\right), \quad S_2 = \sum_{P < uv \leq P' \atop u \leq P^\frac{4}{3}} c_2(u)e\left(\frac{Q}{uv}\right),$$

$$S_3 = \sum_{P < uv \leq P' \atop u, v > P^\frac{4}{3}} c_3(u)\Lambda(v)e\left(\frac{Q}{uv}\right),$$

where the coefficients $c_2(u)$ and $c_3(u)$ are given by

$$c_2(u) = \sum_{dm = u \atop d, m \leq P^\frac{4}{3}} \mu(d)\Lambda(m), \quad c_3(u) = \sum_{dm = u \atop d > P^\frac{4}{3}} \mu(d).$$

Note that

$$|c_2(u)| \leq \sum_{dm = u} \Lambda(m) = \log u, \quad |c_3(u)| \leq \tau(u).$$

For the sum $S_1$, we start with

$$S_1 = \sum_{u \leq P^\frac{4}{3}} \mu(u) \sum_{P/u < v \leq P'/u} (\log v)e\left(\frac{Q}{uv}\right).$$

We apply Lemma 3 to the inner sum. Note that

$$4 \leq Q^\frac{2}{3} \leq Q/u \leq Q, \quad P/u \geq P^\frac{4}{3} \geq P^\frac{4}{3}$$

for this inner sum. Thus, for $P/u < x \leq 2P/u$, we obtain

$$e\left(\frac{Q}{uv}\right) \ll \left(P\exp\left(-\frac{\gamma}{27} (\log P)^3 (\log Q)^2\right) + P^2Q^{-1}\right) u^{-1}$$

by (18) since $B \geq 3(A/\gamma)^3$. By partial summation,

$$\sum_{P/u < v \leq P'/u} (\log v)e\left(\frac{Q}{uv}\right) \ll (\log Q) \sup_{P/u < x \leq P'/u} \left|\sum_{P/u < v \leq x} e\left(\frac{Q}{uv}\right)\right| \ll (\log Q)^{-A} + P^2Q^{-1} u^{-1}(\log Q)$$

by (19).
We substitute this estimate into (21). Then by using \( PQ^{-1} \leq P^\frac{1}{2}Q^{-\frac{1}{2}} \) and

\[
\sum_{u \leq P^\frac{1}{2}} \frac{1}{u} \ll \log P \ll \log Q,
\]
we arrive at the desired estimate

\[
(23) \quad S_1 \ll \left( P(\log Q)^{-A} + P^\frac{3}{2}Q^{-\frac{1}{2}} \right) (\log Q)^2
\]
for \( S_1 \).

The sum \( S_2 \) is treated similarly to \( S_1 \). We start with

\[
S_2 = \sum_{u \leq P^\frac{1}{2}} c_2(u) \sum_{P/u < v \leq P' / u} e \left( \frac{Q}{u} \right).
\]

For the inner sum, we again have

\[
4 \leq Q/u \leq Q, \quad P/u \geq P^\frac{1}{2}
\]
so that the estimate (22) is available. Hence by using (20) and

\[
\sum_{u \leq P^\frac{1}{2}} \log u \ll (\log P)^2 \ll (\log Q)^2,
\]
we arrive at the desired estimate

\[
(24) \quad S_2 \ll \left( P(\log Q)^{-A} + P^\frac{3}{2}Q^{-\frac{1}{2}} \right) (\log Q)^2
\]
for \( S_2 \).

For the sum \( S_3 \), we can employ dyadic subdivision to obtain

\[
(25) \quad S_3 \ll (\log Q)^2 \sup |S_3(U, U', V, V')|,
\]
where the supremum is taken over real numbers \( U, U', V, V' \) with the conditions

\[
U < U' \leq 2U, \quad V < V' \leq 2V, \quad U, V \geq P^\frac{1}{2}, \quad UV \leq P'
\]
and \( S_3(U, U', V, V') \) is defined by

\[
S_3(U, U', V, V') = \sum_{P < uv \leq P'} c_3(u)\Lambda(v)e \left( \frac{Q}{uv} \right).
\]

We apply Lemma 5 to this double sum. Since our choice of \( B \) gives \( B \geq 3B_1(A) \), by (18), we find that

\[
U, V \geq P^\frac{1}{2} \geq \exp(B_1(A)(\log Q)^{\frac{1}{8}}(\log \log Q)^{\frac{1}{3}}).
\]

Also, we have

\[
\sum_{U < u \leq U'} |c_3(u)|^2 \leq \sum_{U < u \leq U'} \tau(u)^2 \ll U(\log U)^3,
\]
\[
\sum_{V < v \leq V'} \Lambda(v)^2 \leq (\log V) \sum_{V < v \leq V'} \Lambda(v) \ll V \log V.
\]
Thus Lemma 5 and $UV \leq P'$ imply
\[ S_3(U, U', V, V') \ll \left( P^2 (\log Q)^{-A} + PQ^{-\frac{1}{2}} \right) (UV)^{\frac{1}{2}} (\log Q)^{\frac{5}{2}} \]
\[ \ll \left( P(\log Q)^{-A} + P^2 Q^{-\frac{1}{2}} \right) (\log Q)^3. \]

By returning to (25), we find that
\[ S_3 \ll \left( P(\log Q)^{-A} + P^2 Q^{-\frac{1}{2}} \right) (\log Q)^5. \]
Combining this with (19), (23) and (24) we arrive at the desired estimate. \qed

**Lemma 7.** For any real number $A \geq 1$, there exists a real number $B = B(A) \geq 1$ such that for any real numbers $P, P', Q \geq 4$ with $P < P' \leq 2P$,
\[ \sum_{P < p \leq P'} e\left(\frac{Q}{p}\right) \ll \left( P(\log Q)^{-A} + P^2 Q^{-\frac{1}{2}} \right) (\log Q)^5 \]
provided
\[ P \geq \exp(B(\log Q)^{\frac{3}{2}} (\log \log Q)^{\frac{1}{3}}), \]
where the implicit constant depends only on $A$.

**Proof.** We may assume $Q$ is larger than some absolute constant depending only on $A$. Let us take the same constant $B = B(A) \geq 1$ as in Lemma 6. We replace $\Lambda(n)$ by $\log p$. This replacement produces an error
\[ \leq \sum_{2 \leq \nu \leq 2 \log P} \sum_{P \leq (2P)^{\frac{1}{2}}} \log p \ll P^{\frac{1}{2}} (\log P)^3. \]
By (26) and $B \geq 1$, we find that
\[ \log P \geq B(\log Q)^{\frac{3}{2}} (\log \log Q)^{\frac{1}{2}} \geq (\log Q)^{\frac{3}{2}} \]
so that the error (27) is bounded as
\[ P(\log Q)^{-A} \geq P(\log P)^{-\frac{3A}{2}} \gg P^\frac{1}{2} (\log P). \]
Therefore, Lemma 6 implies
\[ \sum_{P < p \leq P'} (\log p) e\left(\frac{Q}{p}\right) \ll \left( P(\log Q)^{-A} + P^2 Q^{-\frac{1}{2}} \right) (\log Q)^5 \]
provided (26). By using partial summation, we obtain the lemma. \qed

4. **Exponential sums with multiplicative functions**

Our next task is to estimate the exponential sum
\[ \sum_{P < n \leq P'} v(n) e\left(\frac{Q}{n}\right), \quad P < P' \leq 2P \]
by using the result of Section 3, where $v(n)$ is a multiplicative function satisfying the conditions (V1), (V2), and (V3).

We first prepare an estimate for the exponential sum over primes with the coefficient $v(p)$. This is the point where the condition (V3) is used, which states $v(p)$ has a bounded variation.
Lemma 8. Let \( v(n) \) be a complex-valued multiplicative function satisfying (V1) and (V3). For any real number \( A \geq 1 \), there exists a real number \( B = B(A) \geq 1 \) such that for any real numbers \( P, P', Q \geq 4 \) with \( P < P' \leq 2P \),

\[
\sum_{P < p \leq P'} v(p)e \left( \frac{Q}{p} \right) \ll \left( P(\log Q)^{-A} + P^{\frac{1}{4}}Q^{-\frac{1}{2}} \right) (\log Q)^{C+5}
\]

provided

\[
(28) \quad P \geq \exp(B(\log Q)^{\frac{1}{4}}(\log \log Q)^{\frac{1}{2}}),
\]

where the implicit constant depends only on \( A \) and \( C \).

Proof. Let \( B_1(A) \) be the constant \( B(A) \) in Lemma 7 and for our current proof, take \( B = B_1(A) \). By (V1), we may assume \( P \leq Q \) since otherwise \( P^{\frac{1}{4}}Q^{-\frac{1}{2}} \gg P \). We may also assume that there exists a prime \( p \) such that \( P < p \leq P' \). Suppose that the prime numbers \( p \) with \( P < p \leq P' \) are given by \( q_1 < \cdots < q_N \). Then we have

\[
(29) \quad \sum_{P < p \leq P'} v(p)e \left( \frac{Q}{p} \right) = \sum_{n=1}^{N} v(q_n)e \left( \frac{Q}{q_n} \right).
\]

We introduce

\[
S(x) = \sum_{P < p \leq x} e \left( \frac{Q}{p} \right), \quad q_0 = 1.
\]

Then by applying partial summation to (29),

\[
\sum_{P < p \leq P'} v(p)e \left( \frac{Q}{p} \right) = \sum_{n=1}^{N} v(q_n) (S(q_n) - S(q_{n-1}))
\]

\[
= \sum_{n=1}^{N-1} (v(q_n) - v(q_{n+1})) S(q_n) + v(q_N)S(q_N).
\]

By (V1) and (V3), this can be estimated as

\[
\left| \sum_{P < p \leq P'} v(p)e \left( \frac{Q}{p} \right) \right|
\]

\[
\leq \left( \sup_{P < x \leq P'} |S(x)| \right) \left( \sum_{n=1}^{N-1} |v(q_{n+1}) - v(q_n)| + |v(q_N)| \right)
\]

\[
\leq \left( \sup_{P < x \leq P'} |S(x)| \right) \left( \sum_{p_n \leq P'} |v(p_{n+1}) - v(p_n)| + |v(q_N)| \right)
\]

\[
\ll \left( \sup_{P < x \leq P'} |S(x)| \right) (\log Q)^{C}.
\]

By (28), we can now apply Lemma 7 to the sum \( S(x) \) to obtain

\[
\sup_{P < x \leq P'} |S(x)| \ll \left( P(\log Q)^{-A} + P^{\frac{1}{4}}Q^{-\frac{1}{2}} \right) (\log Q)^{5}
\]

By substituting this estimate into (30), we obtain the assertion. \( \Box \)

We next prepare a preliminary estimate. The proof of the next lemma includes a Vinogradov-type combinatorial decomposition finer than used by Pétermann [14].
Lemma 9. Let \( v(n) \) be a complex-valued multiplicative function satisfying (V1) and (V3). For any real number \( A \geq 1 \), there exists a real number \( B = B(A) \geq 1 \) such that for any real numbers \( P, P', Q, z \geq 4 \) with \( P < P' \leq 2P \) and any positive integer \( \nu \),
\[
\sum_{P < q \leq P'} v(q) e \left( \frac{Q}{q} \right) \ll C^\nu \left( P(\log Q)^{-A} + P^{2\nu} Q^{-\frac{3}{4}} + Pz^{-1} \right) (\log Q)^{C+6}
\]
provided
\[
(31) \quad P \geq \exp \left( B(\log Q)^{\frac{3}{2}} (\log \log Q)^{\frac{1}{2}} \right)
\]
where the implicit constant depends only on \( A \) and \( C \).

Proof. Let \( B_1(A) \) and \( B_2(A) \) be the constant \( B(A) \) in Lemma 5 and Lemma 8, respectively. For our current lemma, we take \( B = 3 \max(B_1(A) + 1, B_2(A)) \).

We may assume \( Q \) is larger than some absolute constant since otherwise (V1) implies \( |v(q)| \leq C^\nu \) if \( \nu(q) = \nu \) and \( q \) is square-free, so \( P(\log Q)^{-A} \gg P \) implies the assertion immediately. Similarly, we may assume \( P \leq Q \) since otherwise \( P^2 Q^{-\frac{3}{4}} \gg P \). If \( \nu = 1 \), then we can apply Lemma 8. Thus we may further assume \( \nu \geq 2 \).

By considering the prime factorization, we can rewrite the sum as
\[
(32) \quad \sum_{\substack{P < q \leq P' \ \nu \text{square-free} \\ \omega(q) = \nu}} v(q) e \left( \frac{Q}{q} \right) = \sum_{\substack{P < p_1, \ldots, p_\nu \leq P' \ \nu \text{square-free} \\ p_1 > \cdots > p_\nu > z}} v(p_1 \cdots p_\nu) e \left( \frac{Q}{p_1 \cdots p_\nu} \right)
\]
\[
= \frac{1}{\nu!} \sum_{\substack{P < p_1, \ldots, p_\nu \leq P' \ \nu \text{square-free} \\ p_1 > \cdots > p_\nu > z \ \nu \text{distinct}}} v(p_1 \cdots p_\nu) e \left( \frac{Q}{p_1 \cdots p_\nu} \right)
\]

Let us consider the sets
\[
P = \{ (p_1, \ldots, p_\nu) \mid P < p_1 \cdots p_\nu \leq P', \ p_1, \ldots, p_\nu > z \},
\]
\[
Q = \{ (p_1, \ldots, p_\nu) \in P \mid p_1, \ldots, p_\nu: \text{distinct} \}
\]
and
\[
R_{ij} = \{ (p_1, \ldots, p_\nu) \in P \mid p_i = p_j \} \quad \text{for } 1 \leq i < j \leq \nu.
\]

We define a completely multiplicative function \( w(n) \) by \( w(p) = v(p) \). By (V1), we have \( |w(q)| \leq C^\nu \) for positive integers \( q \) with \( \Omega(q) = \nu \). Furthermore, for a given finite set \( T \) of \( \nu \)-tuples of primes, we define \( S(T) \) by
\[
S(T) = \sum_{(p_1, \ldots, p_\nu) \in T} w(p_1 \cdots p_\nu) e \left( \frac{Q}{p_1 \cdots p_\nu} \right).
\]

Then the equation (32) implies that
\[
(33) \quad \sum_{\substack{P < q \leq P' \ \nu \text{square-free} \\ \omega(q) = \nu}} v(q) e \left( \frac{Q}{q} \right) = \frac{1}{\nu!} S(Q) = \frac{1}{\nu!} S(P) + O \left( \frac{C^\nu}{\nu!} |P \setminus Q| \right).
\]
By definition, we have a covering
\[(34) \quad \mathcal{P} \setminus \mathcal{Q} \subset \bigcup_{1 \leq i < j \leq \nu} \mathcal{R}_{ij}.\]
By symmetry, we have \(|\mathcal{R}_{ij}| = |\mathcal{R}_{12}|\) for every \(i\) and \(j\) with \(1 \leq i < j \leq \nu\). By the covering (34), we find that
\[(35) \quad \frac{C^\nu}{\nu!} |\mathcal{P} \setminus \mathcal{Q}| \leq \frac{C^\nu}{\nu!} \sum_{1 \leq i < j \leq \nu} |\mathcal{R}_{ij}| = \frac{C^\nu}{\nu!} \left(\frac{\nu}{2}\right)|\mathcal{R}_{12}| \leq \frac{C^\nu}{2(\nu - 2)!} \sum_{z < p \leq (2P)^{1/2}} \frac{P}{p^2} \sum_{p_3 \cdots p_{\nu} \leq P'/p^2} 1.
\]
For a given integer \(n\) with \(P/p^2 < n \leq P'/p^2\), there are at most \((\nu - 2)!\) ways to express \(n\) as \(n = p_3 \cdots p_{\nu}\). Therefore, the estimate (35) can be continued as
\[(36) \quad \frac{C^\nu}{\nu!} |\mathcal{P} \setminus \mathcal{Q}| \leq \frac{C^\nu}{2} \sum_{z < p \leq (2P)^{1/2}} \frac{P}{p^2} \leq C^\nu P z^{-1}.
\]
Let us next consider the sets
\[\mathcal{P}_r(I) = \{ (p_1, \ldots, p_{\nu}) \in \mathcal{P} \mid p_r > P^{1/2}, p_1, \ldots, p_{r-1} \leq P^{1/2} \} \quad \text{for} \quad 1 \leq r \leq \nu,
\]
\[\mathcal{P}(II) = \{ (p_1, \ldots, p_{\nu}) \in \mathcal{P} \mid p_1, \ldots, p_{\nu} \leq P^{1/2} \}.
\]
Then we have a decomposition
\[\mathcal{P} = \bigcup_{r=1}^{\nu} \mathcal{P}_r(I) \cup \mathcal{P}(II), \quad \mathcal{P}_1(I), \ldots, \mathcal{P}_\nu(I), \mathcal{P}(II): \text{disjoint}.
\]
Therefore, the first term on the most right-hand side of (33) can be decomposed as
\[(37) \quad \frac{1}{\nu!} S(\mathcal{P}) = \frac{1}{\nu!} \sum_{r=1}^{\nu} S(\mathcal{P}_r(I)) + \frac{1}{\nu!} S(\mathcal{P}(II)).
\]
By symmetry among \(p_1, \ldots, p_{\nu}\), we find that
\[(38) \quad \frac{1}{\nu!} S(\mathcal{P}_r(I)) = \frac{1}{\nu!} \sum_{d \leq P'/P^{1/2}} w_r(d) \sum_{p_1 \cdots p_{r-1} \leq z} v(p) \left( \frac{Q/d}{p} \right),
\]
where the arithmetic function \(w_r(d)\) is defined by
\[w_r(d) = \sum_{p_1 \cdots p_{r-1} = d \atop p_1, \ldots, p_{r-1} \leq P^{1/2}} w(p_1 \cdots p_{r-1}).
\]
Since for a given integer \(d\), there are at most \((\nu - 1)!\) ways to express \(d\) in the form \(d = p_1 \cdots p_{\nu-1}\), by (V1), we see that
\[(39) \quad |w_r(d)| \leq (\nu - 1)! C^\nu - 1.
\]
We apply Lemma 8 to the inner sum of (38). By (31) and our choice of \(B\),
\[\max(P/d, P^{1/2}, z) \geq P^{1/2} \geq \exp(B_2(A)(\log Q)^{1/2} (\log \log Q)^{1/2}).
\]
Similarly to the estimate (39), we see that
\[ Q \] for large \( S \).

Then the sum \( P \) Note that the definition of
\[ \nu \]
This implies
\[ \nu \]
We now employ dyadic subdivision in (42). This gives
\[ P \]
\[ \nu \]
We then obtain a decomposition
\[ P \]
\[ \nu \]
where \( \alpha \) are defined by
\[ \nu \]

Also, we find that
\[ P'/d \leq 2P/d \leq 2 \max(P/d, P^{2}, z) \]

for large \( Q \) since \( P \leq Q \). Therefore, we may apply Lemma 8 and use (39) to obtain
\[ \frac{1}{\nu} P(\nu(I)) \ll \frac{C}{\nu} \sum_{d \leq P'/d^{2}} \left( \frac{P}{d} (\log Q)^{-A} + \frac{P^{2}Q^{-1/2}}{d} \right) (\log Q)^{C+5} \]
\[ \ll \frac{C}{\nu} \left( P(\log Q)^{-A} + P^{2}Q^{-1/2} \right) (\log Q)^{C+6}. \]

This implies
\[ \frac{1}{\nu} \sum_{\nu} \nu S(\nu(I)) \ll C \left( P(\log Q)^{-A} + P^{2}Q^{-1/2} \right) (\log Q)^{C+6}. \]

We move on to the sum over \( \nu(I) \). We further introduce the sets
\[ \nu(I) = \left\{ (p_{1}, \ldots, p_{\nu}) \in \nu(I) \left| p_{1} \cdot \cdots \cdot p_{r} > P^{2}, \right. \left. p_{1} \cdot \cdots \cdot p_{\nu-1} \leq P^{2} \right\} \] for \( 2 \leq \nu \leq \nu \).

We obtain a decomposition
\[ \nu(I) = \bigcup_{\nu = 2}^{\nu-1} \nu(I), \quad \nu(2), \ldots, \nu(\nu-1)(I) \text{ disjoint} \]

since if \( (p_{1}, \ldots, p_{\nu}) \in \nu(I) \), then \( p_{1} \cdot \cdots \cdot p_{\nu} > P > P^{2} \) and \( p_{1}, p_{\nu} \leq P^{2} \) by definition. This gives
\[ \frac{1}{\nu} S(\nu(I)) = \frac{1}{\nu} \sum_{\nu = 2}^{\nu-1} S(\nu(I)). \]

For each \( \nu \), we change the variable by
\[ u = p_{1} \cdots p_{\nu}, \quad v = p_{r+1} \cdots p_{\nu}. \]

Note that the definition of \( \nu(I) \) implies
\[ u = p_{1} \cdots p_{\nu} = (p_{1} \cdots p_{\nu-1} \cdot p_{\nu} \leq P^{2} \cdot P^{2} = P^{4}. \]

Then the sum \( S(\nu(I)) \) can be expressed as
\[ \sum_{p_{u} < u \leq P^{2}} \alpha_{1}(u) \beta_{1}(v) \left( \frac{Q}{u^{2}} \right), \]
\[ ]

where \( \alpha_{1}(v) \) and \( \beta_{1}(v) \) are defined by
\[ \alpha_{1}(u) = \sum_{p_{1} = u} w(p_{1} \cdots p_{\nu}), \quad \beta_{1}(v) = \sum_{p_{1} \cdots p_{v-1} = u} w(p_{v+1} \cdots p_{\nu}). \]

Similarly to the estimate (39), we see that
\[ |\alpha_{1}(u)| \leq r^{1}C^{r}, \quad |\beta_{1}(v)| \leq (\nu - r)C^{\nu-r}. \]

We now employ dyadic subdivision in (42). This gives
\[ S(\nu(I)) \ll (\log Q)^{2} \sup |T_{\nu}(U, U', V, V')|, \]
\[ ]
where
\[ T_r(U, U', V, V') = \sum_{P < u < u' \leq P'} \alpha_r(u) \beta_r(v) e\left(\frac{Q}{uv}\right) \]
and the supremum is taken over real numbers \( U, U', V, V' \) in the range
\[ \frac{P}{4} < UV \leq 2P, \quad U < U' \leq 2U, \quad V < V' \leq 2V, \quad P^{\frac{1}{2}} \leq U \leq P^{\frac{3}{2}}. \]
In this range, we have \( V > \frac{P}{4}U \geq \frac{P}{4} \geq \exp\left(B_1(A)(\log Q)^{\frac{1}{2}}(\log \log Q)^{\frac{1}{2}}\right) \) for large \( Q \) by (31) and our choice of \( B \). Therefore, we can apply Lemma 5 to the sum \( T_r(U, U', V, V') \). By combining with (43), Lemma 5 gives
\[ T_r(U, U', V, V') \ll r! (\nu - r)! C_{\nu} \left( P(\log Q)^{-A} + P^{\frac{3}{2}}Q^{-\frac{1}{2}} \right) (\log Q). \]
Substituting this estimate into (44), we obtain
\[ \frac{1}{P!} S(P_r, (II)) \ll \frac{1}{(r!)} C_{\nu} \left( P(\log Q)^{-A} + P^{\frac{3}{2}}Q^{-\frac{1}{2}} \right) (\log Q)^3 \]
provided \( 2 \leq r \leq \nu - 1 \). By combining with (41), this implies
\[ (45) \quad \frac{1}{P!} S(P_r, (II)) \ll C_{\nu} \left( P(\log Q)^{-A} + P^{\frac{3}{2}}Q^{-\frac{1}{2}} \right) (\log Q)^3. \]

Before proceeding to the exponential sums with the multiplicative functions, we further prepare two lemmas. The first one provides some estimates for \( v(n) \).

**Lemma 10.** Let \( v(n) \) be a complex-valued multiplicative function satisfying \((V2)\). Then for \( x \geq 4 \), we have
\[ \sum_{n \leq x} |v(n)| \ll (\log x)^C, \quad \sum_{n \leq x} \frac{|v(n)|}{n} \ll (\log x)^C, \]
where the implicit constants depend only on \( C \).

**Proof.** By the Cauchy–Schwarz inequality and \((V2)\),
\[ (46) \quad \sum_{n \leq x} |v(n)| \ll x^{\frac{1}{2}} \left( \sum_{n \leq x} |v(n)|^2 \right)^{\frac{1}{2}} \ll x(\log x)^C. \]
This proves the former estimate. We dissect the latter sum dyadically to obtain
\[ \sum_{n \leq x} \frac{|v(n)|}{n} \ll (\log x)^C \sup_{1 \leq U \leq x} U^{-1} \sum_{U \leq n \leq 2U} |v(n)|. \]
By substituting (46) here, we obtain the latter estimate since \( C \geq 2 \).

The next one is a well-known estimate for the number of smooth numbers.
Lemma 11. For a sufficiently large real number $x$ and a real number $y$, we have

$$\psi(x, y) \leq x \exp \left( -\frac{1}{2} \log x \frac{\log x}{\log y} \right)$$

provided

$$\log x^2 \leq y \leq \exp \left( \frac{\log x}{\log \log x} \right).$$

Proof. Let

$$u = \frac{\log x}{\log y}$$

Then (47) implies

$$u \geq \log \log x$$

so that $u \to \infty$ as $x \to \infty$. By Theorem 7.6 of [12],

$$\psi(x, y) < x \exp \left( -(1 + o(1))u \log u + \log \log y + O \left( \frac{u^2 \log u}{y} \right) \right)$$

as $x \to \infty$. By (48), it follows that

$$\frac{\log \log y}{u \log u} \leq \frac{1}{\log u} \to 0 \quad \text{as} \quad x \to \infty$$

and by (47),

$$\frac{u}{y} \leq \frac{1}{\log x} \to 0 \quad \text{as} \quad x \to \infty.$$

Therefore,

$$\log \log y = o(u \log u), \quad \frac{u^2 \log u}{y} = o(u \log u) \quad (x \to \infty).$$

On inserting these estimate into (49), we arrive at the lemma. \qed

After the above preparations, we can now prove the exponential sum estimate with our multiplicative function $v(n)$.

Lemma 12. Let $v(n)$ be a complex-valued multiplicative function satisfying (V1), (V2), and (V3). For any real number $A \geq 1$, there exists a real number $B = B(A) \geq 1$ such that for any real numbers $P, P', Q \geq 4$ with $P < P' \leq 2P$, we have

$$\sum_{P < n \leq P'} v(n) e \left( \frac{Q}{n} \right) \ll \left( P(\log Q)^{-A} + P^\frac{2}{3} Q^{-\frac{1}{2}} \right) (\log Q)^{6C+6}$$

provided

$$P \geq \exp(B(\log Q)^{\frac{2}{3}} (\log \log Q)^\frac{1}{2}),$$

where the implicit constant depends only on $A$ and $C$.

Proof. Let $B_1(A)$ be the constant $B(A)$ in Lemma 9 and we take $B = 2B_1(A)$ for the current proof. We may assume that $Q$ is larger than some constant depending only on $A$ since otherwise Lemma 10 and $P(\log Q)^{-A} \gg P$ imply the assertion trivially. Also, we may assume $2P \leq Q$ since otherwise Lemma 10 and $P^\frac{2}{3} Q^{-\frac{1}{2}} \gg P$ gives the assertion.
Let
\[ z := \exp\left(\frac{\log P}{2 \log \log Q}\right) \]
so that
\[ z \geq \exp\left( \frac{B}{2} \left( \frac{\log Q}{\log \log Q} \right)^\frac{1}{30} \right) > (\log Q)^{2A} \]
by (50). For an integer \( n \) with \( P < n \leq P' \), we have a unique expression
\[ n = mq, \quad P < mq \leq P', \quad p_{\text{max}}(m) \leq z, \quad p_{\text{min}}(q) > z. \]
Therefore, by using this expression for the change of variables, we have
\[ \sum_{P < n \leq P'} v(n) e \left( \frac{Q}{n} \right) = \sum_{P < mq \leq P'} \sum_{\substack{m < q \leq P' / m \\ p_{\text{max}}(m) \leq z \\ p_{\text{min}}(q) > z}} v(m) v(q) e \left( \frac{Q/m}{q} \right). \]
Hence
\[ \sum_1 = \sum_{m \leq P^{\frac{1}{2}}} v(m) \sum_{\substack{P/m < q \leq P' / m \\ p_{\text{max}}(m) \leq z \\ p_{\text{min}}(q) > z}} v(q) e \left( \frac{Q/m}{q} \right). \]
Since the conditions \( p_{\text{max}}(m) \leq z \) and \( p_{\text{min}}(q) > z \) imply \( (m, q) = 1 \).
The sum \( \sum_1 \) can be expressed as
\[ \sum_{11} = \sum_{q: \text{square-free}} v(q) e \left( \frac{Q/m}{q} \right) \]
for the former sum \( \sum_{11} \), we apply Lemma 9. We first classify \( q \) according to the value of \( \omega(q) \). Since \( p_{\text{min}}(q) > z \), we have
\[ P' \geq q > 2^\omega(q) \]
so that
\[ \omega(q) < \frac{\log P'}{\log z} \leq \frac{2 \log P}{\log z} = 4 \log \log Q. \]
Also, since \( q > P/m \geq P^{\frac{1}{2}} > 1 \) provided \( m \leq P^{\frac{1}{2}} \), we have \( \omega(q) \geq 1 \). Therefore,
\[ \sum_{11} = \sum_{\nu = 1}^{[4 \log \log Q]} \sum_{\substack{P/m < q \leq P' / m \\ p_{\text{min}}(q) > z \\ \omega(q) = \nu \\ q: \text{square-free}}} v(q) e \left( \frac{Q/m}{q} \right). \]
We now apply Lemma 9 to the inner sum. Since
\[ P/m \geq P^{\frac{1}{2}} \geq \exp(B_1(A) (\log Q)^{\frac{1}{5}} (\log \log Q)^{\frac{1}{5}}), \quad Q/m \geq Q P^{-\frac{1}{2}} \geq Q^{\frac{1}{2}} \geq 4 \]
for large $Q$ provided $m \leq P^2$, we may apply Lemma 9 to obtain
\[
\sum_{\substack{P/m < q \leq P'/m \\ p_{\min}(q) > z \\ \omega(q) = \nu}} v(q)e\left(\frac{Q/m}{q}\right) \lesssim C^\nu\left(\frac{P}{m} (\log Q)^{-A} + \frac{P^2 Q^{-\frac{1}{2}}}{m} + \frac{P}{m} z^{-1}\right) (\log Q)^{C+6}
\]
\[
\lesssim e^{C\nu}\left(\frac{P}{m} (\log Q)^{-A} + \frac{P^2 Q^{-\frac{1}{2}}}{m}\right) (\log Q)^{C+6},
\]
where we used (51). By substituting this estimate into (56) and by using
\[
\sum_{\nu = 1}^{[4 \log \log Q]} e^{C\nu} \leq e^{4C \log \log Q} (1 + e^{-C/2} + \cdots) \ll (\log Q)^{4C},
\]
we obtain
\[
(57) \quad \sum_1 \ll \left(\frac{P}{m} (\log Q)^{-A} + \frac{P^2 Q^{-\frac{1}{2}}}{m}\right) (\log Q)^{5C+6}
\]
For the latter sum $\sum_2$, by the Cauchy-Schwarz inequality and (V2),
\[
\left(\sum_2\right)^2 \ll \frac{(P(\log P))^C}{m} \sum_{\substack{P/m < q \leq P'/m \\ p_{\min}(q) > z}} 1
\]
\[
\lesssim \frac{(P(\log P))^C}{m} \sum_{z < p \leq (2P)^{\frac{1}{2}}} \sum_{P/m < q \leq P'/m} 1
\]
\[
\ll \left(\frac{P(\log P)^C}{m}\right)^2 \sum_{z < p \leq (2P)^{\frac{1}{2}}} \frac{1}{p^2} \ll \frac{1}{z} \left(\frac{P(\log P)^C}{m}\right)^2
\]
Thus, by using (51),
\[
(58) \quad \sum_2 \ll \frac{P(\log P)^C}{m} z^{-\frac{1}{2}} \ll \frac{P}{m} (\log Q)^{C-A}.
\]
By (55), (57) and (58), we obtain
\[
\sum_{\substack{P/m < q \leq P'/m \\ p_{\min}(q) > z}} v(q)e\left(\frac{Q/m}{q}\right) \ll \left(\frac{P}{m} (\log Q)^{-A} + \frac{P^2 Q^{-\frac{1}{2}}}{m}\right) (\log Q)^{5C+6}
\]
By substituting this into (54) and by using Lemma 10, we obtain
\[
(59) \quad \sum_1 \ll (P(\log Q)^{-A} + P^2 Q^{-\frac{1}{2}}) (\log Q)^{5C+6}.
\]
We next estimate $\sum_2$. We first estimate trivially
\[
\sum_2 \ll \sum_{\substack{P < mq \leq P' \\ p_{\text{max}}(m) \leq z \\ p_{\min}(q) > z \atop m > P^2}} |v(mq)|.
\]
Then by the Cauchy–Schwarz inequality,

\[(60) \quad \left(\sum_2\right)^2 \ll \left(\sum_{P < mq \leq P'} |v(mq)|^2 \right) \left(\sum_{P < mq \leq P'} 1\right) = \left(\sum_{21}\right) \left(\sum_{22}\right), \text{ say.}\]

For the former sum \(\sum_{21}\), we recall our change of variables (52) and trace back the substitution to obtain

\[(61) \quad \sum_{21} = \sum_{P < n \leq P'} |v(n)|^2 \ll P(\log P)^C,\]

where we used (V2). For the latter sum \(\sum_{22}\), we use Lemma 11 with the parameters \(P^{1/2} < x \leq P'\) and \(y = z\). We can check (47) with this choice of parameters as

\[(\log x)^2 \leq (\log P')^2 \leq (\log Q)^2 \leq z = \exp \left(\frac{\log P}{2 \log \log Q}\right) \leq \exp \left(\frac{\log x}{\log \log x}\right).\]

Thus, we may apply Lemma 11 to obtain

\[
\psi(x, z) \ll x \exp \left(\frac{1}{2} \log x \log \frac{\log x}{\log z}\right) \\
\ll x \exp \left(-\frac{1}{4} \log \frac{P}{\log \log Q}\right) \\
\ll x \exp \left(-\frac{1}{2} \log \log Q \log \frac{\log \log Q}{\log x}\right) \ll x(\log Q)^{-2A} \ll P(\log Q)^{-2A}
\]

for \(P^{1/2} \leq x \leq P'\) provided \(Q\) is larger than some constant depending on \(A\). Thus,

\[
\sum_{22} = \sum_{q \leq P'/P^{1/2}} \sum_{P/q < m \leq P'/q \atop p_{\text{min}}(m) > z} 1 \ll P(\log Q)^{-2A} \sum_{q \leq P'/P^{1/2}} \frac{1}{q} \ll P(\log Q)^{-2A + 1}.
\]

Combining this estimate with (60) and (61), we obtain

\[(62) \quad \sum_2 \ll P(\log Q)^{C - A}.\]

By substituting (59) and (62) into (53), we arrive at the lemma. \(\square\)

5. Error term estimate

In this section, we prove Theorem 1 by using the exponential sum estimate obtained in Section 4. We start with a standard translation of sums with \(\psi(x)\) to exponential sums.

Lemma 13. Let \((x_n)_{n \in I}\) be a sequence of real numbers defined over integers in a set of integers \(I\), \(g(n)\) be a complex-valued function defined on \(I\), \(G(n)\) be a positive function defined on \(I\) such that

\[(63) \quad |g(n)| \leq G(n)\]

for all integer \(n \in I\), and \(H \geq 1\) be a real number. Then

\[
\left|\sum_{n \in I} g(n)\psi(x_n)\right| \ll \sum_{0 \leq |h| \leq H} \frac{1}{|h|} \left|\sum_{n \in I} g(n) e(hx_n)\right| + \frac{1}{H} \sum_{0 \leq |h| \leq H} \left|\sum_{n \in I} G(n) e(hx_n)\right|,
\]
where the implicit constant is absolute.

Proof. Let \( N = [H] \geq 1 \). Then we use Vaaler’s approximation

\[
(64) \quad \psi_N(x) = \sum_{0 < |h| \leq N} c_N(h) e(hx),
\]

\[
(65) \quad |\psi(x) - \psi_N(x)| \leq \frac{1}{2(N+1)} \sum_{|h| \leq N} \left( 1 - \frac{|h|}{N+1} \right) e(hx),
\]

where the complex coefficients \( c_N(h) \) satisfies

\[
(66) \quad |c_N(h)| \leq \frac{1}{2\pi|h|}.
\]

For the details and the proof of this approximation, see [17, p.210, Theorem 18] or [4, p.116, Theorem A.6]. Then, we decompose the sum as

\[
(67) \quad \sum_{n \in I} g(n) \psi(x_n) = \sum_{n \in I} g(n) \psi_N(x_n) + \sum_{n \in I} g(n) (\psi(x_n) - \psi_N(x_n)).
\]

For the former sum, by (64) and (66), we have

\[
(68) \quad \left| \sum_{n \in I} g(n) \psi_N(x_n) \right| \ll \sum_{0 < |h| \leq N} |c_N(h)| \sum_{n \in I} g(n) e(hx_n).
\]

For the latter sum, by (63),

\[
\left| \sum_{n \in I} g(n) (\psi(x_n) - \psi_N(x_n)) \right| \leq \sum_{n \in I} G(n) |\psi(x_n) - \psi_N(x_n)|.
\]

Then we use (65) to obtain

\[
(69) \quad \left| \sum_{n \in I} g(n) (\psi(x_n) - \psi_N(x_n)) \right| \leq \frac{1}{2(N+1)} \sum_{|h| \leq N} \sum_{n \in I} G(n) e(hx_n).
\]

Since \( G(n) \) is real valued, by taking the complex conjugate, we find that

\[
\left| \sum_{n \in I} G(n) e(hx_n) \right| = \left| \sum_{n \in I} G(n) e(-hx_n) \right|.
\]

Thus, by (69), we have

\[
(70) \quad \left| \sum_{n \in I} g(n) (\psi(x_n) - \psi_N(x_n)) \right| \ll \frac{1}{H} \sum_{0 \leq h \leq H} \left| \sum_{n \in I} G(n) e(hx_n) \right|.
\]

Combining (67), (68) and (70), we arrive at the lemma. \( \square \)

By Lemma 13, we can now translate the exponential estimate given in Lemma 12 to the estimate of the sum involving \( \psi(x) \).
Lemma 14. Let \( v(n) \) be a complex-valued multiplicative function satisfying (V1), (V2), and (V3). For any real number \( A \geq 1 \), there exists a real number \( B = B(A,C) \geq 1 \) such that for any real numbers \( P, P', Q \geq 4 \) with \( P < P' \leq 2P \), we have
\[
\sum_{P < n \leq P'} v(n) \psi \left( \frac{Q}{n} \right) \ll P(\log Q)^{-A}
\]
provided
\[
\exp(B(\log Q)^{1/2}(\log \log Q)^{1/2}) \leq P \leq Q(\log Q)^{-2A-12C-14}.
\]
where the implicit constant depends only on \( A \) and \( C \).

Proof. We may assume that \( Q \) is larger than some absolute constant depending on \( A \) and \( C \). Let \( B_1(A) \) be the constant \( B(A) \) in Lemma 12 and for the current proof, we take \( B(A) = 2B_1(A + 6C + 7) \). We use Lemma 13 with
\[
x_n = \frac{Q}{n}, \quad I = (P, P'] \cap \mathbb{Z}, \quad g(n) = v(n), \quad G(n) = |v(n)|, \quad H = (\log Q)^{A+C}.
\]
For the assumption of Lemma 12, it suffices to check that
\[
P \geq \exp(2B_1(A + 6C + 7)(\log Q)^{1/2}(\log \log Q)^{1/2})
\]
\[
\geq \exp(B_1(A + 6C + 7)(\log hQ)^{1/2}(\log \log hQ)^{1/2}),
\]
which holds for \( 1 \leq h \leq H \) and sufficiently large \( Q \). Note that the multiplicative function \( v(n) \) also satisfies (V1), (V2), and (V3). Thus, we may use Lemma 12 with \( g(n) = v(n) \) and \( v(n) \). This gives
\[
\sum_{0 < |h| \leq H} \frac{1}{|h|} \left| \sum_{P < n \leq P'} v(n)e \left( \frac{hQ}{n} \right) \right| = \sum_{1 \leq h \leq H} \frac{1}{|h|} \left| \sum_{P < n \leq P'} v(n)e \left( \frac{hQ}{n} \right) \right| + \sum_{1 \leq h \leq H} \frac{1}{|h|} \left| \sum_{P < n \leq P'} \overline{v(n)}e \left( \frac{hQ}{n} \right) \right|
\]
\[
\ll \left( P(\log Q)^{-A-6C-7} + P^{1/2}Q^{-1/2} \right)(\log Q)^{6C+7} \ll P(\log Q)^{-A}
\]
by (71). For the sum with \( G(n) = |v(n)| \), we use Lemma 10 to the term \( h = 0 \). By Lemma 10, this gives
\[
\frac{1}{H} \sum_{0 \leq h \leq H} \left| \sum_{P < n \leq P'} |v(n)|e \left( \frac{hQ}{n} \right) \right| \ll P(\log Q)^{-A} + \sum_{1 \leq h \leq H} \frac{1}{|h|} \left| \sum_{P < n \leq P'} |v(n)|e \left( \frac{hQ}{n} \right) \right|.
\]
The function \( |v(n)| \) trivially satisfies (V1) and (V2) with the same value of \( C \) as for \( v(n) \). For (V3), by the triangle inequality, we can see
\[
\sum_{p_n \leq x} ||v(p_{n+1})| - |v(p_n)|| \leq \sum_{p_n \leq x} |v(p_{n+1}) - v(p_n)| \leq C(\log x)^C
\]
for \( x \geq 4 \). Therefore we can estimate the second term on the right-hand side of (73) similarly to (72). Therefore,

\[
(74) \quad \frac{1}{H} \sum_{0 \leq h < H} \left| \sum_{P < n \leq P'} |v(n)| e \left( \frac{hQ}{n} \right) \right| \ll P(\log Q)^{-A}.
\]

On inserting (72) and (74) into Lemma 13 with our choice, we obtain the lemma. \( \square \)

We can now prove Theorem 1.

**Proof of Theorem 1.** We may assume that \( x \) is larger than some constant depending only on \( \theta, C \), and the implicit constant in (V). Let us take \( B = B(1, C) \) in Lemma 14 and let

\[
z = \exp(B(\log x)^{\frac{3}{2}}(\log \log x)^{\frac{1}{3}}).
\]

If \( y \leq z \), then the assertion immediately follows by (V). Thus we may assume \( z < y \). Then we dissect the sum at \( n = z \) as

\[
(75) \quad \sum_{n \leq y} \frac{|v(n)|}{n} \psi \left( \frac{x}{n} \right) = \sum_{n \leq z} + \sum_{z < n \leq y} = \sum_1 + \sum_2, \quad \text{say.}
\]

By (V), the former sum \( \sum_1 \) is just

\[
(76) \quad \sum_1 \ll \sum_{n \leq z} \frac{|v(n)|}{n} \ll (\log z)^{\kappa} \ll (\log x)^{\frac{2\kappa}{3}}(\log \log x)^{\frac{\kappa}{3}}.
\]

For the latter sum \( \sum_2 \), we employ dyadic subdivision and partial summation

\[
(77) \quad \sum_2 \ll (\log x) \sup_{\substack{z \leq P \leq y \\ P < P' \leq 2P}} \left| \sum_{P < n \leq P'} \frac{|v(n)|}{n} \psi \left( \frac{x}{n} \right) \right| 
\]

\[
= (\log x) \sup_{\substack{z \leq P \leq y \\ P < P' \leq 2P}} P^{-1} \left| \sum_{P < n \leq P'} v(n) \psi \left( \frac{x}{n} \right) \right|.
\]

If \( x \) is larger than some constant depending on \( C \) and \( \theta \), then \( z < P \leq y \) implies \( \exp(B(1, C)(\log x)^{\frac{3}{2}}(\log \log x)^{\frac{1}{3}}) \leq P \leq x \exp(-\log x^{\theta}) \leq x(\log x)^{-12C-16} \).

Hence, we may apply Lemma 14 with \( A = 1 \) to the right-hand side of (77). Then

\[
(78) \quad \sum_2 \ll 1.
\]

On inserting (76) and (78) into (75), we arrive at the theorem. \( \square \)

6. **The Balakrishnan–Pétermann method**

We next prove Theorem 2 by using Theorem 1. Thus, in this section, we discuss under the hypothesis in Theorem A. It is not clear whether or not the hypothesis of Theorem A immediately implies the assumptions of Theorem 1. Indeed, the hypothesis of Theorem A does not assure the multiplicativity of \( v(n) \). Therefore, we decompose \( v(n) \) into \( \tau_\alpha(n) \) and \( b(n) \) by using the definition (5) and apply Theorem 1 to the function \( \tau_\alpha(n) \). We start with recalling basic properties of \( \tau_\alpha(n) \).
Lemma 15. For any complex number \( \alpha \), we have the Dirichlet series expansion

\[
\zeta(s)^\alpha = \sum_{n=1}^{\infty} \frac{\tau_\alpha(n)}{n^s},
\]

where the multiplicative function \( \tau_\alpha(n) \) is defined by

\[
\tau_\alpha(1) = 1, \quad \tau_\alpha(p^\nu) = \left( \frac{\alpha + \nu - 1}{\nu} \right) = \prod_{\ell=1}^{\nu} \left( \frac{\alpha + \ell - 1}{\ell} \right)
\]

and the series (79) converges absolutely for \( \sigma > 1 \). Moreover, we have

\[
\tau_\alpha(n) \ll n^\varepsilon
\]

for every \( \varepsilon > 0 \), where the implicit constant depends on \( \alpha \) and \( \varepsilon \).

Proof. We have the Taylor expansion

\[
(1 - z)^{-\alpha} = \sum_{\nu=0}^{\infty} \left( \frac{\alpha + \nu - 1}{\nu} \right) z^\nu,
\]

which has the radius of convergence 1. Thus, for \( \sigma > 1 \) and \( p \geq 2 \), we have

\[
\sum_{\nu=1}^{\infty} \left| \frac{\alpha + \nu - 1}{\nu} \right| \frac{1}{p^\nu} \ll p^{-\sigma}.
\]

Therefore, by using the Euler product,

\[
\sum_{n=1}^{\infty} \frac{\tau_\alpha(n)}{n^s} = \prod_p \left( 1 + \sum_{\nu=1}^{\infty} \left| \frac{\alpha + \nu - 1}{\nu} \right| \frac{1}{p^\nu} \right) = \prod_p (1 + O(p^{-\sigma})) < +\infty
\]

for \( \sigma > 1 \). This proves that the series (79) converges absolutely for \( \sigma > 1 \). Then by comparing the Euler product of the both sides of (79) and checking that their argument coincides for \( \sigma > 1 \), we obtain (79) and (80). Finally, by (80),

\[
|\tau_\alpha(p^\nu)| \leq \prod_{\ell=1}^{\nu} \left( 1 + \frac{|\alpha|}{\ell} \right) \leq \exp \left( |\alpha| \sum_{\ell=1}^{\nu} \frac{1}{\ell} \right) \leq (3\nu)^{|\alpha|} \leq (\nu + 1)^{2|\alpha|} \leq \tau(p^\nu)^{2|\alpha|}.
\]

Therefore, by the well-known bound \( \tau(n) \ll n^\varepsilon \), we arrive at

\[
|\tau_\alpha(n)| \leq \tau(n)^{2|\alpha|} \ll n^\varepsilon.
\]

This completes the proof. \( \square \)

Lemma 16. For any complex number \( \alpha \), the multiplicative function \( \tau_\alpha(n) \) satisfies (V1), (V2), (V3) for some constant \( C \) and the estimate

\[
\sum_{n \leq x} \frac{|\tau_\alpha(n)|}{n} \ll (\log x)^{|\alpha|}
\]

for \( x \geq 4 \).
**Proof.** By Lemma 15, we have \( \tau(p) = \alpha \) for every prime \( p \). Thus the conditions (V1), (V3) trivially holds. For (V2), we start with

\[
\sum_{n \leq x} |\tau_\alpha(n)|^2 \leq x \sum_{n \leq x} \frac{|\tau_\alpha(n)|^2}{n} 
\leq x \prod_{p \leq x} \left( 1 + \frac{|\tau_\alpha(p)|^2}{p} + \frac{|\tau_\alpha(p^2)|^2}{p^2} + \ldots \right) 
\leq x \prod_{p \leq x} \left( 1 + \frac{\alpha^2}{p} \right) \left( 1 + \frac{|\tau_\alpha(p^2)|^2}{p^2} + \frac{|\tau_\alpha(p^3)|^2}{p^3} + \ldots \right).
\]

Then by using Mertens’ theorem and (81), this gives

\[
\sum_{n \leq x} |\tau_\alpha(n)|^2 \ll x (\log x)^\alpha \prod_{p} \left( 1 + O(p^{-\frac{3}{2}}) \right) \ll x (\log x)^\alpha.
\]

Therefore, taking \( C \geq 2 \) larger than some constant depending only on \( \alpha \), we find that \( \tau_\alpha(n) \) satisfies (V1), (V2), and (V3) with the same constant \( C \). For the estimate (82), we proceed similarly to obtain

\[
\sum_{n \leq x} |\tau_\alpha(n)| \leq \prod_{p \leq x} \left( 1 + \frac{|\tau_\alpha(p)|}{p} + \frac{|\tau_\alpha(p^2)|}{p^2} + \ldots \right) 
\leq \prod_{p \leq x} \left( 1 + \frac{\alpha}{p} \right) \left( 1 + \frac{|\tau_\alpha(p^2)|}{p^2} + \frac{|\tau_\alpha(p^3)|}{p^3} + \ldots \right) \ll (\log x)^\alpha.
\]

This completes the proof. \( \square \)

**Proof of Theorem 2.** By Theorem A, it suffices to prove

\[
\sum_{n \leq y} \frac{v(n)}{n} \psi \left( \frac{x}{n} \right) \ll (\log x)^{\frac{2\alpha}{3}} (\log \log x)^{\frac{\alpha}{3}}
\]

for sufficiently large \( x \). By (5), we have

\[
v(n) = \sum_{d|m=n} \tau_\alpha(d)b(m).
\]

Therefore,

\[
\sum_{n \leq y} \frac{v(n)}{n} \psi \left( \frac{x}{n} \right) = \sum_{m \leq y} b(m) \frac{m}{m} \sum_{d \leq y/m} \frac{\tau_\alpha(d)}{d} \psi \left( \frac{x}{m} \frac{d}{d} \right).
\]

We apply Theorem 1 to the inner sum. We have already checked in Lemma 16 that \( \tau_\alpha(n) \) satisfies the assumptions on \( v(n) \) of Theorem 1 with \( \kappa = |\alpha| \). For the assumption on \( y \), we have

\[
\frac{y}{m} = \frac{x}{m} \exp(- (\log x)^{\frac{3}{2}}) \leq \frac{x}{m} \exp \left( - \left( \log \frac{x}{m} \right)^{\frac{3}{2}} \right)
\]

so this assumption is satisfied with \( \theta = 1/6 \). Also, we have \( x/m \geq x/y > 4 \) for sufficiently large \( x \). Thus, by Theorem 1,

\[
\sum_{d \leq y/m} \frac{\tau_\alpha(d)}{d} \psi \left( \frac{x}{m} \frac{d}{d} \right) \ll (\log x)^{\frac{2|\alpha|}{3}} (\log \log x)^{\frac{|\alpha|}{3}} \ll (\log x)^{\frac{2|\alpha|}{3}} (\log \log x)^{\frac{|\alpha|}{3}}.
\]
By substituting this estimate into (83) and using the convergence of (3),
\[
\sum_{n \leq y} \frac{v(n)}{n} \psi \left( \frac{x}{n} \right) \ll (\log x)^{\frac{2|\alpha|}{3}} (\log \log x)^{\frac{|\alpha|}{3}} \sum_{m \leq y} \frac{|b(m)|}{m}
\ll (\log x)^{\frac{2|\alpha|}{3}} (\log \log x)^{\frac{|\alpha|}{3}}.
\]
This completes the proof. \(\square\)

7. Examples

We now give some examples of Theorem 2. For most of the examples below, the preparation has been already done by Balakrishnan and Péterman [1], so we just refer [1] to check the hypothesis of Theorem A for such examples. We also include the singular series of the Goldbach conjecture

\begin{align*}
(84) \quad & \mathcal{G}(n) := \begin{cases} 
\mathcal{G}_2 \prod_{p \mid n, p > 2} \left( \frac{p - 1}{p - 2} \right) & \text{(if } n \text{ is even)}, \\
0 & \text{(if } n \text{ is odd)},
\end{cases}
\mathcal{G}_2 := 2 \prod_{p > 2} \left( 1 - \frac{1}{(p - 1)^2} \right),
\end{align*}

as an example which was not mentioned in [1]. This example was considered by Friedlander and Goldston [3] and by Languasco [8, 9]. We introduce a multiplicative function

\begin{align*}
(85) \quad & s(n) := \prod_{p \mid n, p > 2} \left( \frac{p - 1}{p - 2} \right)
\end{align*}
in order to deal with the singular series (84).

**Lemma 17.** For any complex number \(\alpha\), we have
\[
\sum_{n=1}^{\infty} \frac{1}{n^s} \left( \frac{\sigma(n)}{n} \right)^\alpha = \zeta(s) \zeta(s + 1)^\alpha f_1^{(1)}(s + 1),
\]
\[
\sum_{n=1}^{\infty} \frac{1}{n^s} \left( \frac{n}{\varphi(n)} \right)^\alpha = \zeta(s) \zeta(s + 1)^\alpha f_2^{(2)}(s + 1),
\]
\[
\sum_{n=1}^{\infty} \frac{1}{n^s} \left( \frac{\sigma(n)}{\varphi(n)} \right)^{\frac{\alpha}{2}} = \zeta(s) \zeta(s + 1)^\alpha f_3^{(3)}(s + 1),
\]
and
\[
\sum_{n=1}^{\infty} \frac{s(n)^\alpha}{n^s} = \zeta(s) \zeta(s + 1)^\alpha f_4^{(4)}(s + 1),
\]
where \(f_i^{(k)}(s) (1 \leq i \leq 4)\) are Dirichlet series convergent for \(\alpha > 1/2\).

**Proof.** For the first two examples, see Lemma 5.1 and Lemma 5.2 of [1, p.59–60]. In [1], real \(\alpha\) is mainly considered, but there is no difficulty to obtain the same result for complex \(\alpha\). The other two examples can be dealt with in the same way. For the ease of the reader, we prove the third example, for which the special case \(\alpha = 2\) is stated in [1, p.40, (3)].

Let
\[
Z(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \left( \frac{\sigma(n)}{\varphi(n)} \right)^{\frac{\alpha}{2}}, \quad \zeta(s)^{-1} Z(s) = \sum_{n=1}^{\infty} \frac{v(n)}{n^{s+1}},
\]
for \( \sigma > 1 \). Then \( v(n) \) and \( b(n) \) become multiplicative functions and

\[
\frac{v(n)}{n} = \sum_{d|m} \mu(d) \left( \frac{\sigma(m)}{\varphi(m)} \right)^{\frac{\sigma}{2}}, \quad b(n) = \sum_{d|m} \tau_{-\sigma}(d)v(m).
\]

For every prime \( p \) and every integer \( \nu \geq 1 \), we have an identity

\[
\left( \frac{\sigma(p^\nu)}{\varphi(p^\nu)} \right)^{\frac{\sigma}{2}} = \left( 1 + \frac{p^{-1} + \cdots + p^{-\nu}}{1 - p^{-1}} \right)^{\frac{\sigma}{2}}.
\]

By using the binomial expansion, we obtain

\[
\left( \frac{\sigma(p^\nu)}{\varphi(p^\nu)} \right)^{\frac{\sigma}{2}} = 1 + \left( \frac{\sigma(p^\nu)}{\varphi(p^\nu)} \right) O \left( \frac{1}{p^2} \right) = 1 + \frac{\alpha}{p} + O \left( \frac{1}{p^2} \right)
\]

and

\[
\left( \frac{\sigma(p^\nu)}{\varphi(p^\nu)} \right)^{\frac{\sigma}{2}} = \left( 1 + p^{-1} + \cdots + p^{-(\nu-1)} \right)^{\frac{\sigma}{2}} + O \left( \frac{1}{p^\nu} \right),
\]

where the implicit constant depends on \( \alpha \). Therefore, by (86),

\[
\frac{v(p)}{p} = \left( \frac{\sigma(p)}{\varphi(p)} \right)^{\frac{\sigma}{2}} = 1 + \frac{\alpha}{p} + O \left( \frac{1}{p^2} \right).
\]

By Lemma 15 and (86),

\[
b(p) = v(p) + \tau_{-\sigma}(p) = O \left( \frac{1}{p} \right), \quad b(p^\nu) \ll \sum_{d|m=p^\nu} |\tau_{-\sigma}(d)| \ll \varepsilon p^{\sigma \varepsilon}.
\]

Thus, for \( \sigma > 1/2 \), by taking \( \varepsilon > 0 \) sufficiently small so that \( \sigma - \varepsilon > 1/2 \),

\[
\sum_{n=1}^{\infty} \frac{|b(n)|}{n^\sigma} = \prod_p \left( 1 + \sum_{\nu=1}^{\infty} \frac{|b(p^\nu)|}{p^{\nu \sigma}} \right)
\]

\[
= \prod_p \left( 1 + O \left( \frac{1}{p^{\sigma+1}} + \sum_{\nu=2}^{\infty} \frac{1}{p^{\nu(\sigma-\varepsilon)}} \right) \right)
\]

\[
= \prod_p \left( 1 + O \left( \frac{1}{p^{\sigma+1}} + \frac{1}{p^{2(\sigma-\varepsilon)}} \right) \right) < +\infty.
\]

This completes the proof. \( \square \)
Lemma 17. For the last asymptotic formula, by (84) and (85), we obtain

Proof. The first three asymptotic formulas immediately follow by Theorem 2 and Lemma 17. For any complex number \( \alpha \) and \( x \geq 4 \), we have

\[
\sum_{n \leq x} \left( \frac{\sigma(n)}{n} \right)^\alpha = A^{(1)}_r x + \sum_{r=0}^{\lceil \Re \alpha \rceil} A^{(1)}_r (\log x)^{\alpha-r} + O \left( (\log x)^{2\lceil \alpha \rceil / 3} (\log \log x)^{\lceil \alpha \rceil / 3} \right),
\]

\[
\sum_{n \leq x} \left( \frac{n}{\varphi(n)} \right)^\alpha = A^{(2)}_r x + \sum_{r=0}^{\lceil \Re \alpha \rceil} A^{(2)}_r (\log x)^{\alpha-r} + O \left( (\log x)^{2\lceil \alpha \rceil / 3} (\log \log x)^{\lceil \alpha \rceil / 3} \right),
\]

\[
\sum_{n \leq x} \left( \frac{\sigma(n)}{\varphi(n)} \right)^\frac{\alpha}{2} = A^{(3)}_r x + \sum_{r=0}^{\lceil \Re \alpha \rceil} A^{(3)}_r (\log x)^{\alpha-r} + O \left( (\log x)^{2\lceil \alpha \rceil / 3} (\log \log x)^{\lceil \alpha \rceil / 3} \right),
\]

\[
\sum_{n \leq x} \sum_{n : \text{even}} \mathcal{E}(n)^\alpha = A^{(4)}_r x + \sum_{r=0}^{\lceil \Re \alpha \rceil} A^{(4)}_r (\log x)^{\alpha-r} + O \left( (\log x)^{2\lceil \alpha \rceil / 3} (\log \log x)^{\lceil \alpha \rceil / 3} \right),
\]

where \( (A_r^{(i)}) \) are complex coefficients computable from the Laurent expansions of the Dirichlet series in Lemma 17 at \( s = 1 \) and the implicit constants depend only on \( \alpha \).

Proof. The first three asymptotic formulas immediately follow by Theorem 2 and Lemma 17. For the last asymptotic formula, by (84) and (85), we obtain

\[
\sum_{n \leq x} \mathcal{E}(n)^\alpha = \sum_{n \leq x} s(n)^\alpha = \sum_{n \leq x} s(2n)^\alpha = \sum_{n \leq x} s(n)^\alpha
\]

since \( s(2^\nu) = 1 \). By Theorem 2 and Lemma 17, we obtain

\[
\sum_{n \leq x} \sum_{n : \text{even}} s(n)^\alpha = A^{(4)}_r x + \sum_{r=0}^{\lceil \Re \alpha \rceil} A^{(4)}_r (\log x)^{\alpha-r} + O \left( (\log x)^{2\lceil \alpha \rceil / 3} (\log \log x)^{\lceil \alpha \rceil / 3} \right).
\]

On inserting the expansion

\[
(\log x)^{\alpha-r} = (\log x)^{\alpha-r} \left( 1 - \frac{\log 2}{\log x} \right)^{\alpha-r}
\]

\[
= \sum_{\ell=0}^{\infty} (-1)^\ell \binom{\alpha-r}{\ell} (\log 2)^\ell (\log x)^{\alpha-r-\ell}
\]

\[
= \sum_{\ell=0}^{\lceil \Re \alpha \rceil - r} (-1)^\ell \binom{\alpha-r}{\ell} (\log 2)^\ell (\log x)^{\alpha-r-\ell} + O(1),
\]

we obtain the last asymptotic formula.

\[\square\]

Remark 8. To the knowledge of the author, the asymptotic formulas in Theorem 3 provide the error term estimates better than the previous results except the cases

- \( \alpha = +1 \) of the formula (87) (by Walfisz [18]),
- \( \alpha = +1 \) of the formula (88) (by Sitaramachandrarao [15]),
- \( \alpha = -1 \) of the formula (88) (by Liu [10]),
- \( \alpha = +1 \) of the formula (89) (probably well known to the experts),
- \( \alpha = +1 \) of the formula (90) (by Friedlander and Goldston [3]).

These cases except Liu’s result provide the error term estimate

\[ O((\log x)^{\frac{1}{2}}) \]
which is better than Theorem 3. The source of this phenomenon is the fact that there is no log-power in Lemma 3 similar to $(\log Q)^{6C+6}$ in Lemma 12.

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