A THREE-COUNTRY KALDORIAN BUSINESS CYCLE MODEL WITH FIXED EXCHANGE RATES: A CONTINUOUS TIME ANALYSIS

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Abstract. This paper analyses a three-country, fixed exchange rates Kaldorian nonlinear macroeconomic model of business cycles. The countries are connected through international trade, and international capital movement with imperfect capital mobility. Our model is a continuous time version of the discrete time three-country Kaldorian model of Inaba and Asada [22]. Their paper provided numerical studies of the dynamics of the three countries under fixed exchange rates. This paper provides analytical examinations of the local stability of the model’s equilibria, and of the existence of business cycles. The results are illustrated by numerical simulations.

1. Introduction. This paper investigates a dynamic nonlinear macroeconomic model of business cycles that describes the interaction of three countries connected through international trade and international capital movement by fixed exchange rates. For example, this could be applied to the case of Japan, the United States, and the United Kingdom during the Bretton Woods system of fixed exchange rates. The model could also be applied to the dynamic interaction of three Eurozone countries.

Our investigation of this topic builds on earlier studies by many economists and mathematicians in the field of dynamic economic theory. The mainstream models
of neoclassical dynamic economic theory assume full employment, full equilibrium, and full optimizing models with perfect foresight, or ‘rational’ expectation by economic agents (Barro and Sala-i-Martin [13] and Romer [42]). But there is also a long-standing, strong tradition that employs an alternative Keynesian modelling strategy for macrodynamics. This approach stems from Keynes’s classical work [25], and it formalizes the disequilibrium adjustment process and the adaptive behaviors of economic agents when there is underemployment of labor and underutilization of capital stocks. Typical examples of such ‘Keynesian’ macrodynamic models are Kaldor [23], Kalecki [24] and Goodwin [20]. Later examples of models in this tradition include Torre [47], Semmler [45], Dohtani, Misawa, Inaba, Yokoo and Owase [16], Asada, Chiarella, Flaschel and Franke [4], and Asada, Demetrian and Zimka [5], [6].

All of the above-mentioned Keynesian macrodynamic models assume a closed economy. However, international transactions were incorporated in the Keynesian so-called ‘Mundell-Fleming model’ (Mundell [35], [36], Fleming [17], and for textbook treatments Niehans [39] and Frenkel and Razin [18]). But the model is static, ignoring inter-country dynamics.

Dynamic interaction between countries or regions were studied by several authors such as Hamada [21], Sarantis [44], Krugman [26], Sethi [46], Rosser Jr. [43], Nijkamp and Reggiani [40], Puu [41] and Asada, Chiarella, Flaschel and Franke [3] mainly from the perspective of Keynesian disequilibrium dynamics with underemployment of labor and underutilization of capital stock. In addition a group of researchers, including the authors of the present paper, developed a series of Keynesian disequilibrium international macrodynamic models that are international extensions of Kaldor’s nonlinear business cycle model [23]. See, for example, Asada [1], [2], Asada, Douskos, Kalantonis and Markellos [7], Asada, Douskos and Markellos [8], Asada, Inaba and Misawa [9], [10], Asada, Kalantonis, Markellos and Markellos [11], Asada, Misawa and Inaba [12], Maličky and Zimka [32], [33], Medvedová [34] and Nakao [37]. However, all of the above macrodynamic models are models of small open economies, or two-country models, with fixed or flexible exchange rates. Although the dynamic interaction between more than two countries is a more realistic situation that warrants detailed studies, mathematical analyses of the dynamic interaction of three countries are rare. Probably this reflects the analytical difficulty of the problem.

Lorenz [30], [31] and Inaba and Asada [22] are two of a few exceptions. Lorenz [30] studied a six-dimensional continuous time dynamic model of three countries with fixed exchange rates that are connected through international trade. He showed that the model may generate a strange attractor in the sense of Newhouse, Ruelle and Takens [38]’s theorem, leading to a chaotic movement. In [31] Lorenz presented an example of six-dimensional continuous time three-country Kaldorian business cycle model with fixed exchange rates and showed the existence of a strange attractor by numerical simulations. But Lorenz in these papers did not consider international capital movements. On the other hand, Inaba and Asada [22] explored an eight-dimensional discrete time dynamic three-country Kaldorian business cycle model with fixed exchange rates considering both international trade and international capital movements. Using the numerical simulations, they also reported some chaotic dynamic movements.

In this paper we explore a continuous time version of Inaba and Asada [22]’s eight-dimensional, three-country model, using both analytical and numerical techniques.
The earlier discrete time model had the advantage that iterative techniques can generate its numerical simulation relatively easily. But there are two drawbacks to this approach. Taken together they suggest the discrete time model should be replaced by a continuous time version.

The first drawback concerns the realism of the economic modelling. As Asada, Chiarella, Flaschel and Franke [4] pointed out, the discrete time economic modelling strategy has the highly questionable implication that all economic decisions “occur in clustered or completely synchronized fashion at the beginning and the end of each considered period (the beginning of the next period)” (p. 191). They note that “the assumption of a quasi-continuous, time like behavior is more realistic for macroeconomic time series” (ibid.).

The second drawback of the discrete time dynamic model is the difficulty of the mathematical analysis of high dimensional nonlinear models. It is important to note that the behavior of the continuous time version of a dynamic model is not necessarily the same as that of the corresponding discrete time dynamic model. The mathematical analysis of Inaba and Asada’s complicated eight-dimensional nonlinear discrete time dynamic model is extremely difficult. Hence their choice to use only numerical simulations to study its dynamics. In contrast the mathematical analysis of the corresponding continuous time version, while not easy, is more manageable. In this paper, we study the dynamic behavior of the corresponding eight-dimensional continuous time model, both analytically and numerically. In particular we provide a mathematical proof for the existence of the Hopf bifurcation, and a numerical example of the subcritical Hopf bifurcation.

The subcritical Hopf bifurcation that is detected in this paper has an important economic implication. It corresponds to the so called ‘corridor stability’ that was described by Leijonhufvud [28]. This means that the equilibrium point is immune to a ‘small’ shock inside the ‘corridor’ (closed orbit), because it is locally stable. But it is vulnerable to a ‘large’ shock because an initial position of the system that starts from outside the ‘corridor’ never converges to the equilibrium point. This concept of ‘corridor stability’ was first described by Leijonhufvud [28] verbally and later formalized by Benhabib and Miyao [14] using the concept of the subcritical Hopf bifurcation.

The paper is arranged as follows. In Section 2, we introduce the model. Section 3 is devoted to the analysis of the model. As the model is a system of eight nonlinear differential equations, it would be very difficult to investigate its qualitative properties with respect to all its free parameters. Therefore, we gave fixed economic values to all its parameters, except the adjustment speed parameters \( \alpha_1, \alpha_2, \alpha_3 \), which do not influence the model’s equilibria. On the specific model, we then examine the whole process leading to its bifurcation equation. The equation provides all-important information on the qualitative properties of the model around its equilibrium. In Section 4, we present numerical simulations to illustrate our theoretical results. The simulations also help to reveal the dependence of real national incomes, real physical capital stocks, and nominal money stocks in the analyzed countries on the size of the export mobility parameter \( \delta \), and on the degree of capital mobility parameter \( \beta \). It turns out that parameter \( \delta \) has a strong influence on the development of real national incomes, real physical capital stocks, and nominal money stocks in the analyzed countries. But the influence of parameter \( \beta \) on these developments is negligible. We summarize the main results in Section 5.
2. **Formulation of the model.** Our three-country model is supposed to have fixed exchange rates. Denoting the exchange rate of the second country as $E_2$ and of the third country as $E_3$ (they express the values of a unit of their currencies measured in terms of the currency of the first country), we assume without loss of generality that these exchange rates are fixed at the level $E_2 = E_3 = 1$. We also assume that the price levels $p_i$, $i = 1, 2, 3$, of these countries are fixed at such a level that $p_1 = p_2 = p_3 = 1$.

The model consists of the following system of equations:

Disequilibrium adjustment process of the goods market in country $i$ is given by the equation

$$\dot{Y}_i = \alpha_i(C_i + I_i + G_i + J_i - Y_i), \quad \alpha_i > 0,$$

where

$$C_i = c_i(Y_i - T_i) + C_{0i}, \quad 0 < c_i < 1, \quad C_{0i} \geq 0,$$

$$T_i = \tau_i Y_i - T_{0i}, \quad 0 < \tau_i < 1, \quad T_{0i} \geq 0,$$

$$I_i = I_i(Y_i, K_i, r_i), \quad \frac{\partial I_i}{\partial Y_i} > 0, \quad \frac{\partial I_i}{\partial K_i} < 0, \quad \frac{\partial I_i}{\partial r_i} < 0,$$

$Y_i$ - real national income, $C_i$ - real consumption expenditure, $I_i$ - real net private expenditure on physical capital, $G_i$ - real government expenditure (fixed), $J_i$ - real net export, $K_i$ - real physical capital stock, $T_i$ - real income tax, $r_i$ - nominal rate of interest, $\alpha_i$ - adjustment speed in the goods market; the subscript $i$ is the index number of a country, $i = 1, 2, 3$, and a “dot” above variables signals their time derivative. As all endogenous variables in the model are functions of time, $t$, for simplicity this symbol of time is omitted throughout the paper.

Net export (current account) functions are given by the relations:

$$J_1 = \delta H_1(Y_1, Y_2, Y_3), \quad \frac{\partial H_1}{\partial Y_1} < 0, \quad \frac{\partial H_1}{\partial Y_2} > 0, \quad \frac{\partial H_1}{\partial Y_3} > 0, \quad \delta > 0,$$

$$J_2 = \delta H_2(Y_1, Y_2, Y_3), \quad \frac{\partial H_2}{\partial Y_1} > 0, \quad \frac{\partial H_2}{\partial Y_2} < 0, \quad \frac{\partial H_2}{\partial Y_3} > 0,$$

$$J_3 = -J_1 - J_2,$$

$\delta$ - degree of international trade.

Capital accumulation in country $i$ is given by the equation

$$\dot{K}_i = I_i, \quad i = 1, 2, 3.$$

The equilibrium condition of the money market in country $i$ is given by the relation

$$M_i = L_i(Y_i, r_i), \quad \frac{\partial L_i}{\partial Y_i} > 0, \quad \frac{\partial L_i}{\partial r_i} < 0,$$

$M_i$ - nominal money stock, $L_i$ - liquidity preference, $i = 1, 2, 3$.

Capital accounts functions in the three countries are given by relations

$$Q_1 = \beta(r_1 - r_2) + \beta(r_1 - r_3) = \beta(2r_1 - r_2 - r_3), \quad \beta > 0,$$

$$Q_2 = \beta(r_2 - r_1) + \beta(r_2 - r_3) = \beta(2r_2 - r_1 - r_3),$$

$$Q_3 = -Q_1 - Q_2 = \beta(2r_3 - r_1 - r_2),$$

$\beta$ - degree of capital mobility.

The real total balance of payments $A_i$ in country $i$ is defined as

$$A_i = J_i + Q_i, \quad i = 1, 2, 3.$$
Equations that describe the international movement of the money stock between the three countries are given by:

\[ \dot{M}_1 = A_1 \tag{14} \]
\[ \dot{M}_2 = A_2 \tag{15} \]
\[ \dot{M}_3 = A_3 = -A_1 - A_2. \tag{16} \]

Eq. (1) is the Keynesian/Kaldorian quantity adjustment process in the goods market (see Kaldor [23] and Keynes [25]). This equation implies that the level of real output fluctuates according to the excess demand in the goods market. Functions (2), (3) and (4) are the Keynesian consumption function, income tax function, and investment function, respectively. Functions (5), (6) and (7) are the standard Keynesian net export functions of the three countries. They always sum to zero, and investment function, respectively. Functions (5), (6) and (7) are the standard Keynesian money demand function. Equations (10), (11) and (12) are the standard capital account equations of the three countries. In this model, “capital movement” does not mean the movement of physical stocks of capital \( K_i \) among countries. It is assumed that only monetary capital moves between countries. By definition it is always true that \( Q_1 + Q_2 + Q_3 = 0 \). Equation (13) expresses the total balance of payments. The equation \( A_1 + A_2 + A_3 = 0 \) is always satisfied. Equations (14), (15) and (16) ensure that the money stock of a country changes as its overall balance of payments alters. By definition the restrictions \( M_1 + M_2 + M_3 = \bar{M} > 0, \ M_i > 0, \ i = 1, 2, 3, \) always hold. This is a specification of the monetary policy, which means that the total money stock is fixed. It follows from this that \( M_3 = \bar{M} - M_1 - M_2 \).

Solving equation (9) with respect to \( r_i \), we obtain

\[ r_i = r_i(Y_i,M_i), \quad \frac{\partial r_i}{\partial Y_i} > 0, \quad \frac{\partial r_i}{\partial M_i} < 0. \]

Consider now the functions

\[ r_i = \mu_i Y_i^{q_i} - \nu_i M_i + r_{0i}, \quad I_i(Y_i, K_i, r_i) = \gamma_i Y_i^{q_i} - \delta_i K_i - \sigma_i r_i + i_{0i}, \]
\[ i = 1, 2, 3, \]
\[ H_1 = (-\varphi_1 Y_1 + \psi_1 Y_2 + \chi_1 Y_3), \quad H_2 = (\varphi_2 Y_1 - \psi_2 Y_2 + \chi_2 Y_3), \]
\[ H_3 = (\varphi_1 - \varphi_2) Y_1 - (\psi_1 - \psi_2) Y_2 - (\chi_1 + \chi_2) Y_3. \tag{17} \]

On the basis of the assumed policies, definitions and behavioral links, we can reduce the model given by relations (1) – (16) and specifications (17) to the following eight-dimensional system of differential equations

\[ \dot{Y}_1 = \alpha_1 \{ c_1(1 - \tau_1)Y_1 + c_1 T_{01} + C_{01} + G_1 \]
\[ + [\gamma_1 Y_1^{q_1} - \delta_1 K_1 - \sigma_1 (\mu_1 Y_1^{q_1} - \nu_1 M_1 + r_{01}) + i_{01}] \]
\[ + \delta (-\varphi_1 Y_1 + \psi_1 Y_2 + \chi_1 Y_3) - Y_1 \} \equiv F_1(Y_1, Y_2, Y_3, K_1, M_1, \alpha_1, \delta) \]
\[ \dot{K}_1 = \gamma_1 Y_1^{q_1} - \delta_1 K_1 - \sigma_1 (\mu_1 Y_1^{q_1} - \nu_1 M_1 + r_{01}) + i_{01} \equiv F_2(Y_1, K_1, M_1) \]
\[ \dot{Y}_2 = \alpha_2 \{ c_2(1 - \tau_2)Y_2 + c_2 T_{02} + C_{02} + G_2 \]
dynamical system, see Maličký and Zimka [32], [33], or Asada et al. [5], [6].

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From a mathematical point of view model (18) is a dynamic system consisting of eight nonlinear differential equations. We aim to discover whether this system has equilibria, whether they are stable, and whether it can generate business cycles. Business cycles in model (18) are expressed by limit cycles. These can arise only if among the eigenvalues of the Jacobian matrix of model (18), at the equilibrium in question, there is at least one pair of purely imaginary eigenvalues. Liu’s criterion [29] gives a necessary and sufficient condition for the existence of one pair of purely imaginary eigenvalues, with the others having negative real parts. But to show that Liu’s criterion can be satisfied under the specified conditions, with respect to model (18), with its free parameters, would be very demanding and laborious. For example, the scalar term in the characteristic polynomial of its Jacobian matrix, which is the simplest coefficient in this polynomial, consists of $n!$ terms\(^\text{1}\). Therefore, we will continue the investigation of model (18) by making specific assumptions about the values of its parameters. Only the adjustment speed parameters $\alpha_i$, $i = 1, 2, 3$, whose values have no influence on the values of equilibria, are free to vary.

\[^\text{1}\text{For detailed examples of how Liu’s criterion can be satisfied in a high-dimensional nonlinear dynamical system, see Maličký and Zimka [32], [33], or Asada et al. [5], [6].}\]

\[\begin{align*}
+ [\gamma_2 Y_2^{q_2} - \delta_2 K_2 - \sigma_2 (\mu_2 Y_2^{q_2} - \nu_2 M_2 + r_{02}) + i_{02}] \\
+ \delta (\varphi_1 Y_1 - \psi_1 Y_2 + \chi_1 Y_3) - Y_2 \equiv F_3(Y_1, Y_2, Y_3, K_2, M_2, \alpha_2, \delta) \\
\dot{K}_2 = \gamma_2 Y_2^{q_2} - \delta_2 K_2 - \sigma_2 (\mu_2 Y_2^{q_2} - \nu_2 M_2 + r_{02}) + i_{02} \equiv F_4(Y_2, K_2, M_2) \\
\dot{Y}_3 = \alpha_3 \{c_3(1 - \tau_3) Y_3 + c_3 T_{03} + C_{03} + G_3 \\
+ [\gamma_3 Y_3^{q_3} - \delta_3 K_3 - \sigma_3 (\mu_3 Y_3^{q_3} - \nu_3 (\bar{M} - M_1 - M_2) + r_{03}) + i_{03}] \\
+ \delta ((\varphi_1 - \varphi_2) Y_1 - (\psi_1 - \psi_2) Y_2 - (\chi_1 + \chi_2) Y_3) - Y_3 \equiv F_5(Y_1, Y_2, Y_3, K_3, M_1, M_2, \alpha_3, \delta) \\
\dot{K}_3 = \gamma_3 Y_3^{q_3} - \delta_3 K_3 - \sigma_3 (\mu_3 Y_3^{q_3} - \nu_3 (\bar{M} - M_1 - M_2) + r_{03}) + i_{03} \equiv F_6(Y_3, K_3, M_1, M_2) \\
\dot{M}_1 = \delta (-\varphi_1 Y_1 + \psi_1 Y_2 + \chi_1 Y_3) + \beta \{2[\mu_1 Y_1^{q_1} - \nu_1 M_1 + r_{01}] - [\mu_2 Y_2^{q_2} - \nu_2 M_2 + r_{02}] - [\mu_3 Y_3^{q_3} - \nu_3 (\bar{M} - M_1 - M_2) + r_{03}] \} \equiv F_7(Y_1, Y_2, Y_3, M_1, M_2, \alpha, \varphi_1, \psi_1, \chi_1, j = 1, 2, 3, k = 1, 2, \text{ are positive constants.} \\
\dot{M}_2 = \delta (\varphi_2 Y_1 - \psi_2 Y_2 + \chi_2 Y_3) + \beta \{2[\mu_2 Y_2^{q_2} - \nu_2 M_2 + r_{02}] - [\mu_1 Y_1^{q_1} - \nu_1 M_1 + r_{01}] - [\mu_3 Y_3^{q_3} - \nu_3 (\bar{M} - M_1 - M_2) + r_{03}] \} \equiv F_8(Y_1, Y_2, Y_3, M_1, M_2, \alpha, \varphi_2, \psi_2, \chi_2, j = 1, 2, 3, k = 1, 2, \text{ are positive constants.}
\]

The parameters $\alpha_i$, $i = 1, 2, 3$, $\delta$ and $\beta$ in model (18) play important roles. The larger is $\alpha_i$, the quicker is the adjustment in the goods market of country $i$ in response to the excess demand. The larger is $\delta$, the greater is international trade between the three countries. The larger is $\beta$, the higher is the international mobility in money capital between the countries. It will be interesting to observe how changes of these parameter values affect the dynamic properties of the model.

3. Analysis of model (18). From a mathematical point of view model (18) is a dynamic system consisting of eight nonlinear differential equations. We aim to discover whether this system has equilibria, whether they are stable, and whether it can generate business cycles. Business cycles in model (18) are expressed by limit cycles. These can arise only if among the eigenvalues of the Jacobian matrix of model (18), at the equilibrium in question, there is at least one pair of purely imaginary eigenvalues. Liu’s criterion [29] gives a necessary and sufficient condition for the existence of one pair of purely imaginary eigenvalues, with the others having negative real parts. But to show that Liu’s criterion can be satisfied under the specified conditions, with respect to model (18), with its free parameters, would be very demanding and laborious. For example, the scalar term in the characteristic polynomial of its Jacobian matrix, which is the simplest coefficient in this polynomial, consists of $n!$ terms\(^1\). Therefore, we will continue the investigation of model (18) by making specific assumptions about the values of its parameters. Only the adjustment speed parameters $\alpha_i$, $i = 1, 2, 3$, whose values have no influence on the values of equilibria, are free to vary.

\[^\text{1}\text{For detailed examples of how Liu’s criterion can be satisfied in a high-dimensional nonlinear dynamical system, see Maličký and Zimka [32], [33], or Asada et al. [5], [6].}\]
3.1. A concrete example of model (18). Assume the values of the parameters in model (18) are as follows:\footnote{Some the assumed values of the variables, with the respective subscripts $1, 2, 3$, roughly correspond to actual Slovak, Czech and Austrian data. But this paper is not a specific study of those economies.}

\[
\begin{align*}
c_1 &= 0.4, \quad c_2 = 0.502, \quad c_3 = 0.563, \quad \tau_1 = 0.274, \quad \tau_2 = 0.323, \quad \tau_3 = 0.255, \\
C_{01} &= 14.7212, \quad C_{02} = 37.981314, \quad C_{03} = 83.91105, \quad T_{01} = 3.097, \\
T_{02} &= 6.109, \quad T_{03} = 46, \quad G_1 = 15, \quad G_2 = 35.7, \quad G_3 = 180, \quad \gamma_1 = \frac{390.02}{\sqrt{85}}, \\
\gamma_2 &= \frac{650.015}{\sqrt{192}}, \quad \gamma_3 = \frac{725.04}{\sqrt{370}}, \quad \delta_1 = 1, \quad \delta_2 = 0.5, \quad \delta_3 = 0.5, \quad \sigma_1 = \sigma_2 = \sigma_3 = 1, \\
\mu_1 &= \frac{60.20}{\sqrt{85}}, \quad \mu_2 = \frac{160.015}{\sqrt{192}}, \quad \mu_3 = \frac{330.04}{\sqrt{370}}, \quad \nu_1 = \nu_2 = \nu_3 = 1, \quad r_{01} = r_{02} = r_{03} = 0, \quad i_{01} = i_{02} = i_{03} = 0, \quad s_i = q_i = 0.5, \quad i = 1, 2, 3, \quad \varphi_1 = 0.15, \\
\psi_1 &= 0.1, \quad \chi_1 = \frac{371}{7400}, \quad \varphi_2 = 0.1, \quad \psi_2 = 0.15, \quad \chi_2 = 0.19, \quad M = 550, \\
\delta &= 1, \quad \beta = \frac{5000}{3}. \\
\end{align*}
\]

Financial values are in billions. Putting values (19) into model (18), we get the model

\[
\begin{align*}
\dot{Y}_1 &= \alpha_1 \left\{ 0.4(1-0.274)Y_1 + 0.4 \times 3.097 + 14.7212 + 15 + \frac{390.02}{\sqrt{85}} Y_{1,0.5} \right\} \\
-K_1 &= \left( \frac{60.20}{\sqrt{85}} Y_{1,0.5} - M_1 \right) - 0.15Y_1 + 0.1Y_2 + \frac{371}{7400} Y_3 - Y_1 \\
K_1 &= \frac{390.02}{\sqrt{85}} Y_{1,0.5} - K_1 - \left( \frac{60.20}{\sqrt{85}} Y_{1,0.5} - M_1 \right) \\
\dot{Y}_2 &= \alpha_2 \left\{ 0.502(1-0.323)Y_2 + 0.502 \times 6.109 + 37.981314 + 35.7 \right. \nonumber \\
+ &\left. \frac{650.015}{\sqrt{192}} Y_{2,0.5} - 0.5K_2 - \left( \frac{160.015}{\sqrt{192}} Y_{2,0.5} - M_2 \right) \right\} \\
+ 0.1Y_1 - 0.15Y_2 + 0.19Y_3 - Y_2 \\
K_2 &= \frac{650.015}{\sqrt{192}} Y_{2,0.5} - 0.5K_2 - \left( \frac{160.015}{\sqrt{192}} Y_{2,0.5} - M_2 \right) \\
\dot{Y}_3 &= \alpha_3 \left\{ 0.563(1-0.255)Y_3 + 0.563 \times 46 + 83.91105 + 180 \right. \nonumber \\
+ &\left. \frac{725.04}{\sqrt{370}} Y_{3,0.5} - 0.5K_3 - \left( \frac{330.04}{\sqrt{370}} Y_{3,0.5} - (550 - M_1 - M_2) \right) \right\} \\
+(0.15 - 0.1)Y_1 - (0.1 - 0.15)Y_2 - \left( \frac{371}{7400} + 0.19 \right) Y_3 - Y_3 \right\} \\
K_3 &= \frac{725.04}{\sqrt{370}} Y_{3,0.5} - 0.5K_3 - \left( \frac{330.04}{\sqrt{370}} Y_{3,0.5} - (550 - M_1 - M_2) \right) \\
\dot{M}_1 &= -0.15Y_1 + 0.1Y_2 + \frac{371}{7400} Y_3 + \beta \left\{ \frac{60.20}{\sqrt{85}} Y_{1,0.5} - M_1 \right\}
\end{align*}
\]
3.3. Bifurcation equation of model (20). The equilibrium of model (20) because it can tell us if model (20) has limit values \( \tilde{\alpha} \) of (21), with respect to a small neighborhood of the bifurcation value \( \alpha \). This subsection focuses on deriving the bifurcation equation of model (20). It is very important to investigate the bifurcation parameter of model (20) and its origin \( E \). In model (20) we transform the equilibrium \( x = (y_1, K_1, Y_2, K_2, Y_3, K_3, M_1, M_2) \), \( y = (y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8) \), so far, the parameters \( \alpha_1, \alpha_2, \alpha_3 \) in the Jacobian matrix \( J(E, \alpha_1, \alpha_2, \alpha_3) \) of model (20) at the equilibrium (21) is \(^3\):

\[
\begin{pmatrix}
1.142\alpha_1 & -\alpha_1 & 0.1\alpha_1 & 0 & \frac{371}{400}\alpha_1 & 0 & \alpha_1 & 0 \\
2.002 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0.1\alpha_2 & 0 & 0.468\alpha_2 & -0.5\alpha_2 & 0.19\alpha_2 & 0 & 0 & \alpha_2 \\
0 & 0 & 1.278 & -0.5 & 0 & 0 & 0 & 1 \\
0.05\alpha_3 & 0 & 0.05\alpha_3 & 0 & -0.287\alpha_3 & -0.5\alpha_3 & -\alpha_3 & -\alpha_3 \\
0 & 0 & 0 & 0 & 0.534 & -0.5 & -1 & -1 \\
1217.88 & 0 & -695.72 & 0 & -743.644 & 0 & -5000 & 0 \\
-608.917 & 0 & 1391.49 & 0 & -743.504 & 0 & 0 & -5000
\end{pmatrix}
\]

(22)

So far, the parameters \( \alpha_1, \alpha_2, \alpha_3 \) in the Jacobian matrix \( J(E, \alpha_1, \alpha_2, \alpha_3) \) are free to vary. We say that a triple value \( \tilde{\Gamma} = (\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3) \) of the parameters \( \alpha_1, \alpha_2, \alpha_3 \) is critical, if the eigenvalues of the Jacobian matrix (22) corresponding to this triple have one pair of purely imaginary eigenvalues, with the others having negative real parts. It can be shown that the triple \( \Gamma = (\tilde{\alpha}_1 = 0.71904, \tilde{\alpha}_2 = 1/3, \tilde{\alpha}_3 = 1/3) \) is critical. The eigenvalues corresponding to this critical triple \( \Gamma \) are as follows:

\[
\begin{align*}
\lambda_1 &= 0.7878i, \quad \lambda_2 = -0.7878i, \quad \lambda_3 = -0.1312 + 0.3492i, \\
\lambda_4 &= -0.1312 - 0.3492i, \quad \lambda_5 = -0.2442 + 0.2608i, \quad \lambda_6 = -0.2442 - 0.2608i, \\
\lambda_7 &= -5000.22, \quad \lambda_8 = -5000.14, \quad i = \sqrt{-1}.
\end{align*}
\]

(23)

Let us take the parameter \( \alpha_1 \) as the bifurcation parameter of model (20) and its value \( \alpha_1^* = \tilde{\alpha}_1 = 0.71904 \) as the bifurcation value of this model. Later we will investigate the behavior of solutions in a small neighborhood of the equilibrium (21), with respect to a small neighborhood of the bifurcation value \( \alpha_1^* \) at fixed values \( \tilde{\alpha}_2 = 1/3, \tilde{\alpha}_3 = 1/3 \).

3.3. Bifurcation equation of model (20). It is very important to investigate the bifurcation equation of model (20) because it can tell us if model (20) has limit cycles around its equilibrium, if these cycles are stable or unstable, and whether they are on the left or right side of the bifurcation value \( \alpha_1^* = 0.71904 \). Therefore this subsection focuses on deriving the bifurcation equation of model (20).

In model (20) we transform the equilibrium \( E \), which is given by (21), into the origin \( E_0 \) by shifting

\[
x = y + E, \quad x = (Y_1, K_1, Y_2, K_2, Y_3, K_3, M_1, M_2), \quad y = (y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8),
\]

\(^3\)The numbers in this matrix are rounded to three decimal digits.
and the bifurcation value $\alpha_1^*$ into the zero, by shifting $\alpha_1 = \epsilon + \alpha_1^*$, and then performing the Taylor expansion of the obtained model at equilibrium $E_0$ after a small arrangement we get the model

$$\dot{y} = J(E_0, \alpha_1^*) y + \mathcal{Y}(y, \epsilon), \quad (24)$$

where the matrix $J(E_0, \alpha_1^*)$ is gained from the Jacobian matrix (22) by substituting the parameters $\alpha_1, \alpha_2, \alpha_3$ with the values $\alpha_1^* = 0.71904, \bar{\alpha}_2 = 1/3, \bar{\alpha}_3 = 1/3$, respectively.

Then performing the following transformation in system (24)

$$y = M z, \quad \dot{z} = \tilde{J}(E_0, \alpha_1^*) z + Z(z, \epsilon), \quad (25)$$

where $\tilde{J}(E_0, \alpha_1^*)$ is the Jordan form of the matrix $J(E_0, \alpha_1^*)$ of model (24), $Z(z, \epsilon) = (Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8)$, $\lambda_1 = 0.7878i$, $\lambda_2 = \bar{\lambda}_1$, $z_2 = \bar{z}_1$, $Z_2 = \bar{Z}_1$, where the bar indicates a complex conjugate expression.

The polynomial transformation

$$z = u + h(u_1, u_2, \epsilon), \quad u = (u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8), \quad h = (h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8)$$

where

$$h_j(u_1, u_2, \epsilon) = \sum_{m_1 + m_2 + m \geq 2, m \in \{0, 1\}} h_j^{(m_1, m_2, m)} u_1^{m_1} u_2^{m_2} \epsilon^m, \quad 1 \leq j \leq 8,$$

transforms model (25) to a partial normal form on the invariant surface (see for example Bibikov [15], or Maličký and Zimka [32])

$$\begin{align*}
\dot{u}_1 &= \lambda_1 u_1 + \eta_1 u_1 \epsilon + \eta_2 u_2^2 + U^0(u, \epsilon) + U^0(u, \epsilon) \\
\dot{u}_2 &= \lambda_2 u_2 + \eta_1 u_2 \epsilon + \eta_2 u_1 u_2 + U^0(u, \epsilon) + U^0(u, \epsilon) \\
\dot{u}_3 &= \lambda_3 u_3 + H_3^0(u, \epsilon) + H_3^0(u, \epsilon) \\
\dot{u}_4 &= \lambda_4 u_4 + H_4^0(u, \epsilon) + H_4^0(u, \epsilon) \\
\dot{u}_5 &= \lambda_5 u_5 + H_5^0(u, \epsilon) + H_5^0(u, \epsilon) \\
\dot{u}_6 &= \lambda_6 u_6 + H_6^0(u, \epsilon) + H_6^0(u, \epsilon) \\
\dot{u}_7 &= \lambda_7 u_7 + H_7^0(u, \epsilon) + H_7^0(u, \epsilon) \\
\dot{u}_8 &= \lambda_8 u_8 + H_8^0(u, \epsilon) + H_8^0(u, \epsilon),
\end{align*} \quad (26)$$

where

$$\begin{align*}
(\otimes)^0(u_1, u_2, 0, 0, 0, 0, 0, 0, 0, \epsilon) &= 0, \\
(\otimes)^* \left( \sqrt{\epsilon}|u_1|, \sqrt{\epsilon}|u_2|, \sqrt{\epsilon}|u_3|, \sqrt{\epsilon}|u_4|, \sqrt{\epsilon}|u_5|, \sqrt{\epsilon}|u_6|, \sqrt{\epsilon}|u_7|, \sqrt{\epsilon}|u_8|, \epsilon \right) &= \mathcal{O}(\epsilon^{5/2}).
\end{align*}$$

The resonant terms $\eta_1$ and $\eta_2$ in (26), which are determined by the formulae
While all derivatives are calculated at $z_1$, \( \zeta \) exists infinitely many other critical triples \( \Gamma = (\tilde{\alpha}, \tilde{\alpha}, \alpha) \) and denote the resulting Jacobian as

\[
J_\Gamma = \begin{pmatrix}
\frac{\partial^2 Z_1}{\partial z_1 \partial \epsilon} & \frac{\partial^2 Z_1}{\partial z_1 \partial \eta} & \frac{\partial^2 Z_1}{\partial z_1 \partial \lambda} \\
\frac{\partial^2 Z_2}{\partial z_2 \partial \epsilon} & \frac{\partial^2 Z_2}{\partial z_2 \partial \eta} & \frac{\partial^2 Z_2}{\partial z_2 \partial \lambda} \\
\frac{\partial^2 Z_3}{\partial z_3 \partial \epsilon} & \frac{\partial^2 Z_3}{\partial z_3 \partial \eta} & \frac{\partial^2 Z_3}{\partial z_3 \partial \lambda}
\end{pmatrix}
\]

while all derivatives are calculated at $z_i = 0$, $i = 1, 2, \ldots, 8$, and $\epsilon = 0$, are

\[
\eta_1 = 0.6897 + 0.5449i, \quad \eta_2 = 1.11 \times 10^{-5} - 5.0486 \times 10^{-6}i.
\]

In polar coordinates $u_1 = re^{i\phi}$, $u_2 = re^{-i\phi}$, $u_j = v_j$, $j = 3, 4, \ldots, 8$, model (26) has the form

\[
\dot{r} = r(\zeta r^2 + \vartheta e) + R^0(r, \phi, v, \epsilon) + R^*(r, \phi, v, \epsilon)
\]

\[
\dot{\phi} = \omega_0 + \kappa e + \rho r^2 + \frac{1}{r}(\Phi(r, \phi, v, \epsilon) + \Phi^*(r, \phi, v, \epsilon))
\]

\[
\dot{v}_j = \lambda_j v_j + V_j^0(r, \phi, v, \epsilon) + V_j^*(r, \phi, v, \epsilon),
\]

\[
j = 3, 4, 5, 6, 7, 8,
\]

where $\zeta = \text{Re} \eta_1$, $\vartheta = \text{Re} \eta_2$, $\kappa = \text{Im} \eta_1$, $\rho = \text{Im} \eta_2$, $v = (v_3, v_4, v_5, v_6, v_7, v_8)$, and the symbols \((\otimes)^0, (\otimes)^*\) have the same meaning as in model (26).

The equation \(\zeta r^2 + \vartheta e = 0\) is the bifurcation equation of model (20), which on the basis of (27) is

\[
1.11 \times 10^{-5}r^2 + 0.6897e = 0.
\]

As $\zeta > 0$ and $\vartheta > 0$, we get (see, for example, Wiggins [48]) that the equilibrium $E$ of model (20) is locally stable at $\alpha_1 < \alpha^*_1$ and unstable at $\alpha_1 \geq \alpha^*_1$ in a small neighborhood of the bifurcation value $\alpha^*_1 = 0.71904$, and a simple Hopf bifurcation arises with unstable (subcritical) limit cycles at $\alpha_1 < \alpha^*_1$.

3.4. On the existence of other simple Hopf bifurcations in a neighborhood of the critical triple $\Gamma = (\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3) = (1/3, 1/3, 1/3)$. In the previous subsection, we showed that among the parameters $\alpha_2, \alpha_3$ of model (20) there exists the critical triple $\Gamma = (\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3) = (1/3, 1/3, 1/3)$ which enables the birth of a simple Hopf bifurcation. In this subsection, we will show that there exist infinitely many other critical triples $\Gamma = \Gamma_1, \Gamma_2, \Gamma_3$ in a small neighborhood of the critical triple $\Gamma$ that enable the birth of simple Hopf bifurcations.

Let us substitute the values $\alpha_2, \alpha_3$ in Jacobian (22) for the values $\tilde{\alpha}_2, \tilde{\alpha}_3$ respectively, and denote the resulting Jacobian as $J(E, \alpha_1, \tilde{\alpha}_2, \tilde{\alpha}_3)$ and its characteristic equation as

\[
\lambda^8 + a_1\lambda^7 + a_2\lambda^6 + a_3\lambda^5 + a_4\lambda^4 + a_5\lambda^3 + a_6\lambda^2 + a_7\lambda + a_8 = 0,
\]

where $a_j = a_j(\alpha_1, \tilde{\alpha}_2, \tilde{\alpha}_3)$, $j = 1, 2, \ldots, 8$. Let us recall that Jacobian $J(E, \alpha^*_1, \tilde{\alpha}_2, \tilde{\alpha}_3)$ has one pair of purely imaginary eigenvalues with the others having negative real parts.
Consider now a matrix

$$
\Delta(\alpha_1, \tilde{\alpha}_2, \tilde{\alpha}_3) = \begin{pmatrix}
a_7 & a_8 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_5 & a_6 & a_7 & a_8 & 0 & 0 & 0 & 0 \\
a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & 0 & 0 \\
a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \\
0 & 1 & a_1 & a_2 & a_3 & a_4 & 0 & 0 \\
0 & 0 & 0 & 1 & a_1 & a_2 & a_3 & a_4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
$$

where $a_j$ are the coefficients of characteristic equation (30), and its principal sub-
determinants

$$
\Delta_1 = a_7, \quad \Delta_2 = \begin{vmatrix} a_7 & a_8 \\ a_5 & a_6 \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} a_7 & a_8 \\ a_5 & a_6 \end{vmatrix}, \quad \ldots,
$$

$$
\Delta_7 = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ 0 & 1 & a_1 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & 1 & a_1 & a_2 & a_3 \\ 0 & 0 & 0 & 0 & 0 & 1 & a_1 \end{vmatrix}.
$$

Liu’s criterion (Liu [29]) is

$$
a_8(\alpha_1^*, \tilde{\alpha}_2, \tilde{\alpha}_3) > 0, \quad \Delta_j(\alpha_1^*, \tilde{\alpha}_2, \tilde{\alpha}_3) > 0, \quad j = 1, \ldots, 6,
$$

$$
\Delta_7(\alpha_1^*, \tilde{\alpha}_2, \tilde{\alpha}_3) = 0, \quad \frac{\partial \Delta_7(\alpha_1^*, \tilde{\alpha}_2, \tilde{\alpha}_3)}{\partial \alpha_1} \neq 0.
$$

This criterion gives the necessary and sufficient condition for Jacobian $J(E, \alpha_1, \tilde{\alpha}_2, \tilde{\alpha}_3)$ to have at the value $\alpha_1 = \alpha_1^* = 0.71904$ one pair of purely imaginary
eigenvalues with others having negative real parts. In subsection 3.2, we showed
that Jacobian $J(E, \alpha_1, \tilde{\alpha}_2, \tilde{\alpha}_3)$ has this property at the value $\alpha_1^* = 0.79104$. This
means that Liu’s criterion (32) is satisfied.

Consider now a function

$$
F(\alpha_1, \alpha_2, \alpha_3) = \Delta_7(\alpha_1, \alpha_2, \alpha_3),
$$

where $\Delta_7(\alpha_1, \alpha_2, \alpha_3)$ is the expression from (31) in which the values $\tilde{\alpha}_2, \tilde{\alpha}_3$ were
substituted by free parameters $\alpha_2$ and $\alpha_3$, respectively. The function $F(\alpha_1, \alpha_2, \alpha_3)$
has this property:

1. $F(\alpha_1^*, \tilde{\alpha}_2, \tilde{\alpha}_3) = \Delta_7(\alpha_1^*, \tilde{\alpha}_2, \tilde{\alpha}_3) = 0$

2. $\frac{\partial F(\alpha_1^*, \tilde{\alpha}_2, \tilde{\alpha}_3)}{\partial \alpha_1} = \frac{\partial \Delta_7(\alpha_1^*, \tilde{\alpha}_2, \tilde{\alpha}_3)}{\partial \alpha_1} \neq 0$.

According to Implicit Function Theorem there exists a function $\alpha_1 = \Psi(\alpha_2, \alpha_3)$
defined in a small neighborhood $\Omega(\tilde{\alpha}_2, \tilde{\alpha}_3)$ of the point $(\tilde{\alpha}_2, \tilde{\alpha}_3)$ such that $\alpha_1^* = \Psi(\tilde{\alpha}_2, \tilde{\alpha}_3)$ and $F(\Psi(\alpha_2, \alpha_3), \alpha_2, \alpha_3) = \Delta_7(\Psi(\alpha_2, \alpha_3), \alpha_2, \alpha_3) = 0$ at points $(\alpha_2, \alpha_3) \in \Omega(\tilde{\alpha}_2, \tilde{\alpha}_3)$. This means that to the arbitrary point $(\tilde{\alpha}_2, \tilde{\alpha}_3) \in \Omega(\tilde{\alpha}_2, \tilde{\alpha}_3)$ there
exists a corresponding bifurcation point $\alpha_1^* = \Psi(\tilde{\alpha}_2, \tilde{\alpha}_3)$ of model (20) such that
\( \Delta_7(\hat{a}_1, \hat{a}_2, \hat{a}_3) = 0 \). Taking into account that the expressions in (32) are derived from the continuous functions \( a_8(\alpha_1, \alpha_2, \alpha_3) \), \( \Delta_j(\alpha_1, \alpha_2, \alpha_3), j = 1, \ldots, 6 \), \( \Delta_7(\alpha_1, \alpha_2, \alpha_3) \) with continuous derivatives, we get that Liu’s criterion (32) is satisfied at the triple \((\hat{a}_1, \hat{a}_2, \hat{a}_3)\). As the point \((\hat{a}_2, \hat{a}_3)\) is from a small neighborhood of the point \((\hat{a}_2, \hat{a}_3)\), we can say that there is a simple Hopf bifurcation at the bifurcation point \(\hat{\alpha}_1\).

In the performed considerations, from the set of parameters \((\alpha_1, \alpha_2, \alpha_3)\) we have chosen parameter \(\alpha_1\) as a bifurcation parameter. Such a procedure enables us to choose the values of adjustment speeds \(\alpha_2\) and \(\alpha_3\) and to calculate the value of the corresponding bifurcation parameter \(\alpha_1\). Applying the results to a three-country example, we can choose two parameters of the adjustment speeds parameters \(\alpha_1, \alpha_2, \alpha_3\), and then calculate the value of the third corresponding parameter, which becomes the bifurcation parameter.

4. **Numerical simulations.** In subsection 4.1, Figures 1 – 3 show that the behavior of solutions in a small neighborhood of the equilibrium is fully compliant with the theoretical results that follow from the properties of bifurcation equation (29). Specifically, the model should give rise to unstable (subcritical) cycles at the values \(\alpha_1 < \alpha_1^*\), making the equilibrium of model (20) stable in its small neighborhood. However if \(\alpha_1 \geq \alpha_1^*\) the equilibrium would be unstable.

Subsection 4.2 shows how parameters \(\delta\) and \(\beta\) influence the values of equilibrium, and consequently also the developments of real national incomes, real physical stocks and nominal money stocks in the selected countries.

4.1. **The stability/instability of equilibrium and the existence of unstable limit cycles.** Here we present the development of three solutions of model (20). In Figure 1, a solution with the value of parameter \(\alpha_1 = 0.98 \times \alpha_1^*\) goes to the equilibrium of model (20). In Figure 2, a solution with the same value of parameter \(\alpha_1 = 0.98 \times \alpha_1^*\), but starting a little further from the equilibrium than the example in Figure 1, goes out of the equilibrium. In Figure 3 a solution with the value of parameter \(\alpha_1 = 1.1 \times \alpha_1^*\) goes out of the equilibrium for model (20).

![Figure 1](image1.png)

**Figure 1.** Solution starts ‘inside’ the unstable cycle and goes to the equilibrium. \(\alpha_1 = 0.98 \times \alpha_1^*\). Initial values: \(Y_{10} = 1.3 \times Y_1^*, Y_{20} = 1.3 \times Y_2^*, Y_{30} = Y_3^*, K_{10} = K_1^*, \ i = 1, 2, 3, \ M_{10} = M_1^*, M_{20} = M_2^*\). We see that all variables of the solution go to the equilibrium. This is in compliance with the result from the bifurcation equation (29) - that the equilibrium is locally stable on the left side of the bifurcation value \(\alpha_1^*\).
Figure 2. Solution starts 'outside' the unstable cycle and goes out of the cycle. $\alpha_1 = 0.98 \times \alpha_1^*$. Initial values: $Y_{10} = 1.395 \times Y_1^*, Y_{20} = 1.395 \times Y_2^*, Y_{30} = Y_3^*, K_{i0} = K_i^*, i = 1, 2, 3$, $M_{10} = M_1^*, M_{20} = M_2^*$. We see that all variables of the solution which starts a little further from the equilibrium as the solution in Figure 1, go out of the equilibrium. Comparing the initial values of both solutions and their subsequent paths, we observe that these two solutions are 'separated' by a cycle that is unstable. This is in compliance with the result obtained from bifurcation equation (29), that the cycle is unstable (subcritical).

Figure 3. Solution goes out of the equilibrium. $\alpha_1 = 1.1 \times \alpha_1^*$. Initial values: $Y_{10} = 1.1 \times Y_1^*, Y_{20} = 1.1 \times Y_2^*, Y_{30} = Y_3^*, K_{i0} = K_i^*, i = 1, 2, 3$, $M_{10} = M_1^*, M_{20} = M_2^*$. It is clear that if $\alpha_1 > \alpha_1^*$ the equilibrium is unstable. This is consistent with our theoretical results. A similar result holds for the value $\alpha_1 = \alpha_1^*$. Though in this case the solution goes out of equilibrium more slowly than when $\alpha_1 > \alpha_1^*$. This follows immediately from the first differential equation in model (28).

4.2. The influence of parameters $\delta$ and $\beta$ on the values of equilibrium.

4.2.1. The influence of parameter $\delta$ on equilibrium values, for a fixed parameter $\beta$. 
4.2.2. The influence of parameter $\beta$ on equilibrium values, for a fixed parameter $\delta$. From the structure of model (20) it is clear that parameter $\beta$ does not have any influence on the equilibrium values $Y_1^*, Y_2^*, Y_3^*$. The equilibrium values $K_1^*, K_2^*, K_3^*, M_1^*, M_2^*$ are given by the following formulae:

\[
\begin{align*}
K_1^* &= -0.05(-171.24 - 7573.12\beta)/\beta \\
K_2^* &= 0.07(495.23 + 19484.39\beta)/\beta \\
K_3^* &= 0.03(-1504.17 + 43522.62\beta)/\beta \\
M_1^* &= -0.05(-171.24 - 1180.57\beta)/\beta \\
M_2^* &= 0.03(495.23 + 4812.08\beta)/\beta.
\end{align*}
\]

Figure 5. The dependence of $K_1^*, K_2^*, K_3^*$ on parameter $\beta$ at $\delta = 1$. The graphs show that the dependence of the equilibrium values $K_1^*, K_2^*, K_3^*$ on $\beta$ is almost negligible. The same holds for the equilibrium values of $M_1^*, M_2^*$. As the dependence of equilibrium values $K_1^*, K_2^*, K_3^*, M_1^*, M_2^*$ on the changes of parameter $\beta$ is negligible, we do not present considerations on the impact of simultaneous changes in both parameters $\delta$ and $\beta$ in this paper.

5. Conclusion. This paper studies a three-country Kaldorian nonlinear macrodynamic model of business cycles with fixed exchange rates. It is assumed that these countries are connected through international trade and international capital movement with imperfect capital mobility. The model is a continuous time version of the discrete time three-country Kaldorian nonlinear macrodynamic model of business cycles with fixed exchange rates of Inaba and Asada [22]. In both continuous and discrete time versions, these models consist of an eight-dimensional dynamical
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system of nonlinear differential/difference equations. The investigation of the qualitative properties of such high-dimensional systems is rather difficult and laborious. Inaba and Asada [22] studied the dynamic behavior of the model only by means of numerical simulations, showing how changes in parameter $\delta$, which affects the intensity of international trade, and in parameter $\beta$, which affects the degree of capital mobility, influence the behavior of solutions around corresponding equilibria.

In our paper, the model is studied analytically and numerically. The results are illustrated by means of numerical simulations. In section 3 we proved the existence of a Hopf bifurcation in the model having its parameters concretized except the three adjustment speed parameters $\alpha_1, \alpha_2, \alpha_3$, which do not have any influence on the values of model equilibria. We found that at the values $\dot{\alpha}_1 = 0.79104, \dot{\alpha}_2 = \dot{\alpha}_3 = 1/3$ the Jacobian matrix of the model has a pair of purely imaginary eigenvalues, with the others having negative real parts. This enabled us, choosing the value $\hat{\alpha}_1 = 0.79104$ as the bifurcation value of the model and show that the equilibrium $E$ of model (20) is locally stable at $\alpha_1 < \alpha_1^*$ but unstable for $\alpha_1 \geq \alpha_1^*$ in a small neighborhood of the bifurcation value $\alpha_1^* = 0.79104$. At the same time unstable (subcritical) limit cycles occurred if $\alpha_1 < \alpha_1^*$. We also showed that there exist other critical triples $(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3)$ in a small neighborhood of the critical triple $(\hat{\alpha}_1 = 0.79104, \hat{\alpha}_2 = 1/3, \hat{\alpha}_3 = 1/3)$ which led to the creation of Hopf bifurcations.

Section 4 presents numerical simulations of the results. Subsection 4.1 sets out the numerical simulations of solutions to model (20). They are shown to be consistent with the theoretical results. Subsection 4.2 shows that parameter $\delta$, which affects the intensity of international trade, has a strong influence on the development of real national incomes, real physical capital stocks and nominal money stocks. But the degree of capital mobility $\beta$ has only negligible effect on these developments.

The model’s operation was explored using concrete data reflecting three economies, plus the three free parameters $\alpha_1, \alpha_2, \alpha_3$, representing speed of adjustment aspects of the model. The results could be extended by selecting different sets of economies that trade at fixed, or effectively fixed exchange rates, and allow capital movements.

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