Multivariate Bernoulli and Euler polynomials via Lévy processes

E. Di Nardo † I. Oliva ‡

February 17, 2022

Abstract

By a symbolic method, we introduce multivariate Bernoulli and Euler polynomials as powers of polynomials whose coefficients involve multivariate Lévy processes. Many properties of these polynomials are stated straightforwardly thanks to this representation, which could be easily implemented in any symbolic manipulation system. A very simple relation between these two families of multivariate polynomials is provided.

keywords: multivariate moment, multivariate Bernoulli polynomial, multivariate Euler polynomial, multivariate Lévy process, umbral calculus.

1 Introduction

Quite recently many authors have obtained a moment representation for various families of polynomials. For the multivariate Hermite polynomials $H_v(x)$, with $v = (v_1, \ldots, v_d) \in \mathbb{N}_0^d$ a multi-index, i.e. a vector of nonnegative integers, the moment representation

$$H_v(x) = E[(x \Sigma^{-1} + iY)^v]$$

has been given by Withers [11] with $E$ the expectation symbol, $i$ the imaginary unit, $Y \sim N(0, \Sigma^{-1})$ and $\Sigma$ a covariance matrix of full rank $d$. Making use of the Laplace distribution and of the Gamma distribution, Sun [10] gives a moment representation for Bernoulli polynomials, Euler polynomials and Gegenbauer polynomials in the univariate case; see also references therein.

By using a symbolic method, known in the literature as the classical umbral calculus [7], a different moment representation for multivariate Hermite polynomials is provided in [1], without using the imaginary unit. Umbral methods are essentially based on a symbolic device consisting in dealing with sequences of numbers, indexed by nonnegative integers, where the subscripts are treated as powers. Under suitable hypothesis (see [2] for a detailed discussion), these sequences of numbers could be interpreted as moments of random variables (r.v.’s).

In this paper we show how the classical umbral calculus allows us to give a moment representation like (1.1) also for the multivariate Bernoulli polynomials $B^{(t)}_v(x)$ and the Euler polynomials $E^{(t)}_v(x)$, where an additional real parameter $t \in \mathbb{R}$ is included. While in (1.1) the random “part” is represented by the multivariate Gaussian random vector $Y$, in the representation here introduced,
the random “part” is represented by a multivariate Lévy process \cite{8}. Thanks to this representation, we point out many similarities and a new relation between these two families of polynomials. Open questions are addressed at the end of the paper.

2 Multivariate umbral calculus

The classical umbral calculus has reached a more advanced level compared with the notions that we resume in this section. We only recall terminology, notation and basic definitions strictly necessary to deal with the topic of the paper. We skip the proofs, the reader interested in is referred to \cite{1,4}.

A univariate umbral calculus consists of an alphabet \( \mathcal{A} = \{ \alpha, \beta, \ldots \} \) of umbrae, and an evaluation linear functional \( E: \mathbb{R}[\mathcal{A}] \mapsto \mathbb{R} \), defined on the polynomial ring \( \mathbb{R}[\mathcal{A}] \) such that

1) \( E[1] = 1 \);

2) \( E[\alpha^i\beta^j \cdots] = E[\alpha^i]E[\beta^j] \cdots \) (uncorrelation property) for distinct umbrae \( \alpha, \beta, \ldots \) and nonnegative integers \( i, j, \ldots \).

A sequence \( a_0 = 1, a_1, a_2, \ldots \in \mathbb{R} \) is umbrally represented by an umbra \( \alpha \) if \( E[\alpha^n] = a_n \), for all nonnegative integers \( n \). The element \( a_n \) is the \( n \)-th moment of the umbra \( \alpha \). The same sequence of moments could be represented by infinitely many and distinct umbrae. More precisely, the umbrae \( \alpha \) and \( \gamma \) are said to be similar if \( E[\alpha^n] = E[\gamma^n] \) for all nonnegative integers \( n \), in symbols \( \alpha \equiv \gamma \).

An umbra looks like the framework of a r.v. with no reference to any probability space. The way to recognize the umbra corresponding to a r.v. is to characterize the sequence of moments \( \{ a_n \} \). When this sequence exists, we can compare the moment generating function (m.g.f.) of the r.v. with the so-called generating function (g.f.) of the umbra, that is the formal power series

\[
f(\alpha, z) = 1 + \sum_{n \geq 1} a_n z^n / n!.
\]  

(2.1)

If the moments of the r.v. are defined only up to some finite \( m \), then one works with sequences of \( m \) elements only; see \cite{3} for further details.

Let us consider a \( d \)-tuple of umbral monomials \( \mu = (\mu_1, \ldots, \mu_d) \) and set \( \mu^v = \mu_1^{v_1} \cdots \mu_d^{v_d} \). A sequence \( \{ g_v \}_{v \in \mathbb{N}_0^d} \in \mathbb{R} \), with \( g_0 = g_{v_1, \ldots, v_d} \) and \( g_0 = 1 \), is umbrally represented by the \( d \)-tuple \( \mu \) if \( E[\mu^v] = g_v \), for all \( v \in \mathbb{N}_0^d \). The elements \( \{ g_v \}_{v \in \mathbb{N}_0^d} \) are the multivariate moments of \( \mu \). If \( \{ \mu_i \}_{i=1}^d \) are umbral monomials with disjoint supports\footnote{The support of an umbral polynomial \( p \in \mathbb{R}[\mathcal{A}] \) is the set of all umbrae in \( \mathcal{A} \) which occur in \( p \).} then \( g_v = E[\mu_1^{v_1}] \cdots E[\mu_d^{v_d}] \). The g.f. of the \( d \)-tuple \( \mu \) is

\[
f(\mu, z) = E[e^{\mu_1 z_1 + \cdots + \mu_d z_d}] = 1 + \sum_{k \geq 1} \sum_{v \in \mathbb{N}_0^d, |v| = k} g_v z^v / v!
\]  

(2.2)

where \( z^v = z_1^{v_1} \cdots z_d^{v_d}, |v| = v_1 + \cdots + v_d \) and \( v! = v_1! \cdots v_d! \). Two \( d \)-tuples \( \mu_1 \) and \( \mu_2 \) are said to be similar, in symbols \( \mu_1 \equiv \mu_2 \), if and only if \( f(\mu_1, z) = f(\mu_2, z) \), that is \( E[\mu_1^v] = E[\mu_2^v] \) for all \( v \in \mathbb{N}_0^d \). They are said to be uncorrelated if and only if \( E[\mu_1^v \mu_2^w] = E[\mu_1^v]E[\mu_2^w] \) for all \( v, w \in \mathbb{N}_0^d \).

Multivariate Bernoulli umbra. Let \( \iota \) be the Bernoulli umbra \cite{7}, that is the umbra with g.f. \( f(\iota, z) = z / (e^z - 1) \), whose \( n \)-th coefficient is the \( n \)-th Bernoulli number.
Definition 2.1. The multivariate Bernoulli umbra \( \mathfrak{b} \) is the \( d \)-tuple \((\ldots, \eta)\), with all elements equal to the Bernoulli umbra \( \eta \).

From Definition 2.1 and 2.2, we have

\[
f(\mathfrak{b}, z) = E[e^{z_1 + \cdots + z_d}] = f(\mathfrak{b}, z_1 + \cdots + z_d) = \frac{z_1 + \cdots + z_d}{e^{z_1 + \cdots + z_d} - 1}. \tag{2.3}
\]

Definition 2.2. The multivariate Bernoulli numbers \( \{B_{\mathfrak{b}}^{(1)}\}_{v \in \mathbb{N}_0^d} \) are the coefficients of the g.f. \(2.3\), that is \( B_{\mathfrak{b}}^{(1)} = E[\mathfrak{b}_v] \).

Since \( E[\mathfrak{b}_v] = E[\mathfrak{b}_1 \mathfrak{b}_2 \cdots \mathfrak{b}_d] \) the following result is proved.

Proposition 2.3. \( B_{\mathfrak{b}}^{(1)} = E[\mathfrak{b}_v] \) for all \( v \in \mathbb{N}_0^d \).

Set \( \binom{v}{k} = \binom{v_1}{k_1} \cdots \binom{v_d}{k_d} \) for \( v = (v_1, \ldots, v_d) \), \( k = (k_1, \ldots, k_d) \in \mathbb{N}_0^d \), and assume \( k \leq v \) if and only if \( k_j \leq v_j \) for all \( j \in \{1, 2, \ldots, d\} \).

Proposition 2.4. \( B_{\mathfrak{b}}^{(1)} = \sum_{k \leq v} \binom{v}{k} B_k^{(1)} \) for all \( v \in \mathbb{N}_0^d \) such that \( |v| > 1 \).

Proof. Let \( \mathfrak{u} \) be the \( d \)-tuple \((u, \ldots, u)\) with all elements equal to the unity umbra \( \mathfrak{u} \), that is the umbra with all moments equal to 1. We have

\[
\sum_{k \leq v} \binom{v}{k} B_k^{(1)} = \sum_{k \leq v} \binom{v}{k} E[\mathfrak{b}_k] = \sum_{k \leq v} \binom{v}{k} E[\mathfrak{b}_k] E[\mathfrak{u}^{v-k}]
\]

and by linearity [1]

\[
\sum_{k \leq v} \binom{v}{k} E[\mathfrak{b}_k] E[\mathfrak{u}^{v-k}] = E \left[ \sum_{k \leq v} \binom{v}{k} \mathfrak{b}_k \mathfrak{u}^{v-k} \right] = E[(\mathfrak{b} + \mathfrak{u})^v] = E[(\mathfrak{b} + \mathfrak{u})^{|v|}],
\]

as \( E[(\mathfrak{b} + \mathfrak{u})^v] = E[\prod_{i=1}^d (\mathfrak{b} + \mathfrak{u})^{v_i}] \). Since \( E[(\mathfrak{b} + \mathfrak{u})^k] = E[(\mathfrak{b} + 1)^k] = E[\mathfrak{b}_k] \) for all nonnegative \( k > 1 \) [7], the result follows from Proposition 2.3 for \( k = |v| \).

Multivariate Euler umbra. Let \( \eta \) be the Euler umbra, that is the umbra with g.f. \( f(\eta, z) = 2e^z/[e^{2z} + 1] \), whose \( n \)-th coefficient is the \( n \)-th Euler number.

Definition 2.5. The multivariate Euler umbra \( \eta \) is the \( d \)-tuple \((\ldots, \eta)\), with all elements equal to the Euler umbra \( \eta \).

From Definition 2.5 and 2.2, we have

\[
f(\eta, z) = E[e^{\eta z_1 + \cdots + \eta z_d}] = f(\eta, z_1 + \cdots + z_d) = \frac{2e^{(z_1 + \cdots + z_d)}}{e^{2(z_1 + \cdots + z_d)} + 1}. \tag{2.4}
\]

Definition 2.6. The multivariate Euler numbers \( \{\mathfrak{c}_v^{(1)}\}_{v \in \mathbb{N}_0^d} \) are the coefficients of the g.f. \(2.4\), that is \( \mathfrak{c}_v^{(1)} = E[\mathfrak{c}_v] \).

Proposition 2.7. \( \mathfrak{c}_v^{(1)} = E[\mathfrak{c}_v] \) for all \( v \in \mathbb{N}_0^d \).
Multivariate Lévy processes. One feature of the classical umbral calculus is the feasibility to extend the alphabet $\mathcal{A}$ by adding new symbols \([7]\), the so-called auxiliary umbrae, whose moments depend on moments of elements in $\mathcal{A}$. A very important example is the so called dot-product $m.\alpha$ of a nonnegative integer $m$ and an umbra $\alpha$. By using the exponential Bell polynomials \([4]\), the moments of $m.\alpha$ can be expressed in terms of moments of $\alpha$, since $m.\alpha$ represents a sum of $m$ uncorrelated umbrae similar to $\alpha$. So we have $f(m.\alpha, z) = [f(\alpha, z)]^m$ and similarly \([1]\) $f(m.\mu, z) = [f(\mu, z)]^m$. Thanks to the notion of auxiliary umbrae, in this last equality the integer $m$ could be replaced by a real number $t \in \mathbb{R}$

$$f(t.\mu, z) = [f(\mu, z)]^t.$$  

(2.5)

Indeed the multivariate moments of $m.\mu$ are \([1]\)

$$E[(m.\mu)^v] = \sum_{\lambda \vdash v} \frac{v!}{m(\lambda)!} \frac{(m)_{l(\lambda)} E[\mu_\lambda]}{\lambda!}$$  

(2.6)

where the sum is over all partitions $\lambda = (\lambda_1^n, \lambda_2^n, \ldots)$ of the multi-index $v \in \mathbb{N}$. $E[\mu_\lambda]$ is a polynomial in $\mu$ of degree $|v|$ in $m$. Then, the symbol $t.\mu$ denotes the auxiliary umbra such that $E[(t.\mu)^v] = q_\mu(t)$, by which (2.5) follows (see Proposition 2.2 in \([1]\)). In particular, for the multivariate Bernoulli and Euler umbrae we have

$$f(t.\eta, z) = \left(\frac{z_1 + \cdots + z_d}{e^{z_1 + \cdots + z_d} - 1}\right)^t \quad \text{and} \quad f(t.\epsilon, z) = \left(\frac{2e^{z_1 + \cdots + z_d} - 1}{e^{2(z_1 + \cdots + z_d)} + 1}\right)^t.$$  

(2.7)

The auxiliary umbrae $t.\eta$ and $t.\epsilon$ are symbolic versions of multivariate Lévy processes. Indeed, let $X = \{X_t\}_{t \geq 0}$ be a Lévy process on $\mathbb{R}^d$, that is, a stochastic process starting from $0$ and with stationary and independent $d$-dimensional increments. According to the multivariate Lévy-Khintchine formula \([8]\), if we assume that $X$ has a convergent m.g.f. $\varphi_X(z)$ in some neighborhood of $0$, then we have $\varphi_X(z) = (\varphi_{X_1}(z))^t$, \([8]\)

with $X_1 = (X_1^{(1)}, \ldots, X_d^{(1)})$. Within the multivariate umbral calculus, if we denote by $\mu$ the $d$-tuple such that $f(\mu, z) = \varphi_{X_1}(z)$, then the auxiliary umbra $t.\mu$ is the umbral counterpart of $X$. The auxiliary umbra $t.\mu$ has various algebraic properties paralleling those of $t.\alpha$ \([1]\). We recall those we will use later on:

$$t.(c.\mu) \equiv c(t.\mu), \quad (t + s).\mu \equiv t.\mu + s.\mu, \quad t.(\mu_1 + \mu_2) \equiv t.\mu_1 + t.\mu_2$$  

(2.9)

for $c, s, t \in \mathbb{R}$, with $s \neq t$, and $\mu_1$ and $\mu_2$ uncorrelated $d$-tuples of umbral monomials. If we replace $s$ with $-t$ in the second equivalence of (2.9), the auxiliary umbra $-t.\mu$ has the remarkable property $-t.\mu + t.\mu \equiv \epsilon$, where $\epsilon$ is an umbra with g.f. $f(\epsilon, z) = 1$. The auxiliary umbra $-t.\mu$ is called the inverse of $t.\mu$ and it is such that $-t.\mu \equiv t.(-1.\mu)$. Then also $-t.\mu$ is a symbolic version of a multivariate Lévy process. As example, to keep the length of the paper within bounds but also for the open questions addressed in the last section, we just show the probabilistic counterpart

\[\lambda \vdash v\] denotes a partition of a multi-index $v$, in symbols $\lambda \vdash v$, is a matrix $\lambda = (\lambda_{ij})$ of nonnegative integers and with no zero columns in lexicographic order such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = v_r$ for $r = 1, 2, \ldots, d$. The length $l(\lambda)$ of $\lambda$ is the number of columns of $\lambda$. The notation $\lambda = (\lambda_1^n, \lambda_2^n, \ldots)$ means that in the matrix $\lambda$ there are $r_1$ columns equal to $\lambda_1$, $r_2$ columns equal to $\lambda_2$ and so on, with $\lambda_1 < \lambda_2 < \cdots$. We set $m(\lambda) = (r_1, r_2, \ldots)$.\footnote{A partition of a multi-index $v$, in symbols $\lambda \vdash v$, is a matrix $\lambda = (\lambda_{ij})$ of nonnegative integers and with no zero columns in lexicographic order such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = v_r$ for $r = 1, 2, \ldots, d$. The length $l(\lambda)$ of $\lambda$ is the number of columns of $\lambda$. The notation $\lambda = (\lambda_1^n, \lambda_2^n, \ldots)$ means that in the matrix $\lambda$ there are $r_1$ columns equal to $\lambda_1$, $r_2$ columns equal to $\lambda_2$ and so on, with $\lambda_1 < \lambda_2 < \cdots$. We set $m(\lambda) = (r_1, r_2, \ldots)$.}
of \(-t.\mathbf{u}\) and \(-t.\mathbf{v}\). Indeed the following propositions give the probabilistic interpretation of the \(d\)-dimensional random vector \(X_1\) in (2.8) corresponding to \(-1.\mathbf{u}\) and \(-1.\mathbf{v}\) respectively.

**Proposition 2.8.** The inverse \(-1.\mathbf{u}\) of the multivariate Bernoulli umbra is the umbral counterpart of a \(d\)-tuple identically distributed to \((U, \ldots, U)\), where \(U\) is a uniform r.v. on the interval \((0, 1)\).

**Proposition 2.9.** The inverse \(-1.\mathbf{v}\) of the multivariate Euler umbra is the umbral counterpart of a \(d\)-tuple identically distributed to \((X, \ldots, X)\), where \(X = 2Y - 1\) with \(Y\) a Bernoulli r.v. of parameter 1/2.

**Definition 2.10.** The \(t\)-th order multivariate Bernoulli numbers \(\{B_v^{(t)}\}_{v \in \mathbb{N}_0^d}\) are the multivariate moments of the multivariate umbra \(t.\mathbf{u}\), that is \(B_v^{(t)} = E[(t.\mathbf{u})^v]\).

**Definition 2.11** generalizes the definition of the multivariate Bernoulli numbers given in [5]. In particular we have \(B_0^{(0)} = E[(0.\mathbf{u})^0] = 1\) and \(B_v^{(0)} = E[(0.\mathbf{u})^v] = 0\) if \(|v| > 0\).

**Proposition 2.11.** \(B_v^{(t)} = \sum_{k \leq v} \left(\begin{array}{c} v \\ k \end{array}\right) B_k^{(s)} B_{v-k}^{(-s)}\), for all \(s, t \in \mathbb{R}\) and \(v \in \mathbb{N}_0^d\).

**Proof.** For \(s = t\), the proof is straightforward. For \(s, t \in \mathbb{R}\) with \(s \neq t\), from the second of (2.9) we have \(t.\mathbf{u} \equiv (t - s).\mathbf{u} + s.\mathbf{u}\), so that \(E[(t.\mathbf{u})^v] = E[(t - s).\mathbf{u} + s.\mathbf{u})^v]\) for all \(v \in \mathbb{N}_0^d\). The result follows from Definition 2.10 since \(E[(t - s).\mathbf{u} + s.\mathbf{u})^v] = \sum_{k \leq v} \left(\begin{array}{c} v \\ k \end{array}\right) E[(t - s).\mathbf{u}^v] E[(s.\mathbf{u})^k]\).

In Proposition 2.11 set \(t = 0\). We have the following corollary.

**Corollary 2.12.** For all \(s \in \mathbb{R}\) we have \(\sum_{k \leq v} \left(\begin{array}{c} v \\ k \end{array}\right) B_k^{(s)} B_{v-k}^{(-s)} = 1\) if \(v = 0\) otherwise being 0.

**Definition 2.13.** The \(t\)-th order multivariate Euler numbers \(\{\mathcal{E}_v^{(t)}\}_{v \in \mathbb{N}_0^d}\) are the multivariate moments of the multivariate umbra \(t.\mathbf{v}\), that is \(\mathcal{E}_v^{(t)} = E[(t.\mathbf{v})^v]\).

**Definition 2.11** generalizes the definition of the multivariate Euler numbers given in [5]. As before we have \(\mathcal{E}_0^{(0)} = E[(0.\mathbf{v})^0] = 1\), and \(\mathcal{E}_v^{(0)} = E[(0.\mathbf{v})^v] = 0\) if \(|v| > 0\). Proposition 2.11 and Corollary 2.12 can be restated also for \(\mathcal{E}_v^{(t)}\) since their proofs depend only on the moment umbral representation and not on the properties of the involved umbras.

### 3 Multivariate Bernoulli and Euler polynomials

In the classical umbral calculus, we can replace the field \(\mathbb{R}\) with \(\mathbb{R}[x_1, \ldots, x_d]\), where \(x_1, \ldots, x_d\) are indeterminates [11]. Then the linear operator \(E\) is defined on the polynomial ring \(\mathbb{R}[x_1, \ldots, x_d][A]\) with values in \(\mathbb{R}[x_1, \ldots, x_d]\). The only hypothesis to be added on the linear operator \(E\) is that if \(x = (x_1, \ldots, x_d)\) then \(E[x^v \mu^w] = x^v E[\mu^w]\), for all \(v, w \in \mathbb{N}_0^d\).

**Definition 3.1.** (Moment representation of multivariate Bernoulli polynomials) The multivariate Bernoulli polynomial of order \(v \in \mathbb{N}_0^d\) is \(B_v^{(t)}(x) = E[(x + t.\mathbf{u})^v]\), where \(t\) is the multivariate Bernoulli umbra and \(t \in \mathbb{R}\).

---

[Probabilistic counterparts of \(t.\mathbf{u}\) and \(t.\mathbf{v}\) could be given, but the involved random variables are less known. This goal goes beyond the aim of the paper.]
Proposition 3.3. Euler polynomials are such that

The former equality follows from the first equality in (3.1), by observing that
Proof. Thanks to Proposition 2.9, we have 

Thanks to Proposition 2.9, we have 

Corollary 3.5. 

Proof. The former equality follows from the first equality in (3.1), by observing that 

For the latter equality, observe that 

by which the result follows.

Definition 3.2. (Moment representation of multivariate Euler polynomials) The multivariate Euler polynomial of order \( v \in \mathbb{N}_d \) is \( E^{(t)}_v(x) = E\{((x + \frac{1}{2}(t.\eta - u))^v \} \) with \( t \in \mathbb{R}, u = (u, \ldots, u) \) a \( d \)-tuple with all elements equal to the unity umbra \( u \) and \( \eta \) the multivariate Euler umbra.

Definition 3.1 and 3.2 means that the multivariate Bernoulli polynomials and the multivariate Euler polynomials are such that

where \( t.\eta \) and \( t.(\eta - u) \) are multivariate Lévy processes. This symbolic moment representation of the coefficients simplifies the calculus and is computationally efficient [11 2]. In particular, the definition of multivariate Euler polynomials is given according to the terminology first introduced by Nörlund [6]. Indeed, from the first equivalence in (2.9) we have 

\[ \left( \frac{1}{2}(\eta - u) \right)^{t} = \frac{1}{2}[-1.\eta + u] \]

and this last symbol represents a \( d \)-tuple identically distributed to \((Y, \ldots, Y)\), where \( Y \) is a Bernoulli r.v. with parameter \( 1/2 \).

Proposition 3.3. \( B_v^{(t)}(x) = \sum_{k \leq v} \binom{v}{k} x^{v-k} E^{(t)}_k \) and \( 2^{|v|} E^{(t)}_v \left( \frac{1}{2} x + \frac{t}{2} 1 \right) = \sum_{k \leq v} \binom{v}{k} x^{v-k} E^{(t)}_k \) with \( 1 \) the \( d \)-tuple with all elements equal to 1.

Proof. The former equality follows from the first equality in (3.1), by observing that 

For the latter equality, observe that

Since 

we have 

by which the result follows.

Corollary 3.4. \( B_v^{(t)} = B_v^{(t)}(0) \) and \( E_v^{(t)} = 2^{|v|} E_v^{(t)} \left( \frac{1}{2} 1 \right) \), with \( 0 \) and \( 1 \) the \( d \)-tuples with all elements equal to 0 and 1 respectively.

Corollary 3.5. 

Taking into account Definitions 3.1, 3.2 and (2.7), the g.f. of the multivariate Bernoulli and Euler polynomials are respectively

\[ f(x + t.\tau, z) = e^{x_1 z_1 + \cdots + x_d z_d} \left( \frac{z_1 + \cdots + z_d}{e^{z_1 + \cdots + z_d} - 1} \right)^t, f \left( x + \frac{1}{2}[t.(\eta - u)], z \right) = 2^t e^{x_1 z_1 + \cdots + x_d z_d} \left( e^{z_1 + \cdots + z_d} + 1 \right)^t. \]

Proposition 3.6. If \( x = (x_1, \ldots, x_d) \) and \( y = (y_1, \ldots, y_d) \) are two \( d \)-tuples of indeterminates, then

\[ B_v^{(t+s)}(x + y) = \sum_{k \leq v} \binom{v}{k} B_k^{(t)}(x) B_{v-k}^{(s)}(y) \]

\[ E_v^{(t+s)}(x + y) = \sum_{k \leq v} \binom{v}{k} E_k^{(t)}(x) E_{v-k}^{(s)}(y). \]
Proof. We replace \( \mathbb{R}[x_1, \ldots, x_d] \) with \( \mathbb{R}[x_1, \ldots, x_d, y_1, \ldots, y_d] \). From the second equivalence in (2.9), we have

\[
B_v^{(t+s)}(x+y) = E\{[(x + t \cdot u) + (y + s \cdot u)]^v\} = \sum_{k \leq v} \binom{v}{k} E[(x + t \cdot u)^k]E[(y + s \cdot u)^{v-k}],
\]

by which the former equality follows. The latter equality follows by the same arguments. \qed

**Corollary 3.7.** \( \sum_{k \leq v} \binom{v}{k} E_v^{(t)}(x)E_v^{(\ell)}(x) = \sum_{k \leq v} \binom{v}{k} B_v^{(t)}(x)B_{v-k}^{(\ell)}(x) = 2^v x^v. \)

**Proof.** In Proposition 3.6 set \( s = -t \). The result follows by observing that \( E_v^{(0)}(2x) = B_v^{(0)}(2x) = E[(2x)^v] = 2^v x^v. \)

**Proposition 3.8.** \( B_v^{(t)}(t1 - x) = (-1)^{|v|} B_v^{(t)}(x), \) and \( E_v^{(t)}(t1 - x) = (-1)^{|v|} E_v^{(t)}(x). \)

**Proof.** The former equality follows by observing that \( B_v^{(t)}(t1 - x) = E[(t \cdot u - x + t \cdot u)^v] \) and

\[
E[(t \cdot u - x + t \cdot u)^v] = (-1)^{|v|} E[(t \cdot u - x + t \cdot (t \cdot u))^v] = (-1)^{|v|} E \{[x + t \cdot (t \cdot u)]^v\}.
\]

Since \( E[(\ell + u)^k] = (-1)^k E\{[u]^k\} \) for all nonnegative integers \( k \) [7], then \( E[\{(\ell + u)^v\}] = (-1)^{|v|} E\{(\ell + u)^{|v|}\} = E\{[u]^v\} = E\{u^v\}. \) Then we have \( -(\ell + u) \equiv \ell \) and \( t \cdot (t - u) \equiv t \cdot u, \) by which the result follows. Similarly we have

\[
E_v^{(t)}(t1 - x) = E\left[\left(t1 - x + t \cdot \frac{n}{2} - t \cdot 1\right)^v\right] = (-1)^{|v|} E\left[\left(x + t \cdot \left(-\frac{n}{2}\right) + t \cdot \left(-\frac{u}{2}\right)\right)^v\right].
\]

Since \( f(-\frac{n}{2}, z) = f(\frac{n}{2}, z) \), the latter result follows. \qed

As it happens for the univariate case, the multivariate Bernoulli and Euler polynomials share many properties. Undoubtedly, this is due to the connection between Bernoulli and Euler numbers that here is emphasized by the similar multivariate moment representation. Therefore it is reasonable to ask for relations between them. We have chosen to show a connection between the \( t \)-th order multivariate Bernoulli and Euler umbrae they are related to, which can be translated in a connection between the \( t \)-th order multivariate Bernoulli and Euler numbers.

**Lemma 3.9.** If \( \ell \) is the multivariate Bernoulli umbra and \( \eta \) is the multivariate Euler umbra, then \( 2 \ell \equiv \frac{1}{2} (\eta - u) + \ell. \)

**Proof.** We have \( f(2 \ell, z) = f(\ell, 2z) = 2(z_1 + \cdots + z_d)/[e^{2(z_1 + \cdots + z_d)} - 1] = f(\eta - u, \frac{z}{2})f(\ell, z). \) \qed

**Theorem 3.10.** (Relation between multivariate Bernoulli and Euler polynomials) We have \( 2^{|v|} B_v^{(t)}(\frac{\ell}{2}) = E[E_v^{(t)}(x + t \cdot u)\] where \( \ell \) is the multivariate Bernoulli umbra.

**Proof.** From Lemma 3.9 we have \( t(2 \ell) \equiv t \cdot \frac{1}{2} (\eta - u) + \ell \equiv t \cdot \frac{1}{2} (\eta - u) + t \cdot u, \) where last equivalence follows form the third equivalence in (2.9). The result follows since \( 2^{|v|} B_v^{(t)}(\frac{\ell}{2}) = E[(x + t(2 \ell))^v]. \) \qed
Conclusions and open questions: multivariate time-space harmonic polynomials. A family of polynomials \( \{P(x, t)\}_{t \geq 0} \) is said to be time-space harmonic with respect to a stochastic process \( \{X_t\}_{t \geq 0} \) if \( E[P(X_t, t) \mid \mathcal{F}_s] = P(X_s, s) \), for all \( s \leq t \), where \( \mathcal{F}_s = \sigma(X_\tau : \tau \leq s) \) is the natural filtration associated with \( \{X_t\}_{t \geq 0} \). Recently [3] the authors have introduced a new family of polynomials which are time-space harmonic with respect to Lévy processes and by which to express all other families of polynomials sharing the same properties. These polynomials are Appell polynomials and have the form \( E[(x + t \alpha)^i] \) for all positive integers \( i \). By generalizing the definition of conditional evaluation given in [3] to the multivariate case, the multivariate Bernoulli and Euler polynomials should result to be time-space harmonic with respect to the multivariate Lévy processes \(-t\cdot\iota\) and \( \frac{1}{2}[t\cdot(u - 1\cdot\eta)] \) respectively, whose probabilistic counterparts could be recovered via Propositions 2.8 and 2.9. Indeed, Corollary 3.5 shows that these polynomials share one of the main properties of time-space harmonic polynomials: when the vector of indeterminates is replaced by the corresponding Lévy process, their overall mean is zero. We believe that the setting here introduced, together with the one given in [3], could be a fruitful way to build a theory of time-space harmonic polynomials with respect to multivariate Lévy processes.

4 Acknowledgements

We are grateful to the referees for a number of helpful suggestions for improvement in the article.

References

[1] Di Nardo, E., Guarino, G., Senato D. (2011) A new algorithm for computing the multivariate Faà di Bruno’s formula. Appl. Math. Comp., 217, 6286–6295.
[2] Di Nardo, E., Oliva, I. (2009) On the computation of classical, boolean and free cumulants. Appl. Math. Comp., 208, No. 2, 347-354.
[3] Di Nardo, E., Oliva, I. (2011) On some applications of a symbolic representation of non-centered Lévy processes. Comm. Statist. Theory Methods. In press.
[4] Di Nardo, E., Senato, D. (2006) An umbral setting for cumulants and factorial moments. European J. Combin., 27, No. 3, 394–413.
[5] Liu, G. (1998) Higher-order multivariable Euler’s polynomial and higher-order multivariable Bernoulli’s polynomial. Appl. Math. Mech. (English Ed.), 19, No. 9, 895 – 906.
[6] Nörlund, N. E. (1954) Vorlesungen Über Differenzenrechnung, Chelsea, New York.
[7] Rota, G.-C., Taylor, B.D. (1994) The Classical Umbral Calculus. SIAM J. Math. Anal. 25, 694–711.
[8] Sato, K.-I. (1999) Lévy processes and infinitely divisible distributions. Cambridge University Press.
[9] Stanley, R.P. (1997) Enumerative Combinatorics, Vol.1, Cambridge University Press.
[10] Sun, P. (2007) Moment representation of Bernoulli polynomial, Euler polynomial and Gegenbauer polynomials. Statist. Probab. Lett. 77, 748–751.
[11] Withers, C.S. (2000) A simple expression for the multivariate Hermite polynomials. Statist. Probab. Lett. 47, 165–169.