Photo-detection using Bose-condensed atoms in a micro trap

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The quantum theory of photo detection based on the absorption of photons and emission of photo electrons represents one of the cornerstones of quantum optics. It serves to obtain the statistics of emitted photo electrons given the quantum statistics of the incident optical field. Various approximations, as described by this theory, lead eventually to the famous photo-electron statistics of the incident optical field. For the purpose of unveiling the different counting formula of Mandel (author?) [1, 2], a quantum version of a previously known semi-classical Poissonian formula (author?) [3, 4].

As already mentioned, to derive the Mandel formula, some approximations have to be made. These approximations are perfectly justifiable for a solid-state detector device that operates at not too low temperatures. One crucial assumption is the distinguishability of the atoms emitting the observable photo electrons. Another approximation is found to consist in the perturbative calculus used to obtain joint probabilities of photo-electron emissions. Together they lead to the Poissonian operator form, rather independent of the underlying absorption dynamics.

Consider now a device that operates in a rather different regime, that is, it may be cooled down to ultra cold temperatures in order to behave more quantum than a typical solid-state photo detector. For example, let us consider a cloud of magnetically trapped Bose-condensed alkaline atoms (author?) [5, 6]. Atoms can now absorb incident photons to receive part of the photon momentum, giving them sufficient kinetic energy to escape from the trap, to subsequently be detected, for instance by ionization.

As such a system is highly degenerate, the emission centers of escaping atoms, i.e. the condensed atoms themselves, are not distinguishable. Furthermore, a perturbative approach to calculate the emission probabilities is hardly appropriate, as we may deal with Rabi cycles, where atoms absorb and stimulatedly emit photons, thereby returning to the condensate. Thus, the crucial approximations that led to the Mandel formula cannot be applied and thus one may expect a rather different counting formula. Such a counting formula connects the statistics of escaping atoms to the statistics of the incident optical field. For the purpose of unveiling the different counting formula we study in the following a model detector system using a Bose-condensed gas. Although, it serves here merely for demonstrating the differences in the resulting counting formula, we suppose that this system also possibly may be realizable in current experiments.

The paper is organized as follows: In Sec. II the model of the photo detector is introduced. The atom-counting statistics is then derived in Sec. III followed by a discussion of its features in Sec. IV. Finally, a summary and conclusions are given in Sec. V.

II. PHOTO DETECTOR MODEL

A. Mechanism of photo-detection

Let us assume that a cloud of bosonic atoms is magnetically trapped in a micro trap implemented on an atom chip and being cooled well below the condensation temperature, \( T \ll T_c \). We suppose that the trapping potential is highly elongated into one direction and therefore approximate the system as being effectively one dimensional. Furthermore, the atoms shall interact nearly resonant with a collimated light field with incidence parallel to the elongated trap axis, see Fig. II. The overlap of the transverse mode structure of the light with the transverse mode structure of the atomic cloud shall be consid-
The resonant electronic transition of the atoms shall be formed by two levels, the lower level (ground state) being subject to the magnetic trapping, whereas the upper level (excited state) being unaffected by the trap. Thus, if all atoms start from the ground state, those being excited by absorption of an incident photon may leave the trap potential to be detected, e.g. by subsequent ionization. Some of the excited atoms, however, will be de-excited by stimulated emission and therefore will be subject to the trapping potential again. The electronic level scheme including the sidebands generated by the trap potential, and the loss of atoms from the trap is depicted in Fig. 2.

Given the atoms being Bose-condensed at the initial time $t_0$, one may ask for the probability $P_a(t, t_0)$ to observe $a$ atoms escaping from the trap in the time interval $[t_0, t]$, during which light is incident on the atomic probe. After this interval the detector is reset, i.e. all atoms are cooled again into the condensate mode (i.e. the lowest trap level), to start the counting again for an identical time interval $\tau = t - t_0$. Thus $\tau$ will be the analogue to the usual photo-detector integration time and the atom-counting statistics $P_a$ will then be related, in a yet unknown way, to the statistics of the incident optical field.

B. Interaction with the incident optical field

The Hamilton operator of the complete system including the detector and the incident optical field can be decomposed into free and interaction part as

$$\hat{H} = \hat{H}_0 + \hat{V},$$

where the free evolution is governed by

$$\hat{H}_0 = \hat{H}_{em} + \hat{H}_{at},$$

with $\hat{H}_{em}$ being the Hamiltonian of the electromagnetic field and $\hat{H}_{at}$ being the Hamiltonian of the atoms.

The electric field of the incident optical beam can be written as a decomposition of monochromatic modes of wave vector $k$ and frequency $\nu$,

$$\hat{E}(x) = \int dkE_k b_k e^{ikx} + \text{H.a.},$$

where $E_k$ denote the rms vacuum fluctuations of the electric-field modes, $b_k$ and $b_k^\dagger$ are the bosonic photon annihilation and creation operators, respectively. As the field polarization is selected by the resonant atomic transition, only one type of polarization is considered here. Using this expansion the free Hamiltonian of the electromagnetic field thus becomes

$$\hat{H}_{em} = \int dk cb_k^\dagger b_k.$$  

The atomic system is described by the bosonic atom-field operators $\hat{\Phi}_i(x)$, where $i = g, e$ denotes the electronic state, and that satisfy the commutation relations

$$[\hat{\Phi}_i(x), \hat{\Phi}_j(x')] = \delta_{ij} \delta(x - x').$$

The atomic Hamilton operator reads

$$\hat{H}_at = \sum_{i=g,e} \int dx \hat{\Phi}_i^\dagger(x) \left[ -\frac{\hbar^2 \partial^2}{2m} + \delta_{i,e}U(x) \right] \hat{\Phi}_i(x),$$

where the trap potential only acts in the electronic ground state and reads

$$U(x) = \frac{mv^2 x^2}{2} - \hbar \omega_{eg},$$

with $v$ and $\omega_{eg}$ being the trap and electronic-transition frequency, respectively, and $m$ is the atomic mass.

For the purpose of diagonalizing the free atomic Hamiltonian, we define the Schrödinger eigen-modes:

$$\left[ -\frac{\hbar^2 \partial^2}{2m} + \delta_{i,e}U(x) \right] \phi_{i,n}(x) = \hbar \omega_{i,n} \phi_{i,n}(x).$$
Thus \( \phi_{g,n}(x) \) with discrete \( n = 0, 1, \ldots \) and eigen-frequencies

\[
\omega_{g,n} = n \nu - \omega_0, \quad (\omega_0 = \omega_g - \frac{\nu}{2}),
\]

are the harmonic-oscillator eigen-modes corresponding to a trapped atom in its electronic ground state. The modes \( \phi_{e,k}(x) \propto \exp(i k x) \) with continuous \( k \in [-\infty, \infty] \) and eigen-frequencies

\[
\omega_{e,k} = \frac{\hbar k^2}{2m}.
\]

are plane waves corresponding to a free atom in its electronic excited state. These modes form two independent orthonormal sets and obey the standard completeness relations

\[
\sum_n \phi_{e,k}^*(x) \phi_{g,n}(x') = \delta(x - x'),
\]

\[
\int dk \phi_{e,k}^*(x) \phi_{e,k}(x') = \delta(x - x').
\]

Each electronic component of the quantized atomic field can now be expanded as

\[
\hat{\Phi}_e(x) = \sum_n \hat{g}_n \phi_{g,n}(x), \quad \hat{\Phi}_g(x) = \int dk \hat{e}_k \phi_{e,k}(x),
\]

where the operators \( \hat{g}_n \) and \( \hat{e}_k \), each satisfy again the bosonic commutation relations,

\[
[\hat{g}_n, \hat{g}^\dagger_m] = \delta_{nm}, \quad [\hat{e}_k, \hat{e}^\dagger_{k'}] = \delta(k - k').
\]

Using the expansion \( (14) \) the free atomic Hamiltonian \( (7) \) reduces to

\[
\hat{H}_{af} = \sum_n \hbar \omega_{g,n} \hat{g}^\dagger_n \hat{g}_n + \int dk \hbar \omega_{e,k} \hat{e}^\dagger_k \hat{e}_k.
\]

The interaction between the atoms and the optical field reads in dipole approximation

\[
\hat{V} = -d \kappa_\perp \int dk \sum_{i,j} \hat{\Phi}_i^\dagger(x) \hat{E}_j(x) \hat{\Phi}_j(x),
\]

where \( d \) is the transition dipole moment of the atoms and \( \kappa_\perp \) is the matching between the transverse modes of electromagnetic and atomic field. Using the expansions of electric and atomic fields, cf. Eqs \( (4) \) and \( (14) \), respectively, this interaction can be rewritten in optical rotating-wave approximation as

\[
\hat{V} = -\int dk' \int d\nu \kappa_\perp \hat{E}_k^\dagger \hat{g}_n \hat{b}_k \phi_{e,n}(k' - k) + \text{H.a.},
\]

where the (vacuum) Rabi frequency has been defined as

\[
\Omega_k = \frac{d \kappa_\perp}{\hbar},
\]

and the Fourier transforms of the trap modes are defined as

\[
\phi_{e,n}(k) = \frac{1}{\sqrt{2\pi}} \int dx e^{-ikx} \phi_{e,n}(x).
\]

Thus the absorption of a photon of wave vector \( k \) transforms a ground-state atom in trap level \( n \) into an excited-state atom with a superposition of wave vectors \( k' \) given by \( \phi_{e,n}(k' - k) \). We may therefore define the annihilation operator of an excited wave packet, created from trap level \( n \) by absorption of a photon of wave vector \( k \):

\[
\hat{e}_{k,n} = \int dk' \hat{e}_{k'} \phi_{e,n}^\dagger(k' - k).
\]

This relation can be inverted to obtain all operators \( \hat{e}_k \) from the set of operators \( \hat{e}_{k_0,n} \) \( (n = 0, 1, 2, \ldots) \) for a specific wave vector \( k_0 \):

\[
\hat{e}_k = \sum_{n=0}^{\infty} \phi_{e,n}^\dagger(k - k_0) \hat{e}_{k_0,n}.
\]

The excited-wavepacket operators satisfy the following commutation relations

\[
[\hat{e}_{k,n}, \hat{e}_{k',n'}^\dagger] = \int dx \phi_{e,n}^\dagger(x) e^{i(k' - k)x} \phi_{e,n'}(x),
\]

and in particular \( [\hat{e}_{k,n}, \hat{e}_{k,n}^\dagger] = 1 \). Using these wave packet operators, the interaction Hamiltonian can be simplified to

\[
\hat{V} = -\sum_n \int d\nu \kappa_\perp \hat{E}_k^\dagger \hat{g}_n \hat{b}_k + \text{H.a.}
\]

C. Single-mode approximation

If we start from a Bose-condensed gas with all atoms being in the lowest trap level \( n = 0 \), a cycled electronic transition will preferably lead again to the lowest trap level by bosonic enhancement. We may thus approximate the ground-state levels by a single mode, corresponding to the lowest trap level and may simplify the interaction Hamiltonian \( (22) \) to

\[
\hat{V} = -\int d\nu \kappa_\perp \hat{E}_k^\dagger \hat{b}_0 \hat{b}_k + \text{H.a.}
\]

Furthermore, we suppose that the incident optical field is quasi monochromatic with wave vector \( k_0 \), so that only the photon operator \( \hat{b}_{k_0} \) has to be kept. Using thus the definitions \( \hat{b}_0 \rightarrow \hat{b} \), \( \hat{g}_0 \rightarrow \hat{g} \), \( \hat{e}_{k_0,0} \rightarrow \hat{e} \), and \( \Omega_{k_0} \rightarrow \Omega \) the interaction further simplifies to

\[
\hat{V} = -\hbar \Omega \hat{e}^\dagger \hat{g} \hat{b} + \text{H.a.}
\]

The free atomic Hamiltonian, on the other hand, can be written in this single-mode approximation as

\[
\hat{H}_{af} = \hbar \omega_0 \hat{e}^\dagger \hat{e} - \hbar \omega_0 \hat{g}^\dagger \hat{g}
\]

with the average frequency of the excited wave packet being determined by the wave vector of the absorbed photon and the momentum spread of the ground-state trap level:

\[
\overline{\omega}_e = \int dk \omega_{g,k} |\phi_{e,0}(k - k_0)|^2 = \frac{\nu}{4} + \frac{\hbar k_0^2}{2m}.
\]
We note that the performed single-mode approximation neglects the dispersion of the excited wave packet, being now considered as a propagating plane wave.

The effective transition frequency between the relevant two atomic levels becomes now

$$\omega_{e'g} = \omega_e + \omega_g = \omega_g + \frac{\hbar k_0^2}{2m} - \frac{\nu}{4}. \quad (29)$$

In resonance this transition frequency is compensated for by the frequency $c k_0$ of the optical field, from which we obtain the resonance condition

$$c k_0 \approx \omega_g - \frac{\nu}{4}. \quad (30)$$

**D. Loss mechanism**

The spatio-temporal mode of the excited wave-packet in the single-mode approximation is obtained as

$$\phi_e(x,t) = e^{i(k_0 x - \omega_e t)} \phi_{x,0}(x). \quad (31)$$

It corresponds to a motion into the direction of the wave vector of the previously absorbed photon with the group velocity $v_g = \omega_e / k_0$ given by

$$v_g = \frac{\hbar k_0}{2m} \left[ 1 + (2\eta)^{-2} \right], \quad (32)$$

where $\eta = k_0 \delta x_0$ is the Lamb–Dicke parameter with $\delta x_0 = \sqrt{\hbar / (2m\nu)}$ being the rms position spread of the trap ground level. For a typical magnetic trap potential the weak binding regime applies, where the Lamb–Dicke parameter is $\eta \gg 1$. Thus the conservation of momentum is approximately granted and the excited atom compensates for the momentum of the absorbed photon.

If the excited wave packet has moved over a distance $\sim \delta x_0$ it may no longer be recycled into the electronic ground state by stimulated emission of a photon, as the corresponding spatial overlap will be close to zero. It thus has escaped from the trap. The time of flight for this to happen is given by $\tau = \delta x_0 / v_g$ and the corresponding rate for this to happen, $\gamma = 2\pi \tau^{-1}$, is therefore obtained as

$$\gamma = \frac{\hbar k_0^2}{m \eta} \left[ 1 + (2\eta)^{-2} \right] \approx \frac{\pi \hbar k_0^2}{m \eta}. \quad (33)$$

The escape of atoms and their subsequent detection may be modeled as an incoherent loss of atoms, described by the master equation

$$\partial_t \hat{\rho} = \frac{1}{i\hbar} [\hat{H}, \hat{\rho}] + \partial_t \hat{\rho}_{esc}, \quad (34)$$

where the atom loss is modeled by the Lindblad-form part

$$\partial_t \hat{\rho}_{esc} = \gamma \left( \hat{\rho} \hat{\rho}^\dagger - \frac{1}{2} \{ \hat{\rho}^\dagger \hat{\rho}, \hat{\rho} \} \right). \quad (35)$$

The master equation (34) can be written in the form

$$\partial_t \hat{\rho} = \frac{1}{i\hbar} \left( \hat{H}_{\text{eff}} \hat{\rho} - \hat{\rho} \hat{H}_{\text{eff}}^\dagger \right) + \gamma \hat{\rho} \hat{\rho}^\dagger, \quad (36)$$

where according to Eqs (5), (26), and (27) the non-Hermitian effective Hamiltonian reads

$$\hat{H}_{\text{eff}} = \hbar c k_0 \hat{b}^\dagger \hat{b} + \hbar c \omega \hat{\rho}^\dagger \hat{\rho} - \hbar \hbar \delta \hat{\rho}^\dagger \hat{\rho} - \frac{i\hbar \gamma}{2} \hat{\rho} \hat{\rho}^\dagger, \quad (37)$$

with $\Omega = |\Omega| e^{i\phi}$.

**E. Atomic pseudo spin**

For the atomic quantum fields we may define the pseudo spin operator $\hat{\mathbf{S}}$ with components $(\hat{S}_z = \hat{S}_x = i \hat{S}_y)$

$$\hat{S}_z = \hat{c}^\dagger \hat{g} \rho \hat{c}, \quad \hat{S}_+ = (\hat{S}_z)^\dagger, \quad (38)$$

$$\hat{S}_z = \frac{1}{2} (\hat{c}^\dagger \hat{c} - \hat{g}^\dagger \hat{g}), \quad (39)$$

and

$$\hat{S}^2 = \hat{A} \left( \frac{\hat{A}}{2} + 1 \right), \quad (40)$$

where the atom-number operator is defined as

$$\hat{A} = \hat{c}^\dagger \hat{c} + \hat{g}^\dagger \hat{g}. \quad (41)$$

The operators $\hat{S}_{z,\pm}$ satisfy the standard $\mathfrak{su}(2)$ commutation relations, $[\hat{S}_+, \hat{S}_-] = 2 \hat{S}_z$, and $[\hat{S}_z, \hat{S}_\pm] = \pm \hat{S}_\pm$. The total number of excitations – excited atoms plus photons – is given by the operator

$$\hat{N} = \hat{c}^\dagger \hat{c} + \hat{g}^\dagger \hat{g} = \hat{S}_z + \frac{\hat{A}}{2} + \hat{b}^\dagger \hat{b}. \quad (42)$$

The atomic system can now be described in the basis of Dicke states ([author?] [38])

$$|A, A_e\rangle_{\text{at}} = |A - A_e\rangle_{\text{g}} \otimes |A_e\rangle_{\text{e}}, \quad (43)$$

where $A$ is the total number of atoms, i.e. $A|A, A_e\rangle_{\text{at}} = A|A, A_e\rangle_{\text{at}}$, and $A_e$ is the number of excited atoms. The corresponding basis states for the total system can then be written as

$$|A, N, n\rangle = |A, N - n\rangle_{\text{at}} \otimes |n\rangle_{\text{em}}, \quad (44)$$

where $N$ is the total number of excitations, i.e. $\hat{N}|A, N, n\rangle = N|A, N, n\rangle$, and $|n\rangle_{\text{em}}$ is a photon-number state.

Using the definitions of the spin operators (38), (42), the effective Hamiltonian (37) can be rewritten as

$$\hat{H}_{\text{eff}} = \hat{H}_0 - \hbar \Delta \hat{S}_z - \frac{i\hbar \gamma}{4} \hat{A} - \hbar \Omega |\hat{S}_z \hat{b} + \hat{S}_- \hat{b}^\dagger|, \quad (45)$$
where for notational simplicity we defined $\Delta' = \Delta + i\gamma/2$ with $\Delta = ck_0 - \omega x_0$, being the detuning from resonance, and the free Hamiltonian is identified as
\[
\hat{H}_0 = \hbar c k_0 \hat{p} + \frac{\hbar (\omega x_0 - \omega - \Delta')}{2} \hat{A}.
\] (46)

As this free part commutes with the remainder of the effective Hamiltonian, we may transform into the interaction picture with respect to $\hat{H}_0$, to obtain the master equation in the interaction picture
\[
\partial_t \hat{\rho} = \frac{i}{\hbar} \left[ \hat{H}_\text{eff}, \hat{\rho} \right] + \gamma \hat{\rho} \hat{e} \hat{e}^\dagger,
\] (47)
where the transformed effective Hamiltonian becomes
\[
\hat{H}_\text{eff} = -\hbar \Delta' \hat{S}_z - \frac{\hbar \gamma}{4} \hat{\Delta} - \hbar |\Omega| (\hat{S}_+, \hat{b} + \hat{S}_- \hat{b}^\dagger).
\] (48)

For notational convenience we omitted here any indication of being in the interaction picture.

In the master equation (47), the last term describes the escape of an excited-wavepacket atom from the trap. The responsible operator $\hat{e}$, that annihilates one such excited atom from the system, can be written in the basis of the states (44) as
\[
\hat{e} = \sum_{A,N,n} \sqrt{N-n-1} |A-1, N-1, n\rangle \langle A, N, n|.
\] (49)

F. Limit of large number of atoms

Let us now consider the action of the operators appearing in the effective Hamiltonian (48) on the basis states (44). Firstly, the actions of the operators $\hat{S}_+, \hat{b}$ and $\hat{S}_z$ on a state $|A, N, n\rangle$ are:
\[
\hat{S}_+ |A, N, n\rangle = \sqrt{(N-n+1)(A-N-n)n} |A, N, n+1\rangle,
\] (50)
\[
\hat{S}_- |A, N, n\rangle = (N-n-1) |A, N, n-1\rangle,
\] (51)
\[
\hat{S}_z |A, N, n\rangle = (N-\frac{A}{2}) |A, N, n\rangle.
\] (52)

Defining a new spin operator $\hat{L}$ with its components being defined by the actions
\[
\hat{L}_+ |A, N, n\rangle = \sqrt{(N-n)(n+1)} |A, N, n+1\rangle,
\] (52)
\[
\hat{L}_- |A, N, n\rangle = \sqrt{(N-n+1)n} |A, N, n-1\rangle,
\] (53)
\[
\hat{L}_z |A, N, n\rangle = (n-\frac{N}{2}) |A, N, n\rangle,
\] (54)
and obeying the usual angular-momentum commutation relations, Eqs (50) and (51) can be rewritten as
\[
\hat{S}_+ \hat{b} |A, N, n\rangle = \hat{L}_- \left( \hat{A} - \hat{N}/2 - \hat{L}_z \right) \frac{1}{2} |A, N, n\rangle,
\] (55)
\[
\hat{S}_- |A, N, n\rangle = \left( \hat{N} - \hat{A}/2 - \hat{L}_z \right) |A, N, n\rangle.
\] (56)

Thus accordingly we may replace the operators in the effective Hamiltonian (48) to obtain
\[
\hat{H}_\text{eff} = -\hbar \Delta' \left( \frac{\hat{N}}{2} - \hat{L}_z \right) + \hbar \Delta \hat{A}/2
\]
\[
-\hbar |\Omega| \left[ \hat{L}_- \sqrt{\hat{M} + \hat{L}_z - \frac{1}{2}} + \text{h.a.} \right].
\] (57)

where the operator
\[
\hat{M} = \hat{A} - \frac{\hat{N}}{2} + \frac{1}{2},
\] (58)

has been introduced.

For a proper functioning of the detector we assume that the number of atoms in the gas is much larger than the maximum number of photons of the incident optical field. Thus the occupied eigenvalues of the operator $\hat{M}$ are very large and consequently we may perform an expansion over a small parameter being proportional to the inverse atom number (author?).

Thus the interaction part of the effective Hamiltonian is expanded as
\[
\hat{L}_- \sqrt{\hat{M} + \hat{L}_z - \frac{1}{2}} = \hat{L}_- \sqrt{\hat{M}} \left( 1 + \frac{\hat{L}_z - \frac{1}{2}}{2\hat{M}} + \ldots \right),
\] (59)

where the expansion parameter $(\hat{L}_z - \frac{1}{2})/\hat{M}$ is chosen to cancel the first-order contribution in (57), obtaining the zero-order Hamiltonian as
\[
\hat{H}_\text{eff}(t) \approx -\hbar \Delta' \left( \frac{\hat{N}}{2} - \hat{L}_z \right) + \hbar \Delta \frac{\hat{A}}{2} - 2\hbar |\Omega| \sqrt{\hat{M}} \hat{L}_z.
\] (60)

III. ATOM-COUNTING STATISTICS

A. Counting statistics

From the solution of the master equation we need to extract the probability for $a$ atoms having escaped in the time interval $[t_0, t]$, starting at the initial time $t_0$ with a perfect Bose-condensed gas with $A$ atoms in the trap ground state. Thus the initial state at time $t_0$ can be written in the form
\[
\hat{\rho}(t_0) = |A, 0\rangle \langle A, 0| \otimes \hat{\rho}_\text{em}(t_0),
\] (61)
where $\hat{\rho}_\text{em}(t_0)$ is the density operator of the incident optical field at the initial time $t_0$. The latter may be expanded in photon-number states as
\[
\hat{\rho}_\text{em}(t_0) = \sum_{n,n'} \rho_{n,n'}(t_0) |n\rangle \langle n'| \langle n'| \langle n',
\] (62)
so that the complete initial density operator can be written in the basis states (44) as
\[
\hat{\rho}(t_0) = \sum_{n,n'} \rho_{n,n'}(t_0) |A, n\rangle \langle A, n| \langle A, n' | \langle n',
\] (63)
Given the initial density operator \( \rho(0) \), the formal solution of the density operator at time \( t \geq t_0 \) can be obtained from the master equation in the form

\[
\rho(t, t_0) = \sum_{a=0}^{\infty} \rho_a(t, t_0), \tag{64}
\]

where \( \rho_a(t, t_0) \) is the (unnormalized) conditional density operator corresponding to the history of the detector system where in total \( a \) atoms have escaped in the time interval \([t_0, t]\). The norm of this conditional density operator is the probability for this history to occur, which is the desired probability to count \( a \) atoms escaping from the trap:

\[
P_a(t, t_0) = \text{Tr}[\rho_a(t, t_0)]. \tag{65}
\]

The conditional density operator \( \rho_a(t, t_0) \) is itself a sum of all possible histories where \( a \) atoms escape in such a way, that the \( j \)th atom escapes at time \( t_j \in [t_0, t] \), where \( j = 1, \ldots, a \),

\[
\rho_a(t, t_0) = \int_{t_0}^{t_1} dt_a \cdots \int_{t_0}^{t_a} dt_j \rho_a(t_0; t_{a}, \ldots, t_j). \tag{66}
\]

It is a non-unitary evolution \( \mathcal{N} \) intermittent by so-called jump operators \( \mathcal{J} \) that describe the escape of a single atom from the trap. The super operator of the non-unitary evolution is defined as

\[
\mathcal{N}(t) \rho = \mathcal{U}_{\text{eff}}(t) \rho \mathcal{U}_{\text{eff}}^\dagger(t), \tag{70}
\]

the effective non-unitary evolution operator being

\[
\mathcal{U}_{\text{eff}}(t) = \exp \left( -\frac{it}{\hbar} \mathcal{H}_{\text{eff}} \right). \tag{71}
\]

The norm of the conditional density operator on the rhs is the joint probability density for \( a \) atoms to escape from the trap at times \( t_1, \ldots, t_a \):

\[
p_a(t, t_0; t_1, \ldots, t_a) = \text{Tr}[\rho_a(t, t_0; t_{a}, \ldots, t_1)]. \tag{67}
\]

Thus the required counting statistics is obtained as

\[
P_a(t, t_0) = \int_{t_0}^{t} dt_a \cdots \int_{t_0}^{t_a} dt_j p_a(t_0; t_{a}, \ldots, t_j). \tag{68}
\]

**B. Quantum trajectories**

Given the initial density operator \( \rho(0) \), the conditional density operator \( \rho_a(t_0; t_{a}, \ldots, t_j) \) is given by the quantum trajectory \( \{11, 12, 13\} \)

\[
\rho_a(t_0; t_{a}, \ldots, t_j) = \mathcal{N}(t - t_a) \mathcal{J} \mathcal{N}(t_a - t_{a-1}) \mathcal{J} \cdots \mathcal{J} \mathcal{N}(t_1 - t_0) \rho(t_0). \tag{69}
\]

and the escape of an atom is described by the super operator

\[
\mathcal{J} \rho = \gamma \rho \mathcal{J}. \tag{72}
\]

Thus the joint probability density \( \{67\} \) can be written using Eqs \( \{69\}, \{72\} \) and the initial state \( \{63\} \) as

\[
p_a(t, t_0; t_{a}, \ldots, t_j) = \sum_{n,n'} \rho_{n,n'}(t_0) \langle \Phi(t_0; t_{a}, \ldots, t_j | n') | \Phi(t_0; t_{a}, \ldots, t_j | n) \rangle, \tag{73}
\]

where the (unnormalized) quantum-trajectory states starting with the initial state \( |A, n, n\rangle \) are

\[
|\Phi(t_0; t_{a}, \ldots, t_j | n) \rangle = \gamma^{j/2} \mathcal{U}_{\text{eff}}(t - t_a) \mathcal{J} \cdots \mathcal{J} \mathcal{U}_{\text{eff}}(t_1 - t_0) |A, n, n\rangle. \tag{74}
\]

Using the representation of the operator \( \mathcal{J} \) in the basis states, Eq. \( \{49\} \), this state vector can be rewritten as

\[
|\Phi(t_0; t_{a}, \ldots, t_j | n) \rangle = \sum_{n_1, \ldots, n_j} \mathcal{U}_{\text{eff}}(t - t_a) |A - a, n - a, a,n\rangle \\
\times \Psi_{n_a,n_{a-1}}^{a+1,n-a+1}(t_a - t_{a-1}) \cdots \Psi_{n_2,n_1}^{a-1,n-1}(t_2 - t_1) \Psi_{n_1,n}^{A,n}(t_1 - t_0), \tag{75}
\]

where the transition amplitudes are

\[
\Psi_{m,n}^{A,N}(t) = \sqrt{\gamma(N - m)} |A, N, m \rangle \langle \mathcal{U}_{\text{eff}}(t) |A, N, n \rangle. \tag{76}
\]

Here we made use of the fact that the effective Hamiltonian preserves both the atom number and the total number of ex-
citations. As the trajectory \((A)\) has exactly \(A - a\) atoms and \(n - a\) remaining excitations, in the sum of Eq. \((A)\) only terms with \(n = n'\) contribute:

\[
p_a(t, t_0; t_2, \ldots, t_1) = \sum_n P_n p_a(t, t_0; t_2, \ldots, t_1 | n),
\]

where \(P_n = p_{a,n}(t_0)\) is the initial photon statistics and the probability density conditioned on initially \(n\) photons is defined as

\[
p_a(t, t_0; t_2, \ldots, t_1 | n) = \left\| \Phi(t, t_0; t_2, \ldots, t_1 | n) \right\|^2.
\]

Thus the atom-counting statistics can be written as

\[
P_n(t, t_0) = \sum_n P_a(t, t_0 | n) P_n,
\]

where the conditional probability for \(a\) atoms to escape in the time interval \([t_0, t]\) given that \(n\) photons are present is

\[
P_a(t, t_0 | n) = \int_{t_0}^t dt_a \ldots \int_{t_0}^{t_2} dt_1 p_a(t, t_0; t_2, \ldots, t_1 | n).
\]

C. Over damped resonant regime

The features of the dynamics of the absorption and stimulated emission of photons and the loss of atoms from the trap, given a state with \(A\) atoms and \(N\) excitations, depends on the saturation parameter (cf. App. \(A\))

\[
S_{A,N} = \frac{4|\Omega|^2 M_{A,N}}{\Delta^2 + (\gamma/2)^2},
\]

where \(M_{A,N}|A, N, n\rangle = M_{A,N}|A, N, n\rangle\) with \(M_{A,N} = A - (N - 1)/2\). Given perfect resonance of the incident monochromatic light field, \(\Delta = 0\), for \(S_{A,N} > 1\) Rabi oscillations occur that involve cycles of absorption and (stimulated) re-emission of photons until an atom is lost from the trap.

In the opposite over damped case, \(S_{A,N} < 1\), however, no cycling transition is observed but photons are absorbed and their excitation is removed from the system by an atom leaving the trap. In other words, the recoil energy is much larger than the effective coupling energy of the atoms with the photon field,

\[
\frac{(\hbar \kappa_0)^2}{2m} > \frac{\hbar}{\pi} [2h|\Omega| \sqrt{M_{A,N}}].
\]

We recall that in our case the Lamb–Dicke parameter \(\eta > 1\).

Thus the number of absorbed photons equals approximately the number of lost atoms, which means that only the transition amplitudes

\[
\Psi_{n-1,n}^{A,n}(t) = \sqrt{\eta(n - m)}|A, n, n - 1| \hat{U}_{eff}(t)|A, n, n\rangle
\]

have to be considered, since only they start from initial states with all excitation being photonic. From Eq. \((A)\) it becomes then clear that among these transition amplitudes we may further consider only those where one photon has been absorbed, i.e.,

\[
\Psi_{n-1,n}^{A,n}(t) = \sqrt{\eta(n, n - 1 - 1)| \hat{U}_{eff}(t)|A, n, n\rangle}.
\]

However, as these are approximations we must re-normalize correctly the transition amplitudes to obtain statistically correct quantum trajectories. Originally the normalization read

\[
\sum_{m=0}^N \int_0^\infty dt |\Psi_{m,n}^{A,n}(t)|^2 = 1,
\]

meaning that starting from a state \(|A, N, n\rangle\) the system eventually will end up with certainty in one of the states \(|A, N, m\rangle\) with \(m = 0, \ldots, N\). Taking into account now only the transition amplitudes \((A)\), we must use the re-normalized transition amplitude

\[
\Psi_{n-1,n}^{A,n}(t) \rightarrow \Psi_{n-1,n}^{A,n}(t) = \Psi_{n-1,n}^{A,n}(t)/\sqrt{P_{n-1,n}^{A,n}}
\]

where the probability for the considered transition is defined as

\[
P_{n-1,n}^{A,n} = \int_0^\infty dt |\Psi_{n-1,n}^{A,n}(t)|^2 \simeq 1.
\]

In this way we obtain an atom waiting-time distribution

\[
w_{A,n}(t) = |\Psi_{A,n}(t)|^2,
\]

that is properly normalized:

\[
\int_0^\infty dt w_{A,n}(t) = 1.
\]

The quantum trajectory in the over damped regime is thus simplified to

\[
|\Phi(t, t_0; t_2, \ldots, t_1 | n)\rangle = \hat{U}_{eff}(t - t_a)|A - a, n - a, n - a\rangle
\]

\[
\times \prod_{k=0}^{a-1} \Psi_{A-k,n-k}(t_k + 1 - t_k),
\]

so that the joint probability density \((A)\) becomes

\[
p_a(t, t_0; t_2, \ldots, t_1 | n) = W_{A-n-a}(t - t_a)
\]

\[
\times \prod_{k=0}^{a-1} W_{A-k,n-k}(t_k + 1 - t_k),
\]

where

\[
W_{A,n}(t) = \langle A, n, n| \hat{U}_{eff}(t)|A, n, n\rangle
\]

is the probability that no atom leaves the trap within the time interval \(t\) starting from the state \(|A, n, n\rangle\).

When approximating the transition amplitudes, cf. Eq. \((A)\), to obtain statistically correct quantum trajectories that lead to a normalized atom-count statistics \(P_a\), also the probability \((A)\) must be consistently approximated. This can be done by using the general relation

\[
W_{A,n}(t) = 1 - \int_0^t dt' w_{A,n}(t'),
\]

employing on the rhs the approximation \((A)\).
Using the above results, the conditional probability for \(a\) atoms to leave the trap given that \(n\) photons are present, Eq. \(60\), becomes now

\[
P_a(t, t_0|n) = \int_{t_0}^{t} dt_1 \ldots \int_{t_0}^{t_{L}} dt_{L} \int_{t_0}^{t_0} dt_1 W_{A-a,n-a}(t-t_0) \times \prod_{k=0}^{a-1} w_{A-k,n-k}(t_{k+1}-t_k).
\]

(94)

This convolution integral is expressed as the inverse Laplace transform

\[
P_a(\tau|n) = \mathcal{L}^{-1}\left\{ \frac{W_{A-a,n-a}(z)}{z} \prod_{k=0}^{a-1} w_{A-k,n-k}(z) \right\},
\]

(95)

where \(\tau = t - t_0\).

In the resonant case \((\Delta = 0)\), low saturation \((S_A \ll 1)\), and assuming numbers of photons much lower than the atom number, \(n \ll A\), we obtain the atom waiting-time distribution as [cf. Eq. (A2)]

\[
w_{A,n}(t) \propto \sinh^{2} \frac{\gamma t}{2} \sqrt{1-S_A} e^{-\frac{1}{2}(n-(n-1))\sqrt{1-S_A}},
\]

(96)

where the saturation parameter \(A1\) has been approximated by \(S_A = 4\Omega^2 A/(\gamma/2)^2\). This form shows a behavior quite similar to over damped Rabi oscillations of a two-level system interacting with a resonant laser field, with the saturation being now dependent on the number of atoms. The individual Laplace transforms become then

\[
w_{A-k,n-k}(z) = \frac{(n-k)(n-k+q)(n-k+2q)}{(n-k+z_A)(n-k+z_A+q)(n-k+z_A+2q)},
\]

(97)

where we defined

\[
q = \frac{\sqrt{1-S_A}}{1-\sqrt{1-S_A}}, \quad \tau_0^{-1} = \frac{\gamma}{2} \left(1 - \sqrt{1-S_A}\right).
\]

(98)

As we deal with low saturation, \(S_A \ll 1\), the introduced parameters behave as \(q \gg 1\) and \(\tau_0^{-1} \approx \gamma S_A/4\).

Thus the product of transformed waiting-time distributions in Eq. \(95\) becomes

\[
\prod_{k=0}^{a-1} w_{A-k,n-k}(z) = \frac{(n)_a (n+q)_a (n+2q)_a}{(n+q+z_A)_a (n+q+2z_A)_a (n+2q+3z_A)_a},
\]

(99)

where the binomial coefficients are defined by means of the Gamma function,

\[
\Gamma(x) = \frac{\Gamma(x+1)}{\Gamma(x+1)} = \frac{\Gamma(x+1)}{\Gamma(x+1)}.
\]

(100)

The Laplace transform of the no-escape probability is according to Eq. \(93\)

\[
W_{A-a,n-a}(z) = \frac{\tau_0}{n-a+z_A z}
\]

(102)

which can be written as the sum

\[
W_{A-a,n-a}(z) = \frac{\tau_0}{n-a+z_A z}
\]

(103)

\[
+ \frac{\tau_0}{(n-a+z_A z)(n-a+z_A z+q)} + \frac{\tau_0}{(n-a+z_A z)(n-a+z_A z+q)}.
\]

(104)

Thus the complete conditional probability \(25\) becomes the sum of the three inverse Laplace transforms

\[
P_a(\tau|n) = \tau_0 e^{-\pi/\tau_0} \mathcal{L}^{-1}\left\{ \frac{\tau_0}{a+1} \right\} + \left(\frac{\tau_0}{a+1}\right)^2.
\]

(105)

Each Laplace transform results as a Meijer G function [author? \(14\)].

\[
P_a(q, p|n) = \frac{(a+1)^2}{n+1} \left(\frac{n+2q}{a+1} \right) \left(1-p\right)^n \left(\frac{n+q}{a} \right) G^{30}_{33} \left(1-p, 1-q+1, 2q+1, 1-q-a+1, 2q-a+1\right)
\]

(106)

where we have introduced the parameter

\[
p = 1 - e^{-\tau/\tau_0}.
\]

(107)

**IV. DISCUSSION**

The effective rate at which an atom leaves the trap is \(\tau_0^{-1} \approx 4|\Omega^2 A/\gamma\) so that the introduced parameter \(p\), cf. Eq. \(105\).

\[
\frac{(a+1)^2}{n+1} \left(\frac{n+2q}{a+1} \right) \left(1-p\right)^n \left(\frac{n+q}{a} \right) G^{30}_{33} \left(1-p, 1-q+1, 2q+1, 1-q-a+1, 2q-a+1\right).
\]

(108)
Thus, only the fraction \( \eta_D / p \) on the saturation-dependent parameter \( q \) for \( n = 1 \) (solid), \( n = 10 \) (dashed), and \( n = 20 \) (dotted) photons. Atom-escape probabilities are \( p = 0.9 \) (a) and \( p = 0.6 \) (b).

Figure 3: Dependence of the scaled quantum efficiency \( \eta_D / p \) on the saturation-dependent parameter \( q \) for \( n = 1 \) (solid), \( n = 10 \) (dashed), and \( n = 20 \) (dotted) photons. Atom-escape probabilities are \( p = 0.9 \) (a) and \( p = 0.6 \) (b).

can be identified as the probability for an atom to escape from the trap by the absorption of a photon. However, there is additionally the large saturation-dependent parameter \( q \simeq 2/S_A \), cf. Eq. (93), that may change the shape of the conditional probability \( (104) \). It is thus not obvious how these two parameters can be merged into a possibly existing single parameter, such as the quantum efficiency.

However, such a quantum efficiency may be defined phenomenologically, demanding that the average atom count conditioned on the presence of \( n \) photons reads

\[
\bar{n} = \eta_D n.
\]

Thus, only the fraction \( \eta_D \) of the \( n \) photons leads on average to \( \bar{n} \) escaping atoms. In general, the above relation is not necessarily linear, as the efficiency may be a function of the photon number, revealing a nonlinear relation between incoming photon and escaping atom numbers. For our specific case, such a phenomenological quantum efficiency additionally depends on the atom-escape probability \( p \) and the saturation-dependent coefficient \( q \), thus we obtain

\[
\eta_D = \eta_D(q, p, n).
\]

In Fig. 3 (a) this dependence is shown for an atom-escape probability \( p = 0.9 \) in dependence of the parameter \( q \) for \( n = 1 \) (solid), \( n = 10 \) (dashed), and \( n = 20 \) (dotted) photons. It can be observed that for large values of \( q \) all these curves converge to the value of the atom-escape probability, i.e.

\[
\lim_{q \to \infty} \eta_D(q, p, n) = p.
\]

The same behavior is also observed for a lower escape probability \( p = 0.6 \), see Fig. 3 (b). Thus for sufficiently low saturation \( S_A \), i.e. for sufficiently large \( q \), a linear regime is attained, where the quantum efficiency becomes independent of the incident photon number \( n \) and coincides with the atom-escape probability \( p \).

Does the statistics become then identical to that known from the Mandel counting formula? To answer this question, we proceed as follows: Given that by use of Eq. (106) we may identify a quantum efficiency \( \eta_D \) for each conditional atom-count statistics \( P_a(q, p, n) \), we may compare it with the corresponding conditional count-statistics that would correspond to the Mandel formula of photo-detection (11). The latter is given by the Binomial statistics,

\[
P_a^{(M)}(\eta_D | n) = \binom{n}{a} \eta_D^a (1 - \eta_D)^{n-a}.
\]

In Fig. 4 (a) the atom-count statistics is shown for an incident light field with \( n = 10 \) photons, with an atom-escape probability \( p = 0.9 \) and \( q = 100 \). From part (c) it can be seen that the deviation from the Binomial statistics of corresponding quantum efficiency \( \eta_D = 0.8862 \) is not too large. However, this changes if the saturation-dependent coefficient is lowered to \( q = 10 \), see the atom-count statistics shown in Fig. 4 (b). Now the deviation from a Binomial statistics with corresponding quantum efficiency \( \eta_D = 0.7730 \) becomes rather large, cf.
It should be noted that the detection efficiency $\eta_D$ determines the mapping photon to atom numbers to all orders of moments for a conventional photodetector. In our case, however, it only connects the mean photon and atom-count numbers. The shape and width of the atom-count statistics additionally depends on a second parameter, e.g. the saturation, which allows for a difference from the Binomial statistics.

The present model assumes an ideal Bose gas in the limit of zero temperature. However, the inclusion of atomic collisions can be consistently made by identifying the lowest trap mode as the solution of the Gross–Pitaevskii equation. In Thomas–Fermi approximation the only modification will be then the increased (decreased) size of the ground-state mode for repulsive (attractive) scattering, which will modify the time of flight of excited atoms to escape from the trap. Thus repulsive (attractive) scattering will decrease (increase) the atom-loss rate $\gamma$.

The treatment of finite temperature (but below the condensation temperature) requires the departure from the single-mode approximation. However, for low saturation as discussed here, the inclusion of non-condensed modes will not lead to a mixing of condensed and non-condensed atoms by photon absorption: Photon absorption will excite both initially condensed as well as non-condensed atoms to immediately leave the trap. Thus they do not return to the electronic ground state, preventing the mixing. The small fraction of non-condensed atoms at finite temperature will represent distinguishable emission centers that will contribute as a small Binomial admixture to the atom-count statistics. With increasing temperature this admixture will deteriorate the enhancement of photon-number resolution made by the condensed atoms, as we expect from the transition beyond the condensation temperature to a gas of distinguishable atoms.

### V. SUMMARY AND CONCLUSIONS

In summary we have introduced a model for a photodetector using a Bose-condensed gas trapped on an atom chip. By use of an atomic pseudo-spin approximation based on large atom numbers, the photon absorption and atom-escape dynamics could be approximated for the over damped case where Rabi oscillations are suppressed. A quantum-trajectory method then led us to a counting formula involving a sum of three Meijer G functions.

The conditional count statistics has been shown to depend on a saturation-dependent coefficient $q$ and an atom-escape probability $p$, to which the quantum efficiency of the detector converges in the limit of extremely low saturation, $q \to 0$. In this limit the differences from a Binomial statistics vanish so that our result coincides with the Mandel counting formula. However, for realistic values of the saturation substantial deviations can be observed as a reduced atom-count fluctuations.

An open question is yet the behavior of our model detector for saturation $S_A \geq 1$, where full Rabi oscillations start to occur. We assume that in this regime we may expect even larger and dramatic deviations from the Mandel formula. Then coherent processes must be included, that describe the tempo-

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**Figure 5:** Conditional atom-count statistics $P_n(q, p; n)$ for $n = 10$ photons, $p = 0.6$, $q = 100$ (a) and $q = 10$ (b). The lower part shows the deviation from the corresponding Binomial statistics with quantum efficiency $\eta_D = 0.5639$ for $q = 100$ (c) and $\eta_D = 0.3968$ for $q = 10$ (d).

| Atom Count | Deviation from Binomial |
|------------|-------------------------|
| 0          | 0                       |
| 2          | 0.04                    |
| 4          | 0.08                    |
| 6          | 0.04                    |
| 8          | -0.04                   |
| 10         | -0.08                   |

| Atom Count | Probability |
|------------|-------------|
| 0          | 0.1         |
| 2          | 0.2         |
| 4          | 0.3         |
| 6          | 0.1         |
| 8          | 0.09        |
| 10         | 0.04        |
rary storage of photon energy in the atomic gas in the absence of correspondingly escaping atoms. This is left for a future investigation.

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Appendix A: TRANSITION AMPLITUDE

The transition amplitude can be simplified to

\[ \langle A, n, n-1 | e^{-i(\alpha L + \beta L^*)} | A, n, n \rangle = \sqrt{n_L} n^{n-1} (1 + z)^{n-1}, \]

where \( M_{A,n} = A - (n - 1)/2 \). The above matrix element can be calculated as

\[ \langle A, n, n-1 | e^{-i(\alpha L + \beta L^*)} | A, n, n \rangle = \sqrt{n_L} n^{n-1} (1 + z)^{n-1}, \]

where

\[ \zeta = \frac{i \alpha \sin(\Delta/2)}{\delta \cos(\Delta/2) - i \beta \sin(\Delta/2)}, \]

\[ z = -\frac{\alpha^2}{\delta^2} \sin^2(\Delta/2), \]

\[ \delta = \sqrt{\alpha^2 + \beta^2}, \]

using \( \alpha = 2\Omega |\tau| \sqrt{M_{A,n}} \) and \( \beta = -i \gamma \tau/2 \) for the resonant case \( (\Delta = 0) \).

Using the explicit values for \( \alpha \) and \( \beta \) we obtain the explicit expressions for the above parameters with \( \delta = i \delta' \),

\[ \zeta = \frac{i \sqrt{S_{A,n}} \sin(\delta'/2)}{\sqrt{1 - S_{A,n}} \cosh(\delta'/2) - \sinh(\delta'/2)}, \]

\[ z = \frac{-S_{A,n}}{\sqrt{1 - S_{A,n}}}, \]

\[ \delta' = \frac{\gamma \tau}{2} \sqrt{1 - S_{A,n}}, \]

where we defined the saturation parameter,

\[ S_{A,n} = \frac{4|\Omega^2 M_{A,n}|}{(\gamma^2)^2}. \] (A1)

In the over damped regime, \( S_{A,n} < 1 \), the transition amplitude becomes thus

\[ \psi_{n-1,n}^{A,n}(\tau) = -i \sqrt{n n_L} e^{-\gamma \tau/4} (-1)^{n-1} \]

\[ \times \left[ \sqrt{1 - S_{A,n}} \cosh(\delta'/2) + \sinh(\delta'/2) \right]^{n-1} \]

\[ \times \left[ \frac{1}{(1 - S_{A,n})^{n/2}} \right] \sqrt{n} \sinh(\delta'/2). \]

In the limit \( S_{A,n} \ll 1 \) this expression reduces in lowest-order approximation to

\[ \psi_{n-1,n}^{A,n}(\tau) = i (-1)^{n} \sqrt{S_{A,n} n n_L} e^{-\gamma \tau/4} [n-(n-1)\sqrt{1-S_{A,n}}] \]

\[ \times \sinh \left( \frac{\gamma \tau}{4} \sqrt{1 - S_{A,n}} \right). \] (A2)
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