DISSIPATION FOR A NON-CONVEX GRADIENT FLOW
PROBLEM OF A PATLACK-KELLER-SEGEL TYPE FOR
DENSITIES ON $\mathbb{R}^n$, $n \geq 3$

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Abstract. We study an evolution equation that is the gradient flow in
the 2-Wasserstein metric of a non-convex functional for densities in $\mathbb{R}^d$
with $d \geq 3$. Like the Patlack-Keller-Segel system on $\mathbb{R}^2$, this evolution
equation features a competition between the dispersive effects of diffu-
sion, and the accretive effects of a concentrating drift. We determine a
parameter range in which the diffusion dominates, and all mass leaves
any fixed compact subset of $\mathbb{R}^n$ at an explicit polynomial rate.

1. Introduction

1.1. The gradient flow equation: Lieb’s sharp form of Hardy-Littlewood-
Sobolev (HLS) inequality [9] states that for a nonnegative measurable func-
tion $f$ on $\mathbb{R}^d$, and all $0 < \lambda < d$,

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{f(x)f(y)}{|x-y|^\lambda} \, dx \, dy \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{h(x)h(y)}{|x-y|^\lambda} \, dx \, dy =: C_{\text{HLS}}$$

where $h = (1 + |x|^2)^{(\lambda-2d)/2}$.

We focus on the case $\lambda = d - 2$ and for $\alpha > 0$ define the functional

$$E_\alpha[f] := \|f\|_{L^2}^2 - \frac{\alpha}{C_{\text{HLS}}} \int_{\mathbb{R}^d} f(x) \left[(-\Delta)^{-1} f \right] (x) \, dx. \quad (1.2)$$

Note that for $\alpha \in (0, 1]$, $E_\alpha$ is non-negative, but is not convex. For $\alpha > 1$, it
is not convex and not bounded below. In fact, for all $\alpha > 0$, $E_\alpha$ is difference
of two convex functions.

Note that $E_1[f] \geq 0$ by the HLS inequality. For $\alpha > 1$, $E_\alpha[f]$ is not
bounded below, while for $\alpha < 1$ it is bounded from below. Then we have

$$E_\alpha[f] = E_1[f] + \frac{(1 - \alpha)}{C_{\text{HLS}}} \int_{\mathbb{R}^d} f(x) \left[(-\Delta)^{-1} f \right] (x) \, dx. \quad (1.3)$$

This functional arises in one of several natural ways to generalize the
Patlack-Keller-Segel system to more than two spatial dimensions. The evolu-
tion equation we consider is the gradient flow for $E_\alpha$ in the two Wasserstein
metric. This gradient flow equations can be written as

\[(1.4) \quad \frac{\partial f}{\partial t} = \text{div} \left( f \nabla \left( \frac{\delta E_\alpha}{\delta f} \right) \right), \]

where \(\frac{\delta E}{\delta f}\) is the first variation of the energy with respect to the \(L^2\)-metric.

In writing this out more explicitly, it will be convenient to define \(\kappa = \alpha / C_{\text{HLS}}\). We obtain:

\[(1.5) \quad \frac{\partial f}{\partial t} = \left( \frac{d - 2}{d} \right) ||f||^{\frac{4}{d - 2}} \Delta \left( f^{\frac{2d}{d - 2}} \right) - \text{div} (f \nabla c), \]

where \(\nabla c\) is given by

\[(1.6) \quad \nabla c = \frac{d\kappa}{(d - 2)||f||^{\frac{4}{d - 2}}} \nabla \left( [(-\Delta)^{-1} f] (x) \right). \]

We consider equation \((1.5)\) for initial data

\[(1.7) \quad f(t = 0, x) = f_0(x), \]

that satisfies

\[(1.8) \quad 0 \leq f_0(x), \quad (1 + |x|^2)f_0(x) \in L^1(\mathbb{R}^d), \quad E_\alpha[f_0] < \infty. \]

For more background on this equation and information on the relation to the Patlack-Keller-Segel system, see the paper \([10]\), where a number of results on existence and blow-up are proved. The non-convexity of the functional \(E_\alpha\) is the source of interesting features in the study of this gradient flow problem.

Before stating our result, we recall some relevant facts about the original two dimensional problem. In the theory of this problem as developed by Dolbeault and Perthame \([6]\), the logarithmic Hardy-Littlewood-Sobolev (log-HLS) inequality play the role corresponding to \((1.2)\). The sharp log-HLS inequality \([1, 4]\) states that for all \(f \geq 0\) on \(\mathbb{R}^2\) such that \(f \log f\) is integrable,

\[(1.9) \quad \int_{\mathbb{R}^2} f \log f \, dx + \frac{2}{M} \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(x) \log |x - y| f(y) \, dx \, dy \geq M(1 + \log \pi - \log M) \]

where \(M = \int_{\mathbb{R}^2} f \, dx\) is the total mass. Thus, the log HLS inequality differs form the HLS inequality in that the sharp constant in it depends on the mass. Building on earlier work of Jäger and Luckhaus \([7]\), Dolbeault and Perthame showed that the mass \(M = 8\pi\) is critical in two dimensions. For initial data \(f_0\) with \(\int_{\mathbb{R}^2} f_0 \, dx < 8\pi\), solutions \(f(t, x)\) exist for all time, and \(\int f(t, x) \log f(t, x) \, dx\) is uniformly bounded in \(t\). Jäger and Luckhaus proved existence of solutions and obtained this global existence results for solutions with \(\int_{\mathbb{R}^2} f_0 \, dx < M_0\), for an explicit \(M_0\) smaller than \(8\pi\). For initial data with \(f_0\) with \(M := \int_{\mathbb{R}^2} f_0 \, dx > 8\pi\), Dolbeault and Perthame showed solutions with finite second moment blow up in a finite time depending only on \(M - 8\pi\) and the second moment of the initial data. The mass \(M = 8\pi\) makes the left side of \((1.9)\) exactly the functional for which the 2-dimensional Patlack-Keller-Segel system is gradient flow.

In the higher dimensional case, the constant \(\alpha\) plays the role of the mass parameter. In this case, for \(\alpha = 1\), the driving functional for our equation
is precisely the functional that is non-negative by the sharp HLS inequality. Therefore, and one may conjecture that $\alpha = 1$ is the critical parameter value for our equation. This has indeed been proved in previous work of one of the authors: When $\alpha < 1$, solutions for a natural class of initial data exist for all time, but for $\alpha > 1$ this need not be the case.

The next question is whether for $\alpha < 1$, the solutions have a particular, simple asymptotic behavior. In the 2-dimensional case, this problem was investigated in [3], which provides an affirmative answer. Theorem 1.1 below provides such a result for our higher dimensional equation, but only for small $\alpha$, as in [2] for related equation in two dimensions. It is an open problem in our case to remove the small $\alpha$ restriction, and to prove that the solutions approach self-similar scaling solutions with a particular profile. However, we do prove that for small $\alpha$, the evolution is diffusion dominated, with all mass leaving any fixed compact set at an explicitly computable rate. With this background explained, we state our main result:

**Theorem 1.1.** There is an explicitly computable $0 < \alpha_0 < 1$ such that for all $\alpha \in (0, \alpha_0)$, a solution of our equation with initial data satisfying (1.3) satisfies

$$E_\alpha[f(\cdot, t)] \leq Ct^{-\frac{d-2}{d}}$$

where $C$ is a constant depending only on the initial mass $\int_{\mathbb{R}^d} f_0(x) \, dx$ and free energy $E_\alpha(f_0)$.

Note that by (1.3), this means that for such solutions $f(x, t)$,

$$||f(\cdot, t)||_{\frac{2d}{d+2}}^2 \leq \frac{C}{1 - \alpha_0} t^{-\frac{d-2}{d}}.$$

Since the total mass is conserved, this means that such solutions spreads out, with the mass in any fixed compact subset of $\mathbb{R}^d$ decaying to zero at a polynomial rate. Thus, for $\alpha < \alpha_0$, the diffusion dominates the concentrating drift.

The key to proving this is the following lemma:

**Lemma 1.2.**

(1.10)

$$\frac{d}{dt} \int f^q(x, t) \, dx \leq (q - 1) \left[ -\frac{8(d-2)q}{(d+1)(q + \frac{d-2}{d+2})^2} \frac{1}{C_{GNS}^{(q+1)(d+2)/q(d+2)}} + \kappa \right] ||f||_{q+1}^{q+1}.$$

where $C_{GNS}$ is the sharp constant in the Gagliardo-Nirenberg-Sobolev inequality

(1.11)

$$||u||_{\frac{q(d+2)+d-2}{q(d+2)+d-2}} \ ||\nabla u||_2^{\frac{q(d+2)+d-2}{q(d+2)+d-2}} \geq \frac{1}{C_{GNS}} ||u||_{\frac{q(q+1)(d+2)}{q(d+2)+d-2}}^{q+1}.$$

We apply the lemma for $q = 2d/(d-2)$, so that $q + 1 = (3d+2)/(d+2)$. Then the right hand side in (1.10) becomes

$$\frac{d-2}{d+2} \left[ -\frac{16d(d^2-4)}{(d+1)(3d-2)^2} \frac{1}{C_{GNS}^{3d(d+2)/(3d-2)}} + \kappa \right] =: -K$$
Then for
\[ \alpha_0 = \frac{16d(d^2 - 4)}{(d + 1)(3d - 2)^2} \frac{C_{HLS}}{C_{GNS}^{2(3d+2)/(3d-2)}} , \]
\[ \int_{\mathbb{R}^d} f^{2d/(d+2)}(x,t) \, dx \]
is monotone decreasing for all \( \alpha < \alpha_0 \). Integrating, we have that for any \( t_0 < t_1 \),
\[ \int_{\mathbb{R}^d} f^{\alpha}((x,t_1) \, dx - \int_{\mathbb{R}^d} f^{\alpha}((x,t_0) \, dx \leq -K \int_{t_0}^{t_1} \int_{\mathbb{R}^d} f^{\alpha}((x,t) \, dx \, dt \]
Therefore,
\[ \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} \int_{\mathbb{R}^d} f^{\alpha}((x,t) \, dx \, dt \leq \frac{1}{(t_1 - t_0)K} \int_{\mathbb{R}^d} f^{\alpha}((x,t_0) \, dx . \]
By (1.3),
\[ \int_{\mathbb{R}^d} f^{\alpha}((x,t_0) \, dx \leq \left( \frac{E_\alpha[f(\cdot, t_0)]}{1 - \alpha} \right) \frac{d}{\pi \alpha} . \]
Then since a minimum cannot exceed an average, there is some \( t \in [t_0, t_1] \) with
\[ \int_{\mathbb{R}^d} f^{\alpha}((x,t) \, dx \leq \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} \int_{\mathbb{R}^d} f^{\alpha}((x,t) \, dx \, dt \]
(1.12)
\[ \leq \frac{1}{(t_1 - t_0)K} \left( \frac{E_\alpha[f(\cdot, t_0)]}{1 - \alpha} \right) \frac{d}{\pi \alpha} . \]
Now define \( \beta \) so that
\[ \frac{2d}{d + 2} = \beta \frac{3d + 2}{d + 2} + (1 - \beta)1 , \]
which means that \( \beta = \frac{d-2}{2d} \). Then by Hölder’s inequality,
\[ \int_{\mathbb{R}^d} f^{\alpha}((x,t) \, dx \leq \left( \int_{\mathbb{R}^d} f^{\alpha}((x,t) \, dx \right)^{\beta} \left( \int_{\mathbb{R}^d} f \, dx \right)^{1-\beta} . \]
Letting \( M \) denote the conserved mass \( M = \int_{\mathbb{R}^d} f \, dx \), we now have that for some \( t \in [t_0, t_1] ,\)
\[ \int_{\mathbb{R}^d} f^{\alpha}((x,t) \, dx \leq \left( \int_{\mathbb{R}^d} f^{\alpha}((x,t) \, dx \right)^{\frac{d-2}{2d}} M^{\frac{d+2}{2d}} \]
(1.13)
\[ \leq \left( \frac{1}{(t_1 - t_0)K} \left( \frac{E_\alpha[f(\cdot, t_0)]}{1 - \alpha} \right) \frac{d}{\pi \alpha} \right) \frac{d-2}{2d} M^{\frac{d+2}{2d}} . \]
Then, using the monotonicity of the free energy, \( E_\alpha[f(\cdot, t_1)]^{\frac{d}{\pi \alpha}} \leq E_\alpha[f(\cdot, t)]^{\frac{d}{\pi \alpha}} \leq \int_{\mathbb{R}^d} f^{\alpha}((x,t) \, dx \). Thus we have
(1.14) \[ E_\alpha[f(\cdot, t_1)] \leq \left( \frac{1}{(t_1 - t_0)K} \right) \frac{d}{2d} M^{1/2} \left( \frac{E_\alpha[f(\cdot, t_0)]}{1 - \alpha} \right) \frac{d-2}{2d} . \]
Now for arbitrary \( t > 0 \), we may choose \( t_1 = t \) and \( t_0 = t/2 \). Again using the monotonicity of the free energy, we obtain

\[
E_\alpha[f(\cdot, t)] \leq \left( \frac{2}{Kt} \right)^{\frac{d-2}{2d}} M^{1/2} \left( \frac{E_\alpha[f_0]}{1 - \alpha} \right)^{\frac{d-2}{2d}}.
\]

Now we can use this estimate in (1.14) once more with \( t_1 = t/2 \) and \( t_0 = t/4 \) to improve the bound. This leads to the decay rate claimed in the theorem.

2. Proof of the main lemma

\[
\frac{d}{dt} \int f^q(x, t) \, dx = q \int f^{q-1}(x, t) f_t(x, t) \, dx
\]

(2.1)

\[
= Aq \int f^{q-1}(x, t) \left( \Delta(f^{\frac{2d}{d+2}}) - \text{div}(f \nabla c) \right)(x, t) \, dx,
\]

where \( A := \left( \frac{d-2}{d} \right)^{\frac{4}{q-2}} ||f||_{\frac{4}{d+2}}^2 \), and \( \Delta c = -\frac{\kappa}{A} f \). A straightforward calculation yields that

\[
\frac{d}{dt} \int f^q \, dx = I + II,
\]

where

\[
I = -8(d - 2)q(q - 1) \left( \frac{1}{d+1} \right) \left( \frac{1}{d+2} \right)^2 ||f||_{\frac{4}{d+2}}^4 \int |\nabla(f^{\frac{2d}{d+2}})|^2 \, dx,
\]

\[
II = (q - 1) \kappa \int f^{q+1} \, dx.
\]

Let \( u = f^{\frac{q(d+2)+d-2}{2(d+2)}} \). Then we rewrite \( I \) and \( II \) in terms of \( u \) as follows

\[
I = -8(d - 2)q(q - 1) \left( \frac{1}{d+1} \right) \left( \frac{1}{d+2} \right)^2 \frac{4}{(q+1)(d+2)} ||\nabla u||_2^4,
\]

\[
II = (q - 1) \kappa ||u||_{\frac{2(q+1)(d+2)}{q(d+2)+d-2}} \frac{2(q+1)(d+2)}{q(d+2)+d-2}.
\]

We now write down the corresponding Gagliardo-Nirenberg-Sobolev type inequality. For the correct powers to be chosen one can take the power of the norms in \( I \) and \( II \).

\[
||u||_{\frac{4}{(q+1)(d+2)+d-2}}^{\frac{4}{q(d+2)+d-2}} ||\nabla u||_{\frac{q(d+2)+d-2}{q(d+2)+d-2}}^{\frac{q(d+1)+d-2}{q(d+2)+d-2}} \geq \frac{1}{C_{GNS}} ||u||_{\frac{2(q+1)(d+2)}{q(d+2)+d-2}} \frac{2(q+1)(d+2)}{q(d+2)+d-2}.
\]

Using (2.3) in (2.2) yields the result.

3. Acknowledgments

The work of E.A.C. is partially supported by U.S. N.S.F. grant DMS 1501007. The work of S. U. was partially supported by BAGEP 2015 award.
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