ON THE MONOTONICITY OF HILBERT FUNCTIONS

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Abstract. In this paper we show that a large class of one-dimensional Cohen-Macaulay local rings \((A, \mathfrak{m})\) has the property that if \(M\) is a maximal Cohen-Macaulay \(A\)-module then the Hilbert function of \(M\) (with respect to \(\mathfrak{m}\)) is non-decreasing. Examples include

1. Complete intersections \(A = Q/(f, g)\) where \((Q, \mathfrak{n})\) is regular local of dimension three and \(f \in \mathfrak{n}^2 \setminus \mathfrak{n}^3\).

2. One dimensional Cohen-Macaulay quotients of a two dimensional Cohen-Macaulay local ring with pseudo-rational singularity.

1. Introduction

Let \((A, \mathfrak{m})\) be a \(d\)-dimensional Noetherian local ring with residue field \(k\) and let \(M\) be a finitely generated \(A\)-module. Let \(\mu(M)\) denote minimal number of generators of \(M\) and let \(\ell(M)\) denote its length. Let \(\text{codim}(A) = \mu(\mathfrak{m}) - d\) denote the codimension of \(A\).

Let \(G(A) = \bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}\) be the associated graded ring of \(A\) (with respect to \(\mathfrak{m}\)) and let \(G(M) = \bigoplus_{n \geq 0} \mathfrak{m}^n M/\mathfrak{m}^{n+1} M\) be the associated graded module of \(M\) considered as a \(G(A)\)-module. The ring \(G(A)\) has a unique graded maximal ideal \(G = \bigoplus_{n \geq 1} \mathfrak{m}^n/\mathfrak{m}^{n+1}\). Set \(\text{depth}(G(M)) = \text{grade}(G, G(M))\). Let \(e(M)\) denote the multiplicity of \(M\) (with respect to \(\mathfrak{m}\)).

The Hilbert function of \(M\) (with respect to \(\mathfrak{m}\)) is the function

\[ H(M, n) = \ell \left( \frac{\mathfrak{m}^n M}{\mathfrak{m}^{n+1} M} \right) \text{ for all } n \geq 0. \]

A natural question is whether \(H(M, n)\) is non-decreasing (when \(\dim M > 0\)). It is clear that if \(\text{depth}G(M) > 0\) then the Hilbert function of \(M\) is non-decreasing, see Proposition 3.2 of [9]. If \(A\) is regular local then all maximal Cohen-Macaulay (= MCM) modules are free. Thus every MCM module of positive dimension over a regular local ring has a non-decreasing Hilbert function. The next case is that of a hypersurface ring i.e., the completion \(\hat{A} = Q/(f)\) where \((Q, \mathfrak{n})\) is regular local and \(f \in \mathfrak{n}^2\). In Theorem 1, [9] we prove that if \(A\) is a hypersurface ring of positive dimension and if \(M\) is a MCM \(A\)-module then the Hilbert function of \(M\) is non-decreasing. See example 3.3, [9] for an example of a MCM module \(M\) over the hypersurface ring \(k[[x, y]]/(y^3)\) with depth \(G(M) = 0\).

Let \((A, \mathfrak{m})\) be a strict complete intersection of positive dimension and let \(M\) be a maximal Cohen-Macaulay \(A\)-module with bounded betti-numbers. In Theorem 1, [8] we prove that the Hilbert function of \(M\) is non-decreasing. We also prove an

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analogous statement for complete intersections of codimension two, see Theorem 2, [8].

In the ring case Elias [1, 2.3], proved that the Hilbert function of a one-dimensional Cohen-Macaulay ring is non-decreasing if embedding dimension is three. The first example of a one dimensional Cohen-Macaulay ring $A$ with not monotone increasing Hilbert function was given by Herzog and Waldi; [3, 3d]. Later Orecchia, [7, 3.10], proved that for all $b \geq 5$ there exists a reduced one-dimensional Cohen-Macaulay local ring of embedding dimension $b$ whose Hilbert function is not monotone increasing. Finally in [2] we can find similar example with embedding dimension four. A long standing conjecture in theory of Hilbert functions is that the Hilbert function of a one dimensional complete intersection is non-decreasing. Rossi conjectures that a similar result holds for Gorenstein rings.

In this paper we construct a large class of one dimensional Cohen-Macaulay local rings $(A, m)$ with the property that if $M$ is an $MCM_A$-module then the Hilbert function of $M$ is non-decreasing. Recall a Cohen-Macaulay local ring $(B, n)$ is said to have minimal multiplicity if

$$e(B) = 1 + \text{codim}(B).$$

Our result is

**Theorem 1.1.** Let $(B, n)$ be a two dimensional Cohen-Macaulay local ring with minimal multiplicity. Let $(A, m)$ be a one-dimensional Cohen-Macaulay local ring which is a quotient of $B$. If $M$ is a maximal Cohen-Macaulay $A$-module then the Hilbert function of $M$ (with respect to $m$) is non-decreasing.

We now give examples where our result holds.

**Example 1.2.** Let $(Q, n)$ be a regular local ring of dimension three. Let $f_1, f_2 \in n^2$ be an $Q$-regular sequence. Assume $f_1 \in n^2 \setminus n^3$. Let $A = Q/(f_1, f_2)$. Then if $M$ is a maximal Cohen-Macaulay $A$-module then the Hilbert function of $M$ (with respect to $m$) is non-decreasing. The reason for this is that $B = Q/(f_1)$ has minimal multiplicity.

**Example 1.3.** Let $(B, n)$ be a two dimensional local ring with pseudo-rational singularity. Then $B$ has minimal multiplicity, see [6, 5.4]. In particular if $A = B/P$, $P$ a prime ideal of height one or if $A = B/(x)$ where $x$ is $B$-regular then if $M$ is a maximal Cohen-Macaulay $A$-module then the Hilbert function of $M$ (with respect to $m$) is non-decreasing.

**Example 1.4.** There is a large class of one dimensional local rings $(R, m)$ with minimal multiplicity. For examples Arf rings have this property, [5, 2.2]. Let $B = R[X]_{(m, X)}$. Then $B$ is a two dimensional Cohen-Macaulay local ring with minimal multiplicity.

Here is an overview of the contents of the paper. In Section two we introduce notation and discuss a few preliminary facts that we need. In section three we prove Theorem [1].

2. Preliminaries

In this paper all rings are Noetherian and all modules considered are assumed to be finitely generated (unless otherwise stated). Let $(A, m)$ be a local ring of
dimension \(d\) with residue field \(k = A/\mathfrak{m}\). Let \(M\) be an \(A\)-module. If \(m\) is a non-zero element of \(M\) and if \(j\) is the largest integer such that \(m \in m^j M\), then we let \(m^*\) denote the image of \(m\) in \(m^j M/m^{j+1} M\).

The formal power series

\[
H_M(z) = \sum_{n \geq 0} H(M, n) z^n
\]

is called the Hilbert series of \(M\). It is well known that it is of the form

\[
H_M(z) = \frac{h_M(z)}{(1 - z)^r}, \quad \text{where} \quad r = \dim M \quad \text{and} \quad h_M(z) \in \mathbb{Z}[z].
\]

We call \(h_M(z)\) the \(h\)-polynomial of \(M\). If \(f\) is a polynomial we use \(f^{(i)}\) to denote its \(i\)-th derivative. The integers \(\epsilon_i(M) = h_M^{(i)}(1)/i!\) for \(i \geq 0\) are called the Hilbert coefficients of \(M\). The number \(\epsilon(0) = \epsilon_0(M)\) is the multiplicity of \(M\).

2.1. Base change: Let \(\phi: (A, \mathfrak{m}) \to (A', \mathfrak{m}')\) be a local ring homomorphism. Assume \(A'\) is a faithfully flat \(A\) algebra with \(\mathfrak{m}' = \mathfrak{m} A'\). Set \(\mathfrak{m}' = \mathfrak{m} A'\) and if \(N\) is an \(A\)-module set \(N' = N \otimes_A A'\). In these cases it can be seen that

1. \(\ell_A(N) = \ell_{A'}(N')\).
2. \(H(M, n) = H(M', n)\) for all \(n \geq 0\).
3. \(\dim M = \dim M'\) and \(\text{depth}_M M = \text{depth}_{A'} M'\).
4. \(\text{depth} G(M) = \text{depth} G(M')\).

The specific base changes we do are the following:

(i) \(A' = A[X] S\) where \(S = A[X] \setminus \mathfrak{m} A[X]\). The maximal ideal of \(A'\) is \(\mathfrak{n} = \mathfrak{m} A'\).

The residue field of \(A'\) is \(K = k(X)\).

(ii) \(A' = \tilde{A}\) the completion of \(A\) with respect to the maximal ideal.

Thus we can assume that our ring \(A\) is complete with infinite residue field.

I: \(L_i(M)\)

Let \((A, \mathfrak{m})\) be a Noetherian local ring and \(M\) a \(A\)-module. We simplify a construction from [2].

2.2. Set \(L_0(M) = \bigoplus_{n \geq 0} M/m^{n+1} M\). Let \(\mathcal{R} = A[\mathfrak{m}]\) be the Rees-algebra of \(\mathfrak{m}\). Let \(\mathcal{S} = A[\mathfrak{u}]\). Then \(\mathcal{R}\) is a subring of \(\mathcal{S}\). Set \(M[\mathfrak{u}] = M \otimes_A \mathcal{S}\) an \(\mathcal{S}\)-module and so an \(\mathcal{R}\)-module. Let \(\mathcal{R}(M) = \bigoplus_{n \geq 0} m^n M\) be the Rees-module of \(M\) with respect to \(\mathfrak{m}\). We have the following exact sequence of \(\mathcal{R}\)-modules

\[
0 \to \mathcal{R}(M) \to M[\mathfrak{u}] \to L_0(M)(-1) \to 0.
\]

Thus \(L_0(M)(-1)\) (and so \(L_0(M))\) is an \(\mathcal{R}\)-module. We note that \(L_0(M)\) is not a finitely generated \(\mathcal{R}\)-module. Also note that \(L_0(M) = M \otimes_A L_0(A)\).

2.3. For \(i \geq 1\) set

\[
L_i(M) = \text{Tor}_i^A(M, L_0(A)) = \bigoplus_{n \geq 0} \text{Tor}_i^A(M, A/m^{n+1}).
\]

We assert that \(L_i(M)\) is a finitely generated \(\mathcal{R}\)-module for \(i \geq 1\). It is sufficient to prove it for \(i = 1\). We tensor the exact sequence \(0 \to \mathcal{R} \to \mathcal{S} \to L_0(A)(-1) \to 0\) with \(M\) to obtain a sequence of \(\mathcal{R}\)-modules

\[
0 \to L_1(M)(-1) \to \mathcal{R} \otimes_A M \to M[\mathfrak{u}] \to L_0(M)(-1) \to 0.
\]

Thus \(L_1(M)(-1)\) is a \(\mathcal{R}\)-submodule of \(\mathcal{R} \otimes_A M\). The latter module is a finitely generated \(\mathcal{R}\)-module. It follows that \(L_1(M)\) is a finitely generated \(\mathcal{R}\)-module.
2.4. Now assume that $A$ is Cohen-Macaulay of dimension $d \geq 1$. Set $N = \text{Syz}_A^1(M)$ and $F = A^\mu(M)$. We tensor the exact sequence

$$0 \to N \to F \to M \to 0,$$

with $L_0(A)$ to obtain an exact sequence of $R$-modules

$$0 \to L_1(M) \to L_0(N) \to L_0(F) \to L_0(M) \to 0.$$

It is elementary to see that the function $n \to \ell(\text{Tor}_1^A(M, A/m^{n+1}))$ is polynomial of degree $\leq d - 1$. By [11] Corollary II if $M$ is non-free then it is polynomial of degree $d - 1$. Thus $\dim L_1(M) = d$ if $M$ is non-free.

II: Superficial sequences.

2.5. An element $x \in \mathfrak{m}$ is said to be superficial for $M$ if there exists an integer $c > 0$ such that

$$(m^nM : Mx) \cap \mathfrak{m}^c M = \mathfrak{m}^{n-1}M \quad \text{for all} \quad n > c.$$

Superficial elements always exist if $k$ is infinite [11, p. 7]. A sequence $x_1, x_2, \ldots, x_r$ in a local ring $(A, \mathfrak{m})$ is said to be a superficial sequence for $M$ if $x_1$ is superficial for $M$ and $x_i$ is superficial for $M/(x_1, \ldots, x_{i-1})M$ for $2 \leq i \leq r$.

We need the following:

**Proposition 2.6.** Let $(A, \mathfrak{m})$ be a Cohen-Macaulay ring of dimension $d$ and let $M$ be a Cohen-Macaulay $A$-module of dimension $r$. Let $x_1, \ldots, x_c$ be an $M$-superficial sequence with $c \leq r$. Assume $x_1^*, \ldots, x_c^*$ is a $G(M)$-regular sequence. Let $R = A[\mu]u$ be the Rees algebra of $\mathfrak{m}$. Set $X_i = x_iu \in R_1$. Then $X_1, \ldots, X_c$ is a $L_0(M)$-regular sequence.

**Proof.** We prove the result by induction. First consider the case when $c = 1$. Then the result follows from [9] 2.2(3)]. We now assume that $c \geq 2$ and the result holds for all Cohen-Macaulay $A$-modules and sequences of length $c - 1$. By $c = 1$ result we get that $X_1$ is $L_0(M)$-regular. Let $N = M/(x_1)$. As $x_1^*$ is $G(M)$-regular we get $G(M)/x_1^*G(M) \cong G(N)$. So $x_2^*, \ldots, x_c^*$ is a $G(N)$-regular sequence. Now also note that $L_0(M)/X_1L_0(M) = L_0(N)$. Thus the result follows. \qed

3. Proof of Theorem 1.1

In this section we give a proof of Theorem 1.1. We also give an example which shows that it is possible for depth $G(M)$ to be zero.

**Proof of Theorem 1.1.** We may assume that the residue field of $A$ is infinite. Let $N = \text{Syz}_1^B(M)$. Then $N$ is a maximal Cohen-Macaulay $B$-module. As $B$ has minimal multiplicity it follows that $N$ also has minimal multiplicity. So $G(N)$ is Cohen-Macaulay and $\deg h_N(z) \leq 1$, see [10] Theorem 16]. Set $r = \mu(M)$, $h_B(z) = 1 + hz$ and as $e(N) = re(A)$ we write $h_N(z) = r + c + (rh - c)z$ (here $c$ can be negative).

Set $F = A^r$. The exact sequence $0 \to N \to F \to M \to 0$ induces an exact sequence

$$0 \to L_1(M) \to L_0(N) \to L_0(F) \to L_0(M) \to 0$$

of $R$-modules. Let $x_1, x_2$ be an $N \oplus B$-superficial sequence. Then $x_1^*, x_2^*$ is a $G(N) \oplus G(B)$-regular sequence. Set $X_i = x_iu \in R_1$. Then by 2.6 it follows that $X_1, X_2$ is a $L_0(N) \oplus L_0(F)$-regular sequence. By 1.1 it follows that $X_1, X_2$ is also
a \( L_1(M) \)-regular sequence. As \( \dim L_1(M) = 2 \) (see [2.3]) it follows that \( L_1(M) \) is a Cohen-Macaulay \( \mathcal{R} \)-module. Let the Hilbert series of \( L_1(M) \) be \( l(z)/(1 - z)^2 \). Then the coefficients of \( l(z) \) are non-negative.

Let \( l(z) = l_0 + l_1 z + \cdots + l_m z^m \) and let \( h_M(z) = h_0 + h_1 z + \cdots + h_p z^p \). By (1) we get

\[
(1 - z)l(z) = h_N(z) - h_F(z) + (1 - z)h_M(z),
\]

\[
= r + c + (rh - c)z - r(1 + hz) + (1 - z)h_M(z),
\]

\[
= c(1 - z) + (1 - z)h_M(z).
\]

It follows that

\[
l(z) = c + h_M(z).
\]

It follows that \( m = p \) and \( h_i = l_i \) for \( i \geq 1 \). In particular \( h_i \geq 0 \) for \( i \geq 1 \). Also \( h_0 = \mu(M) > 0 \). Thus \( h_M(z) \) has non-negative coefficients. It follows that the Hilbert function of \( M \) is non-decreasing. \( \square \)

We now give an example which shows that it is possible for depth \( G(M) \) to be zero.

**Example 3.1.** Let \( K \) be a field and let \( A = K[[t^6, t^7, t^{15}]] \). It can be verified that

\[
A \cong K[[X, Y, Z]]/(Y^3 - XZ, X^5 - Z^2)
\]

and that

\[
G(A) \cong K[X, Y, Z]/(XZ, Y^6, Y^3Z, Z^2).
\]

Note that \( ZY^2 \) annihilates \( (X, Y, Z) \). So depth \( G(A) = 0 \).

Set \( B = K[[X, Y, Z]]/(Y^3 - XZ) \). Then \( B \) is a two-dimensional Cohen-Macaulay ring with minimal multiplicity and \( A \) is a one-dimensional Cohen-Macaulay quotient of \( B \). Set \( M = A \).

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