An Operator-Splitting Finite Element Method for the Numerical Solution of Radiative Transfer Equation

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Abstract An operator-splitting finite element scheme for the time-dependent, high-dimensional radiative transfer equation is presented in this paper. The streamline upwind Petrov-Galerkin finite element method and discontinuous Galerkin finite element method are used for the spatial-angular discretization of the radiative transfer equation, whereas the implicit backward Euler scheme is used for temporal discretization. Error analysis of the proposed numerical scheme for the fully discrete radiative transfer equation is presented. The stability and convergence estimates for the fully discrete problem are derived. Moreover, an operator-splitting algorithm for numerical simulation of high-dimensional equations is also presented. The validity of the derived estimates and implementation is illustrated with suitable numerical experiments.

Keywords Radiative transfer equation · Operator-splitting method · Streamline upwind Petrov Galerkin finite element methods · Backward Euler scheme · Stability and Convergence analysis

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1 Introduction

Radiation plays a significant role, both as a detectable and dominant mechanism for transmitting energy inside and outside a system, in several areas, including optics, as-
trophysics, atmospheric science, and remote sensing. Consequently, the propagation of radiation through a medium is one of the most critical processes studied extensively. Analyzing the released radiation from an object provides insight into the radiative source, the medium between the object and the observer, and its surroundings. Modeling this physical process results in a time-dependent, six-dimensional Partial Differential Equation (PDE). The higher dimension of PDE is one of the challenges associated with the solution of the radiative transfer equation (RTE). Besides, considerable uncertainty is added due to radiation’s ability to affect the medium’s state, which is the source of the radiation itself. Although analytic solutions to RTE exist for simple cases, numerical solutions are often sought for more realistic, complex applications. Therefore, it is exciting and, at the same time challenging to develop numerical schemes for the radiative model. More details on the radiative transfer model can be found in [9, 29]. The design and implementation of the computational method for time-dependent, high-dimensional radiative transfer equations remain a challenging task in computational science, even though tremendous advances have been made in this area over the past few years.

1.1 Model problem

Let Ω be a bounded domain in \( \mathbb{R}^3 \) with a smooth boundary \( \partial \Omega \). Denote by \( n(x) \) the unit outward normal for \( x \in \partial \Omega \). Let the angular space \( S^2 \) be the unit sphere in \( \mathbb{R}^3 \). For each fixed direction \( s(s_1, s_2, s_3) \in S^2 \), we introduce the following subsets of the boundary \( \partial \Omega \):

\[
\partial \Omega_{s,-} = \{ x \in \partial \Omega : s \cdot n(x) < 0 \}, \quad \partial \Omega_{s,+} = \{ x \in \partial \Omega : s \cdot n(x) \geq 0 \}.
\]

Let \( S \) be the whole domain which is the tensor product domain in space and angle. Then the boundary \( \Gamma = \partial \Omega \times S^2 \) can be split into two parts

\[
\Gamma^- = \{ (x,s) : x \in \partial \Omega_{s,-}, s \in S^2 \}, \quad \Gamma^+ = \{ (x,s) : x \in \partial \Omega_{s,+}, s \in S^2 \}
\]

as the inlet and outlet boundaries. In this article, the high-dimensional radiative transfer equation (RTE) is formally defined by an initial-boundary-value problem:

\[
\frac{\partial u}{\partial t} + s \cdot \nabla u + \sigma_t u - \sigma_a \int_{S^2} u(t,x,s') \Phi(s,s') ds' = f, \quad \text{in} \quad [0,T] \times \Omega \times S^2,
\]

\[
\begin{align*}
&u(t=0,x,s) = u_0, \quad \text{in} \quad \Omega \times S^2, \\
&u(t,x,s) = 0, \quad \text{on} \quad [0,T] \times \Gamma^-,
\end{align*}
\]

where \( \sigma_t(x) = \sigma_a(x) + \sigma_s(x) \). Here, \( \sigma_a(x) \) and \( \sigma_s(x) \) are the total absorption and scattering coefficients, respectively. For simplicity, the particle speed is assumed to be one. Here, the scattering phase function \( \Phi(s,s') \) describes the probability of a photon at position \( x \) that originally propagates in the direction \( s \), and \( s' \) as its new propagation direction after the scattering event. Note that the angular variable \( s \) in the spherical coordinate system is denoted as \( s = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)^T \). Also, we make the following assumptions on the data of the model problem (1) as Further, the data \( \sigma_t, \sigma_a, f \) and \( u_0 \) of the model problem (1) are assumed to be sufficiently
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smooth. Note that, the given RTE model can also be viewed as high-dimensional integro-differential equation.

The existing numerical schemes for RTE models can be classified into (i) stochastic approach and (ii) deterministic approach. Among all stochastic approaches, the Monte Carlo method is often used to solve the radiative transfer equation, see, for example, [22, 23], and the references therein. Nevertheless, the Monte-Carlo simulation’s computational cost is very high due to its iterative design, and it increases when the optical depth becomes large.

Several deterministic numerical schemes have been proposed in the literature for the stationary RTE, see for example, [4, 5, 8, 14, 15, 35], and the references therein. A robust numerical $S_n$-DG-approximations for radiation transport has been discussed in [30, 34]. Stabilized finite element scheme with discrete ordinate method has been discussed for steady-state RTE models in [24, 36]. A numerical scheme based on Ad hoc angular discretization and vectorial finite elements for spatial discretization has been studied in [16]. Recently, an adaptive nested source term iteration method for steady-state RTE has been presented in [10]. For a time-dependent RTE model in one-dimensional slab geometry, a semi-analytical numerical method has been presented in [11]. A low-rank approximation for time-dependent radiation transport in one- and two-dimensional Cartesian geometries has been discussed in [33]. In [12], a variable discrete ordinates method has been used to solve the transient radiation heat transfer in a semi-transparent slab. Despite several numerical schemes proposed in the literature, numerical solution of the time-dependent high-dimensional RTE is still challenging and is an active research field.

The operator-splitting finite element methods have been developed in the recent past for many high-dimensional physical and mathematical models. For example, an operator-splitting numerical method for the micro-macro dilute polymeric fluid model has been provided in [31]. For a high-dimensional the convection-diffusion problem, an operator-splitting method with detailed numerical implementation, has been presented in [20]. The high-dimensional population balance equation using the operator-splitting method has been discussed in [17]. An operator-splitting finite element method for an efficient parallel solution of high-dimensional population balance systems has been discussed in [19]. More details on the standard operator-splitting FEM can be found in book [21].

For high-dimensional time-dependent RTE, we present an operator-splitting heterogeneous FEM. Further, a priori error estimate for the proposed numerical scheme is presented, which is the main contribution of this research article. The key idea is to split the RTE model problem concerning the internal (angular) and the external (spatial) directions, resulting in a transient transport problem and a time-dependent integro-differential equation. The transient transport problem is numerically approximated using the streamline upwind Petrov-Galerkin (SUPG) finite element method, whereas the discontinuous Galerkin method with piecewise constant polynomials (DG(0)) is used for the integro-differential equation. The proposed numerical scheme uses this tailor-made spatial discretization method and the implicit backward Euler scheme. The stability estimate of the fully discrete form of the proposed scheme is first derived. A convergence analysis is then established under the assumption of certain regularity conditions on the data and the stabilization parameter. Finally, an array
of numerical experiments is provided to support the theoretical error estimates of the numerical approximation.

The rest of the article is organized as follows. In Section 2, we briefly discuss the weak formulation of the operator-splitting method for the model problem, and later, we discuss the finite element approximation of RTE. Further, a fully discrete form is derived in this section. Next, the stability estimate of the discrete problem and the convergence analysis of the numerical approximation is presented in Section 3. Further, the implementation of the numerical scheme is discussed in Section 4. Finally, a concluding remark is discussed in Section 5.

2 Finite element approximation for RTE

This section starts with prerequisites for the finite element discretization of the model problem \([1]\). Let \(L^2\) and \(H^m\) be the Sobolev spaces. Denote the \(L^2\)-inner product with respect to spatial variable \(x\) over the domain \(\Omega\) as \((\cdot, \cdot)_{\Omega}\). The inner product and \(L^2\)-norm over the entire domain \(\Omega \times S^2\) are defined by

\[
(v, w) := \int_{S^2} (v, w)_{\mathbb{R}^3} ds = \int_{\Omega} \int_{S^2} vw \, dx \, ds, \quad \|v\|_{H^s}^2 = (v, v),
\]

\[
\langle v, w \rangle_{\partial \Omega} := \int_{S^2} \int_{\partial \Omega} (v, w)_{\mathbb{R}^3} \, d\sigma \, ds, \quad \|v\|_{H^s}^2 = \langle v, v \rangle_{\partial \Omega}.
\]

For simplification of mathematical presentation, we have omitted \(ds\) from \(\int_{S^2} (v, w)_{\mathbb{R}^3} ds\) and we simply write \(\int_{S^2} (v, w)_{\mathbb{R}^3}\) throughout this article. We will adopt the notation associated with the operator-splitting technique introduced in [20] for the numerical analysis of this article.

Next, we introduce Bochner spaces. Let \(Z\) be a Banach space associated with the spatial variable \(x\) equipped with the norm \(\| \cdot \|_Z\). For spaces \(Z\) and \(Y\), we use a short notation \(Y(\mathbb{R}^3; Z)\) and define the following spaces

\[
C(S^2; Z) := \left\{ v : \Omega \to Z \mid v \text{ continuous, sup}_{s \in S^2} \|v(s)\|_Z < \infty \right\},
\]

\[
L^2(S^2; Z) := \left\{ v : \Omega \to Z \mid \int_{S^2} \|v\|_Z^2 < \infty \right\},
\]

\[
H^m(S^2; Z) := \left\{ v \in L^2(S^2; Z) \mid \frac{\partial^j v}{\partial s^j} \in L^2(S^2; Z), \ 1 \leq j \leq m \right\},
\]

where the derivatives \(\frac{\partial^j v}{\partial s^j}(j \text{ a multi-index})\) are expressed in the sense of distributional derivative on \(S^2\) and \(m\) is an integer. The norms in the above spaces are given by

\[
\|v\|_{C(Z)} := \sup_{s \in S^2} \|v\|_Z, \quad \|v\|_{L^2(Z)}^2 := \int_{S^2} \|v\|_Z^2, \quad \|v\|_{H^m(Z)}^2 := \int_{S^2} \sum_{|j| \leq m} \left\| \frac{\partial^j v}{\partial s^j} \right\|_Z^2.
\]
Before deriving the numerical approximation of the model problem (1), the analytical properties of the scattering phase function is discussed here. An operator \( \mathcal{K} \) is defined by

\[
\mathcal{K} v(t, x, s) = \int_{S^2} \Phi(s, s') v(t, x, s') ds'.
\]

**Assumption 1** Assume that the scattering kernel \( \Phi \) holds the following conditions:

- \( \Phi \) is a measurable function and positive, i.e., \( \Phi(s, s') \geq 0 \) for \( s, s' \in S^2 \),
- and it satisfies

\[
\int_{S^2} \Phi(s, s') ds' = 1.
\]

**Lemma 1** \( \mathcal{K} : \Omega \times S^2 \to \Omega \times S^2 \) is a self-adjoint and bounded linear operator with \( \| \mathcal{K} v \|_0 \leq \| v \|_0 \) for all \( v \in \Omega \times S^2 \).

**Proof.** The proof is given in [13, Lemma 2.6].

Next, denote the removal operator [13] by

\[
\mathcal{K}_a v(t, x, s) = \sigma_a(x) v(t, x, s) + \sigma_s(x) v(t, x, s) - \mathcal{K} v(t, x, s),
\]

where \( \sigma_a(x) v(t, x, s) \) models the absorption of particles by the medium. The absorption and remission of particles during the scattering process is described by \( (\sigma_s(x) v(t, x, s) - \mathcal{K} v(t, x, s)) \).

**Assumption 2** The absorption and scattering coefficient satisfy following conditions

- \( \sigma_a \) is measurable, non-negative, and uniformly bounded, i.e., there exists \( \overline{\sigma}_a \in \mathbb{R}^+ \) such that \( 0 \leq \sigma_a(x) \leq \overline{\sigma}_a \) for a.e \( x \in \Omega \).
- \( \sigma_s \) is measurable, non-negative, and uniformly bounded, i.e., \( 0 < \underline{\sigma}_s \leq \sigma_s(x) \leq \overline{\sigma}_s \) for a.e \( x \in \Omega \) and \( \underline{\sigma}_s, \overline{\sigma}_s \in \mathbb{R}^+ \). For the convergence analysis, we assume that \( \underline{\sigma}_s \geq 1/8 \).

**Lemma 2** The operator \( \mathcal{K}_a : \Omega \times S^2 \to \Omega \times S^2 \) is a self-adjoint and elliptic bounded linear operator and it satisfies following estimates:

\[
\begin{align*}
(\mathcal{K}_a v, v) & \geq \underline{\sigma}_a \| v \|_0^2, \\
(\mathcal{K}_a v, w) & \leq \| \mathcal{K}_a v \|_0 \| w \|_0 \leq (2\overline{\sigma}_s + \overline{\sigma}_a) \| v \|_0 \| w \|_0.
\end{align*}
\]

**Proof.** The proof of lemma is discussed in [13, Lemma 2.7].

### 2.1 Operator-Splitting Method

The gradient operator in the RTE model (1) is defined for the spatial variable \( x \) only. Thus, we can take advantage of the decomposition of the model problem by decomposing the model problem (1) into a purely convective problem in the space and an integro-differential equation in the angular variable. Let \( 0 = t_0 < t_1 < \ldots < t_N = T \) be the time discretization of the time interval \([0, T]\). Using the Lie’s operator-splitting
method, in the time interval \((t_n, t_{n+1})\), the operator-splitting method of the model problem (1) read:

**Step 1. (s-direction)**

For given \(\tilde{u}(t^n) = u(t^n)\), find \(\hat{u} : (t^n, t^{n+1}) \times \Omega \times S^2 \to \mathbb{R}\) such that,

\[
\frac{\partial \hat{u}}{\partial t} + \mathcal{K}_\sigma \hat{u} = 0, \quad (t^n, t^{n+1}) \times \Omega \times S^2,
\]

\[
\hat{u}(t^n, x, s) = u(t^n, x, s),
\]

by considering \(x\) as a parameter. In this step, the solution is updated in the \(s\)-direction. Then, this solution \(\hat{u}\) is taken as the initial solution for the \(x\)-direction update.

**Step 2. (x-direction)**

For given \(u(t^n) = \tilde{u}(t^{n+1})\), find \(u : (t^n, t^{n+1}) \times \Omega \times S^2 \to \mathbb{R}\) such that

\[
\frac{\partial u}{\partial t} + s \cdot \nabla u = f, \quad (t^n, t^{n+1}) \times \Omega \times S^2,
\]

\[
u = 0, \quad (t^n, t^{n+1}) \times \Gamma_-, \quad u(t^n, x, s) = \tilde{u}(t^{n+1}, x, s),
\]

by considering the variable \(s\) as a parameter. Here, the solution \(u\) in the time step \((t^n, t^{n+1})\) is obtained by first updating in \(s\)-direction (4) and then updating in \(x\)-direction (5).

A weak formulation for both steps (4) and (5) will be introduced below. Let \(\hat{V}\) is defined by

\[
\hat{V} = \{ v | v \in L^2(\Omega), s \cdot \nabla v \in L^2(\Omega), |s \cdot n|^{1/2} v \in L^2(\partial \Omega_{s, \pm}) \}.
\]

We denote by \(V = \{ v \in \hat{V} : v|_{\partial \Omega_{s, \pm}} = 0 \}\) and \(W := L^2(S^2)\). Finally, we introduce \(\mathcal{P} := \{ v \in L^2(\Omega \times S^2) | v \in L^2(\Omega; W) \cap v \in L^2(S^2; V) \}\). From definition of the finite element space \(\mathcal{P}\), any smooth function \(v \in L^2(S^2; V) \subset \mathcal{P}\) satisfies

\[
\int_{S^2} \|v\|^2_{L^2(\Omega)} < \infty, \quad \int_{S^2} \|s \cdot \nabla v\|^2_{L^2(\Omega)} < \infty, \quad \int_{S^2} \|s \cdot n|^{1/2} v\|^2_{L^2(\partial \Omega_{s, \pm})} < \infty.
\]

Now, we introduce the weak form for the operator-splitting method (4) and (5), which is given by

**Step 1.** Find \(\tilde{u} : (t^n, t^{n+1}) \to \mathcal{P}\) with \(\tilde{u}(t^n) = u(t^n)\) such that

\[
\int_{S^2} (\tilde{u}, \nu)_{x} ds + \int_{S^2} (\mathcal{K}_\sigma \tilde{u}, \nu)_{x} ds = 0, \quad \forall \nu \in \mathcal{P},
\]

The norm associated with the weak formulation (6) is simply the inner product norm \(|\cdot|_0\).
Step 2. Find $u : (t^n, t^{n+1}) \to \mathcal{P}$ with $u(t^n) = \tilde{u}(t^{n+1})$ such that

$$
\int_{S^2} (u, v) + \int_{S^2} a(u, v) = \int_{S^2} (f, v), \forall v \in \mathcal{P},
$$

(7)

where the bilinear form $a(u, v) = (s \cdot \nabla u, v)$. It is well documented in [7] that standard FEM are known to produce spurious oscillation. To achieve the coercivity of the bilinear form $a(u, v)$, the test function is taken as $(v + \delta s \cdot \nabla v)$, where $\delta$ is the stabilization parameter. A more detail about the $\delta$ is given later. Finally, the norm is given by

$$
\|v\|^2 = \int_{S^2} \left( \delta \|s \cdot \nabla v\|^2_{L^2(\Omega)} + \|s \cdot n\|^{1/2}_2 \|v\|^2_{L^2(\partial \Omega_s)} \right).
$$

Here, we will briefly discuss the existence and uniqueness of the weak formulation (6) and (7). We start with the weak form (6). By following [21, Theorem 6.1], it is enough to show that

$$
\int_{S^2} (K \sigma, v) \leq M_1 \|\tilde{u}\|_0 \|v\|_0, \text{ a.e. } 0 < t < T, \tilde{u}, v \in \mathcal{P},
$$

(8)

$$
\int_{S^2} (K \sigma, \tilde{u}) \geq \alpha_1 \|\tilde{u}\|^2_0, \text{ a.e. } 0 < t < T, \tilde{u} \in \mathcal{P},
$$

(9)

where $M_1, \alpha_1$ are the positive constants.

By using Lemma 2, the required results (6) and (7) can be easily verified. Next, we prove the existence and uniqueness of (7). It is sufficient to show that

$$
\int_{S^2} a(u, v + \delta s \cdot \nabla v) \leq M_2 \|u\|_0 \|v\|_0, \text{ a.e. } 0 < t < T, \tilde{u}, v \in \mathcal{P},
$$

(10)

$$
\int_{S^2} a(u, u + \delta s \cdot \nabla u) \geq \alpha_2 \|u\|^2_0, \text{ a.e. } 0 < t < T, u \in \mathcal{P},
$$

(11)

The inequality (10) can be easily obtain by using Cauchy–Schwarz (C-S) inequality. And

$$
\int_{S^2} a(u, u + \delta s \cdot \nabla u) = \int_{S^2} \int_{\partial \Omega_s} \frac{1}{2} (s \cdot n) u^2 + \int_{S^2} \int_{\Omega} \delta (s \cdot \nabla u)^2
$$

\begin{align*}
& \geq \frac{1}{2} \|u\|^2_0,
\end{align*}

which proves (11) for $\alpha_2 = 1/2$. This completes the discussion of the existence and uniqueness of the weak formulation (6) and (7).

2.2 Angular and spatial discretization

In this current subsection, we derive a semi-discrete form of the operator-split equations. It is well-known that the standard Galerkin finite element method for convection problems (7) induces spurious oscillation in the numerical solution. Therefore,
we prefer the SUPG method for spatial discretization. Since the numerical approximation of the double integral term in (6) will be compute-intensive, we implement DG(0) for the angular discretization.

Let $S_2^h$ be a subdivision of $S^2$ into a surface mesh, which is obtained by discretizing the unit sphere $S^2$ using hierarchical sectioning of the sphere into spherical triangles. In particular, the subdivision $S_2^h$ is obtained by projecting polyhedra onto the unit sphere. More details on this type of triangulation can be found in [28, Chapter 3] and [32]. Further, the mesh size of the spherical triangles $K_s$ in $S_2^h$ are denoted by

$$h_s := \max_{K_s \in S_2^h} h_{K_s}, \quad h_{K_s} := \text{diameter of cell } K_s.$$ 

Let $W_h \subset W$, a finite element space of piecewise constant polynomial, given by

$$W_h = \{v : v|_{K'} = c_{K_s}, \forall K_s \in S_2^h\}.$$ 

For the spatial discretization, let $\Omega_h$ be a family of shape regular triangulation of the domain $\Omega$. Further, the mesh size is denoted by

$$h_s := \max_{K \in \Omega_h} h_{K_s}, \quad h_{K_s} := \text{diameter of the cell } K.$$ 

And the finite element space of piecewise linear polynomials $V_h$ that vanish on the inlet boundary $\partial \Omega_{s-}$ is defined as

$$V_h := \{v \in C(\overline{\Omega}) : v|_K \in P_1(K), \forall K \in \Omega_h, v|_{\partial \Omega_{s-}} = 0\}.$$ 

For $u, v \in V_h$, the stabilized SUPG bilinear form is given by

$$a_{SUPG}(u, v) = a(u, v) + \sum_{K \in \Omega_h} \delta_K (s \cdot \nabla u, s \cdot \nabla v)_K,$$

where $\delta_K > 0$ is an user chosen stabilization parameter. For the convergence analysis, we assume that

$$0 \leq \delta_K \leq \delta_0 h, \quad \delta_0 > 0. \quad (12)$$

Further, the corresponding SUPG-norm is given by

$$\|v_h\|_{SUPG}^2 := \left(\sum_{K \in \Omega_h} \delta_K \|s \cdot \nabla v_h\|_{L^2(K)}^2 + \||s \cdot n|^{1/2} v_h\|_{L^2(\partial \Omega_{s+})}^2\right).$$

Moreover, the bilinear form associated with the SUPG discretization is coercive with respect to the $\| \cdot \|_{SUPG}$ by means of (11).
2.3 Semi-discrete method

Let \( \{ \phi \} \) and \( \{ \psi \} \) be the basis of the finite dimensional spaces \( W_h \) and \( V_h \), respectively, i.e,

\[
W_h = \text{span}\{\phi\}, \quad i = 1, 2, \ldots, N_s, \quad V_h = \text{span}\{\psi\}, \quad l = 1, 2, \ldots, N_x.
\]

Then, the finite element space \( P_{1,0}^h \) is defined as

\[
P_{1,0}^h := W_h \otimes V_h = \left\{ \zeta : \zeta = \sum_{i=1}^{N_s} \sum_{l=1}^{N_x} \zeta_{il} \phi_i \psi_l, \lambda_{il} \in \mathbb{R} \right\}.
\]

Any discrete function \( v_h \in P_{1,0}^h \) is given by

\[
v_h = \sum_{i=1}^{N_s} \sum_{l=1}^{N_x} v_{il}^h \phi_i(s) \psi_l(x)
\]

and associated advection operator \( s \cdot \nabla v_h \) is expressed as

\[
s \cdot \nabla v_h = s \sum_{i=1}^{N_s} \sum_{l=1}^{N_x} v_{il}^h \phi_i(s) \nabla \psi_l(x).
\]

The efficient way to handle these entries in the associated mass and stiffness matrices from the resulting finite element approximation is presented in Section 4.

Now, by using the finite element space \( P_{1,0}^h \), the respective semi-discrete form (6) and (7) read:

**Step 1.** Find \( \tilde{u}_{h,x}^n : (t^n, t^{n+1}) \to P_{1,0}^h \) with \( \tilde{u}_{h,x}^n(t^n) = u_{h,x}^n(t^n) \) such that

\[
\int_{S^2} (\tilde{u}_{h,x}^n, \zeta)_x + \int_{S^2} \mathbb{K} \tilde{u}_{h,x}^n, \zeta)_x = 0, \quad \forall \zeta \in P_{1,0}^h.
\]

**Step 2.** Find \( u_{h,x}^{n+1} : (t^n, t^{n+1}) \to P_{1,0}^h \) with \( u_{h,x}^{n+1}(t^n) = \tilde{u}_{h,x}^{n+1}(t^n) \) such that

\[
\int_{S^2} (u_{h,x}^{n+1}, \zeta)_x + \int_{S^2} \alpha_{SUPG}(u_{h,x}^{n+1}, \zeta) = \int_{S^2} (f, \zeta)_x + \int_{S^2} \sum_{K \in \Omega_h} \delta_K (f - u_{h,x}^{n+1}, s \cdot \nabla \zeta)_K, \quad \forall \zeta \in P_{1,0}^h.
\]

To simplify notations, we denote \( v_{h,x} \) by \( v_h \) and also use similar notations throughout this paper.
2.4 Temporal discretization

We consider a uniform partition of the time interval $[0, T]$ with $\Delta t = T / N$, i.e., $t_n = n \Delta t$, $n = 0, 1, \ldots, N$. After discretizing the temporal variable by the implicit backward Euler scheme, the fully discrete operator-split form of the model problem (1) reads:

**Step 1.** For a given $u^n h \in P^1_0$, find $\tilde{u}^{n+1} h \in P^1_0$ such that

$$
\int_{S^2} (\partial_{\Delta t} \tilde{u}^{n+1} h, \zeta)_x + \int_{S^2} (F \tilde{u}^{n+1} h, \zeta)_x = 0, \quad \zeta \in P^1_0,
$$

(15)

where $\partial_{\Delta t} \tilde{u}^{n+1} = (\tilde{u}^{n+1} h - u^n h) / \Delta t$.

**Step 2.** Update the solution $\tilde{u}^{n+1}$ from (15) by finding $u^{n+1} h \in P^1_0$ such that

$$
\int_{S^2} (\partial_{\Delta t} u^{n+1} h, \zeta)_x + \int_{S^2} a_{SU PG} (u^{n+1} h, \zeta)_x = \int_{S^2} (f^{n+1}, \zeta)_x + \int_{S^2} \sum_{K \in \Omega_h} \delta_K \left( f^{n+1} - \partial_{\Delta t} u^{n+1} h, s \cdot \nabla \zeta \right)_K, \quad \zeta \in P^1_0,
$$

(16)

where $\partial_{\Delta t} u^{n+1} = (u^{n+1} h - \tilde{u}^{n+1} h) / \Delta t$.

3 A priori error estimate : stability and convergence analysis

We now discuss the stability and the convergence analysis for the proposed numerical scheme. We first establish interpolation error estimates and then discuss the local truncation error of the two-step method. After that, both the local errors are combined to obtain a global error estimate.

3.1 Stability result

The stability estimate of the two-step operator-splitting method (15)-(16) is derived here.

**Theorem 1** Assume that the stabilization parameter $\delta_K$ satisfy

$$
\delta_K \leq \frac{\Delta t}{4}, \quad \delta = \max \{ \delta_K \}, \quad \Delta t \leq \frac{1}{2}.
$$

(17)

Then, the solutions $\tilde{u}^{n+1} h$ and $u^{n+1} h$ of the two-step algorithm (15) and (16) satisfy

$$
\|u_h^0\|^2_0 + \Delta t \sum_{m=0}^{n-1} \|u_h^{m+1}\|^2_{SU PG} \leq e^{2T} \left( \|u_h^0\|^2_0 + 2\Delta t (1 + 4\delta \Delta t) \sum_{m=0}^{n-1} \|f^{m+1}\|^2_0 \right).
$$

(18)
Proof. Setting $\zeta = \bar{u}_h^{n+1}$ in (15) and using $2(a-b)a = a^2 - b^2 + (a-b)^2$ with Lemma 2, we obtain
\begin{equation}
\frac{1}{2} \| \bar{u}_h^{n+1} \|_0^2 + \frac{1}{2} \| u_h^n - \bar{u}_h^n \|_0^2 + \Delta t \| G_h \| \bar{u}_h^{n+1} \|_0^2 \leq \frac{1}{2} \| u_h^n \|_0^2. \tag{19}
\end{equation}
By neglecting the positive terms from the left hand side of above equation to deduce that
\begin{equation}
\| \bar{u}_h^{n+1} \|_0^2 \leq \| u_h^n \|_0^2. \tag{20}
\end{equation}
Next, setting $\zeta = u_h^{n+1}$ in (16), we get
\begin{equation}
\frac{1}{2} \| u_h^{n+1} \|_0^2 - \frac{1}{2} \| u_h^{n+1} \|_0^2 + \frac{1}{2} \| u_h^n - u_h^{n+1} \|_0^2 + \frac{\Delta t}{2} \int_{S^2} \| u_h^{n+1} \|_{SUPG}^2 \leq \Delta t \int_{S^2} (f_{n+1}^{n+1} u_h^{n+1})_x + \Delta t \int_{S^2} \sum_{K \in \Omega_h} \delta_K (f_{n+1}^{n+1}, \nabla u_h^{n+1})_K
\end{equation}
\begin{equation}
+ \int_{S^2} \sum_{K \in \Omega_h} \delta_K (u_h^{n+1} - \bar{u}_h^{n+1}, \nabla u_h^{n+1})_K. \tag{21}
\end{equation}
Using C-S inequality and Young inequality, first two terms of the right hand side are bounded by
\begin{equation}
\Delta t \int_{S^2} (f_{n+1}^{n+1} u_h^{n+1})_x \leq \frac{\Delta t}{2} \| f_{n+1}^{n+1} \|_0^2 + \frac{\Delta t}{2} \| u_h^{n+1} \|_0^2. \tag{22}
\end{equation}
\begin{equation}
\Delta t \int_{S^2} \sum_{K \in \Omega_h} \delta_K (f_{n+1}^{n+1}, \nabla u_h^{n+1})_K \leq 2 \Delta t \int_{S^2} \sum_{K \in \Omega_h} \delta_K \| f_{n+1}^{n+1} \|_{L^2(K)}^2
\end{equation}
\begin{equation}
+ \frac{\Delta t}{8} \int_{S^2} \| u_h^{n+1} \|_{SUPG}^2. \tag{23}
\end{equation}
Again, employing C-S and Young inequalities with assumptions (17) to deduce that
\begin{equation}
\int_{S^2} \sum_{K \in \Omega_h} \delta_K (u_h^{n+1} - \bar{u}_h^{n+1}, \nabla u_h^{n+1})_K \leq \frac{1}{2} \| u_h^{n+1} \|_0^2 - \bar{u}_h^{n+1} \|_0^2 + \frac{\Delta t}{8} \int_{S^2} \| u_h^{n+1} \|_{SUPG}^2. \tag{24}
\end{equation}
Then, combining (22)–(24), we have
\begin{equation}
\| u_h^{n+1} \|_0^2 - \| \bar{u}_h^{n+1} \|_0^2 + \frac{\Delta t}{2} \int_{S^2} \| u_h^{n+1} \|_{SUPG}^2 \leq \Delta t \| u_h^n \|_0^2 + \Delta t \| f_{n+1}^{n+1} \|_0^2 + 2 \Delta t \int_{S^2} \sum_{K \in \Omega_h} \delta_K \| f_{n+1}^{n+1} \|_{L^2(K)}^2. \tag{25}
\end{equation}
This can be reduced as follows
\begin{equation}
(1 - \Delta t) \| u_h^n \|_0^2 + \frac{\Delta t}{2} \int_{S^2} \| u_h^{n+1} \|_{SUPG}^2 \leq \| \bar{u}_h^{n+1} \|_0^2 + \Delta t \| f_{n+1}^{n+1} \|_0^2 + \Delta t \int_{S^2} \sum_{K \in \Omega_h} \delta_K \| f_{n+1}^{n+1} \|_{L^2(K)}^2. \tag{26}
\end{equation}
using $1/(1 - \Delta t) \leq 1 + 2\Delta t \leq 2$. Employ (26) in (25), we deduce that
\[
\|u_{n+1}^m \|^2_0 + \Delta t \int_{S^2} \|u_{n+1}^m \|^2_{\text{SUPG}} \leq (1 + 2\Delta t) \|u_{n+1}^m \|^2_0 + 2\Delta t (1 + 4\delta \Delta t) \|f_{n+1}^m \|^2_0.
\]
Equation (27)

Adding stability results of both the steps (20) and (27), we have
\[
\|u_{n+1}^m \|^2_0 + \Delta t \sum_{m=0}^{n-1} \int_{S^2} \|u_{n+1}^m \|^2_{\text{SUPG}} \leq (1 + 2\Delta t) \|u_{n+1}^m \|^2_0 + 2\Delta t (1 + 4\delta \Delta t) \sum_{m=0}^{n-1} \|f_{n+1}^m \|^2_0.
\]
Equation (28)

Now summing over $m = 0, 1, \ldots, n - 1$, we get that
\[
\|u_{n+1}^m \|^2_0 + \Delta t \sum_{m=0}^{n-1} \int_{S^2} \|u_{n+1}^m \|^2_{\text{SUPG}} \leq 2\Delta t \sum_{m=0}^{n-1} \|u_{n+1}^m \|^2_0 + \|u_0^m \|^2_0 + 2\Delta t (1 + 4\delta \Delta t) \sum_{m=0}^{n-1} \|f_{n+1}^m \|^2_0.
\]
Equation (29)

By using discrete Grownwall’s lemma, we obtain the stated stability result of the lemma.

\[\text{Remark 1} \] In Theorem 1, the stability condition of the discrete method (15) and (16) is established with the stability parameter $\delta_K$ satisfies $\delta_K = O(\Delta t)$. From (12), we would be able to take $\Delta t \sim h$. For more details on the choice of stabilization parameter, one may see the detailed discussion in [26, 7].

3.2 Convergence analysis

In this subsection, error approximation for the numerical solution. To derive the error estimate of the operator-splitting finite element discretization (15)-(16), we denote
\[
\Pi_h = \pi_h^x \pi_h^s = \pi_h^s \pi_h^x.
\]
Here, $\pi_h^x v \in V_h$, the elliptic projection of $v \in V$ and $\pi_h^s w \in W_h$, the angular interpolant of $w \in W$. By applying the argumentation form [3, Lemma 4.2], we have
\[
\|w - \pi_h^s w\|^2_{L^2(S^2)} \leq C h_h^2 \|w\|^2_{H^1(S^2)}, \forall w \in H^1(S^2).
\]
Equation (30)

Using Galerkin orthogonality, $\pi_h^x u$ satisfies
\[
a_{\text{SUPG}}(\pi_h^x u, \nu_h) = a_{\text{SUPG}}(u, \nu_h) \forall \nu_h \in V_h.
\]

Applying as in (27), we have
\[
\|u - \pi_h^x u\|^2_{\text{SUPG}} \leq C h_h^3 \|u\|^2_{H^2(\Omega)}, \forall u \in V \cap H^2(\Omega).
\]
Equation (31)
Now, we discuss the error estimate of the discrete problems (15)-(16). The local truncation error in the first step (15) is denoted by

$$\tilde{E}_h^n = \tilde{u}_h^n - \Pi_h \tilde{u}(t^n),$$

where $\tilde{u}_h^n$ is the fully discrete solution of the first step (15) and $\tilde{u}$ is the weak solution of (6). The error term $\tilde{E}_h^{n+1}$ solves the following equation

$$\int_{S^2} (\partial_t \tilde{E}_h^{n+1}, \zeta)_x + \int_{S^2} \mathcal{A}(\tilde{E}_h^{n+1}, \zeta) = \int_{S^2} (I_1, \zeta)_x + \int_{S^2} \mathcal{A}(I_2, \zeta), \quad \zeta \in \mathcal{P}_h^{1,0}, \quad (32)$$

where

$$I_1 = \tilde{u}(t^{n+1}) - \partial_t \Pi_h \tilde{u}(t^{n+1}), \quad I_2 = \tilde{u}(t^{n+1}) - \Pi_h \tilde{u}(t^{n+1}),$$

and $\mathcal{A}(v, w) = (\mathcal{K}_0 v, w)_x$, for any discrete function $v$ and $w$.

Next, we discuss the error estimates for both steps (15) and (16) subsequently in the upcoming lemmas.

**Lemma 3** The local truncation error $\tilde{E}_h^{n+1}$ associated with the angular discretization satisfies

$$\sum_{m=0}^{n-1} (\|\tilde{E}_h^{m+1}\|_0^2 - \|E_h^m\|_0^2) \leq C \Delta t \int_0^T \|\tilde{u}_h\|_0^2 + \|\tilde{u}\|_{H^1(L^2)}^2 + \|\tilde{u}\|_{L^2(H^2)}^2 \right).$$

(33)

**Proof.** Setting $\zeta = \tilde{E}_h^{n+1}$ in (32), we get

$$\frac{1}{2}\|\tilde{E}_h^{n+1}\|_0^2 - \frac{1}{2}\|E_h^n\|_0^2 + \Delta t \int_{S^2} \mathcal{A}(\tilde{E}_h^{n+1}, \tilde{E}_h^{n+1})$$

$$\leq \Delta t \int_{S^2} |(I_1, \tilde{E}_h^{n+1})_x| + \Delta t \int_{S^2} |\mathcal{A}(I_2, \tilde{E}_h^{n+1})|.$$ 

(34)

By using the argumentation from (19), it can be deduced that

$$\frac{1}{2}\|\tilde{E}_h^{n+1}\|_0^2 - \frac{1}{2}\|E_h^n\|_0^2 \leq \Delta t \int_{S^2} |(I_1, \tilde{E}_h^{n+1})_x| + \Delta t \int_{S^2} |\mathcal{A}(I_2, \tilde{E}_h^{n+1})|.$$ 

(35)
Let us consider the first term on right hand side. Applying C-S and Young inequalities, we deduce that

\[
\Delta t \int_{S^2} |(I_1, E_{h}^{n+1})_x| \leq 4\Delta t \|u_t(t^{n+1}) - \partial_{x_1} u_{t+1}(t^{n+1})\|_0^2 + \frac{\Delta t}{16} \|E_{h}^{n+1}\|_0^2 \\
\leq 8\Delta t \|u_{t+1}(t^{n+1}) - \partial_{x_1} u_{t+1}(t^{n+1})\|_0^2 + \frac{\Delta t}{16} \|E_{h}^{n+1}\|_0^2 \\
+ 8\Delta t \|u_t(t^{n+1}) - \partial_{x_1} u_t(t^{n+1})\|_0^2 + \frac{\Delta t}{16} \|E_{h}^{n+1}\|_0^2 \\
\leq C\Delta t^2 \int_{S^2} \|u_{t+1}(t^{n+1})\|_0^2 + C\Delta t \|u_t(t^{n+1}) - \partial_{x_1} u_t(t^{n+1})\|_0^2 \\
+ C\Delta t \|\partial_{x_1} u_t(t^{n+1}) - \partial_{x_1} u_t(t^{n+1})\|_0^2 + \frac{\Delta t}{16} \|E_{h}^{n+1}\|_0^2 \\
\leq C\Delta t^2 \int_{S^2} \|u_{t+1}(t^{n+1})\|_0^2 + C\Delta t h^2 \|u_t\|_{L^2}^2 \\
+ C\Delta t h^2 \|u_t\|_{L^2}^2 + \frac{\Delta t}{16} \|E_{h}^{n+1}\|_0^2. \tag{36}
\]

Next, the second term is decomposed as

\[
\int_{S^2} \mathcal{A} (I_2, E_{h}^{n+1}) = \int_{S^2} \mathcal{A} (u(t^{n+1}) - \partial_{x_1} u_t(t^{n+1}), E_{h}^{n+1}) \\
+ \int_{S^2} \mathcal{A} (\partial_{x_1} u_t(t^{n+1}) - \partial_{x_1} u_t(t^{n+1}), E_{h}^{n+1})
\]

Using Lemma 2 with the interpolation and projection estimates, the first term can be deduced by means of C-S and Young inequalities,

\[
\Delta t \int_{S^2} |\mathcal{A} (u_t(t^{n+1}) - \partial_{x_1} u_t(t^{n+1}), E_{h}^{n+1})| \\
\leq \Delta t (2\sigma_s + \sigma_a) \|u_t(t^{n+1}) - \partial_{x_1} u_t(t^{n+1})\|_0 \|E_{h}^{n+1}\|_0 \tag{37}
\]

\[
\leq C\Delta t h^2 \|u_t\|_{H^1(L^2)}^2 + \frac{\Delta t}{32} \|E_{h}^{n+1}\|_0^2.
\]

Similarly, one can obtain that

\[
\Delta t \int_{S^2} |\mathcal{A} (\partial_{x_1} u_t(t^{n+1}) - \partial_{x_1} u_t(t^{n+1}), E_{h}^{n+1})| \leq C\Delta t h^2 \|u_t\|_{H^1(L^2)}^2 + C\Delta t h^2 \|u_t\|_{H^1(L^2)}^2 + \frac{\Delta t}{16} \|E_{h}^{n+1}\|_0^2. \tag{38}
\]

Combing estimates (37) and (38), we get

\[
\Delta t \int_{S^2} |\mathcal{A} (I_2, E_{h}^{n+1})| \leq C\Delta t h^2 \|u_t\|_{H^1(L^2)}^2 + C\Delta t h^2 \|u_t\|_{H^1(L^2)}^2 + \frac{\Delta t}{16} \|E_{h}^{n+1}\|_0^2. \tag{39}
\]

Next, employing (36) and (39) in (35), it can be deduced that

\[
\frac{1}{2} \|E_{h}^{n+1}\|_0^2 + \frac{1}{2} \|E_{h}^{n+1}\|_0^2 + \Delta t \sigma_w \|E_{h}^{n+1}\|_0^2 \leq C\Delta t^2 \int_{S^2} \|u_{t+1}(t^{n+1})\|_0^2 \\
+ C\Delta t h^2 \|u_t\|_{L^2}^2 + C\Delta t h^2 \|u_t\|_{L^2}^2 + \frac{\Delta t}{8} \|E_{h}^{n+1}\|_0^2. \tag{40}
\]
By using assumption $2$, it is reduced as
\[ \frac{1}{2} \| \bar{E}_h^{n+1} \|^2_0 - \frac{1}{2} \| E_h^n \|^2_0 \leq C \Delta t^2 \int_0^{n+1} \| \Pi_h \tilde{u}_t(t_{n+1}) \|^2_0 \]
\[ + C \Delta t h_s^2 \| \tilde{u}_t \|^2_{H^1(V)} + C \Delta t h_s^2 \| \tilde{u}_t \|^2_{L^2(V^*)}. \]
(41)

Finally, summing over $m = 1, 2, \ldots, n - 1$, we obtain the required results.

**Remark 2** Note that the coefficient of $\| \bar{E}_h^{n+1} \|^2_0$ in the left side of (41) can be taken smaller as per our convenience by the means of C-S and Young inequalities.

Next, we discuss the bound for the local truncation error term for the second step, which is given by
\[ E_h^n = u_h^n - \Pi_h u(t^n). \]

Let $\pi_h u \in V_h$ be the elliptic projection of $u \in V$, we have
\[ a_{SUPG}(\pi_h u, v_h) = a_{SUPG}(u, v_h), \quad \forall v_h \in V_h. \]

Then, one can claim that
\[ a_{SUPG}(\pi_h u, v_h) = (\pi_h f - \pi_h u, v_h + \delta s \cdot \nabla v_h), \quad \forall v_h \in V_h. \]

And, we have
\[ \int_{S^2} a_{SUPG}(\Pi_h u, \zeta) = \int_{S^2} (\pi_h f - \pi_h u, \zeta + \delta s \cdot \nabla \zeta)_x, \quad \forall \zeta \in \mathcal{H}^{1,0}_h. \]

(42)

The error term $E_{h}^{n+1}$ satisfies the following equation
\[ \int_{S^2} (\partial_M E_{h}^{n+1} , \zeta)_x + \int_{S^2} a_{SUPG}(E_{h}^{n+1}, \zeta) \]
\[ = \int_{S^2} \left( \Lambda_{u}^{n+1} + \Lambda_{f}^{n+1} + \delta s \cdot \nabla \zeta \right)_x - \delta \int_{S^2} (\partial_M E_{h}^{n+1} , s \cdot \nabla \zeta)_x, \quad \forall \zeta \in \mathcal{H}^{1,0}_h, \]
\[ \text{where functions } \Lambda_{u}^{n+1} \text{ and } \Lambda_{f}^{n+1} \text{ are defined as} \]
\[ \Lambda_{u}^{n+1} = (\pi_h u(t_{n+1}) - \partial_M u(t_{n+1})) \quad \text{and} \quad \Lambda_{f}^{n+1} = f_{n+1} - \pi_h f(t). \]

**Lemma 4** The local truncation error $E_{h}^{n+1}$ associated with the spatial discretization satisfies
\[ \sum_{m=0}^{n-1} \left( \| E_{h}^{m+1} \|^2_0 - \| E_{h}^{m+1} \|^2_0 \right) + \frac{\Delta t}{2} \sum_{m=0}^{n-1} \int_{S^2} \| E_{h}^{m+1} \|^2_{SUPG} \]
\[ \leq C \left[ \Delta t^2 \int_0^T (\| u_t \|^2_0 + \| u_{ttt} \|^2_0) \right] + C \Delta t h_s^2 \sum_{m=0}^{n-1} \left( \| u \|^2_{L^2(H^1)} + \| u_t \|^2_{L^2(H^2)} + \| u_{tt} \|^2_{L^2(H^1)} \right) \]
\[ + C \Delta t h_s^2 \sum_{m=0}^{n-1} \left( \| u \|^2_{H^1(V)} + \| u_t \|^2_{H^2(V)} + \| u_{tt} \|^2_{H^1(V)} \right). \]
(44)
Proof. Setting $\zeta = E_{n+1}^h$ in (43), we obtain
\[
\frac{1}{2\Delta t} \left( \|E_{n+1}^h\|_0^2 - \|E_{n+1}^h\|_0^2 \right) + \frac{1}{2} \int_{S^2} \|E_{n+1}^h\|_{SUPG}^2 
\leq \int_{S^2} \left( A_{u,n}^{n+1} + A_f^{n+1}, E_{n+1}^h + \delta s \cdot \nabla E_{n+1}^h \right)_x \right) + \delta \int_{S^2} |(\partial_{\Delta t} E_{n+1}^h \cdot s \cdot \nabla E_{n+1}^h)_x |.
\]
Applying C-S inequality, Young inequality and summing over $m = 0, 1, \ldots, n - 1$, we get
\[
\sum_{m=0}^{n-1} \left( \|E_{m+1}^h\|_0^2 - \|E_{m+1}^h\|_0^2 \right) + \frac{\Delta t}{2} \sum_{m=0}^{n-1} \int_{S^2} \|E_{m+1}^h\|_{SUPG}^2 
\leq 10(1 + \delta) \sum_{m=0}^{n-1} \Delta t \left( ||A_{u,m+1}^n||_0^2 + ||A_f^{m+1}||_0^2 \right) + 10\delta \sum_{m=0}^{n-1} \Delta t ||\partial_{\Delta t} E_{m+1}^h||_0^2.
\]
(45)
Next, we discuss estimate for last term of the right hand side, i.e., $||\partial_{\Delta t} E_{m+1}^h||_0^2$. Consider $\chi^{n+1} = \partial_{\Delta t} E_{n+1}^h$ in (43), it satisfies the following equation
\[
\int_{S^2} (\partial_{\Delta t} \chi^{n+1}, \zeta)_x + \int_{S^2} a_{SUPG}(\chi^{n+1}, \zeta)
\]
(46)
\[
= \int_{S^2} \left( (\partial_{\Delta t} A_{u}^{n+1} + \partial_{\Delta t} A_f^{n+1}, \zeta + \delta s \cdot \nabla \zeta)_x + \delta \int_{S^2} (\partial_{\Delta t} \chi^{n+1}, s \cdot \nabla \chi^{n+1})_x, \right)
\]
where $\partial_{\Delta t} \chi^{n+1} = (\chi^{n+1} - \chi^n)/\Delta t$ and $\partial_{\Delta t} A_{u}^{n+1}, \partial_{\Delta t} A_f^{n+1}$ are defined in a similar way.
Further, assigning $\zeta = \chi^{n+1} + \delta \partial_{\Delta t} \chi^{n+1}$ in (46) and after small simplification, we deduce
\[
\frac{1}{2\Delta t} \left( \|\chi^{n+1}\|_0^2 - \|\chi^n\|_0^2 \right) + \delta \|\partial_{\Delta t} \chi^{n+1}\|_0^2 + \delta \|s \cdot \nabla \chi^{n+1}\|_0^2 + \frac{1}{2} ||(s \cdot n) \chi^{n+1}||_{L^2(F_n)}^2
\]
\[
+ \frac{\delta^2}{2\Delta t} \left( ||s \cdot \nabla \chi^{n+1}||_0^2 - ||s \cdot \nabla \chi^n||_0^2 \right) + 2\delta \int_{S^2} \left( \partial_{\Delta t} \chi^{n+1}, s \cdot \nabla \chi^{n+1} \right)_x 
\]
\[
\leq \int_{S^2} \left( (\partial_{\Delta t} A_{u}^{n+1} + \partial_{\Delta t} A_f^{n+1}, \chi^{n+1} + \delta s \cdot \nabla \chi^{n+1})_x \right) 
\]
\[
+ \delta \int_{S^2} \left( (\partial_{\Delta t} A_{u}^{n+1} + \partial_{\Delta t} A_f^{n+1}, \partial_{\Delta t} \chi^{n+1} + \delta s \cdot \nabla \chi^{n+1})_x \right) 
\]
\[
- \delta^2 \int_{S^2} \left( \partial_{\Delta t} \chi^{n+1}, s \cdot \nabla (\partial_{\Delta t} \chi^{n+1}) \right)_x.
\]
(47)
Next, we define the norm $||\cdot||_{SS}$ as follows:
\[
||\chi^{n+1}||_{SS} = \delta \left( ||s \cdot \nabla \chi^{n+1} + \partial_{\Delta t} \chi^{n+1}||_0^2 + \frac{1}{2} ||(s \cdot n) \chi^{n+1}||_{L^2(F_n)}^2 \right).
\]
By using the above norm definition, we deduce

\[
\frac{1}{2\Delta t} (\|\mathbf{X}^{n+1}\|_0^2 - \|\mathbf{X}^n\|_0^2) + \|\mathbf{X}^{n+1}\|_\infty^2 + \frac{\delta^2}{2\Delta t} (\|\mathbf{s} \cdot \nabla \mathbf{X}^{n+1}\|_0^2 - \|\mathbf{s} \cdot \nabla \mathbf{X}^n\|_0^2)
\leq \int_{\Omega} \left( \partial_{\Delta t} A_{\mathbf{u}}^{n+1} + \partial_{\Delta t} A_{\mathbf{f}}^{n+1}, \mathbf{X}^{n+1} + \delta \mathbf{s} \cdot \nabla \mathbf{X}^{n+1} \right)_x + \delta \int_{\Omega} \left( \partial_{\Delta t} A_{\mathbf{u}}^{n+1} + \partial_{\Delta t} A_{\mathbf{f}}^{n+1}, \partial_{\Delta t} \mathbf{X}^{n+1} + \delta \mathbf{s} \cdot \nabla \partial_{\Delta t} \mathbf{X}^{n+1} \right)_x
\]

(48)

By using C-S and Young inequalities, we have estimated the right side terms of equation (48). After small mathematical simplification, we obtain the following inequality

\[
\|\mathbf{X}^{n+1}\|_0^2 - \|\mathbf{X}^n\|_0^2 + \frac{\Delta t}{2} \|\mathbf{X}^{n+1}\|_\infty^2 + \delta^2 (\|\mathbf{s} \cdot \nabla \mathbf{X}^{n+1}\|_0^2 - \|\mathbf{s} \cdot \nabla \mathbf{X}^n\|_0^2)
\leq \Delta t \|\mathbf{X}^{n+1}\|_0^2 + \Delta t \|\partial_{\Delta t} A_{\mathbf{u}}^{m+1}\|_0^2 + \|\partial_{\Delta t} A_{\mathbf{f}}^{m+1}\|_0^2.
\]

(49)

By summing over \( m = 1, \ldots, n-1 \) and repeating the argument from Theorem 1, we deduce that

\[
\|\mathbf{X}^n\|_0 + \Delta t \sum_{m=1}^{n-1} \|\mathbf{X}^{m+1}\|_\infty^2 + \delta^2 \|\mathbf{s} \cdot \nabla \mathbf{X}^n\|_0^2 \leq C \exp^{\Delta t} \left( \|\mathbf{X}^0\|_0^2 + C \delta^2 \|\mathbf{X}^n\|_0^2 + \sum_{m=1}^{n-1} \left( \|\partial_{\Delta t} A_{\mathbf{u}}^{m+1}\|_0^2 + \|\partial_{\Delta t} A_{\mathbf{f}}^{m+1}\|_0^2 \right) \right).
\]

(50)

Now substituting the estimate of \( \|\partial_{\Delta t} E_{h}^{m+1}\|_0^2 \) from (50) in (45) and using the above inequality to get

\[
\sum_{m=0}^{n-1} \left( \|E_{h}^{m+1}\|_0^2 - \|E_{h}^{m+1}\|_0^2 \right) + \frac{\Delta t}{2} \sum_{m=0}^{n-1} \int_{\Omega} E_{h}^{m+1} \|\mathbf{s}\|_{\text{UPG}}^2 \leq C \delta \|\partial_{\Delta t} E_{h}^{1}\|_0^2
\]

\[
+ C \Delta t \left[ \sum_{m=0}^{n-1} \left( \|A_{\mathbf{u}}^{m+1}\|_0^2 + \|A_{\mathbf{f}}^{m+1}\|_0^2 \right) + \delta \sum_{m=0}^{n-1} \left( \|\partial_{\Delta t} A_{\mathbf{u}}^{m+1}\|_0^2 + \|\partial_{\Delta t} A_{\mathbf{f}}^{m+1}\|_0^2 \right) \right].
\]

(51)

Applying the standard interpolation results and Taylor’s theorem with integral remainder term, we deduce

\[
\Delta t \sum_{m=0}^{n-1} \|A_{\mathbf{u}}^{m+1}\|_0^2 \leq C \Delta t^2 \sum_{m=0}^{n-1} \int_{\Omega} \|\Pi_h u_{t\Delta t}\|_0^2 + C \Delta t \|\pi_h u_{t\Delta t}\|_0^2 \sum_{m=0}^{n-1} \|\pi_h u_{t\Delta t}\|_0^2.
\]

(52)

\[
\Delta t \sum_{m=0}^{n-1} \|\partial_{\Delta t} A_{\mathbf{u}}^{m+1}\|_0^2 \leq C \Delta t^2 \sum_{m=0}^{n-1} \int_{\Omega} \|\Pi_h u_{t\Delta t}\|_0^2 + C \Delta t \|\pi_h u_{t\Delta t}\|_0^2 \sum_{m=0}^{n-1} \|\pi_h u_{t\Delta t}\|_0^2.
\]

(53)
In a similar way, we have
\[
\Delta t \sum_{m=0}^{n-1} \| \Lambda_f^{m+1} \|^2_0 \leq C \Delta t h^2_{\omega} \sum_{m=0}^{n-1} \| \partial \kappa f(t^{m+1}) \|^2_0. \tag{54}
\]
\[
\Delta t \sum_{m=0}^{n-1} \| \partial_x A^{m+1} \|^2_0 \leq C \Delta t^2 \int_0^T \| f_{\kappa} \|^2_0 + C \Delta t h^2_{\omega} \sum_{m=0}^{n-1} \| \partial_x f(t^{m+1}) \|^2_0. \tag{55}
\]

Next, we need to evaluate bounds for term \(\| \partial \Delta t E_h^1 \|^2_0\). By using the argument from [1] and following similar technique as in (50), we devise that
\[
\delta \| \partial \Delta t E_h^1 \|^2_0 \leq C(\Delta t^2 + h^3_{\omega} + h^2_{\omega}). \tag{56}
\]

By combing the estimates (52)-(56) in (51), we obtain the desired result (44). This completes the proof.

Next, we present the convergence estimate of the operator-splitting finite element method (15)-(16) for the model problem (1).

**Theorem 2** The global error \(e^n = u - u^n_h\) satisfies
\[
\| e^n \|^2_0 + \frac{\Delta t}{2} \sum_{m=0}^{n-1} \int_{\Delta t} \| e^{m+1} \|^2_{SUPG} \leq C(\| u, \tilde{u} \|)(\Delta t^2 + h^3_{\omega} + h^2_{\omega}). \tag{57}
\]

**Proof.** By combining the local error estimates (43) and (44), we obtain that
\[
\| E_h^n \|^2_0 + \frac{\Delta t}{2} \sum_{m=0}^{n-1} \int_{\Delta t} \| E_h^{m+1} \|^2_{SUPG} \\
\leq \| E_h^n \|^2_0 + C \Delta t \left[ \Delta t \int_0^T \| \bar{u} \|^2_0 + h^2_{\omega} \| \bar{u} \|^2_{H^1(\Omega^2)} + h^3_{\omega} \| \bar{u} \|^2_{L^2(\Omega^2)} \right] \\
+ C \left[ \Delta t^2 \int_0^T (\| u_\kappa \|^2_0 + \| u_{tt} \|^2_0) \right] \\
+ \Delta t h^2_{\omega} \sum_{m=0}^{n-1} \left( \| u \|^2_{L^2(\Omega^2)} + \| u_\kappa \|^2_{H^1(\Omega^2)} + \| u_{tt} \|^2_{H^1(\Omega^2)} \right) \\
+ \Delta t h^2_{\omega} \sum_{m=0}^{n-1} \left( \| u \|^2_{H^1(\Omega^2)} + \| u_\kappa \|^2_{H^1(\Omega^2)} + \| u_{tt} \|^2_{H^1(\Omega^2)} \right).
\]

Noting that \( E_h^0 = 0 \). Then, the estimate (58) can be further simplified as
\[
\| E_h^n \|^2_0 + \frac{\Delta t}{2} \sum_{m=0}^{n-1} \int_{\Delta t} \| E_h^{m+1} \|^2_{SUPG} \leq C(\| u, \tilde{u} \|)(\Delta t^2 + h^3_{\omega} + h^2_{\omega}), \tag{59}
\]
where \( C(u, \tilde{u}) \) is the positive constant, depending upon \( u, \tilde{u} \) in (53). Finally, by employing the approximation results (30) and (31) and using above argumentation, the main convergence result (52) is devised. This completes the proof.
4 Computational Results

In this section, we present the numerical algorithm for the proposed discrete scheme and a validation of the theoretical estimates.

4.1 Numerical implementation

We briefly discuss the operator-splitting algorithm for the radiative transfer equation solution, whereas more details on implementation for general scalar equations can be found in [20] and [6]. All the numerical experiments are performed in our in-house finite element package [18, 37].

In Section 2, the finite element space \( P_{1,0}^h \) is given as follows

\[
\mathcal{P}_{1,0}^h := W_h \otimes V_h = \left\{ \lambda : \lambda = \sum_{i=1}^{N_s} \sum_{l=1}^{N_x} \lambda_{il} \phi_i \psi_l, \lambda_{il} \in \mathbb{R} \right\}.
\]

Further, the discrete solution \( u_n^h \in \mathcal{P}_{1,0}^h \) and its gradient are expressed by

\[
u_n^h(x,s) = \sum_{i=1}^{N_s} \sum_{l=1}^{N_x} u_{il}^n \phi_i(s) \psi_l(x), \quad \nabla u_n^h = \sum_{i=1}^{N_s} \sum_{l=1}^{N_x} u_{il}^n \phi_i(s) \nabla \psi_l(x),
\]

where \( u_{il} \) are the unknown degrees of freedom (DOFs). Define the mass matrices \( M_1^k, M_2^k \in \mathbb{R}^{N_s \times N_s} \), where the \((i,j)\)th entries of these matrices are given by

\[
(M_1^k)_{ij} = \int_{\Omega} \phi_i \phi_j ds, \quad (M_2^k)_{ij} = \int_{\Omega} \nabla \phi_i \phi_j ds.
\]

Further, the \((l,m)\)th entries of the matrices \( M_k, A_k, M_2^k, A_2^k \) are given by

\[
(M_k)_{lm} = \int_{\Omega} \psi_l \psi_m d\mathbf{x}, \quad (M_2^k)_{lm} = \sum_{K \in \Omega_h} \delta_K(\psi_l, \nabla \psi_m) d\mathbf{x},
\]

\[
(A_k)_{lm} = \int_{\Omega} s \cdot \nabla \psi_l \psi_m d\mathbf{x}, \quad (A_2^k)_{lm} = \sum_{K \in \Omega_h} \delta_K(s \cdot \nabla \psi_l, s \cdot \nabla \psi_m) d\mathbf{x},
\]

and \(m\)th component of the load vectors \( f_n^h, f_2^h \) are given by

\[
(F_n^h)_{m} = \int_{\Omega} f_n^h \psi_m d\mathbf{x}, \quad (F_2^h)_{m} = \sum_{K \in \Omega_h} \delta_K(f_n^h, s \cdot \nabla \psi_m) d\mathbf{x}.
\]

Here the matrices \( M_2^k, A_2^k \) and \( f_2^h \) are due to the SUPG stabilization terms.

Further, we use \( \tilde{u}_{k+1}^n \) to denote an array of unknown solution coefficients (DOFs) \( \{\tilde{u}_{k+1}^n\}_k \), \( k = 1, 2, \ldots, N_x, i = 1, 2, \ldots, N_s \). With these notations, the system matrix of the \( s \)-direction step in the time interval \( [t^n, t^{n+1}] \) becomes:
For given \( u^n_k \), solve
\[
M_s \tilde{u}^{n+1}_k = M^1_s u^n_k, \quad M_s = \left( M^1_s + \Delta t \sigma_\ell M^1_s - \Delta t \sigma_m M^2_s \right),
\]
for \( k = 1, 2, \ldots, N_x \). In this way, we solve numerical solution in the \( s \)-direction.

We next discuss the \( x \)-direction step, where the updated solution from the \( s \)-direction is used to compute the solution of (15). In the \( x \)-direction step, we first need to transpose the vector \( \tilde{u}^{n+1}_k \) to obtain \( \tilde{u}^{n+1}_k \) and then solve the linear system
\[
\left( M_x + M^0_x + \Delta t (A_x + A^0_x) \right) u^{n+1}_x = \Delta t (F^n_x + F^0_x) + \left( M_x + M^0_x \right) \tilde{u}^{n+1}_k,
\]
for \( \ell = 1, 2, \ldots, N_x \). Though the mass matrix \( M_x \) is independent of \( s \), all other matrices in (61) depend on \( s \). Therefore, all these \( s \)-dependent matrices need to be assembled for each \( \ell \) in all time steps. However, the matrix assembling can be avoided for every \( \ell \) by assembling and storing the \( s \)-dependent matrices in a component form. For example, \( A_x \) can be split as
\[
(A_x)_{lm} = s_1 \int_{\Omega_h} \frac{\partial \psi_l}{\partial x} \psi_m dx + s_2 \int_{\Omega_h} \frac{\partial \psi_l}{\partial y} \psi_m dx + s_3 \int_{\Omega_h} \frac{\partial \psi_l}{\partial z} \psi_m dx,
\]
\[
= s_1 (A_x^{III})_{lm} + s_2 (A_x^{II})_{lm} + s_3 (A_x^{I})_{lm}.
\]

Hence, it is enough to assemble matrices \( A^{III}_x \), \( A^{II}_x \) and \( A^{I}_x \) only once and then multiply it with \( s_1 \), \( s_2 \) and \( s_3 \), respectively, for every \( \ell \) in each time step. Following a similar technique for \( s \)-dependent matrices, matrix assembling for each \( \ell \) in every time step can completely be avoided, and it is enough to assemble all these component matrices only once at the beginning of the computation. In this way, we can solve the linear system (61) very efficiently.

4.2 Validation

To validate the theoretical estimates discussed in the previous sections, we consider multiple test examples using manufactured solutions. The scattering phase function \( \Phi \) is taken from the previous studies [5, 36]. For time discretization, the backward Euler time-stepping method is applied with a final time \( T = 1 \) and time step \( \Delta t = \Delta h \) in all the numerical experiments. Furthermore, the value of absorption and scattering coefficients are taken as \( \sigma_\ell = 2 \), \( \sigma_m = 1/2 \) for all the test problems. For validation purposes, we discuss the error estimate of the numerical approximation in the spatial domain \( \mathcal{G} \) and time-domain \((0, 1)\) by using the following norm:
\[
\ell_2(0, 1; L^2(\mathcal{G})) = \left( \Delta t \sum_{n=1}^N \| u(t^n) - u^n_h \|_{L^2(\mathcal{G})}^2 \right)^{1/2},
\]
where \( L^2 \) error is calculated in \( x \)- and \( s \)-directions, i.e.
\[
\| u(t^n) - u^n_h \|_{L^2(\mathcal{G})}^2 = \int_{\mathcal{G}} (u(t^n) - u^n_h)^2 dx ds.
\]
For all the numerical experiments, the mesh mesh parameters are taken as $N_x = 27, 125, 729, 4913$ and $N_s = 48, 192, 768, 3072$.

**Example 1** Consider the model problem (1) with the exact solution as

$$u(x, s, t) = e^{-\alpha t} \sin(\pi x_1) \sin(\pi x_2) \sin(\pi x_3), \quad \alpha = 0.1,$$

where the source term $f$ is given by

$$f(x, s, t) = (\sigma_e - \alpha - \sigma_s) u + \pi e^{-\alpha t} s_1 (\cos(\pi x_1) \sin(\pi x_2) \sin(\pi x_3))$$
$$+ \pi e^{-\alpha t} s_2 (\sin(\pi x_1) \cos(\pi x_2) \sin(\pi x_3))$$
$$+ \pi e^{-\alpha t} s_3 (\sin(\pi x_1) \sin(\pi x_2) \cos(\pi x_3)).$$

And the scattering phase function $\Phi(s, s') = \frac{2 + 2 s \cdot s'}{4\pi}$.

To verify the accuracy of the numerical approximation, we discuss the discretization errors in the solution of the same example. In particular, we have presented the discretization error in the above-defined norm to authenticate the theoretical results. From Table 1, we can see the order of converges, as expected, with the exact solution’s sufficient regularity.

| Level | $L^2$ order | $L^2(0,1;L^2(\theta))$ order |
|-------|-------------|-----------------------------|
| 1     | 2.6088e-01  | 2.6855e-01                  |
| 2     | 6.3417e-02  | 2.0404 8.1076e-02 1.7278    |
| 3     | 1.9910e-02  | 1.7313 2.7095e-02 1.5812    |
| 4     | 8.1660e-03  | 1.2259 9.6150e-03 1.4947    |

**Example 2** Consider the model problem (1) with the exact solution

$$u(x, s, t) = e^{-\alpha t} s_3 \sin(\pi x_1) \sin(\pi x_2) \sin(\pi x_3), \quad \alpha = 0.1,$$

where the source term $f$ is given by

$$f(x, s, t) = (\sigma_e - \alpha - \sigma_s \eta \cos \theta) u + \pi e^{-\alpha t} s_1 s_3 (\cos(\pi x_1) \sin(\pi x_2) \sin(\pi x_3))$$
$$+ \pi e^{-\alpha t} s_2 s_3 (\sin(\pi x_1) \cos(\pi x_2) \sin(\pi x_3))$$
$$+ \pi e^{-\alpha t} s_3^2 (\sin(\pi x_1) \sin(\pi x_2) \cos(\pi x_3)).$$

And, the Henyey-Greenstein phase function is considered as $\Phi(s, s') = \frac{1 - \eta^2}{4\pi (1 + \eta^2 - 2\eta s \cdot s')^{3/2}}$, where the anisotropy factor $\eta \in (-1, 1)$. 
We have discussed the convergence estimate of this test example with the anisotropy factor $\eta = 0.5$. The discretization error with the convergence order is presented in Table 2. These findings again confirm the error estimates of the numerical approximation achieved in the theoretical findings.

One can see that the convergence order is less than 1.5, and it is due to the dependence of the angular variable on the exact solution in both the test problem. Since DG(0) element is used, the optimal discretization error is almost first-order. The numerical results conclude that we can use tailored numerical methods in the operator-splitting finite element methods. It also explains that the convergence error is not affected by the consistency error induced by the Lie–Trotter splitting technique in the backward Euler heterogeneous finite element method.

### 5 Conclusion and Discussion

An operator-splitting finite element method for the time-dependent, high-dimensional radiative transfer equation is proposed in this paper. The numerical scheme combines the backward Euler scheme, SUPG method, and DG(0) for time, space, and angular discretization. The stability and consistency are established for the fully discrete scheme. Further, the convergence estimate with optimal order is derived. Moreover, the operator-splitting algorithm to compute the solution is also presented. An array of numerical experiments are performed to support the theoretical estimates and validate the proposed algorithm. The computed numerical results validate the implementation and confirm the derived error estimate.

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**Declarations**

**Conflict of interest** The authors declare no competing interests.
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