Exact general solutions for cosmological scalar field evolution in a background-dominated expansion

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We derive exact general solutions (as opposed to attractor particular solutions) and corresponding first integrals for the evolution of a scalar field \( \phi \) in a universe dominated by a background fluid with equation of state parameter \( w_B \). In addition to the previously-examined linear \([V(\phi) = V_0\phi]\) and quadratic \([V(\phi) = V_0\phi^2]\) potentials, we show that exact solutions exist for the power law potential \( V(\phi) = V_0\phi^n \) with \( n = 4(1 + w_B)/(1 - w_B) + 2 \) and \( n = 2(1 + w_B)/(1 - w_B) \). These correspond to the potentials \( V(\phi) = V_0\phi^6 \) and \( V(\phi) = V_0\phi^2 \) for matter domination and \( V(\phi) = V_0\phi^{10} \) and \( V(\phi) = V_0\phi^4 \) for radiation domination. The \( \phi^6 \) and \( \phi^{10} \) potentials can yield either oscillatory or non-oscillatory evolution, and we use the first integrals to determine how the initial conditions map onto each form of evolution. The exponential potential yields an exact solution for a stiff/kination \((w_B = 1)\) background. We use this exact solution to derive an analytic expression for the evolution of the equation of state parameter, \( w_\phi \), for this case.

I. INTRODUCTION

Scalar fields providing a significant component of the energy density of the universe have frequently been invoked in cosmology. They were first introduced as the main component of models for inflation (see, e.g., Refs. \([1, 2]\) for reviews). Later, under the name “quintessence,” scalar fields were introduced as an alternative to the cosmological constant as a mechanism to drive the observed accelerated expansion of the universe \([3, 4, 5–8, 10, 11]\). (See Refs. \([12, 13]\) for reviews). More recently, the possibility that a scalar field might contribute subdominantly to the expansion has been proposed as a possible solution to the “Hubble tension,” the discrepancy between direct local measurements of the Hubble parameter and the value inferred from measurements of the cosmic microwave background \([14–16]\). (The viability of this proposal remains controversial; see, e.g., Refs. \([17–20]\) for arguments against it, and Refs. \([21–23]\) for counterarguments).

The equation governing the evolution of a scalar field \( \phi \) in a potential \( V(\phi) \) is

\[
\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0,
\]

where the Hubble parameter \( H \) is given by

\[
H \equiv \frac{\dot{a}}{a} = \sqrt{\rho/3}.
\]

In this equation, \( a \) is the scale factor, \( \rho \) is the total density, and we take \( 8\pi G = c = \hbar = 1 \) throughout.

For most choices of \( V(\phi) \), Eq. (1) is nonlinear, and exact solutions cannot be derived. On the other hand, particular solutions can often be determined for specific potentials of interest. These particular solutions lack arbitrary constants and so cannot be fit to a given set of initial conditions on \( \phi \). However, these solutions can often be shown, under certain conditions, to act as attractors, so that they describe the asymptotic behavior of \( \phi \) for a wide range of initial conditions. These attractor solutions have been exhaustively studied \([3, 5–8, 10, 11]\).

Here we are concerned with a more difficult issue: are there any known exact solutions or first integrals for Eq. (1)? In the context of inflation, the appropriate choice for \( \rho \) in Eq. (2) is the energy density of the scalar field itself:

\[
\rho_\phi = \frac{\dot{\phi}^2}{2} + V(\phi).
\]

In this case, the Hamilton-Jacobi formulation \([24]\), which involves taking \( H \) as the dependent variable and \( \phi \) as the independent variable, allows Eq. (1) to be rewritten as a first-order equation:

\[
\left( \frac{dH}{d\phi} \right)^2 - \frac{3}{2} H(\phi)^2 = -\frac{1}{2} V(\phi).
\]

Using Eq. (1), it is straightforward to begin with a desired choice for \( H(\phi) \) and derive a corresponding potential \( V(\phi) \); however, the reverse is not true. Eq. (4) remains nonlinear and does not, in general, yield exact solutions for
arbitrary choices of $V(\phi)$. However, there is one exception: for the case where $V(\phi)$ is an exponential potential, it is possible to derive exact (parametric) solutions [25–28]. (See also the approach in Ref. [29]).

When analyzing quintessence models instead of inflation, the choice for $\rho$ in Eq. (2) becomes more complicated. Now we must include both the density of the scalar field as well as the density of any additional background (radiation or nonrelativistic matter). Denoting the latter by $\rho_B$, we have

$$\rho = \rho_B + \frac{\dot{\phi}^2}{2} + V(\phi).$$

In general, Eq. (1) is intractable in this case, although approximate solutions have been derived for certain conditions on the potential when $\rho_B$ represents nonrelativistic matter [30–33]. An exact solution was claimed for the case of nonrelativistic matter plus a scalar field with an exponential potential in Ref. [34], but this solution was later shown to be flawed [35].

However, in some physically-interesting cases, the contribution of the scalar field to the energy density can be neglected in comparison to the density of the background component. It is these models which are the subject of this paper. For quintessence, this will be the case whenever the quintessence density is initially subdominant and becomes important only at late times. (This is effectively the case for which attractor solutions are derived in Refs. [3, 5–8, 10, 11]). Furthermore, the scalar field models invoked to resolve the Hubble tension [14–16] assume a transient density contribution from the scalar field that is always subdominant relative to the background (radiation or matter) density.

While a handful of exact solutions have previously been discussed for constant, linear, and quadratic potentials, we show here that there are exact solutions for a variety of other potentials. While almost none of these solutions are “new” in the sense of being unknown in the mathematics literature, they have not been previously applied to cosmological scalar field evolution. In the next section, we review the general evolution for a scalar field in a barotropic background. In Sec. III, we re-examine the previously-derived exact solutions for constant, linear, and quadratic potentials. In Sec. IV, we derive exact solutions and first integrals for power-law potentials, and we use the first integrals to solve an interesting question for potentials that support both oscillatory and non-oscillatory behavior. In Sec. V, we examine exact solutions for exponential potentials. We discuss our results briefly in Sec. VI.

II. SCALAR FIELD EQUATION OF MOTION IN A BAROTROPIC BACKGROUND

We will assume that the expansion of the universe is dominated by a barotropic fluid with an equation of state parameter $w_B \equiv p_B/\rho_B$, where $p_B$ and $\rho_B$ are the fluid pressure and density, respectively. The most important cases are radiation, with $w_B = 1/3$ and matter, with $w_B = 0$. However, we will consider arbitrary $w_B$; as we will see below, the case of stiff matter ($w_B = 1$) provides some particularly interesting results. Then for $-1 < w_B \leq 1$, the background density evolves as

$$\rho_B \propto a^{-3(1+w_B)},$$

and the Hubble parameter is given by

$$H = \frac{2}{3(1+w_B)} \frac{1}{t},$$

so that the evolution equation for $\phi$ in a background-dominated expansion is

$$\ddot{\phi} + \frac{2}{1+w_B} \frac{1}{t} \dot{\phi} + \frac{dV}{d\phi} = 0.$$  

This is the equation for which we seek exact solutions. These exact solutions will yield two free parameters, which are determined by specifying the values of $\phi$ and $\dot{\phi}$ at some fiducial initial time $t_i$; we will denote these by $\phi_i$ and $\dot{\phi}_i$, respectively.

The physically-observable quantity is not the value of $\phi$, but the density $\rho_\phi$ given by Eq. (3). It is conventional to parametrize the evolution of $\phi$ in terms of the equation of state parameter $w_\phi \equiv p_\phi/\rho_\phi$, where the scalar field pressure is given by

$$p_\phi = \frac{\dot{\phi}^2}{2} - V(\phi).$$

Note that the evolution of $\phi$ is unaffected by addition of a constant to the density in Eq. (3). However, this will affect $\rho_\phi$ and $w_\phi$, so we consider this possibility in discussing our solutions below.
III. CONSTANT, LINEAR, AND QUADRATIC POTENTIALS

Here we examine potentials of the form

\[ V(\phi) = V_0 + V_1 \phi + V_2 \phi^2, \tag{10} \]

where \( V_0, V_1, \) and \( V_2 \) are constants. Background-dominated solutions for constant, linear, and quadratic potentials have all been derived previously, but we include them here for completeness, and we consider in addition linear combinations of such potentials as in Eq. (10). In all of these cases, Eq. (8) reduces to a linear differential equation with straightforward solutions.

Consider first the case of a constant potential \( V(\phi) = V_0. \) Then \( dV/d\phi = 0, \) and Eq. (1) can be solved for \( \dot{\phi}(a) \) for arbitrary \( H \) (not just the background-dominated case). We obtain \( \dot{\phi}(a) = C_1 a^{-3}, \) where \( C_1 \) is a constant of integration. Then the density as a function of the scale factor is just

\[ \rho_\phi(a) = (C_1^2/2)a^{-6} + V_0, \tag{11} \]

i.e., the density evolves as the sum of a constant-density component and a stiff-matter component. Models of this sort were dubbed “skating” models and investigated previously in Refs. [36, 37].

Now consider the linear potential \( V(\phi) = V_1 \phi. \) For the background-dominated case, the evolution of \( \phi \) was first derived in Ref. [10]; in this case we have

\[ \phi = C_1 + C_2 t^{(w_B-1)/(w_B+1)} \left( \frac{1 + w_B}{6 + 2w_B} \right) V_1 t^2, \tag{12} \]

for \( w_B \neq 1. \) For the case of a stiff matter background \( (w_B = 1) \) we have instead

\[ \phi = C_1 + C_2 \ln(t) - \frac{1}{4} C_2 V_1 t^2. \]

Now consider the quadratic potential \( V(\phi) = V_2 \phi^2. \) In this case, Eq. (8) becomes

\[ \ddot{\phi} + 2 \frac{\phi}{1 + w_B} \frac{\dot{\phi}}{t} + 2V_2 \phi = 0, \tag{14} \]

As noted in Ref. [10], this is just a form of Bessel’s equation, with general solution

\[ \phi = t^{-\alpha} [C_1 J_\alpha(\sqrt{2V_2} t) + C_2 Y_\alpha(\sqrt{2V_2} t)], \tag{15} \]

where \( J_\alpha \) and \( Y_\alpha \) are Bessel functions of the first and second kind, and \( \alpha \) is given by

\[ \alpha = \frac{1}{1 + w_B} - \frac{1}{2}. \tag{16} \]

When \( \alpha \) is not an integer, the solution can be written in the simpler form

\[ \phi = t^{-\alpha} [C_1 J_\alpha(\sqrt{2V_2} t) + C_2 J_{-\alpha}(\sqrt{2V_2} t)], \tag{17} \]

For the matter-dominated case, this solution takes the particularly simple form [38]

\[ \phi = \frac{C_1 \sin(\sqrt{2V_2} t) + C_2 \cos(\sqrt{2V_2} t)}{t}. \tag{18} \]

Finally consider the full potential given by Eq. (10). In this case, the solutions given by Eqs. (16), (17), and (18) are simply modified by the addition of a constant:

\[ \phi = t^{-\alpha} [C_1 J_\alpha(\sqrt{2V_2} t) + C_2 Y_\alpha(\sqrt{2V_2} t)] - \frac{V_1}{2V_2}, \tag{19} \]

\[ \phi = t^{-\alpha} [C_1 J_\alpha(\sqrt{2V_2} t) + C_2 J_{-\alpha}(\sqrt{2V_2} t)] - \frac{V_1}{2V_2}, \tag{20} \]

and

\[ \phi = \frac{C_1 \sin(\sqrt{2V_2} t) + C_2 \cos(\sqrt{2V_2} t)}{t} - \frac{V_1}{2V_2}. \tag{21} \]

As an example, consider evolution in a matter-dominated background with the boundary conditions \( \phi = \phi_i \) and \( \dot{\phi} = 0 \) in the limit \( t \to 0. \) Then Eq. (18) yields

\[ \phi = \left( \phi_i + \frac{V_1}{2V_2} \right) \frac{\sin(\sqrt{2V_2} t)}{\sqrt{2V_2} t} - \frac{V_1}{2V_2}. \tag{22} \]
IV. POWER-LAW POTENTIALS

A. Particular solutions

Consider the evolution of a scalar field in a background-dominated expansion with a power-law potential of the form

\[ V = V_0 \phi^n, \]  

with \( V_0 > 0 \). Unlike the linear and quadratic potentials of the previous section, the exact solutions we will derive shortly have not been previously discussed in the context of cosmological scalar fields. For these power-law potentials, Eq. (8) becomes

\[ \ddot{\phi} + \frac{2}{1 + w_B} \frac{1}{t} \dot{\phi} + nV_0 \phi^{n-1} = 0. \]  

Note that if \( \phi(t) \) is a solution to Eq. (24), then so is \( \frac{C^2}{(n-2)} \phi(Ct) \), where \( C \) is an arbitrary constant. Thus, any general solution to this equation must be of the form

\[ \phi = \frac{C^2}{(n-2)} f(Ct), \]  

where \( C \) accounts for one of the two arbitrary constants that must occur in the solution.

Power-law potentials in a background-dominated universe were first studied by Ratra and Peebles \[3\] and further examined in Refs. \[10, 11\]. Eq. (24) has a well-known particular solution, namely

\[ \phi = \left[ -\frac{1}{nV_0} \left( \frac{2}{2-n} \right) \left( \frac{n}{2-n} + \frac{2}{1 + w_B} \right) \right]^{1/(n-2)} t^{2/(2-n)}. \]  

for \( n < 0 \) or \( n > 2 \).

Because Eq. (26) gives only a particular solution, there is no guarantee that this solution is an attractor of the equation of motion, and when it is an attractor, there is no way to determine the evolution of \( \phi \) as this attractor is achieved. Ratra and Peebles \[3\] showed that this particular solution is, in fact, an attractor for negative \( n \) in a radiation or matter-dominated background. Liddle and Scherrer \[10\] extended this result to all values of \( w_B \) for \( n < 0 \).

However, for \( n > 2 \) the result is more complicated. First, the particular solution for \( n > 2 \) is valid only for

\[ n > \frac{4}{1 - w_B}. \]  

Second, it can be shown that this solution is an attractor only for \[10\]

\[ n > \frac{2(3 + w_B)}{1 - w_B}. \]  

Note that Eq. (27) is satisfied whenever Eq. (28) is true. The two cases of greatest interest are radiation domination, for which \( w_B = 1/3 \) and the condition for the particular solution to be an attractor is \( n > 10 \), and matter domination, for which \( w_B = 0 \) and the attractor condition gives \( n > 6 \). When the condition given by Eq. (28) is satisfied, \( \phi \) evolves smoothly to zero as given by Eq. (26), while when the particular solution is not an attractor, the solution is oscillatory about \( \phi = 0 \) (assuming \( n \) is an even integer or \( \phi \) is replaced by \( |\phi| \) in Eq. (24)). See Refs. \[15, 39\] for a further discussion of these points. When \( n = 2(3 + w_B)/(1 - w_B) \), the solution is neutrally stable, allowing both oscillating and non-oscillating trajectories depending on the initial conditions \[10\]. We will see that it is precisely these cases that yield two of our exact solutions below.

B. Exact solutions

To search for exact solutions to Eq. (24), we make the change of variables

\[ t = f(\tau), \]  

and

\[ \phi(\tau) = g(\tau) \psi(\tau), \]  

where

\[ \psi = \frac{n}{1 + w_B} \phi. \]  

This transformation leads to a new equation of motion with

\[ \ddot{\psi} + \frac{2}{1 + w_B} \frac{1}{t} \dot{\psi} + nV_0 \psi^{n-1} = 0. \]  

This equation is the same as Eq. (24) with \( \phi \) replaced by \( \psi \). Therefore, any solution to Eq. (24) is also a solution to Eq. (29). The general solution to Eq. (29) is

\[ \phi = C^2/(n-2) f(Ct), \]  

where \( C \) accounts for one of the two arbitrary constants that must occur in the solution.
where the functions $f(\tau)$ and $g(\tau)$ will be chosen to produce an exactly-solvable differential equation for $\psi(\tau)$. With these substitutions, Eq. \((24)\) becomes

$$
\psi'' + \left[\frac{2g''}{g} + \frac{2}{1 + w_B} \frac{f'}{f} - \frac{f''}{f'}\right] \psi' + \left[\frac{g''}{g} - \frac{2}{1 + w_B} \frac{f'g'}{g} - \left(1 + \frac{w_B}{2}\right) \frac{f''}{g'}\right] \psi + nV_0f'^2g^{n-2}\psi^{n-1} = 0,
$$

where the prime denotes derivative with respect to the new independent variable $\tau$. In order to derive a first integral, and hence an exact solution, we seek functions $f(\tau)$ and $g(\tau)$ for which the factor multiplying $\psi'$ is zero, and the factors multiplying $\psi$ and $\psi^{n-1}$ are constants. For each value of $w_B$, there are two values of $n$ that allow us to derive such functions. We will refer to these as Case (A) and Case (B).

Case A:

$$
n = 4 \left(\frac{1 + w_B}{1 - w_B}\right) + 2,
$$

\[ f(\tau) = \exp(\tau), \]  
\[ g(\tau) = \exp\left(\frac{2}{2 - n}\tau\right). \]

Case B:

$$
n = 2 \left(\frac{1 + w_B}{1 - w_B}\right),
$$

\[ f(\tau) = \tau^{n/2}, \]  
\[ g(\tau) = 1/\tau. \]

For matter ($w_B = 0$) and radiation ($w_B = 1/3$) dominated expansions, Case A corresponds to $n = 6$ and $n = 10$, respectively. As noted above, these are precisely the cases demarcating the boundary between the oscillating and attractor (non-oscillating) solutions. Case B corresponds to $n = 2$ for matter domination and $n = 4$ for radiation domination. The former was already discussed in Sec. III, while the latter will yield a new exact solution.

First consider Case A. Under the transformation above, we obtain

$$
\psi'' - \frac{4}{(n-2)^2} \psi + nV_0\psi^{n-1} = 0,
$$

which can be integrated to yield

$$
\frac{1}{2} \psi'^2 - \frac{2}{(n-2)^2} \psi^2 + V_0\psi^n = C,
$$

where $C$ is a constant determined by the boundary conditions. In terms of the original variables $\phi$ and $t$, Eq. \((39)\) is

$$
\frac{1}{2} \phi^{2n/(n-2)} + \frac{2}{n-2} \phi^{(n+2)/(n-2)} + \frac{2n}{n-2}V_0\phi^n = C.
$$

Eq. \((40)\) provides a first integral for the evolution of the scalar field for Case A.

We can now use Eq. \((39)\) to derive an exact solution. This equation can be rewritten as

$$
\tau = \int \frac{d\psi}{\sqrt{4\psi'^2/(n-2)^2 - 2V_0\psi^n + C_1}} + \ln C_2
$$

The relation between $\tau$, $\psi$, $t$, and $\phi$ then gives us an exact solution in parametric form:

$$
t = C_2 \exp\left(\int \frac{d\psi}{\sqrt{4\psi'^2/(n-2)^2 - 2V_0\psi^n + C_1}}\right),
$$

$$
\phi = C_2^{2/(2-n)} \psi \exp\left(\frac{2}{2 - n} \int \frac{d\psi}{\sqrt{4\psi'^2/(n-2)^2 - 2V_0\psi^n + C_1}}\right).
$$

Note that this solution has the form of the general solution in Eq. \((25)\), with $C_2$ corresponding to $C$ in that equation. As expected, our exact solution contains two arbitrary constants, which are determined by the initial conditions on $\phi(t)$ and $\phi(t)$. 

Now consider Case B. The indicated transformation gives

$$\psi'' + \frac{n^3}{4} V_0 \psi^{n-1} = 0,$$

which integrates to

$$\frac{1}{2} \psi'^2 + \frac{n^2}{4} V_0 \psi^n = C,$$

with $C$ a constant. Reexpressing this equation in terms of $\phi$ and $t$ leads to the corresponding first integral for case B:

$$\frac{1}{2} \dot{\phi}^2 + \frac{2}{n} t \dot{\phi} \phi + \frac{2}{n^2} \phi^2 + t^2 V_0 \phi^n = C.$$

Again, we can use Eq. (45) to derive an exact parametric solution:

$$t = C_2 \left( \int \frac{d\theta}{\sqrt{1 - \theta^n}} + C_1 \right)^{n/2},$$

$$\phi = C_2^{2/(2-n)} \left( n^2 V_0 / 2 \right)^{1/(2-n)} \theta \left( \int \frac{d\theta}{\sqrt{1 - \theta^n}} + C_1 \right)^{-1},$$

where our new parametric variable is $\theta = C^{-1/n} \psi$. As in the previous case, our solution take the general form given by Eq. (25). Note that both sets of first integrals and exact solutions can be found, in a slightly different form, in Ref. [40].

C. Applications

Consider first the set of solutions given by Case A. As noted, these correspond to $n = 10$ for evolution in a radiation-dominated background and $n = 6$ in a matter-dominated background. The latter is the more interesting case, since it is close to the value of $n$ required in models that use the additional energy density from an evolving scalar field to resolve the Hubble tension [14–16]. Further, a scalar field initially at rest in this potential could serve as a "thawing" model for quintessence.

Note that Eq. (24) with $w_B = 0$ and $nV_0 = 1$ corresponds exactly to the Lane-Emden equation, with $n = 6$ resulting in a set of previously-investigated particular solutions. In addition to the particular solution corresponding to Eq. (26), there is a classic solution of the form

$$\phi = \frac{1}{\sqrt{1 + t^2/3}}.$$

(See, e.g., Ref. [41] for a detailed discussion of this solution). Later, the solution

$$\phi = \frac{\sin(\ln \sqrt{t})}{3t - 2 \sin^2(\ln \sqrt{t})}$$

was discovered independently by Srivastava [42] and Sharma [43]. All of these solutions can be extended to a one-parameter family of solutions using Eq. (25). Finally, the full general solution was derived by Mach [44], who showed that particular discrete values of $C$ correspond to the previously-discovered exact solutions. For example, $C = 0$ gives the solution in Eq. (49). Mach's solutions for other values of $C$ are given in terms of Jacobi and Weierstrass elliptic functions.

Because our own exact solution is in integral parametric form, it is difficult to use, and the first integral (Eq. 40) actually provides more useful information. For $n = 6$, we obtain

$$\frac{1}{2} t^3 \dot{\phi}^2 + \frac{1}{2} t^2 \dot{\phi} \phi + t^2 V_0 \phi^6 = C.$$

This first integral is given by Leach [45] who cites several earlier references to it. One of the more interesting aspects of scalar field evolution in this case is the fact (emphasized by Refs. [10, 15, 39]) that this potential can support both
oscillatory behavior as well as smooth evolution to $\phi = 0$. One might naively assume that a sufficiently small initial value of $\dot{\phi}$ would allow $\phi$ to evolve smoothly to zero, while a large negative value of $\dot{\phi}$ would always lead to oscillatory behavior. Surprisingly, the opposite is the case.

Eq. (51) provides a simple condition to determine which form of evolution takes place for a given set of initial conditions. For simplicity, we will take the initial time to correspond to $t = 1$, with initial values of $\phi = \phi_i$ and $\dot{\phi} = \dot{\phi}_i$. Then it is easy to see that the condition for oscillatory behavior is

$$C > 0,$$

corresponding to

$$\frac{1}{2} \dot{\phi}_i^2 + \frac{1}{2} \phi_i \dot{\phi}_i + V_0 \phi_i^6 > 0.$$  \hspace{1cm} (52)

Thus, for a scalar field initially at rest ($\dot{\phi}_i = 0$), all solutions produce oscillatory behavior. The only way for $\phi$ to evolve smoothly to zero is for $\phi_i$ and $\dot{\phi}_i$ to have opposite signs; i.e., the field must initially be rolling downhill! In order to achieve $C \leq 0$, the initial values for $\phi$ and $\dot{\phi}$ must satisfy

$$\phi_i < (8V_0)^{-1/4},$$  \hspace{1cm} (53)

and

$$\frac{\dot{\phi}_i}{2} \left( -1 - \sqrt{1 - 8V_0 \phi_i^6} \right) < \phi_i < \frac{\dot{\phi}_i}{2} \left( -1 + \sqrt{1 - 8V_0 \phi_i^6} \right),$$  \hspace{1cm} (54)

where we have taken $\phi_i > 0$. When these conditions are satisfied, the field does not oscillate but instead evolves smoothly to zero. The regions in parameter space for which these two types of behavior occur are illustrated in Fig. 1 for the case $V_0 = 1$. Fig. 1 suggests that oscillatory behavior for the $\phi^6$ potential during matter domination is in some sense more “generic” than $\phi \rightarrow 0$ evolution. Although both types of evolution are possible, the latter corresponds to a special set of initial conditions with a finite range in both $\phi_i$ and $\dot{\phi}_i$.

The other Case A solution of physical interest is a radiation-dominated background with $n = 10$. In this case Eq. (40) gives the first integral

$$\frac{1}{2} t^{5/2} \ddot{\phi}^2 + \frac{1}{4} t^{3/2} \dot{\phi}^2 + t^{5/2} V_0 \phi^{10} = C.$$  \hspace{1cm} (55)
FIG. 2: As Fig. 1, for \( \phi \) evolving in a radiation-dominated background with the potential \( V = V_0 \phi^6 \) with \( V_0 = 1 \) and the indicated initial values of \( \phi \) and \( \dot{\phi} \) at \( t_i = 1 \).

The analysis here is qualitatively similar to the \( n = 6 \) case for matter domination. This potential supports both oscillatory behavior and smooth \( \phi \to 0 \) evolution, but the latter requires \( \dot{\phi}_i < 0 \) for \( \phi_i > 0 \) and, for an initial time \( t = 1 \),

\[
\phi_i < (32V_0)^{-1/8},
\]

and

\[
\frac{\dot{\phi}_i}{4}[1 - \sqrt{1 - 32V_0\phi_i^8}] < \dot{\phi}_i < \frac{\phi_i}{4}[1 + \sqrt{1 - 32V_0\phi_i^8}].
\]

The regions in parameter space corresponding to these two types of behavior are illustrated in Fig. 2 for the case \( V_0 = 1 \). As in the case of matter domination, the region in which the field evolves smoothly to zero represents a small fraction of parameter space, suggesting that oscillatory behavior in this case is the more generic behavior.

Now consider Case B. Note that the first integral for Case B can be rewritten in the form

\[
\frac{1}{2} \left( t\dot{\phi} + \frac{2}{n}\phi \right)^2 + t^2V_0\phi^n = C.
\]

For even \( n \), we have \( C > 0 \), so these solutions always oscillate.

For a matter-dominated background, Case B corresponds to \( n = 2 \), which gives the trigonometric exact solution already discussed in Sec. III. For a radiation-dominated background, Case B corresponds to a quartic potential \( (n = 4) \). For \( n = 4 \), Eq. (44) can be solved to express \( \psi(\tau) \) in terms of a Jacobi elliptic function; the resulting expression for \( \phi(t) \) is

\[
\phi = \frac{C_1}{\sqrt{t}} \text{cn}[4C_1\sqrt{V_0}(\sqrt{t} - C_2), 1/\sqrt{2}].
\]

where \( C_1 \) and \( C_2 \) are constants of integration. This solution was derived by Greene et al. [46] for evolution in a scalar-field dominated background in the context of inflation. The reason that Eq. (59) gives the same answer for a radiation-dominated background is that Greene et al. assumed a rapidly oscillating \( (\nu \gg H) \) scalar field. In this limit, with a potential of the form \( V(\phi) = V_0\phi^4 \), the scalar field energy density driving the expansion decays as \( \frac{1}{(\nu t)^2} \).
\( \rho_\phi \propto a^{-4} \). Maso et al. give an approximate analytic solution in terms of Jacobi elliptic functions for the evolution of \( \phi(t) \) in the \( \phi^4 \) potential for arbitrary \( w_B \). It is clear from our derivation that their solution is exact for \( w_B = 1/3 \) and approximate for all other cases.

The first integral for \( n = 4 \) is

\[
\frac{1}{2} t^2 \dot{\phi}^2 + \frac{1}{2} t \phi \dot{\phi} + \frac{1}{8} \phi^2 + t^2 V_0 \phi^4 = C. \tag{60}
\]

The limit for which the oscillations are rapid in comparison to the expansion rate corresponds to \( t \gg 1 \), so that the first and last terms on the left-hand side of Eq. (60) dominate, giving \( t^2 \rho_\phi = C \), which corresponds to \( \rho_\phi \propto a^{-4} \) in a radiation-dominated background, as expected. How does \( \rho_\phi \) evolve when the oscillation rate is not large compared to the expansion rate? In that case, we can use Eq. (59) to calculate directly the deviation from \( a^{-4} \) evolution, but it is easier to use the first integral to gain some insight into this evolution. If we consider the value of \( \rho_\phi \) when the field is at the minimum in the potential (\( \phi = 0 \)), then we once again have \( t^2 \rho_\phi = C \), i.e., the value of the kinetic component of the energy when \( \phi \) is at the minimum of the potential scales exactly like \( a^{-4} \) even when the oscillation frequency is not large compared to \( H \). In contrast, if we examine \( \rho_\phi \) when the field achieves its maximum value, \( \phi = \phi_m \), at \( \dot{\phi} = 0 \), we have \( t^2 \rho_\phi = C - (1/8) \phi_m^2 \). Thus \( t^2 \rho_\phi \) is always smaller than its asymptotic value, but this difference decays away as \( \phi_m \) decreases with each oscillation.

V. EXPOSENTIAL POTENTIALS

A. Particular solutions

Now we assume a scalar field in a background-dominated expansion with an exponential potential of the form

\[
V = V_0 e^{-\lambda \phi}, \tag{61}
\]

with \( V_0 > 0 \) and \( \lambda > 0 \). Potentials of this form were among the first to be examined as possible models for quintessence. For these exponential potentials, Eq. (8) gives

\[
\ddot{\phi} + \frac{2}{1 + w_B} \frac{1}{t} \dot{\phi} - \lambda V_0 e^{-\lambda \phi} = 0. \tag{62}
\]

This has a well-known particular solution

\[
\phi = \frac{2}{\lambda} \ln t + \frac{1}{\lambda} \ln \left[ \frac{\lambda^2 V_0}{2} \frac{1 + w_B}{1 - w_B} \right], \tag{63}
\]

for \(-1 < w_B < 1\). Substituting this into Eq. (63) to determine \( \rho_\phi \), we see that this expression for \( \phi \) corresponds to a background-dominated universe with \( \rho_B \gg \rho_\phi \) only in the limit where \( \lambda \gg 1 \); in this limit, Eq. (63) is an attractor for all values of \( w_B \) in the range \(-1 < w_B < 1 \).

B. Exact solutions

Now, however, we will see seek an exact solution for Eq. (62). We make the change of variables

\[
t = e^\tau \tag{64}
\]

and

\[
\psi = \lambda \phi - 2 \tau \tag{65}
\]

and Eq. (62) becomes

\[
\psi'' + \left( \frac{2}{1 + w_B} - 1 \right) (\psi' + 2) - \lambda^2 V_0 e^{-\psi} = 0, \tag{66}
\]

where prime denotes the derivative with respect to \( \tau \). For the case of stiff matter with \( w_B = 1 \) (only), the \( \psi' \) term vanishes, giving

\[
\psi'' - \lambda^2 V_0 e^{-\psi} = 0. \tag{67}
\]
This equation can then be integrated to give the first integral
\[ \frac{1}{2} \psi'^2 + \lambda^2 V_0 e^{-\psi} = C, \] (68)
with constant \( C \). In terms of \( \phi \) and \( t \), this corresponds to
\[ \frac{1}{2}(\lambda t \dot{\phi} - 2)^2 + \lambda^2 t^2 V_0 e^{-\lambda \phi} = C. \] (69)

Now note that Eq. (67) can be integrated exactly, giving [40]
\[ \psi = 2 \ln \left[ \sqrt{\frac{V_0}{2}} \frac{\lambda}{C_1} \cosh(C_1 \tau + C_2) \right], \] (70)
with \( C_1 \) and \( C_2 \) the constants of integration. We can reexpress this exact solution in terms of \( t \) and \( \phi \) to obtain an exact solution to Eq. (62) with \( w_B = 1 \):
\[ \phi = \frac{2}{\lambda} \ln \left[ \sqrt{\frac{V_0}{2}} \frac{\lambda}{C_1} \cosh(C_1 \ln t + C_2) \right], \] (71)
\[ \phi = \frac{2}{\lambda} \ln \left[ \sqrt{\frac{V_0}{8}} \frac{\lambda}{C_1} (C_3 t^{C_1+1} + C_3^{-1} t^{-C_1}) \right], \] (72)
where \( C_3 = \exp(C_2) \). A similar solution for \( \lambda = 1 \) appears to have first been derived by Sajben [48] for the equation describing the electron density near hot filaments in cylindrical coordinates.

C. Applications

In the simplest standard cosmological model, the universe undergoes periods of radiation and matter domination, but the background energy density is never dominated by stiff matter with \( w_B = 1 \). However, there has been increasing interest in the possibility of such a period of stiff matter domination, also called “kination” (since \( w_B = 1 \) could be driven by the kinetic energy of a scalar field). The effects of a kination/stiff-matter dominated era have been studied in relation to baryogenesis [49], Big Bang nucleosynthesis [50], the relic abundance of dark matter [51–54], and the propagation of gravitational radiation ([55] and references therein). Hence, it is not unreasonable to also examine scalar field evolution in a background with \( w_B = 1 \).

Unlike the power-law solutions, the exact solution for a scalar field with an exponential potential evolving in a stiff-matter background is simpler and more useful than the corresponding first integral. To determine the values for \( C_1 \) and \( C_2 \) corresponding to a given set of initial conditions, we will take, for simplicity, \( t = 1 \) to be our initial value of \( t \), and \( \phi = 0 \) to be our initial value of \( \phi \). However, we will allow the initial value of \( \dot{\phi} \) to be a free parameter, \( \dot{\phi}_i \). Then the exact solution above gives the corresponding values for \( C_1 \) and \( C_2 \), namely
\[ C_1 = \frac{\lambda}{2} \sqrt{2V_0 + \left( \frac{\dot{\phi}_i}{\lambda} - \frac{2}{\lambda} \right)^2}, \] (73)
\[ C_2 = \sinh^{-1} \left[ \left( \frac{1}{\sqrt{2V_0}} \right) \left( \frac{\dot{\phi}_i}{\lambda} - \frac{2}{\lambda} \right) \right]. \] (74)

Using our exact solution with Eq. (3), we can derive an expression for the scalar field energy density,
\[ \rho_\phi = \frac{2}{\lambda^2 t^2} [1 + C_1^2 + 2C_1 \tanh(C_1 \ln t + C_2)], \] (75)
and for the equation of state parameter, \( w_\phi \),
\[ w_\phi = 1 - \frac{2C_1^2}{(1 + C_1^2) \cosh^2(C_1 \ln t + C_2) + 2C_1 \cosh(C_1 \ln t + C_2) \sinh(C_1 \ln t + C_2)}. \] (76)

It is clear from Eq. (70) that the equation of state parameter for the scalar field evolves asymptotically to \( w_\phi \to 1 \), so that the scalar field energy density evolves toward that of the background stiff matter. This is illustrated in Fig.
FIG. 3: The evolution of the scalar field equation of state parameter, \( w_\phi \), as a function of \( \ln t \) as given by the exact solution (Eq. 76) for the exponential potential \( V(\phi) = V_0 \exp(-\lambda \phi) \), with \( \phi_i = 0 \) at initial time \( t = 1 \), and \( V_0 = 1 \). Curves correspond to \( \dot{\phi}_i = 0, \lambda = 1 \) (green), \( \dot{\phi}_i = 0, \lambda = 2 \) (blue), \( \dot{\phi}_i = 1, \lambda = 1 \) (black), and \( \dot{\phi}_i = 1, \lambda = 2 \) (red), where \( \dot{\phi}_i \) is the initial value of \( \dot{\phi} \) at \( t = 1 \).

This shows the time evolution of \( w_\phi \) from Eq. (76) for a variety of model parameters. While the evolution to \( w_\phi = 1 \) is inevitable in this model, Eq. (76) gives insight into the rate at which this evolution occurs. Larger values of \( C_1 \) correspond to a more rapid increase in \( w_\phi \), as is apparent in Fig. 3. For fixed \( \dot{\phi}_i \), this corresponds to larger values of both \( V_0 \) and \( \lambda \), while for fixed \( V_0 \) and \( \lambda \), the minimum rate of increase of \( w_\phi \) occurs for \( \dot{\phi}_i = 2/\lambda \).

In the late-time limit, \( t \to \infty \), the scalar field density given by Eq. (75) becomes

\[
\rho_\phi = \frac{2}{\lambda^2 t^2} (1 + C_1)^2. 
\] 

(77)

In comparison, the density of the background stiff matter is \( \rho_B = 1/(3t^2) \). Thus, the background matter will dominate the expansion, and our exact solution for the evolution of \( \phi \) will remain valid arbitrarily long when \( 2(1+C_1)^2/\lambda^2 < 1/3 \). When this equation is not satisfied, the scalar field energy density will eventually come to dominate the expansion, and our results will no longer be valid. Beyond that point the evolution will be given instead by the exact solution of Refs. [25, 27, 28] for the exponential potential with a scalar field dominated expansion.

VI. DISCUSSION

The results presented here can be generalized in several straightforward ways. Adding a constant to any potential leaves Eq. (8) unchanged, so we obtain the same solution \( \phi(t) \), but with a constant added to \( \rho_\phi \). Furthermore, translating any of the potentials by a constant value of \( \phi \) simply translates the corresponding solution by that same constant.

One set of models we have not chosen to examine are those in which the expansion is dominated by vacuum energy. In this case, we have \( H = H_0 \) (a constant), and instead of Eq. (5), the evolution of \( \phi \) is given by

\[
\ddot{\phi} + 3H_0 \dot{\phi} + \frac{dV}{d\phi} = 0. 
\] 

(78)

This would correspond, for example, to the evolution of a scalar field in a universe dominated by a cosmological...
constant. However, from a mathematical point of view, it is clear that Eq. (78) is qualitatively different from Eq. (8). Hence, we have chosen to leave these models for future investigation.

In contrast to the well-explored particular attractor solutions, relatively less attention has been paid to exact solutions of Eq. (8). We have found a variety of such solutions for both power-law and exponential potentials. While we cannot rule out exact solutions for more complicated potentials, we have likely exhausted the simplest exactly-solvable cases. Our exact solutions are applicable to a much narrower range of potentials than is the case for the particular attractor solutions; unfortunately, one cannot pick and choose which potentials yield exact solutions. However, since it is difficult to predict precisely which scalar field potentials will be of interest to future investigators, it seems worthwhile to provide the general catalog given in this paper.

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