NUCLEAR DIMENSION AND $\mathcal{Z}$-STABILITY

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Abstract. Simple, separable, unital, monotracial and nuclear $C^*$-algebras are shown to have finite nuclear dimension whenever they absorb the Jiang-Su algebra tensorially. This completes the proof of the Toms-Winter conjecture in the unique trace case.

The structure theory of simple nuclear $C^*$-algebras is currently undergoing revolutionary progress, driven by the discovery of regularity properties of various flavours: topological, functional analytic and algebraic. Despite the diverse nature of these regularity properties, they are simultaneously satisfied by classes of algebras which have been successfully classified by $K$-theoretic data, and they fail spectacularly for the “exotic” algebras in [28, 30] which provide counterexamples to Elliott’s classification conjecture. The observation that there are deep connections between these disparate properties was crystallised in the following conjecture of Toms and the third named author.

Conjecture A. Let $A$ be a simple, separable, unital, nuclear and non-elementary $C^*$-algebra. Then the following are equivalent:

(i) $A$ has finite nuclear dimension;
(ii) $A$ absorbs the Jiang-Su algebra $\mathcal{Z}$ tensorially ($A$ is $\mathcal{Z}$-stable);
(iii) $A$ has strict comparison.

The Toms-Winter conjecture has been verified for various naturally occurring classes of algebras: in particular $C^*$-algebras of the form $C(X) \rtimes \mathbb{Z}$ arising from uniquely ergodic, minimal homeomorphisms of compact metric spaces satisfy the conjecture; furthermore the subclass of “regular” algebras of this form are classified by their $K$-theory (see [39], and [33, 43] for the case of $\mathbb{Z}^d$ actions). Moreover, the implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) have been established in general by the third named author [42, 44] (see [34] for an extension to the stably projectionless case) and Rørdam [29] respectively; recent progress of Matui and the first named author [23] establishes (iii) $\Rightarrow$ (ii) for $C^*$-algebras with a unique tracial state (this has subsequently been extended to somewhat more general trace simplices in [19, 32, 37]). In the traceless case, (iii) $\Rightarrow$ (ii) boils down to Kirchberg’s celebrated $O_\infty$-stability theorem for purely infinite nuclear $C^*$-algebras, see [16]. (ii) $\Rightarrow$ (i) follows in the UCT case from Kirchberg-Phillips classification in connection with [44] and was shown in general in [24] (see also [2]).

The focus of this paper is the implication from (ii) to (i) in the tracial case. We should discuss this alongside the decomposition rank, the precursor to nuclear dimension from [24]. Having finite decomposition rank is a stronger condition than

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(i) which implies quasidiagonality, and so (in contrast to nuclear dimension) is only applicable to stably finite algebras. Classification results provide one route towards the implication (ii) \(\implies\) (i) via the strategy of comparing algebras with directly constructed models which have finite topological dimension. As well as passing through vast amounts of technical machinery, this approach is inevitably restricted to \(C^*\)-algebras satisfying the still mysterious UCT, so sheds no light on how finite covering dimension actually arises from \(\mathcal{Z}\)-stability. A direct approach was pioneered by Tikuisis and the third named author in [35] (heavily based on the results of Kirchberg and Rørdam in [18]), which proves that \(\mathcal{Z}\)-stable locally homogeneous algebras have finite decomposition rank (even outside the simple setting); however this too essentially relies on the existence of a concrete inductive limit structure. A recent breakthrough was achieved by Matui and the first named author in [24], which, in the presence of a unique tracial state establishes finite decomposition rank from quasidiagonality and \(\mathcal{Z}\)-stability. Our main theorem, stated below, does not require a quasidiagonality assumption (which can often be hard to verify outside a concrete inductive limit setting) to establish finite nuclear dimension from \(\mathcal{Z}\)-stability.

**Theorem B.** Let \(A\) be a simple, separable, unital, nuclear and \(\mathcal{Z}\)-stable \(C^*\)-algebra with a unique tracial state. Then \(\dim_{\text{nuc}}(A) \leq 3\).

Upon combining Theorem [3] with the main results of [23], and the general implications (i) \(\implies\) (ii) of [44] and (ii) \(\implies\) (iii) of [29] this confirms the Toms-Winter conjecture in the form stated above for \(C^*\)-algebras with a unique tracial state. Thus the difference between finite decomposition rank and finite nuclear dimension for unital, simple, separable and monotracial \(\mathcal{Z}\)-stable algebras is precisely quasidiagonality, and it remains an open question from [4] as to whether such algebras are automatically quasidiagonal.

**Corollary C.** Conjecture [4] holds under the additional assumption that \(A\) has at most one tracial trace.

Each of the statements in the Toms-Winter conjecture is a natural analogue of a corresponding property for \(\text{II}_1\) factors: strict comparison corresponds to the fact that the Murray-von Neumann lattice of projections in a \(\text{II}_1\) factor is determined by the trace; tensorial absorption of \(\mathcal{Z}\) to being a McDuff factor (absorbing the hyperfinite \(\text{II}_1\) factor tensorially); and finite nuclear dimension to hyperfiniteness. For (separably acting) injective \(\text{II}_1\) factors these properties are theorems (of course comparison holds for all \(\text{II}_1\) factors), and all play roles in carrying through the implication from injectivity to hyperfiniteness in Connes’ celebrated paper [7]. From this viewpoint, carrying through the implications (iii) \(\implies\) (ii) \(\implies\) (i) in the Toms-Winter conjecture provides a \(C^*\)-version of Connes’ characterisations of injectivity.

In his work Connes relies on a detailed analysis of automorphisms of \(\text{II}_1\) factors, and the notion of an approximately inner flip (the automorphism \(x \otimes y \mapsto y \otimes x\) on the tensor square can be approximated by inner automorphisms) plays a key role. A crucial step in Connes’ proof that injectivity implies hyperfiniteness is to show that any automorphism of an injective \(\text{II}_1\) factor with separable predual is approximately inner [7, Corollary 3.2, Theorem 5.3] in the strong topology. In particular, it has strongly approximately inner flip [7, Theorem 5.1 (7) \(\implies\) (3)] and therefore is McDuff [7, Theorem 5.1 (3) \(\implies\) (2)] in a particularly strong fashion; from this Connes obtains hyperfiniteness of \(M\) [7, Theorem 5.1 (2) \(\implies\) (1)]. Of
course, by now two beautiful alternative proofs without using automorphisms have been obtained by Haagerup and by Popa, but for our purposes it will be more suitable to focus on the original strategy involving approximately inner flips.

Effros and Rosenberg initiated the study of approximately inner flips in the C$^*$-setting in [9]. As they observe, the existence of an approximately inner flip is a strong requirement on a C$^*$-algebra (implying simplicity, nuclearity and at most one trace), but several prominent examples enjoy such a flip, like UHF-algebras, the Jiang-Su algebra $\mathcal{Z}$ and the Cuntz algebras $\mathcal{O}_2$ and $\mathcal{O}_\infty$, all of which play key roles in the modern study of nuclear C$^*$-algebras. Using the approximately inner flip, Effros and Rosenberg characterised the universal UHF algebra $\mathcal{Q}$ as the unique separable unital C$^*$-algebra which is quasidiagonal, has an approximately inner flip and tensorially absorbs $\mathcal{Q}$ [9, Theorem 5.1], using a C$^*$-analogue of Connes’ implication [7, Theorem 5.1 (2) $\Rightarrow$ (1)]. With the benefit of hindsight, this result can be thought of as the precursor to the strategy of [24] and our work here. Effros and Rosenberg compare the first factor embedding $A \mapsto (A \otimes \mathcal{Q})_\omega$ of $A$ into the ultrapower of $A \otimes \mathcal{Q}$ with an embedding of $A$ into $1_A \otimes \mathcal{Q}_\omega \subset (A \otimes \mathcal{Q})_\omega$ obtained from quasidiagonality. They then conjugate these two embeddings using the approximately inner flip on $A$, and obtain an approximately finite dimensional (AF) structure on $A$ directly from this, whence $A$ is isomorphic to $\mathcal{Q}$ by Elliott’s classification theorem [10]. From today’s perspective, Effros and Rosenberg’s approach can be used to directly give approximations witnessing decomposition rank zero (equivalent to being AF by [20]).

Here, it should be noted that there is a significant difference between approximately inner flips in von Neumann algebras and C$^*$-algebras, most notably since the existence of such in the C$^*$-setting is a very strong requirement. A key idea here is the “2-coloured approximately inner flip” (in the norm sense) as introduced in [24]. Starting from Connes’ approximately inner flip on an injective II$_1$ factor, this 2-coloured approximation was established by using a 2-coloured approximate Murray-von Neumann equivalence, which in turn is inspired by Haagerup’s approach [11] to injectivity implies hyperfiniteness. This ingredient is structured so that the flipping strategy gives the estimate $\text{dr}(A) \leq 1$ – the two colours arising from those in the approximately inner flip. The estimate $\text{dr}(A) \leq 3$ of [24, Theorem 1.1] in the $\mathcal{Z}$-stable case arises from using the UHF-stable approximations twice.

In both [9] and [24] quasidiagonality provides *-homomorphisms from a quasidiagonal C$^*$-algebra $A$ into the ultrapowers $U_\omega$ of suitable UHF algebras (analogous to the asymptotic embeddings into the hyperfinite II$_1$ factor which arise from Condition (2) in [7, Theorem 5.1]). As Voiculescu shows in [40], the cone over any C$^*$-algebra $A$ is quasidiagonal, so nontrivial order zero maps from $A$ into these ultrapowers always exist. We aim to use such maps in place of the *-homomorphisms used in [9, 24]. As a byproduct we obtain an alternative proof of Voiculescu’s result when $A$ is nuclear without elementary quotients, and has a separating family of tracial states. The argument uses recent work of Kirchberg and Rørdam on the tracial ideal in an ultrapower [10], which also shows that the order zero maps $\phi : A \to U_\omega$ can be taken to preserve a fixed trace on $A$ (cf Lin’s work [21] on AF-embeddability of crossed products), and are unital modulo the tracial ideal. We perform the flipping strategy working in an algebra of the form $(A \otimes A \otimes V)_\omega$, the presence of a UHF algebra $V$ providing the 2-coloured approximately inner flip. We then use the extra properties of the map $\phi$ to obtain order zero maps $\Psi^{(0)}, \Psi^{(1)} : (U \otimes V)_\omega \to \mathcal{Z}$.
so that $\sum_{i=0}^{1} \Psi(i)(\phi(1_{A}) \otimes 1_{V}) = 1_{Z_{\phi}}$ (for this we use a uniqueness result for certain positive contractions in $Z_{\phi}$ proved using recent Cuntz-semigroup classification theorems [9, 25]); thus the resulting factorisation takes values in $A \otimes Z$. This strategy of proving results for $Z$-stable algebras via the UHF-stable case has its origins in [41]. The use of two maps $\Psi(i)$ doubles the number of colours in the final nuclear dimension approximation, leading to a 4-coloured factorisation: two colours from the $\Psi(i)$, multiplied by two colours from the 2-coloured approximately inner flip. Accordingly when $A$ additionally has an approximately inner flip, for example when it is strongly self-absorbing, we get an estimate $\dim_{\text{nuc}}(A) \leq 1$ in Theorem [3] (see Theorem [5.3]).

In Section 1 we set out some background and examine asymptotically order zero maps, establishing some technical preliminaries. Section 2 provides our “trace-preserving”-quasidiagonality of cones over tracial nuclear C$^*$-algebras. In Section 3 we extract the 2-coloured approximately inner flip for monotracial, simple, unital, nuclear and UHF-stable C$^*$-algebras from the proof of [24, Theorem 4.2], and record some of its consequences. Section 4 provides the uniqueness theorem used in Section 5 to produce the two maps $\Psi(i)$ and then prove Theorem 13.

1. ASYMMOTICALLY ORDER ZERO MAPS

Let $A$ and $B$ be C$^*$-algebras. Recall that a completely positive (cp) map $\psi : A \to B$ is said to be order zero if it preserves orthogonality, i.e. $\psi(e)\psi(f) = 0$ whenever $e,f \in A_+^*$ have $ef = 0$. The structure theorem from [40] for order zero maps $\psi : A \to B$ provides a supporting $^*$-homo-morphism $\pi$ from $A$ into the multiplier algebra $\mathcal{M}(C^*(\psi(A))) \subseteq B^*$ and a positive element $h \in \mathcal{M}(C^*(\psi(A))) \cap \psi(A)'$ such that $\psi(a) = \pi(a)h = h\pi(a)$ for all $a \in A$. In the case when $A$ is unital (as it will be throughout the paper) one can take $h = \psi(1_{A})$ so that $\psi(a) = \psi(1_{A})\pi(a)$ for $a \in A$ and it follows that a cp map $\psi : A \to B$ is order zero if and only if

$$\psi(x)\psi(y) = \psi(xy)\psi(1_{A}), \quad x, y \in A.$$  

From the structure theorem one obtains two further key properties:

- the functional calculus for cpc order zero maps [40, Corollary 4.2]: given $f \in C_0(0, 1]$, define an order zero map $f(\psi) : A \to B$ by $f(\psi)(a) = \pi(a)f(h)$ which is $\pi(a)f(\psi(1_{A}))$ when $A$ is unital.
- the duality between cpc order zero maps $\phi : A \to B$ and $^*$-homomorphisms $C_0(0, 1] \otimes A \to B$ on the cone over $A$, [40, Corollary 4.1].

The nuclear dimension of a C$^*$-algebra from [47] is defined in terms of completely positive approximations which are uniformly decomposable into sums of order zero maps (c.f. [14], which shows that approximations as below can be found for any nuclear C$^*$-algebra when is $n$ allowed to vary with $i$).

**Definition 1.1** ([47] Definition 2.1, [20] Definition 3.1].) A C$^*$-algebra $A$ has nuclear dimension at most $n \in \mathbb{N}$, written $\dim_{\text{nuc}}(A) \leq n$, if there exists a net $(F_i, \psi_i, \phi_i)_{i \in I}$ where $F_i$ are finite dimensional C$^*$-algebras, which decompose as $F_i = F_i^{(0)} \oplus \cdots \oplus F_i^{(n)}$, $\psi_i : A \to F_i$ are cp and contractive (cpc) and $\phi_i : F_i \to A$ are cp such that $\lim_i \phi_i(\psi_i(a)) = a$ for $a \in A$, and each $\phi_i|_{F_i^{(j)}}$ is contractive and order zero. If additionally each $\phi_i$ is contractive, then $A$ is said to have decomposition rank at most $n$. 


We will be repeatedly concerned with sequences of asymptotically order zero maps inducing order zero maps into ultrapowers. Firstly we set out our notation for working with the latter.

**Notation 1.2.** Throughout the paper ω will denote a free ultrafilter on \( \mathbb{N} \). Given a C*-algebra \( A \), write \( A_ω \) for the ultrapower obtained as the quotient \( \ell^∞(A)/c_ω(A) \), where \( c_ω(A) = \{ (x_n)_{n=1}^∞ ∈ \ell^∞(A) : \lim_{n \to ω} \|x_n\| = 0 \} \). We shall generally suppress the quotient map \( \ell^∞(A) \to A_ω \) and instead say that \( (x_n)_{n=1}^∞ \) represents or lifts \( x \in A_ω \) if it quotients to \( x \). It is well known that positive contractions can be lifted to sequences of positive contractions, and unitaries to sequences of unitaries.

We regularly regard \( A \) as embedded in \( A_ω \), where \( a ∈ A \) is represented by the sequence with constant value \( a \). We also work regularly with ultrapowers of tensor products of the form \( (A ⊗ B)_ω \) and throughout the paper \( ⊗ \) denotes the spatial tensor product.

Given C*-algebras \( A \) and \( B \), a uniformly bounded sequence \( (ψ_n)_{n=1}^∞ \) of bounded linear maps \( A → B \) induces a bounded linear map \( Ψ : A_ω → B_ω \) defined at the level of representatives by \( (x_n)_{n=1}^∞ → (ψ_n(x_n))_{n=1}^∞ \). When each \( ψ_n \) is completely positive, contractive, a *-homomorphism, or order zero, then \( Ψ \) enjoys the same property. When each \( ψ_n \) is cp then the induced map \( Ψ \) is order zero if and only if \( \lim_{n → ω} \|ψ_n(x_n)ψ_n(y_n)\| = 0 \) whenever \( \lim_{n → ω} \|x_ny_n\| = 0 \) for bounded sequences \( (x_n)_{n=1}^∞, (y_n)_{n=1}^∞ \in A_+ \). Sometimes the map \( Ψ \) will only be order zero when restricted to a subalgebra, say \( A ⊆ A_ω \). This is the case if and only if the sequence is asymptotically order zero with respect to \( A \) in the sense that \( \lim_{n → ω} \|ψ_n(x)ψ_n(y)\| = 0 \) whenever \( x, y ∈ A_+ \) satisfy \( xy = 0 \); when \( A \) is unital we can describe this by asking for (1.1) to hold asymptotically, i.e.

\[
\lim_{n → ω} \|ψ_n(xy)ψ_n(1) − ψ_n(x)ψ_n(y)\| = 0, \quad x, y ∈ A.
\]

Note too that when \( A \) is nuclear, by the Choi-Effros lifting theorem \[5\] any cpc order zero map \( A → B_ω \) arises from a sequence of cpc maps \( A → B \) which is asymptotically order zero with respect to \( A \). Sequences of asymptotically order zero maps arise naturally from approximations witnessing finite nuclear dimension; indeed, given a system of maps \( (F_i, ψ_i, φ_i)_{i ∈ I} \) as in Definition \[11\], one can modify the approximation by removing unnecessary components so that the maps \( ψ_i \) are asymptotically order zero \[17\] Proposition 3.2).

Sequences of asymptotically order zero maps enjoy a slight variant of the structure theorem using a supporting order zero map in place of the supporting *-homomorphism. A crucial point here is that the supporting order zero map can be chosen to take values in the ultrapower \( B_ω \), rather than the multiplier algebra \( \mathcal{M}(C^*(ψ(A))) \subset (B_ω)^* \).}

**Lemma 1.3.** Let \( A \) and \( B \) be a separable unital C*-algebras, and \( (ψ_n)_{n=1}^∞ \) a sequence of cpc maps \( A → B \) inducing a cpc map \( Ψ : A_ω → B_ω \). Let \( X \) be a separable C*-subalgebra of \( A_ω \) with \( 1_{A_ω} = 1_X \) and suppose that \( Ψ|_X : X → B_ω \) is order zero. Then there exists a sequence \( (π_n)_{n=1}^∞ \) of cpc maps \( π_n : A → C^*(ψ_n(A)) \subset B \) such that the induced cpc map \( Π : A_ω → B_ω \) restricts to an order zero map on \( X \) satisfying

\[
Ψ(x) = Π(x)Ψ(1_X) = Ψ(1_X)Π(x), \quad x ∈ X.
\]

Suppose additionally that \( A \) and \( B \) factorise as \( A = A_0 ⊗ C \) and \( B = B_0 ⊗ C \), for some separable unital C*-algebra \( C \), and each \( ψ_n \) is of the form \( ψ_n ⊗ id_C \) for some
cpc map $\hat{\psi}_n : \hat{A} \to \hat{B}$. Then each $\pi_n$ can be taken of the form $\hat{\pi}_n \otimes \text{id}_C$ for some cpc map $\hat{\pi}_n : \hat{A} \to C^*(\hat{\psi}_n(A))$.

**Proof.** For $m \in \mathbb{N}$, define piecewise linear functions $g_m \in C_0(0,1]$ by

$$g_m(t) = \begin{cases} 1, & t \geq 1/m; \\ 2mt - 1, & 1/(2m) \leq t \leq 1/m; \\ 0, & 0 \leq t \leq 1/(2m). \end{cases}$$

Define $h_m \in C_0(0,1)_+$ to satisfy $h_m(t) = g_m(t)$ for all $t \in (0,1]$. The cpc order zero map $g_m(\Psi|_X) : X \to B_\sigma \cap \Psi(1_X)'$ takes the form

$$g_m(\Psi|_X)(x) = h_m(\Psi(1_X))^{1/2}\Psi(x)h_m(\Psi(1_X))^{1/2}, \quad x \in X$$

since if $\rho$ is the supporting homomorphism of $\Psi|_X$, then

$$g_m(\Psi|_X)(\cdot) = \rho(\cdot)g_m(\Psi|_X(1_X))$$

$$\quad = \rho(\cdot)\Psi|_X(1_X)h_m(\Psi|_X(1_X)) = \Psi|_X(\cdot)h_m(\Psi|_X(1_X)).$$

Fix a countable dense sequence $(x^{(i)})_{i=1}^\infty$ in the unit ball of $X$ and lift each $x^{(i)}$ to a sequence $(\tilde{x}^{(i)})_{n=1}^\infty$ of contractions in $A$. Since $(h_m(\psi_n(1_A)))^{1/2}_{n=1}$ is a lift of $h_m(\Psi(1_X))^{1/2}$ the map $g_m(\Psi|_X)$ is (the restriction to $X$ of the cpc map) induced by the sequence of cpc maps $(\tilde{\psi}_{m,n})_{n=1}^\infty$ from $A$ to $B$ given by

$$\tilde{\psi}_{m,n}(a) = h_m(\psi_n(1_A))^{1/2}\psi_n(a)h_m(\psi_n(1_A))^{1/2}, \quad a \in A.$$ It then follows that for each $i,j,m \in \mathbb{N}$,

$$\lim_{n \to \omega} \| \tilde{\psi}_{m,n}(x^{(i)}_n)\tilde{\psi}_{m,n}(1_A) - \tilde{\psi}_{m,n}(x^{(i)}_n)\tilde{\psi}_{m,n}(x^{(j)}_n) \| = 0$$

As $g_m(\Psi|_X)(X) \subset B_\sigma \cap \Psi(1_X)'$, for all $i,m \in \mathbb{N}$ we also have

$$\lim_{n \to \omega} \| \tilde{\psi}_{m,n}(x^{(i)}_n)\tilde{\psi}_{n}(1_A) - \psi_n(1_A)\tilde{\psi}_{m,n}(x^{(i)}_n) \| = 0.$$

For each $n \in \mathbb{N}$, let $m_n$ be the maximum value of $m \in \{1,2,\ldots,n\}$ such that

$$\min_{n \to \omega} \| \tilde{\psi}_{m,n}(x^{(i)}_n)\tilde{\psi}_{m,n}(1_A) - \psi_n(1_A)\tilde{\psi}_{m,n}(x^{(i)}_n) \| \leq \frac{1}{m}, \quad 1 \leq i,j \leq m,$$

and

$$\| \tilde{\psi}_{m,n}(x^{(i)}_n)\tilde{\psi}_{n}(1_A) - \tilde{\psi}_{m,n}(x^{(i)}_n) \| \leq \frac{1}{m}, \quad 1 \leq i \leq m.$$ The limits in (1.8) and (1.9) demonstrate that $\lim_{n \to \omega} m_n = \infty$.

Define $\pi_n = \psi_{m,n} : A \to B$. As $\lim_{n \to \omega} m_n = \infty$, (1.10) and the order zero relation (1.1) show that the sequence $(\pi_n)_{n=1}^\infty$ induces a cpc order zero map $\Pi : X \to B_\sigma$ and (1.11) shows that $\Pi(X) \subset \Psi(1_X)'$. Let $e \in B_\sigma$ be the contraction represented by the sequence $(g_m(\psi_n(1_A)))_{n=1}^\infty$. As $\|g_m(\psi_n(1_A) - \psi_n(1_A)) \| \leq 2/m$ for all $n$ and $\lim_{n \to \omega} m_n = \infty$, it follows that $e\Psi(1_X) = \Psi(1_X) = \Psi(1_X)e$. Since $\Psi(X)$ lies in the hereditary subalgebra $\Psi(1_X)B_\sigma\Psi(1_X)$, we have

$$e\Psi(x) = \Psi(x)e = \Psi(x), \quad x \in X.$$ Now, given $x \in X$ with lift $(x_n)_{n=1}^\infty$, we have

$$\psi_n(1_A)^{1/2}\pi_n(x_n)\psi_n(1_A)^{1/2} = g_m(\psi_n(1_A))^{1/2}\psi_n(x_n)g_m(\psi_n(1_A))^{1/2},$$

so (1.12) gives

$$\psi(1_X)^{1/2}\Psi(x)\psi(1_X)^{1/2} = \Psi(x).$$
As $\Pi(X)$ commutes with $\Psi(1_X)$, this establishes \textbf{1.3}.

Now suppose $A$, $B$ and each $\psi_n$ factorise as described in the second paragraph of the lemma. Then $h_m(\psi_n(1_A))^{1/2} = h_m(\hat{\psi}_n(1_A))^{1/2} \otimes 1_C$ and so each $\hat{\psi}_{m,n}$ factorises as $\hat{\psi}_{m,n} \otimes \text{id}_C$, with $\hat{\psi}_{m,n} : \hat{A} \to C^*(\hat{\psi}_n(A))$ given by

$$\hat{\psi}_{m,n}(a) = h_m(\hat{\psi}_n(1_A))^{1/2}\hat{\psi}_n(a)h_m(\hat{\psi}_n(1_A))^{1/2}, \quad a \in \hat{A}.\quad (1.15)$$

Thus each $\pi_n$ also enjoys the specified factorisation. \hfill \Box

The functional calculus for order zero maps $\Psi : X \to B_\omega$ arising as in Lemma \textbf{1.3} can be recaptured using the supporting order zero map $\Pi$; for $f \in C_0(0,1)_+$ and $x \in X$, we have $\Pi(\Psi(x)) = \Pi(f(0)) f(\Psi(1_X))$. When $f$ is a polynomial with $f(0) = 0$, then $f(t) = t \cdot g(t)$ for some polynomial $g$ and we have $f(\Psi(x)) = \Pi(x) f(\Psi(1_X)) = \Pi(x) g(\Psi(1_X)) = \Pi(x) g(\Psi(1_X)) = \Pi(x) f(\Psi(1_X))$, where $\Psi$ denotes the supporting $^*$-homomorphism of $\Psi$. In general the claim follows by approximating $f$ uniformly by polynomials. In particular, this shows that $f(\Psi)$ is induced by a sequence of asymptotically order zero maps, whose ranges lie in the same $C^*$-algebras as the maps used to obtain $\Psi$.

**Lemma 1.4.** Let $A$ and $B$ be separable unital $C^*$-algebras and $(\psi_n)_{n=1}^\infty$ a sequence of cpc maps $A \to B$ inducing a cpc map $\Psi : A_\omega \to B_\omega$. Let $X$ be a separable $C^*$-subalgebra of $A_\omega$ with $1_X = 1_{A_\omega}$, and suppose that $\Psi|_X : X \to B_\omega$ is order zero. Given any function $f \in C_0(0,1)_+$, the order zero map $f(\Psi|_X) : X \to B_\omega$ is (the restriction to $X$ of the cpc map) induced by a sequence $(\psi_n)_{n=1}^\infty$ of cp maps $\psi_n : A \to C^*(\psi_n(A)) \subseteq B$.

Suppose additionally that $A$ and $B$ factorise as $A = \hat{A} \otimes C$ and $B = \hat{B} \otimes C$, for some separable unital $C^*$-algebra $C$, and each $\psi_n$ is of the form $\hat{\psi}_n \otimes \text{id}_C$ for some cpc map $\hat{\psi}_n : \hat{A} \to \hat{B}$. Then each $\hat{\psi}_n : A \to B$ can be taken of the form $\hat{\psi}_n \otimes \text{id}_C$ for some cpc map $\hat{\psi}_n : \hat{A} \to C^*(\hat{\psi}_n(A))$.

**Proof.** This follows from the expression $f(\Psi|_X)(x) = \Pi(x) f(\Psi(1_X))$ for $x \in X$, where $\Pi$ is the supporting order zero map of Lemma \textbf{1.3} induced by cpc maps $\pi_n : A \to C^*(\psi_n(A))$. \hfill \Box

We end the section by recording a fact about polar decompositions in ultrapowers. The proof that unitary polar decompositions exist given in \textbf{22} is stated for quotients $\prod B_n/\sum B_n$, but applies without modification to ultrapowers. It then follows immediately that $B_\omega$ has stable rank one ($x = u|x|$ can be approximated by invertibles of the form $u([x] + c1_{B_\omega})$).

**Lemma 1.5 (c.f. \textbf{22} Lemma 19.2.2(1)).** Let $B$ be a separable unital $C^*$-algebra with stable rank one. Then for every $x \in B_\omega$, there exists a unitary $u \in B_\omega$ with $x = u|x|$ and so $B$ has stable rank one.

2. **Tracially large UHF embeddings of cones**

In this section we provide the tracially large order zero maps $A \to U_\omega$ used to obtain the finite dimensional algebras in the finite nuclear dimension factorisations of our main result. The proof works when $U$ is replaced by any separable, unital, nuclear and stably finite $C^*$-algebra.
Notation 2.1. Let $A$ be a separable unital $C\sp*$-algebra with a fixed tracial state $\tau$, and let $\tau_{A,\omega}$ be the induced tracial state on $A_\omega$ defined on a representative sequence $(x_n)_{n=1}^\infty$ by $\lim_{n \to \infty} \tau(x_n)$. Following [19], we define the trace kernel ideal of $\tau_{A,\omega}$ as $I_{A,\tau} := \{(a_n)_n \in A_\omega \mid \lim_{n \to \infty} \tau(a_n^*a) \} \triangleleft A_\omega$. (We do not keep track of the ultrafilter in our notation for the ideal. However, this should not cause confusion; in fact, we think of the free ultrafilter as being fixed throughout the paper.)

We use Kirchberg’s notion of a $\sigma$-ideal from [17], working with the formulation of [19] Definition 4.4]: an ideal $I$ in a $C\sp*$-algebra $D$ is a $\sigma$-ideal if for every separable $C\sp*$-subalgebra $C$ of $D$ there exists a positive contraction $e \in I \cap C\sp{'}$ such that $ee = c$ for all $c \in I \cap C$. A key observation of Kirchberg and Rørdam is that $I_{A,\tau}$ is a $\sigma$-ideal in $A_\omega$ whenever $A$ is unital, and $\tau$ is a tracial state on $A$. [19] Proposition 4.6 and Remark 4.7]. (This is a special case of the results of [19, Section 4]; the trace-kernel ideal $I_P$ obtained from those sequences which converge to zero uniformly on all traces is also a $\sigma$-ideal in $A_\omega$, and so too are the ideals $(I_{A \cap \omega}) \triangleleft (A_\omega \cap A')$ and $(I_{A,\omega}) \triangleleft (A_\omega \cap A')$ in the central sequence algebras). The proof below is based on the argument for surjectivity given at the end of Section 4 of [19]. In forthcoming work we will provide some uniqueness statements for such maps up to a finitely coloured decomposition as a sum of order zero maps arising from conjugation by a suitable sequence of contractions.

Proposition 2.2. Let $A$ and $B$ be separable unital and nuclear $C\sp*$-algebras without elementary quotients, and with extremal traces $\tau_A$ and $\tau_B$ respectively. Then there exists a cpc order zero map $\Psi : A \to B_\omega$ with $1_{B_\omega} - \Psi(1_A) \in J_{B,\tau_B}$ and $\tau_{B,\omega}(\Psi(x)) = \tau_A(x)$ for all $x \in A$.

Proof. The strong operator closures of $A$ and $B$ in the GNS representations corresponding to $\tau_A$ and $\tau_B$ are injective $\text{II}_1$ factors, so by Connes’ uniqueness theorem [7], both these strong closures are isomorphic to the hyperfinite $\text{II}_1$ factor $R$. Let $R^\omega$ be the von Neumann ultrapower, namely the quotient of $L^\infty(R)$ by those sequences $(x_n)_{n=1}^\infty$ with $\lim_{n \to \infty} \tau_B(x_n) = 0$. By Kaplansky’s density theorem, the canonical embeddings $A_\omega/J_{A,\tau_A} \to R^\omega$ and $B_\omega/J_{B,\tau_B} \to R^\omega$ are surjective, so give a surjective $*$-isomorphism $\tilde{\Psi} : A_\omega/J_{A,\tau_A} \to B_\omega/J_{B,\tau_B}$. Let $\pi : A \to A_\omega/J_{A,\tau_A}$ be the canonical unital $*$-homomorphism. Applying the Choi-Effros lifting theorem [3], we obtain a unital cpc map $\hat{\Psi} : A \to B_\omega$ lifting $\tilde{\Psi} \circ \pi$, i.e. $\hat{\Psi} \circ \pi = q \circ \hat{\Psi}$, where $q : B_\omega \to B_\omega/J_{B,\tau_B}$ is the quotient map. As $J_{B,\tau_B}$ is a $\sigma$-ideal in $B_\omega$, let $e \in J_{B,\tau_B} \cap C_0(1_{B_\omega}, \hat{\Psi}(A))'$ be a positive contraction with $ee = c$ for all $c \in J_{B,\tau_B} \cap C_0(1_{B_\omega}, \hat{\Psi}(A))$.

Define a cpc map $\Psi : A \to B_\omega$ by $\Psi(a) = (1_{B_\omega} - e)\hat{\Psi}(x)(1_{B_\omega} - e)$ for $x \in A$. We have $1_{B_\omega} - \Psi(1_A) = 1_{B_\omega} - (1_{B_\omega} - e)^2 \in J_{B,\tau_B}$ and

$$\tau_{B,\omega}(\Psi(x)) = \tau_{B,\omega}(\hat{\Psi}(x)) = \tau_{B,\omega}(\hat{\Psi} \circ \pi(x)) = \tau_{A,\omega} \circ \pi(x) = \tau_A(x), \quad x \in A;$$

for the third equation note that since $R^\omega$ has a unique normal tracial state we have $\tau_{B,\omega} = \tau_{A,\omega}$ for any $x \in A_\omega/J_{A,\tau_A}$.

Finally, if $x, y \in A_+$ satisfy $xy = 0$, then $\hat{\Psi}(x)\hat{\Psi}(y) \in \ker q = J_{B,\tau_B}$, and hence $(1_{B_\omega} - e)\hat{\Psi}(x)\hat{\Psi}(y) = 0$. Since $(1_{B_\omega} - e)$ commutes with $\hat{\Psi}(A)$, we have

$$\Psi(x)\Psi(y) = (1_{B_\omega} - e)\hat{\Psi}(x)(1_{B_\omega} - e)\hat{\Psi}(y)(1_{B_\omega} - e) = (1_{B_\omega} - e)^2 \hat{\Psi}(x)\hat{\Psi}(y)(1_{B_\omega} - e)^2 = 0,$$

according to (2.2).
isomorphism of $R$ a trace preserving embedding $A \to R^\omega \cap R'$ (which is easily obtained from the isomorphism of $R$ with its infinite tensor product), and using the fact that $(B_\omega \cap B')/(J_B,\tau_B \cap B') \cong R^\omega \cap R'$ (E1 Lemma 2.1) when $B$ is nuclear and E1 Theorem 3.3] for general $B$), the same proof allows us to insist that the order zero map $\Psi$ maps into the central sequence algebra $B_\omega \cap B'$.

Note too that the previous proposition recaptures Voiculescu’s result [10] that the cone over a $C^*$-algebra $A$ is quasidiagonal in the special case when $A$ is unital and nuclear without elementary quotients, and has a separating family of tracial states (so in particular when $A$ is also simple and stably finite by the work of Blackadar, Handelman [13, 3] and Haagerup [12]).

**Corollary 2.3.** Let $A$ be a separable, unital and nuclear $C^*$-algebra without elementary quotients, and with the property that for every positive element $x \in A$ there is a tracial state $\tau$ on $A$ with $\tau(x) \neq 0$. Then $C_0(0,1] \otimes A$ is quasidiagonal.

**Proof.** Suppose first that $\tau_A$ is an extremal tracial state on $A$ and take $B$ to be a UHF algebra in Proposition 2.2, and let $\Psi : A \to B_\omega$ be the order zero map given there. The duality between order zero maps from $A$ and $^*$-homomorphisms from $C_0(0,1] \otimes A$ [46 Corollary 4.1], gives a $^*$-homomorphism $\Pi : C_0(0,1] \otimes A \to B_\omega$ which has $\Pi(id_{[0,1]} \otimes x) = \Psi(x)$. Note that

$$\tau_{B,\omega}(\Psi(1_A^n)) = 1, \quad n \in \mathbb{N}$$

as $1_{B_\omega} - \Psi(1_A) \in J_B$. Then the measure on $(0,1]$ induced by $\tau_{B,\omega}(\Pi(\cdot \otimes 1_A))$ is supported at 1 so that $\tau_{B,\omega} \circ \Pi$ vanishes on $C_0(0,1] \otimes A$, and hence factors through $A \cong (C_0(0,1] \otimes A)/(C_0(0,1] \otimes A)$. Accordingly $\tau_{B,\omega} \circ \Pi = \delta_1 \otimes \tau_A$.

Given $t \in (0,1]$, follow the restriction $C_0(0,1] \to C_0(0,t]$ with a map $C_0(0,t] \to C_0(0,1]$ induced by rescaling to obtain a $^*$-homomorphism $\Pi_t : C_0(0,1] \otimes A \to B_\omega$ with $\tau_{B,\omega} \circ \Pi_t = \delta_t \otimes \tau_A$. Then if $\tau_A$ is faithful, the sequence of maps $(\Pi_n)_{n \in \mathbb{N}}$ given by

$$\Pi_n(f) = \Pi_{1/n}(x) \oplus \Pi_{2/n}(f) \oplus \cdots \oplus \Pi_1(f) : A \to B_\omega^{\otimes n} \cong (B_\omega^{\otimes n})$$

can be used to witness quasidiagonality, as asymptotically

$$\frac{1}{n} \sum_{i=1}^n \tau_{B,\omega}(\Pi_{i/n}(f)) \to \int_0^1 \tau_A(f(t))dt, \quad f \in C_0((0,1], A) \cong C_0(0,1] \otimes A$$

where the integral on the right is performed with respect to Lebesgue measure and so is non-zero when $f$ is positive and non-zero, as $\tau_A$ was assumed faithful.

In general, note that for every non-zero positive element $f \in C_0((0,1], A)$, the hypothesis gives an extremal trace $\tau$ on $A$ such that $\int_0^1 \tau(f(t))dt \neq 0$. Thus sums of the maps in (2.4) taken over finite subsets of the extremal boundary of the tracial state space of $A$ can be used to witness quasidiagonality of $C_0(0,1] \otimes A$.

3. 2-coloured approximately inner flips

The next lemma extracts the 2-coloured approximately inner flip for monotracial nuclear UHF-stable $C^*$-algebras from the proof of [24 Theorem 4.2] in the form we need for the sequel. We give the proof for completeness (note that our notation differs from that in [24, particularly for ultrapowers).
Lemma 3.1. Let $A$ be a simple, separable, unital, non-elementary and nuclear C*-algebra with unique tracial state $\tau_A$, and let $V$ be a UHF algebra. Then there exist contractions $v^{(0)}, v^{(1)} \in (A \otimes A \otimes V)^\omega$ with

\[
\sum_{i=0}^1 v^{(i)}(a \otimes b \otimes 1_V)v^{(i)*} = (b \otimes a \otimes 1_V), \quad a, b \in A,
\]

\[
\sum_{i=0}^1 v^{(i)*}v^{(0)} = (1_A \otimes 1_A \otimes V)^\omega \quad \text{and} \quad \sum_{i=0}^1 v^{(i)*}v^{(1)} = 1_A \otimes 1_A \otimes 1_V.
\]

Proof. As $A$ has unique trace its strong closure in the GNS-representation of this trace is an injective $\text{II}_1$ factor, and so has a $\| \cdot \|_{2,\tau}$-approximately inner flip by the implication $(7) \implies (3)$ from Connes’ Theorem [7, Theorem 5.1]). In particular, using Kaplansky’s density theorem, there exists a sequence $(u_n)\in\omega$ of unitaries in $A \otimes A$ with

\[
\lim_{n \to \omega} \| u_n x u_n^* - \sigma(x) \|_{2,\tau \otimes \tau} = 0, \quad x \in A \otimes A,
\]

where $\sigma : A \otimes A \to A \otimes A$ is the flip automorphism of $A \otimes A$.

Write $B = M_2 \otimes A \otimes A \otimes V$ and let $q : B_\omega \to B_\omega / J_{B,\tau_B}$ denote the quotient map. Consider the embedding $\pi : A \otimes A \to B_\omega \cong M_2((A \otimes A \otimes V)^\omega)$, given by

\[
\pi(x) = \begin{bmatrix} \iota(x) \otimes 1_V & 0 \\ 0 & \iota \circ \sigma(x) \otimes 1_V \end{bmatrix},
\]

where $\iota : A \otimes A \to (A \otimes A)^\omega$ is the canonical unital inclusion. Let $u \in B_\omega$ be the partial isometry represented by

\[
u = \left( \begin{bmatrix} 0 & 0 \\ u_n \otimes 1_V & 0 \end{bmatrix} \right)_{n=1}^\infty
\]

so that $q(u) \in q(B_\omega) \cap q(\pi(A \otimes A))'$ from (3.3). Then we have projections

\[
e = \begin{bmatrix} 1_{(A \otimes A \otimes V)^\omega} & 0 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 0 & 1_{(A \otimes A \otimes V)^\omega} \end{bmatrix}
\]

in $B_\omega \cap \pi(A \otimes A)'$ satisfying $e = uu^*$ and $f = uu^*$. By construction, the partial isometry $q(u)$ witnesses Murray-von Neumann equivalence of $q(e)$ and $q(f)$ in $q(B_\omega) \cap q(\pi(A \otimes A))'$.

The relative SI results of [24, Section 3] apply to the inclusion $\pi(A \otimes A) \subset B_\omega$. In particular every trace on $B_\omega \cap \pi(A \otimes A)'$ is of the form $\tilde{\tau} \circ q$ for some trace $\tilde{\tau}$ on $q(B_\omega) \cap q(\pi(A \otimes A))'$ by [24, Proposition 3.3(i)]. Thus $\tau(e) = \tau(f)$ for all traces $\tau \in T(B_\omega \cap \pi(A \otimes A)')$. Further, as $B$ has strict comparison, [24, Proposition 3.3(ii)] applies to show that $B_\omega \cap \pi(A \otimes A)'$ has strict comparison of projections. Then, for each $n \in \mathbb{N}$, apply the 2-coloured Cuntz-Pedersen theorem ([24, Lemma 2.1]) to obtain contractions $\tilde{w}_n^{(0)}, \tilde{w}_n^{(1)} \in (B_\omega \cap \pi(A \otimes A)') \otimes M_n$, such that

\[
\| \tilde{w}_n^{(0)*}\tilde{w}_n^{(0)} + \tilde{w}_n^{(1)*}\tilde{w}_n^{(1)} - e \otimes 1_n \| \leq \frac{4}{n}, \quad \| \tilde{w}_n^{(0)*}\tilde{w}_n^{(0)} + \tilde{w}_n^{(1)*}\tilde{w}_n^{(1)} - f \otimes 1_n \| \leq \frac{4}{n}
\]

and

\[
\text{dist}(\tilde{w}_n^{(0)*}\tilde{w}_n^{(0)}, e \otimes M_n) \leq \frac{2}{n}, \quad \text{dist}(\tilde{w}_n^{(1)*}\tilde{w}_n^{(1)}, e \otimes M_n) \leq \frac{2}{n}.
\]
By regarding $M_n$ (for suitable $n$) as being unitally embedded in $V$, we have \( \tilde{\omega}_n^{(0)}, \tilde{\omega}_n^{(1)} \in B_2 \cap \pi(A \otimes A)' \). A standard reindexing argument gives contractions \( \tilde{v}^{(0)}, \tilde{v}^{(1)} \in B_\omega \cap \pi(A \otimes A)' \) with

\[
(3.9) \quad \sum_{i=0}^1 \tilde{v}^{(i)*} \tilde{v}^{(i)} = \begin{bmatrix} 1_{(A \otimes A \otimes V)_\omega} & 0 \\ 0 & 0 \end{bmatrix}, \quad \sum_{i=0}^1 \tilde{v}^{(i)*} \tilde{v}^{(i)} = \begin{bmatrix} 0 & 0 \\ 0 & 1_{(A \otimes A \otimes V)_\omega} \end{bmatrix}
\]

and

\[
(3.10) \quad \tilde{v}^{(0)*} \tilde{v}^{(0)} \in \begin{bmatrix} 1_A \otimes 1_A \otimes V \omega & 0 \\ 0 & 0 \end{bmatrix}.
\]

From (3.9), there are contractions \( v^{(0)}, v^{(1)} \in (A \otimes A \otimes V)_\omega \) such that

\[
(3.11) \quad \tilde{v}^{(0)} = \begin{bmatrix} 0 & 0 \\ v^{(0)} & 0 \end{bmatrix}, \quad \tilde{v}^{(1)} = \begin{bmatrix} 0 & 0 \\ v^{(1)} & 0 \end{bmatrix}.
\]

These are readily seen to satisfy (3.1) and (3.2).

Since UHF algebras of infinite type are strongly self-absorbing, if a separable C*-algebra $A$ tensorially absorbs a UHF-algebra $U$ of infinite type, then there is an isomorphism $A \to A \otimes U$ which is approximately unitarily equivalent to the first factor embedding $\iota : A \to A \otimes U$ [27, Theorem 7.2.2 and Remark 7.2.3]. Applying this statement to $A \otimes A$, one obtains an isomorphism $\Theta : (A \otimes A \otimes U)_\omega \to (A \otimes A)_\omega$ with $\Theta(a \otimes b \otimes 1_U) = a \otimes b$ for $a, b \in A$. Following Proposition 3.1, this isomorphism gives the following statement, the conclusion of which one might take as the definition that $A$ has “2-coloured approximately inner flip”.

**Proposition 3.2.** Let $A$ be a simple, separable, unital and nuclear C*-algebra with a unique tracial state which absorbs a UHF-algebra of infinite type. Then there exist contractions \( v^{(0)}, v^{(1)} \in (A \otimes A)_\omega \) with \( v^{(i)*} v^{(i)} \in (A \otimes A)_\omega \cap (A \otimes A)' \) for \( i = 0, 1 \), \( \sum_{i=0}^1 v^{(i)*} v^{(i)} = 1_{A \otimes A} \), and

\[
(3.12) \quad \sum_{i=0}^1 v^{(i)}(a \otimes b) v^{(i)*} = b \otimes a, \quad a, b \in A.
\]

Having an approximately inner flip is a strong requirement to impose on a separable C*-algebra $A$, forcing $A$ to be simple and nuclear, and to have at most one tracial state [23, Proposition 2.7, Proposition 2.8, Lemma 2.9]. Essentially the same arguments apply to C*-algebras satisfying the conclusions of Proposition 3.2. We sketch this below for completeness. In particular this shows that in using Lemma 3.1 to prove Theorem 3.1, the unique trace hypothesis really is essential.

**Proposition 3.3 (cf [23]).** Let $A$ be a separable unital C*-algebra with 2-coloured approximately inner flip, i.e., satisfying the conclusions of Proposition 3.2. Then $A$ is simple, nuclear, and has at most one tracial state.

**Proof.** For simplicity, if $I$ is a proper ideal in $A$, then $I \otimes A$ and $A \otimes I$ are distinct ideals in $A \otimes A$. On the other hand, if the flip automorphism on $A$ takes the form (3.12), then it follows that $I \otimes A = A \otimes I$, a contradiction. For the tracial statement, suppose $\tau_1$ and $\tau_2$ are traces on $A$, inducing a trace $\tau_1 \otimes \tau_2 _\omega$ on $(A \otimes A)_\omega$. Noting that $\sum_{i=0}^1 v^{(i)*} v^{(i)} = 1_{(A \otimes A)_\omega}$, applying $(\tau_1 \otimes \tau_2)_\omega$ to (3.12) with $b = 1_A$ shows that $\tau_1(a) = \tau_2(a)$ for $a \in A$. Finally, for nuclearity, given (5.12), lift each $v^{(i)}$.
isometrically to a sequence $(v_n^{(i)})_{n=1}^\infty$ of finite combinations of elementary tensors, $v_n^{(i)} = \sum_{j \in A_n} c_{j,n}^{(i)} \otimes d_{j,n}^{(i)}$. Fix a state $\phi$ on $A$, then the maps

\begin{equation}
A \to A; \quad a \mapsto \sum_{i=0}^1 \sum_{j,k \in A_n} \phi(c_{j,n}^{(i)} a c_{k,n}^{(i)} *) d_{j,n}^{(i)} d_{k,n}^{(i)} *
\end{equation}

are cp, uniformly bounded, finite rank, and converge to $id_A$ as $n \to \infty$ in point norm topology, so that $A$ is nuclear. \hfill \Box

4. Tracially Large Purely Positive Elements in $\mathcal{Z}_\omega$

Our objective in this section is to establish the following uniqueness result for certain positive contractions of full spectrum which we shall apply when $A = B = \mathcal{Z}$.

Lemma 4.1. Let $A$ and $B$ be a simple, unital and exact C$^*$-algebras with unique tracial state $\tau_A$ and $\tau_B$ respectively such that $A \otimes B$ has strict comparison and stable rank one. Let $h \in B$ be a positive contraction with spectrum $[0,1]$. Given positive contractions $e^{(0)}, e^{(1)} \in J_{A,\tau_A} \triangleleft A_\omega$, there is a unitary $u \in (A \otimes B)_\omega$ with $u((1_{A_\omega} - e^{(0)}) \otimes h) u^* = (1_{A_\omega} - e^{(1)}) \otimes h$.

Recall that $C_0(0,1)$ is the universal C$^*$-algebra generated by a positive contraction, so positive contractions in a C$^*$-algebra $D$ are in bijective correspondence with $\star$-homomorphisms $C_0(0,1) \to D$. To prove Lemma 4.1 we use Ciuperca and Elliott’s classification of such $\star$-homomorphisms up to approximate unitary equivalence when $D$ has stable rank one via the Cuntz semigroup from [6] (stated in the form we use as Theorem 4.2 below; see also [22]).

For $D$ a C$^*$-algebra with positive elements $x, y \in D \otimes K$ write $x \lesssim y$ when there exists a sequence $(v_n)_{n=1}^\infty$ in $D \otimes K$ with $v_n y v_n \to x$. Define $x \sim y$ when $x \lesssim y$ and $y \lesssim x$. The Cuntz semigroup is defined to be $\text{Cu}(D) = (D \otimes K)_+ / \sim$, and we write $[x]$ for the class of $x \in (D \otimes K)_+$. This is equipped with an addition arising from identification of $K \cong K \otimes M_2$ to take an orthogonal sum of two positive elements of $D \otimes K$, an order $\leq$ induced from $\lesssim$, and has the property that every increasing sequence in $\text{Cu}(D)$ has a supremum (8). An abstract category $\text{Cu}$ containing the Cuntz semigroups of C$^*$-algebras was set out in [8], which shows that the assignment $\text{Cu}(\cdot)$ is functorial and preserves sequential inductive limits. Given a positive element $x \in A$ and $\varepsilon > 0$, write $(x - \varepsilon)_+$ for $h_\varepsilon(x)$ where $h_\varepsilon(t) = \max(t - \varepsilon, 0)$. Positive elements $x, y \in D \otimes K$ satisfy $x \lesssim y$ if and only if $(x - \varepsilon)_+ \lesssim y$ for all $\varepsilon > 0$. This last result can be found in the survey article [1] (as Proposition 2.17), to which we refer for a full account of the Cuntz semigroup and the category $\text{Cu}$.

Theorem 4.2 (Ciuperca-Elliott [6] Theorem 4]). Let $D$ be a unital C$^*$-algebra with stable rank one, and for $i = 0, 1$, let $\rho^{(i)} : C_0(0,1) \to D$ be $\star$-homomorphisms. Then $\rho^{(0)}$ and $\rho^{(1)}$ are approximately unitarily equivalent if and only if the induced maps $\text{Cu}(\rho^{(i)} : C_0(0,1) \to \text{Cu}(D))$ are equal.

In the proof of Lemma 4.1 we use the property of strict comparison in order to show that the $\star$-homomorphisms of Theorem 4.2 do indeed induce the same map at the level of the Cuntz semigroup. A (2-quasi)trace $\tau$ on a unital C$^*$-algebra $D$ defines a functional $d_\tau$ on $\text{Cu}(D)$ by $d_\tau([a]) = \mu_a((0, \infty))$ for $a \in (D \otimes K)_+$, where $\mu_a$ is the measure on $[0, \infty)$ induced by $\mu_a(f) = \tau(f(a))$ for $f \in C_0[0, \infty)_+$. In particular, note that $d_\tau(f(a)) = \mu_a(\text{Supp}(f))$ where $\text{Supp}(f) = \{t : f(t) \neq 0\}$
is the open support of $f \in C(\text{Sp}(a))$. (We only need the definition of $d_\tau$ when $a \in D$, but in general $\tau$ is extended to a lower semicontinuous (2-quasi)trace on $D \otimes \mathcal{K}$ to make this definition.) A simple unital $C^*$-algebra $D$ is said to have strict comparison if $d_\tau(a) < d_\tau(b)$ for all 2-quasitraces $\tau$ implies that $a \preceq b$.

Note that when $\tau$ is a faithful trace on $D$, and $h \in D_+$ is a positive contraction with full spectrum, then $d_\tau((h - \varepsilon)_+) < d_\tau(h)$ for all $\varepsilon > 0$.

Proof of Lemma 4.1. It suffices to prove the lemma with $e^{(1)} = 0$. For $i = 0, 1$, write $\rho^{(i)}$ for the induced $^*$-homomorphism $C_0([0, 1]) \rightarrow (A \otimes B)_\omega$, i.e.,

\begin{equation}
\rho^{(0)}(\text{id}_{[0, 1]}) = (1_{A_\omega} - e^{(0)}) \otimes h
\end{equation}

and

\begin{equation}
\rho^{(1)}(\text{id}_{[0, 1]}) = 1_{A_\omega} \otimes h.
\end{equation}

We shall show that

\begin{equation}
\langle \rho^{(0)}(f) \rangle = \langle \rho^{(1)}(f) \rangle \text{ in } Cu((A \otimes B)_\omega), \quad f \in C_0([0, 1]).
\end{equation}

Since $Cu(C_0([0, 1]))$ is isomorphic to $\text{lsc}((0, 1), \{0, 1, 2, \ldots, \infty\})$, the lower semicontinuous functions from $(0, 1]$ into the extended natural numbers (a result of Coward which can be found as [6, Theorem 10.1]), it is generated (under the operations of addition and taking suprema) by the equivalence classes of positive elements in $C_0([0, 1])$. Thus, once we have established (4.3) it follows that $Cu(\rho^{(0)}) = Cu(\rho^{(1)})$, and hence $\rho^{(0)}$ and $\rho^{(1)}$ are approximately unitarily equivalent by Theorem 4.2, noting that $(A \otimes B)_\omega$ inherits stable rank one from $A \otimes B$ by Lemma 4.2. Then either a reindexing argument, or the “$\varepsilon$-test” [17, Lemma A.1] can be used to show that $\rho^{(0)}$ and $\rho^{(1)}$ are actually unitarily equivalent.

We now establish (4.3). Write $\tau_{A \otimes B} = \tau_A \otimes \tau_B$. Lift $e^{(0)}$ to a sequence $(e_n)_{n=1}^\infty$ of positive contractions in $A$, which satisfy $\lim_{n \rightarrow \omega} \tau_A(e_n) = 0$. Write $\mu_n$ for the measure on $(0, 1]$ induced by

\begin{equation}
\mu_n(f) = \tau_{A \otimes B}(f((1_A - e_n) \otimes h)), \quad f \in C_0([0, 1])
\end{equation}

and $\mu$ for the measure induced by

\begin{equation}
\mu(f) = \tau_{A \otimes B}(f(1_A \otimes h)), \quad f \in C_0([0, 1]).
\end{equation}

As $e^{(0)} \in J_A$, we have

\begin{equation}
\lim_{n \rightarrow \omega} \tau_{A \otimes B}((1_A - e_n)^m \otimes h^m) = \tau_{A \otimes B}(1_A \otimes h^m), \quad m \in \mathbb{N}
\end{equation}

and so, by approximating $f \in C_0([0, 1])$ uniformly by polynomial functions,

\begin{equation}
\lim_{n \rightarrow \omega} \tau_{A \otimes B}(f((1_A - e_n) \otimes h)) = \tau_{A \otimes B}(f(1_A \otimes h)), \quad f \in C_0([0, 1]).
\end{equation}

Given an open set $U \subset (0, 1]$, take a sequence $(g_m)_{m=1}^\infty$ of continuous functions in $C_0([0, 1])$ with $0 \leq g_m \nrightarrow \chi_U$, where $\chi_U$ is the indicator function of $U$. Then

\begin{equation}
\mu_n(\chi_U) \geq \mu_n(g_m) \overset{n \rightarrow \omega}{\rightarrow} \mu(g_m),
\end{equation}

so that

\begin{equation}
\lim_{n \rightarrow \omega} \mu_n(\chi_U) \geq \mu(g_m) \overset{m \rightarrow \infty}{\rightarrow} \mu(U),
\end{equation}

by the monotone convergence theorem. Likewise for closed subsets $F \subseteq (0, 1]$, we have

\begin{equation}
\lim_{n \rightarrow \omega} \mu_n(F) \leq \mu(F).
\end{equation}
Now, for $f \in C_0(0, 1]$, 
\begin{equation}
(4.10) \quad d_{\tau_{A \otimes B}}(f((1_A - e_n) \otimes h)) = \mu_n(\text{Supp}(f)).
\end{equation}
Thus $4.9$ and $(4.10)$ combine to show
\begin{equation}
(4.11) \quad \lim_{n \to \infty} d_{\tau_{A \otimes B}}(f((1_A - e_n) \otimes h)) \geq d_{\tau_{A \otimes B}}(f(1_A \otimes h)), \quad f \in C_0(0, 1]+.
\end{equation}

Fix $f \in C_0(0, 1]+$ with $\|f\| = 1$. For notational purposes write $\tilde{h} = f(1_A \otimes h)$, and $\tilde{h}_n = f((1_A - e_n) \otimes h)$. As $h$ has full spectrum, so too has $\tilde{h}$. Since $A$ and $B$ (and hence $A \otimes B$, since we work with the spatial tensor norm) is simple, it follows that for a fixed, sufficiently small $\varepsilon > 0$, we have
\begin{equation}
(4.12) \quad d_{\tau_{A \otimes B}}((\tilde{h} - \varepsilon)_+) < d_{\tau_{A \otimes B}}((\tilde{h} - \varepsilon/2)_+) < d_{\tau_{A \otimes B}}(\tilde{h}).
\end{equation}
Define
\begin{equation}
(4.13) \quad r_{\varepsilon/2}(t) = \begin{cases} 2t/\varepsilon & 0 \leq t \leq \varepsilon/2; \\ 1, & t \geq \varepsilon/2. \end{cases}
\end{equation}
Then $r_{\varepsilon/2}(t)(t - \varepsilon/2)_+ = (t - \varepsilon/2)_+$. By $(4.11)$ and $(4.12)$, we have
\begin{equation}
(4.14) \quad \lim_{n \to \infty} d_{\tau_{A \otimes B}}((\tilde{h}_n - \varepsilon/2)_+) \geq d_{\tau_{A \otimes B}}((\tilde{h} - \varepsilon/2)_+) > d_{\tau_{A \otimes B}}((\tilde{h} - \varepsilon)_+).
\end{equation}
Thus, the set $\Lambda = \{n \in \mathbb{N} : d_{\tau_{A \otimes B}}((\tilde{h}_n - \varepsilon/2)_+) > d_{\tau_{A \otimes B}}((\tilde{h} - \varepsilon)_+)\}$ lies in $\omega$, and for $n \in \Lambda$, $(\tilde{h}_n - \varepsilon)_+ < (\tilde{h} - \varepsilon/2)_+$ by strict comparison (exactness of $A$ and $B$, and hence of $A \otimes B$, plays a role here ensuring via Haagerup’s work [12] that the unique trace is the only 2-quasitrace on $A \otimes B$). For $n \in \Lambda$, find $v_n \in A \otimes B$ (it is easy to see that $v_n$ can be taken in $A \otimes B$ regarded as a subalgebra of $A \otimes B \otimes K$ with $\|v_n(\tilde{h}_n - \varepsilon/2)_+v_n^* - (\tilde{h} - \varepsilon)_+\| < 1/n$ and for $n \notin \Lambda$, define $v_n$ arbitrarily with $\|v_n\| \leq 1$. Then
\begin{equation}
(4.15) \quad \lim_{n \to \infty} \|v_n(\tilde{h}_n - \varepsilon/2)_+^{1/2}r_{\varepsilon/2}(\tilde{h}_n)(\tilde{h}_n - \varepsilon/2)_+^{1/2}v_n^* - (\tilde{h} - \varepsilon)_+\| = 0.
\end{equation}
As the sequence $(v_n(\tilde{h}_n - \varepsilon/2)_+^{1/2})_{n=1}^{\infty}$ is uniformly bounded, this witnesses
\begin{equation}
(4.16) \quad \langle f(1_A \otimes h) - \varepsilon)_+ < r_{\varepsilon/2}(f((1_A - e^{(0)}) \otimes h)) \sim f((1_A - e^{(0)}) \otimes h)
\end{equation}
in $Cu((A \otimes B))$. Since $\varepsilon > 0$ can be taken arbitrarily small, this shows that
\begin{equation}
(4.17) \quad \langle f(1_A \otimes h) \rangle < \langle f((1_A - e^{(0)}) \otimes h) \rangle \text{ in } Cu((A \otimes B)).
\end{equation}
The reverse inequality is similar, working with closed supports. Fix an $\varepsilon > 0$ small enough that $\text{Supp}(f(\varepsilon/2)_+) \supseteq \text{Supp}(f(\varepsilon)_+)$, and
\begin{equation}
(4.18) \quad \mu(\text{Supp}(f(\varepsilon)_+)) < d_{\tau_{A \otimes B}}((\tilde{h} - \varepsilon/2)_+).
\end{equation}
By $(4.9)$,
\begin{equation}
(4.19) \quad \lim_{n \to \infty} \mu_n(\text{Supp}(f(\varepsilon)_+)) < d_{\tau_{A \otimes B}}((\tilde{h} - \varepsilon/2)_+),
\end{equation}
so that
\begin{equation}
(4.20) \quad \lim_{n \to \infty} d_{\tau_{A \otimes B}}((\tilde{h}_n - \varepsilon)_+) < d_{\tau_{A \otimes B}}((\tilde{h} - \varepsilon/2)_+).
\end{equation}
Then, arguing in just the same way that $(4.17)$ is obtained from $(4.14)$, we have
\begin{equation}
(4.21) \quad \langle f(1_A \otimes h) \rangle < \langle f((1_A - e^{(0)}) \otimes h) \rangle \text{ in } Cu((A \otimes B)).
\end{equation}
Combining this with $(4.17)$ establishes $(4.13)$, and so completes the proof. \qed
Note that when $B = \mathcal{Z}$ is the Jiang-Su algebra, then when $A$ is simple unital, exact and stably finite, $A \otimes \mathcal{Z}$ has strict comparison and stable rank one by [29], leading to the following version of the lemma, in which we can also replace $\mathcal{Z}$ by any UHF-algebra.

**Lemma 4.3.** Let $A$ be a simple, unital and exact $C^*$-algebra with unique tracial state $\tau_A$. Let $h \in \mathcal{Z}$ be a positive contraction with spectrum $[0,1]$. Given positive contractions $e^{(0)}, e^{(1)} \in J_{A,\tau_A} \subset A_\omega$, there is a unitary $u \in (A \otimes \mathcal{Z})_\omega$ with $u((1_A - e^{(0)}) \otimes h)u^* = (1_A - e^{(1)}) \otimes h$.

5. $\mathcal{Z}$-STABILITY AND NUCLEAR DIMENSION

In this section we establish the main result of the paper: Theorem 5.1. We start by setting out the properties of $\mathcal{Z}$ that we use. Jiang and Su construct the algebra $\mathcal{Z}$ as an inductive limit of prime dimension drop intervals, directly witnessing that it is simple and has a unique tracial state. Their construction also yields that $\mathcal{Z}$ is strongly self-absorbing in the language of [38], so that $\mathcal{Z} \cong \mathcal{Z} \otimes \mathcal{Z}$ via an isomorphism $\theta : \mathcal{Z} \to \mathcal{Z} \otimes \mathcal{Z}$ which is approximately unitarily equivalent to the first factor embedding $a \mapsto a \otimes 1_{\mathcal{Z}}$. [15] Theorem 7.6, Theorem 8.7]. The Jiang-Su algebra has strict comparison, and stable rank one ([29], which shows the same for all simple, separable, unital, exact and stably finite $\mathcal{Z}$-stable $C^*$-algebras). Thus Lemma 4.1 can be applied with $A = B = \mathcal{Z}$.

From the original inductive limit construction one can see that the Jiang-Su algebra is almost divisible (again see [29] for the more general statement in the presence of $\mathcal{Z}$-stability); in particular given $\epsilon > 0$ and $k \in \mathbb{N}$, there exists a cpc order zero map $\psi : M_k \to \mathcal{Z}$ with $\tau_\mathcal{Z}(\psi(1_k)) > 1 - \epsilon$. We essentially use this last fact in the first of two technical lemmas, which allow us to adjoin the UHF tensor factor required to employ Lemma 3.1 and factorise this through a sum of two order zero maps back into $\mathcal{Z}$. The two order zero maps used here double the number of colours used in the final approximation: two colours arise from the 2-coloured approximately inner flip, and another two from Lemma 5.2, leading to a 4-colour approximation in Theorem 5.1.

**Lemma 5.1.** Let $U$ be a UHF algebra. Then there exists a sequence $(\tilde{\psi}_n)^{\infty}_{n=1}$ of cpc order zero maps $U \to \mathcal{Z}$ inducing an order zero map $\tilde{\Psi} : U_\omega \to \mathcal{Z}_\omega$ such that for any positive contraction $e \in J_{U,\tau_U}$, we have $1_{\mathcal{Z}_\omega} - \tilde{\Psi}(1_{U_\omega} - e) \in J_{\mathcal{Z},\tau_\mathcal{Z}}$.

**Proof.** Let $p$ and $q$ be supernatural numbers which are relatively prime and such that $U$ embeds unitally into $M_p \otimes M_q$. As pointed out in [30] Proposition 3.3], the generalized dimension drop interval $\mathcal{Z}_{p,q}$ embeds unitally into $\mathcal{Z}$. Careful inspection of the proof (which follows that of [29] Proposition 2.2]) shows that this embedding can be chosen so that the trace on $\mathcal{Z}$ induces the trace on $\mathcal{Z}_{p,q}$ associated to the Lebesgue integral on $[0,1]$. Note that $\mathcal{Z}_{p,q} \subset C([0,1], M_p \otimes M_q)$, and that there is an order zero map $M_p \otimes M_q \to \mathcal{Z}_{p,q}$ such that the Lebesgue trace on $\mathcal{Z}_{p,q}$ is at most $1 - 1/n$ on the image of the unit of $M_p \otimes M_q$. (Embed the cone $C_0([0,1]) \otimes M_p \otimes M_q$ into $C([0,1]) \otimes M_p \otimes M_q \cong C([0,1], M_p \otimes M_q)$ in such a way that $i_{[0,1]}$ maps to a function which vanishes at the endpoints and is constant 1 on the interval $[1/2n, 1 - 1/2n]$.)

We may now compose these maps to obtain a sequence $(\tilde{\psi}_n)^{\infty}_{n=1}$ of order zero maps $U \to \mathcal{Z}$ with $\tau_\mathcal{Z}(\tilde{\psi}_n(1_U)) > 1 - 1/n$ for each $n \in \mathbb{N}$. Define $\tilde{\Psi}$ to be the cpc order zero map $U_\omega \to \mathcal{Z}_\omega$ induced by the sequence $(\tilde{\psi}_n)^{\infty}_{n=1}$. Given a positive
contraction \( e \in J_{U,\tau_U} \), lift \( e \) to a sequence \((e_n)_{n=1}^{\infty}\) of positive contractions with \( \lim_{n \to \infty} \tau_U(e_n) = 0 \). Since \( \tau_U \circ \tilde{\psi}_n \) is a trace on \( U \) (by Corollary 4.4), we have \( \tau_Z(\tilde{\psi}_n(e_n)) = \tau_U(e_n)\tau_Z(\tilde{\psi}_n(1_U)) \). Then

\[
\tau_Z(1 - \tilde{\psi}_n(1_U - e)) = 1 - \tau_Z(\tilde{\psi}_n(1_U)) + \tau_Z(\tilde{\psi}_n(e_n))
\]

(5.1)

\[= 1 + \tau_Z(\tilde{\psi}_n(1))((\tau_U - 1) \rightarrow 1 + 1(0 - 1) = 0. \]

Thus \( 1 - \tilde{\psi}_n(1_U - e) \in J_{Z,\tau_Z} \), as claimed. \( \square \)

**Lemma 5.2.** Let \( U \) be a UHF algebra, and let \( e \in J_{U,\tau_U} \) be a positive contraction. Then there exist two sequences \((\lambda_n^{(0)})_{n=1}^{\infty}\) and \((\lambda_n^{(1)})_{n=1}^{\infty}\) of cpc order zero maps \( U \to Z \) inducing cpc order zero maps \( \Lambda^{(0)}, \Lambda^{(1)} : U_\omega \to Z_\omega \) such that \( \sum_{j=0}^{1} \Lambda^{(j)}(1_U - e) = 1_{Z_\omega} \).

**Proof.** Fix a positive contraction \( d^{(0)} \in Z \) with spectrum \([0,1]\), and write \( d^{(1)} = 1 - d^{(0)} \) which is also a positive contraction with spectrum \([0,1]\). By Lemma 5.1, there is a sequence \((\tilde{\lambda}_n)_{n=1}^{\infty}\) of cpc order zero maps \( U \to Z \) inducing a cpc order zero map \( \tilde{\Lambda} : U_\omega \to Z_\omega \) such that \( 1_{Z_\omega} - \tilde{\Lambda}(1_U - e) \in J_{Z,\tau_Z} \). Then consider the cpc order zero maps \( \lambda_n^{(j)} : U \to Z \otimes Z \) given by \( \tilde{\lambda}_n^{(j)}(x) = \tilde{\lambda}_n(x) \otimes d^{(j)} \), which induce cpc order zero maps \( \Lambda^{(j)} : U_\omega \to Z_\omega \otimes Z_\omega \) with \( \Lambda^{(j)}(1_U - e) = (\tilde{\Lambda}(1_U - e) \otimes d^{(j)}) \).

By Lemma 4.4 there exist unitaries \( u^{(j)}(z) \in (Z \otimes Z)_\omega \), with \( u^{(j)}(1) = (1_{Z_\omega} \otimes d^{(j)}) \), which we can lift to a sequence \((u^{(j)}_n)_{n=1}^{\infty}\) of unitaries in \( Z \otimes Z \). Define cpc order zero maps \( \lambda_n^{(j)} = \text{ad}(u^{(j)}_n) \circ \tilde{\lambda}_n^{(j)} : U \to Z \otimes Z \), with induced map \( \Lambda^{(j)} \) so that \( \sum_{j=0}^{1} \Lambda^{(j)}(1_U - e) = \sum_{j=0}^{1}(1_{Z_\omega} \otimes d^{(j)}) = 1_{Z_\omega} \otimes 1_Z = 1_{(Z \otimes Z)_\omega} \). Identifying \( Z \otimes Z \) with \( Z \), the lemma is established. \( \square \)

The next lemma packages a factorisation for the order zero maps obtained from the 2-coloured approximately inner flip (cf. Proposition 3.2).

**Lemma 5.3.** Let \( A \) and \( B \) be separable unital \( C^* \)-algebras with \( B \) simple and stably finite and let \( U \) and \( V \) be UHF algebras. Given a contraction \( v \in (B \otimes A \otimes V)_\omega \) with \( v^*v \in (1_B \otimes 1_A \otimes V)_\omega \), let \((\psi_n)_{n=1}^{\infty}\) be a sequence of cpc maps \( A \to U \), such that the cpc map \( \Psi : (B \otimes A \otimes V)_\omega \to (B \otimes U \otimes V)_\omega \) induced by the sequence \((\text{id}_B \otimes \psi_n \otimes \text{id}_V)_{n=1}^{\infty}\) is order zero on \( C^*(B \otimes A \otimes V) \subset (B \otimes A \otimes V)_\omega \). Then the cpc map \( \Theta : A \to (B \otimes U \otimes V)_\omega \) given by \( \Theta(a) = \Psi(v(1_B \otimes a \otimes 1_V)v^*) \) is order zero and can be factorised as follows: there are finite dimensional \( C^* \)-algebras \( G_n \), cpc maps \( \phi_n : A \to G_n \), and \( * \)-homomorphisms \( \theta_n : G_n \to B \otimes U \otimes V \) such that \( \Theta(a) \) is represented by \( (\theta_n(\phi_n(a)))_{n=1}^{\infty} \) for \( a \in A \).

**Proof.** Since \( v^*v \in (1_B \otimes 1_A \otimes V)_\omega \), the map \( a \mapsto v(1_B \otimes a \otimes 1_V)v^* \) is order zero, and hence so too is \( \Theta \).

By following each \( \psi_n \) by a suitable conditional expectation from \( U \) to a finite dimensional \( C^* \)-subalgebra, we can assume that \( C^*(\psi_n(A)) = E_n \subset U \) is finite dimensional for each \( n \) without changing the values of the induced map \( \Psi \) on the separable \( C^* \)-algebra \( X = C^*(B \otimes A \otimes V) \). Then Lemma 1.3 gives a sequence of cpc maps \((\pi_n)_{n=1}^{\infty}, \pi_n : A \to E_n \) such that \((\text{id}_B \otimes \pi_n \otimes \text{id}_V)_{n=1}^{\infty}\) induces a cpc order zero map \( \Pi : X \to (B \otimes U \otimes V)_\omega \) satisfying

\[
\Psi(x) = \Pi(x)\Psi(1_{(B \otimes A \otimes V)_\omega}) = \Psi(1_{(B \otimes A \otimes V)_\omega})\Pi(x), \quad x \in X.
\]
In particular
\[ \Theta(a) = \Psi((1_B \otimes a \otimes 1_V) v^*) = \Pi((1_B \otimes a \otimes 1_V) v^*) \Psi(1_B \otimes a \otimes 1_V) \]
(3.3)
\[ = \Pi^{1/3}(v) \Psi((1_B \otimes a \otimes 1_V) \Psi(1_B \otimes a \otimes 1_V))^{1/2} \Pi^{1/3}(v^*) \]
\[ , \quad a \in A. \]

Write \( s = \Pi^{1/3}(v) \). By Lemma 3.1, \( \Pi^{1/3} \) is induced by a sequence of cpc maps of the form \((\id_B \otimes \pi_a \otimes \id_V)\), where \( \pi_a : A \to E_n \). Since \( v^*v \in (1_B \otimes 1_A \otimes 1_V) \omega \), we can lift \( \pi_a : A \to E_n \to F_n \) such that \( t_n \in 1_B \otimes E_n \otimes F_n \) for all \( n \). As \( B \) is simple and stably finite, \( B \otimes U \otimes V \) has stable rank one by Theorem 5.7, so Lemma 5.5 provides a unitary \( u \in (B \otimes U \otimes V)_\omega \) with \( s = u|s| \). Lift \( u \) to a sequence of unitaries \((u_n)\) in \( B \otimes U \otimes V \), then \((u_n t_n)\) is a lift of \( s \).

Define \( G_n = 1_B \otimes E_n \otimes F_n \) and cpc maps \( \phi_n : A \to G_n \) by
\[ \phi_n(a) = t_n(1_B \otimes \psi_n(1_A) \otimes \psi_n(1_A))^{1/2} \otimes 1_V t_n, \quad a \in A, \]
noting that the choices have been made so that \( \phi_n(a) \subseteq G_n \). Let \( \theta_n : G_n \to B \otimes U \otimes V \) be the \( * \)-homomorphism \( \theta_n = \ad(u_n) \). As \((u_n t_n)\) is a lift of \( \Pi^{1/3}(v) \), (3.3) and (3.4) show that for every \( a \in A \), \((\theta_n(\phi_n(a)))\) is a lift of \( \Theta(a) \), as required.

All the pieces are now in place to prove our main result. As explained in Proposits 2.4 and 2.5, when \( A \) is a \( Z \)-stable C*-algebra, \( \dim_{\text{nuc}}(A) \) can be computed by only approximating the first factor embedding \( \iota : A \to (A \otimes Z)_\omega \); this is essentially used at the end of the proof.

**Proof of Theorem 3.4** Fix UHF algebras \( U \) and \( V \). Since \( A \) has unique tracial state, we can apply Lemma 3.1 to find contractions \((v^{(0)}, v^{(1)}) \in (A \otimes A \otimes V)_\omega \) satisfying (3.1) and (3.2). By Proposition 2.2, we can find an order zero map \( \Psi_2 : A \to U_\omega \) with \( 1_{U_\omega} = \Psi_2(1_A) \in J_{U_\omega} \), which, by the Choi-Effros lifting theorem of [5], is induced by a sequence \((\psi_n(1_A))\) of cpc maps \( A \to U \). Letting \( (x^{(l)}(n))_{l=1}^{n} = 1 \) be a countable dense sequence in \( C^*(A \otimes A \otimes V, v^{(0)}, v^{(1)}) \subset (A \otimes A \otimes V)_\omega \), with fixed isometric lifts \((x^{(l)}(n))_{l=1}^{n} = 1 \) of natural numbers such that
\[ \| (\id_A \otimes \psi_{m_n} \otimes \id_V)(x^{(k)}(l, n)) (\id_A \otimes \psi_{m_n} \otimes \id_V)(1_A \otimes 1_A \otimes 1_V) \]
\[ - (\id_A \otimes \psi_{m_n} \otimes \id_V)(x^{(k)}(l, n)) (\id_A \otimes \psi_{m_n} \otimes \id_V)(1_A \otimes 1_A \otimes 1_V) \]
\[ \leq \frac{1}{n}, \quad 1 \leq k, l \leq n, \]
and \( \tau_U(\psi_{n}(1_A)) \geq 1 - 1/n \) for each \( n \). Let \( \psi_n = \psi_{m_n} \) and let \( \Psi : (A \otimes A \otimes V)_\omega \to (A \otimes U \otimes V)_\omega \) be the cpc map induced by the sequence \((\id_A \otimes \psi_{n} \otimes \id_V)\) which is constructed to be order zero on \( C^*(A \otimes A \otimes V, v^{(0)}, v^{(1)}) \). Write \( \Psi_2 : A \to U_\omega \) for the cpc order zero map induced by \((\psi_n(1_A))_{n=1}^{\infty} \) and note that \( 1_{U_\omega} - \Psi_2(1_A) \in J_{U_\omega} \).

Embedding \( U_\omega \) into \((U \otimes V)_\omega \) as \( U_\omega \otimes 1_V \),
\[ 1_{U_\omega \otimes 1_V} - \Psi_2(1_A) \otimes 1_V = (1_{U_\omega} - \Psi_2(1_A)) \otimes 1_V \in J_{U_\omega \otimes V, \omega}, \]
As \( U \otimes V \) is UHF, Lemma 3.2 provides two sequences \((\lambda_j^{(n)}(1_A))_{n=1}^{\infty} ; j = 0, 1 \), of cpc order zero maps \( U \otimes V \to Z \) such that the induced order zero maps \( \lambda_j^{(n)} : (U \otimes V)_\omega \to Z_\omega \) satisfy \( \sum_{j=0}^{n} \lambda_j^{(n)}(\Psi_2(1_A) \otimes 1_V) = 1_{Z_\omega} \). For each \( n \) and \( j \), let \( \lambda_j^{(n)} = \id_A \otimes \lambda_j^{(n)} ; \ldots \)
\(A \otimes U \otimes V \to A \otimes \mathcal{Z}\), which is cpc and order zero, and so the two sequences \((\lambda^{(j)}_n)_{n=1}^\infty\) induce cpc order zero maps \(\Lambda^{(j)} : (A \otimes U \otimes V)_{\omega} \to (A \otimes \mathcal{Z})_{\omega}\) with

\[
\sum_{j=0}^1 \Lambda^{(j)}(a \otimes \Psi_2(1_A) \otimes 1_V) = a \otimes 1_{\mathcal{Z}}, \quad a \in A.
\]

Let \(\Theta^{(i)} : A \to (A \otimes U \otimes V)_{\omega},\ i = 0,1\), be the cpc order zero map given by \(\Theta^{(i)}(a) = \Psi(v^{(i)}(1_A \otimes a \otimes 1_V)v^{(i)*})\). Hence, for \(a \in A\),

\[
\sum_{i,j=0}^1 \Lambda^{(j)}(\Theta^{(i)}(a)) = \sum_{i,j=0}^1 \Lambda^{(j)}(\Psi(v^{(i)}(1_A \otimes a \otimes 1_V)v^{(i)*}))
\]

\[
= \sum_{j=0}^1 \Lambda^{(j)}(\Psi(a \otimes 1_A \otimes 1_V))
\]

\[
= \sum_{j=0}^1 \Lambda^{(j)}(a \otimes \Psi_2(1_A) \otimes 1_V) = a \otimes 1_{\mathcal{Z}},
\]

using (5.7) for the second equality, and (5.9) for the last.

We can factorise the order zero map \(\Theta^{(i)}\) by Lemma 5.3 (with \(A\) in place of \(B\)), so there are finite dimensional \(C^*\)-algebras \(G^{(i)}_n\), cpc maps \(\phi^{(i)}_n : A \to G^{(i)}_n\) and \(^*\)-homomorphisms \(\theta^{(i)}_n : G^{(i)}_n \to (A \otimes U \otimes V)\) so that for each \(a \in A\), \(\Theta^{(i)}(a)\) is represented by the sequence \((\theta^{(i)}_n(\phi^{(i)}_n(a)))_{n=1}^\infty\). This shows that the first factor embedding \(A \to A \otimes \mathcal{Z}\) has nuclear dimension at most 3. For the nuclear dimension of \(A\), we now follow [35]: As \(A\) is \(\mathcal{Z}\)-stable, and \(\mathcal{Z}\) is strongly self-absorbing, there is a sequence \((\sigma_n)_{n=1}^\infty\) of \(^*\)-isomorphisms \(\sigma_n : A \otimes \mathcal{Z} \to A\) such that \(\sigma_n(a \otimes 1_{\mathcal{Z}}) \to a\) for all \(a \in A\) (see for example [33] Theorem 2.2)). Then the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{id_A} & A \\
\Theta^{(i)}_{i,j=0} \phi^{(i)}_n & \downarrow & \sum_{i,j=0}^1 \sigma_n \circ \lambda^{(j)}_n \circ \theta^{(i)}_n \\
G^{(i)}_n & \xrightarrow{\sum_{i,j=0}^1 \sigma_n \circ \lambda^{(j)}_n \circ \theta^{(i)}_n} & \end{array}
\]

is point-norm asymptotically commutative as \(n \to \omega\), and so witnesses \(\dim_{\text{nuc}}(A) \leq 3\), as required.

When \(A\) has an approximately inner flip we can use this in place of Lemma 3.4 and reduce the bound on the nuclear dimension. In the proof below the factor \(V\) is unnecessary and is included only so that we can quote our earlier results without notational modification; a formally simpler proof can be given just working with \(A \otimes A\).

**Theorem 5.4.** Let \(A\) be a separable, unital and stably finite \(C^*\)-algebra, which has approximately inner flip and is \(\mathcal{Z}\)-stable. Then \(\dim_{\text{nuc}}(A) \leq 1\).

**Proof.** As \(A\) has approximately inner flip it is simple and nuclear \([9]\). Let \((u_n)_{n=1}^\infty\) be a sequence of unitaries in \(A \otimes A\) witnessing the approximate inner flip. We can then follow the proof of Theorem [13] with \(v^{(0)}_n = u_n \otimes 1_V\) and \(v^{(1)}_n = 0\), to obtain a factorisation (5.9) consisting of two maps (the maps with \(i = 1\) in (5.8) vanish), so that \(\dim_{\text{nuc}}(A) \leq 1\). \(\Box\)
In particular the previous result applies to the class of strongly self-absorbing algebras formalised in \cite{38}.

**Corollary 5.5.** Let $A$ be a stably finite strongly self-absorbing C*-algebra. Then $\dim_{\text{nuc}}(A) \leq 1$.

**Proof.** Strongly self-absorbing C*-algebras are separable by definition \cite{38}, have approximately inner flip essentially by definition \cite{38 Proposition 1.5}, and are $\mathcal{Z}$-stable by \cite{43}. Thus Theorem 5.4 applies. □

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