On the multiplication operator by an independent variable in matrix Sobolev spaces

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Received: 20 May 2022 / Accepted: 7 September 2022 / Published online: 19 September 2022
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Abstract
We study the operator $A$ of multiplication by an independent variable in a matrix Sobolev space $W^2(M)$. In the cases of finite measures on $[a, b]$ with $(2 \times 2)$ and $(3 \times 3)$ real continuous matrix weights of full rank it is shown that the operator $A$ is symmetrizable. Namely, there exist two symmetric operators $B$ and $C$ in a larger space such that $Af = CB^{-1}f, f \in D(A)$. As a corollary, we obtain some new orthogonality conditions for the associated Sobolev orthogonal polynomials. These conditions involve two symmetric operators in an indefinite metric space.

Keywords Multiplication operator · Sobolev space · A symmetrizable operator · Sobolev orthogonal polynomials

Mathematics Subject Classification 47B37 · 42C05

1 Introduction

The theory of Sobolev orthogonal polynomials has got a powerful impulse for development during the past 30 years, see the survey [3] and references therein. However, many aspects of the theory are still hidden. One of intriguing topics is an investigation of the associated multiplication operator by an independent variable, which acts in the underlying Sobolev type space. Let us recall basic definitions.
Fix an arbitrary Borel subset $K$ of the complex plane and an arbitrary $\rho \in \mathbb{N}$. Let $M(\delta) = (m_{k,l}(\delta))_{k,l=0}^0$ be a $\mathbb{C}_{(\rho+1)\times(\rho+1)}$-valued function on $\mathfrak{B}(K)$, which entries are countably additive on $\mathfrak{B}(K)$ ($\delta \in \mathfrak{B}(K)$) (it is called a non-negative Hermitian-valued measure on $(K, \mathfrak{B}(K))$). Denote by $\tau(\delta)$ the trace measure, $\tau(\delta) := \sum_{k=0}^0 m_{k,k}(\delta)$, $\delta \in \mathfrak{B}(K)$. By $M'_l := dM/d\tau = (dm_{k,l}/d\tau)_{k,l=0}^0$, we denote the trace derivative of $M$, see [4]. One means by $L^2(M)$ a set of all (classes of the equivalence of) measurable vector-valued functions $f(z) : K \rightarrow \mathbb{C}_{\rho+1}$, $f = (f_0(z), f_1(z), \ldots, f_\rho(z))$, such that

$$
\|f\|^2_{L^2(M)} := \int_K f(z)M'_l(z)f^*(z)d\tau < \infty.
$$

It is known that $L^2(M)$ is a Hilbert space with the following scalar product:

$$
(f, g)_{L^2(M)} := \int_K f(z)M'_l(z)(g(z))^*d\tau, \quad f, g \in L^2(M). \tag{1.1}
$$

It is also known (see [4, p. 294], [5, Lemma 2.1]) that one can consider an arbitrary $\sigma$-finite (non-negative) measure $\mu$, with respect to which all $m_{k,l}$ are absolutely continuous, and set $M_0(z) = M_{0,\mu}(z) := dM/d\mu$ (the Radon–Nikodym derivative of $M$ with respect to $\mu$). The integral in (1.1) exists if and only if the following integral exists:

$$
\int_K f(z)M_{0,\mu}(z)(g(z))^*d\mu. \tag{1.2}
$$

If the integrals exist, they are equal. Such measures $\mu$ we shall call admissible (for $M$). The matrix function $M_0(z) = M_{0,\mu}(z)$ is said to be the (matrix) weight, corresponding to an admissible measure $\mu$. If $\det M_0(z) > 0, \forall z \in K$, we shall say that the matrix weight $M_0(z)$ has full rank.

Denote by $A^2(M)$ a linear manifold in $L^2(M)$ including those classes of the equivalence $[\cdot]$ which possess a representative of the following form:

$$
f(z) = (f(z), f'(z), \ldots, f^{(\rho)}(z)). \tag{1.3}
$$

By $f'(z)$ ($f^{(k)}(z)$) we mean the usual derivative (respectively, the derivative of order $k$) of a complex-valued function $f$ at a point $z \in \mathbb{C}$. In particular, this means that $f(z)$ is defined in an open disc containing $z$.

By $W^2(M)$ we denote the closure of $A^2(M)$ in the norm of $L^2(M)$. The subspace $W^2(M)$ is said to be the Sobolev space with the matrix measure $M$, see [6]. Elements of $A^2(M)$ will be also denoted by their first components.

Let $\mu$ be an admissible measure and $M_0(z)$ be the corresponding matrix weight. Suppose that $1, z, z^2, \ldots$, all belong to $W^2(M)$. Assume that

$$
(p, p)_{W^2(M)} > 0, \tag{1.4}
$$

for an arbitrary non-zero $p \in \mathbb{P}$. Then one can apply the Gram-Schmidt...
orthogonalization process to construct a system \( \{y_n(z)\}_{n=0}^{\infty} \), deg\( y_n = n \), of the associated Sobolev orthogonal polynomials:

\[
\int_K (y_n(z), y'_n(z), \ldots, y_n^{(\rho)}(z))M_0(z) \begin{pmatrix} y_m(z) \\ y'_m(z) \\ \vdots \\ y_m^{(\rho)}(z) \end{pmatrix} \, d\mu \quad (1.5)
\]

\[= A_n \delta_{n,m}, \quad A_n > 0, \quad n, m \in \mathbb{Z}_+.
\]

In this paper we shall only consider the case of \( K = [a, b], -\infty < a < b < + \infty \). We fix an admissible measure \( \mu \) and assume that the entries of the corresponding matrix weight \( M_0(z) \) are real-valued continuous functions. In this case the matrix weight \( M_0(z) \) is said to be real-valued continuous. Denote by \( C(K) \) the set of all classes of equivalence of functions in \( L^2(M) \), which include a continuous representative on \([a, b]\) (i.e. each entry of the representative is continuous on \([a, b]\)). We denote by \( A \) the operator on the whole \( A^2(M) \), which sends \([f(z)]\) to \([f(z)]\).

The main purpose of this paper is to show that under some general conditions there exist two symmetric operators \( B \) and \( C \) in \( L^2(M) \) \((D(B) = D(C) = C(K))\) such that \( Af = CB^{-1}f, f \in D(A) \). As a corollary, we obtain some new orthogonality conditions for the associated Sobolev orthogonal polynomials. These conditions involve two symmetric operators in an indefinite metric space. Observe that some other interesting questions concerning matrix measures and Sobolev type inner products were studied in [1, 2].

**Notations.** As usual, we denote by \( \mathbb{R}, \mathbb{C}, \mathbb{N}, \mathbb{Z}, \mathbb{Z}_+ \), the sets of real numbers, complex numbers, positive integers, integers and non-negative integers, respectively. By \( \mathbb{Z}_{k,l} \) we mean all integers \( j \) satisfying the following inequality: \( k \leq j \leq l \); \((k, l) \in \mathbb{Z}\). By \( \mathbb{C}_{m \times n} \) we denote the set of all \((m \times n)\) matrices with complex entries, \( \mathbb{C}_n := \mathbb{C}_{1 \times n}, \mathbb{C}^n := \mathbb{C}_{n \times 1}, m, n \in \mathbb{N} \). By \( \mathbb{C}_{n \times n}^\geq \) we denote the set of all non-negative Hermitian matrices from \( \mathbb{C}_{n \times n}, n \in \mathbb{N} \). For \( A \in \mathbb{C}_{m \times n} \) the notation \( A^* \) stands for the adjoint matrix \((m, n \in \mathbb{N})\), and \( A^T \) means the transpose of \( A \). By \( \mathbb{P} \) we denote the set of all polynomials with complex coefficients. For an arbitrary Borel set \( K \) of the complex plane we denote by \( \mathfrak{B}(K) \) the set of all Borel subsets of \( K \). Let \( \mu \) be an arbitrary (non-negative) measure on \( \mathfrak{B}(K) \). By \( L^2_{\mu} = L^2_{\mu,K} \) we denote the usual space of (the classes of the equivalence of) complex Borel measurable functions \( f \) on \( K \) such that \( \|f\|^2_{L^2_{\mu,K}} := \int_K |f(z)|^2 \, d\mu < \infty \).

By \( (\cdot, \cdot)_H \) and \( \| \cdot \|_H \) we denote the scalar product and the norm in a Hilbert space \( H \), respectively. The indices may be omitted in obvious cases. For a set \( M \) in \( H \), by \( \overline{M} \) we mean the closure of \( M \) in the norm \( \| \cdot \|_H \). For a linear operator \( A \) in \( H \), by \( D(A) \) we mean its domain, and \( A|_M \) means the restriction of \( A \) to \( M \). By \( A^{-1} \) we denote the inverse of \( A \), if it exists.
2 Matrix multiplication operators and symmetrizability

Our main objective here is to deduce the following theorem.

**Theorem 2.1** Let \( \rho \in \{1, 2\} \), \( K = [a, b] \), \( -\infty < a < b < +\infty \), and \( M(\delta) = (m_{k,l}(\delta))_{k,l=0}^{\rho} \) be a non-negative Hermitian-valued measure on \((K, \mathcal{B}(K))\). Suppose that \( \mu \) is a finite admissible measure for \( M \), such that the corresponding matrix weight \( M_0(z) \) is real-valued continuous and of full rank. Then the operator \( A \), with \( D(A) = A^2(M) \):

\[
A[f(z)] = [zf(z)], \quad [f(z)] \in A^2(M),
\]

is well-defined, and it has the following representation:

\[
A = CB^{-1}|_{A^2(M)}.
\]

Here \( B, C \) are some linear symmetric operators in \( L^2(M) \), and \( B \) is invertible.

**Proof** At first we shall consider an arbitrary \( \rho \in \mathbb{N} \), and \( K, M(\delta), \mu, M_0(z) \) like in the statement of Theorem 2.1. To construct the required operators \( B, C \), we shall use matrix multiplication operators in \( L^2(M) \).

**Lemma 2.2** Let \( \rho \in \mathbb{N} \), and \( K, M(\delta), \mu, M_0(z) \) be like in the statement of Theorem 2.1. Let \( D(z) = (d_{ij}(z))_{i,j=0}^{\rho} \) be an arbitrary \((\rho + 1) \times (\rho + 1)\) matrix function, which entries are real-valued continuous functions on \( K \). Then the following operator \( D \):

\[
Du = [(f_0(z), ..., f_\rho(z))D(z)], \quad u = [(f_0(z), ..., f_\rho(z))] \in C(K),
\]

is a well-defined linear operator in \( L^2(M) \) with the domain \( C(K) \).

**Proof of Lemma** Suppose that an element \( u \in C(K) \) has two continuous representatives \( f, g \):

\[
\|f - g\|_{L^2(M)} = 0.
\]

Observe that \( (M_0(z))^{1/2} \) has \( \mu \)-measurable entries (cf. [4] for the case of the trace derivative). In fact, the norm of the operator \( M_0(z) \) in \( C^{\rho+1} \) is a continuous function in \( z \), and, therefore, it attains its maximum \( L \) on \( K \). Thus, the spectra of all of \( M_0(z) \) lie in \([0, L]\). On the segment \([0, L]\) we can approximate \( \sqrt{x} \) by a polynomial \( p_k(x) \) in the uniform norm \( \| \cdot \|_U \):

\[
\|\sqrt{x} - p_k(x)\|_U < \frac{1}{k}, \quad k \in \mathbb{N}.
\]

Then
\[
\|(M_0(z))^{1/2}x - p_k(M_0(z))x\|_{C^{\alpha+1}}^2 \leq \frac{1}{k^2} \|x\|^2,
\]
where \(E_\lambda(z)\) is the orthogonal resolution of the identity for \(M_0(z)\). Thus, elements of \((M_0(z))^{1/2}\) are \(\mu\)-measurable, as they are the limits of \(\mu\)-measurable functions.

By the structure of the inner product it follows that (cf. [4]):

\[
(f - g)M_0^1(z) = 0,
\]
\(\mu\)-a.e. on \(K\). Then

\[
(f - g)D(z) = (f - g)M_0(z)M_0^{-1}(z)D(z) = 0,
\]
\(\mu\)-a.e. on \(K\). Therefore,

\[
\|fD(z) - gD(z)\|_{L^2(M)} = 0.
\]

Consequently, the operator \(D\) is well-defined. The linearity is obvious. The lemma is proved.

Let us return to the proof of the theorem. By the Leibniz rule we may write:

\[
(zf(z))^{(r)} = zf^{(r)}(z) + rf^{(r-1)}(z), \quad [f] \in A^2(M), \ 0 \leq r \leq \rho.
\]

Therefore,

\[
(zf(z), (zf(z))', \ldots, (zf(z))^{(\rho)}) = (f(z), f'(z), \ldots, f^{(\rho)}(z))A(z),
\]
where

\[
A(z) := \begin{pmatrix}
z & 1 & 0 & \cdots & 0 & 0 \\
0 & z & 2 & \cdots & 0 & 0 \\
0 & 0 & z & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & z & \rho \\
0 & 0 & 0 & \cdots & 0 & z
\end{pmatrix}.
\]

Denote by \(\hat{A}\) the operator in \(L^2(M)\), with \(D(\hat{A}) = C(K)\), which maps \([f_0(z), \ldots, f_\rho(z)]\) to \([f_0(z), \ldots, f_\rho(z)]A(z)\). Observe that \(\hat{A} \supseteq A\). Thus the operator \(A\) is well-defined. Let \(B(z) = (b_{ij}(z))_{i,j=0}^\rho\) be an arbitrary \((\rho + 1) \times (\rho + 1)\) matrix function, which entries are real-valued continuous functions on \(K\). Set

\[
C(z) := B(z)A(z), \quad z \in K.
\]

By \(B\), \(C\) we mean the operators in \(L^2(M)\), with \(D(B) = D(C) = C(K)\), such that:
\[ B[\{f_0(z), \ldots, f_\rho(z)\}] = [(f_0(z), \ldots, f_\rho(z))B(z)], \]
\[ C[\{f_0(z), \ldots, f_\rho(z)\}] = [(f_0(z), \ldots, f_\rho(z))C(z)]. \]

By (2.5) we see that (notice the reversed order due to the right multiplication)
\[ C = \widehat{AB}. \tag{2.6} \]

Our aim now is to specify functions \( b_{i,j}(z) \) to get relation (2.2), as well as to guarantee the symmetry of \( B \) and \( C \). The structure of the inner product in \( L^2(M) \) shows that the following conditions provide the symmetry of \( B \) and \( C \):
\[ B(z)M_0(z) = M_0(z)B^*(z), \quad C(z)M_0(z) = M_0(z)C^*(z), \quad z \in K. \tag{2.7} \]

Observe that
\[ C(z) = zB(z) + \widehat{B}(z), \quad \widehat{B}(z) := (jb_{i,j-1}(z))_{i,j=0}^\rho, \quad z \in K, \]
where \( b_{k,i} \) with negative indices are zeros. Thus conditions (2.7) are satisfied, if and only if
\[ B(z)M_0(z) = M_0(z)B^*(z), \quad \widehat{B}(z)M_0(z) = M_0(z)\widehat{B}^*(z), \quad z \in K. \tag{2.8} \]

Conditions (2.8) are equivalent to the following conditions:
\[ B(z)M_0(z), \widehat{B}(z)M_0(z) \quad \text{are Hermitian matrices for all} \quad z \in K. \tag{2.9} \]

Let \( M_0(z) = (\tilde{m}_{i,j})_{i,j=0}^\rho \). Conditions (2.9) in turn are equivalent to the following conditions:
\[ \sum_{k=0}^\rho b_{i,k}\tilde{m}_{k,j}(z) = \sum_{k=0}^\rho b_{j,k}\tilde{m}_{k,i}(z), \quad 0 \leq j \leq \rho - 1, \quad j < i \leq \rho; \tag{2.10} \]
\[ \sum_{k=1}^\rho kb_{i,k-1}\tilde{m}_{k,j}(z) = \sum_{k=1}^\rho kb_{j,k-1}\tilde{m}_{k,i}(z), \quad 0 \leq j \leq \rho - 1, \quad j < i \leq \rho; \quad z \in K. \tag{2.11} \]

If one can find a real continuous solution \( B(z) \) to Eqs. (2.10), (2.11), such that
\[ \det B(z) \neq 0, \quad z \in K, \]
then the operator \( B \) is invertible and defined on \( C(K) \) as the multiplication operator by \( B^{-1}(z) \). Multiplying relation (2.6) by \( B^{-1}(z) \) from the right and restricting to elements of \( \Lambda^2(M) \), we obtain the required representation for \( A \).

Case \( \rho = 1 \). Equations (2.10), (2.11) now have the following form:
\[ \begin{cases} b_{1,0}\tilde{m}_{0,0}(z) + b_{1,1}\tilde{m}_{1,0}(z) = b_{0,0}\tilde{m}_{0,1}(z) + b_{0,1}\tilde{m}_{1,1}(z), \\ b_{1,0}\tilde{m}_{1,0}(z) = b_{0,0}\tilde{m}_{1,1}(z), \end{cases} \quad z \in K. \tag{2.12} \]

Then \( b_{1,0}, b_{1,1} \) are arbitrary real continuous functions on \( K \), while
\[ b_{0,0} = b_{1,0} \frac{\tilde{m}_{1,0}}{m_{1,1}}, \]
\[ b_{0,1} = \frac{1}{m_{1,1}} \left( b_{1,0} \left( \frac{\tilde{m}_{0,0}}{m_{1,1}} - \frac{\tilde{m}_{1,0}^2}{m_{1,1}} \right) + b_{1,1} \tilde{m}_{1,0} \right). \]

In particular, choose \( b_{1,0} = \tilde{m}_{1,1}, b_{1,1} = 0 \), to get
\[
B(z) = \begin{pmatrix}
\tilde{m}_{1,0}(z) & \tilde{m}_{0,0}(z) - \frac{\tilde{m}_{1,0}^2(z)}{m_{1,1}(z)} \\
\tilde{m}_{1,1}(z) & 0
\end{pmatrix}, \quad z \in K.
\]

It is clear that \( \det B(z) \neq 0, z \in K \).

**Case \( \rho = 2 \).** Equations (2.10), (2.11) lead to the following six equations:
\[
b_{1,0} \tilde{m}_{0,0} + b_{1,1} \tilde{m}_{1,0} + b_{1,2} \tilde{m}_{2,0} = b_{0,0} \tilde{m}_{0,1} + b_{0,1} \tilde{m}_{1,1} + b_{0,2} \tilde{m}_{2,1}, \tag{2.13}
\]
\[
b_{1,0} \tilde{m}_{1,0} + 2b_{1,1} \tilde{m}_{2,0} = b_{0,0} \tilde{m}_{1,1} + 2b_{0,1} \tilde{m}_{2,1}, \tag{2.14}
\]
\[
b_{2,0} \tilde{m}_{0,0} + b_{2,1} \tilde{m}_{1,0} + b_{2,2} \tilde{m}_{2,0} = b_{0,0} \tilde{m}_{0,2} + b_{0,1} \tilde{m}_{1,2} + b_{0,2} \tilde{m}_{2,2}, \tag{2.15}
\]
\[
b_{2,0} \tilde{m}_{1,0} + 2b_{2,1} \tilde{m}_{2,0} = b_{0,0} \tilde{m}_{1,2} + 2b_{0,1} \tilde{m}_{2,2}, \tag{2.16}
\]
\[
b_{2,0} \tilde{m}_{0,1} + b_{2,1} \tilde{m}_{1,1} + b_{2,2} \tilde{m}_{2,1} = b_{1,0} \tilde{m}_{0,2} + b_{1,1} \tilde{m}_{1,2} + b_{1,2} \tilde{m}_{2,2}, \tag{2.17}
\]
\[
b_{2,0} \tilde{m}_{1,1} + 2b_{2,1} \tilde{m}_{2,1} = b_{1,0} \tilde{m}_{1,2} + 2b_{1,1} \tilde{m}_{2,2}. \tag{2.18}
\]

Equations (2.15)–(2.18) are equivalent to the following equations:
\[
b_{0,2} = b_{2,0} \frac{\tilde{m}_{0,0}}{m_{2,2}} + b_{2,1} \frac{\tilde{m}_{1,0}}{m_{2,2}} + b_{2,2} \frac{\tilde{m}_{2,0}}{m_{2,2}} - b_{0,0} \frac{\tilde{m}_{0,2}}{m_{2,2}} - b_{0,1} \frac{\tilde{m}_{1,2}}{m_{2,2}}, \tag{2.19}
\]
\[
b_{0,1} = b_{2,0} \frac{\tilde{m}_{1,0}}{2m_{2,2}} + b_{2,1} \frac{\tilde{m}_{2,0}}{2m_{2,2}} - b_{0,0} \frac{\tilde{m}_{1,2}}{2m_{2,2}}, \tag{2.20}
\]
\[
b_{1,2} = b_{2,0} \frac{\tilde{m}_{0,1}}{m_{2,2}} + b_{2,1} \frac{\tilde{m}_{1,1}}{m_{2,2}} + b_{2,2} \frac{\tilde{m}_{2,1}}{m_{2,2}} - b_{1,0} \frac{\tilde{m}_{0,2}}{m_{2,2}} - b_{1,1} \frac{\tilde{m}_{1,2}}{m_{2,2}}, \tag{2.21}
\]
\[
b_{1,1} = b_{2,0} \frac{\tilde{m}_{1,1}}{2m_{2,2}} + b_{2,1} \frac{\tilde{m}_{2,1}}{2m_{2,2}} - b_{1,0} \frac{\tilde{m}_{1,2}}{2m_{2,2}}. \tag{2.22}
\]

Substitute the expression for \( b_{0,1} \) from (2.20) into (2.19) to get
\[ b_{0,2} = b_{2,0} \left( \frac{\tilde{m}_{0,0}}{m_{2,2}} - \frac{\tilde{m}_{1,2} \tilde{m}_{1,0}}{2m_{2,2}^2} \right) + b_{2,1} \left( \frac{\tilde{m}_{1,0}}{m_{2,2}} - \frac{\tilde{m}_{1,2} \tilde{m}_{2,0}}{m_{2,2}^2} \right) + b_{2,2} \frac{\tilde{m}_{2,0}}{m_{2,2}} + b_{0,0} \Delta_1, \]

where

\[ \Delta_1 := -\frac{\tilde{m}_{0,2}}{m_{2,2}} + \frac{\tilde{m}_{1,2}^2}{2m_{2,2}^2}. \]

Substitute for \( b_{1,1} \) from (2.22) into (2.21) to get

\[ b_{1,2} = b_{2,0} \left( \frac{\tilde{m}_{0,1}}{m_{2,2}} - \frac{\tilde{m}_{1,2} \tilde{m}_{1,1}}{2m_{2,2}^2} \right) + b_{2,1} \left( \frac{\tilde{m}_{1,1}}{m_{2,2}} - \frac{\tilde{m}_{1,2} \tilde{m}_{2,1}}{m_{2,2}^2} \right) + b_{2,2} \frac{\tilde{m}_{2,1}}{m_{2,2}} + b_{1,0} \Delta_1. \]

(2.24)

Observe that Eqs. (2.13)–(2.18) are equivalent to Eqs. (2.13), (2.14), (2.23), (2.20), (2.24), (2.22). Notice that elements of row 0 are expressed in terms of \( b_{0,0} \) and elements of row 2, while elements of row 1 are expressed in terms of \( b_{1,0} \) and elements of row 2.

Substitute for \( b_{0,1}, b_{0,2}, b_{1,1}, b_{1,2} \) from (2.23), (2.20), (2.24), (2.22) into Eq. (2.13). After simplifications we shall get:

\[ b_{1,0} \left( \tilde{m}_{0,0} - \frac{\tilde{m}_{1,2} \tilde{m}_{1,0}}{2m_{2,2}} + \tilde{m}_{2,0} \Delta_1 \right) = b_{0,0} \left( \tilde{m}_{0,1} - \frac{\tilde{m}_{1,2} \tilde{m}_{1,1}}{2m_{2,2}} + \tilde{m}_{2,1} \Delta_1 \right) + b_{2,0} \frac{\tilde{m}_{2,0} \tilde{m}_{0,0} - \tilde{m}_{2,0} \tilde{m}_{0,1}}{\tilde{m}_{2,2}} + \frac{\tilde{m}_{1,2} \tilde{m}_{2,0} \tilde{m}_{1,1} - \tilde{m}_{1,2} \tilde{m}_{2,1} \tilde{m}_{1,0}}{2m_{2,2}^2}. \]

Substitute the same expressions into Eq. (2.14) to get

\[ b_{1,0} \left( \tilde{m}_{1,0} - \frac{\tilde{m}_{2,0} \tilde{m}_{1,2}}{m_{2,2}} \right) = b_{0,0} \left( \tilde{m}_{1,1} - \frac{\tilde{m}_{2,1} \tilde{m}_{1,2}}{m_{2,2}} \right) + b_{2,0} \frac{\tilde{m}_{2,1} \tilde{m}_{1,0} - \tilde{m}_{2,0} \tilde{m}_{1,1}}{m_{2,2}}. \]

(2.26)

Thus, Eqs. (2.13)–(2.18) are equivalent to Eqs. (2.25), (2.26), (2.23), (2.20), (2.24), (2.22). Equation (2.26) is equivalent to the following equation:

\[ b_{0,0} = b_{1,0} \frac{\tilde{m}_{2,2}}{\Delta_2} \left( \tilde{m}_{1,0} - \frac{\tilde{m}_{2,0} \tilde{m}_{1,2}}{m_{2,2}} \right) - b_{2,0} \frac{\tilde{m}_{2,2}}{\Delta_2} \left( \tilde{m}_{2,1} \tilde{m}_{1,0} - \tilde{m}_{2,0} \tilde{m}_{1,1} \right) \tilde{m}_{2,2}, \]

(2.27)

where

\[ \Delta_2 := \tilde{m}_{2,2} \tilde{m}_{1,1} - \tilde{m}_{1,2}^2 > 0. \]

Finally, substitute for \( b_{0,0} \) from (2.27) into (2.25) to get
\[ b_{1,0}c_{1,0} = b_{2,0}c_{2,0}, \quad (2.28) \]

where
\[ c_{1,0} := \tilde{m}_{2,2}\tilde{m}_{0,0}\Delta_2 - \frac{1}{2}\tilde{m}_{1,2}\tilde{m}_{1,0}\Delta_2 + \tilde{m}_{2,0}\tilde{m}_{2,2}\Delta_1 \Delta_2 \]
\[ - (\tilde{m}_{2,2}\tilde{m}_{0,1} - \frac{1}{2}\tilde{m}_{1,2}\tilde{m}_{1,1} + \tilde{m}_{2,2}\tilde{m}_{2,1}\Delta_1)(\tilde{m}_{2,2}\tilde{m}_{0,0} - \tilde{m}_{2,0}\tilde{m}_{1,0}). \]
\[ c_{2,0} := \Delta_2(\tilde{m}_{2,1}\tilde{m}_{0,0} - \tilde{m}_{2,0}\tilde{m}_{0,1}) + \frac{(\tilde{m}_{1,2}\tilde{m}_{2,0}\tilde{m}_{1,1} - \tilde{m}_{1,2}\tilde{m}_{2,1}\tilde{m}_{1,0})}{2\tilde{m}_{2,2}} \Delta_2 \]
\[ - (\tilde{m}_{2,1}\tilde{m}_{1,0} - \tilde{m}_{2,0}\tilde{m}_{1,1})(\tilde{m}_{2,2}\tilde{m}_{0,1} - \frac{1}{2}\tilde{m}_{1,2}\tilde{m}_{1,1} + \tilde{m}_{2,2}\tilde{m}_{2,1}\Delta_1). \]

After simplifications we see that there appear \( \det M_0(z) \):
\[ c_{2,0} = \tilde{m}_{2,1} \det M_0(z), \quad c_{1,0} = \tilde{m}_{2,2} \det M_0(z). \]

Consequently, Eq. (2.28) is equivalent to the following equation:
\[ b_{1,0} = b_{2,0}\frac{\tilde{m}_{2,1}}{\tilde{m}_{2,2}}. \quad (2.29) \]

Thus, the original system (2.13)–(2.18) is equivalent to Eqs. (2.29), (2.27), (2.23), (2.20), (2.24), (2.22). We can choose arbitrary real continuous on \( K \) functions \( b_{2,0}, b_{2,1}, b_{2,2} \). The rest of \( b_{1,j} \) can be found from Eqs. (2.29), (2.27), (2.23), (2.20), (2.24), (2.22).

In particular, choose \( b_{2,0} = 2\tilde{m}_{2,2}^2, b_{2,1} = b_{2,2} = 0 \), to get
\[ B(z) = \begin{pmatrix}
2\tilde{m}_{2,0}\tilde{m}_{2,2} & \tilde{m}_{1,0}\tilde{m}_{2,2} - \tilde{m}_{2,0}\tilde{m}_{1,2} & 2\tilde{m}_{2,2}\tilde{m}_{0,0} - \tilde{m}_{1,2}\tilde{m}_{1,0} + \frac{\tilde{m}_{2,0}\tilde{m}_{1,2}^2}{\tilde{m}_{2,2}} - 2\tilde{m}_{2,0}^2 \\
2\tilde{m}_{2,1}\tilde{m}_{2,2} & \tilde{m}_{1,1}\tilde{m}_{2,2} - \tilde{m}_{2,1}^2 & 2\tilde{m}_{2,2}\tilde{m}_{0,1} - \tilde{m}_{1,2}\tilde{m}_{1,1} + \frac{\tilde{m}_{1,2}^3}{\tilde{m}_{2,2}} - 2\tilde{m}_{2,1}\tilde{m}_{0,2} \\
2\tilde{m}_{2,2}^2 & 0 & 0
\end{pmatrix}. \]

It is easily checked that \( \det B(z) \neq 0, z \in K \). The proof is complete. \( \square \)

**Example** Suppose that \( \rho = 1, K = [a, b], -\infty < a < b < +\infty \),
\[ M_0(z) = \begin{pmatrix}
1 & 0 \\
0 & \tilde{m}_{1,1}(z)
\end{pmatrix}, \]
where \( \tilde{m}_{1,1} \) is a positive continuous function on \( K \), and \( \mu \) is an arbitrary finite measure on \( \mathfrak{B}(K) \). Notice that the matrix measure \( M \) is now given by
\[ M(\delta) = \int_\delta M_0(z)d\mu, \quad \delta \in \mathfrak{B}(K). \]

In this case we have (see definitions in the above proof of Theorem 2.1):
\[ B(z) = \begin{pmatrix} 0 & 1 \\ \tilde{m}_{1,1}(z) & 0 \end{pmatrix}, \]
\[ B^{-1}(z) = \begin{pmatrix} 0 & \frac{1}{\tilde{m}_{1,1}(z)} \\ \frac{1}{\tilde{m}_{1,1}(z)} & 0 \end{pmatrix}, \quad C(z) = \begin{pmatrix} 0 & z \\ z\tilde{m}_{1,1}(z) & \tilde{m}_{1,1}(z) \end{pmatrix}. \]

Notice that
\[ B^{-1}(z)C(z) = A(z) = \begin{pmatrix} z & 1 \\ 0 & z \end{pmatrix}, \]
but
\[ C(z)B^{-1}(z) = \begin{pmatrix} z & 0 \\ \tilde{m}_{1,1}(z) & z \end{pmatrix}. \]

We have
\[ B(z)M_0(z) = \begin{pmatrix} 0 & \tilde{m}_{1,1}(z) \\ \tilde{m}_{1,1}(z) & 0 \end{pmatrix}, \quad C(z)M_0(z) = \begin{pmatrix} 0 & z\tilde{m}_{1,1}(z) \\ z\tilde{m}_{1,1}(z) & \tilde{m}_{1,1}^2(z) \end{pmatrix}, \]
and
\[ B^{-1}(z)M_0(z) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]

**Corollary 2.3** In conditions of Theorem 2.1, suppose additionally that condition (1.4) holds for arbitrary non-zero \( p \in \mathbb{P}. \) Let
\[ \sigma(f, g) = [f, g]_{B^{-1}} := (B^{-1}f, g)_{L^2(M)}, \quad f, g \in C(K). \] (2.30)

Then the associated Sobolev orthogonal polynomials \( \{y_n(z)\}_{n=0}^{\infty}, \) as in (1.5), satisfy the following relations:
\[ [B y_n(A), y_m(A)]_{B^{-1}} = A_n \delta_{n,m}, \quad n, m \in \mathbb{Z}_+. \] (2.31)

Operators \( A, B \) are symmetric with respect to \( \sigma: \)
\[ \sigma(Af, g) = \sigma(f, Ag), \quad \sigma(Bu, v) = \sigma(u, Bv), \]
for all \( f, g \in A^2(M), u, v \in C(K). \)

**Proof** The proof is straightforward. \( \square \)

It looks natural to state the following conjecture.

**Conjecture 1** Theorem 2.1 is true, if we replace the assumption \( \rho \in \{1, 2\} \) by \( \rho \in \mathbb{N}. \)
As we have seen in the proof of Theorem 2.1, there already appeared huge expressions for $\rho = 2$. On the other hand, after proper sorting of the equations and lots of simplifications there appeared $\det M_0(z)$. It looks promising for the validity of the general case.

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