Anosov rational forms in Lie algebras associated to graphs

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Abstract

Anosov diffeomorphisms are an important class of dynamical systems with many peculiar properties. Ever since they were introduced in the sixties, it has been an open question which manifolds can admit them, with for example a positive answer for tori of dimension greater than or equal to two. A natural generalization for tori is to consider nilmanifolds of nilpotency class two, for which the existence of Anosov diffeomorphisms is completely determined by the associated rational 2-step nilpotent Lie algebra. From a given simple undirected graph, one can construct a complex 2-step nilpotent Lie algebra, which in general contains different non-isomorphic rational forms. In this paper we use the theory of Galois descent to give a description of all these rational forms in Lie algebras associated to graphs and determine precisely which ones correspond to a nilmanifold admitting an Anosov diffeomorphism. This is the first class of complex nilpotent Lie algebras having several non-isomorphic rational forms and for which all the ones that are Anosov are described.

1 Introduction

An Anosov diffeomorphism on a closed manifold $M$ is a diffeomorphism $f : M \to M$ that preserves a continuous splitting $TM = E^s \oplus E^u$ of the tangent bundle such that $df$ exponentially contracts the elements of $E^s$ and exponentially expands the elements in $E^u$. The easiest example of such a map is the so-called Arnold’s cat map, i.e. the map induced by the matrix \[
\begin{pmatrix}
2 & 1 \\
1 & 1 
\end{pmatrix}
\] on the torus $\mathbb{R}^2/\mathbb{Z}^2$. In his seminal paper [20], S. Smale introduced the first non-toral example of a manifold admitting an Anosov diffeomorphism, raising the question which manifolds can have an Anosov diffeomorphism. This new example was given on a nilmanifold $N/\Gamma$, i.e. the compact quotient of a 1-connected nilpotent Lie group $N$ by a lattice $\Gamma \subset N$.

It has been conjectured that every closed manifold admitting an Anosov diffeomorphism is, up to homeomorphism, finitely covered by a nilmanifold. Hence an important first step to understand the manifolds with an Anosov diffeomorphism is to study nilmanifolds. For tori, which are quotients of the abelian Lie group $N = \mathbb{R}^n$, the problem is completely solved, namely an Anosov diffeomorphism exists if and only if $n > 1$. The natural next step is to consider nilmanifolds constructed from nilpotent Lie groups of nilpotency class 2. Corresponding to every nilmanifold, there is a unique rational nilpotent Lie algebra $\mathfrak{n}_\mathbb{Q}$, which is a rational form of the Lie algebra corresponding to $N$. Conversely, by the work of Mal’cev [16], every rational form corresponds to a lattice $\Gamma$ of $N$ that is uniquely determined up to commensurability, i.e. up to having an isomorphic subgroup of finite index. Moreover, the existence of an Anosov diffeomorphism depends only on this rational Lie algebra $\mathfrak{n}_\mathbb{Q}$, and more specifically on the existence of a certain automorphism as described in [6]. In the remainder we hence focus on rational forms of (complex) 2-step nilpotent Lie algebras to study Anosov diffeomorphisms on the corresponding nilmanifolds.

In this class, there are several instances for which the existence of Anosov diffeomorphisms is known. Free 2-step nilpotent Lie algebras have a unique rational form, and this form admits an Anosov diffeomorphism if and only if the number of generators is greater than or equal to 3. This was later generalized to one specific rational form of nilpotent Lie algebras associated to a simple undirected graph $G$ in [2], we refer to Section [41] for the exact definition of these Lie algebras. Note that the aforementioned example by S. Smale is a rational form in such a real Lie algebra associated to a graph, but it does not fall under the main result of [2].

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There is also a classification of Anosov diffeomorphisms on Lie algebras up to dimension 8 in [13], containing two families of different rational forms in a fixed real Lie algebra of nilpotency class 2, which is in both cases a Lie algebra associated to a graph. Later, this classification was slightly corrected in [8], where the close relation between Galois theory and Anosov diffeomorphisms was explored for the first time. In conclusion, there are only sparse examples in the literature of real Lie algebras which contain more than one rational form and in which all Anosov rational forms are classified.

In this paper we fully characterize the existence of Anosov diffeomorphisms on quotients of 2-step nilpotent Lie algebras associated to graphs. Our main result gives the first class of Lie algebras having distinct rational forms and with a general characterization of all Anosov rational forms. The paper consists of two parts, the first in which we describe all possibilities for the rational Lie algebra $\mathfrak{n}^2$ using Galois descent, the second were we use this description to characterize the ones which have an Anosov diffeomorphism.

Given a graph $\mathcal{G}$, one can write it as the reduced graph $\mathcal{G}_{\text{red}}$ on the coherent components $\Lambda$, for which we give the exact definitions in Section 4.1. In the same section, we recall how one can construct a 2-step nilpotent Lie algebra $\mathfrak{n}^2_{\mathcal{G}}$ from $\mathcal{G}$ over a field $L \subset \mathbb{C}$ and how the Galois group $\text{Gal}(L/\mathbb{Q})$ naturally acts on this Lie algebra by semi-linear maps. In Section 4.2, we define a group morphism $i : \text{Aut}(\mathcal{G}_{\text{red}}) \to \text{Aut}(\mathfrak{n}^2_{\mathcal{G}})$ which is in fact not natural but easy to construct and allows us to describe all rational forms of $\mathfrak{n}^2_{\mathcal{G}}$. All fields in the theorem are considered as subsets of $\mathbb{C}$.

**Theorem A.** Let $\mathcal{G}$ be a simple undirected graph and $\mathfrak{n}^2_{\mathcal{G}}$ the associated 2-step nilpotent complex Lie algebra. Up to $\mathbb{Q}$-isomorphism, all rational forms of $\mathfrak{n}^2_{\mathcal{G}}$ are given by

$$\mathfrak{n}^2_{\mathcal{G},\rho} = \{ v \in \mathfrak{n}^2_{\mathcal{G}} \mid \forall \sigma \in \text{Gal}(L/\mathbb{Q}) : i(\rho)(\sigma v) = v \}$$

where $L/\mathbb{Q}$ is a finite Galois extension and $\rho : \text{Gal}(L/\mathbb{Q}) \to \text{Aut}(\mathcal{G}_{\text{red}})$ is an injective group morphism. If $K/\mathbb{Q}$ is another finite Galois extension with an injective group morphism $\eta : \text{Gal}(K/\mathbb{Q}) \to \text{Aut}(\mathcal{G}_{\text{red}})$, then $\mathfrak{n}^2_{\mathcal{G},\rho}$ and $\mathfrak{n}^2_{\mathcal{G},\eta}$ are isomorphic if and only if $L = K$ and there exists a $\varphi \in \text{Aut}(\mathcal{G}_{\text{red}})$ such that $\varphi(\rho(\sigma))\varphi^{-1} = \eta(\sigma)$ for all $\sigma \in \text{Gal}(L/\mathbb{Q})$. The rational forms of $\mathfrak{n}^2_{\mathcal{G}}$ are exactly those $\mathfrak{n}^2_{\mathcal{G},\rho}$ for which $L \subset \mathbb{R}$.

In order to prove this result, we introduce the basic notions of Galois cohomology in Section 2 including some preliminary results on computations for semi-direct products. Next, we apply these results to general linear algebraic groups containing all diagonal matrices, in order to relate the Galois cohomology of the automorphism group to the one of a permutation group on the basis vectors. Finally, we apply this to the specific case of Lie algebras associated to graphs to prove Theorem A.

In the second part of the paper, we determine which rational forms of $\mathfrak{n}^2_{\mathcal{G}}$ are Anosov by conditions on the action of the Galois group on the reduced graph. Again, all fields in the following theorem are considered as subsets of $\mathbb{C}$.

**Theorem B.** Let $L/\mathbb{Q}$ be a finite Galois extension and $\rho : \text{Gal}(L/\mathbb{Q}) \to \text{Aut}(\mathcal{G}_{\text{red}})$ be an injective morphism. By $\tau \in \text{Gal}(L/\mathbb{Q})$ we denote the complex conjugation automorphism in $\text{Gal}(L/\mathbb{Q})$ (which is trivial if $L \subset \mathbb{R}$). Then the associated rational form $\mathfrak{n}^2_{\mathcal{G},\rho}$ of $\mathfrak{n}^2_{\mathcal{G}}$ is Anosov if and only if for all $\lambda \in \Lambda$ the following are true:

(i) if $\rho(\tau)$ has a fixed point in $\text{Orb}_\rho(\lambda)$, then $|\lambda| \cdot |\text{Orb}_\rho(\lambda)| \geq 2$ and if equality holds, $\text{Orb}_\rho(\lambda)$ spans an empty subgraph of $\mathcal{G}_{\text{red}},$

(ii) if $\rho(\tau)$ has no fixed points in $\text{Orb}_\rho(\lambda)$ then $|\lambda| \cdot |\text{Orb}_\rho(\lambda)| \geq 4$ and if equality holds, $\text{Orb}_\rho(\lambda)$ spans an empty subgraph of $\mathcal{G}_{\text{red}}$ or a subgraph of $\mathcal{G}_{\text{red}}$ of the form

\[ \begin{array}{ccc}
1 & \lambda_1 & \mu_1 \\
\lambda_2 & 1 & \mu_2 \\
\end{array} \]

where $\rho(\lambda_1) = \lambda_2$, $\rho(\lambda_2) = \lambda_1$, $\rho(\mu_1) = \mu_2$ and $\rho(\mu_2) = \mu_1$.

Note that the condition only depends on how the orbits of the action look like, making it straightforward to check the condition for concrete actions, as we illustrate in the examples. The proof combines properties of
hyperbolic algebraic units with the description of rational forms in Theorem A. Note that a reader interested in Anosov diffeomorphisms could start reading Section A since this part only uses Theorem A and no other results on Galois cohomology from the first part.

2 Galois cohomology

In this section we introduce the tools from Galois cohomology that we will use throughout this paper. We first give an overview of the main definitions in relation to the study of $K$-forms of Lie algebras. Next, we present some technical results on the Galois cohomology of semi-direct products, in particular also for wreath products, which will be fruitful for the computations in the next section. This is the only section in the paper where we do not require the fields to be subfields of $C$ in general.

2.1 Definitions and relation to $K$-forms

First we recall some notions from Galois theory of possibly infinite degree field extensions. If $L/K$ is an extension of fields i.e. $K \subset L$, then we denote with $\text{Aut}(L/K)$ the group of field automorphisms of $L$ that fix every element of $K$. For any subgroup $H \leq \text{Aut}(L/K)$, we write $L^H$ for the field fixed by $H$, i.e. $L^H = \{ l \in L \mid \forall \sigma \in H : \sigma(l) = l \}$.

**Definition 2.1.** A field extension $L/K$ is called Galois if it is algebraic over $K$ and if $L^{\text{Aut}(L/K)} = K$. In this case we write $\text{Gal}(L/K)$ for $\text{Aut}(L/K)$.

Note that the field extension in the definition above is not required to be of finite degree. We can put a topology on $\text{Gal}(L/K)$ turning it into a topological group and more specific into a profinite group. This topology is called the Krull topology and a basis of opens for it is given by

$$\left\{ \sigma \text{Gal}(L/N) \mid \sigma \in \text{Gal}(L/K) \text{ and } N/K \text{ is an intermediate extension of finite degree} \right\}.$$

In the special case when $L/K$ is a finite degree extension, the topology on $\text{Gal}(L/K)$ is just the discrete topology. We can now formulate the Galois correspondence for field extensions of infinite degree.

**Theorem 2.2** (Galois correspondence). Let $L/K$ be a Galois extension. The assignment $H \mapsto L^H$ gives a bijection between

(i) the closed subgroups of $\text{Gal}(L/K)$ and the intermediate field extensions of $L/K$.

(ii) the open subgroups of $\text{Gal}(L/K)$ and the finite degree intermediate extensions of $L/K$.

(iii) the normal open subgroups of $\text{Gal}(L/K)$ and the finite degree intermediate Galois extensions of $L/K$, in which case we have an isomorphism $\text{Gal}(L/K)/H \rightarrow \text{Gal}(L^H/K) : \sigma H \mapsto \sigma|_{L^H}$.

Now let $G$ be a group and $L/K$ a Galois extension. Whenever we have a group morphism $\phi : \text{Gal}(L/K) \rightarrow G$ we will say it is continuous if it is so for the Krull topology on $\text{Gal}(L/K)$ and the discrete topology on $G$. Note that this implies that for any subgroup $H \leq G$, the inverse image $\phi^{-1}(H)$ is an open subgroup of $\text{Gal}(L/K)$ and thus by the Galois correspondence fixes a finite degree intermediate extension of $L/K$.

Next, we give an overview of the notions of Galois cohomology that will be used throughout this paper. In general, given a Galois extension of fields $L/K$ and an object $X$ defined over $L$, Galois cohomology can be used to classify the objects defined over $K$ which become isomorphic to $X$ when extended to $L$. This method is also known as Galois descent. The objects in question can be many structures among which Lie algebras. For the purpose of this paper, we will define everything for Lie algebras, but for a general discussion, we refer to the books [1] and [19].

We will always assume the fields to be of characteristic zero and the Lie algebras of finite dimension. We do allow field extensions to be of infinite degree. Recall that given a field extension $L/K$ and a Lie algebra $\mathfrak{n}^K$ defined over the field $K$, the tensor product of $K$-vector spaces $\mathfrak{n}^K := \mathfrak{n}^K \otimes_K L$ has a natural Lie algebra structure (over the field $L$) defined by

$$[v \otimes l, v' \otimes l'] = [v, v'] \otimes ll'.$$
for any $v,v' \in n^K$ and $l,l' \in L$.

**Definition 2.3.** Let $L/K$ be a field extension and $n^L$ a Lie algebra defined over the field $L$. Then we call a Lie algebra $m^K$ defined over the field $K$ a $K$-form of $n^L$ if the Lie algebra $m^L = m^K \otimes L$ is isomorphic to $n^L$. Two $K$-forms of $n^L$ are called equivalent if they are isomorphic over $K$. We write $\mathcal{F}_K(n^L)$ for the set of equivalence classes of $K$-forms of $n^L$.

Let $L/K$ be a Galois extension and $n^K$ a Lie algebra defined over the ground field $K$. Note that we have a natural inclusion $n^K \to n^L : v \mapsto v \otimes 1$ and an action of the Galois group $\text{Gal}(L/K)$ on $n^L$, which fixes $n^K$ seen as a subset of $n^L$. This action is for any $\sigma \in \text{Gal}(L/K)$, $v \in n^K$ and $l \in L$ defined by $\sigma(v \otimes l) := v \otimes \sigma(l)$ and by extending this additively to any element of $n^L$. As one can check, this action satisfies for all $v,w \in n^L$, $l \in L$ and $\sigma \in \text{Gal}(K/L)$:

1. $\sigma(v) = \sigma(l) \sigma v$,
2. $\sigma(v + w) = \sigma v + \sigma w$,
3. $\sigma[v,w] = [\sigma v,\sigma w]$.

Conditions (i) and (ii) tell us that the map $v \mapsto \sigma v$ is a so-called semi-linear map on $n^L$. We can thus say that $\text{Gal}(L/K)$ has an action on $n^L$ by semi-linear maps.

If $n^K$ and $m^K$ are two Lie algebras defined over the field $K$, we denote the set of Lie algebra homomorphisms (over $L$) from $n^L = n^K \otimes_K L$ to $m^L = m^K \otimes_K L$ by $\text{Hom}(n^L,m^L)$. The actions of $\text{Gal}(L/K)$ on $n^L$ and $m^L$ as described above now also induce an action of $\text{Gal}(L/K)$ on $\text{Hom}(n^L,m^L)$ by setting for any $\varphi \in \text{Hom}(n^L,m^L)$, $\sigma \in \text{Gal}(L/K)$ and $v \in n^L$:

$$(\sigma \varphi)(v) := \sigma \left( \varphi \sigma^{-1} v \right).$$

From the properties (i) (ii) and (iii) above it follows that $\sigma \varphi$ is indeed again a Lie algebra homomorphism. Note that if $p^K$ is a third Lie algebra over $K$, the equality $\sigma(\phi \varphi) = \sigma \phi \sigma \varphi$ holds for any $\varphi \in \text{Hom}(n^L,m^L)$ and $\phi \in \text{Hom}(m^L,p^K)$. In particular, the action of $\text{Gal}(L/K)$ on the invariant subset $\text{Aut}(n^L) \subset \text{Hom}(n^L,n^L)$ is one by group automorphisms, where the group operation on $\text{Aut}(n^L)$ is composition. In general, when $\text{Gal}(L/K)$ has an action on a group by group automorphisms, we will call this group a $\text{Gal}(L/K)$-group.

**Definition 2.4.** Let $L/K$ be a Galois extension and $G$ a $\text{Gal}(L/K)$-group. A continuous map

$$\rho : \text{Gal}(L/K) \to G,$$

where we write $\rho_\sigma = \rho(\sigma)$ for simplicity, is called a cocycle if it satisfies the relation $\rho_{\sigma \tau} = \rho_\sigma \rho_\tau$ for all $\sigma, \tau \in \text{Gal}(L/K)$. The set of cocycles is denoted with $Z^1(L/K,G)$. Two cocycles $\rho, \eta \in Z^1(L/K,G)$ are said to be equivalent if there exists a $g \in G$ such that $g \rho_\sigma \sigma^{-1} g^{-1} = \eta_\sigma$ for all $\sigma \in \text{Gal}(L/K)$. The set of equivalence classes of cocycles is denoted with $H^1(L/K,G)$ and is called the first Galois cohomology set.

**Remark 2.5.** The first Galois cohomology set is never empty, as we always have the equivalence class of the trivial cocycle $\text{Gal}(L/K) \to G : \sigma \mapsto e_G$. In fact this turns $H^1(L/K,G)$ into a pointed set, with the class of the trivial cocycle being the distinguished element. We will call $H^1(L/K,G)$ trivial if it only consists of this one element.

Note that, by our previous discussion, if we start with a Lie algebra $n^K$ defined over $K$, we have a natural action of $\text{Gal}(L/K)$ on $\text{Aut}(n^L)$ by group automorphisms. Therefore we can talk about the associated first Galois cohomology set $H^1(L/K,\text{Aut}(n^L))$. Now we discuss the connection between the first Galois cohomology set and the $K$-forms of $n^L$.

We can associate to each cocycle $\rho : \text{Gal}(L/K) \to \text{Aut}(n^L)$ a $K$-form $n^K_\rho \subset n^L$ by defining

$$n^K_\rho := \{ v \in n^L \mid \forall \sigma \in \text{Gal}(L/K) : \rho_\sigma(\sigma v) = v \}. \quad (1)$$

From properties (i) (ii) (iii) and the fact that each $\rho_\sigma$ is an automorphism of $n^L$, it follows that $n^K_\rho$ is closed under $K$-scalar multiplication, addition and taking the Lie bracket. Therefore we have that $n^K_\rho$ is indeed a Lie
algebra over $K$. We still need to check that $n^K \otimes L \cong n^L$. For this consider the map $n^K \otimes L \to n^L : v \otimes l \mapsto lv$. It is a standard result that this map is an $L$-vector space isomorphism as proven for example in [1 Lemma III.8.21]. It is then also straightforward to check this map preserves the Lie bracket. As it turns out, the $K$-Lie algebras constructed in this way are all the possible $K$-forms of $n^L$ up to $K$-isomorphism.

**Theorem 2.6 (Galois descent for Lie algebras).** Let $L/K$ be a Galois extension and $n^K$ a Lie algebra defined over $K$. The map

$$H^1(L/K, \text{Aut}(n^L)) \to \mathcal{F}_K(n^L) : [\rho] \mapsto [n^K_{\rho}]$$

is a bijection, which sends the trivial cocycle to $[n^K]$.

**Proof.** See [11 Theorem 1.3 and 1.4] or [1] Proposition III.9.1., Remark III.9.2. and Remark III.9.8.] for a proof of this statement. \hfill □

To see how the inverse of this map works, let $m^K$ be a $K$-form of $n^L$. By definition there exists an isomorphism $f : m^K \otimes_K L \to n^L$ of Lie algebras defined over $L$. Then we can associate to $m^K$ a cocycle $\rho^{m^K} \in Z^1(L/K, \text{Aut}(n^L))$ defined by

$$\rho^{m^K}_\sigma = f(\sigma f)^{-1}.$$ 

for all $\sigma \in \text{Gal}(L/K)$ and $v \in n^L$. Of course the cocycle $\rho^{m^K}$ depends on the choice of isomorphism $f$, but its class $[\rho^{m^K}]$ in $H^1(L/K, \text{Aut}(n^L))$ does not. In fact this class only depends on the $K$-isomorphism equivalence class of $m^K$. The inverse of the map from Theorem 2.6 is then given by the assignment $[m^K] \mapsto [\rho^{m^K}]$.

**Remark 2.7.** If $\rho, \eta \in Z^1(L/K, \text{Aut}(n^L))$ are two cocycles and $f$ is and automorphism, then a straightforward computation shows that $f(n^K_\rho) = n^K_\eta$ if and only if $\forall \sigma \in \text{Gal}(L/K) : f \rho_{\sigma} = \eta_{\sigma} \sigma f$.

We now present two well-known examples of groups for which the first Galois cohomology set is trivial. First, consider the general linear group $GL_n(L)$ with $\text{Gal}(L/K)$-action $\sigma(a_{ij})_{ij} := (\sigma(a_{ij}))_{ij}$ for any $(a_{ij})_{ij} \in GL_n(L)$ and $\sigma \in \text{Gal}(L/K)$. From the fact that $\sigma$ preserves the addition and multiplication in $L$, it follows that this is an action by group automorphisms on $GL_n(L)$.

**Theorem 2.8 (Generalized Hilbert’s theorem 90).** Let $L/K$ be a Galois extension and consider the general linear group $GL_n(L)$. Then $H^1(L/K, GL_n(L))$ is trivial, i.e. it contains only one element.

**Proof.** This is exactly [19 p.122, Lemma 1]. \hfill □

Note that $GL_n(L)$ is also the automorphism group of the abelian Lie algebra of dimension $n$ defined over $L$. Using Theorem 2.6, this agrees exactly with the fact that an abelian Lie algebra over $L$ has only one $K$-form up to $K$-isomorphism.

Secondly, consider the additive group $L_a$ of the field $L$. We can define a $\text{Gal}(L/K)$-action on $L_a$ by $\sigma l := \sigma(l)$ for any $l \in L$ and $\sigma \in \text{Gal}(L/K)$. It is clear that this is an action by group automorphisms.

**Theorem 2.9.** Let $L/K$ be a Galois extension and $L_a$ the additive group of the field $L$. Then $H^1(L/K, L_a)$ is trivial.

**Proof.** This is exactly [19 p.72, Proposition 1]. \hfill □

So far we only considered Galois extensions $L/K$ which are by definition algebraic. The following result tells us that it is enough to consider algebraic extensions in order to describe all $K$-forms. We will denote by $\overline{K}$ the algebraic closure of $K$. In the remainder of the paper, every algebraic extension of $K$ will be taken as a subfield of $\overline{K}$. This convention in particular implies that isomorphic field extensions of $K$ are in fact identical.

**Proposition 2.10.** Let $L/K$ be any field extension and $n^K$, $m^K$ Lie algebras over $K$. If $n^K \otimes L \cong m^K \otimes L$, then so is $n^K \otimes \overline{K} \cong m^K \otimes \overline{K}$ and moreover, there exists a finite degree field extension $N/K$ such that $n^K \otimes N \cong m^K \otimes N$.

**Proof.** This is an application of Hilbert’s Nullstellensatz, see [11 p.124, (i)] or [21 Lemma 3.3.] for more details on how to prove this. \hfill □
Let us apply this result for \( L = \mathbb{C} \), \( K = \mathbb{Q} \) and a rational Lie algebra \( n^\mathbb{Q} \). Let \( n^\mathbb{Q} \), \( n^\mathbb{R} \) and \( n^\mathbb{C} \) denote the Lie algebras \( n^\mathbb{Q} \otimes \mathbb{Q} \), \( n^\mathbb{Q} \otimes \mathbb{R} \) and \( n^\mathbb{Q} \otimes \mathbb{C} \), respectively. Proposition \( \ref{prop:completion} \) now gives us a bijection \( \mathcal{F}_Q(n^\mathbb{Q}) \to \mathcal{F}_Q(n^\mathbb{C}) : [m^\mathbb{Q}] \mapsto [m^\mathbb{C}] \). Together with Theorem \( \ref{thm:completion} \) this gives us a bijection

\[
H^1\left( \mathbb{Q}/\mathbb{Q}, \text{Aut} \left( n^\mathbb{Q} \right) \right) \to \mathcal{F}_Q (n^\mathbb{C}) : [\rho] \mapsto [\rho_\mathbb{Q}^\mathbb{C}]
\]

(2)

which is a useful tool to classify the rational forms of a complex Lie algebra.

If we want a similar result for the rational forms of a real Lie algebra, the question arises which equivalence classes of \( H^1(\mathbb{Q}/\mathbb{Q}, \text{Aut}(n^\mathbb{Q})) \) are mapped into \( \mathcal{F}_Q(n^\mathbb{R}) \) under the bijection (2) where we use that \( \mathcal{F}_Q(n^\mathbb{R}) \) is a subset of \( \mathcal{F}_Q(n^\mathbb{C}) \). Note that if we view \( \mathbb{Q} \) as a subset of \( \mathbb{C} \), we have a continuous morphism \( \nu : \text{Gal}(\mathbb{C}/\mathbb{R}) \to \text{Gal}(\mathbb{Q}/\mathbb{Q}) : \sigma \mapsto \sigma|_{\mathbb{Q}} \). This gives a map

\[
\omega : H^1\left( \mathbb{Q}/\mathbb{Q}, \text{Aut} \left( n^\mathbb{Q} \right) \right) \to H^1\left( \mathbb{C}/\mathbb{R}, \text{Aut} \left( n^\mathbb{C} \right) \right) : [\rho] \mapsto [\rho \circ \nu].
\]

(3)

Now take any cocycle \( \rho \in Z^1(\mathbb{Q}/\mathbb{Q}, \text{Aut}(n^\mathbb{Q})) \). From [11] Theorem 1.4, it follows that we have the following equivalences

\[
[n_\rho^\mathbb{Q}] \in \mathcal{F}_Q(n^\mathbb{R}) \iff n_\rho^\mathbb{Q} \otimes \mathbb{R} \simeq n^\mathbb{R}
\]

\[
\iff \omega([\rho]) = [1].
\]

(4)

This tells us exactly what elements of \( H^1(\mathbb{Q}/\mathbb{Q}, \text{Aut}(n^\mathbb{Q})) \) are being mapped into \( \mathcal{F}_Q(n^\mathbb{R}) \) under the bijection (2), namely the inverse image under \( \omega \) of the class of the trivial cocycle \( [1] \in H^1(\mathbb{C}/\mathbb{R}, \text{Aut}(n^\mathbb{C})) \). This can thus help us classify the rational forms of a given real Lie algebra.

### 2.2 Galois cohomology of semi-direct products

In what follows, we prove two results on maps between the cohomology of subgroups which satisfy some normality relations, implying that the group can be written as a semi-direct product. This result will be useful when calculating the cohomology of a certain class of groups in Section \( \ref{sec:examples} \). Let \( L/K \) be a Galois extension. Note that if \( G_1 \) and \( G_2 \) are two Gal(\( L/K \))-groups and \( f : G_1 \to G_2 \) is a Gal(\( L/K \))-equivariant group morphism, then we get a well-defined induced map on cohomology

\[
f_* : H^1(L/K, G_1) \to H^1(L/K, G_2) : [\rho] \mapsto [f \circ \rho].
\]

Let \( G \) be a Gal(\( L/K \))-group and \( A, B \) Gal(\( L/K \))-subgroups of \( G \), i.e. \( \sigma A = A \) and \( \sigma B = B \). Assume as well that \( A \leq N_G(B) \) and \( A \cap B = \{1\} \), then the subgroup \( AB \) is isomorphic to the semi-direct product \( B \rtimes A \) and under this identification, the Gal(\( L/K \))-action on \( B \times A \) is component-wise, i.e. \( \sigma (b, a) = (\sigma b, \sigma a) \) for all \( a \in A, b \in B \) and \( \sigma \in \text{Gal}(L/K) \). We thus also get a natural projection map \( \pi : AB \cong B \rtimes A \to A \) which is a Gal(\( L/K \))-equivariant group morphism. The same holds for the inclusion \( i : A \hookrightarrow AB \).

**Lemma 2.11.** Let \( L/K \) be a Galois extension and \( G \) a Gal(\( L/K \))-group. Let \( A, B, C \) be Gal(\( L/K \))-subgroups of \( G \) such that \( A \cap BC = \{1\}, B \cap C = \{1\}, A \leq N_G(B) \) and \( AB \leq N_G(C) \). We have the following two commutative diagrams of inclusions and projections and their induced maps on cohomology:

\[
\begin{array}{ccc}
A & \xrightarrow{i_1} & AB \\
\pi_2 \downarrow & & \downarrow \pi_{1*} \\
AC & \xleftarrow{i_3} & ABC
\end{array}
\quad
\begin{array}{ccc}
H^1(L/K, A) & \xleftarrow{i_1*} & H^1(L/K, AB) \\
\pi_2 \downarrow & & \downarrow \pi_{1*} \\
H^1(L/K, AC) & \xleftarrow{i_3*} & H^1(L/K, ABC)
\end{array}
\]

(5)

If \( i_{1*} \) is a bijection, then so is \( i_{3*} \).

**Proof.** Note that the projections \( \pi_1, \pi_2, \pi_3, \pi_4 \) are well-defined morphisms by the assumptions and the discussion right before the lemma. Assume \( i_{1*} \) is a bijection. It is clear that \( i_{3*} \) is injective since \( \pi_{3*} \circ i_{3*} = (\pi_3 \circ i_3)_* = (\text{Id}_{C \rtimes A})_* = \text{the identity on } H^1(L/K, AC) \).
Now take any $\rho \in H^1(L/K, ABC)$. Since $i_{1*}$ is surjective by assumption, we can find a $\eta \in H^1(L/K, A)$ such that $i_{1*}[\eta] = \pi_4[A\rho]$. This means that there exists an $g \in AB$ such that $g \pi_4(\rho_\sigma) \sigma g^{-1} = \eta_\sigma$ for all $\sigma \in \text{Gal}(L/K)$. For all $\sigma \in \text{Gal}(L/K)$, let $c_\sigma \in C$ denote the unique element such that $\rho_\sigma = \pi_4(\rho_\sigma)c_\sigma$. Then it follows that

$$g \rho_\sigma \sigma g^{-1} = g \pi_4(\rho_\sigma) \sigma g^{-1} \sigma g c_\sigma \sigma g^{-1} = \eta_\sigma (\sigma g c_\sigma \sigma g^{-1})$$

for all $\sigma \in \text{Gal}(L/K)$. Note that since $C$ is normal in $ABC$, we get that $\sigma g c_\sigma \sigma g^{-1}$ lies in $C$. By consequence we get a cocycle $[\nu] \in H^1(L/K, AC)$ defined by $\nu_\sigma = \eta_\sigma \sigma g c_\sigma \sigma g^{-1}$ for all $\sigma \in \text{Gal}(L/K)$. From equation (6) above it follows that $[\rho] = i_{3*}[\nu]$ and thus that $i_{3*}$ is indeed surjective.

This implies the following result for a sequence of subgroups.

**Lemma 2.12.** Let $L/K$ be a Galois extension and $G$ a Gal($L/K$)-group. Let $A_1, \ldots, A_n$ be Gal($L/K$)-subgroups of $G$ such that $A_i$ normalizes $A_j$ for all $i \leq j$ and $A_i \cap (A_{i-1} \ldots A_n) = \{1\}$ for all $1 \leq i < n$. If for all $1 \leq i \leq n$, the maps

$$H^1(L/K, A_1) \to H^1(L/K, A_i),$$

induced by the inclusions $A_1 \hookrightarrow A_1 A_i$, are bijections, then so is the map

$$H^1(L/K, A_1) \to H^1(L/K, A_1 A_2 \ldots A_n),$$

induced by the inclusion $A_1 \hookrightarrow A_1 A_2 \ldots A_n$.

**Proof.** Consider the following diagram of inclusions:

```
A_1  ---->  A_1 A_2  \\
|          |          |
V          V          V
A_1  ---->  A_1 A_3  ---->  A_1 A_2 A_3  \\
|          |          |          |
V          V          V          V
A_1  ---->  A_1 A_{i-1}  ---->  \ldots  ---->  A_1 A_i  ---->  A_1 A_3 \ldots A_{i-1}  ---->  A_1 A_2 \ldots A_{i-1}  \\
|          |          |          |          |          |          |
V          V          V          V          V          V          V
A_1  ---->  A_1 A_n  ---->  A_1 A_{n-1} A_n  ---->  \ldots  ---->  A_1 A_4 \ldots A_{n-1}  ---->  A_1 A_3 \ldots A_n  ---->  A_1 A_2 \ldots A_n
```

Note that every square in the diagram satisfies the conditions of Lemma 2.11. Since by assumption all the horizontal maps at the top of each column induce a bijection on cohomology, we find by applying Lemma 2.11 inductively that the horizontal maps on the bottom of each column induce bijections on the cohomology. Composing these bijections now gives the bijection $H^1(L/K, A_1) \to H^1(L/K, A_1 A_2 \ldots A_n)$.

In the last part of this section, we describe the first Galois cohomology set of a wreath product with a base which has trivial Galois cohomology. Let us start by recalling the definition of a wreath product.

**Definition 2.13.** Let $G$ and $A$ be groups such that $A$ is finite and has a left action on $\{1, \ldots, n\}$ for some positive integer $n$. The **wreath product** $G \wr A$ is defined as the semi-direct product $G^n \rtimes A$ where the left action of $A$ on $G^n$ is defined by

$$a \cdot (g_1, \ldots, g_n) := (g_{a^{-1}1}, \ldots, g_{a^{-1}n})$$

for any $g_1, \ldots, g_n \in G$ and $a \in A$. The group $G^n$ is called the **base** of the wreath product.

Now, consider a Galois extension $L/K$ and assume $G$ is a Gal($L/K$)-group, i.e. Gal($L/K$) has a left action on $G$ by group automorphisms. We can turn the wreath product $G \wr A$ into a Gal($L/K$)-group as follows. We endow $A$ with the trivial Gal($L/K$) action and $G^n$ with the induced component wise left
Gal($L/K$)-action, i.e. $\sigma(g_1, \ldots, g_n) = (\sigma g_1, \ldots, \sigma g_n)$ for all $g_1, \ldots, g_n \in G$ and $\sigma \in \text{Gal}(L/K)$. Clearly these are actions by group automorphisms. Note that the actions of $A$ and $\text{Gal}(L/K)$ on $G^n$ commute:

$$\sigma (a \cdot (g_1, \ldots, g_n)) = \sigma ((g_{a^{-1}1}, \ldots, g_{a^{-1}n})) = (\sigma g_{a^{-1}1}, \ldots, \sigma g_{a^{-1}n}) = a \cdot (\sigma g_1, \ldots, \sigma g_n) = a \cdot (\sigma (g_1, \ldots, g_n)).$$

At last we define a $\text{Gal}(L/K)$-action on the wreath product $G \wr A$ by $\sigma (g, a) = (\sigma g, \sigma a) = (\sigma g, a)$ for all $g \in G^n, a \in A$ and $\sigma \in \text{Gal}(L/K)$. This is an action by automorphisms since

$$\sigma ((g, a)(h, b)) = \sigma (ga \cdot h, ab)$$

$$= (\sigma g a \cdot h, ab)$$

$$= (\sigma g a^\sigma h, ab)$$

$$= (\sigma g, a)(\sigma h, b)$$

$$= \sigma (g, a)^\sigma (h, b)$$

for any $g, h \in G^n$ and $a, b \in A$.

We can thus talk about the first Galois cohomology set $H^1(L/K, G \wr A)$. Let $\pi : G \wr A \to A : (g, a) \mapsto a$ denote the projection morphism and $i : A \to G \wr A : a \mapsto ((1, \ldots, 1), a)$ the natural injection. Note that both $\pi$ and $i$ are $\text{Gal}(L/K)$-equivariant. By consequence we have the well-defined maps on cohomology:

$$\pi_* : H^1(L/K, G \wr A) \to H^1(L/K, A) : [\alpha] \mapsto [\pi \circ \alpha]$$

$$i_* : H^1(L/K, A) \to H^1(L/K, G \wr A) : [\alpha] \mapsto [i \circ \alpha].$$

The set $H^1(L/K, A)$ is relatively easy to understand since $A$ is a group with trivial Galois action, so it is given by all continuous group morphisms from $\text{Gal}(L/K)$ to $A$ up to conjugation by an element of $A$.

**Theorem 2.14.** Let $L/K$ be a Galois extension and $G$ a $\text{Gal}(L/K)$-group such that for any finite degree intermediate extension $N/K$ the first Galois cohomology set $H^1(L/N, G)$ is trivial. For every finite group $A$ with a left action on $\{1, \ldots, n\}$ with induced $\text{Gal}(L/K)$-action on the wreath product $G \wr A$ as above, we have that

$$i_* : H^1(L/K, A) \to H^1(L/K, G \wr A) : [\rho] \mapsto [i \circ \rho]$$

is a bijection with inverse $\pi_*$. 

**Proof.** Since $\pi \circ i = \text{Id}_A$, it follows that $\pi_* \circ i_*$ is the identity on $H^1(L/K, A)$ as well. For the other direction, we need to prove that $[\rho] = [i \circ \pi \circ \rho]$ for an arbitrary $[\rho] \in H^1(L/K, G \wr A)$.

Take any cocycle $\rho : \text{Gal}(L/K) \to G \wr A$. Let us write $\rho_\sigma = (g_\sigma, a_\sigma)$ for all $\sigma \in \text{Gal}(L/K)$. We then have that

$$\rho_\tau \rho = \rho_\sigma \rho = (g_\sigma a_\sigma \cdot \rho \tau, a_\sigma a_\tau).$$

Thus we have $g_\tau = g_\sigma a_\sigma \cdot \rho \tau$ and $a_\tau = a_\sigma a_\tau$. Let us denote for $h \in G^n$ by $(h)_i$ the $i$-th entry of $h$. Then we have

$$(g_\sigma)_i = (g_\sigma a_\sigma \cdot \rho \tau)_i = (g_\sigma)_i \rho \tau_\sigma = (g_\tau)_i \sigma.$$ 

Note that the group morphism $\pi \circ \rho : \text{Gal}(L/K) \to A$ induces an action of the Galois group on the set $\{1, \ldots, n\}$. The stabilizers stab($i$) for some $i \in \{1, \ldots, n\}$ are subgroups of $\text{Gal}(L/K)$. Moreover they are open subgroups since they can be written as the inverse image of $\{a \in A \mid a \cdot i = i\}$ under the continuous map $\pi \circ \rho$ and $A$ is endowed with the discrete topology. By consequence we have $\text{stab}(i) = \text{Gal}(L/L^{\text{stab}(i)})$.

Now note that for $\sigma, \tau \in \text{stab}(i)$, we have

$$(g_\sigma)_i = (g_\sigma)_i \rho \tau_\sigma = (g_\tau)_i \sigma.$$ 

This shows that the assignment $\sigma \mapsto (g_\sigma)_i$ is a cocycle from $\text{Gal}(L/L^{\text{stab}(i)})$ to $G$. By assumption, $H^1(L/L^{\text{stab}(i)}, G)$ is trivial and thus there exists a $h_i \in G$ such that $h_i (g_\sigma)_i \rho \tau_\sigma = 1$ for all $\sigma \in \text{stab}(i)$. This gives an element $h = (h_1, \ldots, h_n) \in G^n$. Then define a new cocycle $\tilde{\rho} : \text{Gal}(L/K) \to G \wr A : \sigma \mapsto (h, 1)(g_\sigma, a_\sigma)^\sigma (h, 1)^{-1}$. By the way this $\tilde{\rho}$ is defined, it is clear that $[\rho] = [\tilde{\rho}]$. We also have that

$$\tilde{\rho}_\sigma = (h \sigma, a_\sigma \cdot h^{-1}, a_\sigma).$$
Note that now, for \( \sigma \in \text{stab}(i) \) we have

\[
\tilde{g}_\sigma i = h_i(g_\sigma)i^{\sigma h_i^{-1}} = h_i(g_\sigma)i^{\sigma h_i^{-1}} = 1.
\]

Next, let us choose from each orbit of the action defined by \( \pi \circ \rho \equiv \pi \circ \tilde{\rho} \) on \( \{1, \ldots, n\} \), exactly one element \( m_i \), giving a subset \( \{m_1, \ldots, m_k\} \subset \{1, \ldots, n\} \). For \( j \in \text{orb}(m_i) \), we now define the element \( r_j := (g_\sigma)^{-1}_j \) where \( \sigma \in \text{Gal}(L/K) \) is chosen such that \( a_\sigma, m_i = j \). This does not depend on the choice of \( \sigma \). Indeed, if \( \tau \in \text{Gal}(L/K) \) also satisfies \( a_\tau, m_i = j \), then we get

\[
(g_\tau)^{-1}_j = (g_\sigma)^{-1}_j (g_\sigma)^{-1} \sigma g_\sigma, m_i = (g_\sigma)^{-1}_j
\]

where we used that \( \sigma^{-1} \tau \in \text{stab}(m_i) \) and thus \( (g_\sigma^{-1} \tau)_i = 1 \). This gives an element \( r = (r_1, \ldots, r_n) \in G^n \).

We now have that

\[
(r, 1) \tilde{\rho}_\sigma \sigma (r, 1)^{-1} = (r, 1)(\tilde{g}_\sigma, a_\sigma)(\sigma r^{-1}, 1) = (r \tilde{g}_\sigma, a_\sigma, \sigma r^{-1}, a_\sigma).
\]

At last we show that \( r \tilde{g}_\sigma, a_\sigma, \sigma r^{-1} = (1, \ldots, 1) \) for all \( \sigma \in \text{Gal}(L/K) \). Let \( j \in \text{orb}(m_i) \) and let \( \tau \in \text{Gal}(L/K) \) such that \( \tau(m_i) = j \). Then we have for any \( \sigma \in \text{Gal}(L/K) \):

\[
(r, 1) \tilde{\rho}_\sigma \sigma (r, 1)^{-1} = (1, \ldots, 1, a_\sigma)
\]

By consequence we have that \( (r, 1) \tilde{\rho}_\sigma \sigma (r, 1)^{-1} = ((1, \ldots, 1), a_\sigma) \) for any \( \sigma \in \text{Gal}(L/K) \). Note that \( (r \circ \pi \circ \rho)_\sigma = ((1, \ldots, 1), a_\sigma) \) and thus that we have shown that \( [\tilde{\rho}] = [\rho] \).

3 Galois cohomology of linear algebraic groups containing \( D_S \)

From here onwards, we will only consider fields that are subfields of \( \mathbb{C} \). So, let \( L/K \) be a Galois extension lying in \( \mathbb{C} \) and assume that \( G \leq \text{GL}(V) \) is a linear algebraic group defined over \( L \) on some vector space \( V \) with basis \( S \). In this section we first determine the first Galois cohomology set of the linear algebraic groups \( G \) that contain the group \( D_S \) of invertible diagonal matrices with respect to that basis. We start by recalling some results of [7] about the structure of \( G \); then we give a decomposition of \( G \) as an iterated semidirect product of subgroups. By using the results of the previous section, we can then compute the first Galois cohomology set.

3.1 The structure of \( G \)

We will use the same notation as in [7]. For \( \alpha, \beta \in S \), let \( E_{\alpha \beta} \in \text{End}(V) \) denote the linear map which satisfies

\[
E_{\alpha \beta}(\gamma) = \begin{cases} 
\alpha & \text{if } \gamma = \beta \\
0 & \text{else}
\end{cases}
\]

for all \( \gamma \in S \). Next, define a relation \( \prec \) on \( S \) by

\[
\alpha \prec \beta \iff \forall t \in L : I_V + tE_{\alpha \beta} \in G
\]

for all \( \alpha, \beta \in S \). Here \( I_V \) denotes the identity map on \( V \). This relation gives rise to an equivalence relation \( \sim \) on \( S \) by defining for all \( \alpha, \beta \in S \):

\[
\alpha \sim \beta \iff \alpha \prec \beta \text{ and } \alpha \succ \beta.
\]
The equivalence classes are called the *coherent components* and denoted by $\Lambda = \{ [\alpha] \mid \alpha \in S \} = S/\sim$. The relation $\prec$ on $S$ now gives a partial order relation $\preceq$ on the coherent components $\Lambda$ by defining for all $\lambda, \mu \in \Lambda$:

$$\lambda \preceq \mu \iff \exists \alpha \in \lambda, \exists \beta \in \mu : \alpha \prec \beta.$$ 

In fact, if $\lambda \preceq \mu$, then for any $\alpha \in \lambda, \beta \in \mu$ it holds that $\alpha \prec \beta$. Since $\preceq$ is a partial order, we can fix a linear ordering of the coherent components $\Lambda = \{ \lambda_1, \ldots, \lambda_k \}$ such that if $\lambda_i \preceq \lambda_j$, then $i \leq j$. Now, define the subgroup

$$M = \left\langle \{ I_V + t E_{\alpha \beta} \mid t \in \mathbb{R}, \alpha \prec \beta, \alpha \neq \beta \} \right\rangle,$$

which is in fact the unipotent radical of $G$ by [7].

For any relation $R$ on a set $X$, we denote with $\text{Perm}(X, R)$ the group of permutations of $X$ which preserve the relation $R$. If in addition there is a map $f : X \to Y$, where $Y$ is another set, then we denote by $\text{Perm}(X, R, f)$ the subgroup of $\text{Perm}(X, R)$ of those permutations $\theta$ which satisfy $f \circ \theta = f$.

Let $P : \text{Perm}(S, \prec) \to \text{GL}(V)$ be the group morphism which maps a permutation $\theta \in \text{Perm}(S, \prec)$ to the linear map $P_{\theta}$ defined by $P_{\theta}(\alpha) = \theta(\alpha)$ for all $\alpha \in S$. The subgroup $F \leq \text{Perm}(S, \prec)$ is then defined as the inverse image of $G$ under $P$. If we denote with $V_\lambda$ the vector space spanned by the elements in $\lambda$, it is proven in [7] that

$$G = M \left( \prod_{\lambda \in \Lambda} \text{GL}(V_\lambda) \right) P(F).$$

(8)

Note that $\text{GL}(V_\lambda)$ is seen as a subgroup of $\text{GL}(V)$ by letting $A \in \text{GL}(V_\lambda)$ correspond to the element in $\text{GL}(V)$ that maps any $\alpha \in S \setminus \lambda$ to $\alpha$ and any $\beta \in \lambda$ to $A(\beta)$. In general the intersection $P(F) \cap (\prod_{\lambda \in \Lambda} \text{GL}(V_\lambda))$ is not trivial, therefore we will define a subgroup of $P(F)$ in the next section such that the intersection with $\prod_{\lambda \in \Lambda} \text{GL}(V_\lambda)$ is trivial.

Define the map $\Phi : \Lambda \to \mathbb{N} : \lambda \mapsto |\lambda|$. Since any permutation $\theta \in \text{Perm}(S, \prec)$ preserves $\prec$, it also preserves the equivalence relation $\sim$. Therefore $\theta$ induces a permutation $\overline{\theta}$ on the set of coherent components $\Lambda$. Moreover $\overline{\theta}$ preserves the order relation $\preceq$ and the size of the coherent components. By consequence $\overline{\theta} \in \text{Perm}(\Lambda, \preceq, \Phi)$. This gives a morphism $q : \text{Perm}(S, \prec) \to \text{Perm}(\Lambda, \preceq, \Phi) : \theta \mapsto \overline{\theta}$ that fits in the short exact sequence:

$$1 \longrightarrow \prod_{\lambda \in \Lambda} \text{Perm}(\lambda) \xrightarrow{\pi} \text{Perm}(S, \prec) \xrightarrow{q} \text{Perm}(\Lambda, \preceq, \Phi) \longrightarrow 1.$$ 

(9)

In fact, this sequence is right-split, i.e. there exists a morphism $r : \text{Perm}(\Lambda, \preceq, \Phi) \to \text{Perm}(S, \prec)$ such that $q \circ r = \text{Id}$. Such a morphism $r$ is not unique, but let us show how to construct one. First choose an ordering of the vertices inside each coherent component $\lambda = \{ \alpha_{\lambda,1}, \alpha_{\lambda,2}, \ldots, \alpha_{\lambda,|\lambda|} \}$. Then for $\varphi \in \text{Perm}(\Lambda, \preceq, \Phi)$, define $r(\varphi) \in \text{Perm}(S, \prec)$ by $r(\varphi)(\alpha_{\lambda,i}) = \alpha_{\varphi(\lambda),i}$ for all $\lambda \in \Lambda$ and $1 \leq i \leq |\lambda|$. One can check that $r$ is well-defined and satisfies $q \circ r = \text{Id}$. Moreover, if $\varphi \in \text{Perm}(\Lambda, \preceq, \Phi)$ has a fixed point $\lambda \in \Lambda$, then $r(\varphi)|_\lambda = \text{Id}_\lambda$. This is not necessarily true for any choice of $r$ that makes the short exact sequence right-split, but is true for the one we constructed above. Let us for the remainder of this paper always fix such a morphism $r$ which does enjoy this additional property.

We can now also write $\text{Perm}(S, \prec)$ as a semi-direct product

$$\text{Perm}(S, \prec) \cong \left( \prod_{\lambda \in \Lambda} \text{Perm}(\lambda) \right) \rtimes_r \text{Perm}(\Lambda, \preceq, \Phi).$$

(10)

We can define a morphism $\overline{P} : \text{Perm}(\Lambda, \preceq, \Phi) \to \text{GL}(V)$ by $\overline{P} := P \circ r$ and a subgroup $\overline{F} \leq \text{Perm}(\Lambda, \preceq, \Phi)$ as the inverse image of $G$ under $\overline{P}$. We summarize this in the following (non-commutative) diagram:

$$F \leq \text{Perm}(S, \prec) \xrightarrow{P} \text{GL}(V) \geq G$$

$$\overline{F} \leq \text{Perm}(\Lambda, \preceq, \Phi).$$

(11)
We now show that also the subgroup $F$ decomposes as a semi-direct product by showing that $q(F) = \overline{F}$.

If $\theta \in F$, we can write it by the decomposition in (10) uniquely as $\theta = \theta' r(\varphi)$ with $\theta' \in \prod_{\lambda \in \Lambda} \text{Perm}(\lambda)$ and $\varphi \in \text{Perm}(\Lambda, \overline{\alpha}, \overline{\Phi})$. Note that $\prod_{\lambda \in \Lambda} \text{Perm}(\lambda)$ is always a subgroup of $F$, since under $P$ it is mapped into $\prod_{\lambda \in \Lambda} \text{GL}(V_\lambda) \subset G$. By consequence, it follows that $r(\varphi) \in F$ and thus that $\varphi \in \overline{F}$. Hence we get that $q(\theta) = q(\theta') q(r(\varphi)) = \varphi \in \overline{F}$ and thus, since $\theta \in F$ was chosen arbitrarily, that $q(F) \leq \overline{F}$. The other inclusion follows straightforward from the fact that $q \circ r = \text{Id}$. We thus have $q(F) = \overline{F}$, where the kernel of $q : F \rightarrow \overline{F}$ is exactly given by $\prod_{\lambda \in \Lambda} \text{Perm}(\lambda)$. We conclude that $F \cong \prod_{\lambda \in \Lambda} \text{Perm}(\lambda) \rtimes r \overline{F}$.

By applying the map $P$ to $F$, we find

$$P(F) = P \left( \prod_{\lambda \in \Lambda} \text{Perm}(\lambda) \right) \overline{P(F)}.$$ 

Using this and equation (8), we can now write

$$G = M \left( \prod_{\lambda \in \Lambda} \text{GL}(V_\lambda) \right) \overline{P(F)}$$

where the subgroups $M$, $(\prod_{\lambda \in \Lambda} \text{GL}(V_\lambda))$ and $\overline{P(F)}$ do have pairwise trivial intersection. Moreover, since $\overline{P(F)} \leq N_G \left( \prod_{\lambda \in \Lambda} \text{GL}(V_\lambda) \right)$ and $(\prod_{\lambda \in \Lambda} \text{GL}(V_\lambda)) \overline{P(F)} \leq N_G(M)$ we have a semi-direct product decomposition of $G$:

$$G \cong M \times \left( \prod_{\lambda \in \Lambda} \text{GL}(V_\lambda) \right) \rtimes \overline{F}.$$ 

Note that now also $\overline{F} \cong G/\overline{G}^0$ where $\overline{G}^0 := M \left( \prod_{\lambda \in \Lambda} \text{GL}(V_\lambda) \right)$ is the Zariski-connected component of the identity in $G$.

3.2 Writing $G$ as an iterated semi-direct product

Our next goal is to decompose both $M$ and $\prod_{\lambda \in \Lambda} \text{GL}(V_\lambda)$ into an iterated semi-direct product of subgroups which have a simpler structure and are each also normalized by $\overline{P(F)}$, but first we prove two lemma’s about the action of $\overline{F}$ on $\Lambda$.

**Lemma 3.1.** Let $\lambda, \mu \in \Lambda$ be coherent components which lie in the same $\overline{F}$-orbit. Then $\lambda \preceq \mu$ implies $\lambda = \mu$.

**Proof.** Let $\lambda, \mu$ be in the same $\overline{F}$-orbit with $\lambda \preceq \mu$. Then there exists a $\varphi \in \overline{F}$ such that $\varphi(\lambda) = \mu$. Note that $\varphi$ preserves the order relation $\preceq$. Let $m$ be the order of the permutation $\varphi$. We now have $\lambda \preceq \varphi(\lambda)$ and by applying $\varphi$ to this inequality $m$ times we get

$$\lambda \preceq \varphi(\lambda) \preceq \varphi^2(\lambda) \preceq \ldots \preceq \varphi^{m-1}(\lambda) \preceq \lambda.$$ 

This shows that $\lambda = \varphi(\lambda)$ and thus that $\lambda = \mu$. $\square$

**Lemma 3.2.** The relation $\preceq$ on the orbit space $\overline{F} \setminus \Lambda$ defined by

$$O_1 \preceq O_2 \iff \exists \lambda \in O_1, \exists \mu \in O_2 : \lambda \preceq \mu$$

is a partial order relation.

**Proof.** Reflexivity is clear since for any $O \in \overline{F} \setminus \Lambda$ and $\lambda \in O$, we have $\lambda \preceq \lambda$ and thus $O \preceq O$.

For antisymmetry, let $O_1, O_2 \in \overline{F} \setminus \Lambda$ with $O_1 \preceq O_2$ and $O_2 \preceq O_1$. Then there exist $\lambda_1, \mu_1 \in O_1$ and $\lambda_2, \mu_2 \in O_2$ such that $\lambda_1 \preceq \lambda_2$ and $\mu_1 \preceq \mu_2$. Since $\lambda_1, \mu_1$ are in the same $\overline{F}$-orbit, there exists a $\varphi \in \overline{F}$ such that $\varphi(\lambda_1) = \mu_1$. Applying $\varphi$ to $\lambda_1 \preceq \lambda_2$ gives $\mu_1 \preceq \varphi(\lambda_2)$. Combining this with $\mu_2 \preceq \mu_1$ gives that $\mu_2 \preceq \mu_1 \preceq \varphi(\lambda_2)$. Since $\mu_2$ and $\varphi(\lambda_2)$ are in the same $\overline{F}$-orbit, Lemma 3.1 implies that $\mu_2 = \varphi(\lambda_2)$ and thus also that $\mu_2 = \mu_1$, which in turn implies that $O_1 = O_2$.

For transitivity let $O_1 \preceq O_2$ and $O_2 \preceq O_3$. Then there exist $\lambda_1 \in O_1, \lambda_2, \mu_2 \in O_2, \lambda_3 \in O_3$ with $\lambda_1 \preceq \lambda_2$ and $\mu_2 \preceq \mu_3$. Since $\lambda_2$ and $\mu_2$ are in the same $\overline{F}$-orbit, there exists a $\varphi \in \overline{F}$ such that $\varphi(\lambda_2) = \mu_2$. Then we have $\varphi(\lambda_1) \preceq \varphi(\lambda_2) = \mu_2 \preceq \mu_3$ and thus by transitivity of $\preceq$ that $\varphi(\lambda_1) \preceq \mu_3$. This shows that $O_1 \preceq O_3$. $\square$
Since we showed $\preceq$ is a partial order, there is an ordering of the orbits $F \setminus \Lambda = \{ O_1, \ldots, O_l \}$ such that if $O_i \preceq O_j$, then $i \leq j$. We use this ordering to define the following subgroups of $M$:

$$M_i := \langle IV + tE_{\alpha\beta} \mid t \in L, (\alpha, \beta) \in \chi_i \rangle$$

where

$$\chi_i := \{ (\alpha, \beta) \mid \alpha \prec \beta, \alpha \not\prec \beta, [\beta] \in O_i \} \subset S \times S.$$  

Note that $\chi_1$ is empty and thus that $M_1$ is the trivial group. Indeed, assume that the exists an $(\alpha, \beta) \in \chi_1$ and let $[a] \in O_i$ for some $1 \leq i \leq l$. Then since $\alpha \prec \beta$, it follows that $[a] \npreceq [\beta]$ and that $O_i \npreceq O_1$. By the choice of the ordering of the orbits, this implies $i \leq 1$ and thus $i = 1$. Thus $[a]$ and $[\beta]$ lie in the same orbit and Lemma 3.1 implies that $[a] = [\beta]$. This is in contradiction with $\alpha \not\prec \beta$, finishing the proof.

In what follows, we use the notation of the Kronecker-Delta:

$$\delta_{\alpha\beta} = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \not= \beta \end{cases}$$

and we denote with $Y^X$ the set of maps from the set $X$ to the set $Y$. Note that if $Y$ is equipped with a group structure, we have a natural group structure on $Y^X$ as well, defined by $(f \cdot g)(x) = f(x) \cdot g(x)$ for all $f, g \in Y^X$ and $x \in X$.

**Lemma 3.3.** For all $1 \leq i \leq l$, the groups $M_i$ are abelian. Moreover, we have an isomorphism of groups

$$(L_a)^{\chi_i} \to M_i : f \mapsto \prod_{(\alpha, \beta) \in \chi_i} (IV + f(\alpha, \beta)E_{\alpha\beta}).$$

**Proof.** Let us first show that $M_i$ is abelian. Take any $(\alpha, \beta), (\gamma, \epsilon) \in \chi_i$ and any $t, s \in L$. We have

$$(IV + tE_{\alpha\beta})(IV + sE_{\gamma\epsilon}) = (IV + sE_{\gamma\epsilon})(IV + tE_{\alpha\beta})$$

Let us show that the terms on the right-hand side vanish. For contradiction, assume that $\alpha = \epsilon$. Then we have that $[\alpha] = [\epsilon] \in O_i$. We also have $\alpha \prec \beta$ and thus $[\alpha] \npreceq [\beta]$. Since $[\alpha]$ and $[\beta]$ lie in the same orbit $O_i$, Lemma 3.1 implies that $[\alpha] = [\beta]$ which contradicts the fact that $\alpha \not\prec \beta$. It follows that $\alpha \not= \epsilon$. Analogously one proves that $\beta \not= \gamma$ and thus that the right hand side of equation (12) indeed vanishes. This shows that $(IV + tE_{\alpha\beta})$ and $(IV + sE_{\gamma\epsilon})$ commute for any $(\alpha, \beta), (\gamma, \epsilon) \in \chi_i$ and any $t, s \in L$. Since $M_i$ is spanned those elements, it follows that $M_i$ is abelian.

This implies as well that the map from the lemma is well defined. To see it is a morphism, first note that for $(\alpha, \beta) \in \chi_i$, we can not have $\alpha = \beta$ since $\alpha \not\prec \beta$. By consequence we have that $E^2_{\alpha\beta} = 0$. We now get for any $f, f' \in (L_a)^{\chi_i}$ that

$$f + f' \mapsto \prod_{(\alpha, \beta) \in \chi_i} (IV + (f(\alpha, \beta) + f'(\alpha, \beta))E_{\alpha\beta})$$

$$= \prod_{(\alpha, \beta) \in \chi_i} (IV + f(\alpha, \beta)E_{\alpha\beta})(IV + f'(\alpha, \beta)E_{\alpha\beta})$$

$$= \prod_{(\alpha, \beta) \in \chi_i} (IV + f(\alpha, \beta)E_{\alpha\beta}) \prod_{(\alpha, \beta) \in \chi_i} (IV + f'(\alpha, \beta)E_{\alpha\beta})$$

where we use $E^2_{\alpha\beta} = 0$ for $(\alpha, \beta) \in \chi_i$ in the first equality and the fact that $M_i$ is abelian in the second equality. This proves it is a morphism. It is clearly surjective since all the generators of $M_i$ are in the image.

**Lemma 3.4.** For any $1 \leq i \leq j \leq l$, we have that $M_i$ normalizes $M_j$.

**Proof.** It satisfies to prove that if we conjugate a generator of $M_j$ by a generator of $M_i$, we get an element in $M_i$. Take any $(\alpha, \beta) \in \chi_i$, $(\gamma, \epsilon) \in \chi_j$ and $t, s \in L$. Note that

$$(IV + tE_{\alpha\beta})(IV + sE_{\gamma\epsilon})(IV + tE_{\alpha\beta})^{-1} = IV + sE_{\gamma\epsilon} + ts\delta_{\beta\gamma}E_{\alpha\epsilon}$$

$$- t^2\delta_{\alpha\beta}E_{\alpha\beta} - ts\delta_{\epsilon\gamma}E_{\gamma\epsilon} - t^2s\delta_{\beta\gamma}\delta_{\epsilon\gamma}E_{\alpha\beta}.$$ (13)
First, note that $\alpha \neq \beta$ since $\alpha \not\equiv \beta$. Next, assume that $\epsilon = \alpha$. Then we have $\epsilon \prec \beta$ which implies $[\epsilon] \not\equiv [\beta]$ and in turn implies $O_j \not\subseteq O_i$. From the way we chose the ordering of the orbits, it follows that $j \leq i$. Since we assumed $i \leq j$, this gives $i = j$. Using Lemma 3.1, we thus find that $[\epsilon] = [\beta]$. This gives $\epsilon \succ \beta$ and thus $\alpha \succ \beta$, a contradiction. We thus have $\epsilon \neq \alpha$. From this it follows that equation (13) simplifies to

$$(I_V + tE_{\alpha\beta})(I_V + sE_{\gamma\epsilon})(I_V + tE_{\alpha\beta})^{-1} = I_V + sE_{\gamma\epsilon} + ts\delta_{\beta\gamma}E_{\alpha\epsilon}.$$

If $\beta = \gamma$, it follows that $\alpha \prec \epsilon$ and thus that $(\alpha, \epsilon) \in \chi_j$. This shows that $(I_V + sE_{\gamma\epsilon})(I_V + ts\delta_{\beta\gamma}E_{\alpha\epsilon})$ is an element of $M_j$, thus finishing the proof.

**Lemma 3.5.** For any $1 \leq i \leq l$, we have that $P(F)$ normalizes $M_i$.

**Proof.** It satisfies to prove that if we conjugate a generator of $M_j$ by an element of $P(F)$, we get an element in $M_j$. Take any $(\alpha, \beta) \in \chi_i$, $t \in L$ and $\theta \in F$. Note that

$$P_\theta(I_V + tE_{\alpha\beta})P_\theta^{-1} = I_V + tE_{\theta(\alpha)\theta(\beta)}$$

and since $[\beta] \in O_i$, we have $[\theta(\beta)] = P([\beta]) \in O_i$. Using as well that $\theta$ preserves $\prec$, it follows that $(\theta(\alpha), \theta(\beta)) \in \chi_i$ and thus that $I_V + tE_{\theta(\alpha)\theta(\beta)} \in M_i$. This finishes the proof.

**Lemma 3.6.** For any $1 \leq i \leq l$, we have that $\prod_{\lambda \in \Lambda} GL(V_\lambda)$ normalizes $M_i$.

**Proof.** It satisfies to prove that if we conjugate a generator of $M_j$ by an element of $\prod_{\lambda \in \Lambda} GL(V_\lambda)$, we get an element in $M_j$. Take any $(\alpha, \beta) \in \chi_i$, $t \in L$ and $A \in \prod_{\lambda \in \Lambda} GL(V_\lambda)$. We get that

$$A(I_V + tE_{\alpha\beta})A^{-1} = I_V + tAE_{\alpha\beta}A^{-1}. \quad (14)$$

For any $B \in \text{End}(V)$ and $\gamma, \epsilon \in S$, let us write $(B)_{\gamma\epsilon}$ for the $\gamma, \epsilon$-entry in the matrix representation of $B$ with respect to the basis $S$. Calculating the above products in their matrix representation, we find that

$$AE_{\alpha\beta}A^{-1} = \sum_{\gamma \in S} \sum_{\epsilon \in S} (A)_{\gamma\alpha}(A^{-1})_{\beta\epsilon}E_{\gamma\epsilon}. $$

Since $A, A^{-1} \in \prod_{\lambda \in \Lambda} GL(V_\lambda)$ we have that $(A)_{\gamma\alpha} = 0$ if $\gamma \notin [\alpha]$ and $(A^{-1})_{\beta\epsilon} = 0$ if $\epsilon \notin [\beta]$. By consequence we can now rewrite (14) as

$$A(I_V + tE_{\alpha\beta})A^{-1} = I_V + \sum_{\gamma \in [\alpha]} \sum_{\epsilon \in [\beta]} t(A)_{\gamma\alpha}(A^{-1})_{\beta\epsilon}E_{\gamma\epsilon}. $$

Note that $[\alpha] \neq [\beta]$ since $\alpha \not\equiv \beta$. By consequence if we take $\gamma, \gamma' \in [\alpha]$ and $\epsilon, \epsilon' \in [\beta]$, we get that $E_{\gamma\epsilon}E_{\gamma'\epsilon'} = 0$. This implies that

$$A(I_V + tE_{\alpha\beta})A^{-1} = \prod_{\gamma \in [\alpha]} \prod_{\epsilon \in [\beta]} (I_V + t(A)_{\gamma\alpha}(A)_{\beta\epsilon}E_{\gamma\epsilon}). $$

Since $(\alpha, \beta) \in \chi_i$, it follows that $(\gamma, \epsilon) \in \chi_i$ whenever $\gamma \in [\alpha]$ and $\epsilon \in [\beta]$. This proves that $A(I_V + tE_{\alpha\beta})A^{-1}$ is indeed an element of $M_i$.

Next, we decompose $\prod_{\lambda \in \Lambda} GL(V_\lambda)$ into a direct product of subgroups which are each also normalized by $P(F)$. For each $1 \leq i \leq l$, defined the subgroups of $G$:

$$Q_i := \prod_{\lambda \in O_i} GL(V_\lambda).$$

Note that $\prod_{\lambda \in \Lambda} GL(V_\lambda)$ is a direct product of the subgroups $Q_i$, $1 \leq i \leq l$ and thus we have that $Q_i$ normalizes $Q_j$ for any $1 \leq i, j \leq l$. Of course $Q_i$ is also a direct product of the subgroups $GL(V_\lambda)$ for $\lambda \in O_i$ and since for $\mu, \lambda$ in the same orbit $O_i$ we have that $|\lambda| = |\mu|$, we must conclude that also all these factors of $Q_i$ are mutually isomorphic. This is shown by the following explicit isomorphism.
Lemma 3.7. For all \(1 \leq i \leq l\) and \(\lambda \in \mathcal{O}_i\), we have an isomorphism groups

\[
\text{GL}(V_\lambda)^{O_i} \rightarrow Q_i : f \mapsto \prod_{\varphi \in F/\text{Stab}(\lambda)} T_\varphi f(\varphi(\lambda)) T_\varphi^{-1}.
\]

Proof. It is immediate from our assumption on \(r\) that the right-hand side does not depend on the choice of representative \(\varphi\). Take any \(f, f' \in \text{GL}(V_\lambda)^{O_i}\). Then we have

\[
f \cdot f' \mapsto \prod_{\varphi \in F/\text{Stab}(\lambda)} T_\varphi f(\varphi(\lambda)) f'(\varphi(\lambda)) T_\varphi^{-1} = \prod_{\varphi \in F/\text{Stab}(\lambda)} T_\varphi f(\varphi(\lambda)) T_\varphi^{-1} T_\varphi f'(\varphi(\lambda)) T_\varphi^{-1}
\]

Note that whenever \([\varphi] \neq [\phi]\) in \(F/\text{Stab}(\lambda)\) it implies that \(\varphi(\lambda) \neq \phi(\lambda)\). In this case, the elements \(T_\varphi f(\varphi(\lambda)) T_\varphi^{-1}\) and \(T_\phi f'(\phi(\lambda)) T_\phi^{-1}\) commute since they lie in \(\text{GL}(V_{\varphi(\lambda)})\) and \(\text{GL}(V_{\phi(\lambda)})\), respectively. Therefore we can rearrange the factors in the product to get

\[
f \cdot f' \mapsto \left( \prod_{\varphi \in F/\text{Stab}(\lambda)} T_\varphi f(\varphi(\lambda)) T_\varphi^{-1} \right) \left( \prod_{\varphi \in F/\text{Stab}(\lambda)} T_\varphi f'(\varphi(\lambda)) T_\varphi^{-1} \right)
\]

which proves the map is a morphism of groups. One can check as well that the map is a bijection. \(\square\)

Lemma 3.8. For each \(1 \leq i \leq l\), we have that \(P(F)\) normalizes \(Q_i\).

Proof. Take any \(A \in Q_i\) and \(\theta \in F\). We can write \(A = \prod_{\lambda \in \mathcal{O}_i} A_\lambda\) with \(A_\lambda \in \text{GL}(V_\lambda)\) for all \(\lambda \in \mathcal{O}_i\). We then have

\[
P_\theta AP_\theta^{-1} = P_\theta \left( \prod_{\lambda \in \mathcal{O}_i} A_\lambda \right) P_\theta^{-1} = \prod_{\lambda \in \mathcal{O}_i} P_\theta A_\lambda P_\theta^{-1}.
\]

Note that \(P_\theta A_\lambda P_\theta^{-1}\) lies in \(\text{GL}(V_{\theta(\lambda)})\) and that \(\mathcal{O}_i = \mathcal{O}_i\). It thus follows that

\[
P_\theta AP_\theta^{-1} \in \prod_{\lambda \in \mathcal{O}_i} \text{GL}(V_{\theta(\lambda)}) = \prod_{\lambda \in \mathcal{O}_i} \text{GL}(V_\lambda) = Q_i,
\]

which finishes the proof. \(\square\)

### 3.3 Galois cohomology of \(G\)

The previous results allow us to compute the first Galois cohomology set of \(G\), formulated in Theorem 3.9 below. Recall that we have a Galois extension \(L/K\), a vector space \(V\) defined over \(L\) with basis \(S\) and a linear algebraic group \(G \leq \text{GL}(V)\) containing the subgroup of diagonal matrices \(D_S\).

Denote by \((g_{\alpha\beta})_{\alpha\beta}\) the matrix representation of an element \(g \in G\) with respect to this basis \(S\). We can define a left-action of \(\text{Gal}(L/K)\) on \(G\) by letting \(\sigma g\) be the element in \(G\) represented by the matrix \((\sigma(g_{\alpha\beta}))_{\alpha\beta}\). Note that since each \(\sigma \in \text{Gal}(L/K)\) preserves both addition and multiplication of the field \(L\), it can easily be seen that this action of \(\text{Gal}(L/K)\) on \(G\) is one by group automorphisms, thus turning \(G\) into a \(\text{Gal}(L/K)\)-group. By consequence it makes sense to talk about its first Galois cohomology set \(H^1(L/K, G)\).

If we endow \(F\) with the trivial \(\text{Gal}(L/K)\)-action, we get a \(\text{Gal}(L/K)\)-equivariant morphism \(\overline{F} : F \rightarrow G\). This follows from the fact that the elements in the image \(\overline{F}(F)\) are represented by matrices which only have coefficients equal to 0 or 1 with respect to the basis \(S\) and thus it is clear that they are fixed by the action of \(\text{Gal}(L/K)\) on \(G\). This section is devoted to proving the following result:

Theorem 3.9. The map \(\overline{F} : F \rightarrow G\) induces a bijection

\[
\overline{F}_* : H^1(L/K, \overline{F}) \rightarrow H^1(L/K, G) : [\rho] \mapsto [\overline{F} \circ \rho] .
\]
Note that the set $H^1(L/K, \mathcal{F})$ is well-understood since the Galois action on $\mathcal{F}$ is trivial. The set $H^1(L/K, \mathcal{F})$ is thus given by equivalence classes of group morphisms from $\text{Gal}(L/K)$ to $\mathcal{F}$ where two group morphisms $\rho, \eta$ are equivalent if there exists a $\varphi \in \mathcal{F}$ such that $\varphi \rho \varphi^{-1} = \eta \sigma$ for all $\sigma \in \text{Gal}(L/K)$.

The subgroups $M_2, \ldots, M_l, Q_1, \ldots, Q_l$ and $\overline{P}(\mathcal{F})$ as defined in the previous section, are all $\text{Gal}(L/K)$-subgroups of $G$. To prove the above theorem we will apply Lemma 2.12 to these subgroups with $A_1 = \overline{P}(\mathcal{F})$, $A_2 = Q_1$, $\ldots$, $A_{l+1} = Q_l$, $A_{l+2} = M_2$, $\ldots$, $A_{2l} = M_l$. From Lemma’s 3.4, 3.5, 3.6 and 3.8 it follows that the normality relations between the $A_i$’s as required by Lemma 2.12 are satisfied. We are thus only left to prove that the inclusions $M_i \hookrightarrow M_i \overline{P}(\mathcal{F})$, $2 \leq i \leq l$ and $Q_i \hookrightarrow Q_i \overline{P}(\mathcal{F})$, $1 \leq i \leq l$ induce bijections on cohomology, for which we will mainly use Theorem 2.14.

**Proposition 3.10.** For any $2 \leq i \leq l$, let $\overline{P}(\mathcal{F}) \to M_i \overline{P}(\mathcal{F})$ be the inclusion map. Then the induced maps

$$H^1(L/K, \overline{P}(\mathcal{F})) \to H^1(L/K, M_i \overline{P}(\mathcal{F}))$$

are bijections.

**Proof.** Take any $2 \leq i \leq l$. Let us denote the isomorphism from Lemma 3.3 by $\psi : (L_0)^{\chi_i} \to M_i$. Since by Lemma 3.3 $P(\mathcal{F})$ normalizes $M_i$, so does $\overline{P}(\mathcal{F})$ since it is a subgroup of $P(\mathcal{F})$. Together with the fact that $M_i \cap \overline{P}(\mathcal{F}) = 1$, we find that $M_i \overline{P}(\mathcal{F})$ is as a group isomorphic to the semi-direct product $(L_0)^{\chi_i} \rtimes \mathcal{F}$. An explicit group isomorphism can be given by the map

$$(L_0)^{\chi_i} \rtimes \mathcal{F} \to M_i \overline{P}(\mathcal{F}) : (f, \varphi) \mapsto \psi(f) \overline{P}(\varphi). \quad (15)$$

Under this isomorphism we can also endow $(L_0)^{\chi_i} \rtimes \mathcal{F}$ with the corresponding Gal$(L/K)$-action by group automorphisms. It can be easily checked that this action is given by $\sigma (f, \varphi) = (\sigma \circ f, \varphi)$ for all $\sigma \in \text{Gal}(L/K)$, $f \in (L_0)^{\chi_i}$ and $\varphi \in \mathcal{F}$.

Note that for any $\theta \in \text{Perm}(S, \chi)$, the map $\theta \circ \chi : \chi_i \subset S \times S \to \chi$ is a bijection. One can check that under the correspondence of $(15)$, the action of $\mathcal{F}$ on $(L_0)^{\chi_i}$ is given by $\varphi \cdot f = f \circ (r(\varphi)^{-1}) \tau(r(\varphi)^{-1})$ and thus it is an action by permuting the components of the direct product $(L_0)^{\chi_i}$. Therefore, the semi-direct product $(L_0)^{\chi_i} \rtimes \mathcal{F}$ is actually a wreath product $L_0 \wr \mathcal{F}$. Finally, from Theorem 2.14 it follows that $H^1(L/N, L_0)$ is trivial for any finite degree intermediate extension $N/K$. Therefore we can apply Theorem 2.14 to $(L_0)^{\chi_i} \rtimes \mathcal{F}$ and find that

$$H^1(L/K, \overline{P}(\mathcal{F})) \to H^1(L/K, (L_0)^{\chi_i} \rtimes \mathcal{F})$$

induced by the inclusion $\overline{P}(\mathcal{F}) \to (L_0)^{\chi_i} \rtimes \mathcal{F}$, is a bijection. Since $H^1(L/K, \mathcal{F})$ is a functor and we have the isomorphism $(14)$, it now also follows that

$$H^1(L/K, \overline{P}(\mathcal{F})) \to H^1(L/K, M_i \overline{P}(\mathcal{F}))$$

induced by the inclusion, is a bijection as well.

**Proposition 3.11.** For any $1 \leq i \leq l$, let $\overline{P}(\mathcal{F}) \to Q_i \overline{P}(\mathcal{F})$ be the inclusion map. Then the induced maps

$$H^1(L/K, \overline{P}(\mathcal{F})) \to H^1(L/K, Q_i \overline{P}(\mathcal{F}))$$

are bijections.

**Proof.** Take any $1 \leq i \leq l$. Fix any $\lambda \in O_i$. Let us denote the isomorphism from Lemma 3.7 by $\psi : \text{GL}(V_\lambda)^{O_i} \to Q_i$. Since by Lemma 3.8 $P(\mathcal{F})$ normalizes $Q_i$, so does $\overline{P}(\mathcal{F})$ since it is a subgroup of $P(\mathcal{F})$. Together with the fact that $Q_i \cap \overline{P}(\mathcal{F}) = 1$, we find that $Q_i \overline{P}(\mathcal{F})$ is as a group isomorphic to the semi-direct product $\text{GL}(V_\lambda)^{O_i} \rtimes \mathcal{F}$. An explicit group isomorphism can be given by the map

$$\text{GL}(V_\lambda)^{O_i} \rtimes \mathcal{F} \to Q_i \overline{P}(\mathcal{F}) : (f, \varphi) \mapsto \psi(f) \overline{P}(\varphi). \quad (16)$$

Under this isomorphism we can also endow $\text{GL}(V_\lambda)^{O_i} \rtimes \mathcal{F}$ with the corresponding Gal$(L/K)$-action by group automorphisms. Note that $\text{Gal}(L/K)$ acts on $\text{GL}(V_\lambda)$ since it is a Gal$(L/K)$-subgroup of $G$. In fact, the action is given by applying the element of the Galois group to each coefficient of the matrix representation of the element of $\text{GL}(V_\lambda)$ with respect to the basis $\lambda$. This action can be extended to $\text{GL}(V_\lambda)^{O_i}$ by defining

$$f \cdot \lambda = f(\lambda') = f(\lambda(\sigma)) \quad (\sigma \in \text{Gal}(L/K), \lambda' \in \text{GL}(V_\lambda)^{O_i}, \lambda(\sigma) \in \text{GL}(V_\lambda)^{O_i}$$
Precomposing this map with the one from (17), we get the bijection

\[
\sigma(f)(\mu) := \sigma(f(\mu)) \quad \text{for any } \sigma \in \text{Gal}(L/K), \ f \in \text{GL}(V_\lambda)^{O_{\lambda}}, \ \text{and } \mu \in O_{\lambda}.
\]

As one can check the action of \text{Gal}(L/K) on the semi-direct product \text{GL}(V_\lambda)^{O_{\lambda}} \times \mathcal{T} is given by \(\sigma(f, \varphi) = (\sigma f, \varphi).

One can check that the action of \(\mathcal{T}\) on \text{GL}(V_\lambda)^{O_{\lambda}} in the semi-direct product is given by \(\varphi \cdot f = f \circ (\varphi(\sigma_{\lambda}))^{-1}\) and thus it is an action by permuting the components of the direct product \text{GL}(V_\lambda)^{O_{\lambda}} and thus that \text{GL}(V_\lambda)^{O_{\lambda}} \times \mathcal{T}\) is actually a wreath product \text{GL}(V_\lambda) \wr \mathcal{T}. Finally from Theorem 2.3 it follows that \(H^1(L/N, \text{GL}(V_\lambda))\) is trivial for any finite degree intermediate extension \(N/K\). Therefore we can apply Theorem 2.14 to \text{GL}(V_\lambda)^{O_{\lambda}} \wr \mathcal{T}\) and find that

\[
H^1(L/K, \mathcal{T}) \to H^1(L/K, \text{GL}(V_\lambda) \wr \mathcal{T})
\]

induced by the inclusion \(\mathcal{T} \hookrightarrow \text{GL}(V_\lambda) \wr \mathcal{T}\) is a bijection. Since \(H^1(L/K, \bullet)\) is a functor and we have the isomorphism (16), it now also follows that

\[
H^1(L/K, \mathcal{T}(\mathcal{T})) \to H^1(L/K, Q, \mathcal{T}(\mathcal{T}))
\]

induced by the inclusion, is a bijection.

\[\Box\]

\textbf{Proof of Theorem 4.3.} As mentioned below Theorem 3.9 we will apply Lemma 2.12 to the subgroups \(A_1 = \mathcal{T}(\mathcal{T}), A_2 = Q_1, \ldots, A_{n+1} = Q_l, A_{n+2} = M_2, \ldots, A_2 = M_l\). Lemma’s 3.4, 3.5, 3.6 and Propositions 3.10, 3.11 imply that all the requirements of Lemma 2.12 are satisfied, thus giving that the map

\[
H^1(L/K, \mathcal{T}(\mathcal{T})) \to H^1(L/K, G)
\]

(17)

induced by the inclusion, is a bijection. The \text{Gal}(L/K)-group isomorphism \(\mathcal{T} : \mathcal{T} \to \mathcal{T}(\mathcal{T})\) (both are endowed with the trivial \text{Gal}(L/K)-action) induces a bijection on cohomology \(H^1(L/K, \mathcal{T}) \to H^1(L/K, \mathcal{T}(\mathcal{T}))\). Precomposing this map with the one from (17), we get the bijection

\[
\mathcal{T}_*: H^1(L/K, \mathcal{T}) \to H^1(L/K, G) : [\rho] \mapsto [\mathcal{T} \circ \rho]
\]

which finishes the proof.

\[\Box\]

\section{Rational forms in 2-step Lie algebras associated to graphs}

In this section we apply the results of the previous section to the 2-step nilpotent Lie algebras associated to a graph, for which the automorphism group contains a linear algebraic group as in the previous section. Finally, we relate the first Galois cohomology set to the rational forms of these Lie algebras.

\subsection{Lie algebra \(n_G^L\) associated to a graph \(G\)}

We start by recalling the construction from 2 for Lie algebras associated to a simple undirected graph \(G\). A simple undirected graph is a pair \(G = (S, E)\) with \(S\) a finite set and \(E\) a subset of \(\{\{s, t\} \subset S \mid s \neq t\}\). Take any field subfield \(K \subset \mathbb{C}\) and let \(f_2^k(S)\) be the free 2-step Lie algebra generated by \(S\) defined over \(K\). Consider \(I_2^k\) the ideal in \(f_2^k(S)\) generated by the set \(\{[v, w], v \in E, w \in E\}\). We define the 2-step Lie algebra \(n_2^K\) associated to the graph \(G\) defined over \(K\) as the quotient Lie algebra \(n_2^K := f_2^k(S)/I_2^k\).

Note that for a Galois extension \(L/K\) of subfields of \(\mathbb{C}\), we have a natural inclusion \(f_2^k(S) \hookrightarrow f_2^l(S)\) which descends to the quotient \(n_2^k \hookrightarrow n_2^l\). We can thus view \(n_2^k\) as a subset of \(n_2^l\) and get a natural Lie algebra isomorphism \(n_2^k \otimes L \to n_2^l : v \otimes l \mapsto lv\). Therefore we also have a natural action of \text{Gal}(L/K) on \(n_2^l\), leaving \(n_2^k\) fixed and a corresponding action of \text{Gal}(L/K) on \text{Aut}(n_2^l) as discussed in section 2.

We also have a direct sum decomposition as vector spaces

\[
n_2^l = \text{span}_{\mathbb{K}}(S) \oplus [n_2^l, n_2^l] := V \oplus W.
\]

This decomposition allows us to define the following closed subgroups of \text{Aut}(n_2^l):

\[
T = \{f \in \text{Aut}(n_2^l) \mid f(T) = T\}
\]

\[
U = \{f \in \text{Aut}(n_2^l) \mid \forall v \in n_2^l : f(v) - v \in W\}.
\]
As shown in [7], we have that Aut\((n^L_D)\) = \(UT\) and moreover since the intersection \(U \cap T\) is trivial and \(T\) normalizes \(U\), there is an isomorphism \(U \times T \cong Aut(n^L_D) : (u, t) \mapsto ut\). Let
\[
p : GL(n^L_D) \rightarrow GL(V) \cong GL(n^L_D/W)
\]
be the natural projection, then \(p : T \rightarrow p(T)\) is an isomorphism of linear algebraic groups and the image \(p(T)\) contains \(D_S\). Therefore we can apply the results from section 3 to \(G = p(T)\).

The relation \(\prec\) on the set of vertices \(S\) is completely determined by the graph \(G = (S, E)\) in the following way. To each vertex \(v \in S\), we can associate two sets
\[
\Omega'(s) = \{t \in S \mid \{t, s\} \in E\} \quad \text{and} \quad \Omega(s) = \Omega'(s) \cup \{s\},
\]
which are called the open and closed neighbourhoods of \(s\), respectively. The relation \(\prec\) from section 3 then satisfies the equivalence: \(\alpha \prec \beta \iff \Omega'(\alpha) \subset \Omega(\beta)\). Note that the subgraph spanned by a coherent component \(\lambda \subset S\) is either an empty graph or a complete graph. Also, between two different coherent components, either all edges are present or no edges are present. This gives rise to the notion of a reduced graph.

**Definition 4.1.** A simple undirected vertex-weighted graph with loops is a triple \((S, E, \Phi)\) where \(S\) is a set, \(E\) is a subset of \(\{\{s, t\} \mid s, t \in S\}\) and \(\Phi : S \rightarrow \mathbb{R}\) is a map to \(\mathbb{R}\). Its automorphism group is defined as
\[
Aut(S, E, \Phi) = \{\varphi \in \text{Perm}(S) \mid e \in E \iff \varphi(e) \in E, \ \Phi \circ \varphi = \Phi\}.
\]

We say \((S, E, \Phi)\) is empty if \(E = \emptyset\).

Let \(G = (S, E)\) be a simple undirected graph and let \(\Lambda\) denote its set of coherent components, then associated to \(G\) is a simple undirected vertex-weighted graph with loops \(G_{\text{red}}\) which we will call the reduced graph of \(G\). It is defined as \(G_{\text{red}} = (\Lambda, \mathcal{E}, \Phi)\) with
\[
\mathcal{E} := \{\{\lambda, \mu\} \mid \exists s \in \lambda, \exists t \in \mu : \{s, t\} \in E\}
\]
and \(\Phi : \Lambda \rightarrow \mathbb{R} : \lambda \mapsto |\lambda|\). Note that \(G\) is the empty graph if and only if \(G_{\text{red}}\) is the empty graph.

**Example 4.2.** We will make a visual representation of the reduced graph by simply putting the values of the weights near every vertex and drawing a loop at those \(\lambda\) for which \(\{\lambda\} \in \mathcal{E}\). Below we have drawn a concrete example.

![Graph](image)

4.2 Classification of \(K\)-forms of \(n^L_D\)

Let \(L/K\) be a Galois extension, \(G = (S, E)\) a simple undirected graph and \(n^L_D\) the associated 2-step nilpotent Lie algebra. Let \(T\) and \(U\) be the subgroups of \(Aut(n^L_D)\) as defined by [8]. As discussed in the previous part, \(G = p(T) \subset GL(V)\) is linear algebraic group containing \(D_S\). In section 3 we associated to such \(G\) the finite groups \(F\) and \(\overline{F}\). In [7], it was proven that for \(G = p(T)\), these groups \(F\) and \(\overline{F}\) are equal to the groups \(\text{Aut}(G)\) and \(\text{Aut}(G_{\text{red}})\), respectively. From our discussion in section 3 we thus have
\[
p(T) = M \left( \prod_{\lambda \in \Lambda} GL(V_\lambda) \right) \overline{F}(\text{Aut}(G_{\text{red}})).
\]

Recall that \(p|_T : T \rightarrow p(T)\) is a group isomorphism, and thus we can define an injective group morphism
\[
i : \text{Aut}(G_{\text{red}}) \rightarrow \text{Aut}(n^L_D) : \varphi \mapsto ((p|_T)^{-1} \circ \overline{F})(\varphi).
\]
As one can check, for any $\alpha \in S$ we have $i(\varphi)(\alpha) = r(\varphi)(\alpha)$ where $r$ is the chosen right-splitting morphism in the short exact sequence \((\ref{eq:short_exact})\) and for any $\alpha, \beta \in S$ we have $i(\varphi)([\alpha, \beta]) = [r(\varphi)(\alpha), r(\varphi)(\beta)]$. This completely determines $i(\varphi)$ as an automorphism of $\mathfrak{g}_L^L$.

We can also define a morphism

$$\pi : \text{Aut}(\mathfrak{g}_L^L) \to \text{Aut}(\mathfrak{g}_{\text{red}}) : f \mapsto \pi(f)$$ (21)

where $\pi(f)$ is the unique element in $\text{Aut}(\mathfrak{g}_{\text{red}})$ such that $p(f) = m \cdot \mathcal{P}(\pi(f))$ where $m \in M$ and $A \in \prod_{\lambda \in \Lambda} \text{GL}(V_\lambda)$. It is immediate that $\pi \circ i$ is equal to the identity on $\text{Aut}(\mathfrak{g}_{\text{red}})$. Moreover, both $i$ and $\pi$ are $\text{Gal}(L/K)$-equivariant maps for the trivial $\text{Gal}(L/K)$-action on $\text{Aut}(\mathfrak{g}_{\text{red}})$. We now show that these maps induce bijection on the first Galois cohomology set.

**Proposition 4.3.** Let $i, \pi$ be the morphisms from above, then the induced maps on cohomology

$$H^1(L/K, \text{Aut}(\mathfrak{g}_{\text{red}})) \xrightarrow{i_*} H^1(L/K, \text{Aut}(\mathfrak{g}_L^L))$$ (22)

are bijections and each others inverses.

**Proof.** Note that we have the commutative diagram of $\text{Gal}(L/K)$-equivariant group morphisms

$$
\begin{array}{ccc}
i(\text{Aut}(\mathfrak{g}_{\text{red}})) & \longrightarrow & T \\
| & | \\
\mathcal{P}(\text{Aut}(\mathfrak{g}_{\text{red}})) & \longrightarrow & p(T).
\end{array}
$$ (23)

The vertical maps are bijections and as proven in Theorem 3.3, the lower horizontal map induces a bijection on cohomology. By consequence we also have a bijection $H^1(L/K, i(\text{Aut}(\mathfrak{g}_{\text{red}}))) \to H^1(L/K, T)$ induced by the inclusion $i(\text{Aut}(\mathfrak{g}_{\text{red}})) \to T$.

Because $i(\text{Aut}(\mathfrak{g}_{\text{red}}))$ is a subgroup of $T$ and $T$ normalizes $U$, so does $i(\text{Aut}(\mathfrak{g}_{\text{red}}))$. By consequence we can apply Lemma 2.11 to the subgroups $A = i(\text{Aut}(\mathfrak{g}_{\text{red}}))$, $B = (p|T)^{-1} \left( M \prod_{\lambda \in \Lambda} \text{GL}(V_\lambda) \right)$ and $C = U$ which gives the diagram of inclusions:

$$i(\text{Aut}(\mathfrak{g}_{\text{red}})) \longrightarrow T \longrightarrow U \longleftarrow i(\text{Aut}(\mathfrak{g}_{\text{red}})) \longrightarrow \text{Aut}(\mathfrak{g}_L^L).$$

Since $i(\text{Aut}(\mathfrak{g}_{\text{red}})) \to T$ induces a bijection on cohomology, Lemma 2.11 tells us that also the inclusion $U \longleftarrow i(\text{Aut}(\mathfrak{g}_{\text{red}})) \to \text{Aut}(\mathfrak{g}_L^L)$ induces a bijection on cohomology.

Next, we prove that the inclusion $i(\text{Aut}(\mathfrak{g}_{\text{red}})) \to U$ induces a bijection on cohomology. Note that $U$ as a group is isomorphic to the additive group $\text{End}(V, W)$. An explicit morphism $\psi : \text{End}(V, W) \to U$ can be defined by letting $\psi(f)(v) = f(v)$ for all $v \in V$ and $\psi(f)(w) = 0$ for all $w \in W$. The group $\text{End}(V, W)$ is isomorphic to a direct product of $\dim(V) \cdot \dim(W)$ copies of the additive group $\mathbb{Z}$. As one can check, the $\text{Gal}(L/K)$-action on $\text{End}(V, W)$ inherited via $\psi$ from the one on $U$, is given by the component-wise action of $\text{Gal}(L/K)$ on each $\mathbb{Z}$. The action of $i(\text{Aut}(\mathfrak{g}_{\text{red}}))$ on $U$ by conjugation permutes the $\mathbb{Z}$-components of $\text{End}(V, W)$ and thus $(\mathbb{Z})^{\dim V \cdot \dim W} \times \text{Aut}(\mathfrak{g}_{\text{red}})$ is actually a wreath product $L_a \wr \text{Aut}(\mathfrak{g}_{\text{red}})$.

Now we use Theorem 2.9 to apply Theorem 2.14 to $U \longleftarrow i(\text{Aut}(\mathfrak{g}_{\text{red}})) \cong U \times \text{Aut}(\mathfrak{g}_{\text{red}}) \cong (L_a)^{\dim V \cdot \dim W} \times \text{Aut}(\mathfrak{g}_{\text{red}})$ and we find that the map

$$H^1(L/K, i(\text{Aut}(\mathfrak{g}_{\text{red}}))) \to H^1(L/K, U \longleftarrow i(\text{Aut}(\mathfrak{g}_{\text{red}})))$$

induced by the inclusion is a bijection. Composing this map with

$$H^1(L/K, U \longleftarrow i(\text{Aut}(\mathfrak{g}_{\text{red}}))) \to H^1(L/K, \text{Aut}(\mathfrak{g}_L^L)),$$
which we already showed to be a bijection, yields the bijection
\[ H^1(L/K, i(\text{Aut}(G_{\text{red}}))) \rightarrow H^1(L/K, \text{Aut}(n_{\rho}^K)). \]

At last, since \( i : \text{Aut}(G_{\text{red}}) \rightarrow i(\text{Aut}(G_{\text{red}})) \) is a \( \text{Gal}(L/K) \)-equivariant group isomorphism, we get a bijection on cohomology \( i_* : H^1(L/K, \text{Aut}(G_{\text{red}})) \rightarrow H^1(L/K, i(\text{Aut}(G_{\text{red}}))) \) and thus also a bijection
\[
i_* : H^1(L/K, \text{Aut}(G_{\text{red}})) \rightarrow H^1(L/K, \text{Aut}(n_{\rho}^K)) : [\rho] \mapsto [i \circ \rho].
\]
Since \( \pi \circ i \) is the identity on \( \text{Aut}(G_{\text{red}}) \), it follows that \( \pi_* \circ i_* \) is the identity on \( H^1(L/K, \text{Aut}(G_{\text{red}})) \). As we just proved that \( i_* \) is a bijection, we get that \( \pi_* \) is the inverse of \( i_* \). This completes the proof. \qed

Note that combining the proposition above with Theorem 2.16 we get the following result:

**Theorem 4.4.** Let \( L/K \) be a Galois extension and \( G \) a simple undirected graph. The map
\[
H^1(L/K, \text{Aut}(G_{\text{red}})) \rightarrow \mathcal{F}_K(n_{\rho}^K) : [\rho] \mapsto [n_{\rho, i, \rho}^K]
\]
is a bijection.

For notational purposes, we will from now on simply write \( n_{\rho, i, \rho}^K \) for the \( K \)-form \( n_{\rho}^K \). Recall from equation (1), that this form is given by
\[
n_{\rho, i, \rho}^K = \{ v \in n_{\rho}^K \mid \forall \sigma \in \text{Gal}(L/K) : i(\rho)(\sigma v) = v \}.
\]
The bijection above gives a full classification of all \( K \)-forms of \( n_{\rho}^K \) since the set \( H^1(L/K, \text{Aut}(G_{\text{red}})) \) is relatively easy to understand. It is equal to the set of equivalence classes of actions of \( \text{Gal}(L/K) \) on \( G_{\text{red}} \) by automorphisms of the reduced graph, where two actions are equivalent if they are conjugated by a fixed automorphism of \( G_{\text{red}} \). We use this observation in the next section to prove Theorem A.

**4.3 Rational forms of \( n_{\rho}^C \) and \( n_{\rho}^R \)**

So far we have classified all \( K \)-forms of in \( n_{\rho}^K \) for any simple undirected graph \( G \) where \( L/K \) is a Galois extension of subfields of \( \mathbb{C} \). If we want to apply this for \( K = \mathbb{Q} \), we need \( L \) to be an algebraic extension of \( \mathbb{Q} \). This excludes the cases \( L = \mathbb{R} \) or \( L = \mathbb{C} \). In this section we will resolve this and give a description of the rational forms of \( n_{\rho}^R \) and \( n_{\rho}^C \) for any simple undirected graph \( G \).

Let us write \( \tau \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) for the complex conjugation automorphism on \( \overline{\mathbb{Q}} \).

**Theorem 4.5.** Let \( G \) be a simple undirected graph. We have bijections
\[
H^1(\overline{\mathbb{Q}}/\mathbb{Q}, \text{Aut}(G_{\text{red}})) \rightarrow \mathcal{F}_Q(n_{\rho}^C) : [\rho] \mapsto [n_{\rho, \tau, \rho}^C] \quad (24)
\]
and
\[
\left\{ [\rho] \in H^1(\overline{\mathbb{Q}}/\mathbb{Q}, \text{Aut}(G_{\text{red}})) \mid \tau \in \ker(\rho) \right\} \rightarrow \mathcal{F}_Q(n_{\rho}^R) : [\rho] \mapsto [n_{\rho, \tau, \rho}^R]. \quad (25)
\]

**Proof.** The first bijection follows immediately from combining Theorem 4.4 with Proposition 2.10 or with the bijection derived in 2. For the second one, we will apply the equivalence given in 4.

If we view \( \overline{\mathbb{Q}} \) as a subset of \( \mathbb{C} \), we get a continuous morphism \( \nu : \text{Gal}(\mathbb{C}/\mathbb{R}) \rightarrow \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) : \sigma \mapsto \sigma|_{\overline{\mathbb{Q}}} \). This allows us to define the maps \( \omega_1 \) and \( \omega_2 \) with domain and codomain as in the diagram below and which send \( [\rho] \) to \( [\rho \circ \nu] \). Note that we have also two maps \( i_* \) in the diagram, induced by the map \( i : \text{Aut}(G_{\text{red}}) \rightarrow \text{Aut}(n_{\rho}^K) \) as defined in 20 for a general field \( L \).

\[
\begin{array}{ccc}
H^1(\overline{\mathbb{Q}}/\mathbb{Q}, \text{Aut}(G_{\text{red}})) & \xrightarrow{\omega_1} & H^1(\mathbb{C}/\mathbb{R}, \text{Aut}(G_{\text{red}})) \\
\downarrow i_* & & \downarrow i_* \\
H^1(\overline{\mathbb{Q}}/\mathbb{Q}, \text{Aut}(n_{\rho}^K)) & \xrightarrow{\omega_2} & H^1(\mathbb{C}/\mathbb{R}, \text{Aut}(n_{\rho}^K))
\end{array}
\]
The diagram is commutative as $i \circ (\rho \circ \nu) = (i \circ \rho) \circ \nu$ on the level of representatives. From the equivalence in \cite{Emery}, we know that the classes in $H^1(\mathbb{C}/\mathbb{C}, \text{Aut}(\mathbb{C}))$ that give a rational form of $\mathbb{C}_G^\nu$ are exactly the classes in the inverse image of the class of the trivial cocycle under $\omega_2$. Now since the diagram above commutes and since by Proposition \ref{prop:primes}, the induced maps $i_\nu$ are bijections, we get that those classes $[\rho] \in \omega_2^{-1}([1])$ are exactly the ones for which $[n_{G,\rho}^Q] \in F_Q(n_G^2)$. Since we have that

$$[\rho] \in \omega_2^{-1}([1]) \iff [\rho \circ \nu] = [1]$$

$$\iff \exists \phi \in \text{Aut}(\mathcal{G}_{\text{red}}) : \forall \sigma \in \text{Gal}(\mathbb{C}/\mathbb{R}) : \rho_{\sigma(\nu)} = \phi \text{Id}_A \varphi^{-1}$$

$$\iff \rho_\tau = 1$$

$$\iff \tau \in \ker(\rho),$$

this proves that the second map is a bijection.

We are now ready to prove one of the main results of our paper.

**Proof of Theorem 4.1** Take any rational form of $n_G^C$. Then it follows from Theorem 1.4 that up to $\mathbb{Q}$-isomorphism, the form is given by $n_{G,\rho}^Q$ for some continuous morphism $\rho : \text{Gal}((\overline{\mathbb{Q}})/\mathbb{Q}) \to \text{Aut}(\mathcal{G}_{\text{red}})$. Note that $\ker(\rho)$ is an open normal subgroup of $\text{Gal}((\overline{\mathbb{Q}})/\mathbb{Q})$. By Theorem 2.2 we get that $L_\rho := \overline{\mathbb{Q}}^{\ker(\rho)}$ is a finite degree Galois extension of $\mathbb{Q}$ together with a natural isomorphism of groups

$$\text{Gal}((\overline{\mathbb{Q}})/\mathbb{Q})/\ker(\rho) \to \text{Gal}(L_\rho/\mathbb{Q}) : \sigma \ker(\rho) \mapsto \sigma|_{L_\rho}.$$ 

We therefore get an induced injective morphism of groups

$$\overline{\rho} : \text{Gal}(L_\rho/\mathbb{Q}) \cong \text{Gal}((\overline{\mathbb{Q}})/\mathbb{Q})/\ker(\rho) \to \text{Aut}(\mathcal{G}_{\text{red}})$$

which gives a class $[\overline{\rho}] \in H^1(L_\rho/\mathbb{Q}, \text{Aut}(\mathcal{G}_{\text{red}}))$. Note that we have a natural injection $n_{G,\rho}^{L_\rho} \hookrightarrow n_{G,\rho}^{\overline{\rho}}$ and that this injection restricts to a $\mathbb{Q}$-Lie algebra isomorphism $n_{G,\rho}^{L_\rho} \cong n_{G,\rho}^{\overline{\rho}}$. Following Theorem 4.5 we also have that $[n_{G,\rho}^Q] \in F_Q(n_G^2)$ if and only if $\tau \in \ker(\rho)$ and thus if and only if $L_\rho$ is a real extension of $\mathbb{Q}$.

For the second statement, if $\eta : \text{Gal}((\overline{\mathbb{Q}})/\mathbb{Q}) \to \text{Aut}(\mathcal{G}_{\text{red}})$ is another continuous morphism, we have the equivalences

$$n_{G,\overline{\rho}}^Q \cong n_{G,\overline{\rho}}^{\overline{\rho}} \iff n_{G,\rho}^Q \cong n_{G,\eta}^Q$$

$$\iff [\rho] = [\eta]$$

$$\iff \ker(\rho) = \ker(\eta) \text{ and } [\overline{\rho}] = [\overline{\eta}]$$

$$\iff L_\rho = L_\eta \text{ and } [\overline{\rho}] = [\overline{\eta}].$$

At last, note that if $L'/\mathbb{Q}$ is any finite Galois extension and $\rho' : \text{Gal}(L'/\mathbb{Q}) \to \text{Aut}(\mathcal{G}_{\text{red}})$ is any injective group morphism, we can define the continuous morphism $\rho : \text{Gal}((\overline{\mathbb{Q}})/\mathbb{Q}) \to \text{Aut}(\mathcal{G}_{\text{red}}) : \sigma \mapsto \rho'|_{L'}$, which now satisfies $\overline{\rho} = \rho'$. This shows all that needed to be proven.

**4.4 Application: number of different $\mathbb{Q}$-forms**

We apply Theorem 4.1 to show that the Lie algebras $n_G^2$ and $n_G^C$ have either exactly one or infinitely many rational forms. In order to prove this, we need a lemma that ensures the existence of enough non-isomorphic cyclic Galois extensions of a certain prime degree.

**Lemma 4.6.** For every prime $p$, there exist infinitely many real Galois extensions $L_i/\mathbb{Q}$ with $i \in \mathbb{N}$ such that $\text{Gal}(L_i/\mathbb{Q}) \cong \mathbb{Z}/p\mathbb{Z}$ and $L_i \cap L_j = \mathbb{Q}$ for all $i, j \in \mathbb{N}$ with $i \neq j$.

**Proof.** By Dirichlet’s theorem \cite{Dirichlet}, there are infinitely many different primes $q_i$ for $i \in \mathbb{N}$ such that $q_i = 1 \mod 2p$ for all $i \in \mathbb{N}$. If we denote by $\zeta_k = e^{2\pi i/k}$ the primitive $k$-th root of unity, we can define the cyclotomic field extensions $K_i = \mathbb{Q}(\zeta_{q_i})$, for which the Galois group $\text{Gal}(K_i/\mathbb{Q})$ is cyclic of order $q_i - 1$. Since $2p \mid q_i - 1$, there exists a (unique) cyclic subgroup $H_i \subset \text{Gal}(K_i/\mathbb{Q})$ of order $(q_i - 1)/2p$. Let $K_i^{H_i}$ denote
the field which is fixed under $H_i$. Since $\text{Gal}(K_i/\mathbb{Q})$ is abelian, $H_i$ is a normal subgroup and thus we have $\text{Gal}(K_i^H_i/\mathbb{Q}) \cong \frac{\text{Gal}(K_i/\mathbb{Q})}{\text{Gal}(K_i/\mathbb{Q})} \cong \mathbb{Z}/2p\mathbb{Z}$.

Note that $\text{Gal}(K_i^H_i/\mathbb{Q})$ has a unique element $\sigma$ of order 2 and in case $K_i^H_i$ is not totally real, this must be the complex conjugation automorphism. Let $L_i$ be the subfield of $K_i^H_i$ which is fixed by $\{1, \sigma\}$. As before we have that $\{1, \sigma\}$ is a normal subgroup of $\text{Gal}(K_i^H_i/\mathbb{Q})$ and thus that $\text{Gal}(L_i/\mathbb{Q}) \cong \frac{\text{Gal}(K_i^H_i/\mathbb{Q})}{\{1, \sigma\}} \cong \mathbb{Z}/p\mathbb{Z}$. Note that, even if $K_i$ was not totally real, the fields $L_i$ must be real since they are fixed by complex conjugation. We have thus constructed infinitely many real Galois extensions $L_i/\mathbb{Q}$, $i \in \mathbb{N}$ such that $\text{Gal}(L_i/\mathbb{Q}) \cong \mathbb{Z}/p\mathbb{Z}$. Moreover since for $i \neq j$, the primes $q_i$ and $q_j$ are different, we know that $K_i \cap K_j = \mathbb{Q}((\zeta_{q_i} \cap \mathbb{Q}((\zeta_{q_j} = \mathbb{Q}((\zeta_{\text{gcd}(q_i, q_j)}) = \mathbb{Q}$. By consequence also $L_i \cap L_j = \mathbb{Q}$.

**Theorem 4.7.** The Lie algebras $\mathfrak{n}^R_i$ and $\mathfrak{n}^C_i$ associated to a simple undirected graph $\mathcal{G}$ have either exactly one or infinitely many rational forms up to $\mathbb{Q}$-isomorphism. The former is true if and only if $\text{Aut}(\mathcal{G}_{\text{red}})$ is trivial.

**Proof.** If $\text{Aut}(\mathcal{G}_{\text{red}})$ is trivial, then clearly $H^1(\mathbb{Q} \cap \mathbb{R}, \mathbb{Q}, \text{Aut}(\mathcal{G}_{\text{red}}))$ is trivial as well which implies by the discussion above that both $\mathcal{F}_0(n^R)$ and $\mathcal{F}_0(n^C)$ count only one element. So from now on we assume that $\text{Aut}(\mathcal{G}_{\text{red}})$ is non-trivial and show that there infinitely many rational forms.

Since $\text{Aut}(\mathcal{G}_{\text{red}})$ is not trivial, there exists an element $\varphi \in \text{Aut}(\mathcal{G}_{\text{red}})$ of prime order $p$. Let $L_i$ with $i \in \mathbb{N}$ be finite Galois extensions of $\mathbb{Q}$ with Galois group $\mathbb{Z}/p\mathbb{Z}$ as in Lemma 4.6. Choose for all $i \in \mathbb{N}$ a generator $\sigma_i \in \text{Gal}(L_i/\mathbb{Q})$ and define the injective morphisms

$$\rho_i : \text{Gal}(L_i/\mathbb{Q}) \to \text{Aut}(\mathcal{G}_{\text{red}}) : \sigma_i^k \mapsto \varphi^k.$$ 

The fields $L_i$ are all different and hence the corresponding rational forms are non-isomorphic by Theorem 4.1. Since each $L_i$ is a real field, complex conjugation lies in the kernel of each $\rho_i$ and thus $\mathfrak{n}^R_{\rho_i}$ is a rational form of $\mathfrak{n}^R_i$ for all $i \in \mathbb{N}$. This proves that $\mathcal{F}_0(\mathfrak{n}^R_i)$ counts infinitely many elements. Because we have an injection $\mathcal{F}_0(\mathfrak{n}^R_i) \to \mathcal{F}_0(\mathfrak{n}^C_i) : [m^R] \mapsto [m^C]$, this proves as well that $\mathcal{F}_0(\mathfrak{n}^C_i)$ counts infinitely many elements.

As a consequence, we present a family of graphs such that the corresponding real and complex Lie algebras have a unique rational form.

**Example 4.8.** Let $p, q$ be two non-negative integers with $q > 1$. Take two disjoint sets $S_1$ and $S_2$ which have cardinalities $p$ and $q$, respectively. We can define a simple undirected graph $\mathcal{G} = (S_1 \cup S_2, E)$ (where $E = \{(\alpha, \beta) \mid \alpha \in S_1, \beta \in S_2\} \cup \{(\alpha, \beta) \mid \alpha, \beta \in S_1, \alpha \neq \beta\}$). These type of graphs are called *magnet graphs*.

We say $S_1$ is the core of $\mathcal{G}$. The reduced graph of $\mathcal{G}$ is equal to

$$p \quad \circ \quad q$$

From this it is clear that $\text{Aut}(\mathcal{G}_{\text{red}})$ is trivial and thus that the Lie algebras over $\mathbb{R}$ (or $\mathbb{C}$) which are associated to magnet graphs have only one rational form up to $\mathbb{Q}$-isomorphism. Note that these graphs were also considered in [2] in the study of Anosov diffeomorphisms.

Theorem 4.7 raises the question whether it holds for all real and complex Lie algebras. The authors do not know any example of a Lie algebra having at least two non-isomorphic rational forms, but only a finite number of them.

## 5 Anosov diffeomorphisms on nilmanifolds associated to a graph

In the final part of this paper, we apply Theorem 4.1 to find all rational forms of a Lie algebra associated to a graph that are Anosov. In order to do so, we first recall the most important results about Anosov diffeomorphisms on nilmanifolds.
5.1 Background

Let \( N/\Gamma \) be a nilmanifold, i.e. the quotient space of a simply connected nilpotent Lie group \( N \) by a cocompact discrete subgroup \( \Gamma \subset N \). In particular, if \( N = \mathbb{R}^n \) is abelian, then a nilmanifold is exactly the \( n \)-dimensional torus \( \mathbb{R}^n/\mathbb{Z}^n \). If \( n^\mathbb{R} \) denotes the (real) Lie algebra corresponding to \( N \), then \( \Gamma \) determines a rational form of \( n^\mathbb{R} \) given by \( n^\mathbb{Q} = \text{span}_\mathbb{Q}(\log(\Gamma)) \subset n^\mathbb{R} \), see [18]. Moreover, every rational form \( n^\mathbb{Q} \) corresponds to at least one lattice \( \Gamma \) of \( N \). Note that this correspondence is not faithful, there are many examples of non-isomorphic lattices which give rise to isomorphic rational forms, for example in the Heisenberg group. It is known that for two lattices \( \Gamma_1, \Gamma_2 \leq N \) the rational forms \( n^\mathbb{Q}_{\Gamma_1} \) and \( n^\mathbb{Q}_{\Gamma_2} \) are isomorphic if and only if \( \Gamma_1 \) and \( \Gamma_2 \) are abstractly commensurable by the work of Malcev [16].

By combining the results [3, Corollary 3.5] and [17, Theorem C], it follows that the existence of an Anosov diffeomorphism on \( N/\Gamma \) only depends on the isomorphism class of \( n^\mathbb{Q}_{\Gamma} \). The exact characterisation on the Lie algebra motivates the following definition, but for this we still need some terminology. An invertible linear map on a rational vector space is called hyperbolic if all its eigenvalues (over \( \mathbb{C} \)) have modulus different from 1 and is called integer-like if its characteristic polynomial has integer coefficients and constant term equal to 1 or \(-1\).

**Definition 5.1.** Let \( n^\mathbb{Q} \) be a rational Lie algebra. We call a hyperbolic inter-like automorphism of \( n^\mathbb{Q} \) an Anosov automorphism. A rational Lie algebra which admits an Anosov automorphism will be called Anosov.

By the results in [3] and [17], the nilpotent Anosov rational Lie algebras are then exactly the ones coming from a nilmanifold admitting an Anosov diffeomorphism.

Since it is conjectured that every closed manifold admitting an Anosov diffeomorphism is finitely covered by a manifold homeomorphic to a nilmanifold, Anosov Lie algebras have a long history in the literature, see for example the references in [8]. One instance is the classification of Anosov Lie algebras in low dimensions in [15, 12, 13, 14], which contains some families of real Lie algebras containing several non-isomorphic rational forms. Another example is the case of free nilpotent Lie algebras, treated in [3, 5], where these have only rational form up to isomorphism. For 2-step nilpotent Lie algebras associated to graphs, the paper [2] gives an answer for one specific type of rational forms. In general, there are only very few examples of real Lie algebras having more than 1 rational form and for which the ones with an Anosov automorphism are classified.

In what follows, we will answer this question for the rational forms of a real (or complex) 2-step nilpotent Lie algebra associated to a graph \( \Gamma \) and determine which ones are Anosov. From Theorem A we know that every rational form is equal to \( n^\mathbb{Q}_{\Gamma, \rho} \) with \( \rho : \text{Gal}(L, \mathbb{Q}) \to \text{Aut}(\Gamma_{\text{red}}) \) an injective group morphism. The eigenvalues of the Anosov automorphisms we consider always lie in the field \( \mathbb{Q} \), but not necessarily in the finite field extension \( L \) corresponding to the rational form. Hence it will be more convenient to keep on working with \( \mathbb{Q} \) and the extension of the map \( \rho \) to \( \text{Gal}(\mathbb{Q}/\mathbb{Q}) \) to simplify notations.

5.2 Reduction to algebraic integers

Before proving Theorem B we first need a technical result which relates the existence of an Anosov automorphism to having certain algebraic integers for every vertex of our graph. We first introduce some notation.

Let \( \overline{\mathbb{Q}} \) be the algebraic closure of \( \mathbb{Q} \) and \( \Gamma = (S, E) \) a simple undirected graph. We write \( \overline{\mathbb{Q}}^S \) for the set of maps from \( S \) to \( \overline{\mathbb{Q}} \). Note that \( \prod_{\lambda \in \Lambda} \text{Perm}(\lambda) \) has an action on \( \overline{\mathbb{Q}}^S \) by \( \theta : \Psi := \Psi \circ \theta^{-1} \) for all \( \theta \in \prod_{\lambda \in \Lambda} \text{Perm}(\lambda) \) and \( \Psi \in \overline{\mathbb{Q}}^S \). Let us write \( \overline{\mathcal{H}}^S_G \) for the orbit space \( \overline{\mathcal{H}}^S_G := \left( \prod_{\lambda \in \Lambda} \text{Perm}(\lambda) \right) / \overline{\mathbb{Q}}^S \).

Note that the groups \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) and \( \text{Aut}(\Gamma_{\text{red}}) \) have a well-defined left, respectively right action on \( \overline{\mathcal{H}}^S_G \) by

\[
\sigma \cdot [\Psi] := [\sigma \circ \Psi] \quad [\Psi] \cdot \varphi := [\Psi \circ r(\varphi)]
\]
for all $\sigma \in \text{Gal}(\mathbb{Q}/K)$, $\varphi \in \text{Aut}(G_{\text{red}})$ and $\Psi \in \mathbb{Q}^S$. Recall that $r$ denotes the chosen morphisms to make the exact sequence in \((9)\) right-split. To see that the action by $\text{Aut}(G_{\text{red}})$ is well defined, one uses the fact that $\prod_{\lambda \in \Lambda} \text{Perm}(\lambda)$ is a normal subgroup of $\text{Aut}(G)$.

**Theorem 5.2.** Let $G = (S, E)$ be a simple undirected graph and $\rho : \text{Gal}(\mathbb{Q}/\mathbb{Q}) \to \text{Aut}(G_{\text{red}})$ a continuous morphism. The rational form $n_{\rho}^G$ is Anosov if and only if there exists a map $\Psi : S \to \mathbb{Q}$ such that $\Psi(\alpha)$ is a hyperbolic algebraic integer for any vertex $\alpha \in S$, for all $\sigma \in \text{Gal}(\mathbb{Q}/\mathbb{Q})$ it holds that in $H_{\Psi}^G$:

$$\sigma \cdot [\Psi] = [\Psi] \cdot \rho_{\sigma}$$

and for any edge $\{\alpha, \beta\} \in E$ we have $|\Psi(\alpha)\Psi(\beta)| \neq 1$.

Note that the conditions in this theorem do not depend on the choice of representative $\Psi$.

**Proof.** First assume that $n_{\rho}^G \subseteq n_{\Psi}^G$ is Anosov with Anosov automorphism $f : n_{\rho}^G \to n_{\Psi}^G$. Recall that $p$, $i$ and $\pi$ are the morphisms as defined by \([19]\), \([20]\) and \([21]\), with

$$G = p(T) = M \prod_{\lambda \in \Lambda} \text{GL}(V_{\lambda}) \overline{P}(\text{Aut}(G_{\text{red}})).$$

The semi-simple part of $f$ is again Anosov automorphism of $n_{\rho}^G$ and thus we can assume without loss of generality that $f$ is semi-simple. Since $n_{\rho}^G$ is a rational form of $n_{\Psi}^G$, we can naturally extend $f$ to an automorphism of $n_{\Psi}^G$. It follows that $f$ lies in some maximal torus of $\text{Aut}(n_{\Psi}^G)$. We know as well that the subgroup of automorphisms on $n_{\Psi}^G$ which are diagonal on the vertices $S$, i.e. the set $p^{-1}(D_S) \cap T$, is a maximal torus of $\text{Aut}(n_{\Psi}^G)$. Since $\text{Aut}(n_{\Psi}^G)$ is a linear algebraic group over an algebraically closed field, all its maximal tori are conjugate and thus there exists an $h \in \text{Aut}(n_{\Psi}^G)$ and an $\tilde{f} \in p^{-1}(D_S) \cap T$ such that $h \tilde{f} h^{-1} = f$. Moreover, since $i(\text{Aut}(G_{\text{red}}))$ normalizes $p^{-1}(D_S) \cap T$, we can assume that $\pi(h) = 1$.

Let us define $\Psi : S \to \mathbb{Q}$ by assigning to a vertex $\alpha \in S$ its corresponding eigenvalue under $\tilde{f}$. Since $\tilde{f}$ is an Anosov automorphism, it follows that $\Psi(\alpha)$ is a hyperbolic algebraic unit for all $\alpha \in S$. Moreover, for any $\{\alpha, \beta\} \in E$, we have that $[\alpha, \beta] \neq 0$ is also an eigenvector of $\tilde{f}$ with eigenvalue $\Psi(\alpha)\Psi(\beta)$. This implies that $|\Psi(\alpha)\Psi(\beta)| \neq 1$ for any edge $\{\alpha, \beta\} \in E$.

The only thing left to prove now is that equation \([28]\) holds. Since $f\left(n_{\rho}^G\right) = n_{\Psi}^G$, it follows from Remark \([27]\) that $f i(\rho_{\sigma}) = i(\rho_{\sigma})^\sigma f$ for all $\sigma \in \text{Gal}(\mathbb{Q}/\mathbb{Q})$. Substituting $h \tilde{f} h^{-1}$ for $f$, we find the equality

$$\tilde{f} a_{\sigma} = a_{\sigma}^\sigma \tilde{f} \quad \text{where} \quad a_{\sigma} := h^{-1} i(\rho_{\sigma})^\sigma h.$$  

Note that $\pi(a_{\sigma}) = \pi(h^{-1})\pi(i(\rho_{\sigma}))\pi(h) = \pi(i(\rho_{\sigma})) = \rho_{\sigma}$, and hence $p(a_{\sigma}) = m_{\sigma} A_{\sigma} \overline{P}(\rho_{\sigma})$ for some unique $m_{\sigma} \in M, A_{\sigma} \in \prod_{\lambda \in \Lambda} \text{GL}(V_{\lambda})$. Applying the morphism $p : \text{Aut}(n_{\Psi}^G) \to G$ to equation \([28]\), we get that

$$p(\tilde{f}) m_{\sigma} A_{\sigma} \overline{P}(\rho_{\sigma}) = m_{\sigma} A_{\sigma} \overline{P}(\rho_{\sigma})^\sigma p(\tilde{f}).$$

Note that $p(\tilde{f})$ and $^\sigma p(\tilde{f})$ are elements in $\prod_{\lambda \in \Lambda} \text{GL}(V_{\lambda})$, since $\tilde{f}$ is diagonal on $S$. Rearranging the equation above to

$$\left(p(\tilde{f}) m_{\sigma} p(\tilde{f})^{-1}\right)\left(p(\tilde{f}) A_{\sigma}\right) = \left(m_{\sigma}\right)\left(A_{\sigma} \overline{P}(\rho_{\sigma})^\sigma p(\tilde{f}) \overline{P}(\rho_{\sigma})^{-1}\right)\left(\overline{P}(\rho_{\sigma})\right)$$

we can find the equality on $\prod_{\lambda \in \Lambda} \text{GL}(V_{\lambda})$ to be

$$A_{\sigma}^{-1} p(\tilde{f}) A_{\sigma} = \overline{P}(\rho_{\sigma})^\sigma p(\tilde{f}) \overline{P}(\rho_{\sigma})^{-1}$$

Since both sides are elements of $\prod_{\lambda \in \Lambda} \text{GL}(V_{\lambda})$, we can look at their projection onto $\text{GL}(V_{\lambda})$ for any $\lambda \in \Lambda$. We then find that $\tilde{f}|_{V_{\lambda}}$ and $^\sigma \left(\tilde{f}|_{V_{\rho_{\sigma}^{-1}(\lambda)}}\right)$ are similar linear maps. By consequence, their eigenvalues, counted
with multiplicities, coincide. This shows exactly that $[\Psi] = [\sigma \circ \Psi \circ r(\rho^{-1})]$ for any $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. This proves that $\Psi$ satisfies all the required properties.

Conversely, assume such a map $\Psi : S \to \overline{\mathbb{Q}}$ exists and consider the action of $\text{Aut}(G_{\text{red}})$ on $\Lambda$. Choose for each orbit $\mathcal{O}_i \subset \Lambda$ one element $\lambda_i \in \mathcal{O}_i$. Let $g_i(X) \in \overline{\mathbb{Q}}[X]$ be the polynomial defined by

$$g_i(X) = \prod_{\alpha \in \lambda_i} (X - \Psi(\alpha)).$$

Let us fix an action of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\overline{\mathbb{Q}}[X]$ by acting on the coefficients of the polynomials. As one can check this is an action by ring automorphisms. Take an arbitrary $\sigma \in \text{stab}_\mu(\lambda_i)$. By the assumption, we have that $[\sigma \circ \Psi] = [\Psi \circ r(\rho)]$. By consequence, there exists $\theta \in \prod_{\lambda \in \Lambda} \text{Ferm}(\lambda)$ such that $\sigma \circ \Psi = \Psi \circ r(\rho_\sigma) \circ \theta$. We now have that

$$\sigma g_i(X) = \prod_{\alpha \in \lambda_i} \sigma(X - \Psi(\alpha))$$

$$= \prod_{\alpha \in \lambda_i} (X - (\sigma \circ \Psi)(\alpha))$$

$$= \prod_{\alpha \in \lambda_i} (X - (\Psi \circ r(\rho_\sigma)(\alpha)(\alpha))$$

$$= \prod_{\alpha \in \lambda_i} (X - (\Psi \circ r(\rho_\sigma))(\alpha))$$

$$= \prod_{\alpha \in \rho_\sigma(\lambda_i)} (X - \Psi(\alpha))$$

$$= \prod_{\alpha \in \lambda_i} (X - \Psi(\alpha)) = g_i(X) \quad (29)$$

So the coefficients of $g_i(X)$ lie in the (finite) field extension $\overline{\mathbb{Q}}^{\text{stab}_\mu(\lambda_i)}/\mathbb{Q}$.

Next let $B_i : V_{\lambda_i} \to V_{\lambda_i}$ be the linear map given by the companion matrix of $g_i(X)$ in a basis of vertices of $V_{\lambda_i}$, where the order of the basis does not matter to us. Now define the linear map $A : V \to V$ by setting for $v \in V_{\mu}$ with $\mu \in \mathcal{O}_i$:

$$Av = i(\rho_\sigma)^{\tau} B_i i(\rho_\sigma)^{-1} v$$

where $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is chosen such that $\sigma(\lambda_i) = \mu$. Let us first show this is well-defined and independent of the choice of $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Say $\tau \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is another element which also satisfies $\tau(\lambda_i) = \mu$, then $\sigma^{-1} \tau \in \text{stab}_\mu(\lambda_i)$ and we get

$$i(\rho_\tau)^{\tau} B_i i(\rho_\tau)^{-1} v = i(\rho_\sigma) i(\rho_{\sigma^{-1} \tau})^{\sigma} = (\sigma^{-1} \tau B_i)i(\rho_{\sigma^{-1} \tau})^{-1} i(\rho_\tau)^{-1} v$$

since $i(\rho_{\sigma^{-1} \tau})|_{V_{\lambda_i}} = \text{Id}_{V_{\lambda_i}}$. We thus have a well-defined linear map $A : V \to V$ and as one can check $A \in \prod_{\lambda \in \Lambda} \text{GL}(V_{\lambda}) \subset p(T)$. This gives a unique automorphism $f \in T \subset \text{Aut}(n_{G,\rho})$ with $p(f) = A$.

We claim that $f$ induces an Anosov automorphism on $n_{G,\rho}$. To check that $f(n_{G,\rho}) = n_{G,\rho}$, we need to check that $i(\rho_\sigma)^{-1} f i(\rho_\sigma) = \sigma f$ for any $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Since both $f$ and $i(\rho_\sigma)$ lie in $T \subset \text{Aut}(n_{G,\rho})$, it suffices to check this on $V$. Take a $v \in V_{\mu}$ with $\mu \in \mathcal{O}_i$. Let $\tau \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be an element such that $\rho_\tau(\lambda_i) = \rho_\mu(\mu)$ or equivalently $\rho_{\sigma^{-1} \tau}(\lambda_i) = \mu$. Then we have

$$i(\rho_\sigma)^{-1} f i(\rho_\sigma)v = i(\rho_\sigma)^{-1} i(\rho_\tau)^{\tau} B_i i(\rho_\tau)^{-1} i(\rho_\sigma)v$$

$$= i(\rho_{\sigma^{-1} \tau})^{\sigma} (\sigma^{-1} \tau B_i)(\sigma^{-1} \tau i(\rho_{\sigma^{-1} \tau})v)$$

$$= \sigma (i(\rho_{\sigma^{-1} \tau})^{\sigma^{-1} \tau} B_i i(\rho_{\sigma^{-1} \tau})^{-1} v)$$

$$= \sigma f v.$$

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The eigenvalues of \( f \) on \( V \) all lie in the image of \( \Psi(S) \) and are therefore hyperbolic algebraic units. The eigenvalues of \( f \) on \( \mathbb{Q}(\xi_1, \ldots, \xi_n) \) are of the form \( \Phi(\alpha)\Phi(\beta) \) for any edge \( \{\alpha, \beta\} \in E \), which by assumption are hyperbolic algebraic integers as well. This proves that \( f \) induces an Anosov automorphism on \( \mathbb{Q}(\xi_1, \ldots, \xi_n) \).

\[ \square \]

### 5.3 Proof of Theorem B

In what follows, we give an easier to check condition for a rational form \( \mathbb{Q}(\xi_1, \ldots, \xi_n) \) to be Anosov. We recall that an algebraic unit \( \xi_1 \) with conjugates \( \xi_2, \ldots, \xi_n \) is called \( c \)-hyperbolic if and only if

\[
\forall k \in \{1, \ldots, c\} : \forall i_1, \ldots, i_k \in \{1, \ldots, n\} : |\xi_{i_1} \cdots \xi_{i_k}| \neq 1.
\]

The following result guarantees the existence of such algebraic units under certain conditions.

**Proposition 5.3.** Let \( L/\mathbb{Q} \) be a finite field extension of degree \( d \).

(i) If \( L \) is not totally imaginary, then there exists a \( c \)-hyperbolic algebraic unit in \( L \) for all \( 1 \leq c \leq (d-1) \).

(ii) If \( L \) is totally imaginary, then there exists a \( c \)-hyperbolic algebraic unit in \( L \) for all \( 1 \leq c \leq \frac{d}{2} - 1 \).

**Proof.** See [4, Proposition 3.6 and 3.7]. \( \square \)

In particular, since there exist field extensions of arbitrary degree, there exists a \( c \)-hyperbolic unit for every \( c > 0 \).

Any action of a group \( A \) on the reduced graph \( G_{\text{red}} \) by automorphisms is given by a group morphism \( \rho : A \to \text{Aut}(G_{\text{red}}) \). For any \( \lambda \in \Lambda \), we will denote the stabilizer and the orbit of \( \lambda \) for this action by \( \text{Stab}_\rho(\lambda) \) and \( \text{Orb}_\rho(\lambda) \), respectively. The following lemma establishes a connection between a property of an action of \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) on \( G_{\text{red}} \) and the field fixed by the stabilizer being totally imaginary.

**Lemma 5.4.** Let \( \rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{Aut}(G_{\text{red}}) \) be a continuous morphism and let \( \tau \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) denote the complex conjugation automorphism. Then for any \( \lambda \in \Lambda \), the field \( \overline{\mathbb{Q}}(\text{Stab}_\rho(\lambda)) \) is totally imaginary if and only if \( \tau \rho(\lambda) \) has no fixed points in \( \text{Orb}_\rho(\lambda) \).

**Proof.** By definition, the field \( L := \overline{\mathbb{Q}}(\text{Stab}_\rho(\lambda)) \) is totally imaginary if and only if it has no embedding into \( \mathbb{R} \). This is equivalent to \( \sigma(L) \) not being contained in \( \mathbb{R} \) for any \( \sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \), which is in turn equivalent to \( \tau \notin \text{Gal}(\overline{\mathbb{Q}}/\sigma(L)) \) for all \( \sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \). Note that

\[
\text{Gal}(\overline{\mathbb{Q}}/\sigma(L)) = \text{Gal}(\overline{\mathbb{Q}}/\overline{\mathbb{Q}}(\text{Stab}_\rho(\lambda)\sigma^{-1})) = \sigma \text{Stab}_\rho(\lambda)\sigma^{-1} = \text{Stab}_\rho(\rho(\lambda)).
\]

Thus we have shown that \( \overline{\mathbb{Q}}(\text{Stab}_\rho(\lambda)) \) is totally imaginary if and only if \( \tau \notin \text{Stab}_\rho(\rho(\lambda)) \) for any \( \sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \). This is clearly equivalent with the statement in the lemma and thus finishes the proof. \( \square \)

We are now ready to state the main theorem of this section.

**Theorem 5.5.** Let \( \rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{Aut}(G_{\text{red}}) \) be a continuous morphism and let \( \tau \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) denote the complex conjugation automorphism. Then the associated rational form \( \mathbb{Q}(\xi_1, \ldots, \xi_n) \) of \( \mathbb{Q}(\xi_1, \ldots, \xi_n) \) is Anosov if and only if for all \( \lambda \in \Lambda \) the following are true:

(i) if \( \rho(\lambda) \) has a fixed point in \( \text{Orb}_\rho(\lambda) \), then \( |\lambda| \cdot |\text{Orb}_\rho(\lambda)| \geq 2 \) and if equality holds, \( \text{Orb}_\rho(\lambda) \) spans an empty subgraph of \( G_{\text{red}} \),

(ii) if \( \rho(\lambda) \) has no fixed points in \( \text{Orb}_\rho(\lambda) \), then \( |\lambda| \cdot |\text{Orb}_\rho(\lambda)| \geq 4 \) and if equality holds, \( \text{Orb}_\rho(\lambda) \) spans an empty subgraph of \( G_{\text{red}} \) or a subgraph of \( G_{\text{red}} \) of the form

\[
\begin{array}{c}
\lambda_1 \\
\downarrow \\
\lambda_2 \\
\downarrow \\
\mu_1 \\
\downarrow \\
\mu_2
\end{array}
\]

where \( \rho(\lambda_1) = \lambda_2, \rho(\lambda_2) = \lambda_1, \rho(\mu_1) = \mu_2 \) and \( \rho(\mu_2) = \mu_1 \) (and hence \( |\text{Orb}_\rho(\lambda)| = 4 \)).

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Note that if \( \rho_r \) has no fixed points in \( \text{Orb}_r(\lambda) \), then this orbit must have an even number of elements.

**Proof.** First let us show that if all orbits of \( \rho \) satisfy conditions \([i]\) and \([ii]\) then there is a map \( \Psi : S \to \bar{\mathbb{Q}} \) which satisfies the conditions of Theorem \([5.2]\). This will imply that \( n_{\lambda,i}^{\mathbb{Q},\rho} \) is Anosov. Let us order the orbits of the action defined by \( \rho \) on \( \Lambda \) as \( \rho(\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})) \Lambda = \{J_1, \ldots, J_m\} \) and choose from each orbit a coherent component \( \lambda_i \in J_i \).

We will first construct for each \( 1 \leq i \leq m \) a map

\[
\Psi_i : \bigcup_{\lambda \in J_i} \lambda \to \bar{\mathbb{Q}},
\]

such that the image of \( \Psi_i \) consists of hyperbolic algebraic units, for any \( \{\alpha, \beta\} \in E \) with \( [\alpha], [\beta] \in J_i \), we have \( |\Psi_i(\alpha)\Psi_i(\beta)| \neq 1 \) and condition \([27]\) from Theorem \([5.2]\) is satisfied on \( \bigcup_{\lambda \in \lambda_i} \lambda \). Afterwards, we then show how to construct the map \( \Psi \) from these maps \( \Psi_i \). We split up the construction of the maps \( \Psi_i \) into several cases, where many are analogous:

- **|\( \lambda_i \)| \( \geq 2 \)**:
  
  Choose any \( 2 \)-hyperbolic algebraic unit \( \xi \in \bar{\mathbb{Q}} \) of degree \( d := |\lambda_i| \). Such an algebraic unit exists since \( d > 2 \). Let \( \xi_1, \ldots, \xi_d \) denote the conjugates of \( \xi \) and let us order the vertices in \( \lambda_i = \{\alpha_1, \ldots, \alpha_d\} \) Then define

\[
\Psi_i : \bigcup_{\lambda \in J_i} \lambda \to \bar{\mathbb{Q}} : r(\rho_r)(\alpha_j) \mapsto \xi_j \quad \text{for any } 1 \leq i \leq d, \sigma \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}).
\]

To see this is a well-defined expression let \( \sigma_1, \sigma_2 \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \) such that \( r(\rho_r)(\alpha_j) = r(\rho_r)(\alpha_k) \). Then it follows that \( \sigma_1^{-1}\sigma_2 \in \text{Stab}_r(\lambda_i) \) and thus also that \( r(\rho_r)^{-1}r(\rho_r) = r(\rho_r)^{-1}\rho_r) \mid \lambda_i = \text{Id}_{\lambda_i} \) and thus that \( j = k \). This proves \( \Psi_i \) is well-defined. Since \( \xi \) is \( 2 \)-hyperbolic, we have as well that for any \( \{\alpha, \beta\} \in E \) with \( [\alpha], [\beta] \in J_i \) it holds that \( |\Psi_i(\alpha)\Psi_i(\beta)| \neq 1 \).

- \( |J_i| \geq 2 \) and \( \rho_r \) has a fixed point in \( J_i \):

  The stabilizer \( H := \text{Stab}_r(\lambda_i) \) is an open subgroup of \( \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \). Let \( \bar{\mathbb{Q}}^H \) denote the subfield of \( \bar{\mathbb{Q}} \) which is fixed by \( H \). Then \( \bar{\mathbb{Q}}^H/\mathbb{Q} \) is a finite degree extension which is not totally imaginary by Lemma \([6.4]\). Note that

\[
[\bar{\mathbb{Q}}^H : \mathbb{Q}] = [\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) : H] = |J_i| > 2.
\]

Thus by Proposition \([5.3]\) there exists a \( 2 \)-hyperbolic algebraic unit \( \xi \in \bar{\mathbb{Q}}^H \). Then define

\[
\Psi_i : \bigcup_{\lambda \in J_i} \lambda \to \bar{\mathbb{Q}} : r(\rho_r)(\alpha) \mapsto \sigma(\xi) \quad \text{for any } \alpha \in \lambda_i, \sigma \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}).
\]

Let us show this is a well-defined expression. Say we have \( \alpha, \beta \in \lambda_i \) and \( \sigma_1, \sigma_2 \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \) with \( r(\rho_r)(\alpha) = r(\rho_r)(\beta) \). Then it follows that \( \sigma_1^{-1}\sigma_2 \in \text{Stab}_r(\lambda_i) = H \) and that \( \alpha = \beta \). Since \( \xi \in \bar{\mathbb{Q}}^H \), it we get that \( \sigma_2(\xi) = \sigma_1^{-1}\sigma_2(\xi) = \sigma_1(\xi) \) and thus that \( \Psi_i \) is indeed well-defined. Since \( \xi \) is \( 2 \)-hyperbolic, we have as well that for any \( \{\alpha, \beta\} \in E \) with \( [\alpha], [\beta] \in J_i \) it holds that \( |\Psi_i(\alpha)\Psi_i(\beta)| \neq 1 \).

- \( |J_i| > 2 \) and \( \rho_r \) has no fixed points in \( J_i \):

  Note that since \( \rho_r \) has no fixed points in \( J_i \), it follows that \( |J_i| \) is an even number. By consequence we have \( |J_i| \geq 6 \). The stabilizer \( H := \text{Stab}_r(\lambda_i) \) is an open subgroup of \( \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \). Let \( \bar{\mathbb{Q}}^H \) denote the subfield of \( \bar{\mathbb{Q}} \) which is fixed by \( H \). Then \( \bar{\mathbb{Q}}^H/\mathbb{Q} \) is a finite degree extension which is not totally imaginary by Lemma \([6.4]\) Since

\[
[\bar{\mathbb{Q}}^H : \mathbb{Q}] = [\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) : H] = |J_i| \geq 6,
\]

Proposition \([5.3]\) implies that there exists a \( 2 \)-hyperbolic algebraic unit \( \xi \in \bar{\mathbb{Q}}^H \). Then define

\[
\Psi_i : \bigcup_{\lambda \in J_i} \lambda \to \bar{\mathbb{Q}} : r(\rho_r)(\alpha) \mapsto \sigma(\xi) \quad \text{for any } \alpha \in \lambda_i, \sigma \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}).
\]

Analogous to the previous case, \( \Psi_i \) is well-defined and satisfies the necessary conditions.
• \(|J_i| = 2, |\lambda_i| = 2\): and \(\rho_x\) has a fixed point in \(J_i\):

Note that since \(J_i\) counts only two elements, \(\rho_x\) is the identity. Thus the stabilizer \(H := \text{stab}_\rho(\lambda_i)\) is an open subgroup of \(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\) which contains the complex conjugation automorphism. Let \(\overline{\mathbb{Q}}^H\) denote the subfield of \(\overline{\mathbb{Q}}\) which is fixed by \(H\). Then \(\overline{\mathbb{Q}}^H/\mathbb{Q}\) is a real extension of finite degree. Note that

\[
\overline{\mathbb{Q}}^H : \mathbb{Q} = [\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) : H] = |\text{Orb}_\rho(\lambda)| = 2.
\]

From Lemma 4.6 with \(p = 2\), there exists a real degree 2 extension \(L/\mathbb{Q}\) such that \(\overline{\mathbb{Q}}^H \cap L = \mathbb{Q}\). It follows that \(\text{Gal}(\overline{\mathbb{Q}}^H/L) \cong \text{Gal}(\overline{\mathbb{Q}}^H/\mathbb{Q}) \oplus \text{Gal}(L/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}\) is an abelian group and that \(\overline{\mathbb{Q}}^H L/\mathbb{Q}\) is a real extension of degree 4. Let \(\sigma_1, \sigma_2 \in \text{Gal}(\overline{\mathbb{Q}}^H/L)\) be generators for \(\text{Gal}(\overline{\mathbb{Q}}^H/\mathbb{Q})\) and \(\text{Gal}(L/\mathbb{Q})\), respectively. Now let \(\xi \in \overline{\mathbb{Q}}^H L\) be a 2-hyperbolic algebraic unit, which exists by Proposition 5.3.

Let us write \(J_i = \{\lambda_i = \{\alpha_1, \alpha_2\}, \{\beta_1, \beta_2\}\}\). Then define

\[
\Psi_1 : \bigcup_{\lambda \in J_i} \lambda \mapsto \overline{\mathbb{Q}} : \alpha_1 \mapsto \xi_1, \alpha_2 \mapsto \xi_2.
\]

Since \(\xi\) is 2-hyperbolic, we have as well that for any \(\{\alpha, \beta\} \in E\) with \([\alpha], [\beta] \in J_i\) it holds that \(|\Psi_1(\alpha)\Psi_1(\beta)| \neq 1\).

• \(|J_i| = 1, |\lambda_i| = 2\) (and thus \(\rho_x\) has a fixed point in \(J_i\)):

Take any hyperbolic algebraic unit \(\xi_1 \in \overline{\mathbb{Q}}\) of degree 2 and denote its conjugate with \(\xi_2\). Write \(\lambda_i = \{\alpha_1, \alpha_2\}\) and define

\[
\Psi_1 : \bigcup_{\lambda \in J_i} \lambda \mapsto \overline{\mathbb{Q}} : \alpha_1 \mapsto \xi_1, \alpha_2 \mapsto \xi_2.
\]

By the assumption, we know that \([\alpha_1, \alpha_2] \notin E\). Therefore, for any \(\{\alpha, \beta\} \in E\) with \([\alpha], [\beta] \in J_i\) it trivially holds that \(|\Psi_1(\alpha)\Psi_1(\beta)| \neq 1\).

• \(|J_i| = 2, |\lambda_i| = 1\) and \(\rho_x\) has a fixed point in \(J_i\):

Note that since \(J_i\) counts only two elements, \(\rho_x\) is the identity. Thus the stabilizer \(H := \text{stab}_\rho(\lambda_i)\) is an open subgroup of \(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\) which contains the complex conjugation automorphism. Let \(\overline{\mathbb{Q}}^H\) denote the subfield of \(\overline{\mathbb{Q}}\) which is fixed by \(H\). Then \(\overline{\mathbb{Q}}^H/\mathbb{Q}\) is a real extension of finite degree. Note that

\[
\overline{\mathbb{Q}}^H : \mathbb{Q} = [\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) : H] = |\text{Orb}_\rho(\lambda)| = 2.
\]

Thus by Proposition 5.3, there exists a 1-hyperbolic algebraic unit \(\xi_1\) of degree 2 in \(\overline{\mathbb{Q}}^H\). Let \(\xi_2\) denote the conjugate of \(\xi\) and write \(J_i = \{\lambda_i = \{\alpha\}, \{\beta\}\}\). Then define

\[
\Psi_1 : \bigcup_{\lambda \in J_i} \lambda \mapsto \overline{\mathbb{Q}} : \alpha \mapsto \xi_1, \beta \mapsto \xi_2.
\]

By the assumption, we have that \(\{\alpha, \beta\} \notin E\). Therefore, for any \(\{\alpha, \beta\} \in E\) with \([\alpha], [\beta] \in J_i\) it trivially holds that \(|\Psi_1(\alpha)\Psi_1(\beta)| \neq 1\).

• \(|J_i| = 2, |\lambda_i| = 0\) and \(\rho_x\) has no fixed points in \(J_i\):

The stabilizer \(H := \text{Stab}_\rho(\lambda_i)\) is an open subgroup of \(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\). Let \(\overline{\mathbb{Q}}^H\) denote the subfield of \(\overline{\mathbb{Q}}\) which is fixed by \(H\). By Lemma 5.4, \(\overline{\mathbb{Q}}^H/\mathbb{Q}\) is a totally imaginary extension of finite degree. Note that

\[
\overline{\mathbb{Q}}^H : \mathbb{Q} = [\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) : H] = |\text{Orb}_\rho(\lambda)| = 2.
\]

This implies as well that \(\overline{\mathbb{Q}}^H/\mathbb{Q}\) is Galois. By Lemma 4.6 with \(p = 2\), there exists a real degree 2 extension \(L/\mathbb{Q}\) such that \(\overline{\mathbb{Q}}^H \cap L = \mathbb{Q}\). It follows that \(\text{Gal}(\overline{\mathbb{Q}}^H/L) \cong \text{Gal}(\overline{\mathbb{Q}}^H/\mathbb{Q}) \oplus \text{Gal}(L/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}\) is an abelian group and that \(\overline{\mathbb{Q}}^H L/\mathbb{Q}\) is a totally imaginary extension of degree 4.
Let $\sigma_1, \sigma_2 \in \text{Gal}(\overline{\mathbb{Q}}^H L/\mathbb{Q})$ be generators for $\text{Gal}(\overline{\mathbb{Q}}^H /\mathbb{Q})$ and $\text{Gal}(L/\mathbb{Q})$, respectively. Now let $\xi \in \overline{\mathbb{Q}}^H L$ be a 1-hyperbolic algebraic unit, which exists by Proposition 5.3. Let us write $\mathcal{J}_i = \{\lambda_i = \{\alpha_i, \beta_i, \beta_j\}\}$. Then define

$$
\Psi_i : \bigcup_{\lambda \in \mathcal{J}_i} \lambda \rightarrow \overline{\mathbb{Q}} : \alpha \mapsto \xi, \alpha \mapsto \sigma_2(\xi), \beta \mapsto \sigma_1(\xi), \beta \mapsto \sigma_2\sigma_1(\xi).
$$

By assumption, $\mathcal{J}_i$ spans an empty subgraph of $G_{\text{red}}$ and thus it trivially holds that $|\Psi_i(\alpha)\Psi_i(\beta)| \neq 1$ for all $[\alpha], [\beta] \in \mathcal{J}_i$.

- $|\mathcal{J}_i| = 4$, $|\lambda_i| = 1$ and $\rho_\tau$ has no fixed points in $\mathcal{J}_i$.

The stabilizer $H := \text{Stab}_R(\lambda_i)$ is an open subgroup of $\text{Gal}(\overline{\mathbb{Q}}^H /\mathbb{Q})$. Let $\overline{\mathbb{Q}}^H$ denote the subfield of $\overline{\mathbb{Q}}$ which is fixed by $H$. Then $\overline{\mathbb{Q}}^H /\mathbb{Q}$ is a finite degree extension which is totally imaginary by Lemma 5.4. Note that

$$
\overline{\mathbb{Q}}^H = [\text{Gal}(\overline{\mathbb{Q}} /\mathbb{Q}) : H] = |\mathcal{J}_i| = 4.
$$

Thus by Proposition 5.3 there exists a 1-hyperbolic algebraic unit $\xi \in \overline{\mathbb{Q}}^H$. Then define

$$
\Psi_i : \bigcup_{\lambda \in \mathcal{J}_i} \rightarrow \overline{\mathbb{Q}} : r(\rho_\tau)(\alpha) \mapsto \sigma(\xi) \quad \text{for all } \alpha \in \lambda_i, \sigma \in \text{Gal}(\overline{\mathbb{Q}} /\mathbb{Q}).
$$

Analogously to one of the previous cases, $\Psi_i$ is well-defined. By assumption, $\mathcal{J}_i$ spans either an empty subgraph of $G_{\text{red}}$ or a subgraph of the form given by (30). In the first case, it trivially holds that $|\Psi_i(\alpha)\Psi_j(\beta)| \neq 1$ for all $[\alpha], [\beta] \in \mathcal{J}_i$. In the second case, for any $[\alpha], [\beta] \in \mathcal{J}_i$, we have that $|\Psi_i(\alpha)\Psi_j(\beta)| = |\Psi_i(\alpha)\Psi_j(r(\rho_\tau)(\alpha))| = |\Psi(\alpha)\tau(\Psi(\alpha))| = |\Psi(\alpha)|^2$ which is not equal to 1, since $\xi$ is a 1-hyperbolic unit.

Now, let us construct the map $\Psi : S \rightarrow \overline{\mathbb{Q}}$. Consider for any $[\alpha], [\beta] \in \mathcal{J}_i$ and $[\beta] \in \mathcal{J}_j$, the additive group morphisms

$$
h_{\alpha, \beta} : \mathbb{Z}^m \rightarrow \mathbb{R} : (z_1, \ldots, z_m) \mapsto \log |\Psi_i(\alpha)^{z_1}\Psi_j(\beta)^{z_j}|.
$$

Note that these maps can never be identically 0. Indeed, if $i = j$, then $h_{\alpha, \beta}(z_1, \ldots, z_m) = z_1 \log |\Psi_i(\alpha)\Psi_i(\beta)|$. We know that $|\Psi_i(\alpha)\Psi_j(\beta)| \neq 1$ by construction of $\Psi_i$, thus proving it in this case. If $i \neq j$, then $h_{\alpha, \beta}$ being identically zero would imply $\log |\Psi_i(\alpha)^{z_1}\Psi_j(\beta)^{z_j}| = \log |\Psi_i(\alpha)| = 0$ which contradicts the fact that $\Psi_i(\alpha)$ is a hyperbolic algebraic unit. By consequence, the kernels $\ker(h_{\alpha, \beta}) \subset \mathbb{Z}^m$ are subgroups of rank strictly smaller than $m$. Since there are only finitely many edges in $E$, this implies that there exists an element

$$(z'_1, \ldots, z'_m) \in \mathbb{Z}^m \setminus \bigcup_{\{\alpha, \beta\} \in E} \ker(h_{\alpha, \beta}).$$

Then define $\Psi : S \rightarrow \overline{\mathbb{Q}}$ as

$$
\Psi(\alpha) := \Psi_i(\alpha)^{z'_i} \quad \text{for } [\alpha] \in \mathcal{J}_i.
$$

One can check that by the way we defined the $\Psi_i$’s, it holds that $\Psi$ satisfies the conditions of Theorem 5.2.

Conversely, assume for a contradiction that there exists a map $\Psi : S \rightarrow \overline{\mathbb{Q}}$ which satisfies the conditions of Theorem 5.2 and a $\lambda \in \Lambda$ with $\mathcal{J} := \text{Orb}_R(\lambda)$, which does not satisfy condition [3] from the theorem. Then we are in one of the following cases:

- $|\lambda| = 1$ and $|\mathcal{J}| = 1$.

Write $\lambda = \{z\}$. It follows that $\sigma(\Psi(\alpha)) = \Psi(\rho_\tau(\alpha)) = \Psi(\alpha)$ for all $\sigma \in \text{Gal}(\overline{\mathbb{Q}} /\mathbb{Q})$. This implies that $\Psi(\alpha) \in \mathbb{Q}$. The only algebraic units in $\mathbb{Q}$ are 1 and $-1$, which both are not hyperbolic, a contradiction with the assumption on the map $\Psi$. 28
Now assume that there is a \( \lambda \in \Lambda \) with \( \mathcal{J} := \text{Orb}_\rho(\lambda) \), which does not satisfy condition [iii] from the theorem, hence in particular \(|\mathcal{J}|\) is even. Then we are in one of the following cases.

- \(|\lambda| \cdot |\mathcal{J}| = 2 \) and \( \rho_\tau \) has no fixed points in \( \mathcal{J} \):
  
  Note that in this case we have \(|\lambda| = 1 \) and \(|\mathcal{J}| = 2 \). Let us write \( \lambda = \{\alpha\} \). This gives us that \( \Psi(\alpha) \) is an algebraic unit of degree 2, since degree one is excluded by a contradiction analogous to previous case. Now, since \( \mathcal{J} \) spans a non-empty subgraph of \( \mathcal{G}_{\text{red}} \), we also know that \( \bigcup_{\lambda \in \mathcal{J}} \lambda \) spans a non-empty subgraph of \( \mathcal{G} \). If we write \( \bigcup_{\lambda \in \mathcal{J}} \lambda = \{\alpha, \beta\} \), it follows that \( \{\alpha, \beta\} \) is an edge of \( \mathcal{G} \). The conjugate of \( \Psi(\alpha) \) is given by \( \Psi(\beta) \) and since they are algebraic units, we have that they are up to sign each others inverses. Therefore we get that \(|\Psi(\alpha)\Psi(\beta)| = |1| = 1 \) which contradicts the assumptions on the map \( \Psi \).

This shows no such map \( \Psi : S \to \overline{\mathbb{C}} \) can exist when a \( \lambda \in \Lambda \) does not satisfy the conditions [i] or [iii] from the theorem. This concludes the proof.

**Remark 5.6.** Note that if \( \rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{Aut}(\mathcal{G}_{\text{red}}) \) is the trivial morphism, the corresponding rational form \( n_{\mathcal{G}_{\text{red}}}^\mathbb{Q} \) is just the standard one \( n_{\mathcal{G}}^\mathbb{Q} \). The orbits of the action defined by \( \rho \) all count only one coherent component. Thus we recover the result proven in [2] Theorem 1.1, namely that \( n_{\mathcal{G}}^\mathbb{Q} \) is Anosov if and only if for all \( \lambda \in \Lambda \) it holds that either \(|\lambda| \geq 3 \) or it holds that \(|\lambda| = 2 \) and \( \{\lambda, \lambda\} \notin \mathcal{E} \).

If we only consider rational forms of the real Lie algebra \( n_{\mathcal{G}}^\mathbb{R} \), then \( \rho_\tau \) is trivial and thus always has a fixed point in every orbit. Hence the theorem simplifies to the following statement.

**Corollary 5.7 (Real version).** Let \( \rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{Aut}(\mathcal{G}_{\text{red}}) \) be a continuous morphism with complex conjugation in the kernel. Then the associated rational form \( n_{\mathcal{G}_{\text{red}}}^\mathbb{R} \) of \( n_{\mathcal{G}}^\mathbb{R} \) is Anosov if and only if there are no orbits of \( \rho \) which span a (reduced) subgraph of the form:

\[
1 \bullet \rightarrow 1, \quad 2 \circ \rightarrow 1 \bullet
\]

Our final corollary shows that from all rational forms in \( n_{\mathcal{G}}^\mathbb{R} \) that can be Anosov, the standard one \( n_{\mathcal{G}}^\mathbb{Q} \) leads to the strongest condition.

**Corollary 5.8.** Let \( \mathcal{G} \) be a simple undirected graph and \( n_{\mathcal{G}}^\mathbb{Q} \) the associated rational Lie algebra. If \( n_{\mathcal{G}}^\mathbb{Q} \) admits an Anosov automorphism, then so does any other rational form of \( n_{\mathcal{G}}^\mathbb{Q} \).
Proof. Assume that $n_G^C$ is Anosov. Note that $n_G^C$ is the rational form of $n_G^R$ corresponding to the trivial representation $\text{Gal}(\overline{Q}/Q) \to \text{Aut}(G_{\text{red}}) : \sigma \mapsto \text{Id}_\Lambda$. By consequence all orbits of the action of this representation on $G_{\text{red}}$ are singletons. Using Remark 5.6 and the assumption that $n_G^Q$ is Anosov, we find that $G_{\text{red}}$ has no coherent components of size 1 and if a coherent component is of size 2, there is no edge between the vertices that make up that coherent component. But this implies immediately that for any representation $\rho : \text{Gal}(\overline{Q}/Q) \to \text{Aut}(G_{\text{red}})$ the reduced subgraphs 1--2 and 1• can not be spanned by an orbit of $\rho$. By consequence Corollary 5.7 tells us that $n_G^C,\rho$ is Anosov for any representation $\rho$. By Theorem 4.4 we thus have that all rational forms of $n_G^C$ are Anosov.

Remark 5.9. Note that Corollary 5.8 does not generalize to the rational forms of the complex Lie algebra $n_G^C$. Example 5.15 will illustrate this.

5.4 Applications

Let us now apply Theorem 5.5 to certain classes of simple undirected graphs. Before starting, we want to mention a classical result by Erdős and Reny stating that almost every graph has a trivial automorphism group, see [10]. In particular, these graphs have coherent components of size 1 and thus by Theorem 5.5 and Theorem 1 a unique rational form that is not Anosov. This shows that having an Anosov rational form is a rare condition for Lie algebras associated to graphs.

First, we consider trees. A tree is a graph in which any two vertices are connected by at most one path. Equivalently, a graph is a tree if and only if it has no cycles. For two vertices $\alpha, \beta$ in a connected graph $G = (S,E)$, let $d(\alpha, \beta)$ denote the distance between $\alpha$ and $\beta$, given by the minimal number of edges needed to go from $\alpha$ to $\beta$ in the graph. The eccentricity $e(\alpha)$ of a vertex $\alpha$ is defined as $e(\alpha) = \max\{d(\alpha, \beta) \mid \beta \in S\}$. The centre of $G$ is then defined as the set of vertices of $G$ which have minimal eccentricity. It is a standard result that a connected tree has a centre consisting of either one or two vertices. To illustrate this, two trees with their centre are drawn in Figure 1 and 2.

![Figure 1: A tree with a centre consisting of one vertex, drawn in red.](image1.png)

![Figure 2: A tree with a centre consisting of two adjacent vertices, drawn in red.](image2.png)

Proposition 5.10. If $G$ is a connected tree, then $n_G^C$ has no Anosov rational forms.

Proof. Consider any continuous morphism $\rho : \text{Gal}(\overline{Q}/Q) \to \text{Aut}(G_{\text{red}})$. Let $C \subset S$ be the centre of $G = (S,E)$. Note that vertices in the same coherent component can be mapped onto each other by an automorphism of $G$ and must thus have the same eccentricity. By consequence the coherent component of any element from the centre lies completely in the centre. Since $G$ is a tree, we have two cases: either $C$ consists of one vertex or $C$ consists of two adjacent vertices. In the first case, $C$ must itself be a coherent component and must be preserved under any automorphism of $G_{\text{red}}$, implying that $\rho$ has an orbit of the form 1•. If $|C| = 2$, we either have that $C$ is a coherent component preserved under any automorphism of $G_{\text{red}}$, thus implying that $\rho$ has an orbit of the form 2• or we have that $C$ is the union of two coherent components of size 1, in which case $\rho$ has an orbit of the form 1--1. In any case, we see that Theorem 5.5 tells us that $n_G^C,\rho$ is not Anosov. Using Theorem 4.4 and the fact that $\rho$ was an arbitrarily chosen continuous morphism, we get that $n_G^C$ has no Anosov rational forms.

As a second class, let us consider the cycle graphs. The cycle graph of size $n$ is given by vertices $S = \{1, \ldots, n\}$ and edges $E = \{(1,2), (2,3), \ldots, (n-1,n), (n,1)\}$. If $n \geq 5$, then the coherent components are all singletons as illustrated below in Figure 3 for $n = 6$. It follows that for $n \geq 5$, the automorphism group...
of the reduced graph is isomorphic to the dihedral group of order $2n$. Let $a$ be a generator of the rotation subgroup of $\text{Aut}(\mathcal{G}_{\text{red}})$ and $b$ a reflection of $\text{Aut}(\mathcal{G}_{\text{red}})$. Then $\text{Aut}(\mathcal{G}_{\text{red}}) = \{\text{Id}, a, \ldots, a^{n-1}, b, ab, \ldots a^{n-1}b\}$. Let us call a rational form of $n^\mathcal{G}$ of reflection type if the corresponding representation $\text{Gal}(\mathbb{F}/\mathbb{Q}) \to \text{Aut}(\mathcal{G}_{\text{red}})$ has image $\{\text{Id}, a^i b\}$ for some $1 \leq i \leq n$. Then using Corollary 5.7 it is not hard to prove following statement.

**Proposition 5.11.** If $\mathcal{G}$ is a cycle graph of size $n \geq 5$, then $n^\mathcal{G}$ and all reflection-type rational forms of $n^\mathcal{G}$ do not admit an Anosov automorphism, while all other rational forms of $n^\mathcal{G}$ do admit an Anosov automorphism.

**Proof.** Since the coherent components of $\mathcal{G}$ are all singletons, it follows by Remark 5.5 that $n^\mathcal{G}$ is not Anosov. If $n^\mathcal{G}_{\rho}$ is a reflection-type rational form of $n^\mathcal{G}$, then the action induced by $\rho$ on the reduced graph must have an orbit of the form $1 \bullet$ or of the form $1 \longrightarrow 1$. Corollary 5.8 thus implies that all reflection type rational forms are not Anosov. If $n^\mathcal{G}_{\rho}$ is any other rational form of $n^\mathcal{G}$, then the image of $\rho$ contains a non-trivial rotation $a^i$ where $i \in \{1, \ldots, n-1\}$. Note that the orbits under the action of $\rho$ must have size at least equal to the order of $a^i$ in $\text{Aut}(\mathcal{G}_{\text{red}})$. By consequence if $a^i$ has order at least 3, we can conclude that $n^\mathcal{G}_{\rho}$ is Anosov, thus leaving only the case where $a^i$ has order 2. If this is the case and there is an orbit of size 2 it follows that it must be of the form $1 \bullet \bullet 1$, thus showing it must be Anosov as well.

**Proposition 5.12.** If $\mathcal{G}$ is a graph for which there is a non-negative integer $k \geq 0$ such that $\mathcal{G}$ has a unique vertex of degree $k$, then $n^\mathcal{G}$ has no Anosov rational forms.

**Proof.** Note that all vertices in a coherent component have the same degree. By consequence, if $\mathcal{G}$ has a vertex $v$ for which there are no other vertices of the same degree, then $\{v\}$ is a coherent component of $\mathcal{G}$. We also have that $v$ must be fixed under any automorphism of $\mathcal{G}$. In particular, we have for any $\varphi \in \text{Aut}(\mathcal{G}_{\text{red}})$ that $r(\varphi)(v) = v$. From this it follows that $\varphi(\{v\}) = \{r(\varphi)(v)\} = \{v\}$. This shows that the coherent component $\{v\}$ must be fixed under any automorphism of the reduced graph as well. By consequence any action on the reduced graph by automorphisms must have an orbit of size 1 which by Theorem 5.5 proves all rational forms of $n^\mathcal{G}$ are not Anosov. □

The following examples show that our methods can be used to simplify certain classifications of Lie algebras.

**Example 5.13** (Direct sum of two 3-dimensional Heisenberg Lie algebras). Consider the graph $\mathcal{G} = (S, E)$ defined by $S = \{\alpha_1, \alpha_2, \beta_1, \beta_2\}$ and $E = \{\{\alpha_1, \beta_1\}, \{\alpha_2, \beta_2\}\}$. The set of coherent components is then given by $\lambda = \{\lambda_1 := \{\alpha_1, \beta_1\}, \lambda_2 := \{\alpha_2, \beta_2\}\}$. A figure of the graph and reduced graph are given below.

There are only two automorphisms of the reduced graph, namely the identity and $\varphi \in \text{Aut}(\mathcal{G}_{\text{red}})$ which is defined by $\varphi(\lambda_1) = \lambda_2$ and $\varphi(\lambda_2) = \lambda_1$. We can define the morphism $r : \text{Aut}(\mathcal{G}_{\text{red}}) \to \text{Aut}(\mathcal{G})$ by letting $r(\varphi)(\alpha_1) = \alpha_2$, $r(\varphi)(\alpha_2) = \alpha_1$, $r(\varphi)(\beta_1) = \beta_2$ and $r(\varphi)(\beta_2) = \beta_1$. The associated 2-step nilpotent Lie
algebra \( n_2^Q \) is then isomorphic to a direct sum of two 3-dimensional Heisenberg Lie algebras with basis \( \{\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2\} \) where \( \gamma_1 := [\alpha_1, \beta_1] \) and \( \gamma_2 := [\alpha_2, \beta_2] \).

It is clear that if \( \text{Gal}(L/Q) \to \text{Aut}(G_{\text{red}}) \) is an injective group morphism, \( L/Q \) must have degree 2 or 1. All non-isomorphic degree 2 or 1 Galois extensions of \( Q \) are given by \( Q(\sqrt{d}) \) for \( d \) a square free non-zero integer. Note that if \( d = 1 \), \( Q(\sqrt{d}) = Q \). For all square free non-zero integers \( d \), let \( \rho_d \) denote the uniquely determined injective group morphism \( \rho_d : \text{Gal}(Q(\sqrt{d})/Q) \to \text{Aut}(G_{\text{red}}) \). For simplicity, let us write \( n_{\rho_d}^Q \) for the associated rational form \( n_{\rho_d}^Q \). From Theorem \( \ref{thm:main} \) we then get that the sets
\[
\{ n_{\rho_d}^Q | d \neq 0 \text{ square free} \} \quad \text{and} \quad \{ n_{\rho_d}^Q | d \geq 1 \text{ square free} \}
\]
give us a complete set of pairwise non-isomorphic rational forms of \( n_2^Q \) and \( n_2^Q \), respectively. Note that \( n_{\rho,1}^Q \cong n_{\rho}^Q \) is the standard rational form of \( n_2^Q \). We thus get a simpler proof of \( \ref{prop:main} \) Proposition 3.2.] without using the Pfaffian form on 2-step nilpotent Lie algebras.

For square-free \( d \neq 0, 1 \), a basis for \( n_{\rho,d}^Q \subset n_2^Q \) can be given by
\[
X_1 := \alpha_1 + \alpha_2 \quad Y_1 := \beta_1 + \beta_2 \quad Z_1 := \gamma_1 + \gamma_2
\]
\[
X_2 := \sqrt{d}(\alpha_1 - \alpha_2) \quad Y_2 := \sqrt{d}(\beta_1 - \beta_2) \quad Z_2 := \sqrt{d}(\gamma_1 - \gamma_2).
\]

The bracket relations of the rational Lie algebra \( n_{\rho,d}^Q \) in this basis are then given by
\[
[X_1, Y_1] = Z_1 \quad [X_2, Y_2] = Z_2
\]
\[
[X_1, Y_2] = Z_2 \quad [X_2, Y_1] = d Z_1.
\]

From Theorem \( \ref{thm:main} \), we can now easily see that for a square free non-zero integer \( d \), the rational form \( n_{\rho,d}^Q \) is Anosov if and only if \( d > 1 \). This result was already known from the classification of Anosov Lie algebras of dimension \( \leq 8 \) (see \( \ref{ex:anosov} \) Example 2.7. and Theorem 4.2.).

**Definition 5.14.** Let \( G = (S, E) \) be a simple undirected graph. The graph \( G^* := (S, E^*) \) with
\[
E^* = \{ (\alpha, \beta) | \alpha, \beta \in S, \alpha \neq \beta, \{ \alpha, \beta \} \notin E \}
\]
is called the complement graph of \( G \). If \( G = (S, E, \Phi) \) is a simple vertex-weighted undirected graph with loops, then we define its complement graph by \( G^* = (S, E^*, \Phi) \) with
\[
E^* = \{ (\alpha, \beta) | \alpha, \beta \in S, \{ \alpha, \beta \} \notin E \}.
\]

Note that the coherent components of a simple undirected graph and its complement graph coincide. Moreover it follows that \( (G^*)_{\text{red}} = (G_{\text{red}})^* \). We can thus simply write \( G_{\text{red}}^* \). For both simple undirected graphs and simple vertex-weighted undirected graphs with loops, it also holds that the automorphism group of the graph and the complementary graph are equal. This being said, let us look at the Lie algebra associated with the complement graph of the one from Example \( \ref{ex:anosov} \).

**Example 5.15.** Let \( G = (S, E) \) be the graph from Example \( \ref{ex:anosov} \) and \( G^* \) its complement graph. A figure of \( G^* \) and its reduced graph are given below.

\[\text{Graphs} \]

It is now easy to verify that the only injective group morphisms \( \text{Gal}(L/Q) \to \text{Aut}(G^*_{\text{red}}) = \text{Aut}(G_{\text{red}}) \) are the morphisms \( \rho_d : \text{Gal}(Q(\sqrt{d})/Q) \to \text{Aut}(G_{\text{red}}) \) from Example \( \ref{ex:anosov} \) where \( d \) is any square free non-zero integer. Again, let us simply write \( n_{\rho,d}^Q \) for the associated rational form \( n_{\rho,d}^Q \). From Theorem \( \ref{thm:main} \) it thus follows that the sets
\[
\{ n_{G^*,d}^Q | d \neq 0 \text{ square free} \} \quad \text{and} \quad \{ n_{G^*,d}^Q | d \geq 1 \text{ square free} \}
\]
give us a complete set of pairwise non-isomorphic rational forms of $n^G_0$ and $n^G_b$, respectively. A basis for $n^G_0$ can be given by $\{\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ where $\gamma_1 := [\alpha_1, \beta_2]$, $\gamma_2 := [\alpha_2, \beta_1]$, $\gamma_3 = [\alpha_1, \alpha_2]$, $\gamma_4 = [\beta_1, \beta_2]$. A basis for the form $n^G_{\ast, d}$ can then be given by:

$$
\begin{align*}
X_1 &:= \alpha_1 + \alpha_2 & Y_1 &:= \beta_1 + \beta_2 & Z_1 &:= \gamma_1 + \gamma_2 & Z_3 &= -2\sqrt{d}\gamma_3 \\
X_2 &:= \sqrt{d}(\alpha_1 - \alpha_2) & Y_2 &:= \sqrt{d}(\beta_1 - \beta_2) & Z_2 &:= \sqrt{d}(\gamma_2 - \gamma_1) & Z_4 &= -2\sqrt{d}\gamma_4.
\end{align*}
$$

The bracket relations of the rational Lie algebra $n^G_{\ast, d}$ in this basis are then given by

$$
\begin{align*}
[X_1, X_2] &= Z_3 & [X_2, Y_1] &= Z_2 \\
[X_1, Y_1] &= -Z_1 & [X_2, Y_2] &= dZ_1 \\
[X_1, Y_2] &= -Z_2 & [Y_1, Y_2] &= Z_4.
\end{align*}
$$

From Theorem 5.5, we can now easily see that for a square free non-zero integer $d$, the rational form $n^G_{\ast, d}$ is Anosov if and only if $d \geq 1$ (note how this condition is different from the one we got for $G$, namely that $n^G_0$ is Anosov if and only if $d > 1$). This result was already partially known from the classification of real Anosov Lie algebras of dimension $\leq 8$ (see [13 Theorem 4.2.]).

When studying Anosov automorphisms on rational Lie algebras, the question arises if, whenever the Lie algebra is Anosov, we can always take an Anosov automorphism with real eigenvalues. The answer turns out to be negative. The results in this paper allow us to present the following counterexample.

**Example 5.16** (Anosov rational form which does not admit Anosov automorphism with real eigenvalues). Let $G = (S, E)$ be the cycle graph on 6 vertices as drawn in Figure 3. Let us write the vertices and edges as $S = \{\alpha_1, \ldots, \alpha_6\}$ and $E = \{\{\alpha_1, \alpha_2\}, \ldots, \{\alpha_5, \alpha_6\}, \{\alpha_6, \alpha_1\}\}$. The coherent components of $G$ are then simply all the singletons $\Lambda = \{\lambda_i := \{\alpha_i\} \mid 1 \leq i \leq 6\}$. By consequence there is a natural bijection $h : S \to \Lambda : \alpha \mapsto \{\alpha\}$ which gives us a splitting morphism $r : \text{Aut}(G_{\text{red}}) \to \text{Aut}(G) : \varphi \mapsto h^{-1} \circ \varphi \circ h$. Let us write $\text{Aut}(G_{\text{red}}) = \{1, a, \ldots, a^5, b, ab, \ldots, a^5b\}$ where $a$ and $b$ are defined by $a(\lambda_1) = \lambda_2$, $a(\lambda_2) = \lambda_3$, $b(\lambda_1) = \lambda_1$ and $b(\lambda_2) = \lambda_6$. Thus $a$ is a generator for the rotations and $b$ is a reflection, like in our general discussion of cycle graphs.

Now let $L$ be the splitting field of the polynomial $X^3 - 2$ over $\mathbb{Q}$. The roots of this polynomial are given by $\sqrt[3]{2}$, $\omega\sqrt[3]{2}$ and $\overline{\omega}\sqrt[3]{2}$ where $\omega = e^{2\pi i/3}$. The Galois group $\text{Gal}(L/\mathbb{Q})$ is generated by the elements $\sigma$ and $\tau$, defined by

$$
\begin{align*}
\sigma(\sqrt[3]{2}) &= \omega\sqrt[3]{2}, & \sigma(\omega\sqrt[3]{2}) &= \overline{\omega}\sqrt[3]{2}, & \sigma(\overline{\omega}\sqrt[3]{2}) &= \sqrt[3]{2}, \\
\tau(\sqrt[3]{2}) &= \sqrt[3]{2}, & \tau(\omega\sqrt[3]{2}) &= \overline{\omega}\sqrt[3]{2}, & \tau(\overline{\omega}\sqrt[3]{2}) &= \omega\sqrt[3]{2}.
\end{align*}
$$

Note that $\tau$ is just the complex conjugation automorphism on $L$ and that $\text{Gal}(L/\mathbb{Q})$ is isomorphic to the dihedral group of order 6. It follows that we have an injective group morphism $\rho : \text{Gal}(L/\mathbb{Q}) \to \text{Aut}(G_{\text{red}}) : \sigma \mapsto a^2, \tau \mapsto b$, with corresponding rational form $n^G_{3, \rho}$ of $n^G_b$. Using Theorem 5.5 it is straightforward to verify that $n^G_{3, \rho}$ is Anosov. Indeed, there are two orbits each counting 3 coherent components and $\rho_\tau$ has a fixed point on both orbits. Now define the vectors $\beta_i = [\alpha_i, \alpha_{i+1}]$ for $1 \leq i \leq 5$ and $\beta_6 = [\alpha_6, \alpha_1]$. We can write down the following basis for $n^G_{3, \rho}$:
Let us prove by contradiction that \( \tilde{\xi} \) is fixed under \( \tilde{\sigma} \). Now let \( \tilde{\sigma} \) be any element in \( \text{Stab}_{\tilde{\rho}}(\lambda_1) \). It follows that \( \tilde{\sigma}(\xi) = (\sigma \circ \Psi)(\alpha_1) = (\Psi \circ r(\tilde{\rho})) \circ \theta(\alpha_1) \), where \( \theta \in \prod_{L \in \Lambda} \text{Perm}(L) \). Since all the coherent components are singletons, it follows that \( \theta = \text{Id}_L \) and we get that \( \tilde{\sigma}(\xi) = (\Psi \circ r(\tilde{\rho}))(\alpha_1) = \Psi(\alpha_1) = \xi \). By consequence \( \xi \) is fixed under \( \text{Stab}_{\tilde{\rho}}(\lambda_1) \) and in particular under \( \ker(\tilde{\rho}) \). Note that \( Q_{\ker(\tilde{\rho})} \) is exactly equal to \( L \) and thus that \( \xi \in L \). Now \( \sigma(\xi) = \Psi(\alpha_3) \) is also an eigenvalue of \( f \). Since \( \xi \) and \( \sigma(\xi) \) are both real, we have \( \tau(\xi) = \xi \) and \( \tau(\sigma(\xi)) = \sigma(\xi) \). This implies \( \sigma(\xi) = \tau \sigma(\xi) = \sigma^2(\xi) = \sigma^2(\xi) \) and thus that \( \sigma(\xi) = \xi \). By consequence \( \xi \) is fixed under \( \text{Gal}(L/Q) \), which in turn tells us that \( \xi \in Q \). The only algebraic units in \( Q \) are \( 1 \) and \( -1 \) which are not hyperbolic. This gives us the contradiction.

References

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