A conical deficit in the AdS$_4$/CFT$_3$ correspondence

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Received 24 August 2010, in final form 15 November 2010
Published 10 December 2010
Online at stacks.iop.org/CQG/28/015011

Abstract

Inspired by the AdS/CFT correspondence, we propose a new duality that allows the study of strongly coupled field theories living in a $(2+1)$ conical spacetime. Solving the 4D Einstein equations in the presence of an infinite static string and negative cosmological constant, we obtain a conical AdS$_4$ spacetime whose boundary is identified with the $(2+1)$ cone found by Deser, Jackiw and 't Hooft. Using the AdS$_4$/CFT$_3$ correspondence, we calculate the retarded Green’s functions of scalar operators living in the cone.

PACS numbers: 11.25.Tq, 04.62.+v, 04.60.Kz, 11.27.+d

1. Introduction

The AdS$_{d+1}$/CFT$_d$ correspondence, discovered by Maldacena [1], has motivated in recent years the study of conformal gauge theories arising from 10D string theory or 11D M-theory. Much effort has been concentrated on the $d = 4$ case where a configuration of $N$ coincident D3-branes leads to a duality between type IIB string theory in AdS$_5 \times S^5$ and $\mathcal{N} = 4$ SU($N$) super Yang–Mills theory in $\mathbb{R}^{1,3}$ (some reviews can be found in [2–4]). The AdS$_4$/CFT$_3$ correspondence also brought new approaches to the strong coupling problem in QCD (see for instance [5, 6]).

Recently, the $d = 3$ case has gained a lot of attention due to the following two reasons. The first reason is the important progress made in the understanding of D2- and M2-branes configurations. It has been conjectured that M-theory in AdS$_4 \times S^3/\mathbb{Z}_4$ is the dual of $\mathcal{N} = 6$ $U(N) \times U(N)$ super-Chern–Simons gauge theory with level $k$ living in $(2+1)$ Minkowski spacetime [7] (for a recent review see [8]). The second reason is the possible application of the AdS$_4$/CFT$_3$ correspondence to the phenomenology of $(2+1)$ strongly coupled systems in condensed matter [9, 10]. An interesting example is the graphene (a monolayer of carbon crystal) whose electrical conductivity has recently been predicted [10].
The AdS$_4$/CFT$_3$ correspondence allows the study of (2 + 1) systems assuming a flat spacetime. However, in real experiments (like that of graphene) a (2 + 1) system may present structural defects [11]. These kinds of defects can be modeled by considering field theories in curved spaces such as spheres and cones and may also modify the electronic and transport properties of the system (see for instance [12–14]).

In this paper we propose, motivated in the AdS$_4$/CFT$_3$ correspondence, a new duality that allows the study of strongly coupled field theories living in a conical spacetime. First, we find a conical AdS$_4$ spacetime by solving the (3+1) Einstein equations in the presence of an infinite static string and a negative cosmological constant. We identify the boundary of this spacetime with the conical spacetime of Deser, Jackiw and ‘t Hooft obtained as a solution of the (2+1) Einstein equations in the presence of a massive point particle and zero cosmological constant [15]. In order to investigate the effect of the conical deficit on a (2 + 1) strongly coupled field theory, we calculate the retarded Green’s functions of scalar operators $O$. Assuming a conical AdS$_4$/CFT$_3$ correspondence, a scalar operator $O$ couples to a scalar field $\phi$ living in the conical AdS$_4$ spacetime so that the real time retarded Green’s functions of the boundary scalar operator $O$ can be obtained from the on-shell action of the bulk scalar field $\phi$ using a prescription similar to that found by Son and Starinets [16].

The conical AdS$_4$ spacetime found in this paper can be compared in the limit of weak gravitational coupling to that obtained in [17] when considering a vortex line configuration. Black hole solutions in AdS$_4$ may also contain conical deficits [18, 19].

2. A conical deficit in 4D AdS spacetime

2.1. The static string configuration

Consider an infinite static string living in a 4D spacetime with metric $g_{\mu\nu}$ in the presence of a negative cosmological constant $\Lambda = -3/L^2$. This system is described by the Einstein–Hilbert–Nambu–Goto action

$$ S = S_G + S_{NG} = \frac{1}{16\pi G_4} \int dt dz dr d\theta \sqrt{-g} \left[ R + \frac{6}{L^2} \right] - \mu \int d\sigma^0 d\sigma^1 \sqrt{-\det P[g_{ab}]} ,$$

(1)

where

$$ P[g_{ab}] = g_{\mu\nu}(X) \frac{\partial X^\mu(\sigma)}{\partial \sigma^a} \frac{\partial X^\nu(\sigma)}{\partial \sigma^b} $$

(2)

is the 2D induced metric of the Nambu–Goto string with $g_{\mu\nu}(X)$ being the 4D spacetime metric. Note that the embedding string coordinates $X^\mu(\sigma)$ are in general different from the spacetime coordinates $\{t, z, r, \theta\}$. Now we consider the configuration of a static string extended in the $z$ direction. The appropriate gauge for this scenario is $\{\sigma^0, \sigma^1\} = \{t, z\}$ and the embedding string coordinates are $X^\mu(t, z) = \{t, z, X^1(z), X^2(z)\}$.

Varying the Nambu–Goto action with respect to $X^1(z)$ and $X^2(z)$, we obtain the equations of motion

$$ \partial_z \left[ \frac{g_{tt}(X) g_{11}(X) \partial_z X^1}{\sqrt{-g_{tt}(X) [g_{zz}(X) + g_{11}(X) (\partial_z X^1)^2 + g_{22}(X) (\partial_z X^2)^2]}} \right] = 0, $$

$$ \partial_z \left[ \frac{g_{tt}(X) g_{22}(X) \partial_z X^2}{\sqrt{-g_{tt}(X) [g_{zz}(X) + g_{11}(X) (\partial_z X^1)^2 + g_{22}(X) (\partial_z X^2)^2]}} \right] = 0. $$

(3)
These equations are very general and can be solved in any \((X^1, X^2)\) coordinate chart. If we choose the cylindrical chart \(X^1 = r\) and \(X^2 = \theta\), we find the simple solutions  
1. \(r = r_0 = \text{const} > 0\), \(\theta = \theta_0 = \text{const}\),  
2. \(r = 0\), \(g_{\theta \theta}(X) = 0\).

Here, we will consider the second solution corresponding to an infinite static string extended in the \(z\) coordinate and localized in the origin of the \((r, \theta)\) plane. The corresponding on-shell action is

\[
S_{\text{on-shell}}^{\text{NG}} = -\mu \int dt \, dz \sqrt{-g_{tt} g_{zz}}|_{r=0}.
\]

(4)

Assuming that \(g_{tt}\) and \(g_{zz}\) do not depend on \(r\) and \(\theta\), the Nambu–Goto action takes the form

\[
S_{\text{on-shell}}^{\text{NG}} = -\mu \int dt \, dz \int d\theta \sqrt{-g_{\theta \theta}} \frac{\delta(r)}{\pi}.
\]

(5)

A variation in the metric implies the variation

\[
\delta S_{\text{on-shell}}^{\text{NG}} = \frac{1}{2} \int dt \, dz \, d\theta \sqrt{-g} T^{\mu \nu} \delta g_{\mu \nu},
\]

(6)

where

\[
T^{\mu \nu} = -\frac{\mu}{\sqrt{g_{rr} g_{\theta \theta}}} \frac{\delta(r)}{\pi} \begin{pmatrix}
(g_{tt})^{-1} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

(7)

is the stress–energy tensor of the static string.

The variation of the gravity action leads to the Einstein equations of motion

\[
R^{\mu \nu} - \frac{R}{2} g^{\mu \nu} - \frac{3}{L^2} g^{\mu \nu} = 8\pi G_4 T^{\mu \nu}.
\]

(8)

2.2. The conical \(\text{AdS}_4\) solution

Assuming that an static rigid string can be described by a static cylindrical symmetric metric, we begin with the most general static cylindrical symmetric metric in anti-de Sitter (AdS) background:

\[
ds^2 = e^{zL} \left[-e^{2\nu(r)} dr^2 + e^{2\lambda(r)} dr^2 + e^{2\psi(r)} d\theta^2ight] + e^{2\lambda(r)} dz^2,
\]

(9)

where \(-\infty < z < \infty\). This ansatz is motivated by the case of zero cosmological constant [20]. The Einstein equations take the form

\[
e^{2\lambda} e^{-2\lambda} \left[\nu'' + \psi'^2 + \lambda''\right] + \frac{3}{L^2} [e^{-2\lambda} - 1] = -8G_4 \mu e^{2\lambda} e^{-\lambda} e^{-\psi} \delta(r),
\]

(10)

\[
e^{2\lambda} e^{-2\lambda} \left[\nu' \lambda' + \lambda' \psi' + \nu' \psi'\right] + \frac{3}{L^2} [e^{-2\lambda} - 1] = 0,
\]

(11)

\[
e^{2\lambda} e^{-2\lambda} \left[\nu'' + \nu'^2 + \lambda''\right] + \frac{3}{L^2} [e^{-2\lambda} - 1] = 0,
\]

(12)

\[
e^{2\lambda} e^{-2\lambda} \left[-\nu' \lambda' - \lambda' \psi' + \nu'' + \nu' \psi' + \psi'' + \psi'^2\right] + \frac{3}{L^2} [e^{-2\lambda} - 1]
\]

\[= -8G_4 \mu e^{2\lambda} e^{-\lambda} e^{-\psi} \delta(r),
\]

(13)
\[ \frac{\lambda'}{L} = 0, \quad (14) \]

where \( f' \equiv df/dr \). Equations (11), (12) and (14) imply that \( \lambda = 0 \), and \( \nu \) is a constant that can be chosen zero by a rescaling of the time coordinate. Then, equations (10) and (13) reduce to the equation

\[ [\psi'' + \psi^2] e^\psi = [e^\psi]' = -8G_\lambda \delta(r), \quad (15) \]

with the solution

\[ e^\psi = ar + b + \frac{4G_\lambda \mu}{\pi} \int_{-\infty}^{\infty} \frac{e^{ikr}}{k^2} \, dk \]

\[ = (a - 4G_\lambda \mu)r + b \quad (16) \]

We must have \( a = 1 \) and \( b = 0 \) in order to recover the AdS spacetime in the absence of the string (\( \mu = 0 \)). This way, we have obtained the metric

\[ ds^2 = e^{-2z} L \left[ -dt^2 + dr^2 + (1 - 4G_\lambda \mu)^2 r^2 d\theta^2 \right] + dz^2 \quad (17) \]

This metric describes a 4D AdS spacetime with a conical deficit as can be seen by redefining the angular coordinate

\[ \tilde{\theta} = (1 - 4G_\lambda \mu) \theta \quad (18) \]

so that

\[ ds^2 = e^{-2\tilde{\theta}} \left[ -dt^2 + dr^2 + r^2 d\tilde{\theta}^2 \right] + dz^2, \quad (19) \]

with \( 0 \leq \tilde{\theta} < 2\pi(1 - 4G_\lambda \mu) \) corresponding to an angular deficit of \( 8\pi G_\lambda \mu \). A similar angular deficit was obtained earlier in [17] considering a vortex line configuration in the limit of a weak gravitational field.

In the limit of zero cosmological constant (\( L \to \infty \)), our solution reduces to the 4D metric of a cosmic string [20] (see also [21]). Note that the angular deficit must be lower than \( 2\pi \) so that we have a bound for the string tension:

\[ 0 < \mu < \frac{1}{4G_4}. \quad (20) \]

### 2.3. The \((2+1)\) conical boundary

Redefining the transverse radial coordinate

\[ r = \frac{\rho^{1-4G_\lambda \mu}}{(1 - 4G_\lambda \mu)} \quad (21) \]

the metric (17) takes the form

\[ ds^2 = e^{-2\tilde{\theta}} \left[ -dt^2 + \rho^{-8G_\lambda \mu} (dr^2 + \rho^2 d\tilde{\theta}^2) \right] + dz^2. \quad (22) \]

Taking the limit \( z \to -\infty \), we get the boundary of 4D conical AdS spacetime:

\[ ds^2 = -dt^2 + \rho^{-8G_\lambda \mu} (dr^2 + \rho^2 d\tilde{\theta}^2). \quad (23) \]

This metric can be identified with the conical solution obtained by Desser, Jackiw and ’t Hooft [15] when solving the \((2+1)\) Einstein equations with a zero cosmological constant in the presence of a point particle of mass \( M \) localized at the origin. This identification implies the relation

\[ M = \frac{G_\lambda \mu}{G_3} \quad (24) \]
where $G_3$ is the $(2+1)$ Newton constant. Note that in the limit of small $G_4 \mu$, we can approximate the metric (23) by
\begin{equation}
\text{d}s^2 = -\text{d}t^2 + (1 - 8G_4 \mu \ln \rho)(\text{d}\rho^2 + \rho^2 \text{d}\theta^2).
\end{equation}
The logarithmic term is associated with the spatial components of the graviton and is a characteristic of $(2+1)$ fields.

This way, solving the $(3+1)$ Einstein equations in the presence of an infinite static string and negative cosmological constant, we have found a conical AdS$_4$ spacetime whose boundary can be identified with a $(2+1)$ conical spacetime. This result suggests a duality between these spaces similar to the usual AdS$_4$/CFT$_3$ correspondence. Below, we investigate this duality for the case of massive bulk scalar fields in order to see the effect of the conical deficit on boundary scalar operators constructed in the $(2+1)$ strongly coupled field theory.

3. Scalar Green’s function in the $(2+1)$ cone

In this section, we calculate the retarded Green’s function of a scalar operator living in a $(2+1)$ conical geometry assuming a duality similar to the AdS$_4$/CFT$_3$ correspondence. Following the anti-de Sitter/conformal field theory (AdS/CFT) dictionary, a scalar operator of dimension $\Delta$ living in a $(2+1)$ conical boundary would be dual to a scalar field with mass $m$ living in the conical AdS$_4$ bulk. The holographic relation between $m$ and $\Delta$ is given by
\begin{equation}
\Delta = \frac{3}{2} + \sqrt{\frac{9}{4} + \frac{m^2}{L^2}},
\end{equation}
where $L$ is the AdS$_4$ radius.

Before any calculation it is convenient define the coordinate $\tilde{z} = L \exp(\frac{z}{L})$ so that metric (19) takes the Poincaré form
\begin{equation}
\text{d}s^2 = \frac{L^2}{\tilde{z}^2}[-\text{d}t^2 + \text{d}r^2 + r^2 \text{d}\tilde{\theta}^2 + \text{d}\tilde{z}^2],
\end{equation}
where $0 < \tilde{z} < \infty$, and $0 \leq \tilde{\theta} \leq \tilde{\theta}_0$ with $\tilde{\theta}_0 = 2\pi(1 - 4G_4 \mu)$. The boundary is given by the region $\tilde{z} = 0$.

Consider a massive scalar living in the conical AdS$_4$ geometry with action
\begin{equation}
S = -\kappa \int \text{d}^4x \sqrt{-g}[g^{\mu\nu}\partial_\mu \phi \partial_\nu \phi + m^2 \phi^2]
\end{equation}
with $\kappa = \frac{1}{16\pi G_4}$. The equation of motion takes the form
\begin{equation}
\tilde{z}^2 \partial_z[\tilde{z}^{-2} \partial_z \phi] - \partial_\tilde{\theta}^2 \phi + \frac{1}{r} \partial_r [r \partial_r \phi] + \frac{1}{r^2} \partial_\tilde{\theta}^2 \phi - \frac{m^2 L^2}{\tilde{z}^2} \phi = 0.
\end{equation}
The solution for the scalar field regular at $r \to \infty$ can be expanded as
\begin{equation}
\phi(t, r, \tilde{\theta}, \tilde{z}) = (2\pi)^{-3/2} \sum_\lambda \int \text{d}p \text{d}\lambda \phi_p(\omega, \lambda) J_\lambda(pr) \mathcal{Z}(k, \tilde{z}) \mathcal{Z}(\lambda, \tilde{\theta}, \omega, p),
\end{equation}
where $J_\lambda$ is a Bessel function and $\mathcal{Z}(k, \tilde{z})$ satisfies the equation
\begin{equation}
\tilde{z}^2 \partial_z[\tilde{z}^{-2} \partial_z \mathcal{Z}(k, \tilde{z})] + \left(k^2 - \frac{m^2 L^2}{\tilde{z}^2}\right) \mathcal{Z}(k, \tilde{z}) = 0,
\end{equation}
with $k = \sqrt{\omega^2 - p^2}$. Note that we are assuming that the dual operator creates time-like particles and the scalar field is periodic in $\tilde{\theta}$ even in the presence of an angular deficit, i.e.
\begin{equation}
\phi(\tilde{\theta} + \tilde{\theta}_0) = \phi(\tilde{\theta}) \quad \rightarrow \quad \lambda = \frac{2n\pi}{\tilde{\theta}_0} = \frac{n}{(1 - 4G_4 \mu)}.
\end{equation}
The solution to equation (31) which is real at the boundary and satisfies the incoming wave condition at the horizon is

\[ Z(k, \bar{z}) = -iD(k)\bar{z}^{3/2} \left[ J_v(k\bar{z}) \pm iY_v(k\bar{z}) \right] = \begin{cases} -iD(k)\bar{z}^{3/2} H^{(i)}_v(k\bar{z}) & \text{for } \omega > 0 \\ -iD(k)\bar{z}^{3/2} H^{(0)}_v(k\bar{z}) & \text{for } \omega < 0, \end{cases} \tag{33} \]

where \( H^{(i)}_v, H^{(0)}_v \) are the Hankel functions, \( D(k) \) is a real function and

\[ \nu = \sqrt{\frac{3}{2} + m^2 L^2} = \Delta - \frac{3}{2}. \tag{34} \]

Near the boundary, the function \( Z(k, \bar{z}) \) diverges as \( \bar{z}^{3/2-\nu} \) so it is convenient to define the boundary at \( \bar{z} = \epsilon \) and make \( \epsilon \to 0 \) at the very end. The boundary condition for the scalar field can be written as

\[ \phi(t, r, \tilde{\theta}, \bar{z}) = \epsilon^{3/2-\nu} \phi_0(t, r, \tilde{\theta}) \tag{35} \]

where

\[ \phi_0(t, r, \tilde{\theta}) = (2\pi)^{-3/2} \sum_{\lambda} \int d\omega \, dp \, e^{-i\omega t} e^{i\lambda \tilde{\theta}} J_{\lambda}(pr)\phi_\lambda(\omega, p) \tag{36} \]

is the boundary field that couples to a scalar operator \( \mathcal{O} \) with a conformal dimension \( \Delta \) satisfying (26). From (30), (33), (35) and (36), we obtain

\[ Z(k, \bar{z}) = \frac{\bar{z}^{3/2} H^{(i)}_v(k\bar{z})}{\epsilon^{3/2} H^{(0)}_v(k\epsilon)} \epsilon^{3/2-\nu}. \tag{37} \]

The boundary term of the on-shell action can be written as

\[ S_{\text{Boundary}} = -\kappa \lim_{\epsilon \to 0} \int dt \, dr \, d\tilde{\theta} \sqrt{-g} g^{\mu \nu} \partial_{\mu} \phi_\lambda \partial_{\nu} \phi_\lambda |_{z = \epsilon}. \tag{38} \]

Substituting the expansion (30) into this action, we find

\[ S_{\text{Boundary}} = -\kappa \frac{L^2}{2\pi} (1 - 4G_4\mu) \sum_{\lambda, \lambda'} \int_0^\infty d\omega \int_{-\infty}^\infty d\omega' \int_0^\infty dp \int_0^\infty d\omega' dp' \delta(\omega + \omega') \delta(\omega + \omega') \int_0^\infty dr J_{\lambda}(pr) J_{\lambda'}(p'r) F(\omega, p, \omega', p') \phi_\lambda(\omega, p) \phi_{\lambda'}(\omega', p'), \tag{39} \]

where

\[ f_{\lambda\rightarrow\lambda'}(p, p') = \int dr \, r J_{\lambda}(pr) J_{\lambda'}(p'r), \tag{40} \]

\[ F(\omega, p, \omega', p') = \lim_{\epsilon \to 0} [\bar{z}^{-2} Z(k', \bar{z}) \partial_{\bar{z}} Z(k, \bar{z})] |_{z = \epsilon}, \tag{41} \]

and \( k = \sqrt{\omega^2 - p^2}, k' = \sqrt{\omega'^2 - p'^2} \). Using a real time prescription similar to [16], we obtain the retarded Green’s function

\[ G_{\lambda\lambda'}^{R}(\omega, p, \omega', p') = -\frac{\kappa L^2}{\pi} (1 - 4G_4\mu) \delta(\omega + \omega') f_{\lambda\rightarrow\lambda'}(p, p') F(\omega, p, \omega', p'). \tag{42} \]

We can integrate the Bessel functions in (40) finding

\[ f_{\lambda\rightarrow\lambda'}(p, p') = \cos(\pi \lambda) \delta(p - p') \frac{\delta(p - p')}{p} - \sin(\pi \lambda) h_{\lambda}(p, p'), \tag{43} \]

with

\[ h_{\lambda}(p, p') \equiv -\frac{2}{\pi(p^2 - p'^2)} \left( \frac{p}{p'} \right)^\lambda + \frac{1}{\pi p(p - p') \delta(0)} \tag{44} \]
On the other hand,
\[
\mathcal{F}(\omega, p, \omega', p') = \lim_{\epsilon \to 0} \left[ \epsilon^{1-2\nu} \frac{\partial_\nu \left[ \frac{\Gamma^{\nu \mu} \left( k \hat{\nu}^\mu \right) \Gamma \left( 2\nu \right) }{\Gamma \left( 1+\nu \right) } \right]}{\epsilon^{1/2} H^{1/2} \left( k \epsilon \right)} \right]
\]
\[
= \lim_{\epsilon \to 0} \left[ \left( \frac{3}{2} - \nu \right) \epsilon^{-2\nu} + k \epsilon^{1-2\nu} \frac{H^{1/2} \left( k \epsilon \right)}{H^{1/2} \left( k \epsilon \right)} \right].
\]

Using the relation [22]
\[
H^{1/2} \left( k \epsilon \right) = \pm i \csc \pi \nu \left[ -J_{-\nu} (\zeta) + e^{-i \nu \pi} J_{\nu} (\zeta) \right],
\]
and the Bessel expansions
\[
J_{\pm\nu} (\zeta) = \left( \frac{\zeta}{2} \right)^{\pm\nu} \sum_{\ell=0}^\infty \frac{\left( -1 \right)^\ell}{\Gamma(\pm \nu + \ell + 1) \ell!} \left( \frac{\zeta}{2} \right)^{2\ell},
\]
we obtain
\[
\mathcal{F}(\omega, p, \omega', p') = \begin{cases}
- (\omega^2 - p^2)^\nu a(v) [\cos(\pi v) - i \sin(\pi v) \text{sgn}(\omega)] + \ldots & \nu \notin \mathbb{Z} \\
- (\omega^2 - p^2)^\nu b(v) [\ln(\omega^2 - p^2) - i \pi \text{sgn}(\omega)] + \ldots & \nu \in \mathbb{Z},
\end{cases}
\]
where
\[
a(v) = \frac{2^{1-2\nu} \pi}{\sin(\pi v) \Gamma^2(v)}, \quad b(v) = \frac{2^{1-2\nu}}{\Gamma^2(v)},
\]
and the dots indicate the divergent terms that are canceled using holographic renormalization.

In this way, we obtain the retarded Green’s function
\[
G^R_{\lambda\lambda'}(\omega, p, \omega', p') = \frac{\kappa L^2}{\pi} a(v)(1 - 4G_{4\mu})\delta_{\nu, 1-\lambda} \delta(\omega + \omega')
\]
\[
\times \left[ \cos(\pi \lambda) \frac{\delta(p - p')}{p} - \sin(\pi \lambda) h_{\lambda}(p, p') \right]
\]
\[
\times (\omega^2 - p^2)^v [\cos(\pi v) - i \sin(\pi v) \text{sgn}(\omega)]
\]
for \( \nu \notin \mathbb{Z} \) and
\[
G^R_{\lambda\lambda'}(\omega, p, \omega', p') = \frac{\kappa L^2}{\pi} b(v)(1 - 4G_{4\mu})\delta_{\nu, 1-\lambda} \delta(\omega + \omega')
\]
\[
\times \left[ \cos(\pi \lambda) \frac{\delta(p - p')}{p} - \sin(\pi \lambda) h_{\lambda}(p, p') \right]
\]
\[
\times (\omega^2 - p^2)^v [\ln(\omega^2 - p^2) - i \pi \text{sgn}(\omega)]
\]
for \( \nu \in \mathbb{Z} \), where \( h_{\lambda}(p, p') \) is given in equation (44) and \( v = \Delta - 3/2 \).

These results are valid as long as the time-like condition \( \omega^2 - p^2 > 0 \) is satisfied. In the space-like case \( \omega^2 < p^2 \), the imaginary terms in (50) and (51) disappear, while the real terms can be obtained inverting the sign of \( \omega^2 - p^2 \) and substituting \( \cos(\pi v) \) by 1.

The conical deficit, represented by \( G_{4\mu} \), not only modifies the global factor of the retarded Green’s function, but also introduces a term proportional to \( \sin(\pi \lambda) \) with \( \lambda = n/(1 - 4G_{4\mu}) \) that not only breaks conformal invariance but makes the retarded Green’s function now depend on both momenta \( p \) and \( p' \). We will see that even in the absence of this term any Green’s function in the 3D conical spacetime would depend separately on the space vectors \( \vec{x} = (r \cos \theta, r \sin \theta) \) and \( \vec{x}' = (r' \cos \theta', r' \sin \theta') \) and not only on the difference \( \vec{x} - \vec{x}' \).
4. Conical Green’s functions in space coordinates

As we mentioned before, the conical deficit breaks not only conformal symmetry but also translation invariance. Below, we will investigate how this breaking occurs. For simplicity, we will consider discrete values for the string tension

$$\mu = \frac{1}{4G_k \left(1 - \frac{1}{k}\right)} , \quad k = 1, 2, \ldots ,$$

so that we have integer values for \(\lambda\) and \(\lambda'\). Note that the term proportional to \(\sin(\pi \lambda)\) vanishes. Now we need to know how the Green’s function depends on the space coordinates \(r\) and \(\theta\).

For this purpose, we perform the Bessel–Fourier transformation

$$\delta(\omega + \omega')G^R(\omega, r, \theta, r', \theta') = \frac{1}{(2\pi)^2} \sum_{\lambda} \sum_{\lambda'} e^{i\lambda \theta} e^{i\lambda' \theta'} \int dp \int dp' J_{\lambda}(p r) J_{\lambda'}(p' r') G^R_{\lambda\lambda'}(\omega, p, \omega', p').$$

The Green’s function takes the form

$$G^R(\omega, r, \theta, r', \theta') = \begin{cases} \frac{\kappa L^2}{4\pi^2 k} \alpha(\nu) [\cos(\pi \nu) - i \sin(\pi \nu) \text{sgn}(\omega)] \Omega(k, v, \omega, r, r', \theta - \theta') & v \notin \mathbb{Z} \\ \frac{\kappa L^2}{4\pi^2 k} \beta(v) [\partial_v - i\pi \text{sgn}(\omega)] \Omega(k, v, \omega, r, r', \theta - \theta') & v \in \mathbb{Z} \end{cases},$$

where

$$\Omega(k, v, \omega, r, r', \theta - \theta') = \sum_n e^{i \alpha_n (\theta - \theta')} \int dp J_{\kappa n}(p r) J_{\kappa n}(p' r') (\omega^2 - p^2)^v.$$ 

Using the representation

$$J_{\kappa n}(p r) J_{\kappa n}(p' r') = \frac{1}{2\pi} \int_0^{2\pi} \cos(\kappa \alpha r) J_0(p \sqrt{r^2 + r'^2 - 2rr' \cos \alpha})$$

and the integral

$$\int_0^\infty dp J_0(pa)(\omega^2 - p^2)^v = \frac{2v+1}{\pi} \Gamma(v+1) \sin(\pi v) [i \cos(\nu \omega)]^{v+1} K_{\nu+1}(-i\omega a),$$

we obtain

$$\Omega(k, v, \omega, r, r', \theta - \theta') = \frac{2v}{\pi^2} \Gamma(v+1) \sin(\pi v) [i \cos(\nu \omega)]^{v+1} \int_0^{2\pi} d\alpha \sum_n \cos[k n (\theta - \theta')] \cos(k \alpha) \times (r^2 + r'^2 - 2rr' \cos \alpha)^{-\frac{1+i}{4}} K_{\nu+1}(-i\omega \sqrt{r^2 + r'^2 - 2rr' \cos \alpha}),$$

where \(K_n(w)\) is the modified Bessel function. Using this result in (54), we find the retarded Green’s function

$$G^R(\omega, r, r', \theta - \theta') = \frac{\kappa L^2}{4\pi^2 k} 2^{1-v} \nu \cos(\pi \nu) - i\pi \sin(\pi \nu) \text{sgn}(\omega) [i \cos(\nu \omega)]^{v+1}$$

$$\times \int_0^{2\pi} d\alpha \left[ \sum_n \cos[k n (\theta - \theta')] \cos(k \alpha) \right]$$

$$\times (r^2 + r'^2 - 2rr' \cos \alpha)^{-\frac{1+i}{4}} K_{\nu+1}(-i\omega \sqrt{r^2 + r'^2 - 2rr' \cos \alpha}).$$
expressed as an integral of the extra angular coordinate $\alpha$. This result is valid for integer and non-integer $\nu$. In the absence of deficit ($k = 1$), we can use the completeness relation

$$\sum_n \cos(n\alpha) \cos(n\beta) = 2\pi \delta(\alpha - \beta)$$  \hfill (60)

to obtain

$$G^R_{(k=1)}(\omega, r', \theta - \theta') = \frac{\kappa L^2}{2\pi^4 k} \frac{2^{1-v}}{\Gamma(v)} [\cos(\pi \nu) - i\pi \sin(\pi \nu) \text{sgn}(\omega)]$$

$$\times (i|\omega|)^{v+1} |\vec{r} - \vec{r}'|^{v-1} K_v(\kappa r - \bar{r} - \bar{r}')$$, \hfill (61)

where $|\vec{x} - \vec{x}'| = \sqrt{r^2 + r'^2 - 2rr' \cos(\theta - \theta')}$ is the distance between two points in the 2D cone. After a Fourier transform in the $\omega$ coordinate takes the form

$$G^R_{(k=1)}(t - t', r, r', \theta - \theta') \sim (|t - t'|^2 - |\vec{r} - \vec{r}'|^2)^{-\Delta}$$  \hfill (62)

which is the expected result for a $(2 + 1)$ CFT [3].

In the presence of a conical deficit ($k > 1$), it is very difficult to solve the integral in $\alpha$ at any energy $\omega$. However, this can be done in the low-energy regime ($\omega \ll 1$) and for integer $\nu$. Using the approximation

$$K_{v+1}(\xi \to 0) \approx 2^v \Gamma(v + 1) \xi^{-v-1},$$  \hfill (63)

equation (59) takes the form

$$G^R(\omega \to 0) = -\frac{\kappa L^2}{2\pi^4 k} v^2 \pi^2 \frac{2^{v-1}}{\cos(\pi \nu) - i\pi \sin(\pi \nu) \text{sgn}(\omega)}$$

$$\times (i|\omega|)^{v+1} |\vec{r} - \vec{r}'|^{v-1} K_v(\kappa r - \bar{r} - \bar{r}')$$, \hfill (64)

where $\xi = r/r'$ and we used the integral

$$\int_0^{2\pi} \frac{\cos(m\alpha)}{(1 - 2\xi \cos(\alpha) + \xi^2)^{v+1}} d\alpha = 2\pi \frac{\xi^{2v+m}}{(1 - 2\xi \cos(\alpha) + \xi^2)^{v+1}} \sum_{i=0}^{\nu} \frac{(v + m)! (2v - i)!}{i!(v + m - i)! v!(v - i)!} \left( \frac{1 - \xi^2}{\xi^2} \right)^i$$  \hfill (65)

valid for $\xi^2 \leq 1$, $m \geq 0$, and the series

$$\sum_{n=1}^{\infty} \cos(n\alpha)y^n = \frac{1}{2} \left( \frac{1 - y^2}{1 - 2y \cos x + y^2} - 1 \right)$$  \hfill (66)

valid for $y^2 \leq 1$ [23]. It is very interesting to analyze the case $\nu = 1$ (corresponding to $\Delta = 5/2$) that gives

$$G^R(\omega \to 0)|_{\nu=1} = -\frac{\kappa L^2}{\pi^4 k} (r^2 - r'^2)^{-3}[\nu^{2k} - 2\nu^kr^k \cos[k(\bar{\theta} - \bar{\theta}')] + r^{2k}]^{-2}$$

$$\times (r^2 + r'^2)(r^{4k} - r'^{4k}) - 4k(r^2 - r'^2)r^{2k}r'^{2k}$$

$$+ [2(k - 1)r^2 - 2(k + 1)r^2]r'^{2k} \cos[k(\bar{\theta} - \bar{\theta}')]$$

$$+ [2(k + 1)r^2 - 2(k - 1)r^2]r'^{2k} \cos[k(\bar{\theta} - \bar{\theta}')].$$  \hfill (67)
This Green’s function (for \( k > 1 \)) also depends separately on the variables \( r, r', (\theta - \theta') \) and not only on the distance \( |\vec{x} - \vec{x}'| = \sqrt{r^2 + r'^2 - 2rr' \cos(\theta - \theta')} \). This means that a scalar perturbation on a point \( \vec{x} = (r \cos \theta, r \sin \theta) \) produces an effect on a point \( \vec{x}' = (r' \cos \theta', r' \sin \theta') \) that depends not only on the distance between them but also on the position of the perturbation. This is a consequence of the breaking of translation invariance. A similar result can be obtained for other values of \( \nu \). In the \( k = 1 \) case (corresponding to \( \mu = 0 \)), we recover result (62).

Note that in the limit \( \theta \to \theta', r \to r' \), the Green’s function reduces to

\[
G^R(\omega \to 0)|_{\nu = 1} \approx -\frac{k L^2}{\pi^2 k^2} (r' - r)^{-4} = \frac{1}{k^2} G^R_{k=1}(\omega \to 0)|_{\nu = 1}
\] (68)

which means that the conical Green’s function \( (k > 1) \) diverges in the same way as the flat Green’s function \( (k = 1) \). The only difference between the singularities is a factor of \( 1/k^2 \). This result is very different from that obtained by solving the Green’s function of a \((2+1)\) massless scalar field [24] that would correspond to a case \( \nu = -1 \) which is forbidden in the duality. The Green’s function obtained in [24] has a logarithmic dependence so that the singularity can be removed. In our case, the singularity cannot be removed because of the power dependence of the Green’s functions of operators with dimension \( \nu > 0 \).

5. Conclusions

In this work, we found a conical AdS4 spacetime by solving the 4D Einstein equations in the presence of an infinite string and negative cosmological constant. We identified its boundary with the 3D conical spacetime found by Deser, Jackiw and ’t Hooft. Our results suggest a correspondence between fields living in a conical AdS4 spacetime and operators living in a \((2+1)\) cone. We used this correspondence to calculate the retarded Green’s function of scalar operators in the 3D conical spacetime.

It is important to remark that the duality proposed in this paper is phenomenological because the conical AdS4 solution is not derived from M theory. Therefore, we do not know the details of the bulk and boundary theories. Nevertheless, we believe that it is worth to explore this non-conformal duality since the conical space is a natural solution of 3D gravity. A conical AdS4 spacetime may arise from the near horizon limit of a 11D solution to supergravity which is the low energy limit of M theory. Since the sources of a 4-form flux in 11D supergravity are M2- and M5-branes, a possible scenario would involve the intersection of two sets of these branes (one of them associated with the AdS4 and the other associated with the conical deficit) but this scenario may not be easy to construct. For a review of brane intersections in M theory see [25].

As we mentioned in the introduction, conical defects appear in real experiments like the carbon monolayer (graphene). For instance, if carbon atoms are removed from the graphene lattice without affecting the threefold coordination of the other atoms, the lattice surface is warped (fullerene). The graphene sheet with this kind of defect takes the form of a cone with the angular deficit related to the number of removed atoms. An effective low-energy description of the electronic states involves the study of a Dirac theory for massless electrons living in a \((2+1)\) conical spacetime. In this context, there are some results indicating the influence of a conical defect on the density of electronic states [12–14]. It is then interesting to ask if a conical defect can modify the graphene electrical conductivity in the case of finite temperature and chemical potential. This conductivity can be calculated from holography because it is directly related to the retarded Green’s functions of electromagnetic currents and
the AdS/CFT correspondence maps a $U(1)$ current living in the boundary to an Abelian gauge field living in the bulk. Then, a natural continuation of this work should involve the inclusion of temperature and chemical potential in the conical boundary field theory which corresponds to the inclusion of a black brane and a background gauge field in the bulk. This can be done in two different ways. If the background gauge field is strong, we solve the equations arising from the Einstein–Maxwell–Nambu–Goto action with negative cosmological constant. A possible guess for the solution would be a charged AdS$_4$ black brane with a conical deficit (for a recent review on charged AdS black branes see [26]). On the other hand, if the gauge field is weak, we need to solve only the Einstein–Nambu–Goto equations (with negative cosmological constant) and the solution would involve a neutral AdS$_4$ black brane with a conical deficit dual to a finite temperature conical CFT and a gauge field living in a probe brane with an asymptotic value related to the chemical potential. It is important to remark that any of these backgrounds will reduce to the conical AdS$_4$ spacetime obtained here. These interesting issues will be developed in a future paper.

It would also be interesting to explore the effect of a conical defect in the recent proposed mechanisms for chiral symmetry breaking in $(2+1)$ condensed matter systems (see for instance [27]).

Acknowledgment

The authors are financially supported by CNPq.

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