Some Extended Berezin Number Inequalities

Satyajit Sahoo, Mojtaba Bakherad

Abstract. We present generalized extensions of Berezin number inequalities involving the Euclidean Berezin number and $f$-connection of operators.

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $| \cdot |$. Let $\mathcal{L}(\mathcal{H})$ be the $C^*$-algebra of all bounded linear operators from $\mathcal{H}$ into itself. In the case when $\dim \mathcal{H} = n$, we identify $\mathcal{L}(\mathcal{H})$ with the matrix algebra $M_n$ of all $n \times n$ complex matrices. An operator $A \in \mathcal{L}(\mathcal{H})$ is said to be positive, and denoted $A \geq 0$, if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathcal{H}$.

The numerical range of $T \in \mathcal{L}(\mathcal{H})$ is defined as

$$W(T) = \{\langle Tx, x \rangle : x \in \mathcal{H}, |x| = 1\}$$

and the numerical radius of $T$, denoted by $w(T)$, is defined by $w(T) = \sup \{|z| : z \in W(T)\}$.

It is well-known that the set $W(T)$ is a convex subset of the complex plane and that the numerical radius $w(\cdot)$ is a norm on $\mathcal{L}(\mathcal{H})$; being equivalent to the usual operator norm $|T| = \sup\{|Tx| : x \in \mathcal{H}, |x| = 1\}$. In fact, for every $T \in \mathcal{L}(\mathcal{H})$,

$$\frac{1}{2} |T| \leq w(T) \leq |T|. \tag{1}$$

Obtaining sharper lower and upper bounds of (1) have attracted numerous researchers due to its applications in the operator theory and other fields. For example, bounds for the zeros of polynomials is an interesting application of the numerical radius inequalities (see [7]). We refer the reader to [9, 11, 18, 23, 24, 28] as a sample of references treating numerical radius inequalities.

Another interesting set of applications of the quantity $w(A)$ includes the study of perturbation, convergence and approximation problems as well as iterative methods, etc; [2].

A Hilbert space $\mathcal{H} = \mathcal{H}(\Omega)$ of complex valued functions on a nonempty open set $\Omega \subset \mathbb{C}$ which has the property that point evaluations are continuous, is called a functional Hilbert space. The point evaluations are

2010 Mathematics Subject Classification. Primary: 47A63, secondary: 15A60.

Keywords. Young inequality; Euclidean Berezin number; Positive operator; $f$-connection; Berezin number; Reproducing kernel.

Received: 21 May 2020; Accepted: 18 July 2020

Communicated by Fuad Kittaneh

Corresponding author: Satyajit Sahoo

Email addresses: satyajitsahoo2010@gmail.com (Satyajit Sahoo), mojtaba.bakherad@yahoo.com (Mojtaba Bakherad)
of certain scalar ones. For example, the classical Young inequality which states that if $f(\lambda) = \langle f, k_\lambda \rangle$ for all $f \in \mathcal{H}$ by Riesz representation theorem. The collection $\{k_\lambda : \lambda \in \Omega\}$ is known as the reproducing kernel of $\mathcal{H}$. Problem 37 of [14] states that the reproducing kernel of a functional Hilbert space $\mathcal{H}$ with $\{e_n\}$ as an orthonormal basis is $k_\lambda(z) = \sum_n e_n(\lambda)e_n(z)$. Let $\hat{k_\lambda} = k_\lambda/[k_\lambda]$ be the normalized reproducing kernel of $\mathcal{H}$, where $\lambda \in \Omega$. The function $\hat{A}(\lambda) = \langle \hat{A}k_\lambda, \hat{k_\lambda} \rangle$ is the Berezin symbol of a bounded linear operator $A$ on $\mathcal{H}$. The Berezin set and the Berezin number of the operator $A$ are defined by

$$\text{Ber}(A) = \{\hat{A}(\lambda) : \lambda \in \Omega\} \quad \text{and} \quad \text{ber}(A) = \sup\{|\hat{A}(\lambda)| : \lambda \in \Omega\},$$

respectively. These definitions are named in honor of Felix Berezin, who introduced these notions in [8]. For our purpose, we set the Berezin norm as $\text{ber}(\beta T) = |\beta|\text{ber}(T)$ for all $\beta \in \mathbb{C}$.

During the last decades several generalizations, reverses, refinements and applications of the Young inequality have been given, see [3, 19–21] and the references therein. A refinement of inequality (2) is presented by Kittaneh and Manasrah [19] as follows:

$$a^\beta b^{1-\beta} \leq \beta a + (1 - \beta)b,$$

is an example of such important scalar inequalities.

During the last decades several generalizations, reverses, refinements and applications of the Young inequality in various setting have been given, see [3, 19–21] and the references therein. A refinement of inequality (2) is presented by Kittaneh and Manasrah [19] as follows:

$$a^\beta b^{1-\beta} \leq \beta a + (1 - \beta)b - r_0(a^2 - b^2)^{1/2}, \quad \text{where} \quad r_0 = \min(\beta, 1 - \beta).$$

Later, the same authors in [1] presented the general form of (3) as follows:

$$(a^\beta b^{1-\beta}r_0^{1/2} - b) \leq (\beta a + (1 - \beta)b)^{m/2}, \quad \text{where} \quad r_0 = \min(\beta, 1 - \beta)$$

and for any positive integer $m$. Recently, Choi [10] gave a further refinement of the Young inequality as follows:

$$(a^\beta b^{1-\beta}r_0^{1/2} + 2r_0) \leq (\beta a + (1 - \beta)b)^{m},$$

$$\left(a^\beta b^{1-\beta}r_0^{1/2} - b\right) \geq (\beta a + (1 - \beta)b)^{m},$$
Lemma 1.2. Following celebrated McCarthy inequality.

2.1. Berezin number inequalities for $f$

(i) $\beta a + (1 - \beta) b \leq (\beta a^r + (1 - \beta) b^r)^{\frac{1}{r}}$, \hspace{1cm} (7)

It follows from (7) and inequality (5) that

$$
(a^\beta b^{1-\beta})^m + (2r_0)^m \left( \frac{a + b}{2} \right)^m - (ab)^{\frac{m}{r_0}} \leq (\beta a^r + (1 - \beta) b^r)^{\frac{m}{r_0}},
$$

(8)

where $r_0 = \min\{\beta, 1 - \beta\}$. In particular, for $\beta = \frac{1}{2}$, we get

$$
(a^{\frac{1}{2}} b^{\frac{1}{2}})^m + \left( \frac{a + b}{2} \right)^m - (ab)^{\frac{m}{r_0}} \leq \frac{1}{2^r} (a^r + b^r)^{\frac{m}{r_0}}.
$$

(9)

In 1952, Kato [16] showed the mixed Schwarz inequality, which asserts

$$
|\langle Ax, y \rangle|^2 \leq \left| \langle A^{\frac{1}{2}} x, y \rangle \right|^2 \left| \langle A^{\frac{1}{2}} (1 - \beta) y, y \rangle \right|,
$$

(10)

for the operator $A \in \mathcal{L}(\mathcal{H})$ and the vectors $x, y \in \mathcal{H}$, where $|A| = (A^* A)^{1/2}$.

The objective of this paper is to present some results of Berezin number inequalities involving $f$-connections of operators. Finally, we present a generalized Euclidean Berezin number inequality and refine the inequality (13).

Many related results that extend known results from the literature will be presented with an emphasis on the relation with known results in the literature. The first needed inequality is the following generalization of the mixed Cauchy-Schwarz inequality [17, Theorem 1].

Lemma 1.1. Let $A \in \mathcal{L}(\mathcal{H})$ and let $f$ and $g$ be non-negative continuous functions on $[0, \infty)$ satisfying the identity $f(t) g(t) = t$ for all $t \in [0, \infty)$. Then

$$
|\langle Ax, y \rangle| \leq |\langle A x \rangle||\langle A^* y \rangle|,
$$

for all $x, y \in \mathcal{H}$.

When dealing with inner product inequalities, the following inequality becomes handy [12, Theorem 1.2]:

$$
f (\langle Ax, x \rangle) \leq (f(A)^{\frac{1}{2}} x, x),
$$

valid for the convex function $f : J \rightarrow \mathbb{R}$, the self-adjoint operator $A$ with spectrum in $J$ and the unit vector $x \in \mathcal{H}$. The inequality (11) is reversed when $f$ is concave. As a consequence of this inequality, we obtain the following celebrated McCarthy inequality.

Lemma 1.2. Let $T \in \mathcal{L}(\mathcal{H})$, $T \geq 0$ and $x \in \mathcal{H}$ be a unit vector. Then

(i) $\langle Tx, x \rangle \leq \langle T^r x, x \rangle$ for $r \geq 1$;

(ii) $\langle T^r x, x \rangle \leq \langle T x, x \rangle^r$ for $0 < r \leq 1$.

2. Main Results

2.1. Berezin number inequalities for $f$-connections of operators

For positive definite operators $T, S \in \mathcal{L}(\mathcal{H})$, the operator geometric mean is defined by

$$
T \# S = T^{1/2} (T^{-1/2} ST^{-1/2})^{1/2} T^{1/2}.
$$
Let $f$ be a continuous function defined on the real interval $J$ containing the spectrum of $T^{-1/2}ST^{-1/2}$, where $S$ is a self-adjoint operator and $T$ is a positive invertible operator. By using the continuous functional calculus, the $f$-connection $\sigma_f$ is defined as follows

$$T\sigma_f S = T^{1/2} f(T^{-1/2}ST^{-1/2}) T^{1/2}. \quad (12)$$

Note that for the functions $(1 - \beta) + \beta t^\beta$, the definition (12) leads to the arithmetic and geometric operator means, respectively; see [12]. The aim of this subsection is to extend and generalize main result of [6, Theorem 2].

**Theorem 2.1.** Let $T, S, X \in \mathcal{L}(\mathcal{H})$ be such that $T, S$ are positive invertible. Then for $m \in \mathbb{N}$ and $r \geq 1$,

$$ber^m((T\sigma_f S)X) \leq 2^{-m}ber^m((X^* T^{1/2} f^2(T^{-1/2}ST^{-1/2}) T^{1/2}X) + T^r) - \inf_{\lambda \in \Omega} \xi(\hat{k}_1),$$

where

$$\xi(\hat{k}_1) = \left\{ \frac{1}{2} (X^* T^{1/2} f^2(T^{-1/2}ST^{-1/2}) T^{1/2}X + T)\hat{k}_1, \hat{k}_1 \right\}^m - \left( (X^* T^{1/2} f^2(T^{-1/2}ST^{-1/2}) T^{1/2}X\hat{k}_1, \hat{k}_1)(T\hat{k}_1, \hat{k}_1) \right)^{m/2}. \quad (13)$$

**Proof.** We have

\begin{align*}
&\left\{|(T\sigma_f S)X\hat{k}_1, \hat{k}_1)\right\|^m \\
&= \left\{|(T^{1/2} f(T^{-1/2}ST^{-1/2}) T^{1/2}X\hat{k}_1, \hat{k}_1)\right\|^m \\
&= \left\{|(f(T^{-1/2}ST^{-1/2}) T^{1/2}X\hat{k}_1, \hat{k}_1)\right\|^m \\
&\leq \left\{|(f(T^{-1/2}ST^{-1/2}) T^{1/2}X\hat{k}_1, \hat{k}_1)^\lambda \right\}| T^{1/2}\hat{k}_1 |^m \\
&= \left( (f(T^{-1/2}ST^{-1/2}) T^{1/2}X\hat{k}_1, f(T^{-1/2}ST^{-1/2}) T^{1/2}X\hat{k}_1)^{1/2}(T\hat{k}_1, \hat{k}_1)^{1/2})^m \\
&= \left( (X^* T^{1/2} f^2(T^{-1/2}ST^{-1/2}) T^{1/2}X\hat{k}_1, \hat{k}_1)^{1/2}(T\hat{k}_1, \hat{k}_1)^{1/2})^m \\
&\leq 2^{-m} \left( (X^* T^{1/2} f^2(T^{-1/2}ST^{-1/2}) T^{1/2}X\hat{k}_1, \hat{k}_1)^{1/2}(T\hat{k}_1, \hat{k}_1)^{1/2})^m \\
&- \left( \left( (X^* T^{1/2} f^2(T^{-1/2}ST^{-1/2}) T^{1/2}X\hat{k}_1, \hat{k}_1) + (T\hat{k}_1, \hat{k}_1) \right)^m \\
&- (X^* T^{1/2} f^2(T^{-1/2}ST^{-1/2}) T^{1/2}X\hat{k}_1, \hat{k}_1)(T\hat{k}_1, \hat{k}_1) \right)^{m/2} \\
&\leq 2^{-m} \left( (X^* T^{1/2} f^2(T^{-1/2}ST^{-1/2}) T^{1/2}X)^r + T^r \right)\hat{k}_1, \hat{k}_1) ^{m/2} \\
&\leq \left( \left( (X^* T^{1/2} f^2(T^{-1/2}ST^{-1/2}) T^{1/2}X + T)\hat{k}_1, \hat{k}_1 \right)^m \\
&- (X^* T^{1/2} f^2(T^{-1/2}ST^{-1/2}) T^{1/2}X\hat{k}_1, \hat{k}_1)(T\hat{k}_1, \hat{k}_1) \right)^{m/2}.
\end{align*}

Taking supremum over $\lambda \in \Omega$, we get the desired result. \(\square\)

Putting $m = 1 = r$ in Theorem 2.1 and using the fact that $ber(T) \leq |T|$, we get the following result as follows.

**Corollary 2.2.** Let $T, S, X \in \mathcal{L}(\mathcal{H})$ be such that $T, S$ are positive invertible. Then

$$ber((T\sigma_f S)X) \leq \frac{1}{2} ber(X^* T^{1/2} f^2(T^{-1/2}ST^{-1/2}) T^{1/2}X + T) - \inf_{\lambda \in \Omega} \xi(\hat{k}_1),$$

where

$$\xi(\hat{k}_1) = \left\{ \frac{1}{2} (X^* T^{1/2} f^2(T^{-1/2}ST^{-1/2}) T^{1/2}X + T)\hat{k}_1, \hat{k}_1 \right\}^m - \left( (X^* T^{1/2} f^2(T^{-1/2}ST^{-1/2}) T^{1/2}X\hat{k}_1, \hat{k}_1)(T\hat{k}_1, \hat{k}_1) \right)^{m/2}. \quad (14)$$
where
\[ \tilde{\xi}(\hat{k}_1) = \left( \frac{X^* T^{1/2} f^2(T^{-1/2} ST^{-1/2}) T^{1/2} X + T}{2} \right) \]
\[ - \left( (X^* T^{1/2} f^2(T^{-1/2} ST^{-1/2}) T^{1/2} X \hat{k}_1, \hat{k}_1) (T \hat{k}_1, \hat{k}_1) \right)^{1/2}. \]

Taking \( f(t) = t^{1/2} \) and \( m = 1 = r \) in Theorem 2.1, obtain the following result.

**Corollary 2.3.** Let \( T, S, X \in \mathcal{L}(\mathcal{H}) \) be such that \( T, S \) are positive invertible. Then
\[ \mathrm{ber}(T \| S) X) \leq \frac{1}{2} \mathrm{ber}(X^* S X + T) - \inf_{\lambda \in \mathbb{R}} \tilde{\xi}(\hat{k}_1), \]
where
\[ \tilde{\xi}(\hat{k}_1) = \left( \frac{(X^* S X + T)}{2} \right) \hat{k}_1, \hat{k}_1 \]
\[ - \left( (X^* S X \hat{k}_1, \hat{k}_1) (T \hat{k}_1, \hat{k}_1) \right)^{1/2}. \]

Taking \( X = I \) in Theorem 2.1 we get the following.

**Corollary 2.4.** Let \( T, S \in \mathcal{L}(\mathcal{H}) \) be positive invertible. Then for \( m \in \mathbb{N}, r \geq 1 \)
\[ \left| T^{1/2} S \right|^{m} \preceq 2^{-m/2} \left( (T^{1/2} f^2(T^{-1/2} ST^{-1/2}) T^{1/2})^r + T \right|^{m/2} - \inf_{\lambda \in \mathbb{R}} \tilde{\xi}(\hat{k}_1), \]
where
\[ \tilde{\xi}(\hat{k}_1) = \left( \frac{(T^{1/2} f^2(T^{-1/2} ST^{-1/2}) T^{1/2} + T) \hat{k}_1, \hat{k}_1}{} \right)^{m/2} \]
\[ - \left( (T^{1/2} f^2(T^{-1/2} ST^{-1/2}) T^{1/2} \hat{k}_1, \hat{k}_1) (T \hat{k}_1, \hat{k}_1) \right)^{m/2}. \]

Taking \( r = 1 \), \( X = I \) and \( f(t) = t^{1/2} \), we have the following simplified form.

**Corollary 2.5.** Let \( T, S \in \mathcal{L}(\mathcal{H}) \) be such that \( T, S \) are positive invertible. Then for \( m \in \mathbb{N}, r \geq 1 \)
\[ \left| T \| S \right|^{m} \preceq 2^{-m/2} \left| S + T \right|^{m/2} - \inf_{\lambda \in \mathbb{R}} \tilde{\xi}(\hat{k}_1), \]
where
\[ \tilde{\xi}(\hat{k}_1) = \left( \frac{S + T}{2} \hat{k}_1, \hat{k}_1 \right)^{m/2} - \left( S \hat{k}_1, \hat{k}_1 \right)^{m/2} \hat{k}_1, \hat{k}_1 \]
\[ - \left( T \hat{k}_1, \hat{k}_1 \right)^{m/2}. \]

**Proposition 2.6.** Let \( T, S, X \in \mathcal{L}(\mathcal{H}) \) such that \( T, S > 0 \) and \( r > 1, m \in \mathbb{N} \). Then
\[ \left| (T^{1/2} S)^{m} \right| \preceq 2^{-m/2} \left( X^* T^{1/2} f^2(T^{-1/2} ST^{-1/2}) T^{1/2} X \right|^{m/2} - \inf_{\lambda, \mu \in \mathbb{R}} \tilde{\xi}(\hat{k}_1, \hat{k}_1), \]
where
\[ \tilde{\xi}(\hat{k}_1, \hat{k}_1) = \left( \frac{(X^* T^{1/2} f^2(T^{-1/2} ST^{-1/2}) T^{1/2} \hat{k}_1, \hat{k}_1)}{2} \right)^{m/2} \]
\[ - \left( (X^* T^{1/2} f^2(T^{-1/2} ST^{-1/2}) T^{1/2} \hat{k}_1, \hat{k}_1) (T \hat{k}_1, \hat{k}_1) \right)^{m/2}. \]

**Proof.** Let \( \hat{k}_1, \hat{k}_1 \in \mathcal{H}(\mathcal{H}), \) then
\[ \left| (T^{1/2} S)^{m} \right| = \left| (T^{1/2} f(T^{-1/2} ST^{-1/2}) T^{1/2} \hat{k}_1, \hat{k}_1) \right| \]
\[ = \left| (f(T^{-1/2} ST^{-1/2}) T^{1/2} \hat{k}_1, \hat{k}_1) \right| \]
\[ \leq \left( X^* T^{1/2} f^2(T^{-1/2} ST^{-1/2}) T^{1/2} \hat{k}_1, \hat{k}_1 \right)^{m/2} \hat{k}_1, \hat{k}_1 \]
\[ = \left( X^* T^{1/2} f^2(T^{-1/2} ST^{-1/2}) T^{1/2} \hat{k}_1, \hat{k}_1 \right)^{m/2} (T \hat{k}_1, \hat{k}_1) \]}^{m/2}. \]
Using similar technique as in Theorem 2.1, we get

\[
\langle (T\sigma S)X\bar{k}_1,\bar{k}_\mu \rangle^m \leq 2^{-m/r}(\langle X^* T^{1/2} \hat{f}(T^{-1/2}ST^{-1/2})T^{1/2}X\bar{k}_1,\bar{k}_1 \rangle + \langle T\bar{k}_\mu,\bar{k}_\mu \rangle)^{m/r} \\
- \left\{ \left( \frac{\langle X^* T^{1/2} \hat{f}(T^{-1/2}ST^{-1/2})T^{1/2}X\bar{k}_1,\bar{k}_1 \rangle + \langle T\bar{k}_\mu,\bar{k}_\mu \rangle}{2} \right)^m \\
- (\langle X^* T^{1/2} \hat{f}(T^{-1/2}ST^{-1/2})T^{1/2}X\bar{k}_1,\bar{k}_1 \rangle(T\bar{k}_\mu,\bar{k}_\mu) \rangle)^{m/2} \right\}
\]

Taking supremum over \( \lambda, \mu \in \Omega \), we have

\[
|(T\sigma S)X|^m \leq 2^{-m/r}(\langle X^* T^{1/2} \hat{f}(T^{-1/2}ST^{-1/2})T^{1/2}X \rangle^r + |T|)^{m/r} - \inf_{\lambda, \mu \in \Omega} \xi(\bar{k}_1,\bar{k}_\mu),
\]

where

\[
\xi(\bar{k}_1,\bar{k}_\mu) = \left( \frac{\langle X^* T^{1/2} \hat{f}(T^{-1/2}ST^{-1/2})T^{1/2}X\bar{k}_1,\bar{k}_1 \rangle + \langle T\bar{k}_\mu,\bar{k}_\mu \rangle}{2} \right)^m \\
- (\langle X^* T^{1/2} \hat{f}(T^{-1/2}ST^{-1/2})T^{1/2}X\bar{k}_1,\bar{k}_1 \rangle(T\bar{k}_\mu,\bar{k}_\mu) \rangle)^{m/2}.
\]

\( \square \)

In particular, letting \( f(t) = t^{1/2}, \ m = 1,2,\ldots \) we have the following simplified form.

**Corollary 2.7.** Let \( T,S,X \in \mathcal{L}(\mathcal{H}) \) such that \( T,S > 0 \) and let \( r > 1 \). Then

\[
|(T\# S)X|_{\text{ber}}^m \leq 2^{-m/r}(\langle X^* S X \rangle_{\text{ber}}^r + |T|)^{m/r} \\
- \inf_{\lambda, \mu \in \Omega} \left\{ \left( \frac{\langle X^* S X\bar{k}_1,\bar{k}_1 \rangle + \langle T\bar{k}_\mu,\bar{k}_\mu \rangle}{2} \right)^m \\
- (\langle X^* S X\bar{k}_1,\bar{k}_1 \rangle(T\bar{k}_\mu,\bar{k}_\mu) \rangle)^{m/2} \right\}.
\]

2.2. Generalized Euclidean Berezin number inequalities.

In this subsection, we show our main results; starting with the generalized Euclidean Berezin number. Our first result is a generalized refinement of [6, Theorem 9].

**Theorem 2.8.** Let \( \mathcal{H} = \mathcal{H}(\Omega) \) be a reproducing kernel Hilbert space on \( \Omega \) and \( A_i, B_i, S_i \in \mathcal{L}(\mathcal{H}) \) \((i = 1,2,\ldots,n)\) and let \( f \) and \( g \) be non negative continuous functions on \([0,\infty)\) such that \( f(t)g(t) = t \) for all \( t \in [0,\infty) \). Then for \( m = 1,2,\ldots \) and \( p, r \geq m, \)

\[
\text{ber}_m^r(A_1^* S_1 B_1,\ldots,A_n^* S_n B_n) \leq \frac{n^{1-\frac{m}{r}}}{2^m} \text{ber}_m^r \left( \sum_{i=1}^n \langle [B_i^* \hat{f}(|S_i|) B_i]^r + [A_i^* \hat{g}(|S_i|) A_i]^r \rangle \right) \\
- \inf_{\lambda \in \Omega} \xi(\bar{k}_1),
\]

where

\[
\xi(\bar{k}_1) = \sum_{i=1}^n \left[ \frac{1}{2^n} \left( \langle [B_i^* \hat{f}(|S_i|) B_i]^\frac{m}{r} + [A_i^* \hat{g}(|S_i|) A_i]^\frac{m}{r} \rangle \bar{k}_1,\bar{k}_1 \rangle \right)^m \\
- (\langle [B_i^* \hat{f}(|S_i|) B_i]^\frac{m}{r} \bar{k}_1,\bar{k}_1 \rangle \langle [A_i^* \hat{g}(|S_i|) A_i]^\frac{m}{r} \bar{k}_1,\bar{k}_1 \rangle)^{m/2} \right].
\]
Proof. Let \( \hat{k}_1 \) is the normalized reproducing kernel of \( \mathcal{H}(\Omega) \), then
\[
\sum_{i=1}^{n} |f(S_i)B_i \hat{k}_1, \hat{k}_1)|^p = \sum_{i=1}^{n} |(S_iB_i \hat{k}_1, \hat{k}_1)|^p \\
\leq \sum_{i=1}^{n} \left( \frac{1}{2} \left( \left( |B_i^{*} f^2(|S_i|)B_i|^{\hat{2}} \hat{k}_1, \hat{k}_1) + |A_i^{*} g^2(|S_i^*|)A_i|^{\hat{2}} \hat{k}_1, \hat{k}_1) \right)^m - \left( \left( |B_i^{*} f^2(|S_i|)B_i|^{\hat{2}} \hat{k}_1, \hat{k}_1) + |A_i^{*} g^2(|S_i^*|)A_i|^{\hat{2}} \hat{k}_1, \hat{k}_1) \right)^m \right) \right)^{\frac{p}{m}} (9) \]
\[
\leq \frac{1}{2} \left( \left( |B_i^{*} f^2(|S_i|)B_i|^{\hat{2}} \hat{k}_1, \hat{k}_1) + |A_i^{*} g^2(|S_i^*|)A_i|^{\hat{2}} \hat{k}_1, \hat{k}_1) \right)^m \right)^{\frac{p}{m}} \]
where the last inequality follows from (11), noting concavity of the mapping \( t \mapsto t^{\frac{p}{m}} \), as we have \( m \leq r \). Taking the supremum over \( \lambda \in \Omega \), we get the desired inequality. \( \Box \)

Letting \( m = 1 \) in Theorem 2.8, we get [6, Theorem 9].

**Corollary 2.9.** Let \( A_i, B_i, S_i \in L(C) \) \( (i = 1, \ldots, n) \) and let \( f \) and \( g \) be non negative continuous functions on \([0, \infty)\) such that \( f(t)g(t) = t \) for all \( t \in [0, \infty) \). Then for all \( r, p \geq 1 \),
\[
\text{ber}_p^r(A_i^* S_i B_i, \ldots, A^n S_n B_n) \leq \frac{n^{1+\frac{1}{r}}}{2^{1+p}} \text{ber}^r \left( \sum_{i=1}^{n} \left( |B_i^{*} f^2(|S_i|)B_i|^{\hat{2}} + |A_i^{*} g^2(|S_i^*|)A_i|^{\hat{2}} \hat{k}_1, \hat{k}_1) \right) \right) - \inf_{\lambda \in \Omega} \xi(\hat{k}_1),
\]
where
\[
\xi(\hat{k}_1) = \sum_{i=1}^{n} \left( \frac{1}{2} \left( |B_i^{*} f^2(|S_i|)B_i|^{\hat{2}} \hat{k}_1, \hat{k}_1) + |A_i^{*} g^2(|S_i^*|)A_i|^{\hat{2}} \hat{k}_1, \hat{k}_1) \right)^m \right)^{\frac{p}{m}} \]
Choosing \( f(t) = g(t) = t^\frac{1}{2} \) and \( S_i = I \) for \( i = 1, 2, \ldots, n \) in Theorem 2.8 we obtain the following simpler form.
Corollary 2.10. Let $A_i, B_i \in \mathcal{L}(\mathcal{H})$ ($i = 1, 2, \ldots, n$) and let $f$ and $g$ be non negative continuous functions on $[0, \infty)$ such that $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then for $m = 1, 2, \ldots, r, p \geq m$,

$$\text{ber}_p^m(A_1 B_1, \ldots, A_n B_n) \leq \frac{n^{1-\beta}}{2^\beta} \text{ber}_p^m \left( \sum_{i=1}^{n} (|B_i|^{\frac{\beta}{p}} + |A_i|^{\frac{\beta}{p}}) \right) - \inf_{k \in \Omega} \varepsilon(k_i),$$

where

$$\varepsilon(k_i) = \sum_{i=1}^{n} \left[ \frac{1}{2} \left( |B_i|^{\frac{\beta}{p}} + |A_i|^{\frac{\beta}{p}} \right) k_i, k_i \right] - \left( \langle |B_i|^{\frac{\beta}{p}} k_i, k_i \rangle \langle |A_i|^{\frac{\beta}{p}} k_i, k_i \rangle \right) \frac{1}{2}.$$

Following theorem of this article, we present an upper bound for the generalized Berezin number.

Theorem 2.11. Let $S_i \in \mathcal{L}(\mathcal{H})$ ($1 \leq i \leq n$). Then for $0 \leq \beta \leq 1, m \in \mathbb{N}$ and $p \geq 2m$,

$$\text{ber}_p^m(S_1, \ldots, S_n) \leq \text{ber}_p^m \left( \sum_{i=1}^{n} \left( |S_i|^{\frac{\beta}{p}} + (1 - \beta)|S_i|^{\frac{\beta}{p}} \right) \right) - \inf_{k \in \Omega} \varepsilon(k_i),$$

where

$$\varepsilon(k_i) = \left( 2 \min(\beta, 1 - \beta) \right)^m \sum_{i=1}^{n} \left( \frac{|S_i|^{\frac{\beta}{p}} + |S_i|^{\frac{\beta}{p}}}{2} k_i, k_i \right) - \left( \langle |S_i|^{\frac{\beta}{p}} k_i, k_i \rangle \langle |S_i|^{\frac{\beta}{p}} k_i, k_i \rangle \right) \frac{1}{2}.$$

Proof. Let $k_i$ is the normalized reproducing kernel of $\mathcal{H}(\Omega)$, then

$$\sum_{i=1}^{m} \langle S_i k_i, k_i \rangle^p = \sum_{i=1}^{m} \langle |S_i|^{\frac{\beta}{p}} k_i, k_i \rangle^{\frac{\beta}{p}} \langle |S_i|^{\frac{\beta}{p}} (1-\beta) k_i, k_i \rangle^{\frac{(1-\beta)}{p}} \text{ (by (10))}$$

$$= \sum_{i=1}^{m} \langle |S_i|^\beta k_i, k_i \rangle ((1-\beta)|S_i|^\beta k_i, k_i) \frac{1}{2} \text{ (by Lemma 1.2)}$$

$$\leq \sum_{i=1}^{m} \left( \langle |S_i|^\beta k_i, k_i \rangle (|S_i|^\beta k_i, k_i) \right)^{\frac{1}{2}} \text{ (by (5))}$$

$$= \sum_{i=1}^{m} \left( \langle |S_i|^\beta k_i, k_i \rangle + (1 - \beta) (|S_i|^\beta k_i, k_i) \right)^m - \sum_{i=1}^{m} \left( 2 \min(\beta, 1 - \beta) \right)^m$$

$$\times \left( \frac{|S_i|^\beta k_i, k_i} {2} - \langle |S_i|^\beta k_i, k_i \rangle \langle |S_i|^\beta k_i, k_i \rangle \right) \frac{1}{2} \text{ (by Lemma 1.2)}$$

$$= \sum_{i=1}^{m} \left( \langle |S_i|^\beta + (1 - \beta)|S_i|^\beta \rangle k_i, k_i \right) - \sum_{i=1}^{m} \left( 2 \min(\beta, 1 - \beta) \right)^m$$

$$\times \left( \frac{|S_i|^\beta + |S_i|^\beta}{2} k_i, k_i \right) - \langle |S_i|^\beta k_i, k_i \rangle \langle |S_i|^\beta k_i, k_i \rangle \frac{1}{2} \text{ (by Lemma 1.2)}$$

$$= \sum_{i=1}^{m} \left( \langle |S_i|^\beta + (1 - \beta)|S_i|^\beta \rangle k_i, k_i \right)^m - \sum_{i=1}^{m} \left( 2 \min(\beta, 1 - \beta) \right)^m$$

$$\times \left( \frac{|S_i|^\beta + |S_i|^\beta}{2} k_i, k_i \right) - \langle |S_i|^\beta k_i, k_i \rangle \langle |S_i|^\beta k_i, k_i \rangle \frac{1}{2} \text{ (by Lemma 1.2)}$$

$$= \sum_{i=1}^{m} \left( \langle |S_i|^\beta + (1 - \beta)|S_i|^\beta \rangle \right)^m k_i, k_i - \sum_{i=1}^{m} \left( 2 \min(\beta, 1 - \beta) \right)^m$$

$$\times \left( \frac{|S_i|^\beta + |S_i|^\beta}{2} k_i, k_i \right) - \langle |S_i|^\beta k_i, k_i \rangle \langle |S_i|^\beta k_i, k_i \rangle \frac{1}{2}.$$
Taking supremum over \( \lambda \in \Omega \), we get

\[
ber_p^\beta(S_1, \ldots, S_n) \leq ber \left( \sum_{i=1}^{n} \left( \beta \|S_i\|^p + (1 - \beta) \|S_i^*\|^p \right) \right) - \inf_{\lambda \in \Omega} \xi(\hat{k}_1),
\]

where

\[
\xi(\hat{k}_1) = (2 \min(\beta, 1 - \beta)) \frac{1}{m} \sum_{i=1}^{n} \left( \left( \|S_i\|^p + \|S_i^*\|^p \right) \hat{k}_{1i} - \left( \|S_i\| \hat{k}_1, \hat{k}_1 \right) \right). 
\]

\[ \Box \]

The following simpler form follows from Theorem 2.11 by letting \( m = 1 \).

**Corollary 2.12.** Let \( S_i \in \mathcal{L}(\mathcal{H}) \) \((1 \leq i \leq n)\). Then for \( 0 \leq \beta \leq 1 \) and \( p \geq 2 \),

\[
ber_p^\beta(S_1, \ldots, S_n) \leq ber \left( \sum_{i=1}^{n} \beta \|S_i\|^p + (1 - \beta) \|S_i^*\|^p \right) - \inf_{\lambda \in \Omega} \xi(\hat{k}_1),
\]

where

\[
\xi(\hat{k}_1) = 2 \min(\beta, 1 - \beta) \frac{1}{m} \sum_{i=1}^{n} \left( \left( \|S_i\|^p + \|S_i^*\|^p \right) \hat{k}_{1i} - \left( \|S_i\| \hat{k}_1, \hat{k}_1 \right) \right). 
\]

Letting \( \beta = \frac{1}{2} \) and \( m = 1 \) in Theorem 2.11, we obtain the following corollary.

**Corollary 2.13.** Let \( A, B \in \mathcal{L}(\mathcal{H}) \). Then for \( p \geq 2 \),

\[
ber_p^\beta(A, B) \leq \frac{1}{2} ber(|A|^p + |A^*|^p + |B|^p + |B^*|^p) - \inf_{\lambda \in \Omega} \xi(\hat{k}_1),
\]

where

\[
\xi(\hat{k}_1) = \left( \left( \frac{|A|^p + |A^*|^p}{2} \right) \hat{k}_{1i} \right) + \left( \left( \frac{|B|^p + |B^*|^p}{2} \hat{k}_{1i} \right) - \left( |A|^p \hat{k}_1, \hat{k}_1 \right) \right) - \left( |A^*|^p \hat{k}_1, \hat{k}_1 \right) \right) \left( |A|^p \hat{k}_1, \hat{k}_1 \right) \right) - \left( |B|^p \hat{k}_1, \hat{k}_1 \right) \right). 
\]

In particular, we have

\[
ber^2(A) \leq \frac{1}{2} ber(|A|^2 + |A^*|^2)
\]

\[
\quad - \inf_{\lambda \in \Omega} \left\{ \left( \left( \frac{|A|^2 + |A^*|^2}{2} \hat{k}_{1i} \right) - |A|^2 \hat{k}_1, \hat{k}_1 \right) - \left( \frac{A^* A + A A^*}{2} \hat{k}_{1i} \right) - |A|^2 \hat{k}_1, \hat{k}_1 \right) \right\}
\]

\[
= \frac{1}{2} ber(A^* A + A A^*) - \inf_{\lambda \in \Omega} \left\{ \left( \frac{A^* A + A A^*}{2} \hat{k}_{1i} \right) - |A|^2 \hat{k}_1, \hat{k}_1 \right) \right\} - \inf_{\lambda \in \Omega} \left\{ \left( \frac{1}{2} |A|^2 \hat{k}_1, \hat{k}_1 \right) - |A|^2 \hat{k}_1, \hat{k}_1 \right) \right\};
\]

which is a refinement of the inequality [5, Corollary 3.2].

\[
ber^2(A) \leq \frac{1}{2} ber(A^* A + A A^*).
\]

Once we finish studying the Euclidean Berezin number, we show a Berezin number inequality. Hajmohamadi et al. [13] established that

\[
ber^\beta(A^\beta B^{1-\beta}) \leq \|X\| \left( ber(\beta A^\beta + (1 - \beta) B^{1-\beta}) - \inf_{\lambda \in \Omega} \xi(\hat{k}_1) \right) \right),
\]

where \( A, B, X \in \mathcal{L}(\mathcal{H}) \), with \( A, B \geq 0, r \geq 2 \), \( \xi(\hat{k}_1) = r_0 \left( A^\beta \hat{k}_1, \hat{k}_1 \right)^{1/2} - (B^{1-\beta} \hat{k}_1, \hat{k}_1)^{1/2}, r_0 = \min(\beta, 1 - \beta) \) and \( 0 < \beta < 1 \). The following result is the generalized improvement of (13).
Theorem 2.14. Let $A, B, X \in \mathcal{L}(\mathcal{H})$ such that $A, B$ are positive. Then
\[
\text{ber}'(A^\beta XB^{1-\beta}) \leq |X|^r \left[ \text{ber}(\beta A^\frac{\lambda}{r} + (1 - \beta) B^\frac{\lambda}{r}) - \inf_{\lambda \in \Omega} \xi(\tilde{k}_1) \right],
\]
where
\[
\xi(\tilde{k}_1) = (2r_0)^m \left( \frac{(A^\tilde{\xi}_{\tilde{k}_1, \tilde{k}_1}) + (B^\tilde{\xi}_{\tilde{k}_1, \tilde{k}_1})}{2} - \langle A^\tilde{\xi}_{\tilde{k}_1, \tilde{k}_1}, B^\tilde{\xi}_{\tilde{k}_1, \tilde{k}_1} \rangle \right)^\frac{m}{2},
\]
where $r_0 = \min\{\beta, 1 - \beta\}$, $r \geq 2m$ and $0 \leq \beta \leq 1$.

Proof. Let $\hat{k}_1$ is the normalized reproducing kernel of $\mathcal{H}(\Omega)$, then we have
\[
\{(A^\beta XB^{1-\beta})\hat{k}_1, \hat{k}_1)\}^r
= \{X \hat{B}^{-\beta} \hat{k}_1, A^\beta \hat{k}_1)\}^r
\leq |X|^r \| \{ B^{-\beta} \hat{k}_1, A^\beta \hat{k}_1 \}^r \| \text{ (by the Cauchy Schwartz inequality)}
= |X|^r \left( \langle B^{-\beta} \hat{k}_1, \hat{k}_1 \rangle^\frac{\lambda}{r} \langle A^\beta \hat{k}_1, \hat{k}_1 \rangle^\frac{\lambda}{r} \right)^m
\leq |X|^r \left( \langle A^{-\beta} \hat{k}_1, \hat{k}_1 \rangle^\beta \langle B^{-\beta} \hat{k}_1, \hat{k}_1 \rangle^{1-\beta} \right)^m \text{ (by Lemma 1.2)}
\leq |X|^r \left[ \beta(\langle A^{-\beta} \hat{k}_1, \hat{k}_1 \rangle + (1 - \beta) \langle B^{-\beta} \hat{k}_1, \hat{k}_1 \rangle \right]^m
- (2r_0)^m \left( \frac{(A^\tilde{\xi}_{\tilde{k}_1, \tilde{k}_1}) + (B^\tilde{\xi}_{\tilde{k}_1, \tilde{k}_1})}{2} - \langle A^\tilde{\xi}_{\tilde{k}_1, \tilde{k}_1}, B^\tilde{\xi}_{\tilde{k}_1, \tilde{k}_1} \rangle \right)^\frac{m}{2},
\]
where the last inequality follows from (5). Taking supremum over $\lambda \in \Omega$, we deduce the desired inequality. □

Acknowledgements

The authors thank the referee for the valuable suggestions and comments on an earlier version. Incorporating appropriate responses to these in the article has led to a better presentation of the results. We also thank the Government of India for introducing the work from home initiative during the COVID-19 pandemic.

ORCID

Satyajit Sahoo http://orcid.org/0000-0002-1363-0103
Mojtaba Bakherad https://orcid.org/0000-0003-0323-6310

References

[1] Y. Al-Manasrah, F. Kittaneh, A generalization of two refined Young inequalities, Positivity 19 (2015), 757–768.
[2] O. Axelsson, H. Lu, B. Polman, On the numerical radius of matrices and its application to iterative solution methods, Linear Multilinear Algebra 37 (1994), 225–238.
[3] M. Bakherad, M. Krnic, M. S. Moslehian, Reverse Young-type inequalities for matrices and operators, Rocky Mountain J. Math., 46 (2016), no. 4, 1089–1105.
[4] M. Bakherad, Some Berezin number inequalities for operator matrices, Czechoslovak Math. J. 68, 143 (2018), 997–1009.
[5] M. Bakherad, M. T. Garayev, Berezin number inequalities for operators, Concrete Operators 6 (2019), 33–43.
[6] M. Bakherad, U. Yamanci, New estimations for the Berezin number inequality, J. Inequal. Appl. 2020, 40 (2020).
https://doi.org/10.1186/s13660-020-2307-0.
[7] W. Bani-Dom, F. Kittaneh, Numerical radius inequalities for operator matrices, Linear Multilinear Algebra 57 (2009), 421–427.
[8] F. A. Berezin, Covariant and contravariant symbols of operators, Math. USSR, Izv. 6 (1972) (1973), 1117–1151. (In English. Russian original.); translation from Russian Izv. Akad. Nauk SSSR, Ser. Mat. 36 (1972), 1134–1167.
[9] R. Bhatia, *Matrix analysis*, Graduate Texts in Mathematics, 169, Springer-Verlag, New York, 1997.

[10] D. Choi, *A generalization of Young-type inequalities*, Math Ineqaul Appl. 21 (2018), 99–106.

[11] M. El-Haddad, F. Kittaneh, *Numerical radius inequalities for Hilbert space operators. II*, Studia Math. 182 (2007), 133–140.

[12] T. Furuta, J. Mičić, J. Pečarić and Y. Seo, *Mond–Pečarić method in operator inequalities*, Element, Zagreb, 2005.

[13] M. Hajmohamadi, R. Lashkaripour, M. Bakherad, *Improvements of Berezin number inequalities*, Linear and Multilinear Algebra, 68 (2020), no. 6, 1218–1229.

[14] P. R. Halmos, *A Hilbert space problem book*, Second edition. Graduate Texts in Mathematics, 19. Encyclopedia of Mathematics and its Applications, 17. Springer-Verlag, New York-Berlin, 1982.

[15] G. H. Hardy, J. E. Littlewood, G. Pólya, *Inequalities*, 2nd ed., Cambridge Univ. Press, Cambridge, 1988.

[16] T. Kato, *Notes on some inequalities for linear operators*, Math. Ann., 125 (1952), 208–212.

[17] F. Kittaneh, *Notes on some inequalities for Hilbert space operators*, Publ. Res. Inst. Math. Sci. 24 (1988), 283–293.

[18] F. Kittaneh, *Numerical radius inequalities for Hilbert space operators*, Studia Math. 168 (2005), 73–80.

[19] F. Kittaneh, Y. Manasrah, *Improved Young and Heinz inequalities for matrices*, J. Math. Anal. Appl. 361 (2010), 262-269.

[20] M. Sababheh, D. Choi, *A complete refinement of Young’s inequality*, J. Math. Anal. Appl. 440 (2016) no. 1, 379–393.

[21] M. Sababheh, M. S. Moslehian, *Advanced refinements of Young and Heinz inequalities*, J. Number Theory, 172 (2017), 178–199.

[22] M. Sababheh, *Graph indices via the AM-GM inequality*, Disc. Appl. Math.230 (2017), 100–111.

[23] S. Sahoo, N. Das, D. Mishra, *Numerical radius inequalities for operator matrices*, Adv. Oper. Theory 4 (2019), 197–214.

[24] S. Sahoo, N. C. Rout, M. Sababheh, *Some extended numerical radius inequalities*, Linear and Multilinear Algebra, 69 (2021), no. 5, 907–920.

[25] S. Sahoo, N. Das, D. Mishra, *Berezin number and numerical radius inequalities for operators on Hilbert spaces*, Adv. Oper. Theory 5 (2020), no. 3, 714–727.

[26] U. Yamanci, M. Tapdigoglu, *Some results related to the Berezin number inequalities*, Turkish Journal of Mathematics 43, (4) (2019), 1940–1952.

[27] U. Yamanci, T. Remziye, M. Gurdal, *Berezin Number, Grüss-Type Inequalities and Their Applications*, Bulletin of the Malaysian Mathematical Sciences Society (2019) 1–10.

[28] A. Zamani, *A-numerical radius inequalities for semi-Hilbertian space operators*, Linear Algebra and its Applications 578 (2019), 159-183.