Conformal invariant interaction of a scalar field with the higher spin field in $AdS_D$

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**Abstract**

The explicit form of linearized gauge invariant interactions of scalar and general higher even spin fields in the $AdS_D$ space is obtained. In the case of general spin $\ell$ a generalized 'Weyl' transformation is proposed and the corresponding 'Weyl' invariant action is constructed. In both cases the invariant actions of the interacting higher even spin gauge field and the scalar field include the whole tower of invariant actions for couplings of the same scalar with all gauge fields of smaller even spin. For the particular value of $\ell = 4$ all results are in exact agreement with [1].
1 Introduction

After discovering the $AdS_4/CFT_3$ correspondence of the critical $O(N)$ sigma model \cite{2}, interest in the interacting theory of an arbitrary even high spin field drastically increased. So in the center of our attention is a theory of Fradkin-Vasiliev type \cite{3,4,5,6} in Fronsdal's metric formulation \cite{7,8}. This case of $AdS_D/CFT_{D-1}$ correspondence is also of great interest because on the one hand supersymmetry and BPS arguments are absent and on the other hand both conformal points of the boundary theory (i.e. unstable free field theory and critical interacting theory, in the large $N$ limit) correspond to the same higher spin theory, and are connected on the boundary by a Legendre transformation which corresponds to different boundary conditions (regular dimension one or shadow dimension two) in the quantization of the bulk scalar field \cite{9,10,11}. Existence of this scalar field in higher spin gauge theory is also an interesting and important phenomenon and supports the spontaneous symmetry breaking mechanism and mass creation for initially massless gauge fields due to corresponding possible interactions (see for example \cite{12,13,14,15}).

From this point of view any construction of a reasonable even linearized interaction is an interesting and important task in this reconstruction of the higher spin gauge theory from the holographic dual CFT and can be controlled by the corresponding information about the anomalous dimensions of the dual global symmetry currents that fulfill the conservation conditions in the large $N$ limit. Therefore we see that a construction of the conformal coupling of the scalar with a general even higher spin gauge field appears as an interesting example of an interaction which is applicable to many different quantum one-loop calculations such as the trace anomaly of the scalar in the external higher spin gauge field and so on \cite{16,17,18}.

In this article we construct a generalization of the well known action for the conformally coupled scalar field in $D$ dimensions in external gravity

$$ S = \frac{1}{2} \int d^Dz \sqrt{-G} \left[ G^{\mu \nu} \nabla_\mu \phi \nabla_\nu \phi - \frac{(D-2)}{4(D-1)} R(G) \phi^2 \right] . \quad (1) $$

to the coupling with the linearized external higher spin $\ell$ gauge field. Actually in this article we accomplished to generalize the result of \cite{1} for spin four, obtained four years ago, to the general spin $\ell$ case. We show that the gauge and 'Weyl' invariant interaction of the scalar with the spin $\ell$ Fronsdal gauge field can be constructed only if we add the same type of interaction with all lower even spin gauge fields. In other words we can construct a self-consistent interaction of a gauge field with the conformally coupled scalar only with the whole finite tower of gauge fields with spins in the range $2 \leq s \leq \ell$. In the next section we fix the notation and conventions and briefly review the results of \cite{1}. In section 3 we explicitly construct a linearized interaction Lagrangian of the conformal scalar field with the spin $\ell$ gauge field using Noether’s procedure for higher spin
gauge invariance. In section 4 we extend our investigation including Noether’s procedure for generalized Weyl invariance and obtain a unique interacting action after nontrivial and tedious calculations summarized in several appendices. Note also that nonlinear gauge invariant couplings of the scalar field on the level of the equation of motion were under consideration in [19], [20] and on the level of the BRST formalism for higher spin fields in [21]. Summarizing the introduction we can say that this is a linearized interaction with the scalar field for conformal higher spin theory of the type discussed by [22], [23]-[25].

2 The cases of spin two and spin four

We work in Euclidian AdS$_D$ with the following metric, curvature and covariant derivatives:

\[ ds^2 = g_{\mu\nu}(z) dz^\mu dz^\nu = \frac{L^2}{(z^0)^2} \delta_{\mu\nu} dz^\mu dz^\nu, \quad \sqrt{-g} = \frac{L^D}{(z^0)D}, \]

\[ [\nabla_\mu, \nabla_\nu] V^\rho_\lambda = R^\rho_{\mu\lambda\nu} V^\rho_\sigma - R^\rho_{\mu\nu\sigma} V^\rho_\lambda, \]

\[ R^\rho_{\mu\nu\lambda} = -\frac{1}{(z^0)^2} \left( \delta_{\mu\lambda} \delta^\rho_\nu - \delta_{\nu\lambda} \delta^\rho_\mu \right) = -\frac{1}{L^2} \left( g_{\mu\lambda}(z) \delta^\rho_\nu - g_{\nu\lambda}(z) \delta^\rho_\mu \right), \]

\[ R^\rho_{\mu\nu} = -\frac{D-1}{(z^0)^2} \delta^\rho_{\mu\nu} = -\frac{D-1}{L^2} g_{\mu\nu}(z), \quad R = -\frac{(D-1)D}{L^2}. \]

In [1] the authors constructed gauge and generalized Weyl invariant actions for spin two and four gauge fields interacting with a scalar field. Here we review these results in the form suitable for a generalization to arbitrary higher even spin fields. We work with double traceless higher spin fields in Fronsdal’s formulation [7], [8] where the free field equation of motion for the higher spin $\ell$ field $h_{\mu_1...\mu_\ell}$ reads

\[ F_{\mu_1...\mu_\ell} = \Box h_{\mu_1...\mu_\ell} - \ell \nabla_{(\mu_1} \nabla^\rho h_{\mu_2...\mu_\ell)\rho} + \frac{\ell(\ell - 1)}{2} \nabla_{(\mu_1} \nabla_{\mu_2} h_{\mu_3...\mu_\ell)\rho}^\rho + \frac{\ell^2 + \ell(D-6) - 2(D-3)}{L^2} h_{\mu_1...\mu_\ell} + \frac{\ell(\ell - 1)}{L^2} g_{(\mu_1\mu_2} h_{\mu_3...\mu_\ell)\rho}^\rho = 0 \]

This equation is invariant under gauge transformation$^c$

\[ \delta h_{\mu_1...\mu_\ell} = \ell \nabla_{(\mu_1} \epsilon_{\mu_2...\mu_\ell)} = \nabla_{\mu_1} \epsilon_{\mu_2...\mu_\ell} + c.p. \]

where

$^c$We denote symmetrization of indices by round brackets.
\[ h_{\mu_1...\mu_{\ell-4}\rho} = 0, \]
\[ \epsilon_{\mu_1...\mu_{\ell-3}\rho} = 0. \]  

(4)  

(5)  



The trace of Fronsdal’s tensor reads as

\[ r^{(\ell)}_{\mu_1...\mu_{\ell-2}} = -\frac{1}{2} Tr F(h^{\ell}) = \nabla_\alpha \nabla_\beta h^{(\ell)\alpha\beta\mu_1...\mu_{\ell-2}} - \Box h^{(\ell)\alpha\mu_1...\mu_{\ell-2}}\]
\[ - \frac{\ell - 2}{2} \nabla^{(\mu_1} \nabla_\alpha h^{(\ell)\mu_2...\mu_{\ell-2)}\alpha\beta} - \frac{(\ell - 1)(D + \ell - 3)}{L^2} h^{(\ell)\alpha\mu_1...\mu_{\ell-2}}. \]  

(6)  

For the case \( \ell = 2 \) one can see [1] that a Weyl invariant action is

\[ S^{WI}(\phi, h^{(2)}) = S_0(\phi) + S_1^{\Psi(2)}(\phi, h^{(2)}) + S_1^{r(2)}(\phi, h^{(2)}). \]  

(7)  

where

\[ S_0(\phi) = \frac{1}{2} \int d^Dz \sqrt{-g} [\nabla_\mu \phi \nabla^\mu \phi + \frac{D(D - 2)}{4L^2} \phi^2], \]  

(8)  

\[ S_1^{\Psi(2)}(\phi, h^{(2)}) = \frac{1}{2} \int d^Dz \sqrt{-g} h^{(2)\mu\nu} \Psi^{(2)}_{\mu\nu}(\phi), \]  

(9)  

\[ \Psi^{(2)}_{\mu\nu}(\phi) = -\nabla_\mu \phi \nabla_\nu \phi + \frac{g_{\mu\nu}}{2} (\nabla_\lambda \phi \nabla^\lambda \phi + \frac{D(D - 2)}{4L^2} \phi^2), \]  

(10)  

\[ S_1^{r(2)}(\phi, h^{(2)}) = \frac{1}{8} \frac{D - 2}{D - 1} \int d^Dz \sqrt{-g} r^{(2)}(h^{(2)}) \phi^2, \]  

(11)  

\[ r^{(2)}(h^{(2)}) = \nabla_\mu \nabla_\nu h^{(2)\mu\nu} - \Box h^{(2)\mu} - \frac{D - 1}{L^2} h^{(2)\mu}. \]  

(12)  

which is of course the linearized form of (1) and is invariant with respect to the gauge and Weyl transformations [1].

\[ \delta^1_\phi = \epsilon^\mu(z) \nabla_\mu \phi, \quad \delta^0_\phi = 2 \nabla_\mu \epsilon^\mu; \]  

(13)  

\[ \delta^1_\phi = \Delta \sigma(z) \phi(z), \quad \delta^0_\phi = 2 \sigma(z) g_{\mu\nu}. \]  

(14)  

\[ \Delta = 1 - \frac{D}{2}. \]  

(15)  

Now we turn to the case \( \ell = 4 \). In [1] the authors started from the action (8) and applied Noether’s procedure using the following higher spin ’reparametrization’ of the scalar field with a traceless third rank symmetric tensor parameter

\[ \delta^1_\phi(z) = \epsilon^{\mu\nu\lambda}(z) \nabla_\mu \nabla_\nu \nabla_\chi \phi(z), \quad \epsilon^\alpha_{\mu} = 0. \]  

(16)

\[ ^1\Delta \text{ is so-called conformal weight of the scalar and gets fixed by conformal invariance condition} \]
The variation of (8) is

\[ \delta_\epsilon S_0(\phi) = \frac{1}{2} \int d^D z \sqrt{-g} \left\{ -\nabla^\alpha (\epsilon^{\mu \nu \lambda}) \nabla_\mu \nabla_\nu \nabla_\lambda \phi + \epsilon^{\mu \nu} \frac{1}{2} \nabla_\mu \nabla_\nu \phi \nabla_\lambda \phi + \frac{D(D+2)}{8L^2} \nabla_\mu \phi \nabla_\nu \phi - \nabla^\mu (\epsilon^{\mu \nu}) \nabla_\mu \phi \nabla_\nu \phi + \frac{g_{\mu \nu}}{2} (\nabla_\lambda \nabla^\lambda \phi + \frac{D(D-2)}{4L^2} \phi^2) \right\].

(18)

We see immediately that the first two lines of (18) produce interactions with the spin four and two currents. From the other hand, the last line in (18) is proportional to the equation of motion following from \( S_0(\phi) \) and therefore can be absorbed after gauging by the trace of the spin four gauge field (\( 2 \epsilon^{\mu \nu} \rightarrow h^{(4)}_{\alpha \mu \nu} \)) performing the following field redefinition of \( \phi \)

\[ \phi \rightarrow \phi + \frac{1}{2} \nabla_\mu (h^{(4)}_{\alpha \mu \nu}) \nabla_\nu \phi \]  

(19)

Such a type of field redefinition is a standard correction of Noether’s procedure and means that we always can drop from the cubic part of the action terms proportional to the equation of motion following from the quadratic part of the initial action.

So finally we see that the action \( S^{GI}(\phi, h^{(2)}, h^{(4)}) = S_0(\phi) + S_1^{\Psi(2)}(\phi, h^{(2)}) + S_1^{\Psi(4)}(\phi, h^{(4)}) \)

(20)

where \( S_0(\phi) \), \( S_1^{\Psi(2)}(\phi, h^{(2)}) \), and \( S_1^{\Psi(4)}(\phi, h^{(4)}) \) are defined in (8)-(10) and

\[ S_1^{\Psi(4)}(\phi, h^{(4)}) = \frac{1}{4} \int d^D z \sqrt{-g} h^{(4)\mu \nu \alpha \beta} \Psi^{(4)}_{\mu \nu \alpha \beta}(\phi) \]  

(21)

\[ \Psi^{(4)}_{\mu \nu \alpha \beta}(\phi) = \nabla_\mu \nabla_\nu \nabla_\alpha \nabla_\beta) \phi - g_{\mu \nu} [\nabla_\alpha \nabla_\gamma \phi \nabla_\beta] \nabla_\gamma \phi + \frac{D(D+2)}{4L^2} \nabla_\alpha \phi \nabla_\beta) \phi, \]  

(22)

is invariant with respect to the gauge transformations of the spin four field with an additional spin two field gauge transformation inspired by the second divergence of the spin four gauge parameter\(^\dagger\)

\[ \delta_\epsilon^1 \phi(z) = \epsilon^{\mu \nu \lambda} \nabla_\mu \nabla_\nu \nabla_\lambda \phi(z), \]  

\[ \delta_\epsilon^0 h^{(4)\mu \nu \alpha \beta} = 4 \nabla_\mu (\epsilon^{\nu \alpha \beta}), \]  

\[ \delta_\epsilon^0 h^{(4)\alpha \mu \nu} = 2 \epsilon^{\mu \nu}, \]  

\[ \delta_\epsilon^0 h^{(2)\mu \nu} = 2 \nabla_\mu (\epsilon^{\nu}), \]  

(23-25)

\(^\dagger\)For compactness we introduce shortened notations for divergences of the tensor’s symmetry parameters

\[ \epsilon^{\mu \nu \ldots} = \nabla_\lambda \epsilon^{\lambda \mu \nu \ldots}, \quad \epsilon^{\mu \ldots} = \nabla_\nu \nabla_\lambda \epsilon^{\nu \lambda \mu \ldots}, \ldots \]  

(17)

\(^\S\)Note that the spin two part of our action continues to be invariant in respect of usual linearized reparametrization \(^\S\)
Thus we introduced a gauge invariant interaction of the scalar with the spin four gauge field $h_{\mu \alpha \beta}^{(4)}$ in the minimal way. The next step is the spin four Weyl invariant interaction.

We write the generalized Weyl transformation law for the spin four case as in the [1]

\[ \delta \phi(z) = 12 \sigma^{(\mu \nu)} g^{\alpha \beta} \nabla_\mu \nabla_\nu \phi, \]

\[ \delta h_{\alpha \beta}(z) = \Delta_4 \sigma^{\alpha \beta} \nabla_\alpha \nabla_\beta \phi, \]

where we introduced a generalized "conformal" weight $\Delta_4$ for the scalar field. Then following [1] one can make (20) Weyl invariant introducing the following terms

\[ S_1^{(4)} = \frac{1}{2} \xi_4 \int d^D z \sqrt{-\tilde{g}} r^{(4)\mu \nu} \nabla_\mu \phi \nabla_\nu \phi + \frac{1}{2} \xi_4 \int d^D z \sqrt{-\tilde{g}} \nabla_\mu \nabla_\nu r^{(4)\mu \nu} \phi^2, \]

where

\[ r^{(4)\mu \nu} = \nabla_\alpha \nabla_\beta h_{\alpha \beta}^{(4)\mu \nu} \]

\[ \delta r^{(4)\mu \nu} = 0, \quad \delta^1 r^{(4)\mu} = 0, \]

\[ \xi_4 = -\frac{1}{4} \frac{D}{D+3}, \quad \xi_4 = \frac{1}{32} \frac{D(D-2)}{(D+1)(D+3)}, \quad \Delta_4 = \Delta + \frac{D}{2}. \]

Thus the linearized action for a scalar field interacting with the spin two and four fields in a conformally invariant way is

\[ S_1^{(4)}(\phi, h^{(2)}, h^{(4)}) = S_1^{(4)}(\phi, h^{(2)}) + S_1^{(4)}(\phi, h^{(4)}) + S_1^{(4)}(\phi, h^{(4)}), \]

which is invariant with respect to gauge and generalized Weyl transformations

\[ \delta \phi = \epsilon^{a} \nabla_\mu \phi + \epsilon^{a} \sigma^{(\mu \nu)} \nabla_\mu \nabla_\nu \phi + \Delta \sigma \phi + \Delta \sigma^{(\mu \nu)} \nabla_\mu \nabla_\nu \phi, \]

\[ \delta h^{(2)\mu \nu} = 2 \nabla^{(\mu} \epsilon^{\nu)} + 2 \nabla^{(\mu} \sigma^{(2)} \nu) + 2 (1 - \Delta - 4 \Delta_4) \nabla^{(\mu} \sigma^{(1)} \nu), \]

\[ \delta h^{(4)\mu \nu \alpha \beta} = 4 \nabla^{(\mu} \epsilon^{\nu \alpha \beta)} + 12 \sigma^{(\mu \nu)} g^{\alpha \beta}. \]

### 3 Gauge invariant interaction for the spin $\ell$ case

Here we generalize our construction to the general spin $\ell$ case. Again following [1] we apply the following gauge transformation

\[ \delta \phi(z) = \epsilon^{\mu_1 \mu_2 \ldots \mu_{\ell-1}}(z) \nabla_{\mu_1} \nabla_{\mu_2} \ldots \nabla_{\mu_{\ell-1}} \phi(z), \]

\[ \delta h^{(\ell)\mu_1 \ldots \mu_{\ell}} = 1 \nabla^{(\mu_1} \epsilon^{\mu_2 \mu_3 \ldots \mu_{\ell-1}}), \quad \delta h^{(\ell)\alpha \mu_1 \ldots \mu_{\ell-2}} = 2 \epsilon^{(\mu_1 \ldots \mu_{\ell-2}}, \]

\[ \epsilon^{\alpha}_{\mu_3 \ldots \mu_{\ell-1}} = 0 \]

\[ \Delta_4 \neq \Delta \] in [1] because of field redefinition which is the reason why $S_1^{(4)}$ from [1] turned into (22). When we make field redefinition, we add to the Lagrangian terms which are not Weyl invariant, and in order to restore Weyl invariance we have to change the coefficient $\Delta_4$.
to the action \( \mathcal{S} \) and obtain the following starting variation for Noether’s procedure

\[
\delta_1^1 \mathcal{S}_0(\phi) = \int d^Dz \sqrt{-g} \left\{ \nabla^\alpha \epsilon^{\mu_1 ... \mu_{\ell-1}} \nabla_{\alpha} \phi \nabla_{\mu_1} ... \nabla_{\mu_{\ell-1}} \phi + \epsilon^{\mu_1 ... \mu_{\ell-1}} \nabla_{\alpha} \phi \nabla^\alpha \nabla_{\mu_1} ... \nabla_{\mu_{\ell-1}} \phi + \frac{D(D-2)}{4L^2} \epsilon^{\mu_1 ... \mu_{\ell-1}} \phi \nabla_{\mu_1} ... \nabla_{\mu_{\ell-1}} \phi \right\}. \tag{38}
\]

Using the following notations

\[
T(n, k) = \nabla^\alpha \epsilon^{\mu_1 ... \mu_{n-1}} \nabla_{\mu_1} ... \nabla_{\mu_{n-1}} \nabla_{\alpha} \phi \nabla_{\mu_1} ... \nabla_{\mu_{n-1}} \phi, \tag{39}
\]

\[
M(n, k) = \epsilon^{\mu_1 ... \mu_n} \nabla_{\mu_1} ... \nabla_{\mu_n} \nabla_{\alpha} \phi \nabla_{\mu_1} ... \nabla_{\mu_n} \nabla^\alpha \phi, \tag{40}
\]

\[
N(n, k) = \epsilon^{\mu_1 ... \mu_n} \nabla_{\mu_1} ... \nabla_{\mu_n} \phi \nabla_{\mu_{k+1}} ... \nabla_{\mu_n} \phi, \tag{41}
\]

and commutation relation (B.1) from Appendix B we rewrite \( \mathcal{S} \) in the form

\[
\delta_1^1 \mathcal{S}_0(\phi) = \int d^Dz \sqrt{-g} \left\{ T(\ell, 1) + M(\ell - 1, 0) + \frac{(\ell - 1)(\ell - 2)}{2L^2} N(\ell - 1, 1) + \frac{D(D - 2)}{4L^2} N(\ell - 1, 0) \right\}. \tag{42}
\]

Then using relations between \( T(m, n) \), \( M(m, n) \) and \( N(m, n) \) from Appendix A and after some algebra we ‘diagonalize’ \( \mathcal{S} \).

\[
\delta_1^1 \mathcal{S}_0(\phi) = \sum_{m=1}^{4} (-1)^m \left[ \frac{(\ell - m - 1)}{m - 1} \right] \int d^Dz \sqrt{-g} \left\{ -T(2m, m) + \frac{1}{2} M(2m - 2, m - 1) + \frac{(D + 2m - 2)(D + 2m - 4)}{8L^2} N(2m - 2, m - 1) \right. \\
- \frac{m - 1}{\ell - 2m + 1} \epsilon^{\mu_1 ... \mu_{2m-2}} \nabla_{\mu_1} ... \nabla_{\mu_{m-1}} \left( \nabla^2 \phi - \frac{D(D - 2)}{4L^2} \phi \right) \nabla_{\mu_m} ... \nabla_{\mu_{2m-2}} \phi \right\} \tag{43}
\]

Further performing a final symmetrization in \( \mathcal{S} \), we obtain the following elegant expression

\[
\delta_1^1 \mathcal{S}_0(\phi) = \int d^Dz \sqrt{-g} \left\{ \sum_{m=1}^{4} \left[ \frac{(\ell - m - 1)}{m - 1} \right] \left[ -\nabla^{(\mu_{2m} \mu_{1} ... \mu_{2m-1})} \Psi^{(2m)}_{\mu_{1} ... \mu_{2m}} + \left( \nabla^2 \phi - \frac{D(D - 2)}{4L^2} \phi \right) \sum_{m=2}^{4} \left[ \frac{(\ell - m - 1)}{m - 2} \right] \nabla_{\mu_1} ... \nabla_{\mu_{m-1}} \left( \epsilon^{\mu_{1} ... \mu_{2m-2}} \nabla_{\mu_m} ... \nabla_{\mu_{2m-2}} \phi \right) \right] \right\}. \tag{44}
\]

where

\[
\Psi^{(2m)}_{\mu_{1} ... \mu_{2m}} = (-1)^m \left\{ \nabla_{\mu_1} ... \nabla_{\mu_m} \phi \nabla_{\mu_{m+1}} ... \nabla_{\mu_{2m}} \phi \right\}
\]

\[
- \frac{m^2}{8L^2} g_{\mu_{2m-1} ... \mu_{2m}} \phi^{\alpha \beta} \nabla_{\mu_1} ... \nabla_{\mu_{m-1}} \phi \nabla_{\mu_{m}} ... \nabla_{\mu_{2m-2}} \phi \right\} \tag{45}
\]
and we admitted symmetrization for the set \( \mu_1, \ldots, \mu_{2m} \) of indices. So we see that miraculously the coefficients in (45) don’t depend on \( \ell \) ! All \( \ell \)-dependence is concentrated in the second line of (44) proportional to the equation of motion for the action (8). This part like in the spin four case can be removed by an appropriate field redefinition (see [50], [51], (B.6))

\[
\phi \rightarrow \phi + \sum_{m=2}^{\ell} \frac{m-1}{2(\ell-2m+1)} \nabla_{\mu_1} \cdots \nabla_{\mu_{2m-1}} (h^{(2m)\alpha_{\mu_1} \cdots \mu_{2m-2}}_{\alpha \mu_1} \cdots \mu_{2m-2})
\]

and we can drop these terms from our consideration. Thus we arrive at the following spin \( \ell \) gauge invariant action

\[
S^{GI}(\phi, h^{(2)}, h^{(4)}, \ldots, h^{(\ell)}) = S_0(\phi) + \sum_{m=1}^{\ell} S_1^{(2m)}(\phi, h^{(2m)})
\]

where

\[
S_1^{(2m)}(\phi, h^{(2m)}) = \frac{1}{2m} \int d^Dz \sqrt{-g} h^{(2m)\mu_1 \cdots \mu_{2m}} \Psi^{(2m)}_{\mu_1 \cdots \mu_{2m}}
\]

\[
= \frac{(-1)^m}{2m} \int d^Dz \sqrt{-g} \left\{ h^{(2m)\alpha_{\mu_1} \cdots \mu_{2m-1}}_{\alpha \mu_1} \cdots \mu_{2m-1} \phi \nabla_{\mu_{2m}} \phi \right\}
\]

and the final form of the improved gauge transformations

\[
\delta^1_\epsilon \phi(z) = \epsilon_{\mu_1 \mu_2 \cdots \mu_{\ell-1}} (z) \nabla_{\mu_1} \cdots \nabla_{\mu_{\ell-1}} \phi(z),
\]

\[
\delta^0_\epsilon h^{(2m)\mu_1 \cdots \mu_{2m}} = 2m \nabla_{(\mu_{2m}} \epsilon^{(2m)\mu_1 \cdots \mu_{2m-1})}, \quad \delta^0_\epsilon h^{(2m)\alpha_{\mu_1} \cdots \mu_{2m-2}}_{\alpha \mu_1} \cdots \mu_{2m-2} = 2 \epsilon^{(2m)\alpha_{\mu_1} \cdots \mu_{2m-2}},
\]

Now we can insert \( m = \frac{\ell}{2} \) into (45) and compare our general expression for \( S_1^{(\ell)}(\phi, h^{(\ell)}) \) with the already known cases of spin two (the energy momentum tensor for the scalar field) [10] and spin four [22]. We can easily see that for these cases \( S_1^{(\ell=2,4)}(\phi, h^{(\ell)}) \) exactly reproduces [10] and [22] respectively. So we found the gauge invariant action for a general spin \( l \) gauge field coupled to a scalar and this action has the following property:

The gauge invariant action \( S^{GI}(\phi, h^{(2)}, h^{(4)}, \ldots, h^{(\ell)}) \) for a spin \( \ell \) gauge field coupled to a scalar includes gauge invariant actions of the tower of all smaller even spin gauge fields coupled to the same scalar in an analogous way.
Note that this statement holds true only if we think of an even number of
divergencies applied to the gauge parameter as a possible redefinition of gauge
parameter of smaller even spin gauge fields, in that case this amazing hierarchy of
all smaller even spin currents appear. Another possibility is to regard divergencies
of the gauge parameter as gauge transformation for divergencies of the trace of
the spin \( \ell \) field and make an appropriate field redefinition. In that case we
don’t need to introduce smaller spin currents, but the field redefinition will be of
another form. The current of spin \( \ell \) is the same in both approaches, it is unique,
and in the flat space limit reproduces currents constructed in [26], [27] and [21]
applying a partial integration and field redefinition. The interesting point is that
this symmetric form of currents is unique, and the natural generalization of the
energy-momentum tensor of the scalar field [10].

4 Weyl invariant action for a higher spin field
coupled to a scalar

In this section we introduce generalized Weyl transformations for higher spin
fields and derive a Weyl invariant action for a higher spin field coupled to a scalar
field. Following [1] we write the generalized Weyl transformation for the even spin
\( l \) field in the form

\[
\delta^0_0 h^{(\ell)}_{\mu_1...\mu_\ell} = \ell(\ell - 1) \sigma^{(\mu_1...\mu_{\ell-2}} g^{g_{\mu_{\ell-1}\mu_\ell})}, \tag{52}
\]

\[
\delta^0_\alpha h^{(\ell)}_{\alpha\mu_1...\mu_{\ell-2}} = 2(D + 2\ell - 4) \sigma^{\mu_1...\mu_{\ell-2}}, \tag{53}
\]

\[
\delta^1_0 \phi = \Delta_\ell \sigma^{\mu_1...\mu_{\ell-2}} \nabla_{\mu_1}...\nabla_{\mu_{\ell-2}} \phi. \tag{54}
\]

Then we assume that the Weyl invariant action for a spin \( \ell \) field should be
accompanied with similar Weyl invariant actions for smaller spin gauge fields
and therefore can be constructed from (47) adding \( \frac{\ell}{2} \) additional terms

\[
S^{WI}(\phi, h^{(2)}, h^{(4)}, ..., h^{(\ell)}) = S^{GI}(\phi, h^{(2)}, ..., h^{(\ell)}) + \sum_{m=1}^{\ell/2} S^{r(2m)}_1(\phi, h^{(2m)}), \tag{55}
\]

where each \( S^{r(2m)}_1 \) is gauge invariant itself. In the case of spin two we had only
the linearized Ricci scalar (see [11]) and for the spin four case we had two terms
constructed from the spin four generalization of the Ricci scalar (see [27]). Now
we will see that the generalization of the Ricci scalar for a higher spin field
namely the trace of Fronsdal’s operator [6] (see [7], [1]) is the only gauge invariant
combination of two derivatives and a higher spin field which we need to construct
the Weyl invariant action (55) starting from (47). We will use the following
strategy for solving our problem: We apply transformation (52)-(54) to (47) and
try to compensate it with the variation of
\[
\sum_{m=1}^{\ell/2} S_{1}^{(2m)}(\phi, h^{(2m)}), \text{ where } S_{1}^{(f)}(\phi, h^{(2)}, \ldots, h^{(f)}) =
\int d^{D}z \sqrt{-g} \nabla_{\mu_{2m+1}} \cdots \nabla_{\mu_{2}} r(\phi, h^{(2m)})_{1}(\phi, h^{(2m)})_{2} \cdots (\phi, h^{(2m)})_{\ell/2-1} \nabla_{\mu_{m}} \phi \nabla_{\mu_{m+1}} \cdots \nabla_{\mu_{2m}} \phi \tag{56}
\]

introducing necessarily gauge and Weyl transformations for lower spin gauge fields:
\[
\delta_{\sigma} h^{(2m)} = 2m(2m - 1)C_{\ell/2}^{m} \sigma(\phi, h^{(2)}), \quad m = 1, \ldots, \ell/2, \tag{57}
\]
\[
C_{\ell/2}^{0} = 1. \tag{58}
\]

In other words we solve the equation
\[
\delta_{\sigma}^{1} S_{W}^{I}(\phi, h^{(2)}, \ldots, h^{(f)}) = \delta_{\sigma}^{1} S_{0}^{I} + \sum_{s=1}^{\ell/2} \delta_{\sigma}^{0} S_{1}^{(2s)} + \sum_{s=1}^{\ell/2} \delta_{\sigma}^{0} S_{r(2s)}^{I} = 0 \tag{59}
\]

which consists of a system of \(\ell + 1\) equations for \((\ell/2 + 1)(\ell/2 + 2)/2\) dependent variables:
\[
\Delta_{\ell}, \tag{60}
\]
\[
C_{\ell}^{m}, \quad m = 1, 2, \ldots, \ell/2, \tag{61}
\]
\[
\xi^{n}_{2s}, \quad n = 0, 1, \ldots, s - 1; \quad s = 1, \ldots, \ell/2. \tag{62}
\]

but when we find \(\xi_{\ell/2-k}^{s}\) we also find \(\xi_{s-k}^{s}\) for any \(s \geq k\). In other words we find a whole diagonal of this triangle matrix
\[
\begin{pmatrix}
C_{\ell}^{1} & C_{\ell}^{2} & \cdots & C_{\ell/2}^{\ell/2-1} & C_{\ell/2}^{\ell/2} & \Delta_{\ell} \\
\xi_{\ell}^{0} & \xi_{\ell}^{1} & \cdots & \xi_{\ell/2}^{\ell/2-2} & \xi_{\ell/2}^{\ell/2-1} & \xi_{\ell}^{\ell/2-1} \\
\xi_{\ell}^{0} & \xi_{\ell}^{1} & \cdots & \xi_{\ell/2}^{\ell/2-2} & \xi_{\ell/2}^{\ell/2-1} & \xi_{\ell}^{\ell/2-1} \\
\vdots & \vdots & \ddots & \vdots & \cdots & \xi_{\ell/2}^{\ell/2-1} \\
\xi_{\ell}^{0} & \xi_{\ell}^{1} & \cdots & \xi_{\ell/2}^{\ell/2-2} & \xi_{\ell/2}^{\ell/2-1} & \xi_{\ell}^{\ell/2-1} \\
\xi_{\ell}^{0} & \xi_{\ell}^{1}
\end{pmatrix} \tag{63}
\]

\[\|\text{Note that (56) is zero on-shell, when the higher spin gauge field satisfies Fronsdal’s free equation of motion, but here the higher spin gauge field is an external off-shell field, and this interaction is a natural generalization of well known nonminimal Weyl invariant coupling with gravity.} \|
\]
\[\|\text{This system includes also (58) as an equation for } C_{\ell/2}^{\ell/2}. \|
\]
which helps us to solve the whole system. We have two equations for any column of this matrix besides the last, for which we have one equation for \( \Delta \). We start from the last column and go to the left. When we take any column and two equations for that column of variables, we have only two variables to find if we already solved all columns to the right of that one\[1\] That means that our system has a unique solution. Placing all complicated Weyl variations of (59) into the Appendix C, we present here the resulting system of equations for the unknown variables (60)-(62):

\[
\Delta_\ell = 1 - \frac{D}{2} \tag{64}
\]

\[
\frac{(-1)^{\ell/2}}{2}(\Delta_\ell - \frac{\ell - 2}{2}) - (D + 2\ell - 5)\xi_\ell^{\ell/2-1} = 0 \tag{65}
\]

\[
(-1)^m C_\ell^m + \sum_{s=m+1}^{\ell/2} m C_s^m s_2^m = 0, \quad (m = 1, ..., \ell/2 - 1) \tag{66}
\]

\[
\frac{(-1)^{m-1}}{2}(m - 1)C_\ell^m - C_\ell^m(D + 4m - 5)\xi_2^{m-1}
\]

\[+ \frac{1}{2} \sum_{s=m+1}^{\ell/2} C_s^m (-m(m - 1)\xi_2^s - (2s - 2m + 2)(D + 2s + 2m - 5)\xi_2^{m-1}] = 0 \tag{67}
\]

The solution of this system is universal \( \Delta_\ell = \Delta = 1 - \frac{D}{2} \) and

\[
\xi_\ell^m = \frac{(-1)^m}{2^{\ell-2m}(\ell/2)} \binom{\ell/2}{m} \frac{(D/2 + m - 1)_{\ell/2-m}}{(D_{\ell} - 1)_{\ell/2-m}} \tag{68}
\]

\[
C_\ell^m = \frac{(-1)^{\ell/2-m}}{2^{\ell-2m}} \binom{\ell/2 - 1}{m - 1} \frac{(D/2 + m - 1)_{\ell/2-m}}{(D_{\ell} - 1 + 2m)_{\ell/2-m}}. \tag{69}
\]

These expressions completely fix (56) and therefore the full Weyl invariant action (55), and also determine the transformation law for the whole tower of higher spin gauge fields (57)\[1\]

\[1\]It is easy to see that the first two rows of (63) are all we need to find out. The second row gives the solution for any spin \( \ell \). \( \xi \)-s in lower rows are just particular case and can be determined by putting concrete spin value in a general solution, which means that the independent variables are only first two rows of the (63) and the number of variables in these two rows is \( \ell + 1 \), just as much as equations we have. This right-to-left method can be used only due to the fact that we solve system for general spin case. This is a deductive method which we use. Another approach is an inductive method - one could solve equations for concrete cases of spin 2,4,6... and obtaining all rows lower than second (and therefore whole Weyl invariant Lagrangian for lower spins) solve first two rows. Of course this is impossible for general spin \( \ell \).

\[1\]It is easy to see from formula (C.4) that we get also a redefinition of the gauge parameters for all lower even spin fields which in the spin 4 case coincides with formula (63)
Conclusion

We constructed a gauge and generalized Weyl invariant interacting Lagrangian for a linearized higher even spin gauge field and a conformally coupled scalar field in $AdS_D$ space. These interactions are unique and nontrivial (nontriviality of the interactions means that they can’t be absorbed by a field redefinition from the free action (8)). The resulting Lagrangian for the spin $\ell$ field includes all lower even spin gauge fields also with the same type of interaction with the scalar. These results can be used for constructions of more complicated interactions between different higher spin gauge fields in $AdS$ space (see [28]-[30]).

Appendix A

Here we present the basic relations between different $T$-s, $M$-s and $N$-s which we use in section 3.

$$T(n, k) = (-1)^m \sum_{i=0}^{m} \binom{m}{i} T(n-i, k+m-i), \quad (A.1)$$

$$T(n, k) = (-1)^m \sum_{i=0}^{m} \binom{m}{i} T(n-i, k-m), \quad (A.2)$$

and the same for $M$ and $N$. There is another important relation

$$T(n, k) = -M(n-1, k) - \frac{k(k-1)}{2L^2} N(n-1, k-1)$$

$$-\left[ \frac{(n-k-1)(2D+n-k-4)}{2L^2} + \frac{D(D-2)}{4L^2} \right] N(n-1, k)$$

$$-\epsilon_{(\ell-n)}^{\mu_1...\mu_{m-1}} \nabla_{\mu_1}...\nabla_{\mu_k} \phi \nabla_{\mu_{k+1}}...\nabla_{\mu_{n-1}} (\Box - \frac{D(D-2)}{4L^2}) \phi, \quad (A.3)$$

and the 'symmetrization' relations

$$M(2k+1, k) = M(2k+1, k+1) = -\frac{1}{2} M(2k, k), \quad (A.4)$$

$$N(2k+1, k) = N(2k+1, k+1) = -\frac{1}{2} N(2k, k), \quad (A.5)$$

$$T(2m, m) = \nabla^{(\alpha}_{(\ell=2m)} \nabla_{\mu_1}...\nabla_{\mu_{m-1}} \nabla_{\alpha} \phi \nabla_{\mu_m}...\nabla_{\mu_{2m-1}} \phi$$

$$+\frac{(m-1)(m-2)}{6L^2} N(2m-2, m-1) - \frac{(m-1)(m-2)}{12L^2} N(2m-4, m-2), \quad (A.6)$$

$$M(2m-2, m-1) = \epsilon^{\mu_1...\mu_{m-1}}_{(\ell-2m+1)} \nabla_{\mu_1}...\nabla_{\mu_{m-1}} \nabla_{\alpha} \phi \nabla_{\mu_m}...\nabla_{\mu_{2m-2}} \nabla_{\alpha} \phi$$

$$+\frac{(m-1)(m-2)}{3L^2} N(2m-2, m-1, m-1) - \frac{(m-1)(m-2)}{6L^2} N(2m-4, m-2) \quad (A.7)$$
We must mention here that these relations are satisfied up to full derivatives and therefore admit integration.

**Appendix B**

We use the following commutation relations in $AdS_D$

$$
\epsilon^{\mu_1 \cdots \mu_{\ell-1}} [\nabla^\mu, \nabla_{\mu_1} \cdots \nabla_{\mu_k}] \phi = \frac{k(k-1)}{2L^2} \epsilon^{\mu_2 \cdots \mu_{\ell-1} \mu} \nabla_{\mu_2} \cdots \nabla_{\mu_k} \phi, \quad (B.1)
$$

$$
[\nabla_{\mu_1} \cdots \nabla_{\mu_k}, \nabla^\mu] \epsilon^{\mu_1 \cdots \mu_{\ell-1}} = \frac{2k(D+\ell-2) - k(k+1)}{2L^2} \epsilon^{\mu_2k+1 \cdots \mu_{\ell-1}}, \quad (B.2)
$$

$$
\epsilon^{\mu_1 \cdots \mu_{\ell-1}} [\nabla_{\mu}, \nabla_{\mu_1} \cdots \nabla_{\mu_k}] \nabla^\mu \phi = \frac{k(2D+k-3)}{2L^2} \epsilon^{\mu_1 \mu_{2 \cdots \ell-1} \mu} \nabla_{\mu_1} \cdots \nabla_{\mu_k} \phi, \quad (B.3)
$$

$$
\epsilon^{\mu_1 \cdots \mu_{\ell-1}} [\nabla^2, \nabla_{\mu_1} \cdots \nabla_{\mu_k}] \phi = \frac{k(D+k-2)}{L^2} \epsilon^{\mu_1 \mu_{2 \cdots \ell-1} \mu} \nabla_{\mu_1} \cdots \nabla_{\mu_k} \phi, \quad (B.4)
$$

where $\epsilon^{\mu_1 \cdots \mu_{\ell-1}}$ is the symmetric and traceless tensor. Finally we list all necessary binomial identities

$$
\sum_{k=0}^{n-m} (-1)^k \binom{n}{k} = (-1)^{n-m} \binom{n-1}{m-1}, \quad \sum_{k=0}^{n-m} (-1)^k \binom{n}{m+k} = \binom{n-1}{m-1}, \quad (B.5)
$$

$$
\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}, \quad \binom{\ell-m-1}{m-2} = \frac{m-1}{\ell-2m+1} \binom{\ell-m-1}{m-1}, \quad (B.6)
$$

**Appendix C**

Here we present all Weyl variations necessary for the derivation of (64)-(67)

$$
\delta^1_{\alpha} S_0 = \Delta_\epsilon \int d^D z \sqrt{-g} \left\{ \sum_{m=1}^{\ell-1} \binom{\ell-m-2}{m-1} \nabla_{(\mu_{2m}} \sigma^{\mu_1 \cdots \mu_{2m-1})}_{(\ell-2m-1)} \psi_{(2m)} \right. \\
+ \sum_{m=1}^{\ell-1} \frac{(-1)^{m-1}}{2} \binom{\ell-m-3}{m-1} \square \sigma^{\mu_1 \cdots \mu_{2m}}_{(\ell-2m-2)} \nabla_{\mu_1} \cdots \nabla_{\mu_m} \phi \nabla_{\mu_{m+1}} \cdots \nabla_{\mu_{2m}} \phi \\
+ \sum_{m=1}^{\ell-1} (-1)^m \binom{\ell-m-3}{m-1} \sigma^{\mu_1 \cdots \mu_{2m}}_{(\ell-2m-2)} \nabla_{\mu_1} \cdots \nabla_{\mu_m} \nabla_{\alpha} \phi \nabla_{\mu_{m+1}} \cdots \nabla_{\mu_{2m}} \nabla_{\alpha} \phi \\
+ O(\frac{1}{L^2}) \right\}. \quad (C.1)
$$

We don’t have to calculate $O(\frac{1}{L^2})$ terms because they can be fixed from flat space considerations and gauge invariance of Fronsdal’s operator in AdS. The first term in (C.1) can be cancelled by an additional gauge transformation of all
gauge fields with spin less than \( \ell \). To cancel other terms we calculate the variation of \( \sum_{m=1}^{\ell/2} S_1^{(2m)}(\phi, h^{(2m)}) \):

\[
\delta_\sigma^0 S_1^{(2m)}(\phi, h^{(2)}, \ldots, h^{(2m)})
= C_\ell^m \int d^D z \sqrt{-g} \{ -(m - 1) [\nabla^{(\mu_2m - 2) \sigma_{\mu_1 \ldots \mu_2m - 2}}(\ell - 2m + 1)] \Psi^{(2m - 2)}_{\mu_1 \ldots \mu_2m - 2} \\
+ \frac{(-1)^m}{2} \Box \sigma_{(\ell - 2m)}^{\mu_1 \ldots \mu_2m - 2} \nabla_{\mu_1} \ldots \nabla_{\mu_{m - 1}} \phi \nabla_{\mu_m} \ldots \nabla_{\mu_{2m - 2}} \phi \} \\
+ (-1)^m (1 - \frac{D}{2}) \sigma_{(\ell - 2m)}^{\mu_1 \ldots \mu_2m - 2} \nabla_{\mu_1} \ldots \nabla_{\mu_{m - 1}} \nabla_{\mu_m} \ldots \nabla_{\mu_{2m - 2}} \nabla_{\alpha} \phi \}.
\] (C.2)

and the variation of \( \sum_{m=1}^{\ell/2} S_1^{(2m)}(\phi, h^{(2m)}) \):

\[
\delta_\sigma^0 S_1^{(\ell)} = \frac{1}{2} \sum_{m=1}^{\ell/2 - 1} \int d^D z \sqrt{-g} \{ [2m(2m - 1) \xi_\ell^m - 2(2m - 1)(D + 4m - 8) \xi_\ell^{m - 1}] \times \\
\times \nabla^{(\mu_2m - 2) \sigma_{(\ell - 2m + 1)}^{\mu_1 \ldots \mu_2m - 2}}(\Psi^{(2m - 2)}_{\mu_1 \ldots \mu_2m - 2}) \\
- \frac{1}{2} \int d^D z \sqrt{-g} \{ (\ell - 2)(D + 2\ell - 8) \xi_\ell^{\ell/2 - 1} \nabla^{(\mu_2m - 2) \sigma_{(\ell - 2m)}^{\mu_1 \ldots \mu_2m - 2}}(\Psi^{(\ell - 2)}_{\mu_1 \ldots \mu_2m - 2}) \\
+ \frac{1}{2} \sum_{m=1}^{\ell/2 - 1} \int d^D z \sqrt{-g} \{ 2m(1 - \frac{D}{2}) \sigma_{(\ell - 2m)}^{\mu_1 \ldots \mu_2m - 2} \nabla_{\mu_1} \ldots \nabla_{\mu_{m - 1}} \nabla_{\mu_m} \ldots \nabla_{\mu_{2m - 2}} \nabla_{\alpha} \phi \} \\
+ \frac{1}{2} \sum_{m=1}^{\ell/2 - 1} \int d^D z \sqrt{-g} \{ [-m(m - 1) \xi_\ell^m - (\ell - 2m + 2)(D + \ell + 2m - 5) \xi_\ell^{m - 1}] \times \\
\times \Box \sigma_{(\ell - 2m)}^{\mu_1 \ldots \mu_2m - 2} \nabla_{\mu_1} \ldots \nabla_{\mu_{m - 1}} \phi \nabla_{\mu_m} \ldots \nabla_{\mu_{2m - 2}} \phi \} \\
- \frac{1}{2} \int d^D z \sqrt{-g} \{ 2(D + 2\ell - 5) \xi_\ell^{\ell/2 - 1} \Box \sigma^{\mu_1 \ldots \mu_2m - 2} \nabla_{\mu_1} \ldots \nabla_{\mu_{\ell/2 - 1}} \phi \nabla_{\mu_{\ell/2}} \ldots \nabla_{\mu_\ell} \phi \} \\
+ O(\frac{1}{L^2})
\] (C.3)
Then finally we get

\[ \delta_s S_{W1}(\phi, h^{(2)}, \ldots, h^{(\ell)}) = \delta_s^{1} S_0 + \sum_{s=1}^{\ell/2} \delta_s^{0} S_{\psi^{(2s)}} + \sum_{s=1}^{\ell/2} \delta_s^{0} S_1^{(s)} \]

\[ = \sum_{m=1}^{\ell/2-1} \int d^D z \sqrt{-g} \left\{ \left( \frac{\ell - m - 2}{m - 1} \right) \Delta_\ell - m C^{m+1}_\ell [1 + (D + 4m - 4)\xi^{m}_{2m+2}] \right\} \]

\[ + \frac{1}{2} \sum_{s=m+2}^{\ell/2} \left[ (2m + 2)(2m + 1)\xi^{m+1}_{2m} - 2m(D + 4m - 4)\xi^{m}_{2m} \right] \times \]

\[ \times \nabla_\mu_{\sigma^{(m)} \Psi^{1, \ldots, \Psi^{(m-1)}}}(2m) \]

\[ + \int d^D z \sqrt{-g} \left\{ \frac{(1 - \ell/2)}{2} (\Delta_\ell - \ell - 2/2) - (D + 2\ell - 5)\xi^{\ell/2-1}_\ell \right\} \times \]

\[ \times \Delta_\mu \Delta_{\mu_{\ell/2-1}} \phi \Delta_{\mu_{\ell/2-2}} \phi \]

\[ + \sum_{m=1}^{\ell/2} \int d^D z \sqrt{-g} \left\{ \frac{(1 - \ell/2)}{2} \left( \frac{\ell - m - 2}{m - 2} \right) \Delta_\ell - (m - 1)C^{m}_\ell \right\} - C^{m}_\ell (D + 4m - 5)\xi^{m-1}_{2m-1} \]

\[ + \frac{1}{2} \sum_{s=m+1}^{\ell/2} C^{s}_\ell \left[ -m(m - 1)\xi^{m}_{2m} - (2s - 2m + 2)(D + 2m - 2m - 5)\xi^{m-1}_{2m-1} \right] \times \]

\[ \times \Delta_{\mu_{(\ell-2m)}} \Delta_{\mu_{m-1}} \phi \Delta_{\mu_{m}} \Delta_{\mu_{m-2}} \phi \]

\[ + \int d^D z \sqrt{-g} \left\{ \frac{(1 - \ell/2)}{2} \left( 1 - \frac{D}{2} \right) - \Delta_\ell \right\} \sigma^{\mu_1 \ldots \mu_{\ell/2-1}} \Delta_{\mu_{\ell/2-1}} \Delta_{\mu_{\ell/2-2}} \phi \]

\[ + \sum_{m=1}^{\ell/2-1} \int d^D z \sqrt{-g} \left\{ \frac{(1 - m - 1)}{2} \left( \frac{\ell - m - 2}{m - 2} \right) \Delta_\ell + (1 - \frac{D}{2})C^{m}_\ell \right\} \]

\[ + (1 - \frac{D}{2}) \sum_{s=m+1}^{\ell/2} m C^{s}_\ell \xi^{m}_{2s} \sigma^{\mu_1 \ldots \mu_{2m-2}} \Delta_{\mu_{2m-2}} \Delta_{\mu_{m-1}} \Delta_{\mu_{m}} \Delta_{\mu_{m-2}} \Delta_{\mu_{m-1}} \phi \] (C.4)

From this expression we can derive our system of equations (64)-(67).

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References

[1] R. Manvelyan and W. Rühl, ”Conformal coupling of higher spin gauge fields to a scalar field in $AdS_4$ and generalized Weyl invariance”, Phys. Lett. B 593 (2004) 253 [arXiv:hep-th/0403241].

[2] I. R. Klebanov and A. M. Polyakov, “AdS dual of the critical O(N) vector model,” Phys. Lett. B 550 (2002) 213 [arXiv:hep-th/0210114].

[3] E. S. Fradkin and M. A. Vasiliev, “On The Gravitational Interaction Of Massless Higher Spin Fields,” Phys. Lett. B 189 (1987) 89.

[4] E. S. Fradkin and M. A. Vasiliev, “Cubic Interaction In Extended Theories Of Massless Higher Spin Fields,” Nucl. Phys. B 291 (1987) 141.

[5] M. A. Vasiliev, “Higher-spin gauge theories in four, three and two dimensions,” Int. J. Mod. Phys. D 5 (1996) 763 [arXiv:hep-th/9611024].

[6] M. A. Vasiliev, “Nonlinear equations for symmetric massless higher spin fields in (A)dS(d),” [arXiv:hep-th/0304049].

[7] C. Fronsdal, “Singletons And Massless, Integral Spin Fields On De Sitter Space (Elementary Particles In A Curved Space Vii),” Phys. Rev. D 20, (1979)848;

[8] C. Fronsdal, “Massless Fields With Integer Spin,” Phys. Rev. D 18 (1978) 3624.

[9] E. Witten, “Multi-trace operators, boundary conditions, and AdS/CFT correspondence,” [arXiv:hep-th/0112258].

[10] S. S. Gubser and I. R. Klebanov, “A universal result on central charges in the presence of double-trace deformations,” Nucl. Phys. B 656 (2003) 23 [arXiv:hep-th/0212138].

[11] I. R. Klebanov and E. Witten, “AdS/CFT correspondence and symmetry breaking,” Nucl. Phys. B 556 (1999) 89 [arXiv:hep-th/9905104].

[12] R. Manvelyan, K. Mkrtchyan and W. Rühl, ”Ultraviolet behaviour of higher spin gauge field propagators and one loop mass renormalization”, Nucl. Phys. B 803, (2008) 405 [arXiv:hep-th/0804.1211].

[13] R. Manvelyan and W. Rühl, “The off-shell behaviour of propagators and the Goldstone field in higher spin gauge theory on AdS(d+1) space,” Nucl. Phys. B 717 (2005) 3, [arXiv:hep-th/0502123].
[14] R. Manvelyan and W. Rühl, “The masses of gauge fields in higher spin field theory on the bulk of AdS(4),” Phys. Lett. B 613 (2005) 197; [arXiv:hep-th/0412252].

[15] T. Leonhardt, R. Manvelyan and W. Rühl, “Coupling of higher spin gauge fields to a scalar field in AdS(d+1) and their holographic images in the d-dimensional sigma model,” [arXiv:hep-th/0401240].

[16] R. Manvelyan and W. Rühl, “Generalized Curvature and Ricci Tensors for a Higher Spin Potential and the Trace Anomaly in External Higher Spin Fields in AdS4 Space,” Nucl. Phys. B 796 (2008) 457 [arXiv:0710.0952 [hep-th]].

[17] R. Manvelyan and W. Rühl, “The structure of the trace anomaly of higher spin conformal currents in the bulk of AdS(4),” Nucl. Phys. B 751 (2006) 285.

[18] R. Manvelyan and W. Rühl, “The quantum one loop trace anomaly of the higher spin conformal conserved currents in the bulk of AdS(4),” Nucl. Phys. B 733 (2006) 104 [arXiv:hep-th/0506185].

[19] E. Sezgin and P. Sundell, “Holography in 4D (super) higher spin theories and a test via cubic scalar couplings,” [arXiv:hep-th/0305040]

[20] E. Sezgin and P. Sundell, “Analysis of higher spin field equations in four dimensions,” JHEP 0207, 055 (2002) [arXiv:hep-th/0205132].

[21] A. Fotopoulos, N. Irges, A. C. Petkou and M. Tsulaia, “Higher-Spin Gauge Fields Interacting with Scalars: The Lagrangian Cubic Vertex,” JHEP 0710 (2007) 021, [arXiv:0708.1399 [hep-th]].

[22] A. Y. Segal, “Conformal higher spin theory,” Nucl. Phys. B 664 (2003) 59 [arXiv:hep-th/0207212].

[23] E. S. Fradkin and V. Y. Linetsky, “Superconformal Higher Spin Theory In The Cubic Approximation,” Nucl. Phys. B 350, 274 (1991).

[24] E. S. Fradkin and V. Y. Linetsky “Cubic Interaction In Conformal Theory Of Integer Higher Spin Fields In Four-Dimensional Space-Time,” Phys. Lett. B 231, 97 (1989).

[25] E. S. Fradkin and V. Y. Linetsky “A Superconformal Theory Of Massless Higher Spin Fields In D = (2+1),” Mod. Phys. Lett. A 4, 731 (1989) [Annals Phys. 198, 293 (1990)].

[26] D. Anselmi, Class. Quant. Grav. 17, 1383 (2000) [arXiv:hep-th/9906167].
[27] F. A. Berends, G. J. H. Burgers and H. van Dam, “Explicit Construction Of Conserved Currents For Massless Fields Of Arbitrary Spin,” Nucl. Phys. B 271 (1986) 429;

[28] R. Manvelyan, K. Mkrtchyan and W. Rühl, ”Off-shell construction of some trilinear higher spin gauge field interactions”, [arXiv:hep-th/0903.0243] Nucl.Phys.B 826 (2010), 1-17.

[29] R. Manvelyan, K. Mkrtchyan and W. Rühl, “Direct construction of a cubic selfinteraction for higher spin gauge fields,” [arXiv:hep-th/1002.1358].

[30] R. Manvelyan, K. Mkrtchyan and W. Rühl, ”General trilinear interaction for arbitrary even higher spin gauge fields”, [arXiv:hep-th/1003.2877]