Network Regression and
Supervised Centrality Estimation

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Abstract

The centrality in a network is often used to measure nodes’ importance and model network effects on a certain outcome. Empirical studies widely adopt a two-stage procedure, which first estimates the centrality from the observed noisy network and then infers the network effect from the estimated centrality, even though it lacks theoretical understanding. We propose a unified modeling framework to study the properties of centrality estimation and inference and the subsequent network regression analysis with noisy network observations. Furthermore, we propose a supervised centrality estimation methodology, which aims to simultaneously estimate both centrality and network effect. We showcase the advantages of our method compared with the two-stage method both theoretically and numerically via extensive simulations and a case study in predicting currency risk premiums from the global trade network.

Keywords: Hub centrality, Authority centrality, Network effect, Global trade network, Currency risk premium
1 Introduction

In many disciplines such as economics, finance, and sociology, there has been great interest in studying the network effect, that is, the effect of a network on certain outcomes of interest due to relationships among agents (e.g., individuals, firms, industries, and countries). One popular approach is to bridge the outcome and network via an intermediary or a sufficient statistics – the centrality of the network.

As a low-rank summary of a network, centrality is a common metric to measure agents’ importance in the network, which in turn induces a wide range of agent behaviors that consequently shapes certain outcomes of theirs. A strong motivation for centrality is that many real-world networks exhibit a low-rank structure, i.e., the leading singular value dominates the rest in magnitude (Allen et al., 2019, Yang and Zhu, 2020, Liu and Tsyvinski, 2020). Centrality itself has rich implications for studying human capital investment (Jackson et al., 2017), information sharing and advertising (Banerjee et al., 2019, Breza and Chandrasekhar, 2019), firms’ investment decision-making (Allen et al., 2019), the identification of banks that are too-connected-to-fail (Gofman, 2017), and stock returns (Ahern, 2013, Richmond, 2019), among many others.

To be specific, researchers often regress the outcome of interest on the network centrality to study the network effect. This approach has been implemented in many fields including portfolio management, finance, and social networks. In portfolio management, Hochberg et al. (2007), Ahern (2013) and Richmond (2019) demonstrated that, for a trade network of firms or countries, a strategy that shorts portfolios with high centralities and longs those with low centralities yields a significant excess return, and regressing risk metrics on the centrality of the financial institutions helps to understand the amplification of severe adversarial shocks to the central institutions in the network. Liu (2019) examined the effect of centrality in the production network on the government’s investment in strategic
industries to illustrate the effectiveness of industrial policies. For social networks, Ozsoylev et al. (2014) and Rossi et al. (2018) regressed the excess returns of investment managers on the centrality of their social networks to study trading behaviors; Kornienko and Granger (2018) and Mojzisch et al. (2021) studied the network effect on mental health by regressing the stress level on the network centrality.

Network centrality, however, is not directly observable. In practice, researchers often follow a two-stage procedure: in Stage 1, they compute the centrality from a given network adjacency matrix using an algorithm; in Stage 2, the computed centrality is then used as an input in the regression analysis. Such a practice will be referred to as the two-stage procedure throughout.

The validity of the two-stage procedure, however, hinges upon one critical assumption that the centrality is computed from a noiseless observed adjacency matrix in Stage 1 so that it is accurate. In reality, a network is often observed with noise due to the cost of data collection (Lakhina et al., 2003). There are numerous examples of such noise: the friendship network on Facebook or Twitter is far from a perfect measure of real-life social connections; using self-reported friendships to measure social ties suffers from subjective biases (Banerjee et al., 2013); using patent citations to measure the knowledge flow between companies neglects the communication among workers or executives (Yang and Zhu, 2020). Overlooking noise in networks has demonstrable consequences for network analysis (Borgatti et al., 2006, Frantz et al., 2009, Wang et al., 2012, Martin and Niemeyer, 2019, Candelaria and Ura, 2022).

Given a noisy observed network, one has two goals in understanding the network effect:

(G1) Estimate centrality accurately from the observed noisy network.

(G2) Estimate and conduct valid inference of the network effect through the centrality.

The two-stage procedure attempts to achieve these two goals in a sequential manner, yet it has the following drawbacks. First, Stage 1 only uses the information from the
noisy network to estimate centrality without incorporating the auxiliary information from
the regression on the centrality, which can result in inaccurate estimation of the centrality
due to large observational errors in the network. Second, Stage 2 is contingent upon
Stage 1 – regressing the outcome on the inaccurately estimated centrality exacerbates
an inaccurate estimation of the regression coefficients, thereby invalidating the follow-up
statistical inference.

To remedy the shortcomings of the two-stage procedure, we first propose a unified
framework that fuses two models to achieve the two goals: one network generation model
based on the centralities for (G1) and one network regression model for the dependency
of the outcome on the centralities for (G2). We then propose a novel supervised network
centrality estimation (SuperCENT) methodology that accomplishes both (G1) and (G2)
simultaneously, instead of sequentially.

SuperCENT exploits information from the two models – the network regression model
contains auxiliary information on the centrality in addition to the network, and thus
provides supervision to the centrality estimation. The supervision effect improves the
centrality estimation, which in turn benefits the network regression. Therefore, the
centrality estimation and the network regression complement and empower each other.
Under the unified framework, we derive the theoretical convergence rates and asymptotic
distributions of the centralities and regression coefficients estimators, for both the two-stage
and SuperCENT methods, which can be used to construct confidence intervals.

We summarize our contributions as follows. First, to the best of our knowledge, despite
the popular adoption of the two-stage procedure, we are the first to provide a unified frame-
work to study properties of centrality estimation and inference, and the subsequent network
regression analysis when the observed network is noisy.

Second, we are the first to study the properties of the common practice of the two-stage
procedure and demonstrate that it can be problematic when the network noise is large. The
accuracy of the two-stage centrality estimates in Stage 1 depends on the network noise. When
the network noise is large, the centrality estimates are inaccurate, which results in inaccurate centrality coefficient estimates with invalid ad-hoc inference in Stage 2.

Thirdly, we show theoretically and empirically that the proposed SuperCENT dominates the two-stage procedure universally. Specifically, for (G1), SuperCENT yields a more accurate centrality estimation, especially under large network noise; for (G2), SuperCENT boosts the accuracy of the regression coefficient estimation and provides confidence intervals that are valid and narrower than the ad-hoc two-stage confidence intervals.

Lastly, we apply both SuperCENT and the two-stage procedure to predict the currency risk premium, based on an economic theory that links a country’s currency risk premium with its importance within the global trade network (Richmond, 2019). We show that a long-short trading strategy based on SuperCENT centrality estimates yields a return double that of the two-stage procedure. Furthermore, SuperCENT can verify the economic theory via a rigorous statistical test while the two-stage fails.

Our paper contributes to several strands of literature, including network modeling, network regression with centralities, covariate-assisted network modeling, and network effect modeling. First, the proposed unified framework bridges the gap between research on noisy networks and network regression with centralities. Most of the existing network literature focuses on one of these two aspects. On one hand, in studies involving noisy networks, many empirical works have estimated the true network without incorporating centrality measures (e.g., Lakhina et al. (2003), Handcock and Gile (2010), Banerjee et al. (2013), Le et al. (2018), Rohe (2019), Breza et al. (2020)). On the other hand, numerous work, including those mentioned earlier, have focused on the network regression model with centralities while ignoring the estimation error of the centralities inherited from the noise of the network.

Our unified framework also relates to the line of research on networks with covariates
supervision (Zhang et al., 2016, Li et al., 2016, Fan et al., 2016, Binkiewicz et al., 2017, Yan et al., 2019, Ma et al., 2020). One major difference is that SuperCENT uses both the covariates and the response to supervise the estimation, instead of only the covariates. In addition, the existing literature has focused mostly on network formation or community detection.

In econometrics, there has been significant effort to model the network effect on an outcome of interest through regression (De Paula, 2017). One popular approach follows the pioneering work of Manski (1993) and his “reflection model” (Lee, 2007, Bramoullé et al., 2009, Lee et al., 2010, Hsieh and Lee, 2016, Zhu et al., 2017). This approach models the network effect through the observed adjacency matrix itself, not through the centralities like ours. There has also been a recent surge of literature in network recovery based on the reflection model (De Paula et al., 2019, Battaglini et al., 2021). This literature focuses on the issue of identifiability of the network effect, while our work attends to both estimation and inference of the network effect. Another popular approach assumes that the outcome depends on individual fixed effects, and casts the role of the network through the Laplacian matrix, such that connected nodes share similar individual fixed effects (Li et al., 2019, Le and Li, 2020). This approach emphasizes network homophily, while ours concentrates on the nodes’ position or importance in the network using the centralities.

It is worth mentioning that there is an extensive body of literature discussing the concept of centrality in networks with negative-weight edges. The foundational work on these networks stems from social balance theory in sociology (Harary, 1953, Cartwright and Harary, 1956). Bonacich and Lloyd (2004) extends the concept of centrality to such networks, providing interpretations grounded in balance theory. Several subsequent studies have built on this foundation (Chiang et al., 2014, Everett and Borgatti, 2014, Singh, 2019, Ma et al., 2019, Gromov, 2025). Empirical research has also explored these networks, including studies on workplace dynamics (Labianca and Brass, 2006) and alliance-enemy
networks in wartime (König et al., 2017). Our method is applicable to these networks, and centrality can be interpreted within the framework established by the literature.

The rest of this article is organized as follows. Section 2 provides the background and formally introduces the unified framework. Descriptions of the two-stage procedure and SuperCENT are given in Section 3. Theoretical properties are studied in Section 4 and the simulation study is shown in Section 5. Section 6 presents the case study of the relationship between currency risk premiums and the global trade network centralities. Section 7 concludes with a summary and future work. The supplementary materials contain additional background information on network and centralities, detailed descriptions of the algorithms for undirected networks, more simulation results, additional information of the case study, some concrete mathematical expressions, and the proofs. We developed an R package, SuperCENT, that implements the methods (https://jh-cai.com/SuperCENT).

2   A unified framework

2.1   Set-up and background of network

We observe a sample of \( n \) observations \((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\) where \( y_i \in \mathbb{R} \) is the response and \( x_i \in \mathbb{R}^{p-1} \) is the vector of \( p-1 \) covariates for the \( i \)-th observation as in the multivariate regression setting. Let \( y \in \mathbb{R}^n \) denote the column vector of outcome and \( X \in \mathbb{R}^{n \times p} \) denote the design matrix including the intercept, which is assumed to be fixed.

In a network, the nodes are agents and the edges represent relationships between the agents. The edges can be directed or undirected depending on whether the relationships are reciprocal. This article focuses on directed networks; the Supplement provides the results for undirected ones. A weighted directed network with \( n \) nodes can be represented by an asymmetric adjacency matrix \( A \in \mathbb{R}^{n \times n} \) where \( a_{ij} \)'s represent the weighted edges.

Researchers have used multiple versions of network centrality. We refer to Chapter 2 of Jackson (2010) for a comprehensive introduction to centrality. We focus on the hub and authority centralities (Kleinberg, 1999), which extend the well-known eigenvector
centrality associated with the undirected network to the directed network.

For directed networks, there is a distinction between the giver and the recipient, such as the citee-citor in citation networks or web-page networks, the exporter-importer in trade networks, and the investor-investee in investment networks. The hub and authority centralities take into account the different roles of the giver and the recipient, and thus measure the importance of nodes from these two different perspectives. The concept of “hubs and authorities” originated from web searching. Intuitively, the hub centrality of a web page depends on the total level of authority centrality of the web pages it links to, while the authority centrality of a web page depends on the total level of hub centrality of the web pages it receives links from. Supplement S1 provides an example to further illuminate this intuition.

Let \( u_i \) denote the hub centrality and \( v_i \) denote the authority centrality for node \( i \), and let \( u = (u_1, u_2, \ldots, u_n)^\top \), \( v = (v_1, v_2, \ldots, v_n)^\top \). Their relationship hence satisfies \( u = Au \), \( v = A^\top u \). Given \( A \), to calculate the centralities, Kleinberg (1999) proposes iterating with proper normalization as follows until convergence, for \( k = 1, 2, 3, \ldots \),

\[
\begin{align*}
    u^{(k)} &\leftarrow Av^{(k-1)}, \\
    v^{(k)} &\leftarrow A^\top u^{(k)}.
\end{align*}
\]

This iterative algorithm is also well known as the power method to compute the singular value decomposition (SVD) of \( A \) (Van Loan and Golub, 1996). Therefore, the hub and authority centralities are the leading left and right singular vectors of \( A \) respectively. It is worth mentioning that such definition of centrality and the algorithm essentially assume that the adjacency matrix \( A \) is noiseless.

2.2 A unified framework

We propose the following unified modelling framework that encapsulates (G1)-(G2),

\[
\begin{align*}
    A &= A_0 + E = UDV^\top + E = duv^\top + \sum_{l=2}^r d_l u_l v_l^\top + E, \\
    y &= X\beta_x + u\beta_u + v\beta_v + \epsilon,
\end{align*}
\]

(2a) (2b)
where $D$ is a diagonal matrix of dimension $r \times r$ with the singular values $d > d_2 \geq \ldots \geq d_r \geq 0$ as the diagonal entries, and $U = (u, u_2, \ldots, u_r)$ and $V = (v, v_2, \ldots, v_r)$ are two matrices of size $n \times r$ with orthogonal columns of length $\sqrt{n}$. The intuitions of the unified framework are as follows. The hub and authority centralities are calculated as the leading left and right singular vectors of the observed adjacency matrix. As such, it is natural to consider the generative model (2a) for the observed adjacency matrix, where $A_0$ is the true adjacency matrix, the true centralities $u, v \in \mathbb{R}^n$ are the parameters of interest to be estimated, $(u_2, \ldots, u_r)$ and $(v_2, \ldots, v_r)$ are the non-leading singular vectors orthogonal to $u, v$, and $E$ is the additive noise of mean zero. Then, (2b) naturally models the relationship between the centralities and the response variable. Here, $\beta_x \in \mathbb{R}^p$ is the vector of the regression coefficients, $\beta_u, \beta_v \in \mathbb{R}$ are the coefficients of the hub and authority centralities, and the regression error $\epsilon$ has mean zero. Note that in (2b) it is the true centralities, not the estimated ones, that have direct impacts on the response and only $u$ and $v$ are included instead of the entire $U$ and $V$ because it is common practice to consider the network effect via only the centralities.

Under the unified framework (2) with observed data $\{A, X, y\}$, our original two goals (G1)-(G2) become concrete: (i) estimate the true centralities $u, v$; (ii) estimate the regression coefficients $\beta_x, \beta_u, \beta_v$; and (iii) construct valid confidence intervals (CIs) for the centralities and the regression coefficients.

The low-rank mean plus noise model (2a) has been commonly adopted for matrix estimation or denoising (Shabalin and Nobel, 2013, Yang et al., 2016, Cai and Zhang, 2018), matrix completion (Candes and Plan, 2010), and network community detection with slight modifications (Rohe et al., 2011, Zhao et al., 2012, Lei and Rinaldo, 2015, Le et al., 2016, Gao and Ma, 2021). There is a strand of literature on latent variables network models that can be rewritten as (2a) (Hoff, 2009, Soufiani and Airoldi, 2012, Fosdick and Hoff, 2015).

The unified framework unites our estimation goals and provides a theoretical framework to study the behaviors of the two-stage procedure and motivates our new methodology.
Under Model (2a) and some extra assumptions on the noise, Shabalin and Nobel (2013) proves that if the noise-to-signal ratio is large, the leading singular vector of $A$ and that of $A_0$ converge to orthogonal as $n$ goes to infinity. This implies that the naive estimation of the centralities by implementing SVD on the observed network will fail in the presence of large noise, which invalidates the common practice of two-stage. Furthermore, unifying the two models motivates our supervised network centrality estimation (SuperCENT) methodology, which we will describe formally in the next section. We name it the “supervised” centrality estimation because $(X, y)$ in the regression (2b) can be thought of as the supervisors that offer additional supervision to the centrality estimation. It is expected that if the centralities indeed have strong predictive power (that is, the centrality regression coefficients $\beta_u, \beta_v$ are large compared with the regression noise level), the estimation of the centralities will be better when considering both (2a) and (2b) instead of only (2a). With the improved estimation of the centralities, SuperCENT can further improve the estimation and inference of the regression model.

Note that $u, v$ are only identifiable up to a scalar. SVD assumes $u$ and $v$ have unit length. However, we assume $\|u\|_2 = \|v\|_2 = \sqrt{n}$, because the network can grow and consequently the centralities should roughly be on the same scale with the network. This prevents the centrality regression coefficients from exploding as the network grows.

3 Methodology

Sections 3.1 and 3.2 formally introduce the two-stage procedure and SuperCENT, respectively. In Supplement S3, we will derive SuperCENT algorithm and prove its algorithmic convergence, discuss the prediction procedure and tuning parameter selection, and provide an algorithm for undirected networks with eigenvector centrality.

3.1 The two-stage procedure

As mentioned in the introduction, given the unified framework (2) and the observed data $\{A, X, y\}$, a natural and ad-hoc procedure is the two-stage estimator, which can
serve as a benchmark. In view of (2a), the first stage is to perform SVD on the observed adjacency matrix $A$ and take its leading left and right singular vectors and rescale them to have length $\sqrt{n}$, denoted as $\hat{u}^{ts}$ and $\hat{v}^{ts}$, as the estimates for the centralities $u$ and $v$, respectively. The superscript $ts$ stands for two-stage. In view of (2b), given the estimates $\hat{u}^{ts}$ and $\hat{v}^{ts}$, the second stage performs the ordinary least square (OLS) regression of $y$ on $X$ and $\hat{u}^{ts}, \hat{v}^{ts}$, treating $\hat{u}^{ts}, \hat{v}^{ts}$ as fixed covariates.

Hence, the two-stage procedure solves the following two optimizations sequentially,

$$
\begin{align*}
(d^{ts}, \hat{u}^{ts}, \hat{v}^{ts}) & := \arg\min_{d, \|u\|_2=\|v\|_2=\sqrt{n}} \|A - dv^\top\|^2_F, \\
\hat{\beta}^{ts} & := ((\hat{\beta}_x^{ts})^\top, \hat{\beta}_u^{ts}, \hat{\beta}_v^{ts})^\top := \arg\min_{\beta_x, \beta_u, \beta_v} \|y - X\beta_x - \hat{u}^{ts}\beta_u - \hat{v}^{ts}\beta_v\|^2_2.
\end{align*}
$$

(3a)

(3b)

It follows that $\hat{\beta}^{ts} = (\hat{W}^\top\hat{W})^{-1}\hat{W}^\top y$, where $\hat{W} = (X, \hat{u}^{ts}, \hat{v}^{ts})$.

**Remark 1.** (Two-stage “ad-hoc” CI) Besides the estimation of the unknown parameters, valid inference is necessary to evaluate the network effect. Empirical studies usually construct CIs of the regression coefficients from the second-stage regression by assuming that $\hat{u}^{ts}$ and $\hat{v}^{ts}$ are fixed and noiseless. This assumption simplifies the inferential statement, because it follows that $\text{cov}(\hat{\beta}^{ts}) = \sigma_y^2(\hat{W}^\top\hat{W})^{-1}$, where $\hat{W} = (X, \hat{u}^{ts}, \hat{v}^{ts})$. However, the observed network $A$ is one realization from $A_0 + E$ as in Model (2a), which makes its singular vectors $\hat{u}^{ts}, \hat{v}^{ts}$ random. If one ignores this randomness, the inference becomes invalid. We refer to such “ad-hoc” CI as the “two-stage-adhoc” method. To account for the randomness of the estimated singular vectors $\hat{u}^{ts}, \hat{v}^{ts}$ and obtain valid inference, Section 4 derives the asymptotic distribution of the two-stage estimators, which depends on the network noise $E$ as well as the singular values and singular vectors of $A_0$, and discusses the theoretical property of the naive two-stage-adhoc CI. Section 5 shows that the two-stage-adhoc CI is either conservative or invalid, depending on the network noise level.
3.2 SuperCENT methodology

In the two-stage procedure, the estimation of the regression model in Step 2 depends on the centrality estimation in Step 1. The more accurate the centrality estimates are, the better we are able to make inference in the regression model. On the other hand, the centralities are incorporated into the regression model as regressors, so \((X, y)\) can supervise centrality estimation and thus boost the estimation accuracy.

Motivated by the intuition above, we propose to optimize the following objective function

\[
L(p_u, v, \beta, d) := \arg \min_{\beta, u, v} \frac{1}{n} \| y - X \beta - u \beta_u - v \beta_v \|_2^2 + \frac{\lambda}{n^2} \| A - duv \|_F^2,
\]

where \(\beta = (\beta_x^T, \beta_u, \beta_v)^T\), \(\beta_u, \beta_v\) are uniquely identifiable. Without additional structure, the sign is irrelevant since with proper flipping of \((u, v, u, v)\), the objective function will have the same value and the flipped sequence will remain a valid sequence generated by the algorithm. When additional information is available (e.g.,
Algorithm 1: SuperCENT algorithm with a given tuning parameter.

Result: $\hat{d}$, $\hat{u}$, $\hat{v}$, and $\hat{\beta}$.

Input: $A \in \mathbb{R}^{n \times n}$, $X \in \mathbb{R}^{n \times p}$, $y \in \mathbb{R}^{n}$, tuning penalty parameter $\lambda$, tolerance parameter $\rho > 0$, maximum number of iteration $T$;

Initiate $t = 0$;

$(u^{(0)}, v^{(0)}) = (u^{ts}, v^{ts})$;

$W^{(0)} = (X, u^{(0)}, v^{(0)})$;

$\beta^{(0)} = (W^{(0)})^T \frac{W^{(0)}}{\beta^{(0)}} y$;

$d^{(0)} = u^{(0)} A v^{(0)} / (\|u^{(0)}\|_2 \|v^{(0)}\|_2)$;

while $t \leq 1$ or $(\max(\|P_{u^{(t-1)}} - P_{u^{(0)}}\|_2, \|P_{v^{(t-1)}} - P_{v^{(0)}}\|_2) > \rho$ and $t < T$) do

1. $t \leftarrow t + 1$;

2. $u^{(t)} = \left((\beta_u^{(t-1)})^2 + \frac{1}{n} \lambda (d^{(t-1)})^2 \|v^{(t-1)}\|_2^2\right)^{-1}$
   \[ \times \left[\beta_u^{(t-1)} (y - X \beta_x^{(t-1)} - v^{(t-1)} \beta_v^{(t-1)}) + \frac{1}{n} \lambda d^{(t-1)} A v^{(t-1)}\right] ;

3. $v^{(t)} = \left((\beta_v^{(t-1)})^2 + \frac{1}{n} \lambda (d^{(t-1)})^2 \|u^{(t)}\|_2^2\right)^{-1}$
   \[ \times \left[\beta_v^{(t-1)} (y - X \beta_x^{(t-1)} - u^{(t)} \beta_u^{(t-1)}) + \frac{1}{n} \lambda d^{(t-1)} A^T u^{(t)}\right] ;

4. Normalize $u^{(t)}$, $v^{(t)}$ to have norm $\sqrt{n}$:
   \[ u^{(t)} = \sqrt{n} u^{(t)} / \|u^{(t)}\|_2, v^{(t)} = \sqrt{n} v^{(t)} / \|v^{(t)}\|_2 ;

5. $W^{(t)} = (X, u^{(t)}, v^{(t)})$;

6. $\beta^{(t)} = (W^{(t)})^T \frac{W^{(t)}}{\beta^{(t)}} y$;

7. $d^{(t)} = u^{(t)} A v^{(t)} / (\|u^{(t)}\|_2 \|v^{(t)}\|_2)$;

end

$\hat{u} = u^{(t)}, \hat{v} = v^{(t)}, \hat{\beta} = \beta^{(t)}, \hat{d} = d^{(t)}$. 

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positive entries), one can determine the sign as follows: identify the entry that has the largest magnitude in \((\hat{u}^T, \hat{v}^T)\), make that entry positive, and adjust the signs of the remaining entries in \(\hat{u}, \hat{v}\), as well as \(\hat{\beta}_u, \hat{\beta}_v\) accordingly.

**Remark 3** (Full-rankness of \((X, u^{(t)}, v^{(t)})\)). Further note that Step 6 requires \((X, u^{(t)}, v^{(t)})\) to be full-rank and this condition is satisfied with high probability under assumptions to be introduced in Section 4. Please refer to Remark S4 in Supplement S7.1.1 for more discussions.

**Remark 4** (Algorithmic convergence of Algorithm 1). Note that Algorithm 1 will converge to stationary points whose objective function value is smaller than that of the initialization. Precisely, \(\lim_{t \to \infty} \|\partial \mathcal{L}(u^{(t)}, v^{(t)}, \beta^{(t)}, d^{(t)})\|_2 = 0\) and \(\sup_{t \geq 1} \mathcal{L}(u^{(t)}, v^{(t)}, \beta^{(t)}, d^{(t)}) \leq \mathcal{L}(u^{(0)}, v^{(0)}, \beta^{(0)}, d^{(0)})\). Details of the proof are deferred to Supplement S3.1. The theoretical results of the output from this algorithm is given in Supplement S7.1.1, which depends on the stationarity of the estimators.

The tuning parameter can be chosen via cross-validation, for which prediction procedure has to be introduced. The prediction is a non-trivial task, since it involves the network expansion and justification to use the same regression coefficients for centralities with more nodes introduced into the model. Given the length constraint, the detailed prediction procedure and why it works are given in Supplement S3.3, and the cross-validation procedure is provided in Supplement S3.4.

**4 Theoretical properties**

We investigate the statistical properties of SuperCENT and compare it with the two-stage procedure in this section. The proofs are deferred to Supplement S7. We start with introducing notations and assumptions.

Denote SuperCENT estimators, i.e., the minimizer of the objective function (4) with a given tuning parameter \(\lambda\), as \(\hat{d}, \hat{u}, \hat{v}\), and \(\hat{\beta} = (\hat{\beta}_x)^T, \hat{\beta}_u, \hat{\beta}_v)^T\) and the two-stage counterparts as \(\hat{d}^{ts}, \hat{u}^{ts}, \hat{v}^{ts}\), and \(\hat{\beta}^{ts}\). We further denote \(A^\perp = U^\perp D^\perp V^\perp^T\) where
\[ U^\perp = (u_2, \ldots, u_r), \ V^\perp = (v_2, \ldots, v_r) \] and \( D^\perp = \text{diag}(d_2, \ldots, d_r), \ \Omega = \begin{pmatrix} \sigma_q^2 I_n & 0_{n \times n^2} \\ 0_{n^2 \times n} & \sigma_p^2 I_{n^2} \end{pmatrix} \), 

\( P(Xuv) \) as the projection matrix that projects onto the column space of \((X, u, v)\), and similarly for \( P_X, P_u \) and \( P_v \). Define \( \hat{u} = (I - P_X)u \), \( \hat{v} = (I - P_X)v \), which are the centralities projected onto the orthogonal space of \( X \), and \( C_{\hat{u}\hat{v}} = (\hat{u}, \hat{v})^\top (\hat{u}, \hat{v}) \). In the following, the theorems are for \( \text{arg max}_{h \in (u, -u)} \text{sign}(h^\top u) \) and \( \text{arg max}_{g \in (v, -v)} \text{sign}(g^\top v) \), and we continue to use \( \hat{u}, \hat{v} \) to denote them. While these notions are a bit of an abuse of notation, it is reasonable since both the objective function and algorithm are sign-invariant (i.e., proper flipping of signs gives the same value or another valid iteration sequence).

The same notation applies in the simulations as well.

**Assumption 1.** The network noise \( E \) and regression noise \( \epsilon \) have independent normal entries with mean 0 and variance \( \sigma_a^2 \) and \( \sigma_y^2 \) respectively, and they are independent.

**Assumption 2.** The fixed design matrix in the regression \( X \in \mathbb{R}^{n \times p} \) satisfies \( n > p + 2 \), and the dimension \( p \) is non-diverging. We further assume \((X, u, v)\) is full rank and the condition number of \((X, u, v)^\top (X, u, v)\) is smaller or equal to \( 1/\tau^2 \) for some positive constant \( \tau \).

**Assumption 3.** The scaled network noise-to-signal ratio \( \kappa := \frac{\sigma_a^2}{(d - d_2)^2} \rightarrow 0 \).

In Assumption 1, the independence is assumed for simplicity. If the network noises \( e_{ij} \)'s or the regression noises \( \epsilon_i \)'s are dependent with known covariance, the theorems and the corollaries below still hold with slight modifications by simply plugging their covariance matrices into appropriate places; if they are dependent with unknown covariance, extra assumptions on the covariance structure need to be made and new methodologies and theories should be developed. Assumption 2 simply states that the regression is in the conventional low-dimensional fixed-design regime. Assumption 3 is required for the consistency of the two-stage and SuperCENT, which essentially requires the signal-to-noise ratio (SNR) of the network and the gap between the leading and second singular values of the network to be large enough.
Remark 5 (Verification of the full-rank assumption of \((X, u, v)\)). Unlike classical regression settings where the rank of \(X\) can be easily checked, determining the rank of \((X, u, v)\) is more complex because the true centralities \((u, v)\) are not directly observable. However, this full-rank assumption can be verified as follows. Under certain high signal-to-noise ratio assumptions for the observed network, a simple SVD of the noisy adjacency matrix \(A\) generates consistent estimates \((\hat{u}^t, \hat{v}^t)\) of their population counterparts, and the error bound for \((\hat{u}^t, \hat{v}^t)\) is easily obtainable. One can bound the quantities \(|u^\top (I - PX)v|\), \(\|I - PX\|_2\), and \(\|(I - PX)v\|_2\) by their sample counterparts and the error bound for \((\hat{u}^t, \hat{v}^t)\). These bounds collectively lead to an upper bound on \[\frac{|u^\top (I - PX)v|}{\|I - PX\|_2 \|I - PX\|_2},\] thereby proving the full-rank assumption of \((X, u, v)\).

Remark 6 (Practical implication of the full-rank assumption of \((X, u, v)\)). The widespread use of two-stage procedures suggests that researchers and practitioners find centralities useful for predicting responses, even after accounting for the effects of predictors \(X\). This is because centralities typically provide additional information beyond what the covariates offer.\(^1\) This widespread use implies that they consider \((X, \hat{u}^t, \hat{v}^t)\) to be full rank. Often, the noise in the network is overlooked, as noted in the literature in Section 1, leading to an implicit assumption that the estimated centralities are the true centralities, i.e., \(\hat{u}^t = u\) and \(\hat{v}^t = v\). Therefore, researchers and practitioners implementing two-stage procedures essentially assume the full-rankness of \((X, u, v)\).

Theorem 1. Under the unified framework (2) and Assumptions 1-3, suppose \[\sqrt{n} \frac{\sigma_y}{\sqrt{\beta_u^2 + \beta_v^2}} = o(1), \quad \sigma_y = o \left( \sqrt{\frac{n}{\log n}} \right), \quad \text{and} \quad \frac{\beta_u}{\beta_v} \in [\alpha, \bar{\alpha}] \] for positive constants \(\bar{\alpha} > \alpha > 0\), then the

\(^{1}\)When all nodes are equally important, the centralities will not provide additional information and thus should not be included in the regression. In such cases, \(u\) or \(v\) is constant and the full-rank condition of \((X, u, v)\) will be violated due to collinearity with the intercept. This scenario occurs when the row sums or column sums of \(A_0\) are equal, a condition often modeled by a homogeneous Erdős-Rényi model. In practice, one can test whether a network is generated from a homogeneous Erdős-Rényi model (Bubeck et al., 2016, Bickel and Sarkar, 2016, Lei, 2016, Zhang and Chen, 2017, Banerjee and Ma, 2017, Gao and Lafferty, 2017, Ouadah et al., 2020, Hu et al., 2021, Brune et al., 2024).
SuperCENT estimators, defined as the minimizer of the objective function (4) with a given tuning parameter $\lambda$ satisfying $\frac{1}{n} \frac{n \sigma^2_u}{\sigma^2} \left( 1 + \frac{\sigma_u}{\sqrt{\beta_u^2 + \beta_v^2}} \right)^2 = o(1)$, have the following asymptotic distributions,

1. Centralities:

$$\hat{u} - u = \eta u + o(\eta u) \quad \text{and} \quad \hat{v} - v = \eta v + o(\eta v),$$

2. Network effect:

$$\hat{\beta} - \beta = \eta \beta + o(\eta \beta) = \left( \eta \beta_x, \eta \beta_u, \eta \beta_v \right)^\top + o \left( \left( \eta \beta_x, \eta \beta_u, \eta \beta_v \right)^\top \right),$$

where

$$\begin{pmatrix} \eta u \\ \eta v \\ \eta \beta \end{pmatrix} \sim N \left( 0_{(2n+2+p)\times 1}, C \begin{pmatrix} \sigma_u^2 I_n & 0_{n \times n^2} \\ 0_{n^2 \times n} & \sigma_v^2 I_n \end{pmatrix} C^\top \right), \quad \frac{\|o(\eta u)\|}{|\eta u|} \xrightarrow{P} 0, \quad \frac{\|o(\eta v)\|}{|\eta v|} \xrightarrow{P} 0,$$

are as follows.

The matrices related to $\hat{u}$ and $\hat{v}$ are

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \left[ \frac{\lambda d}{n} \left( \frac{d n I}{(-A)\top} - A \right) + \left( \begin{array}{cc} \beta_u^2 & \beta_u \beta_v \\ \beta_u \beta_v & \beta_v^2 \end{array} \right) \otimes \left( I - P(Xuv) \right) \right]^{-1},$$

the matrices related to $\hat{\beta}_u$ and $\hat{\beta}_v$ are

$$\begin{pmatrix} C_{41} & C_{42} \\ C_{51} & C_{52} \end{pmatrix} = C_{\hat{u}\hat{v}}^{-1} \begin{pmatrix} \hat{u} \top \\ \hat{v} \top \end{pmatrix} \begin{pmatrix} -\beta_u I_n & -\beta_v I_n & I_n \end{pmatrix} \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \\ I_n & 0_{n \times n^2} \end{pmatrix},$$

and the matrices related to $\hat{\beta}_x$ are

$$\begin{pmatrix} C_{31} & C_{32} \end{pmatrix} = (X\top X)^{-1} X\top \begin{pmatrix} -\beta_u I_n & -\beta_v I_n & -u & -v & I_n \end{pmatrix} \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \\ C_{41} & C_{42} \\ C_{51} & C_{52} \\ I_n & 0_{n \times n^2} \end{pmatrix}.$$
presents the asymptotic distribution of the SuperCENT estimators as the global minimizer
of the objective function (4). However, this global minimizer may or may not be achievable
using Algorithm 1. The same asymptotic distribution holds for the estimators produced
by Algorithm 1, as long as these estimators are consistent. As a matter of fact, they
are indeed consistent under slightly more conditions; see Supplement Sections S7.1.1 and
S7.1.5 for the details.

Similarly, we derive the asymptotic distribution of the two-stage estimator in Theorem
S2. Comparing the covariance matrices with those of the two-stage in Theorem S2, all
\( \Sigma_u, \Sigma_v \) and \( \Sigma_\beta \) involve both \( \sigma_a^2 \) and \( \sigma_y^2 \) due to the simultaneous estimation, while \( \Sigma_{tu} \)
and \( \Sigma_{tv} \) of the two-stage only involve \( \sigma_a^2 \) and \( \Sigma_{t\beta} \) involves both. Specifically, \( C_u \) and
\( C_v \) are functions of \((\sigma_a, D, U, V, \sigma_y, X, \beta_u, \beta_v, \lambda)\), while the two-stage counterparts \( C_{tu} \)
and \( C_{tv} \) only involve \((\sigma_a, D, U, V)\). Therefore, the difference between SuperCENT and
the two-stage estimators of \( u \) and \( v \) lies in the tuning parameter \( \lambda \) as well as the SNRs of
the network and regression. Following Theorem 1, Proposition 1 provides the convergence
rates of \( \hat{u} \) and \( \hat{v} \). We focus on the rank-one scenario where \( A_0 = duv^T \) in model (2a)
to provide clearer insights for understanding the difference between SuperCENT and the
two-stage estimators.

**Proposition 1.** (Convergence rates of \( \hat{u}, \hat{v} \)) Under the same assumptions and conditions
as in Theorem 1 and further assume \( A_0 \) to be rank-one, the SuperCENT estimators satisfy
the following,

\[
\frac{1}{n} \mathbb{E} \left\| \hat{u} - u \right\|_2^2 = \left( \frac{\sigma_a^2(n-1)}{d^2n^2} - \frac{n-p-2}{n} \beta_u^2 \delta_{ts,sc} \right) (1 + o(1)) \quad (10)
\]

\[
= \kappa(1 + o(1)) - \beta_u^2 \delta_{ts,sc}(1 + o(1)), \quad (11)
\]

\[
\frac{1}{n} \mathbb{E} \left\| \hat{v} - v \right\|_2^2 = \left( \frac{\sigma_a^2(n-1)}{d^2n^2} - \frac{n-p-2}{n} \beta_v^2 \delta_{ts,sc} \right) (1 + o(1)) \quad (12)
\]

\[
= \kappa(1 + o(1)) - \beta_v^2 \delta_{ts,sc}(1 + o(1)), \quad (13)
\]
where
\[
\delta_{ts,sc} = \left( \lambda d^2 + \beta_u^2 + \beta_v^2 \right)^{-2} \left[ \frac{2 \lambda d^2 + \beta_u^2 + \beta_v^2}{d^2 n} \sigma_a^2 - \sigma_y^2 \right].
\] (14)

**Remark 8.** (The role of \(\delta_{ts,sc}\)) Comparing Corollaries S2 and 1, the discrepancies between the two-stage and SuperCENT estimators of the centralities are all proportional to \(\delta_{ts,sc}\) since \(E[\hat{u}^{ts} - \hat{u}]^2/n - E[\hat{u} - \hat{u}]^2/n = \beta_u^2 \delta_{ts,sc}\). It can be seen that, whenever \(\delta_{ts,sc} > 0\), SuperCENT always outperforms the two-stage.

The positiveness of \(\delta_{ts,sc}\) requires \(\frac{2 \lambda d^2 + \beta_u^2 + \beta_v^2}{d^2 n} \sigma_a^2 - \sigma_y^2 > 0\), which depends on the interplay of \((\sigma_a, d, \sigma_y, \beta_u, \beta_v, n, \lambda)\). Specifically, \(\delta_{ts,sc}\) is positive, when the signal of the regression \(\beta_u, \beta_v\) is large, the regression noise \(\sigma_y\) is small, the signal of the network \(d\) is small, or the network noise \(\sigma_a\) is large. This exactly verifies our intuition: when the regression SNR is high, we gain information from the regression to assist centrality estimation; and the advantage is more pronounced when the network SNR is weak. Moreover, \(\delta_{ts,sc}\) involves a tuning parameter \(\lambda\), and is positive when \(\lambda\) is large enough. This is especially true when \(\lambda\) takes the optimal value \(\lambda_0 = n \sigma_y^2 / \sigma_a^2\) given in the remark below.

**Remark 9.** (Optimal \(\lambda\)) Minimizing the convergence rates (11) or (13) with respect to \(\lambda\) leads to the optimal tuning parameter \(\lambda_0 = n \sigma_y^2 / \sigma_a^2\). With the optimal \(\lambda_0\), SuperCENT achieves its best performance and obtains the most improvement over the two-stage.

Plugging the optimal \(\lambda_0\) into (14), we obtain the discrepancy \(\delta_{ts,sc} = \frac{\kappa^2}{\sigma_y^2} \frac{1}{1 + \kappa \left( \frac{\sigma_a^2}{\sigma_y^2} + \frac{\sigma_y^2}{\sigma_a^2} \right)}\), which is always positive. This implies that as long as the tuning parameter is properly selected, SuperCENT will always be superior over the two-stage. Note that \(\lambda_0\) satisfies the condition for \(\lambda\) because when plugging \(\lambda_0\) into the condition, we still have \(\kappa \left( 1 + \frac{\sigma_y}{\sqrt{\sigma_a^2 + \sigma_y^2}} \right)^2 = o(1)\) under Assumption 3.

**Remark 10.** (SuperCENT-\(\hat{\lambda}_0\) and SuperCENT-\(\hat{\lambda}_{cv}\)) The benefit of the optimal value \(\lambda_0\) is two-fold: 1) to benchmark the cross-validation (CV) procedure in Algorithm S6; 2) to provide a candidate for the tuning parameter \(\lambda\) by plugging in the two-stage estimates
of $\sigma_y^2$ and $\sigma_a^2$, i.e., $\hat{\lambda}_0 = n(\hat{\sigma}^{ts}_y)^2/(\hat{\sigma}^{ts}_a)^2$, instead of the time-consuming cross-validation.

We refer to SuperCENT using $\hat{\lambda}_0$ as SuperCENT-$\hat{\lambda}_0$. Furthermore, $\hat{\lambda}_0$ can be used as a guide to lay out the cross-validation grid points in Algorithm S6, to obtain $\hat{\lambda}_{cv}$ and SuperCENT-$\hat{\lambda}_{cv}$.

**Remark 11.** (Estimation comparison) We further compare the estimation of $u, v$ of two-stage and SuperCENT. Plugging in the optimal $\lambda_0$, the standardized MSEs $E\|\hat{u} - u\|_2^2/n$ in (11) and $E\|\hat{v} - v\|_2^2/n$ in (13) respectively become

$$\kappa \frac{1 + \kappa \frac{\beta_u^2}{\sigma_y^2}}{1 + \kappa \left(\frac{\beta^2}{\sigma_y^2} + \frac{\beta^2}{\sigma_y^2}\right)} (1 + o(1)) \quad \text{and} \quad \kappa \frac{1 + \kappa \frac{\beta_v^2}{\sigma_y^2}}{1 + \kappa \left(\frac{\beta^2}{\sigma_y^2} + \frac{\beta^2}{\sigma_y^2}\right)} (1 + o(1)).$$

(Note that the standardized MSEs of two-stage are $E\|\hat{u}^{ts} - u\|_2^2/n = E\|\hat{v}^{ts} - v\|_2^2/n = \kappa(1 + o(1))$ in (S61). Given (15), the performance of SuperCENT boils down to how the regression SNRs regarding $u$ and $v$ compare with the network SNR, i.e., $\kappa \frac{\beta_u^2}{\sigma_y^2}$ and $\kappa \frac{\beta_v^2}{\sigma_y^2}$. Intuitively, when $\kappa \frac{\beta_u^2}{\sigma_y^2}$ is larger than $\kappa \frac{\beta_v^2}{\sigma_y^2}$, $\hat{u}$ converges faster and $\hat{\beta}_u$ also performs better due to the supervision effect; and vice versa.

**Remark 12.** (Two-stage CI versus “two-stage-ad-hoc” CI of $\beta_u$) Based on Theorem S2, we can construct a valid confidence interval for $\beta_u$. Specifically, the asymptotic variance of $\hat{\beta}_u^{ts}$ when $A_0$ is rank-one has the following two-terms: (S62)-(S63) where the first term is the same as the variance in the classical regression results and the second term is due to the randomness nature of $\hat{u}^{ts}, \hat{v}^{ts}$. Compared with the “two-stage-ad-hoc” CI, i.e., the CI that obtained via software directly from the regression in Stage 2, this “two-stage-ad-hoc” CI uses (S62) alone and is thus invalid unless $\sigma_a = 0$.

## 5 Simulation

In this section, we investigate the empirical performances, including the estimation and inference properties of the two-stage and SuperCENT estimators under various settings. Section 5.1 describes the simulation setups and Section 5.2 shows the results. Additional simulations, including a phase-transition experiment, are deferred to Supplement S5.
5.1 Simulation setup

We generate the network following model (2a). We consider the case of $r = 10$ where
the leading singular value $d = 1$ and the non-leading ones as $d_2 = \ldots = d_r = 2^{-1}$. All
entries of $U$ are first generated from i.i.d. $N(0, 1)$ and $V = 0.5U + \epsilon V$ where $\epsilon V$ are
generated from i.i.d. $N(0, 1)$. We then apply Gram-Schmidt to ensure orthogonality
between columns of $U$ and $V$, and finally rescale each column to have length $\sqrt{n}$. For the
regression model (2b), the regression coefficients are $\beta_v = (1, 3, 5)^\top$, the design matrix $X$
consists of a column of 1’s and $p - 1$ columns whose entries follow $N(0, 1)$ independently.

For the properties of the estimators and inference, only the network SNR $\kappa$ and the
regression SNR $\beta_u \sigma_y$, $\beta_v \sigma_y$, matter. Hence, we fix $n = 2^8$, $d = 1$, $d_2 = \ldots = d_r = 2^{-1}$,
and $\beta_v = 1$ and vary $\sigma_a, \sigma_y$, and $\beta_u$. To study the effect of the regression SNR, we consider
$\sigma_y \in 2^{-4,-2,0}$ and $\beta_u \in 2^{0,2,4}$, while ensuring $\frac{\beta_u^2}{\sigma_y^2} \geq \frac{\beta_v^2}{\sigma_y^2}$. As the network SNR is controlled by
$\sigma_a$, we vary $\sigma_a \in 2^{0,2}$. We study the effects of the non-leading singular values $d_2, \ldots, d_r$ in
additional simulations in Supplement S5.

For estimation property, we compare the following procedures: 1. Two-stage; 2. SuperCENT-$\lambda_0$, which implements Algorithm 1 with oracle $\lambda_0 = n\sigma_y^2/\sigma_a^2$ using the
true $\sigma_y, \sigma_a$ and serves as the benchmark; 3. SuperCENT-$\lambda_0$ is SuperCENT with estimated tuning parameter $\hat{\lambda}_0 = n(\hat{\sigma}_y^2)/(\hat{\sigma}_a^2)^2$, where $(\hat{\sigma}_y^2)^2 = \frac{1}{n-p-2} \|y^t - \hat{\beta}^t \|_2^2$ and $(\hat{\sigma}_a^2)^2 = \frac{1}{n-p-2} \| \hat{A}^t - A_0 \|_F^2$.
are estimated from the two-stage procedure; and 4. SuperCENT-$\lambda_{cv}$ is
SuperCENT with tuning parameter $\hat{\lambda}_{cv}$ chosen by cross-validation as in Algorithm S6.

For inference property, we consider the following procedures to construct the confidence
intervals (CIs) for the regression coefficient: 1. Two-stage-adhoc: $\hat{\beta}^t \pm z_{1 - \alpha/2} \hat{\sigma}_{OLS}(\hat{\beta}^t)$,
where $z_{1 - \alpha/2}$ denote the $(1 - \alpha/2)$-quantile of the standard normal distribution, $\hat{\beta}^t$ is the
two-stage estimate of $\beta$ and $\hat{\sigma}_{OLS}(\hat{\beta}^t)$ is the standard error from OLS, assuming $\hat{u}^t, \hat{v}^t$
are fixed predictors; 2. Two-stage-oracle: $\hat{\beta}^t \pm z_{1 - \alpha/2} \sigma(\hat{\beta}^t)$, where $\sigma(\hat{\beta}^t)$ is the standard
error of \( \hat{\beta}^{ts} \), whose mathematical expressions are given in (S62)-(S63) or (S65)-(S66) with the true parameters plugged in; 3. **Two-stage-plugin**: \( \hat{\beta}^{ts} \pm z_{1-\alpha/2} \hat{\sigma}(\hat{\beta}^{ts}) \), where \( \hat{\sigma}(\hat{\beta}^{ts}) \) is the standard error of \( \hat{\beta}^{ts} \) by plugging all the two-stage estimators into (S62)-(S63) or (S65)-(S66); 4. **SuperCENT-\( \lambda_0 \)-oracle**: \( \hat{\beta}^{\lambda_0} \pm z_{1-\alpha/2} \sigma(\hat{\beta}^{\lambda_0}) \), where \( \hat{\beta}^{\lambda_0} \) is the estimate of \( \beta \) by SuperCENT-\( \lambda_0 \) and \( \sigma(\hat{\beta}^{\lambda_0}) \) follows (S167) with the true parameters plugged in; and 5. **SuperCENT-\( \hat{\lambda} \)-cv**: \( \hat{\beta}^{\hat{\lambda}_{cv}} \pm z_{1-\alpha/2} \hat{\sigma}(\hat{\beta}^{\hat{\lambda}_{cv}}) \), where \( \hat{\beta}^{\hat{\lambda}_{cv}} \) is the estimate of \( \beta \) by SuperCENT-\( \hat{\lambda} \)-cv and \( \hat{\sigma}(\hat{\beta}^{\hat{\lambda}_{cv}}) \) is obtained by plugging the SuperCENT-\( \hat{\lambda} \)-cv estimates into (S167). Note that for the **Two-stage-plugin** and **SuperCENT-\( \hat{\lambda} \)-cv**, \( \hat{\sigma}(\hat{\beta}^{ts}) \) and \( \hat{\sigma}(\hat{\beta}^{\hat{\lambda}_{cv}}) \) involve estimation for \( A^\top = U^\top D^\top V^\top \). For the **Two-stage-plugin**, we plug in the estimate from SVD; for **SuperCENT-\( \hat{\lambda} \)-cv**, we perform SVD on \( A \) and then plug in the estimates. The experiments are repeated 500 times.

### 5.2 Simulation results

From the perspective of estimation, we compare the following metrics: the estimation accuracy for the centralities, the network, and the regression coefficients. Let \( P \) denote the projection matrix. Figure 1 shows the loss \( l(\hat{u}, u) = \|P\hat{u} - Pu\|^2 \) and \( \hat{\beta}_u - \beta_u \), respectively, across different \( \sigma_a, \sigma_y \) and \( \beta_u \) with \( d = 1 \) and \( \beta_v = 1 \). Losses such as \( l(\hat{v}, v) = \|P\hat{v} - Pv\|^2 \), \( l(\hat{A}, A_0) = \|\hat{A} - A_0\|^2_{F}/\|A_0\|^2_{F}, l(\hat{\beta}_u, \beta_u) = (\hat{\beta}_u - \beta_u)^2/\beta_u^2, l(\hat{\beta}_v, \beta_v) = (\hat{\beta}_v - \beta_v)^2/\beta_v^2 \), and \( \hat{\beta}_v - \beta_v \) are given in Supplement S5.1.

Figure 1a shows the boxplot of \( \log_{10}(l(\hat{u}, u)) \). The rows correspond to \( \log_2(\sigma_a) \) and the columns correspond to \( \log_2(\beta_u) \). For each panel, the x-axis is \( \log_2(\sigma_y) \) and the y-axis is \( \log_{10}(l(\hat{u}, u)) \). The super-imposed red symbols show the theoretical rates of \( \hat{u}^{ts} \) and \( \hat{u} \) calculated from Theorems S2 and 1 respectively. As expected, three SuperCENT-based methods estimate \( u \) much more accurately than the two-stage procedure. In particular, the supervision effect of \( (X, y) \) is more pronounced when the noise of the outcome regression \( (\sigma_y) \) is small, or when the signal of the outcome regression \( (\beta_u) \) is large, or when the network noise-to-signal \( (\frac{\sigma_a}{\beta} = \sigma_a) \) is large. The numerical comparison validates Remarks 8 and 11.
Figure 1. Boxplot of $\log_{10}(l(\hat{u}, u))$ for the four estimators across different $\sigma_a$, $\sigma_y$ and $\beta_u$ with fixed $d = 1$, $\beta_v = 1$. The super-imposed red symbols show the theoretical rates of the two-stage and SuperCENT calculated from Theorems S2 and 1 respectively in Figure 1a and the median of $\hat{\beta}_u - \beta_u$ in Figure 1b respectively.

on the theoretical comparison of the estimators. Comparing the three SuperCENT-based methods, SuperCENT-$\hat{\lambda}_{cv}$ and SuperCENT-$\hat{\lambda}_0$ are sometimes worse than the benchmark SuperCENT-$\hat{\lambda}_0$, but still better than the two-stage. SuperCENT-$\hat{\lambda}_0$ is typically comparable to or worse than SuperCENT-$\hat{\lambda}_{cv}$, because SuperCENT-$\hat{\lambda}_0$ fails to locate the optimal $\lambda_0$ due to inaccurate estimate of $\sigma_a$ and $\sigma_y$ from the two-stage procedure.

Figure 1b shows $\hat{\beta}_u - \beta_u$. With large $\sigma_a$ or large $\beta_u$, the two-stage estimates are inaccu-
(a) Empirical coverage of $CI_{\beta_u}$. The dashed lines show the nominal confidence level 0.95.

(b) $\log_{10}$ of the width of $CI_{\beta_u}$.

**Figure 2.** Empirical coverage and $\log_{10}$ of the width of $CI_{\beta_u}$ across different $\sigma_a$, $\sigma_y$ and $\beta_u$ with $d = 1$ and $\beta_v = 1$. SuperCENT variants are labelled as circles (○ ●) and the two-stage variants are labelled as triangles (▵ ▼ ▲). The hollow ones are for oracles and the solid ones are for non-oracles.

From the perspective of inference property, Figure 2 shows the empirical coverage probability (CP) and the average width of the 95% confidence interval for $\beta_u$ respectively. The CP and width for the centralities, the network, and $\beta_v$ are given in the Supplement.
Figure 2a shows how the inaccurate estimation of $\beta_u$ by the two-stage further affects its confidence interval. Regarding empirical coverage, when $\beta_u$ is small (leftmost column), all methods are above the nominal level. As $\beta_u$ increases and $\sigma_a$ remains small (top right two panels), most methods (except for two-stage-oracle) remain valid, but for different reasons: the two SuperCENT-based methods remain valid due to the accurate estimation of both $\beta_u$ and the standard error, whereas two-stage and two-stage-ad-hoc remain valid mainly because they over-estimate $\sigma_y^2$, and this conservativeness masks the issue of the inaccurate estimation. Two-stage-oracle uses the true $\sigma_y^2$ and the issue of the inaccurate estimate cannot be hidden, hence the corresponding intervals undercover. When $\beta_u$ increases and $\sigma_a$ gets large too (bottom right panel), the over-estimation of $\sigma_y^2$ can no longer hide the issue of inaccurate estimation, causing all two-stage-related methods to become invalid. On the other hand, the empirical coverage of SuperCENT remains closer to the nominal level.

As for the width of $CI_{\beta_u}$, Figure 2b shows that the confidence intervals by the SuperCENT-based methods have better coverage and are narrower than those by the two-stage methods. The improvement in width becomes more pronounced with larger $\beta_u$ and $\sigma_a$.

6 Global trade network and currency risk premium

In this case study, we demonstrate that SuperCENT can provide a more accurate estimation of the centralities using the global trade network. This has a profound and lucrative implication on portfolio management because the centrality is closely related to currency risk premium, i.e., the excess return from holding foreign currency compared to the US dollar. We further show the advantage of SuperCENT over the two-stage in the inference of regression coefficients, and thus strengthens a related economic theory.

In international finance literature, economists have studied extensively the currency risk premium and remain puzzled by its driving forces. One recent theory, developed by Richmond (2019) using a general equilibrium, shows that countries’ positions in the trade
network can explain the difference in currency premiums and countries that are central in the trade network exhibit lower currency risk premiums. This theory has two implications: (i) the regression coefficients for the centralities should be negative; and (ii) international investors can leverage and profit from a long-short strategy for foreign exchange by taking a long position in currencies of countries with low centralities and a short position in currencies of countries with high centralities. Therefore, if the centralities can be estimated accurately, one can yield a significant investment return based on the strategy.

Motivated by Richmond (2019), we investigate how the global trade network drives the currency risk premium by regressing the currency risk premium on the centrality of the international trade network. To be specific, we consider a triplet of \( \{ A, X, y \} \), where \( A \) is the country-level trade network, \( y \) is the currency risk premium, and \( X \) is the share of the world’s GDP. Since all these quantities are not directly available, we compute them following Richmond (2019). It is worth mentioning that the trade linkage in \( A \) is defined as the trade volume normalized by the pair-wise total GDP, which represents the relative trade (export/import) intensity between two countries. We use a five-year moving average: when considering year \( t \), the average is taken from year \( t - 4 \) to year \( t \). More details are provided in Supplement S6.1. We focus on the period between 1999 and 2013 and include the 24 countries/regions whose exchange rates are available during this period.\(^2\) In Figure 3, the dotted line shows the time series plot of the rank of the five-year moving average of risk premium from 2003 to 2012 for the 24 countries/regions.\(^3\) In each year, we rank the 24 countries/regions’ risk premiums from the largest to the smallest as the 1st to 24th. We show a circular plot to visualize the average trade volume (2003-2012) in Figure S19.

\(^2\)The euro was first adopted in 1999. The bilateral trade data is available until 2013.
\(^3\)The list of country abbreviations is provided in Supplement S6.
\(^4\)We leave the last available year 2013 for the validation purposes.
Centrality estimation. Since neither the two-stage nor SuperCENT is applicable for panel data, we will repeat the analysis for each year from 2003 to 2012. Besides the network and the response variable, we also include the GDP share as a predictor, which is defined as the percentage of country/region GDP among the total GDP of all available countries in the sample for that year. In summary, the unified framework is, for each \( t \),

\[
a_{ijt} = d \cdot \text{Hub}_{it} \times \text{Authority}_{jt} + e_{ijt},
\]

\[
y_{it} = \alpha + \beta_{ut} \cdot \text{Hub}_{it} + \beta_{vt} \cdot \text{Authority}_{it} + \beta_{xt} \cdot \text{GDP share}_{it} + \epsilon_{it}.
\]

In Sections 4 and 5, we have demonstrated that the two-stage is problematic under large network noise. In this case study, the observational error of the network comes from two sources: GDPs and the trade volumes, because each entry of the observed network \( a_{ijt} \) is defined as the trade volume normalized by their GDPs. The accounting of GDP has been a challenge in macroeconomics (Landefeld et al., 2008). For the trade volume, measurement errors are mostly due to (i) underground or illegal import and export; (ii) excluding service trade; (iii) trade cost like transportation or taxes (Lipsey, 2009). Consequently, the observed trade network can be very noisy and the two-stage will perform badly.

On the other hand, SuperCENT can significantly improve over the two-stage when the network noise is large. In what follows, we focus on SuperCENT-\( \hat{\lambda}_{cv} \) using 10-fold cross-validation. We will refer to SuperCENT-\( \hat{\lambda}_{cv} \) as SuperCENT for simplicity and use the superscript \( sc \) for all the SuperCENT-\( \hat{\lambda}_{cv} \)-related estimates. We determine the signs of the centrality estimates by the empirical rule described at the end of Section 3.2. Figure 3 shows the time series plots of the ranking of the hub centrality estimated by two-stage and SuperCENT for the 24 countries/regions, together with the ranking of the currency risk premium. Figure S18 is for the authority centrality. We rank the centrality in ascending order and the risk premium in descending order. Based on the negative relationship between centralities and risk premium established in Richmond (2019), the closer the trends of rankings between centralities and risk premium are, the better the centralities capture the
Figure 3. Time series of ranking of risk premium in descending order and ranking of hub centrality estimated by two-stage and SuperCENT in ascending order from 2003 to 2012. The vertical dashed line indicates 2008, the year of the financial crisis.

The time variation in the risk premium. The centrality estimated by the two-stage procedure is relatively more stable over time compared to SuperCENT. This is because SuperCENT incorporates information of both the GDP share and the currency risk premium, which is more volatile than the trade network itself. Asian trade hubs such as Hong Kong (HKG) and Singapore (SGP) are the most central; while countries like South Africa (ZAF) and New Zealand (NZL) are peripheral. Comparing the ranking of risk premium, the time variation is not reflected in the centrality estimated by the two-stage procedure, while it is well captured by SuperCENT. For the 2008 financial crisis, the SuperCENT centralities fluctuate together with the risk premium while the two-stage centralities mostly remain unchanged.

To emphasize the importance of accurate centrality estimation for portfolio manage-
Figure 4. Time series of the next-year return from 2004 to 2013 based on a strategy that takes a long position on the currencies with the lowest 3 centralities and a short position on the currencies with the highest 3 centralities estimated from 2003 to 2012 respectively.

For either two-stage or SuperCENT, we take a long position on the currencies with the lowest 3 centralities (bottom 10%) and a short position on the currencies with the highest 3 centralities (top 10%). We obtain a return based on the estimated centrality of the period between year $t-4$ and $t$. Similarly, we include a naive long-short strategy based on the return of year $t-1$ as a baseline. Figure 4 shows the year $t+1$ return based on this strategy. The centrality-based portfolios both outperform the naive strategy. The return based on the centrality estimated by SuperCENT is much higher than that of the two-stage procedure. Table 1 shows the 10-year average annualized return and Sharpe ratio (Sharpe, 1994) based on this strategy with the top and bottom 3, 4, and 5 currencies, respectively. The 10-year average return based on SuperCENT centralities doubles that of the two-stage procedure. SuperCENT achieves Sharpe ratios ranging from 0.27 to 0.39, compared to the Sharpe ratios of 0.28 for the Dow Jones, 0.42 for the S&P 500, and 0.39 for the NASDAQ over the sample period (2004-2013).
Table 1: The 10-year average annualized return.

|                      | Top/Bottom 3 |                      | Top/Bottom 4 |                      | Top/Bottom 5 |                      |
|----------------------|--------------|----------------------|--------------|----------------------|--------------|----------------------|
|                      | Hub Authority |                      | Hub Authority |                      | Hub Authority |                      |
| Naive                | -0.13%       | -0.13%               | -0.86%       | -0.86%               | -1.86%       | -1.86%               |
| SuperCENT CV         | 3.49%        | 0.03%                | 2.17%        | 0.65%                | 3.12%        | 0.28%                |
| Two-stage            | 0.53%        | -1.56%               | 1.28%        | -1.16%               | 1.50%        | -0.65%               |
| SuperCENT Sharpe ratio | 0.38 | -0.02 | 0.27 | 0.07 | 0.39 | 0.02 |
| Relative difference of return: | | | | | | |
| SuperCENT vs Two-stage | 554.5% | 102.0% | 70.3% | 156.5% | 108.1% | 142.6% |

Table 2. The summary table of the regression comparing three methods in terms of coefficient estimation, standard error (in parenthesis) and the significant level (by asterisks).

|                      | Two-stage-adhoc |                      | Two-stage |                      | SuperCENT-λ_{cv} |                      |
|----------------------|-----------------|----------------------|-----------|----------------------|------------------|----------------------|
| GDP share            | -0.0159*        | (0.0083)             | -0.0159*  | (0.0083)             | -0.0157***       | (0.0039)            |
| Hub                  | -0.0011         | (0.0006)             | -0.0011*  | (0.0007)             | -0.0020***       | (0.0001)            |
| Authority            | -0.0005         | (0.0006)             | -0.0005   | (0.0006)             | -0.0004*         | (0.0003)            |

Note: *p<0.1; **p<0.05; ***p<0.01

Inference of regression. We further demonstrate the superiority of SuperCENT in inference. Again since our method is not directly applicable to longitudinal data, we take the 10-year average of trade volume and GDP to construct a 10-year trade network and GDP share. Similarly, we take the 10-year average of risk premium as the response.

To better understand the behavior of the two-stage and SuperCENT estimators and how much improvement SuperCENT can potentially achieve, we compare the noise-to-signal ratio $\kappa$ of the trade network and the SNR of the regression. Since both quantities are unknown, we estimate using results from SuperCENT. For the noise-to-signal ratio, $\hat{\kappa}^{sc} = 0.36 \approx 2^{-1.5}$, which is larger than $\kappa = 2^{-8}$ in the simulation where the accuracy of the two-stage estimators is already low. For the SNR of the regression: $(\hat{\beta}_u^{sc}/\hat{\sigma}_y^{sc})^2 = 1.8 \times 10^7 \approx 2^{24}$ and $(\hat{\beta}_v^{sc}/\hat{\sigma}_y^{sc})^2 = 6.1 \times 10^5 \approx 2^{19}$. Compared with the simulation settings where $\kappa = 2^{-4}$, $\beta_u^2/\sigma_y^2 \leq 2^{16}$ and $\beta_v^2/\sigma_y^2 \leq 2^8$, We expect SuperCENT to significantly outperform the two-stage method in both the estimation and inference of $\beta_u$, due to the large value of $(\hat{\beta}_u^{sc}/\hat{\sigma}_y^{sc})^2$ and the fact that $|\hat{\beta}_u^{sc}| \gg |\hat{\beta}_v^{sc}|$ under a relatively large $\hat{\kappa}^{sc}$, while the improvement of $\beta_v$ is less pronounced.
Table 2 shows the coefficient estimation, the standard error, and the significant level for the two-stage-adhoc, two-stage, and SuperCENT, respectively. The standard errors of two-stage and SuperCENT are based on the trade network being rank-one as we tested using the rank inference by Han et al. (2023) in Supplement S2. For the hub centrality $\beta_u$, (i) the estimate from the two-stage methods is $-0.0011$, while the estimate from SuperCENT is $-0.0020$, which is consistent with the inaccuracy we observed in the simulation; (ii) the standard errors from the two-stage methods are close to $0.0007$, much larger than $0.0001$ from SuperCENT, which reinforces the problem of overestimation of $\sigma^2_y$ in two-stage; (iii) the above two facts combined make the confidence intervals by two-stage-adhoc and two-stage unnecessarily wide, yet still invalid: consequently the hub centrality $\beta_u$ is barely significant at level 0.1 using two-stage and is insignificant using two-stage-adhoc; (iv) the two facts in (i) and (ii) also lead to a valid but narrower confidence interval for SuperCENT, making the hub centrality a significant factor at level 0.01 for the currency risk premium; and (v) conclusions drawn from the two-stage-adhoc and two-stage methods contradict the theory in Richmond (2019), while SuperCENT supports the theory. Other regression coefficients’ significance can be also explained by Remark 12; the details are given in Supplement S6.2.

7 Conclusion and discussion

Motivated by the rising use of centrality in empirical literature, we examined centrality estimation and inference on a noisy network (G1) as well as network effect through the centralities in the subsequent network regression (G2). We proposed a unified framework that incorporates the network generation model and the network regression model to achieve both goals. Under the unified framework, we showed that the properties of the commonly used two-stage procedure and that it could yield inaccurate centrality estimates and regression coefficient estimates, as well as invalid inference when the noise-to-signal ratio of
the network is large. We proposed SuperCENT which incorporates the two models and simultaneously estimates the centralities and the effects of the centralities on the outcome. We further derived the convergence rate and the distribution of the SuperCENT estimator and provided valid confidence intervals for all the parameters of interest. We showed that SuperCENT dominates the two-stage universally and improves over the two-stage in terms of centrality estimation, regression coefficient estimations, and inference. The theoretical results are corroborated with extensive simulations and a real case study in predicting currency risk premiums from the global trade network.

The unified framework and SuperCENT methodology can be extended in multiple directions. One can consider a generalized linear model for the outcome model and a generalized network model for networks with noncontinuous edges via link functions to generalize SuperCENT. In the case when only a subset of covariates and outcomes are observed, semi-supervised SuperCENT can be developed. In the case when the network is partially observed, we can perform matrix completion with supervision. SuperCENT can also be extended to a longitudinal model with additional assumptions by using techniques from tensor decomposition as well as functional data analysis to obtain centralities that are smooth over time. For ultra-high-dimensional problems, sparsity can be imposed on centralities due to the existence of abundant peripheral nodes.

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Supplement to “Network Regression and Supervised Centrality Estimation”

This supplementary material contains more details and proofs that are deferred from the main text and is organized as follows. Section S1 demonstrates the intuition behind the hub and authority centralities using a toy example. Section S2 presents the spectral properties of four real networks as empirical evidence to support our network model. Section S3 provides the detailed SuperCENT algorithm and a modified version for undirected networks with the eigenvector centrality along with their derivations and the proof of algorithmic convergence; we further justify and show the prediction algorithm. Section S4 explicates the theoretical properties of the two-stage and provide the explicit mathematical expressions of the asymptotic covariances of the two-stage estimators. Section S5 shows additional results for the simulation in Section 5 and a phase-transition experiment to demonstrate the behaviors of SuperCENT and the two-stage estimators under different network signal-to-noise ratios. Section S6 provides details on data construction, additional results, and information for the case study. Finally, the proofs of the theoretical results are in Section S7.

S1 A toy example for the hub and authority centralities

In this section, we use a toy example to further illuminate the intuition behind the hub and authority centralities. Consider a citation network where each paper is a node and an edge from Paper A to Paper B indicates Paper A cites Paper B. Figure S5a shows an example of the adjacency matrix of such network. Figures S5b and S5c show the same network with different node sizes: the node sizes in Figure S5b are proportional to the hub centralities while those in Figure S5c are proportional to the authority centralities.

To understand the hub centrality, note that Papers 1 and 4 are the major citors: they both cite three papers with Paper 2 being the common one. Except for the common one, Paper 4 cites Papers 5 and 6, which are only cited by Paper 4, and Paper 1 cites Papers 4 and 3, among which Paper 3 is cited twice. Therefore, compared with Paper 4, Paper 1 cites the same number of papers with one being cited more than the others. This makes the hub centrality of Paper 1 larger than that of Paper 4. One can think of Paper 1 as a better survey paper than Paper 4. Paper 7 cites only one paper, which makes its hub centrality smaller than Papers 1 and 4. The rest of the papers have small hub centrality since they do not cite other papers.

As for the authority centrality, attention should be given to citees. Papers 2 and 3 both have two citations, but Paper 2 is cited by Papers 1 and 4 while Paper 3 is cited by Papers 1 and 7. Observe that Paper 4 as a hub is more influential than Paper 7. So the authority
(a) The adjacency matrix of the citation network. For readability, we denote 0 as .

(b) Node size by hub centralities.

(c) Node size by authority centralities.

Figure S5: A toy network to illustrate the hub and authority centrality.

centrality for Paper 2 is the highest, followed by Paper 3. For the same reason, Paper 4 has higher authority centrality than Papers 5 and 6, since Paper 4 is cited by Paper 1 while Papers 5 and 6 are cited by Paper 4.

S2 Spectral properties of four empirical networks in Section 1

In this section, we justify the low-rank assumption of our network model (2a) by demonstrating that many (economic) networks are, in fact, low-rank and each of their leading singular value dominates the non-leading ones. Specifically, we examine the spectral properties of four empirical networks: (A) global trade network, (B) innovation network, (C) production network, and (D) equity network. We will first describe each network and show the top 20 singular values of each network in 2013.

(A) Trade network. The global trade network is the country-level trade network as described in Section 6. We include all 35 countries\(^5\) here in the bilateral trade data from the correlates of war project (COW). The trade network is defined as the proportion of imports from other countries, i.e., the proportion of imports from country \(i\) among all the imports of country \(j\). We can also define the trade network as the import-export amount in US dollars or as the proportion of exports to other countries. The results are similar.

(B) Innovation network. The innovation network is an industry-level network of knowledge flow based on patent citations in the US. The industry is classified by the 3-digit

\(^5\)In addition to the 24 countries that have exchange rates, we further include Austria, Belgium, Germany, Spain, Finland, France, Greece, Ireland, Italy, Netherlands, and Portuguese.
Figure S6. The leading 20 squared singular values of (A) global trade network, (B) innovation network, (C) production network, and (D) equity network in 2013. All networks have a low-rank structure with the leading singular value dominates the non-leading ones.
code of the North American Industry Classification System (NAICS) and we cover 87 industries in 2013. Please refer to the Data Appendix of ? for a detailed description of the construction of the innovation network.

(C) **Production network.** The production network is a network of the input-output flow at the industry level in the US. Particularly, the input-output flow from industry $i$ to $j$ is defined as the proportion of input from industry $i$ among all the inputs of industry $j$. Same as the innovation network, we use the 3-digit NAICS code and cover 87 industries in 2013. Please refer to the Data Appendix of ? for a detailed description of the construction of the production network.

(D) **Equity network.** The equity network is a network of firm-level investor-investee shareholding relationships in China, i.e., the percentage of shares of firm $j$ owned by firm $i$. In 2013, we include 3.6 million firms that have at least one firm as their shareholders\(^6\) and we focus on the largest connected component\(^7\) which includes 1.3 million firms. Please refer to ? for a detailed description of the construction of the equity network.

Figure S6 shows the top 20 squared singular values of each network, because the proportion of the squared singular values among their sum measures the percentage of the variance explained by each singular component. As a common pattern across four networks, the leading singular value dominates the non-leading ones. Since the leading singular vectors correspond to the hub and authority centralities, the centralities are, therefore, able to capture most information of the networks. This consistent pattern across these empirical networks, along with their corresponding theoretical or empirical studies, motivates our network model and lays the foundation of our formulation. In addition, as mentioned in Section 2.2, there exist numerous empirical studies that demonstrate the importance of centralities and there is a rising amount of literature that develops information theory or economic theory based on centralities in recent years.

Another feature that shares across the four networks is low-rankness. Each network has a low-rank structure since the first few leading singular values “explain” most of the variability and thus only the first few leading singular vectors are needed to represent the whole network. In particular, if we use the “elbow” rule to determine the rank of a network, the innovation network appears to be rank-one while the ranks of the rest are less than 10.

To further confirm the low-rankness of the networks, we adopted the rank inference via residual subsampling (RIRS), a universal approach for testing rank proposed by Han et al. (2023). We follow the transformations in their real data analysis to handle asymmetric

\(^6\)Shareholders can be firms or individuals and we only consider firm-to-firm equity holding.

\(^7\)A connected component is defined as a network that is weakly connected, i.e., each pair of nodes has at least one path regardless of edge directions.
networks:

Method 1: \( A + A^\top; \)  Method 2: \( \begin{pmatrix} 0 & A \\ A^\top & 0 \end{pmatrix} \).

Table S3 shows the test statistics and \( p \)-values based on the two transformations for different null hypotheses of (A) global trade network, (B) innovation network, and (C) production network\(^8\). Using \( \alpha = 0.05 \) as the significance level, RIRS estimates the number of rank as 1 for all three networks based on Method 2. While based on Method 1, the only difference is that the rank of the innovation network is estimated as 2.

| Network     | Null hypothesis | Method 1 | Method 2 |
|-------------|-----------------|----------|----------|
| Trade       | \( r = 1 \)     | 0.56     | 0.29     |
| Innovation  | \( r = 1 \)     | -4.51    | 0.00     |
|             | \( r = 2 \)     | -1.12    | 0.13     |
| Production  | \( r = 1 \)     | -1.37    | 0.09     |

\( \textbf{Table S3:} \) Test statistics and \( p \)-value from RIRS.

In short, real networks often have a low-rank structure and the gap between the leading singular value and the rest is large. These phenomena support our low rank assumption on the network model and demonstrates the importance of the hub and authority centralities.

S3 SuperCENT algorithms for different models and their derivations

In this section, we first derive the SuperCENT algorithm and prove the algorithmic convergence of SuperCENT algorithm in Section S3.1. We then derive the SuperCENT algorithm for undirected networks in Section S3.2. We further provide the detailed algorithm for prediction in Section S3.3. Finally, we propose a cross-validation algorithm for selecting the tuning parameter \( \lambda \) in Section S3.4.

S3.1 Derivation of SuperCENT Algorithm 1 for directed network Model (2)

In Section 3, the proposed SuperCENT obtains estimates by optimizing the objective function (4) which combines the network model and the network regression model in the unified framework (2). To solve (4), we use a block descent algorithm by updating \((\hat{u}, \hat{v}, \hat{\beta}, \hat{d})\) where \( \hat{\beta} = (\hat{\beta}_x^\top, \hat{\beta}_u, \hat{\beta}_v)^\top \). The detailed algorithm is shown in Algorithm 1.

The derivation of each step in each iteration of Algorithm 1 is described below. Denote \( W = (X, u, v), \beta = (\beta_x, \beta_u, \beta_v), \) and \( L(u, v, \beta, d) := \frac{1}{n} \| y - X \beta_x - u \beta_u - v \beta_v \|_2^2 + \frac{1}{n} \| A - duv^\top \|_F^2 \). Given \( \lambda \), we minimize the objective function (4) by setting the partial derivatives
of all the parameters as zeros. The partial derivatives are as follows.

\[
\hat{\beta} \mathcal{L} = -\frac{2}{n} \mathbf{W}^\top (\mathbf{y} - \mathbf{W} \mathbf{\beta}), \\
\hat{u} \mathcal{L} = \frac{2}{n^2} \lambda d \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - \frac{2}{n^2} \lambda \mathbf{u}^\top \mathbf{A} \mathbf{v}, \\
\hat{v} \mathcal{L} = \frac{2}{n^2} \beta u (\mathbf{y} - \mathbf{X} \mathbf{\beta}_x - \mathbf{u} \mathbf{\beta}_u - \mathbf{v} \mathbf{\beta}_v) + \frac{2}{n^2} \lambda d \mathbf{u}^\top \mathbf{v} - \frac{2}{n^2} \lambda \mathbf{dA}^\top \mathbf{u},
\]

Setting the partial derivatives above as zeros yields the estimates

\[
\hat{\beta} = (\mathbf{W}^\top \mathbf{W})^{-1} \mathbf{W}^\top \mathbf{y}, \\
\hat{d} = \frac{\hat{u}^\top \mathbf{A} \hat{v}}{\|\hat{u}\|^2 \|\hat{v}\|^2},
\]

with constraints

\[
\hat{u}^\top \hat{u} = n \quad \text{and} \quad \hat{v}^\top \hat{v} = n,
\]

where \(\hat{W} = (\mathbf{X}, \hat{u}, \hat{v})\). Denote \((\hat{u}^{(t)}, \hat{v}^{(t)}, \hat{\beta}^{(t)}, \hat{d}^{(t)})\) as the estimations from the \(t\)-th iteration. Combining (S21)-(S25) and substituting the corresponding estimates from the previous updates, we obtain each update step in each iteration.

To further obtain the estimates of \(\mathbf{u}_t\) and \(\mathbf{v}_t\) with their singular values \(d_l\) for \(l = 2, \ldots, r\), one may perform SVD on \(\mathbf{A} - \hat{d} \hat{\mathbf{u}} \hat{\mathbf{v}}^\top\) and scale appropriately.

### S3.1.1 Proof of algorithmic convergence of SuperCENT Algorithm 1

In Remark 4, we claim that SuperCENT will converge to stationary points that have smaller objective values than the objective function value of our initial point. Precisely, we have \(\lim_{t \to \infty} \|\hat{\mathcal{L}}(\mathbf{u}^{(t)}, \mathbf{v}^{(t)}, \beta^{(t)}, d^{(t)})\|_2 = 0\) and \(\sup_{t \geq 1} \mathcal{L}(\mathbf{u}^{(t)}, \mathbf{v}^{(t)}, \beta^{(t)}, d^{(t)}) \leq \mathcal{L}(\mathbf{u}^{(0)}, \mathbf{v}^{(0)}, \beta^{(0)}, d^{(0)})\).

**Proof.** First note that with given \(\mathbf{u}\) and \(\mathbf{v}\), the \(\beta\) and \(d\) that minimize the objective function have explicit expression, provided in (S23) and (S24) with the hats removed. Therefore, minimization of \(\mathcal{L}(\mathbf{u}, \mathbf{v}, \beta, d)\) with respect to \((\mathbf{u}, \mathbf{v}, \beta, d)\) reduces to minimization of another appropriate function \(\tilde{\mathcal{L}}(\mathbf{u}, \mathbf{v})\) with respect to \((\mathbf{u}, \mathbf{v})\). The detailed expression of \(\tilde{\mathcal{L}}\) can be obtained by plugging (S23) and (S24) without the hats into the expression of \(\mathcal{L}(\mathbf{u}, \mathbf{v}, \beta, d)\). Furthermore, due to the existence of \(\beta\) and \(d\), the scaling of \(\mathbf{u}\) and \(\mathbf{v}\) do not affect the optimal value of the objective function. In other words, \(\tilde{\mathcal{L}}(\mathbf{u}, \mathbf{v}) = \tilde{\mathcal{L}}(c_u \mathbf{u}, c_v \mathbf{v})\), where \(c_u, c_v\) are two non-zero constants.

In the \((t+1)\)-th iteration, Steps 2 and 3, before the normalization in Step 4, decrease the value of the objective function

\[
\mathcal{L}(\mathbf{u}^{(t)}, \mathbf{v}^{(t)}, \beta^{(t)}, d^{(t)}) \geq \mathcal{L}(\mathbf{u}^{(t+1, \text{inter}}), \mathbf{v}^{(t+1, \text{inter}}), \beta^{(t)}, d^{(t)}),
\]

(S26)
where \( u^{(t+1, \text{inter})} \) and \( v^{(t+1, \text{inter})} \) are the intermediate outputs from Steps 2 and 3 in iteration \( t + 1 \). The superscript \( \text{inter} \) indicates the intermediate output in the algorithm.

Because of the scaling properties of \( duv^T, \beta_u \), and \( \beta_v \), there exist \( \beta^{(t, \text{inter})} \) and \( d^{(t, \text{inter})} \) such that

\[
L(u^{(t+1, \text{inter})}, v^{(t+1, \text{inter})}, \beta^{(t)}, d^{(t)}) = L(u^{(t+1)}, v^{(t+1)}, \beta^{(t, \text{inter})}, d^{(t, \text{inter})}),
\]

where \( u^{(t+1)} \) and \( v^{(t+1)} \) are from Step 4, which are root-\( n \) normalization of \( u^{(t+1, \text{inter})} \) and \( v^{(t+1, \text{inter})} \). Since Steps 5, 6, 7 optimize the objective function with respect to \( \beta \) and \( d \), we have

\[
L(u^{(t+1)}, v^{(t+1)}, \beta^{(t, \text{inter})}, d^{(t, \text{inter})}) \geq L(u^{(t+1)}, v^{(t+1)}, \beta^{(t+1)}, d^{(t+1)}).
\]

Combining the three displays above, we have proved that

\[
L(u^{(t)}, v^{(t)}, \beta^{(t)}, d^{(t)}) = \tilde{L}(u^{(t)}, v^{(t)}) \geq L(u^{(t+1)}, v^{(t+1)}, \beta^{(t+1)}, d^{(t+1)}) = \tilde{L}(u^{(t+1)}, v^{(t+1)}).
\]

If \( \partial_u L(u^{(t)}, v^{(t)}, \beta^{(t)}, d^{(t)}) \neq 0 \) or \( \partial_v L(u^{(t)}, v^{(t)}, \beta^{(t)}, d^{(t)}) \neq 0 \), the inequality is strict, meaning that the algorithm will never revisit point \((u^{(t)}, v^{(t)}, \beta^{(t)}, d^{(t)})\) again.

Recall that the scaling of \( u^{(t)}, v^{(t)} \) does not affect \( \tilde{L}(u^{(t)}, v^{(t)}) \), let us focus on the unit sphere \( S \). Note that since \( u^{(t)}/\|u^{(t)}\|_2 \) and \( v^{(t)}/\|v^{(t)}\|_2 \) are on the unit sphere \( S \), there are limiting points for the pair. Clearly, the limiting points are stationary points (otherwise, for any point within a sufficiently small neighborhood, by continuity and continuous differentiability of \( L \) together with the descending nature of the algorithm at the limiting point, one iteration will lead to a point that has a smaller objective function value than any point in the neighborhood). Next, we use common techniques in mathematical analysis to show that \( \lim_{t \to \infty} \|\partial L(u^{(t)}, v^{(t)}, \beta^{(t)}, d^{(t)})\|_2 = 0 \).

For simplicity of notation, we denote

\[
\beta(u, v), d(u, v) = \arg \min_{\beta, d} L(u, v, \beta, d).
\]

Clearly, the partial derivative of objective function \( L \) w.r.t. \( \beta \) and \( d \) are zeros:

\[
\partial_{\beta, d} L(u, v, \beta, d) \bigg|_{\beta = \beta(u, v), d = d(u, v)} = 0.
\]

For any small \( \eta > 0 \), we construct the following covering of \( S \times S \). For any \( z_u, z_v \in S \), such that \( \partial_u, \partial_v, \beta, d \), there exist \( z_u, z_v \) such that

\[
\|\partial_u v, \beta, d \tilde{L}(u, v, \beta, d) \bigg|_{\beta = \beta(z_u, z_v), d = d(z_u, z_v)} \leq \eta.
\]

For any \( z_u, z_v \in S \), such that \( \partial_u, \partial_v, \beta, d \), there exist \( z_u, z_v \) such that

\[
\|\partial_u v, \beta, d \tilde{L}(u, v, \beta, d) \bigg|_{\beta = \beta(z_u, z_v), d = d(z_u, z_v)} \leq \eta.
\]

Clearly, \( S \times S \subset \bigcup_{(z_u, z_v) \in S \times S} C(z_u, z_v) \). By the compactness of \( S \times S \), it can be covered by finite many sets, denoted as \( C(z_u^1, z_v^1), \ldots, C(z_u^n, z_v^n) \).

Since the limiting points are all stationary points, the sequence of pairs, \( u^{(t)}/\|u^{(t)}\|_2 \) and
\( \frac{v^{(t)}}{\|v^{(t)}\|_2} \), will never visit the balls centered around non-stationary points after a certain number of iterations. That is, there exists an integer \( T_\eta > 1 \) such that after \( T_\eta \) iterations, all the pairs, \( \frac{u^{(t)}}{\|u^{(t)}\|_2} \) and \( \frac{v^{(t)}}{\|v^{(t)}\|_2} \), generated by the algorithm are in one of the balls centered around some stationary points. In other words,
\[
\| \partial_u u, \beta, d \mathcal{L}(u, v, \beta, d) \|_{u=\frac{u^{(t)}}{\sqrt{n}}, v=\frac{v^{(t)}}{\sqrt{n}}, \beta=\sqrt{\eta} \beta^{(t)}, d=nd^{(t)}} \|_2 \leq \eta, \text{ for } t > T_\eta,
\]
which implies
\[
\| \partial_u u, \beta, d \mathcal{L}(u, v, \beta, d) \|_{u=\frac{u^{(t)}}{\sqrt{n}}, v=\frac{v^{(t)}}{\sqrt{n}}, \beta=\sqrt{\eta} \beta^{(t)}, d=nd^{(t)}} \|_2 \leq \eta, \text{ for } t > T_\eta, \quad (S33)
\]
which establishes the claim \( \lim_{t \to \infty} \| \partial \mathcal{L}(u^{(t)}, v^{(t)}, \beta^{(t)}, d^{(t)}) \|_2 = 0 \).

\[\square\]

### S3.2 SuperCENT algorithm for undirected networks

When the network is undirected with the eigenvector centrality, it can be represented by a symmetric matrix \( A \). Denote \( u \) as the eigenvector centrality. The objective function of SuperCENT estimation is a special case of (4), i.e.,
\[
(\hat{\beta}_x, \hat{\beta}_u, \hat{d}, \hat{u}) := \arg \min_{\beta_x, \beta_u, d, \|u\|_2=\sqrt{n}} \frac{1}{n} \| y - X \beta_x - u \beta_u \|_2^2 + \frac{\lambda}{n^2} \| A - d u u^\top \|_F^2. \quad (S34)
\]

To solve (S34), we adopt a similar strategy as Algorithm 1 – a partial block descent algorithm by updating \((\hat{\beta}, \hat{d}, \hat{u})\) iteratively until convergence, where \( \hat{\beta} = (\hat{\beta}_x, \hat{\beta}_u)^\top \). To update \( \hat{u} \), we use gradient descent with backtracking line search instead. The initialization can be obtained from the eigen decomposition of \( A \).

Given a tuning parameter \( \lambda \), Algorithm S2 describes the algorithm for a symmetric matrix \( A \). Similarly, the tuning parameters \( \lambda \) can be chosen using cross-validation as described in Section S3.4.

First, our results now apply to multi-rank networks, which require handling non-leading vectors. Second, we provide results not only for the solution of the objective function but also for the solution produced by the algorithm, which adds complexity to the analysis.

The derivation of Algorithm S2 is similar to Algorithm 1. Denote \( W = (X, u), \beta = (\beta_x, \beta_u), \) and \( \mathcal{L}_{sym} := \frac{1}{n} \| y - X \beta_x - u \beta_u \|_2^2 + \frac{\lambda}{n^2} \| A - d u u^\top \|_F^2 \) where the subscript \( sym \) denotes the objective function for a symmetric matrix \( A \). Given \( \lambda \), we minimize the objective function (4) by setting the partial derivatives with respect to \( \beta \) and \( d \) as zero and then applying gradient descent with line backtracking search for \( u \). The partial derivatives are as follows.
\[
\partial_\beta \mathcal{L}_{sym} = -\frac{2}{n} W^\top (y - W \beta), \quad (S35)
\]
\[
\partial_d \mathcal{L}_{sym} = \frac{2}{n^2} \lambda d \| u \|_2^4 - \frac{2}{n^2} \lambda u^\top A u, \quad (S36)
\]
\[
\partial_u \mathcal{L}_{sym} = -\frac{2}{n} \beta_u (y - X \beta_x - u \beta_u) + \frac{4}{n^2} \lambda d^2 \| u \|_2^2 u - \frac{4}{n^2} \lambda d A u. \quad (S37)
\]
Algorithm S2: SuperCENT($A, X, y, \lambda$) to solve (S34) for a symmetric $A$.

**Result:** $\hat{\beta}, \hat{u}$ and $\beta$.

**Input:** the observed network $A \in \mathbb{R}^{n \times n}$, the design matrix $X \in \mathbb{R}^{n \times p}$, the response vector $y \in \mathbb{R}^n$, the tuning penalty parameter $\lambda$, the tolerance parameter $\rho > 0$, the maximum number of iteration $T$.

Initiate $t = 0$,
\[
\begin{aligned}
& u^{(0)} = \arg\min\|u\|_2 = \sqrt{n} \left\| A - d u u^\top \right\|_F, \\
& W^{(0)} = (X, u^{(0)}), \\
& \beta^{(0)} = (W^{(0)}^\top W^{(0)})^{-1} W^{(0)}^\top y, \\
& d^{(0)} = u^{(0)} A u^{(0)}/\|u^{(0)}\|_2^4; \\
\end{aligned}
\]

while $t \leq 1$ or ($\|P u^{(t-1)} - P u^{(t)}\|_2 > \rho$ and $t < T$) do
\[
\begin{aligned}
& 1. \quad t \leftarrow t + 1; \\
& 2. \quad u^{(t)} \leftarrow \text{BLS}(A, X, y, \lambda, u^{(t-1)}, \beta^{(t-1)}, d^{(t-1)}) \text{ of Algorithm S3;} \\
& 3. \quad \text{Normalize } u^{(t)} \text{ to have norm } \sqrt{n}: u^{(t)} = \sqrt{n} u^{(t)}/\|u^{(t)}\|_2; \\
& 4. \quad W^{(t)} = (X, u^{(t)}); \\
& 5. \quad \beta^{(t)} = (W^{(t)}^\top W^{(t)})^{-1} W^{(t)}^\top y; \\
& 6. \quad d^{(t)} = u^{(t)} A u^{(t)}/\|u^{(t)}\|_2^4; \\
\end{aligned}
\]
end
\[
\hat{u} = u^{(t)}, \hat{\beta} = \beta^{(t)}, \hat{d} = d^{(t)}.
\]

Algorithm S3: BLS($A, X, y, \lambda, u^{(t-1)}, \beta^{(t-1)}, d^{(t-1)}$): backtracking line search (BLS) to obtain $u^{(t)}$.

**Result:** $u^{(t)}$

**Input:** the observed network $A \in \mathbb{R}^{n \times n}$, the design matrix $X \in \mathbb{R}^{n \times p}$, the response vector $y \in \mathbb{R}^n$, the tuning penalty parameter $\lambda$, the tolerance parameter $\rho_2 > 0$, the maximum number of iteration $T_2$, the step size shrinkage $\gamma \in (0, 1)$, $u^{(t-1)}$, $\beta^{(t-1)} = (\beta_x^{(t-1)}, \beta_u^{(t-1)})$, $d^{(t-1)}$, $\alpha = 1$;

Let $\mathcal{L}_{\text{sym}, t}(u) = \frac{1}{n} \|y - X \beta_x^{(t-1)} - u^{\beta_u^{(t-1)}}\|_2^2 + \frac{\lambda}{n^2} \|A - d^{(t-1)} u u^\top\|_F^2$.

Initiate $u_0 = u^{(t-1)}, \tau = 1$;

while $\tau \leq 2$ or ($\|P u_{\tau-1} - P u_{\tau-2}\|_2 > \rho_2$ and $\tau < T_2$) do
\[
\begin{aligned}
& 1. \quad \alpha_\tau = \alpha; \\
& 2. \quad \nabla_\tau = -\frac{2}{n} \beta_u^{(t-1)} (y - X \beta_x^{(t-1)} - u_{\tau-1} \beta_u^{(t-1)}) \\
& \quad + \frac{4}{n^2} \lambda d^{(t-1)^2} u_{\tau-1}^2 u_{\tau-1}^\top - \frac{4}{n^2} \lambda d^{(t-1)} A u_{\tau-1}; \\
& 3. \quad \text{while } \mathcal{L}_{\text{sym}, t}(u_{\tau-1} - \alpha_\tau \nabla_\tau) > \mathcal{L}_{\text{sym}, t}(u_{\tau-1}) - \frac{1}{2} \alpha_\tau \|\nabla_\tau\|_2^2 \text{ do} \\
& \quad \quad \text{Update the step size: } \alpha_\tau = \gamma \alpha_\tau; \\
& \quad \text{end} \\
& 4. \quad u_{\tau} = u_{\tau-1} - \alpha_\tau \nabla_\tau; \\
& 5. \quad \tau \leftarrow \tau + 1; \\
\end{aligned}
\]
end
\[
 u^{(t)} = u_{\tau-1}.
\]
Setting the partial derivatives above as zero yields the estimates
\[ \hat{\beta} = (\hat{W}^\top \hat{W})^{-1} \hat{W}^\top y, \]  
\[ \hat{d} = \hat{u}^\top A\hat{u}, \]
where \( \hat{W} = (X, \hat{u}) \). Similarly, denote \( (\hat{\beta}^{(t)}, \hat{d}^{(t)}, \hat{u}^{(t)}) \) as the estimations from the \( t \)-th iteration. Taking together (S38)-(S39) with Algorithm S3 and substituting corresponding estimates from the previous update, we obtain each update step in each iteration.

### S3.3 Prediction algorithm

In this section, we explain and demonstrate that under the unified framework, the fitted model can be used for prediction.

Assume that there are \( n_1 \) training observations, \( n_2 \) testing observations, and in total \( n = n_1 + n_2 \) observations. All of these \( n \) observations form an adjacency matrix \( A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \), which has block structure. In what follows, we will assume \( r = 1 \) for simplicity. At the end of this section, we will provide the algorithm for the multi-rank cases.

Given \( A, X, \) and \( y \), suppose the unified framework holds for all these \( n \) training and testing observations altogether, i.e.,
\[ \begin{align*}
A &= duv^\top + E, \\
y &= X\beta_x + u\beta_u + v\beta_v + \epsilon,
\end{align*} \tag{S40a} \tag{S40b} \]
where we make assumptions on the length of the centralities that \( \|u\| = \|v\| = \sqrt{n} \).

Denote the sub-blocks of \( u, v, E, y, X, \epsilon \) appropriately with subscripts 1 and 2 corresponding to the training and testing parts respectively. Then, the model for the training part of (S40) becomes
\[ \begin{align*}
A_{11} &= du_1v_1^\top + E_{11}, \\
y_1 &= X_1\beta_x + u_1\beta_u + v_1\beta_v + \epsilon_1,
\end{align*} \tag{S41a} \tag{S41b} \]
And the testing part of (S40) becomes
\[ \begin{align*}
A_{22} &= du_2v_2^\top + E_{22}, \\
y_2 &= X_2\beta_x + u_2\beta_u + v_2\beta_v + \epsilon_2.
\end{align*} \tag{S42a} \tag{S42b} \]

Note that because of the joint model (S40) for both training and testing data, the regression coefficients \( \beta_x, \beta_u, \beta_v \) in the training (S41b) and in the testing (S42b) are the same. Therefore, if the estimations of the regression coefficients \( \hat{\beta}_x, \hat{\beta}_u, \hat{\beta}_v \) are obtained from the training data \( A_{11}, X_1, y_1 \) based on model (S41) by our SuperCENT algorithm, and \( u_2, v_2 \) are available, we can plug them into (S42b) for prediction and obtain \( \hat{y}_2 \).

Although such prediction seems straightforward, a few subtle points worth clarification. First, the model (S41) for the training data does not satisfy the norm constraints, \( \|u_1\| = \|v_1\| = \sqrt{n} \),
\[ \|v_1\| = \sqrt{n}. \] An equivalent model can be expressed which satisfies the norm constraints,

\[
\begin{align*}
A_{11} &= \left( \frac{d \|u_1\| \|v_1\|}{n_1} \right) \left( \frac{\sqrt{n_1}}{\|u_1\|} u_1 \right) \left( \frac{\sqrt{n_1}}{\|v_1\|} v_1 \right)^\top + E_{11}, \\
y_1 &= X_1 \beta_x + \left( \frac{\sqrt{n_1}}{\|u_1\|} u_1 \right) \left( \frac{\sqrt{n_1}}{\|v_1\|} v_1 \right) \beta_u + \left( \frac{\sqrt{n_1}}{\|v_1\|} v_1 \right) \beta_v + \epsilon_1.
\end{align*}
\] (S43a)

Therefore, implicitly, we assume the following model for the training data

\[
\begin{align*}
\tilde{A}_{11} &= \tilde{d} \tilde{u}_1 \tilde{v}_1^\top + E_{11}, \\
\tilde{y}_1 &= X_1 \tilde{\beta}_x + \tilde{u}_1 \tilde{\beta}_u + \tilde{v}_1 \tilde{\beta}_v + \epsilon_1,
\end{align*}
\] (S44a)

where \( \tilde{d} = \frac{d \|u_1\| \|v_1\|}{n_1}, \tilde{u}_1 = \frac{\sqrt{n_1}}{\|u_1\|} u_1, \tilde{v}_1 = \frac{\sqrt{n_1}}{\|v_1\|} v_1, \tilde{\beta}_u = \frac{\|u_1\|}{\sqrt{n_1}} \beta_u, \) and \( \tilde{\beta}_v = \frac{\|v_1\|}{\sqrt{n_1}} \beta_v. \) Hence, applying SuperCENT algorithm 1 to the observed training data \( A_{11}, X_1, y_1 \) will produce estimates \( \hat{u}_1, \hat{v}_1, \hat{\beta}_x, \hat{\beta}_u, \hat{\beta}_v, \hat{d} \) for \( u_1, v_1, \beta_x, \beta_u, \beta_v, d \) in model (S44), but not for \( u_1, v_1, \beta_x, \beta_u, \beta_v, d \) in (S41). Therefore, these estimates cannot be plugged into (S42b) for prediction, because there is a mismatch of the scaling.

The model for the testing data that depends on the tilde version of the coefficients is

\[
\begin{align*}
A_{22} &= \left( \frac{d \|u_1\| \|v_1\|}{n_1} \right) \left( \frac{\sqrt{n_1}}{\|u_1\|} u_2 \right) \left( \frac{\sqrt{n_1}}{\|v_1\|} v_2 \right)^\top + E_{22}, \\
y_2 &= X_2 \beta_x + \left( \frac{\sqrt{n_1}}{\|u_1\|} u_2 \right) \left( \frac{\sqrt{n_1}}{\|v_1\|} v_2 \right) \beta_u + \left( \frac{\sqrt{n_1}}{\|v_1\|} v_2 \right) \beta_v + \epsilon_1,
\end{align*}
\] (S45a)

which is equivalent to

\[
\begin{align*}
A_{22} &= \tilde{d} \tilde{u}_2 \tilde{v}_2^\top + E_{22}, \\
y_2 &= X_2 \tilde{\beta}_x + \tilde{u}_2 \tilde{\beta}_u + \tilde{v}_2 \tilde{\beta}_v + \epsilon_2,
\end{align*}
\] (S46a)

where \( \tilde{d}, \tilde{\beta}_u, \tilde{\beta}_v \) are defined above and \( \tilde{u}_2 = \frac{\sqrt{n_1}}{\|u_1\|} u_2, \tilde{v}_2 = \frac{\sqrt{n_1}}{\|v_1\|} v_2. \)

It is clear that to make prediction for \( y_2, \) we need estimates \( \hat{\beta}_x, \hat{\beta}_u, \hat{\beta}_v \) for model (S44) from training data \( A_{11}, X_1, y_1 \) via our Algorithm 1, as well as estimates \( \hat{u}_2, \hat{v}_2. \)

The discussion above (S40)-(S46) first assumes the root-\( n \) constraints for the centralities from all \( n \) observations including training and testing data in model (S40), and then adjusts the training and testing models accordingly to satisfy the root-\( n_1 \) constraints for the training data. Following this discussion, there are two conclusions: (i) the same \( \hat{\beta}_u, \hat{\beta}_v \) can be used for both training and testing data, as long as the \( \tilde{u}_1, \tilde{u}_2, \tilde{v}_1, \tilde{v}_2 \) are scaled appropriately; (ii) the concatenation of the scaled centralities for both training data \( \tilde{u}_1 \) and testing data \( \tilde{u}_2 \) together is proportional to original leading singular vectors for all data, i.e., \( (\tilde{u}_1^\top, \tilde{u}_2^\top)^\top \propto (u_1^\top, u_2^\top)^\top = u, \) similarly for \( v. \)

The discussion above (S40)-(S46) still follows, if one starts with root-\( n_1 \) constraints for the centralities for the training data, and adjust the models for all data and testing data accordingly. Similar conclusions can be obtained. As we mentioned in Section 3.2, the scales of \( u_1, v_1, \beta_u, \beta_v \) are not identifiable (one can multiply the centralities by non-zero constant and divide the regression coefficients by the same non-zero constant), only the products \( u_1 \beta_u, v_1 \beta_v \) are identifiable. Hence, it does not matter what norm-constraint is
imposed on the centralities of the training data, that is $\|\hat{u}_1\|, \|\hat{v}_1\|$ can be anything, as long as $\hat{\beta}_u, \hat{\beta}_v$ are adjusted accordingly. Despite this flexibility, we still assume root-$n_1$ constraints above for clarity purpose. The main take-away is that, for any pair of estimates of $\hat{u}_1, \hat{\beta}_u$ (respectively, $\hat{v}_1, \hat{\beta}_v$), whose product is fully determined, it is not necessary for them to be $\hat{u}_1, \hat{\beta}_v$, what matters for proper prediction is to ensure that $(\hat{u}_1^\top, \hat{u}_2^\top) \propto (u_1^\top, u_2^\top)^\top = u$ (respectively, $(\hat{v}_1^\top, \hat{v}_2^\top)^\top \propto (v_1^\top, v_2^\top)^\top = v$).

The only remaining question is how to obtain estimate $\hat{u}_2$. The procedure for $\hat{v}_2$ is similar and will be omitted. Two approaches will be introduced. The first approach is relatively simple, which is based on the SVD of the large network matrix $A$ and is described in Algorithm S4. Since $\hat{u}_2 = \sqrt{\frac{\beta}{\|u_1\|}}u_2$, where $(u_1^\top, u_2^\top)^\top = u$ is the leading left singular vector of the signal in (S40a). Suppose $\hat{u}$ is the leading left singular vector of $A$. Assume that $\hat{u}_{1:n_1}$ and the SuperCENT estimate $\hat{u}_1$ have an angle less than 90 degrees; otherwise, flip the sign of $\hat{u}_{1:n_1}$. Naturally, we have

$$\hat{u}_2 = \frac{\sqrt{\beta}}{\|u_{1:n_1}\|} \hat{u}_{(n_1+1):n}. \quad (S47)$$

**Algorithm S4**: Modified SVD of $A$ to obtain $\hat{u}_2$ and $\hat{v}_2$.

**Result**: $\hat{u}_2$ and $\hat{v}_2$.

**Input**: The augmented network $A \in \mathbb{R}^{n \times n}$, $\hat{u}_1$, and $\hat{v}_1$.

1. $(\hat{u}, \hat{v})$ are the leading left and right singular vectors of $A$;
2. $\hat{u} = \text{sign}(\hat{u}_1^\top \hat{u}_{1:n_1}) \hat{u}$ and $\hat{v} = \text{sign}(\hat{u}_1^\top \hat{u}_{1:n_1}) \hat{v}$;
3. Rescale $\hat{u}_2 = \frac{\sqrt{\beta}}{\|u_{1:n_1}\|} \hat{u}_{(n_1+1):n}$ and $\hat{v}_2 = \frac{\sqrt{\beta}}{\|v_{1:n_1}\|} \hat{v}_{(n_1+1):n}$.

The first approach only uses the adjacency matrix $A$ to estimate $\hat{u}_2$. The second approach leverages the better estimate from SuperCENT comparing to plain SVD and is described in Algorithm S5. It is based on the following observation. Focusing on the bottom left block of the observed network matrix $A$ in (S40a), we have

$$A_{21} = d\hat{u}_2 \hat{v}_1^\top + E_{21} = \hat{d}\hat{u}_2 \hat{v}_1^\top + E_{21},$$

where $\hat{d}, \hat{u}_2, \hat{v}_1$ were defined the same as above in (S44) and (S46). Since $\hat{d}, \hat{v}_1$ are provided by applying SuperCENT on the training data, we can estimate $\hat{u}_2$ by minimizing $\|A_{21} - \hat{d}\hat{u}_2 \hat{v}_1^\top\|_F^2$. This leads to an estimate

$$\hat{u}_2 = \hat{d}^{-1}(\hat{v}_1^\top \hat{v}_1)^{-1} A_{21} \hat{v}_1. \quad (S48)$$

Plugging in the model of $A_{21}$ into the estimation, it is easily seen that $\hat{u}_2 \approx \hat{u}_2$ plus a term related to noise $E_{21}$.

Moving to the multi-rank case, the same reasoning follows and the only difference is the model for the network for all training and testing changes from (S40a) to

$$A = UDV^\top + E = \hat{U}\hat{D}\hat{V}^\top + E, \quad (S49)$$

where for the terms without tilde, $U = (u, u_2, \cdots, u_r) \in \mathbb{R}^{n \times r}, V = (v, v_2, \cdots, v_r) \in \mathbb{R}^{r \times n}$. The estimation of $\hat{u}_2$ is similar to the previous case.
\[ \mathbb{R}^{n \times r}, D = \text{diag}(d_1, d_2, \ldots, d_r) \in \mathbb{R}^{r \times r}, \text{which satisfy } U^T U = V^T V = nI; \text{and for the terms with}\] tilde, they are defined as follows

\[ UDV^\top = \begin{pmatrix} U_1DV_1^\top & U_1DV_2^\top \\ U_2DV_1^\top & U_2DV_2^\top \end{pmatrix} \] \hspace{1cm} (S50)

\[ = \begin{pmatrix} \tilde{U}_1\Lambda_u\Lambda_u^{-1}D\Lambda_v^{-1}\Lambda_vV_1^\top & U_1\Lambda_u\Lambda_u^{-1}D\Lambda_v^{-1}\Lambda_vV_2^\top \\ U_2\Lambda_u\Lambda_u^{-1}D\Lambda_v^{-1}\Lambda_vV_1^\top & U_2\Lambda_u\Lambda_u^{-1}D\Lambda_v^{-1}\Lambda_vV_2^\top \end{pmatrix} \] \hspace{1cm} (S51)

\[ = \begin{pmatrix} \tilde{U}_1\tilde{D}\tilde{V}_1^\top & \tilde{U}_1\tilde{D}\tilde{V}_2^\top \\ \tilde{U}_2\tilde{D}\tilde{V}_1^\top & \tilde{U}_2\tilde{D}\tilde{V}_2^\top \end{pmatrix} \] \hspace{1cm} (S52)

\[ = \tilde{U}\tilde{D}\tilde{V}^\top, \] \hspace{1cm} (S53)

where \( \Lambda_u = \sqrt{n_1}(\text{diag}(U_1^\top U_1))^{-1/2}, \Lambda_v = \sqrt{n_1}(\text{diag}(V_1^\top V_1))^{-1/2}, \tilde{U}_1 = U_1\Lambda_u, \tilde{U}_2 = U_2\Lambda_u, \tilde{V}_1 = V_1\Lambda_v, \tilde{V}_2 = V_2\Lambda_v, \tilde{D} = \Lambda_v^{-1}D\Lambda_v^{-1}, \tilde{U} = (\tilde{U}_1^\top, \tilde{U}_2^\top)^\top, \tilde{V} = (\tilde{V}_1^\top, \tilde{V}_2^\top)^\top. \) The coefficients \( \hat{\beta}_u, \hat{\beta}_v \) are the same as in the rank one case.

Again, for the training data part, estimations \( \hat{U}_1, \hat{V}_1, \hat{D} \) can be obtained via SuperCENT. To estimate \( \hat{U}_2 \) and \( \hat{V}_2 \), first note that \( A_{21} = \hat{U}_2\hat{D}\hat{V}_1^\top + E_{21} \) and \( A_{12} = \hat{U}_1\hat{D}\hat{V}_2^\top + E_{12} \). Given \( \hat{U}_1, \hat{V}_1, \hat{D} \), minimizing the Frobenius norm of the approximation errors of \( A_{21} \) and \( A_{12} \) yields the estimates \( \hat{U}_2 \) and \( \hat{V}_2 \) respectively, i.e.,

\[ \hat{U}_2 = \arg\min_{\hat{U}_2} \| A_{21} - U_2\hat{D}\hat{V}_1^\top \|_F^2 \quad \text{and} \quad \hat{V}_2 = \arg\min_{\hat{V}_2} \| A_{12} - U_1\hat{D}\hat{V}_2^\top \|_F^2. \] \hspace{1cm} (S54)

Take the first columns of \( \hat{U}_2 \) and \( \hat{V}_2 \) as \( \hat{u}_2 \) and \( \hat{v}_2 \), which are the estimated centralities of the testing data.

Finally, with the testing covariates \( X_2 \), the estimated coefficient \( \hat{\beta}_x, \hat{\beta}_u, \hat{\beta}_v \) from SuperCENT on training data, we predict \( \hat{y}_2 = X_2\hat{\beta}_x + \hat{u}_2\hat{\beta}_u + \hat{v}_2\hat{\beta}_v. \)

**Algorithm S5:** An improved method using SuperCENT estimate of \( U_1 \) and \( V_1 \) to obtain \( \hat{u}_2 \) and \( \hat{v}_2 \).

**Result:** \( \hat{u}_2 \) and \( \hat{v}_2 \).

**Input:** The augmented network \( A \in \mathbb{R}^{n \times n}, \hat{D}, \hat{U}_1, \) and \( \hat{V}_1. \)

1. \( \hat{U}_2 = \left((\hat{V}_1^\top\hat{D}\hat{V}_1)^{-1}\hat{D}\hat{V}_1^\top A_{21}\right)^\top; \hat{V}_2 = \left((\hat{U}_1^\top\hat{D}\hat{U}_1)^{-1}\hat{D}\hat{U}_1^\top A_{12}\right)^\top; \)

2. Take the first columns of \( \hat{U}_2 \) and \( \hat{V}_2 \) as \( \hat{u}_2 \) and \( \hat{v}_2 \) respectively.

### S3.4 Selection of the tuning parameter \( \lambda \)

The tuning parameter \( \lambda \) can be selected using the \( K \)-fold cross-validation. Given the prediction procedure in Section S3.3, the cross-validation procedure can be easily carried out as follows. For each fold of validation data, we first fit the model using the remaining \( K - 1 \) folds with the corresponding induced subnetwork and obtain the estimates for the regression coefficients by implementing Algorithm 1; we then obtain the estimates of the centralities for the validation data by applying Algorithm S4; we last obtain the total
prediction error for the validation data by combining the outcomes from the first two steps. The best tuning parameter \( \lambda \) is set to be the minimizer of the total cross-validation error that sums over all folds. Algorithm S6 outlines this procedure in more detail.

Algorithm S6: The cross-validation algorithm for SuperCENT to choose \( \lambda \).

Result: \( \lambda_{\text{min}} \).

Input: \( A, X, y \).

for \( \lambda \) on a exponentially regular grid do

for each fold do

0. Split the covariates and response into training \( X_{\text{fold}, \text{train}}, y_{\text{fold}, \text{train}} \) and validation \( X_{\text{fold}, \text{val}}, y_{\text{fold}, \text{val}} \) and denote the the induced sub-network corresponding to the training data \( A_{\text{fold}, \text{train}} \);

1. \( (\hat{\beta}_{\text{fold}, \lambda}, \hat{u}_{\text{fold}, \lambda}, \hat{v}_{\text{fold}, \lambda}) \leftarrow \text{SuperCENT}(A_{\text{fold}, \text{train}}, X_{\text{fold}, \text{train}}, y_{\text{fold}, \text{train}}, \lambda) \) by Algorithm 1;

2. \( \hat{u}_{\text{fold}, \text{val}}, \hat{v}_{\text{fold}, \text{val}} \leftarrow \text{SVD}(A) \) and re-scale by Algorithm S4;

3. \( SSE_{\text{fold}, \lambda} = \|y_{\text{fold}, \text{val}} - (X_{\text{fold}, \text{val}}, \hat{u}_{\text{fold}, \text{val}}, \hat{v}_{\text{fold}, \text{val}}) \hat{\beta}_{\text{fold}, \lambda}\|_2^2; \)

end

\( \lambda_{\text{min}} = \min_{\lambda} \sum_{\text{fold}} SSE_{\text{fold}, \lambda} \)

As another strategy, Remark 10 offers an alternative way to choose the tuning parameter based on the theoretical analysis of SuperCENT, which is less time-consuming than cross-validation. However, we recommend using the cross-validation strategy for the best performance based on the simulation results.

Remark S1. The cross-validation procedure will not change the underlying network structure under our unified framework if all the entries of the network noise \( E \) are assumed to be i.i.d., given that the centralities \( U \) and \( V \) are fixed parameters. This is because the generative distribution of a subset of a matrix of low-rank mean with i.i.d. noise remains the same. For cross-validation under other model assumptions, one may consider the edge sampling procedure proposed by \( ? \) or an “out-of-sample” procedure based on embeddings \( (? \).

S4 Theoretical properties of the two-stage procedure

In this section, we present the theoretical properties for the two-stage estimator. Under Assumptions 1-3, the two-stage estimators are consistent, with their asymptotic distributions given in Theorem S2 and their convergence rates given in Propositions S2 and S3. The proof is deferred to Section S7.2.

We first introduce the following notations. Let \( K \) be the \( n^2 \times n^2 \) commutation matrix such that \( \text{vec}(E^\top) = K \text{vec}(E) \) and \( \otimes \) denote the Kronecker product. Define \( \tilde{u} = (I - \)
$P_X u, \hat{v} = (I - P_X) v$, which are the centralities projected onto the orthogonal space of $X$. Denote $c = \tilde{u}^\top \tilde{w}^\top v - (\tilde{u}^\top \tilde{v})^2$ and $C_{\tilde{u}\tilde{v}} = (\tilde{u}, \tilde{v})^\top (u, \bar{v})$, which will show up in the asymptotic expressions.

Recall that the two-stage estimates are denoted as $\hat{\beta}^{ts}$, $\hat{u}^{ts}$, $\hat{v}^{ts}$ and $\hat{\beta}^{ts} = ((\hat{\beta}^{ts})^\top, \hat{u}^{ts}, \hat{v}^{ts})^\top$.

**Theorem S2.** Under the unified framework (2) and Assumptions 1-3, the two-stage estimates have the following asymptotic distribution,

1. Centralities:
   \[
   \hat{u}^{ts} - u = \eta_u^{ts} + o(\eta_u^{ts}) \quad \text{and} \quad \hat{v}^{ts} - v = \eta_v^{ts} + o(\eta_v^{ts}),
   \]  
   \[(S55)\]

2. Network effect:
   \[
   \hat{\beta}^{ts} = \beta + \eta_\beta^{ts} + o(\eta_\beta^{ts}) = \left( \begin{array}{c} \eta_{\beta_1}^{ts} \\ \eta_{\beta_2}^{ts} \\ \eta_{\beta_3}^{ts} \end{array} \right) + o \left( \begin{array}{c} \eta_{\beta_1}^{ts} \\ \eta_{\beta_2}^{ts} \\ \eta_{\beta_3}^{ts} \end{array} \right),
   \]  
   \[(S56)\]

where \[
\eta_\beta^{ts} = \eta_\beta^{ts} \sim N(0, (\sigma^2 I_n \otimes I_{p+1}) \cdot C_{u} \cdot (\sigma^2 I_n \otimes I_{p+1})^\top), \frac{||\eta_{\beta_u}^{ts}||}{\eta_{\beta_u}^{ts}} \to P, \frac{||\eta_{\beta_v}^{ts}||}{\eta_{\beta_v}^{ts}} \to P.
\]

\[
\frac{||\eta_{\beta_u}^{ts}||}{\eta_{\beta_u}^{ts}} \to P, \frac{||\eta_{\beta_v}^{ts}||}{\eta_{\beta_v}^{ts}} \to P,
\]

and $C_u = \left( \begin{array}{cc} C_{12} & C_{12} \\ C_{22} & C_{22} \end{array} \right)$ whose specific forms are as follows.

The matrices related to $\hat{u}^{ts}$ and $\hat{v}^{ts}$ are

\[
\begin{pmatrix} C_{12}^{ts} \\ C_{22}^{ts} \end{pmatrix} = (dn)^{-1} \begin{pmatrix} u^\top \otimes (I - P_u) \\ v^\top \otimes (I - P_v) \end{pmatrix} K,
\]  
   \[(S57)\]

the matrices related to $\hat{\beta}_u^{ts}$ and $\hat{\beta}_v^{ts}$ are

\[
\begin{pmatrix} C_{11}^{ts} & C_{31}^{ts} \\ C_{21}^{ts} & C_{32}^{ts} \\ C_{41}^{ts} & C_{42}^{ts} \\ C_{51}^{ts} & C_{52}^{ts} \end{pmatrix} = C_{u\tilde{u}}^{-1} \begin{pmatrix} \tilde{u}^\top \\ \tilde{v}^\top \end{pmatrix} \left( -\beta_u I_n - \beta_v I_n \right) \begin{pmatrix} 0 & C_{12}^{ts} \\ 0 & C_{22}^{ts} \\ I_n & 0 \end{pmatrix},
\]  
   \[(S58)\]

and the matrices related to $\hat{\beta}_x^{ts}$ are

\[
\begin{pmatrix} C_{31}^{ts} & C_{32}^{ts} \end{pmatrix} = (X^\top X)^{-1} X^\top \left( -\beta_u I_n - \beta_v I_n - u - v \right) \begin{pmatrix} 0 & C_{12}^{ts} \\ 0 & C_{22}^{ts} \\ C_{31}^{ts} & C_{32}^{ts} \\ C_{41}^{ts} & C_{42}^{ts} \\ I_n & 0 \end{pmatrix},
\]  
   \[(S59)\]

Recall the two-stage procedure first estimates the centralities $u$ and $v$ and then plugs the estimated centralities into the regression model. Therefore, the asymptotic distributions for $\hat{u}^{ts}$ and $\hat{v}^{ts}$ only depend on the noise $E$ from the network model, not the regression noise $\epsilon$. This can also be seen in the definitions of $\Sigma_u^{ts}$ and $\Sigma_v^{ts}$, all of which involve only
\( \sigma_a^2 \), but not \( \sigma_y^2 \). On the other hand, the asymptotic variance of \( \hat{\beta}^{ts} \), \( \Sigma_{\beta}^{ts} \), involves both \( \sigma_y^2 \) and \( \sigma_a^2 \), not just \( \sigma_y^2 \). We will highlight this fact in Remarks S2 below.

**Remark S2.** (Non-classical covariance of \( \hat{\beta}^{ts} \)) One important fact to emphasize is that the covariance of \( \hat{\beta}^{ts} \) is not \( \sigma_y^2(\hat{W}^T \hat{W})^{-1} \) where \( \hat{W} = (X, \hat{u}^{ts}, \hat{v}^{ts}) \), which would be the classical results of regression if \( X, \hat{u}^{ts}, \hat{v}^{ts} \) were considered given and fixed. This makes sense, as in our model, the observed network contains noise, which makes the estimated centralities \( \hat{u}^{ts}, \hat{v}^{ts} \) from the first stage random quantities and invalidates the classical result.

Following Theorem S2, we present the convergence rates of the estimated centralities \( \hat{u}^{ts} \) and \( \hat{v}^{ts} \) in Proposition S2. To measure the difference between any estimate \( \hat{u} \) and the true \( u \), one typically uses the loss function \( \|P_u - P_u\|^2 \), which equals the squared sine of the angle between \( \hat{u} \) and \( u \), \( \sin^2 \angle(\hat{u}, u) \). However, the exact form of this loss function is not clean mathematically. Instead, we use the scaled Euclidean distance \( \|\hat{u} - \text{sign}(\hat{u}^T u)u\|^2/n = 2 - 2\cos^2 \angle(\hat{u}, u) \), which has a cleaner expression and is connected to the squared sine through \( \|P_u - P_u\|^2 = (\|\hat{u} - \text{sign}(\hat{u}^T u)u\|^2/n)[1 - (\|\hat{u} - \text{sign}(\hat{u}^T u)u\|^2/n)/4] \). These two losses are approximately equivalent when the estimator is consistent, i.e., the loss goes to zero. In the following, the theorems are for \( \arg\max_{\beta \in \{\hat{u}^{ts}, \hat{v}^{ts}\}} \text{sign}(h^T u) \) and \( \arg\max_{\beta \in \{\hat{u}^{ts}, \hat{v}^{ts}\}} \text{sign}(v^T v) \), and we still use \( \hat{u}^{ts}, \hat{v}^{ts} \) to denote them. While these notions are a bit of an abuse of notation, it is reasonable since both the objective function and algorithm are sign-invariant (i.e., proper flipping of signs gives the same value or another valid iteration sequence). The same notation applies in the simulations as well.

**Proposition S2.** *(Convergence rates of \( \hat{u}^{ts} \) and \( \hat{v}^{ts} \)) Under the unified framework (2) assume \( A_0 \) to be rank-one and Assumptions 1-3, the two-stage estimators satisfy the following,

\[
\frac{1}{n} \mathbb{E} \|\hat{u}^{ts} - u\|^2_2 = \frac{1}{n} \mathbb{E} \|\hat{v}^{ts} - v\|^2_2 = \frac{\sigma_a^2(n - 1)}{d^2n^2}(1 + o(1)) \quad (S60)
\]

\[
= \frac{\sigma_a^2}{d^2n}(1 + o(1)) = \kappa(1 + o(1)). \quad (S61)
\]

The convergence rate of the two-stage depends on the interplay of three parameters: \( d, \sigma_a, n \). When the signal strength \( d \) and the noise level \( \sigma_a \) are of constant order while \( n \) diverges, the two-stage converges at a fast rate. However, it is highly possible that the two-stage may converge slowly for the following reasons: 1) it is understandable that the observed network might become noisier with more nodes, i.e., large \( \sigma_a \), because it may become exponentially costly to collect data for a larger network of the same level of noise; 2) the signal strength \( d \) decays as more nodes are included into the network, since the network edge density might decay with more nodes.

We next present the convergence rates of the regression coefficients \( \hat{\beta}^{ts}_u, \hat{\beta}^{ts}_v, \hat{\beta}^{ts}_z \) in Proposition S3.
Proposition S3. (Asymptotic property of $\hat{\beta}^{ts}$) Under the unified framework (2) and Assumptions 1-3, the two-stage estimators satisfy

$$\mathbb{E}(\hat{\beta}^{ts}_u - \beta_u)^2 = \left(\frac{\sigma_y^2}{c}\hat{v}^\top\hat{v}\right) + \frac{\sigma_y^2}{c^2} \frac{1}{d^2 n} \left[\beta_u^2 \hat{v}^\top(I - P_v)\hat{u}\hat{v}^\top + \beta_u^2 \hat{u}^{\top} \hat{u}^\top(I - P_u)\hat{u}\hat{v}^\top\hat{v}\right] \left[1 + o(1)\right] \quad (S62)$$

$$\mathbb{E}(\hat{\beta}^{ts}_v - \beta_v)^2 = \left(\frac{\sigma_y^2}{c}\hat{u}^\top\hat{u}\right) + \frac{\sigma_y^2}{c^2} \frac{1}{d^2 n} \left[\beta_v^2 \hat{u}^\top(I - P_u)\hat{v}\hat{u}^\top + \beta_v^2 \hat{v}^{\top} \hat{v}^\top(I - P_v)\hat{v}\hat{u}^\top\hat{u}\right] \left[1 + o(1)\right] \quad (S63)$$

$$\text{Cov}(\hat{\beta}^{ts}_x - \beta_x) = \sigma_y^2 \left[(X^\top X)^{-1} + (X^\top X)^{-1} X^\top \begin{pmatrix} u & v \end{pmatrix} C^{-1}_{uv} \begin{pmatrix} u^\top \vspace{1em} v^\top \end{pmatrix} X (X^\top X)^{-1}\right] + \frac{\sigma_y^2}{d^2 n} [(X^\top X)^{-1} X^\top \left[\beta_u^2 (I - P_u) + \beta_v^2 (I - P_v)\right] \begin{pmatrix} u & v \end{pmatrix} C^{-1}_{uv} \begin{pmatrix} u^\top \vspace{1em} v^\top \end{pmatrix} X (X^\top X)^{-1} (1 + o(1))]. \quad (S68)$$

Remark S3. (Comments on (S62)-(S64) for $\hat{\beta}^{ts}$) For the variance of $\hat{\beta}^{ts}_u$, the first term (S62) is the same as the variance in the classical regression results with deterministic predictors by treating $\hat{u}^{ts}$ and $\hat{v}^{ts}$ as given and fixed. The additional term (S63) is due to the randomness nature of $\hat{u}^{ts}$ and $\hat{v}^{ts}$. Note that the second term (S63) is non-negative and it becomes zero if $\sigma_u = 0$, or $\hat{u} \perp \hat{v}$. Therefore, if ad-hoc inference were made by only considering the first term (S62) as in Remark 1, it would be valid only if the network is noiseless or the hub and authority centralities are completely orthogonal after accounting for the covariates $X$, which are rarely the case in real networks. Similar messages can be obtained for the variance of $\hat{\beta}^{ts}_v$ and covariance of $\hat{\beta}^{ts}_x$.

S5 More simulation results

In this section, we show more simulation results that are deferred from Section 5. Section S5.1 provides additional results for Section 5.2. We then show a phase-transition experiment in Section S5.2 to demonstrate the behaviors of SuperCENT and the two-stage estimators under different network signal-to-noise ratios.

To give an overview of the simulation results, Table S4 summarizes the comparison of the two-stage and SuperCENT from the perspectives of both estimation and inference. SuperCENT universally outperforms the two-stage in terms of centrality estimation, regression coefficients estimation, and inference.
Table S4. Comparison of the two-stage and SuperCENT when the network signal-to-noise ratio is low and $\beta_u^2/\sigma_y^2 \gg \beta_v^2/\sigma_y^2$. If $\beta_u^2/\sigma_y^2 \gg \beta_v^2/\sigma_y^2$, the results for $u$ and $v$, $\beta_u$ and $\beta_v$ will be switched. In the estimation panel, ✓ indicates accurate estimation, ✗ indicates inaccurate estimation. In the inference panel, ✓ indicates that the empirical coverage of confidence interval is no less than the nominal level, ✗ indicates that the confidence interval fails to reach the nominal level. In each row, SuperCENT is underlined whenever it outperforms the two-stage.

| Two-stage | SuperCENT |
|-----------|-----------|
| Estimation |           |
| $u$ | ✗ ✓ |
| $v$ | ✗ ✓ (Slightly Improved) |
| $A_0$ | ✗ ✓ |
| $\beta_u$ | ✗ ✓ |
| $\beta_v$ | ✗ ✓ |
| Inference |           |
| $CI_{\beta_u}$ | ✗ ✓ |
| $CI_{\beta_v}$ | ✗ ✓ |
| $CI_{a_{ij}}$ | ✗ ✓ |

Figure S7. The boxplot of $\log_{10}(l(\hat{v}, \hat{v}))$ for the four estimators across different $\sigma_a$, $\sigma_y$ and $\beta_u$ with $d = 1$ and $\beta_v = 1$. where $l(\hat{v}, \hat{v}) = \| P_{\hat{v}} - P_v \|^2$. The super-imposed red symbols show the theoretical rates of the two-stage in Proposition S2 and SuperCENT in Proposition 1.

S5.1 Additional results for Section 5.2

Section 5.2 shows that SuperCENT greatly improves over two-stage in terms of estimation of $u$ and $\beta_u$ as well as the inference of $\beta_u$. In this section, we demonstrate the behaviors of the SuperCENT-based and the two-stage-based estimators of $v$, $A_0$, and $\beta_v$ and their corresponding confidence intervals.

For the estimation of $v$ shown in Figure S7, the improvement of SuperCENT over two-stage is not as large as that of the estimation of $u$ when $\beta_u \in 2^{2,4}$, because $\beta_u^2/\sigma_y^2 \gg \beta_v^2/\sigma_y^2$. But
Figure S8. The boxplot of $\log_{10}(l(\hat{A}, A_0))$ for four estimators across different $\sigma_u$, $\sigma_y$ and $\beta_u$ with $d = 1$ and $\beta_v = 1$ where $l(\hat{A}, A_0) = \|\hat{A} - A_0\|_F^2/\|A_0\|_F^2$. The super-imposed red symbols show the theoretical rates of the two-stage in Proposition S2 and SuperCENT in Proposition 1.

Figure S9. The boxplot of $\log_{10}(l(\hat{\beta}_u, \beta_u))$ for four estimators across different $\sigma_u$, $\sigma_y$ and $\beta_u$ with $d = 1$ and $\beta_v = 1$ where $l(\hat{\beta}_u, \beta_u) = (\hat{\beta}_u - \beta_u)^2/\beta_u^2$. The super-imposed red points show the median of $\log_{10}(l(\hat{\beta}_u, \beta_u))$.

The improvement is still quite significant when $\beta_u = 2^0 \approx \beta_v = 1$. It is worth noting that the supervised effect to $\hat{v}$ shrinks as $\beta_u$ increases, leading to a different trend comparing Figures 1a and S7. This phenomena aligns with Remark 11 where we discuss the estimation of $u$ and $v$ comparing SuperCENT and the two-stage. Specifically, the roles of $u$ and $v$ are not exchangeable, because here we have $\beta_v \leq \beta_u$ by fixing $\beta_v = 1$ and varying $\beta_u \in 2^{0,2,4}$. On the other hand, when $\beta_v \gg \beta_u$ we should expect the improvement in estimating $v$ to increase.

The conclusion for the estimation of $A_0$ is similar to that of $u$ as shown in Figure S8. With the improvement from estimating $u$ and $v$, SuperCENT-$\hat{\lambda}_{cv}$ estimates $A_0$...
Figure S10. The boxplot of $\hat{\beta}_v - \beta_v$ across different $\sigma_a$, $\sigma_y$ and $\beta_u$ with fixed $d = 1$ and $\beta_v = 1$. The dashed lines show $\hat{\beta}_v - \beta_v = 0$. The super-imposed red points show the median of $\hat{\beta}_v - \beta_v$.

more accurately across all the settings. As claimed in Remark 11, the convergence of $\hat{A}$ in this regime only requires $\frac{\sigma^2}{\sigma_a^2} \to \infty$ or $\frac{\sigma^2}{\sigma_y^2} \to \infty$. Therefore, with $\beta_v = 1$, $\beta_u > 1$, $\hat{A}$ converges and $l(\hat{A}, A_0) < l(\hat{A}^{ts}, A_0)$. Comparing Figures 1a, S7 and S8 altogether, when $\beta_u = 2^0$, SuperCENT improves the estimation of both $u$ and $v$ significantly; when $\beta_u \in 2^{2,4}$, SuperCENT improves the estimation of $u$ a lot; therefore, SuperCENT improves the estimation of $A_0$ a lot for all the ranges of $\beta_u$.

Figure S10 shows $\hat{\beta}_v - \beta_v$. We observe an over-estimation of $\beta_v$ when $\beta_u$ is large, which is different from the under-estimation in $\hat{\beta}_u$. The inaccuracy is larger as $\beta_u$ increases. SuperCENT in this case can still improve the accuracy of the estimation of $\beta_v$ but the improvement is not as large as that of $\beta_u$ since the improvement in estimation of $v$ when $\beta_u$ is large is limited as shown in Figure S7, thereby slightly improving over the two-stage from the perspective of $\beta_v$ estimation in Figure S10 as well as the squared error loss in Figure S11.

Similar to $\beta_u$, the inaccurate estimation of $\beta_v$ further affects its confidence interval. Figures S12 and S13 show the empirical coverage and $\log_{10}$ of the average width, respectively, of the 95% confidence interval for $\beta_v$. For the empirical coverage, when $\sigma_a$ remains small (top panels), all methods remain close to the nominal level, again with different reasons for different methods as discussed for the coverage of $CI_{\beta_u}$. When $\sigma_a$ remains small, the inaccuracy in $\hat{\beta}_v$ is relatively small, leaving a relatively small impact on the coverage. As $\beta_u$ increases, the oracles tend to under cover since the inaccuracy increases. Two-stage is still conservative because $\sigma_y^2$ is overestimated, covering up the issue of inaccurate estimation. Similar to the phenomena we observed in the consistent regime, the two-stage-adhoc is on par with the two-stage-oracle even with $\sigma_y^2$ being overestimated and thus below the nominal level. When $\sigma_a$ increases and the inaccuracy is not too severe with $\beta_u = 2^2$ (the
**Figure S11.** The boxplot of \( \log_{10}(l(\hat{\beta}_v, \beta_v)) \) across different \( \sigma_a, \sigma_y \) and \( \beta_u \) with \( d = 1 \) and \( \beta_v = 1 \) where \( l(\hat{\beta}_v, \beta_v) = (\hat{\beta}_v - \beta_v)^2 / \beta_v^2 \). The super-imposed red points show the median of \( \log_{10}(l(\hat{\beta}_v, \beta_v)) \).

**Figure S12.** Empirical coverage of \( CI_{\hat{\beta}_v} \) across different \( \sigma_a, \sigma_y \) and \( \beta_u \) with \( d = 1 \) and \( \beta_v = 1 \). SuperCENT variants are labelled as circles (○●) and the two-stage variants are labelled as triangles (△▲▼▲). The hollow ones are for oracles and the solid ones are for non-oracles. The dashed lines show the nominal confidence level 0.95.

mid-bottom panel), the two-stage is still close to the nominal level while two-stage-oracle and two-stage-adhoc are no longer valid. SuperCENT does not improve much from two-stage as the improvement of estimating \( \beta_v \) is limited. The trade-off between \( \beta_u \) and \( \beta_v \) for SuperCENT is desirable — SuperCENT provides valid and shorter intervals for both \( \beta_u \) and \( \beta_v \) if \( \beta_u \) and \( \beta_v \) are similar; if \( \beta_u \) and \( \beta_v \) differ a lot, SuperCENT provides a valid and shorter interval for the larger effect which is more of one’s interest.

As for the width of the CI for \( \beta_v \), Figure S13 shows that when the SuperCENT methods reach the nominal level, the widths are shorter than two-stage.

Finally, we investigate the average coverage and the average width of confidence intervals...
Figure S13. Width of $CI_{\beta_v}$ across different $\sigma_a$, $\sigma_y$ and $\beta_u$ with $d = 1$ and $\beta_v = 1$. SuperCENT variants are labelled as circles (○•) and the two-stage variants are labelled as triangles (△▲). The hollow ones are for oracles and the solid ones are for non-oracles.

for all the entries $a_{ij}$ of $A_0$ respectively. The average coverage probability of all the methods, $\text{Average}_{ij}(\text{CP}(CI_{a_{ij}}))$, achieves the nominal level of 95%. The coverage tends to be slightly below the nominal coverage as $\sigma_a$ increases, because the estimation becomes worse and the theorem only holds up to $1 + o(1)$. SuperCENT-$\hat{\lambda}_{cv}$ is the closest to the nominal coverage in all the settings compared to the others. Figure S15 shows the log$_{10}$ of the average width of the CIs, $\text{Average}_{ij}(\text{Width}(CI_{a_{ij}}))$. SuperCENT-$\lambda_0$-oracle provides the shortest width among the four methods, followed by SuperCENT-$\hat{\lambda}_{cv}$. The widths of the confidence intervals of both SuperCENT-based methods are shorter than those of the two-stage methods. Again, the improvement of SuperCENT over the two-stage increases as $\sigma_a$ and $\beta_u$ increase or $\sigma_y$ decreases.

S5.2 Results of a phase-transition experiment

To further study the behaviors of SuperCENT and the two-stage and understand the advantages of SuperCENT under different network signal-to-noise ratios (SNRs), we perform a phase-transition experiment.

The simulation setup is similar to that in Section 5.1, except that we only vary $\sigma_a \in 2^{1, 1.25, \ldots, 5}$ as well as the gap between the leading singular values and the non-leading ones, while fixing all the other parameters to investigate the continuous impact of the network noise level or equivalently the network SNR. Specifically, we generate a network with $n = 256 = 2^8$ following the network model (2a), i.e., $A = A_0 = UDV^T + E$ where $U, V \in \mathbb{R}^{n \times r}$, $D$ is a diagonal matrix of dimension $r \times r$ with the singular values $d > d_2 \geq \ldots \geq d_r \geq 0$ as the diagonal entries, and all the entries of $E$ follow $N(0, \sigma_2^2)$ independently. All entries of $U$ and $V$ are first generated from i.i.d. $N(0, 1)$, then applied Gram-Schmidt to ensure orthogonality between columns, and finally each column of $U$ and $V$ is rescaled to have
### Figure S14.

The average empirical coverage $\text{Average}_{ij}(\text{CP}(C_{a_{ij}}))$ across different $\sigma_a$, $\sigma_y$ and $\beta_u$ with $d = 1$ and $\beta_v = 1$. SuperCENT variants are labelled as circles (○ ▪) and the two-stage variants are labelled as triangles (△ ▲). The hollow ones are for oracles and the solid ones are for non-oracles. The dashed lines show the nominal confidence level 0.95.

### Figure S15.

$\log_{10}$ of the average width of $C_{I_{a_{ij}}}$ across different $\sigma_a$, $\sigma_y$ and $\beta_u$ with $d = 1$ and $\beta_v = 1$. SuperCENT variants are labelled as circles (○ ▪) and the two-stage variants are labelled as triangles (△ ▲). The hollow ones are for oracles and the solid ones are for non-oracles.

length $\sqrt{n}$. We consider the case of $r = 10$ where the leading singular value $d_1 = 1$ and the non-leading ones as $d_2 = \ldots = d_r \in \{0, 2^{-1}, 2^{-0.5}\}$. Note that when $d_2 = \ldots = d_{10} = 0$, it is a rank-one setting. We include this setting so as to compare the single rank setting with the multiple ranks in the network model. For the regression model, $y = X\beta_x + u\beta_u + v\beta_v + \epsilon$, we include the covariate matrix $X$ and the hub and authority centralities, namely the leading singular vectors $u = u_1$ and $v = v_1$ instead of the entire $U$ and $V$.

We compare the performance of SuperCENT and the two-stage in terms of estimation accuracy and inference property. For estimation accuracy, we compare the estimation error...
of the hub centrality $u$ and the hub effect $\beta_u$ between the two-stage and SuperCENT-$\lambda_0$ (labeled as SuperCENT); for inference property, we compare the coverage probability and the width of $CI_{\beta_u}$ between the two-stage-adhoc (labeled as two-stage) and SuperCENT-$\lambda_0$-oracle (labeled as SuperCENT).

Before comparing the performance of the two-stage and SuperCENT, we first show the top 20 squared singular values of the observed network adjacency matrix $A$ with fixed leading singular value $d_1 = 1$ for the noiseless component $A_0$, fixed noise level $\sigma_a = 2$ and various non-leading singular values for $A_0$ in Figure S16. The purpose of these plots are to make connections and comparisons to the spectral properties of the four real networks in Section S2. Under the rank-one setting when the noiseless singular values of $A_0$ as $d_1 = 1$ and $d_2 = \ldots = d_{10} = 0$ (the plot on the left), the leading singular value $\hat{d}_{ts}^1$ of the observed network $A$, obtained via simple SVD or two-stage, dominates the non-leading ones, and the non-leading ones slowly decay, which resembles the spectral structure of the innovation network in Figure S6B. For the multi-rank settings when $d = 1$ and $d_2 = \ldots = d_{10} \in \{2^{-1}, 2^{-0.5}\}$ (the two plots on the right), the leading singular value $\hat{d}_{ts}^1$ still dominates the non-leading ones, and the non-leading ones are separated into two groups: $\{\hat{d}_{ts}^2, \ldots, \hat{d}_{ts}^{10}\}$ as the “signal” group and the rest as the “noise” group. As $d_2, \ldots, d_{10}$ get larger, the two-stage estimates of the signal group are also larger as expected, and the gap between the last signal singular value $\hat{d}_{ts}^{10}$ and the first noise singular value $\hat{d}_{ts}^{11}$ is larger. The multi-rank setting with small noiseless non-leading singular values $d_2, \ldots, d_{10}$, especially the middle plot, resembles the global trade network, production network, and equity network in Figures S6A, C, and D.

Figure S17A shows the estimation error of the hub centrality $\sin(\angle(\hat{u}, u))$, i.e., sine of the angle between the true hub centrality $u$ and the estimate $\hat{u}$. The relationship between $\sin(\angle(\hat{u}, u))$ and $\|P_u - P\|_2^2$ has been discussed in Section S4. We use $\sin(\angle(\hat{u}, u))$ for the propose of demonstration. The red curves correspond to the two-stage and the green ones correspond to SuperCENT; while the solid curves correspond to the rank-one setting and the dashed and dotted curves correspond to the multi-rank setting. For the two-stage, the estimation error increases as the network noise $\sigma_a$ increases and $\hat{u}^{ts}$ eventually becomes orthogonal to $u$, i.e., $\sin(\angle(\hat{u}^{ts}, u))$ approaching 1. Comparing settings with different non-
leading singular values among the two-stage estimates (red), the estimation error of the rank-one setting (solid) is smaller than those of the multi-rank settings (dashed, dotted), and the estimation error is larger as $d_2 = \ldots = d_{10}$ get closer to $d_1$. The SuperCENT estimates $\hat{u}$, on the other hand, have much smaller estimation errors in all settings and the supervision effect persists regardless of the rank of the network. Comparing settings with different non-leading singular values among the SuperCENT estimates (green), the estimation accuracy is quite similar when $\sigma_a$ is smaller. As $\sigma_a$ gets larger, the SuperCENT estimates of the multi-rank settings (dashed, dotted) are a bit less accurate than that of

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Figure S17. Comparison between the two-stage and SuperCENT in terms of the estimations of $u$ and $\beta_u$ as well as the coverage probability and the width of $CI_{\beta_u}$ varying the network noise level $\sigma_a$ and the gap between the leading singular value and the non-leading ones. The performance of the two-stage is shown in the red and SuperCENT in the green. The solid line corresponds to the rank-one setting with $d_1 = 1$ and $d_2 = \ldots = d_{10} = 0$, while the dashed and dotted lines correspond to the multi-rank setting with $d_1 = 1, d_2 = \ldots = d_{10} = 2^{-1} = 0.5$ and $d_1 = 1, d_2 = \ldots = d_{10} = 2^{-0.5} \approx 0.7$ respectively. Subfigure A shows the estimation error of the hub centrality $\sin(\angle(\hat{u}, u))$, i.e., sine of the angle between the true hub centrality $u$ and the estimate $\hat{u}$. Subfigure B shows $\hat{\beta}_u - \beta_u$. Subfigures C and D show the coverage probability and the width of the 95% confidence interval of the hub centrality coefficient, $CI_{\beta_u}$, respectively.
the rank-one setting (solid).

Figure S17B shows $\hat{\beta}_u - \beta_u$. The hub effect estimate of two-stage $\hat{\beta}^{ts}_u$ is inaccurate due to the inaccurate centrality estimation as shown in Figure S17A. The larger the network noise $\sigma_a$ or noise-to-signal ratio $\kappa$, the larger the estimation inaccuracy. Comparing settings with different non-leading singular values, the inaccuracy is larger as $d_2 = \ldots = d_{10}$ get closer to $d$. For SuperCENT, the estimate of the hub effect $\hat{u}$ is accurate until around $\sigma_a = 2^{-3.75}$. The estimation inaccuracy are slightly larger under the multi-rank settings than that of the rank-one setting, but still much smaller than that of the two-stage.

As of the inference property, Figures S17C-D show the coverage probability and the width of the 95% confidence interval of the hub centrality coefficient, $CI\beta_u$, respectively. When $\sigma_a \in 2^{1.25,1.5}$ or the network noise is relatively small, the two-stage confidence intervals are conservative and wider than SuperCENT. As $\sigma_a$ increases, the coverage probability of the two-stage confidence intervals sharply drop to zero. The SuperCENT confidence intervals, on the contrary, remain valid until $\sigma_a$ becomes large and are narrower than the two-stage confidence interval. In particular, the confidence intervals under the multi-rank settings are slightly more conservative than that under the rank-one setting and the coverage probability is closer to the nominal 95% as the gaps between the leading singular value $d_1$ and the non-leading ones become larger.

In sum, SuperCENT outperforms the two-stage in terms of both estimation accuracy and inference property and thus SuperCENT should be preferred over the two-stage regardless under different network SNRs.

S6 Details on the case study in Section 6

In this section, we provide details on data construction in Section S6.1, additional results and interpretations on the authority centrality in Section S6.2, with more information on the case study in Section S6.3.

S6.1 Data construction

In the case study, we consider a triplet of $\{A, X, y\}$, where $A$ is the country-level trade network, $y$ is the currency risk premium, and $X$ is the share of world’s GDP. All these quantities are not directly available, and we compute them according to as follows.

To compute the currency risk premium, we obtain the interest rates and the exchange rates from DataStream. The currency risk premium can be calculated as follows. For an investor going long in a country/region $i$, the log risk premium “rx” at time $t + 1$ is $y_{i,t+1} := r_{it} - r_t - \Delta q_{i,t+1}$, where $r_{it}$ is log interest rate of country/region $i$, $r_t$ is the log interest rate of the U.S. and $\Delta q_{i,t+1}$ is the appreciation of U.S. dollar. Only 25 countries/regions have exchange rates available during the period of interest. We exclude the region of Europe as it is not comparable to the others in the trade network, resulting in 24 countries/regions.\footnote{The list of country abbreviations is provided in Section S6.3.}
We use a 5-year moving average of the currency risk premium: when considering year $t$, average is taken from year $t - 4$ to year $t$.

? defined the trade linkage as the trade amount normalized by the pair-wise total GDP, which represents the relative trade (export/import) intensity between two countries. Specifically, the trade linkage between two countries is computed as $a_{ijt} = \frac{S_{ijt}}{GDP_i + GDP_j}$, where $S_{ijt}$ is the dollar value of goods and commodities exported from country $i$ to $j$ at time $t$, and $GDP_i$ is the GDP of country $i$ at time $t$ in U.S. dollar.\(^\text{10}\) Same as the currency risk premium, we use the 5-year moving average in the analysis.

### S6.2 Additional results on the authority centrality

In Section 6, we focus on the hub centrality $\mathbf{u}$ and the its coefficient $\beta_u$. In the following, we show the corresponding results for the authority centrality $\mathbf{v}$ and the its coefficient $\beta_v$.

Figure S18 shows the time series plots of the ranking of the authority centrality estimated by two-stage and SuperCENT for the 24 countries/regions, together with the ranking of the currency risk premium. We rank the centrality in ascending order and the risk premium in descending order. Based on the negative relationship between centralities and risk premium established in ?, the closer the trends of rankings between centralities and risk premium are, the better the centralities capture the time variation in the risk premium. Similar to the hub centrality, the authority centrality estimated by the two-stage procedure is relatively more stable over time compared to SuperCENT, since SuperCENT incorporates information of both the GDP share and currency risk premium, which is more volatile than the trade network itself.

For the coefficient of the authority centrality $\beta_v$ in Table 2, the estimate from two-stage-adhoc and two-stage is $-0.0005$, while the estimate from SuperCENT is $-0.0003$, which is consistent with the inaccuracy we observed in the simulation due to $|\hat{\beta}_v^{sc}| < |\hat{\beta}_u^{sc}|$. SuperCENT still improves its estimation and confidence interval, even though the improvement is not as large as $\beta_u$ due to $(\hat{\beta}_v^{sc})^2 \ll (\hat{\beta}_u^{sc})^2$ and the nonexchangeable roles of $\mathbf{u}$ and $\mathbf{v}$. For $\beta_x$, the estimates from two-stage and SuperCENT are comparable, but the widths of the confidence intervals from two-stage-adhoc and two-stage are much larger than that from SuperCENT, again as a result of the over-estimation of $\sigma_y^2$. Hence, $\beta_x$ is barely significant when using two-stage-adhoc and two-stage, while very significant by SuperCENT.

### S6.3 Additional information for the case study in Section 6

Table S5 provides the country abbreviations and full names. Figure S19 shows the average trade volume from 2003 to 2012 among the 24 countries/regions. The arrows

\(^{10}\)The bilateral trade data comes from the correlates of war project (COW) (?) and the International Monetary Fund (IMF) Direction of Trade Statistics: https://data.imf.org/?sk=9D6028D4-F14A-464C-A2F2-59B2CD424B85. Current U.S. dollar GDP (using 2015 as the base year) data are from the World Bank’s World Development Indicators: https://databank.worldbank.org/source/world-development-indicators.
**Figure S18.** Time series of authority centrality ranking in ascending order from 2003 to 2012. Similar to the hub centrality, if the trend of centralities is close to the trend of risk premium, then the centralities capture the time variation of risk premium, based on the negative relationship between the two as claimed in ?. The vertical dashed line indicates 2008, the year of the financial crisis.

**S7 Additional theoretical results and proof of all theoretical results**

Section S7.1 is devoted to theoretical results and proof of all theoretical results for SuperCENT and Section S7.2 for the two-stage.

We begin by providing some basic properties of the Kronecker product and the commutation matrix. The Kronecker product of \( M = (m_{ij}) \in \mathbb{R}^{m \times n} \) and \( N = (n_{ij}) \in \mathbb{R}^{p \times q} \), denoted by \( M \otimes N \), is defined as

\[
M \otimes N = \begin{pmatrix}
    m_{11}N & \cdots & m_{1n}N \\
    \vdots & \ddots & \vdots \\
    m_{m1}N & \cdots & m_{mn}N
\end{pmatrix} \in \mathbb{R}^{mp \times nq}.
\]  

(S72)

Denote \( K_{mn} \in \{0, 1\}^{mn \times mn} \) as the commutation matrix such that

\[
\text{vec}(M^\top) = K_{mn} \text{vec}(M).
\]  

(S73)
| Code | Country        | Code | Country      | Code | Country       |
|------|---------------|------|--------------|------|---------------|
| AUS  | Australia     | JPN  | Japan        | SGP  | Singapore     |
| CAN  | Canada        | KOR  | Korea        | SWE  | Sweden        |
| CHE  | Switzerland   | KWT  | Kuwait       | THA  | Thailand      |
| CZE  | Czech Republic| MEX  | Mexico       | ZAF  | South Africa  |
| DNK  | Denmark       | MYS  | Malaysia     |      |               |
| GBR  | United Kingdom| NOR  | Norway       |      |               |
| HKG  | Hong Kong     | NZL  | New Zealand  |      |               |
| HUN  | Hungary       | PHL  | Philippines  |      |               |
| IDN  | Indonesia     | POL  | Poland       |      |               |
| IND  | India         | SAU  | Saudi Arabia |      |               |

**Table S5:** List of country abbreviations.

![Figure S19](image-url)

**Figure S19.** The average trade volume from 2003 to 2012 among the 24 countries/regions. Each country is in different color.

We list the following facts about the Kronecker product and the commutation matrix, which are used in the section without specific references. Proofs of these facts can be found in 

Let $M \in \mathbb{R}^{m \times n}$, $N \in \mathbb{R}^{p \times q}$, $P \in \mathbb{R}^{n \times t}$, $Q \in \mathbb{R}^{q \times s}$, and $Z \in \mathbb{R}^{n \times p}$.

(i) $(M \otimes N)^\top = M^\top \otimes N^\top$.

(ii) $(M \otimes N)(P \otimes Q) = (MP) \otimes (NQ)$.

(iii) vec$(MZN) = (N^\top \otimes M) \text{vec}(Z)$. 

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(iv) $\text{vec}(MP) = (I \otimes M)\text{vec}(P) = (P^\top \otimes I)\text{vec}(M)$.

(v) $K_{mn}^\top = K_{nm}$.

(vi) $K_{mn}^\top K_{mn} = K_{mn}^\top K_{mn} = I$.

(vii) $K_{mp}(M \otimes N)K_{qn} = (N \otimes M)$. Equivalently, $K_{mp}(M \otimes N) = (N \otimes M)K_{qn}$.

(viii) $(M \otimes N)K_{nq}(P \otimes Q) = ((MP) \otimes (NQ))K_{st} = K_{mp}((NQ) \otimes (MP))$.

(ix) $\text{tr}(K_{mn}(M \otimes N)) = \text{tr}(MN)$ where $M, N \in \mathbb{R}^{m \times n}$.

### S7.1 Additional theoretical results and proof of all theoretical results for SuperCENT

We start by introducing additional definitions and notations. Recall that Section 4 is devoted for the theoretical results for the SuperCENT estimators, which are the optimizers of the objective function (4). In particular, the main Theorem 1 states the asymptotic distribution of the SuperCENT optimization estimators. In this section, we first present Theorem S3 in Section S7.1.1, which is in parallel of Theorem 1, but for the SuperCENT estimators as the output of Algorithm 1. The proofs for both Theorem 1 and Theorem S3 are provided in Section S7.1.2. Subsequently, Section S7.1.3 provides the proof of Proposition 1, i.e., the convergence rates of $\hat{u}$ and $\hat{v}$, and Section S7.1.4 provides the statement and proof of Proposition S4, i.e., the convergence rate of $\hat{\beta}$. Section S7.1.5 is dedicated to the consistency results of the SuperCENT estimators, which are needed to establish the asymptotic distribution in Theorems 1 and S3. Specifically, we provide two versions of the consistency results: one for the solution of the objective function (4) and one for the solution by Algorithm 1, which are later used to prove Theorem 1 and Theorem S3, respectively. Section S7.1.6 is devoted to Proposition S5, which is essential to prove the consistency results in Section S7.1.5, and its proof. Section S7.1.7 contains all the technical lemmas, which are needed for Proposition S5, along with their proofs.

In the following, $d_1$ and $d$, $u_1$ and $u$, $v_1$ and $v$ are interchangeable, denoting the leading singular value and the leading singular vectors (hub and authority centralities). The relevant notations and definitions are given as follows.

**Definition 1.** Let $\tilde{\lambda} = \frac{\sigma^2}{n \sigma_y} \lambda$.

**Definition 2.** Define function $f_0$ that takes in arguments $\bar{u} \in \mathbb{R}^n, \bar{v} \in \mathbb{R}^n, \tilde{\beta}_x \in \mathbb{R}^p, \tilde{\beta}_u \in \mathbb{R}, \tilde{\beta}_v \in \mathbb{R}, d \in \mathbb{R}^+$,

$$f_0 : (\bar{u}, \bar{v}, \tilde{\beta}_x, \tilde{\beta}_u, \tilde{\beta}_v, d) \mapsto \frac{1}{\sigma_y} \| y - X \tilde{\beta}_x - \bar{u} \tilde{\beta}_u - \bar{v} \tilde{\beta}_v \|^2 + \frac{\tilde{\lambda}}{\sigma^2} \| d \bar{u} \bar{v}^\top - A \|^2_F.$$  \hfill (S74)

**Definition 3.** Define function $\tilde{\beta}$ that takes in arguments $\bar{u} \in \mathbb{R}^n, \bar{v} \in \mathbb{R}^n$ such that $(X, \bar{u}, \bar{v})$ is full rank as follows
\[ \overline{\beta} : (\bar{u}, \bar{v}) \mapsto [(X, \bar{u}, \bar{v})^\top (X, \bar{u}, \bar{v})]^{-1} (X, \bar{u}, \bar{v})^\top y. \] (S75)

Let \( \overline{\beta}_x, \overline{\beta}_u, \overline{\beta}_v \) be functions of \((\bar{u}, \bar{v})\) such that

\[ \overline{\beta}_x(\bar{u}, \bar{v}) \in \mathbb{R}^n, \overline{\beta}_u(\bar{u}, \bar{v}) \in \mathbb{R}, \overline{\beta}_v(\bar{u}, \bar{v}) \in \mathbb{R}, \] (S76)

and

\[ \begin{pmatrix} \overline{\beta}_x(\bar{u}, \bar{v}) \\ \overline{\beta}_u(\bar{u}, \bar{v}) \\ \overline{\beta}_v(\bar{u}, \bar{v}) \end{pmatrix} = \beta(\bar{u}, \bar{v}). \] (S77)

Definition 4. Define function \( \overline{d} \) that takes arguments \( \bar{u} \in \mathbb{R}^n, \bar{v} \in \mathbb{R}^n \)

\[ \overline{d} : (\bar{u}, \bar{v}) \mapsto \frac{\bar{u}^\top A \bar{v}}{\| \bar{u} \|^2} . \] (S78)

Definition 5. Define function \( f_1 \) that takes in arguments \( \bar{u} \in \mathbb{R}^n, \bar{v} \in \mathbb{R}^n \)

\[ f_1 : (\bar{u}, \bar{v}) \mapsto f_0 \left( \bar{u}, \bar{v}, \overline{\beta}_x(\bar{u}, \bar{v}), \overline{\beta}_u(\bar{u}, \bar{v}), \overline{\beta}_v(\bar{u}, \bar{v}), \overline{d}(\bar{u}, \bar{v}) \right). \] (S79)

Definition 6. Define function \( f_2 \) that takes arguments \( \bar{u} \in \mathbb{R}^n, \bar{v} \in \mathbb{R}^n \) as

\[ f_2 : (\bar{u}, \bar{v}) \mapsto \frac{1}{\sigma^2_y} \left\| P_{\bar{u}, \bar{v}} \cdot (\tilde{\varepsilon} + \tilde{u}_1 \beta_u + \tilde{v}_1 \beta_v) \right\|^2 - \frac{\lambda}{n^2 \sigma^2} \left( \bar{u}^\top \left( \sum_{i=1}^r d_i u_i v_i^\top + E \right) \bar{v} \right)^2 . \] (S80)

Definition 7. \( \forall z \in \mathbb{R}^n, \) let \( \tilde{z} = (I - P_X)z. \) Specifically, \( \tilde{u} = (I - P_X)u, \tilde{v} = (I - P_X)v, \) and \( \tilde{\varepsilon} = (I - P_X)\varepsilon. \)

S7.1.1 Theorem S3: Asymptotic distribution for SuperCENT estimator via Algorithm 1

We present the asymptotic distribution for the SuperCENT estimator as the output produced by Algorithm 1. Comparing to Theorem 1, which shows the asymptotic distribution of the minimizer of the objective function (4), Theorem S3 shows that the solution of Algorithm 1 converges to the same asymptotic distribution under slightly different conditions.

Theorem S3. Under the unified framework (2) and Assumptions 1-3, suppose \((X, \bar{u}, \bar{v})\) is full-rank, \( \frac{\sigma_y}{\sqrt{\beta_x^2 + \beta_u^2}} \sqrt{\frac{\log n}{n}} = o(1), \) \( \frac{\sigma_u^2}{n(d-d_2)} \) \( \frac{\sigma_v^2}{\beta_x^2 + \beta_v^2} = o(1), \) and \( \left| \frac{\beta_u}{\beta_v} \right| \in [\alpha, \tilde{\alpha}] \) for positive constants \( \tilde{\alpha} > \alpha > 0, \) then the SuperCENT estimators, defined as the solution of Algorithm 1 with the
two-stage estimators for initialization and a given tuning parameter $\lambda$ satisfying $\frac{1}{\lambda} \frac{n\sigma_u^2}{\sigma_v^2} = o(1)$, $\frac{1}{\lambda^2} \frac{\sigma_v^2}{\beta_u^2 + \beta_v^2} = o(1)$, and $\frac{1}{\lambda (d-d_2)} (\beta_u^2 + \beta_v^2) = o(1)$, converge to the following normal distributions asymptotically.

1. Centralities:

$$\hat{u} - u = \eta_u + o(\eta_u) \quad \text{and} \quad \hat{v} - v = \eta_v + o(\eta_v),$$

(S81)

2. Network effect:

$$\hat{\beta} - \beta = \eta_\beta + o(\eta_\beta) = \left( \eta_{\beta_x}^\top, \eta_{\beta_u}, \eta_{\beta_v} \right) ^\top + o \left( \left( \eta_{\beta_x}^\top, \eta_{\beta_u}, \eta_{\beta_v} \right) ^\top \right),$$

(S82)

where

$$\begin{pmatrix} \eta_u \\ \eta_v \\ \eta_\beta \end{pmatrix} \sim N \left( \mathbf{0}_{(2n+2+p)\times 1}, C \begin{pmatrix} \sigma_y^2 I_n & 0_{n \times n} \\ 0_{n \times n} & \sigma_v^2 I_n^2 \end{pmatrix} C^\top \right), \quad \frac{|o(\eta_u)|}{|\eta_u|} \xrightarrow{P} 0, \quad \frac{|o(\eta_v)|}{|\eta_v|} \xrightarrow{P} 0, \quad \text{and} \quad C \text{ has the same form as in Theorem 1.}$$

**Remark S4.** We would like to point out that, under our model assumptions, with high probability, the full-rank condition for $(X, \hat{u}, \hat{v})$ is satisfied by all $(X, \hat{u}, \hat{v}) = (X, u^{(t)}(\hat{u}), v^{(t)})$, generated by Algorithm 1, including $t = \infty$ which is the output of Algorithm 1. Moreover, with high probability, the full-rank condition for $(X, \hat{u}, \hat{v})$ is also satisfied by the optimizer of the objective function $(4)$. The reasoning is as follows. Under Assumption 3, the two-stage estimator is close to the truth $(u, v)$ with high probability. By using the two-stage estimator as the initialization of our algorithm, the objective function value is already small and is smaller than that of any non-full-rank $(X, \hat{u}, \hat{v})$. This is partly due to the assumption of the condition number, i.e., Assumption 2, which implies a large gap between the minimum objective function value achievable by non-full-rank $(X, \hat{u}, \hat{v})$ and that of the truth, $(X, u, v)$ with high-probability under reasonable noise level. Our algorithm is a descending algorithm, hence all the $(X, u^{(t)}, v^{(t)})$ generated by the algorithm gives a smaller objective function value than that of the minimum objective function achievable by non-full-rank $(X, \hat{u}, \hat{v})$, which implies that they are all full rank. The minimizer of the objective function will have the smallest objective function value and thus it is impossible to be non-full-rank. The details of this proof are relatively tedious, so we have omitted them here.

### S7.1.2 Proof of Theorem 1 and Theorem S3

In this section, we provide the proofs of Theorem 1 in Section S7.1.2 and Theorem S3 in Section S7.1.2, which state the asymptotic normality of the SuperCENT estimators as the minimizer of the objective function $(4)$ and as the solution produced by Algorithm 1, respectively.

**Proof of Theorem 1**

**Proof.** In Section S3, we derive the estimates of $(\hat{u}, \hat{v}, \hat{\beta})$ as (S23)-(S25). Recall that $A^\perp = U^\perp D^\perp V^\perp^\top$ where $U^\perp = (u_2, \ldots, u_r)$, $V^\perp = (v_2, \ldots, v_r)$ and $D^\perp = diag(d_2, \ldots, d_r)$. Then, $A^\perp^\top u = 0$ and
\( A\mathbf{v} = 0 \). Let \( \hat{E} = A^\perp + E \) and \( \Delta_y = u\delta\beta_u + \delta_u\beta_u + v\delta\beta_v + \delta_v\beta_v + \delta_u\delta\beta_u + \delta_v\delta\beta_v \). Together they lead to the first order expansion:

\[
(u + \delta u)^\top (u + \delta u) = n, 
\]

\[
(v + \delta v)^\top (v + \delta v) = n, 
\]

\[
\beta_u + \delta\beta_u = \frac{1}{n}(u + \delta u)^\top (u\beta_u + \epsilon - X\delta\beta_x - v\delta\beta_v - \delta v\beta_v - \delta v\delta\beta_u), 
\]

\[
\beta_v + \delta\beta_v = \frac{1}{n}(v + \delta v)^\top (v\beta_v + \epsilon - X\delta\beta_x - u\delta\beta_u - \delta u\beta_u - \delta u\delta\beta_u), 
\]

\[
\beta_x + \delta\beta_x = (X^\top X)^{-1}X^\top (X\beta_x + \epsilon - \Delta_y), 
\]

\[
d + \delta_d = (u + \delta u)^\top (duv^\top + \hat{E})(v + \delta v)/n^2, 
\]

\[
(\beta_u + \delta\beta_u)(-\epsilon + X\delta\beta_x + \Delta_y) 
\]

\[
+ \lambda(d + \delta_d)^2(u + \delta u) - \lambda(d + \delta_d)(duv^\top + \hat{E})(v + \delta v)/n = 0, 
\]

\[
(\beta_v + \delta\beta_v)(-\epsilon + X\delta\beta_x + \Delta_y) 
\]

\[
+ \lambda(d + \delta_d)^2(v + \delta v) - \lambda(d + \delta_d)(dvu^\top + \hat{E}^\top)(u + \delta u)/n = 0. 
\]

Plugging (S87) into (S85)-(S86) and using (S83)-(S84) to eliminate \( \delta\beta_x \), we have

\[
(u + \delta u)^\top (I - P_X)\delta u\beta_u + (u + \delta u)^\top (I - P_X)\delta v\beta_v 
\]

\[
+ (u + \delta u)^\top (I - P_X)(u + \delta u)\delta\beta_u + (u + \delta u)^\top (I - P_X)(v + \delta v)\delta\beta_u 
\]

\[
= (u + \delta u)^\top (I - P_X)\epsilon 
\]

\[
(v + \delta v)^\top (I - P_X)\delta u\beta_u + (v + \delta v)^\top (I - P_X)\delta v\beta_v 
\]

\[
+ (v + \delta v)^\top (I - P_X)(u + \delta u)\delta\beta_u + (v + \delta v)^\top (I - P_X)(v + \delta v)\delta\beta_u 
\]

\[
= (v + \delta v)^\top (I - P_X)\epsilon. 
\]

Let \( \tilde{u} = (I - P_X)(u + \delta u) \), \( \tilde{v} = (I - P_X)(v + \delta v) \) and \( C\tilde{u}\tilde{v} = \begin{pmatrix} \tilde{u}^\top u, \tilde{u}^\top \tilde{v} \end{pmatrix} \tilde{v}^\top \tilde{v} \). Then (S91)-(S92) can be written as

\[
\begin{pmatrix} \tilde{u}^\top \\ \tilde{v}^\top \end{pmatrix} (\beta_u I_n \beta_v I_n) \begin{pmatrix} \delta u \\ \delta v \end{pmatrix} + C\tilde{u}\tilde{v} \begin{pmatrix} \delta\beta_u \\ \delta\beta_v \end{pmatrix} = \begin{pmatrix} \tilde{u}^\top \\ \tilde{v}^\top \end{pmatrix} \epsilon. 
\]

Solving for \( \delta\beta_u \), \( \delta\beta_v \) gives

\[
\begin{pmatrix} \delta\beta_u \\ \delta\beta_v \end{pmatrix} = C^{-1}\tilde{u}\tilde{v} \begin{pmatrix} \tilde{u}^\top \\ \tilde{v}^\top \end{pmatrix} (-\beta_u I_n - \beta_v I_n I_n) \begin{pmatrix} \delta u \\ \delta v \end{pmatrix}. 
\]
Plugging (S87), (S88) and (S94) into (S89) and (S90) using (S83)-(S84), we have

\[
\begin{bmatrix}
\frac{\lambda d}{n} \left( d u^\top \left( I - \frac{1}{2n} \hat{u} \delta u^\top \right) \right) & -(I - P_u) A^\top (-A^\perp)^\top (I - P_v) \\
(I - P_u) A^\top (I - P_v) & d u^\top \left( I - \frac{1}{2n} \hat{v} \delta v^\top \right)
\end{bmatrix}
\begin{bmatrix}
\hat{\beta}_u \hat{\beta}_v \\
\hat{\beta}_u \hat{\beta}_v
\end{bmatrix} \otimes \left( I - P_{(X \hat{u} \hat{v})} \right)
\begin{bmatrix}
\delta u \\
\delta v
\end{bmatrix}
\]  

\[M_1 = \begin{bmatrix}
\frac{\lambda d}{n} \hat{u}^\top \otimes (I - P_u) \\
\frac{\lambda d}{n} \hat{v}^\top \otimes (I - P_v) K
\end{bmatrix} \begin{bmatrix}
e \\
\text{vec}(E)
\end{bmatrix}.
\]

Then

\[
\begin{bmatrix}
\delta u \\
\delta v
\end{bmatrix} = M_1^{-1} M_2 \begin{bmatrix}
e \\
\text{vec}(E)
\end{bmatrix}.
\]

(S95)

For \(\delta\beta_u\) and \(\delta\beta_v\), plugging (S96) into (S94)

\[
\begin{bmatrix}
\delta u \\
\delta v
\end{bmatrix} = C_{\hat{u} \hat{v}} \begin{bmatrix}
\hat{u}^\top \\
\hat{v}^\top
\end{bmatrix} \begin{bmatrix}
-\beta_u I_n \\
-\beta_v I_n
\end{bmatrix}
\begin{bmatrix}
\delta u \\
\delta v \\
e
\end{bmatrix} (S97)

\[
= C_{\hat{u} \hat{v}} \begin{bmatrix}
\hat{u}^\top \\
\hat{v}^\top
\end{bmatrix} \begin{bmatrix}
-\beta_u I_n \\
-\beta_v I_n
\end{bmatrix}
\begin{bmatrix}
M_1^{-1} M_2 \\
I_n, 0_{n \times n^2}
\end{bmatrix} \begin{bmatrix}
e \\
\text{vec}(E)
\end{bmatrix}.
\]

(S98)

Lastly for \(\delta\beta_x\), we plug (S96) and (S98) into (S97)

\[
\begin{bmatrix}
\delta u \\
\delta v \\
\delta v
\end{bmatrix} = (X^\top X)^{-1} X^\top \begin{bmatrix}
\hat{u} - u \\
\hat{v} - v
\end{bmatrix} \begin{bmatrix}
\delta u \\
\delta v \\
\delta v
\end{bmatrix} (S100)

\[
= (X^\top X)^{-1} X^\top \begin{bmatrix}
-\beta_u I_n \\
-\beta_v I_n - u - v I_n
\end{bmatrix}
\begin{bmatrix}
M_1^{-1} M_2 \\
M_3 \\
I_n, 0_{n \times n^2}
\end{bmatrix} \begin{bmatrix}
e \\
\text{vec}(E)
\end{bmatrix}.
\]

(S101)

Based on the consistency result in Corollary S1, we have \(\min \left\{ \frac{\|\hat{u} - u\|}{\sqrt{n}}, \frac{\|\hat{u} + u\|}{\sqrt{n}} \right\} \to 0\).
Finally, recall that we assume
\[
\begin{align*}
\delta u &= \eta u + o(\eta u), \\
\delta v &= \eta w + o(\eta v), \\
\delta \beta_u &= \eta \beta_u + o(\eta \beta_u), \\
\delta \beta_v &= \eta \beta_v + o(\eta \beta_v), \\
\delta \beta_z &= \eta \beta_z + o(\eta \beta_z),
\end{align*}
\]
(S102)
where
\[
\begin{align*}
\eta u &= \left( \frac{\lambda d}{n} \begin{pmatrix} -A^T d_n I & -I \end{pmatrix} + \begin{pmatrix} \beta_u^2 \beta_u & \beta_v \beta_v \beta_u \beta_v \end{pmatrix} \otimes (I - P(Xwv)) \right)^{-1} \\
\eta v &= \left( \begin{pmatrix} \beta_u & \beta_v \end{pmatrix} \left( I - P(Xwv) \right) \right) \lambda d n \otimes \left( I - P(uv) / n \right) K \\
\eta \beta_z &= \left( \begin{pmatrix} \beta_u & \beta_v \end{pmatrix} \left( I - P(Xwv) \right) \right) \lambda d n \otimes \left( I - P(uv) / n \right) K \\
\eta \beta_u &= \left( \begin{pmatrix} \beta_u & \beta_v \end{pmatrix} \left( I - P(Xwv) \right) \right) \lambda d n \otimes \left( I - P(uv) / n \right) K \\
\eta \beta_v &= \left( \begin{pmatrix} \beta_u & \beta_v \end{pmatrix} \left( I - P(Xwv) \right) \right) \lambda d n \otimes \left( I - P(uv) / n \right) K \\
\eta \beta_z &= \left( \begin{pmatrix} \beta_u & \beta_v \end{pmatrix} \left( I - P(Xwv) \right) \right) \lambda d n \otimes \left( I - P(uv) / n \right) K
\end{align*}
\]
(S103) \(\text{def} \) (S104) \(\text{def} \) (S105) \(\text{def} \) (S106) \(\text{def} \) (S107) \(\text{def} \) (S108) \(\text{def} \) (S109)

and
\[
\frac{\|o(\eta u)\|}{\|\eta u\|} P \to 0, \frac{\|o(\eta v)\|}{\|\eta v\|} P \to 0, \frac{\|o(\eta \beta_u)\|}{\|\eta \beta_u\|} P \to 0, \frac{\|o(\eta \beta_v)\|}{\|\eta \beta_v\|} P \to 0, \frac{\|o(\eta \beta_z)\|}{\|\eta \beta_z\|} P \to 0.
\]

Finally, recall that we assume
\[
\begin{pmatrix} \epsilon \\ \text{vec}(E) \end{pmatrix} \sim N \begin{pmatrix} 0_{(n+n^2) \times 1} \\ \begin{pmatrix} \sigma_y^2 I_n & 0_{n^2 \times n} \\ 0_{n^2 \times n} & \sigma_a^2 I_n \end{pmatrix} \end{pmatrix}.
\]
By (S102) and putting (S105), (S107) and (S109) together, we have

\[
\begin{pmatrix}
\hat{u} - u \\
\hat{v} - v \\
\hat{\beta}_x - \beta_x \\
\hat{\beta}_u - \beta_u \\
\hat{\beta}_v - \beta_v
\end{pmatrix} = \begin{pmatrix}
\eta u + o(\eta u) \\
\eta v + o(\eta v) \\
\eta_{\beta u} + o(\eta_{\beta u}) \\
\eta_{\beta v} + o(\eta_{\beta v}) \\
\eta_{\beta z} + o(\eta_{\beta z})
\end{pmatrix}
\]

(S111)

where

\[
\begin{pmatrix}
\eta u \\
\eta v \\
\eta_{\beta u} \\
\eta_{\beta v} \\
\eta_{\beta z}
\end{pmatrix} = \begin{pmatrix}
C_{11} & C_{12} \\
C_{21} & C_{22} \\
C_{31} & C_{32} \\
C_{41} & C_{42} \\
C_{51} & C_{52}
\end{pmatrix} \begin{pmatrix}
\epsilon \\
\text{vec}(E)
\end{pmatrix}
\]

(S112)

\[
\sim N\left(0_{(2n+2+p)\times 1}, C \begin{pmatrix}
\sigma_g^2 I_n & 0_{n^2 \times n} \\
0_{n^2 \times n} & \sigma_a^2 I_n^2
\end{pmatrix} C^\top \right).
\]

(S113)

Proving Theorem S3

Proof. The proof of Theorem S3 largely parallels that of Theorem 1. However, to derive (S102) from (S96), (S98) and (S101), we need to invoke Corollary S2, i.e., the consistency results for the Super-CENT estimators as the solution of Algorithm 1, instead of Corollary S1. This change follows from the slightly different conditions in Theorem S3 that satisfy the conditions (S252)-(S254) stated in Corollary S2.

S7.1.3 Proof of Proposition 1

Proof. We establish the error bound of \( \hat{u} \) and \( \hat{v} \) through the stronger statement given in the proof for establishing the limiting distribution in Theorem 1 and the fact that \( \| \hat{u} \|_2 = \sqrt{n} \) and \( \| \hat{v} \|_2 = \sqrt{n} \).

Equations (S102) and (S105) in the proof of Theorem 1 give

\[
\begin{cases}
\delta u = \eta u + o(\eta u), \\
\delta v = \eta v + o(\eta v),
\end{cases}
\]

(S115)

where

\[
\begin{pmatrix}
\eta u \\
\eta v
\end{pmatrix} = \begin{pmatrix}
C_{11} & C_{12} \\
C_{21} & C_{22}
\end{pmatrix} \begin{pmatrix}
\epsilon \\
\text{vec}(E)
\end{pmatrix},
\]

(S116)
\[
\frac{\|o\(n u\)}{\|n u\|} \xrightarrow{P} 0, \text{ and } \frac{\|o\(n v\)}{\|n v\|} \xrightarrow{P} 0 \text{ with probability at least } 1 - 6e^{-n} \text{ as in Theorem S4. Therefore,}
\]
\[
\frac{1}{n} \mathbb{E} \|\hat{u} - u\|^2 \leq \frac{1}{n} \text{tr}(\sigma_y^2 C_{11} C_{11}^\top + \sigma_a^2 C_{12} C_{12}^\top) (1 + o(1)) + 6e^{-n}. \tag{S117}
\]

Next, we compute \(\text{tr}(\sigma_y^2 C_{11} C_{11}^\top + \sigma_a^2 C_{12} C_{12}^\top)\). Note that, when \(A_0\) is rank-one,
\[
\begin{pmatrix}
C_{11} \\
C_{21}
\end{pmatrix} = (\lambda d^2 + \beta_u^2 + \beta_v^2)^{-1} \begin{pmatrix}
\beta_u \\
\beta_v
\end{pmatrix} \otimes \left( I - P_{(Xuv)} \right) \tag{S118}
\]
and
\[
\begin{pmatrix}
C_{12} \\
C_{22}
\end{pmatrix} = \frac{1}{dn} \left[ \begin{pmatrix}
v^\top \otimes (I - P_u) \\
u^\top \otimes (I - P_u)
\end{pmatrix} K \right] \\
- \frac{1}{\lambda d^2 + \beta_u^2 + \beta_v^2} \begin{pmatrix}
\beta_u^2 v^\top \otimes \left( I - P_{(Xuv)} \right) + \beta_u \beta_v u^\top \otimes \left( I - P_{(Xuv)} \right) + \beta_v^2 u^\top \otimes \left( I - P_{(Xuv)} \right)
\end{pmatrix} \right] \left( I - P_{(Xuv)} \right) \tag{S119}
\]
Then,
\[
\text{tr}(\sigma_y^2 C_{11} C_{11}^\top + \sigma_a^2 C_{12} C_{12}^\top) \tag{S120}
\]
\[
= \frac{\sigma_y^2}{\lambda d^2 + \beta_u^2 + \beta_v^2} \text{tr} \left( \beta_u^2 \left( I - P_{(Xuv)} \right) \right) \tag{S121}
\]
\[
+ \frac{\sigma_a^2}{n d^2} \text{tr} \left( v^\top v \right) \text{tr} \left( I - P_u \right) \tag{S122}
\]
\[
+ \frac{\sigma_a^2}{n d^2 (\lambda d^2 + \beta_u^2 + \beta_v^2)} \text{tr} \left( \beta_u^4 v^\top v \left( I - P_{(Xuv)} \right) + 2 \beta_u^3 \beta_v u^\top \otimes \left( I - P_{(Xuv)} \right) \right) \tag{S123}
\]
\[
- \frac{2 \sigma_a^2}{n d^2 (\lambda d^2 + \beta_u^2 + \beta_v^2)} \text{tr} \left( \beta_u^2 \left( v^\top \otimes (I - P_u) \right) \right) \left( I - P_{(Xuv)} \right) \tag{S124}
\]
\[
= \frac{\sigma_y^2}{\lambda d^2 + \beta_u^2 + \beta_v^2} \left( \frac{\sigma_a^2}{d^2} - \frac{\sigma_a^2}{d^2 n} \right) \tag{S127}
\]
\[
+ \frac{\sigma_a^2}{n d^2 (\lambda d^2 + \beta_u^2 + \beta_v^2)} \left( \beta_u^2 n + 2 \beta_u \beta_v \text{tr}(v^\top u) + \beta_v n \right) \tag{S128}
\]
\[
- \frac{2 \sigma_a^2}{n d^2 (\lambda d^2 + \beta_u^2 + \beta_v^2)} \beta_u^2 \tag{S129}
\]
\[
= \left( \frac{\sigma_y^2}{\lambda d^2 + \beta_u^2 + \beta_v^2} - \frac{\sigma_a^2}{d^2 n} \right) \tag{S130}
\]
\[
- \beta_u^2 (n - p - 2) \left( \frac{2 \lambda d^2 + \beta_u^2 + \beta_v^2}{d^2 n} \right) \frac{\sigma_a^2}{\sigma_a^2 - \sigma_y^2}. \tag{S131}
\]
Therefore,

\[
\frac{1}{n} \mathbb{E} \| \hat{\mathbf{u}} - \mathbf{u} \|^2 = \left( \frac{\sigma_a^2(n-1)}{d^2n^2} - \frac{\beta_u^2(n-p-2)}{n(\lambda d^2 + \beta_u^2 + \beta_v^2)^2} \left[ \frac{2\lambda d^2 + \beta_u^2 + \beta_v^2}{d^2n} \sigma_a^2 - \sigma_y^2 \right] \right) (1 + o(1)) \tag{S132}
\]

\[
\frac{\sigma_a^2(n-1)}{d^2n^2} - \frac{\beta_u^2(n-p-2)}{n(\lambda d^2 + \beta_u^2 + \beta_v^2)^2} \left[ \frac{2\lambda d^2 + \beta_u^2 + \beta_v^2}{d^2n} \sigma_a^2 - \sigma_y^2 \right] = O \left( \frac{\sigma_a^2}{d^2n} - \beta_u^2 \delta_{ts,sc} \right) \tag{S133}
\]

where \( \delta_{ts,sc} = \frac{1}{(\lambda d^2 + \beta_u^2 + \beta_v^2)^2} \left[ \frac{2\lambda d^2 + \beta_u^2 + \beta_v^2}{d^2n} \sigma_a^2 - \sigma_y^2 \right] \).

To get the optimal \( \lambda \) in Remark 9, we take the partial derivative of \( \ell_u \) with respect to \( \lambda \) yields

\[
\frac{\partial \ell_u}{\partial \lambda} = \frac{\beta_u^2(n-p-2)}{(\lambda d^2 + \beta_u^2 + \beta_v^2)^2} \left[ 2d^2\sigma_y^2 + \frac{\sigma_a^2}{n} \left( 2(\lambda d^2 + \beta_u^2 + \beta_v^2) - 4\lambda d^2 - 2\beta_u^2 - 2\beta_v^2 \right) \right] \tag{S135}
\]

\[
= \frac{\beta_u^2(n-p-2)}{(\lambda d^2 + \beta_u^2 + \beta_v^2)^2} \left[ 2d^2\sigma_y^2 - \frac{2d^2}{n} \sigma_a^2 \lambda \right]. \tag{S136}
\]

Setting \( \frac{\partial \ell_u}{\partial \lambda} = 0 \) yields

\[
\lambda_0 = \frac{n\sigma_y^2}{\sigma_a^2}. \tag{S137}
\]

When \( \lambda \in (0, \lambda_0] \), \( \ell_u \) increases as \( \lambda \) increases; \( \lambda \in (\lambda_0, \infty) \), \( \ell_u \) decreases and converges to 0 as \( \lambda \) increases. The maximum of \( \ell_u \) is then taken at \( \lambda_0 \).

Similarly, we derive the rate of \( \hat{\mathbf{v}} \) as

\[
\frac{1}{n} \mathbb{E} \| \hat{\mathbf{v}} - \mathbf{v} \|^2 \leq \left( \frac{\sigma_a^2(n-1)}{d^2n^2} - \frac{\beta_v^2(n-p-2)}{n(\lambda d^2 + \beta_u^2 + \beta_v^2)^2} \left[ \frac{2\lambda d^2 + \beta_u^2 + \beta_v^2}{d^2n} \sigma_a^2 - \sigma_y^2 \right] \right) (1 + o(1)) \tag{S138}
\]

\[
= O \left( \frac{\sigma_a^2}{d^2n} - \beta_v^2 \delta_{ts,sc} \right). \tag{S140}
\]

**S7.1.4 Proposition S4**

We now provide the convergence rates of \( \hat{\beta} = (\hat{\beta}_x, \hat{\beta}_u, \hat{\beta}_v)^\top \).

**Proposition S4.** Under the assumptions and conditions in Theorem 1 and further assume \( A_0 \)
to be rank-one, the SuperCENT estimators satisfy the following,

\[
\begin{align*}
\mathbb{E}(\hat{\beta}_u - \beta_u)^2 &= \mathbb{E}(\hat{\beta}_u^t - \beta_u)^2 = O\left(\frac{\sigma_u^2 + \sigma_u^2(\beta_u^2 + \beta_u^2)}{n}\right), \\
\mathbb{E}(\hat{\beta}_v - \beta_v)^2 &= \mathbb{E}(\hat{\beta}_v^t - \beta_v)^2 = O\left(\frac{\sigma_v^2 + \sigma_v^2(\beta_v^2 + \beta_v^2)}{d^2n^2}\right), \\
\text{Cov}\left(\hat{\beta}_x - \beta_x\right) &= \text{Cov}\left(\hat{\beta}_x^t - \beta_x\right).
\end{align*}
\]

PROOF:

(1) Rate of \(\hat{\beta}_u\) and \(\hat{\beta}_v\). From Theorem 1, we have

\[
\begin{align*}
\epsilon &- (\delta_u \beta_u + \delta_v \beta_v) \\
= (I - \beta_u C_{11} + \beta_u C_{12})\epsilon + (\beta_u C_{12} + \beta_v C_{22}) \text{vec}(E) \\
= \left[I - \frac{\beta_u^2 + \beta_v^2}{\lambda d^2 + \beta_u^2 + \beta_v^2} \left(I - P(Xuv)\right)\right] \epsilon \\
- \frac{1}{dn} \left[\beta_u v^T \otimes (I - P_u) + \beta_v (u^T \otimes (I - P_v)) K\right] \text{vec}(E) \\
+ \frac{\beta_u^2 + \beta_v^2}{dn(\lambda d^2 + \beta_u^2 + \beta_v^2)} \beta_v \left[u^T \otimes \left(I - P(Xuv)\right)K\right] \text{vec}(E)
\end{align*}
\]

\[
\begin{align*}
&\overset{\text{def}}{=} A_1 \epsilon + (C_1 + C_2) \text{vec}(E).
\end{align*}
\]

Plugging (S147) into (S107), we obtain

\[
\begin{cases}
\eta_{\beta_u} = C_{uv}^{-1} \left(\hat{u}_{-v}^T \hat{v}_{-v}^T\right) \left[\epsilon - (\delta_u \beta_u + \delta_v \beta_v)\right] \\
\eta_{\beta_v} = C_{uv}^{-1} \left(\hat{u}_{-v}^T \hat{v}_{-v}^T\right) \left[A_1 \epsilon + (C_1 + C_2) \text{vec}(E)\right]
\end{cases}
\]

\[
\overset{\text{def}}{=} (B_1 \epsilon + B_2 \text{vec}(E)).
\]

To get the rate of \(\hat{\beta}_u\) and \(\hat{\beta}_v\), we next calculate \(B_1 B_1^T\) and \(B_2 B_2^T\).

(a) \(B_1 B_1^T\).

Since

\[
A_1 A_1^T = I - \frac{(\beta_u^2 + \beta_v^2)(\lambda d^2 + \beta_u^2 + \beta_v^2)}{(\lambda d^2 + \beta_u^2 + \beta_v^2)^2} \left(I - P(Xuv)\right)
\]
and
\[(I - P_{Xuv})\hat{u} = 0,\]  \hspace{1cm} (S152)

consequently
\[B_1B_1^\top = C_{u\hat{v}}^{-1} \begin{pmatrix} \hat{u}^\top \\ \hat{v}^\top \end{pmatrix} A_1A_1^\top \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} C_{u\hat{v}}^{-1} = C_{u\hat{v}}^{-1}.\]  \hspace{1cm} (S153)

(b) $B_2B_2^\top$.

Since
\[C_1C_1^\top = \frac{1}{(d\mu)^2} \left[ \beta_u^2 (v^\top \otimes (I - P_u))(v \otimes (I - P_u)) + \beta_v^2 (u^\top \otimes (I - P_v))(u \otimes (I - P_v)) + \beta_u\beta_v (v^\top \otimes (I - P_u))K(u \otimes (I - P_v)) + \beta_u\beta_v (u^\top \otimes (I - P_v))K(v \otimes (I - P_u)) \right],\]  \hspace{1cm} (S154)
\[= \frac{1}{d^2\mu} \left[ \beta_u^2 (I - P_u) + \beta_v^2 (I - P_v) \right],\]  \hspace{1cm} (S155)
\[C_2C_2^\top = n \left( \frac{\beta_u^2 + \beta_v^2}{d\mu(\lambda d^2 + \beta_u^2 + \beta_v^2)} \right)^2 \left[ \beta_u^2 (I - P_{Xuv}) + \beta_v^2 (I - P_{Xuv}) \right]\]  \hspace{1cm} (S156)

and
\[C_1C_2^\top = C_2C_1^\top = -\frac{1}{d^2\mu} \left[ \beta_u^2 + \beta_v^2 \right] \left[ \beta_u^2 (I - P_{Xuv}) + \beta_v^2 (I - P_{Xuv}) \right],\]  \hspace{1cm} (S157)
consequently
\[\begin{aligned}(C_1 + C_2)(C_1 + C_2)^\top &= \frac{1}{d^2\mu} \left[ (\beta_u^2 (I - P_u) + \beta_v^2 (I - P_v)) - \right. \\
 & \left. \frac{(\beta_u^2 + \beta_v^2)^2 (2\lambda d^2 + \beta_u^2 + \beta_v^2)}{(\lambda d^2 + \beta_u^2 + \beta_v^2)^2} (I - P_{Xuv}) \right].\]  \hspace{1cm} (S158)

Therefore,
\[B_2B_2^\top = C_{u\hat{v}}^{-1} \begin{pmatrix} \hat{u}^\top \\ \hat{v}^\top \end{pmatrix} (C_1 + C_2)(C_1 + C_2)^\top \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} C_{u\hat{v}}^{-1}\]  \hspace{1cm} (S159)
\[= \frac{1}{d^2\mu} C_{u\hat{v}}^{-1} \begin{pmatrix} \hat{u}^\top \\ \hat{v}^\top \end{pmatrix} (\beta_u^2 (I - P_u) + \beta_v^2 (I - P_v)) \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} C_{u\hat{v}}^{-1}\]  \hspace{1cm} (S160)
\[= \frac{1}{d^2\mu} C_{u\hat{v}}^{-1} \begin{pmatrix} \beta_u^2 (I - P_u) \hat{u} \\ 0 \\ \beta_v^2 (I - P_v) \hat{v} \end{pmatrix},\]  \hspace{1cm} (S161)
where the first equality is due to \((S152)\).
Combining (S153) and (S163) as well as (S102),

\[
Cov \left( \frac{\delta \beta_u}{\delta \beta_v} \right) \approx \sigma^2_y B_1 B_1^\top + \sigma^2_x B_2 B_2^\top
\]  
\[
= \sigma^2_y C^{-1} \hat{u} \hat{v} \]  
\[
+ \sigma^2_x \frac{1}{d^2 n} C^{-1} \hat{u} \hat{v} \right) \left[ \beta_u^2 (I - P_u) + \beta_v^2 (I - P_v) \right] \left( \hat{u} \hat{v} \right) C^{-1} \hat{u} \hat{v} \]  
\[
= \sigma^2_y C^{-1} \hat{u} \hat{v} + \sigma^2_x \frac{1}{d^2 n} C^{-1} \hat{u} \hat{v} \left( \beta_u^2 \hat{u} (I - P_v) \hat{u} \right) \left( \beta_v^2 \hat{v} (I - P_u) \hat{v} \right) C^{-1} \hat{u} \hat{v} \]  
\[
Therefore,
\[
E(\hat{\beta}_u - \beta_u)^2 = E(\hat{\beta}_v - \beta_v)^2 \quad \text{and} \quad E(\hat{\beta}_v - \beta_v)^2 = E(\hat{\beta}_v - \beta_v)^2. \]  
\[
(2) \text{ Rate of } \hat{\beta}_x. \quad \text{Recall that}
\[
\tilde{G} = \begin{pmatrix} u \\ v \end{pmatrix} C^{-1} \hat{u} \hat{v} \left( \hat{u} \hat{v} \right) \quad \text{and} \quad G = \begin{pmatrix} u \\ v \end{pmatrix} C^{-1} \hat{u} \hat{v} \left( \hat{u} \hat{v} \right). \]  
\[
By \text{ plugging (S96), (S98), and (S147) into (S109), we have}
\[
\eta \beta_x = (X^\top X)^{-1} X^\top (I - u \delta_x - v \delta_x - \delta_x \delta_v - \delta_v \delta_v) \]  
\[
= \left( (X^\top X)^{-1} X^\top \left[ I - \begin{pmatrix} u \\ v \end{pmatrix} C^{-1} \hat{u} \hat{v} \left( \hat{u} \hat{v} \right) \right] \left[A_1 \epsilon + (C_1 + C_2) \text{vec}(E)\right] \right) \]  
\[
= \left( (X^\top X)^{-1} X^\top (I - \tilde{G}) \left[A_1 \epsilon + (C_1 + C_2) \text{vec}(E)\right] \right) \]  
\[
def \quad (F_1 \epsilon + F_2 \text{vec}(E)). \]  
\[
To \text{ get the rate of } \hat{\beta}_x, \text{ we next calculate } F_1 F_1^\top \text{ and } F_2 F_2^\top. \]  
\[
(a) \quad F_1 F_1^\top. \]  
\[
Since
\[
(I - \tilde{G})(I - \tilde{G})^\top = I - \tilde{G} - \tilde{G}^\top + G
\]  
\[
and
\[
\left( I - P_{Xuv} \right) \hat{u} = 0 \quad \text{and} \quad \left( I - P_{Xuv} \right) u = 0. \]  
\[
consequently
\[
(I - \tilde{G}) A_1 A_1^\top (I - \tilde{G})^\top = (I - \tilde{G})(I - \tilde{G})^\top - \frac{(\beta_u^2 + \beta_v^2) (2 \lambda_d^2 + \beta_u^2 + \beta_v^2)}{(\lambda_d^2 + \beta_u^2 + \beta_v^2)^2} \left( I - P_{Xuv} \right). \]  
\]
Further because \( (I - P_{Xuv})X = 0 \) and \( \bar{u}^T X = 0 \),

\[
F_1 F_1^\top = (X^\top X)^{-1} + (X^\top X)^{-1} (u \ v) C_{\bar{u}\bar{v}}^{-1} \begin{pmatrix} u^\top \\ v^\top \end{pmatrix} X (X^\top X)^{-1}. 
\]  

(S177)

(b) \( F_2 F_2^\top \).

Note that due to (S175)

\[
(I - \bar{G})(I - P_{Xuv}) (I - \bar{G})^\top = (I - P_{Xuv}).
\]  

(S178)

Combining with (S160), we have

\[
(I - \bar{G})(C_1 + C_2)(C_1 + C_2)^\top (I - \bar{G})^\top
= \frac{1}{d^2n} \left[ \beta_u^2 (I - Pu) + \beta_v^2 (I - Pv) \\
- \left( \beta_u^2 (I - Pv) \bar{u} \quad \beta_u^2 (I - Pu) \bar{v} \right) C_{\bar{u}\bar{v}}^{-1} \begin{pmatrix} u^\top \\ v^\top \end{pmatrix} - \left( u \ v \right) C_{\bar{u}\bar{v}}^{-1} \begin{pmatrix} \beta_u^2 (I - Pv) \bar{u} \\ \beta_u^2 (I - Pu) \bar{v} \end{pmatrix} \\
+ \left( u \ v \right) C_{\bar{u}\bar{v}}^{-1} \begin{pmatrix} \beta_v^2 \bar{u}^\top (I - P_v) \bar{u} \\ 0 \\ \beta_u^2 \bar{v}^\top (I - Pu) \bar{v} \end{pmatrix} C_{\bar{u}\bar{v}}^{-1} \begin{pmatrix} u^\top \\ v^\top \end{pmatrix} \\
- \frac{(\beta_u^2 + \beta_v^2)(2\lambda d^2 + \beta_u^2 + \beta_v^2)}{(\lambda d^2 + \beta_u^2 + \beta_v^2)^2} \left( I - P_{Xuv} \right) \right].
\]  

(S182)

Because \( (I - P_{Xuv})X = 0, X^\top \bar{u} = 0 \) and \( X^\top \bar{v} = 0 \),

\[
F_2 F_2^\top
= \frac{1}{d^2n} (X^\top X)^{-1} (u \ v) C_{\bar{u}\bar{v}}^{-1} \begin{pmatrix} \beta_v^2 \bar{u}^\top (I - P_v) \bar{u} \\ 0 \\ \beta_u^2 \bar{v}^\top (I - Pu) \bar{v} \end{pmatrix} C_{\bar{u}\bar{v}}^{-1} \begin{pmatrix} u^\top \\ v^\top \end{pmatrix} X (X^\top X)^{-1}.
\]  

(S186)

Together with (S177) and (S186), we obtain the variance-covariance matrix of \( \delta \beta \) as follows.

\[
Cov(\delta \beta)
\approx \sigma_y^2 \left( X^\top X)^{-1} + (X^\top X)^{-1} (u \ v) C_{\bar{u}\bar{v}}^{-1} \begin{pmatrix} u^\top \\ v^\top \end{pmatrix} X (X^\top X)^{-1} \right]
+ \frac{1}{d^2n} \left( X^\top X)^{-1} (u \ v) C_{\bar{u}\bar{v}}^{-1} \begin{pmatrix} \beta_v^2 \bar{u}^\top (I - P_v) \bar{u} \\ 0 \\ \beta_u^2 \bar{v}^\top (I - Pu) \bar{v} \end{pmatrix} C_{\bar{u}\bar{v}}^{-1} \begin{pmatrix} u^\top \\ v^\top \end{pmatrix} X (X^\top X)^{-1}\right).
\]  

(S190)
S7.1.5 Consistency of the SuperCENT estimators

We show the non-asymptotic high probability bounds for the SuperCENT estimators \( \hat{d}, \hat{u}, \hat{v} \) and \( \hat{\beta} \), along with the corresponding consistency results. We provide two versions of results: one for the solution of the objective function, presented in Section S7.1.5, as well as one for the solution produced by the algorithm, presented in Section S7.1.5.

Consistency results for the solution of the objective function. We first establish the high probability bounds for the SuperCENT estimators as the minimizer of the objective function \((4)\) in Theorem S4, followed by the corresponding consistency results in Corollary S1. The proofs are provided subsequently.

Theorem S4. Under the unified framework \((2)\) and Assumptions 1-3, suppose \((X, \hat{u}, \hat{v})\) is full rank where \((X, \hat{u}, \hat{v})\) is the minimizer of the objective function \((4)\), then with probability at least \(1 - 6e^{-n}\),

\[
\min \left\{ \frac{\| \hat{u} - u \|_2^2}{\sqrt{n}}, \frac{\| \hat{u} + u \|_2^2}{\sqrt{n}} \right\} + \min \left\{ \frac{\| \hat{v} - v \|_2^2}{\sqrt{n}}, \frac{\| \hat{v} + v \|_2^2}{\sqrt{n}} \right\} \leq \frac{200\sigma_a^2}{\lambda(d-d_2)dn} + \frac{2176\sigma_a^2}{n(d-d_2)^2}, \tag{S191}
\]

\[
\left| \frac{\hat{d} - d}{d} \right| \leq \frac{4\sigma_a}{\sqrt{n}d} + \frac{100\sigma_a^2}{\lambda(d-d_2)dn} + \frac{1088\sigma_a^2}{n(d-d_2)^2}. \tag{S192}
\]

If we further have \((X, u, v)^\top (X, u, v)\) has condition number smaller or equal to \(1/\tau^2\) for some positive constant \(\tau\), and when \(\tau^2 \geq \frac{8}{n(d-d_2)} \left( 2176 + \frac{120}{\lambda} \right) \), we have with probability at least \(1 - e^{-3n} - \frac{1}{n^2}\),

\[
\| \hat{\beta} - \beta \| \leq \left( 8\tau^{-4} \left( 2 + \frac{1}{2} \tau \right) \left( \frac{120\sigma_a^2}{\lambda nd(d-d_2)} + \frac{2176\sigma_a^2}{n(d-d_2)^2} \right) \right) \tag{S193}
\]

\[
+ 4\sqrt{2}\tau^{-2} \left( 1 + \frac{1}{2} \tau \right) \sqrt{\frac{120\sigma_a^2}{\lambda nd(d-d_2)} + \frac{2176\sigma_a^2}{n(d-d_2)^2}} \left( \frac{1}{\sqrt{\beta_u^2 + \beta_v^2}} \right) \tag{S194}
\]

\[
+ \sqrt{\frac{120\sigma_a^2}{\lambda nd(d-d_2)} + \frac{2176\sigma_a^2}{n(d-d_2)^2}} \cdot \sqrt{10} \cdot \tau^{-2} \left( 1 + \frac{8\tau^{-2} + 2\tau_1}{\sqrt{n}} \right) \sigma_y \tag{S195}
\]

\[
+ 2\sigma_y \tau^2 \sqrt{\frac{1 + 2\log n}{n}}. \tag{S196}
\]
Corollary S1. Under the assumptions and conditions in Theorem S4, if

\[ \frac{\sigma_a}{\sqrt{n (d - d_2)}} \sqrt{1 + \frac{1}{\lambda}} \left( 1 + \frac{\sigma_y}{\sqrt{\beta_u^2 + \beta_v^2}} \right) \to 0, \]  
(S197)

\[ \sigma_y \sqrt{\frac{\log n}{n}} \to 0, \]  
(S198)

then we have

\[ \min \left\{ \left\| \hat{u} - u \right\|, \left\| \hat{u} + u \right\| \right\} \to 0, \]  
(S199)

\[ \min \left\{ \left\| \hat{v} - v \right\|, \left\| \hat{v} + v \right\| \right\} \to 0, \]  
(S200)

\[ \left| \frac{\hat{d} - d}{d} \right| \to 0, \]  
(S201)

\[ \left| \frac{\hat{\beta} - \beta}{\sqrt{\beta_u^2 + \beta_v^2}} \right| \to 0. \]  
(S202)

If we further assume \( |\beta_u| \in [\alpha, \bar{\alpha}] \) for positive constants \( \bar{\alpha} > \alpha > 0 \), then with high probability

\[ \left| \frac{\hat{\beta}_u - \beta_u}{\beta_u} \right| \to 0 \] and \[ \left| \frac{\hat{\beta}_v - \beta_v}{\beta_v} \right| \to 0. \]  
(S203)

Proof. We start to prove the bounds for \( \hat{u}, \hat{v} \), and \( \hat{d} \), then proceed to \( \hat{\beta} \).

(1) Bounds for \( \hat{u}, \hat{v} \). Clearly, \( f_1(\hat{u}, \hat{v}) \leq f_1(u, v) \). Let \( \delta = 0 \) in Proposition S5 gives

\[ \left( 1 - \frac{\langle \hat{u}, u \rangle^2}{2n^2} - \frac{\langle \hat{v}, v \rangle^2}{2n^2} \right) \leq \frac{30\sigma_a^2}{\lambda (d - d_2) d n} + \frac{544\sigma_a^2}{n (d - d_2)^2}. \]  
(S204)

Note that

\[ 1 - \frac{\langle \hat{u}, u \rangle^2}{2n^2} - \frac{\langle \hat{v}, v \rangle^2}{2n^2} \geq \]

\[ \frac{1}{4} \left( \min \left\{ \left\| \frac{\hat{u} - u}{\sqrt{n}} \right\|^2, \left\| \frac{\hat{u} + u}{\sqrt{n}} \right\|^2 \right\} + \min \left\{ \left\| \frac{\hat{v} - v}{\sqrt{n}} \right\|^2, \left\| \frac{\hat{v} + v}{\sqrt{n}} \right\|^2 \right\} \right). \]  
(S205)

Combine the two inequalities above, we obtain the statement for \( \hat{u} \) and \( \hat{v} \).
(2) Bounds for $\hat{d}$. Note that

$$ |\hat{d} - d| = \left| \frac{\hat{u}^\top A \hat{v}}{n^2} - \frac{u^\top A_0 v}{n^2} \right| $$

(S206)

$$ = \left| \frac{\hat{u}^\top E \hat{v}}{n^2} \right| + \left| \frac{(\hat{u} - u^\top) A_0 \hat{v}}{n^2} \right| + \left| \frac{u^\top A_0 (\hat{v} - v)}{n^2} \right| $$

(S207)

$$ \leq \frac{\|E\|_{op}}{n} + d \left( \frac{\|\hat{u} - u\|^2 + \|\hat{v}_1 - v\|^2}{2n} \right) $$

(S208)

Note that, under the high probability event with probability $1 - 3e^{-n}$, the bounds for $\hat{u}$ and $\hat{v}$ hold in Proposition S5, we have

$$ \|E\|_{op} \leq 4\sqrt{n}\sigma_a. $$

(S209)

Therefore, under the event that the bounds for $\hat{u}, \hat{v}$ hold, we have

$$ |\hat{d} - d| \leq \frac{4\sigma_a}{\sqrt{n}} + d \left( \frac{60\sigma_a^2}{\lambda nd (d - d_2)} + \frac{1088\sigma_a^2}{n (d - d_2)^2} \right). $$

(S210)

(3) Bounds for $\hat{\beta}$. Consider the high probability event that the bounds for $\hat{u}, \hat{v}$ in Proposition S5 holds. Let $g(\omega) = (\omega^\top \omega)^{-1} \omega^\top$, then

$$ \hat{\beta} = g(X, \hat{u}, \hat{v})[\beta + \varepsilon_y, \beta_0] $$

(S211)

$$ = \begin{pmatrix} \beta_x \\ 0 \\ 0 \end{pmatrix} + g(X, \hat{u}, \hat{v}) \begin{pmatrix} 0 \\ \beta_u \\ \beta_v \end{pmatrix} + \varepsilon_y. $$

(S212)

Let

$$ \begin{cases} \omega = (X, u, v), \\ \Delta = (\hat{u} - u, \hat{v} - v), \\ \hat{\beta} = \begin{pmatrix} 0 \\ \beta_u \\ \beta_v \end{pmatrix}. \end{cases} $$

(S213)

Then

$$ \|\hat{\beta} - \beta\| \leq \|[g(\omega + \Delta) - g(\omega)] \omega \hat{\beta}\| + \|g(\omega + \Delta) \varepsilon_y\| $$

(S214)

$$ \leq \|[g(\omega + \Delta) - g(\omega)] \omega \hat{\beta}\| + \|g(\omega + \Delta) - g(\omega)\|_{op} \|\varepsilon_y\|_2 + \|g(\omega)\varepsilon_y\|. $$

(S215)
We start with boundary the first term in Inequality (S214). Note that

\[ g(\omega + \Delta) - g(\omega) = (\omega^T \omega)^{-1} \Delta^T + \left[ (\omega + \Delta)^T (\omega + \Delta) \right]^{-1} - \left[ \omega^T \omega \right]^{-1} \omega^T. \]  
(S216)

Then,

\[
\begin{aligned}
[g(\omega + \Delta) - g(\omega)] \omega \tilde{\beta} \\
= (\omega^T \omega)^{-1} \Delta^T \omega \tilde{\beta} + \left[ (\omega + \Delta)^T (\omega + \Delta) \right]^{-1} \\
\left[ I - (\omega^T \omega + \Delta^T \omega + \omega^T \Delta + \Delta^T \Delta) (\omega^T \omega)^{-1} \right] \omega^T \omega \tilde{\beta} \\
= (\omega^T \omega)^{-1} \Delta^T \omega \tilde{\beta} - ( (\omega + \Delta)^T (\omega + \Delta))^{-1} (\Delta^T \omega \tilde{\beta} + \omega^T \Delta \tilde{\beta} + \Delta^T \Delta \tilde{\beta}) \\
= \left[ (\omega + \Delta)^T (\omega + \Delta) \right]^{-1} (\Delta^T \omega + \omega^T \Delta + \Delta^T \Delta) (\omega^T \omega)^{-1} \Delta^T \omega \tilde{\beta} \\
- \left[ (\omega + \Delta)^T (\omega + \Delta) \right]^{-1} \omega^T \Delta \tilde{\beta} - \left[ (\omega + \Delta)^T (\omega + \Delta) \right]^{-1} \Delta^T \Delta \tilde{\beta}.
\end{aligned}
\]  
(S217)

Let’s consider the smallest eigenvalue of \((\omega + \Delta)^T (\omega + \Delta)\),

\[
\tilde{\lambda}_n ( (\omega + \Delta)^T (\omega + \Delta) ) = \min_{z \in \mathbb{R}^n} \| (\omega + \Delta) z \|^2.
\]  
(S221)

Note that

\[
\| \omega z \| \geq \sqrt{\tilde{\lambda}_n (\omega^T \omega)} \geq \tau \| \omega \|_{\text{op}} \geq \tau \sqrt{n},
\]  
(S222)

and

\[
\| \Delta z \| \leq \sqrt{2n} \max \left\{ \frac{\| \dot{u} - u \|}{\sqrt{n}}, \frac{\| \dot{v} - v \|}{\sqrt{n}} \right\} \leq \sqrt{2n} \left( \frac{120 \sigma_a^2}{\lambda d (d - d_2)} + \frac{2176 \sigma_a^2}{n (d - d_2)^2} \right). \]  
(S223)

Then we have

\[
\tilde{\lambda}_n ( (\omega + \Delta)^T (\omega + \Delta) ) \geq \frac{1}{4} \tau^2 \| \omega \|_{\text{op}}^2.
\]  
(S224)
Therefore,
\[
\| (g(\omega + \Delta) - g(\omega)) \omega \beta \| \leq 4\tau^{-4} \| \omega \|_{op}^{-2} \left( 2 + \frac{1}{2} \tau \right) \| \Delta \|_{op}^2 \| \omega \|_{op} \| \omega \|_{op} \| \beta \|_2 \quad (S225)
\]
\[
+ 4\tau^{-2} \| \omega \|_{op}^{-2} \left( 2 + \frac{1}{2} \tau \right) \| \Delta \|_{op} \| \Delta \|_2 \| \beta \|_2 \quad (S226)
\]
\[
\leq 4\tau^{-4} \| \omega \|_{op}^{-2} \left( 2 + \frac{1}{2} \tau \right) \| \Delta \|_{op}^2 \| \omega \|_{op} \| \beta \|_2 \quad (S227)
\]
\[
+ 4\tau^{-2} \| \omega \|_{op}^{-2} \left( 1 + \frac{1}{2} \tau \right) \| \Delta \|_{op} \| \beta \|_2 \quad (S228)
\]
\[
\leq 4\tau^{-4} \left( 2 + \frac{1}{2} \tau \right) 2 \left[ \frac{120\sigma^2}{\lambda n d (d - d_2)} + \frac{2176\sigma^2}{n (d - d_2)^2} \right] \| \beta \|_2 \quad (S229)
\]
\[
+ 4\tau^{-2} \left( 1 + \frac{1}{2} \tau \right) \sqrt{2} \sqrt{2} \left[ \frac{120\sigma^2}{\lambda n d (d - d_2)} + \frac{2176\sigma^2}{n (d - d_2)^2} \right] \| \beta \|_2 \quad (S230)
\]

Now we turn to the second term in Inequality (S214).
\[
\| (g(\omega + \Delta) - g(\omega)) \omega \|_{op} \| \varepsilon_y \| \leq \left[ \tau^{-2} \| \omega \|_{op}^{-2} \| \Delta \|_{op} + 4\tau^{-2} \| \omega \|_{op}^{-2} \| \Delta \|_{op} \| \omega \|_{op} \| \Delta \|_{op} \| \omega \|_{op} \| \beta \|_2 \right] \| \omega \|_{op} \| \varepsilon_y \|_2 \quad (S231)
\]
\[
\leq \tau^{-2} \| \omega \|_{op}^{-2} \| \Delta \|_{op} \left[ 1 + 4\tau^{-2} \| \Delta \|_{op} \left( 2 + \frac{7}{2} \right) \right] \| \varepsilon_y \|_2 \quad (S232)
\]
\[
\leq \sqrt{\frac{120\gamma^2}{\lambda n d (d - d_2)}} + \frac{2176\sigma^2}{n (d - d_2)^2} \sqrt{2} \tau^{-2} \| \varepsilon_y \| \left( 1 + \frac{8\tau^{-2} + 2\tau^{-1}}{\sqrt{n}} \right). \quad (S233)
\]

Under the high probability event that the bounds for \( \hat{u}, \hat{v} \) in Proposition S5 hold, we have
\[
\| g(\omega + \Delta) - g(\omega) \|_{op} \| \varepsilon_y \| \leq \sqrt{\frac{120\gamma^2}{\lambda n d (d - d_2)}} + \frac{2176\sigma^2}{n (d - d_2)^2} \sqrt{10} \tau^{-2} \left( 1 + \frac{8\tau^{-2} + 2\tau^{-1}}{\sqrt{n}} \right) \sigma_y. \quad (S234)
\]

Now we turn to the third term in Inequality (S214). First note that
\[
g(\omega) \cdot \varepsilon_y \sim N \left( 0, \sigma^2_y : (\omega^\top \omega)^{-1} \right). \quad (S235)
\]

Let \( h = g(\omega) \cdot \varepsilon_y \), then for \( \xi > 0 \),
\[
\mathbb{E}e^{\xi \|h\|^2} = \int e^{\langle \xi \omega, \omega \rangle - \frac{1}{2\sigma_y^2} \|\xi\|^2} \frac{1}{(\sqrt{2\pi\sigma_y})^{p+2}} d\xi \tag{S239}
\]
\[
\leq \int e^{\langle \xi \tau^4 \|\omega\|_{op}^2 - \frac{1}{2\sigma_y^2} \|\xi\|^2} \frac{1}{(\sqrt{2\pi\sigma_y})^{p+2}} d\xi \tag{S240}
\]
\[
= \left( \frac{1}{\sqrt{1 - 2\sigma_y^2 \xi \cdot \tau^4 \|\omega\|_{op}^2}} \right)^{p+2} \tag{S241}
\]
\[
\leq \left( \frac{1}{\sqrt{1 - 2\sigma_y^2 \tau^4 \cdot \frac{1}{n} \xi}} \right)^{p+2} \tag{S242}
\]

and for \( \xi < \frac{n}{2\sigma_y^2 \tau^4} \),

\[
P \left( \|h\|^2 > k \right) \leq \left( \frac{1}{\sqrt{1 - 2\sigma_y^2 \tau^4 \cdot \frac{1}{n} \xi}} \right)^{p+2} e^{-\xi k}. \tag{S243}
\]

Let \( \xi = \frac{n}{4\sigma_y^2 \tau^4}, k = \frac{4\sigma_y^2 \tau^4}{n} (1 + 2 \log n) \), we have

\[
P \left( \|g(\omega)\xi_y\| > \frac{2\sigma_y \tau^2}{\sqrt{n}} \sqrt{1 + 2 \log n} \right) < \frac{1}{n^2}. \tag{S244}
\]

Combining the above results for the three terms in (S214) gives, with probability at least \( 1 - e^{3n} - \frac{1}{n^2} \),

\[
\|\hat{\beta} - \beta\| \leq \left[ 8\tau^{-4} \left( 2 + \frac{1}{2} \right) \frac{120\sigma_a^2}{\lambda nd (d - d_2)} + \frac{2176\sigma_a^2}{n (d - d_2)^2} \right] \tag{S245}
\]
\[
+ 4\sqrt{2\tau^{-2}} \left( 1 + \frac{1}{2} \right) \sqrt{\frac{120\sigma_a^2}{\lambda nd (d - d_2)} + \frac{2176\sigma_a^2}{n (d - d_2)^2}} \sqrt{\frac{\beta_a^2}{\beta_c}} \tag{S246}
\]
\[
+ \sqrt{\frac{120\sigma_a^2}{\lambda nd (d - d_2)} + \frac{2176\sigma_a^2}{n (d - d_2)^2}} \sqrt{10\tau^{-2}} \left( 1 + \frac{8\tau^{-2} + 2\tau^{-1}}{\sqrt{n}} \right) \sigma_y \tag{S247}
\]
\[
+ 2\sigma_y \tau^2 \sqrt{\frac{1 + 2 \log n}{n}}. \tag{S248}
\]

Finally, Corollary S1 directly follows from Theorem S4 under the conditions (S197)-(S198).

\( \square \)

**Consistency results for the solution of Algorithm 1.** We first establish the high probability bounds for the SuperCENT estimators as the solution of Algorithm 1 in Theorem S5, followed by the corresponding consistency results in Corollary S2. The proofs are provided subsequently.
Theorem S5. Under the unified framework (2) and Assumptions 1-3, suppose \((X, u, v)\) is full rank and \((\hat{u}, \hat{v})\) is produced by Algorithm 1, then with probability at least \(1 - 6e^{-n}\),

\[
\min \left\{ \frac{\|\hat{u} - u\|}{\sqrt{n}}, \frac{\|\hat{u} + u\|}{\sqrt{n}} \right\} + \min \left\{ \frac{\|\hat{v} - v\|}{\sqrt{n}}, \frac{\|\hat{v} + v\|}{\sqrt{n}} \right\} \leq 200\sigma_u^2 \lambda (d - d_2) dn + 2176\sigma_u^2 \sigma_y^2 \frac{\sigma_a^2}{n(d - d_2)^2 n^2\lambda d(d - d_2)}, \tag{S249}
\]

\[
\frac{|\hat{d} - d|}{d} \leq \frac{4\sigma_a}{\sqrt{nd}} + \frac{100\sigma_a^2}{\lambda (d - d_2) dn} + \frac{1088\sigma_a^2}{n(d - d_2)^2} + 2048\frac{\beta_u^2 + \beta_v^2}{\sigma_y^2} \frac{\sigma_a^2}{n(d - d_2)^2 n^2\lambda d(d - d_2)}. \tag{S250}
\]

If we further have \((X, u, v)^T (X, u, v)\) has condition number smaller or equal to \(1/\tau^2\) for some positive constant \(\tau\), then with probability \(1 - 6e^{-n} - \frac{1}{n^2}\), the above holds and

\[
\|\hat{\beta} - \beta\| \leq \left( 8\tau^{-4} \left( 2 + \frac{1}{2}\tau \right) \eta + 4\sqrt{2}\tau^{-2} \left( 1 + \frac{1}{2}\tau \right) \sqrt{\eta} \right) \sqrt{\frac{\beta_u^2 + \beta_v^2}{\sigma_a^2}} + \sqrt{10\eta \sigma_y \tau^{-2} \left( 1 + \frac{8\tau^{-2} + 2\tau^{-1}}{\sqrt{n}} \right)} + 2\sigma_y \tau^2 \sqrt{\frac{1 + p/2 + 2\log n}{n}}. \tag{S251}
\]

Corollary S2. Under the assumptions and conditions of Theorem S5, if

\[
\left( 1 + \frac{1}{\lambda} \right) \frac{\sigma_a^2}{n(d - d_2)^2} \left( 1 + \frac{\sigma_y^2}{\beta_u^2 + \beta_v^2} \right) \rightarrow 0, \tag{S252}
\]

\[
\frac{\sqrt{\beta_u^2 + \beta_v^2}}{\sigma_y} \sqrt{\frac{1}{\lambda}} \frac{\sigma_a^2}{n(d - d_2)^2} \rightarrow 0, \tag{S253}
\]

\[
\frac{\sigma_y}{\sqrt{\beta_u^2 + \beta_v^2}} \sqrt{\frac{\log n}{n}} \rightarrow 0, \tag{S254}
\]

then with high probability

\[
\min \left\{ \frac{\|\hat{u} - u\|}{\sqrt{n}}, \frac{\|\hat{u} + u\|}{\sqrt{n}} \right\} \rightarrow 0, \tag{S255}
\]

\[
\min \left\{ \frac{\|\hat{v} - v\|}{\sqrt{n}}, \frac{\|\hat{v} + v\|}{\sqrt{n}} \right\} \rightarrow 0, \tag{S256}
\]

\[
\frac{|\hat{d} - d|}{d} \rightarrow 0, \tag{S257}
\]

\[
\frac{\|\hat{\beta} - \beta\|}{\sqrt{\beta_u^2 + \beta_v^2}} \rightarrow 0. \tag{S258}
\]
If we further assume \( \frac{\beta_u}{\beta_y} \in [\alpha, \tilde{\alpha}] \) for positive constants \( \tilde{\alpha} > \alpha > 0 \), then with high probability

\[
\left| \frac{\hat{\beta}_u - \beta_u}{\beta_u} \right| \to 0, \quad (S259)
\]

\[
\left| \frac{\hat{\beta}_v - \beta_v}{\beta_v} \right| \to 0. \quad (S260)
\]

**Proof.** Denote the leading singular vector of network \( A \) as \( \tilde{u} \) and \( \tilde{v} \) with their corresponding leading singular value as \( \tilde{d} \) for simplicity in notation.

(1) **Bounds for \( \hat{u}, \hat{v} \).** Plugging the unified model (2) into \( f_1(\hat{u}, \hat{v}) - f_1(u, v) \), we have

\[
f_1(\hat{u}, \hat{v}) - f_1(u, v) = \frac{1}{\sigma_y^2} \left\| P(X, u, v)^\perp (\tilde{\varepsilon} + \tilde{u}_1 \beta_u + \tilde{v}_1 \beta_v) \right\|^2 - \frac{\tilde{\lambda}}{\sigma^2 \alpha} \left( \hat{u}^\top \left( \sum_{i=1}^r d_i u_i v_i^\top + E \right) \hat{v} / n \right)^2 (S261)
\]

\[
- \left( \frac{1}{\sigma_y^2} \left\| P(X, u, v)^\perp (\tilde{\varepsilon} + \hat{u}_1 \beta_u + \hat{v}_1 \beta_v) \right\|^2 - \frac{\tilde{\lambda}}{\sigma^2 \alpha} \left( u^\top \left( \sum_{i=1}^r d_i u_i v_i^\top + E \right) v / n \right)^2 \right) (S262)
\]

\[
\leq \frac{1}{\sigma_y^2} \left\| P(X, u, v)^\perp \tilde{\varepsilon} \right\|^2 - \left\| P(X, u, v)^\perp \tilde{\varepsilon} \right\|^2 (S264)
\]

\[
+ 2 \left\| P(X, u, v)^\perp \tilde{\varepsilon} \right\| \left\| P(X, u, v)^\perp (\hat{u}_1 \beta_u + \hat{v}_1 \beta_v) \right\|^2 + \left\| P(X, u, v)^\perp (\hat{u}_1 \beta_u + \hat{v}_1 \beta_v) \right\|^2 \right) \leq 2 \left\| \tilde{\varepsilon} \right\|^2 + 2 \left\| P(X, u, v)^\perp (\hat{u}_1 \beta_u + \hat{v}_1 \beta_v) \right\|^2 (S265)
\]

\[
\leq 2 \left( \frac{\| \beta_u \|}{\sigma_y} \right)^2 + 4n \left( \beta_u \right)^2 \left( 1 - \left( \frac{\langle u, \hat{u} \rangle}{n} \right)^2 \right) + 4n \left( \beta_v \right)^2 \left( 1 - \left( \frac{\langle v, \hat{v} \rangle}{n} \right)^2 \right) \right). (S267)
\]

We next focus on bounding the following terms,

\[
1 - \left( \frac{\langle u, \hat{u} \rangle}{n} \right)^2 \quad \text{and} \quad 1 - \left( \frac{\langle v, \hat{v} \rangle}{n} \right)^2. \quad (S268)
\]

Recall that \( \hat{u} \) and \( \hat{v} \) are the leading singular vectors of the observed network \( A \), where \( A = \sum_{i=1}^r d_i u_i v_i^\top + E \) as defined in our network model (2a). By definition of \( \hat{u}, \hat{v} \), we have

\[
\hat{u}^\top \left( \sum_{i=1}^r d_i u_i v_i^\top + E \right) \hat{v} \geq u^\top \left( \sum_{i=1}^r d_i u_i v_i^\top + E \right) v. \quad (S269)
\]
Rearrange with some algebra,
\[
\text{tr} \left[ E (\bar{v} \bar{u}^T - v u^T) \right] \geq n^2 d \left( 1 - \frac{\langle v, \bar{v} \rangle}{n} \frac{\langle u, \bar{u} \rangle}{n} \right) \]
\[
- n^2 d_2 \sqrt{1 - \left( \frac{\langle v, \bar{v} \rangle}{n} \right)^2} \sqrt{1 - \left( \frac{\langle u, \bar{u} \rangle}{n} \right)^2}. \tag{S270}
\]

Then
\[
\sqrt{2} \| E \|_{op} \| \bar{v} \bar{u}^T - v u^T \|_F \geq n^2 (d - d_2) \left( 1 - \frac{\langle v, \bar{v} \rangle}{n} \frac{\langle u, \bar{u} \rangle}{n} \right). \tag{S271}
\]

Note that
\[
\| \bar{v} \bar{u}^T - v u^T \|_F^2 \leq n^2 \left( 2 - 2 \frac{\langle v, v \rangle}{\| u \|_2} \right) \leq 4n^2 \left( 1 - \frac{\langle v, v \rangle}{\| u \|} \right). \tag{S272}
\]

Therefore,
\[
\left( 1 - \frac{\langle v, \bar{v} \rangle}{n} \frac{\langle u, \bar{u} \rangle}{n} \right) \leq \frac{32 \| E \|_{op}^2}{(d - d_2)^2 n^2}. \tag{S273}
\]

Further,
\[
1 - \frac{\langle v, \bar{v} \rangle}{n} \frac{\langle u, \bar{u} \rangle}{n} \geq \max \left\{ 1 - \left| \frac{\langle v, \bar{v} \rangle}{n} \right|, 1 - \left| \frac{\langle u, \bar{u} \rangle}{n} \right| \right\} \tag{S274}
\]
\[
\geq \max \left\{ \min \left( \frac{\| v - \bar{v} \|}{\sqrt{n}}, \frac{\| v + \bar{v} \|}{\sqrt{n}} \right), \min \left( \frac{\| u - \bar{u} \|}{\sqrt{n}}, \frac{\| u + \bar{u} \|}{\sqrt{n}} \right) \right\}. \tag{S275}
\]

Plug (S273) and (S275) into (S267) gives
\[
f_1(\bar{u}, \bar{v}) - f_1(u, v) \leq 2 \| \varepsilon \|^2 / \sigma_y^2 + 4n \left( \frac{\beta_u}{\sigma_y} \right)^2 \frac{32 \| E \|_{op}^2}{(d - d_2)^2 n^2} + 4n \left( \frac{\beta_v}{\sigma_y} \right)^2 \frac{32 \| E \|_{op}^2}{(d - d_2)^2 n^2}. \tag{S276}
\]

Using Lemma S3 and Lemma S4, we know that with probability at least 1 - 3e^{-n},
\[
f_1(\bar{u}, \bar{v}) - f_1(u, v) \leq 10n + 512 \frac{\beta_u^2 + \beta_v^2}{\sigma_y^2} \frac{\sigma_u^2}{n (d - d_2)^2}. \tag{S277}
\]

Note that the algorithm descends in \( f_1 \), so
\[
f_1(\bar{u}, \bar{v}) - f_1(u, v) \leq 10n + 512 \frac{\beta_u^2 + \beta_v^2}{\sigma_y^2} \frac{\sigma_u^2}{n (d - d_2)^2} \tag{S278}
\]
with probability at least 1 - 3e^{-n}.

Invoking Proposition S5, we have
Bounds for $\hat{p}$

Combining (S279) and (S280), we have

$$1 - \frac{\langle \hat{u}_1 u \rangle^2}{2n^2} - \frac{\langle \hat{v}_1 v \rangle^2}{2n^2} \leq \frac{50\sigma_a^2}{\lambda (d - d_2) dn} + \frac{544\sigma_a^2}{n (d - d_2)^2} + 1024\frac{\beta_u^2 + \beta_v^2}{\sigma_y^2} \frac{\sigma_a^2}{n (d - d_2)^2} n^3 \lambda d (d - d_2).$$ (S279)

Note that

$$1 - \frac{\langle \hat{u}, u \rangle^2}{2n^2} - \frac{\langle \hat{v}, v \rangle^2}{2n^2} \geq \frac{1}{4} \left( \min \left\{ \frac{\| \hat{u} - u \|}{\sqrt{n}}, \frac{\| \hat{u} + u \|}{\sqrt{n}} \right\} \right) + \min \left\{ \frac{\| \hat{v} - v \|}{\sqrt{n}}, \frac{\| \hat{v} + v \|}{\sqrt{n}} \right\}.$$ (S280)

Combining (S279) and (S280), we have

$$\min \left\{ \frac{\| \hat{u} - u \|}{\sqrt{n}}, \frac{\| \hat{u} + u \|}{\sqrt{n}} \right\} + \min \left\{ \frac{\| \hat{v} - v \|}{\sqrt{n}}, \frac{\| \hat{v} + v \|}{\sqrt{n}} \right\} \leq \frac{200\sigma_a^2}{\lambda (d - d_2) dn} + \frac{2176\sigma_a^2}{n (d - d_2)^2} + 4096\frac{\beta_u^2 + \beta_v^2}{\sigma_y^2} \frac{\sigma_a^2}{n (d - d_2)^2} n^3 \lambda d (d - d_2).$$ (S281)

(2) Bounds for $\hat{d}$. Now we turn to $\hat{d}$. Under the high probability event that the bounds for $\hat{u}$ and $\hat{v}$ hold in Proposition S5,

$$|\hat{d} - d| = \left| \frac{\hat{u}^\top A \hat{v}}{n \cdot n} - \frac{u^\top A_0 v}{n \cdot n} \right|$$ (S282)

$$= \left| \frac{\hat{u}^\top E \hat{v}}{n \cdot n} \right| + \left| \frac{(\hat{u} - u^\top) A_0 \hat{v}}{n \cdot n} \right| + \left| \frac{u^\top A_0 (\hat{v} - v)}{n \cdot n} \right|$$ (S283)

$$\leq \frac{\| E \|_{op}}{n} + d \left( \frac{\| \hat{u} - u \|^2 + \| \hat{v} - v \|^2}{2n} \right)$$ (S284)

$$\leq \frac{4\sigma_a}{\sqrt{n}} + d \left( \frac{100\sigma_a^2}{\lambda (d - d_2) dn} + \frac{1088\sigma_a^2}{n (d - d_2)^2} \right) + 2048\frac{\beta_u^2 + \beta_v^2}{\sigma_y^2} \frac{\sigma_a^2}{n (d - d_2)^2} n^3 \lambda d (d - d_2).$$ (S285)

(3) Bounds for $\hat{\beta}$. Denote

$$\eta = \frac{200\sigma_a^2}{\lambda (d - d_2) dn} + \frac{2176\sigma_a^2}{n (d - d_2)^2} + 4096\frac{\beta_u^2 + \beta_v^2}{\sigma_y^2} \frac{\sigma_a^2}{n (d - d_2)^2} n^3 \lambda d (d - d_2).$$ (S286)
Define $g(\omega) = (\omega^T \omega)^{-1} \omega^T$, then
\[
\hat{\beta} = g(X, \hat{u}, \hat{v})[(X, u, v)\beta + \varepsilon_y] \\
= \begin{pmatrix} \beta_x \\ 0 \\ 0 \end{pmatrix}^T + g(X, \hat{u}, \hat{v}) \begin{pmatrix} 0 \\ \beta_u \\ \beta_v \end{pmatrix} + \varepsilon_y \quad (\text{S287})
\]

Let
\[
\begin{cases}
\omega = (X, u, v), \\
\Delta = (\hat{u} - u, \hat{v} - v), \\
\tilde{\beta} = \begin{pmatrix} 0 \\ \beta_u \\ \beta_v \end{pmatrix},
\end{cases}
\]

then
\[
\hat{\beta} - \beta = [g(\omega + \Delta) - g(\omega)] \omega \tilde{\beta} + g(\omega + \Delta) \varepsilon_y. \quad (\text{S290})
\]

Note that
\[
\|\hat{\beta} - \beta\| \leq \|g(\omega + \Delta) - g(\omega)\| \omega \tilde{\beta} + \|g(\omega + \Delta)\| \varepsilon_y \quad (\text{S291})
\]
\[
\leq \|g(\omega + \Delta) - g(\omega)\| \omega \tilde{\beta} + \|g(\omega + \Delta) - g(\omega)\| \|\varepsilon_y\|_2 + \|g(\omega)\| \varepsilon_y \|. \quad (\text{S292})
\]

We start with boundary the first term in Inequality (S292). Note that
\[
g(\omega + \Delta) - g(\omega) = (\omega^T \omega)^{-1} \Delta^T + \left[\left(\omega + \Delta\right)^T (\omega + \Delta)\right]^{-1} - \left[\omega^T \omega\right]^{-1} \omega^T. \quad (\text{S293})
\]

Then
\[
\left[\frac{g(\omega + \Delta) - g(\omega)}{\omega^T \omega}\right] \omega \tilde{\beta} = (\omega^T \omega)^{-1} \Delta^T \omega \beta + \left[\left(\omega + \Delta\right)^T (\omega + \Delta)\right]^{-1} \\
\left[I - (\omega^T \omega + \Delta^T \omega + \omega^T \Delta + \Delta^T \Delta) \left(\omega^T \omega\right)^{-1}\right] \omega^T \omega \beta = (\omega^T \omega)^{-1} \Delta^T \omega \beta - \left[\frac{(\omega + \Delta)^T (\omega + \Delta)}{\omega^T \omega}\right]^{-1} \left(\Delta^T \omega \beta + \omega^T \Delta \beta + \Delta^T \Delta \beta\right) \quad (\text{S294})
\]
\[
\left[\frac{(\omega + \Delta)^T (\omega + \Delta)}{\omega^T \omega}\right]^{-1} \left(\Delta^T \omega \beta + \omega^T \Delta \beta + \Delta^T \Delta \beta\right) - \left[\frac{(\omega + \Delta)^T (\omega + \Delta)}{\omega^T \omega}\right]^{-1} \Delta^T \Delta \beta. \quad (\text{S297})
\]

Let’s consider the smallest eigenvalue of $(\omega + \Delta)^T (\omega + \Delta)$ as
\[
\lambda_n (\omega + \Delta)^T (\omega + \Delta) = \min_{z \in \mathbb{R}^n} \|(\omega + \Delta)z\|^2. \quad (\text{S298})
\]
Note that
\[ \|\omega z\| \geq \sqrt{\lambda_n (\omega^\top \omega)} \geq \tau \|\omega\|_{op} \geq \tau \sqrt{n}, \]  
(S299)
and
\[ \|\Delta z\| \leq \sqrt{2n} \max\left\{ \frac{\|\hat{u} - u\|}{\sqrt{n}}, \frac{\|\hat{v} - v\|}{\sqrt{n}} \right\} \leq \sqrt{2n} \sqrt{\eta}. \]  
(S300)

Then we have
\[ \tilde{\lambda}_n (\omega + \Delta)^\top (\omega + \Delta) \geq \frac{1}{4} \tau^2 \|\omega\|_{op}^2. \]  
(S301)

Therefore,
\[ \| (g(\omega + \Delta) - g(\omega)) \omega \beta \| \leq 4\tau^{-4} \|\omega\|_{op}^{-4} \cdot (2 \|\Delta\|_{op} \cdot \|\omega\|_{op} + \|\Delta\|_{op}^2) \|\Delta\|_{op} \|\omega\|_{op} \|\beta\|_2 \]  
(S302)
\[ \leq 4\tau^{-4} \|\omega\|_{op}^{-4} \cdot (2 + \frac{1}{2} \tau) \|\omega\|_{op}^2 \cdot \|\Delta\|_{op}^2 \|\beta\|_2 \]  
(S303)
\[ + 4\tau^{-2} \|\omega\|_{op}^{-2} \left( 1 + \frac{1}{2} \tau \right) \|\omega\|_{op} \|\Delta\|_{op} \|\beta\|_2 \]  
(S304)
\[ \leq 4\tau^{-4} \left( 2 + \frac{1}{2} \tau \right) \eta \|\beta\|_2 + 4\tau^{-2} \left( 1 + \frac{1}{2} \tau \right) \sqrt{2} \sqrt{\eta} \|\beta\|_2. \]  
(S305)
\[ \| (g(\omega + \Delta) - g(\omega)) \omega \varepsilon_y \|. \]  
(S306)
\[ \leq \left[ \tau^{-2} \|\omega\|_{op}^{-2} \left( 2 + \frac{1}{2} \tau \right) \|\omega\|_{op}^{-2} \left( \tau^{-2} \|\Delta\|_{op} \|\omega\|_{op} + \|\Delta\|_{op}^2 \right) \right] \|\omega\|_{op} \|\varepsilon_y\|_2 \]  
(S307)
\[ \leq \tau^{-2} \|\omega\|_{op}^{-1} \|\Delta\|_{op} \left[ 1 + 4\tau^{-2} \|\omega\|_{op}^{-1} \left( 2 + \frac{\tau}{2} \right) \right] \|\varepsilon_y\|_2 \]  
(S308)
\[ \leq \sqrt{\eta} \cdot \sqrt{2} \tau^{-2} \frac{\varepsilon_y}{\sqrt{n}} \left( 1 + \frac{8\tau^{-2} + 2\tau^{-1}}{\sqrt{n}} \right). \]  
(S309)

Under the high probability event that the bounds for \( \hat{u}, \hat{v} \) in Proposition S5 hold, we have
\[ \| (g(\omega + \Delta) - g(\omega)) \omega \varepsilon_y \| \leq \sqrt{\eta} \sqrt{10} \tau^{-2} \left( 1 + \frac{8\tau^{-2} + 2\tau^{-1}}{\sqrt{n}} \right) \sigma_y. \]  
(S310)

Now we turn to the third term in Inequality (S292). First note that,
\[ g(\omega) \cdot \varepsilon_y \sim N \left( 0, \sigma_y^2 \cdot (\omega^\top \omega)^{-1} \right). \]  
(S311)

Let \( h = g(\omega) \cdot \varepsilon_y \), then for \( \xi > 0 \),
\[ g(\omega) \cdot \varepsilon_y \sim N \left( 0, \sigma_y^2 \cdot (\omega^\top \omega)^{-1} \right). \]  
(S312)
for $\xi < \frac{n}{2\sigma_y\tau}$,

$$\mathbb{P}(\|h\|^2 > k) \leq \left( \frac{1}{\sqrt{1 - 2\sigma_y^2 \cdot \tau^4 \cdot \frac{1}{n}}} \right)^{p+2} e^{-\xi k}. \quad (S318)$$

S7.1.6 Proposition S5

**Proposition S5.** Suppose $(X, u, v)$ is full rank, then for any given $\delta > 0$, with probability at least $1 - 3e^{-n}$, the following holds for all $(\hat{u}, \hat{v})$ such that $(X, u, v)$ is full rank and $f_1(\hat{u}, \hat{v}) \leq f_1(u, v) + \delta$:

$$\left(1 - \frac{\langle \hat{u}, u \rangle^2}{2n^2} - \frac{\langle \hat{v}, v \rangle^2}{2n^2}\right) \leq \frac{30\sigma_a^2}{\lambda(d-d_2) d n} + \frac{544\sigma_a^2}{n (d-d_2)^2} + \frac{2\delta\sigma_a^2}{\lambda n^2 d (d-d_2)}. \quad (S322)$$
Proof. By Definitions 5-6 and the unified model (2), \( f_1(\hat{u}, \hat{v}) \leq f_1(u, v) + \delta \) gives \( f_2(\hat{u}, \hat{v}) \leq f_2(u, v) + \delta \). Therefore,

\[
\mathbb{E} f_2(\hat{u}, \hat{v}) - \mathbb{E} f_2(u, v) \leq f_2(u, v) - \mathbb{E} f_2(u, v) - f_2(\hat{u}, \hat{v}) + \delta.
\]

(S323)

By Lemma S1 and Lemma S2,

\[
\left( 1 - \frac{\langle \hat{u}, u \rangle^2}{2n^2} - \frac{\langle \hat{v}, v \rangle^2}{2n^2} \right) \leq \frac{\sigma_u^2}{(d - d_2) d} \frac{6}{\lambda n^2 \sigma_y^2} \| \hat{\epsilon} \|^2 + 32 \left( \frac{\| E \|_{op}}{n (d - d_2)} \right)^2 + \frac{2 \| E \|_{op}^2}{n^2 d (d - d_2)} + \frac{2 \sigma_y^2 \delta}{\lambda d (d - d_2) n^2}.
\]

(S324)

Note that \( \| \hat{\epsilon} \| \sim \chi_n^2. \) By Lemma S3 and Lemma S4, we have with probability at least \( 1 - 3e^{-n}, \)

\[
\left( 1 - \frac{\langle \hat{u}, u \rangle^2}{2n^2} - \frac{\langle \hat{v}, v \rangle^2}{2n^2} \right) \leq \frac{30 \sigma_u^2}{\lambda (d - d_2) d n} + \frac{544 \sigma_y^2}{n (d - d_2)^2} + \frac{2 \delta \sigma_y^2}{\lambda n^2 d (d - d_2)}.
\]

(S326)

\[ \square \]

S7.1.7 Technical Lemmas

We first present the following tail bounds that are frequently used in Sections S7.1.6 and S7.1.5, and then provide the corresponding proofs in Sections S7.1.7-S7.1.7.

Lemma S1. For any \( \hat{u} \in \mathbb{R}^n, \hat{v} \in \mathbb{R}^n, \) such that \( \| \hat{u} \| = \| \hat{v} \| = \sqrt{n} \), and \( (X, \hat{u}, \hat{v}) \) being full rank,

\[
\mathbb{E} f_2(\hat{u}, \hat{v}) - \mathbb{E} f_2(u, v) \geq \frac{1}{\sigma_y^2} \left\| P(X, u, v) (\hat{u} \beta_u + \hat{v} \beta_v) \right\|^2 + \frac{\hat{\lambda}}{\sigma_y^2 n} \left( 1 - \frac{\langle \hat{u}, u \rangle^2}{2n^2} \right) \left( 1 - \frac{\langle \hat{v}, v \rangle^2}{2n^2} \right) (d - d_2) d.
\]

(S327)

Lemma S2. For any \( \hat{u} \in \mathbb{R}^n, \hat{v} \in \mathbb{R}^n, \) such that \( \| \hat{u} \| = \| \hat{v} \| = \sqrt{n} \), and \( (X, \hat{u}, \hat{v}) \) being full rank,

\[
\left[ f_2(u, v) - \mathbb{E} f_2(u, v) \right] - \left[ f_2(\hat{u}, \hat{v}) - \mathbb{E} f_2(\hat{u}, \hat{v}) \right] \leq \frac{3}{\sigma_y^2} \| \hat{\epsilon} \|^2 + \frac{1}{2 \sigma_y^2} \left\| P(X, u, v) (\hat{u} \beta_u + \hat{v} \beta_v) \right\|^2 + \frac{\hat{\lambda}}{2 \sigma_y^2} \left( 1 - \frac{\langle \hat{u}, u \rangle^2}{2n^2} \right) \left( 1 - \frac{\langle \hat{v}, v \rangle^2}{2n^2} \right) + \frac{\hat{\lambda}}{\sigma_y^2} \| E \|_{op}^2.
\]

(S329)

Lemma S3. For \( E \in \mathbb{R}^{n \times n} \) where \( E_{ij} \overset{i.i.d}{\sim} N(0, \sigma_a^2) \), then for \( t \geq 0, \)

\[
P \left( \frac{E}{\sigma_a} \right)_{op} \geq 2 \sqrt{n} + t \leq 2e^{-t^2/2}.
\]

(S332)
Lemma S4. Let $\chi^2_{n-p}$ follow $\chi^2$-distribution with degree of freedom $n - p$, then

$$P \left( \chi^2_{n-p} \geq n - p + 2\sqrt{n - p}\sqrt{x + 2x} \right) \leq e^{-x}, \quad (S333)$$

$$\mathbb{E} (\chi^2_{n-p}) = n - p, \quad (S334)$$

$$P \left( \chi^2_{n} \leq n - t \right) \leq e^{-t^2/2n}. \quad (S335)$$

Proof of Lemma S1

Proof. $\forall \|\tilde{u}\|_2 = \|\tilde{v}\|_2 = \sqrt{n}$ and $(X, \tilde{u}, \tilde{v})$ full rank,

$$E_{f_2}(\tilde{u}, \tilde{v}) = n - (p + 2) + \frac{1}{\sigma^2_y} \left\| P_\perp (Xu.v) (\tilde{u}_1 \beta_u + \tilde{v}_1 \beta_v) \right\|^2 \quad (S336)$$

$$- \frac{\lambda}{\sigma^2} \left[ \left( \tilde{u}^\top \left( \sum_{i=1}^{r} d_i u_i v_i^\top \right) \tilde{v}/n \right)^2 \right] - \tilde{\lambda}. \quad (S337)$$

For simplicity of notation, denote

$$\tilde{u} = a_u u + b_u u^\perp, \quad (S338)$$

where $u^\perp \perp u$, $\|u^\perp\| = \sqrt{n}$, and

$$\tilde{v} = a_v v + b_v v^\perp, \quad (S339)$$
where $\mathbf{v}^\perp \perp \mathbf{v}$, $\|\mathbf{v}^\perp\| = \sqrt{n}$, then

$$Ef_2(\mathbf{u}, \mathbf{v}) - Ef_2(\mathbf{u}, \mathbf{v})$$

$$\geq \frac{1}{\sigma_y} \left\| P_{(X, \tilde{\mathbf{u}}, \tilde{\mathbf{v}})}(\tilde{\mathbf{u}}_1 \beta_u + \tilde{\mathbf{v}}_1 \beta_v) \right\|^2$$

$$- \frac{\bar{\lambda}}{\sigma_a^2} \left[ \left( a_u a_v d + b_u b_v (\mathbf{u}^\perp)^\top \left( \sum_{i=2}^{r} d_i \mathbf{u}_i \mathbf{v}_i^\top \right) \mathbf{v}^\perp / n \right)^2 \right] + \frac{\bar{\lambda}}{\sigma_a^2} d^2 n^2$$

$$\geq \frac{1}{\sigma_y} \left\| P_{(X, \tilde{\mathbf{u}}, \tilde{\mathbf{v}})}(\tilde{\mathbf{u}}_1 \beta_u + \tilde{\mathbf{v}}_1 \beta_v) \right\|^2$$

$$+ \frac{\bar{\lambda}}{\sigma_a^2} n^2 \left[ d^2 - \left( |a_u a_v| d + \sqrt{(1 - a_u^2)(1 - a_v^2)} d_2 \right)^2 \right]$$

$$\geq \frac{1}{\sigma_y} \left\| P_{(X, \tilde{\mathbf{u}}, \tilde{\mathbf{v}})}(\tilde{\mathbf{u}}_1 \beta_u + \tilde{\mathbf{v}}_1 \beta_v) \right\|^2$$

$$+ \frac{\bar{\lambda}}{\sigma_a^2} n^2 \left[ d^2 - \left( \frac{a_u^2 + a_v^2}{2} d + 2 - a_u^2 - a_v^2 \right)^2 \right]$$

$$\geq \frac{1}{\sigma_y} \left\| P_{(X, \tilde{\mathbf{u}}, \tilde{\mathbf{v}})}(\tilde{\mathbf{u}}_1 \beta_u + \tilde{\mathbf{v}}_1 \beta_v) \right\|^2$$

$$+ \frac{\bar{\lambda}}{\sigma_a^2} n^2 \left( 2 - a_u^2 - a_v^2 \right) (d - d_2) \left[ 1 + \frac{a_u^2 + a_v^2}{2} d + 2 - a_u^2 - a_v^2 \right]$$

$$\geq \frac{1}{\sigma_y} \left\| P_{(X, \tilde{\mathbf{u}}, \tilde{\mathbf{v}})}(\tilde{\mathbf{u}}_1 \beta_u + \tilde{\mathbf{v}}_1 \beta_v) \right\|^2 + \frac{\bar{\lambda}}{\sigma_a^2} n^2 \left( 2 - a_u^2 - a_v^2 \right) (d - d_2) . d$$

Proof of Lemma S2

Proof. For $\forall \mathbf{u} \in \mathbb{R}^n$, $\forall \mathbf{v} \in \mathbb{R}^n$, $\|\mathbf{u}\| = \|\mathbf{v}\| = \sqrt{n}$, and $(X, \tilde{\mathbf{u}}, \tilde{\mathbf{v}})$ full rank, let

$$\Delta(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) = [f_2(\mathbf{u}, \mathbf{v}) - Ef_2(\mathbf{u}, \mathbf{v})] - [f_2(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) - Ef_2(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})].$$

(S351)
Then

\[
\Delta = \frac{1}{\sigma_y^2} \left[ \| P_{(X,u,v) \perp} \bar{\xi} \|^2 - \| P_{(X,u,v)} \perp \bar{\xi} \|^2 \right] - \frac{2}{\sigma_y^2} \left( \bar{u}^T E \bar{v} / n \right)^2 - \frac{\lambda}{\sigma_a^2} (nd) \left( \bar{u}^T E v / n \right) + \frac{2\lambda}{\sigma_a^2} \left( \bar{u}^T \left( \sum_{i=1}^r d_i u_i v_i^\top \right) \bar{v} / n \right) (\bar{u}^T E \bar{v} / n)
\]

\[
\leq \frac{1}{\sigma_y^2} \| \bar{\xi} \|^2 + \frac{2}{\sigma_y^2} \| \bar{\xi} \| \cdot \left( \| P_{(X,u,v) \perp} \bar{\xi} \| \right) + \frac{\lambda}{\sigma_a^2} \| E \|_{op}^2
\]

\[
+ \frac{2\lambda}{\sigma_a^2} \left[ - (nd) \left( \bar{u}^T E v / n \right) + (nd) (c_1 c_2 \bar{u}^T E \bar{v} / n) \right]
\]

\[
+ \left( \bar{u}^T E \bar{v} / n \right) \text{tr} \left( \sum_{i=1}^r d_i u_i v_i^\top, \frac{\bar{w} v^\top}{n} - \frac{w v^\top c_1 c_2}{n} \right)
\]

\[
\leq \frac{3}{\sigma_y^2} \| \bar{\xi} \|^2 + \frac{2}{\sigma_a^2} \left( \| P_{(X,u,v) \perp} \bar{\xi} \| \right)\| E \|_{op} \cdot nd \cdot 2 \left( \left\| \frac{w v^\top - \bar{w} v^\top c_1 c_2}{n} \right\|_{nuc} + \frac{\lambda}{\sigma_a^2} \| E \|_{op}^2 \right).
\]

where

\[
c_1 = \text{sign} \left( \langle \bar{u}, u \rangle \right) \quad \text{(S361)}
\]

\[
c_2 = \text{sign} \left( \langle \bar{v}, v \rangle \right). \quad \text{(S362)}
\]

Note that

\[
\left\| \frac{w v^\top - \bar{w} v^\top c_1 c_2}{n} \right\|_{nuc} \leq \sqrt{2} \left\| \frac{w v^\top - \bar{w} v^\top c_1 c_2}{n} \right\|_F \quad \text{(S363)}
\]

\[
= \sqrt{2} \sqrt{1 - \frac{c_1 \langle u, u \rangle \cdot c_2 \langle v, v \rangle}{n^2}} \quad \text{(S364)}
\]

\[
\leq \sqrt{2} \sqrt{1 - \frac{\langle u, \bar{u} \rangle^2}{2n^2} - \frac{\langle v, \bar{v} \rangle^2}{2n^2}}. \quad \text{(S365)}
\]
Therefore,

\[
\Delta \leq \frac{3}{\sigma_y} \|\varepsilon\|^2 + \frac{1}{2} \left\| P_{(X,u,v)^\perp} (\hat{u}_1 \beta_u + \hat{v}_1 \beta_v) \right\|^2 + \frac{4\sqrt{2}\lambda}{\sigma_a^2} \varepsilon \left( \frac{2\sqrt{2}\|E\|^2_{op} + \frac{n(d-d_2)}{8\sqrt{2}} \left( 1 - \frac{\langle u, \hat{u}\rangle^2}{2n^2} - \frac{\langle v, \hat{v}\rangle^2}{2n^2} \right)}{n(d-d_2)} \right)
\]

\[
+ \frac{\lambda}{\sigma_a^2} \|E\|^2_{op}
\]

\[
\leq \frac{3}{\sigma_y} \|\varepsilon\|^2 + \frac{1}{2} \left\| P_{(X,u,v)^\perp} (\hat{u}_1 \beta_u + \hat{v}_1 \beta_v) \right\|^2 + \frac{4\sqrt{2}\lambda}{\sigma_a^2} \varepsilon \left( \frac{2\sqrt{2}\|E\|^2_{op} + \frac{n(d-d_2)}{8\sqrt{2}} \left( 1 - \frac{\langle u, \hat{u}\rangle^2}{2n^2} - \frac{\langle v, \hat{v}\rangle^2}{2n^2} \right)}{n(d-d_2)} \right)
\]

\[
+ \frac{\lambda}{\sigma_a^2} \|E\|^2_{op}.
\]

\[\text{(S366)} \]

\[\text{(S367)} \]

\[\text{(S368)} \]

\[\text{(S369)} \]

\[\text{(S370)} \]

\[\square\]

**Proof of Lemma S3**

**Proof.** Let

\[
Z_{u,v} = u^\top \frac{E}{\sigma_a} v,
\]

\[\text{(S371)} \]

\[
Y_{u,v} = \langle g, u \rangle + \langle h, v \rangle, \text{ where } g, h \in N(0, I_n).
\]

\[\text{(S372)} \]

Clearly,

\[
\left\| \frac{E}{\sigma_a} \right\|_{op} = \sup_{\|u\| = \|v\| = 1} Z_{u,v}.
\]

\[\text{(S373)} \]

For any pairs \((\hat{u}, \hat{v}), (\tilde{u}, \tilde{v})\) such that \(\|\hat{u}\| = \|\hat{v}\| = \|\tilde{u}\| = \|\tilde{v}\| = 1,

\[
\begin{align*}
\mathbb{E} \left( (Z_{\hat{u}, \hat{v}} - Z_{\tilde{u}, \tilde{v}})^2 \right) &= 2 - 2\langle \hat{u}, \hat{v} \rangle \langle \tilde{u}, \tilde{v} \rangle \\
\mathbb{E} \left( (Y_{\hat{u}, \hat{v}} - Y_{\tilde{u}, \tilde{v}})^2 \right) &= \|\hat{u} - \tilde{u}\|^2 + \|\hat{v} - \tilde{v}\|^2 \\
&= 4 - 2\langle \hat{u}, \hat{v} \rangle - 2\langle \tilde{u}, \tilde{v} \rangle.
\end{align*}
\]

\[\text{(S374)} \]

\[\text{(S375)} \]

\[\text{(S376)} \]

Therefore,

\[
\mathbb{E} \left( (Z_{\hat{u}, \hat{v}} - Z_{\tilde{u}, \tilde{v}})^2 \right) \leq \mathbb{E} \left( (Y_{\hat{u}, \hat{v}} - Y_{\tilde{u}, \tilde{v}})^2 \right).
\]

\[\text{(S377)} \]

Let \(w_i\) be a maximum \(2^{-i}\)-packing of \(S^{n-1}\), then
\[ E \left( \frac{E}{\sigma_a}_{op} \right) = E \left( \sup_{\|u\|=\|v\|=1} u^T \frac{E}{\sigma_a} v \right) \] (S378)

\[ \overset{(a)}{=} \lim_{i \to +\infty} E \left( \max_{u,v \in w_i} u^T \frac{E}{\sigma_a} v \right) \] (S379)

\[ \overset{(b)}{\leq} \lim_{i \to +\infty} E \left( \max_{u,v \in w_i} \langle g, u \rangle + \langle h, v \rangle \right) \] (S380)

\[ = \lim_{i \to +\infty} 2\sqrt{n} = 2\sqrt{n}. \] (S381)

where (a) follows from \( \|E\|_{op} \leq \max_{u,v \in w_i} u^T \frac{E}{\sigma_a} v + 2^{1-i} \|E\|_{op} \), and (b) follows from Sudakov-Fernique inequality.

Note that \( \sup \|u\|=\|v\|=1 \) \( z_{u,v} \) is a 1-Lipschitz function of vector \( \left( \frac{E}{\sigma_a} \right) \), by Theorem 2.26 in ?

\[
P \left( \sup_{\|u\|=\|v\|=1} Z_{u,v} - E \left( \sup_{\|u\|=\|v\|=1} Z_{u,v} \right) \geq t \right) \leq 2e^{-\frac{t^2}{2}}. \] (S382)

Therefore

\[ P \left( \sup \|E\|_{\sigma_a}_{op} \geq 2\sqrt{n} + t \right) \leq 2e^{-\frac{t^2}{2}} \] (S383)

for \( t \geq 0 \).

\[ \square \]

**Proof of Lemma S4**

**Proof.** We only need to prove that for any positive integer \( n \geq 1 \),

\[ P \left( \chi_n^2 \geq n + 2\sqrt{n} \sqrt{\pi} + 2x \right) \leq e^{-x}, \] (S384)

\[ E \left( \chi_n^2 \right) = n, \] (S385)

\[ P \left( \chi_n^2 \leq n - t \right) \leq 2e^{-t^2/8n}. \] (S386)

Suppose \( z_1, z_2, \ldots, z_n \stackrel{i.i.d.}{\sim} N(0, 1) \). Let \( Y := \sum_{i=1}^n z_i^2 \sim \chi_n^2 \). By definition of \( \chi_n^2 \)-distribution, we have

\[ E \left( \chi_n^2 \right) = E(Y) = n, \] (S387)
\[ P \left( \chi_n^2 \geq n + 2\sqrt{n} \sqrt{x} + 2x \right) = P(Y \geq n + 2\sqrt{n} \sqrt{x} + 2x) \] (S388)
\[ \leq \inf_{\frac{1}{2} > \xi > 0} \mathbb{E} e^{\xi(Y - n - 2\sqrt{n} \sqrt{x} - 2x)} \] (S389)
\[ = \inf_{\frac{1}{2} > \xi > 0} e^{-\frac{n}{2} \log^1 - 2\xi} e^{\xi(-n - 2\sqrt{n} \sqrt{x} - 2x)} \] (S390)
\[ \overset{(a)}{=} e^{-\frac{n}{2} \log \frac{n + 2\sqrt{n} \sqrt{x} + 2x}{n}} + (-\sqrt{n} \sqrt{x} - x) \] (S391)
\[ = e^\frac{n}{2} (\log 1 + 2\sqrt{\frac{x}{n}} + 2\frac{\xi}{n} - 2\sqrt{\frac{x}{n}} - x) \] (S392)
\[ \overset{(b)}{\leq} e^{-x} \] (S393)

where Step (a) follows from analysis derivative of the exponent w.r.t. \( \xi \), and Step (b) follows from the fact \( \log(1 + 2t + 2t^2 - 2t) \leq 0 \) for \( t \geq 0 \).

\[ P \left( \chi_n^2 \leq n - t \right) = P(Y \leq n - t) \] (S394)
\[ \leq \inf_{\xi > 0} \mathbb{E} e^{-\xi Y + \xi(n - t)} \] (S395)
\[ = \inf_{\xi > 0} e^{-\frac{n}{2} \log 1 + 2\xi} + \xi(n - t) \] (S396)
\[ = e^{-\frac{n}{2} \log \frac{n - t + t}{n - t}} \] (S397)
\[ = e^\frac{n}{2} \log \frac{n - t + t^2}{8n - 8t} . \] (S398)

Note that
\[ \frac{n}{2} \log \frac{n - t}{n} + \frac{t^2}{2} + \frac{t^2}{8n} \] (S399)
\[ = \frac{n}{2} \left( \log(1 - \frac{t}{n}) + \frac{t}{n} + \frac{1}{4} \left( \frac{t}{n} \right)^2 \right) \] (S400)
\[ \leq \frac{n}{2} \sup_{1 > t \geq 0} \log(1 - t) + t + \frac{1}{4} t^2 \] (S401)
\[ \leq 0, \] (S402)

we have \( P \left( \chi_n^2 \leq n - t \right) \leq e^{-\frac{t^2}{8n}} \).

\[ \square \]

S7.2 Proof of the theoretical results of two-stage

In this section, we prove that the asymptotic distribution of the two-stage estimators as stated in Theorem S2 and show the convergence rates of \( \hat{u}_{ts} \) and \( \tilde{v}_{ts} \) as in Proposition S2.

S7.2.1 Proof of Theorem S2

Proof. The naive two-stage procedure first estimates the centralities \( u \) and \( v \) by the leading left and right singular vectors, rescaled to have norm \( \sqrt{n} \) and denoted as \( \hat{u}_{ts}, \tilde{v}_{ts} \), from the SVD on the ob-
served adjacency matrix $A$, and then performs ordinary least square (OLS) regression of $y$ on $X$ and $\hat{u}^{ts}, \hat{v}^{ts}$, treating $\hat{u}^{ts}, \hat{v}^{ts}$ as given covariates. It is, therefore, equivalent to solve the following two optimization problems sequentially,

$$
\begin{align}
(\hat{u}^{ts}, \hat{v}^{ts}) := & \arg \min_{d \in \mathbb{R}^n, |v| = \sqrt{\eta}} \| A - d u v^\top \|_F^2, \tag{S403a} \\
\hat{\beta}^{ts} = & (\hat{\beta}_x^{ts}, \beta_u^{ts}, \beta_v^{ts})^\top := \arg \min_{\beta_x, \beta_u, \beta_v} \| y - X \beta_x - \hat{u}^{ts} \beta_u - \hat{v}^{ts} \beta_v \|_2^2. \tag{S403b}
\end{align}
$$

Recall that $A^\perp = U^\perp D^\perp V^\perp$ where $U^\perp = (u_2, \ldots, u_r)$, $V^\perp = (v_2, \ldots, v_r)$ and $D^\perp = \text{diag}(d_2, \ldots, d_r)$. Then, $A^\perp u = 0$ and $A^\perp v = 0$. The proof strategy is similar to that of Theorem 1 in Section S7.1.2.

For the first stage, we minimize the objective function (S403a) and we obtain the following result for $\hat{u}^{ts}$ and $\hat{v}^{ts}$:

$$
\begin{align}
\delta_{\hat{u}}^{ts} = & \eta_{\hat{u}}^{ts} + o(\eta_{\hat{u}}^{ts}), \\
\delta_{\hat{v}}^{ts} = & \eta_{\hat{v}}^{ts} + o(\eta_{\hat{v}}^{ts}),
\end{align}
$$

where

\begin{align}
\begin{pmatrix} \eta_{\hat{u}}^{ts} \\ \eta_{\hat{v}}^{ts} \end{pmatrix} &= \begin{pmatrix} nd I & -A^\perp \\ -(A^\perp)^\top & nd I \end{pmatrix}^{-1} \begin{pmatrix} v^\top \otimes (I - P_u) \\ (u^\top \otimes (I - P_v)) K \end{pmatrix} \text{vec}(E), \\
& \overset{\text{def}}{=} \begin{pmatrix} C_{12}^{ts} \\ C_{22}^{ts} \end{pmatrix} \text{vec}(E),
\end{align}

\begin{align}
\frac{||o(\eta_{\hat{u}}^{ts})||}{||\eta_{\hat{u}}^{ts}||} & \xrightarrow{P} 0, \text{ and } \frac{||o(\eta_{\hat{v}}^{ts})||}{||\eta_{\hat{v}}^{ts}||} \xrightarrow{P} 0.
\end{align}

For the second stage, we plug in $\hat{u}^{ts}$ and $\hat{v}^{ts}$ from the first stage into the objective function (S403b) of the second stage and then minimize the objective function. For $\delta_{\hat{u}}^{ts}$ and $\delta_{\hat{v}}^{ts}$, we have

$$
\begin{align}
\delta_{\hat{u}} = & \eta_{\hat{u}}^{ts} + o(\eta_{\hat{u}}^{ts}), \\
\delta_{\hat{v}} = & \eta_{\hat{v}}^{ts} + o(\eta_{\hat{v}}^{ts}),
\end{align}
$$

where

\begin{align}
\begin{pmatrix} \eta_{\hat{u}}^{ts} \\ \eta_{\hat{v}}^{ts} \end{pmatrix} &= \begin{pmatrix} C_{1u}^{-1} \tilde{u}^\top \\ C_{1v}^{-1} \tilde{v}^\top \end{pmatrix} (-\beta_u I_n - \beta_v I_n) \begin{pmatrix} 0_{n \times n} & C_{ts}^{12} \\ 0_{n \times n} & C_{ts}^{22} \end{pmatrix} \begin{pmatrix} \epsilon \text{vec}(E) \end{pmatrix}, \\
& \overset{\text{def}}{=} \begin{pmatrix} C_{41}^{ts} \\ C_{51}^{ts} \end{pmatrix} \begin{pmatrix} \epsilon \text{vec}(E) \end{pmatrix},
\end{align}

where explicitly

\begin{align}
\begin{pmatrix} C_{41}^{ts} \\ C_{51}^{ts} \end{pmatrix} &= C_{1u}^{-1} \begin{pmatrix} \tilde{u}^\top \\ \tilde{v}^\top \end{pmatrix}.
\end{align}

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Putting (S420), (S409) and (S413) together, we have

\[
\begin{pmatrix}
C_{42}^{ts} \\
C_{52}^{ts}
\end{pmatrix} = C^{-1}\begin{pmatrix}
\hat{u}^\top \\
\hat{v}^\top
\end{pmatrix}
\begin{pmatrix}
\beta_u I_n & -\beta_v I_n
\end{pmatrix}
\begin{pmatrix}
nd I & -A^\top \\
(-A^\top) & nd I
\end{pmatrix}^{-1}
\begin{pmatrix}
v^\top \otimes (I - Pu) \\
u^\top \otimes (I - Pv)
\end{pmatrix} K
\]  

(S411)

\[
\frac{|o(o_{\eta_{\beta}^{ts}})|}{|o_\eta^{ts}_{\bar{u}}|} \xrightarrow{P} 0, \text{ and } \frac{|o(o_{\eta_{\beta}^{ts}})|}{|o_\eta^{ts}_{\bar{v}}|} \xrightarrow{P} 0.
\]

For \(\delta_{\beta}^{ts}\), we have \(\delta_{\beta}^{ts} = \eta_{\beta}^{ts} + o(\eta_{\beta}^{ts})\) where \(\|o(\eta_{\beta}^{ts})\|/\|\eta_{\beta}^{ts}\| \xrightarrow{P} 0\) and

\[
\eta_{\beta}^{ts} = (X^\top X)^{-1}X^\top (-\beta_u I_n - \beta_v I_n - u - v) I_n
\]

\[
\begin{pmatrix}
0_{n \times n} \\
0_{n \times n}
\end{pmatrix} C_{12}^{ts} \\
0_{n \times n} C_{22}^{ts}
\]

\[
\begin{pmatrix}
C_{41}^{ts} \\
C_{51}^{ts}
\end{pmatrix} C_{42}^{ts} \\
C_{52}^{ts} C_{52}^{ts}
\]

\[
I_n \ 0_{n \times n^2}
\]

(S412)

\[
\eta_{\beta}^{ts} = \begin{pmatrix}
C_{31}^{ts} \\
C_{32}^{ts}
\end{pmatrix} \begin{pmatrix}
\epsilon \\
\text{vec}(E)
\end{pmatrix}
\]

(S413)

Finally, recall that we assume

\[
\begin{pmatrix}
\epsilon \\
\text{vec}(E)
\end{pmatrix} \sim N\left(0_{(n+n^2) \times 1}, \begin{pmatrix}
\sigma^2_2 I_n & 0_{n \times n^2} \\
0_{n^2 \times n} & \sigma^2_a I_{n^2}
\end{pmatrix}\right).
\]

(S414)

Putting (S420), (S409) and (S413) together, we have

\[
\begin{pmatrix}
\hat{u}^{ts} - u \\
\hat{v}^{ts} - v
\end{pmatrix} = \begin{pmatrix}
\eta_{\hat{u}}^{ts} + o(\eta_{\hat{u}}^{ts}) \\
\eta_{\hat{v}}^{ts} + o(\eta_{\hat{v}}^{ts})
\end{pmatrix}
\]

\[
\begin{pmatrix}
\hat{\beta}_x^{ts} - \beta_x \\
\hat{\beta}_u^{ts} - \beta_u
\end{pmatrix} = \begin{pmatrix}
\eta_{\hat{\beta}_x}^{ts} + o(\eta_{\hat{\beta}_x}^{ts}) \\
\eta_{\hat{\beta}_u}^{ts} + o(\eta_{\hat{\beta}_u}^{ts})
\end{pmatrix}
\]

(S415)

where

\[
\begin{pmatrix}
\eta_{\hat{u}}^{ts} \\
\eta_{\hat{v}}^{ts} \\
\eta_{\hat{\beta}_x}^{ts} \\
\eta_{\hat{\beta}_u}^{ts}
\end{pmatrix} = C^{ts} \begin{pmatrix}
\epsilon \\
\text{vec}(E)
\end{pmatrix}
\]

(S416)

\[
\sim N\left(0_{(2n+2+p) \times 1}, C^{ts} \begin{pmatrix}
\sigma^2_2 I_n & 0_{n \times n^2} \\
0_{n^2 \times n} & \sigma^2_a I_{n^2}
\end{pmatrix} C^{ts \top}\right).
\]

(S417)
S7.2.2 Proof of Proposition S2

Proof. From Theorem S2, we have

\[
\begin{pmatrix}
\eta^{ts}_u \\
\eta^{ts}_v \\
\end{pmatrix} = \begin{pmatrix}
ndI & -A \perp \\
(-A \perp)^T & ndI \\
\end{pmatrix}^{-1} \begin{pmatrix}
v^\top \otimes (I - Pu) \\
(u^\top \otimes (I - Pv))K \\
\end{pmatrix} \text{vec}(E) \\
\] (S419)

\[
def \begin{pmatrix}
C^{ts}_{12} \\
C^{ts}_{22} \\
\end{pmatrix} \text{vec}(E).
\] (S420)

When \(A_0\) is rank one, then \(C^{ts}_{12} = (dn)^{-1}v^\top \otimes (I - Pu)\), \(C^{ts}_{22} = (dn)^{-1}(u^\top \otimes (I - Pv))\) \(K\) and \(C^{ts}_{32} = [(I - Pv) \otimes Pu + P_v \otimes (I - Pu) + P_v \otimes Pu]\). Note that

\[
\frac{1}{n} \text{tr}(\sigma^2_a C^{ts}_{12} C^{ts}_{12}^\top) = \frac{1}{n} \frac{\sigma_a^2}{(dn)^2} \text{tr}\left( (v^\top \otimes (I - Pu)) (v \otimes (I - Pu)) \right) \\
= \frac{1}{n} \frac{\sigma_a^2}{(dn)^2} \text{tr}\left( v^\top v \otimes (I - Pu) \right) \\
= \frac{1}{n} \frac{\sigma_a^2}{(dn)^2} \text{tr}\left( nI - uu^\top \right) \\
= \frac{\sigma_a^2}{d^2n^2} (n - 1).
\] (S421)

Then for the rate of \(\hat{\eta}^{ts}_u\),

\[
\frac{1}{n} \mathbb{E} \|\hat{\eta}^{ts}_u - \eta^{ts}_u\|^2_2 = \left( \frac{\sigma_a^2}{d^2n^2} (n - 1) \right) (1 + o(1)) = O(\kappa).
\] (S425)

Similarly for the rate of \(\hat{\eta}^{ts}_v\),

\[
\frac{1}{n} \text{tr}(\sigma^2_a C^{ts}_{22} C^{ts}_{22}^\top) = \frac{1}{n} \frac{\sigma_a^2}{(dn)^2} \text{tr}\left( (u^\top \otimes (I - Pv)) (u \otimes (I - Pv)) \right) \\
= \frac{\sigma_a^2}{d^2n^2} (n - 1)
\] (S426)

where the first equality is due to \(KK^\top = I\). Hence

\[
\frac{1}{n} \mathbb{E} \|\hat{\eta}^{ts}_v - \eta^{ts}_v\|^2_2 = \left( \frac{\sigma_a^2}{d^2n^2} (n - 1) \right) (1 + o(1)) = O(\kappa).
\] (S428)

\[
\square
\]

S7.2.3 Proof of Proposition S3

Proof. We now prove the rate of \(\hat{\beta}^{ts}\) of two-stage.
(1) Rate of $\hat{\beta}_{ts}^l$ and $\hat{\beta}_{ts}^q$. Recall that

$$
\begin{align*}
\begin{pmatrix} \eta_{\beta_u}^{ts} \\ \eta_{\beta_v}^{ts} \end{pmatrix} &= C^{-1}_{uv} \begin{pmatrix} \hat{u}^T \\ \hat{v}^T \end{pmatrix} \left[ \epsilon - (\delta_{ts}^{\mathbf{B}} \beta_u + \delta_{ts}^{\mathbf{V}} \beta_v) \right] \\
&= C^{-1}_{uv} \begin{pmatrix} \hat{u}^T \\ \hat{v}^T \end{pmatrix} \left[ \epsilon - \frac{1}{dn} \left( \beta_u v^T \otimes (\mathbf{I} - P_u) + \beta_v u^T \otimes (\mathbf{I} - P_v) \mathbf{K} \right) \text{vec}(\mathbf{E}) \right] \\
&\overset{\text{def}}{=} C^{-1}_{uv} \begin{pmatrix} \hat{u}^T \\ \hat{v}^T \end{pmatrix} \left[ \epsilon + B \text{vec}(\mathbf{E}) \right] \\
&\overset{\text{def}}{=} D_1 \epsilon + D_2 \text{vec}(\mathbf{E}).
\end{align*}
$$

Note that

$$
D_1 D_1^T = C^{-1}_{uv} \begin{pmatrix} \hat{u}^T \\ \hat{v}^T \end{pmatrix} \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} C^{-1}_{uv} = C^{-1}_{uv}
$$

and

$$
D_2 D_2^T = C^{-1}_{uv} \begin{pmatrix} \hat{u}^T \\ \hat{v}^T \end{pmatrix} \mathbf{B} \mathbf{B}^T \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} C^{-1}_{uv}
$$

where

$$
\mathbf{B} \mathbf{B}^T = \frac{1}{(dn)^2} \left[ \beta_u^2 (v^T \otimes (\mathbf{I} - P_u))(v \otimes (\mathbf{I} - P_u)) + \beta_v^2 (u^T \otimes (\mathbf{I} - P_v))(u \otimes (\mathbf{I} - P_v)) \right] \\
+ \beta_u \beta_v (v^T \otimes (\mathbf{I} - P_u)) \mathbf{K}^T (u \otimes (\mathbf{I} - P_v)) \\
+ \beta_u \beta_v (u^T \otimes (\mathbf{I} - P_v)) \mathbf{K} (v \otimes (\mathbf{I} - P_u))
$$

$$
\overset{\text{def}}{=} \frac{1}{dn^2} \left[ \beta_u^2 (I - P_u) + \beta_v^2 (I - P_v) \right]
$$

since $(v^T \otimes (I - P_u)) \mathbf{K}^T (u \otimes (I - P_v)) = (v^T (I - P_v) \otimes (I - P_u) u) K = 0$ and $(u^T \otimes (I - P_v)) \mathbf{K} (v \otimes (I - P_u)) = (u^T (I - P_u) \otimes (I - P_v) v) K = 0$.

Plugging in (S433), (S434) and (S438),

$$
\text{Cov} \begin{pmatrix} \delta_{ts}^{\beta_u} \\ \delta_{ts}^{\beta_v} \end{pmatrix} \approx \sigma_x^2 D_1 D_1^T + \sigma_z^2 D_2 D_2^T
$$

$$
= \sigma_y^2 C^{-1}_{uv} \\
+ \sigma_z^2 \frac{1}{dn^2} C^{-1}_{uv} \begin{pmatrix} \hat{u}^T \\ \hat{v}^T \end{pmatrix} \left[ \beta_u^2 (I - P_u) + \beta_v^2 (I - P_v) \right] \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} C^{-1}_{uv}
$$

$$
= \sigma_y^2 C^{-1}_{uv} + \sigma_z^2 \frac{1}{dn^2} C^{-1}_{uv} \begin{pmatrix} \beta_u^2 \hat{u}^T (I - P_u) \hat{u} \\ 0 \end{pmatrix} C^{-1}_{uv}.
$$
Therefore, we obtain the rate of $\hat{\beta}_u^{ts}$ and $\hat{\beta}_v^{ts}$ from the diagonal entries of $Cov \left( \begin{bmatrix} \delta_{\beta_u}^{ts} \\ \delta_{\beta_v}^{ts} \end{bmatrix} \right)$ as

\[
E(\hat{\beta}_u^{ts} - \beta_u)^2 = \frac{\sigma_u^2}{c} \tilde{v}^{\top} \tilde{v} + \frac{\sigma_u^2}{c^2} \frac{1}{dn} \left[ \beta_u^{2} \tilde{v}^{\top} \tilde{u}^{\top} (I - P_v) \tilde{u} \tilde{v} + \beta_v^{2} \tilde{v}^{\top} \tilde{v} (I - P_u) \tilde{v}^{\top} \tilde{v} \right] (1 + o(1)),
\]

\[
E(\hat{\beta}_v^{ts} - \beta_v)^2 = \frac{\sigma_v^2}{c} \tilde{u}^{\top} \tilde{u} + \frac{\sigma_v^2}{c^2} \frac{1}{dn} \left[ \beta_u^{2} \tilde{u}^{\top} \tilde{u} (I - P_u) \tilde{u} \tilde{v} + \beta_v^{2} \tilde{v}^{\top} \tilde{v} (I - P_v) \tilde{v}^{\top} \tilde{u} \right] (1 + o(1)),
\]

where $c = \tilde{u}^{\top} \tilde{v}^{\top} \tilde{v} - (\tilde{u}^{\top} \tilde{v})^2$.

(2) Rate of $\hat{\beta}_z^{ts}$. Recall that

\[
\delta_{\beta_z}^{ts} \approx (X^{\top} X)^{-1} X^{\top} (\epsilon - u \delta_{\beta_u}^{ts} - v \delta_{\beta_v}^{ts} - \beta_u \beta_v + v \delta_{\beta_u}^{ts} + \delta_{\beta_v}^{ts} - \delta_{\beta_u}^{ts} \beta_v).
\]

From (S431), we have

\[
\epsilon - (\delta_{\beta_u}^{ts} \beta_u + \delta_{\beta_v}^{ts} \beta_v) \overset{def}{=} \epsilon + B \text{ vec}(E)
\]

and

\[
\left( \begin{bmatrix} \delta_{\beta_u}^{ts} \\ \delta_{\beta_v}^{ts} \end{bmatrix} \right) \approx C_{\tilde{u} \tilde{v}}^{-1} \left( \begin{bmatrix} \tilde{u}^{\top} \\ \tilde{v}^{\top} \end{bmatrix} \right) \left[ \epsilon + B \text{ vec}(E) \right].
\]

Then we have

\[
u \delta_{\beta_u}^{ts} + v \delta_{\beta_v}^{ts} = \left( \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} \delta_{\beta_u}^{ts} \\ \delta_{\beta_v}^{ts} \end{bmatrix} \right) \approx \left( \begin{bmatrix} u & v \end{bmatrix} C_{\tilde{u} \tilde{v}}^{-1} \left( \begin{bmatrix} \tilde{u}^{\top} \\ \tilde{v}^{\top} \end{bmatrix} \right) \left[ \epsilon + B \text{ vec}(E) \right].
\]

Let

\[
\tilde{G} = \left( \begin{bmatrix} u & v \end{bmatrix} C_{\tilde{u} \tilde{v}}^{-1} \left( \begin{bmatrix} \tilde{u}^{\top} \\ \tilde{v}^{\top} \end{bmatrix} \right) \right) \text{ and } G = \left( \begin{bmatrix} u & v \end{bmatrix} C_{\tilde{u} \tilde{v}}^{-1} \left( \begin{bmatrix} \tilde{u}^{\top} \\ \tilde{v}^{\top} \end{bmatrix} \right) \right).
\]

Plugging (S431) and (S450) into (S447) yields

\[
\delta_{\beta_z}^{ts} \approx (X^{\top} X)^{-1} X^{\top} \left[ I - \left( \begin{bmatrix} u & v \end{bmatrix} C_{\tilde{u} \tilde{v}}^{-1} \left( \begin{bmatrix} \tilde{u}^{\top} \\ \tilde{v}^{\top} \end{bmatrix} \right) \right) \right] \left[ \epsilon + B \text{ vec}(E) \right) \]

\[
= (X^{\top} X)^{-1} X^{\top} (I - \tilde{G})(\epsilon + B \text{ vec}(E)) \]

\[
\overset{def}{=} F_1 \epsilon + F_2 \text{ vec}(E).
\]

Hence, the variance-covariance matrix of $\delta_{\beta_z}^{ts}$ is

\[
Cov \left( \begin{bmatrix} \delta_{\beta_u}^{ts} \\ \delta_{\beta_v}^{ts} \end{bmatrix} \right) \approx \sigma_u^2 F_1 F_1^{\top} + \sigma_v^2 F_2 F_2^{\top}
\]

where we will derive the explicit form of $F_1 F_1^{\top}$ and $F_2 F_2^{\top}$ in the following.
(a) $F_1 F_1^\top$.
Since
\[
(I - \tilde{G})(I - G)^\top = I - G - \tilde{G}^\top + G
\]
and
\[
\begin{pmatrix} \tilde{u}^\top \\ \tilde{v}^\top \end{pmatrix} X = 0 \text{ and thus } \tilde{G}X = 0,
\]
consequently
\[
F_1 F_1^\top = (X^\top X)^{-1} + (X^\top X)^{-1} \tilde{G}X(X^\top X)^{-1}.
\]

(b) $F_2 F_2^\top$.
Recall (S438) where
\[
BB^\top = \frac{1}{d^2 n} \left[ \beta_u^2 (I - P_u) + \beta_v^2 (I - P_v) \right].
\]
Plugging in we have,
\[
(I - \tilde{G})BB^\top (I - \tilde{G})^\top = \frac{1}{d^2 n} \left[ \beta_u^2 (I - P_u) + \beta_v^2 (I - P_v) - \tilde{G}BB^\top - BB^\top \tilde{G}^\top \right.
\]
\[
+ \begin{pmatrix} u & v \end{pmatrix} \beta_u^2 \left( \tilde{u}^\top (I - P_u) \tilde{u}^\top \\ 0 \beta_v^2 \right) \begin{pmatrix} u \\ v \end{pmatrix} \beta_v^2 \left( \tilde{v}^\top (I - P_v) \tilde{v}^\top \right) \left[ C_{\tilde{u} \tilde{v}}^{-1} \begin{pmatrix} u^\top \\ v^\top \end{pmatrix} \right].
\]
Together with (S458) and using (S457), we obtain the variance-covariance matrix of $\delta_x^\beta$ as follows.
\[
Cov \left( \begin{pmatrix} \hat{\beta}_x^\beta \\ \beta_x \end{pmatrix} \right) \approx \sigma_y^2 \left[ (X^\top X)^{-1} + (X^\top X)^{-1} \begin{pmatrix} u & v \end{pmatrix} \right. \left. \beta_u^2 \left( \tilde{u}^\top (I - P_u) \tilde{u}^\top \\ 0 \beta_v^2 \right) \begin{pmatrix} u \\ v \end{pmatrix} \left. \beta_v^2 \left( \tilde{v}^\top (I - P_v) \tilde{v}^\top \right) \left[ C_{\tilde{u} \tilde{v}}^{-1} \begin{pmatrix} u^\top \\ v^\top \end{pmatrix} \right] \right] X(X^\top X)^{-1}
\]
\[
+ \sigma_y^2 \frac{1}{d^2 n} \left( X^\top X \right)^{-1} \begin{pmatrix} u & v \end{pmatrix} \beta_u^2 \left( \tilde{u}^\top (I - P_u) \tilde{u}^\top \\ 0 \beta_v^2 \right) \begin{pmatrix} u \\ v \end{pmatrix} \beta_v^2 \left( \tilde{v}^\top (I - P_v) \tilde{v}^\top \right) \left[ C_{\tilde{u} \tilde{v}}^{-1} \begin{pmatrix} u^\top \\ v^\top \end{pmatrix} \right] X(X^\top X)^{-1}
\]