Abstract

We examine the leading Regge string states relevant for semi-classical spinning string solutions. Using elementary RNS techniques, quadratic terms in an effective Lagrangian are constructed which describe massive NSNS strings in a space-time with five-form flux. We then examine the specific case of AdS$_5 \times S^5$, finding the dependence of AdS "energy" ($E_0$) on spin in AdS ($S$), spin on the sphere ($J$), and orbital angular momentum on the sphere ($\nabla_\alpha \nabla^\alpha$).
1 Introduction

Many of the most remarkable developments in M-theory may be traced to the study of various dualities. Although there is as yet no complete understanding of M-theory in its entirety, it is nevertheless clear that many aspects of the theory are very well captured through dualities of various sorts. In particular, the notion of AdS/CFT duality only achieves its full potential when connecting the various pieces—the open string worldsheet, D-brane worldvolume actions, supergravity $p$-brane solutions, low energy effective theories, etc.—into a complete whole. In this manner, AdS/CFT duality becomes more than a general statement on isometries, but can be applied to very specific cases such as the duality between $\mathcal{N} = 4$ super-Yang Mills theory and strings on $\text{AdS}_5 \times S^5$.

Even within this single context of $\mathcal{N} = 4$ super-Yang Mills, different methods have been applied to enlarge the regime of investigation. The gauge/gravity duality itself relates the 't Hooft coupling of the gauge theory, $\lambda \equiv g_{\text{YM}}^2 N$, to the string theory through $\sqrt{\lambda} = L^2 / \alpha'$ where $L$ is the 'radius' of $\text{AdS}_5$ (or equivalently $S^5$). In general, perturbative results on the gauge theory side may be trusted in the limit $\lambda \ll 1$, while the supergravity limit yields the opposite regime, $\lambda \gg 1$. Thus explicit tests of AdS/CFT in this context have been hindered by the strong/weak coupling nature of the duality. Fortunately, techniques have been devise to move beyond the supergravity limit, where long strings start playing an important rôole.

In particular, Berenstein, Maldacena and Nastase [1] were able to explore the large $R$-charge sector by taking a Penrose limit of $\text{AdS}_5 \times S^5$. The resulting expansion is then controlled by $\tilde{\lambda} = \lambda / J^2$ where $J^2 = J_1^2 + J_2^2 + J_3^2$ is the sum of the three commuting $R$-charges resulting from angular momentum on $S^5$. The resulting string admits an exact treatment in the Green-Schwarz formalism in the plane-wave background [2, 3].

At the same time, one may also explore the far from BPS sector of strings with large spin in $\text{AdS}_5$. This was demonstrated by Gubser, Klebanov and Polyakov [4], who took a semi-classical spinning string in $\text{AdS}_5$ on the leading Regge trajectory. For short strings (spin $S \ll \sqrt{\lambda}$), they obtained $E \approx ML$ where the closed string mass is given by $\alpha' M^2 = 2(S - 2)$. On the other hand, this relation turns over to $E \approx S + (\sqrt{\lambda} / \pi) \ln(S / \sqrt{\lambda})$ in the long string limit. Many subsequent investigations have led to refinements of these results as well as additional progress in obtaining anomalous dimension relations for operators carrying various spin quantum numbers in $\text{AdS}_5$ and $R$-charge on $S^5$ [5–14]. Much of the stringy analysis, however, have been classical or at most semi-classical in nature, as quantization of the string worldsheet is rather unpleasant in a RR background [15–18] (except in special cases such as the Green-Schwarz string in an plane-wave).

In this paper, we investigate the use of covariant worldsheet (RNS) methods to study highly excited string states on the leading Regge trajectory in the $\text{AdS}_5 \times S^5$ background. Our approach is to apply elementary techniques to first covariantly quantize the IIB string in a ten-dimensional flat background, and then to derive a Minkowski-space effective action governing massive string states up to a particular mass level $n$ given by $\alpha' M^2 = 4(n - 1)$. In this manner, we are in principle able to deduce the interactions of any particular massive state with the massless IIB fields (i.e. the metric and RR 5-form) as well as the other states in the massive tower. Although this effective action is constructed perturbatively about a flat background, we note that general covariance fixes its form, at least up to terms proportional to the equations of motion (which vanish on-shell in the Minkowski background). As a
result, this effective action also governs the interactions of massive string states with any background, and in particular the $\text{AdS}_5 \times S^5$ background with non-trivial metric and 5-form turned on. Furthermore, any undetermined terms proportional to the equation of motion computed in the Minkowski background will likewise vanish on the $\text{AdS}_5 \times S^5$ background (as both backgrounds satisfy identical equations of motion). In this manner, we are able to perturbatively investigate the behavior of massive string states (which, for simplicity, we take on the leading Regge trajectory) in $\text{AdS}_5 \times S^5$ using straightforward RNS worldsheet techniques.

In general, for an effective action to properly describe string states up to a particular mass level $n$, it must contain all fields up to and including the level of interest. From an effective field theory point of view, this corresponds to integrating out massive modes above a cutoff scale set by $n$. While this is in principle a straightforward task, the enormous number of string fields would turn this into a hopelessly long endeavor. Thus, in practice, we focus only on the propagation of a single massive field $\Phi$ on the leading Regge level interacting with the background. This leads to an effective (tree-level string) Lagrangian of the schematic form

$$e^{-1} \mathcal{L} = R - \frac{1}{2} \mathcal{F}^2 + \cdots - \frac{1}{2} \nabla_\mu \Phi \nabla^\mu \Phi - \frac{1}{2} M^2 \Phi^2 + \sum_{k \geq 1} (\alpha')^{k-1} \Phi \{ R, \mathcal{F}^{2(5)}, \nabla^2 \}^k \Phi + \cdots, \quad (1)$$

where we have only highlighted the terms of interest in the $\alpha'$ expansion. In general, of course, the string expansion yields an effective action expanded in both $\alpha'$ and $g_s$. However, we content ourselves with working in the large $N$ limit, corresponding to tree level on the string side of the duality.

By omitting cubic and higher interactions of $\Phi$ with lower mass fields in (1), we are essentially postulating that $\Phi$ is sufficiently stable so that it may be effectively treated as a single particle state as opposed to a broad resonance. In the underlying string picture, this requires that the closed string on the leading Regge trajectory has a sufficiently narrow width. At first, this appears a somewhat dangerous assumption, because of the large number of stringy decay products. However, many potential decays turn out to be exponentially suppressed, and in particular it was shown in Refs. [19, 20] that (at least in a Minkowski background) such leading Regge states of mass $M$ have a lifetime on the order $\tau = \mathcal{O}(\alpha' M)$. Thus highly massive states are in fact long lived, and it is therefore valid to only examine the particular massive state $\Phi$ without bringing in the entire tower of string states below mass $M$. Semiclassical decay of strings in an $\text{AdS}_5 \times S^5$ background were studied in [21], and the dual gauge theory phenomena in [22].

In the perturbative expansion of the effective Lagrangian (1), the higher $\alpha'$ terms correspond to the interaction of a string of finite extent with the background curvature and 5-form flux. From this point of view, we may estimate the size of the $k$-th order term as $(n \alpha' / L^2)^k$ where $n$ is the string level and $L$ is the radius of $\text{AdS}_5$ (or $S^5$). This estimate arises because the length of a highly spinning string essentially grows as $\sqrt{n \alpha'}$. As a result, validity of the perturbative expansion requires that $L^2 \gg n \alpha'$, or equivalently that $S \ll \sqrt{\lambda}$, corresponding to short strings spinning in $\text{AdS}_5$.

Technically we calculate only the first non-trivial interactions of the massive state $\Phi$ with the background. This requires a stringy 3-point computation in the NSNS sector to
obtain the $\Phi R\Phi$ interaction as well as a 4-point computation with two Ramond-emission vertex operators to obtain the $\Phi F^{2}_{(5)}\Phi$ term. Although the computations are not difficult, they are somewhat involved, especially when extracting the $\Phi F^{2}_{(5)}\Phi$ contact term from the 4-point function. Thus we have also performed a check on our results by taking the massless graviton limit $\Phi \to h_{\mu\nu}$ to verify that we reproduce the expected interactions in the massless sector.

At this order, we are able to extract the first correction to the flat-space Regge behavior. For a state on the leading Regge trajectory spinning entirely in AdS, we obtain $\alpha'\mu^2 = 2(S - 2) - \alpha'S^2/2L^2$, which is valid so long as the correction term is much less than the flat-space term $2(S - 2)$. (Here $\mu^2$ is related to mass in AdS, and will be properly defined below.) Measuring the energy in AdS units results in the short string Regge relation

$$E = 2 + \sqrt{\frac{2(S - 2)L^2}{\alpha'}} - \frac{S(S - 2)}{2} + 4 + \cdots \approx \sqrt{2\sqrt{\lambda}S \left( 1 - \frac{S}{8\sqrt{\lambda}} + \cdots \right)}, \quad (2)$$

where the second expression is valid in the semi-classical regime $1 \ll S \ll \sqrt{\lambda}$. The leading term is simply the flat-space Regge behavior of short spinning strings, while the correction at least suggests that the Regge trajectories flatten out in the limit of large spin. Of course, the actual $S \to \infty$ limit corresponds to the long string limit, which is unattainable by this perturbative method.

In the next section, we give an overview of the method of extracting the effective action $\mathcal{L}$ from the string $S$-matrix and in particular lay out the actual structure of the interaction Lagrangian. Following that, in section 3 we compute the $\Phi h_{\mu\nu}\Phi$ three-point function and in section 4 we work out the $\Phi F^{2}_{(5)} F^{2}_{(5)} \Phi$ four-point function. Technically, this is the most involved step of the calculation, as it requires pole subtractions to remove terms that may be factorized on three-point functions. We then work out the implications for short-string Regge behavior and conclude with a discussion of these results in section 5.

## 2 Extracting the effective action

In string theory, the basic objects of the underlying CFT are vertex operators, which encode the asymptotic states of the theory. With these, one may compute $S$-matrix elements corresponding to the scattering of on-shell string states. In computing $n$-point $S$-matrix elements, however, one is faced with divergences for certain values of external momenta. These are interpreted as the exchange of on shell degrees of freedom, with the divergences coming about from physical poles in the propagators $\sim 1/\mu^2$. However, String theory (in a first quantized approach) has no string creation or annihilation operators, and so defining a “string propagator” is not possible. Nevertheless, by considering all $n$-point string scattering amplitudes, one may construct an effective particle field theory description, with each vertex operator being associated with a field in the theory. The effective field theory may be encoded in a Lagrangian whose $S$-matrix elements correctly reproduce the string $S$-matrix. In this description, the pole terms can be calculated, and directly correspond to the divergences in the string $S$-matrix elements. In practice, of course, one only calculates the first few scattering amplitudes, and then appeals to further symmetry arguments to determine the effective Lagrangian to the desired order.
As indicated above, we explore massive states on the leading Regge trajectory in the NSNS sector of the IIB string. For a given mass level \( n \), these closed string states have mass \( \alpha' M^2 = 4(n - 1) \) and carry spin \( S = 2n \) (in the flat ten-dimensional background). For a closed bosonic string they would be generated by vertex operators of the form

\[
V(\zeta, k) = \zeta_{\mu_1 \cdots \mu_{2n}} \partial X^{\mu_1} \cdots \partial X^{\mu_n} \partial X^{\nu_1} \cdots \partial X^{\nu_{2n}} e^{ik \cdot X},
\]

where the polarization tensor \( \zeta_{\mu_1 \cdots \mu_{2n}} \) is transverse and trace-free. The actual superconformal vertex operators are only slightly more complicated, and will be presented below. In the following two sections, we compute the appropriate scattering amplitudes to find the effective field theory couplings to the background curvature and 5-form (which are the only massless IIB fields needed for the \( \text{AdS}_5 \times S^5 \) background).

To describe the effective Lagrangian, we may associate a field \( \Phi^{\mu_1 \cdots \mu_{2n}} \) to the above leading interacting massive higher spin Lagrangian for \( \Phi \), as there are well-known difficulties that must be overcome to avoid the presence of ghosts and other unphysical behavior. However, for our purposes, we will not require the complete Lagrangian, as in the end we only need to extract the on-shell behavior of \( \Phi \), and may arrange by hand to place it in an appropriate physical helicity state. Essentially, we never need to worry directly about the propagation of \( \Phi \), but are only interested in its static quantities such as spin and mass. Thus, to make \( \Phi \) correspond to a leading Regge field, we simply take it to be transverse and trace-free by assumption.

The effective Lagrangian that we wish to reconstruct is of the form \( \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_\Phi \) where

\[
e^{-1} \mathcal{L}_0 = R - \frac{1}{4!} F^2 - \cdots
\]

is the usual IIB supergravity Lagrangian (where 5-form self-duality must be imposed by hand after obtaining the equations of motion). For \( \mathcal{L}_\Phi \), we only consider its coupling to the 5-form and gravity. Thus we may write down a diffeomorphism invariant Lagrangian for \( \Phi \) of the form

\[
e^{-1} \mathcal{L}_\Phi = \frac{1}{4 \kappa^2_1} \left[ \frac{1}{2} \Phi^{\mu_1 \cdots \mu_{2n}} \nabla_\lambda \nabla^\lambda \Phi^{\mu_1 \cdots \mu_{2n}} - \frac{1}{2} M_\Phi^2 \Phi^{\mu_1 \cdots \mu_{2n}} \Phi^{\nu_1 \cdots \nu_{2n}} \right.
\]

\[
+ \frac{1}{2} \alpha(n) \Phi^{\mu_1 \cdots \mu_{2n}} R_{\mu_1 \nu_1 \mu_2 \nu_2} \Phi^{\nu_1 \nu_2} \mu_3 \cdots \mu_{2n} + \frac{1}{2} \beta(n) \Phi^{\mu_1 \cdots \mu_{2n}} R^{\mu_1 \nu_1 \mu_2 \nu_2} \Phi^{\mu_3 \cdots \mu_{2n}}
\]

\[
+ \frac{1}{2} \gamma(n) \Phi^{\mu_1 \cdots \mu_{2n}} R^{\mu_1 \cdots \mu_{2n}} R \Phi^{\mu_1 \cdots \mu_{2n}}
\]

\[
+ \frac{1}{2} \delta(n) \Phi^{\mu_1 \cdots \mu_{2n}} F_{\mu_1 \nu_1 \cdots \alpha_1} F_{\nu_1 \cdots \alpha_2} \Phi^{\mu_2 \cdots \mu_{2n}}
\]

\[
+ \frac{1}{2} \epsilon(n) \Phi^{\mu_1 \cdots \mu_{2n}} F_{\mu_1 \cdots \alpha_1} F_{\nu_1 \cdots \alpha_2} \Phi^{\mu_2 \cdots \mu_{2n}}
\]

\[
+ \frac{1}{2} \theta(n) \Phi^{\mu_1 \cdots \mu_{2n}} F^{\alpha_1 \cdots \alpha_5} F_{\alpha_1 \cdots \alpha_5} \Phi^{\mu_1 \cdots \mu_{2n}} + \cdots \right].
\]

Once again, we do not claim this to be a complete description for a massive higher spin field \( \Phi \). All that we require is that it reproduces the equation of motion

\[
[\nabla_\lambda \nabla^\lambda - M_\Phi^2 + \cdots] \Phi^{\mu_1 \cdots \mu_{2n}} = 0,
\]

valid when we impose transverse-tracelessness on \( \Phi \). In particular, the equation of motion is always unambiguous, even in a curved background.
Our goal is to extract the coefficients $\alpha(n), \ldots, \theta(n)$ by comparison with the string $S$-matrix. To do so, we make use of the $\Phi-h_{\mu\nu}$-$\Phi$ three point scattering amplitude as well as the $\Phi-F_{(5)}-F_{(5)}-\Phi$ four point scattering amplitude. At the three-point function level, we will see no contributions that will allow us to determine the coupling constants $\beta(n)$ and $\gamma(n)$ as they vanish on-shell and do not contribute to the scattering process. Furthermore, the $\theta(n)$ term (which is included because we only impose self-duality on $F_{(5)}$ after obtaining the equations of motion) will also vanish once the on-shell conditions are imposed.

In any case, after obtaining the effective Lagrangian (5), we will only use it to study the propagation of massive modes on a background that solves the classical equations of motion for the massless fields. There, whether in $\text{AdS}_5 \times S^5$ or the Minkowski background, both the $\theta(n)$ and $\gamma(n)$ terms are unimportant, and hence they will be omitted in the remainder of this work. The $\beta(n)$ term may at first appear troubling since $R_{\mu\nu}$ is non-vanishing on the $\text{AdS}_5 \times S^5$ background, but in fact it also conveniently cancels out on the full equations of motion. To see this, we first note that the $\beta(n)$ term does not contribute to a three point scattering process because $R_{\mu\nu} \propto \partial^2 h_{\mu\nu}$ when expanded about flat space, and so the external graviton leg, which is on shell, gives $\partial^2 h_{\mu\nu} = 0$. However, this three point function will contribute to the four point $S$-matrix $\Phi-F_{(5)}-F_{(5)}-\Phi$ through a single graviton exchange diagram obtained by combining the $\Phi-\partial^2 h_{\mu\nu}-\Phi$ and $F_{(5)}-h_{\mu\nu}-F_{(5)}$ three point functions with a graviton propagator. The propagator and $\partial^2$ from the graviton vertex cancel, leading to a non-pole contribution to the four-point function which must be canceled by the $\epsilon(n)$ contact term in (5). The result is that $\epsilon(n)$ must be related to $\beta(n)$ in just the combination

$$\frac{1}{2} \beta(n) \Phi^{\mu_1\mu_2 \ldots \mu_{2n}} \Phi_{\mu_2 \ldots \mu_{2n}} \left( R_{\mu_1\nu_1} - \frac{1}{4} \frac{1}{4!} F_{\mu_1\alpha_2 \ldots \alpha_5} F_{\nu_1 \alpha_2 \ldots \alpha_5} \right),$$

which vanishes on the Einstein equation of motion. This renders the term unimportant for our purposes.

Given the effective Lagrangian (5), we may then consider its effect on the propagation of $\Phi$ in the Freund-Rubin background. At this quadratic order in $\Phi$, interactions with the background curvature simply shift the effective mass of $\Phi$ away from its flat ten-dimensional value. In this manner, we may explore the short-string correction to the flat-space Regge behavior $m \sim \sqrt{S}$. Interestingly, this shift will depend on the orientation of the ten-dimensional spin of the string. For example, in a maximally symmetric space, $R_{\mu\nu\rho\sigma} = \pm L^{-2} (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho})$ where $L$ is the radius. Inserting this in (5), and assuming an $\text{AdS}_5 \times S^5$ background, different components of $\Phi$ will get opposite sign contributions to their effective mass terms, depending on whether the spin indices are in the $\text{AdS}_5$ or $S^5$ directions.

One may of course worry about the validity of the above Lagrangian, and think that other higher point interactions are as important as the ones we have calculated. We will address this issue below. However, the end result is that the regime of validity for our calculations is $n \alpha' \ll L^2$ where $\alpha' M_\phi^2 = 4(n - 1)$ is the flat space mass squared of the $\Phi$ particle, and $L$ is the radius of $\text{AdS}_5$. In this way, the perturbative results are valid in the large radius limit, where the string theory should be well approximated by the flat space case.
3 The $\Phi-h_{\mu\nu}-\Phi$ three-point function

We begin with the simpler of the two scattering computations, namely the $\Phi-h_{\mu\nu}-\Phi$ three-point function. This interaction was previously worked out in [23] when obtaining the corrections to the gravitational quadrupole moment of massive string states. Hence we will be suitably brief, although we do present some of our string normalization conventions with care.

3.1 The string SCFT calculation

We use the standard procedure of computing closed string scattering amplitudes by first working with one side of the string at a time, and then “doubling” the calculation to obtain the closed string result. Working in a convenient picture, we use the vertex operators for one side of the string given by

\begin{align}
V^{(n)}_{\{-\mu_1\ldots\mu_n\}} &= \left(i\sqrt{\frac{2}{\alpha'}}\right)^{(n-1)} \zeta_{\mu_1\ldots\mu_n} e^{-\phi} \phi^{\mu_1} \partial X^{\mu_2} \ldots \partial X^{\mu_n} e^{ik \cdot X}, \\
V^{(m)}_{\{\nu_1\ldots\nu_m\}} &= \xi_{\nu_1} i \left(\sqrt{\frac{2}{\alpha'}} \partial X^{\nu_1} + i \sqrt{\frac{\alpha'}{2}} \psi^{\nu_1} k \cdot \psi\right) e^{ik \cdot X},
\end{align}

where we use $\alpha'/2$ rather than $2\alpha'$ because we will only be using our results for closed string amplitudes. The polarization tensors are symmetric, transverse and trace free:

\begin{align}
\zeta_{\mu_1\ldots\mu_i\ldots\mu_n} &= \zeta_{\mu_1\ldots\mu_j\ldots\mu_i\ldots\mu_n}, \\
k^{\mu_1} \zeta_{\mu_1\ldots\mu_i\ldots\mu_n} &= 0, \\
\eta^{\mu_1} \zeta_{\mu_1\ldots\mu_i\ldots\mu_j\ldots\mu_n} &= 0,
\end{align}

and satisfy the closed string mass shell condition

\begin{equation}
-k^2 = M^2 = \frac{2(n-1)}{\alpha'/2}.
\end{equation}

On a single side of the closed string, the three-point function is of the form

\begin{equation}
\mathcal{A} = g_c g' c' (V^{\{\alpha_1\}} \langle V^{\{\mu_1\}}(z_1)V^{\{\nu_1\}}(z_2) V^{\{\rho_1\}}(z_3)\rangle),
\end{equation}

and in general contains terms up to high order in $\alpha'$. However, we are only interested in the leading order in $\alpha'$. In this case, one must contract the $\partial X$ operators together and avoid contractions with exponentials. The reason it is consistent to ignore the higher order terms is because contractions yielding such terms will bring down factors of momentum from one massive and one massless vertex operator. Since this is all performed on shell, momentum conservation and transversality always allows us to replace the massive momentum with a light one (i.e. that of the graviton). As a result, $\sqrt{\alpha' k}$ where $k$ is the graviton momentum serves as the expansion parameter; for curvatures on the scale of $1/L$, this is equivalent to expanding in $\lambda^{-1/4}$. Dropping the higher order terms then corresponds to scattering in the low energy limit.
Paying close attention to the various factors that arise in the calculation, we recall that closed string scattering is conventionally normalized by an overall factor of

$$\frac{8\pi}{g_c^2\alpha'} = \frac{4\pi}{g_c^2} \sqrt{\frac{2}{\alpha'}} \sqrt{\frac{2}{\alpha'}}$$

We find it convenient to absorb one factor of $\sqrt{2/\alpha'}$ into each side of the calculation, leaving only the more geometric normalization $4\pi/g_c^2$ as an overall factor. In this case, the one-sided three point amplitude breaks up into two pieces, $A = A_1 + A_2$, each of which factors into bosonic, fermionic and ghost contributions

$$A_1 = i \left( \frac{2}{\alpha'} \right)^n \langle \partial X^\alpha_1 e^{ik_1 \cdot X(z_1)} \partial X^\alpha_2 \cdots \partial X^\alpha_n e^{ik_n \cdot X(z_n)} \rangle$$

$$A_2 = -i \left( \frac{2}{\alpha'} \right)^{n-1} \langle e^{ik \cdot X(z_1)} \partial X^\alpha_2 \cdots \partial X^\alpha_n e^{ik_n \cdot X(z_n)} \rangle$$

In the $A_2$ term, one may contract all of the $\partial X$ operators with each other, and so get the leading $\alpha'$ behavior relatively easily. For the first term, on the other hand, one must contract at least one of the $\partial X$ operators with an exponential, thus bringing down a momentum factor. Thus to leading order in $\alpha'$, the resulting amplitude is linear in momenta, and ends up having the form

$$A = \frac{1}{2} n! \left( k_{23}^{\alpha_1} \eta_4^{\mu_1 \nu_1} + n k_{12}^{\nu_1} \eta_4^{\mu_1 \alpha_1} + n k_{31}^{\alpha_1} \eta_4^{\nu_1 \alpha_1} \right) \prod_{i=2}^n \eta_i^{\mu_i \nu_i},$$

where we have introduced the notation $k_{ij} = k_i - k_j$, and additionally dropped the overall momentum conserving delta function. Note that, although we have stripped the polarization tensors, reserving these for the closed string amplitude, the above indices must be taken fully symmetric, $(\mu_1 \cdots \mu_n)$ and $(\nu_1 \cdots \nu_n)$. This expression reduces to the well known trilinear open string gauge boson coupling for the case of $n = 1$. In general, there are of course higher order in momentum terms of the form $\alpha' k^2$ in the three-point amplitudes. However, any heavy momenta can be traded off for a light one through momentum conservation and transversality of the polarization, leading only to contributions of higher order in $\alpha'/L^2$, which we may consistently drop.

To convert the above into a closed string amplitude, we “square” the open string result and include a fully-symmetric transverse traceless polarization tensor with $2n$ indices ($n$ on each side). Including all factors of $g_c$, as well as the overall normalization, yields the three-point amplitude with the graviton

$$A^{\Phi-h_{\mu\nu}-\Phi} = i(2\pi)^{10} \delta^{10} (\Sigma k_i) \left( \frac{1}{2 \cdot 2 \cdot (2\pi)^2 g_c^2} \right)$$

$$\times \left( 4\pi g_c \Sigma_{\alpha_1 \alpha_2} \right) \left( 4\pi g_c (n-1)! \Sigma_{\mu_1 \cdots \mu_{2n}} \right) \left( 4\pi g_c (n-1)! \Sigma_{\nu_1 \cdots \nu_{2n}} \right) \prod_{i=3}^{2n} \eta_i^{\mu_i \nu_i}$$

$$\times \frac{1}{2} \left( k_{23}^{\alpha_1} \eta^{\mu_1 \nu_1} + n k_{12}^{\nu_1} \eta^{\mu_1 \alpha_1} + n k_{31}^{\alpha_1} \eta^{\nu_1 \alpha_1} \right) \frac{1}{2} \left( k_{23}^{\alpha_2} \eta^{\mu_2 \nu_2} + n k_{12}^{\nu_2} \eta^{\alpha_2 \nu_2} + n k_{31}^{\alpha_2} \eta^{\alpha_2 \nu_2} \right),$$

(16)
where the corresponding polarizations and momenta are labeled by 1, 2 and 3 for the graviton, first, and second heavy field, respectively. We have also grouped in canonical factors of $g_c$ and $\pi$ along with the polarization tensors; this association corresponds to the canonical normalization used in the effective Lagrangian, below. We also note here that the factors of $(n - 1)!$ are natural because one may determine $g'_c$ in terms of $g_c$ from unitarity and the operator/state isomorphism, giving the relation $g'_c = g_c/(n - 1)!$.

### 3.2 The effective Lagrangian

We now seek the structure of the effective Lagrangian which reproduces the three-point scattering amplitude, $\mathcal{L}_3$. Expanding the last line of (17), while making use of the symmetries and transversality of the polarization tensors, we find

$$
\frac{1}{2} (k_3^{\alpha_1} \eta^{\mu_1 \nu_1} + n k_1^{\mu_1} \phi^{\alpha_1} + n k_3^{\alpha_1}) \frac{1}{2} (k_3^{\alpha_2} \eta^{\mu_2 \nu_2} + n k_1^{\mu_2} \phi^{\alpha_2} + n k_3^{\alpha_2}) 
\cong k_3^{\alpha_1} k_3^{\alpha_2} \eta^{\mu_1 \nu_1} \phi^{\mu_2 \nu_2} + 2 n k_1^{\mu_1} k_3^{\alpha_1} \eta^{\alpha_2} \phi^{\mu_2} - 2 n k_1^{\nu_1} k_3^{\alpha_1} \eta^{\alpha_2} \phi^{\nu_2} - 2 n k_3^{\mu_1} \eta^{\alpha_1} \phi^{\alpha_2} \phi^{\mu_2} 
- 2 n M^2 \eta^{\alpha_1} \phi^{\alpha_2} \phi^{\mu_2} 
+ n^2 k_3^{\alpha_1} k_1^{\alpha_2} (\eta^{\nu_1} \phi^{\mu_2} - \eta^{\mu_1} \phi^{\nu_2}) (\eta^{\nu_2} \phi^{\mu_2} - \eta^{\mu_2} \phi^{\nu_2}),
$$

where we have made use of $k_3^2 = -M^2$ to introduce the mass term. Here, we use $\cong$ to indicate the expressions are equal when the polarizations are included. This may then be identified with the terms in an effective Lagrangian (for $n > 1$)

$$
e^{-1} \mathcal{L}_\Phi = \frac{1}{4 \kappa_{10}^2} \left[ \frac{1}{2} \Phi_{\mu_1 \ldots \mu_2n} \nabla_\lambda \nabla^\lambda \Phi_{\mu_1 \ldots \mu_2n} - \frac{1}{2} M^2 \Phi_{\mu_1 \ldots \mu_2n} \Phi_{\mu_1 \ldots \mu_2n} 
+ n^2 \Phi_{\mu_1 \ldots \mu_2n} R^{\mu_1 \nu_1 \mu_2 \nu_2 \mu_3 \ldots \mu_2n} + \beta(n) \Phi_{\mu_1 \ldots \mu_2n} R^{\mu_1 \nu_1} \Phi_{\nu_1 \mu_2 \ldots \mu_2n} + \ldots \right],
$$

where we have identified $\kappa_{10} = 2\pi g_c$ as usual. The term proportional to the Ricci tensor is undetermined by on-shell scattering, and is included with an undetermined coefficient $\beta(n)$. Note that the three lines on the right-hand side of (17) correspond directly to the first three terms in the effective Lagrangian.

The $n = 1$ case is different because now all three particles are indistinguishable. In fact, this is just the 3 graviton scattering, and is given by the expansion of the Einstein-Hilbert action to third order

$$
\frac{1}{2 \kappa_{10}^2} \int d^{10}x \sqrt{-g} R \bigg|_{3^{rd \text{order}}} = 
- \frac{1}{2 \kappa_{10}^2} \int d^{10}x \frac{1}{4} \left( h^{\mu_1 \nu_1} h^{\mu_2 \nu_2} \partial_{\mu_1} \partial_{\mu_2} h_{\nu_1 \nu_2} + 2 \partial_{\nu_2} h^{\mu_1 \mu_2} \partial_{\mu_2} h_{\mu_1 \nu_1} h^{\nu_1 \nu_2} + h^{\mu_1 \lambda} h^{\mu_2} \partial_{\sigma} \partial_{\mu_1} h_{\mu_2} \right),
$$

where we have taken $g_{\mu \nu} = \eta_{\mu \nu} + h_{\mu \nu}$ and furthermore assumed that the graviton is transverse and traceless. The last term in the above action gives no contribution to the on-shell 3-point scattering amplitude. It is, however, important in determining the contact term extracted from the four point scattering, as will be explained below.
4 The \(\Phi - F_{(5)} - F_{(5)} - \Phi\) four-point function

We now move on to the interactions of the higher spin field \(\Phi\) with the Ramond-Ramond background. Due to the underlying fermionic nature of the Ramond-Ramond states, they only enter pairwise in interactions with the NSNS field \(\Phi\). As a result, the first non-trivial interaction is at the four-point level. Here, an important new feature arises in that the four-point amplitude factorizes on \(s, t\) and \(u\) channel poles, and that these underlying three-point terms must be removed. This, along with field redefinition ambiguities, somewhat complicate the calculation, although the methods are standard, at least since the work of [24,25], which computed the \(R^4\) corrections from closed strings.

4.1 The SCFT calculation

For the four-point function, we will again make use of the massive vertex operators for \(\Phi\) in the \(-1\) picture given by (8). The Ramond-Ramond five-form vertex is composed of fermion emission vertex operators on each side of the string. On a single side, we make use of both 1/2 and \(-1/2\) pictures. In this case, the relevant vertex operators are

\[
\begin{align*}
V_{-1/2} &= u_{\alpha} S_{-1/2} e^{ik \cdot X}, \\
V_{1/2} &= \sqrt{\frac{1}{2}} \sqrt{\frac{2}{\alpha'}} \left[ i \partial X^\mu + \frac{1}{4} \alpha' k \cdot \psi \psi^\mu \right] u_{\dot{\alpha}} \Gamma_{\mu \beta} S_{1/2} e^{ik \cdot X},
\end{align*}
\]

where the subscripts on \(S_q\) and \(\psi_q\) specify the ghost charge, \(e^{q \phi}\). On a single side, the four-point amplitude given by

\[
\mathcal{A} = g_0 g_1 g_2 g_3 \langle V_{1/2}(z_1)V_{-1}(z_2)V_{1/2}(z_3)V_{-1}(z_4) \rangle.
\]

We again find it useful to push one factor of \(\sqrt{2/\alpha'}\) from the normalization into the calculation for left movers, and again calculate only to leading order in \(\alpha'\).

This expansion proceeds in much the same way as the last section, noting again that heavy momenta may only be brought down by \(\partial X\) operators from the light vertex operators. The particulars of the calculation, while straightforward, are somewhat tedious. One performs the \(\partial X\) correlators in much the same way as the previous calculation. To find the contribution from the \(\psi\) CFT, one may use the \(SO(10)\) current algebra \(\psi^\mu \psi^\nu \sim J^{\mu \nu}\), applying it to the \(\psi\) and \(S\) operators, which transform as the \(10\) and \(16\) respectively. This reduces all \(\psi\) CFT amplitudes to the form \(\langle S(z_1)\psi_{-1}(z_2)\psi_{-1}(z_3)S(z_4) \rangle\), which are known from scattering of massless string states. Using these results gives the following amplitude for the left-movers:

\[
\mathcal{A} = \zeta_{\mu_1 \cdots \mu_n}^2 \zeta_{\nu_1 \cdots \nu_n}^3 u_{\alpha}^1 u_{\dot{\alpha}}^1 a^{\frac{(n-1)!}{\sqrt{2}}} (-1)^n \prod_{i=2}^n \eta^{\mu_{i-1}} \eta^{\nu_i}
\times (1-z_3)^{-t/2-1} (z_3)^{-s/2+n-2} \left[ (1-z_3) A_{(k)}^{\mu_{k-1}} + z_3 B_{(k)}^{\mu_{k-1}} \right]^{\dot{\alpha}} \beta ,
\]

where we have taken \(z_1 \to \infty, z_2 = 1, z_4 \to 0, k = k_3 + k_4\), and again \(k_{ij} = k_i - k_j\). We will leave the \(z_3\) integration until after including the right-movers. We have defined a unitless set of Mandelstam variables

\[
\begin{align*}
s &= -\frac{\alpha'}{2} (k_1 + k_2)^2, \\
t &= -\frac{\alpha'}{2} (k_2 + k_3)^2, \\
u &= -\frac{\alpha'}{2} (k_1 + k_3)^2.
\end{align*}
\]
and have introduced the kinematical coefficients

\[
A_{(k)}^{\mu_1 \nu_1} = -\frac{1}{2} (k^\lambda) (\Gamma_{\mu_1} \Gamma_{\nu_1} - (n-1) (\Gamma_{\nu_1} k_{\mu_1} + \Gamma_{\mu_1} k_{\nu_1})),
\]

\[
B_{(k_{23})}^{\mu_1 \nu_1} = \frac{1}{2} (2\Gamma_{\mu_1} k_{23} + 2\Gamma_{\nu_1} k_{23} - k_{23} \cdot \Gamma_{\mu_1} \Gamma_{\nu_1}) + (n-1) (\Gamma_{\nu_1} k_{23} + \Gamma_{\mu_1} k_{23}),
\]

(24)

to write the scattering amplitude more succinctly.

For doubling the CFT calculation into a closed string amplitude, one must take a transpose of the Dirac matrices coming from the right movers. This is accounted for by switching the order of the \( \mu \) and \( \nu \) in the above coefficients. The NS polarizations are changed over to closed string polarizations in the same way as the last calculation. For the Ramond vertex operators, on the other hand, we switch to closed string polarizations by replacing spinor bilinears \( u \bar{u} \) with \( f_{\{\alpha_1\}} \Gamma_{(5)}^{\{\alpha_5\}} \) to obtain \( F_{(5)} \) polarization states. Doubling the calculation in this way gives the closed string amplitude

\[
A^{\Phi-F-F-\Phi} = i(2\pi)^{10} \delta (\Sigma k_i) \left( g'_c (n-1) ! \right)^2 \frac{(4\pi)}{2} \left( \zeta_2^{\mu_1-\mu_2} \right) \left( \zeta_3^{3 \mu_1-3 \mu_2} \right) \left( \hat{f}_1^{\nu_1-\nu_2} \right) \left( \hat{f}_1^{\nu_1-\nu_2} \right) \prod_{i=3}^{2n} \eta_{\mu_i \nu_i}
\]

\[
\times \left[ \Gamma (\frac{a}{2}) \Gamma (\frac{a - n - 1}{2}) \Gamma (\frac{b}{2} + n - 1) \Gamma (1 - (\frac{b}{2} + n - 1)) \Gamma (1 - \frac{c}{2}) \right] \times \text{Tr} \left[ \Gamma^{\alpha_1 \cdots \alpha_5} \left( A^{\nu_1 \mu_2} - B^{\nu_2 \mu_2} \right) \Gamma^{\beta_1 \cdots \beta_5} \left( \frac{t}{2} A_{(k)}^{\mu_1} + \left( -\frac{a}{2} + n - 1 \right) B_{(k_{23})}^{\mu_1 \nu_1} \right) \right].
\]

(25)

Again, we are concerned with the limit where the massless fields have zero momentum. In this limit, the Mandelstam variables go to their minimum physical values. This allows us to see the pole structure arising from the exchange of on shell intermediate particles, and for doing so we take the convenient variables

\[
a = \frac{s}{2} - (n-1), \quad b = \frac{t}{2}, \quad c = \frac{u}{2} - (n-1),
\]

(26)

which satisfy \( a + b + c = 0 \). In the limit of low energy scattering \( \{a, b, c\} \to 0 \). After some simple manipulation, we that in this limit the scattering completely factorizes

\[
A^{\Phi-F-F-\Phi} = i(2\pi)^{10} \delta (\Sigma k_i) \left( g'_c (n-1) ! \right)^2 \frac{(4\pi)}{2} \left( \zeta_2^{\mu_1-\mu_2} \right) \left( \zeta_3^{3 \mu_1-3 \mu_2} \right) \left( \hat{f}_1^{\nu_1-\nu_2} \right) \left( \hat{f}_1^{\nu_1-\nu_2} \right) \prod_{i=3}^{2n} \eta_{\mu_i \nu_i}
\]

\[
\times \text{Tr} \left[ \Gamma^{\alpha_1 \cdots \alpha_5} A_{(k)}^{\mu_1 \nu_1} \Gamma^{\beta_1 \cdots \beta_5} A_{(k_{23})}^{\nu_2 \mu_2} + \Gamma^{\alpha_1 \cdots \alpha_5} B_{(k_{23})}^{\mu_1 \nu_1} \Gamma^{\beta_1 \cdots \beta_5} B_{(k_{23})}^{\nu_2 \mu_2} + \Gamma^{\alpha_1 \cdots \alpha_5} \left( A_{(k)}^{\mu_1 \nu_1} - B_{(k_{23})}^{\mu_1 \nu_1} \right) \Gamma^{\beta_1 \cdots \beta_5} \left( A_{(k_{23})}^{\nu_2 \mu_2} - B_{(k_{23})}^{\nu_2 \mu_2} \right) \right].
\]

(27)
We have written the scattering in this form because \( A_{(k_3 + k_4)}^\mu - B_{(k_3)}^\mu = A_{(k_2 + k_4)}^\mu \). This demonstrates explicitly that the first and third terms map into each other by \( s \leftrightarrow u \) interchange, namely \( 2 \leftrightarrow 3 \), and \( \mu \leftrightarrow \nu \), along with \( a \leftrightarrow c \). The \( t \)-channel term (with the \( b \) denominator) is distinct, however.

In field theory, the tree level four-point scattering amplitude arises as a sum of terms

\[
\begin{array}{c}
\text{Term 1} \\
\text{Term 2} \\
\text{Term 3} \\
\text{Term 4} \\
\text{Term 5}
\end{array}
\]

where we use curly lines for \( \Phi \), dotted lines for \( h_{\mu\nu} \), solid lines for \( F^{(5)} \) and hatched lines to be massive RR fields. (In the massless case, the curly lines become dotted lines, and the hatches are removed.) We wish to extract the contact term which comes directly from the effective Lagrangian, represented by the term with no intermediate particles. To do so, we take the \( S \)-matrix element calculated in string theory, and subtract off the “pole” terms implied by the three point functions. This should in principle completely remove the divergent terms from \( (27) \).

### 4.2 Contact terms from the \( s \) and \( u \) channels

To properly obtain the four-point contact term from the \( S \)-matrix, one should subtract scattering corresponding to the exchange of all intermediate particles present in the effective Lagrangian. This will in principle cancel the poles in the above scattering amplitude, \( (27) \). Unfortunately, this proves to be a difficult calculation for the \( s \) and \( u \) channels, primarily because the propagator for off-shell higher mass RR fields are difficult to write down. Because of this, we use an alternate method to isolate the contact terms from these poles.

Basically, we assume that the field \( \Phi \) does not couple via a \( \Phi F^{(5)} \Box f_m \) term, where \( f_m \) denotes a massive RR field, and \( \Box \) indicates the equation of motion for \( f_m \). Given this, the underlying single particle exchange diagrams in the \( s \) and \( u \) channel will not lead to constant (non-residue) terms. As a result, the \( s \) and \( u \) channel contributions to the contact term are fully contained in the constant parts of the corresponding terms in \( (27) \). (Furthermore, the residues of the \( 1/a \) and \( 1/c \) poles must necessarily be canceled, but this is what we do not check.) For such a constant term, there must be a momentum squared in the numerator to cancel the denominator. Such a term occurs in the \( s \) channel pole because \( A_{(k)}^\mu \) contains a \( k \cdot \Gamma \) piece. The \( u \) channel piece is identical to the \( s \) channel, simply switching \( \mu \leftrightarrow \nu \).

Our procedure to identify the contact terms is therefore as follows. First, we identify terms in the \( s \) channel that are proportional to \( k^2/2 \) in the numerator, and expect that this \( k \) dependence cancels the denominator. This process introduces a factor of \( \alpha'/2 \) because
$k^2$ has units, while $a$ does not. This is the usual $\alpha'$ dependence that is included in the normalization of the $\hat{f}$ polarization tensors, $\hat{f} = \sqrt{\alpha'/2} f$. Taking this as the contact piece in the $s$ channel, one may include the $u$ channel by adding the $\mu \leftrightarrow \nu$ contribution by hand. This process gives the contact terms to be

$$A_{s,u}^{4pt} = i(2\pi)^{10} \delta(\Sigma k_i) \frac{4\pi^2}{2} (g_c'(n-1)!)^2 \left( \zeta^2_{\mu_1 \cdots \mu_2} \right) \left( \zeta^3_{\nu_1 \cdots \nu_2} \right) (f^1_{\alpha_1 \cdots \alpha_5})(f^4_{\beta_1 \cdots \beta_5}) \prod_{i=3}^{2n} \eta_{\mu_1 \nu_i}$$

$$\times \text{Tr} \left[ \Gamma^{\alpha_1 \cdots \alpha_5} \frac{1}{2}(k^\lambda)(\Gamma^{\mu_1} \Gamma^{\nu_1}) \Gamma^{\beta_1 \cdots \beta_5} \frac{1}{2}(k^\lambda)(\Gamma^{\nu_2} \Gamma^{\mu_2}) + \mu \leftrightarrow \nu \right]^{k^2/2 \text{ factor}}$$

$$= i(2\pi)^{10} \delta(\Sigma k_i) \frac{4\pi^2}{2} (g_c'(n-1)!)^2 \left( \zeta^2_{\mu_1 \cdots \mu_2} \right) \left( \zeta^3_{\nu_1 \cdots \nu_2} \right) (f^1_{\alpha_1 \cdots \alpha_5})(f^4_{\beta_1 \cdots \beta_5}) \prod_{i=3}^{2n} \eta_{\mu_1 \nu_i}$$

$$\times \text{Tr} \left[ (\Gamma^{\mu_2} \Gamma^{\alpha_1 \cdots \alpha_5} \Gamma^{\mu_1}) (\Gamma^{\nu_1} \Gamma^{\beta_1 \cdots \beta_5} \Gamma^{\nu_2}) + \mu \leftrightarrow \nu \right]. \quad (29)$$

We take the above trace, remembering that certain terms vanish because of self-duality of the five form, and that many terms are equivalent because of the symmetries of the polarization tensors. Our final result for the scattering due to the contact piece is given by

$$A_{s,u}^{4pt} = i(2\pi)^{10} \delta(\Sigma k_i) \frac{1}{2(2\pi g_c)^2 \cdot 4 \cdot 5!} \left( (4\pi g_c'(n-1)!) \zeta^2_{\mu_1 \cdots \mu_2} \right) \left( (4\pi g_c'(n-1)!) \zeta^3_{\nu_1 \cdots \nu_2} \right)$$

$$\times \left( 4\pi g_c \cdot \sqrt{32} \cdot 5! f^1_{\alpha_1 \cdots \alpha_5} \right) \left( 4\pi g_c \cdot \sqrt{32} \cdot 5! f^4_{\beta_1 \cdots \beta_5} \right) \prod_{i=3}^{2n} \eta_{\mu_1 \nu_i} \prod_{j=3}^{5} \eta_{\alpha_i \beta_j}$$

$$\times \left[ 5\eta_{\mu_1 \alpha_1} \eta_{\beta_1 \alpha_2} \eta_{\nu_1 \beta_2} \eta_{\nu_2 \alpha_2} + 5\eta_{\nu_1 \alpha_1} \eta_{\mu_1 \beta_2} \eta_{\nu_2 \beta_2} \eta_{\mu_2 \alpha_2} - 40\eta_{\nu_1 \alpha_1} \eta_{\mu_1 \beta_1} \eta_{\nu_2 \beta_2} \eta_{\mu_2 \alpha_2} \right]. \quad (30)$$

We will take this as the $s$ and $u$ channel contributions to contact terms. This gives the correct result for the massless case where the subtraction can be done, the intermediate field being the massless five form.

### 4.3 Contact term from the $t$ channel

As mentioned above, the difficulty in addressing the exact $s$ or $u$ channel subtractions is a result of not knowing massive higher spin RR propagators. However, in the $t$ channel, the only intermediate particle allowed in the low energy limit is the graviton. The dilaton does not enter because of self duality of the five-form. The fact that this subtraction can be done leads to a cancellation of the $\beta(n)$ term in our final answer, as we will show in the following.

First, the field theoretic quantities needed are the graviton propagator, the Feynman rule for the $\Phi-h_{\mu\nu}-\Phi$ scattering, and the Feynman rule for the $F_{(5)}-h_{\mu\nu}-F_{(5)}$ scattering. The graviton propagator is

$$\langle h^{\mu\nu}(p) h_{\rho\sigma}(q) \rangle = \Delta^{\mu\nu}_{\rho\sigma}(p,q)$$

$$= -i(2\pi)^{10} \delta^{10}(p + q) \frac{4\kappa_{10}^2}{p^2} \frac{1}{2} \left( P^\mu P^\nu + P^\mu P^\nu - \frac{2}{9} P^{\mu\nu} P_{\rho\sigma} \right), \quad (31)$$

where we have used

$$P^\mu(p) \equiv \left( \eta^\mu_\rho - \frac{p^\mu p_\rho}{p^2} \right), \quad (32)$$
so that $\Delta$ satisfies the transverse traceless conditions. We are evaluating these between conserved currents, one of which is traceless ($F_{(5)}^3 = 0$), so we are left using only

$$\langle h_{\mu\nu}(p) h_{\rho\sigma}(q) \rangle = -i (2\pi)^{10} \delta^{10} (p + q) \frac{4\kappa_0^2}{p^2} \frac{1}{2} \left( \eta_\mu^\nu \eta_\sigma^\rho + \eta_\mu^\rho \eta_\sigma^\nu + \cdots \right). \quad (33)$$

The Feynman rule for $\Phi_\mu^2 h_{\alpha_1 \alpha_2}^{-(2+3)} \Phi_\nu^3$ scattering can be read off from (18),

$$\Gamma = \frac{i}{4(\kappa_0)^2} \prod_{\alpha=3}^{2n} \eta_{\alpha_1 \alpha_2} \left[ \beta(n)(k_2 + k_3)^2 \eta_{\mu_1 \mu_2} (\eta_{\mu_1 \alpha_1} \eta_{\nu_1 \alpha_2}) + \frac{1}{2} (k_{23} \eta_{\mu_1 \nu_1} - 2n k_{23} \eta^{\alpha_1 \mu_1} - 2nk_{23} \eta^{\alpha_1 \nu_1}) \frac{1}{2} (k_{23} \eta_{\mu_2 \nu_2} - 2n k_{23} \eta^{\alpha_2 \mu_2} - 2nk_{23} \eta^{\alpha_2 \nu_2}) \right], \quad (34)$$

where we have replaced all graviton momenta with momenta coming from the $\Phi$ fields, and have taken advantage of the transversality of all polarizations and propagators involved. Note that in transverse traceless gauge $R_{\mu\nu} = -\frac{1}{2} \Box h_{\mu\nu}$, which gives the $\beta(n)$ dependent piece.

We now make the useful definition

$$T^{\alpha\mu\nu} (k_{23}) \equiv (k_{23} \eta_{\mu\nu} - 2n k_{23} \eta^{\alpha\mu} - 2nk_{23} \eta^{\alpha\nu}) \quad (35)$$

to more easily express scattering amplitudes and Feynman rules. One should note that $B^{\mu\nu} = (1/2) T^{\alpha\mu\nu} \Gamma_\alpha$, which will make the field theoretic and string theoretic calculations easier to compare. From the $F_{(5)}^2$ term in the IIB Lagrangian (4), one can read off the Feynman rule for $F_{\rho_1}^1 h_{\alpha_1 \alpha_2}^{-(1+4)} F_{\sigma_1}^3$,

$$\Gamma = -i \frac{1}{2\kappa_0^2} \frac{10}{4 \cdot 5} \eta_{\rho_1 \rho_2} \eta_{\sigma_2 \sigma_1} \prod_{i=2}^{5} \eta_{\sigma_i \rho_i}. \quad (36)$$

which can also be verified by examining the string $F_{(5)} h_{\mu\nu} F_{(5)}$ three-point function. Computing the field theoretic scattering, remembering the factor $1/2!$ because it is second order in perturbation theory, one finds the $t$ channel pole contribution to the scattering amplitude

$$\mathcal{A}_{\text{sub}} = -i (2\pi)^{10} \delta (\Sigma k_i) \left[ \frac{1}{2\kappa_0^2} \frac{5}{4 \cdot 5} \left( \frac{5}{2(k_2 + k_3)^2} \right) \Phi_{\rho_1 \rho_2}^{\mu_1 \mu_2} \Phi_{\nu_1 \nu_2}^{\nu_3 \cdots \nu_n} \Phi_{\sigma_1}^{3 \nu_3 \cdots \nu_n} \times T^{\rho_1 \mu_1 \nu_1} T^{\nu_2 \nu_2} (F_{\rho_1 \rho_2 \cdots \rho_5}^3 (F_{\sigma_1}^3)^{\rho_2 \cdots \rho_5}) \right] + \frac{\beta(n)}{8\kappa_0^2} \frac{1}{4 \cdot 5} \left( \Phi_{\rho_1 \rho_2 \cdots \rho_5}^{\mu_1 \mu_2 \cdots \mu_n} \Phi_{\nu_1}^{3 \nu_3 \cdots \nu_n} \right) \left( F_{\rho_1 \rho_2 \cdots \rho_5}^1 F_{\nu_1}^{4 \rho_3 \cdots \rho_5} + F_{\rho_1 \rho_2 \cdots \rho_5}^4 F_{\nu_1}^{1 \rho_2 \cdots \rho_5} \right) \right]. \quad (37)$$

To compare this with the string theory calculation, we take the $t$ channel $(1/b)$ term from (27) and evaluate the trace of Dirac matrices, again making use of the symmetries of the
polarizations, and noting that certain terms are zero due to self duality of the five-form. The final result for the $t$ channel string amplitude is

$$
\mathcal{A}_{\text{str}} = -i (2\pi)^{10} \delta(\Sigma k_i) \left[ \frac{1}{2\kappa_1^2} \frac{5}{4 \cdot 4! \cdot (k_2 + k_3)^2} \Phi_{\mu_1 \mu_2 \nu_1 \nu_2 n} \Phi_{\mu_3 \mu_4 \nu_3 \nu_4 n} \right. \\
\left. \times T^{\mu_1 \mu_2 \nu_1 \nu_2} T^{\nu_3 \nu_4 \rho_3 \rho_4} (F_{\rho_1 \rho_2 ... \rho_5}^3 (F_{\sigma_1}^4)_{\rho_2 ... \rho_5}) \right]. \quad (38)
$$

As a result, subtracting the underlying graviton exchange contribution $\mathcal{A}_{\text{sub}}$ cancels the pole completely, while introducing a $\beta(n)$ dependent term. This leads to a contact term with undetermined coefficient $\beta(n)$

$$
\mathcal{A}_{\text{4pt}}^{4\text{pt}} = i (2\pi)^{10} \delta(\Sigma k_i) \frac{\beta(n)}{8 \kappa_1^2} \left[ \frac{1}{4 \cdot 4!} 2(\Phi_{\mu_1 \mu_2 ... \mu_2 n}^2 (\Phi_{\nu_1 \nu_2})_{\mu_3 \mu_4 ... \mu_2 n} \right. \\
\left. \times (F_{\mu_1 \mu_2 ... \rho_5}^1 F_{\nu_1 \rho_2 ... \rho_5}^4 + F_{\mu_1 \mu_2 ... \rho_5}^4 F_{\nu_1 \rho_2 ... \rho_5}^4). \quad (39)
$$

We will discuss the total Lagrangian and connection to the massless case in the next subsection.

### 4.4 Complete effective action

Reading off the three-point effective action from (38), and adding in the four-point terms from (39) gives the following effective Lagrangian

$$
e^{-1} \mathcal{L}_{\Phi} = \frac{1}{4 \kappa_1^2} \left[ \frac{1}{2} \frac{1}{2} \Phi_{\mu_1 \cdots \mu_2 n} \nabla_{\lambda} \nabla^{\lambda} \Phi_{\mu_1 \cdots \mu_2 n} - \frac{1}{2} M_\Phi^2 \Phi_{\mu_1 \cdots \mu_2 n} \Phi_{\mu_1 \cdots \mu_2 n} \\
+ n^2 \Phi_{\mu_1 \cdots \mu_2 n} R_{\mu_1 \nu_1 \mu_2 \nu_2} \Phi_{\nu_1 \nu_2} \Phi_{\mu_3 \cdots \mu_2 n} \\
- \frac{1}{4 \cdot 5!} 20 \Phi_{\mu_1 \cdots \mu_2 n} F_{\mu_1 \nu_1 \alpha_3 \cdots \alpha_5} F_{\mu_2 \nu_2 \alpha_3 \cdots \alpha_5} \Phi_{\nu_1 \nu_2} \Phi_{\mu_3 \cdots \mu_2 n} \\
- \frac{1}{4 \cdot 5!} 5 \Phi_{\mu_1 \cdots \mu_2 n} F_{\mu_1 \alpha_3 \cdots \alpha_5} F_{\nu_1 \nu_2 \alpha_3 \cdots \alpha_5} \Phi_{\mu_3 \cdots \mu_2 n} \\
+ \frac{1}{2} \beta(n) \Phi_{\mu_1 \cdots \mu_2 n} \left( R_{\mu_1 \nu_1} - \frac{1}{4 \cdot 4!} (F_{\mu_1 \nu_1})^2 \right) \Phi_{\mu_2 \cdots \mu_2 n} \\
+ \frac{1}{2} \gamma(n) \Phi_{\mu_1 \cdots \mu_2 n} R F_{\mu_1 \cdots \mu_2 n} \\
+ \frac{1}{2} \theta(n) \Phi_{\mu_1 \cdots \mu_2 n} F_{\alpha_1 \cdots \alpha_5} F_{\alpha_1 \cdots \alpha_5} \Phi_{\mu_1 \cdots \mu_2 n} + \ldots \right]. \quad (40)
$$

Note that the last three lines are couplings to terms that vanish on shell, and in particular $\beta(n)$ describes the coupling of $\Phi$ to the Einstein equation. One may think of this as generalizing an on shell condition, linearized about flat space, $\Box h_{\mu \nu} = 0$, to the full non-linear equations $R_{\mu_1 \nu_1} - \frac{1}{4 \cdot 4!} (F^2)_{\mu_1 \nu_1} = 0$ describing a curved background.

We now check that the above Lagrangian indeed reproduces the standard IIB supergravity action for the massless case. First, remember that in the massless case that the kinetic $\Phi$ terms and the Riemann coupling are instead identified with on shell parts of $R$ expanded to third order. Next, for the off shell part, note that the combinatoric factor of $2!$ arising...
from assigning particle 2 or 3 to each of the \( \Phi \) fields in \( \Phi^{\mu\lambda_2\cdots\lambda_2n} \Phi^{\nu\lambda_2\cdots\lambda_2n} R_{\mu\nu}^n \) is the same 2! for that of assigning 2 or 3 to \(-h^{0\alpha_1\lambda_2\cdots\lambda_2n} \partial h_{\mu\nu} \). This is because 2 or 3 may be assigned to the first two \( h \) factors, but not the third because the external legs are on shell. This gives that the massless case is an analog of \( \beta(1) = 2 \). Plugging this in, one finds that the scattering Lagrangian becomes

\[
L_{hhh} + L_{hhFF} = \frac{1}{4 \kappa_{10}^2} \sqrt{-g} \left[ -\frac{1}{2} h^{0\mu_1\nu_1} h_{\mu_1\nu_1} \partial_{\mu_0} \partial_{\nu_0} h_{\mu_1\nu_1} - h_{\nu_1} h_{\mu_1\nu_2\lambda} \lambda h^{\nu_2\nu_1} + h^{\mu_1\mu_2} h_{\nu_2} \left( -\frac{1}{2} \Box h_{\mu_1\nu_1} - \frac{1}{2} (F_\mu)_{\mu_1\nu_1} \right) \right. \\
- \frac{1}{8 \cdot 5!} 40 h^{\mu_1\mu_2} h_{\nu_1\nu_2} (F_{\mu_1\nu_1\alpha_3\cdots\alpha_5} F_{\mu_2\nu_2\alpha_3\cdots\alpha_5}) \left. - \frac{1}{8 \cdot 5!} 10 h^{\mu_1\mu_2} h_{\nu_1\nu_2} (F_{\mu_1\alpha_3\cdots\alpha_5} F_{\nu_2\alpha_3\cdots\alpha_5}) \right] \\
= \frac{1}{2 \kappa_{10}^2} \left[ (\sqrt{-g}) R|_{hhh} - (\sqrt{-g}) \frac{1}{4 \cdot 5!} F^2|_{hhFF} \right] F = *F \text{ imposed}. \quad (41)
\]

The \( \beta(1) = 2 \) term contributes to one of the \( F^2_{(5)} \) terms in just the way needed to give the combinatoric factors of \( \binom{5}{1} \) and \( \binom{5}{2} \), understood as expanding either one of the inverse metrics in \( F^2_{(5)} \) to second order, or two of them to first order.

The unknowns \( \beta(n), \gamma(n) \) and \( \theta(n) \) all multiply terms which vanish on-shell (and which hence are undetermined from the S-matrix computation). Of course, taking (40) and evaluating \( R \) and \( F_{(5)} \) as a background solution, we find all unknown coefficients have dropped out. Such a statement may require more care for other background form fields. For example, without using equations of motion, \( \sqrt{-g} \Phi^2 R \) zero for a \( \Phi \Phi h \) scattering, the first order expansion of \( \sqrt{-g} R \) vanishing in transverse traceless gauge. This term is zero, therefore, for a different reason than \( \sqrt{-g} (\Phi^2)^\mu\nu R_{\mu\nu} \) is zero, the latter vanishing on shell. As a result, one cannot relate the terms \( \sqrt{-g} \Phi^2 R \) and \( \sqrt{-g} F_{(5)}^2 \Phi^2 \) as one may relate \( \sqrt{-g} (\Phi^2)^\mu\nu R_{\mu\nu} \) and \( \sqrt{-g} (\Phi^2)^\mu\nu (F_{(5)}^2)_{\mu\nu} \), although it is tempting to think that these also come in the combination dictated by the trace of the Einstein equations. To find these terms, one would use a \( \Phi \Phi hh \) scattering, which will not vanish in transverse traceless gauge. As a further consideration, one must also consider exchange of dilatons in the case of the three form. We were able to ignore this term because of the self duality condition on \( F_{(5)} \).

### 4.5 Regime of Validity

We now are prepared to discuss the regime of validity for which the above Lagrangian terms are valid. First, let us note that we are restricting our discussion to terms that are quadratic in \( \Phi \), and coupled to fields with massless quanta. In this sense, we are focusing on the terms in the Lagrangian that will change the “free” equations of motion in a background. Also, we will consider only sphere (tree) level amplitudes, and only backgrounds with metric and five form.

We have used \( k_0 \sqrt{\alpha'} \) (light momenta) as expansion parameters, and we will show that we also need to take \( n \alpha' \ll L^2 \). We use \( L \) to denote the smallest length scale associated with
the space, in the case of AdS$_5 \times S^5$, this is simply the radius of either space. Expansions in $\alpha'$ usually fail because when one considers higher derivatives of massive fields one finds large factors, as these momenta are inherently larger than 1 in $\alpha'$ units. We consider higher order interactions schematically of the kind $R^i F^2_j \Phi \Phi$. Terms with the Ricci scalar are zero for our purposes, and terms with the Ricci tensor will couple to the equation of motion as before. Therefore, we only consider the Riemann tensor in for $R$. The above coupling can be determined by a $h^{i(0,0)} (F^{\mp,+) F^{(-,\mp)}_j)^2 \Phi_{(-1,-1)} \Phi_{(-1,-1)}$ scattering where we keep the $\Phi$ terms in their canonical ghost picture. One might think that the extra $\partial X$ operators on the $\Phi$ vertex operators will cause most of the problem. However, these may only bring down momenta from the other $\Phi$ vertex operator, which can then be replaced by massless momenta thanks to the transversality of the polarization of $\Phi$. The $\partial X$ operators from massless vertex operators will lead to heavy derivatives, which might cause a problem. The maximum number of heavy derivatives is $2(i+j)$, simply from examining the structure of the vertex operators (count one $\partial X$ and one $\bar{\partial} X$ per vertex operator with a ghost picture +1 of its canonical value). Therefore, in the worst case, one may find $R^i F^2_j \partial^{2(i+j)} \Phi \Phi$ type interactions. Taking the center of mass energy to be small is the same as putting the $\Phi$ particle close to being at rest, so that the derivatives essentially give $\sqrt{M_5^2} \sim \sqrt{n \alpha'}$. Also, $R$ and $F^2$ are both of the order $1/L^2$. This gives that the overall order of the term is $(n \alpha'/L^2)^{(i+j)}$. Thus, to take this term as being small, we must have $n \alpha' \ll L^2$ as advertised. One may think of this as quantizing the string in the vicinity of its classical geodesic. In the massive case we are considering, the geodesic is simply a particle sitting at a point in a smooth manifold, and so the space near the geodesic is flat space.

5 AdS$_5 \times S^5$ and spinning strings

Now that we have constructed the effective Lagrangian (40) up to first non-trivial order in $\alpha'/L^2$, we may finally investigate its consequences for spinning strings in an AdS$_5 \times S^5$ background. For a direct product space, we now switch notations so that $M,N,...$ denote ten-dimensional indices, while $\mu, \nu ...$ and $m,n,...$ denote AdS$_5$ and $S^5$ indices, respectively. The AdS$_5 \times S^5$ background is then a solution of the Freund-Rubin ansatz

$$R_{\mu \nu} = -\frac{4}{L^2} g_{\mu \nu}, \quad R_{m n} = \frac{4}{L^2} g_{m n},$$

$$F_{\mu \nu \rho \lambda} = \frac{4}{L} \epsilon_{\mu \nu \rho \lambda}, \quad F_{m n p q r} = \frac{4}{L} \epsilon_{m n p q r},$$

(42)

solving the equations of motion

$$R_{M N} = \frac{1}{4 \cdot 4!} F_{M P Q R} F_N^{P Q R S}, \quad F = * F, \quad d F = 0.$$  

(43)

For the maximally symmetric AdS$_5 \times S^5$ solution, we may furthermore write

$$R_{M N I J} = \frac{1}{4 \cdot 4!} F_{M N P Q R} F_{I J}^{P Q R},$$

(44)

where

$$R_{\mu \nu \rho \sigma} = -\frac{1}{L^2} (g_{\mu \rho} g_{\nu \sigma} - g_{\mu \sigma} g_{\nu \rho}), \quad R_{m n p q} = \frac{1}{L^2} (g_{m p} g_{n q} - g_{m q} g_{n p}).$$

(45)
This allows us to replace the five-form terms by curvature terms when evaluating \( \text{10} \) on the \( \text{AdS}_5 \times S^5 \) background

\[
e^{-1} \mathcal{L}_\Phi = \frac{1}{4\kappa_{10}^2} \left[ \frac{1}{2} \Phi_{M_1 \ldots M_{2n}} \nabla_L \nabla^L \Phi^{M_1 \ldots M_{2n}} - \frac{1}{2} M_\Phi^2 \Phi_{M_1 \ldots M_{2n}} \Phi^{M_1 \ldots M_{2n}} \\
+ (n^2 - 4) \Phi_{M_1 M_2 M_3 \ldots M_{2n}} R^{M_1 N_1 M_2 N_2} \Phi_{N_1 N_2} \Phi^{M_3 \ldots M_{2n}} \\
- \Phi_{M_1 M_2 \ldots M_{2n}} R^{M_1 N_1} \Phi_{N_1} \Phi^{M_2 \ldots M_{2n}} + \ldots \right]. \tag{46}
\]

Using \( \text{13} \), as well as (ten-dimensional) tracelessness of \( \Phi \), results in a relatively simply effective Lagrangian

\[
e^{-1} \mathcal{L}_\Phi = \frac{1}{4\kappa_{10}^2} \left[ \frac{1}{2} \Phi_{M_1 \ldots M_{2n}} \nabla_L \nabla^L \Phi^{M_1 \ldots M_{2n}} - \frac{1}{2} M_\Phi^2 \Phi_{M_1 \ldots M_{2n}} \Phi^{M_1 \ldots M_{2n}} \\
+ \frac{n^2}{L^2} \left( \Phi_{\mu_1 \mu_2 M_3 \ldots M_{2n}} \Phi_{\mu_1 \mu_2 M_3 \ldots M_{2n}} - \Phi_{\mu_1 \mu_2 M_3 \ldots M_{2n}} \Phi^{\mu_1 \mu_2 M_3 \ldots M_{2n}} \right) + \ldots \right]. \tag{47}
\]

Note that the background curvature terms on \( \text{AdS}_5 \) and \( S^5 \) contribute equally, but with opposite signs.

A symmetric tensor \( \Phi_{M_1 \ldots M_{2n}} \) in ten dimensions decomposes into a direct sum of symmetric tensors on \( \text{AdS}_5 \), transforming in symmetric representations of the \( R \)-symmetry group \( SO(6) \). Focusing on a single term in this decomposition, we consider in general a field \( \Phi \) with \( S \) indices along \( \text{AdS} \) and \( K \) indices along the sphere, and of course the condition \( S + K = 2n \). The fields are symmetrized, and so we may only choose the number of \( \text{AdS}_5 \) versus \( S^5 \) indices, and not the order in which they come. We consider the field as \( \Phi_{\{M_j\}} = (1/(2n)!) \sum \Phi_{M_i} \), where the \( M_i \) are given as either \( \text{AdS}_5 \) or \( S^5 \) indices, and the right hand side gives the full symmetrized field. In this expression, there will be a given number of times that the \( \text{AdS}_5 \) indices appear in the first two entries, and a given number of times that the \( S^5 \) indices appear in the first two entries. This breakdown is given by a relatively elementary combinatorics problem. Of the total \( (2n)! \) terms in the symmetrized sum, the first two entries break down as: \( (2n - 2)!S(S - 1) \) of them are both \( \text{AdS}_5 \); \( (2n - 2)!2SK \) entries come as mixed; and \( (2n - 2)!K(K - 1) \) of them are both \( S^5 \). It is easy to check that \( S(S - 1) + K(K - 1) + 2SK = 2n(2n - 1) \). Therefore, for a particular mixed index component, we have

\[
g^{\mu_1 \nu_1} g^{\mu_2 \nu_2} \Phi_{\mu_1 \mu_2 \ldots \Phi_{\nu_1 \nu_2} \ldots} = g^{\mu_1 \nu_1} g^{\mu_2 \nu_2} \Phi_{\mu_1 \mu_2 \ldots \Phi_{\nu_1 \nu_2} \ldots} = \frac{S(S - 1)}{2n(2n - 1)} g^{M_1 N_1} g^{M_2 N_2} \Phi_{M_1 M_2 \ldots \Phi_{N_1 N_2} \ldots}, \tag{48}
\]

where in the first step we may generalize the indices because the metric has no mixed components. We do this for the sphere metrics as well, replacing \( S \) by \( K = 2n - S \) in the above. A second way to arrive at the same answer is by expanding all metrics as \( g^{MN} = g^{\mu \nu} + g^{mn} \) and keeping track of combinatorial terms. In either case, the Lagrangian \( \text{(17)} \) becomes

\[
e^{-1} \mathcal{L}_\Phi = \frac{1}{4\kappa_{10}^2} \sum_{S=0}^{2n} \left[ \frac{1}{2} (\Phi_{(S)})_{M_1 \ldots M_{2n}} \nabla_L \nabla^L (\Phi_{(S)})^{M_1 \ldots M_{2n}} \\
- \frac{1}{2} \left( M_\Phi^2 + \frac{(K + S)(K - S)}{2L^2} \right) (\Phi_{(S)})_{M_1 \ldots M_{2n}} (\Phi_{(S)})^{M_1 \ldots M_{2n}} + \ldots \right]. \tag{49}
\]
where the notation \( \Phi(S) \) indicates that there are \( S \) indices along the AdS\(_5\) directions.

The above Lagrangian results in the AdS\(_5\) equation of motion

\[
[\nabla_\mu \nabla^\mu - \mu^2] \Phi(k) = 0, \tag{50}
\]

where the effective mass is

\[
\mu^2 = M_\Phi^2 + \frac{(K + S)(K - S)}{2L^2} - \nabla_m \nabla^m. \tag{51}
\]

The angular Laplacian on \( S^5 \) gives rise to an angular momentum contribution. While this is straightforward for scalar harmonics (and gives \( J(J + 4) \) in the angular momentum \((J, 0, 0)\) representation), the spectrum is somewhat more involved for general tensor harmonics. We focus on a subset of decompositions possible, namely when the field can be written in terms of a symmetric transverse traceless (ST\(^2\) henceforth) part on AdS times a ST\(^2\) part on the sphere. Sphere tensor harmonics of this form were discussed in [26] but also see [27, 28]. In particular, we need the eigenvalues of the d’Alembertian operator on the sphere and in [26] these were found for general ST\(^2\) spherical harmonics. It was also shown that the quadratic Casimir operator of the \( O(n_s + 1) \) \((n_s = \text{dimension of sphere})\) for these harmonics was that of the \((J, K, 0, \cdots, 0)\) representation (these label the length of each row in the Young tableau, and so \( J \geq K \)). It was shown that for the first and second rank ST\(^2\) harmonics \((K = 1, 2)\) that this is indeed the representation. We take this to be the case for the higher rank harmonics as well. In any case, the eigenvalue of the d’Alembertian acting on these harmonics was found to be

\[
\nabla_m \nabla^m = -\frac{J(J + n_s - 1) - K}{L^2}, \tag{52}
\]

and \( J > K \) as above. This gives

\[
\mu^2 = M_\Phi^2 + \frac{(K + S)(K - S)}{2L^2} + \frac{J(J + 4) - K}{L^2}. \tag{53}
\]

We stress here that while it is tempting to think of \( J = K \) as the intrinsic angular momentum and the ‘extra’ coming from \( J \neq K \) as the ‘orbital’ angular momentum, this is not a meaningful concept on the sphere (as \( E_0 \) is the only meaningful ‘energy’ on AdS). The charges are simply \((J, K, 0)\) which label the representation. The analog of this in AdS will be addressed next.

### 5.1 Energy in AdS\(_5\)

To relate this effective mass to energy in AdS\(_5\), we recall that states in AdS\(_5\) may be labeled by their \( SU(2, 2|4) \supset SO(2, 4) \times SO(6) \supset SO(2) \times [SU(2) \times SU(2)] \times SO(6) \) quantum numbers \((E, S_1, S_2; J_1, J_2, J_3)\) with \( S_1 \) and \( S_2 \) corresponding to spacetime angular momentum and \( J_1, J_2, J_3 \) corresponding to the three commuting angular momenta \((R\text{-charges})\) on the sphere. Unitary representations in AdS\(_5\) are then labeled by \( D(E_0, S_1, S_2; J_1, J_2, J_3) \) with \( E_0 \) the lowest energy eigenvalue and \((J_1, J_2, J_3)\) are Dynkin labels for the \( SO(6) \) representation. Using \( SU(2) \times SU(2) \) notation, the symmetric tracefree field \( \Phi(k) \) corresponds
to $D(E_0, S/2, S/2; J, K, 0)$ where $S$ may be thought of as conventional ‘angular momentum’, and $J \geq K = 2n - S$ label the symmetric representation for ‘$R$-charge ($J, K, 0$)’ on the sphere.

Mass in Anti-de Sitter (or any curved) space is a somewhat ambiguous concept, as it does not directly correspond to a Casimir of the symmetry algebra. Nevertheless, we choose the standard convention where masslessness corresponds to the propagation of a reduced number of helicity states. In this case, for symmetric rank-$S$ tensors in $\text{AdS}_{d+1}$ we have [29–35]

$$\left[\nabla_\mu \nabla^\mu - m^2 + (2(d - 2) - (d - 5)S - S^2)\right] \Phi(S) = 0,$$

along with transverse-tracelessness, $\nabla^\mu \Phi_{\mu...} = 0$ and $\Phi_{\mu...} = 0$. This equation is then compared to the eigenvalues of the d’Alembertian operator on AdS via the equation

$$\left[\nabla_\mu \nabla^\mu - E_0(E_0 - d) + S\right] \Phi(S) = 0. \quad (55)$$

The value of $E_0$ corresponding to this definition of mass is then simply [34]

$$E_0 = \frac{d}{2} + \sqrt{m^2 L^2 + \left(\frac{d}{2} + S - 2\right)^2}. \quad (56)$$

Comparing (50) and (54), we determine

$$E_0 = \frac{d}{2} + \sqrt{\mu^2 L^2 + (d/2)^2 + S}, \quad (57)$$

resulting in the AdS$_5$ ($d = 4$) energy relation

$$E_0 = 2 + \sqrt{2\sqrt{\lambda}(K + S - 2) + \frac{1}{2}(K + S - 2)(K - S) + J(J + 4) + 4 + \mathcal{O}(1/\sqrt{\lambda})}, \quad (58)$$

where we have introduced the usual definition $\sqrt{\lambda} = L^2/\alpha'$. In this context, we wish to go as close to ‘$s$-wave’ as possible, and so we take $J = K$, its minimum value. In the limit of large spin or $R$-charge, but still in the short string limit, $1 \ll K + S \ll \sqrt{\lambda}$, we may expand $E_0$ to give

$$E_0 \approx \sqrt{2\sqrt{\lambda}(K + S)} \left(1 + \frac{K - S}{8\sqrt{\lambda}} + \frac{K(K + 4)}{2(K + S)\sqrt{\lambda}} + \cdots \right). \quad (59)$$

A comparable expression was derived in [5] for a semi-classical string spinning in AdS$_5$ and carrying only orbital angular momentum on $S^5$. For vanishing orbital angular momentum (which corresponds to the above discussion), the short-string result was given as [5]

$$E \approx \sqrt{2\sqrt{\lambda}S} \left(1 + \frac{S}{\sqrt{\lambda}} + \cdots \right), \quad (60)$$

which differs from the $K = 0$ limit of our result, (59). It is not clear to us where this discrepancy arises from. However, it is interesting to note that the deviation from flat-space Regge behavior differs in sign (and not just by a factor) between these two results. One important difference to note is that for us the wave function of the string is spread over the
entire sphere. In [5] the string is placed at a single point on the sphere, and so it may be that this state corresponds to a large sum of KK states on the sphere. For example, the wave function being a delta function at the north pole (for scalar harmonics) is given by

$$\sum C^{(2)}_\ell (\cos(\theta)),$$

where $$C^{(2)}_\ell (x)$$ are the Gegenbauer (ultraspherical) polynomials on $$S^5$$. One should do likewise for placing the string at the center of AdS. Using the resulting average $$E_0 = E_0((O))$$ may allow for more direct comparison with results from [5].

From our above computation of mass on AdS$_5$, we may also consider strings with angular momentum, but not spin, on the sphere (i.e. $$K = 0$$). In this case, $$-\nabla_m \nabla^m$$ has eigenvalue $$J(J + 4)$$ for scalar harmonics on $$S^5$$. The resulting expression for $$E_0$$ is

$$E_0 = 2 + \sqrt{2\sqrt{\lambda}(S - 2) + J(J + 4) - \frac{1}{2}S(S - 2) + 4 + O(1/\sqrt{\lambda})} \approx \sqrt{2\sqrt{\lambda}S + J^2 - \frac{1}{2}S^2},$$

(61)

in agreement with the leading behavior given in [5]. Note that, in general, that the Kaluza-Klein spectrum of massive string fields was investigated in [28], based on the identification of higher spin symmetry in the $$\lambda \to 0$$ limit. For leading Regge states, the resulting spectrum was simply $$E_0 = 2\ell + n$$ where $$\ell$$ is the string level and $$n$$ denotes the Kaluza-Klein harmonic on $$S^5$$. As this is outside the scope of the short string case that we investigate here, the results do not appear to be directly comparable.

### 5.2 Conclusions

Although the validity of this perturbative approach depends on taking $$K + S \ll \sqrt{\lambda}$$, correspond to short strings, we feel the results are robust within this limit. Although the string effective action can only be determined up to terms vanishing on-shell, this is all that is required for the Freund-Rubin background. Note, however, that we do not prove that AdS$_5 \times S^5$ is a consistent background for string propagation, although this has been shown at the perturbative level in the pure spinor formalism [36].

We also wish to point out that the gravitational quadrupole interaction of massive string states was previously investigated in [23], where the effective Lagrangian was obtained as an intermediate step towards computing the gravitational ‘$$h$$-factor’ $$h = S_LS_R/2S(S - 1)$$ for NSNS higher spin states. Here the ten-dimensional spin contributions $$S_L$$ and $$S_R$$ arise from the left and right movers, respectively, and the $$h$$ factor is defined by the coupling term

$$(\nabla_\mu \nabla^\mu - m^2 + hR_{\mu\nu\lambda\sigma} \frac{1}{2} \Sigma^{\mu\nu} \frac{1}{2} \Sigma^{\lambda\sigma}) \Phi + \cdots = 0,$$

(62)

where $$\Sigma^{\mu\nu}$$ are Lorentz generators in the spin-$$S$$ representation. As shown in [37], tree-level unitarity is violated for gravitating states with $$h \neq 1$$, and in particular the work of [23] demonstrates this to be the case for massive string states. While this is not a serious issue for the full string theory, as it merely indicates that the scattering becomes strongly coupled at the string scale, it may nevertheless signify a breakdown in the effective field theory description of the massive state $$\Phi$$, despite the indications of a narrow width [19, 20].

Curiously, we note that, while much effort was expended on computing the $$\Phi$$-$F_5$-$F_5$-$\Phi$$ contact terms the previous section, in the end they do not contribute at all to the mass...
shift in a maximally symmetric background. This is because of the particular kinematical combination of $F^2_{(5)}$ which enters (10)

$$F^{MN...}F^{PQ...} + \frac{1}{4}F^{M...}F^{P...} \equiv \mathcal{T}^{MNQ} + \frac{1}{4}g^{PQ}\mathcal{T}^{MLN}_L.$$  \hspace{1cm} (63)

This vanishes identically for a symmetrical tensor $\mathcal{T}^{MNQ} \sim g^{MPNQ} - g^{MQNP}$ contracted between symmetric tracefree fields $\Phi$. From this point of view, the effects of a closed string spinning in $\text{AdS}_5 \times S^5$ are due entirely to its interactions with the NSNS background, and (at least to this order) the Ramond-Ramond background can be completely ignored. While we are uncertain as to whether this holds at all perturbative orders, it nevertheless suggests the possibility that semi-classical bosonic sigma-model approaches to spinning strings in fact do not pick up any corrections from the Ramond-Ramond background.

Essentially, what we have explored from the world-sheet point of view is the coupling of massive higher spin states to the Type IIB supergravity multiplet (or at least the metric and five-form components). In principle, if a manifestly supersymmetric formalism such as a superfield description were to exist, then one would presumably be able to determine all relevant couplings by the supersymmetric completion of the (Riemann) curvature induced terms. However we are not aware of any such approaches pertaining to the IIB massive higher spin fields in long representations of the superalgebra.

Nevertheless, the structure of the IIB supergravity multiplet clearly demonstrates the relation between Riemann and $F^2_{(5)}$ terms. For example, integrability of the IIB supercovariant derivative

$$D_M = \left[ \nabla_M + \frac{i}{16 \cdot 5!} F_{NPQRS} \Gamma^{NPQRS} \Gamma_M + \cdots \right] \epsilon$$  \hspace{1cm} (64)

may be expressed as $[D_M, D_N] \epsilon = \frac{1}{4} \mathcal{R}_{MN} \epsilon$ where [38]

$$\mathcal{R}_{MN} = \left[ R_{MNQP} - \frac{1}{48} F_{MPABC} F_{NQ}^{\ ABC} \right] \Gamma^{PQ}$$

$$\left. + \frac{1}{24} \left[ i\nabla_{[M} F_{N]PQRS} + \frac{1}{4} F_{MNPAB} F_{QRS}^{\ AB} - \frac{1}{4} g_{P[M} F_{N]QABC} F_{QR}^{\ ABC} \right] \Gamma^{PQRS} \right.$$

$$\left. + \cdots . \right.$$

(65)

From this point of view, supersymmetry indicates that any term proportional to Riemann will be related to similar terms containing both $F^2_{(5)}$ and $\nabla F_{(5)}$. The latter are unimportant for our case, as covariant derivatives of $F_{(5)}$ vanish for the maximally supersymmetric backgrounds that we consider. It would be interesting to see if this supercovariant derivative, while originally acting on spinors, can be extended to provide a means of constructing supersymmetric actions for higher-rank bosonic fields $\Phi$. This is perhaps most promising for the case of $p$-form actions, as they may be constructed out of spinor bilinears. These ideas may end up involving the geometry and hidden symmetries of the supercovariant derivative, an idea which has attracted some recent interest [39–44].

Finally, we note that, while it has not been extensively studied, it is entirely feasible to work with Ramond-Ramond emission vertex operators in the RNS formalism. In this fashion, it is for example possible to go beyond linearized superfields [45] to explore the supersymmetric completion of $R^4$ terms from the world-sheet point of view; this was examined
for $R^2(\nabla F)^2$ in [46]. Of course, a more direct approach to Ramond-Ramond backgrounds may nevertheless lie within a Green-Schwarz [16,47] or pure spinor formulation [36,48–51], or ultimately within string field theory.

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