Triangles in $C_5$-free graphs and hypergraphs of girth six

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Abstract
We introduce a new approach and prove that the maximum number of triangles in a $C_5$-free graph on $n$ vertices is at most
\[
(1 + o(1)) \frac{1}{\sqrt[3]{3}} n^{3/2}.
\]

We show a connection to $r$-uniform hypergraphs without (Berge) cycles of length less than six, and estimate their maximum possible size. Using our approach, we also (slightly) improve the previous estimate on the maximum size of an induced-$C_4$-free and $C_5$-free graph.

KEYWORDS
Berge hypergraphs, generalized Turán, triangles

1 | INTRODUCTION

Motivated by a conjecture of Erdős [3] on the maximum possible number of pentagons in a triangle-free graph, Bollobás and Győri [2] initiated the study of the natural converse of this problem. Let $ex(n, K_3, C_5)$ denote the maximum possible number of triangles in a graph on $n$ vertices without containing a cycle of length five as a subgraph. Bollobás and Győri [2] showed that
\[
(1 + o(1)) \frac{1}{\sqrt[3]{3}} n^{3/2} \leq ex(n, K_3, C_5) \leq (1 + o(1)) \frac{5}{4} n^{3/2}.
\]
Their lower bound comes from the following example: Take a $C_4$-free bipartite graph $G_0$ on $n/3 + n/3$ vertices with about $(n/3)^{3/2}$ edges and double each vertex in one of the color classes and add an edge joining the old and the new copy to produce a graph $G$. Then, it is easy to check that $G$ contains no $C_5$ and the number of triangles in $G$ is equal to the number of edges in $G_0$.

Recently, Füredi and Özkahya [7] gave a simpler proof showing a slightly weaker upper bound of $\sqrt{3} n^{3/2} + O(n)$. Alon and Shikhelman [1] improved these results by showing that

$$ex(n, K_3, C_5) \leq (1 + o(1))\frac{\sqrt{3}}{2} n^{3/2}. \quad (2)$$

Since the publication of [1], there have been several papers in this area. See, for example, [8-11,13,15,16]. Ergemlidze, Győri, Methuku, and Salia [6] recently showed that

$$ex(n, K_3, C_5) \leq (1 + o(1))\frac{1}{2\sqrt{2}} n^{3/2}. \quad (3)$$

In this paper our aim is to introduce a new approach and use it to improve two old results and prove a new result. Our approach consists of carefully bounding the number of paths of length 5 (or paths of length 3) by exploiting the structure of certain subgraphs. Roughly speaking, we are able to efficiently bound the number of 5-paths whose middle edge lies in a dense subgraph (e.g., in a $K_4$). We expect this approach to have further applications.

Our first result improves the previous estimates (1)–(3), on the maximum possible number of triangles in a $C_5$-free graph, as follows.

**Theorem 1.** We have,

$$ex(n, K_3, C_5) < (1 + o(1))\frac{1}{3\sqrt{2}} n^{3/2}.$$ 

Given a hypergraph $H$, its 2-shadow is the graph consisting of the edges $\{ab \mid \exists e \in E(H) \text{ such that } ab \subseteq e\}$. A Berge cycle of length $\ell$ in a hypergraph is a set of $\ell$ distinct vertices $\{v_1, \ldots, v_\ell\}$ and $\ell$ distinct hyperedges $\{e_1, \ldots, e_\ell\}$ such that $\{v_i, v_{i+1}\} \subseteq e_i$ with indices taken modulo $\ell$. We say a hypergraph $H$ is of girth $k$ if the smallest length of a Berge cycle in $H$ is $k$.

Applying our approach to the 2-shadow of a hypergraph of girth at least six, we prove the following result.

**Theorem 2.** Let $H$ be an $r$-uniform hypergraph of girth at least six on $n$ vertices. Then

$$|E(H)| \leq (1 + o(1))\frac{n^{3/2}}{r^{3/2}(r - 1)}.$$ 

Let us mention a related result of Lazebnik and Verstraëte [12] which states the following. If $H$ is an $r$-uniform hypergraph of girth at least five, then

$$|E(H)| \leq (1 + o(1))\frac{n^{3/2}}{r(r - 1)}.$$
Note that Theorem 2 shows that if a (Berge) cycle of length 5 is also forbidden, then the above bound can be improved by a factor of $\sqrt{7}$. It would be interesting to determine whether there is a matching construction for the bound in Theorem 2, at least when $r = 3$.

In Section 3.2, we show a close connection between Theorems 1 and 2, and prove that the estimate in Theorem 1 can be slightly improved using Theorem 2. However, to illustrate the main ideas of the proof of Theorem 1, we decided to state Theorem 1 in a slightly weaker form.

Loh, Tait, Timmons, and Zhou [14] introduced the problem of simultaneously forbidding an induced copy of a graph and a (not necessarily induced) copy of another graph. A graph is called induced-F-free if it does not contain an induced copy of $F$. They asked the following question: What is the largest size of an induced-C$_4$-free and C$_5$-free graph on $n$ vertices? They noted that the example showing the lower bound in (1) is in fact induced-C$_4$-free and C$_5$-free, thus it gives a lower bound of $(1 + o(1)) \frac{2}{3} \sqrt[3]{n}$. (If the “induced-C$_4$-free” condition is replaced by “C$_4$-free” condition, then Erdős and Simonovits [4] showed that the answer is $(1 + o(1)) \frac{1}{2} \sqrt[3]{n}$.) In [5], Győri and the current authors determined (asymptotically) the maximum size of an induced-K$_{s,t}$-free and C$_{2k+1}$-free graph on $n$ vertices whenever $s = 2$ and $t \geq 2$, or $s = t = 3$ except in the case when $s = t = k = 2$ (i.e., the question stated above), and in this case an upper bound of only $n^{3/2}/2$ was proven [5]. Here we show that using our approach one can slightly improve this upper bound.

**Theorem 3.** If $G$ is a C$_5$-free and induced-C$_4$-free graph on $n$ vertices, then

$$|E(G)| \leq (1 + o(1)) \frac{n^{3/2}}{2^{\sqrt[3]{2}}}.$$  

1.1 Organization of the paper

In Section 2, we prove Theorem 1. In Section 3, we prove Theorem 2 and show how it can be used to slightly improve Theorem 1. Finally in Section 4, we prove Theorem 3.

**Notation.** Given a graph $G$ and a vertex $v$ of $G$, let $N_1(v) := \{x \in V(G) | \exists y \in E(G) \}$ and $N_2(v) := \{x \in V(G) \setminus (N_1(v) \cup \{v\}) | \exists y \in N_1(v) \text{ such that } xy \in E(G)\}$ denote the first neighborhood and the second neighborhood of $v$, respectively.

For a vertex $v$ of $G$, let $d(v)$ be the degree of $v$. The average degree of a graph $G$ is denoted by $d(G)$, or simply $d$ if it is clear from the context. The maximum degree of a graph $G$ is denoted by $d_{\text{max}}(G)$ or simply $d_{\text{max}}$.

A walk or path usually refers to an unordered one, unless specified otherwise. That is, a walk or path $v_1v_2v_3 \cdots v_k$ is considered equivalent to $v_kv_{k-1}v_{k-2} \cdots v_1$.

2 NUMBER OF TRIANGLES IN A C$_5$-FREE GRAPH: PROOF OF THEOREM 1

Let $G$ be a C$_5$-free graph on $n$ vertices with maximum possible number of triangles. We may assume that each edge of $G$ is contained in a triangle, because otherwise, we can delete it without changing the number of triangles. Two triangles $T$, $T'$ are said to be in the same block if they either share an edge or if there is a sequence of triangles $T, T_1, T_2, ..., T_s, T'$ where each triangle of this sequence shares an edge with the previous one (except the first one of course). It
is easy to see that all the triangles in $G$ are partitioned uniquely into blocks. Notice that triangles from two different blocks of $G$ are edge-disjoint. Below we will characterize the blocks of $G$.

A block of the form \{abc, abc, ..., abc\} where $k \geq 1$, is called a crown-block (i.e., a collection of triangles containing the same edge) and a block consisting of all triangles contained in the complete graph $K_4$ is called a $K_4$-block. See Figure 1.

The following claim was proved in [6]. We repeat its proof for completeness.

Claim 1. Every block of $G$ is either a crown-block or a $K_4$-block.

Proof. If a block contains only one or two triangles, then it is easy to see that it is a crown-block. So we may assume that a block of $G$ contains at least three triangles and let $abc, abc$ be some two triangles in it. We claim that if $bc_1x$ or $ac_1x$ is a triangle in $G$ which is different from $abc$, then $x = c_2$. Indeed, if $x \neq c_2$, then the vertices $a, x, c_1, b, c_2$ contain a $C_5$, a contradiction. Similarly, if $bc_2x$ or $ac_2x$ is a triangle in $G$ which is different from $abc$, then $x = c_1$.

Therefore, if $ac_i$ or $bc_i$ (for $i = 1, 2$) is contained in two triangles, then $abc_i$ forms a $K_4$. However, then there is no triangle in $G$ which shares an edge with this $K_4$ and is not contained in it because if there is such a triangle, then it is easy to find a $C_5$ in $G$, a contradiction. So in this case, the block is a $K_4$-block, and we are done.

So we can assume that whenever $abc_1, abc_2$ are two triangles then the edges $ac_1, bc_1, ac_2, bc_2$ are each contained in exactly one triangle. Therefore, any other triangle which shares an edge with either $abc_1$ or $abc_2$ must contain $ab$. Let $abc_3$ be such a triangle. Then applying the same argument as before for the triangles $abc_1, abc_3$ one can conclude that the edges $ac_3, bc_3$ are contained in exactly one triangle and so, any other triangle of $G$ which shares an edge with one of the triangles $abc_1, abc_2, abc_3$ must contain $ab$ again. So by induction, it is easy to see that all of the triangles in this block must contain $ab$. Therefore, it is a crown-block, as needed. □

![Figure 1](image_url) An example of a crown-block and a $K_4$-block
2.1 | Edge decomposition of $G$

We define a decomposition $\mathcal{D}$ of the edges of $G$ into paths of length 2, triangles and $K_4$'s, as follows: Since each edge of $G$ belongs to a triangle, and all the triangles of $G$ are partitioned into blocks, it follows that the edges of $G$ are partitioned into blocks as well. Moreover, by Claim 1, edges of $G$ can be decomposed into crown-blocks and $K_4$-blocks. We further partition the edges of each crown-block $\{abc_1, abc_2, ..., abc_k\}$ (for some $k \geq 1$) into the triangle $abc_i$ and paths $ac_ib$ where $2 \leq i \leq k$. This gives the desired decomposition $\mathcal{D}$ of $E(G)$.

Claim 2. Let $u, v$ be two nonadjacent vertices of $G$. Then the number of paths of length 2 between $u$ and $v$ is at most two. Moreover, if $uvw$ and $uvw$ are paths of length 2 between $u$ and $v$, then $x$ and $y$ are adjacent.

Proof. First let us prove the second part of the claim. Since we assumed every edge is contained in a triangle and $u$ and $v$ are not adjacent, there is a vertex $w \neq v$ such that $uwv$ is a triangle. If $w \neq y$, then $uxwv$ is a $C_5$, a contradiction. So $w = y$, so $x$ and $y$ are adjacent, as desired.

Now suppose that there are three distinct vertices $x, y, z$ such that $uxv, uyv, uzv$ are paths of length 2 between $u$ and $v$. Then $x$ and $y$ are adjacent by the discussion in the previous paragraph. Therefore $uxvz$ is a $C_5$ in $G$, a contradiction, proving the claim. □

Let $t(v)$ be the number of triangles containing a vertex $v$ and let $t(G) = t = \sum_{v \in V(G)} \frac{t(v)}{n}$. Observe that the number of triangles in $G$ is $nt/3$. Our goal is to bound $t$ from above.

First we claim that for any vertex $v$ of $G$,

$$t(v) \leq d(v) \leq 2t(v). \quad (4)$$

Indeed, $d(v) \leq 2t(v)$ simply follows by noting that every edge is in a triangle. Now notice that $t(v)$ is equal to the number of edges contained in the first neighborhood of $v$ (denoted by $N_1(v)$). Moreover, there is no path of length three in the subgraph induced by $N_1(v)$ because otherwise there is a $C_5$ in $G$. So by Erdős–Gallai theorem, the number of edges contained in $N_1(v)$ is at most $\frac{3 - 1}{2}d(N_1(v)) = d(v)$. Therefore, $t(v) \leq d(v)$.

Note that by adding up (4) for all the vertices $v \in V(G)$ and dividing by $n$, we get

$$t \leq d \leq 2t. \quad (5)$$

Suppose there is a vertex $v$ of $G$, such that $t(v) < t/3$. Then we may delete $v$ and all the edges incident to $v$ from $G$ to obtain a graph $G'$ such that $t(G') > 3(nt/3 - t/3)/(n - 1) = t(G)$. Then it is easy to see that if the theorem holds for $G'$, then it holds for $G$ as well. Repeating this procedure, we may assume that for every vertex $v$ of $G$, $t(v) \geq t/3$. Therefore, by (4), we may assume that the degree of every vertex of $G$ is at least $t/3$.

Claim 3. We may assume that $d_{\max}(G) \leq 12\sqrt{n}$.

Proof. Suppose that there is a vertex $v$ such that $d(v) > 12\sqrt{n}$. The sum of degrees of the vertices in $N_1(v)$ is at least $\frac{1}{12}(n^2 - d(v)) = \frac{d(v)}{3}$ as we assumed that the degree of every vertex is
at least $t/3$. The number of edges inside $N_1(v)$ is $t(v)$, which is at most $d(v)$ by (4). Therefore the number of edges between $N_1(v)$ and $N_2(v)$ is at least $\frac{d(v) - 3d(v)}{2}$. Now notice that any vertex in $N_2(v)$ is incident to at most two of these edges by Claim 2. Therefore, $N_2(v) \geq \frac{d(v) - 3d(v)}{2}$.

Thus we have,

$$n > |N_1(v)| + |N_2(v)| \geq d(v) + \frac{d(v) - 3d(v)}{2} = \frac{d(v)(t - 3)}{6} > 2\sqrt{n} (t - 3),$$

which implies $t < \frac{\sqrt{n}}{2} + 3 < \sqrt{\frac{n}{3}}$ for large enough $n$. Therefore, the total number of triangles in $G$ is less than $(1 + o(1))\frac{n^{3/2}}{3\sqrt{n}}$, proving Theorem 1. \hfill \Box

By the Blakley–Roy inequality, the number of (unordered) walks of length five in $G$ is at least $n d^5/2$. First let us show that most of these walks are paths. Let $v_0v_1v_2v_3v_4$ be a walk that is not a path. Then $v_i = v_j$ for some $i < j$. Fix some $i < j$. Then there are $n$ choices for $v_0$, and then at most $d_{\text{max}}$ choices for every $v_k$ with $k \leq j - 1$, and since $v_i = v_j$, there is only one choice for $v_j$ and again at most $d_{\text{max}}$ choices for every $v_k$ with $k \geq j + 1$. So in total the number of walks that are not paths is at most $\binom{6}{2} n (d_{\text{max}})^4$ as there are $\binom{6}{2} = 15$ choices for $i, j$. Thus the number of (unordered) paths of length five in $G$ is at least $n^2 d^5/2 - 15n (d_{\text{max}})^4$. From now, we refer to a path of length five as a 5-path.

We say a 5-path $v_0v_1v_2v_3v_4$ is bad if there exists an $i$ such that $v_i v_{i+1}v_{i+2}$ is a triangle of $G$; otherwise it is called good. Our aim is to show that the number of bad 5-paths is very small. Let $v_0v_1v_2v_3v_4v_5$ be a bad 5-path. Then there is an $i$ so that $v_i v_{i+1}v_{i+2}$ is a triangle. If we fix an $i$, there are at most $2nt$ choices for $v_i v_{i+1}v_{i+2}$ as each of the $nt/3$ triangles can be ordered in $3! = 6$ ways, and there are at most $d_{\text{max}}$ choices for every vertex $v_k$ with $k < i$ or $k > i + 2$. There are four choices for $i$. Therefore, the total number of 5-paths that are bad is at most $8nt (d_{\text{max}})^3$. This means that the number of good 5-paths is at least $n^2 d^5/2 - 15n (d_{\text{max}})^4 - 8nt (d_{\text{max}})^3$. By (1), the number of triangles of $G$ is at most $(1 + o(1))\frac{n^{3/2}}{3\sqrt{n}}$. Since the number of triangles of $G$ is $nt/3$, we have $t \leq \frac{15}{4} (1 + o(1))n^{1/2}$. Now using Claim 3, it follows that the number of good 5-paths is at least

$$\frac{nd^5}{2} - 15n(12\sqrt{n})^4 - (1 + o(1))8n\frac{15}{4} n^{1/2}(12\sqrt{n})^3 \geq \frac{nd^5}{2} - Cn^3,$$

where $C$ is some positive constant.

Now we seek to bound the number of good 5-paths from above. Recall that we defined a decomposition $D$ of the edges of $G$ into three types of subgraphs: paths of length 2, triangles, and $K_4$'s. We distinguish three cases depending on which type of subgraph the middle edge of a good 5-path belongs to, and bound the number of good 5-paths in each of those cases separately in the following three claims.

A path of length two (or a 2-path) $xyz$ is called good if $x$ and $z$ are not adjacent.

Claim 4. Let $abc$ be a 2-path of the edge-decomposition $D$. Then the number of good 5-paths in $G$ whose middle edge is either $ab$ or $bc$ is at most $n^2$. 

Proof. A good 5-path $xypqzw$ whose middle edge is $ab$ or $bc$ contains good 2-paths, $xyp, qzw$ as subpaths (where $pq$ is either $ab$ or $bc$). Moreover, since $xypqzw$ is a good 5-path and the 2-path $abc$ is contained in the triangle $abc$ (because of the way we defined the decomposition $D$), it follows that $x, y \notin \{a, b, c\}$ and $z, w \notin \{a, b, c\}$.

Let $n_a$ be the number of good 2-paths in $G$ of the form $axy$ where $x, y \notin \{a, b, c\}$, and let $n_b$ be the number of good 2-paths in $G$ of the form $bxy$ where $x, y \notin \{a, b, c\}$. We define $n_c$ similarly. Then the number of good 5-paths whose middle edge is either $ab$ or $bc$ is at most

$$n_a n_b + n_b n_c = n_b (n_a + n_c) \leq \left( \frac{n_a + n_b + n_c}{2} \right)^2.$$ 

We claim that for any fixed vertex $y \notin \{a, b, c\}$, there are at most two good 2-paths of the form $pxy$ with $p \in \{a, b, c\}$ and $x \notin \{a, b, c\}$. If this claim is true, then $n_a + n_b + n_c \leq 2n$, so the right-hand side of the above inequality is at most $n^2$, proving Claim 4.

It remains to prove this claim. Suppose for a contradiction that there are three such good 2-paths, say, $p_1 x_i y, p_2 x_j y, p_3 x_k y$. Notice that if $p_i x_i$ is disjoint from $p_j x_j$ for some $i, j \in \{1, 2, 3\}$, then $p_i p_j x_i y_i$ forms a $C_3$ in $G$, a contradiction (note that here we used that $p_i$ and $p_j$ are adjacent even when $\{p_i, p_j\} = \{a, c\}$ because of the way we defined $D$). Thus the edges $p_1 x_1, p_2 x_2, p_3 x_3$ pairwise intersect, which implies that either $p_1 = p_2 = p_3 = p$ or $x_1 = x_2 = x_3 = x$ (since $p_1, p_2, p_3 \in \{a, b, c\}$ and $x_1, x_2, x_3 \notin \{a, b, c\}$). The former case is impossible by Claim 2 and in the latter case, note that $a, b, c, x$ forms a $K_4$, but this contradicts the definition of $D$ since $abc$ was assumed to be a 2-path component of $D$ and no 2-path of $D$ comes from a $K_4$-block of $G$.

\[ \Box \]

Claim 5. Let $abc$ be a triangle of the edge-decomposition $D$. Then the number of good 5-paths in $G$ whose middle edge is either $ab$, $bc$, or $ca$ is at most $\frac{3n^2}{3}$.

Proof. The proof is very similar to that of the proof of Claim 4. A good 5-path $xypqzw$ whose middle edge is $ab$, $bc$, or $ca$ contains good 2-paths, $xyp, qzw$, as subpaths. Moreover, since $xypqzw$ is a good 5-path, it follows that $x, y \notin \{a, b, c\}$ and $z, w \notin \{a, b, c\}$.

Let $n_a$ be the number of good 2-paths in $G$ of the form $axy$ where $x, y \notin \{a, b, c\}$, and let $n_b, n_c$ be defined similarly. Then the number of good 5-paths whose middle edge is $ab$, $bc$, or $ca$ is at most

$$n_a n_b + n_b n_c + n_c n_a \leq \frac{(n_a + n_b + n_c)^2}{3}.$$ 

By the same argument as in the proof of Claim 4, it is easy to see that $n_a + n_b + n_c \leq 2n$, so the above inequality finishes the proof. \[ \Box \]

Claim 6. Let $abcd$ be a $K_4$ of the edge-decomposition $D$. Then the number of good 5-paths in $G$ whose middle edge belongs to the $K_4$ is at most $\frac{3n^2}{2}$.

Proof. Notice that any good 5-path $xypqzw$ contains good 2-paths, $xyp, qzw$, as subpaths. Suppose the middle edge of $xypqzw$ belongs to the $K_4, abcd$. Then since $xypqzw$ is a good 5-path, it follows that $x, y \notin \{a, b, c, d\}$ and $z, w \notin \{a, b, c, d\}$. 

\[ \Box \]
Let \( n_a \) be the number of good 2-paths in \( G \) of the form \( axy \) where \( x, y \notin \{a, b, c, d\} \), and let \( n_b, n_c, n_d \) be defined similarly. Then the number of good 5-paths whose middle edge belongs to the \( K_4, abcd \), is at most

\[
\sum_{i,j \in \{a,b,c,d\}} n_in_j \leq \frac{3}{8}(n_a + n_b + n_c + n_d)^2. \tag{7}
\]

To see that the above inequality is true one simply needs to expand and rearrange the inequality \( \sum_{i,j \in \{a,b,c,d\}} (n_i - n_j)^2 \geq 0 \).

Using a similar argument as in the proof of Claim 4, it is easy to see that for any fixed vertex \( y \notin \{a, b, c, d\} \), there are at most two good 2-paths of the form \( pxy \) with \( p \in \{a, b, c, d\} \) and \( x \notin \{a, b, c, d\} \). This implies that \( n_a + n_b + n_c + n_d \leq 2n \), so using (7), the proof is complete.

Now we are ready to bound the number of good 5-paths in \( G \) from above. Suppose the number of edges of \( G \) is \( e(G) \), and let \( \alpha e(G) \) and \( \alpha e(G) \) be the number of edges of \( G \) that are contained in triangles and 2-paths of the edge-decomposition \( D \) of \( G \), respectively. Let \( \alpha_1 + \alpha_2 = \alpha \). In other words, \((1 - \alpha)e(G)\) edges of \( G \) belong to the \( K_4 \)'s in \( D \). Then the number of triangles and 2-paths in \( D \) is at most \( \frac{\alpha_1}{3} e(G) \) and \( \frac{\alpha_2}{2} e(G) \), respectively, and the number of \( K_4 \)'s in \( D \) is at most \( \frac{(1 - \alpha)}{6} e(G) \). Therefore, using Claims 4–6, the total number of good 5-paths in \( G \) is at most

\[
\frac{\alpha_1}{3} e(G) \frac{4n^2}{3} + \frac{\alpha_2}{2} e(G)n^2 + \frac{(1 - \alpha)}{6} e(G) \frac{3n^2}{2} \leq \frac{\alpha}{2} e(G) n^2 + \frac{(1 - \alpha)}{4} e(G) n^2 = \frac{(1 + \alpha)}{8} n^3 d.
\]

Combining this with the fact that the number of good 5-paths is at least \( nd^5/2 - Cn^3 \) (by Equation 6), we get

\[
\frac{nd^5}{2} - Cn^3 \leq \frac{(1 + \alpha)}{8} n^3 d,
\]

which simplifies to \( \frac{d^5}{2} \leq \frac{(1 + \alpha)}{8} n^2 d + Cn^2 = (1 + o(1)) \frac{(1 + \alpha)}{8} n^2 d \). Here we used that \( d \geq t = \Omega(\sqrt{n}) \) (by Equation 5). Therefore,

\[
d \leq (1 + o(1)) \left( \frac{1 + \alpha}{4} \right)^{1/4} \sqrt{n}. \tag{8}
\]

Recall that when defining \( D \) we decomposed the edges of each crown-block into a triangle and 2-paths. This means that the number of triangles of \( G \) that belong to crown-blocks of \( G \) is at most \( \frac{\alpha e(G)}{3} + \frac{\alpha e(G)}{2} \leq \frac{ae(G)}{2} \), and the number of triangles that belong to \( K_4 \)-blocks of \( G \) is at most \( \frac{4(1 - \alpha)e(G)}{6} \). Therefore, the total number of triangles in \( G \) is at most

\[
\frac{\alpha e(G)}{2} + \frac{4(1 - \alpha)e(G)}{6} = \frac{4 - \alpha}{6} e(G) = \frac{(4 - \alpha)nd}{12}. \tag{9}
\]
Now using (8), we obtain that the number of triangles in $G$ is at most
\[
(1 + o(1))\left(\frac{1 + \alpha}{4}\right)^{1/4}\frac{(4 - \alpha)}{12}n^{3/2}.
\]

Now optimizing the coefficient of $n^{3/2}$ over $0 \leq \alpha \leq 1$, one obtains that it is maximized at $\alpha = 0$, giving the desired upper bound of $(1 + o(1))\frac{1}{3\sqrt{2}}n^{3/2}$.

3 | ON HYPERGRAPHS OF GIRTH AT LEAST SIX AND FURTHER IMPROVEMENT

In this section we will first study $r$-uniform hypergraphs of girth at least six, and prove Theorem 2. Then we use Theorem 2 to further (slightly) improve the estimate in Theorem 1 on the number of triangles in a $C_5$-free graph.

3.1 | Hypergraphs of girth at least six: Proof of Theorem 2

Let $d$ be the average degree of $H$. Our aim is to show that $d \leq (1 + o(1))\frac{\sqrt{r}}{\sqrt{(r-1)}}$. If a vertex has degree less than $\frac{d}{r}$, then we may delete it and the edges incident to it without decreasing the average degree. So we may assume that the minimum degree $d_{\text{min}}$ of $H$ satisfies $d_{\text{min}} \leq \frac{d}{r}$. If $\alpha$ is an arbitrary vertex of degree $c\sqrt{r}$ for some constant $c$, then the first neighborhood $N_1(v) := \{x \in V(H) \setminus \{v\} : x \in h \text{ for some } h \in E(H)\}$ has size more than $c\sqrt{r}(r-1)$ (since $H$ is linear), and the second neighborhood $N_2(v) := \{x \in V(H) \setminus (N_1(v) \cup \{v\}) : \exists h \in E(H) \text{ such that } x \in h \text{ and } h \cap N_1(v) \neq \emptyset\}$ has size more than
\[
c\sqrt{r}(r-1) \times (d_{\text{min}} - 1)(r-1) \geq c\sqrt{r}(r-1) \times \frac{(d - r)(r-1)}{r} = c\sqrt{r}(r-1)^2(d - r).
\]
Note that here we used that $H$ has no cycles of length at most four. On the other hand, since $|N_2(v)| \leq n$, we have $c\sqrt{r}(r-1)^2d(n-r) \leq n$, implying that $d \leq \frac{r}{r-1}c\sqrt{r} + r$. So if $c > \frac{r}{r-1}$, we have the desired bound on $d$. Thus, we may assume $c \leq \frac{r}{r-1}$, which implies that the maximum degree $d_{\text{max}}$ of $H$ satisfies $d_{\text{max}} \leq \frac{r^{3/2}}{r-1}\sqrt{n}$.

Let $\partial H$ denote the 2-shadow graph of $H$. Let $d^{\partial H}$ and $d_{\text{max}}^{\partial H}$ denote the average degree and maximum degree of $\partial H$, respectively. Note that since $H$ is linear, $d^{\partial H} = (r-1)d$ and $d_{\text{max}}^{\partial H} = (r-1)d_{\text{max}} \leq r^{3/2}/\sqrt{n}$.

We say a 3-path $v_0v_1v_2v_3$ in $\partial H$ is bad if either $\{v_0, v_1, v_2\} \subseteq h$ or $\{v_1, v_2, v_3\} \subseteq h$ for some hyperedge $h \in E(H)$; otherwise it is good.

By the Blakley–Roy inequality the total number of (ordered) 3-walks in $\partial H$ is at least $n(d^{\partial H})^3$. We claim that at most $3n(d_{\text{max}}^{\partial H})^2$ of these 3-walks are not 3-paths. Indeed, suppose $v_0v_1v_2v_3$ is a 3-walk that is not a 3-path. Then there exists a repeated vertex $v$ in the walk such that either $v_0 = v_2 = v$ or $v_1 = v_3 = v$ or $v_0 = v_1 = v$. Since $v$ can be chosen in $n$ ways and the other two vertices of the walk are adjacent to $v$, we can choose them in at most $(d_{\text{max}}^{\partial H})^2$ different ways. Therefore, the number of (ordered) 3-paths in $\partial H$ is at least $n(d^{\partial H})^3 - 3n(d_{\text{max}}^{\partial H})^2 \geq n(d^{\partial H})^3 - 3n(r^{3/2}/\sqrt{n})^2 = n(d^{\partial H})^3 - 3r^3n^2$. 
We will show that most of these 3-paths are good by bounding the number of bad 3-paths. Suppose \( v_0v_1v_2v_3 \) is a bad 3-path. Then either \( \{v_0, v_1, v_2\} \) or \( \{v_1, v_2, v_3\} \) is contained in some hyperedge \( h \in E(H) \). In the first case, the number of choices for \( v_0v_1v_2 \) is \( |E(H)| \binom{r}{3} \) as there are \( \binom{r}{3} \) ways to choose the vertices \( v_0, v_1, v_2 \) from a hyperedge of \( H \) and then \( 3! \) ways to order them. And there are at most \( d H_{\text{max}} \) choices for \( v_3 \). The second case is similar. Therefore, in total, the number of bad 3-paths in \( \partial H \) is at most \( 2|E(H)| \binom{r}{3} d_{\text{max}} H < 2n d d_{\text{max}} H \leq 2n r^2 d_{\text{max}} H \). So the number of (ordered) good 3-paths in \( \partial H \) is at least

\[
3r^3 n^2 n^2 - 2 \frac{r^5}{r-1} n^2 = n (d H) - c_r n^2 = (r-1) d H n - c_r n^2,
\]

where \( c_r = 3r^3 + \frac{2r^5}{r-1} \).

The following claim is useful for upper bounding the number of (ordered) good 3-paths in \( \partial H \).

**Claim 7.** If \( C \) is a cycle of length at most five in \( \partial H \), then its vertex set is contained in some hyperedge of \( H \).

**Proof.** Let \( v_1, v_2, \ldots, v_k, v_1 \) be a cycle of length \( k \) in \( \partial H \) (for some \( k \leq 5 \)). For each \( i \), let \( h_i \) be the hyperedge of \( H \) containing \( v_i, v_{i+1} \) (addition in the subscripts is taken modulo \( k \)). If these \( k \) hyperedges are not all the same, there exist \( j, j' \) such that \( h_j, h_{j+1}, \ldots, h_{j'} \) are all distinct but \( h_{j'+1} = h_j \). So these hyperedges form a cycle in \( H \) of length at most \( k \leq 5 \), a contradiction. Therefore, \( h_1 = h_2 = \cdots = h_k = h \); then \( v_1, v_2, \ldots, v_k \in h \), as desired. \qed

To upper bound the number of (ordered) good 3-paths in \( \partial H \), let us first fix a hyperedge \( h \) of \( H \), and bound the number of good 3-paths \( v_0v_1v_2v_3 \) such that \( v_0, v_1 \in h \).

**Claim 8.** For any vertex \( v \notin h \), there are at most \( (r-1) \) good 3-paths \( v_0v_1v_2v \) such that \( v_0, v_1 \in h \).

**Proof.** Suppose \( v_0v_1v_2v \) and \( v'_0v'_1v'_2v \) are good 3-paths with \( v_0, v_1, v'_0, v'_1 \in h \). Then \( v_2, v'_2 \notin h \) because it would contradict the definition of a good 3-path. We will prove that \( v_1 = v'_1 \) and \( v_2 = v'_2 \).

Suppose \( v_1 \neq v'_1 \). Notice that \( v_1v'_1 \in E(\partial H) \) since \( v_1, v'_1 \in h \). Then depending on whether \( v_2 = v'_2 \) or not, either \( v_1v'_2v_2v_1 \) forms a four-cycle or \( v_1v'_2v'2 \) forms a triangle in \( \partial H \). Then by Claim 7, \( v_1, v'_1, v'_2 \in h' \) for some hyperedge \( h' \in E(H) \). (Note that \( h' \neq h \), since \( v'_2 \notin h \).) But then \( h \) and \( h' \) are two different hyperedges of \( H \) that share at least two vertices, namely, \( v_1, v'_1 \), contradicting the fact that \( H \) is linear. Thus \( v_1 = v'_1 \).

Now if \( v_2 \neq v'_2 \), then \( v_2v_1v'_2 \) is a four-cycle in \( \partial H \), so it must be contained in a hyperedge of \( H \), but this means the 3-path \( v_0v_1v_2v_3 \) is bad, a contradiction. Thus \( v_2 = v'_2 \).

In summary, any two good 3-paths \( v_0v_1v_2v \) and \( v'_0v'_1v'_2v \) with \( v_0, v_1, v'_0, v'_1 \in h \) can only differ in their first vertex, of which there are at most \( h \setminus \{v_1\} \) choices, proving the claim. \qed
Claim 8 implies that for any fixed hyperedge \( h \in E(H) \), there are at most \((r-1)n\) good 3-paths \( v_0v_1v_2v_3 \) with \( v_0, v_1 \in h \). Therefore, the total number of good 3-paths in \( H \) is at most \(|E(H)|(r-1)n = \frac{(r-1)n!}{r^3} \).

Combining this with (10), we obtain \((r-1)^3d^2n - c_rn^2 \leq \frac{(r-1)n!}{r} \). Dividing through by \( d \) and using that \( d = \Omega(\sqrt{n}) \), we get \((r-1)^3d^2n \leq (1 + o(1))\frac{(r-1)n!}{r^3} \) and upon simplification and rearranging, we get

\[
d \leq (1 + o(1))\frac{\sqrt{n}}{\sqrt{r}(r-1)},
\]

and using \(|E(H)| = nd/r \) completes the proof.

3.2 Further improving the estimate on \( ex(n, K_3, C_5) \)

Here we slightly improve Theorem 1, by establishing a connection to hypergraphs of girth at least six and using Theorem 2.

Recall that in the proof of Theorem 1, \( G \) denotes a \( C_5 \)-free graph, and \((1 - \alpha)e(G) \) edges of \( G \) belong to the \( K_4 \)’s in the edge-decomposition \( \Delta \) of \( G \). Let us note that the vertex sets of two different \( K_4 \)’s of \( G \) do not share more than one vertex, since \( G \) is \( C_5 \)-free. Consider the 4-uniform hypergraph \( H \) formed by taking the vertex sets of all the \( K_4 \)’s of \( G \). Then notice that \( H \) is linear and if \( H \) contains a (Berge) cycle of length at most 5, then \( G \) contains a \( C_5 \). Therefore, \( H \) is of girth at least six. Therefore, by Theorem 2, \( H \) contains at most \((1 + o(1))n^{3/2}/24 \) hyperedges. Thus at most \((1 + o(1))n^{3/2}/24 \times \binom{4}{2} = (1 + o(1))n^{3/2}/4 \) edges of \( G \) belong to the \( K_4 \)’s in the edge-decomposition \( \Delta \). Therefore, \((1 - \alpha)e(G) \leq (1 + o(1))n^{3/2}/4 \), which implies \( d \leq (1 + o(1))\frac{\sqrt{n}}{\sqrt{2(1-\alpha)}} \). Combining this with (8), we get

\[
d \leq (1 + o(1))\min\left\{\frac{1}{2(1-\alpha)}, \left(\frac{1 + \alpha}{4}\right)^{1/4}\right\}\sqrt{n},
\]

so using (9), we obtain that the number of triangles in \( G \) is at most

\[
(1 + o(1)) \left(\frac{4 - \alpha}{12}\right) \min\left\{\frac{1}{2(1-\alpha)}, \left(\frac{1 + \alpha}{4}\right)^{1/4}\right\} n^{3/2}.
\]

The above function is maximized at \( \alpha = 0.343171 \), which shows that \( ex(n, K_3, C_5) \leq (1 + o(1)) 0.231975n^{3/2} \).

4 C_5-FREE AND INDUCED-C_4-FREE GRAPHS: PROOF OF THEOREM 3

Let \( G \) be a \( C_5 \)-free graph on \( n \) vertices having no induced copies of \( C_4 \). Let \( G_\alpha \) be the subgraph of \( G \) consisting of the edges that are contained in triangles of \( G \), and let \( G_\delta \) be the subgraph of \( G \) consisting of the remaining edges of \( G \). Since \( G_\alpha \) is \( C_5 \)-free and every edge of it is contained in a
triangle, by the same argument of the proof of Theorem 1, the triangles of $G_\Delta$ can be partitioned into crown-blocks and $K_4$-blocks. So there is a decomposition $\mathcal{D}$ of the edges of $G_\Delta$ into paths of length 2, triangles and $K_4$’s. First let us note that Claim 2 in the proof of Theorem 1 still holds for $G$ (not just for $G_\Delta$), as shown below.

**Claim 9.** Let $u, v$ be two nonadjacent vertices of $G$. Then the number of paths of length 2 between $u$ and $v$ is at most two. Moreover, if $uxv$ and $uyv$ are the paths of length 2 between $u$ and $v$, then $x$ and $y$ are adjacent.

**Proof.** The second part of the claim is trivial since $G$ does not contain an induced copy of $C_4$. To see the first part of the claim, suppose $uxv uyv uzv$, are three distinct paths of length 2 in $G$. Then $x$ and $y$ are adjacent, so $uxyvz$ is a $C_5$ in $G$, a contradiction. $\square$

Our goal is to bound the average degree $d$ of $G$. If a vertex has degree less than $d/2$, then it may be deleted without decreasing the average degree of $G$, so we may assume that $G$ has minimum degree at least $d/2$. Now using this fact and Claim 9, one can show that the maximum degree of $G$ is at most $10\sqrt{n}$ by repeating the same argument as in the proof of Claim 3.

We say a 5-path $v_0v_1v_2v_3w$ is *bad* if there exists an $i$ such that $v_iv_{i+1}v_{i+2}$ is a triangle of $G$; otherwise it is called *good*. Similarly, a 2-path $abc$ is *good* if $a$ and $c$ are not adjacent. By the same argument as in the proof of Theorem 1, the number of (unordered) good 5-paths in $G$ is at least

$$\frac{nd^5}{2} - Cn^3$$

for some constant $C > 0$. Now we bound the number of good 5-paths in $G$ from above. Let $|E(G_\Delta)| = \alpha|E(G)|$ for some $\alpha \geq 0$, so $|E(G_S)| = (1 - \alpha)|E(G)|$.

**Claim 10.** The number of good 5-paths in $G$ whose middle edge is contained in $G_S$ is at most $|E(G_S)|n^2$.

**Proof.** The proof is very similar to that of the proof of Claim 4. A good 5-path $xyabzw$ whose middle edge $ab$ is in $G_S$ contains good 2-paths, $xya$, $bzw$ as subpaths.

Let us fix $ab \in E(G_S)$ and let $n_a$ be the number of good 2-paths in $G$ of the form $axy$ where $x, y \neq b$, and let $n_b$ be the number of good 2-paths in $G$ of the form $bxy$ where $x, y \neq a$. Then the number of good 5-paths whose middle edge is $ab$ is at most $n_an_b \leq (n_a + n_b)^2/4$. By the same argument as in the proof of Claim 4, it is easy to see that $n_a + n_b \leq 2n$, so the number of good 5-paths whose middle edge is $ab \in E(G_S)$ is at most $n^2$. Adding these estimates for all the edges $ab \in E(G_S)$ finishes the proof of the claim. $\square$

Let us further assume that the number of edges of $G_\Delta$ that belong to paths of length 2, triangles and $K_4$’s in its edge-decomposition $\mathcal{D}$ is $\alpha_1|E(G)|, \alpha_2|E(G)|, \alpha_3|E(G)|$, respectively. (Of course, $\alpha_1 + \alpha_2 + \alpha_3 = \alpha$.) Since Claim 9 holds, one can easily check that the proofs of Claims 4–6 are still valid, so these claims hold in the current setting too. These claims, together with Claim 10, imply that the number of good 5-paths in $G$ is at most
We will now bound the right-hand side of the above inequality by carefully selecting a $C_5$-free, and $C_4$-free subgraph $G'$ of $G$, as follows: We select all the edges of $G_S$ and the following edges from $G_{\Delta}$: From each crown-block $\{abc_1, abc_2, ..., abc_k\}$ of $G_{\Delta}$, we select the edges $ac_1, ac_2, ..., ac_k$ to be in $G'$. From each $K_4$-block $abcd$ we select the edges $ab, bc, ac, ad$ to be in $G'$.

To show that $G'$ is $C_4$-free we use the following claim.

\textbf{Claim 11.} All four edges of any $C_4$ in $G$ belong to only one block of $G_{\Delta}$.

\textbf{Proof.} Let $xyzw$ be a 4-cycle in $G$. Then since $G$ does not contain an induced copy of $C_4$, either $xz$ or $yw$ is an edge of $G$. In the first case, $xyz$, $xzw$ are triangles of $G$, and in the second case $ywx, yw$ are triangles of $G$. In both cases, the two triangles share an edge, so they belong to the same block of $G_{\Delta}$. Hence, all four edges of $xyzw$ lie in the same block of $G_{\Delta}$. \hfill \Box

By Claim 11, the edge set of every $C_4$ in $G$ is completely contained in some block of $G_{\Delta}$, and it is easy to check that the selected edges in each block of $G_{\Delta}$ form a $C_4$-free graph. Therefore, $G'$ is $C_4$-free. Since it is a subgraph of $G$, it is also $C_5$-free. Therefore, by a theorem of Erdős and Simonovits [4], $|E(G')| \leq (1+o(1))\frac{1}{2\sqrt{2}}n^{3/2}$. On the other hand, since all the edges of $G_S$ and at least half the edges of $G_{\Delta}$ are selected, we have $|E(G')| \geq |E(G_S)| + \frac{|E(G_{\Delta})|}{2} = (1-\alpha)|E(G)| + \frac{\alpha|E(G)|}{2}$. Therefore,

$$\frac{\alpha|E(G)|}{2} + (1-\alpha)|E(G)| \leq (1+o(1))\frac{1}{2\sqrt{2}}n^{3/2}.$$ 

Therefore, by the discussion above, the number of good 5-paths in $G$ is at most $(1+o(1))\frac{1}{2\sqrt{2}}n^{3/2} \times n^2 = (1+o(1))\frac{1}{2\sqrt{2}}n^{7/2}$. Combining this with (11), we get

$$\frac{nd^5}{2} - Cn^3 \leq (1+o(1))\frac{1}{2\sqrt{2}}n^{7/2},$$

so $\frac{nd^5}{2} \leq (1+o(1))\frac{1}{2\sqrt{2}}n^{7/2}$, implying that $d \leq (1+o(1))\frac{n}{\sqrt[7]{2}}$. Therefore, using that $|E(G)| = nd/2$ finishes the proof.

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