Separation profiles, coarse embeddability and inner expansion

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Abstract

Using separation profiles we prove that there are uncountably many coarse equivalence classes of finitely generated groups which coarsely contain expanders. We prove that the separation profile detects the presence of expanding subgraphs and give an upper bound on the separation profile of any group with finite asymptotic dimension. The bounds obtained are sharp for virtually abelian groups, RAAGs and mapping class groups of sufficient complexity. As a consequence we bound the Cheeger constant of finite subgraphs of Cayley graphs of such groups.

1 Introduction

Any finitely generated group which admits a coarse embedding into a Hilbert space satisfies two long-standing conjectures in topology: the Novikov and coarse Baum-Connes conjectures [Yu00]. The only currently known method of constructing groups which do not admit such an embedding is to use small cancellation techniques to embed a family of expander graphs into a group [Gro00, AD08, Osa14].

A natural and interesting problem arising from this is to quantify how expanding an infinite graph is. One may consider the Cheeger constant $h(\cdot)$ as a measure of “outer expansion” which detects amenability of a group; in this paper we study the separation profile defined by Benjamini-Schramm-Timár as a measure of “inner expansion” [BSTT2].

Given a finite graph $\Gamma$ with $n$ vertices, the cut size of $\Gamma$ is the minimal size of a set of vertices $S$ with the property that any connected component of $\Gamma \setminus S$ has at most $n/2$ vertices.

The separation profile $\text{sep}_X$ of an infinite graph $X$ evaluated at $n$ is the maximum of the cut sizes of all subgraphs of $X$ with at most $n$ vertices. Up

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to affine equivalence this function is a coarse invariant of infinite graphs with uniformly bounded degree, so is well-defined for a finitely generated group.

By other names this profile has a longer history, for instance, the Lipton-Tarjan theorem states that the cut size of any \( n \) vertex planar graph is \( O(\sqrt{n}) \) \cite{LT79}. More recently, Shchur proved that separation occurs naturally as an obstruction to quasi–isometrically embedding a metric space into a tree \cite{Shc14}.

Our first goal is to formalise the relationship between separation profiles and expanders. We start by considering finite graphs.

**Proposition 1.1.** Let \( \Gamma \) be a finite graph with \( n \geq 2 \) vertices. There is a subgraph \( \Gamma' \) of \( \Gamma \) with at least \( \frac{n}{2} \) vertices such that

\[
\frac{nh(\Gamma)}{3} \leq \text{cut}(\Gamma) \leq 2nh(\Gamma').
\]

Applying this to infinite graphs, we prove the following equivalence between expanders and sublinear separation.

**Theorem 1.2.** Let \( X \) be an infinite graph. Then \( \text{sep}_X(n)/n \to 0 \) if and only if for every collection of finite subgraphs \( (\Gamma_n)_{n \in \mathbb{N}} \) of \( X \) such that \( |\Gamma_n| \to \infty \) we have \( h(\Gamma_n) \to 0 \).

This means that the separation profile detects collections of expanding subgraphs of a given graph and that admitting an expander family as subgraphs is a coarse invariant of graphs with bounded geometry.

We prove directly (Proposition 3.7) that any graph of uniformly bounded degree which coarsely contains expanders cannot have sublinear separation. This answers negatively a part of \cite[Question 1.1]{BST12}. As a consequence of the above results we deduce

**Proposition 1.3.** Let \( X \) be an infinite graph of uniformly bounded degree. If \( X \) coarsely embeds into a Hilbert space then \( \text{sep}_X(n)/n \to 0 \).

However, recent results of Arzhantseva-Tessera give examples of box spaces with sublinear separation - because they do not weakly contain expanders - which do not coarsely embed into any uniformly convex Banach space \cite{AT14}.

Combining the ideas of this section of the paper and a construction of groups containing expanders due to Osajda \cite{Osa14} we use separation profiles to prove

**Theorem 1.4.** There are uncountably many coarse equivalence classes of finitely generated groups which coarsely contain expanders.

Theorem 1.4 will show that there are uncountably many different separation profiles of finitely generated groups which are not sublinear. Our next topic of interest - as highlighted by \cite[Question 1.1]{BST12} - concerns the possible sublinear functions which can occur as separation profiles.

In the paper \cite{BST12} the authors give bounds on the separation function for various classes of groups: hyperbolic, virtually nilpotent and direct products of free groups; discovering the following possible separation profiles: bounded,
log(n), polynomial $n^{1-1/d}$ for each $d \in \{2,3,\ldots\}$ and $n/\log(n)$. These were the only known profiles before this paper.

Extending the ideas of [BST12] we provide upper bounds on the separation profiles of groups with finite asymptotic dimension (cf. Definition 4.1) which is optimal for groups with finite Assouad-Nagata dimension. Spaces with finite Assouad-Nagata have $\ell^p$ compression exponent 1, which can be thought of as a very strong form of coarse embeddability [Gal08].

The class of groups with finite Assouad-Nagata dimension includes: polycyclic groups; mapping class groups and relatively hyperbolic groups whose parabolic subgroups have finite Assouad-Nagata dimension [Hum12]; fundamental groups of compact 3-manifolds [MST13]; right-angled Coxeter and Artin groups [DJ99, DJ00]; virtually special groups [HW08] and lamplighter groups [BDS07]; but, for example, Thompson’s groups and wreath products of any two infinite finitely generated groups do not lie in this collection [BDL14].

We prove a more general theorem 4.2 for bounded degree graphs with finite asymptotic dimension, but the key conclusion is

**Theorem 1.5.** Let $G$ be a group with Assouad-Nagata dimension at most $m$, and let $\gamma(n)$ be the growth function of $G$. Then

$$\text{sep}_G(n) \preceq n/(f(n/(2m+2)))$$

where $f$ is the inverse growth function $f(n) = \max \{ k \mid \gamma(k) \leq n \}$.

As a consequence, if $G$ is virtually abelian, and hence virtually $\mathbb{Z}^k$ for some $k$, then we recover the optimal bound $\text{sep}_G(n) \preceq n^{(k-1)/k}$ found in [BST12 Proposition 4.1].

Combining this with Proposition 1.1 we obtain bounds on the Cheeger constant of finite subgraphs of Cayley graphs, we present a sample result below.

**Corollary 1.6.** Let $X$ be a Cayley graph of a group with finite Assouad-Nagata dimension. There exists a constant $C$ such that any $n$-vertex subgraph $\Gamma$ of $X$ satisfies $h(\Gamma) \leq C/\log(n)$.

The remainder of the paper reads as follows: following some preliminaries, section 3 discusses the links between separation, inner expansion and coarse embeddings. In section 4 we bound the separation of groups with finite asymptotic dimension and prove Theorem 1.5. Finally, in section 5 we discuss Corollary 1.6 and other consequences of Theorem 1.5 for various classes of groups. Throughout the paper we will raise a number of questions and possible extensions of the results.

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2 Preliminaries

Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a finite simplicial graph. As a shorthand we will write $|\Gamma|$ for $|V(\Gamma)|$. For a given $c \in (0, 1)$ we define the $c$-cut size of $\Gamma$ to be the smallest $k$ such that there is a set $C \subset V(\Gamma)$ with $|C| = k$ such that any connected component of $\Gamma \setminus C$ - the graph obtained by removing all vertices of $C$ and any edge with an end vertex in $C$ - contains at most $c|\Gamma|$ vertices. We call such $C$ a $c$-cutset for $\Gamma$.

This idea is extended to an infinite graph $X$ by defining the $c$-separation profile, $\text{sep}_c^X : \mathbb{N} \to \mathbb{N}$ as follows:

$$\text{sep}_c^X(n) = \max \{ \text{cut}^c(\Gamma) \mid \Gamma \subseteq X, |\Gamma| \leq n \}.$$  

We will consider separation profiles up to the natural equivalence $f \preceq g$ if there exists a constant $C$ such that $f(n) \leq Cg(n) + C$ and $f \asymp g$ if and only if $f \preceq g$ and $g \preceq f$.

The following results are all found in [BST12]. We will use them in the remainder of the paper without reference.

**Theorem 2.1.** For any graph $X$ and any $c, c' \in (0, 1)$, $\text{sep}_c^X \asymp \text{sep}_{c'}^X$. If $X, X'$ are coarsely equivalent graphs with bounded geometry then $\text{sep}_c^X \asymp \text{sep}_{c'}^{X'}$, so separation is well-defined for a finitely generated group. Finally, if $X'$ is a subgraph of $X$, then $\text{sep}_{c'}^{X'} \preceq \text{sep}_c^X$.

As a shorthand we will use the convention that $c = 1/2$ unless otherwise stated.

Next we recall the definition of Cheeger constant and of a family of expander graphs.

**Definition 2.2.** Let $\Gamma$ be a graph with $|\Gamma| = n$. The vertex-boundary of a subset $A \subseteq V(\Gamma)$ - denoted $\partial A$ - is the set of all vertices in $V(\Gamma) \setminus A$ which are neighbours of some vertex of $A$. The (vertex) Cheeger constant of $\Gamma$ is given by

$$h(\Gamma) = \min \left\{ \frac{|\partial A|}{|A|} \mid |A| \leq n/2 \right\}.$$

Fix some $\varepsilon > 0$. A collection of finite graphs $(\Gamma_n)_{n \in \mathbb{N}}$ is said to be $\varepsilon$-expanding if $|\Gamma_n| \to \infty$ as $n \to \infty$ and $\inf_n h(\Gamma_n) \geq \varepsilon$. If, in addition, each $\Gamma_n$ has maximal degree at most $d$ then $(\Gamma_n)_{n}$ is said to be a $(d, \varepsilon)$-expander.

3 Separation and inner expansion

3.1 Cut sets and the Cheeger constant

In this section we prove Proposition [1.1]

**Proposition 3.1.** Let $\Gamma$ be a graph with $n \geq 2$ vertices. Then

$$\text{cut}(\Gamma) \geq n.h(\Gamma)/3.$$
Proof. Suppose \( h(\Gamma) = \varepsilon \).

Let \( C \) be any cutset of \( \Gamma \) with \(|C| \leq 2n/3\), so any component of \( \Gamma \setminus C \) contains at most \( n/2 \) points. Define \( D \) to be a union of such components with cardinality between \( n/3 \) and \( n/2 \). A greedy search will suffice to do this. As \( \partial D \subseteq C \) we see that \(|C| \geq (n/3)\varepsilon\). Hence, \( \text{cut}(\Gamma) \geq n. h(\Gamma)/3 \).

For the other bound, we will require a more sensitive type of cut.

**Definition 3.2.** Let \( \Gamma \) be a graph with \( n \) vertices. A \( k \)-good cut of \( \Gamma \) is denoted \( \Gamma \to C \Gamma' \) where \( C \) is a non-empty set of vertices of \( \Gamma \), \( \Gamma' \) is some largest connected component of \( \Gamma \setminus C \) and

\[
|\Gamma| - |\Gamma'| > k.
\]

From the definition of a cut set it is clear that \(|\Gamma| > |\Gamma'| > |\Gamma|/2\) whenever \( k \geq c_\Gamma := |\Gamma|/\text{cut}(\Gamma) \) so there is a unique largest component in this case. As we are working with finite graphs every sequence of \( k \)-good cuts terminates.

**Proposition 3.3.** Let \( \Gamma \) be a graph with \( n \geq 2 \) vertices and let \( \Gamma \to C_1 \to C_2 \to \cdots \to C_m \) be any maximal sequence of \( 3c_\Gamma/2 \)-good cuts. Then \( |\Gamma_m| \geq n/2 \) and \( h(\Gamma_m) \geq \text{cut}(\Gamma)/2n \).

**Proof.** Suppose \( |\Gamma_m| < n/2 \). Then \( \bigcup C_i \) is a cut set for \( \Gamma \) containing at most \( 2\text{cut}(\Gamma)/3 \) points, which is a contradiction.

Now let \( A \subset \Gamma_m \) with \(|A| \leq |\Gamma_m|/2\). As \( \Gamma_m \) admits no \( 3c_\Gamma/2 \)-good cuts, we know that

\[
|A| + |\partial A| \leq 3c_\Gamma/2.
\]

Rearranging this, we see that \(|\partial A| \geq |A|/2c_\Gamma\). Hence,

\[
|\partial A| / |A| \geq \text{cut}(\Gamma)/2n.
\]

\[ \square \]

### 3.2 Expanders and sublinear separation

We first prove Theorem 1.2.

**Proposition 3.4.** Let \( X \) be an infinite graph which contains some \( \varepsilon \)-expanding family of subgraphs \((\Gamma_n)_{n \in \mathbb{N}}\). For every \( n \),

\[
\text{sep}_X(\Gamma_n) \geq |\Gamma_n|\varepsilon/3.
\]

In particular, \( \text{sep}_X(n)/n \not\to 0 \).

**Proof.** This follows immediately from Proposition 3.1. \[ \square \]

**Proposition 3.5.** Let \( X \) be an infinite graph and suppose \( \text{sep}_X(n)/n \not\to 0 \). Then there exists some \( \varepsilon > 0 \) such that \( X \) contains an \( \varepsilon \)-expanding family \((\Gamma_n)_{n \in \mathbb{N}}\) of subgraphs.
Proof. Let \( \varepsilon > 0 \) be such that \( \text{sep}_X(n) \geq 2\varepsilon n \) for all \( n \) in some infinite subset \( I \subseteq \mathbb{N} \).

For each \( n \in I \) let \( \Gamma'_n \) be a subgraph of \( X \) with at most \( n \) vertices such that \( \text{cut}(\Gamma'_n) \geq 2\varepsilon n \).

By Proposition 3.3 each \( \Gamma'_n \) has a subgraph \( \Gamma_n \) with \( h(\Gamma_n) \geq \varepsilon \). Moreover, \( |\Gamma_n| \to \infty \) as \( n \to \infty \).

Now we concentrate on graphs with uniformly bounded degree, where the separation profile is a coarse invariant. We obtain one immediate consequence of Proposition 3.3.

**Corollary 3.6.** Let \( X \) be a graph of bounded degree which admits a coarse embedding into an \( L_p \) space. Then \( \text{sep}_X(n)/n \to 0 \) as \( n \to \infty \).

**Proof.** We argue by contradiction. If \( \text{sep}_X(n)/n \not\to 0 \), then \( G \) injectively contains a \((d, \varepsilon)\)-expander family for some \( d \) and \( \varepsilon > 0 \), so it weakly contains expanders. Hence, \( G \) cannot coarsely embed into an \( L_p \) space.

This is of course untrue without the bounded degree hypothesis, we can just consider an infinite complete graph.

**Proposition 3.7.** Let \( X \) be an infinite graph with uniformly bounded degree which coarsely contains expanders. Then \( \text{sep}_X(n)/n \not\to 0 \).

**Proof.** By hypothesis, there exists a family \((Y_m)_{m\in\mathbb{N}}\) of subgraphs with increasing number of vertices, (vertex) Cheeger constant at least \( \varepsilon \) and \((K-2)\)-Lipschitz maps \( \rho_m : Y_m \to X \) where pre-images of points in \( X \) under \( \rho_m \) have cardinality at most \( M \). The constants \( M \), \( K \) and \( K' \) can be chosen to be independent of the choice of \( m \).

Taking a direct product of \( X \) with an \( M \)-clique we may assume that each \( \rho_m \) is injective and \( K \)-Lipschitz.

Let \( X^K \) be the graph obtained from \( X \) by adding edges \( uv \) whenever \( 2 \leq d(v, w) \leq K \). The graphs \( X^K \) and \( X \) are quasi-isometric, so have equivalent separation function. The map \( \rho_m : Y_m \to X^K \) is a graph monomorphism so the Cheeger constant of \( Z_m \), the induced subgraph of \( X^K \) with vertex set \( \rho_m(Y_m) \) is at least \( \varepsilon \). Thus \( \text{sep}_{X^K}(n)/n \not\to 0 \) by Lemma 3.1. Now, as \( X^K \) is quasi-isometric to \( X \) we deduce that \( \text{sep}_X(n)/n \not\to 0 \).

From Propositions 3.5 and 3.7 we deduce the following implications:

\[
\begin{align*}
G \text{ coarsely contains expanders} & \quad \downarrow \\
\text{sep}_G(n)/n \not\to 0 & \quad \downarrow \\
G \text{ weakly contains expanders.} & \quad \\
\end{align*}
\]

The existence of a group which weakly (but not coarsely) contains expanders is still open.
3.3 Separation profiles of groups containing expanders

Osajda’s construction of $C'(1/6)$ small cancellation labellings of graphs satisfying certain girth - $g(\cdot)$, the length of the shortest simple loop - and diameter restrictions gives a method for constructing finitely generated groups which isometrically contain a family of expander graphs.

We let $\mathbf{\Gamma} = (\Gamma_n)_{n \in \mathbb{N}}$ be such a family, which we think of as $(d, \varepsilon)$-expander graphs whose edges are labelled by a finite set $S$. Given any collection of finite graphs $\mathbf{A}$ with a small cancellation labelling, we define a group $G(\mathbf{A})$ which is generated by $S$ and satisfies precisely the set of relations obtained by reading the labels of simple loops in the graphs.

By [Oll06], we know that $G(\mathbf{A})$ is hyperbolic whenever $\mathbf{A}$ is finite. More information on this construction - graphical small cancellation theory - can be found in [Gro00] where it was introduced, and in [Oll06, Osa14].

We impose two additional conditions on this family of graphs which are both satisfied by taking a suitably sparse subsequence of $\mathbf{\Gamma}$.

- For every $k$, and every subset $\mathbf{A} \subseteq \{\Gamma_1, \ldots, \Gamma_k\}$ we have
  \[
  \operatorname{sep}_{G(\mathbf{A})}(n) < n/k \quad \text{for every} \quad n \geq |\Gamma_{k+1}|.
  \]

- For every $k$, $g(\Gamma_{k+1}) > 2|\Gamma_k|$.

The first is possible as for each $k$ we consider finitely many hyperbolic groups, which all have at most polynomial separation $Cn^{(d-1)/d}$ for some $C > 0$ and $d(k) \geq 2$. For the second, we just use the fact that Osajda’s construction assumes that the girth of the sequence $\Gamma_n$ is unbounded.

Now we can prove Theorem 1.4.

**Theorem 3.8.** Let $A, B$ be two infinite subsets of $\mathbb{N}$ with infinite symmetric difference. Define $\mathbf{A}(A), \mathbf{A}(B) = \{\Gamma_n \mid n \in A, B\}$ respectively. Then

\[
\operatorname{sep}_{G(\mathbf{A}(A))}(n) \neq \operatorname{sep}_{G(\mathbf{A}(B))}(n).
\]

In particular, $G(\mathbf{A}(A))$ and $G(\mathbf{A}(B))$ are not coarsely equivalent.

**Proof.** Without loss of generality, we assume that $C = A \setminus B$ is infinite.

If $k \in C$ then $\Gamma_k$ is an isometrically embedded subgraph of $G(\mathbf{A}(A))$ so $\operatorname{sep}_{G(\mathbf{A}(A))}(|\Gamma_k|) \geq \epsilon/3 |\Gamma_k|$, by Lemma 3.1.

Now let $\Gamma$ be a subgraph of $G(\mathbf{A}(B))$ with at most $|\Gamma_k|$ vertices. Using the assumption that $g(\Gamma_{k+1}) > 2|\Gamma_k|$, we see that $\Gamma$ isometrically embeds in some $G(\mathbf{A})$ with $\mathbf{A} \subseteq \{\Gamma_1, \ldots, \Gamma_{k-1}\}$. Hence, $\operatorname{sep}_{G(\mathbf{A}(B))}(|\Gamma_k|) < |\Gamma_k|/k$.

As $C$ is infinite we deduce that $\operatorname{sep}_{G(\mathbf{A}(A))}(n) \nleq \operatorname{sep}_{G(\mathbf{A}(B))}(n)$.

Theorem 1.4 follows by noticing that there are uncountable families of infinite subsets of $\mathbb{N}$ with infinite pairwise symmetric difference.
### 3.4 Inner Cheeger function

One could hope to prove a statement of the following type.

**"Theorem".** Let $X$ be an infinite graph. For each $n$ define $h^\circ_X(n)$ to be the maximal Cheeger constant of any $n$-vertex subgraph of $X$. Then

$$\text{sep}_X(n) \asymp n h^\circ_X(n).$$

Indeed, the inequality $\text{sep}_X(n) \geq n h^\circ_X(n)$ follows immediately from Proposition 3.1.

However, this “inner Cheeger function” $h^\circ$ is too sensitive, as the following example shows.

**Proposition 3.9.** There is an infinite graph $X$ with bounded geometry such that $\text{sep}_X(n) \not\geq n h^\circ_X(n)$.

**Proof.** The construction is as follows. Let $(\Delta_m)$ and $(\Lambda_m)$ be two families of expanders of unbounded girth and suppose $|\Delta_m| = o(|\Lambda_m|)$. We will also assume that $|\Delta_m|$ grows sufficiently quickly, so the girth of $\Delta_{m+1}$ is at least $2|\Lambda_m|$. Define $\Gamma_m$ to be a graph obtained from the disjoint union of $\Lambda_m$ and $\Delta_m$ adding a single edge to connect these graphs and set $X$ to be the graph obtained from the disjoint union of the $\Gamma_m$ adding an edge $e_m$ connecting $\Gamma_m$ to $\Gamma_{m+1}$.

Evaluating $\text{sep}_X(n)$ and $n h^\circ_X(n)$ for $n = |\Gamma_m|$, we see that $\text{sep}_X(n) > cn$ for some $c > 0$ - to find a cutset for $\Gamma_m$ we must - at least - find a $2/3$-cutset for $\Lambda_m$. However, $h^\circ(n)$ achieves maximal value on $\Gamma_m$ - this requires the high girth and quick growth of $|\Delta_m|$ assumptions - which is at most $1/|\Delta_m|$. 

This is a very silly example, but it highlights one side of the real problem. The value of the separation function $\text{sep}_X(n)$ need not be realised by a graph with $n$ vertices, indeed, the fact that it may be realised by relatively small graphs is the cornerstone of the proof of Theorem 1.4 above. Conversely, we cannot allow the inner Cheeger function to consider all subgraphs with at most $n$ vertices - we would obtain the constant function 1 as soon as the graph has an edge - so we are left with more cumbersome statements like those below. The proofs are straightforward consequences of previous results in the paper.

**Proposition 3.10.** Let $X$ be an infinite graph and let $I \subseteq \mathbb{N}$ be such that $\text{sep}_X(n-1) < \text{sep}_X(n)$ for all $n \in I$. Then we have the following equivalence of functions $I \to \mathbb{N}$.

$$\text{sep}_X(n) \asymp n h^\circ_X(n).$$

**Proposition 3.11.** Let $X$ be an infinite graph and for every $n$, let $\Gamma_n$ be a subgraph of $X$ with $|\Gamma_n| \leq n$ and $\text{cut}(\Gamma_n) = \text{sep}_X(n)$. Then

$$\text{sep}_X(n) \geq |\Gamma_n| h^\circ_X(|\Gamma_n|).$$

**Proposition 3.12.** Let $X$ be an infinite graph and suppose that for every $n$ there is some subgraph $\Gamma_n$ of $X$ with $2cn \leq |\Gamma_n| \leq n$ and $\text{cut}(\Gamma_n) \asymp \text{sep}_X(n)$. Then

$$\text{sep}_X(n) \asymp n \max \{ h(\Gamma) \mid \Gamma \leq X, \text{cn} \leq |\Gamma| \leq n \}.$$
4 Separation and asymptotic dimension

In this section we obtain bounds on the separation profile using finite asymptotic dimension.

**Definition 4.1.** Let $X$ be a metric space. We say $X$ has **asymptotic dimension** at most $m$ if there exists a function $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $r > 0$ we can decompose $X$ into $m + 1$ subsets $X_0, \ldots, X_m$ and further decompose each $X_i$ into sets $X_{i,j}$ such that

$$d(X_{i,j}, X_{i,j'}) > r \quad \text{whenever} \quad j \neq j',$$

and $\sup \{\text{diam}(X_{i,j})\} \leq g(r)$.

We say $X$ has **Assouad-Nagata dimension** at most $m$ if the above holds with $g(r) \leq Cr$ for some constant $C > 0$.

We now prove Theorem 1.5 as a consequence of the much more general result stated below.

**Theorem 4.2.** Let $X$ be a graph with asymptotic dimension at most $m$, let $g$ be a function provided by the above definition and let $\gamma(n)$ be the growth function of $X$. Then

$$\text{sep}_X(n) \leq m n / (f_g(n/(2m + 2)))$$

where $f_g(n) = \max \{k \mid \gamma(g(k)) \leq n\}$ is the inverse growth function of $g \circ \gamma$.

**Proof.** Let $\Gamma$ be an $n$-vertex subgraph of $X$, then setting $r = f_g(n/(2m + 2))$, we obtain a cover of $X$ by $m + 1$ subsets $B_0, \ldots, B_m$ such that each $B_i$ decomposes into subsets $B_{i,j}$ of diameter at most $g(r)$ which are $r$ disjoint.

It follows immediately that for some $i$, $\Gamma \cap B_m$ contains at least $n/(m + 1)$ vertices, without loss we assume this is true for $i = 0$. Now, each $B_{i,j}$ meets at most $n/(2m + 2)$ vertices in $\Gamma$, so $V(\Gamma)$ meets at least two such $B_{0,j}$.

Let $U$ be a union of sets $B_{0,j} \cap \Gamma$ with between $n/(4m + 4)$ and $n/(2m + 2)$ vertices - a greedy search will achieve this. Notice that the complement of the $r$ neighbourhood of $U$ in $B_0$ contains at least $n/(2m + 2)$ vertices. Denote this set by $V$.

Consider the following sets:

$$C_l = \{v \in v(A) \mid d(v, U) = l\} \quad \text{where} \quad 1 \leq l \leq r.$$ 

It is clear that for each $l$, $U$ and $V$ lie in different connected components of $\Gamma \setminus C_l$ and that for some $l$, $|C_l| \leq n/r$.

Therefore, $\text{sep}_X(n) \leq n/r$ where $c = 1/(2m + 2)$.  

Proposition 1.3 shows that the separation profile is sublinear for any group which coarsely embeds into a Hilbert space. It is natural to wonder if this can be strengthened to the following.
Question 4.3. Let $\mathcal{H}$ be a Hilbert space (or, more generally, a uniformly convex Banach space). Let $X$ be a graph and let $\phi : X \to \mathcal{H}$ be a Lipschitz map. Is it true that

$$\text{sep}_X(n) \preceq n/f_\phi(n)$$

where $f_\phi(n) = \sup \{ k \mid \sup_{x \in X} |\phi(X) \cap B(x;k)| \leq n \}$.

A positive answer to this question would give a bound on separation in terms of the compression exponent.

5 Consequences and Questions

5.1 Virtually special groups

A finitely generated group is said to be special if it is the fundamental group of a compact special cube complex [HW08]. Examples of virtually special groups include right-angled Artin groups; Coxeter groups [HW10]; fundamental groups of closed hyperbolic 3-manifolds [AGM12, Wis11]; random groups [OW11, MP14] and free-by-cyclic groups [HW14].

Every virtually special group is virtually an undistorted subgroup of a right-angled Artin group, so can be quasi-isometrically embedded into a finite product of trees [DJ99, DJ00]. Hence the separation profile of any such group is bounded from above by $n / \log(n)$. In more specific cases we can give good lower bounds.

Corollary 5.1. Let $\Gamma$ be a finite graph with associated right-angled Artin group $A(\Gamma)$.

- If $\Gamma$ has no edges then $A(\Gamma)$ is free, so $\text{sep}_{A(\Gamma)}$ is bounded.
- If $\Gamma$ has a $k$-clique then $\mathbb{Z}^k \leq A(\Gamma)$, so $\text{sep}_{A(\Gamma)}(n) \geq n^{(k-1)/k}$.
- $F_2 \times F_2 \leq A(\Gamma)$ if and only if $\Gamma$ has an induced square [Kam09], so in this situation, $\text{sep}_{A(\Gamma)}(n) \asymp n / \log(n)$.

Two very natural cases for further study are the pentagon and the path with 4 vertices.

Corollary 5.2. Let $\Gamma$ be a finite graph with associated right-angled Coxeter group $R(\Gamma)$.

- If every connected component of $\Gamma$ is a clique, then $\text{sep}_{R(\Gamma)}$ is bounded.
- If $\Gamma$ has an induced square, then $\text{sep}_{R(\Gamma)}(n) \geq n^{1/2}$, if it has no induced square, then $R(\Gamma)$ is hyperbolic and $\text{sep}_{R(\Gamma)}(n) \leq n^{(k-1)/k}$ for some $k$.
- If $\Gamma$ has an induced $K_{3,3}$, then $\text{sep}_{R(\Gamma)}(n) \asymp n / \log(n)$.

Much more information can be found in [BHSC13, DT14] and references therein.
5.2 Mapping class groups

Given a compact orientable surface, $\Sigma = \Sigma_{g,n}$ of genus $g$ with $n$ boundary components, the mapping class group of $\Sigma$ - $\text{MCG}(\Sigma)$ - is the group of isotopy classes of orientation preserving diffeomorphisms which fix the boundary. It is generated by Dehn twists and is finitely presented. These groups have close connections with geometry, topology and group theory and share striking similarities with lattices in higher rank semisimple Lie groups and outer automorphism groups of free groups.

It was previously shown in [Hum12] that every mapping class group admits a quasi-isometric embedding into a finite product of trees, so we deduce that

$$\text{sep}_{\text{MCG}(\Sigma)}(n) \preceq \frac{n}{\log(n)}.$$  

In low complexity cases, $3g + n - 3 \leq 0$, the mapping class group is virtually free, so separation is bounded [Beh04]. When $3g + n - 3 \geq 3$, the mapping class group has $F_2 \times F_2$ as a subgroup, so $\text{sep}_{\text{MCG}(\Sigma)}(n) \asymp \frac{n}{\log(n)}$, by [BST12, Theorem 3.5].

**Question 5.3.** What is the behaviour of the separation profile of the mapping class groups of the four-holed and five-holed sphere, and the torus with one or two holes?

The rank theorem for mapping class groups states that $\text{MCG}(\Sigma)$ has a free abelian subgroup of rank $3g + n - 3$, so in the case of the five-holed sphere and torus with two holes, $n^{1/2}$ is a lower bound on separation [BLM83].

5.3 Relatively hyperbolic groups

Introduced by Gromov [Gro87] as a generalisation of hyperbolic groups which included geometrically finite Kleinian groups, the class of relatively hyperbolic groups includes: amalgamated products and HNN-extensions over finite subgroups; fully residually free (limit) groups [Dah03, Ali05] - which are key objects in solving the Tarski conjecture [Sel01, KM10], and fundamental groups of non-geometric closed 3-manifolds with at least one hyperbolic component [Dah03].

In [MS13] it is shown that a relatively hyperbolic group quasi-isometrically embeds into a finite product of trees if and only if its peripheral subgroups admit such embeddings. This is extended in [Hum12] to: a relatively hyperbolic group has finite Assoud-Nagata dimension if and only if its peripheral subgroups do. Hence, we have the following upper bounds:

**Corollary 5.4.** Let $G$ be hyperbolic relative to a finite collection of subgroups $\{H_i\}$. If each $H_i$ quasi-isometrically embeds into a finite product of trees then

$$\text{sep}_G(n) \preceq \frac{n}{\log(n)}.$$  

In particular, this holds for all fundamental groups of compact 3-manifolds with no Nil component.
If each $H_i$ has finite Assouad-Nagata dimension, then there exists some $\varepsilon \in (0, 1)$ such that
\[ \text{sep}_G(n) \preceq n/(\log(n))^{\varepsilon}. \]
This holds for all fundamental groups of compact 3-manifolds.

By [Sap11] and Proposition 3.7, such a result cannot be extended to closed aspherical 4-manifolds.

The results of [MS13, Hum12] prove that given a group $G$ which is hyperbolic relative to some $H = \{H_i\}$ there is a quasi-isometric embedding of $G$ into a product of finitely many trees and a tree-graded space $T(H)$. The separation of $T(H)$ is exactly $\max_i \{\text{sep}_{H_i}(n)\}$, so the following bound seems plausible.

**Question 5.5.** Is it true that given any group $G$ which is hyperbolic relative to some $H = \{H_i\}$, we have
\[ \text{sep}_G(n) \preceq \max_i \{\text{sep}_{H_i}(n), n/\log(n)\}? \]

### 5.4 Separation of $\text{Out}(F_n)$

For such groups, we start with a dichotomy: $\text{Out}(F_2)$ is virtually free, so has bounded separation, while for $m \geq 3$, $\text{Out}(F_m)$ has $F_2 \times F_2$ as a subgroup - consider the subgroup generated by left and right multiplication of the first generator by the following two - so the separation is bounded from below by $n/\log(n)$.

**Question 5.6.** Is it true that for $m \geq 3$,
\[ \text{sep}_{\text{Out}(F_m)}(n) \preceq n/\log(n)? \]

At the moment there is no way of obtaining any non-trivial upper bound on separation for such groups.

### 5.5 Linear separation

A long-standing open problem of Benjamini asks whether there exists a finitely generated group whose Cayley graph has uniformly expanding balls.

Such a group would have a linear separation function. The best hope of obtaining such a group at the moment would be to use Osajda’s techniques - we would require a family of expanders $\Gamma_n$ with $|\Gamma_n|$ growing exponentially and such that the girth of $\Gamma_n$ is proportional to $\log |\Gamma_n|$. It is unlikely that such a group would have uniformly expanding balls however, as the embedded expander graphs would occupy an exponentially decreasing proportion of the corresponding metric ball.
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