ON A NEW FORMULAS FOR A DIRECT AND INVERSE CAUCHY PROBLEMS OF HEAT EQUATION

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Abstract. In this paper a solution of the direct Cauchy problems for heat equation is founded in the Hermite polynomial series form. A well-known classical solution of direct problem is represented in the Poisson integral form. The author shows the formulas for the solution of the inverse Cauchy problems have a symmetry with respect to the formulas for the corresponding direct problems. The obtained solution formulas for the inverse problems can serve as a basis for regularizing computational algorithms while well-known classical formula for the solution of inverse problem did not possess such properties and can’t be a basis for regularizing computational algorithms.

Keywords: heat equation, direct/inverse Cauchy problem, well-posed /ill-posed problem, Hermite polynomials, Poisson integral.

Mathematics Subject Classification 2010: 65Rxx Integral equations, integral transforms; 12E10 Special polynomials.

1. INTRODUCTION.

In this paper a direct and inverse Cauchy problems for the heat equation are solved at Cartesian and polar coordinates. The inverse Cauchy problem for the heat equation consists of to reconstruct a priori unknown initial condition of the dynamic system from its known final condition. In 1939 French mathematician Jacques Hadamard defined the problem is called well-posed if a solution exists, the solution is unique, the solution’s behavior hardly changes when there’s a slight change in the initial condition. The problems are called ill-posed or not well-posed if at least one of these three conditions is not fulfilled. The most often, the third condition so called the stability condition of solution is violated for ill-posed problems. In this case there is a paradoxical situation: the problem is mathematically defined but it’s solution cannot be obtained by conventional methods. In mathematics the vast majority of inverse problems is not well-posed: small perturbations of the initial data (observations) may correspond to an arbitrarily large perturbations of the solution. A classic example of ill-posed problem is the inverse Cauchy problem (retrospective problem) for heat equation. The direct Cauchy problem for the heat equation most frequently is well-posed.

In this work the solution for direct Cauchy problems found in the form of the Hermite polynomial series. A well-known classical solution for the direct Cauchy problem is represented in the Poisson integral form. We noticed the solution of inverse Cauchy problem possess the symmetry with respect to the solution of the direct Cauchy problem. The formulas obtained in this paper for the inverse problem can serve as a basis for regularizing computational algorithm. Previously known
classical formula for the solution of the inverse problem did not have such properties that’s why classical formula can’t serve as a basis for convergent algorithm.

The main result for the direct problems is the formulas (5), (6), (8). The main result for the inverse problems is the formulas (12), (13), (14), (26), (27), (28).

2. THE CAUCHY PROBLEM (DIRECT PROBLEM) FOR THE HEAT EQUATION.

A solution \( u(\tau, x) \) of the Cauchy problem for an infinite bar with the initial thermal field \( f(x) \) we will get in the Hermite polynomial series form. For this we use the well-known analytic solution \( u(\tau, x) \) in the Fourier integral form, [1]:

\[
\begin{align*}
  u(\tau, x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\lambda^2 \tau} e^{i\lambda x} \left( \int_{-\infty}^{\infty} e^{-i\lambda \xi} f(\xi) d\xi \right) d\lambda,
\end{align*}
\]

where \( f(x) \) - the initial thermal field, \( u(\tau, x) \) - thermal field at the time \( \tau \) and at the point \( x \).

Write the last equality in the form

\[
\begin{align*}
  u(\tau, x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\lambda^2 (\tau + \beta)} e^{i\lambda x} \left( \int_{-\infty}^{\infty} e^{\lambda^2 \beta} e^{-i\lambda \xi} f(\xi) d\xi \right) d\lambda,
\end{align*}
\]

where \( \beta > 0 \).

The function \( e^{\lambda^2 \tau + i\lambda x} \) is a generating function for the Hermite polynomials, [1], this means that

\[
\begin{align*}
  e^{\lambda^2 \tau + i\lambda x} &= \sum_{j=0}^{\infty} \frac{(-i\lambda)^j \beta^j}{j!} H_j \left( \frac{\xi}{2\sqrt{\beta}} \right),
\end{align*}
\]

where

\[
\begin{align*}
  H_j(z) &= (-1)^j e^{z^2} \frac{d^j}{dz^j} \left(e^{-z^2}\right)
\end{align*}
\]

- the Hermite polynomials.

In accordance with (2) the formula [13] takes the form

\[
\begin{align*}
  u(\tau, x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\lambda^2 (\tau + \beta)} e^{i\lambda x} \sum_{j=0}^{\infty} \frac{(-i\lambda)^j \beta^j}{j!} \int_{-\infty}^{\infty} H_j \left( \frac{\xi}{2\sqrt{\beta}} \right) f(\xi) d\xi d\lambda.
\end{align*}
\]

If we may change the order of integration, then compute the inner integral on the variable \( \lambda \). We use for this Poisson integral, [13]. We get

\[
\begin{align*}
  \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\lambda^2 (\tau + \beta)} e^{i\lambda x} d\lambda &= \frac{e^{-\frac{x^2}{4(\tau + \beta)}}}{2\sqrt{\pi(\tau + \beta)}},
\end{align*}
\]

Differentiating both sides of (4) with respect to \( x \) we obtain the value of the desired integral

\[
\begin{align*}
  \frac{1}{2\pi} \int_{-\infty}^{\infty} (-i\lambda)^j e^{-\lambda^2 (\tau + \beta)} e^{i\lambda x} d\lambda = (-1)^j \frac{d^j}{dx^j} \left[ \frac{e^{-\frac{x^2}{4(\tau + \beta)}}}{2\sqrt{\pi(\tau + \beta)}} \right].
\end{align*}
\]

On the basis of formula (3) for the Hermite polynomials the last formula can be written

\[
\begin{align*}
  \frac{1}{2\pi} \int_{-\infty}^{\infty} (-i\lambda)^j e^{-\lambda^2 (\tau + \beta)} e^{i\lambda x} d\lambda &= \frac{e^{-\frac{x^2}{4(\tau + \beta)}}}{2\sqrt{\pi(\tau + \beta)}(2\sqrt{\tau + \beta})^j} H_j \left( \frac{x}{2\sqrt{\tau + \beta}} \right).
\end{align*}
\]
Finally, we obtain an analytic representation of the thermal field at the time \( \tau \) and at the point \( x \)

\[
(5) \quad u(\tau, x) = e^{-\frac{x^2}{4(\tau + \beta)}} \sum_{j=0}^{\infty} \frac{1}{(2\sqrt{\tau + \beta})^j} H_j \left( \frac{x}{2\sqrt{\tau + \beta}} \right) \frac{\beta^j}{j!} f_j,
\]

where

\[
f_j = \int_{-\infty}^{\infty} H_j \left( \frac{\xi}{2\sqrt{\beta}} \right) f(\xi) d\xi.
\]

**Remark.** By interchanging the variables \( x \) and \( \xi \) at the formula (5) we get another version of the thermal field:

\[
(6) \quad u(\tau, x) = e^{-\frac{x^2}{4\tau}} \sum_{j=0}^{\infty} \frac{1}{(2\sqrt{\tau})^j} H_j \left( \frac{x}{2\sqrt{\tau}} \right) \frac{(\tau + \beta)^j}{j!} f_j,
\]

Where

\[
f_j = \int_{-\infty}^{\infty} H_j \left( \frac{\xi}{2\sqrt{\tau + \beta}} \right) f(\xi) d\xi.
\]

Now we get the third new formula. To do this, the formula (1) can be written in the form

\[
(7) \quad u(\tau, x) = \int_{-\infty}^{\infty} e^{-\frac{\lambda^2}{2(\tau + \beta)}} \left( \int_{-\infty}^{\infty} e^{i\lambda(\xi - \tau)} f(\xi) d\xi \right) d\lambda,
\]

here \( \beta > 0 \).

In view of (2) we get

\[
(8) \quad u(\tau, x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\lambda^2(\tau + \beta)} \sum_{j=0}^{\infty} \frac{(-i\lambda)^j}{j!} \beta^j \int_{-\infty}^{\infty} H_j \left( \frac{x - \xi}{2\sqrt{\beta}} \right) f(\xi) d\xi d\lambda.
\]

To simplify last formula change the order of integration and compute the inner integral on the variable \( \lambda \), substitute \( x = 0 \) in (4). We obtain

\[
(7) \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} (-i\lambda)^j e^{-\lambda^2(\tau + \beta)} d\lambda = \frac{1}{(2\sqrt{\tau + \beta})^{j+1}} H_j(0).
\]

Taking into account the well-known formula from [1]

\[
H_{2j}(0) = \frac{(-1)^j (2j)!}{2j}, \quad H_{2j+1}(0) = 0; \quad j = 0, 1, 2, ...
\]

finally we get an analytical representation of the thermal field

\[
(8) \quad u(\tau, x) = \sum_{j=0}^{\infty} \frac{1}{(2\sqrt{\tau + \beta})^{2j+1}} \frac{(-1)^j \beta^j}{2j!} f_{2j},
\]

Where

\[
f_{2j} = \int_{-\infty}^{\infty} H_{2j} \left( \frac{x - \xi}{2\sqrt{\beta}} \right) f(\xi) d\xi.
\]
2.1. The inverse Cauchy problem for the heat equation.

The inverse heat equation problem for an infinite bar is to find the unknown initial distribution \( f(x) \) of thermal field by the known temperature field \( u(\tau, x) \) \([7],[8],[13]\). This inverse heat equation problem leads to the solving of first-type Fredholm integral equation:

\[
\int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi \tau}} \exp \left( -\frac{(x-\xi)^2}{4\tau} \right) f(\xi) \, d\xi = u(\tau, x),
\]

The left-hand side of equation (9) is the Poisson integral, \([15]\). As it is shown in \([1],[2]\) the solution of equation (9) may be given on the form:

\[
f(x) = \frac{1}{\sqrt{\pi}} \sum_{j=0}^{\infty} \frac{u^{(j)}(0)}{(2\sqrt{\tau})^{n+1} j!} H_j \left( \frac{x}{2\sqrt{\tau}} \right),
\]

where \( H_j(z) \) — Hermite polynomials, \((3)\).

Formula (10) contains a derivatives of arbitrarily high order so the formula (10) can’t serve as a basis for the regularizing computational algorithm. Consequently, it is actual to find the new formulas without derivatives for the solving of the equation (9).

As in chapter-1, we obtain three new formulas.

We get the solution of equation (9) by the Fourier transform integral method from \([1],[9],[11],[12]\)

\[
f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\lambda^2 \tau} e^{i\lambda x} \left( \int_{-\infty}^{\infty} e^{-i\lambda \xi} u(\tau, \xi) \, d\xi \right) \, d\lambda
\]

If \( \beta > 0 \), then the last formula takes the form

\[
f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\lambda^2 \beta} e^{\lambda^2 (\tau + \beta)} e^{i\lambda x} \left( \int_{-\infty}^{\infty} e^{-i\lambda \xi} u(\tau, \xi) \, d\xi \right) \, d\lambda,
\]

Because of the formula (2) we get

\[
f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\lambda^2 \beta} e^{i\lambda x} \sum_{j=0}^{\infty} \frac{(-i\lambda)^j}{j!} (\tau + \beta)^{\frac{j}{2}} \int_{-\infty}^{\infty} H_j \left( \frac{\xi}{2\sqrt{\tau + \beta}} \right) u(\tau, \xi) \, d\xi \, d\lambda
\]

We will change the order of integration and calculate the inner integral on the variable \( \lambda \). The Poisson integral from \([15]\) is used in this calculations. We get

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\lambda^2 \beta} e^{i\lambda x} d\lambda = \frac{e^{-\frac{x^2}{2\beta}}}{2\sqrt{\pi} \beta}
\]

We calculate the value of an integral by \( j \) time differentiating on the variable \( x \):

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} (-i\lambda)^j e^{-\lambda^2 \beta} e^{i\lambda x} d\lambda = (-1)^j \frac{d^j}{dx^j} \left[ \frac{e^{-\frac{x^2}{2\beta}}}{2\sqrt{\pi} \beta} \right].
\]

On the basis of the formula (3) we can write

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} (-i\lambda)^j e^{-\lambda^2 \beta} e^{i\lambda x} d\lambda = \frac{e^{-\frac{x^2}{2\beta}}}{2\sqrt{\pi} \beta (2\sqrt{\beta})^j} H_j \left( \frac{x}{2\sqrt{\beta}} \right).
\]
Finally, first new formula for the initial distribution of the thermal field takes the form

\[ f(x) = \frac{e^{-\frac{x^2}{2\sqrt{\pi\beta}}}}{2\sqrt{\pi\beta}} \sum_{j=0}^{\infty} \frac{1}{(2\sqrt{\beta})^j} H_j \left( \frac{x}{2\sqrt{\beta}} \right) \frac{(\tau + \beta)^j}{j!} u_j, \]

where

\[ u_j = \int_{-\infty}^{\infty} H_j \left( \frac{\xi}{2\sqrt{\tau + \beta}} \right) u(\tau, \xi) d\xi. \]

**Remark.** By interchanging the variables \( x \) and \( \xi \) at the formula (12) we get a second new formula for the solution of the equation (9):

\[ f(x) = \sum_{j=0}^{\infty} \frac{1}{(2\sqrt{\beta})^j} H_j \left( \frac{x}{2\sqrt{\tau + \beta}} \right) \frac{(\tau + \beta)^j}{j!} u_j, \]

where

\[ u_j = \int_{-\infty}^{\infty} e^{-\frac{\xi^2}{2\sqrt{\pi\beta}}} H_j \left( \frac{\xi}{2\sqrt{\beta}} \right) u(\tau, \xi) d\xi. \]

Finally we prove the third new formula for the inverse Cauchy problem solving. We use the integral representation of the solution (11) which can be written as

\[ f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\lambda^2 \beta} e^{\lambda^2(\tau + \beta)} \left( \int_{-\infty}^{\infty} e^{i\lambda(x - \xi)} u(\tau, \xi) d\xi \right) d\lambda, \]

where \( \beta > 0. \)

Because of the formula (12) we get

\[ f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\lambda^2 \beta} \sum_{j=0}^{\infty} \frac{(-i\lambda)^j}{j!} \left( \tau + \beta \right)^j \int_{-\infty}^{\infty} H_j \left( \frac{x - \xi}{2\sqrt{\tau + \beta}} \right) u(\tau, \xi) d\xi d\lambda. \]

If we use the formula (13), then the initial distribution of the thermal field takes the form

\[ f(x) = \frac{1}{2\sqrt{\pi\beta}} \sum_{j=0}^{\infty} \frac{1}{(2\sqrt{\beta})^j} H_j(0) \frac{(\tau + \beta)^j}{j!} u_j, \]

where

\[ u_j = \int_{-\infty}^{\infty} H_j \left( \frac{x - \xi}{2\sqrt{\tau + \beta}} \right) u(\tau, \xi) d\xi. \]

Because of the formula for \( H_j(0) \) as a result we get

\[ f(x) = \frac{1}{\sqrt{\pi}} \sum_{j=0}^{\infty} \frac{1}{(2\sqrt{\beta})^{2j+1}} \frac{(-1)^j (\tau + \beta)^j}{2^{j}(2j)!} u_{2j}, \]

Where

\[ u_j = \int_{-\infty}^{\infty} H_j \left( \frac{x - \xi}{2\sqrt{\tau + \beta}} \right) u(\tau, \xi) d\xi. \]
3. Cauchy problem for the heat equation at polar coordinates.

3.1. Auxiliary propositions. We define polynomials $W_j(z)$ with the help of the generating function $e^{-t^2}I_0(2tz)$

$$e^{-t^2}I_0(2tz) = \sum_{j=0}^{\infty} \frac{t^{2j}}{(2j)!} W_j(z),$$

where $I_0(x)$ - the zero-order Bessel function of the first kind.

It follows from (15) that the polynomials $W_j(z)$ have the form

$$W_j(z) = \frac{d^{2j}}{dt^{2j}} \left[ e^{-t^2}I_0(2tz) \right]_{t=0}$$

We get another view of polynomials $W_j(z)$. From the definition of the Bessel operator, $[1]$,

$$B = \frac{d^2}{dz^2} + \frac{1}{z} \frac{d}{dz}$$

it follows that

$$B^j[I_0(2tz)] = (2t)^{2j}I_0(2tz),$$

Therefore

$$\exp \left( -\frac{B}{4} \right)[I_0(2tz)] = e^{-t^2}I_0(2tz),$$

In this formula we equate the coefficients of the $\xi^{2j}$ degree at the left and right sides. We get

$$\exp \left( -\frac{B}{4} \right) \left[ \frac{2^{2j}z^{2j}}{2^{2j}j!^2} \right] = \frac{W_j(z)}{(2j)!},$$

hence the polynomials $W_j(z)$ have the form

$$W_j(z) = \frac{(2j)!}{j!^2} \exp \left( -\frac{B}{4} \right) \left[ z^{2j} \right].$$

With the help of the polynomials $W_j(z)$ we will obtain the new formulas for solutions of direct and inverse Cauchy problems at the polar coordinates.

3.2. New formulas for solution of the Cauchy problem at polar coordinates. We deduce the new formulas for the solution of the Cauchy problem for the heat equation at polar coordinates $(r, \phi)$ if the thermal regime depends only on the variable $r$. We use the explicit formula for the Cauchy problem solution at polar coordinates $[6]$

$$u(\tau, r) = \int_0^\infty \lambda e^{-\lambda^2 \tau} J_0(\lambda r) \left( \int_0^\infty \xi J_0(\lambda \xi) f(\xi) d\xi \right) d\lambda,$$

where $J_0(\lambda \xi)$ - the zero-order modified Bessel functions, $[1]$.

We write the last equation in the form

$$u(\tau, r) = \int_0^\infty \lambda e^{-\lambda^2(\tau+\beta)} e^{\lambda^2 \beta} J_0(\lambda r) \left( \int_0^\infty \xi J_0(\lambda \xi) f(\xi) d\xi \right) d\lambda,$$

where $\beta > 0$.

In (15) we make the substitutions: $t = i\lambda \sqrt{\beta}$; $z = \frac{x}{2\sqrt{\beta}}$. Then we obtain

$$e^{\lambda^2 \beta} J_0(\lambda x) = \sum_{j=0}^{\infty} (-1)^j \frac{\lambda^{2j} \beta^j}{(2j)!} W_j \left( \frac{x}{2\sqrt{\beta}} \right).$$
The formula (17) according to the relation (18) takes the form

\[ u(\tau, r) = \int_{-\infty}^{\infty} e^{-2\tau^2(\tau+\beta)} J_0(\lambda r) \sum_{j=0}^{\infty} (-1)^j \frac{\lambda^{2j}}{(2j)!} \beta^j \int_0^\infty W_j \left( \frac{\xi}{2\sqrt{\beta}} \right) f(\xi) d\xi d\lambda \]

At the last formula change the integration order and compute the inner integral on the variable \( \lambda \). The Weber integral [6] is used in calculation.

\[ \int_0^\infty \lambda e^{-\lambda^2(\tau+\beta)} J_0(\lambda r) J_0(\lambda \xi) d\lambda = \frac{e^{-\frac{\xi^2}{4(\tau+\beta)}}}{2(\tau+\beta)^\frac{3}{2}} I_0 \left( \frac{r\xi}{2(\tau+\beta)} \right). \]

By equating the coefficients before the \( \xi^{2j} \) degree we get the value of an integral (20)

\[ \frac{1}{2^{2j}2^j} \int_{-\infty}^{\infty} (-1)^j e^{-2\tau^2(\tau+\beta)} J_0(\lambda r) \lambda^{2j} d\lambda = \frac{1}{(2j)!} \frac{d^{2j}}{d\xi^{2j}} \left[ \frac{e^{-\frac{\xi^2}{4(\tau+\beta)}}}{2(\tau+\beta)^\frac{3}{2}} I_0 \left( \frac{r\xi}{2(\tau+\beta)} \right) \right]_{\xi=0}. \]

Due to formula (16) the last formula can be written

\[ \frac{e^{-\frac{\xi^2}{4(\tau+\beta)}}}{2(\tau+\beta)^\frac{3}{2}} \frac{1}{2^j(\tau+\beta)^j} W_j \left( \frac{r}{2\sqrt{\tau+\beta}} \right) = \frac{1}{(2j)!} \frac{d^{2j}}{d\xi^{2j}} \left[ \frac{e^{-\frac{\xi^2}{4(\tau+\beta)}}}{2(\tau+\beta)^\frac{3}{2}} I_0 \left( \frac{r\xi}{2(\tau+\beta)} \right) \right]_{\xi=0}. \]

Then (20) becomes

\[ \int_{-\infty}^{\infty} (-1)^j e^{-2\tau^2(\tau+\beta)} J_0(\lambda r) \lambda^{2j} d\lambda = \frac{e^{-\frac{\xi^2}{4(\tau+\beta)}}}{2(\tau+\beta)^\frac{3}{2}} \frac{j!^2}{(\tau+\beta)^j(2j)!} W_j \left( \frac{r}{2\sqrt{\tau+\beta}} \right). \]

Finally, formula (19) for the thermal field \( u(\tau, r) \) with (21) takes the form

\[ u(\tau, r) = \frac{e^{-\frac{\xi^2}{4(\tau+\beta)}}}{2(\tau+\beta)^\frac{3}{2}} \sum_{j=0}^{\infty} \frac{1}{(\tau+\beta)^j} W_j \left( \frac{r}{2\sqrt{\tau+\beta}} \right) \frac{j!^2}{(2j)!^2} f_j, \]

where \( f_j = \int_{-\infty}^{\infty} W_j \left( \frac{\xi}{2\sqrt{\beta}} \right) f(\xi) d\xi. \)

**Remark.** From the last equality for \( u(\tau, r) \) at \( \tau = 0 \) the expansion theorem on the eigenfunctions \( \left\{ e^{-\frac{\xi^2}{4\tau}} W_j \left( \frac{r}{2\sqrt{\beta}} \right) \right\} \) can be obtained

\[ f(r) = \frac{e^{-\frac{\xi^2}{4\beta}}}{2\beta} \sum_{j=0}^{\infty} \frac{1}{\beta^j} W_j \left( \frac{r}{2\sqrt{\beta}} \right) \frac{j!^2}{(2j)!^2} f_j, \]

where

\[ f_j = \int_{-\infty}^{\infty} W_j \left( \frac{\xi}{2\sqrt{\beta}} \right) f(\xi) d\xi. \]

Similarly to section 1 we can get two formulas for the thermal field. If we replace in the formula (22): \( \beta \leftrightarrow \tau + \beta \), then the new formula takes the form

\[ u(\tau, r) = \frac{e^{-\frac{\xi^2}{4(\tau+\beta)}}}{2\beta} \sum_{j=0}^{\infty} \frac{1}{\beta^j} W_j \left( \frac{r}{2\sqrt{\tau+\beta}} \right) \frac{j!^2}{(2j)!^2} f_j, \]

where

\[ f_j = \int_{-\infty}^{\infty} W_j \left( \frac{\xi}{2\sqrt{\tau+\beta}} \right) f(\xi) d\xi. \]
The third new formula is proved similarly to the section-1. We apply the formula from [10]
\[ J_0(\lambda x)J_0(\lambda y) = \frac{1}{\pi} \int_0^\pi J_0(\lambda \sqrt{x^2 + y^2 - 2xy \cos \phi}) d\phi \]
to the formula (17).

Then
\[ e^{\lambda^2 \tau} J_0(\lambda x)J_0(\lambda y) = \frac{1}{\pi} \sum_{j=0}^{\infty} (-1)^j \frac{\lambda^{2j} \tau^j}{(2j)!} \int_0^\pi \int_0^\pi W_j \left( \frac{\sqrt{x^2 + y^2 - 2xy \cos \phi}}{2\sqrt{\tau}} \right) d\phi. \]

Write equation (17) as
\[ u(\tau, r) = \int_0^\infty \lambda e^{-\lambda^2 (\tau + \beta)} \left( \int_0^\infty e^{\lambda^2 \beta} J_0(\lambda r)J_0(\lambda \xi) f(\xi) d\xi \right) d\lambda, \]
where \( \beta > 0 \).

Next, on the basis of (24) we have
\[ u(\tau, r) = \frac{1}{\pi} \int_0^\infty e^{-\lambda^2 (\tau + \beta)} \sum_{j=0}^{\infty} (-1)^j \frac{\lambda^{2j} \beta^j}{(2j)!} \int_0^\pi \int_0^\pi W_j \left( \frac{\sqrt{r^2 + \xi^2 - 2r\xi \cos \phi}}{2\sqrt{\beta}} \right) d\phi \xi f(\xi) d\xi d\lambda. \]

We change the order of integration and use the definition of the Gamma-function [10] to compute the inner integral on the variable \( \lambda \).
\[ \int_0^\infty (-1)^j e^{-\lambda^2 (\tau + \beta)} \lambda^{2j} d\lambda = \frac{(-1)^j \Gamma \left( j + \frac{1}{2} \right)}{2(\tau + \beta)^{j + \frac{1}{2}}}. \]

Finally, we obtain
\[ u(\tau, r) = \frac{1}{2} \sum_{j=0}^{\infty} (-1)^j \beta^j \frac{\Gamma \left( j + \frac{1}{2} \right)}{(\tau + \beta)^{j + \frac{1}{2}}} f_j, \]
where
\[ f_j = \int_0^\pi \int_0^\pi W_j \left( \frac{\sqrt{r^2 + \xi^2 - 2r\xi \cos \phi}}{2\sqrt{\beta}} \right) d\phi \xi f(\xi) d\xi d\lambda. \]

4. The inverse Cauchy problem for the heat equation at polar coordinates.

The inverse Cauchy problem [3],[4],[5] in polar coordinates leads to the solution of the first-kind integral Fredholm equation
\[ \int_0^\infty e^{-\frac{r^2 + \xi^2}{2\tau}} I_0 \left( \frac{r\xi}{2\tau} \right) f(\xi) d\xi = u(\tau, \xi). \]
To find three previously unknown expressions for the solution of inverse Cauchy problem in the series on the polynomials \( W_j(z) \) [10] we will do the same in section-1.

The first expression is
\[ f(r) = \frac{e^{-r^2}}{2\beta} \sum_{j=0}^{\infty} \frac{1}{\beta^j} W_j \left( \frac{r}{2\sqrt{\beta}} \right) \frac{j!^2 (\beta + \tau)^j}{(2j)!^2} u_j, \]
where

\[ u_j = \int_0^\infty W_j \left( \frac{\xi}{2\sqrt{\tau + \beta}} \right) u(\tau, \xi) \, d\xi. \]

The second expression is

\[ f(r) = e^{-\frac{r^2}{2(\beta + \tau)}} \sum_{j=0}^{\infty} \frac{1}{(\beta + \tau)^j} W_j \left( \frac{r}{2\sqrt{(\beta + \tau)}} \right) \frac{j^{2j}}{(2j)!} u_j, \]

where

\[ u_j = \int_0^\infty W_j \left( \frac{\xi}{2\sqrt{\beta}} \right) u(\tau, \xi) \, d\xi. \]

The third expression is

\[ f(r) = \frac{1}{2} \sum_{j=0}^{\infty} (-1)^j \left( \frac{\tau + \beta}{\tau + \frac{1}{2}} \right)^j \frac{\Gamma \left( j + \frac{1}{2} \right)}{(2j)!} u_j, \]

where

\[ u_j = \int_0^\infty \int_0^\pi W_j \left( \frac{\sqrt{r^2 + \xi^2} - 2r \xi \cos \phi}{2\sqrt{\tau + \beta}} \right) \, d\phi \xi u(\tau, \xi) \, d\xi. \]

REFERENCES

[1] F.M. Mors, G. Fishbah, Methods of theoretical physics, 1958.
[2] Yaremko, O.E. Matrix integral Fourier transforms for problems with discontinuous coefficients and transformation operators (2007) Doklady Mathematics, 76 [II], pp. 323-325.
[3] O.M. Alifanov, Inverse problems of heat exchange, M, 1988, p. 279.
[4] O.M. Alifanov, B.A. Artyukhin, S.V. Rumyancev, The extreme methods of solution of ill-posed problems, M, 1988, p. 288.
[5] J.V. Beck, V. Blackwell, C.R. Clair, Inverse Heat Conduction. Ill-Posed Problems , M, 1989, p. 312.
[6] Sneddon I., Beri D. S., (2008) The classical theory of elasticity. University book, p. 215.
[7] M.M. Lavrentev, Some ill-posed problems of mathematical physics, Novosibirsk, AN SSSR, 1962, p. 92.
[8] A.N. Tikhonov, V. Ya. Arsenin, Methods of solution of ill-posed problems, M, 1979, p. 288.
[9] M.M. Dzhrbashyan, Integral Transforms and Representations of Functions in the Complex Domain, M, 1966.
[10] Watson, G.N., A Treatise on the Theory of Bessel Functions, Second Edition, (1995) Cambridge University Press. ISBN 0-521-48391-3.
[11] Yaremko O.E. Transformation operator and boundary value problems Differential Equation. Vol.40, No. 8, 2004, pp.1149-1160
[12] Bavrin, I.I., Yaremko, O.E. Transformation Operators and Boundary Value Problems in the Theory of Harmonic and Biharmonic Functions (2003) Doklady Mathematics, 68 (3), pp. 371-375.
[13] Arfken, G. B.; Weber, H. J. (2000), Mathematical Methods for Physicists (5th ed.), Boston, MA: Academic Press

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