On $p$-adic actions raising dimension by 2

Michael Levin

Abstract

Raymond and Williams constructed an action of the $p$-adic integers $A_p$ on an $n$-dimensional compactum $Z$, $n \geq 2$, with the orbit space $Z/A_p$ of dimension $n + 2$. A simplified construction of such an action was presented in [1]. In this paper we generalize the approach of [1] to show that for every $n \geq 2$, an $(n + 2)$-dimensional compactum $X$ can be obtained as the orbit space $X = Z/A_p$ of an action of $A_p$ on an $n$-dimensional compactum $Z$ if and only if $\dim_{\mathbb{Z}[\frac{1}{p}]} X \leq n$ where $\dim_{\mathbb{Z}[\frac{1}{p}]} X$ is the cohomological dimension of $X$ with coefficients in $\mathbb{Z}[\frac{1}{p}]$.

Keywords: Cohomological Dimension, Transformation Groups

Math. Subj. Class.: 55M10, 22C05 (54F45)

1 Introduction

Interest in (continuous) actions of the $p$-adic integers $A_p$ on compacta (=compact metric spaces) is inspired by the Hilbert-Smith conjecture that asserts that a compact group acting effectively on a manifold must be a Lie group. This conjecture is equivalent to the following one: the group $A_p$ of the $p$-adic integers cannot act effectively on a manifold. Yang [3] showed that if $A_p$ acts effectively on a manifold $M$ then either $\dim M/A_p = \infty$ or $\dim M/A_p = \dim M + 2$. In order to verify if the latter dimensional relation ever occurs in a more general setting Raymond and Williams [2] constructed an action of $A_p$ on an $n$-dimensional compactum (=compact metric space) $Z$, $n \geq 2$, with $\dim Z/A_p = n + 2$. The author presented in [1] a simpler construction of such an example. In this paper we generalize the approach of [1] to provide a full characterization of finite-dimensional compacta $X$ with $\dim X \geq 4$ that can be obtained as the orbit spaces $X = Z/A_p$ of an action of $A_p$ on a compactum $Z$ with $\dim X = \dim Z + 2$.

Recall that the cohomological dimension of a compactum $X$ with respect to an abelian coefficient group $G$ is the least integer $n$ such that the Čech cohomology $H^{n+1}(X, F; G)$ vanishes for every closed $F \subset X$. Clearly $\dim_G X \leq \dim X$ if $X$ is finite-dimensional. It is well-known that for an action of $A_p$ on a compactum $Z$ we have that $\dim_{\mathbb{Z}[\frac{1}{p}]} Z = \dim_{\mathbb{Z}[\frac{1}{p}]} Z/A_p$. Thus every $(n + 2)$-compactum $X$ that can be represented as the orbit space of an action of $A_p$ on a $n$-dimensional compactum must satisfy $\dim_{\mathbb{Z}[\frac{1}{p}]} X \leq n$. The goal

*This research was supported by THE ISRAEL SCIENCE FOUNDATION (grant No. 522/14)
of this paper is to show that the inequality \( \dim_{Z_p} X \leq n \) is not only necessary but also sufficient for such a representation of \( X \).

**Theorem 1.1** Let \( X \) be an \( (n + 2) \)-dimensional compactum with \( n \geq 2 \). Then \( X \) can be obtained as the orbit space \( X = Z/A_p \) of an action of \( A_p \) on an \( n \)-dimensional compactum \( Z \) if and only if \( \dim_{Z_p} X \leq n \).

The paper is organized as follows. Sections 1 and 2 are devoted to constructions and properties needed for proving Theorem 1.1 in Section 3. A few related remarks are given in Section 4.

## 2 Auxiliary constructions and properties

Consider the unit \((m + 2)\)-sphere \( S^{m+2} \) in \( \mathbb{R}^{m+3} \), \( n \geq 2 \). Represent \( \mathbb{R}^{m+3} \) as the product \( \mathbb{R}^{m+1} \times \mathbb{R}^2 \) of a coordinate plane \( \mathbb{R}^{m+1} \) and its orthogonal complement \( \mathbb{R}^2 \) and let \( S^m = S^{m+2} \cap \mathbb{R}^{m+1} \) and \( S^1 = S^{m+2} \cap \mathbb{R}^2_1 \). Take the closed \( \epsilon \)-neighborhood \( F \) of \( S^m \). Then \( F \) and \( F_\perp = S^{m+2} \setminus \text{int} F \) can be represented as the products \( F = S^m \times D \) and \( F_\perp = D_\perp \times S^1_1 \) of an \( m \)-sphere \( S^m \) and a circle \( S^1_1 \) with a 2-disk \( D \) and an \((m + 1)\)-ball \( D_\perp \) respectively. Thus we have that \( \partial F = \partial F_\perp = S^m \times \partial D = \partial D_\perp \times S^1_1 \). Let \( T = \mathbb{R}/\mathbb{Z} \) freely act on the circle \( \partial D \) by rotations. Then this action induces the corresponding free action on \( \partial F = \partial F_\perp \) and the later action obviously extends over \( F_\perp \) as a free action and it extends over \( F \) by the rotations of \( D \) induced by the rotations of \( \partial D \). This way we have defined an action of \( T \) on \( S^{m+2} \) whose fixed point set is \( S^m \) and \( T \) acts freely on \( S^{m+2} \setminus S^m \). Note that for every subgroup \( \mathbb{Z}/p^k \mathbb{Z} \) of \( T \) the space \( S^{m+2}/\mathbb{Z}/p^k \) is homeomorphic to an \((m + 2)\)-sphere.

**Proposition 2.1** Let \( \Delta \) be an \((m + 3)\)-simplex. There is an unknotted sphere \( S^m \) in \( S^{m+2} = \partial \Delta \) such that for each \((m + 2)\)-simplex \( \Delta' \) the intersection \( S^m \cap \Delta' \) is an \( m \)-ball whose boundary is in \( \partial \Delta' \) and whose interior is in the interior of \( \Delta' \).

**Proof.** The case \( m = 1 \) was proved in [1]. We will proceed by induction on \( m \). Fix an \((m + 2)\)-simplex \( \Delta_s \) of \( \Delta \) and let \( v \) be the vertex of \( \Delta \) that does not belong to \( \Delta_s \) and let \( v_s \) be the barycenter of \( \Delta_s \). Consider a sphere \( S^{m-1} \) embedded in \( \partial \Delta_s \) as required in the proposition with \( m \) replaced by \( m - 1 \) and \( \Delta \) by \( \Delta_s \). Then the union of the cones over the sphere \( S^{m-1} \subset \Delta \) with the vertices \( v_s \) and \( v \) respectively forms a sphere \( S^m \) that satisfies the conclusions of the proposition. \( \blacksquare \)

From now we identify \( S^{m+2} \) with the boundary of \((m + 3)\)-simplex \( \Delta \) as described in Proposition 2.1. Recall that \( F \) is represented as the product \( F = S^m \times D \), take \((m + 4)\) disjoint closed \( m \)-balls \( B'_1, \ldots, B'_{m+4} \) in \( S^m \) and denote \( B_i = B'_i \times D \). Then each closed \((m + 2)\)-ball \( B_i \) is invariant under the action of \( T \) and hence \( M = S^{m+2} \setminus \text{int} (B_1 \cup \cdots \cup B_{m+4}) \) is invariant as well. By Proposition 2.1 we may assume that the balls \( B_i \) lie in the interior of different \((m + 2)\)-simplexes \( \Delta_i \) of \( \Delta \).

**Proposition 2.2**
(i) For every \( g \in T \) the map \( x \to gx, x \in M \), is ambiently isotopic to a homeomorphism \( \phi : M \to M \) by an isotopy of \( M \) that does not move the points of \( S^m \cup \partial M \) and such that \( \phi \) restricted to the \( m \)-skeleton \( \Delta^{(m+1)} \) of \( \Delta \) coincides with the inclusion of \( \Delta^{(m+1)} \) into \( M \) and \( \phi(\Delta \cap M) = \Delta' \cap M \) for every \( (m+2) \)-simplex \( \Delta' \) of \( \Delta \).

(ii) There is a retraction of \( r : M \to \Delta^{(m+1)} \) such that for every \( (m+2) \)-simplex \( \Delta' \) of \( \Delta \) we have that \( r(\Delta' \cap M) \subset \partial \Delta' \).

Proof.

(i) Since \( T \) is path-connected \( g : M \to M \) is isotopic to the identity map of \( M \) by an isotopy that does not move the points of \( M \cap S^m \) (the fixed point set of \( T \)). For each \( \partial B_i \subset \partial M \) we can change this isotopy on a neighborhood of \( \partial B_i \) in \( M \) to get in addition that the points of \( \partial B_i \) are not moved.

(ii) Define \( r \) as on each \( (m+2) \)-simplex \( \Delta' \) of \( \Delta \) as the restriction of a radial projection to \( \partial \Delta' \) from a point of \( \Delta' \) that does not belong to \( M \).

For a prime number \( p \) and a positive integer \( k \) consider the subgroup \( \mathbb{Z}_{p^k} \) of \( T \) and let \( \Gamma_M = \Gamma_M(p^k) = M/\mathbb{Z}_{p^k} \). Note that \( \Gamma_M \) is homeomorphic to \( M \). Denote by \( M_+ \) the space which is the union of \( p^k \) copies \( M_a, a \in \mathbb{Z}_{p^k} \), of \( M \) with \( \partial M \) being identified in all the copies by the identity map. Thus all \( M_a \subset M_+ \) intersect each other at \( \partial M \). Define the action of \( \mathbb{Z}_{p^k} \) on \( M_+ \) by sending \( x \in M_a \) to \( gx \in M_{g+a} \). Note that \( \partial M \subset M_+ \) is invariant under the action of \( \mathbb{Z}_{p^k} \) on \( M_+ \) and the natural projection \( \alpha_+ : M_+ \to M \) sends each \( M_a \) to \( M \) by the identity map is equivariant.

Let \( \Gamma_+ = \Gamma(p^k) = \Gamma_+/\mathbb{Z}_{p^k}, \gamma_+ : \Gamma_+ \to \Gamma \) the projection and \( \Gamma_{\partial M} \) and \( \Gamma_M \) the images in \( \Gamma \) of the bottom set \( M_+ \) and the top set \( M_{\partial M} \) in \( \Gamma_+ \) under the map \( \gamma_+ \). Note that \( \alpha_+ : M_+ \to M \) induces the corresponding map \( \alpha : \Gamma_{\partial M} \to \Gamma_M \) for which \( \Gamma \) is the mapping cylinder of \( \alpha \). Also note that \( \Gamma_M = M/\mathbb{Z}_{p^k} \) and \( \Gamma_{\partial M} \) is the space obtained from \( M \) by collapsing to singletons the orbits of the action of \( \mathbb{Z}_{p^k} \) on \( M \) lying in \( \partial M \). Thus there is a natural projection \( \mu : M \to \Gamma_{\partial M} \) and \( \Delta^{(m+1)} \subset M \) can be considered as a subset of \( \Gamma_{\partial M} \) as well (we can identify \( M_+ \) with one of the spaces \( M_a \subset M_+ \) and regard \( \mu \) as the projection \( M_+ \to \Gamma_{\partial M} = M_+/\mathbb{Z}_{p^k} \) restricted to \( M_a \)).

Let us define a map \( \delta : \Gamma \to \Delta \) by sending \( \Gamma_{\partial M} \) to \( \partial \Delta \) and \( \Gamma_M \) to the barycenter of \( \Delta \) such that \( \delta \) is the identity map on \( \Delta^{(m+1)} \), \( \delta(x) \in \Delta' \) for every \( (m+2) \)-simplex \( \Delta' \) of \( \Delta \) and \( x \in \mu(M \cap \Delta') \) and \( \delta \) is linearly extended along the intervals of the mapping cylinder \( \Gamma \) (in our descriptions we abuse notations regarding simplexes of \( \Delta \) also as subsets of other spaces involved). Denote \( \delta_+ = \gamma_+ \circ \delta : \Gamma_+ \to \Delta \).

In the notation below an element \( g \in \mathbb{Z}_{p^k} \) that appears in a composition with other maps is regarded as a homeomorphism of the space on which \( \mathbb{Z}_{p^k} \) acts.

**Proposition 2.3** For every \((m+2)\)-simplex \( \Delta_i \) of \( \Delta \) there is a map \( r_i^+ : \delta_+^{-1}(\Delta_i) \to \partial \Delta_i \) such that

(i) \( r_i^+ \) coincides with \( \delta_+ \) on \( \delta_+^{-1}(\partial \Delta_i) \);
(ii) for every triangulation of $\Gamma_+$ and every $g_i \in \mathbb{Z}_{p^k}$ there is a map $r_+ : \Gamma_+ \to \partial \Delta$ such that $r_+ (\gamma_+^{(m+2)}) \subset \Delta^{(m+1)}$ and $r_+$ restricted to $\delta_+^{-1}(\Delta_i)$ coincides with $r_i^+ \circ g_i$ for each $i$.

**Proof.** Recall that $\Gamma_+$ is the mapping cylinder of $\alpha_+ : M_+ \to M$ and let $\beta_+ : \Gamma_+ \to M$ be the projection to the top set $M$ of $\Gamma_+$. Also recall that $M_+$ is the union of $p$ copies $M_a$, $a \in \mathbb{Z}_{p^k}$, of $M$ intersecting each other at $\partial M$.

Denote $\Delta_a^{(m+1)} = \delta_+^{-1}(\Delta^{(m+1)}) \cap M_a$, $(\Delta_i \cap M)_a = \delta_+^{-1}(\Delta_i) \cap M_a$, $(\partial \Delta_i)_a = \delta_+^{-1}(\partial \Delta_i) \cap M_a$ and $(S^m \cap M)_a = \alpha_+^{-1}(S^m \cap M) \cap M_a$. For each $a$ and $i$ we will fix an interval $(S_i)_a$ in $(S^m \cap M)_a \cap (\Delta_i \cap M)_a$ that connects the sets $\partial M_i$ and $(\partial \Delta_i)_a$.

By (i) of Proposition 2.2 the map $\beta_+$ can be isotoped relative to $\beta_+^{-1}((S^m \cap M) \cup \partial M)$ into a map $\omega_+ : \Gamma_+ \to M$ such that $\omega_+$ and $\delta_+$ restricted to each $\Delta_a^{(m+1)} \subset M_+$ coincide. Consider the map $r$ from (ii) of Proposition 2.2 and note that $(r \circ \omega_+)(\Gamma_+) \subset \Delta^{(m+1)}$. Define the map $r_+^i$ as the map $r \circ \omega_+$ restricted to $\delta_+^{-1}(\Delta_i)$.

(i) follows from (ii) of Proposition 2.2

(ii) For a subset $A \subset M_+$ by the mapping cylinder of $\alpha_+$ over $A$ we mean the mapping cylinder of $\alpha_+ : A \to \alpha_+(A)$ which is a subset of $\Gamma_+$. Let $S \subset M_+$ be the union of $(S_i)_a$ for all $i$ and $a$, and $B \subset M_+$ the union of all the balls $B_i \subset M_1$. Consider a CW-structure of $\Gamma_+$ for which the interiors of the $(m+3)$-cells are the interiors in $\Gamma_+$ of the mapping cylinders of $\alpha_+$ over $(\Delta_i \cap M)_a \setminus (S_i)_a$ without the points belonging to $M_+$ and $M$. Thus the $(m+2)$-skeleton of this CW-structure is the union of $M_+ \cup M$ and the mapping cylinder of $\alpha_+$ over $\delta_+^{-1}(\Delta_i) \cup S \cup B$.

Let be the map $\phi : M_+ \to \partial \Delta$ be defined by the maps $r_i^+ \circ g_i$ on each $\delta_+^{-1}(\Delta_i)$. Note that the maps $\phi$ and $r \circ \omega_+$ are homotopic on $B$ by a homotopy relative to $B \cap S$. Thus $r \circ \omega_+$ restricted to the union of $M \subset \Gamma_+$ with the mapping cylinder of $\alpha_+$ over $\delta_+^{-1}(\Delta^{(m+1)}) \cup B \cup S$ can be homotoped to a map $\Phi$ such that $\phi$ and $\Phi$ restricted to $\delta_+^{-1}(\Delta^{(m+1)}) \cup B \cup S$ coincide. Thus we can extend $\phi$ to a map $\phi_+$ from the $(m+2)$-skeleton of the CW-structure of $\Gamma_+$ to $\Delta^{(m+1)}$.

Then, since $\Delta^{(m+1)}$ is contractible inside $\partial \Delta$, we may extend $\phi_+$ to a map $r_+ : \Gamma_+ \to \partial \Delta$. For any triangulation of $\Gamma_+$, the $(m+2)$-skeleton of the triangulation can be pushed off the interiors of the $(m+3)$-cells of $\Gamma_+$ relative to the $(m+2)$-skeleton of the CW-structure of $\Gamma_+$ and (ii) follows.

Denote by $\Gamma_*$ the space obtained from $\Gamma_+$ by collapsing the fibers of $\gamma_+$ to singletons over the set $\delta_+^{-1}(\Delta^{(m+1)})$ and let the maps $\gamma_* : \Gamma_* \to \Gamma$, $\delta_* : \Gamma_* \to \Delta$, $r_*^i : \delta_*^{-1}(\Delta_i) \to \partial \Delta_i$ and $r_* : \Gamma_* \to \partial \Delta$ be induced by $\gamma_+$, $\delta_*$, $r_+^i$ and $r_+$ respectively and consider $\Gamma_*$ with the action of $\mathbb{Z}_{p^k}$ induced by the action of $\mathbb{Z}_{p^k}$ on $\Gamma_+$.

**Proposition 2.4**

(i) The conclusions of Proposition 2.3 hold with the subscript “+” being replaced everywhere by the subscript “*”.

(ii) The fixed point set of the action of $\mathbb{Z}_{p^k}$ on $\Gamma_*$ is $(m+1)$-dimensional, the action of $\mathbb{Z}_{p^k}$ is free outside the fixed point set and there are triangulations of $\Gamma_*$ and $\Gamma$ and a subdivision of $\Delta$ for which the action of $\mathbb{Z}_{p^k}$ on $\Gamma_*$ is simplicial and the maps $\gamma_*$ and $\delta$ are simplicial.

4
Proof. (i) is obvious and (ii) can be derived from the construction of the spaces and the maps involved. ■

Let \( L \) be a finite \((m + 3)\)-dimensional simplicial complex and \( \Lambda : L \to \Delta \) a simplicial map such that \( \Lambda \) is 1-to-1 on each simplex of \( L \). Such a map \( \Lambda \) will be called a **sample map**. Denote by \( L' \) the pull-back space of the maps \( \lambda \) and \( \delta : \Gamma = \Gamma(p^k) \to \Delta \) and by \( \Omega : L' \to L \) the pull-back of \( \delta \).

**Proposition 2.5** The map \( \Omega : L' \to L \) induces an isomorphism of \( H_{m+3}(L'; \mathbb{Z}_p) \) and \( H_{m+3}(L; \mathbb{Z}_p) \).

**Proof.** Recall that for every \((m + 2)\)-simplex \( \Delta' \) of \( \Delta \), \( \delta^{-1}(\Delta') \) is the mapping cylinder a map of degree \( p^k \) from \( \partial \Delta' \) to a \((m+1)\)-sphere \( S^{m+1} \) (the boundary of one of the \((m+2)\)-balls \( B_i \)).

Also recall that \( \Gamma = \delta^{-1}(\Delta) \) is the mapping cylinder of the map \( \alpha \) and from the definition of \( \alpha \) one can also observe that \( H_{m+3}(\delta^{-1}(\Delta), \delta^{-1}(\partial \Delta); \mathbb{Z}_p) = \mathbb{Z}_p \) and \( \delta \) induces an isomorphism between \( H_{m+3}(\delta^{-1}(\Delta), \delta^{-1}(\partial \Delta); \mathbb{Z}_p) \) and \( H_{m+3}(\Delta, \partial \Delta; \mathbb{Z}_p) = \mathbb{Z}_p \).

And finally recall that \( \delta \) is 1-to-1 over the \((m+1)\)-skeleton of \( \Delta \). Consider the long exact sequences of the pairs \((L', L'(m+2))\) and \((L, L(m+2))\) for the homology with coefficients in \( \mathbb{Z}_p \). The facts above imply that \( \Omega \) induces isomorphisms \( H_{m+2}(L'(m+2); \mathbb{Z}_p) \to H_{m+2}(L'(m+2); \mathbb{Z}_p) \) and \( H_{m+3}(L', L'(m+2); \mathbb{Z}_p) \to H_{m+3}(L, L(m+2); \mathbb{Z}_p) \). Then, by the 5-lemma, \( \Omega \) induces an isomorphism \( H_{m+3}(L'; \mathbb{Z}_p) \to H_{m+3}(L; \mathbb{Z}_p) \) as well. ■

**Proposition 2.6** Let \( G = \mathbb{Z}_{p^k} \) act simplicially on a finite \((m + 3)\)-dimensional simplicial complex \( K \) such that the action of \( G \) is free on \( K \setminus K^{(m+1)} \). Fix a triangulation of the space \( L = K/G \) for which the projection \( K \to L \) is simplicial with respect to some barycentric subdivision of the triangulation of \( K \) and \( L \) admits a sample map \( \Lambda : L \to \Delta \) to an \((m + 3)\)-simplex \( \Delta \).

Then for every \( k' > k \) there is a finite \((m + 3)\)-dimensional simplicial complex \( K' \), a simplicial action of \( G' = \mathbb{Z}_{p^{k'}} \) on \( K' \) and a map \( \omega : K' \to K \) such that the action of \( G' \) is free on \( K' \setminus K'^{(m+1)} \) and

(i) the actions of \( G \) and \( G' \) agree with \( \omega \) and an epimorphism \( h : G' \to G \). By this we mean that \( \omega(g'x) = h(g')\omega(x) \) for for every \( x \in K' \) and \( g' \in G' \);

(ii) there is a map \( \kappa : K' \to K^{(m+2)} \) such that \( \kappa(K'^{(m+2)}) \subset K^{(m+1)} \) and \( \kappa(\omega^{-1}(\Delta_K)) \subset \Delta_K \) for every simplex \( \Delta_K \) of \( K \);

(iii) the space \( L' = K'/G' \) and the map \( \Omega : L' \to L = L \) determined by \( \omega \) can be obtained as the pull-back space and the pull-back map of the sample map \( \Lambda : L \to \Delta \) and the map \( \delta : \Gamma = \Gamma(p^{k'-k}) \to \Delta \). Thus, by Proposition 2.5, the map \( \Omega \) induces an isomorphism \( H_{m+3}(L'; \mathbb{Z}_p) \to H_{m+3}(L; \mathbb{Z}_p) \).

(iv) Moreover, the action of \( G' \) on \( K' \) is free over \( \Lambda'^{-1}(\Gamma \backslash \gamma_*(S)) \subset L' \) where \( \Lambda' : L' \to \Gamma \) is the pull-back of \( \Lambda \) from (iii) and \( S \subset \Gamma_* = \Gamma_*(p^{k'-k}) \) is the fixed point set of the action of \( \mathbb{Z}_{p^{k'-k}} \) on \( \Gamma_* \).

**Proof.** Replacing the triangulation of \( K \) by its subdivision we assume that the projection \( \pi : K \to L \) is a simplicial map. For every \((m + 3)\)-simplex \( \Delta_L \) of \( L \) fix an \((m + 3)\)-simplex
\( \Delta_K \) of \( K \) such that \( \pi(\Delta_K) = \Delta_L \) and denote by \( K_- \) the union of the \((m+2)\)-skeleton \( K^{(m+2)} \) of \( K \) with all the \((m+3)\)-simplexes of \( K \) that we fixed. Let \( K'_+ \) be the pull-back space of the maps \( \Lambda \circ \pi|K_- : K_- \to \Delta \) and \( \delta_+ = \delta \circ \gamma_* : \Gamma_* = \Gamma_s(p^{k-1}) \to \Delta, \omega_- : K'_+ \to K \) the pull-back map of \( \delta_+ \) and \( \lambda_- : K'_+ \to \Gamma \), the pull back of \( \Lambda \circ \pi|K_- \).

Let \( g \) be a generator of \( G' \) and \( l : G' \to \mathbb{Z}_{p^{k-1}} \) an epimorphism. We will first define the action of \( G' \) on \( \omega_-^{-1}(K^{(m+2)}) \). For each \((m+2)\)-simplex \( \Delta_L \) of \( L \) define the action of \( G' \) on \( \omega_-^{-1}(\pi^{-1}(\Delta_L)) \) as follows. Fix a \((m+2)\)-simplex \( \Delta_K \) of \( \pi^{-1}(\Delta_L) \) and let \( x \in \omega_-^{-1}(\Delta_K) \).

Define \( y = g^tx \) for \( 1 \leq t \leq p^k - 1 \) as the point \( y \in K' \) such that \( \omega_-(y) \in h(g^t)(\Delta_K) \) and \( \lambda_-(y) = \lambda_-(x) \), and for \( t = p^k \) define \( y = g^tx \) as the point \( y \in \Delta_K \) such that \( \lambda_-(y) = l(g)\lambda_-(x) \). We do this independently for every \((m+2)\)-simplex \( \Delta_L \) of \( L \) and this way define the action of \( g \) on \( \omega_-^{-1}(K^{(m+2)}) \). It is easy to see that the action of \( g \) is well-defined, \( g^t \) for \( t = p^k \) is the identity map of \( \omega_-^{-1}(K^{(m+2)}) \) and hence the action of \( g \) defines the action of \( G' \) on \( \omega_-^{-1}(K^{(m+2)}) \). Note that

\[ (*) \] for \( g^t \in G', t = p^k \) and \( g_s = l(g) \in \mathbb{Z}_{p^k-1} \) we have that \( \lambda_- \circ g^t \omega_-^{-1}(\partial\Delta_K) = g_s \circ \lambda_-|\omega_-^{-1}(\partial\Delta_K) \) for every \((m+1)\)-simplex \( \Delta_K \) of \( K \).

Now we will enlarge \( K'_+ \) to a space \( K' \) and extend the action of \( G' \) over \( K' \). Let \( \Delta_L \) be a \((m+3)\)-simplex of \( L \). Recall that we fixed a \((m+3)\)-simplex \( \Delta_K \) in \( \pi^{-1}(\Delta_L) \). For every \( g^t = g^t(\omega_-^{-1}(\partial\Delta_K) \) a copy of the space \( \omega_-^{-1}(\Delta_K) \) (which is in its turn a copy of \( \Gamma_* \)) by identifying \( g^t(\omega_-^{-1}(\partial\Delta_K) \) with \( \omega_-^{-1}(\partial\Delta_K) \) according to \( g \) and for \( x \in \omega_-^{-1}(\Delta_K) \) define \( g^t x \) as the as the point corresponding to \( x \) in the attached space. We will define the action of \( g^t, t = p^k, \) on \( \omega_-^{-1}(\Delta_K) \) by \( y = g^t x, x \in \omega_-^{-1}(\Delta_K) \) such that \( y \in \omega_-^{-1}(\Delta_K) \) and \( \lambda_-(y) = l(g)\lambda_-(x) \). By \( (*) \) the action of \( g \) on \( \omega_-^{-1}(\Delta_K) \) agrees with the action of \( g \) on \( \omega_-^{-1}(K^{(m+2)}) \). We do the above procedure independently for every \((m+3)\)-simplex \( \Delta_L \) of \( L \) and this way we define the space \( K' \) and the action of \( G' \) on \( K' \).

We extend \( \omega_- \) and \( \lambda_- \) to the maps \( \omega : K' \to K \) and \( \lambda' : K' \to \Gamma \) by \( \omega(g^t x) = \omega_-(x) \) and \( \lambda'(g^t x) = \gamma_s(\lambda_-(x)) \) for \( x \) in a fixed \((m+3)\)-simplex \( \Delta_K \) and \( 1 \leq t \leq p^k \).

It is easy to verify that the action of \( G' \) on \( K' \) and the maps \( \omega \) and \( \lambda' \) are well-defined and the conclusion (i) of the proposition holds. Moreover \( \lambda' \circ g^t = \lambda' \) for every \( g^t \in G' \) and hence \( \lambda' \) defines the corresponding map \( \Lambda' : L' = K'/G' \to \Gamma \). Then \( L' \) is the pull-back of the maps \( \Lambda : L \to \Delta \) and \( \delta : \Gamma \to \Delta \) with \( \Lambda' \) being the pull-back of \( \Lambda \) and the map \( \Omega : L' \to L \) induced by \( \omega \) being the pull-back of \( \delta \). Thus, by Proposition 2.3, the conclusion (ii) of the proposition holds as well.

Consider any triangulation of \( K' \) for which the preimages under \( \omega \) of the simplexes of \( K' \) are subcomplexes of \( K' \). Then the map \( r_* : \Gamma_* \to \Delta^{(m+2)} \) provided by Propositions 2.4 and 2.3 for \( g_i = 0 \in \mathbb{Z}_p \) for all \( i \) defines the corresponding map \( \kappa_- : K'_- \to K^{(m+2)} \) such that \( \kappa(\omega_-^{-1}(K^{(m+2)})) \subset K^{(m+1)} \). The construction above and Propositions 2.4 and 2.3 allow us to extend the map \( \kappa_- \) to a map \( \kappa : K' \to K^{(m+2)} \) satisfying the conclusion (ii) of the proposition. Recall that the triangulation of \( K \) is a subdivision of the original triangulation of \( K \). Replacing \( \kappa \) by its composition with the simplical approximation of the identity map of \( K \) with respect to the new and original triangulations of \( K \) we get that the conclusion (ii) of the proposition holds.

The rest of the conclusions of the proposition follows from (ii) of Proposition 2.4.
3 Properties related to cohomological dimension and light maps

A CW-complex $K$ is said to be an absolute extensor for a space $X$ if every map $f : X \to K$ from a closed subset $A \subset X$ continuously extends over $X$. The extension criterion of Dranishnikov says that a simply connected CW-complex is an absolute extensor for a finite dimensional compactum $X$ if and only if $\dim_{H_n(K)} X \leq n$ for every $n > 1$.

Recall the construction of $\Gamma$ and note that for $k' > k$ there is a natural map $\Gamma_{\partial M} = \Gamma_{\partial M}(p^k) \to \Gamma'_{\partial M} = \Gamma_{\partial M}(p^{k'})$ induced by the identity map of $M_0$ in $M_+(p^{k'})$ and $M_0$ in $M_+(p^{k'})$. Also note that the identity map of $M$ induces a natural map $\Gamma_M = \Gamma_M(p^k) \to \Gamma'_M = \Gamma_M(p^{k'})$. Then these natural maps index the corresponding map $\tau = \tau(p^k, p^{k'}) : \Gamma = \Gamma(p^k) \to \Gamma' = \Gamma(p^{k'})$. Also note that from the construction of $\delta : \Gamma \to \Delta$ it follows that for $\delta' = \delta(p^{k'}) : \Gamma' = \Gamma(p^{k'}) \to \Delta$ we have that $\delta = \delta' \circ \tau$.

Let $C = C(p^k)$ be the infinite telescope of the maps $\tau(p^t, p^{t+1}) : \Gamma(p^t) \to \Gamma(p^{t+1})$ for $t \geq k$ and let $\delta_C : C \to \Delta$ be the map determined by the maps $\delta(p^t) : \Gamma(p^t) \to \Delta$.

**Proposition 3.1** For every simplex $\Delta'$ of $\Delta$ the space $\delta_{\Gamma}^{-1}(\Delta')$ is an absolute extensor for every finite dimensional compactum $X$ with $\dim_{[\frac{1}{p}]_{H_1}} X \leq m + 1$.

**Proof.**

If $\dim \Delta' \leq m + 1$ then $(\delta(p^t))^{-1}(\Delta')$ is contractible and hence the reduced homology $H_n(\delta_{\Gamma}^{-1}(\Delta'))$ vanishes for every $n$.

If $\dim \Delta' = m + 2$ then $(\delta(p^t))^{-1}(\Delta')$ is homotopy equivalent to an $(m+1)$-sphere $S^{m+1}$ and the map $\tau(p^t, p^{t+1})$ restricted to $(\delta(p^t))^{-1}(\Delta')$ and $(\delta(p^{t+1}))^{-1}(\Delta')$ acts as a map of degree $p$ between $(m+1)$-spheres and hence $H_{m+1}(\delta_{\Gamma}^{-1}(\Delta')) = \mathbb{Z}[\frac{1}{p}]$ and $H_n(\delta_{\Gamma}^{-1}(\Delta')) = 0$ if $n \neq m + 1$.

If $\Delta' = \Delta$: then $(\delta(p^t))^{-1}(\Delta) = \Gamma(p^t)$. Recall that $\Gamma(p^t)$ is the mapping cylinder of $\alpha : \Gamma_{\partial M}(p^t) \to \Gamma_M(p^t)$ and hence $\Gamma(p^t)$ is homotopy equivalent to $\Gamma_M(p^t)$. Note that $\Gamma_M(p^t)$ is homeomorphic to $M$ and $M$ is homotopy equivalent to the wedge of $m+3$ spheres $S^{m+1}$ and $\tau(p^t, p^{t+1})$ acts on these spheres as a map of degree $p$. Thus $H_{m+1}(\Gamma_C)$ is the direct sum of $m+3$ copies of $\mathbb{Z}[\frac{1}{p}]$ and $H_n(\Gamma_C) = 0$ if $n \neq m + 1$.

Then the proposition follows from Dranishnikov’s extension criterion. ■

A map is said to be light or 0-dimensional if its fibers are 0-dimensional. We will say that a space $N$ has the $n$-approximation property if $N$ is a metric compact ANR and every map $f : X \to N$ from a compactum $X$ with $\dim X \leq n$ can be arbitrarily closely approximated by a light map.

By an extended mapping cylinder $E$ of a map $g : N \to N'$ we will understand the mapping cylinder of $g$ with the product $N' \times [0, 1]$ attached to the mapping cylinder by identifying the top $N'$ of the mapping cylinder with the set $N' \times \{0\}$ of the product $N' \times [0, 1]$. We will refer to the mapping cylinder of $g$ as the proper part of $E$ and to $N' \times [0, 1]$ as the extension part of $E$. 
Proposition 3.2

(i) Every compact n-manifold (possibly with boundary) has the n-approximation property.

(ii) If a compactum $N$ is covered by the interiors of closed subsets of $N$ having the n-approximation property then $N$ has the n-approximation property as well.

(iii) If $N$ has the n-approximation property then any map from an n-dimensional compactum $f : X \to N$ that is light on a closed subset $A$ of $X$ can be arbitrarily closely approximated by a light map that coincides with $f$ on $A$.

(iv) If $N$ has the n-approximation property then $N \times [0, 1]$ has the $(n+1)$-approximation property.

(v) If $N$ and $N'$ have the n-approximation property and $g : N \to N'$ is a light surjective map then the extended mapping cylinder $E$ of $g$ has the $(n+1)$-approximation property.

Proof.

(i) follows from the Baire category theorem applied to mapping spaces and the well-known fact that a closed n-ball has the n-approximation property.

(ii) follows from the Baire category theorem applied to mapping spaces.

(iii) again follows from the Baire category theorem applied to mapping spaces.

(iv) Let $f : X \to N \times [0, 1]$ be a map from a compactum $X$ with $\dim X \leq n + 1$ and let $f = (f_N, f_I)$ be the coordinate maps $f_N : X \to N$ and $f_I : X \to [0, 1]$ of $f$. Decompose $X$ into $X = A \cup B$ with $A$ being $(n-1)$-dimensional and $\sigma$-compact and $B$ being 0-dimensional. Approximate $f_N$ by a map $f'_N : X \to N$ such that $f'_N$ restricted to $A$ is 0-dimensional. Denote by $Y$ the subset of $X$ which is the union of all the non-trivial compact connected sets contained in the fibers of the map $(f'_N, f_I) : X \to N \times [0, 1]$. Then $Y$ is $\sigma$-compact and $Y \cap A$ is 0-dimensional and hence $\dim Y \leq 1$. Approximate $f_I$ by a map $f'_I : X \to [0, 1]$ such that $f'_I$ restricted to $Y$ is 0-dimensional. Then $f' = (f'_N, f'_I) : X \to N \times [0, 1]$ is a 0-dimensional approximation of $f$.

(v) Let $E = E_1 \cup E_2$ be the proper and the extension parts of $E$ respectively. Note that for every extended mapping cylinder the identity map of $E$ can be arbitrarily closely approximated by a map $\phi : E \to E$ such that $\phi^{-1}(E_1)$ is contained in the interior of $E_1$ and $\phi(E_2)$ is contained in the interior of $E_2$. Moreover, if $g$ is light than $\phi$ can be assumed to be light as well. Let $f : X \to E$ be a map from a compactum $X$ with $\dim X \leq n + 1$. By (iv) the map $f : X \to E$ can be approximated by a map $f' : X \to E$ so that $f'$ is 0-dimensional on a closed neighborhood of $f^{-1}(\phi^{-1}(E_1))$ in $X$. Then taking $f'$ to be sufficiently close to $f$ we can also assume that $f'$ is 0-dimensional on $(f')^{-1}(\phi^{-1}(E_1))$. Since $\phi$ is 0-dimensional, the map $\phi \circ f'$ is also 0-dimensional on $(f')^{-1}(\phi^{-1}(E_1))$. Clearly $X \setminus (f')^{-1}(\phi^{-1}(E_1)) \subset (f')^{-1}(\phi^{-1}(E_2))$. Then, by (iii) and (iv), the map $\phi \circ f' : X \to E$ restricted to $(f')^{-1}(\phi^{-1}(E_1))$ can be extended arbitrarily closely to $\phi \circ f'$ to a light map $f'' : X \to E$. ■

Let us return again to the construction of $\Gamma = \Gamma(p^k)$ and $\delta : \Gamma \to \Delta$. For every $(m + 2)$-
simplex $\Delta_i$ of $\Delta$ the set $\delta^{-1}(\Delta_i) \subset \partial M$ can be represented as the mapping cylinder of a map of degree $p^k$ from $\partial \Delta'$ to the sphere $S_i^{m+1} = \gamma_+(\partial B_i) \subset \partial M$, $\partial B_i \subset M$. Enlarge this mapping cylinder to the extended one by attaching $\partial S_i^{m+1} \times [0,1]$ to the top $S_i^{m+1}$ of the mapping cylinder. Let us also attach $S_i^{m+1} \times [0,1]$ to $S_i^{m+1} = \gamma_+(\partial B_i) \subset \partial M$, $\partial B_i \subset M$. Denote by $\Gamma_{\partial M}^E$ the space obtained from $\Gamma_{\partial M}$ after the attachments for all $i$ and denote by $\Gamma_M^E$ the space obtained from $\Gamma_M$ after the attachments for all $i$. Note that $\Gamma_M^E$ is an $(m+2)$-manifold with boundary (actually both $\Gamma_{\partial M}^E$ and $\Gamma_M^E$ are homeomorphic to $M$). Clearly the map $\alpha : \Gamma_{\partial M} \to \Gamma_M$ extends to $\alpha_E : \Gamma_{\partial M}^E \to \Gamma_M^E$ by the identity map on the attached parts. Recall that $\Gamma$ is the mapping cylinder of $\alpha$ and denote by $\Gamma_E$ the extended mapping cylinder $\alpha_E$. Note that the natural rejections of $\Gamma_{\partial M}^E$ to $\Gamma_{\partial M}$ and of $\Gamma_M^E$ to $\Gamma_M$ induce the corresponding retraction from the mapping cylinder of $\alpha_E$ to $\Gamma$, and composing the last retraction with the natural retraction from the extended mapping cylinder $\Gamma_E$ to the mapping cylinder of $\alpha_E$ we obtain the retraction $\gamma_E : \Gamma_E \to \Gamma$. Denote $\delta_E = \delta \circ \gamma_E : \Gamma_E \to \Delta$.

**Proposition 3.3**

(i) $\dim \Gamma_E = m + 3$ and $\dim \gamma_E^{-1}(F) = m + 1$ where $F$ is the image under $\gamma_*$ of the fixed point set of the action of $\mathbb{Z}_{p^k}$ on $\Gamma_*$. 

(ii) $\Gamma_E$ has the $(m+3)$-approximation property. Moreover for every $(m+2)$-simplex $\Delta'$ of $\Delta$ the space $\delta_E^{-1}(\Delta') \subset \Gamma_E$ has the $(m+2)$-approximation property.

**Proof.**

(i) $\dim \Gamma_E = m + 3$ is obvious. Note that $F$ consists of $F_1 = \delta^{-1}(\Delta^{(m+1)})$, $F_2$ is the union of $(S_i^{m+1} \cap S^m) \times [0,1]$ for all $i$ and $F_3 = S^m \subset \Gamma_M$ where $S^m$ is the fixed point set of the action of $T$ on $S_i^{m+2}$. Then $\gamma_E^{-1}(F)$ can be represented as the union of the following parts homeomorphic to $F_1^E = F_1$, $F_2^E = F_2 \times [0,1]$ and $F_3^E = (F_2 \cup F_3) \times [0,1]$. Note that $\dim F_1 = m + 1$ and $\dim F_2 = m$ and $\dim F_3 = m$. Then $\dim F_1^E = m + 1$, $\dim F_2^E = m + 1$ and $\dim F_3^E = m + 1$ and hence $\dim \gamma_E^{-1}(F) = m + 1$.

(ii) By Proposition 3.2 the spaces $\delta_E^{-1}(\Delta')$, $\Gamma_{\partial M}^E$ and $\Gamma_M^E$ have the $(m+3)$-approximation property and hence, again by Proposition 3.2, the space $\Gamma_E$ has the $(m+3)$-approximation property.

Let $L'$ be a CW-complex and $L$ a simplicial complex. We say that a map $\Omega : L' \to L$ is combinatorial if the preimage under $\Omega$ of every simplex of $L$ is a subcomplex of $L'$. We also say that for a map $f : X \to L$ a map $f' : X \to L'$ is a combinatorial lifting of $f$ if $\Omega(f'(f^{-1}(\Delta_L))) \subset \Delta_L$ for every simplex $\Delta_L$ of $L$. It is easy to see that if for a space $X$ and a combinatorial map $\Omega : L' \to L$ we have that $\Omega^{-1}(\Delta_L)$ is an absolute extensor for $f^{-1}(\Delta_L)$ for every simplex $\Delta_L$ of $L$ then any map $f : X \to L$ admits a combinatorial lifting $f' : X \to L'$. Moreover, if $A$ is a closed subset of $X$ and $f_A' : A \to L'$ is a combinatorial lifting of $f$ restricted to $A$ then $f_A'$ extends to a combinatorial lifting $f' : X \to L'$ of $f$.

Let $\Lambda : L \to \Delta$ be a sample map of a finite $(m+3)$-dimensional simplicial complex $L$ to an $(m+3)$-simplex $\Delta$.

**Proposition 3.4** Let $f : X \to L$ be a map from a finite dimensional compactum $X$ with $\dim_{\mathbb{Z}[k]} \leq m + 1$. Then for every natural number $k$ there is a natural number $k' > k$ such
that for the pull-back space $L'$ of the maps $\Lambda$ and $\delta' = \delta(p^k') : \Gamma' = \Gamma(p^k) \to \Delta$ there is a combinatorial lifting $f' : X \to L'$ of $f$ with respect to the pull-back map $\Omega : L' \to L$ of $\delta'$.

**Proof.** Consider the infinite telescope $C(p^k)$ with $\Gamma(p^k)$ being naturally embedded in $C(p^k)$ and let $L_C$ be the pull-back space of the sample map $\Lambda : L \to \Delta$ and $\delta_C : C(p^k) \to \Delta$ and let $\Omega_C : L_C \to L$ and $\Lambda_C : L_C \to C(p^k)$ be the pull-back maps of $\delta_C$ and $\Lambda$ respectively.

Then, by Proposition 3.4 the map $f$ admits a combinatorial lifting $f_C : X \to C(p^k)$ to $C(p^k)$ with respect to $\Omega_C$. Since $X$ is compact there is $k' > k$ such that $\Lambda_C(f_C(X)) \subset C(p^k)$ is contained in the finite part $C'$ of the telescope $C(p^k)$ that ends at $\Gamma(p^k')$. Let $\pi : C' \to \Gamma(p^k')$ be the natural projection and let $\Pi : L_C \to L'$ be the map induced by $\pi$. Then $f' = \Pi \circ f_C : X \to L'$ is the required combinatorial lifting of $f$. ■

Let $L$ be a finite $(m+3)$-dimensional simplicial complex and $\Lambda : L \to \Delta$ a sample map to an $(m+3)$-simplex $\Delta$. Consider the pull-back spaces $L'$ and $L_E'$ of the map $\Lambda$ and the maps $\delta : \Gamma \to \Delta$ and $\delta_E = \delta \circ \gamma_E : \Gamma_E \to \Delta$ respectively and let $\Omega : L' \to L$ and $\Omega_E : L_E' \to L$ be the pull-backs of $\delta$ and $\delta_E$ respectively and let $r_E : L_E' \to L'$ be the retraction induced by the retraction $\gamma_E : \Gamma_E \to \Gamma$.

**Proposition 3.5** Assume that a map $f : X \to L$ from a compactum $X$ is such that $\dim X \leq m + 3$, $\dim_{\mathbb{Z}_p}[X] \leq m + 1$ and $\dim f^{-1}(\Delta_L) \leq \dim \Delta_L$ for every simplex $\Delta_L$ of $L$. Also assume that $f$ admits a combinatorial lifting $f'_\Lambda : X \to L'_\Lambda$ with respect to $\Omega$. Then $f$ also admits a combinatorial lifting $f'_E : X \to L'_E$ with respect to $\Omega_E$ such that $f'_E$ is a light map.

**Proof.** The proposition follows from (ii) of Proposition 3.3. ■

### 4 Proof of Theorem 1.1

**Proposition 4.1** Let $X$ be a compactum with $\dim X \leq m + 3$ and $\dim_{\mathbb{Z}_p}[X] \leq m + 1$. Then there is a compactum $Y$ and an action of $A_p$ on $Y$ such that $\dim Y \leq m + 3$, $\dim Y/A_p \leq m + 1$ and $X$ admits a light map to $Y/A_p$.

**Proof.** We will construct by induction finite simplicial complexes $K_i$, a simplicial action of $\mathbb{Z}_p\mathbb{k}_{ki}$ on $K_i$, bonding maps $\omega_{i+1} : K_{i+1} \to K_i$, $\eta_{i+1} : K_{i+1} \to K_i$ and maps $f_i : X \to L_i = K_i/\mathbb{Z}_p\mathbb{k}_i$ such that $k_{i+1} \geq k_i$, the map $\omega_{i+1}$ agrees with the actions of $\mathbb{Z}_p\mathbb{k}_{i+1}$ and $\mathbb{Z}_p\mathbb{k}_i$ on $K_{i+1}$ and $K_i$ respectively with respect to an epimorphism $\mathbb{Z}_p\mathbb{k}_{i+1} \to \mathbb{Z}_p\mathbb{k}_i$.

Set $L_0 = K_0$ to be an $(m+3)$-dimensional sphere, $f_0 : X \to L_0$ to be any light map and let the trivial group $\mathbb{Z}_p0 = 0$ trivially act on $K_0$. Assume that the construction is completed for $i$. To proceed to $i + 1$ we will choose one of the following procedures.

**Procedure 1.** Let $K = K_i$, $L = L_i$, $k = k_i$ and $\Lambda : L \to K$ the sample map satisfying the assumptions of Proposition 2.6. Assume that $f_i : X \to L_i$ is light, apply Proposition 3.3 for $f = f_i$ to produce $k'$ and set $k_{i+1} = k_i + k'$. Apply Proposition 2.6 with $K = K_i$, $L = L_i$, $k = k_i$, $k' = k_{i+1}$ to construct $K_{i+1} = K'$, $L_{i+1} = L'$, $\omega_{i+1} = \omega$, $\eta_{i+1} = \eta$ and the action of $\mathbb{Z}_p\mathbb{k}_{i+1}$ on $K_{i+1}$ satisfying the conclusion of Proposition 2.6. By (iii) of Proposition
2.6 and Proposition 3.4 the map $f_i$ admits a combinatorial lifting $f_{i+1} : X \to L_{i+1}$ with respect to $\Omega = \Omega_{i+1} : L_{i+1} \to L_i$.

**Procedure 2.** Assume that we proceeded from $i-1$ to $i$ according to Procedure 1. Let $L = L_{i-1}$ and $\Lambda : L = L_{i-1} \to \Delta$ be the sample map used in Procedure 1 for proceeding from $i-1$ to $i$. Since the map $f_{i-1}$ in Procedure 1 was assumed to be light we have that $\dim f_{i-1}^{-1}(\Delta_L) \leq \dim \Delta_L$ for every simplex $\Delta_L$ of $L = L_{i-1}$. Apply Proposition 3.3 for $f = f_{i-1}, f' = f_i, L = L_{i-1}$ to construct $L_{i+1} = L'_E$ and a light map $f_{i+1} = f'_E$. Define $K_{i+1}$ as the pull-back space of the projection $K_i \to L_i = K_i / \mathbb{Z}^{p_i}$ and and $\Omega_{i+1} = \Omega_E : L_{i+1} = L'_E \to L_i = L$, and define $\omega_{i+1} : K_{i+1} \to K_i$ to be the pull-back map of $\Omega_{i+1}$. Set $k_{i+1} = k_i$ and define the action of $\mathbb{Z}^{p_{i+1}}$ on $K_{i+1}$ as the pull-back action of $\mathbb{Z}^{p_i}$ on $K_i$. By (i) of Proposition 3.3 (iii) and (iv) of Proposition 2.6 and (ii) of Proposition 2.4 one can find a triangulation of $L_{i+1}$ whose preimage under the projection $K_{i+1} \to L_{i+1} = K_{i+1} / \mathbb{Z}^{p_{i+1}}$ provides a triangulation of $K_{i+1}$ for which the action of $\mathbb{Z}^{p_{i+1}}$ on $K_{i+1}$ is simplicial and free outside the $(m+1)$-skeleton of $K_{i+1}$. Set $\kappa_{i+1} : K_{i+1} \to K_i$ to be a simplicial approximation of $\omega_{i+1}$.

Carry out the construction applying Procedure 1 for $i = 2t$ and Procedure 2 for $i = 2t+1$. Let $\Omega_{i+1} : L_{i+1} \to L_i$ be the map determined by $\omega_{i+1} : K_{i+1} \to K_i$. Denote $Y = \lim_{\Leftarrow}(K_i, \omega_i)$ and consider $Y$ with the action of $A_p = \lim_{\Leftarrow} \mathbb{Z}^{p_i}$ determined by the actions of $\mathbb{Z}^{p_i}$ on $K_i$ for all $i$. Then $Y/A_p = \lim_{\Leftarrow}(L_i, \Omega_i)$. Note that we can replace the triangulation of $K_{2t}$ by any of its barycentric subdivisions and assume that the triangulation of $K_{2t}$ is as fine as we wish. Then (ii) of Proposition 2.6 guarantees that $\dim Y \leq m+1$ and Procedure 2 allow us to achieve that there is a map $f : X \to Y/A_p$ determined by the maps $f_i$ such that for each $t > 0$ the map $f$ followed by the projection of $Y/A_p$ to $L_{2t}$ is as close to $f_{2t}$ as we wish. Recall that $f_{2t}$ is light. Then the construction can be carried out so that $f$ is light map. Clearly $\dim Y/A_p \leq m+3$ and the proposition follows. ■

**Proof of Theorem 1.1.** Let $f : X \to Y/A_p$ be the light map produced by Proposition 1.1 for $m = n - 1$. Denote by $Z$ be the pull-back space of $f$ and the projection $Y \to Y/A_p$ and endow $Z$ with the pull-back action of $A_p$ on $Y$. Then $X = Z/A_p$ and the pull-back map $Z \to Y$ of $f : X \to Y/A_p$ is light since $f$ is light. Thus we have $\dim Z \leq \dim Y \leq m+1 = n$. ■

## 5 Concluding remarks

Yang [3] showed that for an action of $A_p$ on a compactum $Z$ we have $\dim Z/A_p \leq \dim Z + 3$, $\dim_{z_p} Z/A_p \leq \dim_{z_p} Z + 2$ and $\dim_{z_p} Z/A_p \leq \dim_{z_p} Z + 2$. It is still an open problem if the equality $\dim_{Z/A_p} = \dim Z + 3$ can be achieved. One can show that both Raymond-Williams’ example and the example in [11] satisfy $\dim_{z_p} Z/A_p = \dim_{z_p} Z + 2$. Theorem 1.1 shows that the gap 2 can be also achieved for $\dim_{z_p}$.

**Corollary 5.1.** There is an action of $A_p$ on an $n$-dimensional compactum $Z$, $n \geq 2$, such that $\dim_{z_p} Z/A_p = \dim_{z_p} Z + 2 = n+1$. 

11
Proof. Take any compactum $X$ with $\dim X = n + 2$, $\dim_{\mathbb{Z}_p^{1/p}} X = n - 1$ and $\dim_{\mathbb{Z}_p^{\infty}} X = n + 1$. Note that the dimensional restrictions on $X$ satisfy Bockstein’s inequalities and hence, by Dranishnikov’s realization theorem, such a compactum $X$ exists. Apply Theorem 1.1 to construct an $n$-dimensional compactum $Z$ and an action of $A_p$ on $Z$ such that $X = Z/A_p$. Recall that $\dim_{\mathbb{Z}_p^{1/p}} Z = \dim_{\mathbb{Z}_p^{1/p}} X = n - 1$. Then, by Bockstein inequalities, we have that $\dim_{\mathbb{Z}_p^{\infty}} Z = n - 1$. ■

A similar argument based on Yang’s relations and Bockstein’s inequalities also shows that

**Corollary 5.2** No $(n+3)$-dimensional compactum $X$ with $\dim_{\mathbb{Z}_p^{1/p}} X \leq n - 1$ can be obtained as the orbit space of an action of $A_p$ on an $n$-dimensional compactum $Z$.

We will end with a couple of related problems.

**Problem 5.3** Does there exist a 3-dimensional compactum that can be obtained as the orbit space of an action of $A_p$ on a 1-dimensional compactum? Can any 3-dimensional compactum $X$ with $\dim_{\mathbb{Z}_p^{1/p}} X = 1$ be obtained as the orbit space of an action of $A_p$ on a 1-dimensional compactum?

**Problem 5.4** Let $n \geq 1$. Does there exist an $(n + 2)$-dimensional compactum $X$ that can be obtained as the orbit space of an action of $A_p$ on an $n$-dimensional compactum such that the action of $A_p$ is free over a subset $X'$ of $X$ of $\dim X \setminus X' \leq n - 1$? Can any $(n + 2)$-dimensional compactum $X$ with $\dim_{\mathbb{Z}_p^{1/p}} X \leq n$ be obtained as the orbit space of an action of $A_p$ on an $n$-dimensional compactum such that the action of $A_p$ is free over a subset of $X'$ of $X$ with $\dim X \setminus X' \leq n - 1$?

Note that the action of $A_p$ constructed in Theorem 1.1 is not free over a subset of $X$ of $\dim = n$.

**References**

[1] Levin, Michael. *On Raymond-Williams’ example.* Preprint, http://arxiv.org/abs/1909.12660

[2] Raymond, Frank; Williams, R. F. *Examples of p-adic transformation groups.* Ann. of Math. (2) 78 (1963) 92-106.

[3] Yang, Chung-Tao. *p-adic transformation groups.* Michigan Math. J. 7 (1960) 201-218.

Michael Levin
Department of Mathematics
Ben Gurion University of the Negev
P.O.B. 653

12
