Quantum Mechanics Associated with a Finite Group

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Abstract

I describe, in the simplified context of finite groups and their representations, a mathematical model for a physical system that contains both its quantum and classical aspects. The physically observable system is associated with the space containing elements $f \otimes f$ for $f$ an element in the regular representation of a given finite group $G$. The Hermitian portion of $f \otimes f$ is the Wigner distribution of $f$ whose convolution with a test function leads to a mathematical description of the quantum measurement process. Starting with the Jacobi group that is formed from the semidirect product of the Heisenberg group with its automorphism group $SL(2, \mathbb{F}_N)$ for $N$ an odd prime number I show that the classical phase space is the first order term in a series of subspaces of the Hermitian portion of $f \otimes f$ that are stable under $SL(2, \mathbb{F}_N)$. I define a derivative that is analogous to a pseudodifferential operator to enable a treatment that parallels the continuum case. I give a new derivation of the Schrödinger-Weil representation of the Jacobi group. Keywords: quantum mechanics, finite group, metaplectic. PACS: 03.65.Fd; 02.10.De; 03.65.Ta.

1 Introduction

We consider the following construction: For a given finite group $G$:

1. Form the group ring over the complex numbers.
2. Decompose the group ring into subspaces $V_i$ that are invariant under $G$.
3. For a vector $f$ in subspace $V_i$ form the tensor product $f \otimes f$.
4. Decompose $f \otimes f$ into components that are invariant under the action of $\{g \otimes g \mid g \in S \subset G\}$.

The main idea that we explore is the hypothesis that physical observables reside in the vector space $f \otimes f$. This vector space is closed under operations $g \otimes g$, $g \otimes f$, $f \otimes g$, and $f \otimes f$. The decomposition into subspaces $V_i$ is performed according to the irreducible representations of $G$. The Hermitian portion $f \otimes f$ is constructed from the Wigner distribution of $f$, which is a function of the classical phase space. The convolution with a test function provides a mathematical description of the quantum measurement process. The classical phase space is the first order term in a series of subspaces of the Hermitian portion $f \otimes f$ that are stable under $SL(2, \mathbb{F}_N)$. A new derivation of the Schrödinger-Weil representation of the Jacobi group is given. Keywords: quantum mechanics, finite group, metaplectic. PACS: 03.65.Fd; 02.10.De; 03.65.Ta.
$g \in G$ that take $f \otimes f$ to $gf \otimes gf$. These operations compose, by hypothesis, the allowed transformations of the corresponding physical observables.

In particular, we consider this construction for the semidirect product of the Heisenberg group $H_1(\mathbb{F}_N)$ with the special linear group $SL(2,\mathbb{F}_N)$. $H_1(\mathbb{F}_N)$ is generated by 2 elements whose commutator is an element of the center of the group. The special linear group $SL(2,\mathbb{F}_N)$ is the group of automorphisms of $H_1(\mathbb{F}_N)$ that leaves the center invariant. $SL(2,\mathbb{F}_N)$ is isomorphic to the symplectic group $SP(1,\mathbb{F}_N)$. The semi-direct product $G^J = SL(2,\mathbb{F}_N) \ltimes H_1(\mathbb{F}_N)$ is the Jacobi group (Berndt and Schmidt 1998). We work over the prime field $\mathbb{F}_N$ for $N$ a prime not equal to 2.

By working out the above construction for the Jacobi group $G^J$ we develop a finite model for the quantum system comprised of a single particle with a single spatial degree of freedom whose phase space is a flat torus. The correspondence between the quantum system and our construction is the following: The wavefunction for the quantum system corresponds to a vector $f$ in an invariant subspace of $G^J$. The Hermitian component of $f \otimes f$ corresponds to the Wigner distribution of $f$. The evaluation at the origin of the convolution of the Hermitian component of $f \otimes f$ with a test function corresponds to the expectation value of the test function. We obtain the Weyl map that relates the evaluation of a function over the Wigner distribution with the expectation value of an operator. The transformation of $f \otimes f$ under a 1 parameter family of operations $g \otimes g$ for $g \in SL(2,\mathbb{F}_N) \subset G^J$ corresponds to the evolution of the quantum system under a homogeneous quadratic Hamiltonian. We find subspaces of the Hermitian portion of $f \otimes f$ that are invariant under $g \otimes g$ for $g \in SL(2,\mathbb{F}_N) \subset G^J$. These subspaces correspond to orders in the expansion of an exponential operator expression. The zeroth order subspace corresponds to the quantum total probability; the first order subspace corresponds to the classical phase space. Higher order subspaces are also constructed. The quadratic forms for the 2nd and 4th order subspaces correspond to lesser known ”universal quantum invariants” described by Dodonov 2000 and Dodonov and Man’ko 2000.

The Jacobi group over the real numbers is known in other works as the Schrödinger or Weyl-symplectic group (see, e.g., Guerrero et. al. 1999 and references cited therein and Miller 1977) and also as the inhomogeneous meta-plectic group (de Gosson 2001). The problem of quantizing the classical system whose phase space is a torus has been widely studied. Guerrero et. al. apply the Algebraic Quantization on a Group approach (Aldaya 1989) to the Jacobi (Schrödinger) Group. Rieffel 1989 describes the deformation quantization of toric symplectic manifolds.

Quantum mechanics over a finite field began with work of Weyl and was continued by Schwinger from the viewpoint of approximating quantum systems in infinite dimensional spaces by those associated with finite abelian groups. Varadarajan 1995 reviews this work and links it to the area of deformation quantization.

Athanasiu and Floratos 1994 consider the problem of the classical and quantum evolution on phase space lattices having the structure of $N \otimes N$ for $N$ an
odd prime. They start with the classical phase space with 1 degree of freedom and consider the set of all linear canonical transformations acting on this space. These form the group $SL(2, \mathbb{F}_N)$. From the classical phase space they pass to the Heisenberg group and its Schrödinger representation to construct the metaplectic representation of $SL(2, \mathbb{F}_N)$. The metaplectic representation of $SL(2, \mathbb{F}_N)$ is then used to describe the evolution of the quantum wavefunction.

The present paper incorporates the main ingredients of Athanasiu and Floratos in a more general and rigorous setting. Rather than taking the classical phase space as the starting point we take as starting point the Jacobi group. The classical evolution that these authors start with corresponds to the evolution of the first order subspace in a series of subspaces that we derive.

In order to reach the goals of this paper, we use the discrete Fourier transform (section 2.1.2) to define a derivative operation (section 2.1.3) acting on a function contained in the group algebra of a cyclic group. This allows a treatment in the present discrete case that parallels the continuum case. Our definition corresponds to the definition of a pseudodifferential operator (Folland 1989 p93).

We construct the Schrödinger-Weil representation of the Jacobi group (section 2.3) that has the property that the action of operators $g \in H_1(\mathbb{F}_N) \subset G^J$ coincides with the Schrödinger representation of $H_1(\mathbb{F}_N)$ (see, e.g., Terras 1999 and Grassberger and Hörmann 2001) and the action of $g \in SL(2, \mathbb{F}_N) \subset G^J$ coincides with the metaplectic representation of $SL(2, \mathbb{F}_N)$ (Neuhauser 2002). Additional irreducible representations of the Jacobi group can be formed from the tensor products of the Schrödinger-Weil representation with the representations for which the Heisenberg group acts trivially and $SL(2, \mathbb{F}_N)$ acts according to one of its representations (Berndt and Schmidt 1998). These tensor product representations, though also of interest, are not needed for our chosen model system and will not be considered in this paper.

In section 3 we consider the direct product $f \otimes f$ and show in section 3.1 that $P_+f \otimes f$ corresponds to the Wigner distribution of $f$. We next consider the convolution of $P_+f \otimes f$ with a test function $\sigma_G$. For $f$ appropriately normalized the value of the convolution $\sigma_G P_+f \otimes f$ at the origin is the average value of the test function for the system $P_+f \otimes f$. In section 3.2.2 we show that this leads to the Weyl map that relates the expectation value of an operator with the expectation value of a function evaluated over the Wigner distribution.

We consider the transformation of $P_+f \otimes f$ under $g \otimes g$ for $g \in SL(2, \mathbb{F}_N)$ in section 4 and find subspaces of $P_+f \otimes f$ that are stable under these transformations. These subspaces correspond to orders in a power series expansion of $P_+f \otimes f$. We provide explicit representations of $SL(2, \mathbb{F}_N)$ acting on these subspaces in sections 4.1.1 to 4.1.5 and give their associated invariant bilinear forms.

In order to make explicit contact with two familiar quantum systems in section 4.2 we briefly treat the case of a free particle and the case of the simple harmonic oscillator.

In this paper we apply the construction outlined in the first paragraph only to the Jacobi group. This construction can in principle be applied to an
arbitrary finite group. This work extends similar constructions for particular finite groups in Johnson 1996.

2 The Finite Jacobi Group

The Jacobi group $G^J(F_N)$ for prime field $F_N$ for $N$ a prime number not equal to 2, is formed by the semi-direct product of the Heisenberg Group $H_1(F_N)$ and the special linear group $SL(2,F_N)$: $G^J = SL(2,F_N) \rtimes H_1(F_N)$. A convenient matrix representation is given by Berndt and Schmidt 1998

$$\begin{pmatrix} a & 0 & b & \mu' \\ \lambda & 1 & \mu & \kappa \\ c & 0 & d & -\lambda' \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ with $ad-bc = 1$ and $(\lambda, \mu) = (\lambda', \mu') \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \right)$ (1)

for a general element of $G^J$.

We identify $\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL(2,F_N)$ with $\left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) \in G^J$ (2)

and

$$(\lambda, \mu, \kappa) \in H_1(F_N) with \left( \begin{array}{ccc} 1 & 0 & 0 \\ \lambda & 1 & \mu \\ 0 & 0 & 1 \end{array} \right) \in G^J \right)$ (3)

The Heisenberg group has multiplication

$$((\lambda, \mu, \kappa)(\lambda', \mu', \kappa') = (\lambda + \lambda', \mu + \mu', \kappa + \kappa' + \lambda\mu' - \mu\lambda')$$ (4)

This parametrization can be translated to that of Folland (Folland 1989 p19) by the change of variables $(p = \lambda, q = \mu, t = \kappa/2)$. with multiplication $(p, q, t)(p', q', t') = (p + p', q + q', t + t' + \frac{1}{2}(pq' - qp'))$.

It will be useful in the following to consider a parametrization of the Heisenberg group in terms of generators $t^r_x, t^s_y, t^t_z$

$$t^r_x = \left( \begin{array}{ccc} 1 & 0 & 0 \\ r & 1 & 0 \\ 0 & 0 & 1 \end{array} \right); t^s_y = \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & s \\ 0 & 0 & 1 \end{array} \right); t^t_z = \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{array} \right)$$ (5)

for $r, s, t \in F_N$. A general element of the Heisenberg group can then be written
The group multiplication takes the form

\[ (t^r_x t^s_y t^t_z) (t'^r_x t'^s_y t'^t_z) = t_{x}^{r + r'} t_{y}^{s + s'} t_{z}^{t + t' - 2sr} \]  \hspace{1cm} (7)

and we have the commutation relation

\[ t^r_x t^s_y = t^s_y t^r_x z^{rs} \]  \hspace{1cm} (8)

The following (overcomplete) set of generators for \( SL(2, \mathbb{F}_N) \) will be useful below:

\[ J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad t^a_s = \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix} \quad \text{for } a \neq 0 \]

\[ t^b_u = Jt^{-b}_d J^{-1} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad t^c_d = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \quad \text{for } b, c \in \mathbb{F}_N \]  \hspace{1cm} (9)

with multiplication

\[ t^{a_1}_s t^{a_2}_d = t^{a_1 + a_2}_s, \quad t^{c}_d t^{c}_d = t^{c + c}_d, \quad J^4 = 1 \]  \hspace{1cm} (10)

\[ t^{a+c}_s d = t^{a+c}_s d, \quad t^{a+c}_s u = t^{b+c}_a t^a_s, \quad J t^a_s = t^{1/a}_s J, \quad \text{and,} \]

\[ J t^c_s d = t^{c-c}_s d J^{-1} t^c_s d J \quad \text{for } c \neq 0 \]

\[ J t^b_u = t^{1/b}_s u^{-b} J^{-1} t^{-1/b}_u J^{-1} \quad \text{for } b \neq 0 \]  \hspace{1cm} (13)

We note that \( J^2 = t^{-1}_s \). The automorphisms of the Heisenberg group that are induced by the \( SL(2, \mathbb{F}_N) \) generators are:

\[ J \left(t^r_x t^s_y t^t_z\right)^{-1} = t^{s-r}_x t^{t+2rs}_z \]  \hspace{1cm} (14)

\[ t^a_x \left(t^r_x t^s_y t^t_z\right) t^{1/a}_s = t^{r/a}_x t^{s}_y t^{t}_z \]  \hspace{1cm} (15)

\[ t^c_d \left(t^r_x t^s_y t^t_z\right) t^{-c}_d = t^{r-cs+ct+c}_x t^{s}_y t^{t}_z \]  \hspace{1cm} (16)

\[ t^b_u \left(t^r_x t^s_y t^t_z\right) t^{-b}_u = t^{r-t+br+ct+br}_x t^{s}_y t^{t}_z \]  \hspace{1cm} (17)

### 2.1 Preliminaries

#### 2.1.1 Representation theory for cyclic group of order N

In treating the Jacobi group we will deal with several abelian subgroups that are generated by powers of a single generator. These include the subgroups \( \{ t^k_x \} \), \( \{ t^k_y \} \), \( \{ t^k_z \} \), and \( \{ J t^k_u J^{-1} \} \) for \( k \in \mathbb{F}_N \). For each of these subgroups we form invariant subspaces and irreducible representations in the same way. In order to establish notation, let us review the representation theory for a cyclic group \( G \) of order \( N \) (see, e.g., Terras 1999)
\[ G = \{ g^k \mid g^N = 1, k \in \mathbb{F}_N \} \]

We form the group algebra over the complex numbers with representative element

\[ f = \sum_{k=0}^{N-1} f(k) g^k \]

for \( f(k) \) a complex number. The element

\[ \hat{g}_m = \sum_{k=0}^{N-1} \exp\left(-\frac{2\pi i}{N} mk\right) g^k, \tag{18} \]

determines a 1 dimensional invariant subspace in the group algebra for each \( m \in \mathbb{F}_N \). We have

\[ g^a \hat{g}_m = \exp\left(\frac{2\pi i}{N} ma\right) \hat{g}_m \tag{19} \]

for \( g^a \in G \). The eigenvalue \( \exp\left(\frac{2\pi i}{N} ma\right) \) is the character of \( g^a \) in the representation determined by the action of \( G \) on \( \hat{g}_m \). We also have

\[ \hat{g}_m \hat{g}_{m'} = N \hat{g}_m \delta(m - m') \tag{20} \]

\[ \frac{1}{N} \sum_{m=0}^{N-1} \hat{g}_m = 1 \tag{21} \]

### 2.1.2 Discrete Fourier Transform

Acting on a general element \( f \) in the group algebra

\[ f = \sum_{k=0}^{N-1} f(k) g^k \tag{22} \]

with \( \frac{1}{N} \sum_{m=0}^{N-1} \hat{g}_m = 1 \) we obtain

\[ f = \frac{1}{N} \sum_{m,k=0}^{N-1} f(k) \exp\left(\frac{2\pi i}{N} mk\right) \hat{g}_m \tag{23} \]

\[ = \sum_{m=0}^{N-1} \tilde{f}(m) \hat{g}_m \tag{24} \]

where

\[ \tilde{f}(m) = \frac{1}{N} \sum_{k=0}^{N-1} f(k) \exp\left(\frac{2\pi i}{N} mk\right) \tag{25} \]

is the inverse discrete Fourier transform of \( f(k) \) (see, e.g., Bachman et. al. 2000). Now expanding out \( \hat{g}_m \) in equation 23

\[ f = \sum_{k,m=0}^{N-1} \tilde{f}(m) \exp\left(-\frac{2\pi i}{N} mk\right) g^k \tag{26} \]
and comparing this series with equation 22 we obtain the discrete Fourier transform
\[ f(k) = \sum_{m=0}^{N-1} \tilde{f}(m) \exp \left( -\frac{2\pi i}{N} mk \right) \] (27)

2.1.3 Derivative operation for finite cyclic group

We use the discrete Fourier transform of \( f(k) \) to define a derivative operation and power series expansion of \( f(k) \). In analogy with the case for \( k \) a real number we define
\[ \frac{d^n}{dk^n} f(k) \overset{\text{def}}{=} \sum_{m=0}^{N-1} \left( -\frac{2\pi i}{N} m \right)^n \tilde{f}(m) \exp \left( -\frac{2\pi i}{N} mk \right) \] (28)
\[ = \frac{1}{N} \sum_{h,m=0}^{N-1} \left( -\frac{2\pi i}{N} m \right)^n f(h) \exp \left( \frac{2\pi i}{N} m(h-k) \right) \] (29)

This definition parallels that used to define a pseudodifferential operator (Folland 1989 p93).

Consider the power series expansion of the exponential
\[ \exp \left( l \frac{d}{dk} \right) f(k) = \sum_{n=0}^{\infty} \frac{l^n}{n!} \left( \frac{d^n}{dk^n} \right) f(k) \] (30)

Using equation 29 for the \( n \)th order derivative we can express \( f(k+l) \) in terms of the exponential of a derivative operation:
\[ \exp \left( l \frac{d}{dk} \right) f(k) = \sum_{h,m=0}^{N-1} \sum_{n=0}^{\infty} \frac{l^n}{n!} \exp \left( \frac{2\pi i}{N} m(h-k) \right) \exp \left( \frac{2\pi i}{N} m(h-k-l) \right) \]
\[ = \sum_{h,m=0}^{N-1} \frac{1}{N} f(h) \exp \left( \frac{2\pi i}{N} m(h-k-l) \right) \]
\[ = f(k+l). \] (31)

Applying this formula to the function \( f(k) = k^n \) and identifying terms having the same order of \( l \) on both sides of the equation we obtain, e.g., \( \frac{d}{dk} k^n = nk^{n-1} \).

Applying this formula to a function formed from the product of 2 functions
\[ \exp \left( l \frac{d}{dk} \right) f_1(k) f_2(k) = f_1(k+l) f_2(k+l) \]
\[ = \left( \exp \left( l \frac{d}{dk} \right) f_1(k) \right) \left( \exp \left( l \frac{d}{dk} \right) f_2(k) \right) \]
expanding out the exponentials on both sides of the equality, and identifying
the first order terms in $l$ we obtain the Leibnitz property. We then also obtain
the commutator expression

$$[\frac{d}{dk}, k] = 1.$$ 

Using $\sum_{k=0}^{N-1} \frac{d}{dk} [x(k) y(k)] = 0$ that follows from equation 29 we also have,

$$\sum_{k=0}^{N-1} \left( \frac{d}{dk} x(k) \right) y(k) = - \sum_{k=0}^{N-1} x(k) \frac{d}{dk} y(k) \quad (32)$$

and

$$\sum_{k=0}^{N-1} \left( \frac{d^n}{dk^n} x(k) \right) y(k) = (-1)^n \sum_{k=0}^{N-1} x(k) \frac{d^n}{dk^n} y(k) \quad (33)$$

2.2 Representations of the Heisenberg Group

The representation theory for the Heisenberg group over a finite commutative
ring is presented in Terras 1999 and Grassberger and Hörmann 2001. We
consider the regular representation of $H_1(\mathbb{F}_N)$ over the complex numbers with
general element

$$f = \sum_{r,s,t=0}^{N-1} f(r, s, t) t_y^r t_z^s t_y^t$$

for $f(r, s, t)$ a complex number. The subgroup generated by $\{t_y^r, t_z^s | s, t \in \mathbb{F}_N\}$
is abelian. $H_1(\mathbb{F}_N)$ is the semidirect product of this subgroup with the subgroup
generated by $\{t_z^r | r \in \mathbb{F}_N\}$. We first form invariant subspaces in the $\{t_y^r, t_z^s\}$
group algebra. The product $\hat{y}_\nu \hat{z}_\omega$ for

$$\hat{y}_\nu = \sum_{n=0}^{N-1} \exp\left(\frac{-2\pi i \nu n}{N}\right) t_y^n$$

$$\hat{z}_\omega = \sum_{m=0}^{N-1} \exp\left(\frac{-2\pi i \omega m}{N}\right) t_z^m$$

and $\nu, \omega \in \mathbb{F}_N$ is invariant under the $\{t_y^r, t_z^s\}$ subgroup. Acting on the left of
$\hat{y}_\nu \hat{z}_\omega$ with a general element of the Heisenberg group algebra we obtain

$$f_{\nu \omega} = \sum_{k=0}^{N-1} f(k) t_z^k \hat{y}_\nu \hat{z}_\omega$$

(34)

where $f(k) \in \mathbb{C}$. $f_{\nu \omega}$ is a general element in the left ideal determined by $\hat{y}_\nu \hat{z}_\omega$.
The left action of a general element $t_z^r t_y^s t_z^t \in H_1(\mathbb{F}_N)$ on $f_{\nu \omega}$ is given by

$$t_z^r t_y^s t_z^t f_{\nu \omega} = \sum_{k=0}^{N-1} \exp\left(\frac{2\pi i}{N} (\omega t + s \nu - 2\omega ks + 2\omega rs)\right) f(k-r) t_z^k \hat{y}_\nu \hat{z}_\omega.$$ 

(35)
For $\omega \neq 0$ this representation corresponds to the Schrödinger representation of $t^t_x t^y_y t^z_z \in H_1(\mathbb{F}_N)$. That this representation is irreducible and unique up to equivalence for each $\omega \neq 0$ is described in Grassberger and Hörmann 2001 and Terras 1999.

2.3 A Representation of the Jacobi Group

We wish now to obtain a representation of the Jacobi Group $G^J = SL(2, \mathbb{F}_N) \rtimes H_1(\mathbb{F}_N)$ that behaves under the Heisenberg subgroup $H_1(\mathbb{F}_N)$ according to the Schrödinger representation. For this purpose, consider an element $f$ in the Jacobi group algebra with the following form:

$$f = \sum_{k=0}^{N-1} f(k) t^k_x \hat{y}_0 \hat{s}_- \hat{u}_0 \hat{x}_0 \left[ 1 + \frac{1}{(\frac{\omega}{N}) G(1, N)} \right] \hat d_0$$

In this expression $f(k)$ is a complex number,

$$\hat{y}_0 = \sum_{m=0}^{N-1} t^m_y$$

is the eigenstate of $t_y$ with eigenvalue 1,

$$\hat{s}_- = \sum_{h=0}^{N-1} \exp(-\frac{2\pi i}{N} h \omega) t^h_z$$

is the eigenstate of $t^t_z$ with eigenvalue $\exp(\frac{2\pi i}{N} t \omega)$,

$$\hat{u}_0 = \sum_{m=1}^{N-1} \left( \frac{m}{N} \right) t^m_s$$

is the eigenstate of $t^a_s$ with eigenvalue $\left( \frac{a}{N} \right)$. In this expression $\left( \frac{a}{N} \right)$ is the Legendre symbol. The Legendre symbol has value 1 for $m$ a square modulo $N$ and $-1$ for $m$ a nonsquare modulo $N$.

$$\hat{d}_0 = \sum_{l=0}^{N-1} t^l_d,$$
and \((\mathcal{N}) G(1, N)\) is the product of the Legendre symbol \((\mathcal{N})\) with the Gauss sum (Lang 1994)

\[
G(1, N) = \begin{cases} 
\sqrt{N} & \text{for } N = 1 \mod 4 \\
\sqrt[N]{N} & \text{for } N = 3 \mod 4 
\end{cases}
\] (43)

Let us denote by \(I\) the operator

\[
I = \hat{y}_0 \hat{z} \omega \hat{s} - \hat{u}_0 \hat{x} \omega G(1, N) 
\] (44)

Let us now verify that such an \(f\) is invariant under \(G^f\). By direct calculation we find for the Heisenberg group elements \(\{t_x^r t_y^s t_z^t \mid r, s, t \in \mathbb{F}_N\}\)

\[
t_x^r t_y^s t_z^t f = \sum_k \exp(\frac{2\pi i}{N}(\omega t - 2\omega ks + 2\omega rs)) f(k-r)t_x^k I
\] (45)

This corresponds to the Schrödinger representation equation 35 with \(v = 0\). For the \(SL(2, \mathbb{F}_N)\) group generators we find

\[
t_a^a f = \left(\frac{a}{N}\right) \sum_{k=0}^{N-1} f(ak)t_x^k I
\] (46)

\[
t_b^b f = \sum_{k=0}^{N-1} \exp\left(\frac{2\pi i}{N}b^2 k\omega\right) f(k)t_x^k I
\] (47)

\[
Jf = \left(\frac{\mathcal{N}}{N}\right) G(1, N) \sum_{k,l=0}^{N-1} f(k) \exp\left(\frac{2\pi i}{N}2\omega kl\right)t_x^k I
\] (48)

Equations 46 and 47 are easily calculated since one needs only to pass \(t_a^a\) and \(t_b^b\) through equation 36 until \(t_x^a\) is absorbed in \(\hat{s}\) and \(t_u^a\) is absorbed in \(\hat{u}_0\). The calculation for \(Jf\), though elementary, is involved and I provide details in Appendix A. This representation for the \(SL(2, \mathbb{F}_N)\) subgroup coincides with the metaplectic (or, Weil) representation of \(SL(2, \mathbb{F}_N)\) (Neuhauser 2002). This is an ordinary representation for \(SL(2, \mathbb{F}_N)\). Neuhauser treats the more general case of \(SL(2, K)\) for \(K\) a finite field; Cliff et. al. 2000 consider the case for symplectic groups over rings. The derivation of the metaplectic representation through the action of \(SL(2, \mathbb{F}_N)\) on an ideal in the Jacobi group algebra that I give above is new.

This representation for the Jacobi group is known as the Schrödinger-Weil representation. As described by Berndt and Schmidt 1998 for the real, complex and p-adic cases, additional representations of the Jacobi group can now be obtained by forming the tensor product of the Schrödinger-Weil representation with representations in which the Heisenberg subgroup acts trivially and the \(SL(2)\) subgroup acts according to one of its representations. These representations, though also of interest, are not needed for the description of the quantum system that I model here and will not be considered in this paper.
It will be useful to have an expression available for the action of $t_u$ on $f$

$$t_u f = J t_u^{-1} s^{-1} J f = \sum_{q=0}^{N-1} \exp\left(\frac{-2\pi i cq^2}{N} \frac{e^{iq}}{4\omega}\right) f(q) \tilde{x}_q I$$

(49)

$$= \sum_{k} \exp\left(\frac{-N e^{i k^2}}{2\pi i 4\omega} f(k) t_x^k I. \right.$$

One may verify the second expression by expanding out the exponential and using the definition equation 29 for the derivative.

The parity operator $J^2$ commutes with $SL(2, \mathbb{F}_N)$, intertwines with $t_x^r t_y^s \in H_1(\mathbb{F}_N)$ changing the sign of the translation $J^2 t_x^r t_y^s = t_x^{-r} t_y^{-s} J^2$, and has square equal to 1. We note but will not use the property that the operator

$$P = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{y}_{-k} \omega t_x^k$$

(50)

acts just as $J^2$ within the above ideal of $G_J$.

3 Direct product algebra

3.0.1 Cyclic group of order 4

As a first simple example that will also be useful in the following, let us apply the construction outlined in the introduction to the cyclic group of order 4 $C_4 = \{1, g, g^2, g^3 \mid g^4 = 1\}$. The group ring over the real numbers contains elements

$$a = a_0 1 + a_1 g + a_2 g^2 + a_3 g^3$$

(51)

for $a_i \in \text{real numbers}$. We decompose this algebra into subspaces using the projection operators $p_{\pm} = \frac{1}{2} (1 \pm g^2)$

$$a = [(a_0 + a_2) 1 + (a_1 + a_3) g] p_+$$

$$+ [(a_0 - a_2) 1 + (a_1 - a_3) g] p_-$$

(52)

$g^2$ acts trivially on the $p_+$ subspace and it can be further decomposed into 2 1-

dimensional subspaces that are respectively symmetric and antisymmetric under $g$. We will not consider these further. The $p_-$ subspace that is antisymmetric for $g^2$ is 2-dimensional and irreducible over the real numbers. Mapping $g$ to the unit imaginary $i$ we may identify the $p_-$ subspace with the complex numbers.

Let us consider the direct product of 2 distinct elements $a, b$ in the $p_-$ subspace

$$a \otimes b = (a_0 + ia_1)p_- \otimes (b_0 + ib_1)p_-$$

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Let us introduce the notation $E = i \otimes i$ and form projection operators $P_\pm = 1/2(1 \pm E)$. We may then write
\[
a \otimes b = \{ a_0 (1 \otimes 1) - a_1 b_0 (1 \otimes i) + b_1 (1 \otimes 1) - b_0 (1 \otimes i) \} (p_- \otimes p_-)
\]
\[
= \{ P_+ [(a_0 b_0 + a_1 b_1)(1 \otimes 1) + (a_0 b_1 - a_1 b_0)(1 \otimes i)]
+ P_- [(a_0 b_0 - a_1 b_1)(1 \otimes 1) + (a_0 b_1 + a_1 b_0)(1 \otimes i)] \} (p_- \otimes p_-)
\]

Let us identify $(1 \otimes i)$ with the unit imaginary in the direct product space. The $P_-$ subspace is invariant under the interchange $a \otimes b \rightarrow b \otimes a$. This interchange corresponds to complex conjugation of the $P_+$ subspace. We may identify the $P_+$ subspace with the Hermitian product of 2 complex numbers. The $P_-$ subspace corresponds to the ordinary nonhermitean product of 2 complex numbers.

The group algebra of $G^J$ over the complex numbers coincides with the $p_-$ portion of the group algebra of $C_4 \otimes G^J$ over the real numbers. For $x$ an element of this group algebra, $P_+ x \otimes x$ corresponds to the Hermitian portion of $x \otimes x$. In the following we will mainly be concerned with this Hermitian portion.

### 3.0.2 Useful operators in the direct product algebra

We introduce the translation operators
\[
T^k_x = t^k_x \otimes t^k_x, \quad T^{k -}_x = t^{-k}_x \otimes t^k_x
\]
\[
T^h_y = t^h_y \otimes t^h_y, \quad T^{-h}_y = t^{-h}_y \otimes t^h_y
\]
and form eigenvectors
\[
X_p = \sum_{h=0}^{N-1} \exp(-2\pi i N p h) T^h_x, \quad X_q = \sum_{h=0}^{N-1} \exp(-2\pi i N q h) T^h_x
\]
\[
Y_p = \sum_{h=0}^{N-1} \exp(-2\pi i N p h) T^h_y, \quad Y_q = \sum_{h=0}^{N-1} \exp(-2\pi i N q h) T^h_y
\]

We calculate the commutators
\[
T^r_x T^s_y T^{-r}_x T^{-s}_y (\hat{\omega} \otimes \hat{\omega}) = \exp\left(\frac{2\pi i}{N} 2rs \omega (1 - E)\right) (\hat{\omega} \otimes \hat{\omega})
\]
\[
T^r_x T^s_y T^{-r}_x T^{-s}_y (\hat{\omega} \otimes \hat{\omega}) = \exp\left(\frac{2\pi i}{N} 2rs \omega (1 + E)\right) (\hat{\omega} \otimes \hat{\omega})
\]
3.0.3 Two useful identities

The result that we consider in this paragraph will help to characterize $P_+ (f \otimes f)$. We prove

$$\left( \sum_{r,s=0}^{N-1} (-r)^m (-s)^n T_x^r T_y^s \right) \left( \sum_{q,p=0}^{N-1} q^h p^k X_q Y_p \right) |_{0,0} = \left( \frac{N}{2\pi i} \right)^{m+n} N^2 m! n! \delta(m-h) \delta(n-k)$$

(59)

where the expression on the left is evaluated at the origin of the group ring of the group generated by \{ $T_x^r, T_y^s \mid r, s \in \mathbb{F}_N$ \}. Acting with $T_x^r T_y^s$ on the eigenvectors $X_q Y_p$ in the left-hand side of equation 59 pulls out an exponential term $\exp(\frac{2\pi i}{N}(rq + sp))$. Expand out the eigenvectors $X_q Y_p$ and consider the value of the expression at the origin $T_x^0 T_y^0$. We obtain for the left hand side

$$\sum_{r,s,q,p=0}^{N-1} (-r)^m (-s)^n q^h p^k \exp(\frac{2\pi i}{N}(rq + sp))$$

We write this expression as

$$\sum_{r,s,q,p=0}^{N-1} \left( N \frac{2\pi i}{N} \right)^{m+n} \left( \frac{d}{dq} \right)^m \left( \frac{d}{dp} \right)^n \exp(\frac{2\pi i}{N}(rq + sp))$$

and use equation 33 to reexpress this in the form

$$\sum_{r,s,q,p=0}^{N-1} \left( N \frac{2\pi i}{N} \right)^{m+n} \exp(\frac{2\pi i}{N}(rq + sp)) \left( \frac{d}{dq} \right)^m \left( \frac{d}{dp} \right)^n q^h p^k$$

For $m > h$ or $n > k$ the derivatives lead to a null result. For $m < h$ or $n < k$ the sum over $r, s$ in the exponential leads to a null result. We obtain a nonnull result only for $m = h$ and $n = k$. We conclude

$$\sum_{r,s,q,p=0}^{N-1} (-r)^m (-s)^n q^h p^k \exp(\frac{2\pi i}{N}(rq + sp)) = \left( \frac{N}{2\pi i} \right)^{m+n} N^2 m! n! \delta(m-h) \delta(n-k)$$

In this paragraph we rewrite an operator expression that will allow us to describe subspaces of $f \otimes f$ below in a more transparent way. We show that

$$\mathcal{T}_x^r (I \otimes I) = (P_+ Y_{-4\omega r} + P_- Y_{-4\omega r}) X_0 (I \otimes I) / N$$

(60)
for $I$ given by equation 44. Consider

$$P_+ X_q (I \otimes I) = P_+ Y_q T_y^k (I \otimes I)$$

$$= P_+ T_y^k X_{q-4k\omega} (I \otimes I)$$

For $q - 4k\omega = 0$

$$P_+ X_q (I \otimes I) = P_+ T_y^{q/4\omega} X_0 (I \otimes I).$$

We then have

$$P_+ T_x^r (I \otimes I) = P_+ \sum_{q=0}^{N-1} \exp(2\pi irq/N) X_q (I \otimes I) /N$$

$$= P_+ \sum_{q=0}^{N-1} \exp(2\pi irq/N) T_y^{q/4\omega} X_0 (I \otimes I) /N$$

$$= P_+ Y_{-4\omega r} X_0 (I \otimes I) /N$$

Similarly, we find

$$P_- T_x^r (I \otimes I) = P_- Y_{-4\omega r} X_0 (I \otimes I) /N$$

Summarizing, we have equation 60.

3.1 Direct product $f \otimes f$

We take an element $f$ in the left ideal of the Jacobi group algebra,

$$f = \sum_{k=0}^{N-1} f(k) t_x^k I$$

where $I = \hat{g}_0 \hat{s}_+ \hat{s}_- \hat{u}_0 \hat{d}_0 \left( 1 + \frac{1}{N} \sum_{q} X_q \frac{1}{(G^{(1)} N)} \right) \hat{d}_0$ as described above, and form the direct product of 2 copies of this element

$$f \otimes f = \left( \sum_{k=0}^{N-1} f(k) t_x^k I \right) \otimes \left( \sum_{h=0}^{N-1} f(h) t_x^h I \right)$$

$$= \sum_{h,k=0}^{N-1} (f(k) \otimes f(h)) \left( t_x^k \otimes t_x^h \right) (I \otimes I)$$

$$= \sum_{k,p=0}^{N-1} (f(k) \otimes f(k+p)) T_x^k (1 \otimes t_x^p) (I \otimes I)$$

where $p = h - k$. Acting on $f \otimes f$ with $1 = \frac{1}{N} \sum_q X_q$ and using $T_x^{p/2} T_x^{p/2} = 1 \otimes t_x^p$ we obtain

$$f \otimes f = \sum_{k,p,q=0}^{N-1} \left( f(k) \otimes \exp\left( \frac{2\pi i q(k+p/2)}{N} \right) f(k+p) \right) X_q T_x^{p/2} (I \otimes I) /N$$

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We use equation 31 to express \( f(k + p) \) in terms of the exponential of a derivative operation

\[
f(k + p) = \exp \left( \frac{p}{dk} \right) f(k)
\]

We then express \( f \otimes f \) in the form

\[
f(\otimes f) = \sum_{k,p,q=0}^{N-1} f(k) \otimes \exp \left( \frac{2\pi i}{N} q(k + p/2) \exp \left( \frac{p}{dk} \right) f(k) \right) \cdot X_q \cdot \left( I \otimes I \right) / N
\]

where in the second equality we have used the Campbell-Hausodor expresssion

\[
\exp(A) \exp(B) = \exp(A + B) \exp(1/2[A,B])
\]

valid for the case that \( A \) and \( B \) commute with their commutator. Finally, substituting using equation 60 we obtain

\[
f(\otimes f) = \left\{ P_+ \sum_{k,p,q=0}^{N-1} f(k) \otimes \exp \left( \frac{2\pi i}{N} q(k + p) \left( \frac{1}{2\omega} \frac{d}{2\pi i \frac{d}{dk}} \right) f(k) \right) \cdot X_q \cdot \left( I \otimes I \right) / N^2 \right\}
\]

We identify the \( P_+ \) portion of this expression with the Fourier-Wigner distribution of \( f \) (Folland 1989 p30). Note that in comparing with Folland, our term \( (\frac{1}{2\omega} \frac{d}{dk}) \) corresponds to Folland’s \( D \) and our term \( \frac{1}{2\omega} \) corresponds to Folland’s incorporation of the Heisenberg constant \( h \).

We now rewrite equation 66 in terms of the generators \( \{T_x, T_y\} \) for the \( P_+ \) projection (this amounts to a double Fourier transform) and in terms of the generators \( \{T_x, \overline{T}_y\} \) for the \( P_- \) projection

\[
f(\otimes f) = \left\{ P_+ \sum_{r,s,p=0}^{N-1} f(r) - \left( \frac{1}{2\omega} \frac{p}{2} \right) f(r) + \left( \frac{1}{2\omega} \frac{p}{2} \right) \exp \left( \frac{2\pi i}{N} p \right) \cdot T_x T_y + \right\} \cdot \left( I \otimes I \right) / N
\]

We identify the \( P_+ f \otimes f \) portion of this expression with the Wigner distribution of \( f \) (Folland 1989 p57). This identification is central to the hypothesis outlined in the introduction that associates physical observables with the space \( f \otimes f \).
3.2 Characterization of $f \otimes f$

3.2.1 Convolution with Test Function

Let us now focus on extracting information from $f \otimes f$. Let us specialize to the Hermitian $P_+ f \otimes f$ portion since it is of most direct physical interest.

$P_+ f \otimes f$ in the Wigner form equation 67 composes a function on the regular representation of the abelian group (see, equation 57) \( \{ T_{y}^{T} \, T_{y}^{s} \mid r, s \in \mathbb{F}_{N} \} \). To motivate our viewpoint let us consider a specific test function

$$\sigma_{G} = \sum_{r,s=0}^{N-1} -rT_{x}^{T}T_{y}^{s} = \sum_{r=0}^{N-1} -rT_{x}^{T}Y_{0}$$

and consider the convolution of $\sigma_{G}$ with a function $F$ that is a composed of contributions $a(h)$

$$F = \sum_{h=0}^{N-1} a(h)T_{x}^{h}Y_{0}$$

for $a(h)$ a scaler. For the product we obtain

$$\sigma_{G}F = \sum_{r=0}^{N-1} \left( \sum_{h=0}^{N-1} (h-r)a(h) \right) T_{x}^{r}Y_{0}$$

and the value of the function $\sigma_{G}F$ at $T_{x}^{r}$ is the weighted sum of the displacements from $T_{x}^{r}$ to the positions composing $F$. For $F$ normalized such that $\sum_{h=0}^{N-1} a(h) = 1$, we may interpret $\sigma_{G}F$ evaluated at $T_{x}^{r}$ as the average displacement from $T_{x}^{r}$ to $F$.

In general one may form test functions having a dependence on both $T_{x}^{r}$ and $T_{y}^{s}$

$$\sigma_{G} = \sum_{r,s=0}^{N-1} \sigma(-r,-s)T_{x}^{T}T_{y}^{s}$$

and consider normalized functions $F$

$$F = \sum_{h,k=0}^{N-1} F(h,k)T_{x}^{h}T_{y}^{k}$$

The convolution

$$\sigma_{G}F = \sum_{r,s=0}^{N-1} \left( \sum_{h,k=0}^{N-1} \sigma(h-r,k-s)F(h,k) \right) T_{x}^{r}T_{y}^{s}$$

evaluated at the origin $T_{x}^{0}T_{y}^{0}$ returns the value of $\sigma(h,k)$ averaged over $F$. 

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3.2.2 Evaluation of test functions in the system $P_+ f \otimes f$

Let us now consider some specific examples. Consider the convolution of the test function

$$\sigma_G = X_q Y_p$$

$$= \sum_{m,n=0}^{N-1} \exp\left(\frac{-2\pi i}{N}(mq + np) T_x^m T_y^n\right)$$

with $P_+ f \otimes f$. Forming the product $\sigma_G P_+ f \otimes f$ and evaluating the result at the origin $T_0^0 T_0^0$, we obtain from the Fourier-Wigner form of $P_+ f \otimes f$ (equation 66)

$$\sigma_G P_+ f \otimes f \mid_{0,0} =$$

$$P_+ \left( \sum_{k=0}^{N-1} f(k)^* \exp\left(\frac{2\pi i}{N}(qk + p \left(\frac{-1}{2\omega} \frac{d}{2\pi i \frac{d}{dk}}\right)) f(k)\right) \cdot X_0 (I \otimes I) \right)$$

From the Wigner distribution form equation 67 we obtain

$$\sigma_G P_+ f \otimes f \mid_{0,0} =$$

$$P_+ \left( \sum_{r,s,p=0}^{r,s,p=0} \exp\left(\frac{2\pi i}{N}(rq + sp)\right)f(r + \frac{p}{4\omega})^* f(r - \frac{p}{4\omega}) \exp\left(\frac{-2\pi i}{N}ps\right) \right)$$

$$\cdot X_0 (I \otimes I) / N$$

Comparing these two expressions we see that the expectation value of the operator

$$\exp\left(\frac{2\pi i}{N}(qk + p \left(\frac{-1}{2\omega} \frac{d}{2\pi i \frac{d}{dk}}\right))\right)$$

in equation 72 is associated with the value of the function

$$\exp\left(\frac{2\pi i}{N}(rq + sp)\right)$$

in equation 73 evaluated over the Wigner distribution. This particular correspondence between operator and function in the continuum case is derived from the Weyl map (Weyl 1950 p275, Folland 1989 p81, Wong 1998 p21). This correspondence is rigorous in the present construction.

We next consider the case when $\sigma_G$ is a polynomial in $r$ and $s$ (cf., Folland 1989 p82). Expand the exponential in the expression for $P_+ f \otimes f$ equation 66 in a power series.

$$P_+ f \otimes f =$$

$$P_+ \left\{ \sum_{k,p,q=0,h=0}^{N-1,\infty} f(k)^* \frac{1}{h!} \left(\frac{2\pi i}{N}(qk - \frac{p}{2\omega} \left(\frac{N}{2\pi i \frac{d}{dk}}\right))^h \right) f(k) \right\}$$

$$\cdot X_0 (I \otimes I) / N^2$$
The $h^{th}$ order of this expansion contains terms

$$P_+ \sum_{k=0}^{N-1} f(k)^* \left\{ \sum_{\text{all orderings}} k^j \left( \frac{N}{2\pi i \frac{d}{dk}} \right)^{h-j} \right\} f(k)$$

$$\cdot \frac{1}{h!} \left( \frac{2\pi i}{N} \right)^h \left( \frac{-1}{2\omega} \right)^{h-j} \left( \sum_{p,q=0}^{N-1} q^j p^{h-j} X_q Y_p \right)$$

$$\cdot X_0 (I \otimes I) / N^2$$

where the bracketed term is a sum over all orderings of a product of $j$ factors of $k$ and $h-j$ factors of $\left( \frac{N}{2\pi i \frac{d}{dk}} \right)$. Act on $P_+ f \otimes f$ equation 66 with the test function

$$\sigma_G = \sum_{r,s=0}^{N-1} (-r)^m (-s)^n T^r_x T^s_y$$

and use equation 59 to conclude

$$\sigma_G P_+ f \otimes f |_{0,0} =$$

$$= P_+ \sum_{k=0}^{N-1} f(k)^* \left\{ \frac{m!n!}{(m+n)!} \sum_{\text{all orderings}} k^m \left( \frac{-1}{2\omega} \frac{N}{2\pi i \frac{d}{dk}} \right)^n \right\} f(k)$$

$$\cdot X_0 (I \otimes I)$$

Now act with $\sigma_G$ on the Wigner distribution form of $P_+ f \otimes f$ equation 67 to conclude

$$\sigma_G P_+ f \otimes f |_{0,0} =$$

$$= P_+ \sum_{r,s,p=0}^{N-1} r^m s^n f(r + \frac{p}{4\omega})^* f(r - \frac{p}{4\omega}) \exp(-2\pi i ps)$$

$$\cdot X_0 (I \otimes I) / N$$

The expectation value of the "Weyl-ordered" operator

$$\frac{m!n!}{(m+n)!} \left( \sum_{\text{all orderings}} k^m \left( \frac{-1}{2\omega} \frac{N}{2\pi i \frac{d}{dk}} \right)^n \right)$$

in equation 77 is associated with the evaluation of the function $r^m s^n$ over the Wigner distribution (equation 78).

Based on these results one is led to identify the convolution procedure described above with the quantum measurement process for our system.
4 Transformation of $P_+ f \otimes f$ under $g \otimes g$ for $g \in SL(2, N)$

In this section we consider the transformation of $P_+ f \otimes f$ under $g \otimes g$ for $g \in SL(2, \mathbb{F}_N)$. We decompose $P_+ f \otimes f$ into subspaces that are invariant under these transformations. Since the subgroup $\{g \otimes g : g \in SL(2, \mathbb{F}_N)\}$ does not commute with the subgroup $\{T_r T_s^* = t_r^* t_s^* \otimes t_r^* t_s^* : r, s \in \mathbb{F}_N\}$ the subspaces that we find, except for the trivial case, are not stable under the measurement process described above. They are, however, useful for the description of a system that is evolving under a sequence of transformations that are contained in $SL(2, \mathbb{F}_N)$.

Let us rewrite equation 66 as

$$P_+ f \otimes f = P_+ \sum_{k,q,p=0}^{N-1} f(k)^* \exp \left( \frac{2\pi i}{N} \begin{pmatrix} q & p \end{pmatrix} \begin{pmatrix} k \end{pmatrix} \right) f(k) \begin{pmatrix} X_q Y_p \end{pmatrix}$$

where we view the argument of the exponential as arising from the contraction of the matrix $\begin{pmatrix} q & p \end{pmatrix}$ with the matrix $\begin{pmatrix} k \end{pmatrix} \left( -\frac{1}{2\omega} \frac{d}{dk} \right) \begin{pmatrix} k \end{pmatrix}$. In Appendix B we determine the action of $g \otimes g$ for $g \in SL(2, \mathbb{F}_N)$ on $f \otimes f$. We find that the operation $g \otimes g$ can be implemented in the following way

$$(g \otimes g) P_+ f \otimes f =$$

$$P_+ \sum_{k,q,p=0}^{N-1} f(k)^* \exp \left( \frac{2\pi i}{N} \begin{pmatrix} q & p \end{pmatrix} \begin{pmatrix} k \end{pmatrix} M_{g \otimes g} \left( -\frac{1}{2\omega} \frac{d}{dk} \right) \begin{pmatrix} k \end{pmatrix} \right) f(k) \begin{pmatrix} X_q Y_p \end{pmatrix}$$

where $M_{g \otimes g}$ is a 2x2 matrix. In particular we find

$$M_{t_a \otimes t_a} = \begin{pmatrix} 1/a & 0 \\ 0 & a \end{pmatrix}$$

$$M_{t_b \otimes t_b} = \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix}$$

$$M_{J \otimes J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

From a consideration of the power series expansion of equation 81 we conclude that the subspace containing the $h^{th}$ order term is invariant under $g \otimes g$ for $g \in SL(2, \mathbb{F}_N)$ for each order $h$. The $h^{th}$ order portion of $P_+ f \otimes f$ is evident in equation 74 and contains terms equation 75. Let

$$\tilde{e}_{j,h-j} = P_+ \frac{1}{j!(h-j)!} \left( \frac{2\pi i}{N} \right)^h \sum_{q,p=0}^{N-1} q^j p^{h-j} X_q Y_p$$

$$\cdot \frac{1}{X_0 (I \otimes I) / N^2}$$
and let

\[ \left\langle k^j \left( \frac{-1}{2\omega} \frac{d}{2\pi i \frac{d}{dk}} \right)^{h-j} \right\rangle = \sum_{k=0}^{N-1} f(k)^* \left\{ \frac{j!(h-j)!}{h!} \sum_{\text{all orderings}} k^j \left( \frac{-1}{2\omega} \frac{d}{2\pi i \frac{d}{dk}} \right)^{h-j} \right\} f(k) \]

We may then write the \( h^{th} \) order term \( r_h \) in the expansion of \( P_+ f \otimes f \) as

\[ r_h = \sum_{j=0}^{h} \left\langle k^j \left( \frac{-1}{2\omega} \frac{d}{2\pi i \frac{d}{dk}} \right)^{h-j} \right\rangle \widehat{e}_{j,h-j} \]

The basis vectors \( \{ \widehat{e}_{j,h-j} | 0 \leq j \leq h \} \) span an \( h+1 \) dimensional subspace.

The action of \( g \otimes g \) on \( r_h \) can be obtained from the \( h^{th} \) order term in the power series expansion of \( (g \otimes g) P_+ f \otimes f \). For a given one parameter family of transformations \( g \otimes g \) for \( g \in SL(2, \mathbb{F}_N) \) acting on \( P_+ f \otimes f \) one may determine the evolution of \( r_h \).

### 4.1 Explicit realization of low-order subspaces of \( P_+ (f \otimes f) \)

#### 4.1.1 Zeroth order

Let \( r_0 \) denote the zero order term in the expansion \( P_+ f \otimes f \). We have

\[ r_0 = \sum_{k=0}^{N-1} f(k)^* f(k) \widehat{e}_{0,0} \]  

The action of the generators of \( SL(2, N) \) on \( r_0 \) corresponds to the trivial representation \( \rho_0 \):

\[ \rho_0(J \otimes J) = \rho_0(t^b_u \otimes t^b_u) = \rho_0(t^a_s \otimes t^a_s) = 1. \]

#### 4.1.2 First order

Let \( r_1 \) denote the first order term in the expansion of \( P_+ f \otimes f \). We have

\[ r_1 = \left\langle k \right\rangle \widehat{e}_{1,0} + \left\langle \frac{-1}{2\omega} \frac{d}{2\pi i \frac{d}{dk}} \right\rangle \widehat{e}_{0,1} \]

Writing

\[ r_1 = \left( \begin{array}{c} \left\langle k \right\rangle \\ \left\langle \frac{-1}{2\omega} \frac{d}{2\pi i \frac{d}{dk}} \right\rangle \\ \end{array} \right) \]

the representations \( \rho_1(g \otimes g) \) for the generators \( g \in SL(2, N) \) acting on \( r_1 \) are given by the matrices

\[
\rho_1(t^a_s \otimes t^a_s) = \begin{pmatrix} 1/a & 0 \\ 0 & a \end{pmatrix}, \quad \rho_1(t^b_u \otimes t^b_u) = \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix}, \\
\rho_1(J \otimes J) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \rho_1(t^a_d \otimes t^a_d) = \begin{pmatrix} 1 & -c \\ 0 & 1 \end{pmatrix}
\]
This representation of $g \otimes g$ for $g \in SL(2, F_N)$ is itself equivalent to the standard representation of $SL(2, F_N)$ equation 9. The bilinear form

$$b_1(r'_1, r_1) = (r'_1)^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} r_1,$$  

where $r_1$ and $r'_1$ denote two vectors in the subspace spanned by $\{\hat{e}_{1,0}, \hat{e}_{0,1}\}$ and the superscript $t$ denotes matrix transposition, is invariant under these transformations

$$b_1(\rho_1(g \otimes g)r'_1, \rho_1(g \otimes g)r_1) = b(r'_1, r_1)$$  

The bilinear form $b_1(r'_1, r_1)$ is the standard symplectic form.

### 4.1.3 Second Order

Let $r_2$ denote the second order term in the expansion of $P_+ f \otimes f$. We have

$$r_2 = \langle k^2 \rangle \hat{e}_{2,0} + \left\langle k \left( -\frac{1}{2\omega} \frac{N}{2\pi i} \frac{d}{dk} \right) \right\rangle \hat{e}_{1,1} + \left\langle \left( -\frac{1}{2\omega} \frac{N}{2\pi i} \frac{d}{dk} \right)^2 \right\rangle \hat{e}_{0,2}$$  

Writing $r_2$ as a column matrix

$$r_2 = \begin{pmatrix} \langle k^2 \rangle \\ \left\langle k \left( -\frac{1}{2\omega} \frac{N}{2\pi i} \frac{d}{dk} \right) \right\rangle \\ \left\langle \left( -\frac{1}{2\omega} \frac{N}{2\pi i} \frac{d}{dk} \right)^2 \right\rangle \end{pmatrix}.$$  

the representations $\rho_2(g \otimes g)$ for the generators $g \in SL(2, F_N)$ acting on $r_2$ are

$$\rho_2(t^a_s \otimes t^a_s) = \begin{pmatrix} 1/a^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a^2 \end{pmatrix}, \quad \rho_2(t^b_s \otimes t^b_s) = \begin{pmatrix} 1 & 0 & 0 \\ -b & 1 & 0 \\ b^2 & -2b & 1 \end{pmatrix},$$  

$$\rho_2(J \otimes J) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \rho_2(t^a_d \otimes t^a_d) = \begin{pmatrix} 1 & -2c & c^2 \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix}.$$  

We note that $\rho_2(J^2 \otimes J^2) = 1$ so that $\rho_2$ is a representation of $PSL(2, F_N) = SL(2, F_N)/\{1, J^2\} \sim SO(1, 2)$.

The bilinear form

$$b_2(r'_2, r_2) = (r'_2)^t \begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{pmatrix} r_2$$  

is invariant under $SL(2, F_N)$ transformations. For $r'_2 = r_2$ we find the invariant bilinear form

$$\langle k^2 \rangle \left\langle \left( -\frac{1}{2\omega} \frac{N}{2\pi i} \frac{d}{dk} \right)^2 \right\rangle - \left\langle k \left( -\frac{1}{2\omega} \frac{N}{2\pi i} \frac{d}{dk} \right) \right\rangle^2$$  

21
This invariant corresponds to the invariant \( \langle \hat{p}^2 \rangle \langle \hat{x}^2 \rangle - \frac{1}{4} (\langle \hat{p} \hat{x} + \hat{x} \hat{p} \rangle)^2 \) that Dodonov and Man’ko describe (reviewed in Dodonov 2000 and Dodonov and Man’ko 2000) for a 1d quantum system evolving under a homogeneous quadratic Hamiltonian.

### 4.1.4 Third Order

Writing

\[
r_3 = \langle k^3 \rangle \hat{c}_{3,0} + \langle k^2 \left( \frac{-1 \cdot N \cdot d}{2 \omega \cdot 2 \pi i \cdot dk} \right) \rangle \hat{c}_{2,1} + \langle k \left( \frac{-1 \cdot N \cdot d}{2 \omega \cdot 2 \pi i \cdot dk} \right)^2 \rangle \hat{c}_{1,2} + \langle \left( \frac{-1 \cdot N \cdot d}{2 \omega \cdot 2 \pi i \cdot dk} \right)^3 \rangle \hat{c}_{0,3}
\]

(99)

as a column vector we have the representation

\[
\rho_3 (t^*_a \otimes t^*_a) = \begin{pmatrix}
1/a^3 & 0 & 0 & 0 \\
0 & 1/a & 0 & 0 \\
0 & 0 & a & 0 \\
0 & 0 & 0 & a^3
\end{pmatrix}, \quad \rho_3 (t^*_b \otimes t^*_b) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
-b & 1 & 0 & 0 \\
b^2 & -2b & 1 & 0 \\
-b^3 & 3b^2 & -3b & 1
\end{pmatrix},
\]

(100)

and invariant bilinear form

\[
b_3 (r'_3, r_3) = (r'_3)^t \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & -3 & 0 \\
0 & 3 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix} r_3
\]  

(101)

### 4.1.5 Fourth Order

Writing

\[
r_4 = \langle k^4 \rangle \hat{c}_{4,0} + \langle k^3 \left( \frac{-1 \cdot N \cdot d}{2 \omega \cdot 2 \pi i \cdot dk} \right) \rangle \hat{c}_{3,1} + \langle k^2 \left( \frac{-1 \cdot N \cdot d}{2 \omega \cdot 2 \pi i \cdot dk} \right)^2 \rangle \hat{c}_{2,2} + \\
+ \langle k \left( \frac{-1 \cdot N \cdot d}{2 \omega \cdot 2 \pi i \cdot dk} \right)^3 \rangle \hat{c}_{1,3} + \langle \left( \frac{-1 \cdot N \cdot d}{2 \omega \cdot 2 \pi i \cdot dk} \right)^4 \rangle \hat{c}_{0,4}
\]

(102)
as a column vector we have the representation

\[
\rho_4(t^a_s \otimes t^a_s) = \begin{pmatrix}
\frac{1}{a^4} & 0 & 0 & 0 & 0 \\
0 & \frac{1}{a^2} & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & a^2 & 0 \\
0 & 0 & 0 & 0 & a^4
\end{pmatrix}, \quad \rho_4(t^b_u \otimes t^b_u) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
-b & 1 & 0 & 0 & 0 \\
b^2 & -2r & 1 & 0 & 0 \\
b^3 & 3r^2 & -3r & 1 & 0 \\
b^4 & -4r^3 & 6r^2 & -4r & 1
\end{pmatrix},
\]

\[
\rho_4(J \otimes J) = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

and invariant bilinear form

\[
b_4(r'_4, r_4) = (r'_4)^t \begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -4 & 0 \\
0 & 0 & 6 & 0 & 0 \\
0 & -4 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix} r_4
\]

For \( r_4 = r'_4 \) we obtain the invariant

\[
2 \langle k^4 \rangle \left( \langle \frac{-1}{2\omega} N \frac{d}{2\pi i dk} \rangle \right)^4 - 8 \langle k \langle \frac{-1}{2\omega} N \frac{d}{2\pi i dk} \rangle \rangle ^3 \langle k^3 \langle \frac{-1}{2\omega} N \frac{d}{2\pi i dk} \rangle \rangle + 6 \langle k^2 \langle \frac{-1}{2\omega} N \frac{d}{2\pi i dk} \rangle ^2 \rangle ^2
\]

that corresponds to the invariant

\[
\langle \hat{p}^4 \rangle \langle \hat{x}^4 \rangle + \frac{3}{4} \left( \langle \hat{p}^2 \hat{x}^2 + \hat{x}^2 \hat{p}^2 \rangle \right)^2 - \frac{3}{2} \langle \hat{p}^2 \hat{x}^2 + \hat{x}^2 \hat{p}^2 \rangle - \langle \hat{p}^3 \hat{x} + \hat{x}^3 \hat{p} \rangle - \langle \hat{p}^3 \hat{x} + \hat{x}^3 \hat{p} \rangle
\]

described by Dodonov and Man’ko (Dodonov 2000 and Dodonov and Man’ko 2000).

4.1.6 Higher order

One may continue and obtain higher order representations of \( SL(2, \mathbb{F}_N) \) acting on \( r_h \) for \( h \) odd and of \( PSL(2, \mathbb{F}_N) \) for \( h \) even (cf., e.g., Knapp 1986 p38, Fulton and Harris 1991 p150, and Howe and Tan 1992 pgs. 25, 55).

4.2 2 examples

Let us now consider the dynamics resulting from 2 familiar families of 1-parameter transformations \( g \otimes g \) for \( g \in SL(2, \mathbb{F}_N) \).
4.2.1 Free Particle

Consider the 1-parameter family of transformations determined by \( t_d^m \) for \( t, m \in \mathbb{F}_N \). Using equation 49 for \( t_d^m \) and assigning \( \hbar = \frac{1}{2\omega} \) we obtain

\[
\sum_k \exp(i \frac{\hbar}{2m} d^2 k) f(k) t_k^m I. \tag{107}
\]

Differentiating with respect to \( t \) by expanding out the exponential in equation 107 and using equation 29 results in the Schrödinger equation for a free particle (cf. de Gossen 2001 p273).

The first order subspace equation 90 transforms with \( t \) according to

\[
\rho_1 (t_d^m \otimes t_d^m) \left( \begin{array}{c} \langle k \rangle \\
\langle \hbar \frac{d}{dk} \rangle 
\end{array} \right) = \left( \begin{array}{c} \langle k \rangle + \frac{1}{\hbar} \langle \hbar \frac{d}{dk} \rangle \\
\langle \hbar \frac{d}{dk} \rangle 
\end{array} \right) \tag{108}
\]

corresponding to the motion of a free particle with constant momentum \( \langle \hbar \frac{d}{dk} \rangle \) and velocity \( \langle \hbar \frac{d}{dk} \rangle / m \). One may also readily calculate the evolution of the higher order subspaces.

4.2.2 Simple Harmonic Oscillator

We consider the 1-parameter family of transformations determined by

\[
t_r^n = \left( \begin{array}{cc} a & b \delta \\
b & a 
\end{array} \right)^n \tag{109}
\]

for \( \delta \) a nonsquare in \( \mathbb{F}_N \), \( b \neq 0 \) and \( 0 \leq n < N + 1 \). Powers of this matrix can be calculated by mapping to the isomorphic multiplicative group within the quadratic extension of \( \mathbb{F}_N \)

Writing \( (a + b\sqrt{\delta})^{N+1} = (a + b\sqrt{\delta}) \left( a + b\sqrt{\delta} \right)^N \), and using the binomial expansion of \( (a + b\sqrt{\delta})^N \), where \( \sqrt{\delta} = \sqrt{\delta}^{\frac{N}{2} + \frac{1}{2}} = -\sqrt{\delta} \), \( a^N = a \), and \( b^N = b \), we find \( (a + b\sqrt{\delta})^{N+1} = a^2 - b^2 \delta = 1 \) so that the group generated by \( t_r \) has order \( N + 1 \) (see, e.g., Terras 1999 p307).

For the first order subspace equation 90 we find the evolution with \( n \)

\[
\rho_1 (t_r^n \otimes t_r^n) \left( \begin{array}{c} \langle k \rangle \\
\langle \hbar \frac{d}{dk} \rangle 
\end{array} \right) = \left( \begin{array}{cc} a & -b \delta \\
-b \delta & a 
\end{array} \right)^n \left( \begin{array}{c} \langle k \rangle \\
\langle \hbar \frac{d}{dk} \rangle 
\end{array} \right) \tag{111}
\]

where \( e_+ = \frac{1}{\sqrt{\delta}} \left( \frac{1}{2\omega} \right)\frac{N}{2\pi i} d \) and \( e_- = \frac{1}{\sqrt{\delta}} \left( \frac{1}{2\omega} \right)\frac{N}{2\pi i} d \).
where \( \lambda_{\pm} = a \mp b\sqrt{\delta} \) are eigenvalues corresponding to eigenvectors \( e_{\pm} = \\
\left( \frac{1}{\pm\sqrt{\delta}} \right) \) of \( \rho_{1}(t_{r} \otimes t_{r}) \). We have \( \lambda_{+}\lambda_{-} = 1 \) so that \( \lambda_{+}^{n} = \lambda_{-}^{-n} \). Letting

\[
c(n) = \frac{1}{2}(\lambda_{+}^{n} + \lambda_{+}^{-n})
\]

\[
s(n) = \frac{1}{2\sqrt{\delta}}(\lambda_{+}^{n} - \lambda_{+}^{-n})
\]

we may write

\[
\rho_{1}(t_{r}^{n} \otimes t_{r}^{n}) \left( \frac{\langle k \rangle}{\frac{1}{2\pi} \frac{N}{2} \frac{d}{dx}} \right) = \left( \begin{array}{cc} c(n) & s(n) \\ s(n)\delta & c(n) \end{array} \right) \left( \frac{\langle k \rangle}{\frac{1}{2\pi} \frac{N}{2} \frac{d}{dx}} \right)
\]

One may similarly derive the evolution of the higher order subspaces in the direct product algebra.

Let us defer for now a more detailed treatment of this case. Athanasiu and Floratos 1994 consider the case \( \delta = -1 \). For \( N = 3 \text{ mod } 4 \), \( \delta = -1 \) is a nonsquare in \( \mathbb{F}_{N} \) and so in this instance corresponds to the case that we consider above. Additional related work for the case \( \delta = -1 \) are contained in Balian and Itzykson 1986, Floratos and Leontaris 1997, Athanasiu et al. 1996 and Floratos and Nicolis 2005.

5 Discussion

The main idea guiding this paper is the hypothesis that physical observables reside in a space formed by taking the tensor product of 2 copies of an element in an invariant subspace of an underlying group algebra. The group of transformations leaving this space invariant is identified with the allowed transformations of the physical observables. A methodology for calculating the expectation value for physical observables then follows. The construction that I describe is generic.

In this paper we have considered this construction for the case of the Jacobi group that is the semidirect product of the Heisenberg group \( H_{1}(\mathbb{F}_{N}) \) with its automorphism group \( SL(2, \mathbb{F}_{N}) \). The goal has been to perform this construction for the finite group that provides a model counterpart for the continuum quantum case of a single particle with a single spatial degree of freedom and whose phase space is a flat torus.

We have constructed the Schrödinger-Weil representation of the Jacobi group whose restriction to the Heisenberg subgroup corresponds to the Schrödinger representation and whose restriction to \( SL(2, \mathbb{F}_{N}) \) corresponds to the metaplectic representation. This representation is concretely realized by the left action of the Jacobi group on functions defined in a particular ideal of the Jacobi group algebra.

We have defined a derivative operation acting on a function contained in the regular representation of a cyclic group. This derivative operation allows a treatment for the prime field case that parallels the characteristic zero case.
We take an element \( f \) that is contained within the chosen ideal of the Jacobi group algebra and form the direct product \( f \otimes f \). The Hermitian portion \( P_+ f \otimes f \) resides in the group algebra of the group \( \{ T_r x \otimes T_s y = t_r^x t_s^y \mid r, s \in \mathbb{F}_N \} \); this portion corresponds to the Wigner distribution of \( f \). For normalized \( P_+ f \otimes f \), the convolution of \( P_+ f \otimes f \) with a test function \( \sigma_G = \sum_{r,s \in \mathbb{F}_N} \alpha(-r,-s)T_r^x T_s^y \) has value at the origin \( T_0^x T_0^y \) equal to \( \sigma(r,s) \) evaluated over the Wigner distribution of \( f \). The Weyl map that relates the value of \( \sigma(r,s) \) evaluated over the Wigner distribution with the expectation value of a particular operator is then obtained. The operator expressions that we derive are in the Weyl ordered form and there is no ordering ambiguity.

We next consider the transformation of \( P_+ f \otimes f \) under \( g \otimes g \) for \( g \in SL(2, \mathbb{F}_N) \) and find invariant subspaces. These subspaces correspond to representations of \( SL(2, \mathbb{F}_N) \) that can be realized on spaces of homogeneous polynomials in 2 real variables. The \( n^{th} \) order subspace with dimension \( n + 1 \) is associated with homogeneous polynomials of degree \( n \). The zeroth order subspace is a constant under \( g \otimes g \) and after normalization can be interpreted as the total probability of the quantum system. The first order subspace is 2-dimensional with coordinates \( \langle k \rangle \) and \( \langle \frac{N}{2} \frac{N}{2} \rangle \) and conserved symplectic form. This subspace can be associated with the classical phase space of the physical system. Higher order subspaces and their conserved bilinear forms are described. The quadratic forms for the 2nd and 4th order subspaces correspond to "quantum universal invariants" described by Dodonov 2000 and Dodonov and Man’ko 2000 for quantum systems evolving under a homogeneous quadratic Hamiltonian.

We describe the 1 parameter families of transformations associated with the motion of a free particle and with the simple harmonic oscillator.

We have considered only the Schrödinger-Weil representation of the Jacobi group. It is interesting to consider the nature of the physical systems described by the other representations.

Let us now detail the correspondence between particular constructions in the quantum mechanical treatment of a single particle with a single translational degree of freedom and the construction developed in this paper:
Quantum Mechanics of particle with 1 degree of freedom

| Construction of this paper |
|-----------------------------|
| Configuration space         |
| Invariant subspace of Jacobi group algebra |
| Wavefunction                 |
| Element in invariant subspace of Jacobi group algebra |
| Heisenberg constant         |
| $\frac{-1}{2\omega}$ where $e^{2\pi i N \omega}$ is character of $t_z \in$ center of $G^J$ |
| Wigner function             |
| $P_+ f \otimes f$          |
| Weyl map                    |
| Obtained from convolution of $P_+ f \otimes f$ with test function |
| Expectation value of operator |
| Obtained from convolution with test function and evaluation at origin of algebra |
| Total probability           |
| Zero order subspace of $P_+ f \otimes f$ |
| Classical phase space       |
| First order subspace of $P_+ f \otimes f$ |
| Higher order quantum invariants |
| From bilinear forms for higher order subspaces of $P_+ f \otimes f$ |

A  Action of $J$ on ideal determined by $\hat{y}_0 \hat{z}_\omega \hat{s}_- \hat{u}_0 \hat{x}_0 (1 + \alpha J) \hat{d}_0$

Consider the action of $J$ on an element $f$

$$f = \sum_{k=0}^{N-1} f(k) t_z^k \hat{y}_0 \hat{z}_\omega \hat{s}_- \hat{u}_0 \hat{x}_0 (1 + \alpha J) \hat{d}_0$$

(115)

where $\alpha$ is a number that is to be determined. Let us consider the action of $J$ on the first portion of this expression:

$$J \sum_{k=0}^{N-1} f(k) t_z^k \hat{y}_0 \hat{z}_\omega = \sum_{k,l=0}^{N-1} f(k) J t_z^k t_y^l \hat{z}_\omega$$

(116)

$$= \sum_{k,l=0}^{N-1} f(k) t_z^l t_y^{-k} t_z^{2kl} \hat{z}_\omega J$$

(117)

Inserting $1 = \frac{1}{N} \sum_{\nu} \hat{y}_\nu$ we obtain

$$= \frac{1}{N} \sum_{k,l,\nu=0}^{N-1} f(k) \exp\left(\frac{2\pi i}{N} (2kl\omega - k\nu)\right) t_z^l \hat{y}_\nu \hat{z}_\omega J$$

(118)

Using the identity

$$t_z^{\nu/2\omega} \hat{y}_\nu \hat{z}_\omega = \hat{y}_\nu t_z^{\nu/2\omega} \hat{z}_\omega$$

(119)
we obtain
\[ f(k) \exp(\frac{2\pi i}{N} (2\omega k)(l - \nu \frac{\omega}{2\omega})) \sum_{l'=0}^{N-1} t_{l'}^2 \hat{y}_0 t_{l'}^2/2\omega \hat{z}_w J \] (120)

Letting \( l' = l - \nu \frac{\omega}{2\omega} \) and summing over \( \nu \) we obtain
\[ Jf = \frac{1}{N} \sum_{k, l', r} f(k) \exp(\frac{2\pi i}{N} 2\omega kl') t_{l'}^2 \hat{y}_0 \hat{z}_w \hat{J}_s - \hat{u}_0 \hat{x}_0 (1 + \alpha J) \hat{d}_0 \] (121)

We conclude that the action of \( J \) on the first portion of \( f \) leads to the inverse Fourier transformation of \( f(k) \) and we now have the term \( \hat{J}_s \) (underlined in equation 121) acting on the second part of equation 121.

It is clearer to do the next part of this calculation in two parts. Let us first consider the action of \( \hat{J}_s \) on the term in equation 121 that contains \( \alpha J \) and expand out \( \hat{u}_0 \hat{x}_0 \). We obtain
\[ \hat{y}_0 \hat{z}_w \hat{x}_0 J \hat{J} - \hat{u}_0 \hat{x}_0 \hat{J} \hat{d}_0 = \hat{y}_0 \hat{z}_w \hat{x}_0 J \hat{J} - \hat{u}_0 \hat{x}_0 (1 + \alpha J) \hat{d}_0 \] (122)

Insert 1 = \( JJ^{-1} \) to the left of \( t_{l'}^m \). The resulting term \( JJ^{-1} t_{l'}^m \) can be absorbed into \( \hat{d}_0 \) leaving
\[ \hat{y}_0 \hat{z}_w \hat{x}_0 J \hat{J} - \hat{u}_0 \hat{x}_0 \hat{J} \hat{d}_0 = \hat{y}_0 \hat{z}_w \hat{x}_0 \hat{J} \hat{J} - \hat{u}_0 \hat{x}_0 \hat{J} \hat{d}_0 \] (123)

\( t_{l'}^m r \) passes to the left to be absorbed into \( \hat{x}_0 \) and \( t_{l'}^m r \) converts into the exponential \( \exp(\frac{-2\pi i}{N} mr^2 \omega) \) after acting on \( \hat{z}_w \). The sum over \( m \) of this exponential has support only for \( r = 0 \). We are left with
\[ \hat{y}_0 \hat{z}_w \hat{x}_0 J \hat{J} - \hat{u}_0 \hat{x}_0 \hat{J} \hat{d}_0 = N \hat{y}_0 \hat{z}_w \hat{x}_0 J^2 \hat{J} \hat{d}_0 \] (124)
\[ = N \left( \frac{1}{N} \right) \hat{y}_0 \hat{z}_w \hat{J} - \hat{x}_0 \hat{d}_0 \] (125)

where
\[ J^2 \hat{s}_- = t_{s}^{-1} \hat{s}_- = \left\{ \begin{array}{ll} +1 & \text{for } N = 1 \text{ modulo } 4 \\ -1 & \text{for } N = 3 \text{ modulo } 4 \end{array} \right. \] (126)
Now consider the term \( \hat{y}_0 \hat{x}_0 \hat{s}_- J \hat{u}_0 \hat{x}_0 \hat{d}_0 \) in equation 121. Commute \( J \) through \( \hat{s}_- \) and expand out the resulting term \( J \hat{u}_0 \)

\[
\hat{y}_0 \hat{x}_0 \hat{s}_- J \hat{u}_0 \hat{x}_0 \hat{d}_0 = \hat{y}_0 \hat{x}_0 \hat{s}_- \left( \sum_{m=0}^{N-1} J t_u^m \right) \hat{x}_0 \hat{d}_0 = \\
= \hat{y}_0 \hat{x}_0 \hat{s}_- \left( \sum_{m=1}^{N-1} t_{1/m} \frac{t_{-m} J - 1}{t_{-1/m} J - 1} + J \right) \hat{x}_0 \hat{d}_0 = \\
= \hat{y}_0 \hat{x}_0 \hat{s}_- \left( \sum_{m=1}^{N-1} \left( \frac{-m}{N} \right) t_{-m} + J \right) \hat{x}_0 \hat{d}_0 \tag{127}
\]

where in the second line we use equation 13 to reexpress \( J t_u^m \) and in the third line \( t_{1/m} \) converts into \( \frac{1}{m} \) after acting on \( \hat{s}_- \). In the second line we write \( J^{-1} t_u^{-1} t_{-1} = t_u^{-1} J t_u^{-1} t_{-1} \). \( t_u^{-1} \) then changes the sign of the Legendre symbol \( \left( \frac{m}{N} \right) \) after acting on \( \hat{s}_- \) and \( J t_u^{-1} \) is absorbed into \( \hat{d}_0 \) after commuting through \( \hat{x}_0 \).

It is clearer now to consider separately the two terms in equation 127. For the first term, commute \( \hat{x}_0 \) through \( \hat{s}_- \) and expand \( \hat{x}_0 \) to obtain

\[
\hat{y}_0 \hat{x}_0 \hat{s}_- \left( \sum_{m=1}^{N-1} \left( \frac{-m}{N} \right) t_{-m} \right) \hat{x}_0 \hat{d}_0 = \\
= \hat{y}_0 \hat{x}_0 \hat{s}_- \left( \sum_{m=1}^{N-1} \left( \frac{-m}{N} \right) t_{-m} \right) \hat{x}_0 \hat{d}_0 = \\
= \hat{y}_0 \hat{x}_0 \hat{s}_- \left( \sum_{m=1}^{N-1} \left( \frac{-m}{N} \right) t_{-m} \right) \hat{x}_0 \hat{d}_0 \tag{128}
\]

Now pass \( t_{-m} \) to the left to be absorbed into \( \hat{y}_0 \), pass \( t_{m} \) to the right to be absorbed into \( \hat{x}_0 \) and convert \( t_{m} \) into an exponential by acting on \( \hat{x}_0 \). Sum over \( h \) in this exponential to obtain

\[
\sum_{h=0}^{N-1} \exp(\frac{2\pi i}{N} 2m h^2 \omega) = \left( \frac{-m}{N} \right) \left( \frac{\omega}{N} \right) G(1, N) \tag{129}
\]

Now combining these steps we have for the first term in equation 127

\[
\hat{y}_0 \hat{x}_0 \hat{s}_- \left( \sum_{m=1}^{N-1} \left( \frac{-m}{N} \right) t_{-m} \right) \hat{x}_0 \hat{d}_0 = \\
= \hat{y}_0 \hat{x}_0 \hat{s}_- \left( \sum_{m=1}^{N-1} \left( \frac{-m}{N} \right) \right)^2 \left( \frac{\omega}{N} \right) G(1, N) \hat{x}_0 \hat{d}_0 = \\
= \hat{y}_0 \hat{x}_0 \hat{s}_- \left( \frac{\omega}{N} \right) G(1, N)(\hat{u}_0 - 1) \hat{x}_0 \hat{d}_0 \tag{130}
\]
where in the second line we use \((\frac{m}{N})^2 = 1\) and sum over \(m\) to obtain the third line.

Let us now consider the second term in equation 127. We commute \(\hat{x}_0\) to the right through \(\hat{s}_-\) and use \(\hat{y}_0\) to obtain

\[
\hat{y}_0 \hat{z}_0 \hat{x}_0 \hat{s}_- \hat{x}_0 \hat{d}_0 = \hat{y}_0 \hat{z}_0 \hat{s}_- \hat{x}_0 \hat{y}_0 \hat{d}_0
\]

\[
= \hat{y}_0 \hat{z}_0 \hat{s}_- \sum_{h,l=0}^{N-1} t_{x}^{h} t_{y}^{l} J \hat{d}_0
\]

\[
= \hat{y}_0 \hat{z}_0 \hat{s}_- \sum_{h,l=0}^{N-1} t_{x}^{h} t_{y}^{2hl} J \hat{d}_0
\]

(131)

t_{x}^{l} passes to the left to be absorbed into \(\hat{y}_0\) and \(t_{x}^{2hl}\) converts to the exponential \(\exp(\frac{2\pi i}{N} 2hl \omega)\). The sum over \(l\) of this exponential has support only for \(h = 0\).

We are left with

\[
\hat{y}_0 \hat{z}_0 \hat{x}_0 \hat{s}_- \hat{d}_0 = \hat{y}_0 \hat{z}_0 \hat{s}_- NJ \hat{d}_0
\]

(132)

Combining now the results of equations 124, 130 and 132 we have

\[
Jf = J \sum_{k=0}^{N-1} f(k) t_{x}^{k} \hat{y}_0 \hat{z}_0 \hat{s}_- \hat{\bar{u}}_0 \hat{x}_0 (1 + \alpha J) \hat{d}_0
\]

\[
= \frac{1}{N} \sum_{k,l=0}^{N-1} f(k) \exp(\frac{2\pi i}{N} 2kl \omega) t_{x}^{l} \hat{y}_0 \hat{z}_0 \hat{s}_-
\]

\[
\cdot \left\{ \left( \frac{\omega}{N} \right) G(1, N) (\hat{\bar{u}}_0 - 1) \hat{x}_0 + NJ + \alpha N \left( \frac{-1}{N} \right) \hat{x}_0 \right\} \hat{d}_0
\]

(133)

Using the identity \(\hat{y}_0 \hat{z}_0 \hat{s}_- NJ \hat{d}_0 = \hat{y}_0 \hat{z}_0 \hat{s}_- \hat{\bar{u}}_0 \hat{x}_0 J \hat{d}_0\) to reexpress the \(NJ\) term in the above expression we have

\[
Jf = \frac{\left( \frac{\omega}{N} \right) G(1, N)}{N} \sum_{k,l=0}^{N-1} f(k) \exp(\frac{2\pi i}{N} 2kl \omega) t_{x}^{l} \hat{y}_0 \hat{z}_0 \hat{s}_-
\]

\[
\cdot \left\{ \hat{\bar{u}}_0 \hat{x}_0 (1 + \frac{1}{\left( \frac{\omega}{N} \right) G(1, N)} J) - (1 - \frac{\alpha N \left( \frac{-1}{N} \right) \hat{x}_0}{\left( \frac{\omega}{N} \right) G(1, N)}) \hat{x}_0 \right\} \hat{d}_0
\]

For

\[
\alpha = \frac{1}{\left( \frac{\omega}{N} \right) G(1, N)}
\]

the second term in the above bracket vanishes and we have our result

\[
Jf = \frac{\left( \frac{\omega}{N} \right) G(1, N)}{N} \sum_{k,l=0}^{N-1} f(k) \exp(\frac{2\pi i}{N} 2\omega kl) t_{x}^{l} \hat{y}_0 \hat{z}_0 \hat{s}_- \hat{\bar{u}}_0 \hat{x}_0 \left[ 1 + \frac{1}{\left( \frac{\omega}{N} \right) G(1, N)} J \right] \hat{d}_0
\]
B Action of $g \otimes g$ on $f \otimes f$

We now consider the transformation of $f \otimes f$ under the action of $g \otimes g$ for

$$f = \sum_{k=0}^{N-1} f(k) t_x^k I$$

$$I = \hat{y}_0 \hat{z}_0 \hat{s}_0 \hat{u}_0 \hat{x}_0 \left( 1 + \frac{1}{(N)} G(1, N) \right) \hat{d}_0$$

and $g \in SL(2, \mathbb{F}_N)$. Below I show the calculation for $J \otimes J$ in detail. The calculations for $t_a^p \otimes t_a^p$ and $t_b^p \otimes t_b^p$ are very similar and I merely give the final result.

Consider the action of $J \otimes J$ on $f \otimes f$: Using equation 48 we have

$$(J \otimes J) f \otimes f =$$

$$\kappa \left( \sum_{k,k',l,l'=0}^{N-1} f(k) \exp \left( \frac{2\pi i}{N} (2\omega kk') \right) t_x^l \otimes f(h) \exp \left( \frac{2\pi i}{N} (2\omega hh') \right) t_x^{l'} \right)$$

$$\cdot (I \otimes I) / N$$

where

$$\kappa = \begin{cases} 1 & \text{for } N \equiv 1 \mod 4 \\ i & \text{for } N \equiv 3 \mod 4 \end{cases}$$

Changing variables using $p = h - k$, $p' = h' - k'$, using $T_p^{p'/2} T_x^{p'/2} = 1 \otimes t_x^{p'}$ and inserting $1 = \sum_{q=0}^{N-1} X_q$ we obtain

$$J f \otimes J f = \kappa \sum_{k,k',l,l'=0}^{N-1} f(k) \otimes f(k + p)$$

$$\cdot \exp \left( \frac{2\pi i}{N} (-Ek' + (p + k)(p' + k')2\omega + q(k' + p')/2 \right)$$

$$\cdot X_q T_x^{p'/2} (I \otimes I) / N^2$$

We now sum over $k'$ to obtain $p = \frac{q}{N^2} - k(1 - E)$. Substituting in for $p$ and reexpressing the exponential we obtain

$$J f \otimes J f = \left[ P_+ \sum_{k,l,p=0}^{N-1} f(k) \exp \left( \frac{2\pi i}{N} (p'2\omega k - q \frac{N}{N} \frac{d}{d k}) \right) f(k) \right]$$

$$+ \kappa P_- \sum_{k,l,p=0}^{N-1} f(k) \exp \left( \frac{-2\pi i}{N} (p'2\omega k - q \frac{N}{N} \frac{d}{d k}) \right) f(-k)$$

$$\cdot X_q T_x^{p'/2} (I \otimes I) / N$$

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Let us rewrite equation 64

\[
\sum_{k,l,p=0}^{N-1} f(k) \otimes \exp\left(\frac{2\pi i}{N} (q \ p) \left( -\frac{1}{2}\frac{N}{2\pi i} d \right) \right) f(k) \\
(P_+ X_q Y_p + P_- X_q Y_p) \overline{X}_0 (I \otimes I) / N^2
\]

where we view the argument of the exponential as arising from the contraction of the matrix \((q \ p)\) with the matrix \((-\frac{1}{2}\frac{N}{2\pi i} d \right)\) and we have used the identity equation 60. Then

\[
(J \otimes J) f \otimes f = \sum_{k,l,p,q=0}^{N-1} \exp\left(\frac{2\pi i}{N} (q \ p) M_{J \otimes J} \left( -\frac{1}{2}\frac{N}{2\pi i} d \right) \right) f(k) \quad X_q Y_p
\]

where \(M_{J \otimes J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\) (140)

Similarly, we find,

\[
(t^a_u \otimes t^a_u) f \otimes f = \sum_{k,l,p,q=0}^{N-1} \exp\left(\frac{2\pi i}{N} (q \ p) M_{t^a_u \otimes t^a_u} \left( -\frac{1}{2}\frac{N}{2\pi i} d \right) \right) f(k) \quad X_q Y_p
\]

where \(M_{t^a_u \otimes t^a_u} = \begin{pmatrix} 1/a & 0 \\ 0 & a \end{pmatrix}\) (142)

and

\[
(t^b_u \otimes t^b_u) f \otimes f = \sum_{k,l,p,q=0}^{N-1} \exp\left(\frac{2\pi i}{N} (q \ p) M_{t^b_u \otimes t^b_u} \left( -\frac{1}{2}\frac{N}{2\pi i} d \right) \right) f(k) \quad X_q Y_p
\]

\[
(P_+ \exp(2\pi i (2b k^2 \omega)) \exp\left(\frac{2\pi i}{N} (q \ p) M_{t^b_u \otimes t^b_u} \left( -\frac{1}{2}\frac{N}{2\pi i} d \right) \right) f(k) \quad X_q Y_p
\]

\[
\overline{X}_0 (I \otimes I) / N^2
\]
where

\[ M_{t^b \otimes t^b} = \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix}. \]  

(144)

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