A graph theoretic characterization of the classical
generalized hexagon on 364 vertices

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Abstract

A tetravalent 2-arc-transitive graph of order 728 is either the known
7-arc-transitive incidence graph of the classical generalized hexagon
$GH(3,3)$ or a normal cover of a 2-transitive graph of order 182 de-
noted $A[182,1]$ or $A[182,2]$ in the 2009 list of Potočnik.

1 Preliminaries

In this note, all graphs are finite, simple, connected and undirected. An
ordered pair of adjacent vertices is called an arc. Let $\Gamma$ be a graph. We use
$V \Gamma$, $E \Gamma$, $A \Gamma$ and $\text{Aut}(\Gamma)$ to denote the vertex-set, the edge-set, the arc-set
and the full automorphism group of $\Gamma$, respectively. The distance between
two vertices $u$ and $v$ of $\Gamma$, denoted by $\partial_\Gamma(u,v)$, is the length of a shortest
path connecting $u$ and $v$ in $\Gamma$. The diameter of $\Gamma$, denoted by $\text{diam}(\Gamma)$,
is the maximum distance occurring over all pairs of vertices. Fix a vertex
$v \in V \Gamma$. For $0 \leq i \leq \text{diam}(\Gamma)$, we use $\Gamma_i(u)$ to denote the set of vertex $u$
with $\partial_\Gamma(u,v) = i$. For convenience, we usually use $\Gamma(v)$ to denote $\Gamma_1(v)$. The
degree of $v$, denoted by $\text{deg}_\Gamma(v)$ or simply $\text{deg}(v)$, is the number of vertices
adjacent to $v$ in $\Gamma$, i.e. $\text{deg}_\Gamma(v) = |\Gamma(v)|$. The graph $\Gamma$ is called regular with
valency $k$ (or $k$-regular) if the degree of each vertex of $\Gamma$ is $k$. The girth of $\Gamma$
is the length of a shortest cycle of $\Gamma$.

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Let \( G \leq \text{Aut}(\Gamma) \). The graph \( \Gamma \) is called \( G \)-vertex-transitive (\( G \)-arc-transitive, respectively), if \( G \) is transitive on the vertex-set \( \text{VT} \) (the arc-set \( A\Gamma \), respectively). Let \( s \geq 1 \). An \( s \)-arc of \( \Gamma \) is an \( (s+1) \)-tuple of vertices of \( \Gamma \) in which every two consecutive vertices are adjacent and every three consecutive vertices are pairwise distinct. The graph \( \Gamma \) is called \((G, s)\)-arc-transitive if it is \( G \)-vertex-transitive and \( G \) is also transitive on \( s \)-arcs of \( \Gamma \). The graph \( \Gamma \) is called \((G, s)\)-transitive if it is \((G, s)\)-arc-transitive but not \((G, s+1)\)-arc-transitive. The graph \( \Gamma \) is called \((G, s)\)-distance-transitive, if for each \( 1 \leq i \leq s \) the group \( G \) is transitive on the ordered pairs of form \((u, v)\) with \( \partial_i(u, v) = i \). A \((G, s)\)-distance-transitive graph is called \( G \)-distance-transitive (or distance-transitive), if \( s \) is the diameter of the graph (and \( G \) is the full automorphism group of the graph). By definitions, \( s \)-arc-transitivity implies \( s \)-distance-transitivity.

Take a vertex \( \alpha \) of a vertex-transitive graph \( \Gamma \) and let \( 0 \leq i \leq \text{diam}(\Gamma) \), then the size \( \kappa_i = |\Gamma_i(\alpha)| \) is independent of \( \alpha \).

Let \( \Gamma \) be a \((G, s)\)-distance-transitive graph with valency \( k \geq 3 \). For \( 1 \leq i \leq s \), take \( \alpha \in \text{VT} \) and \( \beta \in \Gamma_i(\alpha) \), then the intersection numbers
\[
\begin{align*}
c_i &= |\Gamma_{i-1}(\alpha) \cap \Gamma(\beta)|, \\
a_i &= |\Gamma_i(\alpha) \cap \Gamma(\beta)|, \\
b_i &= |\Gamma_{i+1}(\alpha) \cap \Gamma(\beta)|
\end{align*}
\]
are independent of \( \alpha \) and \( \beta \), and \( c_i + a_i + b_i = k \). The \( s \)-partial intersection array of \( \Gamma \) is
\[
\iota(\Gamma, s) = \begin{pmatrix}
c_1 & c_2 & \cdots & c_s \\
0 & a_1 & a_2 & \cdots & a_s \\
k & b_1 & b_2 & \cdots & b_s
\end{pmatrix}.
\]
Then \( c_1 = 1 \). For convenience, we let \( b_0 = k \). Take \( \alpha \in \text{VT} \) and let \( 1 \leq i \leq s \). By counting edges between \( \Gamma_{i-1}(\alpha) \) and \( \Gamma_i(\alpha) \), we have
\[
\kappa_{i-1}b_{i-1} = \kappa_ic_i.
\]

From [7, Table 2.4 on pages 135-136] or [5, Chapter 7], one may obtain the following lemma by checking the orders of non-abelian finite simple groups.

**Lemma 1** Let \( N \) be a non-abelian finite simple group with \( |\pi(N)| \geq 3 \) and \( \pi(N) \subseteq \{2, 3, 7, 13\} \). Then \( N \) is one of the groups listed in Table 4.
Let $\Gamma$ be a tetravalent $(G, 2)$-arc-transitive graph of order

$$|V\Gamma| = 728 = 8pq$$

where $G = \text{Aut}(\Gamma)$ and $(p, q) = (7, 13)$. Then $\Gamma$ is $(G, s)$-transitive for some $s \geq 2$. Take a vertex $\alpha \in V\Gamma$. Then the vertex stabilizer $G_\alpha$ and $s$ are listed in Table 2 [9, Lemma 2.6]. Hence $|G_\alpha| \geq e^4 \cdot 3^6$. Since $\Gamma$ is $(G, 2)$-arc-transitive, we have $G_\Gamma^{\alpha(\alpha)} \leq S_4$ is 2-transitive on $\Gamma(\alpha)$, i.e. $G_\Gamma^{\alpha(\alpha)} \cong A_4$ or $S_4$. Then $12 \mid |G_\Gamma^{\alpha(\alpha)}| \mid |G_\alpha|$, and so $|G_\alpha| = 2i \cdot 3j\prime$ where $2 \leq i \leq 4$ and $1 \leq j' \leq 6$. Since $\Gamma$ is $G$-vertex-transitive, by Frattini argument on permutation groups, we have

$$|G| = |G_\alpha| \cdot |V\Gamma| = 2^i \cdot 3^j \cdot p \cdot q = 2^i \cdot 3^j \cdot 7 \cdot 13 \leq 8491392$$

where $5 \leq i = 3 + i' \leq 7$ and $1 \leq j = j' \leq 6$. Note that 2-transitivity implies primitivity. We have that the permutation group $G_\Gamma^{\alpha(\alpha)}$ is primitive on $\Gamma(\alpha)$.

First, we suppose $G$ has a solvable minimal normal subgroup $N$. Then $N$ is an elementary $r$-group with $r \in \{2, 3, p, q\}$. Suppose $|N| = r^e$ for some $e \geq 1$. Note that $N \neq N_\alpha$. Let $|N_\alpha| = r^{e'}$. Then $0 \leq e' < e$. By Frattini argument, we have $|N| = |N_\alpha| \cdot |\alpha^{-1}|$. Hence $|\alpha^{-1}| = r^{e-e'}$. Consider the quotient graph $\Gamma_N$. We have $|V\Gamma| = |\alpha^{-1}| \cdot |V\Gamma_N|$, which means $r^{e-e'} \mid |V\Gamma| = 8pq$. So $r \neq 3$, i.e. $r \in \{2, p, q\}$. The order of the quotient graph $\Gamma_N$ is $|V\Gamma_N| = \frac{|V\Gamma|}{|\alpha^{-1}|} = \frac{8pq}{r^{e-e'}} > 2$. So $N$ is semi-regular on $V\Gamma$, $V\Gamma_N$ is a tetravalent $(G/N, 2)$-arc-transitive graph and $\Gamma$ is a cover of $\Gamma_N$ [11, Theorem 4.1]. Now $N = \mathbb{Z}_p, \mathbb{Z}_q, \mathbb{Z}_2, \mathbb{Z}_2^2$ or $\mathbb{Z}_2^3$. Then $\Gamma_N$ is a 2-arc-transitive graph of order in

| group                | order               |
|----------------------|---------------------|
| $\text{PSL}(3, 2) = \text{PSL}(2, 7)$ | 168 = $2^3 \cdot 3 \cdot 7$ |
| $\text{PSL}(2, 8)$ | 504 = $2^3 \cdot 3^2 \cdot 7$ |
| $R(3) = 2\text{G}_2(3) = \text{PSL}(2, 8) : 3$ | 1512 = $2^3 \cdot 3^3 \cdot 7$ |
| $\text{PSL}(3, 3)$ | 5616 = $2^4 \cdot 3^3 \cdot 13$ |
| $\text{PSU}(3, 3) = 2\text{A}_2(3)$ | 6048 = $2^5 \cdot 3^3 \cdot 7$ |
| $G_2(2) = \text{PSU}(3, 3) : 2$ | 12096 = $2^5 \cdot 3^3 \cdot 7$ |
| $\text{PSL}(2, 27)$ | 9828 = $2^2 \cdot 3^3 \cdot 7 \cdot 13$ |
| $G_2(3)$ | 4245696 = $2^6 \cdot 3^6 \cdot 7 \cdot 13$ |

Table 1: Non-abelian finite simple $\{2, 3, 7, 13\}$-group.

2 The Proof
These two graphs are $2$-transitive, i.e. they are $2$-arc-transitive, but not

$$\Gamma \subseteq \Gamma(N) \subseteq \Gamma(V)$$

We have that $\Gamma \subseteq \Gamma(V)$ for $s \geq 3$.

Now we suppose $G$ has no solvable minimal normal subgroups. Take a minimal normal subgroup $N$ of $G$. Then $N$ is non-solvable. By Burnside’s $p^nq^b$-theorem in [7, Theorem 4.130 on page 239] or [5, page 36], we have $|\pi(N)| \geq 3$. Let $N = T^k$ where $T$ is a non-abelian simple group and $k \geq 1$. Note that $\pi(T)N$ is a non-abelian simple group and $\pi(TN)\subseteq\pi(N)$.

Thus $|\alpha|^N = |\alpha|N = |\alpha|_{N_{\alpha}}$. Consider the quotient graph $\Gamma_N$. Note that $|\alpha|^N = \frac{|\Gamma|}{|\Gamma_N|} = 8pq$. Then $|\alpha|^N_2 = 8$ and $3 \mid |\alpha|^N_3$, which implies $|N|_3 = |N_{\alpha}|_3$. By Table 1 we have $|N|_3 \neq 1$. This implies $N_{\alpha} \neq 1$. So $\Gamma_{\alpha}(\alpha) \neq 1$ and $N$ is not semi-regular on $\Gamma$. Note that $\Gamma_{\alpha}(\alpha) \subseteq \Gamma_{\alpha}(\alpha)$. Thus $\Gamma_{\alpha}(\alpha)$ is transitive on $\Gamma(\alpha)$. We have

$$4 \mid |\alpha|^N_\alpha \mid |\alpha|^N_\alpha.$$ 

If $|\Gamma_{\alpha}(\alpha)| \geq 3$, then $N$ is semi-regular on $\Gamma$. [11, Theorem 4.1]. So we have $|\Gamma_{\alpha}(\alpha)| \leq 2$, i.e. $|\alpha|^N = 4pq$ or $8pq$. This means

$$|\alpha|^N = 4pq, 8pq.$$ 

If $\pi(N) = \{2, 3, r\}$ where $r \in \{p, q\}$, then $|\alpha|^N \leq 8r \leq 8 \max\{p, q\}$, and so $|\Gamma_{\alpha}(\alpha)| \geq 3$. This is a contradiction. Hence

$$\pi(N) = \{2, 3, p, q\},$$

and so $N = PSL(2, 27)$. If $N = PSL(2, 27)$, then $8 \mid |\alpha|$, and so we have $|\alpha| = 4pq$, $|\alpha| = 8pq$, or $|\alpha| = 3$. This is a contradiction. Hence we have $N = G_2(3)$.
Let $N = G_2(3)$. Then $|N| = 4pq \cdot |N_\alpha|$ or $8pq \cdot |N_\alpha|$. So $|N_\alpha| = 11664 = 2^4 \cdot 3^6$ or $|N_\alpha| = 5832 = 2^3 \cdot 3^6$. By Table 2 we have 
\[(s, G_\alpha) = (7, [3^5] : GL(2, 3)).\]

Let $|N| = 8pq \cdot |N_\alpha|$. Then $|N_\alpha| = 2^3 \cdot 3^6$, $|\alpha^N| = 8pq$ and $|VT_N| = 1$. This implies that $N$ is transitive on $VT$. By [8, Theorem 1.1], we get that $\Gamma$ is $(N, 7)$-transitive. By Table 2 we have $|N_\alpha| = 2^4 \cdot 3^6$. This is a contradiction. Hence

\[|N| = 4pq \cdot |N_\alpha|, \quad |\alpha^N| = 4pq \quad \text{and} \quad |VT_N| = 2.\]

So $\Gamma$ is bipartite. Now $|N_\alpha| = 2^4 \cdot 3^6 = |G_\alpha|$. Since $N_\alpha \leq G_\alpha$, we get

\[N_\alpha = G_\alpha.\]

Note that $|G| = |G_\alpha| \cdot |VT| = |N_\alpha| \cdot 8pq = 2|N|$. Thus we have

\[G = N.2 = G_2(3).2.\]

Let $g$ be the girth of $\Gamma$. By [2, Proposition 17.2 on page 131], we get that $g \geq 2s - 2 = 12$. Since $\Gamma$ is bipartite, there is no odd cycles in $\Gamma$. So $g$ is even. Let $g = 2t + 2$. Then $\text{diam}(\Gamma) \geq \frac{g}{2} = t + 1$, $t \geq 5$ and $\Gamma$ is $(G, t)$-arc-transitive. The $t$-partial intersection array of $\Gamma$ is

\[
\iota(\Gamma, t) = \begin{cases} 
* & 1 & 1 & 1 & 1 & 1 & \cdots & c_t = 1 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & a_t = 0 \\
4 & 3 & 3 & 3 & 3 & 3 & \cdots & b_t = 3 
\end{cases}
\]

Hence $3^6 - 1 = 728 = |VT| \geq \sum_{i=0}^{t} |\Gamma_i(\alpha)| = 1 + 4 + 4 \cdot 3 + 4 \cdot 3^2 + \cdots + 4 \cdot 3^{t-1} = 2 \cdot 3^t - 1$. This implies $t < 6$. So we have $t = 5$, $g = 12$ and $\text{diam}(\Gamma) \geq 6$. Now we suppose $\text{diam}(\Gamma) \geq 7$. Then $c_6 \leq 3 = b_5$. We have $|\Gamma_6(\alpha)| = \frac{b_6}{c_6}|\Gamma_5(\alpha)| \geq |\Gamma_5(\alpha)| = 4 \cdot 3^{t-1}$. So $|VT| \geq \sum_{i=0}^{6} |\Gamma_i(\alpha)| = 2 \cdot 3^t - 1 + 4 \cdot 3^{t-1} = 3^6 + 3^4 - 1 > 3^6 - 1 = |VT|$. This is a contradiction. Hence $\text{diam}(\Gamma) = 6$. Then $\Gamma$ is $G$-distance-transitive. By [3, page 222] or [4], $\Gamma$ is the incidence graph of the known generalized hexagon $GH(3, 3)$, i.e. the generalized dodecagon of order $(1, 3)$. We denote this graph by $\Gamma_7$. The graph $\Gamma_7$ is actually 7-arc-transitive by [6, last paragraph on the first page] or [2, 17g on page 137], and it is 7-transitive with smallest order.
The construction of $\Gamma_7$ can be found in [1]. Another construction of $\Gamma_7$ is by orbital graph method. The full automorphism group of $\Gamma_7$ is $G_2(3).2$ with vertex stabilizer of order 11664. The almost simple group $G_2(3).2$ with socle $G_2(3)$ is in the database of almost simple groups of Magma. The group $G_2(3).2$ has only one conjugate class of subgroups of order 11664. The graph $\Gamma_7$ is isomorphic to the orbital graph with arc-transitive group $G_2(3).2$ and an orbit of length four under the action of the stabilizer, and so it can be constructed by Magma. The following is the corresponding Magma code of the construction of $\Gamma_7$.

```
D:=AlmostSimpleGroupDatabase();
NumberOfGroups(D,4245696,8491392);
G:=GroupData(D,203)'permrep;
M:=Subgroups(G:OrderEqual:=11664);
#M;
H:=M[1]'subgroup;
GonH:=CosetImage(G,H);
newH:=Stabilizer(GonH,1);
Orb:=Orbits(newH);
for o in Orb do
  #o;
end for;
Neb:=Set(Orb[2]);
gamma:=OrbitalGraph(GonH,1,Neb);
Aut:=AutomorphismGroup(gamma);
f, := IsIsomorphic(G,Aut);
f;
gamma;
```

The result graph $\Gamma_7$ with vertices numbered $\{1, 2, \ldots, 728\}$ and neighbours for each vertex is in Appendix A. Note that, the labels of the vertices are randomly changed.

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A The graph $\Gamma_7$
| Vertex Neighbours | Vertex Neighbours | Vertex Neighbours | Vertex Neighbours |
|------------------|------------------|------------------|------------------|
| 503 272 546 675  | 502 22 508 613 616 618 620 622 624 626 628 630 632 634 636 638 640 642 644 646 648 650 652 654 656 658 660 662 664 666 668 670 672 674 676 678 680 682 684 686 688 690 692 694 696 698 700 702 704 706 708 710 712 714 716 718 720 722 724 726 728 730 732 734 736 738 740 742 744 746 748 750 752 754 756 758 760 762 764 766 768 770 772 774 776 778 780 782 784 786 788 790 792 794 796 798 800 802 804 806 808 810 812 814 816 818 820 822 824 826 828 830 832 834 836 838 840 842 844 846 848 850 852 854 856 858 860 862 864 866 868 870 872 874 876 878 880 882 884 886 888 890 892 894 896 898 900 902 904 906 908 910 912 914 916 918 920 922 924 926 928 930 932 934 936 938 940 942 944 946 948 950 952 954 956 958 960 962 964 966 968 970 972 974 976 978 980 982 984 986 988 990 992 994 996 998 1000 |
| Vertex | Neighbours | Vertex | Neighbours |
|--------|------------|--------|------------|
| 649    | 186 486 555 620 ; 721 199 284 669 706 ; | | |
| 650    | 34 101 447 544 ; 722 286 586 615 665 ; | | |
| 651    | 304 329 477 522 ; 723 47 102 210 673 ; | | |
| 652    | 36 318 602 631 ; 724 164 375 479 530 ; | | |
| 653    | 442 481 584 597 ; 725 87 460 554 726 ; | | |
| 654    | 46 76 516 645 ; 726 181 341 690 725 ; | | |
| 655    | 475 664 673 719 ; 727 371 486 489 714 ; | | |
| 656    | 406 458 470 476 ; 728 179 588 668 690 ; | | |
| 657    | 98 366 504 551 ; | | |
| 658    | 99 202 455 701 ; | | |
| 659    | 79 162 308 608 ; | | |
| 660    | 303 419 617 619 ; | | |
| 661    | 146 370 620 646 ; | | |
| 662    | 379 621 639 645 ; | | |
| 663    | 55 104 617 676 ; | | |
| 664    | 21 211 215 655 ; | | |
| 665    | 97 247 611 722 ; | | |
| 666    | 314 345 357 589 ; | | |
| 667    | 227 236 307 331 ; | | |
| 668    | 196 365 702 728 ; | | |
| 669    | 304 559 611 721 ; | | |
| 670    | 186 297 386 477 ; | | |
| 671    | 38 209 388 490 ; | | |
| 672    | 62 150 190 714 ; | | |
| 673    | 437 648 655 723 ; | | |
| 674    | 174 177 291 330 ; | | |
| 675    | 81 323 436 556 ; | | |
| 676    | 444 581 596 663 ; | | |
| 677    | 40 265 345 705 ; | | |
| 678    | 43 91 193 680 ; | | |
| 679    | 264 404 407 546 ; | | |
| 680    | 424 470 493 640 ; | | |
| 681    | 136 504 513 519 ; | | |
| 682    | 90 392 569 697 ; | | |
| 683    | 16 386 526 534 ; | | |
| 684    | 166 346 532 552 ; | | |
| 685    | 236 391 407 589 ; | | |
| 686    | 322 342 498 523 ; | | |
| 687    | 182 261 588 628 ; | | |
| 688    | 55 107 264 541 ; | | |
| 689    | 198 412 424 511 ; | | |
| 690    | 212 390 726 728 ; | | |
| 691    | 79 168 513 536 ; | | |
| 692    | 71 526 669 585 ; | | |
| 693    | 197 394 525 697 ; | | |
| 694    | 38 357 564 573 ; | | |
| 695    | 101 102 268 606 ; | | |
| 696    | 31 298 335 563 ; | | |
| 697    | 183 235 682 693 ; | | |
| 698    | 3 346 385 706 ; | | |
| 699    | 118 213 572 709 ; | | |
| 700    | 74 267 280 479 ; | | |
| 701    | 280 369 641 658 ; | | |
| 702    | 172 326 572 668 ; | | |
| 703    | 94 142 188 361 ; | | |
| 704    | 84 361 479 516 ; | | |
| 705    | 88 461 530 677 ; | | |
| 706    | 147 515 698 721 ; | | |
| 707    | 2 462 464 607 ; | | |
| 708    | 28 92 269 615 ; | | |
| 709    | 64 413 619 699 ; | | |
| 710    | 347 414 499 582 ; | | |
| 711    | 19 42 280 572 ; | | |
| 712    | 6 268 463 546 ; | | |
| 713    | 171 310 323 327 ; | | |
| 714    | 381 392 672 727 ; | | |
| 715    | 46 211 393 464 ; | | |
| 716    | 33 128 357 481 ; | | |
| 717    | 222 272 374 450 ; | | |
| 718    | 201 279 322 339 ; | | |
| 719    | 4 389 394 655 ; | | |
| 720    | 7 208 585 644 ; | | |