THE EXTERIOR DEGREE OF A PAIR OF FINITE GROUPS

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Abstract. The exterior degree of a pair of finite groups $(G, N)$, which is a generalization of the exterior degree of finite groups, is the probability for two elements $(g, n)$ in $(G, N)$ such that $g \wedge n = 1$. In the present paper, we state some relations between this concept and the relative commutativity degree, capability and the Schur multiplier of a pair of groups.

1. Introduction

Let $G$ be a group with a normal subgroup $N$, then $(G, N)$ is said to be a pair of groups. Let $G$ and $N$ acting on each other and themselves by conjugation, remember that \([1, 2]\) the non-abelian tensor product $G \otimes N$ is the group generated by the symbols $g \otimes n$ subject to the relations

\[
\begin{align*}
    gg' \otimes n &= (g g' \otimes g n), \\
    g \otimes n' &= (g \otimes n) (n g \otimes n n'),
\end{align*}
\]

for all $g, g'$ in $G$ and $n, n'$ in $N$.

As it mentioned in \([11]\), a relative central extension of the pair $(G, N)$ consists of a group homomorphism $\sigma : M \rightarrow G$ together with an action of $G$ on $M$, such that:

1. $\sigma(M) = N$;
2. $\sigma(gm) = g^{-1} \sigma(m) g$, for all $g \in G$ and $m \in M$;
3. $\sigma(m_1) m = m_1^{-1} mm_1$, for all $m, m_1 \in M$;
4. $\ker \sigma \subseteq Z_G(M)$,

in which

$Z_G(M) = \{m \in M \mid \sigma(m) = m\}$.

The pair $(G, N)$ is called capable if it admits a central extension such that $\ker \sigma = Z_G(M)$. In particular if $M = N$ and $G$ acts on $N$ by conjugation, $Z_G(N)$ is denoted by $Z(G, N)$.

The exterior product $G \wedge N$ is obtained from $G \otimes N$ by imposing the additional relation $x \otimes x$ for all $x$ in $N$ and the image of $g \otimes n$ is denoted by $g \wedge n$ for all $g \in G, n \in N$. By using the notations in \([4]\), the exterior $G$-centre of $N$ is a central subgroup of $N$ which is defined as follows

$Z^\wedge(G, N) = \{n \in N \mid 1 = g \wedge n \in G \wedge N \text{ for all } g \in G\}$.

Already \([4]\) Theorem 3 shows $Z^\wedge(G, N)$ allows us to decide when $(G, N)$ is capable. More precisely a pair $(G, N)$ is capable if and only if $Z^\wedge(G, N) = 1$. It can be
checked that $Z^\wedge(G, N) = \bigcap_{x \in G} N C^\wedge(x)$, in which $N C^\wedge(x)$ is the exterior centralizer of $x$ in the pair of groups $(G, N)$ and it is equal to the set of all elements $n \in N$ such that $1 = x \wedge n \in G \wedge N$. Also for all $x \in N$ we denote $g C^\wedge(x)$ the set of all elements $g \in G$ such that $1 = g \wedge x \in G \wedge N$. In the case for which $N = G$ and $x \in G$, $Z^\wedge(G, G) = Z^\wedge(G)$ and $G C^\wedge(x) = C^\wedge_G(x)$ are the exterior centralizer of $x$ in $G$ and the exterior centre of $G$, respectively.

The commutator map $\kappa : G \wedge N \to [G, N]$ is the group homomorphism defined on the generators by $g \wedge n \mapsto [g, n] = gng^{-1}n^{-1}$ ($g \in G, n \in N$). The Kernel of $\kappa$ is the Schur multiplier of a pair of groups, $M(G, N)$, which was stated by Brown, Loday [1] and Ellis in [5].

2. Known results on the relative commutativity degree and exterior degree

In this section, some known results are remembered for the relative commutativity degree and exterior degree. For any pair of finite groups $(G, N)$, we use the notation of [8] and define the commutativity degree of pair as follows

$$d(G, N) = \frac{1}{|G||N|} |\{(x, y) \in G \times N \mid xy = yx\}|.$$ 

Obviously, if $G = N$, then $d(G, G) = d(G)$, the commutativity degree of a finite group $G$, and $d(G, N) = 1$ if and only if $N \subseteq Z(G)$.

**Theorem 2.1.** [8] Theorem 3.9 Let $H$ and $N$ be two subgroups of $G$ such that $N \subseteq G$ and $N \subseteq H$. Then

$$d(G, H) \leq d(G/N, H/N)d(N),$$

and if $N \cap [H, G] = 1$, then the equality holds.

The concept of exterior degree of finite group, $d^\wedge(G)$, is defined in [12] as the probability for two elements $g$ and $g'$ in $G$ such that $g \wedge g' = 1$.

It is seen that if $k(G)$ is the number of conjugacy classes of $G$, then $|G|^2 d^\wedge(G) = \sum_{i=1}^{k(G)} \sum_{x \in C_i} |C^\wedge_G(x)|$, where $C_1, \ldots, C_{k(G)}$ are the conjugacy classes of $G$.

The following two inequalities in [12] will be generalized in the next section.

**Theorem 2.2.** [12] Theorem 2.3 For every finite group $G$,

$$\frac{d(G)}{|M(G)|} + \frac{|Z^\wedge(G)|}{|G|} (1 - \frac{1}{|M(G)|}) \leq d^\wedge(G) \leq d(G) - \left(\frac{p-1}{p}\right) \frac{|Z(G)| - |Z^\wedge(G)|}{|G|},$$

where $p$ is the smallest prime number dividing the order of $G$.

In the case for which $G$ is a capable group, the above upper bound for $d^\wedge(G)$ can be improved as follows.

**Theorem 2.3.** [12] Theorem 2.8 Let $G$ be a non-abelian capable group and $p$ be the smallest prime number dividing the order of $G$, then $d^\wedge(G) \leq 1/p$. 


3. Exterior degree of a pair of finite groups

In this section, we will define the exterior degree of a pair of finite groups \((G, N)\). In the special case when \(N = G\), the concept of exterior degree of \(G\) is obtained.

**Definition 3.1.** Let \((G, N)\) be a pair of finite groups, set
\[
C := \{(g, n) \in G \times N \mid 1 = g \wedge n \in G \wedge N\}.
\]

We define the exterior degree of the pair of groups \((G, N)\) as the ratio
\[
d^* (G, N) = \frac{|C|}{|G||N|}.
\]

Obviously, if \(N = G\), then \(d^* (G, G) = d^* (G)\) and \(d^* (G, N) = 1\) if and only if \(N \subseteq Z^\wedge(G, N)\).

Let a finite group \(H\) acting on a finite group \(K\) by conjugation. Then we denote the number of conjugacy \(H\)-classes of \(K\) by \(k_H(K)\).

The following lemmas are useful in the future investigation.

**Lemma 3.2.** Let \((G, N)\) be a pair of finite groups, then
\[
d^* (G, N) = \frac{1}{|G||N|} \sum_{x \in G} |NC^\wedge (x)| = \frac{1}{|G||N|} \sum_{x \in N} |GC^\wedge (x)|.
\]

**Proof.** The proof is straightforward. \(\square\)

**Lemma 3.3.** Let \((G, N)\) be a pair of finite groups. Then
(i) If \(k_N(G) = k\) and \(\{x_1, ..., x_k\}\) is a system of representatives for conjugacy \(N\)-classes of a group \(G\), then
\[
d^* (G, N) = \frac{1}{|G|} \sum_{i=1}^{k} \frac{|NC^\wedge (x_i)|}{|C_N(x_i)|}.
\]

(ii) If \(k_G(N) = t\) and \(\{x_1, ..., x_t\}\) is a system of representatives for conjugacy \(G\)-classes of a group \(N\), then
\[
d^* (G, N) = \frac{1}{|N|} \sum_{i=1}^{t} \frac{|GC^\wedge (x_i)|}{|C_G(x_i)|}.
\]

**Proof.** (i) Let \(C_1, ..., C_k\) be the conjugacy \(N\)-classes of \(G\) and \(x_i \in C_i\) for \(1 \leq i \leq k\).

For every \(y \in C_i\), there exists \(n \in N\) such that \(y = x^n_i\), hence \(|NC^\wedge (y)| = |NC^\wedge (x_i)|\).

Therefore
\[
|G||N|d^* (G, N) = \sum_{x \in G} |NC^\wedge (x)| = \sum_{i=1}^{k} \sum_{x \in C_i} |NC^\wedge (x)| = \sum_{i=1}^{k} |N : C_N(x_i)||NC^\wedge (x_i)|
\]
\[
= |N| \sum_{i=1}^{k} \frac{|NC^\wedge (x_i)|}{|C_N(x_i)|},
\]
as asserted.

(ii) The proof is similar to the previous part. \(\square\)

**Proposition 3.4.** Let \((G_1, N_1)\) and \((G_2, N_2)\) be two pairs of finite groups such that \(G = G_1 \times G_2\) and \(N = N_1 \times N_2\) in where \(|G_1|\) and \(|G_2|\) are coprime. Then
\[
d^* (G_1 \times G_2, N_1 \times N_2) = d^* (G_1, N_1)d^* (G_2, N_2).
\]
Proof.

\[ d^\wedge (G, N) = \frac{1}{|G||N|} \sum_{(x,y) \in N} |\alpha C^\wedge ((x, y))| \]

\[ = \frac{1}{|G_1||G_2||H_1||H_2|} \sum_{x \in N_1} \sum_{y \in N_2} |g_1, C^\wedge (x)||g_2, C^\wedge (y)| \]

\[ = \frac{1}{|G_1||H_1|} \sum_{x \in N_1} |g_1, C^\wedge (x)| \frac{1}{|G_2||H_2|} \sum_{y \in N_2} |g_2, C^\wedge (y)| \]

\[ = d^\wedge (G_1, N_1) d^\wedge (G_2, N_2). \]

\[ \square \]

Lemma 3.5. For all \( x \) in \( G \), the factor group \( C_N(x)/_{N} C^\wedge (x) \) is isomorphic to a subgroup of \( M(G, N) \).

Proof. For all \( x \in G \), we can define \( f : C_N(x) \to M(G, N) \) by \( y \mapsto y \wedge x \). Since the elements of \( M(G, N) \) are fixed under the action of \( G \), \( f \) is actually a homomorphism with \( N C^\wedge (x) \) as its kernel. \( \square \)

The following theorem give a generalization of Theorem 2.2.

Theorem 3.6. Let \((G, N)\) be a pair of finite groups and \( p \) be the smallest prime number dividing the order of \( G \). Then

(i) \( \frac{d(G, N)}{|M(G, N)|} + \frac{|Z^\wedge (G, N)|}{|G|} (1 - \frac{1}{|M(G, N)|}) \leq d^\wedge (G, N) \);

(ii) \( d^\wedge (G, N) \leq d(G, N) - (\frac{p-1}{p}) \frac{|Z(G, N)| - |Z^\wedge (G, N)|}{|N|} \).

Proof. (i) By using Lemma 3.3 (i),

\[ d^\wedge (G, N) = \frac{1}{|G|} \sum_{i=1}^{k} \frac{|N C^\wedge (x_i)|}{|C_N(x_i)|} \]

\[ \geq \frac{1}{|G|} \left( |Z^\wedge (G, N)| + \frac{1}{|M(G, N)|} (k - |Z^\wedge (G, N)|) \right) \]

\[ = \frac{k}{|G||M(G, N)|} + \frac{|Z^\wedge (G, N)|}{|G|} \left( 1 - \frac{1}{|M(G, N)|} \right). \]

The results is obtained by the fact that \( d(G, N) = k/|G| \).

(ii) If \( x \notin Z^\wedge (G, N) \), then \( |G :_{G} C^\wedge (x)| \geq p \). Hence by Lemma 3.3 (ii),

\[ d^\wedge (G, N) = \frac{1}{|N|} \sum_{i=1}^{t} \frac{|g C^\wedge (x_i)|}{|C_G(x_i)|} \]

\[ \leq \frac{|Z^\wedge (G, N)|}{|N|} + \frac{1}{p} \frac{|Z(G, N)| - |Z^\wedge (G, N)|}{|N|} + \frac{t - |Z(G, N)|}{|N|} \]

\[ = d(G, N) - (\frac{p-1}{p}) \frac{|Z(G, N)| - |Z^\wedge (G, N)|}{|N|}. \]
Theorem 3.7. If $p$ is the smallest prime number dividing the order of $G$, then for every pair of finite groups $(G, N)$,

\[
\frac{|Z^\wedge(G,N)|}{|N|} + \frac{p(|N| - |Z^\wedge(G,N)|)}{|G||N|} \leq d^\wedge(G,N) \leq \frac{|Z^\wedge(G,N)|}{|N|} + \frac{|N| - |Z^\wedge(G,N)|}{p \cdot |N|}.
\]

Proof. By using Lemma 3.2,

\[
d^\wedge(G,N) = \frac{1}{|G||N|} \sum_{x \in N} |GC^\wedge(x)|
\]

\[
= \frac{1}{|G||N|} \left( \sum_{x \in Z^\wedge(G,N)} |GC^\wedge(x)| + \sum_{x \notin Z^\wedge(G,N)} |GC^\wedge(x)| \right).
\]

Since $p \leq |GC^\wedge(x)| \leq |G|/p$ for all $x \notin Z^\wedge(G,N)$, so

\[
\frac{1}{|G||N|} \left( \sum_{x \in Z^\wedge(G,N)} |GC^\wedge(x)| + \sum_{x \notin Z^\wedge(G,N)} |GC^\wedge(x)| \right)
\]

\[
\leq \frac{|Z^\wedge(G,N)|}{|N|} + \frac{|N| - |Z^\wedge(G,N)|}{p \cdot |N|}.
\]

On the other hand,

\[
\frac{|Z^\wedge(G,N)|}{|N|} + \frac{p(|N| - |Z^\wedge(G,N)|)}{|G||N|}
\]

\[
\leq \frac{1}{|G||N|} \left( \sum_{x \in Z^\wedge(G,N)} |GC^\wedge(x)| + \sum_{x \notin Z^\wedge(G,N)} |GC^\wedge(x)| \right),
\]

hence the result follows.

\[\square\]

Corollary 3.8. Let $(G, N)$ be a pair of finite groups and $p$ be the smallest prime divisor of the order of $G$. If $N \neq Z^\wedge(G,N)$, then $d^\wedge(G,N) \leq (2p - 1)/p^2$. In particular, $d^\wedge(G,N) \leq \frac{3}{4}$.

Proof. In the case $N \not\subseteq Z(G)$, the result is obtained by [3, Theorem 3.9]. Now suppose that $N \subseteq Z(G)$. Then $d(G, N) = 1$, and so by Theorem 3.6 $(ii)$,

\[
d^\wedge(G,N) \leq 1 - \left(\frac{p-1}{p}\right)(1 - \frac{|Z^\wedge(G,N)|}{|N|}) \leq 1 - (p - 1)^2/p^2 = (2p - 1)/p^2.
\]

\[\square\]

The example 4.3 will show that the above upper bound is sharp and the bound we obtained there is the best possible one.

Lemma 3.9. Let $(G, N)$ and $(G, H)$ be pairs of finite groups. Then the following sequence is exact.

\[N \wedge H \times G \wedge N \xrightarrow{1} G \wedge H \rightarrow G/N \wedge H/N \rightarrow 0.\]

Proof. The proof is similar to [1, Proposition 9].

\[\square\]
Theorem 3.10. Let \((G, N)\) and \((G, H)\) be pairs of finite groups. Then

\[ d^\wedge(G, H) \leq d^\wedge(G/N, H/N), \]

and the equality holds when \(N \subseteq Z^\wedge(G, H)\).

Proof. By using Lemma 3.2

\[
d^\wedge(G, H) = \frac{1}{|G||H|} \sum_{x \in G} |hC^\wedge(x)| = \frac{1}{|G||H|} \sum_{x \in G} \sum_{n \in N} |hC^\wedge(xn)|
\]

\[
\leq \frac{1}{|G||H|} \sum_{x \in G} \sum_{n \in N} |\phi C^\wedge(xn)||hC^\wedge(xn) \cap N|
\]

\[
= \frac{1}{|G||H|} \sum_{x \in G} \sum_{n \in N} |\phi C^\wedge(xn)||hC^\wedge(x) \cap N|
\]

\[
\leq \frac{1}{|G||H|} \sum_{x \in G} |\phi C^\wedge(xn)||N| = d^\wedge(G/N, H/N)
\]

and so \(d^\wedge(G, H) \leq d^\wedge(G/N, H/N)\).

In the case for which \(N \subseteq Z^\wedge(G, H)\), Lemma 3.9 implies that \(\text{Im}(l) = 1\), and thus \(G \wedge H \cong G/N \wedge H/N\) which implies that \(d^\wedge(G, H) = d^\wedge(G/N, H/N)\). \(\square\)

Lemma 3.11. Let \((G, N)\) be a pair of finite groups such that \(N \notin Z(G)\), and \(p\) be the smallest prime number dividing the order of \(G\), then

(i) if \([G, N] \cap Z(G, N) = 1\), then \(d(G, N) \leq 1/p\);

(ii) \(d(G, N) \leq 1/p - (1 - 1/p)|Z(G, N)|/|N|\).

Proof. (i) Since the condition of equality in the Theorem 2.1 holds we have

\[ d(G, N) = d(G/Z(G, N), N/Z(G, N)). \]

Moreover \(Z(G/Z(G, N), H/Z(G, N)) = 1\). Thus we can assume that \(Z(G, N) = 1\).

Let \(d(G, N) = k_G(N)/|N| > 1/p\), then

\[
|N| = 1 + \sum_{i=2}^{k_G(N)} |G : C_G(x_i)| \geq 1 + p(k_G(N) - 1) \geq 1 + |N|,
\]

which is a contradiction, hence the result follows.
Proof. (ii) By using Lemma 3.2

\[
\begin{align*}
d(G, N) &= \frac{1}{|G||N|} \sum_{x \in N} |C_G(x)| \\
&= \frac{1}{|G||N|} \sum_{x \in \text{Z}(G, N)} |C_G(x)| + \frac{1}{|G||N|} \sum_{x \in \text{Z}(G, N)} |C_G(x)| \\
&\leq \frac{|G|}{|G||N|} |\text{Z}(G, N)| + \frac{|G|}{p} (|N| - |\text{Z}(G, N)|) \\
&= 1/p - (1 - 1/p) \left( \frac{|\text{Z}(G, N)|}{|N|} \right).
\end{align*}
\]

Now it is easy to see that Theorem 2.3 is obtained by the following theorem when \( N = G \).

**Theorem 3.12.** Let \((G, N)\) be a pair of finite groups such that \( N \not\subset \text{Z}(G) \), and \( p \) is the smallest prime number dividing the order of \( G \). If the pair \((G, N)\) is capable, then

\[
d^\wedge(G, N) \leq 1/p.
\]

**Proof.** We can assume that \([G, N] \cap \text{Z}(G, N) \neq 1\), by Lemma 3.11. Assume by contrary for all \( x \) in \( N - \text{Z}(G, N) \) we have \( C_G^\wedge(x) = C_G(x) \), it follows that

\[
\text{Z}(G, N) \leq \bigcap_{x \in G - \text{Z}(G, N)} C_G(x) \cap N = \bigcap_{x \in G - \text{Z}(G)} C_G^\wedge(x) \cap N.
\]

On the other hand, \( N \cap [G, N] \leq \bigcap_{x \in \text{Z}(G, N)} C_G^\wedge(x) \cap N \) thus \( \text{Z}(G, N) \cap [G, N] \leq \text{Z}^\wedge(G, N) \), which is a contradiction. Hence Lemma 3.3 (ii) follows

\[
\begin{align*}
d^\wedge(G, N) &= \frac{1}{|N|} \sum_{i=1}^{t} \frac{|C_G^\wedge(x_i)|}{|C_G(x_i)|} \\
&\leq \frac{1}{|N|} (1 + 1/p|\text{Z}(G, H)|) - 1 + 1/p + t - \text{Z}(G, N) - 1 \\
&= d(G, N) - (1 - 1/p) \left( \frac{|\text{Z}(G, N)|}{|N|} \right).
\end{align*}
\]

The rest of proof is obtained by Lemma 3.11 (ii).

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Example 4.4 will show that the above bound can be attained by a pair of capable groups, which shows the upper bound is sharp.

4. SOME EXAMPLES
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**Example 4.1.** Consider the dihedral group \( D_{2n} \) with the presentation

\[
D_{2n} = \langle a, b \mid a^n = 1, b^2 = 1, ba = a^{-1}b \rangle.
\]
First assume that $n$ is odd. $D_{2n}$ is the semidirect product of $\langle a \rangle$ by $C_2$. By using (7) of [5] and [9, Proposition 2.11.4], we have $\mathcal{M}(D_{2n}, \langle a \rangle) = 1$. Thus $d^*(D_{2n}, \langle a \rangle) = d(D_{2n}, \langle a \rangle)$ and [8, Example 3.11] states $d(D_{2n}, \langle a \rangle) = \frac{n + 1}{2n}$. In the case $n$ even, $Z^*(D_{2n}, \langle a \rangle) = 1$ when $n > 2$, we have

$$d^*(D_{2n}, \langle a \rangle) = \frac{1}{|D_{2n}|} \sum_{x \in \langle a \rangle} |D_{2n}, C^*(x)|$$

$$= \frac{2n + n^2/2 + n/2(n/2 - 1)}{2n^2}$$

$$= \frac{n + 1}{2n}.$$

**Example 4.2.** Let $Q_{2^n}$ be the generalized quaternion group of order $2^n$ with the presentation

$$Q_{2^n} = \langle a, b \mid a^{2^{n-1}} = 1, \ a^{2^{n-2}} = b^2, \ ba = a^{-1}b \rangle.$$ 

Then

$$d^*(Q_{2^n}, \langle a \rangle) = \frac{1}{|Q_{2^n}|} \sum_{x \in \langle a \rangle} |Q_{2^n}, C^*(x)|$$

$$= \frac{1}{2^n 2^{n-1}} (2^n + (2^{n-1} - 1)2^{n-1})$$

$$= \frac{2^{n-1} + 1}{2^n}.$$

**Example 4.3.** Let $G = C_4$ and $N = 2C_4$. Then [4, Corollary 8] shows that the pair $(C_4, 2C_4)$ is capable, and so

$$d^*(C_4, 2C_4) = \frac{1}{2|C_4|} \sum_{x \in C_4} |2C_4, C^*(x)|$$

$$= \frac{1}{8} (1 + 2 + 1 + 2) = \frac{3}{4}.$$

**Example 4.4.** Let $G = D_8$ and $N$ be the subgroup $\langle a^2, ab \rangle$ of $G$. A computation similar to Example 4.1 shows that the pair $(D_8, \langle a^2, ab \rangle)$ is capable, and

$$d^*(D_8, \langle a^2, ab \rangle) = \frac{1}{|\langle a^2, ab \rangle|} \sum_{x \in D_8} |\langle a^2, ab \rangle, C^*(x)|$$

$$= \frac{1}{32} (4 + 2 + 1 + 2 + 2 + 2 + 1 + 2) = \frac{1}{2}.$$

We remark that all these computations can be done by using the HAP package [9].

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