The Spectra of Arrangement Graphs

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Abstract

Arrangement graphs were introduced for their connection to computational networks and have since generated considerable interest in the literature. In a pair of recent articles by Chen, Ghorbani and Wong, the eigenvalues for the adjacency matrix of an (n,k)-arrangement graph are studied and shown to be integers. In this manuscript, we consider the adjacency matrix directly in terms of the representation theory for the symmetric group. Our point of view yields a simple proof for an explicit formula of the associated spectrum in terms of the characters of irreducible representations evaluated on a transposition. As an application we prove a conjecture raised by Chen, Ghorbani and Wong.

1 Introduction

Since arrangement graphs were first introduced in the seminal paper [3], there has been considerable interest in the literature. In two recent articles, [1] and [2], the eigenvalues of the adjacency matrix are studied. The main result in [2] is that the eigenvalues are integers. In this article we study the adjacency matrix from the perspective of the representation theory of symmetric groups. In particular, we consider the representation associated to the arrangement graph and the corresponding equivariant operator associated to the adjacency matrix. Our approach leads to a simple derivation for an explicit formula of the spectra of (n, k)-arrangement graphs in terms of the characters of irreducible representations evaluated on a transposition.

It seems pertinent to point out that the problem of translating our formula, constructed in representational theoretic language, into a combinatorial algorithm to compute the eigenvalues, has already been solved. In particular, the input into
this algorithm is a finite sequence \( \lambda = (\lambda_1, \ldots, \lambda_j) \) of positive integers, called a partition of \( k \), that satisfies

\[
\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_j \quad \text{and} \quad \lambda_1 + \lambda_2 + \cdots + \lambda_j = k.
\]

A formula due to Frobenius directly calculates the contribution to the eigenvalue associated to the corresponding irreducible representation of the symmetric group \( S_k \). The second part of the algorithm constructs a partition \( \mu \) of \( n \) via a combinatorial rule on the Young diagram \( Y(\lambda) \) called Pieri’s formula. Once again Frobenius’s formula calculates the contribution to the eigenvalue, this time associated to the corresponding irreducible representation of the symmetric group \( S_n \). The one other component of the formula is the binomial coefficient

\[
\binom{n-k}{2}.
\]

As an application of our formula, we prove a conjecture by Chen, Ghorbani and Wong that states for \( k \) fixed and \( n \) large, \(-k\) is the only negative eigenvalue in the spectrum of the \((n, k)\)-arrangement graph.

## 2 The representation associated to an arrangement graph

Suppose \( \Gamma \) is a graph and \( G \) is a subgroup of the automorphism group of \( \Gamma \). Let \( V \) be the set of vertices of \( \Gamma \) and let \( V \) be the complex vector space with basis vectors from \( V \). The group \( G \) acts naturally on \( V \). We refer to this as the associated \( G \)-representation. Let \( U \) denote the adjacency matrix of \( \Gamma \). The matrix \( U \) is directly related to the linear operator \( \Upsilon \) on \( V \) defined by

\[
\Upsilon(v) = \sum_{w \sim v} w
\]

where \( v \) and \( w \) are vertices and \( w \sim v \) if they share an edge. In particular, the matrix for \( \Upsilon \) in a basis of vertices is \( U \). Observe that

\[
\Upsilon(g \cdot v) = g \cdot \Upsilon(v) \quad \text{for each} \quad v \in V.
\]

We call \( \Upsilon \) the equivariant operator associated to \( U \).

We now introduce the graphs to be considered. For each natural number \( n \), let \( I_n = \{1, 2, \ldots, n\} \). It will be convenient for us to identify the symmetric group \( S_n \) with the permutation group of \( I_n \). Suppose \( k \) is a natural number and \( k \leq n \). By definition, a \( k \)-permutation \( \sigma \) is an injective map

\[
\sigma : I_k \to I_n.
\]
Let $V(n, k)$ be the set of $k$-permutations. The arrangement graph $A(n, k)$ is the graph whose vertices are the elements of $V(n, k)$, where two $k$-permutations are adjoined by an edge if they agree as functions on exactly $k - 1$ elements of $I_k$. The symmetric group $G = S_n$, acts naturally on $A(n, k)$ by the formula

$$\pi \cdot \sigma = \pi \circ \sigma$$

where $\pi \in S_n$ and $\sigma \in V(n, k)$. Note that when $k = n$ then $A(n, k)$ is an edgeless graph and the adjacency matrix $U = 0$, so we may assume $k < n$.

Let $V$ be the associated $S_n$-representation. Our first task is to show $V$ is induced from a representation of a subgroup. In particular, the symmetric group $S_k$, of $I_k$, is naturally identified with a subgroup of $S_n$. We also identify the permutations $S((I_n - I_k))$, of the set $I_n - I_k$, with a subgroup of $S_n$. The subgroup $H$ of $S_n$ generated by these two subgroups is naturally isomorphic to the direct product

$$H = S_k \times S((I_n - I_k)).$$

We will also identify $S_k$ with a subset of the vertices: $S_k \subseteq V(n, k)$. If $W$ is the subspace of $V$ with basis vectors from $S_k$ then $W$ is an $H$ invariant subspace. In particular, $W$ is the regular representation of $S_k$ with a trivial action of $S((I_n - I_k))$.

To show that the representation of $G = S_n$ in $V$ is induced from the representation of $H$ in $W$ we use the following lemma. The proof is an exercise in the concept of induced representations and is left to the reader.

**Lemma 2.1** Suppose $V$ is a representation of a finite group $G$. Let $H \subseteq G$ be a subgroup and $W \subseteq V$ an $H$-invariant subspace. Suppose that:

1. The $G$-module generated by $W$ is $V$
2. For each $g \in G - H$ we have $g \cdot W \cap W = \{0\}$

Then there is a natural isomorphism

$$V \cong \text{Ind}^G_H(W).$$

**Proposition 2.2** Suppose $G = S_n$ and $V$ is the representation associated to the arrangement graph. If $W$ and $H$ are defined as above then

$$V \cong \text{Ind}^G_H(W).$$

**Proof.** If $\pi \notin H$ then $\pi(j) \in I_n - I_k$ for some $j \in I_k$. Hence, for each basis vector $\sigma \in S_k$ it follows $\pi \cdot \sigma \notin S_k$ and therefore

$$\pi \cdot W \cap W = \{0\}. $$

Next observe that each basis vector in $V(n, k)$ can be written as $\pi \cdot \sigma$ for some $\sigma \in S_k$. Therefore $G$-module generated by $W$ is $V$ and the result follows from the previous lemma. ■
The irreducible representations of the symmetric group are parametrized by partitions. In particular, a partition \( \lambda \) of a positive integer \( m \) is a finite sequence of positive integers \( \lambda = (\lambda_1, \ldots, \lambda_j) \) such that

\[
\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_j \quad \text{and} \quad \lambda_1 + \lambda_2 + \cdots + \lambda_j = m
\]

If \( \lambda \) is a partition of \( m \) we write \( \lambda \vdash m \) let \( S^\lambda \) denote a realization of the corresponding irreducible representation for \( S_m \). If \( \lambda \vdash k \) we let

\[
\mathbb{W}_\lambda \subseteq \mathbb{W}
\]
denote the corresponding \( S_k \)-isotypic component in \( \mathbb{W} \). Since \( \mathbb{W} \) is the regular representation for \( S_k \) it follows that

\[
\mathbb{W}_\lambda \cong m_\lambda S^\lambda
\]

where \( m_\lambda \) is the dimension of \( S^\lambda \).

**Proposition 2.3**

\[
\mathbb{V} = \bigoplus_{\lambda \vdash k} \text{Ind}^G_H (\mathbb{W}_\lambda)
\]

**Proof.** The result is clear since

\[
\mathbb{V} = \text{Ind}^G_H (\mathbb{W}) = \text{Ind}^G_H \left( \bigoplus_{\lambda \vdash k} \mathbb{W}_\lambda \right) \cong \bigoplus_{\lambda \vdash k} \text{Ind}^G_H (\mathbb{W}_\lambda).
\]

\[\blacksquare\]

### 3 The equivariant operator associated to the adjacency matrix

We need to introduce some notation. If \( p \) is any integer, it will be convenient to define

\[
\binom{p}{2} = \frac{p(p-1)}{2}.
\]

Suppose \( m \) is a positive integer and let \( i, j \) be positive integers with \( i < j \leq m \). We let \((ij)\) denote the corresponding transposition in \( S_m \). Let \( \mathbb{C} \) denote the complex numbers. We define an element of the group algebra \( T_m \in \mathbb{C} [S_m] \) according to the formula

\[
T_m = \sum_{1 \leq i < j \leq m} (ij).
\]

\( T_m \) acts naturally on any representation of \( S_m \). Let \( \Upsilon : \mathbb{V} \to \mathbb{V} \) be the equivariant operator associated to \( U \).
Lemma 3.1 The restriction $\Upsilon : W \to V$ of $\Upsilon$ to $W$ is given by the formula

$$\Upsilon(w) = T_n \cdot w - T_k \cdot w - \left(\frac{n-k}{2}\right)w \text{ for } w \in W.$$ 

Proof. Let $\sigma : I_k \to I_n$ be a $k$-permutation corresponding to a permutation of $I_k$. If a vertex $\gamma \in V(n, k)$ shares an edge with $\sigma$ then there is exactly one value $i \in I_k$ such that $\sigma(i) \neq \gamma(i)$ and $\gamma(i) = j \in I_n - I_k$. In particular $\gamma(i) = j$ with $k < j$ and $\gamma = (ij) \cdot \sigma$. The other type of transpositions that appear in $T_n - T_k$ have the form $(ij)$ with $k < i < j \leq n$. and are from the group $S(I_n - I_k)$. Each of this second type fixes every vector in $W$. Summing up over these two types of transpositions gives the desires result.

If $\lambda \vdash m$, we let $\chi_\lambda : S_m \to \mathbb{C}$ be the character associated to the irreducible representation $S^\lambda$. It is well known that $\chi_\lambda(g)$ is an integer for each $g \in G$.

Lemma 3.2 $T_m$ is in the center of $\mathbb{C}[S_m]$ and acts on the irreducible representation $S^\lambda$ according to the formula

$$T_m \cdot v = \left(\frac{m}{2}\right)\frac{\chi_\lambda(\tau)}{\chi_\lambda(1)}v \text{ for } v \in S^\lambda$$

where $\tau$ is any transposition of $S_m$.

Proof. We use the fact the set $C$ of transpositions is a conjugacy class in $S_m$. Since

$$T_m = \sum_{g \in C} g$$

it follows immediately $T_m$ is in the center of $\mathbb{C}[S_m]$ and therefore acts by a scalar on an irreducible representation. If $\alpha \in \mathbb{C}$ is the eigenvalue for the action of $T_m$ on $S^\lambda$ then taking traces yields

$$\chi_\lambda(1) \alpha = \sum_{g \in C} \chi_\lambda(g)$$

which is the desired result.

Suppose $\lambda \vdash k$ and recall that $W_\lambda = m_\lambda S^\lambda$ is the corresponding isotypic component in $W$.

Proposition 3.3 The restriction of $\Upsilon$ to $\text{Ind}_H^G(W_\lambda)$ is given by the formula

$$\Upsilon(v) = T_n \cdot v - \left(\frac{k}{2}\right)\frac{\chi_\lambda(\tau)}{\chi_\lambda(1)} + \left(\frac{n-k}{2}\right)\right)v \text{ for } v \in \text{Ind}_H^G(W_\lambda).$$
Proof. It follows immediately from the previous two lemmas that the formula is valid for the restriction of $\Upsilon$ to the subspace $W_\lambda$. Now suppose $w \in W_\lambda$ and $g \in S_n$. Since $\Upsilon$ is $S_n$-equivariant we have

$$\Upsilon(g \cdot w) = g \cdot \Upsilon(w) = g \cdot \left( T_n \cdot w - \left( \binom{k}{2} \frac{\chi_\lambda(\tau)}{\chi_\lambda(1)} + \binom{n-k}{2} \right) w \right) =$$

$$T_n \cdot (g \cdot w) - \left( \binom{k}{2} \frac{\chi_\lambda(\tau)}{\chi_\lambda(1)} + \binom{n-k}{2} \right) g \cdot w.$$  

Hence the result follows since every vector in $\text{Ind}_H^G(W_\lambda)$ is a sum of vectors of the form $g \cdot w$ for $g \in S_n$ and $w \in W_\lambda$.  

Observe that the previous lemma implies that the space $\text{Ind}_H^G(W_\lambda)$ is $\Upsilon$-invariant, since it’s clear that $\text{Ind}_H^G(W_\lambda)$ is $T_n$-invariant. Thus, by Proposition 2.3, it is sufficient to calculate $\Upsilon$ on the subspaces $\text{Ind}_H^G(W_\lambda)$ for $\lambda \vdash k$. Hence, we need to understand the action of $T_n$ on $\text{Ind}_H^G(W_\lambda)$. In order to do this, by Lemma 3.2, we need to decompose $\text{Ind}_H^G(W_\lambda)$ into irreducible $S_n$-modules. It is the Littlewood-Richardson rule [6], or rather, the simpler version referred to as Pieri’s formula, that solves this last problem. To make the application of the rule specific, we write $W_\lambda = m_\lambda S^\lambda \otimes \mathbb{C}$ where $\mathbb{C}$ represents the trivial module for $S(I_n - I_k)$ and the group $H = S_k \times S(I_n - I_k)$ acts by the formula $(\sigma, g) \cdot v \otimes z = \sigma \cdot v \otimes g \cdot z$. Since

$$\text{Ind}_H^G(W_\lambda) = \text{Ind}_H^G(m_\lambda S^\lambda \otimes \mathbb{C}) \cong m_\lambda \text{Ind}_H^G(S^\lambda \otimes \mathbb{C})$$

it suffices to decompose $\text{Ind}_H^G(S^\lambda \otimes \mathbb{C})$ into irreducible $S_n$-modules, which is exactly what Pieri’s formula does. To describe the application of the rule in this context, we identify a partition $\lambda$ with the corresponding Young diagram $Y(\lambda)$. Suppose $\lambda \vdash k$ and $\mu \vdash n$. We write $\lambda \prec \mu$ if the Young diagram $Y(\mu)$ for $\mu$ can be obtained from the diagram $Y(\lambda)$ by adding at most one box to each column.

**Theorem 3.4 (Pieri’s formula)** If $\lambda \vdash k$ then

$$\text{Ind}_H^G(S^\lambda \otimes \mathbb{C}) \cong \bigoplus_{\lambda \prec \mu} S^\mu.$$  

This gives us our main result:

**Theorem 3.5** Suppose $\tau_n$ is any transposition of $S_n$ and $\tau_k$ is any transposition of $S_k$. The eigenvalues of the arrangement graph $A(n,k)$ are the numbers of the form

$$\left( \binom{n}{2} \frac{\chi_\mu(\tau_n)}{\chi_\mu(1)} - \binom{k}{2} \frac{\chi_\lambda(\tau_k)}{\chi_\lambda(1)} - \binom{n-k}{2} \right)$$

where $\lambda \vdash k$ and where $\mu \vdash n$ such that $\lambda \prec \mu$.  

6
To see that these numbers are integers and to help calculate their values, one can apply a formula originally attributed to Frobenius [4] and [5]. In particular, suppose \( \lambda = (\lambda_1, \ldots, \lambda_l) \vdash m \). If \( \tau \in S_m \) is any transposition, then we have the following:

\[
\left( \frac{m}{2} \right) \chi_\lambda (\tau) = \sum_{j=1}^l \left( \binom{\lambda_j - j + 1}{2} - \binom{j}{2} \right) = \left( \frac{l}{2} \right) \chi_\lambda (1) - \binom{l+1}{3}
\]

since

\[
\sum_{j=1}^l \binom{j}{2} = \binom{l+1}{3}.
\]

4 A conjecture by Chen, Ghorbani and Wong

In [1, Conjecture 3], Chen Ghorbani and Wong conjecture that for \( k \) fixed and \( n \) large that \(-k\) is the only negative eigenvalue in the spectrum of the \((n, k)\)-arrangement graph. In this section we prove their conjecture. Let \( \mathbb{Q} \) denote the rational numbers.

**Proposition 4.1** There is a polynomial \( p(x) \in \mathbb{Q}[x] \) such that when \( n > p(k) \) then \(-k\) is the only negative eigenvalue in the spectrum of the \((n, k)\)-arrangement graph.

We prove the proposition by establishing two lemmas.

**Lemma 4.2** Suppose \( \lambda = (\lambda_1, \ldots, \lambda_q) \vdash k \), \( \mu = (\mu_1, \ldots, \mu_r) \vdash n \), \( \lambda \prec \mu \) and \( \mu_1 = n - k \). If \( n - k > k \). Then

\[
\left( \frac{n}{2} \right) \chi_\mu (\tau_n) - \left( \frac{k}{2} \right) \chi_\lambda (\tau_k) = \left( \frac{n - k}{2} \right) = -k.
\]

**Proof.** The relation \( \lambda \prec \mu \) means we have to add \( n - k \) boxes to the Young diagram \( Y(\lambda) \), no more than one to each column, to obtain the Young diagram \( Y(\mu) \). Since there are \( \lambda_1 \leq k < n - k \) columns in \( Y(\lambda) \) we must add a positive number of boxes \( a \) to the first row to obtain \( \lambda_1 + a = \mu_1 = n - k \). But then the number of left over boxes to be added to \( Y(\lambda) \) is exactly \( n - k - a = \lambda_1 \) so we must add one box to each column. Thus

\[
\mu_{j+1} = \lambda_j \text{ for } 1 \leq j \leq q.
\]

Using the formula for the characters and the fact that \( r = q + 1 \), we obtain

\[
\left( \frac{n}{2} \right) \chi_\mu (\tau_n) - \left( \frac{k}{2} \right) \chi_\lambda (\tau_k) = \left( \frac{n - k}{2} \right) = -k.
\]

\[
\left( \frac{n - k}{2} \right) - \left( \frac{q + 1}{2} \right) + \left( \frac{\lambda_1 - 1}{2} \right) - \left( \frac{\lambda_1}{2} \right) + \cdots + \left( \frac{\lambda_q - q}{2} \right) - \left( \frac{\lambda_q - q + 1}{2} \right) =
\]
Thus the result follows.

Lemma 4.3 Suppose $\lambda = (\lambda_1, \ldots, \lambda_q) \vdash k$, $\mu = (\mu_1, \ldots, \mu_r) \vdash n$, $\lambda \prec \mu$ and $\mu_1 > n - k$. There is a polynomial $p(x) \in \mathbb{Q}[x]$ such that when $n > p(k)$ then the associated eigenvalue is positive.

Proof. Since

$$\frac{\chi_\lambda(\tau_k)}{\chi_\lambda(1)} \leq 1$$

it follows that

$$-\left(\frac{k}{2}\right) \frac{\chi_\lambda(\tau_k)}{\chi_\lambda(1)} \geq -\left(\frac{k}{2}\right).$$

Using the character formula for the character $\chi_\mu$ and the fact that $\mu_1 > n - k$ we see that

$$\left(\frac{n}{2}\right) \frac{\chi_\mu(\tau_n)}{\chi_\mu(1)} - \left(\frac{n-k}{2}\right) \geq \left(\frac{n-k+1}{2}\right) - \left(\frac{n-k}{2}\right) - \sum_{j=1}^{r} \left(\frac{j}{2}\right) \geq$$

$$n-k - \left(\frac{r+1}{3}\right).$$

Now use the fact that $\lambda \prec \mu$. Since the young diagram $Y(\mu)$ is obtained from the diagram $Y(\lambda)$ by adding at most one box to a column it follows that $r \leq q + 1 \leq k + 1$. Putting this together we have

$$\left(\frac{n}{2}\right) \frac{\chi_\mu(\tau_n)}{\chi_\mu(1)} - \left(\frac{k}{2}\right) \frac{\chi_\lambda(\tau_k)}{\chi_\lambda(1)} - \left(\frac{n-k}{2}\right) \geq n-k - \left(\frac{k+2}{3}\right) - \left(\frac{k}{2}\right).$$

This will be positive if

$$n > \left(\frac{k+2}{3}\right) + \left(\frac{k}{2}\right) + k.$$

Therefore we could use

$$p(k) = \frac{1}{6} k (k+1) (k+5)$$

To finish the proof of the conjecture we only need to see that the condition $\lambda \prec \mu$ implies that $\mu_1 \geq n - k$. But this is clear since $\lambda_1$ is the number of columns in the Young diagram $Y(\lambda)$ and we need to add a total of $n - k$ boxes to obtain the Young
diagram $Y(\mu)$. Hence, if we add $\alpha$ boxes to the first row to obtain $\mu_1 = \lambda_1 + \alpha$ we can at add at most $\lambda_1$ additional boxes. Thus we need $\lambda_1 + \alpha \geq n - k$.

We also make the following observation. Our proof shows that when $n > p(k)$, given $\lambda = (\lambda_1, \ldots, \lambda_q) \vdash k$ then there is exactly one partition $\mu(\lambda) = (n - k, \lambda_1, \ldots, \lambda_q) \vdash n$ such that $\lambda \prec \mu(\lambda)$ and such that the formula for the eigenvalues yields the value $-k$ (this is the partition described in the proof of Lemma 4.2). Since

$$\text{Ind}_H^G(S^\lambda \otimes \mathbb{C}) \cong \bigoplus_{\lambda \prec \mu} S^\mu$$

and since then multiplicity of $S^\lambda$ in $W$ is $\chi_\lambda(1)$ it follows that the multiplicity of the eigenvalue $-k$, for $n > p(k)$, is exactly

$$\sum_{\lambda \vdash k} \chi_{\mu(\lambda)}(1) \chi_\lambda(1).$$

References

[1] Chen, B.F, Ghorbani, E. and Wong, K.B.: Cyclic decomposition of $k$-permutations and eigenvalues of the arrangement graphs. Electron. J. Comb. 20 (4) (2013) #P22

[2] Chen, B.F, Ghorbani, E. and Wong, K.B.: On the eigenvalues of certain Cayley graphs and arrangement graphs. Linear Algebra Appl. 444 (2014) 246-253.

[3] Day, K. and Tripathi, A.: Arrangement graphs: a class of generalized star graphs. Inform. Process. Lett. 42 (1992), 235-241.

[4] Diaconis, P. and Shahshahani, M.: Generating a random permutation with random transpositions. Z. Wahrsch. Verw. Gebiete 57 (1981) 159-179.

[5] Ingram, R.E.: Some Characters of the Symmetric Group. Proc. Amer. Math. Soc. 1 (3) (1950) 358-369.

[6] James, G.D.: The Representation Theory of the Symmetric Groups. Lecture Notes in Mathematics, Volume 682, Springer-Verlag (1978).