TOPOLOGICAL CUBIC POLYNOMIALS WITH ONE PERIODIC RAMIFICATION POINT

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Abstract. For \( n \geq 1 \), consider the space of affine conjugacy classes of topological cubic polynomials \( f : \mathbb{C} \to \mathbb{C} \) with a period \( n \) ramification point. It is shown that this space is a connected topological space.

1. Introduction. Our study of spaces of topological branched coverings \( f : \mathbb{C} \to \mathbb{C} \) is largely motivated by rational dynamics in one complex variable. In the context of affine conjugacy classes of complex cubic polynomials, it is natural to study its two complex dimensional parameter space by considering dynamically relevant one complex dimensional slices (e.g. see [3, 4, 5]). Following Milnor [9], the parameter space of cubic polynomials with marked critical points, modulo affine conjugacy, is naturally identified with \( \mathbb{C}^2 \). For each \( n \geq 1 \), he considered the one dimensional slice \( \mathcal{S}_n \) formed by all such polynomials with a specified critical point periodic of exact period \( n \). Milnor established that \( \mathcal{S}_n \) is smooth and conjectured that \( \mathcal{S}_n \) is connected for all \( n \). Here we give a positive answer to the topological version of this question. More precisely, following Rees [10], we let \( \mathcal{B}_n \) be the space of topological cubic polynomials \( f : \mathbb{C} \to \mathbb{C} \) with marked ramification points such that one of the ramification points has exact period \( n \), modulo affine conjugacy (see Section 2.1 for a precise definition of \( \mathcal{B}_n \)). The aim of this paper is to establish the following:

**Main Theorem.** For all \( n \geq 1 \), the topological space \( \mathcal{B}_n \) is connected.

It is worth to mention that in [10] it is claimed that \( \mathcal{B}_n \) is homotopically equivalent to the complement of finitely many points in \( \mathcal{S}_n \). Although it is reasonable to conjecture that such homotopy equivalence exists, there is a gap in the proof contained in [10, 6.12].

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Let us now outline the strategy of the proof of our main result which relies on complex polynomial dynamics techniques. Following Thurston and Levy [7] or Cui and Tan [6], it is not difficult to show that every connected component of $B_n$ contains elements of $S_n$ (e.g. see Corollary 5.2). Thus to show that $B_n$ is connected we join all components of $S_n$ through paths in $B_n$. In order to join components of $S_n$ we consider its escape locus [2, 9].

The escape locus $E(S_n)$ is the subset of $S_n$ formed by the polynomials which have disconnected Julia sets or equivalently which have a critical point escaping to infinity. Since $E(S_n)$ is a neighborhood of infinity in $S_n$ and every connected component of the algebraic set $S_n$ is not compact, it follows that every component of $S_n$ contains a connected component of $E(S_n)$. The connected components of $E(S_n)$ are called escape regions. Thus, in order to show that $B_n$ is connected we will show that within $B_n$ there exist paths joining all of the escape regions.

Our route in the collection of escape regions is guided by kneading sequences. Associated to a polynomial in the escape locus, the kneading sequence is a periodic sequence of symbols “0” and “1” that encodes the itinerary of the periodic critical value with respect to a suitable partition. This sequence is constant within each escape region. Although there are several escape regions sharing the same kneading sequence, the itinerary $(1^{n-1}0)\infty$ is realized by a unique one which we call the distinguished escape region.

It is convenient to consider the space $F_n$ formed by conjugacy classes of topological polynomials which are conformal near infinity and have a periodic ramification point of period $n$ (see Section 2.2). The notion of kneading sequence extends from $E(S_n)$ to the subspace $E(F_n)$ of $B_n$ formed by the elements of $F_n$ with one escaping ramification point.

We travel towards the distinguished region from another one via the concatenation of two paths in $B_n$: the twisting path and the geometrization path. The twisting path ends at a map in $E(F_n)$ with kneading sequence “closer” to $(1^{n-1}0)\infty$. The geometrization path ends at a polynomial in $E(S_n)$ which is provided by a Theorem of Cui and Tan. Along the geometrization path, which is fully contained in $E(F_n)$, the kneading sequence remains unchanged. We complete the proof by successively moving in the aforementioned manner to reach our distinguished final destination.

Organization.

We start by introducing the relevant spaces of dynamical systems in Section 2. Then we discuss kneading sequences in Section 3. The twisting path and its effect on kneading is the content of Section 4 while the geometrization path is the content of Section 5. We assemble the proof of our Main Theorem in Section 6.

2. Spaces. As discussed in the introduction, the interplay among several spaces of dynamical systems is essential for the proof of our main result. We are mainly concerned with spaces formed by conjugacy classes of topological polynomials $B_n$, of semi-polynomial maps $F_n$ and of cubic polynomials $\text{Poly}_3$. The aim of this section is to introduce these spaces as well as relevant subspaces of them.

2.1. Topological polynomials. Following Rees [10, 1.9], let $B_n$ be the space formed by triples $(f,c,c')$ where

$$f: \mathbb{C} \to \mathbb{C}$$

is a degree 3 topological branched covering with ramification points $\infty, c, c'$ such that all of the following statements hold:
(1) \( \infty \) is a fixed point of \( f \) and \( f \) is locally 3-to-1 around \( \infty \).
(2) \( c \) has period exactly \( n \) under \( f \).
(3) The branch point \( f(c') \) is disjoint from the forward orbit of \( c \).

Let \( B_n \) be the quotient of \( B_n \) under the equivalence relation that identifies \((f, c_f, c'_f)\) and \((g, c_g, c'_g)\) if there exists an affine map \( A \) which is a conjugacy between \( f \) and \( g \) (i.e. \( A^{-1} \circ f \circ A = g \)) such that \( A(c_g) = c_f \) and \( A(c'_g) = c'_f \). We endow \( B_n \) with the topology of uniform convergence and \( B_n \) with the corresponding quotient topology.

2.2. Semi-polynomials. Let us recall the notion of semi-rational map introduced in [6]. Consider a topological branched covering \( f : \mathbb{C} \to \overline{\mathbb{C}} \) and denote by \( P_f \) the closure of

\[ \{ f^m(c) : m > 0, \ c \ \text{ramification point of} \ f \} \]

We say that \( f : \mathbb{C} \to \overline{\mathbb{C}} \) is a semi-rational map if the following statements hold:

1. The set of accumulation points \( P'_f \) of \( P_f \) is finite or empty.
2. If \( P'_f \neq \emptyset \), then \( f \) is holomorphic in a neighborhood of \( P'_f \).
3. Every periodic orbit in \( P'_f \) is either attracting or super-attracting.

Denote by \( F_n \) the subset of \( B_n \) formed by conjugacy classes having semi-rational representatives. Endow \( F_n \) with the subspace topology. An element of \( F_n \) is called a semi-polynomial. Our interest in the set \( E(F_n) \) formed by all the \([ (f, c, c') ] \in F_n \) such that \( P'_f = \{ \infty \} \). That is, \([ (f, c, c') ] \in E(F_n) \) if and only if the orbit of the ramification point \( c' \) tends to \( \infty \). It follows that elements in \([ (f, c, c') ] \in E(F_n) \) have a unique periodic ramification point, so often we will simply write \([ f ] \in E(F_n) \).

2.3. Critically marked cubic polynomials. A critically marked cubic polynomial is a triple \((f, c, c')\) where \( f \) is a cubic polynomial with critical points \( c, c' \in \mathbb{C} \), and \( c = c' \) only when \( f \) has a double critical point in \( \mathbb{C} \). Following Milnor [9], the space of monic centered critically marked cubic polynomials \( \text{Poly}_3^{cm} \) is identified with \( \mathbb{C}^2 \) via the family \( \{ P_{a,v}(z) = z^3 - 3a^2z + 2a^3 + v, \ (a, v) \in \mathbb{C}^2 \} \).

He defines \( S_n \) to be the algebraic subset of \((a, v) \in \mathbb{C}^2 \) for which \(+a\) has exact period \( n \) under the iterations of \( P_{a,v} \). According to [9, Theorem 5.2], \( S_n \) is a smooth affine algebraic set, for all \( n \geq 1 \).

We say that two critically marked cubic polynomials \((f, c_f, c'_f)\) and \((g, c_g, c'_g)\) are affinely conjugated if there exists an affine conjugacy \( A \) between \( f \) and \( g \) (i.e. \( A^{-1} \circ f \circ A = g \)) such that \( A(c_g) = c_f \) and \( A(c'_g) = c'_f \). The moduli space of critically marked cubic polynomials \( \text{Poly}_3 \) is the space of affine conjugacy classes of critically marked cubic polynomial.

The critically marked polynomials \((P_{a,v}, +a, -a)\) and \((P_{a',v'}, +a', -a')\) are affinely conjugated if and only if \((a', v') = (-a, -v) \). Thus, \( \text{Poly}_3 \) is the algebraic surface identified with the quotient of \( \mathbb{C}^2 \) by the action of the involution \( \mathcal{I} : (a, v) \mapsto (-a, -v) \).

That is, \( \text{Poly}_3 \equiv \mathbb{C}^2 / \mathcal{I} \).

We consider the algebraic set \( \mathfrak{S}_n \) formed by all \([ (f, c, c') ] \in \text{Poly}_3 \) such that \( c \) has exact period \( n \) under iterations of \( f \). It follows that \( \mathfrak{S}_n \equiv S_n / \mathcal{I} \).

Recall that the escape locus \( E(\mathfrak{S}_n) \) consists of all \([ (f, c, c') ] \in \mathfrak{S}_n \) such that \( c' \) lies in the basin of infinity. A map \([ (f, c, c') ] \in \mathfrak{S}_n \) is not in \( B_n \) if and only if \( f(c') \)
is in the periodic orbit of $c$. Hence $E_n \setminus B_n$ is finite and disjoint from the escape locus $E(\mathcal{S}_n)$. In particular, $E(\mathcal{S}_n) \subset B_n$.

The topology induced by the uniform convergence of maps $f : \mathbb{C} \to \mathbb{C}$ coincides with the one of Poly3 as a complex orbifold, thus we have the following inclusions of topological spaces:

$$E(\mathcal{S}_n) \subset E(\mathcal{F}_n) \subset B_n.$$ 

3. Kneading sequence. Here we generalize the notion of kneading sequences originally introduced in [9, Definition 5.4] for elements of $E(\mathcal{S}_n)$ to elements of $E(\mathcal{F}_n)$. Then we show that such a sequence is constant on connected components of $E(\mathcal{F}_n)$.

Given $[(f, c, c')] \in E(\mathcal{F}_n)$ we consider a small neighborhood $U$ of $\infty$ such that $f|_U$ is holomorphic. According to the general theory of holomorphic functions with a super-attractive fixed point we may choose $U$ so that $g_f(z) := \lim_{m \to \infty} \frac{1}{3^m} \log |f^m(z)|$, is a well defined continuous function $g_f : U \to [0, +\infty]$ (e.g. see [8, §9]). The level curves $g_f = r$ for $r$ sufficiently large are Jordan curves in $U$ surrounding $\infty$. Moreover, $g_f$ satisfies the functional relation

$$g_f \circ f = 3 \cdot g_f.$$ 

Via this relation, after declaring $g_f(z) = 0$ for all $z$ in the complement of the basin of infinity, we obtain a continuous extension $g_f : \mathbb{C} \to [0, +\infty]$ where $[0, +\infty]$ is endowed with the order topology.

The following lemma shows that some basic features of cubic polynomials with one escaping critical point are also present in the dynamical space of the semipolynomial $f$ (see Figure 1).

**Lemma 3.1.** Let $[(f, c, c')] \in E(\mathcal{F}_n)$. Then

$$D := \{z \in \mathbb{C} : g_f(z) < g_f(f(c'))\}$$

is a Jordan domain and

$$f^{-1}(D) = \{z \in \mathbb{C} : g_f(z) < g_f(c')\}$$

is the disjoint union of two Jordan domains $D_0$ and $D_1$ such that $f : D_0 \to D$ is a degree 2 covering ramifying at $c$ and $f : D_1 \to D$ is a homeomorphism.
Before proving this lemma let us define the **kneading word** \( \kappa^f \in \{0, 1\}^n \) by
\[
\kappa^f := \kappa_1 \ldots \kappa_n \quad \text{where} \quad \kappa_j = i \iff f^j(c) \in D^f_i.
\]
The kneading word does not depend on the representative of \([f, c, c']\) and on the choice of \(U\). Always \(\kappa_n = 0\) since the period of \(c\) is \(n\). The **kneading sequence** of \(f\) is the element of \(\{0, 1\}^\mathbb{N}\) obtained as the infinite repetition of the kneading word. When \(f\) is a polynomial this agrees with the definition introduced by Milnor.

The unramified preimage of the branch point \(f(c)\) (resp. \(f(c')\)) is called the **co-ramification point of \(c\)** (resp. \(c'\)). Note that the co-ramification point of \(c\) always lies in \(D^f_1\).

**Proof of Lemma 3.1.** The Böttcher map furnishes a local conjugacy \(\phi\) between \(f\) and \(z \mapsto z^3\) near infinity (e.g. see [8, §9]). That is, there exist a punctured neighborhood \(V\) of \(\infty\) and a conformal isomorphism \(\phi : V \to \{ z \in \mathbb{C} : |z| > r_0 \}\) for some \(r_0 > 1\) such that \(\phi \circ f(z) = \phi(z)^3\) for all \(z \in V\). It follows that
\[
g_f(z) = \log |\phi(z)|.
\]
Therefore \(\{ z : g_f(z) = r \}\) is a Jordan curve around \(\infty\) for all \(r > r_0\).

Given \(r > 0\), let \(V_r = \{ z : r < g_f(z) < +\infty \}\) and observe that \(f : V_r \to V_{3r}\) is a branched covering of degree 3. For all \(r > 0\), it follows that \(V_r\) is connected and \(\mathbb{C} \setminus V_r\) is a disjoint union of Jordan domains.

Now if \(r > g_f(c')\), then the map \(f : V_r \to V_{3r}\) is branched point free and thus it is a topological covering of degree 3. Hence, we deduce that \(V_r\) is a disk punctured at infinity and bounded by a Jordan curve. Therefore \(D\) is a Jordan domain. By the Riemann-Hurwitz formula, the Euler characteristic of \(f^{-1}(D)\) is 2, since \(f : f^{-1}(D) \to D\) has exactly one ramification point \(c\). Hence \(f^{-1}(D)\) is the union of two disjoint disks \(D^f_0\) and \(D^f_1\), each one mapping onto \(D\). \(\square\)

**Proposition 3.2.** The kneading sequence is constant on each connected component of \(\mathcal{E}(\mathcal{F}_n)\).

In order to prove the proposition we first establish the continuous dependence of \(g_f\) on \(f\):

**Lemma 3.3.** Let \(F_n\) be the subspace of \(B_n\) formed by all \((f, c, c') \in B_n\) such that \(f\) is holomorphic near infinity. Given \([[(f_0, c_0, c'_0)] \in \mathcal{E}(\mathcal{F}_n)\), there exists a neighborhood \(V\) of \((f_0, c_0, c'_0)\) in \(F_n\) such that if \((f, c, c') \in V\) then \([[(f, c, c')] \in \mathcal{E}(\mathcal{F}_n)\) and \(g_f : \overline{\mathbb{C}} \to [0, +\infty]\) depends continuously on \((f, c, c') \in F_n\).

**Proof.** It is easy to check that the conjugacy class \([[(f, c, c')]\) lies in \(\mathcal{E}(\mathcal{F}_n)\) for all \((f, c, c')\) in an open neighborhood \(V \subset F_n\) of \((f_0, c_0, c'_0)\). Thus, \(g_f : \overline{\mathbb{C}} \to [0, +\infty]\) is well defined in \(V\).

The continuous dependence of the Böttcher map on holomorphic functions with a super-attracting fixed point at \(\infty\) shows that, after shrinking \(V\) if necessary, there exists a neighborhood \(U\) of \(\infty\) in \(\overline{\mathbb{C}}\) such that \(g_f : U \to [0, +\infty]\) is well defined and depends continuously on \((f, c, c') \in V\). Then, the functional relation
\[
g_f(z) = \frac{g_f(f(z))}{3}
\]
spreads the continuous dependence of \(g_f : U \to [0, +\infty]\) to the basin of infinity under iterations of \(f_0\).

Now the continuous dependence extends to \(\overline{\mathbb{C}}\) as follows. Given \(\varepsilon > 0\) small and a compact set \(X\) such that \(g_{f_0}(X) \subset [0, \varepsilon]\) it is sufficient to show that for \(f\) close
to \( f_0 \), also \( g_f(X) \subset [0, \varepsilon[. \) In fact, since \( X \) is compact, we may choose \( 0 < \lambda < 1 \) such that \( g_{f_0}(X) \subset [0, \varepsilon \lambda[. \) Now let \( N \) be large enough and \( f \) be sufficiently close to \( f_0 \) such that \( g_{f_0}(f_0^n(z)) < 3^\lambda \varepsilon \lambda < 3^N \varepsilon \lambda' \) for some \( \lambda < \lambda' < 1 \) and such that this latter inequality implies that \( g_f(f^N(z)) < 3^N \varepsilon \), equivalently \( g_f(z) < \varepsilon \). Hence, \( g_f(X) \subset [0, \varepsilon[ \) as desired. \( \square \)

**Proof of Proposition 3.2.** As the kneading word lies in a discrete set, it is sufficient to prove that it is locally constant in \( \mathcal{E}(\mathcal{F}_n) \). Consider \( [(f_0, c_0, c'_0)] \in \mathcal{E}(\mathcal{F}_n) \) and \( 1 \leq m \leq n - 1 \). Assume that \( f_0^m(c_0) \in D_i^0 \) for \( i = 0 \) (resp. \( i = 1 \)). Denote by \( c_0 \) the co-ramification point of \( c_0 \) and recall that \( c_0 \in D_i^0 \). Let \( X \subset D_i^0 \) be a compact connected set containing \( f_0^m(c_0) \) for all \( m \). For some \( \varepsilon > 0 \) small, \( g_{f_0}(z) < g_{f_0}(c'_0) - \varepsilon \), for all \( z \in X \). Hence, there exists a neighborhood \( W \) of \( (f_0, c_0, c'_0) \) in \( \mathcal{E}_n \) such that \( (f_1, c_1, c'_1) \subset W \), then \( f_1^n(c_1) \subset X \) is contained in \( D_i^1 \) and \( \kappa_{f_0}^m = \kappa_{f_0}^m \). \( \square \)

4. **Twisting path.** This section is devoted to the construction of a “twisting” path with the desired effect on the kneading word:

**Proposition 4.1.** Let \( m \) be such that \( 1 \leq m \leq n - 1 \). Given \( [f_0] \in \mathcal{E}(\mathcal{G}_n) \) with kneading word \( \kappa([f_0]) = \kappa_1 \ldots \kappa_{n-1}0 \), there exists \( [f_1] \in \mathcal{E}(\mathcal{F}_n) \) with kneading word \( \kappa([f_1]) = \kappa'_1 \ldots \kappa'_{n-1}0 \) such that the following hold:

1. There exists a path in \( \mathcal{B}_n \) with endpoints \( [f_0] \) and \( [f_1] \).
2. \( \kappa'_j = \begin{cases} \kappa_j & \text{if } j \neq m, \\ 1 - \kappa_m & \text{if } j = m. \end{cases} \)

The rest of this section is devoted to the proof of the proposition which consists first on constructing a path in \( \mathcal{B}_n \) (the twisting path) and then proving that the endpoint of this path has the desired kneading word.

4.1. **Construction of the twisting path.** Consider \( f_0 \) and \( m \) as in the statement of the proposition. First we will construct an appropriate loop in the dynamical space of \( f_0 \) and then we will thicken the loop to an annulus where we will postcompose \( f_0 \) by twists to obtain a twisting path.

Consider \( [(f_0, c_0, c'_0)] \in \mathcal{B}_n \) such that \( [f_0] \in \mathcal{E}(\mathcal{G}_n) \). Let \( \mathcal{O}(c_0) \) denote the orbit of the periodic ramification point. Recall that \( D = \{ z \in \mathcal{C}: g_{f_0}(z) < g_{f_0}(c'_0) \} \) and let \( D_0, D_1 \) be the connected components of \( f_0^{-1}(D) \). We follow the notation of Lemma 3.1. That is, \( D_0 \) contains the ramification point \( c_0 \) and \( D_1 \) contains the co-ramification point \( c_0 \).

**Step 1.** Construction of a twisting loop. This step consists of the construction of a suitable loop \( \tau \) surrounding \( f_0^{m+1}(c_0) \) in \( D \). More precisely a twisting loop \( \tau: [0, 1] \to \mathcal{C} \) is a continuous function such that all of the following statements hold:

1. \( \tau(0) = \tau(1) = f_0(c'_0) \).
2. \( \tau([0, 1]) \subset D \setminus \mathcal{O}(c_0) \).
3. \( \tau \) bounds a Jordan domain \( V \subset D \) such that \( V \cap \mathcal{O}(c_0) = \{ f_0^{m+1}(c_0) \} \).
(L4) The connected component of \( f^{-1}_0(\tau) \) containing \( c'_0 \) bounds two Jordan domains \( V'_0 \subset D_0 \) and \( V'_1 \subset D_1 \) with \( f^{-1}_0(c_0) \in V'_0 \cup V'_1 \).

Such a path \( \tau \) exists. Indeed, let \( Y : [0, 1] \to \overline{D} \) be an arc such that \( Y([0, 1]) \subset D \setminus \mathcal{O}(c_0) \) with initial point \( f_0(c'_0) \) and endpoint \( f_0(c_0) \). Abusing of notation, let \( Y = Y([0, 1]). \) The set \( f^{-1}_0(Y) \) separates \( f^{-1}_0(D) \) into 3 connected components (cf. Figure 2), each mapping bijectively onto \( D \setminus Y \). Let \( U \) denote the one containing \( f^{-1}_0(c_0) \). Let \( \tau' \) be a loop with endpoints at \( c'_0 \) such that \( \tau'([0, 1]) \subset U \setminus \mathcal{O}(c_0) \) and \( \tau' \) bounds a Jordan domain \( V' \) with \( V' \cap \mathcal{O}(c_0) = \{ f^{-1}_0(c_0) \} \) (cf. Figure 2). The path \( \tau := f_0 \circ \tau' \) and the domain \( V := f_0(V') \) satisfy the desired conditions.²

**Figure 2.** Illustration of the construction of the twisting loop corresponding to \( m = 3 \) and kneading word 1000. The exterior curve in black is the level curve \( g_{f_0} = g_{f_0}(c'_0) \). The set \( f^{-1}_0(Y) \) is drawn in gray.

**Step 2.** Twisting. Thicken \( \tau([0, 1]) \) in order to obtain an open annulus \( A \) bounded by Jordan curves such that:

(A1) \( A \subset f_0(D) \).

(A2) \( A \) is disjoint from \( \mathcal{O}(c_0) \).

(A3) \( D \setminus (A \cup V) \) is connected.

**Figure 3.** Illustration of the annulus \( A \) around the twisting loop \( \tau \) (left) and its preimage (right).

²The choice of different orientations for \( \tau([0, 1]) \) may give different twisting paths but does not affect the results in Proposition 4.1.
Observe that each component of $\partial A$ is homotopically equivalent to $\tau([0,1])$ rel $\mathcal{O}(c_0)$. Moreover, $f_0^{-1}(A)$ has exactly two connected components, one containing $c_0'$ and the other containing the corresponding co-ramification point $co_0'$ (see Figure 3).

The loop $\tau$ cuts $A$ into two sub-annuli $A_{ext} := A \setminus \overline{V}$ and $A_{int} := A \cap V$. For $t \in [0,1]$, let $T_t : \mathbb{C} \to \mathbb{C}$ be a continuous family of quasiuniversal maps such that all of the following hold:

(T1) $T_t(\tau(s)) = \tau(s - t \mod 1)$ for all $s \in [0,1]$.
(T2) $T_0 = \text{id}_A$ and $T_t$ is the identity on $\mathbb{C} \setminus A$.
(T3) $T_1$ is the inverse of a Dehn twist in $A_{ext}$ and a Dehn twist in $A_{int}$.

Now we may introduce the twisting path as

$$(f_t, c_t, c'_t) := (T_t \circ f_0, c_0, c'_0)$$

where $t \in [0,1]$.

The action of $f_t$ on the periodic orbit $\mathcal{O}(c_0)$ remains unchanged for all $t$. Thus $(f_t, c_t, c'_t) \in B_n$. Also, $f_0(c'_0) = f_1(c'_1)$ and $f_0(z) = f_1(z)$ for all $z \notin f_0(D)$ and all $t$. Since $f_0'(c'_0) \notin f_0(D)$ for all $k \geq 2$, the orbit of $c'_0$ under $f_0$ and the one of $c'_1$ under $f_1$ coincide. Moreover, $f_1'(c'_1) \to +\infty$ and $f_1$ is a semi-polynomial. That is, $[f_1] \in \mathcal{E}(\mathcal{F}_n)$.

4.2. Kneading after twisting. In order to finish the proof of Proposition 4.1, our aim now is to determine the kneading word of the endpoint $f_1$ of the twisting path constructed above.

By (A1) and (T2), $f_1(z) = f_0(z)$ for all $z \notin D$. In particular, $g_{f_1}(z) = g_{f_0}(z)$ for all $z \notin D$. In particular, $\partial D$ is the $g_{f_1}$-level curve containing the branched value $f_1(c'_1) = T_1 \circ f_0(c'_0) = f_0(c'_0)$. Therefore, in order to determine the kneading word of $f_1$ we have to analyze the location of the orbit of the periodic ramification point in $f_1^{-1}(D) = D_0^1 \cup D_1^1$. (See Figure 4.)

By (A3) the set $E := D \setminus (A \cup V)$ is simply connected. Moreover, by (T2), the maps $f_0$ and $f_1$ agree on the set $f_1^{-1}(E) = f_0^{-1}(T_1^{-1}(E)) = f_0^{-1}(E)$. Since $f_0(c_0) \in E$, we have that $f_0^{-1}(E)$ consists of two connected components $E_0 \subset D_0^0$ and $E_1 \subset D_1^0$ each mapping onto $E$. Since $c_0 = c_0' \in E_0$ we have that $E_0 \subset D_0^1$ and $E_1 \subset D_1^1$. In view of (L3) and (A2), with the exception of $f_0^m(c_0)$ all the periodic critical orbit is in $E_0 \cup E_1$, therefore

$$\kappa'_i = \kappa_i \quad \text{for all} \quad i \neq m.$$  

As in (L4) a connected component of $f_0^{-1}(V)$ with $c'_0$ in its boundary is denoted by $V'_0$ or $V'_1$ according to whether it is contained in $D_0^1$ or in $D_1^0$. Observe that $V'_0$ and $V'_1$ are still contained in $f_1^{-1}(D)$ since

$$f_1(V'_i) = T_1(f_0(V'_i)) = T_1(V) = V \subset D.$$  

However, we will show that $V'_{i-1} \subset D_i^1$ for $i = 0,1$. That is, we will show that the elements of $V'_i$ “switch” label in $\{0,1\}$.

For $t \in [0,1]$, let $\overline{T}_t : \mathbb{C} \to \mathbb{C}$ be the lift of $T_t$ by $f_0$ which is the identity in $\mathbb{C} \setminus D$. That is:

$$T_t \circ f_0 = f_0 \circ \overline{T}_t,$$

and

$$\overline{T}_1(z) = z \quad \text{for all} \quad z \in \mathbb{C} \setminus D.$$  

Denote by $A'_{ext}$ the component of $f_0^{-1}(A_{ext})$ surrounding $V'_0$ and $V'_1$. Consider an arc $\gamma : [0,1] \to A'_{ext}$ in $D$ starting at the inner boundary of $A_{ext}$ (i.e. $\partial V$) and
ending at the outer boundary of $A_{\text{ext}}$. Since $f_0 : A'_{\text{ext}} \to A_{\text{ext}}$ is a regular covering of degree 2, the preimage of $\gamma$ under $f_0$ consists of two arcs (see Figure 4). We denote by $\gamma_0$ the one starting at $\partial V'_0$ and by $\gamma_1$ the one starting at $\partial V'_1$. The lift $\widetilde{T}_1 : A'_{\text{ext}} \to A'_{\text{ext}}$ is a half twist and therefore $\widetilde{T}_1^{-1}(\gamma_i(0)) = \gamma_{1-i}(0) \in \partial V'_{1-i}$. As $f_0(\gamma_i) = f_0 \circ \widetilde{T}_1(\widetilde{T}_1^{-1}(\gamma_i)) = f_1(\widetilde{T}_1^{-1}(\gamma_i))$, we deduce that

$$\widetilde{T}_1^{-1}(\gamma_i) \subset f_1^{-1}(D).$$

That is, $\widetilde{T}_1^{-1}(\gamma_i)$ connects $\partial V'_{1-i}$ to $\widetilde{T}_1^{-1}(\gamma_i(1)) = \gamma_{1-i}(1) \in E_i \subset D_1^{F_i}$ within $f_1^{-1}(D)$ which implies that $V'_{1-i} \subset D_1^{F_i}$. From (L4), $f_0^m(c_0) \in V'_0 \cup V'_1$ and it follows that

$$\kappa_m = 1 - \kappa_m.$$

Thus we have completed the proof of Proposition 4.1.

5. The geometrization path. The aim of this section is to show that the endpoint of the twisting path can be joined through an appropriate path to a polynomial in $\mathcal{E}(S_n)$. With the same effort we establish a slightly more general result:

**Theorem 5.1.** Every path connected component of $\mathcal{E}(F_n)$ contains an element of $\mathcal{E}(S_n)$.

Given an element of $\mathcal{E}(F_n)$, a path within $\mathcal{E}(F_n)$ joining it to an element of $\mathcal{E}(S_n)$ will be called a **geometrization path**. The endpoint of such a geometrization path will be obtained from a Theorem by Cui and Tan [6]. Cui-Tan’s Theorem and its prerequisites are discussed in Section 5.1. To apply this theorem we rule out the presence of Thurston obstructions in Section 5.2. To obtain the geometrization path itself we show that $c$-equivalence classes, defined in Section 5.1, are path-connected. In Section 5.3 we combine the above ingredients to prove Theorem 5.1. In Section 5.4 we obtain the following:

**Corollary 5.2.** Every connected component of $\mathcal{B}_n$ contains an element of $\mathcal{E}(S_n)$.
5.1. Cui-Tan Theorem. In order to state the aforementioned result by Cui and Tan we need to introduce the notions of $c$-equivalence and Thurston obstructions for semi-rational maps (see Section 2.2).

Following [6], we say that two semi-rational maps $f$ and $g$ are $c$-equivalent if there exist homeomorphisms $\varphi, \psi : \mathbb{C} \to \mathbb{C}$ and a neighborhood $U$ of $P_f$ such that all of the following statements hold:

- $\varphi \circ f = g \circ \psi$.
- $\varphi$ is holomorphic in $U$.
- $\varphi$ is isotopic to $\psi$ relative to $U \cup P_f$.

Möbius conjugate semi-rational maps are $c$-equivalent. Hence, the notion of $c$-equivalence is well defined in $\mathcal{E}(\mathcal{F}_n)$. That is, two elements of $\mathcal{E}(\mathcal{F}_n)$ are said $c$-equivalent if they can be represented by $c$-equivalent semi-polynomials.

Consider a semi-rational map $f : \mathbb{C} \to \mathbb{C}$. We say that a Jordan curve $\gamma \subset \mathbb{C} \setminus P_f$ is non-peripheral if each one of the two disks in $\mathbb{C}$ bounded by $\gamma$ contains at least two elements of $P_f$.

A pairwise disjoint and pairwise non-homotopic finite collection $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$ of non-peripheral Jordan curves $\gamma_j \subset \mathbb{C} \setminus P_f$ is called a multicurve in $\mathbb{C} \setminus P_f$.

Given a multicurve $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$ we say that $\Gamma$ is a $f$-stable multicurve if, for all $j = 1, \ldots, k$, each connected component of $f^{-1}(\gamma_j)$ either fails to be non-periodic or is homotopic in $\mathbb{C} \setminus P_f$ to some $\gamma_i \in \Gamma$.

The transition matrix $W_\Gamma$ associated to a $f$-stable multicurve $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$ is the $k \times k$ non-negative matrix $(a_{ij})$ with rational entries:

$$a_{ij} = \sum_{\alpha} \frac{1}{\deg(\alpha \to \gamma_j)}$$

where the sum is taken over all connected components $\alpha$ of $f^{-1}(\gamma_j)$ such that $\alpha$ is homotopic in $\mathbb{C} \setminus P_f$ to $\gamma_i$.

A $f$-stable multicurve is called a Thurston obstruction for $f$ if the leading eigenvalue $\lambda$ of $W_\Gamma$ is such that $\lambda \geq 1$.

**Theorem 5.3** (Cui and Tan). Let $f_0 : \mathbb{C} \to \mathbb{C}$ be a semi-rational map. Then, $f_0$ is $c$-equivalent to a rational map $f_1$ if and only if $f_0$ has no Thurston obstruction. In this case, $f_1$ is unique up to Möbius conjugacy.

5.2. No Thurston obstructions. A straightforward application of the ideas introduced by Levy [7] to show that Thurston obstructions in polynomial dynamics contain “Levy cycles” allows us to establish the following:

**Proposition 5.4.** If $[(f, c, c')] \in \mathcal{E}(\mathcal{F}_n)$ is a class of semi-polynomials then $f$ has no Thurston obstruction.

**Proof.** Let $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$ be a non-empty $f$-stable multicurve, and let $W_\Gamma = (a_{ij})$ be the corresponding transition matrix. If $W_\Gamma = 0$ then $\Gamma$ is not a Thurston obstruction. We may assume that $\Gamma$ is minimal in the sense that for all $i$ there exists $j$ such that $a_{ij} \neq 0$, for otherwise we could erase $\gamma_i$ from $\Gamma$.

For $1 \leq i \leq k$, let $U_i$ be the bounded component of $\mathbb{C} \setminus \gamma_i$. We prove that

$$f^m(c') \notin U_i$$

for all $m \geq 1$ and for all $i$. For otherwise, there would exist $U_i$ and $m \geq 1$ such that $f^m(c') \in U_i$ and $f^{m+1}(c') \notin U_j$ for all $j = 1, \ldots, k$. Then, for all $j$ and for
Lemma 5.7. The set formed by the homeomorphisms \( \varphi : (\overline{\mathbb{C}}, 0) \to (\overline{\mathbb{C}}, 0) \) holomorphic in a neighborhood of 0, endowed with the uniform convergence topology, is path connected.
Proof. Consider such a homeomorphism \( \varphi \). Without loss of generality, we may assume that \( \varphi'(0) = 1 \).

For \( t \in [0, 1] \) and \( r \geq 0 \) let

\[
\rho_t(r) = \begin{cases} 
tr & \text{if } 0 \leq r < 1, \\
(2 - t)r - 2 + 2t & \text{if } 1 \leq r \leq 2, \\
r & \text{if } 2 < r,
\end{cases}
\]

be a linear interpolation between \( r \mapsto tr \) and the identity. Let \( h_t : (\mathbb{T}, 0) \to (\mathbb{T}, 0) \) be given by

\[ h_t(re^{i\theta}) = \rho_t(r)e^{i\theta}. \]

For \( t \in [0, 1] \), it follows that \( \varphi_t = h_t^{-1} \circ \varphi \circ h_t : (\mathbb{T}, 0) \to (\mathbb{T}, 0) \) is a continuous family of homeomorphisms holomorphic in a neighborhood of 0. Moreover, \( \varphi_t \) converges uniformly, as \( t \to 0 \), to a homeomorphism \( \varphi_0 : (\mathbb{T}, 0) \to (\mathbb{T}, 0) \) which is the identity on the unit disk. Now \( \varphi_0 \) is isotopic to the identity rel the closed unit disk by Alexander’s trick (cf. [1]). \( \square \)

5.4. Proof of Corollary 5.2. In view of Theorem 5.1, given an element \((f, c, c')\) of \( B_n \) it is enough to show that the connected component of \( B_n \) containing \([f, c, c']\) also contains elements of \( \mathcal{E}(\mathcal{F}_n) \). We may assume that \( c' \neq f(c') \). By definition of branch points, near infinity, there are local homeomorphisms \( \varphi, \psi \) such that \( \varphi \circ f \circ \psi(z) = z^3 \). We may assume that \( \varphi \) and \( \psi \) extend to topological disks so that restricted to their boundaries are the identity. If follows that after isotopies from \( \varphi \) and \( \psi \) to the identity we may assume that \( (f, c, c') \) is \( z^3 \) in \( U = \mathbb{T} \setminus \{|z| \leq R\} \) for \( R \) large enough. Now consider a path \( \delta : [0, 1] \to \{|z| < R^3\} \) joining \( f(c') \) with a point \( v \) in the fundamental annulus \( R \leq |z| < R^3 \). Choose \( \delta \) so that it avoids \( c' \) and the orbit of \( c \). Consider a small neighborhood \( V \) of \( \delta([0, 1]) \) and let \( h_t : \mathbb{T} \to \mathbb{T} \) be an isotopy so that \( h_0 \) is the identity, \( h_t \) is the identity in \( \mathbb{T}\setminus V \) and \( h_t(c') = \delta(t) \), for all \( t \). It follows that \( (h_t \circ f, c, c') \in B_n \) is a path joining \((f, c, c')\) with the \((h_1 \circ f, c, c')\). Note that \( h_t \circ f \) is a semi-polynomial map since the orbit of \( v = h_1 \circ g(c') \) converges to infinity. Moreover, \([h_1 \circ f, c, c']\) is in the connected component of \( B_n \) containing \([f, c, c']\). \( \square \)

6. Proof of the Main Theorem.

Proof. The idea is to show that all the escape regions of \( \mathcal{S}_n \) lie in the same connected component of \( B_n \). Consider a class of polynomials \([f]\) in an escape region \( \mathcal{U} \subset \mathcal{S}_n \) with kneading word \( \kappa^f = \kappa_1 \ldots \kappa_{n-1} 0 \) such that \( \kappa_m = 0 \) for some \( m \in \{1, \ldots, n-1\} \).

In view of Proposition 4.1, via twisting, there exists a path in \( B_n \) joining \([f]\) and some \([f_1] \in \mathcal{E}(\mathcal{F}_n) \) with kneading word obtained from \( \kappa^f \) by replacing \( \kappa_m = 0 \) with 1. Now according to Theorem 5.1, there exists a geometrization path in \( \mathcal{E}(\mathcal{F}_n) \subset B_n \) from \([f_1]\) to some \([f_2] \in \mathcal{E}(\mathcal{S}_n) \). By Proposition 3.2, the kneading word of \([f_2]\) agrees with the one of \([f_1]\).

By iterating this process a finite number of times we conclude that \( \mathcal{U} \) is in the same connected component of \( B_n \) as the unique distinguished escape component with kneading word \( 1^n - 1 0 \) (cf. [9, Lemma 5.17] for the uniqueness). Hence all the escape regions of \( \mathcal{S}_n \) are in the same connected component of \( B_n \). It follows that \( B_n \) is connected since in view of Corollary 5.2 every connected component of \( B_n \) contains elements of \( \mathcal{E}(\mathcal{S}_n) \). \( \square \)
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