EFFECTIVE DYNAMICS ON A LINE

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Abstract

The effective classical/quantum dynamics of a particle constrained on a closed line embedded in a higher dimensional configuration space is analyzed. By considering explicit examples it is shown how different reduction mechanisms produce unequivalent dynamical behaviors. The relation with a formal treatment of the constraint is discussed. While classically it is always possible to strictly enforce the constraint by setting to zero the energy stored in the motion normal to the constraint surface, the quantum description is far more sensitive to the reduction mechanism. Not only quantum dynamics is plagued by the usual ambiguities inherent to the quantization procedure, but also in some cases the constraint’s equations do not contain all the necessary information to reconstruct the effective motion.

In this paper we would like to discuss a few aspects of reduction of particles motion from a higher dimensional configuration space to a line. This problem –as the more general one of reducing on a submanifold of arbitrary codimension– appears in physics in the most different contexts. From the general theory of constrained systems in mathematical physics, to the construction of confining devices in solid states and plasma physics, to the analysis of dynamics around solitonic solutions in non-linear field theories, to more speculative applications in attempts to unification of fundamental interactions. From the viewpoint of the theory of constrained systems, the problem is trivially solvable –up to ambiguities inherent to the quantization procedure [1]. At the classical level one proceeds by adapting coordinates and imposes constraints by freezing motion in directions normal to the constraint surface. If the system is not subject to forces tangent to the constraint, the procedure yields free dynamics. For a line we trivially obtain $\mathcal{H}_{\text{cons}} = \frac{1}{2} p_{\xi}^2$ in the arc-length parameterization $\xi$. The quantum mechanical problem is slightly more complicated. We may proceed by reducing the classical theory and quantizing the corresponding Dirac brackets or by quantizing the whole theory and imposing constraints as functional conditions on quantum states. The two strategies turn out to be equivalent, yielding the classically expected result plus a quantum gauge connection $A$ and a quantum potential $Q$ depending on the quantization prescriptions, $\mathcal{H}_{\text{cons}} = \frac{1}{2} (-i \partial_{\xi} - eA)^2 + Q$. 

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On the other hand, from a practical viewpoint, effective motion on a submanifold of the original configuration space is dynamically produced in a variety of different physical systems. The corresponding effective dynamics turns in general to be more rich than the one obtained by Dirac’s algorithm. The character depends strongly on the peculiar reduction mechanism. We are going to illustrate this—and eventually its relation with the theory of constrained systems—by considering three different mechanisms. The first one is relevant in a variety of contexts, its most popular application being perhaps in solid state physics [3]. The second one is borrowed from plasma physics [4] and the third one form high energy physics [5]. We consider a line embedded in an \((n+1)\)-dimensional manifold \(M\). In order to retain the maximal amount of information on the induced dynamics we close the line into a loop \(L\). This makes it possible to consider non-trivial gauge interactions in one dimension. Classical/quantum dynamics on \(M\) is tentatively described by a Lagrangian quadratic in the velocities

\[
\mathcal{L} = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j + A_i \dot{x}^i + V
\]

Three kinds of interactions act on the particle

- a gravitational-like force described by a non-trivial metric \(g_{ij}(x)\)
- a magnetic-like force described by a closed antisymmetric two-form \(B_{ij}(x)\) or by the associated vector potential \(A_i(x)\)
- a scalar potential \(V(x)\)

Under appropriate conditions each one of these interactions may produce an effective one dimensional dynamics. Dimensional reduction produced by a nontrivial metric (topology) goes back to the ideas of Kaluza and Klein [5]. Variation on this theme are still a key ingredient in todays attempts to unification [6]. Confinement by a magnetic field was first considered by Alfvén in the fifties and is part of everyday work for plasma physicists [4]. The scalar potential mechanism is used in the construction of 2d quantum Hall devices and is relevant in many other applications [3].

By reconsidering these mechanisms in a single prospective we show how different dynamical behaviors are produced by different reduction procedures. Of course, all these models have been extensively studied in the literature. We concentrate on rather unconventional and sometimes unexplored aspects. Our focus is on the geometry of the reduction mechanism.

1. Reducing by a scalar potential

The first model we consider is obtained by enforcing the constraint by a scalar potential. This mechanism is used in the construction of 2d quantum Hall devices and finds interesting application in molecular and chemical physics [3]. In the context of unification

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\(^1\) The restriction of considering an effective one-dimensional non-relativistic dynamics is not essential. The effective motion on a line retains all the features of the reduction mechanism, avoiding complications produced by a non-trivial intrinsic geometry and the corresponding heaviness in the notation. Moreover, the reduction mechanisms we consider depend only on the spatial structure and not on time, making the relativistic and non-relativistic problems essentially analogous. The generalization to an arbitrary codimension and relativistic dynamics is almost straightforward.
of fundamental interaction it is somehow related to the Rubakov-Shaposhnikov model \([4]\) which has recently attracted new attention in the literature \([8, 9]\). The reduction on a line by a potential has been discussed over the years by many authors \([10, 11]\) and presented in a final form by Takagi and Tanzawa \([11]\). We consider a particle moving in the Euclidean space \(\mathcal{M} = R^{n+1}\) – we first focus on the case \(n = 2\) – under the action of a potential \(V(x)\) satisfying the following conditions: i) \(V\) presents a deep minimum in correspondence of a loop \(L\) and ii) \(V\) depends only on the distance from the loop

\[
\mathcal{L} = \frac{1}{2} \dot{x}^i \dot{x}^i - V \tag{1.1}
\]

The particle experiences a force attracting it toward the loop \(L\). No forces act in the tangent direction. A very deep minimum of the potential \(V(x)\) traps the particle in a narrow neighborhood of the line producing an effective one dimensional motion. In order to find the explicit form of the induced dynamics we proceed by adapting coordinates. Denoting by \(t\) the tangent vector and introducing in every point an orthonormal frame \(\{n_1, n_2\}\) of the normal space, we consider the coordinate transformation \(x \to (\xi, \nu^a; a = 1, 2)\): \(\xi\) is the arc-length on \(L\) measured from some reference point and \(\nu_1, \nu_2\) are the distances along the geodesics leaving the line with velocity \(n_1\) and \(n_2\) respectively. In this coordinate frame the flat \(R^3\) metrics rewrites as

\[
ge_{ij} = \begin{pmatrix}
(1 - \kappa_a \nu^a)^{1/2} + \nu^2 \tau^2 & \tau \varepsilon_{bc} \nu^c \\
\tau \varepsilon_{ac} \nu^c & \delta_{ab}
\end{pmatrix}
\tag{1.2}
\]

where \(\nu = \sqrt{\nu_1^2 + \nu_2^2}\) is the distance from the line, \(\kappa_a = n_a \cdot \nabla_t t\) are the extrinsic curvatures of the line and \(\tau \varepsilon_{ab} = n_a \cdot \nabla_t n_b\) is its generalized torsion \([4]\). Observe that a rotation of the normal frame \(\{n_1, n_2\}\) by a point dependent angle \(\chi(\xi)\) produce \(\tau\) to transform as a \(SO(2)\) gauge connection, \(\tau \to \tau + \partial_\xi \chi\).

Quantum dynamics is described by the univocally defined Hamiltonian \(\mathcal{H} = -\frac{1}{2} \partial_t \partial_\xi + V(\nu)\). To enforce the constraint we expand \(V\) around its minimum keeping only the quadratic term: \(V(\nu) = \nu^2/2\epsilon^2\). \(\mathcal{H}\) expands then in a power series in the small parameter \(\epsilon\). The first term of the expansion is an harmonic oscillator in the normal variables and diverges as \(\epsilon^{-1}\). It represents the fluctuations of the particle when squeezed on the constraint surface. The subsequent term is independent on \(\epsilon\) and have to be identified with the Hamiltonian describing the effective motion along the line. Freezing the normal oscillation in an eigenstate, ordinary perturbation theory or averaging techniques are used to separate the dynamics along the line from the one in the normal directions. The procedure yields \([11]\)

\[
\mathcal{H}_{\text{eff}}^V = \frac{1}{2}(-i \partial_\xi - e \tau)^2 + Q^V \tag{1.3}
\]

The effective dynamics is by no means free. The particle experiences a coupling with a gauge field and a univocally defined quantum potential. The gauge connection is proportional to the torsion of the line and survives in the classical limit. The coupling constant \(e\) corresponds to the angular momentum stored in the normal oscillations and is quantized in integer multiplies of \(\hbar\). The gauge group \(SO(2)\) correspond to the group

\(\text{When } n_1\text{ and } n_2\text{ are chosen as the normal } n\text{ and the binormal } b\text{ to the line, } k_1\text{ correspond to the curvature } k, k_2\text{ is zero and } \tau\text{ equals the torsion } t\). In a generic frame \(k = k = \sqrt{k_1^2 + k_2^2}\) and \(t = \tau - \partial_\xi \chi, \chi(\xi)\) being the angle between \(\{n_1, n_2\}\) and \(\{n, b\}\).
of the normal bundle of the line. Generalizing to an arbitrary codimension \( n \) we obtain in fact the gauge group \( SO(n) \). This is broken in the direct product of orthogonal groups of lower dimensionality if the potential is not completely symmetric –i.e. does not depend only on the distance from the line [12]. The scalar interaction is of pure quantal nature and is proportional to the extrinsic curvature of the line, \( Q^V = -\kappa^2/8 \). Thought not deeply of a geometrical nature, this model is substantially equivalent to the Rubakov-Shaposhnikov unification scenario which has recently attracted new attention [4,8,9]. An interesting feature of the model is that it is possible to generate the grand-unified group \( SO(10) \) in a very natural manner.

2. Reducing by a vector potential

We next consider dimensional reduction produced by a magnetic-like force \( B_{ij} \). This mechanism is commonly employed in plasma physics in confining charged particles inside mirror machines and is based on the so called guiding center approximation [4]. First consider a charged particle moving in a homogeneous magnetic field of strength \( B \). Its trajectory is an helix of radius \( \simeq 1/B \) wrapping around a straight field line. The particle performs a fast rotation in the plane normal to the field and propagates freely along the magnetic line. The conservation of the guiding center position –the center of the circular orbit– prevents the particle from drifting in the directions normal to the line. In the strong field limit the rotational motion becomes undetectable and the particle behaves as effectively confined on the straight field line. Next consider the motion in an inhomogeneous field \( \vec{B}(x) \). Provided the field is strong enough, the motion still decomposes on three different energy scales. A fast rotation of radius \( \simeq 1/|\vec{B}(x)| \) and energy \( \simeq |\vec{B}(x)| \), a drift along the field lines with every of order one and a very slow drift in the directions normal to the field with energy \( \simeq 1/|\vec{B}(x)| \). The conservation of the guiding center position is in general broken. When the magnetic field norm is a constant however, \( |\vec{B}(x)| = 1/\epsilon \), it is still possible to consider the formal limit \( \epsilon \to 0 \) and trap the particle on a magnetic field line.

From a geometrical point of view this model is more appealing than the previous one. An antisymmetric two-form \( B_{ij} \) is a quite common object in geometry and is generally introduced to describe a complex structure on a manifold [13]. In that context however \( B_{ij} \) is required to be non-degenerate while the degenerate directions of the magnetic field are precisely the submanifold on which we are reducing dynamics. We therefore describe the configuration space of the system as a manifold having a somehow mixed real and complex structure. We introduce a \( 2m+1 \) dimensional manifold \( \mathcal{M} \)–to start with we set \( m = 1 \)– endowed with a Riemannian metric \( g_{ij} \) and a closed antisymmetric two-form \( B_{ij} \) of rank \( 2m \). We further assume the norm \( \sqrt{B_{ij}B^{ij}} = 1/\epsilon \) to be constant. This represents a geometry having one real and \( m \) complex directions. The real direction closes up in loops \( L \). Dynamics on \( \mathcal{M} \) is free in the sense of this ‘half-real/half-complex’ geometry

\[
\mathcal{L} = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j + A_i \dot{x}^i
\]

where \( A_i \) is the vector potential representing the two-form \( B_{ij} \). We expect the strong field regime to produce an effective one dimensional dynamics along a real direction of \( \mathcal{M} \). In order to obtain the explicit form we proceed again by adapting coordinates. We introduce an Euler-Darboux frame \( x \to (\xi, \delta_a; a = 1, 2) \): \( \xi \) is the arc-length along
every magnetic field line while $\delta_1$, $\delta_2$ are coordinates bringing $B_{ij}$ in canonical form [14]. Metric and magnetic two-form rewrites as

$$g_{ij} = \begin{pmatrix} 1 & a_b \\ a_a & \gamma_{ab} + a_a a_b \end{pmatrix}, \quad B_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & \varepsilon_{ab} \end{pmatrix}$$

(2.2)

where $\varepsilon_{ab}$ is the completely antisymmetric tensor in two dimensions and $a_a, \gamma_{ab}$ may be related to the quantities characterizing the foliation of $\mathcal{M}$ in its real directions: the extrinsic curvatures $\kappa_a$ and the generalized torsion $\tau$ of every line plus the expansion $\theta_{ab}$ and the vorticity $\omega$ defined as the symmetric and antisymmetric part of $t \cdot \nabla_{n_a} n_b$ respectively.

In the adapted frame the Hamiltonian operator $\mathcal{H} = -\frac{1}{2} g^{ij} (\nabla_i - i A_i) (\nabla_j - i A_j) + \alpha R$ defined up to curvature ambiguities expands naturally in powers of $\epsilon$. A quite complex procedure of averaging allows to separate the various freedoms in the first few terms of the expansion [15]. The first term is an harmonic oscillator representing the fast rotation of the system around the line. As in the previous case it diverges as $\epsilon^{-1}$. The second term is of order one and has to be identified with the effective Hamiltonian describing dynamics on the line. Freezing the system in a harmonic oscillator eigenstate we obtain

$$\mathcal{H}^{\mathcal{A}}_{\text{eff}} = \frac{1}{2} (-i \partial_\xi = \epsilon (\tau + \omega))^2 - \epsilon^2 \left( \frac{1}{4} R + \frac{1}{4} \kappa^2 - \frac{1}{2} \theta^2 + \frac{1}{2} \omega^2 \right) + Q^A$$

(2.3)

Once again the effective dynamics is not free. The particle experiences a gauge as well as scalar force. These depend now not only on the extrinsic properties of the constraint surface but also on the geometry of the foliation of the space $\mathcal{M}$ in its real directions. The gauge connection, as an example, is proportional to the sum of the torsion of the line and the vorticity of the foliation along the line. It survives in the classical limit. The coupling constant $\epsilon$ corresponds to the energy stored in the fast rotation around the line. In the quantum description it is always different from zero and is quantized in half-integer multiples of $\hbar$. The gauge group $U(1)$ corresponds again to the symmetry group of the normal double of the constraint surface. In the present model, however, the normal space carries a complex structure, so that the generalization to an arbitrary codimension $2m$ produces the gauge group $U(1) \times SU(m)$. The scalar interaction consists in a part surviving in the classical limit plus a pure quantum contribution $Q^A = (\alpha - \frac{1}{16}) R - \frac{1}{16} \kappa^2 - \frac{3}{16} \theta^2 + \frac{1}{8} \omega^2$. Both depend on the scalar curvature $R$ of $\mathcal{M}$, the extrinsic curvature $\kappa$, the expansion rate $\theta = \sqrt{\theta_{ab} \theta_{ab}}$ and the vorticity $\omega$ of the foliation.

### 3. Reducing by a metric (topology)

Dimensional reduction produced by a metric –generated indeed by a non-trivial topological background– may be obtained by means of the Kaluza-Klein mechanism [5]. Though this procedure is not directly assimilable to the reduction on a subspace of the original configuration space, we briefly review it here for comparison with other models. In order to obtain an effective one dimensional dynamics we start with a configuration space $\mathcal{M} = L \times \Sigma$ given by the direct product of a loop $L$ times a compact manifold $\Sigma$ with a group of isometries $G$. The original Kaluza-Klein mechanism considers $\Sigma$

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3 The orthonormal frame $\{t, n_1, n_2\}$ is constructed along every line $L$ as in the previous paragraph.
to be the circle $S^1$, $G = U(1)$. For now we restrict our attention to this simple case. Dynamics on $M$ is assumed to be free

$$\mathcal{L} = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j$$

(3.1)

The basic idea is that when the size of $\Sigma$ is shrunk to zero the motion along the directions of the compact manifold becomes undetectable and we are left with an effective one-dimensional configurations space. To find out the explicit form of the effective dynamics it is convenient to parameterize the metric in the form

$$g_{ij} = \begin{pmatrix} 1 + a^2/2v & a/2v \\ a/2v & 1/2v \end{pmatrix}$$

(3.2)

The first coordinate $\xi$ is again the arc-length along $L$ while the second coordinate $\sigma$ parameterizes the circle $S^1$. Observe that a $\xi$ dependent translation of $\sigma$, $(\xi, \sigma) \rightarrow (\xi, \sigma - \chi(\xi))$ produces $a$ to transform as a vector potential, $a \rightarrow a + \partial_\xi \chi$. The ‘covariance’ of (3.2) under this transformation indicates that the embedding of $L$ in $M$ is not relevant for this model.

Quantum dynamics on $M$ is described the Hamiltonian operator $\mathcal{H} = -\frac{1}{2} g^{ij} \nabla_i \nabla_j + \alpha R$ –defined up to curvature ambiguities. Dimensional reduction is implemented by expanding $a$ and $v$ in a Fourier series in the compact direction and retaining only the zero order harmonic –basically by averaging over $\sigma$. This produces the effective Hamiltonian

$$\mathcal{H}^{\text{eff}}_{g_{ij}} = \frac{1}{2} (-i \partial_\xi - e \langle a \rangle)^2 + e^2 \langle v \rangle + Q^{g_{ij}}$$

(3.3)

where $\langle a \rangle = \int_{S^1} a(\xi, \sigma) d\sigma / \int_{S^1} d\sigma$ and $\langle v \rangle = \int_{S^1} v(\xi, \sigma) d\sigma / \int_{S^1} d\sigma$ indicates the average of $a$ and $v$ in the compact direction. The particle experiences again a gauge and a scalar interaction. Since the effective configuration space is not a submanifold embedded in $M$ these are not directly connected to any extrinsic geometrical quantity. The gauge force survives in the classical limit. The coupling constant $e$ corresponds to the momentum stored in the compact direction and is again quantized in integer multiples of $\hbar$. The gauge group $U(1)$ coincides to the group of isometries of the circle $S^1$. Generalizing to an arbitrary codimension –with an appropriate ansatz on the metric– yields in fact the gauge group $G$. The scalar interaction consists of the classical potential $\langle v \rangle$ plus a pure quantum contribution $Q^{g_{ij}}$ depending on the average of first and second derivatives of $a$ and $v$ over $\sigma$.

4. Discussion: reducing vs. constraining

The first two models we considered –the dynamical reduction form a higher dimensional configuration space to a line produced by a scalar and a vector potential– represent two different physical realizations of the same constrained system. Unequivalent dynamical behaviors are generated on the line: the analytic form of the induced gauge connection and of the scalar potential are different, the induced gauge group is different and even the coupling constant $e$ is quantized in a different way. It is therefore natural to wonder about the relation between these models and the formal treatment of the constraint according to the methods of analytical mechanics.
We first consider the classical theory, setting $-i\partial_\xi \rightarrow p_\xi$ and $Q^V, A_{gij} \rightarrow 0$ in (1.3), (2.3) and (3.3). The difference between the effective Hamiltonians $H^V_{eff}, H_{eff}^A, H^{gij}_{eff}$ and the result expected from a formal analysis, $H_{cons} = \frac{1}{2}p_\xi^2$, is then given by the terms proportional to the effective coupling constant $e$ and to its square $e^2$. In all the models we discussed, $e$ keeps somehow track of the energy stored in the motion normal to the constraint’s surface. A strict enforcement of the constraint requires as a compatibility condition the momenta in the directions normal to the constraint to be zero. That is $e = 0$. On the other hand, from a physical viewpoint, the constraint is enforced when it is impossible to detect deviations of the motion form the line, so that the system is actually free to move in the directions normal to the line on length scales less than the experimental resolution. The track of this hidden motion is kept by the induced interactions. There is no contradiction between a physical realization and a formal treatment of the constraint, once we remember that we are working in the hypothesis $e = 0$.

Much more subtle is the situation in quantum mechanics. The uncertainty principle forbids to set simultaneously to zero a coordinate and its conjugate momentum, so that a strict enforcement of the constraint is a priori impossible. This corresponds to the well known fact that we are not allowed to impose simultaneously all the constraints as functional conditions on the quantum states or, equivalently, that the quantization of Dirac’s brackets is defined up to terms of order $\hbar^2$. A track of the mechanism producing the dimensional reduction on the effective constrained dynamics seems therefore to be unavoidable. Not knowing anything about the reduction mechanism –we suppose to start the analysis from the classical theory plus its constraints– it is extremely appealing that the ambiguities inherent to the construction of the quantum theory allow us the freedom to reproduce different situations. There are nevertheless two facts that have to be remarked. The first one is that extremely different dynamical behaviors may be produced by choosing different quantization procedures / reduction mechanisms. As an example, we may be tempted to quantize the motion on the line as free, while in our first model the quantum potential $Q^V$ attracts and possibly confines particles around points of strong curvature. This information, nevertheless, is somehow contained in the constraint’s equations and is therefore predictable to some extent. The second and perhaps more remarkable point that emerges from our analysis, is that in some circumstances the effective constrained dynamics depends on informations which are not contained in the constraint’s equations. In our second model, the effective Hamiltonian (2.3) depends on expansion and vorticity of the foliation of $M$ in its real directions. These quantities are not computable starting from the constraint’s equations.

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