Improved BFT embedding having chain-structure

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ABSTRACT

We newly revisit the gauge non-invariant chiral Schwinger model with $a = 1$ in view of the chain structure. As a result, we show that the Dirac brackets can be easily read off from the exact symplectic algebra of second-class constraints. Furthermore, by using an improved BFT embedding preserving the chain structure, we obtain the desired gauge invariant action including a new type of Wess-Zumino term.

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1 Introduction

Gauge theories which play an important role in modern theoretical physics belong to the class of the singular Lagrangian theories, and there has been unceasingly interested in the quantization of these constrained theories since Dirac’s pioneering work [1]. Batalin, Fradkin, and Vilkovsky [2] had proposed a new kind of quantization method for constraint systems, which is particularly powerful for deriving a covariantly gauge-fixed action in configuration space. Furthermore, Batalin, Fradkin, and Tyutin (BFT) [3] had generalized this method introducing some auxiliary fields in the extended phase space. After their works, several authors [4, 5, 6, 7] have systematically applied this BFT method to various interesting models including the bosonized Chiral Schwinger Model (CSM) [8].

Recently, Shirzad and Monemzadeh [9] have applied a modified BFT method, which preserves the chain structure of constraints [10], to the bosonized CSM. However, their constraint algebra is incorrect from the start. Furthermore, they did not carry out further possible simplification for the constraint algebra. As a result, although they have newly applied the idea of the chain structure, they have not successfully obtained a desired gauge invariant Lagrangian.

In this paper, we shall newly resolve this unsatisfactory situation of the $a = 1$ CSM by improving our BFT method [6] and the non-trivial application of the well-known technique of covariant path integral evaluation [11]. As results, we find the gauge invariant quantum action revealing a new type of Wess-Zumino (WZ) term [4].

2 Chain Structure of a Second-Class System

First, let us consider a given canonical Hamiltonian $H_C$ with a single primary constraint $\phi^{(1)}$ from the motivation of analyzing the CSM to recapitulate the known chain-by-chain method [10] briefly. The total Hamiltonian is defined as

$$H_T = H_C + v\phi^{(1)},$$

where $v$ is a Lagrange multiplier to the primary constraint. Then, from the consistency condition, i.e., $\dot{\phi}^{(1)} = 0$ in the Hamiltonian formulation, we have

$$\dot{\phi}^{(1)} := \{\phi^{(1)}, H_T\} = \{\phi^{(1)}, H_C\} + v\{\phi^{(1)}, \phi^{(1)}\} = 0$$

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whose solutions are among three possible cases: (1) if \( \{ \phi^{(1)}, \phi^{(1)} \} \neq 0 \), the Lagrange multiplier \( v \) is fixed, (2) if \( \{ \phi^{(1)}, H_C \} = 0 \) as well as \( \{ \phi^{(1)}, \phi^{(1)} \} = 0 \), the condition is identically satisfied, and (3) if \( \{ \phi^{(1)}, H_C \} \neq 0 \) as well as \( \{ \phi^{(1)}, \phi^{(1)} \} \neq 0 \), the condition generates a new constraint as \( \phi^{(2)} = \{ \phi^{(1)}, H_C \} \).

Since we are only interested in the last nontrivial case, we continue to require the consistency condition on the constraint as \( \dot{\phi}^{(2)} = 0 \) with the condition of \( \{ \phi^{(2)}, H_C \} \neq 0 \) as well as \( \{ \phi^{(2)}, \phi^{(1)} \} = 0 \) as like in Eq. (2), producing a next stage constraint \( \phi^{(3)} = \{ \phi^{(2)}, H_C \} \), and so forth. Let us assume that after \( (n-1) \)-th step of the procedure, we obtain all constraints which satisfy the following relations

\[
\begin{align*}
\{ \phi^{(i)}, \phi^{(1)} \} &= 0, \\
\{ \phi^{(i)}, H_C \} &= \phi^{(i+1)}, \quad i = 1, 2, \ldots, n - 1.
\end{align*}
\]

While the consistency condition for the last constraint is explicitly written as

\[
\dot{\phi}^{(n)} := \{ \phi^{(n)}, H_T \} = \{ \phi^{(n)}, H_C \} + v \{ \phi^{(n)}, \phi^{(1)} \} = 0.
\]

Among three possible solutions for this consistency condition, let us concentrate on the case that the Lagrange multiplier has been finally determined as

\[
v = -\frac{\{ \phi^{(n)}, H_C \}}{\{ \phi^{(n)}, \phi^{(1)} \}}
\]

with the requirement of \( \{ \phi^{(n)}, \phi^{(1)} \} = \eta \neq 0 \). Note here that all the constraints \( \phi^{(i)}, (i = 2, 3, \ldots, n) \) are generated from the only one primary constraint \( \phi^{(1)} \) through the consistency conditions, i.e., the constraint set has a chain structure.

Furthermore, considering the Jacobi identities for the set of the whole constraints and canonical Hamiltonian, \( \{ \phi^{(i)}, H_C \}, i = 1, 2, \ldots, n \), one can obtain

\[
\begin{align*}
\{ \phi^{(i)}, \phi^{(j)} \} &= 0, \quad i + j \leq n, \\
\{ \phi^{(n-i+1)}, \phi^{(i)} \} &= (-1)^{i+1} \eta, \quad i = 1, 2, \ldots, n.
\end{align*}
\]

From the above relations, we have newly observed that if we define the constraint algebra \( \Delta_{\alpha\beta} \equiv \{ \phi_\alpha, \phi_\beta \} \) as usual, the relations (6) imply that all the matrix elements above the off-diagonal component are simply zero. On the other hand, the relations (7) intimate that the off-diagonal elements are not
vanishing but alternating $\pm \eta$, i.e., $\Delta_{1,n} = -\eta$, $\Delta_{2,n-1} = +\eta$, etc, implying that the system we are considering is of second-class (SC) constraints. Moreover, one can also show that the total number of the constraints are always even: that is to say, comparing two relations both when $i = 1$ and $i = n$, one obtain $\{\phi(n), \phi(1)\} = \eta$ as well as $\{\phi(1), \phi(n)\} = (-1)^{n+1}\eta$. Thus, the antisymmetricity of the Poisson brackets gives rise to the even number of constraints.

Our additional observation is in order: when $i + j \leq n$, among the Jacobi identities

$$\{\phi(i), \{\phi(j), H_C\}\} + \{\phi(j), \{H_C, \phi(i)\}\} + \{H_C, \{\phi(i), \phi(j)\}\} = 0 \quad (8)$$

are reduced to

$$\{\phi(i), \phi(i+1)\} - \{\phi(j), \phi(i+1)\} = 0, \quad (9)$$

where we have used the relations (6) and (7). Therefore, as far as we consider the upper part of the off-diagonal elements of the Dirac matrix $\Delta_{\alpha\beta}$, the knowledge of the first low elements in the matrix, generates all the remaining off-diagonal elements, i.e., downward left, starting from the very elements. That is, when $i + j \leq n - 1$, we have

$$\{\phi(1), \phi(i+j)\} = -\{\phi(2), \phi(i+j-1)\} = \cdots = \{\phi(j), \phi(i+1)\} = \cdots = \{\phi(i+j-1), \phi(2)\} = -\{\phi(i+j), \phi(1)\} = 0. \quad (10)$$

Note that at this stage one can not still say anything on the elements in the lower part, i.e., below the off-diagonal components in the Dirac’s matrix $\Delta_{\alpha\beta}$, which is the subject to discuss in below.

Now, let us consider all the constraints:

$$\{(\phi(1), \phi(2), \cdots, \phi(k), \phi(k+1), \cdots, \phi(n-1), \phi(n))\}, \quad (11)$$

where $k = n/2$ and $n$ is even as proved before. To make the discussion easier let us relabel the superscripts as

$$\phi^{(k)} = (-1)^k \phi^{(n-k+1)}. \quad (12)$$

As a result, we can construct the following ($k$)-pairs Poisson brackets, which are all the off-diagonal elements of the Dirac matrix $\Delta_{\alpha\beta}$ designated from the
sum of the superscripts to be \( n + 1 \), as

\[
\{ \phi^{(1)}, \phi^{(n)} \} = -\eta, \quad \Rightarrow \quad \{ \phi^{(1)}, \phi^{*(1)} \} = \eta, \\
\{ \phi^{(2)}, \phi^{(n-1)} \} = \eta, \quad \Rightarrow \quad \{ \phi^{(2)}, \phi^{*(2)} \} = \eta, \\
\ldots \\
\{ \phi^{(k)}, \phi^{(n-k+1)} \} = (-1)^k \eta, \quad \Rightarrow \quad \{ \phi^{(k)}, \phi^{*(k)} \} = \eta. \quad (13)
\]

Now let us redefine the constraints as

\[
\tilde{\phi}^{(1)} \equiv \phi^{(1)}, \quad \tilde{\phi}^{*(1)} \equiv \phi^{*(1)}. \quad (14)
\]

Then, through the Gram-Schmidt process, we can obtain new set of the constraints

\[
\tilde{\phi}^{(k)} = \phi^{(k)} - \sum_{i=1}^{k-1} \frac{\{ \phi^{(k)}, \tilde{\phi}^{*(i)} \} \tilde{\phi}^{(i)}}{\{ \tilde{\phi}^{(i)}, \tilde{\phi}^{*(i)} \}}, \\
\tilde{\phi}^{*(k)} = \phi^{*(k)} - \sum_{i=1}^{k-1} \frac{\{ \phi^{*(k)}, \tilde{\phi}^{(i)} \} \tilde{\phi}^{(i)}}{\{ \tilde{\phi}^{(i)}, \tilde{\phi}^{*(i)} \}}, \quad (15)
\]

which are satisfied with the following compact relations

\[
\{ \tilde{\phi}^{(k)}, \tilde{\phi}^{*(k')} \} = \delta_{kk'} \eta, \\
\{ \tilde{\phi}^{(i)}, \tilde{\phi}^{(j)} \} = 0, \quad (i \neq j), \\
\{ \phi^{*(i)}, \phi^{*(j)} \} = 0, \quad (i \neq j). \quad (16)
\]

Therefore, we have shown that for any one-chain constrained system we can obtain off-diagonalized Dirac matrix \( \Delta_{\alpha\beta} \), which is symplectic, in general. The inverse of symplectic Dirac matrix is naturally used to define usual Dirac brackets without much efforts. However, note here that we should pay much attention to the Dirac matrix \( \Delta_{\alpha\beta} \) when there exist self-anticommuting matrix components, e.g., \( \{ \phi^{*(i)}, \phi^{*(i)} \} \neq 0, \quad (i = 1, 2, \cdots, n/2 - 1) \). This arises in many cases including the chiral boson, the chiral Schwinger, the Maxwell-Chern-Simons, the self-dual models, and so on. In those cases, the Dirac matrix usually contains a derivative term, which is anticommuting itself, through a certain Poisson bracket in constraint algebra. If then, one should first redefine some of constraints giving a derivative term in constraint algebra to have vanishing self-anticommuting matrix component in the Dirac matrix.
3 CSM with Chain Structure

As is well-known, the fermionic CSM with a regularization ambiguity \( a = 1 \) is equivalent to the following bosonized action \[8\]

\[
S_{CSM} = \int d^2 x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + A_\nu (\eta^{\mu\nu} - \epsilon^{\mu\nu}) \partial_\mu \phi + \frac{1}{2} A_\mu A^\mu \right],
\]

(17)

where \( \eta^{\mu\nu} = \text{diag}(1,-1) \), and \( \epsilon^{01} = 1 \). Comparing with the CSM with \( a > 1 \), the number of constraints in CSM with \( a = 1 \) is four and the former has two. In order to explicitly study the advantages of chain structure, it is very instructive to apply the chain structure idea to the above bosonized CSM with \( a = 1 \).

The canonical momenta are given by

\[
\begin{align*}
\pi^0 &= 0, \\
\pi^1 &= \dot{A}_1 - \partial_1 A_0 \equiv E, \\
\pi_\phi &= \dot{\phi} + A_0 - A_1 \equiv \pi,
\end{align*}
\]

(18)

where the overdot denotes the time derivative. The canonical Hamiltonian is written as

\[
H_C = \int dx \left[ \frac{1}{2} E^2 + \frac{1}{2} \pi^2 + \frac{1}{2} (\partial_1 \phi)^2 + E \partial_1 A_0 + (\pi + A_1 + \partial_1 \phi)(A_1 - A_0) \right].
\]

(19)

Following Dirac’s standard procedure \[1\], one finds one primary constraint

\[
\Phi_1 \equiv \pi^0 \approx 0,
\]

(20)

and three secondary constraints

\[
\begin{align*}
\Phi_2 &\equiv \partial_1 E + \pi + \partial_1 \phi + A_1 \approx 0, \\
\Phi_3 &\equiv E \approx 0, \\
\Phi_4 &\equiv -\pi - \partial_1 \phi - 2A_1 + A_0 \approx 0.
\end{align*}
\]

(21)

These secondary constraints starting from the primary constraint \( \Phi_1 \) are successively obtained by conserving the constraints with respect to the total Hamiltonian in the usual Dirac scheme

\[
H_T = H_C + \int dx \ v\Phi_1,
\]

(22)
where $v$ denotes a Lagrange multiplier, which is fixed as follows

$$v = \partial_1 \pi + \partial_1^2 \phi + 2E + 2\partial_1 A_1.$$  \hfill (23)

No further constraints are generated in this procedure showing that the CSM with $a = 1$ belongs to one-chain, and all the constraints are fully second-class (SC) constraints.

It is appropriate to comment that in the work of Shirzad at al. there should be a term in their constraint algebra [9] in which $\Delta_{44}$ is not simply zero but $2\partial_x \delta(x - y)$. This term is in itself antisymmetric, which case is excluded from the general consideration of chain structure as seen in Sec. [2]. Therefore, before we proceed to further, we have to redefine this constraint by making use of the other constraints.

Now, let us redefine $\Phi_4$ as $\Phi_1$ by using $\Phi_1$ as follows

$$\Phi_1 \equiv \Phi_4' + \partial_1 \Phi_1 = -\pi - \partial_1 \phi - 2A_1 + A_0 + \partial_1 \pi^0.$$  \hfill (24)

Note that the redefined constraint is still SC one. However, the new set of the constraints $\Phi_i (i = 1, \cdots, 4)$ satisfies more simplified form of the SC constraints algebra

$$\Delta_{ij}(x, y) \equiv \{\Phi_i(x), \Phi_j(y)\}$$

$$\begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 2 \\
1 & 0 & -2 & 0
\end{pmatrix} \delta(x - y),$$  \hfill (25)

which is a constant and antisymmetric matrix. Moreover, all the matrix elements above the off-diagonal components are simply zero, and the off-diagonal elements are alternating $\mp 1$.

Furthermore, by following the previous discussion on the chain structure, we can make use of the Gram-Schmidt process to define all the constraints as orthogonalized ones

$$\Omega_i = (\tilde{\Phi}^{(1)}, \tilde{\Phi}^{(2)}, \tilde{\phi}^{*(1)}, \tilde{\phi}^{*(2)}),$$  \hfill (26)

where

$$\begin{align*}
\tilde{\phi}^{(1)} &= \Phi_1, & \tilde{\phi}^{*(1)} &= -\Phi_4, \\
\tilde{\phi}^{(2)} &= \Phi_2, & \tilde{\phi}^{*(2)} &= \Phi_3 + 2\Phi_1.
\end{align*}$$  \hfill (27)
Then, the final form of the Dirac matrix turns out to be symplectic matrix as

\[ \tilde{\Delta}_{ij}(x, y) \equiv \{\Omega_i(x), \Omega_j(y)\} \]

\[ = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \delta(x - y) \equiv J\delta(x - y), \quad (28) \]

where \( I \) represents \((2 \times 2)\) identity matrix.

Therefore, we have completely converted the Dirac matrix of the CSM into the desired symplectic one by analyzing the chain structure of the model systematically, which structure is of one-chain. Now, as we know, the Dirac brackets can be obtained from the symplectic matrix (28) without much efforts.

It seems appropriate to comment on the previous work [9] of Shirzad et al. They used the constraint \( \Phi'_4 \) wrongly to obtain the constraint algebra (25) which constraint gives a derivative term in \( \Delta_{44} \) as stated above. In other words, the constraint \( \Phi'_4 \) gives a self-anticommuting matrix element in the constraint algebra, and in order to eliminate the corresponding self-anticommuting element for further analysis they should have modified the constraint \( \Phi'_4 \) to \( \Phi_4 \) as shown in Eq.(24). Then, one can obtain the correct constraint algebra (25).

4 Improved BFT Embedding

At least for the one-chain system we have shown that the Dirac matrix can be finally reduced to the desired symplectic form using the fact that we can make the SC constraints to be paired with and apply the Gram-Schmidt process to make them off-diagonal. As a result, we have shown that using the chain structure of constrained system has a great advantage of finding the Dirac brackets of SC system easier than finding them based on the standard Dirac scheme. On the other hand, the BFT algorithm is known to systematically convert gauge non-invariant theory into gauge invariant one by introducing auxiliary fields, and with the improved BFT formalism preserving the chain structure we will later quantize the bosonized CSM in the path integral framework.

The basic idea on the usual BFT algorithm is to convert the SC constraints \( \Omega_i \) into fully first-class (FC) ones \( \tau_i \) by introducing auxiliary fields \( \eta^i \)
as
\[ \tau_i = \Omega_i + X_{ij} \eta^j + O(\eta^{(2)}), \] (29)
where \( \Omega_i \) is given by Eq.(26) and requiring that fully FC constraints satisfy the strongly involutive Poisson brackets relations
\[ \{ \tau_i, \tau_j \} = 0. \] (30)
The strongly involutive Poisson brackets relation (30) gives the following set of equations on the zero-th order of \( \eta \) to be solved for \( X_{ij} \) and \( \omega_{ij} \) as
\[ \Delta_{ij}(x,y) + \int dz_1 dz_2 X_{ik}(x,z_1) \omega^{kl}(z_1,z_2) X_{jl}(z_2,y) = 0, \] (31)
where \( \omega_{ij} \) is given by \( \{ \eta^i, \eta^j \} = \omega_{ij} \) for the newly introduced auxiliary fields. Therefore, by choosing a set of proper Ansatz for \( \omega_{ij}, X_{ij} \), we can solve the set of equations (31) which means that we can successfully convert the SC constraints into fully FC ones (29).

However, the practical finding of FC quantities including the Hamiltonian is not so straightforward in real world. The difficulties arise from the fact that there is no guided proper way to choose the matrix elements for \( X_{ij} \) and \( \omega_{ij} \) in Eq. (31). One has learned by experience that for any different choices of \( X_{ij} \) and \( \omega_{ij} \) the corresponding FC functions including the constraints and Hamiltonian are equivalent to each other up to the existing constraints. In this respect, it would have great advantage of choosing \( X_{ij} \) and \( \omega_{ij} \) as simple as possible for further analysis. In our case, since we have obtained the simplest expression for the Dirac matrix \( \Delta_{ij}(x,y) \) in its symplectic form by using the chain structure of constrained system, the natural candidates for the solutions of \( X_{ij} \) and \( \omega_{ij} \) shall be simply given by \( X_{ij} = \Delta_{ij} \) and \( \omega_{ij} = -\Delta_{ij} \) where \( \Delta_{ij} \) is now the symplectic matrix (28). Moreover, the simple choices of \( X_{ij} = \Delta_{ij} \) and \( \omega_{ij} = -\Delta_{ij}, \) i.e., the exactly symplectic matrices, may further simplify the explicit construction of FC quantities since according the usual BFT formalism they would be proportional only to the second powers in the auxiliary fields. This would help to easily construct FC functions for complicated theories including non-abelian models not in infinite but in finite powers in the auxiliary fields.

Now let us come back to the CSM with \( a = 1 \) case, and start from defining the solutions of \( \omega_{ij} = -\Delta_{ij} \) and \( X_{ij} = \Delta_{ij} \) in Eq. (31) as
\[ \omega^{ij}(x,y) = -J\delta(x-y), \]
\[ X_{ij}(x, y) = J\delta(x - y). \]  

(32)

Then, the fully FC constraints are explicitly written by

\[
\begin{align*}
\tau_1 &= \Omega_1 + \eta^3 = \pi^0 + \eta^3, \\
\tau_2 &= \Omega_2 + \eta^4 = \partial_1 E + \partial_1 \phi + \pi + A_1 + \eta^4, \\
\tau_3 &= \Omega_3 - \eta^1 = \pi + \partial_1 \phi + 2A_1 - A_0 - \eta^1, \\
\tau_4 &= \Omega_4 - \eta^2 = E + 2\pi_0 - \eta^2.
\end{align*}
\]  

(33)

Note that the higher order correction terms \(\eta^{(n)}\) \((n \geq 2)\) in Eq. (30) automatically vanish as a consequence of the proper choice (33). Therefore, we have all the FC constraints with only \(\eta^{(1)}\) contributing in the series (30) in the extended phase space.

On the other hand, the construction of FC quantities can be done along similar lines as in the case of the constraints, by representing them as a power series in the auxiliary fields and requiring

\[ \{\tau_i, \tilde{F}\} = 0 \]  

(34)

subject to the condition \(\tilde{F}(\mathcal{O}; \eta^i = 0) = F\). Here \(F\) (or, \(\mathcal{O}\)) is a quantity (or, variables) in the original phase space, while \(\tilde{F}\) is a quantity in the extended phase space.

Then, we obtain the proper BFT physical variables as

\[
\begin{align*}
\tilde{A}^\mu &= A^\mu + A^{\mu(1)} = (A^0 + \eta^1 + \partial_1 \eta^3 + 2\eta^4, A_1 - \partial_1 \eta^2 + \eta^4) \\
\tilde{\pi}^\mu &= \pi^\mu + \pi^{\mu(1)} = (\pi^0 + \eta^3, \pi^1 - \eta^2 - 2\eta^3), \\
\tilde{\phi} &= \phi + \phi^{(1)} = \phi + \eta^2 + \eta^3, \\
\tilde{\pi}_\phi &= \pi_\phi + \pi_{\phi}^{(1)} = \pi_\phi + \partial_1 \eta^2 + \partial_1 \eta^3.
\end{align*}
\]  

(35)

Meanwhile, the non-vanishing Poisson brackets of the physical fields (35) in the extended phase space are directly read off as

\[
\begin{align*}
\{\tilde{A}_0(x), \tilde{A}_0(y)\} &= 2\partial^x_1 \delta(x - y), \quad \{\tilde{A}_0(x), \tilde{\phi}(y)\} = \delta(x - y), \\
\{\tilde{A}_0(x), \tilde{\pi}(y)\} &= -\partial^x_1 \delta(x - y), \quad \{\tilde{A}_1(x), \tilde{A}_1(y)\} = 2\partial^x_1 \delta(x - y), \\
\{\tilde{A}_1(x), \tilde{\phi}(y)\} &= \delta(x - y), \quad \{\tilde{A}_1(x), \tilde{\pi}(y)\} = -\delta(x - y), \\
\{\tilde{\phi}(x), \tilde{\pi}(y)\} &= \delta(x - y).
\end{align*}
\]  

(36)
These are exactly the same Dirac brackets for consistent quantization in the traditional Dirac scheme where the quantum brackets are usually found through a tedious algebraic manipulation.

The FC Hamiltonian can be also obtained either by following the prescription (34) directly or by noting that any functional of FC fields such as $\tilde{F} = (\tilde{A}^{\mu}, \tilde{\pi}^{\mu}, \tilde{\phi}, \tilde{\pi}_{\phi})$ will be FC. As results, we have obtained the FC Hamiltonian from the former method as

$$\tilde{H} = H_{C} + H^{(1)} + H^{(2)},$$

where

$$H^{(1)} = \int dx \left[ -\eta^1(\partial_1 E + \partial_1 \phi + \pi + A_1) - \eta^2 E \\
- \eta^3(2E + \partial_1 A_1 - \partial_1^2 E) - \eta^4(\pi + \partial_1 \phi + 2\partial_1 E + A_0) \right],$$

$$H^{(2)} = \int dx \left[ -\eta^1\eta^4 + \frac{1}{2}(\eta^2)^2 + 2\eta^2\eta^3 - \frac{1}{2}\eta^3\partial_1\eta^4 + \frac{1}{2}\eta^4\partial_1\eta^3 \\
+ 2(\eta^3)^2 - \eta^3\partial_1^2\eta^3 - (\eta^4)^2 \right],$$

where the superscripts denotes the power of the auxiliary fields $\eta^i$ and we note that higher power terms greater than $\eta^{(2)}$ in the auxiliary fields do not appear in the FC Hamiltonian. One can easily show that the FC Hamiltonian (37) is exactly equivalent to the one obtained from the replacement of the original fields in the canonical Hamiltonian (19) by the physical BFT fields.

Finally, one can easily write down the desired Hamiltonian $\tilde{H}_T$, which exactly preserves the chain structure of the constraint system in the extended phase space,

$$\tilde{H}_T = \tilde{H} + \sum_{i=1}^{3} \eta^i\tilde{\Omega}_{i+1}$$

equivalent to the strongly involutive Hamiltonian $\tilde{H}$.

Since in the Hamiltonian formalism the FC constraint system indicates the presence of a local symmetry, this completes the operatorial conversion of the original SC system with Hamiltonian $H_C$ and constraints $\Phi_i$ into the FC ones in the extended phase space with Hamiltonian $\tilde{H}$ and constraints $\tilde{\Omega}_i$ by using the property (34).
5 Gauge Invariant Quantum Lagrangian

In this section, we consider the partition function of the model in order to extract out the gauge invariant quantum Lagrangian corresponding to $\tilde{H}_T$. Our starting partition function is given by the Faddeev-Popov (FP) formula as follows

$$Z = \int D\pi D\phi D\pi D\eta^i D\delta(\hat{\Omega}_i) D\delta(\Gamma_j) \det \left\{ \hat{\Omega}_i, \Gamma_j \right\} e^{iS}, \quad (40)$$

where

$$S = \int d^2x \left( \pi^\mu \dot{A}_\mu + \pi \phi + \frac{1}{2} \sum_{i=1}^4 \eta^i \omega_{ij} \dot{\eta}^j - \tilde{H}_T \right) \quad (41)$$

with the Hamiltonian density $\tilde{H}_T$. The gauge fixing conditions $\Gamma_i$ should be chosen so that the determinant occurring in the functional measure is non-vanishing. Moreover, $\Gamma_i$ may be assumed to be independent of the momenta so that these are considered as the FP type gauge conditions [4, 5, 6].

Following the standard path integral method, one can obtain the following gauge invariant action $S_F$

$$S_F = S_{CSM} + S_{WZ} + S_{NWZ};$$

$$S_{CSM} = \int d^2x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + A_\nu (\eta^{\mu\nu} - \epsilon^{\mu\nu}) \partial_\mu \phi + \frac{1}{2} A^\mu A_\mu \right],$$

$$S_{WZ} = -\int d^2x \frac{1}{2} \eta^3 \epsilon_{\mu\nu} F^{\mu\nu},$$

$$S_{NWZ} = \int d^2x \left[ A_0 (\dot{\eta}^3 - \partial_1 \eta^3 + 2n^4) - A_1 (\partial_1 \eta^3 + \eta^4) + (\dot{\phi} + \partial_1 \phi) (\dot{\eta}^3 + n^4) + \partial_1 \eta^3 (\dot{\eta}^3 - \partial_1 \eta^3 - n^4) + \frac{1}{2} (\dot{\eta}^3 + \eta^4)^2 + (\eta^4)^2 - \frac{1}{2} (\eta^4 - \partial_1 \eta^3 - 2\partial_1 \eta^4)^2 \right] \quad (42)$$

The desired final action $S_F$ is invariant under the gauge transformations as

$$\delta A_\mu = \partial_\mu \Lambda, \quad \delta \phi = \Lambda, \quad \delta \eta^3 = -\Lambda, \quad \delta \eta^4 = 0, \quad (43)$$

where the new type of WZ action $S_{NWZ}$ in itself is invariant under the above transformations, which means that this term is not related to the gauge symmetry. The resultant action has not only the well-known WZ term $S_{WZ}$
canceling the gauge anomaly, but also a new type of WZ term \( S_{NWZ} \), which is irrelevant to the gauge symmetry but is needed to make the SC system into the fully FC one analogous to the case of the CS model [4].

6 Conclusion

In conclusion, we have revisited the chain structure [9] analysis for the non-trivial \( a = 1 \) bosonized CSM, which belongs to one-chain system with one primary and three secondary constraints. In this chain structure formalism, we have newly defined the second-class constraints as the proper orthogonalized ones, and then have successfully converted the Dirac matrix into the symplectic one. As a result, we have resolved the unsatisfactory situation of the \( a = 1 \) CSM in the incomplete previous work [9]. Furthermore, based on our improved BFT method preserving the chain structure in the extended phase space, we have found the desired gauge invariant quantum action.

Through further investigation, it will be interesting to apply this newly improved BFT method to non-Abelian cases [13] as well as an Abelian four-dimensional anomalous chiral gauge theory [14], which seem to be very difficult to analyze within the framework of the original BFT formalism [5].

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References

[1] P. A. M. Dirac, Lectures on quantum mechanics, Belfer graduate School, Yeshiva Univ. Press, New York (1964).
[2] E. S. Fradkin and G. V. Vilkovisky, Phys. Lett. 55B (1975) 224; I. A. Batalin and G. A. Vilkovisky, Phys. Lett. 69B (1977) 309; I. A. Batalin and E. S. Fradkin, Phys. Lett. B 180 (1986) 157.

[3] I. A. Batalin and E. S. Fradkin, Nucl. Phys. B 279 (1987) 514; I. A. Batalin and I. V. Tyutin, Int. J. Mod. Phys. A 6 (1991) 3255.

[4] R. Banerjee, Phys. Rev. D 48 (1993) R5467; Won T. Kim and Y.-J. Park, Phys. Lett. B 336 (1994) 376; Y.-W. Kim, Y.-J. Park, K. Y. Kim, and Y. Kim, Phys. Rev. D 51 (1995) 2943.

[5] R. Banerjee, H. J. Rothe, and K. D. Rothe, Phys. Rev. D 49 (1994) 5438; J. H. Cha, Y.-W. Kim, Y.-J. Park, Y. Kim, S.-K. Kim, and Won T. Kim, Z. Phys. C 69 (1995) 175.

[6] Y.-W. Kim, M.-I. Park, Y.-J. Park, and Sean J. Yoon, Int. J. Mod. Phys. A 12 (1997) 4217; Won T. Kim, Y.-W. Kim, M.-I. Park, Y.-J. Park, and Sean J. Yoon, J. Phys. G 23 (1997) 325.

[7] E. Harikumar and M. Sivakumar, Nucl. Phys. B 565 (2000) 385; J. Ananias Neto, Phys. Lett. B 571 (2003) 105; S.-T. Hong, Y.-W. Kim, Y.-J. Park, and K. D. Rothe, J. Phys. A 36 (2003) 1643; Y.-W. Kim, C.-Y. Lee, S.-K. Kim, Y.-J. Park, Eur. Phys. J. C 34 (2004) 383.

[8] R. Jackiw and R. Rajaraman, Phys. Rev. Lett. 54 (1985) 1219; ibid., 54 (1985) 2060(E); O. Babelon, F.A. Schaposnik, and C.M. Viallet, Phys. Lett. 177B (1986) 385.

[9] A. Shirzad and M. Monemzadeh, Phys. Lett. B 584, 220 (2004).

[10] A. Cabo and D. L. Martinez, Phys. Rev D 42, 2726 (1990); M. Chaichian, D. L. Martinez and L. Lusanna, Ann. Phys. 232, 40 (1994); F. Loran and A. Shirzad, Int. J. Mod. Phys. A17, 625 (2002).
[11] T. Fujiwara, Y. Igarashi and J. Kubo, Nucl. Phys. B 341 (1990) 695;
Y.-W. Kim, S.-K. Kim, Won T. Kim, Y.-J. Park, K. Y. Kim, and Y. Kim, Phys. Rev. D 46 (1992) 4574.

[12] L. D. Faddeev and V. N. Popov, Phys. Lett. 25B (1967) 29;
P. Senjanovic, Ann. Phys. (N. Y.) 100 (1979) 227;
P. Senjanovic, Ann. Phys. (N. Y.) 209 (1991) 248(E).

[13] Y.-W. Kim and K. D. Rothe, Nucl. Phys. B 511 (1998) 510;
E. M. C. Abreu, J. Ananias Neto, and W. Oliveira, Phys. Lett. B 483 (2000) 337.

[14] R. Rajaraman, Phys. Lett. B 184 (1987) 369;
F. S. Otto, Phys. Rev. D 43 (1991) 548.