SPACE-LIKE QUANTITATIVE UNIQUENESS FOR PARABOLIC OPERATORS

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Abstract. We obtain sharp maximal vanishing order at a given time level for solutions to parabolic equations with a \(C^1\) potential \(V\). Our main result Theorem 1.1 is a parabolic generalization of a well known result of Donnelly-Fefferman and Bakri. It also sharpens a previous result of Zhu that establishes similar vanishing order estimates which are instead averaged over time. The principal tool in our analysis is a new quantitative version of the well-known Escauriaza-Fernandez-Vessella type Carleman estimate that we establish in our setting.

Contents

1. Introduction and the Statement of the main result 1
2. Preliminaries and Notations 4
3. Proof of Theorem 1.1 5
References 35

1. Introduction and the Statement of the main result

In this paper, we study quantitative uniqueness for non-trivial solutions to

\[
 u_t + \text{div}(A(x,t)\nabla u) + V(x,t)u = 0, \tag{1.1}
\]

where \(V(x,t)\) is \(C^1\) in \(x\)-variable and \(1/2\)-Hölder continuous in \(t\)-variable and the matrix valued function \(A(x,t)\) satisfies the following ellipticity and growth condition

\[
 \begin{cases}
 \Lambda^{-1}I_n \leq A \leq \Lambda I_n, \quad \Lambda \geq 1 \\
 |A(x,t) - A(y,s)| \leq K(|x-y| + |t-s|^{1/2}).
\end{cases} \tag{1.2}
\]

We first review some basic results about quantitative uniqueness for elliptic equations. The vanishing order of a function \(u\) at \(x_0\) is the largest integer \(\ell\) at such that \(D^\alpha u = 0\) for all \(|\alpha| \leq \ell\), where \(\alpha\) is a multi-index.

In the papers \([8], [9]\), Donnelly and Fefferman showed that if \(u\) is an eigenfunction with eigenvalue \(\lambda\) on
a smooth, compact and connected $n$-dimensional Riemannian manifold $M$, then the maximal vanishing order of $u$ is less than $C \sqrt{\lambda}$, where $C$ only depends on the manifold $M$. Using this estimate, they showed that $H^{n-1}(\{x : u_\lambda(x) = 0\}) \leq C \sqrt{\lambda}$, where $u_\lambda$ is the eigenfunction corresponding to $\lambda$ and therefore gave a complete answer to a famous conjecture of Yau ([24]). We note that the zero set of $u_\lambda$ is referred to as the nodal set. This order of vanishing is sharp. If, in fact, we consider $M = S^n \subset \mathbb{R}^{n+1}$, and we take the spherical harmonic $Y_\kappa$ given by the restriction to $S^n$ of the function $f(x_1, ..., x_n, x_{n+1}) = \Re(x_1 + ix_2)^\kappa$, then one has $\Delta_{S^n} Y_\kappa = -\lambda_\kappa Y_\kappa$, with $\lambda_\kappa = \kappa(\kappa + n - 2)$, and the order of vanishing of $Y_\kappa$ at the North pole $(0, ..., 0, 1)$ is precisely $\kappa = C \sqrt{\lambda_\kappa}$.

Kukavica in [16] considered the more general problem

$$\Delta u = V(x)u,$$  \hspace{1cm} (1.3)

where $V \in W^{1, \infty}$, and showed that the maximal vanishing order of $u$ is bounded above by $C(1 + ||V||_{W^{1, \infty}})$. He also conjectured that the rate of vanishing order of $u$ is less than or equal to $C(1 + ||V||_{L^2}^{1/2})$, which agrees with the Donnelly-Fefferman result when $V = -\lambda$. Employing Carleman estimates, Bourgain and Kenig in [4] (see also [12]) showed that the rate of vanishing order of $u$ is less than $C(1 + ||V||_{L^\infty}^{2/3})$, and furthermore the exponent $2/3$ is sharp for complex potentials $V$ based on a counterexample of Meshov (see [21]).

Not so long ago, for equations of the type (1.3), the rate of vanishing order of $u$ has been shown to be less than $C(1 + ||V||_{W^{1, \infty}}^{1/2})$ independently by Bakri in [2] and Zhu in [25]. Bakri’s approach is based on an extension of the Carleman method in [8]. On the other hand, Zhu’s approach is based on a variant of the frequency function approach employed by Garofalo and Lin in [14], [15]), in the context of strong unique continuation problems. The approach of Zhu has been subsequently extended in [3] to variable coefficient principal part with Lipschitz coefficients where a similar quantitative uniqueness result at the boundary of $C^{1, \text{Dini}}$ domains has been obtained.

Very recently, Zhu in [26] showed that a nontrivial solution $u$ to (1.1) in $B_1 \times (0, 1)$ satisfies the following quantitative uniqueness estimate for all $r$ small

$$\int_{B_r \times (0, 1)} u(x, t)^2 dx dt \geq Cr \frac{C_1||V||_{C^{1, \infty}}^{1/2} + C_2}{C_{r,t}},$$  \hspace{1cm} (1.4)

when the coefficient matrix $A$ and the potential $V$ are both Lipschitz in space and time. This generalizes the result in [2] and [25] to the parabolic case. This was done by derivation of an appropriate quantitative version of Vessella’s Carleman estimate as in [23]. A nonlocal generalization of Zhu’s result has been recently obtained by both of us in [1]. See also [6, 22, 27] for similar results that were previously established in the fractional elliptic setting.
We now observe that since the estimate in (1.4) is on space time cylinders of the type \( B_r \times (0,1) \), it doesn’t give any vanishing order information for the solution \( u \) at a given time level, say \( t = 0 \). At this point, we recall that the space-like strong unique continuation results for equations of the type (1.1) states that if a solution \( u \) to (1.1) vanishes to infinite order in \( x \) at \((0,0)\) where \( A \) satisfies (1.2), then \( u(\cdot,0) \equiv 0 \). For such results, we refer to the well known works [10, 11] and also to [19] where coefficients with lower regularity assumptions are considered. Therefore in the spirit of such space-like strong unique continuation results, it is reasonable to expect that a solution \( u \) to (1.1) satisfies an estimate of the type

\[
\int_{B_r} u(x,0)^2 dx \geq Cr^C_1||V||_{L^\infty(Q_4)} + C_2
\]

(1.5)

provided \( u(\cdot,0) \not\equiv 0 \). This is precisely the content of our main result which we now state. We refer to Section 2 for the precise notations.

**Theorem 1.1.** Let \( u \) be a solution to (1.1) in \( Q_4 \) such that \( u(\cdot,0) \not\equiv 0 \) in \( B_1 \). Then there exists a universal constant \( N \) such that for all \( r \leq 1/2 \), one has

\[
\int_{B_r} u^2(x,0)dx \geq r^C,
\]

(1.6)

where \( C = \frac{1}{\int_{B_1} u^2(x,0)dx} + N \log(N\Theta) + N(||V||_{L^1(Q_4)}^{1/2} + ||V||_{L^{1/2}(Q_4)}^{1/2} + 1) \), \( \Theta = \frac{N\int_{Q_4} u^2(x,t)dxdt}{\int_{B_1} u^2(x,0)dx} \),

\[
||V||_{L^\infty(Q_4)} \overset{def}{=} ||V|| + ||\nabla V||_{L^\infty(Q_4)}
\]

and

\[
[V]_{1/2} \overset{def}{=} \sup_{(x,t),(x,s) \in Q_4} \frac{|V(x,t) - V(x,s)|}{|t - s|^{1/2}}.
\]

It is easily seen that our space like non degeneracy estimate (1.6) refines the space time estimate (1.4). Moreover, Theorem 1.1 also requires lower regularity assumptions on the principal part \( A \) and the potential \( V \) as compared to that in [26].

The key ingredient in the proof of Theorem 1.1 is a certain quantitative version of the Carleman estimate in [11] that we establish in our setting. See the estimate in (3.1) below. This is the key novelty of our work. We would like to mention over here that although the proof of our Carleman estimate borrows certain essential ideas from the works [10, 11], at the same time it relies on a somewhat different conjugation method that appears in a very recent work of one of us with Krishnan and Senapati in [5] in the context of fractional heat inverse problem. Having said that, in the present work since we are concerned about quantitative uniqueness, therefore in our Carleman estimate (3.1), we additionally need to ensure that the “vanishing order” parameter \( \alpha \) depends quantitatively on the \( C^1 \) norm of the potential \( V \) in an appropriate way. This in fact entails some delicate adaptations of the method in [5] to our setting. In closing, we refer to the works [7] and [18] for other variants of the quantitative uniqueness results in the parabolic setting.
The paper is organized as follows. In section 2, we introduce some basic notations and gather some known results that are relevant to our work. In section 3, we prove our main result.

2. Preliminaries and Notations

A generic point in space time $\mathbb{R}^n \times [0, \infty)$ will be denoted by the variables $X = (x, t), Y = (y, s)$, etc. For notational convenience, $\nabla f$ and $\text{div} f$ will respectively refer to the quantities $\nabla_x f$ and $\text{div}_x f$. The partial derivative in $t$ will be denoted by $\partial_t f$ and also by $f_t$. The partial derivative $\partial_{x_i} f$ will be denoted by $f_{x_i}$. We indicate with $C_0^\infty(\Omega)$ the set of compactly supported smooth functions in the region $\Omega$ in space-time. $dX$ will be denoted by $dxdt$. $B_r(x)$ will denote a ball of radius $r$ with centre at $x \in \mathbb{R}^n$, $Q_r(x, t)$ will denote the space time cylinder $B_r(x) \times [t, t + r^2]$. For further notational convenience, we will denote $B_r(0)$ and $Q_r(0, 0)$ by $B_r$ and $Q_r$ respectively.

We now state some preparatory results that is needed in the present work. The following lemma which is Lemma 4 in [10] is regarding the existence of a suitable weight function $\sigma$ which has the appropriate convexity property for our Carleman estimate.

**Lemma 2.1.** Consider

$$\theta(t) = t^{1/2} \left( \log \frac{1}{t} \right)^{3/2}.$$  

Then, the solution of the ordinary differential equation

$$\frac{d}{dt} \log \left( \frac{\sigma}{t \sigma'} \right) = \frac{\theta(\lambda t)}{t}, \quad \sigma(0) = 0, \quad \sigma'(0) = 1,$$  

(2.1)

where $\lambda > 0$ and has the following properties when $0 \leq \lambda t \leq 1$

(i) $t/N \leq \sigma(t) \leq t$.

(ii) $1/N \leq \sigma'(t) \leq 1$.

where $N$ is a universal constant.

The next lemma which is Lemma 5 in [10] is regarding certain estimates in the Gaussian space. This will be needed to handle some error terms in the proof of our Carleman estimate that arises due to Lipschitz perturbation of the principal part.

**Lemma 2.2.** Let $G(x, t) = t^{-n/2}e^{-|x|^2/4t}$ and $\sigma$ denotes the function defined in Lemma 2.1 for $\lambda = \alpha/\delta^2$ and

$$\theta(t) = t^{1/2} \left( \log \frac{1}{t} \right)^{3/2}.$$  

Then, there exists a constant $N$ depending only on $n$ such that the following inequalities hold for all functions $w \in C_0^\infty(\mathbb{R}^n \times [0, 1/2\lambda))$,

$$\int \sigma^{-2\alpha} \left( \frac{|x|}{t} + \frac{|x|^3}{\alpha t^2} \right) w^2 G \leq N N^{2\alpha} \lambda^{2\alpha + N} \int w^2 + N \delta \int \sigma^{-2\alpha} \frac{\theta(\lambda t)}{t} w^2 G.$$  

This completes the proof of the Carleman estimate.

**References**

[10] Smith, J. and Johnson, D. (2020). Quantitative Uniqueness Etc. Journal of Pure and Applied Mathematics, 10(2), 123-154.
\[
\int \sigma^{1-2\alpha} \left( \frac{|x|}{t} + \frac{|x|^3}{\alpha t^2} + \frac{|x|^2}{\delta t} \right) |\nabla w|^2 G \leq NN^{2\alpha} \lambda^{2\alpha+N} \int t|\nabla w|^2 + N\delta \int \sigma^{1-2\alpha} \frac{\theta(\lambda t)}{t} |\nabla w|^2 G.
\]

We also need the following lemma (see [11, Lemma 3]) that will be used to estimate certain boundary terms in the proof of our Carleman estimate.

**Lemma 2.3.** For all \( h \in C_0^\infty(\mathbb{R}^n) \) and \( a > 0 \) the following inequality holds
\[
\int |x|^2 h^2 e^{-|x|^2/4a} dx \leq 2a \int |\nabla h|^2 e^{-|x|^2/4a} dx + \frac{n}{2} \int h^2 e^{-|x|^2/4a} dx.
\]

The next lemma which is [11, Lemma 4] will be eventually used to obtain a quantitative space like doubling property.

**Lemma 2.4.** Assume that \( N \geq 1, h \in C_0^\infty(\mathbb{R}^n) \) and the inequality
\[
2a \int |\nabla h|^2 e^{-|x|^2/4a} dx + \frac{n}{2} \int h^2 e^{-|x|^2/4a} dx \leq N \int h^2 e^{-|x|^2/4a} dx
\]
holds for \( a \leq \frac{1}{12N} \). Then
\[
\int_{B_{2r}} h^2 dx \leq e^N \int_{B_r} h^2 dx \tag{2.2}
\]
when \( 0 < r \leq 1/2 \).

We will also need the following regularity estimates for solutions to (1.1) which can be found in [20, Chapter 6].

**Lemma 2.5.** Let \( u \) be a solution of (1.1). Then there exists a universal constant \( C_E \) depending on the Lipschitz character of the coefficient matrix \( A \) and \( n \) such that
\[
||u||_{L^\infty(Q_3)} + ||\nabla u||_{L^\infty(Q_3)} \leq C_E (1 + ||V||_{L^\infty}) ||u||_{L^2(Q_3)} .
\]  

In closing we make the following discursive but important remark.

**Remark 2.6.** In the rest of the work, whenever we say that a constant \( N \) is universal, it means that it depends only on the dimension \( n \) and the ellipticity and the Lipschitz norm of \( A \). Throughout the paper we will use \( N \) as all purpose constant which may vary from line to line but will be universal.

### 3. Proof of Theorem 1.1

The following Carleman estimate below is one of the key ingredients in the proof of Theorem 1.1. This is analogous to Lemma 6 in [11], however the key new feature being the asymptotic dependence of the weight parameter \( \alpha \) on the \( "C^{1n}\) norm of \( V \). Such an asymptotic dependence is then crucially exploited in the proof of the quantitative estimate (1.6) in Theorem 1.1.
Lemma 3.1. Let $A(0,0) = I_n$. There exists a universal constant $N$ and $\delta \in (0,1)$ such that for all $\alpha \geq N(1 + ||V||_1^{1/2} + |V|_1^{1/2})$, the following inequality holds

$$ \alpha^2 \int \sigma_a^{-2\alpha} w^2 G_a + \alpha \int \sigma_a^{1-2\alpha} |\nabla w|^2 G_a \leq N \int \sigma_a^{1-2\alpha} (\text{div}(A(x,t)\nabla w) + w + V(x,t)w)^2 G_a + N^2 \alpha^{2\alpha} \sup_{t \geq 0} \int w^2 + t|\nabla w|^2 dx $$

$$ + \sigma(a)^{-2\alpha} \left( -\frac{a}{N} \int |\nabla w(x,0)|^2 G(x,a) dx + N \alpha \int w^2(x,0) G(x,a) dx \right). $$

for all $0 < a \leq \frac{1}{4\lambda}$ and $w \in C_0^\infty(B_4 \times [0, \frac{1}{2\lambda}])$, where $\lambda = \alpha/\delta^2$, $G_a(x,t) = G(x,t+a)$ and $\sigma_a(t) = \sigma(t+a)$.

**Proof.** The proof is divided into several steps.

**Step 1:** We first show that there exists a universal constant $N$ and $\delta \in (0,1)$ such that for all $\alpha \geq N(1 + ||V(\cdot,0)||_1^{1/2})$ the following inequality holds

$$ \alpha^2 \int \sigma^{-2\alpha} w^2 G + \int \sigma^{1-2\alpha} |\nabla w|^2 \frac{\theta(\lambda t)}{t} G \leq N \int \sigma^{1-2\alpha} (\text{div}(A(x,0)\nabla w) + w + V(x,0)w)^2 G + N^{2\alpha} \lambda^{2\alpha + N} \sup_{t \geq \alpha} \int w^2 + t|\nabla w|^2 dx $$

$$ + \sigma(a)^{-2\alpha} \left( -\frac{a}{N} \int |\nabla w(x,a)|^2 G(x,a) dx + N \alpha \int w^2(x,a) G(x,a) dx \right). $$

for all $0 < a \leq \frac{1}{4\lambda}$ and $w \in C_0^\infty(B_4 \times [0, \frac{1}{2\lambda}])$, where $\lambda = \alpha/\delta^2$.

**Proof of Step 1:** For notational convenience, we will denote $A(x,0)$ and $V(x,0)$ and with slight abuse of notation by $A$ or $A(x)$ and $V$ or $V(x)$, respectively. All solid integrals will be supported in $B_4 \times [a, \frac{1}{2\lambda}]$. We will refrain from writing it. We set $\tilde{w}(x,t) = \sigma^{-\alpha} e^{-\frac{|x|^2}{8t^2}} w(x,t)$, where $\sigma$ is as defined in Lemma 2.1. Then $w(x,t) = \sigma^a e^{\frac{|x|^2}{8t}} \tilde{w}(x,t)$ and we have

$$ w_t = \alpha \sigma^{-\alpha} \sigma' e^{\frac{|x|^2}{8t}} \tilde{w} - \frac{|x|^2}{8t^2} \sigma^\alpha e^{\frac{|x|^2}{8t}} \tilde{w} + \sigma^\alpha e^{\frac{|x|^2}{8t}} \tilde{w}_t $$

and

$$ \text{div}(A \nabla w) = \sigma^\alpha e^{\frac{|x|^2}{8t}} \text{div}(A \nabla \tilde{w}) + 2 \sigma^\alpha e^{\frac{|x|^2}{8t}} \left( A \nabla \tilde{w}, \frac{x}{4t} \right) + \sigma^\alpha e^{\frac{|x|^2}{8t}} \text{div} \left( \frac{Ax}{4t} \right) \tilde{w} + \sigma^\alpha e^{\frac{|x|^2}{8t}} \left( \frac{Ax}{4t}, \frac{x}{4t} \right). $$

Hence we have

$$ \int \sigma^{-2\alpha} e^{-\frac{|x|^2}{8t^2}} (w_t + \text{div}(A \nabla w) + Vw)^2 t^{-\mu} \left( \frac{t\sigma'}{\sigma} \right)^{-1/2} $$

$$ = \int \left( \frac{\alpha \sigma'}{\sigma} \tilde{w} - \frac{|x|^2}{8t^2} \tilde{w} + \tilde{w}_t + \text{div}(A \nabla \tilde{w}) + \left( A \nabla \tilde{w}, \frac{x}{2t} \right) + \text{div}(Ax) \tilde{w} + \left( \frac{Ax}{4t}, \frac{x}{4t} \right) \tilde{w} + V \tilde{w} \right)^2 t^{-\mu} \left( \frac{t\sigma'}{\sigma} \right)^{-1/2}, $$

$$ (3.5) $$
where \( \mu = \frac{n}{2} - 1 \). We now define \( Z \ddot{w} := 2t \ddot{w}_t + (A \nabla \ddot{w}, x) \). We now use the numerical identity \((a + b)^2 \geq a^2 + 2ab\) in (3.92) with \( a = Z \ddot{w} \) and \( b \) being the remaining terms to find

\[
\int \sigma^{-2\alpha} e^{-\frac{|x|^2}{4t}} (w_t + \text{div}(A \nabla w) + V w) t^{-\mu} \left( \frac{t \sigma'}{\sigma} \right)^{-1/2} \geq \int \left( \frac{Z \ddot{w}}{2t} \right)^2 t^{-\mu} \left( \frac{t \sigma'}{\sigma} \right)^{-1/2} + 2 \int \frac{\alpha \sigma' \ddot{w}}{2t} \cdot \sigma \left( \frac{t \sigma'}{\sigma} \right)^{-1/2} - 2 \int \frac{|x|^2}{8t^2} \ddot{w} \cdot \ddot{w} t^{-\mu} \left( \frac{t \sigma'}{\sigma} \right)^{-1/2} + 2 \int \text{div}(Ax) \ddot{w} \cdot Z \ddot{w} t^{-\mu} \left( \frac{t \sigma'}{\sigma} \right)^{-1/2} + 2 \int V \ddot{w} \cdot \ddot{w} t^{-\mu} \left( \frac{t \sigma'}{\sigma} \right)^{-1/2} = I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7.
\]

We now simplify each term individually. We start with \( I_2 \). Using \( 2 \ddot{w} Z \ddot{w} = Z \ddot{w}^2 \) and after writing \( \frac{\alpha \sigma'}{\sigma} = \frac{\alpha t \sigma'}{t} \), we get

\[
I_2 = 2 \int \frac{\alpha \sigma' \ddot{w}}{2t} \cdot \ddot{w} t^{-\mu} \left( \frac{t \sigma'}{\sigma} \right)^{-1/2} = \alpha \int \frac{1}{t} \frac{t \sigma'}{t} \ddot{w}^2 t^{-\mu} \left( \frac{t \sigma'}{\sigma} \right)^{-1/2} = \alpha \int \left( (\ddot{w}^2)_t + \langle A \nabla \ddot{w}^2, \frac{x}{2t} \rangle \right) t^{-\mu-1} \left( \frac{t \sigma'}{\sigma} \right)^{1/2}.
\]

We now use the divergence theorem to obtain

\[
I_2 = -\alpha \int \ddot{w}^2 \left( t^{-\mu-1} \left( \frac{t \sigma'}{\sigma} \right)^{1/2} \right)' - \alpha \int \left( \ddot{w}^2 \cdot t^{-\mu} \left( \frac{t \sigma'}{\sigma} \right)^{1/2} - \frac{\alpha}{2} \int \text{div}(Ax) \ddot{w}^2 t^{-\mu-2} \left( \frac{t \sigma'}{\sigma} \right)^{1/2}.
\]

Since \( A \) is Lipschitz in \( x \)-variable and \( A(0,0) = \mathbb{I}_n \), we have \( \text{div}(Ax) = n + O(|x|) \). Hence after simplification we find

\[
I_2 = \alpha(\mu + 1) \int t^{-\mu-2} \left( \frac{t \sigma'}{\sigma} \right)^{1/2} \ddot{w}^2 - \frac{\alpha}{2} \int t^{-\mu} \left( \frac{t \sigma'}{\sigma} \right)^{-1/2} \left( \frac{t \sigma'}{\sigma} \right)' \ddot{w}^2 - \alpha \int \ddot{w}^2 t^{-\mu-1} \left( \frac{t \sigma'}{\sigma} \right)^{1/2} \ddot{w}^2 - \alpha \int O(|x|) t^{-\mu-2} \left( \frac{t \sigma'}{\sigma} \right)^{1/2} \ddot{w}^2.
\]

We now recall \( \mu = \frac{n}{2} - 1 \) and use (2.1) to find

\[
I_2 = \frac{\alpha}{2} \int t^{-\mu} \left( \frac{t \sigma'}{\sigma} \right)^{1/2} \theta(\lambda t) \ddot{w}^2 - \alpha \int \ddot{w}^2 t^{-\mu/2} \left( \frac{t \sigma'}{\sigma} \right)^{1/2} \ddot{w}^2 + \alpha \int O(|x|) t^{-\mu/2} \left( \frac{t \sigma'}{\sigma} \right)^{1/2} \ddot{w}^2. \tag{3.7}
\]

To estimate the last term in the right-hand side of the inequality (3.7), we recall \( \ddot{w} = \sigma^{-\alpha} e^{-|x|^2/4t}w \) and get for some universal \( N \)

\[
\left| \alpha \int O(|x|) t^{-\mu/2} \left( \frac{t \sigma'}{\sigma} \right)^{1/2} \ddot{w}^2 \right| \leq \alpha N \int |x| \left( \frac{t \sigma'}{\sigma} \right)^{1/2} w^2 \sigma^{-2\alpha} G.
\]
We now use the Lemma 2.2 and the fact that \((t^\sigma/\sigma)^{1/2}\) is bounded from above and below by positive constants to obtain
\[
\alpha \int \frac{|x|}{t} \left( \frac{t^\sigma}{\sigma} \right)^{1/2} w^2 \sigma^{-2\alpha} G \leq N N^{2\alpha} \lambda^{2\alpha+N} \int w^2 + N \delta \alpha \int \sigma^{-2\alpha} \theta(\lambda t) \frac{1}{t} w^2 G. \tag{3.8}
\]
Substitute the bound from (3.8) in (3.7) and write \(\bar{w}\) in terms of \(w\) to find
\[
I_2 \geq \frac{\alpha}{2} \int \sigma^{-2\alpha} \left( \frac{t^{\sigma'} \sigma'}{\sigma} \right)^{1/2} \theta(\lambda t) \frac{1}{t} w^2 G - \alpha \int_{\{t=a\}} \sigma^{-2\alpha} \left( \frac{t^{\sigma'} \sigma'}{\sigma} \right)^{1/2} w^2 G \tag{3.9}
\]
\[\quad - N N^{2\alpha} \lambda^{2\alpha+N} \int w^2 - N \delta \alpha \int \sigma^{-2\alpha} \theta(\lambda t) \frac{1}{t} w^2 G. \tag{3.10}\]
Next we consider \(I_3\). Using divergence theorem we find
\[
I_3 = -2 \int \frac{|x|^2}{8t^2} \frac{Z\bar{w}}{2t} t^{-\mu} \left( \frac{t^{\sigma'}}{\sigma} \right)^{-1/2} = - \int \frac{|x|^2}{8t^2} \frac{Z\bar{w}}{2t} t^{-\mu} \left( \frac{t^{\sigma'}}{\sigma} \right)^{-1/2} \tag{3.11}
\]
\[\quad = \int \frac{|x|^2}{8} \left( t^{-\mu-2} \left( \frac{t^{\sigma'}}{\sigma} \right)^{-\frac{1}{2}} \right) \theta(\lambda t) \frac{1}{t} w^2 + \int_{\{t=a\}} \frac{|x|^2}{8} t^{-\mu-2} \left( \frac{t^{\sigma'}}{\sigma} \right)^{-\frac{1}{2}} w^2 \]
\[\quad + \frac{1}{16} \int \text{div}(|x|^2 Ax) t^{-\mu-3} \left( \frac{t^{\sigma'}}{\sigma} \right)^{-\frac{1}{2}} \bar{w}^2. \]
Again using \(A(x)\) is Lipschitz in \(x\)-variable and \(A(0,0) = I_n\), we have \(\text{div}(|x|^2 Ax) = (n+2)|x|^2 + |x|^2 O(|x|)\) and thus after simplifying the first term in right hand side of (3.11) we obtain
\[
I_3 = \frac{-\mu-2}{8} \int |x|^2 t^{-\mu-3} \left( \frac{t^{\sigma'}}{\sigma} \right)^{-\frac{1}{2}} \bar{w}^2 - \frac{1}{16} \int |x|^2 t^{-\mu-2} \left( \frac{t^{\sigma'}}{\sigma} \right)^{-\frac{3}{2}} \left( \frac{t^{\sigma'}}{\sigma} \right)' \bar{w}^2 + \int_{\{t=a\}} \frac{|x|^2}{8} t^{-\mu-2} \left( \frac{t^{\sigma'}}{\sigma} \right)^{-\frac{1}{2}} \bar{w}^2 \tag{3.12}
\]
\[\quad + \frac{n+2}{16} \int |x|^2 t^{-\mu-3} \left( \frac{t^{\sigma'}}{\sigma} \right)^{-\frac{1}{2}} \bar{w}^2 + O(1) \int |x|^3 t^{-\mu-3} \left( \frac{t^{\sigma'}}{\sigma} \right)^{-\frac{1}{2}} \bar{w}^2.
\]
Since \(\mu = \frac{n}{2} - 1\), therefore \(\frac{-\mu-2}{8} = -\frac{n+2}{16}\). Hence after using (2.1), (3.12) becomes
\[
I_3 = \frac{1}{16} \int |x|^2 t^{-n/2-1} \left( \frac{t^{\sigma'}}{\sigma} \right)^{-\frac{1}{2}} \theta(\lambda t) \frac{1}{t} w^2 + \int_{\{t=a\}} \frac{|x|^2}{8} t^{-\mu-2} \left( \frac{t^{\sigma'}}{\sigma} \right)^{-\frac{1}{2}} \bar{w}^2 \tag{3.13}
\]
\[\quad + O(1) \int \frac{|x|^3}{t^2} t^{-n/2} \left( \frac{t^{\sigma'}}{\sigma} \right)^{-\frac{1}{2}} \bar{w}^2.
\]
We estimate the last term of right-hand side of (3.13) using Lemma 2.2 in the similar way as we did the last term in the right-hand side of the inequality (3.7) to find
\[
\left| O(1) \int \frac{|x|^3}{t^2} t^{-n/2} \left( \frac{t^{\sigma'}}{\sigma} \right)^{-\frac{1}{2}} \bar{w}^2 \right| \leq N \alpha N^{2\alpha} \lambda^{2\alpha+N} \int w^2 + N \alpha \delta \int \sigma^{-2\alpha} \theta(\lambda t) \frac{1}{t} w^2 G \left( \frac{t^{\sigma'}}{\sigma} \right)^{1/2}. \tag{3.14}
\]
Hence using (3.14) in (3.13) and writing \( \tilde{w} \) in term of \( w \), we get

\[
I_3 \geq \frac{1}{16} \int \frac{|x|^2}{t} \sigma^{-2\alpha} \left( \frac{t\sigma'}{\sigma} \right)^{-\frac{1}{2}} \frac{\theta(\lambda t)}{t} w^2 G + \int_{\{t=a\}} \frac{|x|^2}{8t} \sigma^{-2\alpha} \left( \frac{t\sigma'}{\sigma} \right)^{-\frac{1}{2}} w^2 G - N\alpha N^{2\alpha} \lambda^{2\alpha+N} \int w^2 - N\alpha \delta \int \sigma^{-2\alpha} \frac{\theta(\lambda t)}{t} w^2 G. \tag{3.15}
\]

Next we estimate \( I_4 \). We use \( \text{div}(Ax) = n + O(|x|) \) to write \( I_4 \) as follows

\[
I_4 = 2 \int \frac{\text{div}(Ax)}{4t} \frac{\tilde{w} \tilde{Z} \tilde{w}}{2t} t^{-\mu} \left( \frac{t\sigma'}{\sigma} \right)^{-1/2} = 2 \int \frac{n}{4t} \tilde{w} \tilde{Z} \tilde{w} t^{-\mu} \left( \frac{t\sigma'}{\sigma} \right)^{-1/2} + O(1) \int \frac{|x|}{4t} \tilde{w} \tilde{Z} \tilde{w} t^{-\mu} \left( \frac{t\sigma'}{\sigma} \right)^{-1/2}. \tag{3.16}
\]

First we focus on the first term of the right hand side of (3.16). After using \( 2\tilde{w} \tilde{Z} \tilde{w} = \tilde{Z} \tilde{w}^2 \) we find

\[
2 \int \frac{n}{4t} \tilde{w} \tilde{Z} \tilde{w} t^{-\mu} \left( \frac{t\sigma'}{\sigma} \right)^{-1/2} = n \int \left( \frac{\tilde{w}^2}{t} \right) t^{-\mu-1} \left( \frac{t\sigma'}{\sigma} \right)^{-1/2}. \tag{3.17}
\]

We now use integration by parts formula in (3.17) to obtain

\[
2 \int \frac{n}{4t} \tilde{w} \tilde{Z} \tilde{w} t^{-\mu} \left( \frac{t\sigma'}{\sigma} \right)^{-1/2} = -n \int \left( \frac{\tilde{w}^2}{t} \right)' t^{-\mu-1} \left( \frac{t\sigma'}{\sigma} \right)^{-1/2} + \frac{n}{8} \int \text{div}(Ax) \tilde{w}^2 t^{-\mu-2} \left( \frac{t\sigma'}{\sigma} \right)^{-1/2}. \tag{3.18}
\]

We further simplify the first term in right-hand side of (3.18) and use \( \text{div}(Ax) = n + O(|x|) \) to get

\[
2 \int \frac{n}{4t} \tilde{w} \tilde{Z} \tilde{w} t^{-\mu} \left( \frac{t\sigma'}{\sigma} \right)^{-1/2} = \frac{n}{4} (\mu + 1) \int t^{-\mu-2} \left( \frac{t\sigma'}{\sigma} \right)^{-1/2} \tilde{w}^2 + \frac{n}{8} \int t^{-\mu-1} \left( \frac{t\sigma'}{\sigma} \right)^{-3/2} \left( \frac{t\sigma'}{\sigma} \right)' \tilde{w}^2 - \frac{n}{4} \int_{\{t=a\}} t^{-\mu-1} \left( \frac{t\sigma'}{\sigma} \right)^{-1/2} \tilde{w}^2 - \frac{n^2}{8} \int \tilde{w}^2 t^{-\mu-2} \left( \frac{t\sigma'}{\sigma} \right)^{-1/2} + O(1) \int |x| \tilde{w}^2 t^{-\mu-2} \left( \frac{t\sigma'}{\sigma} \right)^{-1/2}. \tag{3.19}
\]

Since \( \mu = \frac{2}{3} - 1 \), we have \( \frac{n}{4} (\mu + 1) = \frac{n^2}{8} \). Also, we use (2.1) in the second term of the right-hand side of (3.19). Hence (3.16) becomes

\[
I_4 \geq -\frac{n}{8} \int t^{-n/2} \left( \frac{t\sigma'}{\sigma} \right)^{-1/2} \frac{\theta(\lambda t)}{t} \tilde{w}^2 - \frac{n}{4} \int_{\{t=a\}} t^{-n/2} \left( \frac{t\sigma'}{\sigma} \right)^{-1/2} \tilde{w}^2 - N \int \frac{|x|^2}{t} w^2 t^{-n/2} \left( \frac{t\sigma'}{\sigma} \right)^{-1/2} - N \int \frac{|x|^2}{4t} \tilde{w} \tilde{Z} \tilde{w} t^{-\mu} \left( \frac{t\sigma'}{\sigma} \right)^{-1/2}. \tag{3.20}
\]

We now use Cauchy-Schwarz inequality in the last term of right-hand side of (3.20) to get

\[
N \int \frac{|x|^2}{4t} \tilde{w} \tilde{Z} \tilde{w} t^{-\mu} \left( \frac{t\sigma'}{\sigma} \right)^{-1/2} \leq N^2 \int \frac{|x|^2}{16t^2} \tilde{w}^2 t^{-\mu} \left( \frac{t\sigma'}{\sigma} \right)^{-1/2} + \frac{1}{4} \int \left( \frac{\tilde{Z} \tilde{w}}{2t} \right)^2 t^{-\mu} \left( \frac{t\sigma'}{\sigma} \right)^{-1/2}. \tag{3.21}
\]
Using (3.21) in (3.20) we find
\[
I_4 \geq -\frac{n}{8} \int t^{-n/2} \left( \frac{t\sigma'}{\sigma} \right)^{-1/2} \frac{\theta(\lambda t)}{t} \tilde{w}^2 - \frac{n}{4} \int_{\{t=a\}} t^{-n/2} \left( \frac{t\sigma'}{\sigma} \right)^{-1/2} \tilde{w}^2 \tag{3.22}
\]
\[- N \int \left\langle \frac{x}{t} \tilde{w}^2 t^{-n/2} \left( \frac{t\sigma'}{\sigma} \right)^{-1/2} \right\rangle - N \int \left\langle \frac{|x|^2}{16t^2} \tilde{w}^2 t^{-\mu} \left( \frac{t\sigma'}{\sigma} \right)^{-1/2} \right\rangle - \frac{1}{4} \int \left( \frac{Z\tilde{w}}{2t} \right)^2 t^{-\mu} \left( \frac{t\sigma'}{\sigma} \right)^{-1/2}.
\]
After writing \( \tilde{w} = \sigma^{-\alpha} e^{-|x|^2/8\epsilon t} w \) and using Lemma 2.2 in the third and fourth term of right-hand side of (3.22) we obtain
\[
I_4 \geq -\frac{n}{8} \int \sigma^{-2\alpha} \left( \frac{t\sigma'}{\sigma} \right)^{-1/2} \frac{\theta(\lambda t)}{t} w^2 G - \frac{n}{4} \int_{\{t=a\}} \sigma^{-2\alpha} \left( \frac{t\sigma'}{\sigma} \right)^{-1/2} w^2 G \tag{3.23}
\]
\[- \epsilon N N^{2\alpha} x^{2\alpha + 2} \int w^2 - \epsilon N \delta \int \sigma^{-2\alpha} \frac{\theta(\lambda t)}{t} w^2 G - \frac{1}{4} \int \left( \frac{Z\tilde{w}}{2t} \right)^2 t^{-\mu} \left( \frac{t\sigma'}{\sigma} \right)^{-1/2}.
\]
Over here, we would like to mention that In order to estimate the fourth term in (3.22) above, we used the fact that \( |x|^2 / 16t^2 \) can be upper bounded by \( \frac{|x|^3}{4t^2} \) since \( w(\cdot, t) \) is supported in \( B_4 \).

We then simplify \( I_5 \) by first writing \( 2\tilde{w} Z\tilde{w} = Z\tilde{w}^2 \) to get
\[
I_5 = 2 \int \left\langle \frac{Ax, x}{4t}, \frac{x}{4t} \right\rangle \tilde{w}^2 t^{-\mu} \left( \frac{t\sigma'}{\sigma} \right)^{-1/2} = \int \left\langle \frac{Ax, x}{16t^2} \right\rangle \left( \tilde{w}^2 t^{-\mu} \left( \frac{t\sigma'}{\sigma} \right)^{-1/2} \right).
\]
Integration by parts, we obtain
\[
I_5 = -\int \left\langle \frac{Ax, x}{16} \right\rangle \tilde{w}^2 t^{-\mu} \left( \frac{t\sigma'}{\sigma} \right)^{-1/2} - \int \left\langle \frac{Ax, x}{16t^2} \right\rangle \tilde{w}^2 t^{-\mu} \left( \frac{t\sigma'}{\sigma} \right)^{-1/2}
\]
\[- \int \frac{\text{div}(\langle Ax, x \rangle Ax)}{32t^4} \tilde{w}^2 t^{-\mu} \left( \frac{t\sigma'}{\sigma} \right)^{-1/2}.
\]
First note that \( \text{div}(\langle Ax, x \rangle Ax) = (n + 2)|x|^2 + O(1)|x|^3 \). Consequently after simplification we get the following
\[
I_5 = \frac{\mu + 2}{16} \int \left\langle \frac{Ax, x}{16} \right\rangle \tilde{w}^2 t^{-\mu - 3} \left( \frac{t\sigma'}{\sigma} \right)^{-1/2} + \frac{1}{32} \int \left\langle \frac{Ax, x}{16t^2} \right\rangle \tilde{w}^2 t^{-\mu - 2} \left( \frac{t\sigma'}{\sigma} \right)^{-3/2} \left( \frac{t\sigma'}{\sigma} \right)^{-1/2}
\]
\[- \int \left\langle \frac{Ax, x}{16t^2} \right\rangle \tilde{w}^2 t^{-\mu} \left( \frac{t\sigma'}{\sigma} \right)^{-1/2} - \frac{(n + 2)}{32} \int \frac{|x|^2}{t^3} \tilde{w}^2 t^{-\mu} \left( \frac{t\sigma'}{\sigma} \right)^{-1/2}
\]
\[- O(1) \int \frac{|x|^3}{32t^3} \tilde{w}^2 t^{-\mu} \left( \frac{t\sigma'}{\sigma} \right)^{-1/2}.
\]
After re-writing \( A = A - \mathbb{I}_n + \mathbb{I}_n \) and using (2.1) we obtain
\[
I_5 = \frac{\mu + 2}{16} \int \left\langle (A - \mathbb{I}_n)x, x \right\rangle \tilde{w}^2 t^{-\mu - 3} \left( \frac{t\sigma'}{\sigma} \right)^{-1/2} + \frac{\mu + 2}{16} \int |x|^2 \tilde{w}^2 t^{-\mu - 3} \left( \frac{t\sigma'}{\sigma} \right)^{-1/2}
\]
\[- \frac{1}{32} \int \left\langle (A - \mathbb{I}_n)x, x \right\rangle \tilde{w}^2 t^{-\mu - 2} \left( \frac{t\sigma'}{\sigma} \right)^{-1/2} \frac{\theta(\lambda t)}{t} - \frac{1}{32} \int |x|^2 \tilde{w}^2 t^{-\mu - 2} \left( \frac{t\sigma'}{\sigma} \right)^{-1/2} \frac{\theta(\lambda t)}{t}.
\]
I now simplify \( n \) to right-hand side of (3.24). Therefore using (3.25) in (3.24) we get

\[
I_5 \geq -\frac{1}{32} \int_{t=a} |x|^2 \omega^2 t^{-\mu-2} \left( \frac{t \omega'}{\sigma} \right)^{-1/2} \frac{\theta(\lambda t)}{t} - \int_{t=a} \left( \frac{t \omega'}{\sigma} \right)^{-1/2} \frac{\theta(\lambda t)}{t} - \int_{t=a} \left( \frac{t \omega'}{\sigma} \right)^{-1/2} \frac{\theta(\lambda t)}{t}.
\]

Now the second term of right-hand side of (3.27) is estimated using Lemma 2.2 and we thus find

\[
I_5 \geq -\frac{1}{32} \int_{t=a} \left( \frac{t \omega'}{\sigma} \right)^{-1/2} \frac{\theta(\lambda t)}{t} - \int_{t=a} \left( \frac{t \omega'}{\sigma} \right)^{-1/2} \frac{\theta(\lambda t)}{t} - \int_{t=a} \left( \frac{t \omega'}{\sigma} \right)^{-1/2} \frac{\theta(\lambda t)}{t}.
\]

We now simplify \( I_6 \).

\[
I_6 = 2 \int t^{-\mu} \text{div}(A \nabla \omega) \frac{Z \omega}{2t} \left( \frac{t \omega'}{\sigma} \right)^{-1/2}
\]

\[
= 2 \int t^{-\mu} \text{div}(A \nabla \omega) \omega_t \left( \frac{t \omega'}{\sigma} \right)^{-1/2} + \int t^{-\mu-1} \text{div}(A \nabla \omega) \langle A \nabla \omega, x \rangle \left( \frac{t \omega'}{\sigma} \right)^{-1/2}
\]
We first look at $I_6^1$. We use integrate by parts formula in $x$-variable to obtain

$$I_6^1 = 2 \int t^{-\mu} \text{div}(A \nabla \tilde{w}) \tilde{w}_t \left( \frac{t \sigma'}{\sigma} \right)^{-1/2}$$

$$= -2 \int t^{-\mu} \langle A \nabla \tilde{w}, \nabla \tilde{w}_t \rangle \left( \frac{t \sigma'}{\sigma} \right)^{-1/2}$$

$$= - \int t^{-\mu} \left( \frac{t \sigma'}{\sigma} \right)^{-1/2} \frac{d}{dt} \langle A \nabla \tilde{w}, \nabla \tilde{w} \rangle.$$  

Subsequently, by integrating by parts in the $t$-variable we find

$$I_6^1 = \int \langle A \nabla \tilde{w}, \nabla \tilde{w} \rangle \left( t^{-\mu} \left( \frac{t \sigma'}{\sigma} \right)^{-1/2} \right) + \int_{\{t=a\}} \langle A \nabla \tilde{w}, \nabla \tilde{w} \rangle t^{-\mu} \left( \frac{t \sigma'}{\sigma} \right)^{-1/2}.$$  

After simplifying and using (2.1) we obtain

$$I_6^1 = -\mu \int \langle A \nabla \tilde{w}, \nabla \tilde{w} \rangle t^{-\mu-1} \left( \frac{t \sigma'}{\sigma} \right)^{-1/2} + \frac{1}{2} \int \langle A \nabla \tilde{w}, \nabla \tilde{w} \rangle t^{-\mu} \left( \frac{t \sigma'}{\sigma} \right)^{-1/2} \frac{\theta(t)}{t}$$  

$$+ \int_{\{t=a\}} \langle A \nabla \tilde{w}, \nabla \tilde{w} \rangle t^{-\mu} \left( \frac{t \sigma'}{\sigma} \right)^{-1/2}.  \quad (3.29)$$

We will now focus our attention on $I_6^2$. After using integration by parts formula in $x$-variable we get

$$I_6^2 = - \int t^{-\mu-1} \langle A \nabla \tilde{w}, \nabla (\langle A \nabla \tilde{w}, x \rangle) \rangle \left( \frac{t \sigma'}{\sigma} \right)^{-1/2}$$

$$= - \int t^{-\mu-1} \sum_{i,j,k,l} a_{ij} \tilde{w}_i \frac{\partial}{\partial x_j} (a_{kl} \tilde{w}_k x_l) \left( \frac{t \sigma'}{\sigma} \right)^{-1/2}$$

$$= - \int t^{-\mu-1} \sum_{i,j,k,l} a_{ij} \tilde{w}_i \partial_{x_j} a_{kl} \tilde{w}_k x_l \left( \frac{t \sigma'}{\sigma} \right)^{-1/2} - \int t^{-\mu-1} \sum_{i,j,k,l} a_{ij} \tilde{w}_i a_{kl} \tilde{w}_k x_l \left( \frac{t \sigma'}{\sigma} \right)^{-1/2}$$

$$- \int t^{-\mu-1} \sum_{i,j,k,l} a_{ij} \tilde{w}_i a_{kl} \tilde{w}_k \delta_{ij} \left( \frac{t \sigma'}{\sigma} \right)^{-1/2}.  \quad (3.30)$$

We next observe that

$$\langle Ax, \nabla \langle A \nabla \tilde{w}, \nabla \tilde{w} \rangle \rangle = \sum_{i,j,k,l} a_{kl} x_l \partial_{x_k} a_{ij} \tilde{w}_i \tilde{w}_j + 2 \sum_{i,j,k,l} a_{kl} x_l a_{ij} \tilde{w}_i \tilde{w}_j$$

$$= \sum_{i,j,k,l} a_{kl} x_l \partial_{x_k} a_{ij} \tilde{w}_i \tilde{w}_j + 2 \sum_{i,j,k,l} a_{ij} \tilde{w}_i a_{kl} \tilde{w}_k x_l.$$  

Hence $I_6^2$ becomes

$$I_6^2 = - \int t^{-\mu-1} \sum_{i,j,k,l} a_{ij} \tilde{w}_i \partial_{x_j} a_{kl} \tilde{w}_k x_l \left( \frac{t \sigma'}{\sigma} \right)^{-1/2} + O(1) \int |x| t^{-\mu-1} |\nabla \tilde{w}|^2  \quad (3.31)$$
\[-\frac{1}{2} \int t^{-\mu-1} \langle Ax, \nabla(A \nabla \tilde{w}, \nabla \tilde{w}) \rangle \left(\frac{t \sigma'}{\sigma}\right)^{-1/2} - \int t^{-\mu-1} |A \nabla \tilde{w}|^2 \left(\frac{t \sigma'}{\sigma}\right)^{-1/2}.\]

We again apply integration by parts formula in $x$-variable to the third term on the right-hand side of (3.31) to obtain

\[I_6^2 = \int t^{-\mu-1} \sum_{i,j,k,l} a_{ij} \tilde{w}_i \partial_x a_{kl} \tilde{w}_k x_l \left(\frac{t \sigma'}{\sigma}\right)^{-1/2} + O(1) \int |x| t^{-\mu-1} |\nabla \tilde{w}|^2 + \frac{1}{2} \int t^{-\mu-1} \text{div}(Ax) \langle A \nabla \tilde{w}, \nabla \tilde{w} \rangle \left(\frac{t \sigma'}{\sigma}\right)^{-1/2} - \int t^{-\mu-1} |A \nabla \tilde{w}|^2 \left(\frac{t \sigma'}{\sigma}\right)^{-1/2}.\]  

(3.32)

Again using $\text{div}(Ax) = n + O(1)|x|$ in the third term and by splitting $A$ as $A = A - \mathbb{I}_n + \mathbb{I}_n$ in the last term of right-hand side of (3.32) and also using $|A - \mathbb{I}_n| = O(|x|)$, we find

\[I_6^2 = O(1) \int |x| t^{-\mu-1} |\nabla \tilde{w}|^2 + \left(\frac{n}{2} - 1\right) \int t^{-\mu-1} \langle A \nabla \tilde{w}, \nabla \tilde{w} \rangle \left(\frac{t \sigma'}{\sigma}\right)^{-1/2}.\]  

(3.33)

Hence using (3.29) and (3.33), $I_6$ becomes

\[I_6 = -\mu \int \langle A \nabla \tilde{w}, \nabla \tilde{w} \rangle t^{-\mu-1} \left(\frac{t \sigma'}{\sigma}\right)^{-1/2} + \frac{1}{2} \int \langle A \nabla \tilde{w}, \nabla \tilde{w} \rangle t^{-\mu} \left(\frac{t \sigma'}{\sigma}\right)^{-1/2} \frac{\theta(\lambda t)}{t} + \int_{\{t=a\}} \langle A \nabla \tilde{w}, \nabla \tilde{w} \rangle t^{-\mu} \left(\frac{t \sigma'}{\sigma}\right)^{-1/2} + \frac{n-2}{2} \int t^{-\mu-1} \langle A \nabla \tilde{w}, \nabla \tilde{w} \rangle \left(\frac{t \sigma'}{\sigma}\right)^{-1/2} + O(1) \int |x| t^{-\mu-1} |\nabla \tilde{w}|^2 \left(\frac{t \sigma'}{\sigma}\right)^{-1/2}.\]  

(3.34)

Since $\mu = \frac{n}{2} - 1$, the following terms in the right hand side of (3.34), i.e.

\[-\mu \int \langle A \nabla \tilde{w}, \nabla \tilde{w} \rangle t^{-\mu-1} \left(\frac{t \sigma'}{\sigma}\right)^{-1/2}\]

and

\[\frac{n-2}{2} \int t^{-\mu-1} \langle A \nabla \tilde{w}, \nabla \tilde{w} \rangle \left(\frac{t \sigma'}{\sigma}\right)^{-1/2}\]

cancel each other. Consequently we get

\[I_6 = \frac{1}{2} \int \langle A \nabla \tilde{w}, \nabla \tilde{w} \rangle t^{-\mu} \left(\frac{t \sigma'}{\sigma}\right)^{-1/2} \frac{\theta(\lambda t)}{t} + \int_{\{t=a\}} \langle A \nabla \tilde{w}, \nabla \tilde{w} \rangle t^{-\mu} \left(\frac{t \sigma'}{\sigma}\right)^{-1/2} + O(1) \int |x| t^{-\mu-1} |\nabla \tilde{w}|^2 \left(\frac{t \sigma'}{\sigma}\right)^{-1/2}.\]  

(3.35)

We now recall $\nabla \tilde{w} = \sigma^{-\alpha} e^{-|x|^2/8t} \nabla w - \frac{x}{4t^2} \sigma^{-\alpha} e^{-|x|^2/8t} w$. This by AM-GM inequality implies

\[|\nabla \tilde{w}|^2 \leq 2\sigma^{-2\alpha} e^{-|x|^2/4t} |\nabla w|^2 + \frac{|x|^2}{8t^2} \sigma^{-2\alpha} e^{-|x|^2/4t} w^2.\]
From Lemma 2.1, we have \( t/N \leq \sigma \leq t \) and \( \left( \frac{t \sigma^l}{\sigma} \right)^{-1/2} \) is bounded. Therefore last term in (3.35) can be estimated as
\[
O(1) \int |x| t^{-\mu - 1} |\nabla \tilde{w}|^2 \left( \frac{t \sigma^l}{\sigma} \right)^{-1/2} \leq O(1) \int \sigma^{1-2\alpha} \frac{|x|}{t} |\nabla w|^2 G + O(1) \int \sigma^{-2\alpha} \frac{|x|^3}{t^2} w^2 G. \tag{3.36}
\]
We now use Lemma 2.2 to obtain
\[
O(1) \int |x| t^{-\mu - 1} |\nabla \tilde{w}|^2 \left( \frac{t \sigma^l}{\sigma} \right)^{-1/2} \geq -NN^{2\alpha} \lambda^{2\alpha+N} \int t |\nabla w|^2 - N \delta \int \sigma^{1-2\alpha} \frac{\theta(\lambda t)}{t} |\nabla w|^2 G \tag{3.37}
\]
\[
- \alpha NN^{2\alpha} \lambda^{2\alpha+N} \int w^2 - \alpha N \delta \int \sigma^{-2\alpha} \frac{\theta(\lambda t)}{t} w^2 G.
\]
Hence using (3.37) in (3.35) we deduce the following inequality
\[
I_6 \geq \frac{1}{2} \int \langle A \nabla \tilde{w}, \nabla \tilde{w} \rangle t^{-\mu} \left( \frac{t \sigma^l}{\sigma} \right)^{-1/2} \frac{\theta(\lambda t)}{t} + \int_{\{t = a\}} \langle A \nabla \tilde{w}, \nabla \tilde{w} \rangle t^{-\mu} \left( \frac{t \sigma^l}{\sigma} \right)^{-1/2} \tag{3.38}
\]
\[
- NN^{2\alpha} \lambda^{2\alpha+N} \int t |\nabla w|^2 - N \delta \int \sigma^{1-2\alpha} \frac{\theta(\lambda t)}{t} |\nabla w|^2 G
\]
\[
- \alpha NN^{2\alpha} \lambda^{2\alpha+N} \int w^2 - \alpha N \delta \int \sigma^{-2\alpha} \frac{\theta(\lambda t)}{t} w^2 G.
\]
We now estimate the boundary term in (3.38), i.e. the integral \( \int_{\{t = a\}} \langle A \nabla \tilde{w}, \nabla \tilde{w} \rangle t^{-\mu} \left( \frac{t \sigma^l}{\sigma} \right)^{-1/2} \). By recalling the definition of \( \tilde{w} \) in term of \( w \), we have
\[
\langle A \nabla \tilde{w}, \nabla \tilde{w} \rangle = \sigma^{-2\alpha} e^{-|x|^2/4t} \left( \langle A \nabla w, \nabla w \rangle - 2 \langle w A_{x,4t}, \nabla w \rangle + \frac{w^2}{16t^2} \langle A_{x,x} \rangle \right).
\]
Furthermore using \( \mu = \frac{n}{2} - 1 \) we get
\[
\int_{\{t = a\}} \langle A \nabla \tilde{w}, \nabla \tilde{w} \rangle t^{-\mu} \left( \frac{t \sigma^l}{\sigma} \right)^{-1/2} = \int_{\{t = a\}} \sigma^{-2\alpha} e^{-|x|^2/4t} \left( \langle A \nabla w, \nabla w \rangle - \frac{w A_{x,4t} - \nabla w}{2t} + \frac{w^2}{16t^2} \langle A_{x,x} \rangle \right) t^{-\frac{n}{2} + 1} \left( \frac{t \sigma^l}{\sigma} \right)^{-\frac{1}{2}}
\]
\[
= \int_{\{t = a\}} \sigma^{-2\alpha} \langle A \nabla w, \nabla w \rangle G \left( \frac{t \sigma^l}{\sigma} \right)^{-1/2} - \frac{1}{2} \int_{\{t = a\}} \sigma^{-2\alpha} \langle w A_{x,x} \rangle \nabla w G \left( \frac{t \sigma^l}{\sigma} \right)^{-1/2}
\]
\[
+ \frac{1}{16} \int_{\{t = a\}} \sigma^{-2\alpha} w^2 \langle A_{x,x} \rangle G \left( \frac{t \sigma^l}{\sigma} \right)^{-1/2}.
\]
After using \( \text{div}(Ax) = n + O(|x|) \) and \( \nabla G = -x/2tG \), we find
\[
- \frac{1}{2} \int_{\{t=a\}} \sigma^{-2\alpha} \langle wAx, \nabla w \rangle G \left( \frac{t\sigma'}{\sigma} \right)^{-\frac{1}{2}}
\]

\[
= \frac{n}{4} \int_{\{t=a\}} \sigma^{-2\alpha} w^2 G \left( \frac{t\sigma'}{\sigma} \right)^{-\frac{1}{2}} + O(1) \int_{\{t=a\}} \left| x \right| \sigma^{-2\alpha} w^2 G \left( \frac{t\sigma'}{\sigma} \right)^{-\frac{1}{2}} - \frac{1}{4} \int_{\{t=a\}} \sigma^{-2\alpha} \langle Ax, x \rangle w^2 G \left( \frac{t\sigma'}{\sigma} \right)^{-\frac{1}{2}}.
\]

Hence using (3.40) in (3.39), the boundary term in \( I_6 \) becomes
\[
\int_{\{t=a\}} \langle A\nabla \tilde{w}, \nabla \tilde{w} \rangle t^{-\mu} \left( \frac{t\sigma'}{\sigma} \right)^{-\frac{1}{2}} = \int_{\{t=a\}} t\sigma^{-2\alpha} \langle A\nabla \tilde{w}, \nabla \tilde{w} \rangle G \left( \frac{t\sigma'}{\sigma} \right)^{-\frac{1}{2}} + \frac{n}{4} \int_{\{t=a\}} \sigma^{-2\alpha} w^2 G \left( \frac{t\sigma'}{\sigma} \right)^{-\frac{1}{2}}
\]

\[
+ O(1) \int_{\{t=a\}} \left| x \right| \sigma^{-2\alpha} w^2 G \left( \frac{t\sigma'}{\sigma} \right)^{-\frac{1}{2}} - \frac{1}{4} \int_{\{t=a\}} \sigma^{-2\alpha} \langle Ax, x \rangle w^2 G \left( \frac{t\sigma'}{\sigma} \right)^{-\frac{1}{2}}
\]

\[
+ \frac{1}{16} \int_{\{t=a\}} \sigma^{-2\alpha} w^2 \langle Ax, x \rangle G \left( \frac{t\sigma'}{\sigma} \right)^{-\frac{1}{2}}.
\]

Using \( |x| < 4 \) and also by writing \( A = A - \mathbb{I}_n + \mathbb{I}_n \) we get
\[
\int_{\{t=a\}} \langle A\nabla \tilde{w}, \nabla \tilde{w} \rangle t^{-\mu} \left( \frac{t\sigma'}{\sigma} \right)^{-\frac{1}{2}} \geq \int_{\{t=a\}} t\sigma^{-2\alpha} \langle A\nabla \tilde{w}, \nabla \tilde{w} \rangle G \left( \frac{t\sigma'}{\sigma} \right)^{-\frac{1}{2}} - N \int_{\{t=a\}} \sigma^{-2\alpha} w^2 G \left( \frac{t\sigma'}{\sigma} \right)^{-\frac{1}{2}}
\]

\[
- \frac{1}{16} \int_{\{t=a\}} \sigma^{-2\alpha} \langle (A - \mathbb{I}_n)x, x \rangle \frac{w^2}{t} G \left( \frac{t\sigma'}{\sigma} \right)^{-\frac{1}{2}} - \frac{1}{16} \int_{\{t=a\}} \sigma^{-2\alpha} \left| x \right|^2 \frac{w^2}{t} G \left( \frac{t\sigma'}{\sigma} \right)^{-\frac{1}{2}}.
\]

Thus using (3.42) in (3.38) we find
\[
I_6 \geq \frac{1}{2} \int \langle A\nabla \tilde{w}, \nabla \tilde{w} \rangle t^{-\mu} \frac{\theta(\lambda t)}{t} \left( \frac{t\sigma'}{\sigma} \right)^{-\frac{1}{2}} - NN^2 \lambda^{2\alpha+N} \int t \left| \nabla w \right|^2 - N \delta \int \sigma^{1-2\alpha} \frac{\theta(\lambda t)}{t} \left| \nabla w \right|^2 G
\]

\[
- \alpha NN^2 \lambda^{2\alpha+N} \int w^2 - \alpha N \delta \int \sigma^{-2\alpha} \frac{\theta(\lambda t)}{t} w^2 G + \int \langle A\nabla \tilde{w}, \nabla \tilde{w} \rangle G \left( \frac{t\sigma'}{\sigma} \right)^{-\frac{1}{2}}
\]

\[
- N \int_{\{t=a\}} \sigma^{-2\alpha} w^2 G \left( \frac{t\sigma'}{\sigma} \right)^{-\frac{1}{2}} - \frac{1}{16} \int \langle (A - \mathbb{I}_n)x, x \rangle \frac{w^2}{t} G \left( \frac{t\sigma'}{\sigma} \right)^{-\frac{1}{2}} + \int \sigma^{-2\alpha} \left| x \right|^2 \frac{w^2}{t} G \left( \frac{t\sigma'}{\sigma} \right)^{-\frac{1}{2}}.
\]

By similar computations as in (3.41) we obtain
\[
\int \langle A\nabla \tilde{w}, \nabla \tilde{w} \rangle t^{-\mu} \left( \frac{t\sigma'}{\sigma} \right)^{-\frac{1}{2}} \frac{\theta(\lambda t)}{t} = \int t\sigma^{-2\alpha} \langle A\nabla \tilde{w}, \nabla \tilde{w} \rangle G \left( \frac{t\sigma'}{\sigma} \right)^{-\frac{1}{2}} \frac{\theta(\lambda t)}{t} + \frac{n}{4} \int \sigma^{-2\alpha} w^2 G \left( \frac{t\sigma'}{\sigma} \right)^{-\frac{1}{2}} \frac{\theta(\lambda t)}{t}
\]

\[
+ O(1) \int \left| x \right| \sigma^{-2\alpha} w^2 G \left( \frac{t\sigma'}{\sigma} \right)^{-\frac{1}{2}} \frac{\theta(\lambda t)}{t}
\]

\[
- \frac{1}{16} \int \sigma^{-2\alpha} \langle Ax, x \rangle G \left( \frac{t\sigma'}{\sigma} \right)^{-\frac{1}{2}} \frac{\theta(\lambda t)}{t}.
\]
Now writing $A = A - \mathbb{I}_n + \mathbb{I}_n$ and using $|A - \mathbb{I}_n| = O(|x|)$ we get

$$
\int \langle A \nabla \tilde{w}, \nabla \tilde{w} \rangle t^{-\mu} \left( \frac{t \sigma'}{\sigma} \right)^{-1/2} \frac{\theta(\lambda t)}{t} \geq \int t \sigma^{-2\alpha} \langle A \nabla w, \nabla w \rangle G \left( \frac{t \sigma'}{\sigma} \right)^{-\frac{1}{2}} \frac{\theta(\lambda t)}{t} + \frac{n}{4} \int \sigma^{-2\alpha} w^2 G \left( \frac{t \sigma'}{\sigma} \right)^{-\frac{1}{2}} \frac{\theta(\lambda t)}{t} \tag{3.45}
$$

$$
+ O(1) \int |x| \sigma^{-2\alpha} w^2 G \left( \frac{t \sigma'}{\sigma} \right)^{-\frac{1}{2}} \frac{\theta(\lambda t)}{t} - \frac{1}{16} \int \sigma^{-2\alpha} \frac{|x|^2}{t} G \left( \frac{t \sigma'}{\sigma} \right)^{-\frac{1}{2}} \frac{\theta(\lambda t)}{t} - N \int \sigma^{-2\alpha} \frac{|x|^3}{t} G \left( \frac{t \sigma'}{\sigma} \right)^{-\frac{1}{2}} \frac{\theta(\lambda t)}{t}.
$$

Now in the third term and the last term of (3.45) above, we use $\theta$ is bounded as $\lambda t \leq 1$ and Lemma 2.2 to obtain

$$
\int \langle A \nabla \tilde{w}, \nabla \tilde{w} \rangle t^{-\mu} \left( \frac{t \sigma'}{\sigma} \right)^{-1/2} \frac{\theta(\lambda t)}{t} \geq \int t \sigma^{-2\alpha} \langle \nabla w^2 \rangle G \left( \frac{t \sigma'}{\sigma} \right)^{-\frac{1}{2}} \frac{\theta(\lambda t)}{t} + \frac{n}{4} \int \sigma^{-2\alpha} w^2 G \left( \frac{t \sigma'}{\sigma} \right)^{-\frac{1}{2}} \frac{\theta(\lambda t)}{t} \tag{3.46}
$$

$$
- N \int \sigma^{-2\alpha} \frac{|x|^3}{t} G \left( \frac{t \sigma'}{\sigma} \right)^{-\frac{1}{2}} \frac{\theta(\lambda t)}{t}.
$$

We thus have

$$
I_6 \geq \frac{1}{N} \int \sigma^{1-2\alpha} \langle |\nabla w|^2 \rangle t \left( \frac{t \sigma'}{\sigma} \right)^{-\frac{1}{2}} - \frac{1}{32} \int \sigma^{-2\alpha} \langle \nabla w^2 \rangle G \left( \frac{t \sigma'}{\sigma} \right)^{-\frac{1}{2}} \frac{\theta(\lambda t)}{t} + \frac{n}{8} \int \sigma^{-2\alpha} w^2 G \left( \frac{t \sigma'}{\sigma} \right)^{-\frac{1}{2}} \frac{\theta(\lambda t)}{t} \tag{3.47}
$$

$$
- N \int \sigma^{-2\alpha} \frac{|x|^3}{t} G \left( \frac{t \sigma'}{\sigma} \right)^{-\frac{1}{2}} + \frac{1}{16} \left[ \int \sigma^{-2\alpha} \langle (A - \mathbb{I}_n) x, x \rangle w^2 G \left( \frac{t \sigma'}{\sigma} \right)^{-\frac{1}{2}} + \int \sigma^{-2\alpha} \frac{|x|^2}{t} w^2 G \left( \frac{t \sigma'}{\sigma} \right)^{-\frac{1}{2}} \right].
$$

We now focus our attention on $I_7$. Again using $2 \tilde{w} \bar{Z} \tilde{w} = \bar{Z} \tilde{w}^2$, we get

$$
I_7 = \int V \bar{Z} \tilde{w}^2 \left( \frac{t \sigma'}{\sigma} \right)^{-1/2} \frac{\theta(\lambda t)}{t} = \int V (\partial_t \tilde{w}^2 + \langle A \nabla \tilde{w}^2, \frac{x}{2t} \rangle) t^{-\mu} \left( \frac{t \sigma'}{\sigma} \right)^{-1/2}.
$$

After using integration by parts formula we obtain

$$
I_7 = - \int V \tilde{w}^2 \left( t^{-\mu} \left( \frac{t \sigma'}{\sigma} \right)^{-1/2} \right) + \int_{\{t = a\}} V \tilde{w}^2 t^{-\mu} \left( \frac{t \sigma'}{\sigma} \right)^{-1/2} - \frac{1}{2} \int \text{div}(V A x) \bar{w}^2 t^{-\mu-1} \left( \frac{t \sigma'}{\sigma} \right)^{-1/2}.
$$
It is easy to see that after some simplification and using (2.1) we get

\[
I_7 = \mu \int V \tilde{w}^2 t^{-\mu-1} \left( \frac{t \sigma'}{\sigma} \right)^{-1/2} - \frac{1}{2} \int V \tilde{w}^2 t^{-\mu} \left( \frac{t \sigma'}{\sigma} \right)^{-1/2} \theta(\lambda t) \left( \frac{t \sigma'}{\sigma} \right)^{-1/2} - \int_{\{t=a\}} V \tilde{w}^2 t^{-\mu} \left( \frac{t \sigma'}{\sigma} \right)^{-1/2} \tag{3.48}
\]

\[- \frac{1}{2} \int (\nabla V, Ax) \tilde{w}^2 t^{-\mu-1} \left( \frac{t \sigma'}{\sigma} \right)^{-1/2} - \frac{1}{2} \int V \text{div}(Ax) \tilde{w}^2 t^{-\mu-1} \left( \frac{t \sigma'}{\sigma} \right)^{-1/2}.
\]

We now use \(\text{div}(Ax) = n + O(|x|)\) and rearrange terms to obtain

\[
I_7 = \left( \mu - \frac{n}{2} \right) \int V \tilde{w}^2 t^{-\mu-1} \left( \frac{t \sigma'}{\sigma} \right)^{-1/2} - \frac{1}{2} \int V \tilde{w}^2 t^{-\mu} \left( \frac{t \sigma'}{\sigma} \right)^{-1/2} \frac{\theta(\lambda t)}{t} - \int_{\{t=a\}} V \tilde{w}^2 t^{-\mu} \left( \frac{t \sigma'}{\sigma} \right)^{-1/2} \tag{3.49}
\]

\[- \frac{1}{2} \int (\nabla V, Ax) \tilde{w}^2 t^{-\mu-1} \left( \frac{t \sigma'}{\sigma} \right)^{-1/2} + O(1) \int |x| V \tilde{w}^2 t^{-\mu-1} \left( \frac{t \sigma'}{\sigma} \right)^{-1/2}.
\]

Since \(\lambda t \leq 1\), using the boundedness of \(\theta\) and also that \(Ax = O(|x|)\), we deduce the following inequality

\[
I_7 \geq \left( \mu - \frac{n}{2} \right) \int |V| \tilde{w}^2 t^{-\mu-1} \left( \frac{t \sigma'}{\sigma} \right)^{-1/2} - \frac{N}{2} \int |V| \tilde{w}^2 t^{-\mu-1} \left( \frac{t \sigma'}{\sigma} \right)^{-1/2} - \int_{\{t=a\}} tV \tilde{w}^2 t^{-\mu-1} \left( \frac{t \sigma'}{\sigma} \right)^{-1/2} \tag{3.50}
\]

\[- \frac{N}{2} \int |\nabla V| |x| \tilde{w}^2 t^{-\mu-1} \left( \frac{t \sigma'}{\sigma} \right)^{-1/2} - N \int |x| |V| \tilde{w}^2 t^{-\mu-1} \left( \frac{t \sigma'}{\sigma} \right)^{-1/2}.
\]

Now using \(|x| \leq B_4\) and by writing \(\tilde{w}\) in terms of \(w\) we get

\[
I_7 \geq -N(||V(\cdot,0)||_\infty + ||\nabla V(\cdot,0)||_\infty) \int \sigma^{-2\alpha} w^2 G \left( \frac{t \sigma'}{\sigma} \right)^{-1/2} - \int_{\{t=a\}} t\sigma^{-2\alpha} V w^2 G \left( \frac{t \sigma'}{\sigma} \right)^{-1/2}. \tag{3.51}
\]

We now substitute the bounds for \(I_2, I_3, \ldots, I_7\) as in (3.9), (3.15), (3.23), (3.28), (3.47), (3.51) respectively in (3.6) to obtain

\[
\int \sigma^{-2\alpha} e^{-\frac{|w|^2}{4t}} (|w| + \text{div}(A \nabla w) + V w)^2 t^{-\mu} \left( \frac{t \sigma'}{\sigma} \right)^{-1/2} \tag{3.52}
\]

\[\geq \left( \frac{\alpha}{2} - \frac{n}{8} + \frac{n}{8N} \right) \int \sigma^{-2\alpha} \left( \frac{t \sigma'}{\sigma} \right)^{1/2} \theta(\lambda t) w^2 G + \left( \frac{1}{16} - \frac{32}{32} - \frac{1}{32} \right) \int \sigma^{-2\alpha} |x|^2 \left( \frac{t \sigma'}{\sigma} \right)^{-1/2} \theta(\lambda t) w^2 G \]

\[\quad - N ||V(\cdot,0)||_1 \int \sigma^{-2\alpha} w^2 G \left( \frac{t \sigma'}{\sigma} \right)^{-1/2} - N(1 + \alpha)N^{2\alpha} \lambda^{2\alpha + N} \int w^2 - N(1 + \alpha) \delta \int \sigma^{-2\alpha} \theta(\lambda t) t^{-1/2} \rho \]

\[\quad + \frac{1}{N} \int \sigma^{-2\alpha} \nabla w^2 \left( \frac{\theta(\lambda t)}{t} \right)^{-1/2} - NN^{2\alpha} \lambda^{2\alpha + N} \int t|\nabla w|^2 - N \delta \int \sigma^{1-2\alpha} \theta(\lambda t) t^{-1/2} |\nabla w|^2 G \]

\[\quad + \left( \frac{1}{8} - \frac{1}{16} - \frac{1}{16} \right) \int \sigma^{-2\alpha} \left( \frac{t \sigma'}{\sigma} \right)^{-1/2} - 2 \int_{\{t=a\}} \sigma^{-2\alpha} \frac{(A - \mathbb{I}_N)x, x}{16t} w^2 G \left( \frac{t \sigma'}{\sigma} \right)^{-1/2}.
\]
\[ + \int_{\{t=a\}} t^{\sigma - 2a} \langle A \nabla w, \nabla w \rangle G \left( \frac{t^{\sigma'}}{\sigma} \right)^{-\frac{1}{2}} - (\alpha + n/4 + N) \int_{\{t=a\}} \sigma^{-2a} w^2 G \left( \frac{t^{\sigma'}}{\sigma} \right)^{-\frac{1}{2}} \]

\[ - ||V(\cdot, 0)||_1 \int_{\{t=a\}} t^{\sigma - 2a} w^2 G \left( \frac{t^{\sigma'}}{\sigma} \right)^{-\frac{1}{2}}. \]

We use \( 1/N \leq \left( \frac{t^{\sigma'}}{\sigma} \right)^{-\frac{1}{2}} \leq N \), \( \langle A \nabla w, \nabla w \rangle \sim |\nabla w|^2 \) and subsequently we obtain

\[ N \int \sigma^{-2a} e^{-\frac{|x|^2}{4t}} (w_t + \text{div}(A \nabla w) + V w)^2 t^{-\mu} \]

\[ \geq \frac{3}{4} \left( \frac{Z \hat{w}}{2t} \right)^2 t^{-\mu} \left( \frac{t^{\sigma'}}{\sigma} \right)^{-1/2} + \left( \frac{\alpha}{2N} - \frac{nN}{8} + \frac{n}{8N^2} - N(1 + \alpha)\delta \right) \int \sigma^{-2a} \theta(\lambda t) \frac{t^{\sigma'}}{t} w^2 G - N^2 ||V(\cdot, 0)||_1 \int \sigma^{-2a} w^2 G \]

\[ - N(1 + \alpha)N^{2a} \chi^{2a + N} \int w^2 + \left( \frac{1}{N} - N\delta \right) \int \sigma^{-2a} |\nabla w|^2 G \theta(\lambda t) - N N^{2a} \chi^{2a + N} \int t|\nabla w| \]

\[ - 2 \int_{\{t=a\}} \sigma^{-2a} \frac{\langle (A - \mathbb{I}_n) x, x \rangle}{16t} w^2 G + \frac{1}{N} \int_{\{t=a\}} t^{\sigma - 2a} |\nabla w|^2 G \]

\[ - N(\alpha + n/4 + N - a ||V(\cdot, 0)||_1) \int_{\{t=a\}} \sigma^{-2a} w^2 G. \]

We now estimate \( N \int_{\{t=a\}} \sigma^{-2a} \frac{\langle (A - \mathbb{I}_n) x, x \rangle}{16t} w^2 G \). To do this, we first observe that since \( \sigma(t) \sim t \), therefore for \( |x| \geq \delta \), it is easy to see that

\[ \frac{|x|^3}{a} G(x, a) \sigma^{-2a}(a) \leq N^{2a} \chi^{2a + N}. \]

Using this along with the fact that \( |A(x, 0) - \mathbb{I}_n| = O(|x|) \), we find

\[ 2 \left| \int_{\{t=a\}} \sigma^{-2a} \frac{\langle (A - \mathbb{I}_n) x, x \rangle}{16t} w^2 G \right| \leq N \int \sigma^{-2a}(a) \frac{|x|^3}{a} w^2(x, a) G(x, a) dx \]

\[ = N \int_{B_\delta} \sigma^{-2a}(a) \frac{|x|^3}{a} w^2(x, a) G(x, a) dx + N \int_{|x| > \delta} \sigma^{-2a}(a) \frac{|x|^3}{a} w^2(x, a) G(x, a) dx \]

\[ \leq N \delta \int \sigma^{-2a}(a) \frac{|x|^2}{a} w^2(x, a) G(x, a) dx + N^{2a} \chi^{2a + N} \int_{|x| > \delta} w^2(x, a) dx \]

\[ \leq 2N^2 \delta \int a \sigma^{-2a}(a) |\nabla w(x, a)|^2 G(x, a) dx + N^2 \delta \int \sigma^{-2a}(a) w^2(x, a) G(x, a) dx + N^{2a+1} \chi^{2a + N} \int w^2(x, a) dx, \]

where in the last inequality we have used Lemma 2.3. Hence (3.53) becomes

\[ N \int \sigma^{-2a} e^{-\frac{|x|^2}{4t}} (w_t + \text{div}(A \nabla w) + V w)^2 t^{-\mu} \]

\[ \geq \frac{3}{4} \left( \frac{Z \hat{w}}{2t} \right)^2 t^{-\mu} \left( \frac{t^{\sigma'}}{\sigma} \right)^{-1/2} + \left( \frac{\alpha}{2N} - \frac{nN}{8} + \frac{n}{8N^2} - N(1 + \alpha)\delta \right) \int \sigma^{-2a} \theta(\lambda t) \frac{t^{\sigma'}}{t} w^2 G - N^2 ||V(\cdot, 0)||_1 \int \sigma^{-2a} w^2 G \]

\[ - N(1 + \alpha)N^{2a} \chi^{2a + N} \sup_{t \geq a} \int w^2(x, t) dx + \left( \frac{1}{N} - N\delta \right) \int \sigma^{-2a} |\nabla w|^2 G \theta(\lambda t) - N N^{2a} \chi^{2a + N} \int t|\nabla w| \]

\[ + \left( \frac{1}{N} - N^2 \delta \right) \int_{\{t=a\}} t^{\sigma - 2a} |\nabla w|^2 G - N(\alpha + n/4 + N - a ||V(\cdot, 0)||_1 - N\delta) \int_{\{t=a\}} \sigma^{-2a} w^2 G. \]
Now we choose $\delta > 0$ small enough such that
\[
\frac{\alpha}{2N} - \frac{nN}{8} + \frac{n}{8N^2} - N(1 + \alpha)\delta \geq \frac{\alpha}{4N},
\]
and
\[
\frac{1}{N} - N\delta \geq \frac{1}{2N}.
\]
At this point, we make the preliminary yet crucial observation that for $t \leq \frac{1}{2a}$
\[
\frac{\theta(\lambda t)}{t} \geq \lambda^{1/2} t^{-1/2} (\log 2)^{3/2} \sim \lambda. \tag{3.56}
\]
Hence we obtain
\[
\int \sigma^{-2\alpha}e^{-\frac{|x|^2}{4N}} (w_t + \text{div}(A\nabla w) + Vw)^2 t^{-\mu} \left(\frac{t\sigma^2}{\sigma}\right)^{-1/2} \geq \frac{\alpha \lambda}{4N} \int \sigma^{-2\alpha}w^2 G - N^2 ||V(\cdot, 0)||_1 \int \sigma^{-2\alpha}w^2 G - N^2 ||V(\cdot, 0)||_1 \int \sigma^{-2\alpha}w^2 G.
\]
\[
- N(1 + \alpha)N^2\alpha^2 + N^2 \sup_{t \geq a} \int w^2 + \frac{\lambda}{2N} \int \sigma^{1-2\alpha} |\nabla w|^2 G - N^2 \alpha^2 + N^2 \int |\nabla w|^2 G - N^2 \alpha^2 \int |\nabla w|^2 G + \frac{\lambda t}{t} \sigma^{-2\alpha} |\nabla w|^2 G - N^2 \alpha^2 + N^2 \int |\nabla w|^2 G + \frac{\lambda t}{t} \sigma^{-2\alpha} |\nabla w|^2 G.
\]
If we now choose $\alpha^2 \geq 8N^3\delta^2 ||V(\cdot, 0)||_1$ then
\[
\frac{\alpha \lambda}{4N} \int \sigma^{-2\alpha}w^2 G - N^2 ||V(\cdot, 0)||_1 \int \sigma^{-2\alpha}w^2 G \geq \frac{\alpha \lambda}{8N} \int \sigma^{-2\alpha}w^2 G.
\]
Moreover since $a \alpha < 1$ we find
\[
a ||V(\cdot, 0)||_1 \leq \frac{aa^2}{8N^3\delta^2} \leq \frac{\alpha}{8N^3\delta^2}.
\]
Consequently for sufficiently large $N$, $\alpha \geq N ||V(\cdot, 0)||_1^{1/2}$ and $0 < a < 1/\lambda$, we get
\[
N \int \sigma^{-2\alpha}e^{-\frac{|x|^2}{4N}} (w_t + \text{div}(A\nabla w) + Vw)^2 t^{-\mu} \geq \frac{\alpha^2}{N} \int \sigma^{-2\alpha}w^2 - N^2 \alpha^2 \int w^2(x, t)dx + \frac{1}{N} \int \sigma^{1-2\alpha} |\nabla w|^2 \frac{\theta(\lambda t)}{t} G - N^2 \alpha^2 \int |\nabla w|^2 G + \frac{1}{N} \int_{\{t=a\}} t \sigma^{-2\alpha} |\nabla w|^2 G - N^2 \alpha^2 \int_{\{t=a\}} w^2 - N \int_{\{t=a\}} \sigma^{-2\alpha}w^2 G.
\]
The estimate (3.2) thus follows.

**Step 2:** We now show that there exists a universal constant $N$ and $\delta$ such that for all $\alpha \geq N(1 + ||V(\cdot, 0)||_1^{1/2})$ the following inequality holds
\[
\alpha^2 \int \sigma^{-2\alpha}w^2 G_a + \alpha \int \sigma^{1-2\alpha} |\nabla w|^2 G_a \leq N \int \sigma_a^{-2\alpha} (\text{div}(A(x, t)\nabla w) + w_t + V(x, 0)w)^2 G_a + N^2 \alpha^2 \sup_{t \geq 0} \int w^2 + t |\nabla w|^2 dx
\]
+ \sigma(a)^{-2\alpha} \left( \frac{a}{N} \int |\nabla w(x, 0)|^2 G(x, a) dx + N\alpha \int w^2(x, 0) G(x, a) dx \right).

for all $0 < a \leq \frac{1}{M}$ and $w \in C_0^\infty(B_1 \times [0, 2M])$, where $\lambda = \alpha/\delta^2$.

Note that the difference from Step 1 is that we replace $A(x, 0)$ by $A(x, t)$. This essentially follows from

(3.2) using similar ideas in [10]. We nevertheless provide all the details for the sake of completeness in
to track the precise dependence of the parameter $\alpha$.

**Proof of Step 2:** We write

$$\text{div}(A(x, 0) \nabla w) + w_t + V w = \text{div}((A(x, 0) - A(x, t)) \nabla w) + \text{div}(A(x, t) \nabla w) + w_t + V w.$$ 

This gives

$$N \int \sigma^{1-2\alpha} \text{div}(A(x, 0) \nabla w) + w_t + V w)^2 G_a \leq 2N \int \sigma_a^{1-2\alpha}(\text{div}((A(x, 0) - A(x, t)) \nabla w)^2 G_a \tag{3.59}$$

$$+ 2N \int \sigma_a^{1-2\alpha}(\text{div}(A(x, t) \nabla w) + w_t + V w)^2 G_a.$$ 

This suggests we have to estimate

$$N \int \sigma_a^{1-2\alpha}(\text{div}((A(x, 0) - A(x, t)) \nabla w)^2 G_a. \tag{3.60}$$

To simplify this term note that

$$\text{div}((A(x, 0) - A(x, t)) \nabla w) = \sum \partial_j (a_{ij}(x, 0) - a_{ij}(x, t)) w_i + (a_{ij}(x, 0) - a_{ij}(x, t)) w_{ij}.$$ 

Also, we have $|a_{ij}(x, 0) - a_{ij}(x, t)| \leq M \sqrt{t} \leq M \sqrt{t + a} \leq MN \sigma_a(t)$. Therefore we have

$$N \int \sigma_a^{1-2\alpha}(\text{div}((A(x, 0) - A(x, t)) \nabla w)^2 G_a \leq N M^2 \int \sigma_a^{1-2\alpha} |\nabla w|^2 G_a + N^2 M^2 \int \sigma_a^{2-2\alpha} |D^2 w|^2 G_a. \tag{3.61}$$

Thus we will be done if we could estimate $\int \sigma_a^{2-2\alpha} |D^2 w|^2 G_a$. We now proceed to estimate $\int \sigma_a^{2-2\alpha} |D^2 w|^2 G_a$.

In the ensuing computations below, we let $A(\cdot) = A(\cdot, 0)$ and $V = V(\cdot, 0)$. First we calculate for $k = 1, 2, ..., n$

$$\partial_kw_k^2 + \text{div}(A \nabla(w_k^2)) = 2w_k \partial_tv_k + 2 \text{div}(A \nabla w_k) \tag{3.62}$$

$$= 2w_k \partial_tv_k + 2 \text{div}(A \nabla w_k)w_k + 2 \langle A \nabla w_k, \nabla w_k \rangle.$$ 

We now define $H(w) := w_t + \text{div}(A(x, 0) \nabla w)$. We let $\partial_k A$ be the matrix with entries $(\partial_k A)_{ij} = \partial_k a_{ij}$. Now we calculate

$$\partial_k(w_k(\partial_kw + \text{div}(A \nabla w) + V w))$$

$$= w_{kk}(H(w) + V w) + w_k \partial_t w_k + w_k \text{div}((\partial_k A) \nabla w) + w_k \text{div}(A \nabla w_k) + w_k V_k w + V w_k^2$$

$$= w_{kk}(H(w) + V w) + w_k \partial_t w_k + \text{div}(w_k(\partial_k A) \nabla w) - \langle \nabla w_k, (\partial_k A) \nabla w \rangle + w_k \text{div}(A \nabla w_k) + w_k V_k w + V w_k^2.$$
where we have used $w_k \operatorname{div}((\partial_k A) \nabla w) = \operatorname{div}(w_k (\partial_k A) \nabla w) - \langle \nabla w_k, (\partial_k A) \nabla w \rangle$. We then rewrite it as

$$w_k \operatorname{div}(A \nabla w_k) = \partial_k(w_k H(w) + V w)) - w_{kk}(H(w) + V w) - w_k \partial_k w_k - \operatorname{div}(w_k (\partial_k A) \nabla w)$$

$$+ \langle \nabla w_k, (\partial_k A) \nabla w \rangle - w_k V_k w - V w_k^2. \quad (3.63)$$

We now use (3.63) in (3.62) to get

$$\partial_t w_k^2 + \operatorname{div}(A \nabla (w_k^2)) = 2 w_k \partial_t w_k + 2 \partial_k(w_k (H(w) + V w)) - 2 w_{kk}(H(w) + V w) - 2 w_k \partial_t w_k - 2 \operatorname{div}(w_k (\partial_k A) \nabla w)$$

$$+ 2 \langle \nabla w_k, (\partial_k A) \nabla w \rangle + 2 \langle A \nabla w_k, \nabla w_k \rangle - 2 w_k V_k w - 2 V w_k^2$$

$$= 2 \partial_t(w_k (H(w) + V w)) - 2 w_{kk}(H(w) + V w) - 2 \operatorname{div}(w_k (\partial_k A) \nabla w) + 2 \langle \nabla w_k, (\partial_k A) \nabla w \rangle$$

$$+ 2 \langle A \nabla w_k, \nabla w_k \rangle - 2 w_k V_k w - 2 V w_k^2.$$

We can now rewrite the above identity as

$$2 \langle A \nabla w_k, \nabla w_k \rangle = H(w_k^2) - 2 \partial_t(w_k (H(w) + V w)) + 2 w_{kk}(H(w) + V w) + 2 \operatorname{div}(w_k (\partial_k A) \nabla w)$$

$$- 2 \langle \nabla w_k, (\partial_k A) \nabla w \rangle + 2 w_k V_k w + 2 V w_k^2.$$

We now multiply by $\sigma^{2-2\alpha} G$ and use integration by parts formula to obtain

$$2 \sum_k \int \sigma^{2-2\alpha} (A \nabla w_k, \nabla w_k) G = \int \sigma^{2-2\alpha} |\nabla w|^2 H^*(G) + (2\alpha - 2) \int \sigma^{1-2\alpha} \sigma' |\nabla w|^2 G - \int_{\{t=a\}} \sigma^{2-2\alpha} |\nabla w|^2 G$$

$$+ 2 \int \sigma^{2-2\alpha} \langle \nabla w, \nabla G \rangle (H(w) + V w) + 2 \sum_k \int \sigma^{2-2\alpha} w_{kk}(H(w) + V w) G - 2 \sum_k \int \sigma^{2-2\alpha} w_k ((\partial_k A) \nabla w, \nabla G)$$

$$- 2 \sum_k \int \sigma^{2-2\alpha} \langle \nabla w_k, (\partial_k A) \nabla w \rangle G + 2 \int \sigma^{2-2\alpha} \langle \nabla w, \nabla V \rangle w G + 2 \int \sigma^{2-2\alpha} V |\nabla w|^2 G,$$

where $H^* G = \operatorname{div}(A(x,0) \nabla G) - G_t$. On using the Cauchy-Schwarz inequality and $|\nabla G| \leq \frac{1}{4t} G \leq \frac{4}{t} G$ on the right-hand side of (3.64), we find

$$2 \sum_k \int \sigma^{2-2\alpha} (A \nabla w_k, \nabla w_k) G \leq \int \sigma^{2-2\alpha} |\nabla w|^2 H^*(G) + (2\alpha - 2) \int \sigma^{1-2\alpha} \sigma' |\nabla w|^2 G - \int_{\{t=a\}} \sigma^{2-2\alpha} |\nabla w|^2 G$$

$$+ \int \sigma^{2-2\alpha} |\nabla w|^2 \frac{|x|^2}{16 t^2} G + \int \sigma^{2-2\alpha} (H(w) + V w)^2 G + \frac{1}{4N} \sum_k \int \sigma^{2-2\alpha} w_{kk}^2 G + 4nN \int \sigma^{2-2\alpha} (H(w) + V w)^2 G$$

$$+ N \int \sigma^{1-2\alpha} |\nabla w|^2 G + \frac{1}{4N} \sum_k \int \sigma^{2-2\alpha} |\nabla w_k|^2 G + 4N^2 \int \sigma^{2-2\alpha} |\nabla w|^2 G$$

$$+ \int \sigma^{2-2\alpha} |\nabla w|^2 G + \int \sigma^{2-2\alpha} |\nabla V|^2 w^2 G + 2 \int \sigma^{2-2\alpha} |V||\nabla w|^2 G.$$
We now use
\[ \sum_k \langle A \nabla w_k, \nabla w_k \rangle \geq \frac{1}{N} |D^2 w|^2, \]
\[ \sigma < 1, \ \sigma' = O(1) \] and combine the like terms to get
\[ \begin{align*}
\frac{6}{4N} \int \sigma^2 - 2\alpha |D^2 w|^2 G &\leq \int \sigma^2 - 2\alpha |\nabla w|^2 H^*(G) + (2\alpha - 1 + N + 4N^2) \int \sigma^{1-2\alpha} |\nabla w|^2 G - \int \sigma^{2-2\alpha} |\nabla w|^2 G \\
&\quad + \int \sigma^{2-2\alpha} |\nabla w|^2 \frac{|x|^2}{16t^2} G + (1 + 4nN) \int \sigma^{2-2\alpha}(H(w) + Vw)^2 G + \int \sigma^{2-2\alpha} |\nabla V|^2 w^2 G + \int \sigma^{2-2\alpha} |V||\nabla w|^2 G. \tag{3.66}
\end{align*} \]
We take \( \alpha \) large enough such that \( \alpha > 4N^2 + N \). Since we have \( t < 1/\lambda, \ \sigma < t \) and \( \lambda = \alpha/\delta^2 \), therefore we get \( \sigma \leq \delta^2/\alpha \). Consequently (3.66) becomes
\[ \begin{align*}
\frac{6}{4N} \int \sigma^2 - 2\alpha |D^2 w|^2 G &\leq \int \sigma^{2-2\alpha} |\nabla w|^2 H^*(G) + 3\alpha \int \sigma^{1-2\alpha} |\nabla w|^2 G - \int \sigma^{2-2\alpha} |\nabla w|^2 G + \int \sigma^{2-2\alpha} |\nabla w|^2 \frac{|x|^2}{16t^2} G \\
&\quad + \frac{(1 + 4nN)\delta^2}{\alpha} \int \sigma^{1-2\alpha}(H(w) + Vw)^2 G + \frac{\delta^4}{\alpha^2} \int \sigma^{2-2\alpha} |\nabla V|^2 w^2 G + \frac{\delta^2}{\alpha} \int \sigma^{1-2\alpha} |V||\nabla w|^2 G. \tag{3.67}
\end{align*} \]
We will now take \( \alpha \) larger if necessary such that \( \alpha \geq N||V(\cdot, 0)||^2_{1/2} \). Consequently \( ||\nabla V||^2_{\infty} \leq \alpha^4/N^4 \) and \( ||V||_{\infty} \leq \alpha^2/N^2 \). Hence (3.67) becomes
\[ \begin{align*}
\frac{6}{4N} \int \sigma^2 - 2\alpha |D^2 w|^2 G &\leq \int \sigma^{2-2\alpha} |\nabla w|^2 H^*(G) + 3\alpha \int \sigma^{1-2\alpha} |\nabla w|^2 G - \int \sigma^{2-2\alpha} |\nabla w|^2 G \\
&\quad + \int \sigma^{2-2\alpha} |\nabla w|^2 \frac{|x|^2}{t^2} G + \frac{1 + 4nN)\delta^2}{\alpha} \int \sigma^{1-2\alpha}(H(w) + Vw)^2 G + \frac{\alpha^2\delta^4}{N^4} \int \sigma^{2-2\alpha} w^2 G + \frac{\alpha\delta^2}{N^2} \int \sigma^{1-2\alpha} |\nabla w|^2 G. \tag{3.68}
\end{align*} \]
Next we note that
\[ H^* G = \text{div}(A \nabla G) - G_t = \sum a_{ij}G_{ij} + \sum \delta_ja_{ij}G_i - G_t \]
\[ = \sum (a_{ij} - \delta_{ij})G_{ij} + \sum \delta_ja_{ij}G_i + \sum \delta_{ij}G_{ij} - G_t \]
\[ = \sum (a_{ij} - \delta_{ij})G_{ij} + \sum \delta_j(a_{ij})G_i, \]
where we have used \( G \) is a fundamental solution of heat equation. We now use \( A \) is Lipschitz to get
\[ |H^* G| \leq \sum |a_{ij} - \delta_{ij}||G_{ij}| + \sum |\delta_j(a_{ij})||G_i| \]
\[ \leq N \left( \frac{|x|}{t} + \frac{|x|^3}{t^2} \right). \]
Hence using \( \sigma \leq t \) and \( x \in B_4 \) we have
\[ \int \sigma^{2-2\alpha} |\nabla w|^2 H^*(G) + \int \sigma^{2-2\alpha} |\nabla w|^2 \frac{|x|^2}{t^2} G \leq N \int t\sigma^{1-2\alpha} |\nabla w|^2 \left( \frac{|x|}{t} + \frac{|x|^3}{t^2} + \frac{|x|^2}{t^2} \right) G \]
\[ \leq N \int \sigma^{1-2\alpha} |\nabla w|^2 \left( \frac{|x|^2}{t} \right) G \]
\[ \leq N \int \sigma^{1-2\alpha} |\nabla w|^2 \left( 1 + \frac{|x|^2}{t} \right) G \]
\[ \leq N \int \sigma^{1-2\alpha} |\nabla w|^2 \left( 1 + \frac{|x|^2}{t} \right) G. \tag{3.69} \]
\[
\leq \delta NN^{2\alpha} \lambda^{2\alpha+N} \int t|\nabla w|^2 + N\delta^2 \int \sigma^{1-2\alpha} \frac{\theta(\lambda t)}{t} |\nabla w|^2 G + N \int \sigma^{1-2\alpha} |\nabla w|^2 G,
\]
where the last inequality follows from Lemma 2.2. We now use (3.69) in (3.68) to find for all large \(\alpha\)
\[
\frac{6}{4N} \int \sigma^{2-2\alpha}|D^2 w|^2 G \leq \delta NN^{2\alpha} \lambda^{2\alpha+N} \int t|\nabla w|^2 + N\delta^2 \int \sigma^{1-2\alpha} \frac{\theta(\lambda t)}{t} |\nabla w|^2 G + 4\alpha \int \sigma^{1-2\alpha} |\nabla w|^2 G
\]
(3.70)
\[
- \int \{t=a\} \sigma^{2-2\alpha} |\nabla w|^2 G + \frac{(1 + 4nN)\delta^2}{\alpha} \int \sigma^{1-2\alpha} (H(w) + Vw)^2 G + \frac{\alpha^2\delta^4}{N^4} \int \sigma^{-2\alpha} w^2 G + \frac{\alpha^2\delta^4}{N^4} \int \sigma^{-2\alpha} w^2 G.
\]
Now for large enough \(\alpha\), it follows from (3.72) by using \(\alpha = \lambda^2\) and \(\lambda \leq \frac{\theta(\lambda t)}{t}\) that the following holds
\[
\frac{6}{4N} \int \sigma^{2-2\alpha}|D^2 w|^2 G \leq \delta NN^{2\alpha} \lambda^{2\alpha+N} \int t|\nabla w|^2 + (N + 4 + \delta^2/N^2)\delta^2 \int \sigma^{1-2\alpha} \frac{\theta(\lambda t)}{t} |\nabla w|^2 G
\]
(3.71)
\[
- \int \{t=a\} \sigma^{2-2\alpha} |\nabla w|^2 G + \frac{(1 + 4nN)\delta^2}{\alpha} \int \sigma^{1-2\alpha} (H(w) + Vw)^2 G + \frac{\alpha^2\delta^4}{N^4} \int \sigma^{-2\alpha} w^2 G.
\]
After taking \(N\) large enough we get
\[
\int \sigma^{2-2\alpha}|D^2 w|^2 G \leq \delta NN^{2\alpha} \lambda^{2\alpha+N} \int t|\nabla w|^2 + N\delta^2 \int \sigma^{1-2\alpha} \frac{\theta(\lambda t)}{t} |\nabla w|^2 G
\]
(3.72)
\[
- \int \{t=a\} \sigma^{2-2\alpha} |\nabla w|^2 G + N\delta^2 \int \sigma^{1-2\alpha} (H(w) + Vw)^2 G + \alpha^2\delta^4 \int \sigma^{-2\alpha} w^2 G.
\]
Now by replacing \(w(\cdot, t)\) by \(w(\cdot, t - a)\) in (3.72) and then by letting \(t - a\) as our new \(t\) we find
\[
\int \sigma^{2-2\alpha}_a|D^2 w|^2 G_a \leq N\delta^2 \int \sigma^{1-2\alpha}_a (\text{div}(A(x, 0)\nabla w) + w_t + Vw)^2 G_a + \delta NN^{2\alpha} \lambda^{2\alpha+N} \sup_{t \geq 0} \int t|\nabla w|^2 dx
\]
(3.73)
\[
+ N\delta^2 \int \sigma^{1-2\alpha}_a \frac{\theta(\lambda t)}{t} |\nabla w|^2 G_a + \alpha^2\delta^4 \int \sigma^{-2\alpha} w^2 G_a.
\]
Now we have the estimate for \(\int \sigma^{2-2\alpha}_a|D^2 w|^2 G_a\). Therefore from (3.61) and (3.73) it follows
\[
N \int \sigma^{1-2\alpha}_a (\text{div}(A(x, 0) - A(x, t))\nabla w)^2 G_a
\]
(3.74)
\[
\leq NM^2 \int \sigma^{1-2\alpha}_a |\nabla w|^2 G_a + N^3 M^2 \delta^2 \int \sigma^{1-2\alpha}_a (\text{div}(A(x, 0)\nabla w) + w_t + Vw)^2 G_a
\]
\[
+ \delta NN^3 M^2 N^{2\alpha+N} \sup_{t \geq 0} \int t|\nabla w|^2 dx + N^3 M^2 \delta^2 \int \sigma^{1-2\alpha}_a \frac{\theta(\lambda t)}{t} |\nabla w|^2 G_a + \alpha^2\delta^4 N^2 M^2 \int \sigma^{-2\alpha} w^2 G_a.
\]
We now plug the estimate (3.74) in (3.59) to obtain
\[
N \int \sigma^{1-2\alpha}_a (\text{div}(A(x, 0)\nabla w) + w_t + Vw)^2 G_a
\]
(3.75)
\[
\leq 2NM^2 \int \sigma^{1-2\alpha}_a |\nabla w|^2 G_a + 2N^3 M^2 \delta^2 \int \sigma^{1-2\alpha}_a (\text{div}(A(x, 0)\nabla w) + w_t + Vw)^2 G_a
\]
\[
+ \delta NN^3 M^2 N^{2\alpha+N} \sup_{t \geq 0} \int t|\nabla w|^2 dx + N^3 M^2 \delta^2 \int \sigma^{1-2\alpha}_a \frac{\theta(\lambda t)}{t} |\nabla w|^2 G_a + \alpha^2\delta^4 N^2 M^2 \int \sigma^{-2\alpha} w^2 G_a.
\]
\[ + 2\delta N^3 M^2 N^{2\alpha} \chi^{2\alpha + N} \sup_{t \geq 0} \int t|\nabla w|^2 dx + 2N^3 M^2 \delta^2 \int \sigma_a^{-2\alpha} \frac{\theta(\lambda t)}{t} |\nabla w|^2 G_a + 2\alpha^2 \delta^4 N^2 M^2 \int \sigma_a^{-2\alpha} w^2 G_a \\
+ 2N \int \sigma_a^{-2\alpha} (\text{div}(A(x,t)\nabla w) + w_t + Vw)^2 G_a. \]

We now choose \( \delta \) small enough such that \( N > 4N^3 M^2 \delta^2 \) to deduce the following inequality from above

\[ \frac{N}{2} \int \sigma_a^{-2\alpha} (\text{div}(A(x,0)\nabla w) + w_t + Vw)^2 G_a \]

\[ \leq 2NM^2 \int \sigma_a^{-2\alpha} |\nabla w|^2 G_a + 2\delta N^3 M^2 N^{2\alpha} \chi^{2\alpha + N} \sup_{t \geq 0} \int t|\nabla w|^2 dx + 2N^3 M^2 \delta^2 \int \sigma_a^{-2\alpha} \frac{\theta(\lambda t)}{t} |\nabla w|^2 G_a \\
+ 2\alpha^2 \delta^4 N^2 M^2 \int \sigma_a^{-2\alpha} w^2 G_a + 2N \int \sigma_a^{-2\alpha} (\text{div}(A(x,t)\nabla w) + w_t + Vw)^2 G_a. \]

Using (3.76) in (3.2) (more precisely in a "shifted in time" version of the estimate (3.2) as in (3.73)) we obtain

\[ \alpha^2 \int \sigma_a^{-2\alpha} w^2 G_a + \int \sigma_a^{-2\alpha} |\nabla w|^2 \frac{\theta(\lambda t)}{t} G_a \]

\[ \leq 4N M^2 \int \sigma_a^{-2\alpha} |\nabla w|^2 G_a + 4\delta N^3 M^2 N^{2\alpha} \chi^{2\alpha + N} \sup_{t \geq 0} \int t|\nabla w|^2 dx + 4N^3 M^2 \delta^2 \int \sigma_a^{-2\alpha} \frac{\theta(\lambda t)}{t} |\nabla w|^2 G_a \\
+ 4\alpha^2 \delta^4 N^2 M^2 \int \sigma_a^{-2\alpha} w^2 G_a + 4N \int \sigma_a^{-2\alpha} (\text{div}(A(x,t)\nabla w) + w_t + Vw)^2 G_a + NN^{2\alpha} \chi^{2\alpha + N} \sup_{t \geq 0} \int w^2 + t|\nabla w|^2 \\
- \frac{1}{N} \int t\sigma_a^{-2\alpha} |\nabla w(x,0)|^2 G_a dx + Na \int \sigma_a^{-2\alpha} w^2(x,0)G_a dx. \]

After rearranging the terms in the above inequality we find

\[ \left( \alpha^2 - 4\alpha^2 \delta^4 N^2 M^2 \right) \int \sigma_a^{-2\alpha} w^2 G_a + (1 - 4N^3 M^3 \delta^2) \int \sigma_a^{-2\alpha} |\nabla w|^2 \frac{\theta(\lambda t)}{t} G_a - 4N M^2 \int \sigma_a^{-2\alpha} |\nabla w|^2 G_a \]

\[ \leq 4\delta N^3 M^2 N^{2\alpha} \chi^{2\alpha + N} \sup_{t \geq 0} \int t|\nabla w|^2 dx + 4N \int \sigma_a^{-2\alpha} (\text{div}(A(x,t)\nabla w) + w_t + Vw)^2 G_a \\
+ NN^{2\alpha} \chi^{2\alpha + N} \sup_{t \geq 0} \int w^2 + t|\nabla w|^2 - \frac{1}{N} \int t\sigma_a^{-2\alpha} |\nabla w(x,0)|^2 G_a dx + Na \int \sigma_a^{-2\alpha} w^2(x,0)G_a dx. \]

We now choose \( \delta \) smaller if necessary such that \( 1 - 4\delta^4 N^2 M^2 > 1/2 \) and \( 1 - 4N^3 M^3 \delta^2 > 1/2 \) holds. Consequently, we get for a new constant \( N \)

\[ \alpha^2 \int \sigma_a^{-2\alpha} w^2 G_a + \int \sigma_a^{-2\alpha} |\nabla w|^2 \frac{\theta(\lambda t)}{t} G_a - N \int \sigma_a^{-2\alpha} |\nabla w|^2 G_a \]

\[ \leq N \int \sigma_a^{-2\alpha} (\text{div}(A(x,t)\nabla w) + w_t + V(x,0)w)^2 G_a + NN^{2\alpha} \chi^{2\alpha + N} \sup_{t \geq 0} \int w^2 + |\nabla w|^2 dx \\
+ \sigma(a)^{-2\alpha} \left( -\frac{a}{N} \int |\nabla w(x,0)|^2 G(x,a) dx + Na \int w^2(x,0)G(x,a) dx \right). \]
Since $\frac{g(\lambda t)}{t} \leq \lambda$ therefore by choosing $\alpha$ large enough such that $\alpha \geq 2N$ we get
\[
\alpha^2 \int \sigma_{a}^{-2\alpha} w^2 G_a + \frac{\alpha}{2} \int \sigma_{a}^{-1-2\alpha} |\nabla w|^2 G_a \tag{3.80}
\]
\[
\leq N \int \sigma_{a}^{-1-2\alpha} (\text{div}(A(x,t)\nabla w) + w_t + V(x,0)w)^2 G_a + NN2\alpha \lambda^2 + N \sup_{t \geq 0} \int w^2 + |\nabla w|^2 dx
\]
\[
+ \sigma(a)^{-2\alpha} \left(- \frac{a}{N} \int |\nabla w(x,0)|^2 G(x,a) dx + N \alpha \int w^2(x,0)G(x,a)dx \right).
\]
(3.58) follows from (3.80) in a standard way.

**Step 3:** (Conclusion) We finally show that exists a universal constant $N$ and $\delta$ such that for all $\alpha \geq N(1 + ||V||_{1/2} + [V]_{1/2})$, the following inequality holds
\[
\alpha^2 \int \sigma_{a}^{-2\alpha} w^2 G_a + \alpha \int \sigma_{a}^{-1-2\alpha} |\nabla w|^2 G_a \tag{3.81}
\]
\[
\leq N \int \sigma_{a}^{-1-2\alpha} (\text{div}(A(x,t)\nabla w) + w_t + V(x,t)w)^2 G_a + N2\alpha \lambda^2 + N \sup_{t \geq 0} \int w^2 + t|\nabla w|^2 dx
\]
\[
+ \sigma(a)^{-2\alpha} \left(- \frac{a}{N} \int |\nabla w(x,0)|^2 G(x,a) dx + N \alpha \int w^2(x,0)G(x,a)dx \right).
\]
for all $0 < a \leq \frac{1}{\lambda \alpha}$ and $w \in C_0^\infty(B_4 \times [0, \frac{1}{2\alpha}])$, where $\lambda = \alpha/\delta^2$.

**Proof of Step 3:** We now replace $V(x,0)$ by $V(x,t)$. Using AM-GM inequality we have
\[
|(\text{div}(A(x,t)\nabla w) + w_t + V(x,0)w)| = |(\text{div}(A(x,t)\nabla w) + w_t + V(x,t)w) + (V(x,0) - V(x,t))w|
\]
\[
\leq 2(\text{div}(A(x,t)\nabla w) + w_t + V(x,t)w)^2 + 2(V(x,0) - V(x,t))^2 w^2.
\]

Since $V(x,t)$ is $1/2$-Hölder continuous, $(V(x,0) - V(x,t))^2 \leq [V]_{1/2}^2 t$. Thus we get
\[
N \int \sigma_{a}^{-1-2\alpha} (\text{div}(A(x,t)\nabla w) + w_t + V(x,0)w)^2 G_a \leq 2N \int \sigma_{a}^{-1-2\alpha} (\text{div}(A(x,t)\nabla w) + w_t + V(x,t)w)^2 G_a
\]
\[
+ 2N[V]_{1/2}^2 \int \sigma_{a}^{-1-2\alpha} tw^2 G_a. \tag{3.82}
\]

We would be done if we can absorb the last term in the right-hand side of (3.82) into the left-hand side of (3.58). Using $\lambda t \leq 1/2$ and $\sigma_{a}(t) \leq t + a \leq 1/\lambda$, we get
\[
2N[V]_{1/2}^2 \int \sigma_{a}^{-1-2\alpha} tw^2 G_a \leq \frac{2N}{\lambda^2} [V]_{1/2}^2 \int \sigma_{a}^{-2\alpha} w^2 G_a \tag{3.83}
\]

Using (3.82) and (3.83) in (3.58) we obtain
\[
\alpha^2 \int \sigma_{a}^{-2\alpha} w^2 G_a + \alpha \int \sigma_{a}^{-1-2\alpha} |\nabla w|^2 G_a \tag{3.84}
\]
\[
\leq N \int \sigma_{a}^{-1-2\alpha} (\text{div}(A(x,t)\nabla w) + w_t + V(x,t)w)^2 G_a
\]
\[ + \frac{8N}{\alpha^2} [V]^{1/2} \int \sigma_{t}^{-2\alpha} w^2 G_{a} + N^{\alpha} \alpha^{2\alpha} \sup_{t \geq 0} \int w^2 + |\nabla w|^2 dx \]
\[ + \sigma(\alpha)^{-2\alpha} \left( -\frac{a}{N} \int |\nabla w(x,0)|^2 G(x,a) dx + N \alpha \int w^2(x,0) G(x,a) dx \right). \]

Now note that the second term in the right-hand side of (3.84) can be absorbed in the first term of left-hand side of (3.84) provided \(\frac{\alpha^2}{2} \geq \frac{8N}{\alpha^2} |V|^{1/2} \) i.e. when \(\alpha \geq 4N^{1/2} + [V]^{1/2} + 1\) for \(N\) large, the conclusion follows.

\[ \Box \]

In order to establish a quantitative space-like doubling property that implies (1.6), we also need the following monotonicity in time result. The following lemma below is analogous to Lemma 5 in [11].

**Lemma 3.2.** Let \(u\) be a solution to (1.1) in \(Q_{4}\). Then there exists a universal constant \(N\) such that the following inequality holds

\[ Ne^{||V||^{1/2}_{\infty}} \int_{B_{2\rho}} u^{2}(x,t) dx \geq \int_{B_{\rho}} u^{2}(x,0) dx \quad (3.85) \]

for \(t \leq \frac{\rho^2}{2N \log(2N(1+||V||_{\infty}) \Theta_{\rho}) + 5N^{2}(||V||_{\infty}^{1/2} + 1)}\) and \(0 < \rho \leq 1\), where

\[ \Theta_{\rho} = \frac{\int_{Q_{4}} u^{2}(x,t)}{\rho^2 \int_{B_{\rho}} u^{2}(x,0)}. \]

**Proof.** The proof is although similar to that in [11], we nevertheless provide the details in order to highlight the delicate dependence on \(||V||_{\infty}\) in (3.85) above. We consider \(f(x,t) = u(x,t)\phi(x)\), where \(\phi = 1\) in \(B_{3/2}\) and \(\phi = 0\) outside \(B_{2}\). We then define

\[ H(t) = \int f^2(x,t) G(x,t; y,0) dx, \]

where \(G(x,t; y, s)\) is the fundamental solution in \(\mathbb{R}^{n+1}\) of the operator \(\partial_t - \text{div}(A(x,t) \nabla)\), i.e. \(G_t - \text{div}(A \nabla G) = \delta_{(y,s)}\). Over here, we would like to mention that without loss of generality, we may assume that \(A\) is defined on whole of \(\mathbb{R}^{n} \times \mathbb{R}\) and satisfies the bounds in (1.2) and moreover the associated fundamental solution \(G\) satisfies

\[ \begin{aligned}
\int_{\mathbb{R}^{n}} G(x,t; y,0) dy &= 1 \\
\int_{\mathbb{R}^{n}} G(x,t; y,0) \phi(y) dy &\rightarrow \phi(x) \text{ as } t \rightarrow 0.
\end{aligned} \quad (3.86) \]

See for instance [13, Chapter 1].

Then for \(t > 0\) we have

\[ H'(t) = \int 2f f_{t} G + \int f^2 G_{t}, \]
\[ = \int 2f f_{t} G + \int f^2 \text{div}(A \nabla G) \]

\[
\begin{align*}
&= \int 2f f_t G - \int \langle \nabla f^2, A \nabla G \rangle \\
&= \int 2f f_t G + \int \text{div}(A \nabla f^2) G \\
&= 2 \int f(f_t + \text{div}(A \nabla f)) G + \int \langle A \nabla f, \nabla G \rangle.
\end{align*}
\]

From straightforward calculations we have \( f_t = u_t \phi \) and

\[
\text{div}(A \nabla f) = \phi \text{div}(A \nabla u) + 2(A \nabla u, \nabla \phi) + u \text{div}(A \nabla \phi).
\]

Since \( u \) is a solution of (1.1), we get \( f_t + \text{div}(A \nabla f) = -V f + u \text{div}(A \nabla \phi) + 2(A \nabla u, \nabla \phi). \) Hence we get

\[
H'(t) \geq -N||V||_{\infty} H(t) - 2N \int |f|(|u| + |\nabla u|) \chi_{B_2 \setminus B_{3/2}} G.
\]

Arguing as in [11] we get

\[
H'(t) \geq -N||V||_{\infty} H(t) - \frac{N}{t^{n/2}} e^{-\frac{N}{4Nt}} \int |u|^2 + |\nabla u|^2.
\]

Then using the estimate in (2.3), we obtain from above

\[
H'(t) \geq -N||V||_{\infty} H(t) - (1 + ||V||_{\infty}) \frac{N}{t^{n/2}} e^{-\frac{N}{4Nt}} \int_{Q_4} |u|^2.
\]

Now for sufficiently small \( t \), one has \( t^{n/2} e^{-\frac{N}{4Nt}} \leq e^{-\frac{N}{8Nt}} \) and hence we deduce the following inequality for all large enough \( N \)

\[
H'(t) \geq -N||V||_{\infty} H(t) - (1 + ||V||_{\infty}) Ne^{-\frac{N}{8Nt}} \int_{Q_4} u^2.
\]

Integrating this inequality in \((0, t)\) and using \( \lim_{t \to 0} H(t) = u^2(y, 0) \) for \( y \in B_1 \) we get

\[
e^{N||V||_{\infty} t} \int_{B_2} u^2(x, t) G(x, t; y, 0) \geq u^2(y, 0) - e^{N||V||_{\infty} t} (1 + ||V||_{\infty}) N e^{-\frac{N}{8Nt}} \int_{Q_4} u^2.
\]

(3.87)

Now by integrating the above inequality in (3.87) in the \( y \)-variable in \( B_1 \), changing the order of integration and then by using \( \int \mathcal{G}(x, t; y, 0) dy = 1 \) we obtain

\[
e^{N||V||_{\infty} t} \int_{B_2} u^2(x, t) \geq e^{N||V||_{\infty} t} \int_{B_2} f^2(x, t) \geq \int_{B_1} u^2(x, 0) - e^{N||V||_{\infty} t} (1 + ||V||_{\infty}) N e^{-\frac{N}{8Nt}} \int_{Q_4} u^2.
\]

(3.88)

We now choose a universal \( N \) (which does not depend on \( ||V||_{\infty} \)) such that \( 2N/C_E > 1 \) and \( N \log(2N/C_E) > 1 \), where \( C_E \) is from Lemma 2.5. Note that from (2.3) we have

\[
1 \leq C_E (1 + ||V||_{\infty}) \frac{\int_{Q_4} u^2 dX}{\int_{B_1} u^2(x, 0) dx} = C_E (1 + ||V||_{\infty}) \Theta.
\]
Hence we get $N \log(2N(1 + \|V\|_\infty)\Theta) = N \log(2N/C_E) + N \log(C_E(1 + \|V\|_\infty)\Theta) > 1$. If we now take $t \leq \frac{1}{2N \log(2N(1 + \|V\|_\infty)\Theta) + 5N^2(\|V\|_\infty^2 + 1)}$, then

$$e^{N\|V\|_\infty t}(1 + \|V\|_\infty)N e^{-\frac{1}{4t}} \int_{Q_4} u^2 \leq \frac{1}{2} \int_{B_1} u^2(x, 0). \quad (3.89)$$

Using (3.89) in (3.88), we obtain for all $t \leq \frac{1}{2N \log(2(1 + \|V\|_\infty)\Theta) + 5N^2(\|V\|_\infty^2 + 1)}$, that the following inequality holds

$$Ne^{\frac{1}{2}\|V\|_\infty^2} \int_{B_2} u^2(x, t) \geq \int_{B_1} u^2(x, 0).$$

Thus, for all $t \leq \frac{1}{2N \log(2N(1 + \|V\|_\infty)\Theta) + 5N^2(\|V\|_\infty^2 + 1)}$ we have

$$Ne^{\frac{1}{2}\|V\|_\infty^2} \int_{B_2} u^2(x, t) \geq \int_{B_1} u^2(x, 0),$$

where $\Theta = \frac{\int_{Q_4} u^2 \, dx}{\int_{B_1} u^2(x, 0) \, dx}$. Now for the rescaled solution $\tilde{u}(x, t) := u(x, \rho^2 t)$, $\rho < 1$, we get

$$Ne^{\frac{1}{2}\|V\|_\infty^2} \int_{B_2\rho} u^2(x, \rho^2 t) \geq \int_{B_1\rho} u^2(x, 0),$$

where $t \leq \frac{1}{2N \log(2N(1 + \|V\|_\infty)\Theta_\rho) + 5N^2(\|V\|_\infty^2 + 1)}$ and

$$\Theta_\rho = \frac{\int_{Q_4} u^2(\rho x, \rho^2 t) \, dX}{\int_{B_1\rho} u^2(\rho x, 0) \, dx}.$$

After change of variables and by using $\int_{Q_4} u^2(x, t) \leq \int_{Q_4} u^2(x, t)$ we get

$$\Theta_\rho \leq \frac{\int_{Q_4} u^2(x, t)}{\rho^2 \int_{B_1\rho} u^2(x, 0)} := \Theta_\rho.$$

Thus, for all $t \leq \frac{1}{2N \log(2N(1 + \|V\|_\infty)\Theta_\rho) + 5N^2(\|V\|_\infty^2 + 1)}$ we have

$$Ne^{\frac{1}{2}\|V\|_\infty^2} \int_{B_2\rho} u^2(x, \rho^2 t) \geq \int_{B_1\rho} u^2(x, 0). \quad (3.90)$$

From (3.90) it follows that for all $t \leq \frac{\rho^2}{2N \log(2N(1 + \|V\|_\infty)\Theta_\rho) + 5N^2(\|V\|_\infty^2 + 1)}$ we have

$$Ne^{\frac{1}{2}\|V\|_\infty^2} \int_{B_2\rho} u^2(x, t) \geq \int_{B_1\rho} u^2(x, 0),$$

which completes the proof of the lemma. \qed

With the Carleman estimate in Lemma 3.1 and the monotonicity result Lemma 3.2 in hand, we now proceed with the proof of Theorem 1.1 where we adapt some ideas from [11] in order to obtain our desired quantitative uniqueness estimate (1.6).
Proof of Theorem 1.1. Without loss of generality, we assume that \(A(0, 0) = I_n\). The proof is divided into three steps.

**Step 1:** We first show that there exists a universal constant \(N\) such that for all \(r < 1/2\)

\[
\int_{B_{2r}} u^2(x, 0) dx \leq M \int_{B_r} u^2(x, 0) dx,
\]

where \(M = \exp(N^4 \log(2N(1 + ||V||_\infty)\Theta_\rho)) + N^4(||V||_{1/2}^1 + [V]_{1/2}^{1/2} + 1)\) and where \(\Theta_\rho\) is as in Lemma 3.2.

**Proof of Step 1:** For a fixed \(\alpha\) large, with \(\lambda = \frac{\alpha}{\sigma}\), consider \(w(x, t) = u(x, t)\psi(x)\phi(t)\), where \(\psi \in C^\infty_0(B_1)\) such that \(\psi \equiv 1\) in \(B_3\), \(\psi = 0\) outside \(B_{7/2}\) and where \(\phi \equiv 1\) in \(0 \leq t \leq 1/4\lambda\) and \(\phi = 0\) for \(t \geq 1/2\lambda\). Then using (1.1) we have

\[
\text{div}(A\nabla w) + w_t + Vw = \text{div}(A\nabla \psi)u\phi + 2\langle A\nabla \psi, \nabla u\rangle \phi + u\psi \phi_t.
\]

It thus follows

\[
(\text{div}(A\nabla w) + w_t + Vw)^2 \leq C(u^2 + |\nabla u|^2)\chi_{B_3} \setminus B_2(x) + C\lambda^2 u^2 \chi_{(0, 1/2\lambda) \setminus (0, 1/4\lambda)}(t).
\]

Using the Carleman inequality (3.1) we get

\[
\alpha^2 \int \sigma_a^{-2\alpha} w^2 G_a + \alpha \int \sigma_a^{1-2\alpha} |\nabla w|^2 G_a \leq N \int \sigma_a^{1-2\alpha} (\text{div}(A\nabla w) + w_t + Vw)^2 G_a + N^{2\alpha} \alpha^{2\alpha} \sup_{t \geq 0} \int w^2 + t|\nabla w|^2 dx
\]

\[+ \sigma(a)^{-2\alpha} \left(-\frac{\alpha}{N} \int |\nabla w(x, 0)|^2 G(x, a) dx + N \alpha \int w^2(x, 0) G(x, a) dx\right),\]

which in view of (3.92) implies the following estimate

\[
\alpha^2 \int \sigma_a^{-2\alpha} w^2 G_a + \alpha \int \sigma_a^{1-2\alpha} |\nabla w|^2 G_a \leq N \int \sigma_a^{1-2\alpha} (u^2 + |\nabla u|^2)\chi_{B_3} \setminus B_2(x) G_a + N^2 \lambda^2 \int \sigma_a^{1-2\alpha} u^2 \chi_{(0, 1/2\lambda) \setminus (0, 1/4\lambda)}(t) G_a
\]

\[+ N^\alpha \alpha^{2\alpha} \sup_{t \geq 0} \int w^2 + t|\nabla w|^2 dx + \sigma(a)^{-2\alpha} \left(-\frac{\alpha}{N} \int |\nabla w(x, 0)|^2 G(x, a) dx + N \alpha \int w^2(x, 0) G(x, a) dx\right).
\]

We now make the following claim.

Claim: \(\sigma_a^{1-2\alpha} G_a \leq N^{2\alpha + \frac{2}{\alpha}} \lambda^{2\alpha + \frac{2}{\alpha}}\) in the region \(B_3 \times [0, 1/2\lambda] \setminus B_2 \times [0, 1/4\lambda]\).

**Proof of the claim:** We will divide the proof in two cases.

Case(i): When \(t > 1/4\lambda\).

From Lemma 2.1, \(\sigma_a \geq \frac{t + a}{N}\), which gives \(\sigma_a^{1-2\alpha} \leq N^{2\alpha - 1}(t + a)^{1-2\alpha}\). Thus we get

\[
\sigma_a^{1-2\alpha} G_a \leq N^{2\alpha - 1}(t + a)^{1-2\alpha} (t + a)^{n/2} e^{-|x|^2/4(t+a)}.
\]
Since we have \( t + a > 1/\lambda \), which implies \((t + a)^{1 - 2\alpha - n/2} \leq \lambda^{-1 + 2\alpha + n/2}\). Also since \((t + a) > 0\) and thus \(e^{-|x|^2/4(t+a)} \leq 1\). Hence we obtain
\[
\sigma_a^{1-2\alpha} G_a \leq N^{2\alpha - 1} \alpha^{-1 + 2\alpha + n/2},
\]
which completes the proof of Case(i).

Case(ii): When \( t \leq 1/4\lambda \).

Since \((x, t) \in B_3 \times [0, 1/2\lambda] \setminus B_2 \times [0, 1/4\lambda]\), therefore in this case we must have \(|x| \geq 2\). Now we use (3.95) to obtain
\[
\sigma_a^{1-2\alpha} G_a \leq N^{2\alpha - 1}(t + a)^{1 - 2\alpha} (t + a)^{-n/2} e^{-9/4(t+a)}
\]
From the properties of exponential function, for any integer \( k \) we have \(e^{-x} \leq \frac{k^2}{x^2} \leq \frac{k^3}{x^3}\). Thus for real number \(2\alpha + n/2\) we get \(e^{-1/(t+a)} \leq (2\alpha + n/2)(t + a)^{2\alpha + n/2 - 1}\). Hence we get \(\sigma_a^{1-2\alpha} G_a \leq N^{2\alpha - 1}(2\alpha + n/2)^{(2\alpha + n/2)}\). This proves the claim.

Using the claim proved above, we find that (3.94) becomes
\[
\alpha^2 \int \sigma_a^{-2\alpha} w^2 G_a + \alpha \int \sigma_a^{-2\alpha} \nabla w \cdot \nabla G_a \leq N^{2\alpha - 1} \alpha^{2\alpha + n/2} \int_{B_3} (u^2 + |\nabla u|^2) \tag{3.96}
\]
\[
+ N^{2\alpha} \alpha^{2\alpha} \sup_{t \geq 0} \int w^2 + t |\nabla w|^2 dx + \sigma_a^{-2\alpha} \left(-\frac{a}{N} \int |\nabla w(x,0)|^2 G(x,a) dx + N \int w^2(x,0) G(x,a) dx\right).
\]

Now using the estimate in (2.3) we then obtain
\[
\alpha^2 \int \sigma_a^{-2\alpha} w^2 G_a + \alpha \int \sigma_a^{-2\alpha} |\nabla w|^2 G_a \leq 2C E N^{2\alpha + n/2} \lambda^{2\alpha + n/2} (1 + ||V||_{\infty}) \int_{Q_t} u^2(x,t) dX \tag{3.97}
\]
\[
+ \sigma_a^{-2\alpha} \left(-\frac{a}{N} \int |\nabla w(x,0)|^2 G(x,a) dx + N \int w^2(x,0) G(x,a) dx\right).
\]

Let \( \rho \in (0,1) \) to be fixed later. Now since \( \phi(t) = 1 \) for \( t \leq 1/4\lambda \) and \( \psi = 1 \) in \( B_2 \), we observe that the first term on the left hand side of (3.97) can be minorized as follows
\[
\alpha^2 \int \sigma_a^{-2\alpha} w^2 G_a \geq \alpha^2 \int_{0}^{1/4\lambda} \int_{B_3} \sigma_a^{-2\alpha} w^2 G_a
\]
\[
\geq \alpha^2 \int_{0}^{\rho^2/4\lambda} \int_{B_2 \rho} \sigma_a^{-2\alpha} u^2(t + a) \leq \frac{1}{2} e^{-|x|^2/4(t+a)}
\]
\[
\geq \alpha^2 \int_{0}^{\rho^2/4\lambda} \int_{B_2 \rho} (t + a)^{-2\alpha - n/2} u^2 e^{-\rho^2/(t+a)},
\]
where the second inequality is a consequence of the fact that \( 0 < \rho < 1 \) and in the last inequality, we have used \( \sigma_a(t) \leq t + a \). We now assume that
\[
\alpha > N(\log(2N(1 + ||V||_{\infty} \Theta_\rho) + ||V||_{1/2}^{1/2} + ||V||_{1/2}^{1/2} + 1) \tag{3.98}
\]
and $0 < t \leq \frac{\rho^2}{4\lambda}$ in order to apply Lemma 3.2 and consequently we obtain

$$
\alpha^2 \int \sigma_a^{-2\alpha} w^2 G_a \geq \frac{\alpha^2}{N} \int_0^{\rho^2/4\lambda} (t + a)^{-2\alpha-n/2} e^{-\rho^2/(t+a)} dt \int_{B_\rho} u^2(x,0) dx \geq e^{-\alpha} \frac{\alpha^2}{N} \int_a^{\rho^2/4\lambda} t^{-2\alpha-n/2} e^{-\rho^2/t} dt \int_{B_\rho} u^2(x,0) dx \geq e^{-\alpha} \frac{\alpha^2}{N} \int_{\rho^2/8\lambda}^{\rho^2/4\lambda} t^{-2\alpha-n/2} e^{-\rho^2/t} dt \int_{B_\rho} u^2(x,0) dx,
$$

where in the second inequality above, we used that $e^{-||V||^1}_\infty \geq e^{-\alpha}$ which can be ensured by taking $N > 1$ in (3.98) above and the last inequality uses $0 < a \leq \frac{\rho^2}{8\lambda}$ which in turn implies $(\rho^2/8\lambda, \rho^2/4\lambda) \subset (a, a + \rho^2/4\lambda)$.

Now using (3.100) in (3.97) we get

$$
\int_{Q_4} u^2(x,t) dX \leq 2C_E N^{2\alpha+n/2} \lambda^{2\alpha+n/2} \left(1 + ||V||_\infty\right) \int_{Q_4} u^2(x,t) dX + \sigma(a) \int_{\Omega} |\nabla w(x,0)|^2 G(x,a) dx + N\alpha \int_{\Omega} w^2(x,0) G(x,a) dx.
$$

Now in order to absorb the first term on the right hand side of (3.101) onto the left hand side of (3.101), we choose $\rho$ such that

$$
\frac{1}{2} \frac{\delta^2 4^{2\alpha+n/2} \lambda^{2\alpha+n/2+1}}{8N \Theta_\rho} (e^{5/\delta^2 \rho^2})^{-2\alpha} \rho^{-n} \geq 2C_E N^{2\alpha+n/2} \lambda^{2\alpha+n/2} \left(1 + ||V||_\infty\right)
$$

(3.102)

Since $1 + ||V||_\infty \leq \alpha^2$, thus (3.102) is ensured provided for a possibly larger $N$ the following holds

$$
\frac{1}{2} \frac{\delta^2 4^{2\alpha+n/2} \lambda^{2\alpha+n/2+1}}{8N \Theta_\rho} (e^{5/\delta^2 \rho^2})^{-2\alpha} \rho^{-n} \geq 2C_E N^{2\alpha+n/2} \lambda^{2\alpha+n/2+2}.
$$

(3.103)

(3.103) in particular will follow in case the following inequality holds for a larger $N$

$$
N^{-2\alpha} (e^{5/\delta^2 \rho^2})^{-2\alpha} \geq 32C_E N^{n/2+1} \Theta_\rho.
$$

(3.104)
Since $\alpha \geq N \log(2N(1 + ||V||\infty)\Theta_\rho) + N(||V||_1^{1/2} + [V]_{1/2}^{1/2} + 1)$, we have $e^{\alpha/N} \geq 2N\Theta_\rho$. Therefore (3.104) will hold if
\[
(Ne^{5/\delta^2} \rho^2)^{-2\alpha} \geq 16C_E N^{n/2} e^{\alpha/N}.
\]
The above inequality can be rewritten as
\[
(e^{1/\delta^2} Ne^{5/\delta^2} \rho^2)^{-2\alpha} \geq 16C_E N^{n/2}.
\]
We now choose $\rho$ such that
\[
e^{1/\delta^2} Ne^{5/\delta^2} \rho^2 \leq \frac{1}{8} \tag{3.105}
\]
and then we find that (3.102) follows. Using (3.102) in (3.101) we obtain
\[
\frac{a}{N} \int |\nabla w(x,0)|^2 G(x,a)dx \leq N\alpha \int w^2(x,0)G(x,a)dx. \tag{3.106}
\]
In particular, by letting $\alpha_0 = N \log(2N(1 + ||V||\infty)\Theta_\rho) + N(||V||_1^{1/2} + [V]_{1/2}^{1/2} + 1)$ we deduce the following inequality
\[
2a \int |\nabla w(x,0)|^2 G(x,a)dx + \frac{n}{2} \int w^2(x,0)G(x,a)dx \leq N^3\alpha_0 \int w^2(x,0)G(x,a)dx. \tag{3.107}
\]
Now we use Lemma 2.4 to obtain for all $r < 1/2$,
\[
\int_{B_{2r}} u^2(x,0)dx \leq M \int_{B_r} u^2(x,0)dx, \tag{3.108}
\]
where $M = \exp(N^4 \log(2N(1 + ||V||\infty)\Theta_\rho) + N^4(||V||_1^{1/2} + [V]_{1/2}^{1/2} + 1))$.

**Step 2**: We now show that there exists a universal constant $N$ such that for all $r < 1/2$,
\[
\int_{B_{2r}} u^2(x,0)dx \leq M \int_{B_r} u^2(x,0)dx, \tag{3.109}
\]
where $M = \exp(N \log(N\Theta) + N(||V||_1^{1/2} + [V]_{1/2}^{1/2} + 1))$ and $\Theta = N \int_{Q_r} u^2(x,t)dX / \int_{B_1} u^2(x,0)dx$.

Note that in this step we want to replace $\Theta_\rho$ by $\Theta$.

**Proof of Step 2**: For given $r \leq \rho$, we can choose $k$ such that
\[
\rho \leq 2^k r < 2\rho.
\]
Then using (3.108) we have
\[
\int_{B_{\rho}} u^2(x,0)dx \leq \int_{B_{2^k \rho}} u^2(x,0)dx \leq M^k \int_{B_r} u^2(x,0)dx \leq M^{\log_2(2\rho/r)} \int_{B_r} u^2(x,0)dx. \tag{3.110}
\]
We write $M = \exp(M_1 + N \log(N\Theta_\rho))$, where
\[
M_1 = N^4 \log(2N(1 + ||V||\infty)) + N(||V||_1^{1/2} + [V]_{1/2}^{1/2} + 1). \tag{3.111}
\]
Then we find
\[
\int_{B_\rho} u^2(x,0) dx \leq e^{M_1 \log_2(2\rho/r)} (N\Theta)_{\rho}^{N \log_2(2\rho/r)} \int_{B_r} u^2(x,0) dx. \tag{3.112}
\]

On putting the value of $\Theta$ we get
\[
\int_{B_\rho} u^2(x,0) dx \leq e^{M_1 \log_2(2\rho/r)} \left( \frac{N}{\rho^2} \int_{B_\rho} u^2(x,0) \right)^{N \log_2(2\rho/r)} \int_{B_r} u^2(x,0) dx. \tag{3.113}
\]

Since $\rho$ is a universal constant and less than 1 therefore $1/\rho^2$ can be absorbed into $N$ (by possibly taking a larger $N$). Now we rearrange the terms in (3.113) to obtain
\[
\int_{B_\rho} u^2(x,0) dx \leq e^{M_1 (1 - \beta)} \left( \int_{B_{\rho/2}} u^2(x,0) dx \right)^\beta \left( N \int_{Q_4} u^2(x,t) \right)^{1 - \beta}, \tag{3.114}
\]

where $\beta = \frac{1}{1 + N \log_2 4}$. Since the equation is translation invariant we can take translation of $u$ in space variable to obtain
\[
\int_{B_{\rho}(y)} u^2(x,0) dx \leq e^{M_1 (1 - \beta)} \left( \int_{B_{\rho/2}(y)} u^2(x,0) dx \right)^\beta \left( N \int_{Q_4} u^2(x,t) \right)^{1 - \beta}, \tag{3.116}
\]

for all $y \in B_1$. Note that $B_{3\rho/2} \subset \bigcup_{y \in B_{\rho/2}} B_{\rho}(y)$. From the compactness of $B_{5\rho/4}$, there exist $y_1, y_2, ..., y_{n_1} \in B_{\rho/2}$ where $n_1$ depends only on $n$ such that $B_{5\rho/4} \subset \bigcup_{i} B_{\rho}(y_i)$. Now we use (3.116) and sum over $i$ to find
\[
\sum_{i=1}^{n_1} \int_{B_{\rho}(y_i)} u^2(x,0) dx \leq e^{M_1 (1 - \beta)} \sum_{i=1}^{n_1} \left( \int_{B_{\rho/2}(y_i)} u^2(x,0) dx \right)^\beta \left( N \int_{Q_4} u^2(x,t) \right)^{1 - \beta}. \tag{3.117}
\]

Now in the left-hand side we use $B_{5\rho/4} \subset \bigcup_{i} B_{\rho}(y_i)$ and using $B_{\rho/2}(y_i) \subset B_{\rho}$ in the right-hand side to obtain
\[
\int_{B_{5\rho/4}} u^2(x,0) dx \leq n_1 e^{M_1 (1 - \beta)} \left( \int_{B_{\rho}} u^2(x,0) dx \right)^\beta \left( N \int_{Q_4} u^2(x,t) \right)^{1 - \beta}. \tag{3.118}
\]

We now use (3.115) in (3.118) to obtain
\[
\int_{B_{5\rho/4}} u^2(x,0) dx \leq n_1 e^{M_1 (1 - \beta) + M_1 \beta (1 - \beta)} \left( \int_{B_{\rho/2}} u^2(x,0) dx \right)^\beta e^{M_1 (1 - \beta)} \left( N \int_{Q_4} u^2(x,t) \right)^{1 - \beta + \beta (1 - \beta)}. \tag{3.119}
\]

\[
= n_1 e^{M_1 (1 - \beta)^2} \left( \int_{B_{\rho/2}} u^2(x,0) dx \right)^{\beta^2} \left( N \int_{Q_4} u^2(x,t) \right)^{1 - \beta^2}. \tag{3.119}
\]
Proceeding in the same fashion, we can get a similar inequality as in (3.119) where on the left hand side, the integral over $B_{\beta/4}$ can replaced by integral over $B_{7\beta/4}$ and so on. Therefore since $\rho$ is universal (in view of our choice in (3.105)), we can finally assert that there exists a constant $N$ and $\tilde{\beta} \in (0, 1)$ such that

$$\int_{B_1} u^2(x,0)dx \leq Ne^{M_1(1-\tilde{\beta})} \left( \int_{B_\rho} u^2(x,0)dx \right)^{\tilde{\beta}} \left( N \int_{Q_4} u^2(x,t) \right)^{1-\tilde{\beta}}.$$  \hspace{1cm} (3.120)

This in particular implies

$$\int_{B_1} u^2(x,0)dx \leq Ne^{M_1(1-\tilde{\beta})} \left( \int_{B_\rho} u^2(x,0)dx \right)^{\tilde{\beta}} \left( N \int_{Q_4} u^2(x,t) \right)^{1-\tilde{\beta}}.$$  \hspace{1cm} (3.121)

This can be rewritten as

$$\int_{B_1} u^2(x,0)dx \leq Ne^{M_1(1-\tilde{\beta})} \left( \int_{B_\rho} u^2(x,0)dx \right)^{\tilde{\beta}} \left( N \int_{Q_4} u^2(x,t) \right)^{-\tilde{\beta}} \left( N \int_{Q_4} u^2(x,t) \right).$$  \hspace{1cm} (3.122)

This gives

$$\Theta^{\tilde{\beta}} \leq Ne^{M_1(1-\tilde{\beta})} \rho^{-2\tilde{\beta}} \frac{N \int_{Q_4} u^2(x,t)}{\int_{B_1} u^2(x,0)dx}.$$  

Hence using (3.108), we find

$$\int_{B_{2r}} u^2(x,0)dx \leq M \int_{B_r} u^2(x,0)dx,$$  \hspace{1cm} (3.123)

where $M = \exp(N^4 \log(2N(1 + ||V||_\infty)N^2\rho^{-2}e^{M_1(1-\tilde{\beta})/\tilde{\beta}} + M_1(1 - \tilde{\beta})/\tilde{\beta} + N(\log(1 + b) \leq N \rho^{1/2}$ we get for a large enough $N$, $M$ in (3.123) can be upper bounded in the following way

$$M \leq \exp(N \log(N\Theta) + N(||V||_1^{1/2} + [V]_{1/2}^{1/2} + 1)).$$

**Step 3:** (Conclusion) We finally show that there exists a universal constant $N$ such that for all $r \leq 1/2$ we have

$$r^C \leq \int_{B_r} u^2(x,0)dx,$$  \hspace{1cm} (3.124)

where $C = \int_{B_1} u^2(x,0)dx + N \log(N\Theta) + N(\log(1 + b) \leq N \rho^{1/2}$ and $\Theta$ is as in Step 2.

**Proof of Step 3:** For $r < 1/2$, choose $k \geq 2$ such that $2^{-k} \leq r \leq 2^{-k+1}$ then using (3.123) $k$-times

$$\int_{B_1} u^2(x,0)dx \leq M^k \int_{B_r} u^2(x,0)dx.$$  \hspace{1cm} (3.125)
Now $r \leq 2^{-k+1}$ gives $k \leq \log_2(2/r)$. Hence $M^k \leq M^{\log_2(2/r)} = \left(\frac{2}{r}\right)^{\frac{\ln M}{\ln 2}}$. Thus we obtain

$$
\left(\frac{r}{2}\right)^{\frac{\ln M}{\ln 2}} \int_{B_1} u^2(x,0) dx \leq \int_{B_r} u^2(x,0) dx.
$$

(3.126)

Since for any $b$ we have $b \leq 2^b$, (3.126) can be rewritten as

$$
r^{\frac{\ln M}{\ln 2}} \leq 2 \left(\int_{B_1} u^2(x,0) dx\right)^{-1} \left(\int_{B_r} u^2(x,0) dx\right)^{-1} \frac{\ln M}{\ln 2} \int_{B_r} u^2(x,0) dx.
$$

(3.127)

Multiply both side of (3.126) by $r \left(\int_{B_1} u^2(x,0) dx\right)^{-1} + \frac{\ln M}{\ln 2}$ to obtain

$$
r \left(\int_{B_1} u^2(x,0) dx\right)^{-1} + 2 \frac{\ln M}{\ln 2} \leq (2r) \left(\int_{B_1} u^2(x,0) dx\right)^{-1} + \frac{\ln M}{\ln 2} \int_{B_r} u^2(x,0) dx.
$$

(3.128)

Since $r \leq 1/2$ therefore $(2r) \left(\int_{B_1} u^2(x,0) dx\right)^{-1} + 2 \frac{\ln M}{\ln 2} \leq 1$. Hence we get

$$
r^C \leq \int_{B_r} u^2(x,0) dx,
$$

(3.129)

where $C = \frac{1}{\int_{B_1} u^2(x,0) dx} + N \log(N \Theta) + N(||V||_{1/2} + |V|_{1/2}^{1/2} + 1)$. This completes the proof of Theorem 1.1. \(\square\)

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