AN ITERATIVE ALGORITHM FOR PERIODIC SYLVESTER MATRIX EQUATIONS

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Abstract. The problem of solving periodic Sylvester matrix equations is discussed in this paper. A new kind of iterative algorithm is proposed for constructing the least square solution for the equations. The basic idea is to develop the solution matrices in the least square sense. Two numerical examples are presented to illustrate the convergence and performance of the iterative method.

1. Introduction. The common periodic Sylvester matrix equations are the matrix equations given by

\[ A_j X_j + X_{j+1} B_j = C_j, \tag{1} \]

where the matrices \( A_j, B_j, C_j \in \mathbb{R}^{n \times n} \) are given coefficient matrices and the unknown matrices \( X_j \in \mathbb{R}^{n \times n} \) are to be determined. All of the mentioned matrices follow a \( T \) cycle, i.e., \( A_{j+T} = A_j, B_{j+T} = B_j, C_{j+T} = C_j \) and \( X_{j+T} = X_j \).

Compared with linear time-invariant systems, linear time-varying systems are more theoretically challenging since it is generally impossible to know the state transition matrices. Hence, linear time-varying systems, including linear periodic systems as special cases, have received considerable attention during the past several decades [16, 26, 27]. However, the analysis of time varying systems, especially periodic systems, often involves the solution of periodic matrix equations. Periodic matrix equations play fundamental roles in the study of (discrete-time) linear periodic systems. [22] proposes an effective algorithm for the periodic discrete-time Riccati equation arising from a linear periodic time-varying system \((A_k, B_k)\).
which is particularly efficient when $A_k$ is time-invariant and $B_k$ is periodic, and has special property associated with the problem of spacecraft attitude control using magnetic torques. [4] discuss the solvability of the Sylvester-like matrix equation $AX + f(X)B = C$ through an auxiliary standard (or generalized) Sylvester matrix equation. [10] proposes a gradient based iterative method to find the solutions of the general Sylvester discrete-time periodic matrix equations, which is proven that the proposed iterative method can obtain the solutions of the periodic matrix equations for any initial matrices.

For this reason, many researchers have carried out their works for finding analytical and numerical solutions of many kinds of matrix equations. [28] is concerned with iterative methods for solving a class of coupled matrix equations including the well-known coupled Markovian jump Lyapunov matrix equations as special cases. [19] develops a gradient based and a least squares based iterative algorithms for solving matrix equation $AXB + CX^TD = F$. [8] applies a hierarchical identification principle to study solving the Sylvester and Lyapunov matrix equations. [23] presented a family of iterative algorithms for the matrix equation $AX = F$ and the coupled Sylvester matrix equations utilizing the properties of the eigenvalues related to the symmetric positive definite matrices. By constructing an objective function and using the gradient search, a gradient-based iteration is established in [9] for solving the coupled matrix equations $A_iXB_i = F_i, i = 1, 2, ..., p$. By using the hierarchical identification principle and introducing the convergence factor and the iterative matrix, a family of inversion-free iterative algorithms is proposed in [24] for solving nonlinear matrix equations $X + A^TX^{-1}A = I$. M. Dehghan and M. Hajarian proposed some iterative algorithms based on the conjugate gradient (CG) method for solving the system of generalized Sylvester matrix equations([5]), coupled Sylvester matrix equations([6]) and the second-order Sylvester matrix equation $EVF^2 - AVF^2 - CV = BW([7])$, which are applications of CG in the area of solving time-invariant matrix equations. There are still many papers that are available for reference(one can see [25, 15, 1, 2, 3, 17, 18, 20, 21]).

Periodic Sylvester matrix equation, one of the most straightforward extension of time-variant matrix equation, has vital applications for analysis and design of linear periodic systems. Taking advantage of the CG method, M.Hajarian proposed an iterative algorithm for solving the problem in [11]. It can be proved that M.Hajarian’s algorithm is valid. However, the number of iteration of its operation process is relatively large on account of that the computational method of iteration step is not so ideal. Recently, we have proposed a new iterative method for coupled matrix equations in [14], where time-invariant unknown matrices are restricted by several time-invariant matrix equations at the same time. Based on the idea presented in [14] and [12], we could build a new method which can be used to solve the matrix equation in time-variant form. And in this paper, we develop a new iterative algorithm for periodic Sylvester matrix equation (1), which has better convergence property and less computational burden compared with existed methods.

Here, we give descriptions of some symbols which will be encountered in the rest of this paper. $\text{tr}(A)$ means the trace of matrix $A$. For the space $\mathbb{R}^{n \times n}$, an inner product $(A, B)$ indicates $\text{tr}(B^T A)$ for all $A, B \in \mathbb{R}^{n \times n}$. In this sense, norm $\|A\|$ is a Frobenius norm of matrix $A$. By contrast, the symbol $\|A\|_2$ denotes the spectral norm of matrix $A$. For matrices $M$ and $N$, their Kronecker product is expressed as $M \otimes N$. For matrix

$$X = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \in \mathbb{R}^{n \times n},$$
vec($X$) is the column stretching operation of $X$, which is
\[
\text{vec}(X) = \begin{bmatrix} x_1^T & x_2^T & \cdots & x_n^T \end{bmatrix}^T.
\]
According to the property of Kronecker product, for matrices $M$, $N$ and $X$ with appropriate dimension,
\[
\text{vec}(MXN) = (N^T \otimes M) \text{vec}(X)
\]
holds.

2. Main results. For this type of periodic Sylvester matrix equation, least squares method is a widely adopted technique to solve it. Its ultimate goal is seeking the matrix sequence $X_j$ for $j = 0, 1, \cdots, T - 1$ to minimize the following index function:
\[
J = \sum_{j=0}^{T-1} \frac{1}{2} \|C_j - A_j X_j - X_{j+1} B_j\|^2.
\]
(2)
From this point of view, we have:
\[
\frac{\partial J}{\partial X_j} = A_j^T (C_j - A_j X_j - X_{j+1} B_j) + (C_{j-1} - A_{j-1} X_{j-1} - X_j B_{j-1}) B_{j-1}^T.
\]
Immediately, for $j = 0, 1, \cdots, T-1$, the least squares solution ($X_0^*, X_1^*, \cdots, X_T^*$) satisfies
\[
\frac{\partial J}{\partial X_j} \bigg|_{X_j = X_j^*} = 0.
\]
Based on above-mentioned theoretical basis, the iteration method to solve the periodic Sylvester matrix equation via the least squares method can be expressed as the following algorithm.

Algorithm 1. (An iterative algorithm for equation (1))
1. Set allowed error $\varepsilon$, choose initial matrices $X_j(0) \in \mathbb{R}^{n \times n}$ for $j = 0, 1, \cdots, T - 1$, calculate
   
   \[
   Q_j(0) = C_j - A_j X_j(0) - X_{j+1}(0) B_j;
   \]
   \[
   R_j(0) = A_j^T Q_j(0) + Q_{j-1}(0) B_{j-1}^T;
   \]
   \[
   P_j(0) = -R_j(0);
   \]
   
   $k := 0$;

2. If $\sum_{j=0}^{T-1} \|R_j(k)\| \leq \varepsilon$, stop; else, go to next step;
3. For $j = 0, 1, \cdots, T - 1$, calculate
   
   \[
   \alpha(k) = \frac{\sum_{j=0}^{T-1} \text{tr} \left[P_j^T(k) R_j(k)\right]}{\sum_{j=0}^{T-1} \|A_j P_j(k) + P_{j+1}(k) B_j\|^2};
   \]
   \[
   X_j(k+1) = X_j(k) + \alpha(k) P_j(k) \in \mathbb{R}^{n \times n};
   \]
   \[
   Q_j(k+1) = C_j - A_j X_j(k+1) - X_{j+1}(k+1) B_j \in \mathbb{R}^{n \times n};
   \]
   \[
   R_j(k+1) = A_j^T Q_j(k+1) + Q_{j-1}(k+1) B_{j-1}^T;
   \]
   \[
   P_j(k+1) = -R_j(k+1) + \frac{\sum_{j=0}^{T-1} \|R_j(k+1)\|^2}{\sum_{j=0}^{T-1} \|R_j(k)\|^2} P_j(k) \in \mathbb{R}^{n \times n};
   \]
   
   $k = k + 1$;
4. Go to Step 2.

The proposed algorithm needs about $O(Tn^3)$ operations at the most in each iteration step. And in the rest of this section, the convergence of the presented algorithm would be discussed.

Before further discussion, some fundamental lemmas should be put forward.

**Lemma 1.** Consider matrix sequences $\{R_j(k)\}, \{P_j(k)\}$ generated by Algorithm 1. The following equation comes to true:

$$\sum_{j=0}^{T-1} \text{tr} \left[ R_j^T(k + 1)P_j(k) \right] = 0$$

for $k \geq 0$.

**Proof.** According to step 3 of Algorithm 1, we can get

$$R_j(k + 1) = A_j^T (C_j - A_jX_j(k + 1) - X_{j+1}(k + 1)B_j) + (C_{j-1} - A_{j-1}X_{j-1}(k + 1) - X_j(k + 1)B_{j-1})B_{j-1}^T$$

$$= A_j^T (C_j - A_jX_j(k) - X_{j+1}(k)B_j) + (C_{j-1} - A_{j-1}X_{j-1}(k) - X_j(k)B_{j-1})B_{j-1}^T$$

$$- \alpha(k)A_j^T (A_jP_j(k) + P_{j+1}(k)B_j) - \alpha(k) (A_{j-1}P_{j-1}(k) + P_j(k)B_{j-1})B_{j-1}^T$$

$$= R_j(k) - \alpha(k)A_j^T (A_jP_j(k) + P_{j+1}(k)B_j) - \alpha(k) (A_{j-1}P_{j-1}(k) + P_j(k)B_{j-1})B_{j-1}^T.$$  

Further,

$$\text{tr} \left[ R_j^T(k + 1)P_j(k) \right] = \text{tr} \left[ R_j^T(k)P_j(k) \right] - \alpha(k)\text{tr} \left[ (A_jP_j(k) + P_{j+1}(k)B_j)^T A_jP_j(k) \right]$$

$$- \alpha(k)\text{tr} \left[ (A_{j-1}P_{j-1}(k) + P_j(k)B_{j-1})^T P_j(k)B_{j-1} \right].$$

Summing the above equations together gives

$$\sum_{j=0}^{T-1} \text{tr} \left[ R_j^T(k + 1)P_j(k) \right] = \sum_{j=0}^{T-1} \text{tr} \left[ R_j^T(k)P_j(k) \right]$$

$$- \alpha(k) \sum_{j=0}^{T-1} \text{tr} \left[ (A_jP_j(k) + P_{j+1}(k)B_j)^T A_jP_j(k) \right]$$

$$- \alpha(k) \sum_{j=0}^{T-1} \text{tr} \left[ (A_{j-1}P_{j-1}(k) + P_j(k)B_{j-1})^T P_j(k)B_{j-1} \right]$$

$$= \sum_{j=0}^{T-1} \text{tr} \left[ R_j^T(k)P_j(k) \right] - \alpha(k) \sum_{j=0}^{T-1} \|A_jP_j(k) + P_{j+1}(k)B_j\|^2.$$  

Considering the definition of $\alpha(k)$ in Algorithm 1, the following equation comes to true:

$$\sum_{j=0}^{T-1} \text{tr} \left[ R_j^T(k + 1)P_j(k) \right] = 0.$$  

Then, we accomplish the proof. □
Lemma 2. Consider matrix sequences \( \{ R_j(k) \} \), \( \{ P_j(k) \} \) generated by Algorithm 1, the following equation comes to true:

\[
\sum_{j=0}^{T-1} \text{tr} \left[ R_j^T(k) P_j(k) \right] = - \sum_{j=0}^{T-1} \| R_j(k) \|^2
\]

for \( k \geq 0 \).

Proof. Obviously, equation (3) is true for \( k = 0 \). According to the expressions of \( P_j(k+1) \) in Algorithm 1, we get

\[
\text{tr} \left[ R_j^T(k+1) P_j(k+1) \right] = - \| R_j(k+1) \|^2 + \frac{\sum_{j=0}^{T-1} \| R_j(k+1) \|^2}{\sum_{j=0}^{T-1} \| R_j(k) \|^2} \text{tr} \left[ R_j^T(k+1) P_j(k) \right].
\]

Summing the above equations together from \( j = 0 \) to \( j = T-1 \) gives

\[
\sum_{j=0}^{T-1} \text{tr} \left[ R_j^T(k+1) P_j(k+1) \right] = - \sum_{j=0}^{T-1} \| R_j(k+1) \|^2 + \frac{\sum_{j=0}^{T-1} \| R_j(k+1) \|^2}{\sum_{j=0}^{T-1} \| R_j(k) \|^2} \sum_{j=0}^{T-1} \text{tr} \left[ R_j^T(k+1) P_j(k) \right].
\]

According to Lemma 1, we have

\[
\sum_{j=0}^{T-1} \text{tr} \left[ R_j^T(k) P_j(k) \right] = - \sum_{j=0}^{T-1} \| R_j(k) \|^2.
\]

Thus, the proof of lemma 2 has been accomplished.

Lemma 3. For the sequences \( \{ R_j(k) \} \), \( \{ P_j(k) \} \), \( j = 0, 1, \cdots, T-1 \), in Algorithm 1, the following relation comes to true:

\[
\sum_{k \geq 0} \left( \frac{\sum_{j=0}^{T-1} \| R_j(k) \|^2}{\sum_{j=0}^{T-1} \| P_j(k) \|^2} \right)^2 < \infty.
\]

Proof. First of all, let

\[
\pi = \begin{bmatrix}
E \otimes A_0 & B_0^T \otimes E & B_1^T \otimes E & B_2^T \otimes E & \ldots & B_{T-1}^T \otimes E \\
E \otimes A_1 & E \otimes A_0 & B_1^T \otimes E & B_2^T \otimes E & \ldots & B_{T-1}^T \otimes E \\
E \otimes A_2 & E \otimes A_1 & E \otimes A_0 & B_2^T \otimes E & \ldots & B_{T-1}^T \otimes E \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
B_{T-1}^T \otimes E & E \otimes A_{T-2} & E \otimes A_{T-3} & \ldots & E \otimes A_{T-1} & B_{T-1}^T \otimes E
\end{bmatrix},
\]

where \( E \) is a \( n \) order unit matrix.
Via some mathematical derivation, we have

\[
\sum_{j=0}^{T-1} \|A_j P_j(k) + P_{j+1}(k)B_j\|^2 = \sum_{j=0}^{T-1} \|(E \otimes A_j) \text{vec}(P_j(k)) + (B_j^T \otimes E) \text{vec}(P_{j+1}(k))\|^2
\]

\[
= \left\| \begin{array}{ccc}
(E \otimes A_0) \text{vec}(P_0(k)) + (B_0^T \otimes E) \text{vec}(P_1(k)) \\
(E \otimes A_1) \text{vec}(P_1(k)) + (B_1^T \otimes E) \text{vec}(P_2(k)) \\
\vdots \\
(E \otimes A_{T-1}) \text{vec}(P_{T-1}(k)) + (B_{T-1}^T \otimes E) \text{vec}(P_0(k))
\end{array} \right\|^2
\]

\[
= \left\| \begin{array}{ccc}
E \otimes A_0 & B_0^T \otimes E \\
E \otimes A_1 & B_1^T \otimes E \\
\vdots & \vdots \\
B_{T-1}^T \otimes E & E \otimes A_{T-1}
\end{array} \right\|^2
\]

\[
= \sum_{j=0}^{T-1} \|P_j(k)\|^2.
\]

That means the relation

\[
\sum_{j=0}^{T-1} \|A_j P_j(k) + P_{j+1}(k)B_j\|^2 \leq \pi \sum_{j=0}^{T-1} \|P_j(k)\|^2 \tag{5}
\]

holds.

According to Lemma 2, for \( k \geq 0 \), one has

\[
J(k+1) = \frac{1}{2} \sum_{j=0}^{T-1} \|Q_j(k) - \alpha(k) [A_j P_j(k) + P_{j+1}(k)B_j]\|^2
\]

\[
= \frac{1}{2} \sum_{j=0}^{T-1} \|Q_j(k)\|^2 - \alpha(k) \sum_{j=0}^{T-1} \text{tr} \left[ Q_j^T(k) (A_j P_j(k) + P_{j+1}(k)B_j) \right]
\]

\[
+ \frac{1}{2} \alpha^2(k) \sum_{j=0}^{T-1} \|A_j P_j(k) + P_{j+1}(k)B_j\|^2
\]
\[ J(k) = \alpha(k) \sum_{j=0}^{T-1} \text{tr} \left[ P_j^T(k) A_j^T Q_j(k) + P_j^T(k) Q_{j-1}(k) B_j^{T-1} \right] \]
\[ + \frac{1}{2} \alpha(k) \sum_{j=0}^{T-1} \text{tr} \left[ P_j^T(k) R_j(k) \right] \]
\[ = J(k) - \frac{1}{2} \alpha(k) \sum_{j=0}^{T-1} \text{tr} \left[ P_j^T(k) R_j(k) \right]. \]

Then,
\[ J(k+1) - J(k) = -\frac{1}{2} \left( \sum_{j=0}^{T-1} \text{tr} \left[ P_j^T(k) R_j(k) \right] \right)^2 \]
\[ \leq 0. \quad (6) \]
This indicates that \( \{J(k)\} \) is a descent sequence, which means, for all \( k \geq 0 \), there holds
\[ J(k+1) \leq J(0). \]
So we have
\[ \sum_{k=0}^{\infty} [J(k) - J(k+1)] = J(0) - \lim_{k \to \infty} J(k) < \infty. \quad (7) \]
In conclusion, in view of relations (5) and (7), we get
\[ \sum_{k \geq 0} \left( \sum_{j=0}^{T-1} \|R_j(k)\| \right)^2 \leq \pi \sum_{k \geq 0} \left( \frac{\sum_{j=0}^{T-1} \text{tr} \left[ R_j^T(k) P_j(k) \right]}{\sum_{j=0}^{T-1} \|P_j(k)\|^2} \right)^2 \]
\[ \leq 2 \pi (J(0) - \lim_{k \to \infty} J(k)) \]
\[ < \infty. \]
The proof is thus finished. \( \square \)

Based on the above three lemmas, the following conclusion is raised:

**Theorem 1.** Consider periodic Sylvester matrix equation (1). The matrix sequences \( \{R_j(k)\}, j = 0, 1, \cdots, T-1 \), generated by Algorithm 1 satisfies
\[ \lim_{k \to \infty} \|R_j(k)\| = 0. \]
That is, matrix sequences \( X_j(k), j = 0, 1, \cdots, T-1 \), derived from Algorithm 1 converge to a least squares solution of equation 1.
Proof. In the light of Lemma 1, there holds:
\[
\sum_{j=0}^{T-1} \|P_j(k+1)\|^2 = \sum_{j=0}^{T-1} \left| \sum_{j=0}^{T-1} \frac{\|R_j(k+1)\|^2 \|P_j(k)\|^2}{\sum_{j=0}^{T-1} \|R_j(k)\|^2} \right|^2 \tag{8}
\]
\[
= \left( \frac{\sum_{j=0}^{T-1} \|R_j(k+1)\|^2}{\sum_{j=0}^{T-1} \|R_j(k)\|^2} \right)^2 \sum_{j=0}^{T-1} \|P_j(k)\|^2 + \sum_{j=0}^{T-1} \|R_j(k+1)\|^2. \tag{9}
\]
Let
\[
t_k = \frac{\sum_{j=0}^{T-1} \|P_j(k)\|^2}{\left( \sum_{j=0}^{T-1} \|R_j(k)\|^2 \right)^2}.
\]
Then relation (8) can be represented as
\[
t_{k+1} = t_k + \frac{1}{\sum_{j=0}^{T-1} \|R_j(k+1)\|^2}. \tag{10}
\]
Let us suppose
\[
\lim_{k \to \infty} \sum_{j=0}^{T-1} \|R_j(k)\|^2 \neq 0. \tag{11}
\]
Accordingly, there must exist a constant \(\delta > 0\) such that
\[
\sum_{j=0}^{T-1} \|R_j(k)\|^2 \geq \delta
\]
for all \(k \geq 0\). In view of relations (10) and (11), we have
\[
t_{k+1} \leq t_0 + \frac{k + 1}{\delta}.
\]
In other words, the following relation comes to true:
\[
\frac{1}{t_{k+1}} \geq \frac{\delta}{\delta t_0 + k + 1}.
\]
Further, one can get
\[
\sum_{k=1}^{\infty} \frac{1}{t_k} \geq \sum_{k=1}^{\infty} \frac{\delta}{\delta t_0 + k + 1} = \infty.
\]
At the same time, Lemma 3 shows that
\[
\sum_{k=1}^{\infty} \frac{1}{t_k} < \infty.
\]
Obviously, this is a contradiction. Thus, the theorem proof has been achieved.

3. Numerical examples. In the following, two examples are given to illustrate the effectiveness and practical application of the proposed approach.

Example 1. Consider the following 3-periodic Sylvester matrix equation:
\[
A_j X_j + X_{j+1} B_j = C_j
\]
Here, the coefficient matrices are respectively given by

\[
\begin{align*}
A_0 &= \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}, \\
A_1 &= \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, \\
A_2 &= \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}, \\ 
\end{align*}
\]

\[
\begin{align*}
B_0 &= \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}, \\
B_1 &= \begin{bmatrix} 1 & -0.4 \\ 1 & 0.5 \end{bmatrix}, \\
B_2 &= \begin{bmatrix} 3.1 & -1.7 \\ 0.8 & 3.2 \end{bmatrix}, \\
\end{align*}
\]

\[
\begin{align*}
C_0 &= \begin{bmatrix} 4 & 0.6 \\ 0.6 & 8.4 \end{bmatrix}, \\
C_1 &= \begin{bmatrix} 2.4 & 2.2 \\ 1.2 & 1.8 \end{bmatrix}, \\
C_2 &= \begin{bmatrix} 8.6 & 2.1 \\ 1.6 & 4.2 \end{bmatrix}, 
\end{align*}
\]

By applying the iterative algorithm given in Algorithm 1 with \(X_0(0) = X_1(0) = X_2(0) = 10^{-6}\), the iterative solution \(X_j, j = 0, 1, 2\), to this equation is given by:

\[
\begin{align*}
X_0 &= \begin{bmatrix} 0.0027577391 & 2.1909519 \\ -0.34053816 & 0.19495985 \end{bmatrix}, \\
X_1 &= \begin{bmatrix} -1.967749 & 8.2760362 \\ 0.61770874 & -0.6436907 \end{bmatrix}, \\
X_2 &= \begin{bmatrix} -2.1694945 & 3.9517858 \\ 6.8386895 & -4.906358 \end{bmatrix}, 
\end{align*}
\]

Define the relative iteration error

\[
\delta(k) = \sqrt{\frac{\sum_{j=0}^{2} \|X_j(k) - X_j\|^2}{\sum_{j=0}^{2} \|X_j\|^2}}.
\]

In the meantime, we solved this equation by the algorithm raised in [11] with the same initial value and compared the solution with the result we got utilizing the method in this paper. The numerical results are shown in Fig. 1. Obviously, \(\delta(k)\) decreases quickly and converges to zero along with \(k\) increases. By the comparison, it is easy to see that the proposed algorithm has much faster convergence speed than the reference item.

![Figure 1. The residuals for the iterative algorithm](image-url)
Example 2. Consider the linear time-invariant continuous-time reachable system $S$ described by

$$\dot{x}(t) = Ax(t) + Bu(t),$$

where $x(t) \in \mathbb{R}^3$ is the state vector and $u(t) \in \mathbb{R}^2$ is the control input vector and

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Introduce a multirate control scheme of the system $S$ where the state vector $x(t)$ is sampled with a period $T = 1[\text{s}]$, and the component $u_1(t)$ of the control input vector $u(t)$ is connected to a zeroth-order hold circuit with interval $T_1 := N_1T = 1[\text{s}]$, the component $u_2(t)$ of the control input vector $u(t)$ is connected to a zeroth-order hold circuit with interval $T_2 := N_2T = 2[\text{s}]$. The hold devices are synchronized at time $t = 0$. According to the results introduced in [13], the multirate sampled-data system is modeled by a 2-periodic system in the form of

$$x(t + 1) = A(t)x(t) + B(t)u(t)$$

with

$$A_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & e & 0 & 0 & 0 \\ 0 & 0 & e^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & e & 0 & 0 \\ 0 & 0 & e^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$B_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ e - 1 & 0 \\ 0 & 1 - e^{-1} \\ 1 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ e - 1 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Obviously, the open-loop system is unstable. In order to stabilize the system, find 2-periodic control law $u(t) = K(t)x(t)$ such that the poles of the periodic close-loop system are assigned at $\Gamma = \{0.5, 0.6, 0.7, -0.6, -0.7\}$. Based on the previous works, periodic feedback $K(t)$ can be characterized as:

$$\begin{cases} A(t)X(t) - X(t + 1)F(t) + B(t)G(t) = 0 \\ K(t) = G(t)X(t)^{-1} \end{cases}, t = 0, 1,$$

where $G(t) \in \mathbb{R}^{2 \times 5}$ be a given 2-periodic parameter matrix, and $F(t) \in \mathbb{R}^{5 \times 5}$ be a given 2-periodic matrix such that the eigenvalue set of $F(1)F(0)$ is $\Gamma$. In the meantime, periodic pair $(F(t), G(t))$ should be completely observable. Specially, let
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\[ G(t) = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}, \ t = 0,1 \]

\[ F(t) = \begin{cases} \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0.6 & 0 & 0 & 0 \\ 0 & 0 & -0.6 & 0 & 0 \\ 0 & 0 & 0 & 0.7 & 0 \\ 0 & 0 & 0 & 0 & -0.7 \end{bmatrix}, \ t = 0 \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \ t = 1 \end{cases} \]

The proposed Algorithm 1 applied to the example gives the following 2-periodic matrix:

\[ X(0) = \begin{bmatrix} 1.0000 & 0 & 1.0000 & 0 & 1.0000 \\ 0 & 1.6667 & 0 & 1.4286 & 0 \\ -0.8027 & 0 & -0.4556 & 0 & -0.4287 \\ 0 & 1.8608 & 0 & 1.5313 & 0 \\ -3.0000 & 0 & -0.2500 & 0 & -0.1765 \end{bmatrix}, \]

\[ X(1) = \begin{bmatrix} 2.0000 & 0 & -1.6667 & 0 & -1.4286 \\ 0 & 1.6667 & 0 & 1.4286 & 0 \\ -0.9274 & 0 & -0.7997 & 0 & -0.7898 \\ 0 & 2.1945 & 0 & 1.7078 & 0 \\ -4.0000 & 0 & -1.2500 & 0 & -1.1765 \end{bmatrix}. \]

Accordingly, the value of the periodic feedback is given as:

\[ K(0) = \begin{bmatrix} 1.0000 & 0 & 0 & 0 & 0 \\ 0 & 3.1034 & 0 & -2.2422 & 0 \end{bmatrix}, \]

\[ K(1) = \begin{bmatrix} 0.0284 & 0 & -1.4760 & 0 & 0.1064 \\ 0 & 1.6861 & 0 & -0.8249 & 0 \end{bmatrix}. \]

What can be verified is that the poles assignment is valid.

4. Conclusions. The problem of solving periodic Sylvester matrix equation is addressed in this paper. A numerical iterative algorithm based on conjugate gradient method is proposed. Strict mathematical proof shows the matrix sequence generated by the given algorithm can converge to the exact solution in finite steps. At last, both numerical example and practical application are employed to demonstrate the validity of the proposed algorithm.

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