Gravity with a non-Abelian gauge symmetry in a Hamiltonian way

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Abstract. In this note we study possible derivation of new gravitational actions which are also invariant under a non-Abelian gauge symmetry. We start from a canonical representation of symmetry generators and define the corresponding canonical action which can also contain additional Hamiltonian constraints not related to local symmetries. The obtained class of theories possess between zero and some finite number of degrees of freedom. We discuss three special cases, starting with a theory with maximal number of degrees of freedom and finishing with a theory with zero degrees of freedom.

1. Introduction

We are interested in non-Abelian gauge theories which are also invariant under general coordinate transformations, so that they might include gravitational interactions. These kinds of models are interesting from the point of view of coupling the matter fields to gravity, and recently there has been a lot of interest in construction of higher-spin actions. One example of these theories is Chern-Simons gravity for de Sitter, anti-de Sitter or Poincaré gauge group in any space-time dimension.

We start by constructing the canonical representation of the constraints and defining the canonical action as a pure constraint action. This is, by construction, first order action invariant under non-Abelian gauge transformations and spatial or space-time diffeomorphisms. This method was used in two dimensions to obtain Wess-Zumino-Witten action [1] and its supersymmetric version [2]. The method has never been applied in three dimensions. In this paper we address this question, showing that Chern-Simons action is not the only gauge theory which can be constructed.

2. General setup

Consider a gravitational theory that is gauge-invariant under transformations of a semi-simple Lie group $G$, with the algebra $\mathfrak{g}$. For simplicity, we will consider a three dimensional spacetime, with topology $\mathbb{R} \times \Sigma$ and the signature $(-,+,+)$, but our results are easily generalized to an arbitrary dimension.
Let $T_a$, $a = 1, \ldots, n$, be anti-Hermitian generators of $\mathcal{G}$ with the structure constants $f_{abc}$ and the non-degenerate Cartan metric $g_{ab}$,

$$[T_a, T_b] = f_{abc} T_c, \quad g_{ab} = i \text{Tr}(T_a T_b). \quad (2.1)$$

The fundamental field in the theory is a connection $A^a_\mu(x)$, whose canonically conjugated momentum is $\pi^a_\mu(x)$. The spacetime manifold is parametrized by the coordinates $x^\mu = (t, x^i)$ $(i = 1, 2)$. The canonical action is defined as a Legendre transformation of a Hamiltonian density $\mathcal{H}(A, \pi, u)$,

$$I_c[A, \pi, u] = \int dt d^2 x [A^a_\mu \pi^a_\mu - \mathcal{H}(A, \pi, u)], \quad (2.2)$$

where $u(x)$ are Hamiltonian multipliers. Due to general covariance, $\mathcal{H}$ is a pure constraint, implying that it is a linear combination of all constraints in the theory. Since the theory is invariant under a gauge group $G$, there must be $n$ first class constraints $G_a(A, \pi) = 0$ generating a symmetry with the local parameter $\lambda^a$, such that

$$\{G_a, G^b_b\} = f_{abc} G^c \delta^a_b, \quad (2.3)$$

where we use the short-hand notation $G_a \equiv G_a(t, \vec{x})$, $G^b_b \equiv G_b(t, \vec{x})$ and $\delta \equiv \delta^{(2)}(\vec{x} - \vec{x}')$. We will focus on the case when the theory is invariant under gauge transformations of the form $\delta A^a_\mu = -D_\mu \lambda^a$, where $D_\mu \lambda^a = \partial_\mu \lambda^a + f^{ab}_{\phantom{ab}c} A^c_\mu \lambda^c$. On the other hand, the presence of a time derivative in the transformation law of the fields implies that $\dot{\lambda}^a$ should be treated in the canonical formalism as an independent local parameter, therefore involving an additional set of first class constraints, which we denote by $\dot{G}_a(A, \pi) = 0$, such that their time evolution lead to $G_a(A, \pi) = 0$ $[3]$.

Since the theory should also be invariant under space-time diffeomorphisms, there are additional first class constraints $\mathcal{H}_\mu(A, \pi)$, generating transformations with the parameter $\xi^\mu(x)$. Gauge transformations generated by $\mathcal{H}_\mu$ coincide with diffeomorphisms only on-shell $[4]$. Namely, the diffeomorphism group is a kinematical symmetry of any diffeomorphism invariant action. On the other hand, the constraints depend on the explicit form of the Lagrangian, so the infinitesimal transformations they generate form a dynamical symmetry group. Furthermore, the explicit non-covariance of the Hamiltonian prescription makes the temporal generator $\mathcal{H}_t(A, \pi)$ difficult to obtain. To simplify the problem, we will consider only generators along spatial coordinates, $\mathcal{H}_i(A, \pi)$. Thus, we have $2 \varepsilon$ constraints associated to diffeomorphisms, where $\varepsilon = 1$ (if they are independent on gauge transformations) and $\varepsilon = 0$ (if they are dependent). This generically lead to theories with spatial diffeomorphisms only. We shall discuss this issue later.

There also might exist an even number $2m$ $(m \geq 0)$ of second class constraints $\phi_M(A, \pi) = 0$, not related to any local symmetry.

Taking all constraints into account, that is, $2n + 2 \varepsilon$ first class ones and $2m$ second class ones, we obtain the theory with $n - m - 2 \varepsilon$ degrees of freedom.

To construct the action with these degrees of freedom, we first need a representation of the constraints.

3. Canonical representation of the generators

We seek for a canonical generator

$$G[\lambda, \varepsilon] = \int d^2 x \left( \lambda^a G_a + \varepsilon^a \dot{G}_a \right), \quad (3.1)$$

which produces the gauge transformations through the Poisson brackets,

$$\delta A^a_\mu = \{A^a_\mu, G[\lambda, \varepsilon]\} = \varepsilon^a, \quad \delta A^a_i = \{A^a_i, G[\lambda, \varepsilon]\} = -D_i \lambda^a, \quad (3.2)$$
where $\epsilon^a = -D_t \lambda^a$. One particular solution of the above equations leads to the representations

$$G_a = D_i \pi^i_a, \quad \bar{G}_a = \pi^i_a.$$  \hspace{1cm} (3.3)

Since $G_a$ are linear in momenta, they must be primary constraints, which follow directly from the definition of momenta, $\pi = \delta I/\delta A$. On the other hand, the constraints $G_a$ could be secondary, or a linear combination of primary and secondary constraints, as it happens in Chern-Simons case [5].

To construct the diffeomorphism generators, we first recall the identity

$$D_\mu (\xi^\mu A^a_\mu) = L_\xi A^a_\mu + \xi^\nu F^a_{\mu \nu},$$  \hspace{1cm} (3.4)

where $F^a_{\mu \nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{\alpha \beta}_a A^\alpha_\mu A^\beta_\nu$ is the field-strength. The above relation shows that the gauge transformations and diffeomorphisms can be dependent on-shell, if the equations of motion are $F^a_{\mu \nu} = 0$ (which is true in a theory with zero degrees of freedom, i.e., when $A$ is a pure gauge). In general, gauge transformations and diffeomorphisms are independent.

From the previous identity we can deduce the form of the spatial diffeomorphisms generators,

$$H_i = -A^b_i G_b + F^a_{ik} \pi^k_b.$$  \hspace{1cm} (3.5)

It is straightforward to check that, under spatial diffeomorphisms, $A^a_\mu$ transforms like a vector, while $\pi^i_a$ transforms like a vector density of weight one. Direct calculation shows that $H_i$ close the spatial part of the Dirac algebra [6],

$$\{H_i, H_j\} = H_i \partial_j \delta + H_j \partial_i \delta,$$  \hspace{1cm} (3.6)

and they commute with the gauge generators,

$$\{G_a, H_i\} = G^i_a \partial_i \delta.$$  \hspace{1cm} (3.7)

Since $G_b$ and $H_i$ are first class constraints, it follows that

$$K_i \equiv F^a_{ik} \pi^k_b = H_i + A^b_i G_b,$$  \hspace{1cm} (3.8)

are also first class constraints. If $G_a$ and $H_i$ are independent, the constraints $K_i$ have the form of primary constraints. Their Poisson brackets are

$$\{G_a, K'_i\} = 0,$$  \hspace{1cm} (3.9)

$$\{K_i, K'_j\} = K'_i \partial_j \delta + K'_j \partial_i \delta - F^a_{ij} G_a \delta.$$

Note that the Poisson brackets of $K_i$ do not close without the presence of $G_a$.

The above discussion implies that, when spatial diffeomorphisms and gauge transformations are independent, we have the following constraint structure:

- **Primary I class**: $G_a = \pi^i_a$, $n$.
- **Primary/secondary I class**: $G_a = D_i \pi^i_a$, $n$.
- **Primary I class**: $K_i = F^a_{ij} \pi^j_a$, $2\varepsilon$.
- **Primary/secondary II class**: $\phi_M$, $2m$.

The most general Hamiltonian density reads

$$\mathcal{H} = u^a G_a + v^a G_a + \zeta^i K_i + w^M \phi_M,$$  \hspace{1cm} (3.10)
where \((u, v, \zeta, w)\) are arbitrary multipliers. It is worthwhile noticing that this is an extended Hamiltonian which includes both primary and secondary constraints, obtained from the definition of momenta and evolution of primary constraints, respectively. It is different from the canonical Hamiltonian, a Legendre transformation of the Lagrangian, unique only up to constraints. The resulting equations of motion of these two Hamiltonians are not identical, but the difference is not physical.

In order to obtain the canonical action (2.2), the next step is to determine some multipliers in terms of the fields using the consistency conditions of primary constraints. To this end, we have to specify how many secondary constraints we have, and which constrains are primary and which secondary.

4. Construction of the canonical action

In the previous section we found a particular representation of first class constraints and wrote an extended Hamiltonian describing a theory with \(n = m - 2\epsilon\) degrees of freedom. Now we have to choose \(m\) and \(\epsilon\) where, as we discussed before, we always have \(\epsilon = 1\), except when \(F = 0\). Let us focus on special cases.

**Case 1.** Let us assume that there is a minimal number of constraints, that is, there are no second class constraints and no diffeomorphism constraints. The constraint structure is:

\[
\text{Primary I class} \quad \tilde{G}_a = \pi^a, \quad n,
\]

\[
\text{Secondary I class} \quad G_a = D_i \pi^i_a, \quad n.
\]

The theory has \(n\) degrees of freedom. The Hamiltonian density (3.10) becomes

\[
\mathcal{H}_1 = u^a \pi^a_i + v^a D_i \pi^i_a.
\]

A time evolution of the variable \(X(A, \pi)\) on the phase space is given by \(\dot{X} = \int d^2 x' \{X, \mathcal{H}'\}\). We also have to ensure that \(G_a\) are secondary constraints following from the consistency conditions on \(\tilde{G}_a\). We find that \(\dot{\pi}^a_i = D_i \pi^i_a\) only if \(v^a = -A^a_i\). The Hamiltonian is, therefore,

\[
\mathcal{H}_1 = u^a \pi^a_i - A^a_i D_i \pi^i_a, \quad (4.2)
\]

and the corresponding Hamiltonian equations have the form

\[
\dot{A}^a_i = u^a, \quad \dot{\pi}^a_i = D_i \pi^i_a = 0, \quad \dot{\pi}^i_a = -f^i_{ab} A^b_i \pi^a_c, \quad (4.3)
\]

Furthermore, from (2.2), the canonical action becomes

\[
I_1 = \int d^3 x \left[ \dot{A}^a_i \pi^i_a + A^a_i D_i \pi^i_a + \left( \dot{A}^a_i - u^a \right) \pi^a_i \right]
\]

\[
= \int d^3 x \left( F^{a}_{ab} \pi^b_a - \bar{u}^a \pi^a_i \right) + \text{BT}, \quad (4.4)
\]

where, in the second line, we redefined an arbitrary multiplier as \(\bar{u}^a = u^a - \dot{A}^a_i\), and BT denotes the boundary term. The obtained action is not invariant under general coordinate transformations, what is a consequence of an absence of diffeomorphism constraints. Indeed, \(F_{ab}^{a} = 0\) is not an equation of motion of this theory so, according to (3.4), gauge transformations are not on-shell equivalent to diffeomorphisms.

We should compare this example with a well known case of non-Abelian Yang-Mills theory, since they have the same set of the first class constraints \((\pi^a_i, D_i \pi^i_a)\). The essential difference
between two theories is that in Yang-Mills theory the Hamiltonian density is not a linear combination of constraints, as in (4.2).

The action (4.4) is gauge invariant but not diffeomorphisms invariant. Vanishing Hamiltonian is, therefore, necessary, but not sufficient condition for a theory to be diffeomorphism invariant. This case, therefore, does not give us a good gravitational action. We need to add the diffeomorphism constraints, which is our next case to consider.

Case 2. Let us add the spatial diffeomorphism constraints, \( K_i \), in the Hamiltonian action. In this settings, there are \( 2n + 2 \) first class constraints:

- **Primary I class**: \( G_a = \pi_a^i \), \( K_i = F_{ij}^a \pi_a^j \), \( n + 2 \),
- **Secondary I class**: \( G_a = D_i \pi_a^i \), \( n \).

The number of physical degrees of freedom is \( n - 2 \). The Hamiltonian density is of the form

\[
\mathcal{H}_2 = u^a \pi_a^i + v^a D_i \pi_a^i + \zeta^i F_{ik}^a \pi_a^k,
\]

where again we have to replace \( v^a = -A_t^a \) to ensure that the evolution of \( G_a \) leads to \( G_a \).

Equations of motion read

\[
\begin{align*}
\dot{A}_a^t &= u^a, \\
\dot{A}_i^a &= D_i A_t^a + \zeta^i F_{ji}^a, \\
\dot{\pi}_a^i &= D_i \pi_a^i = 0, \\
\dot{\pi}_a^i &= D_i \pi_a^i = f_{ac} A_t^c \pi_d^i + D_j \left( \zeta^j \pi_a^i - \zeta^i \pi_a^j \right).
\end{align*}
\]

The canonical action becomes

\[
I_2 = \int d^3x \left( F_{li}^a \pi_a^l - \zeta^i F_{ik}^a \pi_a^k - \bar{u}^a \pi_a^i + \bar{u}^a \pi_a^i \right) + BT,
\]

where again we introduced the multiplier \( \bar{u}^a \).

This example illustrates that it is possible to have a nontrivial spatial-diffeomorphism invariant gauge theory with dynamical degrees of freedom in three dimensions. However, the time-like diffeomorphisms are still absent. In order to include them, we need some additional ingredients, and one possibility is to consider a theory with second class constraints. We do it in the next example.

Case 3. So far we had examples of theories with maximal number of degrees of freedom (minimal number of symmetries). Let us study another extreme case, where a number of degrees of freedom is zero. This can happen only if the fields are pure gauge, or if \( F_{\mu\nu}^a = 0 \) is fulfilled dynamically. It implies that the diffeomorphisms should not be an independent symmetry. To have zero degrees of freedom, we need \( 2n \) second class constraints. The simplest possibility is to introduce primary constraints, linear in \( \pi_a^i \), what leads to the following constraints:

- **Primary I class**: \( G_a = \pi_a^i \), \( n \),
- **Secondary I class**: \( G_a = D_i \pi_a^i \), \( n \),
- **Primary II class**: \( \phi_a^i = \pi_a^i + L_a^i(A) \), \( 2n \).

The Hamiltonian density is

\[
\mathcal{H}_3 = u^a \pi_a^i + v^a G_a + w_i^a \phi_a^i.
\]

Let us analyze the form of \( L_a^i(A) \). From the requirement that the constraint \( \phi_a^i \) commute with \( G_a \) and \( \dot{G}_a \) it follows that \( L_a^i(A) \) do not depend of \( A_t^a \). Furthermore, since \( \phi_a^i \) are second class constraints, the matrix \( \{\phi_a^i, \phi_a^j\} \) must be invertible. This matrix is antisymmetric under the
It turns out that the result does not depend on the choice of \( v \) because it is equivalent to a redefinition of the indefinite multipliers.

In order to recover the vanishing brackets between \( \phi \) and \( G \), we redefine these constraints in the following way,

\[
\tilde{G}_a = G_a + h_a, \\
\tilde{\phi}_a^i = \phi_a^i + s_a^i, \tag{4.12}
\]

where \( h_a \) and \( s_a^i \) do not depend on \( A_i^a \) in order to commute with \( \pi_a^i \), and we discard a possibility that they depend on momenta because the constraints are already linear in them. Thus, \( h_a \) and \( s_a^i \) depend on \( A_i^a \) only. Now we require that the constraints of first (\( \tilde{G}_a \)) and second class (\( \tilde{\phi}_a^i \)) are separated, i.e., their commutator weakly vanishes. As a result, we find

\[
h_a = \kappa \epsilon^{ij} \partial_i A_{aj}, \quad s_a^i = \frac{k - 2\kappa}{2} \epsilon^{ij} A_{aj}, \tag{4.13}
\]

where \( \kappa \) is a real constant. Now, the algebra of the constraints becomes

\[
\{\tilde{G}_a, \tilde{G}_b^a\} = f_{ab} \tilde{G}_c^a \delta, \\
\{\tilde{\phi}_a^i, \tilde{G}_b^a\} = f_{ab} \tilde{\phi}_c^i \delta, \\
\{\tilde{\phi}_a^i, \tilde{\phi}_b^j\} = k \epsilon^{ij} g_{ab} \delta. \tag{4.14}
\]

It turns out that the result does not depend on the choice of \( \kappa \) in the redefinition of the constraints (4.13) because it is equivalent to a redefinition of the indefinite multipliers \( v^a \) and \( w_i^a \), so we can choose \( k = 2\kappa \) without loss of generality, so that \( s_a^i = 0 \). To summarize, the constraints are

\[
\tilde{G}_a = D_i \pi_a^i + \kappa \epsilon^{ij} \partial_i A_{aj}, \\
\tilde{\phi}_a^i = \phi_a^i + \pi_a^i - \kappa \epsilon^{ij} A_{aj}. \tag{4.15}
\]

The consistency condition of the primary constraints lead to \( v^a = -A_i^a \) and \( w_i^a = 0 \), and the Hamiltonian density takes the form

\[
\mathcal{H}_3 = u^a \pi_a^i - A_i^a \left( D_i \pi_a^i + \kappa \epsilon^{ij} \partial_i A_{aj}\right). \tag{4.16}
\]

Hamiltonian equations become

\[
\dot{A}_t^a = u^a, \\
\dot{A}_i^a = D_i A_t^a, \\
\dot{\pi}_t^a = D_i \pi_t^a + \kappa \epsilon^{ij} \partial_i A_{aj} = 0, \\
\dot{\pi}_a^i = -f_{ab} A_t^b \pi_a^c + \kappa \epsilon^{ij} \partial_j A_t^a. \tag{4.17}
\]
We also note that $F^a_{\mu} = \dot{A}^a - D_\mu A^a = 0$ and $F^a_{\nu} = \frac{1}{i\pi} \epsilon_{ij}(\bar{G}^a - D_\nu \phi^a) = 0$, which means that on-shell $F^a_{\mu} = 0$, as expected, because the theory has zero degrees of freedom. In consequence, the spatial diffeomorphisms are dependent on the gauge transformations, as we saw in (3.4).

Using the equations for the multipliers $\nu^a$, $w^a_i$ and the constraints $\phi^i_a$ to eliminate the non-physical momenta $\pi^i_a$ in the canonical action (2.2), we obtain

$$I_3 = \int d^3 x \left[ \kappa \epsilon^{ij} \left( \dot{A}^a_i A_{aj} + A^a_i F_{ij} a - \bar{u}^a \pi^i_a \right) \right].$$

(4.18)

We recognize the first part of this action as the Chern-Simons theory,

$$I_{CS}[A] = \kappa \int d^3 x \epsilon^{\mu
u\rho} \left( A^a_{\mu} \partial_{\nu} A^a_{\rho} + \frac{1}{3} f^{abc} A^a_{\mu} A^b_{\nu} A^c_{\rho} \right),$$

(4.19)

with $\epsilon^{ij} \equiv \epsilon^{ij}$, so that

$$I_3 = I_{CS}[A] - \int d^3 x \bar{u}^a \pi^i_a.$$

(4.20)

The actions $I_3$ and $I_{CS}$ are physically equivalent, corresponding to the extended and canonical Hamiltonian, respectively, as discussed at the end of the previous section. Indeed, the term $\bar{u}^a \pi^i_a$ does not change a space of solutions because $\pi^i_a = 0$, $\bar{u}^a = 0$ on-shell in both theories, and they both have zero degrees of freedom. We can therefore say that $I_3 = I_{CS}$.

5. Conclusions

We used Hamiltonian formalism to explore a physical content of a pure constraint action invariant under non-Abelian gauge symmetries and spatial diffeomorphisms. Our primary motivation was to have a gravitational theory coupled to non-Abelian matter and (or) higher spin fields.

The construction was based on a canonical representation of symmetry generators. Due to non-covariance of the formalism, time-like diffeomorphisms were not included in the construction, which, in one of the examples analyzed (Case 2), led to a non-Abelian theory invariant only under spatial diffeomorphisms. Although this model could not describe a gravity theory, it is remarkable that it possesses $n - 2$ degrees of freedom and is not equivalent to Yang-Mills theory. This theory should be better explored in future. In second example (Case 1), the diffeomorphisms constraints were absent and the theory was not invariant under general coordinate transformations. In the third example (Case 3), the time-like diffeomorphisms were not independent on gauge transformations, leading to a fully diffeomorphism-invariant theory, or Chern-Simons model.

Let us mention that a standard Dirac Hamiltonian analysis of Chern-Simons gravity starts from the known action and arrives to so-called total Hamiltonian which contains only primary constraints (see Chapter 6.4 in Ref.[7]). In our approach, however, we start from the extended Hamiltonian that includes all constraints, both primary and secondary, and reconstruct the Chern-Simons action in a different, but physically equivalent way, finding that it is the unique action which can be obtained under given assumptions.

We, therefore, showed that Hamiltonian method can be successfully applied to obtain a diffeomorphism invariant theory possessing non-Abelian gauge symmetry.

One of the open questions to be addressed in future is how to treat time-like diffeomorphisms in this formalism in a general way. Another of the future tasks is to analyze the uniqueness of canonical representations, i.e., discuss non-equivalent representations of the generators. The challenge is also to construct a physically interesting example of a theory that possess degrees of freedom, but is also invariant under space-time diffeomorphisms.

7
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