BPS Explained III: Dimensional Leveraging
or
The Leading Order Behavior of the BBGKY Hierarchy
in a Plasma

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Abstract

This is the third in a series of lectures on the technique of dimensional continuation, employed by Brown, Preston and Singleton (BPS), for calculating Coulomb energy exchange rates in a plasma. Two important examples of such processes are the charged particle stopping power and the temperature equilibration rate between different plasma species. The first lecture was devoted to understanding the machinery of dimensional continuation, and the second concentrated on calculating the electron-ion temperature equilibration rate in the extreme quantum limit where the Born approximation is fully justified. In this lecture, I will examine one of the main theoretical underpinnings of the BPS theory, namely, the dimensional reduction of the BBGKY hierarchy. There are two broad classes of kinetic equations, applicable in complementary regimes, represented by the Boltzmann equation (BE) and the Lenard-Balescu equation (LBE). The BE describes the short-distance effects of 2-body scattering, while the LBE models 2-point long-distance correlations. It is well known that the BE suffers a long-distance logarithmic divergence (in three spatial dimensions), confirming that it is indeed missing long-distance physics (correlations are being ignored). Conversely, the LBE suffers from a short-distance logarithmic divergence (in three dimensions), another indication that relevant physics is being overlooked (the scattering physics). There are multiple industries in plasma physics devoted to regulating these infinities, thereby giving mathematical and physical meaning to the various calculations. To my knowledge, BPS is the only formalism that applies a regularization scheme systematically in a perturbative expansion of the dimensionless plasma coupling parameter $g$, while simultaneously treating short- and long-distance scales consistently and in the same manner. A novel aspect of the BPS formalism is that it employs dimensional continuation to regulate the divergent integrals in the kinetic equations, a procedure first used in quantum field theory to regulate divergent integrals during the renormalization program. The idea of dimensional continuation is that one should perform the integrals in an arbitrary number of spatial dimensions $\nu$, where, remarkably, the integrals become finite (except for $\nu = 3$, where we happen to live). The only remembrance of the three dimensional divergences are simple poles of the form $1/(\nu - 3)$. The BPS formalism hinges on the leading order in $g$ behavior of the BBGKY hierarchy as a function of the spatial dimension $\nu$, both above and below the critical dimension $\nu = 3$. In these notes, I will prove that to leading order in $g$, the BBGKY hierarchy reduces to the BE for $\nu > 3$ and to the LBE for $\nu < 3$. We must eventually return to three dimensions, and the BPS formalism shows that the simple poles associated with the BE and the LBE exactly cancel, rendering the $\nu \to 3$ limit finite. Furthermore, the leading order behavior of the LBE becomes next-to-leading order when $\nu$ is analytically continued from $\nu < 3$ to $\nu > 3$. This provides the leading and next-to-leading order terms in $g$ exactly, which is equivalent to an exact calculation of the so-called Coulomb logarithm with no use of an integral cut-off. Therefore, in this way, BPS takes all Coulomb interactions into account to leading and next-to-leading order in $g$. 
I. INTRODUCTION

This is the third lecture on a novel technique for calculating the charged particle stopping power and the temperature equilibration rate in a weakly coupled fully-ionized plasma [1, 2]. The method is exact to leading and next-to-leading order in the plasma coupling \( g \), and therefore calculates the Coulomb logarithm exactly. In Lecture I [3] of this series, I discussed the basic theoretical machinery of dimensional continuation, and in Lecture II [4], as an example of the method, I calculated the energy exchange rate between electrons and ions in a hot plasma in using the BPS formalism. This formalism can be viewed in the light of convergent kinetic equations, and to my knowledge, it is the only formalism in the literature that uses a systematic expansion in powers of \( g \). It is quite gratifying, therefore, that the BPS stopping power has recently been verified experimentally [5], and this has provided impetus for another lecture. The purpose of these notes is to prove one of the primary claims upon which BPS is based, namely, that to leading order in the plasma coupling \( g \), the BBGKY hierarchy [6] reduces to (i) the Boltzmann equation in dimensions \( \nu > 3 \), and to (ii) the Lenard-Balescu equation [7, 8] in dimensions \( \nu < 3 \). However, for \( \nu = 3 \) (the dimension of interest), things are not so clean: the Boltzmann equation (BE) suffers a long-distance divergence, and the Lenard-Balescu equation (LBE) contains a short-distance divergence. In both cases, the divergences are logarithmic, and this is a crucial observation in regularizing them. Denoting the \( \nu \)-dimensional Coulomb potential by \( \phi_\nu(r) \), we see that the divergences in \( \nu = 3 \) arise because the potential \( \phi_3(r) \sim 1/r \) is the only potential \( \phi_\nu(r) \) whose integral contains both a short- and a long-distance divergence. The dimensional reduction of BBGKY is illustrated schematically in Fig. 1.

The kinetic equations for systems interacting via the Coulomb force diverge in three spatial dimensions, and there have been many attempts to rectify this problem. In these notes, I will concentrate on the method of Brown, Preston, and Singleton (BPS) of Ref. [2]. The method relies on dimensional continuation, which is a regularization technique adopted from quantum field theory calculations in arbitrary spatial dimensions \( \nu \). I will prove rigorously that the BBGKY hierarchy collapses to the Lenard-Balescu equation for \( \nu < 3 \) to leading order in the plasma coupling \( g \). This is quite an involved calculation, and Clemmow and Dougherty [9] is my primary reference. Their calculation breaks down in \( \nu = 3 \) dimensions, but goes through unscathed in dimensions less than three. For completeness, I will also prove that to leading order in \( g \), the BBGKY hierarchy reduces to the Boltzmann equation for \( \nu > 3 \). I will base this calculation on that of Huang in Ref. [10], which breaks down in three dimensions, but becomes rigorous in dimensions greater than three.

As we are concerned with short- and long-distance divergences, we must be clear in our nomenclature. In keeping with the standard usage of quantum mechanics, I will call a short-distance divergence an *ultra-violet* (UV) divergence, and a long-distance divergence an *infra-red* (IR) divergence. This nomenclature arises from the well known episode in the
FIG. 1: For $\nu > 3$ the “textbook derivation” of the Boltzmann equation for a Coulomb potential is rigorous; furthermore, the BBGKY hierarchy reduces to the Boltzmann equation to leading order in $g$. A similar reduction from the BBGKY hierarchy holds for the Lenard-Balescu equation in $\nu < 3$, and the “textbook derivation” is also rigorous in these dimensions. In $\nu = 3$, the derivations of the Boltzmann and Lenard-Balescu equations break down for the Coulomb potential.

history of physics in which classical physics spectacularly failed to calculate the observed black-body spectrum. The classical calculation captured the long-distance infra-red part of the spectrum correctly, but it predicted that the short-distance ultra-violet part of the spectrum would diverge, which is absurd (and contrary to observation). In textbooks this episode is now called the *ultraviolet catastrophe* [11], although more precisely it might be called the Rayleigh-Jeans catastrophe. As we will see, the Lenard-Balescu equation in three dimensions suffers its own UV catastrophe, and for similar reasons. Conversely, it turns out that the Boltzmann equation in three dimensions suffers from an IR divergence. Both are related to the $1/r$ behavior of the Coulomb potential.

It might be of interest to review the history of the UV catastrophe in more details, and to bring out its role in the development of quantum mechanics. The UV catastrophe was indeed a catastrophe for classical physics, and in retrospect can be marked as the birth of quantum mechanics, although in a round about fashion [11]. The classical calculation of the spectral output of a black body is quite simple, involving a single integral over all black-body frequencies $\omega$. It was supposed to be a triumph of classical physics, but embarrassingly, the spectral integral turned out to diverge at small wavelengths or high frequencies. In other words, the classical integral possessed a UV divergence. This was completely unexpected, and is, as we know, cured by the discrete nature of quantum particles of light. Max Planck was examining the divergent classical integral in 1900, and noticed that it became finite if the integral were replaced by a sum over discrete energy states $E_n = n\hbar\omega$, where the angular frequency $\omega$ is that of the light or the electromagnetic radiation emitted from the black body. One of the most radical things in Planck’s scheme is that it required a new
physical constant $h$, sometime written as $\hbar = h/2\pi$, with units of action (energy times time, or equivalently momentum times distance). The constant $h$ is now called Planck’s constant, but at the time, as far as I know, Planck attached no fundamental significance to it. We can, in a certain sense, think of Planck’s method as just another attempt at regulating a divergence, in this case a UV divergence. Then, in 1905, in his work on the photo-electric effect, Einstein proposed that Planck’s energy quanta be taken literally. Einstein reasoned that light with frequency $\nu$ is composed of discrete particles, now called photons, with energy $E = h\nu = \hbar \omega$. The history of physics is rich in attempts to regulate infinite integrals, and the unexpected consequences from doing so. Dimensional continuation is just one of many regularization schemes, and is interesting that quantum mechanics has its roots in one such regularization attempt.

These notes are organized as follows. In Section II we discuss the Coulomb plasma in arbitrary dimensions, showing that the Coulomb force is short-range in dimensions $\nu > 3$ and long-range in $\nu < 3$. The dimension $\nu = 3$ is the critical dimension in which long- and short-range contributions are comparable. This section also discusses the distribution function, and as a warm-up exercise we derive the standard result for the dielectric function in a multi-component plasma. In Section III we discuss how to find energy transfer rates using dimensional continuation, and in Section IV we derive the BBGKY hierarchy in an arbitrary number of dimensions. We show how to define perturbation theory in the plasma coupling constant $g$, and we calculate the BBGKY hierarchy accurate to order $g^2$. We show that a complementary collection of 2-point correlations are dominant in $\nu > 3$ compared to $\nu < 3$, and this leads to the qualitative differences between the Boltzmann equation and the Lenard-Balescu equation. In Section V we derive the Boltzmann equation from BBGKY in $\nu > 3$, and in Section VI we derive the Lenard-Balescu equation in $\nu < 3$. We conclude with Section VII, and cover some supplementary material in the appendices.
II. THE COULOMB PLASMA IN ARBITRARY DIMENSIONS

We start with a plasma composed of multiple species labeled by an index \(a\), the various species being delineated by of a common electric charge \(e_a\) and a common mass \(m_a\). Each species is assumed to be in thermal equilibrium with itself at temperature \(T_a\), with a spatially uniform number density \(n_a\). Since we are working in \(\nu\) spatial dimensions, the engineering units of the number density are \(L^{-\nu}\), and the units of charge \(e_a\) are energy times \(L^\nu\). We will measure temperature in units of energy, setting Boltzmann’s constant to unity \(k_B = 1\), and we will employ the notation \(\beta_a = 1/T_a\) for the inverse temperature.

A. The Coulomb Potential

Before considering a plasma in a general number of dimensions, it is instructive to look at the Coulomb field of a single point charge \(e\) in \(\nu\) dimensions. Since Gauss’s law holds in an arbitrary number of dimensions, then for a point particle at the origin with charge \(e\) we have (in cgs rationalized units)

\[
\nabla \cdot \mathbf{E} = e \delta^\nu(x) ,
\]

(2.1)

where \(\mathbf{E} = (E_1, \cdots, E_\nu)\) is the electric field vector, with \(\nabla = (\partial/\partial x_1, \cdots, \partial/\partial x_\nu)\) being the \(\nu\)-dimensional spatial gradient, and \(\delta^\nu(x)\) being the \(\nu\)-dimensional Dirac \(\delta\)-function centered at the origin. This can be expressed in an integral fashion by integrating any spatial region \(\Sigma\) containing the charge,

\[
\int_{\Sigma} d^\nu x \nabla \cdot \mathbf{E} = e .
\]

(2.2)

To find the electric field we will use Gauss’s theorem,

\[
\int_{\Sigma} d^\nu x \nabla \cdot \mathbf{E} = \int_{\partial \Sigma} dA \cdot \mathbf{E} ,
\]

(2.3)

and exploit the usual symmetry arguments. Let \(\Sigma = B_r\) be the \(\nu\)-dimensional ball of radius \(r\) centered on the point charge \(e\), and therefore the \((\nu-1)\)-dimensional hyperspherical boundary is \(\partial \Sigma = S_r\). By symmetry, the field \(\mathbf{E}(x)\) points radially outward with a magnitude \(E(r)\), along the direction \(\hat{x}\) normal to \(S_r\). The length \(E(r)\) depends only upon the radial distance \(r = |x|\) and not upon its angular location along \(S_r\), and therefore (2.2) gives

\[
e = \int_{B_r} d^\nu x \nabla \cdot \mathbf{E} = \int_{S_r} dA \cdot \mathbf{E} = \Omega_{\nu-1} r^{\nu-1} \cdot E(r) \quad \text{with} \quad \Omega_{\nu-1} = \frac{2\pi^{\nu/2}}{\Gamma(\nu/2)} .
\]

(2.4)

The relation for the solid angle comes from (A9), and the electric field of a point particle at the origin becomes

\[
\mathbf{E}(x) = e \frac{\Gamma(\nu/2)}{2\pi^{\nu/2}} \frac{\hat{x}}{r^{\nu-1}} ,
\]

(2.5)
where \( \hat{x} \) is a unit vector pointing in the direction of \( x \), and the radial variable is \( r = |x| \). Note that \( x = r \hat{x} \), which allows us to write (2.5) in a frequently used alternative form,

\[
E(x) = e \frac{\Gamma(\nu/2)}{2\pi^{\nu/2}} \frac{x}{r^\nu}.
\]

(2.6)

It will often be more convenient to work with the electric potential \( \phi \) defined by

\[
E = -\nabla \phi,
\]

(2.7)

and upon integrating (2.5), we find

\[
\phi(x) = \frac{\Gamma(\nu/2 - 1)}{4\pi^{\nu/2}} \frac{e}{r^{\nu-2}},
\]

(2.8)

where we have chosen the constant of integration so that the potential vanishes at radial infinity (for \( \nu > 2 \)). Generalizing to a multi-component plasma, the electric field of a particle of type \( b \) at the origin is

\[
E_b(x) = e_b \frac{\Gamma(\nu/2)}{2\pi^{\nu/2}} \frac{\hat{x}}{r^{\nu-1}},
\]

(2.9)

and the corresponding potential is

\[
\phi_b(x) = e_b \frac{\Gamma(\nu/2 - 1)}{4\pi^{\nu/2}} \frac{1}{r^{\nu-2}}.
\]

(2.10)

Note that \( E_b(x_a - x_b) \) is the electric field at \( x_a \) produced by a point charge \( e_b \) at \( x_b \), and consequently, the force acting on charge \( a \) from charge \( b \) is

\[
F_a^{(b)} = e_a E_b(x_a - x_b) = e_a e_b \frac{\Gamma(\nu/2)}{2\pi^{\nu/2}} \frac{x_a - x_b}{|x_a - x_b|^\nu}.
\]

(2.11)

This form will appear in the BBGKY kinetic equations for many-body Coulomb systems.

The prefactor of the electric field in these units, known as cgs rationalized units, depends upon the spatial dimension \( \nu \). In three dimensions we find the usual factor of \( 4\pi \),

\[
E_3(x) = \frac{e}{4\pi} \frac{\hat{x}}{r^2}.
\]

(2.12)

\[
\phi_3(x) = \frac{e}{4\pi} \frac{1}{r},
\]

(2.13)

where the numerical subscript denotes \( \nu = 3 \). This potential will turn out to be special, in that its integral diverges logarithmically at both small and short distances. It is the only potential whose integral diverges in the IR and the UV. It will be useful for our intuition to look at the electric field and its potential for dimensions on either side of three. For example, in \( \nu = 4 \), we have

\[
E_4(x) = \frac{e}{2\pi^2} \frac{\hat{x}}{r^3}.
\]

(2.14)

\[
\phi_4(x) = \frac{e}{4\pi^2} \frac{1}{r^2},
\]

(2.15)
and we see that the potential converges more quickly than $\phi_3$ for large values of $r$. The case $\nu = 2$ must be handled with a little care, as we cannot simply substitute $\nu = 2$ into (2.8). The electric field in $\nu = 2$ dimensions is proportional to $1/r$, which integrates to a logarithm for the potential, so that

$$E_2(x) = \frac{e}{2\pi} \frac{\dot{x}}{r}$$

(2.16)

$$\phi_2(x) = -\frac{e}{2\pi} \ln(r/r_0)$$

(2.17)

where $r_0$ is an arbitrary integration constant at which the potential is chosen to vanish. In $\nu = 2$ something different has happened. The potential no longer asymptotes to a constant value at large $r$, but diverges logarithmically. We must therefore choose a finite but arbitrary radius $r_0$ along which the potential vanishes. Note that the 2-dimensional potential also diverges logarithmically at small $r$. Since a logarithmic divergence is an integrable divergence, it is not as severe as the $1/r$ divergence in $\nu = 3$, and this is why the Lenard-Balescu equation in $\nu < 3$ does not contain a short distance divergence, as it does for $\nu \geq 3$. We can also arrive at (2.17) by performing an analytic continuation in $\nu$ near the region $\nu = 2$. In other words, define the small (continuous) parameter $\epsilon = \nu - 2$, and note that (2.8) takes the form

$$\phi(x) = \frac{e}{4\pi} \frac{\Gamma(\epsilon/2)}{(\sqrt{\pi} r)^\epsilon}$$

(2.18)

Using the expansions

$$\Gamma(\epsilon/2) = \frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon)$$

(2.19)

$$a^{-\epsilon} = e^{-\epsilon \ln a} = 1 - \epsilon \ln a + \mathcal{O}(\epsilon)$$

(2.20)

and dropping linear and higher order terms in $\epsilon$, we find

$$\phi_2(x) = \frac{e}{4\pi} \left( \frac{2}{\epsilon} - \gamma \right) \left( 1 - \epsilon \ln \pi^{1/2} r \right) = \frac{e}{4\pi} \left[ \frac{2}{\epsilon} - \gamma - \ln \pi - 2 \ln r \right]$$

(2.21)

In the limit $\nu \to 2$, this potential contains the infinite constant $2/\epsilon - \gamma - \ln \pi$. The physical reason for this (harmless) divergence is that the zero of potential energy in (2.8) vanishes in the asymptotic limit $r \to \infty$, while the logarithmic potential for $\nu = 2$ does not vanish at large $r$, but instead diverges. This is not a problem, as we are free to subtract a constant (even an infinite constant) from any potential. Indeed, as we move from $\nu = 3$ to $\nu = 2$, there is no discontinuity in the $r$-behavior of the electric field. Let us therefore define a shifted potential $\tilde{\phi}_2(x) = \phi_2(x) - \phi_2(x_0)$, that is to say, we subtract the constant value $\phi_2(x_0)$ at an arbitrary $x_0$, and we find $\tilde{\phi}_2(x) = -(e/2\pi) \ln(r/r_0)$, which is just (2.17). Thus, there are no physically measurable discontinuities as we dimensionally continue from $\nu = 3$ to $\nu = 2$. Since we are eventually interested in analytically continuing $\nu$ to complex values in a small
neighborhood around $\nu = 3$, and then taking the limit $\nu \to 3$, we may continue to formally use (2.8). It is, however, important that the electric field and its potential are well defined for all positive integer values of $\nu$. Finally, let us examine $\nu = 1$, for which we obtain

$$E_1(x) = (e/2) \text{sign}(x)$$

$$\phi_1(x) = -(e/2) r,$$

where $r = |x|$. The unit vector pointing away from the origin, $\hat{x}$, can point only left or right, and can therefore be thought of as the sign function $\text{sign}(x)$: plus one for positive $x$ and minus one for negative $x$, located at $x = 0$. Since $x = |x| \text{sign}(x)$, we can still express the spatial point $x$ in the form $x = r\hat{x}$.

**B. The Coulomb Potential and Dimensional Regularization**

Let us explore in more detail how dimensional continuation acts as a regulator for divergent integrals, rendering them finite and therefore algebraically amenable. We shall start by considering the Boltzmann equation for a plasma in three dimensions. We do not require the actual equation at the moment, but we only need to recall that it suffers a long-distance IR divergence. In contrast, it is interesting to note that the far more idealized hard-sphere scattering model is finite. This hard-sphere model is based on the idea that particles in a dilute gas behave like billiard balls. This is of course incorrect, or at least a highly idealized picture, but the model still provides useful insight. The reason that hard-sphere scattering is finite is that it is short-range: the force acts only during the collision, after which the particles move freely in a constant potential (until the next collision). There are no long-distance effects in this model. A gas of neutral particles acts somewhat like billiard balls, so we expect the Boltzmann equation to be finite for a gas. And indeed it is. This is because the force between neutral particles is short-range, and they do not see one another at large distances. In fact, the Boltzmann kernel is finite for any short-range force in three dimensions. The irony, however, is that we are not interested in short-range forces. Instead, it is the Coulomb force that is of relevance to plasma physics, and in three dimensions this is a long-range force. Consequently, we find a long-distance logarithmic divergence in the Boltzmann scattering kernel. This IR divergence essentially arises from the integration of the potential $\phi_3(r) \sim 1/r$ at large-$r$. Since the divergence is only logarithmic, any potential that falls off faster than $1/r$ at large-$r$ will not produce a divergence in the Boltzmann scattering kernel. We will return to this point in the next paragraph. In summary, even though the Boltzmann equation gets the short-distance physics right, it gets the long-distance physics wrong, and

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1 As stated in the introduction, we are using the quantum mechanical nomenclature in which IR stands for *infra-red* long-distance physics, UV stand for short-distance *ultra-violet* physics.
we pay the price through a logarithmic IR divergence. Conversely, when we capture longdistance collective effects in a plasma using the Lenard-Balescu equation, we find a shortdistance UV divergence (in three dimensions). The Lenard-Balescu equation captures the long-distance physics correctly, but models the short-distance physics incorrectly, and once again we pay a price, this time introducing a logarithmic UV divergence. In this case, any potential that diverges less severely than \( \phi_3(r) \sim 1/r \) at small-\( r \) will not suffer a divergence. But as before, we are not interested in such forces. This reasoning, however, will be essential to understanding why the Coulomb potential in \( \nu \) dimensions regulates the various kinetic equations.

We can now show why working in an arbitrary number of dimensions \( \nu \) acts as a regulator. It turns out that the Boltzmann equation (BE) becomes IR finite for \( \nu > 3 \), and that the Lenard-Balescu equation (LBE) becomes UV finite for \( \nu < 3 \), with the only reminder of past divergences being simple poles of the form \( 1/(\nu - 3) \). Now that we know how the Coulomb force works as a function of dimension, we can understand this behavior. Figure 2 shows the Coulomb potential \( \phi_\nu(r) \sim 1/r^{\nu-2} \) for a positive charge at the origin for \( \nu = 1, 2, 3, 4 \) (remembering that \( \nu = 2 \) is actually logarithmic). For aesthetic reasons, the arbitrary integration constants of the potentials have been adjusted so that the graphs for \( \nu = 2, 3, 4 \) intersect at a common point. The Figure shows that by simply dialing the dimension \( \nu \), a potential \( \phi_\nu(r) \) can be selected with the appropriate short- and long-distance behavior. The potential \( \phi_3(r) \sim 1/r \) is special, in that it produces logarithmic divergences in the UV and IR, indicating that short- and long-distance physics are equally important in three dimensions. Thus, \( \phi_3(r) \) is a borderline case, and this is the reason that the BE and the LBE suffer IR and UV divergences, respectively. However, and this is the key point, for \( \nu < 3 \), the left panel shows that the potential diverges less slowly than \( \phi_3(r) \) for small-\( r \), and this is what renders the LBE finite in the UV. Conversely, for \( \nu > 3 \) the right panel of the Fig. 2 shows that the potential converges more rapidly than \( \phi_3(r) \) for large \( r \), and this renders the BE finite in the IR. This is the reason dimensional continuation works as a regulator.

As we have emphasized, the divergences in question are only logarithmic (rather than linear or higher order), and can therefore be rendered finite by slightly adjusting the rate of convergence of the potential \( \phi_\nu(r) \) in the offending region of \( r \) (either at large or small values of \( r \)). This takes us into the domain of convergent kinetic equations. The integral of any potential that diverges less slowly that \( 1/r \) as \( r \to \infty \), even by an infinitesimal amount, will in fact converge at large-\( r \). For example, the potential \( \phi(r) \sim 1/r^{1+\delta_1} \) with \( \delta_1 > 0 \), but otherwise \( \delta_1 \) can be as close to zero as we wish, gives a convergent integral in the IR,

\[
\int \frac{dr}{r^{1+\delta_1}} \sim r^{-\delta_1} \to 0 \quad \text{as } r \to \infty ,
\]

and the BE does not possess an IR divergence for such a potential. Conversely, the integral of any potential that diverges less slowly that \( 1/r \) as \( r \to 0 \) will in fact converge at small-\( r \).
FIG. 2: The Coulomb potential $\phi$ for a positive charge at the origin as a function of radius $r$, for dimensions $\nu = 1, 2, 3, 4$. The zeroes of potential energy have been adjusted for visual clarity. Short-distance ultraviolet (UV) physics is emphasized in dimensions $\nu > 3$ (left panel), and long-distance or infrared (IR) physics dominates when $\nu < 3$ (right panel). For $\nu = 3$, the UV and IR physics are equally important, and the energy rate diverges logarithmically at both large and small distances in three dimensions. The left panel illustrates that for $\nu < 3$, the Coulomb potential diverges less severely than $1/r$ as $r$ gets small, and consequently the LBE does not suffer a short-distance divergence in $\nu < 3$. Similarly, the right panel illustrates that the Coulomb force converges to zero more rapidly than $1/r$ as $r$ gets large, and this renders the Boltzmann kernel finite at large distances when $\nu > 3$.

For example, consider a potential of the form $\phi(r) \sim 1/r^{1-\delta_2}$, where $\delta_2 > 0$ (with $\delta_2$ as close to zero as we wish). Then the integral of the potential is finite in the UV,

$$\int dr \frac{1}{r^{1-\delta_2}} \sim r^{\delta_2} \to 0 \text{ as } r \to 0,$$

and the LBE does not suffer a UV divergence for such a potential. This is the essence of the techniques of convergent kinetic theory, of which the BPS formalism is an example. One must be exceedingly careful, however, as the same physical regularization scheme must be used at short- and long-distances. The quantities $\delta_1$ of (2.24) and $\delta_2$ of (2.25) are not independent! If they are treated independently, then one can produce spurious unphysical constants in the Coulomb logarithm. Another benefit of the BPS formalism is that it treats long and short distances in the same manner. See Lecture I regarding the Lamb Shift, in which the significance of using the same regularization was first realized. To my knowledge, BPS is the only convergent kinetic scheme that treats the long- and short-distance divergences in the same way.
C. The Fourier Transform of the Coulomb Potential

Unlike the spatial representation of the potential, we will show that the Fourier representation takes the same form in any dimension. This is quite useful for calculations. There are a number of conventions for the spatial Fourier transform, and I employ

$$\phi(x) = \int d\nu k (2\pi)^\nu \nu e^{i x \cdot k} \tilde{\phi}(k)$$ \hspace{1cm} (2.26)

$$\tilde{\phi}(k) = \int d^\nu x e^{-i x \cdot k} \phi(x).$$ \hspace{1cm} (2.27)

As a general rule, the factors of $2\pi$ will always be placed with the $k$-integral, as this is analogous to placing factors of $2\pi\hbar$ with the $p$-integral, a convention based in quantum mechanics that we shall also follow. As we now show, the $\nu$-dimensional Coulomb potential (2.8), which is repeated here for convenience,

$$\phi(x) = \frac{\Gamma(\nu/2 - 1)}{4\pi^{\nu/2}} \frac{e}{r^{\nu-2}},$$ \hspace{1cm} (2.28)

has the Fourier transform

$$\tilde{\phi}(k) = \frac{e}{k^2} \quad \text{where} \quad k^2 \equiv k \cdot k = \sum_{\ell=1}^\nu k^2_\ell.$$ \hspace{1cm} (2.29)

As emphasized above, the form of $\tilde{\phi}(k)$ does not depend upon the dimension of space, except in a trivial way though the length of $k^2$.

Expression (2.29) for the Fourier transform of the potential (2.28) can be established in a number of ways. Perhaps the easiest is just to use Laplace’s equation,

$$\nabla^2 \phi(x) = -e \delta^{(\nu)}(x),$$ \hspace{1cm} (2.30)

which is obtained by substituting (2.7) into (2.1). Upon inserting (2.26) for $\phi(x)$ into (2.30), and using the integral representation of the $\delta$-function, we can write Laplace’s equation in the form

$$- \int d^\nu k \frac{k^2}{(2\pi)^\nu} e^{i x \cdot k} \tilde{\phi}(k) = -e \delta^{(\nu)}(x) = -e \int d^\nu k \frac{e^{i x \cdot k}}{(2\pi)^\nu},$$ \hspace{1cm} (2.31)

or

$$\int d^\nu k \frac{k^2}{(2\pi)^\nu} e^{i x \cdot k} \left[ k^2 \tilde{\phi}(k) - e \right] = 0.$$ \hspace{1cm} (2.32)

The quantity in square brackets must vanish, and solving for $\tilde{\phi}(k)$ indeed gives (2.29).

It is also informative to prove this result by taking the Fourier transform directly. The formula to remember is

$$\frac{1}{a} = \int_0^\infty ds e^{-as},$$ \hspace{1cm} (2.33)
where Re \( a > 0 \), from which we can take \( p \) derivatives to obtain another useful expression,

\[
\frac{1}{a^p} = \frac{1}{\Gamma(p)} \int_0^\infty ds \, s^{p-1} e^{-as}.
\]  

(2.34)

We can analytically continue to complex values of \( p \), in particular to \( p = (\nu - 2)/2 \). We can also prove (2.34) by changing variables in the \( s \)-integral to \( u = as \), which produces the correct scaling \( 1/a^p \), while the remaining \( u \)-integral exactly cancels the Gamma-function \( \Gamma(p) \). We now use relation (2.34) to rewrite the term \( 1/r^{\nu-2} \) in the potential. First note that \( r = (x \cdot x)^{1/2} \), which we write as \( r = (x^2)^{1/2} \), and this allows us to express

\[
\frac{1}{r^{\nu-2}} = \frac{1}{(x^2)^{(\nu-2)/2}} = \frac{1}{\Gamma(\nu/2 - 1)} \int_0^\infty ds \, s^{-(\nu-4)/2} e^{-sx^2}.
\]  

(2.35)

The Fourier transform of the Coulomb potential is therefore

\[
\tilde{\phi}(k) = e^{\Gamma(\nu/2 - 1)} \int_0^{\infty} d\nu \, x \, e^{-ix \cdot k} \frac{1}{r^{\nu-2}}
\]  

(2.36)

\[
= \frac{e}{4\pi^{\nu/2}} \int d\nu \, x \int_0^{\infty} ds \, s^{(\nu-4)/2} e^{-sx^2 - ix \cdot k}.
\]  

(2.37)

We can interchange the \( s \) and \( x \) integrals because the integrand is uniformly convergent. We then perform the \( x \)-integrals by completing the square, and we find

\[
\tilde{\phi}(k) = \frac{e}{4\pi^{\nu/2}} \int_0^{\infty} ds \, s^{(\nu-4)/2} \int d\nu \, x \, e^{-s(x + i\nu/2s)} e^{-k^2/4s}
\]  

(2.38)

\[
= \frac{e}{4\pi^{\nu/2}} \int_0^{\infty} ds \, s^{(\nu-4)/2} \left( \frac{\pi}{s} \right)^{\nu/2} e^{-k^2/4s}
\]  

(2.39)

\[
= \frac{e}{4} \int_0^{\infty} ds \, s^{-2} e^{-k^2/4s} = \frac{e}{4} \int_0^{\infty} dt \, e^{-tk^2/4} = \frac{e}{k^2}.
\]  

(2.40)

D. The Distribution Function

For each component \( a \) of the plasma, there is a distribution function \( f_a \) define by

\[
f_a(x, p, t) \frac{d^\nu x \, d^\nu p}{(2\pi \hbar)^{\nu}} \equiv \text{number of particles of type } a \text{ in a hypervolume } \\
d^\nu x \text{ about } x \text{ and } d^\nu p \text{ about } p \text{ at time } t,
\]  

(2.41)

where where \( \hbar \) is Planck’s constant, and \( \hbar = \hbar/2\pi \). The phase-space factor \( (2\pi \hbar)^{\nu} = \hbar^{\nu} \) ensures that \( f_a \) counts the number of semi-classical quantum states in a phase-space volume \( d^\nu x \, d^\nu p \). The factor \( \hbar \) makes the volume element in (2.41) dimensionless, thereby rendering \( f_a \) dimensionless. Using \( \hbar \) in place of \( h \) is merely a convention. More critically, the factor of \( \hbar \) makes the classical to quantum transition more transparent, and the normalization (2.41) implies

\[
\int \frac{d^\nu p}{(2\pi \hbar)^{\nu}} \, f_a(x, p, t) = n_a(x, t),
\]  

(2.42)
where \( n_a(x,t) \) is the number density of \( a \)-type particles at position \( x \) and time \( t \). That is to say, \( n_a \, d^x \! x \) is the number of particles of species \( a \) in a hypervolume \( d^x \! x \) about position \( x \) at time \( t \). When performing long calculations, it is often convenient to combine the space and momentum variables into a single phase-space variable \( X = (x,p) \). Distribution functions are then written \( f = f(X,t) \), and the corresponding integration measure becomes

\[
dX = \frac{d^x \! x \, d^p \! p}{(2\pi\hbar)^\nu} . \tag{2.43}
\]

Definition (2.41) now means that \( f_a(X,t) \, dX \) is the number of particles of type \( a \) in a volume element \( dX \) about phase-space location \( X \) at time \( t \). This notation will be particularly useful when considering multi-particle distributions.

Throughout these notes, we primarily consider plasmas in local thermodynamic equilibrium, so that \( n_a \) is a slowly varying function of space and time, with engineering units of \( L^{-\nu} \). We will often consider special cases in which the plasma is completely uniform, and the number density is constant in space. As a general rule, we will leave time dependence implicit. From (2.42), we see that a normalized Maxwell-Boltzmann distribution at temperature \( T_a \) and number density of \( n_a \) takes the form

\[
f_a(p) = n_a \left( \frac{2\pi\hbar^2 \beta_a}{m_a} \right)^{\nu/2} \exp \left\{ -\beta_a \frac{p^2}{2m_a} \right\} = n_a \lambda_a^\nu e^{-\beta_a E_a} , \tag{2.44}
\]

where \( \beta_a = 1/T_a \) is the inverse temperature in energy units, and \( E_a = p_a^2/2m_a \) is the kinetic energy of an individual particle of species \( a \). The thermal de Broglie wavelength for particle \( a \) is defined to be

\[
\lambda_a = \hbar \left( \frac{2\pi \beta_a}{m_a} \right)^{1/2} = \frac{\hbar}{\sqrt{2\pi m_a T_a}} . \tag{2.45}
\]

Expression (2.44) shows explicitly that \( f_a \) is dimensionless in accordance with our normalization, as \( n_a \lambda_a^\nu \) is dimensionless. However, one might rightly ask why would a quantum parameter, \( \hbar \), appear in a classical distribution? In fact, physical averages do not depend on \( \hbar \), as the factors of \( \hbar \) in the integration measure (2.43) are exactly canceled by those in the normalization term \( \lambda_a^\nu \). However, this normalization is more than a mere convention, for \( n_a \lambda_a^\nu \) counts the number of quantum states available to system \( a \). Therefore, this normalization is the only one that gets questions about entropy correct. In fact, upon writing the distribution function in terms of the chemical potential \( \mu_a \),

\[
f_a = e^{-\beta_a (E_a - \mu_a)} , \tag{2.46}
\]

we see that

\[
e^{\beta_a \mu_a} = n_a \lambda_a^\nu . \tag{2.47}
\]
This gives the correct chemical potential of a free gas,

\[ \mu_a = T_a \ln n_a \lambda^\nu = T_a \ln \left\{ n_a \left[ \frac{\hbar}{m_a T_a} \right]^{1/2} \right\}^\nu, \]  

(2.48)
a standard result from quantum statistical mechanics, with a trivial generalization to multiple dimensions.

Two-particle correlations are described by a two-component correlation function, defined by

\[ f_{ab}(x_a, p_a, x_b, p_b, t) \frac{d^\nu x_a}{(2\pi\hbar)^\nu} \frac{d^\nu p_a}{(2\pi\hbar)^\nu} \equiv \text{number of particles of type } a \text{ in a hypervolume } d^\nu x_a \text{ about } x_a \text{ and } d^\nu p_a \text{ about } p_a, \text{ and } d^\nu x_b \text{ about } x_b \text{ and } d^\nu p_b \text{ about } p_b, \text{ at time } t. \]

(2.49)

We often write this as \( f_2(x_1, p_1, x_2, p_2, t) \) in a single-species plasma. We can go on to define higher order distribution functions, and in fact show that they satisfy a system of connected equations known as the BBGKY hierarchy. We will discuss this in much more detail in future sections. For now, let us stick to the basic properties of a multi-component plasma, and let us see what we can do with just the single-particle distribution.

### E. The Dielectric Function of a Plasma

As these notes are also a tutorial, it is useful to take a detour and to describe how kinetic theory allows us to calculate the induced charge density \( \rho_{\text{ind}} \) of a plasma when a small external electric field \( \mathbf{D} \) is applied. This in turn allows us to calculate the dielectric function of the plasma. Consider a multi-species plasma in which species \( b \) has charge \( e_b \) and mass \( m_b \). Furthermore, suppose that each species \( b \) is in thermal equilibrium with itself at temperature \( T_b \) and Maxwell-Boltzmann distribution \( f_b \). We now use basic kinetic theory to prove that the dielectric function in a general number of dimensions takes the form

\[ \epsilon(k, \omega) = 1 + \sum_b \frac{e_b^2}{k^2} \int \frac{d^\nu p_b}{(2\pi\hbar)^\nu} \frac{1}{\omega - k \cdot v_b + i\eta} \cdot \frac{\partial f_b(p_b)}{\partial p_b}, \]

(2.50)

where the limit \( \eta \to 0^+ \) is understood, and \( p_b = m_b v_b \). Reference [12] derives this well known result in three spatial dimensions, and this section is a simple extension to a general number of spatial dimensions \( \nu \). It serves mainly as a refresher to the reader who is not an expert, and it is very beautiful physics. It is quite astounding that one can get so much from so little.
1. The Induced Charge Density and the Dielectric Function

Let us start with a neutral equilibrium plasma, and apply a small external electric field \( \mathbf{D} \). Since the charges in the plasma are free to move, even the smallest field creates an induced charge density \( \rho_{\text{ind}} \), as the ions and electrons are separated by the field. The charge separation creates an induced field \( \mathbf{P} \) that weakens the applied field. The observed field \( \mathbf{E} \) is a combination of both of these [12],

\[
\mathbf{E} = \mathbf{D} - \mathbf{P} \quad \text{where} \\
\nabla \cdot \mathbf{D} = \rho_{\text{ext}} \tag{2.52} \\
\nabla \cdot \mathbf{P} = -\rho_{\text{ind}} . \tag{2.53}
\]

We are free to set the density \( \rho_{\text{ext}} \) to anything that creates the desired applied field \( \mathbf{D} \). The observed field \( \mathbf{E} \) satisfies Gauss’s law,

\[
\nabla \cdot \mathbf{E} = \rho , \tag{2.54}
\]

where \( \rho \) is the total charge density of the medium, which consists of the external charge density and the induced charge density

\[
\rho = \rho_{\text{ext}} + \rho_{\text{ind}} . \tag{2.55}
\]

The field \( \mathbf{D}(\mathbf{x},t) = \mathbf{E}(\mathbf{x},t) + \mathbf{P}(\mathbf{x},t) \) depends not just on the electric field \( \mathbf{E}(\mathbf{x},t) \) at time \( t \), but also on the electric fields at earlier times as well, through the polarization

\[
\mathbf{P}(\mathbf{x},t) = \int_{-\infty}^{t} dt' \int d^{d'}x' \chi(\mathbf{x} - \mathbf{x}', t - t') \mathbf{E}(\mathbf{x}', t') . \tag{2.56}
\]

It is understood that the kernel satisfies causality, \( \chi(\mathbf{x},t) = 0 \) when \( t < 0 \), and we can therefore extend the \( t' \) integral from \( t \) to \( \infty \), giving

\[
\mathbf{D}(\mathbf{x},t) = \mathbf{E}(\mathbf{x},t) + \int_{-\infty}^{\infty} dt' \int d^{d'}x' \chi(\mathbf{x} - \mathbf{x}', t - t') \mathbf{E}(\mathbf{x}', t') . \tag{2.57}
\]

Using the convolution theorem, the Fourier transform takes a particularly simple form,

\[
\tilde{\mathbf{D}}(\mathbf{k},\omega) = \epsilon(\mathbf{k},\omega) \tilde{\mathbf{E}}(\mathbf{k},\omega) \tag{2.58} \\
\epsilon(\mathbf{k},\omega) = 1 + \chi(\mathbf{k},\omega) \tag{2.59},
\]

where the spatial and temporal Fourier transform of the susceptibility is

\[
\tilde{\chi}(\mathbf{k},\omega) = \int_{-\infty}^{\infty} dt \int d^{d}x \chi(\mathbf{x},t) e^{-i\mathbf{k}\cdot\mathbf{x} + i\omega t} , \tag{2.60}
\]
and the inverse transform is
\[ \chi(x, t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int \frac{d^3k}{(2\pi)^3} \tilde{\chi}(k, \omega) e^{ik \cdot x - i\omega t}. \] (2.61)

This sign convention is consistent with the sign conventions of quantum mechanics: \(-i\omega t\) and \(i k \cdot x\) in (2.61).^2 Returning to (2.51), we can write the Fourier transform of the polarization vector as
\[ \tilde{P}(k, \omega) = \tilde{D}(k, \omega) - \tilde{E}(k, \omega) = \chi(k, \omega) \tilde{E}(k, \omega). \] (2.62)

The spatial form of Gauss’s law \(\nabla \cdot P = -\rho_{\text{ind}}\) translates into \(i k \cdot \tilde{P} = -\tilde{\rho}_{\text{ind}}\) in Fourier space, and this allows us to write
\[ \tilde{\rho}_{\text{ind}}(k, \omega) = -i \chi(k, \omega) k \cdot \tilde{E}(k, \omega) = -\chi(k, \omega) k^2 \tilde{\phi}(k, \omega) = -\chi(k, \omega) \tilde{\rho}(k, \omega). \] (2.63)

The last form has been expressed in terms of the potential \(\phi\), defined by \(E = -\nabla \phi\), which in Fourier space becomes \(\tilde{E} = -i k \tilde{\phi}(k, \omega)\). The Fourier transform of the dielectric is therefore
\[ \chi(k, \omega) = -\frac{\tilde{\rho}_{\text{ind}}(k, \omega)}{k^2 \tilde{\phi}(k, \omega)}, \] (2.64)

and the problem reduces to calculating \(\tilde{\rho}_{\text{ind}}(k, \omega)\). The tool for performing such calculations is kinetic theory.

2. Calculation of the Dielectric Function of a Plasma

Let us concentrate on an individual plasma component \(a\). In the absence of an applied field, we assume that species \(a\) is in thermal equilibrium with itself, specified by a Maxwell-Boltzmann distribution \(\bar{f}_a(p)\) with inverse temperature \(\beta_a = 1/T_a\) and charge \(e_a\). When an external electric field is applied, this induces a charge density \(\rho_{\text{ind}}\). The distribution function consequently departs from equilibrium, and the system is then specified by a new distribution \(f_a(x, p, t)\). Because collisions are unimportant to the induced charges, the distribution \(f_a\) satisfies the collisionless Maxwell-Boltzmann equation
\[ \frac{\partial f_a}{\partial t} + v \cdot \frac{\partial f_a}{\partial x} + e_a E \cdot \frac{\partial f_a}{\partial p} = 0, \] (2.65)

where \(v = p/m_a\), and \(E = D - P\) is the total electric field seen by \(a\). The electric field \(E\) is the sum of the applied field \(D\) and an induced contribution \(-P\). Note that the kinetic

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^2 The quantum energy and momentum operators are \(\hat{E} = i\hbar \partial/\partial t\) and \(\hat{p} = -i\hbar \partial/\partial x\), so that \(\hat{E} e^{-i\omega t} = \hbar \omega e^{-i\omega t}\) and \(\hat{p} e^{ik \cdot x} = \hbar k e^{ik \cdot x}\); therefore, the energy and momentum Eigenvalues are \(E = \hbar \omega\) and \(p = \hbar k\).
equation (2.65) implies charge conservation. That is to say, if we define the charge density and electric current by

\[ \rho_a(x, t) = \int \frac{d^3 p}{(2\pi\hbar)^3} e_a f_a(x, p, t) \]

\[ J_a(x, t) = \int \frac{d^3 p}{(2\pi\hbar)^3} e_a v_a f_a(x, p, t), \]

then these quantities satisfy the continuity equation

\[ \frac{\partial \rho_a}{\partial t} + \nabla \cdot J_a = 0. \]

This is because the last term in the kinetic equation (2.65) is a total divergence in momentum space, and therefore integrates to zero by the divergence theorem. It is reassuring to see charge conservation arising directly from the kinetic equation. Indeed, all of hydrodynamics can be recovered from kinetic theory, although this would take us well beyond the scope of these notes. For this calculation, we start by expressing \( f_a \) in terms of a small perturbation \( h_a \),

\[ f_a(x, p, t) = \bar{f}_a(p) + h_a(x, p, t). \]

Upon substituting (2.69) back into (2.65) and work to first order. Since the induced electric field \( \mathbf{E} \) is first order, we can neglect the small second-order term \( e \mathbf{E} \cdot \partial h / \partial \mathbf{p} \), and write the kinetic equation as

\[ \frac{\partial h_a}{\partial t} + \mathbf{v} \cdot \nabla h_a + e_a \mathbf{E} \cdot \frac{\partial \bar{f}}{\partial \mathbf{p}} = 0. \]

It is often convenient to express the electric field in terms of a potential,

\[ \mathbf{E} = -\nabla \phi \equiv -\frac{\partial \phi}{\partial \mathbf{x}}, \]

in which case the transport equation (2.70) takes the form

\[ \frac{\partial h_a}{\partial t} + \mathbf{v} \cdot \frac{\partial h_a}{\partial \mathbf{x}} = e_a \frac{\partial \phi}{\partial \mathbf{x}} \cdot \frac{\partial \bar{f}}{\partial \mathbf{p}}. \]

To solve this equation we take the space and time Fourier, thereby giving

\[ -i\omega \tilde{h}_a + i \mathbf{k} \cdot \mathbf{v} \tilde{h}_a = e_a \tilde{\phi}(\mathbf{k}, \omega) i \mathbf{k} \cdot \frac{\partial \bar{f}}{\partial \mathbf{p}}, \]

where \( \tilde{h}_a \) is a function of \( \mathbf{k}, \mathbf{p}, \) and \( p \). We can now solve (2.73) for the perturbation,

\[ \tilde{h}_a(\mathbf{k}, \mathbf{p}, \omega) = -\frac{e_a \tilde{\phi}(\mathbf{k}, \omega)}{\omega - \mathbf{k} \cdot \mathbf{v}} \mathbf{k} \cdot \frac{\partial \bar{f}(\mathbf{p})}{\partial \mathbf{p}}. \]
Note that $\tilde{\phi}(k)$ is the Fourier transform of the applied potential, and it not given by (2.29). In performing the inverse Fourier transform to recover the correlation function $h(x, p, t)$ in space and time, we must integrate over $k$ and $\omega$. By convention, we hold $k$ fixed and integrate over the variable $\omega$ first. The integration contour for $\omega$ lies in the complex $\omega$-plane slightly above the real axis. This avoids the pole at $\omega = k \cdot v$ when integrating over $\omega$, and establishes the proper causality for $h(x, p, t)$. This choice of contour is equivalent to integrating over real values of $\omega$, but adding a small complex term $i\eta$ to the numerator in (2.74). We can therefore write the Fourier transform of the correlation function as

$$\tilde{h}_a(k, p, \omega) = -\frac{e_a}{\omega - k \cdot v + i\eta} \frac{k}{k^2} \partial_f(p) \partial p, \quad (2.75)$$

where limit $\eta \to 0^+$ is understood. We can always restore the correlation function to space and time variables by performing the inverse Fourier transform,

$$h_a(x, p, t) = -\int d^\nu k \frac{d\omega}{2\pi} e^{i(k \cdot x - i\omega t)} e_a \frac{1}{k^2} \frac{k}{\omega - k \cdot v_a + i\eta} k \cdot \partial_f_a(p) \times k^2 \tilde{\phi}(k, \omega), \quad (2.76)$$

where we have factored out the term $k^2 \tilde{\phi}(k)$ for convenience. Note that the form of $\tilde{\phi}(k)$ is unknown, but it will cancel from the dielectric function. The induced charge density therefore becomes

$$\rho_{\text{ind}}(x, t) = \sum_b \int \frac{d^\nu p}{(2\pi\hbar)^\nu} e_b h_b(x, p, t). \quad (2.77)$$

It is actually more convenient to continue working in Fourier space, and using (2.75) allows us to express the induced charge density as

$$\tilde{\rho}_{\text{ind}}(k, \omega) = \sum_b \int \frac{d^\nu p}{(2\pi\hbar)^\nu} e_b \tilde{h}_b(k, p, \omega) \quad (2.78)$$

$$= -\sum_b \int \frac{d^\nu p}{(2\pi\hbar)^\nu} e_b^2 \frac{1}{k^2} \frac{k}{\omega - k \cdot v_b + i\eta} k \cdot \partial_f_b(p) \times k^2 \tilde{\phi}(k, \omega). \quad (2.79)$$

The susceptibility is therefore

$$\tilde{\chi}(k, \omega) = -\frac{\tilde{\rho}_{\text{ind}}(k, \omega)}{k^2 \tilde{\phi}(k, \omega)} = \sum_b \frac{e_b^2}{k^2} \int \frac{d^\nu p}{(2\pi\hbar)^\nu} \frac{1}{\omega - k \cdot v_b + i\eta} k \cdot \partial_f_b(p) \partial p, \quad (2.80)$$

which gives (2.50) for the dielectric function $\epsilon(k, \omega) = 1 + \chi(k, \omega)$. 

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III. COULOMB ENERGY TRANSFER RATES IN ARBITRARY DIMENSIONS

This section is a review of the basic BPS formalism, presented here for completeness, with an emphasis on the role of analytic continuation of the spatial dimension $\nu$. We turn now to calculating Coulomb energy exchange rates in the multi-component plasma described in the previous section. The charged particle stopping power and the temperature equilibration rate between plasma species of different temperatures are the two canonical examples I have in mind.

A. Coulomb Energy Exchange

The single-particle distribution function for plasma species $a$ satisfies a general kinetic equation of the form

$$\frac{\partial f_a}{\partial t} + v_a \cdot \frac{\partial f_a}{\partial x} + F_a \cdot \frac{\partial f_a}{\partial p} = \left( \frac{\partial f_a}{\partial t} \right)_c \equiv \sum_b K_{ab}^\nu[f] ,$$

(3.1)

where the velocity is given by $v_a = p/m_a$, and $F_a$ is the total force acting on $a$ at $x$, e.g. $F_a = e_a E(x)$ in the case of an external electric field. The scattering rate $(\partial f_a/\partial t)_c$ is a generic expression that accounts for the effects of scattering or collisions. It is calculated in kinetic theory textbooks under various conditions, the most relevant being for the Boltzmann kernel $B_{ab}[f]$ and the Lenard-Balescu kernel $L_{ab}[f]$. For now, we will keep the form of the kernel generic and simply write $K_{ab}^\nu[f]$. For stopping power calculations and other Coulomb energy exchange processes, we will set the external force to zero, so the distribution function $f_a$ satisfies

$$\frac{\partial f_a}{\partial t} + v_a \cdot \frac{\partial f_a}{\partial x} = \sum_b K_{ab}^\nu[f] .$$

(3.2)

The kinetic energy density of plasma species $a$ is defined by

$$E_a = \int \frac{d^\nu p_a}{(2\pi\hbar)^\nu} \frac{p_a^2}{2m_a} f_a(p_a,t) ,$$

(3.3)

where $f_a$ is the corresponding distribution function. The stopping power is related to the rate of energy loss by

$$\frac{d E_a}{dt} = \int \frac{d^\nu p_a}{(2\pi\hbar)^\nu} \frac{p_a^2}{2m_a} \frac{\partial f_a(p_a,t)}{\partial t} .$$

(3.4)

Using the kinetic equation (3.2), the divergence over $x$ integrates to zero, and we find

$$\frac{d E_a}{dt} = \int \frac{d^\nu p_a}{(2\pi\hbar)^\nu} \frac{p_a^2}{2m_a} \frac{\partial f_a(p_a,t)}{\partial t} \equiv \sum_b \int \frac{d^\nu p_a}{(2\pi\hbar)^\nu} \frac{p_a^2}{2m_a} K_{ab}^\nu[f] .$$

(3.5)

We can therefore identify the rate of change in the kinetic-energy density of species $a$ resulting from its Coulomb interactions with species $b$ by

$$\frac{d E_{ab}}{dt} = \int \frac{d^\nu p_a}{(2\pi\hbar)^\nu} \frac{p_a^2}{2m_a} K_{ab}^\nu[f] .$$

(3.6)
B. Dimensional Reduction of BBGKY

As we have seen, moving to an arbitrary dimension $\nu$ acts as a regulator, rendering the kinetic equations, both the Boltzmann equation (BE) and the Lenard-Balescu equation (LBE), finite in their respective dimensional regimes. Dimensional regularization, however, does far more than this. A more subtle advantage of working in a general dimension is that it acts as a “physics sieve”, in that it selects the proper scattering kernel to leading order (LO) in $g$ in the dimension at hand:

$$\text{BBGKY in } \nu > 3 \Rightarrow \frac{\partial f_a}{\partial t} + v_a \cdot \frac{\partial f_a}{\partial x} = \sum_b B_{ab}[f] \text{ to LO in } g ,$$ (3.7)

where $B_{ab}$ is the $\nu$-dimensional Boltzmann scattering kernel, and

$$\text{BBGKY in } \nu < 3 \Rightarrow \frac{\partial f_a}{\partial t} + v_a \cdot \frac{\partial f_a}{\partial x} = \sum_b L_{ab}[f] \text{ to LO in } g ,$$ (3.8)

where $L_{ab}[f]$ is the $\nu$-dimensional scattering kernel for the Lenard-Balescu equation. Proving this statement, which I call the dimensional reduction theorem, is the main purpose of these notes. Figure 1 serves as a useful pictorial representation of the theorem.

1. The Boltzmann Kernel

In this subsection I will review the Boltzmann scattering kernel in some detail. In formal work I will write the Boltzmann equation in schematic form as

$$\frac{\partial f_a}{\partial t} + v_a \cdot \frac{\partial f_a}{\partial x} = \sum_b B_{ab}[f] \quad : \nu > 3 ,$$ (3.9)

or in calculations I will use the form

$$B_{ab}[f] = \int \frac{d^\nu p_b}{(2\pi \hbar)^\nu} |v_a - v_b| d\sigma_{ab} \left\{ f_a(p'_a)f_b(p'_b) - f_a(p_a)f_b(p_b) \right\} .$$ (3.10)

BPS included the quantum effects of two-body Coulomb scattering by replacing the classical cross section by the corresponding quantum transition amplitude $T(ab \rightarrow a'b') \equiv T_{a'b';ab}(W,q^2)$, where $W$ is the center-of-mass energy and $q^2$ is the square of the momentum exchange during the collision. The cross section $d\sigma_{ab}$ and the square of the scattering amplitude $|T_{a'b';ab}(W,q^2)|^2$ are related by

$$|v_a - v_b| d\sigma_{ab} \equiv \int \frac{d^\nu p'_a}{(2\pi \hbar)^\nu} \frac{d^\nu p'_b}{(2\pi \hbar)^\nu} |T_{a'b';ab}(W,q^2)|^2 (2\pi \hbar)^\nu \delta^{\nu}(p'_a + p'_b - p_a - p_b) \times$$

$$(2\pi \hbar)^\nu \delta \left( E'_a + E'_b - E_a - E_b \right) ,$$ (3.11)
and the Boltzmann equation can then be written

\[ B_{ab}[f] = \int \frac{d^\nu p_a'}{(2\pi \hbar)^\nu} \frac{d^\nu p_b'}{(2\pi \hbar)^\nu} \frac{d^\nu p_b}{(2\pi \hbar)^\nu} \left| T_{a'b';ab}(W, q^2) \right|^2 \left\{ f_a(p_a')f_b(p_b') - f_a(p_a)f_b(p_b) \right\} \]

\[ (2\pi \hbar)^\nu \delta^\nu(p_a' + p_b' - p_a - p_b) (2\pi \hbar) \delta^\nu(E_a' + E_b' - E_a - E_b). \]  

(3.12)

The latter expression is more useful for formal manipulations, even in the classical regime, where one can define a classical “transition amplitude” \( T_{a'b';ab} \) from (3.11) by using the classical Rutherford cross section for \( d\sigma_{ab} \). Surprisingly, the classical amplitude is identical to quantum Born amplitude. When \( \nu > 3 \), expression (3.9) allows us to write the rate of change of the energy density resulting from the now finite Boltzmann kernel as

\[ \frac{d\mathcal{E}_{ab}^\nu}{dt} = \int \frac{d^\nu p_a}{(2\pi \hbar)^\nu} \frac{p_a^2}{2m_a} \ B_{ab}[f] \ : \ \nu > 3. \]  

(3.13)

I have used a “greater than” superscript to remind us that we should calculate (3.13) in dimensions greater than three.

As we have discussed, in dimensions greater than three the derivation of the Boltzmann equation for Coulomb scattering is rigorous and finite. This is because the short distance physics of the Coulomb potential is dominant in dimensions \( \nu > 3 \), and the Boltzmann equation is designed to capture short distance scattering physics. Furthermore, the long distance physics, where the Boltzmann equation breaks down in three dimensions, falls off faster than \( 1/r \) at large distances, thereby rendering the scattering finite for \( \nu > 3 \). It should not be a surprise that a simple scaling argument shows why \( B_{ab}[f] \) is finite for \( \nu > 3 \). Write \( \epsilon = \nu - 3 > 0 \), and note that the amplitude scales as \( |T|^2 \sim 1/q^2 \), for momentum transfer \( q \). Finally, a \( \delta \)-function in \( q \) contributes a power \( q^{-1} \), so that

\[ \nu > 3 : B_{ab} \sim \int dq \ q^{\nu - 1} \cdot \frac{1}{q^2} \cdot \frac{1}{q} \sim \int dq \ q^{\nu - 4} \sim \int dq \ q^{-1+\epsilon} \]  

(3.14)

\[ \sim q' \to 0 \ \text{as} \ q \to 0. \]  

(3.15)

The momentum transfer is related to the corresponding wavenumber by \( q = \hbar k \), and we see that small values of \( q \) correspond to large distances. This means that the Boltzmann equation does not possess an IR divergence for \( \nu > 3 \).
2. The Lenard-Balescu Kernel

I usually write the Lenard-Balescu equation in schematic form,

$$\frac{\partial f_a}{\partial t} + \mathbf{v}_a \cdot \frac{\partial f_a}{\partial \mathbf{x}} = \sum_b L_{ab}[f] \quad : \nu < 3 ,$$

although for calculations, we will use the explicit form

$$L_{ab}[f] = -\frac{\partial}{\partial \mathbf{p}_a} \cdot \mathbf{J}(\mathbf{p}_a)$$

$$\mathbf{J}(\mathbf{p}_a) = \int \frac{d^\nu p_b}{(2\pi)^{\nu}} \frac{d^\nu k}{(2\pi)^{\nu}} k \left| \frac{e_a e_b}{k^2}\epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v}_a) \right|^2 \pi \delta(\mathbf{k} \cdot \mathbf{v}_a - \mathbf{k} \cdot \mathbf{v}_b)$$

$$\left[ \mathbf{k} \cdot \frac{\partial}{\partial \mathbf{p}_a} - \mathbf{k} \cdot \frac{\partial}{\partial \mathbf{p}_b} \right] f_a(\mathbf{p}_a) f_b(\mathbf{p}_b) ,$$

where \( \mathbf{v}_a = \mathbf{p}_a/m_a \) is really an integration variable. The dielectric function \( \epsilon \) is given by (2.50), which we repeat here for convenience with a change in summation index,

$$\epsilon(\mathbf{k}, \omega) = 1 + \sum \frac{e_c^2}{k^2} \int \frac{d^\nu p_c}{(2\pi)^\nu} \frac{1}{\omega - \mathbf{k} \cdot \mathbf{v}_c + i\eta} \mathbf{k} \cdot \frac{\partial f_c(\mathbf{p}_c)}{\partial \mathbf{p}_c} ,$$

and the prescription \( \eta \to 0^+ \) is implicit, defining the correct retarded time response. Therefore, when \( \nu < 3 \), the rate (3.8) allows us to express

$$\frac{d\mathcal{E}_{ab}}{dt} = \int \frac{d^\nu p_a}{(2\pi)^\nu} \frac{p_a^2}{2m_a} L_{ab}[f] \quad : \nu < 3 .$$

I have used a “less than” superscript to remind us that we should calculate (3.20) in dimensions less than three, where it is finite and well defined.

In dimensions less than three one finds a complementary situation to the Boltzmann equation, namely, the derivation of the Lenard-Balescu equation is rigorous and finite when \( \nu < 3 \). This is because the long distance physics of the Coulomb potential is dominant in dimensions \( \nu < 3 \), and the Lenard-Balescu equation is designed to capture such long distance physics. Furthermore, the Coulomb potential falls off faster than \( 1/r \) at large distances, where the LBE breaks down in three dimensions, and this renders the kernel finite in \( \nu < 3 \). A scaling argument shows why \( L_{ab}[f] \) is finite for \( \nu < 3 \). Since \( \epsilon = \nu - 3 < 0 \), we will work with the quantity \( |\epsilon| > 0 \). From (6.3), the kernel contains an obvious linear term \( k \), and a factor \( k^{-4} \) arising from the Fourier transform of the potential. Note that \( \epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v}) \to 1 \) for large values of \( k \), so the dielectric function does not change the \( k \to \infty \) scaling behavior.

\(^3\) Note that Eq. (3.57) for \( L_{ab}[f] \) in Ref. [4] contains a spurious integration over the momentum \( \mathbf{p}_a \). Fortunately, this typo was innocuous and did not affect the results that followed.
The $\delta$-function gives a factor $k^{-1}$, and the $p$-derivative terms provide a compensating factor of $k$, so that

$$\nu < 3 : L_{ab} \sim \int dk k^{\nu-1} \cdot k \cdot \frac{1}{k^4} \cdot \frac{1}{k} \cdot k \sim \int dk k^{\nu-4} \sim \int dk k^{-1-|\epsilon|}$$  \hfill (3.21)

$$\sim k^{-|\epsilon|} \rightarrow 0 \text{ as } k \rightarrow \infty.$$  \hfill (3.22)

Since large values of $k$ correspond to small distances, this means that the LBE is finite at small distances for $\nu < 3$.

C. Completing the Picture: The Rate and Analytic Continuation

As a matter of completeness, let us finish the calculation of the rate $dE_{ab}/dt$. Recall that we have calculated the rates the rates $dE_{ab}^/>/dt$ and $dE_{ab}^</dt$ to leading order (LO) in $g$, and they contain simple poles $1/(\nu - 3)$, and they take the general form

$$\frac{dE_{ab}^>}{dt} = H(\nu) \frac{g^2}{\nu - 3} + O(\nu - 3) \quad \text{is LO in } g \text{ when } \nu > 3$$  \hfill (3.23)

$$\frac{dE_{ab}^<}{dt} = G(\nu) \frac{g^{\nu-1}}{3 - \nu} + O(\nu - 3) \quad \text{is LO in } g \text{ when } \nu < 3,$$  \hfill (3.24)

where $H(\nu)$ and $G(\nu)$ are coefficients that depend upon $\nu$. The heavy lifting for a real process is in calculating the functions $H(\nu)$ and $G(\nu)$ using the exact expressions for $B_{ab}$ and $L_{ab}$. Once these calculations have been completed, in order to compare the rates (3.23) and (3.24), we must then analytically continue to a common value of the dimension $\nu$ (and then take the limit $\nu \rightarrow 3$). Analytically continuing the spatial dimension makes sense because we can view the quantities $dE_{ab}^>/dt$ and $dE_{ab}^</dt$ as functions of a complex parameter $\nu$, even though they were only calculated for positive integer values of $\nu$. This is analogous to analytically continuing the factorial function on the positive integers to the Gamma function on the complex plane. For definiteness, I will analytically continue (3.24) to $\nu > 3$, in which case $g^{\nu-1} = g^{2+(\nu-3)}$ becomes subleading relative to the $g^2$ dependence of (3.23), so that

$$\frac{dE_{ab}^<}{dt} = -G(\nu) \frac{g^{2+(\nu-3)}}{\nu - 3} + O(\nu - 3) \quad \text{is NLO in } g \text{ when } \nu > 3.$$  \hfill (3.25)

This is illustrated in Fig. 3. To finish calculating the rates, we need to work consistently to linear order in the small parameter $\epsilon = \nu - 3$; therefore, we should expand $H(\nu)$ and $G(\nu)$ to first order in $\epsilon$,

$$H(\nu) = -A + \epsilon H_1 + O(\epsilon^2)$$  \hfill (3.26)

$$G(\nu) = -A + \epsilon G_1 + O(\epsilon^2).$$  \hfill (3.27)
analytically continue around the \( \nu = 3 \) pole

\[
\begin{align*}
\frac{d\mathcal{E}_\nu^/>}{dt} & \quad g^{\nu-1} = g^{2-3(1-\nu)} \\
\downarrow & \\
\frac{d\mathcal{E}_\nu^/>}{dt} & \quad g^{\nu-1} = g^{2+(\nu-3)} \\
\downarrow & \\
\frac{d\mathcal{E}_\nu^/>}{dt} & \quad g^{2-|\nu-3|} \\
\downarrow & \\
\frac{d\mathcal{E}_\nu^/>}{dt} & \quad g^{2+|\nu-3|}
\end{align*}
\]

LO: large when \( g \ll 1 \) \quad NLO: small when \( g \ll 1 \)

FIG. 3: The analytic continuation of \( \frac{d\mathcal{E}_\nu^/>}{dt} \) from \( \nu < 3 \) to the region \( \nu > 3 \) in the complex \( \nu \)-plane. The same expression can be used for \( \frac{d\mathcal{E}_\nu^/>}{dt} \) throughout the complex plane since the pole at \( \nu = 3 \) can easily be avoided. The quantity \( \frac{d\mathcal{E}_\nu^/>}{dt} \sim g^{2+(\nu-3)} \) is leading order in \( g \) for \( \nu < 3 \). However, upon analytically continuing to \( \nu > 3 \) we find that \( \frac{d\mathcal{E}_\nu^/>}{dt} \sim g^{2+|\nu-3|} \) is next-to-leading order in \( g \) relative to \( \frac{d\mathcal{E}_\nu^/>}{dt} \sim g^2 \).

It is crucially important here that \( H(\nu) \) and \( G(\nu) \) give the same value at \( \nu = 3 \), a term that I have called \( -A \) in (3.26) and (3.27), otherwise the divergent poles will not cancel. Finally, upon writing \( g^\epsilon = \exp\{\epsilon \ln g\} \) in (3.25), and expanding to first order in \( \epsilon \), we find

\[
\frac{g^\epsilon}{\epsilon} = \frac{1}{\epsilon} + \ln g + O(\epsilon) . \tag{3.28}
\]

This is where the nonanalyticity in \( g \) arises, i.e. the \( \ln g \) term, and we can now express the rates as

\[
\frac{d\mathcal{E}_\nu^/>}{dt} = -\frac{A}{\nu - 3} g^2 + H_1 g^2 + O(\nu - 3; g^3) \quad \nu > 3 \tag{3.29}
\]

\[
\frac{d\mathcal{E}_\nu^/>}{dt} = -\frac{A}{\nu - 3} g^2 - G_1 g^2 - A g^2 \ln g + O(\nu - 3; g^3) \quad \nu > 3 . \tag{3.30}
\]

These expressions hold in the common dimension \( \nu > 3 \), and to obtain the leading and next-to-leading order result in three dimensions, we add and take the limit:

\[
\frac{d\mathcal{E}_\nu}{dt} = \lim_{\nu \to 3^+} \left[ \frac{d\mathcal{E}_\nu^/>}{dt} + \frac{d\mathcal{E}_\nu^/>}{dt} \right] + O(g^3) = -Ag^2 \ln g +Bg^2 + O(g^3) , \tag{3.31}
\]

with \( B = H_1 - G_1 \). This gives the energy exchange rate from Coulomb interactions between plasma species, accurate to leading order and next-to-leading order in \( g \), in terms of the Coulomb logarithm,

\[
\frac{d\mathcal{E}_\nu}{dt} = -Ag^2 \ln C g + O(g^3) , \tag{3.32}
\]

where \( \ln C = -B/A \). The quantity \( L = \ln C g \) is known as the Coulomb logarithm.
IV. THE BBGKY HIERARCHY IN ARBITRARY DIMENSIONS

Now that we have reviewed statistical mechanics and Coulomb physics in \( \nu \)-dimensions, we can start addressing the main claim of these notes, namely, that to leading order in the plasma coupling \( g \), the BBGKY hierarchy reduces to the Boltzmann equation in \( \nu > 3 \), and to the Lenard-Balescu equation in \( \nu < 3 \). To prove this, we first derive the BBGKY hierarchy in a general number of dimensions, primarily to establish notation, and because it will be the starting point in our derivation. We then develop perturbation theory in powers of the coupling \( g \), and we calculate the BBGKY equations to order \( g^2 \). To solve these equations, we must make some approximations, and we discuss how the two regimes \( \nu < 3 \) and \( \nu > 3 \) affect the validity of the approximations. More precisely, we will show that complementary 2-point correlations should be dropped in these respective dimensional regimes. This will serve as a starting point for Section V on the Boltzmann equation and Section VI on the Lenard-Balescu equation. The derivation of the BBGKY hierarchy presented here was adapted from a three dimensional argument given in Clemmow and Dougherty [9], which will be our primary reference throughout this section.

A. Liouville’s Theorem and Ensemble Averages

For simplicity, we consider a plasma with a single species of particle. We can add a charge neutralizing background if desired; for example, one might consider an electron plasma with fixed ions as the background. Multiple plasma species can (and soon will) be added. There are several ways of representing the state of many-particle systems such as plasmas. The first is through the use of a phase-spaced called \( \mu \), which is the \( 2\nu \)-dimensional phase space \((x, p)\) of a single particle. The state of a plasma with \( N \) particles is given by specifying the \( \nu \)-dimensional positions \( x_i \) and momenta \( p_i \) for every particle \( i = 1, 2, \ldots, N \) in the plasma. Each particle is represented by a point \((x_i, p_i)\) in \( \mu \)-space, and the system looks like a swarm of \( N \) particles, each interacting with all of the particles of the system. The \( \nu \)-dimensional spatial slice of \( \mu \)-space is the part of phase space that we observe with our eyes in the laboratory. The single particle distribution function \( f_1(x, p) \) lives in \( \mu \)-space, and specifies the number of particles within a phase space element \( d^\nu x \, d^\nu p \). A second way of representing the state of a multi-particle system is through the \( 2\nu N \)-dimensional phase space defined by \((x_1, p_1, \ldots, x_N, p_N)\). This larger phase space is called \( \Gamma \)-space, and by design, the entire system is represented by a single point in \( \Gamma \), rather than the swarm of points in \( \mu \). To simplify notation, we denote the \( 2\nu \)-dimensional phase space for particle \( i \) by \( X_i = (x_i, p_i) \), so that points in \( \Gamma \)-space are specified by coordinate values \((X_1, \ldots, X_N)\). Once the initial condition of the \( N \)-body system is specified in \( \Gamma \)-space, that is to say, once the locations and velocities of all \( N \) particles are specified at some initial time \( t = 0 \), then the subsequent evolution at any future time \( t \) is uniquely determined. The system therefore traces out a
path in Γ-space as it evolves in time. There are only two possible types of paths: either the path is periodic or it never intersects itself. This is related to the ergodic properties of the system.

Let us now consider a large ensemble of systems in Γ-space, each evolving in time, tracing out a unique path for every member of the ensemble. We can think of building the ensemble by initially populating Γ-space uniformly among all possible initial conditions. None of the systems in Γ-space interact, and every system evolves along its own private non-intersecting trajectory in this $2\nu N$-dimensional space. The world-lines in Γ look like a tangle of non-intersecting spaghetti, with each strand oriented forward in time, never looping back on itself. Since we have populated Γ-space with all possible initial configurations, then by the ergodic principle, ensemble averages in Γ-space give time averaged quantities as measured by experiment. For an ensemble of systems in Γ-space, the ensemble density is defined by

$$\rho(X_1, \ldots, X_N, t) \, dX_1 \cdots dX_N \equiv \text{probability that a system selected from the ensemble lies within } dX_i \text{ of } X_i \text{ for } i = 1, \ldots, N, \text{ at time } t,$$  \hspace{1cm} (4.1)

where the measures are defined by

$$dX_i = \frac{d^\nu x_i \, d^\nu p_i}{(2\pi \hbar)^\nu}. \hspace{1cm} (4.2)$$

The density is of course normalized to unity,

$$\int dX_1 \cdots dX_N \rho = 1,$$ \hspace{1cm} (4.3)

and $\rho$ is symmetric in each of its arguments $X_i$ (again, this relates to the ergodic mixing properties of the system).

Since the individual systems in Γ are non-interacting, the density satisfies the conservation equation

$$\frac{\partial \rho}{\partial t} + \sum_{i=1}^{N} \nabla_{X_i} \cdot (\rho \dot{X}_i) = 0. \hspace{1cm} (4.4)$$

It is more convenient to break the variables $X_i = (x_i, p_i)$ into their space and velocity components, thereby giving

$$\frac{\partial \rho}{\partial t} + \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \cdot (\rho \dot{x}_i) + \sum_{i=1}^{N} \frac{\partial}{\partial p_i} \cdot (\rho \dot{p}_i) = 0,$$ \hspace{1cm} (4.5)

which we write in the form

$$\frac{\partial \rho}{\partial t} + \sum_{i=1}^{N} \dot{x}_i \cdot \frac{\partial \rho}{\partial x_i} + \sum_{i=1}^{N} \dot{p}_i \cdot \frac{\partial \rho}{\partial p_i} + \sum_{i=1}^{N} \rho \left( \frac{\partial}{\partial x_i} \cdot \dot{x}_i + \frac{\partial}{\partial p_i} \cdot \dot{p}_i \right) = 0. \hspace{1cm} (4.6)$$
Using Hamilton’s equations of motion,

\[ \dot{x}_i = \frac{\partial H}{\partial p_i}, \]  
\[ \dot{p}_i = -\frac{\partial H}{\partial x_i}, \]

where \( H \) is the Hamiltonian, we see that the last term in Eq. (4.6) vanishes,

\[ \left( \frac{\partial}{\partial x_i} \cdot \dot{x}_i + \frac{\partial}{\partial p_i} \cdot \dot{p}_i \right) = \sum_{\ell=1}^{\nu} \left( \frac{\partial^2 H}{\partial x_i^\ell \partial p_i^\ell} - \frac{\partial^2 H}{\partial p_i^\ell \partial x_i^\ell} \right) = 0. \]  

The density in \( \Gamma \)-space therefore satisfies Liouville’s equation

\[ \frac{\partial \rho}{\partial t} + \sum_{i=1}^{N} v_i \cdot \frac{\partial \rho}{\partial x_i} + \sum_{i=1}^{N} F_i \cdot \frac{\partial \rho}{\partial p_i} = 0, \]

where we have substituted \( v_i = \dot{x}_i \) and \( F_i = \dot{p}_i \). The force \( F_i \) includes the effects from all the other particles, and therefore \( F_i \) depends upon all of the coordinates \( X_1, \cdots, X_N \). Note that (4.10) takes the same form as the collisionless Boltzmann equation. This is because for non-interacting particles, the function \( f(x, p, t) \) acts like the ensemble density \( \rho(X_1, \cdots, X_N) \).

### B. The Hierarchy of Distribution Functions

We now show that there is a hierarchy of distributions functions that measure successively higher-order correlations in the plasma. Note that we can define the average value of a general quantity \( Q = Q(X_1, \cdots, X_N) \), where we have not allowed for a possible explicit time dependence in \( Q \) (although we could), by

\[ \langle Q(X_1, \cdots, X_N, t) \rangle = \int dX_1 \cdots dX_N \rho(X_1, \cdots, X_N, t)Q(X_1, \cdots, X_N). \]

Note that the time dependence of the average is due to that in \( \rho(t) \). Let us now define the one-particle function by

\[ F(X) = \sum_{i=1}^{N} \delta(X - X_i), \]

where the particles are located at \( X_1, \cdots, X_N \). Writing \( X = (x, p) \), we can recover the single-particle distribution by performing the ensemble average

\[ f_1(x, p, t) = \langle F(X) \rangle. \]

To see this, we explicitly perform the ensemble average,

\[ f_1(x, p, t) = \int dX_1 dX_2 \cdots dX_N \rho(X_1, X_2, \cdots, X_N, t) \sum_{i=1}^{N} \delta(X - X_i) \]
\[ = N \int dX_2 \cdots dX_N \rho(X, X_2, \cdots, X_N, t). \]
The factor of $N$ occurs because $\rho$ is symmetric in its arguments, and therefore each term in the sum over $\delta$-functions is identical. Using the normalization (4.3) we see that

$$\int \frac{d^\nu x \, d^\nu p}{(2\pi\hbar)^\nu} \, f_1(x, p, t) = N \int dX dX_2 \cdots dX_N \, \rho(X, X_2, \cdots, X_N, t) = N \, ,$$

(4.16)
as required. Therefore the single-particle distribution $f_1$ is automatically normalized correctly. This motivates the definition of the $s$-particle correlation function

$$f_s(x_1, \cdots, x_s) = \frac{N!}{(N-s)!} \int dX_{s+1} \cdots dX_N \, \rho(x_1, \cdots, x_s, x_{s+1}, \cdots, x_N) \, ,$$

(4.17)

which has the normalization

$$\int dX_1 \cdots dX_s \, f_s(x_1, \cdots, x_s) = \frac{N!}{(N-s)!} \, .$$

(4.18)

We have now defined a hierarchy of distribution functions $f_1, f_2, \cdots, f_s, \cdots, f_{N-1}, f_N$, where $f_N \equiv \rho$. In the next section we will find a set of coupled kinetic equations for each $f_s$ by integrating over successive sub-spaces of the Liouville’s equation.

C. The BBGKY Hierarchy of Kinetic Equations

Let us express the forces $F_i$ in Liouville’s equation (4.10) in terms of external forces $F_i^{(0)}$ and the Coulomb interaction force $F_i^{(j)}$,

$$F_i = F_i^{(0)} + \sum_{j=1}^{N} F_i^{(j)} = \sum_{j=0}^{N} F_i^{(j)} \, ,$$

(4.19)

where the Coulomb force on $i$ from $j$ takes the form

$$F_i^{(j)} = e^2 \frac{\Gamma(\nu/2)}{2\pi^{\nu/2}} \frac{x_i - x_j}{|x_i - x_j|^{\nu}} \, .$$

(4.20)

We must exclude the $j = i$ term from the sum in (4.19); or equivalently we can include the value $j = i$ in the sum by formally setting $F_i^{(i)} = 0$. We now show that the distribution functions $f_s$ satisfy the following coupled set of kinetic equations called the BBGKY hierarchy,

$$\frac{\partial f_s}{\partial t} + \sum_{i=1}^{s} v_i \cdot \frac{\partial f_s}{\partial x_i} + \sum_{i=1}^{s} \sum_{j=0}^{s} F_i^{(j)} \cdot \frac{\partial f_s}{\partial p_i} = -\sum_{i=1}^{s} \int dX_{s+1} \, F_i^{(s+1)} \cdot \frac{\partial f_{s+1}}{\partial p_i} \, ,$$

(4.21)

where $s = 1, \cdots N-1$. The BBGKY hierarchy is the the “$F = ma$” of kinetic theory. These equations are completed by Liouville’s equation (4.10) for $x = N$,

$$\frac{\partial f_N}{\partial t} + \sum_{i=1}^{N} v_i \cdot \frac{\partial f_N}{\partial x_i} + \sum_{i=1}^{N} \sum_{j=0}^{N} F_i^{(j)} \cdot \frac{\partial f_N}{\partial p_i} = 0 \, .$$

(4.22)
We thus have a complete set of $N$ equations in $N$ variables $f_1, f_2, \ldots, f_s, \ldots, f_N$, and a unique solution will exist. It should be emphasized that the BBGKY hierarchy is time reversal invariant. It is only upon closing the equations at some level, usually $s = 1$ or $s = 2$, that we introduce time non-invariance. In other words, it is closing the hierarchy of kinetic equations that introduces the arrow of time. Except under the most contrived of conditions, we cannot hope to find an exact solution, or even a numerical solution, as $N$ is macroscopically large. However, the BBGKY hierarchy is still an extremely useful piece of theoretical machinery, particularly in more formal arguments, and provides for a deeper understanding of kinetic theory.

To prove (4.21), let us integrate (4.10) over $dX_{s+1} \cdots dX_N$, and multiply by the normalization factor of $f_s$, thereby giving the exact equation

$$\frac{N!}{(N-s)!} \int dX_{s+1} \cdots dX_N \left( \frac{\partial \rho}{\partial t} + \sum_{i=1}^{N} v_i \cdot \frac{\partial \rho}{\partial x_i} + \sum_{i=1}^{N} \sum_{j=0}^{N} F_i^{(j)} \cdot \frac{\partial \rho}{\partial p_i} \right) = 0 . \quad (4.23)$$

The first two terms of (4.23) are rather trivial to evaluate, and they correspond to the first two terms of (4.21),

$$\text{term1} = \frac{N!}{(N-s)!} \int dX_{s+1} \cdots dX_N \frac{\partial \rho}{\partial t} = \frac{\partial}{\partial t} \frac{N!}{(N-s)!} \int dX_{s+1} \cdots dX_N \rho = \frac{\partial f_s}{\partial t} \quad (4.24)$$

$$\text{term2} = \frac{N!}{(N-s)!} \sum_{i=1}^{N} \int dX_{s+1} \cdots dX_N v_i \cdot \frac{\partial \rho}{\partial x_i} = \sum_{i=1}^{s} v_i \cdot \frac{\partial f_s}{\partial x_i} . \quad (4.25)$$

In expression (4.25), note that the sum over $i$ has been truncated from $N$ to $s$. This is because the terms $i = s + 1, \ldots, N$ vanish by the use of divergence theorem, and the fact that $\rho$ vanishes on the distant surface at infinity. To see that such terms explicitly vanish, let $i \geq s + 1$, and consider the integral

$$\int dX_{s+1} \cdots dX_N v_i \cdot \frac{\partial \rho}{\partial x_i} = \int dX_{s+1} \cdots dX_N \frac{\partial}{\partial x_i} \left( \rho v_i \right) \quad (4.26)$$

$$= \int dX_{s+1} \cdots dX_{i-1} dX_{i+1} \cdots dX_N \int \frac{d^3 p_i}{(2\pi \hbar)^3} \int_V d^\nu x_i \frac{\partial}{\partial x_i} \left( \rho v_i \right)$$

$$= \int dX_{s+1} \cdots dX_{i-1} dX_{i+1} \cdots dX_N \int \frac{d^3 p_i}{(2\pi \hbar)^3} \oint_{\partial V} dS_i \cdot \rho v_i = 0 ,$$

which vanishes because $\rho$ vanishes on the surface at infinity, the boundary $\partial V$. Note that we have enclosed the $\nu$-space space $x_i$ in a very large but finite volume $V$ (the volume will be taken to infinity in the limit). The boundary of $V$, denoted $\partial V$, is often called the surface at infinity. This is all standard, but it might be useful for the novice to have seen such a calculation all the way through.
At this point in the derivation, the equation for $f_s$ is

$$\frac{\partial f_s}{\partial t} + \sum_{i=1}^{s} v_i \cdot \frac{\partial f_s}{\partial x_i} + \frac{N!}{(N-s)!} \sum_{i=1}^{N} \sum_{j=0}^{N} \int dX_{s+1} \cdots dX_N F_i^{(j)} \cdot \frac{\partial \rho}{\partial p_i} = 0 . \quad (4.27)$$

We must now consider the last term in (4.27), which we decompose about the $i = s$ contribution,

$$\sum_{j=0}^{N} F_i^{(j)} = \sum_{j=0}^{s} F_i^{(j)} + \sum_{j=s+1}^{N} F_i^{(j)} . \quad (4.28)$$

The first sum in (4.28) is handled as before, and we can write (4.27) in the form

$$\frac{\partial f_s}{\partial t} + \sum_{i=1}^{s} v_i \cdot \frac{\partial f_s}{\partial x_i} + \sum_{i=1}^{s} \sum_{j=0}^{s} F_i^{(j)} \cdot \frac{\partial f_s}{\partial p_i} +$$

$$\frac{N!}{(N-s)!} \sum_{i=1}^{N} \sum_{j=s+1}^{N} \int dX_{s+1} \cdots dX_N F_i^{(j)} \cdot \frac{\partial \rho}{\partial p_i} = 0 . \quad (4.29)$$

The final step in the calculation is to address the last term in (4.29). Recall that the distribution function $\rho$ is symmetric in its arguments $X_1, \cdots, X_N$. This means that every term of the $j$-sum in the last term of (4.29) is identical. Therefore, let us represent the sum by arbitrarily choosing the first term $j = s + 1$, and multiplying by $(N - s)$ to account for the remaining terms in the sum. This allows us to express the last term in (4.29) as

$$\frac{N!}{(N-s)!} \sum_{i=1}^{N} \sum_{j=s+1}^{N} \int dX_{s+1} \cdots dX_N F_i^{(s+1)} \cdot \frac{\partial \rho}{\partial p_i} ,$$

where we have, for the usual reasons, truncated the $i$-sum at $i = s$. We can express (4.30) as

$$\sum_{i=1}^{s} \int dX_{s+1} F_i^{(s+1)} \cdot \frac{\partial}{\partial p_i} \frac{N!}{(N-s-1)!} \int dX_{s+2} \cdots dX_N \rho$$

$$= \sum_{i=1}^{s} \int dX_{s+1} F_i^{(s+1)} \cdot \frac{\partial f_{s+1}}{\partial p_i} , \quad (4.31)$$

and substituting (4.32) back into (4.29). This establishes the BBGKY hierarchy (4.21), which is repeated again for convenience,

$$\frac{\partial f_s}{\partial t} + \sum_{i=1}^{s} v_i \cdot \frac{\partial f_s}{\partial x_i} + \sum_{i=1}^{s} \sum_{j=0}^{s} F_i^{(j)} \cdot \frac{\partial f_s}{\partial p_i} = - \sum_{i=1}^{s} \int dX_{s+1} F_i^{(s+1)} \cdot \frac{\partial f_{s+1}}{\partial p_i} . \quad (4.33)$$

for $s = 1, \cdots, N - 1$. The system is closed with Liouville’s equation,

$$\frac{\partial f_N}{\partial t} + \sum_{i=1}^{N} v_i \cdot \frac{\partial f_N}{\partial x_i} + \sum_{i=1}^{N} \sum_{j=0}^{N} F_i^{(j)} \cdot \frac{\partial f_N}{\partial p_i} = 0 . \quad (4.34)$$
D. Dimensionless Variables

We have now developed the BBGKY hierarchy (4.33) and (4.34) in the quite general setting of a non-equilibrium but single-component plasma. The plasma could be generalized to have multiple components, but at the expense of increasing the complexity of the counting arguments and the simplicity of the formalism. In these notes, it turns out to be quite easy to generalize the results of a single-component calculation to that of a multi-component plasma. For the sake of simplicity, we continue with a single-component plasma, whose constituents have charge $e$ and mass $m$. In equilibrium, the plasma is characterized by temperature $T$ and number density $n$. We measure $T$ in energy units, while $n$ is the number of particles per unit hypervolume. The Debye wavenumber $\kappa$, and the plasma frequency $\omega_p$, are given by

$$\kappa^2 = \frac{e^2 n}{T} \quad (4.35)$$

$$\omega_p^2 = \frac{e^2 n}{m}. \quad (4.36)$$

By dimensional analysis, these expressions hold in any spatial dimension $\nu$, and we can therefore use $\kappa$ and $\omega_p$ as defined by (4.35) and (4.36) in any dimension under consideration, as the electric charge absorbs any dimensional factors involving $\nu$. Let us generalize the equilibrium system by imposing a small non-equilibrium background on the equilibrium plasma. This new quasi-equilibrium system is still described by the BBGKY hierarchy, and it is quite informative to express the BBGKY kinetic equations in terms of dimensionless variables. This is possible because the background equilibrium plasma provides natural length and time scales. We shall see that the coupling constant $g$ emerges quite naturally, and that a consistent perturbation theory in powers of $g$ can be developed.

We first express the basic kinematic variables in dimensionless form. We do this with the following scale transformation, where the over-bar denotes the dimensionless form of the corresponding variable,

$$x = \bar{x}/\kappa \quad t = \bar{t}/\omega_p \quad (4.37)$$

$$v = (\omega_p/\kappa) \bar{v} \quad p = (m\omega_p/\kappa) \bar{p} = (T\kappa/\omega_p) \bar{p} \quad (4.38)$$

$$dX = \left(\frac{m\omega_p}{\hbar\kappa^2}\right)^\nu d\bar{X} \quad F^{(0)} = \kappa T \bar{F}^{(0)} \quad (4.39)$$

$$f_s(X_1, \cdots, X_s, t) = \left(\frac{\hbar\kappa^2}{m\omega_p}\right)^{\nu s} \bar{f}_s(\bar{X}_1, \cdots, \bar{X}_s, t) \quad (4.40)$$

Motivated by the scaling $\kappa T$ for the external force $F^{(0)}$, we are immediately led to express the Coulomb force as

$$F_i^{(j)} = e^2 \frac{\Gamma(\nu/2)}{2\pi^{\nu/2}} \frac{x_i - x_j}{|x_i - x_j|^{\nu}} \quad (4.41)$$

$$= g \kappa T \bar{F}_i^{(j)} \quad (4.42)$$
where the dimensionless Coulomb force is defined by
\[ \bar{F}_i^{(j)} = \frac{\bar{x}_i - \bar{x}_j}{|\bar{x}_i - \bar{x}_j|^{\nu}}, \]  
(4.43)
and the remaining factors combine to form the plasma coupling constant,
\[ g = \frac{\Gamma(\nu/2)}{2\pi^{\nu/2}} \frac{e^2 \kappa^{\nu-2}}{T}. \]  
(4.44)

We see that the expansion parameter \( g \) simply falls out of the algebra. Finally, the BBGKY hierarchy (4.33) and (4.34) can be expressed in the dimensionless form
\[ \frac{\partial \bar{f}_s}{\partial \bar{t}} + \sum_{i=1}^{s} \bar{v}_i \cdot \frac{\partial \bar{f}_s}{\partial \bar{x}_i} + \sum_{i=1}^{s} \bar{F}_i^{(0)} \cdot \frac{\partial \bar{f}_s}{\partial \bar{p}_i} + g \sum_{i=1}^{s} \sum_{j=1}^{s} \bar{F}_i^{(j)} \cdot \frac{\partial \bar{f}_s}{\partial \bar{p}_i} = -g \sum_{i=1}^{s} \int d\bar{X}_{s+1} \bar{F}_i^{(s+1)} \cdot \frac{\partial \bar{f}_{s+1}}{\partial \bar{p}_i}, \]  
(4.45)
for \( i = 1, \ldots, N - 1 \), in the square bracket along with the \( i = N \) equation
\[ \frac{\partial \bar{f}_N}{\partial \bar{t}} + \sum_{i=1}^{N} \bar{v}_i \cdot \frac{\partial \bar{f}_N}{\partial \bar{x}_i} + \sum_{i=1}^{N} \bar{F}_i^{(0)} \cdot \frac{\partial \bar{f}_N}{\partial \bar{p}_i} + g \sum_{i=1}^{N} \sum_{j=1}^{N} \bar{F}_i^{(j)} \cdot \frac{\partial \bar{f}_N}{\partial \bar{p}_i} = 0. \]  
(4.46)

It should be emphasized again that the coupling constant \( g \) as defined by (4.44) emerges quite naturally, and when \( \nu = 3 \), the coupling takes the usual form \( g = e^2 \kappa / 4\pi T \) (in rationalized cgs units).

In the next section, we will develop a method that permits us to solve the BBGKY equations perturbatively as an expansion in powers of \( g \). Before doing this, it is instructive to work through the algebra establishing (4.45) and (4.46). We start by measuring space in units of inverse \( \kappa \) and time in units of inverse \( \omega_p \),
\[ x = \bar{x}/\kappa \]  
(4.47)
\[ t = \bar{t}/\omega_p, \]  
(4.48)
where the bared quantities are dimensionless. Note that
\[ \omega_p/\kappa = \sqrt{T/m} = v_{th} \]  
(4.49)
is the thermal velocity of the plasma, which we use to form a dimensionless velocity
\[ v = (\omega_p/\kappa) \bar{v}. \]  
(4.50)
Since \( \omega_p/\kappa \) has units of velocity, we see that \( m\omega_p/\kappa \) has units of momentum. The relation for the thermal velocity (4.49) implies that \( \omega_p/\kappa = \kappa T/\omega_p \), and we can thus scale the momentum in two separate but equivalent ways
\[ p = (m\omega_p/\kappa) \bar{p} = (\kappa T/\omega_p) \bar{p}. \]  
(4.51)
Both forms of the momentum scaling will be used interchangeably. Since the temperature $T$ has energy units, the quantity $\kappa T$ has units of force, and we define the dimensionless external force as

$$F^{(0)} = \kappa T \tilde{F}^{(0)}.$$  

(4.52)

This motivates expressing the Coulomb force by

$$F^{(j)} = e^2 \frac{\Gamma(\nu/2)}{2\pi^{\nu/2}} \frac{x_i - x_j}{|x_i - x_j|^{\nu}} = e^2 \frac{\Gamma(\nu/2)}{2\pi^{\nu/2}} \frac{\kappa^{-2}}{T} \cdot \frac{x_i - x_j}{|x_i - x_j|^{\nu}}$$  

(4.53)

$$= g \kappa T \tilde{F}^{(j)}.$$  

(4.54)

The next quantity that we consider is the phase space measure, and it transforms as

$$dX = d^\nu x d^\nu p \left( \frac{2\pi}{\hbar} \right)^{\nu} = \left( \frac{m\omega_p}{\hbar \kappa^2} \right)^{\nu} \left( \frac{2\pi}{\hbar^2} \right)^{\nu} d\tilde{X}. $$  

(4.55)

The final quantity to consider is the distribution function itself, which transforms by a constant factor

$$f_s(X_1, \cdots, X_s, t) = N_s \tilde{f}_s(\tilde{X}_1, \cdots, \tilde{X}_s, \tilde{t}).$$  

(4.56)

We can find $N_s$ by the requirement that

$$\int d\tilde{X}_1 \cdots d\tilde{X}_s \tilde{f}_s(\tilde{X}_1, \cdots, \tilde{X}_s) = \int dX_1 \cdots dX_s f_s(X_1, \cdots, X_s)$$  

(4.57)

$$= \left( \frac{m\omega_p}{\hbar \kappa^2} \right)^{\nu s} N_s \int d\tilde{X}_1 \cdots d\tilde{X}_s \tilde{f}_s(\tilde{X}_1, \cdots, \tilde{X}_s),$$  

(4.58)

which implies

$$N_s = \left( \frac{\hbar \kappa^2}{m\omega_p} \right)^{\nu s}. $$  

(4.59)

We will later require the ratio

$$\frac{N_{s+1}}{N_s} = \left( \frac{\hbar \kappa^2}{m\omega_p} \right)^{\nu} = N_1,$$

(4.60)

but for now we will express our results in terms of $N_s$. Upon changing to dimensionless variables, the first two terms of the dimensional BBGKY hierarchy (4.33) become

$$\text{term1} \equiv \frac{\partial f_s}{\partial t} = \omega_p N_s \frac{\partial \tilde{f}_s}{\partial \tilde{t}}$$  

(4.61)

$$\text{term2} \equiv \sum_{i=1}^{s} \frac{\omega_p}{\kappa} \tilde{v}_i \cdot \left( \frac{\partial \tilde{f}_s}{\partial \tilde{x}_i} \right) = \omega_p N_s \sum_{i=1}^{s} \tilde{v}_i \cdot \frac{\partial \tilde{f}_s}{\partial \tilde{x}_i}.$$  

(4.62)
The third term of (4.33) can be decomposed into an external force and an internal Coulomb contribution,

\[
\text{term3a} \equiv \sum_{i=1}^{s} F_i^{(0)} \cdot \frac{\partial f_s}{\partial p_i} = \sum_{i=1}^{s} \left( \kappa T F_i^{(0)} \right) \cdot \left( \frac{\omega_p N_s}{\kappa T} \frac{\partial f_s}{\partial p_i} \right) = \omega_p N_s \sum_{i=1}^{s} F_i^{(0)} \cdot \frac{\partial f_s}{\partial p_i} \quad (4.63)
\]

\[
\text{term3b} \equiv \sum_{i=1}^{s} \sum_{j=1}^{s} F_i^{(j)} \cdot \frac{\partial f_s}{\partial p_i} = \omega_p N_s g \sum_{i=1}^{s} \sum_{j=1}^{s} \bar{F}_i^{(j)} \cdot \frac{\partial f_s}{\partial p_i} \quad (4.64)
\]

The BBGKY equations now become

\[
\frac{\partial \bar{f}_s}{\partial t} + \sum_{i=1}^{s} \bar{v}_i \cdot \frac{\partial \bar{f}_s}{\partial \bar{x}_i} + \sum_{i=1}^{s} F_i^{(0)} \cdot \frac{\partial \bar{f}_s}{\partial \bar{p}_i} + g \sum_{i=1}^{s} \sum_{j=1}^{s} F_i^{(j)} \cdot \frac{\partial \bar{f}_s}{\partial \bar{p}_i} + \frac{1}{\omega_p N_s} \sum_{i=1}^{s} \int d\bar{X}_{s+1} \bar{F}_i^{(s+1)} \cdot \frac{\partial f_{s+1}}{\partial p_i} = 0 \quad (4.65)
\]

Finally, the last term in (4.65) involving \( f_{s+1} \) can be written

\[
\text{term4} \equiv \frac{1}{\omega_p N_s} \sum_{i=1}^{s} \int d\bar{X}_{s+1} \bar{F}_i^{(s+1)} \cdot \frac{\partial f_{s+1}}{\partial p_i} \quad (4.66)
\]

\[
= \frac{1}{\omega_p N_s} \frac{N_{s+1}}{N_s} \left( \frac{m \omega_p}{\hbar \kappa^2} \right)^\nu \cdot g \kappa T \cdot \frac{\omega_p}{\kappa T} \sum_{i=1}^{s} \int d\bar{X}_{s+1} \bar{F}_i^{(s+1)} \cdot \frac{\partial \bar{f}_{s+1}}{\partial \bar{p}_i} \quad (4.67)
\]

\[
= g \sum_{i=1}^{s} \int d\bar{X}_{s+1} \bar{F}_i^{(s+1)} \cdot \frac{\partial \bar{f}_{s+1}}{\partial \bar{p}_i} \quad (4.68)
\]

where we have used (4.60) for \( N_{s+1}/N_s = N_1 \). We have now established (4.45) and (4.46), which we reproduce below for convenience,

\[
\frac{\partial \bar{f}_s}{\partial t} + \sum_{i=1}^{s} \bar{v}_i \cdot \frac{\partial \bar{f}_s}{\partial \bar{x}_i} + \sum_{i=1}^{s} F_i^{(0)} \cdot \frac{\partial \bar{f}_s}{\partial \bar{p}_i} + g \sum_{i=1}^{s} \sum_{j=1}^{s} F_i^{(j)} \cdot \frac{\partial \bar{f}_s}{\partial \bar{p}_i} =
\]

\[-g \sum_{i=1}^{s} \int d\bar{X}_{s+1} \bar{F}_i^{(s+1)} \cdot \frac{\partial \bar{f}_{s+1}}{\partial \bar{p}_i} \quad (4.69)
\]

for \( s = 1, \cdots, N - 1 \), and

\[
\frac{\partial \bar{f}_N}{\partial t} + \sum_{i=1}^{N} \bar{v}_i \cdot \frac{\partial \bar{f}_N}{\partial \bar{x}_i} + \sum_{i=1}^{N} F_i^{(0)} \cdot \frac{\partial \bar{f}_N}{\partial \bar{p}_i} + g \sum_{i=1}^{N} \sum_{j=1}^{N} F_i^{(j)} \cdot \frac{\partial \bar{f}_N}{\partial \bar{p}_i} = 0 \quad (4.70)
\]
E. Perturbation Theory

As expressed in the form (4.69) and (4.70), it is unclear how to solve the BBGKY hierarchy perturbatively in powers of $g$. This is because the relation between the distribution functions $\bar{f}_s$ and the coupling constant $g$ is not straightforward. The proper procedure is to expand in powers of the so-called reduced distribution functions $\bar{h}_s = \bar{h}_s(\bar{X}_1, \cdots, \bar{X}_s)$. We define the reduced distribution $\bar{h}_s$ by subtracting all possible lower order correlations from $\bar{f}_s$, a procedure that will be made more precise in just a moment. Consequently, the distribution $\bar{h}_s$ is also called the correlation function, as it encodes the full complement of $s$-body correlations. Perturbation theory is then constructed by expanding in powers of $\bar{h}_s$.

We start this recursive procedure by first constructing the 2-point correlation function $\bar{h}_2$. To do this, let us briefly return to dimensional variables, and define

$$\bar{h}_2(X_1, X_2) = f_2(X_1, X_2) - f_1(X_1)f_1(X_2) .$$

(4.71)

It is clear that $\bar{h}_2(X_1, X_2)$ captures the 2-body correlations, as the uncorrelated piece $f_1(X_1)f_1(X_2)$ has been subtracted from the full 2-body distribution $f_2(X_1, X_2)$: the remainder can only be the correlations. We will assume that $\bar{h}_2$ is of order $g$, and more generally that $g\bar{h}_s \propto g^s$. In dimensionless coordinates, we can therefore express the 2-point function by the expansion

$$\bar{f}_2(\bar{X}_1, \bar{X}_2) = \bar{f}_1(\bar{X}_1)\bar{f}_1(\bar{X}_2) + g\bar{h}_2(\bar{X}_1, \bar{X}_2) .$$

(4.72)

We will justify this perturbative assumption by proving that we can expand (4.69) and (4.70) to second order in $g$ (in principle we could work to any desired order in $g$). In a similar manner, the reduced 3-point function $\bar{h}_3$ is defined by the expansion

$$\bar{f}_3(\bar{X}_1, \bar{X}_2, \bar{X}_3) = \bar{f}_1(\bar{X}_1)\bar{f}_1(\bar{X}_2)\bar{f}_1(\bar{X}_3) + g\left[\bar{h}_2(\bar{X}_1, \bar{X}_2)\bar{f}_1(\bar{X}_3) + \bar{h}_2(\bar{X}_2, \bar{X}_3)\bar{f}_1(\bar{X}_1) + \bar{h}_2(\bar{X}_1, \bar{X}_3)\bar{f}_1(\bar{X}_2) + g^2\bar{h}_3(\bar{X}_1, \bar{X}_2, \bar{X}_3)\right] + g^2\bar{h}_3(\bar{X}_1, \bar{X}_2, \bar{X}_3) .$$

(4.73)

We have removed the following lower order correlations from $\bar{f}_3$: (i) a completely uncorrelated piece consisting of the product of three 1-point functions $\bar{f}_1\times\bar{f}_1\times\bar{f}_1$, and (ii) three 2-point correlations involving $\bar{h}_2\times\bar{f}_1$, evaluated on the cyclic permutations of $X_1$, $X_2$, and $X_3$, and (iii) the 3-point correlation $\bar{h}_3$. Note that $\bar{h}_3$ is of order $g^2$, or in dimensional form, $g\bar{h}_3 \propto g^3$. To the order $g^2$ in which we are working, the $\bar{h}_3$ term must therefore be dropped from (4.73) for consistency. Although we will not do so in these notes, one may press onward and calculate the order $g^3$ terms. To do this, we would keep the $g^2\bar{h}_3$ contribution to $\bar{f}_3$. We would also need to construct the 4-point correlation function $\bar{h}_4$ by subtracting off the lower order correlations from $\bar{f}_4$, which schematically takes the form

$$g^3\bar{h}_4(X_1, X_2, X_3, X_3) = \bar{f}_4(X_1, X_2, X_3, X_4) - \bar{f}_1(X_1)\bar{f}_1(X_2)\bar{f}_1(X_3)\bar{f}_1(X_4) - g^2\left[\bar{h}_3(\bar{X}_1, \bar{X}_2, \bar{X}_3)\bar{f}_1(\bar{X}_4) + \cdots\right] - g^2\left[\bar{h}_2(\bar{X}_1, \bar{X}_2)\bar{h}_2(\bar{X}_3, \bar{X}_4) + \cdots\right] - g\left[\bar{h}_2(\bar{X}_1, \bar{X}_2)\bar{f}_1(\bar{X}_3)\bar{f}_1(\bar{X}_4) + \cdots\right] .$$

(4.74)
As usual, we subtract off the completely uncorrelated piece, this time consisting of the product of four 1-point functions. At this order, there are several more combinations of lower-order correlations that must be removed. For example, there are terms like $\bar{h}_3 \times \bar{f}_1$, in addition to pair-wise 2-point contributions like $\bar{h}_2 \times \bar{h}_2$. Finally, there are contributions of the form $\bar{h}_2 \times \bar{f}_1 \bar{f}_1$. The next higher-order contribution would have even more combinations of lower-order correlations, and we see that higher-order calculations become quite involved very rapidly.

We now show that one can work consistently to order $g^2$, dropping terms of order $g^3$ and higher. We must prove that the $s = 1$ and $s = 2$ equations contain terms of order $g^2$ or lower, and that the $s \geq 3$ equations contain terms of order of $g^3$ and higher. For consistency, we must therefore work only with the $s = 1$ equation, and a truncated version of the $s = 2$ equation (as we have seen, we must also drop the $h_3$-contribution in the $\bar{f}_3$ term, as this contribution is of order $g^3$). Writing the factors of $g$ explicitly, we will show that the $s = 1$ equation takes the form

$$\frac{\partial}{\partial t} + V_A + gV_b[\bar{f}_1] \quad \bar{f}_1 = g^2 K[\bar{h}_2], \quad (4.75)$$

where $K[\bar{h}_2]$ is a homogeneous integration kernel, while $V_A$ and $V_b[\bar{f}_1]$ are differential and integro-differential operators on $\bar{X}_1$ space, respectively. Note that the operator $V_b[\bar{f}_1]$ contains a functional dependence on $\bar{f}_1$. The truncated $s = 2$ equation can be expressed in the form

$$\frac{\partial}{\partial t} + V_C + gV_d[\bar{f}_1] \quad g\bar{h}_2 = g S[\bar{f}_1] + O(g^3), \quad (4.76)$$

where $S[\bar{f}_1]$ is a source term depending upon $\bar{f}_1$, while $V_C$ is a differential operator, and $V_D[\bar{f}_1]$ is an integro-differential operator. Both operators act on $\bar{X}_1-\bar{X}_2$ space, of which $\bar{h}_2$ is a function. The precise form of the source term, the operators, and the kernel are not important to this perturbative argument, although we shall calculate these quantities explicitly in the next paragraph. The point here is that both (4.75) and (4.76) are of order $g^2$, and that higher-$s$ equations are of order $g^3$ and higher. Since the kernel $K[\bar{h}_2]$ is homogeneous, note that the $g$-dependence on the right-hand-side of (4.75) may be recast in the more suggestive form $gK[g\bar{h}_2]$, so that (4.75) and (4.76) is a system of coupled integro-differential equations for $\bar{f}_1$ and $g\bar{h}_2$. These equations are accurate to order $g^2$, with error of order $g^3$. With a lot of work, one can show that the $s = 3$ equation is of order $g^3$, and consistency demands that we neglect it as well (and all higher order equations). This justifies the assumption that $\bar{h}_2$ is of order $g$, and that we are indeed working consistently with an accuracy of order $g^2$, and an absolute error of order $g^3$. 

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1. General Number of Spatial Dimensions $\nu$

Let us now verify equations (4.75) and (4.76). We shall drop the bar from the dimensionless quantities for ease of notation, and the $s = 1$ equation of (4.69) becomes

$$
\left( \frac{\partial}{\partial t} + v_1 \cdot \frac{\partial}{\partial x_1} + F_{1}^{(0)} \cdot \frac{\partial}{\partial p_1} \right) f_1(x_1) = -g \int dX_2 F_{1}^{(2)} \cdot \frac{\partial}{\partial p_1} f_2(x_1, X_2) .
$$

When working with the Boltzmann equation in $\nu > 3$, this form will be particularly useful. For the perturbative analysis, however, it is better to expand $f_2$ (and $f_3$ in the $s = 2$ equation) in terms of the 2-point correlation $h_2$. For convenience we repeat here the expansions (4.72) and (4.73), but in an annotated form,

$$
f_2 = f_1(X_1) f_1(X_2) + g h_2(X_1, X_2) .
$$

(4.78)

$$
f_3 = f_1(X_1) f_1(X_2) f_1(X_3) +
\text{uncorrelated}
\left[ g h_2(X_1, X_2) f_1(X_3) + h_2(X_2, X_3) f_1(X_1) + h_2(X_1, X_3) f_1(X_2) \right] + \text{higher-order} .
$$

(4.79)

Using (4.78) in (4.77) gives the coupled integro-differential equation

$$
\frac{\partial f_1(X_1)}{\partial t} + v_1 \cdot \frac{\partial f_1(X_1)}{\partial x_1} + F_{1}^{(0)} \cdot \frac{\partial f_1(X_1)}{\partial p_1} + g \int dX_3 f_1(X_3) F_{1}^{(3)} \cdot \frac{\partial f_1(X_1)}{\partial p_1} =
\left[ -g^2 \int dX_3 F_{1}^{(3)} \cdot \frac{\partial h_2(X_1, X_3)}{\partial p_1} \right].
$$

(4.80)

We have replaced the integration variable $X_2$ in (4.77) by $X_3$ to avoid conflicts with the variable $X_2$ when we turn to the $s = 2$ equation. We can recast the above equation in a more compact form by defining the self-consistent electric field at $x_1$ by

$$
F_{1}[f_1] = \int dX_3 f_1(X_3) F_{1}^{(3)} = \int dX_3 f_1(X_3) F(x_1 - x_3) ,
$$

(4.81)

so that (4.80) becomes

$$
\left( \frac{\partial}{\partial t} + v_1 \cdot \frac{\partial}{\partial x_1} + F_{1}^{(0)} \cdot \frac{\partial}{\partial p_1} \right) f_1(x_1) = -g^2 \int dX_3 F_{1}^{(3)} \cdot \frac{\partial h_2(X_1, X_3)}{\partial p_1} .
$$

(4.82)

We have identified the quantities $V_{\lambda}$, $V_{\delta}[f_1]$, and $K[h_2]$ in (4.75) by the under-braces. Note that there is a factor of $g$ for every Coulomb interaction $F_{1}^{(3)}$, and a factor of $g$ for the
correlation $\bar{h}_2$. In dimensionless units, there is no difference between the electric force and the electric field, as the factors of electric charge have been collected in the coupling constant $g$.

Let us now turn to the $s = 2$ equation of the BBGKY hierarchy (4.69), which we write in the form

$$
\left[ \frac{\partial}{\partial t} + \sum_{i=1}^{2} \left( v_i \cdot \frac{\partial}{\partial x_i} + F_i^{(0)} \cdot \frac{\partial}{\partial p_i} \right) \right] f_2 + gF_1^{(2)} \cdot \left[ \frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_2} \right] f_2 =
$$

$$
- g \sum_{i=1}^{2} \int dX_2 \frac{\partial f_3}{\partial p_i} \cdot F_i^{(3)} f_2,
$$

(4.83)

Note that we have expanded the 1-2 scattering term as

$$
\sum_{i=1}^{2} \sum_{j=1}^{2} gF_i^{(j)} \cdot \frac{\partial f_2}{\partial p_i} = gF_1^{(2)} \cdot \left[ \frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_2} \right] f_2
$$

(4.84)

by using Newton’s third law $F_1^{(2)} = -F_2^{(1)}$. Expression (4.83) can be recast into an equation for $h_2$ by expanding $f_2$ and $f_3$ in terms of the 2-point correlation $h_2$. We will do this in stages, emphasizing the role played by the spatial dimension $\nu$ at each step, showing how the physics changes depending upon whether $\nu < 3$ or $\nu > 3$. Using (4.78) for $f_2$ in the 1-2 scattering term allows us to express (4.83) as

$$
\left[ \frac{\partial}{\partial t} + \sum_{i=1}^{2} \left( v_i \cdot \frac{\partial}{\partial x_i} + F_i^{(0)} \cdot \frac{\partial}{\partial p_i} \right) \right] f_2 + gF_1^{(2)} \cdot \left[ \frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_2} \right] gh_2 +
$$

$$
g \sum_{i=1}^{2} \int dX_3 \frac{\partial f_3}{\partial p_i} \cdot F_i^{(3)} f_1 = gS[f_1],
$$

(4.85)

where the source term is defined by

$$
S[f_1] = -F_1^{(2)} \cdot \left[ \frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_2} \right] f_1(X_1)f_1(X_2).
$$

(4.86)

It is instructive to contrast equation (4.83) with (4.85). The latter form is more amenable to the perturbative analysis we are performing. It will also be used in deriving the Lenard-Balescu equation for $\nu < 3$, where the Coulomb forces become long-range and 2-body scattering becomes soft. In this regime, we can drop the correlation term

$$
gF_1^{(2)} \cdot \left[ \frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_2} \right] gh_2
$$

(4.87)

from (4.85). In contrast, this term must be kept when $\nu > 3$. This is because the scatter becomes short-range, and momentum exchange can be become quite large. In this case, it is best not to make the substitution for $f_2$ in the 1-2 scattering term, and to use (4.83) instead.
This is justified, however, only after perturbation theory has been established. We will have more say about this in the next section. For now, we retain all terms for completeness.

Let us return to the general perturbative argument. I will present the detailed algebraic manipulations, since this calculation provides a template for proving that the $s = 3$ kinetic equation is indeed higher order. Upon expanding the remaining $f_2$-term in (4.85), the $s = 1$ equation can now be written as

\[
\left[ \frac{\partial}{\partial t} + \sum_{i=1}^{2} \left( v_i \cdot \frac{\partial}{\partial x_i} + F_i^{(0)} \cdot \frac{\partial}{\partial p_i} \right) \right] gh_2 + gF_1^{(2)} \cdot \left[ \frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_2} \right] gh_2 +
\]

(4.88)

\[
\left[ \frac{\partial}{\partial t} + \sum_{i=1}^{2} \left( v_i \cdot \frac{\partial}{\partial x_i} + F_i^{(0)} \cdot \frac{\partial}{\partial p_i} \right) \right] f_1(X_1) f_1(X_2) +
\]

\[
g \sum_{i=1}^{2} \int dX_3 F_i^{(3)} \cdot \frac{\partial f_3}{\partial p_i} = gS[f_1].
\]

The contribution from the uncorrelated piece of $f_2$ is written in the second line of (4.88), which breaks up into two collections of terms, one proportional to $f_1(X_2)$ and the other proportional to $f_1(X_1)$:

\[
\left[ \frac{\partial f_1(X_1)}{\partial t} + v_1 \cdot \frac{\partial f_1(X_1)}{\partial x_1} + F_i^{(0)} \cdot \frac{\partial f_1(X_1)}{\partial p_1} \right] f_1(X_2) +
\]

(4.89)

\[
\left[ \frac{\partial f_1(X_2)}{\partial t} + v_2 \cdot \frac{\partial f_1(X_2)}{\partial x_2} + F_i^{(0)} \cdot \frac{\partial f_1(X_2)}{\partial p_2} \right] f_1(X_1).
\]

Our strategy will be to expand $f_3$ in (4.88) using (4.79), and then to collect terms that reproduce the $s = 1$ equation (4.82) within the square brackets. This equation will be evaluated at $X_1$ and $X_2$ in each square bracket, respectively, but they will otherwise vanish. This leaves only an equation involving $gh_2$ on the left-hand-side (which is explicitly of order $g^2$).

To perform this calculation, we express the $f_3$ scattering term by

\[
g \sum_{i=1}^{2} \int dX_3 F_i^{(3)} \cdot \frac{\partial f_3}{\partial p_i} =
\]

(4.90)

\[
g \sum_{i=1}^{2} \int dX_3 F_i^{(3)} \cdot \frac{\partial}{\partial p_i} \left[ f_1(X_1) gh_2(X_2, X_3) + f_1(X_2) gh_2(X_1, X_3) \right] +
\]

\[
g \sum_{i=1}^{2} F_i[f_1] \cdot \frac{\partial}{\partial p_i} f_1(X_1) f_1(X_2) + g \sum_{i=1}^{2} F_i[f_1] \cdot \frac{\partial}{\partial p_i} gh_2(X_1, X_2),
\]

The second line of (4.90) can be traced to the 2-3 and 1-3 correlations in (4.79), while the terms in the third line come from the uncorrelated piece of $f_3$ and the 1-2 correlation,
respectively. We have generalized to definition of the self-consistent field to any position \( x_i \),

\[
F_i[f_1] = \int dX_3 \; f_1(X_3) F_i^{(3)} = \int dX_3 \; f_1(X_3) F(x_i - x_3) .
\]  

(4.91)

We now express equation (4.88) in the form

\[
\left[ \frac{\partial}{\partial t} + \sum_{i=1}^{2} \left( v_i \cdot \frac{\partial}{\partial x_i} + F_i^{(0)} \cdot \frac{\partial}{\partial p_i} + gF_i[f_1] \cdot \frac{\partial}{\partial p_i} \right) \right] gh_2 + gF_2^{(2)} \cdot \left[ \frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_2} \right] gh_2 +
\]

\[
\sum_{i=1}^{2} \int dX_3 \; gF_i^{(3)} \cdot \frac{\partial}{\partial p_i} \left[ f_1(X_1) \; gh_2(X_2, X_3) + f_1(X_2) \; gh_2(X_1, X_3) \right] +
\]

\[
\left[ \frac{\partial f_1(X_1)}{\partial t} + v_1 \cdot \frac{\partial f_1(X_1)}{\partial x_1} + F_1^{(0)} \cdot \frac{\partial f_1(X_1)}{\partial p_1} + gF_1[f_1] \cdot \frac{\partial f_1(X_1)}{\partial p_1} \right] f_1(X_2) +
\]

\[
\left[ \frac{\partial f_1(X_2)}{\partial t} + v_2 \cdot \frac{\partial f_1(X_2)}{\partial x_2} + F_2^{(0)} \cdot \frac{\partial f_1(X_2)}{\partial p_2} + gF_2[f_1] \cdot \frac{\partial f_1(X_2)}{\partial p_2} \right] f_1(X_1) = gS[f_1] .
\]  

(4.92)

Note that there are terms directly proportional to \( f_1(X_2) \), and others proportional to \( f_1(X_1) \), which have been grouped together in the square brackets. As mentioned above, each of the square brackets will turn out to vanish upon using the \( s = 1 \) equation at \( X_1 \) and \( X_2 \), respectively. Note that \( i = 1 \) term of the sum in the second line (from the 2-3 and 1-3 correlations) takes the form

\[
g^2 \int dX_3 \; F_1^{(3)} \cdot \frac{\partial f_1(X_1)}{\partial p_1} \; h_2(X_2, X_3) + g^2 \int dX_3 \; F_1^{(3)} \cdot \frac{\partial h_2(X_1, X_3)}{\partial p_1} \times f_1(X_2) ,
\]

(4.93)

and the \( i = 2 \) term is

\[
g^2 \int dX_3 \; F_2^{(3)} \cdot \frac{\partial f_1(X_2)}{\partial p_2} \; h_2(X_1, X_3) + g^2 \int dX_3 \; F_2^{(3)} \cdot \frac{\partial h_2(X_2, X_3)}{\partial p_2} \times f_1(X_1) .
\]

(4.94)

The second term in (4.93) marked by an under-brace is the kernel of the \( s = 1 \) equation (4.82), evaluated at the default position \( X_1 \). When combined with the terms in the first square bracket, those proportional to \( f_1(X_2) \), we find the \( s = 1 \) equation evaluated at \( X_1 \), and these terms vanish. Note that (4.82) is evaluated at the phase space position \( X_1 \), and since \( X_1 \) is just a free variable (in the formal mathematical sense), we can make the replacement \( X_1 \rightarrow X_2 \) in (4.82). Thus the \( s = 1 \) equation can also be evaluated at \( X_2 \). We see that the second term in (4.94) contains the kernel of the \( s = 1 \) equation at \( X_2 \), and the second square bracket also vanishes. The truncated \( s = 2 \) equation therefore becomes
\[ \left[ \frac{\partial}{\partial t} + \sum_{i=1}^{2} \left( \mathbf{v}_i \cdot \frac{\partial}{\partial \mathbf{x}_i} + F_i^{(0)} \cdot \frac{\partial}{\partial \mathbf{p}_i} + g F_i[f_1] \cdot \frac{\partial}{\partial \mathbf{p}_i} \right) \right] gh_2 + \]

\[ g \int dX_3 F_1^{(3)} \cdot \frac{\partial f_1(X_1)}{\partial \mathbf{p}_1} gh_2(X_2, X_3) + g \int dX_3 F_2^{(3)} \cdot \frac{\partial f_1(X_2)}{\partial \mathbf{p}_2} gh_2(X_1, X_3) + \]

\[ g F_1^{(2)} \cdot \left[ \frac{\partial}{\partial \mathbf{p}_1} - \frac{\partial}{\partial \mathbf{p}_2} \right] gh_2 = gS[f_1] , \]

which is in the form given by (4.76). Also note that the absolute error incurred by dropping the \( h_3 \)-contribution from \( gf_3 \) is of order \( g^3 \).

To fully complete the argument, we must show that the \( s = 3 \) equation, expressed here for completeness,

\[ \left( \frac{\partial}{\partial t} + \sum_{i=1}^{3} \left( \mathbf{v}_i \cdot \frac{\partial}{\partial \mathbf{x}_i} + F_i^{(0)} \cdot \frac{\partial}{\partial \mathbf{p}_i} + g \sum_{j=1}^{3} F_i^{(j)} \cdot \frac{\partial}{\partial \mathbf{p}_i} \right) \right) f_3(X_1, X_2, X_3) \]

\[ = -g \int dX_4 \sum_{i=1}^{3} F_i^{(4)} \cdot \frac{\partial}{\partial \mathbf{p}_i} f_4(X_1, X_2, X_3, X_4) , \]

is of order \( g^3 \) or higher. This is performed in complete analogy to the \( s = 2 \) case just presented. We first express (4.96) in terms of \( gh_3 \), up to order order \( g^3 \). The definition of \( h_3 \) and \( h_4 \) ensure that the accuracy of (4.96) is of order order \( g^3 \). There will be terms analogous to the square brackets in (4.92), proportional to factors of \( f_1 \), but these terms will vanish by using the lower-order equations for \( f_1 \) and \( gh_2 \). The final result will be of order \( g^3 \), and must therefore be dropped for consistency.

2. Coulomb Physics in \( \nu < 3 \) and \( \nu > 3 \)

We have expanded the BBGKY hierarchy to order \( g^2 \) in a general number of spatial dimensions \( \nu \), with little regard to the behavior of the Coulomb physics as a function \( \nu \). The equations have been quite general, but we must make some approximations to proceed, and the validity of the approximations depends upon whether the scattering is hard or soft, that is to say, upon whether \( \nu > 3 \) or \( \nu < 3 \), respectively. We have already addressed the 1-2 correlation and how it must be kept in \( \nu > 3 \), and how can it can, in part, be dropped in \( \nu < 3 \). It will turn out that complementary collections of 2-body correlations dominate in dimensions \( \nu < 3 \) compared with \( \nu > 3 \), and vice verse. In \( \nu = 3 \), both collection of terms contribute equally to the transport equations, but they have the misfortune of diverging
logarithmically in at long- and short-distances. Recall from Section II A, that the electric electric field at position \( x \) from a point charge \( e \) at the origin is

\[
E(x) = e \frac{\Gamma(\nu/2)}{2\pi^{\nu/2}} \frac{\hat{x}}{r^{\nu-1}},
\]

(4.97)

where \( \hat{x} \) is the unit vector at the origin, and \( r = |x| \) is the distance to \( x \). It is often more convenient to work with the Coulomb potential

\[
\phi(x) = e \frac{\Gamma(\nu/2 - 1)}{4\pi^{\nu/2}} \frac{1}{r^{\nu-2}}.
\]

(4.98)

These two expressions are just equations (2.5) and (2.8), and they produce a qualitative difference in the Coulomb field for \( \nu > 3 \) and \( \nu < 3 \). The reader is encouraged to revisit Fig. 2 for details. Short-distance ultraviolet (UV) physics is dominant in dimensions \( \nu > 3 \), and conversely, long-distance infrared (IR) physics dominates when \( \nu < 3 \). The dimension \( \nu = 3 \) is a critical case, in which the UV and IR physics are equally dominant. For \( \nu < 3 \), the Coulomb potential diverges less severely that \( 1/r \) as \( r \to 0 \). In this regime, the Lenard-Balescu scattering kernel does not suffer a UV divergence. In like manner, for \( \nu > 3 \), the Coulomb force converges to zero faster than \( 1/r \) as \( r \to \infty \). In this regime, and the Boltzmann scattering kernel does not suffer an IR divergence.

The primary results from the previous section are the \( s = 1 \) equation (4.82), repeated here for convenience,

\[
\left( \frac{\partial}{\partial t} + v_1 \cdot \frac{\partial}{\partial x_1} + F_{i}^{(0)} \cdot \frac{\partial}{\partial p_i} + gF_1[f_1] \cdot \frac{\partial}{\partial p_1} \right) f_1(X_1) = -g \int dX_3 F_{i}^{(3)} \cdot \frac{\partial gh_2(X_1, X_3)}{\partial p_1},
\]

(4.99)

and the truncated \( s = 2 \) equation (4.95),

\[
\left[ \frac{\partial}{\partial t} + \sum_{i=1}^{2} \left( v_i \cdot \frac{\partial}{\partial x_i} + F_{i}^{(0)} \cdot \frac{\partial}{\partial p_i} + gF_1[f_1] \cdot \frac{\partial}{\partial p_1} \right) \right] g h_2 +
\]

\[
g \sum_{i=1}^{2} F_i[gh_2] \cdot \frac{\partial f_1}{\partial p_i} + g F_1^{(2)} \cdot \left[ \frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_2} \right] g h_2 = gS[f_1] .
\]

(4.100)

We have written (4.100) in a compact form, using the self consistent field \( F_i[f_1] \) defined in (4.91), and the self-consistent field induced by \( h_2 \),

\[
F_i[h_2] = \int dX_3 h_2(X_3, X_j) F_i^{(3)} .
\]

(4.101)

Here, \( j = 2 \) when \( i = 1 \), and \( j = 1 \) when \( i = 2 \). Note that long-distance 2-body Coulomb scattering is soft in dimensions \( \nu < 3 \), and the momentum exchange is small. We can therefore drop the 1-2 correlation term

\[
g F_1^{(2)} \cdot \left[ \frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_2} \right] g h_2(X_1, X_2) .
\]

(4.102)
which we assume is formally of order $g^3$. This assumption does not mean that the momentum difference in the collision is being neglected, i.e. we are not dropping the term

$$-gF_1^{(2)} \cdot \left[ \frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_2} \right] f_1(X_1)f_1(X_2) = gS[f_1]. \quad (4.103)$$

In $\nu < 3$, we can therefore neglect (4.102) from (4.100), giving

$$\left[ \frac{\partial}{\partial t} + \sum_{i=1}^{2} \left( v_i \cdot \frac{\partial}{\partial x_i} + F_i^{(0)} \cdot \frac{\partial}{\partial p_i} + gF_i[f_1] \cdot \frac{\partial}{\partial p_i} \right) \right] gh_2(X_1, X_2) +$$

$$\int dX_3 gh_2(X_3, X_2) gF_1^{(3)} \cdot \frac{\partial f_1(X_1)}{\partial p_1} + \int dX_3 gh_2(X_3, X_1) gF_2^{(3)} \cdot \frac{\partial f_1(X_2)}{\partial p_2} = gS[f_1],$$

where we have expanded the $F_i[h_2]$ term for clarity. The equation for $h_2$ can be recast in the form

$$\frac{\partial h_2}{\partial t} + V_1 h_2 + V_2 h_2 = S[f_1], \quad (4.105)$$

with source

$$S[f_1] = -F_1^{(2)} \cdot \left[ \frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_2} \right] f_1(X_1)f_1(X_2). \quad (4.106)$$

Here, $V_1$ is a linear integro-differential operator defined in $X_1$-space by

$$V_1 h_2(X_1, X_2) = v_1 \cdot \frac{\partial h_2}{\partial x_1} + F_1^{(0)} \cdot \frac{\partial h_2}{\partial p_1} + gF_1 \cdot \frac{\partial h_2}{\partial p_1} + g \int dX_3 h_2(X_3, X_2) F_1^{(3)} \cdot \frac{\partial f_1(X_1)}{\partial p_1},$$

and $V_2$ is the corresponding operator in $X_2$-space,

$$V_2 h_2(X_1, X_2) = v_2 \cdot \frac{\partial h_2}{\partial x_2} + F_2^{(0)} \cdot \frac{\partial h_2}{\partial p_2} + gF_2 \cdot \frac{\partial h_2}{\partial p_2} + g \int dX_3 h_2(X_3, X_1) F_2^{(3)} \cdot \frac{\partial f_1(X_2)}{\partial p_2}. \quad (4.107)$$

These expressions simplify marginally in the case of a uniform plasma, and this will be our starting point in Section VI on the Lenard-Balescu equation.

In dimensions $\nu > 3$, the behavior of the Coulomb field is quite different. The potential becomes short-range, and 2-body scattering is not always soft. This means that we cannot drop the 2-body momentum exchange term in (4.102). This term arose from the expansion of $f_2$, so it is convenient not to make this expansion, and to express the $s = 1$ equation as

$$\left( \frac{\partial}{\partial t} + v_1 \cdot \frac{\partial}{\partial x_1} + F_1^{(0)} \cdot \frac{\partial}{\partial p_1} \right) f_1 = -g \int dX_2 F_1^{(2)} \cdot \frac{\partial f_2}{\partial p_1}. \quad (4.109)$$
We can, however, drop the short-range contributions to the scattering in $f_3$, and use the truncated $s = 2$ equation

$$
\left[ \frac{\partial}{\partial t} + \sum_{i=1}^{2} \left( v_i \cdot \frac{\partial}{\partial x_i} + F_i^{(0)} \cdot \frac{\partial}{\partial p_i} + F_i[f_1] \cdot \frac{\partial}{\partial p_i} \right) \right] f_2 + g F_i^{(2)} \cdot \left[ \frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_2} \right] f_2 = 0.
$$

(4.110)

This will be our starting point for Section V on the Boltzmann equation. Note that the term $f_2$ implicitly contains a functional dependence on the 2-point correlation $h_2$.

3. The Uniform Plasma

In the calculations that follow, we make one more significant assumption, namely, we take the plasma to be spatially uniform in that the 1-point function $f_1 = f(p_1)$ depends only upon the momentum. By Galilean invariance, the 2-point function can only be of the form $h_2 = h(x_1 - x_2, p_1, p_2)$. This plasma conforms to the experimental situations involved in inertial confinement fusion (ICF), the testing ground of charged particle stopping power. I did not wish to introduce this assumption sooner, as I wanted to prove the validity of perturbation theory for the BBGKY hierarchy in a more general setting. The kinetic equations simplify somewhat, in that the self-consistent forces $F_i[f]$ vanish. Keeping the external force $F^{(0)}$ for generality, in $\nu < 3$, the $s = 1$ equation becomes

$$
\left( \frac{\partial}{\partial t} + F_1^{(0)} \cdot \frac{\partial}{\partial p_1} \right) f(p_1, t) = -g^2 \int dX_3 F_1^{(3)} \cdot \frac{\partial h(X_1, X_3, t)}{\partial p_1}.
$$

(4.111)

We shall change spatial integration variables from $x_3$ to $x = x_1 - x_3$. Setting $p = p_3$ and $X = (x, p)$, we can express (4.111) as

$$
\left( \frac{\partial}{\partial t} + F_1^{(0)} \cdot \frac{\partial}{\partial p_1} \right) f(p_1, t) = -g \int dX F(x) \cdot \frac{\partial}{\partial p_1} g h_2(x, p_1, p, t).
$$

(4.112)

In a similar manner, the truncated $s = 2$ equation in $\nu < 3$ becomes

$$
\frac{\partial h_2}{\partial t} + V_1 h_2 + V_2 h_2 = S[f_1],
$$

(4.113)

with source

$$
S[f_1] = -F_i^{(2)} \cdot \left[ \frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_2} \right] f(p_1)f_1(p_2).
$$

(4.114)
Here, $V_1$ is a linear integro-differential operator defined in $X_1$-space by

$$V_1 h_2(X_1, X_2) = v_1 \cdot \frac{\partial h_2}{\partial x_1} + F^{(0)}_1 \cdot \frac{\partial h_2}{\partial p_1} + g \int dX_3 h_2(X_3, X_2) F^{(3)}_1 \cdot \frac{\partial f_1(p_1)}{\partial p_1},$$

(4.115)

and $V_2$ is the corresponding operator in $X_2$-space,

$$V_2 h_2(X_1, X_2) = v_2 \cdot \frac{\partial h_2}{\partial x_2} + F^{(0)}_2 \cdot \frac{\partial h_2}{\partial p_2} + g \int dX_3 h_2(X_3, X_1) F^{(3)}_2 \cdot \frac{\partial f_1(p_2)}{\partial p_2}.$$

(4.116)

This will be our starting point when $\nu < 3$ of Section VI on the Lenard-Balescu equation. In contrast, when $\nu > 3$ in the Boltzmann analysis in Sec V, we shall start with

$$\left( \frac{\partial}{\partial t} + F^{(0)}_1 \cdot \frac{\partial}{\partial p_1} \right) f_1 = -g \int dX_2 F^{(2)}_1 \cdot \frac{\partial f_2}{\partial p_1},$$

(4.117)

and

$$\left[ \frac{\partial}{\partial t} + \sum_{i=1}^2 \left( v_i \cdot \frac{\partial}{\partial x_i} + F^{(0)}_i \cdot \frac{\partial}{\partial p_i} \right) \right] f_2 + g F^{(2)}_1 \cdot \left[ \frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_2} \right] f_2 = 0.$$

(4.118)

The latter equation is just the $s = 2$ equation in which $f_3 \to 0$. 
V. THE BOLTZMANN EQUATION FROM BBGKY IN $\nu > 3$

In this section will prove that in spatial dimensions $\nu > 3$, the Boltzmann equation (BE) follows from the BBGKY hierarchy to leading order in the plasma coupling $g$. For simplicity we work with a single-component plasma, for which the Boltzmann equation takes the form

$$\frac{\partial f}{\partial t} + \mathbf{v}_1 \cdot \frac{\partial f}{\partial \mathbf{x}_1} = B[f],$$

with scattering kernel

$$B[f] = \int \frac{d^\nu p_2}{(2\pi \hbar)^\nu} |\mathbf{v}_1 - \mathbf{v}_2| d\sigma_{12} \left\{ f(p'_1)f(p'_2) - f(p_1)f(p_2) \right\},$$

where $p_1 = m\mathbf{v}_1$. The $\nu$-dimensional cross section $d\sigma_{12}$ is defined in Appendix A3. The argument of this section is based on that of Huang from Section 3.5 of Ref. [10]. Huang’s argument in fact breaks down for the Coulomb force in $\nu = 3$ spatial dimensions (the actual case of physical interest) because of a long-distance infra-red (IR) divergence. However, the argument goes through unscathed when generalized to arbitrary spatial dimensions $\nu > 3$. This is because the Coulomb force goes like $1/r^{\nu-1}$, which falls off faster than $1/r$ at large $r$ for $\nu > 3$, thereby rendering finite any potential IR divergence. Since the short distance physics of the BE is correct, the scattering kernel does not, as we expect, suffer a short-distance ultra-violet (UV) divergence.

In Section IV E, we showed that for short-range interactions, in particular the Coulomb force in dimensions $\nu > 3$, the BBGKY hierarchy to order $g^2$ can be expressed as

$$\left( \frac{\partial}{\partial t} + \mathbf{v}_1 \cdot \frac{\partial}{\partial \mathbf{x}_1} \right) f_1(x_1) = - \int dX_2 F_1^{(2)} \frac{\partial}{\partial p_1} f_2(X_1, X_2)$$

and

$$\left( \frac{\partial}{\partial t} + \mathbf{v}_1 \cdot \frac{\partial}{\partial \mathbf{x}_1} + \mathbf{v}_2 \cdot \frac{\partial}{\partial \mathbf{x}_2} + F_1^{(2)} \left[ \frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_2} \right] \right) f_2(X_1, X_2) = 0,$$

where $f_1$ is the single-particle distribution function, and $f_2$ is the 2-point distribution. Expression (5.3) is the first BBGKY equation, while (5.4) is the second BBGKY equation, except that the 3-point function has been dropped from the right-hand-side of the full $s = 2$ equation (4.83). To continue, let us express (5.4) in center-of-mass coordinates. Since our final goal is to apply this formalism to a multi-species plasma, in this calculation let us temporarily suppose particle-1 has mass $m_1$ and particle-2 has mass $m_2$. We will denote the total mass by $M = m_1 + m_2$, and the reduced mass by $m_{12} = m_1 m_2 / M$. We then define the total and relative momentum, and the center-of-mass and relative position by

$$\mathbf{x} = x_1 - x_2, \quad \mathbf{p} = m_{12}(\mathbf{v}_1 - \mathbf{v}_2)$$

and

$$\mathbf{R} = \frac{m_1 x_1 + m_2 x_2}{M}, \quad \mathbf{P} = m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2.$$
Recalling that $P = 0$ in the center-of-mass frame, in Appendix A 3 we show that
\begin{equation}
\frac{\partial}{\partial x_1} x_1 + \frac{\partial}{\partial x_2} x_2 = (v_1 - v_2) \cdot \frac{\partial}{\partial x} \tag{5.7}
\end{equation}
\begin{equation}
\frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_2} = \frac{\partial}{\partial p} \tag{5.8}
\end{equation}
where $p_1 = m_1 v_1$ and $p_2 = m_2 v_2$. To find the BE, we must consider the asymptotic time limit $t \to \infty$. This is because, as per Bogoliubov’s hypothesis, the 2-point correlation $h_2$ comes into equilibrium much sooner than the single particle distribution $f_1$. We may consequently set the time derivative in (5.4) to zero, giving the static equation
\begin{equation}
\left( (v_1 - v_2) \cdot \frac{\partial}{\partial x} + F^{(2)}_1(x) \cdot \frac{\partial}{\partial p} \right) f_2 = 0 \tag{5.9}
\end{equation}
where we have used (5.7) and (5.8) to write (5.4) in terms of relative coordinates. Let us now express the first BBGKY equation (5.3) in terms of a scattering kernel,
\begin{equation}
\left( \frac{\partial}{\partial t} + v_1 \cdot \frac{\partial}{\partial x_1} \right) f_1 = B[f] \tag{5.10}
\end{equation}
where the kernel is defined by
\begin{equation}
B[f] \equiv -\int dX_2 \ F^{(2)}_1 \cdot \frac{\partial f_2}{\partial p_1} \tag{5.11}
\end{equation}
\begin{equation}
= -\int dX_2 \ F(x_1 - x_2) \cdot \left[ \frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_2} \right] f_2 = -\int dX_2 \ F^{(2)}_1(x) \cdot \frac{\partial f_2}{\partial p} \tag{5.12}
\end{equation}
We have added zero in the form of the total derivative $\partial/\partial p_2$ in (5.12), and we have used (5.8) to express the resulting difference in momentum derivatives in terms of the derivative of the relative momentum. We can now use (5.9) to write the scattering kernel in the form
\begin{equation}
B[f] = \int \frac{d^n p_2}{(2\pi \hbar)^n} \int d^n x_2 \ (v_1 - v_2) \cdot \frac{\partial f_2}{\partial x} \tag{5.13}
\end{equation}
It is understood that (5.13) is to be evaluated in the limit $t \to \infty$, or rather, at asymptotic times compared to the time scale of the 2-point correlations $h_2$.

We shall now express (5.13) in terms of the $\nu$-dimensional cross section $d\sigma_{12}$. See Appendix A 3 of these notes for a detailed treatment of the cross section in a general number of dimension. As illustrated in Fig. 4, let the beam-line of the 1+2 collision define the $x$-axis, so that $v_1 - v_2 = |v_1 - v_2| \hat{x}$. In two-body scattering, the velocity vectors $v_1$ and $v_2$ are directed toward one another along the beam-line, but they are offset (in a normal direction to $x$) by a distance $b$ called the impact parameter. Using expression (A17), the $\nu$-dimensional volume element in cylindrical coordinates about particle-2 can be expressed as
FIG. 4: Two-body scattering for a short-range force. Particle-1 has velocity \( v_1 \) and particle-2 has velocity \( v_2 \), although for simplicity particle-2 is pictured at rest. The particle velocities are directed towards one another, with the beam-line defining the \( x \)-axis. Therefore, \( v_1 - v_2 = |v_1 - v_2|\hat{x} \). The cross section is given in terms of the impact parameter \( b \) by \( d\sigma_{12} = d\Omega_{\nu-2} b^{\nu-2} db \), and therefore the volume element about particle-2 can be written \( d^\nu x_2 = d\sigma_{12} dx \).

\[
\frac{1}{2} \left( \frac{2\pi \hbar}{\nu} \right)^\nu d\nu x_2 = d\Omega_{\nu-2} b^{\nu-2} db \ dx.
\] (5.14)

Section A 3 of these notes proves that \( \nu \)-dimensional differential scattering cross section takes the form

\[
d\sigma_{12} = d\Omega_{\nu-2} b^{\nu-2} db,
\] (5.15)

and therefore the spatial volume element can be written

\[
\frac{1}{2} \left( \frac{2\pi \hbar}{\nu} \right)^\nu d^\nu x_2 = d\sigma_{12} dx.
\] (5.16)

Since the Coulomb force in \( \nu > 3 \) is short range with a characteristic distance scale \( r_0 \sim \kappa^{-1} \), we can choose points \( x_1 \) and \( x_2 \) on either side of \( x \) such that the force virtually vanishes for \( x < x_1 \) and \( x > x_2 \), although the force cannot be neglected for \( x_1 < x < x_2 \). In other words, we can choose the points \( x_1 \) and \( x_2 \) right after and right before the collision of interest. This is illustrated in Fig. 4. We can therefore write (5.13) in the form

\[
B[f] = \int \frac{d^\nu p_2}{(2\pi \hbar)^\nu} \int d\sigma_{12} \int_{x_1}^{x_2} dx |v_1 - v_2| \frac{\partial f_2}{\partial x}
\] (5.17)

\[
= \int \frac{d^\nu p_2}{(2\pi \hbar)^\nu} |v_1 - v_2| d\sigma_{12} \left[ f_2(x_2) - f_2(x_1) \right].
\] (5.18)
Recall that the 2-point function \( f_2 \) is the product of two factors of \( f_1 \) and a correlation function \( h_2 \). Since the Coulomb force is short range in \( \nu > 3 \), the function \( h_2 \) vanishes at \( x_1 \) and \( x_2 \), and we have

\[
f_2 = f_1 \times f_1 + h_2 \rightarrow f_1 \times f_1 , \tag{5.19}
\]

so that

\[
f_2(x_1) = f_1(p_1) f_1(p_2) \tag{5.20}
\]
\[
f_2(x_2) = f_1(p'_1) f_1(p'_2) . \tag{5.21}
\]

Here, \( p_1 = m_1 v_1 \) and \( p_2 = m_2 v_2 \) are the momenta before the collision, and \( p'_1 \) and \( p'_2 \) are the momenta after the collision. In standard derivations of the Boltzmann equation, the assumption of *molecular chaos* is invoked at this juncture. This principle states that the momenta before and after a collision are uncorrelated, and we see that the short-range nature of the Coulomb force in \( \nu > 3 \) justifies this assumption. We finally arrive at the Boltzmann equation

\[
B[f] = \int \frac{d^\nu p_2}{(2\pi \hbar)^\nu} \int d\Omega |v_1 - v_2| \frac{d\sigma_{12}}{d\Omega} \left[ f_1(p'_1) f_1(p'_2) - f_1(p_1) f_1(p_2) \right] . \tag{5.22}
\]

In terms of a *quantum* transition amplitude \( T \), we can express the Boltzmann scattering kernel in the form

\[
B[f] = \int \frac{d^\nu p'_1}{(2\pi \hbar)^\nu} \frac{d^\nu p'_2}{(2\pi \hbar)^\nu} \frac{d^\nu p_2}{(2\pi \hbar)^\nu} |T_{12}^\nu| \left\{ f_1(p'_1) f_1(p'_2) - f_1(p_1) f_1(p_2) \right\} (2\pi \hbar)^{\nu} \delta(p'_1 + p'_2 - p_1 - p_2) (2\pi \hbar)^{\nu} \delta(E'_1 + E'_2 - E_1 - E_2) . \tag{5.23}
\]

The time non-invariance of the BE happens in two places in this argument: (i) using the \( s = 2 \) equation at asymptotic times, and (ii) the molecular chaos assumption. The generalization to multi-species is easy. Two-point functions are still uncorrelated at \( x_1 \) and \( x_2 \), so that

\[
f_2(x_1) = f_a(p_a) f_b(p_b) \tag{5.24}
\]
\[
f_2(x_2) = f_a(p'_a) f_b(p'_b) , \tag{5.25}
\]

and the Boltzmann scattering kernel becomes

\[
B_{ab}[f] = \int \frac{d^\nu p'_a}{(2\pi \hbar)^\nu} \frac{d^\nu p'_b}{(2\pi \hbar)^\nu} \frac{d^\nu p_b}{(2\pi \hbar)^\nu} |T_{ab}^\nu| \left\{ f_a(p'_a) f_b(p'_b) - f_a(p_a) f_b(p_b) \right\} (2\pi \hbar)^{\nu} \delta(p'_a + p'_b - p_a - p_b) (2\pi \hbar)^{\nu} \delta(E'_a + E'_b - E_a - E_b) . \tag{5.26}
\]
VI. THE LENARD-BALESCU EQUATION FROM BBGKY IN $\nu < 3$

We now derive the Lenard-Balescu equation (LBE) from the BBGKY hierarchy in spatial dimensions $\nu < 3$. We rely on Chapter 12 of Clemmow and Dougherty [9] as our primary source in this section, since it is so clearly written and easily generalizes to multiple dimensions. The calculation is very long, but quite informative. The calculation of Ref. [9] actually breaks down in $\nu = 3$ spatial dimensions because of a short-distance ultra-violet (UV) divergence. However, all quantities become finite when $\nu < 3$, and the calculation can proceed as presented. This is because the Coulomb force falls off like $1/r^{\nu-1}$, and this renders the UV divergence finite in $\nu < 3$. Since the long distance physics of the LBE is correct, the kernel does not suffer a long-distance infra-red (IR) divergence. In a single-component plasma, the LBE takes the form

$$\frac{\partial f}{\partial t} + \mathbf{v}_1 \cdot \frac{\partial f}{\partial \mathbf{x}_1} = L[f], \tag{6.1}$$

with scattering kernel

$$L[f] = -\frac{\partial}{\partial \mathbf{p}} \cdot \mathbf{J}(\mathbf{p}) \tag{6.2}$$

$$\mathbf{J}(\mathbf{p}) = \int \frac{d^\nu p_2}{(2\pi)^\nu} \frac{d^\nu k}{(2\pi)^\nu} \mathbf{k} \left| \frac{e^2}{k^2 \epsilon(\mathbf{k}, \mathbf{v}_1)} \right|^2 \pi \delta(\mathbf{k} \cdot \mathbf{v}_1 - \mathbf{k} \cdot \mathbf{v}_2) \left[ \mathbf{k} \cdot \frac{\partial}{\partial \mathbf{p}_1} - \mathbf{k} \cdot \frac{\partial}{\partial \mathbf{p}_2} \right] f(\mathbf{p}_1)f(\mathbf{p}_2).$$

As always, we take $\mathbf{v}_i = \mathbf{p}_i/m$ for $i = 1, 2$. For a single component plasma, the dielectric function $\epsilon$ is given by

$$\epsilon(\mathbf{k}, \omega) = 1 + \frac{e^2}{k^2} \int \frac{d^\nu p}{(2\pi)^\nu} \frac{1}{\omega - \mathbf{k} \cdot \mathbf{v} + i\eta} \mathbf{k} \cdot \frac{\partial f(\mathbf{p})}{\partial \mathbf{p}}, \tag{6.3}$$

and the prescription $\eta \rightarrow 0^+$ is implicit, defining the correct retarded time response.

A. Formal Solution to the Perturbative Equations

We now explicitly assume the plasma to be uniform, in the sense that the 1-point function is constant in space, being a function only of momentum,

$$f_1(\mathbf{X}, t) = f(\mathbf{p}, t), \tag{6.4}$$

with $\mathbf{X} = (\mathbf{x}, \mathbf{p})$. Galilean invariance then constrains the 2-point function to take the form

$$h_2(\mathbf{X}_1, \mathbf{X}_2, t) = h(\mathbf{x}_1 - \mathbf{x}_2, \mathbf{p}_1, \mathbf{p}_2, t). \tag{6.5}$$

The time dependence $t$ will often be left implicit, and we will employ a slight abuse of notation by writing $h(\mathbf{X}_1, \mathbf{X}_2)$. The so called self-consistent fields $\mathbf{F}_i[f]$ defined in (4.91)
vanish under the condition of uniformity, and the kinetic equations take slightly simpler forms. In Section IV E, we showed that for long-range interactions, in particular for the Coulomb force in \( \nu < 3 \), the coupled system integro-differential equations is

\[
\frac{\partial}{\partial t} f(p_1, t) = -g \int dX F(x) \cdot \frac{\partial}{\partial p_1} gh_2(x, p_1, p, t) .
\]

(6.6)

and

\[
\frac{\partial h}{\partial t} + V_1 h + V_2 h = S[f] ,
\]

(6.7)

where the source term is

\[
S[f] = -F^{(2)}_1 \cdot \left[ \frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_2} \right] f(p_1)f(p_2) .
\]

(6.8)

Note that \( F^{(2)}_1 = eE(x_1 - x_2) \) is the Coulomb force at \( x_1 \) from a point charge at \( x_2 \). These equations are accurate to order \( g^2 \) in the plasma coupling. The quantity \( V_1 \) is an integro-differential operator defined in \( X_1 \)-space by

\[
V_1 h(X_1, X_2) = v_1 \cdot \frac{\partial h(X_1, X_2)}{\partial x_1} + \int dX_3 h(X_3, X_2) F^{(3)}_1 \cdot \frac{\partial f(p_1)}{\partial p_1} ,
\]

(6.9)

and \( V_2 \) is the corresponding operator in \( X_2 \)-space,

\[
V_2 h(X_1, X_2) = v_2 \cdot \frac{\partial h(X_1, X_2)}{\partial x_2} + \int dX_3 h(X_3, X_1) F^{(3)}_2 \cdot \frac{\partial f(p_2)}{\partial p_2} ,
\]

(6.10)

with \( F^{(3)}_i \) (for \( i = 1, 2 \)) being the Coulomb force at \( x_i \) from \( x_3 \). When the correlation function \( h \) is written without arguments, it is assumed to be \( h(X_1, X_2) \). The variable \( X_2 \) in (6.9) “just goes along for the ride”, and we may regard \( V_1 \) as an operator in \( X_1 \)-space acting on a functions \( h(X_1) \). Similarly, the variable \( X_1 \) is free in the operator \( V_2 \). The operators \( V_1 \) and \( V_2 \) therefore commute when acting on functions \( h(X_1, X_2) \) of two variables, and \( V_1 \) and \( V_2 \) may consequently be treated as numbers when solving the differential equation (6.7) for \( h(t) \).

In deriving the Lenard-Balescu equation, we require the asymptotic time limit \( t \to \infty \) of \( h(t) \). This is because of Bogoliubov’s hypothesis, which states that \( h(x, p, t) \) quickly relaxes to its asymptotic value \( h(x, p, \infty) \), relative to \( f(p, t) \). We may therefore treat \( t \) as a parameter in the source term \( S \), and the operators \( V_1 \), and \( V_2 \).

Let us now find a formal solution for \( h(t) \). We leave the phase-space variables \( x \) and \( p \) implicit. We will employ the method of Laplace transforms, where the Laplace transform and its inverse are related by

\[
\tilde{h}(p) = \int_0^\infty dt e^{-pt} h(t) \]

(6.11)

\[
h(t) = \frac{1}{2\pi i} \int_C dp e^{pt} \tilde{h}(p) ,
\]

(6.12)
where the contour $C$ runs parallel to the imaginary axis with all poles of $\tilde{h}(p)$ lying to the left of $C$. The analytic structure of $\tilde{h}(p)$ in the complex $p$-plane determines the function $h(t)$ for all values of $t$ greater than zero. Let us multiply (6.7) by $e^{-pt}$ and integrate over $t$, giving the equation

$$
\int_0^\infty dt \ e^{-pt} \left( \frac{\partial h}{\partial t} + V_1 h + V_2 h \right) = \int_0^\infty dt \ e^{-pt} S = p^{-1} S .
$$

(6.13)

Upon integrating by parts,

$$
\int_0^\infty dt \ e^{-pt} \frac{\partial h}{\partial t} = e^{-pt} h(t) \bigg|_0^\infty + \int_0^\infty dt \ p \ e^{-pt} h = -h(0) + p \tilde{h}(p) ,
$$

(6.14)

we can express this as

$$(p + V_1 + V_2)\tilde{h}(p) = p^{-1} S + h(0) .
$$

(6.15)

Solving for the Laplace transform $\tilde{h}$ gives the formal solution

$$
\tilde{h}(p) = (p + V_1 + V_2)^{-1} \left( p^{-1} S + h(0) \right) .
$$

(6.16)

We can find the asymptotic value $h(\infty)$ from (6.16) in the following manner. From (6.16) we see that $\tilde{h}(p)$ has a pole at $p = 0$, in addition to the other poles lying in the left half-plane with $\text{Re} \ p < 0$. As $t \to \infty$, the dominant contribution to the integral (6.11) comes from the $p = 0$ pole. This means we can replace the contour $C$ by a circular contour $C_r$ of radius $r$ about the origin, and we can evaluate $h(\infty)$ by integrating around $C_r$ and taking the limit $r \to 0^+$. Points on $C_r$ are given by $p = r e^{i\theta}$. Therefore $dp = ir \, d\theta$, and we can change variables from $p$ to $\theta$. Since we are interested in the $r \to 0^+$ limit, we can replace factors of $p$ in the integrand by factors of $r$ (there is no $\theta$-dependence at the origin), giving

$$
h(\infty) = \lim_{t \to \infty} \lim_{r \to 0^+} \frac{1}{2\pi i} \oint_{C_r} dp \ e^{pt} \tilde{h}(p) = \lim_{t \to \infty} \lim_{r \to 0^+} \frac{1}{2\pi} \int_0^{2\pi} d\theta \ r \ e^{rt} \tilde{h}(r)
$$

(6.17)

$$
= \lim_{r \to 0^+} \frac{1}{2\pi} \cdot 2\pi \cdot r \cdot 1 \cdot \tilde{h}(r) ,
$$

(6.18)

where the factor of unity comes from $e^{rt} \to 1$ as $r \to 0^+$ (note that $t$ is fixed while the $r$-limit is taken). Consequently, we find the elegant and compact result

$$
h(\infty) = \lim_{p \to 0^+} p \tilde{h}(p) ,
$$

(6.19)

where we have changed variables from $r$ back to $p$ in the limit. Upon using (6.16) for the Laplace transform $\tilde{h}(p)$, we can write the asymptotic form as

$$
h(\infty) = \lim_{p \to 0^+} (p + V_1 + V_2)^{-1} S .
$$

(6.20)
Note that the initial condition $\tilde{h}(0)$ does not appear in the asymptotic form.

Recall that the Laplace transform of $e^{-at}$ is $(p + a)^{-1}$, and we can therefore write

$$
(p + V_1 + V_2)^{-1} = \int_0^\infty dt e^{-(p+V_1+V_2)t} 
= \frac{1}{(2\pi i)^2} \int_0^\infty dt e^{-pt} \int_{C_1} dp_1 \frac{e^{p_1 t}}{p_1 + V_1} \int_{C_2} dp_2 \frac{e^{p_2 t}}{p_2 + V_2},
$$

where we have expressed $e^{-V_1 t}$ and $e^{-V_2 t}$ as inverse Laplace transforms defined by contours $C_1$ and $C_2$, respectively. These two contours are suitable inverse Laplace transform contours parallel to the imaginary axis, with all poles lying to their left. Upon performing the $t$-integral, we find

$$
(p + V_1 + V_2)^{-1} = \frac{1}{(2\pi i)^2} \int_{C_1} dp_1 \int_{C_2} dp_2 \frac{1}{p - p_1 - p_2} \frac{1}{p_1 + V_1} \frac{1}{p_2 + V_2},
$$

where $\text{Re} \, p > \text{Re}(p_1 + p_2)$ for the $t$-integral convergence at large $t$. The asymptotic form of $h$ can now be written

$$
h(\infty) = \lim_{p \to 0^+} \frac{1}{(2\pi i)^2} \int_{C_1} dp_1 \int_{C_2} dp_2 \frac{1}{p - p_1 - p_2} \frac{1}{p_1 + V_1} \frac{1}{p_2 + V_2} S[f].
$$

The source $S[f]$ is defined by (4.106). The problem now reduces to an exercise in complex analysis, albeit a rather involved exercise. The next step involves calculating the action of the operators $(p + V_2)^{-1}$ and $(p + V_1)^{-1}$ on the source $S$.

B. Preliminary Example

As a prelude to finding the inverse operators above, let us consider a simpler problem in which $h$ is a function of only one phase-space variable $X$ (rather than $X_1$ and $X_2$), so that $h(X, t) = h(x, \mathbf{p}, t)$. Suppose now that $h$ satisfies the simplified equation

$$
\frac{\partial h}{\partial t} + Vh = 0,
$$

where the operator is defined by

$$
Vh(x, \mathbf{p}) = \mathbf{v} \cdot \frac{\partial h}{\partial x} + \int dX_3 h(x_3, \mathbf{p}_3) F_{3}^{(3)} \cdot \frac{\partial f(\mathbf{p})}{\partial \mathbf{p}}.
$$

As usual, $F_{3}^{(3)} = eE(x - x_3)$ is the Coulomb force at $x$ from a point charge at $x_3$, and the integration variable is $X_3 = (x_3, \mathbf{p}_3)$. We shall express the operator $V$ in the more suggestive form

$$
Vh = \mathbf{v} \cdot \frac{\partial h}{\partial x} + eE[h] \cdot \frac{\partial f(\mathbf{p})}{\partial \mathbf{p}},
$$
where we define the electric field functional by
\[
E[h](x) = \int dX_3 h(x_3, p_3) E(x - x_3) .
\] (6.28)
The quantity \(E[h]\) is analogous to the self-consistent electric field \(E[f]\), although it does not vanish in a uniform plasma. To solve (6.25) for \(h\), let us take the spatial Fourier transform and the temporal Laplace transform of (6.25),
\[
(p + V) \tilde{h}(k, p) = \tilde{h}(k, p, 0) ,
\] (6.29)
which has the formal solution
\[
\tilde{h}(k, p, p) = (p + V)^{-1} \tilde{h}(k, p, 0) .
\] (6.30)
The tilde over a function is used to denote both the Fourier and Laplace transforms. The transform of relevance should be clear from context (and from the presence of the variable \(x\) vs. \(k\) or \(t\) vs. \(p\)). In other words, we are using a mixed notation in which \(\tilde{h}(k, p, p)\) is the spatial Fourier transform and the temporal Laplace transform of \(h(x, p, t)\), while \(\tilde{h}(k, p, 0)\) is the spatial Fourier transform of \(h(x, p, t = 0)\). The momentum variable “just goes along for the ride,” so we will keep it implicit.

To find an expression for \((p + V)^{-1}\) that we can use in a calculation, let us repeat the steps leading to the formal expression (6.30), except that now we shall employ the explicit form (6.27) for \(V\). Note that we are using the Fourier conventions given by (2.60) and (2.61), or equivalently by (2.26) and (2.27). Clemmow and Dougherty [9] use a convention with the opposite sign of \(k\) and different factors of \(2\pi\), so care must be taken when comparing the results from these notes to Ref. [9]. Note that the spatial integral in (6.28) is a convolution of \(h(x)\) and the Coulomb field \(E(x)\) of a point charge. We can therefore use the convolution theorem when taking the spatial Fourier transform of (6.27), giving
\[
p \tilde{h} + i k \cdot v \tilde{h} + e \tilde{E}[h] \cdot \frac{\partial f}{\partial p} = \tilde{h}(k, p, 0) .
\] (6.31)
As stated above, the spatial Fourier transform \(\tilde{E}[h]\) of the self-induced field \(E[h]\) is performed by applying the convolution theorem, so that (being explicit with the arguments)
\[
\tilde{E}[h](k, p) = \int \frac{d^\nu p_3}{(2\pi \hbar)^\nu} \tilde{h}(k, p_3, p) \tilde{E}(k) ,
\] (6.32)
where \(\tilde{E}(k)\) is the Fourier transform of the static Coulomb field,
\[
\tilde{E}(k) = -i k \tilde{\phi}(k) = i k \frac{e}{k^2} .
\] (6.33)
For notational simplicity, we drop the functional dependence on \( h \) from (6.32), and write \( \tilde{E}(\mathbf{k}, p) \). We can now solve (6.31) for \( \tilde{h} \), giving

\[
(p + V)^{-1} \tilde{h}(\mathbf{k}, p, 0) \equiv \tilde{h}(\mathbf{k}, p, p) = \frac{1}{p + i\mathbf{k} \cdot \mathbf{v}} \left[ \tilde{h}(\mathbf{k}, p, 0) - e \tilde{E}(\mathbf{k}, p) \cdot \frac{\partial f(p)}{\partial p} \right].
\]

(6.34)

As with the point charge in (6.33), the self-consistent electric field \( \tilde{E}(\mathbf{k}, p) \) can be expressed in terms of a self-consistent potential \( \tilde{\phi}(\mathbf{k}, p) \) defined by

\[
\tilde{E}(\mathbf{k}, p) = -i\mathbf{k} \tilde{\phi}(\mathbf{k}, p).
\]

(6.35)

In terms of this self-consistent potential, we have

\[
\tilde{h}(\mathbf{k}, p, p) = \frac{1}{p + i\mathbf{k} \cdot \mathbf{v}} \left[ \tilde{h}(\mathbf{k}, p, 0) + e \tilde{\phi}(\mathbf{k}, p)(i\mathbf{k}) \cdot \frac{\partial f(p)}{\partial p} \right]
\]

(6.36)

\[
\tilde{\phi}(\mathbf{k}, p) = \frac{e}{k^2} \int \frac{d^4p'}{(2\pi\hbar)^4} \frac{1}{p + i\mathbf{k} \cdot \mathbf{v}' \cdot \mathbf{v}} \left[ \tilde{h}(\mathbf{k}, p', 0) + e \tilde{\phi}(\mathbf{k}, p) (i\mathbf{k}) \cdot \frac{\partial f(p')}{\partial p'} \right],
\]

(6.37)

where we have changed integration variables from \( p_3 \) to \( p' \).

Let us now substitute (6.36) for \( \tilde{h} \) into (6.37) for the potential,

\[
\tilde{\phi}(\mathbf{k}, p) = \frac{e}{k^2} \int \frac{d^4p'}{(2\pi\hbar)^4} \frac{1}{p + i\mathbf{k} \cdot \mathbf{v}} \left[ \tilde{h}(\mathbf{k}, p', 0) + e \tilde{\phi}(\mathbf{k}, p) (i\mathbf{k}) \cdot \frac{\partial f(p')}{\partial p'} \right],
\]

(6.38)

where \( \mathbf{v}' = p'/m \). Note that \( \tilde{\phi}(\mathbf{k}, p) \) appears on both sides of this equation, and upon isolating the \( \tilde{\phi}(\mathbf{k}, p) \) term, we find

\[
\left[ 1 - \frac{e^2}{k^2} \int \frac{d^4p'}{(2\pi\hbar)^4} \frac{1}{p + i\mathbf{k} \cdot \mathbf{v}} (i\mathbf{k}) \cdot \frac{\partial f(p')}{\partial p'} \right] \tilde{\phi}(\mathbf{k}, p) = \frac{e}{k^2} \int \frac{d^4p'}{(2\pi\hbar)^4} \frac{\tilde{h}(\mathbf{k}, p', 0)}{p + i\mathbf{k} \cdot \mathbf{v}'},
\]

(6.39)

Solving (6.39) for the self-consistent potential therefore gives

\[
\tilde{\phi}(\mathbf{k}, p) = \frac{e}{\bar{\epsilon}(\mathbf{k}, p) k^2} \int \frac{d^4p'}{(2\pi\hbar)^4} \frac{\tilde{h}(\mathbf{k}, p', 0)}{p + i\mathbf{k} \cdot \mathbf{v}'},
\]

(6.40)

where the “dielectric function” in Laplace space is defined by

\[
\bar{\epsilon}(\mathbf{k}, p) = 1 - \int \frac{d^4p'}{(2\pi\hbar)^4} \frac{e^2}{k^2} \frac{1}{p + i\mathbf{k} \cdot \mathbf{v}} (i\mathbf{k}) \cdot \frac{\partial f(p')}{\partial p'},
\]

(6.41)

with \( p \) lying on the contour \( C \). For future reference, we record the following identities:

\[
\int \frac{d^4p'}{(2\pi\hbar)^4} \frac{e^2}{k^2} \frac{i\mathbf{k} \cdot \partial f(p')/\partial p'}{p + i\mathbf{k} \cdot \mathbf{v}'} = 1 - \bar{\epsilon}(\mathbf{k}, p)
\]

(6.42)

\[
\int \frac{d^4p'}{(2\pi\hbar)^4} \frac{e^2}{k^2} \frac{i\mathbf{k} \cdot \partial f(p')/\partial p'}{p - i\mathbf{k} \cdot \mathbf{v}'} = \bar{\epsilon}(-\mathbf{k}, p) - 1.
\]

(6.43)
We will use these expressions throughout. Now, upon substituting (6.40) back into (6.36), we find

\[(p + V)^{-1} \tilde{h}(k, p, 0) \equiv \tilde{h}(k, p, p)\]

\[
= \frac{1}{p + i\mathbf{k} \cdot \mathbf{v}} \left[ \tilde{h}(k, p, 0) + \frac{e^2}{\epsilon(k, p) k^2} i\mathbf{k} \cdot \frac{\partial f(p)}{\partial p} \int \frac{d^3 p'}{(2\pi \hbar)^3} \frac{\tilde{h}(k, p', 0)}{p + i\mathbf{k} \cdot \mathbf{v}'} \right]. \tag{6.44}
\]

We will generalize this result to the operators \(V_1\) and \(V_2\) shortly. To identify the quantity \(\bar{\epsilon}(k, p)\) physically, we can analytically continue (6.41), allowing \(p\) to lie anywhere in the complex plane. When \(p = -i\omega\), we note that

\[\tilde{\epsilon}(k, -i\omega) = \epsilon(k, \omega) \quad \tag{6.45}\]

Thus, the analytically continued dielectric function in Laplace space is just the ordinary dielectric function in temporal Fourier space. We also note that

\[\epsilon(-k, -\omega) = \epsilon^*(k, \omega), \tag{6.46}\]

and therefore

\[\tilde{\epsilon}(-k, i\omega) = \epsilon(-k, -\omega) = \epsilon^*(k, \omega). \tag{6.47}\]

These complex conjugation properties will be useful in the forthcoming calculation.

**C. The Lenard-Balescu Equation**

We now return to the Lenard-Balescu formalism in \(X_1\)-\(X_2\) space, and to the 2-point correlation \(h(X_1, X_2, t)\). Bogoliubov’s hypothesis means that the time scale of \(h(X_1, X_2, t)\) is much shorter than the time scale of \(f(X, t)\), so we can replace \(h\) by its \(t \to \infty\) limit relative to \(f\). Therefore, we shall assume that \(h(X_1, X_2, t)\) relaxes to its asymptotic value \(h(X_1, X_2, \infty)\), and that the LBE kernel is

\[
L[h] \equiv -\int dX_2 e\mathbf{E}_1^{(2)} \cdot \frac{\partial h(X_1, X_2, \infty)}{\partial p_1} = -\frac{\partial}{\partial p_1} \cdot \mathbf{J} \tag{6.48}
\]

\[
\mathbf{J}(X_1) \equiv \int dX_2 e\mathbf{E}_1^{(2)} h(X_1, X_2, \infty). \tag{6.49}
\]

In the single-particle distribution \(f(p, t)\), the time \(t\) is treated as a parameter. It is noteworthy that the approximation of replacing \(h(t)\) by its asymptotic value of \(h(\infty)\) is where the non-reversibility in time enters the LBE kinetic equation. Because the distribution \(f\) is uniform, the 2-point correlation function \(h(x_1, x_2, , p_1, p_2, \infty)\) reduces to a function of
only a single spatial coordinate, $h(x, p_1, p_2, \infty)$. The Fourier transform of this function is

$$\tilde{h}(k, p_1, p_2, \infty) = \int \frac{d^\nu k}{(2\pi)^\nu} e^{i(x_1-x_2)\cdot k} h(x_1, x_2, p_1, p_2, \infty).$$

(6.50)

In expression (6.49), let us write the inter-molecular as $F^{(2)}_1 = eE(x_1 - x_2)$, where $E(x)$ is the electric field of a point charge at the origin. Recall that the Fourier transforms of $E(x)$ and the corresponding potential $\phi(x)$ are

$$e\tilde{E}(k) = -ik\tilde{\phi}(k)$$

(6.51)

$$\tilde{\phi}(k) = \frac{e^2}{k^2},$$

(6.52)

where I have temporarily changed conventions by including an extra factor of $e$ into the electric potential $\phi$. This is to avoid factors of $e\phi$, as the stray electric charge is cumbersome to track. For a multi-species plasma, we would change $e^2$ in (6.52) to $e_a e_b$. The Lenard-Balescu current (6.49) now becomes

$$J = \int \frac{d^\nu p_2}{(2\pi \hbar)^\nu} \int d^\nu x_2 eE(x_1 - x_2) h(x_1, x_2, p_1, p_2, \infty)$$

$$= \int \frac{d^\nu p_2}{(2\pi \hbar)^\nu} \int d^\nu x eE(x) h(x, p_1, p_2, \infty),$$

(6.54)

where we have made the change of variables $x = x_1 - x_2$ in (6.54), thereby illustrating that $J$ is constant in space (independent of $x_1$). We can express the $x$-integration in (6.54) as

$$\int d^\nu x eE(x) h(x) = \int d^\nu x \int \frac{d^\nu k_1}{(2\pi)^\nu} e^{ik_1 \cdot x} \tilde{E}(k_1) \int \frac{d^\nu k_2}{(2\pi)^\nu} e^{ik_2 \cdot x} \tilde{h}(k_2)$$

$$= \int \frac{d^\nu k_2}{(2\pi)^\nu} \int \frac{d^\nu k_1}{(2\pi)^\nu} (2\pi)^\nu \delta^{(\nu)}(k_2 + k_1) (-ik_1)\tilde{\phi}(k_1) \tilde{h}(k_2)$$

$$= \int \frac{d^\nu k_2}{(2\pi)^\nu} ik_2 \tilde{\phi}(-k_2) \tilde{h}(k_2),$$

(6.55)

and since $\tilde{\phi}(k)$ is even in $k$, the current (6.54) becomes

$$J(p_1) = \int \frac{d^\nu k}{(2\pi)^\nu} \frac{d^\nu p_2}{(2\pi \hbar)^\nu} ik\tilde{\phi}(k) \tilde{h}(k, p_1, p_2, \infty)$$

$$= -\int \frac{d^\nu k}{(2\pi)^\nu} k\tilde{\phi}(k) \text{Im} I(k, p_1, p_2, \infty),$$

(6.56)

where

$$I(k, p_1) \equiv \int \frac{d^\nu p_2}{(2\pi \hbar)^\nu} \tilde{h}(k, p_1, p_2, \infty).$$

(6.60)
In (6.59) we have used the fact that $J$ must be real, although we will continue to employ the form (6.58) until the end of the calculation. Note that we only need to find a momentum integral of the correlation function, $I(k, p_1)$, and not the correlation function itself. This is quite fortunate, since the integral (6.60) turns out to simplify considerably relative to the full perturbation $\tilde{h}(k, p_1, p_2, \infty)$. Using expression (6.24) relating the perturbation to the source term, the spatial Fourier transform of the correlation function becomes

$$\tilde{h}(k, p_1, p_2, \infty) = \int d^3 xe^{-ik \cdot x} h(x, p_1, p_2, \infty)$$  \hspace{1cm} (6.61)$$

$$= \lim_{p \to 0^+} \frac{1}{(2\pi i)^2} \int_{C_1} dp_1 \int_{C_2} dp_2 \frac{1}{(p - p_1 - p_2)(p_1 + V_1)(p_2 + V_2)} S(k, p_1, p_2),$$  \hspace{1cm} (6.62)$$

where the source term is

$$S(k, p_1, p_2) = \tilde{\phi}(k) i k \cdot \left[ \frac{\partial f}{\partial p_1} f(p_2) - \frac{\partial f}{\partial p_2} f(p_1) \right].$$  \hspace{1cm} (6.63)$$

For future reference, I have labeled the two terms of $S$ by the names termA and termB, and (6.60) becomes

$$I(k, p_1)$$

$$= -\frac{1}{(2\pi)^2} \lim_{p \to 0^+} \int \frac{d^3p_2}{(2\pi\hbar)^3} \int_{C_1} dp_1 \int_{C_2} dp_2 \frac{1}{(p - p_1 - p_2)(p_1 + V_1)(p_2 + V_2)} S(k, p_1, p_2).$$  \hspace{1cm} (6.64)$$

We must now calculate $(p_1 + V_1)^{-1}$ and $(p_2 + V_2)^{-1}$ on $S$. Expression (6.44) is the solution to the inverse problem in the simpler context of a single space-variable, and with this result in hand, let us return to the full equation involving $V_1$ and $V_2$. Since $x_1$ and $x_2$ appear with opposite signs in (6.50), the value of $k$ in $V_2$ must be of the opposite sign as the corresponding value in $V_1$, and by referring back to (6.44) we can express

$$(p_1 + V_1)^{-1} S(p_1, p_2) = \frac{1}{p_1 + ik \cdot v_1} \left[ S(p_1, p_2) + \frac{\tilde{\phi}(k)}{\epsilon(k, p_1)} ik \cdot \int \frac{d^3p_1'}{(2\pi\hbar)^3} \frac{S(p_1', p_2)}{p_1 + ik \cdot v_1} \right]$$  \hspace{1cm} (6.65)$$

$$(p_2 + V_2)^{-1} S(p_1, p_2) = \frac{1}{p_2 - ik \cdot v_2} \left[ S(p_1, p_2) - \frac{\tilde{\phi}(k)}{\epsilon(-k, p_2)} ik \cdot \int \frac{d^3p_2'}{(2\pi\hbar)^3} \frac{S(p_1, p_2')}{p_2 - ik \cdot v_2} \right].$$  \hspace{1cm} (6.66)$$
Returning to (6.64), we find
\[
\int \frac{d^\nu p_2}{(2\pi \hbar)^\nu} (p_2 + V_2)^{-1} S(p_1, p_2) = \int \frac{d^\nu p_2}{(2\pi \hbar)^\nu} \frac{S(p_1, p_2)}{p_2 - i\mathbf{k} \cdot \mathbf{v}_2} \tag{6.67}
\]
\[
= \frac{1}{\epsilon(-\mathbf{k}, p_2)} \int \frac{d^\nu p_2}{(2\pi \hbar)^\nu} \frac{\tilde{\phi}(\mathbf{k}) \cdot \partial f(p_2)}{p_2 - i\mathbf{k} \cdot \mathbf{v}_2} \int \frac{d^\nu p_2'}{(2\pi \hbar)^\nu} \frac{S(p_1', p_2')}{p_2' - i\mathbf{k} \cdot \mathbf{v}_2'} \tag{6.68}
\]
Thus, upon changing variables in the last integral from $p_1'$ to $p_2$, we can write
\[
\int \frac{d^\nu p_2}{(2\pi \hbar)^\nu} (p_2 + V_2)^{-1} S(p_1, p_2) = \frac{1}{\epsilon(-\mathbf{k}, p_2)} \int \frac{d^\nu p_2}{(2\pi \hbar)^\nu} \frac{S(p_1, p_2)}{p_2 - i\mathbf{k} \cdot \mathbf{v}_2}, \tag{6.69}
\]
and (6.64) becomes
\[
I(\mathbf{k}, p_1) = -\frac{1}{(2\pi i)^2} \lim_{p \to 0^+} \int \frac{d^\nu p_2}{(2\pi \hbar)^\nu} \int \frac{dp_1 \int \frac{dp_2}{p_2 + p_1 - p} \frac{1}{\epsilon(-\mathbf{k}, p_2)} \frac{1}{p_2 - i\mathbf{k} \cdot \mathbf{v}_2}}{p_2 + p_1 - p} \frac{\tilde{\phi}(\mathbf{k}) \cdot \partial f(p_2)}{p_2 - i\mathbf{k} \cdot \mathbf{v}_2} \times (p_1 + V_1)^{-1} S(\mathbf{k}, p_1, p_2). \tag{6.70}
\]
We now perform the integral over $p_2$. There are poles at $p_2 = i\mathbf{k} \cdot \mathbf{v}_2$ and $p_2 = p - p_1$, and the zeros of $\epsilon(\mathbf{k}, p_2)$. Recall that the contour $C_2$ runs parallel to the imaginary axis with all singularities of $(p_2 + V_2)^{-1}$ lying to the left of $C_2$. As illustrated in Fig. 5, we can complete the contour $C_2$ to include a large semicircle at infinity (as the integrand vanishes there). The contour now encloses the pole $p_2 = p - p_1$ and is clock-wise oriented, and so the $p_2$ integral may be performed using the residue theorem,
\[
I(\mathbf{k}, p_1) = \frac{1}{2\pi i} \lim_{p \to 0^+} \int \frac{d^\nu p_2}{(2\pi \hbar)^\nu} \int \frac{dp_1}{\epsilon(-\mathbf{k}, p - p_1)} \frac{1}{p - p_1 - i\mathbf{k} \cdot \mathbf{v}_2} \frac{1}{p + i\mathbf{k} \cdot \mathbf{v}_2} (p_1 + V_1)^{-1} S(\mathbf{k}, p_1, p_2). \tag{6.71}
\]
Using (6.65) to express $(p + V_1)^{-1} S$, we can now write (6.71) in the form
\[
I(\mathbf{k}, p_1) = \frac{1}{2\pi i} \lim_{p \to 0^+} \int \frac{d^\nu p_2}{(2\pi \hbar)^\nu} \int \frac{dp_1}{\epsilon(-\mathbf{k}, p - p_1)} \frac{1}{p - p_1 - i\mathbf{k} \cdot \mathbf{v}_2} \frac{1}{p_1 + i\mathbf{k} \cdot \mathbf{v}_1} \frac{\tilde{\phi}(\mathbf{k}) \cdot \partial f(p_1)}{p_1 + i\mathbf{k} \cdot \mathbf{v}_1} \tag{6.72}
\]
\[
\left[ S(p_1, p_2) + \frac{\tilde{\phi}(\mathbf{k}) \cdot \partial f(p_1)}{\epsilon(\mathbf{k}, p_1)} \frac{1}{\epsilon(\mathbf{k}, p_1)} \frac{1}{p_1 + i\mathbf{k} \cdot \mathbf{v}_1} \times \frac{\tilde{\phi}(\mathbf{k}) \cdot \partial f(p_1)}{\epsilon(\mathbf{k}, p_1)} \frac{1}{\epsilon(\mathbf{k}, p_1)} \right] \quad \text{term1}
\]
\[
\frac{\partial f(p_1)}{\epsilon(\mathbf{k}, p_1)} \frac{1}{\epsilon(\mathbf{k}, p_1)} \frac{1}{p_1 + i\mathbf{k} \cdot \mathbf{v}_1} \times \frac{\partial f(p_1)}{\epsilon(\mathbf{k}, p_1)} \frac{1}{\epsilon(\mathbf{k}, p_1)} \quad \text{term2}
\]
FIG. 5: Contour $C_2$ in the $p_2$-plane. The contour can be closed in the right half-plane, oriented in the clockwise direction and enclosing the simple pole $p_2 = p - p_1$.

where I have labeled the two terms of (6.72) by term1 and term2. The source $S$ given by (6.63) contains two terms, termA and termB, and we must therefore examine a total of four terms:

$$I(k, p_1) = I_{1A} + I_{1B} + I_{2A} + I_{2B},$$

where

$$I_{1A} = -\frac{1}{2\pi i} \lim_{p \to 0^+} \int_{C_1} dp_1 \int \frac{dp_2}{(2\pi \hbar)^{\nu}} \frac{1}{\epsilon(-k, p - p_1)} \frac{1}{p - p_1 - i k \cdot v_2} \frac{1}{p_1 + i k \cdot v_1}$$

$$\left[ \frac{\phi(k) i k \cdot \partial f(p_2)}{\partial p_2} f(p_1) \right],$$

(6.74)

$$I_{1B} = \frac{1}{2\pi i} \lim_{p \to 0^+} \int_{C_1} dp_1 \int \frac{dp_2}{(2\pi \hbar)^{\nu}} \frac{1}{\epsilon(-k, p - p_1)} \frac{1}{p - p_1 - i k \cdot v_2} \frac{1}{p_1 + i k \cdot v_1}$$

$$\left[ \frac{\tilde{\phi}(k) i k \cdot \partial f(p_1)}{\partial p_1} f(p_2) \right],$$

(6.75)

$$I_{2A} = -\frac{1}{2\pi i} \lim_{p \to 0^+} \int_{C_1} dp_1 \int \frac{dp_2}{(2\pi \hbar)^{\nu}} \frac{1}{\epsilon(-k, p - p_1)} \frac{1}{p - p_1 - i k \cdot v_2} \frac{1}{p_1 + i k \cdot v_1}$$

$$\left[ \frac{\phi(k) i k \cdot \partial f}{\partial p_1} \int \frac{dp_1'}{(2\pi \hbar)^{\nu}} \frac{\phi(k) i k \cdot \partial f}{\partial p_2} f(p_1') \right],$$

(6.76)

and

$$I_{2B} = \frac{1}{2\pi i} \lim_{p \to 0^+} \int_{C_1} dp_1 \int \frac{dp_2}{(2\pi \hbar)^{\nu}} \frac{1}{\epsilon(-k, p - p_1)} \frac{1}{p - p_1 - i k \cdot v_2} \frac{1}{p_1 + i k \cdot v_1}$$

$$\left[ \frac{\tilde{\phi}(k) i k \cdot \partial f}{\partial p_1} \int \frac{dp_1'}{(2\pi \hbar)^{\nu}} \frac{\tilde{\phi}(k) i k \cdot \partial f}{\partial p_2} f(p_1') \right].$$

(6.77)
We combine the $A$-terms together and the $B$-terms together. Using (6.43) in (6.74), we can perform the $p_2$-integral to give

\[
I_{1A} = -\frac{1}{2\pi i} \lim_{p \to +0} \int_{C_1} dp_1 \frac{f(p_1)}{p_1 + ik \cdot \nu_1} \frac{1}{\bar{\epsilon}(-k, p - p_1)} \times \left[ \bar{\epsilon}(-k, p - p_1) - 1 \right]
\]

\[
= \frac{1}{2\pi i} \lim_{p \to +0} \int_{C_1} dp_1 \frac{1}{p_1 + i k \cdot \nu_1} \left[ \frac{1}{\bar{\epsilon}(-k, p - p_1)} - 1 \right] f(p_1),
\]  

(6.78)

Similarly, we use (6.43) in (6.76) to perform the $p_2$-integral, and after rearranging terms and changing the remaining integration variable from $p_1'$ to $p_2$, we find

\[
I_{2A} = \frac{1}{2\pi i} \lim_{p \to +0} \int_{C_1} dp_1 \frac{1}{p_1 + i k \cdot \nu_1} \left[ \frac{1}{\bar{\epsilon}(-k, p - p_1)} - 1 \right]
\]

\[
\frac{1}{\bar{\epsilon}(k, p_1)} \left[ \phi(k) i k \cdot \frac{\partial f}{\partial p_1} \int \frac{d^\nu p_2}{(2\pi \hbar)^\nu} \frac{f(p_2)}{p_1 + i k \cdot \nu_2} \right].
\]  

(6.79)

Upon reorganizing the terms in (6.75) we can write

\[
I_{1B} = \frac{1}{2\pi i} \lim_{p \to +0} \int_{C_1} dp_1 \frac{1}{p_1 + i k \cdot \nu_1} \frac{1}{\bar{\epsilon}(-k, p - p_1)} \left[ \frac{1}{\bar{\epsilon}(k, p_1)} - 1 \right]
\]

\[
\left[ \phi(k) i k \cdot \frac{\partial f}{\partial p_1} \int \frac{d^\nu p_2}{(2\pi \hbar)^\nu} \frac{f(p_2)}{p_1 - p_1 - i k \cdot \nu_2} \right].
\]  

(6.80)

Finally, upon using (6.42) to perform the $p_1'$-integral, expression (6.77) becomes

\[
I_{2B} = \frac{1}{2\pi i} \lim_{p \to +0} \int_{C_1} dp_1 \frac{1}{p_1 + i k \cdot \nu_1} \frac{1}{\bar{\epsilon}(-k, p - p_1)} \left[ \frac{1}{\bar{\epsilon}(k, p_1)} - 1 \right]
\]

\[
\left[ \phi(k) i k \cdot \frac{\partial f}{\partial p_1} \int \frac{d^\nu p_2}{(2\pi \hbar)^\nu} \frac{f(p_2)}{p_1 - p_1 - i k \cdot \nu_2} \right].
\]  

(6.81)

Note that the $k \cdot \nu_2$ term in the $p_2$ integrals of (6.79) and (6.81) have opposite signs, a fact that will be critical as the calculation proceeds. Note that (6.78) and (6.79) give

\[
I_{1A} + I_{2A} = \frac{1}{2\pi i} \lim_{p \to +0} \int_{C_1} dp_1 \frac{1}{p_1 + i k \cdot \nu_1} \left[ \frac{1}{\bar{\epsilon}(-k, p - p_1)} - 1 \right] \left[ f(p_1) + \phi(k) i k \cdot \frac{\partial f}{\partial p_1} \int \frac{d^\nu p_2}{(2\pi \hbar)^\nu} \frac{f(p_2)}{p_1 + i k \cdot \nu_2} \right],
\]  

(6.82)

while the final two terms become

\[
I_{1B} + I_{2B} = \frac{1}{2\pi i} \lim_{p \to +0} \int_{C_1} dp_1 \frac{1}{p_1 + i k \cdot \nu_1} \frac{1}{\bar{\epsilon}(-k, p - p_1) \bar{\epsilon}(k, p_1)}
\]

\[
\left[ \phi(k) i k \cdot \frac{\partial f}{\partial p_1} \int \frac{d^\nu p_2}{(2\pi \hbar)^\nu} \frac{f(p_2)}{p_1 - i k \cdot \nu_2} \right].
\]  

(6.83)
FIG. 6: The Laplace contour $C_1$ and associated poles $p_1 = -i k \cdot v_1$, $p_1 = -i k \cdot v_2$, and $p_1 = p - i k \cdot v_2$, along with the singularities arising from the zeros of $\bar{\epsilon}(k, p_1)$ and $\bar{\epsilon}(-k, p - p_1)$. Not every pole or singularity in the Figure is associated with every term in (6.84). Since $\text{Re}(p - p_1) > 0$, for real $p > 0$ we must have $0 < \eta < p$. This allows us to take the limit $\eta \to 0^+$ before taking $p \to 0^+$ (the order of limits cannot be reversed).

Upon adding (6.82) and (6.83) we can write $I(k, p_1)$ in the form

$$I = \lim_{p \to 0^+} \frac{1}{2\pi i} \int_{C_1} dp_1 \frac{1}{p_1 + i k \cdot v_1} \left\{ \frac{\tilde{\phi}(k) i k \cdot \partial f / \partial p_1}{\bar{\epsilon}(-k, p - p_1) \bar{\epsilon}(k, p_1)} \int \frac{d^\nu p_2}{(2\pi)^\nu} \frac{f(p_2)}{p_2 - p_1 - i k \cdot v_2} + \right.$$  

$$\left. \left[ \frac{1}{\bar{\epsilon}(-k, p - p_1)} - \frac{1}{(b)} \right] \left[ f(p_1) \right]_{(c)} + \frac{\tilde{\phi}(k)(i k) \cdot \partial f / \partial p_1}{\bar{\epsilon}(k, p_1)} \int \frac{d^\nu p_2}{(2\pi)^\nu} \frac{f(p_2)}{p_1 + i k \cdot v_2} \right\} ,$$  

(6.84)

where I have labeled the terms as in Ref. [9]. From (6.59), the Lenard-Balescu current can be expressed as

$$J(p_1) = -\int \frac{d^\nu k}{(2\pi)^\nu} k\bar{\phi}(k) \left[ \text{Im} I_{[a+b]xc} + \text{Im} I_{b\times d} + \text{Im} I_{[a \times d] + e} \right] ,$$  

(6.85)

where $I_{[a+b]xc}$ is the result of the (a)+(b) term times the (c) term in (6.84), $I_{b\times d}$ is (b) times (d), and $I_{[a \times d] + e}$ is (a) times (d) plus term (e).

The contour $C_1$ lies parallel to the imaginary axis in complex $p_1$-plane such that the singularities from $(p_1 + V_1)^{-1}$ lie to the left of $C_1$. There are simple poles at $p_1 = -i k \cdot v_1$, $p_1 = -i k \cdot v_2$, $p_1 = p - i k \cdot v_2$. The latter pole, however, is not associated with $(p + V_1)^{-1}$, as it is arose from the term $1/(p - p_1 - p_2)$ in (6.64). There are also singularities arising from the zeros of $\bar{\epsilon}(k, p_1)$ (the singularities and corresponding branch cut lie in the left half-plane for plasma stability) and the zeros of $\bar{\epsilon}(-k, p - p_1)$ (whose singularities and branch cut lie in the right half-plane). As shown in Fig. 6, we can offset $C_1$ by a small amount $\eta$ in the
real direction, with $0 < \eta < \rho$. We must therefore take the $\eta \to 0^+$ limit before taking $p \to 0^+$. Not every term in (6.85) will involve every singularity, so care must be taken when evaluating (6.84). Reference [9] emphasizes that the guiding principle in closing the contour $C_1$ is to avoid enclosing a singularity arising from the zeros of the dielectric function $\varepsilon$. In this way, we avoid crossing a branch-cut when closing the contour at infinity. The first term we consider is

$$I_{[a+b]c} = \frac{1}{2\pi i} \lim_{p \to 0^+} \int_{C_1} dp_1 \frac{1}{p_1 + i\mathbf{k} \cdot \mathbf{v}_1} \left[ \frac{1}{\varepsilon(-\mathbf{k}, p - p_1)} - 1 \right] f(p_1).$$  

(6.86)

There is a simple pole at $p_1 = -i\mathbf{k} \cdot \mathbf{v}_1$, so we complete the $C_1$ contour by a large semi-circle in the left-hand $p_1$-plane to form a closed contour $C_L$, as illustrated in the left panel of Fig. 7. This closed contour has a counter-clockwise orientation and encircles the pole $p_1 = -i\mathbf{k} \cdot \mathbf{v}_1$. The path $C_L$ avoids the singularity in the right half-plan arising from the zeros of the dielectric function $\varepsilon(-\mathbf{k}, p - p_1)$, and there are no such singularities in the left-half plane from $\varepsilon(\mathbf{k}, p_1)$, so upon applying the residue theorem we find

$$I_{[a+b]c} = \frac{1}{2\pi i} \int_{C_L} dp_1 \frac{1}{p_1 + i\mathbf{k} \cdot \mathbf{v}_1} \left[ \frac{1}{\varepsilon(-\mathbf{k}, p - p_1)} - 1 \right] f(p_1)$$  

(6.87)

$$\left[ \frac{1}{\varepsilon(-\mathbf{k}, i\mathbf{k} \cdot \mathbf{v}_1)} - 1 \right] f(p_1) = \left[ \frac{1}{\varepsilon(-\mathbf{k}, -\mathbf{k} \cdot \mathbf{v}_1)} - 1 \right] f(p_1).$$  

(6.88)

In the last equality we have used (6.45) to express the result in terms of the ordinary dielectric function $\epsilon(\mathbf{k}, \omega)$ in temporal Fourier space. We only need the imaginary component,

$$\text{Im} I_{[a+b]c} = \frac{\text{Im} \epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v}_1)}{\epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v}_1)^2} f(p_1),$$  

(6.89)

where we have used the fact that $\epsilon(-\mathbf{k}, -\omega) = \epsilon^*(\mathbf{k}, \omega)$. Recall that the sing-component dielectric function takes the form

$$\epsilon(\mathbf{k}, \omega) = 1 + \int \frac{d^3p_2}{(2\pi \hbar)^3} \frac{\tilde{\phi}(\mathbf{k}) \mathbf{k} \cdot \partial f / \partial \mathbf{p}_2}{\omega - \mathbf{k} \cdot \mathbf{v}_2 + \imath \eta}.$$  

(6.90)

$$I(\mathbf{k}, p_1) = \frac{1}{2\pi i} \lim_{p \to 0^+} \int \frac{d^3p_2}{(2\pi \hbar)^3} \int_{C_1} dp_1 \frac{1}{\epsilon(-\mathbf{k}, p - p_1)} \frac{1}{p - p_1 - i\mathbf{k} \cdot \mathbf{v}_2} (p_1 + V_1)^{-1} S(\mathbf{k}, p_1, p_2).$$  

(6.91)

To find the imaginary part, we use the functional relation

$$\frac{1}{\omega - \mathbf{k} \cdot \mathbf{v}_2 + \imath \eta} = P \frac{1}{\omega - \mathbf{k} \cdot \mathbf{v}_2} - \imath \pi \delta(\omega - \mathbf{k} \cdot \mathbf{v}_2),$$  

(6.92)

where the first term in (6.92) gives the principle-part integral. We can therefore express

$$\text{Im} \epsilon(\mathbf{k}, \omega) = -\pi \int \frac{d^3p_2}{(2\pi \hbar)^3} \delta(\omega - \mathbf{k} \cdot \mathbf{v}_2) \tilde{\phi}(\mathbf{k}) \mathbf{k} \cdot \frac{\partial f}{\partial \mathbf{p}_2},$$  

(6.93)
FIG. 7: Closed contours $C_L$ and $C_R$ for the integrals $I_{[a+b]xC}$ of Eq. (6.86) and $I_{b\times d}$ of Eq. (6.97), respectively.

and hence

$$\text{Im} I_{[a+b]xC} = -\pi \int \frac{d^\nu p_2}{(2\pi \hbar)^\nu} \frac{\tilde{\phi}(k) \delta(k \cdot v_1 - k \cdot v_2)}{|\epsilon(k, k \cdot v_1)|^2} \frac{k \cdot \partial f(p_2)}{\partial p_2} f(p_1). \quad (6.94)$$

This gives the corresponding contribution to the Lenard-Balescu current

$$J_{[a+b]xC} = - \int \frac{d^\nu k}{(2\pi)^\nu} k \tilde{\phi}(k) \text{Im} I_{[a+b]C(k, p_1, p_2)} \quad (6.95)$$

$$= \pi \int \frac{d^\nu k}{(2\pi)^\nu} \frac{d^\nu p_2}{(2\pi \hbar)^\nu} \frac{[\tilde{\phi}(k)]^2 k \delta(k \cdot v_1 - k \cdot v_2)}{|\epsilon(k, k \cdot v_1)|^2} \frac{k \cdot \partial f(p_2)}{\partial p_2} f(p_1). \quad (6.96)$$

We now evaluate $b \times d$ contribution. Interestingly, this term vanishes upon closing the contour $C_1$ to the right to form a closed contour $C_R$ lying in the right half-plane,

$$I_{b \times d} \propto \int_{C_R} dp_1 \frac{1}{p_1 + ik \cdot v_1} \frac{1}{p_1 + ik \cdot v_2} \frac{1}{\tilde{\epsilon}(k, p_1)} = 0. \quad (6.97)$$

This is because the contour $C_R$ does not enclose the simple poles $p_1 = -ik \cdot v_1$ and $p_1 = -ik \cdot v_2$ on the imaginary axis, nor the zeros of $\tilde{\epsilon}(k, p_1)$ in the left half-plane. The residue theorem therefore gives a vanishing integral. The final term involves $[a \times d] + e$. We can in fact take the limit $p \to 0^+$ inside $\tilde{\epsilon}(-k, p - p_1)$ to give $\tilde{\epsilon}(-k, -p_1) \tilde{\epsilon}(k, p_1) = |\tilde{\epsilon}(k, p_1)|^2$, and therefore

$$I_{[a \times d] + e} = \frac{1}{2\pi i} \lim_{p \to 0^+} \int \frac{d^\nu p_2}{(2\pi \hbar)^\nu} f(p_2) \tilde{\phi}(k) ik \cdot \frac{\partial f}{\partial p_1} \int_{C_1} dp_1 \frac{1}{p_1 + ik \cdot v_1} \times \frac{1}{|\tilde{\epsilon}(k, p_1)|^2} \left[ \frac{1}{p_1 + ik \cdot v_2} + \frac{1}{p - p_1 - ik \cdot v_2 - p} \right] \quad (6.98)$$

$$= \lim_{p \to 0^+} \int \frac{d^\nu p_2}{(2\pi \hbar)^\nu} \tilde{\phi}(k) k \cdot \frac{\partial f}{\partial p_1} f(p_2) I_C(k, p_1, p_2), \quad (6.99)$$
where we define the contour integral

$$I_c = \frac{1}{2\pi} \int_{C_1} dp_1 \frac{1}{\bar{\epsilon}(k, p_1)^2} \frac{1}{p_1 + i k \cdot v_1} \left[ \frac{1}{p_1 + i k \cdot v_2} - \frac{1}{p_1 + i k \cdot v_2 - p} \right].$$ \hfill (6.100)

All factors of $i$ from and all signs in (6.98) have been placed in the contour integral $I_c$ of (6.100). We can parameterize points $p_1 \in C_1$ by $p_1 = -i \omega + \eta$ for arbitrary real $\omega$ and fixed real $\eta$, with $0 < \eta < p$. We must therefore take the $\eta \rightarrow 0^+$ limit before the $p \rightarrow 0^+$ limit. The contour integral over $C_1$ in (6.98) can now be expressed as an integral over real $\omega$,

$$I_c = -\frac{i}{2\pi} \int_{-\infty}^{\infty} d\omega \left[ \frac{1}{\bar{\epsilon}(k, \omega)^2} \frac{1}{\omega - k \cdot v_2 + i \eta} \right] \left[ \frac{1}{\omega - k \cdot v_2 - ip} \right] \left[ \frac{1}{\omega - k \cdot v_1 + i \eta} - \frac{1}{\omega - k \cdot v_1 - i \eta} \right],$$ \hfill (6.101)

where we taken the $\eta \rightarrow 0^+$ limit inside the dielectric function, and then used (6.45) to set $\bar{\epsilon}(k, -i \omega) = \epsilon(k, \omega)$. Furthermore, since the $p$-limit is taken at the end of the calculation, in performing the $\eta$-limit we have $p \neq 0$. Therefore, in the last term of (6.101), we have taken $\eta \rightarrow 0^+$, leaving the small imaginary piece $-ip$ in the denominator. Since $\eta$ and $p$ are both infinitesimal, we have

$$\frac{1}{\omega - k \cdot v_2 + i \eta} - \frac{1}{\omega - k \cdot v_2 - ip} = -2\pi i \delta(\omega - k \cdot v_2),$$ \hfill (6.102)

where the principles parts cancel. Similarly, the second term in (6.101) becomes

$$\frac{1}{\omega - k \cdot v_1 + i \eta} = P \frac{1}{\omega - k \cdot v_1} - \pi i \delta(\omega - k \cdot v_1).$$ \hfill (6.103)

The principle part is real and does not contribute to the imaginary piece of $I_{[\omega\delta]^{+e}}$ (it also integrates to zero when performing the $k$-integration), and therefore

$$I_c = \pi i \int_{-\infty}^{\infty} d\omega \frac{1}{\epsilon(k, \omega)^2} \delta(\omega - k \cdot v_1) \delta(\omega - k \cdot v_2) = \pi \frac{\delta(k \cdot v_1 - k \cdot v_2)}{[\epsilon(k \cdot v_1)^2].}$$ \hfill (6.104)

We therefore arrive at

$$\text{Im} I_{[\omega\delta]^{+e}} = \pi \int \frac{d^\nu p_2}{(2\pi\hbar)^\nu} \tilde{\phi}(k) k \cdot \frac{\partial f}{\partial p_1} f(p_2) \frac{\delta(k \cdot v_1 - k \cdot v_2)}{\epsilon(k \cdot v_1)^2},$$ \hfill (6.105)

which gives a contribution to the current

$$J_{[\omega\delta]^{+e}} = -\int \frac{d^\nu k}{(2\pi)^\nu} k \tilde{\phi}(k) \text{Im} I_{[\omega\delta]^{+e}}(k, p_1, p_2)$$

$$= -\pi \int \frac{d^\nu k}{(2\pi)^\nu} \frac{d^\nu p_2}{(2\pi\hbar)^\nu} \frac{[\tilde{\phi}(k)]^2 k \delta(k \cdot v_1 - k \cdot v_2)}{\epsilon(k, k \cdot v_1)^2} k \cdot \frac{\partial f(p_1)}{\partial p_1} f(p_2).$$ \hfill (6.106)

Upon adding (6.96) and (6.107) we find the total Lenard-Balescu current,

$$J = \pi \int \frac{d^\nu k}{(2\pi)^\nu} \frac{d^\nu p_2}{(2\pi\hbar)^\nu} \frac{[\tilde{\phi}(k)]^2 k \delta(k \cdot v_2 - k \cdot v_1)}{(\epsilon^2/k^2)^2} \left[ k \cdot \frac{\partial}{\partial p_2} - k \cdot \frac{\partial}{\partial p_1} \right] f(p_1)f(p_2),$$ \hfill (6.107)

and the proof is complete for a single-component plasma.
D. Generalization to a Multi-species Plasma

It is easy to generalize the previous result to a multi-species plasma. The variables \( X_i \) and the distribution functions must contain species indices, e.g. \( f_1^{(a)}(X_a, t) \) and \( f_2^{(ab)}(X_a, X_b, t) \). The latter gives the joint probability of finding species \( a \) at \( X_a \) and species \( b \) at \( X_b \). The first order correction becomes

\[
f_2^{(ab)}(X_a, X_b, t) = f_a(X_a, t)f_b(X_b, t) + h_{ab}(X_a, X_b, t) .
\] (6.109)

The Lenard-Balescu kernel becomes

\[
\sum_b L_{ab}[f] = -\sum_b \frac{\partial}{\partial p_a} \cdot J_{ab}[f](p_a) ,
\] (6.110)

where

\[
J_{ab}[f](p_a) = \pi \int \frac{d^6 k}{(2\pi)^{\nu}} \frac{d^6 p_b}{(2\pi \hbar)^{\nu}} \left( \frac{e_a e_b}{k^2} \right)^2 k \frac{\delta[k \cdot (v_b - v_a)]}{\epsilon(k, k \cdot v_a)^2} \left[ k \cdot \frac{\partial}{\partial p_b} - k \cdot \frac{\partial}{\partial p_a} \right] f_a(p_a)f_b(p_b) ,
\] (6.111)

and \( \epsilon(k, \omega) \) is the multi-component dielectric function.
VII. CONCLUSIONS

Calculating the rate of Coulomb energy exchange in a plasma is notoriously difficult, even for the case of a fully ionized weakly coupled plasma. Two examples of experimental relevance are the charged particle stopping power and the temperature equilibration rate between electrons and ions in a non-equilibrium plasma. Naive calculations of these processes suffer from logarithmic divergences at both long- and short-distance scales, and we must therefore resort to more sophisticated methods of calculation. Corresponding to these divergences are two broad classes of kinetic equations, applicable in complementary regimes, represented by the Boltzmann equation (BE) and the Lenard-Balescu equation (LBE). The BE describes the short-distance effects of 2-body scattering, including large angle scattering, while the LBE models 2-point long-distance correlations. It is well known that the BE suffers a long-distance logarithmic divergence for Coulomb scattering (in three spatial dimensions), confirming that it is indeed missing long-distance physics (correlations are being ignored). Conversely, the LBE suffers from a short-distance logarithmic divergence for Coulomb interactions (in three dimensions), another indication that relevant physics is being overlooked (the short-distance scattering physics).

The fact that the BE and the LBE are relevant in complementary regimes allows us to capitalize on the lessons physicists have learned from quantum field theory, a formalism developed by particle physicists for understanding the fundamental interactions of nature. In quantum field theory, an array of divergences are encountered, from logarithmic to quadratic and higher, and the so called renormalization program was developed to form meaningful and finite predictions from these divergent results. The first ingredient is to temporarily regularize the theory by rendering the integrals finite. At the end of the calculation, the regularization will be removed, but in the interim, the finite expressions can be algebraically manipulated in a meaningful fashion. After regularization has been performed, one then renormalizes the theory by reinterpreting physical properties like the electric charge and mass in such a way as to give finite predictions as the regularization scheme is removed. There are many regularization schemes in use, each with their own strengths and weaknesses. The simplest one is to choose arbitrary large- and small-distance cutoffs in the integrals, such as the Debye wavelength and the classical distance of closest approach for the electrons or ions in the plasma. This is the regularization scheme first adopted by Landau and Spitzer, and it produces a scaling factor called the Coulomb logarithm, which is defined to be the natural logarithm of the ratio of the large- to small-distance scales. Much effort has been devoted to determining the precise value of the Coulomb logarithm. However, such a crude regularization method is inherently uncertain in determining the exact value of the Coulomb logarithm, and this exercise is doomed to failure from the start. For example, one could just as correctly take twice the Debye length as the long distance cutoff, thereby leaving the constant inside the logarithm undetermined by this regularization method. It is interesting
to note that if the divergence had been higher order rather than logarithmic, this crude
cutoff method would not have been acceptable to plasma physicists. It is only because the
divergence in Coulomb exchange processes is logarithmic that one can get away with such a
naive regularization scheme for so long.

The most pertinent feature of relativistic quantum field theory is that it is a many-body
theory. The non-relativistic limit of these theories provides a rigorous treatment of plasma
physics from which the framework of a non-relativistic many-body field theory [14, 15]. One
of the subtleties of the renormalization program is that the regularization scheme often breaks
the symmetries of the system, thereby changing the structure of the theory. For example,
the cut-off method breaks Lorentz invariance, which is essential for electrodynamics. While
the symmetries must return when the regularization is removed, the system becomes more
complex and when its symmetries are broken, and the restoration of the symmetries as the
regularization is removed can often be nontrivial. Indeed, some symmetries remain broken.
For example, gauge symmetry and particle number conservation cannot both be preserved in
the standard model of particle physics. As gauge invariance is essential for defining the the-
ory, it turns out that matter is not stable and decays by so-called nonperturbative sphaleron
processes (albeit the proton is quite long lived, with a decay rate many times the age of
the universe). Since a plasma is a many-body system, it is not surprising that we encounter
divergences similar to those in quantum field theory. Furthermore, since the renormalization
program makes experimental predictions, we must take it seriously, and it is not surprising
that techniques developed in field theory are applicable to plasma theory. Regularization
methods are often chosen in such a way as to preserve as many symmetries as possible. The
method of dimensional regularization is one such method that stands apart from most others
in that it preserves the essential symmetries, such as Lorentz invariance and gauge invari-
ce. The dimensional continuation formalism of Brown-Preston-Singleton (BPS) relies on
a technique adopted from quantum field theory called dimensional regularization. Processes
that are divergent in $\nu = 3$ spatial dimensions can often be regularized by looking at the
system in a general number of dimensions $\nu$. The three-dimensional divergences show up as
simple poles of the form $1/(\nu - 3)$. Dimensional regularization soon led to the insight that
the physics of a system is critically dependent upon the dimension $\nu$ space. The general
rule is that long distance fluctuations are greater in lower dimensions, while short distance
physics is more important in higher dimensions. In fact, in $\nu = 1$, it has been shown that
the quantum fluctuations are so large that spontaneous symmetry breaking cannot occur,
even if it is permitted classically [16]. Another interesting result is that in $\nu = 1$ dimensions ,
the photon acquires a mass via quantum loop corrections [17]. Other phenomena are unique
to $\nu = 2$ dimensions, such as high temperature superconductivity. BPS does not require
the introduction of a Coulomb logarithm, as the regularization is performed by changing the
dimension $\nu$ of space. The BPS method uses dimensional continuation to find the Coulomb
energy exchange at the integer values $\nu = 1, 2, 4, 5, \cdots$ (except for $\nu = 3$). By applying
Carlson’s theorem [18], we can define an analytically continued quantity for complex $\nu$, in a similar way that the factorial function on the positive integers can be analytically continued to the gamma function over the complex plane. For Coulomb energy exchange processes, the continuation to complex $\nu$ allows us to take the $\nu \to 3$ limit to obtain a finite result valid in $\nu = 3$ dimensions. In this way, the BPS formalism regularizes the traditional $\nu = 3$ divergences, and allows us to define the theory in three dimensions. In this third installment of the BPS Explained lecture series, we have proven a pivotal result of process of dimensional continuation upon which the BPS formalism resides. Namely, that to leading order in the plasma coupling $g$, the BBGKY hierarchy of kinetic equations reduces to (i) the Boltzmann equation for spatial dimensions $\nu > 3$, and (ii) the Lenard-Balescu equation for $\nu < 3$. The bulk of these notes were devoted to proving the latter.

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Appendix A: The Cross Section and Hyperspherical Coordinates

To make these notes self contained, and to establish some notation, I shall give a quick review of the material required from Lectures I and II.

1. Hyperspherical Coordinates

Kinematic quantities such as $\nu$-dimensional momentum or position vectors are elements of Euclidean space $\mathbb{R}^\nu$. We can decompose any element $x \in \mathbb{R}^\nu$ in terms of a rectilinear orthonormal basis $\hat{e}_\ell$, so that $x = \sum_{\ell=1}^\nu x_\ell \hat{e}_\ell$, or in component notation $x = (x_1, \cdots, x_\nu)$. Each component is given by $x_\ell = \hat{e}_\ell \cdot x$, and a change $dx$ in the vector $x$ corresponds to a change $dx_\ell = \hat{e}_\ell \cdot dx$ in the rectilinear coordinate $x_\ell$. Letting $x$ vary successively along each independent direction $\hat{e}_\ell$, we can trace out a small $\nu$-dimensional hypercube with sides of length $dx_\ell$; therefore, the rectilinear volume element is given by the simple form

$$d\nu x = \prod_{\ell=1}^\nu dx_\ell = dx_1 dx_2 \cdots dx_\nu . \tag{A1}$$

Similar considerations hold for momentum volume element $d\nu p$. In performing integrals over the kinematic variables, however, symmetry usually dictates the use of hyperspherical coordinates rather than rectilinear coordinates. I will therefore review the hyperspherical coordinate system in this section, deriving the measure for a $\nu$-dimensional volume element $d\nu x$ in terms of hyperspherical coordinates. It should be emphasized again that this formalism also holds in momentum space for the momentum volume element. For our purposes, the primary utility of hyperspherical coordinates is that the volume element $d\nu x$ can be written as a product of certain conveniently chosen dimensionless angles, which I will collectively refer to as $d\Omega_{\nu-1}$, and an overall dimensionfull radial factor $r^{\nu-1} dr$, so that

$$d\nu x = d\Omega_{\nu-1} r^{\nu-1} dr . \tag{A2}$$

To prove this, let us recall why the three dimensional volume element takes the form $d^3x = \sin \theta d\theta d\phi r^2 dr$ (with $0 \leq \theta \leq \pi$, $0 \leq \phi < 2\pi$, and $0 \leq r < \infty$). As depicted in Fig. 8, the three dimensional vector $x$ has length $r$, and subtends a polar angle $\theta$ relative to the $z$-axis, while its projection onto the $x$-$y$ plane subtends an azimuthal angle $\phi$ relative to the $x$-axis. The two angles $\theta$ and $\phi$ specify completely the direction of the unit vector $\hat{x}$, while an additional coordinate $r$ determines the total vector $x = r \hat{x}$. As we increase the polar angle $\theta$ by a small amount $d\theta$, the vector $x$ sweeps out an arc of length $dR_1 = rd\theta$. Similarly, a change $d\phi$ in the azimuthal angle will cause $x$ to sweep out an arc in the $x$-$y$ plane of length $dR_2 = r \sin \theta d\phi$, where the factor of $\sin \theta$ in $dR_2$ arises from the projection of $x$ onto the $x$-$y$ plane. Moving along the radial direction gives the final independent
displacement \( dR_3 = dr \). For small displacements in \( d\theta \), \( d\phi \), and \( dr \), the vector \( \mathbf{x} \) sweeps out a small cubic volume element with sides of length \( dR_1 \), \( dR_2 \), and \( dR_3 \), and therefore
\[
d^3x = dR_1 dR_2 dR_3 = r d\theta \cdot r \sin \theta d\phi \cdot dr.
\]

Let us now consider the volume element \( d^4x \) in four dimensional space, and denote the coordinate axes by \( x, y, z, w \). Since we cannot visualize four dimensional space, let us examine this problem in two steps, each of which can be visualized in three dimensions. As shown in Fig. 9a, let \( \theta_1 \) be the angle between the \( w \)-axis and the four dimensional vector \( \mathbf{x} \). The \( w \)-axis and the vector \( \mathbf{x} \) lie in a plane, and \( \theta_1 \) can therefore be visualized. Let us now project \( \mathbf{x} \) onto the \( w = 0 \) hyperplane (a three dimensional slice of four-space), denoting the projected three-vector by \( \mathbf{x}_w \). Since this vector lies in three-space, it too can be visualized. Since the three-plane \( w = 0 \) lies perpendicular to each of the axes \( x, y, \) and \( z \), the vector \( \mathbf{x}_w \) lies in the three dimensional space shown in Fig. 9b, and its length is \( |\mathbf{x}_w| = r \sin \theta_1 \).

Let the angle \( \theta_2 \) be the polar angle between the \( z \)-axis and the vector \( \mathbf{x}_w \), while \( \theta_3 \) is the usual azimuthal angle \( \phi \). The last angle \( \theta_3 \) runs between \( 0 \) and \( 2\pi \), while all previous angles run between \( 0 \) and \( \pi \). As we vary the three angles and the radial coordinate, we sweep out a four-dimensional cube (or an approximate cube) with sides of length \( dR_1 = r d\theta_1 \), \( dR_2 = r \sin \theta_1 d\theta_2 \), \( dR_3 = r \sin \theta_1 \sin \theta_2 d\theta_3 \), and \( dR_4 = dr \). This gives a four dimensional volume element
\[
d^4x \equiv dR_1 dR_2 dR_3 dR_4 = \sin^2 \theta_1 d\theta_1 \sin \theta_2 d\theta_2 d\theta_3 r^3 dr , \tag{A3}
\]
where \( 0 \leq \theta_\ell \leq \pi \) for \( \ell = 1, 2 \) and \( 0 \leq \theta_3 < 2\pi \). It is amusing to calculate the four-volume of a four-dimensional ball of radius \( r \) by integrating the volume element over the appropriate angles,
\[
B_4 = \int_0^\pi d\theta_1 \sin^2 \theta_1 \int_0^\pi d\theta_2 \sin \theta_2 \int_0^{2\pi} d\theta_3 \int_0^r dr' r'^3 = \frac{1}{2} \pi^2 r^4 . \tag{A4}
\]
The derivative of \( B_4 \) with respect to \( r \) gives the hypersurface area of the enclosing three-sphere,
\[
S_3 = \frac{dB_4}{dr} = 2\pi^2 r^3 . \tag{A5}
\]
Let us now project \( x \) onto the orthogonal three dimensional space, so that \( x = (w, x, y, z) \rightarrow x_w = (0, x, y, z) \). The length of the projection \( x_w \) is \( r_w = r \sin \theta_1 \). (b) The vector \( x_w \) can be viewed as a three dimensional vector \( x = (x, y, z) \), which then defines the usual polar and azimuthal angles of Fig. 8, denoted here by \( \theta_2 \) and \( \theta_3 \) respectively.

These are well known results, analogous to a three dimensional ball of radius \( r \) and volume \( B_3 = 4\pi r^3/3 \), which is of course bounded by the two-sphere of area \( S_2 = 4\pi r^2 \).

We can readily generalize this procedure to an arbitrary number of dimensions. Consider a point \( \mathbf{x} \in \mathbb{R}^\nu \) given by the rectilinear coordinates \( \mathbf{x} = (x_1, x_2, \ldots, x_\nu) \). Let \( \theta_1 \) be the angle between the vector \( \mathbf{x} \) and the \( x_1 \)-axis, in a manner similar to that of Figs. 8 and 9a. Note that \( dR_1 = r \, d\theta_1 \) is the arc length swept out by \( \mathbf{x} \) as the angle \( \theta_1 \) is incremented by \( d\theta_1 \). Let us now project \( \mathbf{x} \) onto the hyperplane \( x_1 = 0 \), the \((\nu - 1)\)-plane normal to the \( x_1 \)-axis and passing through the origin. Denote this projection by \( \mathbf{x}_1 \), that is to say, let \( \mathbf{x} \rightarrow \mathbf{x}_1 = (0, x_2, \ldots, x_\nu) \), and note that the length of \( \mathbf{x}_1 \) is \( r_1 = r \sin \theta_1 \). We proceed to the next step and define \( \theta_2 \) as the angle between the \( x_2 \)-axis and the projection \( \mathbf{x}_1 \). Note that as the angle \( \theta_2 \) is incremented by \( d\theta_2 \), the vector \( \mathbf{x}_1 \) sweeps out an arc of length \( dR_2 = r_1 \, d\theta_2 = r \sin \theta_1 \, d\theta_2 \).

In a similar fashion, project \( \mathbf{x}_1 \) onto the \( x_2 \)-plane, that is, the plane described by \( x_1 = 0 \) and \( x_2 = 0 \). This projection is given by \( \mathbf{x} \rightarrow \mathbf{x}_2 = (0, 0, x_3, \ldots, x_\nu) \), and the length of \( \mathbf{x}_2 \) is \( r_2 = r \sin \theta_2 = r \sin \theta_1 \sin \theta_2 \). For the general \( \ell \)-th iteration, let \( \theta_\ell \) be the angle between the \( x_\ell \)-axis and \( \mathbf{x}_{\ell-1} \), so that \( dR_\ell = r_{\ell-1} \, d\theta_\ell = r \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{\ell-1} \, d\theta_\ell \), where we have used the fact that \( r_{\ell-1} = r \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{\ell-1} \). This gives the \( \nu \)-dimensional volume element

\[
d^\nu x = \prod_{\ell=1}^{\nu} dR_\ell = \sin^{\nu-2} \theta_1 \, d\theta_1 \cdot \sin^{\nu-3} \theta_2 \, d\theta_2 \cdots \sin \theta_{\nu-2} \, d\theta_{\nu-2} \cdot d\theta_{\nu-1} \cdot r^{\nu-1} \, dr . \tag{A6}
\]

The angles \( \theta_1, \ldots, \theta_{\nu-2} \) run from 0 to \( \pi \), while \( \theta_{\nu-1} \) runs from 0 to \( 2\pi \). For notational convenience, I will write the angular measure in (A6) as

\[
d\Omega_{\nu-1} = \sin^{\nu-2} \theta_1 \, d\theta_1 \sin^{\nu-3} \theta_2 \, d\theta_2 \cdots \sin \theta_{\nu-2} \, d\theta_{\nu-2} \, d\theta_{\nu-1} , \tag{A7}
\]
so that \( d^\nu x = d\Omega_{\nu-1} r^{\nu-1} dr \), which establishes (A2). Also note that

\[
d\Omega_{\nu-1} = d\Omega_{\nu-2} \sin^{\nu-2} \theta_1 d\theta_1.
\] (A8)

It is easy to show that the integration over all angles gives the total solid angle

\[
\Omega_{\nu-1} \equiv \int d\Omega_{\nu-1} = \frac{2\pi^{\nu/2}}{\Gamma(\nu/2)}.\] (A9)

To prove this, first consider the one-dimensional Gaussian integral

\[
\int_{-\infty}^{\infty} dx \ e^{-x^2} = \sqrt{\pi}.
\] (A10)

If we multiply both sides together \( \nu \) times, we find

\[
\left( \sqrt{\pi} \right)^\nu = \int_{-\infty}^{\infty} dx_1 e^{-x_1^2} \int_{-\infty}^{\infty} dx_2 e^{-x_2^2} \cdots \int_{-\infty}^{\infty} dx_\nu e^{-x_\nu^2} = \int d^\nu x \ e^{-x^2},
\] (A11)

where the vector \( x \) in the exponential of the last expression is the \( \nu \)-dimensional vector \( x = (x_1, x_2, \cdots, x_\nu) \), and \( x^2 = x \cdot x = \sum_{\ell=1}^{\nu} x_\ell^2 \). As in (A2), we can factor the angular integrals out of the right-hand-side of (A11), and the remaining one-dimensional integral can be converted to a Gamma function with the change of variables \( t = r^2 \),

\[
\pi^{\nu/2} = \int d\Omega_{\nu-1} \cdot \int_0^{\infty} dr \ r^{\nu-1} e^{-r^2} = \int d\Omega_{\nu-1} \cdot \frac{1}{2} \Gamma(\nu/2),
\] (A12)

and solving for \( \int d\Omega_{\nu-1} \) gives (A9).

A few general remarks on calculating physical quantities in the BPS program are in order. When we calculate the temperature equilibration rate between plasma species or the charged particle stopping power, we encounter integrals of the form

\[
I_1(\nu) \equiv \int d^\nu x \ F_1(r) = \Omega_{\nu-1} \int_0^{\infty} dr \ r^{\nu-1} F_1(r)
\] (A13)

\[
I_2(\nu) \equiv \int d^\nu x \ F_2(r, \theta) = \Omega_{\nu-2} \int_0^{\infty} dr \ r^{\nu-1} \int_0^{\pi} d\theta \ \sin^{\nu-2} \theta \ F_2(r, \theta),
\] (A14)

respectively. The exact expressions for \( F_1 \) and \( F_2 \) are not important here, except that their angular dependence is determined by the following considerations. The integral (A13) is spherically symmetric because the energy exchange between plasma species is isotropic, while in integral (A14), the motion of the charged particle defines a preferred direction around which one must integrate, thereby leaving a single angular dependence. The integrals \( I_1(\nu) \) and \( I_2(\nu) \) can be viewed as functions defined on the integers \( \nu \in \mathbb{N} \), and as discussed at length in Lecture I [3], Carlson’s Theorem [18] ensures that there are unique analytic continuations of \( I_1(\nu) \) and \( I_2(\nu) \) for \( \nu \in \mathbb{C} \). This is similar to extending the factorial function \( n! \) on the integers to the gamma function \( \Gamma(\nu) \) on the complex \( \nu \)-plane. Let us examine more closely how this analytic continuation to complex \( \nu \) works in practice. First, the solid
angles $\Omega_{\nu-1}$ and $\Omega_{\nu-2}$ are well defined for complex arguments $\nu$, as they are composed of simple exponential factors like $\pi^{\nu/2}$ and Gamma functions, whose analytic properties are well known. As for the integrals, simply treat $\nu$ as an arbitrary integer dimension, and perform the integral for general $\nu$. The integral will of course depend upon the value of $\nu$, and once the integral has been performed exactly (not approximately and not numerically), we are free to set the value of $\nu$ to a complex number (presumably in a small neighborhood about $\nu = 3$). This provides functions $I_1(\nu)$ and $I_2(\nu)$ with complex argument $\nu \in \mathbb{C}$.

2. The Hypervolume of Spheres, Disks, and Cylinders

We shall now calculate the hypervolume of several useful geometric objects. Let us first consider a $\nu$-dimensional ball of radius $r$, defined by the set of points $\mathbf{x} \in \mathbb{R}^\nu$ for which $|\mathbf{x}| \leq r$. We will denote this object by $B_\nu(r)$, and in two and three dimensions this is a disk and a ball, both volume centered at the origin. We can find the $\nu$-dimensional hypervolume of the ball $B_\nu(r)$ by simply integrating (A6) over all permissible values of the coordinates. It should cause no confusion to denote the hypervolume of the region $B_\nu(r)$ by the same symbol, and using (A9), together with $x \Gamma(x) = \Gamma(x+1)$, we find

$$B_\nu(r) = \int d\Omega_{\nu-1} \int_0^r dr' r'^{\nu-1} = \frac{\pi^{\nu/2}}{\Gamma(\nu/2 + 1)} r^{\nu}. \quad (A15)$$

The boundary of $B_\nu(r)$ is a $(\nu-1)$-dimensional sphere $S_{\nu-1}(r)$ defined by $|\mathbf{x}| = r$, or $\sum_{\ell=1}^\nu x_\ell^2 = r^2$. By differentiating (A15) with respect to the radius $r$, we can also find the hyperarea of a $(\nu-1)$-dimensional sphere $S_{\nu-1}(r)$ of radius $r$ in $\mathbb{R}^\nu$,

$$S_{\nu-1}(r) = \frac{dB_\nu(r)}{dr} = \frac{2\pi^{\nu/2}}{\Gamma(\nu/2)} r^{\nu-1} = \Omega_{\nu-1} r^{\nu-1}. \quad (A16)$$
FIG. 11: The hyperarea of a hypercylinder $C_{\nu-1}(r, L)$ of length $L$ and radius $r$ is $C_{\nu-1}(r, L) = S_{\nu-2}(r) \cdot L$, and the hypervolume bounded by the cylinder is $V_{\nu}(r, L) = B_{\nu-1}(r) \cdot L$.

For brevity, I have denoted the hyperarea by the same symbol $S_{\nu-1}(r)$ as the sphere itself, which is simply the $(\nu-1)$-dimensional boundary of the region $B_{\nu}(r)$. This is illustrated in Fig. 10. The distinction I am making between “hypervolume” and “hyperarea” is somewhat arbitrary, since these are both terms involving regions in a higher dimensional space. When I wish to talk about a $\nu$-dimensional subregion of the hyperspace $\mathbb{R}^\nu$, such as $B_{\nu}(r)$, I will use the term hypervolume. On the other hand, when I wish to emphasize a boundary region of a hypervolume, such as $S_{\nu-1}(r)$, I will use the term “hyperarea.” Regarding the usage of the term “solid angle,” suppose we keep the radius $r$ fixed but vary the angles $\theta_i$ over ranges $d\theta_i$. The region swept out by this procedure lies on the $(\nu-1)$-dimensional sphere $S_{\nu-1}(r)$ with a hyperarea $dS_{\nu-1} = d\Omega_{\nu-1} r^{\nu-1}$. We are therefore justified in calling $d\Omega_{\nu-1}$ the solid angle in $\nu$ dimensions.

Finally, let us discuss the $(\nu-1)$-dimensional cylindrical $C_{\nu-1}(r, L)$ of radius $r$ and length $L$. Again, it is easiest to argue from analogy in three dimensions. To form a two-cylinder $C_2(r, L)$ in $\mathbb{R}^3$, we let a two dimensional disk $B_2(r)$ sweep out a volume as it moves a distance $L$ in the orthogonal direction, as illustrated in Fig. 11. Similarly, a $(\nu-1)$-dimensional cylinder can be formed by letting a $(\nu-1)$-dimensional ball $B_{\nu-1}(r)$ sweep out a distance $L$ along the orthogonal axis. Therefore, the hyperarea of the $(\nu-1)$-dimensional cylinder is

$$C_{\nu-1}(r, L) = S_{\nu-2}(r) \cdot L,$$  \hspace{1cm} (A17)

and the $\nu$-dimensional hypervolume enclosed by this cylinder is

$$V_{\nu}(r, L) = B_{\nu-1}(r) \cdot L.$$  \hspace{1cm} (A18)

Note that this is the natural geometry of a scattering experiment to measure the cross section in $\nu$ dimensions, which leads us to the next section.
### 3. The Cross Section

Now that we have examined the Coulomb plasma in some detail, we should address two-body scattering and the cross section. This is necessary formalism, since the Boltzmann equation contains the differential cross section for two-body scattering. For continuity, we review the notion of “cross section” in \( \nu \)-dimensions. As illustrated in Fig. 12, we consider a beam of projectiles 1 with flux \( I_0 \) striking a fixed target 2, although we can perform a similar analysis in the lab frame in which the scattering centers are also moving. In \( \nu \) dimensions, the spatial region normal to the beam axis is a \((\nu-1)\)-dimensional hyperplane, and the flux \( I_0 \) is the number of particles per second per unit hyperarea passing through this plane. The engineering units of \( I_0 \) are therefore \( L^{1-\nu} \cdot T^{-1} \). In other words, the number of particles in a time interval \( dt \) passing through a hyperarea \( dA \) normal to the beam is \( dN = I_0 \cdot dA \cdot dt \); therefore, the differential rate through the normal area \( dA \) is \( dR = I_0 \cdot dA \). Let us now place a particle counter along position \( \hat{\Omega} \) some distance away from the scattering center, and let us measure the rate \( dR_{12}(\hat{\Omega}) \) at which the 1-particles enter a given solid angle centered about direction \( \hat{\Omega} \). We can therefore define the differential cross section \( d\sigma_{12} \) in the usual way,

\[
d\sigma_{12} \cdot I_0 = dR_{12}.
\] (A19)

Note that the engineering of \( d\sigma_{12} \) are \( L^{\nu-1} \). The cross section is usually quite sensitive to the details of any given physical theory, thereby making it a good experimental probe. Indeed, in high energy physics, it is the primary diagnostic.

![Fig. 12: Definition of the cross section in a general number of dimensions. The incident flux \( I_0 \) of species 1 is the rate of particles per unit hyperarea normal to the beam. The units of \( I_0 \) are \( L^{1-\nu} \cdot T^{-1} \), where \( L \) and \( T \) denote the units of space and time. By definition, the differential cross section \( d\sigma_{12} \) is related to the rate \( dR_{12} \), each at angular position \( \hat{\Omega} \), by \( dR_{12}(\hat{\Omega}) = I_0 \cdot d\sigma_{12}(\hat{\Omega}) \). The cross section per unit solid angle about the direction \( \hat{\Omega} \) is denoted by \( d\sigma_{12}/d\Omega \). The engineering units of \( d\sigma_{12} \) are \( L^{\nu-1} \).](image-url)
Suppose the scattering center arises from a central potential force, such as the \( \nu \)-dimensional Coulomb field. Then two-body particle motion is confined along a two-dimensional plane, and this holds true even in \( \nu \) dimensions. Let \( b \) denote the impact parameter of the projectile relative to the scattering center. As the particle traverses its plane of motion, its position is uniquely characterized by a function \( b = b(\theta) \), where \( \theta \) is the angle between the beam direction and the projectile (with the scattering center defining the origin). The rate at which particles pass through the hyperannulus of width \( db \) and radius \( b \) is
\[
\frac{dR}{\Omega} = \Omega_{\nu-2} b^{\nu-2} db \cdot I_0,
\]
and by particle number conservation, the same number of scattered particles reaches the hyperannulus around \( \Omega \). This is the analog of \( dR = 2\pi b db \cdot I_0 \) in three dimensions. It is actually better to consider a differential \( d\Omega_{\nu-2} \) rather than the total angular extent \( \Omega_{\nu-2} \). Again, this corresponds to \( dR = d\theta b db \cdot I_0 \) in three dimensions. The cross section in a \( \nu \)-dimensional central potential is therefore given by
\[
d\sigma_{12} = d\Omega_{\nu-2} b^{\nu-2} db . \tag{A20}
\]
This is the differential form of Eq. (8.31) of Ref. [2]. However, for include two-body quantum scattering effects, it is more convenient to replace the cross section \( d\sigma_{12} \) by the quantum scattering amplitude \( T(1 + 2 \rightarrow 1' + 2') \equiv T_{1'2';12}(W, q^2) \) by using the relation
\[
|v_1 - v_2| d\sigma_{12} = \int \frac{d^\nu p_1}{(2\pi \hbar)^\nu} \frac{d^\nu p_2}{(2\pi \hbar)^\nu} |T_{1'2';12}(W, q^2)|^2 (2\pi \hbar)^\nu \delta'(P_1' + P_2' - P_1 - P_2) \times
\]
\[
(2\pi \hbar)\delta\left(E_1' + E_2' - E_1 - E_2\right). \tag{A21}
\]
In the amplitude, \( W \) is the total center-of-mass energy and \( q^2 \) is the square of the momentum exchange. This is just a rewriting of the expression \( I_0 \cdot d\sigma_{12} = dR_{12} \), since \( |v_1 - v_2| \) is proportional to the flux \( I_0 \), and the rate \( dR_{12} \) is proportional to the square of the scattering amplitude \( |T_{1'2';12}|^2 \). The integration is over all values of \( p_1 \) and \( p_2 \) consistent with energy and momentum conservation. In three dimensions, for example, there are six momentum integrals and four \( \delta \)-functions, leaving two differential degrees of freedom, namely, the cross sectional area. This is why I continue to use the differential cross section \( d\sigma_{12} \) on the left-hand-side of (A21), to imply that some of the angular coordinates have not been integrated over.
Appendix B: Center-of-Momentum Coordinates

In calculating the convective terms in (5.9), it is useful to transform to center-of-momentum coordinates. We will generalize to two species for the scattering. We define the total and relative moment, and the center-of-mass and the relative position by

\[ P = m_1 v_1 + m_2 v_2 = p_1 + p_2 \]  
\[ p = m_{12} (v_1 - v_2) = \frac{m_2 p_1 - m_1 p_2}{M} \]  
\[ R = \frac{m_1 x_1 + m_2 x_2}{M} \]  
\[ x = x_1 - x_2, \]

where

\[ M = m_1 + m_2 \]  
\[ m_{12} = \frac{m_1 m_2}{m_1 + m_2}. \]

I am using the notation \( x \) for the relative position, rather than usual notation \( r \), because in the text, the beam-axis in two-body scattering has been called \( x \). The inverse transforms are

\[ p_2 = \frac{m_2}{M} P - p \quad p_1 = \frac{m_1}{M} P + p \]  
\[ x_2 = R - \frac{m_1}{M} x \quad x_1 = R + \frac{m_2}{M} x, \]

and it is easy to check that the gradients transform as,

\[ \frac{\partial}{\partial p_2} = \frac{\partial}{\partial P} - \frac{m_1}{M} \frac{\partial}{\partial p} \quad \frac{\partial}{\partial p_1} = \frac{\partial}{\partial P} + \frac{m_2}{M} \frac{\partial}{\partial p} \]  
\[ \frac{\partial}{\partial x_2} = \frac{m_2}{M} \frac{\partial}{\partial R} - \frac{\partial}{\partial x} \quad \frac{\partial}{\partial x_1} = \frac{m_1}{M} \frac{\partial}{\partial R} + \frac{\partial}{\partial x}, \]

and

\[ \frac{\partial}{\partial p} = \frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_2} \quad \frac{\partial}{\partial P} = \frac{m_1}{M} \frac{\partial}{\partial p_1} + \frac{m_2}{M} \frac{\partial}{\partial p_2} \]  
\[ \frac{\partial}{\partial x} = \frac{m_2}{M} \frac{\partial}{\partial x_1} - \frac{m_1}{M} \frac{\partial}{\partial x_2} \quad \frac{\partial}{\partial R} = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}. \]

I have recorded these formulae here for convenience.
Appendix C: The Multi-component Poisson-Vlasov Equation

Let us restore the species index in this section, and consider a collisionless plasma,
\[
\frac{\partial f_a}{\partial t} + \mathbf{v} \cdot \frac{\partial f_a}{\partial \mathbf{x}} + e_a \mathbf{E} \cdot \frac{\partial f_a}{\partial \mathbf{p}} = 0 \quad (C1)
\]
where \( \mathbf{E}(\mathbf{x}, t) \) is the self-consistent electric field associated with the \( f_b \). There is a charge \( e_a \) at position \( \mathbf{x} \), and the quantity \( \mathbf{E}_x^{(b)} \) inside the integral is the static Coulomb field at \( \mathbf{x} \) originating from a point-charge of type \( b \) at position \( \mathbf{x}_b \), so that
\[
\mathbf{E}_x^{(b)} = \mathbf{E}_b(\mathbf{x} - \mathbf{x}_b) = e_b \frac{\Gamma(\nu/2)}{2\pi^{\nu/2}} \frac{\mathbf{x} - \mathbf{x}_b}{|\mathbf{x} - \mathbf{x}_b|^{\nu}}. \quad (C3)
\]
This means that the divergence of the self-consistent electric field is
\[
\nabla \cdot \mathbf{E}(\mathbf{x}, t) = \sum_b \int \frac{d^\nu p}{(2\pi \hbar)^\nu} e_b f_b(\mathbf{x}, \mathbf{p}, t). \quad (C4)
\]
To emphasize that \( \mathbf{E} \) in (C1) is a functional of the distributions \( f_b \), we shall often write \( \mathbf{E}[f] \) in place of \( \mathbf{E}(\mathbf{x}, t) \). It is actually more convenient to write (C1) and (C4) in terms of the electric potential \( \phi \), where \( \mathbf{E} = -\partial \phi / \partial \mathbf{x} \), so that the kinetic equations become
\[
\frac{\partial f_a}{\partial t} + \mathbf{v} \cdot \frac{\partial f_a}{\partial \mathbf{x}} - e_a \frac{\partial \phi}{\partial \mathbf{x}} \cdot \frac{\partial f_a}{\partial \mathbf{p}} = 0 \quad (C5)
\]
\[
\nabla^2 \phi(\mathbf{x}, t) = -\sum_b \int \frac{d^\nu p}{(2\pi \hbar)^\nu} e_b f_b(X, t). \quad (C6)
\]
This form of the kinetic equations is known as the Poisson-Vlasov equations, and we shall use it interchangeably with (C1)–(C4). For visual clarity, I am using a mixed notation in which the Laplacian is denoted by
\[
\frac{\partial}{\partial \mathbf{x}} \cdot \frac{\partial \phi}{\partial \mathbf{x}} = \nabla^2 \phi. \quad (C7)
\]
Let us now perform a perturbative analysis on the Vlasov equation (C1), or equivalently (C5). Rather than perturbing about an equilibrium configuration, let us take the 0-th order starting point as a solution to the Vlasov equation itself. In other words, suppose \( f_{a1} \) is a solution to
\[
\frac{\partial f_{a1}(X, t)}{\partial t} + \mathbf{v} \cdot \frac{\partial f_{a1}(X, t)}{\partial \mathbf{x}} + e_a \mathbf{E}^0 \cdot \frac{\partial f_{a1}(X, t)}{\partial \mathbf{p}} = 0, \quad (C8)
\]
where the 0-th order self-consistence electric field at \( \mathbf{x} \) is
\[
\mathbf{E}^0(\mathbf{x}, t) = \sum_b \int dX_b f_{b1}(\mathbf{x}_b, \mathbf{p}_b, t) \mathbf{E}_b(\mathbf{x} - \mathbf{x}_b). \quad (C9)
\]
Now suppose that the perturbation
\[ f_a(x, p, t) = f_{a1}(x, p, t) + h_a(x, p, t) \] (C10)
satisfies the Vlasov equation (C1), and let us find the corresponding equation satisfied by \( h_a \). Upon substituting (C10) for \( h_a \) into the electric field (C2), the self-consistent electric field receives a 0-th order contribution from \( f_{a1} \) and a 1-st order contribution from \( h_a \),
\[
E(x, t) = \sum_b \int dX_b \left[ f_{b1}(X_b, p_b, t) + h_b(X_b, p_b, t) \right] E_b(x - X_b) \\
= E^0(x, t) + E^1(x, t) ,
\] (C11)
where the first-order self-consistent field is
\[
E^1(x, t) = \sum_b \int dX_b h_b(x, p_b, t) E_b(x - X_b) .
\] (C12)

We also substitute the perturbation (C10) back into the kinetic equation (C1), using the electric field (C11) and working only to first order in \( h_a \). The 0-th order terms vanish because \( f_{a1} \) is a solution to equation (C8), and after some algebra we find that the perturbation satisfies
\[
\frac{\partial h_a}{\partial t} + v \cdot \frac{\partial h_a}{\partial x} + e_a E^0[f_{1}] \cdot \frac{\partial h_a}{\partial p} + e_a E^1[h] \cdot \frac{\partial f_{a1}}{\partial p} = 0 .
\] (C13)

It should be reiterated that we have dropped the second-order term \( E^1[h] \cdot (\partial h_a/\partial p_a) \). We can also write (C13) in the form
\[
\frac{\partial h_a}{\partial t} + V h_a = 0 ,
\] (C14)
where the operator \( V \) is defined by
\[
V h_a = v \cdot \frac{\partial h_a}{\partial x} + e_a E^0[f_{1}] \cdot \frac{\partial h_a}{\partial p} + e_a E^1[h] \cdot \frac{\partial f_{a1}}{\partial p}
\] (C15)
\[
= v \cdot \frac{\partial h_a}{\partial x} + e_a \sum_b \left[ \int dX_b f_{b1}(X_b, t) E^{(b)}_x \cdot \frac{\partial h_a}{\partial p} + \int dX_b h_b(X_b, t) E^{(b)}_x \cdot \frac{\partial f_{a1}}{\partial p} \right] .
\] (C16)

From Bogoliubov’s hypothesis, the time dependence of \( f_{a1} \) is much slower than that of \( h_a \), and we can regard the operator \( V \) as constant in time as far as its action on a perturbation \( h_a \) is concerned. Note that equations (C15) and (C16) serve a definition of the operator \( V \) on any function \( h_a(X, t) \), whether \( h_a \) is the perturbation or not.

As in Section VI, to solve (C14) we take the temporal Laplace transform and the spatial Fourier transform,
\[
(p + V) \tilde{h}(k, p, p) = \tilde{h}(k, p, 0) ,
\] (C17)
giving the formal solution

\[ \tilde{h}(\mathbf{k}, \mathbf{p}, p) = (p + V)^{-1} \tilde{h}(\mathbf{k}, \mathbf{p}, 0) . \]  

(C18)

We now preform the inversion (C18) for a specific example in which the unperturbed plasma is a function of momentum only, \( f_{a1} = f_a(\mathbf{p}) \). We then see that the 0-th order self-consistent field vanishes,

\[ E^0(\mathbf{x}, t) = \sum_b \int \frac{d^\nu p_b}{(2\pi \hbar)^\nu} f_b(\mathbf{p}_b) \int d^\nu x_b E_b(\mathbf{x}_a - \mathbf{x}_b) = 0 , \quad \]  

(C19)

since the electric field

\[ E_b(\mathbf{x}_a - \mathbf{x}_b) = -\frac{\partial \phi(\mathbf{x})}{\partial \mathbf{x}} \bigg|_{\mathbf{x} = \mathbf{x}_a - \mathbf{x}_b} = \frac{\partial \phi(\mathbf{x}_a - \mathbf{x}_b)}{\partial \mathbf{x}_b} \]  

(C20)

is a total derivative of the potential \( \phi(\mathbf{x}) \). The operator \( V \) now take the form

\[ Vh_a = \mathbf{v} \cdot \frac{\partial h_a}{\partial \mathbf{x}} + e_a \mathbf{E} \cdot \frac{\partial f_a}{\partial \mathbf{p}} \]  

(C21)

\[ = \mathbf{v} \cdot \frac{\partial h_a}{\partial \mathbf{x}} + e_a \sum_b \int dX_b h_b(\mathbf{x}_b, \mathbf{p}_b, t) E_b(\mathbf{x} - \mathbf{x}_b) \cdot \frac{\partial f_a(\mathbf{p})}{\partial \mathbf{p}} . \]  

(C22)

Since \( E^0 \) vanishes, the first-order field \( E^1 \) defined in (C12) is the total self-consistent electric field, and I have therefore dropped the 1-superscript. Upon taking the temporal-Laplace transform and the spatial-Fourier transform of (C14) and (C21), we find

\[ p\tilde{h}_a + ik \cdot \mathbf{v}_a \tilde{h}_a + e_a \tilde{\mathbf{E}} \cdot \frac{\partial f_a}{\partial \mathbf{p}} = \tilde{h}_a(0) , \]  

(C23)

where the Fourier transform \( \tilde{\mathbf{E}} \) can be calculated from (C22) by the convolution theorem,

\[ \tilde{\mathbf{E}}(\mathbf{k}, \mathbf{p}) = \sum_b \int \frac{d^\nu p_b}{(2\pi \hbar)^\nu} \tilde{h}_b(\mathbf{k}, \mathbf{p}_b, p) \tilde{\mathbf{E}}_b(\mathbf{k}) . \]  

(C24)

By way of notation, the tilde over a function is used to denote both Laplace and Fourier transforms, so care must be taken when interpreting such terms. In other words, we take \( \tilde{h}_a = \tilde{h}_a(\mathbf{k}, \mathbf{p}, p), \tilde{\mathbf{E}} = \tilde{\mathbf{E}}(\mathbf{k}, \mathbf{p}), \) and \( \tilde{h}_a(0) = \tilde{h}_b(\mathbf{k}, \mathbf{p}, t = 0) \). That is to say, \( \tilde{h}_a(0) \) is the spatial Fourier transform of \( h_a(\mathbf{x}, \mathbf{p}, t) \) evaluated at \( t = 0 \), while \( \tilde{h}_a \) and \( \tilde{\mathbf{E}} \) are spatial Fourier transforms and temporal Laplace transforms. We can now solve (C23) for the perturbation, giving

\[ \tilde{h}_a(\mathbf{k}, \mathbf{p}, p) = (p + V)^{-1} \tilde{h}_a(\mathbf{k}, \mathbf{p}, 0) \]  

(C25)

\[ = \frac{1}{p + ik \cdot \mathbf{v}_a} \left[ \tilde{h}_a(\mathbf{k}, \mathbf{p}, 0) - e_a \tilde{\mathbf{E}}(\mathbf{k}, \mathbf{p}) \cdot \frac{\partial f_a(\mathbf{p})}{\partial \mathbf{p}} \right] . \]  

(C26)
We can now write (C25) and (C24) in the Poisson-Vlasov form

\[ \tilde{E}_b(k) = -i k \tilde{\phi}_b(k) = -i k \frac{e_b}{k^2} . \] (C27)

In like manner, the self-consistent electric field \( \tilde{E}(k, p) \) can be expressed in terms of a self-consistent potential \( \tilde{\phi}(k, p) \) defined by

\[ \tilde{E}(k, p) = -i k \tilde{\phi}(k, p) . \] (C28)

We can now write (C25) and (C24) in the Poisson-Vlasov form

\[ \tilde{h}_a(k, p, p) = \frac{1}{p + i k \cdot v_a} \left[ \tilde{h}_a(k, p, 0) + e_a \tilde{\phi}(k, p)(i k) \cdot \frac{\partial f_a(p)}{\partial p} \right] \] (C29)

\[ \tilde{\phi}(k, p) = \sum_b \frac{e_b}{k^2} \int \frac{d^\nu p_b}{(2\pi \hbar)^\nu} \tilde{h}_b(k, p_b, p) . \] (C30)

Let us substitute (C29) for the perturbation \( \tilde{h}_a \) into (C30) for the potential, thereby giving

\[ \tilde{\phi}(k, p) = \sum_b \frac{e_b}{k^2} \int \frac{d^\nu p_b}{(2\pi \hbar)^\nu} \frac{1}{p + i k \cdot v_b} \left[ \tilde{h}_b(k, p_b, 0) + e_b \tilde{\phi}(k, p)(i k) \cdot \frac{\partial f_b(p_b)}{\partial p_b} \right] . \] (C31)

Note that \( \tilde{\phi}(k, p) \) appears on both sides of this equation, and upon isolating the \( \tilde{\phi}(k, p) \) term, we find

\[ \left[ 1 - \sum_b \frac{e_b^2}{k^2} \int \frac{d^\nu p_b}{(2\pi \hbar)^\nu} \frac{1}{p + i k \cdot v_b} \frac{\partial f_b(p_b)}{\partial p_b} \right] \tilde{\phi}(k, p) = \sum_b \frac{e_b}{k^2} \int \frac{d^\nu p_b}{(2\pi \hbar)^\nu} \frac{\tilde{h}_b(k, p_b, 0)}{p + i k \cdot v_b} . \] (C32)

Solving for the self-consistent potential gives

\[ \tilde{\phi}(k, p) = \sum_b \frac{e_b}{\tilde{\epsilon}(k, p) k^2} \int \frac{d^\nu p_b}{(2\pi \hbar)^\nu} \frac{\tilde{h}_b(k, p_b, 0)}{p + i k \cdot v_b} , \] (C33)

where the “dielectric function” in Laplace space is defined by

\[ \tilde{\epsilon}(k, p) = 1 - \sum_b \int \frac{d^\nu p_b}{(2\pi \hbar)^\nu} \frac{e_b^2}{k^2} \frac{1}{p + i k \cdot v_b} \frac{i k \cdot \partial f_b(p_b)}{\partial p_b} , \] (C34)

with \( p \) lying on the contour \( C \). In fact, we can analytically continue (C34), and allow \( p \) to lie anywhere in the complex plane to the right of \( C \). For future reference, we record the following identities:

\[ \sum_b \int \frac{d^\nu p_b}{(2\pi \hbar)^\nu} \frac{e_b^2}{k^2} \frac{i k \cdot \partial f_b/p}{p + i k \cdot v_b} = 1 - \tilde{\epsilon}(k, p) \] (C35)

\[ \sum_b \int \frac{d^\nu p_b}{(2\pi \hbar)^\nu} \frac{e_b^2}{k^2} \frac{i k \cdot \partial f_b/p}{p - i k \cdot v_b} = \tilde{\epsilon}(-k, p) - 1 . \] (C36)
Upon substituting (C33) back into (C29) we find the inverse
\[
(p + V)^{-1} \tilde{h}_a(k, p, 0) \equiv \tilde{h}_a(k, p, p)
\]
\[
= \frac{1}{p + i k \cdot v} \left[ \tilde{h}(k, p, 0) + \sum_b e_a e_b \frac{\partial f_a(p)}{\partial p} \int \frac{d^\nu p'}{(2\pi \hbar)^\nu} \frac{\tilde{h}_b(k, p', 0)}{p + i k \cdot v'} \right].
\]

We have use expressions (C35), (C36), and (C37) in Section VI C in their single-component forms. This analysis shows that the results in Section VI C also hold for a multi-component plasma.

Let us pause now to understand these results physically. We see from (2.50) that the dielectric function \( \epsilon(k, \omega) \) can be analytically continued to complex values of \( \omega \), thereby taking the form
\[
\epsilon(k, \omega) = 1 + \sum_b \int \frac{d^\nu p}{(2\pi \hbar)^\nu} \frac{e_b^2}{k^2} \frac{1}{\omega - k \cdot v_b} k \cdot \frac{\partial f_b}{\partial p},
\]
where \( \text{Re} \omega > 0 \). The quantity \( \tilde{\epsilon}(k, p) \) in (C34) is just the analytically continued dielectric function \( \epsilon(k, \omega) \) to a complex frequency \( \omega = ip \) (for real \( p \)),\(^6\) and we see that
\[
\epsilon(k, \omega = ip) = \tilde{\epsilon}(k, p),
\]
or equivalently, \( \epsilon(k, p = -i\omega) = \epsilon(k, \omega) \). We can also analytically continue the self-consistent potential (C33) to Fourier space by setting \( p = -i\omega + \eta \), where \( \omega \) is real and \( \eta > 0 \). Upon taking the limit \( \eta \to 0^+ \), we find
\[
\tilde{\phi}(k, \omega) = \sum_b \frac{e_b}{\epsilon(k, \omega) k^2} \int \frac{d^\nu p_b}{(2\pi \hbar)^\nu} \frac{i \tilde{h}_b(k, p_b, 0)}{\omega - k \cdot v_b + i\eta}.
\]

Note that the \( k^2 \) term in the denominator of the self-consistent potential (C40) is accompanied by a factor of \( \epsilon(k, \omega) \) relative to the static Coulomb potential of a point charge, \( \tilde{\phi}_a(k) = e_a/k^2 \). This means that the self-consistent field is accompanied by Landau screening, and in fact we could write the equations using only the screened potential,

\[
\tilde{\phi}_a^{\text{landau}}(k, \omega) = \frac{e_a}{\epsilon(k, \omega) k^2}.
\]

We choose, however, to keep the factors of \( \epsilon(k, \omega) \) explicit. The Fourier components of the electric field are determined by \( \tilde{E}(k, \omega) = -i k \tilde{\phi}(k, \omega) \), so that

\(^6\) We should actually set \( \omega = ip + \eta \) with \( \eta > 0 \), so that \( \text{Re} \omega > 0 \). We can then take the limit \( \eta \to 0^+ \), which gives (C39).
\[
\tilde{E}(k, \omega) = \sum_b \frac{k}{\epsilon(k, \omega)} \frac{e_b}{k^2} \int \frac{d^\nu p_b}{(2\pi \hbar)^\nu} \frac{\tilde{h}_b(k, p_b, 0)}{\omega - k \cdot v_b + i\eta}, \tag{C42}
\]

where \(v_b = p_b/m_b\). If we wanted to find the electric field \(E(x, t)\) as a function of space and time, it is more convenient to revert back to Laplace space by setting \(p = -i\omega\) in (C42),

\[
\tilde{E}(k) = \sum_b \frac{-i k}{\bar{\epsilon}(k, p)} \frac{e_b}{k^2} \int \frac{d^\nu p_b}{(2\pi \hbar)^\nu} \frac{\tilde{h}_b(k, p_b, 0)}{p + i k \cdot v_b}, \tag{C43}
\]

and then taking the inverse Laplace transform. We will, however, not work through this algebra, and remain content to have found the electric field and the perturbation in Laplace and Fourier space. Chapter 24 of Ref. [13] does a good job of finding \(E(x, t)\) in various physical cases of interest.
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