Base change and $K$-theory for $\operatorname{GL}(n, \mathbb{R})$

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Abstract

We investigate base change $\mathbb{C}/\mathbb{R}$ at the level of $K$-theory for the general linear group $\operatorname{GL}(n, \mathbb{R})$. In the course of this study, we compute in detail the $C^*$-algebra $K$-theory of this disconnected group. We investigate the interaction of base change with the Baum-Connes correspondence for $\operatorname{GL}(n, \mathbb{R})$ and $\operatorname{GL}(n, \mathbb{C})$. This article is the archimedean companion of our previous article [9].

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1 Introduction

In the general theory of automorphic forms, an important role is played by base change. Base change has a global aspect and a local aspect [1]. In this article, we focus on the archimedean case of base change for the general linear group $\operatorname{GL}(n, \mathbb{R})$, and we investigate base change for this group at the level of $K$-theory.

For $\operatorname{GL}(n, \mathbb{R})$ and $\operatorname{GL}(n, \mathbb{C})$ we have the Langlands classification and the associated $L$-parameters [7]. We recall that the domain of an $L$-parameter of $\operatorname{GL}(n, F)$ over an archimedean field $F$ is the Weil group $W_F$. The Weil groups are given by

$$W_{\mathbb{C}} = \mathbb{C}^\times$$

and

$$W_{\mathbb{R}} = \mathbb{C}^\times \rtimes \mathbb{Z}/2\mathbb{Z}$$

where the generator of $\mathbb{Z}/2\mathbb{Z}$ sends a complex number $z$ to its conjugate $\overline{z}$. Base change is defined by restriction of $L$-parameter from $W_{\mathbb{R}}$ to $W_{\mathbb{C}}$.

An $L$-parameter $\phi$ is tempered if $\phi(W_F)$ is bounded. Base change therefore determines a map of tempered duals.

In this article, we investigate the interaction of base change with the Baum-Connes correspondence for $\operatorname{GL}(n, \mathbb{R})$ and $\operatorname{GL}(n, \mathbb{C})$. 
Let $F$ denote $\mathbb{R}$ or $\mathbb{C}$ and let $G = G(F) = \text{GL}(n, F)$. Let $C^*_r(G)$ denote the reduced $C^*$-algebra of $G$. The Baum-Connes correspondence is a canonical isomorphism [8][5]

$$\mu_F : K^*_r(G(F))(EG(F)) \to K^*_r(C^*_r(G(F)))$$

where $EG(F)$ is a universal example for the action of $G(F)$.

The noncommutative space $C^*_r(G(F))$ is strongly Morita equivalent to the commutative $C^*$-algebra $C_0(A^t_n(F))$ where $A^t_n(F)$ denotes the tempered dual of $G(F)$, see [10] section 1.2[11]. As a consequence of this, we have

$$K^*_r(C^*_r(G(F))) \cong K^*_r(A^t_n(F)).$$

This leads to the following formulation of the Baum-Connes correspondence:

$$K^*_r(G(F))(EG(F)) \cong K^*_r(A^t_n(F)).$$

Base change $\mathbb{C}/\mathbb{R}$ determines a map

$$b_{\mathbb{C}/\mathbb{R}} : A^t_n(\mathbb{R}) \to A^t_n(\mathbb{C}).$$

This leads to the following diagram

$$\begin{array}{ccc}
K^*_r(G(\mathbb{C}))(EG(\mathbb{C})) & \xrightarrow{\mu_{\mathbb{C}}} & K^*_r(A^t_n(\mathbb{C})) \\
\downarrow & & \downarrow b^*_{\mathbb{C}/\mathbb{R}} \\
K^*_r(G(\mathbb{R}))(EG(\mathbb{R})) & \xrightarrow{\mu_{\mathbb{R}}} & K^*_r(A^t_n(\mathbb{R})).
\end{array}$$

where the left-hand vertical map is the unique map which makes the diagram commutative.

In section 2 we describe the tempered dual $A^t_n(F)$ as a locally compact Hausdorff space.

In section 3 we recall base change for archimedean fields.

In section 4 we compute the $K$-theory for the reduced $C^*$-algebra of $\text{GL}(n, \mathbb{R})$. We show that the $K$-theory depends on essentially one parameter $q$ given by the maximum number of $2'$s in the partitions of $n$ into $1'$s and $2'$s. There are precisely $\lfloor \frac{n}{2} \rfloor + 1$ such partitions. If $n$ is even then $q = n/2$ (Theorem 4.7) and if $n$ is odd then $q = (n - 1)/2$ (Theorem 4.8). The real reductive Lie group $\text{GL}(n, \mathbb{R})$ is, of course, not connected. If $n$ is even our formulas show that we always have non-trivial $K^0$ and $K^1$.

In section 5 we recall the $K$-theory for the reduced $C^*$-algebra of the complex reductive group $\text{GL}(n, \mathbb{C})$, see [11].
In section 6 we compute the base change map $BC : A_{n}^t(\mathbb{R}) \to A_{n}^t(\mathbb{C})$ and prove that $BC$ is a continuous proper map. At level of $K$-theory, base change is the zero map for $n > 1$ (Theorem 6.2).

In section 7, where we study the case $n = 1$, base change for $K^1$ creates a map

$$\mathcal{R}(\mathbb{T}) \to \mathcal{R}(\mathbb{Z}/2\mathbb{Z})$$

where $\mathcal{R}(\mathbb{T})$ is the representation ring of the circle group $\mathbb{T}$ and $\mathcal{R}(\mathbb{Z}/2\mathbb{Z})$ is the representation ring of the group $\mathbb{Z}/2\mathbb{Z}$. This map sends the trivial character of $\mathbb{T}$ to $1 \oplus \varepsilon$, where $\varepsilon$ is the nontrivial character of $\mathbb{Z}/2\mathbb{Z}$, and sends all the other characters of $\mathbb{T}$ to zero.

This map has an interpretation in terms of $K$-cycles. The $K$-cycle

$$(C_0(\mathbb{R}), L^2(\mathbb{R}), id/dx)$$

is equivariant with respect to $\mathbb{C}^\times$ and $\mathbb{R}^\times$, and therefore determines a class $\varnothing_\mathbb{C} \in K_1^{\mathbb{C}^\times}(E\mathbb{C}^\times)$ and a class $\varnothing_\mathbb{R} \in K_1^{\mathbb{R}^\times}(E\mathbb{R}^\times)$. On the left-hand-side of the Baum-Connes correspondence, base change in dimension 1 admits the following description:

$$\varnothing_\mathbb{C} \mapsto (\varnothing_\mathbb{R}, \varnothing_\mathbb{R})$$

This extends the results of [9] to archimedean fields.

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2 On the tempered dual of $GL(n)$

Let $F = \mathbb{R}$. In order to compute the $K$-theory of the reduced $C^*$-algebra of $GL(n, F)$ we need to parametrize the tempered dual $A_{n}^t(\mathbb{R})$ of $GL(n, F)$.

Let $M$ be a standard Levi subgroup of $GL(n, F)$, i.e. a block-diagonal subgroup. Let $^0M$ be the subgroup of $M$ such that the determinant of each block-diagonal is $\pm 1$. Denote by $X(M) = M/^0M$ the group of unramified characters of $M$, consisting of those characters which are trivial on $^0M$.

Let $W(M) = N(M)/M$ denote the Weyl group of $M$. $W(M)$ acts on the discrete series $E_2(^0M)$ of $^0M$ by permutations.

Now, choose one element $\sigma \in E_2(^0M)$ for each $W(M)$-orbit. The isotropy subgroup of $W(M)$ is defined to be

$$W_\sigma(M) = \{ \omega \in W(M) : \omega \cdot \sigma = \sigma \}.$$
Form the disjoint union

\[
\bigsqcup_{(M,\sigma)} X(M)/W_\sigma(M) = \bigsqcup_{M} \bigsqcup_{\sigma \in E_2(0,M)} X(M)/W_\sigma(M). \tag{1}
\]

The disjoint union has the structure of a locally compact, Hausdorff space and is called the Harish-Chandra parameter space. The parametrization of the tempered dual $A^f_n(\mathbb{R})$ is due to Harish-Chandra, see [6].

**Proposition 2.1** (Harish-Chandra). [6] There exists a bijection

\[
\bigsqcup_{(M,\sigma)} X(M)/W_\sigma(M) \rightarrow A^f_n(\mathbb{R})
\]

\[\chi^\sigma \mapsto i_{GL(n),MN}(\chi^\sigma \otimes 1),\]

where $\chi^\sigma(x) := \chi(x)\sigma(x)$ for all $x \in M$.

In view of the above bijection, we will denote the Harish-Chandra parameter space by $A^f_n(\mathbb{R})$.

We will see now the particular features of the archimedean case, starting with $GL(n, \mathbb{R})$. Since the discrete series of $GL(n, \mathbb{R})$ is empty for $n \geq 3$, we only need to consider partitions of $n$ into 1’s and 2’s. This allows us to decompose $n$ as $n = 2q + r$, where $q$ is the number of 2’s and $r$ is the number of 1’s in the partition. To this decomposition we associate the partition

\[n = (2,...,2,1,...,1),\]

which corresponds to the Levi subgroup

\[M \cong GL(2, \mathbb{R}) \times ... \times GL(2, \mathbb{R}) \times GL(1, \mathbb{R}) \times ... \times GL(1, \mathbb{R}).\]

Varying $q$ and $r$ we determine a representative in each equivalence class of Levi subgroups. The subgroup $0M$ of $M$ is given by

\[0M \cong SL^\pm(2, \mathbb{R}) \times ... \times SL^\pm(2, \mathbb{R}) \times SL^\pm(1, \mathbb{R}) \times ... \times SL^\pm(1, \mathbb{R}).\]

where

\[SL^\pm(m, \mathbb{R}) = \{g \in GL(m, \mathbb{R}) : |\det(g)| = 1\}\]

is the unimodular subgroup of $GL(m, \mathbb{R})$. In particular, $SL^\pm(1, \mathbb{R}) = \{\pm1\} \cong \mathbb{Z}/2\mathbb{Z}$. 
The representations in the discrete series of $GL(2, \mathbb{R})$, denoted $D_\ell$ for $\ell \in \mathbb{N}$ ($\ell \geq 1$) are induced from $SL(2, \mathbb{R})$ \cite[p.399]{7}:

$$D_\ell = \text{ind}_{SL^\pm(2, \mathbb{R}), SL(2, \mathbb{R})} (D^\pm_\ell),$$

where $D^\pm_\ell$ acts in the space

$$\{ f : \mathcal{H} \rightarrow \mathbb{C} | f \text{ analytic}, \|f\|^2 = \int \int |f(z)|^2 y^{\ell-1} dx dy < \infty \}.$$ 

Here, $\mathcal{H}$ denotes the Poincaré upper half plane. The action of $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is given by

$$D^\pm_\ell(g)(f(z)) = (bz + d)^{-(\ell+1)} f\left(\frac{az + c}{bz + d}\right).$$

More generally, an element $\sigma$ from the discrete series $E_2^0(M)$ is given by

$$\sigma = i_{G,M,N} (D^\pm_{\ell_1} \otimes ... \otimes D^\pm_{\ell_q} \otimes \tau_1 \otimes ... \otimes \tau_r \otimes 1),$$

(2)

where $D^\pm_{\ell_i}$ ($\ell_i \geq 1$) are the discrete series representations of $SL^\pm(2, \mathbb{R})$ and $\tau_j$ is a representation of $SL^\pm(1, \mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$, i.e. $id = (x \mapsto x)$ or $sgn = (x \mapsto \frac{x}{|x|})$.

Finally we will compute the unramified characters $X(M)$, where $M$ is the Levi subgroup associated to the partition $n = 2q + r$.

Let $x \in GL(2, \mathbb{R})$. Any character of $GL(2, \mathbb{R})$ is given by

$$\chi(det(x)) = (sgn(det(x)))^\varepsilon |det(x)|^t \chi$$

($\varepsilon = 0, 1, t \in \mathbb{R}$) and it is unramified provided that

$$\chi(det(g)) = \chi(\pm 1) = (\pm 1)^\varepsilon = 1,$$

for all $g \in SL^\pm(2, \mathbb{R})$. This implies $\varepsilon = 0$ and any unramified character of $GL(2, \mathbb{R})$ has the form

$$\chi(x) = |det(x)|^t,$$

(3)

for some $t \in \mathbb{R}$.

Similarly, any unramified character of $GL(1, \mathbb{R}) = \mathbb{R}^\times$ has the form

$$\xi(x) = |x|^t,$$

(4)

for some $t \in \mathbb{R}$.

Given a block diagonal matrix $\text{diag}(g_1, ..., g_q, \omega_1, ..., \omega_r) \in M$, where $g_i \in GL(2, \mathbb{R})$ and $\omega_j \in GL(1, \mathbb{R})$, we conclude from (3) and (4) that any unramified character $\chi \in X(M)$ is given by

$$\chi(\text{diag}(g_1, ..., g_q, \omega_1, ..., \omega_r)) = |det(g_1)|^{t_1} \times ... \times |det(g_q)|^{t_q} \times |\omega_1|^{t_{q+1}} \times ... \times |\omega_r|^{t_{q+r}},$$

for some $(t_1, ..., t_{q+r}) \in \mathbb{R}^{q+r}$. We can denote such element $\chi \in X(M)$ by $\chi(t_1, ..., t_{q+r})$. We have the following result.
Proposition 2.2. Let $M$ be a Levi subgroup of $GL(n, \mathbb{R})$, associated to the partition $n = 2q + r$. Then, there is a bijection

$$X(M) \rightarrow \mathbb{R}^{q+r}, \chi(t_1, ..., t_{q+r}) \mapsto (t_1, ..., t_{q+r}).$$

Let us consider now $GL(n, \mathbb{C})$. The tempered dual of $GL(n, \mathbb{C})$ comprises the unitary principal series in accordance with Harish-Chandra [10, p. 277]. The corresponding Levi subgroup is a maximal torus $T \cong (\mathbb{C}^\times)^n$. It follows that $^0T \cong \mathbb{T}^n$ the compact $n$-torus.

The principal series representations are given by

$$\pi_{\ell,i} = i_{G,TV}(\sigma \otimes 1), \quad (5)$$

where $\sigma = \sigma_1 \otimes ... \otimes \sigma_n$ and $\sigma_j(z) = (\frac{z}{|z|^\ell_j})^{|z|t_j}$ ($\ell_j \in \mathbb{Z}$ and $t_j \in \mathbb{R}$).

An unramified character is given by

$$\chi \left( \begin{array}{c} z_1 \\ \vdots \\ z_n \end{array} \right) = |z_1|^{|t_1|} \times ... \times |z_n|^{|t_n|}$$

and we can represent $\chi$ as $\chi(t_1, ..., t_n)$. Therefore, we have the following result.

Proposition 2.3. Denote by $T$ the standard maximal torus in $GL(n, \mathbb{C})$. There is a bijection

$$X(T) \rightarrow \mathbb{R}^n, \chi(t_1, ..., t_n) \mapsto (t_1, ..., t_n).$$

3 Base change for archimedean fields

The Weil group attached to a local field $F$ will be denoted $W_F$ as in [13]. We may state the base change problem for archimedean fields in the following way. Consider the archimedean base change $\mathbb{C}/\mathbb{R}$. We have $W_\mathbb{C} \subset W_\mathbb{R}$ and there is a natural map

$$Res_{W_\mathbb{C}}^{W_\mathbb{R}} : G_n(\mathbb{R}) \rightarrow G_n(\mathbb{C}) \quad (6)$$

called restriction. By the local Langlands correspondence for archimedean fields [3 Theorem 3.1, p.236][7], there is a base change map

$$BC : A_n(\mathbb{R}) \rightarrow A_n(\mathbb{C}) \quad (7)$$
such that the following diagram commutes

\[
\begin{array}{ccc}
\mathcal{A}_n(\mathbb{R}) & \xrightarrow{BC} & \mathcal{A}_n(\mathbb{C}) \\
\downarrow \mathcal{L}_n & & \downarrow \mathcal{L}_n \\
\mathcal{G}_n(\mathbb{R}) & \xrightarrow{Res_{W_C}} & \mathcal{G}_n(\mathbb{C})
\end{array}
\]

Arthur and Clozel’s book [1] gives a full treatment of base change for $GL(n)$. The case of archimedean base change can be captured in an elegant formula [1, p. 71]. We briefly review these results.

Given a partition $n = 2q + r$ let $\chi_i (i = 1, ..., q)$ be a ramified character of $\mathbb{C}^\times$ and let $\xi_j (j = 1, ..., r)$ be a ramified character of $\mathbb{R}^\times$. Since the $\chi_i$’s are ramified, $\chi_i(z) \neq \chi_i^\sigma(z) = \chi_i(\overline{z})$. By Langlands classification [7], each $\chi_i$ defines a discrete series representation $\pi(\chi_i)$ of $GL(2, \mathbb{R})$, with $\pi(\chi_i) = \pi(\chi_i^\sigma)$.

Denote by $\pi(\chi_1, ..., \chi_q, \xi_1, ..., \xi_r)$ the generalized principal series representation of $GL(n, \mathbb{R})$

\[
\pi(\chi_1, ..., \chi_q, \xi_1, ..., \xi_r) = i_{GL(n, \mathbb{R}), MN}(\pi(\chi_1) \otimes ... \otimes \pi(\chi_q) \otimes \xi_1 \otimes ... \otimes \xi_r \otimes 1). \tag{8}
\]

The base change map for the general principal series representation is given by induction from the Borel subgroup $B(\mathbb{C})$ [1, p. 71]:

\[
BC(\pi) = \Pi(\chi_1, ..., \chi_q, \xi_1, ..., \xi_r) = i_{GL(n, \mathbb{C}), B(\mathbb{C})}(\chi_1, \chi_1^\sigma, ..., \chi_q, \chi_q^\sigma, \xi_1 \circ N, ..., \xi_r \circ N), \tag{9}
\]

where $N = N_{\mathbb{C}/\mathbb{R}} : \mathbb{C}^\times \longrightarrow \mathbb{R}^\times$ is the norm map defined by $z \mapsto z\overline{z}$.

We illustrate the base change map with two simple examples.

**Example 3.1.** For $n = 1$, base change is simply composition with the norm map

\[
BC : \mathcal{A}_1^1(\mathbb{R}) \rightarrow \mathcal{A}_1^1(\mathbb{C}) , \ BC(\chi) = \chi \circ N.
\]

**Example 3.2.** For $n = 2$, there are two different kinds of representations, one for each partition of $2$. According to (3), $\pi(\chi)$ corresponds to the partition $2 = 2 + 0$ and $\pi(\xi_1, \xi_2)$ corresponds to the partition $2 = 1 + 1$. Then the base change map is given, respectively, by

\[
BC(\pi(\chi)) = i_{GL(2, \mathbb{C}), B(\mathbb{C})}(\chi, \chi^\sigma),
\]

and

\[
BC(\pi(\xi_1, \xi_2)) = i_{GL(2, \mathbb{C}), B(\mathbb{C})}(\xi_1 \circ N, \xi_2 \circ N).
\]
4 \textit{K-theory for } GL(n, \mathbb{R}) \\

Using the Harish-Chandra parametrization of the tempered dual of \textit{GL}(n) (recall that the Harish-Chandra parameter space is a locally compact, Hausdorff topological space) we can compute the \textit{K}-theory of the reduced \textit{C}*-algebra \textit{C}*_r GL(n, \mathbb{R}).

We have

\[ K_j(\textit{C}*_r GL(n, \mathbb{R})) = K_j(\bigsqcup_{(M,\sigma)} X(M)/W_\sigma(M)) = \bigoplus_{(M,\sigma)} K^j(X(M)/W_\sigma(M)) \]

(10)

where \( n_M = q + r \) if \( M \) is a representative of the equivalence class of Levi subgroup associated to the partition \( n = 2q + r \). Hence the \textit{K}-theory depends on \( n \) and on each Levi subgroup.

To compute (11) we have to consider the following orbit spaces:

(i) \( \mathbb{R}^n \), in which case \( W_\sigma(M) \) is the trivial subgroup of the Weil group \( W(M) \);

(ii) \( \mathbb{R}^n/S_n \), where \( W_\sigma(M) = W(M) \) (this is one of the possibilities for the partition of \( n \) into 1’s);

(iii) \( \mathbb{R}^n/(S_{n_1} \times \cdots \times S_{n_k}) \), where \( W_\sigma(M) = S_{n_1} \times \cdots \times S_{n_k} \subset W(M) \) (see the examples below).

\textbf{Definition 4.1.} An orbit space as indicated in (ii) and (iii) is called a closed cone.

The \textit{K}-theory for \( \mathbb{R}^n \) may be summarized as follows

\[ K^j(\mathbb{R}^n) = \begin{cases} \mathbb{Z}, & n = j \text{(mod 2)} \\ 0, & \text{otherwise}. \end{cases} \]

The next results show that the \textit{K}-theory of a closed cone vanishes.

\textbf{Lemma 4.2.} \( K^j(\mathbb{R}^n/S_n) = 0, j = 0, 1 \).

\textit{Proof.} We need the following definition. A point \((a_1, \ldots, a_n) \in \mathbb{R}^n\) is called normalized if \( a_j \leq a_{j+1} \), for \( j = 1, 2, \ldots, n - 1 \). Therefore, in each orbit there is exactly one normalized point and \( \mathbb{R}^n/S_n \) is homeomorphic to the subset of \( \mathbb{R}^n \) consisting of all normalized points of \( \mathbb{R}^n \). We denote the set of all normalized points of \( \mathbb{R}^n \) by \( N(\mathbb{R}^n) \).
In the case of \( n = 2 \), let \((a_1, a_2)\) be a normalized point of \( \mathbb{R}^2 \). Then, there is a unique \( t \in [1, +\infty[ \) such that \( a_2 = ta_1 \) and the map

\[
\mathbb{R} \times [1, +\infty[ \to N(\mathbb{R}^2), (a, t) \mapsto (a, ta)
\]

is a homeomorphism.

If \( n > 2 \) then the map

\[
N(\mathbb{R}^{n-1}) \times [1, +\infty[ \to N(\mathbb{R}^n), (a_1, ..., a_{n-1}, t) \mapsto (a_1, ..., a_{n-1}, ta_n)
\]

is a homeomorphism. Since \([1, +\infty[ \) kills both the \( K \)-theory groups \( K^0 \) and \( K^1 \), the result follows by applying Künneth formula.

The symmetric group \( S_n \) acts on \( \mathbb{R}^n \) by permuting the components. This induces an action of any subgroup \( S_{n_1} \times ... \times S_{n_k} \) of \( S_n \) on \( \mathbb{R}^n \). Write

\[
\mathbb{R}^n \cong \mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_k} \times \mathbb{R}^{n-n_1-...-n_k}.
\]

If \( n = n_1 + ... + n_k \) then we simply have \( \mathbb{R}^n \cong \mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_k} \).

\( S_{n_1} \times ... \times S_{n_k} \) acts on \( \mathbb{R}^n \) as follows.

\( S_{n_1} \) permutes the components of \( \mathbb{R}^{n_1} \) leaving the remaining fixed;

\( S_{n_2} \) permutes the components of \( \mathbb{R}^{n_2} \) leaving the remaining fixed;

and so on. If \( n > n_1 + ... + n_k \) the components of \( \mathbb{R}^{n-n_1-...-n_k} \) remain fixed.

This can be interpreted, of course, as the action of the trivial subgroup. As a consequence, one identify the orbit spaces

\[
\mathbb{R}^n/(S_{n_1} \times ... \times S_{n_k}) \cong \mathbb{R}^{n_1}/S_{n_1} \times ... \times \mathbb{R}^{n_k}/S_{n_k} \times \mathbb{R}^{n-n_1-...-n_k}.
\]

**Lemma 4.3.** \( K^j(\mathbb{R}^n/(S_{n_1} \times ... \times S_{n_k})) = 0, j = 0, 1, \) where \( S_{n_1} \times ... \times S_{n_k} \subset S_n \).

**Proof.** It suffices to prove for \( \mathbb{R}^n/(S_{n_1} \times S_{n_2}) \). The general case follows by induction on \( k \).

Now, \( \mathbb{R}^n/(S_{n_1} \times S_{n_2}) \cong \mathbb{R}^{n_1}/S_{n_1} \times \mathbb{R}^{n-n_1}/S_{n_2} \). Applying the Künneth formula and Lemma 4.2, the result follows.

We give now some examples by computing \( K^*_j GL(n, \mathbb{R}) \) for small \( n \).

**Example 4.4.** We start with the case of \( GL(1, \mathbb{R}) \). We have:

\[
M = \mathbb{R}^\times, \ 0M = \mathbb{Z}/2\mathbb{Z}, \ W(M) = 1 \text{ and } X(M) = \mathbb{R}.
\]

Hence,

\[
\mathcal{A}_1^1(\mathbb{R}) \cong \bigcup_{\sigma \in (\mathbb{Z}/2\mathbb{Z})} \mathbb{R}/1 = \mathbb{R} \sqcup \mathbb{R}, \tag{11}
\]

and the \( K \)-theory is given by

\[
K_j^* GL(1, \mathbb{R}) \cong K^j(\mathcal{A}_1(\mathbb{R})) = K^j(\mathbb{R} \sqcup \mathbb{R}) = K^j(\mathbb{R}) \oplus K^j(\mathbb{R}) = \left\{ \begin{array}{ll}
\mathbb{Z} \oplus \mathbb{Z}, & j = 1 \\
0, & j = 0.
\end{array} \right.
\]
Example 4.5. For $GL(2, \mathbb{R})$ we have two partitions of $n = 2$ and the following data

| Partition | $M$ | $^0M$ | $W(M)$ | $X(M)$ | $\sigma \in E_2(^0M)$ |
|-----------|-----|-------|--------|--------|------------------|
| $2+0$     | $GL(2, \mathbb{R})$ | $SL^+(2, \mathbb{R})$ | 1       | $\mathbb{R}$ | $\sigma = i_{G,P}(D^+_\ell)$, $\ell \in \mathbb{N}$ |
| $1+1$     | $\mathbb{R}^2$       | $(\mathbb{Z}/2\mathbb{Z})^2$ | $\mathbb{Z}/2\mathbb{Z}$ | $\mathbb{R}^2$ | $\sigma = i_{G,P}(id \otimes sgn)$ |

Then the tempered dual is parameterized as follows

$$A^1_3(\mathbb{R}) \cong \bigsqcup_{(M, \sigma)} X(M)/W_\sigma(M) = (\bigsqcup_{\ell \in \mathbb{N}} \mathbb{R}) \cup (\mathbb{R}^2/S_2) \cup (\mathbb{R}^2/S_2) \cup \mathbb{R}$$

and the $K$-theory groups are given by

$$K_jC^*_r GL(2, \mathbb{R}) \cong K^j(A^1_3(\mathbb{R})) \cong (\bigoplus_{\ell \in \mathbb{N}} K^j(\mathbb{R})) \oplus K^j(\mathbb{R}^2) = \begin{cases} \bigoplus_{\ell \in \mathbb{N}} \mathbb{Z} & , j = 1 \\ \mathbb{Z} & , j = 0. \end{cases}$$

Example 4.6. For $GL(3, \mathbb{R})$ there are two partitions for $n = 3$, to which correspond the following data

| Partition | $M$ | $^0M$ | $W(M)$ | $X(M)$ |
|-----------|-----|-------|--------|--------|
| $2+1$     | $GL(3, \mathbb{R}) \times \mathbb{R}^3$ | $SL^+(2, \mathbb{R}) \times (\mathbb{Z}/2\mathbb{Z})^3$ | 1       | $\mathbb{R}^2$ |
| $1+1+1$   | $\mathbb{R}^3$       | $(\mathbb{Z}/2\mathbb{Z})^3$       | $S_3$   | $\mathbb{R}^3$ |

For the partition $3 = 2 + 1$, an element $\sigma \in E_2(^0M)$ is given by

$$\sigma = i_{G,P}(D^+_\ell \otimes \tau) , \ell \in \mathbb{N} \text{ and } \tau \in (\mathbb{Z}/2\mathbb{Z}).$$

For the partition $3 = 1 + 1 + 1$, an element $\sigma \in E_2(^0M)$ is given by

$$\sigma = i_{G,P}(\bigotimes_{i=1}^3 \tau_i) , \tau_i \in (\mathbb{Z}/2\mathbb{Z}).$$

The tempered dual is parameterized as follows

$$A^1_3(\mathbb{R}) \cong \bigsqcup_{(M, \sigma)} X(M)/W_\sigma(M) = \bigsqcup_{N \times (\mathbb{Z}/2\mathbb{Z})} (\mathbb{R}^2/1) \bigsqcup_{(\mathbb{Z}/2\mathbb{Z})^3} (\mathbb{R}^3/S_3).$$

The $K$-theory groups are given by

$$K_jC^*_r GL(3, \mathbb{R}) \cong K^j(A^1_3(\mathbb{R})) \cong \bigoplus_{N \times (\mathbb{Z}/2\mathbb{Z})} K^j(\mathbb{R}^2) \oplus 0 = \begin{cases} \bigoplus_{N \times (\mathbb{Z}/2\mathbb{Z})} \mathbb{Z} & , j = 0 \\ 0 & , j = 1. \end{cases}$$
The general case of $GL(n, \mathbb{R})$ will now be considered. It can be split in two cases: $n$ even and $n$ odd.

- $n = 2q$ even

Suppose $n$ is even. For every partition $n = 2q + r$, either $W_{\sigma}(M) = 1$ or $W_{\sigma}(M) \neq 1$. If $W_{\sigma}(M) \neq 1$ then $\mathbb{R}^{n_{\sigma}}/W_{\sigma}(M)$ is a cone and the $K$-groups $K^0$ and $K^1$ both vanish. This happens precisely when $r > 2$ and therefore we have only two partitions, corresponding to the choices of $r = 0$ and $r = 2$, which contribute to the $K$-theory with non-zero $K$-groups.

| Partition | $M$ | $^0M$ | $W(M)$ |
|-----------|-----|-------|--------|
| $2q$      | $GL(2, \mathbb{R})^q$ | $SL^\pm(2, \mathbb{R})^q$ | $S_q$ |
| $2(q - 1) + 2$ | $GL(2, \mathbb{R})^{q-1} \times (\mathbb{R}^\times)^2$ | $SL^\pm(2, \mathbb{R})^{q-1} \times (\mathbb{Z}/2\mathbb{Z})^2$ | $S_{q-1} \times (\mathbb{Z}/2\mathbb{Z})$ |

We also have $X(M) \cong \mathbb{R}^q$ for $n = 2q$, and $X(M) \cong \mathbb{R}^{q+1}$, for $n = 2(q - 1) + 2$.

For the partition $n = 2q$ $(r = 0)$, an element $\sigma \in E_2(0M)$ is given by

$$\sigma = i_{G,P}(\mathcal{D}_{\ell_1}^+ \otimes ... \otimes \mathcal{D}_{\ell_q}^+), (\ell_1, ... \ell_q) \in \mathbb{N}^q \text{ and } \ell_i \neq \ell_j \text{ if } i \neq j.$$ 

For the partition $n = 2(q - 1) + 2$ $(r = 2)$, an element $\sigma \in E_2(0M)$ is given by

$$\sigma = i_{G,P}(\mathcal{D}_{\ell_1}^+ \otimes ... \otimes \mathcal{D}_{\ell_{q-1}}^+ \otimes \text{id} \otimes \text{sgn}), (\ell_1, ... \ell_{q-1}) \in \mathbb{N}^{q-1} \text{ and } \ell_i \neq \ell_j \text{ if } i \neq j.$$ 

Therefore, the tempered dual has the following form

$$\mathcal{A}_{\text{t}}^i(\mathbb{R}) = \mathcal{A}_{\text{t}}^i(\mathbb{R}) = \bigsqcup_{\ell \in \mathbb{N}^q} \mathbb{R}^q \sqcup \bigsqcup_{\ell \in \mathbb{N}^{q-1}} \mathbb{R}^{q+1} \sqcup \mathcal{C}$$

where $\mathcal{C}$ is a disjoint union of closed cones as in Definition 4.1.

**Theorem 4.7.** Suppose $n = 2q$ even. Then the $K$-groups are

$$K_j C^*_{\text{r}} GL(n, \mathbb{R}) \cong \begin{cases} \bigoplus_{\ell \in \mathbb{N}^q} \mathbb{Z} & j \equiv q (\text{mod} 2) \\ \bigoplus_{\ell \in \mathbb{N}^{q-1}} \mathbb{Z} & \text{otherwise.} \end{cases}$$

If $q = 1$ then the direct sum $\bigoplus_{\ell \in \mathbb{N}^{q-1}} \mathbb{Z}$ will denote a single copy of $\mathbb{Z}$.

- $n = 2q + 1$ odd

If $n$ is odd only one partition contributes to the $K$-theory of $GL(n, \mathbb{R})$ with non-zero $K$-groups:
An element \( \sigma \in E_{2q}(0M) \) is given by

\[
\sigma = \iota_{G,P}(D_{\ell_1}^+ \otimes \cdots \otimes D_{\ell_q}^+ \otimes \tau), \quad (\ell_1, \ldots, \ell_q, \tau) \in \mathbb{N}^q \times (\mathbb{Z}/2\mathbb{Z}) \text{ and } \ell_i \neq \ell_j \text{ if } i \neq j.
\]

The tempered dual is given by

\[
A_t(n) = A_{2q+1}(R) = \bigsqcup_{\ell \in (\mathbb{N}^q \times (\mathbb{Z}/2\mathbb{Z}))} \mathbb{R}^{q+1} \sqcup C
\]

where \( C \) is a disjoint union of closed cones as in Definition 4.1.

Theorem 4.8. Suppose \( n = 2q + 1 \) is odd. Then the \( K \)-groups are

\[
K_jC^*GL(n, \mathbb{R}) \cong \begin{cases} \bigoplus_{\ell \in (\mathbb{N}^q \times (\mathbb{Z}/2\mathbb{Z}))} \mathbb{Z} & , \, j \equiv q + 1 \,(\text{mod}\,2) \\ 0 & , \, \text{otherwise} \end{cases}
\]

Here, we use the following convention: if \( q = 0 \) then the direct sum is \( \bigoplus_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z} \cong \mathbb{Z} \oplus \mathbb{Z} \).

We conclude that the \( K \)-theory of \( C^*GL(n, \mathbb{R}) \) depends on essentially one parameter \( q \) given by the maximum number of 2’s in the partitions of \( n \) into 1’s and 2’s. If \( n \) is even then \( q = \frac{n}{2} \) and if \( n \) is odd then \( q = \frac{n-1}{2} \).

5  **\( K \)-theory for \( GL(n, \mathbb{C}) \)**

Let \( T^0 \) be the maximal compact subgroup of the maximal compact torus \( T \) of \( GL(n, \mathbb{C}) \). Let \( \sigma \) be a unitary character of \( T^0 \). We note that \( W = W(T), \) \( W_\sigma = W_\sigma(T) \). If \( W_\sigma = 1 \) then we say that the orbit \( W \cdot \sigma \) is generic.

Theorem 5.1. The \( K \)-theory of \( C^*GL(n, \mathbb{C}) \) admits the following description. If \( n = j \mod 2 \) then \( K_j \) is free abelian on countably many generators, one for each generic \( W \)-orbit in the unitary dual of \( T^0 \), and \( K_{j+1} = 0 \).

Proof. We have a homeomorphism of locally compact Hausdorff spaces:

\[
A_t^i(\mathbb{C}) \cong \bigsqcup X(T)/W_\sigma(T)
\]

by the Harish-Chandra Plancherel Theorem for complex reductive groups, and the identification of the Jacobson topology on the left-hand-side with the natural topology on the right-hand-side, as in [11]. The result now follows from Lemma 4.3.
6 The base change map

In this section we define base change as a map of topological spaces and study the induced $K$-theory map.

**Proposition 6.1.** The base change map $BC: \mathcal{A}_n^t(\mathbb{R}) \to \mathcal{A}_n^t(\mathbb{C})$ is a continuous proper map.

**Proof.** First, we consider the case $n = 1$.

As we have seen in Example 3.1, base change for $GL(1)$ is the map given by $BC(\chi) = \chi \circ N$, for all characters $\chi \in \mathcal{A}_1^t(\mathbb{R})$, where $N: \mathbb{C}^\times \to \mathbb{R}^\times$ is the norm map.

Let $z \in \mathbb{C}^\times$. We have

$$BC(\chi)(z) = \chi(|z|^2) = |z|^{2it}. \quad (12)$$

A generic element from $\mathcal{A}_1^t(\mathbb{C})$ has the form

$$\mu(z) = \left(\frac{z}{|z|}\right)^\ell |z|^{it}, \quad (13)$$

where $\ell \in \mathbb{Z}$ and $t \in S^1$, as stated before. Viewing the Pontryagin duals $\mathcal{A}_1^t(\mathbb{R})$ and $\mathcal{A}_1^t(\mathbb{C})$ as topological spaces by forgetting the group structure, and comparing (12) and (13), the base change map can be defined as the following continuous map

$$\varphi: \mathcal{A}_1^t(\mathbb{R}) \cong \mathbb{R} \times (\mathbb{Z}/2\mathbb{Z}) \quad \mapsto \quad \mathcal{A}_1^t(\mathbb{C}) \cong \mathbb{R} \times \mathbb{Z}$$

$$\chi = (t, \varepsilon) \quad \mapsto \quad (2t, 0)$$

A compact subset of $\mathbb{R} \times \mathbb{Z}$ in the connected component $\{\ell\}$ of $\mathbb{Z}$ has the form $K \times \{\ell\} \subset \mathbb{R} \times \mathbb{Z}$, where $K \subset \mathbb{R}$ is compact. We have

$$\varphi^{-1}(K \times \{\ell\}) = \begin{cases} \emptyset & \text{if } \ell \neq 0 \\ \frac{1}{2}K \times \{\varepsilon\} & \text{if } \ell = 0, \end{cases}$$

where $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$. Therefore $\varphi^{-1}(K \times \{\ell\})$ is compact and $\varphi$ is proper.

The Case $n > 1$

Base change determines a map $BC: \mathcal{A}_n^t(\mathbb{R}) \to \mathcal{A}_n^t(\mathbb{C})$ of topological spaces. Let $X = X(M)/W_\sigma(M)$ be a connected component of $\mathcal{A}_n^t(\mathbb{R})$. Then, $X$ is mapped under $BC$ into a connected component $Y = Y(T)/W_\sigma'(T)$ of $\mathcal{A}_n^t(\mathbb{C})$. Given a generalized principal series representation

$$\pi(\chi_1, \ldots, \chi_q, \xi_1, \ldots, \xi_r)$$
where the $\chi_i$'s are ramified characters of $\mathbb{C}^\times$ and the $\xi$'s are ramified characters of $\mathbb{R}^\times$, then

$$BC(\pi) = i_{G,B}(\chi_1, \chi_1^\tau, ..., \chi_q^\tau, \xi_1 \circ N, ..., \xi_r \circ N).$$

Here, $N = N_{\mathbb{C}/\mathbb{R}}$ is the norm map and $\tau$ is the generator of $Gal(\mathbb{C}/\mathbb{R})$.

We associate to $\pi$ the usual parameters uniquely defined for each character $\chi$ and $\xi$. For simplicity, we write the set of parameters as a $(q+r)$-uple:

$$(t, t') = (t_1, ..., t_q, t'_1, ..., t'_r) \in \mathbb{R}^{q+r} \cong X(M).$$

Now, if $\pi(\chi_1, ..., \chi_q, \xi_1, ..., \xi_r)$ lies in the connected component defined by the fixed parameters $(t, \varepsilon) \in \mathbb{Z}^q \times (\mathbb{Z}/2\mathbb{Z})^r$, then

$$(t, t') \in X(M) \mapsto (t, t, 2t') \in Y(T)$$

is a continuous proper map.

It follows that

$$BC : X(M)/W_\sigma(M) \to Y(T)/W_\sigma'(T)$$

is continuous and proper since the orbit spaces are endowed with the quotient topology. \qed

**Theorem 6.2.** The functorial map induced by base change

$$K_j(C_\tau^*GL(n, \mathbb{C})) \xrightarrow{K_j(BC)} K_j(C_\tau^*GL(n, \mathbb{R}))$$

is zero for $n > 1$.

*Proof.* The case $n > 2$

We start with the case $n > 2$. Let $n = 2q + r$ be a partition and $M$ the associated Levi subgroup of $GL(n, \mathbb{R})$. Denote by $X_{\mathbb{R}}(M)$ the unramified characters of $M$. As we have seen, $X_{\mathbb{R}}(M)$ is parametrized by $\mathbb{R}^{q+r}$. On the other hand, the only Levi subgroup of $GL(n, \mathbb{C})$ for $n = 2q+r$ is the diagonal subgroup $X_{\mathbb{C}}(M) = (\mathbb{C}^\times)^{2q+r}$.

If $q = 0$ then $r = n$ and both $X_{\mathbb{R}}(M)$ and $X_{\mathbb{C}}(M)$ are parametrized by $\mathbb{R}^n$. But then in the real case an element $\sigma \in E_2(0,M)$ is given by

$$\sigma = i_{GL(n,\mathbb{R}),P}(\chi_1 \otimes ... \otimes \chi_n),$$

with $\chi_i \in \mathbb{Z}/2\mathbb{Z}$. Since $n \geq 3$ there is always repetition of the $\chi_i$'s. It follows that the isotropy subgroups $W_\sigma(M)$ are all nontrivial and the quotient spaces $\mathbb{R}^n/W_\sigma$ are closed cones. Therefore, the $K$-theory groups vanish.
If \( q \neq 0 \), then \( X_R(M) \) is parametrized by \( \mathbb{R}^{q+r} \) and \( X_C(M) \) is parametrized by \( \mathbb{R}^{2q+r} \) (see Propositions 2.2 and 2.3).

Base change creates a map

\[
\mathbb{R}^{q+r} \rightarrow \mathbb{R}^{2q+r}.
\]

Composing with the stereographic projections we obtain a map

\[
S^{q+r} \rightarrow S^{2q+r}
\]

between spheres. Any such map is nullhomotopic [4, Proposition 17.9]. Therefore, the induced \( K \)-theory map

\[
K^j(S^{2q+r}) \rightarrow K^j(S^{q+r})
\]

is the zero map.

The Case \( n = 2 \)

For \( n = 2 \) there are two Levi subgroups of \( GL(2, \mathbb{R}) \), \( M_1 \cong GL(2, \mathbb{R}) \) and the diagonal subgroup \( M_2 \cong (\mathbb{R}^\times)^2 \). By Proposition 2.2 \( X(M_1) \) is parametrized by \( \mathbb{R} \) and \( X(M_2) \) is parametrized by \( \mathbb{R}^2 \). The group \( GL(2, \mathbb{C}) \) has only one Levi subgroup, the diagonal subgroup \( M \cong (\mathbb{C}^\times)^2 \). From Proposition 2.3 it is parametrized by \( \mathbb{R}^2 \).

Since \( K^1(\mathcal{A}_2^i(\mathbb{C})) = 0 \) by Theorem 5.1, we only have to consider the \( K^0 \) functor. The only contribution to \( K^0(\mathcal{A}_2^i(\mathbb{R})) \) comes from \( M_2 \cong (\mathbb{R}^\times)^2 \) and we have (see Example 4.5)

\[
K^0(\mathcal{A}_2^i(\mathbb{R})) \cong \mathbb{Z}.
\]

For the Levi subgroup \( M_2 \cong (\mathbb{R}^\times)^2 \), base change is

\[
BC : \mathcal{A}_2^i(\mathbb{R}) \rightarrow \mathcal{A}_2^i(\mathbb{C})
\]

\[
\pi(\xi_1, \xi_2) \mapsto i_{GL(2, \mathbb{C}), B(\mathbb{C})}(\xi_1 \circ N, \xi_2 \circ N),
\]

Therefore, it maps a class \([t_1, t_2]\), which lies in the connected component \((\varepsilon_1, \varepsilon_2)\), into the class \([2t_1, 2t_2]\), which lies in the connect component \((0, 0)\). In other words, base change maps a generalized principal series \( \pi(\xi_1, \xi_2) \) into a nongeneric point of \( \mathcal{A}_2^i(\mathbb{C}) \). It follows from Theorem 5.1 that

\[
K^0(BC) : K^0(\mathcal{A}_2^i(\mathbb{R})) \rightarrow K^0(\mathcal{A}_2^i(\mathbb{C}))
\]

is the zero map. \( \square \)
7 Base change in one dimension

In this section we consider base change for $GL(1)$.

**Theorem 7.1.** The functorial map induced by base change

$$K_1(C^*_r GL(1, \mathbb{C})) \xrightarrow{K_1(BC)} K_1(C^*_r GL(1, \mathbb{R}))$$

is given by $K_1(BC) = \Delta \circ Pr$, where $Pr$ is the projection of the zero component of $K^1(\mathcal{A}_1'(\mathbb{C}))$ into $\mathbb{Z}$ and $\Delta$ is the diagonal $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$.

**Proof.** For $GL(1)$, base change

$$\chi \in \mathcal{A}_1'(\mathbb{R}) \mapsto \chi \circ N_{\mathbb{C}/\mathbb{R}} \in \mathcal{A}_1'(\mathbb{C})$$

induces a map

$$K^1(BC) : K^1(\mathcal{A}_1'(\mathbb{C})) \to K^1(\mathcal{A}_1'(\mathbb{R})).$$

Any character $\chi \in \mathcal{A}_1'(\mathbb{R})$ is uniquely determined by a pair of parameters $(t, \varepsilon) \in \mathbb{R} \times \mathbb{Z}/2\mathbb{Z}$. Similarly, any character $\mu \in \mathcal{A}_1'(\mathbb{C})$ is uniquely determined by a pair of parameters $(t, \ell) \in \mathbb{R} \times \mathbb{Z}$. The discrete parameter $\varepsilon$ (resp., $\ell$) labels each connected component of $\mathcal{A}_1'(\mathbb{R}) = \mathbb{R} \sqcup \mathbb{R}$ (resp., $\mathcal{A}_1'(\mathbb{C}) = \bigsqcup_{\mathbb{Z}} \mathbb{R}$).

Base change maps each component $\varepsilon$ of $\mathcal{A}_1'(\mathbb{R})$ into the component 0 of $\mathcal{A}_1'(\mathbb{C})$, sending $t \in \mathbb{R}$ to $2t \in \mathbb{R}$. The map $t \mapsto 2t$ is homotopic to the identity. At the level of $K^1$, the base change map may be described by the following commutative diagram

$$\begin{array}{ccc}
\oplus_{\mathbb{R}} \mathbb{Z} & \xrightarrow{K^1(BC)} & \mathbb{Z} \oplus \mathbb{Z} \\
\downarrow Pr & & \downarrow \Delta \\
\mathbb{Z} & & \\
\end{array}$$

Here, $Pr$ is the projection of the zero component of $K^1(\mathcal{A}_1'(\mathbb{C})) \cong \oplus_{\mathbb{R}} \mathbb{Z}$ into $\mathbb{Z}$ and $\Delta$ is the diagonal map. \qed

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