CONSTRUCTION OF HAMILTONIAN-STATIONARY
LAGRANGIAN SUBMANIFOLDS OF CONSTANT CURVATURE \( \varepsilon \)
IN COMPLEX SPACE FORMS \( \tilde{M}^n(4\varepsilon) \)

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Abstract. Lagrangian submanifolds of a Kaehler manifold are called Hamiltonian-stationary (or \( H \)-stationary for short) if it is a critical point of the area functional restricted to compactly supported Hamiltonian variations. In [11] an effective method to constructing Lagrangian submanifolds of constant curvature \( \varepsilon \) in complex space form \( M^n(4\varepsilon) \) was introduced. In this article we survey recent results on construction of Hamiltonian-stationary Lagrangian submanifolds in complex space forms using this method.

1. Introduction.

Let \( \tilde{M}^n(4\varepsilon) \) denote the complex projective \( n \)-space \( CP^n(4\varepsilon) \), the complex Euclidean \( n \)-space \( C^n \) or the complex hyperbolic \( n \)-space \( CH^n(4\varepsilon) \) according to \( \varepsilon > 0, \varepsilon = 0 \) or \( \varepsilon < 0 \), respectively.

The Kaehler 2-form \( \omega \) is defined by \( \omega(\cdot,\cdot) = \langle J\cdot,\cdot \rangle \), where \( J \) is the complex structure. An isometric immersion \( \psi: M^n \to \tilde{M}^n(4\varepsilon) \) of an \( n \)-manifold \( M \) into \( \tilde{M}^n(4\varepsilon) \) is called Lagrangian if \( \psi^*\omega = 0 \) on \( M \). A vector field \( X \) on \( \tilde{M}^n(4\varepsilon) \) is called Hamiltonian if \( L_X \omega = f\omega \) for some smooth function \( f \) on \( \tilde{M}^n(4\varepsilon) \), where \( L \) is the Lie derivative. Thus, there exists a smooth real-valued function \( \varphi \) on \( \tilde{M}^n(4\varepsilon) \) such that \( X = J\nabla \varphi \), where \( \nabla \) is the gradient. The diffeomorphisms of the flux \( \phi_t \) of \( X \) transform Lagrangian submanifolds into Lagrangian submanifolds.

A normal vector field \( \xi \) to a Lagrangian immersion \( \psi: M^n \to \tilde{M}^n(4\varepsilon) \) is called Hamiltonian if \( \xi = J\nabla f \), where \( f \) is a smooth function on \( M^n \) and \( \nabla f \) is the gradient of \( f \) with respect to the induced metric. If \( f \in C_0^\infty(M) \) and \( \psi_t: M \to \tilde{M}^n(4\varepsilon) \) is a variation of \( \psi \) with \( \psi_0 = \psi \) and variational vector field \( \xi \), then the first variation of the volume functional is

\[
\frac{d}{dt}_{|t=0} \text{vol}(M, \psi_t^*g) = -\int_M f \text{div}JHdM,
\]

where \( H \) is the mean curvature vector of \( \psi \) and \( \text{div} \) is the divergence on \( M^n \). Critical points of this variational functional are called \( H \)-stationary or Hamiltonian-stationary (cf. [19]). Among others, \( H \)-stationary Lagrangian submanifolds in complex space forms have been studied in [1]-[10], [12]-[19].

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An effective method using twisted products for constructing Lagrangian immersions of a real space form $M^n(\varepsilon)$ into a complex space form $\tilde{M}^n(4\varepsilon)$ was developed by Chen, Dillen, Verstraelen and Vrancken in [11].

One main result of [11] states that if the twistor form of a twisted product decomposition $TP^n_{f_1\ldots f_\ell}(\varepsilon)$ of a simply-connected real space form $M^n(\varepsilon)$ of constant curvature $\varepsilon$ is twisted closed, then it admits a "unique" adapted Lagrangian immersion:

$$L_{f_1\ldots f_\ell} : TP^n_{f_1\ldots f_\ell}(\varepsilon) \rightarrow \tilde{M}^n(4\varepsilon).$$

Conversely, if $L : M^n(\varepsilon) \rightarrow \tilde{M}^n(4\varepsilon)$ is a non-totally geodesic Lagrangian immersion, then $M^n(\varepsilon)$ admits a twisted product decomposition with twisted closed twistor form; moreover, the Lagrangian immersion is given by the adapted Lagrangian immersion of the twisted product decomposition. A twisted product decomposition of a real space form is called a warped product decomposition if it is a warped product.

In this article we survey recent results concerning construction of Hamiltonian-stationary Lagrangian submanifolds in complex space forms using this effective method of [11].

2. Preliminaries.

2.1. Basic notation and formulas. Let $L : M \rightarrow \tilde{M}^n(4\varepsilon)$ be an isometric immersion of a Riemannian $n$-manifold $M$ into $\tilde{M}^n(4\varepsilon)$. Denote the Riemannian connections of $M$ and $\tilde{M}^n(4\varepsilon)$ by $\nabla$ and $\tilde{\nabla}$, respectively; and by $D$ the connection on the normal bundle of the submanifold. Let $R$ denote the curvature tensor of $\nabla$.

The formulas of Gauss and Weingarten are

\begin{align*}
\tilde{\nabla}X Y &= \nabla X Y + h(X,Y), \\
\tilde{\nabla}X \xi &= -A_\xi X + D_X \xi
\end{align*}

for tangent vector fields $X,Y$ and normal vector field $\xi$.

If $L : M \rightarrow \tilde{M}^n(4\varepsilon)$ is a Lagrangian immersion, then the equations of Gauss and Codazzi are given respectively by

\begin{align*}
\langle R(X,Y)Z,W \rangle &= \langle h(X,W), h(Y,Z) \rangle - \langle h(X,Z), h(Y,W) \rangle \\
&\quad + \varepsilon \{ \langle X, W \rangle \langle Y, Z \rangle - \langle X, Z \rangle \langle Y, W \rangle \} ,
(2.3)
\end{align*}

\begin{align*}
\langle \nabla h \rangle(\nabla h)(X,Y,Z) &= (\nabla h)(Y,X,Z) ,
(2.4)
\end{align*}

where

\begin{align*}
\langle \nabla h \rangle(X,Y,Z) &= D_X h(Y,Z) - h(\nabla_X Y,Z) - h(Y,\nabla_X Z).
\end{align*}

For the Lagrangian immersion we also have (cf. [14])

\begin{align*}
D_X JY &= J\nabla_X Y ,
(2.5)
\langle h(X,Y), JZ \rangle &= \langle h(Y,Z), JX \rangle = \langle h(Z,X), JY \rangle .
(2.6)
\end{align*}

At a given point $p$ on the Lagrangian submanifold $M$, the relative null space $N_p$ at $p$ is the subspace of the tangent space $T_pM$ defined by

\begin{align*}
N_p &= \{ X \in T_pM : h(X,Y) = 0 \forall Y \in T_pM \}. 
\end{align*}
The dimension of $\mathcal{N}_p$ is called the \textit{relative nullity} at \( p \).

2.2. \textbf{Lagrangian and Legendrian submanifolds}. We recall a general method from [21] for constructing Lagrangian submanifolds via Hopf’s fibration.

\textbf{Case (1):} \( CP^n(4) \). Let 
\[ S^{2n+1}(1) = \{(z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} : \langle z, z \rangle = 1 \} \]
be the unit hypersphere in \( \mathbb{C}^{n+1} \) centered at the origin. On \( S^{2n+1}(1) \) we consider the canonical Sasakian structure consisting of \( \hat{\psi} \) induced from the complex structure \( J \) and the structure vector field \( \xi = Jx \) with \( x \) being the position vector.

An isometric immersion \( \psi: M \to S^{2n+1}(1) \) is called \textit{Legendrian} if \( \xi \) is normal to \( f_\ast(TM) \) and \( \langle \phi(\psi_\ast(TM)), \psi_\ast(TM) \rangle = 0 \), where \( \langle , \rangle \) denotes the inner product on \( \mathbb{C}^{n+1} \). The vectors of \( S^{2n+1}(1) \) normal to \( \xi \) at a point \( z \) define the horizontal subspace \( \mathcal{H}_z \) of the Hopf fibration:
\[ \pi: S^{2n+1}(1) \to CP^n(4). \]

Let \( \hat{\psi}: M \to CP^n(4) \) be a Lagrangian isometric immersion. Then there is an isometric covering map \( \tau: \hat{M} \to M \) and a Legendrian immersion \( \hat{\psi}: \hat{M} \to S^{2n+1}(1) \) such that \( \hat{\psi}(\tau) = \pi(\hat{\psi}) \). Hence every Lagrangian immersion can be lifted locally (or globally if we assume the manifold is simply connected) to a Legendrian immersion of the same Riemannian manifold.

Conversely, suppose that \( \psi: \hat{M} \to S^{2n+1}(1) \) is a Legendrian immersion. Then \( \hat{\psi} = \pi(\hat{\psi}): M \to CP^n(4) \) is a Lagrangian isometric immersion. Under this correspondence, the second fundamental forms \( h^\psi \) and \( h^{\hat{\psi}} \) of \( \psi \) and \( \hat{\psi} \) satisfy \( \pi_\ast h^\psi = h^{\hat{\psi}} \). We shall denote \( h^\psi \) and \( h^{\hat{\psi}} \) simply by \( h \).

\textbf{Case (2):} \( CH^n(-4) \). Consider the complex number space \( \mathbb{C}_1^{n+1} \) with the pseudo Euclidean metric: \( g_0 = -dz_1d\bar{z}_1 + \sum_{j=2}^{n+1} dz_j d\bar{z}_j \). Put 
\[ H_2^{2n+1}(-1) = \{ z = (z_1, z_2, \ldots, z_{n+1}) : \langle z, z \rangle = -1 \}, \]
where \( \langle , \rangle \) is the inner product on \( \mathbb{C}_1^{n+1} \) induced from \( g_0 \).

Put
\[ T_z^1 = \{ z \in \mathbb{C}^{n+1} : \Re \langle u, z \rangle = \Re \langle u, iz \rangle = 0 \} \]
and
\[ H_1^1 = \{ \lambda \in \mathbb{C} : |\lambda| = 1 \}. \]
Then we have an \( H_1^1 \)-action on \( H_2^{2n+1}(-1) \), \( z \mapsto \lambda z \) and at each point \( z \in H_2^{2n+1}(-1) \), the vector \( iz \) is tangent to the flow of the action. Since the metric \( g_0 \) is Hermitian, we have \( \Re g_0(iz, iz) = -1 \). The orbit lies in the negative definite plane spanned by \( z \) and \( iz \). The quotient space \( H_2^{2n+1}/\sim \), under the identification from the action, is the complex hyperbolic space \( CH^n(-4) \) with holomorphic sectional curvature \(-4\), with the complex structure \( J \) induced from the canonical complex structure \( J \) on \( \mathbb{C}_1^{n+1} \) via the following pseudo-Riemannian submersion:
\[ \pi: H_2^{2n+1}(-1) \to CH^n(-4). \]

Just as in Case (1), let \( g: M \to CH^n(-4) \) be a Lagrangian isometric immersion. Then there exists an isometric covering map \( \tau: \hat{M} \to M \), and a Legendrian
isometric immersion \( f : \hat{M} \to H^1_{2n+1}(-1) \) such that \( g(\tau) = \pi(f) \). Hence every Lagrangian immersion can be lifted locally to a Legendrian immersion.

Conversely, let \( f : \hat{M} \to H^1_{2n+1}(-1) \) be a Legendrian immersion. Then \( g = \pi(f) : M \to CH^n(-4) \) is again a Lagrangian isometric immersion. Similarly, under this correspondence, the second fundamental forms \( h^f \) and \( h^g \) of \( f \) and \( g \) satisfy \( \pi_* h^f = h^g \). We shall also denote \( h^f \) and \( h^g \) simply by \( h \).

Assume that \( M \) is a submanifold of \( S^{2n+1}(1) \) or \( H^1_{2n+1}(1) \). Denote by \( \hat{\nabla} \) and \( \nabla \) the Levi-Civita connections of \( C^n+1 \) or \( C^1_{n+1} \) and of \( M \), respectively. Let \( h \) be the second fundamental form of \( M \) in \( S^{2n+1}(1) \) or \( H^1_{2n+1}(1) \). Then we have

\[
\hat{\nabla}_X Y = \nabla_X Y + h(X, Y) - \varepsilon \langle X, Y \rangle x,
\]

where \( x \) is the position vector of \( M \) in \( C^n+1 \) or in \( C^1_{n+1} \), and \( \varepsilon = 1 \) or \( -1 \), according to the ambient space being \( C^n+1 \) or being \( C^1_{n+1} \), respectively.

3. Warped product decompositions and \( H \)-stationary.

Let \((M_j, g_j), j = 1, \ldots, m, \) be \( m \) Riemannian manifolds, \( f_i \) a positive function on \( M_1 \times \cdots \times M_m \) and \( \pi_i : M_1 \times \cdots \times M_m \to M_i \) the \( i \)-th canonical projection for \( i = 1, \ldots, m \). The twisted product

\[
f, M_1 \times \cdots \times f, M_m
\]
is the product manifold \( M_1 \times \cdots \times M_m \) equipped with the twisted product metric \( g \) defined by

\[
g(X, Y) = f^2_1 g_1(\pi_1 X, \pi_1 Y) + \cdots + f^2_m g_m(\pi_m X, \pi_m Y).
\]

Let \( N^{n-\ell}(\varepsilon) \) be an \((n-\ell)\)-dimensional real space form of constant curvature \( \varepsilon \). For \( \ell < n-1 \) we consider the following twisted product:

\[
f, I_1 \times \cdots \times f, I_\ell \times 1 N^{n-\ell}(\varepsilon)
\]

with twisted product metric given by

\[
g = f^2_1 dx_1^2 + \cdots + f^2_\ell dx_\ell^2 + g_0,
\]

where \( g_0 \) is the canonical metric of \( N^{n-\ell}(\varepsilon) \) and \( I_1, \ldots, I_\ell \) are open intervals. When \( \ell = n-1 \), we shall replace \( N^{n-\ell}(\varepsilon) \) by an open interval. If the twisted product is a real-space-form \( M^n(\varepsilon) \), it is called a twisted product decomposition of \( M^n(\varepsilon) \). We denote such a decomposition by \( TP^n_{f_1, \ldots, f_\ell}(\varepsilon) \).

Coordinates \( x_1, \ldots, x_n \) on \( TP^n_{f_1, \ldots, f_\ell}(\varepsilon) \) are called adapted coordinates if \( \partial / \partial x_j \) is tangent to \( I_j \) for \( j = 1, \ldots, \ell \), the last \( n-\ell \) coordinate vectors are tangent to \( N^{n-\ell}(\varepsilon) \), and if the metric takes the form \( 3.3 \).

The twistor form \( \Phi(T P) \) on \( TP^n_{f_1, \ldots, f_\ell}(\varepsilon) \) is defined by

\[
\Phi(T P) = f^2_1 dx_1 + \cdots + f^2_\ell dx_\ell.
\]

The twistor form is called twisted closed if we have (cf. \[11\])

\[
\sum_{i,j=1}^\ell \frac{\partial f^2_i}{\partial x_j} dx_j \wedge dx_i = 0.
\]
Obviously, if $\ell = 1$, the twisted form $\Phi(TP)$ is twisted closed automatically.

**Theorem 3.1.** [11] Let $TP^n_{f_1,\ldots,f_\ell}(\varepsilon)$, $\ell \in [1,n]$, be a twisted product decomposition of a simply-connected real-space-form $M^n(\varepsilon)$. If the twistor form $\Phi(TP)$ is twisted closed, then, up to rigid motions of $\tilde{M}^n(4\varepsilon)$, there is a unique Lagrangian immersion:

\[
L_{f_1,\ldots,f_\ell} : TP^n_{f_1,\ldots,f_\ell}(\varepsilon) \to \tilde{M}^n(4\varepsilon),
\]

whose second fundamental form satisfies

\[
h\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_j}\right) = J \frac{\partial}{\partial x_j}, \quad j = 1,\ldots,\ell; \quad h\left(\frac{\partial}{\partial x_r}, \frac{\partial}{\partial x_i}\right) = 0, \text{ otherwise},
\]

for any adapted coordinate system $\{x_1,\ldots,x_n\}$.

Conversely, if $L : M^n(\varepsilon) \to M^n(4\varepsilon)$ is a non-totally geodesic Lagrangian immersion of a real-space-form $M^n(\varepsilon)$ of constant curvature $\varepsilon$ into a complex-space-form $\tilde{M}^n(4\varepsilon)$, then $M^n(\varepsilon)$ admits an appropriate twisted product decomposition with twisted closed twistor form and, moreover, the Lagrangian immersion $L$ is given by the corresponding adapted Lagrangian immersion of the twisted product.

The $H$-stationary condition for the adapted Lagrangian immersion $L_{f_1,\ldots,f_\ell}$ have been computed by Dong and Han in [15].

**Proposition 3.1.** Let $L_{f_1,\ldots,f_\ell} : TP^n_{f_1,\ldots,f_\ell}(\varepsilon) \to \tilde{M}^n(4\varepsilon)$ be an adapted Lagrangian immersion given in Theorem 3.1. Then $L_{f_1,\ldots,f_\ell}$ is $H$-stationary if and only if the twistor functions $f_1,\ldots,f_\ell$ satisfy

\[
\sum_{j=1}^\ell \frac{1}{f_j^2} \frac{\partial f_j^2}{\partial x_j} = \sum_{1 \leq i < j \leq \ell} \frac{1}{f_i^2 f_j^2} \frac{\partial f_i^2}{\partial x_j}.
\]

An immediate consequence of this proposition is the following

**Corollary 3.1.** [15] Any adapted Lagrangian immersion $L_{f_f} : TP^n_{f_f}(\varepsilon) \to \tilde{M}^n(4\varepsilon)$ (with $k = 2$ and $f_1 = f_2 = f$) is $H$-stationary.

**Definition 3.1.** A twisted product decomposition $TP^n_{f_1,\ldots,f_\ell}(\varepsilon)$ of a real space form $M^n(\varepsilon)$ is called a warped product decomposition if $\ell < n$ and $f_1,\ldots,f_\ell$ are independent of the adapted coordinates $x_1,\ldots,x_\ell$.

By applying Theorem 3.1 and Proposition 3.1, we also have the following (cf. [10, 12]).

**Proposition 3.2.** Let $TP^n_{f_1}(\varepsilon)$ be a twisted product decomposition of a real space form $M^n(\varepsilon)$ of constant curvature $\varepsilon$. Then the adapted Lagrangian immersion $L_{f_1} : TP^n_{f_1}(\varepsilon) \to \tilde{M}^n(4\varepsilon)$ is $H$-stationary if and only if $TP^n_{f_1}(\varepsilon)$ is a warped product decomposition.

**Proposition 3.3.** If $TP^n_{f_1,\ldots,f_\ell}(\varepsilon)$ is a warped product decomposition of a simply-connected real space form $M^n(\varepsilon)$, then up to rigid motions, there exists a unique $H$-stationary Lagrangian immersion

\[
L_{f_1,\ldots,f_\ell} : TP^n_{f_1,\ldots,f_\ell}(\varepsilon) \to \tilde{M}^n(4\varepsilon)
\]

whose second fundamental form satisfies (3.7).
4. \textit{H}-stationary Lagrangian submanifolds arisen from warped product decompositions.

It follows from Proposition 3.3 that each warped decomposition of a real space space of constant curvature \( \varepsilon \) gives rise a Hamiltonian-stationary Lagrangian submanifold in a complex space form of constant holomorphic sectional curvature \( 4\varepsilon \).

Hamiltonian-stationary Lagrangian submanifolds in complex space forms arisen from warped product decompositions have been completely classified by Chen and Dillen in [10].

\textbf{Theorem 4.1.} [10] There exist two families of non-totally geodesic Hamiltonian-stationary Lagrangian submanifolds in \( \mathbb{C}^n \) arisen from warped product decompositions:

(a) Flat Lagrangian submanifolds defined by

\begin{equation}
L(x_1, \ldots, x_n) = (a_1 e^{ix_1}, \ldots, a_\ell e^{ix_\ell}, x_{\ell+1}, \ldots, x_n)
\end{equation}

with \( a_1, \ldots, a_\ell > 0 \) and \( \ell \in [0, n-1] \).

(b) Flat Lagrangian submanifolds defined by

\begin{equation}
L(x_1, \ldots, x_n) = \left( \frac{\sqrt{1 + 4a_1^2 + 1}}{\sqrt{2(1 + 4a_1^2)^{1/2}}} e^{\frac{1}{2}(1 - \sqrt{1 - 4b_1})} x_1 x_{\ell+1}, \ldots, \frac{\sqrt{1 + 4b_\ell - 1}}{\sqrt{2(1 + 4b_\ell)^{1/2}}} e^{\frac{1}{2}(1 + \sqrt{1 - 4b_\ell})} x_{\ell+1} x_{\ell+k}, \ldots, \frac{\sqrt{1 + 4a_1^2 - 1}}{\sqrt{2(1 + 4a_1)^{1/2}}} e^{\frac{1}{2}(1 - \sqrt{1 - 4b_k})} x_k x_{\ell+k}, \ldots, \frac{\sqrt{1 + 4a_\ell^2 - 1}}{\sqrt{2(1 + 4a_\ell)^{1/2}}} e^{\frac{1}{2}(1 + \sqrt{1 - 4b_\ell})} x_\ell x_{\ell+k} \right)
\end{equation}

where \( b_1, \ldots, b_k, a_{k+1}, \ldots, a_\ell > 0 \) and \( \ell \in [1, n-1] \).

\textbf{Theorem 4.2.} There exist two families of non-totally geodesic Hamiltonian-stationary Lagrangian submanifolds of constant curvature one in \( \mathbb{C}P^n(4) \) arisen from warped product decompositions:

(a) Lagrangian submanifolds defined by

\begin{align*}
\hat{L} &= \left( 2a_1 e^{\frac{ix_1}{2}} \sin \left( \frac{i}{2} \sqrt{1 + 4a_1^2} x_1 \right) \cos x_{\ell+1}, \ldots, 2a_\ell e^{\frac{ix_\ell}{2}} \sin \left( \frac{i}{2} \sqrt{1 + 4a_\ell^2} x_\ell \right) \cos x_{\ell+1}, \right. \\
e^{\frac{ix_1}{2}} \left( \cos \left( \frac{i}{2} \sqrt{1 + 4a_1^2} x_1 \right) - \frac{i}{\sqrt{1 + 4a_1^2}} \sin \left( \frac{i}{2} \sqrt{1 + 4a_1^2} x_1 \right) \right) \sin \theta_{\ell+1}, \ldots, \\
e^{\frac{ix_\ell}{2}} \left( \cos \left( \frac{i}{2} \sqrt{1 + 4a_\ell^2} x_\ell \right) - \frac{i}{\sqrt{1 + 4a_\ell^2}} \sin \left( \frac{i}{2} \sqrt{1 + 4a_\ell^2} x_\ell \right) \right) \sin \theta_{2\ell} \prod_{j=1}^{\ell-1} \cos \theta_{\ell+j}, \\
sin \theta_{2\ell+1} \prod_{r=\ell+1}^{2\ell} \cos \theta_r, \ldots, \sin \theta_n \prod_{r=\ell+1}^{n-1} \cos \theta_r, \quad \ell \leq \frac{1}{2}(n + 1) \quad \text{and} \quad a_1, \ldots, a_\ell > 0.
\end{align*}

with \( \ell \leq \frac{1}{2}(n + 1) \) and \( a_1, \ldots, a_\ell > 0 \).
(b) Lagrangian submanifolds defined by
\[ \mathcal{L}(x_1, \ldots, x_{\ell}, \theta_1, \ldots, \theta_{\ell-1}) = \left( \frac{2a_1 e^{\frac{a}{2} x_1}}{\sqrt{1 + 4a_1^2}} \sin \left( \frac{1}{2} \sqrt{1 + 4a_1^2} x_1 \right) \sin \theta_{\ell+1}, \ldots, \right. \\
\frac{2a_{\ell-1} e^{\frac{a}{2} x_{\ell-1}}}{\sqrt{1 + 4a_{\ell-1}^2}} \sin \left( \frac{1}{2} \sqrt{1 + 4a_{\ell-1}^2} x_{\ell-1} \right) \sin \theta_{2\ell-1} \prod_{j=1}^{\ell-2} \cos \theta_{j+1}, \ldots, \\
\frac{2ae^{\frac{a}{2} x_{\ell}}}{\sqrt{1 + 4a_{\ell}^2}} \sin \left( \frac{1}{2} \sqrt{1 + 4a_{\ell}^2} x_{\ell} \right) \prod_{j=1}^{\ell-1} \cos \theta_{j+1}, \ldots, \\
e^{\frac{a}{2} x_1} \left( \sin \left( \frac{1}{2} \sqrt{1 + 4a_1^2} x_1 \right) - i \cos \left( \frac{1}{2} \sqrt{1 + 4a_1^2} x_1 \right) \right) \sin \theta_1, \ldots, \\
e^{\frac{a}{2} x_{\ell-1}} \left( \sin \left( \frac{1}{2} \sqrt{1 + 4a_{\ell-1}^2} x_{\ell-1} \right) - i \cos \left( \frac{1}{2} \sqrt{1 + 4a_{\ell-1}^2} x_{\ell-1} \right) \right) \prod_{j=1}^{\ell-2} \cos \theta_{j+1}, \ldots, \\
e^{\frac{a}{2} x_{\ell}} \left( \sin \left( \frac{1}{2} \sqrt{1 + 4a_{\ell}^2} x_{\ell} \right) - i \cos \left( \frac{1}{2} \sqrt{1 + 4a_{\ell}^2} x_{\ell} \right) \right) \prod_{j=1}^{\ell-1} \cos \theta_{j+1} \right), \\
with\ n = 2\ell - 1 \geq 3 \text{ and } a_1, \ldots, a_{\ell} > 0.

**Theorem 4.3.** There exist twenty-one families of non-totally geodesic Hamiltonian-stationary Lagrangian submanifolds of constant curvature $-1$ in $CH^n(-4)$ arisen from warped product decompositions:

1. $n = 2$ and Lagrangian submanifolds defined by
\[ \mathcal{L}(x_1, \theta_2) = \left( \cosh \theta_2, \frac{2ae^{\frac{a}{2} x_1}}{\sqrt{1 + 4a^2}} \sin \left( \frac{1}{2} \sqrt{1 + 4a^2} x_1 \right) \sin \theta_2, \right. \]
\[ \left. e^{\frac{a}{2} x_1} \left( \cos \left( \frac{\sqrt{1 + 4a^2} x_1}{2} \right) - \frac{2a}{\sqrt{1 + 4a^2}} \sin \left( \frac{1}{2} \sqrt{1 + 4a^2} x_1 \right) \right) \sin \theta_2 \right), \ a > 0. \]

2. $n = 2$ and Lagrangian submanifolds defined by
\[ \mathcal{L} = \left( e^{\frac{a}{2} x_1} \left( 1 - \frac{i x_1}{2} \right) \cosh \theta_2, \frac{x_1}{2} e^{\frac{a}{2} x_1} \cosh \theta_2, \sinh \theta_2 \right). \]

3. $n = 2$ and Lagrangian submanifolds defined by
\[ \mathcal{L} = \left( e^{\frac{a}{2} x_1} \left( \cosh \left( \frac{\sqrt{1 - 4a^2}}{2} \right) - \frac{i}{\sqrt{1 - 4a^2}} \sin \left( \frac{x_1}{2} \sqrt{1 - 4a^2} \right) \right) \cosh \theta_2, \right. \]
\[ \frac{2ae^{\frac{a}{2} x_1}}{\sqrt{1 - 4a^2}} \cosh \theta_2 \sin \left( \frac{1}{2} \sqrt{1 - 4a^2} x_1 \right), \sinh \theta_2 \right), \ 4a^2 < 1. \]

4. $n = 2$ and Lagrangian submanifolds defined by
\[ \mathcal{L} = \left( e^{\frac{a}{2} x_1} \left( \cosh \left( \frac{\sqrt{4a^2 - 1}}{2} \right) - \frac{i}{\sqrt{4a^2 - 1}} \sin \left( \frac{x_1}{2} \sqrt{4a^2 - 1} \right) \right) \cosh \theta_2, \right. \]
\[ \frac{2ae^{\frac{a}{2} x_1}}{\sqrt{4a^2 - 1}} \cosh \theta_2 \sinh \left( \frac{1}{2} \sqrt{4a^2 - 1} x_1 \right), \sinh \theta_2 \right), \ 4a^2 > 1. \]
(5) \( n = 3 \) and Lagrangian submanifolds defined by

\[
\hat{L} = \left( e^{\frac{x_1}{2}} (2i + x_1) \cosh \theta_3, \frac{x_1 e^{\frac{x_1}{2}}}{2} \cosh \theta_3, \frac{2be^{\frac{x_2}{2}} \sin \left( \frac{x_2}{2} \sqrt{1 + 4b^2} \right)}{\sqrt{1 + 4b^2}} \sinh \theta_3 \right)
\]

\[
e^{\frac{x_2}{2}} \left\{ \cos \left( \frac{1}{2} \sqrt{1 + 4b^2} x_2 \right) - \frac{i}{\sqrt{1 + 4b^2}} \sin \left( \frac{1}{2} \sqrt{1 + 4b^2} x_2 \right) \right\} \sinh \theta_3 \right).
\]

(6) \( n = 3 \) and Lagrangian submanifolds defined by

\[
\hat{L}(x_1, x_2, \theta_3) = \left( e^{\frac{x_1}{2}} \cosh \left( \frac{\sqrt{4a^2 - 1}}{2} x_1 \right) \sin \theta_3, e^{\frac{x_1}{2}} \sinh \left( \frac{\sqrt{4a^2 - 1}}{2} x_1 \right) \cosh \theta_3, \right.
\]

\[
e^{\frac{x_2}{2}} \left( \cos \left( \frac{\sqrt{1 + 4b^2}}{2} x_2 \right) - \frac{i}{\sqrt{1 + 4b^2}} \sin \left( \frac{\sqrt{1 + 4b^2}}{2} x_2 \right) \right) \sinh \theta_3,
\]

\[
\frac{2be^{\frac{x_2}{2}}}{\sqrt{1 + 4b^2}} \sin \left( \frac{\sqrt{1 + 4b^2}}{2} x_2 \right) \sinh \theta_3 \right), \quad 4a^2 > 1.
\]

(7) \( n = 3 \) and Lagrangian submanifolds defined by:

\[
\hat{L}(x_1, x_2, \theta_3) = \left( e^{\frac{x_1}{2}} \left( \cos \left( \frac{1}{2} \sqrt{1 - 4a^2} x_1 \right) - \frac{i}{\sqrt{1 - 4a^2}} \sin \left( \frac{1}{2} \sqrt{1 - 4a^2} x_1 \right) \right) \cosh \theta_3, \right.
\]

\[
e^{\frac{x_2}{2}} \left( \cos \left( \frac{1}{2} \sqrt{1 + 4b^2} x_2 \right) - \frac{i}{\sqrt{1 + 4b^2}} \sin \left( \frac{1}{2} \sqrt{1 + 4b^2} x_2 \right) \right) \sinh \theta_3
\]

\[
\frac{2ae^{\frac{x_1}{2}}}{\sqrt{1 - 4a^2}} \sin \left( \frac{1}{2} \sqrt{1 - 4a^2} x_1 \right) \cosh \theta_3, \frac{2be^{\frac{x_2}{2}}}{\sqrt{1 + 4b^2}} \sin \left( \frac{1}{2} \sqrt{1 + 4b^2} x_2 \right) \sinh \theta_3 \right), \quad 4a^2 < 1.
\]

(8) Lagrangian submanifolds defined by

\[
\hat{L}(x_1, \ldots, x_{n-1}, \theta_n) = \left( ae^{\theta_n} + \frac{e^{-\theta_n} + 2ie^{\theta_n} \sum_{j=1}^{n-1} a_j^2 x_j}{2a}, a_1 e^{ix_1 + \theta_n}, \ldots, \right.
\]

\[
a_{n-1} e^{ix_{n-1} + \theta_n} \left( \frac{e^{-\theta_n} + 2ie^{\theta_n} \sum_{j=1}^{n-1} a_j^2 x_j}{2a} \right)
\]

with \( a = \sqrt{a_1^2 + \cdots + a_{n-1}^2} \).
(9) $n > 2\ell \geq 2$ and Lagrangian submanifolds defined by

\[
\mathcal{L} = \sinh \theta_{\ell+1} \left( \coth \theta_{\ell+1}, \frac{2a_1 e^{\frac{x_1}{2}}}{\sqrt{1 + 4a_1^2}} \sin \left( \frac{1}{2} \sqrt{1 + 4a_1^2} \right) \sin \theta_{\ell+2}, \ldots, \\
\frac{2a_1 e^{\frac{x_1}{2}}}{\sqrt{1 + 4a_1^2}} \sin \left( \frac{1}{2} \sqrt{1 + 4a_1^2} x_1 \right) \sin \theta_{2\ell+1} \prod_{j=2}^{\ell} \cos \theta_{\ell+j}, \\
e^{\frac{x_1}{2}} \left( \cos \left( \frac{1}{2} \sqrt{1 + 4a_1^2} x_1 \right) - \frac{i}{\sqrt{1 + 4a_1^2}} \sin \left( \frac{1}{2} \sqrt{1 + 4a_1^2} x_1 \right) \right) \sin \theta_{\ell+2}, \ldots, \\
e^{\frac{x_1}{2}} \left( \cos \left( \frac{1}{2} \sqrt{1 + 4a_1^2} x_1 \right) - \frac{i}{\sqrt{1 + 4a_1^2}} \sin \left( \frac{1}{2} \sqrt{1 + 4a_1^2} x_1 \right) \right) \sin \theta_{2\ell+1} \prod_{j=2}^{\ell} \cos \theta_{\ell+j}, \\
\cos \theta_{\ell+2} \cdots \cos \theta_{2\ell+1} \sin \theta_{2\ell+2}, \ldots, \cos \theta_{\ell+2} \cdots \cos \theta_{n-1} \sin \theta_n, \cos \theta_{\ell+2} \cdots \cos \theta_n \right) .
\]

(10) $n > 2\ell \geq 2$ and Lagrangian submanifolds defined by

\[
\hat{\mathcal{L}} = \sinh \theta_{\ell+1} \left( e^{\frac{x_1}{2}} (1 - \frac{i}{2} x_1) \coth \theta_{\ell+1}, \frac{1}{2} e^{\frac{x_1}{2}} x_1 \coth \theta_{\ell+1}, \\
\frac{2a_2 e^{\frac{x_2}{2}}}{\sqrt{1 + 4a_2^2}} \sin \left( \frac{1}{2} \sqrt{1 + 4a_2^2} x_2 \right) \sin \theta_{\ell+2}, \ldots, \\
\frac{2a_2 e^{\frac{x_2}{2}}}{\sqrt{1 + 4a_2^2}} \sin \left( \frac{1}{2} \sqrt{1 + 4a_2^2} x_2 \right) \sin \theta_{2\ell+1} \prod_{r=\ell+2}^{2\ell-1} \cos \theta_r, \\
e^{\frac{x_2}{2}} \left( \cos \left( \frac{1}{2} \sqrt{1 + 4a_2^2} x_2 \right) - \frac{i}{\sqrt{1 + 4a_2^2}} \sin \left( \frac{1}{2} \sqrt{1 + 4a_2^2} x_2 \right) \right) \sin \theta_{\ell+2}, \ldots, \\
e^{\frac{x_2}{2}} \left( \cos \left( \frac{1}{2} \sqrt{1 + 4a_2^2} x_2 \right) - \frac{i}{\sqrt{1 + 4a_2^2}} \sin \left( \frac{1}{2} \sqrt{1 + 4a_2^2} x_2 \right) \right) \sin \theta_{2\ell+1} \prod_{r=\ell+2}^{2\ell-1} \cos \theta_r, \\
\cos \theta_{\ell+2} \cdots \cos \theta_{2\ell+1} \sin \theta_{2\ell+2}, \ldots, \cos \theta_{\ell+2} \cdots \cos \theta_{n-1} \sin \theta_n, \cos \theta_{\ell+2} \cdots \cos \theta_n \right) .
\]
(11) $n > 2\ell \geq 2$ and Lagrangian submanifolds defined by

$$\hat{L} = \sinh \theta_{\ell+1} \left( e^{\frac{1}{2} x_1} \cosh \theta_{\ell+1} \left[ \cosh \left( \frac{\sqrt{4a_0^2 - 1} x_1}{2} \right) - \frac{i \sinh \left( \frac{\sqrt{4a_0^2 - 1} x_1}{2} \right)}{\sqrt{4a_0^2 - 1}} \right] \right),$$

$$\frac{2a_2 e^{\frac{1}{2} x_1}}{\sqrt{4a_0^2 - 1}} \cosh \theta_{\ell+1} \sinh \left( \frac{1}{2} \sqrt{4a_0^2 - 1} x_1 \right), \quad \frac{2a_2 e^{\frac{1}{2} x_2}}{\sqrt{1 + 4a_0^2}} \sin \left( \frac{1}{2} \sqrt{1 + 4a_0^2} x_2 \right) \sin \theta_{\ell+2},$$

$$e^{\frac{2}{2} x_2} \left[ \cos \left( \frac{1}{2} \sqrt{1 + 4a_0^2} x_2 \right) - \frac{i}{\sqrt{1 + 4a_0^2}} \sin \left( \frac{1}{2} \sqrt{1 + 4a_0^2} x_2 \right) \right] \sin \theta_{\ell+2},$$

$$\ldots, \quad \frac{2a_2 e^{\frac{1}{2} x_{2\ell}}}{\sqrt{1 + 4a_0^2}} \sin \left( \frac{1}{2} \sqrt{1 + 4a_0^2} x_2 \right) \sin \theta_{2\ell} \prod_{r=\ell+2}^{2\ell-1} \cos \theta_r,$$

$$e^{\frac{2}{2} x_{2\ell}} \left( \cos \left( \frac{1}{2} \sqrt{1 + 4a_0^2} x_2 \right) - \frac{i}{\sqrt{1 + 4a_0^2}} \sin \left( \frac{1}{2} \sqrt{1 + 4a_0^2} x_2 \right) \right) \sin \theta_{2\ell} \prod_{r=\ell+2}^{2\ell-1} \cos \theta_r,$$

$$\cos \theta_{\ell+2} \cdots \cos \theta_{2\ell} \sin \theta_{2\ell+1}, \ldots, \cos \theta_{\ell+2} \cdots \cos \theta_{n-1} \sin \theta_n, \cos \theta_{\ell+2} \cdots \cos \theta_n.$$
(13) \( n \geq 3, n > 2\ell \geq 6 \) and Lagrangian submanifolds defined by

\[
\hat{L} = \sinh \theta_{\ell+1} \left( 1 + \frac{a_1^2}{2} i a_1 x_1 \right) \coth \theta_{\ell+1} - a_1^2 \left( \frac{1}{2} + i x_1 \right) \sin \theta_{\ell+2},
\]

\[
a_1^2 \left( \frac{1}{2} + i x_1 \right) \coth \theta_{\ell+1} + \left( 1 - \frac{a_1^2}{2} - i a_1^2 x_1 \right) \sin \theta_{\ell+2}, i a_1 e^{ix_1} (\coth \theta_{\ell+1} - \sin \theta_{\ell+2}),
\]

\[
\frac{2a_1 e^{ix_1}}{\sqrt{1 + 4a_2^2}} \sin \left( \frac{1}{2} \sqrt{1 + 4a_2^2} x_2 \right) \cos \theta_{\ell+2} \sin \theta_{\ell+3},
\]

\[
e^{\frac{i}{2} x_2} \left( \cos \left( \frac{1}{2} \sqrt{1 + 4a_2^2} x_2 \right) - \frac{i}{\sqrt{1 + 4a_2^2}} \sin \left( \frac{1}{2} \sqrt{1 + 4a_2^2} x_2 \right) \right) \cos \theta_{\ell+2} \sin \theta_{\ell+3},
\]

\[
\ldots
\]

\[
\frac{2a_1 e^{ix_1}}{\sqrt{1 + 4a_2^2}} \sin \left( \frac{1}{2} \sqrt{1 + 4a_2^2} x_\ell \right) \sin \theta_{2\ell+1} \prod_{j=2}^{\ell} \cos \theta_{\ell+j},
\]

\[
e^{\frac{i}{2} x_\ell} \left( \cos \left( \frac{1}{2} \sqrt{1 + 4a_2^2} x_\ell \right) - \frac{i}{\sqrt{1 + 4a_2^2}} \sin \left( \frac{1}{2} \sqrt{1 + 4a_2^2} x_\ell \right) \right) \sin \theta_{2\ell+1} \prod_{t=\ell+2}^{2\ell} \cos \theta_t,
\]

\[
\cos \theta_{\ell+2} \cdots \cos \theta_{2\ell+1} \sin \theta_{2\ell+2}, \ldots, \cos \theta_{\ell+2} \cdots \cos \theta_{n-1} \sin \theta_n, \cos \theta_{\ell+2} \cdots \cos \theta_n.
\]

(14) \( n = 2\ell \geq 4 \) and Lagrangian submanifolds defined by

\[
\hat{L} = \sinh \theta_{\ell+1} \left( \coth \theta_{\ell+1}, \frac{2a_1 e^{ix_1}}{\sqrt{1 + 4a_1^2}} \sin \left( \frac{1}{2} \sqrt{1 + 4a_1^2} x_1 \right) \sin \theta_{\ell+2},
\]

\[
e^{\frac{i}{2} x_1} \left( \cos \left( \frac{1}{2} \sqrt{1 + 4a_1^2} x_1 \right) - \frac{i}{\sqrt{1 + 4a_1^2}} \sin \left( \frac{1}{2} \sqrt{1 + 4a_1^2} x_1 \right) \right) \cos \theta_{\ell+2},
\]

\[
\ldots
\]

\[
\frac{2a_{\ell-1} e^{ix_{\ell-1}}}{\sqrt{1 + 4a_{\ell-1}^2}} \sin \left( \frac{1}{2} \sqrt{1 + 4a_{\ell-1}^2} x_{\ell-1} \right) \sin \theta_{2\ell} \prod_{j=2}^{\ell-1} \cos \theta_{\ell+j},
\]

\[
e^{\frac{i}{2} x_{\ell-1}} \left( \cos \left( \frac{1}{2} \sqrt{1 + 4a_{\ell-1}^2} x_{\ell-1} \right) - \frac{i}{\sqrt{1 + 4a_{\ell-1}^2}} \sin \left( \frac{1}{2} \sqrt{1 + 4a_{\ell-1}^2} x_{\ell-1} \right) \right) \sin \theta_{2\ell} \prod_{t=\ell+2}^{2\ell-1} \cos \theta_t,
\]

\[
\frac{2a_{\ell} e^{ix_\ell}}{\sqrt{1 + 4a_\ell^2}} \sin \left( \frac{1}{2} \sqrt{1 + 4a_\ell^2} x_\ell \right) \cos \theta_{\ell+2} \cdots \cos \theta_{2\ell},
\]

\[
e^{\frac{i}{2} x_\ell} \left( \cos \left( \frac{1}{2} \sqrt{1 + 4a_\ell^2} x_\ell \right) - \frac{i}{\sqrt{1 + 4a_\ell^2}} \sin \left( \frac{1}{2} \sqrt{1 + 4a_\ell^2} x_\ell \right) \right) \cos \theta_{\ell+2} \cdots \cos \theta_{2\ell}.
\]
(15) $n = 2\ell \geq 4$ and Lagrangian submanifolds defined by
\[
\hat{L} = \sinh \theta_{\ell+1} \left( e^{\frac{x_{x_1}}{2}(1 - \frac{i}{2}x_1)} \coth \theta_{\ell+1} \right) \frac{e^{\frac{x_{x_1}}{2}x_1}}{2} \coth \theta_{\ell+1},
\]
\[
\frac{2a_2 e^{\frac{x_{x_2}}{2}x_2}}{\sqrt{1 + 4a_2^2}} \sin \theta_{\ell+2}, \ldots, \frac{2a_2 e^{\frac{x_{x_2}}{2}x_2}}{\sqrt{1 + 4a_2^2}} \sin \theta_{2\ell} \prod_{r=\ell+2}^{n-1} \cos \theta_r,
\]
\[
e^{\frac{x_{x_1}}{2}} \left( \cos \left( \frac{i}{2}\sqrt{1 + 4a_2^2}x_1 \right) - \frac{i}{\sqrt{1 + 4a_2^2}} \right) \sin \left( \frac{i}{2}\sqrt{1 + 4a_2^2}x_1 \right) \sin \theta_{\ell+2}, \ldots,
\]
\[
e^{\frac{x_{x_1}}{2}x_1} \left( \cos \left( \frac{i}{2}\sqrt{1 + 4a_2^2}x_1 \right) - \frac{i}{\sqrt{1 + 4a_2^2}} \right) \sin \left( \frac{i}{2}\sqrt{1 + 4a_2^2}x_1 \right) \sin \theta_{\ell+2}, \ldots
\]
\[
e^{\frac{x_{x_1}}{2}x_1} \left( \cos \left( \frac{i}{2}\sqrt{1 + 4a_2^2}x_1 \right) - \frac{i}{\sqrt{1 + 4a_2^2}} \right) \sin \left( \frac{i}{2}\sqrt{1 + 4a_2^2}x_1 \right) \sin \theta_{\ell+2}, \ldots
\]

(16) $n = 2\ell \geq 4$ and Lagrangian submanifolds defined by
\[
\hat{L} = \sinh \theta_{\ell+1} \left( e^{\frac{x_{x_1}}{2}x_1} \coth \theta_{\ell+1} \right) \left[ \cosh \left( \sqrt{1 + 4a_2^2}x_1 \right) - i \sinh \left( \frac{i}{2}\sqrt{1 + 4a_2^2}x_1 \right) \right],
\]
\[
\frac{2a_2 e^{\frac{x_{x_2}}{2}x_2}}{\sqrt{4a_2^2 - 1}} \sinh \left( \frac{i}{2}\sqrt{4a_2^2 - 1}x_1 \right), \frac{2a_2 e^{\frac{x_{x_2}}{2}x_2}}{\sqrt{4a_2^2 - 1}} \sinh \left( \frac{i}{2}\sqrt{4a_2^2 - 1}x_1 \right) \sin \theta_{\ell+2},
\]
\[
e^{\frac{x_{x_1}}{2}x_1} \left( \cos \left( \sqrt{1 + 4a_2^2}x_1 \right) - \frac{i}{\sqrt{1 + 4a_2^2}} \right) \sin \left( \sqrt{1 + 4a_2^2}x_1 \right) \sin \theta_{\ell+2}, \ldots,
\]
\[
e^{\frac{x_{x_1}}{2}x_1} \left( \cos \left( \sqrt{1 + 4a_2^2}x_1 \right) - \frac{i}{\sqrt{1 + 4a_2^2}} \right) \sin \left( \sqrt{1 + 4a_2^2}x_1 \right) \sin \theta_{\ell+2}, \ldots
\]
\[
e^{\frac{x_{x_1}}{2}x_1} \left( \cos \left( \sqrt{1 + 4a_2^2}x_1 \right) - \frac{i}{\sqrt{1 + 4a_2^2}} \right) \sin \left( \sqrt{1 + 4a_2^2}x_1 \right) \sin \theta_{\ell+2}, \ldots
\]

(17) $n = 2\ell \geq 4$ and Lagrangian submanifolds defined by
\[
\hat{L} = \sinh \theta_{\ell+1} \left( e^{\frac{x_{x_1}}{2}x_1} \coth \theta_{\ell+1} \right) \left[ \cosh \left( \frac{1}{2}\sqrt{1 - 4a_1^2}x_1 \right) - i \sinh \left( \frac{i}{2}\sqrt{1 - 4a_1^2}x_1 \right) \right],
\]
\[
\frac{2a_1 e^{\frac{x_{x_1}}{2}x_1}}{\sqrt{1 - 4a_1^2}} \sinh \left( \frac{i}{2}\sqrt{1 - 4a_1^2}x_1 \right), \frac{2a_1 e^{\frac{x_{x_1}}{2}x_1}}{\sqrt{1 - 4a_1^2}} \sinh \left( \frac{i}{2}\sqrt{1 + 4a_1^2}x_1 \right) \sin \theta_{\ell+2},
\]
\[
e^{\frac{x_{x_1}}{2}x_1} \left( \cos \left( \frac{a_1}{2}\sqrt{1 - 4a_1^2}x_1 \right) - \frac{i}{\sqrt{1 - 4a_1^2}} \right) \sin \left( \frac{a_1}{2}\sqrt{1 - 4a_1^2}x_1 \right) \sin \theta_{\ell+2}, \ldots,
\]
\[
e^{\frac{x_{x_1}}{2}x_1} \left( \cos \left( \frac{a_1}{2}\sqrt{1 - 4a_1^2}x_1 \right) - \frac{i}{\sqrt{1 - 4a_1^2}} \right) \sin \left( \frac{a_1}{2}\sqrt{1 - 4a_1^2}x_1 \right) \sin \theta_{\ell+2}, \ldots
\]
\[
e^{\frac{x_{x_1}}{2}x_1} \left( \cos \left( \frac{a_1}{2}\sqrt{1 - 4a_1^2}x_1 \right) - \frac{i}{\sqrt{1 - 4a_1^2}} \right) \sin \left( \frac{a_1}{2}\sqrt{1 - 4a_1^2}x_1 \right) \sin \theta_{\ell+2}, \ldots
\]

(18) $n = 2\ell \geq 6$ and Lagrangian submanifolds defined by
\[
\hat{L} = \sinh \theta_{\ell+1} \left( \frac{1 + a_1^2}{2} + i a_1^2 x_1 \right) \coth \theta_{\ell+1} - a_1^2 \left( \frac{1}{2} + i x_1 \right) \sin \theta_{\ell+2},
\]
\[ a_1^2 \left( \frac{1}{2} + ix_1 \right) \coth \theta_{\ell+1} + \left( 1 - \frac{a_1^2}{2} - ia_1^2x_1 \right) \sin \theta_{\ell+2}, \quad ia_1e^{ix_1} \left( \coth \theta_{\ell+1} - \sin \theta_{\ell+2} \right), \]

\[ \frac{2a_2e^{ix_2}}{\sqrt{1 + 4a_2^2}} \sin \left( \frac{i}{2} \sqrt{1 + 4a_2^2x_2} \right) \sin \theta_{\ell+3}, \]

\[ e^{\frac{ix_2}{2}} \left( \cos \left( \frac{i}{2} \sqrt{1 + 4a_2^2x_2} \right) - \frac{i}{\sqrt{1 + 4a_2^2}} \sin \left( \frac{i}{2} \sqrt{1 + 4a_2^2x_2} \right) \right) \cos \theta_{\ell+2} \sin \theta_{\ell+3}, \]

\[ \ldots \]

\[ 2a_{\ell-1}e^{\frac{ix_{\ell-1}}{2}} \sin \left( \frac{x_{\ell-1}}{2} \sqrt{1 + 4a_{\ell-1}^2} \right) \sin \theta_{2\ell} \prod_{j=2}^{\ell-1} \cos \theta_{\ell+j}, \]

\[ e^{\frac{ix_{\ell-1}}{2}} \left( \cos \left( \frac{x_{\ell-1}}{2} \sqrt{1 + 4a_{\ell-1}^2} \right) - \frac{i}{\sqrt{1 + 4a_{\ell-1}^2}} \sin \left( \frac{x_{\ell-1}}{2} \sqrt{1 + 4a_{\ell-1}^2} \right) \right) \sin \theta_{2\ell} \prod_{\ell=\ell+2}^{2\ell-1} \cos \theta_{\ell}, \]

\[ 2a_{\ell}e^{\frac{ix_{\ell}}{2}} \sin \left( \frac{1}{2} \sqrt{1 + 4a_{\ell}^2x_{\ell}} \right) \cos \theta_{\ell+2} \ldots \cos \theta_{2\ell}, \]

\[ e^{\frac{ix_{\ell}}{2}} \left( \cos \left( \frac{1}{2} \sqrt{1 + 4a_{\ell}^2x_{\ell}} \right) - \frac{ix_{\ell}}{\sqrt{1 + 4a_{\ell}^2}} \sin \left( \frac{1}{2} \sqrt{1 + 4a_{\ell}^2x_{\ell}} \right) \right) \cos \theta_{\ell+2} \ldots \cos \theta_{2\ell}. \]

(19) \( n = 2\ell - 1 \geq 5 \) and Lagrangian submanifolds defined by

\[ \dot{L} = \sinh \theta_{\ell+1} \left( e^{\frac{ix_1}{2}} \left( 1 - \frac{i}{2} x_1 \right) \coth \theta_{\ell+1}, \frac{x_1}{2} e^{\frac{ix_1}{2}} \coth \theta_{\ell+1}, \right. \]

\[ \frac{2a_2e^{ix_2}}{\sqrt{1 + 4a_2^2}} \sin \left( \frac{i}{2} \sqrt{1 + 4a_2^2x_2} \right) \sin \theta_{\ell+2}, \ldots, \]

\[ 2a_{\ell-1}e^{\frac{ix_{\ell-1}}{2}} \sin \left( \frac{x_{\ell-1}}{2} \sqrt{1 + 4a_{\ell-1}^2x_{\ell-1}} \right) \sin \theta_{2\ell-1} \prod_{r=\ell+2}^{2\ell-2} \cos \theta_{\ell}, \]

\[ e^{\frac{ix_{\ell-1}}{2}} \left( \cos \left( \frac{x_{\ell-1}}{2} \sqrt{1 + 4a_{\ell-1}^2x_{\ell-1}} \right) - \frac{i}{\sqrt{1 + 4a_{\ell-1}^2}} \sin \left( \frac{x_{\ell-1}}{2} \sqrt{1 + 4a_{\ell-1}^2x_{\ell-1}} \right) \right) \sin \theta_{2\ell} \prod_{r=\ell+2}^{2\ell-1} \cos \theta_{\ell}, \]

\[ 2a_{\ell}e^{\frac{ix_{\ell}}{2}} \sin \left( \frac{1}{2} \sqrt{1 + 4a_{\ell}^2x_{\ell}} \right) \prod_{r=\ell+2}^{2\ell-1} \cos \theta_{\ell}, \]

\[ e^{\frac{ix_{\ell}}{2}} \left( \cos \left( \frac{1}{2} \sqrt{1 + 4a_{\ell}^2x_{\ell}} \right) - \frac{i}{\sqrt{1 + 4a_{\ell}^2}} \sin \left( \frac{1}{2} \sqrt{1 + 4a_{\ell}^2x_{\ell}} \right) \right) \prod_{r=\ell+2}^{2\ell-1} \cos \theta_{\ell}, \]
\[\begin{align*}
(20) \quad n = 2\ell - 1 \geq 5 \text{ and Lagrangian submanifolds defined by} \\
\hat{L} &= \sinh \theta_{\ell+1} \left( e^{\frac{\pi}{4} x_1} \coth \theta_{\ell+1} \left[ \cosh \left( \frac{\sqrt{4a_1^2-1} x_1}{2} \right) - i \frac{\sinh \left( \frac{\sqrt{4a_1^2-1} x_1}{2} \right)}{\sqrt{4a_1^2-1}} \right] \right), \\
\frac{2a_1 e^{\frac{\pi}{4} x_1}}{\sqrt{4a_1^2-1}} \coth \theta_{\ell+1} \sin \left( \frac{\sqrt{4a_1^2-1} x_1}{2} \right), \quad \frac{2a_2 e^{\frac{\pi}{4} x_2}}{\sqrt{1+4a_2^2}} \sin \left( \sqrt{1+4a_2^2} x_2 \right) \sin \theta_{\ell+2}, \\
&\quad \ldots, \quad \frac{2a_{\ell-1} e^{\frac{\pi}{4} x_{\ell-1}}}{\sqrt{1+4a_{\ell-1}^2}} \sin \left( \sqrt{1+4a_{\ell-1}^2} x_{\ell-1} \right) \sin \theta_{2\ell-1} \prod_{r=\ell+2}^{2\ell-2} \cos \theta_r, \\
&\quad e^{\frac{\pi}{4} x_2} \left( \cos \left( \frac{\sqrt{1+4a_2^2} x_2}{2} \right) - i \frac{\sin \left( \frac{\sqrt{1+4a_2^2} x_2}{2} \right)}{\sqrt{1+4a_2^2}} \right) \sin \theta_{\ell+2}, \ldots, \\
&\quad e^{\frac{\pi}{4} x_{\ell-1}} \left( \cos \left( \sqrt{1+4a_{\ell-1}^2} x_{\ell-1} \right) - i \frac{\sin \left( \sqrt{1+4a_{\ell-1}^2} x_{\ell-1} \right)}{\sqrt{1+4a_{\ell-1}^2}} \right) \sin \theta_{2\ell-1} \prod_{r=\ell+2}^{2\ell-2} \cos \theta_r, \\
&\quad \left( \frac{2a_{\ell-1} e^{\frac{\pi}{4} x_{\ell-1}}}{\sqrt{1+4a_{\ell-1}^2}} \sin \left( \sqrt{1+4a_{\ell-1}^2} x_{\ell-1} \right) \right) \prod_{r=\ell+2}^{2\ell-1} \cos \theta_r.
\end{align*}\]

\[\begin{align*}
(21) \quad n = 2\ell - 1 \geq 5 \text{ and Lagrangian submanifolds defined by} \\
\hat{L} &= \sinh \theta_{\ell+1} \left( e^{\frac{\pi}{4} x_1} \coth \theta_{\ell+1} \left[ \cos \left( \frac{\sqrt{1-4a_1^2} x_1}{2} \right) - i \frac{\sin \left( \frac{\sqrt{1-4a_1^2} x_1}{2} \right)}{\sqrt{1-4a_1^2}} \right] \right), \\
\frac{2a_1 e^{\frac{\pi}{4} x_1}}{\sqrt{1-4a_1^2}} \coth \theta_{\ell+1} \sin \left( \sqrt{1-4a_1^2} x_1 \right), \quad \frac{2a_2 e^{\frac{\pi}{4} x_2}}{\sqrt{1+4a_2^2}} \sin \left( \sqrt{1+4a_2^2} x_2 \right) \sin \theta_{\ell+2}, \\
&\quad \ldots, \quad \frac{2a_{\ell-1} e^{\frac{\pi}{4} x_{\ell-1}}}{\sqrt{1+4a_{\ell-1}^2}} \sin \left( \sqrt{1+4a_{\ell-1}^2} x_{\ell-1} \right) \sin \theta_{2\ell-1} \prod_{r=\ell+2}^{2\ell-2} \cos \theta_r, \\
&\quad e^{\frac{\pi}{4} x_2} \left( \cos \left( \sqrt{1+4a_2^2} x_2 \right) - i \frac{\sin \left( \sqrt{1+4a_2^2} x_2 \right)}{\sqrt{1+4a_2^2}} \right) \sin \theta_{\ell+2}, \ldots, \\
&\quad e^{\frac{\pi}{4} x_{\ell-1}} \left( \cos \left( \sqrt{1+4a_{\ell-1}^2} x_{\ell-1} \right) - i \frac{\sin \left( \sqrt{1+4a_{\ell-1}^2} x_{\ell-1} \right)}{\sqrt{1+4a_{\ell-1}^2}} \right) \sin \theta_{2\ell-1} \prod_{r=\ell+2}^{2\ell-2} \cos \theta_r, \\
&\quad \left( \frac{2a_{\ell-1} e^{\frac{\pi}{4} x_{\ell-1}}}{\sqrt{1+4a_{\ell-1}^2}} \sin \left( \sqrt{1+4a_{\ell-1}^2} x_{\ell-1} \right) \right) \prod_{r=\ell+2}^{2\ell-1} \cos \theta_r.
\end{align*}\]

In above, all \(a_1, \ldots, a_\ell\) are positive numbers.
5. Type I Hamiltonian-stationary Lagrangian surfaces in $\mathcal{M}^2(4\varepsilon)$.

For a twisted product decomposition $TP_{fk}^2(\varepsilon)$ of a simply-connected surface of constant curvature $\varepsilon$, we have

$$
\left( \frac{f_x}{f} \right)_y + \left( \frac{k_y}{f} \right)_x = -\varepsilon f_k.
$$

The twistor form $\Phi(TP) = f^2 dx^2 + k^2 dy^2$ is twisted closed if and only if we have

$$
f f_y = kk_x.
$$

For $n = \ell = 2$, Proposition 5.1 reduces to

**Proposition 5.1.** \[15\] Let $L_{fk} : TP_{fk}^2(\varepsilon) \to \mathcal{M}^2(4\varepsilon)$ be an adapted Lagrangian immersion. Then $L_{fk}$ is Hamiltonian-stationary if and only if we have

$$
k^3 f_x + f^3 k_y = f^2 k f_y + f k^2 k_x \quad \text{(or equivalently, (5.1) and (5.2)).}
$$

Proposition 5.1 implies that the adapted Lagrangian immersion $L_{fk}$ is always Hamiltonian-stationary whenever $f^2 = k^2$. We call such Hamiltonian-stationary Lagrangian surfaces to be of **type I**.

It was proved in [11] that Hamiltonian-stationary Lagrangian surfaces of type $I$ in $CP^2(4)$ are congruent to

$$
L(x, y) = \frac{1}{\alpha} \left( \frac{ib}{2} + \tanh(x + y), e^{\frac{1}{2} k_y} \sec,(x + y) \cos(\alpha(x - y)) \right) \quad e^{\frac{1}{2} k_y} \sec,(x + y) \sin(\alpha(x - y))
$$

$$
\alpha = \frac{1}{2} \sqrt{4 + b^2}, \quad b > 0.
$$

It is also known in [11] that type I Hamiltonian-stationary Lagrangian immersions in $C^2$ are congruent to one of the following two immersions:

$$
L = a(\varepsilon^x, \varepsilon^y), \quad a > 0;
$$

$$
L = \sqrt{2a} \varepsilon^{\frac{1}{2} k_y} \left( \cos \left( \frac{\sqrt{1 + 4b^2}}{2} (x - y) \right), \sin \left( \frac{\sqrt{1 + 4b^2}}{2} (x - y) \right) \right).
$$

For type I Hamiltonian-stationary Lagrangian immersions in $CH^2(-4)$, it is known in [11] that they are congruent to a Lagrangian surfaces obtained from one of the following five families:

$$
L = \frac{1}{\alpha} \left( \frac{ib}{2} - \tan s, \frac{e^{\frac{1}{2} k_y} \cos(\alpha t)}{\cos s}, \frac{e^{\frac{1}{2} k_y} \sin(\alpha t)}{\cos s} \right), \quad \alpha^2 = \frac{k^2}{4} - 1, \quad b > 2;
$$

$$
L = \frac{1}{\alpha} \left( \frac{ib}{2} - \tan s, \frac{e^{\frac{1}{2} k_y} \cos(\alpha t)}{\cos s}, \frac{e^{\frac{1}{2} k_y} \sin(\alpha t)}{\cos s} \right), \quad \alpha^2 = 1 - \frac{b^2}{4}, \quad b \in (0, 2);
$$

$$
L = e^{i s} \sec s \left( \frac{t^2}{2} + \frac{3}{4} + \frac{e^{-2i s}}{4} - \frac{i s}{2}, i \left( \frac{t^2}{2} - 1 + \frac{e^{-2i s}}{4} - \frac{i s}{2} \right) \right);
$$

$$
L = \frac{1}{\alpha} \left( \frac{ib}{2} + \coth s, \frac{e^{\frac{1}{2} k_y} \cos(\alpha t)}{\cosh s}, \frac{e^{\frac{1}{2} k_y} \sin(\alpha t)}{\cosh s} \right), \quad \alpha^2 = \frac{k^2}{4} + 1, \quad b > 0;
$$

$$
L = \left( \frac{2}{x + y} + i \frac{\sqrt{2} e^{i x}}{x + y}, \frac{\sqrt{2} e^{i y}}{x + y} \right),
$$

where $s = x + y$ and $t = x - y$.  

6. Type II Hamiltonian-stationary Lagrangian surfaces in $\widetilde{M}^2(4\varepsilon)$.

In this and next two sections, I present my recent joint work with O. J. Garay and Z. Zhou \[13\] concerning Hamiltonian-stationary Lagrangian surfaces of type II.

If the two twistor functions $f$ and $k$ are unequal, then the twisted product decomposition $T^2\Pi f_k(\varepsilon)$ of a surface of constant curvature $\varepsilon$ gives rise to a $H$-stationary Lagrangian surface in $\widetilde{M}^2(4\varepsilon)$ if and only if $f$ and $k$ satisfy the following over-determined PDE system:

\[
\begin{align*}
\left(\frac{k}{f}\right)_x + \left(\frac{f}{k}\right)_y &= 0, \\
\frac{k}{f} &= \frac{f}{k}, \\
\left(\frac{f}{k}\right)_y + \left(\frac{k}{f}\right)_x &= -\varepsilon fk, \quad (\varepsilon = 1, 0 \text{ or } -1).
\end{align*}
\]

Hamiltonian-stationary Lagrangian immersions obtained from such functions $f,k$ with $f^2 \neq k^2$ are said to be of type II.

In order to find non-trivial solutions of this over-determined PDE system (6.1)-(6.3), first we give the following two lemmas from \[13\].

**Lemma 6.1.** If $f(x,y)$ and $k(x,y)$ satisfy Eqs. (6.1) and (6.2), then

\[
F(x,y) = cmf(m^2x,y), \quad K(x,y) = ckm(m^2x,y), \quad 1 \neq m \in \mathbb{R}^+,
\]

satisfy Eqs. (6.1) and (6.2) automatically for any constant $c \neq 0$.

The easiest way to obtain a solution of system (6.1)-(6.3) with $f^2 \neq k^2$ is to start from $f,k$ with $f^2 = k^2$ since (6.1) is apparently satisfied, and (6.2) and (6.3) are in much simpler form.

**Lemma 6.2.** If $f(x,y)$ is a solution of system (6.1)-(6.3) with $f^2 = k^2$, then

\[
f(x,y) = \hat{f}(x+y),
\]

for some function $\hat{f}(x+y)$ which is a traveling wave solution with unit traveling speed. In addition, we have a family of solutions of the system (6.1)-(6.3) given by

\[
\begin{align*}
F(x,y) &= \frac{m\sqrt{1+m^2}}{\sqrt{2}}\hat{f}(m^2x+y), \\
K(x,y) &= \pm \frac{\sqrt{1+m^2}}{\sqrt{2}}\hat{f}(m^2x+y),
\end{align*}
\]

where $m$ is any positive constant.

Now, we can apply Lemma 6.2 to construct some nontrivial traveling wave solutions of the over-determined PDE system (6.1)-(6.3) with $f^2 \neq k^2$.

**Example 6.1.** When $\varepsilon = 1$ and $f = k = \hat{f}(x+y)$, Eq. (6.3) becomes

\[
2(\ln \hat{f})'' + \hat{f}^2 = 0.
\]
Since \( \hat{f} \neq 0 \), (6.6) implies \( \hat{f} \) is non-constant. Thus, Eq. (6.6) yields

\[
2 \frac{\hat{f}''}{f^2} + \hat{f}^2 = c_1^2 = \text{constant}, \quad c_1 > 0. \tag{6.7}
\]

After solving (6.7), we know that, up to translations and sign, \( \hat{f} \) is

\[
\hat{f}(u) = c_1 \operatorname{sech} \left( \frac{c_1 u}{\sqrt{2}} \right). \tag{6.8}
\]

For notational simplicity, taking \( c_1 = \sqrt{2} \sqrt{1 + m^2} \) and using Lemma 6.2, we obtain the following solutions of system (6.1)-(6.3):

\[
f(x, y) = cm \operatorname{sech} \left( \frac{c(m^2 x + y)}{\sqrt{1 + m^2}} \right), \quad k(x, y) = \pm c \operatorname{sech} \left( \frac{c(m^2 x + y)}{\sqrt{1 + m^2}} \right),
\]

where \( c, m \) are positive real numbers.

**Example 6.2.** When \( \varepsilon = 0 \) and \( f = \pm k = \hat{f}(x + y) \), Eq. (6.3) becomes \((\ln \hat{f})'' = 0\), which implies that \( \hat{f} = ae^{bu} \) for some real numbers \( a, b \) with \( a \neq 0 \). Thus, we have the following solutions of system (6.1)-(6.3):

\[
f(x, y) = aime^{b(m^2 x + y)}, \quad k(x, y) = \pm ae^{b(m^2 x + y)}, \quad a \neq 0. \tag{6.10}
\]

**Example 6.3.** When \( \varepsilon = -1 \) and \( f = \pm k = \hat{f}(x + y) \), (6.3) becomes

\[
2 \left( \frac{\hat{f}''}{\hat{f}} \right)' = \hat{f}^2. \tag{6.11}
\]

Since \( \hat{f} \neq 0 \), (6.11) implies that \( \hat{f} \) is non-constant. Thus, (6.11) yields

\[
2 \frac{\hat{f}''}{f^2} - \hat{f}^2 = \pm c^2 = \text{constant}, \quad c \geq 0. \tag{6.12}
\]

**Case (i):** \(-c^2 < 0\). Solving (6.12) gives \( \hat{f}(u) = c \sec \left( \frac{cu}{\sqrt{2}} \right) \). As in Example 6.1, we find the following solutions of system (6.1)-(6.3):

\[
f(x, y) = cm \sec \left( \frac{c(m^2 x + y)}{\sqrt{1 + m^2}} \right), \quad k(x, y) = \pm c \sec \left( \frac{c(m^2 x + y)}{\sqrt{1 + m^2}} \right),
\]

where \( c, m \) are positive real numbers.

**Case (ii):** \(-c^2 < 0\). \( c^2 > 0 \). In this case, after solving (6.12) we get another family of solutions:

\[
f(x, y) = cm \operatorname{csch} \left( \frac{c(m^2 x + y)}{\sqrt{1 + m^2}} \right), \quad k(x, y) = \pm c \operatorname{csch} \left( \frac{c(m^2 x + y)}{\sqrt{1 + m^2}} \right).
\]

**Case (iii):** \( c > 0 \). In this case, after solving (6.12) we get another family of solutions:

\[
f(x, y) = \frac{m \sqrt{1 + m^2}}{m^2 x + y}, \quad k(x, y) = \frac{\sqrt{1 + m^2}}{m^2 x + y}, \quad m \neq 1.
\]

**Example 6.2.** When \( \varepsilon = 0 \) and \( f = \pm k = \hat{f}(x + y) \), Eq. (6.3) becomes \((\ln \hat{f})'' = 0\), which implies that \( \hat{f} = ae^{bu} \) for some real numbers \( a, b \) with \( a \neq 0 \). Thus, we have the following solutions of system (6.1)-(6.3):

\[
f(x, y) = aime^{b(m^2 x + y)}, \quad k(x, y) = \pm ae^{b(m^2 x + y)}, \quad a \neq 0. \tag{6.10}
\]

**Example 6.3.** When \( \varepsilon = -1 \) and \( f = \pm k = \hat{f}(x + y) \), (6.3) becomes

\[
2 \left( \frac{\hat{f}''}{\hat{f}} \right)' = \hat{f}^2. \tag{6.11}
\]

Since \( \hat{f} \neq 0 \), (6.11) implies that \( \hat{f} \) is non-constant. Thus, (6.11) yields

\[
2 \frac{\hat{f}''}{f^2} - \hat{f}^2 = \pm c^2 = \text{constant}, \quad c \geq 0. \tag{6.12}
\]

**Case (i):** \(-c^2 < 0\). Solving (6.12) gives \( \hat{f}(u) = c \sec \left( \frac{cu}{\sqrt{2}} \right) \). As in Example 6.1, we find the following solutions of system (6.1)-(6.3):

\[
f(x, y) = cm \sec \left( \frac{c(m^2 x + y)}{\sqrt{1 + m^2}} \right), \quad k(x, y) = \pm c \sec \left( \frac{c(m^2 x + y)}{\sqrt{1 + m^2}} \right),
\]

where \( c, m \) are positive real numbers.

**Case (ii):** \(-c^2 < 0\). \( c^2 > 0 \). In this case, after solving (6.12) we get another family of solutions:

\[
f(x, y) = cm \operatorname{csch} \left( \frac{c(m^2 x + y)}{\sqrt{1 + m^2}} \right), \quad k(x, y) = \pm c \operatorname{csch} \left( \frac{c(m^2 x + y)}{\sqrt{1 + m^2}} \right).
\]

**Case (iii):** \( c > 0 \). In this case, after solving (6.12) we get another family of solutions:

\[
f(x, y) = \frac{m \sqrt{1 + m^2}}{m^2 x + y}, \quad k(x, y) = \frac{\sqrt{1 + m^2}}{m^2 x + y}, \quad m \neq 1.
\]
7. H-STATIONARY SURFACES ARISING FROM TRAVELING WAVE SOLUTIONS.

We are able to construct type II Hamiltonian-stationary Lagrangian surfaces in $M^2(4\varepsilon)$ using the traveling wave solutions.

First, let us consider the twisted product decomposition with

\begin{equation}
\hat{L} = \frac{1}{\sqrt{2 + m^2}} \left( \frac{2m \sqrt{2 + m^2}}{\sqrt{1 + 5m^2}} e^{i(x+y)} \sin \left( \frac{\sqrt{1 + 5m^2}}{2\sqrt{1 + m^2}} (x - y) \right), \right.
\end{equation}

\begin{equation}
\left. e^{-i(x+y)} \left[ \frac{\sqrt{1 + m^2} \cos \left( \frac{\sqrt{1 + 5m^2}}{2\sqrt{1 + m^2}} (x - y) \right)}{\sqrt{1 + 5m^2}} \sin \left( \frac{\sqrt{1 + 5m^2}}{2\sqrt{1 + m^2}} (x - y) \right) \right) \right]
\end{equation}

Next, let us consider the following twisted product flat metric:

\begin{equation}
g = e^{2b(m^2x+y)} (m^2dx^2 + dy^2), \ m \neq 0, 1; \ b \in \mathbb{R}.
\end{equation}

The corresponding H-stationary Lagrangian immersion $L$ satisfies the following PDE system:

\begin{equation}
L_{xx} = (i + bm^2) L_x - bm^2 L_y,
\end{equation}

\begin{equation}
L_{xy} = bL_x + bm^2 L_y,
\end{equation}

\begin{equation}
L_{yy} = -bL_x + (i + b) L_y.
\end{equation}

Solving this system yields the following new family of flat H-stationary Lagrangian surfaces in $\mathbb{C}^2$:

\begin{equation}
L(x, y) = \frac{e^{b(x+y)+b(m^2x+y)} \left( \frac{2m \sin \left( \frac{1}{2} \sqrt{1 + 4b^2m^2} (x - y) \right)}{\sqrt{1 + 4b^2m^2}} \right)}{\sqrt{1 + b^2(1 + m^2)^2}},
\end{equation}

\begin{equation}
(1 + m^2) \cos \left( \frac{1}{2} \sqrt{1 + 4b^2m^2} (x - y) \right)
\end{equation}

\begin{equation}
- \frac{i(1 - m^2) \sin \left( \frac{1}{2} \sqrt{1 + 4b^2m^2} (x - y) \right)}{\sqrt{1 + 4b^2m^2}},
\end{equation}

\begin{equation}
(1 + 4b^2m^2). \end{equation}
where \(b, m\) are real numbers with \(0 < m \neq 1\). In particular, if \(b = 0\), we obtain the following new family of \(H\)-stationary surfaces:

\[
L = \frac{e^{\frac{1}{2}(x+y)}}{\sqrt{1+m^2}} \sin \left( \frac{x-y}{2} \right) \left( 2m, (1+m^2) \cot \left( \frac{x-y}{2} \right) - i(1-m^2) \right).
\]

For \(CH^2(-4)\) we obtain the following:

**Theorem 7.1.** [13] There exist five families of Hamiltonian-stationary surfaces of type II in \(CH^2(-4)\) arisen the traveling wave solutions:

(a) \(\hat{L} = \left( 1 - \frac{i(1+m^2)}{m^2x+y}, \frac{m\sqrt{1+m^2}}{m^2x+y} e^{ix}, \frac{\sqrt{1+m^2} e^{iy}}{m^2x+y} \right)\);

(b) \(\hat{L} = \text{sech} \left( \frac{x+3y}{2\sqrt{3}} \right) \left( \frac{x-y + 4i}{2} e^{\frac{1}{2}(x+y)}, \frac{x-y - i}{2} e^{\frac{1}{2}(x+y)}, \sqrt{3} + 2\tan \left( \frac{x+3y}{2\sqrt{3}} \right) \right)\);

(c) \(\hat{L} = \left( \frac{\sqrt{3m^4+2m^2-1} \cosh(\alpha(x-y)) + i(m^2-1) \sinh(\alpha(x-y))}{m \sqrt{3m^2-1} e^{-\frac{1}{2}(x+y)}} \right) \sec \left( \frac{m^2x+y}{\sqrt{1+m^2}} \right) \frac{1}{\sqrt{3m^2-1}} \sec \left( \frac{m^2x+y}{\sqrt{1+m^2}} \right)\);

(d) \(\hat{L} = \left( \frac{\sqrt{1-2m^2-3m^4} \cos(\beta(x-y)) + i(1-m^2) \sin(\beta(x-y))}{m \sqrt{1-3m^2} e^{-\frac{1}{2}(x+y)}} \right) \frac{2m e^{\frac{1}{2}(x+y)}}{\sqrt{1-3m^2}} \sec \left( \frac{m^2x+y}{\sqrt{1+m^2}} \right) \frac{1}{m} \frac{1+\sqrt{1+m^2}}{\sqrt{1+m^2}} \tan \left( \frac{m^2x+y}{\sqrt{1+m^2}} \right)\);

(e) \(\hat{L} = \frac{1}{\sqrt{1+m^2}} \cosh \left( \frac{m^2x+y}{\sqrt{1+m^2}} \right) \left( \sinh \left( \frac{m^2x+y}{\sqrt{1+m^2}} \right) - i \sqrt{1+m^2} \cosh \left( \frac{m^2x+y}{\sqrt{1+m^2}} \right) \right)\).

\(e^{\frac{1}{2}(x+y)} \left\{ \sqrt{1+m^2} \cos \left( \frac{\sqrt{1+5m^2}}{2\sqrt{1+m^2}} (x-y) \right) + i(m^2-1) \sin \left( \frac{\sqrt{1+5m^2}}{2\sqrt{1+m^2}} (x-y) \right) \right\}, \)

\(\frac{2m \sqrt{2+m^2}}{\sqrt{1+3m^2}} e^{\frac{1}{2}(x+y)} \sin \left( \frac{\sqrt{1+5m^2}}{2\sqrt{1+m^2}} (x-y) \right) \).

where \(\alpha = \sqrt{\frac{3m^2-1}{2\sqrt{1+m^2}}}\) and \(\beta = \sqrt{\frac{1-3m^2}{2\sqrt{1+m^2}}}\).

8. Complete solutions of system [11] for \(\varepsilon = 0\).

It is quite difficult to find all exact solutions of the over-determined system. Fortunately, we are able to solve it for the case \(\varepsilon = 0\).

**Theorem 8.1.** [13] The solutions \(\{f, k\}\) of the following over-determined PDE system:

\[
\left( \frac{k}{f} \right)_x + \left( \frac{f}{k} \right)_y = 0, \quad \frac{f_y}{k} = \frac{k_x}{f}, \quad \left( \frac{f}{k} \right)_y + \left( \frac{k}{f} \right)_x = 0,
\]
are the following:

\[ f(x, y) = \pm k(x, y) = ae^{b(x+y)}; \]
\[ f(x, y) = \pm ae^{b(m^2x+y)}; \]
\[ f(x, y) = \frac{a}{\sqrt{-y}} e^{\arctan \sqrt{-y/x}}, \quad k(x, y) = \pm \frac{a}{\sqrt{-y}} e^{\arctan \sqrt{-y/x}}, \]

where \( a, b, c, m \) are real numbers with \( a, c, m \neq 0 \) and \( m \neq \pm 1 \).

The \( H \)-stationary Lagrangian surfaces of type II in \( \mathbb{C}^2 \) corresponding to solutions (8.3) can also be completely determined as follows:

\[
L = \frac{\sqrt{2\pi}ar^2}{\cosh(\pi/2)} \left( iJ_{\frac{1}{4}(1+i\nu)}(r^2)T_c^+(r, \theta) + J_{\frac{1}{4}(1-i\nu)}(r^2)T_c^-(r, \theta) \right) + \frac{1}{r^2} \int_0^r re^{ir^2} J_{\frac{1}{4}(1+i\nu)}(r^2) dr + \frac{i}{r^2} \int_0^r re^{ir^2} J_{\frac{1}{4}(1-i\nu)}(r^2) dr;
\]

\[
ir^2 J_{\frac{1}{4}(1+i\nu)}(r^2)T_c^+(r, \theta) - r^2 J_{\frac{1}{4}(1-i\nu)}(r^2)T_c^-(r, \theta);
\]

\[
L = \frac{1}{r^2} \int_0^r re^{ir^2} J_{\frac{1}{4}(1+i\nu)}(r^2) dr - \frac{i}{r^2} \int_0^r re^{ir^2} J_{\frac{1}{4}(1-i\nu)}(r^2) dr
\]

for \( a > 0, c \neq 0 \), where \( r, \theta \) and \( x, y \) are related by

\[
x = 2r^2 \cos^2 \theta, \quad y = -2r^2 \sin^2 \theta,
\]

and \( J_\nu(z) \) is the Bessel function of the first kind with index \( \nu \), which can be expressed in the following infinite series:

\[
J_\nu(z) = \left( \frac{z}{2} \right) ^\nu \sum_{j=0}^{\infty} \frac{(-1)^j(z/2)^{2j}}{j! \Gamma(\nu + j + 1)}.
\]

9. SOME APPLICATIONS.

As an application of the results in section 7, we mention the following classification result from [12].

**Theorem 9.1.** There exist five families of Hamiltonian-stationary Lagrangian submanifolds of constant curvature with positive relative nullity in the complex projective 3-space \( \mathbb{CP}^3(4) \):

1. A totally geodesic Lagrangian submanifold given by \( L: \mathbb{RP}^3(1) \rightarrow \mathbb{CP}^3 \);
2. A Lagrangian submanifold defined by

\[
L(x, y, s) = \left( \frac{\phi e^{2s} \sin (\delta s)}{\delta}, \frac{\phi e^{2s} \{2 \delta \cos (\delta s) - i \sin (\delta s)\}}{\delta \sqrt{4a^2 + \delta^2}}, \frac{2\hat{c}x + i b(1 - x^2 - y^2)}{\hat{c}(1 + x^2 + y^2)}, \frac{2b(\hat{c} + 2ia)\phi}{\hat{c}(4a^2 + \delta^2)}, \frac{2\hat{c}y + ic(1 - x^2 - y^2)}{\hat{c}(1 + x^2 + y^2)}, \frac{-2c(\hat{c} + 2ia)\phi}{\hat{c}(4a^2 + \delta^2)} \right),
\]
where \(a, b, c\) are real numbers and
\[
\hat{c} = \sqrt{b^2 + c^2} \neq 0, \quad \delta = \frac{1}{2} \sqrt{1 + 4a^2 + \hat{c}^2}, \quad \phi = \frac{a(1 - x^2 - y^2) + bx + cy}{1 + x^2 + y^2};
\]

(3) A Lagrangian submanifold defined by
\[
L(x, s, t) = \cos x \left( \tan x, \frac{2b \tanh s + i \sqrt{2b}}{\sqrt{2 + 4b}} \sqrt{2be^{is}/\sqrt{2b}} \text{sech} s \left( \frac{\sqrt{1 + 2b}}{\sqrt{2b}} \right), \right.
\]
\[
\frac{\sqrt{2be^{is}/\sqrt{2b}} \text{sech} s \left( \frac{\sqrt{1 + 2b}}{\sqrt{2b}} \right) \sin \left( \frac{\sqrt{1 + 2b}}{\sqrt{2b}} t \right)}{\sqrt{1 + 2b}} \left), \right.
\]
where \(b\) is a positive number;

(4) A Lagrangian submanifold defined by
\[
L(x, y, z) = \left( e^{ix} \left( \cos \left( \frac{i}{2} \sqrt{1 + 4a^2} y \right) + \frac{i}{\sqrt{1 + 4a^2}} \sin \left( \frac{i}{2} \sqrt{1 + 4a^2} y \right) \right) \cos x, \right.
\]
\[
\frac{2ae^{i\frac{\pi}{4}} \cos x}{\sqrt{1 + 4a^2}} \sin \left( \frac{i}{2} \sqrt{1 + 4a^2} y \right), \quad \frac{2be^{i\frac{\pi}{4}} \cos x}{\sqrt{1 + 4b^2}} \sin \left( \frac{i}{2} \sqrt{1 + 4b^2} z \right),
\]
\[
e^{i\frac{3\pi}{4}} \left( \cos \left( \frac{i}{2} \sqrt{1 + 4b^2} z \right) + \frac{i}{\sqrt{1 + 4b^2}} \sin \left( \frac{i}{2} \sqrt{1 + 4b^2} z \right) \right) \sin x,
\]
where \(a, b\) are positive numbers.

(5) A Lagrangian submanifold defined by
\[
L(x, y, z) = \left( \sin x, \tilde{L}(y, z) \cos x \right),
\]
where \(\tilde{L}\) is a horizontal lift of a type II Hamiltonian-stationary Lagrangian surface \(L : TP^2(1) \rightarrow CP^2(4)\).

Conversely, locally every Hamiltonian-stationary Lagrangian submanifold of constant curvature in \(CP^3\) with positive relative nullity is congruent to an open portion of a Lagrangian submanifold from one of the above five families.

For Hamiltonian-stationary Lagrangian submanifolds in \(CH^3\), we have the following result from [S].

**Theorem 9.2.** There exist ten families of Hamiltonian-stationary Lagrangian submanifolds of constant curvature in \(CH^3(-4)\) with positive relative nullity:

1. A totally geodesic Lagrangian submanifold \(L : H^3(-1) \rightarrow CH^3(-4)\);
2. A Lagrangian submanifold defined by
\[
L(s, y, z) = \frac{1}{2(1 - y^2 - z^2)} \left( (2i + s)e^{i\frac{\pi}{4}} (2by + \sqrt{1 + b^2(1 + y^2 + z^2)}), \right.
\]
\[
se^{i\frac{\pi}{4}} (2by + \sqrt{1 + b^2(1 + y^2 + z^2)}), 4\sqrt{1 + b^2} y + 2b(1 + y^2 + z^2), 4z \right), \quad b \in \mathbb{R}.
\]
(3) A Lagrangian submanifold defined by
\[ L(s, y, z) = \frac{1}{1 - y^2 - z^2} \left( \frac{e^{\frac{2}{4\alpha^2 - 4b^2}}(by + a(1 + y^2 + z^2))\{2\delta \cosh \delta s - i \sinh \delta s\}}{\delta \sqrt{4\alpha^2 - 4b^2}} \right. \\
\left. \frac{e^{\frac{2}{4\alpha^2 - 4b^2}}(by + a(1 + y^2 + z^2)) \sinh \delta s}{\delta \sqrt{4\alpha^2 - 4b^2}} \right), \]
where \( a, b, \delta \) are real numbers satisfying \( 4\alpha^2 - b^2 > 1 \) and \( 2\delta = \sqrt{4\alpha^2 - b^2 - 1} \).

(4) A Lagrangian submanifold defined by
\[ L(s, y, z) = \frac{1}{1 - y^2 - z^2} \left( \frac{e^{\frac{2}{4\alpha^2 - 4b^2}}(by + a(1 + y^2 + z^2))\{2\gamma \cos \gamma s - i \sin \gamma s\}}{\gamma \sqrt{4\alpha^2 - 4b^2}} \right. \\
\left. \frac{e^{\frac{2}{4\alpha^2 - 4b^2}}(by + a(1 + y^2 + z^2)) \sin \gamma s}{\gamma \sqrt{4\alpha^2 - 4b^2}} \right), \]
where \( a, b, \gamma \) are real numbers satisfying \( 4\alpha^2 < 1 + b^2 \), \( 2\gamma = \sqrt{1 + b^2 - 4\alpha^2} \) and \( 4\alpha^2 \neq b^2 \).

(5) A Lagrangian submanifold defined by
\[ L(s, y, z) = \left( \frac{2y - a^2(1 + is)((1 + y)^2 + z^2)}{\sqrt{a^2 - 1}(1 - y^2 - z^2)} \right) \frac{2z}{1 - y^2 - z^2}, \]
\[ \frac{1 + y^2 + z^2 + ia^2s((1 + y)^2 + z^2)}{\sqrt{a^2 - 1}(1 - y^2 - z^2)} \frac{ae^{is((1 + y)^2 + z^2)}}{1 - y^2 - z^2}, \]
where \( a^2 \neq 0, 1 \).

(6) A Lagrangian submanifold defined by
\[ L(s, y, z) = \left( \frac{is}{2} + \frac{3}{2} - i + \frac{2i - 3 - is + (2i - 2 - is)y}{1 - y^2 - z^2} \right) \frac{2z}{1 - y^2 - z^2}, \]
\[ \frac{is}{2} - \frac{1}{2} - i + \frac{1 + 2i - is + (2 + 2i - is)y}{1 - y^2 - z^2} \frac{e^{is((1 + y)^2 + z^2)}}{1 - y^2 - z^2}. \]

(7) A Lagrangian submanifold defined by
\[ L(x, s, t) = \frac{\cosh x}{\sqrt{1 - 2b}} \left( \sqrt{2b} \tan s - i, \sqrt{2b}e^{is}\sqrt{2b} \sec s \cos \left( \frac{1 - 2b}{\sqrt{2b}} \right) \right), \]
\[ \sqrt{2b}e^{is}/\sqrt{2b} \sec s \sin \left( \frac{1 - 2b}{\sqrt{2b}} \right), \sqrt{1 - 2b} \tanh x \right), \]
where \( 0 < 2b < 1 \).

(8) A Lagrangian submanifold defined by
\[ L(x, s, t) = \frac{\cosh x}{\sqrt{1 - 2b}} \left( \sqrt{2b}e^{is}/\sqrt{2b} \sec s \cosh \left( \frac{\sqrt{2b} - 1}{\sqrt{2b}} \right), \sqrt{2b} \tan s - i, \right. \]
\[ \sqrt{2b}e^{is}/\sqrt{2b} \sec s \sinh \left( \frac{\sqrt{2b} - 1}{\sqrt{2b}} \right), \sqrt{2b} \tanh x \left. \right) \right), \]
where \( 2b > 1 \).
(9) A Lagrangian submanifold defined by
\[ L(x, s, t) = \frac{\cosh x}{\sqrt{2}(1 + e^{2is})} \left( i + 2e^{2is}(s + it^2), i + 2e^{2is}(s + it^2), \right. \left. \sqrt{2}(1 + e^{2is}) \tanh x, 2\sqrt{2}e^{2is}t \right), \]

(10) A Lagrangian submanifold defined by
\[ L(x, y, z) = (\tilde{P}(y, z) \cosh x, \sinh x), \]
where \( \tilde{P} \) is a horizontal lift of a type II Hamiltonian-stationary Lagrangian surface \( L : TP_{f^2k^2}(-1) \to CH^2(-4) \) via the Hopf fibration \( \pi : H^1_1(-1) \to CH^2(-4) \).

Conversely, locally every Hamiltonian-stationary Lagrangian submanifold of constant curvature in \( CH^3(-4) \) with positive relative nullity is congruent to an open portion of a Lagrangian submanifold from one of the above families.

Remark 9.1. (Added on July 13, 2013) The PDE system (A) given in Theorem 8.1 was completely solved in [22]. In particular, it was proved in [22] that the PDE system admits only traveling wave solutions, whenever \( \varepsilon \neq 0 \).

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