A variant of Cowling–Price’s and Miyachi’s theorems for the Bessel–Struve transform on \( \mathbb{R}^d \)

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**ABSTRACT**

In this paper, we consider the Bessel–Struve transform on \( \mathbb{R}^d \). We establish new versions of Cowling–Price’s and Miyachi’s theorems for the Bessel–Struve transform on \( \mathbb{R}^d \). The techniques of the proofs are based on the properties of the Bessel–Struve kernel, the relation between the Bessel–Struve transform with the classical Fourier transform and on the positivity of the multi-variables Weyl and multi-variables Sonine transforms. The results of this paper are new, and they have novelty and generalize some results exist in the literature.

1. Introduction

An old Theorem of Hardy [1] proved way back in 1933 says that a function \( f \) and its Fourier transform \( \hat{f} \) cannot both have arbitrary gaussian decay unless \( f \) is identically zero. Defining \( \psi_\alpha(x) = e^{-\alpha|x|^2} \), we can state Hardy’s Theorem more precisely as follows: if both \( f/\psi_\alpha \) and \( \hat{f}/\psi_\beta \) are in \( L^\infty(\mathbb{R}) \) for some positive numbers \( \alpha, \beta \) then \( f = 0 \) whenever \( \alpha > 1/4 \). Moreover, when \( \alpha = 1/4 \) the function \( f \) is a constant multiple of \( \psi_\alpha \) and when \( \alpha < 1/4 \) there are infinitely many linearly independent functions satisfying both conditions. In 1983, Cowling and Price [2] generalized Hardy’s Theorem by replacing the \( L^\infty \) estimates by \( L^p \) estimates. They proved that if \( f/\psi_\alpha \in L^p(\mathbb{R}) \) and \( \hat{f}/\psi_\beta \in L^q(\mathbb{R}) \) for some \( p \) and \( q \) satisfying \( 1 < p, q < \infty \), then \( f = 0 \) whenever \( \alpha > 1/4 \). The same conclusion holds even when \( \alpha = 1/4 \) provided \( \alpha \) is finite. In 1997, Miyachi [3] proved the following generalization of Hardy’s Theorem. If \( f \) is a measurable function on \( \mathbb{R} \) such that \( f/\psi_\alpha \in L^1(\mathbb{R}) + L^\infty(\mathbb{R}) \) and

\[
\int_{\mathbb{R}} \log^+ \frac{\hat{f}(\xi)}{\lambda} \frac{1}{\lambda} \, d\xi < \infty
\]

for some positive constants \( \alpha \) and \( \lambda \), then \( f \) is a constant multiple of \( \psi_\alpha \).

The Cowling–Price’s principle and their variants have been studied by many authors for various Fourier transforms, for examples (cf. [4–7]) and others.

In this paper we consider the differential operator \( \Delta_\gamma, \gamma > 0, \) on \( \mathbb{R}^n \) given by

\[
\Delta_\gamma u(x) = D^2 u(x) + \frac{2\gamma}{x} [Du(x) - Du(0)].
\]

This operator is called Bessel–Struve operator. It is connected with Dunkl theory [8, 9]. In fact, Trimèche in [10] has introduced this operator and has built the harmonic analysis associated with this operator. In particular, the generalized Fourier transform associated with \( \Delta_\gamma \), called Bessel–Struve transform was studied.

The purpose of the present paper is twofold. On one hand, we want to prove a variant of Cowling–Price’s theorem for the Bessel–Struve transform on \( \mathbb{R}^d \). We note that the analogue of the Cowling–Price’s theorem for the Bessel–Struve transform on the real line was studied in [11]. Our version is different and it is motivated by the version of the Cowling–Price given by Ray and Sarkar in the cadre of symmetric spaces (cf. [12]). On the other hand, we want to prove a variant of Miyachi’s theorem for the Bessel–Struve transform on \( \mathbb{R}^d \). We note also that the analogue of Hardy’s theorem was proved for the Bessel–Struve transform on \( \mathbb{R} \) in [11], and the analogue of Beurling’s theorem was proved for the Bessel–Struve transform on \( \mathbb{R} \) in [13].

The structure of this paper is the following. In §2, we recall some basic facts about the harmonic analysis results related to the operator \( \Delta_\gamma \). §3 is devoted
to give variants of Cowling–Price’s theorem for the Bessel–Struve transform on $\mathbb{R}^d$. The last section of the paper aims to prove an analogue of Miyachi’s theorem for the Bessel–Struve transform on $\mathbb{R}^d$.

2. Preliminaries

In order to confirm the basic and standard notations, we briefly overview the theory of Bessel–Struve operator and related harmonic analysis. Main reference is [10, 11, 13].

2.1. The Bessel–Struve kernel

The Bessel–Struve kernel is the function $K_{\gamma,d}$ given by

$$K_{\gamma,d}(ix, \lambda) = V_{\gamma,d}(e^{i\langle \cdot, \cdot \rangle})_{\mathbb{R}^d},$$

for all $x, \lambda \in \mathbb{R}^d$, (1)

where $V_{\gamma,d}$ is the multi-variable Sonine transform defined by

$$V_{\gamma,d}(f)(x) = \frac{2\Gamma(\gamma + \frac{d}{2})}{\Gamma(\gamma)\Gamma(\frac{d}{2})} \int_0^1 f(tx)(1-t^2)^{-\frac{d}{2}-1} dt,$$

for all $x \in \mathbb{R}^d$. (2)

From the relations (1) and (2) we have

$$K_{\gamma,d}(ix, \lambda) = \frac{2\Gamma(\gamma + \frac{d}{2})}{\Gamma(\gamma)\Gamma(\frac{d}{2})} \int_0^1 e^{i\langle x, \lambda \rangle}(1-t^2)^{-\frac{d}{2}-1} dt,$$

for all $x, \lambda \in \mathbb{R}^d$. (3)

This kernel has a holomorphic extension to $\mathbb{C}^d \times \mathbb{C}^d$ and possesses the following properties:

(i) For all $z, t \in \mathbb{C}^d$, for all $\lambda \in \mathbb{C}$, $K_{\gamma,d}(z, t) = K_{\gamma,d}(t, z)$, $K_{\gamma,d}(z, 0) = 1$ and

$$K_{\gamma,d}(\lambda z, t) = K_{\gamma,d}(z, \lambda t).$$

(ii) For all $v \in \mathbb{N}^d, x \in \mathbb{R}^d$ and $z \in \mathbb{C}^d$,

$$|D^n_z K_{\gamma,d}(x, z)| \leq \|x\|^n \exp(|x||\text{Re} z|),$$

where

$$D^n_z = \frac{\partial |v|}{\partial z_1^{v_1} \cdots \partial z_d^{v_d}}$$

and $|v| = v_1 + \cdots + v_d$.

In particular,

$$|K_{\gamma,d}(x, iy)| \leq 1, \quad \text{for all } x, y \in \mathbb{R}^d.$$

(iii) For all $\lambda, x \in \mathbb{R}^d$

$$K_{\gamma,d}(\lambda x) = i^{1-d} \sqrt{\pi} \Gamma(\gamma + \frac{d}{2}) \frac{\Gamma(\gamma + \frac{d}{2})}{\Gamma(\gamma + \frac{d}{2})} \prod_{1 \leq j \leq d} j_{\gamma - \frac{1}{2}}(\lambda x) h_{\gamma - \frac{1}{2}}(\lambda x),$$

where $j_{\gamma - \frac{1}{2}}$ and $h_{\gamma - \frac{1}{2}}$ are respectively the normalized Bessel and Struve functions of order $\gamma - \frac{1}{2}$.

2.2. The Bessel–Struve transform

In the following, we denote by $L^p(\mathbb{R}^d), 1 \leq p \leq \infty$, the space of measurable functions $f$ on $\mathbb{R}^d$ with finite $L^p$-norm $\|f\|_{L^p(\mathbb{R}^d)}$ with respect to the Lebesgue measure $dx$.

$$d\mu_{\gamma,d}(x) := \|x\|^2 dx \quad \text{with } \gamma \geq 0 \text{ the weighted measure}.$$

$L^p(d\mu_{\gamma,d}), 1 \leq p \leq \infty$, the space of measurable functions $f$ on $\mathbb{R}^d$:

$$\|f\|_{L^p(d\mu_{\gamma,d})} = \left( \int_{\mathbb{R}^d} |f(x)|^p d\mu_{\gamma,d}(x) \right)^{1/p} < \infty,$$

if $1 \leq p < \infty$,

$$\|f\|_{L^\infty(d\mu_{\gamma,d})} = \text{ess sup}_{x \in \mathbb{R}^d} |f(x)| < \infty.$$

$C_b(\mathbb{R}^d)$ the space of continuous bounded functions on $\mathbb{R}^d$.

$P(\mathbb{R})$ the set of polynomials on $\mathbb{R}$ and $P_m(\mathbb{R})$ the one of degree $m$.

The Bessel–Struve transform $F_{\gamma,d}$ on $L^1(d\mu_{\gamma,d})$ is given by

$$\forall x \in \mathbb{R}^d, \quad F_{\gamma,d}(f)(\xi) = \int_{\mathbb{R}^d} f(x)K_{\gamma,d}(x - i\xi) d\mu_{\gamma,d}(x).$$

The generalized heat kernel related to the Bessel–Struve transform $N_{\gamma,d}(\cdot, s), s > 0$, is given by

$$\forall x \in \mathbb{R}^d, \quad N_{\gamma,d}(x, s) = \frac{1}{(2\pi s)^{d/2}} e^{-|x|^2/4s}.$$ (8)

It satisfies

$$\forall \xi \in \mathbb{R}^d, \quad F_{\gamma,d}(N_{\gamma,d}(-s, \cdot))(\xi) = C e^{-s|\xi|^2},$$

where $C$ is a positive constant.

Some basic properties of the Bessel–Struve transform are the following (cf. [10])

(i) For all $f \in L^1(d\mu_{\gamma,d}),$

$$\|F_{\gamma,d}(f)\|_{L^\infty(d\mu_{\gamma,d})} \leq \|f\|_{L^1(d\mu_{\gamma,d})}.$$

(ii) For all $f \in L^1(d\mu_{\gamma,d}),$

$$F_{\gamma,d}(f) = F \circ \gamma_{\gamma,d}(f),$$

where $F$ is the classical Fourier transform on $\mathbb{R}^d$ and the operator $\gamma_{\gamma,d}$ is multi-variable Weyl transform defined by

$$\gamma_{\gamma,d}(f)(y) = 2\Gamma\left(\gamma + \frac{d}{2}\right) \frac{\Gamma(\gamma + \frac{d}{2})}{\Gamma(\gamma + \frac{d}{2})} \int_0^\infty f(r) \left(\frac{y}{\|y\|} \right)^{2} (\|y\|^2 - \|y\|^2)^{\gamma - 1} r dr,$$

for all $y \in \mathbb{R}^d \setminus \{0\}$. (11)
Proposition 2.1: For \( f \in L^1(d\mu_{\gamma, d}) \) and \( g \in C_b(\mathbb{R}^d) \) we have
\[
\int_{\mathbb{R}^d} \mathcal{V}_{\gamma, d}(f)(y)g(y) \, dy = \int_{\mathbb{R}^d} \mathcal{V}_{\gamma, d}(g)(x) \, d\mu_{\gamma, d}(x). 
\]
(12)

Remark 2.1: By taking \( g \equiv 1 \) in (12) we can deduce that for all \( f \in L^1(d\mu_{\gamma, d}) \),
\[
\int_{\mathbb{R}^d} \mathcal{V}_{\gamma, d}(f)(y) \, dy = \int_{\mathbb{R}^d} f(x) \, d\mu_{\gamma, d}(x). 
\]
(13)

Using the relations (10) and (11), we obtain the following.

Proposition 2.2: Let \( \psi \in \mathcal{P}_m(\mathbb{R}) \) be homogeneous. Then for all \( \delta > 0 \), there exists a homogeneous \( Q \in \mathcal{P}_m(\mathbb{R}) \) such that
\[
\forall \xi \in \mathbb{R}^d, \quad \mathcal{V}_{\gamma, d}(\psi(\| \cdot \|)) \leq C(\| \xi \|) \, e^{-\| \xi \|^2/4d}. 
\]
(14)

3. Cowling–Price’s theorem for the Bessel–Struve transform

We shall prove a generalization of Cowling–Price’s theorem for the Bessel–Struve transform \( \mathcal{F}_{\gamma, d} \) on \( \mathbb{R}^d \).

Theorem 3.1: Let \( f \) be a measurable function on \( \mathbb{R}^d \) which satisfies for \( p, q \in [1, \infty) \) and \( a, b > 0 \)
\[
\int_{\mathbb{R}^d} \frac{e^{ap|\xi|^2} \|f(x)\|^p}{(1 + \|x\|^2)^{n}} \, d\mu_{\gamma, d}(x) < \infty 
\]
and
\[
\int_{\mathbb{R}^d} \frac{e^{bq|\xi|^2} \|\mathcal{V}_{\gamma, d}(f)(\xi)\|^q}{(1 + \|\xi\|^2)^m} \, d\xi < \infty, 
\]
for any \( n \in (d + 2\gamma, d + 2\gamma + p) \) and \( m \in (d, d + q) \). Then

(i) If \( ab > \frac{1}{4}, \) then \( f = 0 \) almost everywhere.
(ii) If \( ab = \frac{1}{4}, \) then \( f \) is a constant multiple of \( N_{\gamma, d}(\cdot, b) \).
(iii) If \( ab < \frac{1}{4}, \) then for all \( \delta \in (b, 1/4a) \), all functions of the form \( f(x) = P(\|x\|\|N_{\gamma, d}(x, \delta)\|, P \in \mathcal{P}(\mathbb{R}), \) satisfy (15) and (16).

Proof: Clearly (15) implies that \( f \) belongs to \( L^1(d\mu_{\gamma, d}) \) and thus, \( \mathcal{V}_{\gamma, d}(f)(\xi) \) exists for all \( \xi \in \mathbb{R}^d \). Moreover, it has an entire holomorphic extension on \( \mathbb{C}^d \) satisfying for some \( s > 0 \),
\[
|\mathcal{F}_{\gamma, d}(f)(z)| \leq C e^{a|\text{Im} z|^2/4d} (1 + |\text{Im} z|)^s. 
\]
(17)

Actually, it follows from (7) and (5) that for all \( z = \xi + i\eta \in \mathbb{C}^d \),
\[
|\mathcal{F}_{\gamma, d}(f)(\xi + i\eta)| 
\leq \int_{\mathbb{R}^d} |f(x)||K_{\gamma, d}(x, -i\xi + \eta)| \, d\mu_{\gamma, d}(x) 
\leq e^{\eta^2/4a} \left( \int_{\mathbb{R}^d} \frac{e^{a|\xi|^2} |f(x)|}{(1 + \|x\|^2)^{n/p}} (1 + \|x\|)^{n/p} \right)^{1/p} \times \left( \int_{\mathbb{R}^d} e^{-a(|\xi| - \frac{\eta}{2a})^2} \, d\mu_{\gamma, d}(x) \right)^{1/p}. 
\]

Using Hölder’s inequality and (15) we can obtain that
\[
|\mathcal{F}_{\gamma, d}(f)(\xi + i\eta)| 
\leq C e^{\eta^2/4a} \left( \int_{\mathbb{R}^d} (1 + \|x\|)^{n/p} e^{-ap(\|x\| - \|\xi\|)^2} \, d\mu_{\gamma, d}(x) \right)^{1/p} 
\leq C e^{\eta^2/4a} \left( \int_{0}^{\infty} (1 + r)^{n/p - 1} e^{-ap(r - \|\xi\|)^2} \, dr \right)^{1/p} \leq C e^{\eta^2/4a} (1 + \|\eta\|)^{n/p + 2\gamma + d - 1}/p'. 
\]

If \( ab = \frac{1}{4}, \) then
\[
|\mathcal{F}_{\gamma, d}(f)(\xi + i\eta)| \leq C e^{\|\eta\|^2} (1 + \|\eta\|)^{n/p + 2\gamma + d - 1}/p'. 
\]

Therefore, if we let \( g(z) = e^{b|\text{Re} z|^2 + \cdots + b|z|^2} \mathcal{V}_{\gamma, d}(f)(z) \), then
\[
|g(z)| \leq C e^{b|\text{Re} z|^2} (1 + |\text{Im} z|)^{n/p + 2\gamma + d - 1}/p'. 
\]

Hence it follows from (16) that
\[
\int_{\mathbb{R}^d} \frac{|g(\xi)|^q}{(1 + \|\xi\|^2)^m} \, d\xi < \infty. 
\]

Here we use the following lemma.

Lemma 3.1: ([7]) Let \( h \) be an entire function on \( \mathbb{C}^d \) such that
\[
|h(z)| \leq C e^{a|\text{Re} z|^2} (1 + |\text{Im} z|)^l 
\]
for some \( l > 0, a > 0 \) and
\[
\int_{\mathbb{R}^d} \frac{|h(x)|^q}{(1 + |x|^2)^m} \, dx < \infty, 
\]
for some \( q \geq 1, m > 1 \) and \( Q \in \mathcal{P}_M(\mathbb{R}^d) \). Then \( h \) is a polynomial with \( \deg h \leq \min(l, (m - M - d)/q) \) and, furthermore if \( m \geq q + M + d \), then \( h \) is a constant.

Hence by this lemma \( g \) is a polynomial, we say \( P_{b, \gamma} \) with \( \deg P_{b, \gamma} \leq \min(n/p + (2\gamma + d - 1)/p', (m - d)/q) \). Furthermore, if \( m \leq d + q \), then \( g \) is a constant by Lemma 3.1 and thus, \( \mathcal{F}_{\gamma, d}(f)(\xi) \) is \( \mathcal{C} e^{-b|x|^2} \) and hence \( f(x) = CN_{\gamma, d}(x, b) = C e^{-a|x|^2} \). When \( n > d + 2\gamma \) and \( m > d \), these functions satisfy (16) and (15) respectively. This proves (ii).

If \( ab > \frac{1}{4}, \) then we can choose positive constants, \( a_1, b_1 \) such that \( a > a_1 = 1/(4b_1) > 1/4b \). Then \( f \) and \( \mathcal{F}_{\gamma, d} \) also satisfy (15) and (16) with \( a \) and \( b \) replaced.
by $a_1$ and $b_1$ respectively. Therefore, it follows that $F_{\gamma,d}(f)(\xi) = P_{b_1}(x) e^{-b_1|\xi|^2}$. But then $F_{\gamma,d}(f)$ cannot satisfy (16) unless $P_{b_1} \equiv 0$, which implies $f \equiv 0$. This proves (i).

If $ab < \frac{1}{4}$, then for all $\delta \in (b, 1/4a)$, the functions of the form $f(x) = P((|x|)N_{\gamma,d}(x, \delta))$, where $P \in \mathcal{P}(\mathbb{R})$, satisfy (15) and (16). This proves (iii).

4. Miyachi’s theorem for the Bessel–Struve transform

**Theorem 4.1:** Let $f$ be a measurable function on $\mathbb{R}^d$ such that

$$e^{\xi^2/2} f \in L^p(\mu_{\gamma,d}) + L^q(\mu_{\gamma,d})$$

and

$$\int_{\mathbb{R}^d} \frac{\log^+ |F_{\gamma,d}(f)(\xi)| e^{\frac{1}{\xi^2}}}{\lambda} \, d\xi < \infty,$$

for some constants $a, b, \lambda > 0$ and $1 \leq p, q \leq \infty$.

(i) If $ab > \frac{1}{4}$, then $f \equiv 0$ almost everywhere.

(ii) If $ab = \frac{1}{4}$, then $f = CN_{\gamma,d}((b, b))$ with $|C| \leq \lambda$.

(iii) If $ab < \frac{1}{4}$, then for all $\delta \in (b, 1/4a)$, all functions of the form $f(x) = P((|x|)N_{\gamma,d}(x, \delta))$, $P \in \mathcal{P}(\mathbb{R})$, satisfy (18) and (19).

To prove this result we need the following lemmas.

**Lemma 4.1:** ([5]) Let $h$ be an entire function on $\mathbb{C}^d$ such that

$$|h(z)| \leq A e^{8|x|^2}$$

and

$$\int_{\mathbb{R}^d} \log^+ |h(y)| \, dy < \infty,$$

for some positive constants $A, B$. Then $h$ is a constant.

**Lemma 4.2:** Let $r \in [1, \infty], a > 0$. Then for $g \in L^r(\mu_{\gamma,d})$, there exists $C > 0$ such that

$$\|e^{\xi^2/2} t_{\gamma,d}(e^{-\alpha|y|^2} g)\|_{L^r(\gamma,d)} \leq C \|g\|_{L^r(\mu_{\gamma,d})}.$$  

**Proof:** From the hypothesis it follows that $e^{-\alpha|y|^2} g$ belongs to $L^1(\mu_{\gamma,d}(\mathbb{R}^d))$. Then by Proposition 2.1, we deduce that $t_{\gamma,d}(e^{-\alpha|y|^2} g)$ is defined almost everywhere on $\mathbb{R}^d$. Here we consider two cases.

(i) If $r \in [1, \infty)$, then

$$\|e^{\xi^2/2} t_{\gamma,d}(e^{-\alpha|y|^2} g)\|_{L^r(\gamma,d)} \leq C \int_{\mathbb{R}^d} e^{-\alpha|y|^2} \left(\int_{|\xi|}^{\infty} e^{-\eta|y|^2} \frac{(x \cdot \eta)^{-r}}{(t^2 - |\eta|^2)^{r-1}} \, dt\right) \, dy \leq C \|g\|_{L^r(\mu_{\gamma,d})}.$$  

where $r'$ is the conjugate exponent of $r$. Since

$$t_{\gamma,d}(e^{-\alpha|y|^2})(x) = C e^{-t|x|^2},$$

for $t > 0$ (cf. [4]), it follows from (13) that

$$\|e^{\xi^2/2} t_{\gamma,d}(e^{-\alpha|y|^2} g)\|_{L^r(\gamma,d)} \leq C \int_{\mathbb{R}^d} t_{\gamma,d}(\|g\|_r)(x) \, dx \leq C \int_{\mathbb{R}^d} |g(x)| t_{\mu_{\gamma,d}}(x) \, dx < \infty.$$  

(ii) If $r = \infty$, then it follows from (21) that

$$e^{\xi^2/2} t_{\gamma,d}(e^{-\alpha|y|^2} g)(x) \leq e^{\xi^2/2} t_{\gamma,d}(e^{-\alpha|y|^2}(x) g\|_{L^\infty(\mu_{\gamma,d})} \leq C \|g\|_{L^\infty(\mu_{\gamma,d})} < \infty.$$  

This completes the proof.

**Lemma 4.3:** Let $p, q$ in $[1, \infty]$ and $f$ a measurable function on $\mathbb{R}^d$ such that

$$e^{\xi^2/2} f \in L^p(\mu_{\gamma,d}) + L^q(\mu_{\gamma,d}),$$

for some $a > 0$. Then for all $z \in \mathbb{C}^d$, the integral

$$F_{\gamma,d}(f)(z) = \int_{\mathbb{R}^d} f(x) \mathcal{H}_{\gamma,d}(z, x) \, dx$$

is well-defined. $F_{\gamma,d}(f)(z)$ is entire and there exists $C > 0$ such that for all $\xi, \eta \in \mathbb{R}^d$,

$$|F_{\gamma,d}(f)(\xi + i\eta)| \leq C e^{\frac{|\eta|^2}{4a}}.$$  

**Proof:** The first assertion easily follows from (5) and Hölder’s inequality.

We shall prove (23). The relation (22) implies that $f$ belongs to $L^1(\mu_{\gamma,d})$ and thus, $t_{\gamma,d}(f) \in L^1(\mathbb{R}^d)$ by (13). Hence by (10), for all $\xi, \eta \in \mathbb{R}^d$,

$$F_{\gamma,d}(f)(\xi + i\eta) = \int_{\mathbb{R}^d} t_{\gamma,d}(f)(x) e^{-i(x \cdot \xi + \eta)} \, dx.$$  

Thus

$$|F_{\gamma,d}(f)(\xi + i\eta)| \leq e^{\frac{|\eta|^2}{2a}} \int_{\mathbb{R}^d} e^{\xi^2/2} t_{\gamma,d}(f)(x) e^{-\alpha|x|^2 + (x \cdot \eta)} \, dx \leq e^{\frac{|\eta|^2}{2a}} \int_{\mathbb{R}^d} e^{\xi^2/2} t_{\gamma,d}(f)(x) e^{-\alpha|x|^2 - 2\alpha |\eta|^2} \, dx.$$  

Since (22) implies that there exist $u \in L^p_{\gamma,d}(\mathbb{R}^d)$ and $v$ belongs to $L^q_{\gamma,d}(\mathbb{R}^d)$ such that

$$f(x) = e^{-\alpha|x|^2} u(x) + e^{-\alpha|x|^2} v(x),$$

it follows from Lemma 4.2 that

$$\int_{\mathbb{R}^d} e^{\xi^2/2} t_{\gamma,d}(f)(x) e^{-\alpha|x|^2 - 2\alpha |\eta|^2} \, dx \leq C(\|u\|_{\gamma,d}^p + \|v\|_{\gamma,d}^q) < \infty.$$  

Therefore, the desired result follows.
**Proof of Theorem 4.1:** We will divide the proof in each case.

(i) \( ab > \frac{1}{4} \). Let \( h \) be a function on \( \mathbb{C}^d \) defined by

\[
h(z) = \left( \prod_{j=1}^{d} e^{\xi_j^2/4a} \right) \mathcal{F}_{\gamma,d}(f)(z). \tag{24}
\]

This function is entire on \( \mathbb{C}^d \) and by (23) we see that

\[
|h(\xi + in)| \leq C e^{\xi^2/4a}, \quad \text{for all } \xi \in \mathbb{R}^d \text{ and } n \in \mathbb{R}^d.
\]

On the other hand, we note that

\[
\int_{\mathbb{R}^d} \log^+ |h(y)| \, dy = \int_{\mathbb{R}^d} \log^+ \left| e^{\xi |y|^2/4a} \mathcal{F}_{\gamma,d}(f)(y) \right| \, dy
\]

\[
= \int_{\mathbb{R}^d} \log^+ \left( \frac{e^{\xi |y|^2} |\mathcal{F}_{\gamma,d}(f)(y)| |\mathcal{F}_{\gamma,d}(f)(y)|}{\lambda} \right) \, dy
\]

\[
\leq \int_{\mathbb{R}^d} \log^+ \left( e^{\xi |y|^2} |\mathcal{F}_{\gamma,d}(f)(y)| \right) \, dy
\]

\[
+ \int_{\mathbb{R}^d} \log^+ \left( e^{(1/4a) - b}|y|^2 \right) \, dy,
\]

because \( \log^+ (cd) \leq \log^+ (c) + d \) for all \( cd > 0 \).

Since \( ab > \frac{1}{4} \), (19) implies that

\[
\int_{\mathbb{R}^d} \log^+ |h(y)| \, dy < \infty. \tag{26}
\]

Then it follows from (25) and (26) that \( h \) satisfies the assumptions in Lemma 4.1 and thus, \( h \) is a constant and

\[
\mathcal{F}_{\gamma,d}(f)(y) = C e^{-(1/4a)|y|^2}.
\]

Since \( ab > \frac{1}{4} \), (19) holds whenever \( C = 0 \) and the injectivity of \( \mathcal{F}_{\gamma,d} \) implies that \( f = 0 \) almost everywhere.

(ii) \( ab = \frac{1}{4} \). As in the previous case, it follows that \( \mathcal{F}_{\gamma,d}(f)(\xi) = C e^{-\xi^2/2a} \). Then (19) holds whenever \( |C| \leq \lambda \). Hence \( f = CN_{\gamma,d}(b \cdot) \) with \( |C| \leq \lambda \).

(iii) \( ab < \frac{1}{4} \). If \( f \) is a given form \( f(x) = P(\|x\|)N_{\gamma,d}(x, \delta), \) where \( P \in \mathcal{P}(\mathbb{R}) \) and \( \delta \in (b, 1/4a) \), then \( \mathcal{F}_{\gamma,d}(f)(\xi) = Q(\|\xi\|) e^{-\delta|\xi|^2} \) for some \( Q \in \mathcal{P} \). Then \( f \) and \( \mathcal{F}_{\gamma,d}(f) \) satisfy (18) and (19) for all \( \delta \in (b, 1/4a) \).

The following is an immediate consequence of Theorem 4.1.

**Corollary 4.1:** Let \( f \) be a measurable function on \( \mathbb{R}^d \) such that

\[
e^{a|x|^2} f \in L^p(d\mu_{\gamma,d}) + L^q(d\mu_{\gamma,d}), \tag{27}
\]

and

\[
\int_{\mathbb{R}^d} |\mathcal{F}_{\gamma,d}(f)(\xi)| e^{b|\xi|^2} \, d\xi < \infty, \tag{28}
\]

for some constants \( a, b > 0, \quad 1 < p, q < \infty \) and \( 0 < r \leq \infty \).

(i) If \( ab \geq \frac{1}{4} \), then \( f = 0 \) almost everywhere.

(ii) If \( ab < \frac{1}{4} \), then for all \( \delta \in (b, 1/4a) \), all functions of the form \( f(x) = P(\|x\|)N_{\gamma,d}(\delta, x), \) \( P \in \mathcal{P}(\mathbb{R}) \), satisfy (27) and (28).

**Corollary 4.2:** Let \( f \) be an integrable function on \( \mathbb{R}^d \) and \( p, q \in [1, \infty] \) such that \( e^{2b|\xi|^2} f \) belongs to \( L^p(d\mu_{\gamma,d}) + L^q(d\mu_{\gamma,d}) \), for some positive \( a \). Further assume that

\[
\int_{\mathbb{R}^d} e^{2b|\xi|^2} |\mathcal{F}_{\gamma,d}(f)(\xi)|^2 \log^+ \left( \frac{e^{2b|\xi|^2} |\mathcal{F}_{\gamma,d}(f)(\xi)|}{c} \right) \, d\xi < \infty,
\]

for some positive numbers \( b \) and \( c \). If \( ab = \frac{1}{4} \), then we get \( f(x) = A e^{-a|x|^2}, \) \( x \in \mathbb{R}^d \) and \( A \) is a positive constant. If \( ab < \frac{1}{4} \), there are infinitely many linearly independent functions meeting the hypotheses. Otherwise \( ab > \frac{1}{4} \), \( f \) vanishes almost everywhere on \( \mathbb{R}^d \).

**Remark 4.1:** In (18) and (27), \( L^1(d\mu_{\gamma,d}) + L^\infty(d\mu_{\gamma,d}) \) is essential, because

\[
L^p(d\mu_{\gamma,d}) + L^q(d\mu_{\gamma,d}) \subset L^1(d\mu_{\gamma,d}) + L^\infty(d\mu_{\gamma,d}).
\]

Indeed, for \( f = f_1 + f_2 \in L^p(d\mu_{\gamma,d}) + L^q(d\mu_{\gamma,d}) \), we put \( f_\infty(x) = f_1(x) \) if \( |f_1(x)| \leq 1 \) and 0 otherwise, and \( f_{1+} = f_1 - f_{1\infty} \). Then

\[
f = (f_\infty + f_2\infty) + (f_{1+} + f_{2+}) = f_\infty + f_1.
\]

It is easy to see that

\[
\|f_{1+}\|_{L^1(d\mu_{\gamma,d})} \leq \|f_1\|_{L^p(d\mu_{\gamma,d})} \leq \|f_1\|_{L^p(d\mu_{\gamma,d})}
\]

and

\[
\|f_{2+}\|_{L^1(d\mu_{\gamma,d})} \leq \|f_2\|_{L^q(d\mu_{\gamma,d})} \leq \|f_2\|_{L^q(d\mu_{\gamma,d})},
\]

respectively. Therefore, \( f_\infty \in L^\infty(d\mu_{\gamma,d}) \) and \( f_\infty \in L^1(d\mu_{\gamma,d}) \).

5. Conclusions

In the present paper, we have successfully studied two qualitative uncertainty principles for the Bessel–Struve transform on \( \mathbb{R}^d \). The obtained results have a novelty and contribution to the literature, and they improve and generalize the results of Hamem et al. [11] and Negzaoui [13]. It is our hope that this work motivate the researchers to study the quantitative uncertainty principles for the Bessel–Struve transform on \( \mathbb{R}^d \).

**Acknowledgments**

The authors are deeply indebted to the referees for providing constructive comments and helps in improving the contents of this article.
Disclosure statement
No potential conflict of interest was reported by the authors.

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