1. Introduction

Chiral symmetry plays an essential role in particle physics. In QCD it is a global symmetry which forbids an additive quark mass renormalization and whose spontaneous breakdown provides us with (nearly) Goldstone bosons with their specific interactions.

On the other hand, in electroweak interactions chiral symmetry plays an even more fundamental role: it is a local gauged symmetry. Its presence is necessary to provide a renormalizable theory.

Accordingly, when trying to regularize these theories on the lattice (which is the only non-perturbative regularization known for gauge theories) one meets two basic problems. As it is (or rather was) well known one cannot realize chiral symmetry exactly on the lattice without sacrificing some even more important principles, like locality or absence of extra physical particles. This is the content of the famous no-go theorem by Nielsen and Ninomiya \[1\]. In Wilson’s formulation of lattice QCD chiral symmetry is explicitly broken by an irrelevant term which does not affect the continuum limit. It causes, however, $O(a)$ lattice artifacts and other unwanted effects at finite lattice spacing.

For the electroweak theory, on the other side, the problem of chiral symmetry on the lattice is a principal one: is it possible to define a chiral gauge theory non-perturbatively? This second problem is much more difficult, and assumes that the first one – a solution for global chirality – has been successfully solved.

Several ways to solve these problems have been proposed. These were mostly covered on previous lattice conferences. Here I mention only the domain wall fermions \[2,3\] and the related overlap formalism \[4,5\]. One can state that these approaches provide a solution to the problem of global chirality. The numerical results using domain wall fermions \[3\] are encouraging. In the overlap formalism also important steps were made towards the solution of the gauged chiral symmetry (for references see Ref. \[5\]).

It was unclear, however, to what extent are the theoretical constructions used in these approaches necessary, or what is the basic ingredient behind the chiral properties observed. As it became clear recently, the basic relation which has to be satisfied to realize chiral symmetry at finite lattice spacing, is the Ginsparg-Wilson (GW) relation \[6\] suggested a long time ago!

In this talk I shall concentrate on the role of the GW relation and its consequences. Some of the results were obtained earlier in the approaches mentioned above, but even these are more transparent within the new framework.

1.1. The Nielsen-Ninomiya no-go theorem

Consider the free fermionic action

$$ S_F = a^4 \sum_{x,y} \bar{\psi}(x) D(x-y) \psi(y) , \quad (1) $$

describing a massless fermion on the lattice. (Colour, Dirac and flavour indices are suppressed.)

The desirable properties of the Dirac operator $D$ should be

(a) $D(x)$ is local (bounded by $Ce^{-\gamma|x|}$)
(b) \( \tilde{D}(p) = i\gamma_{\mu}p_{\mu} + O(ap^2) \) for \( p \ll \pi/a \)

(c) \( \tilde{D}(p) \) is invertible for \( p \neq 0 \) (no massless doublers)

(d) \( \gamma_5D + D\gamma_5 = 0 \)

Here \( \tilde{D}(p) \) is the Fourier transform of \( D(x) \).

The theorem by Nielsen and Ninomiya states that properties (a)-(d) cannot hold simultaneously.

Following Wilson, to avoid the massless doublers present in the naive discretization of the Dirac operator, one introduces a chiral symmetry breaking, irrelevant operator. The corresponding Wilson-Dirac operator is given by

\[
D_w = \frac{1}{2}\gamma_{\mu}(\nabla_{\mu} + \nabla_{\mu}^*) - \frac{1}{2}a\nabla_\mu \nabla_{\mu},
\]

(2)

where \( \nabla_{\mu} (\nabla_{\mu}^*) \) is the forward (backward) lattice derivative.

Due to the explicit breaking of chiral invariance, however, some extra problems appear:

1. Chiral symmetry is broken at finite lattice spacing by \( O(a) \) lattice artifacts and it is recovered only in the continuum limit.

2. One has to fine tune the bare quark mass since there is an additive quark mass renormalization.

3. Operators of different chiral representations get mixed, etc.

### 1.2. Locality

One of the conditions of the no-go theorem was locality. The natural definition (used here) is that the Dirac operator in an arbitrary gauge field background falls off exponentially i.e. satisfies the bound

\[
||D(x, y; U)|| \leq Ce^{-\gamma|x-y|},
\]

(3)

where \( C \) and \( \gamma > 0 \) are independent of the gauge field \( U \). In addition, it is also assumed that \( D(x, y; U) \) depends negligibly on \( U \)’s far apart from \( x \) and \( y \), i.e. that

\[
\delta D(x, y; U)/\delta U_{\mu}(z)
\]

(4)

also falls exponentially in \( |x-z|, |y-z| \) and \( |x-y| \). It is generally assumed that such interactions satisfy universality.

Note that it is physically too restrictive to consider only nearest neighbour interactions or even those with strictly finite support. In a real system the interaction coefficients die away exponentially rather then having a finite number of strictly nonzero terms.

On the other hand, interactions decreasing more slowly then exponentially should be considered as non-local and non-acceptable, since universality is not expected to hold for these. Some non-local interaction can give the same correlations as a local one (e.g. when a bad RG transformation is used, or when some physical fields are integrated out). This is, however, no sufficient excuse to use them – a slight change in the interaction could modify the picture drastically.

Theoretically any decay rate \( \gamma \) in eq. (3) is allowed. In practice, however, one works at finite lattice spacing and \( \gamma > ma \) must hold. For that reason it is important to have a sufficiently small interaction range.

In principle it could happen that the decay rate of \( D(x, y; U) \) is not uniformly bounded by an exponential, but there exists a bound \( \gamma(U) \) (with \( \inf_U \gamma(U) = 0 \)). In general, such interactions should be considered non-local. However, it may happen that those configurations when \( \gamma(U) \ll \gamma_0 \) are so strongly suppressed that they practically ‘never’ occur in a MC simulation. This is a potentially dangerous possibility, but could perhaps be accepted with extra caution.

### 2. The Ginsparg-Wilson relation

Back in 1982 Ginsparg and Wilson suggested a way to avoid the no-go theorem and preserve consequences of the chiral symmetry. They suggested to require instead of relation

\[
\gamma_5D + D\gamma_5 = 0
\]

the following milder condition

\[
\gamma_5D^{-1} + D^{-1}\gamma_5 = a2R\gamma_5.
\]

(5)

Here \( a \) is the lattice spacing and \( R \) is a local operator. For simplicity it will be assumed also that \( R \) is trivial in Dirac indices. That \( R \) commutes with \( \gamma_5 \) follows from eq. (3). The locality of \( R \) expresses the requirement that the propagating states are effectively chiral, so at distances larger than the range of \( R \) its presence is not felt.
viously, this is a highly nontrivial condition since the propagator $D^{-1}$ on the l.h.s. is non-local. (The coefficient 2 on the r.h.s. is for historical reasons only.)

Accordingly, the Dirac operator $D$ should satisfy the Ginsparg-Wilson relation

$$\gamma_5 D + D \gamma_5 = a D 2 R \gamma_5 D \quad (6)$$

In Ref. the relation originates from RG considerations applied to the free fermionic action and has been generalized to fermions in a gauge field. The local operator $R$ comes from a chiral symmetry breaking term in the block transformation only, not from the original action of the RG procedure. For that reason the physics described by $D$ is expected to be chirally invariant. (For a chirally symmetric block transformation $D$ becomes chirally invariant but non-local, in accordance with the no-go theorem.)

Two comments are in order here:

1. The naive commutation property for $D$ is recovered in the continuum limit $a \to 0$.

2. The r.h.s. of eq. is zero on solutions, i.e. for $D \psi = 0$.

Ginsparg and Wilson suggested that a solution to eq. is the mildest way to break chiral invariance: it should maintain the physical consequences of chiral invariance (soft pion theorem, etc.). However, no solution was found for the interacting case (QCD), and the paper has been practically forgotten for 15 years!

Last year Peter Hasenfratz realized that the fixed point (FP) action (or classically perfect action) for QCD satisfies the GW relation. This observation revived the interest in this relation.

### 2.1. The fermionic FP action

Let me recall the definition of the FP action. For the pure gauge part of the action one has the saddle point (i.e. classical) equation

$$S_{\gamma_5}^{FP}(V) = \min_U \left\{ S_{\gamma_5}^{FP}(U) + T_{\gamma_5}(V, U) \right\} \quad (7)$$

Here the gauge field $V$ lives on the coarse lattice, $U$ on the fine one, and $T_{\gamma_5}(V, U)$ is the blocking kernel. This is a recursive definition, but its recursive evaluation converges rapidly since the minimizing field $U = \overline{U}(V)$ is much smoother than the original $V$.

In the classical approximation the fermionic part remains quadratic, and the corresponding propagator satisfies the recursion relation

$$D_{FP}^{-1}(n, n'; V) = \frac{1}{\kappa} \delta_{nn'} \quad (8)$$

$$+ \sum_{x, y} \omega(x, n; U) D_{FP}^{-1}(x, y; U) \omega(y, n'; U)$$

Here $n, n'$ label the sites on the coarse lattice, $x, y$ on the fine lattice, while on the r.h.s. $U = \overline{U}(V)$, the minimizing configuration of the pure gauge problem is taken. The term proportional to $1/\kappa$ comes from the chiral breaking term in the blocking kernel for fermions; $\omega(x, n; U)$ characterizes the averaging procedure for the fermions in RG transformation – it is assumed to be trivial in Dirac indices and usually it only extends over the hypercube. Applying eq. recursively and using the fact that the original $D$ (in the continuum) anti-commutes with $\gamma_5$, it is easy to see that $D_{FP}^{-1}$ satisfies eq. with

$$R(U) = \frac{1}{\kappa} (\text{local operator}) \quad (9)$$

(In fact, for the RG transformation used, $R_{nn'} \propto \delta_{nn'}$ or it lives on the hypercube, i.e. is even ultra-local.)

In Ref. Hasenfratz has demonstrated that, indeed, the GW relation implies the chiral Ward identities. The argument is the following: When the symmetry breaking part appears in some expressions (integrated out in fermionic fields)

$$\langle \ldots \overline{\psi} DR \gamma_5 D \psi \ldots \rangle \quad (10)$$

then the $D^{-1}$ factors from contractions of the fermionic fields are cancelled by the $D$’s on both sides in the breaking term, leaving only the local operator $R$. As a consequence, the symmetry relation $\partial_\mu J_{5\mu} = 0$ is satisfied up to a contact term – i.e. one obtains the Ward identities. In this paper it has also been shown that the proper order parameter for spontaneous chiral symmetry breaking is given by

$$\langle \overline{\psi}_x \gamma_4 \psi_x \rangle_{\text{sub}} \equiv \langle \overline{\psi}_x \gamma_4 \psi_x - R_{xx} \rangle \quad (11)$$

It has been proven as well that the current is not renormalized ($Z_A = Z_V = 1$) and operators in
different chiral representations do not mix. I shall discuss these issues later, in the light of newer developments.

In Ref. [10] it has been shown that there is an exact Atiyah-Singer index theorem on the lattice, and the fermionic spectrum (in arbitrary gauge background) has nice chiral properties. The discussion was based on the FP action, but it used mostly only the GW relation to establish these facts.

As mentioned in the introduction, the domain wall and the related overlap formalism also produced Dirac operators with nice chiral properties. Neuberger [11] has pointed out that the Dirac operator obtained earlier [12] also satisfies the GW relation. To be on more general ground, I discuss first the overlap Dirac operator.

2.2. Neuberger’s Dirac operator

This is an explicit construction. Introduce \[ A = 1 - aD_w, \] (12)

where \( D_w \) is the standard massless Wilson-Dirac operator given by eq. (3). Note that

\[ \gamma_5 D_w \gamma_5 = D_w^\dagger. \] (13)

Then define

\[ D = \frac{1}{a} \left( 1 - A \frac{1}{\sqrt{A^\dagger A}} \right). \] (14)

It satisfies the GW relation

\[ \gamma_5 D + D \gamma_5 = aD \gamma_5 D \] (15)

(i.e. eq. (6) with \( R = 1/2; \) sometimes this special case will be referred to as the GW relation). Introducing \( V \) through

\[ V = 1 - aD, \] (16)

together with the relation \( \gamma_5 D \gamma_5 = D^\dagger \) one concludes that eq. (15) is equivalent to

\[ V^\dagger V = 1, \] (17)

i.e. \( V \) is unitary. In other words,

\[ D = \frac{1}{a} \left( 1 - V \right) \] (18)

with a unitary \( V \) defines a general solution to eq. (13). Of course, eq. (18) in itself is not enough to define an acceptable Dirac operator. For the free case the overlap Dirac operator has the properties (a)-(c) listed in the no-go theorem.

It will be argued later that \( D \) remains local in a gauge field background as well. As a consequence, \( D(x,y,U) \) defines an acceptable lattice Dirac operator (properties (a)-(c) of no-go theorem).

Let me make a few simple comments. (Some of these straightforward observations in this or modified form have been made independently by several people and I shall not always give a reference. See, however, Refs. [13,14].)

(1) Instead of \( D_w \) in eq. (12) one could start with any acceptable Dirac operator \( D_0 \) (massless, no doublers, local).

(2) If the original \( D_0 \) satisfies the GW relation eq. (4) then \( A^\dagger A = 1 \) hence \( D = D_0 \), i.e. the operator reproduces itself.

(3) \( H = \gamma_5 A = \gamma_5 (1 - D_w) \) is hermitian and \( A^\dagger A = H^2 \). As a consequence

\[ D = \frac{1}{a} \left( 1 - \gamma_5 \epsilon(H) \right), \] (19)

where \( \epsilon(x) \) is the sign function.

(4) For \( A = A(\mu) = \mu - aD_w \) (where \( 0 < \mu < 2 \)) one has

\[ D = \mu \frac{1}{a} \left( 1 - A \frac{1}{\sqrt{A^\dagger A}} \right), \] (20)

which satisfies a slightly modified relation

\[ \gamma_5 D + D \gamma_5 = \frac{1}{\mu} aD \gamma_5 D \] (21)

The choice of \( \mu \neq 1 \) will be useful as discussed later. Note that for \( \mu < 0 \) there are no massless fermions at all, while for \( \mu > 2 \) there are too many, since the modes around \( \lambda(D_w) = 2 \) (the massive doublers) also become massless modes of \( D \). Therefore only \( 0 < \mu < 2 \) produces an acceptable Dirac operator.

(5) It is also easy to generalize to the case of arbitrary local \( R \):

\[ A = 1 - \sqrt{2R} D_0 \sqrt{2R}, \] (22)

and

\[ D = \frac{1}{a} \frac{1}{\sqrt{2R}} \left( 1 - A \frac{1}{\sqrt{A^\dagger A}} \right) \frac{1}{\sqrt{2R}}. \] (23)

This satisfies eq. (3). For simplicity, however, we shall consider mostly the case \( R = 1/2 \).
3. The index theorem on the lattice

For a massless Dirac operator in the continuum theory the Atiyah-Singer index theorem [15] holds:

\[ Q_{\text{top}} = \text{index}(D). \] (24)

Here

\[ Q_{\text{top}} = \frac{1}{32\pi^2} \int d^4x \varepsilon_{\mu\nu\rho\sigma} \text{tr}(F_{\mu\nu}F_{\rho\sigma}) \] (25)

is the topological charge of the gauge field, and

\[ \text{index}(D) \equiv n_+ - n_- , \] (26)

where \( n_+ , n_- \) is the number of solutions to \( Du = 0 \) with +/− chiralities, i.e. \( \gamma_5 u = \pm u \).

The spectrum of \( D \) in the continuum consists of

- \( \lambda = 0 \); the corresponding modes have definite chirality
- \( \lambda = \pm i\alpha \) (\( \alpha \) real, non-zero); for the corresponding modes \( (u_\lambda, \gamma_5 u_\lambda) = 0 \).

Obviously, \( n_+ - n_- \) is topological invariant: an unpaired eigenvalue at \( \lambda = 0 \) cannot be continuously produced or destroyed.

In the lattice formulation one has problems with both sides of eq. (24). On the lhs. there are several ways to assign \( Q_{\text{top}}(U) \) to a given gauge configuration (field theoretical definition, geometrical definition). To avoid ambiguities for strongly fluctuating gauge fields, cooling is often applied. An additional problem is that, in general, dislocations could be present, i.e. configurations which violate the inequality

\[ S_{\text{g}}(U) \geq S_{\text{inst}}^{\text{cont}} \cdot |Q_{\text{top}}(U)|. \] (27)

On the rhs. in an arbitrary gauge field \( D_w \) has no exactly zero eigenvalues − they are distorted by lattice artifacts. The spectrum of \( D_w \) consists of

- real eigenvalues, with \( (u_\lambda, \gamma_5 u_\lambda) \neq 0 \) but not exactly \( \pm 1 \)
- complex eigenvalues, in cc. pairs, \( \lambda \) and \( \lambda^* \) with \( (u_\lambda, \gamma_5 u_\lambda) = 0 \).

For smooth instanton background \( (\rho \gg a) D_w \) has an isolated real, nearly zero eigenvalue with \( (u_\lambda, \gamma_5 u_\lambda) \approx +1 \) or \(-1 \). However, for a small instanton, or a typical MC configuration (say at \( \beta = 6.0 \)) there are a lot of small real eigenvalues. Therefore the definition of topological charge based on investigating the spectrum of \( D_w \) is quite ambiguous [14,17].

Let me remind you that the RG approach has a theoretically appealing (although difficult in practice) way to define the topological charge of a given configuration, in spin models [18] and in gauge theories [19]. The solution is the following: take the minimizing configuration on the fine lattice

\[ U_{\text{fine}} = U(U_{\text{coarse}}) \] (28)

and define the FP topological charge on \( U_{\text{coarse}} \) through

\[ Q_{\text{top}}(U_{\text{coarse}}) = Q_{\text{top}}^{\text{naive}}(U_{\text{fine}}). \] (29)

It is easy to show that with this definition there are no dislocations, i.e.

\[ S_{\text{FP}}(U) \geq S_{\text{inst}}^{\text{cont}} \cdot |Q_{\text{top}}^{\text{FP}}(U)| \] (30)

holds for any configuration.

As has been shown in Ref. [19] the FP fermion operator has nice chiral properties which allow one to establish the exact index theorem on the lattice.

Let me discuss the main points. The GW relation eq. (17) (with \( a = 1 \)) and \( \gamma_5 D \gamma_5 = D^\dagger \) implies

\[ D + D^\dagger = DD^\dagger = D^\dagger D, \] (31)

hence \( D \) and \( D^\dagger \) commute, i.e. \( D \) is normal. (As a consequence, the eigenvectors corresponding to different eigenvalues are orthogonal to each other.) The spectrum of \( D \) (cf. eq. (18)) is on the circle \( \lambda = 1 - e^{i\alpha} \). There are three types of eigenvalues

- \( \lambda = 0 \), with \( \gamma_5 u_\lambda = \pm u_\lambda \) (denote their numbers by \( n_+, n_- \)),
- \( \lambda = 2 \), with \( \gamma_5 u_\lambda = \pm u_\lambda \) (with multiplicities \( n'_+, n'_- \)),
- \( \lambda = \pm i \), with \( \gamma_5 u_\lambda = \pm u_\lambda \) (to be assigned).
\[ \lambda = \text{complex, with } \gamma_5 u_\lambda = u_\lambda^* \text{ and } (u_\lambda, \gamma_5 u_\lambda) = 0. \]

Note that \([14]\) since \(\text{Tr}(\gamma_5) = 0\) the index of \(D\) and \(2 - D\) is opposite:

\[ n'_+ - n'_- = n_- - n_+ . \quad (32) \]

Define the topological charge density by

\[ q(x) = \frac{1}{2} \text{tr} (\gamma_5 D(x, x)). \quad (33) \]

For the topological charge \(Q_{\text{top}} = \sum_x q(x)\) we have

\[ Q_{\text{top}} = \frac{1}{2} \text{Tr}(\gamma_5 D) = -\frac{1}{2} \text{Tr}(\gamma_5 (2 - D)) = -\frac{1}{2} \sum_\lambda (2 - \lambda)(u_\lambda, \gamma_5 u_\lambda) \quad (34) \]

\[ = n_- - n_+ = \text{index}(D). \]

Here we have used the properties of eigenvectors of \(D\) listed above.

Obviously, \(Q_{\text{top}}\) is topological invariant since the integer \(n_- - n_+\) cannot change smoothly. This fact can also be obtained directly from the GW relation.

Equation (34) is the index theorem on the lattice: it states that one can define a (gauge invariant, real) topological charge density \(q(x; U)\) on a given gauge configuration which leads to an integer topological charge. The topological charge defined this way equals to the index of \(D\). Both sides are defined through \(D\). Is this a tautology? No: the important point here is that one has a local topological charge density not just an integer number defining the charge.

It is easy to see that for smooth configurations

\[ q(x) = \frac{1}{32\pi^2} \epsilon_{\mu
u\rho\sigma} \text{tr} (F_{\mu\nu} F_{\rho\sigma}) + O(a^2) \quad (35) \]

since \(q(x)\) is a gauge invariant, pseudoscalar, dimension 4 operator and the prefactor is fixed by the fact that in the continuum limit \(a \rightarrow 0\) one should recover the continuum topological charge.

I would like to stress that any acceptable solution to the GW relation defines a topological charge for which the exact index theorem holds. Note, however, that different solutions will give, in general, different values for the topological charge – they need to agree only for sufficiently smooth configurations. From a practical point of view the operator \(D\) has to provide a topological charge which is robust enough, giving a reasonable answer also for relatively rough configurations. To illustrate this point, consider Neuberger’s operator in eq. (20) with a small \(\mu\), say \(\mu = 0.01\). For this choice only very large instantons are counted as such when defined through eq. (34) – perhaps only those with \(\rho/a > 100\). Smaller instantons “fall through the lattice” since the smallest eigenvalue of \(D\) would shift beyond \(\mu\). Obviously, this would be a bad Dirac operator and a bad topological charge – in spite of the fact that they satisfy the exact index theorem, eq. (34).

Any acceptable discretization of the Dirac operator (without requiring to be a solution to the GW relation) could be used to determine the correct topological charge for sufficiently small lattice spacing \(a\) – the exactly zero eigenvalue of the continuum Dirac operator will move only slightly away from zero, remaining real, and no other nearly zero eigenvalues will appear. The extra requirement of GW relation forbids the eigenvalue to move away from zero for such configurations. Hence no extra conditions (beyond acceptability and GW relation) are required for the index theorem to hold. However, as pointed out above, these conditions do not guarantee that the topological charge defined this way is the ‘correct’ one – except for sufficiently large instantons. Of course, there is no unique way to define the topological charge on the lattice, but for not-too-small instantons (with a radius of a few lattice spacing, say) the geometrical definition is natural. For practical reasons, one would like to have a GW Dirac operator \(D\) which reproduces the natural topological charge of not too large instantons as well. I discuss these points perhaps more than necessary, because there is some controversy in the literature \([3][4]\).

In Ref. (10) it has also been shown that for the FP Dirac operator one has in addition

\[ Q_{\text{top}}^{\text{gaugeFP}} = \sum_x \text{tr} (\gamma_5 R D^{\text{FP}})_{xx} = Q_{\text{top}}^{\text{termFP}}, \quad (36) \]
4. The chiral condensate

Restoring the lattice spacing \( a \), the spectrum of \( D \) is given by a circle \( \lambda = \frac{1}{2} (1 - e^{i\alpha}) \) with \( -\pi \leq \alpha < \pi \), crossing the real axis at \( \lambda = 0 \) and \( \lambda = 2/a \). Correspondingly, the spectrum of \( D^{-1} \) is a straight line parallel to the imaginary axis, of the form \( 1/\lambda = a/2 + i\gamma \), \( -\infty < \gamma < +\infty \) for arbitrary gauge field.

Denote the unnormalized expectation value of \( O(\overline{\psi}, \psi) \) in a fixed gauge background by

\[
\langle O(\overline{\psi}, \psi) \rangle_F \equiv \int d\overline{\psi} d\psi e^{-S_F(\overline{\psi}, \psi)} O(\overline{\psi}, \psi). 
\]  

Then we have

\[
a^4 \sum_x \langle \overline{\psi}_x \psi_x \rangle_F = (1)_F \sum_\lambda \frac{1}{\lambda}. \tag{38}
\]

Obviously, the \( a/2 \) part of \( 1/\lambda \) is independent of the dynamics and has to be subtracted to define a proper chiral order parameter. This yields to

\[
\langle \overline{\psi}_x \psi_x \rangle_{\text{sub}} = \langle \overline{\psi}_x \psi_x - 2 N_f N_c a^2 1 \rangle_F, \tag{39}
\]

the same as suggested in Ref. \[10\].

Adding a mass term \( m \) to \( D \) would shift the whole circle of eigenvalues. Since the real eigenvalues at \( \lambda = 2/a \) also contribute to chiral relations it is better to leave them unchanged. This is simply achieved by the replacement

\[
D \rightarrow \left( 1 - \frac{1}{2} a m \right) D + m \tag{40}
\]

which not only shifts the circle but also rescales its radius appropriately. With this definition of \( m \) one obtains

\[
\frac{\partial}{\partial m} (1)_F = \left\langle \overline{\psi} \left( 1 - \frac{1}{2} a D \right) \psi \right\rangle_F. \tag{41}
\]

This expression is identical to eq. (39), i.e. the proper order parameter. This equivalent form on the rhs. has been suggested first by Chandrasekharan [23], who obtained it from Lüscher’s exact symmetry.

5. Exact chiral symmetry on the lattice

At this moment we are in a peculiar situation: all signs of a symmetry are present (Ward identities, index theorem, chiral spectrum, . . .) but the action is not chirally invariant.

Lüscher made an important observation that in fact the action possesses an exact symmetry. Perform an infinitesimal change of variables \( \psi \rightarrow \psi + i \epsilon \delta \psi \) and \( \bar{\psi} \rightarrow \bar{\psi} + i \epsilon \delta \bar{\psi} \) with a local flavour singlet transformation

\[
\delta \psi \ = \ \gamma_5 \left( 1 - \frac{1}{2} a D \right) \psi \tag{42}
\]

\[
\delta \bar{\psi} \ = \ \bar{\psi} \left( 1 - \frac{1}{2} a D \right) \gamma_5.
\]

i.e. the topological charge defined by \( D^{FP} \) coincides with the old definition in pure gauge theory. In this case, for example, there are no dislocations, and one expects small lattice artifacts.

An important point is that \( D(x, y; U) \) is discontinuous in the gauge field. The reason is that on the lattice one can connect continuously any two configurations, but index(\( D(U) \)) has a jump on the path connecting, say, \( Q_{\text{top}} = 0 \) and 1.

With the FP action the origin of this discontinuity is that the minimizing configuration on the path connecting, say, \( Q \leq \pi \) (4) The chiral condensate does not imply a loss of locality. \( \lambda \) value of \([23,24]\), remains also local when an isolated eigenvalues, and one expects small lattice artifacts. In this case, for example, there are no dislocations, but index(\( D(U) \)) has a jump on the path connecting, say, \( Q_{\text{top}} = 0 \) and 1.

Near a degenerate minimum (when the instanton “falls through the lattice”). For Neuberger’s Dirac operator the discontinuity appears when a real eigenvalue of \( H = \gamma_5 (1 - D_w) \) changes sign, and \( \epsilon (H) \) in eq. (13) is discontinuous.

Note that a similar discontinuity appears when the fermions are defined in the continuum, in a continuous gauge background obtained by interpolating the lattice gauge field [22]. In this case the process of interpolation becomes discontinuous, similarly to the situation with the FP action.

An important question: does \( D \) become non-local when this discontinuity occurs? This would be a very bad news for the present approach. Fortunately, this is not true in general. In the FP action the two solutions \( U_1 \) and \( U_2 \) corresponding to the degenerate minima are expected to differ only locally. Neuberger’s \( D \), as shown by Lüscher [23][24], remains also local when an isolated eigenvalue of \( H \) passes zero, so a discontinuity of \( D \) does not imply a loss of locality.
From the GW relation eq. (17) it follows that the action is invariant
\[ \delta (\bar{\psi} D \psi) = 0. \] (43)

The corresponding flavour non-singlet transformation is given by
\[ \delta \psi = T \gamma_5 \left( 1 - \frac{1}{2} a D \right) \psi \] (44)
\[ \delta \bar{\psi} = \bar{\psi} \left( 1 - \frac{1}{2} a D \right) \gamma_5 T \] (45)

Now it seems that we have more symmetry than possible – the flavour singlet transformation should be anomalous. Where is the anomaly hidden? As pointed out by Lüscher, the fermionic integration measure is not invariant under the singlet transformation in eq. (43).
\[ \delta [d\bar{\psi}d\psi] = \text{Tr}(a\gamma_5 D)[d\bar{\psi}d\psi] = 2N_t (n_- - n_+) [d\bar{\psi}d\psi]. \] (46)

That is, the measure breaks the flavour singlet chiral symmetry in a topologically non-trivial gauge field where index(D) = n_- - n_+ ≠ 0. The corresponding (global) Ward identity for the flavour singlet transformation is
\[ \langle \delta \mathcal{O} \rangle_F = 2N_t \nu \langle \mathcal{O} \rangle_F, \] (47)
where \( \nu = Q_{top}(U). \)

Note that the non-singlet symmetry is not anomalous, \( \langle \delta \mathcal{O} \rangle_F = 0 \) since eq. (17) for this case contains \( \text{Tr}(a\gamma_5 DT) \propto \text{tr}(T) = 0. \)

Using these exact symmetries of the action, the Ward identities follow, of course, directly. By taking
\[ \mathcal{O} = \sum_x \bar{\psi}_x t_a \gamma_5 \psi_x, \] (48)
and a non-singlet transformation one has
\[ \langle \delta \mathcal{O} \rangle_F = 0 \] i.e.
\[ \langle \bar{\psi} \left( 1 - \frac{1}{2} a D \right) \psi \rangle_F = 0 \quad (m = 0, \ V \text{ finite}). \] (49)
This is the order parameter which is expected to be broken spontaneously in the thermodynamic limit [1]:
\[ \lim_{m \to 0} \lim_{V \to \infty} \langle \bar{\psi} \left( 1 - \frac{1}{2} a D \right) \psi \rangle = \Sigma \neq 0. \] (50)

In Ref. [2] there is also a relation connected with the U(1) problem (massive \( \eta' \) meson) – this will be discussed later.

Whether the chiral symmetry is indeed broken spontaneously in QCD is a delicate dynamical question. The chiral invariant Dirac operator provides a firm basis to investigate this question by avoiding an explicit breaking.

Note that the Nielsen-Ninomiya no-go theorem is avoided here by the non-standard form of the lattice chiral symmetry: D is not invariant under the naive \( \gamma_5 \) symmetry, nevertheless it is invariant under the generalized transformation.

6. Chiral decomposition

In the continuum, chiral symmetry allows one to decompose the fermion field into left and right chiral components, transforming independently. Besides yielding a convenient formalism for vector theories like QCD, this decomposition is a necessary first step towards the chiral gauge theories.

The matrix \( \gamma_5 \) anti-commutes with the continuum Dirac operator and \( \gamma_5^2 = 1 \). This allows to introduce the chiral projectors
\[ P_\pm = \frac{1}{2} (1 \pm \gamma_5), \] (51)
and the action decomposes as \( D_{cont} = P_+ D_{cont} P_- + P_- D_{cont} P_+ \).

On the lattice it is useful to introduce [26,27] the operator
\[ \gamma_5 = \gamma_5 (1 - a D) = \gamma_5 V, \] (52)
where D satisfies the GW relation eq. (15). Remember that V is unitary and \( \gamma_5 V \gamma_5 = V^T \). Then
\[ \gamma_5 D = - D \gamma_5, \quad \text{and} \quad \gamma_5^2 = 1. \] (53)

One can introduce the projectors
\[ \hat{P}_\pm = \frac{1}{2} (1 \pm \gamma_5). \] (54)

They satisfy the relations
\[ D \hat{P}_+ = P_- D, \quad D \hat{P}_- = P_+ D, \] (55)
and allow the decomposition
\[ D = P_+ D \hat{P}_+ + P_- D \hat{P}_-. \] (56)
To conclude the definitions let us introduce the chiral components of ψ:

\[ \psi_L = \bar{P}_- \psi, \quad \psi_L = \bar{P}_+ \psi, \]
\[ \psi_R = \bar{P}_+ \psi, \quad \psi_R = \bar{P}_- \psi. \tag{56} \tag{57} \]

In terms of these components we have

\[ \bar{\psi} D \psi = \bar{\psi}_L D \psi_L + \bar{\psi}_R D \psi_R, \tag{58} \]

analogously to the continuum case. Note that eqs. (56-57) are not symmetric in ψ and \( \bar{\psi} \). As discussed later, this asymmetric definition seems to play an essential role.

Analogously to the continuum case, the mass term should couple the L and R components (but not the L.L or R.R components):

\[ S \equiv \bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L = \bar{\psi} \left( 1 - \frac{1}{2} aD \right) \psi. \tag{59} \]

The corresponding pseudoscalar is given by

\[ P \equiv \bar{\psi}_L \psi_R - \bar{\psi}_R \psi_L = \bar{\psi} \gamma_5 \left( 1 - \frac{1}{2} aD \right) \psi. \tag{60} \]

The scalar (pseudoscalar) densities \( S \) and \( P \) will often appear in the formulae below.

Performing a left chiral transformation means

\[ \delta \psi_L = -\psi_L, \quad \delta \psi_L = \bar{\psi}_L, \tag{61} \]
\[ \delta \psi_R = 0, \quad \delta \psi_R = 0, \tag{62} \]

Using this transformation one has

\[ \delta P = S, \quad \delta S = P. \tag{63} \]

The combinations in eqs. (59,60) are the natural generalizations of the corresponding continuum expressions. Note that the \( 1 - \frac{1}{2} aD \) factor cancels the contribution of the unphysical real eigenmode at \( \lambda = 2/a \). Using these combinations of fields, one obtains very simple, compact relations, for example:

\[ \sum_x \langle P_x \rangle_F = \frac{m}{V} \sum_{x,y} \langle P_x P_y \rangle_F = \frac{1}{V} \sum_x \langle S_x \rangle_F + \frac{N_f^2}{m} \nu^2 \langle 1 \rangle_F. \tag{65} \]

\[ \frac{1}{V} \langle \nu^2 \rangle = -\frac{m}{N_f} \langle S \rangle + O(m^2). \tag{66} \]

Note that only the \( \lambda = 0 \) eigenmodes contribute when evaluating the trace.

It is instructive to consider the flavour singlet Ward identity which leads to a relation for the \( \eta' \) propagator:

\[ \frac{1}{V} \sum_{x,y} \langle P_x P_y \rangle_F = \frac{1}{V} \sum_x \langle S_x \rangle_F + \frac{N_f^2}{m} \nu^2 \langle 1 \rangle_F. \tag{65} \]

If the \( \eta' \) stays massive in the chiral limit \( m \rightarrow 0 \) then the lhs. (after averaging over the gauge fields) should go to zero. This leads to the interesting relation [25] between the topological susceptibility and chiral condensate in full QCD

\[ \frac{1}{V} \langle \nu^2 \rangle = -\frac{m}{N_f} \langle S \rangle + O(m^2). \tag{66} \]

This relation has been obtained previously in the continuum [25,23], but it is important to have its lattice version as well. It shows, for example that the topological susceptibility must be a well-behaved physical quantity on the lattice.

For the flavour non-singlet variant of eq. (65) the term proportional to \( \nu^2 \) is absent since this symmetry is not anomalous. Therefore a non-vanishing chiral condensate, eq. (34), yields to (quasi) Goldstone bosons with mass \( m_{\pi}^2 \propto m \Sigma \).

We conclude this section with a few remarks. Note that \( \gamma_5 \) coincides with \( \epsilon(H) \) introduced in the overlap formalism. Another point worth mentioning is that for the modes with \( \lambda = 0 \) one has \( \hat{\gamma}_5 = -\gamma_5 \) while for the modes at \( \lambda = 2/a \) they differ in sign, \( \hat{\gamma}_5 = -\gamma_5 \), as seen from the definition eq. (51). This observation will be important when we consider chiral fermions.

7. The \( \theta \) parameter

We can now introduce the \( \theta \) parameter, as in the continuum. Here I shall restrict the discussion to some basic points only.

Define the lattice Lagrangean with a complex mass \( m \) and the \( \theta \)-parameter:

\[ \mathcal{L} = \mathcal{L}_g + \bar{\psi} D \psi + m \bar{\psi}_R \psi_L + m \bar{\psi}_L \psi_R - i \theta q(x), \tag{67} \]

where \( \mathcal{L}_g \) is the gauge action and \( q(x) \) is defined through \( D \) by eq. (33). By changing the variables \( \psi_L \rightarrow e^{-i\alpha} \psi_L, \quad \bar{\psi}_L \rightarrow e^{i\alpha} \bar{\psi}_L \).
one can restore the invariance of the action by $m \to e^{\alpha}m$ and the non-invariance of the measure can be compensated by $\theta \to \theta - N_t \alpha$. Therefore the free energy should depend only on the combination $m \exp(i\theta/N_t)$, as in the continuum. One can proceed now e.g. by repeating (now on the lattice) the steps of Ref. [28] where e.g. useful sum rules have been obtained for the spectrum of the Dirac operator. To illustrate this, consider the partition function

$$Z \equiv Z(\theta) = \sum_\nu e^{i\theta \nu} Z_\nu = e^{-F(m, \theta)}.$$  \hfill (69)

Under some natural assumptions [28] one has for the free energy

$$F(m, \theta) = -VN_t \Sigma \Re \left( me^{i\theta/N_t} \right) + O(m^2).$$  \hfill (70)

Taking then the second derivative with respect to $\theta$ (for real $m$) one obtains

$$\langle \nu^2 \rangle = V \Sigma m \frac{1}{N_t}$$  \hfill (71)

which is equivalent to eq. (67).

As another example, it is easy to show that the fermion determinant is given by

$$\det D(U) = (2m^*)^\nu \prod_\lambda \left( |\lambda|^2 + \left( 1 - \frac{|\lambda|^2}{4} \right) |m|^2 \right),$$  \hfill (72)

for $\nu \geq 0$, and the product is over the complex eigenvalues only. For $\nu < 0$ the prefactor is replaced by $(2m^*)^{-\nu}$. Eq. (72) differs from its continuum counterpart by the presence of the factor $1 - |\lambda|^2/4$.

8. Cut-off effects with GW Dirac operators

An important consequence of the exact chiral symmetry on the lattice is that there are no $O(a)$ lattice artifacts, to any order of $g$. The reason is the following. The $O(a)$ lattice artifacts of any action could be compensated by adding an extra term which is a lattice version of $\overline{\psi} \sigma_{\mu \nu} F_{\mu \nu} \psi$ with an appropriately chosen coefficient. The original action in our case has the exact (generalized) chiral symmetry of eq. (12) and its artifacts could be compensated only by a similarly invariant extra term. Any lattice regularization of $\overline{\psi} \sigma_{\mu \nu} F_{\mu \nu} \psi$ will violate, however, this symmetry, hence the $O(a)$ artifacts should be absent for the original lattice action as well. Note also that the clover term needed for $O(a)$ improvement has been determined by requiring that some Ward identity holds independently of the gauge field. This is, however, automatically satisfied here for $m = 0$ hence there is no need for extra clover term.

In addition, in the presence of a mass $m$ there are also no $O(am)$ artifacts, if the mass is introduced properly, as in eq. (71). Then the symmetry $m \to -m$ forbids such term. Indeed, under a finite chiral transformation

$$\psi \to \psi' = \gamma_5 \psi, \quad \overline{\psi} \to -\overline{\psi} \gamma_5$$  \hfill (73)

$\overline{\psi} D \psi$ is invariant while the mass term $\overline{\psi}(1 - \frac{1}{2}a D) \psi$ changes sign. Although due to the anomaly the partition function is not invariant for $\nu \neq 0$, this fact is not relevant for our considerations. Indeed, for $m \neq 0$ and sufficiently large volume one can restrict the system to the topologically trivial sector $\nu = 0$.

Note, however, that the $O(a^2)$ lattice artifacts could still be large at the relevant lattice spacings, and could be quite different for different solutions of the GW relation.

Finally, it is interesting to see (although it is not necessary after the previous discussion) that the absence of $O(a)$ artifacts on the tree level follows directly from the GW relation: expanding $D$ in powers of $a$ it implies that

$$D = i\gamma_\mu \nabla_\mu + ac \left( \nabla^2 + \frac{i}{2} \sigma_{\mu \nu} F_{\mu \nu} \right) + O(a^2),$$  \hfill (74)

(with some constant $c$) i.e. a term with $c_{SW} = 1$ is automatically generated.

9. Properties of Neuberger’s Dirac operator

Here we shall discuss some basic properties of Neuberger’s Dirac operator and its straightforward generalizations.

Consider first the free case. The spectrum $E(\vec{p})$ of $D$ defined in eq. (14) is shown in fig. 5 together with the spectrum for the standard Wilson-Dirac
operator for $\vec{p} = (p, 0, 0)$. The latter is denoted by $W$ and given by

$$\cosh E = \frac{3 - 2 \cos p}{2 - \cos p},$$

(75)

while $N$ denotes the two branches for Neuberger’s Dirac operator where $\cosh E(\vec{p})$ is given by

$$\sqrt{1 + \sin^2 p}, \quad \text{and} \quad \frac{3 - 2 \cos p}{2(1 - \cos p)}.$$  

(76)

(Note that the lower curve extends only up to the point where the two branches join each other – this peculiar behaviour is due to the non-analyticity in the square root.) The spectrum for $\alpha p > 1$ is worse than that for $D_w$. This illustrates that a solution of GW relation can have large $O(\alpha^2)$ artifacts – the fact that $D$ is spread over some distance does not necessarily make the artifacts small!

The locality of $D$ for the free case is guaranteed by the inequality $||D(x - y)|| \leq C e^{-\gamma |x - y|_0}$ where $|x - y|_0 = \max_\mu |x - y|_\mu$ and $\gamma = 0.693 \ldots$

This asymptotic behaviour is shown on fig. 3 by dashed line. The decay of $||D(x)||$ for Neuberger’s $D$ is shown by circles. Two more solutions to GW relation are added. Crosses show the case with $A = 1 - D_{HC}$ where $D_{HC}$ is a FP action truncated to the hypercube. Since there is a non-trivial $R$ in this case the more natural choice, $A = 1 - \sqrt{2R}D_{HC}\sqrt{2R}$, (cf. eqs. (22,23)) is also shown (+ signs). Finally, the true FP action is also indicated (triangles). It is obvious that there are rather large differences in the range of interactions for different GW Dirac operators, and this is an important factor in making a choice for numerical simulations (see also Ref. [13]).

A less trivial point is the question of locality in a gauge field background. In Ref. [24] it is proven that for small gauge fields

$$||1 - U_{pl}|| < \epsilon \quad \text{for all plaquettes}$$

(77)

with $\epsilon < 1/30$, the corresponding $D(x, y; U)$ is local with exponentially decaying tail, and the localization length is uniformly bounded.

Although eq. (77) should hold sufficiently close to the continuum limit, it is violated at practical values of $a$. Theoretically one can restrict the gauge fields to satisfy eq. (77). This is sufficient to show that there exists an exactly chiral invariant, acceptable lattice regularization of QCD. However, the restriction leads to an astronomical correlation length, hence it is only of theoretical interest. As mentioned before, in Ref. [24] it is
also shown that an isolated zero of $A$ does not spoil locality.

An important question is the locality in gauge fields generated by MC simulations. First results on this are presented in Refs. [24,32]. In fig. 3 results from Ref. [24] are shown, for SU(3) gauge fields generated at $\beta = 6.4$, 6.2 and 6.0 (triangles, squares, circles). The full symbols are for $\mu = 1$ while the open ones for (roughly) optimized values, $\mu = 1.2$ for $\beta = 6.4$ and $\mu = 1.4$ for $\beta \leq 6.2$.

It is also important to have a sufficiently effective way of evaluating the inverse square root. This issue is discussed in refs. [14,24,32–36]. I only mention here that it seems to be a feasible task but still much has to be done in this direction.

A few comments are in order:

(1) Unlike evaluating $D^{-1}$ for massless quarks, taking the inverse square root is not a critical problem – appearance of zero modes of $A$ is accidental.

(2) Starting with a good approximation of the FP action instead of $D_w$ (as in eq. (22)) the probability to have a zero mode of $A$ is strongly suppressed, since with the exact FP action $A$ would be unitary. Note, however, that for any continuous parametrization of the FP action, $A$ must have a zero mode for specific configurations, when an instanton “falls through the lattice”. This happens on the boundary separating regions of gauge field configurations with different topological charges.

(3) Using a GW Dirac operator there will be no “exceptional configurations” in the usual sense [37] – these would correspond to eigenvalues $\lambda(D) < 0$. (One should distinguish these from configurations where $\lambda(A) = 0$ appearing in this approach.)

(4) One can construct, in principle, configurations when many “instantons fall through” simultaneously at different sites of a large lattice. This would produce a situation when $A$ has many nearly zero eigenvalues. Will $D$ be local on such configurations? One expects that such configurations have negligible weight in the partition function. Will their effect be also negligible? I think, these questions have to be investigated.

10. Measuring the topological charge

There are several equivalent ways to measure the topological charge defined by $D$ through eqs. (33,34):

(1) find the zero eigenmodes of $D$ (or of hermitian $\gamma_5 D$),

(2) count the real eigenvalues of $D_w$ in the region $0 \leq \lambda(D_w) < \mu$, with chirality signs,

(3) trace the “level crossings” of $H(\mu') = \gamma_5 (\mu' - D_w)$ while $\mu'$ runs from 0 to $\mu$.

Note that although we rely here on the exact index theorem on the lattice, the value of the topological charge determined through $D$ with $A = \mu - D_w$ is the same as the method using $D_w$ suggested a long time ago [16,17]. The last method has also been used in Ref. [4] in the framework of the overlap formalism.

This definition of the topological charge works well for small instantons, as expected. But how does it work for typical MC configurations? Does $D$ (generated using $D_w$) give a ‘better’ definition than the geometrical one? Hernandez [38] argues that the answer is no. Gattringer and Hipp [39] simulated SU(2) on a $12^4$ lattice at $\beta = 2.4$ and found that the value of the topological charge,
ν on individual configurations depends strongly on the parameter μ since there are usually many real eigenvalues in the region of interest, and only very few configurations have a gap in the spectrum where ν does not depend on μ chosen in a reasonable range. They also note that a naive inclusion of the clover term makes the situation worse. The same conclusion is found by DeGrand, A. Hasenfratz and Kovács [20]. This last fact is not too surprising since the corresponding artifact is not an O(μ) effect.

Note, however, that in the average this corresponding definition of charge works better: Edwards, Heller and Narayanan [35] found that the topological susceptibility stays roughly constant (independent of μ) for μ ≥ 1.0. Here I would like to point out that the μ-independence of ⟨ν²⟩ does not necessarily mean the ‘true’ value: if ν = νtrue + noise then ⟨ν²⟩ > ⟨ν²true⟩. A good definition of the topological charge (at a given lattice spacing) should give relatively unambiguous definition of ν on individual configurations. In other words, the value of the lattice spacing a when a GW Dirac operator gives a ‘good’ definition of the topological charge could strongly depend on the choice of D. With the presently used lattice spacings the original choice of Neuberger (using Dw) seems to produce a topological charge which is too ambiguous.

On the other hand, measuring the topological charge with the FP action in the Schwinger model [19] – mostly because of technical problems to perform the required multigrid minimization on the gauge configurations.

For QCD a good strategy would be a compromise: take a Dirac operator D0 with good properties (small cut-off effects, nearly satisfying the GW relation, . . . ) and build D from it, which in turn could be used in the definition of the topological charge. It would have smaller O(α²) artifacts, better locality, fewer ‘dangerous’ configurations, larger gap in the real eigenvalues hence less ambiguity of Qtop, but will also need more programming effort and extra computational overhead.

Measuring the topological charge density using q(x) = \frac{1}{2}tr(\gamma_5 D(x,x;U)) is perhaps too time consuming to be feasible.

11. On chiral gauge theories

The question whether it is possible to define chiral gauge theories (like the Standard Model) on the lattice is a purely theoretical problem, but of great importance. Much has been done in this direction and has been reported earlier [41]. Nevertheless, there remain several open questions.

A GW Dirac operator seems to be a good starting point: it has exact gauge invariance and one can define Weyl fermions (cf. eq. (60)). Here I shall report briefly about new developments (partially yet unpublished) by Lüscher [27] based on the properties of GW Dirac operators. The discussion will touch only a few interesting points which could be easily understood on the basis of previous sections.

Consider fermion fields which are purely left handed (see eq. (47))

\[ \hat{P}_- \psi = \psi, \quad \bar{\psi} P_+ = \bar{\psi}. \]  

These are local, gauge invariant conditions. Introduce a basis for \( \psi(x) \) and \( \bar{\psi}(x) \) through relations

\[ \hat{P}_- v_j = v_j, \quad \bar{\psi}(x) = \sum_j v_j(x)c_j, \]  

\[ \bar{\tau}_k P_+ = \tau_k, \quad \bar{\psi}(x) = \sum_k \bar{\tau}_k \tau_k(x), \]  

where \( c_j \) and \( \tau_k \) are Grassmann variables. In this basis the fermion action reads

\[ S_F = \bar{\psi} D \psi = \sum_{kj} \tau_k M_{kj} c_j \]  

with \( M_{kj} = \sum_x \tau_k D v_j \).

The fermion propagator (when there are no zero modes of D) is given by

\[ \langle \psi(x) \bar{\psi}(y) \rangle_F = \langle 1 \rangle_F \left( \hat{P}_- D^{-1} P_+ \right)_{xy}. \]  

It is easy to show that this is chiral in the generalized sense. Note that

\[ \hat{P}_- D^{-1} P_+ = P_- D^{-1} P_+ + \frac{1}{2} P_+. \]  

The first term on the rhs. is chiral in the usual sense (with \( \gamma_5 \)) while \( P_+ \) is a contact term.

Consider a chiral gauge theory with fermion multiplet \( \alpha = 1, 2, \ldots , N \), with integer charges
For the number of zero modes \((n_-)_\alpha, (n_+)_\alpha\) the index theorem implies

\[
(n_-)_\alpha - (n_+)_\alpha = e_\alpha^2 \nu
\]

since \(FF\) is replaced by \(e_\alpha^2 FF\) for smooth gauge fields when \(e_\alpha \neq 1\).

A simple but important observation is that the number of \(\psi\) and \(\bar{\psi}\) fields are not necessarily equal. They are given by \(\text{Tr}(\hat{P}_-)\) and \(\text{Tr}(P_+)\), respectively. For their difference we have

\[
\text{Tr}(\hat{P}_-) - \text{Tr}(P_+)= \frac{1}{2} \text{Tr}(\gamma_5 D) = \nu \sum \alpha e_\alpha^2 , \quad (85)
\]

To illustrate this, consider a background field with \(\nu = 1\), where \(n_- = 1, n_+ = 0\) and \(n'_+ = 0\). Then for the \(\lambda = 0\) mode

\[
\gamma_5 v = \hat{\gamma}_5 v = -v
\]

while for the eigenvector \(v'\) at \(\lambda = 2\)

\[
\gamma_5 v' = -\hat{\gamma}_5 v' = v'
\]

According to eqs. (78-80), at \(\lambda = 0\) we have only \(\psi\) (but not \(\bar{\psi}\)) while at \(\lambda = 2\) both \(\psi\) and \(\bar{\psi}\) are present since \(\gamma_5\) and \(\hat{\gamma}_5\) differ by sign at this point. Obviously for complex values of \(\lambda\) both \(\psi\) and \(\bar{\psi}\) are present again. Altogether there is one more \(\psi\) than \(\bar{\psi}\). Of course, this statement depends on the topological charge of the gauge field.

As a consequence the fermionic average \(\langle \mathcal{O} \rangle_F\) of any operator in a given background gauge field vanishes unless the fermionic number of type \(\alpha\) (the number of \(\psi_\alpha\)'s minus the number of \(\bar{\psi}_\alpha\)'s) in \(\mathcal{O}\) equals to \((n_-)_\alpha - (n_+)_\alpha = \nu \sum \alpha e_\alpha^2\).

Perform now (for the U(1) case) a global gauge transformation \(q(x) = \exp(i\phi)\). This modifies \(\mathcal{O}\) by a phase factor \(\exp(i\phi)\) where \(\phi = \omega \sum \alpha e_\alpha (n_- - n_+)_\alpha = \omega \sum \alpha e_\alpha^3\). From the other side a global U(1) gauge transformation does not change the gauge fields at all, hence invariance under this transformation requires

\[
\sum \alpha e_\alpha^3 = 0 , \quad (88)
\]

i.e. the condition of anomaly cancellation of the continuum theory.

These observations, however, constitute only the starting point of a difficult mathematical problem – to find a satisfactory choice of the fermionic integration measure (“chiral determinant”), i.e. of the basis \(v_j(x; U)\).

Lüscher has proven that it is possible to construct a basis for the U(1) case that satisfies the following requirements:

1. The fermionic expectation values \(\langle \mathcal{O} \rangle_F\) are smooth functions of \(U\).

2. The field equations (obtained by some change of variables) are local:

\[
\langle \delta \mathcal{L}(y) \mathcal{O}(x_1) \mathcal{O}(x_2) \ldots \rangle = 0 \quad (89)
\]

if \(y\) is far from \(x_1, x_2, \ldots\).

3. \(\langle \mathcal{O} \rangle_F\) is gauge invariant if \(\mathcal{O}\) is.

Note that the last property may not be necessary to require. In principle, it is possible that after the leading violation of gauge invariance is cancelled by the condition \(\sum \alpha e_\alpha^3 = 0\), the rest is small and goes away in the continuum limit, without the need of extra counter terms. This possibility is based on a simple but surprising result of Foerster, Nielsen and Ninomiya [12], who show that gauge invariance on the lattice is robust in the sense that the effect of a sufficiently small violating term vanishes in the continuum limit.

The constructions of the overlap and gauge fixing formalisms assume this scenario. It is clear, however, that it is much safer – if possible – to preserve chiral gauge invariance at finite lattice spacing.

### 12. Summary

It has been illustrated that Dirac operators satisfying the GW relation provide an elegant solution to the chirality problem in lattice QCD. In particular, there is no additive quark mass renormalization, and Ward identities, soft pion theorem, etc. are valid, there are no O(a) artifacts.

It is pointed out that there is a large freedom in choosing a GW Dirac operator, and the choice should be optimized for locality, smallness of artifacts, etc.

For practical implementations of these ideas new algorithms are needed. That the task is feasible is illustrated by MC calculations in the domain wall approach [13] which could be considered as a particular approximation to a GW Dirac op-
erator. Not restricting itself to the given ideology could open ways to more efficient algorithms.

Finally, as shown by Lüscher’s construction for the U(1) case, it seems to be possible to define chiral gauge theories on the lattice based on a GW Dirac operator.

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