Abstract

A formalism of gauge and Lorentz covariant Schwinger-Dyson equation is built up for fermion propagator in presence of arbitrary external gauge field within ladder approximation. Different external electromagnetic field dependent trial solutions are investigated which spontaneously break chiral symmetry.

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Fermion propagator, once external gauge field sets in, may exhibit new phenomena. In literature, people focus on whether and how external field will trigger chiral symmetry breaking (CSB)\cite{1, 2, 3}. We further show next that fermion propagator in presence of external gauge field will provides us an alternative way to realize the gauge invariance of the theory and leads to well known Ward-Takahashi identity (WTI)\cite{4}.

It is well-known that fermion propagator is not explicitly gauge covariant. This is due to the fact that gauge transformation relates different Green’s functions, just fermion propagator itself cannot realize gauge covariance. Although the fermion propagator is not a gauge invariant quantity, it satisfies constraints of gauge covariance. It obeys WTI which relates it to the fermion-boson vertex. Moreover, it follows the gauge covariance relation dictated by its Landau-Khalatnikov-Fradkin transformation law\cite{5}. Now with fermion propagator in presence of external gauge field, we present a way to realize the gauge covariance for fermion propagator itself. Suppose $S(x, y; A)$ is a fermion propagator in presence of external gauge field $A^\mu$ which can be either Abelian or non-abelian,

$$S(x, y; A) = -i\langle 0|T\psi(x)\bar{\psi}(y)|0\rangle_A ,$$ (1)

then $S(x, y; 0)$ is usual fermion propagator without external gauge field, which is function of $x - y$ due to invariance of space-time translation. With its Fourier transformation $S(p)$,

$$S(x, y; 0) = \int\frac{d^4p}{(2\pi)^4}e^{-ip(x-y)} S(p) = S(i\partial_x)\delta^4(x-y) ,$$ (2)

we can construct a gauge covariant fermion propagator $S(x, y; A)$,

$$S(x, y; A) = S(i\nabla_x)\delta(x-y) , \quad \nabla^\mu_x = \partial^\mu_x - igA^\mu(x) ,$$ (3)

which is covariant under gauge transformation $gA^\mu(x) \rightarrow V(x)gA^\mu(x)V^\dagger(x) - iV(x)[\partial^\mu V^\dagger(x)]$, or in terms of covariant derivative $\nabla^\mu_x \rightarrow V(x)\nabla^\mu_x V^\dagger(x)$, it transforms as

$$S(x, y; A) \rightarrow S'(x, y; A) = V(x)S(i\nabla_x)V^\dagger(x)\delta^4(x-y) = V(x)S(x, y; A)V^\dagger(y) .$$ (4)

Note here what we have introduced into theory is gauge potential $A^\mu$, it has a special non-homogeneous term $-(i/g)V\partial^\mu V^\dagger$ under gauge transformation which play the key role in covariant differential and then can make the theory covariant under gauge transformations. This is different as those discussions given by Leung et. al.\cite{6}, where external magnetic field is introduced into theory, since magnetic field as field strength of Abelian gauge theory is a gauge invariant quantity, add it into theory can not make theory gauge covariant.
There is ambiguity when we naively generalize from \( S(p) \) to \( S(i\nabla_x) \): the different component of \( p^\mu \) are commutative, but \( \nabla^\mu \) are not with \( [\nabla^\mu, \nabla^\nu] = -iF^\mu_\nu \). Which cause uncertainties in the arrangement of different sequences of \( S(i\nabla_x) \)’s arguments. For example, a term of \( (p^2)^2 \) can have different generalizations such as \( (-\nabla^2_x)^2 \) or \( \nabla^\mu_x \nabla^\nu_x \nabla^\mu_x \nabla^\nu_x \), which are not the same due to their nonzero difference

\[
(-\nabla^2_x)^2 - \nabla^\mu_x \nabla^\nu_x \nabla^\mu_x \nabla^\nu_x = \nabla^\mu_x [\nabla^\mu_x, \nabla^\nu_x] \nabla^\nu_x = -i\nabla^\mu_x F^\mu_\nu \nabla^\nu_x.
\]

Although it is not one to one correspondence when we generalize from \( S(p) \) to \( S(i\nabla_x) \), the inverse process is unique. So as long as there is a way to determine \( S(i\nabla_x) \), we can uniquely fix conventional fermion propagator \( S(p) \) by just vanishing external gauge field \( A^\mu \).

Once fermion propagator with external gauge field is known, we can obtain all Green’s functions for two fermions and fermion currents by differentiating \( S(x, y; A) \) with respect to \( A^\mu_\alpha \),

\[
- i\langle 0| T\psi(x)\bar{\psi}(y)g\psi(x)1\gamma^\mu\gamma^\nu\psi(x)\cdots g\psi(x_n)1\gamma^\mu\gamma^\nu\psi(x_n)|0\rangle = \frac{\delta S(x, y; A)}{\delta A^\mu_\alpha(x)} \bigg|_{A^\mu_\alpha = 0},
\]

where \( t_\alpha \) is gauge group generator. This imply that an infinite series of Green’s functions are known through \( S(x, y; A) \). With it, original relations build up by WT identities from gauge invariance of the theory now are explicitly exhibited through external gauge field dependence of the fermion propagator. To verify our statement, as a simplified example, we apply relation (4) to abelian gauge theory in the following to explicitly derive WT identities.

Consider the transformation matrix \( V(x) = e^{i\beta(x)} \) in abelian gauge theory, then (4) imply \( S(x, y; A') = V(x)S(x, y; A)V^\dagger(y) \), which for infinitesimal \( \beta(x) \) become

\[
S(x, y; A - \frac{1}{g} \partial \beta) = S(x, y; A) + i\beta(x)S(x, y; A) - iS(x, y; A)\beta(y)
\]

(6)

Expand the l.h.s. of above equation as

\[
S(x, y; A - \frac{1}{g} \partial \beta) = S(x, y; A) - \frac{1}{g} \int d^4z \frac{\delta S(x, y; A)}{\delta A^\alpha_{\sigma}(z)} \frac{\partial \beta_{\alpha}(z)}{\partial z_{\sigma}}
\]

(7)

Ignoring total derivative term, compare order \( \beta \) terms in (6) and (7), we find

\[
i[\delta(x - z) - \delta(x - y)]S(x, y; A) = \frac{1}{g} \frac{\partial}{\partial z_{\sigma}} \frac{\delta S(x, y; A)}{\delta A^\alpha_{\sigma}(z)}
\]

(8)

Combine with definition (1) and expression (5), above equation reduce to the standard WT identity

\[
[\delta(x - z) - \delta(x - y)]\langle 0| T\psi(x)\bar{\psi}(y)|0\rangle_A = -i\frac{\partial}{\partial z_{\sigma}} \langle 0| T\psi(x)\bar{\psi}(y)\bar{\psi}(z)t_\alpha\gamma_{\sigma}\psi(z)|0\rangle_A
\]

(9)
One can obtain a series WT identities by further differentiating with external gauge fields and then vanishing them.

To obtain fermion propagator with external gauge field, we need to solve the corresponding dynamical equation. As we know, fermion propagator in the field theory is determined dynamically by Schwinger-Dyson equation (SDE), now what we need to do is to generalize traditional SDE for fermion propagator to the case including external gauge field and build up a covariant formalism which can keep the gauge covariance lead by external gauge field. This gauge covariance of external field is very important since it is directly related the validity of WT identities. It is purpose of this paper to build up a gauge covariant SDE and then search for its gauge covariant solutions. This work can be seen as a generalization of our previous result on gauge and Lorentz covariant computation program for free fermion propagator in presence of external gauge field\[7, 8\]. For simplicity, we only consider massless Abelian gauge theory as a prototype of the discussion. Since bare mass of fermion vanishes, the Lagrangian of this theory exhibit chiral symmetry. Our plan is first to show that, in contrast to conventional case that ladder approximation SDE is not gauge invariant, include in external electromagnetic potential $A_\mu$ will really make it gauge invariant under gauge transformation of $A_\mu$ and we construct systematic method to solve this equation gauge and Lorentz covariantly. Then we will try to search for possible CSB solutions. Since this paper mainly focus on the issue of constructing gauge and Lorentz covariant SDE for fermion propagator in presence of arbitrary external gauge field, for the CSB solution we limit ourself in emphasizing that our formalism could include various possible solutions and do not make further detail discussions.

We start by considering following standard ladder approximation SDE for fermion propagator in massless QED in Minkowski space,

\[
S^{-1}(x, y; 0) - i\partial_x \delta^4(x - y) = g^2 D^{\mu\nu}(x, y) \gamma_\mu S(x, y; 0) \gamma_\nu, \tag{10}
\]

with photon propagator $D^{\mu\nu}(x, y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} D^{\mu\nu}(p)$. With (2), SDE (10) can be written in terms of momentum space fermion propagator $S(p)$,

\[
[S^{-1}(i\partial_x) - i\partial_x] \delta^4(x - y) = g^2 D^{\mu\nu}(x, y) \gamma_\mu S(i\partial_x) \delta(x - y). \tag{11}
\]

Now we generalize SDE (10) to the case in presence of external electromagnetic field by
replacing $\phi$ in l.h.s. of (10) with covariant derivative $\nabla_x$,

$$S^{-1}(x, y; A) - i \nabla_x \delta^4(x - y) = g^2 D^{\mu\nu}(x, y)\gamma_\mu S(x, y; A)\gamma_\nu,$$  \hspace{1cm} (12)

and we are looking for its type (3) solution. Before constructing covariant computation formalism, we first show that this equation is invariant under gauge transformation (4). This can be done with help of transformation law for $S \rightarrow V(x)S^{-1}(x, y; A)V^\dagger(y)$, since it will result correct relation

$$\int d^4 y \, S^{-1}(x, y; A)S'(y, z; A) = \int d^4 y \, V(x)S^{-1}(x, y; A)S(y, z; A)V^\dagger(z) = \delta^4(x - z).$$  \hspace{1cm} (13)

Then (12) can be rewritten by multiply $V(x)$ from l.h.s. and $V^\dagger(y)$ from r.h.s. of the equation

$$V(x)S^{-1}(x, y; A)V^\dagger(y) - iV(x)\nabla_x \delta^4(x - y)V^\dagger(y) = g^2 D^{\mu\nu}(x, y)\gamma_\mu V(x)S(x, y; A)V^\dagger(y)\gamma_\nu.$$  \hspace{1cm} (14)

Since the second term of l.h.s. of (14) is equal to $-iV(x)\nabla_x \delta^4(x - y)V^\dagger(x) = -i\nabla'_x \delta^4(x - y)$, it is easy to see (14) is just SDE for fermion propagator after gauge transformation

$$S^{-1}(x, y; A) - i\nabla'_x \delta^4(x - y) = g^2 D^{\mu\nu}(x, y)\gamma_\mu S'(x, y; A)\gamma_\nu.$$ \hspace{1cm} (15)

Which achieve the gauge invariance of SDE (12).

To obtain gauge covariant solution of (12), note that (3) implies that $S(x, y; A)$ expressed in terms of $S(i\nabla_x)$ can keep the gauge covariance, so we can use (12) to fix the form of $S(i\nabla_x)$. To realize this idea, we need first know how to express $S^{-1}(x, y; A)$ appeared in (12) in terms of $S(i\nabla_x)$. We make our attempt by introducing $S^{-1}(i\nabla_x)$ similar as (3)

$$S^{-1}(x, y; A) = S^{-1}(i\nabla_x)\delta(x - y),$$ \hspace{1cm} (16)

which has an alternative expression

$$S^{-1}(x, y; A) = \int \frac{d^4 q}{(2\pi)^4} e^{-i q \cdot (x - y)} S^{-1}(i\nabla_x + q) = \int \frac{d^4 q}{(2\pi)^4} e^{-i q \cdot (x - y)} S^{-1}(i\nabla_y + q - i\partial_y) \bigg|_{i\partial_y \text{moved to l.h.s.}},$$  \hspace{1cm} (17)

where we have used property that

$$\int \frac{d^4 q}{(2\pi)^4} e^{-i q \cdot (x - y)} f(q)g(x) = f(-i\partial_y) \int \frac{d^4 q}{(2\pi)^4} e^{-i q \cdot (x - y)} g(x) = f(-i\partial_y)[g(y)\delta^4(x - y)] = \int \frac{d^4 q}{(2\pi)^4} e^{-i q \cdot (x - y)} f(q - i\partial_y)g(y).$$
With (17), we can further express 

$$S^{-1}(x, y; A) = \int \frac{d^4q}{(2\pi)^4} S^{-1}(-i\nabla_y + q)e^{-iq(x-y)} = \delta(x - y)S^{-1}(-i\nabla_y),$$

(18)

where \(\nabla_x^\mu = \partial_x^\mu + igA^\mu(x)\). With help of expression (18),

$$\delta(x - y) = \int d^4x' S(x, x')S^{-1}(x', y) = \int d^4x'S(i\nabla_x)\delta(x - x')\delta(x' - y)S^{-1}(-i\nabla_y)$$

$$= S(i\nabla_x)\delta(x - y)S^{-1}(-i\nabla_y) = S(i\nabla_x)S^{-1}(i\nabla_x)\delta(x - y),$$

(19)

which imply the constraint for \(S^{-1}(i\nabla_x)\) is

$$S(i\nabla_x)S^{-1}(i\nabla_x) = 1.$$  

(20)

With this type of \(S^{-1}(i\nabla_x)\) and (20), SDE (12) can be expressed as an equation to fix \(S(i\nabla_x)\)

$$[S^{-1}(i\nabla_x) - i\nabla_x]\delta^4(x - y) = g^2D^{\mu\nu}(x, y)\gamma_\mu S(i\nabla_x)\delta^4(x - y)\gamma_\nu,$$

(21)

it goes back to SDE (11) when we switching off external electromagnetic potential \(A^\mu\).

To solve SDE (21), a direct method is to expand the solution in terms of powers of external electromagnetic potential \(A^\mu\) and take special Schwinger-Fock gauge to relate gauge potential \(A^\mu(x)\) directly with gauge field strength \(F^{\mu\nu}_x\) [2]. This expansion is not gauge covariant, since it decompose covariant derivative apart and the operation for various parts of the covariant derivative are different which will violet the gauge covariance of the computation program. So this method will generally result a non-covariant \(S(x, y; A)\) due to the non-covariant computation procedure. To keep gauge covariance of \(S(x, y; A)\), we need to construct a covariant solution formalism. Inspired by the method proposed in Ref. [4], we write \(S(i\nabla_x)\) as

$$S(x, y; A) = S(i\nabla_x)\delta(x - y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)}S(i\nabla_x + p)1$$

$$= \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)}e^{i\nabla_x \cdot \frac{q}{\partial p} S(p - \tilde{F}_{p,x})e^{-ip(x-y)}\frac{q}{\partial p} 1} = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)}e^{i\nabla_x \cdot \frac{\partial}{\partial p} S(p - \tilde{F}_{p,x})1}. $$

(22)

\(S^{-1}(i\nabla_x)\) has the similar result. \(\tilde{F}_{p,x}^{\mu}\) is defined as

$$\tilde{F}_{p,x}^{\mu} = p^\mu - e^{-i\nabla_x \cdot \frac{\partial}{\partial p}}(p^\mu + i\nabla_x \cdot \frac{\partial}{\partial p} e^{i\nabla_x \cdot \frac{\partial}{\partial p}})$$

$$= -\frac{1}{2} [\nabla_x^{\nu}, \nabla_x^{\mu}] \frac{\partial}{\partial p^\nu} + \frac{i}{3} \partial^3 [\nabla_x^{\nu}, \nabla_x^{\mu}] \frac{\partial^2}{\partial p^\lambda \partial p^\nu} + \frac{1}{8} \partial^p \partial^\nu [\nabla_x^{\nu}, \nabla_x^{\mu}] \frac{\partial^3}{\partial p^\mu \partial p^\nu \partial p^\rho} + O(p^5),$$

(23)
in which for convenience of the calculation we assign powers to $\nabla^\mu$ and it is taken as order of $p^1$ and then $O(p^5)$ denote terms with at least five covariant derivatives. Note $\tilde{F}_{\mu\nu}^{\mu}$ commute with $\tilde{F}_{\mu\nu}^{\nu}$, but this property no longer holds when we goes to nonabelian gauge theory. Further

\[ [p^\mu - \tilde{F}_{\mu\nu}^{\mu}, p^\nu - \tilde{F}_{\mu\nu}^{\nu}] = e^{-i\nabla_x \frac{\partial}{\partial p^\nu}}[p^\mu + i\nabla_x p^\nu + i\nabla^\nu]e^{i\nabla_x \frac{\partial}{\partial p^\mu}} = \text{abelian} \implies iF^{\mu\nu}_x. \tag{24} \]

The l.h.s. of (21) now become

\[ [S^{-1}(i\nabla_x) - i\nabla_x] \delta^4(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip \cdot (x-y)} \left[ e^{i\nabla_x \frac{\partial}{\partial p}} S^{-1}(p - \tilde{F}_{\mu\nu}) 1 - i\nabla_x \hat{p} \right] \]

\[ = \int \frac{d^4p}{(2\pi)^4} e^{-ip \cdot (x-y)} e^{i\nabla_x \frac{\partial}{\partial p}} [S^{-1}(p - \tilde{F}_{\mu\nu}) 1 - \hat{p}] . \]

While r.h.s. of (21) become

\[ g^2 D^{\mu\nu}(x, y) \gamma_\mu S(i\nabla_x) \delta^4(x-y) \gamma_\nu \]

\[ = g^2 \int \frac{d^4p d^4q}{(2\pi)^8} e^{-i(q+p) \cdot (x-y)} D^{\mu\nu}(q) \left[ e^{i\nabla_x \frac{\partial}{\partial p}} S(p - \tilde{F}_{\mu\nu}) 1 \gamma_\nu \right] \]

\[ = g^2 \int \frac{d^4p}{(2\pi)^4} e^{-ip \cdot (x-y)} \left[ e^{i\nabla_x \frac{\partial}{\partial p}} \int \frac{d^4q}{(2\pi)^4} D^{\mu\nu}(q) \gamma_\mu S(p - q - \tilde{F}_{\mu\nu}) 1 \gamma_\nu \right] . \]

Combine l.h.s. and r.h.s. of (21) results together, subtract out factor $\int \frac{d^4p}{(2\pi)^4} e^{-ip \cdot (x-y)} e^{i\nabla_x \frac{\partial}{\partial p}}$ in both sides of the equation, we obtain

\[ S^{-1}(p - \tilde{F}_{\mu\nu}) 1 - \hat{p} = g^2 \int \frac{d^4q}{(2\pi)^4} D^{\mu\nu}(p-q) \gamma_\mu S(q - \tilde{F}_{\mu\nu}) 1 \gamma_\nu . \tag{25} \]

The advantage of above version of SDE in presence of external electromagnetic field is that external field appeared in the equation is through field strength not original gauge potential, it is this feature of the formalism making our computation program gauge covariant. To solve above equation, we notice that (24) now imply constraint

\[ S(p - \tilde{F}_{\mu\nu})S^{-1}(p - \tilde{F}_{\mu\nu}) = 1. \tag{26} \]

with it, in the following we try to search for its solutions.

### A. Trivial solution

it is easy to see that conventional fermion propagator which satisfy traditional SDE

\[ [Z(p^2) - 1] \hat{p} - \Sigma(p^2) = g^2 \int \frac{d^4q}{(2\pi)^4} D^{\mu\nu}(p-q) \gamma_\mu \frac{1}{Z(q^2)\hat{q} - \Sigma(q^2)} \gamma_\nu . \tag{27} \]
just coincide (25) and (26) if we identify

\[ S^{-1}(p - \tilde{F}_{p,x})1 = Z(p^2)\dot{p} - \Sigma(p^2) \quad \text{and} \quad S(p - \tilde{F}_{p,x})1 = \frac{1}{Z(p^2)\dot{p} - \Sigma(p^2)} , \tag{28} \]

which seems strange at first sight, since l.h.s. of (28) depend on argument \( p^\mu - \tilde{F}^\mu \) which require r.h.s of (28) should be written as some function of argument \( p^\mu - \tilde{F}^\mu \), or one can finish operations of momentum differentials inside \( \tilde{F}^\mu \). Since from (23), these differentials has coefficients depend on external field strength, finishing operations of momentum differentials in general will leave some momentum and external field strength dependent terms, but now r.h.s. of (28) only depend on momentum, external field strength donot show up. There must be some special reason. Notice that since \([p^\mu, \tilde{F}^\nu_{p,x}] \neq 0\), as we discussed before for \( S(i \nabla_x) \), in general there will be arrangement ambiguity for different sequences of \( S^{-1}(p - \tilde{F}) \)'s and \( S(p - \tilde{F}) \)'s argument \( p^\mu - \tilde{F}^\mu \). Now r.h.s. of (28) in fact offers a definition of the operator orders. We show next that carefully arrange the order of argument of \( p^\mu - \tilde{F}^\mu \), it is possible to realize above strange result. As an example, if we perform momentum expansion, for \( p^2 \) term, there is no arrangement order problem

\[(p - \tilde{F}_{p,x})^2 1 = \left[ p^2 + \frac{1}{2} [\nabla_x^\nu, \nabla_x^\mu] (\frac{\partial}{\partial p^\nu} p^\mu + p^\mu \frac{\partial}{\partial p^\nu}) + \frac{1}{4} [\nabla_x^\nu, \nabla_x^\mu][\nabla_x^{\nu'}, \nabla_x^{\mu'}] \frac{\partial^2}{\partial p^\nu \partial p^{\nu'}} \right] 1 = p^2 . \]

While for \( p^4 \) term we have three different kinds of arrangements, the first is

\[(p - \tilde{F}_{p,x})^2 (p - \tilde{F}_{p,x})^2 1 = \left[ p^2 + \frac{1}{2} [\nabla_x^\nu, \nabla_x^\mu] (\frac{\partial}{\partial p^\nu} p^\mu + p^\mu \frac{\partial}{\partial p^\nu}) + \frac{1}{4} [\nabla_x^\nu, \nabla_x^\mu][\nabla_x^{\nu'}, \nabla_x^{\mu'}] \frac{\partial^2}{\partial p^\nu \partial p^{\nu'}} \right]^2 1 = p^4 + \frac{1}{2} [\nabla_x^\nu, \nabla_x^\mu][\nabla_x^{\nu'}, \nabla_x^{\mu'}] , \tag{29} \]

the second is

\[(p^\mu - \tilde{F}^\mu_{p,x})(p^\nu - \tilde{F}^\nu_{p,x})(p_\mu - \tilde{F}_\mu_{p,x})(p_\nu - \tilde{F}_\nu_{p,x})1 \]
\[= (p - \tilde{F}_{p,x})^2 (p - \tilde{F}_{p,x})^2 1 + (p - \tilde{F}_{p,x})[(p^\nu - \tilde{F}^\nu_{p,x}), (p_\mu - \tilde{F}_\mu_{p,x})](p_\nu - \tilde{F}_\nu_{p,x})1 \]
\[= (p - \tilde{F}_{p,x})^2 (p - \tilde{F}_{p,x})^2 1 - \frac{1}{2} [\nabla_x^\nu, \nabla_x^\mu][\nabla_x^{\nu'}, \nabla_x^{\mu'}] = p^4 , \tag{30} \]

where we have used result (24). The third is

\[(p^\mu - \tilde{F}^\mu_{p,x})(p - \tilde{F}_{p,x})^2 (p_\mu - \tilde{F}_\mu_{p,x})1 \]
\[= (p^\mu - \tilde{F}^\mu_{p,x})(p^\nu - \tilde{F}^\nu_{p,x})(p_\mu - \tilde{F}_\mu_{p,x})(p_\nu - \tilde{F}_\nu_{p,x})1 - (p^\mu - \tilde{F}^\mu_{p,x})(p^\nu - \tilde{F}^\nu_{p,x})[(p_\mu - \tilde{F}_\mu_{p,x}), (p_\nu - \tilde{F}_\nu_{p,x})]1 \]
\[= p^4 - \frac{1}{2} [\nabla_x^\nu, \nabla_x^\mu][\nabla_x^{\nu'}, \nabla_x^{\mu'}] . \tag{31} \]
Therefore there are two ways of arrangement matching our result \((28)\), i.e. make field strength disappear, we can adjust its relative ratio to cancel the external field dependence
\[
\left[ \frac{1}{2} - a \right] (p - \tilde{F}_{p,x})^2 (p - \tilde{F}_{p,x})^2 + (p^\mu - \tilde{F}_{p,x}^\mu)(p - \tilde{F}_{p,x})^2 (p^\mu - \tilde{F}_{p,x}^\mu) \\
+ a (p^\mu - \tilde{F}_{p,x}^\mu)(p^\nu - \tilde{F}_{p,x}^\nu)(p^\mu - \tilde{F}_{p,x}^\mu)(p^\nu - \tilde{F}_{p,x}^\nu) \right] 1 = p^4 ,
\]
(32)
in which parameter \(a\) is an arbitrary constant which can be tuned to make both \(p^4\) terms appeared in \(S(p - \tilde{F}_{p,x})\) and \(S^{-1}(p - \tilde{F}_{p,x})\) be all independent on external field. One can continue to discuss all high orders which make us believe the correctness of solution \((28)\).

Now we already explain that why \((28)\) is the solution of SDE with presence of external field. This solution at level of \(S(p - \tilde{F}_{p,x})\) and \(S^{-1}(p - \tilde{F}_{p,x})\) is independent of external field.

With result \((28)\) and similar derivation as \((22)\), we obtain explicit expression for \(S(x, y; A)\) in terms of commutators as
\[
S^{-1}(x, y; A) = \int \frac{d^4p}{(2\pi)^4} e^{-ip \cdot (x - y)} e^{i\nabla_x \cdot \frac{\phi}{2}} [Z(p^2)\dot{\phi} - \Sigma(p^2)] \\
S(x, y; A) = \int \frac{d^4p}{(2\pi)^4} e^{-ip \cdot (x - y)} e^{i\nabla_x \cdot \frac{\phi}{2}} \frac{1}{Z(p^2)\dot{\phi} - \Sigma(p^2)} .
\]
(33)
Exploit the function \(a[x - y; A(x)] = e^{-z \nabla_x \cdot \frac{\phi}{2}} \mid_{z = x - y}\) introduced in Ref.\[7\], using partial differential rules changing the object of differential \(\frac{\partial}{\partial p}\) from \([Z(p^2)\dot{\phi} - \Sigma(p^2)]^{-1}\) to \(e^{-ip \cdot (x - y)}\) and note that \(e^{-i\nabla_x \cdot \frac{\phi}{2}} e^{-ip \cdot (x - y)} = a[x - y; A(x)] e^{-ip \cdot (x - y)}\), we can rewrite \((33)\) as
\[
S(x, y; A) = a[x - y; A(x)] S(x, y; 0) + \text{momentum space total derivative terms} ,
\]
(34)
where phrase momentum space total derivative terms appeared in above formula denote terms like
\[
\int \frac{d^4p}{(2\pi)^4} \frac{\partial}{\partial \mu} \left[ e^{-ip \cdot (x - y)} f[-z \cdot \nabla_x, \nabla_x \cdot \frac{\partial}{\partial p}] \nabla^\mu g[-z \cdot \nabla_x, \nabla_x \cdot \frac{\partial}{\partial p}] [Z(p^2)\dot{\phi} - \Sigma(p^2)]^{-1} \right] ,
\]
with \(f(k, l)\) and \(g(k, l)\) are \(k^n l^m\) type power functions. For \(x \neq y\), it is expected that the integration over total derivative term will equivalent to the integration on the boundary of the space-time for which the \(e^{-ip \cdot (x - y)}\) in will cause strong oscillation interference cancellation. While for the coincidence limit of the propagator, with definition \((1)\) and result \((33)\), we find
\[
\langle 0 | T \psi(x) \overline{\psi}(x) | 0 \rangle_A = -\frac{1}{4} \langle \psi \overline{\psi} \rangle 1 + i \int \frac{d^4p}{(2\pi)^4} [e^{i\nabla_x \cdot \frac{\phi}{2}} - 1] \frac{1}{Z(p^2)\dot{\phi} - \Sigma(p^2)} ,
\]
(35)
in which the first term $\langle \psi \psi \rangle$ is the standard fermion condensate constant which is external electromagnetic field and space-time coordinates independent, the second term is the momentum space total derivative terms. Taking trace for spinor index, we find the difference of fermion condensate in presence of external electromagnetic field $\langle \psi \psi \rangle_A$ with conventional fermion condensate $\langle \psi \psi \rangle$ is

$$\langle \psi \psi \rangle_A - \langle \psi \psi \rangle = -4i \int \frac{d^4p}{(2\pi)^4} \left[ \cosh(i\nabla_x \cdot \frac{\partial}{\partial p}) - 1 \right] \frac{\Sigma(p^2)}{Z^2(p^2)p^2 - \Sigma^2(p^2)} .$$  \hspace{1cm} (36)

If we further ignore the total differential terms in the integrand of (36), r.h.s. of above equation vanishes. Then external electromagnetic field will play no role in the fermion condensate, this is the reason that why we call this solution the trivial solution.

B. Solution with weak external electromagnetic field

Beyond trivial solution discussed above, there are many nontrivial solutions which in contrast to trivial solution, will depend on external electromagnetic field at level of $S^{-1}(p - \tilde{F}_{p,x})1$ and $S^{-1}(p - \tilde{F}_{p,x})1$. We focus our attention on the solution for which external electromagnetic field is weak. We take following ansatz for $S^{-1}(p - \tilde{F}_{p,x})$ and $S(p - \tilde{F}_{p,x})$

$$S^{-1}(p - \tilde{F}_{p,x}) = \hat{p} - \hat{\Sigma}(p^2, F^2_x) \hspace{1cm} S(p - \tilde{F}_{p,x})1 = \frac{1}{\hat{p} - \hat{\Sigma}(p^2, F^2_x)} .$$  \hspace{1cm} (37)

With definition $F^2_x \equiv -[\nabla_{\nu,x}, \nabla_{\mu,x}][\nabla_{\nu,x}, \nabla_{\mu,x}]$. Compare it to (28) in last subsection, $Z(p^2)$ is replaced to 1 which is just for simplicity and fermion self energy $\Sigma(p^2)$ is replaced to external electromagnetic field strength dependent $\hat{\Sigma}(p^2, F^2_x)$. This is due to fact that the most general arrangement on the argument of $p - \tilde{F}_{p,x}$ in $S^{-1}(p - \tilde{F}_{p,x})1$ and $S(p - \tilde{F}_{p,x})1$ should be dependent on $F^2_x$. For example, the most general $p^4$ order arrangement of the argument is not (32), but

$$\left[ (1 - a - b)(p - \tilde{F}_{p,x})^2(p - \tilde{F}_{p,x})^2 + b(p^\mu - \tilde{F}^\mu_{p,x})(p - \tilde{F}_{p,x})^2(p_{\mu} - \tilde{F}_{\mu,p,x}) \right] \left[ a(p^\mu - \tilde{F}^\mu_{p,x})(p^\nu - \tilde{F}^\nu_{p,x})(p_{\mu} - \tilde{F}_{\mu,p,x})(p_{\nu} - \tilde{F}_{\nu,p,x}) \right] 1 = p^4 + \frac{1}{2}(1 - a - 2b)F^2_x ,$$  \hspace{1cm} (38)

which is dependent on external field with arbitrary coefficients $a$ and $b$. With ansatz (37), (25) now become

$$- \Sigma(p^2, F^2_x) = g^2 \int \frac{d^4q}{(2\pi)^4} D^{\mu\nu}(p - q)\gamma_{\mu} \frac{1}{\hat{p} - \hat{\Sigma}(q^2, F^2_x)} \gamma_{\nu} ,$$  \hspace{1cm} (39)
which can be decomposed into two equations

\[ 0 = g^2 \int \frac{d^4q}{(2\pi)^4} D^{\mu\nu}(p-q)\gamma_\mu \frac{q^\nu}{q^2 - \Sigma^2(q^2, F_x^2)} \gamma^\nu \quad (40) \]

\[ -\Sigma(p^2, F_x^2) = g^2 \int \frac{d^4q}{(2\pi)^4} D^\mu_\nu(p-q) \frac{\Sigma(q^2, F_x^2)}{q^2 - \Sigma^2(q^2, F_x^2)} \cdot (41) \]

Equation (40) can be satisfied by suitable choice of gauge \[ \theta ] . (41) is same as traditional SDE for fermion self energy, except the feature that it depends on external electromagnetic field strength \( F_x^2 \). With this property, conventional result for SDE can also be applied here, i.e., there exist a critical coupling, nonzero solution happens if the coupling is stronger than its critical value. Since we are interested in the case of weak external field, we can further expand \( \Sigma(p^2, F_x^2) \) as

\[ \Sigma(p^2, F_x^2) = \Sigma_1(p^2) F_x^2 , \quad (42) \]

weak field assumption make it legal of dropping out high order external field terms and external field independent term in the expansion is ignored, (41) in this case become

\[ -\Sigma_1(p^2) = g^2 \int \frac{d^4q}{(2\pi)^4} D^\mu(p-q) \frac{\Sigma_1(q^2)}{q^2} , \quad (43) \]

which is just linearized SDE for fermion self energy , then we get solution

\[ \Sigma_1(p^2) \propto \Sigma(p^2) \bigg|_{\text{linearized}} . \quad (44) \]

With it the fermion condensate now become

\[ \langle 0 | T\bar{\psi}(x)\psi(x) | 0 \rangle_A = i \int \frac{d^4p}{(2\pi)^4} \frac{\Sigma_1(p^2)}{p^2} F_x^2 . \quad (45) \]

We find that the fermion condensate is nonzero and linear in \( F_x^2 \) at weak external field situation. This result is similar as that obtained in Ref.\[ 11 \] . The feature of this kind solution is that it has same critical coupling as that in conventional case without external field \[ 2 \] and condensate is proportional to \( F_x^2 \) which imply that external field, when it is weak, tends to enhance the value of the condensate.

C. Possibility of finding solution with strong external electromagnetic field

The solution discussed in last subsection has feature that it will not generate CSB when coupling of the theory below its critical value. In literature, people are interested in the case
that CSB happens at any couplings as long as the external electromagnetic field is strong enough \([1, 3]\). Now we investigate this type solution by taking following ansatz

\[
S^{-1}(p - \tilde{F}_{p,x}) 1 = \left[ 1 + C_{\mu\nu}(p, F_x) \frac{\partial^2}{\partial p_\mu \partial p_\nu} \right] \left[ \hat{p} - \tilde{\Sigma}(p^2, F_x^2) \right] \tag{46}
\]

\[
\begin{align*}
\hat{p} = & \hat{p} - \tilde{\Sigma}(p^2, F_x^2) - 2C_\mu^\mu(p, F_x)\tilde{\Sigma}'(p^2, F_x^2) - 4C_{\mu\nu}(p, F_x)p^\mu p^\nu \tilde{\Sigma}''(p^2, F_x^2),
\end{align*}
\]

where prime on the \(\tilde{\Sigma}\) denote the differential with respect to variable \(p^2\). The difference of above ansatz with trivial solution (28) and weak external electromagnetic field solution (37) is that now we include in the effect of momentum differentials which in trivial solution is forced to be cancelled and in weak external field solution is represented through its \(F_x^2\) dependence by arranging orders of the arguments for \(S^{-1}\). The ansatz we take here is just as a prototype to explain the effects from momentum differentials, a relatively simple differential term is taken in (46) and we donot discuss how to realize it through arranging the order of the argument \(p^\mu - \tilde{F}_{p,x}^\mu\) as which we have done in last two subsections.

We parameterize \(C_{\mu\nu}(p, F_x)\) as \(C_{\mu\nu}(p, F_x) = F_{\mu'\nu'} F_{\mu'\nu'} D(p, F_x)\) and define

\[
\tilde{\Sigma}(p^2, F_x^2) + 2C_\mu^\mu(p, F_x)\tilde{\Sigma}'(p^2, F_x^2) + 4C_{\mu\nu}(p, F_x)p^\mu p^\nu \tilde{\Sigma}''(p^2, F_x^2) = e^{-C(p, F_x)} \tilde{\Sigma}(p^2, F_x^2), \tag{47}
\]

which leads

\[
D(p, F_x) = \frac{[e^{-C(p, F_x)} - 1]\tilde{\Sigma}(p^2, F_x^2)}{2F^2\tilde{\Sigma}'(p^2, F_x^2) + 4p^\mu p^\nu F_{\mu'\nu'} F_{\mu'\nu'} \tilde{\Sigma}''(p^2, F_x^2)}. \tag{48}
\]

With (46)

\[
S(p - \tilde{F}_{p,x}) 1 = \frac{1}{\hat{p} - \tilde{\Sigma}(p^2, F_x^2)} \left[ 1 + C_{\mu\nu}(p, F_x) \frac{\partial^2}{\partial p_\mu \partial p_\nu} \right]^{-1} \frac{1}{\hat{p} - \tilde{\Sigma}(p^2, F_x^2)}. \tag{49}
\]

(25) then become

\[
- \tilde{\Sigma}(p^2, F_x^2) = g^2 e^{C(p, F_x)} \int \frac{d^4q}{(2\pi)^4} D^{\mu\nu}(p - q) \gamma_\mu \frac{1}{\hat{p} - \tilde{\Sigma}(p^2, F_x^2)} \gamma_\nu, \tag{50}
\]

which is just SDE (39) discussed in last subsection with coupling constant \(g^2\) being replaced with \(g^2 e^{C(p, F_x)}\). As long as factor \(e^{C(p, F_x)}\) become large when external field is strong, effective coupling constant \(g^2 e^{C(p, F_x)}\) will become big enough larger than its critical value which will then trigger CSB. This offers the possibility that strong external field trigging CSB.

In conclusion, we have built up general gauge invariant SDE for fermion propagator after introducing external gauge field into traditional SDE and set up systematic method.
to solve this equation gauge and Lorentz covariantly in abelian gauge theory and with
ladder approximation. With help of this method, we find trivial solution and weak external
electromagnetic field solution which have the same critical coupling as case of conventional
CSB in absence of external electromagnetic field. We point out possibility to construct strong
external electromagnetic field solution which will trigger CSB when external electromagnetic
field is strong enough.

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