Factored coset models: A unifying approach to different bosonization schemes

A N Theron, F G Scholtz and H B Geyer

Institute of Theoretical Physics,
University of Stellenbosch, 7600 Stellenbosch, South Africa

We discuss various bosonization schemes from a path integral perspective. Our analysis shows that the existence of different bosonization schemes, such as abelian bosonization of non-abelian models and non-abelian bosonization of fermions with colour and flavour indices, can be understood as different ways of factoring out a dynamically trivial coset which contains the fermions. From this perspective follows the importance of the coset model in ensuring the correct superselection rules on the bosonic level.

Keyword abstract: Path integral, Wess-Zumino-Witten model, abelian bosonization, non-abelian bosonization, Thirring model

I. INTRODUCTION

Bosonization has a long history in two dimensional quantum field theory. The subject originated with the work of Coleman and Mandelstam on abelian models in the middle seventies (see also Ref. [3]), and was soon applied to the non-abelian case [3]. This generalization, referred to hereafter as “earlier non-abelian bosonization”, however, obscures the non-abelian symmetries of the original fermionic model. Since Witten’s introduction of a non-abelian bosonization scheme which maintains in a manifest way the fermionic symmetries on the bosonic level [4], the “earlier non-abelian” bosonization has remained somewhat in the background.

The bosonization scheme of Witten is based on the equivalence of free fermions to the Wess-Zumino-Witten (WZW) model and has been applied with great success to a number of field theoretical problems, in particular to massive fermions carrying both flavour and colour indices [3]. Such fermions may be bosonized in terms of either a $U(N_f) \times U(N_c) \times SU(1)$ WZW model or in terms of a $SU(N_f) \times SU(N_c) \times U(1)$ WZW model. A subtlety arises here since, although the last option is desirable because it separates the colour and flavour degrees of freedom, it has been pointed out in Refs. [3], that the resulting bosonic image of the fermionic mass term is incorrect.

Bosonization has been extensively discussed from the path integral point of view. An approach has been pioneered in Ref. [3] where interacting fermionic models are bosonized by “decoupling” the fermions via a chiral transformation. More recently, interest in deriving the bosonization rules or dictionary with path integrals has been revived [3]. In Ref. [3] it was shown that bosonization can be viewed as a type of duality transformation, and in Ref. [3] it was derived by first introducing a gauge symmetry, followed by choosing an appropriate gauge. In Ref. [3] it was shown that the bosonization rules can be derived by factoring the fermions out as part of a trivial coset model. In this manner the complete bosonization dictionary for both abelian and non-abelian bosonization was derived. A particular advantage of this approach is that the superselection rules are embodied in the dynamically trivial coset model.

Despite extensive work on bosonization some gaps still exist in the literature. “Earlier non-abelian bosonization”, the equivalence of the massless $SU(2)$ Thirring model to the sine-Gordon and free boson model (not to be confused with the equivalence of the massive Thirring model and the sine-Gordon model [3]), and the different non-abelian bosonization schemes for coloured and flavoured fermions have not been derived with path integrals. The above mentioned subtlety in the $SU(N_f) \times SU(N_c) \times U(1)$ non-abelian scheme for mass bilinears has not been resolved at all. In this paper we present a path integral derivation of the “earlier non-abelian bosonization” and show the equivalence of the $SU(2)$ Thirring model with the sine-Gordon model. We also discuss non-abelian bosonization for fermions with colour and flavour indices and show how to derive both the $U(N_f,N_c)$ and $SU(N_f) \times SU(N_c) \times U(1)$ schemes with path integrals. Our method also leads to the correct form for the boson image of the fermionic mass terms in the $SU(N_f) \times SU(N_c) \times U(1)$ scheme.

These results are obtained using the formalism of Ref. [3], where the fermions are factored out as part of a coset model which is dynamically trivial. Indeed, we show that the above mentioned bosonization schemes correspond to factoring out the fermions as part of different coset models. The formalism of Ref. [3] therefore provides us not only with a convenient way of deriving the bosonization dictionary but also with a way of understanding the relation between different bosonization schemes and thus act as a unifying framework for the different schemes. Furthermore the coset model explicitly encapsulates the superselection rules of the different schemes. On the canonical level the coset model that is factored out corresponds to either a finite dimensional or a discrete Hilbert space that must be retained to ensure the correct superselection rules. We show that
II. BOSONIZATION BY FACTORING OUT THE FERMIONS

We briefly review the main results of Ref. [11]. Working in two-dimensional Minkowski space we start with the generating functional of free Dirac fermions with source terms (or external fields) for the currents and mass terms.

\[ Z[A_+, \eta, \bar{\eta}] = \int D\bar{\psi} D\psi \exp\{i \int d^2x [\bar{\psi}\gamma^\mu \partial_\mu \psi + \psi_+^\dagger A_+ \psi_- + \eta \psi_+^\dagger \psi_- + \bar{\eta} \psi_-^\dagger \psi_+] \} . \]  

\[ (1) \]

Light cone components are used and for simplicity we first consider an \( A_+ \) dependence only, returning to the more general case later. The first step is the introduction of bosonic degrees of freedom in the path integral. This is accomplished by inserting the following identity in the path integral

\[ 1 = \int DB_+ \delta(B_+) \exp\{i \int d^2x \psi_+^\dagger B_+ \psi_- \} \]
\[ = \int DB_+ D\lambda_- \exp\{i \int d^2x [\psi_+^\dagger B_+ \psi_- + B_+ \lambda_-] \} . \]  

\[ (2) \]

Inserting this identity in eq. (1) and shifting \( B_+ \to B_+ + A_+ \), we have

\[ Z[A_+, \bar{\eta}, \eta] = \int D\bar{\psi} D\psi DB_+ D\lambda_- e^{i \int d^2x L} \]

\[ (3) \]

with

\[ L = \psi_+^\dagger i \partial_- \psi_+ + \psi_-^\dagger i \partial_+ \psi_- + \psi_+^\dagger B_+ \psi_- + B_+ \lambda_- - A_+ \lambda_- + \eta \psi_+^\dagger \psi_- + \bar{\eta} \psi_-^\dagger \psi_+ . \]  

\[ (4) \]

The aim is to factor the fermionic part of the generating functional out as part of a constrained fermionic model which is topological and thus contains no dynamics. This is achieved by the following change of variables in the path integral

\[ \psi_- \to e^{-i\theta(x)} \psi_- \quad \psi_+ \to \psi_+ \]
\[ \psi_-^\dagger \to \psi_-^\dagger e^{i\theta(x)} \quad \psi_+^\dagger \to \psi_+^\dagger , \]

\[ (5) \]

\[ \lambda_- = \alpha \partial_- \theta , \]

where \( \alpha \) is a number to be specified later.

The change of variables \( (5) \) is a chiral transformation and the fermionic measure is not invariant under this transformation. This results from the chiral anomaly which appears in the path integral formalism as a non-invariance of the fermionic measure, as first shown by Fujikawa [13] for an infinitesimal transformation. For the finite rotation required in expression \( (3) \), the Jacobian reads [14]

\[ J_F = \exp\left( \frac{i}{\pi} \int d^2x \left( \frac{1}{2} \partial_- \theta \partial_+ \theta + \partial_- \theta B_+ \right) \right) . \]  

\[ (7) \]

The determinant for the change of variables \( (5) \) is \( \det(\alpha \partial_-) \) and it may be written as a functional integral over Grassmann variables (ghosts)

\[ \int Db_+ Dc_+ e^{i \int d^2x \bar{b}_+ \partial_- c_+} . \]  

\[ (8) \]

This determinant is actually irrelevant for evaluating the generating functional \( (1) \) as it only affects the normalization. It is, however, important if we are interested in correlation functions of the energy momentum tensor and the central charge of the model.
The generating functional (8) now reads
\[ Z[A_+, \bar{\eta}, \eta] = \int D\bar{\psi} D\psi DB_+ D\theta Db_+ Dc_+ e^{\int d^2x \left( L_{cl} + L_B + L_{source} \right)} \] (9)
where
\[ L_{cl} = \psi_+^\dagger i\partial_- \psi_+ + \psi_-^\dagger i\partial_+ \psi_- + \psi_+^\dagger (B_+ - \partial_+ \theta) \psi_- + b_+ \partial_- c_+ , \] (10)
\[ L_B = \frac{1}{2\pi} \partial_+ \theta \partial_- \theta + \frac{1}{\pi} B_+ \partial_- \theta + \alpha B_+ \partial_+ \theta , \] (11)
\[ L_{source} = -\alpha A_+ \partial_+ \theta + \eta \psi_+^\dagger e^{i\theta} \psi_- + \bar{\eta} \psi_-^\dagger e^{-i\theta} \psi_+ . \] (12)

\( L_B \) originates from the fermionic Jacobian (8) and the \( \lambda_\beta B_+ \) term of eq. (8). We note that if we choose \( \alpha = \frac{1}{\pi} \) the \( B_+ \) dependence dissapears completely from \( L_B \) and the generating functional (9) factorises into that of a constrained fermionic model and a free massless boson model. (The \( \theta \) dependence in the fermionic part dissapears under the shift \( B_+ \rightarrow B_+ - \partial_+ \theta \).)

The fermionic and bosonic Lagrangians are still coupled through the sources, but upon functional differentiation and setting the sources (\( \eta \) and \( \bar{\eta} \)) equal to zero we achieve complete decoupling. To summarize, the generating functional factorizes as
\[ Z[A_+, \bar{\eta}, \eta]_{\bar{\eta}, \eta = 0} = Z_{cl} Z_B[A_+, \eta, \bar{\eta}]_{\bar{\eta}, \eta = 0} \] (13)
where
\[ Z_{cl} = \int D\bar{\psi} D\psi DB_+ Db_+ Dc_+ e^{iS_{cl}} \] (14)
with
\[ S_{cl} = \int d^2x [\psi_+^\dagger i\partial_- \psi_+ + \psi_-^\dagger i\partial_+ \psi_- + \psi_+^\dagger B_+ \psi_- + b_+ \partial_- c_+] \] (15)
and
\[ Z_B[A_+, \bar{\eta}, \eta] = \int D\theta e^{iS_B} \] (16)
with
\[ S_B = \int d^2x \left[ \frac{1}{2\pi} \partial_+ \theta \partial_- \theta + \frac{1}{\pi} A_+ \partial_- \theta + \eta \psi_+^\dagger e^{i\theta} \psi_- + \bar{\eta} \psi_-^\dagger e^{-i\theta} \psi_+ \right] . \] (17)

Eqs. (13) to (17) are the desired results. If we wish to calculate current-current correlation functions in the generating functional (8), we may calculate these correlators instead with the generating functional (17). The fermions are decoupled as part of a constrained fermionic model which contains no dynamics. It is a topological model that only contains information about the superselection rules of the original fermions.

To calculate correlators of mass bilinears in (8) we have to differentiate with respect to \( \eta \) and \( \bar{\eta} \) and put these sources equal to zero. Using the result (14) we obtain for example
\[ \frac{\delta^2 Z[A_+, 0, \bar{\eta}, \eta]}{\delta \bar{\eta}(x_1) \delta \eta(x_2)} \bigg|_{\bar{\eta}, \eta = 0} = \langle \psi_+^\dagger \psi_+ (x_1) \psi_-^\dagger \psi_- (x_2) \rangle_{\text{free fermion}} \]
\[ = \langle \psi_+^\dagger \psi_+ (x_1) \psi_-^\dagger \psi_- (x_2) \rangle_{\text{coset}} \left( e^{i\theta(x_1)} e^{i\theta(x_2)} \right)_{\text{free boson}} , \] (18)
where \( \langle \rangle_{\text{coset}} \) denotes the correlation function of mass bilinears evaluated in the coset model. Because of the trivial nature of the coset this quantity is a constant, so that we obtain the result that the free fermion correlator is proportional to the correlators of exponentials of free boson fields. In this way one obtains the standard bosonization rule that fermionic mass terms may be represented by the exponentials of free boson fields.

The fermionic coset model (8) is a model with a discrete Hilbert space. It is a conformally invariant model with central charge \( c = 0 \) (9), and therefore has only \( h = 0 \) primary fields. This is another way of noting that the correlators of primary fields are constant.

We would like to stress that eq. (14) should be interpreted as a gauge fixed fermionic coset model in the light cone gauge, that is, both left and right currents are constrained to zero. The light cone gauge was induced by the choice of the identity (8). Indeed, an alternative choice of the identity (8) leads to the Lorentz gauge, as explained below. Alternatively, as explained in Ref. (11), a coset model that is not gauge fixed may be obtained by inserting the quantity
\[
\int DB_{\mu}\delta(\epsilon^{\mu\nu}\partial_{\nu}B_{\mu}) \exp\{i \int d^{2}x \bar{\psi}\gamma^{\mu}B_{\mu}\psi\}
= \int DB_{\mu}D\lambda \exp\{i \int d^{2}x [\bar{\psi}\gamma^{\mu}B_{\mu}\psi + \lambda \epsilon^{\mu\nu}\partial_{\nu}B_{\mu}]\}
\]
into the generating functional
\[
Z[A_{\mu}] = \int D\bar{\psi}D\psi \exp\{i \int d^{2}x \bar{\psi}\gamma^{\mu}(\partial_{\mu} + A_{\mu})\psi\}.
\]
(19)

This quantity may be inserted into the path integral without altering the physics \[11\]. It amounts to gauging the fermions and constraining the connection \(B_{\mu}\) to be flat \[9\].

The fermions are now factored out by the \(\gamma_{5}\) transformation
\[
\psi \rightarrow e^{-\gamma_{5}\theta(x)}\psi, \\
\bar{\psi} \rightarrow \bar{\psi}e^{-\gamma_{5}\theta(x)},
\]
(21)
\[
\lambda(x) \rightarrow \frac{1}{\pi}\theta(x).
\]
(22)

The Jacobian associated with the change of variables (21) reads
\[
J_{\psi} = \exp\{\frac{i}{\pi} \int d^{2}x(\frac{1}{2\pi}\partial_{\mu}\theta\partial^{\mu}\theta + \epsilon^{\mu\nu}\partial_{\mu}\theta(B_{\mu} + A_{\mu}))\},
\]
(23)
while we ignore the Jacobian associated with (22) which is just a divergent constant. We obtain
\[
Z[A_{\mu}] = \int D\bar{\psi}D\psi DB_{\mu}D\theta \ e^{i \int d^{2}x L}
\]
(24)
with
\[
L = \bar{\psi}\gamma^{\mu}\partial_{\mu}\psi + \bar{\psi}\gamma^{\mu}(A_{\mu} + B_{\mu} + \epsilon_{\mu\nu}\partial^{\nu}\theta)\psi + \frac{1}{2\pi}\partial_{\mu}\theta\partial^{\mu}\theta + \frac{1}{\pi}A_{\mu}\epsilon^{\mu\nu}\partial_{\nu}\theta.
\]
(25)
Shifting the \(B_{\mu}\) field to \(B_{\mu} \rightarrow B_{\mu} + A_{\mu} + \epsilon_{\mu\nu}\partial_{\nu}\theta\) yields the desired result – a coset model which has not been gauge fixed and a free bosonic field.

To obtain a coset model in the Lorentz gauge, one replaces the identity (2) by
\[
1 = \int DB_{\mu}\delta(B_{\mu}) \exp\{i \int d^{2}x \bar{\psi}\gamma^{\mu}B_{\mu}\psi\}
= \int DB_{\mu}D\lambda_{\mu} \exp\{i \int d^{2}x [\bar{\psi}\gamma^{\mu}B_{\mu}\psi + B_{\mu}\lambda^{\mu}]\},
\]
(26)
which is manifestly covariant, and insert it into the generating functional (20). We again perform the transformation (21) but replace transformation (22) by
\[
\lambda_{\mu} = \partial_{\mu}\eta + \frac{1}{\pi}\epsilon_{\mu\nu}\partial^{\nu}\theta
\]
(27)
with Jacobian \(\text{det}(\frac{1}{\pi}\partial_{\mu}\partial^{\nu})\). This yields the result
\[
Z[A_{\mu}] = \int D\bar{\psi}D\psi DB_{\mu}D\eta D\theta \det(\partial_{\mu}\partial^{\nu})e^{i \int d^{2}x L}
\]
(28)
with
\[
L = \bar{\psi}\gamma^{\mu}\partial_{\mu}\psi + \bar{\psi}\gamma^{\mu}(B_{\mu} + A_{\mu} + \epsilon_{\mu\nu}\partial^{\nu}\theta)\psi + \partial^{\mu}\eta B_{\mu} + \frac{1}{2\pi}\partial^{\mu}\theta\partial_{\mu}\theta + \frac{1}{\pi}\epsilon^{\mu\nu}A_{\mu}\partial_{\nu}\theta.
\]
(29)
Shifting the \(B_{\mu}\) field to \(B_{\mu} \rightarrow B_{\mu} + \epsilon_{\mu\nu}\partial^{\nu}\theta\), and performing a partial integration it is realized that \(\eta\) is just the Lagrange multiplier that enforces the Lorentz gauge condition \(\partial^{\mu}B_{\mu}\). The determinant \(\text{det}(\frac{1}{\pi}\partial_{\mu}\partial^{\nu})\) is the Fadeev-Popov determinant associated with the Lorentz gauge and may be written as an integral with ghost fields similar to eq. (8). The appearance of \(A_{\mu}\) in the coset sector is spurious; functional differentiation at \(A_{\mu} = 0\) yields no contribution from this sector, because correlators of coset currents vanish.

This concludes our review of abelian bosonization in the factored coset approach. The trivial model may be factored out in the light cone gauge \[15\], the Lorentz gauge \[28\], or in a form \[24\] that has not been gauge fixed at all.
III. EARLIER NON-ABELIAN BOSONIZATION

We now consider the non-abelian case and apply the formalism of the previous section. For simplicity we only consider a model with fermions in the fundamental representation of $SU(2)$,

$$Z[A^a_i] = \int D\bar{\psi}D\psi \exp\{i\int d^2x \left[ \bar{\psi}i\gamma^\mu\partial_\mu \psi + \psi^+_i A^a_i t^a \psi^-_i \right] \} ,$$  \hspace{1cm} (30)

with $t^a = \sigma^a/2$ and $\sigma^a$ the Pauli matrices. We call this additional index a flavour index. We constrain each flavour of fermion separately into a $U(1)$ coset model by introducing the identity

$$1 = \int DB^+_i \delta(B^+_i) DB^+_2 \delta(B^+_2) \exp\{i \int d^2x \left[ \psi^+_i A^1_i + \psi^+_2 B^1_i \psi^+_2 \right] \} ,$$  \hspace{1cm} (31)

where we have written the flavour index explicitly. We introduce two Lagrange multipliers $\lambda^a_i$ to lift the Dirac deltas into the action, perform on each flavour a $U(1)$ chiral transformation

$$\psi^-_i \rightarrow e^{-i\theta^a(x)} \psi^-_i \quad \psi^+_i \rightarrow \psi^+_i ,$$  \hspace{1cm} (32)

and then transform the Lagrange multipliers to

$$\lambda^a = \frac{1}{\pi} \partial_\theta \theta^a \hspace{1cm} (33)$$

Each flavour has an associated anomalous Jacobian \([3]\), so that we obtain the result (after lifting the Jacobians associated with \([3]\) with ghosts and making the appropriate shifts on the $B$ fields)

$$Z[A_\mu] = \int D\bar{\psi}D\psi DB^+_1 DB^+_2 D\theta^1 D\theta^2 DB^1_1 DC^1_1 DC^2_1 DC^1_2 DC^2_2 \exp\{i \int d^2x [L_{\text{cf}} + L_B + L_{\text{source}}] \} .$$  \hspace{1cm} (34)

Here the various Lagrangians are defined as

$$L_{\text{cf}} = \sum_{j=1,2} \left[ \bar{\psi}^j i\gamma^\mu \partial_\mu \psi^j + B^j_\mu \bar{\psi}^j \gamma^\mu \psi^j + b^j_\mu \partial_- c^j_+ \right] ,$$  \hspace{1cm} (35)

$$L_B = \sum_{j=1,2} \frac{1}{2\pi} \partial_- \theta^j \partial^\theta \theta^j \hspace{1cm} (36)$$

$$L_{\text{source}} = \frac{1}{2\pi} A^3_\pm (\partial_+ \theta_1 - \partial_- \theta_2)$$

$$+ \frac{i}{2} (\psi^1_+ \psi^2_+ e^{i\theta^1_+ + i\theta^2_+} + \psi^1_- \psi^2_- e^{-i\theta^1_- - i\theta^2_-}) A^{(1)}_\pm$$

$$+ \frac{i}{2} (\psi^1_+ \psi^2_- e^{-i\theta^1_- + i\theta^2_-} - \psi^1_- \psi^2_+ e^{i\theta^1_+ - i\theta^2_+}) A^{(2)}_\pm \hspace{1cm} (37)$$

The constrained model \([33]\) is dynamically trivial. The central charge of this model is zero and correlators of the fermion fields are constant.

The first two terms of $L_{\text{source}}$ originate from the Jacobian \([3]\) because $A^3_\pm$ couples to the $u(1)$ currents. Since current correlators are obtained by differentiating with respect to the sources and setting them equal to zero, one notes that the bosonic images of the fermionic currents may be read off immediately from $L_{\text{source}}$. These rules coincide with the “old non-abelian bosonization rules” of Banks et al. \([3]\), thus yielding a first path integral derivation. Of importance is the appearance of the coset fields $\psi^j$ in these rules. As mentioned, the correlators of these fields are constant – on the second quantized level these fields are the path integral analogue of fermionic operators without a space-time dependence.

In writing down the bosonic image of the fermionic currents, the authors of Ref. \([3]\) need to introduce fermionic operators with no space-time dependence, denoted by $\chi$ in Ref. \([3]\). In our approach the coset fields play the role of these constant fermion operators. The constrained fermion approach to bosonization therefore provides a natural framework to understand the need to introduce these constant fermionic operators in the bosonization dictionary.

The boson images of the mass bilinears are likewise obtained by introducing the appropriate source terms in \([37]\).
IV. EQUIVALENCE OF THE SU(2) THIRRING AND SINE-GORDON MODELS

In this section we apply the preceding formalism to the SU(2) massless Thirring model. It was shown in Ref. [2] from “earlier non-abelian bosonization” that this model is equivalent to a free massless boson and a sine-Gordon model. We discuss here this important result from a path integral point of view.

The Lagrangian of the massless SU(2) Thirring model is given by

\[ L_T = \bar{\psi} i \gamma^\mu \partial_\mu \psi - g^2 j_\mu^a j^{a\mu} , \]  

where \( \psi \) is in the fundamental representation of SU(2) and \( j_\mu^a = \bar{\psi} \gamma_\mu t^a \psi \) with \( t^a \) given below eq. (30).

The four-point interaction is eliminated by introducing an auxiliary field \( A_\mu^a \) to yield equivalently

\[ L_T = \bar{\psi} i \gamma^\mu \partial_\mu \psi + 2g j_\mu^a A^{a\mu} + A_\mu^a A^{a\mu} . \]  

Since both left and right components of the currents are present we find it convenient to use the formalism based on inserting the quantity (40) for both flavours into the generating functional. The Lagrangian then reads (we write the flavour indices explicitly)

\[
\begin{align*}
L_T &= \bar{\psi} i \gamma^\mu \partial_\mu \psi + \bar{\psi} \gamma^\mu (B_\mu^1 + g A_{\mu}^0) \psi^1 \\
&+ \bar{\psi} \gamma^\mu (B_\mu^2 - g A_{\mu}^0) \psi^2 + \bar{\psi} \gamma^\mu A_{\mu}^{(+)\mu} \psi^1 + g \bar{\psi} \gamma^\mu A_{\mu}^{(-)\mu} \psi^2 \\
&+ A_\mu^{(0)} A_{\mu}^{(0)} + A_\mu^{(+)\mu} A_{\mu}^{(-)\mu} + \frac{1}{\pi} \theta^1 \epsilon^{\mu\nu} \partial_\nu B_\mu^1 + \frac{1}{\pi} \theta^2 \epsilon^{\mu\nu} \partial_\nu B_\mu^2 
\end{align*}
\]

where \( A_{\mu}^{(\pm)} = A_{\mu}^{(1)} \mp i A_{\mu}^{(2)} \). The \( B_\mu^a \) and \( \theta^a \) fields carry a flavour index as in the previous section. The auxiliary field \( A_\mu \) plays a similar role as the source \( A_\mu \) of the previous section.

The fermions are constrained by the chiral rotation

\[ \psi^1 \rightarrow e^{-i\gamma^5 \theta^1} \psi^1 , \quad \bar{\psi}^1 \rightarrow \bar{\psi}^1 e^{-i\gamma^5 \theta^1} , \]

(41)
to obtain, (taking again the Jacobian (23) into account)

\[
\begin{align*}
L_T &= \bar{\psi} i \gamma^\mu \partial_\mu \psi + \bar{\psi} \gamma^\mu (B_\mu^1 + g A_{\mu}^0 + \epsilon_{\mu\nu} \partial^\nu \theta^1) \psi^1 \\
&+ \bar{\psi} \gamma^\mu (B_\mu^2 - g A_{\mu}^0 + \epsilon_{\mu\nu} \partial^\nu \theta^2) \psi^2 + g \bar{\psi} \gamma^\mu \epsilon_{\mu\nu} \partial^\nu \theta^1 \psi^1 A_{\mu}^{(-)} \\
&+ g \bar{\psi} \gamma^\mu \epsilon_{\mu\nu} \epsilon_{\gamma\delta} \partial^\nu \theta^2 \psi^2 A_{\mu}^{(+)} + A_{\mu}^{(0)} A_{\mu}^{(0)} + A_{\mu}^{(+)} A_{\mu}^{(-)} \\
&+ \frac{1}{4\pi} \partial_\mu \theta^1 \partial^\mu \theta^1 + \frac{1}{2\pi} \partial_\mu \theta^2 \partial^\mu \theta^2 + \frac{g}{\pi} \epsilon^{\mu\nu} \partial_\nu \theta^1 A_{\mu}^{(0)} - \frac{g}{\pi} \epsilon^{\mu\nu} \partial_\nu \theta^2 A_{\mu}^{(0)} .
\end{align*}
\]

(42)

Shifting, as before, the \( B_\mu^a \) field and performing the integration over the auxiliary field \( A_\mu \), we obtain

\[
\begin{align*}
L_T &= L_{\text{cost}} + \bar{\psi} \gamma^\mu \partial_\mu \psi + \bar{\psi} \gamma^\mu (B_\mu^1 \psi^1 + \bar{\psi} \gamma^\mu \gamma^\mu \psi^2 \\
&+ \frac{1}{2\pi} + \frac{g^2}{4\pi^2} \partial_\mu \theta^1 \partial^\mu \theta^1 + \frac{1}{2\pi} + \frac{g^2}{4\pi^2} \partial_\mu \theta^2 \partial^\mu \theta^2 \\
&- \frac{g^2}{2\pi^2} \partial_\mu \theta^1 \partial^\mu \theta^2 
\end{align*}
\]

(43)

where

\[ L_{\text{cost}} = \bar{\psi} i \gamma^\mu \partial_\mu \psi + \bar{\psi} \gamma^\mu B_\mu^1 \psi^1 + \bar{\psi} \gamma^\mu B_\mu^2 \psi^2 . \]

(44)

This may be simplified somewhat by the change of variables

\[ \eta = \theta^1 + \theta^2 \quad \rho = \theta^1 - \theta^2 , \]

(45)
to obtain

\[
\begin{align*}
L_T &= L_{\text{cost}} + \bar{\psi} \gamma^\mu e^{-i\gamma^5 \rho} \gamma^\mu \psi^1 e^{i\gamma^5 \rho} e^{i\gamma_\mu \gamma^\mu \psi^2} \\
&+ \frac{1}{4\pi} \partial_\mu \eta \partial^\mu \eta + \left( \frac{1}{4\pi} + \frac{g^2}{4\pi^2} \right) \partial_\mu \rho \partial^\mu \rho .
\end{align*}
\]

(46)
The coset model is factored out. From now on we use the notation $B$ where the coset model is discussed in the previous section. Apart from this difference the procedure is similar. We briefly review the main

$$L_{\text{int}} = 2g^2 \psi_+^2 \psi_-^2 \psi_+^2 e^{2i\rho} .$$

(We ignore terms independent of $\rho$ as they contribute only a constant.) Making an expansion of the generating functional in terms of $L_{\text{int}}$ gives

$$Z_T = \int D\bar{\psi} D\psi DB_\mu DB_\rho D_\rho \exp \left( iS_{\text{const}} + iS_{\text{free boson}} \right) \times \left( 1 + \int d^2x \left( 2g^2 (\psi_+^2 \psi_-^2 e^{2i\rho} + \psi_+^2 \psi_-^2 e^{2i\rho}) + \ldots \right) \right) ,$$

where

$$S_{\text{free boson}} = \int d^2x \left[ \frac{1}{4\pi} \partial_\mu \eta \partial^\mu \eta + \left( \frac{1}{4\pi} + \frac{g^2}{4\pi^2} \right) \partial_\mu \rho \partial^\mu \rho \right] .$$

The fermionic correlators that appear in the perturbative expansion (48) yield a constant, because they are evaluated in the coset model. Using chiral selection rules we note that the series (13) is just the expansion of an exponent with argument constant $\times \cos(2\rho)$ in the free bosonic model – see also Refs. [3][2][1]. This establishes the equivalence between the $SU(2)$ Thirring model and the sine-Gordon model together with a free boson field (the $\eta$ field).

V. NON-ABELIAN BOSONIZATION

We now turn our attention to the non-abelian bosonization of Witten [3], where the free fermions (with a $U(N)$ chiral symmetry) are bosonized in terms of a WZW model which realizes the non-abelian symmetry in a manifest way. In ref. [1] it was shown that this non-abelian dictionary may be obtained by factoring out a $U(N)$ coset model instead of $N$ $U(1)$ coset models as discussed in the previous section. Apart from this difference the procedure is similar. We briefly review the main results of [1] concerning non-abelian bosonization.

We start again with the generating functional for the currents and fermionic mass terms. The mass terms are chosen to be invariant under $U(N)$ gauge transformation, this implies that it is general enough to consider a generating functional that only depends on one component $A_+$ for the current because the $A_-$ may be gauge transformed to zero as explained in [1].

$$Z[A_+, \bar{\eta}, \eta] = \int D\bar{\psi} D\psi \exp \left\{ i \int d^2x \left[ \bar{\psi}_+^2 i \partial_+ \psi_- + \psi_+^2 i \partial_- \psi_- ight. ight.$$

$$\left. - \bar{\psi}_+^2 A_+ \bar{\psi}_- + \eta \bar{\psi}_+^2 + \bar{\eta} \bar{\psi}_-^2 \right] \right\} ,$$

where $A_+ = A_+^a t^a$ with $t^a$ the generators of $SU(N)$ and $\psi$ in the fundamental representation of $SU(N)$. We factor out a $U(N)$ coset model by introducing the following identity

$$1 = \int DB_\mu D_\rho \exp \left\{ i \int d^2x \left[ \bar{\psi}_- [B_+^a \lambda^a + C_+ \psi_-] \right] \right\}$$

$$= \int DB_\mu D_\rho \exp \left\{ i \int d^2x \left[ \bar{\psi}_- [B_+^a t^a + C_+ \psi_-] \right] \right\} .$$

The fields $C_+$ and $\rho_-$ are introduced to factor out the $U(1)$ symmetry of the fermions, if this is neglected, a $U(N)/SU(N)$ coset model is factored out. From now on we use the notation $B_+ = B_+^a t^a$ and $\lambda_+ = \lambda_+^a t^a$. The procedure is formally the same as in the abelian case. We shift $B_+ \to B_+ + A_+$ and perform the non-abelian analogue of the change of variables [3]

$$\psi_- \to e^{-ig} \psi_- \qquad \psi_+ \to \psi_+$$

$$\psi_-^\dagger \to \psi_-^\dagger \ge e^{ig} \qquad \psi_+^\dagger \to \psi_+^\dagger$$

$$\lambda_+ = \frac{1}{4\pi} g^{-1} \partial_- g$$

(53)
\[ \rho_\pm = \frac{1}{\pi} \partial_\theta \]  

The contribution to the measure is the non-abelian anomaly which may be iterated to yield (37),

\[ J_\rho = \exp[iS[g]] + \frac{i}{4\pi} \int d^2x [\text{Tr}(B_+(\partial_- g)g^{-1}) + \frac{N}{2\pi} \partial_- \theta \partial_+ \theta + \frac{N}{\pi} C_+ \partial_+ \theta] \]  

where \( S[g] \) is the WZW action

\[ S[g] = -\frac{1}{8\pi} \int d^2x \text{Tr}(g\partial_\mu g^{-1}g\partial^\mu g^{-1}) - \frac{1}{12\pi} \int d^3x \epsilon_{\mu\nu\rho} \text{Tr}(g\partial^\mu g^{-1}g\partial^\nu g^{-1}g\partial^\rho g^{-1}) \]  

Eq. (55) is the non-abelian analogue of (7). The Jacobian associated with transformation (53) is (see refs. [11,16])

\[ J = \int Db_+ Dc_+ e^{i\int d^2x \text{Tr}(b_+ \partial_+ c_+)}, \]  

where Grassmann fields have been introduced in the adjoint of the coset model, For the change of variables (54) we have the Jacobian (5).

Transformations (53) and (54) were chosen so that the \( B^a_+ \lambda^a_+ \) term of eq. (37) cancels against the \( B_+ g^{-1} \partial_- g \) term of eq. (53). Shifting the \( B_+ \) field to \( B_+ \to B_+ - ig^{-1} \partial g \) and \( C_+ \to C_+ - \partial_+ \theta \), we obtain a completely decoupled fermionic coset model with \( B_+ \) constraining the \( SU(N) \) currents to be zero and \( C_+ \) constraining the \( U(1) \) current to be zero. The dynamics is contained in the WZW and free boson model,

\[ Z[A_+, \bar{\eta}, \eta]|_\rho, \eta = 0 = Z_{\text{coset}} Z_B[A_+, \bar{\eta}, \eta]|_\rho, \eta = 0 \]  

where

\[ Z_{\text{coset}} = \int D\bar{\psi} D\psi DB_+ DC_+ Db_+ Dc_+ Db_+ Dc_+ e^{iS_{\text{coset}}}, \]  

\[ S_{\text{coset}} = \int d^2x [\psi_i \gamma^\mu i\partial_\mu \psi_i + \psi_j^\dagger B_+ \psi_j - B_+ \partial_+ c_j \]  

\[ + \psi_j^\dagger C_+ \psi_- + \bar{b}_+ \partial_+ \bar{c}_+], \]  

and

\[ Z_B[A_+, \bar{\eta}, \eta] = \int Dg D\theta e^{iS_B} \]  

with

\[ S_B = WZW[g] + \int d^2x [\frac{1}{4\pi} \text{Tr}(A_+(\partial_- g)g^{-1}) + \frac{N}{2\pi} \partial_- \theta \partial_+ \theta \]  

\[ + \eta \psi_i^j g_j k e^{\psi_i^j \phi - \bar{\eta} \psi_i^j g_j k e^{-\psi_i^j \phi}]} \]  

The only coupling between the coset model and the boson model is through the sources of the mass bilinears. Functional differentiation with respect to these sources at zero gives a complete decoupling between the coset and boson models, as indicated in expression (58). As before, the fermionic correlators are constants (because the coset model has \( c = 0 [14] \)), which corresponds on the second quantized level to fermionic operators independent of space-time. The Hilbert space of a \( SU(N) \) coset model is finite dimensional, and the \( U(1) \) coset model has a discrete Hilbert space. We thus retain this fermionic Hilbert space in our bosonization prescription. The role of this space is to ensure the correct superselection rules, while the dynamics is in the bosonic sector.

If we have both colour and flavour indices on the fermions we may proceed in two different ways. Consider the Lagrangian

\[ L = \psi_i^\alpha \gamma^\mu i\partial_\mu \psi_i^\alpha + \eta \psi_j^\dagger \alpha \psi_i^\dagger - \alpha + \bar{\eta} \psi_i^\dagger \alpha \psi_i^\dagger - \alpha, \]  

where \( j = 1 \ldots N_f \) is called a flavour index and \( \alpha = 1 \ldots N_c \) a colour index. We focus our attention here on source terms for fermionic masses as the bosonization rule for the relevant currents is straightforward. The model actually has an \( U(N_f N_c) \) symmetry which is larger than the \( U(1) \times SU(N_f) \times SU(N_c) \) symmetry where we limit ourselves to separate rotations of the flavour and colour indices. The \( U(N_f N_c) \) bosonization scheme of (5) is reproduced by factoring out an \( U(1) \times SU(N_f N_c) \).
coset model. This is done precisely as described above with \( g \) taking values in \( SU(N_f N_c) \) as may be seen by relabelling the fermions with a single index \( r = 1 \ldots N_f N_c \).

Alternatively we may factor out an \( U(1) \times SU(N_f) \times SU(N_c) \) fermionic coset by generalizing the identity \( [\ref{coset1}] \) to

\[
1 = \int DB^a_\mu \delta(B^a_\mu)DE^b \delta(E^b) DC \delta(C) \exp\{ i \int d^2 x [\bar{\psi}^+ B^a_\mu T^a \psi - \bar{\psi}^+ E^a \gamma ^5 T^a \psi + \bar{\psi}^+ C \psi] \} \tag{64}
\]

where \( T^a \) are generators of \( SU(N_f) \) and \( t^a \) are generators of \( SU(N_c) \). The Dirac deltas are handled, as usual, by introducing Lagrange multipliers, and the chiral transformation in the present case reads

\[
\begin{align*}
\psi_+ & \to e^{-i\theta} g^{-1} h^{-1} \psi_+ \\
\psi^\dagger_+ & \to \psi^\dagger_+ h g e^{i\theta}
\end{align*}
\tag{65}
\]

with \( g \in SU(N_f) \) and \( h \in SU(N_c) \). By performing again the relevant changes of variables on the Lagrange multipliers (transforming from the algebra to the group) and shifting the auxiliary fields, the final result is easily seen to be

\[
Z[\eta, \bar{\eta}]_{\eta, \bar{\eta} = 0} = Z_{\text{coset}} Z_B[\eta, \bar{\eta}]_{\eta, \bar{\eta} = 0}
\tag{66}
\]

where the coset action is given by

\[
S_{\text{coset}} = \int d^2 x [\bar{\psi}^\mu i \partial_\mu \psi + \psi^\dagger_+ B^a_\mu T^a \psi_+ + \bar{\psi}^+ E^a \gamma ^5 T^a \psi + \text{ghosts}] \tag{67}
\]

and the bosonic action by

\[
S_B = N_c W ZW[g] + N_f W ZW[h] + \int d^2 x \frac{N_c N_f}{2\pi} \partial_\mu \partial_\nu \theta + \eta \psi^\dagger_+ \alpha g_{ij} \partial_\mu \partial_\nu \theta e^{i\theta} \psi_+ \gamma_+ \psi_+ \gamma_+ \beta \\
\tag{68}
\]

The only new aspect in the derivation of \( [\ref{coset1}] \) is to remember that we pick up \( N_f \) times the colour anomaly, \( N_c \) times the flavour anomaly and \( N_f N_c \) times the abelian anomaly under transformation \( [\ref{coset2}] \).

As indicated in \( [\ref{coset3}] \), functional differentiation with respect to the sources and equating them to zero give us complete decoupling. For example the two point mass correlator reads

\[
\begin{align*}
\langle \psi^\dagger_+ \psi_-(x) \psi^\dagger_+ \psi_+(y) \rangle_{\text{free fermion}} &= \langle \psi^\dagger_+ \alpha \psi^\gamma_+ \beta (x) \psi^\dagger \gamma_+ \gamma (y) \rangle_{\text{coset}} \\
&\times \langle g_{ij}(x) g_{ij}(y) \rangle \langle h_{\alpha \beta}(x) h_{\gamma \delta}(y) \rangle \langle e^{i\theta(x)} e^{-i\theta(y)} \rangle_{\text{free boson}}
\end{align*}
\tag{69}
\]

The fermionic coset part is again a trivial model with central charge \( c = 0 \) \( [\ref{coset4}] \), enforcing only superselection rules.

The result \( [\ref{coset5}] \) differs from the bosonic representation of the fermionic mass bilinears given in Refs. \( [\ref{coset6}] \) by the presence of the coset operator \( \psi^\dagger_+ \). These operators still carry both flavour and colour indices and therefore we do not have the complete decoupling of colour and flavour degrees of freedom. In Refs. \( [\ref{coset7}] \) it was noted that the omission of these factors leads to an incorrect result for the four point function (of mass bilinears) and in Ref. \( [\ref{coset8}] \) it was suggested that a finite dimensional Hilbert space should be retained on the bosonic level to solve this problem. Our approach automatically provides the coset Hilbert space that plays this role.

The coset model is topological as it contains only the zero modes of the free fermions. The result \( [\ref{coset9}] \) therefore suggests the following boson image for the mass bilinear on the second quantize level

\[
\psi^\dagger_+ \gamma \psi_-
\]

where \( c \) and \( d \) are fermion creation and annihilation operators

\[
\begin{align*}
\{ c_{\alpha \beta}^+, c_{\alpha \beta} \} &= \delta_{\alpha \beta} \\
\{ d_{\alpha \beta}^+, d_{\alpha \beta} \} &= \delta_{\alpha \beta}
\end{align*}
\tag{71}
\]

with \( a \) and \( b \) a left or right index. Other anticommutators vanish, and the fermion Fock vacuum is defined by \( c|0\rangle = d|0\rangle = 0 \).

In the case of the \( U(N_f N_c) \) bosonization scheme \( [\ref{coset10}] \) the fermionic coset operators should, in principal, also be retained in the bosonization dictionary. However, in this case the \( SU(N_f N_c) \) selection rules are simpler and little harm is done ignoring this factor, as the \( SU(N_f N_c) \) selection rules are also present for the WZW model.
VI. CONCLUSIONS

In this paper we have shown how various bosonization schemes that exist in two dimensional quantum field theory may be understood from the path integral point of view in terms of a single formalism where a trivial fermionic coset model is factored out. We showed how the old non-abelian bosonization may be incorporated in the path integral approach and applied this to the massless $SU(2)$ Thirring model to show its equivalence to the sine-Gordon model and a free boson model.

It is known from the old bosonization dictionary that constant fermionic operators are required on the bosonic level. The only way of incorporating this notion of constant operators in a path integral is in terms of a field theoretic model that is dynamically trivial or topological. We are naturally lead in our approach to fermionic coset models that play this role.

This path integral approach to bosonization forces us to keep a finite dimensional fermionic Hilbert space (apart from the $U(1)$ which is discrete), which enforces superselection rules while the dynamics decouples into the bosonic part. The fermionic coset Hilbert space may be neglected when the selection rules are obvious, but in the general case it must be retained. Two particular cases where this is important is in the old non-abelian bosonization (abelian bosonization applied to non-abelian models) and in the $SU(N_f) \times SU(N_c)$ non-abelian bosonization scheme of coloured and flavoured fermions.

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