EQUIVARIANT EMBEDDING OF METRIZABLE $G$-SPACES IN LINEAR $G$-SPACES

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Abstract. Given a Lie group $G$ we study the class $M_G$ of proper metrizable $G$-spaces with metrizable orbit spaces, and show that any $G$-space $X \in M_G$ admits a closed $G$-embedding into a convex $G$-subset $C$ of some locally convex linear $G$-space, such that $X$ has some $G$-neighborhood in $C$ which belongs to the class $M_G$. As a corollary we see that any $G$-ANR for $M_G$ is a $G$-ANE for $M_G$.

1. Introduction

In this paper we study spaces in the class $M_G$ of proper metrizable $G$-spaces with metrizable orbit spaces, where $G$ is a Lie group. In the classical theory of retracts the Wojdyslawski embedding theorem (see [Hu, Chapter III, Theorem 2.1]) ensures that any metrizable space can be embedded as a closed subset of a convex subspace of some Banach space.

In the equivariant case, S. Antonyan has proved that for any topological group $G$, any $G$-space $X$ with a $G$-invariant metric admits a $G$-embedding as a closed $G$-subset of a convex $G$-subspace $C$ of some Banach $G$-space $B$ with a $G$-invariant norm [An1 Proposition 8 and Theorem 1]. However, the motivation for our study lies in the theory of extensors and retracts for the class $M_G$, and hence we would like to know that $B$ belongs to $M_G$, or at least that a neighborhood of the image of $X$ in $C$ does.

It was shown by E. Elfving [E1 Theorem 3.11] that for a linear Lie group $G$, any Palais proper metrizable $G$-space $X$ which is locally compact, separable and finite-dimensional, and which has finitely many orbit types, admits a closed $G$-embedding in a linear $G$-space such that $G$ acts Palais properly on a $G$-neighborhood of the image. In [E2] the linearity assumption on the group $G$ is dropped and the assumptions on the space $X$ weakened; see Theorem 5.4.

Here we will prove the following theorem:

Theorem 5.1. Let $G$ be a Lie group and let $X \in M_G$. Then there exists a $G$-embedding $e : X \to L$ where $L$ is a locally convex linear $G$-space such that $e(X)$ is a closed subset of some $G$-invariant convex subset $C$ of $L$ and $e(X)$ has some...
$G$-neighborhood $V$ in $C$ such that $V \in \mathcal{M}_G$.

Using Theorem 6.1, we show that the classes $\mathcal{M}_G \cap \text{G-ANE-M}_G$ and $\text{G-ANR-M}_G$ are the same.

2. Preliminaries

Throughout the paper $G$ will denote an arbitrary Lie group unless otherwise stated, where Lie groups are defined to be Hausdorff and second countable.

A $G$-space is a Hausdorff topological space $X$ with a continuous action of the group $G$, namely a continuous map $\Phi: G \times X \to X$ such that $\Phi(e, x) = x$ and $\Phi(g_1, \Phi(g_2, x)) = \Phi(g_1g_2, x)$ for all $g_1, g_2 \in G$ and all $x \in X$, where $e \in G$ is the unit element. We usually denote $\Phi(g, x)$ by $gx$.

Let $X$ be a $G$-space. Given any set $S \subset X$ and any subgroup $H$ of $G$, denote $HS = \{gs \mid g \in H, s \in S\}$. A subset $S \subset X$ is said to be a $G$-subset if $GS = S$, and a neighborhood which is a $G$-subset is called a $G$-neighborhood. Given $x \in X$ we define the isotropy subgroup of $G$ at $x$ as the subgroup $G_x = \{g \in G \mid gx = x\}$.

Suppose $X$ and $Y$ are two $G$-spaces. A continuous map $f: X \to Y$ is a $G$-map if $f(gx) = gf(x)$ for every $g \in G$ and every $x \in X$. $G$-maps which are homeomorphisms, embeddings, retractions, etc., are called $G$-homeomorphisms, $G$-embeddings, $G$-retractions and so on.

A completely regular $G$-space $X$ is said to be Cartan if every point has a neighborhood $V$ such that the closure of the set $\{g \in G : gV \cap V \neq \emptyset\}$ is compact. The action of $G$ on a completely regular $G$-space $X$ is proper if for any pair of points $x, y \in X$ there exist neighborhoods $V_x, V_y$ of $x$ and $y$ such that the closure of the set $\{g \in G : gV_x \cap V_y \neq \emptyset\}$ is compact. We say that $X$ is a proper $G$-space. The action of $G$ on a completely regular $G$-space $X$ is Palais proper if for any point $x \in X$ there exists a neighborhood $V_x$ such that any point $y \in X$ has a neighborhood $V_y$ for which the closure of $\{g \in G : gV_x \cap V_y \neq \emptyset\}$ is compact. Then we say that $X$ is a Palais proper $G$-space.

Clearly a Palais proper $G$-space is proper, and any proper $G$-space must be a Cartan $G$-space.

We will denote by $\mathcal{M}_G$ the class of proper metrizable $G$-spaces $X$ which have a metrizable orbit space $X/G$. By [An-Ne, Theorem B], for a proper metrizable $G$-space $X$, the metrizability of $X/G$ is equivalent to $X/G$ being paracompact, or $X$ admitting a $G$-invariant metric, where a metric $d$ on a $G$-space $X$ is said to be $G$-invariant if $d(gx, gy) = d(x, y)$ for all $x, y \in X$ and all $g \in G$.

Let $H$ be a closed subgroup of $G$. A subset $S$ of a $G$-space $X$ is an $H$-slice if $GS$ is open in $X$ and there exists a $G$-map $f: GS \to G/H$ such that $S = f^{-1}(eH)$.

The set $S$ is a slice at the point $x \in X$ if $x \in S$ and $S$ is a $G_x$-slice.

By [Pa2, Theorem 2.3.3] we know that in a Cartan $G$-space, there exists a slice at every point and the isotropy subgroup of $G$ at any point is compact.

The following lemma describes how a slice at $x \in X$ induces an $H$-slice for any subgroup $H$ of $G$ which is conjugate to $G_x$.

**Lemma 2.1.** Suppose that $X$ is a Cartan $G$-space and let $S$ be a slice at $x \in X$.

Let $H$ be a subgroup of $G$ which is conjugate to $G_x$ by an element $\bar{g} \in G$; that is, $H = \bar{g}G_x\bar{g}^{-1} = G_{\bar{g}x}$. Then $\bar{g}S$ is a slice at $\bar{g}x$, and in particular $\bar{g}S$ is an $H$-slice.

**Proof.** By [Pa2, Proposition 1.1.5] the maps $G/G_x \to Gx$ and $G/G_{\bar{g}x} \to Gx$ given by $gG_x \mapsto gx$ and $gG_{\bar{g}x} \mapsto g\bar{g}x$, respectively, are $G$-homeomorphisms. Thus there
is a $G$-homeomorphism $h: G/G_x \to G/G\bar{g}_x$ given by $h: G/G_x \cong G \xrightarrow{\approx} G\bar{g}_x = G\bar{g}_x$. Hence an open refinement by $G$ be a compact subgroup of $S$ be an

**Theorem 2.2** ([Pa2 Theorem 2.1.4]). Suppose $X$ is a Cartan $G$-space and let $H$ be a compact subgroup of $G$. A subset $S \subset X$ is an $H$-slice if and only if the following conditions hold:

1. $S$ is closed in $GS$,
2. $S = HS$,
3. $gS \cap S \neq \emptyset$ implies $g \in H$,
4. $GS$ is open in $X$,
5. $S$ has a neighborhood $V$ in $GS$ such that the closure of $\{g \in G : gV \cap V \neq \emptyset\}$ is compact.

We say that the open set $GS$ is a tubular neighborhood (of $x$) if $S$ is an $H$-slice (a slice at $x$).

**Definition 2.3** (Tubular covering). A tubular covering of a $G$-space $X$ is a covering of $X$ by tubular neighborhoods.

**Lemma 2.4.** An open $G$-subset of a tubular neighborhood is a tubular neighborhood. Hence an open refinement by $G$-sets of a tubular covering is a tubular covering.

**Proof.** Suppose $H$ is a closed subgroup of $G$, let $S$ be an $H$-slice and let $f: GS \to G/H$ be a corresponding $G$-map. Let $U$ be an open $G$-invariant subset of $GS$. Now $S' = S \cap U$ is an $H$-slice since $GS' = U$ is open and $f' = f|U: U \to G/H$ is a $G$-map where $(f')^{-1}(eH) = f^{-1}(eH) \cap U = S \cap U = S'$.

**Lemma 2.5.** If $\{GS_k\}_{k \in K}$ is a family of pairwise disjoint tubular neighborhoods in $X$, where $S_k$ is an $H$-slice for all $k \in K$, then $S = \bigcup_{k \in K} S_k$ is an $H$-slice.

**Proof.** Here a corresponding map $f: GS = \bigcup_{k \in K} GS_k \to G/H$ is defined by $f|GS_k = f_k$ where $f_k: GS_k \to G/H$ is a map corresponding to the slice $S_k$.

Given a closed subgroup $H$ of $G$ and an $H$-space $S$, there is an action of $H$ on the product $G \times S$ given by $h(g, s) = (gh^{-1}, hs)$. We denote by $G \times H S$ the quotient space $(G \times S)/H$, which is called the twisted product of $G$ and $S$ with respect to $H$. There is an action of $G$ on $G \times_H S$ defined by the formula $\bar{g}[g, s] = [\bar{g}g, s].$

**Proposition 2.6** ([El Proposition 1.18]). Let $H$ be a closed subgroup of $G$ and let $S$ be an $H$-slice in a $G$-space $X$. Then $G \times_H S \cong G \times H S$.

3. Countability of tubular coverings

The following lemma makes use of a technique originating with J. Milnor; see [Pa1 Theorem 1.8.2].

**Lemma 3.1.** Let $X$ be a Cartan $G$-space and suppose that $X/G$ is paracompact. Then $X$ admits a countable locally finite tubular covering.

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1 We assume that paracompact spaces are Hausdorff.
Proof. By [An4] Proposition 3.6 we know that $G$ has at most countably many compact conjugacy types, represented by compact subgroups $H_n$ of $G$ where $n \in \mathbb{N}$. Suppose that $\{G_S\}_{i \in I}$ is a tubular covering of $X$ such that for each $i \in I$, $S_i$ is a slice at some point $x_i \in X$, where $G_{x_i}$ is a compact subgroup of $G$. Then $gG_{x_i}g^{-1} = H_{x_i}$ for some $g \in G$ and some $n_i \in \mathbb{N}$, and by Lemma 2.4 $gS_i$ is a slice at $gx_i$ and $G_{gx_i} = gG_{x_i}g^{-1} = H_{n_i}$. Thus we may assume from the beginning that each $S_i$ is an $H_{n_i}$-slice for some $n_i \in \mathbb{N}$. Let $\{\varphi_i : X \to [0, 1]\}_{i \in I}$ be a $G$-invariant partition of unity subordinate to $\{G_S\}_{i \in I}$. Such a partition of unity exists because $X/G$ is paracompact.

For each finite $T \subset I$, denote

$$W(T) = \{x \in X : \varphi_i(x) > \varphi_j(x) \text{ for all } i \in T \text{ and for all } j \in I \setminus T\}.$$ 

Denote by $u_T : X \to [0, 1]$ the continuous $G$-invariant map

$$u_T(x) = \max\{0, \min_{i \in I, j \in I \setminus T} (\varphi_i(x) - \varphi_j(x))\}.$$ 

Then $W(T) = u_T^{-1}(0, 1]$ is open and $G$-invariant in $X$.

If $x \in W(T)$, then $\varphi_i(x) > \varphi_j(x) \geq 0$ for all $i \in T, j \in I \setminus T$, so in particular $x \in \varphi^{-1}_i(0, 1]$ for all $i \in T$. Thus $W(T)$ is an open $G$-invariant subset of $\varphi^{-1}_i(0, 1] \subset GS_i$ for each $i \in T$; hence $W(T)$ is a tubular neighborhood by Lemma 2.4.

If Card $T = \text{Card } T'$ and $T \neq T'$, then $W(T) \cap W(T') = \emptyset$, because if $i \in T \setminus T'$, $j \in T' \setminus T$ and $x \in W(T) \cap W(T')$, then simultaneously $\varphi_i(x) > \varphi_j(x)$ and $\varphi_j(x) < \varphi_i(x)$, which is impossible.

Define

$$W^m_n = \bigcup\{W(T) : \text{Card } T = n \text{ and } W(T) \text{ is an } H_m\text{-tubular neighborhood}\}$$

for all $m, n \in \mathbb{N}$. Now $W^m_n$ is an $H_m$-tubular neighborhood by Lemma 2.5 because it is a disjoint union of $H_m$-tubular neighborhoods. It follows that $\{W^m_n\}_{(m,n) \in \mathbb{N} \times \mathbb{N}}$ is a countable tubular covering of $X$.

Denote by $\pi_X : X \to X/G$ the canonical projection. Since $X/G$ is paracompact, the open covering $\{\pi_X(W^m_n)\}_{(m,n) \in \mathbb{N} \times \mathbb{N}}$ of $X/G$ admits a precise locally finite refinement by [Du] Chapter VIII, Theorem 1.4], and in particular this refinement is countable. Denote it by $\{V^m_n\}_{n \in \mathbb{N}}$. Now $\pi^{-1}_X(V^m_n)$ is a countable locally finite refinement of $\{W^m_n\}_{(m,n) \in \mathbb{N} \times \mathbb{N}}$ by $G$-neighborhoods; hence it is a countable locally finite tubular covering of $X$ by Lemma 2.4 and we are done. \hfill \square

4. Homeomorphism properties of isovariant $G$-maps

The main result in this section is an important homeomorphism property of isovariant $G$-maps between Cartan $G$-spaces, which we will use in proving our main theorem. Recall that a map $f : X \to Y$ between $G$-spaces is isovariant if $G_x = G_{f(x)}$ for all $x \in X$. Our lemma builds on the following result for compact transformation groups:

Lemma 4.1 ([Br] Exercise 10 of Chapter I]). Let $H$ be a compact Hausdorff topological group, and let $f : X \to Y$ be an isovariant $H$-map between $H$-spaces. Then $f$ is an open map if and only if the induced map $\bar{f} : X/H \to Y/H$ is an open map.
Lemma 4.2. Suppose that $X$ and $Y$ are Cartan $G$-spaces and that $f: X \to Y$ is an isovariant $G$-map. Then $f$ is a $G$-homeomorphism if and only if the induced map $\bar{f}: X/G \to Y/G$ is a homeomorphism.

Proof. We have a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\pi_X & \downarrow & \pi_Y \\
X/G & \xrightarrow{\bar{f}} & Y/G
\end{array}
$$

where $\pi_X$ and $\pi_Y$ denote the canonical projections. Since $\pi_X$ is continuous and $\pi_Y$ is open, it is clear that if $f$ is open, then so is $\bar{f}$. Hence if $f$ is a homeomorphism, so is $\bar{f}$.

Now suppose $\bar{f}$ is a homeomorphism; it then easily follows from the bijectivity of $\bar{f}$ and the fact that any isovariant $G$-map restricts to a bijection on the orbits that $f$ is a bijection. It remains to show that $f$ is open. Being Cartan, $Y$ has a covering $\{GS^Y_i\}_{i \in I}$ where each $S^Y_i$ is a slice at a point $y_i \in Y$ such that the isotropy subgroups $G_{y_i}$ are compact for all $i \in I$. Let $i \in I$. Then $S^X_i = f^{-1}(S^Y_i)$ is a slice at $x_i = f^{-1}(y_i)$. Note that by isovariance $G_{y_i} = G_{x_i}$. We consider now only the restrictions

$$
\begin{array}{ccc}
GS^X_i & \xrightarrow{f} & GF(S^X_i) = GS^Y_i \\
\pi_X & \downarrow & \pi_Y \\
GS^X_i/G & \xrightarrow{\bar{f}} & GS^Y_i/G = GS^Y_i/G.
\end{array}
$$

Since $GS^X_i/G \approx S^X_i/G_{x_i}$ and $GS^Y_i/G \approx S^Y_i/G_{x_i}$ by [2] Lemma 1.23, we see that the restriction $f[S^X_i]: S^X_i \to S^Y_i$ induces a map $\bar{f}: S^X_i/G_{x_i} \to S^Y_i/G_{x_i}$ given by $\bar{f}(G_{x_i}s) = G_{x_i}(f(s))$, which makes the following diagram commutative:

$$
\begin{array}{ccc}
S^X_i & \xrightarrow{f} & S^Y_i \\
\pi & \downarrow & \pi' \\
S^X_i/G_{x_i} & \xrightarrow{\bar{f}} & S^Y_i/G_{x_i} \\
\approx & \approx & \approx \\
GS^X_i/G & \xrightarrow{\bar{f}} & GS^Y_i/G.
\end{array}
$$

Hence since $\bar{f}$ is open, so is $\bar{f}$. The isotropy subgroup $G_{x_i}$ is compact by [2] Theorem 2.3.3 since the $G$-action on $X$ is Cartan and thus by Lemma [1] the restriction $f[S^X_i]$ is open.
The map $\text{id}_G \times f|S_i^X$ induces a map $G \times_{G_{x_i}} f|S_i^X: G \times_{G_{x_i}} S_i^X \to G \times_{G_{x_i}} S_i^Y$ as in [Br] Chapter II, Proposition 2.1] which makes the following diagram commutative:

$$
\begin{array}{ccc}
G \times S_i^X & \xrightarrow{\text{id}_G \times f|S_i^X} & G \times S_i^Y \\
\downarrow & & \downarrow \\
G \times_{G_{x_i}} S_i^X & \xrightarrow{G \times_{G_{x_i}} f|S_i^X} & G \times_{G_{x_i}} S_i^Y \\
\downarrow & & \downarrow \\
GS_i^X & \xrightarrow{f} & GS_i^Y
\end{array}
$$

and since $f|S_i^X$ is open, $G \times_{G_{x_i}} f|S_i^X$ is open by [Br Proposition II 2.1]. Hence $f|GS_i^X$ is open onto its image; i.e., it is a homeomorphism onto its image.

Since the $S_i^Y$ are slices, the set $GS_i^Y$ is open in $Y$ for each $i \in I$. Hence $f$ is open; i.e., it is a homeomorphism.

5. Embedding theorem

Now we are able to prove the following theorem, which is the main result of this note:

**Theorem 5.1.** Let $G$ be a Lie group and let $X \in \mathcal{M}_G$. Then there exists a $G$-embedding $e: X \to L$ where $L$ is a locally convex linear $G$-space such that $e(X)$ is a closed subset of some $G$-invariant convex subset $C$ of $L$ and $e(X)$ has some $G$-neighborhood $V$ in $C$ such that $V \in \mathcal{M}_G$.

We will prove Theorem 5.1 by finding an isovariant $G$-map $e: X \to L$ which induces an embedding $X/G \to L/G$ under suitable conditions, and by applying Lemma 4.2. The same method has been used by J. Jaworowski [Ja] and by G. Bredon [Br] Chapter II.10 for compact Lie group actions.

**Definition 5.2** (LCL $G$-space). A locally convex linear $G$-space (for short, an LCL $G$-space) is a $G$-space $L$ which is a locally convex topological vector space where each element $g \in G$ represents a linear map $L \to L$.

**Lemma 5.3.** A product of LCL $G$-spaces with diagonal action is an LCL $G$-space.

**Proof.** Let $X_i$ be a family of LCL $G$-spaces where the indices $i$ run through some set $I$. Set $X = \prod_{i \in I} X_i$; now it is clear that $X$ is a topological vector space and a $G$-space where each $g \in G$ represents a linear map $X \to X$. Furthermore $X$ is locally convex.

Indeed, suppose $U$ is an open neighborhood of $x \in X$. Then there is some basic open set $V = \prod_{i \in I} V_i$ such that $x \in V \subset U$. For $V$ to be an element in the basis, we know that each $V_i$ is open in $X_i$ and $V_i = X_i$ for all but finitely many $i \in I$. Now for each $V_i \neq X_i$ there exists a convex neighborhood $W_i$ of $\text{pr}_i(x)$ such that $W_i \subset V_i$. Whenever $V_i = X_i$, set $W_i = X_i$. Define $W = \prod_{i \in I} W_i$. Now $W$ is a convex neighborhood of $x$ and $W \subset V \subset U$. □
We are going to use the following result, which is obtained in \[\text{[E2]}\] Section 3, although there it is not explicitly stated as a theorem:

**Theorem 5.4.** Let $G$ be a Lie group and assume that a $G$-space $X \in \mathcal{M}_G$ admits a metric $d$ which satisfies

$$(*) \quad \forall r > 0 \forall x \in X : \bar{B}_r^d(x) \text{ is compact,}$$

where $\bar{B}_r^d(x)$ denotes the closed ball of radius $r$ about $x$ with respect to the metric $d$.

Then there exists a Banach $G$-space $B$ and a closed $G$-embedding $i : X \to B$ such that $i(X) \subset B \setminus \{0\}$ and $G$ acts properly on $B \setminus \{0\}$.

**Remark 5.5.**

i) Note that the metric $d$ satisfying $(*)$ can be any metric on $X$, independent of the $G$-action.

ii) A normed vector space is locally convex.

iii) From the construction of the space $B$ one easily sees that the norm in $B$ is $G$-invariant, inducing a $G$-invariant metric on $B$ and, by restriction, on $B \setminus \{0\}$. Hence, $B \setminus \{0\} \in \mathcal{M}_G$.

Suppose that the $G$-space $X$ has a global $H$-slice $S$ with $H$ a compact subgroup of $G$, i.e. $X = GS$. Then there exists an isovariant $H$-map

$$\varphi : S \to \prod_{n \in \mathbb{N}} E_n$$

where each $E_n$ is a Euclidean representation space for $H$ (see \[\text{[An2]}\] Lemma 5). The twisted product $G \times_H E_n$ is a $G$-space, and there is an $H$-embedding $i_n : E_n \to G \times_H E_n$ defined by $i_n(x) = [e, x]$. This defines an isovariant $H$-map

$$\tilde{\varphi} : S \to \prod_{n \in \mathbb{N}} E_n \xrightarrow{\prod i_n} \prod_{n \in \mathbb{N}} G \times_H E_n.$$ 

Using this we obtain a $G$-map $\psi : X = GS \to \prod_{n \in \mathbb{N}} G \times_H E_n$ by setting $\psi(gx) = g\tilde{\varphi}(x)$.

**Lemma 5.6.** The map $\psi$ is isovariant.

**Proof.** By equivariance $G_{gs} \subset G_{\psi(gs)}$ for all $gs \in GS$. Thus we only need to show that $G_{gs} \supset G_{\psi(gs)}$.

Assume that $g\tilde{\varphi}(S) \cap \tilde{\varphi}(S) \neq \emptyset$ for some $g \in G$. Then $[g, \varphi_n(s_1)] = [e, \varphi_n(s_2)]$ for all $n \in \mathbb{N}$ and for some $s_1, s_2 \in S$, where $\varphi_n = \text{pr}_n \circ \varphi$. But then $g \in H$.

If we now assume that $g \in G_{\psi(gs)}$ for some $gs \in GS$, then $g\tilde{\psi}(gs) = \psi(gs)$, giving $g\tilde{\varphi}(s) = g\tilde{\varphi}(s)$; hence $g^{-1}g\varphi(s) = \varphi(s)$ and thus by the previous argument $g^{-1}gs = H_s$. But then $g^{-1}gs = s$; hence $gs = gs$ and thus $g \in G_{gs}$. It follows that $G_{\psi(gs)} = G_{gs}$ for all $gs \in GS$ and thus $\psi$ is isovariant.

Now $G \times_H E_n$ is a proper $G$-space by \[\text{[E1]}\] Proposition 1.3, it is second countable and it is a manifold because $G \times_H E_n \to G/H$ is a vector bundle by \[\text{[Kaw]}\] Theorem 2.26. It has a metric with the property $(*)$ because any second countable manifold can be embedded as a closed subset of some Euclidean space (see \[\text{[H-W]}\] Theorem V.3) for the embedding theorem, and see \[\text{[Br]}\] Chapter III, Corollary 10.2.
for the closedness). Note that this embedding does not need to be a $G$-embedding, as the condition (*) on the metric is independent of the action of $G$.

Furthermore, $(G \times_H E_n)/G$ is metrizable by [Pa2] Theorem 4.3.4, giving $G \times_H E_n \in \mathcal{M}_G$. Hence, by Theorem 5.3, we obtain a $G$-embedding $G \times_H E_n \rightarrow B_n$, where $B_n$ is a Banach $G$-space and $G$ acts properly on $B_n \setminus \{0\}$. This gives an isovariant $G$-map

$$\tilde{\psi}: X = GS \rightarrow N \rightarrow \prod_{n \in N} G \times_H E_n \rightarrow \prod_{n \in N} B_n =: \tilde{Z},$$

where $\tilde{\psi}(GS) \subset \tilde{Z} \setminus \{0\}$.

**Lemma 5.7.** There is a $G$-invariant metric $d$ on $\tilde{Z}$, which induces a pseudometric $\tilde{d}$ on $\tilde{Z}/G$.

**Proof.** We have $\tilde{Z} = \Pi_{n \in N} B_n$ where each $B_n$ is a Banach $G$-space with a $G$-invariant metric $d_n$ induced by the norm as noted in Remark 5.5. In each $B_n$ we define a new metric $e_n$ by setting $e_n(x, y) = \min\{d_n(x, y), \frac{1}{n}\}$. The metric $e_n$ is equivalent to $d_n$ by [Du] Chapter IX, Theorem 3.3] and $e_n$ is $G$-invariant because each $d_n$ is so.

Denote by $\pi_m : \tilde{Z} = \Pi_{n \in N} B_n \rightarrow B_m$ the $m$th projection; we define a metric $d : \tilde{Z} \times \tilde{Z} \rightarrow \mathbb{R}$ by setting

$$d(z, z') = \sup_{m \in N} e_m(\pi_m(z), \pi_m(z')).$$

The map $d$ metrizes the product topology on $\tilde{Z}$ by [Du] Chapter IX, Theorem 7.2] since $e_n(B_n) \rightarrow 0$ as $n \rightarrow \infty$, and $d$ is $G$-invariant because each $e_n$ is so.

The metric $d$ induces a pseudometric $\tilde{d}$ on $\tilde{Z}/G$ by $\tilde{d}(\tilde{x}, \tilde{y}) = d(Gx, Gy)$, where $\tilde{x} = \pi(x)$, $\tilde{y} = \pi(y)$, and $\pi : \tilde{Z} \rightarrow \tilde{Z}/G$ is the canonical projection. The pseudometric $\tilde{d}$ induces the quotient topology on $\tilde{Z}/G$ since the arguments

$$y \in B_d(x, r) \Rightarrow \tilde{d}(\tilde{x}, \tilde{y}) \leq d(x, y) < r \Rightarrow \pi(y) = \tilde{y} \in B_{\tilde{d}}(\tilde{x}, r)$$

and

$$\tilde{y} \in B_{\tilde{d}}(\tilde{x}, r) \Rightarrow \inf_{g \in G} d(x, gy) = \tilde{d}(\tilde{x}, \tilde{y}) < r \Rightarrow gy \in B_d(x, r) \text{ for some } g \in G$$

$$\Rightarrow \tilde{y} = \pi(gy) \in \pi B_d(x, r)$$

imply that $\pi B_d(x, r) = B_{\tilde{d}}(\tilde{x}, r)$.

The pseudometric $\tilde{d}$ restricted to the set $(\tilde{Z} \setminus \{0\})/G$ is the same as the pseudometric induced by the metric $d$ restricted to $\tilde{Z} \setminus \{0\}$; hence the restriction of $\tilde{d}$ is a pseudometric inducing the quotient topology on $(\tilde{Z} \setminus \{0\})/G$, denoted by $\tilde{d}$.

**Lemma 5.8.** $G$ acts properly on $\tilde{Z} \setminus \{0\}$.

**Proof.** $G$ acts properly on $B_n \setminus \{0\}$ for each $n \in N$ and hence the space $\tilde{Z}_n = \left( \prod_{i=1}^{n-1} B_i \right) \times (B_n \setminus \{0\}) \times \left( \prod_{i=n+1}^{\infty} B_i \right)$ is a proper $G$-space for all $n \in N$. Furthermore $\tilde{Z} \setminus \{0\} = \bigcup_{n \in \mathbb{N}} \tilde{Z}_n$. Now $\tilde{Z} \setminus \{0\}$ is a Cartan $G$-space since any completely regular $G$-space which is the union of open Cartan $G$-subspaces is a Cartan $G$-space. Since $(\tilde{Z} \setminus \{0\})/G$ admits a pseudometric $\tilde{d}$, it is regular, and it follows that the action of $G$ on $\tilde{Z} \setminus \{0\}$ is Palais proper by [Pa2] Proposition 1.2.5], so it is certainly proper. \qed
Lemma 5.9. The pseudometric $\bar{d}$ on $(\tilde{Z} \setminus \{0\})/G$ is a metric.

Proof. Since $G$ acts properly on $\tilde{Z} \setminus \{0\}$ by Lemma 5.8, the quotient space $(\tilde{Z} \setminus \{0\})/G$ is Hausdorff by [Pa2, Proposition 1.1.4], and we have seen that $\bar{d}$ induces this topology. But then $\bar{d}$ must be a metric.

Next we pass from the global slice situation above to the more complicated situation with arbitrary spaces from $\mathcal{M}_G$.

Lemma 5.10. Let $X$ be as in Theorem 5.1. There exists an LCL G-space $Z$ and an isovariant G-map $\tilde{f}: X \to Z$ such that $\tilde{f}(X) \subset Z \setminus \{0\}$ and $Z \setminus \{0\} \in \mathcal{M}_G$.

Proof. By Lemma 3.1 we may assume that $\{GS_n\}_{n \in \mathbb{N}}$ is a locally finite tubular covering of $X$. By Lemma 2.4 and by the normality of $X/G$ we may let $\{GR_n\}_{n \in \mathbb{N}}$ and $\{GT_n\}_{n \in \mathbb{N}}$ be similar coverings such that $GT_n \subset GR_n \subset GR_n \subset GS_n$ for each $n \in \mathbb{N}$. According to the discussion above each $GS_n$ admits an isovariant G-map $\tilde{f}_n$ into an LCL G-space $Z_n$, where $\tilde{f}_n(GS_n) \subset Z_n \setminus \{0\}$, and $G$ acts properly on $Z_n \setminus \{0\}$. For each $n \in \mathbb{N}$ we may construct a $G$-invariant map $\lambda_n: X \to I$ such that $\lambda_n(GT_n) = \{1\}$ and $\lambda_n(X \setminus GR_n) = \{0\}$, since $X/G$ is normal.

We obtain a continuous and isovariant G-map $\tilde{f}: X \to \prod_{n \in \mathbb{N}} Z_n =: Z$ by setting $\tilde{f}(x) = (\lambda_1(x)\tilde{f}_1(x), \ldots, \lambda_i(x)\tilde{f}_i(x), \ldots)$.

Clearly $\tilde{f}(X) \subset Z \setminus \{0\}$, and $Z$, being a product of LCL G-spaces, is an LCL G-space by Lemma 5.9.

Furthermore, $Z$ admits a $G$-invariant metric which induces a pseudometric on $Z/G$ just as in Lemma 5.7. $G$ acts properly on $Z \setminus \{0\}$ as in Lemma 5.8 and the pseudometric on $Z/G$ restricts to a metric on $(Z \setminus \{0\})/G$ as in Lemma 5.9. In other words, $Z \setminus \{0\} \in \mathcal{M}_G$.

Lemma 5.11. With $X$ as in Theorem 5.1, there exists a G-embedding $e: X \to L$ where $L$ is an LCL G-space, $e(X) \subset C$ for some convex G-subset $C$ of $L$ and $e(X)$ has a $G$-neighborhood $V$ in $C$ such that $V \in \mathcal{M}_G$.

Proof. Let $h: X/G \to B$ be an embedding of the metrizable space $X/G$ into a Banach space $B$ such that $h(X/G)$ is a closed subset of $C'$ where $C'$ is a convex subset of $B$. Such a map exists by the Wojdyslawski embedding theorem. With $Z$ as in Lemma 5.10 define a map $e: X \to X \times B =: X$ by setting $e = (\tilde{f}, h \circ \pi_X)$ where $\tilde{f}$ is the isovariant G-map obtained in Lemma 5.10 and $\pi_X: X \to X/G$ is the natural projection. Let $G$ act trivially on $B$. Now we have

i) $e$ is an isovariant G-map because $\tilde{f}$ is isovariant.

ii) $e$ is injective since the induced map $\bar{e}: X \to Z/G \times B$ and $e$ is injective. The map $\bar{e}$ is injective because $h$ is injective.

iii) The map $\bar{e}$ is a homeomorphism onto its image, whose inverse $\bar{e}^{-1}: \bar{e}(X/G) \to X/G$ is given by $h^{-1} \circ pr_2$, where $\pi_Z: Z \to Z/G$ is the natural projection and $pr_2: Z \times B \to B$ is the projection to the second coordinate.

iv) $e(X) \subset Z \times C' = C$, which is convex, and $G$ acts properly on $Z \setminus \{0\} \times C' = V$, which is a neighborhood of $e(X)$ in $C$.

Now $e(X) \subset V$, which is proper, and in particular $e(X)$ is Cartan. Thus by Lemma 4.2 the map $e$ is a homeomorphism onto its image. The space $V/G = (Z \setminus \{0\})/G \times C'$ is metrizable; thus $V \in \mathcal{M}_G$. □
Lemma 5.12. The image $e(X)$ is a closed subset of $Z \times C'$.

Proof. Assume that $y = (u, v) \in (Z \times C') \setminus e(X)$. In case $v \not\in h(X/G)$, since $h$ is a closed embedding, there exists a neighborhood $U$ of $v$ in $C'$ such that $U \cap h(X/G) = \emptyset$. Thus $Z \times U$ is a neighborhood of $(u, v)$ which does not intersect $e(X)$.

In case $v \in h(X/G)$ we have $v = h(\pi_X(x))$ for some $x \in X$ and then we must have $Gu \cap G\tilde{f}(x) = \emptyset$ (if not, then $u = g\tilde{f}(x) = \tilde{f}(gx)$ for some $g \in G$, giving $v = h(\pi_X(x)) = h(\pi_X(gx))$, which implies $y = (u, v) = e(gx) \in e(X)$).

If $u \neq 0$, then $u$ and $\tilde{f}(x)$ are points in the open subset $Z \setminus \{0\}$ of $Z$, where $(Z \setminus \{0\})/G$ is metrizable. Thus there exist disjoint $G$-neighborhoods $U$ and $W$ of $Gu$ and $G\tilde{f}(x)$ in $Z \setminus \{0\}$ and hence in $Z$. If $u = 0$, then $\pi_Z(u)$ is closed in $Z/G$, which is pseudometrizable and hence regular, so there exist again disjoint $G$-neighborhoods $U$ and $W$ of $Gu$ and $G\tilde{f}(x)$ in $Z$. Thus we see that in any case there exist disjoint $G$-neighborhoods $U$ and $W$ of $Gu$ and $G\tilde{f}(x)$ in $Z$.

Since $\tilde{f}$ is continuous and equivariant there exists a $G$-neighborhood $\tilde{W}$ of $Gx$ in $X$ such that $\tilde{f}(\tilde{W}) \subset W$. Since $\pi_X$ is an open map and $h$ is an embedding there exists an open neighborhood $M$ of $h(\pi_X(x))$ in $C'$ such that $M \cap h(\pi_X(X)) = h(\pi_X(\tilde{W}))$.

Clearly $U \times M$ is an open neighborhood of $y$ in $Z \times C'$; we show that it is disjoint from $e(X)$. If $(x, u) \in e(X) \cap (U \times M)$, then $x = \tilde{f}(z) \in U$ and $w = h(\pi_X(z)) \in M$ for some $z \in X$. But then $\tilde{f}(z) \not\in \tilde{W}$ giving $z \not\in \tilde{W}$ and since $\tilde{W}$ is $G$-invariant we have $\pi_X(z) \not\in \pi_X(\tilde{W})$, which gives $h(\pi_X(z)) \not\in h(\pi_X(\tilde{W}))$ since $h$ is injective.

However, $Z \times X$ and $h(\pi_X(z)) \in M$ implies that $h(\pi_X(z)) \in M \cap h(\pi_X(X)) = h(\pi_X(\tilde{W}))$, which gives a contradiction. Hence we must have $e(X) \cap (U \times M) = \emptyset$.

It follows that $e(X)$ is closed in $Z \times C'$. □

The rest of the proof of Theorem 5.1 If we now set $L = Z \times B$, then $L$ is an LCL $G$-space. Set $C = Z \times C'$ and $V = (Z \setminus \{0\}) \times C'$ as before. Then $V$ is a $G$-neighborhood of $e(X)$ in $C$, $C$ is a $G$-invariant convex subset of $L$, $e(X)$ is closed in $V$, and $V \in \mathcal{M}_G$. Hence the theorem is true. □

6. Application

We say that a $G$-space $Y$ is a $G$-equivariant absolute neighborhood extensor ($G$-ANE) for $\mathcal{M}_G$, written $Y \in G$-ANE-$\mathcal{M}_G$, if for any $G$-space $X \in \mathcal{M}_G$ and any closed $G$-invariant subset $A$ in $X$ with a $G$-map $f: A \to Y$ there exists a $G$-extension $F: U \to Y$ of $f$ over some $G$-neighborhood $U$ of $A$ in $X$.

We say that a $G$-space $X \in \mathcal{M}_G$ is a $G$-equivariant absolute neighborhood retract ($G$-ANR) for $\mathcal{M}_G$, written $X \in G$-ANR-$\mathcal{M}_G$, if, whenever there exists a closed $G$-embedding $i: X \to Y$ of $X$ into some $G$-space $Y \in \mathcal{M}_G$, then there exists a $G$-neighborhood retraction $r: U \to i(X)$ where $U$ is a $G$-neighborhood of $i(X)$ in $Y$.

It is easy to show that then $\mathcal{M}_G \cap G$-ANE-$\mathcal{M}_G \subset G$-ANR-$\mathcal{M}_G$. Here we show that the two classes are the same, following the classical proof from the non-equivariant case (see, for instance, [Hu Chapter III, Theorem 3.2]).

Corollary 6.1. $G$-ANR-$\mathcal{M}_G = \mathcal{M}_G \cap G$-ANE-$\mathcal{M}_G$.

Proof. We should show that $G$-ANR-$\mathcal{M}_G \subset \mathcal{M}_G \cap G$-ANE-$\mathcal{M}_G$. Suppose that $Y \in G$-ANR-$\mathcal{M}_G$. Then $Y \in \mathcal{M}_G$ by definition. By Theorem 5.1 we may assume that $Y$ is a closed $G$-subset of a convex $G$-subset $C$ of some LCL $G$-space, where $Y$
EMBEDDING OF METRIZABLE $G$-SPACES IN LINEAR $G$-SPACES

has a $G$-neighborhood $U$ in $C$ such that $U \in \mathcal{M}_G$. Since $Y \in G$-ANR-$\mathcal{M}_G$, there exists a $G$-neighborhood $V$ of $Y$ in $U$ (and hence in $C$) and a $G$-retraction $r: V \to Y$. Let $X \in \mathcal{M}_G$ and let $A$ be a closed $G$-invariant subset of $X$ with a $G$-map $f: A \to Y$. By the equivariant Dugundji extension theorem [An3 Corollary 1] we know that $C \in G$-ANE-$\mathcal{M}_G$; hence the map $i \circ f: A \to Y \hookrightarrow C$ admits a $G$-extension $h: W \to C$, where $W$ is a $G$-neighborhood of $A$ in $X$. Set $W' = W \cap h^{-1}(V)$. Now $W'$ is a $G$-neighborhood of $A$ in $X$ and the map $r \circ h|W': W' \to V \hookrightarrow Y$ is a neighborhood $G$-extension of $f$. It follows that $Y \in G$-ANE-$\mathcal{M}_G$. □

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REFERENCES

[An1] S. Antonyan. Equivariant embeddings into $G$-ARs. Glas. Mat. Ser. III, 22(42)(2):503–533, 1987. MR097632 (89k:54041)

[An2] S. Antonyan. Retraction properties of an orbit space. Math. USSR-Sb., 65(2):305–321, 1990. MR0976513 (89k:54042)

[An3] S. Antonyan. Extensorial properties of orbit spaces of proper group actions. Topology Appl., 98(1-3):35–46, 1999. MR1719992

[An4] S. Antonyan. Universal proper $G$-spaces. Topology Appl., 117(1):23–43, 2002. MR1874002

[An-Ne] S. Antonyan and S. de Neymet. Invariant pseudometrics on Palais proper $G$-spaces. Acta Math. Hung., 98(1-2):59–69, 2003. MR1958466 (2003m:22025)

[Br] G. Bredon. Introduction to compact transformation groups. Academic Press, New York, 1972. MR0413144 (54:1265)

[Du] J. Dugundji. Topology. Allyn and Bacon Inc., Boston, Mass., 1966. MR0193606 (33:1824)

[E1] E. Elfving. The $G$-homotopy type of proper locally linear $G$-manifolds. Ann. Acad. Sci. Fenn. Math. Diss., 108, 1996. MR1413841 (97g:57055)

[E2] E. Elfving. The $G$-homotopy type of proper locally linear $G$-manifolds. II. Manuscripta Math., 105(2):235–251, 2001. MR1846619 (2002c:57053)

[Hu] S. T. Hu. Theory of retracts. Wayne State University Press, Detroit, 1965. MR0181977 (31:6202)

[H-W] W. Hurewicz and H. Wallman. Dimension Theory. Princeton University Press, Princeton, NJ, 1941. MR0006493 (3:312b)

[Ja] J. Jaworowski. $G$-spaces with a finite structure and their embedding in $G$-vector spaces. Acta Math. Acad. Sci. Hungar., 39(1-3):175–177, 1982. MR0653689 (83h:57040)

[Kaw] K. Kawakubo. The theory of transformation groups. The Clarendon Press, Oxford University Press, New York, 1991. MR1150492 (93g:57044)

[Pa1] R. Palais. The classification of $G$-spaces. Mem. Amer. Math. Soc. No. 36, 1960. MR01177401 (31:1664)

[Pa2] R. Palais. On the existence of slices for actions of non-compact Lie groups. Ann. of Math. (2), 73:295–323, 1961. MR0126506 (23:A3802)

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