Viscosity Solutions of Hamilton–Jacobi Equations for Neutral-Type Systems

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Abstract

The paper deals with path-dependent Hamilton–Jacobi equations with a coinvariant derivative which arise in investigations of optimal control problems and differential games for neutral-type systems in Hale’s form. A viscosity (generalized) solution of a Cauchy problem for such equations is considered. The existence, uniqueness, and consistency of the viscosity solution are proved. Equivalent definitions of the viscosity solution, including the definitions of minimax and Dini solutions, are obtained. Application of the results to an optimal control problem for neutral-type systems in Hale’s form are given.

Keywords Neutral-type systems · Hamilton–Jacobi equations · Coinvariant derivatives · Viscosity solutions · Minimax solutions · Optimal control problems

Mathematics Subject Classification 49L20 · 49L25 · 34K40

1 Introduction

The paper aims to develop the viscosity solution theory for path-dependent Hamilton–Jacobi (HJ) equations with a coinvariant derivative which arise in optimal control problems and differential games for neutral-type systems in Hale’s form [12].

Initially, the notion of coinvariant derivatives and path-dependent HJ equations with such derivatives were considered in [15] to describe infinitesimal properties of value functionals in optimal control problems for time-delay systems. Also, similar

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derivative notions are known such as Clio derivatives [1] and horizontal and vertical derivatives [7] (the connection between these notions was addressed in [10]). As with HJ equations with partial derivatives, path-dependent HJ equations with coinvariant derivatives may not have a differentiable solution and various approaches to the definition of a generalized solution needed to be examined.

The minimax approach [30, 31, 33] and its application to differential games [16, 17] for time-delay systems were developed in [18–20] (see also [2] for an extension of these results to an infinite dimensional setting). Investigations of the viscosity approach [5, 6] to the generalized solution definition for path-dependent HJ equations associated with time-delay systems can be roughly divided into two groups. In the first group [2, 8, 13, 14, 21, 25, 29, 34], modified definitions of the viscosity solution based on some parameterizations were researched. In the second group [26, 27, 35], more natural definitions of the viscosity solutions are used, but the path-dependent HJ equations are considered on the wider spaces of discontinuous functions.

In contrast to path-dependent HJ equations arising in optimal control and differential game problems for time-delay systems, path-dependent HJ equations associated with the problems for more general neutral-type systems in Hale’s form have a new term. The fact that this term is not defined at all points of the functional space and discontinuous at some points of the domain does not allow us to apply the above results and requires us to construct the new theory of generalized solutions.

The minimax solution theory for such equations and the application of this theory to differential games for neutral-type systems in Hale’s form were investigated in [11, 22, 23, 28]. The present paper develops the viscosity solution theory for such equations. Following [26, 27], we consider the path-dependent HJ equations on the space of discontinuous functions and introduce the definition of a viscosity solution of a Cauchy problem for this equation. We prove that the viscosity solution exists and is unique (see Theorem 1). We obtain additional equivalent definitions of the viscosity (generalized) solution including the definitions of minimax and Dini solutions (see Theorem 4). We also establish that a coinvariant differentiable solution of the Cauchy problem coincides with the viscosity solution and, on the other hand, the viscosity solution at the points of coinvariant differentiability satisfies the HJ equation (see Theorem 5). Moreover, we consider an application of the results to an optimal control problem for a neutral-type system in Hale’s form (see Theorem 6).

The main idea of the proofs is to obtain the existence and uniqueness of the minimax solution based on results from [28] and to establish the equivalence of the minimax and viscosity solutions using the scheme from [4, 32, 33] (see also [26]). However, there are several obstacles to implement that. Firstly, since the minimax solution from [28] is defined on the space of Lipschitz continuous functions, we prove the certain Lipschitz continuous property of this solution and, using that, extend the minimax solution to the space of piecewise Lipschitz continuous functions (see Theorems 2 and 3 (a) ⇔ (b)). Secondly, as noted above, the path-dependent HJ equations under consideration have the special term which is not defined on the whole space of piecewise Lipschitz continuous functions. Therefore, we introduce the definitions of a viscosity solution (see Definition 1) only on a certain subspace on which this term is defined. Such a definition seems one of the main features of the present paper, since usually the viscosity solution definitions for even more particular path-dependent HJ equations
were considered on whole functional spaces. This definition does not allow us to directly apply the scheme from [32, 33], but we overcome this obstacle by introducing an additional auxiliary definition of minimax solution on the space of continuously differentiable functions (C¹-minimax solution in Definition 4). The fact that the C¹-minimax solution coincides with the usual minimax solution (see Theorem 3 (b) ⇔ (c)) completes the proof of the equivalence of the minimax and viscosity solutions (see Theorem 4). Thirdly, the extension to the space of piecewise Lipschitz continuous functions does not allow us to expect a continuous solution, as opposed to [11, 22, 28]. Nevertheless, the class of, generally speaking, discontinuous functionals, suggested in the paper, is suitable for obtaining the existence and uniqueness results within this class.

2 Main Results

2.1 Functional Spaces

Let \( \mathbb{R}^n \) be the \( n \)-dimensional Euclidean space with the inner product \( \langle \cdot, \cdot \rangle \) and the norm \( \| \cdot \| \). A function \( x(\cdot): [a, b) \mapsto \mathbb{R}^n \) (or \( x(\cdot): [a, b] \mapsto \mathbb{R}^n \)) is called piecewise Lipschitz continuous if there exist points \( a = \xi_1 < \xi_2 < \ldots < \xi_k = b \) such that the function \( x(\cdot) \) is Lipschitz continuous on the interval \([\xi_i, \xi_{i+1})\) for each \( i \in 1, k - 1 \). Note that such a function \( x(\cdot) \) is right continuous on the interval \( \xi \in [a, b) \) and has a finite left limit \( x(\xi - 0) \) for any \( \xi \in (a, b] \). Denote by \( \text{PLip}(\{-h, 0\}, \mathbb{R}^n) \) (or \( \text{PLip}([a, b], \mathbb{R}^n) \)) the linear space of piecewise Lipschitz continuous functions \( x(\cdot): [a, b) \mapsto \mathbb{R}^n \) (or \( x(\cdot): [a, b] \mapsto \mathbb{R}^n \)).

Let \( \vartheta, h > 0 \). Denote

\[
\text{PLip} = \text{PLip}((-h, 0), \mathbb{R}^n), \quad \text{Lip} = \text{Lip}((-h, 0), \mathbb{R}^n), \quad \text{C}^1 = \text{C}^1((-h, 0), \mathbb{R}^n). 
\]

where \( \text{Lip}((-h, 0), \mathbb{R}^n) \) and \( \text{C}^1((-h, 0), \mathbb{R}^n) \) are the linear spaces of Lipschitz continuous and continuously differentiable functions \( x(\cdot): [a, b) \mapsto \mathbb{R}^n \) with a finite limit \( x(-0) \). Denote

\[
\text{PLip}_* = \{ w(\cdot) \in \text{PLip}: \text{there exists } \delta_w > 0 \text{ such that } w(\cdot) \text{ is continuously differentiable on } [-h, -h + \delta_w] \}. 
\]

Note that the following inclusions are valid:

\[
\text{C}^1 \subset \text{Lip} \subset \text{PLip}, \quad \text{C}^1 \subset \text{PLip}_* \subset \text{PLip}. 
\]

For the sake of brevity, for any \( w(\cdot) \in \text{PLip} \), we denote

\[
\| w(\cdot) \|_1 = \int_{-h}^{0} \| w(\xi) \| d\xi, \quad \| w(\cdot) \|_\infty = \sup_{\xi \in [-h, 0)} \| w(\xi) \|, \quad w(-0) = w(0 - 0). 
\]
Without loss of generality of the results presented below, we can suppose the existence of \( I \in \mathbb{N} \) such that \( \vartheta = Ih \). Define the spaces

\[
\mathbb{G} = [0, \vartheta] \times \mathbb{R}^n \times \text{PLip}, \quad \mathbb{G}_\vartheta = \bigcup_{i=0}^{Ih-1} (ih, (i+1)h) \times \mathbb{R}^n \times \text{PLip}_\vartheta.
\]

### 2.2 Hamilton–Jacobi Equation

For each \((\tau, z, w(\cdot)) \in \mathbb{G}\), denote

\[
\Lambda(\tau, z, w(\cdot)) = \{ x(\cdot) \in \text{PLip}([\tau - h, \tau], \mathbb{R}^n) : x(\tau) = z, x(t) = w(t - \tau), t \in [\tau - h, \tau) \}.
\]

Following \([26, 27]\) (see also \([15, 18]\)), a functional \( \varphi : \mathbb{G} \mapsto \mathbb{R} \) is called coinvariantly (ci-) differentiable at a point \((\tau, z, w(\cdot)) \in \mathbb{G}, \tau < \vartheta \) if there exist

\[
\partial^ci_{\tau,w} \varphi(\tau, z, w(\cdot)) \in \mathbb{R} \quad \text{and} \quad \nabla_z \varphi(\tau, z, w(\cdot)) \in \mathbb{R}^n
\]

such that, for every \( t \in [\tau, \vartheta] \), \( y \in \mathbb{R}^n \), and \( x(\cdot) \in \Lambda(\tau, z, w(\cdot)) \), the relation below holds

\[
\varphi(t, y, x_t(\cdot)) - \varphi(\tau, z, w(\cdot)) = (t - \tau) \partial^ci_{\tau,w} \varphi(\tau, z, w(\cdot)) + (y - z, \nabla_z \varphi(\tau, z, w(\cdot))) + o(|t|\vartheta)
\]

where \( x_t(\cdot) \) denotes the function from PLip such that \( x_t(\xi) = x(t + \xi), \xi \in [-h, 0) \) and the value \( o(\delta) \) can depend on \( x(\cdot) \) and \( o(\delta) / \delta \to 0 \) as \( \delta \to 0 \). Then \( \partial^ci_{\tau,w} \varphi(\tau, z, w(\cdot)) \) is the ci-derivative of \( \varphi \) with respect to \( \{t, w(\cdot)\} \) and \( \nabla_z \varphi(\tau, z, w(\cdot)) \) is the gradient of \( \varphi \) with respect to \( z \).

Similarly, the mapping \( \mathbb{G} \ni (\tau, z, w(\cdot)) \mapsto \phi = (\phi_1, \ldots, \phi_n) \in \mathbb{R}^n \) is called ci-differentiable at a point \((\tau, z, w(\cdot)) \in \mathbb{G}, \tau < \vartheta \), if the functionals \( \phi_i : \mathbb{G} \mapsto \mathbb{R} \), \( i = 1, n \) are ci-differentiable at this point.

Let us fix the functions \( g : [0, \vartheta] \times \mathbb{R}^n \mapsto \mathbb{R}^n \) and \( H : [0, \vartheta] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R} \) and the mapping \( \sigma : \mathbb{R}^n \times \text{PLip} \mapsto \mathbb{R} \) satisfying the following conditions:

1. (g) The function \( g \) is continuously differentiable.
2. (H1) The function \( H \) is continuous.
3. (H2) There exists a constant \( c_H > 0 \) such that

\[
|H(\tau, z, r, s) - H(\tau, z, r, s')| \leq c_H(1 + \|z\| + \|r\|)\|s - s'\|
\]

for any \( \tau \in [0, \vartheta], z, r, s, s' \in \mathbb{R}^n \).
4. (H3) For every \( \alpha > 0 \), there exists \( \lambda_H = \lambda_H(\alpha) > 0 \) such that

\[
|H(\tau, z, r, s) - H(\tau, z', r', s)| \leq \lambda_H(\|z - z'\| + \|r - r'\|)(1 + \|s\|)
\]

for any \( \tau \in [0, \vartheta], z, r, z', r', s \in \mathbb{R}^n: \max(\|z\|, \|r\|, \|z'\|, \|r'\|) \leq \alpha \).
5. (s) For every \( \alpha > 0 \), there exists \( \lambda_\sigma = \lambda_\sigma(\alpha) > 0 \) such that

\[
|\sigma(z, w(\cdot)) - \sigma(z', w'(\cdot))| \leq \lambda_\sigma(\|z - z'\| + \|w(\cdot) - w'(\cdot)\|)
\]
for any \((z, w(\cdot)), (z’, w’(\cdot)) \in P(\alpha)\), where

\[
P(\alpha) = \{(z, w(\cdot)) \in \mathbb{R}^n \times \text{PLip}: \|z\| \leq \alpha, \|w(\cdot)\|_\infty \leq \alpha\}.
\] (5)

Consider the mapping \(g_*(\tau, z, w(\cdot)) = g(\tau, w(-h)), (\tau, z, w(\cdot)) \in \mathbb{G}\). Due to condition \((g)\), \(g_*\) is \(C^1\)-differentiable on \(\mathbb{G}_*\) (see (3)) and

\[
\partial_{\tau, w}^c g_*(\tau, z, w(\cdot)) = G(\tau, w(-h)), \quad \partial^+ w(-h)/d\xi, \quad \nabla z g_*(\tau, z, w(\cdot)) = 0
\] (6)

for any \((\tau, z, w(\cdot)) \in \mathbb{G}_*\), where \(d^+ w(-h)/d\xi\) is the right derivative of the function \(w(\xi), \xi \in [-h, 0]\) at the point \(\xi = -h\) and \(G(\tau, x, y) = \partial g(\tau, x)/\partial x + \nabla_x g(\tau, x)y\).

Since the mapping \(g_*\) is determined by the function \(g\) and does not depend on \(z\), for brevity, we denote

\[
\partial_{\tau, w}^c g(\tau, w(\cdot)) = \partial_{\tau, w}^c g_*(\tau, z, w(\cdot)).
\] (7)

For the functional \(\varphi: \mathbb{G} \mapsto \mathbb{R}\), let us consider the Cauchy problem for the HJ equation

\[
\begin{align*}
\partial_{\tau, w}^c \varphi(\tau, z, w(\cdot)) + \langle \partial_{\tau, w}^c g(\tau, w(\cdot)), \nabla z \varphi(\tau, z, w(\cdot)) \rangle + H(\tau, z, w(-h), \nabla z \varphi(\tau, z, w(\cdot))) = 0, \\
\varphi(\vartheta, z, w(\cdot)) = \sigma(z, w(\cdot)), \\
(\tau, z, w(\cdot)) \in \mathbb{G}_*,
\end{align*}
\] (8)

and the terminal condition

\[
\varphi(\vartheta, z, w(\cdot)) = \sigma(z, w(\cdot)), \quad (z, w(\cdot)) \in \mathbb{R}^n \times \text{PLip}.
\] (9)

**Remark 1** Such Cauchy problems arise in investigations of optimal control problems and differential games for neutral-type systems in Hale’s from (see, e.g., [11]). In contrast to Cauchy problems corresponding to time-delay systems [18–21], the new term \(\partial_{\tau, w}^c g(\tau, w(\cdot))\) appears. Since the functional \(\partial_{\tau, w}^c g(\tau, w(\cdot))\) depends on the derivative \(d^+ w(-h)/d\xi\) (see (6), (7)), it is not defined at all points of \(\mathbb{G}\) and discontinuous with respect to the uniform norm. Thus, we can not apply results from [18–21] to Cauchy problem (8), (9). Moreover, we need to consider the HJ equation only on a set on which \(\partial_{\tau, w}^c g(\tau, w(\cdot))\) is defined. We choose \(\mathbb{G}_*\) as such a set. Due to condition \((g)\), the value \(\partial_{\tau, w}^c g(\tau, w(\cdot))\) is defined on \(\mathbb{G}_*\).

We search a solution of this problem among the functionals \(\varphi: \mathbb{G} \mapsto \mathbb{R}\) satisfying the following conditions:

1. (\(\varphi_1\)) For each \((\tau, w(\cdot)) \in [0, \vartheta] \times \text{Lip}\), the function \(\overline{\varphi}(t) = \varphi(t, w(-0), w(\cdot))\), \(t \in [\tau, \vartheta]\) is continuous.
2. (\(\varphi_2\)) For every \(\alpha > 0\), there exists \(\lambda_{\varphi} = \lambda_{\varphi}(\alpha) > 0\) such that

\[
|\varphi(\tau, z, w(\cdot)) - \varphi(\tau, z’, w’(\cdot))| \leq \lambda_{\varphi} v(\tau, z - z’, w(\cdot) - w’(\cdot))
\] (10)

for any \(\tau \in [0, \vartheta]\) and \((z, w(\cdot)), (z’, w’(\cdot)) \in P(\alpha)\) (see (5)), where

\[
v(\tau, z, w(\cdot)) = \|z\| + \|w(\cdot)\|_1 + \|w(-h)\| + \|w(\cdot) - \tau\|
\] (11)
Remark 2  Note that if \( \varphi: \mathbb{G} \mapsto \mathbb{R} \) satisfies conditions \((\varphi_1), (\varphi_2)\), then the functional 
\[ \hat{\varphi}(\tau, w(\cdot)) = \varphi(\tau, w(-0), w(\cdot)), (\tau, w(\cdot)) \in [0, \vartheta] \times \text{Lip} \]
is continuous with respect to uniform norm. Thus, these conditions are consistent with prior works \([11, 22, 28]\) in which solutions of HJ equations for neutral-type systems on the space of Lipschitz continuous functions were considered in the class of continuous functionals. Moreover, these conditions are an analog of the conditions suggested in \([26, 27]\), devoted to viscosity solutions of HJ equations for time-delay systems, with additional terms \(\|w(-h)\|\) and \(\|w(ih - \tau)\|\).

2.3 Viscosity Solution

Denote

\[ A_0(\tau, z, w(\cdot)) = \{x(\cdot) \in A(\tau, z, w(\cdot)) : x(t) = z, t \in [\tau, \vartheta] \}, \quad (\tau, z, w(\cdot)) \in \mathbb{G}. \]

Remark 3  Similar to \([27]\), note that if a functional \( \varphi: \mathbb{G} \mapsto \mathbb{R} \) satisfies conditions \((\varphi_1), (\varphi_2)\), is ci-differentiable at a point \((\tau, z, w(\cdot)) \in \mathbb{G}_s\), and satisfies HJ equation \((8)\) at this point then the function

\[ \tilde{\varphi}(t, x) = \varphi(t, x, \kappa_t(\cdot)), \quad (t, x) \in [\tau, \vartheta] \times \mathbb{R}^n, \quad \kappa(\cdot) \in A_0(\tau, z, w(\cdot)), \]

has a right partial derivative \(\partial^+ \tilde{\varphi}(\tau, z) / \partial \tau\) and a gradient \(\nabla_z \tilde{\varphi}(\tau, z)\) at the point \((\tau, z)\), and satisfies the following HJ equation:

\[ \partial^+ \tilde{\varphi}(\tau, z) / \partial \tau + (\partial_{\tau,w}^ig(\tau, w(\cdot)), \nabla_z \tilde{\varphi}(\tau, z)) + H(\tau, z, w(-h), \nabla_z \tilde{\varphi}(\tau, z)) = 0. \]

Thus, we might say that HJ equation with a ci-derivative \((8)\) is locally this HJ equation with partial derivatives.

Then, based on the classical definition of viscosity solutions \([5]\), it leads us in a natural way to the following definition.

Definition 1  A functional \( \varphi: \mathbb{G} \mapsto \mathbb{R} \) is called a viscosity solution of problem \((8), (9)\) if \( \varphi \) satisfies conditions \((\varphi_1), (\varphi_2), (9)\) and conditions

\begin{align*}
\text{for every } (\tau, z, w(\cdot)) \in \mathbb{G}_s, \quad \psi \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), \quad \text{and } \delta > 0, \\
&\text{if } \varphi(\tau, z, w(\cdot)) - \psi(\tau, z) \leq \varphi(t, x, \kappa_t(\cdot)) - \psi(t, x) \\
&\text{for any } (t, x) \in O^+_\delta(\tau, z), \quad \kappa(\cdot) \in A_0(\tau, z, w(\cdot)), \\
&\text{then } \partial \psi(\tau, z) / \partial t + (\partial_{\tau,w}^ig(\tau, w(\cdot)), \nabla_z \psi(\tau, z)) + H(\tau, z, w(-h), \nabla_z \psi(\tau, z)) \leq 0, \tag{13a}
\end{align*}

\begin{align*}
\text{for every } (\tau, z, w(\cdot)) \in \mathbb{G}_s, \quad \psi \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), \quad \text{and } \delta > 0, \\
&\text{if } \varphi(\tau, z, w(\cdot)) - \psi(\tau, z) \geq \varphi(t, x, \kappa_t(\cdot)) - \psi(t, x) \\
&\text{for any } (t, x) \in O^+_\delta(\tau, z), \quad \kappa(\cdot) \in A_0(\tau, z, w(\cdot)), \\
&\text{then } \partial \psi(\tau, z) / \partial t + (\partial_{\tau,w}^ig(\tau, w(\cdot)), \nabla_z \psi(\tau, z)) + H(\tau, z, w(-h), \nabla_z \psi(\tau, z)) \geq 0. \tag{13b}
\end{align*}
Here $O^+(\tau, z) = \{(t, x) \in [\tau, \tau + \delta] \times \mathbb{R}^n : \|x - z\| \leq \delta\}$, and $C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ is the space of continuously differentiable functions $\psi : \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}$.

The main result of the paper is the following theorem.

**Theorem 1** There exists a unique viscosity solution $\varphi$ of problem $(8), (9)$.

In order to prove the theorem, we consider the minimax approach to the definition of generalized solutions of problem $(8), (9)$ (see [28]).

### 2.4 Minimax Solution

Taking the constant $c_H > 0$ from $(H_2)$, denote

$$F^\eta(x, y) = \{l \in \mathbb{R}^n : \|l\| \leq c_H(1 + \|x\| + \|y\|) + \eta\}, \quad x, y \in \mathbb{R}^n, \quad \eta \in [0, 1].$$

(14)

Let $(\tau, z, w(\cdot)) \in \mathbb{G}$. Denote by $X^\eta(\tau, z, w(\cdot))$ the set of the functions $x(\cdot) \in \Lambda(\tau, z, w(\cdot))$ such that the function $y(t) = x(t) - g(t, x(t - h)), \ t \in [\tau, \vartheta]$ is Lipschitz continuous and the neutral-type differential inclusion

$$\frac{d}{dt}(x(t) - g(t, x(t - h))) \in F^\eta(x(t), x(t - h)) \text{ for a.e. } t \in [\tau, \vartheta],$$

(15)

holds. Note that the set $X^\eta(\tau, z, w(\cdot))$ is not empty. At least, the function $x(\cdot) \in \Lambda(\tau, z, w(\cdot))$ satisfying $x(t) = g(t, x(t - h)) + z - g(\tau, w(-h)), \ t \in (\tau, \vartheta)$, belongs to $X^\eta(\tau, z, w(\cdot))$.

The following inequalities for functionals $\varphi : \mathbb{G} \mapsto \mathbb{R}$ are key to define minimax solutions under consideration below:

$$\inf_{x(\cdot) \in X^\eta(\tau, z, w(\cdot))} \left(\varphi(t, x(t), x_r(\cdot)) + \omega(\tau, t, x(\cdot), s)\right) \leq \varphi(\tau, z, w(\cdot)), \quad (16a)$$

$$\sup_{x(\cdot) \in X^\eta(\tau, z, w(\cdot))} \left(\varphi(t, x(t), x_r(\cdot)) + \omega(\tau, t, x(\cdot), s)\right) \geq \varphi(\tau, z, w(\cdot)), \quad (16b)$$

where $(\tau, z, w(\cdot)) \in \mathbb{G}$, $\tau < \vartheta$, $t \in (\tau, \vartheta)$, $s \in \mathbb{R}^n$, $\eta \in [0, 1]$, and

$$\omega(\tau, t, x(\cdot), s) = \int_{\xi}^{t} H(\xi, x(\xi), x(\xi - h), s)d\xi$$

$$- \left(\left(x(t) - g(t, x(t - h))\right) - \left(x(\tau) - g(\tau, x(\tau - h))\right), s\right).$$

(17)

**Definition 2** A functional $\varphi : \mathbb{G} \mapsto \mathbb{R}$ is called a minimax solution of problem $(8), (9)$, if $\varphi$ satisfies conditions $(\varphi_1), (\varphi_2), (9)$ and the inequalities $(16a), (16b)$ for any $(\tau, z, w(\cdot)) \in \mathbb{G}$, $\tau < \vartheta$, $t \in (\tau, \vartheta)$, $s \in \mathbb{R}^n$, and $\eta = 0$.

This definition of minimax solution seems to be natural for considered problem $(8), (9)$. Nevertheless, recently, the theory of minimax solution of problems similar to $(8), (9)$.
(9) was developed on the space of Lipschitz continuous functions (see [28]). To apply these results, it is convenient to give the following auxiliary definition of a minimax solution.

**Definition 3** A functional $\varphi : G \mapsto \mathbb{R}$ is called a Lip-minimax solution of problem (8), (9), if $\varphi$ satisfies conditions $(\varphi_1), (\varphi_2)$ and, taking $z = w(-0)$, satisfies condition (9) for any $w(\cdot) \in \text{Lip}$ and inequalities (16a), (16b) for any $(\tau, w(\cdot)) \in [0, \vartheta) \times \text{Lip}$, $t \in (\tau, \vartheta]$, $s \in \mathbb{R}^n$, and $\eta = 0$.

The first step to prove Theorem 1 is the following theorem which follows from Lemmas 8, 12 taking Remark 2 into account.

**Theorem 2** There exists a unique Lip-minimax solution $\varphi$ of problem (8), (9).

Then, we introduce one more auxiliary definition of a minimax solution.

**Definition 4** A functional $\varphi : G \mapsto \mathbb{R}$ is called a C$^1$-minimax solution of problem (8), (9), if $\varphi$ satisfies conditions $(\varphi_1), (\varphi_2)$, and, taking $z = w(-0)$, satisfies condition (9) for any $w(\cdot) \in \text{C}^1$ and inequalities (16a), (16b) for any $i \in 0, I - 1$, $(\tau, w(\cdot)) \in [ih, (i + 1)h) \times \text{C}^1$, $s \in \mathbb{R}^n$, and $\eta \in (0, 1]$.

The following theorem establishes the equivalence of these three definitions.

**Theorem 3** The following statements are equivalent:

(a) The functional $\varphi : G \mapsto \mathbb{R}$ is a minimax solution of problem (8), (9).

(b) The functional $\varphi : G \mapsto \mathbb{R}$ is a Lip-minimax solution of problem (8), (9).

(c) The functional $\varphi : G \mapsto \mathbb{R}$ is a C$^1$-minimax solution of problem (8), (9).

The implications $(a) \Rightarrow (b) \Rightarrow (c)$ are valid due to inclusions (2) and inclusion $X^0(\tau, z, w(\cdot)) \subset X^\eta(\tau, z, w(\cdot))$ for any $\eta \in (0, 1]$. The implication $(c) \Rightarrow (a)$ is proved in Lemma 13.

Finally, in order to prove Theorem 1, we follow scheme from [4, 32, 33] (see also [26]). For that, we define the corresponding notions of lower and upper right directional derivatives as well as the notions of subdifferential and superdifferential of a functional $\varphi : G \mapsto \mathbb{R}$.

By analogy with [27] (see also [20]), lower and upper right directional derivatives of a functional $\varphi : G \mapsto \mathbb{R}$ along $(l_0, l) \in [0, +\infty) \times \mathbb{R}^n$ at $(\tau, z, w(\cdot)) \in G$, $\tau < \vartheta$ are defined by

$$
\frac{\partial^-}{\partial_{(l_0, l)}} \varphi(\tau, z, w(\cdot)) = \liminf_{\delta \downarrow 0} \frac{\varphi(\tau + l_0 \delta, z + l \delta, \kappa_{\tau + l_0 \delta}(\cdot)) - \varphi(\tau, z, w(\cdot))}{\delta}, \quad (18a)
$$

$$
\frac{\partial^+}{\partial_{(l_0, l)}} \varphi(\tau, z, w(\cdot)) = \limsup_{\delta \downarrow 0} \frac{\varphi(\tau + l_0 \delta, z + l \delta, \kappa_{\tau + l_0 \delta}(\cdot)) - \varphi(\tau, z, w(\cdot))}{\delta}, \quad (18b)
$$

where $\kappa(\cdot) \in A_0(\tau, z, w(\cdot))$.

The subdifferential of a functional $\varphi : G \mapsto \mathbb{R}$ at a point $(\tau, z, w(\cdot)) \in G$, $\tau < \vartheta$ is a set, denoted by $D^-(\tau, z, w(\cdot))$, of pairs $(p_0, p) \in \mathbb{R} \times \mathbb{R}^n$ such that

$$
\lim_{\delta \to 0} \inf_{(t, x) \in O^+_{\delta}(\tau, z)} \frac{\varphi(t, x, \kappa_t(\cdot)) - \varphi(\tau, z, w(\cdot)) - (t - \tau)p_0 - (x - z, p)}{|t - \tau| + \|x - z\|} \geq 0. \quad (19)
$$
The superdifferential of a functional \( \varphi : G \mapsto \mathbb{R} \) at a point \((\tau, z, w(\cdot)) \in G, \tau < \vartheta \) is a set, denoted by \( D^+(\tau, z, w(\cdot)) \), of pairs \((q_0, q) \in \mathbb{R} \times \mathbb{R}^n\) such that
\[
\lim_{\delta \to 0} \sup_{(t, x) \in D^+(\tau, z)} \frac{\varphi(t, x, \kappa_t(\cdot)) - \varphi(\tau, z, w(\cdot)) - (t - \tau)q_0 - \langle x - z, q \rangle}{|t - \tau| + \|x - z\|} \leq 0. \tag{20}
\]

The theorem below together with Theorem 2 completes the proof of Theorem 1.

**Theorem 4** The following statements are equivalent:

(a) The functional \( \varphi : G \mapsto \mathbb{R} \) is a minimax solution of problem (8), (9).
(b) The functional \( \varphi : G \mapsto \mathbb{R} \) is a \( C^1 \)-minimax solution of problem (8), (9).
(c) The functional \( \varphi : G \mapsto \mathbb{R} \) satisfies conditions (\( \varphi_1 \)), (\( \varphi_2 \)), (9) and, for every \((\tau, z, w(\cdot)) \in G_* \) and \( s \in \mathbb{R}^n \), the following inequalities hold:
\[
\inf_{l \in F^0(\tau, w(\cdot)) + \partial_{\tau, w}^c(\tau, w(\cdot))} \left( \partial_{l, 1}^{-} \varphi(\tau, z, w(\cdot)) + \langle \partial_{l, 2}^{c_1} g(\tau, w(\cdot)), s \rangle \right) + H(\tau, z, w(-h), s) - \langle l, s \rangle \leq 0, \tag{21a}
\]
\[
\sup_{l \in F^0(\tau, w(\cdot)) + \partial_{\tau, w}^c(\tau, w(\cdot))} \left( \partial_{l, 1}^{+} \varphi(\tau, z, w(\cdot)) + \langle \partial_{l, 2}^{c_1} g(\tau, w(\cdot)), s \rangle \right) + H(\tau, z, w(-h), s) - \langle l, s \rangle \geq 0. \tag{21b}
\]

(d) The functional \( \varphi : G \mapsto \mathbb{R} \) satisfies conditions (\( \varphi_1 \)), (\( \varphi_2 \)), (9) and, for every \((\tau, z, w(\cdot)) \in G_* \), \((p_0, p) \in \mathbb{R} \times D^- \varphi(\tau, z, w(\cdot))\), and \((q_0, q) \in \mathbb{R} \times \mathbb{R}^n \) the following inequalities hold:
\[
p_0 + \langle \partial_{l, 2}^{c_1} g(\tau, w(\cdot)), p \rangle + H(\tau, z, w(-h), p) \leq 0, \tag{22a}
\]
\[
q_0 + \langle \partial_{l, 2}^{c_1} g(\tau, w(\cdot)), p \rangle + H(\tau, z, w(-h), q) \geq 0. \tag{22b}
\]

(e) The functional \( \varphi : G \mapsto \mathbb{R} \) is a viscosity solution of problem (8), (9).

The implications (a) \( \Rightarrow \) (e) \( \Rightarrow \) (d) \( \Rightarrow \) (c) \( \Rightarrow \) (b) \( \Rightarrow \) (a) follow from Lemmas 15, 17, 21, 22, and Theorem 3, respectively.

**Remark 4** Besides the proof of Theorem 1, Theorem 4 establishes other equivalent definitions of generalized solution of (8), (9) which have analogues in the classical theory [33]. Namely, (c) is similar to the infinitesimal variant of the definitions of minimax solutions (see [30]) or Dini solutions according to another terminology (see [3]). (d) is an analog of the equivalent notion of the viscosity solutions considered in [6].

### 2.5 Consistency

**Remark 5** Note that, directly from definitions of ci-differentiability (see (4)) and sub- and superdifferentials (see (19) and (20)), if a functional \( \varphi : G \mapsto \mathbb{R} \) satisfies condi-
tions \((φ_1), (φ_2)\) and is ci-differentiable \((τ, z, w(\cdot)) \in G_s\), then

\[
D^-φ(τ, z, w(\cdot)) = \{(p_0, p): p_0 \leq \partial^{ci}_{t,w}φ(τ, z, w(\cdot)), p = \nabla_zφ(τ, z, w(\cdot))\},
\]

\[
D^+φ(τ, z, w(\cdot)) = \{(q_0, q): q_0 \geq \partial^{ci}_{t,w}φ(τ, z, w(\cdot)), q = \nabla_zφ(τ, z, w(\cdot))\}.
\]

Hence, the following statement about consistency of the viscosity solution definition (see Definition 1) and problem (8), (9) can be obtained from Theorem 4.

**Theorem 5** (a) Let a functional \(φ: G \mapsto \mathbb{R}\) be the viscosity solution of problem (8), (9). If \(φ\) is ci-differentiable at a point \((τ, z, w(\cdot)) \in G_s\), then it satisfies HJ equation (8) at this point. (b) Let a functional \(φ: G \mapsto \mathbb{R}\) be ci-differentiable at every \((τ, z, w(\cdot)) \in G_s\), satisfy HJ equation (8) on \(G_s\) and satisfy conditions \((φ_1), (φ_1),\) and (9). Then \(φ\) is the viscosity solution of problem (8), (9).

### 2.6 Application for Optimal Control Problems

In spite of the fact that prior works (see, e.g., [11, 23]) consider dynamical optimization problems for neutral-type systems on the space of Lipschitz continuous functions, we can also apply the results of this paper to get characterizations of value functionals in such problems.

For example, consider the following optimal control problem: for each \((τ, w(\cdot)) \in [0, θ] \times \text{Lip}\), it is required to minimize the Bolza cost functional

\[
J(τ, w(\cdot), u(\cdot)) = \sigma (x(θ), x_0(\cdot)) + \int_τ^{θ} f^0(t, x(t), x(t-h), u(t))dt,
\]

over all measurable functions \(u(\cdot): [τ, θ] \mapsto \mathbb{U}\), where \(x(\cdot) \in \Lambda(τ, w(-0), w(\cdot))\) is a Lipschitz continuous function satisfying the neutral-type equation

\[
\frac{d}{dt}(x(t) - g(t, x(t-h))) = f(t, x(t), x(t-h), u(t)) \text{ for a.e. } t \in [τ, θ].
\]

Here \(\mathbb{U} \subset \mathbb{R}^m\) is the compact set; the function \(g\) and the functional \(σ\) satisfy conditions \((g)\) and \((σ)\); the functions \(f: [0, θ] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{U} \mapsto \mathbb{R}\) and \(f^0: [0, θ] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{U} \mapsto \mathbb{R}\) satisfy the following conditions:

\(f_1\) The functions \(f\) and \(f^0\) are continuous.

\(f_2\) There exists a constant \(c_f > 0\) such that

\[
\|f(t, z, r, u)\| + |f^0(t, z, r, u)| \leq c_f (1 + \|z\| + \|r\|)
\]

for any \(t \in [0, θ]\), \(z, r \in \mathbb{R}^n\), and \(u \in \mathbb{U}\).

\(f_3\) For every \(α > 0\), there exists \(α_f = \lambda_f(α) > 0\) such that

\[
\|f(t, z, r, u) - f(t, z', r, u)\| + |f^0(t, z, r, u) - f^0(t, z', r, u)| \leq α_f (\|z - z'\| + \|r - r\|)
\]

for any \(t \in [0, θ]\), \(z, r, z', r' \in \mathbb{R}^n\): \(\max\{\|z\|, \|r\|, \|z'\|, \|r'\|\} \leq α\), and \(u \in \mathbb{U}\).
Note that, under these conditions, the function
\[
H(t, z, r, s) = \min_{u \in \mathcal{U}} \left( f(t, z, r, u) + f^0(t, z, r, u) \right), \quad t \in [0, \vartheta], \quad z, r, s \in \mathbb{R}^n,
\]  
\tag{25}
\]
satisfies conditions $(H_1)$–$(H_3)$. The value functional of this problem is
\[
\hat{\varphi}(\tau, w(\cdot)) = \inf_{u(\cdot)} J(\tau, w(\cdot), u(\cdot)).
\]  
\tag{26}

Applying Lemma 23 to show that $\hat{\varphi}$ is the functional from Lemma 8 and next, using
Lemma 12 and Theorem 4, taking Remark 2 into account, we can obtain the following
result.

**Theorem 6** There exists a unique functional $\varphi : \mathbb{G} \mapsto \mathbb{R}$ satisfying conditions $(\varphi_1)$, $(\varphi_2)$ and the equality
\[
\varphi(\tau, w(0), w(\cdot)) = \hat{\varphi}(\tau, w(\cdot)), \quad (\tau, w(\cdot)) \in [0, \vartheta] \times \text{Lip}.
\]

Such a functional $\varphi$ is a unique viscosity solution of problem $(8)$, $(9)$.

### 3 Proofs

#### 3.1 Auxiliary Statements

**Lemma 1** For every $\alpha > 0$, there exists $\alpha_X = \alpha_X(\alpha) > 0$, $\alpha^*_X = \alpha^*_X(\alpha) > 0$, and $\lambda^*_X = \lambda^*_X(\alpha) > 0$ such that, for each $\tau \in [0, \vartheta]$ and $(z, w(\cdot)) \in P(\alpha)$ (see (5)), the functions $x(\cdot) \in X^1(\tau, z, w(\cdot))$ and $y(t) = x(t) - g(t, x(t - h))$, $t \in [\tau, \vartheta]$, satisfy the relations
\[
(x(t), x(\cdot)) \in P(\alpha_X), \quad \|y(t)\| \leq \alpha^*_X, \quad \|y(t) - y(t')\| \leq \lambda^*_X|t - t'|, \quad t, t' \in [\tau, \vartheta].
\]

**Proof** The existence of $\alpha_X$ satisfying the first relation can be proved in the same way as in Lemma 6.1 from [22]. Due to condition $(g)$, there exists $\alpha_g > 0$ such that $\|g(\tau, w(-h))\| \leq \alpha_g$ for any $\tau \in [0, \vartheta]$ and $(z, w(\cdot)) \in P(\alpha)$. Then, from (14), (15), the second relation is obtained by
\[
\|y(t)\| \leq \|z\| + \|g(\tau, w(\cdot))\| + \int^t_{\tau} \left(c_H(1 + \|x(\xi)\| + \|x(\xi - h)\|) + 1\right)d\xi \\
\leq \alpha + \alpha_g + (c_H(1 + 2\alpha_X) + 1)\vartheta := \alpha^*_X, \quad t \in [\tau, \vartheta],
\]
and, taking $\lambda^*_X = c_H(1 + 2\alpha_X)$, the third relation is derived by
\[
\|y(t') - y(t)\| \leq c_H \int^t_{t'} \left(c_H(1 + \|x(\xi)\| + \|x(\xi - h)\|) + 1\right)d\xi \leq \lambda^*_X(t' - t),
\]
where, without loss of generality, we assume $\tau \leq t \leq t' \leq \vartheta$.  \(\square\)
Lemma 2 Let the function $g : [0, \vartheta] \times \mathbb{R}^n \mapsto \mathbb{R}^n$ satisfy (g). Then, for every $\alpha > 0$ there exists $\lambda_g = \lambda_g(\alpha) > 0$ such that

$$|g(t, x) - g(t', x')| \leq \lambda_g(|t - t'| + \|x - x'\|)$$

for any $(t, x), (t', x') \in [0, \vartheta] \times \mathbb{R}^n$: $\max\{\|x\|, \|x'\|\} \leq \alpha$.

Lemma 3 Let $\alpha_0, \lambda_0 > 0$. There exists $\lambda_X = \lambda_X(\alpha_0, \lambda_0) > 0$ with the following property. Let $(\tau, z, w(\cdot)) \in \mathcal{G}_e$. Let $\delta_w > 0$ be such that $w(\cdot)$ is continuously differentiable on $[-h, -h + \delta_w]$ (see (3)). Let the relations

$$(z, w(\cdot)) \in P(\alpha_0), \quad \|w(\xi) - w(\xi')\| \leq \lambda_0|\xi - \xi'|, \xi, \xi' \in [-h, -h + \delta_w].$$

(27)

hold. Then, every function $x(\cdot) \in X^1(\tau, z, w(\cdot))$ satisfies the inequality

$$\|x(t) - x(t')\| \leq \lambda_X|t - t'|, \quad t, t' \in [\tau, \tau + \delta_w].$$

(28)

Proof According to Lemmas 1, 2, define $\lambda^*_X = \lambda^*_X(\alpha_0)$ and $\lambda_g = \lambda_g(\alpha_0)$. Put $\lambda_X = \lambda^*_X + \lambda_g(1 + \lambda_0)$. Then the inequality (28) follows from the estimates

$$\|x(t) - x(t')\| \leq \|y(t) - y(t')\| + \|g(t, w(t - \tau - h)) - g(t', w(t' - \tau - h))\|$$

$$\leq \lambda_X|t - t'|.$$

The lemma has been proved. $\Box$

Let $(z, w(\cdot)) \in \mathbb{R}^n \times \text{PLip}$. Denote by $\Gamma(z, w(\cdot))$ the set of sequences $\{w^j(\cdot)\}_{j \in \mathbb{N}} \subset \text{Lip}$ such that

$$\|w(\cdot) - w^j(\cdot)\|_1 \to 0, \quad \|z - w^j(-0)\| \to 0, \quad \|w(\xi) - w^j(\xi)\| \to 0, \quad \xi \in [-h, 0],$$

as $j \to \infty$. (29)

Lemma 4 For each $(z, w(\cdot)) \in \mathbb{R}^n \times \text{PLip}$, there exists a sequence $\{w^j(\cdot)\}_{j \in \mathbb{N}} \subset \Gamma(z, w(\cdot))$ such that

$(w^j_1)$ The inclusion $w^j(\cdot) \in C^1$ holds for any $j \in \mathbb{N}$.

$(w^j_2)$ The inequality $\|w^j(\cdot)\|_\infty \leq \max\{\|z\|, \|w(\cdot)\|_\infty\}$ holds for any $j \in \mathbb{N}$.

$(w^j_3)$ For every $\xi \in (-h, 0)$ satisfying the equality $w(\xi - 0) = w(\xi)$, there exists $\delta > 0$ such that

$$\max_{\xi \in [-\delta, \delta]} \|w(\xi + \zeta) - w^j(\xi + \zeta)\| \to 0 \text{ as } j \to \infty.$$

Proof Let us take a continuously differentiable function $\beta(\cdot) : (-\infty, +\infty) \mapsto [0, +\infty)$ such that

$$\beta(\xi) = 0, \quad \xi \in (-\infty, 0] \cup [1, +\infty), \quad \int_{-\infty}^{+\infty} \beta(\zeta) d\zeta = 1.$$
Let \( \overline{w}(\cdot): [-h, 1] \mapsto \mathbb{R}^n \) be such that \( \overline{w}(\xi) = w(\xi), \xi \in [-h, 0) \) and \( \overline{w}(\xi) = z, \xi \in [0, 1] \). Then one can show the functions

\[
w_j(\xi) = j \int_{-\infty}^{+\infty} \beta(j\xi) \overline{w}(\xi + \xi) d\xi, \quad j \in \mathbb{N},
\]

satisfy the statements of the lemma.

**Lemma 5** Let \( (\tau, z, w(\cdot)) \in \mathcal{G} \). Let \( \{w_j(\cdot)\}_{j \in \mathbb{N}} \subset \Gamma(z, w(\cdot)) \) satisfy conditions (\( w_1^j \))--(\( w_2^j \)). Let \( x_j(\cdot) \in X^{1/2}(\tau, w^j(0), w^j(\cdot)), \ j \in \mathbb{N} \). Then there exist a subsequence \( x_{jm}^j(\cdot), m \in \mathbb{N} \) and a function \( x(\cdot) \in X^0(\tau, z, w(\cdot)) \) such that \( \{x_{jm}^j(\cdot)\}_{m \in \mathbb{N}} \in \Gamma(x(t), x_t(\cdot)) \) for any \( t \in [\tau, \min\{\tau + h, \vartheta\}] \).

**Proof** Let \( \alpha_0 = \max\{\|z\|, \|w(\cdot)\|\}_{\infty} \). In accordance with Lemma 1, define \( \alpha^*_X = \alpha^*_X(\alpha_0) \) and \( \lambda^*_X = \lambda^*_X(\alpha_0) \). Then, for the functions \( y_j(t) = x_j(t) - g(t, x_j(t - h)), t \in [\tau, \vartheta], j \in \mathbb{N} \), we have

\[
\|y_j(t)\| \leq \alpha^*_X, \quad \|y_j(t) - y_j(t')\| \leq \lambda^*_X |t - t'|, \ t, t' \in [\tau, \vartheta].
\]

Due to these estimates and Arzela-Ascoli theorem (see, e.g., [24, p. 207]), without loss of generality, we can suppose that there exists a continuous function \( y(\cdot): [\tau, \vartheta] \mapsto \mathbb{R}^n \) such that

\[
\max_{t \in [\tau, \vartheta]} \|y(t) - y_j(t)\| \to 0 \text{ as } j \to \infty.
\]

From (30) and (31), one can establish the Lipschitz continuity of \( y(\cdot) \). Denote \( \tau_h = \min\{\tau + h, \vartheta\} \). Define the function \( x(\cdot) \) so that

\[
x(t) = \begin{cases} 
  w(t - \tau), \text{ if } t \in [\tau - h, \tau), \\
  y(t) + g(t, w(t - \tau - h)), \text{ if } t \in [\tau, \tau_h), \\
  g(t, x(t - h)), \text{ if } t \in [\tau_h, \vartheta].
\end{cases}
\]

Choose \( \lambda_g = \lambda_g(\alpha_0) \) according to Lemma 2. Then, taking into account definitions of \( x_j(\cdot), y_j(\cdot), \) and \( x(\cdot) \) and condition (\( w^j_2 \)), we have

\[
\|x(\xi) - x_j(t)\| \leq \|y(t) - y_j(t)\| + \lambda_g \|w(t - \tau - h) - w^j(t - \tau - h)\|, \ \xi \in [\tau, \tau_h). \]

\[
\|x(t) - x_j(t)\| = \|w(t - \tau) - w^j(t - \tau)\|, \ t \in [\tau - h, \tau),
\]

From these estimates, taking into account the inclusion \( \{w^j(\cdot)\}_{j \in \mathbb{N}} \subset \Gamma(z, w(\cdot)) \) and (31), we obtain \( \{x_t^j(\cdot)\}_{j \in \mathbb{N}} \in \Gamma(x(t), x_t(\cdot)) \) for any \( t \in [\tau, \tau_h) \).

Let us show the inclusion \( x(\cdot) \in X^0(\tau, z, w(\cdot)) \). Firstly, according to (32), (33), the inclusion \( x(\cdot) \in \Lambda(\tau, z, w(\cdot)) \) holds. Note that, due to (32), the function \( x(\cdot) \)
satisfies (15) for every \( t \in (\tau, \vartheta] \) in which \( \eta = 0 \). Let \( t \in (\tau, \tau_h) \) be such that there exists \( dy(t)/dt \) and \( w(t - \tau - h - 0) = w(t - \tau - h) \). Let \( \varepsilon > 0 \). Then, due to the inclusion \( \{w^j(\cdot)\}_{j \in \mathbb{N}} \in \Gamma(z, w(\cdot)) \) and relations (30), (31), there exist \( \delta \in (0, \min\{t - \tau, \tau_h - t, \varepsilon/(9c_H\lambda_g)\}) \) and \( j_* > 0 \) such that

\[
\|w(t - \tau - h) - w^j(t - \tau - h + \xi)\| \leq \varepsilon/(3c_H \max\{1, 3\lambda_g\}), \\
\|y(t) - y^j(t + \xi)\| \leq \varepsilon/(9c_H), \quad 1/j \leq \varepsilon/3, 
\]

(34)
for any \( \xi \in [-\delta, \delta] \) and \( j \geq j_* \), where \( c_H \) is from condition \((H2)\). From these estimates, choice of \( \lambda_g \), and definitions of \( x^j(\cdot), y^j(\cdot) \), and \( x(\cdot) \), we derive

\[
\|x(t) - x^j(t + \xi)\| \leq \|y(t) - y^j(t + \xi)\| \\
+ \lambda_g (\xi + \|w(t - \tau - h) - w^j(t - \tau - h + \xi)\|) \leq \varepsilon/(3c_H).
\]

Then, due to the inclusion \( x^j(\cdot) \in X^{1/j}(\tau, w^j(-0), w^j(\cdot)) \), and the first and the third inequalities in (34), we derive

\[
\|d y^j(t + \xi)/dt\| \leq c_H (\|x(t)\| + \|w(t - \tau - h)\|) + \varepsilon \text{ for a.e. } \xi \in [-\delta, \delta].
\]

Using Lemma 12 from [9, p. 63], we obtain

\[
\|dy(t)/dt\| \leq c_H (\|x(t)\| + \|w(t - \tau - h)\|) + \varepsilon.
\]

This estimate holds for every \( \varepsilon > 0 \) that means for \( \varepsilon = 0 \). Thus, the inclusion \( x(\cdot) \in X^0(\tau, z, w(\cdot)) \) is proved.

\[
\text{Lemma 6 Let } (\tau, z, w(\cdot)) \in \mathbb{G}_s. \text{ Let } \delta_w \in (0, h) \text{ be such that } w(\cdot) \text{ is continuously differentiable on } [-h, -h + \delta_w]. \text{ Then the inclusion } (t, x(t), x_t(\cdot)) \in \mathbb{G}_s \text{ holds for any } t \in [\tau, \tau + \delta_w] \text{ (see (3)), and the following functions coincide and are continuous:}
\]

\[
\overline{g}(t) := \frac{d}{dt} \left( g(t, x(t - h)) = \partial_{t,w}^i g(t, x_t(\cdot)) = \partial_{t,w}^i g(t, \kappa_t(\cdot)), \quad t \in [\tau, \tau + \delta_w]. \right)
\]

for any \( x(\cdot) \in \Lambda(\tau, z, w(\cdot)) \) and \( \kappa(\cdot) \in \Lambda_0(\tau, z, w(\cdot)) \).

\[
\text{Proof is directly following from definition (7) of } \partial_{t,w}^i g(t, w(\cdot)).
\]

\[
\text{Lemma 7 Let } (z, w(\cdot)) \in \mathbb{R}^n \times \text{PLip}. \text{ Then, for every } \varepsilon > 0, \text{ there exists } v = v(\varepsilon) > 0 \text{ such that, for every } \tau \in [0, \vartheta] \text{ and } t, t' \in [\tau, \vartheta] \text{ satisfying } |t - t'| \leq v \text{ the inequality below holds:}
\]

\[
\|\kappa_t(\cdot) - \kappa_{t'}(\cdot)\|_1 \leq \varepsilon, \quad \kappa(\cdot) \in \Lambda_0(\tau, z, w(\cdot)).
\]

\[
\text{Proof can be obtained using approximation of } (z, w(\cdot)) \text{ by Lipschitz continuous functions (see Lemma 4).}
\]

\[
\text{Springer}
\]
3.2 Proof of Theorem 2

According to Theorem 3 from [28], the following lemma holds:

**Lemma 8** There exists a unique continuous (with respect to uniform norm) functional \( \hat{\phi} : [0, \vartheta] \times \text{Lip} \mapsto \mathbb{R} \) satisfying condition \( \hat{\phi}(\vartheta, w(\cdot)) = \sigma(w(-0), w(\cdot)) \) for any \( w(\cdot) \in \text{Lip} \) and the inequalities

\[
\inf_{x(\cdot) \in X^0(\tau, w(-0), w(\cdot))} \left( \hat{\phi}(t, x_t(\cdot)) + \omega(t, t, x(\cdot), s) \right) \leq \hat{\phi}(\tau, w(\cdot)), \\
\sup_{x(\cdot) \in X^0(\tau, w(-0), w(\cdot))} \left( \hat{\phi}(t, x_t(\cdot)) + \omega(t, t, x(\cdot), s) \right) \geq \hat{\phi}(\tau, w(\cdot)),
\]

for any \( (\tau, w(\cdot)) \in [0, \vartheta) \times \text{Lip}, \ t \in (\tau, \vartheta], \) and \( s \in \mathbb{R}^n \), where \( \omega \) is from (17).

Lemmas 9 and 10 below can be proved similar to Lemmas 1 and 3 from [28]. Let \( \alpha > 0 \). Define \( \lambda_g = \lambda_g(\alpha) > 1 \) and \( \lambda_H = \lambda_H(\alpha) > 1 \) according to Lemma 2 and condition \((H_3)\). For every \( \gamma, \varepsilon > 0 \) and \( (\tau, z, w(\cdot)) \in \mathbb{G} \), denote

\[
\theta_{\gamma, \varepsilon}^\alpha(\tau) = \left( e^{-(4\lambda_H + 2\lambda_g^g/\varepsilon)\tau} - \gamma \right) / \gamma, \\
\nu_{\gamma, \varepsilon}^\alpha(\tau, z, w(\cdot)) = \theta_{\gamma, \varepsilon}^\alpha(\tau) \left( \sqrt{\varepsilon^4 + \|z\|^2} + 2\lambda_H \int_{-\varepsilon}^0 \left( 1 - \frac{2\lambda_g^g}{\varepsilon} \right) \|w(\xi)\|d\xi \right).
\]

**Lemma 9** Let \( \alpha, \varepsilon > 0 \) and \( \tau \in [0, \vartheta] \). Let \( \gamma > 0 \) be such that \( \theta_{\gamma, \varepsilon}^\alpha(t) > 1 \) for any \( t \in [0, \vartheta] \). Let \( x(\cdot), x'(\cdot) \) be Lipschitz continuous functions from \( [\tau - h, \vartheta] \) to \( \mathbb{R}^n \) satisfying the inequality

\[
\|x(t)\| \leq \alpha, \quad \|x'(t)\| \leq \alpha, \quad t \in [\tau - h, \vartheta].
\]

Then the following inequality holds:

\[
\nu_{\gamma, \varepsilon}^\alpha(t, \Delta y(t), \Delta x_t(\cdot)) - \nu_{\gamma, \varepsilon}^\alpha(\tau, \Delta y(\tau), \Delta x_t(\cdot)) \leq \int_\tau^t \Delta H_{\gamma, \varepsilon}^\alpha(\xi)d\xi, \quad t \in [\tau, \vartheta],
\]

where \( \Delta x(t) = x(t) - x'(t), \Delta y(t) = \Delta x(t) - g(t, x(t - h)) + g(t, x'(t - h)), \) and

\[
\Delta H_{\gamma, \varepsilon}^\alpha(t) = H(t, x(t), x(t - h), \nabla_x \nu_{\gamma, \varepsilon}^\alpha(t, \Delta y(t), \Delta x_t(\cdot))) \\
- H(t, x'(t), x'(t - h), \nabla_x \nu_{\gamma, \varepsilon}^\alpha(t, \Delta y(t), \Delta x_t(\cdot))) \\
+ (\Delta y(t), \nabla_z \nu_{\gamma, \varepsilon}^\alpha(t, \Delta y(t), \Delta x_t(\cdot))).
\]

**Lemma 10** Let \( \hat{\phi} : [0, \vartheta] \times \text{Lip} \mapsto \mathbb{R} \) be taken from Lemma 8. Let \( \alpha, \gamma, \varepsilon > 0 \), \( (\tau, w(\cdot)), (\tau, w'(\cdot)) \in [0, \vartheta] \times \text{Lip} \) and \( t \in [\tau, \vartheta] \). Then there exist functions \( x(\cdot) \in X^0(\tau, w(-0), w(\cdot)) \) and \( x'(\cdot) \in X^0(\tau, w'(-0), w'(\cdot)) \) such that

\[
\hat{\phi}(t, x_t(\cdot)) - \hat{\phi}(t, x'_t(\cdot)) + \int_\tau^t \Delta H_{\gamma, \varepsilon}^\alpha(\xi)d\xi \leq \hat{\phi}(\tau, w(\cdot)) - \hat{\phi}(\tau, w'(\cdot)) + (t - \tau)\varepsilon,
\]

where we use denotations from Lemma 9.
Lemma 11 Let \( \hat{\phi} : [0, \theta] \times \text{Lip} \mapsto \mathbb{R} \) be taken from Lemma 8. For every \( \alpha > 0 \), there exists \( \lambda_\varphi = \lambda_\varphi(\alpha) > 0 \) such that

\[
|\hat{\phi}(\tau, w(\cdot)) - \hat{\phi}(\tau, w'(\cdot))| \leq \lambda_\varphi v(\tau, w(-0) - w'(-0), w(\cdot) - w'(\cdot))
\]

for any \( \tau \in [0, \theta] \) and \((z, w(\cdot)), (z', w'(\cdot)) \in P(\alpha) \cap (\mathbb{R}^n \times \text{Lip})\), where \( P(\alpha) \) and \( v \) are defined according to (5) and (11), respectively.

**Proof** Let us prove that, for each \( i \in \overline{0, I} \) and \( \alpha > 0 \), there exists \( \lambda_i = \lambda_i(\alpha) > 0 \) such that

\[
\hat{\phi}(\tau, w(\cdot)) - \hat{\phi}(\tau, w'(\cdot)) \leq \lambda_i v(\tau, w(-0) - w'(-0), w(\cdot) - w'(\cdot)) \tag{38}
\]

for any \( \tau \in [ih, \theta] \) and \((w(-0), w(\cdot)), (w'(-0), w'(\cdot)) \in P(\alpha) \cap (\mathbb{R}^n \times \text{Lip})\). After that, taking \( \lambda_\varphi = \lambda_0 \), we will get the statement of the lemma.

Note that, for \( i = I \) and each \( \alpha > 0 \), inequality (38) holds due to (11), conditions (9), and (\( \alpha > 0 \)) if we take \( \lambda_I = \lambda_I(\alpha) = \lambda_\alpha(\alpha) \).

Assume that inequality (38) holds for \( i = j + 1 \leq I \) and prove it for \( i = j \). Let \( \alpha > 0 \). According to Lemmas 1 and 2 and condition (H2), define \( \alpha_X = \alpha_X(\alpha) \), \( \lambda_g = \lambda_g(\alpha_X) \), and \( \lambda_H = \lambda_H(\alpha_X) > 1 \). Due to our assumption, there exists \( \lambda_{j+1} = \lambda_{j+1}(\alpha_X) \) such that (38) holds for \( i = j + 1 \). In accordance with (36), there exist \( \gamma > 0 \) such that

\[
\theta_{\gamma}^{\alpha_X}(0) \geq \theta_{\gamma}^{\alpha_X}(t) \geq \max(\lambda_{j+1}, 1), \quad t \in [0, \theta]. \tag{39}
\]

Put

\[
\lambda_j = 2\theta_{\gamma}^{\alpha_X}(0)\lambda_H(1 + 2\lambda_g) + \lambda_{j+1}(2 + \lambda_g). \tag{40}
\]

Since \( \lambda_j \geq \lambda_{j+1} \), then inequality (38) already holds for \( \lambda_j \) and \( \tau \in [(j + 1)h, \theta] \). Let \( \tau \in [(j + 1)h, (j + 2)h] \) and \((w(-0), w(\cdot)), (w'(-0), w'(\cdot)) \in P(\alpha) \cap (\mathbb{R}^n \times \text{Lip})\). Let us show, for every \( \zeta > 0 \), the following estimate:

\[
\hat{\phi}(\tau, w'(\cdot)) - \hat{\phi}(\tau, w(\cdot)) \leq \lambda_j v(\tau, w'(-0) - w(-0), w'(\cdot) - w(\cdot)) + \zeta. \tag{41}
\]

Let \( \zeta > 0 \). Denote \( \vartheta_*= (j + 1)h \). Choose \( \varepsilon > 0 \) such that

\[
\theta_{\gamma}^{\alpha_X}(\tau)\varepsilon^2 + (\vartheta_* - \tau)\varepsilon \leq \zeta. \tag{42}
\]

According to Lemma 10, where we take \( \alpha = \alpha_X \), define the functions \( x(\cdot) \in X^0(\tau, w(-0), w(\cdot)) \) and \( x'(\cdot) \in X^0(\tau, w'(-0), w'(\cdot)) \). Due to the choice of \( \alpha_X \), these functions satisfy (37) for \( \alpha = \alpha_X \). Then, using Lemma 9, we have

\[
\hat{\phi}(\tau, w'(\cdot)) - \hat{\phi}(\tau, w(\cdot)) \leq \hat{\phi}(\vartheta_*, x^{\alpha_X}_{\vartheta_*}(\cdot)) - \hat{\phi}(\vartheta_*, x^{\alpha_X}_{\vartheta_*}(\cdot)) + v_{\gamma, \epsilon}^{\alpha_X}(\tau, \Delta y(\tau), \Delta x(\cdot)) - v_{\gamma, \epsilon}^{\alpha_X}(\vartheta_*, \Delta y(\vartheta_*), \Delta x_{\vartheta_*}(\cdot)) + (\vartheta_* - \tau)\varepsilon. \tag{43}
\]
Due to the choice of $\lambda_g, \lambda_H > 1$, and (11), (36), (37), (39), we derive

$$
\nu^{\alpha x}_y(\tau, \Delta y(\tau), \Delta x_\tau(\cdot)) \\
\leq \theta^{\alpha x}_y(\tau) (\varepsilon^2 + \|\Delta x(\tau)\| + \lambda_g \|\Delta x(\tau - h)\| + 2\lambda_H(1 + 2\lambda_g) \|\Delta x_\tau(\cdot)\|_1) \\
\leq \theta^{\alpha x}_y(\tau) \varepsilon^2 + 2\theta^{\alpha x}_y(0) \lambda_H (1 + 2\lambda_g) \nu(\tau, w(0) - w'(0), w(\cdot) - w'(\cdot))
$$

(44)

and, taking into account the choice of $\lambda_{j+1}$ and (39), we obtain

$$
\hat{\nu}(\vartheta_\ast, x_\vartheta(\cdot)) - \hat{\nu}(\vartheta_\ast, x_{\vartheta}(\cdot)) = \lambda_{j+1} \nu(\vartheta_\ast, \Delta x(\vartheta_\ast), \Delta x_{\vartheta}(\cdot)) \\
\leq \lambda_{j+1} (\|\Delta y(\vartheta_\ast)\| + \|\Delta x_{\vartheta}(\cdot)\|_1 + (2 + \lambda_g) \|\Delta x(\vartheta_\ast)\|)

\leq \nu^{\alpha x}_y(\vartheta_\ast, \Delta y(\vartheta_\ast), \Delta x_{\vartheta}(\cdot)) + \lambda_{j+1} (2 + \lambda_g) \|\Delta x(\vartheta_\ast)\|.
$$

(45)

Due to (11), the inequality $\|\Delta x(\vartheta_\ast)\| \leq \nu(\tau, w(0) - w'(0), w(\cdot) - w'(\cdot))$ in both cases of $\tau = jh$ and $\tau \in (jh, \vartheta_\ast)$ is then true. Thus, from (40), (42)–(45), we get (41). \square

**Lemma 12** Let $\hat{\nu} : [0, \vartheta] \times \text{Lip} \mapsto \mathbb{R}$ be taken from Lemma 8. Then there exists a unique functional $\varphi : \mathbb{G} \mapsto \mathbb{R}$ satisfying conditions (ϕ₁), (ϕ₂) and the equality

$$
\varphi(\tau, w(0), w(\cdot)) = \hat{\nu}(\tau, w(\cdot)), \quad (\tau, w(\cdot)) \in [0, \vartheta] \times \text{Lip}.
$$

(46)

**Proof** Let $(\tau, z, w(\cdot)) \in \mathbb{G}$ and $\alpha_0 = \{\|z\|, \|w(\cdot)\|\}$. Due to Lemma 4, there exists $\{w^i(\cdot)\}_{j \in \mathbb{N}} \in \Gamma(z, w(\cdot))$ satisfying condition (w₂). Let us consider the sequence $A^j = \hat{\nu}(\tau, w^j(\cdot))$, $j \in \mathbb{N}$. Then, taking $\lambda_\varphi = \lambda_\varphi(\alpha_0)$ according to Lemma 11 and (11), we have

$$
|A^j| \leq |\hat{\nu}(\tau, w^j(\cdot)) - \hat{\nu}(\tau, 0(\cdot))| + |\hat{\nu}(\tau, 0(\cdot))| \leq \lambda_\varphi(3 + h) \alpha_0 + |\hat{\nu}(\tau, 0(\cdot))|.
$$

Therefore, the sequence $A^j$ is bounded. Hence, without loss of generality, we can assume the existence of $A^\ast$ such that $A^j \rightarrow A^\ast$ as $j \rightarrow \infty$. Let us show this limit does not depend on the choice of $\{w^j(\cdot)\}_{j \in \mathbb{N}} \in \Gamma(z, w(\cdot))$ satisfying condition (w₂). Let $\{r^j(\cdot)\}_{j \in \mathbb{N}} \in \Gamma(z, w(\cdot))$ satisfy condition (w₂). Then, for defined above $\lambda_\varphi$, using the definition of $\Gamma(z, w(\cdot))$ (see (29)) and (11), we derive

$$
|\hat{\nu}(\tau, w^j(\cdot)) - \hat{\nu}(\tau, r^j(\cdot))| \leq \lambda_\varphi \nu(\tau, w^j(0) - r^j(0), w^j(\cdot) - r^j(\cdot)) \rightarrow 0
$$

as $j \rightarrow +\infty$. Thus, $\hat{\nu}(\tau, r^j(\cdot)) \rightarrow A^\ast$ as $j \rightarrow +\infty$ for any $\{r^j(\cdot)\}_{j \in \mathbb{N}} \in \Gamma(z, w(\cdot))$ satisfying condition (w₂). Put $\varphi(\tau, z, w(\cdot)) = A^\ast$. By the similar way, we can define the values $\varphi(\tau, z, w(\cdot))$ for any $\tau \in [0, \vartheta] \times \text{Lip}$.

The equality (46) directly follows from the inclusion $\{w^j(\cdot) \equiv w(\cdot)\}_{j \in \mathbb{N}} \in \Gamma(w(0), w(\cdot))$ for any $w(\cdot) \in \text{Lip}$. Condition (ϕ₁) holds for $\varphi$ due to (46) and the continuity of $\hat{\nu}$.

Let us show the condition (ϕ₂). Let $\alpha > 0$. Define $\lambda_\varphi = \lambda_\varphi(\alpha)$ according to Lemma 11. Let $\tau \in [0, \vartheta]$ and $(z, w(\cdot)), (z', w'(\cdot)) \in P(\alpha)$. Due to Lemma 4, there exist $\{w^j(\cdot)\}_{j \in \mathbb{N}} \in \Gamma(z, w(\cdot))$ and $\{r^j(\cdot)\}_{j \in \mathbb{N}} \in \Gamma(z', w'(\cdot))$ satisfying condition
(\omega_j, \lambda_j). Then we have
\[ |\hat{\varphi}(\tau, w^j(\cdot)) - \hat{\varphi}(\tau, \lambda_j(\cdot))| \leq \lambda_{\varphi} \psi(\tau, w^j(-0) - \lambda_j(-0), w^j(\cdot) - \lambda_j(\cdot)), \quad j \in \mathbb{N}. \]
Passing to the limit as \( j \to +\infty \), taking into account the construction of \( \varphi \), we obtain (10).

Let us prove the uniqueness. Let functionals \( \varphi: \mathbb{G} \mapsto \mathbb{R} \) and \( \varphi': \mathbb{G} \mapsto \mathbb{R} \) satisfy (\( \varphi_1 \), (\( \varphi_2 \)), and (46). Let \((\tau, z, w(\cdot)) \in \mathbb{G} \). According to Lemma 4, there exists \( \{ w^j(\cdot) \}_{j \in \mathbb{N}} \in \Gamma(z, w(\cdot)) \) satisfying condition (\( \omega_j \)). Let \( \alpha_0 = \{ \|z\|, \|w(\cdot)\|_{\infty} \} \). Then, due to (\( \varphi_2 \)), there exists \( \lambda_{\varphi}(\alpha_0) \) such that, taking (46) into account, the following estimates hold:
\[
|\varphi(\tau, z, w(\cdot)) - \varphi'(\tau, z, w(\cdot))| \leq |\varphi(\tau, z, w(\cdot)) - \varphi(\tau, w^j(\cdot))| + |\varphi'(\tau, z, w^j(\cdot)) - \varphi'(\tau, w^j(\cdot))| \leq 2\lambda_{\varphi}(\tau, z - w^j(-0), w(\cdot) - w^j(\cdot)).
\]
Passing to the limit as \( j \to +\infty \), we get \( \varphi(\tau, z, w(\cdot)) = \varphi'(\tau, z, w(\cdot)) \). \( \square \)

### 3.3 Proof of Theorem 3

**Lemma 13** If the functional \( \varphi: \mathbb{G} \mapsto \mathbb{R} \) is a \( \mathbb{C}_1 \)-minimax solution of problem (8), (9), then it is the minimax solution of problem (8), (9).

**Proof** Let us show that \( \varphi \) satisfies inequalities (16a), (16b) for any \((\tau, z, w(\cdot)) \in \mathbb{G} \), \( \tau < \vartheta \), \( t \in (\tau, \vartheta] \), \( s \in \mathbb{R}^n \), and \( \eta = 0 \), where, without loss of generality, we can suppose that \( i\eta \leq t < t + (i+1)h \) for some \( i \in 0, T - 1 \) and \( t < t + h \).

Let \( i \in 0, T - 1 \), \((\tau, z, w(\cdot)) \in [i\eta, (i + 1)\eta) \times \mathbb{R}^n \times \operatorname{PLip}, t \in (\tau, (i + 1)h) \), and \( s \in \mathbb{R}^n \). Due to Lemma 4, there exists a sequence \( \{ w^j(\cdot) \}_{j \in \mathbb{N}} \in \Gamma(z, w(\cdot)) \) such that conditions (\( \omega_j \))-\( \omega_j \) hold. Since \( \varphi \) is a \( \mathbb{C}_1 \)-minimax solution of problem (8), (9), for each \( j \in \mathbb{N} \), there exists \( x^j(\cdot) \in X^{1/j}(\tau, w^j(-0), w^j(\cdot)) \) such that
\[
\varphi(t, x^j(t), x^j(\cdot)) + \omega(\tau, t, x^j(\cdot), s) \leq \varphi(\tau, w^j(-0), w^j(\cdot)) + 1/j.
\]
In accordance with Lemma 5, without loss of generality, we can suppose the existence of \( x(\cdot) \in X^0(\tau, z, w(\cdot)) \) such that \( \{ x^j(\cdot) \}_{j \in \mathbb{N}} \in \Gamma(x(t), x^j(\cdot)) \). Let \( \alpha_0 = \max\{\|z\|, \|w(\cdot)\|_{\infty} \} \). According Lemmas 1, 2 and conditions (\( H_3 \)), (\( \varphi_2 \)), define \( \alpha_X = \alpha_X(\alpha_0), \lambda_g = \lambda_g(\alpha_X), \lambda_H = \lambda_H(\alpha_X) \), and \( \lambda_{\varphi} = \lambda_{\varphi}(\alpha_X) \). Denote \( \lambda_{\omega} = \lambda_H(1 + \|s\|) + (1 + \lambda_g)\|s\| \). Then, using (17) and (29), we derive
\[
\begin{align*}
|\varphi(\tau, z, w(\cdot)) - \varphi(\tau, w^j(-0), w^j(\cdot))| & \leq \lambda_{\varphi} \psi(\tau, z - w^j(-0), w(\cdot) - w^j(\cdot)), \\
|\varphi(t, x(t), x^j(\cdot)) - \varphi(t, x^j(t), x^j(\cdot))| & \leq \lambda_{\varphi} \psi(t, x(t) - x^j(t), x^j(\cdot) - x^j(\cdot)), \\
|\omega(\tau, t, x(\cdot), s) - \omega(\tau, t, x^j(\cdot), s)| & \leq \lambda_{\omega} \psi(t, x(t) - x^j(t), x^j(\cdot) - x^j(\cdot)), \quad + \lambda_{\omega} \psi(\tau, z - w^j(-0), w(\cdot) - w^j(\cdot)).
\end{align*}
\]
Thus, passing to the limit as \( j \to +\infty \), form (47), we derive
\[
\varphi(t, x(t), x^j(\cdot)) + \omega(\tau, t, x(\cdot), s) \leq \varphi(\tau, z, w(\cdot)).
\]
This estimate implies (16a). Inequality (16b) can be proved by the similar way.

The functional $\varphi$ satisfies condition (9) for any $w(\cdot) \in PLip$ since $\varphi$ satisfies condition (9) for any $w(\cdot) \in C^1$ and due to Lemma 4 and condition $(\sigma)$. □

### 3.4 Proof of Theorem 4

**Lemma 14** Let $(\tau, z, w(\cdot)) \in G_*$. Let $i \in 0, I - 1$ satisfy $\tau \in (ih, (i + 1)h)$. Then there exists $\delta_* \in (0, h)$ such that $[\tau, \tau + \delta_*] \subset (ih, (i + 1)h)$ and the function $w(\cdot)$ is continuously differentiable on $[-h, -h + \delta_*]$.

**Proof** According to definition (3) of $G_*$, there exists $\delta_w > 0$ such that $w(\cdot)$ is continuously differentiable on $[-h, -h + \delta_w]$. Let $\delta_i > 0$ be such that $\tau + \delta_i < (i + 1)h$. Then, taking $\delta_* = \min\{\delta_w, \delta_i\}$, we obtain the statement of the lemma. □

**Lemma 15** If the functional $\varphi : G \mapsto \mathbb{R}$ is a minimax solution of problem (8), (9), then it is the viscosity solution of problem (8), (9).

**Proof** Let us prove that $\varphi$ satisfy (13a). Let $(\tau, z, w(\cdot)) \in G_*$, $\psi \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$, and $\delta > 0$ be such that

$$
\varphi(\tau, z, w(\cdot)) - \psi(\tau, z) \leq \varphi(t, x, \kappa_t(\cdot)) - \psi(t, x), \quad (t, x) \in O^+_\delta(\tau, z),
$$

where $\kappa(\cdot) \in \Lambda_0(\tau, z, w(\cdot))$. Let $\alpha_0 = \max\{\|z\|, \|w(\cdot)\|_{\infty}\}$. According to Lemmas 1, 3, 14 and condition $(\varphi_2)$, define $\alpha_X = \alpha_X(\alpha_0), \lambda_X = \lambda_X(\alpha_0), \delta_*$, and $\lambda_\varphi = \lambda_\varphi(\alpha_X)$, respectively. Denote $t_* = \tau + \min\{\delta_*, \delta, \delta/\lambda_X\}$. Then, we have

$$
|\varphi(t, x(t), x_1(\cdot)) - \psi(t, x(t), \kappa_t(\cdot))| \leq \lambda_\varphi \int_\tau^t \|x(\xi) - z\| d\xi \leq \lambda_\varphi \lambda_X (t - \tau)^2,
$$

$$(t, x(t)) \in O^+_\delta(\tau, z), \quad t \in [\tau, t_*], \quad x(\cdot) \in X^1(\tau, z, w(\cdot)). \quad (48)$$

Let $t_j = \tau + (t_* - \tau)/j$, $j \in \mathbb{N}$. Due to the inclusion $\psi \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$, there exist $\varepsilon_j > 0$, $j \in \mathbb{N}$ such that $\varepsilon_j \to 0$ as $j \to \infty$ and

$$
\frac{\psi(t_j, x(t_j)) - \psi(\tau, z)}{t_j - \tau} - \frac{\partial}{\partial \tau} \psi(\tau, z) - \left(\frac{x(t_j) - x(\tau)}{I_j - \tau}, \nabla_z \psi(\tau, z)\right) \geq -\varepsilon_j \quad (49)
$$

for any $x(\cdot) \in X^1(\tau, z, w(\cdot))$. Since $\varphi$ is the minimax solution of problem (8), (9), for each $j \in \mathbb{N}$, there exists $x^j(\cdot) \in X^1(\tau, z, w(\cdot))$ such that

$$
\varphi(t_j, x_j^j(t_j), x_j^j(\cdot)) + \omega(\tau, t_j, x_j^j(\cdot), \nabla_z \psi(\tau, z)) \leq \varphi(\tau, z, w(\cdot)) + (t_j - \tau)\varepsilon_j.
$$

(50)
Thus, taking (17) into account, we derive

\[
\frac{1}{t_j - \tau} \int_{\tau}^{t_j} H(\xi, x^j(\xi), x^j(\xi - h), \nabla_z \psi(\tau, z))d\xi
\]

\[
+ \frac{1}{t_j - \tau} \left\{ g(t_j, x(t_j - h)) - g(\tau, x(\tau - h)), \nabla_z \psi(\tau, z) \right\}
\]

\[
\leq -\frac{\partial}{\partial \tau} \psi(\tau, z) + 2\varepsilon_j + \lambda \chi(t_j - \tau).
\]

Passing to the limit as \( j \to \infty \), taking into account condition \((H_1)\), inequalities (27), (28), and Lemma 6, we obtain

\[
\partial \psi(\tau, z) / \partial \tau + (\partial^c t, w)g(\tau, w(\cdot)), \nabla_z \psi(\tau, z) + H(\tau, z, w(\cdot), \nabla_z \psi(\tau, z)) \leq 0.
\]

Thus, condition (13a) is proved. Condition (13b) can be proved similarly. \( \square \)

**Lemma 16** Let \( \varphi : \mathbb{G} \mapsto \mathbb{R} \) satisfies \((\varphi_1)\), \((\varphi_2)\) and \((\tau, z, w(\cdot)) \in \mathbb{G}_n\). Let \( \delta_1 > 0 \) be taken from Lemma 14. Then the function \( \tilde{\varphi} \) defined by

\[
\tilde{\varphi}(t, x) = \varphi(t, x, \kappa_1(\cdot)), \quad \kappa(\cdot) \in \Lambda_0(\tau, z, w(\cdot)),
\]

is continuous at every \((t, x) \in [\tau, \tau + \delta] \times \mathbb{R}^n\).

**Proof** Let \((t, x) \in [\tau, \tau + \delta] \times \mathbb{R}^n\) and \( \varepsilon > 0 \). Let \( \alpha_0 = \max\{||x||, ||z||, ||w(\cdot)||_\infty\} \).

Due to condition \((\varphi_2)\), define \( \lambda = \lambda_0(1) \). Let \( \varepsilon_* = \varepsilon/(32\lambda) \). Then, applying Lemma 4 to \((x, \kappa_1)\), we obtain the existence of \( w^*(\cdot) \in \mathbb{C}^1 \) such that

\[
||w^*(\cdot)||_\infty \leq \alpha_0, \quad ||x - w^*(-t)|| \leq \varepsilon_*, \quad ||\kappa_1 \kappa(h) - w^*(-h)|| \leq \varepsilon_*, \quad ||\kappa(ih) - w^*(ih - t)|| \leq \varepsilon_*. \tag{52}
\]

where \( i \in 0, \frac{T - 1}{h} \) satisfies \( \tau \in (ih, (i + 1)h) \) which implies \( t \in (ih, (i + 1)h) \) according to the choice \( \delta_1 \). Since \( w^*(\cdot) \in \mathbb{C}^1 \), there exist \( \lambda > 0 \) such that

\[
||w^*(\xi) - w^*(\xi')|| \leq \lambda||\xi - \xi'||, \quad \xi, \xi' \in [-h, 0). \tag{53}
\]

Due to Lemma 7 and condition \((\varphi_1)\), there exists \( \nu_1 > 0 \) such that

\[
||\kappa_1(\cdot) - \kappa_1(\cdot)||_1 \leq \nu_1, \quad ||\kappa(ih) - w^*(ih - t)|| \leq \nu_1. \tag{54}
\]

for every \( t' \in [\tau, \tau + \delta] \); \( |t' - t| \leq \nu_1 \). Due to the choice of \( \delta_1 \), there exists \( \lambda_0 > 0 \) such that

\[
||w(\xi - h) - w(\xi' - h)|| \leq \lambda_0||\xi - \xi'||, \quad \xi, \xi' \in [-h, -h + \delta]. \tag{55}
\]
Define

\[ v = \min\{1, \varepsilon_*, \varepsilon_*/\lambda_0, \varepsilon_*/\lambda^*, \nu_\bullet\}. \quad (56) \]

For proving the lemma, it suffices to establish the inequality

\[ |\tilde{\phi}(t, x) - \tilde{\phi}(t', x')| \leq c, \quad (t', x') \in O_* := O_v(t, x) \cap ([\tau, \tau + \delta_0] \times \mathbb{R}^n). \quad (57) \]

Firstly, due to the choice of \(\delta_*, \alpha_0, \) and \(\lambda_\varphi,\) from (52), (53), the first inequality in (54), and (55)-(57), we derive

\[ |\varphi(t', x', \kappa_\tau(\cdot)) - \varphi(t', w^*(-0), w^*(\cdot))| \leq \lambda_\varphi \left(4\varepsilon_\bullet + \|x' - x\| + \|\kappa_\tau'(\cdot) - \kappa_\tau(\cdot)\|_1 + \|\kappa_\tau'(-h) - \kappa_\tau(-h)\| + \|w^*(ih - t) - w^*(ih - t')\| \right) \leq 8\lambda_\varphi \varepsilon_\bullet \]

for any \((t', x') \in O_*\). Applying this estimate for \((t, x)\) and arbitrary \((t', x') \in O_*\), taking (51) into account, we obtain

\[ |\tilde{\phi}(t, x) - \tilde{\phi}(t', x')| \leq |\varphi(t, w^*(-0), w^*(\cdot)) - \varphi(t', w^*(-0), w^*(\cdot))| + 16\lambda_\varphi \varepsilon_\bullet. \]

In accordance with the choice of \(\varepsilon_\bullet,\) the second inequality in (54), and (56), from this inequality we conclude (57).

\[ \square \]

**Lemma 17** If the functional \(\varphi: \mathbb{G} \mapsto \mathbb{R}\) is a viscosity solution of problem (8), (9), then it satisfies (22a) and (22b).

**Proof** Let \((\tau, z, w(\cdot)) \in \mathbb{G}_\bullet\) and \((p_0, p) \in D^- \varphi(\tau, z, w(\cdot))\). Choose the number \(\delta_\bullet\) in accordance with Lemma 14. Taking the function \(\tilde{\phi}\) by (51), define consistently the function \(\tilde{\phi}_\bullet\) as follows

\[ \tilde{\phi}_\bullet(t, x) = \min\{0, \tilde{\phi}(t, x) - \tilde{\phi}(\tau, z) - (t - \tau)p_0 - \langle x - z, p \rangle\} \]

for \((t, x) \in [\tau, \tau + \delta_\bullet] \times \mathbb{R}^n, \tilde{\phi}_\bullet(t, x) = \tilde{\phi}_\bullet(\tau + \delta_\bullet, x)\) for \((t, x) \in (\tau + \delta_\bullet, +\infty) \times \mathbb{R}^n,\)

and \(\tilde{\phi}_\bullet(t, x) = \tilde{\phi}_\bullet(\tau - (t - \tau), x)\) for \((t, x) \in (-\infty, \tau) \times \mathbb{R}^n.\) Then, according to Lemma 16, this function is continuous on \(\mathbb{R} \times \mathbb{R}^n.\) The further proof of the fact that \(\varphi\) satisfies (22a) can be carried out in the same way as in Lemma 8.1 from [27]. In a similar way one can prove that (13b) implies (22b).

\[ \square \]

**Lemma 18** Let \(\varphi: \mathbb{G} \mapsto \mathbb{R}\) satisfy (\(\varphi_2\)) and \((\tau, z, w(\cdot)) \in \mathbb{G}_\bullet.\) Let \(L \subset \mathbb{R}^n\) be a nonempty compact set. Suppose that

\[ \partial_{1+l}^- \varphi(\tau, z, w(\cdot)) > 0, \quad l \in L. \quad (58) \]

Then, there exist \(\varepsilon_\circ, \delta_\circ > 0\) such that

\[ \varphi(t, z + l(t - \tau), \kappa_l(\cdot)) - \varphi(\tau, z, w(\cdot)) > \varepsilon_\circ(\tau - t) \quad (59) \]

for any \(t \in (\tau, \tau + \delta_\circ]\) and \(l \in L,\) where \(\kappa(\cdot) \in \Lambda_0(\tau, z, w(\cdot)).\)
Proof The proof of the lemma is based on definition (18a) of lower directional derivatives and condition \((\varphi_2)\).

Lemma 19 Let \(\varphi : \mathbb{G} \mapsto \mathbb{R}\) satisfy condition \((\varphi_2)\). Let \((\tau, z, w(\cdot)) \in \mathbb{G}_\ast\). Let \(L \subset \mathbb{R}^n\) be a nonempty convex compact set. Suppose that \((58)\) holds. Then, for every \(\delta > 0\), there exist

\[ (t_\ast, x_\ast) \in \Omega_\delta := \{(t, x) \in [\tau, \tau + \delta] \times \mathbb{R}^n : \min_{l \in L} \|x - z - l(t - \tau)\| \leq \delta\}, \]

\[ (p_0, p) \in D^-\varphi(t_\ast, x_\ast, \kappa_\ast(\cdot)), \quad \kappa(\cdot) \in \Lambda_0(\tau, z, w(\cdot)), \]

such that

\[ p_0 + \langle l, p \rangle > 0, \quad l \in L. \tag{61} \]

Proof Take \(\delta_\ast > 0\) from Lemma 18. According to \((60)\), one can take \(\alpha_\Omega > 0\) such that \(\|\|x\|, \|w(\cdot)\|_\infty\| \leq \alpha_\Omega\) for any \((t, x) \in \Omega_{\delta_\ast}\). Due to condition \((\varphi_2)\), define \(\lambda_\varphi = \lambda_\varphi(\alpha_\Omega)\). Then, for every \((t, x), (t_\ast, x_\ast) \in \Omega_{\delta_\ast}\) satisfying \(t \geq t_\ast\), and \(\chi(\cdot) \in \Lambda_0(t_\ast, x_\ast, \kappa_\ast(\cdot))\), we derive

\[ |\varphi(t, x, \chi_t(\cdot)) - \varphi(t, x, \kappa_t(\cdot))| \leq \lambda_\varphi \|\chi_t(\cdot) - \kappa_t(\cdot)\| \leq \lambda_\varphi\|x_\ast - z\|(t - t_\ast). \tag{62} \]

According to Lemma 18, there exist \(\epsilon_\ast, \delta_\ast > 0\) such that \((59)\) holds. Then, without loss of generality, we can suppose that

\[ \delta \leq \min\{\delta_\ast, \delta_\ast\}, \quad \delta < \epsilon_\ast/(\lambda_\varphi(1 + \lambda_L)), \quad \lambda_L = \max\{\|l\| : l \in L\}. \tag{63} \]

For each \(k \in \mathbb{N}\), define the function

\[ \gamma_k(t, x, \xi, y) = \varphi(t, x, \kappa_t(\cdot)) + k\|x - y\|^2 + k(t - \xi)^2 - \epsilon_\ast(\xi - \tau), \tag{64} \]

where \((t, x) \in \Omega_\delta\) and \((\xi, y) \in \Omega_\delta^* := \{(\xi, y) \in \Omega_\delta : \min_{l \in L} \|y - z - l(\xi - \tau)\| = 0\}.

According to the choice of \(\delta\) and Lemma 16, \(\gamma_k\) is continuous. The set \(\Omega_\delta \times \Omega_\delta^*\) is compact. Therefore, there exists \((t_k, x_k, \xi_k, y_k) \in \Omega_\delta \times \Omega_\delta^*\) such that

\[ \gamma_k(t_k, x_k, \xi_k, y_k) = \min_{(t, x, \xi, y) \in \Omega_\delta \times \Omega_\delta^*} \gamma_k(t, x, \xi, y). \tag{65} \]

Furthermore, without loss of generality, we suppose that \((t_k, x_k, \xi_k, y_k) \to (\bar{t}, \bar{x}, \bar{\xi}, \bar{y}) \in \Omega_\delta \times \Omega_\delta^*\) as \(k \to \infty\). Due to \((65)\), we have

\[ \gamma_k(t_k, x_k, \xi_k, y_k) \leq \gamma_k(\tau, z, \tau, z) = \varphi(\tau, z, w(\cdot)). \tag{66} \]

Hence, we obtain

\[ \bar{t} = \bar{\xi}, \quad \bar{x} = \bar{y}. \tag{67} \]
Let us show that $\bar{t} < \tau + \delta$. For the sake of a contradiction, suppose that $\bar{t} = \tau + \delta$.

Then, applying Lemma 16 and (59), (64), we derive

$$\liminf_{k \to \infty} \gamma_k(t_k, x_k, \xi_k, y_k) \geq \lim_{k \to \infty} \left( \varphi(t_k, x_k, \kappa_k(\cdot)) - \varepsilon_0(\xi_k - \tau) \right)$$

$$= \varphi(\tau + \delta, \bar{y}, \kappa_{\tau + \delta}(\cdot)) - \varepsilon_0\delta > \varphi(\tau, z, w(\cdot)).$$

This inequality contradicts (66).

In accordance with $\bar{t} < \tau + \delta$ and (67), one can take $k \in \mathbb{N}$ so that

$$t_k < \tau + \delta, \quad \xi_k < \tau + \delta, \quad \|x_k - y_k\| \leq \delta/4, \quad \lambda_L|t_k - \xi_k| \leq \delta/4,$$

where the number $\lambda_L$ is defined in (63). Put

$$p_0 = -2k(t_k - \xi_k) - \lambda_{\varphi}\|x_k - z\|, \quad p = -2k(x_k - y_k).$$

Let us prove the inclusion $(p_0, p) \in D^{-}\varphi(t_k, x_k, \kappa_k(\cdot))$. Since $(\xi_k, y_k) \in \Omega_{\delta}^p$, there exists $l_k$ such that $y_k = z + l_k(\xi_k - \tau)$. Then, due to definition $\lambda_L$ in (63) and (68), we have

$$\|x - z - l_k(\tau - \tau)\| \leq \|x - x_k\| + \|x_k - y_k\| + \lambda_L|t - t_k| + \lambda_L|t_k - \xi_k| \leq \delta$$

for any $(t, x) \in O_{\delta(1 + \lambda_L)/4}^+(t_k, x_k)$. It means that $(t, x) \in \Omega_{\delta}$ for any $(t, x) \in O_{\delta(1 + \lambda_L)/4}^+(t_k, x_k)$. Applying (64), (65), we obtain

$$0 \leq \gamma_k(t, x, \xi_k, y_k) - \gamma_k(t_k, x_k, \xi_k, y_k) = \varphi(t, x, \kappa_k(\cdot)) - \varphi(t_k, x_k, \kappa_k(\cdot))$$

$$+ k\|x - x_k\|^2 + 2k(x - x_k, x_k - y_k) + k|t - t_k|^2 + 2k(t - t_k)(t_k - \xi_k)$$

for any $(t, x) \in O_{\delta(1 + \lambda_L)/4}^+(t_k, x_k)$. Then, taking into account (19), (62) for $(t_a, x_a) = (t_k, x_k)$, and (69), we conclude $(p_0, p) \in D^{-}\varphi(t_k, x_k, \kappa_k(\cdot))$.

Let us prove (61). Let $l \in L$. Since $L$ is convex, then we have

$$l_v = (lv + l_k(\xi_k - \tau))/(v + \xi_k - \tau) \in L, \quad v \in (0, \delta + \tau - \xi_k).$$

From this inclusion and $(\xi_k, y_k) \in \Omega_{\delta}^p$, we derive $\|y_k + lv - z - l_k(\xi_k + v - \tau)\| = 0$ that means the inclusion $(\xi_k + v, y_k + lv) \in \Omega_{\delta}^p, v \in (0, \delta + \tau - \xi_k)$. Then, according to (64), (65), for every $v \in (0, \delta + \tau - \xi_k)$, we obtain

$$0 \leq \gamma_k(t_k, x_k, \xi_k + v, y_k + lv, \gamma_k(t_k, x_k, \xi_k, y_k)$$

$$= k\|l\|^2v^2 - 2k(l, x_k - y_k)v + kv^2 - 2k(t_k - \xi_k)v - \varepsilon_0v.$$

Dividing this inequality by $v$ and passing to the limit as $v \to +0$, we get

$$\varepsilon_0 \leq -2k(x_k - y_k, l) - 2k(t_k - \xi_k).$$

(70)
Since \((t_k, x_k) \in \Omega_\delta\), there exists \(l_x \in L\) such that \(\|x_k - z - l_x(t_k - \tau)\| \leq \delta\). Then, using (63), we derive
\[
\|x_k - z\| \leq \|x_k - z - l_x(t_k - \tau)\| + \|l_x\|(t_k - \tau) \leq (1 + \lambda L)\delta < \epsilon_0/\lambda\phi. \tag{71}
\]
From (69)–(71), we conclude (61). Thus, taking \((t_\eta, x_\eta) = (t_k, x_k)\) we obtain statement of the lemma.

\textbf{Lemma 20} Let \(\phi : \mathbb{G} \mapsto \mathbb{R}\) satisfy \((\varphi_2)\) and \((\tau, z, w(\cdot)) \in \mathbb{G}_\eta\). If there exists \(l_\eta \in \mathbb{R}^n\) such that \(\partial_{1,l_\eta} \phi(t, z, w(\cdot)) \in \mathbb{R}\), then \(\partial_{1,l_\eta} \phi(t, z, w(\cdot)) \in \mathbb{R}\) for every \(l \in \mathbb{R}^n\), and the function \(\phi(l) = \partial_{1,l_\eta} \phi(t, z, w(\cdot)), l \in \mathbb{R}^n\), is continuous. If there exists \(l_\eta \in \mathbb{R}^n\) such that \(\partial_{1,l_\eta} \phi(t, z, w(\cdot)) = +\infty\), then \(\partial_{1,l_\eta} \phi(t, z, w(\cdot)) = +\infty\) for every \(l \in \mathbb{R}^n\).

\textbf{Proof} follows directly from condition \((\varphi_2)\) and definition (18).

\textbf{Lemma 21} If the functional \(\phi : \mathbb{G} \mapsto \mathbb{R}\) satisfies \((22a)\) and \((22b)\), then it satisfies \((21a)\) and \((21b)\).

\textbf{Proof} Let us prove \((21a)\). For the sake of a contradiction, suppose that there exist \((\tau, z, w(\cdot)) \in \mathbb{G}_\eta\) and \(s \in \mathbb{R}^n\) such that
\[
\partial_{1,l} \phi(\tau, z, w(\cdot)) + \langle \partial_{1,l} g(\tau, w(\cdot), s) + H(\tau, z, w(-h), s) - \langle l, s \rangle > 0
\]
for any \(l \in F(z, w(-h)) + \partial_{1,l} g(\tau, w(\cdot))\). In the case if there exists \(l_\eta \in F(z, w(-h)) + \partial_{1,l_\eta} g(\tau, w(\cdot))\) such that \(\partial_{1,l_\eta} \phi(\tau, z, w(\cdot)) \in \mathbb{R}\), then, taking into account Lemma 20, one can take \(\eta, \epsilon > 0\) so that
\[
\partial_{1,l} \phi(\tau, z, w(\cdot)) + \langle \partial_{1,l} g(\tau, w(\cdot), s) + H(\tau, z, w(-h), s) - \langle l, s \rangle > \epsilon \tag{72}
\]
for any \(l \in F^n(z, w(-h)) + \partial_{1,l} g(\tau, w(\cdot))\). If there exists \(l_0 \in F(z, w(-h))\) such that \(\partial_{1,l_0} \phi(\tau, z, w(\cdot)) = +\infty\), then, in accordance with Lemma 20, inequality (72) also holds.

Put \(L = F^n(z, w(-h)) + \partial_{1,l} g(\tau, w(\cdot))\). According to conditions \((H_1)\) and (14), there exists \(\delta_2 > 0\) such that
\[
|H(t, x, w(t - \tau - h), s) - H(\tau, z, w(-h), s)| \leq \epsilon, \quad F^0(x, w(t - \tau - h)) \subseteq F^n(z, w(-h)), \tag{73}
\]
Define the functional \(\phi_\eta : \mathbb{G} \mapsto \mathbb{R}\) by
\[
\phi_\eta(t, x, r(\cdot)) = \phi(t, x, r(\cdot)) + \langle \partial_{1,l} g(\tau, w(\cdot), s) + H(\tau, z, w(-h), s) - \langle l, s \rangle > (t - \tau) - \langle x, s \rangle, \quad (t, x, r(\cdot)) \in \mathbb{G}.
\]
Since \(\phi\) satisfy \((\varphi_2)\), then \(\phi_\eta\) satisfy \((\varphi_2)\). Moreover, from (72), we derive \(\partial_{1,l} \phi_\eta(\tau, z, w(\cdot)) > 0, l \in L\). Applying Lemma 19 to the functional \(\phi_\eta\), the set
Let us define
\[ p'_0 = p_0 - H(t, z, w(-h), s) - \langle \partial_{t,w}^i g(\tau, w(\cdot)), s \rangle + \varepsilon, \quad p' = p + s. \]

Then, we have \((p'_0, p') \in D^{-}\varphi(t_s, x_s, \kappa(\cdot))\). Thus, taking the choice of \(\delta\) into account, from (14), (22a), (73), and condition \((H_3)\), we obtain
\[
0 \geq p'_0 + \langle \partial_{t,w}^i g(\tau, w(\cdot)), p' \rangle + H(t_s, x_s, w(t_s - \tau - h), p') \\
\geq p_0 + \langle \partial_{t,w}^i g(\tau, w(\cdot)), p \rangle - c_H\left(1 + \|x_s\| + \|w(t_s - \tau - h)\|\right)\|p\| \\
= p_0 + \langle \partial_{t,w}^i g(\tau, w(\cdot)), p \rangle + \min_{l \in F^0(t_s, w(t_s - \tau - h))} \langle l, p \rangle \\
\geq p_0 + \langle \partial_{t,w}^i g(\tau, w(\cdot)), p \rangle + \min_{l \in F^0(t, w(-h))} \langle l, p \rangle.
\]

This estimate contradicts (74). Thus, (21a) holds. For \(\partial_t^+ \varphi(t, z, w(\cdot))\) and \(D^+ \varphi(t, z, w(\cdot))\), one can establish statements similar to Lemmas 19, 20 and prove (21b).

\[\-boxed{Lemma 22}\] If the functional \(\varphi: \mathbb{G} \mapsto \mathbb{R}\) satisfies \((\varphi_1), (\varphi_2), (21a),\) and \((21b)\), then it is \(C^1\)-minimax solution of problem \((8), (9)\).

\[\text{Proof}\] Let us prove (16a) for any \(i \in 0, I - 1, (\tau, w(\cdot)) \in [ih, (i + 1)h) \times C^1, t \in (\tau, (i + 1)h)\), \(\eta \in (0, 1)\), and \(s \in \mathbb{R}^n\).

For the sake of a contradiction, suppose that there exist \(i \in 0, I - 1, (\tau, w(\cdot)) \in [ih, (i + 1)h) \times C^1, t \in (\tau, (i + 1)h)\), \(\eta \in (0, 1)\), \(s \in \mathbb{R}^n\), and \(\varepsilon > 0\) such that
\[
\varphi(t, x(t), x_t(\cdot)) - \varphi(\tau, w(-0), w(\cdot)) + \omega(\tau, t, x(\cdot), s) > \varepsilon
\]
for any \(x(\cdot) \in X^\eta := X^\eta(\tau, w(-0), w(\cdot))\). Let
\[
\xi_* = \max \left\{ \xi \in [\tau, t]: \min_{x(\cdot) \in X^\eta} \left( \varphi(\xi, x(\xi), x_\xi(\cdot)) + \omega(\xi, \tau, x(\cdot), s) \right) \leq \beta(\xi) \right\},
\]
where \(\beta(\xi) = \varphi(\tau, w(-0), w(\cdot)) + \varepsilon(\xi - t)/(t - \tau), \xi \in [\tau, t]\). Similar to Assertion 2 from [28], one can show that the set \(X^\eta\) is a compact on the space of continuous functions and there exist \(\alpha_X, \lambda_X > 0\) such that
\[
\|x(\xi)\| \leq \alpha_X, \quad \|x(\xi) - x(\xi')\| \leq \lambda_X|\xi - \xi'|, \quad \xi, \xi' \in [\tau - h, \vartheta], \quad x(\cdot) \in X^\eta.
\]

Using conditions \((\varphi_1)\) and \((\varphi_2)\), one can state the functional \(\hat{\varphi}(\tau, w(\cdot)) = \varphi(\tau, w(-0), w(\cdot))\) is continuous at any \((\tau, w(\cdot)) \in [0, \vartheta] \times \text{Lip}\). Thus, taking into account conditions \((g), (H_1)\), and definition (17) of \(\omega\), we can establish that these
maximum and minimum are archived. In accordance with (75), we have \( \xi_* < t \). Let the function \( x(\cdot) \in X^0 \) be such that

\[
\varphi(\xi_*, x(\xi_*), x_{\xi_*}(\cdot)) + \omega(\tau, \xi_*, x(\cdot), s) \leq \beta(\xi_*). \tag{76}
\]

Denote \( \varepsilon_* = \varepsilon/(5(t - \tau)) \). Due to (21a), there exists \( l_* \in F^0(x(\xi_*), x(\xi_* - h)) + \partial^{ci}_{\tau, w} g(\xi_*, x_{\xi_*}(\cdot)) \) satisfying

\[
\partial_{1, l_*} \varphi(\xi_*, x(\xi_*), x_{\xi_*}(\cdot)) + (\partial^{ci}_{\tau, w} g(\xi_*, x_{\xi_*}(\cdot)), s) + H(\xi_*, x(\xi_*), x(\xi_* - h), s - (l_*, s) \leq \varepsilon_* \tag{77}
\]

Redefine \( x(\cdot) \) on the interval \([\xi_*, \vartheta]\) so that \( dx(\xi)/d\xi = l_*, \xi \in [\xi_*, \vartheta] \). According to condition (\(\varphi_2\)), define \( \lambda_\varphi = \lambda_\varphi(\alpha_X) \). Then, due to (14), (18a), condition (\(H_1\)), and Lemma 6 in which, since \( w(\cdot) \in C^1 \), we can take \( \delta_w \in (\xi_* - \tau, t - \tau) \), there exist \( t_* \in (\xi_*, \min\{t + \delta_w, \xi_* + \varepsilon_*/(\lambda_\varphi \lambda_X)\}) \) such that

\[
\frac{\varphi(t_*, x(t_*), \kappa_{t_*}(\cdot)) - \varphi(\xi_*, x(\xi_*), x_{\xi_*}(\cdot))}{t_* - \xi_*} \leq \partial_{1, l_*} \varphi(\xi_*, x(\xi_*), x_{\xi_*}(\cdot)) + \varepsilon_*, \tag{78}
\]

where \( \kappa(\cdot) \in A_0(\xi_*, x(\xi_*), x_{\xi_*}(\cdot)), \) and

\[
F^0(x(\xi_*), x(\xi_* - h)) \subset F_{1/2}^0(x(\hat{\xi}_t), x(\hat{\xi}_t - h)),
\]

\[
\left| H(\xi, x(\xi), x_{\xi}(\cdot), s) - H(\xi_*, x(\xi_*), x_{\xi_*}(\cdot), s) \right| \leq \varepsilon_*,
\]

\[
\left| \frac{d}{d\xi}(g(\xi, x_{\xi}(\cdot)) - \partial^{ci}_{\tau, w} g(\xi_*, x_{\xi_*}(\cdot)) \right| \leq \min \left\{ \eta_2, \frac{\varepsilon_*}{2 ||s|| + 1} \right\} \tag{79}
\]

for any \( \xi \in [\xi_*, t_*] \). Then, we have

\[
\frac{d}{d\xi}\left(x(\xi) - g(\xi, x(\xi - h))\right) \in F^0(x(\xi), x(\xi - h)) \quad a.e. \xi \in [\xi_*, t_*].
\]

Redefining the function \( x(\cdot) \) on the interval \([t_*, \vartheta]\) such that \( x(\cdot) \in X^0 \), according to the choice of \(\alpha_X, \lambda_X, \lambda_\varphi,\) and \( t_* \), we derive

\[
|\varphi(t_*, x(t_*), \kappa_{t_*}(\cdot)) - \varphi(t_*, x(t_*), x_{t_*}(\cdot))| \leq \lambda_\varphi \lambda_X (t_* - \xi_*)^2 \leq \varepsilon_*(t_* - \xi_*).
\]

Using this estimate, from (76)–(79), taking the definition of \( \varepsilon_* \) and \( \beta \) into account, we obtain

\[
\varphi(t_*, x(t_*), x_{t_*}(\cdot)) + \omega(\tau, t_*, x(\cdot), s) \leq \beta(t_*).
\]

This inequality contradict the choice of \( \xi_* \). □
3.5 Proof of Theorem 6

**Lemma 23**  The value functional $\hat{\varphi}$ defined in (26) is the functional from Lemma 8.

**Proof** Let us prove (35a). Let $(\tau, w(\cdot)) \in [0, \vartheta) \times \text{Lip}, t \in (\tau, \vartheta], s \in \mathbb{R}^n$, and $\varepsilon > 0$. Due to definition (26) of $\hat{\varphi}$, one can show (see, e.g., [3, Chapter VIII, Theorem 1.9]) that there exists $u(\cdot) : [\tau, \vartheta] \mapsto \mathbb{U}$ such that

$$
\hat{\varphi}(t, x_t(\cdot)) + \int_{\tau}^{t} f^0(\xi, x(\xi), x(\xi - h), u(\xi)) d\xi \leq \hat{\varphi}(\tau, x_\tau(\cdot))
$$

where $x(\cdot) \in \Lambda(\tau, w(-0), w(\cdot))$ is the Lipschitz continuous function satisfying neutral-type equation (24). From this estimate, using (25), we get (35a).

Let us prove (35b). Let $(\tau, w(\cdot)) \in [0, \vartheta) \times \text{Lip}, t \in (\tau, \vartheta], s \in \mathbb{R}^n$, and $\varepsilon > 0$. Due to Assertion 2 from [28], there exist $\alpha_X, \lambda_X > 0$ such that

$$
\|x(\xi)\| \leq \alpha_X, \quad \|x(\xi) - x(\xi')\| \leq \lambda_X|\xi - \xi'|, \quad \xi, \xi' \in [\tau - h, \vartheta]
$$

for any Lipschitz continuous function $x(\cdot) \in \Lambda(\tau, w(-0), w(\cdot))$ satisfying neutral-type equation (24). Then, taking into account condition $(H_1)$ and continuity of the functions $f$ and $f^0$, there exist $\delta > 0$ such that

$$
\left| H(\xi, x(\xi), x(\xi - h), s) - H(\xi', x(\xi'), x(\xi' - h), s) \right| \leq \varepsilon,
$$
$$
\left| f(\xi, x(\xi), x(\xi - h), u) - f(\xi', x(\xi'), x(\xi' - h), u) \right| \leq \varepsilon,
$$
$$
\left| f^0(\xi, x(\xi), x(\xi - h), u) - f^0(\xi', x(\xi'), x(\xi' - h), u) \right| \leq \varepsilon
$$

for any $\xi, \xi' \in [\tau, \vartheta]$ satisfying $|\xi - \xi'| \leq \delta$, any $u \in \mathbb{U}$, and any Lipschitz continuous $x(\cdot) \in \Lambda(\tau, w(-0), w(\cdot))$ satisfying (24). Let $k \in \mathbb{N}$ be such that $\Delta t := (t - \tau)/k \leq \delta$ and $t_i = \tau + i \Delta t, i \in \overline{0, k}$. Let the function $u(\cdot) : [\tau, \vartheta] \mapsto \mathbb{U}$ and the Lipschitz continuous function $x(\cdot)$ satisfy (24) and the following feedback rule:

$$
u(\xi) = u_i, \quad \xi \in [t_i, t_{i+1}), \quad i \in \overline{0, k - 1}
$$

where, in accordance with (25), the value $u_i \in \mathbb{U}$ can be taken satisfying $H(t_i, x(t_i), x(t_i - h), s) = (f(t_i, x(t_i), x(t_i - h), u_i), s) + f^0(t_i, x(t_i), x(t_i - h), u_i)$. Then, taking into account (17) and (26), we derive

$$
\hat{\varphi}(t_{i+1}, x_{t_{i+1}}(\cdot)) + \omega(t_i, t_{i+1}, x(\cdot), s) + 3(t_{i+1} - t_i) \varepsilon
$$

$$
\geq \hat{\varphi}(t_{i+1}, x_{t_{i+1}}(\cdot)) + \int_{t_i}^{t_{i+1}} f^0(\xi, x(\xi), x(\xi - h), u(\xi)) d\xi \geq \hat{\varphi}(t_i, x_t(\cdot))
$$

for any $i \in \overline{0, k - 1}$. Using this estimate for each $i \in \overline{0, k - 1}$ and $\varepsilon > 0$, we conclude (35b). \hfill \square

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References

1. Aubin, J.P., Haddad, G.: History path dependent optimal control and portfolio valuation and management. Positivity 6(3), 331–358 (2002). https://doi.org/10.1023/A:1020244921138
2. Bayraktar, E., Keller, C.: Path-dependent Hamilton-Jacobi equations in infinite dimensions. J. Funct. Anal. 275(8), 2096–2161 (2018). https://doi.org/10.1016/j.jfa.2018.07.010
3. Bardi, M., Capuzzo-Dolcetta, I.: Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations. Birkhäuser, Boston (1997)
4. Clarke, F.H., Ledyaev, Yu.S.: Mean value inequalities in Hilbert space. Trans. Am. Math. Soc. 344(1), 307–324 (1994). https://doi.org/10.2307/2154718
5. Crandall, M.G., Lions, P.-L.: Viscosity solutions of Hamilton-Jacobi equations. Trans. Am. Math. Soc. 277(1), 1–42 (1983). https://doi.org/10.2307/1999343
6. Crandall, M.G., Evans, L.C., Lions, P.-L.: Some properties of viscosity solutions of Hamilton-Jacobi equations. Trans. Am. Math. Soc. 282(2), 487–502 (1984). https://doi.org/10.2307/1999247
7. Dupire, B.: Functional Itô calculus. https://ssrn.com/abstract=1435551 (2009). Accessed 25 July 2009
8. Ekren, I., Touzi, N., Zhang, J.: Viscosity solutions of fully nonlinear parabolic path dependent PDEs: part I. Ann. Probab. 44(2), 1212–1253 (2016). https://doi.org/10.1214/15-AOP1027
9. Filippov, A.F.: Differential Equations with Discontinuous Righthand Sides. Springer, Berlin (1988)
10. Gomoyunov, M.I., Lukoyanov, N.Yu., Plaksin, A.R.: Path-dependent Hamilton-Jacobi equations: the minimax solutions revised. Appl. Math. Optim. 84(1), S1087–S1117 (2021)
11. Gomoyunov, M.I., Plaksin, A.R.: On basic equation of differential games for neutral-type systems. Mech. Solids 54(2), 131–143 (2019)
12. Hale, J.: Theory of Functional Differential Equations. Springer-Verlag, New York (1977)
13. Kaise, H.: Path-dependent differential games of inf-sup type and Isaacs partial differential equations. In: Proceedings of the 54th IEEE Conference on Decision and Control (CDC). 1972–1977 (2015). https://doi.org/10.1109/CDC.2015.7402496
14. Kaise, H., Kato, T., Takahashi, Y.: Hamilton-Jacobi partial differential equations with path-dependent terminal costs under superlinear Lagrangians. In: Proceedings of the 23rd International Symposium on Mathematical Theory of Networks and Systems (MTNS), pp. 692–699 (2018)
15. Kim, A.V.: Functional Differential Equations: Application of i-Smooth Calculus. Kluwer Academic Publishers, Dordrecht (1999)
16. Krasovskii, N.N., Subbotin, A.I.: Game-Theoretical Control Problems. Springer, New York (1988)
17. Krasovskii, A.N., Krasovskii, N.N.: Control Under Lack of Information. Birkhäuser, Berlin (1995)
18. Lukoyanov, N.Yu.: A Hamilton-Jacobi type equation in control problems with hereditary information. J. Appl. Math. Mech. 64, 243–253 (2000). https://doi.org/10.1016/S0021-8928(00)00046-0
19. Lukoyanov, N.Yu.: Functional Hamilton-Jacobi type equation in ci-derivatives for systems with distributed delays. Nonlinear Funct. Anal. Appl. 8(3), 365–397 (2003)
20. Lukoyanov, N.Yu.: On optimality conditions in control problems for time-delay systems. Proc. Steklov Inst. Math. 1, 175–187 (2010)
21. Lukoyanov, N.Yu.: Minimax and viscosity solutions in optimization problems for hereditary systems. Proc. Steklov Inst. Math. 2, 214–225 (2010). https://doi.org/10.1134/S0081543810060179
22. Lukoyanov, N.Yu., Plaksin, A.R.: Hamilton-Jacobi equations for neutral-type systems: inequalities for directional derivatives of minimax solutions. Minimax Theory Appl. 5(2), 369–381 (2020)
23. Lukoyanov, N.Yu., Plaksin, A.R.: On the theory of positional differential games for neutral-type systems. Proc. Steklov Inst. Math. 309(1), 583–592 (2020). https://doi.org/10.1134/S0081543820040100
24. Natanson, I.P.: Theory of Functions of a Real Variable, vol. 2. Frederick Ungar Publishing Co., New-York (1960)
25. Pham, T., Zhang, J.: Two person zero-sum game in weak formulation and path dependent Bellman-Isaacs equation. SIAM J. Control. Optim. 52(4), 2090–2121 (2014). https://doi.org/10.1137/120894907
26. Plaksin, A.: Minimax and viscosity solutions of Hamilton-Jacobi-Bellman equations for time-delay systems. J. Optim. Theory Appl. 187(1), 22–42 (2020). https://doi.org/10.1007/s10957-020-01742-6
27. Plaksin, A.: Viscosity solutions of Hamilton-Jacobi-Bellman-Isaacs equations for time-delay systems. SIAM J. Control. Optim. 59(3), 1951–1972 (2021). https://doi.org/10.1137/20M1311880
28. Plaksin, A.: On the minimax solution of the Hamilton-Jacobi equations for neutral-type systems: the case of an inhomogeneous Hamiltonian. Differ. Equ. 57(11), 1516–1526 (2021). https://doi.org/10.1134/S0012266121110100
29. Soner, H.M.: On the Hamilton-Jacobi-Bellman equations in Banach spaces. J. Optim. Theory Appl. 57(3), 429–437 (1988). https://doi.org/10.1007/BF02346162
30. Subbotin, A.I.: A generalization of the basic equation of the theory of differential games. Sov. Math.-Doklady. 22, 358–362 (1980)
31. Subbotin, A.I.: Generalization of the main equation of differential game theory. J. Optim. Theory Appl. 43(1), 151–162 (1984)
32. Subbotin, A.I.: On a property of the subdifferential. Math. USSR - Sbornik. 74(1), 63–78 (1993)
33. Subbotin, A.I.: Generalized Solutions of First Order PDEs: The Dynamical Optimization Perspective. Birkhäuser, Boston (1995)
34. Zhou, J.: Delay optimal control and viscosity solutions to associated Hamilton-Jacobi-Bellman equations. Int. J. Control 92(10), 2263–2273 (2019). https://doi.org/10.1080/00207199.2018.1436769
35. Zhou, J.: A notion of viscosity solutions to second-order Hamilton-Jacobi-Bellman equations with delays. Int. J. Control (2021). https://doi.org/10.1080/00207199.2021.1921279

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