Dimension formula for the space of relative symmetric polynomials of $D_n$ with respect to any irreducible representation

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Abstract
We provide an alternative formula along with the generating function for the dimension of the vector space of relative symmetric polynomials of $D_n$ with respect to any irreducible character of $D_n$.

1 Introduction

Every complex representation of a finite group has a canonical decomposition into direct sum of isotypical components. Serre’s textbook [2, page 21] gives the formula for the projection map to all these components. We recall the formula here:

Given the isotypical decomposition $V = \bigoplus_{\chi \in \text{Irr}(G)} V_{\chi}$, the projection to the component $V_{\chi}$ is given by

$$p = \frac{\deg \chi}{|G|} \sum_{g \in G} \bar{\chi}(g) \rho(g) \quad (1.1)$$

When $\chi$ is the trivial 1-dimensional representation this projection is the Reynold’s operator.

In this paper we focus on the natural action of $S_n$ (and its subgroups) on the complex polynomial algebra of $n$ variables, by permuting the variables.

We denote by $H_d(x_1, x_2, \ldots, x_n)$ the complex vector space of all homogeneous polynomials of degree $d$ in the $n$ variables, $x_1, x_2, \ldots, x_n$, sometimes denoted simply by $H_d$.

The image of the Reynold’s operator will be the space of all symmetric polynomials of degree $d$.

M. Shahryari [3] has introduced the notion of relative symmetric polynomials for any subgroup $G \subset S_n$ with respect to any irreducible character $\chi$ of $G$. 


The vector space of relative symmetric polynomials of $G$ relative to $\chi$, denoted by $H_d(G,\chi)$ is defined as the image of the projection operator defined in Equation (1.1) in this case:

$$T(G,\chi) = \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g)g$$ (1.2)

Finding the dimension of this vector space of relative symmetric polynomials for various subgroups of the symmetric group of $S_n$ is a fundamental question.

In later papers Babaei, Zamani and Shahryari found the dimension of the space of relative invariants for $S_n$ and its subgroup $A_n$ [3] and for Young subgroup [4].

In a series of papers Babaei and Zamani have given corresponding formula for the cyclic group in [7], for the dicyclic group in [8] and for the dihedral group $D_n$ [9].

In our work here we relook at the formula for dihedral groups. Babaei-Zamani formula is a summation involving cosine values which are in general irrational numbers. So they may be inconvenient to calculate dimensions which are non-negative integers. By using appropriate theorems from elementary number theory along with combinatorial counting arguments, we have provided an alternative dimensional formula as a summation of integer terms. Another advantage of our formula is that it allows us to write down the generating function. See theorems 4.1 and 4.2 for the precise statements.

A notable feature is that this formula gives this dimension as a summation over the divisors of $n$ involving the Môbius and Euler $\varphi$-functions, even though this problem is not apparently connected with number theory.

Our paper is organized this way: after this introduction, in the second section a few number-theoretic preliminaries are assembled, in Section 3 the character table of the dihedral group is given following Serre [2]. In the fourth section we state and prove the formula. Finally Section 5 illustrates how easy it is to compute with these formulae.

## 2 Preliminaries

First we set up the notations:

- We use the standard notations $\phi(n)$ and $\mu(n)$ respectively for the Euler’s totient function and Môbius function.

- For $n$ any positive integer and $r$ a divisor of $n$ we denote by $S_r(n)$ the set of integers between 1 and $n$ having $r$ as their gcd.

$$S_r(n) := \{k : 1 \leq k \leq n, \gcd(k,n) = r\}$$

We state below, without proofs, some well known facts from elementary number theory as lemmas. These results were known to Ramanujan [1] and von Sterneck [6].

**Lemma 2.1** With the notation as above, we have $S_r(n) = \{rk : k \in S_1(n/r)\}$. I.e., $S_r(n) = rS_1(n/r)$. In particular, $|S_r(n)| = |S_1(n/r)|$. 


Lemma 2.2 The sum of all the primitive \( n \)th roots of unity is \( \mu(n) \).

\[
\text{(ie) } \sum_{\substack{k=1 \\gcd(k,n)=1}}^{n} \exp\left(\frac{2\pi ik}{n}\right) = \mu(n)
\]

In fact we need the following variation of Lemma 2:

Lemma 2.3

\[
\sum_{k=1 \atop \gcd(k,n)=1}^{n} \cos\left(\frac{2\pi k}{n}\right) = \mu(n)
\]

Lemma 2.4 For any two positive integers \( n \) and \( m \)

\[
\sum_{k=1}^{n} \exp\left(\frac{2\pi imk}{n}\right) = \mu\left(\frac{n}{\gcd(m,n)}\right) \frac{\phi(n)}{\phi(n/\gcd(m,n))}
\]

3 Characters of the Dihedral group \( D_n \)

We write the elements of \( D_n \), as \( D_n = \{1, \sigma, \sigma^2, \ldots, \sigma^{n-1}, \tau, \tau\sigma, \tau\sigma^2, \ldots, \tau\sigma^{n-1}\} \). The dihedral group \( D_n \) has only degree 1 and degree 2 irreducible representations.

3.1 One-dimensional representations:

- When \( n \) is odd there are two irreducible representations of degree 1 namely \( \chi_1 \) and \( \chi_2 \) and the character table for those representations is given below:

| Character | \( \sigma^k \) | \( \tau\sigma^k \) |
|-----------|----------------|-------------------|
| \( \chi_1 \) | 1 | 1 |
| \( \chi_2 \) | 1 | \(-1\) |

- When \( n \) is even there are four irreducible representations of degree 1 namely \( \chi_1, \chi_2, \chi_3 \) and \( \chi_4 \) and the character table for those representations is given below:

| Character | \( \sigma^k \) | \( \tau\sigma^k \) |
|-----------|----------------|-------------------|
| \( \chi_1 \) | 1 | 1 |
| \( \chi_2 \) | 1 | \(-1\) |
| \( \chi_3 \) | \((-1)^k\) | \((-1)^k\) |
| \( \chi_4 \) | \((-1)^k\) | \((-1)^{k+1}\) |

3.2 Two-dimensional representations

Let \( h \) be a positive integer with \( h < n/2 \). A representation \( \rho_h \) of \( D_n \) has the character given by \( \psi_h(\sigma^k) = 2 \cos\frac{2\pi hk}{n} \) and \( \psi_h(\tau\sigma^k) = 0 \).

In our paper we follow the convention that whenever \( n \) or \( r \) is not a positive integer the binomial coefficient \( \binom{n}{r} \) is interpreted as zero. Now we can state the main result of our paper:

3
4 Dimension formulæ and generating functions

Theorem 4.1 Let \( \psi_h \) be the irreducible character of degree 2 of the dihedral group \( D_n \) as above. Then the dimension of \( H_d(D_n, \psi) \), the vector space of relative symmetric polynomials is described in two cases:

Case (i) \( h \) is coprime to \( n \):

\[
\dim H_d(D_n, \psi_h) = \frac{2}{n} \sum_{r|n} \left(r + \frac{d}{n/r} - 1\right) \mu \left(\frac{n}{r}\right)
\]

The generating function in this case is given by

\[
\sum_{d=0}^{\infty} \dim H_d(D_n, \psi_h) t^d = \frac{2}{n} \sum_{r|n} \mu \left(\frac{n}{r}\right) \left(1 - t^{\frac{n}{r}}\right)^{-r}
\]

Case (ii) \( h \) is not coprime to \( n \):

\[
\dim H_d(D_n, \psi_h) = \frac{2}{n} \sum_{r|n} \left(r + \frac{d}{n/r} - 1\right) \mu \left(\frac{n}{r}\right) \frac{\phi \left(\frac{n}{r}\right)}{\phi \left(\frac{n}{r}/g\right)}
\]

where \( \mu(n) \) is the Möbius function and \( g = \gcd(h, \frac{n}{r}) \).

The generating function in this case is given by

\[
\sum_{d=0}^{\infty} \dim H_d(D_n, \psi_h) t^d = \frac{2}{n} \sum_{r|n} \mu \left(\frac{n}{r}\right) \frac{\phi \left(\frac{n}{r}\right)}{\phi \left(\frac{n}{r}/g\right)} \left(1 - t^{\frac{n}{r}}\right)^{-r}
\]

Proof:

It suffices to prove the formula for the dimension of \( H_d(D_n, \psi) \) for a general \( d \). The formula for the generating function is a straightforward consequence.

Case (i): For definiteness we fix the embedding of \( D_n \) in \( S_n \) with the generators of \( D_n \) as below: \( D_n = \langle \sigma, \tau \rangle \) where \( \sigma \) is the \( n \)-cycle given by \( (1 2 3 \ldots n) \) and \( \tau(j) = n + 1 - j \) is the reversal permutation. In fact, \( D_n \) can be embedded in \( S_n \) uniquely up to conjugacy. Now in the case of a 2-dimensional irreducible character \( \chi \) of \( D_n \), \( \psi(\tau \sigma^k) = 0 \) for all \( k \). So the dimension formula reduces to the summation over the cyclic subgroup of all rotations in \( D_n \).

\[
\dim H_d(D_n, \psi) = \frac{\psi(1)}{|D_n|} \sum_{k=1}^{n} \psi(\sigma^k) \operatorname{Tr}(\sigma^k) = \frac{2}{2n} \sum_{k=1}^{n} 2 \cos \frac{2\pi k}{n} \operatorname{Tr}(\sigma^k) \quad (4.1)
\]

Note that \( \operatorname{Tr}(\sigma^k) \) is the trace of the \( k^{th} \) power of the \( n \)-cycle \( \sigma \) in the vector space of homogeneous polynomials in \( n \) variables, with \( \sigma \) permuting the variables cyclically. As this vector space has all monomials of degree \( d \) in \( n \) variables as basis its dimension is \( \binom{n+d-1}{n-1} \). Being a permutation action \( \operatorname{Tr}(\sigma^k) \) is the number of monomials of degree \( d \) in \( n \) variables fixed by \( \sigma^k \). So the calculation boils down to finding the number of invariant monomials of degree \( d \). To calculate \( \operatorname{Tr}(\sigma^k) \), let \( r = \gcd(n, k) \). Then \( \sigma^k \) decomposes into a product of \( r \) number of disjoint cycles of length \( \frac{n}{r} \). For a monomial to be invariant under
\(\sigma^k\), degree of all the variables within an \(\frac{n}{r}\)-cycle should be constant. Call these degrees \(d_1, d_2, \ldots, d_r\).

\[
d = \frac{n}{r}d_1 + \frac{n}{r}d_2 + \ldots + \frac{n}{r}d_r
\]

Therefore,

\[
d_1 + d_2 + \ldots + d_r = \frac{d}{n/r}
\]

A necessary condition is \(d\) must be a multiple of \(\frac{n}{r}\). Let us assume this holds.

So \(\text{Tr}(\sigma^k) = \) number of ordered partitions of \(\frac{d}{n/r}\) into \(r\) parts.

This is well known to be \(\left( r + \frac{d}{n/r} - 1 \right)\). Substituting this value of trace in to (4.1), we get

\[
\dim H_d(D_n, \psi) = \frac{1}{n} \sum_{k=1}^{n} \sum_{a \in S_1(n/r)} 2 \cos \left( \frac{2\pi k}{n} \right) \left( r + \frac{d}{n/r} - 1 \right)
\]

Note that for two terms of the summation on right hand side if \(1 \leq k_1, k_2 \leq n\) are such that \(\gcd(k_1, n) = \gcd(k_2, n)\), then the coefficient of \(\cos \frac{2\pi k_1}{n}\) equals that of \(\cos \frac{2\pi k_2}{n}\). As the gcd of any number with \(n\) is a divisor of \(n\), the dimension formula can be rewritten as a summation over the divisors of \(n\). We treat the above summation as a sum of binomial coefficients \(\binom{r + \frac{d}{n/r} - 1}{r - 1}\), one for each divisor \(r\) of \(n\) with some weights. These weights are sums of cosine values. So the dimensional formula takes the form

\[
\dim H_d(D_n, \psi) = \frac{2}{n} \sum_{r|n} \mu \left( \frac{n}{r} \right) \left( r + \frac{d}{n/r} - 1 \right) \quad \text{by lemma} 2.1
\]

Using lemma 2.3 the above equation reduces to

\[
\dim H_d(D_n, \psi) = \frac{2}{n} \sum_{r|n} \mu \left( \frac{n}{r} \right) \left( r + \frac{d}{n/r} - 1 \right) \quad \text{(4.4)}
\]

Proof of case (ii): Proceeding as in case (i), we have

\[
\dim H_d(D_n, \psi) = \frac{2}{n} \sum_{r|n} \sum_{a \in S_1(n/r)} \cos \left( \frac{2\pi ah}{n/r} \right) \left( r + \frac{d}{n/r} - 1 \right) \quad \text{(4.5)}
\]

The inner summation inside the square brackets in the above equation is actually the sum of the real parts of \(h^{th}\) powers of all primitive \(\left( \frac{n}{r} \right)^{th}\) roots of unity. Defining \(g = \gcd(h, n/r)\), we see that the above is same as the sum of all the
real parts of \( g \)th powers of all primitive \( \left( \frac{n}{r} \right) \)th roots of unity. Now we can apply Lemma 2.4 which makes equation (4.5) to become

\[
\dim H_d(D_n, \psi) = \frac{2}{n} \sum_{r|n} \left[ \sum_{\alpha \in S_1(n/\sqrt{r})} \cos \frac{2\pi \alpha}{n/\sqrt{r}} \phi(n/r) \left( r + \frac{d}{n/\sqrt{r}} - 1 \right) \right]
\]

Again using lemma 2.3

\[
\dim H_d(D_n, \psi) = \frac{2}{n} \sum_{r|n} \mu \left( \frac{n/r}{g} \right) \phi(n/r) \left( r + \frac{d}{n/r} - 1 \right)
\]

\[\text{Theorem 4.2 Case (i), } n \text{ is odd:}
\]

Let \( \chi_1 \) and \( \chi_2 \) be the two irreducible characters of degree 1 with \( \chi_1 \) being the trivial character and \( \psi_2 \) taking +1 on rotations and −1 on reflections. The dimensions of \( H_d(D_n, \chi_1) \) and \( H_d(D_n, \chi_2) \) are given by

\[
\dim H_d(D_n, \chi_1) = \frac{1}{2n} \left[ \sum_{r|n} \phi \left( \frac{n}{r} \right) \left( r + \frac{d}{n/r} - 1 \right) + n \sum_{l=0}^{[d/2]} \left( \frac{n-1}{r} + l - 1 \right) \right]
\]

and

\[
\dim H_d(D_n, \chi_2) = \frac{1}{2n} \left[ \sum_{r|n} \phi \left( \frac{n}{r} \right) \left( r + \frac{d}{n/r} - 1 \right) - n \sum_{l=0}^{[d/2]} \left( \frac{n-1}{r} + l - 1 \right) \right]
\]

The generating functions for the above two cases are given by

\[
\sum_{d=0}^{\infty} \dim H_d(D_n, \chi_1) t^d = \frac{1}{2n} \left[ \sum_{r|n} \phi \left( \frac{n}{r} \right) \left( 1 - t^{\frac{n}{r}} \right)^{-r} + \frac{n \left( 1 - t^{2n} \right)^{-(n-1)/2}}{1-t} \right]
\]

\[
\sum_{d=0}^{\infty} \dim H_d(D_n, \chi_2) t^d = \frac{1}{2n} \left[ \sum_{r|n} \phi \left( \frac{n}{r} \right) \left( 1 - t^{\frac{n}{r}} \right)^{-r} - \frac{n \left( 1 - t^{2n} \right)^{-(n-1)/2}}{1-t} \right]
\]

Case (ii) when \( n \) is even:

Let \( \chi_1, \chi_2, \chi_3 \) and \( \chi_4 \) be the four irreducible characters of degree 1. Let \( H_d(D_n, \psi_1), H_d(D_n, \psi_2), H_d(D_n, \psi_3) \) and \( H_d(D_n, \chi_4) \) be the space of Relative symmetric polynomials with respect to \( \psi_1, \psi_2, \psi_3 \) and \( \psi_4 \). Then the dimensions \( H_d(D_n, \chi_1), H_d(D_n, \chi_2), H_d(D_n, \chi_3) \) and \( H_d(D_n, \chi_4) \) are given by

\[
\dim H_d(D_n, \chi_1) = \frac{1}{2n} \left\{ \sum_{r|n} \phi \left( \frac{n}{r} \right) \left( r + \frac{d}{n/r} - 1 \right) \right\}
\]

\[
+ \frac{n}{2} \left[ \left( \frac{n}{2} + \frac{d}{2} - 1 \right) + \sum_{l=0}^{[d/2]} \left( \frac{n-2}{2} + l - 1 \right) (d-2l+1) \right]
\]
The generating functions for the above four cases are given by:

\[ \sum_{d=0}^{\infty} \dim H_d(D_n, \chi_1) t^d = \frac{1}{2n} \left( \sum_{r|n} \phi \left( \frac{n}{r} \right) \right) \left( 1 - t^{\frac{n}{2}} \right)^{-r} + \frac{n}{2} \left( 1 - t^{\frac{n}{2}} \right)^{-\left( n+2 \right)/2} \left( 2 + t^2 \right)(1 + t) \]

\[ \sum_{d=0}^{\infty} \dim H_d(D_n, \chi_2) t^d = \frac{1}{2n} \left( \sum_{r|n} \phi \left( \frac{n}{r} \right) \right) \left( 1 - t^{\frac{n}{2}} \right)^{-r} - \frac{n}{2} \left( 1 - t^{\frac{n}{2}} \right)^{-\left( n+2 \right)/2} \left( 2 + t^2 \right)(1 + t) \]

\[ \sum_{d=0}^{\infty} \dim H_d(D_n, \chi_3) t^d \]

\[ = \frac{1}{2n} \left( \sum_{r|n} \phi \left( \frac{n}{r} \right) \right) (-1)^r \left( 1 - t^{\frac{n}{2}} \right)^{-r} - \frac{n}{2} \left( 1 - t^{\frac{n}{2}} \right)^{-\left( n+2 \right)/2} \left( 1 + t + t^2 \right) \]

\[ \sum_{d=0}^{\infty} \dim H_d(D_n, \chi_4) t^d \]

\[ = \frac{1}{2n} \left( \sum_{r|n} \phi \left( \frac{n}{r} \right) \right) (-1)^r \left( 1 - t^{\frac{n}{2}} \right)^{-r} + \frac{n}{2} \left( 1 - t^{\frac{n}{2}} \right)^{-\left( n+2 \right)/2} \left( 1 + t + t^2 \right) \]

Proof:

Case (i) (when \( n \) is odd):

By definition:

\[ \dim H_d(D_n, \chi_1) = \frac{1}{2n} \left( \sum_{k=1}^{n} \text{Tr}(\sigma^k)\psi_1(\sigma^k) + \sum_{k=1}^{n} \text{Tr}(\tau\sigma^k)\psi_1(\tau\sigma^k) \right) \]
(ie)
\[ \dim H_d(D_n, \chi_1) = \frac{1}{2n} \left[ \sum_{k=1}^{n} \text{Tr}(\sigma^k) + \sum_{k=1}^{n} \text{Tr}(\tau\sigma^k) \right] \]

When \( n \) is odd all the reflections of \( D_n \) falls into a single conjugacy class. Hence the above summation becomes
\[ \dim H_d(D_n, \chi_1) = \frac{1}{2n} \left[ n \text{Tr}(\tau) + \sum_{k=1}^{n} \text{Tr}(\sigma^k) \right] \]

Now calculation of \( \text{Tr}(\sigma^k) \) is the same as in Theorem 1. It remains to find \( \text{Tr}(\tau) \). Since \( n \) is odd, \( \tau \) is the product of \( \frac{n-1}{2} \) transpositions and has one fixed point. Now \( \text{Tr}(\tau) \) is the count of monomials fixed by \( \tau \). We denote the variables by \( x_1, x_2, \ldots, x_{(n-1)/2}, y_1, y_2, \ldots, y_{(n-1)/2}, z \). Without loss of generality, let us assume that \( \tau(z) = z \), \( \tau(x_i) = y_i \) and \( \tau(y_i) = x_i \) for \( 1 \leq i \leq (n-1)/2 \). Now a monomial of degree \( d \) invariant under \( \tau \) has to be of the form
\[ z^{d_0} x_1^{d_1} y_1^{d_1} x_2^{d_2} y_2^{d_2} \cdots \]

Number of tuples \( (d_0, d_1, \ldots, d_{(n-1)/2}) \) such that
\[ d_0 + 2(d_1 + d_2 + \ldots + d_{(n-1)/2}) = d \]
is the total number of ordered partitions of \( (d-d_0)/2 \) into \( (n-1)/2 \) parts with all \( d_0 \) satisfying \( d-d_0 \) is an even non negative integer. This is easily verified to be \( \sum_{l=0}^{[d/2]} \left( \binom{n}{d/2} \right) \). Hence the formula becomes
\[ \dim H_d(D_n, \chi_1) = \frac{1}{2n} \left[ \sum_{r|n} \left( r + \frac{d}{r} - 1 \right) - n \sum_{l=0}^{[d/2]} \left( \binom{n-1}{2l} - 1 \right) \right] \]

Using the same arguments as above, we have
\[ \dim H_d(D_n, \chi_2) = \frac{1}{2n} \left[ \sum_{r|n} \left( r + \frac{d}{r} - 1 \right) + n \sum_{l=0}^{[d/2]} \left( \binom{n-1}{2l} - 1 \right) \right] \]
case (ii): By definition
\[ \dim H_d(D_n, \chi_1) = \frac{1}{2n} \left[ \sum_{k=1}^{n} \text{Tr}(\sigma^k)\psi_1(\sigma^k) + \sum_{k=1}^{n} \text{Tr}(\tau\sigma^k)\psi_1(\tau\sigma^k) \right] \]

(ie)
\[ \dim H_d(D_n, \chi_1) = \frac{1}{2n} \left[ \sum_{k=1}^{n} \text{Tr}(\sigma^k) + \sum_{k=1}^{n} \text{Tr}(\tau\sigma^k) \right] \]

Now \( \text{Tr}(\sigma^k) \) is the same as in Theorem 1. It remains to find \( \text{Tr}(\tau\sigma^k) \). Since \( n \) is even, all the reflections \( \tau\sigma^k \) fall into two conjugacy classes according as \( k \) is even or odd. Hence the formula becomes
\[ \dim H_d(D_n, \chi_1) = \frac{1}{2n} \left[ \sum_{k=1}^{n} \text{Tr}(\sigma^k) + \frac{n}{2} \text{Tr}(\tau) + \frac{n}{2} \text{Tr}(\tau) \right] \]

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Using the same argument as we did in case (i) of this theorem, we can find the trace of the reflections in both the cases (when \( n \) is odd or even). Hence the formula becomes

\[
\dim H_d(D_n, \chi_1) = \frac{1}{2n} \left\{ \sum_{r|n} \left( r + \frac{d}{n/r} - 1 \right) \phi \left( \frac{n}{r} \right) \right. \\
+ \frac{n}{2} \sum_{l=0}^{d/2} \left[ \left( \frac{n^2}{2} + l - 1 \right) (d - 2l + 1) + \left( \frac{n}{2} + l - 1 \right) \right] \left. \sum_{r|n} \phi \left( \frac{n}{r} \right) \right\}
\]

In the same way

\[
\dim H_d(D_n, \chi_2) = \frac{1}{2n} \left\{ \sum_{r|n} \left( r + \frac{d}{n/r} - 1 \right) \phi \left( \frac{n}{r} \right) \\
- \frac{n}{2} \sum_{l=0}^{\lfloor d/2 \rfloor} \left[ \left( \frac{n^2}{2} + l - 1 \right) (d - 2l + 1) + \left( \frac{n}{2} + l - 1 \right) \right] \left. \right\}
\]

Now again using the definition, we have

\[
\dim H_d(D_n, \chi_3) = \frac{1}{2n} \left\{ \sum_{r|n} \left( r + \frac{d}{n/r} - 1 \right) \phi \left( \frac{n}{r} \right) + \sum_{k=1}^{n} \text{Tr} (\sigma^k) \psi_3(\sigma^k) \right. \\
+ \sum_{k=1}^{n} \left( -1 \right)^k \left. \text{Tr} (\tau \sigma^k) \psi_3(\tau \sigma^k) \right\}
\]

(i.e.

\[
\dim H_d(D_n, \chi_3) = \frac{1}{2n} \left[ \sum_{k=1}^{n} \left( -1 \right)^k \text{Tr} (\sigma^k) + \sum_{k=1}^{n} \left( -1 \right)^k \text{Tr} (\tau \sigma^k) \right]\]

Using the same result for \( \text{Tr} (\sigma^k) \) from Theorem 1, we have

\[
\dim H_d(D_n, \chi_3) = \frac{1}{2n} \left\{ \sum_{r|n} \left( \sum_{k \in S_r(n)} \left( -1 \right)^k \left( r + \frac{d}{n/r} - 1 \right) \right) + \sum_{k=1}^{n} \left( -1 \right)^k \text{Tr} (\tau \sigma^k) \right\}
\]

Since \( n \) is even, all \( k \)'s in the inner summation are even or odd according as \( r \) is even or odd and it sums up to \( \left( -1 \right)^r \phi \left( \frac{n}{r} \right) \). Hence the above equation becomes

\[
\dim H_d(D_n, \chi_3) = \frac{1}{2n} \left\{ \sum_{r|n} \left( -1 \right)^r \phi \left( \frac{n}{r} \right) \left( r + \frac{d}{n/r} - 1 \right) \right. \\
+ \left( -1 \right)^k \sum_{k=1}^{n} \sum_{r|n} \left( -1 \right)^k \text{Tr} (\tau \sigma^k) \left. \right\}
\]

It remains to find \( \text{Tr} (\tau \sigma^k) \). Using the same argument as we did in case (i), we can calculate it. Hence the above equation reduces to

\[
\dim H_d(D_n, \chi_3) = \frac{1}{2n} \left\{ \sum_{r|n} \left( -1 \right)^r \phi \left( \frac{n}{r} \right) \left( r + \frac{d}{n/r} - 1 \right) \right. \\
+ \frac{n}{2} \left( \frac{n^2}{2} + \frac{d}{2} - 1 \right) \sum_{l=0}^{d/2} \left[ \left( \frac{n^2}{2} + l - 1 \right) (d - 2l + 1) \right] \left. \right\}
\]
In the same way, we have

$$\dim H_\delta(D_n, \chi_4) = \frac{1}{2n} \left\{ \sum_{r|n} (-1)^r \phi \left( \frac{n}{r} \right) \left( r + \frac{d}{r} - 1 \right) \right\} - \frac{n}{2} \left[ \left( \frac{n}{2} + \frac{d}{l} - 1 \right) - \sum_{l=0}^{\lfloor d/2 \rfloor} \left( \frac{n-2}{2} + l - 1 \right)(d - 2l + 1) \right]$$

To find the generating functions for \( \dim H_\delta(D_n, \chi_1), \dim H_\delta(D_n, \chi_2), \dim H_\delta(D_n, \chi_3) \) and \( \dim H_\delta(D_n, \chi_4) \):

Let \( G_1(t) \) be the generating function for \( \sum_{r|n} \phi \left( \frac{n}{r} \right) \left( r + \frac{d}{r} - 1 \right) \) which is easily seen to be \( \sum_{r|n} (-1)^r \phi \left( \frac{n}{r} \right) (1 - t^r)^{-r} \). Let \( G_2(t) \) and \( G_3(t) \) be the generating functions corresponding to \( (\frac{n}{2} + \frac{d}{l} - 1) \) and \( \sum_{l=0}^{\lfloor d/2 \rfloor} \left( \frac{n-2}{2} + l - 1 \right)(d - 2l + 1) \) respectively. Now \( G_2(t) \) is very easily verified to be \((1 - t^2)^{\frac{n}{2}}\). Now the generating functions for \( \dim(H_\delta, \chi_1) \), \( \dim(H_\delta, \chi_2) \), \( \dim(H_\delta, \chi_3) \) and \( \dim(H_\delta, \chi_4) \) are given by:

$$\sum_{d=0}^{\infty} \dim H_\delta(D_n, \chi_1)t^d = \frac{1}{2n} \left\{ G_1(t) + \frac{n}{2} [G_2(t) + G_3(t)] \right\}$$

$$\sum_{d=0}^{\infty} \dim H_\delta(D_n, \chi_2)t^d = \frac{1}{2n} \left\{ G_1(t) - \frac{n}{2} [G_2(t) + G_3(t)] \right\}$$

$$\sum_{d=0}^{\infty} \dim H_\delta(D_n, \chi_3)t^d = \frac{1}{2n} \left\{ G_1(t) + \frac{n}{2} [G_2(t) - G_3(t)] \right\}$$

$$\sum_{d=0}^{\infty} \dim H_\delta(D_n, \chi_4)t^d = \frac{1}{2n} \left\{ G_1(t) - \frac{n}{2} [G_2(t) - G_3(t)] \right\}$$

Now evaluation of \( G_3(t) \) given in the following lemma.

Lemma 4.3 For

$$\sum_{l=0}^{\lfloor d/2 \rfloor} \left( \frac{n-2}{2} + l - 1 \right)(d - 2l + 1)$$

the generating function is

$$G_3(t) = (1 - t^2)^{-\frac{n}{2}}(1 + 2t + t^2 + t^3)$$

Proof of Lemma:

First consider

$$\sum_{l=0}^{\lfloor d/2 \rfloor} \left( \frac{n-2}{2} + l - 1 \right)(d - 2l + 1)$$

Let

$$b_d = \sum_{l=0}^{\lfloor d/2 \rfloor} \left( \frac{m}{l} + l - 1 \right)(d - 2l + 1)$$

One easily verifies
Now again induction gives

\[
\sum_{l=0}^{d} \binom{m + l - 1}{l} = \binom{m + d}{d}
\]

Therefore,

\[
b_{2d+1} = b_{2d} + \binom{m + d}{d}
\]

Let

\[
\sum_{d=0}^{\infty} b_{2d} t^{2d} = G_5(t)
\]

Now

\[
\sum_{d=0}^{\infty} b_{2d+1} t^{2d+1} = t \sum_{d=0}^{\infty} \left[ b_{2d} t^{2d} + \binom{m + d}{d} t^{2d} \right] = tG_5(t) + t(1 - t^2)^{-(m+1)}
\]

Hence

\[
G_4(t) = \sum_{d=0}^{\infty} b_{2d} t^{d} = \sum_{t, \text{even}} + \sum_{t, \text{odd}} = G_5(t) + tG_5(t) + t(1 - t^2)^{-(m+1)}
\]

To evaluate \(G_5(t)\)

\[
G_5(t) = \sum_{d=0}^{\infty} \sum_{l=0}^{d} \binom{m + l - 1}{l} (2d - 2l + 1) t^{2d}
\]

\[
= \sum_{d=0}^{\infty} \sum_{l=0}^{d} \binom{m + l - 1}{l} (2d + 1) t^{2d} - 2 \sum_{l=0}^{d} \binom{m + l - 1}{l} t^{2d}
\]

Now using induction one can show that

\[
\sum_{l=0}^{d} \binom{m + l - 1}{l} = m \sum_{l=0}^{d} \binom{m + l - 1}{l} - m \sum_{l=0}^{d} \binom{m + l - 1}{l}
\]

\[
\text{(iv)} \quad \sum_{l=0}^{d} \binom{m + l - 1}{l} = m \sum_{l=0}^{d} \frac{m + l (m + l - 1)!}{l!(m - 1)!} - m \sum_{l=0}^{d} \binom{m + l - 1}{l}
\]

\[
= m \sum_{l=0}^{d} \binom{m + l}{l} - m \sum_{l=0}^{d} \binom{m + l - 1}{l}
\]
After simplification, we have

\[ \sum_{l=0}^{d} l \binom{m+l-1}{l} = m \binom{m+d}{d-1} \]

Hence,

\[ G_5(t) = \sum_{d=0}^{\infty} \binom{m+d}{d} (2d+1)t^{2d} - 2 \sum_{d=0}^{\infty} m \binom{m+d}{d-1} t^{2d} \]

\[ = \sum_{d=0}^{\infty} \binom{m+d}{d} 2dt^{2d} + \sum_{d=0}^{\infty} \binom{m+d}{d} t^{2d} - 2 \sum_{d=0}^{\infty} m \binom{m+d}{d-1} t^{2d} \]

Now using

\[ \sum_{d=0}^{\infty} \binom{m+d}{d} 2dt^{2d} = \frac{d}{dt} \left[ (1-t^2)^{-(m+1)} \right] \]

and

\[ \sum_{d=0}^{\infty} m \binom{m+d}{d} t^{2d} = 2mt^2(1-t^2)^{-(m+2)} \]

Simplifying

\[ G_5(t) = (1-t^2)^{-(m+2)}(2t^2 + 1) \]

Finally

\[ G_4(t) = (1-t^2)^{-(n+2)}(1 + 2t + 2t^2 + t^3) \]

Replacing \( m \) by \( \frac{n-2}{2} \) in \( G_4(t) \), we have

\[ G_3(t) = (1-t^2)^{-(n+2)/2}(1 + 2t + 2t^2 + t^3) \]

5 Examples

5.1 The dihedral group \( D_{10} \)

Consider the group \( D_{10} \). It has 4 one-dimensional irreducible representations and 4 two-dimensional irreducible representations. Here we give the generating functions for all the eight irreducible representations.

Two-dimensional representations: As \( 0 < h < \frac{n}{2} \), \( h \) can assume 1, 2, 3 and 4

- case (i) when \( h \) is coprime with \( n \) i.e., \( h = 1, 3 \).

  The generating function is

  \[ \frac{1}{5} \left\{ \frac{1}{1-t^{10}} - \frac{1}{(1-t^5)^2} - \frac{1}{(1-t^2)^5} + \frac{1}{(1-t)^{10}} \right\} \]

- case (i) when \( h \) is not coprime with \( n \) i.e., \( h = 2, 4 \)

  The generating function is

  \[ \frac{1}{5} \left\{ -\frac{1}{1-t^{10}} - \frac{1}{(1-t^5)^2} + \frac{1}{(1-t^2)^5} + \frac{1}{(1-t)^{10}} \right\} \]

One-dimensional representations:
The generating functions for \( \chi_1 \) and \( \chi_2 \) are given by
\[
\frac{1}{20} \left\{ \left[ \frac{1}{(1 - t^{10})} + \frac{4}{(1 - t^5)^2} + \frac{1}{(1 - t^2)^5} + \frac{1}{(1 - t)^{10}} \right] \pm 10 \left[ \frac{(2 + t^2)(1 + t)}{(1 - t^2)^6} \right] \right\}
\]
The generating functions for \( \chi_3 \) and \( \chi_4 \) are given by
\[
\frac{1}{20} \left\{ \left[ -\frac{1}{(1 - t^{10})} + \frac{4}{(1 - t^5)^2} - \frac{1}{(1 - t^2)^5} + \frac{1}{(1 - t)^{10}} \right] \pm 10 \left[ \frac{(1 + t + t^2)}{(1 - t^2)^6} \right] \right\}
\]

6 Existence of relative invariants

The above formulæ give that the dimension of the space of relative symmetric polynomials of degree 1 is 2 whatever be the character. In particular relative invariants for \( D_n \) exist in degree 1 always.

One can easily see that the dimension is positive for \( D_n \) for any degree \( d \) when \( n \) a prime number. It seems all the dimensions are always positive though we are unable to prove this.

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