A NOVEL GALERKIN METHOD FOR SOLVING PDES ON THE SPHERE USING HIGHLY LOCALIZED KERNEL BASES

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Abstract. The main goal of this paper is to introduce a novel meshless kernel Galerkin method for numerically solving partial differential equations on the sphere. Specifically, we will use this method to treat the partial differential equation for stationary heat conduction on $S^2$, in an inhomogeneous, anisotropic medium. The Galerkin method used to do this employs spatially well-localized, “small footprint”, robust bases for the associated kernel space. The stiffness matrices arising in the problem have entries decaying exponentially fast away from the diagonal. Discretization is achieved by first zeroing out small entries, resulting in a sparse matrix, and then replacing the remaining entries by ones computed via a very efficient kernel quadrature formula for the sphere. Error estimates for the approximate Galerkin solution are also obtained.

1. Introduction

The main goal of this paper is to introduce and analyze a novel meshless kernel Galerkin method for numerically solving partial differential equations on the sphere. Specifically, we will use this method to treat the partial differential equation for stationary heat conduction on $S^2$, the unit sphere in $\mathbb{R}^3$, in an inhomogeneous, anisotropic medium. The equation for this heat-flow is

$$Lu = -\text{div}(a \cdot \nabla u) + bu = f,$$

where $\text{div}$ and $\nabla$ are the divergence and gradient on $S^2$, $a$ is a rank 2 positive definite tensor on $S^2$, $f$ is in the Sobolev space $H^s$, $s \geq 0$, and $b : S^2 \to \mathbb{R}$ is $C^\infty$ and strictly positive. The analysis includes error estimates when the exact stiffness matrix is used, and also when various quadrature-based discretizations of that matrix are employed.

The kernels that we employ here are surface splines,

$$\phi_m(x \cdot y) = (-1)^m (1 - x \cdot y)^{m-1} \log(1 - x \cdot y), \quad m \geq 2.$$
These are conditionally positive definite spherical basis functions (SBF). Their reproducing kernel Hilbert spaces (native spaces) are equivalent to the Sobolev spaces \( H_m \approx W^{m}_{2}(S^2) \). The associated approximation spaces involve spans of \( \{\phi_m((\cdot) \cdot \xi)\}_{\xi \in X} \), with \( X \) being a discrete, finite set of quasi-uniformly distributed centers or nodes, along with spherical harmonics of order \( m \). We denote them by \( V_{\phi,m,X} \).

It is well known that, under mild conditions on an SBF \( \phi \), the spaces \( V_{\phi,X} \) have excellent approximation power [23]. This makes them an obvious choice for use in meshless methods for solving PDEs. Kernel Galerkin methods using radial basis functions (RBFs) were employed in [38] to theoretically treat elliptic partial differential equations on \( \mathbb{R}^n \). On \( S^2 \), SBF Galerkin methods for \( -\Delta u = f \) have been studied in [18,19]. More recently, Le Gia et al. [20,21] used collocation and multi-level SBF methods for the purpose.

There are some drawbacks to these methods. Although convergence to the solution was established in [20,21] under fairly mild assumptions, the rates obtained were not optimal. At each level, the rescaling required to ensure stability precludes optimality. For Galerkin methods on \( S^2 \), there is an additional problem that arises in connection with discretization. Entries in the stiffness matrices have to be numerically computed via quadrature. For instance, the method used in [18] requires solving an optimization problem to find the weights involved. This is a computationally expensive process. Applying it to large numbers of nodes is problematic.

The new meshless Galerkin method that we present here overcomes these difficulties. There are two novel features of our Galerkin approach to numerically approximating solutions to (1.1). First, for the surface splines, Fuselier et al. [8] recently showed in the Lagrange basis for \( V_{\phi,m,X} \) each Lagrange function \( \chi_{\xi}(x) \) is highly localized spatially; indeed, \( \chi_{\xi}(x) \) decays exponentially fast as \( x \) moves away from \( \xi \). Moreover, when \( \chi_{\xi} \) is expressed in the kernel basis for \( V_{\phi,m,X} \), the coefficient \( \alpha_{\xi,\eta} \) of \( \phi_m(x \cdot \eta) \) also decays exponentially fast as \( \eta \) moves away from \( \xi \). Thus \( \chi_{\xi} \) has a “small footprint” in the kernel basis. These features make the Lagrange basis robust. It is for this basis that we will build our initial theory.

\( L^2 \)-error estimates for the Galerkin solutions constructed using the thin-plate splines are derived in section 6.1. Indeed, we derive the error estimates for Galerkin solutions constructed using an arbitrary SBF \( \phi \), restricted only by the condition that coefficients in its expansion in spherical harmonics satisfy (6.1). (These conditions hold for \( \phi_m \), with \( \tau = m \).) If \( f \in H_s(S^2) \), the \( L^2 \)-error bounds derived in section 6.1 are \( C\|f\|_s h^{s+2}_X \), for \( 0 \leq s \leq 2m - 2 \), and \( C\|f\|_{2m-2} h^{2m}_X \), for \( s > 2m - 2 \). These error estimates differ from those found in works cited above in two ways: they hold for operators of the form \( L \) in (1.1), not just \( \Delta \), and they apply even when \( s \) is fractional.

Constructing the \( \chi_{\xi} \)’s requires all of the points in \( X \). In this sense, the \( \chi_{\xi} \)’s form a global Lagrange basis; finding them is computationally expensive. In [8] a local Lagrange basis was also introduced. Each basis element \( \chi_{\xi}^{loc} \) is constructed using only \( O((\log N)^2) \) in the neighborhood of \( \xi \), and it approximates \( \chi_{\xi} \) well. These elements have very fast spatial decay, although not exponential. They have the advantage that computing them is fast and parallelizable.

The computational advantages of these local bases make them a good choice for the implementation of our meshless method. We use our initial theory for the global Lagrange basis to further develop this method when a local Lagrange basis is used.
The stiffness matrix $A$ in the global Lagrange basis is the key to the whole method. Our technique relies on having high quality approximations to this matrix. $A$ itself has a number of very attractive properties in the global basis: (1) The entries $A_{\xi,\eta}$ decay exponentially fast in the distance between $\xi$ and $\eta$. Thus $A$ is essentially sparse and, as we shall see, zeroing out small entries will provide a sparse approximation. (2) If $N_X = \text{card}(X)$, then the number of non-negligible entries in each row is $O((\log N_X)^2)$. (3) The condition number of $A$ is $O(q_X^{-2})$, where $q_X$ is half the separation distance for $X$.

Discretization of the stiffness matrix is essential to the Galerkin method. After zeroing out small entries, the remaining matrix is sparse, having $O(N_X (\log N_X)^2)$ entries. These entries are integrals that have to be computed via quadrature. To do this, we use kernel-based quadrature formulas for the sphere [9,15,32], with our kernels being surface splines. The special bases available to us for the approximation spaces corresponding to the $\phi_m$’s enable us to efficiently construct the quadrature formulas [9]. Unlike the quadrature formula used in [18], the weights are obtained by solving a linear system of equations. Finding the weights follows readily by applying the techniques of [8]. These quadrature formulas are accurate, optimally so in many cases, even in the presence of noise, and they are stable when the number of nodes, which we denote by $Y$, increases. Indeed, tests run with this method handled over two-thirds of a million nodes [9].

When the problem is discretized, we get the continuous error plus a quadrature error. The quadrature error comes from replacing the stiffness matrix $A$ by $A_Y$, whose entries are computed from those of $A$ via Theorem 7.5 gives a theoretical bound on $\|A - A_Y\|_2$. If we ignore logarithms and similar terms, this is $\|A - A_Y\|_2 \sim \left(\frac{N_X}{N_Y}\right)^M$, where $\phi_M$, $2 \leq M \leq m$, is used to obtain weights in the quadrature formula. The $L^2$-error estimate for the discretized Galerkin solution is given in Theorem 7.9.

There is related work for $\mathbb{R}^n$. As mentioned earlier, Wendland [38] explored RBF Galerkin for domains in $\mathbb{R}^n$. However, problems with discretizing the stiffness matrix entries were limiting factors in implementing the method. For $\mathbb{S}^n$, the quadrature methods discussed above avoid these problems. In [2], Bond et al. successfully employed an RBF Galerkin method, using an approach based on the one used here, in numerical experiments for a peridynamic model of a non-local heat equation (See [4] for a discussion of peridynamics.)

We now discuss an outline of the paper and its organization. In section 2 we discuss background information on quasi-uniform sets of centers, geometry, and Sobolev spaces. In section 3 we discuss the relevant approximation spaces for the Galerkin method that we introduce. Section 4 describes the major tools used here: the highly localized Lagrange bases we employ throughout the work (section 4.1) and the quadrature formulas essential for discretizing the stiffness matrix (section 4.2). Section 5 discusses general properties of (1.1), including regularity of weak solutions and a useful application of the “Nitsche trick” [27]. Section 6.1 gives Galerkin error estimates in the case where the approximation spaces are generated by SBFs satisfying only (6.1). The next section, section 6.2 is key. It sets forth the properties of the stiffness matrix in the Lagrange basis: its quasi-bandedness, good conditioning, decay of matrix elements away from the diagonal, and general robustness. Section 7 discusses aspects of discretizing the problem. Theorem 7.9 is the main result of the section; it contains the theoretical $L^2$-bound on the difference between discretized Galerkin solution and the exact solution. In section 8...
we discuss reducing the computational expense of numerically finding the Galerkin solution to the problem. There are two aspects of this. The first is obtaining a truncated approximation $\tilde{A}Y$ to the discretized stiffness matrix $A_Y$. Each row in $\tilde{A}Y$ has $O((\log N)^2)$ non-zero entries, as opposed to $N_X$ in $A_Y$. The second is to replace the global Lagrange basis with a local one, which is much easier to numerically find. The error estimates from making these approximations are virtually unchanged. In section 9, the results of numerical experiments that we did are presented. In terms of rates of convergence, the numerical results were actually better than the theory predicted. Finally, we wish to mention a few new results discussed in the appendix. In section A, we establish a generalized version of the so-called “doubling trick”, which applies to SBF interpolation of functions twice as smooth as those in the native space of the SBF. The result, which is given in Theorem A.3, holds for functions smoother than those in the native space for $\phi$, but not having “double” the smoothness. In addition, it applies for SBFs that are conditionally positive definite.

2. Preliminaries

2.1. Geometry of sets of centers. Although $S^2$ is the underlying space for the Galerkin methods treated in this paper, much of what we will discuss in the next few sections applies to $S^n$. In view of this, we will work in $S^n$.

Let $\text{dist}(x,y)$ be the geodesic (great circle) distance between two points on $S^n$. We will let $X = \{x_j\}_{j=1}^N \subset S^n$ be a set of $N$ distinct points; we will call $X$ a set of centers. We remark that, apart from, say the vertices of the Platonic solids in $S^2$ and similar quantities in $S^n$, $n > 2$, there are no uniformly distributed sets of points in $S^n$. We can, however, obtain quasi-uniform sets of points. We will explain this below.

There are three geometrical quantities associated with $X$. The first is the separation radius, $q_X$. For every $x \in X$, the radius of the largest ball whose interior contains no other point of $X$ is given by $\frac{1}{2}\text{dist}(x, X \setminus \{x\})$; $q_X$ is defined to be the smallest of these radii:

$$q_X := \min_{x \in X} \frac{1}{2}\text{dist}(x, X \setminus \{x\}).$$

Obviously $2q_X$ is the minimum distance between any two points in $X$. The second and third are the mesh norm, $h_X$, and the mesh ratio, $\rho_X$. The mesh norm $h_X$ is the radius of the largest ball in $S^n$ whose interior contains no point of $X$. It also can be characterized as the largest distance of any point in $S^n$ from $X$. The mesh ratio is the ratio of $h_X$ to $q_X$:

$$h_X := \max_{x \in S^n} \text{dist}(x, X) \quad \text{and} \quad \rho_X := h_X/q_X.$$

The mesh norm, which is also called the fill distance, measures how tightly packed the centers are in $S^n$. The mesh ratio measures how uniformly the centers are placed. When it is close to 1, the distribution of the points in $X$ is said to be quasi-uniform.

For $\rho \geq 1$, let $F_\rho = F_\rho(S^n)$ be the family of all sets of centers $X$ with $\rho_X \leq \rho$; we will say that the family $F_\rho$ is $\rho$-uniform. Unless confusion would arise, we will not indicate either $\rho$ or $S^n$, and just use $F$ to designate the family $F_\rho(S^n)$. The specific $\rho$ or sphere $S^n$ will be clear from the context.
On $\mathbb{S}^2$, there are three important quasi-uniform sets of nodes (centers): Fibonacci nodes, icosahedral nodes, quasi-minimum energy nodes. All three of these families of nodes are quite popular in applications; see, for example, [12, 22, 28, 34] for the icosahedral nodes, [17, 31, 35] for the Fibonacci nodes, and [5, 6, 30, 40] for the quasi-minimum energy nodes. Similar considerations apply to $\mathbb{S}^n$, $n > 2$.

2.2. **Spherical harmonics and Sobolev spaces.** The sphere $\mathbb{S}^n$ is of course a Riemannian manifold with metric tensor $g_{ij}$ and invariant measure $d\mu = \sqrt{\det(g_{ij})} dx^1 \cdots dx^n$, where $x^1, x^2, \ldots, x^n$ is a smooth set of local coordinates. For $\mathbb{S}^2$, the metric tensor in spherical coordinates $(\theta, \varphi)$, with $\theta$ being the colatitude and $\varphi$ being the longitude, has the form

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 \theta \end{pmatrix}.$$

The metric tensor for $\mathbb{S}^n$ also can be expressed in a similar set of coordinates. On any Riemannian manifold there are two important operators: the covariant derivative $\nabla$, powers of covariant derivatives $\nabla^k$, which is an operator on tensors, and the Laplace-Beltrami operator $\Delta = -\nabla^* \nabla$. The covariant derivative operating on a function is the usual gradient, expressed appropriately. Other powers are tensor operators. For example, $\nabla^2$ plays the role of a Hessian. In local coordinates, $\Delta$ has the form

$$\Delta u = \frac{1}{\sqrt{\det(g_{ij})}} \sum_{i,j} \frac{\partial}{\partial x^i} \sqrt{\det(g_{ij})} g^{ij} \frac{\partial u}{\partial x^j},$$

where $g^{ij} = (g_{ij})^{-1}$. For $\mathbb{S}^2$, in spherical coordinates, the Laplace-Beltrami operator is given by

$$\Delta u = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 u}{\partial \varphi^2}.$$

We now turn to a discussion of spherical harmonics; the details may be found in [25]. Spherical harmonics are eigenfunctions of $\Delta$. On $\mathbb{S}^n$, the eigenvalues of $-\Delta$ are $\lambda_\ell = \ell(\ell + n - 1)$. The eigenspace corresponding to $\lambda_\ell$ is degenerate, and has dimension

$$d_\ell = \begin{cases} 1, & \ell = 0, \\ \frac{(2\ell + n - 1)\Gamma(\ell + n - 1)}{\Gamma(\ell + 1)\Gamma(n)} \sim \ell^{n-1}, & \ell \geq 1. \end{cases}$$

We note that $d_\ell = O(\ell^{n-1})$. The eigenfunctions corresponding to $\lambda_\ell$ are denoted by $Y_{\ell,k}$, where $k = 1, \ldots, d_\ell$. We will use the real-valued versions of the spherical harmonics. The eigenspace of $\lambda_\ell$ will be denoted by $\mathcal{H}_\ell$. The space of all spherical harmonics of order $L$ or less will be denoted by $\Pi_L = \bigoplus_{\ell=0}^L \mathcal{H}_\ell$. In addition, we mention the well-known addition formula. Let $x, y \in \mathbb{S}^n$ and let $x \cdot y$ denote the usual dot product from $\mathbb{R}^{n+1}$. Then,

$$\sum_{k=1}^{d_\ell} Y_{\ell,k}(x)Y_{\ell,k}(y) = \frac{2\ell + n - 1}{(n - 1)\omega_n} P^{n-1}_\ell(x \cdot y),$$

where $\omega_n$ is the volume of $\mathbb{S}^n$, and $P^{n-1}_\ell$ is the degree $\ell$ ultraspherical polynomial of order $\frac{n-1}{2}$. 

The spherical harmonics form a complete orthonormal set in $L^2(S^n)$. Given $f$ in $L^2(S^n)$, we can expand $f$ in the series $f = \sum_{\ell=0}^{\infty} \sum_{k=1}^{d_{\ell}} \hat{f}_{\ell,k} Y_{\ell,k}$. As usual, given $f,g \in L^2$, we have $\langle f,g \rangle_{L^2} = \sum_{\ell=0}^{\infty} \sum_{k=1}^{d_{\ell}} \hat{f}_{\ell,k} \hat{g}_{\ell,k}$.

In this paper we will work with fractional order Sobolev spaces defined in terms of Bessel potentials [33,37]. The Sobolev space of order $\tau \geq 0$ is

$$H_\tau := \{ f \in L^2 : \| f \|_{H_\tau} := \| (I - \Delta)^{\tau/2} f \|_{L^2} < \infty \}.$$ 

This is a Hilbert space in the inner product

$$(2.4) \quad \langle f,g \rangle_{H_\tau} = \langle (I - \Delta)^{\tau/2} f, (I - \Delta)^{\tau/2} g \rangle_{L^2} = \sum_{\ell=0}^{\infty} \sum_{k=1}^{d_{\ell}} (1 + \lambda \ell)^\tau \hat{f}_{\ell,k} \hat{g}_{\ell,k}.$$ 

When $\tau = m$ is an integer, these spaces agree, up to norm equivalence, with $W^{m,2}(S^n)$, which are defined in terms of covariant derivatives [1,37].

### 3. Spherical basis functions and approximation spaces

In the following, we will be working on $S^n$. We start with zonal functions. A continuous function $\phi : [-1,1] \to \mathbb{R}$ is said to be zonal if it has the expansion,

$$\phi(t) := \sum_{\ell=0}^{\infty} \phi_{\ell} \frac{2\ell + n - 1}{(n - 1)\omega_n} \Lambda_{n-1}^{\frac{n-1}{2}}(t),$$

where $\Lambda_{n-1}^{\frac{n-1}{2}}$ is a degree $\ell$ ultra spherical polynomial [36 §4.7]. Zonal functions give rise to kernels on the sphere in the following way. Let $t = x \cdot y, x,y \in S^n$. Using the addition formula (2.3) in (3.1), we see that

$$\phi(x \cdot y) = \sum_{\ell=0}^{\infty} \phi_{\ell} \sum_{k=1}^{d_{\ell}} Y_{\ell,k}(x)Y_{\ell,k}(y),$$

which is a kernel mapping $S^n \times S^n$ to $\mathbb{R}$.

SBFs are zonal functions having $\phi_{\ell} > 0$ for all $\ell \geq 0$. They are strictly positive definite functions on $S^n$. This means that the matrix $A = (\phi(\xi \cdot \eta))_{\xi,\eta \in X}$ is positive definite for every choice of $X$. Equivalently,

$$\sum_{\xi,\eta \in X} c_\xi c_\eta \phi(\xi \cdot \eta) > 0,$$

except when the $c$’s are all 0. $A$ being positive definite allows us to interpolate arbitrary continuous functions (or data, for that matter) on $S^n$, using functions from the approximation space

$$V_{\phi,X} := \text{span}\{\phi((\cdot) \cdot \xi) : \xi \in X\}.$$ 

This follows because the existence of $c = A^{-1} f|_X$ implies that

$$I_X f(x) = \sum_{\xi \in X} c_\xi \phi(x \cdot \xi)$$

interpolates $f$ on $X$, and does so uniquely. The SBF $\phi$ is also a reproducing kernel for the Hilbert space

$$\mathcal{N} := \{ f \in L^2 : \sum_{\ell,k} \hat{\phi}_{\ell}^{-1} |\hat{f}_{\ell,k}|^2 < \infty \}.$$
This space is often called the native space of $\phi$; it has the inner product

$$
\langle f,g \rangle_N = \sum_{\ell=0}^{\infty} \sum_{k=1}^{d_\ell} \hat{\phi}_{\ell}^{-1} \hat{f}_{\ell,k} \overline{g_{\ell,k}}.
$$

The SBF $\phi_\tau$ for which $\hat{\phi}_{\tau,\ell} = (1 + \lambda_\ell)^{-\tau}$, with $\tau > n/2$, is especially important. The native space for $\phi_\tau$ is the Sobolev space $H_\tau$, since $\|f\|_N = \|(I - \Delta)^{\tau/2} f\|_{L^2} = \|f\|_{H_\tau}$. Making use of this observation yields a fractional order “zeros lemma”, similar to integer order ones proved in [14, Appendix A]. This will be important in the sequel.

**Lemma 3.1** (Zeros lemma). Let $\sigma, \tau \in \mathbb{R}$ satisfy $\tau > n/2$, $0 \leq \sigma \leq \tau$. In addition, let $X \subset S^n$ be quasi-uniform. If $u \in H_\sigma$ satisfies $u|_X = 0$, then, for $h_X$ sufficiently small, we have

$$
\|u\|_{H_\sigma} \leq C h_X^{\tau-\sigma} \|u\|_{H_\tau}.
$$

**Proof.** Let $I_{X,\phi_\tau}$ be the interpolation operator corresponding to $\phi_\tau$. By [26, Theorem 5.5], we have that

$$
\|u - I_{X,\phi_\tau} u\|_{H_\sigma} \leq C h_X^{\tau-\sigma} \|u\|_{H_\tau}.
$$

Note that $u|_X = 0$ implies that $I_{X,\phi_\tau} u \equiv 0$. Using this in the previous equation then yields the result. $\square$

The SBFs discussed above are all strictly positive definite. We will also need to make use of conditionally positive definite SBFs. These SBFs have the form given in (3.1), but the $\hat{\phi}_{\ell}$’s need only be positive for $\ell > L$. For $0 \leq \ell \leq L$, $\hat{\phi}_\ell$ is arbitrary. Conditionally positive definite SBFs are employed to interpolate scattered data, with the requirement that the interpolants reproduce $\Pi_L$, the space of spherical harmonics of degree $L$ or less. (Other spaces are also possible.) For a conditionally positive definite SBF $\phi$, the corresponding approximation space is defined to be

$$
V_{\phi,L,X} := \left\{ \sum_{\xi \in X} a_\xi \phi((\cdot) \cdot \xi) : \sum_{\xi \in X} a_\xi p(\xi) = 0 \, \forall \, p \in \Pi_L \right\} + \Pi_L.
$$

The interpolation operator that both interpolates continuous functions and reproduces $\Pi_L$ is

$$
I_{X,L} f = \sum_{\xi \in X} a_{\xi, L} \phi(x \cdot \xi) + p_{X,L}, \quad p_{X,L} \in \Pi_L,
$$

and

$$
\sum_{\xi \in X} a_{\xi, L} p(\xi) = 0, \quad p \in \Pi_L.
$$

The coefficients $a_{\xi, L}$ and the polynomial $p_{X,L}$ are determined by the requirements that the interpolation condition $I_{X,L} f|_X = f|_X$ hold and also that the coefficients satisfy the condition on the right above, provided $X$ is unisolvent. Again, the interpolant is unique. There is also a semi-Hilbert space $\mathcal{N}$ associated with $\phi$. This is defined to be

$$
\mathcal{N} := \{ f \in L^2 : \sum_{\ell=L+1}^{\infty} \sum_{k=1}^{d_\ell} \hat{\phi}_{\ell}^{-1} |\hat{f}_{\ell,k}|^2 < \infty \},
$$

$$
\langle f,g \rangle_{\mathcal{N}} = \sum_{\ell=L+1}^{\infty} \sum_{k=1}^{d_\ell} \hat{\phi}_{\ell}^{-1} \hat{f}_{\ell,k} \overline{g_{\ell,k}}.
$$

In addition, we will need the following well-known fact, which we state without proof.

**Proposition 3.2.** Suppose that $\tau > n/2$ and that $\phi$ is an SBF such that there are constants $c$, $C$ and $L \in \mathbb{N}$ for which $c(1 + \lambda_\ell)^{-\tau} \leq \hat{\phi}_\ell \leq C(1 + \lambda_\ell)^{-\tau}$ holds either for all $\ell \geq 0$ or for all $\ell \geq L+1$. If $\epsilon > 0$, then $\phi \in H_{2\tau - \frac{n}{2} - \epsilon}$. 
4. Highly localized “small footprint” bases

Surface splines, or polyharmonic kernels, are special conditionally positive definite SBFs, and are a key ingredient in the kernel methods presented here. While they can be defined for any $\mathbb{S}^n$ [14], we will restrict our attention to the case of $\mathbb{S}^2$ [8]. Their explicit forms are given below:

$$\phi_m(t) = (-1)^m(1-t)^{m-1}\log(1-t), \quad 1 < m \in \mathbb{N},$$

$$\hat{\phi}_{m,\ell} = C_m \frac{\Gamma(\ell-m+1)}{\Gamma(\ell+m+1)} \sim \ell^{-2m} \sim \lambda_{\ell}^{-m}, \quad \ell > m-1.$$  

Here $C_m = 2^{m+1}\pi\Gamma(m)^2$. These kernels are conditionally positive definite, and interpolation with them will reproduce $\Pi_{m-1}$. The space $\mathcal{N}$ associated with $\phi_m$ is, up to norm equivalence, the Sobolev space $H_m(\mathbb{S}^2)$ modulo $\Pi_{m-1}$. Also, since $\hat{\phi}_{m,\ell} \sim \ell^{-2m}$, it is easy to show that $\phi_m \in H_{2m-1-\epsilon}(\mathbb{S}^2)$, $\epsilon > 0$. Furthermore, this implies that the approximation space for $\phi_m$ satisfies

$$V_{\phi_m} := V_{\phi_{m-1}} \subset H_{2m-1-\epsilon}(\mathbb{S}^2), \quad \forall \epsilon > 0.$$  

4.1. Lagrange functions. We can form a basis for $V_{\phi_m}$ using Lagrange functions or cardinal functions. A Lagrange function $\chi_\xi$ is defined as the unique interpolant from $V_{\phi_{m-1}}$ that satisfies $\chi_\xi(\eta) = \delta_{\xi,\eta}$. Since $\chi_\xi \in V_{\phi_m}$, it has the form

$$\chi_\xi = \sum_{\zeta \in X} \alpha_{\xi,\zeta} \phi_m((\cdot) \cdot \zeta) + p_\xi, \quad p_\xi \in \Pi_{m-1}, \quad \sum_{\zeta \in X} \alpha_{\xi,\zeta} |p(\zeta)| = 0 \quad \forall \ p \in \Pi_{m-1}.$$  

Interpolation using the $\chi_\xi$’s is simple: If $f$ is a continuous function on $\mathbb{S}^2$, with $f|_X$ given, then $I_{\phi_m}f = \sum_{\xi \in X} f(\xi)\chi_\xi$.

There are two important properties of the Lagrange functions constructed from the $\phi_m$’s. First, they are well localized in space. Indeed, $\chi_\xi(x)$ decays exponentially in $\text{dist}(x,\xi)$. Second, they have a small “footprint” in the set of basis elements. Again, the coefficients $\alpha_{\xi,\zeta}$ decay exponentially in $\text{dist}(\xi,\zeta)$. Each $\chi_\xi$ is effectively using only a small number of kernels from the set $\{\phi_m((\cdot) \cdot \xi): \xi \in X\}$; i.e, $\chi_\xi$ has a small “footprint” in the set of kernels. The precise result is stated below:

**Theorem 4.1** ([8] Theorem 5.3). Let $\rho > 0$ be a fixed mesh ratio and let $\mathbb{N} \ni m \geq 2$. There exist constants $h^*$, $\nu$, $c_1$, $c_2$ and $C$, depending only on $m$ and $\rho$, so that if $h_X \leq h^*$, then $\chi_\xi$ given in (4.2) has these properties:

$$|\chi_\xi(x)| \leq C \exp\left(-\nu \frac{\text{dist}(x,\xi)}{h_X}\right),$$

$$|\alpha_{\xi,\zeta}| \leq C q_X^{2-2m} \exp\left(-\nu \frac{\text{dist}(\xi,\zeta)}{h_X}\right),$$

$$c_1 q_X^{2/p} \|\nabla\chi_\xi\|_{L^p(\mathbb{S}^2)} \leq \left\| \sum_{\xi \in X} \chi_\xi \right\|_{L_p(\mathbb{S}^2)} \leq c_2 q_X^{2/p} \|\nabla\chi_\xi\|_{L_p(\mathbb{S}^2)}.$$  

In addition to the various bounds above, we will also need a bound on $\nabla \chi_\xi$, the covariant derivative of $\chi_\xi$. The lemma below will be needed to obtain this bound, as well as several others in the sequel.
Lemma 4.2. Let \( x \in \mathbb{S}^2 \) be fixed. Then, there is a constant \( C \) that is independent of \( \nu \) and \( X \) for which we have

\[
\sum_{\xi \in X} e^{-\frac{\nu}{\rho_X^2} \text{dist}(x,\xi)} < \frac{C \rho_X^2}{(1 - e^{-\nu})^2}.
\]

In addition, if \( B(x, r_0) \) is the ball of radius \( r_0 \) and center \( x \), then

\[
\sum_{\xi \in X \cap B(x, r_0)^c} e^{-\frac{\nu}{\rho_X^2} \text{dist}(x,\xi)} < C \rho_X^2 n_0 e^{-(n_0 - 1)\nu}, \quad n_0 = \lceil r_0 / h_X \rceil.
\]

Proof. Divide the sphere into bands of width \( \sim h_X \), center \( x \), and outer radius \( \sim nh_X, n \geq 1 \). The sum then satisfies the inequality

\[
\sum_{\xi \in X} e^{-\frac{\nu}{\rho_X^2} \text{dist}(x,\xi)} = \sum_{n=1}^{n_{\text{max}}} \sum_{x \in \text{band}_n \cap X} e^{-\frac{\nu}{\rho_X^2} d(x,\xi)} < \sum_{n=1}^{n_{\text{max}}} \#(\text{band}_n \cap X) e^{-(n-1)\nu},
\]

where \( n_{\text{max}} \sim \pi / h_X \). The area of \( \text{band}_n \) is \( \sim nh_X^2 \). Consequently, we have that cardinality \( \#(\text{band}_n \cap X) \sim nh_X^2 / q_X^2 = n \rho_X^2 \). Using this in the equation above yields

\[
\sum_{\xi \in X} e^{-\frac{\nu}{\rho_X^2} d(x,\xi)} < C \rho_X^2 \sum_{n=1}^{\infty} n e^{-(n-1)\nu}.
\]

Summing the series on the right above yields (4.6). To obtain (4.7), we sum the series \( \sum_{n=n_0}^{\infty} n \rho_X^2 \) and use the fact that \( n_0 - (n_0 - 1)e^{-\nu} < n_0 \).

Theorem 4.3. Adopt the notation of Theorem 4.1. There exists a constant \( C = C(\rho, m) \) such that

\[
|\nabla \chi_\xi(x)| \leq C q_X^{-1} e^{-\frac{\nu}{\rho_X^2} d(x,\xi)}.
\]

In addition, \( \|\nabla \chi_\xi\|_{L^\infty} \leq C q_X^{-1} \). Finally,

\[
\Lambda_1 := \max_{x \in \mathbb{S}^2} \sum_{\xi \in X} |\nabla \chi_\xi(x)| < C \rho_X^2 q_X^{-1} \frac{1}{(1 - e^{-\nu})^2}.
\]

Proof. The Hölder estimate given in [14] Theorem 5.3, with \( \epsilon = 1 \), is

\[
|\chi_\xi(x) - \chi_\xi(y)| \leq C d(x, y) e^{-\frac{\nu}{\rho_X^2} d(x,\xi)}.
\]

Fixing \( x \) and dividing by \( d(x, y) \) yields

\[
\left| \frac{\chi_\xi(x) - \chi_\xi(y)}{d(x, y)} \right| \leq C q_X^{-1} e^{-\frac{\nu}{\rho_X^2} d(x,\xi)}.
\]

Let \( \hat{t} \) be a unit tangent vector based at \( x \). Choose \( y \) to be a point along the geodesic starting at \( x \) with tangent \( \hat{t} \). Then,

\[
\lim_{d(x, y) \to 0} \left| \frac{\chi_\xi(x) - \chi_\xi(y)}{d(x, y)} \right| = |D_t(\chi_\xi)(x)| \leq C q_X^{-1} e^{-\frac{\nu}{\rho_X^2} d(x,\xi)}.
\]

This holds for every direction \( \hat{t} \). Since \( \max_{\hat{t}} |D_t(\chi_\xi)(x)| = |\nabla \chi_\xi(x)| \), the bound (4.8) follows immediately. Obviously, we also have \( \|\nabla \chi_\xi\|_{L^\infty} \leq C q_X^{-1} \).
Proposition 4.4. Adopt the notation and assumptions of Theorem 4.1 and suppose that $a, b \in C^\infty$, $m \geq 2$ and $0 < \epsilon < 2m - 3$. Then, $b\chi \xi \chi \eta \in H_{2m-1-\epsilon} \cap L^\infty$. Moreover, for $h\chi$ sufficiently small, there exists $C = C(p,m)$ such that

$$
\|b\chi \xi \chi \eta\|_{H_{2m-1-\epsilon}} \leq Ch_X^{2+\epsilon-2m}\|b\|_{H_{2m}}. 
$$

(4.10)

$$
\|a\nabla \chi \xi \cdot \chi \eta\|_{H_{2m-\epsilon-2}} \leq Ch_X^{1+\epsilon-2m}\|a\|_{H_{2m}}.
$$

(4.11)

Proof. In the proof below we will need the inequalities $2m - \epsilon - 1 > 2$ and $2m - \epsilon - 2 > 1$, which follows easily from $0 < \epsilon < 2m - 3$.

Theorem [B.1] applies to $b\chi \xi$, because $b \in C^\infty \subset H_{2m-1-\epsilon} \cap L^\infty$ and, by Theorem [4.1], $\chi \xi \in H_{2m-1-\epsilon} \cap L^\infty$. Consequently, the three products $b\chi \xi \chi \eta$, $b\chi \xi$ and $\chi \xi \chi \eta$ are in $H_{2m-1-\epsilon} \cap L^\infty$. A straightforward application of Theorem [B.1] to the various products then results in this bound:

$$
\|b\chi \xi \chi \eta\|_{H_{2m-1-\epsilon}} \leq C\|b\|_{L^\infty}\left(\|\chi \xi\|_{L^\infty}\|\chi \eta\|_{H_{2m-1-\epsilon}} + \|\chi \xi\|_{H_{2m-1-\epsilon}}\|\chi \eta\|_{L^\infty}\right)
$$

$$
+ C\|b\|_{H_{2m-1-\epsilon}}\|\chi \xi\|_{L^\infty}\|\chi \eta\|_{L^\infty}.
$$

By (4.5), with $p = \infty$, we have that both $\|\chi \xi\|_{L^\infty}$ and $\|\chi \eta\|_{L^\infty}$ are bounded by the constant $c_2$, because the corresponding $\beta$’s have a single entry, 1. Moreover, since $2m - 1 - \epsilon > 1$, the Sobolev embedding theorem and a standard inclusion inequality imply that $\|b\|_{L^\infty} \leq C\|b\|_{H_{2m-1-\epsilon}} \leq C\|b\|_{H_{2m}}$. Inserting these in the previous inequality then yields

$$
\|b\chi \xi \chi \eta\|_{H_{2m-1-\epsilon}} \leq Cc_2\|b\|_{H_{2m}}\left(\|\chi \xi\|_{H_{2m-1-\epsilon}} + \|\chi \eta\|_{H_{2m-1-\epsilon}} + c_2\right).
$$

(4.12)

We will now employ a Bernstein inequality [23, Theorem 6.1] that holds for functions in $V_{\phi_m,X}$. The parameters in the theorem are $\beta = 2m$, from (4.1), $p = 2$, $n = 2$, $\gamma = 2m-1-\epsilon > 2$ and $g = \chi \xi$. The theorem then implies that $\|\chi \xi\|_{H_{2m-1-\epsilon}} \leq Cq_X^{1+\epsilon-2m}\|\chi \xi\|_{L^2}$. Moreover, if we set $p = 2$ in (4.5), we also have both $\|\chi \xi\|_{L^2}$ and $\|\chi \eta\|_{L^2}$ bounded by $c_2q_X$. Thus, $\|\chi \xi\|_{H_{2m-1-\epsilon}} \leq Cq_X^{2+\epsilon-2m}$. Combining the various bounds above we arrive at $\|b\chi \xi \chi \eta\|_{H_{2m-1-\epsilon}} \leq Cc_2\|b\|_{H_{2m}}q_X^{2+\epsilon-2m}(2 + c_2q_X^{2m-2-\epsilon})$. Since $q_X \ll 1$ and $2m - 2 - \epsilon > 1$, we have $\|b\chi \xi \chi \eta\|_{H_{2m-1-\epsilon}} \leq Cc_2\|b\|_{H_{2m}}, q_X^{2+\epsilon-2m}$. From this, (4.10) follows on observing that $q_X \sim h_X$.

To obtain the second bound, note that, by Corollary [B.3], the conditions on $\chi \xi,\chi \eta$ imply that $\nabla \chi \xi \cdot \nabla \chi \eta \in H_{2m-\epsilon-2} \cap L^\infty$, since $2m - \epsilon - 2 > 1$, and that those on $a$ are the ones used for $b$. Consequently,

$$
\|a\nabla \chi \xi \cdot \chi \eta\|_{H_{2m-\epsilon-2}} \leq C\|a\|_{H_{2m}}\left(\|\nabla \chi \xi \cdot \chi \eta\|_{H_{2m-\epsilon-2}} + \|\nabla \chi \xi \cdot \chi \eta\|_{L^\infty}\right).
$$

(4.13)

Since $2m - 2 - \epsilon > 1$, we may again apply the Sobolev embedding theorem to obtain $\|\nabla \chi \xi \cdot \chi \eta\|_{L^\infty} \leq C\|\nabla \chi \xi \cdot \chi \eta\|_{H_{2m-\epsilon-2}}$. Combining this with the previous inequality results in

$$
\|a\nabla \chi \xi \cdot \chi \eta\|_{H_{2m-\epsilon-2}} \leq C\|a\|_{H_{2m}}\|\nabla \chi \xi \cdot \chi \eta\|_{H_{2m-\epsilon-2}}.
$$

To estimate the norm on the right we will use Corollary [B.3]. This implies that $\|\nabla \chi \xi \cdot \chi \eta\|_{H_{2m-\epsilon-2}} \leq C(\|\chi \xi\|_{H_{2m-1-\epsilon}} + \|\chi \eta\|_{H_{2m-1-\epsilon}})(\|\nabla \chi \xi\|_{L^\infty} + \|\nabla \chi \eta\|_{L^\infty}).$

---

1The precise version of the theorem holds for a positive definite SBF. However, it is easy to modify it so that it will hold for a conditionally positive definite SBF.
We may use Theorem 4.3 and the bounds on $\|\chi_\xi\|_{H^{2m-1-\epsilon}}$, $\|\chi_\eta\|_{H^{2m-1-\epsilon}}$ found above to obtain this:

$$\|\nabla \chi_\xi \cdot \nabla \chi_\eta\|_{H^{2m-2}} \leq Ch_{X}^{2+\epsilon-2m}q_{X}^{-1} \leq Ch_{X}^{1+\epsilon-2m}.$$  \hspace{1cm} (4.14)

Finally, using the bound from (4.14) in (4.13) yields (4.11). \hfill \Box

4.2. Quadrature formulas. Numerically computing the integrals that arise in any Galerkin method ultimately requires a quadrature formula. In the setting of a sphere and other homogeneous manifolds, kernel quadrature formulas \cite{9, 15, 32} have been developed and analyzed. Let $f$ be continuous and consider the surface spline $\phi_m$ given in (4.1). In addition, let $Y$ be a quasi-uniform set of points on $S^2$, which may be different from $X$. The quantities $q_Y$, $h_Y$, and $\rho_Y$, and cardinality $N_Y$ have their usual meanings. Using $\phi_m$, form the Lagrange functions $\tilde{\chi}_\zeta$, $\zeta \in Y$ corresponding to $Y$ and the interpolant $I_Yf = \sum_{\zeta \in Y} f(\zeta)\tilde{\chi}_\zeta$. The quadrature formula is obtained integrating $I_Yf$:

$$Q_Y(f) = \int_{S^2} I_Yf(x)d\mu(x) = \sum_{\zeta \in Y} f(\zeta)w_\zeta, \quad w_\zeta := \int_{S^2} \tilde{\chi}_\zeta(x)d\mu(x).$$

We point out that a few of the weights $w_\zeta$ can be near zero or become slightly negative in the case of arbitrary $Y$. This is usually not the case for most quasi-uniform sets $Y$. (See the discussion in \cite{9, Section 2.2.1; 32}.) In fact, not only are the weights positive for most sets, but they also satisfy the lower bound

$$w_\zeta \geq Ch_{Y}^{2}. \hspace{1cm} (4.15)$$

In the rest of our discussion, we will assume that (4.15) holds. The only situation where this assumption comes into play will be in stability considerations of the discretized version of the stiffness matrix.

Positive or not, the weights all satisfy an upper bound; namely,

$$|w_\zeta| \leq Ch_{Y}^{2}. \hspace{1cm} (4.16)$$

Since $w_\zeta = \int_{S^2} \tilde{\chi}_\zeta(x)d\mu(x)$, we have that $|w_\zeta| \leq \|\tilde{\chi}_\zeta\|_{L_1(S^2)}$. To estimate the right side, use (4.3), with $p = 1$, $\beta_\zeta = 1$ and all of the other $\beta$’s equal to 0. This gives us $\|\beta\|_{L_1} = 1$, and so $\|\tilde{\chi}_\zeta\|_{L_1(S^2)} \leq c_2q_{Y}^{2} \leq c_2\rho_{Y}^{-2}h_{Y}^{2}$.

The salient feature of this quadrature formula is that the weights can be obtained by solving a linear system of equations that is stable and, while not sparse, has entries that decay rapidly as they move away from the diagonal. For $m = 2$, weights for a set $Y$ having 600,000 points were easily computed \cite{9, section 5}.

Error estimates for the quadrature formula $Q_Y$ have been derived for functions in various integer valued Sobolev spaces. However, we will need stronger results. We begin with the proposition below, which holds on $S^n$, $n \geq 2$. Consider a (conditionally) positive definite SBF $\phi$ that satisfies $c(1+\lambda_\ell)^{-\tau} \leq \phi_\ell \leq C(1+\lambda_\ell)^{-\tau}$ for all $\ell \geq L + 1$. The novel feature of this result is that it uses a new version of the “doubling trick”, which is established in Theorem A.3 to obtain higher convergence rates for functions smoother than ones in the native space of $\phi$. The result is this:

**Proposition 4.5.** Let $\tau > n/2$, $2\tau \geq \mu > n/2$, and $f \in H_\mu$. If $h_Y$ is sufficiently small, then

$$\int_{S^n} f(x)d\mu - Q_Y(f) \leq Ch_{Y}^{\mu}\|f\|_{H_\mu}. \hspace{1cm} (4.17)$$

Finally, using the bound from (4.14) in (4.13) yields (4.11).
Proof. Note that \( \left| \int_{S^n} f d\mu - Q_Y(f) \right| \leq \int_{S^n} |f - I_Y f| d\mu \leq \omega_n^{1/2} \|f - I_Y f\|_{L^2} \), where \( \omega_n \) is the volume of \( S^n \). In addition, by Theorem \[A.3\] with \( \beta = 0 \), we have \( \|f - I_Y f\|_{L^2} \leq Ch_Y^\alpha \|f\|_{H^\alpha} \). Combining the two inequalities yields \((4.17)\). \(\square\)

5. WEAK AND STRONG SOLUTIONS TO \( Lu = f \)

In this section we will lay out the properties, assumptions and various aspects of weak and strong solutions to \((1.1)\). In local coordinates on \( S^2 \), this equation has the form,

\[
Lu = -\frac{1}{\sqrt{\det(g_{ij})}} \sum_{i,j} \frac{\partial}{\partial x^i} \sqrt{\det(g_{ij})} a^{ij}(x) \frac{\partial u}{\partial x^j} + b(x) u = f.
\]

Here, \( g_{ij} \) is the covariant form of the standard metric tensor \( g \) on \( S^2 \); as usual, \( g^{ij} = [g_{ij}]^{-1} \) are the contravariant components of \( g \). The \( a^{ij} \)'s are contravariant components of a \( C^\infty \), symmetric rank 2 tensor \( a \) that is positive definite in the sense that there exist positive constants \( c_1, c_2 \) such that

\[
c_1 \sum_{i,j} g^{ij}(x) v_i v_j \leq \sum_{i,j} a^{ij}(x) v_i v_j \leq c_2 \sum_{i,j} g^{ij}(x) v_i v_j
\]

holds for all vectors \( v \) in the tangent space at \( x \in S^2 \). The function \( b(x) \) is \( C^\infty \). In addition, we assume that there are constants \( b_1, b_2 \) such that, for all \( x \in S^2 \), \( 0 < b_1 \leq b(x) \leq b_2 \). We note that in the case that \( a = g \), this reduces to the case \( Lu = -\Delta u + bu \).

With \( L \) as given in \((5.1)\) and \( f \in L^2(S^2) \), we can place \( Lu = f \) into weak form by multiplying by \( v \in H_1 \) and integrating by parts to arrive at

\[
\langle u, v \rangle := \int_{S^2} \left( \sum_{i,j=1}^2 a^{ij} \frac{\partial u}{\partial x^i} \frac{\partial v}{\partial x^j} + buv \right) d\mu = \int_{S^2} fvd\mu := \ell(v).
\]

By \((5.2)\) and the assumptions on \( b(x) \), the bilinear form \( \langle \cdot, \cdot \rangle_a \) satisfies

\[
M_1 \|u\|_{H_1}^2 = (c_1 + b_1) \langle u, u \rangle_{H_1} \leq \langle u, u \rangle_a \leq (c_2 + b_2) \langle u, u \rangle_{H_1} = M_2 \|u\|_{H_1}^2.
\]

A straightforward application of the Lax-Milgram theorem, together with \( \ell(v) \) being a bounded linear functional on \( H_1 \), then yields the following result:

**Proposition 5.1.** The bilinear form \( \langle \cdot, \cdot \rangle_a \) is coercive and bounded on \( H_1 \) and defines an inner product on \( H_1 \), with the norms \( \| \cdot \|_{a} \) and \( \| \cdot \|_{H_1} \) being equivalent. In addition, for \( f \in L^2 \), there is a unique \( u \in H_1 \) such that \((5.3)\) is satisfied; that is, \( u \in H_1 \) weakly solves \( Lu = f \). Finally, \( \|u\|_{L^2} \leq \|f\|_{L^2} \).

We now turn to the regularity of the weak solution to \( Lu = f \). A priori estimates of the general type needed here may be found in the survey article by Mikhailets and Murach \[24\] Theorem 6.6], along with the references. They are, however, given for pseudo-differential operators. A simpler approach is to use the local regularity theorems in \[7\] pgs. 261-269, which apply to open sets in \( \mathbb{R}^n \), and so to coordinate patches on \( S^n \). Since the sphere is compact, they apply globally to \( Lu = f \), and so we have the (standard) regularity result that we will use in the sequel.
Proposition 5.2. Let \( L \) be as described above. If \( u \) is a distributional solution to \( Lu = f \), where \( f \in H_s, 0 \leq s, s \in \mathbb{R} \), then \( u \in H_{s+2} \). In addition, for any \( t < s + 1 \) there is a constant \( C_t > 0 \) such that \( \|u\|_{H_{s+2}} \leq C_t(\|Lu\|_{H_s} + \|u\|_{H_t}) \) and \( \|u\|_{H_{s+2}} \leq C\|Lu\|_{H_s} \) all hold.

Proof. Let \( \{(\Omega_\alpha, \phi_\alpha)\} \) be an atlas for \( \mathbb{S}^n \). Since \( \mathbb{S}^n \) is a compact, \( C^\infty \) manifold, we may cover \( \mathbb{S}^n \) with a finite number of \( \Omega_\alpha \)'s. When \( \phi_\alpha(\Omega_\alpha) \cap \phi_\beta(\Omega_\beta) \neq \emptyset \), then \( \phi_\beta \circ \phi_\alpha^{-1} \) is \( C^\infty \). In a fixed chart \( (\Omega_\alpha, \phi_\alpha) \), the points \( p \in \Omega_\alpha \) are parametrized by \( p = \phi_\alpha^{-1}(x) \), where \( x = (x^1, \ldots, x^d) \in U_\alpha = \phi_\alpha(\Omega_\alpha) \). In addition, we choose the atlas so that in local coordinates the operator \( L \) is strongly elliptic.

Choose \( \{\rho_\alpha\} \) to be a \( C^\infty \) partition of unity subordinate to \( \{(\Omega_\alpha, \phi_\alpha)\} \). A distribution on \( \mathbb{S}^n \) is a continuous linear functional on \( D(\mathbb{S}^n) := C^\infty(\mathbb{S}^n) \). As usual, we denote the set of distributions by \( D'(\mathbb{S}^n) \). Distributions may be characterized in terms of local charts and a subordinate partition of unity: \( u \in D'(\mathbb{S}^n) \) if and only if \( \rho_\alpha u \circ \phi_\alpha^{-1} \) is compactly supported in \( U_\alpha \subseteq \mathbb{R}^n \) [16] §1.8. Next, because of the properties of a partition of unity, \( \rho_\alpha \circ \phi_\alpha^{-1} \) will be 1 in a neighborhood \( \Omega_0 \) of some point \( x_0 \in U_\alpha \). It follows that in local \( \Omega_\alpha \)-coordinates on \( \Omega_0 \), we have \( L\rho_\alpha u \circ \phi_\alpha^{-1} = Lu \circ \phi_\alpha^{-1} = f \circ \phi_\alpha^{-1} \in H^1_s(\Omega_0) \). By [7] Theorem 6.30, we then have \( u \circ \phi_\alpha^{-1}|_{\Omega_0} \in H^1_{s+2}(\Omega_0) \). If we use a different partition of unity \( \{\rho'_\alpha\} \), we can find another point \( x_1 \in U_\alpha \) and a corresponding neighborhood \( \Omega_1 \) for which \( u \circ \phi_\alpha^{-1}|_{\Omega_1} \in H^1_{s+2}(\Omega_1) \) holds. Continuing in this way will yield \( u \circ \phi_\alpha^{-1} \in H^1_{s+2}(U_\alpha) \) and \( Lu \circ \phi_\alpha^{-1} = f \circ \phi_\alpha^{-1} \) on \( U_\alpha \). Since \( \rho_\alpha \circ \phi_\alpha^{-1} \) is \( C^\infty \), with support within \( U_\alpha \), \( \rho_\alpha u \circ \phi_\alpha^{-1} \) also has its support within \( U_\alpha \). Thus, \( u \circ \phi_\alpha^{-1} \) being in \( H^1_{s+2}(U_\alpha) \) implies that \( \rho_\alpha u \circ \phi_\alpha^{-1} \in H^1_{s+2}(U_\alpha) \). Obviously \( \rho_\alpha u \circ \phi_\alpha^{-1} \) may be extended to all of \( \mathbb{R}^n \) by setting it equal to 0 outside its support in \( U_\alpha \). Let \( u_\alpha \) be this extension; note that \( u_\alpha \) is in \( H^1_{s+2}(\mathbb{R}^n) \). Since there are only finitely many \( \alpha \)'s, \( \sum_\alpha \|u_\alpha\|_{H^1_{s+2}(\mathbb{R}^n)} < \infty \). By [11] Proposition, p. 35, we have, up to equivalent norms, that \( \|u\|_{H^1_{s+2}(\mathbb{S}^n)} = \sum_\alpha \|u_\alpha\|_{H^1_{s+2}(\mathbb{R}^n)} < \infty \); consequently, \( u \in H^1_{s+2}(\mathbb{S}^n) \).

We now turn to the inequalities. The support of \( u_\alpha \) is contained in \( U_\alpha \), where \( L \) is strongly elliptic. By [7] Corollary 6.27, \( \Omega = \Omega_\alpha \), we have, with \( s \to s + 2 \) and \( t < s - 1 \to t < s + 1 \),

\[
\|u_\alpha\|_{H^1_{s+2}(\mathbb{R}^n)} \leq C_t(\|L(u_\alpha)\|_{H^1_s(\mathbb{R}^n)} + \|u_\alpha\|_{H_t(\mathbb{R}^n)}).
\]

The last part of the argument in step 2 of [7] Theorem 6.25 starts with \( L(u_\alpha) = L(\rho_\alpha u) = \rho_\alpha Lu + [L, \rho_\alpha]u \), in local coordinates; the commutator term is a differential operator having order lower than \( L \). Repeating the rest of the arguments in [7] Theorem 6.25 and [7] Corollary 6.27, we arrive at

\[
\|u_\alpha\|_{H^1_{s+2}(\mathbb{R}^n)} \leq C_t(\|\rho_\alpha Lu\|_{H^1_s(\mathbb{R}^n)} + \|u_\alpha\|_{H^1(\mathbb{R}^n)}).
\]

Summing over \( \alpha \) then yields \( \|u\|_{H^1_{s+2}(\mathbb{S}^n)} \leq C_t(\|Lu\|_{H^1_s(\mathbb{S}^n)} + \|u\|_{H^1(\mathbb{S}^n)}) \), which in the notation \( \|\cdot\|_s = \|\cdot\|_{H^1_s(\mathbb{S}^n)} \) is the first inequality. To obtain the second inequality, start by setting \( t = 0 \) in the first. Also note that, from (5.3), we have \( \min(b)\|u\|_{L^2}^2 \leq \|u\|_{L^2}^2 = \langle Lu, u \rangle_{L^2} \leq \|Lu\|_{L^2}\|u\|_{L^2} \). Dividing by \( \|u\|_{L^2} \), we obtain \( \|u\|_{L^2} \leq C\|Lu\|_{L^2} \), since \( \min(b) > 0 \). Thus \( \|u\|_{s+2} \leq C(\|Lu\|_{H^1} + \|Lu\|_{L^2}) \). The inequality that we want then follows from the observation that \( \|Lu\|_{L^2} \leq \|Lu\|_{H^1} \). □

Remark 5.3. The proof above holds for any \( C^\infty \) compact Riemannian manifold \( \mathbb{M} \) and any strongly elliptic differential operator on \( \mathbb{M} \).
We close this section with a corollary to the regularity result above for \( L \) defined in (6.1). The corollary forms the basis of the “Nitsche trick” [27] that we will use later.

**Corollary 5.4.** Let \( V \) be a closed subspace of \( H_1 \) and let \( P_V \) be the orthogonal projection of \( V \) onto \( H_1 \), relative to the inner product \( \langle \cdot, \cdot \rangle_a \). If \( u \in H_1 \) and \( Lw = u - P_V u \), then

\[
\|u - P_V u\|_{L^2}^2 \leq \|w - P_V w\|_a \|u - P_V u\|_a.
\]

**Proof.** Since we have \( Lw = u - P_V u \), the regularity result above implies that \( w \) is in \( H_3 \). Integrating by parts in \( \langle w, u - P_V u \rangle_a \) yields

\[
\langle w, u - P_V u \rangle_a = \langle Lw, u - P_V u \rangle_{L^2} = \langle u - P_V u, u - P_V u \rangle_{L^2} = \|u - P_V u\|^2_{L^2}.
\]

Next, note that \( P_V w \) is in \( V \), and so \( P_V w \) is orthogonal to \( u - P_V u \), relative to \( \langle \cdot, \cdot \rangle_a \). Consequently, \( \langle w, u - P_V u \rangle_a = \langle w - P_V w, u - P_V u \rangle_a \). It follows that \( \|u - P_V u\|^2_{L^2} = \langle w - P_V w, u - P_V u \rangle_a \). Applying Schwarz’s inequality then yields (5.5). \( \square \)

6. Galerkin approximation for \( Lu = f \)

### 6.1. Error estimates

We will use spaces of spherical basis functions to obtain approximate solutions to \( Lu = f \); specifically, the \( V_{\phi, x} \)'s and the \( V_{\phi, L, X} \)'s defined earlier. Let \( \phi \) be an SBF on \( S^2 \) that is positive definite or conditionally positive definite. For \( \tau > 1 \), we will make the assumption that the Fourier-Legendre coefficients of \( \phi \) satisfy

\[
(1 + \lambda \ell)^{-\tau} \leq \hat{\phi}_{\ell} \leq C(1 + \lambda \ell)^{-\tau}, \quad \forall \ell \geq L + 1,
\]

where \( c \) and \( C \) are positive constants and \( L \) is the highest order special harmonic reproduced by interpolation from \( V_{\phi, L, X} \). In later sections, when we will be concerned with the stability of discretizing the problem, we will restrict the SBFs to the thin-plate splines, because they have highly localized bases. For obtaining error estimates, this is unnecessary, since the estimates depend only on the space and not the basis employed.

Let \( P_X := P_{V_X} \) be the orthogonal projection of \( H_1 \) onto the finite dimensional space \( V_X \), in the \( \langle \cdot, \cdot \rangle_a \) inner product, and let \( I_X \) be the interpolation operator associated with \( V_X \). Since \( \|u - P_X u\|_a = \min_{v \in V_X} \|u - v\|_a \), we have that

\[
\|u - P_X u\|^2_a \leq \|u - I_X u\|^2_a \leq C\|u - I_X u\|_{H^1},
\]

where the last inequality follows from the equivalence of the norms \( \| \cdot \|_a \) and \( \| \cdot \|_{H^1} \). The same reasoning applies to the solution \( w \) to \( Lw = u - P_X u \), so \( \|w - P_X w\|_a \leq C\|w - I_X w\|_{H^1} \). Combining these estimates with the one from Corollary 5.3 then yields this:

\[
\|u - P_X u\|^2_{L^2} \leq C\|w - I_X w\|_{H^1} \|u - I_X u\|_{H^1}.
\]

The regularity results in Proposition 5.2 imply that if \( f \in H_s \), then the solution \( u \) to \( Lu = f \) is in \( H_{s+2} \). Moreover, if \( \tau > 1 \), then the projection \( P_X u \) exists and is in \( H_{s+\alpha} \), for any \( \alpha < \tau - 1 \). Thus, \( u - P_X u \) belongs to \( H_{\sigma}, \sigma := \min(s+2, \tau+\alpha) \). Applying the elliptic regularity result to \( Lw = u - P_X u \) then gives us \( w \in H_{\sigma+2} \).

We are mainly interested in the case of \( S^2 \) – i.e., \( n = 2 \). For that case, we have the following lemma, which will be needed to obtain error estimates.
Lemma 6.1. Let \( n = 2 \). In the notation used above,
\[
\|w - I_X w\|_{H_1} \leq Ch^2 \|u - I_X u\|_{H_1}.
\]

Proof. The solution to \( Lu = u - P_X u \) is in \( w \in H_{\sigma+2} \), where \( \sigma := \min(s+2, \tau + \alpha) \), \( \alpha < \tau - n/2 = \tau - 1 \), and \( \tau > n/2 = 1 \). It follows that \( \sigma > 1 \) and so \( w \) is in \( H_{\sigma+2} \subset H_3 \). Applying Theorem A.3 then yields
\[
\|w - I_X w\|_{H_1} \leq Ch^2 \|w\|_{H_3}.
\]
Furthermore, by Proposition 5.2 \( \|w\|_{H_3} \leq C \|Lu\|_{H_1} = \|u - P_X u\|_{H_1} \). Since the usual norm for \( H_1 \) is equivalent to the \( \|\cdot\|_a \) norm, we have \( \|w\|_{H_3} \leq C \|u - P_X u\|_a \). Then, by this inequality and (6.2), we see that \( \|w\|_{H_3} \leq C \|u - I_X u\|_{H_1} \). Combining this with (6.4) gives us (6.6).

Theorem 6.2. Let \( n = 2 \) and \( Lu = f, f \in H_s, s \geq 0 \). In the notation used above,
\[
\|u - P_X u\|_{L^2} \leq C \left\{ \begin{array}{ll}
Ch^{s+2} \|u\|_{H^{s+2}} & \text{if } s \leq 2\tau - 2, \\
Ch^{2\tau} \|u\|_{H^{2\tau}} & \text{if } 2\tau - 2 < s.
\end{array} \right.
\]

Proof. By Theorem A.3 we have that
\[
\|u - I_X u\|_{H_1} \leq C \left\{ \begin{array}{ll}
Ch^{s+1} \|u\|_{H^{s+2}} & \text{if } s \leq 2\tau - 2, \\
Ch^{2\tau - 1} \|u\|_{H^{2\tau}} & \text{if } 2\tau - 2 < s.
\end{array} \right.
\]
The estimate in terms of the Sobolev norms of \( u \) then follows from (6.3) and (6.4). Proposition 5.2 implies that \( \|u\|_{s+2} \leq C \|Lu\|_s \). Since \( f = Lu, \|u\|_{s+2} \leq C \|f\|_s \). When \( s > 2\tau - 2 \), then the previous argument applies, with \( s + 2 \) being replaced by \( 2\tau \).

6.2. The stiffness matrix in the Lagrange basis. The error estimates obtained above are, as we noted, independent of the basis used for the Galerkin approximation space. However, to actually solve for the Galerkin approximation, we must pick a suitable \( \phi \) for which there is a good basis for \( V_{\phi,L,X} \), one that results in a numerically robust method for finding the Galerkin solution. We will show that a surface spline \( \phi_m \) and the corresponding Lagrange basis \( \{\chi_\xi: \xi \in X\} \) will provide the required robustness.

Let \( m \geq 2 \) and set \( V_{m,X} = V_{\phi_m,m-1,X} \). Take the basis for \( V_{m,X} \) to be \( \{\chi_\xi: \xi \in X\} \). The Galerkin approximation to the solution \( Lu = f \) is the orthogonal projection \( u_X := P_{m,X} u \) of \( u \) onto \( V_{m,X} \), in the \( \langle \cdot, \cdot \rangle_a \) inner product. If \( u_X = \sum_{\xi \in X} \alpha_\xi \chi_\xi \), then, from the weak form of \( Lu = f \) and the usual normal equations, we obtain the stiffness matrix:
\[
A \alpha = \tilde{f}, \text{ where } A_{\xi,\eta} = \langle \chi_\xi, \chi_\eta \rangle_a, \quad \alpha = (\alpha_\xi), \quad \tilde{f} = (\langle f, \chi_\xi \rangle_{L^2}).
\]
Eventually, we will discretize the problem by using quadrature methods to approximate \( A \). For now, we will restrict our attention to \( A \).

6.2.1. Stability of \( A \). We want to estimate \( \kappa_2(A) \), the condition number for \( A \). We begin with the observation that \( A \) is a real, self-adjoint matrix. It is also a Gram matrix for the linearly independent set, \( \{\chi_\xi: \xi \in X\} \), and is therefore positive definite as well. Consequently, \( \kappa_2(A) = \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)} \).

We will begin by estimating \( \lambda_{\min}(A) \). First of all, the operator \( L \) is self-adjoint and positive definite. Standard variational methods then imply that
\[
\min_{v \in H_1} \langle v, v \rangle_A = \lambda_{\min}(L), \quad \|v\|_{L^2} = 1.
\]
Proof. Let \(\alpha^T A\alpha = \langle v, v \rangle_a\). By (6.9), we have that
\[
\alpha^T A\alpha \geq \lambda_{\min}(L)\|v\|_{L^2}^2.
\]
Next, from (4.5), with \(p = 2\), we have
\[
\|v\|_{L^2} = \|\sum_{\xi \in X} \alpha_\xi \chi_\xi\|_{L^2} \geq c_1 q_X \|\alpha\|_{\ell_2(X)}.
\]
From the inequality above, we then have
\[
\alpha^T A\alpha \geq \lambda_{\min}(L)\|v\|_{L^2}^2 \geq c_1^2 q_X^2 \lambda_{\min}(L)\|\alpha\|_{\ell_2(X)}^2,
\]
which holds for all \(\alpha \in \mathbb{R}^{|X|}\). Hence, we have that
\[
\lambda_{\min}(A) \geq c_1^2 q_X^2 \lambda_{\min}(L).
\]
Estimating \(\lambda_{\max}(A)\) requires the Bernstein inequality from [23, Theorem 6.1]. In the case at hand, the theorem cited \(^2\) holds with \(n = 2, p = 2, 0 < \gamma < 2m - 1\), and \(g = v\). For every \(v \in \mathcal{V}_{m,X}\) and every \(0 < \gamma < 2m - 1\), we have that there is a constant \(C > 0\), where \(C = C(m, \rho)\), such that
\[
\|g\|_{H_\gamma} \leq C q_X^{-\gamma} \|g\|_{L^2}.
\]
Since \(\alpha^T A\alpha = \langle v, v \rangle_a\) and, by (6.4), \(\langle v, v \rangle_a \leq M_2 \|v\|_{H_1}^2\), then, from (6.11) (\(\gamma = 1\)) and (4.5) (\(p = 2, n = 2\)), we have that \(\alpha^T A\alpha \leq C q_X^{-2} \|v\|_{L^2}^2 \leq C c_2 \|\alpha\|_{\ell_2(X)}^2\).
Consequently,
\[
\lambda_{\max}(A) \leq C, \quad C = C(m, \rho, \|\cdot\|_a).
\]
Combining (6.12) and (6.10) results in the following:

**Theorem 6.3.** Let \(A\) be the stiffness matrix in the basis \(\{\chi_\xi : \xi \in X\}\) for \(\mathcal{V}_{m,X}\). If \(q_X\) is sufficiently small and \(\rho\) is fixed, then there is an constant \(C = C(m, \rho, \|\cdot\|_a)\) for which the condition number \(\kappa_2(A)\) satisfies \(\kappa_2(A) \leq C q_X^{-2}\).

### 6.2.2. Exponential decay of the entries of \(A\).

At this point, we turn to the behavior of the entries \(A_{\xi,\eta}\) in \(A\). What we will see is that the entries in \(A\) decay exponentially in \(\text{dist}(\xi, \eta)\), making \(A\) nearly sparse. In section 8 we will use this decay to construct a sparse discretization for \(A\). Establishing decay requires the following lemma.

**Lemma 6.4.** Let \(f, g\) be in \(C^1(S^2)\), and let \(\nabla f, \nabla g\) be their covariant derivatives. Then, there is a constant \(C\) such that for all \(x \in S^2\) we have
\[
|\sum_{i,j} a_{ij}^x(x) \nabla_i f(x) \nabla_j g(x)| \leq C|\nabla f(x)| \|\nabla g(x)\|.
\]

**Proof.** The matrix \(a_{ij}^x\) is positive definite, so we may use it as an inner product. Schwarz’s inequality applied to this inner product implies that
\[
|\sum_{i,j} a_{ij}^x \nabla_i f \nabla_j h| \leq (\sum_{i,j} a_{ij}^x \nabla_i f \nabla_j f)^{1/2} (\sum_{i,j} a_{ij}^x \nabla_i g \nabla_j g)^{1/2}.
\]
By (6.2), we have that \(\sum_{i,j} a_{ij}^x \nabla_i f \nabla_j f \leq \sum_{i,j} a_{ij}^x \nabla_i f \nabla_j f = |\nabla f|^2\). This also holds for \(g\) as well. Applying these inequalities to the previous one yields (6.13). \(\square\)

**Proposition 6.5.** For \(h_X\) sufficiently small and \(\nu\) as in Theorem 4.1
\[
|A_{\xi,\eta}| \leq C h_X^{-2} e^{-\frac{\nu}{h_X} \text{dist}(\xi, \eta)}.
\]

\(^2\)The precise version of the theorem holds for a positive definite SBF. However, it is easy to modify it so that it will hold for a conditionally positive definite SBF.
Proof. Since we have $A_{\xi,\eta} = \langle \chi_\xi, \chi_\eta \rangle_a$, we have that
\[
|A_{\xi,\eta}| \leq \int_{S^2} \left| \sum_{i,j} a^{ij} \nabla_i \chi_\xi \nabla_j \chi_\eta + b(x) \chi_\xi \chi_\eta \right| d\mu(x).
\]
By Lemma 6.4 and the boundedness of $b$, we have that
\[
\left| \sum_{i,j} a^{ij} \nabla_i \chi_\xi \nabla_j \chi_\eta + b(x) \chi_\xi \chi_\eta \right| \leq C |\nabla \chi_\xi| |\nabla \chi_\eta| + \|b\| L^\infty |\chi_\xi| |\chi_\eta|.
\]
Moreover, using (4.8) and (4.3), we see that
\[
C |\nabla \chi_\xi| |\nabla \chi_\eta| + \|b\| L^\infty |\chi_\xi| |\chi_\eta| \leq (C_1 q_X^{-2} + C_2) \exp \left(-\frac{\nu \text{dist}(x, \xi) + \text{dist}(x, \eta)}{h_X}\right).
\]
Because $\text{dist}(\cdot, \cdot)$ is a metric on $S^2$, we may use the triangle inequality: $\text{dist}(x, \xi) + \text{dist}(x, \eta) \geq \text{dist}(\xi, \eta)$. In addition, $h_X$ small implies that $q_X = h_X/\rho_X$ is also small. Combining these facts yields, uniformly in $x$, this inequality:
\[
\left| \sum_{i,j} a^{ij} \nabla_i (\chi_\xi) \nabla_j (\chi_\eta) + b(x) \chi_\xi \chi_\eta \right| \leq C h_X^{-2} e^{-\frac{\nu}{\rho_X} \text{dist}(\xi, \eta)}.
\]
Integrating both sides above then establishes (6.14). \qed

7. The discretized Galerkin solution

The discretized solution to the Galerkin problem is obtained simply by replacing the stiffness matrix $A$ from the original problem with a discretized version, which is obtained via quadrature, and then solving as usual. In order to carry out a complete analysis of this method, in the sequel we will restrict the tensor $A^{ij}$ to have the form $A^{ij} = a g^{ij}$, where $a \in C^\infty(S^2)$.

7.1. Discretizing the stiffness matrix. We now turn to the task of discretizing the stiffness matrix. Our approach is to approximate the $(\xi, \eta)$ entry $A_{\xi,\eta} = \int_{S^2} (a \nabla \chi_\xi \cdot \nabla \chi_\eta + b \chi_\xi \chi_\eta) d\mu$ by means of the quadrature formulas discussed in section 4.2. In doing so, we will allow for the surface spline used in the Galerkin method, $\phi_m$, to differ from the one used in the quadrature formula. We will denote the latter by $\phi_M$, with $M \geq 2$.

7.1.1. Discretization error for the stiffness matrix. To discretize the stiffness matrix, we will employ a set of nodes $Y$ that is chosen independently of $X$. In general, $Y$ will be much larger than $X$ and need not contain $X$ as a subset. That said, the discretization of $A_{\xi,\eta} = \int_{S^2} (a \nabla \chi_\xi \cdot \nabla \chi_\eta + b \chi_\xi \chi_\eta) d\mu$ is $Q_Y (a \nabla \chi_\xi \cdot \nabla \chi_\eta + b \chi_\xi \chi_\eta)$. In explicit form, this is given by
\[
A_{\xi,\eta}^Y := \sum_{\xi \in Y} (a \nabla \chi_\xi \cdot \nabla \chi_\eta + b \chi_\xi \chi_\eta) |_\xi w_\xi.
\]
We will need the following lemma to obtain bounds on the error $|A_{\xi,\eta} - A_{\xi,\eta}^Y|$:

Lemma 7.1. Suppose that $m > M$ are positive integers and that $\epsilon > 0$. Let $\sigma = m - M$. If $M \geq \sigma + 1$ and $\epsilon < 2\sigma - 1$, then $\epsilon < m - 2$ and $2m - 1 - \epsilon > 2M$.

Proof. We will begin by showing the first inequality. By assumption, $\epsilon < 2\sigma - 1$, so $\epsilon < \sigma + \sigma + 1 - 2 \leq \sigma + M - 2 = m - 2$. To get the second, note that $2m - \epsilon - 1 = 2M + 2\sigma - 1 - \epsilon$. Since $2\sigma - 1 - \epsilon > 0$, we have that $2m - \epsilon - 1 > 2M$. \qed
Corollary 7.2. Suppose that $a(x), b(x) \in C^\infty(\mathbb{S}^2)$, and adopt the notation and assumptions from Lemma 4.1. In addition, let $\{\chi_\xi : \xi \in X\}$ and $\{\tilde{\chi}_\xi : \xi \in Y\}$ be the Lagrange bases for $V_{m,X} := V_{\phi_m,X}$ and for $V_{m,Y} := V_{\phi_m,Y}$, respectively. Then, if $\delta := 2\sigma - 1 - \epsilon > 0$, we have:

$$
(7.2) \quad |A_{\xi,\eta} - A_{\xi,\eta}^Y| \leq C(h_Y/h_X)^{2M}h_X^{-\delta}, \quad C = C(||a||_{H_{2m}}, ||b||_{H_{2m}}),
$$

where $A_{\xi,\eta}^Y = Q_Y(a\nabla \chi_\xi \cdot \nabla \chi_\eta + b\chi_\xi \chi_\eta)$. Moreover, $\delta \leq m - 2 - \epsilon$, so it may be made as small as we wish by taking $\epsilon$ close to $m - 2$.

Proof. By Proposition 4.4, $b\chi_\xi \chi_\eta$ is in $H_{2m-1-\epsilon}$ for all $0 < \epsilon < m - 2$. By Lemma 7.1 if $M \geq \sigma + 1$ and $\epsilon < 2\sigma - 1$, then we have both $2m - 1 - \epsilon > 2M$ and $\epsilon < m - 2$. It follows that we may use (4.17) with $\tau = M$ and $\mu = 2M < 2m - 1 - \epsilon = 2M + 2\sigma - 1 - \epsilon$. From the bound in (4.10), and from $||b\chi_\xi \chi_\eta||_{H_{2M}} \leq ||b\chi_\xi \chi_\eta||_{H_{2m-1-\epsilon}}$, we see that

$$
(7.3) \quad \left| \int_{\mathbb{S}^2} b\chi_\xi \chi_\eta d\mu - Q_Y(b\chi_\xi \chi_\eta) \right| \leq C(h_Y/h_X)^{2M} ||b\chi_\xi \chi_\eta||_{H_{2m-1-\epsilon}}^2
\quad \leq C(h_Y/h_X)^{2M} h_X^{1+\epsilon-(2\sigma-1)-2M} ||b||_{H_{2m}}
\quad \leq C(h_Y/h_X)^{2M} h_X^{1-\delta} ||b||_{H_{2m}}.
$$

Using the same argument, but with the bounds from (4.10) instead of (4.10), we have

$$
(7.4) \quad |\int_{\mathbb{S}^2} a\nabla \chi_\xi \cdot \nabla \chi_\eta d\mu - Q_Y(a\nabla \chi_\xi \cdot \nabla \chi_\eta) | \leq C(h_Y/h_X)^{2M} h_X^{-\delta} ||a||_{H_{2m}}.
$$

If we combine (7.3) and (7.4) and note that $h_X < \pi$, we obtain (7.2). To prove the statement concerning $\delta$, observe that $\delta = \sigma - 2 - \epsilon + (\sigma + 1) \leq m - M - 2 + \epsilon + M = m - 2 - \epsilon$. \hfill \square

Remark 7.3. If $M \geq m$ and $0 < \epsilon < m - 2$ , then, obviously, $1 < m + 1 < 2m - \epsilon - 1 < 2M$. Thus, we may use (4.17) with $\tau = M$ and $\mu = 2m - 1 - \epsilon$. The same arguments employed above then imply that the error estimate in (7.2) becomes

$$
(7.5) \quad |A_{\xi,\eta} - A_{\xi,\eta}^Y| \leq C \max(||a||_{H_{2m}}, ||b||_{H_{2m}})(h_Y/h_X)^{2m-1-\epsilon}.
$$

The right side in this inequality depends on $m$ and $\epsilon$, but not on $M$. It follows that there is no advantage in choosing $M > m$.

One important fact is that $A_{\xi,\eta}^Y$ decays in dist$(\xi, \eta)$ in the same way as $A_{\xi,\eta}$. We will establish this below. Before carrying out the proof, we mention that, although we use the assumption (4.15) in our proof, it is not necessary to do so. We also wish to point out that the denominator of the fraction in the exponent is $h_X$, and not $h_Y$, as one would first suppose.

Proposition 7.4. The discretized entry $A_{\xi,\eta}^Y$ satisfies the bound

$$
|A_{\xi,\eta}^Y| \leq C h_X^{-2} \exp(-\frac{\nu}{h_X} \text{dist}(\xi, \eta)).
$$

Proof. The same argument used to establish (6.14) yields

$$
|A_{\xi,\eta}^Y| \leq C h_X^{-2} \left( \sum_{\zeta \in Y} |w_{\zeta}| \right) e^{-\frac{\nu}{h_X} \text{dist}(\xi, \eta)}.
$$
By our assumption that the weights are positive, we have that \( \sum_{\zeta \in Y} w_\zeta = \sum_{\zeta \in Y} w_\zeta = 4\pi \), from which the result is immediate.

The theorem below gives us the desired bound on the error \( \|A - A^Y\|_2 \) that is made in using quadrature to compute the entries in the stiffness matrix.

**Theorem 7.5.** Let \( m > M \geq 2 \) and let \( \phi_M \) be the surface spline used for quadrature. If \( 0 < \delta < 2(m - M) - 1 \), then, for \( h_X \) and \( h_Y \) sufficiently small, we have

\[
\|A - A^Y\|_2 \leq C (\log(h_Y))^2 (h_Y/h_X)^{2M} h_X^{-\delta}.
\]

**Proof.** Recall that, for a self-adjoint matrix \( C \), \( \|C\|_2 \leq \|C\|_1 = \|C\|_\infty \). Applying this to the self-adjoint matrix \( A - A^Y \) yields

\[
\|A - A^Y\|_2 \leq \max_{\eta \in X} \left( \sum_{\xi \in X} |A_{\xi,\eta} - A^Y_{\xi,\eta}| \right).
\]

The task is now to bound the sums on the right above. Let \( r_0 > h_x \) and let \( B(\eta, r_0) \) be the ball with center \( \eta \) and radius \( r_0 \). We may break up the sum over \( X \) into a sum over centers inside \( B(\eta, r_0) \) and those in \( B(\eta, r_0) \):

\[
\sum_{\xi \in X} |A_{\xi,\eta} - A^Y_{\xi,\eta}| = \sum_{\xi \in X \cap B(\eta, r_0)} |A_{\xi,\eta} - A^Y_{\xi,\eta}| + \sum_{\xi \in X \cap B(\eta, r_0)^g} |A_{\xi,\eta} - A^Y_{\xi,\eta}|.
\]

By Propositions 6.5 and 7.4, we have that \( |A_{\xi,\eta} - A^Y_{\xi,\eta}| \leq C h_X^{-2} e^{-\frac{\rho^2}{h_X^2} \text{dist}(\xi, \eta)} \). Using this and (4.7) yields

\[
\sum_{\xi \in X \cap B(\eta, r_0)^g} |A_{\xi,\eta} - A^Y_{\xi,\eta}| \leq C h_X^{-2} \sum_{\xi \in X \cap B(\eta, r_0)^g} e^{-\frac{\rho^2}{h_X^2} \text{dist}(\xi, \eta)}
\]

\[
< C \rho^2 h_X^{-2} e^{-\frac{(n_0 - 1)\nu}{\nu} \frac{n_0}{1 - e^{-\nu} - 1}}, \quad n_0 = \left\lceil \frac{r_0}{h_X} \right\rceil.
\]

The set of remaining centers is \( X \cap B(\eta, r_0) \), whose cardinality may be bounded by \( \text{vol}(B(\eta, r_0))/\text{vol}(B(\eta, q_X)) \sim \frac{r_0^2}{q_X^2} = \frac{h_X}{r_0} h_X^2 < h_X^2 n_0^2 \). From this fact and the uniform estimate on \( |A_{\xi,\eta} - A^Y_{\xi,\eta}| \) in (7.2), we see that

\[
\sum_{\xi \in X \cap B(\eta, r_0)} |A_{\xi,\eta} - A^Y_{\xi,\eta}| \leq C \rho^2 h_X^{-2} (h_Y/h_X)^{2M} h_X^{-\delta}.
\]

Choose a constant \( K_0 \) so that \( K_0 \nu > 2M \), where \( \nu \) is as in Theorem 4.1 and pick \( r_0 = K_0 h_X \log(h_Y) \) so that \( n_0 \sim K_0 \log(h_Y) \). (The \( h_Y \) is not a mistake.) The bounds in (7.7) and (7.2) are then, respectively, \( C h_X^{-2} h_Y^{K \nu} \log(h_Y) \) and \( C (\log(h_Y))^2 (h_Y/h_X)^{2M} h_X^{-\delta} \). Note that \( h_X h_Y \leq h_X^{2M} h_X^{-2M} \leq (h_Y/h_X)^{2M} \). Since \( h_Y \) and \( h_X \) are small and \( \delta > 0 \), we also have that \( |\log(h_Y)| \leq (\log(h_Y))^2 h_X^{-\delta} \). Combining the various bounds above results in

\[
\sum_{\xi \in X} |A_{\xi,\eta} - A^Y_{\xi,\eta}| \leq C (\log(h_Y))^2 (h_Y/h_X)^{2M} h_X^{-\delta}.
\]

This holding uniformly in \( \eta \) immediately implies (7.6).
7.1.2. Stability of the discretized stiffness matrix. We now turn to the question of how stable, numerically, the discretized stiffness matrix $A^Y$ is. Answering this question requires the following lemma, which relates certain $L^2$ norms. We will need the two lemmas below.

**Lemma 7.6.** Let $X$, $Y$ be quasi-uniform, with $\rho_X, \rho_Y \leq \rho$ and suppose that $u \in V_X$ and $I_Y u$ is the interpolant of $u$ relative to the kernel $\phi_m$, $m \geq 2$ and its associated space $V_Y$. Then, there exists a constant $C(\rho)$ such that \( \frac{1}{2} \| u \|_{L^2} \leq \| I_Y u \|_{L^2} \leq \frac{3}{2} \| u \|_{L^2} \), provided $h_Y \leq C(\rho) q_X$.

**Proof.** Let $\tilde{u} = I_Y u$. Since $\| u \|_{L^2} - \| u - \tilde{u} \|_{L^2} \leq \| \tilde{u} \|_{L^2} \leq \| u \|_{L^2} + \| u - \tilde{u} \|_{L^2}$, we need only find $C(\rho)$ such that $\| u - \tilde{u} \|_{L^2} \leq \frac{1}{2} \| u \|_{L^2}$. From [9, Theorem 4.6], we have that

$$\| u - \tilde{u} \|_{L^2} = \| u - I_Y u \|_{L^2} \leq C_1 h_Y^2 \| u \|_{W^2}$$

since $u \in W^3 \subset W^2$ and $m \geq \mu = 2$. The constant $C_1$ only depends on $Y$ through $\rho$. We now apply the Bernstein inequality\(^3\) from [23, Theorem 6.1]. First, $u \in V_X$, the space associated with $\phi_3$. This is essentially the Green’s function associated with $\beta = 6$ in the Bernstein inequality. In addition, we may take $\gamma = 2$. Thus, we have

$$\| u \|_{W^2} \approx \| u \|_{H^2} \leq C_2 q_X^{-2} \| u \|_{L^2}, \text{ where } C_2 = C(\rho).$$

Combining this with the previous inequality yields $\| u - \tilde{u} \|_{L^2} \leq C_1 C_2 (h_Y/q_X)^2 \| u \|_{L^2}$. The result immediately follows on choosing $C(\rho) \leq (2 C_1 C_2)^{-1/2}$. \(\square\)

**Theorem 7.7.** Let $a, b \in C^\infty$ satisfy $a(x) \geq a_0 > 0$ and $b(x) \geq b_0 > 0$. Then $\lambda_{\min}(A^Y) \geq C q_X^2$, provided $h_Y \leq C(\rho) q_X$.

**Proof.** Let $u \in V_X$, and so, $u = \sum_{\xi \in X} u(\xi) \lambda_\xi$ and $\nabla u = \sum_{\xi \in X} u(\xi) \nabla \lambda_\xi$. Moreover, $u|_X$ is an arbitrary vector in $\mathbb{R}^{N_X}$. It follows that

$$(u|_X)^T A^Y u|_X = \sum_{\zeta \in Y} (a(\zeta) \nabla u(\zeta) \cdot \nabla u(\zeta) + b(\zeta) u(\zeta)^2) w_\zeta$$

$$\geq b_0 \sum_{\zeta \in Y} u(\zeta)^2 w_\zeta \geq C b_0 N_Y^{-1} \sum_{\zeta \in Y} u(\zeta)^2 \quad \text{(by (4.15))}$$

$$\geq C b_0 h_Y^2 \| u \|_{L^2}. \quad \text{Let } \tilde{u} = I_Y u = \sum_{\zeta \in Y} u(\zeta) \tilde{\lambda}_\zeta,$$

which is the interpolant of $u$ relative to $V_Y$, the space associated with $\phi_m$ and $Y$. By Corollary 3.11, (3.1) and (3.3) in [13], $\| u \|_{L^2} = \| \tilde{u} \|_{L^2} \geq C q_Y^{-1} \| \tilde{u} \|_{L^2}$. In addition, Lemma 7.6 implies that $\| \tilde{u} \|_{L^2} = \| I_Y u \|_{L^2} \geq \frac{1}{2} \| u \|_{L^2}$. Again applying the results from [13] then yields $\| u \|_{L^2} \geq C q_X \| u \|_{L^2}$. Consequently, $\| u \|_{L^2} \geq C q_Y^{-1} q_X \| u \|_{L^2}$. Combining this with the lower bound on the quadratic form $(u|_X)^T A^Y u|_X$ then gives us $(u|_X)^T A^Y u|_X \geq C b_0 (h_Y/q_Y)^2 q_X^2 \| u \|_{L^2} = C b_0 \rho^2 q_X^2 \| u \|_{L^2}$, so $\lambda_{\min}(A^Y) \geq C q_X^2$. \(\square\)

**Corollary 7.8.** If $h_Y$ is chosen so that $(\log(h_Y))^2 (h_Y/h_X)^{2M} h_X^{-\delta} \leq C$, then the condition number $\kappa_2(A^Y)$ satisfies $\kappa_2(A^Y) \leq C q_X^{-2}$.\(^3\)

\(^3\)The theorem actually requires the SBFs involved to be strictly positive definite. However, by a simple adaptation of the argument used to prove [9, Theorem 4.6], one can establish the result needed here.
Proof. Because \( \lambda_{\text{max}}(A^Y) = \|A^Y\|_2 \leq \|A\|_2 + \|A - A^Y\|_2 = \lambda_{\text{max}}(A) + \|A - A^Y\|_2 \), we have, from \((6.12), (7.6)\) and the condition on \(h_Y\), that \(\lambda_{\text{max}}(A^Y) \leq C\). Applying Theorem \(7.7\) then yields the result. \(\square\)

7.2. Error estimates for the discretized Galerkin solution. Let \(\tilde{f}\) be the column vector \(\tilde{f}^T\) defined in \(6.8\) and let \(u_h := u_{h_X} = \sum_\xi \alpha_\xi \chi_\xi\) be the Galerkin approximation to the solution \(u\) of \(Lu = f\). The coefficient vector \(\alpha\) is given by \(\alpha = A^{-1}\tilde{f}\). The discretized solution \(u_h^Y := u_{h_Y}^Y\) is obtained by replacing the stiffness matrix \(A\) by \(A^Y\) in the problem. The solution that results is \(u_h^Y := \sum_\xi \alpha_\xi^Y \chi_\xi\), where \(\alpha^Y = (A^Y)^{-1}\tilde{f}\).

Our goal is to analyze the \(L^2\)-error between \(u\) and \(u_h^Y\). The triangle inequality implies that \(\|u - u_h^Y\|_{L^2} \leq \|u - u_h\|_{L^2} + \|u_h - u_h^Y\|_{L^2}\). We can estimate \(\|u - u_h\|_{L^2}\) using \((6.7)\):

\[
\|u - u_h\|_{L^2} \leq Ch^{s+2}\|f\|_{H^s}.
\]

We also have, by \((4.5)\) and \(A^{-1} - (A^Y)^{-1} = (A^Y)^{-1}(A^Y - A)A^{-1}\), that

\[
\|u_h - u_h^Y\|_{L^2} = \|\sum_\xi (\alpha_\xi - \alpha_\xi^Y) \chi_\xi\|_{L^2} \\
\leq c_2 q_X \|\alpha - \alpha^Y\|_{\ell^2} \\
\leq c_2 q_X \|(A^Y)^{-1}\| A^Y - A \|A^{-1}\tilde{f}\|_{\ell^2}.
\]

Using \((4.5)\) again, we have \(\|\alpha\|_{\ell^2} \leq c_1^{-1} q_X^{-1} \|\sum_\xi a_\xi \chi_\xi\|_{L^2} = c_1^{-1} q_X^{-1} \|u_h\|_{L^2}\). In addition, from Theorem \(7.5\), \(\|(A^Y)^{-1}\| \leq Cq_X^{-2}\). Combining these inequalities results in

\[
\|u_h - u_h^Y\|_{L^2} \leq Cq_X^{-2}\|A^Y - A\|\|u_h\|_{L^2}.
\]

Because \(\|u - u_h\|_{L^2} \leq Ch^{s+2}\|f\|_{H^s}\), we have \(\|u_h\|_{L^2} \leq \|u\|_{L^2} + Ch^{s+2}\|f\|_{H^s}\). Moreover, by Proposition \(5.1\), \(\|u\|_{L^2} \leq C\|f\|_{L^2} \leq C\|f\|_{H^s}\). Thus, for \(h_X \sim q_X\) small,

\[
\|u_h - u_h^Y\|_{L^2} \leq Cq_X^{-2}\|A^Y - A\| (\|f\|_{H^s} + Ch^{s+2}\|f\|_{H^s}) \leq Cq_X^{-2}\|A^Y - A\|\|f\|_{H^s}.
\]

From this and \((7.9)\), it follows that

\[
\|u - u_h^Y\|_{L^2} \leq C(h_X^{s+2} + q_X^{-2}\|A^Y - A\|_{L^2})\|f\|_{H^s}.
\]

The above discussion together with Theorem \(7.5\) yields \((7.11)\) below. Note that the second term in \((7.11)\) measures the quadrature error. Ideally, the “fine set” \(Y\) can be chosen so that the second term is comparable to the optimal error \(O(h_X^{s+2})\).

**Theorem 7.9.** Let \(f, s\) be as in Theorem \(6.2\) and let \(m, M, \delta\) be as in Theorem \(7.5\). Then, for \(h_X\) and \(h_Y\) sufficiently small, we have

\[
\|u - u_h^Y\|_{L^2} \leq C\left(h_X^{s+2} + \frac{\left(\log(h_Y)h_Y^M\right)^2}{h_X^{2M+2+\delta}}\right)\|f\|_{H^s}.
\]

\(^4\)We assume that \(\tilde{f}\) is known exactly here. In practice, it can be computed very accurately via quadrature. See section \(7\).
8. Sparse approximation and local Lagrange functions

This section discusses reducing the computational expense of numerically finding the Galerkin solution to the problem. There are two aspects of this. The first is obtaining a truncated approximation \( \tilde{A}Y \) to the discretized stiffness matrix \( AY \). Each row in \( \tilde{A}Y \) has \( O((\log(NX))^2) \) non-zero entries, as opposed to \( NX \) in \( AY \). The second is to replace the global Lagrange basis with a local one. As mentioned in the introduction, computing this basis requires inverting a number small matrices, a task that is parallelizable. The error estimates from making these approximations are virtually unchanged.

8.1. Sparse approximation. So far, we have not addressed the question of how well a sparse approximation to the stiffness matrix would perform. Suppose that, in \( AY \), we discard all entries \( AY_{\xi,\eta} \) that satisfy \( \text{dist}(\xi,\eta) \geq Kh_X|\log h_X| \), where \( K\nu > 2 \). Let the matrix we get in this way be \( \bar{A}Y \), where

\[
\bar{A}Y_{\xi,\eta} := \begin{cases} 
0, & \text{dist}(\xi,\eta) > Kh_X|\log h_X|, \\
AY_{\xi,\eta}, & \text{dist}(\xi,\eta) \leq Kh_X|\log h_X|.
\end{cases}
\]

The matrix \( \bar{A}Y_{\xi,\eta} \) is symmetric. The number of non-zero elements in each row is approximately the ratio of the areas of caps having radii \( Kh_X|\log h_X| \) and \( h_X \), respectively. If we make use of this and of the fact that, since \( X \) is quasi-uniform, \( h_X \sim N_X^{-1/2} \), then we see that

\[
\#\{\text{row } \eta\} \approx \frac{(Kh_X|\log h_X|)^2}{h_X^2} = K^2(\log(h_X))^2 \approx \frac{1}{4} K^2(\log(NX))^2,
\]

as opposed to \( NX \) for \( AY \) itself.

**Proposition 8.1.** Let \( K\nu > 2 \) and \( \bar{A}Y \) be defined by (8.1). Then,

\[
\|AY - \bar{A}Y\|_2 \leq \frac{2C K e^{-\nu}}{(1 - e^{-\nu})^2} h_X^{K\nu - 2}|\log(h_X)|.
\]

Proof. We will follow the proof of Theorem (7.5). Because \( AY \) and \( \bar{A}Y \) are symmetric, the norm \( \|AY - \bar{A}Y\|_2 \) satisfies the bound

\[
\|AY - \bar{A}Y\|_2 \leq \|AY - \bar{A}Y\|_1 = \|AY - \bar{A}Y\|_{\infty} = \max_{\eta \in X} (\sum_{\xi \in X} |AY_{\xi,\eta} - \bar{A}Y_{\xi,\eta}|).
\]

We again want to estimate each term in the sums above. Let \( B_\eta \) be the ball centered at \( \eta \) and having radius \( r_h = Kh_X|\log(h_X)| \). From Proposition (7.4) and the definition of \( \bar{A}Y \), we have that

\[
\sum_{\xi \in X} |AY_{\xi,\eta} - \bar{A}Y_{\xi,\eta}| = \sum_{\xi \in X \cap B_\eta^0} |AY_{\xi,\eta}|
\]

\[
\leq Ch_X^{-2} \sum_{\xi \in X \cap B_\eta^0} e^{-\frac{\nu}{h_X} \text{dist}(\xi,\eta)}.
\]
Next, divide $B_X^\eta$ into bands of width $\sim h_X$, the $n^{th}$ band being a distance approximately $r_n + nh_X$ from $\eta$. Repeating the derivation of (7.7), mutatis mutandis, we obtain
\[
\sum_{\xi \in X} |A_{\xi,\eta}^{\nu} - \tilde{A}_{\xi,\eta}^{\nu}| \leq C h_X^{-2} e^{-\nu K |\log(h_X)|} \sum_{n=1}^{\infty} (K |\log(h_X)| + n) e^{-\nu n} \\
\leq \frac{2CKe^{-\nu}}{(1 - e^{-\nu})^2} h_X^{K-2} |\log(h_X)|.
\]
Combining the inequalities above yields (8.3). \qed

8.2. Local Lagrange functions. The result above quantifies the error made in zeroing out the entries $A_{\xi,\eta}^{\nu}$ corresponding to all $\xi, \eta$ such that $\text{dist}(\xi, \eta) > Kh_X |\log(h_X)|$. To obtain the rest of the entries we still need to use (7.1); this entails finding the $\chi_{\xi,\eta}$’s, which are global in the sense that they require all of the points in $X$ for their computation.

There is a way around this. In [8], Fuselier et al. introduced a basis for $V_{\phi_m,X}$ composed of local Lagrange functions, $\{\chi_{\xi,\eta}^{\text{loc}} : \xi \in X\}$. These basis functions are simply Lagrange functions for the points in $X$ that lie in a ball of radius $Kh_X |\log(h_X)|$ about $\xi \in X$, where $K$ is appropriately chosen. A detailed description of their construction and properties is given in [8, §6.3]. We will simply list what we need here, in the theorem below.

**Theorem 8.2** ([8, Theorem 6.5]). Let the notation and assumptions of Theorem 4.4 hold; define $\Upsilon_{\xi} := X \cap B(\xi, Kh_X |\log(h_X)|)$, where $K > 0$ is as above. There exist $\mu = \mu(m)$ such that for $K$ chosen to satisfy $J := K \nu - 4m + 2 - 2\mu > 0$ these hold:
\[
\|\chi_{\xi,\eta}^{\text{loc}} - \chi_{\eta}^{\nu}\|_{L^\infty} \leq C h_X^{J}, \\
|\chi_{\xi,\eta}^{\text{loc}}(x)| \leq C(1 + \text{dist}(x, \xi)/h_X)^{-J}.
\]
Furthermore, when $J > 2$, the set $\{\chi_{\xi,\eta}^{\text{loc}}\}$ is $L^p$ stable: there are $C_1, C_2 > 0$ for which
\[
C_1 4_X^{2/p} \|\beta\|_{L^p(\Upsilon_{\xi})} \leq \|\sum_{\xi \in X} \beta_{\xi} \chi_{\xi,\eta}^{\text{loc}}\|_{L^p(\mathbb{R}^d)} \leq C_2 4_X^{2/p} \|\beta\|_{L^p(\Upsilon_{\xi})}.
\]
Local Lagrange functions may be expanded in global ones. Because $\chi_{\xi,\eta}^{\text{loc}}$ is a Lagrange function for $\Upsilon_{\xi}$, it satisfies $\chi_{\xi,\eta}^{\text{loc}}(\eta) = \delta_{\xi,\eta}$, for all $\eta \in \Upsilon_{\xi}$. Of course, we also have $\chi_{\xi}(\eta) = \delta_{\xi,\eta}$, for all $\eta \in X$. Expanding $\chi_{\xi,\eta}^{\text{loc}}(x)$ in terms of the basis $\{\chi_{\eta}\}_{\eta \in X}$ results in
\[
\chi_{\xi,\eta}^{\text{loc}}(x) = \chi_{\xi}(x) + \sum_{\eta \notin \Upsilon_{\xi}} \chi_{\xi,\eta}^{\text{loc}}(\eta) \chi_{\eta}(x) = \chi_{\xi}(x) + \sum_{\eta \notin \Upsilon_{\xi}} (\chi_{\xi,\eta}^{\text{loc}}(\eta) - \chi_{\xi}(\eta)) \chi_{\eta}(x),
\]
since, for $\xi \neq \eta \in X$, $\chi_{\xi}(\eta) = \delta_{\xi,\eta} = 0$. Taking the covariant derivative in the equation above yields
\[
\nabla \chi_{\xi,\eta}^{\text{loc}}(x) = \nabla \chi_{\xi}(x) + \sum_{\eta \notin \Upsilon_{\xi}} \chi_{\xi,\eta}^{\text{loc}}(\eta) \nabla \chi_{\eta}(x) = \nabla \chi_{\xi}(x) + \sum_{\eta \notin \Upsilon_{\xi}} (\chi_{\xi,\eta}^{\text{loc}}(\eta) - \chi_{\xi}(\eta)) \nabla \chi_{\eta}(x).
\]
From (8.6) and (8.4), it easily follows that
\[
|\nabla \chi_{\xi,\eta}^{\text{loc}}(x) - \nabla \chi_{\xi}(x)| \leq \|\chi_{\xi,\eta}^{\text{loc}} - \chi_{\xi}\|_{L^\infty} \sum_{\eta \notin \Upsilon_{\xi}} |\nabla \chi_{\eta}(x)| \leq C h_X^{J} \sum_{\eta \in X} |\nabla \chi_{\eta}(x)|.\]

\[^5\text{One may take } \mu = \iota, \text{ where } \iota \text{ is constructed in [8, Lemma 6.4].}\]
Applying (4.9) to the rightmost inequality then yields the following result:

**Lemma 8.3.** If \( J > 1 \), then \( \| \nabla \chi^\text{loc}_\xi - \nabla \chi^\text{loc}_\eta \|_{L^\infty} \leq C \rho_X^2 h_X^{J-1} \).

The result we are aiming at is estimating the error made in replacing the exact Lagrange functions by the **local** Lagrange functions in computing \( A^Y_{\xi, \eta} \). Specifically, define

\[
A^Y_{\text{loc}, \xi, \eta} := \sum_{\zeta \in Y} \left( a \nabla \chi^\text{loc}_\xi \cdot \nabla \chi^\text{loc}_\eta + b \chi^\text{loc}_\xi \chi^\text{loc}_\eta \right) |_\zeta w_\zeta.
\]

We want to estimate \( |A^Y_{\xi, \eta} - A^Y_{\text{loc}, \xi, \eta}| \). This we do in the proposition below.

**Proposition 8.4.** Let \( A^Y_{\xi, \eta} \) be given by (7.1) and let \( A^Y_{\text{loc}, \xi, \eta} \) be as above. Then,

\[
|A^Y_{\xi, \eta} - A^Y_{\text{loc}, \xi, \eta}| \leq Ch_X^{J-2}
\]

holds uniformly for \( \xi, \eta \in X \) and the set \( Y \).

**Proof.** Note that, at \( x \in \mathcal{S}^2 \), we have

\[
|\nabla \chi^\text{loc}_\xi \cdot \nabla \chi^\text{loc}_\eta - \nabla \chi^\text{loc}_\xi \cdot \nabla \chi^\text{loc}_\eta| \leq |\nabla \chi^\text{loc}_\xi - \nabla \chi^\text{loc}_\eta| |\nabla \chi^\text{loc}_\xi| + |\nabla \chi^\text{loc}_\xi - \nabla \chi^\text{loc}_\eta| |\nabla \chi^\text{loc}_\eta| + |\nabla \chi^\text{loc}_\xi| |\nabla \chi^\text{loc}_\eta - \nabla \chi^\text{loc}_\eta|.
\]

By this inequality and Theorem 4.3, we see that

\[
|A^Y_{\xi, \eta} - A^Y_{\text{loc}, \xi, \eta}| \leq C_{h_X} h_X^{J-1} q_X^{-1} + C_{h_X} h_X^{J-2}
\]

which is (8.9). Specifically, applying (8.2) then yields

\[
|A^Y_{\xi, \eta} - A^Y_{\text{loc}, \xi, \eta}| \leq C_{h_X} h_X^{J-2} \sim Ch_X^{J-2}.
\]

**Distance estimates.** We have already dealt with a bound on \( \| A^Y - \tilde{A}^Y \|_2 \) in Proposition 8.1. We are really only interested in the “chopped” version of \( A^Y_{\text{loc}} \) i.e., \( \tilde{A}^Y_{\text{loc}} \), which is defined analogously to \( A^Y \) in (8.1).

**Proposition 8.5.** For \( h_X \) sufficiently small and \( J > 2 \), we have that

\[
\| \tilde{A}^Y_{\text{loc}} - \tilde{A}^Y \|_2 \leq CK^2 (\log(h_X))^2 h_X^{J-2}.
\]

**Proof.** As before, we have \( \| \tilde{A}^Y_{\text{loc}} - \tilde{A}^Y \|_2 \leq \max_{\eta \in X} \sum_{\xi \in X} |\tilde{A}^Y_{\xi, \eta} - \tilde{A}^Y_{\text{loc}, \xi, \eta}| \). For fixed \( \eta \), all terms with \( d(\xi, \eta) > Kh_X |\log(h_X)| \) are 0. It follows that

\[
\sum_{\xi \in X} |\tilde{A}^Y_{\xi, \eta} - \tilde{A}^Y_{\text{loc}, \xi, \eta}| = \sum_{\xi \in B(\eta, r_h) \cap X} |A^Y_{\xi, \eta} - A^Y_{\text{loc}, \xi, \eta}|, \quad r_h = Kh_X |\log(h_X)|.
\]

From (8.8), the difference in the right sum is uniformly bounded by \( Ch_X^{J-2} \). Consequently, applying (8.2) then yields

\[
\| \tilde{A}^Y_{\text{loc}} - \tilde{A}^Y \|_2 \leq \max_{\eta \in X} \sum_{\xi \in X} |\tilde{A}^Y_{\xi, \eta} - \tilde{A}^Y_{\text{loc}, \xi, \eta}| \leq Ch_X^{J-2} \max_{\eta \in X} \# \{ \text{row } \eta \} \leq CK^2 (\log(h_X))^2 h_X^{J-2},
\]

which is (8.9).
Assume that the hypotheses of Theorem 7.5, Proposition 8.1 and Proposition 8.5 hold. Then,

\[
\begin{align*}
\|A - \tilde{A}\|_2 \leq & C((\log(h_Y))^2(h_Y/h_X)^{2M}h_X^{-\delta} + |\log(h_X)|\|K\|_{X^{-2}}), \\
\|A - \tilde{A}_{loc}\|_2 \leq & C((\log(h_Y))^2(h_Y/h_X)^{2M}h_X^{-\delta} + (\log(h_X))^2 h_X^{-2}).
\end{align*}
\]

**Proof.** The first result follows from first applying the triangle inequality and the three distance estimates from Theorem 7.5, Proposition 8.1 and Proposition 8.5 to \(\|A - \tilde{A}\|_2\). Establishing the second is done in a similar way, employing the additional fact that \(h_X^{K\nu - 2}\log(h_X) < (\log(h_X))^2 h_X^{-2}\), since \(K\nu - 2 \geq J - 2\).

**Stability.** The matrices \(\tilde{A}\) and \(\tilde{A}_{loc}\) both have roughly the same stability properties as \(A\) and \(A_Y\). We will establish them in Theorem 8.8 below. To do this, we will need the following elementary result from linear algebra, which we state without proof.

**Lemma 8.7.** Let \(S\) and \(T\) be Hermitian \(n \times n\) matrices and let \(S\) be positive definite. If there exists \(0 \leq \varepsilon < 1\) such that \(\|S - T\|_2 \leq \varepsilon\lambda_{\min}(S)\), then \(T\) is positive definite, and, in addition, these hold:

\[
(1 - \varepsilon)\lambda_{\min}(S) \leq \lambda_{\min}(T) \leq (1 + \varepsilon)\lambda_{\min}(S),
\]

\[
\frac{1 - \varepsilon}{1 + \varepsilon}\kappa_2(S) \leq \kappa_2(T) \leq \frac{1 + \varepsilon}{1 - \varepsilon}\kappa_2(S).
\]

**Theorem 8.8.** Suppose that the conditions of Theorem 7.7 and Corollary 7.8 are satisfied and that \(J\) is as in Theorem 8.2. If \(J > 4\), then both \(\tilde{A}\) and \(\tilde{A}_{loc}\) are positive definite, have \(\lambda_{\min}(\tilde{A}) \sim \lambda_{\min}(\tilde{A}_{loc}) \sim q_X^2\), and also have \(\kappa_2(\tilde{A}) \sim \kappa_2(\tilde{A}_{loc}) \sim q_X^{-2}\).

**Proof.** Note that \(J > 4\) implies that \(K\nu > 4\), so that, as long as \(q_X\) is small, (8.3) holds, and so, using \(h_X = \rho_X q_X\) and applying Theorem 7.7 we have

\[
\|A^Y - \tilde{A}^Y\|_2 \leq C q_X^{K\nu - 4} |\log(q_X)| q_X^2 \leq \underbrace{C q_X^{K\nu - 4}}_{\varepsilon_1} |\log(q_X)| \lambda_{\min}(A^Y) = \varepsilon_1 \lambda_{\min}(A^Y).
\]

Since \(K\nu - 4 > 0\), we may choose \(q_X\) so small that \(\varepsilon_1 < 1\). Lemma 8.7 then implies the results stated for \(\tilde{A}^Y\). Using this result and (8.9), we get

\[
\|\tilde{A}^Y - \tilde{A}_{loc}^Y\|_2 \leq \underbrace{C q_X^{J - 4} |\log(q_X)|^2}_{\varepsilon_2} \lambda_{\min}(\tilde{A}^Y) = \varepsilon_2 \lambda_{\min}(\tilde{A}^Y).
\]

Because \(J > 4\), we may choose \(q_X\) sufficiently small so that \(\varepsilon_2 < 1\). Applying Lemma 8.7 then yields the result for \(\tilde{A}_{loc}^Y\).

**Sparse and sparse local Lagrange Galerkin error estimates.** We conclude by giving errors for the \(L^2\)-Galerkin approximations to \(u\), \(\tilde{u}_b^Y\) and \(\tilde{u}_{loc,b}^Y\), which are obtained by discretizing with the chopped matrices \(\tilde{A}\) and \(\tilde{A}_{loc}\). The estimates below are
gotten in very nearly the same way as the one in Theorem 7.9. The only change is that \( \|(A^Y)^{-1}||A^Y - A\| = \lambda_{\min}(A^Y)^{-1}||A^Y - A\| \) gets replaced by

\[
\lambda_{\min}(\tilde{A}^Y)^{-1}||A - \tilde{A}^Y\|_2 \sim q_X^{-2}||A - \tilde{A}^Y\|_2
\]

in the first instance, and by

\[
\lambda_{\min}(\tilde{A}_{loc}^Y)^{-1}||A - \tilde{A}_{loc}^Y\|_2 \sim q_X^{-2}||A - \tilde{A}_{loc}^Y\|_2
\]

in the second.

**Theorem 8.9.** Let \( J \) and \( K \) be as defined in Theorem 8.2. If the conditions of Theorem 7.7 and Corollary 7.8 are satisfied, then

\[
\| u - \tilde{u}_h \|_{L^2} \leq C \left(h_X^{s+2} + \frac{(\log(h_Y)h_Y^M)^2}{h_X^{2M+2+\delta}} + |\log(h_X)|h_X^{K_\nu-4}\right)\|f\|_{H_\sigma}, \quad K_\nu > 4,
\]

(8.10)

\[
\| u - \tilde{u}_{loc,h} \|_{L^2} \leq C \left(h_X^{s+2} + \frac{(\log(h_Y)h_Y^M)^2}{h_X^{2M+2+\delta}} + (\log(h_X))^2h_X^{J_\nu-4}\right)\|f\|_{H_\sigma}, \quad J > 4.
\]

(8.11)

9. Implementation and numerical experiments

This section discusses the practical aspects of implementation and presents numerical experiments. We discuss the construction of point sets for the discrete approximation space and quadrature points, the assembly of the stiffness matrix, and the assembly of the right hand side vector.

The numerical scheme requires two sets of points on the sphere. The coarse set \( X \) is used to build a basis for the approximation space \( V_{\phi_m,X} \) discussed in section 4. This space is spanned by the Lagrange functions \( \{\chi_\xi\}_{\xi \in X} \) defined in (4.2), which have this form:

\[
\chi_\xi(x) = \sum_{\eta \in X} \alpha_{\eta,\xi}\phi_m(x \cdot \eta) + \sum_{\ell=0}^{m-1} \sum_{k=1}^{2\ell+1} \beta_{\ell,k,\xi}Y_{\ell,k}(x),
\]

\[
\sum_{\eta} \alpha_{\eta,\xi}Y_{\ell,k}(\eta) = 0, \quad 0 \leq \ell \leq m - 1, \quad 1 \leq k \leq 2\ell + 1.
\]

The \( \alpha_{\xi,\eta} \) and \( \beta_{\ell,k,\xi} \) coefficients must be computed for each \( \xi \in X \). They are determined by \( \chi_\xi(\zeta) = \delta_{\xi,\zeta} \) and the second set of equations above. Solving for them is not very difficult, even though each Lagrange function requires all of the points in \( X \) for its computation. However, for large data sets, there is a very efficient, parallelizable way to numerically compute local Lagrange functions. These functions, which were introduced in [8], require relatively few nodes from \( X \) and give very good approximations for the \( \chi_\xi \)'s.

The second, finer set \( Y \) is used, in the quadrature formulas, to discretize entries in the stiffness matrix (7.1). By Theorem 7.25 the mesh norm of the set \( Y \) determines the error in the discrete stiffness matrix and should be chosen appropriately to obtain a desired accuracy in numerically approximating it. The quadrature weights \( \{w_\zeta\}_{\zeta \in Y} \) satisfy

\[
\int_{S^2} \tilde{\chi}_\zeta(x) d\mu(x) = w_\zeta.
\]
where $\tilde{\chi}_\zeta$ is the Lagrange function centered at $\zeta \in Y$ for the kernel $\phi_M$. The kernel $\phi_M$ need not be the same kernel as the one used in the construction of the approximation space $V_X$, which was $\phi_m$. The weights can be computed efficiently by solving a single linear system that can be preconditioned by the local Lagrange functions; see [9] for details. Solving the system with Generalized Minimum Residual method (GMRES) and the local Lagrange preconditioner requires few iterations. Experiments performed in [9] demonstrate that the number of iterations required seems to be independent of the number of points in $Y$.

The stiffness matrix assembly requires computing the quadrature nodes $Y$ and quadrature weights $\{w_\zeta\}_{\zeta \in Y}$ and the coefficients $\{\alpha_{\eta,\zeta}\}_{\xi,\eta \in X}$ and $\{\beta_{\xi,\zeta}\}_{\xi \in X}$. We recall the discrete stiffness matrix entries found via quadrature:

$$A_{\xi,\eta}^Y = \sum_{\zeta \in Y} (a\nabla \chi_\xi \cdot \nabla \chi_\eta + b\chi_\xi \chi_\eta)w_\zeta.$$

We provide some details of the computation of $\nabla \chi_\xi \cdot \nabla \chi_\eta(\zeta)$. We expand the Lagrange functions in terms of the surface splines $\phi_m$ (denoted $\phi$) as $\chi_\xi(\zeta) = \sum_{\tau \in X} \alpha_{\tau,\xi} \phi(\zeta, \tau) + p_\xi(\zeta)$ and $\chi_\eta(\zeta) = \sum_{\gamma \in X} \alpha_{\gamma,\eta} \phi(\zeta, \gamma) + p_\eta(\zeta)$. Let $x = \sin(\theta) \cos(\varphi), y = \sin(\theta) \sin(\varphi), z = \cos(\theta)$ where $0 \leq \theta \leq \pi$ and $0 \leq \varphi \leq 2\pi$. On $S^2$, $\nabla f = \frac{\partial f}{\partial \theta} + \frac{1}{\sin(\theta)} \frac{\partial f}{\partial \varphi} \hat{\varphi}$. Let $\phi'(\zeta, \tau) = (m - 1)(1 - \zeta \cdot \tau)^{m-2} \log(1 - \zeta \cdot \tau) + \zeta \cdot \tau - 1$.

Let $\tau = (\tau_x, \tau_y, \tau_z)$ in Cartesian coordinates. We note that $\frac{\partial \phi'(\zeta, \tau)}{\partial x} |_{\zeta} = \phi'(\zeta, \tau) \tau_x$, and similarly for the $y$ and $z$ partial derivatives. Evaluating the covariant derivative of the restricted surface spline in Cartesian coordinates then yields

$$\nabla \phi'(\zeta, \tau) |_{\zeta} = \phi'(\zeta, \tau) \left((1 - \zeta_x^2) \tau_x - \zeta_x \zeta_y \tau_y - \zeta_x \zeta_z \tau_z \right)i$$

$$+ \left(-\zeta_x \zeta_y \tau_x + (1 - \zeta_y^2) \tau_y - \zeta_y \zeta_z \tau_z \right)j$$

$$+ \left(-\zeta_x \zeta_z \tau_x - \zeta_y \zeta_z \tau_y + (1 - \zeta_z^2) \tau_z \right)k.$$

The evaluation of $\nabla \chi_\xi \cdot \nabla \chi_\eta(\zeta)$ then reduces to

$$\nabla \chi_\xi(\zeta) \cdot \nabla \chi_\eta(\zeta) = \left(\sum_{\tau} \alpha_{\tau,\xi} \nabla \phi(\zeta, \tau) + \nabla p_\xi(\zeta)\right) \cdot \left(\sum_{\gamma} \alpha_{\gamma,\eta} \nabla \phi(\zeta, \gamma) + \nabla p_\eta(\zeta)\right).$$

One final comment. In section 7.2, we noted that the column vector $\tilde{f}_\xi = \langle f, \chi_\xi \rangle$, had to be numerically computed via quadrature, which is what was done in the numerical experiments described below, where $\tilde{f}_\xi = \sum_{\zeta \in Y} \tilde{f}(\zeta) \chi_\xi(\eta)w_\zeta$.

9.1. Numerical experiments. In this section, we discuss numerical results of various experiments that explore the computational properties of the Galerkin method. We consider different differential operators, explore the effects of the quadrature node density on the $L^2$-error of the discrete solution, and compute condition numbers for the discrete stiffness matrix. We also demonstrate that local Lagrange spaces, as discussed in [8], provide a computationally less expensive approximation space and yield comparable error and condition numbers as the approximation space generated by the Galerkin functions. We choose the spherical basis function $\phi_3(t) = (1 - t)^2 \log(1 - t)$ to construct the approximation space and $\phi_2(t) = (1 - t) \log(1 - t)$ for the quadrature weights. We use the minimum energy points for the centers $X$ used in the approximation space $V_{\phi_3, X}$. For the quadrature nodes, we use the icosahedral nodes and quasi-minimum energy points. These
points are available for download; see [39]. For each experiment, the $L^2$-error is computed by evaluating the discrete solution on a set of evaluation points $E$ and applying the Lagrange function quadrature rule. The set $E$ is 62,500 quasi-minimum energy points, which is used for each experiment independent of $X$ and $Y$. Let $N_X$ and $N_Y$ denote the number of points in $X$ and $Y$ respectively. We approximate $h_Y$ by $\sqrt{N_Y}$.

We first consider the problem $-\Delta u + u = f$ with $u = \exp(\cos(\theta))$ and $f = \exp(\cos(\theta))(\cos^2(\theta) + 2\epsilon \cos(\theta))$. In the second and third columns of Table 1 we display the relative $L^2$-errors of the discrete solution for two separate experiments. To obtain the discrete stiffness matrix, we first fixed 961 centers for this basis. The $\chi$ function, the local Lagrange function can be made to satisfy $\|\chi - \chi_{\xi}^{loc}\|_{L^\infty} \sim h_X^{2m}$, where $m$ is the smoothness of the kernel $\phi$. For the anisotropic problem, the fifth

$$f = \left(-\frac{1}{2} \cos^3(\theta) + \cos^2(\theta) - 5 \cos(\theta) + 1 \right) \exp(\cos(\theta)).$$

We also consider the possibility of using a local Lagrange basis to discretize the PDE. In this case, the approximation space is $V_X = \text{span}\{\chi_{\xi}^{loc} : \xi \in X\}$, where the $\chi_{\xi}^{loc}$ functions are constructed using only kernels $\phi(\cdot, \eta)$ such that $\text{dist}(\xi, \eta) \leq 7h_X|\log(h_X)|$. See [8] for a detailed description of the theoretical properties of this basis. The $\chi_{\xi}^{loc}$’s may be constructed in parallel by solving a small linear system. This reduces computational complexity associated with assembling the $a_{\xi,\eta}$ coefficients. By appropriately tuning the number of kernels used per Lagrange function, the local Lagrange function can be made to satisfy $\|\chi_{\xi} - \chi_{\xi}^{loc}\|_{L^\infty} \sim h_X^{2m}$.

Table 1. Both $-\Delta u + u = f$ and $-\text{div}(a \cdot \nabla u) + u = f$ were numerically solved using minimum energy point sets for $X$ and icosahedral point sets for $Y$. The $L^2$-error for all cases was $O(\|\log(h_Y)\|^2 h_Y^{5+})$.

Here, $h_Y = N_Y^{-1/2}$. For the first equation, a Lagrange basis was used, and, for the second, a local Lagrange basis.

| $-\Delta u + u = f$ | $-\text{div}(a \cdot \nabla u) + u = f$ |
|---------------------|---------------------|
| **Lagrange Basis**  | **Local Lagrange Basis** |
| $N_Y$   | $N_X = 961$ | $N_X = 3721$ | $N_Y$   | $N_X = 961$ | $N_X = 3721$ |
| 2,562  | 7.86e-5 | 2.19e-2 | 2,500  | 8.00e-5 | 2.10e-2 |
| 10,242 | 2.22e-6 | 3.76e-5 | 10,000 | 2.46e-6 | 3.23e-5 |
| 23,042 | 3.34e-7 | 3.83e-6 | 22,500 | 3.02e-7 | 4.78e-6 |
| 40,962 | 8.96e-8 | 9.32e-7 | 40,000 | 7.80e-8 | 1.04e-6 |
| 92,162 | 1.50e-8 | 1.27e-7 | 90,000 | 1.10e-8 | 1.49e-7 |
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(a) Adjusted $L^2$-error for $-\Delta u + u = f$. (b) Adjusted $L^2$-error for $-\text{div}(a \cdot \nabla u) + u = f$.

(c) Adjusted $L^2$-error for $Y$ fixed. (d) Condition number vs. $q_X^{-1}$.

Figure 1. In (a) and (b), semi-log plots of the errors (adjusted by removing log factors) for $-\Delta u + u = f$ and $-\text{div}(a \cdot \nabla u) + u = f$ are shown. The minimum energy points were used for $X$ and icosahedral points were used for $Y$. In (c), a loglog plot of the $L^2$-error vs. $h_X$ is plotted. For this experiment, the number of quadrature points is fixed and the number of centers used for the approximation space varies. In (d), the log of the condition number for the stiffness matrix for $-\Delta u + u = f$ is plotted.

and sixth columns in Table 1 display the results of the experiment using the local Lagrange basis. For $N_X = 961$, each local Lagrange function is constructed using about 423 centers and for $N_X = 3,721$, each local Lagrange function is constructed using around 776 centers, where the number of centers used per kernel is chosen to be all centers with distance at most $7h_X |\log(h_X)|$ from the center. The computed $L^2$-errors from using the local basis versus the full basis are negligible, confirming the results in section 8.2. Since the local bases offer comparable $L^2$-error while being computationally simpler, they offer no drawbacks when compared to the full basis and certainly are a good choice for doing the discretization step. The results of the two experiments are plotted in Figure 1(a) and Figure 1(b).

A third experiment was conducted keeping $Y$ with fixed and varying $X$. The result is displayed in Figure 1(c). In this experiment, the error increases with decreasing $h_X$. This is counter intuitive, but in complete agreement with the theory. What this illustrates is that the dominant term in the $L^2$-error comes from quadrature. This is no surprise and is a well-known phenomenon in Galerkin methods.

The condition number of the discrete stiffness matrix is dependent primarily on the separation radius of the centers, $q_X$. We theoretically predicted the condition
number to be $O(q_X^{-2})$, which we validated numerically. See Figure II(d). In addition, the theory predicts that changing the quadrature nodes should not significantly alter the condition number of the stiffness matrix. Again, this result was validated.

**Appendix A. Interpolation errors and the “doubling trick”**

In this section we will discuss interpolation errors for spherical basis functions. Previous work on error estimates concentrated on interpolating functions not smooth enough to be in the reproducing Hilbert space $\mathcal{N}$ corresponding to an SBF $\phi$. We will also need error estimates for interpolating functions smoother than those in $\mathcal{N}$. Results of this kind have been developed by Schaback [29] for positive definite functions on $\mathbb{R}^n$ and on manifolds. In addition, Fuselier and Wright [10, Proposition 11] give a thorough treatment of the topic. For SBFs, the main result is that if $N$ is equivalent to Sobolev space $H^{\tau, n/2}$, then, for functions in $H^{2\tau}$, the error rate is double the one obtained for functions in $\mathcal{N}$. This result is known as the “doubling trick”.

Throughout this section we will assume that an SBF $\phi$ has coefficients $\hat{\phi}_\ell$ that satisfy the following condition. There are constants $C$, $L \in \mathbb{N}$ such that

\begin{equation}
(\text{A.1}) \quad C(1 + \lambda_\ell)^{-\tau/2} \leq \hat{\phi}_\ell \leq C(1 + \lambda_\ell)^{-\tau},
\end{equation}

holds either for all $\ell \geq 0$ or for all $\ell \geq L+1$. Here $\lambda_\ell = \ell(\ell + n - 1)$ is an eigenvalue of $-\Delta_{S^n}$.

**A.1. Positive definite SBFs.** In this section, we will deal with positive definite SBFs, so $\hat{\phi}_\ell > 0$ for all $\ell$. The proposition below is a statement of the “doubling trick” in the case where $f \in H^{\tau, \alpha}$, $0 \leq \alpha \leq \tau$. We follow this up with a general result combining the doubling trick with estimates from [26, Theorem 5.5]. We separate the two so that the doubling trick itself is clearly stated.

**Proposition A.1.** Let $\alpha, \beta, \tau \in \mathbb{R}$, with $\tau > n/2$ and $\alpha, \beta \in [0, \tau]$. Suppose that $f \in H^{\tau, \alpha}(\mathbb{S}^n)$ and that (A.1) holds for all $\ell \geq 0$. If $h_X$ is sufficiently small, then

\begin{equation}
(\text{A.2}) \quad \|f - I_X f\|_{H^\beta} \leq C h_X^{\tau + \alpha - \beta} \|f\|_{H^{\tau, \alpha}}.
\end{equation}

**Proof.** We will first deal with the $\beta = \tau$ case. The interpolant $I_X f$ being the projection of $f$ onto $V_X$ in the native space $\mathcal{N}$ implies that $(f - I_X f, v)_{\mathcal{N}} = 0$ for all $v \in V_X$. Consequently, we have $(f - I_X f, I_X f)_{\mathcal{N}} = 0$ and so $\|f - I_X f\|_{\mathcal{N}}^2 = \langle f - I_X f, f \rangle_{\mathcal{N}}$. Let $g := f - I_X f$. The previous equation then takes the form $\|g\|_{\mathcal{N}}^2 = \langle g, f \rangle_{\mathcal{N}}$. From (A.2) and the bounds on $\hat{\phi}_\ell$, we see that

\begin{align*}
\|g\|_{\mathcal{N}}^2 &= \langle g, f \rangle_{\mathcal{N}} = \sum_{\ell=0}^{\infty} \sum_{m=1}^{d_\ell} \langle \hat{\phi}_\ell \rangle^{-1} \hat{f}_{\ell m} \hat{g}_{\ell m} \\
&\leq C \sum_{\ell=0}^{\infty} \sum_{m=1}^{d_\ell} (1 + \lambda_\ell)^{(\tau + \alpha)/2} |\hat{f}_{\ell m}| (1 + \lambda_\ell)^{(\tau - \alpha)/2} |\hat{g}_{\ell m}| \\
&\leq C \left( \sum_{\ell, m} (1 + \lambda_\ell)^{\tau + \alpha} |\hat{f}_{\ell m}|^2 \right)^{1/2} \left( \sum_{\ell, m} (1 + \lambda_\ell)^{\tau - \alpha} |\hat{g}_{\ell m}|^2 \right)^{1/2} \\
&= C \|f\|_{H^{\tau, \alpha}} \|g\|_{H^{\tau, -\alpha}}.
\end{align*}

(A.3)
Applying Lemma 3.1 to \( g = f - I_X f \) gives us \( \|g\|_{H^{r-\alpha}} \leq Ch^{\alpha}_X \|g\|_{H^r} \). Combining this and (A.3) then yields

\[
\|g\|^2_N \leq Ch^{\alpha}_X \|f\|_{H^{r+\alpha}} \|g\|_{H^r}.
\]

In addition, the conditions on \( \hat{\phi}_\ell \) imply that \( \|g\|^2_N \geq c\|g\|^2_{H^r} \), and so

\[
c\|g\|^2_{H^r} \leq Ch^{\alpha}_X \|f\|_{H^{r+\alpha}} \|g\|_{H^r}.
\]

Dividing both sides above by \( \|g\|_{H^r} \) and replacing \( g \) by \( f = f - I_X f \) then yields the \( \beta = \tau \) case. If \( 0 < \beta \leq \tau \), Lemma 3.1 implies that \( \|f - I_X f\|_{H^\beta} \leq Ch^{\tau-\beta}\|f - I_X f\|_{H^r} \). Since we have already shown that \( \|f - I_X f\|_{H^\alpha} \leq Ch^{\tau+\alpha-\beta}\|f\|_{H^{r+\alpha}} \), we have \( \|f - I_X f\|_{H^\beta} \leq Ch^{\tau+\alpha-\beta}\|f\|_{H^{r+\alpha}} \). □

We want to combine this with the result found in [26, Theorem 5.5], which deals with estimates for \( f \in H^\mu \), \( \tau \geq \mu > n/2 \), to get the following general result.

**Theorem A.2.** Let \( \beta, \mu, \tau \in \mathbb{R} \), with \( \tau > n/2 \), \( n/2 < \mu \leq 2\tau \), and \( \beta \leq \min(\mu, \tau) \). Suppose that \( f \in H^\mu(\mathbb{S}^n) \) and that (A.1) holds for all \( \ell \geq 0 \). If \( h_X \) is sufficiently small, then

\[
(A.4) \quad \|f - I_X f\|_{H^\beta} \leq Ch^{\mu-\beta}_X \|f\|_{H^\mu}.
\]

**Proof.** If \( \mu \geq \tau \), the result then follows from (A.2), with \( \alpha = \mu - \tau \). If \( \mu \leq \tau \), then (A.4) follows [26, Theorem 5.5]. □

### A.2. Conditionally positive definite SBFs.

The SBFs dealt with above are all strictly positive definite. We will also need to obtain interpolation estimates for the conditionally positive definite SBFs discussed in section 3. Recall that for these SBFs, the \( \hat{\phi}_\ell \)'s need only be positive for \( \ell > L \). For \( 0 \leq \ell \leq L \), the \( \hat{\phi}_\ell \)'s can be arbitrary. The interpolation operator for a conditionally positive definite SBF \( \phi \) that reproduces \( \Pi_L \) is given in (3.3). Note that the coefficients \( a_{\xi, L} \) and the polynomial \( p_{X, L} \) are determined by the requirements that the interpolation condition \( I_{X, L} f|_X = f|_X \) hold and also that the coefficients satisfy the condition on the right above. This condition also implies that changing the \( \hat{\phi}_\ell \)'s, with \( 0 \leq \ell \leq L \), will not change \( I_{X, L} f \), because

\[
\sum_{\xi \in X} \sum_{\ell=0}^L a_{\ell, L} \phi_{\ell, \ell+L}(x) \phi_{\ell, \ell}(\xi) = \sum_{\ell=0}^L \sum_{\xi \in X} a_{\ell, L} \phi_{\ell, \ell+L}(x) \phi_{\ell, \ell}(\xi) = 0.
\]

The same reasoning further gives us that the terms \( \sum_{\xi \in X} a_{\xi, L} \phi(x \cdot \xi) \) and \( p_{X, L} \) are orthogonal. Thus, letting \( P_{\Pi_L} \) be the orthogonal projection onto \( \Pi_L \), we have

\[
(A.5) \quad P_{\Pi_L} I_{X, L} f = p_{X, L}.
\]

These remarks above allow us to assume that \( \hat{\phi}_\ell = 1 \) for \( 0 \leq \ell \leq L \), with no loss of generality. We will thus make this assumption. Doing so turns \( \phi \) into a strictly positive definite SBF and thus, consequently, makes possible forming the standard SBF interpolant \( I_X f(x) = \sum_{\xi \in X} a_{\xi} \phi(x \cdot \xi) \), with the \( a_{\xi} \)'s determined by \( I_X f|_X = f|_X \).

The two interpolants \( I_X f \) and \( I_{X, L} f \) are related in several ways. First of all, the difference of the two is given by

\[
I_X f - I_{X, L} f = \sum_{\xi \in X} (a_{\xi} - a_{\xi, L}) \phi((\cdot) \cdot \xi) - p_{X, L}.
\]
Since $\langle I_X f - I_X L f \rangle_X = 0$, $\sum_{\xi \in X} (a_{\xi, L} - a_{\xi}) \phi(\cdot) \cdot \xi = 0$. Or, put another way, $\sum_{\xi \in X} (a_{\xi} - a_{\xi, L}) \phi(\cdot) \cdot \xi = I_X p_{X, L}$. Rewriting the equation above using this fact yields

$$I_X f - I_X L f = I_X p_{X, L} - p_{X, L}.$$  \hspace{1cm} (A.6)

**Theorem A.3.** Let $\beta, \mu, \tau \in \mathbb{R}$, with $\tau > n/2$, $n/2 < \mu \leq 2\tau$, and $\beta \leq \min(\mu, \tau)$. Suppose that $f \in H_\mu(\mathbb{S}^n)$ and that $(A.1)$ holds for all $\ell \geq L + 1$. If $h_X$ is sufficiently small, then

$$\|f - I_X L f\|_{H_\beta} \leq C h_X^{\mu - \beta} \|f\|_{H_\mu}. \hspace{1cm} (A.7)$$

**Proof.** From (A.6) and Theorem A.2 we see that

$$\|I_X L f - f\|_{H_\beta} \leq \|I_X f - f\|_{H_\beta} + \|I_X p_{X, L} - p_{X, L}\|_{H_\beta} \leq C h_X^{\mu - \beta} \|f\|_{H_\mu} + \|I_X p_{X, L} - p_{X, L}\|_{H_\beta}.$$  \hspace{1cm} (A.8)

Because $p_{X, L}$ is a degree $L$ polynomial, it is analytic, so of course it is in $H_\mu$. It follows from Theorem A.2 that $\|I_X p_{X, L} - p_{X, L}\|_{H_\beta} \leq C h_X^{\mu - \beta} \|p_{X, L}\|_{H_\mu}$. Furthermore, because $p_{X, L} \in \Pi_L$,

$$\lambda_L^{(\mu - \beta)/2} \|p_{X, L}\|_{H_\mu} \leq \|p_{X, L}\|_{H_\beta} \leq \lambda_L^{(\mu - \beta)/2} \|p_{X, L}\|_{H_\mu}.$$  \hspace{1cm} (A.9)

Consequently, $\|I_X p_{X, L} - p_{X, L}\|_{H_\beta} \leq C h_X^{\mu - \beta} \|p_{X, L}\|_{H_\beta}$. Rewriting (A.5) as

$$P_{\Pi_L} (I_X L f - f) + P_{\Pi_L} f = p_{X, L},$$

we obtain

$$\|p_{X, L}\|_{H_\beta} = \|P_{\Pi_L} (I_X L f - f) + P_{\Pi_L} f\|_{H_\beta} \leq \|I_X L f - f\|_{H_\beta} + \|f\|_{H_\beta},$$

from which it follows that

$$\|I_X p_{X, L} - p_{X, L}\|_{H_\beta} \leq C h_X^{\mu - \beta} (\|I_X L f - f\|_{H_\beta} + \|f\|_{H_\beta}).$$  \hspace{1cm} (A.10)

From (A.8) and the previous inequality, we have

$$\|I_X L f - f\|_{H_\beta} \leq C h_X^{\mu - \beta} \|f\|_{H_\mu} + C h_X^{\mu - \beta} \|I_X L f - f\|_{H_\beta}.$$  \hspace{1cm} (A.11)

Choose $h_X$ so small that $C h_X^{\mu - \beta} < \frac{1}{2}$ yields

$$\|I_X L f - f\|_{H_\beta} \leq C h_X^{\mu - \beta} \|f\|_{H_\mu} + \frac{1}{2} \|I_X L f - f\|_{H_\beta}.$$  \hspace{1cm} (A.12)

Subtracting $\frac{1}{2} \|I_X L f - f\|_{H_\beta}$ from both sides and manipulating the result gives us (A.7). $\square$

**Appendix B. Sobolev space algebras**

In addition to the estimates on interpolation error estimates derived above, we will need to deal with bounds on Sobolev norms of products of functions. Fortunately, Coulhon et al. have established the requisite results. We will state these results for $\mathbb{S}^n$, in our notation, and only for the “$p=2$” cases. Before we state these results, we point out that $L^2_\tau = H_\tau$ and that the norm $\|f\|_{H_\tau} = \|(-\Delta)^{\tau/2} f\|_{L^2} = \|f\|_{L^2}$, which is defined in p. 286, is equivalent to $\|f\|_{H_\tau}$.

**Theorem B.1** (Theorem 27). Let $f, g \in H_\tau \cap L^\infty$, where $\tau \in [0, \infty)$. Then, $fg \in H_\tau \cap L^\infty$ and there exists $C > 0$ such that

$$\|fg\|_{H_\tau} \leq C (\|f\|_{L^\infty} \|g\|_{H_\tau} + \|g\|_{L^\infty} \|f\|_{H_\tau}).$$  \hspace{1cm} (B.1)
Proof. We just need to verify that the conditions in [3, Theorem 27] are satisfied. The parameters in [3, Theorem 27] connect with ours this way: \( \tau := \alpha, \ p_1 = q_2 = \infty, \ p_2 = q_1 = 2 \). If we take \( M = \mathbb{S}^n \), then all the conditions imposed on the manifold, including that of bounded geometry and positive injectivity radius, are satisfied. Thus, (B.1) holds for \( \tau \in [0, \infty) \).

Using Banach space interpolation methods, Coulhon et al.\(^6\) showed that the following holds:

**Proposition B.2** ([3, p. 334]). Let \( \tau > 0 \). Then \( f \) is in \( H_{\tau+1} \) if and only if \( f \) and \( |\nabla f| \) are in \( H_\tau \). In addition, \( \|f\|_{H_{\tau+1}} \sim \|f\|_{H_\tau} + \|\nabla f\|_{H_\tau} \).

**Corollary B.3.** Let \( f, g \in H_{\tau+1} \) and suppose that \( f, g, |\nabla f|, |\nabla g| \in L^\infty \). Then \( \nabla f \cdot \nabla g \in H_\tau \cap L^\infty \) and

\[
\|\nabla f \cdot \nabla g\|_{H_\tau} \leq C\left(\|f\|_{H_{\tau+1}} + \|g\|_{H_{\tau+1}}\right)\left(\|\nabla f\|_{L^\infty} + \|\nabla g\|_{L^\infty}\right).
\]

**Proof.** We will first prove the result for \( f = g \). By Proposition B.2, \( \nabla f \in H_\tau \) and \( \|\nabla f\|_{H_\tau} \leq C\left(\|f\|_{H_{\tau+1}} + \|f\|_{H_{\tau+1}}\right) \leq C\|f\|_{H_{\tau+1}} \). Moreover, by Theorem B.1, \( \|\nabla f\|_{H_\tau} \leq C \|\nabla f\|_{H_{\tau+1}} \|\nabla f\|_{L^\infty} \\|\nabla f\|_{L^\infty} \). The remarks above and (B.1) then imply that

\[
\|\nabla f\|_{H_\tau} \leq C \|\nabla f\|_{H_{\tau+1}} \|\nabla f\|_{L^\infty} \\|\nabla f\|_{L^\infty} \leq C \|f\|_{H_{\tau+1}} \|\nabla f\|_{L^\infty}.
\]

The result for \( f = g \) then follows immediately. For the general case, just use \( |\nabla(f + g)|^2 - |\nabla(f - g)|^2 = 4\nabla f \cdot \nabla g \) and apply the result for the \( f = g \) case. \( \square \)

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\(^6\)As stated, the theorem [3, Theorem 30], which was employed in the interpolation process, requires that the manifold be unbounded. However, examining the result as stated in [37, Theorem 7.4.5] does not make this assumption, and so the interpolation result holds for compact manifolds as well.
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