Supersymmetric many-body Euler-Calogero-Moser model

Sergey Krivonos\textsuperscript{a,b}, Olaf Lechtenfeld\textsuperscript{c} and Anton Sutulin\textsuperscript{a}

\textsuperscript{a} Bogoliubov Laboratory of Theoretical Physics, JINR, 141980 Dubna, Russia
\textsuperscript{b} St. Petersburg Department of V.A. Steklov Institute of Mathematics of the Russian Academy of Sciences, 27 Fontanka, St. Petersburg, Russia
\textsuperscript{c} Institut f"ur Theoretische Physik and Riemann Center for Geometry and Physics, Leibniz Universit"at Hannover, Appelstrasse 2, D-30167 Hannover, Germany

krivonos@theor.jinr.ru, lechtenf@itp.uni-hannover.de, sutulin@theor.jinr.ru

Abstract

We explicitly construct a supersymmetric $so(n)$ spin-Calogero model with an arbitrary even number $N$ of supersymmetries. It features $\frac{1}{2}Nn(n+1)$ rather than $Nn$ fermionic coordinates and a very simple structure of the supercharges and the Hamiltonian. The latter, together with additional conserved currents, form an $osp(N|2)$ superalgebra. We provide a superspace description for the simplest case, namely $N=2$ supersymmetry. The reduction to an $N$-extended supersymmetric goldfish model is also discussed.

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1 Introduction

In recent years notable progress was achieved in the supersymmetrization of the bosonic matrix models [12, 3, 4, 5]. It has been known for a long time that matrix models are an efficient tool of constructing conformally invariant systems (see e.g. [7] and refs. therein) For example, the Calogero model as well as its different extensions [8, 10, 11, 12] are closely related to matrix models and can be obtained from them by a reduction procedure. The supersymmetrization of matrix models consists in replacing the bosonic matrix entries by superfields [1, 2, 3, 4, 5]. It has been known for a long time that matrix models are an efficient tool of constructing conformally invariant systems (see e.g. [7] and refs. therein)

It depends on the coordinates $x_i(t)$ and momenta $p_i(t)$ of each particle as well as on the internal degrees of freedom encoded in the angular momenta $\ell_{ij} = -\ell_{ji}$. The coordinates and momenta satisfy the standard Poisson brackets

$$\{x_i, p_j\} = \delta_{ij}, \quad \{\ell_{ij}, \ell_{km}\} = \frac{1}{2}(\delta_{ik}\ell_{jm} + \delta_{jm}\ell_{ik} - \delta_{jk}\ell_{im} - \delta_{im}\ell_{jk}).$$

The ECM model with the Hamiltonian (1.1) possesses conformal invariance. Indeed, if we define the conserved currents of the dilatation $D$ and conformal boost $K$ as

$$D = -\frac{1}{2} \sum_{i=1}^{n} x_i p_i + tH \quad \text{and} \quad K = \frac{1}{2} \sum_{i=1}^{n} x_i^2 - t \sum_{i=1}^{n} x_i p_i + t^2 H,$$

then it is easy to demonstrate that they generate the one-dimensional conformal algebra $so(1,2)$:

$$\{H, K\} = 2D, \quad \{H, D\} = H, \quad \{K, D\} = -K.$$  

The equations of motion which follow from the Hamiltonian (1.1),

$$\ddot{x}_i = 2 \sum_{k \neq i}^n \frac{\ell_{ik}^2}{(x_i - x_k)^3} \quad \text{and} \quad \dot{\ell}_{ij} = -\sum_{k \neq i, j} \ell_{ik} \ell_{kj} \left(\frac{1}{(x_i - x_k)^2} - \frac{1}{(x_k - x_j)^2}\right),$$

consistently reduce to (see e.g. [13, 11, 12])

$$\ddot{x}_i = 2 \sum_{j \neq i} \frac{\dot{x}_j x_j}{x_i - x_j}$$

upon setting

$$\ell_{ij} = - (x_i - x_j) \sqrt{x_i x_j}.$$  

This maximally superintegrable system is known as the goldfish model [14, 15].

In what follows we will construct an $\mathcal{N}$-extended supersymmetric generalization of the Hamiltonian (1.1) and demonstrate an $Osp(N|2)$ invariance of this $\mathcal{N} = 2M$ supersymmetric ECM model. We also provide a superfield description for the simplest case of $\mathcal{N} = 2$ supersymmetry. Finally, we will perform the supersymmetric version of the reduction (1.8), ending up with an $\mathcal{N}$-extended supersymmetric goldfish model.

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1 An up to now unique example of a matrix system with $\mathcal{N} = 8$ supersymmetry has appeared in [5] in $\mathcal{N} = 4$ superspace.
2 \( \mathcal{N} \)-extended supersymmetric Euler–Calogero–Moser model

2.1 Extended super Poincaré algebra

The bosonic ECM model \([1, 4]\) can be obtained from a free ensemble of real symmetric matrices. This feature is parallel to the descendence of the \(su(n)\) spin-Calogero model \([9]\) from the Hermitian matrix model (for details see \([11]\)), for which a supersymmetrization has been constructed in \([6]\). In full analogy with that case, to construct \(\mathcal{N}\) supercharges \(Q^a\) and \(\overline{Q}_a\) generating an \(\mathcal{N}=2M\) superalgebra

\[
\{Q^a, \overline{Q}_b\} = -2i \delta^a_b H \quad \text{and} \quad \{Q^a, Q^b\} = \{\overline{Q}_a, \overline{Q}_b\} = 0 \quad \text{for} \quad a, b = 1, 2, \ldots, M ,
\]

one has to introduce two types of fermions:

- \(\mathcal{N} \times n\) fermions \(\psi^a_i\) and \(\overline{\psi}_i^a\) with \(i = 1, \ldots, n\). These fermions can be combined with the bosonic coordinates \(x_i(t)\) into \(\mathcal{N} = 2M\) supermultiplets.

- \(\frac{1}{2} \mathcal{N} \times (n-1)\) additional fermions \(\rho^a_{ij}\) and \(\overline{\rho}_{ija}\) subject to \(\rho^a_{ii} = \overline{\rho}_{ii} = 0\) (no sum).

In total, we thus utilize \(\frac{1}{2} \mathcal{N} n(n+1)\) fermions of type \(\psi\) and \(\rho\), which we demand to obey the following Poisson brackets

\[
\{\psi^a_i, \overline{\psi}_j^b\} = -i \delta^a_b \delta_{ij} \quad \text{and} \quad \{\rho^a_{ij}, \overline{\rho}_{km}^b\} = -\frac{i}{2} \delta^a_b \left(1 - \delta_{ij}\right) \left(1 - \delta_{km}\right) \left(\delta_{ik} \delta_{jm} + \delta_{im} \delta_{jk}\right).
\]

Using these fermions one can construct the composite objects

\[
\Pi_{ij} = -\Pi_{ji} = -i \left[\left(\psi^a_i - \psi^a_j\right) \rho_{ij} + \left(\overline{\psi}_i^a - \overline{\psi}_j^a\right) \overline{\rho}_{ij} + \sum_{k=1}^n \left(\rho^a_{ik} \overline{\rho}_{kj} - \rho^a_{kj} \overline{\rho}_{ik}\right)\right],
\]

which satisfy the \(so(n)\) Poisson brackets \(\{\Pi_{ij}, \Pi_{km}\} = \frac{1}{2} \left(\delta_{ik} \Pi_{jm} + \delta_{jm} \Pi_{ik} - \delta_{jk} \Pi_{im} - \delta_{im} \Pi_{jk}\right)\),

\[
\{\Pi_{ij}, \psi^e_k\} = \left(\delta_{ik} - \delta_{jk}\right) \rho^e_{ij},
\]

\[
\{\Pi_{ij}, \rho^e_{km}\} = -\frac{1}{2} \left(1 - \delta_{km}\right) \left[\delta_{ik} \delta_{jm} + \delta_{im} \delta_{jk}\right] \left(\psi^a_i - \psi^a_j\right)
\]

\[
+ \left(\delta_{nm} \delta_{ik} + \delta_{kn} \delta_{jm}\right) \rho^a_{in} - \left(\delta_{nm} \delta_{ik} + \delta_{kn} \delta_{jm}\right) \rho^a_{in}\right].
\]

The key idea for constructing the supercharges \(Q^a, \overline{Q}_a\) generating \(\{2.1\}\) is to “prolong” \(\ell_{ij}\) to \(\ell_{ij} + \Pi_{ij}\) in all expressions, leading to

\[
Q^a = \sum_{i=1}^n \rho^a_i \psi^a_i - \sum_{i \neq j} \left(\ell_{ij} + \Pi_{ij}\right) \rho^a_{ij} \overline{x}_i - x_j \quad \text{and} \quad \overline{Q}_a = \sum_{i=1}^n \rho^a_i \overline{\psi}_i^a - \sum_{i \neq j} \left(\ell_{ij} + \Pi_{ij}\right) \overline{\rho}_{ij} a \overline{x}_i - x_j
\]

which, together with the Hamiltonian

\[
H = \frac{1}{2} \sum_{i=1}^n \rho^2_i + \frac{1}{2} \sum_{i \neq j} \left(\ell_{ij} + \Pi_{ij}\right)^2 \overline{x}_i - x_j
\]

indeed obey the \(\mathcal{N} = 2M\) super Poincaré algebra \(\{2.1\}\) and thus describe an \(\mathcal{N} = 2M\) supersymmetric extension of the \(n\)-particle Euler–Calogero–Moser model. To confirm this fact it is most convenient to treat \(\Pi_{ij}\) as independent objects, which by themselves span the \(so(n)\) algebra \(\{2.1\}\) and Poisson-commute with the fermions as in \(\{2.5\}\). Due to these properties, our construction is valid for an arbitrary number of supersymmetries, in a full analogy with the extended supersymmetric \(su(n)\)–spin Calogero model \([5]\).
2.2 Superconformal invariance

The bosonic $n$-particle ECM model admits a dynamical conformal symmetry. Our $\mathcal{N} = 2M$ supersymmetric extension with the supercharges (2.6) and Hamiltonian (2.7) possesses a dynamical superconformal symmetry. Indeed, starting from the conserved conformal boost current

$$K = \frac{1}{2} \sum_{i=1}^{n} x_i^2 - t \sum_{i=1}^{n} x_i p_i + t^2 H,$$

(2.8)

the remaining conserved currents can easily be obtained by successively Poisson-commuting the super Poincaré generators with $K$. In this way one finds the full list of conserved currents:

$$D = -\frac{1}{2} \sum_{i=1}^{n} x_i p_i + tH,$$

$$J^a_b = -\sum_{i=1}^{n} \psi_i^a \tilde{\psi}_{i,b} - \sum_{i \neq j} \rho^a_{ij} \bar{\rho}_{ij,b},$$

$$I^{ab} = -\sum_{i=1}^{n} \psi_i^a \bar{\psi}_i^b - \sum_{i \neq j} \rho^a_{ij} \bar{\rho}_{ij,b},$$

$$T_{ab} = \sum_{i=1}^{n} \bar{\psi}_{i,a} \psi_{i,b} + \sum_{i \neq j} \tilde{\rho}_{ij,a} \bar{\rho}_{ij,b},$$

$$S^a = \sum_{i=1}^{n} x_i \psi_i^a - t Q^a,$$

$$\mathcal{S}_a = \sum_{i=1}^{n} x_i \bar{\psi}_{i,a} - t \bar{Q}_a.$$

(2.9)

Together with the supercharges $Q^a, \bar{Q}_a$ (2.6), the Hamiltonian $H$ (2.7) and the conformal boost current $K$ (2.8) they form an $\text{osp}(\mathcal{N}|2)$ superalgebra:

\begin{align*}
\{H, K\} &= 2D, \quad \{H, D\} = H, \quad \{K, D\} = -K, \\
\{J^a_b, J^c_d\} &= i(\delta^a_c J^d_b - \delta^a_d J^c_b), \quad \{J^a_b, I^{cd}\} = i(\delta^a_c I^{bd} - \delta^a_d I^{bc}), \quad \{J^a_b, T_{cd}\} = -i(\delta^a_c T_{bd} - \delta^a_d T_{bc}), \\
\{J^{ab}, T_{cd}\} &= i(\delta^a_c J^{bd} - \delta^a_d J^{bc} - \delta^a_b J^{cd} + \delta^a_d J^{bc}), \\
\{D, Q^a\} &= -\frac{1}{2} Q^a, \quad \{D, \bar{Q}_a\} = -\frac{1}{2} \bar{Q}_a, \quad \{D, S^a\} = \frac{1}{2} S^a, \quad \{D, \mathcal{S}_a\} = \frac{1}{2} \mathcal{S}_a, \\
\{H, S^a\} &= -Q^a, \quad \{H, \bar{Q}_a\} = -\bar{Q}_a, \quad \{K, S^a\} = S^a, \quad \{K, \bar{Q}_a\} = \mathcal{S}_a, \\
\{J^a_b, Q^c\} &= i(\delta^a_c Q^b - \delta^a_b Q^c), \quad \{J^a_b, \bar{Q}_c\} = i(\delta^a_c \bar{Q}_b - \delta^a_b \bar{Q}_c), \quad \{J^a_b, \bar{S}_c\} = i \delta^a_c \bar{S}_b, \\
\{I^{ab}, \bar{Q}_c\} &= -i(\delta^a_b Q^c - \delta^a_c Q^b), \quad \{I^{ab}, \mathcal{S}_c\} = -i(\delta^a_b \bar{S}_c - \delta^a_c \bar{S}_b), \\
\{T_{ab}, Q^c\} &= i(\delta^a_b \bar{Q}_c - \delta^a_c \bar{Q}_b), \quad \{T_{ab}, \bar{S}_c\} = i(\delta^a_b \bar{S}_c - \delta^a_c \bar{S}_b), \\
\{Q^a, \bar{Q}_b\} &= 2i \delta^a_b D + J^a_b, \quad \{S^a, \bar{Q}_b\} = 2i \delta^a_b D - J^a_b, \\
\{Q^a, \bar{S}_b\} &= I^{ab}, \quad \{\bar{Q}_a, \bar{S}_b\} = -T_{ab}. 
\end{align*}

(2.10)

A $\mathcal{N}$ subalgebra is generated by $J^a_b$ and extended to an $\mathfrak{so}(2M)$ subalgebra by adding $I^{ab}$ and $T_{ab}$.

3 \(\mathcal{N} = 2\) supersymmetric Euler–Calogero–Moser model in superspace

With the Hamiltonian description of an $\mathcal{N}$-extended supersymmetric ECM model at hand, it is quite instructive to construct the superfield description of the simplest case with $\mathcal{N} = 2$ supersymmetry. Such a description may be useful for understanding the general structure of the given supersymmetric construction, especially the role played by the additional $\rho$-type fermions and the currents $\ell_{ij}$.

To obtain a superspace representation of the $\mathcal{N} = 2$ supersymmetric Euler–Calogero–Moser model, defined with $M=1$ by the supercharges $Q, \bar{Q}$ (2.6) and the Hamiltonian (2.7), one firstly has to solve two tasks:

- assemble the physical components $x_i, \psi_i, \bar{\psi}_i, \rho_{ij}$ and $\bar{\rho}_{ij}$ into appropriate $\mathcal{N}=2$ superfields,
- introduce auxiliary bosonic superfields $\nu_i, \bar{\nu}_i$ whose leading components realize $\ell_{ij}$ via bilinear combinations.

Let us start with the first task. From the structure of the supercharges $Q, \bar{Q}$ (2.6) it is clear that $\mathcal{N} = 2$ supersymmetry transforms the coordinates $x_i$ into the fermions $\psi_i, \bar{\psi}_i$. Thus, one must introduce $n$ bosonic $\mathcal{N}=2$ superfields $x_i$ with the following components,

$$x_i = x_i, \quad \psi_i = -iDx_i, \quad \bar{\psi}_i = -i\bar{D}\bar{x}_i, \quad A_i = \frac{1}{2} [D, D] x_i.$$

(3.1)
Here, $|\theta = 0$ projection, while $D$ and $\overline{D}$ are $\mathcal{N}=2$ covariant derivatives obeying the relations
\begin{equation}
\{D, \overline{D}\} = 2i\partial_i \quad \text{and} \quad \{D, D\} = \{\overline{D}, \overline{D}\} = 0. \tag{3.2}
\end{equation}

The fermions $\rho_{ij}$, $\bar{\rho}_{ij}$ are put into $n(n-1)$ fermionic superfields $\rho_{ij}$, $\bar{\rho}_{ij}$, symmetric and of zero diagonal in the indices $i,j$, i.e.
\begin{equation}
\rho_{ij} = \rho_{ji}, \quad \bar{\rho}_{ij} = \bar{\rho}_{ji}, \quad \rho_{ii} = \bar{\rho}_{ii} = 0 \quad (\text{no sum}). \tag{3.3}
\end{equation}
As $\mathcal{N}=2$ superfields the $\rho_{ij}$ and $\bar{\rho}_{ij}$ contain a lot of components. However, their leading components $\rho_{ij}$ and $\bar{\rho}_{ij}$ transform under the $\mathcal{N}=2$ supersymmetry generated by $Q$ and $\overline{Q}$ (2.2) as follows,
\begin{equation}
\begin{split}
\delta_Q \rho_{ij} &\sim i \left[ \frac{\psi_i - \psi_j}{x_i - x_j} \rho_{ij} - \sum_{k \neq i,j} \frac{x_i - x_j}{(x_i - x_k)(x_j - x_k)} \rho_{ik} \rho_{jk} \right], \\
\delta_{\overline{Q}} \bar{\rho}_{ij} &\sim i \left[ \frac{\bar{\psi}_i - \bar{\psi}_j}{x_i - x_j} \bar{\rho}_{ij} - \sum_{k \neq i,j} \frac{x_i - x_j}{(x_i - x_k)(x_j - x_k)} \bar{\rho}_{ik} \bar{\rho}_{jk} \right].
\end{split} \tag{3.4}
\end{equation}
To realize these transformations in superspace we are forced to impose the following nonlinear chirality conditions,
\begin{equation}
D \rho_{ij} = i \left[ \frac{\psi_i - \psi_j}{x_i - x_j} \rho_{ij} - \sum_{k \neq i,j} \frac{x_i - x_j}{(x_i - x_k)(x_j - x_k)} \rho_{ik} \rho_{jk} \right], \\
\overline{D} \bar{\rho}_{ij} = i \left[ \frac{\bar{\psi}_i - \bar{\psi}_j}{x_i - x_j} \bar{\rho}_{ij} - \sum_{k \neq i,j} \frac{x_i - x_j}{(x_i - x_k)(x_j - x_k)} \bar{\rho}_{ik} \bar{\rho}_{jk} \right]. \tag{3.5}
\end{equation}
These conditions leave in the superfields $\rho_{ij}$ and $\bar{\rho}_{ij}$ only the components
\begin{equation}
\rho_{ij} = |\rho_{ij}|, \quad B_{ij} = |\overline{D} \rho_{ij}|, \quad \bar{\rho}_{ij} = |\bar{\rho}_{ij}|, \quad \overline{D} \bar{\rho}_{ij} = D \rho_{ij}. \tag{3.6}
\end{equation}
To get the correct Poisson brackets for $\psi_i$, $\bar{\psi}_i$ and $\rho_{ij}$, $\bar{\rho}_{ij}$ (2.2) after passing to the Hamiltonian formalism, the kinetic terms for these fermionic components must read
\begin{equation}
\mathcal{L}_{\text{kin}}^\psi = \frac{i}{2} \sum_{i=1}^n (\psi_i \bar{\psi}_i - \psi_i \bar{\psi}_i) \quad \text{and} \quad \mathcal{L}_{\text{kin}}^{\bar{\psi}} = \frac{i}{2} \sum_{i,j} (\bar{\rho}_{ij} \bar{\rho}_{ij} - \rho_{ij} \bar{\rho}_{ij}). \tag{3.7}
\end{equation}
Altogether, we arrive at the following superfield action for the purely $\mathcal{N}=2$ supersymmetric system with $l_{ij} = 0$,
\begin{equation}
S_0 = \int dt \ d^2 \theta \left[ - \frac{1}{2} \sum_{i=1}^n D x_i \overline{D} x_i - \frac{1}{2} \sum_{i,j}^n \rho_{ij} \bar{\rho}_{ij} \right], \quad d^2 \theta \equiv D \overline{D}. \tag{3.8}
\end{equation}
Now we come to the second task: realize the $\ell_{ij}$ in terms of auxiliary semi-dynamical variables. As $so(n)$ generators the $\ell_{ij}$ possess the standard realization
\begin{equation}
\ell_{ij} = \frac{i}{2} (v_i \bar{v}_j - v_j \bar{v}_i) \tag{3.9}
\end{equation}
in terms of $2n$ bosonic variables $v_i, \bar{v}_i$ subject to
\begin{equation}
\{v_i, \bar{v}_j\} = -\delta_{ij}. \tag{3.10}
\end{equation}
To implement these new semi-dynamical variables $v_i, \bar{v}_i$ at the superfield level, we have to introduce $2n$ bosonic superfields $v_i, \bar{v}_i$. Additional information about these superfields again comes from the transformation of their first components under $\mathcal{N}=2$ supersymmetry. These transformations can be learned from the explicit structure of the supercharges $Q, \overline{Q}$ (2.6), with the $\ell_{ij}$ being replaced by their realization $\ell_{ij}$ (3.10):
\begin{equation}
\delta_Q v_i \sim i \epsilon \sum_{j \neq i}^n \frac{\rho_{ij} v_j}{x_i - x_j} \quad \text{and} \quad \delta_{\overline{Q}} \bar{v}_i \sim i \epsilon \sum_{j \neq i}^n \frac{\bar{\rho}_{ij} \bar{v}_j}{x_i - x_j}. \tag{3.11}
\end{equation}
This form of the transformations implies that, like $\rho_{ij}$ and $\bar{\rho}_{ij}$, also the superfields $v_i$ and $\bar{v}_i$ are subject to nonlinear chirality conditions,

$$Dv_i = i \sum_{j \neq i}^n \frac{\rho_{ij} v_j}{x_i - x_j} \quad \text{and} \quad \cD \bar{v}_i = i \sum_{j \neq i}^n \frac{\bar{\rho}_{ij} \bar{v}_j}{x_i - x_j}. \quad (3.12)$$

These conditions leave in the superfields $v_i$ and $\bar{v}_i$ only the components

$$v_i = v_{i|}, \quad C_i = -i Dv_i|, \quad \bar{v}_i = \bar{v}_{i|}, \quad \cC_i = -i D\bar{v}_i|. \quad (3.13)$$

Finally, to have the brackets $\{3.10\}$, the kinetic terms for $v_i, \bar{v}_i$ must take the form

$$\mathcal{L}_{\text{kin}}^v = -\frac{i}{2} \sum_{i=1}^n (\dot{v}_i \ddot{v}_i - v_i \dddot{v}_i). \quad (3.14)$$

Therefore, the interaction part ($l_{ij} \neq 0$) of the superfield action reads

$$S_1 = -\frac{1}{2} \int \text{d}t \text{d}^2 \theta \sum_{i=1}^n v_i \bar{v}_i. \quad (3.15)$$

Combining everything, we conclude that the superfield action should have the form

$$S = S_0 + S_1 = \int \text{d}t \text{d}^2 \theta \left[ -\frac{1}{2} \sum_{i=1}^n Dx_i \bar{D}x_i + \frac{1}{2} \sum_{i,j}^n \rho_{ij} \bar{\rho}_{ij} - \frac{1}{2} \sum_{i=1}^n v_i \bar{v}_i \right], \quad (3.16)$$

where the superfields $\rho_{ij}, \bar{\rho}_{ij}, v_i$ and $\bar{v}_i$ are subject to the constraints $\{3.5\}$ and $\{3.12\}$, respectively.

Despite the extremely simple form of the superfield action $\{3.11\}$, its component version looks quite complicated due to the nonlinear chirality constraints $\{3.5\}$ and $\{3.12\}$. We will write the corresponding component Lagrangian as the sum of a kinetic term $\mathcal{L}_{\text{kin}}$, auxiliary-field terms $\mathcal{L}_{\text{aux}}^A, \mathcal{L}_{\text{aux}}^B, \mathcal{L}_{\text{aux}}^C$ and a “matter” term $\mathcal{L}_{\text{matter}}$.

$$\mathcal{L} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{aux}}^A + \mathcal{L}_{\text{aux}}^B + \mathcal{L}_{\text{aux}}^C + \mathcal{L}_{\text{matter}}. \quad (3.17)$$

The explicit form of these terms is

$$\begin{align*}
\mathcal{L}_{\text{kin}} & = \frac{1}{2} \sum_{i=1}^n \dddot{x}_i x_i + \frac{1}{2} \sum_{i=1}^n \left( \dot{\psi}_i \dddot{\psi}_i - \dddot{\psi}_i \dot{\psi}_i \right) + \frac{i}{2} \sum_{i,j} (\dddot{\rho}_{ij} \dddot{\bar{\rho}}_{ij} - \dddot{\rho}_{ij} \dddot{\bar{\rho}}_{ij}) - \frac{i}{2} \sum_{i=1}^n (\dot{v}_i \dddot{v}_i - \dddot{v}_i \dot{v}_i), \\
\mathcal{L}_{\text{aux}}^A & = \frac{1}{2} \sum_{i=1}^n A_i A_i - \sum_{j \neq i}^n \frac{A_j - A_i}{x_i - x_j} \rho_{ij} \bar{\rho}_{ij}, \\
\mathcal{L}_{\text{aux}}^B & = \frac{1}{2} \sum_{i,j=1}^n B_{ij} \bar{B}_{ij} + \frac{1}{2} \sum_{i,j=1}^n \left( \frac{\dddot{\psi}_i - \dddot{\psi}_j}{x_i - x_j} B_{ij} \bar{\rho}_{ij} + \frac{\dddot{\bar{\psi}}_i - \dddot{\bar{\psi}}_j}{x_i - x_j} \bar{B}_{ij} \rho_{ij} \right) + \frac{1}{2} \sum_{i,j=1}^n \left( \bar{B}_{ij} \dddot{v}_j \dddot{v}_i - \bar{B}_{ij} \dddot{v}_i \dddot{v}_j \right), \\
\mathcal{L}_{\text{aux}}^C & = -\frac{1}{2} \sum_{i=1}^n \frac{C_i \bar{C}_i}{x_i} + \frac{1}{2} \sum_{i,j=1}^n \frac{1}{x_i - x_j} \left[ \rho_{ij} \bar{C}_j \dddot{v}_i - \bar{\rho}_{ij} \dddot{C}_j v_i \right], \\
\mathcal{L}_{\text{matter}} & = \frac{1}{2} \sum_{i,j \neq k} \frac{\rho_{ij} \dddot{\bar{\rho}}_{ik}}{(x_i - x_j)(x_k - x_j)} v_j \dddot{\varepsilon}_k - \frac{1}{2} \sum_{i,j \neq k} \left( \frac{\dddot{\psi}_i - \dddot{\psi}_j}{x_i - x_j} \bar{\rho}_{ij} v_j \dddot{v}_i - \frac{\dddot{\bar{\psi}}_i - \dddot{\bar{\psi}}_j}{x_i - x_j} \rho_{ij} v_j \dddot{v}_i \right) \\
& \quad + \frac{1}{2} \sum_{j \neq i}^n \left( \frac{\dddot{\psi}_i - \dddot{\psi}_j}{x_i - x_j} \right) \rho_{ij} \dddot{\bar{\rho}}_{ij} + \frac{1}{2} \sum_{i,j \neq k,l} \frac{(x_i - x_j)^2}{x_i - x_k} \left( \rho_{ik} \rho_{jk} \dddot{\rho}_{ij} \dddot{\bar{\rho}}_{ij} \right) \dddot{v}_j (\dddot{v}_j - \dddot{\psi}_j) - (\dddot{\psi}_j - \dddot{\psi}_j) \rho_{ik} \rho_{jk} \rho_{ij} \\
& \quad + \frac{1}{2} \sum_{i,j \neq k} \frac{1}{x_i - x_k}(x_j - x_k) \left( \dddot{v}_j (\dddot{\psi}_j - \dddot{\psi}_k) - (\dddot{\psi}_j - \dddot{\psi}_k) \right) \left( \dddot{v}_j (\dddot{\psi}_j - \dddot{\psi}_k) - (\dddot{\psi}_j - \dddot{\psi}_k) \right) \rho_{ik} \rho_{jk} \rho_{ij}. \quad (3.18)
\end{align*}$$
To go on-shell we eliminate the auxiliary fields \( A_i, B_{ij}, \overline{B}_{ij}, C_i, \overline{C}_i \) using their equations of motion,

\[
A_i = 2 \sum_{j \neq i} \frac{\rho_{ij} \tilde{\rho}_{ij}}{x_i - x_j}, \quad C_i = \sum_{j \neq i} \frac{\tilde{\rho}_{ij} v_j}{x_i - x_j}, \quad \overline{C}_i = \sum_{j \neq i} \frac{\rho_{ij} \tilde{v}_i}{x_i - x_j},
\]

\[
B_{ij} = \frac{1}{2} \frac{v_i \tilde{v}_j - v_j \tilde{v}_i}{x_i - x_j} - i \frac{(\tilde{\psi}_i - \tilde{\psi}_j)}{x_i - x_j} \rho_{ij} + i \sum_{k \neq i, j} \frac{1}{x_i - x_j} \left( \frac{x_i - x_k}{x_k - x_j} \rho_{ik} \tilde{\rho}_{jk} - \frac{x_j - x_k}{x_k - x_i} \rho_{jk} \tilde{\rho}_{ik} \right),
\]

\[
\overline{B}_{ij} = \frac{1}{2} \frac{v_i \tilde{v}_j - v_j \tilde{v}_i}{x_i - x_j} - i \frac{(\tilde{\psi}_i - \tilde{\psi}_j)}{x_i - x_j} \rho_{ij} - i \sum_{k \neq i, j} \frac{1}{x_i - x_j} \left( \frac{x_i - x_k}{x_k - x_j} \rho_{ik} \tilde{\rho}_{jk} - \frac{x_j - x_k}{x_k - x_i} \rho_{jk} \tilde{\rho}_{ik} \right).
\]

After the substitution of the auxiliary components by the expressions (3.19), a straightforward but slightly tedious calculation brings the Lagrangian (3.17) to the extremely simple form

\[
\mathcal{L} = \frac{1}{2} \sum_{i=1}^{n} \dot{x}_i \dot{x}_i - \frac{i}{2} \sum_{i=1}^{n} \left( \tilde{\psi}_i \dot{\tilde{\psi}}_i - \psi_i \dot{\psi}_i \right) + \frac{1}{2} \sum_{i,j} \left( \dot{\rho}_{ij} \rho_{ij} - \rho_{ij} \dot{\rho}_{ij} \right) - \frac{i}{2} \sum_{i=1}^{n} \left( \tilde{\psi}_i \dot{\psi}_i - \psi_i \dot{\tilde{\psi}}_i \right) - \frac{n}{2} \left( \tilde{\ell}_{ij} + \Pi_{ij} \right)^2,
\]

where \( \Pi_{ij} \) is still defined as in (2.3) for \( a = 1 \) and \( \tilde{\ell}_{ij} \) is expressed in terms of semi-dynamical variables as in (3.19).

Thus, the superfield action (3.16), with the superfields \( \rho_{ij}, \tilde{\rho}_{ij}, v_i, \tilde{v}_i \) nonlinearly constrained by (3.5) and (3.12), indeed describes the \( \mathcal{N}=2 \) supersymmetric Euler–Calogero–Moser model.

To conclude, let us make a few comments:

- The nonlinear chirality conditions (3.5) can be slightly simplified by passing to different superfields

\[
\xi_{ij} = \frac{\rho_{ij}}{x_i - x_j}, \quad \tilde{\xi}_{ij} = \frac{\tilde{\rho}_{ij}}{x_i - x_j} \Rightarrow D \xi_{ij} + \frac{1}{n} \sum_{k=1}^{n} \xi_{ik} \xi_{jk} = 0, \quad D \tilde{\xi}_{ij} + \frac{1}{n} \sum_{k=1}^{n} \tilde{\xi}_{ik} \tilde{\xi}_{jk} = 0.
\]

However, the Lagrangian, Hamiltonian and Poisson brackets will look more complicated in terms of \( \xi_{ij} \) and \( \tilde{\xi}_{ij} \), despite the fact that the constraints for these new superfields do no longer involve the superfields \( x_i \).

- The auxiliary superfields \( v_i, \tilde{v}_i \) cannot be redefined in a similar manner. Thus, the nonlinear chirality constraints (3.12) are unavoidable.

- The superfield action (3.16) looks like a free action for all superfields involved. However, all interactions are hidden inside the nonlinear chirality constraints (3.5) and (3.12). This feature makes our construction quite different from most \( \mathcal{N}=2 \) supersymmetric mechanics where the interactions are generated via superpotentials. We are curious whether our mechanism to turn on interactions may be applied elsewhere for constructing new interacting superfield models.

## 4 Supersymmetric goldfish model

To construct an \( \mathcal{N} = 2M \) supersymmetric extension of the bosonic \( n \)-particle goldfish model (1.7) one has to impose a modified version of the constraints (1.8). It is not too hard to guess such constraints to be

\[
\tilde{C}_{ij} \equiv \ell_{ij} + (x_i - x_j) \sqrt{x_i x_j} + \Pi_{ij} \approx 0.
\]

One may check that these constraints weakly commute with the Hamiltonian (2.4), with the supercharges (2.6) and with each other, hence they are first class.

To get the equations of motion, one has to evaluate the brackets of all component fields involved with the
Hamiltonian (2.7) and then to impose the constraints (4.1). This results in the following equations of motion:

\[
\dot{x}_i = p_i, \quad \dot{p}_i = 2 \sum_{j \neq i}^{n} \frac{p_j p_j}{x_i - x_j},
\]

\[
\dot{\psi}_i^a = 2 \sum_{j \neq i}^{n} \frac{\sqrt{p_i p_j}}{x_i - x_j} \phi_j^a, \quad \dot{\bar{\psi}}_i^a = 2 \sum_{j \neq i}^{n} \frac{\sqrt{p_i p_j}}{x_i - x_j} \bar{\phi}_j^a,
\]

\[
\dot{\phi}_{ij} = -\sqrt{p_i p_j} \left( \psi_i^a \psi_j^a + \sum_{k \neq i,j}^{n} \frac{\sqrt{p_i p_k} \phi_{jk}^a + \sqrt{p_j p_k} \phi_{kj}^a}{x_i - x_k} \rho_{ik}^a - 2 \delta_{ij} \frac{\sqrt{p_i p_k}}{x_i - x_k} \phi_{ik}^a \right),
\]

\[
\dot{\bar{\phi}}_{ij} = -\sqrt{p_i p_j} \left( \bar{\psi}_i^a \bar{\psi}_j^a + \sum_{k \neq i,j}^{n} \frac{\sqrt{p_i p_k} \bar{\phi}_{jk}^a + \sqrt{p_j p_k} \bar{\phi}_{kj}^a}{x_i - x_k} \bar{\rho}_{ik}^a - 2 \delta_{ij} \frac{\sqrt{p_i p_k}}{x_i - x_k} \bar{\phi}_{ik}^a \right). \quad (4.2)
\]

The $N$-extended supersymmetry transformations, generated by Poisson-commuting $i (\bar{\epsilon}_a Q^a + \epsilon^a \bar{Q}_a)$ with all components fields and then by imposing the constraints (4.1), have the form

\[
\begin{align*}
\delta x_i &= i (\bar{\epsilon}_a \psi_i^a + \epsilon^a \bar{\psi}_i^a), \quad \delta p_i = 2i \sum_{j \neq i}^{n} \frac{\sqrt{p_i p_j}}{x_i - x_j} (\bar{\epsilon}_a \phi_{ij}^a + \epsilon^a \bar{\phi}_{ij}^a), \\
\delta \psi_i^a &= 2i \sum_{j \neq i}^{n} \frac{\phi_{ij}^a}{x_i - x_j} (\bar{\epsilon}_b \rho_{ij}^b + \epsilon^b \bar{\rho}_{ij}^b) - \epsilon^a p_i, \quad \delta \bar{\psi}_i^a = 2i \sum_{j \neq i}^{n} \frac{\bar{\phi}_{ij}^a}{x_i - x_j} (\epsilon_b \rho_{ij}^b + \epsilon^b \bar{\rho}_{ij}^b) - \bar{\epsilon}_a p_i, \\
\delta \phi_{ij}^a &= -\epsilon^a \sqrt{p_i p_j} + \epsilon^a \delta_{ij} p_i - i \frac{\phi_{ij}^a}{x_i - x_j} (\epsilon_b \rho_{ij}^b + \epsilon^b \bar{\rho}_{ij}^b) + i \sum_{k \neq i,j}^{n} \frac{\rho_{jk}^a}{x_i - x_k} (\epsilon_b \rho_{jk}^b + \epsilon^b \bar{\rho}_{jk}^b) \\
&\quad + i \frac{\rho_{jk}^b}{x_j - x_k} (\epsilon_b \rho_{jk}^b + \epsilon^b \bar{\rho}_{jk}^b) - 2 i \delta_{ij} \frac{\rho_{jk}^a}{x_i - x_k} (\epsilon_b \rho_{jk}^b + \epsilon^b \bar{\rho}_{jk}^b), \quad (4.3)
\end{align*}
\]

One may verify that these transformations form the $N=2$ superalgebra and leave the equations of motion (4.2) invariant.

After imposing the constraints (4.1), the Hamiltonian (2.7) and the supercharges (2.6) acquire the form

\[
H_{red} = \frac{1}{2} \left( \sum_i p_i \right)^2 \quad \text{and} \quad (Q^a)_{red} = \sum_i p_i \psi_i^a + \frac{1}{2} \sum_{i \neq j}^{n} \sqrt{p_i p_j} \rho_{ij}^a, \quad (\bar{Q}_a)_{red} = \sum_i p_i \bar{\psi}_i^a + \frac{1}{2} \sum_{i \neq j}^{n} \sqrt{p_i p_j} \bar{\rho}_{ij}^a. \quad (4.4)
\]

It is clear that the correct equations of motion require a deformation of the basic Poisson brackets (1.2), (2.2), similarly to the purely bosonic case [12]. We plan to analyze the corresponding deformation of the Poisson brackets elsewhere.
5 Conclusion

We proposed a novel $N$-extended supersymmetric $so(n)$ spin-Calogero model by a direct supersymmetrization of the bosonic Euler–Calogero–Moser system [8]. The constructed model contains

- $n$ bosonic coordinates $x_i$ which stem from the diagonal part of a real symmetric matrix,
- the off-shell elements of this symmetric matrix, which enter the supercharges and the Hamiltonian only through $so(n)$ currents $\ell_{ij}$,
- $Nn$ fermions $\psi^a_i$ and $\bar{\psi}^a_i$, which combine with the $x_i$ to $n$ supermultiplets,
- $\frac{1}{2}N \times n(n-1)$ additional fermions $\rho_{ij}^a = \rho_{ij}^a$ and $\bar{\rho}_{ij}^a = (\rho_{ij}^a)^\dagger$ for $i \neq j$.

The supercharges $Q^a$ and $\bar{Q}^a_b$ and the Hamiltonian form an $N$-extended Poincaré superalgebra and have the standard structure up to cubic in the fermions. Additional conserved currents enlarge this superalgebra to a dynamical $osp(N|2)$ superconformal symmetry of the ECM model. Having performed the Hamiltonian reduction of the ECM model, we obtained the $N$-supersymmetric goldfish system for $n$ particles.

The structure of the $so(n)$ spin-Calogero supercharges (2.6) and Hamiltonian (2.7) is quite similar to the supercharges and the Hamiltonian of the extended supersymmetric $su(n)$ spin-Calogero model [6]. Indeed, the former can be obtained from the latter by restricting the $su(n)$ currents $\ell_{ij}$ to the $so(n)$ subalgebra, imposing antisymmetry in their indices, and likewise restricting the matrix fermions $\rho_{ij}^a$ and $\bar{\rho}_{ij}^a$ to be symmetric in their indices. Upon such a reduction, the composite object $\Pi_{ij}$ also becomes antisymmetric in $(i,j)$ and generates an $so(n)$ algebra. The first-class constraints $\ell_{ii} + \Pi_{ij} \approx 0$ present in the $su(n)$ spin-Calogero model [6] are then satisfied automatically, and the reduced supercharges and Hamiltonian will coincide with the supercharges (2.6) and Hamiltonian (2.7). However, the compatibility of this reduction with the extended supersymmetry is not a priori evident and has to be checked explicitly.

The superfield description of our model in the simplest case of $N=2$ supersymmetry features

- coordinates $x_i$ and fermions $\psi_i, \bar{\psi}_j$ forming standard unconstrained bosonic superfields of type $(1, 2, 1)$,
- fermionic symmetric matrices $\rho_{ij}, \bar{\rho}_{ij}$ (with vanishing diagonal), subject to nonlinear chirality constraints,
- $2n$ bosonic $N=2$ semi-dynamical superfields $v_i, \bar{v}_i$ also obeying some nonlinear chirality constraints.

The superspace action contains only the standard kinetic terms for all superfields. It is only the nonlinear constraints which result in a rather complicated component action. However, after eliminating the auxiliary components via their equations of motion, the action acquires quite a simple form again, with an interaction quadratic and quartic in the fermions.

The presented $N=2$ supersymmetric case is not too illuminating, because it can also be constructed without matrix fermions $\rho_{ij}$ and $\bar{\rho}_{ij}$, in analogy with the $N=2$ supersymmetric Calogero model [16, 17]. One may discard the terms quadratic in $\rho_{ij}$ and $\bar{\rho}_{ij}$ in the nonlinear chirality constraints [47]. Thus, the generic superfield structure of the $N$-extended ECM model becomes visible at $N=4$ only. We are planning to address this elsewhere.

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