Virasoro and W-constraints for the $q$-KP hierarchy

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Abstract. Based on the Adler-Shiota-van Moerbeke (ASvM) formula, the Virasoro constraints and W-constraints for the $p$-reduced $q$-deformed Kadomtsev-Petviashvili ($q$-KP) hierarchy are established.

Keywords: $q$-KP hierarchy, Virasoro constraints, W-constraints

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INTRODUCTION

The origin of $q$-calculus (quantum calculus) [1, 2] traces back to the early 20th century. Many mathematicians have important works in the area of $q$-calculus and $q$-hypergeometric series. The $q$-deformation of classical nonlinear integrable system (also called $q$-deformed integrable system) started in 1990’s by means of $q$-derivative $\partial_q$ instead of usual derivative $\partial$ with respect to $x$ in the classical system. As we know, the $q$-deformed integrable system reduces to a classical integrable system as $q$ goes to 1.

Several $q$-deformed integrable systems have been presented, for example, $q$-deformation of the KdV hierarchy [3, 4, 5], $q$-Toda equation [6], $q$-Calogero-Moser equation [7]. Obviously, the $q$-deformed Kadomtsev-Petviashvili ($q$-KP) hierarchy is also a subject of intensive study in the literature from [8] to [13].

The additional symmetries, string equations and Virasoro constraints [14, 15, 16, 17, 18, 19] of the classical KP hierarchy are important since they are involved in the matrix models of the string theory [20]. For example, there are several new works [21, 22, 23, 24, 25] on this topic. It is quite interesting to study the analogous properties of $q$-deformed KP hierarchy by this expanding method. In [11], the additional symmetries of the $q$-KP hierarchy were provided. Recently, additional symmetries and the string equations associated with the $q$-KP hierarchy have already been reported in [11, 13]. The negative Virasoro constraint generators $\{L_{-n}, n \geq 1\}$ of the $2-$reduced $q$-KP hierarchy are also obtained in [13] by the similar method of [18].

Our main purpose of this article is to give the complete Virasoro constraint generators $\{L_n, n \geq -1\}$ and W-constraints $\{w_m, m \geq -2\}$ for the $p$-reduced $q$-KP hierarchy by the different process with negative part of Virasoro constraints given in [13]. The method of this paper is based on Adler-Shiota-van Moerbeke (ASvM) formula.

This paper is organized as follows. We give a brief description of $q$-calculus and $q$-KP hierarchy in Section 2 for reader’s convenience. The main results are stated and proved
in Section 3, which are the Virasoro constraints and W-constraints on the \( \tau \) function for the \( p \)-reduced \( q \)-KP hierarchy. Section 4 is devoted to conclusions and discussions.

**q-CALCULUS AND q-KP HIERARCHY**

At the beginning of this section, let us recall some useful facts of \( q \)-calculus [2] in the following to make this paper be self-contained.

The Euler-Jackson \( q \)-difference \( \partial_q \) is defined by

\[
\partial_q (f(x)) = \frac{f(qx) - f(x)}{x(q - 1)}, \quad q \neq 1
\]

and the \( q \)-shift operator is \( \theta (f(x)) = f(qx) \). It is worth pointing out that \( \theta \) does not commute with \( \partial_q \), indeed, the relation \( (\partial_q \theta (f)) = q^k \theta (\partial_q f) \), \( k \in \mathbb{Z} \) holds. The limit of \( \partial_q (f(x)) \) as \( q \) approaches 1 is the ordinary differentiation \( \partial_x (f(x)) \). We denote the formal inverse of \( \partial_q \) as \( \partial_q^{-1} \). The following \( q \)-deformed Leibnitz rule holds

\[
\partial_q^n \circ f = \sum_{k \geq 0} \binom{n}{k}_q \theta^{n-k}(\partial_q f) \partial_q^{n-k}, \quad n \in \mathbb{Z}
\]

where the \( q \)-number \( (n)_q = \frac{q^n - 1}{q - 1} \) and the \( q \)-binomial is introduced as

\[
\binom{n}{k}_q = 1, \quad \binom{n}{k}_q = \frac{(n)_q(n-1)_q \cdots (n-k+1)_q}{(1)_q(2)_q \cdots (k)_q}.
\]

Let \( (n)_q! = (n)_q(n-1)_q(n-2)_q \cdots (1)_q \), the \( q \)-exponent \( e_q (x) \) is defined by

\[
e_q (x) = \sum_{n=0}^{\infty} \frac{x^n}{(n)_q!} = \exp \left( \sum_{k=1}^{\infty} \frac{(1 - q)^k}{k(1 - q^k)} x^k \right).
\]

Similar to the general way of describing the classical KP hierarchy [14, 19], we will give a brief introduction of \( q \)-KP hierarchy and its additional symmetries based on [10, 11].

The Lax operator \( L \) of \( q \)-KP hierarchy is given by

\[
L = \partial_q + u_0 + u_{-1} \partial_q^{-1} + u_{-2} \partial_q^{-2} + \cdots.
\]

where \( u_i = u_i(x, t_1, t_2, t_3, \cdots) \), \( i = 0, -1, -2, -3, \cdots \). The corresponding Lax equation of the \( q \)-KP hierarchy is defined as

\[
\frac{\partial L}{\partial t_n} = [B_n, L], \quad n = 1, 2, 3, \cdots,
\]

here the differential part \( B_n = (L^n)_+ = \sum_{i=0}^{n} b_i \partial_q^i \) and the integral part \( L_n^+ = L^n - L_n^+ \). \( L \) in eq.(3) can be generated by dressing operator \( S = 1 + \sum_{k=1}^{\infty} s_k \partial_q^{-k} \) in the following way

\[
L = S \partial_q S^{-1}.
\]
Dressing operator $S$ satisfies Sato equation
\[
\frac{\partial S}{\partial t_n} = -(L^n)_-, \quad n = 1, 2, 3, \ldots .
\] (6)

The $q$-wave function $w_q(x,t;z)$ and the $q$-adjoint function $w^*_q(x,t;z)$ of $q$-KP hierarchy are given by
\[
w_q(x,t;z) = S e_q(xz) \exp \left( \sum_{i=1}^{\infty} t_i z^i \right), \quad w^*_q(x,t;z) = (S^*)^{-1}|_{x/q}\exp(-\sum_{i=1}^{\infty} t_i z^i),
\]
which satisfies following linear $q$-differential equations
\[
Lw_q = zw_q, \quad L^*_q|_{x/q}w^*_q = zw^*_q,
\]
where the notation $P|_{x/t} = \sum_i P_i(x/t)t^i\partial^i_q$ is used for a $q$-pseudo-differential operator of the form $P = \sum_i p_i(x)\partial^i_q$, and the conjugate operation “$*$” for $P$ is defined by $P^* = \sum (\partial^*_p)^i p_i(x)$ with $\partial^*_p = -\partial_q\theta^{-1} = -\frac{1}{q}\partial_q$, $(\partial_q^{-1})^* = (\partial_q^*)^{-1} = -\theta\partial_q^{-1}$, $(PQ)^* = Q^* P^*$ for any two $q$-PDOs.

Furthermore, $w_q(x,t;z)$ and $w^*_q(x,t;z)$ of $q$-KP hierarchy can be expressed by sole function $\tau_q(x,t)$ [10] as
\[
w_q = \frac{\tau_q(x; t - [z^{-1}])}{\tau_q(x; t)} e_q(xz) e^{\xi(t,z)} = \frac{e_q(xz) e^{\xi(t,z)} e^{-\sum_{i=1}^{\infty} \frac{1}{i} \theta \partial_q} \tau_q}{\tau_q},
\]
\[
w^*_q = \frac{\tau_q(x; t + [z^{-1}])}{\tau_q(x; t)} e_{1/q}(x) e^{-\xi(t,z)} = \frac{e_{1/q}(x) e^{-\xi(t,z)} e^{\sum_{i=1}^{\infty} \frac{1}{i} \theta \partial_q} \tau_q}{\tau_q},
\]
where $\xi(t,z) = \sum_{i=1}^{\infty} t_i z^i$ and $[z] = \left( z, \frac{z^2}{2}, \frac{z^3}{3}, \ldots \right)$. The operator $G(z)$ is introduced as $G(z)f(t) = f(t - [z^{-1}])$, then
\[
w_q = \frac{G(z) \tau_q}{\tau_q} e_q(xz) e^{\xi(t,z)} \equiv w_q e_q(xz) e^{\xi(t,z)}.
\] (8)

The following Lemma shows there exist an essential correspondence between $q$-KP hierarchy and KP hierarchy.

**Lemma 1.** [10] Let $L_1 = \partial + u_{-1} \partial^{-1} + u_{-2} \partial^{-2} + \cdots$, where $\partial = \partial/\partial x$, be a solution of the classical KP hierarchy and $\tau$ be its tau function. Then $\tau_q(x,t) = \tau(t + [x]_q)$ is a tau function of the $q$-KP hierarchy associated with Lax operator $L$ in eq. (5), where $[x]_q = (x, \frac{(1-q)^2}{2(1-q^2)} x^2, \frac{(1-q)^3}{3(1-q^3)} x^3, \ldots, \frac{(1-q)^i}{i(1-q^i)} x^i, \ldots )$.

Define $\Gamma_q$ and Orlov-Shulman’s $M$ operator [11] for $q$-KP hierarchy as $M = S\Gamma_q S^{-1}$ and $\Gamma_q = \sum_{i=1}^{\infty} \left( it_i + \frac{1-q^i}{(1-q^i)} x^i \right) \partial_q^{-1}$. The the additional flows for each pair $\{m,n\}$ are
Defined as follows
\[ \frac{\partial S}{\partial t_{m,n}} = -(M^m L^n)_{-} S, \]

or equivalently
\[ \frac{\partial L}{\partial t_{m,n}} = -[(M^m L^n)_{-}, L], \quad \frac{\partial M}{\partial t_{m,n}} = -[(M^m L^n)_{-}, M]. \]

The additional flows \( \partial_{mn} = \frac{\partial}{\partial t_{m,n}} \) commute with the hierarchy \( \partial_k = \frac{\partial}{\partial t_k} \), i.e. \([\partial_{mn}, \partial_k] = 0\) but do not commute with each other, so they are additional symmetries [12]. \((M^m L^n)_{-}\) serves as the generator of the additional symmetries along the trajectory parametrized by \( t_{m,n}^{*} \).

**Theorem 1.** [13] If an operator \( L \) does not depend on the parameters \( t_n \) and the additional variables \( t_{1-n+1}^{*} \), then \( L^n \) is a purely differential operator, and the string equations of the \( q \)-KP hierarchy are given by
\[ [L^n, \frac{1}{n} (ML^{-n+1})_{+}] = 1, \quad n = 2, 3, 4, \cdots \]

**VIRASORO AND W-CONSTRAINTS FOR THE \( q \)-KP HIERARCHY**

In this section, we mainly study the Virasoro constraints and W-constraints on \( \tau \)-function of the \( p \)-reduced \( q \)-KP hierarchy. To this end, two useful vertex operators \( X_q(\mu, \lambda) \) and \( Y_q(\mu, \lambda) \) would be introduced.

The vertex operator \( X_q(\mu, \lambda) \) is defined in [11] as
\[ X_q(\mu, \lambda) = e_q(x \mu)e_q^{-1}(x \lambda)exp(\sum_{i=1}^{\infty} t_i(\mu^i - \lambda^i))exp(- \sum_{i=1}^{\infty} \frac{\mu^i - \lambda^i}{i} \partial_i). \]

We can also denote the vertex operator \( X_q(\mu, \lambda) \) by
\[ X_q(\mu, \lambda) =: \exp(\alpha(\lambda) - \alpha(\mu)) ; \]

where the symbol :: means that we keep \( t_i \) be always left side of \( \partial_j \), and \( \alpha(\lambda) = \sum \alpha_n \frac{\lambda^{-n}}{n}, \) here \( \alpha_0 = 0, \alpha_n = \partial_n = \frac{\partial}{\partial t_n} \) for \( n > 0, \alpha_n = |n|t_{|n|} + (1-q)^{|n|} x_{|n|} \) for \( n < 0. \)

The following lemma is given without proof.

**Lemma 2.** Taylor expansion of the \( X_q(\mu, \lambda) \) on \( \mu \) at the point of \( \lambda \) is
\[ X_q(\mu, \lambda) = \sum_{m=0}^{\infty} \frac{(\mu - \lambda)^m}{m!} \sum_{n=-\infty}^{\infty} \lambda^{-m-n} W_n^{(m)}, \]

where \( \sum_{n=-\infty}^{\infty} \lambda^{-m-n} W_n^{(m)} = \frac{\partial^m}{\partial \mu^m} X_q(\mu, \lambda)|_{\mu=\lambda}. \)
The first items of $W_n^{(m)}$ are

\[
W_n^{(0)} = \delta_{n,0},
\]

\[
W_n^{(1)} = \alpha_n,
\]

\[
W_n^{(2)} = (-n - 1)\alpha_n + \sum_{i+j=n} :\alpha_i\alpha_j:
\]

\[
W_n^{(3)} = (n + 1)(n + 1)\alpha_n + \sum_{i+j+k=n} :\alpha_i\alpha_j\alpha_k: - \frac{3}{2}(n + 2) \sum_{i+j=n} :\alpha_i\alpha_j:
\]

There is Adler-Shiota-van Moerbeke (ASvM) formula \[1\] for $q$-KP hierarchy as

\[
X_q(\mu, \lambda)w_q(x, r; z) = (\lambda - \mu)Y_q(\mu, \lambda)w_q(x, t; z), \quad (14)
\]

where the operator $Y_q(\mu, \lambda)$ is the generators of additional symmetry of $q$-KP hierarchy as

\[
Y_q(\mu, \lambda) = \sum_{m=0}^{\infty} \frac{\mu - \lambda)^m}{m!} \sum_{n=-\infty}^{\infty} \lambda^{-m-n-1}(M^mL^{m+n})_. \quad (15)
\]

ASvM formula is equivalent to the following equation

\[
\partial_{m,n+m}\tau_q = \frac{W_n^{(m+1)}(\tau_q)}{m+1}. \quad (16)
\]

The following theorem holds by virtue of the ASvM formula.

**Theorem 2.**

\[
(W_n^{(m+1)} - c)\tau_q = 0, \quad m = 0, 1, 2, 3 \ldots . \quad (17)
\]

**Proof.** Consider the condition $\partial_{m,n+m}\hat{\tau}_q = 0$, from eq.(18), and denote $\hat{\tau}_q = G(z)\tau_q$,

\[
\partial_{m,n+m}\hat{\tau}_q = \partial_{m,n+m}\tau_q = \frac{\hat{\tau}_q}{\tau_q} \left( \frac{\partial_{m,n+m}\tau_q}{\tau_q} - \frac{\partial_{m,n+m}\tau_q}{\tau_q} \right) = \hat{\tau}_q (G(z) - 1) \frac{\partial_{m,n+m}\tau_q}{\tau_q} = 0.
\]

The operator $G(z)$ has the property, which is $(G(z) - 1)f(t) = 0$ implies $f(t)$ is a constant, from this we can get

\[
\frac{\partial_{m,n+m}\tau_q}{\tau_q} = c \quad (18)
\]

where $c$ is constant. Combining eq.(16) with eq.(18) finishes the proof. $\square$

Now we consider the $p$-reduced $q$-KP hierarchy, by setting $(L^p)_- = 0$, i.e. $L^p = (L^p)_+$. From Lax equation of $q$-KP hierarchy, the $p$-reduced condition means that $L$ is independent on $t_{jp}$ as $\partial_{jp}L = 0, j = 1, 2, 3, \ldots$ and $\tau_q$ is independent on $t_{jp}$ as $\partial_{jp}\tau_q = 0, j = 1, 2, 3, \ldots$.

Based on theorem 2, the Virasoro constraints and W-constraints for the $p$-reduced $q$-KP hierarchy will be obtained. Let $n = kp$ in theorem 2 and denote

\[
\tilde{t}_i = t_i + \frac{(1 - q)^i}{i(1 - q^i')}x', \quad i = 1, 2, 3, \ldots . \quad (19)
\]
First of all, for \( m = 0 \), eq. (17) in theorem 2 becomes

\[
(W_{kp}^{(1)} - c)\tau_q = 0.
\]  

(20)

Let \( c = 0 \), we have that \( \alpha_{kp}\tau_q = \frac{\partial \tau_q}{\partial \tilde{t}_{kp}} = 0 \), it is just the condition \( L^p = (L^p)_+ \) for \( p \)-reduced \( q \)-KP hierarchy.

For \( m = 1 \), it is

\[
(W_{kp}^{(2)} - c)\tau_q = 0
\]

(21)

**Theorem 3.** The Virasoro constraints imposed on the tau function \( \tau_q \) of the \( p \)-reduced \( q \)-KP hierarchy are

\[
L_k\tau_q = 0, \quad k = -1, 0, 1, 2, 3, \ldots,
\]

here

\[
L_{-1} = \frac{1}{p} \sum_{\substack{n = p+1 \\ n \neq 0(\text{mod} \ p)}} n\tilde{t}_n \frac{\partial}{\partial \tilde{t}_{n-p}} + \frac{1}{2p} \sum_{i+j=p} ij\tilde{t}_i\tilde{t}_j,
\]

\[
L_0 = \frac{1}{p} \sum_{\substack{n = 1 \\ n \neq 0(\text{mod} \ p)}} n\tilde{t}_n \frac{\partial}{\partial \tilde{t}_n} + \left( \frac{p}{24} - \frac{1}{24p} \right),
\]

\[
L_k = \frac{1}{p} \sum_{\substack{n = 1 \\ n \neq 0(\text{mod} \ p)}} n\tilde{t}_n \frac{\partial}{\partial \tilde{t}_{n+kp}} + \frac{1}{2p} \sum_{i+j=kp \ i,j \neq 0(\text{mod} \ p)} ij\tilde{t}_i\tilde{t}_j, \quad k \geq 1,
\]

and \( L_n \) satisfy Virasoro algebra commutation relations

\[
[L_n, L_m] = (n-m)L_{(n+m)}, \quad m,n = -1, 0, 1, 2, 3, \ldots.
\]  

(22)

**Proof.** Following the results in eq. (20) and eq. (21), we have

\[
\left( \frac{W_{kp}^{(2)}}{2} - c \right)\tau_q = \left( \frac{1}{2} \sum_{i+j=kp} : \alpha_i \alpha_j : - c \right)\tau_q = 0.
\]

(23)

Define \( L_k = \frac{W_{kp}^{(2)}}{p} \), let \( c = \frac{p}{24} - \frac{1}{24p} \) in \( L_0 \), otherwise \( c = 0 \). The \( p \)-reduced condition \( n \neq 0(\text{mod} \ p) \) can be naturally added without destroying the algebra structure, because of \( \tilde{t}_{mp} \) is presented together with \( \frac{\partial}{\partial \tilde{t}_{mp+kp}} \).

By a straightforward and tedious calculation, the Virasoro commutation relations

\[
[L_n, L_m] = (n-m)L_{(n+m)}, \quad m,n = -1, 0, 1, 2, 3, \ldots
\]
can be verified.

For $m = 2$, it is

$$\frac{W_{k_p}^{(3)}}{3} - c) \tau_q = \left( \frac{1}{3} \sum_{i+j+h=k_p} : \alpha_i \alpha_j \alpha_h : - c \right) \tau_q = 0. \quad (24)$$

**Theorem 4.** Let

$$w_m = \sum_{i+j+h=mp, \ i, j, h \neq 0 (\text{mod} p)} : \alpha_i \alpha_j \alpha_h : , \ m \geq -2,$$

the W-constraints on the tau function $\tau_q$ of the $p$-reduced $q$-KP hierarchy are

$$w_m \tau_q = 0, m \geq -2,$$

and they satisfy following algebra commutation relations

$$[L_n, w_m] = (2n - m)w_{n+m}, n \geq -1, m \geq -2.$$

For $m \geq 3$, using the similar technique in theorem 3 and 4, we can deduce the higher order algebraic constrains on the tau function $\tau_q$ of the $p$-reduced $q$-KP hierarchy.

**Remark 1.** As we know, the $q$-deformed KP hierarchy reduces to the classical KP hierarchy when $q \to 1$ and $u_0 = 0$. The parameters $(\tilde{t}_1, \tilde{t}_2, \cdots, \tilde{t}_i, \cdots)$ tend to $(t_1 + x, t_2, \cdots, t_i, \cdots)$ as $q \to 1$. One can further identify $t_1 + x$ with $x$ in the classical KP hierarchy, i.e. $t_1 + x \to x$. The deformation as $q$ goes to 1 of Virasoro constraints and W-constraints for the $p$-reduced $q$-KP hierarchy are identical with the results of the classical KP hierarchy given by L.A.Dickey [16] and S.Panda, S.Roy [18].

**CONCLUSIONS AND DISCUSSIONS**

To summarize, we have derived the Virasoro constraints and W-constraints of the $p$-reduced $q$-KP hierarchy in theorem 3 and 4 respectively. The results of this paper show obviously that the Virasoro constraint generators $\{L_n, n \geq -1\}$ and W-constraints $\{w_m, m \geq -2\}$ for the $p$-reduced $q$-KP hierarchy are different with the form of the KP hierarchy. Furthermore, we also would like to point out the following interesting relation between the $q$-KP hierarchy and the KP hierarchy

$$L_n = \tilde{L}_n \bigg|_{t_i \to \tilde{t}_i = t_i + \frac{(1-q)^i}{i(1-q^i)} x^i}$$

and it seems to demonstrate that $q$-deformation is a non-uniform transformation for coordinates $t_i \to \tilde{t}_i$, which is consistent with results on $\tau$ function [10] and the $q$-soliton [12] of the $q$-KP hierarchy. Here $\tilde{L}_n$ [16, 18] are Virasoro generators of the KP hierarchy.
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