L$^p$-INTEGRABILITY OF THE GRADIENT OF SOLUTIONS TO QUASILINEAR SYSTEMS WITH DISCONTINUOUS COEFFICIENTS

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Abstract. The Dirichlet problem for a class of quasilinear elliptic systems of equations with small-BMO coefficients in Reifenberg-flat domain $\Omega$ is considered. The lower order terms supposed to satisfy controlled growth conditions in $u$ and $Du$. It is obtained $L^p$-integrability with $p > 2$ of $Du$ where $p$ depends explicitly on the data. An analogous result is obtained also for the Cauchy-Dirichlet problem for quasilinear parabolic systems.

1. Introduction

In the present work we study the integrability properties of the weak solutions of the following Dirichlet problem

\begin{equation}
\begin{cases}
D_{\alpha}( \sigma_{ij}^{\alpha\beta}(x) D_{\beta} u^j(x) + a_i^\alpha(x, u)) = b_i(x, u, Du) & \text{a.a. } x \in \Omega \\
u(x) = 0 & \text{on } \partial \Omega
\end{cases}
\end{equation}

where $\Omega \subset \mathbb{R}^n$, $n \geq 2$ is a bounded Reifenberg flat domain (see Definition 2). The principal coefficients are discontinuous with "small" discontinuity expressed in terms of their bounded mean oscillation (BMO) in $\Omega$ (cf. [20]). The matrix $A(x) = \{A_{ij}^{\alpha\beta}(x)\}_{i,j \leq N}^{\alpha, \beta \leq n}$ verifies

\begin{equation}
A_{ij}^{\alpha\beta}(x) \xi^i \xi^j \geq \lambda |\xi|^2 \quad \forall \xi \in \mathbb{M}^{N \times n}, \quad ||A||_{\infty, \Omega} \leq M
\end{equation}

with some positive constants $\lambda$ and $M$. The non linear terms

\begin{align*}
& a(x, u) = \{a_i^\alpha(x, u)\}_{i \leq N}^{\alpha \leq n} \quad \text{and} \quad b(x, u, z) = \{b_i(x, u, z)\}_{i \leq N} \\
& \text{supposed to be Carathéodory functions for } x \in \Omega, \ u \in \mathbb{R}^N, \ z \in \mathbb{M}^{N \times n} \text{ and satisfy controlled growth conditions. Namely, for } |u|, |z| \to \infty \text{ we have}
\end{align*}

\begin{equation}
\begin{align*}
& a_i^\alpha(x, u) = O(\varphi_1(x) + |u|^\frac{n}{n+2}) \quad \text{and} \quad b_i(x, u, z) = O(\varphi_2(x) + |u|^\frac{n+2}{n-2}) \\
& \text{with } \varphi_1 \in L^p(\Omega), \ p > 2 \text{ and } \varphi_2 \in L^q(\Omega), \ q > \frac{2n}{n+2}.
\end{align*}
\end{equation}

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Our aim is to show that the problem (1) satisfies the Calderón–Zygmund property when \( \Omega \) is \((\delta, R)\)-Reifenberg flat and the coefficients are \((\delta, R)\)-vanishing in \( \Omega \). Precisely, each bounded weak solution \( u \in W^{1,2}_0 \cap L^\infty(\Omega; \mathbb{R}^N) \) of (1) gains better regularity from the data \( \varphi_1 \) and \( \varphi_2 \) and belongs to \( W^{1, \min\{p, q^*\}}_0 \cap L^\infty(\Omega; \mathbb{R}^N) \) where \( q^* \) is the Sobolev conjugate of \( q \) (see (4)).

Similar result is obtained also for the Cauchy-Dirichlet problem for the parabolic quasilinear system

\[
\begin{aligned}
&u_i' - D_\alpha (A^{\alpha\beta}_{ij}(x,t)D_\beta u^j + a^{\alpha}_i(x,t,u, Du)) = b_i(x,t,u, Du) \quad \text{a.a. } (x,t) \in Q \\
&u(x,t) = 0 \quad (x,t) \in \partial Q
\end{aligned}
\]

in a cylinder \( Q = \Omega \times (0, T) \) where \( \Omega \) is \((\delta, R)\)-Reifenberg flat and \( \partial Q = \Omega \cup \{ \partial \Omega \times (0, T) \} \) is the parabolic boundary.

The problem of integrability and regularity of the solutions of linear and quasilinear elliptic/parabolic equations and systems is widely studied. Let us start with the classical results concerning equations/systems with smooth coefficients presented in the monographs [24, 25]. In the scalar case, \( N = 1 \), the notorious results of De Giorgi [10] and Nash [30] assert H"older regularity of the solutions of linear divergence form equations with only \( L^\infty \) principal coefficients. One remarkable result that permits to obtain higher integrability of the weak solutions is due to Gehring [16]. He studied integrability properties of functions satisfying the reverse H"older inequality. It was noticed that some power of the gradient of the weak solutions satisfies local reverse H"older inequality. Modifying the Gehring lemma, Giaquinta and Modica [18] firstly obtain higher integrability of solutions of divergence form quasilinear elliptic equations. For the sake of completeness we give this result as it is presented in the monograph by Giaquinta [17, Theorem V.2.3].

**Theorem 1.** Suppose that \( g \in L^q(\Omega) \), \( F \in L^{q+\delta}(\Omega) \), \( g, F \geq 0 \), \( q > 1 \), \( \delta > 0 \) and

\[
\frac{1}{p} \left( \int_{B_{2R}(x)} g^p \, dx \right)^{1/p} \leq C \left\{ \left( \int_{B_{2R}(x)} g^q \, dx \right)^{1/q} + \left( \int_{B_{2R}(x)} F^p \, dx \right)^{1/p} \right\}
\]

for any ball \( B_{2R} \subset \Omega \), \( 2R < R_0 \), \( p \in [q, p_0) \) where \( C > 0 \), \( p_0 > q \) depend only on \( B, \theta, q, n \).
There are various generalizations of the above theorem permitting to study elliptic and parabolic problems with Dirichlet and Neumann boundary conditions (see [2, 3, 4, 5, 11, 15, 27]). Other results concerning higher integrability of divergence form quasilinear equations and variational equations could be found in [9, 17, 19, 21, 28]. The \( L^p \)-estimates of derivatives obtained such way laid the foundation to the so-called ”direct method” of proving partial regularity of solutions. Recently, the method of A-harmonic approximation permits to study the regularity of the solutions without the use of the Gehring lemma. For more details we refer the reader to [1, 12, 13, 14], see also the references therein.

The regularity theory for linear operators with smooth data was extended on operators with discontinuous coefficients defined in rough domains. In [6, 7, 8] the authors consider divergence form elliptic and parabolic equations and systems with \( BMO \) coefficients in Reifenberg flat domain with Dirichlet boundary conditions extending such way the known results on operators with \( VMO \) coefficients too (see also [28, 34, 35, 22, 23] and the references therein). In [15] a reverse Hölder inequality is established for quasilinear elliptic systems with principal coefficient being \( VMO \) in \( x \) and under controlled growth conditions over the lower order terms. It permits the authors to obtain interior Hölder continuity of solutions to scalar equations as well as partial Hölder regularity of solutions to systems. In [31, 32] global Hölder regularity of solutions to elliptic quasilinear equations with \( VMO \) in \( x \) principal coefficients is proved under \textit{strictly controlled growth} conditions. Later this result is extended for quasilinear elliptic and parabolic equations in Reifenberg flat domains supposing \textit{controlled growth} conditions and Dirichlet boundary data (see [11, 33, 36, 37]).

In the present work we extend the results from [37] to elliptic and parabolic systems with discontinuous data. Making use of the linear \( L^p \)-theory for systems, developed in [7, 8] and the bootstrap method we prove \( Du \in L^r \) with \( r \) depending explicitly on the data \( \varphi_1 \) and \( \varphi_2 \) in (3).

2. \textbf{Elliptic systems, definitions and main result}

In the following we use the standard notations:

- \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \rho > 0 \) and \( \mathcal{B}_\rho(x) = \{ y \in \mathbb{R}^n : |x - y| < \rho \} \).
- Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain, \( x \in \Omega \) and denote \( \Omega_\rho(x) = \Omega \cap \mathcal{B}_\rho(x) \).
- \( \mathbb{M}^{N \times n} \) is the set of \( N \times n \)-matrices.
- For a vector function \( u = (u^1, \ldots, u^N) : \Omega \to \mathbb{R}^N \) we write
  \[
  |u|^2 = \sum_{j \leq N} |u^j|^2, \quad D_\alpha u^j = \frac{\partial}{\partial x_\alpha} u^j,
  \]
\[ Du = \{ D_\alpha u^j \}_{j \leq N} \in M^{N \times n}, \quad |Du|^2 = \sum_{\alpha \leq n} |D_\alpha u^j|^2. \]

- Let \( f : \Omega \to \mathbb{R} \) and \( |\Omega| \) be the Lebesgue measure of \( \Omega \), then
  \[
  \int_\Omega f(y)dy = \frac{1}{|\Omega|} \int_\Omega f(y)dy, \quad \|f\|_{p,\Omega}^p = \|f\|_{L^p(\Omega)}^p = \int_\Omega |f(y)|^pdy.
  \]
- For \( u \in L^p(\Omega; \mathbb{R}^N) \) write \( \|u\|_{p,\Omega} \) instead of \( \|u\|_{L^p(\Omega; \mathbb{R}^N)} \).
- For each \( s \in (1, \infty) \) recall that \( s^* \) means the Sobolev conjugate of \( s \)
  \[ s^* = \begin{cases} \frac{ns}{n-s} & \text{if } s < n \\ \text{arbitrary large number} > 1 & \text{if } s \geq n. \end{cases} \]

For the function spaces we follow the notions of the monographs [24, 28]. Through all the paper the standard summation convention on repeated upper and lower indexes is adopted. The letter \( C \) is used for various constants and may change from one occurrence to another.

In [38] Reifenberg introduced a class of domains with rough boundary that can be approximated by hyperplanes at every point and at every scale. Namely

**Definition 2.** The domain \( \Omega \) is \((\delta, R)\)-Reifenberg flat if there exist positive constants \( R, \delta < 1 \) such that for each \( x \in \partial \Omega \) and each \( \rho \in (0, R) \) there is a local coordinate system \( \{y_1, \ldots, y_n\} \) with the property
  \[ B_\rho(x) \cap \{y_n > \delta \rho\} \subset \Omega_\rho(x) \subset B_\rho(x) \cap \{y_n > -\delta \rho\}. \]

Reifenberg arrived at that concept of flatness in his studies on Plateau’s problem in higher dimensions and he proved that such a domain is locally a topological disc when \( \delta \) is small enough. It is easy to see that a \( C^1 \)-domain is a Reifenberg flat with \( \delta \to 0 \) as \( R \to 0 \). A domain with Lipschitz boundary with a Lipschitz constant less than \( \delta \) also verifies the condition (5) if \( \delta \) is small enough (say \( \delta < 1/8 \)). But the class of Reifenberg’s domains is much more wider and contain domains with fractal boundaries. For instance, consider a self-similar snowflake \( S_\beta \). It is a flat version of the Koch snowflake \( S_{\pi/3} \) where the angle of the spike with respect to the horizontal is \( \beta \). A domain \( \Omega \subset \mathbb{R}^2 \) with \( S_\beta \subset \partial \Omega \) is a Reifenberg flat if \( 0 < \sin \beta < \delta < 1/8 \). This kind of flatness exhibits minimal geometrical conditions necessary for some natural properties in analysis and potential theory to hold. For more detailed overview of the properties of these domains we refer the reader to the papers [29, 39].

From (5) it follows that \( \partial \Omega \) satisfies the \((A)\)-property (cf. [9, 17, 24]). Precisely, the measure \( |\Omega_\rho(x)| \) is \( \delta \)-comparable to \( |\mathcal{B}_\rho(x)| \), that is there exists a positive constant
Definition 3. We say that a function \( a(x) \) is a \((\delta, R)\)-vanishing if there exist positive constants \( R \) and \( \delta < 1 \) such that

\[
A(\delta) < 1/2 \quad \text{such that} \\
(A) \\
A(\delta)|B_\rho(x)| \leq |\Omega_\rho(x)| \leq (1 - A(\delta))|B_\rho(x)|
\]

for any fixed \( x \in \partial \Omega \), \( \rho \in (0, R) \) and \( \delta \in (0, 1) \). This condition excludes that \( \Omega \) may have sharp outward and inward cusps. Moreover, for small \( \delta \) they can be approximated in a uniform way by Lipschitz domains with a Lipschitz constant less then \( \delta \) (see \([8, \text{Lemma 5.1}]\)). As consequence, they are \( W^{1,p}\)-extension domains, \( 1 \leq p \leq \infty \), hence the usual extension theorems, the Sobolev and Sobolev–Poincaré inequalities are valid in \( \Omega \).

To describe the discontinuity of the principal coefficients we need of the following

\[
\int_\Omega A_{ij}^{\alpha \beta}(x) D_\beta w^i(x) D_\alpha \chi^i(x) dx + \sum_{\alpha \leq n} \int_\Omega a_\alpha(x, u(x)) D_\alpha \chi^i(x) dx \\
+ \int_\Omega b_i(x, u(x), Du(x)) \chi^i(x) dx = 0, \quad j = 1, \ldots, N
\]

for all \( \chi \in W^{1,p'}(\Omega; \mathbb{R}^n) \), \( p' = p/(p - 1) \). The conditions (7) and (8) are the natural ones that ensure convergence of the integrals above. Moreover, they are optimal since a growth of the gradient greater than \( \frac{n+2}{n} \) leads to unbounded solutions as it is seen from the following example (cf. \([25, 32]\)). The function \( u(x) \in W^{1,2}(\mathcal{B}_1(0)) \), \( u(x) = |x|^{\frac{n+2}{n}} \) is a solution of the equation \( \Delta u = C|Du|^r \) in \( \mathcal{B}_1(0) \). Note that \( u(x) \notin L^\infty(\mathcal{B}_1(0)) \) for \( \frac{n+2}{n} < r < 2 \).
In generally we cannot expect boundedness of each solution of (1) unless we add some structural conditions. Consider, for instance, the system

$$D_{\alpha} (A^\alpha_i (x, u, Du)) = b_i (x, u, Du) \quad x \in \Omega$$

where

$$A^\alpha_i (x, u, Du) = \sum_{\beta \leq n} \sum_{j \leq N} (A^{\alpha \beta}_{ij} (x) D_{\beta} u^j + a^\alpha_i (x, u))$$

are measurable in $x \in \Omega$. Assume a pointwise coercive and sign conditions, both of them for large values of the corresponding component of $u$, precisely: for every $i \in \{1, \ldots, N\}$ there exist constants $\theta^i, M^i, \nu \in (0, +\infty)$ such that for $u_i \geq \theta^i$ we have

$$\begin{cases} \nu |\xi_i|^2 - M^i \leq \sum_{\alpha \leq n} A^\alpha_i (x, u, \xi) \xi_i^\alpha \\ 0 \leq b_i (x, u, \xi) \quad \text{for a.a.} \; x \in \Omega, \; \forall \xi \in \mathbb{M}^{N \times n}. \end{cases} \quad (9)$$

Suppose (7), (8) and (9) and let $u \in W^{1,2} \cap L^{\frac{2n}{n-2}} (\Omega; \mathbb{R}^N)$ be a weak solution of (1) then for each $i \in \{1, \ldots, N\}$

$$\sup_{\Omega} u^i \leq \theta^i + K^i$$

where $K^i$ depend on $M^i, n, |\Omega|$ and $\nu$ (see [26]).

**Theorem 4.** Let $u \in W^{1,2}_0 \cap L^\infty (\Omega; \mathbb{R}^N)$ be a weak solution of the problem (1) under the conditions (2), (7) and (8). Then there exists a small number $\delta_0 > 0$ such that if $\Omega$ is $(\delta, R)$-Reifenberg flat domain and $A^{\alpha \beta}_{ij} (x)$ are $(\delta, R)$-vanishing with $\delta < \delta_0 < 1$ then

$$u \in W^{1, r}_0 \cap L^\infty (\Omega; \mathbb{R}^N) \quad \text{with} \quad r = \min\{p, q^*\}. \quad (10)$$

**Proof.** In [17, Chapter 5] Giaquinta considers quasilinear strongly elliptic systems with $L^\infty$ principal coefficients, under the conditions (7) and (8). Making use of the reverse Hölder’s inequality and the version of the Gehring lemma it is shown that there exists an exponent $r_0 > 2$ such that $u \in W^{1, r_0}_0 (\Omega, \mathbb{R}^N)$ (cf. [17, Theorem V.2.3],[9, Chapter III] or [28, Lemma 3.2.23]). Since, roughly speaking, Caccioppoli-type inequalities hold up to the boundary, the method for obtaining higher integrability can be carried over up to the boundary. In [17, Chapter 5] it is done for the Dirichlet problem in Lipschitz domain. Since the Reifenberg flat domain can be uniformly approximated by Lipschitz domains the same result still holds true. Precisely, there is $r_0 > 2$ such that

$$\|Du\|_{r, \Omega} \leq N \quad \forall \; r \in [2, r_0) \quad (11)$$

where $N$ and $r_0$ depend on $n, \Lambda, \lambda, \|\varphi_1\|_{p, \Omega}, \|\varphi_2\|_{q, \Omega}, |\Omega|, \|Du\|_{2, \Omega}$. 
Let \( n > 2 \) and \( u \in W^{1,2}_0 \cap L^\infty(\Omega; \mathbb{R}^N) \) be a solution of \((1)\). Fixing that solution in the nonlinear terms we get the linearized problem

\[
\begin{cases}
D_\alpha (A^{ij}_\alpha (x) D_\beta u^j(x)) = f_i(x) - \text{div}(A_i(x)) \quad \text{a.a. } x \in \Omega \\
u(x) = 0
\end{cases}
\]

where

\[
f_i(x) = b_i(x, u, D u), \quad f(x) = b(x, u, D u),
\]

\[
A_i(x) = (a_1^i(x, u), \ldots, a_n^i(x, u)), \quad A(x) = (A_1(x), \ldots, A_N(x))
\]

and by \((7), (8)\) and \((11)\) we get

\[
\begin{align*}
\|A\|_{p, \Omega} & \leq C \left( \|\varphi_1\|_{p, \Omega} + \|u\|_{n+2, \infty, \Omega} \right) \\
\|f\|_{q_1, \Omega} & \leq C \left( \|\varphi_2\|_{q_1, \Omega} + \|u\|_{n+2, \infty, \Omega} + \|Du\|_{n+2, \infty, \Omega} \right)
\end{align*}
\]

with \( p > 2 \) and \( q_1 = \min \{ q, \frac{rn}{n+2} \} \). Further, for all \( f_i \in L^{q_i}(\Omega), \ i = 1, \ldots, N \) there exists a vector field \( \mathbb{F}_i(x) \in L^{q_i}(\Omega, \mathbb{R}^n) \) such that \( f_i(x) = \text{div} \mathbb{F}_i(x) \). Denote \( \mathbb{F}(x) = (\mathbb{F}_1(x), \ldots, \mathbb{F}_N(x)) \), then by \([32, \text{Lemma } 3.1]\) we have

\[
\|\mathbb{F}\|_{q_1, \Omega} \leq C\|f\|_{q_1, \Omega}, \quad q_1 = \min \{ q, \frac{rn}{n+2} \}.
\]

Thus the problem \((12)\) becomes

\[
\begin{cases}
D_\alpha (A^{ij}_\alpha (x) D_\beta u^j(x)) = \text{div}(\mathbb{F}_i(x) - A_i(x)) \quad \text{a.a. } x \in \Omega \\
u(x) = 0, \quad x \in \partial \Omega.
\end{cases}
\]

For linear systems as above we dispose with the regularity result of Byun and Wang [8, Theorem 1.7] that asserts there exists a small positive constant \( \delta = \delta(\lambda, p, n, N) \) such that for each \((\delta, R)\)-vanishing \( A^{ij}_\alpha \), for each \((\delta, R)\)-Reifenberg flat \( \Omega \), and for each matrix function \( \mathbb{F} - A \in L^{r_1}(\Omega; \mathbb{M}^{N \times n}) \), with \( r_1 = \min \{ p, q_1^* \} \), the solution \( u \in W^{1,2}_0 \cap L^\infty(\Omega; \mathbb{R}^N) \) of \((15)\) belongs to \( W^{1,r_1}_0 \cap L^\infty(\Omega; \mathbb{R}^N) \) and the following estimate holds

\[
\|Du\|_{r_1, \Omega} \leq C\|\mathbb{F} - A\|_{r_1, \Omega}, \quad r_1 = \min \{ p, q_1^* \}
\]

with \( C = C(\lambda, p, n, N, |\Omega|) \).

Our goal is to show the inclusion \( Du \in L^r(\Omega; \mathbb{M}^{N \times n}) \) with \( r = \min \{ p, q^* \} \). For this we study the following cases:

1) If \( q \leq \frac{rn}{n+2} \) then \( q_1 = q \) in \((14)\) and \( r_1 \equiv r = \min \{ p, q^* \} \).
2) If \( q > \frac{r_0 n}{n+2} \), then \( q_1 = \frac{r_0 n}{n+2} \) and

\[
q_1^* = \begin{cases} 
\frac{r_0 n}{n+2 - r_0} & \text{if } \frac{r_0 n}{n+2} < n \\
\text{arbitrary large number} & \text{if } \frac{r_0 n}{n+2} \geq n.
\end{cases}
\]

Consider again two sub-cases:

2a) If \( n > \frac{r_0 n}{n+2} \) then \( r_1 = \min\{p, \frac{r_0 n}{n+2 - r_0}\} \). If \( r_1 = p \) then the theorem holds true otherwise \( Du \in L^{r_1}(\Omega : \mathbb{M}^{n \times N}) \) with \( r_1 = \frac{r_0 n}{n+2 - r_0} \).

2b) If \( n \leq \frac{r_0 n}{n+2} \) then \( q_1^* \) is arbitrary large number, that implies \( r_1 = p \) and the theorem holds true once again.

It is easy to see that \( r_1 \equiv r \) unless

\[
\frac{r_0 n}{n+2} < q \quad \text{and} \quad \frac{r_0 n}{n+2} < n
\]

when \( u \in W^{1,r_1} \cap L^\infty(\Omega; \mathbb{R}^N) \) with \( r_1 = \frac{r_0 n}{n+2 - r_0} \). It holds for any solution of the linearized problem (15) including the one fixed in the coefficients in (1).

Consider once again (13) with \( Du \in L^{r_1}(\Omega; \mathbb{M}^{n \times N}) \). Hence \( f_i \in L^{q_2}(\Omega) \) with \( q_2 = \min\{q, \frac{r_0 n^2}{(n+2)^2 - r_0(n+2)}\} \) and the associated vector-field \( F_i \) belongs to \( L^{q_2^*}(\Omega; \mathbb{R}^n) \).

Thus \( F_i - A_i \in L^{r_2}(\Omega; \mathbb{R}^n) \) with \( r_2 = \min\{p, q_2^*\} \). Applying [8, Theorem 1.7] to system (15) and repeating the same procedure as above we get that the theorem holds with \( r_2 \equiv r \) if

\[
(17) \quad \begin{cases} 
(1 + n^2) - r_0(n+2) \geq n \quad \text{or} \quad (n+2)^2 - r_0(n+2) \geq q.
\end{cases}
\]

Otherwise \( r_2 = \frac{r_0 n^2}{(n+2)^2 - r_0(n+2) - r_0 n} \) if

\[
(18) \quad \frac{r_0 n^2}{(n+2)^2 - r_0(n+2)} < q \quad \text{and} \quad \frac{r_0 n^2}{(n+2)^2 - r_0(n+2)} < n
\]

Repeating the same procedure \( k \)-times we get that the assertion holds if

\[
(19) \quad \frac{r_0 n^k}{(n+2)^k - r_0 \sum_{s=0}^{k-2} n^s(n+2)^{k-1-s}} \geq \min\{n, q\}.
\]

Direct calculations give that (19) is equivalent to

\[
k > \min \left\{ \left[ \log \frac{r_0}{r_0 - 2} / \log \frac{n+2}{n} \right], \left[ \log \frac{r_0 (2q + qn + 2)}{q(n+2)(r_0 - 2)} / \log \frac{n+2}{n} \right] + 1 \right\}
\]

where \([x]\) means the integer part of \(x\).

The case \( n = 2 \) is simpler and is left to the reader. □
3. Quasilinear parabolic systems

Let \( Q = \Omega \times (0, T) \) be a cylinder in \( \mathbb{R}^{n+1} \) with \( \Omega \) being \((\delta, R)\)-Reifenberg flat. Denote by \( C_\rho \) the parabolic cylinder

\[
C_\rho(x,t) = B_\rho(x) \times (t - \rho^2, t), \quad Q_\rho(x,t) = Q \cap C_\rho(x,t) \quad \text{for} \ (x,t) \in Q,
\]

and

\[
\bar{u}_{Q_\rho(x,t)} = \frac{1}{|Q_\rho(x,t)|} \int_{Q_\rho(x,t)} a(y, \tau) dy d\tau.
\]

Let \( 1 < r < \infty \) and \( u : Q \rightarrow \mathbb{R}^N \).

1. The space \( W_r^{1,0}(Q; \mathbb{R}^N) \) consists of all functions \( u \in L^r(Q; \mathbb{R}^N) \) having a finite norm

\[
\|u\|_{W_r^{1,0}(Q; \mathbb{R}^N)} = \|u\|_{L^r(Q; \mathbb{R}^N)} + \|Du\|_{r,Q}.
\]

2. The space \( W_r^{1,r}(Q; \mathbb{R}^N) = L^r(0, T; W^{1,r}(\Omega; \mathbb{R}^N)) \cap W^{1,r}(0, T; W^{-1,r'}(\Omega; \mathbb{R}^N)) \), \( r' = r/(r-1) \) consists of the functions \( u \in W_r^{1,0}(Q; \mathbb{R}^N) \) for which there exist vector functions \( g \in L^r(Q; \mathbb{R}^N) \) and \( F \in L^r(Q; M^{N \times n}) \) such that

\[
u_t = \text{div}F - g \quad \text{a.e. in} \ Q
\]

in the sense of distributions, that is, for each vector function \( \chi \in C_0^\infty(Q) \) with \( \chi(x,T) = 0 \) holds

\[
\int_Q u \cdot \chi_t \, dx dt = \int_Q (F \cdot D\chi + g \cdot \chi) \, dx dt.
\]

The space \( W_r^{1,r}(Q; \mathbb{R}^N) \) is endowed by the norm

\[
\|u\|_{W_r^{1,r}(Q; \mathbb{R}^N)} = \|u\|_{W_r^{1,0}(Q; \mathbb{R}^N)} + \inf \left\{ \left( \int_Q |F|^r + |g|^r \right)^{1/r} \right\}
\]

where the infimum is taken over all \( F \) and \( g \) satisfying (20). The closure of \( C_0^\infty(Q) \) with respect to this norm is denoted by \( W_r^{1,r}(Q; \mathbb{R}^N) \).

3. \( V^2(Q; \mathbb{R}^N) \) stands for the Banach space of all functions \( u \in W_2^{1,0}(Q; \mathbb{R}^N) \) for which

\[
\|u\|_{V^2(Q; \mathbb{R}^N)} = \sup_{t \in [0,T]} \|u(\cdot, t)\|_{L^2(\Omega; \mathbb{R}^N)} + \|Du\|_{2,Q} < \infty.
\]

4. \( V_2^{1,0}(Q; \mathbb{R}^N) \) consists of all \( u \in V^2(Q; \mathbb{R}^N) \) that are continuous in \( t \) with respect to the norm of \( L^2(\Omega; \mathbb{R}^N) \)

\[
\lim_{\Delta t \to 0} \|u(\cdot, t + \Delta t) - u(\cdot, t)\|_{L^2(\Omega; \mathbb{R}^N)} = 0.
\]

The norm in \( V_2^{1,0}(Q; \mathbb{R}^N) \) is given by

\[
\|u\|_{V_2^{1,0}(Q; \mathbb{R}^N)} = \max_{t \in [0,T]} \|u(\cdot, t)\|_{L^2(\Omega; \mathbb{R}^N)} + \|Du\|_{2,Q}.
\]
We consider the Cauchy-Dirichlet problem for the strongly parabolic quasilinear system
\[
\begin{align*}
  u_i^t - D_\alpha (A_{ij}^\alpha D_\beta u^j + a_i^\alpha (x, t, u)) &= b_i(x, t, u, Du) \quad \text{a.a. } (x, t) \in Q \\
  u(x, t) &= 0 \quad (x, t) \in \partial Q.
\end{align*}
\] (21)

The principal coefficients satisfy \( A_{ij}^\alpha \in L^\infty(Q) \) and
\[
A_{ij}^\alpha \xi_i \xi_j \geq \nu |\xi|^2 \quad \forall \xi \in \mathbb{M}^{N \times n}, \quad \nu = \text{const} > 0.
\] (22)

In addition we suppose that \( A_{ij}^\alpha \) are \((\delta, R)\)-vanishing, that is
\[
\sup_{0 < \rho \leq R} \sup_{(x, t) \in Q} \int_{Q_{\rho}(x, t)} |A_{ij}^\alpha(y, \tau) - A_{ij}^\alpha(x, t)|^2 dyd\tau \leq \delta^2
\] (23)

which implies small BMO norm of each \( A_{ij}^\alpha \).

The functions \( a_i^\alpha(x, t, u), b_i(x, t, u, Du) \) are Carathéodory ones and verify the controlled growth conditions (see [2, 25])
\[
\sum_{\alpha \leq N} |a_i^\alpha(x, t, u)| \leq \Lambda(\psi_1(x, t) + |u|^{\frac{n+4}{n}})
\] (24)
\[
\sum_{i \leq N} |b_i(x, t, u, z)| \leq \Lambda(\psi_2(x, t) + |u|^{\frac{n+4}{n}} + |z|^{\frac{n+4}{n+2}}).
\] (25)

with \( \psi_1 \in L^p(Q), p > 2 \) and \( \psi_2 \in L^q(Q), q > \frac{2(n+2)}{n+4} \).

A vector function \( u \in \dot{W}^{1,2}_*(Q; \mathbb{R}^N) \) is a weak solution to (21) if for any function \( \chi \in \dot{W}^{1,2}_*(Q; \mathbb{R}^N), \chi(x, T) = 0 \) we have
\[
\int_Q u_i^t(x, t)\chi_t(x, t)dxdt - \int_Q (A_{ij}^\alpha(x, t)D_\beta u^j(x, t) + a_i^\alpha(x, t, u))D_\alpha \chi_i(x, t)dxdt \\
+ \int_Q b_i(x, t, u, Du)\chi_i(x, t)dxdt = 0.
\]

**Theorem 5.** Let \( u \in \dot{W}^{1,2}_* \cap L^\infty(Q; \mathbb{R}^N) \) be a weak solution to (21) under the conditions (22)-(25). Then there exists a small positive constant \( \delta_0 < 1 \) such that if \( \Omega \) is \((\delta, R)\)-Reifenberg flat and \( A_{ij}^\alpha \) are \((\delta, R)\)-vanishing with \( 0 < \delta < \delta_0 \) then
\[
u \in \dot{W}^{1,r}_* \cap L^\infty(Q; \mathbb{R}^N) \quad \text{with } r = \min\{p, q^{**}\}
\]
where \( q^{**} = \begin{cases} \frac{q(n+2)}{n+2-q} & \text{if } q < n+2 \\ \text{arbitrary large number} > 1 & \text{if } q \geq n+2. \end{cases} \)
Proof. The higher integrability of the gradient of the solution follows by the modification of the Gehring lemma due to Arkhipova [4, Theorem 1] which is very efficient for the study of parabolic systems with controlled growth conditions in domains with boundary $\partial \Omega$ satisfying a kind of (A)-property. Recently, similar result is obtained in [11] for domains having strongly Lipschitz boundary. Since the Reifenberg flat domain can be approximated uniformly with Lipschitz domains with a small Lipschitz constant [8, Lemma 5.1]) we have that there exists $r_0 > 2$ such that for any solution $u \in \dot{W}^{1,2}(Q; \mathbb{R}^N)$ holds

$$\|Du\|_{r,Q} \leq N \quad \forall r \in [2, r_0)$$

where $r_0$ and $N$ depend on the data of the problem and $\|Du\|_{2,Q}$. Take a solution $u \in \dot{W}^{1,2} \cap L^\infty(Q; \mathbb{R}^N)$ of (21) and fix it in the lower order terms. Thus we get the linearized problem

$$\begin{cases}
  u^*_i - D_\alpha(A^\alpha_{ij}D_\beta u^j) = f_i(x,t) + \text{div} A_i(x,t) & \text{a.a. } (x,t) \in Q \\
  u(x,t) = 0 & \text{on } \partial Q
\end{cases}$$

(26)

where

$$A_i(x,t) = (a^1_i(x,t,u), \ldots, a^n_i(x,t,u)), \quad A(x,t) = \{a^\alpha_i(x,t,u(x,t))\}_{i \leq N}$$

$$f_i(x,t) = b_i(x,t,u,Du), \quad f = (f_1(x,t), \ldots, f_N(x,t)).$$

Making use of the conditions (24) and (25) we get

$$\sum_{i \leq N} \|A_i\|_{p,Q} \leq C \left( \|\psi_1\|_{p,Q} + \|u\|_{\frac{n+2}{n+4},Q} \right)$$

$$\sum_{i \leq N} \|f_i\|_{q_1,Q} \leq C \left( \|\psi_2\|_{q_1,Q} + \|u\|_{\frac{n}{n},Q} + \|Du\|_{\frac{n+4}{n(n+4)},Q} \right)$$

with $p > 2$, $q_1 = \min \left\{ q : \frac{nq(n+2)}{n+4} \right\}$. Let $\Omega' \subset \mathbb{R}^n$ be $C^2$-domain such that $\Omega \subset \Omega'$ and consider the cylinder $Q' = \Omega' \times (0,T)$. Suppose that $f_1(x,t)$ is extended as zero out of $Q$. It is well known (cf. [25]) that for each $f_i \in L^{q_1}(Q')$ the linear problem

$$\begin{cases}
  F^*_i - \Delta F^i = F^*_i - D_\alpha(\delta^\alpha_{ij}D_\beta F^j) = f_i(x,t) & \text{a.a. } (x,t) \in Q' \\
  F^i(x,t) = 0 & \text{on } \partial Q'
\end{cases}$$

has a unique solution $F^i \in \dot{W}^{2,1}_{q_1}(Q')$ and the estimate holds

$$\|F^i\|_{q_1,Q} \leq \|F^i\|_{\dot{W}^{2,1}_{q_1}(Q')} \leq C \|f_i\|_{q_1,Q}.$$

Denote by $F = (F^1, \ldots, F^N)$, $F^*_i = (A^\alpha_{ij} - \delta^\alpha_{ij}\delta_{ij})D_\beta F^j$ and $F_i = (F^1_i, \ldots, F^N_i)$. Since the Sobolev trace theorem holds for domain with Reifenberg flat boundary we
get that $F_i|_S \in W^{2-\frac{1}{n}, \frac{1}{n}}(S)$ where $S = \partial \Omega \times (0, T)$. Consider the linear system

$$\begin{cases}
(u^i - F^i)_t - D_\alpha(A_{ij}^{\alpha \beta} D_\beta(u^j - F^j)) = D_\alpha((A_{ij}^{\alpha \beta} - \delta_{\alpha \beta} \delta_{ij})D_\beta F^j) + \text{div} (A_i) \\
\text{div} (\bar{F}_i + A_i) = 0 \quad \text{on } \partial \Omega.
\end{cases} \tag{27}$$

By the imbedding theorems, $DF^i \in L^{q_i^{**}}$ (cf. [25, Ch.II,Lemma 3.3]) and

$$\|DF^i\|_{q_i^{**}, Q} \leq C\|f^i\|_{q_1, Q}.$$

Hence $\bar{F}_i + A_i \in L^{r_1}(Q)$ with $r_1 = \min\{p, q_i^{**}\}$. Applying [7, Corollary 2.10] on the linear problem (27) we get

$$\|D(u - F)\|_{r_1, Q} \leq C\left(1 + \|DF\|_{q_i^{**}, Q} + \|A\|_{p, Q}\right) \leq C\left(1 + \|f\|_{q_1, Q} + \|A\|_{p, Q}\right).$$

Hence

$$\|Du\|_{r_1, Q} \leq C(1 + \|f\|_{q_1, Q} + \|A\|_{p, Q}).$$

Applying the bootstrapping arguments, we obtain as in the elliptic case that the theorem holds after $k$ iterations with

$$k \geq \min\left\{\left[\frac{\log(r_0/(r_0 - 2))}{\log((n + 4)/(n + 2))}\right], \left[\frac{\log r_0[q(n + 4) - 2(n + 2)]}{q(r_0 - 2)(n + 4)}\right] + 1\right\}.$$

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