Optimal stopping, randomized stopping and singular control with partial information flow

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18 February 2018

Abstract

The purpose of this paper is two-fold:

\begin{itemize}
  \item We extend the well-known relation between optimal stopping and randomized stopping of a given stochastic process to a situation where the available information flow is a sub-filtration of the filtration of the process. We call these problems optimal stopping and randomized stopping with \textit{partial information}.
  \item Following an idea of Krylov [K] we introduce a special \textit{singular stochastic control problem with partial information} and show that this is also equivalent to the partial information optimal stopping and randomized stopping problems. Then we show that the solution of this singular control problem can be expressed in terms of (partial information) variational inequalities, which in turn can be rewritten as a reflected backward stochastic differential equation (RBSDE) with partial information.
\end{itemize}

MSC [2010]: 93EXX; 93E20; 60J75; 62L15; 60H10; 60H20; 49N30.

Keywords: Optimal stopping; Optimal control; Singular control; RBSDEs; Partial information.

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This research was carried out with support of the Norwegian Research Council, within the research project Challenges in Stochastic Control, Information and Applications (STOCONINF), project number 250768/F20.

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1 Introduction

There are several classic papers in the literature on the relation between optimal stopping, randomized stopping and singular control of a given stochastic process with filtration $\mathbb{F} := \{\mathcal{F}_t\}_{t \geq 0}$. See e.g. Krylov [K], Wang [W], Gyöngy and Šiška [GS] and the references therein.

The purpose of this paper is to extend this relation to a situation where the admissible stopping times are required to be stopping times with respect to a given partial information flow $\mathbb{H} = \{\mathcal{H}_t\}_{t \geq 0}$ with $\mathcal{H}_t \subseteq \mathcal{F}_t$ for all $t$, and the admissible controls are required to be $\mathbb{H}$-adapted. This is a common situation in many applications, and one of our motivations for this paper is to be able to study such more realistic optimal stopping problems. To the best of our knowledge, the only paper in the literature that deals with this type of partial information optimal stopping is Øksendal and Sulem [ØS2], where the study is based on a maximum principle for singular stochastic control of jump diffusions, associated reflected backward differential equations and optimal stopping. In the current paper we extend the result of Øksendal and Sulem [ØS2] to a more general setting, using a more direct approach. Our main idea is based on the following two key elements:

(i) We prove an extension of Lemma 2 (a), p. 36, in Krylov [K] to a partial information flow situation.

(ii) We extend the results in Gyöngy and Šiška [GS] to partial information.

2 Framework and problem formulations

Let $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}, P)$ be a filtered probability space satisfying the usual conditions. Let $T \leq \infty$ be a fixed terminal time and let $\mathbb{H} = \{\mathcal{H}_t\}_{t \geq 0} \subset \mathcal{F}$ be another filtration satisfying the usual conditions. We assume that

$$\mathcal{H}_t \subseteq \mathcal{F}_t,$$

for all $t$. Further, let $\mathcal{T}_\mathbb{H} = \mathcal{T}_\mathbb{H}^{(T)}$ denote the set of all $\mathbb{H}$-stopping times $\tau \leq T$, i.e. the set of all functions

$$\tau : \Omega \to [0, T],$$

such that $\{\omega : \tau(\omega) \leq t\} \in \mathcal{H}_t$ for all $t \in [0, T]$. For example, we could have $\mathcal{H}_t = \mathcal{F}_{(t-\delta)^+}$ for $t \geq 0$.

In the following we let $\{k(t)\}_{t \geq 0}$ be a given $\mathbb{F}$-predictable process which is continuous at $t = 0$ and satisfies

$$\sup_{\tau \in \mathcal{T}_\mathbb{H}} E[|k(\tau)|] =: \kappa < \infty.$$

2.1 Partial information optimal stopping problem

We first consider the following partial information optimal stopping problem:
Problem 2.1
Find $\Phi \in \mathbb{R}$ and $\tau^* \in \mathcal{T}$ such that
\[ \Phi := \sup_{\tau \in \mathcal{T}} E[k(\tau)] = E[k(\tau^*)]. \quad (2.1) \]

Since $\mathcal{H}_t \subseteq \mathcal{F}_t$ for all $t$, we call this a partial information optimal stopping problem. If we had $\mathcal{H}_t \supseteq \mathcal{F}_t$ for all $t$, this would be an inside information optimal stopping problem.

A special inside information optimal stopping problem is studied (and solved) in Hu and Øksendal [HØ], based on Malliavin calculus and forward integration theory.

2.2 Partial information randomized stopping

Next we formulate the corresponding partial information randomized stopping problem:

Problem 2.2 Let $G_{\mathbb{H}}$ be the set of $\mathbb{H}$-adapted, right-continuous and non-decreasing processes $G(t)$ such that
\[ G(0^-) = 0 \text{ and } G(T) = 1 \text{ a.s.} \]

Find $\Lambda \in \mathbb{R}$ and $G^* \in G_{\mathbb{H}}$ such that
\[ \Lambda := \sup_{G \in G_{\mathbb{H}}} E \left[ \int_0^T k(t) dG(t) \right] = E \left[ \int_0^T k(t) dG^*(t) \right]. \]

2.3 Partial information singular control

Finally, we introduce our corresponding partial information singular control problem:

Problem 2.3 Let $A_{\mathbb{H}}$ denote the set of all $\mathbb{H}$-adapted non-decreasing right-continuous processes $\xi(t) : [0, T] \rightarrow [0, \infty]$ such that $\xi(0^-) = 0$ and $\xi(T) = \infty$.

Find $\Psi \in \mathbb{R}$ and $\xi^* \in A_{\mathbb{H}}$ such that
\[ \Psi := \sup_{\xi \in A_{\mathbb{H}}} E \left[ \int_0^T k(t) \exp \left(-\xi(t)\right) d\xi(t) \right] = \sup_{\xi \in A_{\mathbb{H}}} E \left[ \int_0^T k(t) \exp \left(-\xi^*(t)\right) d\xi^*(t) \right]. \]

We will prove that all these 3 problems are equivalent, in the sense that
\[ \Phi = \Lambda = \Psi, \]
and we will find explicit relations between the optimal $\tau^*$, $G^*$ and $\xi^*$. 

3
3 Randomized stopping and optimal stopping with partial information flow

In this section we prove that Problem 2.1 and Problem 2.2 are equivalent. The following result may be regarded as an extension of Theorem 2.1 in Gyöngy and Šisko [GS] to partial information:

**Theorem 3.1**

\[ \Lambda := \sup_{G \in \mathcal{G}_{\mathcal{H}}} \mathbb{E} \left[ \int_0^T k(t) dG(t) \right] = \sup_{\tau \in \mathcal{T}_{\mathcal{H}}} \mathbb{E} [k(\tau)] =: \Phi. \]

**Proof.**

Choose \( \tau \in \mathcal{T}_{\mathcal{H}} \) and define

\[ G^{(n)}(t) = 1_{\{t \geq \tau > 0\}} + (1 - e^{-nt})1_{\{\tau = 0\}}, \quad n = 1, 2, \ldots. \]

Then \( G^{(n)}(t) \in \mathcal{G}_{\mathcal{H}} \) and we see that

\[ \mathbb{E} [k(\tau)] = \lim_{n \to \infty} \mathbb{E} \left[ \int_0^T k(t) dG^{(n)}(t) \right] \leq \sup_{G \in \mathcal{G}_{\mathcal{H}}} \mathbb{E} \left[ \int_0^T k(t) dG(t) \right]. \]

Since \( \tau \in \mathcal{T}_{\mathcal{H}} \) was arbitrary, this proves that

\[ \sup_{\tau \in \mathcal{T}_{\mathcal{H}}} \mathbb{E} [k(\tau)] \leq \sup_{G \in \mathcal{G}_{\mathcal{H}}} \mathbb{E} \left[ \int_0^T k(t) dG(t) \right]. \]

To get the opposite inequality, we define for each \( G \in \mathcal{G}_{\mathcal{H}} \) and \( r \in [0, G(T)) = [0, 1), \) the time change \( \alpha(r) \) by

\[ \alpha(r) = \inf \{ s \geq 0 ; G(s) > r \}. \]

Then \( \{ \omega ; \alpha(r) < t \} = \{ \omega ; G(t) > r \} \in \mathcal{H}_t, \) so \( \alpha(r) \in \mathcal{T}_{\mathcal{H}} \) for all \( r. \) Moreover, \( G(\alpha(t)) = t \) for all \( t \) and hence

\[ \mathbb{E} \left[ \int_0^T k(t) dG(t) \right] = \mathbb{E} \left[ \int_0^{G(T)} k(\alpha(r)) dr \right] \leq \int_0^1 \sup_{\tau \in \mathcal{T}_{\mathcal{H}}} \mathbb{E} [k(\tau)] dr = \sup_{\tau \in \mathcal{T}_{\mathcal{H}}} \mathbb{E} [k(\tau)]. \]

\( \square \)

4 Singular control and optimal stopping with partial information

In this section we prove that Problem 2.1 and Problem 2.3 are equivalent. First we establish an auxiliary result:
Lemma 4.1 Let $\xi \in A_{H}$ and $t \in [0, T]$. Then

$$\int_{0^+}^{t} \exp(-\xi(s)) d\xi(s) \leq 1 - \exp(-\xi(t)) \leq \int_{0^+}^{t} \exp(-\xi(s^-)) d\xi(s); \ t \in [0, T],$$

i.e.,

$$\exp(-\xi(t)) d\xi(t) \leq d(-\exp(-\xi(t))) \leq \exp(-\xi(t^-)) d\xi(t); \ t \in [0, T].$$

Proof.

(i) We first prove the second inequality:

Let $\xi \in A_{H}$. The Itô formula for semimartingales (see e.g. Theorem II.32 in Protter [P]), gives that for all $f \in C^2(\mathbb{R})$ we have

$$f(\xi(t)) = f(0) + \int_{0^+}^{t} f'(\xi(s^-)) d\xi(s) + \frac{1}{2} \int_{0^+}^{t} f''(\xi(s^-)) d[\xi, \xi]_s^c$$

$$+ \sum_{0<s \leq t} \{ f(\xi(s)) - f(\xi(s^-)) - f'(\xi(s^-)) \Delta \xi(s) \},$$

where $\Delta \xi(s) = \xi(s) - \xi(s^-) > 0$. Since $\xi$ has finite variation, we have $d[\xi, \xi]_s^c = 0$. Therefore, applying the Itô formula to the concave function

$$f(x) = 1 - \exp(-x); \ x \in \mathbb{R},$$

we get

$$1 - \exp(-\xi(t)) = \int_{0^+}^{t} \exp(-\xi(s^-)) d\xi(s)$$

$$+ \sum_{0<s \leq t} \{ -\exp(-\xi(s)) + \exp(-\xi(s^-)) - \exp(-\xi(s^-)) \Delta \xi(s) \}$$

$$\leq \int_{0^+}^{t} \exp(-\xi(s^-)) d\xi(s),$$

which proves the second inequality.

(ii) We proceed to prove the first inequality:
From (4.1) we get

\[1 - \exp(-\xi(t)) = \int_{0^+}^{t} \exp(-\xi(s))d\xi(s)\]

\[+ \sum_{0<s\leq t} \{\exp(-\xi(s^-)) - \exp(-\xi(s^+))\}\Delta\xi(s)\]

\[+ \sum_{0<s\leq t} \{-\exp(-\xi(s)) + \exp(-\xi(s^-)) - \exp(-\xi(s^-))\}\Delta\xi(s)\]

\[= \int_{0^+}^{t} \exp(-\xi(s))d\xi(s)\]

\[+ \sum_{0<s\leq t} \{-\exp(-\xi(s)) + \exp(-\xi(s^-)) - \exp(-\xi(s^-))\}\Delta\xi(s)\]

\[= \int_{0^+}^{t} \exp(-\xi(s))d\xi(s)\]

\[+ \sum_{0<s\leq t} \{\exp(-\xi(s^-)) - \exp(-\xi(s)) + \exp(-\xi(s^+))(\xi(s^-) - \xi(s))\}.\]

Since \(x \mapsto e^{-x}\) is convex, we have

\[e^{-x} - e^{-y} \geq -e^{-y}(x - y).\]

Hence

\[\exp(-\xi(s^-)) - \exp(-\xi(s)) + \exp(-\xi(s^+))(\xi(s^-) - \xi(s)) \geq 0,\]

and we conclude that

\[1 - \exp(-\xi(t)) \geq \int_{0^+}^{t} \exp(-\xi(s))d\xi(s).\]

\[\Box\]

We proceed to prove the main result of this section:

**Theorem 4.2** Define \(A_c^\mathbb{H} = \{\xi \in A_\mathbb{H} ; \xi \text{ is continuous}\}. Then

\[
\sup_{\xi \in A^\mathbb{H}} E \left[ \int_0^T k(t) \exp (-\xi(t)) d\xi(t) \right] = \Psi := \sup_{\xi \in A\mathbb{H}} E \left[ \int_0^T k(t) \exp (-\xi(t)) d\xi(t) \right]
\]

\[
= \sup_{G \in G\mathbb{H}} E \left[ \int_0^T k(t) dG(t) \right] = \sup_{\tau \in T\mathbb{H}} E [k(\tau)] =: \Phi.
\]
Proof.
Let $\xi \in A_H$. Then $w(t) := 1 - e^{-\xi(t)} \in G_H$ and hence, by Lemma 4.1 and Proposition ??,

$$E \left[ \int_0^T k(t) \exp(-\xi(t)) d\xi(t) \right] \leq E \left[ \int_0^T k(t) d\left( - \exp(-\xi(t)) \right) \right]$$

$$= E \left[ \int_0^T k(t) dw(t) \right] \leq \sup_{G \in G_H} E \left[ \int_0^T k(t) dG(t) \right]$$

$$= \sup_{\tau \in T_H} E \left[ k(\tau) \right].$$

Therefore,

$$\sup_{\xi \in A_H} E \left[ \int_0^T k(t) \exp(-\xi(t)) d\xi(t) \right] \leq \sup_{\tau \in T_H} E \left[ k(\tau) \right]. \quad (4.3)$$

To get the opposite inequality, choose $\tau \in T_H$ and define, for $n = 1, 2, \ldots$

$$u^{(n)}(t) = \begin{cases} 0, & \text{for } t < \tau, \\ n, & \text{for } t \geq \tau, \end{cases}$$

and

$$G^{(n)}(t) = \begin{cases} 0, & \text{for } t < \tau, \\ 1 - e^{-n(t-\tau)}, & \text{for } t \geq \tau. \end{cases}$$

Then $\xi^{(n)}(t) := \int_0^t u^{(n)}(s) ds \in A_{\overline{H}}$, $G^{(n)}(t) \in G_H$ and for all $\delta = \frac{1}{\sqrt{n}}$,

$$\int_0^T k(t)u^{(n)}(t) \exp \left( - \int_0^t u^{(n)}(s) ds \right) dt = \int_0^T k(t) dG^{(n)}(t) = I_n + J_n + K_n,$$

where

$$I_n = \int_\tau^{\tau+\delta} k(\tau) dG^{(n)}(t), \quad J_n = \int_\tau^{\tau+\delta} (k(t) - k(\tau)) dG^{(n)}(t), \quad K_n = \int_\tau^{T} k(t) dG^{(n)}(t).$$

We see that

$$I_n = k(\tau)(1 - e^{-n\delta}) = k(\tau)(1 - e^{-\sqrt{n}}) \to k(\tau) \text{ when } n \to \infty;$$

$$E \left[ |K(n)| \right] \leq \sup_{t \geq 0} E \left[ |k(t)| \mathbf{1}_{\{t > \tau + \delta\}} \right] n \exp(-n\delta) \leq \kappa n \exp(-\sqrt{n}) \to 0 \text{ when } n \to \infty,$$

and

$$|J_n| \leq \sup_{t \in [\tau, \tau + \delta]} |k(t) - k(\tau)| \to 0 \text{ when } n \to \infty.$$

7
Combining the above we conclude that
\[
\lim_{n \to \infty} E \left[ \int_0^T k(t) u^{(n)}(t) \exp \left( - \int_0^t u^{(n)}(s) \, ds \right) \, dt \right] = E[k(\tau)].
\]
Therefore
\[
\sup_{\xi \in \mathcal{A}_n} E \left[ \int_0^T k(t) \exp(-\xi(t)) \, d\xi(t) \right] \geq E[k(\tau)].
\]
Since \( \tau \in \mathcal{T}_H \) was arbitrary this proves that
\[
\sup_{\xi \in \mathcal{A}_n} E \left[ \int_0^T k(t) \exp(-\xi(t)) \, d\xi(t) \right] \geq \sup_{\tau \in \mathcal{T}_H} E[k(\tau)].
\]
Combining this with (4.3) we get
\[
\sup_{\tau \in \mathcal{T}_H} E[k(\tau)] \leq \sup_{\xi \in \mathcal{A}_n} E \left[ \int_0^T k(t) \exp(-\xi(t)) \, d\xi(t) \right] \\
\leq \sup_{\xi \in \mathcal{A}_n} E \left[ \int_0^T k(t) \exp(-\xi(t)) \, d\xi(t) \right] \\
\leq \sup_{\tau \in \mathcal{T}_H} E[k(\tau)],
\]
and we conclude that we have equality everywhere in this chain of inequalities. By Theorem 3.1 this proves Theorem 4.2.

\[
\square
\]

It is of interest to find the connection between an optimal stopping time \( \tau^* \in \mathcal{T}_H \) for Problem 2.1 and the corresponding optimal singular controls \( G^*, \xi^* \) for Problem 2.2 and Problem 2.3, respectively. The connection is given by the following result:

**Theorem 4.3**

a) Suppose \( \tau^* \in \mathcal{T}_H \) is an optimal stopping time for Problem 2.1. Define
\[
G^*(t) := 1_{\{\tau^* > 0\}} + 1_{\{\tau^* = 0\}}.
\]
Then \( G^* \in \mathcal{G}_H \) is an optimal singular control for Problem 2.2.

b) Conversely, suppose \( G^* \in \mathcal{G}_H \) is an optimal singular control for Problem 2.2. Define
\[
\alpha^*(r) := \inf\{s \geq 0; G^*(s) > r\}; \text{ for } r \in [0, 1).
\]
Then \( \alpha^*(r) \in \mathcal{T}_H \) and \( \alpha^*(r) \) is an optimal stopping time for Problem 2.1 for all \( r \in [0, 1) \).
c) Suppose $\xi^* \in A$ is an optimal control for Problem 2.3. Then

$$G^*(t) := 1 - E\left[\int_t^T \exp(-\xi^*(s))d\xi^*(s) \mid \mathcal{H}_t\right]$$

is an optimal control for Problem 2.2.

d) Conversely, suppose $G^*(t)$ is an optimal control for Problem 2.2. Define $\xi^*(t)$ to be a solution of the differential equation

$$d\xi^*(t) = \exp(\xi^*(t))dG^*(t); \quad \xi^*(0^-) = 0, \xi(T) = \infty.$$ 

Then $\xi^*(t)$ is an optimal control for Problem 2.3.

Proof.

a) Suppose $\tau^* \in T$ is optimal for Problem 2.1 and let $G^*$ be as in (4.4). Then by Theorem 3.1

$$\sup_{\tau \in T} E[k(\tau)] = E[k(\tau^*)] = E\left[\int_0^T k(t)dG^*(t)\right] \leq \sup_{G \in \mathcal{G}} E\left[\int_0^T k(t)dG(t)\right] = \sup_{\tau \in T} E[k(\tau)].$$

Hence we have equality in the above, and therefore

$$E\left[\int_0^T k(t)dG^*(t)\right] = \sup_{G \in \mathcal{G}} E\left[\int_0^T k(t)dG(t)\right],$$

which proves that $G^*$ is optimal for Problem 2.2.

b) Conversely, suppose $G^* \in \mathcal{G}$ is optimal for Problem 2.2. Let $\alpha^*(r)$ be as in (4.5). Then $\alpha^*(r) \in T$ for all $r$ and, by Theorem 3.1

$$\sup_{G \in \mathcal{G}} E\left[\int_0^T k(t)dG(t)\right] = E\left[\int_0^T k(t)dG^*(t)\right] = E\left[\int_0^{G^*(T)} k(\alpha^*(r))dr\right] = \int_0^1 E[k(\alpha^*(r))]dr \leq \int_0^1 \sup_{\tau \in T} E[k(\tau)]dr = \sup_{\tau \in T} E[k(\tau)] \leq \sup_{G \in \mathcal{G}} E\left[\int_0^T k(t)dG(t)\right].$$
We conclude that we have equality everywhere in the above, and therefore
\[ E[k(\alpha^*(r))] = \sup_{\tau \in \mathcal{T}_\mathcal{H}} E[k(\tau)], \text{ for a.a. } r \in [0,1), \ a.s. \]

Therefore \( \alpha^*(r) \) is an optimal stopping time for a.a. \( r \in [0,1) \). Choose arbitrary \( \bar{r} \in [0,1) \). Then since \( \alpha^*(r) \) is right-continuous we can find \( r_n \in (0,1) \) such that \( \alpha^*(\bar{r}) \) is optimal for all \( n \) and \( \alpha^*(r_n) \to \alpha^*(\bar{r}) \) as \( n \to \infty \). This gives
\[ E[k_{\alpha^*(r)}] = \lim_{n \to \infty} E[k(\alpha^*(r_n))] = \sup_{\tau \in \mathcal{T}_\mathcal{H}} E[k(\tau)]. \]

Hence \( \alpha^*(r) \) is an optimal stopping time for all \( r \in [0,1) \).

c),d) If \( G^*(t) \) and \( \xi^*(t) \) are chosen as given in c) and d) respectively, then we see in either case that
\[ E[\int_0^T k(s)dG^*(s)] = E[\int_0^T k(s)\exp(-\xi^*(s))d\xi^*(s)]. \]

The two statements c) and d) follow from this.

\[ \square \]

Remark 4.4 In the case when
\[ k(t) \geq 0, \text{ for all } t \in [0,T], \]
we can extend the sets \( \mathcal{G}_\mathcal{H} \) and \( \mathcal{A}_\mathcal{H} \) of admissible controls to the following:

- \( \mathcal{G}^{(1)}_\mathcal{H} = G; \{G(t) \text{ is } \mathcal{H}-\text{adapted, right-continuous and non-decreasing} \}
  \quad \text{with } G(0^-) = 0 \text{ and } G(T) \leq 1 \}, \]

- \( \mathcal{A}^{(\infty)}_\mathcal{H} = \{\xi; \xi(t) \text{ is } \mathcal{H}-\text{adapted, right-continuous and non-decreasing} \}
  \quad \text{with } \xi(0^-) = 0 \text{ and } \xi(T) \leq \infty \}. \]

Then we can show by the same method as above that
\[ \sup_{G \in \mathcal{G}^{(1)}_\mathcal{H}} E\left[\int_0^T k(t)dG(t)\right] = \sup_{\tau \in \mathcal{T}_\mathcal{H}} E[k(\tau)] = \sup_{G \in \mathcal{G}_\mathcal{H}} E\left[\int_0^T k(t)dG(t)\right], \]

and
\[ \sup_{\xi \in \mathcal{A}^{(\infty)}_\mathcal{H}} E\left[\int_0^T k(t)\exp(-\xi(t))d\xi(t)\right] = \sup_{\tau \in \mathcal{T}_\mathcal{H}} E\left[\int_0^T k(t)\exp(-\xi(t))d\xi(t)\right] = \sup_{\tau \in \mathcal{T}_\mathcal{H}} E[k(\tau)]. \]

Moreover, the optimal \( G^* \in \mathcal{G}^{(1)}_\mathcal{H} \) satisfies
\[ G^*(T) = 1, \]
and the optimal \( \xi \in \mathcal{A}^{(\infty)}_\mathcal{H} \) satisfies
\[ \xi^*(T) = \infty. \]
5 Singular control with partial information flow

We discuss now the singular control problem.

5.1 Variational inequalities

In this section we assume that $T < \infty$.

We now turn to the partial information singular control problem (Problem 2.3):

Problem 5.1 Find $\Psi \in \mathbb{R}$ and $\xi^* \in \mathcal{A}_H$ such that

$$\Psi = \sup_{\xi \in \mathcal{A}_H} J(\xi) = J(\xi^*),$$  \hspace{1cm} (5.1)

where

$$J(\xi) = E \left[ \int_0^T k(t) \exp(-\xi(t)) d\xi(t) \right].$$  \hspace{1cm} (5.2)

We interpret $\exp(-\xi(t))$ as 0 when $\xi(t) = \infty$. Note that if $\xi(t_0) = \infty$ for some $t_0 < T$, then $\xi(t) = \infty$ for all $t \geq t_0$ and $d\xi(t) = 0$ for $t \geq t_0$.

With this interpretation, Problem 5.1 can be considered as a generalization of the singular control problem discussed in Section 2 of Øksendal and Sulem [ØS2], where a singular control version of the maximum principle is used. However, the problem (5.1) - (5.2) is not covered by the results in Øksendal and Sulem [ØS2]. Here we give a direct approach based on a variational argument.

Proceeding as in Øksendal and Sulem [ØS2], for $\xi \in \mathcal{A}_H$ we define $\mathcal{V}(\xi)$ to be the set of càdlàg processes $\zeta(t) : [0, T] \to [0, \infty]$ of finite variation such that there exists $\delta = \delta(\xi) > 0$ such that

$$\xi + y\zeta \in \mathcal{A}_H \text{ for all } y \in [0, \delta].$$

For $\xi \in \mathcal{A}_H$ and $\zeta \in \mathcal{V}(\xi)$ we define $D(\xi, \zeta) \in \mathbb{R}$ by
\[ D(\xi, \zeta) := \lim_{y \to 0^+} \sup \frac{1}{y} \left( J(\xi + y\zeta) - J(\xi) \right) \]

\[ = \lim_{y \to 0^+} \sup \frac{1}{y} \left( E \left[ \int_0^T k(s) \left\{ \exp \left( - (\xi(s) + y\zeta(s)) \right) \left( d\xi(s) + y d\zeta(s) \right) - \exp \left( - \xi(s) \right) d\xi(s) \right\} \right] \right) \]

\[ = \lim_{y \to 0^+} \sup \frac{1}{y} \left( E \left[ \int_0^T k(s) \left\{ \exp \left( - \xi(s) \right) \left( \exp(-y\zeta(s)) - 1 \right) d\xi(s) 
+ y \exp \left( - \xi(s) \right) \exp \left( - y\zeta(s) \right) d\zeta(s) \right\} \right] \right) \]

\[ = E \left[ \int_0^T k(s) \exp \left( - \xi(s) \right) \left\{ -\zeta(s) d\xi(s) + d\zeta(s) \right\} \right]. \quad (5.3) \]

Now suppose \( \xi = \hat{\xi} \) maximizes \( J(\xi) \). Then by (5.3)

\[ E \left[ \int_0^T k(s) \exp \left( - \hat{\xi}(s) \right) \left\{ -\zeta(s) d\hat{\xi}(s) + d\zeta(s) \right\} \right] = D(\hat{\xi}, \zeta) \leq 0, \quad (5.4) \]

for all \( \zeta \in \mathcal{V}(\hat{\xi}) \). In particular, if we for \( \delta > 0 \) choose

\[ \zeta_0(s) = \begin{cases} 0; & s < t, \\ \frac{(s-t)\alpha}{\delta}; & t \leq s \leq t + \delta, \\ \alpha; & s \geq t + \delta, \end{cases} \]

for some \( t \in [0, T] \) and some bounded \( \mathcal{H}_t \)-measurable random variable \( \alpha(\cdot) \geq 0 \), then \( \zeta_0 \in \mathcal{V}(\hat{\xi}) \) and (5.4) gives

\[ E \left[ \int_t^{t+\delta} k(s) \exp \left( - \hat{\xi}(s) \right) \frac{(s-t)\alpha}{\delta} d\hat{\xi}(s) + \int_t^{T+\delta} k(s) \exp \left( - \hat{\xi}(s) \right) a d\hat{\xi}(s) 
- \int_t^{t+\delta} k(s) \exp \left( - \hat{\xi}(s) \right) \frac{\alpha}{\delta} ds \right] \geq 0. \]

Since this holds for all such \( \alpha(\cdot) \) and all \( \delta > 0 \), we conclude that

\[ E \left[ \int_t^T k(s) \exp \left( - \hat{\xi}(s) \right) d\hat{\xi}(s) - k(t) \exp \left( - \hat{\xi}(t) \right) \left| \mathcal{H}_t \right| \right] \geq 0; \ t \in [0, T]. \]

Next, let us choose

1. \( d\zeta_1(s) = d\hat{\xi}(s) \) and
2. \( d\zeta_2(s) = -d\hat{\xi}(s) \).
Then $\zeta_i \in \mathcal{V}(\hat{\xi})$ for $i = 1, 2$ and (5.4) gives

$$E \left[ \int_0^T k(s) \exp \left( -\hat{\xi}(s) \right) \left\{ -\hat{\xi}(s) d\hat{\xi}(s) + d\hat{\xi}(s) \right\} \right] = 0. \quad (5.5)$$

Note that by the Fubini theorem we have

$$\int_0^T \left( \int_t^T k(s) \exp \left( -\hat{\xi}(s) \right) d\hat{\xi}(s) \right) d\hat{\xi}(t) \quad (5.6)$$

$$= \int_0^T \left( \int_0^s d\hat{\xi}(t) \right) k(s) \exp \left( -\hat{\xi}(s) \right) d\hat{\xi}(s)$$

$$= \int_0^T k(s) \exp \left( -\hat{\xi}(s) \right) \hat{\xi}(s) d\hat{\xi}(s).$$

Substituting (5.6) into (5.5) we get

$$E \left[ \int_0^T \left\{ \int_t^T k(s) \exp \left( -\hat{\xi}(s) \right) d\hat{\xi}(s) - k(t) \exp \left( -\hat{\xi}(t) \right) \right\} d\hat{\xi}(t) \right] = 0.$$

This proves part a) of the following theorem:

**Theorem 5.2 (Variational inequalities)**

a) Suppose $\hat{\xi} \in \mathcal{A}_H$ is optimal for (5.1) - (5.2). Then

$$E \left[ \int_t^T k(s) \exp \left( -\hat{\xi}(s) \right) d\hat{\xi}(s) - k(t) \exp \left( -\hat{\xi}(t) \right) \right] \geq 0; \quad t \in [0, T]. \quad (5.7)$$

and

$$E \left[ \int_t^T k(s) \exp \left( -\hat{\xi}(s) \right) d\hat{\xi}(s) - k(t) \exp \left( -\hat{\xi}(t) \right) \right] d\hat{\xi}(t) = 0; \quad t \in [0, T]. \quad (5.8)$$

b) Conversely, suppose (5.7) - (5.8) hold for some $\hat{\xi} \in \mathcal{A}_H$. Then

$$D(\hat{\xi}, \zeta) \leq 0 \text{ for all } \zeta \in \mathcal{V}(\hat{\xi}). \quad (5.9)$$

Proof.

Statement b) is proved by reversing the argument used to prove that (5.9) $\Rightarrow$ (5.7) - (5.8). We omit the details.
5.2 Reflected BSDEs with partial information

We recall a direct approach to optimal stopping with partial information, as presented in e.g. Øksendal and Zhang [ØZ]:

Define the Snell envelope $Y(t)$ for $0 \leq t \leq T$ by

$$Y(t) = \sup_{\tau \in T_{t,T}} E[k(\tau) \mid \mathcal{H}_t],$$

where $T_{t,T}^H$ is the family of $\mathbb{H}$-stopping times $\tau$ such that $t \leq \tau \leq T$.

**Theorem 5.3** If $Y(t)$ is the snell envelope as defined above, then there exists an $\mathbb{H}$-adapted, non-decreasing, right-continuous process $K(t)$ and an $\mathbb{H}$-martingale $M(t)$ such that $(Y(t), K(t), M(t))$ is the unique solution of the RBSDE given by the following equations and inequalities:

- $dY(t) = -dK(t) + dM(t)$;
- $Y(t) \geq E[k(t) \mid \mathcal{H}_t]$;
- $Y(T) = E[k(T) \mid \mathcal{H}_T]$;
- $(Y(t) - E[k(t) \mid \mathcal{H}_t])dK(t) = 0; \hspace{1em} t \in [0,T]$.

5.3 Singular control and related RBSDE under partial information

It is possible to express Theorem 5.2 in terms of a reflected backward stochastic differential equation (RBSDE) with respect to a partial filtration, as follows:

Consider the problem to find $\mathbb{H}$-adapted processes $p(t), \xi(t)$ such that

- $p(\cdot)$ is càdlàg,
- $\xi \in \mathcal{A}_{\mathbb{H}}$,
- $p(t) = E \left[ \int_t^T d\xi(s) \mid \mathcal{H}_t \right]; \hspace{1em} t \in [0,T]$,
- $p(t) - E \left[ k(t) \mid \mathcal{H}_t \right] \geq 0, \hspace{1em} \text{for all } t \in [0,T]$ and
- $(p(t) - E \left[ k(t) \mid \mathcal{H}_t \right]) d\xi(t) = 0, \hspace{1em} \text{for all } t \in [0,T]$.

If such a pair $(p, \xi)$ exists, we call it the solution of the singular RBSDE (5.10) - (5.14) with filtration $\mathbb{H}$. The process $k(t)$ is called a reflecting barrier.
Remark 5.4 Note that the RBSDE (5.12) can be written in the following equivalent way:
\[
\begin{cases}
  dp(t) = -d\xi(t) + dM(t); t \in [0, T] \\
p(T) = 0,
\end{cases}
\tag{5.15}
\]
for some (unique) $\mathbb{H}$-martingale $M$.

It now follows from (5.15) and (4.1) that
\[
d\left(\exp\left(-\hat{\xi}(t)\right)\hat{p}(t)\right) = \exp\left(-\hat{\xi}(t)\right)\left\{-k(t)d\hat{\xi}(t) + dM(t)\right\} + p(t)dZ(t),
\]
where
\[
Z(t) = \sum_{0 \leq l \leq t} \{\exp(-\hat{\xi}(l)) - \exp(-\hat{\xi}(l^-)) + \exp(-\hat{\xi}(l))\Delta\hat{\xi}(l)\}.
\]
Let $\tau$ be the stopping time of jumps for $\xi$. Further let $L_1 := \max\{n \geq 0 : \tau_n \leq T\}$. Define $r_i := \tau_i$ and
\[
g(t) = \exp(-\hat{\xi}(t)) - \exp(-\hat{\xi}(t^-)) + \exp(-\hat{\xi}(t))\Delta\hat{\xi}(t).
\]
Integrating and taking conditional expectation with respect to $\mathcal{H}_t$, we get
\[
\exp\left(-\hat{\xi}(t)\right)\hat{p}(t) = E\left[\int_t^T k(s) \exp\left(-\hat{\xi}(s)\right) d\hat{\xi}(s) \bigg| \mathcal{H}_t\right] - E\left[\sum_{i=1}^n p(\tau_i)g(\tau_i) \bigg| \mathcal{H}_t\right], \quad t \in [0, T].
\]
Now let $\epsilon$ be such that $\epsilon \leq r_1 - r_2$, then
\[
\exp\left(-\hat{\xi}(r_1 - \epsilon)\right)\hat{p}(r_1 - \epsilon) = E\left[\int_{(r_1 - \epsilon)^+}^T k(s) \exp\left(-\hat{\xi}(s)\right) d\hat{\xi}(s) \bigg| \mathcal{H}_{r_1^-}\right] + p(r_1^-)E\left[g(r_1) \bigg| \mathcal{H}_{r_1^-}\right].
\]
Letting $\epsilon \to 0$ gives
\[
\exp\left(-\hat{\xi}(r_1^-)\right)\hat{p}(r_1^-) = E\left[\int_{r_1}^T k(s) \exp\left(-\hat{\xi}(s)\right) d\hat{\xi}(s) \bigg| \mathcal{H}_{r_1^-}\right] - p(r_1^-)E\left[g(r_1) \bigg| \mathcal{H}_{r_1^-}\right],
\]
or
\[
\hat{p}(r_1^-) = E\left[\int_{r_1}^T \frac{k(s) \exp\left(-\hat{\xi}(s)\right) d\hat{\xi}(s)}{p(r_1^-)E\left[g(r_1) \bigg| \mathcal{H}_{r_1^-}\right] + \exp\left(-\hat{\xi}(r_1^-)\right)} \bigg| \mathcal{H}_{r_1^-}\right].
\]
Further,
\[
\exp\left(-\hat{\xi}(r_2 - \epsilon)\right)p(r_2) = \int_{(r_2 - \epsilon)^+}^T k(s) \exp\left(-\hat{\xi}(s)\right) d\hat{\xi}(s)
- \int_{r_2 - \epsilon}^T \exp\left(-\hat{\xi}(s)\right) dM(s)
- p(r_2)g(r_2) - p(r_1^-)g(r_1).
\]
This gives

\[
\exp \left( -\hat{\xi}(r_2) \right) p(r_2) = E \left[ \int_{r_2}^{T} k(s) \exp \left( -\hat{\xi}(s) \right) d\hat{\xi}(s) \bigg| \mathcal{H}_{r_2} \right] - E \left[ p(r_2^-)g(r_2) \bigg| \mathcal{H}_{r_2} \right] - p(r_1^-)E \left[ g(r_1) \bigg| \mathcal{H}_{r_2} \right].
\]

So

\[
p(r_2) = E \left[ \int_{r_2}^{T} \frac{k(s) \exp \left( -\hat{\xi}(s) \right)}{\exp \left( -\hat{\xi}(r_2) \right) + E \left[ g(r_2) \bigg| \mathcal{H}_{r_2} \right]} d\hat{\xi}(s) \bigg| \mathcal{H}_{r_2} \right] - E \left[ \frac{p(r_1^-)g(r_2)}{\exp \left( -\hat{\xi}(r_2) \right) + E \left[ g(r_2) \bigg| \mathcal{H}_{r_2} \right]} \bigg| \mathcal{H}_{r_2} \right].
\]

On the other hand we have that

\[
\hat{p}(t) = E \left[ \int_{t}^{T} \frac{k(s) \exp \left( -\hat{\xi}(s) \right)}{\exp \left( -\hat{\xi}(t) \right)} d\hat{\xi}(s) \bigg| \mathcal{H}_{r_1} \right] - E \left[ \sum_{i=1}^{n} p(r_i^-)g(r_i)1_{(t,T]}(r_i) \bigg| \mathcal{H}_t \right],
\]

with explicitly known \( p(r_i^-) \) for all \( i \).

Substituting this into (5.7) and (5.8) we get the following theorem.

**Theorem 5.5 (RBSDE formulation)**

1. Suppose \( \hat{\xi} \in \mathcal{A}_\mathbb{H} \) is optimal for (5.1) - (5.2). Let \( \hat{p}(t) \) be the solution of the corresponding RBSDE (5.12), i.e.

\[
p(t) = E \left[ \int_{t}^{T} d\hat{\xi}(s) \bigg| \mathcal{H}_t \right]; \quad t \in [0, T].
\]

Then

\[
E \left[ \exp \left( -\hat{\xi}(t) \right) \hat{p}(t) + E \left[ \sum_{i=1}^{n} p(r_i^-)g(r_i)1_{(t,T]}(r_i) \bigg| \mathcal{H}_t \right] \right] - \exp \left( -\hat{\xi}(t) \right) k(t) \bigg| \mathcal{H}_t \right] \geq 0 \text{ for all } t \in [0, T],
\]

16
and

\[
E \left[ \exp \left( -\hat{\xi}(t) \right) \hat{p}(t) + E \left[ \sum_{i=1}^{n} p(r_i) g(r_i) 1_{(t,T)}(r_1) \bigg| \mathcal{H}_t \right] \right] \\
- \exp \left( -\hat{\xi}(t) \right) k(t) \big| \mathcal{H}_t \bigg) d\hat{\xi}(t) = 0 \text{ for all } t \in [0, T].
\]

Thus \((\hat{p}(t), \hat{\xi}(t))\) solves the RBSDE (5.10) - (5.14) up to the time

\[ T_\infty := \inf \left\{ t > 0 ; \hat{\xi}(t) = \infty \right\} \leq T. \]

2. Conversely, suppose \((\hat{p}(t), \hat{\xi}(t))\) is the solution of the RBSDE (5.10) - (5.14) up to the time \(T_\infty\). Then

\[ D(\hat{\xi}, \zeta) \leq 0 \text{ for all } \zeta \in \mathcal{V}(\hat{\xi}). \]

References

[GS] Gyöngy, I., & Šiška, D. (2008). On randomized stopping. Bernoulli, 352-361.

[HØ] Hu, Y., & Øksendal, B. (2008). Optimal stopping with advanced information flow: selected examples. Banach Center Publications, 1(83), 107-116.

[K] Krylov, N. V. (1980). Controlled Diffusion Processes. Springer Science & Business Media.

[ØS2] Øksendal, B., & Sulem, A. (2012). Singular stochastic control and optimal stopping with partial information of Itô–Lévy processes. SIAM Journal on Control and Optimization, 50(4), 2254-2287.

[ØZ] Øksendal, B., & Zhang, T. (2012). Backward stochastic differential equations with respect to general filtrations and applications to insider finance. Communications on Stochastic Analysis, 6(4), 13.

[P] Protter, P. Stochastic Integration and Differential Equations, 2003. Springer (2nd Edition).

[W] Wang, B. (2004). Singular control of stochastic linear systems with recursive utility. Systems & Control Letters, 51(2), 105-122.