On quasinormal modes of small
Schwarzschild-Anti-de-Sitter black hole

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Abstract

We compute the quasinormal modes associated with decay of the massless scalar
field around a small Schwarzschild-Anti-de-Sitter black hole. The computations
shows that when the horizon radius is much less than the anti-de-Sitter radius,
the imaginary part of the frequency goes to zero as $r_+^{d-2}$ while the real part of $\omega$
reduces to its minimum and then goes to $d - 1$. Thus the quasinormal modes
approach the usual AdS modes in the limit $r_+ \to 0$. This agrees with suggestions
of Horowitz et al (Phys.Rev. D62 024027 (2000)).

Original interest to quasinormal modes of black holes arose since they are the char-
acteristics of black holes which do not depend on initial perturbations and are functions
of a black hole parameters only. At present, this interest is renewed since the QN fre-
quencies are in the suggested region of the gravitational wave detectors which are under
construction. In general QNM’s are important in black holes dynamics and appear in such
processes as collisions of two black holes, decay of different fields in a BH background.

All this motivated the investigation of the QNM of black holes in asymptotically
flat space-time (see [1] for a recent review). The quasinormal modes of asymptotically
de-Sitter black holes were studied in [2],[3]. Recently an unexpected application of quasi-
normal modes have appeared due to the AdS/CFT correspondence [4]: it proved out
that a large black hole in AdS space corresponds to an approximately thermal state in
the CFT, and, thereby, perturbation of the black hole corresponds to the perturbation
of the above thermal state, while the decay of the perturbation can be associated with
the return to thermal equilibrium. Thus the QN frequencies give us the thermalization
timescale which is very hard to compute directly. Quasinormal modes for different types
of perturbations of black holes in AdS space have been studied recently in a lot of papers
[5]-[14].

Black holes are considered to be small (large), when its horizon radius is much smaller
(larger) then the anti-de-Sitter radius. When computing the QN frequencies associated
with the decay of massless scalar field in the background of small SAdS BH a striking
conjecture with the black hole critical phenomena was found: $\omega_{Im}$ is proportional to BH
radius $r_+$ to high accuracy, and the slope of the line $\omega_{Im}$ to the $r_+$ axis, 2.66, turned out
to be very close to the special frequency $\lambda = 2.67$ which corresponds to the growing mode
exp $\lambda t$ describing the late time behaviour of the critical solution \cite{15}. Yet, the quasinormal frequencies of the SAdS black hole have been computed only for black holes with horizon radius $r_+ \geq 0.4R$, where $R$ is the anti-de-Sitter radius (except for one mode corresponding to $r_+ = 0.2R$ for which the behavior of the wave function was numerically reproduced). This is not sufficient to study the small black hole limit, but several suggestions were made. In \cite{14} it was stated that both real and imaginary parts of the QN modes for small black holes are very large and proportional to the surface gravity, however, later, it was shown by numerical integration of the wave equation, that at least for $r_+ \geq 0.4R$, in agreement with the first study \cite{3,4}, that both the real and imaginary parts of $\omega$ are decreasing, and, stated that the behaviour $\omega_{Re} \to \text{const}, \omega_{Im} \to 0$ at $r_+ \to 0$ is expected. Note that by considering an approximate symmetry of the SAdS metric in the limit $r_+ \to 0$ it was supposed that $\omega_{Im} \to 0$ as $r_+^2 \to 0$. Here we try to compute the quasinormal modes of black holes with the horizon radius smaller than $0.4R$, to approach the small black holes regime ($r_+ \ll R$) as much as possible, and to obtain more definite hints of very small black hole behaviour. It proves out that computations of quasinormal modes for very small black holes are quite reliable within the method proposed in \cite{8} provided one avoids accumulating of numerical error (see Appendix).

The d-dimensional Schwarzschild-Anti-de-Sitter metric is:

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2d\Omega_{d-2}^2,$$  \hfill (1)

where

$$f(r) = 1 - \frac{r_0^{d-3}}{r} + \frac{r^2}{R^2}. \hfill (2)$$

Here $r_0$ is related to the black hole mass

$$M = \frac{(d - 2)A_{d-2}r_0^{d-3}}{16\pi G_d},$$

and $A_{d-2}$ is the area of a unit (d-2) sphere.

Quasinormal modes of black holes in asymptotically anti-de-Sitter space time are governed by the wave equation

$$\left(\frac{d^2}{dr_*^2} + \omega^2\right)\Psi(r) = U\Psi(r), \hfill (3)$$

where the potential $U$ is given by

$$U(r) = f(r)\left(\frac{l(l + d - 3)}{r^2} - \frac{(2 - d)(d - 4)}{4r^2}f(r) + \frac{2 - d}{2r}f'(r)\right), \hfill (4)$$

and we take $\omega = \omega_{Re} - i\omega_{Im}$. The tortoise coordinate is $dr* = f^{-1}(r)dr$, $l$ is the angular harmonic index. By rescaling of $r$ we can put $R = 1$. It is essential that the effective potential is infinite at spatial infinity. Thus the wave function vanishes at infinity and satisfies the purely in-going wave condition at the black hole horizon.

We managed to computed the quasinormal modes for the $d = 4$ black hole with the radius up to $r_+ = 1/30R$ (see Fig1-2). This reasonably approximates behavior in small black hole regime. It proved out that the oscillation frequency falls down to some
minimum, and then begins to grow approaching \( d - 1 \) when \( r_+ \to 0 \) (see Fig 2). This minimum of the \( \text{Re}\omega \) equals

\[
\min(\omega_{\text{Re}}) \approx 2.362868 \quad \text{at} \quad r_+ = 0.395R, \quad d = 4
\]

\[
\min(\omega_{\text{Re}}) \approx 3.705140 \quad \text{at} \quad r_+ = 0.341R, \quad d = 5,
\]

and (according to our preliminary results) continues to increase for higher dimensions. Herewith the more dimensional AdS space is the less black hole radius \( r_+ \) at which the real oscillation frequency attains its minimum. Upon thorough consideration of the Fig.1 of the paper [9] one can learn that among the real oscillation frequencies corresponding to \( r_+ = 0.2, 0.4, 0.6, 0.8R, \omega_{\text{Re}} \) at \( r_+ = 0.4R \) is the least. This agrees with our computations showing the minimum of \( \omega_{\text{Re}} \) at \( r_+ \approx 0.395R \) for a four dimensional black hole. The imaginary part of \( \omega \) fall down to zero, and the closer \( r_+ \) to zero the better the corresponding plot can be fit by the function \( Ar_+^2 \). For \( d = 4 \) the best fit for the last five points (from \( r_+ = 1/16 \) to \( r_+ = 1/30 \)) in the Tab.1 is \( \omega_{\text{Im}} = 8.06653r_+^2 \). The higher the dimensions are the less \( \omega_{\text{Im}} \) of small black holes is, i.e. the more the damping time of a perturbation.

| \( r_+ \) | \( \omega_{\text{Re}} \)  | \( \omega_{\text{Im}} \) |
|-------|-----------------|-----------------|
|  0.3  | 2.38447         | 0.70413         |
|  0.25 | 2.41945         | 0.54735         |
|   0.2 | 2.47511         | 0.38999         |
| 0.125 | 2.62274         | 0.16392         |
|   0.1 | 2.69282         | 0.10096         |
|1/12  | 2.74472         | 0.06616         |
|1/14  | 2.78341         | 0.04578         |
|1/16  | 2.81289         | 0.03311         |
|1/18  | 2.83574         | 0.02491         |
|1/20  | 2.8539          | 0.01932         |
|1/25  | 2.88584         | 0.01138         |
|1/30  | 2.9065          | 0.0074          |

Tab.1 The fundamental quasi-normal frequencies corresponding to \( d = 4 \) SAdS black hole, \( R = 1 \).

Thus, even though the boundary conditions at \( r = r_+ \) do not reduce to regularity at the origin in the limit \( r_+ \to 0 \), the quasinormal modes approach the usual AdS modes [16] in this limit (we checked it out for \( d = 4, 5 \)), i.e. \( \omega_{\text{Re}} \to d - 1 \) and \( \omega_{\text{Im}} \to 0 \), as was discussed in [3].
Figure 1: Imaginary part of $\omega$ for $d = 4$ black hole, $l = 0$, $n = 0$.

Figure 2: Real part of $\omega$ for $d = 4$ black hole, $l = 0$, $n = 0$.

Figure 3: Real part of $\omega$ for $d = 4$ black hole near its minimum, $l = 0$, $n = 0$. 
Figure 4: Real part of $\omega$ for $d = 5$ black hole near its minimum, $l = 0$, $n = 0$.

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Appendix

When computing the quasinormal mode one has to truncate the sum representing the wave function

$$\psi(x) = \sum_{n=0}^{\infty} a_n (x - x_+)^n, \quad x = \frac{1}{r}$$

with some large $N$ and find the roots of the equation $|\psi(x)| = 0$ at $r = \infty (x = 0)$. After simplification the truncated sum \( (5) \) takes the polynomial in $\omega$ form, and the necessary roots can easily be found by Mathematica. However this reduction to a polynomial form takes a lot of computer time and can not be performed for a sum of order $N \sim 200$ or more. Thus for small black holes we have to use the trial and error method. Herewith there is a danger of missing the fundamental mode we are seeking, and, of ”catching” another overtone. Fortunately, for reasonably small black holes there are no other overtones close to the fundamental one, and the minimums of $|\psi|$ are sufficiently widely separated.

Another difficulty is that for small values of $r_+$ the initial tiny errors when determining the quantities involved in the sum in \( (5) \) (namely $u_i$, $t_i$, and $s_i$ of the paper \( [5] \)) begin to grow when coming to $N$ of order 1000 or greater. Therefore one has to improve precision of these quantities (with the help of a build-in function of Mathematica) up until further increasing of precision will not influence the result. It proved out, that the 50-digital precision of $u_i, t_i, s_i$ is quite enough in this paper. In addition on must set higher precision of recurrence relations for coefficients $a_n$. When approaching the limit $r_+ = 0$ the number $N$ of the truncated sum \( (5) \) at which an approximate frequency converge grows as is shown on the Fig.5 for $d = 4$. The more $d$, the more the number $N$ giving good approximation of the frequency. When following all these receptions one can be sure that the convergence plot will be smooth and that at small changing of $r_+$ the corresponding frequency will not change noisily.