A cohomological proof that real representations of semisimple Lie algebras have $\mathbb{Q}$-forms

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Abstract. A Lie algebra $\mathfrak{g}_\mathbb{Q}$ over $\mathbb{Q}$ is said to be $\mathbb{R}$-universal if every homomorphism from $\mathfrak{g}_\mathbb{Q}$ to $\mathfrak{gl}(n, \mathbb{R})$ is conjugate to a homomorphism into $\mathfrak{gl}(n, \mathbb{Q})$ (for every $n$). By using Galois cohomology, we provide a short proof of the known fact that every real semisimple Lie algebra has an $\mathbb{R}$-universal $\mathbb{Q}$-form. We also provide a description of the $\mathbb{R}$-universal $\mathbb{Q}$-forms of each compact, simple real Lie algebra.

Key words: semisimple Lie algebra; finite-dimensional representation; global field; Galois cohomology; linear algebraic group; Tits algebra

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1 Introduction

Definition 1.1. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{Q}$. (All Lie algebras and all representations are assumed to be finite-dimensional.)

1. $\mathfrak{g}$ is universal for real representations (or $\mathbb{R}$-universal, for short) if every real representation of $\mathfrak{g}$ has a $\mathbb{Q}$-form [6, Defn. 7.1]. This means that if $\rho: \mathfrak{g} \to \mathfrak{gl}(n, \mathbb{R})$ is any (Q-linear) Lie algebra homomorphism, then there exists $M \in \text{GL}(n, \mathbb{R})$, such that $M \rho(x) M^{-1} \in \mathfrak{gl}(n, \mathbb{Q})$, for every $x \in \mathfrak{g}$.

2. $\mathfrak{g}$ is a $\mathbb{Q}$-form of a real Lie algebra $\mathfrak{g}_\mathbb{R}$ if $\mathfrak{g} \otimes_\mathbb{Q} \mathbb{R} \cong \mathfrak{g}_\mathbb{R}$.

This note uses Galois cohomology to present a short proof of the following known result, which was first obtained by M.S. Raghunathan [9, §3] in the important special case where $\mathfrak{g}_\mathbb{R}$ is compact.

Proposition 1.2 ([6, Thm. 1.2]). Every real semisimple Lie algebra has a $\mathbb{Q}$-form that is $\mathbb{R}$-universal.

The proof in [6] constructs an $\mathbb{R}$-universal $\mathbb{Q}$-form explicitly, and is rather tedious, but a much nicer proof was given by G. Prasad and A. Rapinchuk [8, Prop. 3 and Rem. 3]. Assuming some fundamental results of J. Tits [11], our proof in Section 2 is a bit shorter and more direct. (On the other hand, we provide less information about the $\mathbb{Q}$-form than is supplied in [8].)

Section 3 gives an explicit characterization of the $\mathbb{R}$-universal $\mathbb{Q}$-forms of every compact, simple, real Lie algebra.

Due to the well-known correspondence between $\mathbb{Q}$-forms and arithmetic subgroups [11, Defn. 7.11, p. 49], Proposition 1.2 has the following consequence in the theory of discrete subgroups:
Corollary 1.3. Let $G$ be a connected, semisimple Lie group with finite center. Then there is a discrete subgroup $\Gamma$ of $G$, such that

1. $G/\Gamma$ has finite volume (so $\Gamma$ is a “lattice” in $G$), and

2. if $\rho: G \to \text{GL}(n, \mathbb{R})$ is any finite-dimensional representation of $G$, then $\rho(\Gamma)$ is conjugate to a subgroup of $\text{GL}(n, \mathbb{Z})$.

2 Proof of the main result

We begin by recalling a result of J. Tits that uses Galois cohomology to characterize the irreducible representations of semisimple algebraic groups over fields that are not algebraically closed.

Definition 2.1 ([11, §4.2]). Suppose $g$ is a semisimple Lie algebra over a subfield $F$ of $\mathbb{C}$, and let $G$ be the corresponding simply connected, semisimple algebraic group over $F$. It is well known that there is a (unique) quasi-split algebraic group $G^q$ over $F$, and a 1-cocycle $\xi: \text{Gal}(\overline{F}/F) \to \overline{G}^q$, where $\overline{G}^q$ is the adjoint group of $G^q$, such that $G$ is $F$-isomorphic to the Galois twist $\xi G^q$. Although the cocycle $\xi$ is not unique, it represents a well-defined cohomology class $[\xi] \in H^1(F; \overline{G}^q)$.

Letting $Z(G^q)$ be the center of $G^q$, the short exact sequence $e \to Z(G^q) \to G^q \to \overline{G}^q \to e$ yields a corresponding long exact sequence of Galois cohomology sets, including a connecting map $\delta_*: H^1(F; G^q) \to H^2(F; Z(G^q))$. Hence, we have a cohomology class $\delta_*[\xi] \in H^2(F; Z(G^q))$.

Now, fix a maximal $F$-torus $T$ of $G^q$ that contains a maximal $F$-split torus, and suppose $\lambda$ is a weight of $T$ that is invariant under the $*$-action of the Galois group $\text{Gal}(\overline{F}/F)$. Then the restriction of $\lambda$ to $Z(G^q)$ is a $\text{Gal}(\overline{F}/F)$-equivariant homomorphism from $Z(G^q)$ to the group $\mu$ of roots of unity in $\mathbb{C}$, so it induces a homomorphism $\lambda_*: H^2(F; Z(G^q)) \to H^2(F; \mu)$. Therefore, we may define

$$\beta_{\rho,F}(\lambda) = \lambda_* \delta_*[\xi] \in H^2(F; \mu).$$

Proposition 2.2 (Tits [11, Thm. 7.2 and Lem. 7.4]). Suppose $g$ is a semisimple Lie algebra over a subfield $F$ of $\mathbb{C}$, and $\lambda$ is a dominant weight. Then:

1. There is an irreducible representation $F_{\rho,\lambda}: g \to \mathfrak{gl}(n, F)$, for some $n$, such that $F_{\rho,\lambda} \otimes_F \mathbb{C}$ has an irreducible summand with highest weight $\lambda$. Furthermore, $F_{\rho,\lambda}$ is unique up to isomorphism.

2. $F_{\rho,\lambda_1} \cong F_{\rho,\lambda_2}$ if and only if $\lambda_1$ and $\lambda_2$ are in the same orbit of the $*$-action of $\text{Gal}(\overline{F}/F)$.

3. $F_{\rho,\lambda} \otimes_F \mathbb{C}$ is irreducible if and only if:

   (a) $\lambda$ is invariant under the $*$-action of $\text{Gal}(\overline{F}/F)$, and

   (b) $\beta_{\rho,F}(\lambda)$ is the trivial element of $H^2(F; \mu)$. 

Corollary 2.3. Suppose \( g \) is a semisimple Lie algebra over \( \mathbb{Q} \), such that \( g \) splits over a quadratic extension of \( \mathbb{Q} \). Let \( T \) be a maximal \( \mathbb{Q} \)-torus of the corresponding simply connected algebraic group \( G \), such that \( T \) contains both a maximal \( \mathbb{Q} \)-split torus and a maximal \( \mathbb{R} \)-split torus. Then \( g \) is \( \mathbb{R} \)-universal if and only if, for every dominant weight \( \lambda \) of \( T \):

\[
\begin{align*}
\text{if } \lambda &\text{ is invariant under the } *\text{-action of } \text{Gal}(\mathbb{C}/\mathbb{R}), \text{ and } \beta_{g,\mathbb{R}}(\lambda) = 0, \\
\text{then } \lambda &\text{ is also invariant under the } *\text{-action of } \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \text{ and } \beta_{g,\mathbb{Q}}(\lambda) = 0.
\end{align*}
\]

(2.4)

Proof. We prove only (\( \Leftarrow \)), but the argument is reversible. We wish to show that \( \mathbb{Q}g_\lambda \) is a \( \mathbb{Q} \)-form of \( \mathbb{R}g_\lambda \), for every dominant weight \( \lambda \). By combining (2.4) with Proposition 2.2(3), we see that \( \mathbb{Q}g_\lambda \otimes \mathbb{C} \) is irreducible if and only if \( \mathbb{R}g_\lambda \otimes \mathbb{C} \) is irreducible. Furthermore, since \( g_\mathbb{Q} \) splits over a quadratic extension, we know that \( \mathbb{Q}g_\lambda \otimes \mathbb{C} \) is either irreducible or the direct sum of two irreducibles \([6]\) Cor. 3.2(2)]. Therefore, \( \mathbb{Q}g_\lambda \otimes \mathbb{C} \) and \( \mathbb{R}g_\lambda \otimes \mathbb{C} \) have the same number of irreducible constituents. (Namely, either they are both irreducible, or they are both the direct sum of 2 irreducibles.) Since \( \mathbb{R}g_\lambda \) is a summand of \( \mathbb{Q}g_\lambda \otimes \mathbb{R} \), this implies that \( \mathbb{R}g_\lambda \cong \mathbb{Q}g_\lambda \otimes \mathbb{R} \), so \( \mathbb{Q}g_\lambda \) is a \( \mathbb{Q} \)-form of \( \mathbb{R}g_\lambda \). \( \Box \)

We will also use the following (weak form of an) important fact in the theory of Galois cohomology:

Proposition 2.5 (Kneser [3] Thm. 5.1b, p. 77, [7] Prop. 6.17, p. 337]). If \( G \) is a connected, semisimple algebraic group over an algebraic number field \( F \subset \mathbb{R} \), and \( G \) splits over a finite, Galois extension \( L \) of \( F \), with \( L \subset \mathbb{C} \), but \( L \not\subset \mathbb{R} \), then the restriction map

\[
H^1(L/F; G(L)) \to H^1(\mathbb{R}; G)
\]

is surjective.

Proof of Proposition 1.2. Suppose \( g_\mathbb{R} \) is a real semisimple Lie algebra, and let \( G \) be the simply connected, semisimple \( \mathbb{R} \)-algebraic group whose Lie algebra is \( g_\mathbb{R} \). Write \( G = G^\mathbb{C} \), where \( \xi : \text{Gal}(\mathbb{C}/\mathbb{R}) \to G^\mathbb{C} \) is a 1-cocycle and \( G^\mathbb{C} \) is quasi-split. Let \( L = \mathbb{Q}[\bar{\xi}] \). By choosing an appropriate \( \mathbb{Q} \)-form, we may assume that \( G^\mathbb{Q} \) is a quasi-split \( \mathbb{Q} \)-group that splits over \( L \), and that the \( * \)-action of \( \text{Gal}(L/\mathbb{Q}) \) is the same as the \( * \)-action of \( \text{Gal}(\mathbb{C}/\mathbb{R}) \).

Let \( \sigma \) be the nontrivial element of \( \text{Gal}(\mathbb{C}/\mathbb{R}) \), fix a representative \( a \in G^\mathbb{Q} \) of \( \xi(\sigma) \in G^\mathbb{Q} \), and let \( z = a^\sigma a \). Since \( a^2 \) is trivial and \( \xi \) is a 1-cocycle, we know that \( z \) is trivial in \( G^\mathbb{Q} \), which means \( z \in Z(G^\mathbb{Q}) \). This implies that \( a \) commutes with \( \sigma \), so \( \sigma \) fixes \( z \), which means \( z \in Z(G^\mathbb{Q})(\mathbb{R}) \).

Let \( H \) be the product of the almost simple factors of \( G^\mathbb{Q} \) that are absolutely almost simple and of type \( ^2A_n \) (more concretely, \( H \) is the product of the factors that are isomorphic to \( \text{SU}(k, \ell) \), for some \( k \) and \( \ell \)), let \( Z(H)^2 = \langle w^2 \mid w \in Z(H) \rangle \), let \( G^\mathbb{Q} = G^\mathbb{Q}/Z(H)^2 \), and let \( \bar{z} \) be the image of \( z \) in \( G^\mathbb{Q} \). Note that \( G^\mathbb{Q} \) is a \( \mathbb{Q} \)-group (since \( Z(H)^2 \) is a \( \mathbb{Q} \)-subgroup of \( G^\mathbb{Q} \)).

We claim that we may assume \( \bar{z} \in Z(G^\mathbb{Q})(\mathbb{Q}) \). While proving this, we may consider each simple factor individually, so there is no harm in assuming \( G^\mathbb{Q} \) is almost simple. This allows us to furthermore assume that \( G^\mathbb{Q} \) is absolutely almost simple. (Otherwise, since every \( \mathbb{C} \)-group is split, we could assume \( \xi \) is trivial.) Also, since \( |\text{Gal}(\mathbb{C}/\mathbb{R})| = 2 \), we may assume, by replacing \( a \) with \( aw \) for an appropriately chosen \( w \in \langle z \rangle \), that \( |\bar{z}| \) is a power of 2. Assuming, as we may, that \( \bar{z} \) is nontrivial, this implies that \( G^\mathbb{Q} \) is not of type \( ^{1,2}E_6 \).
Then \(Z(G^q)(\mathbb{R}) = Z(G^q)(\mathbb{Q})\): if \(G^q\) is not of type \(^2A_n\), then this can be seen from the table on page 332 of [7], but if not, then the definition of \(G^q\) implies \(|Z(G^q)| \leq 2\), so every element of \(Z(G^q)\) is defined over \(\mathbb{Q}\). This completes the proof of the claim.

The claim of the preceding paragraph implies that the cyclic subgroup \((z)\) generated by \(z\) is defined over \(\mathbb{Q}\). Hence, the quotient \(G^q = G^q/\langle z, Z(H)^2 \rangle\) is a semisimple \(\mathbb{Q}\)-group. Now, since \(a^\omega = z\) is trivial in \(G^q\), we know that \(\xi\) lifts to a 1-cocycle \(\tilde{\xi}: \text{Gal}(\mathbb{C}/\mathbb{R}) \rightarrow \tilde{G}^q\). Then Proposition 2.3 implies that, after replacing \(\tilde{\xi}\) with a cohomologous cocycle, we may assume \(\tilde{\xi}\) is the restriction of a 1-cocycle \(\zeta: \text{Gal}(L/\mathbb{Q}) \rightarrow G^q(L)\). Let \(G^q = \zeta G^q\), so \(G^q\) is a \(\mathbb{Q}\)-group that is \(L\)-split, and is isomorphic to \(G\) over \(\mathbb{R}\). Also, let \(g\) be the Lie algebra of \(G^q\).

To complete the proof, we show that \(g\) is \(\mathbb{R}\)-universal, by verifying (2.4). To this end, let \(\lambda\) be a \(\text{Gal}(\mathbb{C}/\mathbb{R})\)-invariant dominant weight, such that \(\beta_{g, \mathbb{R}}(\lambda) = 0\). Since \(G^q\) is \(L\)-split and the \(*\)-action of \(\text{Gal}(L/\mathbb{Q})\) is the same as the \(*\)-action of \(\text{Gal}(\mathbb{C}/\mathbb{R})\) (by the choice of the \(\mathbb{Q}\)-form of \(G^q\)), we know that \(\lambda\) is invariant under \(\text{Gal}(\mathbb{C}/\mathbb{R})\).

Since \(\zeta\) is a 1-cocycle into \(\tilde{G}^q = G^q/\langle z, Z(H)^2 \rangle\), we know that, in the notation of Definition 2.1 with \(F = \mathbb{Q}\), we have \(\delta_\omega[\zeta] \in H^2(\mathbb{Q}; \langle z, Z(H)^2 \rangle)\). Therefore, in order to show that \(\beta_{g, \mathbb{Q}}(\lambda) = \lambda_\omega \delta_\omega[\zeta] = 0\), it suffices to show that \(\lambda\) is trivial on both \(z\) and \(Z(H)^2\). Note that, in the notation of Definition 2.1 with \(F = \mathbb{R}\), we have \(\lambda_\omega \delta_\omega[\zeta] = \beta_{g, \mathbb{R}}(\lambda) = 0\).

Under the natural identification of \(H^2(\mathbb{R}; Z(G^q))\) with \(\{ w \in Z(G^q) \mid \omega w = w \}/\{ w \omega w \mid w \in Z(G^q) \}\), we have \(\delta_\omega[\zeta] = [z]\), so this means \(\lambda(z) = 1\) (since \(\omega \bar{w} = 1\) for all \(\omega \in \mu\)). Furthermore, since the restriction of \(\lambda\) to \(Z(G^q)\) is a \(\text{Gal}(\mathbb{C}/\mathbb{R})\)-equivariant homomorphism, and \(Z(H)(\mathbb{R}) = Z(H)\) (cf. [7] p. 332)), we have \(\lambda(Z(H)) \subseteq \mu(\mathbb{R}) = \{\pm 1\}\), so \(\lambda\) is also trivial on \(Z(H)^2\).

\section{\(\mathbb{R}\)-universal \(\mathbb{Q}\)-forms of compact, simple Lie algebras}

This section describes the \(\mathbb{R}\)-universal \(\mathbb{Q}\)-forms of each compact, simple Lie algebra. (This problem was proposed by the author in [6, p. 485].) Many cases (including most of the exceptional types) are handled by the corollary of the following observation.

**Proposition 3.1** (cf. [6, §7]). Suppose \(g_\mathbb{R}\) is a compact, simple Lie algebra over \(\mathbb{R}\). There is a \(\mathbb{Q}\)-form \(g_\mathbb{Q}\) of \(g_\mathbb{R}\) such that \(g_\mathbb{Q}\) splits over some quadratic extension of \(\mathbb{Q}\), but is not \(\mathbb{R}\)-universal, if and only if either

\begin{enumerate}
  \item \(g_\mathbb{R} \cong \text{su}(n)\), for some \(n\) that is divisible by 4, or
  \item \(g_\mathbb{R} \cong \text{so}(n)\), for some \(n \neq 3, 5 \pmod{8}\) (with \(n \geq 6\)).
\end{enumerate}

**Erratum 3.2.** Proposition 3.1 is stated incorrectly in [6, §7]. The error is in Prop. 7.3(a), where \(\ell\) is required to only be odd, whereas it actually needs to be \(\equiv 3 \pmod{4}\). This means that \(g\), the compact real form of type \(A_\ell\), is isomorphic to \(\text{su}(n)\), for some \(n\) that is divisible by 4. In [6, §7], it is incorrectly stated that \(n\) only needs to be even, not divisible by 4.

**Corollary 3.3.** Suppose \(g_\mathbb{R}\) is a compact, simple Lie algebra over \(\mathbb{R}\). If \(g_\mathbb{R}\) is of type \(C_n\), \(E_7\), \(E_8\), \(F_4\), or \(G_2\), then every \(\mathbb{Q}\)-form of \(g_\mathbb{R}\) is \(\mathbb{R}\)-universal.
Proof. Every Lie algebra of any of these types (over an algebraic number field) splits over an appropriate quadratic extension \([7, \text{Prop. 6.16, p. 335}]\), and \(g_5\) does not appear in Proposition 3.1. (Lie algebras of type \(B_n\) also split over a quadratic extension, but they are the Lie algebras in 3.1(b) with \(n\) odd.)

In the remainder of this section, we determine exactly which \(\mathbb{Q}\)-forms are \(\mathbb{R}\)-universal for each of the other types of compact simple Lie algebras: \(\mathfrak{su}(n)\) (Proposition 3.7), \(\mathfrak{so}(n)\) (Proposition 3.9), \(\mathfrak{so}(n,1)\) (Lemma 3.12 and Proposition 3.13), and \(\mathfrak{e}_6\) (Proposition 3.14). The results for classical groups can be obtained quite easily from the calculations of \(\beta_g,F(\lambda)\) in \([4, \S 27.B, \text{pp. 378–379}]\), and the answer for \(\mathfrak{e}_6\) is immediate from observations of Tits \([11, \S 6.4]\).

We will use the following concrete interpretation of \(\beta_g,F(\lambda)\):

**Proposition 3.4** (Tits \([11, \text{Cor. 3.5, \S 4.2, and Lem. 7.4}]\)). Suppose \(g\) is a semisimple Lie algebra over a field \(F\) of characteristic 0, and \(\lambda\) is a dominant weight. Let \(L\) be the center of \(D_{g,F}(\lambda) = \text{End}_g(F^\lambda)\) (which, by Schur's Lemma, is a division algebra). Then:

1. \(\lambda\) is invariant under the \(*\)-action of \(\text{Gal}(\overline{L}/L)\), and
2. \(\beta_{g,L}(\lambda) = [D_{g,F}(\lambda)]\), after identifying \(H^2(L; \mu)\) with the Brauer group of \(L\).

Since every root of \(G^q\) is trivial on the center, the following observation is immediate from Definition 2.1.

**Lemma 3.5** (Tits \([11, \text{Thm. 3.3}]\)). Suppose \(\lambda\) is a dominant weight of a semisimple Lie algebra \(g\) over a field \(F\) of characteristic zero. If \(\lambda\) is in the root lattice, then \(\beta_{g,F}(\lambda) = 0\).

### 3A Q-forms of \(\mathfrak{su}(n)\)

**Lemma 3.6.** Suppose \(g = \mathfrak{su}_n(A; L, \tau)\), where

- \(L\) is a quadratic extension of a field \(F\) of characteristic zero,
- \(\tau\) is the nontrivial element of \(\text{Gal}(L/F)\),
- \(A\) is a \(\tau\)-Hermitian matrix in \(\text{GL}_n(L)\), and
- \(n\) is divisible by 4.

Then, for any \(\text{Gal}(L/F)\)-invariant weight \(\lambda\) of \(g\), we have \(\beta_{g,F}(\lambda) = 0\) if and only if either 
\(\text{det} A = 0\) is a norm in \(L\), or \(\lambda\) is in the root lattice.

**Proof.** We may assume that \(\lambda\) is not in the root lattice, for otherwise Lemma 3.5 tells us that \(\beta_{g,F}(\lambda) = 0\).

Let \(L = F[\sqrt{a}]\) and \(b = \text{det} A\). Since \(n\) is divisible by 4, we have \(b = (-1)^{n(n-1)/2} \text{det} A\), so \([4, \text{SU}(B, \tau)\) on p. 378, and Cor. 10.35 on p. 131] (and Proposition 3.4) tells us that \(\beta_{g,F}(\lambda)\) is represented by the quaternion algebra \((a,b)_F\). This is trivial in the Brauer group if and only if it is split, which means that \(b\) is a norm in \(L = F[\sqrt{a}]\).

**Proposition 3.7.** Suppose \(g\) is a \(\mathbb{Q}\)-form of \(\mathfrak{su}(n)\), for some \(n \geq 2\), so \(g = \mathfrak{su}_k(A; D, \tau)\), where
• \( D \) is a central division algebra of some degree \( d \) over an imaginary quadratic extension \( L \) of \( \mathbb{Q} \),

• \( \tau \) is an involution of \( D \), such that the restriction of \( \tau \) to \( L \) is nontrivial, and

• \( A \) is a \( \tau \)-Hermitian matrix in \( \text{GL}_k(D) \), where \( n = kd \).

Then \( g \) is \( \mathbb{R} \)-universal if and only if

1. \( D = \mathbb{R} \) is a field, so \( A \) is a \( \tau \)-Hermitian matrix in \( \text{GL}_n(L) \), and

2. either \( n \) is not divisible by 4, or \( \det A \) is the norm of some element of \( L \).

Proof. \(( \Leftarrow \) From (1), we see that \( g = \mathfrak{su}_n(A; L, \tau) \), so \( g \) is \( L \)-split. Hence, we may assume \( n \) is divisible by 4, for otherwise Proposition 3.1 implies the desired conclusion that \( g \) is \( \mathbb{R} \)-universal. Then, since \( \text{Gal}(\mathbb{C}/\mathbb{R}) \) and \( \text{Gal}(L/\mathbb{Q}) \) have the same \( * \)-action (and \( \det A \) is a norm in \( L \)), Lemma 3.6 implies that \( \beta_{g, \mathbb{Q}}(\lambda) = 0 \) for every \( \text{Gal}(\mathbb{C}/\mathbb{R}) \)-invariant weight \( \lambda \). This establishes (2.4), so \( g \) is \( \mathbb{R} \)-universal.

\(( \Rightarrow \) The natural representation \( \rho \colon g \hookrightarrow \text{Mat}_{k \times k}(D) \) is irreducible over \( \mathbb{Q} \). Since \( g \) is \( \mathbb{R} \)-universal, this representation must remain irreducible over \( \mathbb{R} \). By Schur’s Lemma, this implies that \( D \otimes_{\mathbb{Q}} \mathbb{R} \) has no zero divisors. On the other hand, \( D \otimes_{\mathbb{Q}} \mathbb{R} \) is split (since its center is \( \mathbb{L} \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{C} \)). Therefore \( D \) must be a field, so \( D = \mathbb{R} \). This establishes (1).

Let \( \lambda \) be a \( \text{Gal}(\mathbb{C}/\mathbb{R}) \)-invariant weight that is not in the root lattice. If \( n \) is divisible by 4, then \( (-1)^{n(n-1)/2} \det I = 1 \) is a square in \( \mathbb{R} \), so Lemma 3.6 tells us that \( \beta_{g, \mathbb{R}}(\lambda) = 0 \). Then, since \( g \) is \( \mathbb{R} \)-universal, we must have \( \beta_{g, Q}(\lambda) = 0 \). So Lemma 3.6 tells us that \( \det A \) must be a norm in \( L \). This establishes (2). \( \blacksquare \)

3B \( \mathbb{Q} \)-forms of \( \mathfrak{so}(n) \)

Notation 3.8. Any symmetric matrix \( A \in \text{GL}_k(\mathbb{Q}) \) determines a nondegenerate quadratic form on \( \mathbb{Q}^k \). We use \( \text{Cliff}^0_\mathbb{Q}(A) \) to denote the corresponding even Clifford algebra [5, p. 104].

It is well known that \( \text{Cliff}^0_\mathbb{Q}(A) \) is either a simple algebra or the direct sum of two isomorphic simple algebras over \( \mathbb{Q} \) [5, Thms. 2.4 and 2.5, p. 110]. If \( A \) has been diagonalized, then it is straightforward to determine whether this simple algebra is split (in which case, we also say that \( \text{Cliff}^0_\mathbb{Q}(A) \) is split). Namely, the simple algebra is Brauer equivalent to a quaternion algebra that can be calculated from the eigenvalues of \( A \) (cf. [5, Cor. 3.14, p. 117]).

Proposition 3.9. \( \text{Suppose } g \text{ is a } \mathbb{Q} \text{-form of } \mathfrak{so}(n), \text{ for some odd } n \geq 5, \text{ so } g = \mathfrak{so}_n(A; \mathbb{Q}), \text{ where } A \text{ is a symmetric matrix in } \text{GL}_n(\mathbb{Q}). \text{ Then } g \text{ is } \mathbb{R} \text{-universal if and only if either } n \equiv \pm 3 \pmod{8} \text{ or } \text{Cliff}^0_\mathbb{Q}(A) \text{ is split.} \)

Proof. Since \( g \) is of type \( B_k \) (where \( 2k + 1 = n \)), we know that it splits over a quadratic extension \( L \) of \( \mathbb{Q} \) [7, Prop. 6.16(2), p. 335]. Hence, we may assume \( n \equiv \pm 1 \pmod{8} \), for otherwise Proposition 3.1 implies that \( g \) is \( \mathbb{R} \)-universal.

If \( \lambda \) is any dominant weight of \( g \) that is not in the root lattice, then the proof of Proposition 3.1 tells us that \( \beta_{g, \mathbb{R}}(\lambda) = 0 \). (This can also be deduced from [4, §27B, type \( B_n \), p. 378].) Furthermore, [4, §27B, type \( B_n \), p. 378] shows that \( \beta_{g, \mathbb{Q}}(\lambda) \) is Brauer equivalent to \( \text{Cliff}^0_\mathbb{Q}(A) \). Hence, we conclude from (2.4) that \( g \) is \( \mathbb{R} \)-universal if and only if \( \text{Cliff}^0_\mathbb{Q}(A) \) is split. \( \blacksquare \)
Before discussing the $\mathbb{Q}$-forms of $\mathfrak{so}(n)$ when $n$ is even, we record a few simple observations.

**Lemma 3.10.** The following are equivalent:

1. $\mathfrak{g}$ is $\mathbb{R}$-universal.
2. $\mathbb{Q} \rho_\lambda \otimes_\mathbb{Q} \mathbb{R}$ is irreducible, for every dominant weight $\lambda$.
3. $D_{\mathfrak{g},\mathbb{Q}}(\lambda) \otimes_\mathbb{Q} \mathbb{R} \cong D_{\mathfrak{g},\mathbb{R}}(\lambda)$, for every dominant weight $\lambda$.
4. $\dim_\mathbb{Q} D_{\mathfrak{g},\mathbb{Q}}(\lambda) = \dim_\mathbb{R} D_{\mathfrak{g},\mathbb{R}}(\lambda)$, for every dominant weight $\lambda$.

**Proof.** (1)$\iff$(2) $\mathfrak{g}$ is $\mathbb{R}$-universal if and only if $\mathbb{Q} \rho_\lambda$ is a $\mathbb{Q}$-form of $\mathbb{R} \rho_\lambda$, for every dominant weight $\lambda$.

(2)$\iff$(3) We have $\text{End}_\mathfrak{g}(\mathbb{Q} \rho_\lambda \otimes_\mathbb{Q} \mathbb{R}) = D_{\mathfrak{g},\mathbb{Q}}(\lambda) \otimes_\mathbb{Q} \mathbb{R}$. This is a division algebra if and only if $\mathbb{Q} \rho_\lambda \otimes_\mathbb{Q} \mathbb{R}$ is irreducible (and hence equal to $\mathbb{R} \rho_\lambda$).

(3)$\iff$(4) $D_{\mathfrak{g},\mathbb{Q}}(\lambda) \otimes_\mathbb{Q} \mathbb{R}$ is a matrix algebra over $D_{\mathfrak{g},\mathbb{R}}(\lambda)$.

**Lemma 3.11.** If $\mathfrak{g}$ is $\mathbb{R}$-universal, then every $\text{Gal}(\mathbb{C}/\mathbb{R})$-invariant dominant weight is also $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant.

**Proof.** Suppose $\lambda$ is invariant under the $\ast$-action of $\text{Gal}(\mathbb{C}/\mathbb{R})$, but not the $\ast$-action of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. By replacing $\lambda$ with an appropriate positive integer multiple, we may assume that $\lambda$ is in the root lattice, so $\beta_\lambda(\mathbb{R}) = 0$ (see Lemma 3.5). Therefore $\beta_\lambda(\mathbb{Q})$ must also be trivial, which means that $D_{\mathfrak{g},\mathbb{Q}}(\lambda) = \mathbb{Q}$. However, this contradicts Proposition 3.4(1), since $\lambda$ is not $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant.

**Lemma 3.12.** If $\mathfrak{g}$ is an $\mathbb{R}$-universal $\mathbb{Q}$-form of $\mathfrak{so}(2k)$, with $k \geq 3$, then $\mathfrak{g} \cong \mathfrak{so}_{2k}(A; \mathbb{Q})$, for some symmetric $A \in \text{GL}_{2k}(\mathbb{Q})$.

**Proof.** Lemma 3.11 implies that triality forms are not $\mathbb{R}$-universal. Therefore, if $\mathfrak{g}$ is not as described, then $\mathfrak{g} = \mathfrak{su}_k(A; D, \tau_r)$, where

- $D$ is a quaternion division algebra over $\mathbb{Q}$,
- $\tau_r$ is the reversion anti-involution, and
- $A$ is a $\tau_r$-Hermitian matrix in $\text{GL}_k(D)$.

We know that $\mathfrak{g} \otimes_\mathbb{Q} \mathbb{R}$ is not isomorphic to $\mathfrak{so}_k(\mathbb{H})$ (since, by assumption, it is isomorphic to $\mathfrak{so}(2k)$. Therefore, $D$ splits over $\mathbb{R}$.

We can now argue as in the proof of Proposition 3.7(\Rightarrow). The natural representation $\rho: \mathfrak{g} \to \text{Mat}_{4\times k}(D)$ is irreducible over $\mathbb{Q}$, and it is obvious that $D \subseteq D_{\mathfrak{g},\mathbb{Q}}(\lambda)$. So the division algebra $D_{\mathfrak{g},\mathbb{R}}(\lambda) = D_{\mathfrak{g},\mathbb{Q}}(\lambda) \otimes_\mathbb{Q} \mathbb{R}$ contains contains $D \otimes_\mathbb{Q} \mathbb{R}$, which is split, and therefore has zero divisors. This is a contradiction.

With Lemma 3.12 in hand, the following proposition characterizes the $\mathbb{R}$-universal $\mathbb{Q}$-forms of $\mathfrak{so}(n)$ when $n$ is even.

**Proposition 3.13.** Let $\mathfrak{g} = \mathfrak{so}_{2k}(A; \mathbb{Q})$, with $k \geq 3$, for some symmetric $A \in \text{GL}_{2k}(\mathbb{Q})$ that is positive-definite over $\mathbb{R}$. Then $\mathfrak{g}$ is $\mathbb{R}$-universal if and only if either
1. \( k \) is odd and \( \text{Cliff}^0_{\mathbb{Q}}(A) \) is split, or

2. \( k \equiv 2 \pmod{4} \), and \( \det A \) is a square in \( \mathbb{Q} \), or

3. \( k \) is divisible by 4, \( \det A \) is a square in \( \mathbb{Q} \), and \( \text{Cliff}^0_{\mathbb{Q}}(A) \) is split.

**Proof.** As in [4, §27B, type \( D_n \), p. 379], let \( \lambda, \lambda_+, \lambda_- \) be dominant weights that represent the three nonzero classes modulo the root lattice. The weight \( \lambda \) corresponds to the natural representation of \( g \) on \( \mathbb{Q}^{2k} \), so \( \beta_{g,\mathbb{Q}}(\lambda) = [\mathbb{Q}] = 0 \).

Suppose, first, that \( k \) is even. Then \( \text{so}(2k) \) is an inner form, which means that \( \text{Gal}(\mathbb{C}/\mathbb{R}) \) acts trivially on \( \Lambda_+ \). Hence, Lemma 3.11 tells us that if \( g \) is \( \mathbb{R} \)-universal, then \( g \) must be inner, which (since \( k \) is even) means that \( \det A \) is a square in \( \mathbb{Q} \). Therefore, \( \text{Cliff}^0_{\mathbb{Q}}(A) \) is a direct sum of two algebras \( C^+ \) and \( C^- \) that are Brauer equivalent to the full Clifford algebra [5, Thm. 2.5(3), p. 110]. Furthermore, from [4, §27B, type \( D_n \), p. 379], we know that \( \beta_{g,\mathbb{Q}}(\lambda_+) = [C^\pm] \). Since \( k \) is even, the Clifford algebra of \( \text{so}(2k) \) is split if and only if \( k \) is divisible by 4 [5, p. 123]. Therefore, Lemma 3.11 shows that an inner form \( g \) is:

- automatically \( \mathbb{R} \)-universal, when \( k \equiv 2 \pmod{4} \), but
- \( \mathbb{R} \)-universal if and only if \( \text{Cliff}^0_{\mathbb{Q}}(A) \) is split, when \( k \) is divisible by 4.

Assume, now, that \( k \) is odd. This means that \( \text{so}(2k) \) is an outer form (so \( g \) is obviously also outer). Then \( \text{Gal}(\mathbb{C}/\mathbb{R}) \) interchanges \( \lambda_+ \) and \( \lambda_- \), so Proposition 3.4(1) implies \( D_{g,\mathbb{R}}(\lambda_\pm) = \mathbb{C} \). Let \( L \) be the (unique, imaginary) quadratic extension of \( \mathbb{Q} \) over which \( g \) becomes inner. From [4, §27B, type \( D_n \), p. 379], we see that \( D_{g,\mathbb{Q}}(\lambda_\pm) \) is Brauer equivalent to \( \text{Cliff}^0_{\mathbb{Q}}(A) \), which is central simple over \( L \). Hence, Lemma 3.10 implies that \( g \) is \( \mathbb{R} \)-universal if and only if \( \text{Cliff}^0_{\mathbb{Q}}(A) \) is split. \( \blacksquare \)

### 3C \( \mathbb{Q} \)-forms of \( ^2E_6 \)

**Proposition 3.14.** Suppose \( g \) is a \( \mathbb{Q} \)-form of the compact real Lie algebra of type \( ^2E_6 \), and let \( L \) be the unique quadratic extension of \( \mathbb{Q} \) over which \( g \) is inner. Then \( g \) is \( \mathbb{R} \)-universal if and only if it splits over \( L \).

**Proof.** Let \( \lambda \) be a weight that is not in the root lattice. Then \( \lambda \) is not fixed by the \(*\)-action of \( \text{Gal}(\mathbb{C}/\mathbb{R}) \), so \( \rho_\lambda^g \otimes_{\mathbb{R}} \mathbb{C} \) is the direct sum of two irreducible representations [11, Lem. 7.4]. Hence, \( g \) is \( \mathbb{R} \)-universal if and only if \( \rho_\lambda^g \otimes_{\mathbb{R}} \mathbb{C} \) is also the direct sum of only two irreducible representations; in other words, \( \beta_{g,L}(\lambda) = 0 \). This obviously holds if \( g \) splits over \( L \).

Now, assume \( g \) does not split over \( L \). Since \( g \) is outer over \( \mathbb{R} \), but inner over \( L \), we know that \( L \) is an imaginary extension, so \( g \) obviously splits at the infinite place of \( L \). Then, by inspection of the possible Tits indices of type \( ^1E_6 \) over a nonarchimedean local field [10, p. 58], we see that the central vertex of the Tits index is circled at every place, so it must be circled over \( L \) [2, Satz 4.3.3]. Therefore, \( g \) must be of type \( ^1E_{6,2}^{16} \) over \( L \) (since it is not split). From [11, 6.4.5], we see that this implies \( \beta_{g,L}(\lambda) \neq 0 \). So \( g \) is not \( \mathbb{R} \)-universal. \( \blacksquare \)

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