Cooperative output feedback tracking control of stochastic nonlinear heterogeneous multi-agent systems

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Summary
We study cooperative output feedback tracking control of stochastic nonlinear heterogeneous leader-following multi-agent systems. Each agent has a continuous-time stochastic nonlinear heterogeneous dynamics with an unmeasurable state, and there are additive and multiplicative noises along with information exchange among agents. We propose admissible distributed observation strategies for estimating the leader’s and the followers’ states, and admissible cooperative output feedback control strategies based on the certainty equivalence principle. By output regulation theory and stochastic analysis, we show if the dynamics of each agent satisfies the Lipschitz condition, and the product of the leader’s Lipschitz coefficient, the intensity of multiplicative measurement noises, and the constant related to the leader’s Lipschitz coefficient and dimension is less than $1/4$ the minimum nonzero eigenvalue of graph Laplacian, then there exist admissible distributed observation and cooperative control strategies to ensure mean square bounded output tracking. Finally, the effectiveness of our control strategies is demonstrated by a numerical simulation.

KEYWORDS
additive and multiplicative measurement noise, heterogeneous multi-agent system, mean square bounded output tracking, nonlinear dynamics

1 | INTRODUCTION

In recent years, many scholars have studied the distributed cooperative control of multi-agent systems in precise communication environments. However, when each agent interacts with its neighbors through the communication network, communication processes are inevitably interfered by random noises due to uncertain communication environment. Now, more and more researchers pay attention to distributed cooperative control of multi-agent systems with random communication noises.

The research on cooperative control of linear multi-agent systems with noises has reached a reasonable degree of maturity. Some scholars have studied nonlinear multi-agent systems with additive noises. Yu et al. investigated fixed-time consensus of first-order nonlinear multi-agent systems. By Lyapunov theory, Xiong et al. studied fixed-time consensus of second-order nonlinear multi-agent systems. Li et al. investigated cluster consensus of high-order nonlinear multi-agent systems under the Markovian topology. Compared with additive noises, multiplicative noises play a stabilizing role in the almost sure stability of systems. There have been some results for nonlinear multi-agent systems...
with multiplicative noises.\textsuperscript{15-20} By graph theory and stochastic Lyapunov theory, Zheng et al.\textsuperscript{15} investigated nonlinear multi-agent systems with first-order integrator dynamics and gave sufficient conditions for finite-time consensus in probability under five different topologies. The nonlinear multi-agent systems with second-order integrator dynamics were studied in References \textsuperscript{16,17}. By sliding mode control, Zhao et al.\textsuperscript{16} gave sufficient conditions for mean square exponential tracking. Based on stochastic finite-time stability theory, Zhao et al.\textsuperscript{17} gave a sufficient condition for finite-time consensus in probability. The nonlinear multi-agent systems with high-order integrator dynamics were studied in References \textsuperscript{18,19}. By the back-stepping method, Li et al.\textsuperscript{18} gave sufficient conditions for practical output tracking. By dynamic output feedback control and introducing an adaptive parameter, You et al.\textsuperscript{19} studied the leader-following consensus problem for high-order stochastic nonlinear multi-agent systems. By Lyapunov function and Halanay-type inequalities, Zong et al.\textsuperscript{20} gave sufficient conditions for mean-square and almost consensus of nonlinear multi-agent systems with time delays.

In practical applications, agents may have different dynamics.\textsuperscript{21} The distributed cooperative control problem of linear and nonlinear heterogeneous multi-agent systems have been studied in References \textsuperscript{22-37}, respectively. Li and Li\textsuperscript{27} gave sufficient conditions for the existence of admissible distributed observation and cooperative control strategies to achieve mean square bounded output tracking. Du et al.\textsuperscript{28} studied distributed fixed-time consensus for nonlinear heterogeneous multi-agent systems with first-order integrator dynamics. The nonlinear heterogeneous multi-agent systems with second-order integrator dynamics were investigated in References \textsuperscript{29,30}. By constructing a linear controller, Su et al.\textsuperscript{29} studied the cooperative output regulation problem. By introducing a nonlinear internal model, Wang et al.\textsuperscript{30} studied the robust nonlinear coordination problem with the leader. The nonlinear heterogeneous multi-agent systems in lower triangular form were studied in References \textsuperscript{31,32}. By introducing a common internal model, Su et al.\textsuperscript{31} transformed the global robust output regulation problem into the global robust stability problem. Liu et al.\textsuperscript{32} studied the adaptive output regulation problem under the switching topology with joint connectivity. The heterogeneous high-order nonlinear multi-agent systems in non-lower triangular form were investigated in References \textsuperscript{33-37}. By the input-output feedback linearization method, Bidram et al.\textsuperscript{33} transformed the nonlinear heterogeneous output consensus problem into the linear heterogeneous output consensus problem. Xiang et al.\textsuperscript{34} studied the output regulation problem under switching topologies. By nonlinear output regulation theory, Isidori et al.\textsuperscript{35} investigated the robust output consensus problem. By the nonlinear internal model principle, Liu et al.\textsuperscript{36} studied the robust output regulation problem under switching topologies. By the internal model principle and the adaptive method, Guo et al.\textsuperscript{37} investigated the cooperative output regulation problem with unknown control directions. By integrating the distributed observer approach and the distributed internal model approach, Cai et al.\textsuperscript{38} investigated the robust output regulation problem over switching networks.

Previously, most of literature on heterogeneous nonlinear multi-agent systems assumed that each agent can get its neighboring information precisely. In this paper, we generalize the results in Li and Li\textsuperscript{27} to the nonlinear case and investigate the cooperative output-feedback control of stochastic heterogeneous nonlinear multi-agent systems. Here, each agent has a continuous-time nonlinear heterogeneous dynamics with unmeasurable states, and there are additive and multiplicative noises along with information exchange among agents. We propose admissible distributed observation strategies for estimating each follower’s state and the leader’s state, and admissible cooperative output feedback control strategies based on the certainty equivalence principle. By output regulation theory and stochastic analysis, we give sufficient conditions on the dynamics of agents, the network graph and the noises for the existence of admissible distributed observation and cooperative control strategies to ensure mean square bounded output tracking. The effectiveness of our control strategies is demonstrated by a numerical simulation.

1. Different from the cooperative control problem of linear heterogeneous multi-agent systems in Li and Li,\textsuperscript{27} the dynamics of the leader and the followers are nonlinear. Then the distributed state observers lead to a complex nonlinear stochastic differential equation of the estimation error for the leader's state. For estimating the state of this equation, a sufficient condition is given, which ensures that the generalized Riccati equation related to the leader's Lipschitz coefficient has a unique positive definite solution. By the inverse of this positive definite solution, we construct an appropriate Lyapunov function, which contains an exponential function related to the leader's Lipschitz coefficient, the leader's dimension, the network topology and noises. By Itô's formula, we obtain a differential equation of the Lyapunov function for the estimation error. Compared with Reference \textsuperscript{27}, this equation contains an additional uncertain dynamics due to the unknown nonlinear part of the leader. By taking the norm of the nonlinear term and using the compatibility of the norm and Lipschitz condition, the nonlinear term is transformed into a new term related to the
Lipschitz coefficient of the leader. Then, by differentiating and scaling the Lyapunov function, we obtain the stochastic differential inequality containing the quadratic drift term associated with the estimate error for the leader state. By integrating both sides of the above inequality and taking the mathematical expectation, we obtain the stochastic integral inequality containing the quadratic drift term. By proving the negative definiteness of this quadratic term, we proved the mean square boundedness of the estimation error. Overall, by developing the above method, the influence of the leader’s nonlinear uncertain dynamics on the upper bound of the state estimation error is analyzed. Compared with You et al.,\textsuperscript{19} whose control laws can ensure mean exponential tracking of stochastic nonlinear multi-agent systems, our control laws can ensure mean square exponential tracking.

We show that if the dynamics of each agent satisfies the Lipschitz condition, and the product of the leader’s Lipschitz coefficient, the intensity of multiplicative measurement noises, and constants related to the leader’s Lipschitz coefficient and the leader’s dimension is less than 1/4 the minimum nonzero eigenvalue of graph Laplacian, then there exist admissible distributed observation and cooperative control strategies to ensure mean square bounded output tracking. In addition, we further give the sufficient conditions for the existence of admissible distributed observation and cooperative control strategies to ensure the mean square bounded output tracking under the Markovian switching topology.

The rest of this paper is arranged as follows. The problem is formulated in Section 2. The cooperative control of heterogeneous nonlinear multi-agent systems under the fixed topology and the Markovian switching topology have been studied in Sections 3 and 4, respectively. A numerical simulation is given in Section 5 to demonstrate the effectiveness of our control laws. The conclusion is given in Section 6.

Notation: Throughout this paper, unless otherwise specified, we use the following notations. The symbols $\mathbb{R}^n$ denotes the set of $n$-dimensional real column vectors; $\mathbb{R}^{m \times n}$ denotes the set of $m \times n$ real matrices; $\eta_{N,1}$ denotes the $N$-dimensional zero vector; $0_{m \times n}$ represents the $m \times n$ dimensional zero matrix; $I_m$ denotes the $m \times m$ identity matrix; $\text{diag}(A_1, \ldots, A_N)$ represents the block diagonal matrix with entries being $A_1, \ldots, A_N$. For a given vector or matrix $X$, $X^T$ denotes its transpose, $\text{Tr}(X)$ denotes its trace, and $\|X\|$ represents its 2-norm. For a given real matrix $A \in \mathbb{R}^{m \times n}$, $\sigma(A)$ denotes the spectrum of $A$, and $\lambda_i(A)(i = 1, \ldots, n)$ represents the $i$th eigenvalue of $A$ arranged in order of ascending real part. For a given complex number $Z$, $\text{Re}(Z)$ represents its real part. For a given real symmetric matrix $B \in \mathbb{R}^{n \times n}$, $\lambda_{\text{min}}(B)$ is the minimum eigenvalue of $B$, and $\lambda_{\text{max}}(B)$ is the maximum eigenvalue of $B$. $A > 0$ (or $A \geq 0$) denotes that $A$ is positive definite (or positive semi-definite) and $A < 0$ (or $A \leq 0$) denotes that $A$ is negative definite (or negative semi-definite). For two real symmetric matrices $A$ and $B$, $A > B$ (or $A \geq B$) denotes that $A - B$ is positive definite (or $A - B$ is positive semi-definite), and $A < B$ (or $A \leq B$) denotes that $A - B$ is negative definite (or $A - B$ is negative semi-definite). For two matrices $C$ and $D$, $C \otimes D$ denotes their Kronecker product. Let $(\Omega, \mathcal{F}, \{F_t\}_{t \geq 0}, \mathbb{P})$ a complete probability space with a filtration $\{F_t\}_{t \geq 0}$ satisfying the usual conditions, namely, it is right continuous and increasing while $F_0$ contains all $\mathbb{P}$-null sets; $W(t) = (w_1(t), \ldots, w_m(t))^T$ denotes a $m$-dimensional standard Brownian motion defined in $(\Omega, \mathcal{F}, \{F_t\}_{t \geq 0}, \mathbb{P})$. For a given random variable $X$, the mathematical expectation of $X$ is denoted by $\mathbb{E}[X]$.

2 Problem Formulations

Consider a leader-following multi-agent system consisting of a leader and $N$ followers, where the leader is indexed by 0 and the $N$ followers are indexed by 1, \ldots, $N$, respectively. The dynamics of the leader is given by

\begin{equation}
\begin{aligned}
\dot{x}_{0,1}(t) &= [x_{0,2}(t) + f_{01}(x_{0,1}(t))]dt, \\
\dot{x}_{0,2}(t) &= [x_{0,3}(t) + f_{02}(x_{0,1}(t), x_{0,2}(t))]dt, \\
&\vdots \\
\dot{x}_{0,n_0}(t) &= [f_{0n_0}(x_{0,1}(t), x_{0,2}(t), \ldots, x_{0,n_0}(t))]dt, \\
y_0(t) &= x_{0,1}(t),
\end{aligned}
\end{equation}

where $x_0(t) = (x_{0,1}(t), x_{0,2}(t), \ldots, x_{0,n_0}(t))^T \in \mathbb{R}^{n_0}$ is the state and $y_0(t) \in \mathbb{R}$ is the output of the leader, respectively; $f_{0i}(\cdot)$ is a nonlinear function, $i = 1, \ldots, n_0$. 

The dynamics of the ith follower is given by

\[
\begin{align*}
\dot{x}_{i,1}(t) &= [x_{i,2}(t) + f_{i1}(x_{i,1}(t)) + \beta_{i,1} u_{i,1}(t)] dt + \beta_{i,1} dw_{i,1}(t), \\
\dot{x}_{i,2}(t) &= [x_{i,3}(t) + f_{i2}(x_{i,1}(t), x_{i,2}(t)) + \beta_{i,2} u_{i,2}(t)] dt + \beta_{i,2} dw_{i,2}(t), \\
&\vdots \\
\dot{x}_{i,n_i}(t) &= [u_i(t) + f_{in_i}(x_{i,1}(t), x_{i,2}(t), \ldots, x_{i,n_i}(t)) + \beta_{i,n_i} dw_{i,n_i}(t)], \\
y_i(t) &= x_{i,1}(t), \quad 1, 2, \ldots, N,
\end{align*}
\]

(2)

where \(x_i(t) = (x_{i,1}(t), x_{i,2}(t), \ldots, x_{i,n_i}(t))^T \in \mathbb{R}^{n_i}\) is the state, \(u_i(t) \in \mathbb{R}\) is the input, and \(y_i(t) \in \mathbb{R}\) is the output of the ith follower, respectively, \(f_i(\cdot)\) is a nonlinear function, \(i = 1, 2, \ldots, N\).


denote \(A_i = \begin{bmatrix} 0_{(n_i-1) \times 1} & I_{n_i-1} \\ 0_{1 \times (n_i-1)} & 0 \end{bmatrix}, \quad C_i = \begin{bmatrix} 1 & 0_{1 \times (n_i-1)} \end{bmatrix}, \quad f_i(x_i(t)) = (f_{i1}(x_{i,1}(t)), f_{i2}(x_{i,1}(t), x_{i,2}(t)), \ldots, f_{in_i}(x_{i,n_i}(t)))^T, \quad i = 0, 1, 2, \ldots, N, \beta_i = \text{diag}(\beta_{i,1}, \beta_{i,2}, \ldots, \beta_{i,n_i}), \quad B_i = \begin{bmatrix} 0_{(n_i-1) \times 1} \end{bmatrix} \text{ and } w_i(t) = (w_{i,1}(t), w_{i,2}(t), \ldots, w_{i,n_i}(t))^T, \quad i = 1, 2, \ldots, N.

Then the systems (1) and (2) can be rewritten in the following forms respectively

\[
\begin{align*}
\dot{x}_0(t) &= [A_0 x_0(t) + f_0(x_0(t))] dt, \\
y_0(t) &= C_0 x_0(t),
\end{align*}
\]

(3)

and

\[
\begin{align*}
\dot{x}_i(t) &= [A_i x_i(t) + B_i u_i(t) + f_i(x_i(t))] dt + \beta_i dw_i(t), \\
y_i(t) &= C_i x_i(t), \quad i = 1, 2, \ldots, N.
\end{align*}
\]

(4)

Remark 1. By the definition of \(A_0\) and \(C_0\) in (3), it can be verified that \((A_0, C_0)\) is observable. By the definition of \(A_i, B_i \text{ and } C_i\) in (4), it can be verified that \((A_i, B_i)\) is stabilizable and \((A_i, C_i)\) is detectable. Therefore, the linear part of the system (3) is observable and the linear part of the system (4) is stabilizable and detectable.

3 | COOPERATIVE CONTROL UNDER THE FIXED TOPOLOGY

We use \(\overline{G} = (\overline{V}, \overline{E}, \overline{A})\) to represent a weighted digraph formed by the leader and \(N\) followers, and use \(G = (V, E, A)\) to represent the subdigraph formed by \(N\) followers, where the set of nodes \(\overline{V} = \{0, 1, 2, \ldots, N\}\) and \(V = \overline{V} \setminus \{0\}\), and the set of edges \(\overline{E} \subseteq \overline{V} \times \overline{V}\) and \(E \subseteq V \times V\). Denote the neighbors of the ith follower by \(\mathcal{N}_i\). The adjacency matrix \(\overline{A} = [a_{ij}] \in \mathbb{R}^{N \times N}\), \(\overline{A} = \begin{bmatrix} 0 & 0_{1 \times N} \\ a_0 & \overline{A} \end{bmatrix} \in \mathbb{R}^{(N+1) \times (N+1)}\), and if \(j \in \mathcal{N}_i\), then \(a_{ij} = 1\), otherwise \(a_{ij} = 0\); \(a = [a_{10}, a_{20}, \ldots, a_{N0}]^T\) and if \(0 \in \mathcal{N}_i\), then \(a_{0i} = 1\), otherwise \(a_{0i} = 0\). The Laplacian matrix of \(\overline{G}\) is given by

\[
\overline{L} = \begin{bmatrix} 0 & 0_{1 \times N} \\ -a & \overline{L} + F \end{bmatrix} \in \mathbb{R}^{(N+1) \times (N+1)},
\]

where \(L\) is the Laplacian matrix of \(G\) and \(F = \text{diag}(a_{10}, a_{20}, \ldots, a_{N0})\).

In this section, we formulate the assumptions on the agent’s dynamics, the communication graph and the noises for the existence of admissible distributed observation and cooperative control strategies to achieve mean square bounded output tracking.

The assumptions and lemmas required in this section are given below.

3.1 | Assumptions

Assumption 1. There exists a constant \(q_i\) such that \(\|f_i(x) - f_i(y)\| \leq q_i \|x - y\|\) for any \(x, y \in \mathbb{R}^n\), where \(f_i(0) = 0, i = 0, 1, 2, \ldots, N; j = 1, \ldots, n_i\).

Assumption 2. There are constants \(\rho_1\) and \(\rho_2\), functions \(\theta_i(z)\) and \(\pi_i(z)\) such that

\[
\|\pi_i(x) - \pi_i(y)\| \leq \rho_1 \|x - y\|,
\]

\[
\|\theta_i(x) - \theta_i(y)\| \leq \rho_2 \|x - y\|,
\]

\[
\frac{\partial \pi_i}{\partial z} [A_0 z + f_0(z)] = A_i \pi_i(z) + f_i(\pi_i(z)) + B_i \theta_i(z)
\]

\[
C_i \pi_i(z) = C_0 z, \quad i = 1, 2, \ldots, N,
\]

(5)

for any \(x, y, z \in \mathbb{R}^n\), where \(\pi_i(0) = 0, \theta_i(0) = 0\).
Assumption 3. The Brownian motions \( \{w_{ij}(t), l = 1, 2, i = 1, 2, \ldots, N, j \in \mathcal{N}_i \} \) are independent.

Assumption 4. The digraph \( \overline{G} \) contains a spanning tree and the graph \( G \) is undirected.

Lemma 1. Suppose that the pair \((A, C)\) is observable. For any \( α \in \left[\lambda_0^u(A), \infty\right)\), where \( \lambda_0^u(A) = \sum_{i=1}^{n} \max\{Re(\lambda_i(A)), 0\} \), the generalized algebraic Riccati equation

\[
AP + PA^T - 2αPC^T(\mathbf{I}_p + CPC^T)^{-1}CP + I_n = 0
\]

has a unique positive solution \( P \).

3.2 Admissible distributed observation and cooperative control strategies

Since each agent has a dynamics with an unmeasurable state, we consider the following set of admissible observation strategies to estimate agents’ states. Denote

\[
\mathcal{J} = \{ J = \{ (\Theta, \Xi_i), i = 1, \ldots, N \} \},
\]

where \( \Theta_i \) represents an observer of the \( i \)-th follower to observe its own state, and \( \Xi_i \) represents a distributed observer of the \( i \)-th follower to observe the leader’s state. Here,

\[
\Theta_i : \dot{\hat{x}}_i(t) = A_i\hat{x}_i(t) + f_i(\hat{x}_i(t)) + B_iu_i(t) + \Gamma_{\theta_i}H_i(y_i(t) - C_i\hat{x}_i(t)),
\]

where \( \hat{x}_i(t) = (\hat{x}_{i,1}(t), \hat{x}_{i,2}(t), \ldots, \hat{x}_{i,N}(t))^T \in \mathbb{R}^n \) is the estimate of \( x_i(t) \) at time \( t \), \( f_i(\cdot) \) is given in \((4)\), \( \Gamma_{\theta_i} = \text{diag}(\theta_i, \ldots, \theta_i^N) \), \( \theta_i \in \mathbb{R} \) is the constant to be designed, and \( H_i \in \mathbb{R}^{n} \) is the gain matrix to be designed.

\[
\Xi_i : \dot{\hat{x}}_{i0}(t) = A_{i0}\hat{x}_{i0}(t)dt + f_0(\hat{x}_{i0}(t))dt + G_1 \sum_{j \in \mathcal{N}_i} a_{ij} \left[ C_0(\hat{x}_{j0}(t) - \hat{x}_{i0}(t)) dt + \sigma_{10}dw_{10}(t) \right] + G_2 a_{i0}\left[ (y_0(t) - C_0\hat{x}_{i0}(t))dt + \sigma_{20}dw_{20}(t) \right] + \sigma_{30}(y_0(t) - C_0\hat{x}_{i0}(t))dw_{20}(t),
\]

where \( \hat{x}_{i0}(t) \) is the estimate of \( x_{i0}(t) \) by the \( i \)-th follower at time \( t \); \( f_0(\cdot) \) is given in \((3)\), \( \{w_{ij}(t), l = 1, 2, i = 1, 2, \ldots, N, j \in \mathcal{N}_i \} \) are one-dimensional standard Brownian motions, \( \{y_j(t), i = 1, 2, \ldots, N, j \in \mathcal{N}_i \} \) and \( \{\sigma_{ij}(t), i = 1, 2, \ldots, N, j \in \mathcal{N}_i \} \) represent the intensity coefficients of additive and multiplicative measurement noises. \( G_1 \in \mathbb{R}^{n_0} \) and \( G_2 \in \mathbb{R}^{n_0} \) are the gain matrices to be designed.

Based on the designed state observer \((8), (9)\) and the certainty equivalence principle, we consider the following distributed control law

\[
u_i(t) = \theta_i(\hat{x}_{i0}(t)) + \theta_i^{N+1}K_i\Gamma_{\theta_i}^{-1}(\hat{x}_i(t) - \pi_i(\hat{x}_{i0}(t))), \quad i = 1, 2, \ldots, N,
\]

where \( K_i \in \mathbb{R}^{1 \times n} \) is the gain matrix to be designed, \( \theta_i(\cdot) \) is the constant to be designed, \( \theta_i(\cdot) \) and \( \pi_i(\cdot) \) satisfies the output regulation Equation \((5)\). Based on \((10)\), we consider the set of admissible distributed cooperative control strategies

\[
U^* = \left\{ U = \{ u_i(t) = \theta_i(\hat{x}_{i0}(t)) + \theta_i^{N+1}K_i\Gamma_{\theta_i}^{-1}(\hat{x}_i(t) - \pi_i(\hat{x}_{i0}(t))), \quad i = 1, 2, \ldots, N \} \right\}
\]

Remark 2. The distributed observer proposed in References \(39,40\) requires that each follower know the leader’s output \( y_0(t) \). Compared with References \(39,40\), here, the distributed observer of each follower who is not adjacent to the leader does not need the leader’s output \( y_0(t) \), but only the follower’s own and neighbors’ estimates of the leader’s state. Moreover, the distributed observers proposed in References \(41,42\) do not cover the influence of communication noises. Compared with References \(41,42\), here, the influences of both additive and multiplicative noises are considered in the proposed distributed observers.
The definition of mean square bounded output tracking is given below.

**Definition 1.** The leader-following heterogeneous multi-agent systems (1)–(2) under the distributed control law (8), (9) and (10) is said to achieve mean square bounded output tracking, if for any given initial values $x_0(0), x_i(0), \dot{x}_i(0)$ and $\dot{x}_0(0), i = 1, \ldots, N$, there exists a constant $C > 0$ such that

$$\limsup_{t \to \infty} \mathbb{E} \left[ \| y_i(t) - y_0(t) \|^2 \right] \leq C, \quad i = 1, 2, \ldots, N.$$  

Especially, if $C = 0$, then the leader-following heterogeneous multi-agent systems (1)–(2) is said to achieve mean square output tracking.

Next, we will give conditions for the existence of admissible distributed observation and cooperative control strategies to achieve mean square bounded output tracking.

**Theorem 1.** Suppose that Assumptions 1, 2, 3, and 4 hold and max \( \{2q_0\sqrt{n_0}\|P^{-1}\|^2 \lambda_{\text{max}}^2(P), 1\}\) \(\sigma^2 = \max \{\max_{1 \leq i \leq N} \sigma_i^2, \max_{1 \leq i \leq N} \sigma_i^2\}\), \(P\) is the unique positive solution of the generalized Riccati equation

\[A_0^T + P A_0^T - 2q_0 P C_0^T (I_p + C_0 P C_0^T) C_0^T C_0 + I_n = 0.\]

1. Then there exists an admissible observation strategy \(J \in J\) and an admissible cooperative control strategy \(U \in U\) such that the leader-following heterogeneous multi-agent systems (1)–(2) achieve mean square bounded output tracking.

2. Choose \(H_i, K_i, \theta_i, i = 1, \ldots, N\) such that \(A_i - H_iC_i, A_i + B_iK_i\) are Hurwitz and max \(2q_0\sqrt{n_0}\|\bar{P}\|, 2q_0\sqrt{n_0}\|\bar{P}\|, \in \Theta_i\) where \(\bar{P}\) and \(\tilde{P}\) are the unique positive solution of the Lyapunov equations

\[\frac{(A_i - H_iC_i)^T}{\hat{P}} + \frac{(A_i - H_iC_i)}{\tilde{P}} = -I_{n_i}, \quad \frac{(A_i + B_iK_i)^T}{\hat{P}} + \frac{(A_i + B_iK_i)}{\tilde{P}} = -I_{n_i}, \quad i = 1, \ldots, N, \text{ respectively}; \]

Choose \(G_1 = k_1 \Gamma_{\theta_0} C_0^T (I_p + C_0 P C_0^T) C_0^T\) and \(G_2 = k_2 \Gamma_{\theta_0} C_0^T (I_p + C_0 P C_0^T) C_0^T\), where \(\Gamma_{\theta_0} = \text{diag}(\theta_0, \ldots, \theta_0^n)\), max \(2q_0\sqrt{n_0}\|P^{-1}\|^2 \lambda_{\text{max}}^2(P), 1\) \(\theta_0 < \frac{\lambda_{\text{max}}^2(P)}{\lambda_{\text{min}}^2(\tilde{P})}\), \(k_1, k_2 \in (k, \bar{k})\), \(k\) and \(\bar{k}\) are the solutions of \(\theta_0 \sigma^2 \lambda_1(L + F) k^2 - \lambda_1(L + F) k + q_0 < 0\) in the variable \(k\), then under the distributed control law (8), (9), and (10), the leader-following heterogeneous multi-agent systems (1)–(2) achieve mean square bounded output tracking.

\[\limsup_{t \to \infty} \mathbb{E}[\| y_i(t) - y_0(t) \|^2]^2 \leq \frac{\theta_i^{2n} \text{Tr} \left\{ \hat{P} \Gamma_{\theta_0}^{-1} \tilde{P} \Gamma_{\theta_0}^{-1} \beta_i \right\} \left( 1 + \frac{3\sigma_i^2 \theta_i^{2n}}{\sigma_i^2 \lambda_{\text{min}}^2(\tilde{P})} \right)}{\lambda_{\text{min}}^2(\hat{P}) \min_{i \in \{1, 2, \ldots, N\}} \left\{ \theta_i - 2q_0 \sqrt{n_0}\|P^{-1}\|^2 \lambda_{\text{max}}^2(P) \right\}}, \quad i = 1, \ldots, N, \quad (12)\]

where \(\Gamma_{\theta_0} = \text{diag}(\theta_0, \ldots, \theta_0^n)\), \(s_1 = \sum_{i=1}^N Y_i^T \left[I_N \otimes G_1 \Gamma_{\theta_0}^{-1} P^{-1} \Gamma_{\theta_0}^{-1} G_1 \right] Y_j + Y_0^T \left[I_N \otimes G_2 \Gamma_{\theta_0}^{-1} P^{-1} \Gamma_{\theta_0}^{-1} G_2 \right] Y_0\), \(Y_j = (Y_{1j}, Y_{2j}, \ldots, Y_{Nj})^T\), \(Y_0 = (Y_{10}, Y_{20}, \ldots, Y_{N0})^T\), \(s_2 = \min_{i \in \{1, 2, \ldots, N\}} \left\{ \frac{1}{2 \lambda_{\text{max}}^2(\tilde{P})} \left( \theta_i - 2q_0 \| \tilde{P} \| \right) \right\}\), \(s_3 = \max_{i \in \{1, 2, \ldots, N\}} \left\{ \frac{1}{\sqrt{\lambda_{\text{max}}^2(\tilde{P})}} \| \tilde{P} \| H_i \right\}\), \(s_4 = \max_{i \in \{1, 2, \ldots, N\}} \left\{ \frac{\theta_i}{\theta_i \sqrt{\lambda_{\text{min}}^2(\hat{P})} + \theta_i \sqrt{\lambda_{\text{max}}^2(\tilde{P})} \| K_i \|} \right\}\).

The proof is given in Appendix A.

**Remark 3.** The condition max \(\{2q_0\sqrt{n_0}\|P^{-1}\|^2 \lambda_{\text{max}}^2(P), 1\}\) \(\sigma^2 q_0 < \frac{\lambda_{\text{max}}^2(P)}{4}\) in Theorem 1 shows the influence of multiplicative noises, the leader’s dynamics and the communication graph on the existence of admissible distributed observation and cooperative control strategies to achieve mean square bounded output tracking. It is shown that smaller multiplicative noises, smaller Lipschitz coefficients and dimensions of the leader and more connected communication graphs are all more helpful for the cooperatibility of the system. This is consistent with intuition.
Remark 4. According to (12) in Theorem 1, we know that the upper bounds of the mean square output tracking errors are affected by the intensities of additive noises, the nonlinear uncertain dynamics and the state matrices of the leader and the followers, and so forth. What’s more, the upper bounds are proportional to the intensities of additive noises. Therefore, if the intensities of additive noises are sufficiently small, then the upper bounds of the mean square output tracking errors can be made arbitrarily small. In particular, if there are no additive noises, these upper bounds become zeros.

4 | COOPERATIVE CONTROL UNDER THE MARKOVIAN SWITCHING TOPOLOGY

In the real world, due to the movement of the agent, the environment and other factors, the communication network connecting the agents is randomly interrupted and restored. Here, we describe this changing topology by the Markovian switching topology. Therefore, it is necessary to study cooperative control of heterogeneous nonlinear systems under Markovian switching topology.

In this section, we assume that the topology graph is a Markovian switching topology. Let the switching signal \( r(t) \) be defined in the probability space \((\Omega, \mathcal{F}, \{F_t\}_{t \geq 0}, \mathbb{P})\). The signal \( r(t) \) is a right continuous homogeneous Markov chain and has a finite state space \( S = \{1, 2, \ldots, S\} \). The matrix \( Q = [q_{ij}]_{i,j \in S} \) is the transfer rate matrix of the Markov chain \( r(t) \) and satisfies

\[
P(r(t + \Delta) = j | r(t) = i) = \begin{cases} 
q_{ij} \Delta + o(\Delta), & i \neq j, \\
1 + q_{ii} \Delta + o(\Delta), & i = j,
\end{cases}
\]

where if \( i \neq j, q_{ij} \) is the transition rate of the Markov chain from state \( i \) to state \( j \) with \( q_{ij} \geq 0 \); if \( i = j, q_{ii} = -\sum_{j \neq i} q_{ij} \Delta > 0 \) and \( \lim_{\Delta \to 0} \frac{o(\Delta)}{\Delta} = 0 \).

Similar to the fixed topology, we use \( \bar{G}(r(t)) = (\bar{V}, \bar{E}(r(t)), \bar{A}(r(t))) \) to represent a weighted graph formed by the leader and \( N \) followers. The Laplacian matrix of \( \bar{G}(r(t)) \) is \( \bar{L}(r(t)) = \begin{bmatrix} 0 & 0_{N \times N} \\
-a(r(t)) & \mathcal{L}(r(t)) + F(r(t)) \end{bmatrix} \in \mathbb{R}^{(N+1) \times (N+1)} \), where \( \mathcal{L}(r(t)) \) is the Laplacian matrix \( G(r(t)) \), \( F(r(t)) = \text{diag}(a_{10}(r(t)), a_{20}(r(t)), \ldots, a_{N0}(r(t))) \), \( a(r(t)) = [a_{10}(r(t)), a_{20}(r(t)), \ldots, a_{N0}(r(t))]^T \).

Similar to (7), we consider the admissible distributed observation

\[
\bar{J} = \{(\hat{\Theta}_i, \hat{z}_i), i = 1, \ldots, N \}
\]

where \( \hat{\Theta}_i \) represents an observer of the \( i \)-th follower to observe its own state, and \( \hat{z}_i \) represents a distributed observer of the \( i \)-th follower to observe the leader’s state. Here,

\[
\hat{\Theta}_i : \dot{\hat{x}}_i(t) = A_i \hat{x}_i(t) + f_i(\hat{x}_i(t)) + B_i u_i(t) + \Gamma_{i0} \hat{H}_i(y_i(t) - C_i \hat{x}_i(t)),
\]

where \( \hat{x}_i(t) = (\hat{x}_{i,1}(t), \hat{x}_{i,2}(t), \ldots, \hat{x}_{i,N}(t))^T \in \mathbb{R}^n \) is the estimate of \( x_i(t) \) at time \( t \), \( \Gamma_{i0} = \text{diag}(\theta_i, \ldots, \theta^{n}_i), \theta_i \) is the constant to be designed, \( \hat{H}_i \in \mathbb{R}^{n_i} \) is the matrix to be designed.

\[
\hat{z}_i : d\hat{x}_{i0}(t) = A_0 \hat{x}_{i0}(t)dt + f_0(\hat{x}_{i0}(t))dt + G_1(r(t)) \sum_{j \in \mathcal{N}_i} a_{ij}(r(t)) \left[ C_0 \left( \hat{x}_{i0}(t) - \hat{x}_{j0}(t) \right) \right] dt + Y_{1j} dw_{1j}(t) \\
+ \sigma_{ij} (C_0 \hat{x}_{i0}(t) - C_0 \hat{x}_{j0}(t)) dw_{2j}(t) \] \\
+ G_2(r(t)) a_{i0}(r(t)) \left[ (y_i(t) - C_0 \hat{x}_{i0}(t)) \right] dt \\
+ Y_{2j} dw_{20}(t) + \sigma_{i0} (y_i(t) - C_0 \hat{x}_{i0}(t)) dw_{20}(t),
\]

where \( \hat{x}_{i0}(t) \) is the estimate of \( x_{i0}(t) \) by the \( i \)-th follower at time \( t \), \( \{w_{ij}(t), l = 1, 2, i = 1, 2, \ldots, N, j \in \mathcal{N}_i\} \) are standard one-dimensional Brownian motions, \( \{Y_{ij} \in \mathbb{R}, i = 1, \ldots, N, j \in \mathcal{N}_i\} \) and \( \{\sigma_{ij} \in \mathbb{R}, i = 1, \ldots, N, j \in \mathcal{N}_i\} \) represent the
intensity coefficients of additive and multiplicative measurement noises, respectively; \( G_1(r(t)) \in \mathbb{R}^{n_q} \) and \( G_2(r(t)) \in \mathbb{R}^{n_u} \) are the gain matrices to be designed.

Based on the designed state observer (13), (14) and the certainty equivalence principle, we consider the distributed control law

\[
 u_i(t) = \theta_i(\hat{x}_{i0}(t)) + \theta_i^{0, t+1} \tilde{K}_i \Gamma_{\theta}^{-1} (\hat{x}_i(t) - \pi_i(\hat{x}_{i0}(t))), \quad i = 1, 2, \ldots, N,
\]

(15)

where \( \tilde{K}_i \in \mathbb{R}^{1 \times n_q} \) is the gain matrix to be designed, \( \theta_i \) is the constant to be designed, \( \theta_i(\cdot) \) and \( \pi_i(\cdot) \) satisfies the output regulation Equation (5).

Based on (15), we consider the set of admissible distributed cooperative control strategies

\[
 U^* = \left\{ U = \{ u_i(t) = \theta_i(\hat{x}_{i0}(t)) + \theta_i^{0, t+1} \tilde{K}_i \Gamma_{\theta}^{-1} (\hat{x}_i(t) - \pi_i(\hat{x}_{i0}(t))), i = 1, 2, \ldots, N \} \right\}
\]

(16)

**Assumption 5.** The digraph \( \mathcal{G}_l(I) \) contains a spanning tree and the graph \( \mathcal{G}(I) \) is undirected, where \( I \in \mathbb{S} \).

Next, we will give conditions for the existence of admissible distributed observation and cooperative control strategies to achieve mean square bounded output tracking under the Markovian switching topology.

**Theorem 2.** Suppose that Assumptions 1, 2, 3, and 5 hold and max \( \{ 2q_0 \sqrt{n_0} ||P^{-1}|| \gamma_{\max}^2 (P), 1 \} \times \sigma_0 < \min_{l \in \mathbb{S}} \left\{ \frac{\lambda_2(C_l(F(l)) \Gamma^{-1}(P))}{4}, \right\} \), where \( \sigma_0^2 = \max_{1 \leq j \leq N} \{ \max_{1 \leq i \leq n_q} |\pi_i'_{ij}(P)|^2 \}, P \) is the unique positive solution of the generalized Riccati equation \( A_0 P + P A_0^T - 2q_0 PC_1^T(I_p + C_0 PC_0^T)^{-1} C_0 P + I_n = 0 \).

1. Then there exists an admissible observation strategy \( \hat{I} \in \mathcal{J} \) an admissible cooperative control strategy \( \hat{U} \in \mathcal{U}^* \) such that the leader-following heterogeneous multi-agent systems (1)–(2) achieve mean square bounded output tracking.
2. Choose \( \hat{H}_i, \hat{K}_i, \theta_i, \quad i = 1, \ldots, N \) such that \( A_i - \hat{H}_i C_i \) and \( A_i + B_i \hat{K}_i \) are Hurwitz and max \( \{ 2q_0 \sqrt{n_0} ||\hat{P}||, 1 \} < \theta_i(0), \) where \( \hat{P} \) and \( \hat{P}^t \) are the unique positive solutions of the Lyapunov equations \( A_i - \hat{H}_i C_i \hat{P} + \hat{P}^t (A_i - \hat{H}_i C_i)^t = -\hat{N}_i \) and \( (A_i + B_i \hat{K}_i) \hat{P} + \hat{P}^t (A_i + B_i \hat{K}_i)^t = -\hat{N}_i, \) respectively. Choose \( G_1(l) = k_1(l) \Gamma_{\theta}^{-1} (I_p + C_0 PC_0^T)^{-1} \) and \( G_2(l) = k_2(l) \Gamma_{\theta}^{-1} (I_p + C_0 PC_0^T)^{-1} \), where \( \Gamma_{\theta}^{-1} = \text{ diag } \{ \lambda_1(\mathcal{L}(l) + F(l)), \lambda_2(\mathcal{L}(l) + F(l)), \lambda_3(\mathcal{L}(l) + F(l)) \} \), \( \lambda_1(l), \lambda_2(l), \lambda_3(l) \) are the solutions of \( \theta_0(l) \lambda_1(l) C_l + F(l) \lambda_2(l) C_l \lambda_3(l) C_l \lambda_4(l) + 0 \in \lambda_1(l) \), then under the distributed control law (13), (14), (15), the leader-following heterogeneous multi-agent systems (1)–(2) achieve mean square bounded output tracking.

\[
 \operatorname{lim sup}_{t \to \infty} \mathbb{E} \left[ ||y(t) - \hat{y}(t)||^2 \right] \leq \frac{\theta_i^{2, n_i} T \beta_i^T \Gamma_{\theta}^{-1} (I_p + C_0 PC_0^T) \beta_i}{\lambda_{\min}(\hat{P}^t) \min_{l \in \mathbb{S}} \{ \theta_0(l), \lambda_{\max}(\hat{P}) \}} \left( 1 + \frac{3 \sigma_0^2 \theta_i^{2, n_i}}{\lambda_{\min}(\hat{P})} \right) + \frac{6 \sigma_0^2 \theta_i^{2, n_i} \sigma_1 \lambda_{\max}(\hat{P})}{\lambda_{\min}(\hat{P}) \min_{l \in \mathbb{S}} \{ \theta_0(l), \lambda_{\max}(\hat{P}) \}}, \quad i = 1, 2, \ldots, N,
\]

where \( \Gamma_{\theta} = \text{ diag } \{ \theta_1, \ldots, \theta_N \}, \) \( \sigma_1 = \max_{l \in \mathbb{S}} \left\{ \sum_{j=1}^N Y_j^T \left[ I_N \otimes G_j(l) \Gamma_{\theta}^{-1} (I_p + C_0 PC_0^T) G_j(l) \right] Y_j \right\}, \) \( Y_j = (Y_j^T, Y_j^T, \ldots, Y_j^T)^T, \) \( Y_j = (Y_j^T, Y_j^T, \ldots, Y_j^T)^T, \) \( \sigma_2 = \max_{l \in \mathbb{S}} \left\{ \frac{1}{\lambda_{\min}(\hat{P})} \right\}, \) \( \sigma_3 = \max_{l \in \mathbb{S}} \left\{ \frac{1}{\lambda_{\min}(\hat{P})} \right\}, \) \( \sigma_4 = \max_{l \in \mathbb{S}} \left\{ \frac{1}{\lambda_{\min}(\hat{P})} \right\} \).

The proof is given in Appendix B.
5 | NUMERICAL SIMULATION

In this section, we will use a numerical example to demonstrate the effectiveness of our control laws.

We consider a heterogeneous multi-agent system consisting of a leader pendulum and three follower pendulums, and demonstrate that the angular displacement of followers can track that of the leader under the distributed control law (8), (9), and (10).

Referring to Reference 43, the dynamics of the leader is given by

\[
\begin{align*}
    \dot{x}_{0,1}(t) &= x_{0,2}(t) dt, \\
    \dot{x}_{0,2}(t) &= -\mu_0 x_{0,2}(t) - \zeta \sin(x_{0,1}(t)) dt, \\
    y_0(t) &= x_{0,1}(t).
\end{align*}
\]  

(17)

The dynamics of the \(i\)th follower is given by

\[
\begin{align*}
    \dot{x}_{i,1}(t) &= x_{i,2}(t) dt + \beta_{i,1} dw_{i,1}(t), \\
    \dot{x}_{i,2}(t) &= u_i(t) - \mu_i x_{i,2}(t) - \zeta \sin(x_{i,1}(t)) dt + \beta_{i,2} dw_{i,2}(t), \\
    y_i(t) &= x_{i,1}(t).
\end{align*}
\]  

(18)

where \(\mu_0\) and \(\mu_i\) are the damping factor, \(l_0\) and \(l_i\) are the length of the pendulum, \(\beta_{i,1} = 0.1i\), \(\beta_{i,2} = 0.2i\), \(i = 1, 2, 3\).

By (17) and (18), we have

\[
A_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, C_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}, B_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, i = 1, 2, 3.
\]

The communication topology \(\bar{G} = (\bar{V}, \bar{E}, \bar{A})\) is shown in Figure 1, where \(\bar{A} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\). The additive measurement noises in (9) are given by \(0.03 dw_{i,1}(t)\) and \(0.0005 C_0 (\dot{x}_{0,1}(t) - \dot{x}_{0,0}(t)) dt\), \(i = 1, 2, 3, j \in N_i\). The damping factor \(\mu_0 = 0.1\), \(\mu_1 = 0.2\), \(\mu_2 = 0.2\) and \(\mu_3 = 0.3\). The length of the pendulum \(l_0 = 2\), \(l_1 = 1\), \(l_2 = 2\) and \(l_3 = 5\).

It can be verified that the condition the pair \((A_i, B_i)\) is controllable for \(i = 1, 2, 3\), and the pair \((C_i, A_i)\) is observable for \(i = 0, 1, 2, 3\). Choose

\[
K_1 = K_2 = K_3 = [-1, -2], H_1 = H_2 = H_3 = [2; 2]^T
\]

such that \(A_i + B_i K_i\) and \(A_i - H_i C_i\) are Hurwitz, \(i = 1, 2, 3\).

By (5), we get

\[
\pi_1(z) = \pi_2(z) = \pi_3(z) = z.
\]

\[
\theta_1(z) = [0 \quad 0.1]z + 5 \sin([1 \quad 0]z), \theta_2(z) = [0 \quad 0.1]z, \theta_3(z) = [0 \quad 0.2]z - 3 \sin([1 \quad 0]z).
\]

\[\text{FIGURE 1} \quad \text{The communication topology graph.}\]
By (17) and (18), we get $q_0 = 5\sqrt{2}$, $n_0 = 2$, $q_1 = 10\sqrt{2}$, $q_2 = 2\sqrt{2}$, $q_3 = 2\sqrt{2}$ and $n_i = 2$, $i = 1, 2, 3$. By the generalized algebraic Riccati equation $A_0P + PA_0^T - 2q_0PC_0^T(I_P + C_0PC_0^T)^{-1}C_0P + I_n = 0$, we have $P = \begin{bmatrix} 0.4019 & 0.3149 \\ 0.3149 & 1.2766 \end{bmatrix}$. By Lyapunov equations $(A_i - H_iC_i)\tilde{P}_i + \tilde{P}_i(A_i - H_iC_i)^T = -I_n$, and $(A_i + B_iK_i)\tilde{P}_i + \tilde{P}_i(A_i + B_iK_i)^T = -I_n$, $i = 1, 2, 3$, we get $\tilde{P}_i = \begin{bmatrix} 1.5000 & -0.5000 \\ -0.5000 & 0.5000 \end{bmatrix}$ and $\tilde{P}_i = \begin{bmatrix} 0.3750 & 0.2500 \\ 0.2500 & 1.2500 \end{bmatrix}$, $i = 1, 2, 3$.

Choose $\theta_0 = 128$, $\theta_1 = 69$, $\theta_2 = 35$, $\theta_3 = 14$, $k_1 = 4$ and $k_2 = 6$. By the definitions of $G_1$ and $G_2$, we obtain $G_1 = \begin{bmatrix} 226 \\ 28931 \end{bmatrix}$ and $G_2 = \begin{bmatrix} 339 \\ 43397 \end{bmatrix}$.

The mean square output tracking errors of the leader-following heterogeneous multi-agent systems (17)–(18) under the distributed control law (8), (9), and (10) are shown in Figure 2.

### 6 CONCLUSION

In this paper, we have studied cooperative output feedback tracking control of stochastic nonlinear heterogeneous leader-following multi-agent systems under the fixed topology and the Markovian switching topology, respectively. By output regulation theory and stochastic analysis, we show that if the dynamics of each agent satisfies the Lipschitz condition, and the product of the leader’s Lipschitz coefficient, the intensity of multiplicative measurement noises, and constants related to the leader’s Lipschitz coefficient and dimension is less than $1/4$ the minimum nonzero eigenvalue of graph Laplacian, then there exist admissible distributed observation and cooperative control strategies to ensure mean square bounded output tracking. In addition, we further give sufficient conditions for the existence of admissible distributed observation and cooperative control strategies to ensure the mean square bounded output tracking under the Markovian switching topology. Especially, if there is no additive noise in the dynamics of each follower and additive measurement noise along with information exchange among agents, then there exist admissible distributed observation and cooperative control strategies to achieve mean square output tracking. Efforts can be made to investigate the consensus problem of heterogeneous multi-agent systems under signed topologies.

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**CONFLICT OF INTEREST STATEMENT**

The authors declare no potential conflict of interest.

**DATA AVAILABILITY STATEMENT**

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.
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APPENDIX A

Proof of Theorem 1. Denote \( e_i(t) = x_i(t) - \hat{x}_i(t), \) \( e_j(t) = (e^T_{i1}(t), e^T_{i2}(t), \ldots, e^T_{in}(t)) \), \( \hat{e}_i(t) = \frac{1}{\theta_i} e_i(t), \) \( i = 1, 2, \ldots, N, j = 1, 2, \ldots, n_i, \) \( \hat{e}_i(t) = (\hat{e}^T_{i1}(t), \hat{e}^T_{i2}(t), \ldots, \hat{e}^T_{in}(t)) \), \( \theta_i \) \( \hat{e}_i(t) = \frac{1}{\theta_i} e_i(t), \) \( i = 1, 2, \ldots, N, j = 1, 2, \ldots, n_i, \) \( \theta_i \) \( \hat{e}_i(t) = \frac{1}{\theta_i} e_i(t), \) \( i = 1, 2, \ldots, N, j = 1, 2, \ldots, n_i, \) \( \theta_i \) \( \hat{e}_i(t) = \frac{1}{\theta_i} e_i(t), \) \( i = 1, 2, \ldots, N, j = 1, 2, \ldots, n_i, \) \( \theta_i \)

By the definition of \( \hat{e}_i(t) \), we have \( \hat{e}_i(t) = \Gamma^{-1}_i e_i(t) \). Combining \( \Gamma^{-1}_i A_i - \Gamma^{-1}_i H_i C_i = \theta_i A_i + C_i \Gamma^{-1}_i = \theta_i C_i \) and the above equation, we obtain

\[
\begin{align*}
\hat{d}_i(t) &= \theta_i (A_i - H_i C_i) \hat{e}_i(t)dt + \Gamma^{-1}_i [f_i(x_i(t)) - f_i(\hat{x}_i(t))])dt + \beta_i dw_i(t).
\end{align*}
\]

In the following, we will estimate \( \limsup_{t \to +\infty} \mathbb{E}[\|e_i(t)\|^2] \). Choose \( V_i(t) = \hat{e}_i^T(t) \hat{P}_i \hat{e}_i(t) \), where the positive definite matrix \( \hat{P}_i \) satisfies \( (A_i - H_i C_i)^T \hat{P}_i + \hat{P}_i (A_i - H_i C_i)^T = -I_{n_i}, i = 1, \ldots, N \). By Itô’s formula, the above equation, Assumption 1 and \( \theta_i > 1 \), we get

\[
\begin{align*}
\mathbb{E}[V_i(t)] &= 2 \theta_i \hat{e}_i^T(t) \hat{P}_i (A_i - H_i C_i) \hat{e}_i(t)dt + 2 \hat{e}_i^T(t) \hat{P}_i \Gamma^{-1}_i [f_i(x_i(t)) - f_i(\hat{x}_i(t))])dt \\
&\quad + 2 \hat{e}_i^T(t) \hat{P}_i \Gamma^{-1}_i \beta_i dw_i(t) + \frac{1}{2} \text{Tr} \left\{ \beta_i^T \Gamma^{-1}_i \hat{P}_i \Gamma^{-1}_i \beta_i \right\} dt \\
&= \theta_i \hat{e}_i^T(t) [(A_i - H_i C_i)^T \hat{P}_i + \hat{P}_i (A_i - H_i C_i)^T] \hat{e}_i(t)dt + 2 \hat{e}_i^T(t) \hat{P}_i \Gamma^{-1}_i [f_i(x_i(t)) - f_i(\hat{x}_i(t))])dt
\end{align*}
\]
+ 2\tilde{e}_i^T(t)\tilde{P}_i\Gamma_i^{-1}\beta_td\omega_i(t) + \frac{1}{2} \text{Tr} \left\{ \beta_i^T\Gamma_i^{-1}\tilde{P}_i\Gamma_i^{-1}\beta_i \right\} dt \\
= -\theta_i\tilde{e}_i^T(t)\dot{\theta}_i(t) dt + 2\tilde{e}_i^T(t)\tilde{P}_i\Gamma_i^{-1}\left[f_i(x_i(t)) - f_i(\hat{x}_i(t))\right]dt + 2\tilde{e}_i^T(t)\tilde{P}_i\Gamma_i^{-1}\beta_td\omega_i(t) \\
+ \frac{1}{2} \text{Tr} \left\{ \beta_i^T\Gamma_i^{-1}\tilde{P}_i\Gamma_i^{-1}\beta_i \right\} dt \\
\leq -\theta_i\|\dot{\theta}_i(t)\|^2 dt + 2\|\tilde{e}_i(t)\|\|\tilde{P}_i\|\|\Gamma_i^{-1}\left[f_i(x_i(t)) - f_i(\hat{x}_i(t))\right]\|dt \\
+ 2\tilde{e}_i^T(t)\tilde{P}_i\Gamma_i^{-1}\beta_td\omega_i(t) + \frac{1}{2} \text{Tr} \left\{ \beta_i^T\Gamma_i^{-1}\tilde{P}_i\Gamma_i^{-1}\beta_i \right\} dt \\
= -\theta_i\|\dot{\theta}_i(t)\|^2 dt + 2\|\tilde{e}_i(t)\|\|\tilde{P}_i\|\|\Gamma_i^{-1}\|\left[f_i(x_i(t)) - f_i(\hat{x}_i(t))\right]\|^2 dt \\
+ 2\tilde{e}_i^T(t)\tilde{P}_i\Gamma_i^{-1}\beta_td\omega_i(t) + \frac{1}{2} \text{Tr} \left\{ \beta_i^T\Gamma_i^{-1}\tilde{P}_i\Gamma_i^{-1}\beta_i \right\} dt \\
\leq -\theta_i\|\dot{\theta}_i(t)\|^2 dt + 2\|\tilde{e}_i(t)\|\|	ilde{P}_i\|\|\Gamma_i^{-1}\|\left[\sum_{j=1}^{n_i} \left(\frac{\theta_j^2}{\theta_j^2}\|\tilde{e}_j(t)\|^2 + \frac{\theta_j^2}{\theta_j^2}\|\dot{\theta}_j(t)\|^2\right)\right] dt \\
+ 2\tilde{e}_i^T(t)\tilde{P}_i\Gamma_i^{-1}\beta_td\omega_i(t) + \frac{1}{2} \text{Tr} \left\{ \beta_i^T\Gamma_i^{-1}\tilde{P}_i\Gamma_i^{-1}\beta_i \right\} dt \\
\leq -\theta_i\|\dot{\theta}_i(t)\|^2 dt + 2\|\tilde{e}_i(t)\|\|\tilde{P}_i\|\|\Gamma_i^{-1}\|\left[\sum_{j=1}^{n_i} \left(\frac{\theta_j^2}{\theta_j^2}\|\tilde{e}_j(t)\|^2 + \frac{\theta_j^2}{\theta_j^2}\|\dot{\theta}_j(t)\|^2\right)\right] dt \\
+ 2\tilde{e}_i^T(t)\tilde{P}_i\Gamma_i^{-1}\beta_td\omega_i(t) + \frac{1}{2} \text{Tr} \left\{ \beta_i^T\Gamma_i^{-1}\tilde{P}_i\Gamma_i^{-1}\beta_i \right\} dt \\
\leq -\theta_i\|\dot{\theta}_i(t)\|^2 dt + 2\|\tilde{e}_i(t)\|\|\tilde{P}_i\|\|\Gamma_i^{-1}\|\left[\sum_{j=1}^{n_i} \left(\frac{\theta_j^2}{\theta_j^2}\|\tilde{e}_j(t)\|^2 + \frac{\theta_j^2}{\theta_j^2}\|\dot{\theta}_j(t)\|^2\right)\right] dt \\
+ 2\tilde{e}_i^T(t)\tilde{P}_i\Gamma_i^{-1}\beta_td\omega_i(t) + \frac{1}{2} \text{Tr} \left\{ \beta_i^T\Gamma_i^{-1}\tilde{P}_i\Gamma_i^{-1}\beta_i \right\} dt \\
= (2\|\tilde{e}_i(t)\|\|\tilde{P}_i\| - \theta_i)\|\tilde{e}_i(t)\|^2 dt + 2\tilde{e}_i^T(t)\tilde{P}_i\Gamma_i^{-1}\beta_td\omega_i(t) + \frac{1}{2} \text{Tr} \left\{ \beta_i^T\Gamma_i^{-1}\tilde{P}_i\Gamma_i^{-1}\beta_i \right\} dt \\
= (2\|\tilde{e}_i(t)\|\|\tilde{P}_i\| - \theta_i)\|\tilde{e}_i(t)\|^2 dt + 2\tilde{e}_i^T(t)\tilde{P}_i\Gamma_i^{-1}\beta_td\omega_i(t) + \frac{1}{2} \text{Tr} \left\{ \beta_i^T\Gamma_i^{-1}\tilde{P}_i\Gamma_i^{-1}\beta_i \right\} dt. \hspace{1cm} \text{A1}

Denote \( W_i(t) = e^{\epsilon t}V_i(t) \), for any given \( \gamma \in \left( 0, \min_{i \in \{1, 2, \ldots, N\}} \left\{ \frac{\theta_i - 2\|\tilde{P}_i\|}{\lambda_{\text{min}}(\tilde{P}_i)} \right\} \right) \). By the above inequality and applying Itô’s formula to \( W(t) \), we have

\[
\begin{align*}
\frac{dW_i(t)}{dt} &= \gamma e^{\epsilon t}V_i(t)dt + e^{\epsilon t}dV_i(t) \\
&\leq \gamma e^{\epsilon t}\tilde{P}_i(\gamma)\tilde{e}_i(t)dt + 2e^{\epsilon t}\tilde{P}_i(\gamma)\Gamma_i^{-1}\beta_td\omega_i(t) + \frac{1}{2}e^{\epsilon t}\text{Tr} \left\{ \beta_i^T\Gamma_i^{-1}\tilde{P}_i(\gamma)\Gamma_i^{-1}\beta_i \right\} dt.
\end{align*}
\]

where \( \tilde{P}_i(\gamma) = \gamma \tilde{P}_i + (2\|\tilde{P}_i\| - \theta_i)\lambda_i, i = 1, \ldots, N. \)
Integrating both sides of the above inequality from 0 to \( t \) and taking the mathematical expectation, we obtain

\[
\mathbb{E}[W(t)] \leq \mathbb{E}[W(0)] + \mathbb{E} \left[ \int_0^t \gamma e^{\gamma t} \tilde{e}_i^T(s) \tilde{\Psi}_i(\gamma) \tilde{e}_i(s) ds \right] + \int_0^t \frac{1}{2} e^{\gamma t} \text{Tr} \left\{ \beta_i^T \Gamma_{\theta_i}^{-1} \tilde{P}_i \Gamma_{\theta_i}^{-1} \beta_i \right\} ds. \tag{A2}
\]

By \( 0 < \gamma < \min_{j=1,2,\ldots,N} \left\{ \frac{\theta - 2\sqrt{\pi} \| \Psi_{\lambda_{\min}(P_i)} \|}{\lambda_{\min}(P_i)} \right\} \), we have \( \tilde{\Psi}(\gamma) < 0 \).

By (A2) and \( \tilde{\Psi}(\gamma) < 0 \), we get

\[
\mathbb{E}[W(t)] \leq \mathbb{E}[W(0)] + \frac{1}{2\gamma} \text{Tr} \left\{ \beta_i^T \Gamma_{\theta_i}^{-1} \tilde{P}_i \Gamma_{\theta_i}^{-1} \beta_i \right\} \left[ e^{\gamma t} - 1 \right].
\]

Combining \( \lambda_{\min}(P_i)\| \tilde{e}_i(t) \|^2 \leq V_i(t) \) and the above inequality, we have

\[
\mathbb{E}[\| \tilde{e}_i(t) \|^2] \leq \frac{1}{\lambda_{\min}(P_i)} e^{-\gamma t} \mathbb{E}[V_i(0)] + \frac{1}{2\gamma \lambda_{\min}(P_i)} \text{Tr} \left\{ \beta_i^T \Gamma_{\theta_i}^{-1} \tilde{P}_i \Gamma_{\theta_i}^{-1} \beta_i \right\} \left[ 1 - e^{-\gamma t} \right].
\]

Taking the upper limit on both sides of the above inequality, we get

\[
\lim \sup_{t \to \infty} \mathbb{E}[\| \tilde{e}_i(t) \|^2] \leq \frac{1}{2\gamma \lambda_{\min}(P_i)} \text{Tr} \left\{ \beta_i^T \Gamma_{\theta_i}^{-1} \tilde{P}_i \Gamma_{\theta_i}^{-1} \beta_i \right\}.
\]

Combining \( \tilde{e}_i(t) = \Gamma_{\theta_i}^{-1} e_i(t) \) and the above inequality, we obtain

\[
\lim \sup_{t \to \infty} \mathbb{E}[\| e_i(t) \|^2] \leq \frac{\| \Gamma_{\theta_i} \|^2}{2\gamma \lambda_{\min}(P_i)} \text{Tr} \left\{ \beta_i^T \Gamma_{\theta_i}^{-1} \tilde{P}_i \Gamma_{\theta_i}^{-1} \beta_i \right\} = \frac{\theta_{2n}}{2\gamma \lambda_{\min}(P_i)} \text{Tr} \left\{ \beta_i^T \Gamma_{\theta_i}^{-1} \tilde{P}_i \Gamma_{\theta_i}^{-1} \beta_i \right\}.
\]

By the definition of \( e_i(t) \) and the above inequality, we have

\[
\lim \sup_{t \to \infty} \mathbb{E}[\| x_i(t) - \hat{x}_i(t) \|^2] \leq \frac{\theta_{2n}}{2\gamma \lambda_{\min}(P_i)} \text{Tr} \left\{ \beta_i^T \Gamma_{\theta_i}^{-1} \tilde{P}_i \Gamma_{\theta_i}^{-1} \beta_i \right\}. \tag{A3}
\]

Next, we proceed to estimate \( \lim \sup_{t \to \infty} \mathbb{E}[\| x_{0i}(t) - x_{0i}(t) \|^2], i = 1, \ldots, N \).

By (3) and (9), we get

\[
d\delta(t) = (I_N \otimes A_0 + \Delta f_0(t) - \mathcal{L} \otimes G_1 C_0 - F \otimes G_2 C_0) \delta(t) dt
+ M_1 dw_{1y}(t) + M_2 dw_{10}(t) + M_3(t) dw_{2y}(t) + M_4(t) dw_{20}(t),
\]

where \( \Delta f_0(t) = [(f_0(\tilde{x}_{10}(t)) - f_0(x_0(t)))^T, \ldots, (f_0(\tilde{x}_{N0}(t)) - f_0(x_0(t)))^T]^T \), \( M_1 = \sum_{i,j=1}^N [\tilde{S}_{ij} \otimes G_1] \gamma_j, M_2 = \sum_{i,j=1}^N [\tilde{S}_{ii} \otimes G_2] \gamma_0, M_3(t) = \sum_{i,j=1}^N \sigma_{0i}[S_{2j} \otimes G_1 C_0] \delta(t), M_4(t) = -\sum_{i,j=1}^N \sigma_{0i} [\tilde{S}_{2j} \otimes G_2 C_0] \delta(t), \gamma_j = (\gamma_{1j}, \gamma_{2j}, \ldots, \gamma_{Nj})^T, \gamma_0 = (\gamma_{10}, \gamma_{20}, \ldots, \gamma_{N0})^T, \gamma_{ij} = \theta_{ij} \eta_i \eta_j^T, \tilde{S}_{ii} = a_{i0} \eta_i \eta_i^T, S_{2j} = a_{ij} \eta_i (\eta_j - \eta_i)^T, \tilde{S}_{2j} = a_{i0} \eta_i \eta_i^T, \eta_i \) denotes the \( N \)-dimensional column vector with the \( i \)th element being 1 and others being zero.

From Assumption 4, we know that \( \mathcal{L} + F \) is a positive definite matrix. Hence, there exists a unitary matrix \( \Phi \) such that \( \Phi^T \mathcal{L} \Phi + \Phi^T F \Phi = \text{diag}(\lambda_1(\mathcal{L} + F), \ldots, \lambda_N(\mathcal{L} + F)) = \Lambda \).

Denote \( \tilde{\delta}(t) = (\Phi \otimes \Gamma_{\theta_i})^{-1} \delta(t), \) where \( \Gamma_{\theta_i} = \text{diag}(\theta_0, \ldots, \theta_{0N}) \). By the above inequality, we have

\[
\mathbb{E}[\| \tilde{x}_i(t) - \hat{x}_i(t) \|^2] \leq \frac{\theta_{2n}}{2\gamma \lambda_{\min}(P_i)} \text{Tr} \left\{ \beta_i^T \Gamma_{\theta_i}^{-1} \tilde{P}_i \Gamma_{\theta_i}^{-1} \beta_i \right\}.
\]
where \( M_5 = \sum_{i=1}^{N} \left[ \Phi^T S_{1,ij} \otimes \Gamma_{\theta_0}^{-1} \right] Y_j \), \( M_6 = \sum_{i=1}^{N} \left[ \Phi^T \tilde{S}_{11i} \otimes \Gamma_{\theta_0}^{-1} \right] G_0 Y_0 \), \( M_7(t) = \sum_{i=1}^{N} \sigma_i \left[ \Phi^T S_{2,ij} \otimes \Gamma_{\theta_0}^{-1} \right] C_0 G_0 \Gamma_0 \tilde{\delta}(t) \), \( M_8(t) = -\sum_{i=1}^{N} \sigma_i \Phi^T \tilde{S}_{2,ij} \otimes \Gamma_{\theta_0}^{-1} \Gamma_0 \mbox{d} \bar{\delta}(t) \).

Choose the Lyapunov function \( \bar{V}(t) = \delta^T(t)(I_N \otimes P^{-1}) \tilde{\delta}(t) \). By the above equation and Itô's formula, we get

\[
\begin{align*}
\mbox{d} \bar{V}(t) &= 2\delta^T(t)(I_N \otimes P^{-1})(I_N \otimes \theta_0 A_0 - \Phi^T L \Phi \otimes \Gamma_{\theta_0}^{-1} G_0 C_0 \Gamma_0 - \Phi^T F \Phi \otimes \Gamma_{\theta_0}^{-1} G_0 C_0 \Gamma_0) \tilde{\delta}(t) \mbox{d}t \\
&+ 2\delta^T(t)(I_N \otimes P^{-1})(\Phi \otimes \Gamma_{\theta_0}^{-1})^{-1} \Gamma_0 \mbox{d} \delta(t) + M_0(t) \mbox{d}t + 2\delta^T(t)(I_N \otimes P^{-1}) M_5 \mbox{d} \delta(t) \\
&+ 2\delta^T(t)(I_N \otimes P^{-1}) M_6 \mbox{d} \delta(t) + 2\delta^T(t)(I_N \otimes P^{-1}) M_8(t) \mbox{d} \delta(t),
\end{align*}
\]

where

\[
M_9(t) = \sum_{i,j=1}^{N} Y_j^T \left[ \left( S_{1,ij} \otimes \Gamma_{\theta_0}^{-1} \right) \otimes G_1 \Gamma_{\theta_0}^{-1} \Gamma_0^{-1} \right] Y_j + \sum_{i,j=1}^{N} Y_0^T \left[ \left( S_{1,ij} \otimes \Gamma_{\theta_0}^{-1} \right) \otimes \Gamma_0 C_0 \Gamma_0 G_0 \right] \tilde{\delta}(t) + \sum_{i,j=1}^{N} \sigma_i^2 \delta(t) \left[ \left( \Phi^T S_{2,ij} \otimes \Gamma_{\theta_0}^{-1} \right) \otimes \Gamma_0 C_0 \Gamma_0 G_0 \right] \tilde{\delta}(t)
\]

By Assumption 1 and the norm inequality, similar to (A1), we obtain

\[
2\delta^T(t)(I_N \otimes P^{-1})(\Phi \otimes \Gamma_{\theta_0}^{-1})^{-1} \Gamma_0 \mbox{d} \delta(t) \leq 2\| \delta(t) \| \| \Phi \otimes P^{-1} \| \| I_N \otimes \Gamma_{\theta_0}^{-1} \| \Gamma_0 \mbox{d} \delta(t)
\]

\[
= 2\| \delta(t) \| \| \Phi \| \| P^{-1} \| \| I_N \otimes \Gamma_{\theta_0}^{-1} \| \Gamma_0 \mbox{d} \delta(t)
\]

\[
\leq 2\eta_0 \| \Gamma_0 \| \| P^{-1} \| \| \tilde{\delta}(t) \| \mbox{d} \delta(t).
\]

Noting that \( \Phi \Phi^T = I_N \), \( \sum_{i=1}^{N} \left( S_{1,ij} \right) \leq I_N \), \( 1 \leq j \leq N \), \( \sum_{i=1}^{N} \left( S_{1,ij} \right) \leq I_N \), \( \sum_{i,j=1}^{N} \left( S_{2,ij} \right) = 2\mathcal{L}, \sum_{i=1}^{N} \left( S_{2,ij} \right) = F \leq 2F \).

By the definition of \( M_0(t) \), we get

\[
M_9(t) \leq \sum_{j=1}^{N} \left[ \sum_{i=1}^{N} \left( S_{1,ij} \right) \otimes G_1 \Gamma_{\theta_0}^{-1} \Gamma_0^{-1} \right] Y_j + \sum_{i=1}^{N} \left( S_{1,ij} \right) \otimes G_1 \Gamma_{\theta_0}^{-1} \Gamma_0^{-1} \right] Y_0
\]

\[
+ \sigma^2 \delta \left[ \left( \sum_{i=1}^{N} \left( S_{2,ij} \right) \right) \otimes \Gamma_0 C_0 \Gamma_0 G_0 \right] \tilde{\delta}(t)
\]

\[
\leq \sum_{j=1}^{N} \left[ \sum_{i=1}^{N} \left( S_{1,ij} \right) \right] Y_j + \sum_{i=1}^{N} \left( S_{1,ij} \right) Y_0
\]

\[
+ 2\delta^T(t) \left( \Phi^T \mathcal{L} \Phi \otimes \Gamma_0 C_0 \Gamma_0 G_0 \right) \tilde{\delta}(t)
\]

\[
+ 2\delta^T(t) \left( \Phi^T F \Phi \otimes \Gamma_0 C_0 \Gamma_0 G_0 \right) \tilde{\delta}(t),
\]

where \( \sigma^2 = \max \left\{ \max_{1 \leq i,j \leq N} \sigma_i^2, \max_{1 \leq i \leq N} \sigma_i^3 \right\} \).

Substituting (A5) and the above inequality into (A4), we have

\[
\begin{align*}
\mbox{d} \bar{V}(t) &\leq 2\delta^T(t)(I_N \otimes P^{-1})(I_N \otimes \theta_0 A_0 - \Phi^T L \Phi \otimes \Gamma_{\theta_0}^{-1} G_0 C_0 \Gamma_0 - \Phi^T F \Phi \otimes \Gamma_{\theta_0}^{-1} G_0 C_0 \Gamma_0) \tilde{\delta}(t) \mbox{d}t \\
&+ 2\eta_0 \| \Gamma_0 \| \| P^{-1} \| \| \tilde{\delta}(t) \| \mbox{d} \delta(t) + 2\sigma^2 \delta(t) \left( \Phi^T \mathcal{L} \Phi \otimes \Gamma_0 C_0 \Gamma_0 G_0 \right) \tilde{\delta}(t) \mbox{d}t
\end{align*}
\]
\[\begin{align*}
&+ 2\delta^T(t)(I_N \otimes P^{-1})M_2\delta w_{11}(t) + 2\delta^T(t)(I_N \otimes P^{-1})M_6\delta w_{10}(t) \\
&+ 2\delta^T(t)(I_N \otimes P^{-1})M_7\delta w_{20}(t) + 2\delta^T(t)(I_N \otimes P^{-1})M_8\delta w_{20}(t),
\end{align*}\]

where \( \sigma_1 = \sum_{j=1}^{N} Y_j^T \left[ I_N \otimes \Gamma_1^{T} \Gamma_1^{-1} \Gamma_0 \right] Y_j + Y_0^T \left[ I_N \otimes \Gamma_2^{T} \Gamma_2^{-1} \Gamma_0 \right] Y_0. \)

Denote \( \delta(t) = (I_N \otimes P^{-1}) \delta(t). \) By the above inequality, we obtain

\[\begin{align*}
dV(t) & \leq 2\delta^T(t)[I_N \otimes \theta_0A_0P - \Phi^T L \Phi \otimes \Gamma_0 \Gamma_0^{-1} G_1 C_0 \Gamma_0 \Gamma_0^{-1} P - \Phi^T F \Phi \otimes \Gamma_0 \Gamma_0^{-1} G_2 C_0 \Gamma_0 \Gamma_0^{-1} P] \delta(t) dt \\
&+ 2q_0 \sqrt{\tau_0} \|P^{-1}\| \delta^T(t) (I_N \otimes P^2) \delta(t) dt + 2\sigma^2 \delta^T(t) \left( \Phi^T L \Phi \otimes \Gamma_0 \Gamma_0^{-1} \Gamma_1^{-1} \Gamma_0 \Gamma_0^{-1} P \right) \delta(t) dt + \sigma_1 dt \\
&+ 2\delta^T(t)M_3\delta w_{11}(t) + 2\delta^T(t)M_6\delta w_{10}(t) + 2\delta^T(t)M_{10}(t)\delta w_{20}(t) + 2\delta^T(t)M_{11}(t)\delta w_{20}(t),
\end{align*}\]

where \( M_{10}(t) = \sum_{j=1}^{N} \sigma_0 \left[ \phi_1^T S_2 \phi \otimes \Gamma_0 \Gamma_0^{-1} G_2 C_0 \Gamma_0 \Gamma_0^{-1} P \right] \delta(t), \ M_{11}(t) = - \sum_{j=1}^{N} \sigma_0 \left[ 2 \sigma_0 \phi_1 \otimes \Gamma_0 \Gamma_0^{-1} G_2 C_0 \Gamma_0 \Gamma_0^{-1} P \right] \delta(t). \)

Denote \( \tilde{W}(t) = e^{\delta t} \tilde{V}(t), \) for any given \( \tilde{\gamma} \in \left( 0, \frac{2q_0 \sqrt{\tau_0} \|P^{-1}\| \lambda_{\max}(P)}{\lambda_{\min}(P)} \right). \) By the above inequality, \( C_0 \Gamma_0 = \theta_0 C_0 \) and applying Ito’s formula to \( \tilde{W}(t), \) we get

\[\begin{align*}
d\tilde{W}(t) &= \tilde{\gamma} e^{\delta t} \tilde{W}(t) dt + e^{\delta t} d\tilde{W}(t) \\
&\leq \tilde{\gamma} e^{\delta t} \tilde{W}(t) dt + 2e^{\delta t} \theta_0 \tilde{W}(t) (I_N \otimes A_0 P - \Phi^T L \Phi \otimes \Gamma_0 \Gamma_0^{-1} G_1 C_0 \Gamma_0 \Gamma_0^{-1} P) \delta(t) dt \\
&+ 2e^{\delta t} \theta_0 \tilde{W}(t) \left( \Phi^T L \Phi \otimes \Gamma_0 \Gamma_0^{-1} G_2 C_0 \Gamma_0 \Gamma_0^{-1} P \right) \delta(t) dt + 2q_0 \sqrt{\tau_0} \|P^{-1}\| \tilde{W}(t) \tilde{W}(t) dt \\
&+ 2e^{\delta t} \theta_0 \tilde{W}(t) \left( \Phi^T F \Phi \otimes \Gamma_0 \Gamma_0^{-1} \Gamma_1^{-1} \Gamma_0 \Gamma_0^{-1} G_2 C_0 \Gamma_0 \Gamma_0^{-1} P \right) \delta(t) dt + e^{\delta t} \sigma_1 dt \\
&+ 2e^{\delta t} \tilde{W}(t) \left( M_3 \delta w_{11}(t) + 2e^{\sigma t} \tilde{W}(t) M_6 \delta w_{10}(t) + 2e^{\sigma t} \tilde{W}(t) M_{10}(t) \delta w_{20}(t) + 2e^{\sigma t} \tilde{W}(t) M_{11}(t) \delta w_{20}(t) 
\end{align*}\]

where \( \Psi(\tilde{\gamma}) = \tilde{\gamma} (I_N \otimes P) + \theta_0 (I_N \otimes (PA_0 + A_0^T P) - 2 \theta_0 \Phi^T L \Phi \otimes \Gamma_0 \Gamma_0^{-1} G_1 C_0 \Gamma_0 \Gamma_0^{-1} P - 2 \theta_0 \Phi^T F \Phi \otimes \Gamma_0 \Gamma_0^{-1} G_2 C_0 \Gamma_0 \Gamma_0^{-1} P + 2 \sigma^2 \theta_0^2 (\Phi^T L \Phi \otimes P C_0 \Gamma_0 \Gamma_0^{-1} \Gamma_1^{-1} \Gamma_0 \Gamma_0^{-1} G_2 C_0 \Gamma_0 \Gamma_0^{-1} P) + 2q_0 \sqrt{\tau_0} \|P^{-1}\| (I_N \otimes P^2). \)

Integrating both sides of the above inequality from 0 to t and taking the mathematical expectation, we obtain

\[\begin{align*}
\mathbb{E}[\tilde{W}(t)] & \leq \mathbb{E}[\tilde{W}(0)] + \mathbb{E} \left[ \int_0^t e^{\delta s} [\tilde{\gamma}(s) \tilde{\Psi}(s)] ds \right] + \int_0^t e^{\delta s} \sigma_1 ds.
\end{align*}\]  
(A6)
In the following, we prove that the matrix $Ψ(\hat{\gamma}) < 0$. Noting that $G_1 = k_1 \Gamma_{\theta_0} \sigma_0^{C_h \Gamma_{\theta_0}^{0}}(I_p + C_0 \sigma_0^{C_h \Gamma_{\theta_0}^{0}})^{-1}$, $G_2 = k_2 \Gamma_{\theta_0} \sigma_0^{C_h \Gamma_{\theta_0}^{0}}(I_p + C_0 \sigma_0^{C_h \Gamma_{\theta_0}^{0}})^{-1}$ and the values of $k_1$, $k_2$, we have

$$Ψ(\hat{\gamma}) = \hat{\gamma}(I_N \otimes P) + \theta_0(I_N \otimes (PA_0 + A_0^T P)) - 2\theta_0 \Phi \Lambda \Phi \otimes \Gamma_{\theta_0}^{-1} G_1 C_0 P - 2\theta_0 \Phi \Lambda \Phi \otimes \Gamma_{\theta_0}^{-1} G_2 C_0 P$$

$$+ 2\sigma_0^2 \theta_0 \left( \Phi \Lambda \Phi \otimes PC_0^{\Gamma_{\theta_0}^{0}} G_1^{\Gamma_{\theta_0}^{-1} P \Gamma_{\theta_0}^{-1} G_1 C_0 P} \right)$$

By the generalized algebraic Riccati equation $A_0 P + PA_0^T \leq 2q_0 \sigma_0^{C_h \Gamma_{\theta_0}^{0}}(I_p + C_0 \sigma_0^{C_h \Gamma_{\theta_0}^{0}})^{-1} C_0 P + I_n = 0$, we obtain

$$\hat{\gamma} P + \theta_0(PA_0 + A_0^T P) - \frac{2\theta_0 q_0}{\lambda_1(\Lambda + F_0)} \Phi \Lambda \Phi \otimes PC_0^{\Gamma_{\theta_0}^{0}} G_1^{\Gamma_{\theta_0}^{-1} P \Gamma_{\theta_0}^{-1} G_1 C_0 P} + 2q_0 \sqrt{n_0} \|\| P^{-1}\|P^2$$

Combining (A6)–(A8), we have

$$\mathbb{E}[\hat{W}(t)] \leq \mathbb{E}[\hat{W}(0)] + \frac{\sigma_1}{\hat{\gamma}}[e^{\hat{\gamma} t} - 1].$$
Taking the upper limit on both sides of the above inequality, we obtain

$$\limsup_{t \to \infty} \mathbb{E}[\|\delta(t)\|^2] \leq \frac{\theta^{2n_0} \lambda_{\max}(P)\sigma_1}{\bar{\gamma}}.$$ 

By the definition of $\delta(t)$ and the above inequality, we get

$$\limsup_{t \to \infty} \mathbb{E}[\|\hat{x}_{i0}(t) - x_0(t)\|^2] \leq \frac{\theta^{2n_0} \lambda_{\max}(P)\sigma_1}{\bar{\gamma}}.$$ 

Then, we proceed to estimate $\limsup_{t \to \infty} \mathbb{E}[\|y_i(t) - y_0(t)\|^2], i = 1, \ldots, N$.

Denote $\Delta_i(t) = \hat{x}_i(t) - \pi_i(x_0(t)), i = 1, \ldots, N$. By (8) and Assumption 2, we have

\begin{align*}
\dot{\Delta}_i(t) &= \dot{\hat{x}}_i(t) - \dot{\pi}_i(x_0(t)) \\
&= A_i\hat{x}_i(t) + f_i(\hat{x}_i(t)) + B_iu_i(t) + \Delta_i(t) - \pi_i(x_0(t))) - \frac{\partial \pi_i}{\partial x_0}(A_0x_0(t) + f_0(x_0(t))) \\
&= A_i\hat{x}_i(t) + f_i(\hat{x}_i(t)) + B_i[\pi_i(\hat{x}_i(t)) + \theta_i^{n_i+1}K_i\Gamma_i^{-1}(\hat{x}_i(t) - \pi_i(x_0(t)))]) + \Delta_i(t) - \pi_i(x_0(t))) \\
&\quad - [A_i\pi_i(x_0(t)) + f_i(\pi_i(x_0(t))) + B_i\pi_i(x_0(t))]] \\
&= A_i\Delta_i(t) + \left[f_i(\hat{x}_i(t)) - f_i(\pi_i(x_0(t)))\right] + B_i[\pi_i(\hat{x}_i(t)) - \pi_i(x_0(t))] + \Delta_i(t) - \pi_i(x_0(t))) \\
&\quad + \theta_i^{n_i+1}B_iK_i\Gamma_i^{-1}(\pi_i(x_0(t)) - \pi_i(\hat{x}_i(t))) \\
&= (A_i + \theta_i^{n_i+1}B_iK_i\Gamma_i^{-1})\Delta_i(t) + \left[f_i(\hat{x}_i(t)) - f_i(\pi_i(x_0(t)))\right] + B_i[\pi_i(\hat{x}_i(t)) - \pi_i(x_0(t))] \\
&\quad + \theta_i^{n_i+1}B_iK_i\Gamma_i^{-1}(\pi_i(x_0(t)) - \pi_i(\hat{x}_i(t))). \tag{A9}
\end{align*}

Denote $\Delta_i(t) = \Gamma_i^{-1}\Delta_i(t), i = 1, \ldots, N$. Combing $\Gamma_i^{-1}A_i\Gamma_i = \theta_iA_i; \theta_i^{n_i+1}B_i = \theta_iB_i$ and (A9), we get

\begin{align*}
\dot{\Delta}_i(t) &= \theta_i(A_i + B_iK_i)\Delta_i(t) + \Gamma_i^{-1}\left[f_i(\pi_i(x_0(t))) - f_i(\pi_i(x_0(t)))\right] + \Gamma_i^{-1}B_i[\pi_i(\hat{x}_i(t)) - \pi_i(x_0(t))] \\
&\quad + \dot{\pi}_i(x_0(t)) - \pi_i(x_0(t))) \\
&= \theta_i(A_i + B_iK_i)\Delta_i(t) + \Gamma_i^{-1}\left[f_i(\pi_i(x_0(t))) - f_i(\pi_i(x_0(t)))\right] + \Gamma_i^{-1}B_i[\pi_i(\hat{x}_i(t)) - \pi_i(x_0(t)))] \\
&\quad + \Delta_i(t) - \pi_i(x_0(t))) \\
&\quad + \Gamma_i^{-1}\dot{\pi}_i(x_0(t)) - \pi_i(x_0(t))) \\
&\quad + \theta_iB_iK_i\Gamma_i^{-1}[\pi_i(x_0(t)) - \pi_i(\hat{x}_i(t))]. \tag{A10}
\end{align*}

Choose $\hat{V}_i(t) = \Delta_i^T(t)P_i\Delta_i(t), i = 1, \ldots, N$, where $P_i$ is the unique positive solution of the Lyapunov equation $(A_i + B_iK_i)P_i + P_i(A_i + B_iK_i)^T = -I_{n_i}$.

By differentiating $\hat{V}_i(t)$, we have

\begin{align*}
\dot{\hat{V}}_i(t) &= 2\theta_i\Delta_i^T(t)P_i(A_i + B_iK_i)\Delta_i(t) + 2\Delta_i^T(t)\dot{P}_i\Gamma_i^{-1}\left[f_i(\hat{x}_i(t)) - f_i(\pi_i(x_0(t)))\right] \\
&\quad + 2\Delta_i^T(t)\hat{P}_i\Gamma_i^{-1}B_i[\pi_i(\hat{x}_i(t)) - \pi_i(x_0(t))] + 2\Delta_i^T(t)\hat{P}_iH_iC_i(\hat{x}_i(t) - x_0(t)) \\
&\quad + 2\theta_i\Delta_i^T(t)P_iB_iK_i\Gamma_i^{-1}[\pi_i(x_0(t)) - \pi_i(\hat{x}_i(t)))] \\
&= -\theta_i\|\Delta_i(t)\|^2 + 2\Delta_i^T(t)\dot{P}_i\Gamma_i^{-1}\left[f_i(\hat{x}_i(t)) - f_i(\pi_i(x_0(t)))\right] \\
&\quad + 2\Delta_i^T(t)\hat{P}_i\Gamma_i^{-1}B_i[\pi_i(\hat{x}_i(t)) - \pi_i(x_0(t))] + 2\Delta_i^T(t)\hat{P}_iH_iC_i(\hat{x}_i(t) - x_0(t)) \\
&\quad + 2\theta_i\Delta_i^T(t)P_iB_iK_i\Gamma_i^{-1}[\pi_i(x_0(t)) - \pi_i(\hat{x}_i(t))]. \tag{A11}
\end{align*}
Similar to (A1), we get
\[ 2\Delta_t^T(t)\bar{P}_t\Gamma_{\theta_i}^{-1}[\tilde{f}(\hat{x}_i(t)) - f_t(\pi_i(x_0(t)))] \]
\[ \leq 2\|\Delta_t(t)\|\|\bar{P}_t\|\|\Gamma_{\theta_i}^{-1}[f_t(\xi_i(t)) - f_t(\pi_i(x_0(t)))] \]
\[ \leq 2\sqrt{n_i}\|\bar{P}_t\|\|\Delta_t(t)\|^2. \]

Combining the above inequality, \(2q_i\sqrt{n_i}\|\bar{P}_t\| < \theta_i\) and \(\bar{V}_i(t) \leq \lambda_{\text{max}}(\bar{P}_t)\|\Delta_t(t)\|^2\), we have
\[ -\theta_i\|\Delta_t(t)\|^2 + 2\Delta_t^T(t)\bar{P}_t\Gamma_{\theta_i}^{-1}[\tilde{f}(\hat{x}_i(t)) - f_t(\pi_i(x_0(t)))] \]
\[ \leq (2q_i\sqrt{n_i}\|\bar{P}_t\| - \theta_i)\|\Delta_t(t)\|^2 \]
\[ \leq (2q_i\sqrt{n_i}\|\bar{P}_t\| - \theta_i)\frac{\bar{V}_i(t)}{\lambda_{\text{max}}(\bar{P}_t)}. \quad (A12) \]

Noting that \(\lambda_{\text{min}}(\bar{P}_t)\|\Delta_t(t)\|^2 \leq \bar{V}_i(t)\), by the norm inequality, we obtain
\[ 2\Delta_t^T(t)\bar{P}_tH_tC_i(\hat{x}_i(t) - x_i(t)) \]
\[ \leq 2\|\Delta_t(t)\|\|\bar{P}_t\|\|H_t\|\|C_i\|\|\hat{x}_i(t) - x_i(t)\| \]
\[ \leq \frac{2\sqrt{\bar{V}_i(t)}}{\sqrt{\lambda_{\text{min}}(\bar{P}_t)}}\|\bar{P}_t\|\|H_t\|\|C_i\|\|\hat{x}_i(t) - x_i(t)\|. \quad (A13) \]

Combining Assumption 2, \(\lambda_{\text{min}}(\bar{P}_t)\|\Delta_t(t)\|^2 \leq \bar{V}_i(t)\) and the norm inequality, we have
\[ 2\theta_i\Delta_t^T(t)\bar{P}_tB_tK_i\Gamma_{\theta_i}^{-1}[\pi_i(x_0(t)) - \pi_i(\hat{x}_{0i}(t))] \]
\[ = 2\Delta_t^T(t)\bar{P}_tB_tK_i\theta_i\Gamma_{\theta_i}^{-1}[\pi_i(x_0(t)) - \pi_i(\hat{x}_{0i}(t))] \]
\[ \leq 2\|\Delta_t(t)\|\|\bar{P}_t\|\|B_t\|\|K_i\|\|\theta_i\Gamma_{\theta_i}^{-1}\|\|\pi_i(x_0(t)) - \pi_i(\hat{x}_{0i}(t))\| \]
\[ \leq 2\|\Delta_t(t)\|\|\bar{P}_t\|\|K_i\|\|\pi_i(x_0(t)) - \pi_i(\hat{x}_{0i}(t))\| \]
\[ \leq 2\rho_1\|\Delta_t(t)\|\|\bar{P}_t\|\|\pi_i(x_0(t)) - \pi_i(\hat{x}_{0i}(t))\| \]
\[ \leq \frac{2\rho_1\sqrt{\bar{V}_i(t)}}{\sqrt{\lambda_{\text{min}}(\bar{P}_t)}}\|\bar{P}_t\|\|K_i\|\|\hat{x}_{0i}(t) - x_0(t)\|. \quad (A14) \]

Similar to the above inequality, we get
\[ 2\Delta_t^T(t)\bar{P}_t\Gamma_{\theta_i}^{-1}B_t[\theta_i(\hat{x}_{0i}(t)) - \theta_i(x_0(t))] \]
\[ = \frac{2}{\theta_i}2\Delta_t^T(t)\bar{P}_tB_t[\theta_i(\hat{x}_{0i}(t)) - \theta_i(x_0(t))] \]
\[ \leq \frac{2}{\theta_i}2\|\Delta_t(t)\|\|\bar{P}_t\|\|B_t\|\|\theta_i(\hat{x}_{0i}(t)) - \theta_i(x_0(t))\| \]
\[ \leq \frac{2}{\theta_i^2}2\rho_2\|\Delta_t(t)\|\|\bar{P}_t\|\|\hat{x}_{0i}(t) - x_0(t)\| \]
\[ \leq \frac{2\rho_2\sqrt{\bar{V}_i(t)}}{\theta_i\sqrt{\lambda_{\text{min}}(\bar{P}_t)}}\|\bar{P}_t\|\|\hat{x}_{0i}(t) - x_0(t)\|. \quad (A15) \]
Substituting (A12)–(A15) into (A11) leads to

\[
\dot{V}_i(t) \leq (2q_i\|\bar{P}_i\| - \theta_i)\frac{\bar{V}_i(t)}{\lambda_{\max}(P_i)} + \frac{2\sqrt{V_i(t)}}{\lambda_{\min}(P_i)}\|\dot{x}_i(t) - x_i(t)\| \\
+ \frac{2\rho_2\sqrt{V_i(t)}}{\theta_i^{\alpha_2}}\|P_i\|\|\dot{x}_i(t) - x_i(t)\| + \frac{2\rho_1\sqrt{V_i(t)}}{\lambda_{\min}(P_i)}\|\dot{K}_i\|\|\dot{x}_i(t) - x_i(t)\|.
\]  

(A16)

Denote \(\dot{V}_i(t) = \sqrt{V_i(t)}\). By the above inequality, we obtain

\[
d\dot{V}_i(t) = \frac{1}{2\sqrt{V_i(t)}}dV_i(t) \\
\leq \frac{(2q_i\|\bar{P}_i\| - \theta_i)}{2\lambda_{\max}(P_i)}dV_i(t) + \frac{1}{\lambda_{\min}(P_i)}\|\dot{P}_i\||H_i||\dot{x}_i(t) - x_i(t)\|dt \\
+ \frac{\rho_2}{\theta_i^{\alpha_2}}\|\dot{P}_i\|\|\dot{x}_i(t) - x_i(t)\|dt + \frac{\rho_1}{\lambda_{\min}(P_i)}\|\dot{P}_i\|\|K_i\|\|\dot{x}_i(t) - x_i(t)\|dt \\
= -(\theta_i - 2q_i\|\bar{P}_i\|)\frac{\dot{V}_i(t)}{2\lambda_{\min}(P_i)}dt + \frac{1}{\lambda_{\min}(P_i)}\|\dot{P}_i\|\|H_i||\dot{x}_i(t) - x_i(t)\|dt \\
+ \left[\frac{\rho_2}{\theta_i^{\alpha_2}}\|\dot{P}_i\| + \frac{\rho_1}{\lambda_{\min}(P_i)}\|\dot{P}_i\|\|K_i\|\right]\|\dot{x}_i(t) - x_i(t)\|dt \\
\leq -\sigma_2\dot{V}_i(t)dt + \sigma_3\|\dot{x}_i(t) - x_i(t)\|dt + \sigma_4\|\dot{x}_i(t) - x_i(t)\|dt.
\]

where \(\sigma_2 = \min_{i\in\{1,2,\ldots,N\}} \left\{ \frac{1}{2\lambda_{\max}(P_i)}(\theta_i - 2q_i\|\bar{P}_i\|) \right\} \), \(\sigma_3 = \max_{i\in\{1,2,\ldots,N\}} \left\{ \frac{1}{\lambda_{\min}(P_i)}\|\dot{P}_i\||H_i| \right\} \), \(\sigma_4 = \max_{i\in\{1,2,\ldots,N\}} \left\{ \frac{\rho_2}{\theta_i^{\alpha_2}}\|\dot{P}_i\| + \frac{\rho_1}{\lambda_{\min}(P_i)}\|\dot{P}_i\|\|K_i\| \right\} \).

Integrating both sides of the above inequality from 0 to \(t\), we get

\[
V_i(t) \leq e^{-\sigma_2 t}V_i(0) + \int_0^t \sigma_3 e^{-\sigma_2 (t-s)}\|\dot{x}_i(s) - x_i(s)\|ds + \int_0^t \sigma_4 e^{-\sigma_2 (t-s)}\|\dot{x}_i(s) - x_i(s)\|ds.
\]

By \(\dot{V}_i(t) = \sqrt{V_i(t)}\) and the above inequality, we have

\[
\mathbb{E}[\dot{V}_i(t)] \\
\leq \mathbb{E} \left[ e^{-\sigma_2 t}V_i(0) + \int_0^t \sigma_3 e^{-\sigma_2 (t-s)}\|\dot{x}_i(s) - x_i(s)\|ds + \int_0^t \sigma_4 e^{-\sigma_2 (t-s)}\|\dot{x}_i(s) - x_i(s)\|ds \right]^2 \\
\leq 3e^{-2\sigma_2 t}\mathbb{E}[\dot{V}_i^2(0)] + 3\mathbb{E} \left[ \int_0^t \sigma_3 e^{-\sigma_2 (t-s)}\|\dot{x}_i(s) - x_i(s)\|ds \right]^2 \\
+ 3\mathbb{E} \left[ \int_0^t \sigma_4 e^{-\sigma_2 (t-s)}\|\dot{x}_i(s) - x_i(s)\|ds \right]^2.
\]  

(A17)

By \(\sigma_2 > 0\), we get

\[
\lim_{t \to \infty} e^{-2\sigma_2 t}\mathbb{E}[\dot{V}_i^2(0)] = 0.
\]  

(A18)
Noting that \( \left( \mathbb{E} \left\| \int_0^t X(r) \, dr \right\|^2 \right)^{1/2} \leq f'_0(1) \left( \mathbb{E} \left\| X(r) \right\|^2 \right)^{1/2} \), we obtain
\[
\left( \mathbb{E} \left[ \left( \int_0^t \sigma_3 e^{-\sigma_2(t-s)} \| \dot{X}(s) - x(s) \| ds \right)^2 \right] \right)^{1/2} \\
\leq \int_0^T \left( \mathbb{E} \left[ \left( \sigma_3 e^{-\sigma_2(t-s)} \| \dot{X}(s) - x(s) \| \right)^2 \right] \right)^{1/2} ds \\
= \int_0^T \left( \sigma_3^2 e^{-2\sigma_2(t-s)} \mathbb{E} \left[ \| \dot{X}(s) - x(s) \|^2 \right] \right)^{1/2} ds.
\]
(A19)

By (A3), we know that for any given constant \( \epsilon_1 > 0 \), there exists a positive constant \( T_1 \) such that for any \( t > T_1 \)
\[
\mathbb{E} \left[ \| \dot{X}(t) - x(t) \|^2 \right] \leq \frac{\theta^{2\eta_1}}{2\gamma \lambda_{\min}(P)} \text{Tr} \left\{ \rho^T_i \Gamma^{-1}_\theta P_i \Gamma^{-1}_\theta \beta_i \right\} + \epsilon_1.
\]

By (A19) and the above inequality, we have
\[
\left( \mathbb{E} \left[ \left( \int_0^t \sigma_3 e^{-\sigma_2(t-s)} \| \dot{X}(s) - x(s) \| ds \right)^2 \right] \right)^{1/2} \\
\leq \int_0^T \left( \sigma_3^2 e^{-2\sigma_2(t-s)} \mathbb{E} \left[ \| \dot{X}(s) - x(s) \|^2 \right] \right)^{1/2} ds \\
+ \int_0^T \left( \sigma_3^2 e^{-2\sigma_2(t-s)} \mathbb{E} \left[ \| \dot{X}(s) - x(s) \|^2 \right] \right)^{1/2} ds \\
= \left( \frac{\theta^{2\eta_1}}{2\gamma \lambda_{\min}(P)} \text{Tr} \left\{ \rho^T_i \Gamma^{-1}_\theta P_i \Gamma^{-1}_\theta \beta_i \right\} + \epsilon_1 \right) \int_0^t e^{-\sigma_2(t-s)} ds \\
+ \sigma_3 \left( \frac{\theta^{2\eta_1}}{2\gamma \lambda_{\min}(P)} \text{Tr} \left\{ \rho^T_i \Gamma^{-1}_\theta P_i \Gamma^{-1}_\theta \beta_i \right\} + \epsilon_1 \right) \frac{1}{\sigma_2} \left[ 1 - e^{-\sigma_2(t-T_1)} \right].
\]

Taking the upper limit on both sides of the above inequality, we get
\[
\limsup_{t \to \infty} \left( \mathbb{E} \left[ \left( \int_0^t \sigma_3 e^{-\sigma_2(t-s)} \| \dot{X}(s) - x(s) \| ds \right)^2 \right] \right)^{1/2} \leq \frac{\sigma_3 \left( \frac{\theta^{2\eta_1}}{2\gamma \lambda_{\min}(P)} \text{Tr} \left\{ \rho^T_i \Gamma^{-1}_\theta P_i \Gamma^{-1}_\theta \beta_i \right\} + \epsilon_1 \right)^{1/2}}{\sigma_2}.
\]
Then, by the arbitrariness of $\epsilon_1$ and the above inequality, we obtain
\[
\limsup_{t \to \infty} \left( \mathbb{E} \left[ \left( \int_0^t \sigma_3 e^{-\sigma_2 (t-s)} \| \hat{x}_i(s) - x_i(s) \| ds \right)^2 \right] \right)^{\frac{1}{2}} \leq \frac{\sigma_3}{\lambda_{\min}(\bar{P}_1)} \frac{\pi^{-\frac{1}{2}} \sqrt{2 \gamma}}{\sqrt{\lambda_{\max}(P) \sigma_2^2}}. 
\]
By the above inequality, we have
\[
\limsup_{t \to \infty} \mathbb{E} \left[ \left( \int_0^t \sigma_4 e^{-\sigma_2 (t-s)} \| \hat{x}_i(s) - x_i(s) \| ds \right)^2 \right] \leq \frac{\sigma_4^2 \theta_0^{2n_i} \lambda_{\max}(P) \sigma_1}{\gamma \sigma_2^2}. 
\] (A20)
Similar to the above inequality, we get
\[
\limsup_{t \to \infty} \mathbb{E} \left[ \left( \int_0^t \sigma_4 e^{-\sigma_2 (t-s)} \| \hat{x}_0(s) - x_0(s) \| ds \right)^2 \right] \leq \frac{\sigma_4^2 \theta_0^{2n_i} \lambda_{\max}(P) \sigma_1}{\gamma \sigma_2^2}. 
\] (A21)
Combining (A17)–(A18) and (A20)–(A21), we have
\[
\limsup_{t \to \infty} \mathbb{E}[\bar{V}_i(t)] \leq \frac{3 \sigma_3^2 \theta_0^{2n_i} \pi^{-\frac{1}{2}} \sqrt{2 \gamma}}{\lambda_{\min}(\bar{P}_1) \lambda_{\min}(\bar{P}_1)} + \frac{3 \sigma_4^2 \theta_0^{2n_i} \lambda_{\max}(P) \sigma_1}{\gamma \sigma_2^2}. 
\]
Noting that $\lambda_{\min}(\bar{P}_1) \mathbb{E}[\| \Delta_i(t) \|^2] \leq \mathbb{E}[\bar{V}_i(t)]$, by the above inequality, we obtain
\[
\limsup_{t \to \infty} \mathbb{E}[\| \Delta_i(t) \|^2] \leq \frac{3 \sigma_3^2 \theta_0^{2n_i} \pi^{-\frac{1}{2}} \sqrt{2 \gamma}}{\lambda_{\min}(\bar{P}_1) \lambda_{\min}(\bar{P}_1)} + \frac{3 \sigma_4^2 \theta_0^{2n_i} \lambda_{\max}(P) \sigma_1}{\gamma \sigma_2^2}. 
\]
Combining $\Delta_i(t) = \Gamma_i^{-1} \Delta_i(t)$ and the above inequality, we have
\[
\limsup_{t \to \infty} \mathbb{E}[\| \Delta_i(t) \|^2] \leq \| \Gamma_i \|^2 \mathbb{E}[\| \Delta_i(t) \|^2] \leq \frac{3 \sigma_3^2 \theta_0^{2n_i} \pi^{-\frac{1}{2}} \sqrt{2 \gamma}}{\lambda_{\min}(\bar{P}_1) \lambda_{\min}(\bar{P}_1)} + \frac{3 \sigma_4^2 \theta_0^{2n_i} \lambda_{\max}(P) \sigma_1}{\gamma \sigma_2^2}. 
\] (A22)
By the norm inequality, (4), (5), and (8), we get
\[
\mathbb{E}[\| y_i(t) - y_0(t) \|^2] = \mathbb{E}[\| y_i(t) - C \hat{x}_i(t) + C \hat{x}_i(t) - y_0(t) \|^2] \\
\leq 2 \mathbb{E}[\| y_i(t) - C \hat{x}_i(t) \|^2] + 2 \mathbb{E}[\| C \hat{x}_i(t) - y_0(t) \|^2] \\
\leq 2 \mathbb{E}[\| y_i(t) - C \hat{x}_i(t) \|^2] + 2 \mathbb{E}[\| C \hat{x}_i(t) - C \pi(t) \|^2] \\
= 2 \mathbb{E}[\| y_i(t) - C \hat{x}_i(t) \|^2] + 2 \mathbb{E}[\| C \hat{x}_i(t) - C \pi(t) \|^2] \\
\leq 2 \mathbb{E}[\| y_i(t) - C \hat{x}_i(t) \|^2] + 2 \mathbb{E}[\| y_i(t) - C \pi(t) \|^2] \\
= 2 \mathbb{E}[\| y_i(t) - C \hat{x}_i(t) \|^2] + 2 \mathbb{E}[\| y_i(t) - C \pi(t) \|^2] \\
\leq 2 \mathbb{E}[\| y_i(t) - C \hat{x}_i(t) \|^2] + 2 \mathbb{E}[\| y_i(t) - C \pi(t) \|^2]. 
\]
By (A3), (A22) and the above inequality, we obtain
\[
\limsup_{t \to \infty} \mathbb{E}[\| y_i(t) - y_0(t) \|^2] \\
\leq \frac{\theta_0^{2n_i} \pi^{-\frac{1}{2}} \sqrt{2 \gamma}}{\lambda_{\min}(\bar{P}_1)} + \frac{3 \sigma_3^2 \theta_0^{2n_i} \pi^{-\frac{1}{2}} \sqrt{2 \gamma}}{\lambda_{\min}(\bar{P}_1) \lambda_{\min}(\bar{P}_1)} + \frac{6 \sigma_4^2 \theta_0^{2n_i} \lambda_{\max}(P)}{\gamma \sigma_2^2}. 
\]
\[ \lim_{t \to \infty} \mathbb{E}[[\|y(t) - y_0(t)\|^2]] \leq \frac{\theta_{i0}^2 \text{Tr} \left\{ \left( 1 + \frac{3\sigma_i^2}{\sigma^2 \lambda_{\min}(P_0)} \right) \left( 1 + \frac{3\sigma_i^2}{\sigma^2 \lambda_{\min}(P_1)} \right) \lambda_{\min}(P_1) \right\}}{\lambda_{\min}(P_t)} \cdot \min_{i \in \{1, \ldots, N\}} \left\{ \frac{\theta_i - 2q_i \sqrt{n_i |P_t|}}{\lambda_{\min}(P_1)} \right\}. \]
Choose the Lyapunov function $\hat{V}(t) = \delta^T(t)(I_N \otimes P^{-1})\hat{S}(t)$, where $P$ is the unique positive solution of the generalized Riccati equation $A_0P + PA_0^T - 2q_0PC_0^T(I_p + C_0PC_0^T)^{-1}C_0P + I_N = 0$. By the above equation and Itô's formula, we obtain

\[
d\hat{V}(t) = 2\delta^T(t)(I_N \otimes P^{-1}) \left[ I_N \otimes \theta_0(l)A_0 - \Phi^T(l)\mathcal{L}(l)\Phi(l) \otimes \Gamma^{-1}_{\theta_0}(l)G_1(l)C_0\Gamma_{\theta_0}(l) \right] \delta(t)dt + 2\delta^T(t)(I_N \otimes P^{-1})
\]

\[
\times (\Phi(l) \otimes \Gamma_{\theta_0}(l))^{-1} \Delta f_0(t)dt + M_9(t)dt + \sum_{j=1}^S q_j\hat{V}(t)
\]

\[
+ 2\delta^T(t)(I_N \otimes P^{-1})M_5(t)dw_{1j}(t) + 2\delta^T(t)(I_N \otimes P^{-1})M_6(t)dw_{10}(t)
\]

\[
+ 2\delta^T(t)(I_N \otimes P^{-1})M_7(t)dw_{2j}(t) + 2\delta^T(t)(I_N \otimes P^{-1})M_8(t)dw_{20}(t),
\]

where $M_9(t) = \sum_{i,j=1}^N \gamma_i^T \left[ \left( S_{ij}(l)\Phi(l)\Phi^T(l)S_{ij}(l) \otimes G_1^T(l)\Gamma^{-1}_{\theta_0}(l)P^{-1}\Gamma^{-1}_{\theta_0}(l)G_1(l) \right) Y_j + \sum_{i,j=1}^N \gamma_i^T \left( \tilde{S}_{ij}(l)\Phi(l)\Phi^T(l)\tilde{S}_{ij}(l) \otimes G_1^T(l)\Gamma^{-1}_{\theta_0}(l)P^{-1}\Gamma^{-1}_{\theta_0}(l)G_1(l) \right) Y_j \right]$

\[
+ \sum_{i=1}^N \sigma^2(t) \left[ \left( \Phi(l)G_2^T(l)\Gamma^{-1}_{\theta_0}(l)P^{-1}\Gamma^{-1}_{\theta_0}(l)G_2(l) \otimes \Gamma_{\theta_0}(l)C_0^TG_1^T(l)\Gamma^{-1}_{\theta_0}(l)P^{-1}\Gamma^{-1}_{\theta_0}(l)G_1(l)C_0\Gamma_{\theta_0}(l) \right) \delta(t) + \sum_{i=1}^N \sigma^2(t) \left( \Phi(l)G_2^T(l)\Gamma^{-1}_{\theta_0}(l)P^{-1}\Gamma^{-1}_{\theta_0}(l)G_2(l) \otimes \Gamma_{\theta_0}(l)C_0^TG_1^T(l)\Gamma^{-1}_{\theta_0}(l)P^{-1}\Gamma^{-1}_{\theta_0}(l)G_1(l)C_0\Gamma_{\theta_0}(l) \right) \delta(t), \quad l \in S.
\]

By Assumption 1 and the norm inequality, similar to (A5), we get

\[
2\delta^T(t)(I_N \otimes P^{-1})\Phi(l) \otimes \Gamma_{\theta_0}(l)^{-1} \Delta f_0(t) = 2\delta^T(t)(\Phi^T(l) \otimes P^{-1})(I_N \otimes \Gamma_{\theta_0}(l)^{-1}) \Delta f_0(t) \leq 2\|\delta(t)\| \|\Phi(l)\| \|P^{-1}\| \|(I_N \otimes \Gamma_{\theta_0}(l)^{-1}) \Delta f_0(t)\| \\
\leq 2q_0\sqrt{r_0} \|P^{-1}\| \|\delta(t)\|.
\]

Noting that $\Phi(l)\Phi^T(l) = I_N$, $\sum_{i=1}^N \left( S_{1ij}(l)S_{1ij}(l) \right) \leq I_N, \ 1 \leq j \leq N$, $\sum_{i=1}^N \left( \tilde{S}_{1ij}(l)\tilde{S}_{1ij}(l) \right) \leq I_N, \ \sum_{i=1}^N \left( S_{2ij}(l)S_{2ij}(l) \right) = 2\mathcal{L}(l), \ \sum_{i=1}^N \left( \tilde{S}_{2ij}(l)\tilde{S}_{2ij}(l) \right) = F(l) \leq 2F(l), \ l \in S$, by the definition of $M_9(t)$, we have

\[
M_9(t) \leq \sum_{j=1}^N \gamma_j^T \left[ \sum_{i=1}^N \left( S_{1ij}(l)S_{1ij}(l) \otimes G_1^T(l)\Gamma^{-1}_{\theta_0}(l)P^{-1}\Gamma^{-1}_{\theta_0}(l)G_1(l) \right) Y_j \right] \gamma_j + \gamma_0^T \left[ \sum_{i=1}^N \left( \tilde{S}_{1ij}(l)\tilde{S}_{1ij}(l) \otimes G_2(l)\Gamma^{-1}_{\theta_0}(l)P^{-1}\Gamma^{-1}_{\theta_0}(l)G_2(l) \right) Y_0 \right] \\
+ \sigma^2(t) \left[ \sum_{i=1}^N \left( \Phi(l)S_{2ij}(l)S_{2ij}(l) \otimes \Gamma_{\theta_0}(l)C_0^TG_1^T(l)\Gamma^{-1}_{\theta_0}(l)P^{-1}\Gamma^{-1}_{\theta_0}(l)G_1(l)C_0\Gamma_{\theta_0}(l) \right) \delta(t) \right] \\
+ \sigma^2(t) \left[ \sum_{i=1}^N \left( \Phi(l)\tilde{S}_{2ij}(l)\tilde{S}_{2ij}(l) \otimes \Gamma_{\theta_0}(l)C_0^TG_1^T(l)\Gamma^{-1}_{\theta_0}(l)P^{-1}\Gamma^{-1}_{\theta_0}(l)G_1(l)C_0\Gamma_{\theta_0}(l) \right) \delta(t) \right]
\]

\[
\leq \sum_{j=1}^N \gamma_j^T \left[ I_N \otimes G_1^T(l)\Gamma^{-1}_{\theta_0}(l)P^{-1}\Gamma^{-1}_{\theta_0}(l)G_1(l) \right] Y_j + \gamma_0^T \left[ I_N \otimes G_2^T(l)\Gamma^{-1}_{\theta_0}(l)P^{-1}\Gamma^{-1}_{\theta_0}(l)G_2(l) \right] Y_0 \\
+ 2\sigma^2(t) \left[ \Phi(l)\mathcal{L}(l)\Phi(l) \otimes \Gamma_{\theta_0}(l)C_0^TG_1^T(l)\Gamma^{-1}_{\theta_0}(l)P^{-1}\Gamma^{-1}_{\theta_0}(l)G_1(l)C_0\Gamma_{\theta_0}(l) \right] \delta(t) \\
+ 2\sigma^2(t) \left[ \Phi(l)F(l)\Phi(l) \otimes \Gamma_{\theta_0}(l)C_0^TG_2^T(l)\Gamma^{-1}_{\theta_0}(l)P^{-1}\Gamma^{-1}_{\theta_0}(l)G_2(l)C_0\Gamma_{\theta_0}(l) \right] \delta(t),
\]

where $l \in S$, $\sigma^2 = \max \{ \max_{1 \leq j \leq N} \sigma_j^2, \max_{1 \leq i \leq N} \sigma_i^2 \}$. 
Noting that $\sum_{j=1}^S q_{0j} = 0$, by (B3), (B4) and the above inequality, we obtain

$$d \tilde{V}(t) \leq 2\delta^T(t) (I_N \otimes P^{-1})(I_N \otimes \theta_0(t)A_0 - \Phi^T(t)L(t)\Phi(t) \otimes \Gamma^{-1}_i(l)G_1(l)C_0\Gamma_{0i}(l)$$

$$- \Phi^T(t)F(t)\Phi(t) \otimes \Gamma^{-1}_i(l)G_2(l)C_0\Gamma_{0i}(l)\tilde{b}(t)dt + 2q_0\sqrt{n_0}\|P^{-1}\|\tilde{\sigma}(t)\tilde{\sigma}(t)dt$$

$$+ 2\sigma^2\tilde{\sigma}^T(t) \left[ \Phi^T(t)L(t)\Phi(t) \otimes \Gamma^{-1}_i(l)G_2(l)C_0\Gamma_{0i}(l) \right] \delta(t)dt + \varpi_1 dt + 2\delta^T(t)(I_N \otimes P^{-1})M_5(t)dw_{12}(t)$$

$$+ 2\delta^T(t)(I_N \otimes P^{-1})M_6(t)dw_{10}(t) + 2\delta^T(t)(I_N \otimes P^{-1})M_7(t)dw_{22}(t)$$

$$+ 2\delta^T(t)(I_N \otimes P^{-1})M_8(t)dw_{20}(t),$$

where $\varpi_1 = \max_{l \in S} \left\{ \sum_{j=1}^N \gamma_j^T \left[ I_N \otimes G_1^T(l)\Gamma^{-1}_i(l)P^{-1}\Gamma^{-1}_i(l)G_1(l) \right] Y_j + \gamma_0^T \left[ I_N \otimes G_2^T(l)\Gamma^{-1}_i(l)P^{-1}\Gamma^{-1}_i(l)G_2(l) \right] Y_0 \right\}, I \in S.$

Denote $\tilde{\delta}(t) = (I_N \otimes P^{-1}) \tilde{b}(t).$ By the above inequality, we get

$$d \tilde{V}(t) \leq 2\delta^T(t) \left[ I_N \otimes \theta_0(t)A_0P - \Phi^T(t)L(t)\Phi(t) \otimes \Gamma^{-1}_i(l)G_1(l)C_0\Gamma_{0i}(l)P \right] \tilde{\delta}(t)dt$$

$$+ 2q_0\sqrt{n_0}\|P^{-1}\|\tilde{\sigma}(t)\tilde{\sigma}(t)dt + 2\sigma^2\tilde{\sigma}^T(t) \left[ \Phi^T(t)L(t)\Phi(t) \otimes \Gamma^{-1}_i(l)G_2(l)C_0\Gamma_{0i}(l)P \right] \delta(t)dt + \varpi_1 dt$$

$$+ 2\delta^T(t)M_5(t)dw_{12}(t) + 2\delta^T(t)M_6(t)dw_{10}(t) + 2\delta^T(t)M_7(t)dw_{22}(t)$$

$$+ 2\delta^T(t)M_8(t)dw_{20}(t),$$

where $M_{10}(t) = \sum_{i=1}^N \sigma_{0i}(t)G_1^T(l)S_2(l)\Phi(t)C_0\Gamma_{0i}(l)P \tilde{\delta}(t), M_{11}(t) = -\sum_{i=1}^N \sigma_{0i}(t)G_2^T(l)\Phi(t) \otimes \Gamma^{-1}_i(l)G_2(l)C_0\Gamma_{0i}(l)P \tilde{\delta}(t), I \in S.$

Denote $\tilde{W}(t) = e^{\delta t} \tilde{V}(t), \forall \delta \in \left(0, \frac{\min_{l \in S} \left\{ a_{0l}(t) - 2q_0\sqrt{n_0}\|P^{-1}\|\tilde{\sigma}(t)\tilde{\sigma}(t) \right\}}{\max_{l \in S}(P)} \right).$ Noting that $C_0\Gamma_{0i}(l) = \theta_0(l)C_0, I \in S,$ by the above inequality and applying Itô’s formula, we have

$$d \tilde{W}(t) = \dot{\tilde{W}}(t)dt + e^{\delta t}d \tilde{V}(t)$$

$$\leq \dot{\tilde{W}}(t)dt + 2e^{\delta t} \tilde{\sigma}(t)(I_N \otimes P)\tilde{\delta}(t)dt + 2e^{\delta t} \tilde{\sigma}(t)(I_N \otimes \theta_0(t)A_0P - \Phi^T(t)L(t)\Phi(t) \otimes \Gamma^{-1}_i(l)G_1(l)C_0\Gamma_{0i}(l)P \tilde{\delta}(t)dt$$

$$+ 2q_0\sqrt{n_0}\|P^{-1}\|\tilde{\sigma}(t)\tilde{\sigma}(t)dt + e^{\delta t}\varpi_1 dt + 2e^{\delta t} \tilde{\sigma}(t)(I_N \otimes P^{-1})M_5(t)dw_{12}(t) + 2e^{\delta t} \tilde{\sigma}(t)(I_N \otimes P^{-1})M_6(t)dw_{10}(t)$$

$$+ 2e^{\delta t} \tilde{\sigma}(t)(I_N \otimes P^{-1})M_7(t)dw_{22}(t) + 2e^{\delta t} \tilde{\sigma}(t)(I_N \otimes P^{-1})M_8(t)dw_{20}(t)$$

$$= \dot{\tilde{W}}(t)dt + 2e^{\delta t} \tilde{\sigma}(t)(I_N \otimes P)\tilde{\delta}(t)dt + 2e^{\delta t} \tilde{\sigma}_0(t)\tilde{\delta}(t)(I_N \otimes A_0P - \Phi^T(t)L(t)\Phi(t) \otimes \Gamma^{-1}_i(l)G_1(l)C_0\Gamma_{0i}(l)P \tilde{\delta}(t)dt$$

$$+ 2q_0\sqrt{n_0}\|P^{-1}\|\tilde{\sigma}(t)\tilde{\sigma}(t)dt + 2e^{\delta t}\varpi_1 dt + 2e^{\delta t} \tilde{\sigma}(t)(I_N \otimes P^{-1})M_5(t)dw_{12}(t) + 2e^{\delta t} \tilde{\sigma}(t)(I_N \otimes P^{-1})M_6(t)dw_{10}(t)$$

$$+ 2e^{\delta t} \tilde{\sigma}(t)(I_N \otimes P^{-1})M_7(t)dw_{22}(t) + 2e^{\delta t} \tilde{\sigma}(t)(I_N \otimes P^{-1})M_8(t)dw_{20}(t).$$
\[ \begin{align*}
&\otimes PC_0^T G_1^T(l)\Gamma_{\theta_0}^{-1}(l)G_1(l)C_0 \right) \tilde{\delta}(t)dt + 2e^{\delta t} \sigma^2 \theta_0^2(l)\tilde{\delta}(T(l)) \Theta^T(l) \\
&\times F(l)\Phi(l) \otimes PC_0^T G_2^T(l)\Gamma_{\theta_0}^{-1}(l)G_2(l)C_0 \right) \tilde{\delta}(t)dt + e^{\delta t} \sigma_1 dt \\
&+ 2e^{\delta t} \delta^T(t)M_2(t)dw_{10}(t) + 2e^{\delta t} \delta^T(t)M_6(t)dw_{10}(t) + 2e^{\delta t} \delta^T(t)M_{10}(t)dw_{20}(t) \\
&+ 2e^{\delta t} \delta^T(t)M_{11}(t)dw_{20}(t) \\
&= e^{\delta t} \delta^T(t)\Psi(\tilde{\delta}(t)dt + e^{\delta t} \sigma_1 dt + 2e^{\delta t} \delta^T(t)M_5(t)dw_{10}(t) + 2e^{\delta t} \delta^T(t)M_6(t)dw_{10}(t) \\
&+ 2e^{\delta t} \delta^T(t)M_{10}(t)dw_{20}(t) + 2e^{\delta t} \delta^T(t)M_{11}(t)dw_{20}(t),
\end{align*} \]

where \( \Psi(\tilde{\delta}) = \hat{\gamma}(N \otimes P) + \theta_0(l)(N \otimes (PA_0 + A_0^T P)) - 2\theta_0(l)\Phi^T(l)\mathcal{L}(l)\Phi(l) \otimes \Gamma_{\theta_0}^{-1}(l) \\
\times G_1(l)C_0 P - 2\theta_0(l)\Phi^T(l)F(l)\Phi(l) \otimes \Gamma_{\theta_0}^{-1}(l)G_2(l)C_0 P \\
+ 2\sigma^2 \theta_0^2(l) \left[ \Phi^T(l)\mathcal{L}(l)\Phi(l) \otimes PC_0^T G_1^T(l)\Gamma_{\theta_0}^{-1}(l)P^{-1}\Gamma_{\theta_0}^{-1}(l)G_1(l)C_0 P \right] \\
+ 2\sigma^2 \theta_0^2(l) \left[ \Phi^T(l)F(l)\Phi(l) \otimes PC_0^T G_2^T(l)\Gamma_{\theta_0}^{-1}(l)P^{-1}\Gamma_{\theta_0}^{-1}(l)G_2(l)C_0 P \right] \\
+ 2q_0 \sqrt{\pi_0 ||P^{-1}||} (N \otimes P^2) \]

\[ \leq \hat{\gamma}(N \otimes P) + \theta_0(l)(N \otimes (PA_0 + A_0^T P)) + (2k_1^2(l)\sigma^2 \theta_0^2(l) - 2\theta_0(l)k_2(l)) \\
\times \Phi^T(l)\mathcal{L}(l)\Phi(l) \otimes PC_0^T (I_p + C_0 PC_0^T)^{-1} C_0 P \\
+ (2k_2^2(l)\sigma^2 \theta_0^2(l) - 2\theta_0(l)k_2(l)) \Phi^T(l)F(l)\Phi(l) \otimes PC_0^T \\
\times (I_p + C_0 PC_0^T)^{-1} C_0 P + 2q_0 \sqrt{\pi_0 ||P^{-1}||} (N \otimes P^2) \]

\[ \leq \hat{\gamma}(N \otimes P) + \theta_0(l)(N \otimes (PA_0 + A_0^T P)) + (2k_1^2(l)\sigma^2 \theta_0^2(l) - 2\theta_0(l)k_2(l)) \\
\times \Phi^T(l)\mathcal{L}(l)\Phi(l) \otimes PC_0^T (I_p + C_0 PC_0^T)^{-1} C_0 P \\
+ (2k_2^2(l)\sigma^2 \theta_0^2(l) - 2\theta_0(l)k_2(l)) \Phi^T(l)F(l)\Phi(l) \otimes PC_0^T \\
\times (I_p + C_0 PC_0^T)^{-1} C_0 P + 2q_0 \sqrt{\pi_0 ||P^{-1}||} (N \otimes P^2) \]

\[ \leq \hat{\gamma}(N \otimes P) + \theta_0(l)(N \otimes (PA_0 + A_0^T P)) - \frac{2\theta_0(l)q_0}{\lambda_1(\mathcal{L}(l) + F(l))} \Phi^T(l)\mathcal{L}(l)\Phi(l) \\
\otimes PC_0^T (I_p + C_0 PC_0^T)^{-1} C_0 P + 2q_0 \sqrt{\pi_0 ||P^{-1}||} (N \otimes P^2) \\
- \frac{2\theta_0(l)q_0}{\lambda_1(\mathcal{L}(l) + F(l))} \Phi^T(l)F(l)\Phi(l) \otimes PC_0^T (I_p + C_0 PC_0^T)^{-1} C_0 P, \quad l \in S. \] (B6)

By the generalized algebraic Riccati equation \( A_0 P + PA_0^T = 2q_0 PC_0^T (I_p + C_0 PC_0^T)^{-1} C_0 P + I_0 = 0 \), we get

\[ \hat{\gamma} P + \theta_0(l)(PA_0 + A_0^T P) - \frac{2\theta_0(l)q_0}{\lambda_1(\mathcal{L}(l) + F(l))} \lambda_1(\mathcal{L}(l) + F(l))PC_0^T \\
\times (I_p + C_0 PC_0^T)^{-1} C_0 P + 2q_0 \sqrt{\pi_0 ||P^{-1}||} P^2 \]
\[ \leq 0, \quad i = 1, \ldots, N, \quad l \in S. \]  

(B7)

By (B5)–(B7), we obtain

\[ \mathbb{E}[\tilde{V}(t)] \leq \mathbb{E}[\tilde{V}(0)] + \frac{\sigma_1}{\tilde{\gamma}} [e^{\tilde{\gamma}t} - 1]. \]

By the definition of \( W(t) \) and the above inequality, we get

\[ \mathbb{E}[\tilde{V}(t)] \leq e^{-\tilde{\gamma}t} \mathbb{E}[\tilde{V}(0)] + \frac{\sigma_1}{\tilde{\gamma}} [1 - e^{\tilde{\gamma}t}]. \]

Noting that \( \mathbb{E}[\tilde{V}(t)] = \mathbb{E}[\tilde{\delta}^2(t)(I_N \otimes P^{-1})\tilde{\delta}(t)] \geq \lambda_{\min}(I_N \otimes P^{-1}) \mathbb{E}[\|\tilde{\delta}(t)\|^2] = \frac{1}{\lambda_{\max}(P)} \mathbb{E}[\|\tilde{\delta}(t)\|^2], \) by the above inequality, we have

\[ \mathbb{E}[\|\tilde{\delta}(t)\|^2] \leq e^{-\tilde{\gamma}t} \lambda_{\max}(P) \mathbb{E}[\|\tilde{\delta}(0)\|^2] + \frac{\lambda_{\max}(P) \sigma_1}{\tilde{\gamma}} [1 - e^{\tilde{\gamma}t}]. \]

which together with \( \tilde{\delta}(t) = (\Phi(l) \otimes \Gamma_{\theta}(l))^{-1} \delta(t) \) gives

\[ \mathbb{E}[\|\tilde{\delta}(t)\|^2] \leq \|\Phi(l) \otimes \Gamma_{\theta}(l)\|^2 \mathbb{E}[\|\tilde{\delta}(t)\|^2] \]

\[ \leq \|\Phi(l)\|^2 \|\Gamma_{\theta}(l)\|^2 \mathbb{E}[\|\tilde{\delta}(t)\|^2] \]

\[ = \theta_0^{2n}(l) \mathbb{E}[\|\tilde{\delta}(t)\|^2] \]

\[ \leq e^{-\tilde{\gamma}t} \lambda_{\max}(P) \mathbb{E}[\|\tilde{\delta}(0)\|^2] \max_{l \in S} \{ \theta_0^{2n}(l) \}
\]

\[ + \frac{\lambda_{\max}(P) \sigma_1}{\tilde{\gamma}} [1 - e^{\tilde{\gamma}t}], \quad l \in S. \]

Taking the upper limit on both sides of the above inequality, we get

\[ \lim_{t \to \infty} \mathbb{E}[\|\tilde{\delta}(t)\|^2] \leq \frac{\max_{l \in S} \{ \theta_0^{2n}(l) \} \lambda_{\max}(P) \sigma_1}{\tilde{\gamma}}, \quad l \in S. \]

By the definition of \( \delta(t) \) and the above inequality, we obtain

\[ \lim_{t \to \infty} \mathbb{E}[\|x_0(t) - x_0(t)\|^2] \leq \frac{\max_{l \in S} \{ \theta_0^{2n}(l) \} \lambda_{\max}(P) \sigma_1}{\tilde{\gamma}}, \quad l \in S. \]

Similar to (B1), we have

\[ \lim_{t \to \infty} \mathbb{E}[\|y_i(t) - y_0(t)\|^2] \]

\[ \leq \frac{\theta_i^{2n} \text{Tr} \left\{ \hat{P}_i \Gamma_{\theta}^{-1} \Gamma_{\theta}^{-1} \rho_i \right\}}{\lambda_{\min}(P_i) \min_{i \in \{1, 2, \ldots, N\}} \left\{ \frac{\theta_i^{2n} \sqrt{\sigma_1^2 \lambda_{\min}(P_i)}}{\lambda_{\min}(P_i)} \right\}} \left( 1 + \frac{3 \sigma_2^2 \theta_i^{2n}}{\sigma_1^2 \lambda_{\min}(P_i)} \right) \]

\[ + \frac{6 \sigma_4^2 \theta_i^{2n} \sigma_1^2 \lambda_{\max}(P) \max_{l \in S} \{ \theta_0^{2n}(l) \}}{\sigma_1^2 \lambda_{\min}(P_i) \min_{i \in S} \left\{ \theta_0(l) - 2q_0 \sqrt{n_0} \|P^{-1}\| \lambda_{\max}(P) \right\}}, \quad i = 1, \ldots, N. \]

Similar to Theorem 1, Theorem 2 follows immediately from what we have proved before.