ON THE NON-VANISHING OF THE FIRST BETTI NUMBER OF HYPERBOLIC THREE MANIFOLDS

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Abstract. We show the non-vanishing of cohomology groups of sufficiently small congruence lattices in $SL(1, D)$, where $D$ is a quaternion division algebras defined over a number field $E$ contained inside a solvable extension of a totally real number field. As a corollary, we obtain new examples of compact, arithmetic, hyperbolic three manifolds, with non-torsion first homology group, confirming a conjecture of Thurston. The proof uses the characterisation of the image of solvable base change by the author, and the construction of cusp forms with non-zero cusp cohomology by Labesse and Schwermer.

1. Introduction

Let $D$ be a quaternion division algebra over a number field $E$. Let $G$ denote the connected, semisimple algebraic group $SL_1(D)$ over $E$. Denote by $G_\infty(E)$ the real Lie group $G(E \otimes \mathbb{R})$ and fix a maximal compact subgroup $K_\infty$ of $G_\infty$. Let $s_1$ (resp. $2r_2$) be the number of real (resp. complex) places of $E$ at which $D$ splits, and let $s = s_1 + r_2$. The quotient space $M := G_\infty/K_\infty$ with the natural $G_\infty(E)$-invariant metric, is a symmetric space isomorphic to $\mathcal{H}_2^{s_1} \times \mathcal{H}_3^{r_2}$, where for a natural number $n$, $\mathcal{H}_n$ denotes the simply connected hyperbolic space of dimension $n$.

Let $\mathbb{A}$ (resp. $\mathbb{A}_f$) denote the ring of adeles (resp. finite adeles) of $\mathbb{Q}$. Let $K$ be a compact, open subgroup of $G(\mathbb{A}_f \otimes E)$, and denote by $\Gamma_K$ the corresponding congruence arithmetic lattice in $G_\infty(E)$ defined by the projection to $G_\infty(E)$ of the group $G(E) \cap G_\infty(E)K$. For sufficiently small congruence subgroups $K$, $\Gamma_K$ is a torsion-free lattice and $\Gamma_K \backslash M$ is a (compact) Riemannian manifold.

In this note, we prove

Theorem 1. With the above notation, assume further that $E$ is a finite extension of a totally real number field $F$ contained inside a solvable
extension $L$ of $F$. For sufficiently small congruence subgroups $\Gamma \subset G_{\infty}(E)$, the cohomology groups

$$H^s(\Gamma \backslash M, \mathbb{C})$$

are non-zero.

The theorem was proved by Labesse and Schwermer \cite[Corollary 6.3]{ls}, in the case when there exists a tower of field extensions

$$E = F_i \supset F_{i-1} \supset \cdots \supset F_0 = F,$$

such that $F_{i+1}/F_i$ is either a cyclic extension of prime degree or a non-normal cubic extension. The proof rests on the following two observations: one, that the base change (constructed by Langlands \cite{l}) of the discrete series representations from $SL(2, \mathbb{R})$ to $SL(2, \mathbb{C})$ are cohomologically non-trivial representations of $SL(2, \mathbb{C})$, and thus the base change of cohomologically non-trivial cusp forms (for $SL(2)$) are cohomologically non-trivial. Secondly, when working with the group $SL_2$ over a totally real field, the required cohomologically non-trivial automorphic representations have discrete series as their archimedean components, and an argument using pseudo-coefficients shows the existence of such cusp forms.

The new ingredient that goes into extending the theorem of Labesse and Schwermer to all relatively solvable extensions of a totally real field, is the criterion of base change descent for an invariant cuspidal automorphic representation with respect to a solvable group of automorphisms of the field proved by the author in \cite{r}.

If either $D$ is unramified at all finite places of $E$ or if $F$ is taken to be the field of rationals and the Galois closure $L$ of $E$ over $\mathbb{Q}$ is of odd order over $\mathbb{Q}$, then Theorem \ref{thm} was proved by Clozel \cite{c}. Clozel’s proof uses the construction of algebraic Hecke characters due to Weil, and the automorphic induction of suitable such characters produces the desired cuspidal cohomology with non-zero cohomology. Clozel’s method proves the non-vanishing of the cuspidal cohomology of the split groups $SL_2$ over any number field $E$. But in order to produce cohomological forms on inner forms of $SL_2$ using the Jacquet-Langlands theorem, it is required that the constructed cusp form on $SL_2(\mathbb{A} \otimes E)$ have discrete series components at the places of $E$ where $D$ ramifies, and thus the choice of $D$ has to be suitably restricted.

A particular case of interest is the following:

**Corollary 1.** With notation as in Theorem \ref{thm}, assume further that $E$ has exactly one pair of conjugate complex places, and the quaternion division algebra $D$ is ramified at all the real places of $E$. For sufficiently
A folklore conjecture (attributed to Thurston) is that the first betti number of a compact, hyperbolic three manifold becomes positive upon going to some finite cover. The first examples of compact, hyperbolic arithmetic three manifolds $M_\Gamma$ with non-vanishing rational first homology group are due to Millson [M]. Using geometric arguments, Millson showed the non-vanishing of the first betti number for sufficiently small congruence subgroups, where the arithmetic structure arises from rank 4 quadratic forms over a totally real number field $F$, and of signature $(3,1)$ at one archimedean place and anisotropic at all other real places. As mentioned above, other non-vanishing results confirming Thurston’s conjecture have been proved by Labesse-Schwermer [LS] and Clozel [C].

**Example.** Let $L$ be a non-real Galois extension of $\mathbb{Q}$ with Galois group isomorphic to $S_4$. Consider a subgroup $H$ isomorphic to $S_3$ and containing a complex conjugation $\sigma$ corresponding to some archimedean place. Then the invariant field $E = L^H$ is a degree 4 extension of $\mathbb{Q}$ having the following properties:

- $E$ does not contain any quadratic extension of $\mathbb{Q}$.
- $E$ has at least one real place.
- The subgroup containing the conjugates of $\sigma$ with respect to $S_4$ will not be equal to $S_3$. Hence, $E$ is not totally real.

This gives us an example of a field $E$ satisfying the hypothesis of the corollary. The non-vanishing result for the corresponding hyperbolic three manifolds given by Corollary [C] are not in general covered by the non-vanishing results proved in [C], [LS] or in [M].

2. General coefficients

Theorem [C] can be generalized for suitable non-trivial coefficient systems also. Let $F$ and $E$ be as in the hypothesis of the theorem. Given a finite dimensional complex representation $V$ of $SL_2(\mathbb{R} \otimes F)$, we now define the base change representation $\Psi(V)$ of the group $G_\infty(E)$ [LS]. We define it first when $V$ is irreducible and extend it additively. If $V$ is irreducible, then $V$ can be written as,

$$V \simeq \otimes_{v \in P_{\infty}(F)} V_v,$$

where $P_{\infty}(F)$ is the collection of the archimedean places of $F$, and the component $V_v$ of $V$ at the place $v$ is an irreducible representation of $SL_2(F_v) \simeq SL_2(\mathbb{R})$, say of dimension $k(v)$.

Let $V_k$ (resp. $\bar{V}_k$) denote the irreducible, holomorphic (resp. anti-holomorphic) representation of $SL_2(\mathbb{C})$ of dimension $k$. Restricted to
SU(2) they give raise to isomorphic representations, which we continue to denote by $V_k$. Define the representation $W_k$ of $SL_2(\mathbb{C})$ by $W_k = V_k \otimes \overline{V}_k$.

Suppose $D$ is a quaternion algebra over $E$. We define the base change coefficients $\Psi(V)$ of $G_\infty(E)$, as a tensor product of the representations $\Psi(V)_w$ of the component groups $G(E_w)$, as $w$ runs over the collection of archimedean places of $E$. Suppose $w$ lies over a place $v$ of $F$. Define,

$$\Psi(V)_w \simeq \begin{cases} V_{k(v)} & \text{if } w \text{ is real,} \\ W_{k(v)} & \text{if } v \text{ is complex.} \end{cases}$$

Restricting the representation $\Psi(V)$ to a torsion-free lattice $\Gamma$ gives raise to a well defined local system $L_{\Psi(V)}$ on the manifold $\Gamma \backslash M$. The extension of Theorem 1 to non-trivial coefficients is the following:

**Theorem 2.** Let $F$ be a totally real number field, and $L$ be a solvable finite extension of $F$. Let $E$ be a finite extension of $F$ contained in $L$, and $D$ be a quaternion division algebra over $E$. Let $V$ be a finite dimensional complex representation of $SL_2(\mathbb{R} \otimes F)$. Then, $H^s(\Gamma \backslash M, L_{\Psi(V)}) \neq 0$.

### 3. Proof

In order to prove Theorem 2, it is more convenient to work with the cohomology groups $H^*(G, E; V)$, defined as a direct limit indexed by the compact open subgroups $K \subset G(\mathbb{A}_f \otimes E)$:

$$H^*(G, E; V) = \lim_{\longrightarrow K} H^*(\Gamma_K, \Psi(V)) \simeq \lim_{\longrightarrow K} H^*(\Gamma_K \backslash M, L_{\Psi(V)}).$$

These cohomology groups can be reinterpreted in terms of the automorphic spectrum of $G(\mathbb{A} \otimes E)$. Let $\rho$ denote the representation of $G(\mathbb{A} \otimes E)$ acting by right translations on the space $L^2(G(E) \backslash G(\mathbb{A} \otimes E))$ consisting of square integrable functions on $G(E) \backslash G(\mathbb{A} \otimes E)$. This decomposes as a direct sum of irreducible admissible representations $\pi$ of $G(\mathbb{A} \otimes F)$ with finite multiplicity $m(\pi)$:

$$\rho = \bigoplus_{\pi} m(\pi) \pi,$$

With respect to the decomposition $G(\mathbb{A} \otimes E) = G_\infty(E)G(\mathbb{A}_f \otimes E)$, write $\pi = \pi_\infty \pi_f$, where $\pi_\infty$ (resp. $\pi_f$) is a representation of $G_\infty(E)$ (resp. $G(\mathbb{A}_f \otimes E)$). The cohomology groups $H^*(G, E; V)$ can also be expressed in terms of the relative Lie algebra cohomology (see [BW]) of the automorphic spectrum as,

$$(1) \quad H^*(G, E; V) \simeq \bigoplus_{\pi} m(\pi) H^*(g, K_\infty, \pi_\infty \otimes V) \otimes \pi_f,$$
where $g$ is the Lie algebra of $G_{\infty}(E)$. Hence in order to prove the theorem, it is enough to construct an irreducible representation $\pi$ of $G(\mathbb{A}_E)$ with $m(\pi)$ positive and such that $H^*(g, K_{\infty}, \pi_{\infty} \otimes V)$ is nonzero.

We can assume that $V$ is irreducible of the form $V \simeq \otimes_{v \in \mathcal{P}(F)} V_{k(v)}$. Let $D_{k}^+$ (resp. $D_{k}^-$) be the holomorphic (resp. antiholomorphic) discrete series of $SL_2(\mathbb{R})$ of weight $k + 1$. We have,

$$(2) \quad H^q(sl_2(\mathbb{R}), SO(2), D_{k}^+ \otimes V_k) = \begin{cases} \mathbb{C} & \text{if } q = 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $sl_2(\mathbb{R})$ and $sl_2(\mathbb{C})$ denotes respectively the Lie algebras of $SL_2(\mathbb{R})$ and $SL_2(\mathbb{C})$.

Let $S$ be a finite set of finite places of $F$, containing all the finite places $v$ of $F$ dividing a finite place of $E$ at which $D$ ramifies. By [LS Proposition 2.5], there exists an irreducible, admissible representation of $SL_2(\mathbb{A} \otimes F)$ satisfying the following properties:

- The multiplicity $m_0(\pi)$ of $\pi$ occurring in the cuspidal spectrum $L_0^0(SL_2(F) \backslash SL_2(\mathbb{A} \otimes F))$ consisting of square integrable cuspidal functions on $SL_2(F) \backslash SL_2(\mathbb{A} \otimes F)$ is nonzero. Further $\pi$ is stable in the sense of [LL].
- The local component $\pi_v$ of $\pi$ at an archimedean place $v$ of $F$ is a discrete series representation, with $\pi_v \in \{ D_{k(v)}^+, D_{k(v)}^- \}$.
- For any $v \in S$, the local component $\pi_v$ of $\pi$ is isomorphic to the Steinberg representation of $SL_2(F_v)$. Let $\Pi$ be a cuspidal, automorphic representation of $GL_2(\mathbb{A} \otimes F)$, such that $\pi$ occurs in the restriction of $\Pi$ to $SL_2(\mathbb{A} \otimes F)$. Let $\Pi_L$ be the base change of $\Pi$ to $GL_2(\mathbb{A} \otimes L)$ defined by Langlands in [L]. Since $\pi$ is stable, i.e., $\Pi$ is not automorphically induced from a character of a quadratic extension of $F$, $\Pi_L$ is a cuspidal automorphic representation of $GL_2(\mathbb{A} \otimes L)$.

Let $H$ be the Galois group of $L$ over $E$. Since $\Pi_L$ is $H$-invariant, by the descent criterion proved in [R], there exists an idele class character $\chi$ of $L$, such that the representation $\Pi_L \otimes \chi$ is the base change from $E$ to $L$ of a cuspidal representation $\Pi_E$ of $GL_2(\mathbb{A} \otimes E)$. Let $\pi_E$ be a constituent of the restriction of $\Pi_E$ to $SL_2(\mathbb{A} \otimes E)$, and occurring in the automorphic spectrum $G(\mathbb{A}_E)$ with non-zero multiplicity $m(\pi_E)$.

Base change makes sense at the level of $L$-packets (see [LS]), and let $\pi_{k,E}$ denote the representation of $SL_2(\mathbb{C})$ obtained as base change of the $L$-packet $\{ D_{k}^+, D_{k}^- \}$ ($L$-packets for complex groups consist of only
one element). It is known that (see [LS]),
\[ H^1(\mathfrak{sl}_2(\mathbb{C}), SU(2), \pi_{k,\mathbb{C}} \otimes W_k) \neq 0. \]

Let \( w \) be an archimedean place of \( E \) lying over a real place \( v \) of \( F \). Now twisting by a character does not alter the restriction of an automorphic representation of \( GL_2 \) to \( SL_2 \). Hence if \( w \) is a real place of \( E \), then the local component \( \pi_{E,w} \) of \( \pi_E \) at \( w \) belongs to \( \{ D^+_{k(v)}, D^-_{k(v)} \} \), and if \( w \) is a complex place of \( E \), then \( \pi_{E,w} \) is isomorphic to \( \pi_{k(v),\mathbb{C}} \).

The local components of the base change to \( E \) of \( \pi \) continues to be the Steinberg representation of \( SL_2(E_w) \), at the places of \( E \) where \( D \) ramifies. By the theorem of Jacquet-Langlands ([JL], [LS]) applied to \( L \)-packets of \( SL_2 \) and its inner forms, we get an automorphic representation \( JL(\pi_E) \) of \( G \) over \( E \). At a place \( w \) where \( D \) is ramified, the local component \( JL(\pi_E)_w \) is isomorphic to the restriction of the representation \( V_{k(v)} \) to \( SU(2) \), where \( v \) is a place of \( F \) dividing \( w \). In particular, the zeroth relative Lie cohomology group
\[ H^0(\mathfrak{su}_2, SU_2, V_k \otimes V_k) = (V_k \otimes V_k)^{SU(2)} \neq 0. \]

At a place \( w \) of \( E \) where \( D \) splits, \( JL(\pi_E)_w \simeq \pi_{E,w} \), and hence the first relative Lie algebra cohomology with coefficients in the component of \( \Psi(V) \) at \( w \) is non-zero. It follows from equations (2), (3), (4) and by the Kunneth formula for the relative Lie algebra cohomology that
\[ H^s(\mathfrak{g}, K_{\infty}, JL(\pi_E)_\infty \otimes \Psi(V)) \neq 0. \]

By Equation (1), this proves Theorem 2.

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