Abstract. For complex parallelisable manifolds $\Gamma \backslash G$, with $G$ a solvable or semisimple complex Lie group, the Frölicher spectral sequence degenerates at the second page. In the solvable case, the de Rham cohomology carries a pure Hodge structure. In contrast, in the semisimple case, purity depends on the lattice, but there is always a direct summand of the de Rham cohomology which does carry a pure Hodge structure and is independent of the lattice.

1. Introduction

This article is mainly concerned with the cohomology of complex manifolds of the form $X = \Gamma \backslash G$, where $G$ is a connected complex Lie group and $\Gamma \subseteq G$ a cocompact, discrete subgroup. Every such manifold has trivial holomorphic tangent bundle and vice versa Wang [19] (see also [20, Sect. 1.12]) has shown that every compact complex parallelisable manifold is of this form and we may assume $G$ to be simply connected.

We consider the Frölicher spectral sequence of a compact complex parallelisable manifold $X = \Gamma \backslash G$. If the Frölicher spectral sequence of a compact complex manifold degenerates at the first page, then every holomorphic form is closed. By this fact, we can easily say that the Frölicher spectral sequence of $X$ degenerates at the first page if and only if $G$ is abelian. Hence, unlike compact Kähler manifolds, if $G$ is not abelian $X$ does not satisfy $\partial \bar{\partial}$-Lemma as in [6].

If $G$ is nilpotent, the Dolbeault cohomology and de Rham cohomology can be computed from the Lie algebra of $G$ [16]. In particular, it is independent of the lattice $\Gamma$. Furthermore, it was recently shown in [13] by Popovici, Ugarte and the second named author that they are page-1-$\partial \bar{\partial}$-manifolds, meaning that the Frölicher spectral sequence degenerates at the second page and the Hodge filtration induces a pure Hodge structure on the de Rham cohomology. One of the goals of
this article is to understand to what extend these results hold for not necessarily nilpotent $G$.

Unlike in the nilpotent case, in general the dimensions of Dolbeault and de Rham cohomology can depend on the lattice (see, e.g., [3, Ex. 3.4] for $G$ solvable and [20] or [7] for $G = \text{SL}_2(\mathbb{C})$). Nevertheless, it was shown by the first-named author in [10] that, for $G$ a solvable complex Lie group, regardless of the choice of $\Gamma$, the Frölicher spectral sequence of degenerates at the second page. Our first result generalizes this by also asserting the existence of a pure Hodge structure on the cohomology of these manifolds.

**Theorem A.** If $G$ is solvable, $X$ is a page-$1$-$\partial\bar{\partial}$-manifold.

The proof will be given in Section 3. Similar techniques also yield an analogue of this theorem for certain solvmanifolds of splitting type (see Theorem 9).

In the semisimple case, the situation is more subtle. Computations by Ghys [7] and Winkelmann [20] show that for $G = \text{SL}_2(\mathbb{C})$, there exist lattices $\Gamma, \Gamma' \subset G$ such that $\Gamma \setminus G$ is page-$1$-$\partial\bar{\partial}$ and $\Gamma' \setminus G$ is not. Building on work of Akhiezer, we give a conceptual explanation of this phenomenon.

Suppose $G$ is semisimple. Take a maximal compact subgroup $K \subset G$. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebra of $G$ and $K$ respectively. Then, associated with the locally symmetric space $Y = \Gamma \setminus G/K$, we have the canonical injection $H^k(\mathfrak{g}; \mathfrak{k}; \mathbb{C}) \hookrightarrow H^k(\Gamma, \mathbb{C})$ where $H^k(\mathfrak{g}; \mathfrak{k}; \mathbb{C})$ is the relative Lie algebra cohomology and $H^k(\Gamma, \mathbb{C})$ is the group cohomology (see [17, Intr.], for instance). In general, this injection is not an isomorphism.

**Theorem B.** Let $G$ be semisimple.

(i) The Frölicher spectral sequence of $X$ degenerates at the second page.

(ii) $X$ is a page-$1$-$\partial\bar{\partial}$-manifold if and only if $H^k(\mathfrak{g}; \mathfrak{k}; \mathbb{C}) \cong H^k(\Gamma, \mathbb{C})$ for any $k \in \mathbb{Z}$.

In view of Theorems A and B, it appears natural to make the following:

**Conjecture.** For any compact complex parallelisable manifold $X$, the Frölicher spectral sequence degenerates at the second page.

Apart from the results in this article, indication that this might hold is given by [11], where it is shown that $d_2 : E^{0,1}_2(X) \to E^{2,0}_2(X)$ always vanishes.

### 2. Preliminaries

#### 2.1. Page-$1$-$\partial\bar{\partial}$-manifolds

We briefly recall some of the terminology of [13]. Let $A = (A^*, \partial, \bar{\partial})$ be a bounded double complex of complex vector spaces and $\text{Tot} A^* := \bigoplus_{p+q=\bullet} A^{p,q}$ the associated total complex with differential $d := \partial + \bar{\partial}$. Associated with the filtration $F^r A = \bigoplus_{p \geq r} A^{p,q}$ we get an induced filtration on the total cohomology $H^r_{dR}(A) := H^\bullet(\text{Tot} A)$, still denoted by $F$, and the spectral sequence

$$E_1^{p,q}(A) = H^{p,q}_{\partial}(A) \Rightarrow (H^{p+q}_{dR}(A), F).$$
Analogously, for the filtration $\mathcal{F}^r A = \bigoplus_{q \geq r} A^{p,q}$ we obtain a second spectral sequence converging to $H_{dR}(A)$, which we denote by $\tilde{E}_r(A)$. It has the spaces $H^{p,q}_0(A)$ on its first page. The Bott–Chern cohomology is defined as

$$H^{p,q}_{BC}(A) := \frac{\ker (\partial : A^{p,q} \to A^{p+1,q}) \cap \ker (\bar{\partial} : A^{p,q} \to A^{p,q+1})}{\text{im} (\bar{\partial} : A^{p-1,q-1} \to A^{p,q})}$$

and the Aeppli cohomology as

$$H^{p,q}_A(A) := \frac{\ker (\bar{\partial} : A^{p-1,q-1} \to A^{p,q})}{\text{im} (\partial : A^{p-1,q} \to A^{p,q}) + \text{im} (\bar{\partial} : A^{p,q-1} \to A^{p,q})}.$$ 

All double complexes which we will consider will have finite dimensional cohomology, i.e., the spaces $H^{p,q}_0(A)$ and $H^{p,q}_0(A)$ are finite dimensional for all $p, q \in \mathbb{Z}$ (this also implies that $E^{p,q}_r(A), E^{p,r}_r(A), H^{p,q}_0(A), H^{p,q}_1(A), H^{k}_{dR}(A)$ are finite dimensional [18, Sect. 2]). Moreover, as the notation already suggests, in most cases we will consider double complexes $A$ that are equipped with a real structure, i.e., an antilinear involution $(\cdot)$ such that $A^{p,q} = A^{q,p}$ and $\bar{\partial}a = \bar{\partial}a$. In this case the second spectral sequence is determined by the first and can be ignored. We denote by $H^{p,q}(A) := \text{im}(H^{p,q}_0(A) \to H^{p+q}_{dR}(A))$. We have equalities

$$H^{p,q}(A) = \{[\alpha] \in H^*(\text{Tot } A) : \alpha \in A^{p,q} = (F^p \cap \tilde{F}^q)H^{p+q}_{dR}(A).$$

Recall [6] that a double complex $A$ as above is said to satisfy the $\partial \bar{\partial}$-Lemma (or to have the $\partial \bar{\partial}$-property) if it satisfies one (hence all) of the following equivalent properties:

1. $\ker \partial \cap \ker \bar{\partial} \cap \text{im } d = \text{im } \partial \bar{\partial}$.
2. Both spectral sequences degenerate at page 1 and for all $k \in \mathbb{Z}$, the filtrations $F$ and $\tilde{F}$ on $H^k_{dR}(A)$ induce a pure Hodge structure of degree $k$, i.e., $H^k_{dR}(A) = \bigoplus_{p+q=k} H^{p,q}_{\bar{\partial}}(A)$.
3. Every class in $H^{p,q}_{\bar{\partial}}(A)$ admits a $d$-closed pure-type representative and the map $H^{p,q}_{\bar{\partial}}(A) \to H^{p,q}(A)$ sending a class via such a representative induces a well-defined isomorphism

$$\bigoplus_{p,q \in \mathbb{Z}} H^{p,q}_{\bar{\partial}}(A) \to H^*(A)$$

(and analogously for $H^{p,q}_0(A)$).
4. $A$ is a direct sum of complexes of the following types:

   1 \text{ (‘squares’): complexes with a single pure-bidegree generator $a$, such that $\partial \bar{\partial}a \neq 0$:}

   $$(a) \xrightarrow{\partial a} (\partial \bar{\partial}a) \xrightarrow{\bar{\partial}a} (\partial a)$$

   2 \text{ (‘dots’): complexes concentrated in a single bidegree, with all differentials being zero:}

   $$(a).$$
The main interest of this paper is the following analogue of this property introduced in [13].

**Definition 1.** A bounded double complex $A$ is said to have the page-1-$\partial\bar{\partial}$-property if both its spectral sequences degenerate at page 2 and for all $k \in \mathbb{Z}$, the filtrations $F$ and $\bar{F}$ on $H^k_{dR}(A)$ induce a pure Hodge structure of degree $k$.

There is an obvious extension of this notion to any page of the Frölicher spectral sequence, including the usual $\partial\bar{\partial}$-property as the page-0-$\partial\bar{\partial}$-property, but we will not need this here.

Define $h^p,q_{\#}(A) = \dim H^p,q_{\#}(A)$ and $h^r_{\#}(A) = \sum_{r=p+q} h^p,q_{\#}(A)$ for each $\# = \partial, \bar{\partial}, BC, A$.

**Proposition 2 ([13]).** Let $A$ be a bounded double complex with finite-dimensional cohomology. The following assertions are equivalent:

1. $A$ is page-1-$\partial\bar{\partial}$.
2. Every class in $E^2_{p,q}(A)$ admits a $d$-closed pure-type representative and the map $E^2_{p,q}(A) \to H^{p,q}(A)$ sending a class via such a representative induces a well defined linear isomorphism$
\bigoplus E^2_{p,q}(A) \to H^*_dR(A)
$
and analogously for $\bar{E}_2$.
3. For every $r \in \mathbb{Z}$ the equality $h^r_A(A) + h^r_{BC}(A) = h^r_{\partial}(A) + h^r_{\bar{\partial}}(A)$ holds.
4. $A$ is a direct sum of squares, dots and ‘lines’, i.e., complexes generated by a single pure-bidegree generator $a$, such that exactly one of $\partial a$ or $\bar{\partial} a$ is nonzero:
\[
\langle \partial a \rangle, \quad \langle a \rangle \quad \rightarrow \quad \langle \partial a \rangle.
\]

There is also a characterisation in terms of suitable exactness properties (see [14] for details).

**Remark 3.** Note that for complexes with real structure, property 2 implies the symmetry $\dim E^2_{p,q} = \dim E^2_{q,p}$.

Let $X$ be a complex manifold and $A_X = (A^*_X, \partial, \bar{\partial})$ the Dolbeault double complex, i.e., the double complex of $\mathbb{C}$-valued differential forms. We denote $E^p,q(X) = E^p,q_{\#}(A_X)$ and call it the Frölicher spectral sequence. In this case, the total cohomology is the de Rham cohomology $H^*_dR(X, \mathbb{C})$ of $X$, the $\partial$-cohomology $H^p,q_{\partial}(A_X)$ is the Dolbeault cohomology of $X$. We denote $H^p,q_{\#}(X) = H^p,q_{\#}(A_X)$, $h^p,q_{\#}(X) = h^p,q_{\#}(A_X)$ and $h^r_{\#}(X) = h^r_{\#}(A_X)$ for each $\# = \partial, \bar{\partial}, BC, A$.

**Definition 4.** Let $X$ be a compact complex manifold. $X$ is called a page-1-$\partial\bar{\partial}$-manifold if the Dolbeault double complex $A_X$ is page-1-$\partial\bar{\partial}$.

Proposition 2 and [18, Cor. 13] (see also [2], [3], which treat all cases relevant for us) imply the following.
Corollary 5. Let \( X \) be a compact complex manifold. Given any bounded double complex with \( C \) and a map \( C \rightarrow A_X \) of double complexes, such that for all \( p, q \in \mathbb{Z} \) the induced map \( H^{p,q}_\#(C) \rightarrow H^{p,q}_\#(X) \) is an isomorphism for \( \# = \partial, \bar{\partial} \), then it is also an isomorphism for each \( \# = BC, A \). Moreover, if \( C \) has the page-1-\( \partial \bar{\partial} \)-property, then also \( X \) is a page-1-\( \partial \bar{\partial} \)-manifold.

If \( C \) is equipped with a real structure and \( \varphi \) respects this map, it is sufficient to consider only the induced map in \( H_\partial \).

The following observation gives rise to many examples of page-1-\( \partial \bar{\partial} \)-complexes.

Lemma 6 ([13, Lem. 4.6.]). If \( A = (A^*, d_1), B = (B^*, d_2) \) are (simple) complexes, then the double complex \( A \otimes B = (A^* \otimes B^*, d_1 \otimes \text{Id}_B, \text{Id}_A \otimes d_2) \) is page-1-\( \partial \bar{\partial} \). The spaces \( E^{p,q}_2(A \otimes B) = H^{p,q}(A \otimes B) \) are identified with \( H^p(A) \otimes H^q(B) \).

2.2. Complex Lie groups and lattices

Let \( G \) be a complex Lie group and \( \mathfrak{g} \) its Lie algebra. The complex structure of \( G \) induces a splitting \( \mathfrak{g}_C = \mathfrak{g}_1,0 \oplus \mathfrak{g}_0,1 \) into \( i \) and \(-i\) eigenspaces and we denote by \( \mathfrak{g}^\pm_\partial = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1} \) the splitting of the dual space. Denote by \( \Lambda_\partial := \Lambda^* \mathfrak{g}^\pm_\partial \subseteq A_G \) the space of left invariant forms on \( G \). The restriction of the exterior differential makes it into a sub-double complex of \( A_G \). Denote by \( \Lambda^+_\partial = (\Lambda^* \mathfrak{g}_1,0, \partial) \) and \( \Lambda^-_\partial = (\Lambda^* \mathfrak{g}^{0,1}, \bar{\partial}) \) the simple complexes of left-invariant forms of types \((*, 0)\) and \((0, *)\). The following observation was made in [13] (formulated there for \( G \) nilpotent):

Lemma 7. The complex \( \Lambda_\partial \) is the tensor product of the simple complexes \( \Lambda^+_\partial \) and \( \Lambda^-_\partial \), hence it is a page-1-\( \partial \bar{\partial} \)-complex.

Now assume that there exists a lattice \( \Gamma \subseteq G \) (which, for the purpose of this article, will mean that \( \Gamma \) is a discrete, cocompact subgroup). We will write \( X := \Gamma \backslash G \) for the compact complex manifold obtained as the quotient of \( G \) by the left action of \( \Gamma \). Since \( \Lambda_\partial \) consists of left-invariant forms, it can also be considered as a sub-double complex of \( A_X \). It carries the induced real structure. Hence, by Lemma 7 and Corollary 5 we have the following proposition:

Proposition 8. If the inclusion \( i : \Lambda_\partial \rightarrow A_X \) induces a cohomology isomorphism \( H^{p,q}_\partial(\Lambda_\partial) \cong H^{p,q}_\partial(X) \), then \( X \) is a page-1-\( \partial \bar{\partial} \)-manifold.

In [16], Sakane proved that if \( G \) is nilpotent, then the inclusion \( i : \Lambda_\partial \rightarrow A_X \) induces a cohomology isomorphism \( H^{p,q}_\partial(\Lambda_\partial) \cong H^{p,q}_\partial(X) \). Hence, if \( G \) is nilpotent, then \( X \) is a page-1-\( \partial \bar{\partial} \)-manifold.

If \( G \) is not nilpotent the inclusion \( i : \Lambda_\partial \rightarrow A_X \) does not induce an isomorphism \( H^{p,q}_\partial(\Lambda_\partial) \cong H^{p,q}_\partial(X) \) (e.g., Nakamura manifolds). Since it is known that Lie groups admitting a lattice are unimodular [12, Lem. 6.2], by averaging associated with a bi-invariant Haar measure, we obtain a map of double complexes \( \mu : A^*_X \rightarrow \Lambda_\partial \) such that \( \mu \circ i = \text{Id} \) (see the proof of [4, Thm. 7]). Thus, one obtains a direct sum decomposition \( A_X = \Lambda_\partial \oplus A_X/\Lambda_\partial \) and a corresponding decomposition on all cohomologies considered. In particular, the cohomology groups of \( \Lambda_\partial \) inject into those of \( X \).
3. Solvable case

We prove Theorem A. We keep the notation of the preceding section and assume that $G$ is a complex solvable group. Let $N$ be the nilradical of $G$. We can take a connected simply-connected complex nilpotent subgroup $C \subseteq G$ such that $G = C \cdot N$. Associated with $C$, we define the diagonalizable representation

$$G = C \cdot N \ni c \cdot n \mapsto \text{Ad}_{c} \in \text{Aut}(\mathfrak{g}_{1,0})$$

where $(\text{Ad}_c)_s$ is the semisimple part of the Jordan decomposition of the adjoint operator $(\text{Ad}_c) \in \text{Aut}(\mathfrak{g}_{1,0})$. Denote this representation by $\text{Ad}_s : G \to \text{Aut}(\mathfrak{g}_{1,0})$.

We have a basis $X_1, \ldots, X_n$ of $\mathfrak{g}_{1,0}$ such that

$$\text{Ad}_s(g) = \text{diag}((\alpha_1(g), \ldots, \alpha_n(g))$$

for some characters $\alpha_1, \ldots, \alpha_n$ of $G$. Let

$$B^*_1 := \left\langle \frac{\partial I}{\alpha I} \right| I \subseteq \{1, \ldots, n\} \text{ such that } \left( \frac{\partial I}{\alpha I} \right)_{|I} = 1 \right\rangle,$$

(where we shorten, e.g., $\alpha I := \alpha_{i_1} \cdots \alpha_{i_k}$ for a multi-index $I = (i_1, \ldots, i_k)$).

Then the inclusion $B^*_1 \subset A^*_X$ induces an isomorphism $H^*(B^*_1) \cong H^0(X)$ ([9]). Consider the following subcomplex of $A_X$:

$$C^{1,1} := C^{1,0} \otimes_{\mathbb{C}} B^*_1 \oplus B^{\ast \ast}_{1} \otimes \mathbb{C} \Lambda^2 g^{1,0}\cdot\mathbb{C}.$$

Then $C^{1,1}$ is a double complex with real structure and the inclusion $C^{1,1} \subset A^*_X$ induces an isomorphism $H_{\bar{\partial}}^p (C^{1,1} \otimes \mathbb{C}) \cong H_{\bar{\partial}}^p (X)$ ([3, Sect. 2.6]). By Corollary 5, it suffices to show that $C^{1,1}$ is page-1-$\bar{\partial}$ in order to prove Theorem A.

Take the weight decomposition $\Lambda^* \otimes \mathbb{C} = \bigoplus \lambda V_{\lambda}$ for the representation $\text{Ad}_s : G \to \text{Aut}(\mathfrak{g}_{1,0})$. We have $\Lambda^* \otimes \mathbb{C} = \bigoplus \lambda V_{\lambda}$. (We must distinguish $V_1 \subset \Lambda^* \otimes \mathbb{C}$ from $V_1 \subset \Lambda^* \otimes \mathbb{C}$ for the trivial representation 1). Then we can write

$$\Lambda^* \otimes \mathbb{C} = \bigoplus \lambda V_{\lambda} \otimes \left( \bigoplus (\alpha/\bar{\alpha})_{|I} = 1 \right) \frac{\alpha}{\bar{\alpha}} V_{\bar{\alpha}}$$

and

$$\bigoplus \Lambda^* \otimes \mathbb{C} = \bigoplus \left( \bigoplus (\alpha/\bar{\alpha})_{|I} = 1 \right) \frac{\alpha}{\bar{\alpha}} V_{\bar{\alpha}} \otimes V_{\lambda}.$$

We have

$$\Lambda^* \otimes \mathbb{C} \bigoplus B^*_1 \otimes \mathbb{C} \bigoplus B^{\ast \ast}_{1} \otimes \mathbb{C} \Lambda^2 g^{1,0} = \bigoplus (\alpha/\bar{\alpha})_{|I} = 1 \alpha V_{\lambda} \otimes \frac{1}{\alpha} V_{\bar{\lambda}}$$

and hence

$$C^{1,1} = \bigoplus (\alpha/\bar{\alpha})_{|I} = 1 \left\{ \bigoplus \alpha V_{\lambda} \otimes \frac{1}{\alpha} V_{\bar{\lambda}} + \alpha V_{\lambda} \otimes \frac{1}{\alpha} V_{\bar{\lambda}} + \bigoplus \frac{1}{\alpha} V_{\lambda} \otimes \bar{\lambda} \right\}.$$
3.1. Variant for complex solvmanifolds of splitting type

In this subsection only, $G$ will not be a complex Lie group, but we assume that $G$ is the semidirect product $\mathbb{C}^n \ltimes \varphi N$ so that we have the following:

1. $N$ is a connected simply-connected $2m$-dimensional nilpotent Lie group endowed with an $N$-bi-invariant complex structure $J_N$ (denote the Lie algebras of $\mathbb{C}^n$ and $N$ by $\mathfrak{a}$ and, respectively, $\mathfrak{n}$);
2. for any $t \in \mathbb{C}^n$, it holds that $\phi(t) \in \text{Aut}(N)$ is a holomorphic automorphism of $N$ with respect to $J_N$;
3. $\phi$ induces a semisimple action on $\mathfrak{n}$;
4. $G$ has a lattice $\Gamma$; (then $\Gamma$ can be written as $\Gamma = \Gamma_{\mathbb{C}^n} \ltimes \varphi \Gamma_N$ such that $\Gamma_{\mathbb{C}^n}$ and $\Gamma_N$ are lattices of $\mathbb{C}^n$ and, respectively, $N$, and, for any $t \in \Gamma_{\mathbb{C}^n}$, it holds $\phi(t)(\Gamma_N) \subseteq \Gamma_N$.)

Consider the solvmanifold $X = \Gamma \backslash G$. This satisfies [8, Assumption 1.1] ([3, Assumption 2.11]).

**Theorem 9.** $X$ is a page-1-$\partial \bar{\partial}$-manifold.

This result is a generalization of the observation on completely solvable Nakamura manifolds in [13, Cor. 3.3] based on [3].

**Proof.** Consider the standard basis $\{X_1, \ldots, X_n\}$ of $\mathbb{C}^n$. Consider the decomposition $\mathfrak{n} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{n}_{1,0} \oplus \mathfrak{n}_{0,1}$ induced by $J_N$. By condition (2), this decomposition is a direct sum of $\mathbb{C}^n$-modules. By condition (3), we have a basis $\{Y_1, \ldots, Y_m\}$ of $\mathfrak{n}_{1,0}$ and characters $\alpha_1, \ldots, \alpha_m \in \text{Hom}(\mathbb{C}^n; \mathbb{C}^*)$ such that the induced action $\phi$ on $\mathfrak{n}_{1,0}$ is represented by

$$
\mathbb{C}^n \ni t \mapsto \phi(t) = \text{diag}(\alpha_1(t), \ldots, \alpha_m(t)) \in \text{Aut}(\mathfrak{n}_{1,0}).
$$

For any $j \in \{1, \ldots, m\}$, since $Y_j$ is an $N$-left-invariant $(1,0)$-vector field on $N$, the $(1,0)$-vector field $\alpha_j Y_j$ on $\mathbb{C}^n \ltimes \varphi N$ is $G$-left-invariant. Consider the Lie algebra $\mathfrak{g}$ of $G$ and the decomposition $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g}_{1,0} \oplus \mathfrak{g}_{0,1}$ induced by $J$. Hence we have a basis $\{X_1, \ldots, X_n, \alpha_1 Y_1, \ldots, \alpha_m Y_m\}$ of $\mathfrak{g}_{1,0}$. Let

$$
\{x_1, \ldots, x_n, \alpha_1^{-1} y_1, \ldots, \alpha_m^{-1} y_m\}
$$

be its dual basis of $\mathfrak{g}^{1,0}$. Then we have

$$
\Lambda^p,q \mathfrak{g}_{\mathbb{C}} = \Lambda^p \langle x_1, \ldots, x_n, \alpha_1^{-1} y_1, \ldots, \alpha_m^{-1} y_m \rangle \otimes \Lambda^q \langle \bar{x}_1, \ldots, \bar{x}_n, \bar{\alpha}_1^{-1} \bar{y}_1, \ldots, \bar{\alpha}_m^{-1} \bar{y}_m \rangle
$$

For any $j \in \{1, \ldots, m\}$, there exist unique unitary characters $\beta_j \in \text{Hom}(\mathbb{C}^n; \mathbb{C}^*)$ and $\gamma_j \in \text{Hom}(\mathbb{C}^n; \mathbb{C}^*)$ on $\mathbb{C}^n$ such that $\alpha_j \beta_j^{-1}$ and $\bar{\alpha}_j \bar{\gamma}_j^{-1}$ are holomorphic [8, Lem. 2.2].

Define the differential bi-graded sub-algebra $B^{**}_G \subseteq A^{**}_X$ as

$$
B^{p,q}_G := \mathbb{C} \langle x_l \wedge (\alpha_j^{-1} y_j) y_j \wedge \bar{x}_K \wedge (\bar{\alpha}_L^{-1} \bar{y}_L) \bar{y}_L \rangle \{ |I|+|J|=p \text{ and } |K|+|L|=q \} \text{ such that } (\beta_j \gamma_L)_{|I|=1}
$$

Define

$$
C^{**}_G := B^{**}_G + \overline{B^{**}_G}.
$$
Thus \( C_{r + s} \) is a double complex with real structure and the inclusion \( C_{r + s} \subset A_{r + s} \) induces an isomorphism \( H^p_{\tilde{\partial}}(C_{r + s}) \cong H^p_{\tilde{\partial}}(X) \) and \( H^p_{\partial C}(C_{r + s}) \cong H^p_{\partial C}(X) \) ([3, Sect. 2.5]). Again, it suffices to show that \( C_{r + s} \) is page-1-\( \partial \partial \). Take the weight decomposition \( \Lambda^* n^{1,0} = \bigoplus \alpha V^{1,0}_\alpha \) and \( \Lambda^* n^{0,1} = \bigoplus \alpha V^{0,1}_\alpha \) for the \( \mathbb{C}^n \)-action. Then we have

\[
B_{r + s}^* = \bigoplus_{(\beta, \gamma) | \gamma = 1} \Lambda^* n^{1,0} \otimes (\alpha \beta^{-1} V^{1,0}_\alpha) \otimes (\lambda \gamma^{-1} V^{0,1}_\lambda)
\]

where \( \beta \in \text{Hom}(\mathbb{C}^n; \mathbb{C}^*) \) and \( \gamma \in \text{Hom}(\mathbb{C}^n; \mathbb{C}^*) \) are unitary characters on \( \mathbb{C}^n \) such that \( \alpha \beta^{-1} \) and \( \lambda \gamma^{-1} \) are holomorphic. We have

\[
C_{r + s}^* = \bigoplus_{(\beta, \gamma) | \gamma = 1, \alpha \lambda \neq \beta \gamma} \Lambda^* n^{1,0} \otimes (\alpha \beta^{-1} \lambda \gamma^{-1} V^{1,0}_\alpha) \otimes V^{0,1}_\lambda
\]

\[
+ \bigoplus_{(\beta, \gamma) | \gamma = 1, \alpha \lambda \neq \beta \gamma} \Lambda^* n^{1,0} \otimes V^{1,0}_\lambda \otimes (\alpha \beta^{-1} \lambda \gamma^{-1} V^{0,1}_\alpha)
\]

\[
+ \bigoplus_{(\alpha \lambda) | \gamma = 1} \Lambda^* n^{1,0} \otimes V^{0,1}_\alpha \otimes V^{0,1}_\lambda.
\]

Since \( J_N \) is bi-invariant, we have \( d = \partial : V^{1,0}_\alpha \rightarrow V^{1,0}_\alpha \) and \( d = \bar{\partial} : V^{0,1}_\alpha \rightarrow V^{0,1}_\alpha \). Since \( \alpha \beta^{-1} \lambda \gamma^{-1} \) is holomorphic, we have

\[
d = \partial : \alpha \beta^{-1} \lambda \gamma^{-1} V^{1,0}_\alpha \rightarrow \Lambda^{1,0} \otimes (\alpha \beta^{-1} \lambda \gamma^{-1} V^{1,0}_\alpha)
\]

\[
and
\[
d = \bar{\partial} : \alpha \beta^{-1} \lambda \gamma^{-1} V^{0,1}_\alpha \rightarrow \Lambda^{0,1} \otimes (\alpha \beta^{-1} \lambda \gamma^{-1} V^{0,1}_\alpha).
\]

Thus each each factor of the direct sum of \( C_{r + s}^* \) is the tensor product of simple complexes, hence it is a page-1-\( \partial \partial \)-complex. \( \square \)

4. Semisimple case

In this section, prove Theorem B. We keep the notation from Section 2 and assume \( G \) to be a complex semisimple Lie group, and \( K \subset G \) a maximal compact subgroup. Denote by \( \mathfrak{g} = \text{Lie}(G), \mathfrak{k} = \text{Lie}(K) \) the respective Lie algebras. Note that we have \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{j} \mathfrak{k} \).

As before, we denote by \( X := \Gamma \backslash G \) the quotient by the left action of \( \Gamma \). Later, we will also consider \( Y := G/K \) the quotient of the right action by \( K \) and \( Z := \Gamma \backslash G/K \) the quotient by both. A first observation is the following lemma.

**Lemma 10.** There is a canonical isomorphism \( H(\mathfrak{g}_{1,0}) = H_{dR}(K, \mathbb{C}) \).

**Proof.** This is a consequence of the identification of complex Lie algebras \( \mathfrak{g}_{1,0} \cong \mathfrak{g} = \mathfrak{g} \oplus \mathfrak{j} \mathfrak{g} = \mathfrak{g} \mathbb{C} \) (whence the identification \( \Lambda^* \mathfrak{g}_{1,0} \cong \Lambda^* \mathfrak{g} \mathbb{C} \)) and the fact that the de Rham cohomology of compact Lie groups can be computed from left-invariant forms. \( \square \)
Theorem 11. The Frölicher spectral sequence of $X$ degenerates at page 2, with $E_2^{p,q} = H^p_{dR}(K, \mathbb{C}) \otimes H^q(\Gamma, \mathbb{C})$.

We will give two proofs of this theorem, each one highlighting somewhat different aspects. Both will use the following computation of Dolbeault cohomology by Akhiezer. Note that the left-multiplication of $G$ on $X$ induces a holomorphic representation of $G$ on the Dolbeault cohomology $H^p_{\partial}(X)$.

Theorem 12 (Akhiezer [1]). There is a $G$-module isomorphism

$$H^p_{\partial}(X) = \Lambda^p g^{1,0} \otimes H^q(\Gamma, \mathbb{C})$$

where $\Lambda^p g^{1,0}$ is a $G$-module induced by the adjoint representation and $H^q(\Gamma, \mathbb{C})$ is a trivial $G$-module.

Proof of Theorem 11 via three spectral sequences.

Claim 1. The Hochschild–Serre spectral sequence for $\mathfrak{k} \subset g$ degenerates at the second page.

We notice that $H^*(g_\mathbb{C}) = H^*(g_{1,0}) \otimes H^*(g_{0,1}) \cong H^*(\mathfrak{k}_\mathbb{C}) \otimes \overline{H^*(\mathfrak{k}_\mathbb{C})}$ by Lemma 10. Let $E_{r(HS)}^{p,q}$ be the Hochschild–Serre spectral sequence for $\mathfrak{k} \subset g$. We have $E_2^{p,q} = H^p(g; \mathfrak{k}, \mathbb{C}) \otimes H^q(\mathfrak{k}_\mathbb{C}) = (\Lambda^p(\mathfrak{j}^p\mathfrak{k}_\mathbb{C}))^{\mathfrak{k}} \otimes H^q(\mathfrak{k}_\mathbb{C})$ by the decomposition $g = \mathfrak{k} + \mathfrak{j} \mathfrak{k}$ with $[\mathfrak{j} \mathfrak{k}, \mathfrak{j} \mathfrak{k}] \subset \mathfrak{k}$ and $[\mathfrak{k}, \mathfrak{j} \mathfrak{k}] \subset \mathfrak{j} \mathfrak{k}$. Since $J$ gives a $\mathfrak{k}$-module isomorphism $\mathfrak{k} \cong \mathfrak{j} \mathfrak{k}$, we have $H^*(\mathfrak{k}_\mathbb{C}) = (\Lambda^p(\mathfrak{k}_\mathbb{C}))^{\mathfrak{k}} \otimes (\Lambda^p(\mathfrak{j} \mathfrak{k}_\mathbb{C}))^{\mathfrak{j} \mathfrak{k}}$. Thus the claim follows.

Claim 2. The Serre spectral sequence for the fiber bundle $X \rightarrow Z$ degenerates at the second page.

Let $E_{r(S)}^{p,q}$ be the Serre spectral sequence for the fiber bundle $X \rightarrow Z$. Since $X \rightarrow Z$ is a principal $K$-bundle, we have $E_{2(S)}^{p,q} = H^p_{dR}(Z) \otimes H^q_{dR}(K)$. By the multiplicative structure on the spectral sequence, we have $d_2^{p,q} = \text{Id} H^p_{dR}(Z) \otimes d_2^{0,q}$. We notice that the inclusion $i : \Lambda^* \subset A^* X$ induces a morphism $i_{r(S)}^{p,q} : E_{r(HS)}^{p,q} \rightarrow E_{r(S)}^{p,q}$. By the averaging, we have a map $\mu : A^* X \rightarrow \Lambda^* \mu$ such that $\mu \circ i = \text{Id}$. This implies that $i_{r(S)}^{p,q} : E_{r(HS)}^{p,q} \rightarrow E_{r(S)}^{p,q}$ is injective. By $E_{r(HS)}^{0,q} \cong H^q(\mathfrak{k}) \cong H^q_{dR}(K)$, we can say that $i_{r(S)}^{0,q} : E_{r(HS)}^{0,q} \rightarrow E_{r(S)}^{0,q}$ is an isomorphism. By the first claim, we have $d_{r(S)}^{0,q} = 0$ on $E_{2(HS)}^{0,q}$ and hence $d_{r(S)}^{0,q} = 0$ on $E_{2(S)}^{0,q}$. This implies $d_{r(S)}^{p,q} = 0$ on $E_{r(S)}^{p,q}$. By the same argument, we can say $d_{r(S)}^{p,q} = 0$ on $E_{r(S)}^{p,q}$ for any $r \geq 2$.

Claim 3. The Frölicher spectral sequence of $X$ degenerates at the second page.

Let $E_{r(FS)}^{p,q}$ be the Frölicher spectral sequence of $X$. By Akhiezer’s theorem, we have $E_1^{p,q} = H^p_{\partial}(X) = \Lambda^p g^{1,0} \otimes H^q(\Gamma, \mathbb{C})$. The differential $d_{r(FS)}^{p,q}$ on $E_1^{p,q} = H^p_{\partial}(X)$ is induced by the differential $\partial$ and we can easily check that it is determined by the $g_{1,0}$-module structure of $H^p_{\partial}(X) = H^q(\Gamma, \mathbb{C})$. Since $H^q(\Gamma, \mathbb{C})$ is a trivial $G$-module, $g_{1,0}$ acts trivially on $H^0_{\partial}(X)$ and we can say that the differential $\partial$ induces a trivial differential on $H^q(\Gamma, \mathbb{C})$. Thus we have $E_2^{p,q} \cong H^p(g_{1,0}) \otimes H^q(\Gamma, \mathbb{C})$. By the above argument, we have $E_2^{p,q} \otimes \mathbb{C} = H^p(Z, \mathbb{C}) \otimes H^q(\Gamma, \mathbb{C}) \cong H^p(\Gamma, \mathbb{C}) \otimes H^q(g_{1,0})$. By the $E_2$-degeneration of $E_{r(S)}^{p,q}$, for
any \( r \in \mathbb{Z} \) we have \( \sum_{p+q=r} \dim E_{2(S)}^{p,q} = \sum_{p+q=r} \dim E_{2(S)}^{p,q} = \dim H^{r}_{dR}(X) \) and hence the claim follows. \( \square \)

**Proof of Theorem 11 via lattice-cohomology subcomplex of \( A_X \).** The lattice cohomology \( H(\Gamma, \mathbb{C}) \) can be computed via the complex \( A^{\Gamma}_{Y} \) of \( \Gamma \)-invariant forms on \( Y \) [5, Chap. VII]. By pulling forms back to \( G \) then pushing forward to \( X \), we obtain a map of (simple) complexes

\[
\sigma : A^{\Gamma}_{Y} \rightarrow A_{X}.
\]

We will also consider the projection map

\[
pr : A_{X} \to (A^{0,\bullet}_{X}, \bar{\partial})
\]

which is a map of complexes and induces the ‘edge maps’

\[
H^{q}_{dR}(X) \rightarrow H^{0,q}_{\bar{\partial}}(X)
\]

**Claim.** The composition \( pr \circ \sigma \) induces isomorphisms \( H^{q}(\Gamma, \mathbb{C}) \cong E^{0,q}_{1} \).

Admitting the claim, the edge maps are surjective, hence there can be no differentials starting at \( E^{0,q}_{r} \), for any \( q \) and \( r \geq 1 \). Thus, by the Leibniz-rule, any possible nonzero differential has to live on \( E_{1} \) and be of the form \( \text{Id} \otimes d_{g_{1,0}} \).

In order to prove the claim, we use the following identifications (see [5, Chap. VII]):

1. \( A^{\Gamma}_{Y} \) is identified with the \( K \)-invariants of \( A := \Lambda(\mathfrak{g}/\mathfrak{t})_{\mathbb{C}}^{\wedge} \otimes C^{\infty}(X, \mathbb{C}) \), where the action of \( K \) is induced by right multiplication on \( X \) on the right factor and the adjoint action on the left factor,
2. \( A_{X} \) is identified with \( \Lambda_{\mathfrak{g}} \otimes C^{\infty}(X, \mathbb{C}) \),
3. \( A^{0,\bullet}_{X} \) is identified with \( \Lambda_{\mathfrak{g}}^{\wedge} \otimes C^{\infty}(X, \mathbb{C}) \).

Under these identifications, consider the map \( \tilde{\sigma} : A \rightarrow A_{X} \) induced by the inclusion \( (\mathfrak{g}/\mathfrak{t})_{\mathbb{C}}^{\wedge} \otimes A_{Y} \rightarrow \mathfrak{g}_{\mathbb{C}}^{\wedge} \). Since the composition \( \mathfrak{g}_{0,1} \rightarrow \mathfrak{g}_{\mathbb{C}} \rightarrow (\mathfrak{g}/\mathfrak{t})_{\mathbb{C}}^{\wedge} \) is an isomorphism, so is \( pr \circ \tilde{\sigma} \). On the other hand, \( pr \circ \tilde{\sigma} \) is an equivariant map, therefore it identifies the \( K \)-invariants on the left (i.e., \( A^{\Gamma}_{Y} \)) with the \( K \)-invariants on the right. But by averaging over \( K \), one sees that the complex of \( K \)-invariants is a direct summand in \( A^{0,\bullet} \). Therefore, the edge map \( H^{q}(\Gamma, \mathbb{C}) \rightarrow E^{0,q}_{1} \) induced by \( pr \circ \tilde{\sigma} \) has to be injective. Since by Akhiezer’s result both source and target have the same dimension, the claim follows. \( \square \)

We consider the canonical injection \( H^{k}(\mathfrak{g}; \mathfrak{t}, \mathbb{C}) \hookrightarrow H^{k}(\Gamma, \mathbb{C}) \) associated with the locally symmetric space \( Z := \Gamma\backslash G/K \). In view of Theorem 12 it can be identified with the map induced by the inclusion of left-invariant forms \( H^{0,k}_{\bar{\partial}}(\Lambda_{\mathfrak{g}}) \hookrightarrow H^{0,k}_{\bar{\partial}}(X) \).

**Corollary 13.** \( X \) is a page-1-\( \partial \bar{\partial} \)-manifold if and only if \( H^{k}(\mathfrak{g}; \mathfrak{t}, \mathbb{C}) \cong H^{k}(\Gamma, \mathbb{C}) \) for any \( k \in \mathbb{Z} \).

**Proof.** Suppose \( X \) is a page-1-\( \partial \bar{\partial} \)-manifold. By Remark 3, we have \( \dim E^{p,q}_{2} = \dim E^{q,p}_{2} \). By Theorem 11, this implies \( \dim H^{k}_{dR}(K) = \dim H^{k}(\Gamma) \) for any \( k \in \mathbb{Z} \). As we saw in the first proof of Theorem 11, we have \( H^{k}(\mathfrak{g}; \mathfrak{t}) \cong H^{k}_{dR}(K) \). Thus we
have \( \dim H^k(\mathfrak{g}; \mathfrak{t}, \mathbb{C}) = \dim H^k(\Gamma, \mathbb{C}) \) and so the injection \( H^k(\mathfrak{g}; \mathfrak{t}, \mathbb{C}) \hookrightarrow H^k(\Gamma, \mathbb{C}) \) is an isomorphism.

We assume \( H^k(\mathfrak{g}; \mathfrak{t}, \mathbb{C}) \cong H^k(\Gamma, \mathbb{C}) \) for any \( k \in \mathbb{Z} \). Then by Theorem 12 and Lemma 10, we have

\[
\dim H^p,q_{\bar{\partial}}(X) = \dim \Lambda^p \mathfrak{g}^{1,0} \otimes H^q(g_{1,0}) = \dim H^p,q_{\bar{\partial}}(\Lambda_\mathfrak{g}).
\]

Hence the injection \( H^p,q_{\bar{\partial}}(\Lambda_\mathfrak{g}) \hookrightarrow H^p,q_{\bar{\partial}}(X) \) is an isomorphism and so \( X \) is a page-1-\( \partial \bar{\partial} \)-manifold by Proposition 8.

**Remark 14.** For \( G = \text{SL}_2(\mathbb{C}) \) with \( K = \text{SU}(2) \), \( b_1(K) = b_2(K) = 0 \) but for any \( r \in \mathbb{N} \), there exist lattices \( \Gamma \subseteq G \) such that \( b_1(\Gamma) = b_2(\Gamma) = r \) (see [20, §B], [7, 6.2]). Note that by a theorem of Raghunathan [15], \( b_1(\Gamma) = 0 \) as soon as \( G \) has no \( \text{SL}_2(\mathbb{C}) \)-factor. Moreover, for \( k < \text{rk}_\mathbb{R} \mathfrak{g} \), the injection \( H^k(\mathfrak{g}; \mathfrak{t}, \mathbb{C}) \hookrightarrow H^k(\Gamma, \mathbb{C}) \) is an isomorphism (see [5, VII, Cor. 4.4]).

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