Abstract—Unsourced random-access (U-RA) is a type of grant-free random access with a virtually unlimited number of users, of which only a certain number $K_a$ are active on the same time slot. Users employ exactly the same codebook, and the task of the receiver is to decode the list of transmitted messages. Recently a concatenated coding construction for U-RA on the AWGN channel was presented, in which a sparse regression code (SPARC) is used as an inner code to create an effective outer OR-channel. Then an outer code is used to resolve the multiple-access interference in the OR-MAC. In this work we show that this concatenated construction can achieve a vanishing per-user error probability in the limit of large blocklength and a large number of active users at sum-rates up to the symmetric Shannon capacity, i.e. as long as $K_a R < 0.5 \log_2(1 + K_a \text{SNR})$. This extends previous point-to-point optimality results about SPARCs to the unsourced multiuser scenario. Additionally, we calculate the algorithmic threshold, that is a bound on the sum-rate up to which the inner decoding can be done reliably with the low-complexity AMP algorithm.

I. INTRODUCTION

One of the key new application scenarios of future wireless networks is known as the Internet-of-Things (IoT), where it is envisioned that a very large number of devices (referred to as users) is sending data to a common access point. Typical examples thereof include sensors for monitoring smart infrastructure or biomedical devices. This type of communication is characterized by short messages and sporadic activity.

The large number of users and the sporadic nature of the transmission makes it very wasteful to allocate dedicated transmission resources to all the users. In contrast to this, the traditional information theoretic treatment of the multiple-access uplink channel is focused on few users $K$, large blocklength $n$ and coordinated transmission, in the sense that each user is given an individual distinct codebook, and the $K$ users agree on which rate $K$-tuple inside the capacity region to operate $[1]$. Mathematically, this is reflected by considering the limit of infinite message- and blocklength while keeping the rate and the number of users fixed. Another route, more suited to the IoT requirements, was taken in recent works like $[2, 3]$, where the number of users $K$ is taken to infinity along with the blocklength. It was shown, that the information theoretic limits may be drastically different, when the number of users grows together with the blocklength.

A novel random access paradigm, referred to as unsourced random-access (U-RA), was suggested in $[3]$. In U-RA each user employs the same codebook and the task of the decoder is to recover the list of transmitted messages irrespective of the identity of the users. The number of inactive users in such a model can be arbitrary large and the performance of the system depends only on the number of active users $K_a$. Furthermore, a transmission protocol without the need for a subscriber identity is well suited for mass production. These features make U-RA particularly interesting for the aforementioned IoT applications.

In $[3]$ the U-RA model for the real adder AWGN-MAC was introduced and a finite-blocklength random coding bound on the achievable energy-per-bit over $N_0(E_b/N_0)$ was established. In following works several practical approaches were suggested which successfully reduced the gap to the random coding achievability bound $[2, 4, 5]$. The model has been extended to fading $[6]$ and MIMO channels $[7]$. A concatenated coding approach for the U-RA problem on the real adder AWGN was proposed in $[5]$. The idea is to split each transmission up into $L$ sublots. In each subslot the active users send a column from a common inner coding matrix, while the symbols across all subslots are chosen from a common outer tree code.

The relation of the inner code to sparse regression codes (SPARCs) was pointed out. SPARCs were introduced in $[11]$ as a class of channel codes for the point-to-point AWGN channel, which can achieve rates up to Shannon capacity under maximum-likelihood decoding. Later, it was shown that SPARCs can achieve capacity under approximate message passing (AMP) decoding with either power allocation $[12]$ or spatial coupling $[13]$. AMP is an iterative low-complexity algorithm for solving random linear estimation problems or generalized versions thereof $[14, 15, 16]$. A recent survey on SPARCs can be found in $[17]$.

Based on the connection of the inner code of $[5]$ to SPARCs, in $[10]$ we suggested a modified version of AMP as an inner decoder, which improved the performance compared to the original inner decoder of $[5]$. One of the appealing features of the AMP algorithm is, that it is possible to analyse its asymptotic performance, averaged over certain random matrix ensembles, through the so called state evolution (SE) equations $[16, 18]$. Interestingly the SE equations can also be obtained as the fixed points of the replica symmetric (RS) potential, an expression that was first calculated through the non-rigorous replica method $[19, 20]$. It was shown that in random linear estimation problems the fixed points of the RS potential also characterize the symbols-wise posterior distribution of the input elements and therefore also the error probability of several optimal estimators like the symbol-by-symbol maximum-
a-posteriori (SBS-MAP) estimator \([21, 22]\). The difference between the AMP and the SBS-MAP estimate is, that the SBS-MAP estimate always corresponds to the global minimum of the RS-potential, while the AMP algorithm gets ‘stuck’ in local minima. The rate below which a local minimum appears was called the algorithmic or belief-propagation threshold in \([21, 23, 13]\). It was shown in \([23, 24]\) that, despite the existence of local minima in the RS-potential, the AMP algorithm can still converge to the global minimum when used with spatially coupled matrices. Although the RS-potential was derived by (and named after) the non-rigorous replica method, it was recently proven to hold rigorously \([25, 24]\). The proof of \([24]\) is more general in the sense that it includes the case where the

\[ W = \frac{K_a}{J} \]  

consists of blocks of size \(2^J\) and each block is considered to be drawn iid from some distribution on \(\mathbb{R}^{2^J}\). Initially, the result of \([24]\) relied on the conjecture that the SE equations of the AMP algorithm hold for the case of a block iid distribution. But \([13]\) has shown that the SE equations hold under quite weak assumptions on the rate, which include the block iid case, and therefore has proven the missing conjecture in \([24]\).

Building on these results, in \([10]\), we calculated the RS-potential of the inner decoding problem, which allowed us to calculate the asymptotic error probabilities of the SBS-MAP and the AMP estimate. The results were semi-analytical, in the sense that the fixpoints of the RS-potential could only be evaluated numerically. In this work we show, that in the limit of \(K_a, J \to \infty\) with \(J = a \log_2 K_a\) for some \(a > 1\), the RS-potential converges to a simple form with a sharp threshold on the achievable sum-rates.

We have also shown in \([10]\) that the inner decoding creates an effective outer OR-channel \([26, 27]\) under a specific input constraint and we gave upper bound on the achievable rates on that channel. As pointed out in \([28]\), the outer tree code of \([5]\) is able to achieve that bound exactly in the limit of infinite subslots \(L\) at a decoding complexity exponential in \(L\) or up to a multiplicative constant with a decoding complexity linear in \(L\).

Our main contribution in this work is to show that the concatenated coding scheme of \([5]\) consisting of multiuser SPARCs combined with an outer tree code is reliable, in the sense that it can achieve a vanishing per-user error probability in the limit of large blocklength and infinitely many users, at sum-rates up to the symmetric Shannon capacity \(0.5 \log_2(1 + K_a \text{SNR})\). This also shows that an unsourced random access scheme can, in the considered scaling regime \(a > 1\), achieve the same symmetric rates as a non-unsourced scheme.

The U-RA problem on the real AWGN adder is formally equivalent to the On-Off random access scheme defined in \([29]\), and there are several other works, which analyse the sparse recovery problem, assuming an iid prior on the unknown vector, using either the replica method like \([21, 30]\) or more direct compressed sensing based methods like \([21, 29, 30]\). It is not obvious how the asymptotic result of our Theorem \(1\) below can be obtained directly from replica arguments, since it requires \(J\) to scale proportional to the blocklength \(n\), i.e. the undersampling ratio \(2^J/n\) to go to infinity. Such a behavior is not covered by the available framework. We can obtain this result by first calculating the RS-potential in the limit of large \(n, L\) with fixed \(J\) and then take the limit \(J \to \infty\). Also compressed sensing based results like \([29, 30]\) are insufficient, since they contain unspecified constants, which are necessary to derive an exact capacity.

### II. System Model

Let \(K_a\) denote the number of active users, \(n\) the number of available channel uses and \(B = nR\) the size of a message in bits. The spectral efficiency is given by \(\mu = K_aB/n\). The channel model used is

\[ y = \sum_{i=1}^{K_a} q_i x_i + z, \tag{1} \]

where each \(x_i \in C \subset \mathbb{R}^n\) is taken from a common codebook \(C\) and \(q_i \in \{0, 1\}\) are binary variables indicating whether a user is active. The number of active users is denoted as \(K_a = \sum_{i=1}^{K_a} q_i\). The codewords are assumed to be normalized \(\|x_i\|^2 = nP\) and the noise vector \(z\) is Gaussian iid \(z_i \sim \mathcal{N}(0, N_0/2)\), such that \(\text{SNR} = 2P/N_0\) denotes the real per-user SNR. All the active users pick one of the \(2^B\) codewords from \(C\), based on their message \(W_k \in [1 : 2^B]\). The decoder of the system produces a list \(g(y)\) of at most \(K_a\) messages. An error is declared if one of the transmitted messages is missing in the output list \(g(y)\) and we define the per-user probability of error as:

\[ P_e = \frac{1}{K_a} \sum_{k=1}^{K_a} \mathbb{P}(W_k \notin g(y)). \tag{2} \]

Note that the error is independent of the user identities in general and especially independent of the inactive users. The performance of the system is measured in terms of the standard quantity \(E_b/N_0 := P/(RN_0)\) and the described coding construction is called reliable if \(P_e \to 0\) in the considered limit.

### III. Coding Construction

In this work we focus on a special type of codebook, where each transmitted codeword is created in the following way: First, the \(B\)-bit message \(W_k\) of user \(k\) is mapped to an \(L\)-bit codeword from some common outer codebook. Then each of the \(J\)-bit sub-sequences is mapped to an index \(i_k(l) \in [1 : 2^J]\) for \(l = [1 : L]\) and \(k = [1 : K_a]\). The inner codebook is based on a set of \(L\) coding matrices \(A_l \in \mathbb{R}^{n \times 2^J}\). Let \(a_l^{(i)}\) with \(i = [1 : 2^J]\) denote the columns of \(A_l\). The inner codeword of user \(k\) corresponding to the sequence of indices \(i_k(1), ..., i_k(L)\) is then created as

\[ x_k = \sum_{l=1}^{L} a_l^{(i_k(l))}. \tag{3} \]

The \(A_l\) are assumed to be scaled such that \(\|a_l^{(i)}\|^2 = nP/L\). The above encoding model can be written in matrix form as

\[ y = \sum_{k=1}^{K_a} A m_k + z = A \left( \sum_{k=1}^{K_a} m_k \right) + z. \tag{4} \]
where $A = (A_1, \ldots, A_L)$ and $m_k \in \mathbb{R}^{L \times 2^J}$ is a binary vector satisfying $m_{k,(l-1)2^J+i(l)} = 1$ and zero otherwise, for all $l = [1:L]$. Let $s = \sum_{k=1}^{K_{\alpha}} m_{k}$ can be viewed as concatenation of an inner point-to-point channel $s \rightarrow As + z$ and an outer binary input adder MAC $(m_1, \ldots, m_{K_{\alpha}}) \rightarrow s$. We will refer to those as the inner and outer channel, the corresponding encoder and decoder will be referred to as inner and outer encoder/decoder and the aggregated system of inner and outer encoder/decoder is the concatenated system.

The per-user inner rate in terms of bits/c.u. is given by $R_{in} = LJ/n$ and the outer rate is given by $R_{out} = B/LJ$.

### IV. Main Result

Our main result states that inner and outer codes exist, such that the concatenated coding construction described above is reliable at sum-rates up to the symmetric Shannon capacity.

**Theorem 1.** Let $n, L, J, K_{\alpha} \rightarrow \infty$ and $R, SNR \rightarrow 0$ with fixed $E_b/N_0 = SNR/(2R)$, $S = K_{\alpha}R$ and $J = \alpha \log_2 K_{\alpha}$ for any $\alpha > 1$. In this limit there is a concatenated code as described above that can be decoded with $P_e \rightarrow 0$ if

$$S < \frac{1}{2} \log_2 (1 + K_{\alpha}SNR)$$

Note that within our asymptotic regime $K_{\alpha}SNR = 2SE_b/N_0$ is a constant. As mentioned in Section III the inner decoding is equivalent to a structured sparse recovery problem of finding $s$ from the knowledge of $y$ and $A$, where

$$y = As + z$$

and $s \in \mathbb{R}^{L \times 2^J}$ is generated according to the model described in Section III i.e. $s = \sum_{k=1}^{K_{\alpha}} m_{k}$. We say that $s$ is generated from evenly distributed messages, if the outer encoded sequences $i_k(1), \ldots, i_k(L)$ are distributed evenly, i.e. $P(i_k(s) = j) = 1/2^J$ for all $j \in [1:2^J]$, and so $P(m_{k,(l-1)2^J+i(l)} = 1) = 1/2^J$ for all $l \in [1:L]$. We will show that it is enough to recover the support of $s$. The asymptotic limitations of the problem of support recovery of structured sparse vectors in the considered scaling regime are a novel result on their own, therefore we analyse two types of support estimators. Let $\rho$ be the binary vector indicating the support of $s$, i.e. $\rho_i = 1$ if and only if $s_i \neq 0$. The SBS-MAP estimator of $\rho$

$$\hat{\rho}_i = \arg \max_{\rho \in \{0,1\}} P(\rho_i = \rho | y, A)$$

minimizes the SBS error probability $P(\hat{\rho}_i \neq \rho_i)$ but is typically unfeasible to compute in practice. The second estimator is the low-complexity AMP algorithm, which produces an estimate of $\rho$ by iterating the following equations

$$\rho^{i+1} = \eta_t(A^T z^i + \rho^i)$$

$$z^{i+1} = y - A \rho^{i+1} + \frac{2^JL}{n} z^i (\eta_t(A^T z^i + \rho^i))$$

where the functions $\eta_t : \mathbb{R}^{2^J} \rightarrow \mathbb{R}^{2^J}$ are defined component-wise $\eta_t(x) = (\eta_t_1(x_1), \ldots, \eta_t_{2^J}(x_{2^J}))^T$ and each component is given by

$$\eta_t(x) = \sqrt{\hat{P}} \left( 1 + \frac{p_0}{1-p_0} \exp \left( \frac{\hat{P} - 2\sqrt{\hat{P}x}}{2\tau_t^2} \right) \right)^{-1}$$

with $\tau_t^2 = \|x\|^2_2/\hat{P}$, $\hat{P} = nSNR/L$ and $p_0 = (1 - 2^{-J})K_{\alpha}$. $\langle x \rangle = (\sum_{i=1}^N x_i)/N$ denotes the average of a vector, $\eta'_{t}$ denotes the componentwise derivative of $\eta_t$ and we choose $\rho^0 = 0$ as initial value. Our result on the inner recovery problem is as follows:

**Theorem 2.** Let $A \in \mathbb{R}^{L \times 2^J}$ be a matrix with Gaussian iid entries $A_{ij} \sim \mathcal{N}(0, P/L)$ and let $y$ and $s$ be jointly distributed according to the model with $s$ being generated from evenly distributed messages. Furthermore, let $R_{in} = LJ/n$. In the limit $L, n, K_{\alpha}J \rightarrow \infty$ with $J = \alpha \log_2 K_{\alpha}$ for some $\alpha > 1$ and SNR, $R_{in} \rightarrow 0$ with fixed ratio $\epsilon_{in} = SNR/(2R_{in})$ and fixed inner sum-rate $S_{in} = K_{\alpha}R_{in}$ the following holds: The SBS-MAP detector recovers the support of $s$ reliably if

$$S_{in} \left(1 - \frac{1}{\alpha} \right) < \frac{1}{2} \log_2 (1 + 2S_{in}\epsilon_{in})$$

and the AMP decoder recovers the support of $s$ reliably if

$$S_{in} < \log_2 (1 - \frac{1}{\alpha})^{-1} - 1\epsilon_{in}$$

**Remark 1.** In the case $K_{\alpha} = 1$ no outer code is necessary, so $R_{in} = R$ and furthermore $S_{in} = R$ and $2S_{in}\epsilon_{in} = SNR$. Hence, if $K_{\alpha} = 1$ is fixed and $J \rightarrow \infty$, which corresponds to $\alpha \rightarrow \infty$, then (10) recovers the statements of [11, 24], i.e. that SPARCs are reliable at rates up to the Shannon capacity $0.5 \log_2 (1 + SNR)$ under optimal decoding. Also the algorithmic threshold (11) coincides with the result of [13]. In that sense Theorems 1 and 2 are an extension of [13] and show that SPARCs can achieve the optimal rate limit in the unsourced random access scenario. However, notice that the concept of our proof technique is simpler, since we make use of the result in [10], which states that not only the sections are described by a decoupled channel model, but in the limit $J \rightarrow \infty$ also the individual components. So all the results of Theorem 2 can be derived from the fixpoints of a simple scalar-to-scalar function.

**Remark 2.** The sparse recovery problem (11) is very general and it is possible to describe random coding for several different classical multiple-access variants, where all the users are assumed to have their own codebook. For that, let $K_{\alpha} = 1$ and identify the number of section with the number of users. The matrices $A_1, \ldots, A_L$ are then the codebooks of the individual users:

- Fixed $L$ in the limit $J, n \rightarrow \infty$ describes the classical AWGN adder MAC from [1], where each user has his own codebook.
• $L, J, n \to \infty$, where only a fraction of the sections are non-zero describes the many-access channel treated in [2].
• $J$ fixed and $L, n \to \infty$ describes specific version of the many-access MAC treated in [11, 5].

It is interesting, that in the first case Theorem 2 gives the correct result, after letting $\alpha \to \infty$, $K_a = 1$ and $L = K$. The case of $J, n \to \infty$ at finite $L$ is not directly covered though by our analysis framework. Nonetheless, we believe that an extension of this framework should be able to show that all of the above cases can be derived from a single scalar RS-potential, but this is left for future work.

**Proof of Theorem 1**. Theorem 2 shows that, if condition (10) is fulfilled, there exists an inner coding matrix $A$ such that the power constraint is fulfilled on average and the SBS-MAP estimator (7) recovers the support $\rho$ of $s$ reliably. Then $\rho$ is given as the componentwise OR-combination of the input message vectors $m_k$:

$$\rho = \bigvee_{k=1}^{K_a} m_k$$  \hspace{1cm} (12)

This creates an outer noiseless OR-MAC [26, 27]. Let us assume, that all the message vectors $m_i$ are independently encoded by the same outer code and that the outer encoded symbols are evenly distributed. The per-user rate of this outer code is limited by

$$K_a R_{\text{out}} J < 2^J \mathcal{H}_2((1 - 2^{-J})^{K_a})$$  \hspace{1cm} (13)

where $\mathcal{H}_2$ denotes the binary entropy function. As shown in [28], in the considered limit $K_a, J \to \infty$, inequality (13) implies

$$R_{\text{out}} < 1 - \alpha^{-1}.$$  \hspace{1cm} (14)

Although (14) is formally an upper bound it is shown in [28] that the bound is tight, since it is achievable by the outer tree code described in [5]. Therefore we can assume that a capacity achieving outer code exists if $R_{\text{out}} < 1 - \alpha^{-1}$. Since the total rate is given by $R = R_{\text{in}} R_{\text{out}}$, we have that $S = S_{\text{in}} R_{\text{out}}$ and (5) follows from Theorem 2.

It remains to proof Theorem 2. For that, we build on our results from [10], which characterize the performance of the SBS-MAP estimator (7) in the limit $L, n \to \infty$ with a fixed ratio $L/n$ and fixed $J$. Through a series of approximations it is shown in [10] that for a Gaussian iid $A$ the error statistics of the SBS-MAP estimator (7) converge to the error statistics of an SBS-MAP estimate in $2^L$ decoupled real Gaussian channels:

$$r_i = (\eta \hat{P})^2 s_i + z_i$$  \hspace{1cm} (15)

where $\hat{P} = n \text{SNR}/L = J \text{SNR}/R_{\text{in}} = 2J \bar{E}_{\text{in}}$ and each component $i = [1 : L2^{-J}]$ is considered independently of the others. Furthermore, $s_i \in \{0, 1\}$ with

$$p_0 := \mathbb{P}(s_i = 0) = (1 - 2^{-J})^{K_a}$$  \hspace{1cm} (16)

$$\mathbb{P}(s_i = 1) = 1 - p_0 \text{ and } z_i \sim \mathcal{N}(0, 1).$$

The factor $\eta$ is determined by the minimizer of the function

$$i^{RS}(\eta) = 2^J I(\eta \hat{P}) + \frac{2^J}{2\beta} [\eta - 1] \log_2 e - \log_2 \eta],$$  \hspace{1cm} (17)

where $I(\eta \hat{P})$ is the input-output mutual information of the decoupled model (15) and $\beta = 2^J R_{\text{in}}/J$. The RS potential (17) was introduced in [10] as an approximation of the true RS potential of the recovery problem [5], but it was shown that the error terms in this approximation are of order $K_a/2^J$. In the asymptotic regime that we consider, $K_a, J \to \infty$ with $2^J = K_a^\alpha$ and some $\alpha > 1$ we have that $K_a/2^J \to 0$. Therefore, in this limit, (17) indeed characterizes the performance of the SBS-MAP estimator (7) exactly.

The AMP algorithm (8) is strongly connected to the RS-potential (17) in that the asymptotic error distribution of the AMP estimate at convergence is described by the same decoupled channel model (15), only that the coefficient $\eta$ that determines the effective channel strength is given by the smallest local minimizer of (17) [10]. The next Theorem gives the pointwise limit of (17).

**Theorem 3.** In the limit $K_a, J \to \infty$, $R_{\text{in}}, \text{SNR} \to 0$ with fixed ratios $\bar{E}_{\text{in}} = \text{SNR}/(2R_{\text{in}})$, $S = K R_{\text{in}}$ and $J = \alpha \log_2 K_a$ for some $\alpha > 1$ the pointwise limit of the RS-potential (17) is given by (up to additive or multiplicative terms that are independent of $\eta$ and therefore do not influence the critical points of $i^{RS}(\eta)$):

$$i_0^{RS}(\eta) := \lim_{J \to \infty} i^{RS}(\eta) = \eta \bar{E}_{\text{in}} [1 - \theta(\eta - \tilde{\eta})] + \frac{S}{\log_2 e} \left( \frac{1}{\alpha} - 1 \right) \theta(\eta - \tilde{\eta}) + \frac{1}{2} [\eta(\eta - 1) - \ln \eta]$$  \hspace{1cm} (18)

where

$$\theta(x) := \begin{cases} 1, & \text{if } x > 0 \\ \frac{1}{2}, & \text{if } x = 0 \\ 0, & \text{if } x < 0 \end{cases}$$  \hspace{1cm} (19)

and

$$\tilde{\eta} = 1 - \frac{1}{\bar{E}_{\text{in}} \log_2 e}$$  \hspace{1cm} (20)

**Proof.** The RS-potential (17) rescaled by $\beta/2^J$ takes the form

$$i^{RS}(\eta) = \frac{R_{\text{in}} 2^J}{J} I(\eta \hat{P}) + \frac{\log_2 e}{2} [\eta(\eta - 1) - \ln \eta]$$  \hspace{1cm} (21)

with the mutual information

$$I(\eta \hat{P}) := I(X; Y) = H(Y) - H(Y|X)$$  \hspace{1cm} (22)

for $P(X = 0) = p_0$, $P(X = 1) = 1 - p_0$ and $Y = (\eta \hat{P})^2 X + Z$, for $Z \sim \mathcal{N}(0, 1)$ independent of $X$. The mutual information $I(\eta \hat{P})$ can be evaluated as follows. First, note that in an additive channel $H(Y|X) = H(Z)$, so $H(Y|X)$
is independent of \( \eta \) and therefore we can ignore it. The distribution of \( Y \) is given by

\[
p(y) = p_0 p(y|x=0) + (1-p_0) p(y|x=1) = \frac{p_0}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) + \frac{1-p_0}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} \left(y-(\eta \hat{P})^\frac{1}{2}\right)^2\right),
\]

so the differential output entropy \( H(Y) = -\int p(y) \log_2 p(y) dy \) can be split into the sum of two parts. Define \( H_0 \) and \( H_1 \) respectively by

\[
H_0 := -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{2}\right) \log_2(p(y)) dy
\]

and

\[
H_1 := -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2} \left(y-(\eta \hat{P})^\frac{1}{2}\right)^2\right) \log_2(p(y)) dy
\]

such that the following relation holds:

\[
I(\eta \hat{P}) = p_0 H_0 + (1-p_0) H_1.
\]

Taking into account the scaling factor in (21) and using that

\[
\lim_{J \to \infty} \frac{R_{\text{snr}} 2^J}{J} = K_a \quad \text{and} \quad \lim_{J \to \infty} p_0 = 1
\]

we get that

\[
\lim_{J \to \infty} \frac{R_{\text{snr}} 2^J}{J} I(\eta \hat{P}) = \lim_{J \to \infty} \left( \frac{R_{\text{snr}} 2^J}{J} H_0 + \frac{S}{J} H_1 \right).
\]

Now let us take a closer look at \( \log_2 p(y) = \log_2(e) \log p(y) \) which appears in both \( H_0 \) and \( H_1 \). Let \( x_1, x_2 > 0 \) with \( x_2 > x_1 \). Then for the logarithm of the sum of exponentials it holds that

\[
-\ln(e^{-x_1} + e^{-x_2}) = x_1 + \ln(1 + e^{-(x_2-x_1)}).
\]

The error term \( \ln(1 + e^{-(x_2-x_1)}) \) decays exponentially as the difference \( x_2 - x_1 \) grows. Since \( p(y) \) is the sum of two exponentials we can approximate \( \ln p(y) \) by:

\[
-\ln p(y) = \min \left\{ \frac{y^2}{2} - \ln(p_0), \frac{1}{2} \left(y-(\eta \hat{P})^\frac{1}{2}\right)^2 - \ln(1-p_0) \right\}
\]

This approximation is justified, since the difference of the two exponents in \( p(y) \) is proportional to \( \sqrt{J} \), and so it grows large with \( J \).

First, note that since \( \min\{a,b\} \leq a \) and \( \min\{a,b\} \leq b \) holds for all \( a, b \in \mathbb{R} \), \( -\ln p(y) \leq y^2/2 - \ln(1-p_0) \) as well as \( -\ln p(y + (\eta \hat{P})^\frac{1}{2}) \leq y^2/2 + \ln(2^J/K) \). This means that each of the integrands in \( H_0 \) and \( H_1/J \) resp. is bounded uniformly, for all \( J \), by an integrable function. This allows us to evaluate the integrals by using Lebesgue’s theorem on dominated convergence. For this purpose we need to calculate the pointwise limits of \( \ln p(y) \) and \( \ln p(y + (\eta \hat{P})^\frac{1}{2})/J \). The theorem on dominated convergence then states, that the limit of the integrals is given by the integral of the pointwise limits. The minimum in (29) can be expressed as

\[
-\ln p(y) = \begin{cases} \frac{y^2}{2} & y < \gamma \\frac{1}{2} \left(y-(\eta \hat{P})^\frac{1}{2}\right)^2 + \ln \left(\frac{2^J}{K}\right) & y \geq \gamma \end{cases}
\]

where we neglected \( \ln(p_0) = \ln(1-K/2^J) \approx K/2^J \) and \( \gamma \) is given by

\[
\gamma = \frac{1}{2} \left(\eta \hat{P}\right)^\frac{1}{2} + \ln \left(\frac{2^J}{K}\right) \left(\eta \hat{P}\right)^{-\frac{1}{2}}.
\]

Given the considered scaling constraints and \( \hat{P} = J S N R / R_{\text{snr}} \), \( \gamma \) can be rewritten as

\[
\gamma = \sqrt{\frac{1}{2} \left(\eta E_{\text{snr}} + \frac{1}{2 \log e \sqrt{p_{\text{snr}}}}\right)}
\]

The term in parenthesis is strictly positive for all \( \eta \) so \( \lim_{J \to \infty} \gamma = \infty \) and therefore the pointwise limit of \( \ln p(y) \) is given by \( \lim_{J \to \infty} \ln p(y) = -y^2/2 \). It follows from Lebesgue’s theorem on dominated convergence that

\[
\lim_{J \to \infty} H_0 = \log_2 e
\]

which is independent of \( \eta \), so we can ignore it when evaluating \( I^{S \text{snr}}(\eta) \). For the calculation of \( H_1 \) note that:

\[
-\ln p(y + (\eta \hat{P})^\frac{1}{2}) \begin{cases} \frac{1}{2} \left(y+(\eta \hat{P})^\frac{1}{2}\right)^2 & y < \gamma' \\frac{1}{2} \left(y+(\eta \hat{P})^\frac{1}{2}\right)^2 + \ln \left(\frac{2^J}{K}\right) & y \geq \gamma' \end{cases}
\]

where we defined \( \gamma' := \gamma - (\eta \hat{P})^\frac{1}{2} \), \( \gamma' \) is not non-negative anymore and therefore the asymptotic behavior of \( \gamma' \) depends on \( \eta \) in the following way:

\[
\lim_{J \to \infty} \gamma' = \begin{cases} \infty & \text{if } \eta < \bar{\eta} \\gamma' & \text{if } \eta = \bar{\eta} \\gamma' & \text{if } \eta > \bar{\eta} \end{cases}
\]

where \( \eta \) was defined in (20). This gives the following asymptotic behavior:

\[
-\lim_{J \to \infty} \frac{\ln p(y + (\eta \hat{P})^\frac{1}{2})}{J} = \begin{cases} \eta E_{\text{snr}} \
\frac{1}{1-\alpha^{-1}} / \log_2 e \end{cases}
\]

Finally, using (33), (25), (27), (36) and the \( \theta \) function defined in (19) we get:

\[
\lim_{J \to \infty} \frac{1}{\log_2 e} \left(\frac{R_{\text{snr}} 2^J}{J}\right) = \eta E_{\text{snr}}[1-\theta(\eta - \bar{\eta})] + S \left(1 - \frac{1}{\alpha}\right) \theta(\eta - \bar{\eta}) + \frac{1}{2} [(\eta - 1) - \ln \eta]
\]

This proofs the statement of the theorem. \( \square \)

With Theorem 3 we can prove Theorem 2 and conclude the proof of Theorem 1.
We have discussed that the error probability of the SBS-MAP detector is specified by $\eta^* \tilde{P} = \eta^* 2E_{in}^J$, the effective channel strength in the decoupled model [15], where $\eta^*$ is the global minimizer of $\eta^* S(\eta)$ in the interval $[0, 1]$. In a similar fashion the error probability of the AMP decoder [14] at convergence is described by $\eta_{oc} \tilde{P}$, where $\eta_{oc}$ is the smallest local minimizer of $\eta^* S(\eta)$. By Theorem 3 the derivative of $\eta^* S(\eta)$ in (18) is given by
\[
\frac{\partial \eta^* S}{\partial \eta}(\eta) = S \frac{E_{in}[1 - \theta(\eta - \eta)]}{2} (1 - \frac{1}{\eta})
\]
for $\eta \neq \bar{\eta}$. The critical points of the derivative are
\[
\eta^*_0 = (1 + 2S E_{in})^{-1}
\]
and
\[
\eta^*_1 = 1.
\]
Note that the first point $\eta^*_0$ is critical if and only if $\eta^*_0 < \bar{\eta}$, which, after rearranging, gives precisely condition (11). Also note, that the second derivative of $\eta^* S$ is $(4 \eta)^{-2}$, so it is non-negative for all $\eta > 0$. Therefore the critical points are indeed minima. A local maximum may appear only at $\eta = \bar{\eta}$ where $\eta^* S$ is not differentiable. The values of $\eta^* S$ at the minimal points are
\[
\eta^* S(\eta^*_0) = \frac{S E_{in}}{1 + 2SE_{in}} + \frac{1}{2} \left[ \frac{-2SE_{in}}{1 + 2SE_{in}} + \ln(1 + 2SE_{in}) \right]
\]
\[
= \frac{S}{2 \log_2 e} \left( 1 - \frac{1}{\alpha} \right)
\]
if $\eta^*_0 < \bar{\eta}$, and
\[
\eta^* S(\eta^*_1) = \frac{S}{\log_2 e} \left( 1 - \frac{1}{\alpha} \right)
\]
It is apparent that $\eta^* S(\bar{\eta})$ is the global minimum if and only if condition (10) is fulfilled. We implicitly used here that $\bar{\eta} \leq 1$, that is because condition (10) implies $\bar{\eta} < 1$, which can be seen by solving inequality (10) for $E_{in}^J$. If $\eta^*_1 = 1$ is indeed the global minimizer of (18), the effective power in the decoupled channel (15) is given by $\tilde{P}$. Since $\tilde{P}$ grows proportional to $J$, the effective power in the channel and therefore also the probability of misestimating the support go to zero with $J \rightarrow \infty$. This concludes the proof of Theorem 3.

V. CONCLUSION

We have shown that the concatenated coding construction in [5] is asymptotically optimal as the blocklength $n$, the number of active users $K_n$, the number $L$ and the size $J$ of the subslots go to infinity. This makes the SPARC based coding construction the first of the known U-RA codes to have an asymptotic optimality guarantee. Our result also shows more generally that the achievable trade-off between sumrate and $E_{in}/N_0$ in U-RA converges to the Shannon bound in the considered limit.

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