Lorentz Covariant Distributions with the Spectral Conditions

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Abstract. The Lorentz covariant tempered distributions with the supports in the product of the closed upper light cones are described.

1 Introduction

Gårding and Wightman [1] formulated the physical views of the quantum fields as the axioms system. The vacuum expectation values of the quantum fields products would be the Fourier transforms of the Lorentz covariant tempered distributions with the supports in the product of the closed upper light cones [2]. The Lorentz invariant tempered distributions of one variable are studied in the papers [3], [4]. The description of the Lorentz invariant tempered distributions with the supports for one argument only lying in the hyperboloid \( \{ x \in \mathbb{R}^4 : x^0 > 0, (x, x) \geq \mu > 0 \} \) is obtained in the paper [5]. The authors of the papers [3] - [5] wanted to describe the Lorentz invariant distributions in terms of the distributions given on the Lorentz group orbit space. This orbit space has a complicated structure. It is noted [6] that a tempered distribution with a support in the closed upper light cone may be represented as the action of the wave operator in some power on a differentiable function with a support in the closed upper light cone. For the description of the Lorentz covariant differentiable functions the boundary of the closed upper light cone is not important. The measure of this boundary is zero.

In this paper we obtain the description of the Lorentz covariant tempered distributions with the supports in the product of the closed upper light cones. Thus we obtain the description of the vacuum expectation values of the products of the quantum fields satisfying the properties of Lorentz covariance and spectral condition [2].

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2 Lorentz covariance and spectral condition

For a complex $2 \times 2$ - matrix

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}. \quad (2.1)$$

we define the following $2 \times 2$ - matrices

$$A^T = \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix}, \quad A^* = (\tilde{A})^T. \quad (2.2)$$

The matrix (2.1) is said to be Hermitian if $A^* = A$. Let us choose the basis of the Hermitian $2 \times 2$ - matrices

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.3)$$

We relate with a vector $x \in \mathbb{R}^4$ a Hermitian matrix

$$\tilde{x} = \sum_{\mu=0}^{3} x^\mu \sigma^\mu \quad (2.4)$$

We introduce the bilinear form

$$(x, y) = x^0 y^0 - \sum_{k=1}^{3} x^k y^k. \quad (2.5)$$

$(x, x)$ is the Minkowski metric.

The group $SL(2, \mathbb{C})$ consists of the complex $2 \times 2$ - matrices (2.1) with the determinant that equals one. The group $SU(2)$ is the maximal compact subgroup of the group $SL(2, \mathbb{C})$. It consists of the matrices from the group $SL(2, \mathbb{C})$ satisfying the equation $AA^* = \sigma^0$. Let us describe the irreducible representations of the group $SU(2)$. We consider the half - integers $l \in 1/2\mathbb{Z}_+$, i.e. $l = 0, 1/2, 1, 3/2, \ldots$. We define the representation of the group $SU(2)$ on the space of the polynomials with degree less than $2l$

$$T_l(A) \phi(z) = (A_{12} z + A_{22})^{2l} \phi(\frac{A_{11} z + A_{21}}{A_{12} z + A_{22}}). \quad (2.6)$$

We consider a half - integer $n = -l, -l + 1, \ldots, l - 1, l$. We choose the polynomial basis

$$\psi_n(z) = ((l - n)! (l + n)!)^{-1/2} z^{l-n} \quad (2.7)$$

The definitions (2.6), (2.7) imply

$$T_l(A) \psi_n(z) = \sum_{m=-l}^{l} \psi_m(z) t^l_{mn}(A)$$

where the polynomial

$$t^l_{mn}(A) = ((l - m)! (l + m)! (l - n)! (l + n)!)^{1/2} \times$$

$$\sum_{j=-\infty}^{\infty} \frac{A_{11}^{l-m-j} A_{12}^{m-n+j} A_{21}^{l+n-j} A_{22}^{l}}{\Gamma(j+1) \Gamma(l-m-j+1) \Gamma(m-n+j+1) \Gamma(l+n-j+1)} \quad (2.8)$$
where $\Gamma(z)$ is the gamma-function. The function $(\Gamma(z))^{-1}$ equals zero for $z = 0, -1, -2, \ldots$. Therefore the series (2.8) is a polynomial.

The relation (2.6) defines a representation of the group $SU(2)$. Thus the polynomial (2.8) defines a representation of the group $SU(2)$

$$t^l_{mn}(AB) = \sum_{k=-l}^{l} t^l_{mk}(A)t^l_{kn}(B).$$  \hspace{1cm} (2.9)

This $(2l+1)$-dimensional representation is irreducible ([7], Chapter III, Section 2.3).

The relations (2.8), (2.9) have an analytic continuation to the matrices from the group $SL(2,\mathbb{C})$. By making the change $j \rightarrow j + n - m$ of the summation variable in the equality (2.8) we have

$$t^l_{mn}(A) = t^l_{nm}(A^T).$$  \hspace{1cm} (2.10)

Due to ([7], Chapter III, Section 8.3) we have

$$t^l_{m_1n_1}(A)t^l_{m_2n_2}(A) = \sum_{l_3 \in 1/2\mathbb{Z}^+} \sum_{m_3,n_3=-l_3} C(l_1, l_2, l_3; m_1, m_2, m_3)C(l_1, l_2, l_3; n_1, n_2, n_3)t^l_{m_3n_3}(A)$$  \hspace{1cm} (2.11)

for a matrix $A \in SU(2)$. The Clebsch-Gordan coefficient $C(l_1, l_2, l_3; m_1, m_2, m_3)$ is not zero only if $m_3 = m_1 + m_2$ and the half-integers $l_1, l_2, l_3 \in 1/2\mathbb{Z}^+$ satisfy the triangle condition: it is possible to construct a triangle with the sides of length $l_1, l_2, l_3$ and an integer perimeter $l_1 + l_2 + l_3$. It means that the half-integer $l_3$ is one of the half-integers $|l_1-l_2|, |l_1-l_2| + 1, \ldots, l_1+l_2-1, l_1+l_2$. Let the half-integers $l_1, l_2, l_3 \in 1/2\mathbb{Z}^+$ satisfy the triangle condition. Let the half-integers $m_i = -l_i, -l_i+1, \ldots, l_i-1, l_i, i = 1, 2, m_3 = m_1 + m_2$. Then due to ([7], Chapter III, Section 8.3)

$$C(l_1, l_2, l_3; m_1, m_2, m_3) = (-1)^{l_1-l_3+m_2}(2l_3+1)^{1/2}\left[\frac{(l_1+l_2-l_3)!(l_1+l_3-l_2)!(l_2+l_3-l_1)!(l_3-m_3)!(l_3+m_3)!}{(l_1+l_2+l_3+1)!(l_1-m_1)!(l_1+m_1)!(l_2-m_2)!(l_2+m_2)!}\right]^{1/2}\times \sum_{j=0}^{l_3+l_1-l_2} \frac{(-1)^j(l_1+m_1+j)!}{j!\Gamma(l_3-m_3-j+1)\Gamma(l_1-l_2+m_3+j+1)(l_2+l_3-l_1-j)!}.$$  \hspace{1cm} (2.12)

Let $dA$ be the normalized Haar measure on the group $SU(2)$. Due to ([7], Chapter III, Section 8.3)

$$C(l_1, l_2, l_3; m_1, m_2, m_3)C(l_1, l_2, l_3; n_1, n_2, n_3) = (2l_3+1)\int_{SU(2)} dA t^l_{m_1n_1}(A)t^l_{m_2n_2}(A)t^l_{m_3n_3}(A).$$  \hspace{1cm} (2.13)

The coefficients of the polynomial (2.8) are real. By using the relations (2.10) and $A^* = A^{-1}$ we can rewrite the equality (2.13) as

$$C(l_1, l_2, l_3; m_1, m_2, m_3)C(l_1, l_2, l_3; n_1, n_2, n_3) = (2l_3+1)\int_{SU(2)} dA t^l_{m_1n_1}(A)t^l_{m_2n_2}(A)t^l_{m_3n_3}(A^{-1}).$$  \hspace{1cm} (2.14)
If the half-integers $l_1, l_2, l_3 \in 1/2\mathbb{Z}_+$ satisfy the triangle condition, then due to ([7], Chapter III, Section 8.3) we have

$$C(l_1, l_2, l_3; l_1, -l_2, l_1 - l_2) = \left[ \frac{(2l_3 + 1)(2l_1)!}{(l_1 + l_2 - l_3)!}\right]^{1/2}. \tag{2.15}$$

Let us choose the half-integers $n_1 = l_1, n_2 = -l_2$ in the equality (2.17). Then the relation (2.14) and the invariance of the Haar measure $dA$ imply

$$\sum_{n_1 = -l_1}^{l_1} \sum_{n_2 = -l_2}^{l_2} t_{l_1 n_1}^{l_1} (A) t_{m_2 n_2}^{l_2} (A) C(l_1, l_2, l_3; n_1, n_2, m_3) = \sum_{n_3 = -l_3}^{l_3} C(l_1, l_2, l_3; m_1, m_2, n_3) t_{m_3 n_3}^{l_3} (A). \tag{2.16}$$

The relation (2.16) has an analytic continuation to any matrix from the group $SL(2, \mathbb{C})$.

For any natural numbers $m, n$ and the half-integers $l_1, ..., l_{n+1}, \tilde{l}_1, ..., \tilde{l}_{n+1} \in 1/2\mathbb{Z}_+$; $m_i = -l_i, -l_i + 1, ..., l_i - 1, l_i, \tilde{m}_i = -l_i, -l_i + 1, ..., \tilde{l}_i - 1, \tilde{l}_i, i = 1, ..., n + 1$, we consider the set of the tempered distributions

$$F_{m_1, ..., m_{n+1}; m_1, ..., m_{n+1}}(x_1, ..., x_m) \in S'(\mathbb{R}^{4m}).$$

This set is called a Lorentz covariant distribution if for any matrix $A \in SL(2, \mathbb{C})$

$$F_{m_1, ..., m_{n+1}; m_1, ..., m_{n+1}}(A \tilde{x}, A \tilde{x}) = \prod_{i=1}^{n+1} t_{m_i k_i}^{l_i} (A) t_{\tilde{m}_i \tilde{k}_i}^{\tilde{l}_i} (A) F_{k_1, ..., k_{n+1}; k_1, ..., k_{n+1}}(\tilde{x}, ..., \tilde{x}) \tag{2.17}$$

where $2 \times 2$-matrix $\tilde{x}$ is given by the relation (2.4). The half-integers $l_1, ..., l_{n+1}, \tilde{l}_1, ..., \tilde{l}_{n+1}$ in the relation (2.17) are not arbitrary. Let us choose the matrix $A = -\sigma^0$ in the equality (2.17). For this matrix

$$A \tilde{x} = \tilde{x}, \quad j = 1, ..., m.$$ 

The definition (2.8) implies

$$t_{m_j k_j}^{l_j} (-\sigma^0) = (-1)^{2l_j} \delta_{m_j k_j}.$$ 

Hence the equality (2.17) is valid for the matrix $A = -\sigma^0$ if

$$(-1)^{2(l_1 + \cdots + l_{n+1} + \tilde{l}_1 + \cdots + \tilde{l}_{n+1})} = 1$$

i.e. the sum $l_1 + \cdots + l_{n+1} + \tilde{l}_1 + \cdots + \tilde{l}_{n+1}$ is an integer. This condition is supposed below.

Due to the paper [6] we obtain the representation for a tempered distribution with a support in the closed upper light cone

$$\nabla_+ = \{ x \in \mathbb{R}^4 : x^0 \geq 0, (x, x) \geq 0 \}.$$

**Lemma.** Let a tempered distribution $F(x) \in S'(\mathbb{R}^4)$ have a support in the closed upper light cone. There is a natural number $q$ such that

$$F(x) = (\partial_x, \partial_x)^q f(x), \tag{2.18}$$

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\[(\partial_x, \partial_x) = \left( \frac{\partial}{\partial x^0} \right)^2 - \frac{3}{2} \left( \frac{\partial}{\partial x^k} \right)^2 \]  

(2.19)

where a differentiable function \(f(x)\) with a support in the closed upper light cone is polynomial bounded.

**Proof.** Let us introduce the step function

\[\theta(x) = \begin{cases} 
1, & x \geq 0, \\
0, & x < 0. 
\end{cases} \]  

(2.20)

Due to ([8], Section 30, formulas (126), (146))

\[(8\pi)^{-1}(\partial_x, \partial_x)^2(\theta(x^0)\theta((x,x))) = \delta(x).\]  

(2.21)

The relation (2.21) implies for any natural number \(q \geq 2\)

\[(2\pi 4^{q-1}(q-2)!(q-1)!)^{-1}(\partial_x, \partial_x)^q((x,x)^{q-2}\theta(x^0)\theta((x,x))) = \delta(x).\]  

(2.22)

For any vector \(x \in \overline{V}_+\) the intersection of the support of the function

\[(x - y, x - y)^{q-2}\theta(x^0 - y^0)\theta((x - y, x - y))\]

with respect of the variable \(y\) with the cone \(\overline{V}_+\) is compact. This function is \(2(q-3)\) times differentiable. Let a tempered distribution \(F(x) \in S'(\mathbb{R}^4)\) have a support in the cone \(\overline{V}_+\). Then for a sufficiently large natural number \(q\) the function

\[f(x) = (2\pi 4^{q-1}(q-2)!(q-1)!)^{-1}\int d^4 y F(y)(x-y, x-y)^{q-2}\theta(x^0 - y^0)\theta((x-y, x-y)) \]  

(2.23)

is differentiable and has a support in the cone \(\overline{V}_+\). The function (2.23) is polynomial bounded. The lemma is proved.

Let us consider the relation (2.17) for the simplest case \(m = n = 1\).

**Proposition.** Any tempered distribution \(F_{m_1, m_2; n_1, n_2}(x) \in S'\) with a support in the closed upper light cone satisfying the covariance relation (2.17) for \(m = n = 1\) has the following form

\[\int d^4 x f_{m_1, m_2; n_1, n_2}(x) \phi(x) = \sum_{l_1, l_2, l_3; m_1, m_2} C(l_1, l_2, l_3; m_1, m_2, m_3) C(l_1, l_2, l_3; m_1, m_2, m_3) \times \]

\[\int d^4 x \theta(x^0)\theta((x,x)) f_{l_1, l_2, l_3; m_1, m_2, m_3}(x) \phi(x) \]  

(2.24)

where a test function \(\phi(x) \in S(\mathbb{R}^4)\); \(q\) is a natural number; the coefficient \(C(l_1, l_2, l_3; m_1, m_2, m_3)\) is given by the relation (2.12); \(2 \times 2\) matrix \(\tilde{x}\) is given by the relation (2.4); the polynomial \(t_{mn}^l(A)\) is given by the relation (2.8); the differentiable function \(f_{l_1, l_2, l_3; m_1, m_2, m_3}(x)\) with a support in the positive semi-axis is polynomial bounded.

**Proof.** The coefficient \(C(l_1, l_2, l_3; m_1, m_2, m_3)\) is not zero when the half-integers \(l_1, l_2, l_3 \in 1/2\mathbb{Z}_+\) satisfy the triangle condition. Hence the sums in the right-hand side of the equality (2.24) are finite.
The function \( f^{l_1,l_2,l_3,l_4}(s) \) is polynomial bounded. Hence the integrals in the right - hand side of the equality (2.24) are absolutely convergent. The right - hand side of the equality defines the tempered distribution from \( S'(\mathbb{R}^4) \).

The relations (2.9), (2.16) imply that the right - hand side of the equality (2.24) satisfies the covariance relation (2.17) for \( m = n = 1 \).

Let us prove that any tempered distribution from \( S'(\mathbb{R}^4) \) with a support in the cone \( V + \) satisfying the covariance relation (2.17) for \( m = n = 1 \) has the form (2.24).

If a support of a tempered distribution \( F^{l_1,l_2,l_3,l_4}_{m_1,m_2;m_1,m_2}(x) \in S'(\mathbb{R}^4) \) is in the cone \( V + \), then the representation (2.18) is valid where the differentiable function \( f^{l_1,l_2,l_3,l_4}_{m_1,m_2;m_1,m_2}(x) \) is given by the equality (2.23).

For the open upper light cone

\[
V_+ = \{ x \in \mathbb{R}^4 : x^0 > 0, (x, x) > 0 \}
\]

we introduce the coordinates

\[
\tilde{x} = \mu g(t, z) g(t, z)^* \tag{2.25}
\]

where \( \mu \) is a positive number and \( 2 \times 2 \) - matrix

\[
g(t, z) = \begin{pmatrix} t^{-1} & 0 \\ z & t \end{pmatrix}. \tag{2.26}
\]

\( t \) is a positive number and \( z \) is a complex number. The equality (2.25) is a consequence of Gauss decomposition for a Hermitian positive definite \( 2 \times 2 \) - matrix.

Let us introduce the function

\[
f^{l_1,l_2,l_3,l_4}_{m_1,m_2;m_1,m_2}(\mu, g(t, z)) = \sum_{k_1=-l_1}^{l_1} \sum_{k_2=-l_2}^{l_2} \sum_{k'_1=-l'_1}^{l'_1} \sum_{k'_2=-l'_2}^{l'_2} \left( \prod_{i=1}^{2} t^{l_i}_{m_i,k_i} (g(t, z)^{-1}) t^{l_i}_{m_i,k_i} (g(t, z)^{-1}) \right) f^{l_1,l_2,l_3,l_4}_{k_1,k_2;k'_1,k'_2}(\mu g(t, z) g(t, z)^*). \tag{2.27}
\]

The function \( f^{l_1,l_2,l_3,l_4}_{m_1,m_2;m_1,m_2}(\tilde{x}) \) defined by the relation (2.23) satisfies the covariance relation (2.17) for \( m = n = 1 \). Hence for any matrix \( A \) of type (2.23) we have

\[
f^{l_1,l_2,l_3,l_4}_{m_1,m_2;m_1,m_2}(\mu, A g(t, z)) = f^{l_1,l_2,l_3,l_4}_{m_1,m_2;m_1,m_2}(\mu, g(t, z)). \tag{2.28}
\]

Let us choose the basis of the Lie algebra of the group of the matrices (2.26)

\[
a_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix}, \quad a_3 = 1/2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

The equality (2.28) implies three relations

\[
\frac{\partial}{\partial s} f^{l_1,l_2,l_3,l_4}_{m_1,m_2;m_1,m_2}(\mu, \exp\{sa_1\} g(t, z))|_{s=0} = t^{-1} \frac{\partial}{\partial \text{Re} z} f^{l_1,l_2,l_3,l_4}_{m_1,m_2;m_1,m_2}(\mu, g(t, z)) = 0,
\]

\[
\frac{\partial}{\partial s} f^{l_1,l_2,l_3,l_4}_{m_1,m_2;m_1,m_2}(\mu, \exp\{sa_2\} g(t, z))|_{s=0} = t^{-1} \frac{\partial}{\partial \text{Im} z} f^{l_1,l_2,l_3,l_4}_{m_1,m_2;m_1,m_2}(\mu, g(t, z)) = 0,
\]

\[
\frac{\partial}{\partial s} f^{l_1,l_2,l_3,l_4}_{m_1,m_2;m_1,m_2}(\mu, \exp\{sa_3\} g(t, z))|_{s=0} = -1/2 (\text{Re} z \frac{\partial}{\partial \text{Re} z} + \text{Im} z \frac{\partial}{\partial \text{Im} z} + t \frac{\partial}{\partial t}) f^{l_1,l_2,l_3,l_4}_{m_1,m_2;m_1,m_2}(\mu, g(t, z)) = 0. \tag{2.29}
\]
Due to the relations (2.29) the function (2.27) is independent of the variable $g(t, z)$. Hence the definition (2.27) implies
\[
 f_{l_1, l_2, l_3, l_4}^{\mu_1, g(t, z)} = f_{l_1, l_2, l_3, l_4}^{\mu, \sigma^0} = f_{l_1, l_2, l_3, l_4}^{\mu, \sigma^0}. \tag{2.30}
\]

The relations (2.9), (2.27) and (2.30) imply
\[
f_{l_1, l_2, l_3, l_4}^{\mu g(t, z) g(t, z)^*} = \sum_{k_1 = -l_1}^{l_1} \sum_{k_2 = -l_2}^{l_2} \sum_{k_3 = -l_3}^{l_3} \sum_{k_4 = -l_4}^{l_4} \left( \prod_{i=1}^{2} t_{l_i, k_i}^1 (g(t, z)) \overline{t_{l_i, k_i}^1 (g(t, z))} \right) f_{l_1, l_2, l_3, l_4}^{\mu, \sigma^0}. \tag{2.31}
\]

We choose the natural number $q$ in the relation (2.23) such that the function $f_{l_1, l_2, l_3, l_4}^{\mu_1, g(t, z)} (\tilde{x})$ is $l_1 + l_2 + l_3 + l_4$ times differentiable. (The covariance relation (2.17) supposes that the number $l_1 + l_2 + l_3 + l_4$ is nonnegative integer.) The support of this function lies in the cone $\mathcal{V}_+$. Therefore its first $l_1 + l_2 + l_3 + l_4$ derivatives vanish on the boundary of the cone $\mathcal{V}_+$. Hence the function $f_{l_1, l_2, l_3, l_4}^{\mu_1, g(t, z)} (\mu \sigma^0)$ is $l_1 + l_2 + l_3 + l_4 + 1$ times differentiable and its first $l_1 + l_2 + l_3 + l_4 + 1$ derivatives vanish at the point $\mu = 0$. Due to the relation (2.23) the function $f_{l_1, l_2, l_3, l_4}^{\mu_1, g(t, z)} (\mu \sigma^0)$ is polynomial bounded.

By making use of the coordinate substitution (2.25) and the relation (2.31) we have
\[
 \int d^4 x f_{l_1, l_2, l_3, l_4}^{\mu_1, g(t, z)} (\tilde{x}) \phi(\tilde{x}) = \sum_{k_1 = -l_1}^{l_1} \sum_{k_2 = -l_2}^{l_2} \sum_{k_3 = -l_3}^{l_3} \sum_{k_4 = -l_4}^{l_4} \int_0^{\infty} \mu^3 d\mu \int_0^{\infty} t^{-3} dt \int d\text{Re}z \int d\text{Im}z \left( \prod_{i=1}^{2} t_{l_i, k_i}^1 (g(t, z)) \overline{t_{l_i, k_i}^1 (g(t, z))} \right) f_{l_1, l_2, l_3, l_4}^{\mu_1, g(t, z)} (\mu \sigma^0) \phi(\mu g(t, z) g(t, z)^*). \tag{2.32}
\]

The function $f_{l_1, l_2, l_3, l_4}^{\mu_1, g(t, z)} (\tilde{x})$ defined by the relation (2.23) satisfies the covariance relation (2.17) for $m = n = 1$. We substitute in the equality (2.32) the sequence of the functions $\phi(\tilde{x}) \in S(\mathbb{R}^4)$ converging to the distribution
\[
 2(\mu, \sigma) \delta(\psi((x, x)^{1/2}))
\]
where a vector $\mathbf{x} = (x^1, x^2, x^3)$ and a function $\psi(s) \in S(\mathbb{R})$. Now the covariance relation (2.17) for a matrix $A \in SU(2)$ implies
\[
 f_{l_1, l_2, l_3, l_4}^{\mu_1, g(t, z)} (\mu \sigma^0) = \sum_{k_1 = -l_1}^{l_1} \sum_{k_2 = -l_2}^{l_2} \sum_{k_3 = -l_3}^{l_3} \sum_{k_4 = -l_4}^{l_4} \left( \prod_{i=1}^{2} t_{l_i, k_i}^1 (A) \overline{t_{l_i, k_i}^1 (A)} \right) f_{l_1, l_2, l_3, l_4}^{\mu_1, g(t, z)} (\mu \sigma^0). \tag{2.33}
\]

We integrate the relation (2.33) with the normalized Haar measure $dA$ on the group $SU(2)$. The relations (2.11), (2.13) imply
\[
 f_{l_1, l_2, l_3, l_4}^{\mu_1, g(t, z)} (\mu \sigma^0) = \sum_{l_3 \in \mathbb{Z}_+} \sum_{m_3, k_3 = -l_3}^{l_3} \sum_{k_2 = -l_2}^{l_2} \sum_{k_1 = -l_1}^{l_1} \sum_{k_2 = -l_2}^{l_2} C(l_1, l_2, l_3; m_1, m_2, m_3) \times
 C(l_1, l_2, l_3; m_1, m_2, m_3)(2l_3 + 1)^{-1} C(l_1, l_2, l_3; k_1, k_2, k_3) C(l_1, l_2, l_3; k_1, k_2, k_3) f_{l_1, l_2, l_3, l_4}^{\mu_1, g(t, z)} (\mu \sigma^0). \tag{2.34}
\]
Let us introduce the function

\[ f^{l_1,l_2,l_3;l_1,l_2}(\mu) = \sum_{k_1=-l_1}^{l_1} \sum_{k_2=-l_2}^{l_2} \sum_{k_3=-l_3}^{l_3} (2l_3 + 1)^{-\frac{1}{2}} \times \]

\[ C(l_1,l_2,l_3;k_1,k_2,k_3)C(\hat{l}_1,\hat{l}_2,\hat{l}_3;k_1,k_2,k_3) \mu^{-2l_3} f^{l_1,l_2;l_1,l_2}(\mu \sigma^0). \]  

(2.35)

The function \( f^{l_1,l_2,l_3;l_1,l_2}(\mu^0) \) is \( l_1 + l_2 + \hat{l}_1 + \hat{l}_2 + 1 \) times differentiable and its first \( l_1 + l_2 + \hat{l}_1 + \hat{l}_2 \) vanish at the point \( \mu = 0 \). Hence the function (2.35) is differentiable. The function \( f^{l_1,l_2,l_3;l_1,l_2}(\mu \sigma^0) \) is polynomial bounded. Hence the function (2.35) is polynomial bounded.

By making use of the relations (2.34), (2.35) and the relations (2.10), (2.16) for a matrix (2.26) we can rewrite the equality (2.32) as

\[ \int d^4x f^{l_1,l_2,l_3;l_1,l_2}(\phi)(\hat{x}) = \sum_{l_3 \in \mathbb{Z}_{++}} \int_0^\infty \mu^3 d\mu \int_0^\infty t^{-3} dt \int dRez \int dImz \]

\[ C(l_1,l_2,l_3;m_1,m_2,m_3)C(\hat{l}_1,\hat{l}_2,\hat{l}_3;\hat{m}_1,\hat{m}_2,\hat{m}_3) \times \]

\[ t_{l_3}^{l_1,l_2,l_3}(\mu g(t,z)g(t,z)^*) 2 f^{l_1,l_2;l_1,l_2}(\mu)(\mu g(t,z)g(t,z)^*) \]  

(2.36)

By making use of the coordinate substitution inverse of the substitution (2.25) and the equality (2.18) we obtain the equality (2.24). The proposition is proved.

Let us consider a tempered distribution with a support in the cone \( \nabla_+ \) satisfying the covariance relation (2.17) for \( m = 1, n = 0 \). This case corresponds to the equality (2.24) with \( l_2 = \hat{l}_2 = 0, m_2 = \hat{m}_2 = 0 \). The relation (2.12) implies

\[ C(l_1,0,l_3;m_1,0,m_3) = \delta_{l_1,l_3} \delta_{m_1,m_3}. \]  

(2.37)

Let us consider the general case.

**Theorem.** Any tempered distribution

\[ F^{l_1,...,l_{n+1}}_{m_1,...,m_{n+1}}(x_1,...,x_{n+1}) \in S'(\mathbb{R}^{4m}) \]

with a support in the product \( \nabla_+^{4m} \) of the cones satisfying the covariance relation (2.17) has the following form

\[ \int d^4m x F^{l_1,...,l_{n+1}}_{m_1,...,m_{n+1}}(x_1,...,x_m) \phi(x_1,...,x_m) = \]

\[ \sum_{k_1=-l_1}^{l_1} \sum_{k_{n+1}=-l_{n+1}}^{l_{n+1}} \int_0^\infty \mu^m d\mu \int_0^\infty t^{-3} dt \int dRez \int dImz \int_{SU(2)} dA \]

\[ \int d^4m \rho(\sum_{j=1}^{n+1} \tilde{\mu}_j - \sigma^0) (\prod_{i=1}^{n+1} t_{m_i,k_i}(g(t,z)A) t_{l_{m_i,k_i}}(g(t,z)A)) \times \]

\[ 2 f^{l_1,...,l_{n+1}}_{k_1,...,k_{n+1};l_1,...,l_{n+1}}(\mu p_1,...,\mu p_m) (\prod_{i=1}^{m} \partial x_i)(\partial x_i)^y) \phi(x_1,...,x_m) \]  

(2.38)

where a test function \( \phi(x_1,...,x_m) \in S(\mathbb{R}^{4m}) \); \( dA \) is the normalized Haar measure on the group \( SU(2) \); \( 2 \times 2 \) - matrix \( g(t,z) \) has the form (2.26); the polynomial \( t^{l_1}_{m_1}(A) \) is defined by the
it is sufficient to prove the invariance of the measure

\[ \text{group } Z \]

the matrix (2.39) by any matrix from the group

\[ SU \]

the group

\[ \text{SL} \]

verify that the measure

\[ d \]

is invariant under the left shifts on the group

\[ g \]

second matrix

\[ u \]

the invariance of the measure

\[ \text{du} \]

The Haar measure

\[ \text{on the group } SL(2, C) \]

the group

\[ Z \_ D(2) \]

any matrix from the group

\[ SU \]

by any matrix from the group

\[ SU \] of the matrix (2.26) is called the group

\[ Z \_ D(2) \]. For any matrix from the group

\[ SL(2, C) \] the Gram decomposition is valid

\[
\begin{pmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{pmatrix} = \begin{pmatrix} t^{-1} & 0 \\
z & t
\end{pmatrix} \begin{pmatrix} \alpha & \beta \\
-\overline{\beta} & \overline{\alpha}
\end{pmatrix},
\]

\[
t = (|A_{11}|^2 + |A_{12}|^2)^{-1/2},
\]

\[
z = (\overline{A_{11}}A_{21} + \overline{A_{12}}A_{22})(|A_{11}|^2 + |A_{12}|^2)^{-1/2},
\]

\[
\alpha = A_{11}(|A_{11}|^2 + |A_{12}|^2)^{-1/2},
\]

\[
\beta = A_{12}(|A_{11}|^2 + |A_{12}|^2)^{-1/2}.
\]

(2.39)

The first matrix in the right-hand side of the first equality (2.39) \( g(t, z) \in Z \_ D(2) \) and the second matrix \( u \in SU(2) \). Let \( du \) be the normalized Haar measure on the group \( SU(2) \). We consider the measure \( d(g(t, z)u) = t^{-3}dt\text{Re}zd\text{Im}zdu \) on the group \( SL(2, C) \). If this measure is invariant under the left shifts on the group \( SL(2, C) \), then the right-hand side of the equality (2.38) satisfies the covariance relation (2.17) due to the relation (2.9). It is easy to verify that the measure \( dg(t, z) = t^{-3}dt\text{Re}zd\text{Im}z \) is invariant under the left shifts on the group \( Z \_ D(2) \). Hence the measure \( d(g(t, z)u) \) is invariant under the left multiplication of the matrix (2.39) by any matrix from the group \( Z \_ D(2) \). Due to the Gram decomposition it is sufficient to prove the invariance of the measure \( d(g(t, z)u) \) under the left multiplication of the matrix (2.39) by any matrix from the group \( SU(2) \). Let the complex numbers \( \alpha, \beta \) satisfy the relation \( |\alpha|^2 + |\beta|^2 = 1 \). The Gram decomposition (2.39) implies

\[
\begin{pmatrix}
\alpha & \beta \\
-\overline{\beta} & \overline{\alpha}
\end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\
z & t
\end{pmatrix} = \begin{pmatrix} t_1^{-1} & 0 \\
z_1 & t_1
\end{pmatrix} \begin{pmatrix} \alpha_1 & \beta_1 \\
-\overline{\beta_1} & \overline{\alpha_1}
\end{pmatrix},
\]

\[
t_1 = (|\alpha t^{-1} + \beta z|^2 + |\beta t^2|^2)^{-1/2},
\]

\[
z_1 = ((\overline{\alpha t^{-1} + \beta z})(-\overline{\beta t^{-1}} + \alpha z) + \overline{\alpha t^2})(|\alpha t^{-1} + \beta z|^2 + |\beta|^2 t^2)^{-1/2},
\]

\[
\alpha_1 = (\alpha t^{-1} + \beta z)(|\alpha t^{-1} + \beta z|^2 + |\beta t^2|^2)^{-1/2},
\]

\[
\beta_1 = \beta t(|\alpha t^{-1} + \beta z|^2 + |\beta|^2 t^2)^{-1/2}.
\]

(2.40)

The Haar measure \( du \) is invariant under the shifts on the group \( SU(2) \). In order to prove the invariance of the measure \( d(g(t, z)u) \) under the left multiplication of the matrix (2.39) by any matrix from the group \( SU(2) \) it is sufficient to prove the equality

\[
t_1^{-3}dt_1d\text{Re}z_1d\text{Im}z_1 = t^{-3}dt\text{Re}zd\text{Im}z
\]

(2.41)

where the numbers \( t_1, z_1 \) are defined by the second and the third relations (2.40). We introduce the coordinates

\[
(|x|^2 + 1)^{1/2} \sigma^0 + \sum_{k=1}^{3} x^k \sigma^k = g(t, z)g(t, z)^*,
\]

\[
|x|^2 = \sum_{k=1}^{3} (x^k)^2.
\]

(2.42)
It is easy to calculate
\[
\frac{1}{2}(|x|^2 + 1)^{-1/2} dx^1 dx^2 dx^3 = t^{-3} dt dRez dImz. \tag{2.43}
\]

The mapping \( g(t, z) \rightarrow g(t_1, z_1) \) corresponds with the rotation
\[
\begin{pmatrix}
\alpha & \beta \\
-\beta & \alpha
\end{pmatrix}
\left((|x|^2 + 1)^{1/2} \sigma^0 + \sum_{k=1}^{3} x^k \sigma^k \right)
\begin{pmatrix}
\alpha & \beta \\
-\beta & \alpha
\end{pmatrix}^*\]

of the vector \( x \in \mathbb{R}^3 \) given by the relation (2.42). The invariance of the measure (2.43) under the rotations implies the equality (2.41). Hence the measure \( d(g(t, z)u) = t^{-3} dtdRez dImz du \) is invariant under the left shifts on the group \( SL(2, \mathbb{C}) \). Therefore the distribution (2.38) satisfies the covariance relation (2.17).

Let us prove the absolute convergence of the integral (2.38). The coefficients of the polynomial (2.8) are real. The relations (2.9), (2.10) imply
\[
\sum_{k=-l}^{l} |t_{mk}^{l}(A)|^2 = t_{mm}^{l}(AA^*). \tag{2.44}
\]
The Cauchy inequality and the relation (2.44) imply
\[
| \sum_{k_1=-l_1}^{l_1} \cdots \sum_{k_{n+1}=-l_{n+1}}^{l_{n+1}} \sum_{k_{n+1}=-l_{n+1}}^{l_{n+1}} \left( \prod_{i=1}^{n+1} t_{i_{m_{k}} k_{i}}^{l} (g(t, z)u) t_{i_{m_{k}} k_{i}}^{l} (g(t, z)u) \right) \times
f_{k_1, \ldots, k_{n+1}; k_1, \ldots, k_{n+1}}^{l_1, \ldots, l_{n+1}} (\mu p_1, \ldots, \mu p_m) | \leq
| \sum_{k_1=-l_1}^{l_1} \cdots \sum_{k_{n+1}=-l_{n+1}}^{l_{n+1}} \sum_{k_{n+1}=-l_{n+1}}^{l_{n+1}} \mu^{-2} \sum_{i=1}^{n+1} (l_{i} + l_{i}) \times
f_{k_1, \ldots, k_{n+1}; k_1, \ldots, k_{n+1}}^{l_1, \ldots, l_{n+1}} (\mu p_1, \ldots, \mu p_m) |^{1/2} \times
\left( \prod_{i=1}^{n+1} t_{i_{m_{k}} k_{i}}^{l} (\mu g(t, z)g(t, z)^*) t_{i_{m_{k}} k_{i}}^{l} (\mu g(t, z)g(t, z)^*) \right)^{1/2}. \tag{2.45}
\]

Let us introduce the coordinates
\[
\tilde{x}_j = \mu g(t, z) \tilde{p}_j g(t, z)^*, \quad j = 1, \ldots, m,
\]
\[
\tilde{p}_{m} = \sigma^0 - \sum_{j=1}^{m-1} \tilde{p}_j, \quad m > 1,
\]
\[
\tilde{p}_m = \sigma^0, \quad m = 1. \tag{2.46}
\]

By summing up the equalities (2.46) we obtain the decomposition (2.25) for the matrix \( \tilde{x}_1 + \cdots + \tilde{x}_m \). The function \( f_{m_1, \ldots, m_{n+1}; m_1, \ldots, m_{n+1}} (p_1, \ldots, p_m) \) is polynomial bounded. For the set
\[
\{ p_1, \ldots, p_m \in \nabla_+: \sum_{j=1}^{m} \tilde{p}_j = \sigma^0 \}.
\]
the following estimate is valid

\[ |f_{m_1,\ldots,m_{n+1};l_1,\ldots,l_{n+1}}(\mu p_1, \ldots, \mu p_m)| \leq C(1 + \sum_{\nu=0}^{3} \sum_{i=1}^{m} (\mu p_i^\nu)^2)^N \leq C(1 + 4m\mu^2)^N \]  

(2.47)

where the constant \( C \) does not depend on the variables \( p_1, \ldots, p_m \). The function

\[ f_{m_1,\ldots,m_{n+1};l_1,\ldots,l_{n+1}}(p_1, \ldots, p_m) \]

is differentiable \( l_1 + \cdots + l_{n+1} + \hat{l}_1 + \cdots + \hat{l}_{n+1} + 1 \) times and its support lies in the product \( \mathbb{V}^{\times m}_+ \) of the cones. Hence the function

\[ f_{m_1,\ldots,m_{n+1};l_1,\ldots,l_{n+1}}(\mu p_1, \ldots, \mu p_m) \]

is differentiable with respect to the variable \( \mu l_1 + \cdots + l_{n+1} + \hat{l}_1 + \cdots + \hat{l}_{n+1} + 1 \) times and its first \( l_1 + \cdots + l_{n+1} + \hat{l}_1 + \cdots + \hat{l}_{n+1} \) derivatives vanish at the point \( \mu = 0 \). Now the inequalities (2.45), (2.47) imply the absolute convergence of the integral (2.38). The integral (2.38) defines the tempered distribution from \( S'(R^{4m}) \).

Let us prove the equality (2.38). If a support of a tempered distribution

\[ \mathcal{F}_{m_1,\ldots,m_{n+1};l_1,\ldots,l_{n+1}}(x_1, \ldots, x_{n+1}) \in S'(R^{4m}) \]

lies in the product \( \mathbb{V}^{\times m}_+ \) of the cones, then the relations analogous with the relations (2.18), (2.23) are valid

\[ \mathcal{F}_{m_1,\ldots,m_{n+1};l_1,\ldots,l_{n+1}}(x_1, \ldots, x_{n+1}) = \left( \prod_{i=1}^{m} (\partial x_i, \partial x_i)^q \right) f_{m_1,\ldots,m_{n+1};l_1,\ldots,l_{n+1}}(x_1, \ldots, x_{n+1}), \]  

(2.48)

\[ f_{m_1,\ldots,m_{n+1};l_1,\ldots,l_{n+1}}(x_1, \ldots, x_{n+1}) = \]  

(2\pi)^{q-1}(q-2)!/(q-1)! \times \int d^{4m} y \mathcal{F}_{m_1,\ldots,m_{n+1};l_1,\ldots,l_{n+1}}(y_1, \ldots, y_{n+1}) \times \]  

\[ \prod_{i=1}^{m}(x_i - y_i, x_i - y_i)^{q-2} \theta(x_i^0 - y_i^0) \theta((x_i - y_i, x_i - y_i)). \]  

(2.49)

We can choose the natural number \( q \) in the relations (2.48), (2.49) such that the function (2.49) is differentiable \( l_1 + \cdots + l_{n+1} + \hat{l}_1 + \cdots + \hat{l}_{n+1} + 1 \) times. A support of the function (2.49) lies in the product \( \mathbb{V}^{\times m}_+ \) of the cones. The function (2.49) is polynomial bounded. If the tempered distribution (2.48) satisfies the covariance relation (2.17), then the function (2.49) satisfies the covariance relation (2.17) also. By making use of the proof of Proposition we obtain

\[ f_{m_1,\ldots,m_{n+1};l_1,\ldots,l_{n+1}}(\mu g(t, z)\tilde{p}_1 g(t, z)^*, \ldots, \mu g(t, z)\tilde{p}_m g(t, z)^*) = \]  

\[ \sum_{k_1=-l_1}^{l_1} \cdots \sum_{k_{n+1}=-\hat{l}_{n+1}}^{\hat{l}_{n+1}} (t_{\sum_{i=1}^{n+1} k_i} \tilde{t}_{\sum_{i=1}^{n+1} \hat{k}_i})(g(t, z)) \]  

\[ f_{k_1,\ldots,k_{n+1};k_1,\ldots,k_{n+1}}(\mu \tilde{p}_1, \ldots, \mu \tilde{p}_m), \]

\[ \tilde{p}_m = \sigma^0 - \sum_{j=1}^{m-1} \tilde{p}_j, m > 1, \]

\[ \tilde{p}_m = \sigma^0, m = 1. \]  

(2.50)
By the definition (2.49) the function \( f^1_{l_1},...,l_{n+1};i_1,...,i_{n+1} \) \((\mu p_1,...,\mu p_m)\) is differentiable \( l_1+\cdots+l_{n+1} + i_1 + \cdots + i_{n+1} + 1 \) times with respect to the variable \( \mu \) and its first \( l_1+\cdots+l_{n+1} + i_1 + \cdots + i_{n+1} \) derivatives vanish at the point \( \mu = 0 \). The estimate (2.47) is valid. By using the relations (2.43), (2.50) and the coordinates (2.46) we have

\[
\int d^{4m}x f^1_{l_1},...,l_{n+1};i_1,...,i_{n+1}(x_1,...,x_{n+1})\phi(x_1,...,\bar{x}_m) = \sum_{k=-l_1}^{l_1} \cdots \sum_{k_{n+1}=-l_{n+1}}^{l_{n+1}} \int_0^\infty \mu^{4m-1}d\mu \int_0^\infty t^{-3}dt \int dRez \int dImz \int d^4p \delta(\sum_{j=1}^m \hat{p}_j - \sigma^0)(\prod_{i=1}^{n+1} \tilde{l}_i^{k_{m,k_i}}(g(t,z))t_{m,k_i}^{l_{m,i}}(g(t,z))) \times 

2f^1_{l_1},...,l_{n+1};i_1,...,i_{n+1}(\mu p_1,...,\mu p_m)\phi(\mu g(t,z)\hat{p}_1g(t,z)^*,...,\mu g(t,z)\hat{p}_mg(t,z)^*). \tag{2.51}
\]

Let us insert in the equality (2.51) a sequence of the functions \( \phi_\nu(x_1,...,x_m) \in S(\mathbb{R}^{4m}) \) convergent to the distribution

\[
2\left(\sum_{i=1}^m x_i, \sum_{i=1}^m x_i\right)^{3/2} \delta\left(\sum_{i=1}^m x_i\right) \psi\left(\left(\sum_{i=1}^m x_i, \sum_{i=1}^m x_i\right)^{1/2}, x_1, ..., x_{m-1}\right)
\]

where a function \( \psi(\mu, x_1, ..., x_{m-1}) \in S(\mathbb{R} \times \mathbb{R}^{4(m-1)}) \). Now by using the covariance relation (2.17) for the distribution (2.51) and a matrix \( u \in SU(2) \) we have

\[
f^1_{l_1},...,l_{n+1};i_1,...,i_{n+1}(\mu \tilde{p}_1,...,\mu \tilde{p}_m) = 
\sum_{k_1=-l_1}^{l_1} \cdots \sum_{k_{n+1}=-l_{n+1}}^{l_{n+1}} \prod_{i=1}^{n+1} \tilde{l}_i^{k_{m,k_i}}(u) \tilde{l}_i^{k_{m,k_i}}(u) \cdot f^1_{l_1},...,l_{n+1};i_1,...,i_{n+1}(\mu u^{-1}\hat{p}_1u,...,\mu u^{-1}\hat{p}_mu),
\]

\[
\tilde{p}_m = \sigma^0 - \sum_{j=1}^{m-1} \tilde{p}_j, m > 1,
\]

\[
\tilde{p}_m = \sigma^0, m = 1. \tag{2.52}
\]

We integrate the equality (2.52) with the normalized Haar measure on the group \( SU(2) \) and insert the derived equality in the equality (2.51). Let us change the integration variables

\[
\tilde{p}_j \rightarrow u\tilde{p}_ju^*, j = 1, ..., m-1, m > 1,
\]

\[
u \tilde{p}_mu^* = \tilde{p}_m = \sigma^0, m = 1.
\]

The derived equality and the relation (2.48) yield the equality (2.38). The theorem is proved.

By using the relations (2.11), (2.13) it is possible to calculate the integral over the group \( SU(2) \) in the right-hand side of the equality (2.38) for \( m = n = 1 \). The derived equality may be rewritten in the form (2.24).
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