Propagation of different kinds of non-linear ion-acoustic waves in Earth’s magnetosphere

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Abstract
The non-linear propagation of different kinds of ion-acoustic waves (IAWs) in multi-component plasma involving proton beam, positive ions and isothermal electrons has been studied. Using the derivative expansion method of basic equations, namely the hydrodynamic and Poisson equations, they are reduced to a single evolution equation of the non-linear Schrödinger (NLS)-type equation. By applying this model to plasma formed in Earth’s magnetosphere, different waves can be predicted that express the properties of the plasma. Using the separating variables method and the $G'/G$-expansion method, we derived the exact analytical solutions to the evolution equation by using different solution regions in which non-linear waves are defined. A comparison has been made between the solutions describing the differential equation in each region in which the solution can appear using the data on the Earth’s magnetosphere in the study by Alotaibi et al. (2021).

Keywords Soliton wave · Explosive pulses · Shock-like waves

1 Introduction
The wave process appears in many natural phenomena, such as plasma waves, optical waves, and water waves, which are governed by non-linear partial differential equations. The study of non-linear evolution equations significantly assists in the understanding of some physical properties of waves that appear in several different engineering applications and physical systems. The interest in physical properties of waves has been motivated, on one hand, by observations of the heating of electrons (Honzawa and Kawai 1972), parametric thermal instabilities (Evans et al. 2015), narrowing and stimulated electromagnetic emissions (Thidé et al. 1983), and radial drifts in hot regions (Patel et al. 2018), naturally occurring. Thus, methods for describing and interpreting theoretical studies of physical phenomena depend on finding exact solutions to many types of differential equations. Some well-known evolution equations are non-linear partial differential equations, such as the NLS-type equation and its generalisations are successfully used to describe non-linear IAWs that are commonly observed in the astronomical environments (Gharaee et al. 2011; Moslem et al. 2019b). To understand the behaviour and properties of different types of wave solutions to non-linear differential equations, investigative analysis must occur due to the increased interest in the field of space research in recent years. There are many theoretical studies on the properties of plasma fluid formed in the Earth’s ionosphere or on other planets including Ali et al. (2021), Helling et al. (2021), Tolba et al. (2017, 2021), Salem et al. (2017), and Moslem et al. (2019a). Many different non-linear effects occur on ionospheric plasma under the influence of strong radio emissions according to Levine et al. (2020), Golikov et al. (2016), Pavlov et al. (2000), and Ma et al. (2012). The non-linear interference of strong radio waves in the ionosphere causes plasma medium disturbances, resulting in the presence of a wide spectrum of inhomogeneity, known as artificial ionospheric disturbance (Thompson et al. 2017; Bakhmetieva et al. 2020). With regard to plasma models discussed previously, the fluid constituent is usually composed largely of positive ions with few negative ions, which are treated as strictly hot. Likewise, some of the other models indicated that positive ions and superthermal electrons are abundant in the plasma regime.
predictions (Khan et al. 2020; Malik et al. 2020; Alotaibi have been conducted by means of small amplitudes in ionospheric heaters and that have led to different types of wave predictions (Khan et al. 2020; Malik et al. 2020; Alotaibi et al. 2021). Many different studies have explored methods for finding solutions to the NLS-type equation and its applications in various physical phenomena, for example but are not limited to these.

The importance comes from determining the regions of stability and instability, because in each region the equation must have a solution specific to that region. This is because in each region the differential equation has to have a solution specific to that region. We explain how to find the correlation between exact solutions and some analytic solutions of the NLS-type equation, which is a non-linear partial differential equation. We used the direct integration method which is done by separating the variables. We find that the discovery of the value of the integral depends explicitly on the sign of the coefficients in the differential equation for which the integral must be calculated. Meanwhile, the variable constants depend on changing the physical system variables. Depending on the values of the variable’s physical parameters, the variable constants in the differential equation may be greater than or less than zero.

The aims to study are

1. To probe the stability condition of the IAWs in Earth’s magnetosphere
2. To find out the condition when the existence of a soliton wave is possible
3. To determine the condition when the existence of an explosive wave is possible
4. To describe the electrostatic waves that may appear in Earth’s magnetosphere

This paper is structured as follows. Section 2 describes the basic formalism of the physical problem and the derivation of the NLS-type equation for a plasma model with hot positive ions in the presence of isothermal electrons. From previous studies, we observed that including plasma positive-negative ions thermal effects is paramount in determining the characteristics of solitary wave propagation and other waves (Tolba et al. 2021; Mushinzimana et al. 2021), as well as the dependence on the parametric investigation and delimitation of the solution’s existence domains. In addition, we found that the NLS-type equation has a bright solitary wave solution in the region where each of two variable coefficients (P and Q) are a positive sign. In Sect. 3, some analytical properties of the wave solutions are given, which help us in Sect. 4 with the numerical properties of the waves, existence domains, and variations with compositional parameters and profiles. In Sect. 5, we discuss aspects of the quasi-neutrality assumption that underpin formalism, while Sect. 5 summarises our findings.

2 Governing equations and derivation of evolution equation

In this section, we consider the propagation of non-linear IAWs in an unmagnified, collisionless plasma consisting of three components positive ions, proton beam and electrons which have Maxwellian distribution. Therefore, the dynamics of potential oscillations in our plasma is governed by the following normalised equations: for positive ions (Afify et al. 2022; Alotaibi et al. 2021)

\[
\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x}(n_i u_i) = 0, \quad (1)
\]

\[
\frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x} + \frac{\partial \phi}{\partial x} + 3n_i \frac{\partial n_i}{\partial x} = 0, \quad (2)
\]

for the ion beam

\[
\frac{\partial n_b}{\partial t} + \frac{\partial}{\partial x}(n_b u_b) = 0, \quad (3)
\]

\[
\frac{\partial u_b}{\partial t} + u_b \frac{\partial u_b}{\partial x} + \mu_b \frac{\partial \phi}{\partial x} + 3\sigma_b \mu_b n_b \frac{\partial n_b}{\partial x} = 0. \quad (4)
\]

The Maxwellian electrons are expressed as

\[
n_e = \exp(\sigma_e \phi). \quad (5)
\]

The system of Eqs. (1)–(5) is closed by the Poisson equation

\[
\frac{\partial^2 \phi}{\partial x^2} + (\delta_n n_b + n_i - \delta_e n_e) = 0, \quad (6)
\]

where \(n_{i,b,e}\) is the perturbed number of densities of the plasma species that are normalised by \(n_{i0}, n_{b0}\) and \(n_{e0}\), respectively. The two ions’ velocities \(u_i\) and \(u_b\) are normalised by the speed of ion-acoustic of the first ion \(C_i = (K_B T_e/m_i)^{1/2}\), and \(\phi\) is the electrostatic potential normalised by \(K_B T_e/e\), where \(K_B\) is the Boltzmann constant. The space coordinate \(x\) and the time \(t\) variables are normalised by the Debye length \(\lambda_{Di} = (K_B T_e/4\pi n_{i0} e^2)^{1/2}\) and the inverse of the plasma density \(\omega_{pi}^{-1} = (m_i/4\pi n_{i0} e^2)^{1/2}\) respectively. Here, the physical non-dimensional quantities appearing in the system are \(\sigma_e = (T_i/T_e), \mu_b = m_i/m_b,\)
σb = (Tb/Ti), δb = nio/nio, and δe = nei/nio, where Tb is the temperature of positive ions, Ti is the temperature of electron, and the zeroth moment condition appeases the relation δb - δe + 1 = 0.

For non-linear IAWs analysis, the characteristic variables are expanded as shown by Yahia et al. (2021)

ξ = ε (x - V gt t) and τ = ε 2 t,

where 0 < ε < 1 is a small (real) parameter and V g is the envelope-group velocity. The dependent variables are expanded as

Φ (x, t) = Φ0 + ∑m=1∞ εm ∑L=−m L m Φ L,m (ξ, τ) exp (i L β),

where

Φ L,m = [nI,m nB,m nE,m ϕ uI,m uB,m ϕ u0,m]T,

Φ L,0 = [1 δb δe 0 uI0 0]T

and

β = k x - ω t.

Here k and ω are real variables representing the fundamental (carrier) wave number and the angular frequency; respectively, and T stands for transpose. All functions of Φ L,m satisfy the reality condition Φ−L,m = Φ∗ L,m, where the asterisk denotes the complex conjugate. By substituting Eqs. (7) and (8) into Eqs. (1)–(6) and collecting terms of the same powers of ε, the first-order (m = 1) equations with L = 1 gives

nI11 = s1 ϕ11, uI11 = s2 ϕ11, nB11 = s3 ϕ11,

uB11 = s4 ϕ11, nE11 = s5 ϕ11.

By using Eqs. (11) and (12), we get the linear dispersion relation from the Poisson equation

k2 ω2 + k2 = 0.

Apparentlv, there are four roots for real values of k in the dispersion relation (13), as ω (k), where ω is the angular frequency and k is the wave number. Figure 1(a) shows the frequency values of the relationship between the concentration δb of a proton beam and its velocity ub0. The second-order (m = 2) reduced equations with L = 1 yield

nI12 = s1 ϕ12 + s5 ∂ϕ11 ∂ξ, uI12 = s2 ϕ12 + s6 ∂ϕ11 ∂ξ,

nB12 = s3 ϕ12 + s7 ∂ϕ11 ∂ξ,

uB12 = s4 ϕ12 + s8 ∂ϕ11 ∂ξ, nE12 = s5 ϕ12,

s1, s2, ..., and s8 are given in the Appendix with the compatibility condition

V g = ∂ω ∂k = f1 f2 + f0 f1 ω - f1 ω2 k (f0 f2 - f1 ω).

Fig. 1 (a): The variation of the frequency ω in the uI0 - δb plane for the region PQ > 0, for μb = 0.98, δb = δb + 1, σb = 1, M = 0.03, σe = 0.22, k = 0.77, δb = 0.2, uI0 = 0.5. (b): The variation of the group velocity V g in the uI0 - δb plane for the region PQ > 0, for μb = 0.98, δb = δb + 1, σb = 1.01, M = 0.03, σe = 0.22, k = 0.78, δb = 0.35 uI0 = 0.86.
where \( f_0 = \delta_b \mu_b (k u_{b0} - \omega) \), \( f_1 = (k^2 (u_{b0}^2 - 3 \mu_b \sigma_b) - 2 k u_{b0} \omega + \omega^2)^2 \) and \( f_2 = (\sigma^2 - 3 k^2)^2 \).

This equation represents the group velocity of the IAWs, where the group velocity is normalized by the ion acoustic speed. Figure 1(b) showing the behaviour of the group velocity, from the concentration of the proton beam \( \delta_b \) versus its velocity \( u_{b0} \). The second-order harmonic modes \( m = L = 2 \) arising from the nonlinear self-interaction of the carrier waves are obtained in terms of \( \varphi_{11}^2 \) as

\[
\begin{align*}
n_{122} &= s_9 \varphi_{11}^2, & u_{122} &= s_{10} \varphi_{11}^2, \\
n_{222} &= s_{11} \varphi_{11}^2, & u_{222} &= s_{12} \varphi_{11}^2, \\
n_{e22} &= s_2 s_{13} \varphi_{11}^2, & \varphi_{22} &= s_2 s_{14} \varphi_{11}^2,
\end{align*}
\]

(16)

where \( s_9, \ldots, s_{14} \) are given in the Appendix. The nonlinear self-interaction of the carrier wave also leads to the creation of a zeroth-order harmonic. Its strength is analytically determined by taking the \( L = 0 \) component of the third-order reduced equations, which can be expressed as

\[
\begin{align*}
n_{102} &= s_{15} |\varphi_{11}|^2, & u_{102} &= s_{16} |\varphi_{11}|^2, & n_{b02} &= s_{17} |\varphi_{11}|^2, \\
u_{b02} &= s_{18} |\varphi_{11}|^2, & \varphi_{02} &= s_{19} |\varphi_{11}|^2, & n_{e02} &= s_{20} |\varphi_{11}|^2,
\end{align*}
\]

(17)

where \( s_{15}, \ldots, s_{20} \) are given in the Appendix. Finally, the third-harmonic modes \( m = 3 \) and \( L = 1 \), with the aid of Eqs. (12)–(17), give a set of equations. The compatibility condition for these equations yields the NLS-type equation

\[
i \frac{\partial \varphi}{\partial \tau} + P \frac{\partial^2 \varphi}{\partial \xi^2} + Q |\varphi|^2 \varphi = 0.
\]

(18)

We set \( \varphi_{11} \equiv \varphi \) for simplicity. \( P \) is the coefficient of the dispersion term and \( Q \) is the coefficient of the nonlinear term that is given in the Appendix.

### 3 Construction of the method and the extraction of solutions

The direct method is one of the most substantial methods of finding accurate solutions to the equations of non-linear evolution, through which it is possible to find some solutions to these equations that describe the natural phenomenon. Here, the exact solutions of the non-linear Schrödinger-type equation [NLS-type equation (18)] will be estimated using the method of separating variables. It is followed by the use of the direct integration method to obtain analytical solutions to the Schrödinger equation. To use the separation of variables method, we propose that the solution \( \varphi \) can be expressed as shown in Yahia et al. (2021); and Sabry et al. (2012)

\[
\varphi (\xi, \tau) = \Phi (\xi) \exp (i M \tau),
\]

(19)

where \( \Phi \) is the component of amplitude and \( M < \omega \). By substituting Eq. (19) into Eq. (18), we obtain

\[

P \frac{d^2 \Phi}{d \xi^2} + Q \Phi^3 - M \Phi = 0,
\]

(20)

\[
d^2 \Phi \frac{d \xi^2}{d \xi^2} + H_1 \Phi^3 - H_2 \Phi = 0,
\]

(21)

where \( H_1 = \frac{Q}{M} \) and \( H_2 = \frac{M}{p} \). Let \( \Phi' = F \), and get the following Hamilton system

\[
\frac{d \Phi}{d \xi} = F,
\]

(22)

\[
\frac{d F}{d \xi} = -H_1 \Phi^3 + H_2 \Phi.
\]

(23)

Thus the system (22), (23) \( \Phi' = X_1 (\Phi, F) \) and \( F' = X_2 (\Phi, F) \) is the Hamiltonian if such that \( X_1 = \partial Y/\partial F \) and \( X_2 = -\partial Y/\partial \Phi \) and \( Y (\Phi, F) \) satisfies the condition

\[
\partial X_1/\partial \Phi + \partial X_2/\partial F = 0.
\]

(24)

If we put \( d \Phi/d \xi = d F/d \xi = 0 \) in system (22), (23) and solving it, we obtain the critical (rest or equilibrium) points that are

\[
a_1 = (0, 0), \quad a_2 = i \sqrt{\frac{H_1}{H_2}}, 0 \quad \text{and} \quad a_3 = -i \sqrt{\frac{H_1}{H_2}}, 0.
\]

(26)

The Jacobian matrix of the linear Hamiltonian system is

\[
J = \begin{pmatrix}
0 & -H_2 & -3H_1 \Psi^2 \\
H_2 & 1 & 0 \\
3H_1 \Psi^2 & 0 & 1
\end{pmatrix},
\]

(27)

and the eigenvalues \( \lambda_i \) are determined from the relation

\[
|J - \lambda I| = 0,
\]

(28)

where, \( I \) is the unit matrix, so that the eigenvalues at the first equilibrium point are \( \lambda_{1,2} = \mp i (H_1)^{1/2} \), and the eigenvalues at the second equilibrium point are \( \lambda_{3,4} = \mp (-2H_1)^{1/2} \), the last eigenvalues at the third equilibrium point are \( \lambda_{3,4} = \mp (-2H_1)^{1/2} \), at the same values in the second equilibrium.
point. We investigate the stable and unstable regions in the plasma system. Therefore, we use typical data in the Earth’s magnetosphere that are shown in Alotaibi et al. (2021). Accordingly, we obtain the unstable regions of a modulation solution, i.e. the black region for \( PQ > 0 \), and the yellow region is the stable regions for \( PQ < 0 \), as shown in Fig. 2. The physical parameters in the instability region are \( \mu_b = 0.98, \delta_e = \delta_b + 1, \sigma_b = 1, M = 0.03, \sigma_e = 0.22, k = 0.77, \delta_b = 0.2, u_{b0} = 0.5 \), whereas the physical parameters in the stability region are \( \mu_b = 0.98, \delta_e = \delta_b + 1, \sigma_b = 1.01, M = 0.03, \sigma_e = 0.22, k = 0.78, \delta_b = 0.35 u_{b0} = 0.36 \) where the region of instability is the region of occurrence of the bright soliton wave and the region of stability is the region of occurrence of dark solitons. We note that the instability region is divided into three regions that expand when the initial velocity of protons increases with the increase in proton concentration, and the stability region gradually shrinks as shown in Fig. 1. Checking the parameter polarity of \( H_1 \) using the same data in Fig. 1, we note that the parameter values of \( H_1 \) are greater than zero and less than zero. Where, \( H_1 = P/Q \) from Eq. (21), this leads to \( H_1 \) which is the same behaviour of the \( PQ \), in other words \( H_1 \) is greater than zero in the black region in Fig. 2 and is less than zero in the yellow region in that Figure as well. Remarkably, the region of \( H_1 \) is less than zero when the value of the initial velocity of proton beam \( u_{b0} \) exceeds 0.2, and the proton beam concentration \( \delta_b > 0.4 \) as shown in Fig. 2. In the instability region i.e. in the black region \( H_1 > 0 \), we obtain \( \lambda_1 = -i \sqrt{H_1} \), and \( \lambda_2 = i \sqrt{H_1} \) and this gives us an unstable, centre as shown in Fig. 3. However, when \( H_1 \) is less than zero, i.e. in the blue region, we obtain an unstable saddle, \( \lambda_1 = -i \sqrt{-H_1} \), and \( \lambda_2 = i \sqrt{-H_1} \), where \( \sqrt{-H_1} = \sqrt{H_0} \) in the remind \( H_1 < 0 \), in Fig. 4.

3.1 First exact solutions of the non-linear Schrödinger-type equation

Here, the exact solutions of the non-linear Schrödinger-type equation [NLS-type equation (18)] will be estimated using...
regions of instability (Das 1989; Scott 1999) follows:
in regions that verify that the bright soliton as shown in Fig. 5, because the bright soliton appears in the intersection areas; the physical plasma model as depicted in Fig. 3. As it is clear from this Figure, there must be regions of intersection between \( P \) and \( Q \) in between them, meaning that when it is \( P > 0 \), it intersects and exists with \( Q > 0 \) and \( Q < 0 \) and vice versa. Therefore, it can be concluded that the four intersection areas are the areas where one solution can emerge without the other. So, we see that the integration in Eq. (32), which strictly bases on the sign of the \( P \) and \( Q \), and both of them are instrumental in determining the sign of the \( A_1 \) and \( B_1 \) coefficients. For the occurrence of \( P > 0 \) and \( Q > 0 \) as shown in Figs. 3(a) and 3(b); and by integrating Eq. (32), in these conditions it leads to

\[
\int \frac{d\psi}{\sqrt{\psi^2 - B_1}} = \sqrt{A_1} \xi, \tag{33}
\]

whose solution is given by

\[
\psi = B_1^{1/2} \left( \cosh\sqrt{A_1} \xi \right). \tag{34}
\]

The solution (34) exists in the union regions between the red region and the blue region, in the intersection regions between the parameter region \( P > 0 \) and the parameter region \( Q > 0 \). Through this, we get the general solution of the NLS-type equation (18), which gives the solution of the soliton wave, and it is known that the soliton wave is a bright soliton as shown in Fig. 5, because the bright soliton appears in regions that verify that \( P Q \) is greater than zero, which are regions of instability (Das 1989; Scott 1999) as follows:

\[
\varphi = B_1^{-1/2} \left( \text{sech}\sqrt{A_1} \xi \right) \exp(i M \tau). \tag{35}
\]

We notice that the wave energy decreases as well as the width when the concentration ratio of the proton beam \( \delta_b \) decreases, as shown in Fig. 5(a), where the black dashed curve at \( \delta_b = 0.2 \) and the red dashed curve at \( \delta_b = 0.13 \), and where the decreases in \( \delta_b \) lead to decreases in the concentration ratio of electrons. This comes from the neutrality condition. However, when there are increases in the initial velocity of the proton beam \( u_{ib0} \), this leads to decreases in the amplitude of the bright soliton wave and the width as shown in Fig. 5(b), where the black dashed curve at \( u_{ib0} = 0.5 \) and the black bold curve at \( u_{ib0} = 0.45 \). This occurs because the initial velocity is normalised by the ion-acoustic speed, when the last velocity is increased by the energy of positive ion increasing. This leads to the initial velocity of the proton beam being decreased. Additionally, the amplitude of the bright soliton increases and the width is expanded, when the temperature ratio of proton beam \( g \) is decreased as shown in Fig. 5(c), where the black dashed curve is at \( \sigma_b = 1 \) and the blue dashed curve at \( \sigma_b = 0.95 \), because the temperature of proton beam is normalised by the temperature of the positive ions. Lastly, when the temperature ratio of electrons is decreased the amplitude of the wave is faster and taller and the width is expanded as shown in Fig. 5(d), with the black dashed curve at \( \sigma_e = 0.22 \) and the green dashed curve at \( \sigma_e = 0.17 \). Because the temperature of electrons is normalised by the temperature of the positive ion, as the ratio of temperature increases, it leads to the wave energy is decreasing.

The second solution of the integration is when two conditions are less than zero, in other words \( P < 0 \) and \( Q < 0 \). This leads to \( A_1 < 0 \) and \( B_1 < 0 \); therefore, we put \( A_1 = i^2 A_{10} \) and \( B_1 = i^2 B_{10} \). The integration in Eq. (33) yields

\[
\psi = i^{-1} \sqrt{B_{10}} \left( \csc\sqrt{A_{10}} \xi \right). \tag{36}
\]

Through this, we get the general solution of the NLS-type equation (18), which gives the solution of the periodic explosive wave, and it is known that the periodic explosive wave solution is the instability solution which confirms that the presence in the region of instability \( P Q \) is greater than zero as shown in Figs. 4(a), 4(b) and 4(c).

\[
\varphi = i^{-1} B_{10}^{-1/2} \left( \csc\sqrt{A_{10}} \xi \right) \exp(i M \tau). \tag{37}
\]

The third solution, in which the third condition \( Q < 0 < P \), the integration of Eq. (33) is reduced to the following formula

\[
\int \frac{d\psi}{\sqrt{\psi^2 + B_{10}}} = \sqrt{A_1} \xi, \tag{38}
\]

whose integration yields the following solution

\[
\psi = B_{10}^{1/2} \left( \sinh\sqrt{A_1} \xi \right). \tag{39}
\]
Substituting from Eq. (39) into Eq. (19), we get the general solution of Eq. (18):

\[ \varphi = B_{10}^{-1/2} \left( \text{csch} \sqrt{A_1 \xi} \right) \exp (i M \tau). \]  

(40)
There are two qualitatively different types of detonation processes: those that involve only linearly stable waves in the absence of dissipation and those that embrace two stable waves (in the same sense) and one unstable wave (Shirira et al. 1997).

In this section, we introduce the $G'/G$-expansion method for solving the NLS-type equation. The first crucial step is that the solution we are looking for is expressed in the form set out in Tolba (2021), and Moslem et al. (2022)

$$\Phi(\xi) = a_0 + \sum_{i=1}^{\beta} \left( a_i \left( \frac{G'}{G} \right)^i + b_i \left( \frac{G'}{G} \right)^{-i} \right)$$

The positive integer $\beta$ is the balance between the non-linear term and the dispersion i.e. $\beta = 1$: The unknown constants $a_0$, $a_i$ and $b_i$ are to be determined by substituting Eq. (43) into Eq. (20), where we get a class of solutions:

$$a_0 = -\sqrt{\frac{M}{5Q}}, a_1 = -i \frac{2P}{Q}, b_1 = \frac{i}{15} \sqrt{\frac{2M}{QP}}.$$  

$$a_0 = \sqrt{\frac{M}{5Q}}, a_1 = -i \frac{2P}{Q}, b_1 = \frac{i}{15} \sqrt{\frac{2M}{QP}}.$$  

$$a_0 = -\sqrt{\frac{M}{Q}}, a_1 = -i \frac{2P}{Q}, b_1 = 0.$$  

$$a_0 = 0, a_1 = i \frac{2P}{Q}, b_1 = \frac{-iM}{3\sqrt{2PQ}}.$$  

Therefore, the solution for Eq. (20) is

$$\Phi(\xi) = a_0 + a_1 \frac{G'}{G} + b_1 \left( \frac{G'}{G} \right)^{-1},$$

where the function $G(\xi) \equiv G$ satisfies the Riccati equation

$$\frac{d^2G}{d\xi^2} + h_1 \frac{dG}{d\xi} + h_2 G = 0,$$

where $h_1$ and $h_2$ are arbitrary constants. The possible solutions of Eq. (49) give

1. Hyperbolic family solution

$$G(\xi) = \exp \left( -\frac{h_1}{2} \xi \right) \left( w_1 \cosh[\Delta_1 \xi] + w_2 \sinh[\Delta_1 \xi] \right),$$  

where $w_2 > w_1$, are the arbitrary constants and $(h_1^2 - 4h_2)^{1/2} > 0$, and $\Delta_1 = (h_1^2 - 4h_2)^{1/2}$.

2. Trigonometric family solution

$$G(\xi) = \exp \left( -\frac{h_1}{2} \xi \right) \left( w_3 \sin[\Delta_2 \xi] + w_4 \cos[\Delta_2 \xi] \right),$$

where $w_2 > w_1$, are the arbitrary constants and $(h_1^2 - 4h_2)^{1/2} > 0$, and $\Delta_1 = (h_1^2 - 4h_2)^{1/2}$.
where \( w_3 < w_4 \), are the arbitrary constants and \((h_1^2 - 4h_2^2)^{1/2} < 0\), and \( \Delta_2 = (h_1^2 - 4h_2^2)^{1/2} / i \).

3. Exponential family solution

\[
G(\xi) = \exp \left( \frac{-h_1}{2} \xi \right) (w_5 + w_6),
\]

(52)

where \((h_1^2 - 4h_2^2)^{1/2} = 0\), \( w_5 \) and \( w_6 \) are the arbitrary constants.

Here, we present a numerical analysis of ion acoustic waves, where the value of the non-linear coefficient and the dispersion coefficient are influenced by various plasma parameters such as the concentration of the proton beam (via \( \delta_b \)), the initial velocity of the proton beam (via \( u_{b0} \)), the proton beam temperature ratio (via \( \sigma_b \)) and the electron temperature ratio (via \( \sigma_e \)). We refer to the numerical analysis solutions in the instability region and the stability region, by the same physical parameters above. In the instability region, when we apply the above data in the analytical solution (50) supporting the values of the constants (44), we get the bright soliton as shown in Fig. 9, at \( P > 0 \) and \( Q > 0 \), where the red curve is at \( \sigma_b = 1 \) and the black curve at \( \sigma_b = 1.012 \), \( w_1 = 4 \), \( w_2 = (-115)^{1/2} \), \( h_1 = 7 \), \( h_2 = 1 \). But, when applying this solution in the instability region at \( P < 0 \) and \( Q < 0 \), we obtain the bright soliton as shown in Fig. 10. This wave behaviour cannot be obtained by the exact solution in the above at the same physical parameters, where the red dashed curve is at \( \sigma_b = 1.01 \) and the black dashed curve at \( \sigma_b = 1.021 \), \( w_1 = 3.2 \), \( w_2 = (-.14)^{1/2} \), \( h_1 = 1.33 \), \( h_2 = 0.47 \). When we move to the stability region, i.e. when \( P > 0 \) and \( Q < 0 \), we get the anti-kink wave as shown in Fig. 11, where the red dashed curve is at \( \sigma_b = 1.021 \) and the black dashed curve at \( \sigma_b = 1.01 \), \( w_1 = 3 \), \( w_2 = (-.13)^{1/2} \), \( h_1 = 1.3 \), \( h_2 = 0.47 \), and this wave behaviour is not available from the exact solution. Similarly, the wave behaviour is in the stability region when \( P < 0 \) and \( Q > 0 \) is an anti-entangled wave as shown in Fig. 12, but noting that wave amplitude decreases. Where the red bold curve is at \( \sigma_b = 1.02 \) and the black bold curve at \( \sigma_b = 1.01 \), \( w_1 = 4 \), \( w_2 = (-.13)^{1/2} \), \( h_1 = 1.3 \), \( h_2 = 0.47 \). In both cases of the anti-entanglement wave, we detect that when the ratio of...
Fig. 13 The profile of the absolute value of the ion acoustic periodic wave of the first case $P > 0$ and $Q > 0$, for $\mu_b = 0.98$, $\delta_c = \delta_b + 1$, $\sigma_b = 1$, $M = 0.03$, $\sigma_c = 0.22$, $k = 0.77$, $\delta_b = 0.2$, $u_{b0} = 0.5$

Fig. 14 The profile of the absolute value of the ion acoustic periodic wave of the second case $P < 0$ and $Q < 0$, for $\mu_b = 0.98$, $\delta_c = \delta_b + 1$, $\sigma_b = 1$, $M = 0.03$, $\sigma_c = 0.22$, $k = 0.77$, $\delta_b = 0.2$, $u_{b0} = 0.5$

Fig. 15 The profile of the absolute value of the ion acoustic periodic wave of the third case $P > 0$ and $Q < 0$, for $\mu_b = 0.98$, $\delta_c = \delta_b + 1$, $\sigma_b = 1.01$, $M = 0.002$, $\sigma_c = 0.22$, $k = 0.77$, $\delta_b = 0.4$, $u_{b0} = 0.4$

Fig. 16 The profile of the absolute value of the ion acoustic periodic wave of the forth case $P < 0$ and $Q > 0$, for $\mu_b = 0.98$, $\delta_c = \delta_b + 1$, $\sigma_b = 1$, $M = 0.03$, $\sigma_c = 0.22$, $k = 0.77$, $\delta_b = 0.2$, $u_{b0} = 0.5$

Fig. 17 The profile of the absolute value of the ion acoustic explosive wave of the first case $P > 0$, and $Q > 0$. For variation $u_{b0}$, for $\mu_b = 0.98$, $\delta_c = \delta_b + 1$, $\sigma_b = 1.01$, $M = 0.03$, $\sigma_c = 0.22$, $k = 0.77$, $\delta_b = 0.2$

Remarkably, when the ratio of the initial velocity of the proton beam decreases, this results in an increase in the energy of wave. When we move to the use of the second analytical solution (51), with the constants (44) and its realisation in the regions of stability and instability when using the data that checks each region, we get the following wave behaviour. This solution gives the periodic wave in all cases, in which both the sign of the non-linear modulus and the sign of the dispersion coefficient change, with different values of the amplitude of the wave and the width. When $P > 0$ and $Q > 0$, $w_3 = 1.2$, $w_2 = (-0.4)^{1/2}$, $h_1 = 0.5$, $h_2 = 0.86$, we obtain the periodic wave as shown in Fig. 13, whereas at $P < 0$ and $Q < 0$, we obtain a periodic wave with a high value of the amplitude compared with the case in Fig. 13, as shown in Fig. 14. When our discussion of the behaviour of the second analytic solution in the region in which it is $P Q < 0$, i.e. $Q < 0$, and $P > 0$, at $u_{b0} = 0.4$, $\delta_b = 0.2$, we obtain the periodic wave as shown in Fig. 15, but when $P Q < 0$, i.e. $Q > 0$, and $P < 0$, at $u_{b0} = 0.3$, $\delta_b = 0.2$, we obtain the periodic wave as shown in Fig. 16, with the different values of the amplitude of the wave. Finally, we will investigate the behaviour of the last analytic solution (52) in both cases of stability using the same values of the previous physical parameters, with by the aid constants (44). Firstly, when $P Q > 0$, or more specifically at $P > 0$ and $Q > 0$, the last analytical solution gives the explosive wave as shown in Fig. 17, where the black dashed curve at $u_{b0} = 0.45$, and the red dashed curve at $u_{b0} = 0.47$. Remarkably, when the ratio of the initial velocity of the pro-
ton beam $u_{b0}$ decreases, this leads to the energy of the explosive wave increasing. Secondly, then $PQ > 0$, more specifically at $P < 0$ and $Q < 0$, the last analytical solution gives the explosive wave as shown in Fig. 18, where the black dashed curve is at $u_{b0} = 0.2$, and the red dashed curve at $u_{b0} = 0.25$. Remarkably, when the ratio of the initial velocity of the proton beam $u_{b0}$ decreases this leads to the energy of the explosive wave is increasing. The third case when $PQ < 0$, i.e. $P > 0$ and $Q < 0$, and the fourth case when $PQ < 0$, i.e. $P < 0$ and $Q > 0$, are given as the explosive wave with a different value of the amplitude and the width of the wave, in $w_5 = 0.5, w_6 = 3$, and $h_1 = 0.031$.

### 5 Summary

The non-linear fluid equations describing a plasma model in Earth’s magnetosphere have been presented. We used perturbation theory as effective tool to curtail the basic equations to one equation; the NLS-type equation. We have identified the regions in which each solution can be achieved, depending on the physical variables of the system. We found the solutions of the NLS-type equation by using the traveling-wave solutions (direct integration method) and the $G'/G$-expansion method. A comparison is made between the behaviour of each solution obtained from the first method with the behaviour of each solution obtained from the second method in the same region of stability or instability. Therefore, the separation of variables in the NLS-type equation provides us with many solutions that have different interpretations of physical phenomena, and also the analytical methods such as the expansion method have provided the same explanation, as well as different kinds of explanation in Earth’s magnetosphere and laboratory.

### Appendix

\[
s_1 = \frac{-k^2}{3k^2 - \omega^2}, \quad s_2 = \frac{s_1 \omega}{k}, \quad s_3 = \frac{k^2 \mu_b}{\omega^2 - 2k\mu_b \omega + k^2 (u_{b0}^2 - 3\mu_b \sigma_b)},
\]

\[
s_4 = \frac{k \mu_b (\omega - k u_{b0})}{\omega^2 - 2k\mu_b \omega + k^2 (u_{b0}^2 - 3\mu_b \sigma_b)}, \quad s_5 = \frac{i (k + 3s_1 - k_2 V_g + \omega_2 - s_1 V_g \omega)}{3k^2 - \omega^2},
\]

\[
s_6 = \frac{i (3s_2 - 3s_1 V_g + \omega + 3\omega_1 - s_2 V_g \omega)}{3k^2 - \omega^2},
\]

\[
s_7 = \frac{i (k \mu_b - s_4 V_g + s_3 (u_{b0} - V_g) + 3\mu_b \sigma_b)) + (s_4 + s_3 (u_{b0} - V_g)) \omega)}{\omega^2 - 2k \mu_b \omega + k^2 (u_{b0}^2 - 3\mu_b \sigma_b)},
\]

\[
s_8 = \frac{i (k u_{b0} (s_4 (u_{b0} - V_g) + \mu_b) - 3k (s_4 - s_3 V_g) \mu_b \sigma_b - (s_4 (u_{b0} - V_g) + \mu_b + 3s_3 \mu_b \sigma_b)) \omega)}{\omega^2 - 2k \mu_b \omega + k^2 (u_{b0}^2 - 3\mu_b \sigma_b)},
\]

\[
s_9 = \frac{s_9_1}{s_9_0}, \quad s_9_0 = s_9_0_1 + s_9_0_2, \quad s_9_1 = s_9_1_1 + s_9_1_2 + s_9_1_3,
\]

\[
s_9_0_1 = 2k^4 \left( u_{b0}^2 \left( 1 + 12k^2 3\sigma_e \delta_e \right) - 3\mu_b \left( \delta_b \sigma_b + 12k^2 \sigma_b + 3\sigma_b \sigma_e \delta_e \right) \right) - 4k^3 u_{b0} \omega (1 + 12k^2 - 3\sigma_e \delta_e),
\]

\[
s_9_0_2 = 2k^2 (-1 - \delta_b \mu_b + (u_{b0}^2 - 3 - 3\mu_b \sigma_b) (4k^2 + \sigma_e \delta_e)) \omega^2 + 2 \omega^2 (4k^2 + \sigma_e \delta_e) (2k u_{b0} - \omega),
\]

\[
s_9_1_1 = ik \left( a_1 \delta_b - ik (2s_2 s_4 u_{b0} \delta_b + \delta_e (u_{b0}^2 - 3\mu_b \sigma_b) \sigma_e^2) + a_2 (-\delta_b \mu_b + (u_{b0}^2 - 3\mu_b \sigma_b) (4k^2 + \sigma_e \delta_e)) \right),
\]

\[
s_9_1_2 = 2k (i a_2 u_{b0} (4k^2 + \sigma_e \delta_e) + k (s_3 s_4 \delta_b - s_1 s_2 \delta_b \mu_b + 4k^2 s_1 s_2 (u_{b0}^2 - 3\mu_b \sigma_b) + s_1 s_2 \delta_e (u_{b0}^2 - 3\mu_b \sigma_b) \sigma_e + u_{b0} \delta_e \sigma_e^2)) \omega + \omega^3 (-2s_1 s_2 (4k^2 + \sigma_e \delta_e)),
\]
\[ s_{113} = \omega^3 (k(ia_2(4k^2 + \sigma_e\delta_e)) + k(16k^2s_1s_2u_{b0} + \delta_e\sigma_e(4s_1s_2u_{b0} + \sigma_e)))\omega^2, \]

\[ s_{10} = \frac{s_{101}}{s_{90}}, \quad s_{101} = s_{102} + s_{103} + s_{104} + s_{105}, \quad s_{104} = \omega^3 (k\sigma_e^2\delta_e + i\alpha_2\delta_e\sigma_e + 4i\alpha_2k^2), \]

\[ s_{102} = -2k^{4}s_1s_2(u_{b0}^2(1 + 12k^2 + 3\delta_e\sigma_e) - 3\mu_b(\delta_b + \sigma_b + 12k^2\sigma_b) + 3\delta_e\sigma_e\sigma_e)), \]

\[ s_{103} = k^2(4k^3s_1s_2u_{b0} + 4i\alpha_2k^2(u_{b0}^2 - 3\mu_b\sigma_b))\omega + k4s_1s_2u_{b0}(1 + 3\delta_e\sigma_e)\omega \]

\[ k^2((a_1\delta_b - a_2\delta_b\mu_b + a_3\delta_e(u_{b0}^2 - 3\mu_b\sigma_b)\sigma_e)\omega + k(\delta_e(u_{b0}^2 - 3\mu_b\sigma_b)\sigma_e^2 + 2s_3s_4\delta_b\mu_b)\omega \]

\[ s_{105} = -2k\omega^2(12k^3s_1s_2 + 4i\alpha_2k^2u_{b0}^2 + i\alpha_2u_{b0}\delta_e\sigma_e + k(s_3s_4\delta_b + u_{b0}\delta_e\sigma_e^2 + s_1(s_2 + 3s_2\delta_e\sigma_e)), \]

\[ s_{11} = \frac{s_{111} + s_{112}}{s_{90}}, \]

\[ s_{111} = -k(2s_3s_4(ku_{b0} - \omega)(12k^4 - \delta_e\sigma_e\omega^2 + k^2(1 + 3\delta_e\sigma_e - 4\omega^2)), \]

\[ s_{112} = \frac{i(k(1 + 12k^2 + 3\delta_e\sigma_e) - a_1(4k^2 + \delta_e\sigma_e)\omega^2 - ik\mu_b(-i\alpha_2k + 3k^2\delta_e\sigma_e^2 + 2k\omega^2 - \delta_e\sigma_e^2\omega^2))). \]

\[ s_{12} = \frac{s_{121} + s_{122} + s_{123} + s_{124}}{s_{90}}, \quad s_{124} = \omega^3 (k\delta_e\mu_b\sigma_e^2 + i\alpha_1(4k^2 + \delta_e\sigma_e)) \]

\[ s_{121} = -i\omega^2(4k^2 + 3\delta_e\sigma_e) - k\mu_b(a_2 - 2ik\omega^2s_2u_{b0} + 3ik\delta_e\sigma_e^2)), \]

\[ s_{122} = \frac{\omega^2((i\alpha_1u_{b0} + k(2s_1s_2u_{b0}\delta_e\sigma_e^2 + 2s_3s_4\delta_b + 3\mu_b(4k^2 + \delta_e\sigma_e)))}, \]

\[ s_{13} = \frac{s_{131} + s_{132} + s_{133} + s_{134}}{s_{90}}, \quad s_{131} = -2k\omega^2(s_1s_2 + s_3s_4\delta_b + u_{b0}\delta_e\sigma_e^2), \]

\[ s_{132} = k^2\omega^2(ia_2 + i\alpha_1\delta_b - 2k\omega^2u_{b0}(2s_1s_2 + s_3s_4\delta_b) + k\delta_e\sigma_e^2(u_{b0}^2 - 3 - 3\delta_b\mu_b)), \]

\[ s_{133} = k\omega^2(-i\alpha_2u_{b0} + k(3s_3s_4\delta_b + s_1s_2u_{b0} + 3s_1s_2\delta_b\sigma_b + 3\mu_b\delta_e\sigma_e^2)), \]

\[ s_{134} = k^2(3i\alpha_1\delta_b + 6s_3s_4u_{b0}\delta_b - i\alpha_2(u_{b0} + 3\mu_b\sigma_b) + 3k\delta_e(u_{b0}^2 - 3\mu_b\sigma_b)\sigma_e^2)), \]

\[ s_{14} = \frac{s_{141} + s_{142} + s_{143} + s_{144}}{s_{90}} - \frac{4k^2\sigma_e^2\omega^4}{s_{90}}, \]

\[ s_{141} = 2k\sigma_e(-(s_1s_2 - s_3s_4\delta_b + 4k^2u_{b0}\sigma_e)\omega^3, \]

\[ s_{142} = k\sigma_e^2(-i(a_2 + a_1\delta_b - 2k\omega^2u_{b0}(2s_1s_2 + s_3s_4\delta_b)) + k(4k^2(-3u_{b0}^2 - 3\mu_b\sigma_b) - 1 - \delta_b\mu_b)), \]

\[ s_{143} = k\omega^2\sigma_e(i\alpha_2u_{b0} + k(-3s_3s_4\delta_b + s_1s_2u_{b0} - 3\mu_b\sigma_b) + u_{b0}\sigma_e + 12k^2u_{b0}\sigma_e)), \]

\[ s_{144} = k^3\sigma_e((-3i\alpha_1\delta_b - 6s_3s_4u_{b0}\delta_b + i\alpha_2(u_{b0}^2 - 3\mu_b\sigma_b) + k(1 + 12k^2)u_{b0} - 3\mu_b(\delta_b + \sigma_b + 12k^2\sigma_b))\sigma_e)), \]

\[ s_{15} = \frac{s_{151} + s_{152}}{s_{150}}, \quad s_{16} = \frac{s_{161} + s_{162} + s_{163}}{s_{150}}, \quad s_{17} = \frac{s_{171} + s_{172}}{s_{150}}, \]

\[ s_{150} = (u_{b0} - V_g^2)^2 - \delta_e(V_g^2 - 3)((u_{b0} - V_g^2)^2 - 3\mu_b\sigma_b)\sigma_e + \mu_b((V_g^2 - 3)\delta_b - 3\sigma_b)), \]

\[ s_{151} = \delta_e(-s_4(s_4 + s_1(V_g - u_{b0}))) + s_3^2((u_{b0} - V_g^2 - 3\mu_b\sigma_b))^2, \]

\[ s_{152} = (3s_1^2 + s_2^2 + 2s_1s_2 V_g)^2(-\delta_b\mu_b + \delta_e((u_{b0} - V_g^2)^2 - 3\mu_b\sigma_b)\sigma_e)), \]

\[ s_{161} = 3s_1^2V_g\delta_b\mu_b - \delta_e((u_{b0} - V_g^2 - 3\mu_b\sigma_b)\sigma_e + V_g(-s_1^2\delta_b + s_2^3\delta_b(u_{b0} - V_g^2 - 3\mu_b\sigma_b))), \]

\[ s_{162} = V_g\delta_e((u_{b0} - V_g^2)^2 - 3\mu_b\sigma_b)\sigma_e^2 + s_2^2(\delta_b\mu_b - \delta_e((u_{b0} - V_g^2)^2 - 3\mu_b\sigma_b)\sigma_e)), \]

\[ s_{163} = s_3(s_4(u_{b0} - V_g)V_g\delta_b - 2s_2((u_{b0} - V_g^2)^2 - 3\mu_b(\delta_b + \sigma_b) - \delta_e((u_{b0} - V_g^2)^2 - 3\mu_b\sigma_b)\sigma_e)), \]

\[ s_{171} = -\mu_b(3s_1^2 + s_2^2 + 2s_1s_2 V_g^2) + \mu_b\delta_e\sigma_e^2(V_g^2 - 3)^2 + s_2^2(1 - \delta_e\sigma_e(V_g^2 - 3)), \]

\[ s_{172} = s_3s_4(u_{b0} - V_g)(-1 + \delta_e\sigma_e(V_g^2 - 3)) + s_3(u_{b0} - V_g^2 - 3\mu_b\sigma_b)(1 + \delta_e\sigma_e(V_g^2 - 3)). \]
Propagation of different kinds of non-linear ion-acoustic waves in Earth’s magnetosphere

\[ s_{18} = \frac{s_{181} + s_{182} + s_{183}}{s_{150}}, \quad s_{19} = \frac{s_{191} + s_{192} + s_{193}}{s_{150}}, \quad s_{20} = \frac{s_{201} + s_{202} + s_{203}}{s_{150}}, \]

\[ s_{181} = s_{4}(u_{b0} - V_g)(-1 + \delta_e\sigma_e(V_g^2 - 3)) + \mu_b(2s_1s_2V_g + 3s_1^2 + s_2^2)(u_{b0} - V_g)), \]

\[ s_{182} = \mu_b(s_1^2(3\sigma_b(1 - u_{b0} + V_g) - (V_g^2 - 3)) + 3s_1^2(-1 + u_{b0} - V_g)\delta_e\sigma_e\sigma_b(V_g^2 - 3)), \]

\[ s_{183} = \mu_b((u_{b0} + V_g)(V_g^2 - 3)\delta_e\sigma_e^2) + s_1s_4\mu_b(-\delta_b(V_g^2 - 3) + \sigma_b(3 - 3(V_g^2 - 3)\delta_e\sigma_e)), \]

\[ s_{191} = s_1(u_{b0} - V_g)(2s_2V_g(-u_{b0} + V_g) + s_4(-V_g^2 + 3\delta_b) + 6s_1s_2V_g\mu_b\sigma_b + 3s_1^2(3\mu_b\sigma_b - (u_{b0} - V_g^2)), \]

\[ s_{192} = s_2(-3\mu_b\sigma_b - (u_{b0} - V_g^2)^2 + (V_g^2 - 3)(-\delta_b(s_1^2 + s_3^2(3\mu_b\sigma_b - u_{b0} + V_g))), \]

\[ s_{193} = \delta_e\sigma_e^2(V_g^2 - 3)(u_{b0} - V_g^2 - 3\mu_b\sigma_b), \]

\[ s_{201} = -\sigma_e(s_1(u_{b0} - V_g)(2s_2(u_{b0} - V_g)V_g - s_4(V_g^2 - 3)) - 6s_1s_2V_g\mu_b\sigma_b), \]

\[ s_{202} = -\sigma_e((3s_1^2 + s_2^2)(u_{b0} - V_g^2 - 3\mu_b\sigma_b) + \delta_b(V_g^2 - 3)(s_1^2 + s_3^2(-u_{b0} + V_g + 3\mu_b\sigma_b))), \]

\[ s_{203} = \sigma^2(u_{b0}^2 - 2u_{b0}V_g + V_g^2(1 + \delta_b\mu_b) - 3\mu_b(\delta_b + \sigma_b)), \]

\[ a_1 = \frac{ik^2\mu_b^2(k^2(u_{b0}^2 + 3\mu_b\mu_b\sigma_b) - 2ku_{b0}\omega + \omega^2)}{(k^2(u_{b0}^2 - 3\mu_b\mu_b\sigma_b) - 2ku_{b0}\omega + \omega^2)^2}, \quad a_2 = \frac{ik^3(3k^2 + \omega^2)}{(-3k^2 + \omega^2)^2}. \]

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Declarations

Competing interests The authors declare no competing interests.

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