Mixed-Spin Ladders and Plaquette Spin Chains

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We investigate low-energy properties of a generalized spin ladder model with both of the spin alteration and the bond alteration, which allows us to systematically study not only ladder systems but also alternating spin chains. By exploiting non-linear σ model techniques we study the model with particular emphasis on the competition between gapful and gapless states. Our approach turns out to provide a more consistent semi-classical description of alternating spin chains than that in the previous work. We also study a closely related model, i.e., a spin chain with plaquette structure, and show that frustration causes little effect on its low-energy properties so far as the strength of frustration is weaker than a certain critical value.

I. INTRODUCTION

Recent extensive experimental and theoretical investigations on spin chains and ladders have been providing a variety of interesting topics. A typical example is a spin ladder system with and without impurities. It has been revealed that the impurity effects give rise to drastic changes in the ladder system, e.g., a gapful spin state could be driven to a gapless state, and consequently lead to a magnetic ordered state. Another example is a spin-Peierls system such as CuGeO$_3$, for which the effects of frustration and bond alternation, as well as impurities, play a crucial role. Also, a mixed-spin chain with alternating array of two kinds of spins have stimulated intensive experimental and theoretical studies. The alternating spin chain may be regarded as a high concentration limit of magnetic impurities in ordinary spin chains, so that this problem has close relationship to the above-mentioned spin systems with impurities. A common feature in these problems is how the gapful and gapless states compete with each other, providing a variety of interesting phenomena. In this connection, a spin system with plaquette structure is also interesting, for which one can indeed observe how the spin gap is generated according to a topological nature of the system.

Motivated by the above stimulating topics, we investigate in this paper a quantum spin ladder model with both of the spin alternation and the bond alternation. This system may be referred to as a "mixed-spin ladder". What is remarkable is that this model allows us to systematically study spin ladders, alternating spin chains, and spin chains with periodic array of magnetic impurities. We take a non-linear σ model (NLσM) approach to describe low-energy properties of the model, and focus particular attention on the competition between gapful and gapless states. Also, we study a closely related model, i.e., a spin chain with plaquette structure, which can be naturally constructed from a particular mixed-spin ladder model. This model is also instructive to discuss the gap formation in quantum spin systems.

The paper is organized as follows. In §2, we first introduce the model system, and map it to the NLσM by taking the continuum limit. We discuss in §3 the properties of two- and three-leg ladder systems with particular attention to the relationship to alternating spin chains, and then move to many-leg ladder systems to discuss spin chains with a periodic array of magnetic impurities. In §4 we next investigate a plaquette spin chain with frustration, and show that frustration little affects low-energy properties when its strength is weaker than a certain critical value, being consistent with recent numerical findings. Brief summary is given in the last section.

II. MAPPING TO THE NON-LINEAR σ MODEL

We consider a $n_l$-leg mixed-spin ladder system with the bond alternation, which is described by the following Hamiltonian

$$H = \sum_{j=1}^{N} \left\{ \sum_{a=1}^{n_l} J_a \left[ 1 + (-1)^j \gamma_a \right] S_{a,j} \cdot S_{a,j+1} \right\} + \sum_{a=1}^{n_l-1} J' S_{a,j} \cdot S_{a+1,j} \right\} , \tag{2.1}$$

where $N$ is the number of lattice sites for each chain, and $S_{a,j}$ is the spin operator at the $j$-th site of the $a$-th chain. Here $\gamma_a$ is the bond-alternation parameter, and $J_a$ and $J'$ denote the intralag and interlag coupling constants, respectively. Throughout this paper we assume that the spin have different values for different chains, but the same value $s_0$ in the same chain. In Fig.1, we have drawn the ladder model schematically. Since we deal with the case of $J_a, J' > 0$, and $-1 \leq \gamma_a \leq 1$, the classical minimum of the Hamiltonian eq. (2.1) is realized in the Néel state. What is most distinct from ordinary spin ladders is that the present system includes not only the bond alternation, but also the spin alternation. This generalization
thus allows us to study interesting spin chains in some limiting cases. For example, setting \( J_a = 0 \left(1 < a < n_1 \right) \),
\( \gamma_1 = 1 \), and \( \gamma_{n_1} = -1 \), the model eq. (2.1) reproduces
an alternating-spin chain with the periodic arrangement of spins \( s_1 \circ s_2 \circ ... \circ s_{n_1} \circ s_{n_1} \circ ... \circ s_2 \circ s_1 \). Moreover, if we further take a large \( n_1 \) limit for the ladder composed of a chain with spin-\( s_2 \) and all other chains with spin-\( s_1 \),
the Hamiltonian can be reduced to the spin-\( s_1 \) chain with a periodic array of dilute spin-\( s_2 \) impurities. [10] In this way, by choosing appropriate parameters for the number of legs \( n_1 \), the spin \( s_a \) and strength of bond-alternation, the mixed-spin ladder model naturally interpolates various interesting spin systems which have been intensively studied recently. We note that a similar but different ladder system with spin alternation has been discussed for the two-leg case. [11]

In the following we shall map the system to the NL\( \sigma M \) and discuss its low-energy properties. For this end, it is convenient to exploit techniques in coherent-state path integral formalism. Since the detail of the formulation can be found in standard text books [10] and also in recent papers [13,17,19], we briefly summarize how to apply techniques to the present model. The partition function in this system is given by

\[
Z = \int \mathcal{D}\Omega \exp \left\{ -\int_0^\beta \! d\tau H(\tau) \right\} ,
\]
(2.2)

where

\[
\omega[\Omega] = \int_0^\beta \! d\tau \varphi(1 - \cos \theta), \tag{2.3}
\]

\[
H(\tau) = \sum_{j=1}^N \left\{ \sum_{a=1}^{n_1} J_a s_a^2 \left[ 1 + (-1)^j \gamma_a \right] \Omega_a(j) \cdot \Omega_a(j+1) + \sum_{a=1}^{n_1-1} J' s_a s_{a+1} \Omega_a(j) \cdot \Omega_{a+1}(j) \right\} . \tag{2.4}
\]

In the above expression we have introduced the unit vector \( \Omega_a(j) \) by \( S_{a,j} = s_a \Omega_a(j) \), where \( \Omega = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta) \). The term \( \omega[\Omega] \) is the Berry phase which corresponds to the solid angle enclosed by the vector \( \Omega \) on unit sphere. In our semi-classical approach, it is assumed that the spin wave analysis can correctly describe low-energy modes at wave vectors near 0 and near the ordering wave vector \( \pi \). Then, \( \Omega_a \) may be written as

\[
\Omega_a(j) = (-1)^{a+j} \phi_j \left[ 1 - l_a^2(j) \right]^{1/2} + l_a(j). \tag{2.5}
\]

The staggered field \( \phi_j \) in eq. (2.5), which corresponds to the component around \( \pi \), is slowly varying field on the scale of lattice spacing. We have assumed here that the field \( \phi_j \) does not depend on the index \( a \), which implies that the spin correlation along rungs is sufficiently strong so as to develop the coherence in this direction; i.e., \( \xi \gg n_1 a_0 \), where \( \xi \) is the staggered spin correlation length and \( a_0 \) is the lattice constant. The other field, \( l_a(j) \), is small fluctuation field around \( k = 0 \): \(|l| \ll 1 \). As is seen below, the introduction of the \( a \)-dependence for the field \( l \) improves our semi-classical approximation. The constraint \( \Omega_a^2(j) = 1 \) is now replaced by \( \phi_j^2 = 1 \) and \( \phi_j \cdot l_a(j) = 0 \). Substituting eq. (2.3) into eq. (2.2) and making the expansion up to quadratic order in \( \Omega, \phi \), and \( l \), we obtain the action in the continuum limit

\[
S_H = \frac{J_1 s_1^2}{2} \int dx \int_0^\beta \! d\tau \left\{ K \phi'^2 + \sum_{a,b} l_a l_{a+b} + \sum_a g_a l_a \cdot \phi' \right\} , \tag{2.6}
\]

where the lattice constant is taken as unity. For later convenience, we have introduced the parameters \( K, g \), and \( L \) which are given in terms of microscopic parameters in our model. Their explicit forms will be given in the following sections. We next take the continuum limit of the Berry phase term, which results in an ordinary form,

\[
S_B = i \sum_{a,j} s_a \omega[\Omega_a(j)]
= -i \theta' Q - is_1 \int dx \int_0^\beta \! d\tau \sum_a f_a l_a \cdot \phi \times \dot{\phi} , \tag{2.7}
\]

where \( \theta' = 2\pi \sum_a (-1)^a s_a, f^i = (1, \alpha_2, \alpha_3, ...), \) and \( \alpha_m = s_m/s_1 \). The value \( Q \) in eq. (2.7) is the winding number defined by \( Q = \frac{1}{8} \int dx \int d\phi \cdot (\phi' \times \dot{\phi}) \). Integrating eqs. (2.6) and (2.7) over the fluctuation fields \( l \), we thus end up with the NL\( \sigma M \) with topological term,

\[
Z = \int \mathcal{D}\phi \exp \left\{ i \theta Q - \frac{1}{2 g} \int dx \int_0^\beta \! d\tau \left( \psi_\phi \phi'^2 + \frac{1}{\psi^2} \phi^2 \right) \right\} , \tag{2.8}
\]

where

\[
\theta = \theta' + 2\pi s_1 g' L^{-1} f ,
\]

\[
g = \frac{1}{s_1} \left\{ \left( K - \frac{1}{4} g' L^{-1} g \right) f' L^{-1} f \right\}^{-1/2} ,
\]

\[
\psi_\phi = J_1 s_1 \left\{ \frac{K - \frac{1}{2} g' L^{-1} g}{f' L^{-1} f} \right\}^{1/2} . \tag{2.9}
\]

This completes the basic formulation in our semi-classical approach. Note that the topological term in eq. (2.8) is quite essential to classify the behavior of the system; it is known that the system with \( \theta = \pi(mod 2\pi) \) is gapless, whereas the system with \( \theta \neq \pi(mod 2\pi) \) is gapful. Indeed in our analysis, this topological term plays a central role to specify whether the system is gapful or gapless. In the following sections, we deal with some interesting examples of the model, and discuss their low-energy properties based on the NL\( \sigma M \) approach.
III. SPIN LADDERS AND ALTERNATING SPIN CHAINS

A. Two-leg ladders

Let us start with a two-leg ladder system. For \( n_1 = 2 \), we have the parameters for the NLσM, \( K = 1 + \alpha_2^2 R_2 \), \( g' = (4 \gamma_1, -4 \alpha_2^3 R_2 \gamma_2) \), \( f' = (1, \alpha_2) \), and

\[
L = \begin{pmatrix}
4 + \alpha_2 R' & \alpha_2 R' \\
\alpha_2 R' & 4 \alpha_2 R_2 + \alpha_2 R'
\end{pmatrix},
\]

(3.1)

where \( R_2 = J_2/J_1 \) and \( R' = J'/J_1 \). From these expressions, we obtain

\[
\theta = 2 \pi (s_1 - s_2) + \frac{2 \pi s_1}{R' + 4 \alpha_2 R_2 + \alpha_2 R' R_2} \times \{ R'(1 - \alpha_2)(\gamma_1 + \alpha_2^2 \gamma_2 R_2) + 4 \alpha_2 R_2 (\gamma_1 - \alpha_2 \gamma_2) \}.
\]

(3.2)

In the case of \( \gamma_1 = \gamma_2 = 0 \), the system is reduced to a ladder system without bond alternation, for which the topological term is given by \( \theta = 2 \pi (s_1 - s_2) \). [11] Therefore, if either of the spin in the two chains has a half-integer spin, the system becomes gapless, otherwise the system is gapful. If we include the bond alternation, the system is gapful in general.

Since essential properties for ordinary two-leg ladder systems have been already clarified, we focus our attention on some characteristic cases, and confirm that our results consistently reproduce those derived previously. [10,11] Let us first consider the uniform spin case \( s_1 = s_2 = s \), with a special bond-alternation \( \gamma_1 = -\gamma_2 = \gamma \), \( R' = R \), and \( R_2 = 1 \). Then the system is reduced to an ordinary spin chain with the bond alternation (its structure is similar to a "snake chain"). In this case, we obtain the topological term with \( \theta = 8 \pi s \gamma / (2 + R) \), which agrees with those derived by Delgado et al. [18]

Another interesting limit is the case of \( \gamma_1 = \gamma_2 = 1 \), \( R' = 2 \), \( R_2 = 1 \), and \( J \rightarrow J/2 \). In this case also, the system is reduced to a snake chain, but it becomes a mixed-spin chain with alternating array of spins, \( s_1 \circ s_1 \circ s_2 \circ s_2 \circ s_1 \circ s_1 \circ s_2 \circ s_2 \circ \ldots \). Here we have to change the lattice constant \( 1 \rightarrow 2 \) to obtain the straight chain from the snake chain. Defining the effective spin \( s' = 2 s_1 s_2 / (s_1 + s_2) \), we have \( \theta = 2 \pi s' \), \( g = 2 / s' \), and \( v_s = 2 J s' \). Therefore, the system is gapful in general except for special cases for which \( \theta = \pi \). It is seen that these results agree with those obtained directly from the single chain model. [10] The coincidence is apparent since in the two-leg ladder system the way how the fluctuation \( L_\alpha (j) \) is introduced is the same as in the previous one. [10]

In the following subsection, however, we shall see that our approach based on a generalized ladder model indeed improves a semi-classical description of alternating spin chains.

B. Three-leg ladders

We now turn to a three-leg ladder system. For \( n_1 = 3 \) the parameters for the NLσM read \( K = 1 + \alpha_2^2 R_2 + \alpha_3^2 R_3 \), \( \theta' = -2 \pi (s_1 - s_2 + s_3) \), \( g' = (4 \gamma_1, -4 \alpha_3^2 R_2 \gamma_2, 4 \alpha_3^2 R_3 \gamma_3) \), \( f' = (1, \alpha_2, \alpha_3) \), and

\[
L = \begin{pmatrix}
4 + \alpha_2 R' & \alpha_2 R' & 0 \\
\alpha_2 R' & 4 \alpha_2 R_2 + \alpha_2 (1 + \alpha_3) R' & \alpha_2 \alpha_3 R' \\
0 & \alpha_2 \alpha_3 R' & 4 \alpha_2 R_3 + \alpha_2 \alpha_3 R'
\end{pmatrix},
\]

(3.3)

where \( R_3 = J_3/J_1 \).

Although our three-leg ladder model includes various cases according to the choice of the model parameters, we again focus our discussions on some interesting cases. We start with a simple case for which \( \gamma_1 = \gamma_2 = \gamma_3 = \gamma \): three chains with different spins are simply connected. In this case, we can typically see how the topological nature of mixed spins shows up in low-energy physics. The corresponding topological term is reduced to the form,

\[
\theta = 2 \pi (s_1 - s_2 + s_3) / (\gamma - 1).
\]

(3.4)

An ordinary three-leg ladder is given by \( s_1 = s_2 = s_3 = 1/2 \), for which we have \( \theta = \pi \) (mod \( 2 \pi \)) for \( \gamma = 0 \), and hence the system becomes gapless, as is well known. For general \( \gamma \), \( \theta \) takes values different from \( \pi \) (mod \( 2 \pi \)), resulting in the gapful phase.

A remarkable point we wish to mention here is that there is a particular combination of spins, \( s_1 + s_3 = s_2 \), for which the bond-alternation parameter \( \gamma \) disappears from the expression of the topological term. For example, for the simplest case with \( s_1 = s_3 = 1/2 \), \( s_2 = 1 \), which would be a possible candidate for experimental realization, we always have \( \theta = 0 \) irrespective of the bond alternation. This is the remarkable result for which the topological nature of spins essentially determines whether the system is gapful or gapless, hiding the effects of the bond alternation. This implies that the Berry phase arising from the bond-alternation cancels each other due to the simple spin configuration of the system.

The second example we consider here is again a simple model with \( \gamma_1 = -\gamma_3 = 1 \), \( \gamma_2 = \gamma \), \( R_3 = 1 \), \( \alpha_2 = \alpha \), and \( \alpha_3 = 1 \), which indeed exhibits the interplay of the spin alternation and the bond alternation. In particular, in this model, the three-leg ladder system is reduced to a kind of plaquette chain (Fig.2, see also the next section), for which a square plaquette is connected to its next plaquette sharing one of its corner alternately. We shall see that this system still possesses nontrivial interesting cases including the alternating spin chain. The topological term \( \theta \) has the form

\[
\theta = 2 \pi s_2 - \frac{2 \pi s_1 \alpha_2^2 R_2 (4 - 2 R' + \alpha R')}{{2R'}^2 + 4 \alpha_2 R_2 + \alpha_2^2 R_2 R_2}.
\]

(3.5)

Let us now discuss whether the system is gapless or gapful according to the topological term. One can easily see
from [3,5] that $\theta$ takes values different from $\pi$ in general, resulting in a gapful phase. We find three possibilities to let $\theta$ be equal to $2\pi s_2$, for which the system can be gapless for a half-integer $s_2$: (i) $R_2 = 0$, (ii) $\gamma = 0$, and (iii) $R' = 4/(2 - \alpha)$.

In the case of (i) our model is further reduced to the alternating spin chain, $s_1 \circ s_1 \circ s_2 \circ s_1 \circ s_1 \circ s_2 \ldots$ with singlet ground state. Setting $R' = 2$, $J_1 \rightarrow J_1/2$, and the lattice constant $1 \rightarrow 3$, we obtain the parameters

\[
g = \frac{2}{s_1} \frac{2 + \alpha}{\sqrt{\alpha(8 + \alpha^2)}}
\]
\[
v_s = 6J_1s_1\sqrt{\frac{\alpha}{8 + \alpha^2}}.
\]

We have numerically checked that the velocity $v_s$ coincides with the velocity $v_{sw}$ which is estimated by the spin wave analysis. In contrast to this consistent result, when the NLσM techniques are naively applied to a single chain model, one may encounter a pathological result for which $v_s$ and $v_{sw}$ have different values. This implies that the present approach improves the previous one, and provides a NLσM mapping consistent with the spin wave analysis.

The results in the case (ii) are rather simple, in which the system is decomposed into two parts: One is the dimer part which has singlet state of two $s_1$ spins, and the other is the spin-$s_2$ chain. The critical behavior is thus determined by the $s_2$-spin chain. We have to note, however, that in this case our approximation based on a NLσM becomes worse, because in our approach the spin coherence is assumed to be well developed among chains, whereas in this limiting case such coherence is not developed. So, the present approach may describe only the qualitative properties in this limiting case. Lastly, we point out a novel behavior observed in the case (iii). It may be rather surprising that the topological term in this case is controlled only by the spin $s_2$ in spite of the existence of the bond-alternation $\gamma$, which is quite contrasted to the behavior observed in ordinary spin chains with bond-alternation $\gamma$, where the topological term should depend on $\gamma$. This phenomenon may come from the interplay between the bond alternation and the spin alternation. All the above three examples may give characteristic phenomena inherent in ladder systems with the spin alternation.

C. Spin chains with magnetic impurities

We have seen so far that the mixed-ladder model allows us to naturally interpolate the physics of ladders and that of spin chains. Stimulated by this, we further extend the analysis of the case (i) to a larger $n_l$ case to discuss a spin chain with magnetic impurities. Although the spin chain with periodic impurities seems a little bit peculiar at first glance, it still involves some essential properties expected for ordinary impurities, as claimed in refs. [10,20].

We start with the multiple ladder Hamiltonian (2.1) with $n_l = 2m + 1$ which has the spin $s_{a} = s_1$ ($s_2$) for $a \neq m + 1$ ($a = m + 1$). Setting $J_a = 0$ ($1 < a < n_l$), $2J_1 = 2J_{n_l} = J' = J$, $\gamma_1 = 1$, $\gamma_{n_l} = -1$, and redefining the lattice constant $1 \rightarrow n_l$ in this model, we arrive at the spin-$s_1$ chain with periodic array of spin-$s_2$ impurities. The case (i) in the previous subsection corresponds to $m = 1$. We analyze this model semiclassically as done before. As a result, we arrive at the NLσM with topological term $\theta = 2\pi s_2$. Therefore, we can say that the system may be gapful (gapless), when $s_2$ is integer (half-integer), as is naturally seen from the topological nature of the system. This implies that well-separated $s_2$ spins correlate with each other making a narrow gapless band, even if the background spin $s_1$ forms the Haldane gap.

As mentioned above, our approach yields qualitatively correct results even for the case of dilute impurities. Now, the question is how consistent the present NLσM approach is. For this purpose we have numerically checked again that the velocity $v_s$ in the $m \rightarrow \infty$ limit agrees with $v_{sw}$ deduced from the spin-wave spectrum, although the ways to obtain the spin velocities seem quite different in two formulations. This implies that our treatment based on the NLσM is, at least, a consistent semiclassical approach even for the dilute-impurity case. As mentioned in the previous subsection, this indeed improves the previous results obtained directly from the spin chain model. The reason why our treatment based on the ladder model has such advantage is as follows: In the ladder system the correlation effects between two adjacent $s_2$ spins in the same chain can be naturally taken into account rather well. These two $s_2$ spins are then reduced to two well-separated magnetic impurities in the spin chain model, for which the interference between them is quite essential for low-energy properties. If we start the single chain model, it is not easy to incorporate the correlation between largely separated spins. We would hence say that our approach based on the ladder system may provide a more efficient framework to incorporate such interference among magnetic impurities. It should be noted, however, that in order to discuss low-energy properties more quantitatively, it is necessary to integrate out the gapful degrees of freedom properly (spin sectors). This point should be further improved by taking into account the quantum fluctuations more precisely. We believe that the method proposed recently should provide an efficient way to resolve this problem.

IV. PLAQUETTE SPIN CHAINS

We have discussed so far how the gapful and gapless states compete with each other reflecting the topological nature of the system, the interaction strength, etc. In this connection, we now wish to discuss a closely related model, i.e., a spin chain model with plaquette structure,
which is also instructive to discuss how the spin gap formation occurs in quantum spin systems. As far as the spin gap formation is concerned, our system may be related to two-dimensional plaquette spin systems studied extensively. We show that the universality class of the plaquette chain is the same as that of a mixed-spin ladder.

The plaquette spin chain can be naturally constructed from a three-leg ladder system. To see the way clearly, we start by observing how a $n_l$-leg ladder system with uniform spin $s$, for which exchange coupling along the rungs is ferromagnetic (i.e. we choose $J' < 0$ and $J > 0$), can be unified into an effective single chain. As is well known, low-energy properties of the system is identical with those for the single-spin-$n_ls$ chain. This can be explicitly shown by the NLσM approach, and we arrive at the NLσM with the effective intrachain coupling $J/n_l$. The resulting velocity $v_s = 2Js$ again turns out to be the same as $v_{sw}$ obtained by the spin-wave analysis. It is to be noted that the value of $J'$ does not appear in the parameters in the effective NLσM within the present approximation.

Based on the above observation, we now consider a specific spin ladder system composed of three chains with spins $s$, $2s$, and $s$, respectively (see Fig.3). Applying the idea outlined above, let us decompose the middle spin-$2s$ chain into two spin-$s$ chains with ferromagnetic interchain couplings. The effective four-chain system is now considered in the continuum limit. In order to map the system to the NLσM, we introduce the sigma-model field as

$$\Omega_a(j) = (-1)^{n_a+j} \phi_j [1 - l^2(j)]^{-1/2} + l_a(j) \quad (4.1)$$

where $n_a = 1(0)$ for $a = 1, 4(a = 2, 3)$. The resulting NLσM has following parameters

$$K = 2 + 4R_2, \quad (4.2)$$

$$L = \begin{pmatrix}
4 + 2R' & R' & R' & 0 \\
R' & 8R_2 + 2R' - R_3 & R_3 & R' \\
R' & R_3 & 8R_2 + 2R' - R_3 & R' \\
0 & R' & R' & 4 + 2R'
\end{pmatrix} \quad (4.3)$$

$$g = 4\gamma \begin{pmatrix} 1 \\
0 \\
0 \\
-1 \end{pmatrix}, \quad f = \begin{pmatrix} 1 \\
1 \\
1 \\
1 \end{pmatrix}. \quad (4.4)$$

Now, setting $\gamma = 1$, $R_2 = 0$, and redefining the lattice constant $1 \to 3$ and the coupling constant $J_1 \to J_1/2$, we end up with the plaquette chain schematically shown in Fig.3. The corresponding NLσM has the $\theta = 0$, $g = \sqrt{1 + R'/s}$, and $v_s = 3J's/\sqrt{1 + R'}$. In this way, low-energy properties of our plaquette spin chain is naturally related to those of a mixed-spin ladder discussed so far.

It is to be noticed that $\theta$ is always zero, which implies that the plaquette spin system is in the gapful phase irrespective of the model parameters. Another remarkable point is that $R_3$ does not enter in these parameters whether it is positive or negative. When $R_3$ is negative, namely, in the case of ferromagnetic coupling $J_3$, the resulting gapful phase is naturally understood from the above discussions for the three-leg chain with spins $s$, $2s$, and $s$. Namely, our plaquette spin chain belongs to the same universality class of the above three-leg ladder system with spins $s$, $2s$ and $s$.

On the other hand, in the case of antiferromagnetic coupling $R_3 > 0$, the result is nontrivial, because the system is now subject to frustration in the plaquette. By using the case of $R_3 > 0$, we in turn have an opportunity to discuss the effects of frustration on our plaquette chain. For this purpose, we first note that the above gapful phase should be changed to another phase when the strength of frustration increases. Then, the question is to what extent this gapful phase is stable against the frustration. To check this point, we here recall that the spin wave spectrum should have a linear dispersion in order for the system to be mapped to the NLσM. Keeping this fact in mind, we reexamine the spin wave spectrum for the present system. The behavior of the spin wave dispersion for various values of $R_3$ is shown in Fig.4. There are four modes, one of which indeed shows a linear dispersion in the low-energy regime. Let us now focus on the dispersionless mode, which is decoupled from other collective excitations and depends only on the coupling $J_3$. It may contain key information about the stability of the system. Increasing the coupling $R_3$ from the ferromagnetic to anti-ferromagnetic regime, this mode goes downward uniformly, and finally reaches the $p$ axis when $R_3 = R'$. Beyond this critical value, $R'$, the above spin wave excitations are not well-defined. Therefore we find that our NLσM analysis of the plaquette spin chain holds valid only in the region $R_3 < R'$ where the mapping to the NLσM is allowed.

A remarkable point we wish to stress is that the lowest spin-wave mode is not influenced by the change of the coupling strength, implying that $v_{sw}$ does not change. This is consistent with the above results for the NLσM that $R_3$ does not enter in the model parameters. Summarizing the above facts, we come to the following conclusion: even if the coupling $R_3$ in the plaquette is increased, the effect of frustration little affects the low-energy properties of the model in the region for $R_3 < R'$, at which the system may undergo a phase transition. We find this remarkable result to be consistent with the recent numerical studies on the plaquette spin chain. Although in the present approach, we cannot say what happens beyond the critical $R'$, the numerical study has predicted that the system should undergo a first-order phase transition to enter another gapful phase.
V. SUMMARY

We have investigated low-energy properties of a spin ladder system with both of the spin alternation and the bond alternation, which has been shown to involve various interesting spin systems. In particular, we have discussed how the gapless and gapful states compete with each other according to the topological nature of the system. Starting with the spin-wave analysis on the spin ladder system, and introducing fluctuation fields, we have mapped the system to the NLσM. By using the mixed-spin ladder systems composed of multiple chains, we have discussed characteristic properties of the alternating spin chain as well as the spin chain with magnetic impurities. We have found that our approach in the present study provide a more consistent semi-classical description for such spin chains than the previous work based on the single-chain model.

We have also studied the plaquette spin chain, and shown that the system belongs to the same universality class of the three-leg ladder system composed of chains with spins s, 2s and s. The effects of frustration on the plaquette chain has then been investigated. It has been shown that the frustration little affects low-energy properties when its strength is weaker than a certain critical value.

In this paper, we have restricted our discussions to several specific spin models to demonstrate how the spin alternation and the bond alternation affects the low-energy properties. By extending and improving our treatments, we think that various interesting quantum spin systems can be described systematically in the same framework of the mixed-spin ladder system, which should be done in the future study.

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[22] Note that the calculated spin velocity for $m \to \infty$ is less than that expected for the pure case; $v_s < 2J_s$. This result implies that our approach indeed incorporate interference among impurities even for the dilute limit.

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FIG. 1. Mixed-spin ladder with the bond alternation.

FIG. 2. A special model for the three-leg ladder system.
FIG. 3. (a) Spin ladder composed of three chains with spins $s, 2s$, and $s$. (b) Corresponding four chain system (See the text). (c) Spin chain with plaquette structure. In the system (b) by choosing $J_2 = 0$ and $\gamma = 1$, and redefining the coupling constant $J_1 \rightarrow J_1/2$, we have the plaquette spin chain (c).

FIG. 4. Spin-wave spectrum as functions of momentum $p$ for $J_3 = -0.4(a), 0.8(b)$, and $1.0(c)$ with other parameters being fixed as $s = 1/2, J = J' = 1, \gamma = 1$, and $J_2 = 0$. 
Fig. 1

\[ J_1(1+\gamma_1) \quad J_1(1-\gamma_1) \]

\[ J_2(1+\gamma_2) \quad J_2(1-\gamma_2) \]

\[ J_3(1+\gamma_3) \quad J_3(1-\gamma_3) \]

Fig. 2

\[ 2J_1 \]

\[ J_2(1+\gamma_2) \quad J_2(1-\gamma_2) \]

\[ J_1(1+\gamma_1) \quad J_1(1-\gamma_1) \]
