A Cut-free Sequent Calculus for Bi-Intuitionistic Logic: Extended Version

Linda Buisman and Rajeev Goré

1 The Australian National University
   Canberra ACT 0200, Australia

2 Logic and Computation Programme
   Canberra Research Laboratory, NICTA*, Australia
   {Linda.Buisman|Rajeev.Gore}@anu.edu.au

Abstract. Bi-intuitionistic logic is the extension of intuitionistic logic with a connective dual to implication. Bi-intuitionistic logic was introduced by Rauszer as a Hilbert calculus with algebraic and Kripke semantics. But her subsequent “cut-free” sequent calculus for BiInt has recently been shown by Uustalu to fail cut-elimination. We present a new cut-free sequent calculus for BiInt, and prove it sound and complete with respect to its Kripke semantics. Ensuring completeness is complicated by the interaction between implication and its dual, similarly to future and past modalities in tense logic. Our calculus handles this interaction using extended sequents which pass information from premises to conclusions using variables instantiated at the leaves of failed derivation trees. Our simple termination argument allows our calculus to be used for automated deduction, although this is not its main purpose.

1 Introduction

Propositional intuitionistic logic (Int) has connectives $\rightarrow$, $\land$, $\lor$ and $\neg$, with $\neg \varphi$ often defined as $\neg \varphi := \varphi \rightarrow \bot$. Int has a well-known Kripke semantics, where a possible world $w$ makes $\varphi \rightarrow \psi$ true if every successor $v$ that makes $\varphi$ true also makes $\psi$ true. Int also has an algebraic semantics in terms of Heyting algebras, and there is a well-known embedding from Int into the classical modal logic S4. Int is constructive in that it rejects the Law of Excluded Middle: that is, $\varphi \lor \neg \varphi$ is not a theorem of Int.

Propositional dual intuitionistic logic (DualInt) has connectives $\leftarrow$, $\land$, $\lor$ and $\sim$, with $\sim \varphi$ often defined as $\sim \varphi := \top \leftarrow \varphi$. DualInt also has Kripke semantics, where a possible world $w$ makes $\varphi \leftarrow \psi$ true if there exists a predecessor $v$ where $\varphi$ holds, but $\psi$ does not hold: that is, $\varphi$ excludes $\psi$. Thus, the $\leftarrow$ connective of DualInt is dual to implication in Int. DualInt also has algebraic

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semantics in terms of Brouwer algebras [13]. There is a less well-known embedding from DualInt into S4. DualInt is para-consistent in that it rejects the Law of Non-contradiction: that is, $\varphi \land \neg \varphi$ is DualInt-satisfiable. Various names have been used for $\neg \neg :$ coimplication [24, 23], subtraction [2, 3], pseudo-difference [16], explication [15]. We refer to it as exclusion.

Bi-intuitionistic logic (BiInt), also known as subtractive logic and Heyting-Brouwer logic, is the union of Int and DualInt, and it is a conservative extension of both. BiInt was first studied by Rauszer [15, 16]. BiInt is an interesting logic to study, since it combines the constructive aspects of Int with the para-consistency of DualInt. While every Int-theorem is also a BiInt-theorem, adding DualInt connectives introduces a non-constructive aspect to the logic – the disjunction property does not hold for BiInt formulae if they contain $\neg \neg$.

Note that BiInt differs from intuitionistic logic with constructive negation, also known as constructible falsity [14], where the disjunction property does hold.

While the proof theory of Int and DualInt separately has been studied extensively and there are many cut-free sequent systems for Int (for example, [8, 6, 5]) and DualInt (for example, [20, 4]), the case for BiInt is less satisfactory. Although Rauszer presented a sequent calculus for BiInt in [15] and “proved” it cut-free, Uustalu has recently given a counter-example [21] to her cut-elimination theorem: the formula $p \rightarrow (q \lor (r \rightarrow ((p \neg \neg q) \land r)))$ is BiInt-valid, but cannot be derived in Rauszer’s calculus without the cut rule. Similarly, Uustalu’s counterexample shows that Crolard’s sequent calculus [2] for BiInt is not cut-free. Uustalu’s counterexample fails in both Rauszer’s and Crolard’s calculi because they limit certain sequent rules to singleton succedents or antecedents in the conclusion, and the rules do not capture the interaction between implication and exclusion.

Uustalu and Pinto have also given a cut-free sequent-calculus for BiInt in [23]. Since only the abstract of this work has been published so far, we have not been able to examine their sequent rules, or verify their proofs. According to the abstract [23] and personal communication with Uustalu [22], his calculus uses labelled formulae, thereby utilising some semantic aspects, such as explicit worlds and accessibility, directly in the rules. Hence a traditional cut-free sequent calculus for BiInt is still an open problem.

We present a new purely syntactic cut-free sequent calculus for BiInt. We avoid Rauszer’s and Crolard’s restrictions on the antecedents and succedents for certain rules by basing our rules on Dragalin’s GHPC [5] which allows multiple formulae on both sides of sequents. To maintain intuitionistic soundness, we restrict the premise of the implication-right rule to a singleton in the succedent. Dually, the premise of our exclusion-left rule is restricted to a singleton in the antecedent. But using Dragalin’s calculus and its dual does not give us BiInt completeness. We therefore follow Schwendimann [17], and use sequents which pass relevant information from premises to conclusions using variables instantiated at the leaves of failed derivation trees. We then recompute parts of our derivation trees using the new information, similarly to the restart technique of [11]. Our calculus thus uses a purely syntactic addition to traditional sequents,
rather than resorting to a semantic mechanism such as labels. Our termination argument also relies on two new rules from Švejdar [18].

If we were interested only in decision procedures, we could obtain a decision procedure for BiInt by embedding it into the tense logic KtS 4 [24], and using tableaux for description logics with inverse roles [11]. However, an embedding into KtS 4 provides no proof-theoretic insights into BiInt itself. Moreover, the restart technique of Horrocks et al. [11] involves non-deterministic expansion of disjunctions, which is complicated by inverse roles. Their actual implementation avoids this non-determinism by keeping a global view of the whole counter-model under construction. In contrast, we handle this non-determinism by syntactically encoding it using variables and extended formulae, neither of which have a semantic content. Our purely syntactic approach is preferable for proof-theoretic reasons, since models are never explicitly involved in the proof system: see Remark 3.

The rest of the paper is organized as follows. In Section 2, we define the syntax and semantics of BiInt. In Section 3, we introduce our sequent calculus GBiInt and give an example derivation of Uustalu’s interaction formula. We prove the soundness and completeness of GBiInt in Sections 4 and 5 respectively. In Section 6, we outline further work.

## 2 Syntax and Semantics of BiInt

In this section we introduce the syntax and semantics of BiInt.

**Definition 1 (Syntax).** The formulae of BiInt are defined as:

\[
p ::= \top | \bot | p_0 | p_1 | \cdots \tag{2.1}
\]

\[
\varphi ::= p | \neg \varphi | \varphi \land \varphi | \varphi \lor \varphi | \varphi \to \varphi | \varphi \prec \varphi | \neg \varphi | \neg \varphi \tag{2.2}
\]

We refer to the set of atoms as Atoms, and we refer to the set of BiInt formulae as Fml.

The connectives \(\neg\) and \(\to\) are those of intuitionistic logic, and the connectives \(\sim\) and \(\prec\) are those of dual intuitionistic logic. The connectives \(\lor\) and \(\land\) are from both.

**Definition 2 (Length).** The length of a BiInt formula \(\chi\) is defined as:

\[len(\chi) = \begin{cases} 
1 & \text{if } \chi \in \text{Atoms} \\
len(\varphi) + 1 & \text{if } \chi \in \{\neg \varphi, \sim \varphi\} \\
len(\varphi) + len(\psi) + 1 & \text{if } \chi \in \{\varphi \lor \psi, \varphi \land \psi, \varphi \to \psi, \varphi \prec \psi\}.
\end{cases}\]

We use the language of classical first-order logic when reasoning about BiInt at the meta-level.

**Definition 3 (Frame).** A BiInt frame is a pair \((W, R)\), where:
1. \( W \) is a non-empty set of worlds;
2. \( R \subseteq W \times W \) is the binary accessibility relation;
3. \( R \) is reflexive, i.e., \( \forall u \in W. uR u \);
4. \( R \) is transitive, i.e., \( \forall u, v, w \in W. (uR v \land vR w \Rightarrow uR w) \).

**Definition 4 (Model).** A BiInt model is a triple \( M = \langle W, R, \vartheta \rangle \), where:

1. \( \langle W, R \rangle \) is a BiInt frame;
2. The truth valuation \( \vartheta \) is a function \( W \times \text{Atoms} \rightarrow \{\text{true}, \text{false}\} \), which tells us the truth value of an atom at a world;
3. The persistence property holds:
   \[ \forall u, w \in W, \forall p \in \text{Atoms}. (\vartheta(w, p) = \text{true} \land wR u) \Rightarrow (\vartheta(u, p) = \text{true}) ; \]
4. \( \forall w \in W. \vartheta(w, \top) = \text{true} ; \)
5. \( \forall w \in W. \vartheta(w, \bot) = \text{false} . \)

**Definition 5 (Forcing of atoms).** Given a model \( M = \langle W, R, \vartheta \rangle \), a world \( w \in W \) and an atom \( p \in \text{Atoms} \), we write \( w \models p \) if \( \vartheta(w, p) = \text{true} \). We pronounce \( \models \) as “forces”, and we pronounce \( \not\models \) as “rejects”.

**Definition 6 (Forcing of formulae).** Given a model \( M = \langle W, R, \vartheta \rangle \), a world \( w \in W \) and formulae \( \varphi, \psi \in \text{Fml} \), we write:

\[
\begin{align*}
    w \models \varphi \lor \psi & \quad \text{if} \quad w \models \varphi \lor w \models \psi \\
    w \models \varphi \land \psi & \quad \text{if} \quad w \models \varphi \land w \models \psi \\
    w \models \neg \varphi & \quad \text{if} \quad \forall u \in W. (wRu \Rightarrow (u \not\models \varphi)) \\
    w \models \varphi \rightarrow \psi & \quad \text{if} \quad \forall u \in W. (wRu \Rightarrow (u \not\models \varphi \lor u \models \psi)) \\
    w \models \neg \neg \varphi & \quad \text{if} \quad \exists u \in W. (uRw \land u \not\models \varphi) \\
    w \models \neg \neg \varphi \land \psi & \quad \text{if} \quad \exists u \in W. (uRw \land u \not\models \neg \neg \varphi \land u \models \psi)
\end{align*}
\]

From the semantics, it can be seen that the connectives \( \neg \) and \( \neg \neg \) can be derived from \( \rightarrow \) and \( \neg \neg \) respectively. Therefore from now on we restrict our attention to the connectives \( \rightarrow, \neg \neg, \land, \lor \) only.

**Lemma 1.** The persistence property also holds for formulae, that is:

\[ \forall M = \langle W, R, \vartheta \rangle, \forall u, w \in W. \forall \varphi \in \text{Fml}. (w \models \varphi \land wRu \Rightarrow u \models \varphi) . \]

*Proof.* By induction on the length of \( \varphi \).

**Lemma 2.** The reverse persistence property holds:

\[ \forall M = \langle W, R, \vartheta \rangle, \forall u, w \in W. \forall \varphi \in \text{Fml}. (w \not\models \varphi \land wRu \Rightarrow u \not\models \varphi) . \]

*Proof.* Reverse persistence follows from persistence, because the truth valuation is binary. That is, suppose for a contradiction that

\[ \exists M = \langle W, R, \vartheta \rangle, \exists u, w \in W. \exists \varphi \in \text{Fml}. (w \not\models \varphi \land wRu \land u \models \varphi) . \]

Then \( u \models \varphi \) and \( uRw \) together with the persistence property give us \( w \models \varphi \), which contradicts \( w \not\models \varphi \).
We write $\epsilon$ to mean the empty set. Given two sets of formulae $\Delta$ and $\Gamma$, we write $\Delta, \Gamma$ for $\Delta \cup \Gamma$. Given a set of formulae $\Delta$ and a formula $\varphi$, we write $\Delta, \varphi$ for $\Delta \cup \{\varphi\}$.

**Definition 7.** Given a model $M = \langle W, R, \vartheta \rangle$, a world $w \in W$ and sets of formulae $\Delta$ and $\Gamma$, we write:

\[
\begin{align*}
    w \models \Gamma & \quad \text{if} \quad \forall \varphi \in \Gamma. w \models \varphi \\
    w \not\models \Delta & \quad \text{if} \quad \forall \varphi \in \Delta. w \not\models \varphi.
\end{align*}
\]

As a corollary, for any world $w$, we vacuously have $w \models \epsilon$ and $w \not\models \epsilon$.

**Definition 8 (Consequence).** Given two sets $\Gamma$ and $\Delta$ of formulae, $\Gamma \not\models_{\text{BiInt}} \Delta$ means:

\[
\forall M = \langle W, R, \vartheta \rangle. \forall w \in W. \text{if } w \models \Gamma \text{ then } \exists \varphi \in \Delta. w \not\models \varphi.
\]

We write $\Gamma \not\models_{\text{BiInt}} \Delta$ to mean that it is not the case that $\Gamma \models_{\text{BiInt}} \Delta$, that is:

\[
\exists M = \langle W, R, \vartheta \rangle. \exists w \in W. (w \models \Gamma \& w \not\models \Delta).
\]

Thus $\Gamma \not\models_{\text{BiInt}} \Delta$ means that $\Gamma \models_{\text{BiInt}} \Delta$ is falsifiable.

We wish to prove $\Gamma \models_{\text{BiInt}} \Delta$ by failing to falsify $\Gamma \models_{\text{BiInt}} \Delta$. By Definition 8, $\Gamma \not\models_{\text{BiInt}} \Delta$ means that there exists a BiInt model $M = \langle W, R, \vartheta \rangle$ that contains a world $w_0 \in W$ such that $w_0 \models \Gamma$ and $w_0 \not\models \Delta$. We therefore try to construct the model using a standard counter-model construction approach: see [7]. We shall start with an initial world $w_0$ and assume that $w_0 \models \Gamma$ and $w_0 \not\models \Delta$, and then systematically decompose the formulae in $\Gamma$ and $\Delta$. The procedure will either:

- lead to a contradiction and therefore conclude that it cannot be the case that $w_0 \models \Gamma$ and $w_0 \not\models \Delta$, therefore $\Gamma \models_{\text{BiInt}} \Delta$ holds, OR
- construct the counter-model successfully and therefore demonstrate that it is possible that $w_0 \models \Gamma$ and $w_0 \not\models \Delta$, therefore $\Gamma \not\models_{\text{BiInt}} \Delta$ does not hold.

### 3 Our Sequent Calculus GBiInt

We now present GBiInt, a Gentzen-style sequent calculus for BiInt. The sequents have a non-traditional component in the form of variables that are instantiated at the leaves of the derivation tree, and passed back to lower sequents from premises to conclusion. Note that the variables are not names for Kripke models and have no semantic content.

#### 3.1 Sequents

First, we introduce an extended syntax that will help us in the presentation of some of our sequent rules.

**Definition 9 (Extended Syntax).** The extended BiInt formulae are defined as follows:

\[
\begin{align*}
    & \text{Variables: } x, y, \ldots \\
    & \text{Leaf formulae: } \varphi, \psi, \chi, \ldots \\
    & \text{Sequent rule: } \Gamma \vdash \Delta
\end{align*}
\]
1. If $\varphi$ is a BiInt formula, then $\varphi$ is an extended BiInt formula.
2. If $S$ and $P$ are sets of sets of BiInt formulae, then $\bigvee S$ and $\bigwedge P$ are extended BiInt formulae.

If $S = \{\{\varphi_0^0, \ldots, \varphi_0^m\}, \ldots, \{\varphi_n^0, \ldots, \varphi_n^k\}\}$ and $P = \{\{\psi_0^0, \ldots, \psi_0^m\}, \ldots, \{\psi_m^0, \ldots, \psi_m^k\}\}$, then from every extended BiInt formula we can obtain a BiInt formula as follows:

$$\bigvee S \equiv (\varphi_0^0 \land \cdots \land \varphi_0^m) \lor \cdots \lor (\varphi_n^0 \land \cdots \land \varphi_n^k)$$

$$\bigwedge P \equiv (\psi_0^0 \lor \cdots \lor \psi_0^m) \land \cdots \land (\psi_m^0 \lor \cdots \lor \psi_m^k).$$

From now on, we implicitly treat extended BiInt formulae as their BiInt equivalents. The following semantics follows directly from Definition 9:

**Definition 10 (Semantics of Extended Syntax).** Given a BiInt model $M = \langle W, R, \emptyset \rangle$, and a world $w_0 \in W$, we write:

$$w \models \bigvee S \quad \text{if} \quad \exists \Gamma \in S. w \models \Gamma$$

$$w \models \bigwedge P \quad \text{if} \quad \exists \Delta \in P. w \models \Delta.$$

We can now extend the definition of forcing and rejecting to extended BiInt formulae in the obvious way. If $\Gamma$ and $\Delta$ are sets of extended BiInt formulae viewed as their BiInt equivalents, and $\varphi$ is an extended BiInt formula viewed as its BiInt equivalent, then:

$$w \models \Gamma \quad \text{if} \quad \forall \varphi \in \Gamma. w \models \varphi$$

$$w \models \Delta \quad \text{if} \quad \forall \varphi \in \Delta. w \not\models \varphi.$$

**Definition 11 (Sequent).** A GBiInt sequent is an expression of the form

$$\frac{S \hspace{1cm} P}{\Gamma \vdash \Delta}$$

and consists of the following components:

- **Left hand side (LHS):** $\Gamma$, a set of extended BiInt formulae;
- **Right hand side (RHS):** $\Delta$, a set of extended BiInt formulae;
- **Variables:** $S$, $P$, each of which is a set of sets of formulae.

We shall sometimes use $\Gamma \vdash \Delta$ to refer to sequents, ignoring the variable values for readability. We shall only do that in cases where the values of the variables are not important to the discussion. Note that the variables do not contain extended BiInt formulae.

We now define the meaning of a sequent in terms of the counter-model under construction.

**Definition 12 (Falsifiability).** A sequent

$$\frac{S \hspace{1cm} P}{\Gamma \vdash \Delta}$$

is falsifiable [at $w_0$ in $M$] if and only if there exists a BiInt model $M = \langle W, R, \emptyset \rangle$ and $\exists w_0 \in W$ such that $w_0 \models \Gamma$ and $w_0 \not\models \Delta$. 

6
Definition 13 (Variable conditions). We say the variable conditions of a sequent
\[ \gamma = \frac{S}{P} \vdash \Delta \]
hold if and only if \( \gamma \) is falsifiable at \( w_0 \) in some model \( M = \langle W, R, \vartheta \rangle \) and the following conditions hold:

- **S-condition**: Successor condition
  \[ \exists \Sigma \in S. \forall w \in W. w_0 R w \Rightarrow w \models \Sigma \]

- **P-condition**: Predecessor condition
  \[ \exists \Pi \in P. \forall w \in W. w R w_0 \Rightarrow w = |\Pi| \]

Lemma 3. A sequent \( \Gamma \vdash \Delta \) is not falsifiable if and only if \( \Gamma \vdash_{\text{BiInt}} \Delta \).

Proof. Applying the negation of Definition 12 to \( \Gamma \vdash \Delta \) gives \( \Gamma \vdash_{\text{BiInt}} \Delta \).

3.2 Sequent Rules

Definition 14 (Sequent Rule). A sequent rule is of one of the forms

\[ \frac{\gamma_1 \cdots \gamma_n}{\gamma_0} \]

where \( \gamma_i, 0 \leq i \leq n \) for \( n \geq 0 \), are sequents. The rule consists of the following components:

- **Conclusion**: \( \gamma_0 \), written below the horizontal line;
- **Premise(s)**: Optional, \( \gamma_1, \cdots, \gamma_n \), written above the horizontal line;
- **Name**: Written to the left of the horizontal line;
- **Side conditions**: Optional, written underneath the rule;
- **Branching**: Universal (indicated by a solid line) or existential (indicated by a dashed line); explained shortly.

To achieve completeness and termination for BiInt, we combine a number of ideas from various existing systems for Int, as well as use variables for updating worlds with relevant information received from successors and predecessors. Our rules can be divided into two groups: traditional (Fig. 1) and non-traditional (Fig. 2).

Our traditional rules (Fig. 1) are based on Dragalin’s GHPC [5] for Int because we require multiple formulae in the succedents and antecedents of sequents for completeness; we have added symmetric rules for the DualInt connective \( \rightarrow \). The main difference is that our \( (\rightarrow_L) \) rule and the symmetric \( (\rightarrow_R) \) carry their principal formula and all side formulae into the premises. Our rules for \( \land \) and \( \lor \) also carry their principal formula into their premises to assist with termination. Note that there are other approaches to a terminating sequent calculus for Int, e.g., Dyckhoff’s contraction-free calculi [6], or history methods by Heuerding et al. [10] and Howe [12]. These methods are less suitable when the interaction
for the \((\rightarrow)\) and \((\land)\) rules have two premises instead of one, and they are connected by existential branching as indicated by the dotted horizontal line. Existential branching means that the conclusion is derivable if some premise is derivable; thus it is dual to the conventional universal branching, where the conclusion is derivable if all premises are derivable. We chose existential branching rather than two separate non-invertible rules so the left premise can communicate information via variables to the right premise. This inter-premise communication and the use of variables is crucial to proving interaction formulae of BiInt, and it gives our calculus an operational reading.

When applying an existential branching rule during backward proof search, we first create the left premise. If the left premise is non-derivable, then it returns the variables \(S_1\) and \(P_1\). We then use these variables to create the right premise, which corresponds to the same world as the conclusion, but with updated information via variables to the right premise. This inter-premise communication and the use of variables is crucial to proving interaction formulae of BiInt, and it gives our calculus an operational reading.

For every rule with premises \(\pi_i\) and conclusion \(\gamma\), apply the rule only if:

\[ \forall \pi_i, (LHS_{\pi_i} \not\subseteq LHS_\gamma \text{ or } RHS_{\pi_i} \not\subseteq RHS_\gamma) \]

**Fig. 1.** GBiInt rules - traditional
Our existential branching rules work together with \((\text{Ret})\), which assigns the variables at non-derivable leaves of failed derivation trees, and \((\land_R)\) and \((\lor_L)\), which extract the different variable choices at existential branching rules.

The conclusion of each of our rules assigns the variables based on the variables returned from the premise(s), and we use the indices \(i, 1, 2\) to indicate the premise from which the variable takes its value. For rules with a single premise, the variables are simply passed down from premise to conclusion. For example, the conclusion of \((\land_L)\) in Fig. 1 assigns \(S := S_1\), where \(S_1\) is the value of the variable at the premise. However, for rules with multiple universally branching premises, we take a union of the sets of sets corresponding to each falsifiable premise. For example, the conclusion of \((\land_R)\) in Fig. 2 assigns \(S := \bigcup^n_i S_i\), where \(S_i\) is the value of the variable at the \(i\)-th premise.

This way, the sets of sets stored in our variables determinise the return of formulae to lower sequents – each non-derivable premise corresponds to an open branch, and at this point we do not know whether it will stay open once
processed in conjunction with lower sequents. Therefore, we need to temporarily keep all open branches: see Example 2. Then the intuition behind adding $\land P$ to the right premise of $(\rightarrow_R)$ is that the subsequent application of $(\land_R)$ will create one or more premises, depending on the cardinality of $P$. Since $P$ is a set of sets representing all the open branches, all of the premises of $(\land_R)$ have to be derivable in order to obtain a derivation. On the other hand, if some premises of $(\land_R)$ are non-derivable (open), we form the set that consists of the union of the variables returned by those premises, and pass the union back to lower sequents, and so on. The premises that are derivable contribute only $\epsilon$ and are thus ignored by the union operator. Also, we only create the right premise of $(\rightarrow_R)$ if every member of $P$ introduces new formulae to the current world. Otherwise, the current world already contains one of the open branches, which would still remain open after an application of $(\land_R)$. To summarise, the sets-of-sets concept of variables is critical to the soundness of GBiInt, as it allows us to remember the required choices arising further up the tree.

The extended syntax allows us to syntactically encode the variable choices described above. While the variables $S$ and $P$ are sets of sets when we pass them down the tree and combine them using set union, we use $\lor S$ on the left and $\land P$ on the right of the sequent to reflect these choices when we add $\lor S$ or $\land P$ to the right premise of an existentially branching rule. Then the $(\lor_L)$ and $(\land_R)$ rules break down the extended formulae $\lor S$ and $\land P$ to yield several premises, each corresponding to one variable choice. Thus the extended syntax allows us to give an intuitive syntactic representation of the variable choices.

We have also added the rule $(\rightarrow_I_R)$ for implication on the right (and dually, $(\leftarrow_I_L)$) originally given by Švejdar [18]. Rather than immediately creating the successor for a rejected $\varphi \rightarrow \psi$, the $(\rightarrow_I_R)$ rule first pre-emptively adds $\psi$ to the right hand side of the sequent. Although Švejdar himself does not give the semantics behind this rule, and is unable to explain the precise role it plays in his calculus, it is very useful in our termination proof. The rule effectively uses the reverse persistence property – if some successor $v$ forces $\varphi$ and rejects $\psi$, then the current world $w$ must reject $\psi$ too, for if $w$ forces $\psi$, then by forward persistence so does $v$, thus giving a contradiction.

The side condition on each of our rules is a general blocking condition, where we only explore the premise(s), if they are different from the conclusion. For example, in the $(\land_R)$ case, the blocking condition means that we apply the rule in backward proof search only if $\varphi \not\in \Delta$ and $\psi \not\in \Delta$, since otherwise some premise would be equal to the conclusion.

GBiInt also has the subformula property. This is obvious for all rules, except $(\rightarrow_R)$ and the dual $(\leftarrow_L)$. For these, the right premise “constructs” the formulae $\land P$ and $\lor S$. However, since $P$ and $S$ are sets of sets of subformulae of the conclusion that are again extracted by $(\land_R)$ and $(\lor_L)$, the right premise of $(\rightarrow_R)$ and $(\leftarrow_L)$ effectively only contains subformulae of the conclusion.

**Definition 15 (GBiInt tree).** A GBiInt tree for a sequent

$$\frac{\emptyset \mid \Gamma \vdash \Delta}{\langle \Gamma, \Delta \rangle}$$


is a tree rooted at \( \frac{S}{P} \mid \Gamma \vdash \Delta \), such that:

1. Each child is obtained by a backwards application of a GBiInt rule, and
2. Each leaf is an instance of a \((\bot_L)\), \((\top_R)\), \((Id)\) or \((Ret)\) rule.

**Definition 16.** A GBiInt tree \( T \) rooted at \( \gamma = \frac{S}{P} \mid \Gamma \vdash \Delta \) is a derivation if:

1. \( \gamma \) is the conclusion of a \((\bot_L)\), \((\top_R)\) or \((Id)\) rule application, OR,
2. \( \gamma \) is the conclusion of a universal branching rule application, and all its premises are derivations, OR,
3. \( \gamma \) is the conclusion of an existential branching rule application, and some premise is a derivation.

We say that \( \gamma \) is derivable if there exists a derivation for \( \gamma \).
We say that \( \gamma \) is not derivable if \( \gamma \) has no derivation.

### 3.3 Examples

In the following examples, we use a simplified version of the \((\land_R)\) rule, which discards the principal formula from the premises, merely to save horizontal space. Also, we only show non-empty variable values.

**Example 1.** The following is a derivation tree of Uustalu’s counterexample, the interaction formula \( p \rightarrow (q \lor (r \rightarrow ((p \prec q) \land r))) \), simplified to the sequent \( p \vdash q, r \rightarrow ((p \prec q) \land r) \). We abbreviate \( X := r \rightarrow ((p \prec q) \land r) \). The tree should be read bottom-up while ignoring the variables \( S \) and \( P \). At the leaves, the variables are assigned and transmit information down to parents and across to some siblings. The top left application of \((Ret)\) occurs because an application of the \((\prec R)\) rule to the bolded \( p \prec q \) is blocked, since its left premise would not be different from its conclusion.

Notice that the key to finding the contradiction is the bolded \( p \prec q \) formula that is passed from the left-most leaf node back to the right premise (1) of the \((\rightarrow_R)\) rule. Also, the \((\land_R)\) rule in (1) is unary in this case, since the returned \( \mathcal{P} \) variable contains only one set of formulae.

\[
\begin{align*}
\text{(Ret)} & \quad S = \{(p,r,q)\} \quad P = \{(p \prec q)\} \quad p, r, q \vdash p \prec q \\
\text{\((\prec R)\)} & \quad S = \{(p,r,q)\} \quad P = \{(p \prec q)\} \quad p, r \vdash p \prec q \\
\text{\((\land_R)\)} & \quad S = \{(p,r,q)\} \quad P = \{(p \prec q)\} \quad p, r \vdash (p \prec q) \land r \\
\text{(1)} & \quad p \vdash q, r \rightarrow ((p \prec q) \land r)
\end{align*}
\]

Where (1) is:

\[
\begin{align*}
\text{(Id)} & \quad p, q \vdash q, X, p \prec q \\
\text{(Id)} & \quad p \vdash q, X, p \prec q, p \\
\text{\((\prec R)\)} & \quad p \vdash q, X, p \prec q \\
\text{\((\land_R)\)} & \quad p \vdash q, X, \land \{(p \prec q)\}
\end{align*}
\]
Example 2. The following example is a GBiInt-tree of a falsifiable sequent, and it shows how in the case of multiple choices for the variables, a contradiction caused by one of them does not give us a derivation. We abbreviate $Y := (\top \rightarrow p) \land (\top \rightarrow q)$, and $X := Y \rightarrow \bot$.

\[
\begin{array}{ll}
& (\mathsf{Ret}) \\
& S = \{\{x\}\} \\
& p = \{\{\top \rightarrow p\}\} \\
& \vdash p, X \rightarrow \bot, \top \rightarrow p \\
& (\land_R) \\
& p = \{\{\top \rightarrow q\}\} \\
& q = \{\{\top \rightarrow p\}\} \\
& \vdash q, X \rightarrow \bot, \top \rightarrow q \\
& (\vdash_L) \\
& X, \bot \rightarrow \bot \\
\end{array}
\]

Where (2) is:

\[
\begin{array}{ll}
(\mathsf{Ret}) & \quad \vdash S = \{\{q\}\} \\
& \vdash q, p, X \rightarrow \bot, \top \rightarrow q \\
& \vdash q, X \rightarrow \bot, \top \rightarrow q \\
& (\land_R) \\
& \vdash \{q\}, p, X \rightarrow \bot, \top \rightarrow q \\
& \vdash \{q\}, p, X \rightarrow \bot, \top \rightarrow q \\
& (\vdash_R) \\
& \vdash p, X \rightarrow \bot, \top \rightarrow q \\
& \vdash p, X \rightarrow \bot, \top \rightarrow q \\
& \vdash p, X \rightarrow \bot, \top \rightarrow q \\
\end{array}
\]

In this case, the $(\land_R)$ rule in (2) has two premises, since the returned $P$ variable contains two sets of formulae. Since only the left premise of the $(\land_R)$ rule is derivable, the conclusion is not derivable. Thus, the open branch corresponding to the bolded member $\{\top \rightarrow q\}$ of $P$ remains open. If we did not return both variables choices from the left sibling of (2), then we might mistakenly derive (2) without seeing this open branch.

Lemma 4. If a GBiInt-tree $T$ rooted at $\gamma = S \vdash \Delta$ is a derivation then $S = P = \epsilon$.

Proof. By induction on the longest branch in $T$.

3.4 Termination Proof

We first show that proof search in GBiInt terminates because the subsequent soundness proof relies on our ability to receive the variables from the left premises of transitional rules.

Definition 17. The rules of GBiInt are categorised as follows:

Operational: $(\mathsf{Ret})$;
Function Prove

Input: sequent $\gamma_0$

Output: Derivable (true or false)

1. If $\rho \in \{(Id), (\bot_L), (\top_R)\}$ applicable to $\gamma_0$ then
   (a) Return true
2. Else if any special or static rule $\rho$ applicable to $\gamma_0$ then
   (a) Let $\gamma_1, \ldots, \gamma_n$ be the premises of $\rho$
   (b) Return $\bigwedge Prove(\gamma_i)$
3. Else for each transitional rule $\rho$ applicable to $\gamma_0$ do
   (a) Let $\gamma_1$ and $\gamma_2$ be the premises of $\rho$
   (b) If $\bigvee Prove(\gamma_i) = true$ then return true
4. Endif
5. Return false.

Fig. 3. Proof search strategy. Note that we have left out the variables for simplicity.

$\bigwedge_{i=1}^n Prove(\gamma_i)$ is true iff $Prove(\gamma_i)$ is true for all premises $\gamma_i$ for $1 \leq i \leq n$, and $\bigvee_{i \in \{1,2\}} Prove(\gamma_i)$ is true iff $Prove(\gamma_i)$ is true for some premise $\gamma_i$ for $i \in \{1,2\}$.

Logical:

Static: $(Id)$, $(\bot_L)$, $(\top_R)$, $(\land_L)$, $(\lor_L)$, $(\land_R)$, $(\lor_R)$, $(\rightarrow_L)$, $(\leftarrow R)$,
   $(\neg\leftarrow L)$;

Transitional: $(\rightarrow R)$, $(\leftarrow L)$;

Special: $(\lor_L)$, $(\land_R)$.

The intuition behind the classification of the logical rules is that the static rules add formulae to the current world in the counter-model, the transitional rules create new worlds and add formulae to them, and the special rules decompose variables returned from non-derivable leaves. We shall prove this formally for each rule later. The classification justifies the following search strategy.

Definition 18 (Strategy). The strategy defined in Figure 3 is used when applying the rules of our sequent calculus in backward proof search. Note that we have left out the variables for simplicity.

Definition 19 (Subformulae). For a BiInt formula, we define the subformulae as follows, where $p \in$ Atoms and $\varphi, \psi \in$ Fml:

$$
\begin{align*}
sf(p) & = \{p\} \\
sf(\varphi \lor \psi) & = sf(\varphi) \cup sf(\psi) \cup \{\varphi \lor \psi\} \\
sf(\varphi \land \psi) & = sf(\varphi) \cup sf(\psi) \cup \{\varphi \land \psi\} \\
sf(\varphi \rightarrow \psi) & = sf(\varphi) \cup sf(\psi) \cup \{\varphi \rightarrow \psi\} \\
sf(\varphi \leftarrow \psi) & = sf(\varphi) \cup sf(\psi) \cup \{\varphi \leftarrow \psi\} \\
sf(\lor S) & = \bigcup_{\Sigma \in S} sf(\Sigma) \\
sf(\land P) & = \bigcup_{\Pi \in P} sf(\Pi)
\end{align*}
$$
For a set $\Gamma$ of extended BiInt formulae, we define $sf(\Gamma) = \bigcup_{\chi \in \Gamma} sf(\chi)$.

Note that the subformulae of $\forall S$ and $\exists P$ do not include the conjunctions and disjunctions implicit in their BiInt equivalents.

**Definition 20 (LEN).** Let $\succ_{ten}$ be a lexicographic ordering of sequents:

$$(\Gamma_2 \vdash \Delta_2) \succ_{ten} (\Gamma_1 \vdash \Delta_1) \text{ iff } |\Gamma_2| > |\Gamma_1| \text{ or } |\Gamma_2| = |\Gamma_1| \text{ and } |\Delta_2| > |\Delta_1|$$

**Definition 21.** Given a GBiInt-tree $T$ and a branch $B$ in $T$, we say that $B$ is **forward-only** if $B$ contains only applications of static and special rules, $(\neg_R)$ and the right premises of $(\neg_L)$. Similarly, $B$ is **backward-only** if $B$ contains only applications of static and special rules, $(\neg_L)$ and the right premises of $(\neg_R)$. A branch is **single-directional** if it is either forward-only or backward-only. Finally, a branch contains **interleaved** left premises of transitional rules if it contains a sequence $(\cdots, \gamma_i, \cdots, \gamma_j, \cdots, \gamma_k, \cdots)$ such that $\gamma_i$ is the left premise of $(\neg_R)$, $\gamma_j$ is the left premise of $(\neg_L)$, and $\gamma_k$ is the left premise of $(\neg_R)$.

**Lemma 5.** Every forward-only branch of any GBiInt-tree is finite.

**Proof.** We show that on every such branch, the length of a sequent defined according to $\succ_{ten}$ increases.

Consider a rule $\rho$, and a backwards application of $\rho$ to some $\Gamma \vdash \Delta$, which yields $n$ premises $\Gamma_i \vdash \Delta_i$, where $1 \leq i \leq n$.

We show that if $\rho$ is a static rule, then for all premises $i$, we have $(\Gamma_i \vdash \Delta_i) \succ_{ten} (\Gamma \vdash \Delta)$:

- $\rho = (\bigwedge L), (\bigvee L), (\neg L)$: Then $|\Gamma_i| > |\Gamma|$;
- $\rho = (\neg R)_l$: Then $|\Gamma_i| = |\Gamma|$ and $|\Delta_i| > |\Delta|$;
- $\rho = (\neg R)_r$: Then for the left premise, $|\Gamma_1| = |\Gamma|$ and $|\Delta_1| > |\Delta|$, and for the right premise, $|\Gamma_2| > |\Gamma|$;
- $\rho = (\neg L)$: Then for the left premise, $|\Gamma_1| > |\Gamma|$, and for the right premise, $|\Gamma_2| = |\Gamma|$ and $|\Delta_2| > |\Delta|$.

We now show the cases for $\rho \in \{(\neg R), (\neg L), (\bigwedge R), (\bigvee L)\}$. Even though the right premise of $(\neg R)$ and $(\neg L)$ itself is not greater than the conclusion, we show that the lemma holds on the overall GBiInt branch, since according to the strategy we immediately apply $(\bigwedge R)$ or $(\bigvee L)$, thus increasing the length of the premise according to $\succ_{ten}$.

- $\rho = (\neg R)$: For every $(\neg R)$ rule application:
  1. Consider the left premise $\Gamma_1 \vdash \Delta_1$. We know that according to our strategy, the $(\neg R)_l$ rule has already been applied and thus $\psi \in \Delta$, so $(\neg R)$ is applied only if $\varphi \notin \Gamma$. Therefore, for the left premise, we have $|\Gamma_1| > |\Gamma|$;
2. Consider the right premise $\Gamma_2 \vdash \Delta_2$. It is created only if 
\[ P_1 \neq \epsilon \ & \ & \forall \Pi_i \in P_1. \Pi_i \not\subseteq \{ \Delta, \varphi \rightarrow \psi \}. \] (3.1)

That is, every member of $P_1$ introduces new formulae to the RHS. But recall that $sf(\bigwedge P_1) \subseteq sf(\Gamma \cup \Delta)$. According to our strategy, the $(\bigwedge_R)$ rule will be immediately applied to $\bigwedge P_1$ in $\Delta_2$, giving $n \geq 1$ premises $\Gamma_2^j \vdash \Delta_2^j$ where $1 \leq j \leq n$. By 3.1, we will then have $|\Delta_2^j| > |\Delta|$ for all $j$. We also have $|\Gamma_2^j| = |\Gamma|$ for all $j$. Therefore, according to the lexicographic ordering, we have $(\Gamma_2^j \vdash \Delta_2^j) >_{\text{ten}} (\Gamma \vdash \Delta)$ for all the premises $\Gamma_2^j \vdash \Delta_2^j$.

$\rho = (\bigwedge_R)$: Since the $(\bigwedge_R)$ rule is only used in conjunction with the right premise of the $(\rightarrow_R)$ rule, see case 2 above;

$\rho = (\prec_L)$: For every $(\prec_L)$ rule application:
1. The assumption of the lemma does not apply to the left premise;
2. The case for the right premise is dual to the case for $(\rightarrow_R)$ above.

$\rho = (\bigvee_L)$: By symmetry with the case for $(\bigwedge_R)$ above;

Since the length of a sequent defined according to $>_\text{ten}$ increases on every forward-only branch as shown above, and since GBIInt has the subformula property, eventually no more formulae can be added to a sequent on a forward-only branch, and the branch will terminate.

**Lemma 6.** Every backward-only branch of any GBIInt-tree is finite.

**Proof.** By symmetry with Lemma 5.

**Lemma 7.** If a GBIInt-tree contains an infinite branch, then the branch contains an infinite number of interleaved left premises of transitional rules.

**Proof.** By Lemmas 5 and 6, single-directional branches must eventually terminate. Thus, a potential infinite loop must involve an infinite number of interleaved left premises of transitional rules $(\rightarrow_R)$ and $(\prec_L)$.

**Definition 22 (Degree).** The degree of a BiInt formula $\chi$ is defined as:
\[
\deg(\chi) = \begin{cases} 
0 & \text{if } \chi \in \text{Atoms} \\
\deg(\varphi) + \deg(\psi) & \text{if } \chi \in \{ \varphi \lor \psi, \varphi \land \psi \} \\
\deg(\varphi) + \deg(\psi) + 1 & \text{if } \chi \in \{ \varphi \rightarrow \psi, \varphi \prec \psi \}
\end{cases}
\]

Thus, the degree of $\varphi$ is the number of $\rightarrow$ and $\prec$ connectives in $\varphi$.

The degree of a sequent $\Gamma \vdash \Delta$ is defined as:
\[
\deg(\Gamma \vdash \Delta) = \sum_{\varphi \in sf(\Gamma \cup \Delta)} \deg(\varphi)
\]

Note that we have deliberately defined the degree of a sequent as the sum of the degrees of subformulae, because it allows us to make the following observations, which will be crucial in the main termination proof.
Corollary 1. Since GBiInt has the subformula property, the degree of a sequent can never increase in backward proof search. In other words, no GBiInt rule can increase the degree of a sequent.

Corollary 2. Given two sequents $\gamma_1$ and $\gamma_2$, if $sf(\gamma_2) \subset sf(\gamma_1)$, then $\deg(\gamma_2) < \deg(\gamma_1)$. That is, removing some formula $\varphi$ from a sequent during backward proof search decreases the degree of the sequent if $\varphi$ is not a subformula of any other formula in the sequent, since $\varphi$ no longer contributes to the sum of degrees of subformulas.

Theorem 1 (Termination). Every GBiInt-tree constructed according to the strategy of Definition 18 is finite.

Proof. Suppose for a contradiction that there exists an infinite GBiInt-tree $T$. Since every rule has a finite number of premises, i.e., finite branching, then by König’s lemma an infinite tree can only be obtained by having a branch of infinite length. Thus, $T$ has an infinite branch $B$. By Lemma 7, $B$ must contain an infinite number of interleaved left premises of transitional rules, as shown below:

Let $\chi \in sf(\pi_0)$ be some formula such that $\deg(\chi) = \max\{\deg(\varphi) \mid \varphi \in sf(\pi_0)\}$, that is, $\chi$ is one of the subformulæ with the maximum degree. In particular, this means that $\chi$ is not a subformula of any formula with a larger degree. We shall now show that $\chi \notin sf(\pi_2)$.

There are two cases:

- $\chi \notin sf(\pi_0)$: Then $\chi \in sf(D_0)$ or $\chi = \varphi_0 \to \psi_0$. In both cases, $\chi \notin sf(\pi_2)$.
- $\chi \in sf(\pi_0)$: Then it cannot be the case that $\chi \in sf(\varphi_1)$ or $\chi \in sf(\psi_1)$, since then $\deg(\varphi_1 \to \psi_1) > \deg(\chi)$, contradicting our assumption that $\deg(\chi) = \max\{\deg(\varphi) \mid \varphi \in sf(\pi_0)\})$. Therefore, either:
  - $\chi$ and all its occurrences in subformulae disappear from the sequent at the premise of ($\to L$), in which case $\chi \notin sf(\pi_2)$, or
– \( \chi \) is moved to the RHS of the sequent by applying the \((\to_L)\) rule to some formula \( \chi \to \tau \). However, since \( \text{deg}(\chi \to \tau) > \text{deg}(\chi) \), it again contradicts our assumption that \( \text{deg}(\chi) = \max(\{\text{deg}(\varphi) \mid \varphi \in sf(\pi_0)\}) \).

We have shown that for some formula \( \chi \) we have \( \chi \in sf(\pi_0) \) and \( \chi \notin sf(\pi_2) \). Also, by the subformula property of \( \text{GBiInt} \) we have \( sf(\pi_2) \subseteq sf(\pi_0) \). Together with \( \chi \in sf(\pi_0) \) and \( \chi \notin sf(\pi_2) \), this means \( sf(\pi_2) \subseteq sf(\pi_0) \). Then by Corollary 2 we have \( \text{deg}(\pi_2) < \text{deg}(\pi_0) \). Note that the steps indicated by vertical ellipses (\( \vdots \)) are arbitrary, since by Corollary 1 no rule can increase the degree of a sequent.

Since we have \( \text{deg}(\pi_2) < \text{deg}(\pi_0) \), we know that every sequence of interleaved transitional rule applications must decrease the degree of the sequent. This can only happen a finite number of times, until no more transitional rules are applicable. Therefore our assumption was wrong, and no branch \( B \) can be infinite. Therefore, every \( \text{GBiInt} \)-tree is finite.

4 Soundness

4.1 Proof Outline

Instead of the traditional approach of showing that each rule application preserves validity downwards, we use the notion of falsifiability and show that each rule application preserves falsifiability upwards. We then use Lemma 3 to make the connection between falsifiability and validity.

Also, our addition of variables to the calculus introduces a two-way flow of information in the \( \text{GBiInt} \) trees, and this complicates the usually simple soundness proof.

We separate the notion of soundness into two: local soundness, applicable locally to a single rule application, and global soundness, which takes into account the propagation of variables from the leaves down to some node, and possible instances of the operational \( (\text{Ret}) \) rule. Note that locality here refers to locality in the \( \text{GBiInt} \) trees, not locality in the underlying Kripke models. We use the notions of static and transitional rules to classify the rules according to this latter notion.

4.2 Local soundness

Definition 23 (Local soundness). A logical rule in \( \text{GBiInt} \) is locally sound if and only if:

– For rules with universal branching: if the conclusion is falsifiable, then some premise is falsifiable;
– For rules with existential branching: if the conclusion is falsifiable, then all premises are falsifiable.
We shall now show that each static and special rule is locally sound, and we shall then use induction on the height of a derivation tree to extend our proof to arbitrary trees containing static rules, special rules, transitional rules and the operational (Ret) rule.

**Lemma 8.** Each static and special rule of GBiInt is locally sound.

*Proof.* We consider each static and special rule in turn. We assume that the conclusion is falsifiable, and show that some premise is falsifiable.

1. \((Id)\) \[ S;_e\mid P;_e \models \Gamma, \varphi \vdash \Delta, \varphi \]

   The conclusion of this rule is never falsifiable, because no BiInt model can contain a world \( w \) such that \( w \models \varphi \) and \( w \not\models \varphi \).

2. \((\bot_L)\) \[ S;_e\mid P;_e \models \Gamma, \bot \vdash \Delta \]

   The conclusion of this rule is never falsifiable, because by Property 5 of Definition 4, no BiInt model can contain a world \( w \) such that \( w \models \bot \).

3. \((\top_R)\) \[ S;_e\mid P;_e \models \Gamma \vdash \Delta, \top \]

   The conclusion of this rule is never falsifiable, because by Property 4 of Definition 4, no BiInt model can contain a world \( w \) such that \( w \not\models \top \).

4. \((\wedge_R)\)

   \[ \begin{array}{c}
   S_1;_e\mid P_1;_e \models \Gamma \vdash \Delta, \varphi \wedge \psi, \varphi \\
   S_2;_e\mid P_2;_e \models \Gamma \vdash \Delta, \varphi \wedge \psi, \psi \\
   \end{array} \]

   \[ S;_e\mid P;_e \models \Gamma \vdash \Delta, \varphi \wedge \psi \]

   Since the conclusion is falsifiable by assumption, we know from Definition 12 that there exists a world \( w_0 \) such that:
   (i) \( w_0 \models \Gamma \)
   (ii) \( w_0 \models \Delta, \varphi \wedge \psi \).

   From the semantics of \( \wedge \) in BiInt, (b) implies that either:
   (ii.1) \( w_0 \models \Delta, \varphi \wedge \psi, \varphi \)
   (ii.2) \( w_0 \models \Delta, \varphi \wedge \psi, \psi \).

   To show that some premise of the \((\wedge_R)\) rule is falsifiable, we need to show that there exists a world \( w' \) such that some premise is falsifiable at \( w' \). We let \( w' = w_0 \).

   Then case (ii.1) together with (i) gives us that the left premise is falsifiable, or case (ii.2) together with (i) gives us that the right premise is falsifiable.

5. \((\vee_L)\)

   \[ \begin{array}{c}
   S_1;_e\mid P_1;_e \models \Gamma, \varphi \lor \psi, \varphi \vdash \Delta \\
   S_2;_e\mid P_2;_e \models \Gamma, \varphi \lor \psi, \psi \vdash \Delta \\
   \end{array} \]

   \[ S;_e\mid P;_e \models \Gamma, \varphi \lor \psi \vdash \Delta \]

   By symmetry with the \((\wedge_R)\) rule.

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Since the conclusion is falsifiable by assumption, we know from Definition 12 that there exists a world \( w_0 \) such that:

(i) \( w_0 \models \Gamma \) and

(ii) \( w_0 \models \Delta, \varphi \lor \psi \)

To show that the premise of the \((\lor R)\) rule is falsifiable, we need to show that there exists a world \( w' \) such that the premise is falsifiable at \( w' \). We let \( w' = w_0 \).

From the semantics of \( \lor \) in BiInt, (ii) implies that \( w_0 \models \Delta, \varphi \lor \psi, \varphi \) and \( w_0 \models \Delta, \varphi \lor \psi, \psi \). Together with (i), this means that the premise is falsifiable.

By symmetry with the \((\lor R)\) rule.

Since the conclusion is falsifiable by assumption, we know from Definition 12 that there exists a world \( w_0 \) such that:

(i) \( w_0 \models \Gamma, \varphi \rightarrow \psi \) and

(ii) \( w_0 \models \Delta \).

From the semantics of \( \rightarrow \) in BiInt, (i) implies that for all successors \( w \), we have \( w \not\models \varphi \) or \( w \models \psi \).

By reflexivity of \( R \), this applies to \( w_0 \) too, so we have:

(i.1) \( w_0 \not\models \varphi \) or

(i.2) \( w_0 \models \psi \).

To show that some premise of the \((\rightarrow L)\) rule is falsifiable, we need to show that there exists a world \( w' \) such that some premise is falsifiable at \( w' \). We let \( w' = w_0 \).

Then items (i), (ii) and (i.1) give us that the left premise is falsifiable, or items (i), (ii) and (i.2) give us that the right premise is falsifiable.

By symmetry with \((\rightarrow L)\).
10.

\[
\frac{\frac{S_1 \vdash \Delta, \phi \rightarrow \psi, \psi}{S = S_1 \quad P = P_1}}{\Gamma \vdash \Delta, \phi \rightarrow \psi}
\]

Since the conclusion is falsifiable by assumption, we know from Definition 12 that there exists a world \(w_0\) such that:

(i) \(w_0 \models \Gamma\) and
(ii) \(w_0 \models \Delta, \phi \rightarrow \psi\).

From the semantics of \(\rightarrow\) in BiInt, (ii) implies that there exists a successor \(w_1\) such that:

(iii) \(w_0 R w_1\) and
(iv) \(w_1 \models \phi\) and
(v) \(w_1 \not\models \psi\).

Then, by the reverse persistence property of BiInt, and (iii) and (v), we have:

(vi) \(w_0 \not\models \psi\).

To show that the premise of the \((\rightarrow)\) rule is falsifiable, we need to show that there exists a world \(w\) such that the premise is falsifiable at \(w\). We let \(w = w_0\).

Then items (i), (ii) and (vi) give us that the premise is falsifiable.

11.

\[
\frac{\frac{S_1 \vdash \Gamma, \phi, \phi \rightarrow \psi \vdash \Delta}{S = S_1 \quad P = P_1}}{S \vdash \Gamma, \phi \rightarrow \psi \vdash \Delta}
\]

By symmetry with \((\rightarrow)\).

12.

\[
\frac{\frac{S_1 \vdash \Gamma, \Sigma_1 \vdash \Delta \ldots S_n \vdash \Gamma, \Sigma_n \vdash \Delta}{S = \bigcup_i S_i, P = \bigcup_i P_i}}{\Gamma, \Sigma \vdash \Delta}
\]

Since the conclusion is falsifiable by assumption, we know from Definition 12 that there exists a world \(w_0\) such that:

(i) \(w_0 \models \Gamma, \Sigma\) and
(ii) \(w_0 \models \Delta\).

From the semantics of \(\Sigma\) (recall Definition 10), (i) implies that:

(iii) for some \(\Sigma_i \in \Sigma\), we have \(w_0 \models \Sigma_i\).

To show that some premise of the \((\Sigma)\) rule is falsifiable, we need to show that there exists a world \(u'\) such that this premise is falsifiable at \(u'\). We let \(u' = u\).

Then items (i), (ii) and (iii) give us that the \(i\)-th premise containing \(\Sigma_i\) is falsifiable at \(w_0\).

13.

\[
\frac{\frac{S_1 \vdash \Gamma, \Pi_1 \ldots S_n \vdash \Gamma, \Pi_n}{S = \bigcup_i S_i, P = \bigcup_i P_i}}{\Gamma \vdash \Delta, \Pi}
\]

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By symmetry with ($\lor L$).

**Remark 1.** Note that the static rules also preserve falsifiability downwards: if some premise $\pi$ is falsifiable, then the conclusion $\gamma$ is falsifiable. This is easy to see, since we have $LHS_\pi \supseteq LHS_\gamma$ and $RHS_\pi \supseteq RHS_\gamma$.

### 4.3 Global soundness

We have shown that all the static and special rules preserve falsifiability upwards, in other words, they are *locally sound*. Since the $S$ and $P$ variables propagate downwards, from the leaves to the root, we can only reason about the variable conditions of rules when we consider an entire tree rooted at a rule application. Similarly, since the soundness of the transitional rules relies on the variables, we can only reason about it if we consider an entire tree rooted at a transitional rule application. We shall now show that $\text{GBiInt}$ rules are *globally sound*, that is, they preserve falsifiability upwards and variable conditions downwards.

**Lemma 9 (Global soundness).** Given any $\text{GBiInt}$ tree $T$, for every sequent $\gamma_0 \in T$, the following holds: if $\gamma_0$ is falsifiable, then:

1. Some universally branching, or all existentially branching, premises are falsifiable,
2. The variable conditions hold at $\gamma_0$.

**Proof.** By induction on the length $h(\gamma_0)$ of the longest branch from $\gamma_0$ to a leaf sequent of $T$.

**Base case:** $h(\gamma_0) = 0$. So $\gamma_0$ itself is an instance of ($Id$), ($\bot_L$), ($\top_R$), or ($Ret$).

($Id$), ($\bot_L$), ($\top_R$): The conclusion of these rules is never falsifiable, so there is nothing to show.

($Ret$):
- The conclusion of the ($Ret$) rule is $\Gamma \vdash \Delta$, and there is no premise. From the side condition of the ($Ret$) rule, we know that no other rules are applicable to $\Gamma \vdash \Delta$. We will now show that $\Gamma \vdash \Delta$ is falsifiable, and that it obeys the variable conditions.
- We create a model with a single world $w_0$, and for every atom $p$ in $\Gamma$, we let $\vartheta(w_0, p) = \text{true}$, and for every atom $q$ in $\Delta$, we let $\vartheta(w_0, q) = \text{false}$.
- Note that an atom cannot be both in $\Gamma$ and $\Delta$, since the ($Id$) rule in particular is not applicable to $\Gamma \vdash \Delta$.
- To show that $\Gamma \vdash \Delta$ is falsifiable at $w_0$, we need to show that $w_0 \models \Gamma$ and $w_0 \models \Delta$. For every atom in $\Gamma$ and $\Delta$, the valuation ensures this. For every composite formula $\varphi$, we do a simple induction on its length. The fact that the ($Ret$) rule is applied implies that no other rules are applicable, therefore the required subformula $\psi$ is already in $\Gamma$ or $\Delta$ as appropriate, and $\psi$ falls under the induction hypothesis.
- Thus we know that:
  
  (i) $w_0 \models \Gamma$ and

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Induction step: We assume that the lemma holds for all $w_0 \in \mathcal{W}$: $w_0 \mathcal{R} w \Rightarrow w \models \Gamma$. Similarly, (ii) and the reverse persistence property of $\mathbf{BiInt}$ give us that $\forall w \in \mathcal{W}, w \mathcal{R} w_0 \Rightarrow w \models \Delta$. Then the conclusion of the (Ret) rule obeys the variable conditions:

**$S$-condition:** Successor condition
$$\exists \Sigma \in \{\Gamma\}. \forall w \in \mathcal{W}, w_0 \mathcal{R} w \Rightarrow w \models \Sigma$$

**$P$-condition:** Predecessor condition
$$\exists \Pi \in (\Delta). \forall w \in \mathcal{W}, w \mathcal{R} w_0 \Rightarrow w \models \Pi$$

**Induction step:** We assume that the lemma holds for all $\gamma_0$ with $h(\gamma_0) \leq k$, and show that it holds for all $\gamma_0$ with $h(\gamma_0) \leq k + 1$.

Consider the rule application $\rho$ such that $\gamma_0$ is the conclusion of $\rho$. By the assumption of the lemma, we have that the conclusion $\gamma_0$ of $\rho$ is falsifiable at some $w_0$ in some model $\mathcal{M} = (\mathcal{W}, \mathcal{R}, \mathcal{V})$. The only possibilities are that $\rho$ is a static or a special rule, or that it is a transitional rule:

1. $\rho$ is one of the static or special rules (universally branching). Then Lemma 8 tells us that some premise is falsifiable. We now need to show that the variable conditions hold at $\gamma_0$. There are two cases:
   - $\rho$ is unary: The premise $\gamma_1$ of $\rho$ has $h(\gamma_1) \leq k$, therefore the induction hypothesis applies to $\gamma_1$. By Lemma 8 and the fact that $\gamma_0$ is falsifiable at $w_0$, we know that the premise $\gamma_1$ is falsifiable at $w_0$, so by the induction hypothesis we have that the variable conditions hold at $\gamma_1$. Since $\gamma_1$ has the same variables as $\gamma_0$, and since $\gamma_1$ is falsified by the same world $w_0$ as $\gamma_0$, we then know that $\gamma_0$ also obeys the variable conditions.
   - $\rho$ is $n$-ary with $n > 1$: We show the case for $S$; the case for $P$ is symmetric. The premises $\gamma_1$ to $\gamma_n$ of $\rho$ each have $\gamma_i \leq k$, therefore the induction hypothesis applies to each $\gamma_i$. By Lemma 8 and the fact that $\gamma_0$ is falsifiable at $w_0$, we know that some $\gamma_m$ is falsifiable at $w_0$, too. Therefore the induction hypothesis tells us that the variable conditions hold at $\gamma_m$. That is, we know that:
     $$\exists \Sigma_m \in S_m, \forall w \in \mathcal{W}, w_0 \mathcal{R} w \Rightarrow w \models \Sigma_m.$$  
     To show that the conclusion $\gamma_0$ obeys the variable condition for $S$, we need to show the following:
     $$\exists \Sigma \in \bigcup_{i=1}^n S_i, \forall w \in \mathcal{W}, w_0 \mathcal{R} w \Rightarrow w \models \Sigma.$$  
     Since $\Sigma_m \in S_m$ and $S_m \subseteq \bigcup_i^n S_i$, we have $\Sigma_m \in \bigcup_i^n S_i$ and thus the variable conditions hold for $S$ at the conclusion $\gamma_0$.

2. $\rho$ is one of the transitional rules (existentially branching). We show the case for the ($\rightarrow_R$) rule, the case for the ($\leftarrow_L$) rule is symmetric:

   \[
   s/p = \begin{cases} 
   s_1/p_1 & \text{if } p_1 = \epsilon \\
   s_2/p_2 & \text{if right prem created} \\
   \{\Gamma\}/\{\Delta, \varphi \rightarrow \psi\} & \text{otherwise}
   \end{cases}
   \]
   - right prem created only if $p_1 \neq \epsilon$ & $\forall \Pi \in p_1, \Pi \notin \{\Delta, \varphi \rightarrow \psi\}$
So suppose that the conclusion is falsifiable. Then we know from Definition 12 that there exists a world \( w_0 \) such that:

(i) \( w_0 \models I \) and

(ii) \( w_0 \models \Delta, \varphi \rightarrow \psi \).

From the semantics of \( \rightarrow \) in BiInt, (ii) implies that there exists a successor \( w_1 \) such that:

(iii) \( w_0 R w_1 \) and

(iv) \( w_1 \models \varphi \) and

(v) \( w_1 \not\models \psi \).

(a) To show that the left premise of the \((\rightarrow R)\) rule is falsifiable, we need to show that there exists a world \( w' \) such that this premise is falsifiable at \( w' \). We let \( w' = w_1 \).

Then items (i), (iv) and (v) give us that the left premise is falsifiable. Now, the left premise \( \gamma_1 \) is of distance \( \leq k \) from the furthest leaf node of \( T \), therefore the induction hypothesis applies to \( \gamma_1 \). By the hypothesis assumption, since \( \gamma_1 \) is falsifiable at \( w_1 \), we have that the variable conditions hold at \( \gamma_1 \). In particular, the \( P \) condition holds, giving us:

\[ \exists \Pi \in P_1. \forall w \in W. w \models =_{\Pi} \quad (4.1) \]

Now there are two cases: either the right premise was created, or it was not (and there is nothing to show). If it was created, then we need to show that it is falsifiable by exhibiting a world \( w'' \) such that the right premise is falsifiable at \( w'' \). We let \( w'' = w_0 \). Then, since \( w_0 R w_1 \), we have \( w_0 =_{\Pi} \) by (4.1). Since \( \Pi \in P_1 \), then by Definition 10 we have that \( w_0 =_{\bigwedge P_1} \), together with (i) and (ii), this means that the right premise is falsifiable at \( w_0 \). Moreover, the variable conditions hold at the right premise, since it also is falsifiable, and of distance \( \leq k \) from the furthest leaf node of \( T \), so the induction hypothesis applies to it.

(b) We need to show that the variable conditions hold at the conclusion \( \gamma_0 \) of the \((\rightarrow R)\) rule. We show the case for the variable \( S \); the case for \( P \) is symmetric. We need to show that:

\[ \exists \Sigma \in S. \forall w \in W. w \models =_{\bigwedge P_1} \Rightarrow w \models \Sigma \quad (4.2) \]

Where \( S := \begin{cases} S_1 & \text{if } P_1 = \epsilon \\ S_2 & \text{if right prem created} \\ \{I\} & \text{otherwise} \end{cases} \)

Since we have shown that the variable conditions hold at the left premise, we know that in particular \( P_1 \neq \epsilon \). Therefore there are two cases: either the right premise was created, or it was not:

- If the right premise \( \gamma_2 \) was created, then we know that the variable conditions hold at \( \gamma_2 \), since \( \gamma_2 \) falls under the induction hypothesis. This gives us:

\[ \exists \Sigma_2 \in S_2. \forall w \in W. w \models =_{\bigwedge P_1} \Rightarrow w \models \Sigma_2 \]

Thus \( S := S_2 \) obeys (4.2).
If the right premise was not created, then we need to show that 
\{ \Gamma \} obeys the variable conditions at the conclusion. Now, we 
have \( w_0 \models \Gamma \) by (i), and then the persistence property tells us 
that \( \forall w \in W. w_0 R w \Rightarrow w \models \Gamma \). Thus \( S := \{ \Gamma \} \) obeys (4.2).

4.4 Main Soundness Proof

**Lemma 10.** If \( \Gamma \vdash \Delta \) is derivable then \( \Gamma \vdash \Delta \) is not falsifiable.

*Proof.* By induction on the height \( k \) of the derivation.

**Base case:** For the base case, the height is 1. A derivation of height 1 can 
only be an instance of (\( \bot_L \)), (\( \top_R \)) or (\( Id \)). In each case, \( \gamma \) is not falsifiable, as 
shown in cases 1 to 3 of Lemma 8.

**Inductive step:** We assume that if there is a derivation for \( \gamma \) of height \( \leq k \), 
then \( \gamma \) is not falsifiable. We show that if there is a derivation for \( \gamma \) of height 
\( \leq k + 1 \), then \( \gamma \) is not falsifiable.

For a contradiction, suppose there is a derivation \( T \) for \( \gamma \) of height \( k + 1 \) and 
\( \gamma \) is falsifiable. Consider the bottom-most rule application \( \rho \) in \( T \), then \( \gamma \) is the 
conclusion of \( \rho \).

Then, by Definition 16, since \( T \) is a derivation, then all universally branching 
premises, or some existentially branching premise of \( \rho \) are rooted at derivations 
of height \( \leq k \), so by the induction hypothesis, all universally branching premises 
are, or some existentially branching premise is not falsifiable. But since the 
conclusion \( \gamma \) of \( \rho \) is falsifiable by supposition, then by Lemma 9, some universally 
branching premise, or all existentially branching premises are falsifiable. Now we 
have a contradiction, therefore our assumption was wrong and \( \gamma \) is not falsifiable.

**Theorem 2 (Soundness).** If \( \Gamma \vdash \Delta \) is derivable, then \( \Gamma \vDash_{\text{BiInt}} \Delta \).

*Proof.* By Lemma 10, we have that \( \Gamma \vdash \Delta \) is not derivable. Then by Lemma 3, 
we have \( \Gamma \vDash_{\text{BiInt}} \Delta \).

5 Completeness

5.1 Proof Outline

We wish to prove:

if \( \Gamma \vDash_{\text{BiInt}} \Delta \), then \( \Gamma \vdash \Delta \) is derivable.

Instead, we prove the contrapositive:

if \( \Gamma \vdash \Delta \) is not derivable, then there exists a counter-model for \( \Gamma \vDash_{\text{BiInt}} \Delta \).

Our proof is based on a standard technique for proving completeness of tableau 
calculi: see [9]. We have adapted this technique to a two-sided sequent calculus 
with variables.

We assume that \( \Gamma \vdash \Delta \) is not derivable, meaning that none of the GBiInt-
trees for \( \Gamma \vdash \Delta \) is a derivation. Then we choose formulae from sequents found in 
possibly different GBiInt-trees for \( \Gamma \vdash \Delta \) in order to construct a counter-model 
for \( \Gamma \vDash_{\text{BiInt}} \Delta \). The counter-model is constructed so that it contains a world \( w_0 \) 
such that \( w_0 \models \Gamma \) and \( w_0 \models \Delta \), hence \( \Gamma \vDash_{\text{BiInt}} \Delta \) does not hold.
5.2 Saturated Sets

Definition 24. Given a sequent $\Gamma \vdash \Delta$, we say that:

- $\Gamma \vdash \Delta$ is **consistent** if all of the following hold:
  1. $\bot \notin \Gamma$
  2. $\top \notin \Delta$
  3. $\Gamma \cap \Delta = \epsilon$
- $\Gamma \vdash \Delta$ is **closed** with respect to a GBiInt rule $\rho$ if either:
  1. $\rho$ is not applicable to $\Gamma \vdash \Delta$, or
  2. Whenever $\Gamma \vdash \Delta$ matches the conclusion of an instance of $\rho$, then for some premise $\Gamma_1 \vdash \Delta_1$ of the instance of $\rho$, we have $\Gamma_1 \subseteq \Gamma$ and $\Delta_1 \subseteq \Delta$.
- $\Gamma \vdash \Delta$ is **saturated** if it is consistent and closed with respect to the static rules of GBiInt.

The following corollaries follow directly from the definition of consistent sequents.

Corollary 3. If $\Gamma \vdash \Delta$ is consistent, then none of the rules $(Id)$, $(\bot\,L)$, $(\top\,R)$ is applicable to it.

Corollary 4. If the sequent $\frac{S \parallel \Gamma \vdash \Delta}{P}$ is not derivable, then $\Gamma \vdash \Delta$ is consistent for all values of $S$ and $P$.

Remark 2. As usual, every sequent has a set of one or more “saturations” due to the branching of $(\land\,R)$, $(\lor\,L)$, etc., rules. The usual approach is to non-deterministically choose one of the non-derivable premises of each such rule. However, in the presence of the inverse relation, a branch that appears open may close once we return variables to a lower sequent. Therefore, we need to temporarily keep all the non-derivable premises, since we do not know which of the open branches will stay open when we return to a lower sequent.

Lemma 11. For each finite non-derivable sequent $\Gamma \vdash \Delta$, there is an effective procedure to construct a finite set $\zeta = \{\alpha_1, \cdots, \alpha_n\}$ of finite saturated sequents, with $\Gamma \cup \Delta \subseteq LHS(\alpha_j) \cup RHS(\alpha_j) \subseteq sf(\Gamma) \cup sf(\Delta)$ for all $1 \leq j \leq n$.

Proof. Since $\Gamma \vdash \Delta$ is non-derivable, we know from Corollary 4 that $\Gamma \vdash \Delta$ is consistent. Then from Corollary 3 we know that the $(Id)$, $(\bot\,L)$, $(\top\,R)$ rules are not applicable to $\Gamma \vdash \Delta$. Let $T = \Gamma \vdash \Delta$. While some static rule $\rho$ is applicable to a leaf of $T$, extend $T$ by applying $\rho$ to the leaf to obtain new leaves. Keep the non-derivable leaves only; by Corollary 4 they are consistent. By Theorem 1, the saturation process will eventually terminate; let $\zeta = \{\alpha_1, \cdots, \alpha_n\}$ be the final leaves of $T$. Since the formulae in each premise are always subformulae of the conclusion, we have that $LHS(\alpha_j) \cup RHS(\alpha_j) \subseteq sf(\Gamma) \cup sf(\Delta)$ for all $1 \leq j \leq n$. 

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5.3 Model Graphs and Satisfiability Lemma

We shall use model graphs as an intermediate structure between GBiInt-trees and BiInt models.

**Definition 25.** A model graph for a sequent $\Gamma \vdash \Delta$ is a finite BiInt frame $\langle W, R \rangle$ such that all $w \in W$ are saturated sequents $\Gamma_w \vdash \Delta_w$ and all of the following hold:

1. $\Gamma \subseteq \Gamma_{w_0}$ and $\Delta \subseteq \Delta_{w_0}$ for some $w_0 \in W$, where $w_0 = \Gamma_{w_0} \vdash \Delta_{w_0}$;
2. if $\varphi \rightarrow \psi \in \Delta_w$ then $\exists v \in W$ with $wRv$ and $\varphi \in \Gamma_v$ and $\psi \in \Delta_v$;
3. if $\varphi \leftarrow \psi \in \Gamma_w$ then $\exists v \in W$ with $vRw$ and $\varphi \in \Gamma_v$ and $\psi \in \Delta_v$;
4. if $wRv$ and $\varphi \rightarrow \psi \in \Gamma_w$ then $\psi \in \Gamma_v$ or $\varphi \in \Delta_v$;
5. if $wRv$ and $\varphi \leftarrow \psi \in \Delta_w$ then $\psi \in \Gamma_v$ or $\varphi \in \Delta_w$;
6. if $wRv$ and $\varphi \in \Gamma_w$ then $\varphi \in \Gamma_v$;
7. if $vRw$ and $\varphi \in \Delta_w$ then $\varphi \in \Delta_v$.

We now show that given a model graph, we can use it to construct a BiInt model.

**Lemma 12.** If there exists a model graph $\langle W, R \rangle$ for $\Gamma \vdash \Delta$, then there exists a BiInt model $M = \langle W, R, \vartheta \rangle$ such that for some $w_0 \in W$, we have $w_0 \models \Gamma$ and $w_0 \models \Delta$. We call $M$ the counter-model for $\Gamma \vdash \Delta$.

**Proof.** Since we already have a BiInt frame $\langle W, R \rangle$, we need to define a valuation $\vartheta$ in order to construct a BiInt model $M = \langle W, R, \vartheta \rangle$:

1. For every world $w \in W$ and every atom $p \in \Gamma_w$, let $\vartheta(w, p) = \text{true}$.
2. For every world $w \in W$ and every atom $q \in \Delta_w$, let $\vartheta(w, q) = \text{false}$.

Then properties 6 and 7 of Definition 25 ensure persistence and reverse persistence respectively.

We now need to show that for every world $w \in W$, we have $w \models \Gamma_w$ and $w \models \Delta_w$; we can do this by simple induction on the length of the formulae in $\Gamma \vdash \Delta$.

Now let $w_0$ be the world in the model graph such that $\Gamma \subseteq \Gamma_{w_0}$ and $\Delta \subseteq \Delta_{w_0}$. Since our proof by induction has shown that for every world $w \in W$, we have $w \models \Gamma_w$ and $w \models \Delta_w$, then in particular, we have that $w_0 \models \Gamma_{w_0}$ and $w_0 \models \Delta_{w_0}$. Then, since we have that $\Gamma \subseteq \Gamma_{w_0}$ and $\Delta \subseteq \Delta_{w_0}$, we also have $w_0 \models \Gamma$ and $w_0 \models \Delta$.

5.4 Main Completeness Proof

We now show how to construct a model graph for $\Gamma \vdash \Delta$ from a consistent $\Gamma \vdash \Delta$. Recall from Remark 2 that we need to keep a number of independent versions of worlds because of the choices arising due to disjunctive non-determinism. We do this by storing one or more independent connected-components $\langle W_1, R_1 \rangle, \ldots, \langle W_n, R_n \rangle$ in the constructed model graph $\langle W, R \rangle$, and the indices (sorts) of worlds and
Procedure MGC

Input: sequent \( \Gamma \vdash \Delta \)

Output: model graph \( \langle W^f, R^f \rangle \), variables \( S^f \) and \( P^f \)

1. Let \( \zeta = \{ \alpha_1, \ldots, \alpha_n \} \) be the result of saturating \( \Gamma \vdash \Delta \) using Lemma 11;
2. For each \( \alpha_i \in \zeta \) do
   (a) Let \( \langle W_i, R_i \rangle = \{ \langle \alpha_i \rangle, \{ (\alpha_i, \alpha_i) \} \} \); let \( \text{recompute} := \text{false} \);
   (b) For each non-blocked \( \varphi \rightarrow \psi \in \Delta_{\alpha_i} \) and while \( \text{recompute} = \text{false} \) do
      i. Apply \( \langle \neg \mu \rangle \) to \( \varphi \rightarrow \psi \) and obtain a left premise \( \pi_1 = \Gamma_{\alpha_i}, \varphi \vdash \psi \);
      ii. Let \( \langle W, R \rangle, S, P := \text{MGC}(\pi_1) \);
      iii. If \( \exists H_j \in P : H_j \subseteq \Delta_{\alpha_i} \) then
         A. Let \( u_i \in W_j \) be the root of the connected component \( W_j \) from \( W \);
         B. Let \( G = \langle W_i, R_i \rangle[j := i] \); add \( G \) to \( \langle W_i, R_i \rangle \), and put \( \alpha_i \in P, u_i \).
      iv. else
         A. Let \( \langle W_i, R_i \rangle = (\epsilon, \epsilon) \); let \( \text{recompute} := \text{true} \);
         B. Invoke the right premise of \( \langle \neg \mu \rangle \) to obtain \( \pi_2 = \Gamma_{\alpha_i} \wedge \Delta_{\alpha_i} \);
         C. Apply \( \langle \wedge \mu \rangle \) to \( \pi_2 \) to obtain \( m \geq 1 \) non-derivable premises \( \gamma_1, \ldots, \gamma_m \);
         D. For each \( \gamma_k, 1 \leq k \leq m \), let \( \langle W_k, R_k \rangle, S_k, P_k := \text{MGC}(\gamma_k) \);
         E. Let \( \langle W_i, R_i \rangle := \langle \bigcup W_k, \bigcup R_k \rangle \), and \( S_i := \bigcup S_k \) and \( P_i := \bigcup P_k \);
   (c) For each non-blocked \( \varphi \rightarrow \psi \in \Gamma_{\alpha_i} \) and while \( \text{recompute} = \text{false} \) do
      i. Perform a symmetric procedure to Steps 2(b)i to 2(b)ivE.
   (d) If \( \text{recompute} = \text{false} \) then let \( S_i := \{ \Gamma_{\alpha_i} \} \) and \( P_i := \{ \Delta_{\alpha_i} \} \).
3. Return \( \langle \bigcup W_i, \bigcup R_i \rangle, \bigcup S_i, \bigcup P_i \)

Fig. 4. Model Graph Construction Procedure
When we return from MGC, we form the union of the components of the model graph and the variables from the different “states”, so that the caller of MGC can extract the appropriate component at Step 2(b)iiiA.

Remark 3. Note that while the counter-model construction procedure keeps the whole counter-model in memory, this procedure is only used to prove the completeness of GBiInt. Our procedure for checking the validity of BiInt formulae (Fig. 3) does not need the whole counter-model, and explores one branch at a time, as is usual for sequent/tableaux calculi.

Theorem 3 (Completeness). GBiInt is complete: if \( \Gamma \vdash \Delta \) is not derivable, then there exists a counter-model for \( \Gamma \Vdash_{\text{Int}} \Delta \).

Proof. Suppose \( \Gamma \vdash \Delta \) is not derivable, then by Corollary 4 we have that \( \Gamma \vdash \Delta \) is consistent. We construct a model graph for \( \Gamma \vdash \Delta \) using the procedure given in Figure 4, and obtain \( \langle W^f, R^f \rangle \). We let \( \langle W, R \rangle \) be any connected component of \( \langle W^f, R^f \rangle \). We now show that \( \langle W, R \rangle \) satisfies the properties of a model graph from Definition 25:

1. \( \Gamma \subseteq \Gamma_{w_0} \) and \( \Delta \subseteq \Delta_{w_0} \) for some \( w_0 \in W \): This holds because \( w_0 \) is one of the saturated sequents obtained from \( \Gamma \vdash \Delta \). Moreover, if we delete the original \( w_0 \) at Step 2(b)ivA, a final version of \( w_0 \) is created at Step 2(b)iiiB which is never deleted.
2. if \( \varphi \rightarrow \psi \in \Delta_w \) then \( \exists v \in W \) with \( wRv \) and \( \varphi \in \Gamma_v \) and \( \psi \in \Delta_v \): This holds because we have either created \( v \) using \( (\rightarrow R) \) at Step 2(b)iiiB, or had \( w \) fulfill the role of this successor by reflexivity if \( (\rightarrow R) \) was blocked.
3. if \( \varphi<\psi \in \Gamma_w \) then there exists some \( v \in W \) with \( vRw \) and \( \varphi \in \Gamma_v \) and \( \psi \in \Delta_v \):
   By symmetry with property 2.
4. if \( wRv \) and \( \varphi \rightarrow \psi \in \Gamma_w \) then \( \psi \in \Gamma_v \) or \( \varphi \in \Delta_v \): In our construction, there are three ways of obtaining \( wRv \), so we need to show that for each case, the property holds. We first show that \( \varphi \rightarrow \psi \in \Gamma_v \):
   (a) \( v \) was created by applying \( (\rightarrow R) \) to \( w \) on some \( \alpha \rightarrow \beta \in \Delta_w \). Then \( \Gamma_v \) also contains \( \varphi \rightarrow \psi \).
   (b) \( w \) was created by applying \( (\prec L) \) to some \( \alpha \prec \beta \in \Gamma_v \). Then, when the final version of \( \Gamma_v \) was created, \( \varphi \rightarrow \psi \in \Gamma_w \) was added to the \( S \) variable at Step 2d. There are two cases:
     - The right premise \( \pi_2 \) of \( (\prec L) \) was invoked at \( v \). Then \( S \) was added to \( \pi_2 \) at \( v \) by the symmetric process to Step 2(b)ivB. Thus the updated \( \Gamma_v \) also contains \( \varphi \rightarrow \psi \).
     - The right premise of \( (\prec L) \) was not invoked at \( v \). This means that \( \exists \Sigma_j \subseteq S, \Sigma_j \subseteq \Gamma_v \), and the \( j \)-th version of \( v \)'s predecessor \( w \) is chosen at the symmetric process to Step 2(b)iiiA. But since Step 2d at \( w \) assigns \( \Sigma_j := \Gamma_w \), then we have \( \Gamma_w \subseteq \Gamma_v \) and thus \( \varphi \rightarrow \psi \in \Gamma_v \).
   (c) \( v = w \), and \( wRw \) by reflexivity. Then \( \Gamma_v = \Gamma_w \), so \( \varphi \rightarrow \psi \in \Gamma_v \).

In all cases, saturation for \( v \) will then ensure that \( \psi \in \Gamma_v \) or \( \varphi \in \Delta_v \).
5. if $vRw$ and $\varphi \rightarrow \psi \in \Delta_w$ then $\psi \in \Gamma_v$ or $\varphi \in \Delta_v$:  
By symmetry with property 4.
6. if $wRv$ and $\varphi \in \Gamma_w$ then $\varphi \in \Gamma_v$:  
By similar argument to property 4.
7. if $vRw$ and $\varphi \in \Delta_w$ then $\varphi \in \Delta_v$:  
By symmetry with property 6.

We can obtain a counter-model for $\Gamma \models_{\text{n.int}} \Delta$ from $\langle W, R \rangle$ via Lemma 12.

**Definition 26.** A di-tree is a directed graph such that if the direction of the edges is ignored, it is a tree.

**Theorem 4.** Every falsifiable BiInt sequent can be falsified by a model whose frame is a di-tree, consisting of reflexive points.

**Proof.** From Lemmas 5 and 6, we know that the construction of new successors for $\varphi \rightarrow \psi$ and predecessors for $\varphi \rightarrow \psi$ stops when either there are no rejected $\varphi \rightarrow \psi$-formulae or forced $\varphi \rightarrow \psi$-formulae in the current world, or the current world already forces $\varphi$ and rejects $\psi$. In the latter case, the world itself fulfills the role of the successor or predecessor by reflexivity, and no new successors or predecessors are created.

The reason we are able to avoid proper cycles is the persistence and reverse persistence properties of BiInt, used in the $(-\rightarrow I_R)$ and $(-< I_L)$ rules.

Consider the $\rightarrow$ case. Every time some $\varphi \rightarrow \psi$ appears on the RHS of a sequent $\Gamma \vdash \Delta, \varphi \rightarrow \psi$, we first add $\psi$ to the RHS to obtain $\Gamma \vdash \Delta, \varphi \rightarrow \psi, \psi$ using the $(-\rightarrow I_R)$ rule, since by reverse persistence the current world must reject everything that some successor world rejects. Now that $\psi$ is on the RHS, we need to apply the $(-\rightarrow I_R)$ rule to create the $\varphi \rightarrow \psi$-successor $\Gamma, \varphi \vdash \psi$ only if $\varphi$ is not already on the LHS. For if $\varphi \in LHS$, then the successor $\Gamma \vdash \psi$ that fulfills $\varphi \rightarrow \psi$ can be the current world itself. So there is no point creating it explicitly.

**Corollary 5.** BiInt is characterised by finite rooted reflexive and transitive di-trees of reflexive points.

### 6 Conclusions and Future Work

Our cut-free calculus for BiInt enjoys terminating backward proof-search and is sound and complete w.r.t Kripke semantics. A simple Java implementation of GBiInt is available at http://users.rsise.anu.edu.au/~linda. The next step is to add a cut rule to GBiInt, and prove cut elimination syntactically. We are also extending our work to the modal logic \(S5\), and the tense logic \(Kt.S4\). Our approach of existential branching and inter-premise communication bears some similarities to hypersequents of Pottinger and Avron [1]. It would be interesting to investigate this correspondence further. From an automated deduction perspective, GBiInt is the first step towards an efficient decision procedure for BiInt. The next task is to analyse the computational complexity of GBiInt and investigate which of the traditional optimisations for tableaux systems are still applicable in the intuitionistic case.

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