Moderate maximal inequalities for the Ornstein-Uhlenbeck process

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Abstract

The maximal inequalities for diffusion processes have drawn increasing attention in recent years. However, the existing proof of the $L^p$ maximum inequalities for the Ornstein-Uhlenbeck process was dubious. Here we give a rigorous proof of the moderate maximum inequalities for the Ornstein-Uhlenbeck process, which include the $L^p$ maximum inequalities as special cases and generalize the remarkable $L^1$ maximum inequalities obtained by Graversen and Peskir [P. Am. Math. Soc., 128(10):3035-3041, 2000]. As a corollary, we also obtain a new moderate maximal inequality for continuous local martingales, which can be viewed as a supplement of the classical Burkholder-Davis-Gundy inequality.

Keywords: moderate function, law of the iterated logarithm, good $\lambda$ inequality

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1 Introduction

The Ornstein-Uhlenbeck process, which is the solution to the Langevin equation

$$dX_t = -\alpha X_t dt + dW_t, \quad X_0 = 0,$$

is one of the most important kinetic models in statistical mechanics, where $\alpha > 0$ and $W$ is a standard Wiener process. It describes the velocity of an underdamped Brownian particle or the position of an overdamped Brownian particle driven by the harmonic potential. An important question is how far the Brownian particle can travel before a given time. This problem is closely related to the maximal inequalities in probability theory.

The $L^p$ maximal inequalities for martingales are one of the classical results in probability theory. Let $M$ be a continuous local martingale vanishing at zero. The Burkholder-Davis-Gundy (BDG) inequality [2] claims that for any $p > 0$, there exist two positive constants $c_p$ and $C_p$ such that for any stopping time $\tau$ of $M$,

$$c_p \mathbb{E}[M]\tau^{p/2} \leq \mathbb{E}\left[ \sup_{0 \leq t \leq \tau} |M_t|^p \right] \leq C_p \mathbb{E}[M]\tau^{p/2},$$

where $[M]$ is the quadratic variation process of $M$ (see [2, Chapter IV, Exercise 4.25] for a moderate version of this inequality).
Over the past two decades, significant progress has been made in the maximal inequalities for diffusion processes [3–10]. In particular, Graversen and Peskir [5] proved the following remarkable $L^1$ maximum inequality for the Ornstein-Uhlenbeck process by using Lenglart’s domination principle: there exist two positive constants $c$ and $C$ independent on $\alpha$ such that for any stopping time $\tau$ of $X$,

$$\frac{c}{\alpha^{1/2}} \mathbb{E} \log^{1/2}(1 + \alpha \tau) \leq \mathbb{E} \left[ \sup_{0 \leq t \leq \tau} |X_t| \right] \leq \frac{C}{\alpha^{1/2}} \mathbb{E} \log^{1/2}(1 + \alpha \tau).$$

Subsequently, Peskir [6] established the $L^1$ maximum inequalities for a large class of diffusion processes and obtained satisfactory results. However, the $L^p$ maximum inequalities for diffusion processes turn out to be more difficult, even for the Ornstein-Uhlenbeck process. As an attempt, Yan et al. [7–9] studied the $L^p$ maximum inequalities for a class of diffusion processes. Although their ideas are fairly nice, their detailed proofs are questionable because they mistakenly regarded the random time $T_{I(S<T)}$ as a stopping time, where $S$ and $T$ are two stopping times with $S \leq T$ [8, Page 6, Lines 2 and 10].

In this paper, we give a rigorous proof of the moderate maximum inequalities for the Ornstein-Uhlenbeck process, which include the $L^p$ maximum inequalities as special cases. Our method is based on using “good-$\lambda$” inequalities (Lemma 3.3) introduced by Burkerholder (cf. [1]). As a corollary, we also obtain a new moderate maximal inequality for the Brownian motion and general continuous local martingales, which can be viewed as a good supplement to the classical BDG inequality.

2 Results

Let $X$ be the one-dimensional Ornstein-Uhlenbeck process starting from zero, which is the solution to the stochastic differential equation

$$dX_t = -\alpha X_t dt + dW_t, \quad X_0 = 0,$$

where $\alpha > 0$ and $W$ is a standard Brownian motion defined on some filtered probability space $(\Omega, \{\mathcal{F}_t\}, \mathbb{P})$. We next introduce the conception of moderate function(see also [2, p.164, line 11]).

**Definition 2.1.** A function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is called *moderate* if

(a) it is a continuous increasing function vanishing at zero and
(b) there exists $\lambda > 1$ and $\gamma < \infty$ such that

$$f(\lambda t) \leq \gamma f(t) \quad \text{for all } t > 0. \quad (2.2)$$

In particular, $f(t) = t^p$ is a moderate function for any $p > 0$. Our main result is the following moderate maximum inequality.
Theorem 2.2. For any moderate function $f$, there exists two positive constants $c_{\alpha,f}$ and $C_{\alpha,f}$ such that for any stopping time $\tau$ with respect to $\{\mathcal{F}_t\}$,

$$c_{\alpha,f} \mathbb{E} f \left( \log^{1/2}(1 + \alpha\tau) \right) \leq \mathbb{E} \left[ \sup_{0 \leq t \leq \tau} f \left( |X_t| \right) \right] \leq C_{\alpha,f} \mathbb{E} f \left( \log^{1/2}(1 + \alpha\tau) \right).$$

In particular, for any $p > 0$, there exists two positive constants $c_p$ and $C_p$ independent of $\alpha$ such that for any stopping time $\tau$ of $X$,

$$c_p \mathbb{E} \log^{p/2}(1 + \alpha\tau) \leq \mathbb{E} \left[ \sup_{0 \leq t \leq \tau} |X_t|^p \right] \leq C_p \mathbb{E} \log^{p/2}(1 + \alpha\tau).$$

The above theorem implies two moderate maximum inequalities for the Brownian motion and continuous local martingale which are in line with Corollary 2.7 and 2.8 in [5]. For the reader’s convenience, we provide short proofs here.

Corollary 2.3. For any moderate function $f$, there exists two positive constants $c_f$ and $C_f$ such that for any stopping time $\tau$ of $W$,

$$c_f \mathbb{E} f \left( \log^{1/2}(1 + \log(1 + \tau)) \right) \leq \mathbb{E} \left[ \sup_{0 \leq t \leq \tau} f \left( \frac{|W_t|}{\sqrt{1 + t}} \right) \right] \leq C_f \mathbb{E} f \left( \log^{1/2}(1 + \log(1 + \tau)) \right).$$

In particular, for any $p > 0$, there exists two positive constants $c_p$ and $C_p$ such that for any stopping time $\tau$ of $W$,

$$c_p \mathbb{E} \log^{p/2}(1 + \log(1 + \tau)) \leq \mathbb{E} \left[ \sup_{0 \leq t \leq \tau} \frac{|W_t|^p}{(1 + t)^{p/2}} \right] \leq C_p \mathbb{E} \log^{p/2}(1 + \log(1 + \tau)).$$

Proof. It is easy to check that (2.1) has the following explicit solution:

$$X_t = \int_0^t e^{\alpha(s-t)} dW_s. \tag{2.3}$$

It is well known that there exists a Brownian motion $B$ such that

$$X_t = \frac{1}{\sqrt{2\alpha}} e^{-\alpha t} B_{e^{2\alpha t} - 1} = \frac{1}{\sqrt{2\alpha}} \frac{B_{H(t)}}{\sqrt{H(t) + 1}}, \tag{2.4}$$

where $H(t) = e^{2\alpha t} - 1$. For any stopping time $\tau$ of $B$, it is easy to check that $H^{-1}(\tau) = \log(1 + \tau)/2\alpha$ is a stopping time of $X$. Thus it follows from Theorem 2.2 that

$$\mathbb{E} \left[ \sup_{0 \leq t \leq \tau} f \left( \frac{1}{\sqrt{2\alpha}} \frac{|B_t|}{\sqrt{1 + t}} \right) \right] \sim \mathbb{E} f \left( \log^{p/2}(1 + \alpha H^{-1}(\tau)) \right) = \mathbb{E} f \left( \log^{p/2}(1 + \frac{1}{2} \log(1 + \tau)) \right).$$

The desired result follows from the definition the moderate function and the fact that

$$\log(1 + \frac{1}{2} \log(1 + x)) \sim \log(1 + \log(1 + x))$$

as $x \to 0$ or $x \to \infty$. \qed
Since any continuous local martingale is a time change of the Brownian motion, the above corollary implies a moderate maximal inequality for continuous local martingales.

**Corollary 2.4.** Let $M$ be a continuous local martingale vanishing at zero with quadratic variation process $[M]$. For any moderate function $f$, there exists two positive constants $c_f$ and $C_f$ such that for any stopping time $\tau$ of $M$,

$$c_f \mathbb{E} f \left( \log^{1/2}(1 + \log(1 + [M]_\tau)) \right) \leq \mathbb{E} \left[ \sup_{0 \leq t \leq \tau} f \left( \frac{|M_t|}{\sqrt{1 + [M]_t}} \right) \right] \leq C_f \mathbb{E} f \left( \log^{1/2}(1 + \log(1 + [M]_\tau)) \right).$$

In particular, for any $p > 0$, there exists two positive constants $c_p$ and $C_p$ independent of $M$ such that for any stopping time $\tau$ of $M$,

$$c_p \mathbb{E} \log^{p/2}(1 + \log(1 + [M]_\tau)) \leq \mathbb{E} \left[ \sup_{0 \leq t \leq \tau} |M_t|^p (1 + [M]_t)^{p/2} \right] \leq C_p \mathbb{E} \log^{p/2}(1 + \log(1 + [M])).$$

**Proof.** Since $M$ is a continuous local martingale, there exists a Brownian motion $B$ such that $M_t = B_{[M]_t}$. Thus the desired result follows directly from Corollary 2.3.

According to the BDG inequality, the $L^p$ maximum of a continuous local martingale $M$ in average behaves as $[M]^{p/2}$. The above corollary shows that the $L^p$ maximum of $M$, normalized by $(1 + [M])^{p/2}$, in average behaves as $\log^{p/2}(1 + \log(1 + [M]))$. The relationship between the BDG inequality and our result is rather similar to that between the central limit theorem and the law of iterated logarithm.

### 3 Proof of Theorem 2.2

Let $X^*$ be the maximum process of $|X|$ defined by

$$X^*_t = \sup_{0 \leq s \leq t} |X_s|.$$  

To prove our main result, we first need two lemmas.

**Lemma 3.1.** There exists a function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\phi(\delta) \to 0$ as $\delta \to 0$ such that for any $t \geq 1$ and $\delta > 0$,

$$\mathbb{P}(X^*_t < \delta \log^{1/2} t) \leq \phi(\delta).$$  

**Proof.** For any $x \geq 0$, let $\tau_x = \inf \{ t \geq 0 : |X_t| = x \}$. Following the proof for equation (2.30) in [3], we define

$$u(x) := 2 \int_0^x e^{\alpha y^2} dy \int_0^y e^{-\alpha z^2} dz.$$ 

It is easy to check that $u \in C^2(\mathbb{R})$ and satisfies the ordinary differential equation

$$\frac{1}{2} u''(x) - \alpha xu'(x) = 1, \quad u(0) = u'(0) = 0.$$
By Ito’s formula, we have
\[
\mathbb{E}u(X_{\tau_x \wedge t}) = \mathbb{E}x \wedge t + \mathbb{E} \int_0^{\tau_x \wedge t} u'(X_s)dW_s = \mathbb{E}x \wedge t.
\]
Since \(u\) is an even function, taking \(t \to \infty\) in the above equation gives rise to
\[
\mathbb{E}x = u(x) = 2 \int_0^x e^{\alpha y^2} dy \int_y^\infty e^{-\alpha z^2} dz \leq \sqrt{\frac{\pi}{\alpha}} e^{\alpha x^2} \leq C e^{2\alpha x^2},
\]
where \(C\) is a constant independent of \(x\). By Chebyshev’s inequality, we have
\[
\mathbb{P}(X_t < \delta \log^{1/2} t) = \mathbb{P}(\tau_{\delta \log^{1/2} t} \geq t) \leq t^{-1} \mathbb{E}\tau_{\delta \log^{1/2} t} \leq C t^{2\alpha^2 - 1}.
\]
When \(\delta\) is sufficiently small, for any \(\epsilon > 0\), we can find \(T > 1\) such that
\[
\sup_{t \geq T} \mathbb{P}(X_t < \delta \log^{1/2} t) \leq C T^{-1/2} < \epsilon.
\]
On the other hand, when \(\delta\) is sufficient small,
\[
\sup_{t \leq T} \mathbb{P}(X_t < \delta \log^{1/2} t) \leq \mathbb{P}(X_t < \delta \log^{1/2} T) < \epsilon.
\]
Combining the above two inequalities, we obtain the desired result.

\textbf{Lemma 3.2.} There exists a function \(\phi : \mathbb{R}_+ \to \mathbb{R}_+\) satisfying \(\phi(\delta) \to 0\) as \(\delta \to 0\) such that for any \(t \geq 2\) and \(\delta > 0\),
\[
\mathbb{P}(X_t \geq \delta^{-1} \log^{1/2} t) \leq \phi(\delta).
\]

\textbf{Proof.} By the law of iterated logarithm of the Brownian motion, we have
\[
\limsup_{t \to \infty} \frac{|B_t|}{\sqrt{2t \log \log t}} = 1, \quad \text{a.s.}
\]
Thus it follows from (2.4) that
\[
\limsup_{t \to \infty} \frac{|X_t|}{\log^{1/2} t} = \frac{1}{\sqrt{2\alpha}} \limsup_{t \to \infty} \frac{|B_{\log^{1/2} t}|}{e^{\alpha \log^{1/2} t}} = \frac{1}{\sqrt{\alpha}}, \quad \text{a.s.} \tag{3.3}
\]
Obviously, \(\limsup_{t \to \infty} \frac{X_t}{\log^{1/2} t} \geq \frac{1}{\sqrt{\alpha}}, \varepsilon\text{.a.s.}\). On the other hand, let \(k(t) := \inf\{s : |X_s| = X_t^*\}\), noticing any continuous function can attain its maximum over a closed interval, \(k(t)\) is well-defined and \(k(t) \leq t\). By (3.3), it is easy to see \(k(t) \to \infty\) as \(t \to \infty\), so
\[
\limsup_{t \to \infty} \frac{X_t^*}{\log^{1/2} t} \leq \limsup_{t \to \infty} \frac{X_{k(t)}}{\log^{1/2} k(t)} \leq \limsup_{t \to \infty} \frac{|X_t|}{\log^{1/2} t} = \frac{1}{\sqrt{\alpha}}.
\]
To sum up, we have
\[
\limsup_{t \to \infty} \frac{X_t^*}{\log^{1/2} t} = \frac{1}{\sqrt{\alpha}}, \quad \text{a.s.}\]
When \(\delta\) is sufficiently small, for any \(\epsilon > 0\), we can find \(T \geq 2\) such that
\[
\sup_{t \geq T} \mathbb{P}(X_t^* \geq \delta^{-1} \log^{1/2} t) \leq \mathbb{P} \left( \sup_{t \geq T} \frac{X_t^*}{\log^{1/2} t} \geq \frac{2}{\sqrt{\alpha}} \right) < \epsilon.
\]
On the other hand, when $\delta$ is sufficient small, we have
\[
\sup_{2 \leq t \leq T} \mathbb{P}(X_t^x \geq \delta^{-1} \log^{1/2} t) \leq \mathbb{P}(X_T^x \geq \delta^{-1} \log^{1/2} 2) < \epsilon.
\]
Combining the above two inequalities, we obtain the desired result.

To proceed, we need a classical result, whose proof can be found in [2, Chapter IV, Lemma 4.9].

**Lemma 3.3.** Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfy $\phi(\delta) \to 0$ as $\delta \to 0$. Assume that a pair of nonnegative random variables $(X, Y)$ satisfies the following good $\lambda$ inequality for any $\delta, \lambda > 0$:
\[
\mathbb{P}(X \geq 2\lambda, Y < \delta \lambda) \leq \phi(\delta) \mathbb{P}(X \geq \lambda).
\]
Then for any moderate function $f$, there exists a positive constant $C$ depending on $f$ and $\phi$ such that
\[
\mathbb{E} f(X) \leq C \mathbb{E} f(Y).
\]

In order to use the above lemma to prove our main results, we still need the following two lemmas.

**Lemma 3.4.** There exists a function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\phi(\delta) \to 0$ as $\delta \to 0$ such that for any stopping time $\tau$ with respect to $\mathcal{F}_t$ and any $\delta, \lambda > 0$, the following good $\lambda$ inequality holds:
\[
\mathbb{P}(\log^{1/2}(1 + \alpha \tau) \geq 2\lambda, X_\tau^* < \delta \lambda) \leq \phi(\delta) \mathbb{P}(\log^{1/2}(1 + \alpha \tau) \geq \lambda).
\]

**Proof.** It is easy to see that
\[
\mathbb{P}_0(\log^{1/2}(1 + \alpha \tau) \geq 2\lambda, X_\tau^* < \delta \lambda) \leq \mathbb{P}_0(\tau \geq r, X_\tau^* < \delta \lambda),
\]
where $r = \alpha^{-1}(e^{\lambda^2} - 1)$ and $s = \alpha^{-1}(e^{4\lambda^2} - 1)$. By the Markov property of $X$, we have
\[
\mathbb{P}_0(\log^{1/2}(1 + \alpha \tau) \geq 2\lambda, X_\tau^* < \delta \lambda) \leq \mathbb{E}_0 \left[ 1_{(\tau \geq r)} \mathbb{P}_0(X_s^* < \delta \lambda | \mathcal{F}_r) \right] \leq \mathbb{E}_0 \left[ 1_{(\tau \geq r)} \mathbb{P}_{X_r}(X_{s-r}^* < \delta \lambda) \right] \leq \sup_{|x| < \delta \lambda} \mathbb{P}_x(X_{s-r}^* < \delta \lambda) \mathbb{P}_0(\tau \geq r).
\]

For any $x \in \mathbb{R}$, let $X^x$ be the solution to the following stochastic differential equation:
\[
dX_t^x = -\alpha X_t^x dt + dW_t, \quad X_0^x = x.
\]
Then it is easy to check that $X_t^x = X_0^0 + xe^{-t}$ for any $x \in \mathbb{R}$ and $t \geq 0$. This suggests that
\[
\sup_{|x| < \delta \lambda} \mathbb{P}_x(X_{s-r}^* < \delta \lambda) \leq \mathbb{P}_0(X_{s-r}^* < 2\delta \lambda).
\]
Thus we obtain that
\[
\mathbb{P}_0(\log^{1/2}(1 + \alpha \tau) \geq 2\lambda, X_\tau^* < \delta \lambda) \leq \mathbb{P}_0(X_{s-r}^* < 2\delta \lambda) \mathbb{P}_0(\log^{1/2}(1 + \alpha \tau) \geq \lambda).
\]
For convenience, set $\lambda_0 = \log^{1/2}(\alpha + 1)$. We first consider the case of $\lambda > \lambda_0$. In this case, we have $s - r \geq e^{\lambda^2} \geq 1$. Thus it follows from Lemma 3.1 that

$$
P_0(X^*_{s-r} < 2\delta \lambda) \leq P_0(X^*_{c\lambda^2} < 2\delta \lambda) \leq \phi(2\delta), \tag{3.4}$$

where $\phi$ is the function defined in Lemma 3.1. We next consider the case of $0 < \lambda \leq \lambda_0$. In this case, we have $s - r = \alpha^{-1}(e^{\lambda^2} - e^{\lambda^2}) \geq c\lambda^2$, where $c$ is a positive constant independent of $\lambda$. It is easy to see from (2.1) that for any $t \geq 0$,

$$(1 + \alpha t)X^*_t \geq W^*_t.$$ 

This suggests that

$$
P_0(X^*_{s-r} < 2\delta \lambda) \leq P_0(X^*_{c\lambda^2} < 2\delta \lambda) \leq P_0(W^*_{c\lambda^2} < 2(1 + \alpha c\lambda_0^2)\delta \lambda)$$

$$
\leq P_0([W_{c\lambda^2}] \leq 2(1 + \alpha c\lambda_0^2)\delta \lambda) = P_0([W_{c\lambda^2}] < 2(1 + \alpha c\lambda_0^2)\delta), \tag{3.5}
$$

which tends to zero as $\delta \to 0$. Combining (3.4) and (3.5), we obtain the desired result.

**Lemma 3.5.** There exists a function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\phi(\delta) \to 0$ as $\delta \to 0$ such that for any stopping time $\tau$ with respect to $\{\mathcal{F}_t\}$ and any $\delta, \lambda > 0$, the following good $\lambda$ inequality holds:

$$
P(X^*_\tau \geq 2\lambda, \log^{1/2}(1 + \alpha \tau) < \delta \lambda) \leq \phi(\delta)P(X^*_\tau \geq \lambda).$$

**Proof.** Let $\tau_\lambda = \inf\{t \geq 0 : |X_t| = \lambda\}$. It is easy to see that

$$
P_0(X^*_\tau \geq 2\lambda, \log^{1/2}(1 + \alpha \tau) < \delta \lambda) \leq P_0(X^*_{s\vee \tau_\lambda} \geq 2\lambda, \tau > \tau_\lambda),$$

where $s = \alpha^{-1}(e^{\lambda^2} - 1)$. By the strong Markov property of $X$, we have

$$
P_0(X^*_\tau \geq 2\lambda, \log^{1/2}(1 + \alpha \tau) < \delta \lambda) \leq E_0\left[1_{\{\tau > \tau_\lambda\}}P_0(X^*_{s\vee \tau_\lambda} \geq 2\lambda | \mathcal{F}_{\tau_\lambda})\right]$$

$$\leq E_0\left[1_{\{\tau > \tau_\lambda\}}P_{X_{\tau_\lambda}}(X^*_{s\vee \tau_\lambda-\tau_\lambda} \geq 2\lambda)\right]$$

$$\leq \sup_{|x| = \lambda} P_x(X^*_{s\vee \tau_\lambda-\tau_\lambda} \geq 2\lambda)\ P_0(\tau > \tau_\lambda).$$

Since $X^*_\tau = X^*_0 + xe^{-t}$ for any $x \in \mathbb{R}$ and $t \geq 0$, it is easy to check that

$$\sup_{|x| = \lambda} P_x(X^*_{s\vee \tau_\lambda-\tau_\lambda} \geq 2\lambda) \leq P_0(X^*_{s\vee \tau_\lambda-\tau_\lambda} \geq \lambda) \leq P_0(X^*_s \geq \lambda).$$

This shows that

$$
P_0(X^*_\tau \geq 2\lambda, \log^{1/2}(1 + \alpha \tau) < \delta \lambda) \leq P_0(X^*_s \geq \lambda)\ P_0(X^*_\tau \geq \lambda).$$

For convenience, set $\lambda_0 = \log^{1/2}(2\alpha + \alpha^{-1} + 1)$. We first consider the case of $\delta \lambda > \lambda_0$. In this case, we have $2 \leq s \leq e^{6s^2}\lambda^2$. Thus it follows from Lemma 3.2 that

$$
P_0(X^*_s \geq \lambda) \leq P_0(X^*_{c\delta^2} \geq \lambda) \leq \phi(2\delta), \tag{3.6}$$
where \( \phi \) is the function defined in Lemma 3.2. We next consider the case of \( 0 < \delta \lambda \leq \lambda_0 \). In this case, we have \( s \leq C\delta^2 \lambda^2 \leq C\lambda_0^2 \), where \( C \) is a positive constant independent of \( \delta \) and \( \lambda \). By Gronwall’s inequality, it is easy to see from (2.1) that for any \( t \geq 0 \),

\[
X^*_t \leq e^{\alpha t} W^*_t.
\]

This suggests that

\[
P_0(X^*_s \geq \lambda) \leq P_0(X^*_s \geq \lambda) \leq P_0(W^*_s \geq \lambda e^{-\alpha C\lambda^2_0}) \leq P_0(|W_{C\delta^2\lambda^2}| \geq \lambda e^{-\alpha C\lambda^2_0}) = P_0(|W_C| \geq \delta^{-1} e^{-\alpha C\lambda^2_0}),
\]

which tends to zero as \( \delta \to 0 \). Combining (3.6) and (3.7), we obtain the desired result.

We are now in a position to prove our main theorem.

**Proof of Theorem 2.2** The first part of the theorem follows directly from Lemmas 3.3, 3.4, and 3.5. We next prove the second part of the theorem. Let \( Y \) be the Ornstein-Uhlenbeck process solving the stochastic differential equation

\[
dY_t = -Y_t dt + d\tilde{W}_t, \quad Y_0 = 0,
\]

where \( \tilde{W}_t = \sqrt{\alpha}W_{t/\alpha} \). By the explicit expression (2.3) and basic calculation, one can see \( X_t = Y_{\alpha t}/\sqrt{\alpha} \). And for any stopping time \( \tau \) of \( X \), it is easy to check that \( \alpha \tau \) is a stopping time of \( Y \). If we take \( f(t) = t^p \) with \( p > 0 \), then there exists positive constants \( c_p \) and \( C_p \) independent of \( \alpha \) such that

\[
c_p \mathbb{E} \log^{p/2}(1 + \alpha \tau) \leq \mathbb{E} \left[ \sup_{0 \leq t \leq \tau} |Y_{\alpha t}|^p \right] \leq C_p \mathbb{E} \log^{p/2}(1 + \alpha \tau),
\]

which gives the desired result.

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