Homographic Solutions of the $N$-body Generalized Lennard-Jones System

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Abstract In this paper, we obtain the existence of non-planar circular homographic solutions and non-circular homographic solutions of the $(2+N)$- and $(3+N)$-body problems of the Lennard-Jones system. These results show the essential difference between the Lennard-Jones potential and the Newton’s potential of universal gravitation.

Keywords Homographic solution, Lennard-Jones potential, non-planar circular solution

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1 Introduction

In this paper, we consider the existence of special homographic solutions of the generalized Lennard-Jones (L-J for short) system. The configuration space in $\mathbb{R}^3$ is defined by

$$X = \left\{ q = (q_1, q_2, \ldots, q_N) \in (\mathbb{R}^3)^N \mid \sum_{k=1}^{N} m_k q_k = 0, q_k \neq q_j, k \neq j \right\}.$$ 

The generalized $N$-body L-J potential is given by

$$U(q) = \sum_{k<j} m_k m_j \left( \frac{1}{|q_k - q_j|^\beta} - \frac{1}{|q_k - q_j|^\alpha} \right).$$

For simplicity, we consider the case of $m_i = 1$ for $1 \leq i \leq N$. By the Newton’s second law, the motion equations are given by

$$\ddot{q}_k = -\nabla U(q), \quad (1.1)$$

where $\nabla U(q)$ is the gradient of $U(q)$. Then the motion of $q_k$ satisfies

$$\ddot{q}_k = \sum_{j \neq k} \left( \frac{\beta}{|q_{kj}|^{\beta+2}} - \frac{\alpha}{|q_{kj}|^{\alpha+2}} \right) (q_k - q_j), \quad (1.2)$$

where $q_{kj} = q_k - q_j$ and $q_{jk} = -q_{kj}$. Let $q = q(t)$ be a solution of (1.2). It is flat, if $q(t)$ is contained in a plane $\Pi(t)$ in $\mathbb{R}^3$ for any $t \in \mathbb{R}$. It is planar, if there is a plane $\Pi$ in $\mathbb{R}^3$ which contains $q(t)$ for all $t \in \mathbb{R}$.

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In the literature, researchers obtained many results of the planar solutions of the N-body problems of the L-J potential. In 2004, when \( \alpha = 6, \beta = 12, a = 2 \) and \( b = 1 \), Corbera, Llibre and Pérez-Chavela in [1] and [2] obtained the existence of the constant solutions, the circular solutions and central configurations of the planar 2- and 3-body problems of L-J system. In [3], Jones studied the central configurations and proved N-gon and \((N + 1)\)-gon are the central configurations of the planar L-J system. In [6], Sbano and Southall proved the existence of the Lennard-Jones potential.

In this paper, we focus on the flat solutions and the non-flat solutions. We define that for \( 1 \leq k \leq N, r \in \mathbb{R}^+ \) and \( \omega_k = \frac{2\pi k}{N} \),

\[
Q_1 = (0, 0, r_0), \quad Q_2 = (0, 0, -r_0), \quad Q_3 = (0, 0, 0), \quad q_k = \phi(\lambda)r_0(\cos\omega_k, \sin\omega_k, 0), \quad (1.3)
\]

where \( \phi(\lambda) = \sqrt{\lambda^2 - 1} \) and define

\[
E(t) = \begin{pmatrix}
\cos\omega(t) & -\sin\omega(t) & 0 \\
\sin\omega(t) & \cos\omega(t) & 0 \\
0 & 0 & 1
\end{pmatrix}.
\quad (1.4)
\]

If \( \omega(t) = \omega_0 t \) for some constant \( \omega_0 \neq 0 \), we write \( E(t) \) as \( E_{\omega_0}(t) \) for simplicity.

At first, we consider the circular solution \( q(t) \) of the \((2 + N)\)-body problem satisfying

\[
q(t) = E_{\omega_0}(t)q_0, \quad (1.5)
\]

where \( q_0 = (Q_1, Q_2, q_1, \ldots, q_N) \in (\mathbb{R}^3)^{2+N} \) as in (1.3) and \( E_{\omega_0}(t) \) is given by (1.4). Note that angular momentum is a constant vector parallel with the z-axis, i.e., \( \dot{\omega} \equiv \omega_0 \). In this case, \( q(t) \) is the solution of (1.2) only if there exists a \( \lambda \) such that the following equation holds,

\[
\text{diag}(-\omega_0^2, -\omega_0^2, 0)q_0 = \nabla U(q).
\quad (1.6)
\]

**Theorem 1.1** When \( 0 < \alpha < \beta, N \geq 2 \), there exists \( \lambda_1 \) which is given by (iii) of Lemma 2.1 such that for any \( \lambda \geq \max\{2, \lambda_1\} \), \( q(t) = E_{\omega_0}(t)q_0 \in (\mathbb{R}^3)^{2+N} \) satisfying (1.5) is a solution of (1.2) where \( \omega_0 \) is defined by (1.4) and satisfies

\[
\omega_0 = \pm \left( \frac{\theta_\alpha}{(\sqrt{\lambda^2 - 1}r_0)^{\alpha+2}} - \frac{\theta_\beta}{(\sqrt{\lambda^2 - 1}r_0)^{\beta+2}} + \frac{2\alpha}{(\lambda r_0)^{\alpha+2}} - \frac{2\beta}{(\lambda r_0)^{\beta+2}} \right)^{\frac{1}{2}}, \quad (1.7)
\]

\[
r_0 = \frac{G_1(\lambda)}{\sqrt{\lambda^2 - 1}}, \text{ } \theta_\alpha \text{ and } \theta_\beta \text{ are constants given by (2.1), and } G_1(\lambda) \text{ is given by}
\]

\[
G_1(\lambda) = \frac{\sqrt{\lambda^2 - 1}}{2\lambda} \left( \frac{\beta(2\lambda^{\beta+2} + 2^2 + \beta N)}{\alpha(2\lambda^{\alpha+2} + 2^2 + \alpha N)} \right)^{\frac{1}{\alpha - \beta}} \text{ for } \lambda \geq 2.
\quad (1.8)
\]

By adding another body \( Q_3 \) at the center of mass, there exists a class of special homographic solutions of the \((3 + N)\)-body problem when \( N \geq 2 \). Suppose that

\[
q(t) = E_{\omega_0}(t)q, \quad (1.9)
\]

where \( q = (Q_1, Q_2, Q_3, q_1, \ldots, q_N) \in (\mathbb{R}^3)^{3+N} \) is given by (1.3) and \( E_{\omega_0}(t) \) is given by (1.4).
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Proposition 1.2 When $0 < \alpha < \beta$, there exists a $\lambda_2$ such that for any $\lambda \geq \max\{2, \lambda_2\}$, $q(t) = E_{\omega_0}(t)q$ satisfying (1.9) is the homographic solution of the system (1.2), where $r_0 = \frac{G_2(\lambda)}{\sqrt{N-1}}$, $E_{\omega_0}(t)$ is defined by (1.4), $\omega_0$ satisfies

$$\omega_0 = \pm \left( \frac{\theta_\alpha + \alpha}{(\sqrt{\lambda^2 - 1})^{\alpha+2}} - \frac{\theta_\beta + \beta}{(\sqrt{\lambda^2 - 1})^{\beta+2}} + \frac{2\alpha}{(\lambda r_0)^{\alpha+2}} - \frac{2\beta}{(\lambda r_0)^{\beta+2}} \right),$$

and $G_2(\lambda)$ is given by

$$G_2(\lambda) = \frac{\sqrt{\lambda^2 - 1}}{2\lambda} \left( \frac{\beta(2\beta+2\beta^2+2\lambda^2+2\lambda^2+2\lambda^2+N)}{\alpha(2\alpha+2\alpha^2+2\alpha^2+2\alpha^2+N)} \right)^{\frac{1}{-\alpha}}.$$  

In Theorem 1.3, we prove the existence of the non-circular homographic solutions. Suppose $q = (Q_1, Q_2, q_1, \ldots, q_N) \in (R^3)^2+N$ is given by (1.3), $E(t)$ is defined by (1.4) and $r_0 \equiv 1$ in $q_k$. When $r \in R^+$, we define

$$\Psi(r) = \frac{\beta(N2\beta^2+2\lambda^2+2\lambda^2)}{(\beta+1)(2\lambda)^{\beta+2}r^{\beta+1}} - \frac{\alpha(N2\alpha^2+2\alpha^2)}{(\alpha+1)(2\lambda)^{\alpha+2}r^{\alpha+1}}.$$  

Define $\bar{r}$ as the only positive root $\Psi'(r) = 0$ by

$$\bar{r} = \frac{1}{2\lambda} \left( \frac{\beta(N2\beta^2+2\lambda^2+2\lambda^2)}{\alpha(N2\alpha^2+2\alpha^2)} \right)^{\frac{1}{-\alpha}}.$$  

Theorem 1.3 When $0 < \alpha < \beta$ and $N \geq 2$, there exists $\lambda_0$ given by Lemma 2.3 such that for $\lambda \geq \lambda_0$, $q(t) = r(t)E(t)q_0$ satisfying (1.12) is a homographic solution of (1.2) where $r(t)$ is the solution of following Hamiltonian system with given Hamiltonian energy

$$\begin{cases}
\dot{r} = -\nabla \Psi(r) = \frac{N\beta}{\lambda^{\beta+2}r^{\beta+1}} - \frac{N\alpha}{\lambda^{\alpha+2}r^{\alpha+1}} + \frac{2\beta}{2^{\beta+2}r^{\beta+1}} - \frac{2\alpha}{2^{\alpha+2}r^{\alpha+1}}, \\
H(r, \dot{r}) = h, \\
r(\tau) = r(0), \quad \dot{r}(\tau) = \dot{r}(0),
\end{cases}$$  

where $h$ satisfies $\Psi(\bar{r}) < h < \Psi(\Lambda)$, $\omega(t)$ is defined by $E(t)$ and satisfies

$$\omega(t) = \pm \int_0^t \sqrt{\left( \frac{(N-2)\beta}{(\lambda r)^{\beta+2}} - \frac{(N-2)\alpha}{(\lambda r)^{\alpha+2}} + \frac{2\beta}{(2r)^{\beta+2}} - \frac{2\alpha}{(2r)^{\alpha+2}} - \frac{\theta_\beta}{(\phi r)^{\beta+2}} + \frac{\theta_\alpha}{(\phi r)^{\alpha+2}} \right) dt},$$

$\Lambda$ is given by

$$\Lambda = \frac{1}{2\phi} \left( \frac{\beta((N-2)(2\phi)^{\beta+2}+2(\lambda\phi)^{\beta+2}-\theta_\beta(2\lambda)^{\beta+2})}{\alpha((N-2)(2\phi)^{\alpha+2}+2(\lambda\phi)^{\alpha+2}-\theta_\alpha(2\lambda)^{\alpha+2})} \right)^{\frac{1}{-\alpha}}.$$  

It is well known that in the celestial mechanics, if the configuration is flat, the solution must be planar, i.e., the motion must be in a given fixed plane; if the configuration is 3-dimensional, the corresponding solution must be homothetic. In Theorems 1.1 and 1.3, if $N = 2$, the homographic solution is flat but it is not planar; if $N \geq 3$, the configuration is 3-dimensional, and the corresponding solution is circular but not homothetic. These results show the essential differences between the Newton’s law of universal gravitation and the L-J potential.
2 The Homographic Solutions

2.1 The Circular Homographic Solutions of the (2 + N)- and (3 + N)-body Cases

In this subsection, we consider (1.5). The shape of configuration $q_0$ is uniquely determined by $\lambda$. We define $\theta_\beta$ and $\theta_\alpha$ by

$$
\theta_\beta = \beta \sum_{j=1}^{N-1} \frac{1}{|1 - \epsilon^{i\omega_j}|^\beta}, \quad \theta_\alpha = \alpha \sum_{j=1}^{N-1} \frac{1}{|1 - \epsilon^{i\omega_j}|^\alpha},
$$

where $i^2 = -1$. By direct computations, we have the following lemma.

**Lemma 2.1** For any given $0 < \alpha < \beta$, $N \geq 2$, $G_1(\lambda)$ and $G_2(\lambda)$ have following properties.

(i) $G_1(\lambda)$ and $G_2(\lambda)$ are monotonically increasing functions in $\lambda$ when $\lambda \geq 2$;

(ii) $\lim_{\lambda \to \infty} G_1(\lambda) = \lim_{\lambda \to \infty} G_2(\lambda) = \infty$;

(iii) there exists a $\lambda_1 = \lambda_1(N) > 0$ such that for any $\lambda \geq \lambda_1$,

$$
G_1(\lambda) \geq \left( \frac{\theta_\beta}{\theta_\alpha} \right)^{\frac{1}{\lambda - \alpha}};
$$

(iv) there exists a $\lambda_2 = \lambda_2(N) > 0$ such that for any $\lambda \geq \lambda_2$, we have

$$
G_2(\lambda) \geq \max \left\{ \left( \frac{\theta_\beta}{\theta_\alpha} \right)^{\frac{1}{\lambda - \alpha}}, \left( \frac{\beta}{\alpha} \right)^{\frac{1}{\lambda - \alpha}} \right\}.
$$

Then we give the proof of Theorem 1.1.

**Proof of Theorem 1.1** To prove $q = (Q_1, Q_2, q_1, \ldots, q_N)$ satisfies (1.6), we calculate $\frac{\partial U}{\partial q_k}$ for $j = 1, 2$ and $\frac{\partial U}{\partial q_k}$ for $1 \leq k \leq N$. For $i, j \in \{1, 2\}$, we have

$$
\frac{\partial U}{\partial Q_j} = \sum_{k=1}^{N} \left( \frac{\alpha(Q_j - q_k)}{|Q_j - q_k|^{\alpha + 2}} - \frac{\beta(Q_j - q_k)}{|Q_j - q_k|^{\beta + 2}} \right) + \left( \frac{\alpha(Q_j - Q_i)}{|Q_j - Q_i|^{\alpha + 2}} - \frac{\beta(Q_j - Q_i)}{|Q_j - Q_i|^{\beta + 2}} \right).
$$

Since $Q_1 = (0, 0, r_0), Q_2 = (0, 0, -r_0)$ and $q_k = \phi(\lambda)r_0(\cos \omega_k, \sin \omega_k, 0)$, we have that

$$
\sum_{k=1}^{N} \left( \frac{\alpha}{|Q_j - q_k|^{\alpha + 2}} - \frac{\beta}{|Q_j - q_k|^{\beta + 2}} \right) (Q_j - q_k) = \left( \frac{N\alpha}{|\lambda r_0|^{\alpha + 2}} - \frac{N\beta}{|\lambda r_0|^{\beta + 2}} \right) Q_j
$$

and

$$
\left( \frac{\alpha}{|Q_j - Q_i|^{\alpha + 2}} - \frac{\beta}{|Q_j - Q_i|^{\beta + 2}} \right) (Q_j - Q_i) = \left( \frac{2\alpha}{2|r_0|^{\alpha + 2}} - \frac{2\beta}{|r_0|^{\beta + 2}} \right) Q_j.
$$

Note that (2.5) and (2.6) yield that for $j = 1$ or 2,

$$
\frac{\partial U}{\partial Q_j} = \left( \frac{N\alpha}{|\lambda r_0|^{\alpha + 2}} - \frac{N\beta}{|\lambda r_0|^{\beta + 2}} + \frac{2\alpha}{2|r_0|^{\alpha + 2}} - \frac{2\beta}{|r_0|^{\beta + 2}} \right) Q_j.
$$

Note that for $j = 1, 2$,

$$
\text{diag}(-\omega_0^2, -\omega_0^2, 0) Q_j = 0.
$$

We have the following equality holds.

$$
\frac{\partial U}{\partial Q_j} = \frac{N\alpha}{|\lambda r_0|^{\alpha + 2}} - \frac{N\beta}{|\lambda r_0|^{\beta + 2}} + \frac{2\alpha}{2|r_0|^{\alpha + 2}} - \frac{2\beta}{|r_0|^{\beta + 2}} = 0.
$$

By (2.9), we obtain that

$$
r_0 = r(\lambda) = \frac{1}{2\lambda} \left( \frac{\beta(2\lambda^{\beta + 2} + 2^{2+\beta}N)}{\alpha(2\lambda^{\alpha + 2} + 2^{\alpha+2}N)} \right)^{\frac{1}{\pi - \alpha}}.
$$
Note that $\lambda > 0$ is needed up to now. By (1.8), we have that $r(\lambda) = \frac{G_i(\lambda)}{\sqrt{\lambda - 1}}$ when $\lambda \geq 1$.

Note that $\frac{\partial U}{\partial q_k}$ can be calculated by

\[
\frac{\partial U}{\partial q_k} = \sum_{j \neq k} \left( \frac{\alpha}{|q_{kj}|^{\alpha + 2}} - \frac{\beta}{|q_{kj}|^{\beta + 2}} \right) q_{kj} + \sum_{j=1}^{2} \left( \frac{\alpha (q_k - Q_j)}{|q_k - Q_j|^{\alpha + 2}} - \frac{\beta (q_k - Q_j)}{|q_k - Q_j|^{\beta + 2}} \right).
\]

(2.11)

Because following discussion is in $xy$-plane, we identify the $xy$-plane with $C$. Therefore, $q_k = \phi(\lambda) r_0 e^{i\omega_k}$ and $q_{kj} = \phi(\lambda) r_0 e^{i\omega_{jk}} (1 - e^{i\omega_{jk}})$. Hence, we have that

\[
\sum_{j \neq k} \left( \frac{\alpha}{|q_{kj}|^{\alpha + 2}} - \frac{\beta}{|q_{kj}|^{\beta + 2}} \right) q_{kj}
= \sum_{j \neq k} \left( \frac{\alpha (1 - e^{i\omega_{jk}}) q_k}{(\phi r_0)^{\alpha + 2} |1 - e^{i\omega_{jk}}|^{\alpha + 2}} - \frac{\beta (1 - e^{i\omega_{jk}}) q_k}{(\phi r_0)^{\beta + 2} |1 - e^{i\omega_{jk}}|^{\beta + 2}} \right)
= \sum_{j=1}^{N-1} \left( \frac{\alpha (1 - e^{i\omega_{j}}) q_k}{(\phi r_0)^{\alpha + 2} |1 - e^{i\omega_{j}}|^{\alpha + 2}} - \frac{\beta (1 - e^{i\omega_{j}}) q_k}{(\phi r_0)^{\beta + 2} |1 - e^{i\omega_{j}}|^{\beta + 2}} \right).
\]

(2.12)

Note that $|1 - e^{i\omega_{j}}| = |1 - e^{i\omega_{N-j}}| = \sqrt{2 - 2 \cos \omega_{j}}$ and $(1 - e^{i\omega_{j}}) + (1 - e^{i\omega_{N-j}}) = 2 - 2 \cos \omega_{j}$. Then we have

\[
\sum_{j=1}^{N-1} \frac{1 - e^{i\omega_{j}}}{|1 - e^{i\omega_{j}}|^{\alpha + 2}} = \frac{1}{2} \left( \sum_{j=1}^{N-1} \frac{1 - e^{i\omega_{j}}}{|1 - e^{i\omega_{j}}|^{\alpha + 2}} + \sum_{j=1}^{N-1} \frac{1 - e^{i\omega_{N-j}}}{|1 - e^{i\omega_{N-j}}|^{\alpha + 2}} \right)
= \frac{1}{2} \left( \sum_{j=1}^{N-1} \frac{1}{|1 - e^{i\omega_{j}}|^{\alpha + 2}} \right).
\]

(2.13)

Therefore, we can reduce (2.12) to

\[
\sum_{j \neq k} \left( \frac{\alpha}{|q_{kj}|^{\alpha + 2}} - \frac{\beta}{|q_{kj}|^{\beta + 2}} \right) q_{kj} = \frac{1}{2} \sum_{j=1}^{N-1} \left( \frac{\alpha}{(\phi r_0)^{\alpha + 2} |1 - e^{i\omega_{j}}|^{\alpha + 2}} - \frac{\beta}{(\phi r_0)^{\beta + 2} |1 - e^{i\omega_{j}}|^{\beta + 2}} \right) q_k
= \left( \frac{\theta_\alpha}{(\phi r_0)^{\alpha + 2}} - \frac{\theta_\beta}{(\phi r_0)^{\beta + 2}} \right) q_k,
\]

(2.14)

where $\theta_\beta$ and $\theta_\alpha$ are given by (2.1). The second summation of (2.11) can be simplified as

\[
\sum_{j=1}^{2} \left( \frac{\alpha (q_k - Q_j)}{|q_k - Q_j|^{\alpha + 2}} - \frac{\beta (q_k - Q_j)}{|q_k - Q_j|^{\beta + 2}} \right) = \left( \frac{\alpha}{(\lambda r_0)^{\alpha + 2}} - \frac{\beta}{(\lambda r_0)^{\beta + 2}} \right) (q_k - Q_1 + q_k - Q_2)
= \left( \frac{2 \alpha}{(\lambda r_0)^{\alpha + 2}} - \frac{2 \beta}{(\lambda r_0)^{\beta + 2}} \right) q_k.
\]

(2.15)

Then (2.11) is reduced to

\[
\frac{\partial U}{\partial q_k} = \left( \frac{\theta_\alpha}{(\phi r_0)^{\alpha + 2}} - \frac{\theta_\beta}{(\phi r_0)^{\beta + 2}} \right) + \left( \frac{2 \alpha}{(\lambda r_0)^{\alpha + 2}} - \frac{2 \beta}{(\lambda r_0)^{\beta + 2}} \right) q_k.
\]

(2.16)

By diag($-\omega_0^2, -\omega_0^2, 0$) $q_k + \frac{\partial U}{\partial q_k} = 0$, we have that

\[
\omega_0^2 = \frac{\theta_\alpha}{(\phi r_0)^{\alpha + 2}} - \frac{\theta_\beta}{(\phi r_0)^{\beta + 2}} + \frac{2 \alpha}{(\lambda r_0)^{\alpha + 2}} - \frac{2 \beta}{(\lambda r_0)^{\beta + 2}}.
\]

(2.17)

By $\lambda \geq 2$ and (2.10), we have that

\[
\lambda r_0 = \frac{1}{2} \left( \frac{\beta (2 \lambda^{\beta + 2} + 2 \beta^{\beta + 2} N)}{\alpha (2 \lambda^{\alpha + 2} + 2 \alpha^{\alpha + 2} N)} \right)^{\frac{1}{\beta - \alpha}} \geq \left( \frac{\beta}{\alpha} \right)^{\frac{1}{\beta - \alpha}}.
\]
This yields that
\[
\frac{2\alpha}{(\lambda r_0)^{\alpha+2}} - \frac{2\beta}{(\lambda r_0)^{\beta+2}} \geq 0. 
\]  
(2.18)

By Lemma 2.1, we have that there exists a \( \lambda_1 = \lambda_1(N) \) which depends on \( N \) such that for any \( \lambda \geq \lambda_1 \), \( \phi(\lambda)r_0 = G_1(\lambda) \geq \left( \frac{\theta_\alpha}{\phi_0} \right)^{\alpha+2} \). This yields
\[
\frac{\theta_\alpha}{(\phi r_0)^{\alpha+2}} - \frac{\theta_\beta}{(\phi r_0)^{\beta+2}} \geq 0. 
\]  
(2.19)

Hence, by (2.18) and (2.19), \( \omega_0 \) satisfying (2.17) is well-defined for any \( \lambda \geq \max\{2, \lambda_1\} \).

Then \( q(t) = E_{\omega_0}(t)q \) is a homographic solution of the system (1.2) where \( q \) satisfies (1.5), and \( \omega_0 \) is given by (2.17).

Readers may verify Proposition 1.2 by Lemma 2.1 following the proof of Theorem 1.1. We omit the proof here.

When \( N = 2 \), the solution is a flat non-planar solution. There is a plane \( \Pi(t) \) in \( \mathbb{R}^3 \) such that \( q(t) \in \Pi(t) \) for all \( t \in S_{r} \) pair-wisely different. In the celestial mechanics, there does not exist any flat non-planar solution.

**Corollary 2.2**  
(i) (The \( (2+2) \)-body problem) When \( 0 < \alpha < \beta \), \( N = 2 \) and \( \lambda \geq 2 \), \( q(t) \) satisfying (1.5) is the non-planar solution of (1.2). Furthermore, the configuration is always a rhombus.

(ii) (The \( (3+2) \)-body problem) When \( N = 2 \), \( \lambda \geq \max\{2, \lambda_2\} \), \( q(t) \) satisfying (1.9) is a non-planar solution of (1.2). Especially, the configuration of \( q(t) \) is always a rhombus where \( Q_3 \) is at the center of this rhombus.

### 2.2 One General Periodic Homographic Solution

In this section, we will prove Theorem 1.3. Before the proof, we need the following lemma.

**Lemma 2.3**  
There exists a \( \lambda_0 \) such that for all \( \lambda > \lambda_0 > 2 \), the following inequality holds,
\[
(N2^{\beta+2} + 2\lambda^{\beta+2})((N - 2)2^{\alpha+2}\phi^{\beta+2} + 2\lambda^{\alpha+2}\phi^{\beta+2} - \theta_\alpha(2\lambda)^{\alpha+2}\phi^{\beta-\alpha})
\]
\[
< (N2^{\alpha+2} + 2\lambda^{\alpha+2})((N - 2)(2\phi)^{\beta+2} + 2(\lambda\phi)^{\beta+2} - \theta_\beta(2\lambda)^{\beta+2}). 
\]  
(2.20)

**Proof**  
Note that the highest order of \( \lambda \) on the both sides of (2.20) are \( 4\lambda^{\beta+\alpha+4}\phi^{\beta+2} \). The second highest order of \( \lambda \) on the right hand side of (2.20) is \( N2^{\alpha+3}(\lambda\phi)^{\beta+2} \) and the second highest order of \( \lambda \) on the left hand side of (2.20) is \( (N - 2)2^{\alpha+3}\lambda^{\beta+2}\phi^{\beta+2} - \theta_\alpha2^{\alpha+3}\lambda^{\beta+\alpha+4}\phi^{\beta-\alpha} \). Note
\[
N2^{\alpha+3}(\lambda\phi)^{\beta+2} > (N - 2)2^{\alpha+3}(\lambda\phi)^{\beta+2} - \theta_\alpha2^{\alpha+3}\lambda^{\beta+\alpha+4}\phi^{\beta-\alpha}. 
\]  
(2.21)

is equivalent to
\[
2^{\alpha+4}(\lambda\phi)^{\beta+2} > -\theta_\alpha2^{\alpha+3}\lambda^{\beta+\alpha+4}\phi^{\beta-\alpha}. 
\]  
(2.22)

It holds because the left hand side is positive and the right hand side is negative when \( \lambda > 0 \). Then we obtain this lemma holds when \( \lambda_0 \) is sufficiently large. \( \square \)

Now, we give the proof of Theorem 1.3.

**Proof of Theorem 1.3**  
We plug \( q(t) = r(t)E(t)q_0 \) defined in (1.12) into the both sides of (1.2) and obtain that the left hand side of (1.2) are
\[
\ddot{Q}_j(t) = (\ddot{r}(t)E(t) + 2\dot{r}(t)\dot{E}(t) + r(t)\ddot{E}(t))Q_j, 
\]  
(2.23)
where \(1 \leq j \leq 2\) and \(1 \leq k \leq N\) and on the other hand, by (2.4–2.16), we have
\[
\begin{align*}
\frac{\partial U(q)}{\partial Q_j} &= rE(t) \left( \frac{N\alpha}{(\lambda r)^{\alpha+2}} - \frac{N\beta}{(\lambda r)^{\beta+2}} + \frac{2\alpha}{(2r)^{\alpha+2}} - \frac{2\beta}{(2r(t))^{\beta+2}} \right) Q_j, \\
\frac{\partial U(q)}{\partial q_k} &= rE(t) \left( \frac{\theta\alpha}{(\phi r)^{\alpha+2}} - \frac{\theta\beta}{(\phi r)^{\beta+2}} + \frac{2\alpha}{(\lambda r)^{\alpha+2}} - \frac{2\beta}{(\lambda r)^{\beta+2}} \right) q_k.
\end{align*}
\]
We define \(A(t)\) by
\[
A(t) \equiv r^{-1}(t)E^{-1}(t)(\ddot{r}E(t) + 2\dot{r}\dot{E}(t) + r(t)\ddot{E}(t))q_k,
\]
(2.24)

Then \(\ddot{Q}_j(t) = r(t)E(t)A(t)Q_j = -\frac{\partial U(q(t))}{\partial Q_j}\) and (2.25) yield that
\[
\ddot{r} = \frac{N\beta}{\lambda^{\beta+2}r^{\beta+1}} - \frac{N\alpha}{\lambda^{\alpha+2}r^{\alpha+1}} + \frac{2\beta}{2^{\beta+2}r^{\beta+1}} - \frac{2\alpha}{2^{\alpha+2}r^{\alpha+1}}.
\]
(2.28)
By \(\ddot{q}_k = r(t)E(t)A(t)q_k = -\frac{\partial U(q(t))}{\partial q_k}\), (2.26) and (2.28) yield that
\[
\ddot{r} - r\ddot{\omega}^2 = \frac{\theta\beta}{\phi^{\beta+2}r^{\beta+1}} - \frac{\theta\alpha}{\phi^{\alpha+2}r^{\alpha+1}} + \frac{2\beta}{\lambda^{\beta+2}r^{\beta+1}} - \frac{2\alpha}{\lambda^{\alpha+2}r^{\alpha+1}}.
\]
(2.29)
By (2.28) and (2.29), we have that
\[
\dot{\omega}^2 = \frac{(N - 2)\beta}{(\lambda r)^{\beta+2}} - \frac{(N - 2)\alpha}{(\lambda r)^{\alpha+2}} + \frac{2\beta}{(2r)^{\beta+2}} - \frac{2\alpha}{(2r)^{\alpha+2}} - \frac{\theta\beta}{(\phi r)^{\beta+2}} + \frac{\theta\alpha}{(\phi r)^{\alpha+2}} \geq 0.
\]
(2.30)
If \(q(t)\) is the solution of (1.2), \(\ddot{\omega}^2 \geq 0\) is a necessary condition, i.e.,
\[
\frac{(N - 2)\beta}{(\lambda r)^{\beta+2}} - \frac{(N - 2)\alpha}{(\lambda r)^{\alpha+2}} + \frac{2\beta}{(2r)^{\beta+2}} - \frac{2\alpha}{(2r)^{\alpha+2}} - \frac{\theta\beta}{(\phi r)^{\beta+2}} + \frac{\theta\alpha}{(\phi r)^{\alpha+2}} \geq 0.
\]
(2.31)
Note that (2.31) is equivalent to for all \(t \in \mathbb{R}\),
\[
r \leq \Lambda(\lambda) \equiv \frac{1}{2\lambda\phi} \left( \frac{\beta((N - 2)(2\phi)^{\beta+2} + 2(\lambda\phi)^{\beta+2} - \theta\beta(2\lambda)^{\beta+2})}{\alpha((N - 2)(2\phi)^{\alpha+2} + 2(\lambda\phi)^{\alpha+2} - \theta\alpha(2\lambda)^{\alpha+2})} \right)^{\frac{1}{\beta - \alpha}}.
\]
(2.32)
Note that (2.28) can be simplified to
\[
\ddot{r} = -\Psi'(r) = \frac{\beta(N2^{\beta+2} + 2\lambda^{\beta+2})}{(2\lambda r)^{\beta+2}} - \frac{\alpha(N2^{\alpha+2} + 2\lambda^{\alpha+2})}{(2\lambda r)^{\alpha+2}},
\]
(2.33)
where \(\Psi\) is defined by (1.13). Note that \(\ddot{r}\) given by (1.14) is the only positive root of \(\Psi'(\ddot{r}) = 0\).

As the discussion on Theorem 4.1 of [4], there is a \(\tau\)-periodic oscillating solution of \(r(t) > 0\) for any given Hamiltonian energy \(\Psi(\ddot{r}) < h < 0\). For any solution \(r(t)\) which is not the constant solution, i.e., \(r(t) \neq \ddot{r}\), we must have \(t_1, t_2 \in [0, \tau]\) such that \(r(t_1) \leq \ddot{r}\) and \(r(t_2) \geq \ddot{r}\).

If there exists some \(\lambda_0\) such that \(\Lambda(\lambda_0) > \ddot{r}\), then \(r(t)\), which is the periodic solution of (2.33), satisfies \(\max_{t \in \mathbb{R}} r(t) < \Lambda(\lambda_0)\) for the given Hamiltonian energy \(h \in (\Psi(\ddot{r}), \Psi(\Lambda(\lambda_0)))\). Then for \(\lambda_0\), \(q(t) = r(t)E(\omega(t))q\) is the solution of the system (1.2).

Next, we prove the existence of \(\lambda_0\) which yields for any \(\lambda \geq \lambda_0\), the following inequality holds.
\[
\Lambda(\lambda) > \ddot{r}.
\]
(2.34)
Note that (2.34) holds is equivalent to
\[
\frac{1}{2\lambda\phi} \left( \frac{\beta((N-2)(2\phi)^{\beta+2} + 2(\lambda\phi)^{\beta+2} - \theta_{\beta}(2\lambda)^{\beta+2})}{\alpha((N-2)(2\phi)^{\alpha+2} + 2(\lambda\phi)^{\alpha+2} - \theta_{\alpha}(2\lambda)^{\alpha+2})} \right) \frac{1}{\pi} > \frac{1}{2\lambda} \left( \frac{\beta(N2^{\beta+2} + 2\lambda^{\beta+2})}{\alpha(N2^{\alpha+2} + 2\lambda^{\alpha+2})} \right) \frac{1}{\pi}.
\]
(2.35)

By direct computations, we have that (2.35) holds if and only if (2.20) holds. By Lemma 2.3, we have that (2.35) holds.

Therefore, for such \( \lambda \) which satisfies (2.34), \( q(t) = r(t)E(t)q_0 \) is a solution of the system where \( r(t) \) is the classical solution of the ODE (2.33) and \( \omega \) is given by (1.16).

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