Operational Detection of Entanglement via Quantum Designs

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1. Introduction

Entanglement\[^{1,2}\] is one of the most notable characteristics of quantum theory as compared to classical theory. Apart from its fundamental importance, quantum entanglement also plays a vital role in various tasks in quantum information processing. Then, a basic yet crucial question to ask is how to determine whether a given quantum state is entangled or not. Although considerable results have been obtained, a universal method for checking entanglement is still not available.

A bipartite quantum state \( \rho_{AB} \) is called separable if it can be written as a convex combination of product states, that is

\[
\rho_{AB} = \sum_k p_k |a_k\rangle \langle a_k| \otimes |b_k\rangle \langle b_k| \tag{1}
\]

where the coefficients \( \{p_k\} \) form a probability distribution with \( p_k \geq 0 \) and \( \sum_k p_k = 1 \); Otherwise, \( \rho_{AB} \) is entangled. The well-known positive partial transpose (PPT) criterion\[^{3,4}\] is both necessary and sufficient for detecting entanglement for the simple 2 \( \times \) 2 and 2 \( \times \) 3 systems, but not for higher dimensions. For instance, there exist the so-called bound entangled states\[^{5}\] which are PPT and nondistillable. An equivalent generalization of PPT, namely the entanglement witness,\[^{1,2,4}\] is a universal method for detecting an arbitrary entangled state \( \rho \) with \( \text{tr}(W \rho) < 0 \) where \( W \) is a suitably-chosen (but may not be unique) Hermitian observable.

While the construction of an entanglement witness can be difficult sometimes, a number of nonuniversal but easily applicable methods have been proposed. The most prominent example is the computable cross-norm or realignment (CCNR) criterion,\[^{6-8}\] which is based on the correlations of local orthogonal observables (LOOs). Then, the local uncertainty relation (LUR)\[^{9,10}\] ensures that a witness for CCNR. Another class of linear correlation-based entanglement criteria is constructed using the normalized symmetric informationally complete positive operator-valued measures (SIC POVMs).\[^{11}\] dubbed as the ESIC criterion;\[^{12}\] see also refs.\[^{13,14}\]. Very recently, a general approach for checking separability is taken by considering the linear correlations of specific operators in refs.\[^{15,16}\], which incidentally recovers both the CCNR and the ESIC criteria.

Apart from SIC POVMs, another extension for entanglement detection is via mutually unbiased bases (MUBs),\[^{17-22}\] both of which are special cases of quantum 2-designs. For instance, a measurement-device-independent entanglement detection method is discussed in ref.\[^{23}\] using SIC POVMs and MUBs. Recently, depending on the random moments calculated from quantum designs, a method for characterizing multipartite entanglement is proposed.\[^{24,25}\] Generally speaking, however, the random moments are not directly measurable in experiments. Therefore, here we focus on the correlations defined via quantum designs from an operational point of view. As quantum designs correspond to a single generalized measurement, the correlations can be directly obtained in one go in experiments, rather than one by one as in other criteria like the CCNR. Specifically, we propose several new entanglement criteria using quantum 2-designs by extending the CCNR, ESIC, and LUR criteria.

This paper is organized as follows. We first briefly review the concept of quantum designs in Section 2, with a special emphasis on the discussion of SIC POVMs. In Section 3, upon recalling some well-known entanglement criteria including the CCNR, ESIC, and LUR criteria, we construct several new criteria using quantum designs. Then these criteria are tested using various
bipartite entangled states in Section 4, and we conclude in Section 5.

2. Quantum Designs

Design is an important mathematical concept which can be used to imitate uniform averages over certain groups, which in turn can be regarded as a pseudorandom process. Designs are denominated either unitary or spherical, hinging on which group one chooses. For qubit systems, the local measurement settings can be characterized over the Bloch sphere, then it is more convenient to use spherical designs than unitary designs. A spherical design is a collection of points on the unit sphere for which the th-order polynomials can be averaged over to obtain the same value as that integrating over the surface with certain measures. Formally, a probability distribution over the set of quantum states \( |\psi_i\rangle \) is a quantum spherical design if

\[
\sum_i p_i (|\psi_i\rangle \langle \psi_i|)^{\otimes d} = \int |\psi\rangle \langle \psi|^{\otimes d} \, d\psi
\]  

where the integral over \( |\psi\rangle \) is taken over the Haar measure on the unit sphere.  

In this work, we choose \( t = 2 \) in particular to focus on the investigation of bipartite entanglement. We can associate the complete set of points of a spherical 2-design as a measurement \( \{\Pi_k\}_{k=1}^N \), where \( N \) denotes the number of settings. Then, a quantum state \( \rho \) can be reconstructed as

\[
\rho = (d + 1) \frac{N}{d} \sum_{k=1}^N p_k \Pi_k - 1
\]  

where \( d \) is the dimension, and the probability of obtaining the outcome \( \Pi_k \) is given by the Born rule

\[
\tilde{p}_k = \text{tr}(\rho \Pi_k) = \langle \tilde{\Pi}_k \rangle
\]  

The following inequality can be derived directly from Equation (3), such that

\[
\sum_{k=1}^N \tilde{p}_k^2 \leq \frac{d[1 + \text{tr}(\rho^2)]}{(d + 1)N} \leq \frac{2d}{N(d + 1)}
\]  

where the upper bound is saturated when \( \rho \) is pure. For simplicity and later use, let’s rewrite

\[
\Pi_k = \sqrt{\frac{N(d+1)}{2d}} \tilde{\Pi}_k
\]  

as the normalized version of the quantum 2-design. Then, the constraint in Equation (5) can be reformulated as

\[
\sum_{k=1}^N \tilde{p}_k^2 \leq 1
\]  

where \( \tilde{p}_k = \text{tr}(\rho \Pi_k) = \langle \Pi_k \rangle \) for a given state \( \rho \), and the equality is achieved when \( \rho \) is pure.

Among the typical examples of quantum 2-designs are SIC POVMs and MUBs. Both of them have been demonstrated being useful for entanglement detection. Here we look at SIC POVMs in specific. A SIC POVM in dimension \( d \) comprises of \( d^2 \) subnormalized projectors \( |\psi_i\rangle \langle \psi_i|/d \) with equal pairwise fidelity, such that

\[
|\langle \psi_i | \psi_j \rangle|^2 = \frac{d\delta_{ij} + 1}{d+1}, \quad i, j = 1, 2, \ldots, d^2
\]  

One can check that the normalized version of the SIC POVM that satisfies the condition in Equation (7) takes on the form

\[
E_k = \sqrt{\frac{d+1}{2d}} |\psi_k\rangle \langle \psi_k|
\]  

Although being widely believed and numerically supported, the existence of SIC POVMs in any finite dimension remains as an open problem. For a recent review, see refs. [29–32].

3. Correlation-Based Entanglement Criteria

As discussed early, entanglement criteria constructed using correlations are particularly relevant for easier experimental realizations. To be specific, in this work we first reinvestigate the CCNR and ESIC criteria which are linear, as well as the LUR criterion which is nonlinear. These criteria are extended straightforwardly by utilizing quantum 2-designs, and the question whether the new criteria thus obtained are improved or not naturally follows.

3.1. Linear Criteria

Consider a bipartite quantum state \( \rho_{AB} \) with the dimension \( d = d_A \times d_B \), where \( d_A \) and \( d_B \) represent the local dimensions of the subsystems \( \rho_A = \text{tr}_B(\rho_{AB}) \) and \( \rho_B = \text{tr}_A(\rho_{AB}) \) respectively. Let \( \{M_A^k\}_{k=1}^{d_A} \) and \( \{M_B^k\}_{k=1}^{d_B} \) denote the local operations acting on the two subsystems. Then, the linear correlation matrix between these two measurements can be written as

\[
[C]_{ij} = \langle M_A^i \otimes M_B^j \rangle = \text{tr}(\rho_{AB} M_A^i \otimes M_B^j)
\]  

the size of which is \( K_a \times K_B \). Then, we have the following proposition.

Proposition 1. Let \( \{M_A^k\} \) and \( \{M_B^k\} \) be the properly normalized local measurements acting on the bipartite state \( \rho_{AB} \). If \( \rho_{AB} \) is separable, then

\[
||C||_F \leq 1
\]  

has to hold; Otherwise, it is entangled. The symbol \( || \cdot ||_F \) denotes the trace norm.

The proof can be found in refs. [12, 15]. Depending on the local measurements one chooses, different entanglement criteria can be derived from Proposition 1. For instance, if \( \{M_A^k\} \) and \( \{M_B^k\} \) are LOOs, we get the CCNR criterion. The LOOs can be found by...
invoking the Schmidt decomposition (assuming $d_A \leq d_B$), such that

$$\rho_{AB} = \sum_{i=1}^{d_B^2} \lambda_i G_i^A \otimes G_i^B$$  \hspace{1cm} (12)

where $\lambda_i = \langle G_i^A \otimes G_i^B \rangle$ are the Schmidt coefficients. It is easy to check that the set of orthonormal bases of the Hermitian observables $\{G_i^A\}$ and $\{G_i^B\}$ fulfill the conditions

$$\text{tr}(G_i^A G_j^A) = \text{tr}(G_i^B G_j^B) = \delta_{ij}$$  \hspace{1cm} (13)

and

$$\sum_k \langle G_k^i \rangle^2 = d_A$$.  \hspace{1cm} (14)

Hence, an equivalent form of the CCNR criterion is given by

$$\sum_k \lambda_k \leq 1$$  \hspace{1cm} (15)

as the Schmidt coefficients $\lambda_k$s happen to be the singular values of $\rho$.

If, instead, one chooses the local measurements to be the normalized SIC POVMs as in Equation (9), the ESIC criterion proposed in ref. [12] is recovered. As demonstrated by various examples in ref. [12], the ESIC criterion is more powerful as compared to CCNR. So, here, we take a step further by asking the question whether the criterion as defined in Proposition 3 can be improved again if quantum 2-designs besides SIC POVMs are utilized. To distinguish the ESIC criterion, we dub the one with quantum 2-designs as ESIC; see below.

**Proposition 4 (LSIC).** Let $\{E_i^A\}$ and $\{E_i^B\}$ be the normalized 2-designs acting on two subsystems. If a bipartite state $\rho_{AB}$ is separable, then

$$1 + \sum_k \langle E_k^A \otimes E_k^B \rangle - \frac{1}{2} \sum_k \langle E_k^A \otimes 1 + 1 \otimes E_k^B \rangle^2 \geq 0$$  \hspace{1cm} (19)

has to hold; Otherwise, it is entangled.

Notice the sign differences in Equation (20) as compared to Equation (18). Similarly, here, we are interested in the question whether the criterion as defined in Proposition 3 can be further improved if quadratic terms in Equation (20) as compared to Equation (18). Similarly, here, we are interested in the question whether the criterion as defined in Proposition 3 can be further improved if quantum 2-designs besides SIC POVMs are utilized. To distinguish the ESIC criterion, we dub the one with quantum 2-designs as ESIC; see below.

**Proposition 4 (L2D).** Let $\{\Pi_i^A\}$ and $\{\Pi_i^B\}$ be the normalized 2-designs acting on two subsystems. If a bipartite state $\rho_{AB}$ is separable, then

$$1 + \sum_k \langle \Pi_k^A \otimes \Pi_k^B \rangle - \frac{1}{2} \sum_k \langle \Pi_k^A \otimes 1 + 1 \otimes \Pi_k^B \rangle^2 \geq 0$$  \hspace{1cm} (20)

has to hold; Otherwise, it is entangled.

The proof is similar to that for Proposition 3, which can be found in Appendix A.

### 4. Applications

In this section, we test various entanglement criteria proposed above using simple $2 \times 2$, $3 \times 3$, and $2 \times 3$ entangled quantum states.

#### 4.1. 2 × 2 Entangled States

For the first application, we consider the noisy $2 \times 2$ quantum states with the form

$$\rho_{\text{in}}(p) = p|\psi\rangle\langle\psi| + (1 - p)\rho_s$$  \hspace{1cm} (21)

where the entangled state $|\psi\rangle$ can be set to be one of the Bell states,

$$|\psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$$  \hspace{1cm} (22)

$$|\phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$$  \hspace{1cm} (23)

and the separable noise $\rho_s$ is given by

$$\rho_s = \frac{2}{3}|00\rangle\langle00| + \frac{1}{3}|01\rangle\langle01| + \frac{1}{3}|10\rangle\langle10|$$  \hspace{1cm} (24)

Using the PPT criterion, these four families of states can be checked to be entangled for any $p > 0$. 
The number of elements of the quantum 2-designs that we choose for testing are \( N = 4, 7, 9 \) respectively. Note that when \( N = 4 \), it is simply the SIC POVM. Table 1 shows the threshold values of \( p \) reported by various criteria. The smaller the threshold is, the better the corresponding criterion is.

### Table 1. Threshold values of \( p \) for detecting the entangled states as in Equation (21) using various criteria. Since the states are entangled for any \( p > 0 \) with PPT, the smaller the threshold is, the better the corresponding criterion is.

| Criteria | PPT | CCNR | ESIC | E2D | LUR \(^{(4)}\) | LSIC | L2D |
|----------|-----|------|------|-----|--------------|------|-----|
| \( |\psi^-\rangle \) | 0.2918 | 0.2678 | 0.2678 | 0.2501 | 0.2501 | 0.2501 | |
| \( |\psi^+\rangle \) | 0.2918 | 0.2678 | 0.2678 | 0.2779 | 1 | 1 | |
| \( |\phi^\pm\rangle \) | 0.2164 | 0.2053 | 0.2053 | 0.2028 | 1 | 1 | |

\(^{(4)}\) Note that the LOOs used in the LUR criterion are different for each case by invoking the Schmidt decomposition.

Table 2. For the 50,000 randomly generated 2 \( \times \) 2 entangled states, the values in the table show the proportions that can be detected by various criteria.

| Criteria | PPT | CCNR | ESIC | E2D | LUR | LSIC | L2D |
|----------|-----|------|------|-----|-----|------|-----|
| \( |\psi^+\rangle \) | 100% | 86.39% | 88.52% | 3.86% | 3.86% | |

The number of elements of the quantum 2-designs that we choose for testing are \( N = 4, 7, 9 \) respectively. Note that when \( N = 4 \), it is simply the SIC POVM. Table 1 shows the threshold values of \( p \) reported by various criteria. The smaller the threshold is, the better the corresponding criterion is. Several other features can be observed from the table. First, ESIC is exactly equivalent to E2D, so is the pair of LSIC and L2D. In other words, there is no difference of the detection power between the ESIC and E2D criteria, as well as the LSIC and L2D criteria. This tells us that using quantum 2-designs with more settings like \( N = 7, 9 \) is not helpful for improving the detection power as compared to SIC POVM with \( N = 4 \). With this simple example, we provide additional evidence for the potentially unique role played by SIC POVMs in quantum information processing. Second, except for the case of \( |\psi^+\rangle \), LUR performs better than ESIC. It is worth noting that although both criteria have been proposed previously, the comparison between them is missing until now. This observation, to some extent, refutes our intuition that nonlinear criteria such as LUR are always better than linear ones like ESIC. Next, both the ESIC and LUR criteria are better than CCNR. Finally, the LSIC and L2D criteria are completely ineffective for detecting certain entangled states. This observation proves again that the advantage of nonlinear criteria over linear ones does not always exist.

To go a step further, we generate random two-qubit states according to the Haar measure and keep 50,000 entangled ones that are NPT, then check the proportions that can be detected by various criteria; see the results in Table 2. Apart from the same features that we can draw as those in Table 1, we find that ESIC and E2D are able to detect more states as compared to LUR and CCNR. The LSIC and L2D criteria, however, are the weakest among all. Moreover, all the states that can be detected by LSIC and L2D can also be detected by all the other criteria. Here, we emphasize that SIC POVMs play a fundamental role for detecting entanglement considering the criteria that we investigate in this work. Experimentally, this special feature automatically provides the minimal number of settings that one should choose for the task of entanglement detection.

### 4.2. 3 \( \times \) 3 Entangled States

We move on the consider the 3 \( \times \) 3 entangled quantum states. In dimension \( d = 3 \), there exist three different families of SIC POVMs, from which we arbitrarily choose one for testing. For each SIC POVM, the number of elements is given by \( N = 9 \). For other quantum 2-designs that we employ for the E2D and L2D criteria, we superimpose one set of SIC POVM over another (with proper rotations) to get a measurement with \( N = 18 \) elements. Note that the arbitrariness in choosing quantum 2-designs is ensured by its rotational symmetry.

#### 4.2.1. Bound Entangled States

We first consider the 3 \( \times \) 3 bound entangled states\(^{[21]}\) mixed with white noise

\[
\rho(p) = p \rho_{\text{BE}} + (1 - p) \frac{1}{9}
\]

where

\[
\rho_{\text{BE}} = \frac{1}{4} \left( 1 - \sum_{i=0}^{4} |\psi_i\rangle \langle \psi_i| \right)
\]

with

\[
|\psi_0\rangle = \frac{1}{\sqrt{2}} (|0\rangle(|0\rangle - |1\rangle)) \quad |\psi_1\rangle = \frac{1}{\sqrt{2}} (|1\rangle(|0\rangle - |1\rangle)|2\rangle)
\]

\[
|\psi_2\rangle = \frac{1}{\sqrt{2}} (|2\rangle(|1\rangle - |2\rangle)) \quad |\psi_3\rangle = \frac{1}{\sqrt{2}} (|1\rangle(|1\rangle - |2\rangle)|0\rangle)
\]

\[
|\psi_4\rangle = \frac{1}{3} (|0\rangle + |1\rangle + |2\rangle)(|0\rangle + |1\rangle + |2\rangle)
\]

Table 3 shows the threshold values of \( p \) reported by various criteria. Similar to Table 1, the smaller the threshold is, the better the corresponding criterion is. One finds that ESIC and E2D are equivalent, which are better than LUR, and all of them are better than CCNR. For this particular entangled state, the LSIC and L2D criteria are completely ineffective.

#### 4.2.2. Chessboard States

Next we consider the 3 \( \times \) 3 chessboard states defined as\(^{[34]}\)

\[
\rho_{\text{chess}} = \mathcal{N} \sum_{j=1}^{4} |V_j\rangle \langle V_j|
\]
where $\mathcal{N}$ is the normalization coefficient and the unnormalized vectors $|V_i\rangle$ are

$|V_1\rangle = |v_5, 0, v_1/v_6; 0, v_6, 0, 0, 0, 0\rangle$
$|V_2\rangle = |0, v_1, 0; v_2, 0; v_3, 0, 0, 0\rangle$
$|V_3\rangle = |v_5, 0, 0; 0, -v_2, 0; v_1v_3/v_5, 0, 0\rangle$
$|V_4\rangle = |0, v_2, 0; -v_1, 0; 0, 0, 0\rangle$ \hspace{1cm} (29)

We randomly generate 50 000 chessboard states with the six parameters $v_i$s taking values independently from a Gaussian distribution with standard deviation two and mean zero. Figure 1 illustrates the fractions of states that are detected by various criteria. Within numerical fluctuations, we find that the PPT criterion fails completely. Again, the ESIC and E2D criteria are exactly equivalent, and they can detect roughly 1% more states than that of LUR, and roughly 2% more than that of CCNR.

4.2.3. Horodecki States

The $3 \times 3$ bound entangled states introduced by Horodecki are given by \cite{35}

$$
\rho_{PH}^x = \frac{1}{8x + 1} \begin{pmatrix}
    x & 0 & 0 & 0 & x & 0 & 0 & 0 & x \\
    0 & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & x & 0 & 0 & 0 & 0 & 0 \\
    x & 0 & 0 & 0 & x & 0 & 0 & 0 & x \\
    0 & 0 & 0 & 0 & x & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & \frac{1 + x}{2} & 0 & \frac{\sqrt{1 - x^2}}{2} & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{1 - x^2}}{2} & 0 & \frac{1 + x}{2} \\
    x & 0 & 0 & 0 & x & 0 & 0 & 0 & x \\
\end{pmatrix} \hspace{1cm} (30)
$$

with the parameter $0 < x < 1$. Although these states cannot be detected by the PPT criterion and are not distillable, they are nevertheless all entangled. Consider the mixture of $\rho_{PH}^x$ with white noise

$$
\rho(x, p) = p\rho_{PH}^x + (1 - p)\frac{1}{2}, \hspace{0.5cm} 0 \leq p \leq 1 \hspace{1cm} (31)
$$

In Figure 2, we show the parameter ranges that are detected by various criteria. All the states above the curves can be detected by the corresponding criterion. One finds that the ESIC and E2D criteria are exactly equivalent, and both of them are better than LUR. The CCNR criterion is the worst among all. Moreover, the LSIC and L2D criteria can hardly detect any states, thus are not shown in the figure.

4.3. $2 \times 3$ Entangled States

For the last application, we consider the $2 \times 3$ entangled states that are NPT. In this case, the LUR, LSIC, and L2D criteria simply do not apply since all of them require the balanced dimension of the two subsystems. We randomly generate 50 000 entangled states according to the Haar measure, and the values shown in Table 4 represent the proportions that can be detected by various criteria. Once more, we confirm that the ESIC criterion is exactly equivalent to the E2D criterion, and they are all better than CCNR.

Table 4. For the 50 000 randomly generated $2 \times 3$ entangled states, the values in the table show the proportions that can be detected by various criteria.

| PPT | CCNR | ESIC | E2D |
|-----|------|-----|-----|
| 100% | 38.13% | 41.62% | 41.62% |
5. Summary

From an operational point of view, we constructed several new entanglement criteria in this work. We first generalized the ESIC criterion\(^1\) to the E2D criterion using quantum 2-designs. Then, based on the LUR criterion\(^9,10\) we proposed the LSIC criterion using SIC POVMs, and more generally the L2D criterion using quantum 2-designs. Counter-intuitively, the E2D and L2D criteria with more settings are exactly equivalent to the corresponding ESIC and LSIC criteria respectively. In other words, there is no difference of the detection power between the ESIC and LSIC criteria respectively. In other words, the positive values \(C_A\) and \(C_B\) can be computed by considering all the states. For separable states, the measurement outcomes are uncorrelated, so the variance of \(A_k \otimes 1 \otimes B_k\) is equal to the sum of the local variances, that is

\[
\sum_k d^2(A_k \otimes 1 \otimes B_k) \geq C_A + C_B
\]

As an outlook, it is interesting to reinvestigate the corresponding criteria using quantum \(t\)-designs with \(t > 2\) for multiparticle entanglement detection, for which we would also expect no improvement over that of the simplest setting of SIC POVMs.

Appendix A: Proof of Proposition 3

Here we explicitly derive the LSIC criterion as introduced in Proposition 3. Consider the normalized SIC POVM in a finite dimension \(d\) as in Equation (9), that is

\[
E_k = \sqrt{\frac{d+1}{2d}} |\psi_k\rangle \langle \psi_k |
\]

and the square of it is given by

\[
E_k^2 = \frac{d+1}{2d} |\psi_k\rangle \langle \psi_k |
\]

Because of the completeness condition, we have

\[
\sum_k E_k^2 = \frac{d+1}{2} \mathbb{1}
\]

The corresponding probabilities satisfy the relation

\[
\sum_k e_k^2 \leq 1
\]

where

\[
e_k = \text{tr}(\rho E_k) = \langle E_k \rangle
\]

is the probability for getting the \(k\)th outcome.

Variance of the measurement operator is given by

\[
\sum_k d^2(E_k) = \sum_k (E_k^2) - \sum_k (E_k)^2 = \frac{d+1}{2} - \sum_k e_k^2 \geq \frac{d-1}{2}
\]

Next, we discuss the properties of local uncertainty relations\(^9,10\). In general, a pair of quantum systems \(A\) and \(B\) can be characterized by the operators \(A_k\) and \(B_k\) with the sum uncertainty relations giving by

\[
\sum_k d^2(A_k) \geq C_A, \quad \sum_k d^2(B_k) \geq C_B
\]

The positive values \(C_A\) and \(C_B\) can be computed by considering all the states. For separable states, the measurement outcomes are uncorrelated, so the variance of \(A_k \otimes 1 \otimes B_k\) is equal to the sum of the local variances, that is

\[
\sum_k d^2(A_k \otimes 1 \otimes B_k) \geq C_A + C_B
\]

Then, we have

\[
\sum_k d^2(E_k^A \otimes 1 \otimes E_k^B) = \sum_k (E_k^A \otimes 1 \otimes E_k^B)^2 = \sum_k (E_k^A \otimes 1 \otimes E_k^B)^2
\]

Meanwhile

\[
\sum_k d^2(E_k^A \otimes 1 \otimes E_k^B) = \sum_k (E_k^A \otimes 1 \otimes E_k^B)^2 = \sum_k (E_k^A \otimes 1 \otimes E_k^B)^2
\]

which recovers Equation (20) in the main text.

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Conflict of Interest

The authors declare no conflict of interest.

Data Availability Statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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entanglement detection, quantum designs, quantum information
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