Ensuring both Accurate Convergence and Differential Privacy in Nash Equilibrium Seeking on Directed Graphs

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Abstract—We study in this paper privacy protection in fully distributed Nash equilibrium seeking where a player can only access its own cost function and receive information from its immediate neighbors over a directed communication network. In view of the non-cooperative nature of the underlying decision-making process, it is imperative to protect the privacy of individual players in networked games when sensitive information is involved. We propose an approach that can achieve both accurate convergence and rigorous differential privacy with finite cumulative privacy budget in distributed Nash equilibrium seeking, which is in sharp contrast to existing differential-privacy solutions for networked games that have to trade convergence accuracy for differential privacy. The approach is applicable even when the communication graph is unbalanced and it does not require individual players to have any global structure information of the communication graph. Since the approach utilizes independent noises for privacy protection, it can combat adversaries having access to all shared messages in the network. It is also encryption-free, ensuring high efficiency in communication and computation. Numerical comparison results with existing counterparts confirm the effectiveness of the proposed approach.

I. INTRODUCTION

Nash equilibrium (NE) seeking in game theory addresses the problem where multiple players compete to minimize their individual cost functions under different types of informational constraints (on what each individual player knows and how it interacts with other players) [1], [2], [3]). In recent years, this problem has gained increased attention in various domains, with applications ranging from energy bidding in power grids [4], resource allocation in communication networks [5], [6], business strategies in economics [7], to route management in road networks [8]. Due to practical constraints, in many of these application scenarios, there are no central coordinator/mediator existing to collect and disperse information, and individual players only have access to the decisions of their local neighbors. Such networked games are usually termed as games in the partial-decision information setting [9], [10]. In contrast to the classical full-decision information setting where a player knows the actions of all its competitors, either by observation or via a central coordinator, (see, e.g., [6], [11], [12], [13], [14], [15]), in the partial-decision information setting, individual players cannot compute their cost functions or gradients due to the lack of information. Consequently, players have to exchange information among local neighbors to estimate required global information for NE seeking.

Since the seminal results in [16], [17], significant inroads have been made in fully distributed NE seeking. For example, by combining (projected) gradient and consensus operations, fully distributed algorithms have been proposed for NE seeking in both the continuous-time domain [18], [19] and the discrete-time domain [20]. Algorithms for distributed NE seeking have also been proposed based on ADMM [21] or proximal best-response [22]. Recently, distributed algorithms have been proposed for NE seeking on directed graphs that are not necessarily balanced, both in the presence of global-structure knowledge of the communication network [23] and in the fully distributed scenario without any global-structure knowledge [24]. However, all of these distributed algorithms require players to share explicit (estimated) decisions in every iteration, which is problematic when sensitive information is involved. In fact, given that in noncooperative games all players are opponents, it is important for individual players to protect their private information, which, otherwise, might be exploited by opponents. For example, in Nash-Cournot games, individual players’ cost functions might be market sensitive, and should be kept private [25]. Furthermore, in many scenarios, privacy protection is required by legislation. For example, in routing games [26], California Privacy Rights Act forbids disclosing drivers’ spatiotemporal information, which can be exploited to infer a person’s activities [27]. In addition, privacy protection is also crucial to encourage participation of players in cooperative policies [28].

To address the urgent problem of privacy protection in NE seeking, several results have been reported in recent years (see, e.g., [29], [30], [31]). However, most of these assume the presence of a coordinator. In the fully distributed case, the authors in [32] exploit spatially-correlated noise to protect the privacy of players. However, their approach is only effective when the communication graph satisfies certain properties. Recently, the authors of [28] use a constant uncertain parameter to obfuscate individual players’ pseudo-gradients to achieve privacy protection in continuous-time aggregative games. However, the privacy strength enabled by such a constant scalar is weak in the sense that only the exact value of the cost function is avoided from being uniquely identifiable. In fact, this approach cannot avoid relations among private parameters from being disclosed to opponent players. As differential privacy has emerged as the de facto standard for privacy protection due to its strong resilience against arbitrary post-processing and auxiliary information [33], recent results in [34] and [35] propose independent-noise based differential-privacy mechanisms for aggregative games. To ensure a finite
cumulative privacy budget for \( \epsilon \)-differential privacy (privacy will essentially be lost when the cumulative privacy budget becomes unbounded), these approaches employ summable stepsize sequences, which, however, make it impossible to retain accurate convergence to the exact Nash equilibrium.

In this paper, we propose a distributed NE seeking approach on directed graphs that can ensure both accurate convergence and rigorous \( \epsilon \)-differential privacy with guaranteed finite cumulative privacy budget. This approach is motivated by the observation that differential-privacy noises enter the algorithm through inter-player interaction, which becomes unnecessary after convergence. So we propose to gradually weaken the inter-player interaction to attenuate the effect of differential-privacy noise in shared messages on NE seeking. Note that inter-player interaction is necessary for all players’ convergence to the NE, and thus we judiciously design the weakening factor sequence and the stepsize sequence, under which we prove that our approach can ensure accurate convergence to the exact NE even in the presence of differential-privacy noise. We rigorously prove that the algorithm is \( \epsilon \)-differentially private with a finite cumulative privacy budget, even when the number of iterations tends to infinity. It is worth noting that compared with our recent results on differentially-private distributed optimization [36], [37], the results here for NE seeking are fundamentally different: Firstly, agents in distributed optimization are cooperative in computing a common objective function, whereas players in games are competitive and only mind their own individual cost functions. Secondly, noises in games can easily alter the NE (as evidenced by the loss of accurate convergence in existing differential-privacy solutions for aggregative games [34]), and thus we have to carefully design the noise-adding mechanism and inter-player interaction to ensure accurate convergence to the NE. Finally, our result in [36] can ensure finite cumulative privacy budget in the vanilla single-variable based distributed optimization, whereas its cumulative privacy budget can still grow unbounded in two-variable based distributed optimization. In contrast, the approach in this paper ensures finite cumulative privacy budget for NE seeking involving multiple variables for individual players. Moreover, different from our recent results in [38] which address aggregative games on symmetric communication graphs, this paper addresses general networked games that are not necessarily aggregative on directed communication graphs that could be unbalanced.

**Contributions:** The main contributions of this paper are as follows:

1) We propose a fully distributed approach for NE seeking that can ensure both accurate convergence to the NE and rigorous \( \epsilon \)-differential privacy with bounded cumulative privacy budget, even when the number of iterations tends to infinity. It is in sharp contrast to existing differential-privacy solutions for aggregative games (e.g., [34], [35]) that have to trade accurate convergence for differential privacy. This appears to be the first algorithm able to achieve such goals for general Nash games.

2) The proposed approach is applicable to general directed graphs that are not necessarily balanced. Nor does it require individual players to have access to global-structure parameters of the communication graph. This is in contrast to previous works assuming symmetric or balanced graphs, or knowledge of the global network structure such as the Perron-Frobenius eigenvector of the graph.

3) Our convergence analysis does not require uniformly bounded pseudo gradients, an assumption that is common in existing results for NE seeking in the presence of noises. Note that removing this assumption is significant in that in the presence of unbounded noise (e.g., Gaussian or Laplace noise used in differential privacy), the pseudo gradients will become unbounded in many common games (e.g., the Nash-Cournot game under a linear inverse-demand price). Moreover, our analysis only requires cost functions to be strictly monotone, which is more general than existing results requiring strongly monotone cost functions (see, e.g., [9], [24], [20]).

4) Even without considering privacy protection, our proof techniques are fundamentally different from existing counterparts and are of independent interest in themselves. More specifically, existing proof techniques (in, e.g., [16], [35], [39], [40], [41], [42], [43], [44])) for partial-decision information games rely on the geometric (exponential) decreasing of consensus errors among the players. Such a fast decreasing of the consensus error is crucial for proving exact convergence to the NE, but is only possible under persistent interaction. In the proposed approach, the diminishing interaction makes it impossible to have such geometric decreasing of consensus errors, which entails new proof techniques.

The organization of the paper is as follows. Sec. II provides the problem formulation and some preliminary results. Sec. III presents a distributed NE seeking algorithm in the absence of constraints on decision variables. Sec. IV presents some general convergence results, based on which, Sec. V proves the almost sure convergence of all players to the exact NE, even in the presence of differential-privacy noise. Sec. VI proves that the algorithm can achieve rigorous \( \epsilon \)-differential privacy with a guaranteed finite cumulative privacy budget, even when the number of iterations tends to infinity. Sec. VII extends the approach to the case where individual players’ decisions are constrained to closed and convex sets. Sec. VIII presents numerical comparisons with existing results. Finally, Sec. IX concludes the paper.

**Notations:** We use \( \mathbb{R}^d \) to denote the Euclidean space of dimension \( d \). We write \( I_d \) for the identity matrix of dimension \( d \), and \( 1_d \) for the \( d \)-dimensional column vector with all entries equal to 1; in both cases we suppress the dimension when clear from the context. For a vector \( x, \langle x \rangle_i \) denotes its \( i \)th element. We write \( x > 0 \) (resp. \( x \geq 0 \)) if all elements of \( x \) are positive (resp. non-negative). We use \( \langle \cdot, \cdot \rangle \) to denote the inner product and \( \| x \|_2 \) for the standard Euclidean norm of a vector \( x \). We use \( \| x \|_1 \) to represent the \( \ell_1 \) norm of a vector \( x \). We write \( \| A \| \) for the matrix norm induced by a vector norm \( \| \cdot \| \), and let \( A^T \) denote the transpose of a matrix \( A \). For two vectors \( u \) and \( v \) with the same dimension, we write \( u \leq v \) to mean that each entry of \( u \) is no larger than the corresponding entry of \( v \). Sometimes, we abbreviate \( \text{almost surely by a.s.} \) All vectors are viewed as column vectors unless stated otherwise.
II. PROBLEM FORMULATION AND PRELIMINARIES

A. On Networked Nash Games

We consider a networked Nash game among a set of \( m \) players (or agents), i.e., \([m] = \{1, 2, \ldots, m\}\). We index the players by \(1, 2, \ldots, m\). Player \( i \) is characterized by a feasible action set \( K_i \subseteq \mathbb{R}^{d_i} \) and a cost function \( f_i(x_i, x_{-i}) \) where \( x_i \in K_i \) denotes the decision of player \( i \) and \( x_{-i} \triangleq [x_1^T, \ldots, x_{i-1}^T, x_{i+1}^T, \ldots, x_m^T]^T \in K_{-i} \triangleq K_1 \times \cdots \times K_{i-1} \times K_{i+1} \times \cdots \times K_m \) denotes the joint decisions of all players except player \( i \). Note that we allow different \( x_i \) to have different dimensions \( d_i \).

Traditionally when a mediator/coordinator exists, every player \( i \) can access all other players’ decisions \( x_{-i} \). Then, the game that player \( i \) faces can be formulated as the following parametrized optimization problem:

\[
\min f_i(x_i, x_{-i}) \quad \text{s.t.} \quad x_i \in K_i \quad \text{and} \quad x_{-i} \in K_{-i}. \tag{1}
\]

The constraint set \( K_i \) and the function \( f_i(\cdot) \) are assumed to be known to player \( i \) only.

At the NE \( x^* = [(x_1^*)^T, \ldots, (x_m^*)^T]^T \in \mathbb{R}^D \) with \( D = \sum_{i=1}^m d_i \), each player has

\[
f_i(x_i^*, x_{-i}^*) \leq f_i(x_i, x_{-i}^*), \quad \forall x_i \in K_i.
\]

Namely, at the NE, no player can unilaterally reduce its cost by changing its own decision.

We consider a networked game scenario where no mediator/coordinator exists, and hence a player cannot access all the other players’ decisions. More specifically, to compensate for this lack of global information on others’ decisions, players communicate and share decisions locally among neighbors, which is commonly referred to as the partial-decision information scenario [9]. To describe the local communication among neighboring players, we use a directed graph \( G = ([m], \mathcal{E}) \) where \([m] = \{1, 2, \ldots, m\}\) is the set of nodes (players) and \( \mathcal{E} \subseteq [m] \times [m] \) is the edge set of ordered node pairs describing the interactions among players. To characterize the interactions among the players, we also use the notion of directed graph induced by a weight matrix \( L = \{L_{ij}\} \in \mathbb{R}^{m \times m} \), denoted as \( G_L = ([m], \mathcal{E}_L) \). More specifically, in \( G_L = ([m], \mathcal{E}_L) \), a directed edge \((i, j)\) from agent \( j \) to agent \( i \) exists, i.e., \((i, j) \in \mathcal{E}_L\) if and only if \( L_{ij} > 0 \). For a player \( i \) in \([m]\), its in-neighbor set \( n_i^m \) is defined as the collection of players \( j \) such that \( L_{ij} > 0 \); similarly, the out-neighbor set \( n_i^o \) of player \( i \) is the collection of players \( j \) such that \( L_{ij} > 0 \). Note that we do not consider self loops, and, hence, the induced graph \( G_L = ([m], \mathcal{E}_L) \) is independent of the diagonal entries of \( L \).

To characterize the NE of the networked game (1), we also introduce the following notations:

\[
F_i(x_i, x_{-i}) \triangleq \nabla_{x_i} f_i(x_i, x_{-i}), \quad \phi(x) \triangleq \begin{pmatrix} F_1(x_1, x_{-1}) \\ \vdots \\ F_m(x_m, x_{-m}) \end{pmatrix}. \tag{3}
\]

We make the following assumptions on the constraint sets \( K_i \) and the functions \( f_i \):

**Assumption 1.** Every \( K_i \subseteq \mathbb{R}^d \) is non-empty, closed, and convex. Every function \( f_i(x_i, x_{-i}) \) is convex and differentiable in \( x_i \) over some open set containing the set \( K_i \) for each \( x_{-i} \). The mapping \( \phi(x) \) is strictly monotone over \( K \triangleq K_1 \times \cdots \times K_m \), i.e., for all \( x \neq x' \) in \( K \), we always have

\[
(\phi(x) - \phi(x'))^T(x - x') > 0.
\]

**Remark 1.** It is worth noting that the strictly monotone assumption on \( \phi(x) \) is weaker than the commonly used strongly monotone assumption in, e.g., [9], [20], [34], [35], [44], [45].

Assumption 1 ensures that the game (1) has a unique NE \( x^* \). Moreover, following [24], we also make the following assumption on the mapping \( F_i(x_i, x_{-i}) \):

**Assumption 2.** Each mapping \( F_i(x_i, x_{-i}) \) is Lipschitz continuous in both of its arguments, \( x_i \) and \( x_{-i} \). Namely, for all \( x_i, y_i \in \mathbb{R}^{d_i} \) and \( x_{-i}, y_{-i} \in \mathbb{R}^{D-d_i} \), we always have

\[
\|F_i(x_i, x_{-i}) - F_i(y_i, x_{-i})\|_2 \leq L_1\|x_i - y_i\|_2,
\]

\[
\|F_i(x_i, x_{-i}) - F_i(x_i, y_{-i})\|_2 \leq L_2\|x_{-i} - y_{-i}\|_2
\]

for all \( i \in [m] \), where \( D = \sum_{i=1}^m d_i \), and \( L_1, L_2 \) are some positive constants.

We make the following assumption on the inter-player communication described by the \( L \)-induced directed graph \( G_L \):

**Assumption 3.** The off-diagonal entries of the matrix \( L = \{L_{ij}\} \in \mathbb{R}^{m \times m} \) are non-negative and its diagonal entries \( L_{ii} = -\sum_{j \neq i} L_{ij} \) satisfy \( L_{ii} > -1 \) for all \( i \in [m] \). Moreover, the digraph \( G_L \) is strongly connected, i.e., there exists a (multi-hop) path from every node to every other node.

In the analysis of our approach, we use the following results:

**Lemma 1.** [36] Let \( \{v^k\}, \{\alpha^k\} \), and \( \{p^k\} \) be random non-negative scalar sequences, and \( \{q^k\} \) be a deterministic non-negative scalar sequence satisfying \( \sum_{k=0}^\infty \alpha^k < \infty \) a.s., \( \sum_{k=0}^\infty q^k = \infty \), \( \sum_{k=0}^\infty p^k < \infty \) a.s., and the following inequality:

\[
\mathbb{E}[v^{k+1}\mathbb{I}\mathcal{F}^k] \leq (1 + \alpha^k - q^k)v^k + p^k, \quad \forall k \geq 0 \quad \text{a.s.}
\]

where \( \mathcal{F}^k = \{v^\ell, \alpha^\ell, p^\ell; 0 \leq \ell \leq k\} \). Then, \( \sum_{k=0}^\infty \alpha^k q^k v^k < \infty \) and \( \lim_{k \rightarrow \infty} v^k = 0 \) hold almost surely.

**Lemma 2.** [36] Let \( \{v^k\} \subset \mathbb{R}^d \) and \( \{u^k\} \subset \mathbb{R}^p \) be random nonnegative vector sequences, and \( \{a^k\} \) and \( \{b^k\} \) be random nonnegative scalar sequences such that

\[
\mathbb{E}[v^{k+1}\mathbb{I}\mathcal{F}^k] \leq (V^k + a^k I^T)v^k + b^k - H^k u^k, \quad \forall k \geq 0
\]

holds a.s., where \( \{V^k\} \) and \( \{H^k\} \) are random sequences of nonnegative matrices and \( \mathbb{E}[v^{k+1}\mathbb{I}\mathcal{F}^k] \) denotes the conditional expectation given \( v^\ell, u^\ell, a^\ell, b^\ell, V^\ell, H^\ell \) for \( \ell = 0, 1, \ldots, k \). Assume that \( \{a^k\} \) and \( \{b^k\} \) satisfy \( \sum_{k=0}^\infty a^k < \infty \) and \( \sum_{k=0}^\infty b^k < \infty \) a.s., and that there exists a (deterministic) vector \( \pi^* > 0 \) such that \( \pi^* V^k \leq \pi^* \) and \( \pi^* H^k \geq 0 \) hold a.s. for all \( k \geq 0 \). Then, we have 1) \( \pi^* v^k \) converges to some random variable \( \pi^* v \geq 0 \) a.s.; 2) \( \{v^k\} \) is bounded a.s.; and 3) \( \sum_{k=0}^\infty \pi^* H^k u^k < \infty \) holds almost surely.
B. On Differential Privacy

We adopt the notion of \( \epsilon \)-differential privacy for continuous bit streams [46], which has recently been applied to distributed optimization algorithms (see, e.g., [47] as well as our work [36]). To enable differential privacy, we inject Laplace noise \( \text{Lap}(\nu) \) to all shared messages, where \( \nu > 0 \) is a constant parameter of the probability density function \( \frac{1}{\nu} e^{-\frac{|x|}{\nu}} \). One can verify that \( \text{Lap}(\nu) \) has mean zero and variance \( 2\nu^2 \). For the convenience of differential-privacy analysis, we represent the networked game \( P \) in (1) by three parameters \((K, \mathbb{F}, \mathbb{G}_L)\), where \( K \triangleq K_1 \times \cdots \times K_m \) is the domain of decision variables, \( \mathbb{F} \triangleq \{ f_1, \cdots, f_m \} \), and \( \mathbb{G}_L \) denotes the communication graph. We define “adjacency” between two networked games as follows:

**Definition 1.** Two networked games \( P \triangleq (K, \mathbb{F}, \mathbb{G}_L) \) and \( P' \triangleq (K', \mathbb{F}', \mathbb{G}'_L) \) are adjacent if the following conditions hold:

- \( K = K' \) and \( \mathbb{G}_L = \mathbb{G}'_L \), i.e., the domain of decision variables and the communication graphs are identical;
- there exists an \( i \in [m] \) such that \( f_i \neq f'_i \) but \( f_j = f'_j \) for all \( j \in [m], j \neq i \).

Definition 1 says that two networked games are adjacent if and only if they differ only in the cost function of a single player.

Given a distributed algorithm for NE seeking, we represent an execution of this algorithm as \( A \), which is an infinite sequence of the iteration variable \( \vartheta \), i.e., \( A = \{ \vartheta^0, \vartheta^1, \cdots \} \). We consider adversaries that can observe all communicated messages in the network \( \mathbb{G}_L \). Thus, the observation part of an execution is the infinite sequence of shared messages, which is represented by \( O \). We define the mapping from execution sequence to observation sequence by \( \mathcal{R}(A) \triangleq O \). Given a networked game \( P \), observation sequence \( O \), and an initial state \( \vartheta^0 \), \( \mathcal{R}^{-1}(P, O, \vartheta^0) \) is the set of executions \( A \) that can generate the observation \( O \).

**Definition 2.** \((\epsilon, \delta)\)-differential privacy, adapted from [47]). For a given \( \epsilon > 0 \), an iterative NE seeking algorithm is \((\epsilon, \delta)\)-differentially private if for any two adjacent \( P \) and \( P' \), any set of observation sequences \( O_s \subseteq O \) (with \( O \) denoting the set of all possible observation sequences), and any initial state \( \vartheta^0 \), we always have

\[
P[\mathcal{R}^{-1}(P, O_s, \vartheta^0)] \leq e^\epsilon P[\mathcal{R}^{-1}(P', O_s, \vartheta^0)],
\]

where the probability \( P \) is taken over the randomness over iteration processes.

The definition of \( \epsilon \)-differential privacy guarantees that an adversary having access to all shared messages cannot gain information with a significant probability of any player’s cost function. The definition also implies that a smaller \( \epsilon \) corresponds to a higher level of privacy. It is worth noting that the \( \epsilon \)-differential privacy considered here is more stringent than other relaxed (approximate) notions of differential privacy, including, e.g., \((\epsilon, \delta)\)-differential privacy [48], zero-concentrated differential privacy [49], and Rényi differential privacy [50].

III. A DIFFERENTIALLY-PRIVATE NE seeking algorithm

Next, we present a fully distributed NE seeking algorithm that can guarantee both \( \epsilon \)-differential privacy and accurate convergence. For the convenience of exposition, in this section, we consider the unconstrained case where \( K_i = \mathbb{R}^{d_i} \). The constrained case will be discussed in Sec. VII.

Since at each iteration \( k \), each player \( i \) does not have direct access to other players’ decisions \( x^k_i \), we let each player \( i \) maintain \( m \) variables \( x^k_{(i)1}, x^k_{(i)2}, \cdots, x^k_{(i)m} \), where \( x^k_{(i)j} \) denotes the decision of player \( i \) whereas \( x^k_{(i)j} \) denotes player \( i \)'s estimate of player \( j \)'s decision variable \( x^k_{(j)\ell} \) (\( \ell = 1, \cdots, i - 1, i + 1, \cdots, m \)). Player \( i \) exploits local exchange of information to ensure that its estimates \( x^k_{(i)\ell} \) can track the true decisions \( x^k_{(i)\ell} \) of all other players.

To ensure rigorous differential privacy, every shared message in every iteration must be obfuscated by independent noise, which results in significant reduction in algorithmic accuracy. In fact, to contain the cumulative effect of differential-privacy noise in distributed optimization or NE seeking, existing privacy solutions have to restrict the degree of gradient exploration (by using a fast decreasing stepsize) so as to use a fast decreasing differential-privacy noise [34], [47]. However, the restricted degree of gradient exploration unavoidably leads to the loss of provable convergence to the desired equilibrium point. Motivated by the observation that differential-privacy noise enters the algorithm through inter-player interaction, which gradually becomes unnecessary as players converge to the equilibrium, we propose to gradually weaken inter-player interactions to gradually reduce and eliminate the influence of differential-privacy noise on NE seeking. Interestingly, we prove that by judiciously designing the weakening factor for inter-player interaction, we can ensure almost sure convergence to the NE even under persistent differential-privacy noise.

The detailed algorithm is summarized in Algorithm 1.

Algorithm 1: Distributed NE seeking with guaranteed convergence and differential privacy

Parameters: Stepsize \( \lambda \) > 0 and weakening factor \( \gamma > 0 \). Every player \( i \) maintains one decision variable \( x^k_{(i)j} \) and \( m - 1 \) estimates \( x^k_{(i)j} \triangleq [x^k_{(i)j1}, \cdots, x^k_{(i)j(i-1)}, x^k_{(i)j(i+1)}, \cdots, x^k_{(i)jm}]^T \) of other players’ decision variables. Player \( i \) sets \( x^0_{(i)\ell} \) randomly in \( \mathbb{R}^{d_i} \) for all \( \ell \in [m] \).

for \( k = 1, 2, \cdots \) do

a) For both its decision variable \( x^k_{(j)j} \) and estimate variables \( x^k_{(j)1}, \cdots, x^k_{(j)(j-1)}, x^k_{(j)(j+1)} \), every player \( j \) adds respective persistent differential-privacy noise \( \zeta^k_{(j)1}, \cdots, \zeta^k_{(j)m} \) and then sends the obscured values \( x^k_{(j)j} + \zeta^k_{(j)1}, \cdots, x^k_{(j)jm} + \zeta^k_{(j)m} \) to all players \( i \in [m] \).

b) After receiving \( x^k_{(j)j}, \zeta^k_{(j)1}, \cdots, x^k_{(j)jm} \) from all \( j \in [m] \), player \( i \) updates its decision variable and estimate...
variables $x_{(i),l}^{k+1}$ as follows:

$$x_{(i),l}^{k+1} = x_{(i),l}^{k} + \gamma^{k} \sum_{j \in \mathbb{N}^{n}} L_{ij}(x_{(j),l}^{k} + c_{(j),l}^{k} - x_{(i),l}^{k}) - \lambda^{k} F_{i}(x_{(i),l}^{k}, x_{(i),l}^{k-1}),$$

$$x_{(i),l}^{k+1} = x_{(i),l}^{k} + \gamma^{k} \sum_{j \in \mathbb{N}^{n}} L_{ij}(x_{(j),l}^{k} + c_{(j),l}^{k} - x_{(i),l}^{k}), \quad \forall l \neq i.$$

(5)

c) end

IV. A General Convergence Result

We first have to establish some general results necessary for the convergence analysis of Algorithm 1.

Lemma 3. Under Assumption 3, we have the following properties:

1) the eigenvectors of the matrix $I + \gamma L$ are time-invariant;
2) $I + \gamma L$ has a unique positive left eigenvector $u^{\top}$ (associated with eigenvalue 1) satisfying $u^{\top} \mathbb{1} = m$;
3) the spectral radius of $I + \gamma L - \frac{1}{m} u^{\top} u$ is upper-bounded by $1 - \alpha \gamma$ when $\gamma > 0$ is small enough, where $0 < \alpha < 1$ is determined by the eigenvalues of $I + L - \frac{1}{m} u^{\top} u$;
4) there exists an $L$-dependent matrix norm $\| \cdot \|_{L}$ such that $\| I + \gamma L - \frac{1}{m} u^{\top} u \|_{L} \leq 1 - \alpha \gamma$ for $0 < \alpha < 1$ when $\gamma$ is small enough. Moreover, this norm has an associated inner product $\langle \cdot, \cdot \rangle_{L}$, i.e., $\| x \|_{L}^{2} = \langle x, x \rangle_{L}$.

Proof. 1) Representing the eigenvalues and associated eigenvectors of $L$ as $\{\varphi_{1}, \cdot \cdot \cdot, \varphi_{m}\}$ and $\{v_{1}, \cdot \cdot \cdot, v_{m}\}$, respectively, we can verify that the eigenvalues and associated eigenvectors of $I + \gamma L$ are given by $\{1 + \gamma \varphi_{1}, \cdot \cdot \cdot, 1 + \gamma \varphi_{m}\}$ and $\{v_{1}, \cdot \cdot \cdot, v_{m}\}$, respectively. Hence the result in the first statement is proven.

2) Under Assumption 3, one can obtain from [51] (or Lemma 1 in [52]) that $I + L$ has a unique positive left eigenvector $u^{\top}$ (associated with eigenvalue 1) satisfying $u^{\top} \mathbb{1} = m$, and, hence according to the proven statement 1), $I + \gamma L$ has a unique positive left eigenvector $u^{\top}$ (associated with eigenvalue 1) satisfying $u^{\top} \mathbb{1} = m$.

3) Representing the eigenvalues of $L$ by $\{\varphi_{1}, \cdot \cdot \cdot, \varphi_{m}\}$, the eigenvalues of $I + L$ can be expressed as $\{1 + \varphi_{1}, \cdot \cdot \cdot, 1 + \varphi_{m}\}$. Under Assumption 3, $I + L$ is irreducible. Using Perron–Frobenius theorem for irreducible non-negative matrices, one can obtain that $I + L$ has one unique eigenvalue equal to one and all its other eigenvalues strictly less than one in absolute value, implying that one and only one of $\varphi_{1}$ is zero. Represent this eigenvalue of $L$ as $\varphi_{m} = 0$ without loss of generality. Then we have $|1 + \varphi_{i}| < 1$ for all $1 \leq i \leq m - 1$. One can verify that the eigenvalues of $I + \gamma L - \frac{1}{m} u^{\top} u$ are given by $\{1 + \gamma \varphi_{1}, \cdot \cdot \cdot, 1 + \gamma \varphi_{m-1}, 0\}$. Next, we prove the third statement by showing that there exists an $\alpha$ satisfying $|1 + \gamma \varphi_{1}| < 1 - \alpha \gamma$ for every $i = 1, 2, \cdot \cdot \cdot, m - 1$.

To this end, we represent $\varphi_{1}$ as $\varphi_{1} = a_{i} + ib_{i}$, where $a_{i}$ and $b_{i}$ are real numbers, and $i$ is the imaginary unit. Because $|1 + \varphi_{1}| < 1$ holds for all $1 \leq i \leq m - 1$, we have $a_{i} < 0$ for $i = 1, 2, \cdot \cdot \cdot, m - 1$. Under the new representation of $\varphi_{1}$, $|1 + \gamma \varphi_{1}|$ becomes $\sqrt{(1 - |a_{i}|^{2}) + (b_{i})^{2}}$. So we only have to prove

$$\sqrt{(1 - |a_{i}|^{2}) + (b_{i})^{2}} < 1 - \alpha \gamma$$

(6)

for some $0 < \alpha < 1$ when $\gamma$ is small enough. Taking square on both sides, we can convert (6) to

$$\alpha^{2}(\gamma)^{2} < 2 \alpha \gamma - 2 \gamma |a_{i}|,$$

i.e.,

$$\alpha^{2} - \frac{2}{\gamma} \alpha \gamma < (a_{i}^{2} + b_{i}^{2}) - \frac{2 |a_{i}|}{\gamma}.$$

(7)

When $\gamma$ is less than $\frac{2 |a_{i}|}{\alpha + b_{i}}$ (note $a_{i} < 0$ for $i = 1, \cdot \cdot \cdot, m - 1$), the right hand side of (7) is negative whereas the left hand side is a quadratic function of $\alpha$ with two $x$-intercepts given by $\alpha = 0$ and $\alpha = \frac{b_{i}}{a_{i}}$. So there always exists an $\alpha$ in the interval $(0,1)$ making the left hand side of (7) larger than its right hand side, and hence making (7) hold. Therefore, there always exists an $0 < \alpha < 1$ making $|1 + \gamma \varphi_{1}| < 1 - \alpha \gamma$ hold when $\gamma > 0$ is less than $\frac{2 |a_{i}|}{\alpha + b_{i}}$.

Given that the above derivation is independent of $i$, we have $|1 + \gamma \varphi_{i}| < 1 - \alpha \gamma$ for some $0 < \alpha < 1$ and all $i = 1, \cdot \cdot \cdot, m - 1$, and hence the third statement in the Lemma.

4) According to Lemma 5.6.10 in [51] (and the discussions thereafter), there always exits a matrix norm $\| \cdot \|_{L}$ such that the norm of $I + \gamma L - \frac{1}{m} u^{\top} u$ is arbitrarily close to its spectral radius. From the proven result in the third statement of the lemma, we know that the spectral radius of $I + \gamma L - \frac{1}{m} u^{\top} u$ is always less than $1 - \alpha \gamma$ for some $0 < \alpha < 1$ when $\gamma$ is smaller than $\min \left\{ \frac{2 |a_{i}|^{2}}{a_{i}^{2} + b_{i}^{2}}, \cdot \cdot \cdot, \frac{2 |a_{m-1}|^{2}}{a_{m-1}^{2} + b_{m-1}^{2}} \right\}$. Combining these two statements yields that there always exists a matrix norm $\| \cdot \|_{L}$ such that $\| I + \gamma L - \frac{1}{m} u^{\top} u \|_{L} < 1 - \alpha \gamma$ holds for some $0 < \alpha < 1$ when $\gamma$ is smaller than $\min \left\{ \frac{2 |a_{1}|^{2}}{a_{1}^{2} + b_{1}^{2}}, \cdot \cdot \cdot, \frac{2 |a_{m-1}|^{2}}{a_{m-1}^{2} + b_{m-1}^{2}} \right\}$. Furthermore, still according to Lemma 5.6.10 in [51] (and the discussions thereafter), we known that this matrix norm $\| \cdot \|_{L}$ can be expressed as $\| x \|_{L} = \| x \|_{2}$ for some $L$ determined by $\gamma$. So the norm $\| \cdot \|_{L}$ satisfies the Parallelogram Law and, hence, has an associated inner product $\langle \cdot, \cdot \rangle_{L}$.

Using the time-invariant positive left eigenvector $u \triangleq [u_{1}, \cdot \cdot \cdot, u_{m}]^{T}$ of $I + \gamma L$ guaranteed by Lemma 3, we define a weighted average $x^{k}_{i} \triangleq \frac{1}{m} \sum_{\ell=1}^{m} u_{i}(x^{k}_{\ell})$, of player $i$’s decision variable $x^{k}_{i}$ and other players’ estimates $x^{k}_{(j),l}$ ($j \neq i$) of this decision variable.

For the convenience of analysis, we also define the assembly of the $i$th decision variable $x^{k}_{i}$ as well as the assembly of the weighted average $\tilde{x}^{k}_{i}$ as

$$x^{k}_{i} = \begin{bmatrix} (x^{k}_{i(1)})^{T} \\ \vdots \\ (x^{k}_{i(m)})^{T} \end{bmatrix} \in \mathbb{R}^{m \times d_{i}}, \quad \tilde{x}^{k}_{i} = \begin{bmatrix} (\tilde{x}^{k}_{i(1)})^{T} \\ \vdots \\ (\tilde{x}^{k}_{i})^{T} \end{bmatrix} \in \mathbb{R}^{m \times d_{i}},$$

(8)

respectively.

To measure the distance between matrix variables $x^{k}_{i}$ and $\tilde{x}^{k}_{i}$, we define a matrix norm for an arbitrary vector norm $\| \cdot \|_{P}$ More specifically, for a matrix $X$ in $\mathbb{R}^{m \times d}$, we define $\| X \|_{P} \triangleq \left\| \left\| X_{(1)} \right\|_{P}, \| X_{(2)} \|_{P}, \cdot \cdot \cdot, \| X_{(d)} \|_{P} \right\|_{2}$, where
Since we have 
and 
\(X, Z\) and 
\(Y, Z\) for any real numbers \(a, b\) and real matrices \(X, Y, Z \in \mathbb{R}^{m \times d_i}\).

In addition, we have the following result:

**Lemma 4.** For any norm \(\| \cdot \|_p\), any \(T \in \mathbb{R}^{m \times m}\), and \(X \in \mathbb{R}^{m \times d_i}\), we always have \(\|WX\|_p \leq \|W\|_p \|X\|_p\). Furthermore, there exist constants \(\delta_{2,L}\) and \(\delta_{2,L}\) such that for any \(X \in \mathbb{R}^{m \times d_i}\), we have \(\|X\|_L \leq \delta_{L,2}\|X\|_2\) and \(\|X\|_2 \leq \delta_{2,L}\|X\|_L\).

**Proof.** The result follows from the line of reasoning in Lemma 5 and Lemma 6 in [52], and hence we do not include the proof here.

Based on the above, we have the following convergence result for general distributed algorithms for problem (1):

**Proposition 1.** Let Assumptions 1 and 2 hold, and let \(x^* = (x_1^*, \ldots, x_n^*)^T\) denote the NE of (1). If, under the interaction matrix \(L\), a distributed algorithm generates sequences \(\{x_i^k\}\) for all \(i \in [m]\) such that a.s. we have

\[
\mathbb{E}\left[\sum_{i=1}^m \|\bar{x}_i^{k+1} - x_i^k\|_2^2\right] F_k \leq \left(1\right)^k 11^T \left[\sum_{i=1}^m \|\bar{x}_i^k - x_i^0\|_2^2\right] F_k + b^k 1 - c^k \left((\phi(\bar{x}^k) - \phi(x^*))^T (\bar{x}^k - x^*)\right),
\]

where \(\| \cdot \|_L\) is an L-dependent norm, \(F_k = \{x_i^k, i \in [m]\}, 0 \leq \ell \leq k\), \(\bar{x}_i^k = (\bar{x}_i^k)^T, \ldots, (\bar{x}_m^k)^T\), \(a_k^\ell, k = 0\) satisfy \(\sum_{k=0}^\infty a_k^\ell < \infty\) and \(\sum_{k=0}^\infty b_k^k < \infty\), respectively, a.s., the deterministic non-negative scalar sequences \(\{\kappa_i\}\) and \(\{\gamma_i\}\) satisfy \(\sum_{k=0}^\infty \kappa_k < \infty\) and \(\sum_{k=0}^\infty \gamma_k < \infty\), and the scalars \(\kappa_1\) and \(\kappa_2\) satisfy \(0 < \kappa_1 < 1\) and \(0 < \kappa_2 < 1\), respectively, for all \(k \geq 0\). Then, we have \(\lim_{k \to \infty} \|\bar{x}_i^k - x_i^k\|_2 = 0\) and \(\lim_{k \to \infty} x_i^k = x_i^*\) a.s. for all \(i\), implying \(\lim_{k \to \infty} (x_i^k - x_i^*) = 0\) a.s. for all \(i\).

**Proof.** Since we have \(\phi(\bar{x}^k) - \phi(x^*) > 0\) for all \(k\) from Assumption 1, by letting \(v_k = \left[\sum_{i=1}^m \|\bar{x}_i^k - x_i^k\|_2^2, \sum_{i=1}^m \|\bar{x}_i^k - x_i^k\|_2^2\right]^T\), we can arrive at the following relationship from (10) a.s. for all \(k \geq 0\):

\[
\mathbb{E}\left[\sum_{i=1}^m \|\bar{x}_i^{k+1} - x_i^k\|_2^2\right] F_k \leq \left(1 + \frac{\kappa_1}{\gamma_2} \frac{1}{1 - \kappa_2}\right) + a_k 11^T v_k + b_k 1.
\]

By setting \(a_k = \left[\frac{\kappa_1}{\gamma_2} \frac{1}{1 - \kappa_2}\right]\), we have \(\pi^T = \left[\frac{\kappa_1}{\gamma_2} \frac{1}{1 - \kappa_2}\right] = \pi^T\).

Thus, relation (11) meets all conditions of Lemma 2, implying that \(\lim_{k \to \infty} \pi^T v_k^k\) exists a.s., and that the sequences \(\{\sum_{i=1}^m \|\bar{x}_i^k - x_i^k\|_2^2\}\) and \(\{\sum_{i=1}^m \|\bar{x}_i^k - x_i^k\|_2^2\}\) are bounded almost surely.

Consider the second element of \(v_k^k\) in (11), which should satisfy the following inequality a.s.:

\[
\sum_{i=1}^m \|\bar{x}_i^{k+1} - x_i^k\|_2^2 \leq (1 + \kappa_2) \sum_{i=1}^m \|\bar{x}_i^k - x_i^k\|_2^2 + \beta_k \quad \forall k \geq 0,
\]

where \(\beta_k = a_k \left[\sum_{i=1}^m \|\bar{x}_i^k - x_i^k\|_2^2, \sum_{i=1}^m \|\bar{x}_i^k - x_i^k\|_2^2\right]^T\).

Using the assumption that \(\sum_{k=0}^\infty a_k^\ell < \infty\) holds a.s., and the proven results that \(\sum_{i=1}^m \|\bar{x}_i^k - x_i^k\|_2^2\) and \(\sum_{i=1}^m \|\bar{x}_i^k - x_i^k\|_2^2\) are bounded a.s., one obtains \(\sum_{k=0}^\infty \beta_k < \infty\) almost surely. Thus, under the assumption of the proposition, \(\sum_{k=0}^\infty b_k^k < \infty\) a.s. and \(\sum_{k=0}^\infty \gamma_k = \infty\), (12) satisfies the conditions of Lemma 1 with \(v_k = \sum_{i=1}^m \|\bar{x}_i^k - x_i^k\|_2^2\), \(q_k = \kappa_2\gamma_k\), and \(p_k = \beta_k\). Therefore, we have the following relationship a.s.:

\[
\sum_{k=0}^\infty \kappa_2 \gamma_k \sum_{i=1}^m \|\bar{x}_i^k - x_i^k\|_2^2 < \infty, \quad \lim_{k \to \infty} \sum_{i=1}^m \|\bar{x}_i^k - x_i^k\|_2^2 = 0.
\]

We next proceed to prove \(\sum_{i=1}^m \|\bar{x}_i^k - x_i^k\|_2^2 \to 0\) almost surely. One can verify that under \(\sum_{k=0}^\infty \kappa_k < \infty\) and \(\sum_{k=0}^\infty b_k^k < \infty\), the inequality in (10) satisfies the relationship in Lemma 2 with

\[
\mathbb{E}\left[\sum_{i=1}^m \|\bar{x}_i^k - x_i^k\|_2^2\right] F_k \leq \left(1\right)^k 11^T \left[\sum_{i=1}^m \|\bar{x}_i^k - x_i^k\|_2^2\right] F_k + b^k 1 - c^k \left((\phi(\bar{x}^k) - \phi(x^*))^T (\bar{x}^k - x^*)\right),
\]

where \(\| \cdot \|_L\) is an L-dependent norm, \(F_k = \{x_i^k, i \in [m]\}, 0 \leq \ell \leq k\), \(\bar{x}_i^k = (\bar{x}_i^k)^T, \ldots, (\bar{x}_m^k)^T\), the random nonnegative scalar sequences \(\{\kappa_i\}\) and \(\{\gamma_i\}\) satisfy \(\sum_{k=0}^\infty \kappa_k < \infty\) and \(\sum_{k=0}^\infty \gamma_k = \infty\), respectively, a.s., the deterministic non-negative scalar sequences \(\{\beta_k\}\) and \(\{\gamma_k\}\) satisfy \(\sum_{k=0}^\infty \gamma_k = \infty\), and the scalars \(\kappa_1\) and \(\kappa_2\) satisfy \(0 < \kappa_1 < 1\) and \(0 < \kappa_2 < 1\), respectively, for all \(k \geq 0\). Then, we have \(\lim_{k \to \infty} \|\bar{x}_i^k - x_i^k\|_2 = 0\) and \(\lim_{k \to \infty} x_i^k = x_i^*\) a.s. for all \(i\), implying \(\lim_{k \to \infty} (x_i^k - x_i^*) = 0\) a.s. for all \(i\).

Next, using (14) and the proven a.s. convergence of \(\|\bar{x}_i^k - x_i^k\|_2\), we prove that \(\bar{x}^k\) converges to \(x^*\) almost surely. The condition \(\sum_{k=0}^\infty \kappa_k = \infty\), the property \(\phi(\bar{x}^k) - \phi(x^*)^T (\bar{x}^k - x^*) > 0\) (see Assumption 1), and (14) imply that there
exists a subsequence of \( \{\bar{x}^k\} \), say \( \{\bar{x}^{k_i}\} \), along which 
\((\phi(\bar{x}^k) - \phi(x^*))^T (\bar{x}^k - x^*)\) converges to zero almost surely.

The strictly monotone condition on \(\phi(\cdot)\) in Assumption 1 implies that \(\{\bar{x}^{k_i}\}\) must converge to \(x^*\) almost surely. This and the fact that \(\|\bar{x}^k - x^*\|_2\) converges to zero imply that \(\bar{x}^k\) converges to \(x^*\) almost surely. Further note that \(\|x^k_i - x^*\|_2\) converging to zero implies the convergence of \(x^k_i\) to \(x^*_i\) for all \(i \in [m]\). Therefore, we have \(x^k_{(i)\ell}\) converging to \(x^*_{(i)\ell}\) a.s. for all \(i \in [m]\).

V. CONVERGENCE ANALYSIS FOR ALGORITHM 1

In this section, based on Proposition 1, we establish the convergence of Algorithm 1 to the unique NE under persistent differential-privacy noise satisfying the following assumption:

**Assumption 4.** For every \(i, \ell \in [m] \) and conditional on the state \(x^k_{(i)\ell}\), the random noise \(\zeta^k_{(i)\ell}\) that player \(i\) adds to its shared decision (or estimates of other players’ decisions) satisfies 
\[
\mathbb{E}
\left[
\zeta^k_{(i)\ell} \mid x^k_{(i)\ell}
\right] = 0
\]
and 
\[
\mathbb{E}
\left[
\|\zeta_{(i)\ell}\|_2^2 \mid x^k_{(i)\ell}
\right] = \sigma^2 \leq \infty,
\]
where \(\{\zeta^k\}\) is the weakening sequence in Algorithm 1. Furthermore, \(\mathbb{E}
\left[
\|x^k_{(i)\ell}\|_2^2
\right] < \infty \) holds for all \(i, \ell \in [m]\).

**Remark 2.** Since \(\gamma^k\) decreases with time, the condition (15) can be satisfied even when the sequence \(\{\sigma^2\}\) increases with time. For example, for \(\gamma^k = O(1/t^2)\), if \(\{\sigma^2\}\) increases with time at a rate no larger than \(O(1/t^{0.3})\), the summable condition in (15) still holds. Allowing \(\{\sigma^2\}\) to increase with time is key to enabling strong \(\epsilon\)-differential privacy, which will be detailed later in Theorem 2.

**Theorem 1.** Under Assumptions 1-4, if there exists some \(T \geq 0\) such that the sequences \(\{\gamma^k\}\) and \(\{\lambda^k\}\) satisfy
\[
\sum_{k=T}^{\infty} \gamma^k = \infty, \quad \sum_{k=T}^{\infty} \lambda^k = \infty, \quad \sum_{k=T}^{\infty} \frac{\gamma^k}{\lambda^k} < \infty, \quad \sum_{k=T}^{\infty} \frac{\lambda^k}{\gamma^k} < \infty,
\]
then Algorithm 1 converges to the unique NE of problem (1) almost surely.

**Proof.** The basic idea of the proof is to use Proposition 1. Namely, we will prove that the quantities \(\sum_{i=1}^{m} \|x^k_i - x^*_i\|^2\) and \(\sum_{i=1}^{m} \|x^k_i - x^*_{(i)\ell}\|_2^2\) satisfy the conditions in Proposition 1. To this end, we organize our proof in three parts. In Part I, we analyze the evolution of \(\sum_{i=1}^{m} \|x^k_i - x^*_{(i)\ell}\|_2^2\), and in Part II, we analyze the evolution of \(\sum_{i=1}^{m} \|x^k_i - x^*_i\|^2\). In Part III, we combine Part I and Part II to complete the proof of the theorem. Note that because the results of Proposition 1 are asymptotic, they remain valid when the starting index is shifted from \(k = 0\) to \(k = T\), for an arbitrary \(T \geq 0\).

Part I: The evolution of \(\sum_{i=1}^{m} \|x^k_i - x^*_{(i)\ell}\|_2^2\).

From Algorithm 1, one can verify that the evolution of \(x^k_i\) follows:
\[
x^k_{i}(\ell) = (I + \gamma^k L)x^k_{i} + \gamma^k L_o \zeta^k_{(i)\ell} - \lambda^k e_i F^k_{(i,i)}(x^k_{(i)\ell}, x^k_{(i)\ell - 1}),
\]
where \(e_i \in \mathbb{R}^m\) is a unitary vector with the \(i\)th element equal to 1 and all the other elements equal to zero, \(L_o \in \mathbb{R}^{m \times m}\) is the matrix obtained by replacing all diagonal entries of matrix \(L\) with zero, and \(\zeta^k_{(i)\ell} = \left[\zeta^k_{(i)\ell}, \ldots, \zeta^k_{(m)\ell}\right]^T \in \mathbb{R}^{m \times d}\).

One can obtain that \(\bar{x}^k_{(i)\ell} = \frac{1}{m} u^k x^k_{(i)\ell}\) always holds, which, in combination with (16), yields
\[
\bar{x}^{k + 1}_{(i)\ell} = \frac{1}{m} u^k x^{k + 1}_{(i)\ell}
\]
\[
= \frac{1}{m} u^k L_o \zeta^k_{(i)\ell} - \frac{1}{m} \lambda^k u^k F^k_{(i,i)}(x^k_{(i)\ell}, x^k_{(i)\ell - 1})
\]
(17)

where \(u^k\) is the \(i\)th entry of \(u\). Note that in the last equality we used the property \(u^T (I + \gamma^k L) = u^T\) from Lemma 3.

Combining (16) with (17) yields
\[
x^k_{i}(\ell) = \frac{1}{m} u^k x^k_{i} + \gamma^k \left(I - \frac{1}{m} L_o\right) \zeta^k_{(i)\ell} - \lambda^k e_i F^k_{(i,i)}(x^k_{(i)\ell}, x^k_{(i)\ell - 1}).
\]

Defining new symbols \(W^k \triangleq I + \gamma^k L - \frac{1}{m} L_o\), \(L_o \triangleq I - \frac{1}{m} L_o\), and \(\Pi_{\ell} \triangleq e_i - \frac{u^k}{m}\), we can simply (18) as follows:
\[
x^k_{i}(\ell) = W^k x^k_{i} + \gamma^k \Pi_L \zeta^k_{(i)\ell} - \lambda^k \Pi_{\ell} e_i F^k_{(i,i)}(x^k_{(i)\ell}, x^k_{(i)\ell - 1}).
\]

The second statement of Lemma 3 implies \(W^k = 0\) and further \(W^k \bar{x}^k_{(i)\ell} = 0\). Hence, we can subtract \(W^k \bar{x}^k_{(i)\ell} = 0\) from the right hand side of (19) to obtain
\[
x^k_{i}(\ell) = \bar{x}^k_{(i)\ell} + \frac{1}{m} u^k x^k_{i} + \gamma^k \Pi_L O \zeta^k_{(i)\ell} - \lambda^k \Pi_{\ell} e_i F^k_{(i,i)}(x^k_{(i)\ell}, x^k_{(i)\ell - 1} - (20)
\]
Taking the \(\|\cdot\|_L\) norm on both sides leads to
\[
\|x^k_{i}(\ell) - \bar{x}^k_{(i)\ell}\|^2 \leq \|x^k_{i}(\ell) - \bar{x}^k_{(i)\ell}\|_L^2 + \gamma^k \|L_o \zeta^k_{(i)\ell}\|_L^2 \leq \|x^k_{i}(\ell) - \bar{x}^k_{(i)\ell}\|_L^2 + \gamma^k \|L_o \zeta^k_{(i)\ell}\|_L^2 \]
Further using the inequalities in Lemma 4, we can simplify (22) as

$$\mathbb{E} \left[ \|x_{k+1}^i - x^k_i\|^2_F \right]$$

$$\leq \left( \|W_k\|_L \|x_k^i - x^k_i\|_L + \lambda_k \|\Pi_{L_k}\|_L \|F_i^T(x^k_i, x_{k-1}^i)\|_L \right)^2$$

$$\quad + (\gamma^k)^2 \|\Pi_{L_k}\|_L^2 \delta^2_{L_k,2} m(\sigma_k^i)^2.$$  \hspace{1cm} (23)

According to Lemma 3, we have \(\|W_k\|_L \leq 1 - \alpha \epsilon k\) for some \(0 < \alpha < 1\) when \(\gamma^k\) is small enough. Given that \(\{\gamma^k\}\) is square summable, we have \(\|W_k\|_L \leq 1 - \alpha \epsilon k\) for some \(0 < \alpha < 1\) when \(k\) is larger than some \(T\). Therefore, (23) means that there always exists a \(T \geq 0\) such that we have

$$\mathbb{E} \left[ \|x_{k+1}^i - x^k_i\|^2_F \right]$$

$$\leq \left( (1 - \alpha \epsilon k) \|x_k^i - x^k_i\|_L + \lambda_k \|\Pi_{L_k}\|_L \|F_i^T(x^k_i, x_{k-1}^i)\|_L \right)^2$$

$$\quad + (\gamma^k)^2 \|\Pi_{L_k}\|_L^2 \delta^2_{L_k,2} m(\sigma_k^i)^2.$$  \hspace{1cm} (24)

For \(k \geq T\).

Applying to the first term on the right hand side of (24) the inequality \((a + b)^2 \leq (1 + c)^2 a^2 + (1 + c^{-1}) b^2\), valid for any scalars \(a, b, c > 0\), we can obtain

$$\mathbb{E} \left[ \|x_{k+1}^i - x^k_i\|^2_F \right]$$

$$\leq (1 + \epsilon) (1 - \alpha \epsilon k) \|x_k^i - x^k_i\|_L^2$$

$$\quad + (1 + \epsilon^{-1}) \lambda^2 \|\Pi_{L_k}\|_L^2 \|F_i^T(x^k_i, x_{k-1}^i)\|_L^2$$

$$\quad + (\gamma^k)^2 \|\Pi_{L_k}\|_L^2 \delta^2_{L_k,2} m(\sigma_k^i)^2.$$  \hspace{1cm} (25)

By setting \(c = \frac{\gamma^k}{\gamma^k \epsilon \alpha}\) (which further results in \(1 + \epsilon = \frac{1}{1 + \frac{\gamma^k \epsilon}{\gamma^k \alpha}}\) and \(1 + \epsilon^{-1} = \frac{1}{1 - \frac{\gamma^k \epsilon}{\gamma^k \alpha}}\), we can rewrite (25) as

$$\mathbb{E} \left[ \|x_{k+1}^i - x^k_i\|^2_F \right]$$

$$\leq (1 - \alpha \epsilon k) \|x_k^i - x^k_i\|_L^2 + (\gamma^k)^2 \|\Pi_{L_k}\|_L^2 \delta^2_{L_k,2} m(\sigma_k^i)^2$$

$$\quad + (\lambda^2 \|\Pi_{L_k}\|_L^2 \|F_i^T(x^k_i, x_{k-1}^i)\|_L^2) \gamma^k \alpha.$$  \hspace{1cm} (26)

Next, we use the Lipschitz property in Assumption 2 to bound \(\|F_i^T(x^k_i, x_{k-1}^i)\|_L^2\) by \(\|F_i(x^k_i, x_{k-1}^i)\|_L^2\) at the NE point \(x^*_i = x^k_i - \beta F_i(x^*_i, x^*_i)\) for all \(i \in [m]\) and an arbitrary \(\beta\) (see Proposition 1.5.8 of [53]), implying \(F_i(x^*_i, x^*_i) = 0\) for all \(i \in [m]\). Therefore, we have the following relationship for \(\|F_i(x^k_i, x_{k-1}^i)\|_L^2\):

$$\|F_i(x^k_i, x_{k-1}^i)\|_L^2$$

$$\leq \delta_{L_k,2}^2 \|F_i(x^k_i, x_{k-1}^i)\|_2^2$$

$$\leq \delta_{L_k,2}^2 \|F_i(x^k_i, x_{k-1}^i) - F_i(x^*_i, x^*_i) + F_i(x^*_i, x^*_i) - F_i(x^k_i, x_{k-1}^i)\|_2^2$$

$$\leq 2 \delta_{L_k,2}^2 \|F_i(x^k_i, x_{k-1}^i) - F_i(x^*_i, x^*_i)\|_2^2$$

$$\quad + 2 \delta_{L_k,2}^2 \|F_i(x^*_i, x^*_i) - F_i(x^k_i, x_{k-1}^i)\|_2^2$$

$$\leq 2 \delta_{L_k,2}^2 \|x_{k-1}^i - x^*_i\|_2^2 + 2 \delta_{L_k,2}^2 \|x^*_i - x^k_i\|_2^2,$$  \hspace{1cm} (27)

where in the last inequality we used Assumption 2.

Since the following inequalities always hold:

$$\|x_{k-1}^i - x^*_i\|_2^2 = \|x_{k-1}^i - x^k_i\|_2^2 + \|x^k_i - x^*_i\|_2^2$$

$$\leq 2 \|x_{k-1}^i - x^k_i\|_2^2 + 2 \|x^k_i - x^*_i\|_2^2,$$  \hspace{1cm} (28)

we can plug (28) and (29) into (27) to obtain

$$\|F_i(x^k_i, x_{k-1}^i)\|_L^2$$

$$\leq 4 \delta_{L_k,2}^2 \|x^k_i - x^*_i\|_2^2 + 4 \delta_{L_k,2}^2 \|x^k_i - x^*_i\|_2^2$$

$$+ 4 \delta_{L_k,2}^2 \|x^*_i - x^k_i\|_2^2 + 4 \delta_{L_k,2}^2 \|x^*_i - x^k_i\|_2^2.$$  \hspace{1cm} (30)

Plugging (30) into (26) leads to

$$\mathbb{E} \left[ \|x_{k+1}^i - x^k_i\|^2_F \right]$$

$$\leq (1 - \alpha \epsilon k) \|x_k^i - x^k_i\|_L^2 + (\gamma^k)^2 \|\Pi_{L_k}\|_L^2 \delta^2_{L_k,2} m(\sigma_k^i)^2$$

$$\quad + (\lambda^2 \|\Pi_{L_k}\|_L^2 \|F_i^T(x^k_i, x_{k-1}^i)\|_L^2) \gamma^k \alpha$$

$$\quad + (\gamma^k)^2 \|\Pi_{L_k}\|_L^2 \delta^2_{L_k,2} m(\sigma_k^i)^2,$$  \hspace{1cm} (31)

Summing (31) from \(i = 1\) to \(i = m\) yields

$$\mathbb{E} \left[ \sum_{i=1}^m \|x_{k+1}^i - x^k_i\|^2_F \right]$$

$$\leq (1 - \alpha \epsilon k) \sum_{i=1}^m \|x_k^i - x^k_i\|_L^2 + (\gamma^k)^2 \|\Pi_{L_k}\|_L^2 \delta^2_{L_k,2} m \sum_{i=1}^m (\sigma_k^i)^2$$

$$\quad + (\lambda^2 \|\Pi_{L_k}\|_L^2 \|F_i^T(x^k_i, x_{k-1}^i)\|_L^2) \gamma^k \alpha$$

$$\quad + (\gamma^k)^2 \|\Pi_{L_k}\|_L^2 \delta^2_{L_k,2} m \sum_{i=1}^m (\sigma_k^i)^2,$$  \hspace{1cm} (32)

Defining \(\bar{L} \triangleq \max\{\bar{L}_1, \bar{L}_2\}\) and noting

$$\sum_{i=1}^m \|x_{k}^i - x^k_i\|^2_2 = \sum_{i=1}^m \left( \|x_k^i - x^k_i\|^2_2 + \|x_{k-1}^i - x^k_i\|^2_2 \right)$$

$$\quad + \sum_{i=1}^m \|x^k_i - x^k_i\|^2_2 = (m - 1) \sum_{i=1}^m \|x^k_i - x^k_i\|^2_2,$$
we can further simply (32) as

\[
\mathbb{E} \left[ \sum_{i=1}^{m} \left\| x_{i}^{k+1} - \hat{x}_{i}^{k+1} \right\|_{L}^{2} \right] \leq (1 - \alpha \gamma^{h}) \sum_{i=1}^{m} \left\| x_{i}^{k} - \hat{x}_{i}^{k} \right\|_{L}^{2} + (\gamma h)^{2} \| \Pi_{L_{0}} \|_{\frac{1}{2}} \delta_{L_{0}} \sum_{i=1}^{m} (\sigma_{i}^{k})^{2} \]

\[
+ \frac{4(\lambda k)^{2} \| \Pi_{e_{i}} \|_{L}^{2} \delta_{L_{1}}^{2}}{\gamma \lambda \alpha} \sum_{i=1}^{m} \left\| x_{i}^{k} - \hat{x}_{i}^{k} \right\|_{L}^{2}

+ \frac{4m(\lambda k)^{2} \| \Pi_{e_{i}} \|_{L}^{2} \delta_{L_{1}}^{2}}{\gamma \lambda \alpha} \sum_{i=1}^{m} \left\| x_{i}^{k} - x_{i}^{*} \right\|_{2}^{2}
\]

(33)

From (17), we obtain

\[
\bar{x}_{i}^{k+1} - x_{i}^{*} = \bar{x}_{i}^{k+1} - x_{i}^{*} + \gamma \left( \frac{u^{T} L_{0} \epsilon_{i}^{k}}{m} - \frac{1}{m} \lambda k u_{i} F_{i}(x_{i(i)}, x_{i(i)}) \right).
\]

(34)

Taking \( \| \cdot \|_{2} \) on both sides leads to

\[
\| \bar{x}_{i}^{k+1} - x_{i}^{*} \|_{2}^{2} = \| \bar{x}_{i}^{k+1} - x_{i}^{*} \|_{2}^{2} + \gamma \left( \frac{u^{T} L_{0} \epsilon_{i}^{k}}{m} - \frac{1}{m} \lambda k u_{i} F_{i}(x_{i(i)}, x_{i(i)}) \right)
\]

\[
+ 2 \left( \frac{u^{T} L_{0} \epsilon_{i}^{k}}{m} - \frac{1}{m} \lambda k u_{i} F_{i}(x_{i(i)}, x_{i(i)}) \right) \left( \frac{u^{T} L_{0} \epsilon_{i}^{k}}{m} - \frac{1}{m} \lambda k u_{i} F_{i}(x_{i(i)}, x_{i(i)}) \right) \left( \frac{u^{T} L_{0} \epsilon_{i}^{k}}{m} - \frac{1}{m} \lambda k u_{i} F_{i}(x_{i(i)}, x_{i(i)}) \right)
\]

(35)

Taking the conditional expectation on both sides, with respect to \( \mathcal{F}_{k} = \{ x_{i}^{k} : 0 \leq \ell \leq k, i \in [m] \} \), leads to

\[
\mathbb{E} \left[ \left\| \bar{x}_{i}^{k+1} - x_{i}^{*} \right\|_{2}^{2} \right] \leq \left\| \bar{x}_{i}^{k} - x_{i}^{*} \right\|_{2}^{2} + (2^{\gamma h}) \left( \frac{u^{T} L_{0} \epsilon_{i}^{k}}{m} - \frac{1}{m} \lambda k u_{i} F_{i}(x_{i(i)}, x_{i(i)}) \right) \]

\[
+ \left( \frac{u^{T} L_{0} \epsilon_{i}^{k}}{m} - \frac{1}{m} \lambda k u_{i} F_{i}(x_{i(i)}, x_{i(i)}) \right) \left( \frac{u^{T} L_{0} \epsilon_{i}^{k}}{m} - \frac{1}{m} \lambda k u_{i} F_{i}(x_{i(i)}, x_{i(i)}) \right)
\]

(36)

Next we bound the last two terms in the preceding inequality.

For the second last term, we can bound it using the relationship in (30):

\[
\left\| F_{i}(x_{i(i)}, x_{i(i)}) \right\|_{2} \leq 4 L_{1} \left\| x_{i(i)} - x_{i(i)}^{*} \right\|_{2} \]

\[
+ 4 L_{2} \left\| x_{i(i)}^{k} - x_{i(i)}^{*} \right\|_{2} + 4 L_{1} \left\| x_{i(i)}^{k} - x_{i(i)}^{*} \right\|_{2} + 4 L_{1} \left\| x_{i(i)}^{k} - x_{i(i)}^{*} \right\|_{2}
\]

(37)

For the inner-product term in (36), we bound it using the relationship \( F_{i}(x_{i}, x_{i}^{*}) = 0 \) (since \( x_{i}^{*} = x_{i} - \beta F_{i}(x_{i}, x_{i}^{*}) \) holds for an arbitrary \( \beta \), see Proposition 1.5.8 of [53]). More specifically, we first split it into two inner-product terms:

\[
\begin{align*}
2 \lambda k \left( \langle \bar{x}_{i}^{k} - x_{i}^{*}, F_{i}(x_{i(i)}, x_{i(i)}) \rangle \right) \\
= 2 \lambda k \left( \langle \bar{x}_{i}^{k} - x_{i}^{*}, F_{i}(x_{i(i)}, x_{i(i)}) \rangle - F_{i}(x_{i}^{*}, x_{i}^{*}) \right) \\
&= 2 \lambda k \left( \langle \bar{x}_{i}^{k} - x_{i}^{*}, F_{i}(x_{i(i)}, x_{i(i)}) \rangle - F_{i}(x_{i}^{*}, x_{i}^{*}) \right) \\
&+ 2 \lambda k \left( \langle \bar{x}_{i}^{k} - x_{i}^{*}, F_{i}(x_{i(i)}, x_{i(i)}) \rangle - F_{i}(x_{i}^{*}, x_{i}^{*}) \right) \\
&\geq -2 \lambda k \| x_{i}^{*} - x_{i}^{*} \|_{2} \| F_{i}(x_{i(i)}, x_{i(i)}) - F_{i}(x_{i}^{*}, x_{i}^{*}) \|_{2}
\end{align*}
\]

(38)

For the fist inner-product term on the right hand side of (38), using the Cauchy-Schwarz inequality yields

\[
\begin{align*}
2 \lambda k \left( \langle \bar{x}_{i}^{k} - x_{i}^{*}, F_{i}(x_{i(i)}, x_{i(i)}) \rangle - F_{i}(x_{i}^{*}, x_{i}^{*}) \right) \\
&\geq -2 \lambda k \| x_{i}^{*} - x_{i}^{*} \|_{2} \| F_{i}(x_{i(i)}, x_{i(i)}) - F_{i}(x_{i}^{*}, x_{i}^{*}) \|_{2}
\end{align*}
\]

(39)

The Lipschitz assumption in Assumption 2 implies

\[
\begin{align*}
\| F_{i}(x_{i(i)}, x_{i(i)}) - F_{i}(x_{i}^{*}, x_{i}^{*}) \|_{2} \leq 2 \| F_{i}(x_{i(i)}, x_{i(i)}) - F_{i}(x_{i}^{*}, x_{i}^{*}) \|_{2}
&+ 2 \| F_{i}(x_{i(i)}, x_{i(i)}) - F_{i}(x_{i}^{*}, x_{i}^{*}) \|_{2}
\leq 2 L_{1} \| x_{i}^{*} - x_{i}^{*} \|_{2} + 2 L_{2} \| x_{i}^{*} - x_{i}^{*} \|_{2}
\end{align*}
\]

(40)

Combining (38), (39), and (40) leads to

\[
\begin{align*}
2 \lambda k \left( \langle \bar{x}_{i}^{k} - x_{i}^{*}, F_{i}(x_{i(i)}, x_{i(i)}) \rangle \right) \\
&\geq -2 \lambda k \| x_{i}^{*} - x_{i}^{*} \|_{2} - 2 \lambda k \| x_{i}^{*} - x_{i}^{*} \|_{2}
&+ 2 \lambda k \left( \langle \bar{x}_{i}^{k} - x_{i}^{*}, F_{i}(x_{i(i)}, x_{i(i)}) - F_{i}(x_{i}^{*}, x_{i}^{*}) \rangle \right)
\end{align*}
\]

(41)

Further substituting (37) and (41) into (36) yields

\[
\begin{align*}
2 \lambda k \left( \langle \bar{x}_{i}^{k} - x_{i}^{*}, F_{i}(x_{i(i)}, x_{i(i)}) \rangle \right) \\
&\leq 2 \lambda k \left( \langle \bar{x}_{i}^{k} - x_{i}^{*}, F_{i}(x_{i(i)}, x_{i(i)}) \rangle \right)
&+ 8 \lambda k \| x_{i}^{*} - x_{i}^{*} \|_{2} + 8 \lambda k \| x_{i}^{*} - x_{i}^{*} \|_{2}
&+ 2 \lambda k \left( \langle \bar{x}_{i}^{k} - x_{i}^{*}, F_{i}(x_{i(i)}, x_{i(i)}) \rangle \right)
\end{align*}
\]

(42)

Summing (42) from \( i = 1 \) to \( i = m \), and using the relationship \( \sum_{i=1}^{m} \| \bar{x}_{i}^{k} - x_{i}^{*} \|_{2} = \)
(m - 1) \sum_{i=1}^{m} \| \hat{x}_i^k - x_i^* \|_2^2 \quad \text{and} \quad \sum_{i=1}^{m} \| x_i^k - \hat{x}_i^k \|_2^2 = 
\sum_{i=1}^{m} \left( \| x^k_{(i)} - \hat{x}^k_{(i)} \|_2^2 + \| x^k_{(i)} - \hat{x}_i^k \|_2^2 \right) \text{lead to}
\mathbb{E} \left[ \sum_{i=1}^{m} \| x^k_{i+1} - x_i^* \|_2^2 \right] 
\leq \sum_{i=1}^{m} \| x^k_i - x_i^* \|_2^2 + 2(\gamma_i^k)\|uTL_0\|_2^2 \sum_{i=1}^{m} (\sigma_i^k)^2 
+ \frac{8(\lambda_i^k)^2u_i^2\hat{L}_i^2}{m^{2}} \sum_{i=1}^{m} \| x^k_i - \hat{x}_i^k \|_2^2 + \frac{8(\lambda_i^k)^2u_i^2\hat{L}_i^2}{m^{2}} \sum_{i=1}^{m} \| x^k_i - x_i^* \|_2^2 
+ \frac{u_i(\lambda_i^k)^2 m \sum_{i=1}^{m} \| x^k_i - x_i^* \|_2^2}{m^{2}} + \frac{2u_i\hat{L}_i\lambda_i^k m \sum_{i=1}^{m} \| x^k_i - x_i^* \|_2^2}{m^{2}} 
- \frac{2u_i\lambda_i^k}{m} (\phi(\hat{x}_i^k) - \phi(x_i^*))^T (\hat{x}_i^k - x_i^*)
\right.

where \( \hat{x}_i^k = [(x_1^k)^T, \ldots, (x_m^k)^T]^T \) and \( \bar{L} \triangleq \max\{L_1, L_2\} \).

Part III: Combination of Step I and Step II.
By combining (33) and (43), and using Assumption 4, we have \( \sum_{i=1}^{m} \| x^k_i - x_i^* \|_2^2 \) and \( \sum_{i=1}^{m} \| \hat{x}_i^k - x_i^* \|_2^2 \) satisfying the conditions in Proposition 1 with \( \kappa_1 = \frac{2u_i\hat{L}_i\lambda_i^k}{m^{2}}, \kappa_2 = \alpha, a(k) = \max\{a_1, a_2, a_3, a_4, a_5\}, a_1 \triangleq \frac{4(\lambda_i^k)^2m}{\|u_i TL_0\|_2^2}, a_2 \triangleq \frac{8(\lambda_i^k)^2u_i\hat{L}_i^2}{m^{2}} \sum_{i=1}^{m} (\sigma_i^k)^2, a_3 \triangleq \frac{8(\lambda_i^k)^2u_i\hat{L}_i^2}{m^{2}} \sum_{i=1}^{m} (\sigma_i^k)^2, a_4 \triangleq \frac{8(\lambda_i^k)^2u_i\hat{L}_i^2}{m^{2}} \sum_{i=1}^{m} (\sigma_i^k)^2, a_5 \triangleq \frac{8(\lambda_i^k)^2u_i\hat{L}_i^2}{m^{2}} \sum_{i=1}^{m} (\sigma_i^k)^2, b = \max\{b_1, b_2\}, b_1 \triangleq \frac{\gamma_i^k (\lambda_i^k)^2 \| \Pi L_0 \|_2^2 \gamma_i^k}{m^{2}} m \sum_{i=1}^{m} (\sigma_i^k)^2, b_2 \triangleq \frac{2(\gamma_i^k)^2u_i \hat{L}_i \lambda_i^k}{m^{2}} m \sum_{i=1}^{m} (\sigma_i^k)^2, \) and \( c(k) = \frac{2u_i\lambda_i^k}{m} \).

Remark 3. Since channel noise in wireless communications can be viewed as a special case of the differential-privacy noise considered here, the proposed Algorithm 1 is also resilient to channel noises.

Remark 4. Since \( x^k_{(i)} \) converges to the NE following the dynamics (10) in Proposition 1, we leverage (10) in Proposition 1 to characterize the convergence speed. The first relationship in (13) (i.e., \( \sum_{k=0}^{\infty} \kappa_2 \sum_{i=1}^{m} \| x^k_i - \hat{x}_i^k \|_2^2 < \infty \)) implies that \( \| x^k_i - \hat{x}_i^k \|_2^2 \) decreases to zero no slower than \( \mathcal{O}(\frac{1}{k}) \), implying that the convergence speed of \( x^k_i \) to \( \hat{x}_i^k \) is no slower than \( \mathcal{O}(\frac{1}{k}) \). Therefore, the convergence speed of every decision variable \( x^k_{(i)} \) to the NE \( x_i^* \), which is the combination of the convergence of \( x^k_{(i)} \) to \( \hat{x}_i^k \) and the convergence of \( \hat{x}_i^k \) to \( x_i^* \), is no slower than \( \mathcal{O}(\frac{1}{k}) \). (For example, under \( \gamma_i^k = \mathcal{O}(\frac{1}{m}) \), \( \mathcal{O}(\frac{1}{k} \gamma_i^k)^{m} \) is \( \mathcal{O}(\frac{1}{k}) \).)

VI. PRIVACY ANALYSIS OF ALGORITHM 1

For the convenience of differential-privacy analysis, we first define the sensitivity of a distributed algorithm to problem (1), which is inspired by the distributed-optimization privacy design in [47]:

Definition 3. For any given initial state \( \theta_0 \) and adjacent networked games \( \mathcal{P} \) and \( \mathcal{P}^\prime \), the sensitivity of a NE seeking algorithm at iteration \( k \) is
\[
\Delta^k \triangleq \sup_{\sigma \in \mathbb{E}} \left\{ \sup_{\theta \in \mathbb{E}^{-1}(\mathcal{P}, \theta_0)} \| \theta^{k+1} - \theta^k \|_1 \right\}.
\]

Based on this definition, we obtain the following result:

Lemma 5. At each iteration \( k \), if each player in Algorithm 1 adds a vector noise \( \zeta(t) \in \mathbb{R}^{d_t} \) (consisting of \( d_t \) independent Laplace noises with parameter \( \nu(t) \)) to each of its shared message \( x^k(t) \) such that \( \sum_{k=0}^{\infty} \| \zeta(t) \|_1 \leq \delta \), then Algorithm 1 is \( \epsilon \)-differentially private with the cumulative privacy budget from iterations \( k = 0 \) to \( k = T_0 \) less than \( \epsilon \).

Proof. The result can be obtained following the derivation of Lemma 2 in [47] (see also Theorem 3 in [34]), and hence we do not include the proof here.

As is the case in [47], since the change in the cost function \( f_i \) can be arbitrarily large in the definition of adjacency in Definition 1, we have to introduce the following assumption to ensure a bounded sensitivity:

Assumption 5. \( F_i(x_{(i)}, x_{(i-1)}) \) is bounded for all \( i \in [m] \), i.e., there exists a constant \( \bar{C} \) such that \( \| F_i(x_{(i)}, x_{(i-1)}) \|_1 \leq \bar{C} \) holds for all \( x_{(i)} \in \mathbb{R}^{d_i}, x_{(i-1)} \in \mathbb{R}^{D-d_i}, \) and \( 1 \leq i \leq m \).

Remark 5. Note that the uniform boundedness condition on \( F_i(x_{(i)}, x_{(i-1)}) \) is not required in the convergence analysis.

Theorem 2. Under Assumptions 1, 2, 3, 5, if the sequences \( \{ \lambda^k \} \) and \( \{ \gamma^k \} \) satisfy the conditions in Theorem 1, and all elements of \( c(1), \ldots, c(m) \) are drawn independently from the Laplace distribution \( \text{Lap}(\nu(k)) \) with \( \| \sigma^k \|_2^2 = 2(\nu(k))^2 \) satisfying Assumption 4, then all players will converge to the NE almost surely. Moreover,

1) Algorithm 1 is \( \epsilon \)-differentially private with the cumulative privacy budget from \( k = 0 \) to \( k = T_0 \) bounded by \( \epsilon \leq \sum_{k=0}^{T_0} \| \Delta^k \|_1 \bar{C} \) where \( \bar{C} \) is from Assumption 5. And the cumulative privacy budget is always finite even as \( T_0 \to \infty \) when the sequence \( \{ \lambda^k \} \) is summable;

2) If two non-negative sequences \( \{ \nu(k) \} \) and \( \{ \lambda^k \} \) satisfy \( \Phi_{\lambda, \nu} \triangleq \sum_{k=0}^{\infty} \frac{\lambda^k}{\nu(k)} < \infty \), then picking \( \nu(k) \) of the
Laplace noise parameter as \( \nu^k = \frac{2C\Phi}{\epsilon} \) ensures that Algorithm 1 is \( \epsilon \)-differentially private with any cumulative privacy budget \( \epsilon > 0 \), even when the number of iterations tends to infinity.

3) In the special case \( \lambda^k = \frac{1}{k} \) and \( \gamma^k = \frac{1}{k} \), setting \( \nu^k = \frac{2C\Phi}{\epsilon} \) and \( \Phi \triangleq \sum_{k=1}^{\infty} \frac{1}{k^3} \approx 3.93 \) (which can be shown to satisfy Assumption 4) ensures that Algorithm 1 is \( \epsilon \)-differentially private with any cumulative privacy budget \( \epsilon > 0 \) even when the number of iterations tends to infinity.

**Proof.** Since the injected Laplace noise satisfies the conditions in Assumption 4, Theorem 1 implies that the iterate \( x^k(i)_i \) of every player \( i \) will converge to \( x^*_i \) almost surely.

To prove the statements on differential privacy, we first show that the sensitivity \( \Delta_k \) of Algorithm 1 is no larger than \( 2C\lambda^{k-1} \). According to Definition 3, for any given observation \( O \) and initial state \( \vartheta^0 \), the sensitivity is determined by \( ||R^{-1}(P, O, \vartheta^0) - R^{-1}(P', O, \vartheta^0)||_1 \) for two adjacent networks \( P \) and \( P' \). Since \( P \) and \( P' \) are adjacent, only one of their cost functions is different. Let us pick this different cost function as the \( i \)th one, i.e., \( f_i(\cdot) \), without loss of generality. Given that the observations under \( P \) and \( P' \) are identical, we have \( x^k(i) = x^k(i) \) for all \( k \geq 0 \) and \( \ell \neq i \).

Therefore, by defining \( x^k(i)_j \triangleq ((x^k(j))^T, \ldots, (x^k(j))_{(m)}^T)^T \) for all \( j \in [n] \), we can obtain the following relationship for Algorithm 1’s sensitivity:

\[
||R^{-1}(P, O, \vartheta^0) - R^{-1}(P', O, \vartheta^0)||_1 = \left\| \left[ \begin{array}{c} x^{k+1}(i)_1 \\ \vdots \\ x^{k+1}(i)_m \\ x^{k+1}(i) \\ \vdots \\ x^{k}(i)_i \\ \end{array} \right] - \left[ \begin{array}{c} x^{k+1}(i)_1 \\ \vdots \\ x^{k+1}(i)_m \\ x^{k+1}(i) \\ \vdots \\ x^k(i)_i \\ \end{array} \right] \right\|_1 = \left\| \left[ \begin{array}{c} x^{k+1}(i)_i - x^{k+1}(i)_i \\ \vdots \\ x^{k+1}(i) - x^k(i) \\ \vdots \\ x^{k+1}(i) - x^k(i) \\ \end{array} \right] \right\|_1,
\]

where in the second last equality we used the fact that only the \( i \)th cost function is different, and in the last equality, we used the fact that \( x^{k+1}(i)_i \) and \( x^{k+1}(i)_i \) for \( \ell \neq i \) are updated independently of \( F_{i}(\cdot, \cdot) \) and \( F_{j}(\cdot, \cdot) \), and hence are the same when observations are identical in \( P \) and \( P' \).

Using the update rule in (5), we can further write the above relationship as

\[
||R^{-1}(P, O, \vartheta^0) - R^{-1}(P', O, \vartheta^0)||_1 = \left\| \sum_{j\in\mathbb{N}_m} L_{ij} x^{k}(j)_i + c^k_j - x^{k}(i)_i \right\|_1 - \lambda^k F_i(x^k(i), x^k(i-1)) \] 

\[
- x^k(i)_i - \gamma^k \sum_{j\in\mathbb{N}_m} L_{ij} x^{k}(j)_i + c^k_j - x^{k}(i)_i \right\|_1
\]

\[
+ \lambda^k F'_i(x^k(i), x^k(i-1)) \right\|_1 \leq \left\| \sum_{j\in\mathbb{N}_m} L_{ij} x^{k}(j)_i + c^k_j - x^{k}(i)_i \right\|_1 \]

\[
= \lambda^k F_i(x^k(i), x^k(i-1)) - \lambda^k F'_i(x^k(i), x^k(i-1)) \right\|_1,
\]

where we have used the fact that the shared messages are the same for iterations up to \( k \).

According to Assumption 5, we have

\[
||F_i(x^k(i), x^k(i-1))||_1 \leq C, \quad ||F'_i(x^k(i), x^k(i-1))||_1 \leq C.
\]

Combining the two preceding relations leads to

\[
||R^{-1}(P, O, \vartheta^0) - R^{-1}(P', O, \vartheta^0)||_1 \leq 2\lambda^k C.
\]

Using the result in Lemma 5, the cumulative privacy budget is always less than \( \sum_{k=1}^{T_0} \frac{2C\lambda^k}{\epsilon} \). Therefore, the cumulative privacy budget \( \epsilon \) will always be finite even when the number of iterations \( T_0 \) tends to infinity if the sequence \( \{\lambda^k\} \) satisfies \( \sum_{k=0}^{\infty} \frac{\lambda^k}{\epsilon} \leq \infty \), which concludes the proof for the first statement.

From the preceding derivation, it should be clear that the cumulative privacy budget is inversely proportional to \( \nu^k \). Thus, the second statement can be obtained from the first statement by scaling \( \nu^k \) proportionally. The third statement can be obtained by specializing the selection of \( \lambda^k \), \( \gamma^k \), and \( \nu^k \) sequences as specified in the statement.

It is worth noting that to ensure a finite cumulative privacy budget (since an unbounded privacy budget means complete loss of privacy), the approaches in [34] and [47] resort to summable stepsizes (geometrically-decreasing stepsizes, more specifically), which, however, also make the convergence to the exact desired value impossible. To the contrary, our approach allows the stepsizes to be non-summable, and hence ensures both accurate convergence and finite cumulative privacy budget, even when the number of iterations goes to infinity. This appears to be the first time that accurate convergence and rigorous \( \epsilon \)-differential privacy (with guaranteed finite cumulative privacy budget) is achieved in general networked games on directed communication graphs.

**Remark 6.** It can be seen that to ensure a bounded cumulative privacy budget \( \epsilon = \sum_{k=1}^{\infty} \frac{1}{k^3} \) when \( k \to \infty \), we use Laplace noise with parameter \( \nu^k \) increasing with time (since we require \( \{\frac{\lambda^k}{\epsilon}\} \) to be summable while \( \{\lambda^k\} \) is non-summable). Since the strength of shared signal is always \( x^k(i) \) for \( \ell = 1, \ldots, m \), an increasing \( \nu^k \) implies an increasing relative level between noise \( \zeta^k(i) \) and signal \( s^k(i) \). However, since it is \( \gamma^k \)-Laplace noise \( \nu^k \) that is actually fed into the algorithm, and the increase in the noise level \( \nu^k \) is outpaced by the decrease of \( \gamma^k \) (see Assumption 4), the actual noise fed into the algorithm decays with time, which makes it possible for Algorithm 1 to ensure accurate convergence to the NE. Moreover, Theorem 1 implies that the convergence is not affected by scaling \( \nu^k \) by any constant coefficient \( \frac{1}{\epsilon} > 0 \) so as to achieve any desired level of privacy, as long as the Laplace noise parameter \( \nu^k \) (with associated variance \( \sigma^2 \)) satisfies Assumption 4.

**VII. EXTENSION TO THE CONSTRAINED CASE**

In this section, we extend Algorithm 1 to the case where each decision variable \( x^k(i) \) is constrained to a convex and closed subset \( K_i \subseteq \mathbb{R}^{d_i} \). In this case, besides introducing a projection operator \( \Pi_{K_i} \) for each player’s update of its decision variable, we also remove the influence of other players’ estimate on the evolution of player \( i \)’s decision variable \( x^k(i) \), which is crucial for the convergence analysis. (Note that the evolution of the
estimates is still the same as in Algorithm 1.) The detailed procedure is summarized in Algorithm 2.

Algorithm 2: Distributed NE seeking with guaranteed convergence and differential privacy in the presence of constraints

Parameters: Stepsize \( \lambda^k > 0 \) and weakening factor \( \gamma^k > 0 \).

Every player \( i \) maintains one decision variable \( x^{k}_{(i)} \) which is randomly initialized in \( K_i \), and \( m - 1 \) estimates of other players’ decision variables \( x^{(k,\cdot)}_{(i)} \) for \( k = 1, 2, \ldots \)

a) For both its decision variable \( x^{k}_{(j)} \) and estimate variables \( x^{k}_{(j,\cdot)} \), every player \( j \) sends respective persistent differential-privacy noise \( \zeta^k_{(j)} \) and \( \zeta^k_{(j,\cdot)} \), and then receives the obscured values \( x^{k+1}_{(j)} + \zeta^k_{(j,\cdot)} \) to all players \( j \in \mathbb{N}^{out} \).

b) After receiving \( x^{k}_{(j)} + \zeta^k_{(j,\cdot)} \) from all \( j \in \mathbb{N}^{out} \), player \( i \) updates its decision variable and estimate variables \( x^{k+1}_{(i,\cdot)} \) as follows:

\[
x^{k+1}_{(i,\cdot)} = \Pi_{K_i} \left[ x^{(k)}_{(i)} + \lambda^k F_i(x^{(k)}_{(i)}, x^{(k-1)}_{(i,\cdot)}) \right],
\]

where \( \Pi_{K_i} [\cdot] \) denotes the projection of a vector onto the set \( K_i \).

c) end

Using the notation of (8) for player \( i \)’s decision variable \( x^{k}_{(i)} \), as well as other players’ estimates of this decision variable \( x^{k}_{(j)} \) for \( \ell = 1, \ldots, i-1, i+1, \ldots, m \), we have

\[
x^{k+1}_{(i)} = (I + \gamma^k L_{i-1}) x^{k}_{(i)} + \gamma^k L_{i-1} x^{k}_{(i,\cdot)} + c_{i} J^{T} i \left( x^{(k)}_{(i)}, x^{(k-1)}_{(i,\cdot)} \right),
\]

with \( L_{i-1} \) and \( L_{i-1} \) constructed by replacing the \( i \)th rows of \( L \) and \( L_{-i} \) with zero elements, respectively, and

\[
J_i(x^{(k)}_{(i)}, x^{(k)}_{(i,\cdot)}) = \Pi_{K_i} \left[ x^{(k)}_{(i)} - \lambda^k F_i(x^{(k)}_{(i)}, x^{(k-1)}_{(i,\cdot)}) \right] - x^{(k)}_{(i,\cdot)}.
\]

A. Convergence Analysis of Algorithm 2

Since \( G_L \) is strongly connected according to Assumption 3, \( G_{L_{-i}} \) has at least one spanning tree rooted at player \( i \). Therefore, similar to Lemma 3, we can obtain the following result for \( L_{-i} \):

Lemma 6. Under Assumption 3, every \( L_{-i} \) has the following properties:

1) the eigenvectors of \( I + \gamma^k L_{i-1} \) are time-invariant;
2) \( I + \gamma^k L_{i-1} \) has a unique nonnegative left eigenvector \( u_{L_{-i}}^T \) (associated with eigenvalue 1) satisfying \( u_{L_{-i}}^T 1 = m \);
3) there exists a matrix norm \( \| \cdot \|_{L_{-i}} \) such that \( \| 1 - \alpha \gamma^k L_{-i} \|_{L_{-i}} \leq 1 - \alpha \gamma^k \) for \( 0 < \alpha < 1 \) when \( \gamma^k \) is small enough. Moreover, this \( \| \cdot \|_{L_{-i}} \) has an associated inner product \( \langle \cdot, \cdot \rangle_{L_{-i}} \), i.e., \( \| x \|_{L_{-i}}^2 = \langle x, x \rangle_{L_{-i}} \).

Proof. The proof follows the same lines as in the proof of Lemma 3, and hence is not included here.

Based on Lemma 6, we can define

\[
(x^{k}_{(i)})^T = \frac{u_{L_{-i}} x^{k}_{(i)}}{m},
\]

\[
x^{k}_{(i)} = \left[ \frac{(x^{k}_{(i)})^T}{(x^{k}_{(i)})^T} \right] \in \mathbb{R}^{m \times d_i}.
\]

Then, following a derivation similar to Proposition 1, we can obtain the following proposition:

Proposition 2. Let Assumptions 1 and 2 hold, and let \( x^* = [(x^*_{(i)})^T, (x^*_{(j)})^T, \ldots, (x^*_{(m)})^T]^T \) denote the NE of the networked game (1). If, under the interaction matrix \( L \) (associated with \( L_{-i} \) for \( i = 1, \ldots, m \)), a distributed algorithm generates sequences \( \{x^{k}_{(i)}\} \) for all \( i \in \mathbb{N} \) such that a.s. we have

\[
\left[ \mathbb{E} \left[ \sum_{i=1}^{m} \| x^{k+1}_{(i)} - x^{k}_{(i)} \|_{L_{-i}}^2 \right] \right] \\
\leq \left[ \left( \begin{array}{ccc} 1 & \kappa_1 \gamma^k & 0 \\ 0 & 1 - \kappa_2 \gamma^k & \lambda^k \end{array} \right) + \alpha^k \left( \begin{array}{ccc} m \| x^{k}_{(i)} - x^*_{(i)} \|_{L_{-i}}^2 \end{array} \right) \right] \\
+ b^k \left( \begin{array}{ccc} \phi(x^k) - \phi(x^*) \end{array} \right)^T (x^k - x^*) , \ \forall k \geq 0
\]

where \( \| \cdot \|_{L_{-i}} \) is an \( L_{-i} \)-dependent norm, \( F_k = \{ x^k, i \in \mathbb{N} \}, 0 \leq \ell \leq k \), the random nonnegative scalar sequences \( \{a^k\}, \{b^k\} \) satisfy \( \sum_{k=0}^{\infty} a^k < \infty \) and \( \sum_{k=0}^{\infty} b^k < \infty \), respectively, and the scalars \( \kappa_1 \) and \( \kappa_2 \) satisfy \( \kappa_1 > 0 \) and \( 0 < \kappa_2 \gamma^k < 1 \), respectively, for all \( k \geq 0 \). Then, we have \( \lim_{k \to \infty} \| x^k - x^* \|_{L_{-i}} = 0 \) a.s. for all \( i \).

Proof. The proof follows that of Proposition 1. The only difference is that the original \( \lambda^k F_i(x^{(k)}_{(i)}, x^{(k-1)}_{(i,\cdot)}) \) is replaced with \( J_i(x^{(k)}_{(i)}, x^{(k-1)}_{(i,\cdot)}) \) for all \( k \). More specifically, in the derivation of the inequality for \( \| x^k - x^* \|_{L_{-i}}^2 \) in (27), \( \| F_i(x^{(k)}_{(i)}, x^{(k-1)}_{(i,\cdot)}) \|_{L_{-i}}^2 \) becomes \( \| J_i(x^{(k)}_{(i)}, x^{(k-1)}_{(i,\cdot)}) \|_{L_{-i}}^2 \), and in the derivation of the inequality for \( \| x^{k+1}_{(i)} - x^*_{(i)} \|_{L_{-i}}^2 \), we have \( \| x^{k+1}_{(i)} - x^*_{(i)} \|_{L_{-i}}^2 \) for all \( i \in \mathbb{N} \).
and
\[\|x_{i(i)}^{k+1} - x_i^*\|_2^2 = \|\Pi_{K_i} [x_{i(i)}^k - \lambda^k F_i(x_{i(i)}^k, x_{(i)-i})] - \Pi_{K_i} [x_i^* - \lambda^k F_i(x_i^*, x_{(i)-i})]\|_2^2 \leq \|x_{i(i)}^k - \lambda^k F_i(x_{i(i)}^k, x_{(i)-i}) - (x_i^* - \lambda^k F_i(x_i^*, x_{(i)-i}))\|_2^2.\]

Then, the remaining steps will be similar to those in the proof of Proposition 1. 

Using Proposition 2, we can establish the convergence of Algorithm 2:

**Theorem 3.** Under Assumptions 1-4, if every decision variable is constrained to a convex and closed set \(K_i\), and there exists some \(T \geq 0\) such that the sequences \(\{\gamma^k\}\) and \(\{\lambda^k\}\) satisfy
\[\sum_{k=T}^\infty \gamma^k = \infty, \quad \sum_{k=T}^\infty \lambda^k = \infty, \quad \sum_{k=T}^\infty (\gamma^k)^2 < \infty, \quad \sum_{k=T}^\infty (\lambda^k)^2 < \infty,\]
then Algorithm 2 converges to the NE of problem (1) almost surely.

**Proof.** The proof follows the same lines as in the proof of Theorem 1. 

**Remark 7.** Since \(x_{i(i)}^k\) converges to the NE following the dynamics (50) in Proposition 2, we leverage (50) to characterize the convergence speed. Similar to (13) in Proposition 1, we can obtain that \(\sum_{k=0}^\infty \kappa_2 \gamma^k \sum_{i=1}^m \|x_i^k - \bar{x}_i^k\|_2^2 < \infty\) always holds. Eqn. (50) also implies the following inequality:
\[\mathbb{E} \left[ \sum_{i=1}^m \|x_{i(i)}^{k+1} - x_i^*\|_2^2 \right] \leq (1 + \alpha^k) \sum_{i=1}^m \|x_{i(i)}^k - x_i^*\|_2^2 + \gamma^k (\kappa_1 \gamma^k + \alpha^k) \sum_{i=1}^m \|x_i^k - \bar{x}_i^k\|_2^2 + b^k.\]

Given \(\kappa_1 \gamma^k + \alpha^k > 0\), \(\sum_{i=1}^m \|x_i^k - \bar{x}_i^k\|_2^2 < \infty\) (note that \(\sum_{k=0}^\infty \gamma^k < \infty\)) imply that \(\alpha^k\) decreases faster than \(\{\gamma^k\}\), the following relationship holds a.s.:
\[\mathbb{E} \left[ \sum_{i=1}^m \|x_{i(i)}^{k+1} - x_i^*\|_2^2 \right] \leq (1 + \alpha^k) \sum_{i=1}^m \|x_{i(i)}^k - x_i^*\|_2^2 + b^k,\]
where \(b^k \triangleq \kappa_1 \gamma^k + \alpha^k \sum_{i=1}^m \|x_i^k - \bar{x}_i^k\|_2^2 + b^k\) satisfies \(\sum_{k=0}^\infty b^k < \infty\). Since non-negative summable sequences decrease to zero no slower than \(O(\frac{1}{k})\), we know that \(\mathbb{E} \left[ \sum_{i=1}^m \|x_{i(i)}^{k+1} - x_i^*\|_2^2 \right]\) decreases to zero no slower than \(O(\frac{1}{k})\).

**B. Privacy Analysis for Algorithm 2**

Similar to the privacy analysis for Algorithm 1 in Sec. VI, we can also analyze the strength of enabled differential privacy for Algorithm 2:

**Theorem 4.** Under Assumptions 1, 2, 3, 5, if the sequences \(\{\lambda^k\}\) and \(\{\gamma^k\}\) satisfy the conditions in Theorem 3, and for all \(i \in [m]\), all elements of \(\xi_{i(i)}^k, \cdots, \xi_{i(m)}^k\) are drawn independently from Laplace distribution \(\text{Lap}(\nu^k)\) with \((\nu^k)^2 = 2(\nu^k)^2\) satisfying Assumption 4, then all \(x_{i(i)}^k\) will converge a.s. to the NE \(x_i^*\). Moreover,

1) Algorithm 2 is \(\epsilon\)-differentially private with the cumulative privacy budget bounded by \(\epsilon \leq \sum_{k=0}^\infty \frac{C_2}{\sqrt{k}}\) for iterations from \(k = 0\) to \(k = T_0\), where \(C_2\) is from Assumption 5. Moreover, the cumulative privacy budget is always finite even for \(T_0 \to \infty\) when the sequence \(\{\alpha^k\}\) is summable;

2) If two non-negative sequences \(\{\nu^k\}\) and \(\{\lambda^k\}\) satisfy \(\Phi_{\lambda, \nu} \triangleq \sum_{k=1}^\infty \frac{2\Phi}{\sqrt{k}}\) ensures that Algorithm 2 is \(\epsilon\)-differentially private with any desired cumulative privacy budget \(\epsilon > 0\) even when \(k\) tends to infinity;

3) In the special case where \(\lambda^k = \frac{1}{k}\) and \(\gamma^k = \frac{1}{m}\), setting \(\nu^k = \frac{2C_2}{\sqrt{k}}\), \(\Phi \triangleq \sum_{k=1}^\infty \frac{2\Phi}{\sqrt{k}}\approx 3.93\) (which can be shown to satisfy Assumption 4) ensures that Algorithm 2 is \(\epsilon\)-differentially private for any desired cumulative privacy budget \(\epsilon > 0\) even when \(k\) tends to infinity.

**Proof.** The derivation follows the line of reasoning in the proof of Theorem 2, and hence is not included here. 

**VIII. Numerical Simulations**

In this section, we use a networked Nash-Cournot game to evaluate the proposed NE seeking approach. In a Nash-Cournot game, \(m\) economically rational firms compete over \(N\) markets [9], [16], [24]. The firms produce a homogeneous commodity, and, hence their costs or payoffs are affected by each other because every firm’s product affects the market price.

We consider a setting involving \(N = 7\) markets \(M_1, \cdots, M_7\), and \(m = 20\) firms, as illustrated in Fig. 1, where each firm is represented by a circle. When the \(i\)th firm participates in the \(j\)th market, we draw an edge from circle \(i\) to \(M_j\). We consider the case where each firm can only see information of neighbors through local communications [9], [16]. Namely, no central mediator exists that can gather and disperse the production information across the firms. Instead, each firm can communicate with its immediate neighbors to share production decisions. We let \(x_i \in \mathbb{R}^d\) represent the amount of firm \(i\)'s products, where each entry of \(x_i\) represents the amount of firm \(i\)'s commodity delivery to a market that it participates in. Note that since a firm \(i\) can participate in no more than \(N\) markets, we always have \(1 \leq d_i \leq N\). We can use a projection matrix \(B_i \in \mathbb{R}^{N \times d_i}\) (with entries either one or zero) to specify which markets firm \(i\) participates in. More specifically, the \((m, n)\)th entry of \(B_i\) is 1 if and only if firm \(i\) participates and delivers \(x_{ni}\) amount of product to market \(m\), where \(x_{ni}\) represents the \(n\)th entry of the vector \(x_i\). We use \(C_{ij}\) to represent firm \(i\)'s maximal capacity for market \(j\) that it participates in. Augmenting \(C_i\) as \(C_i \triangleq [C_{i1}, \cdots, C_{id}]^T\), we always have \(x_i \leq C_i\). Augment \(B_i\) as \(B \triangleq [B_1, \cdots, B_N]\). One can verify that \(B x \in \mathbb{R}^N = \sum_{i=1}^N B_i x_i\) describes the total product supply of all firms to all markets, when the production amount of firm \(i\) is given by \(x_i\). Following [9], we suppose that the commodity’s price in every market \(M_i\) follows a linear inverse demand function, i.e., the price in mar-
ket $i$ decreases linearly with the total amount of commodity delivered to the market:

$$p_i(x) = \bar{p}_i - \chi_i[Bx]_i,$$

where $\bar{p}_i$ and $\chi_i$ are positive constants and $[Bx]_i$ denotes the $i$th element of the total-supply vector $Bx$. One can see that the price decreases linearly with an increase in the amount of supplied commodity.

Using $p \triangleq [p_1, \ldots, p_N]^T$ to represent the price vector of all $N$ markets, one can verify that $p$ has the following relationship with the total-supply vector $Bx$:

$$p = \bar{p} - \Xi Bx,$$

where $\bar{p} \triangleq [\bar{p}_1, \ldots, \bar{p}_N]^T$ is the augmented $\bar{p}_i$ and $\Xi \triangleq \text{diag}(\chi_1, \ldots, \chi_N)$ is a diagonal matrix. Under these notations, firm $i$’s total payoff can be expressed as $p^T B_i x_i$.

Following [9], we assume that the production cost of firm $i$ is a quadratic function:

$$c_i(x_i) = x_i^T Q_i x_i + q_i^T x_i,$$

where $Q_i \in \mathbb{R}^{d_i \times d_i}$ is a positive definite matrix and $q_i \in \mathbb{R}^{d_i}$.

Summarizing firm $i$’s production cost $c_i(\cdot)$ and payoff $p^T B_i x_i$, we can represent firm $i$’s cost function as

$$f_i(x_i, x) = c_i(x_i) - (\bar{p} - \Xi Bx)^T B_i^T x_i.$$

The gradient of the cost function is

$$F_i(x_i, x) = 2Q_i x_i + q_i + B_i^T \Xi B_i x_i - B_i (\bar{p} - \Xi Bx).$$

From the preceding expressions, it is clear that firm $i$’s cost function and gradient are both affected by other firms’ actions.

In our numerical simulations, we considered $N = 7$ markets and 20 firms, as indicated earlier. Since all firms can only communicate with their local neighbors, we generated local communication patterns randomly. To ensure that the communication graph is strongly connected, we first connected the 20 firms using a directed ring topology, and then added random links between any pairs of firms with probability 0.1. The communication graph used is depicted in Fig. 2. We set the maximal capacities for firm $i$ (elements in $C_i$) randomly from a uniform distribution on the interval $[8, 10]$. We set $Q_i$ in the production cost function as $\nu I$ and randomly selected $\nu$ from a uniform distribution on the interval $[1, 10]$. We randomly selected $q_i$ in $c_i(x_i)$ from a uniform distribution on $[1, 2]$. The constants $\bar{p}_i$ and $\chi_i$ in the price function were randomly selected from uniform distributions on the intervals $[10, 20]$ and $[1, 3]$, respectively.

To evaluate the proposed approach under differential-privacy design, we injected differential-privacy vector noise $\zeta^{(k)}_{ij\ell}$ $(1 \leq \ell \leq 20)$ in every message $x^{(k)}_{ij\ell}$ that firm $i$ shares with its neighbors in each iteration. Each element of the noise vector $\zeta^{(k)}_{ij\ell}$ follows Laplace distribution with parameter $\nu^k = 1 + 0.1k^{0.2}$. So the magnitude of noise increases with time. We set the stepsize sequence $\{\lambda^k\}$ and decaying sequence $\{\gamma^k\}$ as $\lambda^k = \frac{0.1}{1 + 0.1k}$ and $\gamma^k = \frac{1}{1 + 0.1k}$, respectively, which can be verified to satisfy the conditions in Theorems 3 and 4. To address the stochasticity induced by the differential-privacy noise in evaluation, we ran our Algorithm 2 for 100 times and calculated the average of the gap $\|x^k - x^*\|$ between generated iterate $x^k$ and the NE $x^*$ as a function of the iteration index $k$. We also calculated the variance of the gap of the 100 runs as a function of the iteration index $k$. The trajectories of the average and variance are given by the red curve and error bars in Fig. 3. For comparison, we also ran the existing distributed NE seeking algorithm proposed by Nguyen et al. in [24] under the same level of differential-privacy noise, and the existing differential-privacy approach for networked games proposed by Ye et al. in [34] under the same cumulative budget of privacy $\epsilon$. Note that [34] addresses undirected graphs but its differential-privacy strategy, i.e., using geometrically decreasing stepsizes (to be summable) to ensure a finite privacy budget, can be adapted to the directed-graph scenario. The average error and variance of the existing NE seeking approach for directed graphs in [24] under the same level of noise are given by the blue curve and error bars in Fig. 3. And the average error and variance of applying existing differential-privacy design [34] to the NE seeking approach for directed graphs in [24] are given by the black curve and error bars in Fig. 3. The comparison clears shows that the algorithm introduced in this paper has a much better accuracy.
IX. CONCLUSIONS

As differential privacy has become the de facto standard for privacy protection, achieving differential privacy becomes imperative for networked games due to the competitive relations among players. However, existing differential-privacy solutions for NE seeking have to trade convergence accuracy for differential privacy. This paper has introduced a fully distributed NE seeking approach that can ensure both accurate convergence and rigorous $\epsilon$-differential privacy with bounded cumulative privacy budget, even when the number of iterations tends to infinity. The simultaneous achievement of both goals is in sharp contrast to existing differential-privacy solutions for aggregative games that trade convergence accuracy for privacy, and to the best of our knowledge, has not been achieved before for general networked Nash games. The approach is applicable to general directed graphs that are not necessarily balanced, and it does not require players to know any global structure knowledge of the communication network. Numerical comparison results confirm that in the presence of differential-privacy noise, the proposed approach has a better accuracy than existing counterparts, while maintaining a comparable convergence speed.

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