A generalized backdoor criterion

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Abstract

We generalize Pearl’s backdoor criterion for directed acyclic graphs (DAGs) to more general types of graphs that describe Markov equivalence classes of DAGs with or without arbitrarily many hidden variables. We also give easily checkable necessary and sufficient graphical criteria for the existence of a set of variables that satisfy our generalized backdoor criterion, when considering a single intervention and a single outcome variable. Moreover, if such a set exists, we provide an explicit set that fulfills the criterion. We illustrate the results in several examples. R-code will be available in the R-package pcalg.

1 Introduction

Causal Bayesian networks are widely used for causal reasoning (e.g., Glymour et al. (1987); Koller and Friedman (2009); Pearl (1995, 2000, 2009); Spirtes et al. (1993, 2000)). In particular, when the causal structure is known and represented by a directed acyclic graph (DAG), this framework allows one to deduce post-intervention distributions and causal effects from the pre-intervention (or observational) distribution.

Hence, when the causal DAG is known, one can estimate causal effects from observational data. Covariate adjustment is often used for this purpose. The backdoor criterion (Pearl 1993) is a graphical criterion that is sufficient for adjustment, in the sense that a set of variables can be used for adjustment if they satisfy the backdoor criterion for the given graph.

In practice, there are two important complications. First, the underlying DAG may be unknown. In that case one can try to estimate the DAG, but in
general one cannot identify the underlying DAG uniquely. Instead, one can identify its Markov equivalence class which consists of all DAGs that encode the same conditional independence relationships as the underlying DAG. Such a Markov equivalence class can be represented uniquely by a different type of graph, called a completed partially directed acyclic graph (CPDAG) (Andersson et al., 1997; Meek, 1995; Spirtes et al., 1993). Second, it is often the case that some important variables were not measured, meaning that we do not have causal sufficiency. In this case, one can work with maximal ancestral graphs (MAGs) instead of DAGs (Richardson and Spirtes, 2002, 2003). Finally, the underlying MAG may be unknown, so that it must be estimated from data. Again, there is an identifiability problem here, as we can generally only identify the Markov equivalence class of the underlying MAG, which can be represented uniquely by a partial ancestral graph (PAG) (Ali et al., 2009; Richardson and Spirtes, 2002).

In this paper, we therefore consider generalizations of the backdoor criterion to the following three scenarios:

1. We assume causal sufficiency, and we only know the CPDAG, i.e., the Markov equivalence class of the underlying DAG;

2. We do not assume causal sufficiency, and we know the MAG on the observed variables;

3. We do not assume causal sufficiency, and we only know the PAG, i.e., the Markov equivalence class of the underlying MAG on the observed variables.

In scenarios 2 and 3, we allow for arbitrarily many hidden (or unmeasured) variables. We do not, however, allow for selection variables, that is, for unmeasured variables that determine whether a unit is included in the sample.

Since the backdoor criterion is a simple criterion that is widely used for DAGs, it seems useful to have similar criteria for CPDAGs, MAGs and PAGs. We also hope that our generalized backdoor criterion will make working with MAGs and PAGs less daunting, and more accessible to people in practice.

Our generalized backdoor criterion for DAGs, CPDAGs, MAGs and PAGs is given in Section 3, see especially Definition 3.7 and Theorem 3.1. These results are derived by first formulating invariance conditions that are sufficient for adjustment, and then using the graphical criteria for invariance derived by Zhang (2008a). We also show that the generalized backdoor criterion is equivalent to Pearl’s backdoor criterion for single interventions in
DAGs, and slightly more general for multiple interventions in DAGs (Lemma 3.1). In Section 4, we give necessary and sufficient criteria for the existence of a set that satisfies the generalized backdoor criterion relative to a pair of variables \((X, Y)\) in a given causal DAG, CPDAG, MAG or PAG. Moreover, if there is a set that satisfies the generalized backdoor criterion, we provide an explicit set that does so. These results are summarized in Theorem 4.1 for DAGs, CPDAGs, MAGs and PAGs in general. Corollaries 4.1-4.3 specialize the results for DAGs, CPDAGs and MAGs, respectively. We illustrate the results with several examples in Section 5 and all proofs are given in Section 7.

The R-package `pcalg` (Kalisch et al., 2012) contains relevant R-code.

We close this introduction by discussing related work. For a given causal DAG, identifiability of causal effects in general or via covariate adjustment has been studied by various authors. In particular, there are complete graphical criteria for the identification of causal effects when a causal DAG with unmeasured variables is given (e.g., Huang and Valtorta (2006); Shpitser and Pearl (2006a,b, 2008); Tian and Pearl (2002)). Shpitser et al. (2010a,b) studied effects that are identifiable via covariate adjustment, and provided necessary and sufficient graphical criteria for this purpose, again when the causal DAG is given. These results can be viewed as an improvement on the backdoor criterion, which is only sufficient for adjustment. Textor and Liškiewicz (2011) studied covariate adjustment for a given DAG from an algorithmic perspective. Among other things, they showed that the backdoor criterion and the adjustment criterion of Shpitser et al. (2010b) are equivalent when one is interested in minimal adjustment sets for a certain subclass of graphs.

There are also existing approaches that do not make the assumption that the causal DAG is given. The Prediction Algorithm (Spirtes et al., 2000, Ch 7) roughly starts from a PAG and uses invariance results. In that sense it is probably closest to our work. The main difference between this method and our results is that the Prediction Algorithm is much more complex. In particular, it searches over all possible orderings of the variables, which quickly becomes infeasible for large graphs. The Prediction Algorithm may, however, be more informative, in the sense that certain distributions may be identifiable by the Prediction Algorithm but not by the generalized backdoor criterion. Studying the exact relationship between these two approaches would be an interesting topic for future work.

Other work on data driven methods for selection of adjustment variables for the estimation of causal effects does not assume that the causal structure is known, but does make some assumptions about causal relationships between the variables of interest and/or about the existence of a set of
variables that can be used for covariate adjustment (De Luna et al., 2011; Entner et al., 2013; VanderWeele and Shpitser, 2011)). In the current paper, we do not make any such assumptions. On the other hand, we start from a given DAG, CPDAG, MAG or PAG. We do not see this as a severe restriction of our approach, however, since there are algorithms to estimate CPDAGs and PAGs from data (e.g., the PC algorithm (Spirtes et al., 2000), Greedy Equivalence Search (Chickering, 2002), and versions of the FCI algorithm (Claassen et al., 2013; Colombo et al., 2012; Spirtes et al., 2000)). These algorithms have been shown to be consistent, even in certain sparse high-dimensional settings (Colombo et al., 2012; Kalisch and Bühlmann, 2007). In practice, one could therefore first employ such an algorithm, and then apply the results in the current paper.

2 Preliminaries

Throughout this paper, we denote sets in a bold font (e.g., $X$) and graphs in a calligraphic font (e.g., $\mathcal{D}$ or $\mathcal{M}$).

2.1 Basic graphical definitions

A graph $\mathcal{G} = (V, E)$ consists of a set of vertices $V = \{X_1, \ldots, X_p\}$ and a set of edges $E$. The vertices represent random variables and the edges describe conditional independence and causal (ancestral) relationships. There is at most one edge between every pair of vertices, and the edge set $E$ can contain (a subset of) the following four edge types: → (directed), ↔ (bi-directed), ❌ (non-directed) and ⇔ (partially directed). A directed graph contains only directed edges, a mixed graph can contain directed and bi-directed edges, and a partial mixed graph can contain all four edge types. The endpoints of an edge are called marks and they can be tails, arrowheads or circles. We use the symbol “*” to denote an arbitrary edge mark. When we are only interested in the presence or absence of edges, and not in the edge marks, then we talk about the skeleton of a graph.

Two vertices are adjacent if there is an edge between them. The adjacency set of a vertex $X$ in $\mathcal{G}$, denoted by $\text{adj}(X, \mathcal{G})$, consists of all vertices adjacent to $X$ in $\mathcal{G}$. A path is a sequence of distinct adjacent vertices. The length of a path $p = \langle X_i, X_{i+1}, \ldots, X_{i+\ell} \rangle$ equals the corresponding number of edges, in this case $\ell$. The path $p$ is said to be out of (into) $X_i$ if the edge between $X_i$ and $X_{i+1}$ has a tail (arrowhead) at $X_i$. A sub-path of $p$ from $X_j$ to $X_{j'}$ is denoted by $p(X_j, X_{j'})$. We denote the concatenation of paths by $\oplus$, so that for example $p = p(X_i, X_{i+k}) \oplus p(X_{i+k}, X_{i+\ell})$ for
k \in \{1, \ldots, \ell - 1\}. The path \( p \) is a directed path from \( X_i \) to \( X_{i+\ell} \) if for all \( k \in \{1, \ldots, \ell\} \) the edge \( X_{i+k-1} \rightarrow X_{i+k} \) occurs, and it is a possibly directed path if for all \( k \in \{1, \ldots, \ell\} \), the edge between \( X_{i+k-1} \) and \( X_{i+k} \) is not into \( X_{i+k-1} \). A cycle occurs when there is a path between \( X_i \) and \( X_j \) of length greater than 1, and \( X_i \) and \( X_j \) are adjacent. A directed path from \( X_i \) to \( X_j \) forms a directed cycle together with the edge \( X_j \rightarrow X_i \), and an almost directed cycle together with the edge \( X_j \leftrightarrow X_i \). A directed acyclic graph (DAG) is a directed graph that does not have directed cycles. An ancestral graph is a mixed graph that does not have directed or almost directed cycles.

If \( X_j \rightarrow X_i \), we say that \( X_i \) is a child of \( X_j \), and \( X_j \) is a parent of \( X_i \). The corresponding sets of parents and children are denoted by \( \text{pa}(X_i, G) \) and \( \text{ch}(X_i, G) \). If there is a (possibly) directed path from \( X_i \) to \( X_j \) or if \( X_i = X_j \), then \( X_i \) is a (possible) ancestor of \( X_j \) and \( X_j \) a (possible) descendant of \( X_i \). The sets of ancestors, descendants, possible ancestors, and possible descendants of a vertex \( X_i \) in \( G \) are denoted by \( \text{an}(X_i, G) \), \( \text{de}(X_i, G) \), \( \text{possibleAn}(X_i, G) \), and \( \text{possibleDe}(X_i, G) \) respectively. These definitions are applied disjunctively to a set \( Y \subseteq V \), e.g., \( \text{an}(Y, G) = \{X_i \mid X_i \in \text{an}(X_j, G) \text{ for some } X_j \in Y \} \).

A path \( (X_j, X_j, X_k) \) is an unshielded triple if \( X_i \) and \( X_k \) are not adjacent. A non-endpoint vertex \( X_j \) on a path is a collider on the path if the path contains \( \leftrightarrow X_j \leftrightarrow \). A non-endpoint vertex on a path which is not a collider is a non-collider on the path. A collider path is a path on which every non-endpoint vertex is a collider. A path of length one is a trivial collider path.

## 2.2 Causal Bayesian networks

A Bayesian network for a set of variables \( V = \{X_1, \ldots, X_p\} \) is a pair \( (D, f) \), where \( D = (V, E) \) is a DAG, and \( f \) is a joint probability density for \( V \) (with respect to some dominating measure) that factorizes according to \( D \):

\[
f(V) = \prod_{i=1}^{p} f(X_i|\text{pa}(X_i, D)).
\]

When the DAG is interpreted causally, in the sense that \( X_i \rightarrow X_j \) means that \( X_i \) has a direct causal effect on \( X_j \), then we talk about a causal DAG and a causal Bayesian network.

One can easily derive post-intervention densities when the causal Bayesian network is given and all variables are observed. In particular, we consider atomic interventions \( \text{do}(X = x) \) for \( X \subseteq V \) [Pearl, 2000], which represent outside interventions that set the variables in \( X \) to their respective values in \( x \). We assume that such interventions are effective, meaning that \( X = x \) after the intervention. Moreover, we assume that the interventions are local, meaning that the generating mechanisms of the other variables, and hence
their conditional distributions given their parents, do not change. We then have

\[ f(\mathbf{V}|\text{do}(\mathbf{X} = \mathbf{x})) = \begin{cases} \prod_{X_i \in \mathbf{V} \setminus \mathbf{x}} f(X_i|\text{pa}(X_i, D)) & \text{for values of } \mathbf{V} \text{ consistent with } \mathbf{x}, \\ 0 & \text{otherwise.} \end{cases} \]

This is known as the truncated factorization formula, see, e.g., Pearl (2000); Robins (1986); Spirtes et al. (1993).

In a Bayesian network \((D, f)\), the DAG \(D\) encodes conditional independence relationships in the density \(f\) via d-separation (Pearl (2000), see also Definition 3.5). Several DAGs can encode the same conditional independence relationships. Such DAGs form a Markov equivalence class which can be uniquely represented by a CPDAG. A CPDAG is a graph with the same skeleton as each DAG in its equivalence class, and its edges are either directed \((\rightarrow)\) or non-directed \((\leftrightarrow)\). An edge \(X_i \rightarrow X_j\) in such a CPDAG means that \(X_i \rightarrow X_j\) is present in every DAG in the Markov equivalence class, while an edge \(X_i \leftrightarrow X_j\) represents uncertainty about the edge marks, in the sense that the Markov equivalence class contains at least one DAG with \(X_i \rightarrow X_j\) and at least one DAG with \(X_i \leftarrow X_j\). (Note that many authors use \(X_i \rightarrow X_j\) instead of \(X_i \leftrightarrow X_j\); we use \(\leftrightarrow\) to ensure that the CPDAG is syntactically a PAG; see below.) The CPDAG that represents the Markov equivalence class of a causal DAG is called a causal CPDAG.

When some of the variables in a DAG are unobserved, one can transform the DAG into a unique maximal ancestral graph (MAG) on the observed variables; see Richardson and Spirtes (2002)(p.981) for an algorithm. In particular, two vertices \(X_i\) and \(X_j\) are adjacent in a MAG if and only if no subset of the remaining observed variables makes them conditionally independent. Moreover, a tail mark \(X_i \leftrightarrow X_j\) in a MAG means that \(X_i\) is an ancestor of \(X_j\) in all DAGs represented by \(\mathcal{M}\), while an arrowhead \(X_i \leftarrow X_j\) means that \(X_i\) is not an ancestor of \(X_j\) in all DAGs represented by \(\mathcal{M}\). Several different DAGs can lead to the same MAG, and a MAG represents a class of (infinitely many) DAGs that have the same d-separation and ancestral relationships among the observed variables. The MAG of a causal DAG is called a causal MAG.

A MAG encodes conditional independence relationships via the concept of m-separation (Definition 3.5). Again, several MAGs can encode the same conditional independence relationships. Such MAGs are called Markov equivalent, and can be uniquely represented by a partial ancestral graph (PAG). This is a partial mixed graph with the same skeleton as each
MAG in its Markov equivalence class. A tail mark (arrowhead) at an edge $X_i \rightarrow X_j$ ($X_i \leftarrow X_j$) in such a PAG means that $X_i \rightarrow X_j$ ($X_i \leftarrow X_j$) in every MAG in the Markov equivalence class, while a circle mark at an edge $X_i \leftarrow X_j$ represents uncertainty about the edge mark, in the sense that the Markov equivalence class contains at least one MAG with $X_i \rightarrow X_j$, and at least one MAG with $X_i \leftarrow X_j$. The PAG that represents the Markov equivalence class of a causal MAG is called a causal PAG.

These definitions ensure that, syntactically, every DAG is a MAG and a CPDAG, every CPDAG is a PAG, and every MAG is a PAG (but the interpretation of the edge marks is specific for each graph!). If $f$ is the joint density of the observed variables, and $G$ represents a causal DAG $D / \text{CPDAG} \; \mathcal{C} / \text{MAG} \; \mathcal{M} / \text{PAG} \; \mathcal{P}$, then we call the pair $(G, f)$ a causal model.

3 Generalized backdoor criterion

We will now present our generalized backdoor criterion in Definition 3.7 and Theorem 3.1 where the name “generalized backdoor criterion” is motivated by Lemma 3.1. But we first need to introduce some more specialized definitions.

Zhang (2008a) introduced the concept of (definitely) visible edges in MAGs and PAGs. The reason for this is as follows. A directed edge $X \rightarrow Y$ in a DAG, CPDAG, MAG or PAG always means that $X$ is a cause (or ancestor) of $Y$, because of the tail mark at $X$. However, when we allow for hidden variables, there may be a hidden confounding variable between $X$ and $Y$. Visible edges refer to situations where there cannot be such a hidden confounder between $X$ and $Y$. Invisible edges, on the other hand, are possibly confounded in the sense that there is a DAG represented by the MAG or PAG with $X \leftarrow L \rightarrow Y$ where $L$ is not measured.

**Definition 3.1.** (Visible and invisible edges; cf. Zhang (2008a)) All directed edges in DAGs and CPDAGs are said to be visible. Given a MAG $\mathcal{M} / \text{PAG} \; \mathcal{P}$, a directed edge $A \rightarrow B$ in $\mathcal{M} / \mathcal{P}$ is visible if there is a vertex $C$ not adjacent to $B$, such that there is an edge between $C$ and $A$ that is into $A$, or there is a collider path between $C$ and $A$ that is into $A$ and every non-endpoint vertex on the path is a parent of $B$. Otherwise $A \rightarrow B$ is said to be invisible.

Figure 1 illustrates the different graphical configurations that can lead to a visible edge. We note that Zhang (2008a) used slightly different terminology, referring to definitely visible edges in a PAG, while we simply say
visible for both MAGs and PAGs. Borboudakis et al. (2012) used the term *pure-causal* edges instead of *visible* edges in MAGs.

We can now generalize the concept of a backdoor path:

**Definition 3.2.** *(Backdoor path)* Let \((X, Y)\) be an ordered pair of vertices in \(G\), where \(G\) is a DAG, CPDAG, MAG or PAG. We say that a path between \(X\) and \(Y\) is a backdoor path from \(X\) to \(Y\) if it does not have a visible edge out of \(X\).

In a DAG, this definition reduces to a path between \(X\) and \(Y\) that is into \(X\), which is the usual backdoor path as defined by Pearl (1993). In a CPDAG, a backdoor path from \(X\) to \(Y\) is a path between \(X\) and \(Y\) that is not out of \(X\) (so starting with \(X \leftarrow \) or \(X \xrightarrow{}\)). In a MAG, it is a path between \(X\) and \(Y\) that is into \(X\) or out of \(X\) with an invisible edge (so starting with \(X \leftrightarrow\), \(X \leftarrow\) or an invisible edge \(X \rightarrow\)). Finally, in a PAG, it is a path between \(X\) and \(Y\) that starts with \(X \leftarrow\), \(X \leftrightarrow\) or an invisible edge \(X \rightarrow\).

We also need generalizations of the concept of d-separation in DAGs (Def. 1.2.3 of Pearl (2000)). In MAGs, one can use m-separation (Sec. 3.4 of Richardson and Spirtes (2002)). In CPDAGs and PAGs, there is the additional complication that it may be unclear whether a vertex is a collider or a non-collider on the path. We therefore need the following definitions:

**Definition 3.3.** *(Definite non-collider; Zhang (2008a)) A non-endpoint vertex \(X_j\) on a path \(\ldots, X_i, X_j, X_k, \ldots\) in a partial mixed graph \(G\) is a definite non-collider on the path if (i) there is a tail mark at \(X_j\), i.e., \(X_i \leftarrow X_j\) or \(X_j \xrightarrow{} X_k\), or (ii) \((X_i, X_j, X_k)\) is unshielded and has circle marks at \(X_j\), i.e., \(X_i \leftrightarrow X_j \leftrightarrow X_k\) and \(X_i\) and \(X_k\) are not adjacent in \(G\).
The motivation for conditions (i) and (ii) is straightforward. A tail mark out of $X_j$ on the path ensures that $X_j$ is a non-collider on the path in any graph obtained by orienting any circle marks. Condition (ii) comes from the fact that the collider status of unshielded triples is known in CPDAGs and PAGs. Hence, if the graph contains an unshielded triple that was not oriented as a collider, then it must be a non-collider in all underlying DAGs or MAGs. If $G$ is a DAG or a MAG, then only condition (i) applies and reduces to the usual definition of a non-collider.

**Definition 3.4. (Definite status path)** A non-endpoint vertex $X_j$ on a path $p$ in a partial mixed graph is said to be of a definite status if it is either a collider or a definite non-collider on $p$. The path $p$ is said to be of a definite status if all non-endpoint vertices on the path are of a definite status.

A path of length one is a trivial definite status path. Moreover, in DAGs and MAGs, all paths are of a definite status. We now define m-connection for definite status paths:

**Definition 3.5. (m-connection)** A definite status path $p$ between vertices $X$ and $Y$ in a partial mixed graph is m-connecting given a (possibly empty) set of variables $Z (X, Y \notin Z)$ if

(a) every definite non-collider on the path is not in $Z$; and

(b) every collider on the path is an ancestor of some member of $Z$.

If a definite status path $p$ is not m-connecting given $Z$, then we say that $Z$ blocks $p$.

If $Z = \emptyset$, we usually omit the phrase “given the empty set”. Definition 3.5 reduces to m-connection for MAGs and d-connection for DAGs. We note that Zhang (2008a) used the notions of possible m-connection and definite m-connection in PAGs, where his notion of definite m-connection is the same as our notion of m-connection for definite status paths.

We now define an adjustment criterion for causal DAGs, CPDAGs, MAGs and PAGs.

**Definition 3.6. (Adjustment criterion)** Let $X$, $Y$ and $W$ be disjoint sets of vertices in $G$, where $G$ represents a causal DAG, CPDAG, MAG or PAG. Then we say that $W$ satisfies the adjustment criterion relative to $(X, Y)$.
and if for any causal model \((\mathcal{G}, f)\) we have

\[
f(y|do(x)) = \begin{cases} 
  f(y|x) & \text{if } W = \emptyset, \\
  \int_w f(y|w, x)f(w)dw = E_W\{f(y|w, x)\} & \text{otherwise.}
\end{cases}
\]

(1)

If \(X = \{X\}\) and \(Y = \{Y\}\), we simply say that a set satisfies the criterion relative to \((X, Y)\) (rather than \((\{X\}, \{Y\})\)) and the given graph.

We now propose the following generalized backdoor criterion for DAGs, CPDAGs, MAGs and PAGs. We will show in Theorem 3.1 that this is sufficient for adjustment.

**Definition 3.7.** (Generalized backdoor criterion) Let \(X, Y\) and \(W\) be disjoint sets of vertices in \(\mathcal{G}\), where \(\mathcal{G}\) represents a causal DAG, CPDAG, MAG or PAG. Then \(W\) satisfies the generalized backdoor criterion relative to \((X, Y)\) and \(\mathcal{G}\) if:

- (B-i) \(W\) does not contain possible descendants of \(X\) (along a definite status path) in \(\mathcal{G}\); and
- (B-ii) For every \(X \in X\), the set \(W \cup X \setminus \{X\}\) blocks every definite status backdoor path from \(X\) to any member of \(Y\), if any, in \(\mathcal{G}\).

**Theorem 3.1.** Let \(X, Y\) and \(W\) be disjoint sets of vertices in \(\mathcal{G}\), where \(\mathcal{G}\) represents a causal DAG, CPDAG, MAG or PAG. If \(W\) satisfies the generalized backdoor criterion relative to \((X, Y)\) and \(\mathcal{G}\) (Definition 3.7) then it satisfies the adjustment criterion relative to \((X, Y)\) and \(\mathcal{G}\) (Definition 3.6).

The proof of Theorem 3.1 consists of two steps. First, we formulate invariance criteria that are sufficient for adjustment (Theorem 7.1). Next, we translate the invariance criteria into the graphical criteria given in Definition 3.7 using results of Zhang (2008a) (Theorem 7.4).

We refer to Definition 3.7 as a generalized backdoor criterion, because its conditions are closely related to Pearl’s original backdoor criterion [Pearl, 1993, 2000].

**Definition 3.8.** (Pearl’s backdoor criterion; Definition 3.3.1 of Pearl [2000]) A set of variables \(W\) satisfies the backdoor criterion relative to an ordered pair of variables \((X, Y)\) in a causal DAG \(\mathcal{D}\) if

- (P-i) No vertex in \(W\) is a descendant of \(X\) in \(\mathcal{D}\); and
blocks every path between $X$ and $Y$ in $D$ that is into $X$.

Similarly, if $X$ and $Y$ are two disjoint subsets of vertices in $D$, then $W$ is said to satisfy the backdoor criterion relative to $(X,Y)$ in $D$ if it satisfies the criterion relative to any pair $(X,Y)$ such that $X \in X$ and $Y \in Y$.

In particular, the conditions in Definition 3.7 are equivalent to Pearl’s backdoor criterion for a DAG with a single intervention ($|X| = 1$). For a DAG with multiple interventions, any set that satisfies Pearl’s backdoor criterion also satisfies the generalized backdoor criterion, but not necessarily the other way around. In this sense, our criterion is slightly better; see Lemma 3.1 and Example 1.

Lemma 3.1. Let $X$, $Y$ and $W$ be disjoint sets of vertices in a causal DAG $D$. If $W$ satisfies Pearl’s backdoor criterion (Definition 3.8) relative to $(X,Y)$ and $D$, then $W$ satisfies the generalized backdoor criterion (Definition 3.7) relative to $(X,Y)$ and $D$.

4 Finding a set that satisfies the generalized backdoor criterion

An important reason for the popularity of Pearl’s backdoor criterion is the following. Consider two distinct vertices $X$ and $Y$ in a DAG $D$. Then $pa(X,D)$ satisfies the backdoor criterion relative to $(X,Y)$ and $D$, unless $Y \in pa(X,D)$. In the latter case, there is no set that satisfies the backdoor criterion, but it is easy to see that $f(y|do(x)) = f(y)$ since there cannot be a directed path from $X$ to $Y$ in $D$.

In this section, we formulate similar results for the generalized backdoor criterion. In particular, we consider the following problem. Given two distinct vertices $X$ and $Y$ in a causal DAG, CPDAG, MAG or PAG, can we easily determine if there is a set that satisfies the generalized backdoor criterion relative to $(X,Y)$ and the given graph? Moreover, if this question is answered positively, can we give an explicit set that satisfies the criterion? These questions are addressed in general in Theorem 4.1. Corollaries 4.1-4.3 give specific results for DAGs, CPDAGs and MAGs. We emphasize that throughout this section, we work with a single intervention variable $X$ and a single variable of interest $Y$, rather than sets $X$ and $Y$.

In a DAG, the following result is well-known. If $X$ and $Y$ are not adjacent in a DAG $D$, and $X \notin an(Y,D)$, then $pa(X,D)$ blocks all paths between $X$ and $Y$. In MAGs, we have a similar result, but we need to use
D-SEP\((X, Y, \mathcal{M})\) instead of the parent set; see Definition 4.1 and Lemma 4.1.

**Definition 4.1.** \((D-SEP(X, Y, \mathcal{G}); \text{ cf. p136 of Spirtes et al. (2000)})\) Let \(X\) and \(Y\) be two distinct vertices in mixed graph \(\mathcal{G}\). We say that \(V \in D-SEP(X, Y, \mathcal{G})\) if \(V \neq X\) and there is a collider path between \(X\) and \(V\) in \(\mathcal{G}\), such that every vertex on this path is an ancestor of \(X\) or \(Y\) in \(\mathcal{G}\).

**Lemma 4.1.** \((\text{cf. Spirtes et al. (1999, Lemma 12)})\) Let \(X\) and \(Y\) be two distinct vertices in a MAG \(\mathcal{M}\). If \(X\) and \(Y\) are not adjacent in \(\mathcal{M}\), then \(D-SEP(X, Y, \mathcal{M})\) blocks all paths between \(X\) and \(Y\) in \(\mathcal{M}\).

**Definition 4.2.** \((R\text{ and } R_X)\) Let \(X\) be a vertex in \(\mathcal{G}\), where \(\mathcal{G}\) represents a causal DAG, CPDAG, MAG or PAG.

Let \(R\) be a DAG or MAG represented by \(\mathcal{G}\), in the following sense. If \(\mathcal{G}\) is a DAG or a MAG, we simply let \(R = \mathcal{G}\). If \(\mathcal{G}\) is a CPDAG / PAG, we let \(R\) be a DAG / MAG in the Markov equivalence class described by \(\mathcal{G}\) with the same number of edges into \(X\) as \(\mathcal{G}\).

Let \(R_X\) be the graph obtained from \(R\) by removing all directed edges out of \(X\) that are visible in \(\mathcal{G}\).

We can now present the main result of this section.

**Theorem 4.1.** \((\text{Generalized backdoor set})\) Let \(X\) and \(Y\) be two distinct vertices in \(\mathcal{G}\), where \(\mathcal{G}\) is a causal DAG, CPDAG, MAG or PAG. Let \(R\) and \(R_X\) be defined as in Definition 4.2.

If \(Y \in \text{adj}(X, R_X)\) or \(D-SEP(X, Y, R_X) \cap \text{possibleDe}(X, \mathcal{G}) \neq \emptyset\), then \(f(y|do(x))\) is not identifiable via the generalized backdoor criterion. Otherwise \(D-SEP(X, Y, R_X)\) satisfies the generalized backdoor criterion relative to \((X, Y)\) and \(\mathcal{G}\).

We need the special choice of \(R\) in Theorem 4.1 to ensure that \(D-SEP(X, Y, R_X) \cap \text{possibleDe}(X, \mathcal{G}) \neq \emptyset\) implies that \(f(y|do(x))\) is not identifiable via the backdoor criterion. This statement is not true if one takes an arbitrary DAG or MAG in the Markov equivalence class represented by \(\mathcal{G}\); see also Example 7.
For DAGs, CPDAGs and MAGs we can simplify Theorem 4.1 somewhat, see Corollaries 4.1 - 4.3. Corollary 4.1 is the well-known result for DAGs that we discussed earlier.

Corollary 4.1. (Backdoor set for a DAG) Let \( X \) and \( Y \) be two distinct vertices in a causal DAG \( D \). If \( Y \in \text{pa}(X, D) \), then \( f(y|\text{do}(x)) \) cannot be identified via the generalized backdoor criterion. Otherwise, \( \text{pa}(X, D) \) satisfies the generalized backdoor criterion relative to \( (X, Y) \) and \( D \).

Corollary 4.2. (Backdoor set for a CPDAG) Let \( X \) and \( Y \) be two distinct vertices in a causal CPDAG \( C \). Let \( C_X \) be the graph obtained from \( C \) by removing all directed edges out of \( X \). If \( Y \in \text{pa}(X, C) \) or \( Y \in \text{possibleDe}(X, C_X) \), then \( f(y|\text{do}(x)) \) cannot be identified via the generalized backdoor criterion. Otherwise, \( \text{pa}(X, C) \) satisfies the generalized backdoor criterion relative to \( (X, Y) \) and \( C \).

Corollary 4.3. (Backdoor set for a MAG) Let \( X \) and \( Y \) be two distinct vertices in a causal MAG \( M \). If \( Y \in \text{adj}(X, M) \) or \( \text{D-SEP}(X, Y, M_X) \cap \text{de}(X, M) \neq \emptyset \), then \( f(y|\text{do}(x)) \) is not identifiable via the generalized backdoor criterion. Otherwise \( \text{D-SEP}(X, Y, M_X) \) satisfies the generalized backdoor criterion relative to \( (X, Y) \) and \( M \).

Corollary 4.3 follows straightforwardly from Theorem 4.1 and is given without proof.

5 Examples

We now give several examples to illustrate the theory. We start with an example that shows that a post-intervention distribution \( f(y|\text{do}(x)) \) might be unidentifiable by Pearl’s backdoor criterion (Definition 3.8), but identifiable by the generalized backdoor criterion (Definition 3.7).

Example 1. Consider the DAG \( D \) in Figure 2 and let \( X = \{X_1, X_3, X_4\} \) and \( Y = \{Y\} \). We first show that \( W = \emptyset \) satisfies the generalized backdoor criterion (Definition 3.7) with respect to \( (X, Y) \) and \( D \). Note that we cannot use Theorem 4.1 since \( X \) is a set. We therefore work with Definition 3.7 directly. We only need to check that the backdoor path from \( X_4 \) to \( Y \) is blocked by \( W \cup X \setminus \{X_4\} = \{X_1, X_3\} \), which is the case since \( X_3 \) is a non-collider on the path. Indeed, we have that \( f(y|\text{do}(\{x_1, x_3, x_4\})) = f(y|x_1, x_3, x_4) \) in Figure 2, which can be further simplified to \( f(y|x_3) \).

On the other hand, there is no set that satisfies Pearl’s backdoor criterion (Definition 3.8) with respect to \( (X, Y) \). To see this, note that \( \{X_2, X_3, X_4\} \subseteq \)}
de(\(X_1, \mathcal{D}\)). Hence, the only possible candidate set is \(W = \emptyset\). But this set does not block the backdoor path from \(X_4\) to \(Y\), since there is no collider on this path.

\[\begin{array}{cccc}
  & X_4 \\
 X_1 & \rightarrow & X_2 & \rightarrow & X_3 & \rightarrow & Y \\
\end{array}\]

Figure 2: The DAG \(\mathcal{D}\) for Example 1, where \(X = \{X_1, X_3, X_4\}\) and \(Y = \{Y\}\). There is no set that satisfies Pearl’s backdoor criterion relative to \((X, Y)\) and \(\mathcal{D}\), but the empty set satisfies the generalized backdoor criterion.

Next, we note that not all identifiable post-intervention distributions are identifiable by the generalized backdoor criterion. A trivial example is given below.

**Example 2.** Let \(X\) and \(Y\) be two distinct vertices in \(\mathcal{G}\), where \(\mathcal{G}\) represents a causal DAG, CPDAG, MAG or PAG. If \(X \leftrightarrow Y\) in \(\mathcal{G}\), then there is no subset of the remaining vertices of \(\mathcal{G}\) that satisfies the generalized backdoor criterion. This follows directly from Theorem 4.1 and \(Y \in \text{adj}(X, \mathcal{R}_X)\).

On the other hand, \(f(y \mid \text{do}(x))\) is identifiable and equals \(f(y)\). To see this, note that there cannot be a possibly directed path from \(X\) to \(Y\) in \(\mathcal{G}\), because of Lemma 7.3.

Examples 3 and 4 illustrate the theory for CPDAGs. In Example 4, the post-intervention distribution is identifiable via the generalized backdoor criterion, while in Example 3 it is not.

**Example 3.** In the CPDAG \(\mathcal{C}\) in Figure 3, \(f(y \mid \text{do}(x))\) is not identifiable. To see this, note that the Markov equivalence class represented by this CPDAG contains three DAGs. Without loss of generality, we denote these by \(\mathcal{D}_1, \mathcal{D}_2\) and \(\mathcal{D}_3\), where we assume that \(\mathcal{D}_1\) contains the sub-graph \(X \leftarrow V_2 \rightarrow Y\), \(\mathcal{D}_2\) contains the sub-graph \(X \leftarrow V_2 \leftarrow Y\), and \(\mathcal{D}_3\) contains the sub-graph \(X \rightarrow V_2 \rightarrow Y\). In \(\mathcal{D}_1\) and \(\mathcal{D}_2\) there is no directed path from \(X\) to \(Y\), so that \(f(y \mid \text{do}(x)) = f(y)\). In \(\mathcal{D}_3\), however, there is a directed path from \(X\) to \(Y\), so that \(f(y \mid \text{do}(x))\) generally does not equal \(f(y)\). Hence, \(f(y \mid \text{do}(x))\) is not identifiable, and therefore certainly not identifiable via the generalized backdoor criterion.
We now apply Theorem 4.1 to the CPDAG $C$ to check if this leads to the same conclusion. Note that $G = C$, $R = D_3$ and $R_X = D_3$. We then have $D\text{-SEP}(X, Y, R_X) = \{V_1, V_2, V_3\}$ and $\text{possibleDe}(X, G) = \{V_2, Y\}$. Hence, $D\text{-SEP}(X, Y, R_X) \cap \text{possibleDe}(X, G) = \{V_2\}$, and Theorem 4.1 correctly says that $f(y|\text{do}(x))$ is not identifiable via the generalized backdoor criterion.

Finally, we check if Corollary 4.2 also yields the same result. Note that $C_X = C$ and $Y \in \text{possibleDe}(X, C_X) = \{V_2, Y\}$. Hence, we again find that $f(y|\text{do}(x))$ is not identifiable via the backdoor criterion.

Example 4. In the CPDAG $C'$ in Figure 4, $f(y|\text{do}(x))$ is identifiable and equals $f(y)$, since there is no possibly directed path from $X$ to $Y$ in $C'$.

We now check if we also arrive at this conclusion when we apply Theorem 4.1. Note that there are two DAGs in the Markov equivalence class described by $C'$, namely $D'_1$ with the edge $X \rightarrow V_2$ and $D'_2$ with the edge $X \leftarrow V_2$. Thus, in Theorem 4.1 we have $G = C'$, $R = D'_1$ and $R_X = D'_1$. Moreover, $Y \notin \text{adj}(X, R_X) = \{V_1, V_2, V_3\}$ and $D\text{-SEP}(X, Y, R_X) = \{V_1, V_3\}$ and $\text{possibleDe}(X, G) = \{V_2, V_4\}$. Hence, $D\text{-SEP}(X, Y, R_X) \cap \text{possibleDe}(X, G) = \emptyset$. According to Theorem 4.1, $D\text{-SEP}(X, Y, R_X) = \{V_1, V_3\}$ satisfies the generalized backdoor criterion relative to $(X, Y)$ and $C'$. We can indeed check that $\{V_1, V_3\}$ satisfies the conditions in Definition 5.7.

Finally, we also apply Corollary 4.2. Note that $C'_X = C'$ Moreover, $Y \notin \text{pa}(X, C')$ and $Y \notin \text{possibleDe}(X, C'_X)$. Hence, we again find that $\text{pa}(X, C') = \{V_1, V_3\}$ satisfies the generalized backdoor criterion relative to $(X, Y)$ and $C'$.

Examples 5 and 6 illustrate the theory for MAGs. In Example 5, the post-intervention distribution $f(y|\text{do}(x))$ is not identifiable via the generalized backdoor criterion because $Y \in \text{adj}(X, M_X)$. In Example 6, $f(y|\text{do}(x))$
Figure 4: The CPDAG $C'$ for Example 4. The set $\{V_1, V_3\}$ satisfies the generalized backdoor criterion relative to $(X, Y)$ and $C'$.

is not identifiable via the generalized backdoor criterion because $D\text{-SEP}(X, Y, M_X) \cap \text{de}(X, M) \neq \emptyset$.

**Example 5.** Consider the MAG $M$ consisting of the invisible edge $X \rightarrow Y$. Hence, the underlying DAG could be as in Figure 5 where $L$ is unobserved. This is a well-known example where $f(y|\text{do}(x))$ is not identifiable.

We now apply Corollary 4.3 to check if we indeed find that $f(y|\text{do}(x))$ is not identifiable via the backdoor criterion. We have that $M = M_X$ is the graph $X \rightarrow Y$. Hence, $Y \in \text{adj}(X, M_X)$, which leads to the correct conclusion that $f(y|\text{do}(x))$ is not identifiable via the generalized backdoor criterion.

Figure 5: A possible DAG described by the MAG in Example 5 where $L$ is latent. The post-intervention distribution $f(y|\text{do}(x))$ is not identifiable from this DAG.

**Example 6.** Consider the MAG $M$ in Figure 6 and apply Corollary 4.3. Since the edge $X \rightarrow V_3$ is visible, $M_X$ is constructed from $M$ by removing this edge. We then have $D\text{-SEP}(X, Y, M_X) = \{V_1, V_2, V_3\}$ and $\text{de}(X, M) = \{V_3, V_5, Y\}$. Hence, $D\text{-SEP}(X, Y, M_X) \cap \text{de}(X, M) = \{V_3\}$, and Corollary 4.3 says that $f(y|\text{do}(x))$ is not identifiable via the generalized backdoor criterion.
Indeed, we see that it is impossible to satisfy conditions (B-i) and (B-ii) in Definition 3.7. In order to block the backdoor path \((X, V_2, V_4, Y)\), we must include \(V_2\) or \(V_4\) in our set \(W\). But doing so opens the collider \(V_2\) on the backdoor path \((X, V_2, V_3, V_5, Y)\). Hence, the latter path must be blocked by \(V_3\) or \(V_5\). But both these vertices are descendants of \(X\) in \(M\), and are therefore not allowed by condition (B-i).

Figure 6: The MAG \(\mathcal{M}\) for Example 6. The post-intervention distribution \(f(y \mid \text{do}(x))\) is not identifiable via the generalized backdoor criterion, since \(D-SEP(X, Y, \mathcal{M}_X) \cap \text{de}(X, \mathcal{M}) \neq \emptyset\).

Finally, we give an example of a PAG where a post-intervention distribution is identifiable via the generalized backdoor criterion. This example also illustrates that there may be subsets of \(D-SEP(X, Y, \mathcal{M}_X)\) in Theorem 4.1 that satisfy the generalized backdoor criterion. In other words, we may find a non-minimal set, and if one is interested in a minimal set (i.e., a set such that no strict subset of this set satisfies the generalized backdoor criterion), one could consider all subsets of \(D-SEP(X, Y, \mathcal{M}_X)\). Example 7 also illustrates the importance of the special choice of \(\mathcal{R}_X\).

**Example 7.** Consider the PAG \(\mathcal{P}\) in Figure 7. We want to apply Theorem 4.1 with \(\mathcal{G} = \mathcal{P}, \mathcal{R} = \mathcal{M}\) as given in Figure 8 and \(\mathcal{R}_X\) is as \(\mathcal{M}\) but without the edge \(X \rightarrow Y\) that is visible in \(\mathcal{G}\). We then have \(Y \notin \text{adj}(X, \mathcal{R}_X)\) and \(D-SEP(X, Y, \mathcal{R}_X) \cap \text{possibleDe}(X, \mathcal{G}) = \{V_1, V_2\} \cap \{V_3, V_4, Y\} = \emptyset\). Hence, Theorem 4.1 implies that the set \(\{V_1, V_2\}\) satisfies the generalized backdoor criterion relative to \((X, Y)\) and \(\mathcal{P}\). One can easily verify that all subsets of \(\{V_1, V_2\}\) also satisfy the generalized backdoor criterion relative to \((X, Y)\) and \(\mathcal{P}\), since all backdoor paths are blocked by the collider \(V_4\). This shows that in this example \(D-SEP(X, Y, \mathcal{R}_X)\) is not minimal, in the sense that there are strict subsets of this set that also satisfy the generalized backdoor criterion relative to \((X, Y)\) and \(\mathcal{P}\).

This example also shows the importance of the special choice of \(\mathcal{R}\) as specified in Theorem 4.1 and Definition 4.2. To see this, let \(\mathcal{R}'\) be as \(\mathcal{R}\),
but with the edge $X \leftarrow V_3$ instead of $X \rightarrow V_3$, so that there is an additional edge into $X$. Then $D$-$SEP(X, Y, \mathcal{R}'_X) = \{V_1, V_2, V_3\}$, and we get $D$-$SEP(X, Y, \mathcal{R}'_X) \cap \text{possibleDe}(X, \mathcal{G}) = \{V_3\} \neq \emptyset$. This shows that applying Theorem 4.1 with $\mathcal{R}'$ instead of $\mathcal{R}$ leads to incorrect results.

Figure 7: The PAG $\mathcal{P}$ for Example 7. All subsets of $\{V_1, V_3\}$ satisfy the generalized backdoor criterion relative to $(X, Y)$ and $\mathcal{P}$.

Figure 8: A possible MAG $\mathcal{M}$ for Example 7, constructed so that it has the same number of edges into $X$ as $\mathcal{P}$.

6 Discussion

In this paper, we generalized Pearl’s backdoor criterion [Pearl, 1993] to a generalized backdoor criterion for DAGs, CPDAGs, MAGs and PAGs. When the intervention variable and the variable of interest are both singletons, we also provide easily checkable necessary and sufficient criteria for the existence of a set that satisfies the generalized backdoor criterion. Moreover, if such a set exists, we provide an explicit set that satisfies the generalized
backdoor criterion. This set is not necessarily minimal, so if one is interested in a minimal set, one could consider all subsets.

Although effects that are identifiable via the generalized backdoor criterion are only a subset of all identifiable causal effects, we hope that the generalized backdoor criterion will be useful in practice, and will make it easier to work with CPDAGs, MAGs and PAGs. Moreover, combining our results for CPDAGs and PAGs with fast causal structure learning algorithms such as the PC algorithm (Spirtes et al. (2000)) or the FCI algorithm (Claassen et al. (2013); Colombo et al. (2012); Spirtes et al. (2000)) yields a computationally efficient way to obtain information on causal effects when assuming that the observational distribution is faithful to the true unknown causal DAG with or without hidden variables. To our knowledge, the Prediction Algorithm of (Spirtes et al. (2000)) is the only alternative approach under the same assumptions, but this algorithm is computationally much more complex.

The IDA algorithm (Maathuis et al. (2010, 2009)) has been designed to obtain bounds on causal effects when assuming that the observational distribution is faithful to the true underlying causal DAG without hidden variables. This approach roughly combines the PC algorithm with Pearl’s backdoor criterion. We could now apply a similar approach in the setting with hidden variables, by combining the FCI algorithm with the generalized backdoor criterion for MAGs.

Possible directions for future work include studying the exact relationship between the Prediction Algorithm and our approach, generalizing Pearl’s frontdoor criterion (Pearl, 2000, Section 3.3.2) to CPDAGs, MAGs and PAGs, and generalizing the results in Section 4 to allow for sets \( X \) and \( Y \).

7 Proofs

7.1 Proofs for Section 3

In order to prove Theorem 3.1, we first formulate invariance conditions that will turn out to be sufficient for adjustment (see Theorem 7.1).

**Definition 7.1.** (Invariance criterion) Let \( X \), \( Y \) and \( W \) be disjoint sets of vertices in \( \mathcal{G} \), where \( \mathcal{G} \) is a causal DAG, CPDAG, MAG or PAG. Then \( W \) satisfies the invariance criterion relative to \( (X, Y) \) and \( \mathcal{G} \) if the following two conditions hold for all causal models \((\mathcal{G}, f)\):

\[(I-i) \quad f(w|do(x)) = f(w) \; \text{and} \]

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Note that condition (I-i) is void if $W = \emptyset$. In the language of Zhang (2008a), conditions (I-i) and (I-ii) are equivalent to saying that $f(w)$ and $f(y|x, w)$ are entailed to be invariant under interventions on $X$ given $G$. The conditions are also closely related to the conditions in equation (9) of Pearl (1993).

**Theorem 7.1.** Let $X$, $Y$ and $W$ be disjoint sets of vertices in $G$, where $G$ is a causal DAG, CPDAG, MAG or PAG. If $W$ satisfies the invariance criterion relative to $(X, Y)$ and $G$, then it satisfies the adjustment criterion relative to $(X, Y)$ and $G$.

**Proof.** If $W = \emptyset$, condition (I-ii) immediately gives $f(y|do(x)) = f(y|x)$. Otherwise, we have

$$f(y|do(x)) = \int_w f(y, w|do(x))dw = \int_w f(y|w, do(x))f(w|do(x))dw. \quad (2)$$

Under conditions (I-i) and (I-ii), the right hand side of (2) simplifies to $\int_w f(y|w, x)f(w)dw$. 

Spirtes et al. (1993, 2000); Zhang (2008a) formulated invariance results for DAGs, MAGs and PAGs. We give a similar result for CPDAGs in Theorem 7.2 and then summarize the results for all graphs in Theorem 7.3. First, we briefly define what we mean by invariance (please see Zhang (2008a) for full details). A density $f(y|z)$ is said to be invariant under interventions of $X$ given a DAG $D$ if $f_{X=x}(y|z) = f(y|z)$ for all causal models $(D, f)$, where the subscript $X := x$ denotes $do(X = x)$. (We use this more complicated notation since $X$ and $Z$ are allowed to overlap. In Definition 7.1 we use two special cases: in (I-i) we have $X = X$, $Y = W$ and $Z = \emptyset$, and in (I-ii) we have $X = X$, $Y = Y$ and $Z = X \cup W$.) The density $f(y|z)$ is said to be invariant under interventions on $X$ given a CPDAG $C$, MAG $M$ or PAG $P$ if it is invariant under interventions on $X$ given all DAGs represented by $C$, $M$ or $P$, respectively.

**Theorem 7.2.** (Invariance for CPDAGs) Let $X$, $Y$, $Z$ be three subsets of observed vertices in a causal CPDAG $C$, where $X \cap Y = Y \cap Z = \emptyset$. Then $f(Y|Z)$ is invariant under interventions of $X$ given $C$ if and only if

1. for every $X \in X \cap Z$, every $m$-connecting definite status path, if any, between $X$ and any member of $Y$ given $Z \setminus \{X\}$ is out of $X$;
2. for every \( X \in X \cap \text{possibleAn}(Z,G) \setminus Z \), there is no m-connecting definite status path between \( X \) and any member of \( Y \) given \( Z \); and

3. for every \( X \in X \setminus \text{an}(Z,G) \), every m-connecting definite status path, if any, between \( X \) and any member of \( Y \) given \( Z \) is into \( X \).

Proof. The proof is analogous to the proofs of Zhang (2008a), and we omit it here for reasons of space.

**Theorem 7.3.** (Graphical criteria for invariance) Let \( X, Y, Z \) be three subsets of observed vertices in \( G \), where \( G \) represents a causal DAG, CPDAG, MAG or PAG. Moreover, let \( X \cap Y = Y \cap Z = \emptyset \). Then \( f(y|z) \) is invariant under interventions of \( X \) given \( G \) if and only if

1. for every \( X \in X \cap Z \), every m-connecting definite status path, if any, between \( X \) and any member of \( Y \) given \( Z \setminus \{X\} \) is out of \( X \) with a visible edge;

2. for every \( X \in X \cap \text{possibleAn}(Z,G) \setminus Z \), there is no m-connecting definite status path between \( X \) and any member of \( Y \) given \( Z \); and

3. for every \( X \in X \setminus \text{an}(Z,G) \), every m-connecting definite status path, if any, between \( X \) and any member of \( Y \) given \( Z \) is into \( X \).

Proof. One can easily check that for each type of graph, the conditions reduce to the appropriate conditions.

We also need the following basic property of PAGs and CPDAGs:

**Lemma 7.1.** (Basic property of CPDAGs and PAGs; Lemma 1 of Meek (1995) for CPDAGs, and Lemma 3.5.1 of Zhang (2006) for PAGs) For any three vertices \( A, B, \) and \( C \) in a CPDAG \( C \) or PAG \( P \), the following holds: If \( A \leftrightarrow B \rightarrow C \), then there is an edge between \( A \) and \( C \) with an arrowhead at \( C \), namely \( A \rightarrow C \). Furthermore, if the edge between \( A \) and \( B \) is \( A \rightarrow B \), then the edge between \( A \) and \( C \) is either \( A \rightarrow C \) or \( A \leftrightarrow C \) (i.e., not \( A \leftrightarrow C \)).

We can now show that the invariance conditions in Definition 7.1 are equivalent to the graphical conditions of Definition 3.7.

**Theorem 7.4.** The generalized backdoor criterion (Definition 3.7) is equivalent to the invariance criterion (Definition 7.1).
Proof. We first show that condition (B-ii) of Definition 3.7 is equivalent to condition (I-ii) of Definition 7.1. We use Theorem 7.3 with \((X', Y', Z')\), where 
\[X' = X, \ Y' = Y \text{ and } Z' = X \cup W.\]
Then clause (1) of the theorem applies, yielding that (I-ii) is equivalent to the following: For every \(X \in X\), every m-connecting definite status path, if any, between \(X\) and any member of \(Y\) given \(X \cup W \setminus \{X\}\) is out of \(X\) with a visible edge. This is equivalent to Condition (B-ii) by our definition of a backdoor path (see Definition 3.2).

We now show that condition (B-i) of Definition 3.7 (including the phrase “along a definite status path”) is equivalent to condition (I-i) in Definition 7.1. We use Theorem 7.3 with \((X', Y', Z')\), where 
\[X' = X, \ Y' = W \text{ and } Z' = \emptyset.\]
Then clause (3) of the theorem applies, yielding that (I-i) is equivalent to the following condition (I-i)': For every \(X \in X\), every m-connecting definite status path, if any, between \(X\) and any member of \(W\) is into \(X\). We now show that (I-i)' is equivalent to (B-i).

First suppose that \(W\) violates (B-i). Then there are \(W \in W \text{ and } X \in X\) such that there is a possibly directed definite status path \(p\) from \(X\) to \(W\). Since \(p\) is possibly directed, it is not into \(X\) and it cannot contain colliders. Hence, it is an m-connecting definite status path between \(X\) and \(W\) that is not into \(X\). This violates (I-i)'.

Now suppose that \(W\) violates (I-i)'. Then there are \(W \in W \text{ and } X \in X\) such that there is a m-connecting definite status path between \(X\) and \(W\) that is not into \(X\). Let \(p = (X = X_1, X_2, \ldots, X_k = W)\) be such a path. Then every non-endpoint vertex on \(p\) must be a definite non-collider. Suppose that \(p\) is not a possibly directed path from \(X\) to \(W\), meaning that there exists an \(i \in \{2, \ldots, k\}\) such that the edge between \(X_{i-1}\) and \(X_i\) is into \(X_{i-1}\). If \(i = 2\), this means that the path is into \(X\), which is a contradiction. If \(i > 2\), then the edge between \(X_{i-2}\) and \(X_{i-1}\) must be out of \(X_{i-1}\), since \(X_{i-1}\) is a definite non-collider. But this means that the edge must be into \(X_{i-2}\), since edges of the form \(\rightarrow\) or \(\leftarrow\) are not allowed. Continuing this argument, we find that for all \(j \in \{2, \ldots, i\}\), the edge between \(X_{j-1}\) and \(X_j\) is into \(X_{j-1}\). But this means that the path is into \(X\), which is a contradiction. Hence, \(p\) is a possibly directed path from \(X\) to \(W\). Together with the fact that \(p\) is of a definite status, this violates (B-i).

Finally, Lemma 7.2 (below) shows that the phrase “along a definite status path” can be removed from condition (B-i). \(\square\)

Let \(X\) and \(Y\) be two distinct vertices in \(\mathcal{G}\), where \(\mathcal{G}\) denotes a DAG, CPDAG, MAG or PAG. If \(Y \in \text{possibleDe}(X, \mathcal{G})\), then there is a possibly directed definite status path \(p = (X = U_1, \ldots, U_k = Y)\) from \(X\) to \(Y\). Moreover, if \(U_{i-1} \leftrightarrow U_i\), then \(U_j \rightarrow U_{j+1}\) for all \(j \in \{i, \ldots, k\}\).
Theorem \[3.1\] This follows directly from Theorems \[7.3\] and \[7.4\].
means that (i) no non-collider on \( p \) is in \( W \cup X \setminus \{X\} \), (ii) all colliders on \( p \) have a descendant in \( W \cup X \setminus \{X\} \), (iii) there is at least one collider on \( p \) that has a descendant in \( X \setminus \{X\} \) but not in \( W \). Among all colliders satisfying (iii), let \( Q \) be the one that is closest to \( Y \) on \( p \), and let \( X' \) denote a descendant of \( Q \) in \( X \setminus \{X\} \). Then the directed path \( q(Q, X') \) from \( Q \) to \( X' \) is m-connecting given \( W \), since it is a path consisting of non-colliders and none of its vertices are in \( W \). Moreover, the sub-path \( p(Q, Y) \) of \( p \) is m-connecting given \( W \) by construction. Now consider \( q(X', Q) \oplus p(Q, Y) \) and remove possible loops. Then we have a backdoor path from \( X' \) to \( Y \) that is m-connecting given \( W \). This contradicts (P-ii).

### 7.2 Proofs for Section 4

We first give several lemmas, starting with a result about m-connection in MAGs. This result basically says that replacing condition (b) in Definition 3.5 by “every collider on the path is an ancestor of some member of \( Z \cup \{X, Y\} \)” does not change the m-separation relations in a MAG.

**Lemma 7.3.** (Richardson (2003, Corollary 1)) Let \( X \) and \( Y \) be two distinct vertices and \( Z \) be a subset of vertices in a MAG \( M \), with \( Z \cap \{X, Y\} = \emptyset \). If there is a path between \( X \) and \( Y \) in \( M \) on which no non-collider is in \( Z \) and every collider is in \( \text{an}(Z \cup \{X, Y\}, M) \), then there is a path (not necessarily the same path) m-connecting \( X \) and \( Y \) given \( Z \) in \( M \).

The following lemma says that we can check the existence of m-connecting definite status backdoor paths in \( G \) by checking the existence of m-connecting paths in \( R_X \), where \( G \) is a causal DAG / CPDAG / MAG / PAG. Let \( R_X \) be as defined in Definition 4.2. Then there is a definite status m-connecting backdoor path from \( X \) to \( Y \) given \( Z \) in \( G \) if and only if there is an m-connecting path between \( X \) and \( Y \) given \( Z \) in \( R_X \).

**Lemma 7.4.** Let \( X \) and \( Y \) be two distinct vertices and \( Z \) be a subset of vertices in \( G \), where \( G \) is a causal DAG / CPDAG / MAG / PAG. Let \( R_X \) be as defined in Definition 4.2. Then there is a definite status m-connecting backdoor path from \( X \) to \( Y \) given \( Z \) in \( G \) if and only if there is an m-connecting path between \( X \) and \( Y \) given \( Z \) in \( R_X \).

**Proof.** Let \( R \) be as defined in Definition 4.2. We first prove the “only if” statement. Suppose there is a definite status m-connecting backdoor path \( p \) from \( X \) to \( Y \) given \( Z \) in \( G \). Let \( p' \) and \( p'' \) be the corresponding paths in \( R \) and \( R_X \), consisting of the same sequence of vertices. (Note that \( p'' \) exists by the definition of \( R_X \) and the fact that \( p \) is a backdoor path in \( G \).) The path \( p' \) is m-connecting given \( Z \) in \( R \), since \( R = G \) if \( G \) is a causal DAG or...
MAG, and \( R \) is a DAG or MAG in the Markov equivalence class described by \( \mathcal{G} \) otherwise. The path \( p'' \), however, is not necessarily m-connecting in \( \mathcal{R}_X \), since it may happen that there is a collider \( Q \) on the path such that \( Q \in \text{an}(Z, R) \) but \( Q \notin \text{an}(Z, \mathcal{R}_X) \). Hence, \( p'' \) satisfies the following properties: no non-collider on \( p'' \) is in \( Z \) and every collider on \( p'' \) is in \( \text{an}(Z \cup \{ X \}, \mathcal{R}_X) \). It then follows from Lemma 7.3 that there is an m-connecting path between \( X \) and \( Y \) given \( Z \) in \( \mathcal{R}_X \).

We now prove the “if” statement. Suppose that there is an m-connecting path \( p'' \) between \( X \) and \( Y \) given \( Z \) in \( \mathcal{R}_X \). Let \( p' \) and \( p \) be the corresponding paths in \( R \) and \( \mathcal{G} \), consisting of the same sequence of vertices. Then \( p' \) is also m-connecting given \( Z \) in \( \mathcal{R}_X \). Moreover, \( p \) does not start with a visible edge out of \( X \) in \( \mathcal{G} \), because \( p'' \) exists in \( \mathcal{R}_X \). This means that there exists a path that satisfies the condition in lemma 2’ in the proof of Lemma 5.1.7 of [Zhang (2006)], and this implies that there exists an m-connecting definite status backdoor path between \( X \) and \( Y \) given \( Z \) in \( \mathcal{G} \).

The next lemma is used several times to derive a contradiction.

**Lemma 7.5.** Let \( U \) and \( V \) be two distinct vertices in \( \mathcal{G} \), where \( \mathcal{G} \) denotes a DAG, CPDAG, MAG or PAG. Then \( \mathcal{G} \) cannot have both a possibly directed path from \( U \) to \( V \) and an edge of the form \( V \leftrightarrow U \).

**Proof.** This lemma is trivial for DAGs and MAGs, since they cannot have any (almost) directed cycles. So we only show the result for CPDAGs and PAGs. Let \( \mathcal{G} \) denote the CPDAG or PAG, and suppose that \( \mathcal{G} \) contains an edge of the form \( V \leftrightarrow U \) as well as a possibly directed path from \( U \) to \( V \) in \( \mathcal{G} \). Then there is also a possibly directed definite status path \( p = \langle U = U_1, \ldots, U_k = V \rangle \) from \( U \) to \( V \) in \( \mathcal{G} \), by Lemma 7.2. Moreover, if \( U_{i-1} \leftrightarrow U_i \) for some \( i \in \{2, \ldots, k\} \), then \( U_j \rightarrow U_{j+1} \) for all \( j \in \{i, \ldots, k\} \). The length of \( p \) must be greater than 1, because of the edge \( V \leftrightarrow U \).

If \( p \) is fully directed, we immediately obtain a contradiction, since this implies an (almost) directed cycle in any DAG or MAG in the Markov equivalence class described by \( \mathcal{G} \). Otherwise, if \( p \) contains a directed sub-path, let \( p(U_d, V) \) be the longest directed sub-path. Then the sub-path \( p(U, U_d) \) must be of the form \( U \rightarrow \cdots \rightarrow U_d \) or \( U \leftarrow \cdots \leftarrow U_d \). In either case, repeated application of Lemma 7.1 implies an edge of the form \( V \leftrightarrow U_d \). This gives an (almost) directed cycle together with the directed path \( p(U_d, V) \), in any DAG or MAG in the Markov equivalence class described by \( \mathcal{G} \). Otherwise, \( p \) does not contain a directed sub-path. Let \( T \) be the vertex adjacent to \( V \) on the path. Then the path has one of the following two forms: \( U \leftarrow \cdots \leftarrow T \leftarrow V \)
or \( T \rightarrow V \). Repeated application of Lemma 7.1 yields an edge of the form \( V \rightarrow T \), which contradicts \( T \rightarrow V \) or \( T \leftrightarrow V \).

Theorem 4.1 requires a special DAG or MAG in the equivalence class described by a CPDAG or PAG. The following lemma establishes that such a DAG or MAG exists. This result is closely related to constructions in Ali et al. (2005), Theorem 2 of Zhang (2008b), and Lemma 27 of Zhang (2008a).

**Lemma 7.6.** Let \( \mathcal{G} \) be a PAG (CPDAG) with \( k \) edges into \( X \), \( k \in \{0, 1, \ldots \} \). Then there exists a MAG (DAG) \( \mathcal{R} \) in the Markov equivalence class represented by \( \mathcal{G} \) that has \( k \) edges into \( X \).

**Proof.** Building on the work of Meek (1995), Theorem 2 of Zhang (2008b) gives a procedure to create a MAG (DAG) in the Markov equivalence class represented by a PAG (CPDAG) \( \mathcal{G} \). One first replaces all non-directed (\( \leftrightarrow \)) edges in \( \mathcal{G} \) by directed (\( \rightarrow \)) edges. Next, one considers the circle component \( \mathcal{G}^C \) of \( \mathcal{G} \), that is, the sub-graph of \( \mathcal{G} \) consisting of non-directed (\( \leftrightarrow \)) edges, and orients this into a DAG without unshielded colliders. The first step of this procedure only creates tail marks, and hence cannot yield an additional edge into \( X \). For the second step, we will argue that we can choose a DAG without unshielded colliders that does not have any edges into \( X \).

First, we note that \( \mathcal{G}^C \) is chordal, i.e., any cycle of length 4 or more has a chord, which is an edge joining two vertices that are not adjacent in the cycle (see the proof of Lemma 4.1 of Zhang (2008b)). Any chordal graph with more than one vertex has two simplicial vertices, i.e., vertices \( V \) such that all vertices adjacent to \( V \) are also adjacent to each other (e.g., Golumbic (1980)). Hence, \( \mathcal{G}^C \) must have at least one simplicial vertex that is different from \( X \). We choose such a vertex \( V_1 \) and orient any edges incident to \( V_1 \) into \( V_1 \). Since \( V_1 \) is simplicial, this does not create unshielded colliders. We then remove \( V_1 \) and these edges from the graph. The resulting graph is again chordal (e.g., Golumbic (1980)), and therefore again has at least one simplicial vertex that is different from \( X \). We choose such a vertex \( V_2 \), and orient any edges incident to \( V_2 \) into \( V_2 \). We continue this procedure until all edges are oriented. The resulting ordering is called a perfect elimination scheme for \( \mathcal{G}^C \). By construction, this procedure yields an acyclic graph without unshielded colliders. Moreover, since \( X \) is chosen as the last vertex in the perfect elimination scheme, we do not orient any edges into \( X \).

The special graph \( \mathcal{R}_X \) in Theorem 4.1 (see Definition 4.2) is needed in the following lemma.
**Lemma 7.7.** Let $X$ and $Y$ be two distinct vertices in $\mathcal{G}$, where $\mathcal{G}$ represents a causal DAG, CPDAG, MAG or PAG. Let $\mathcal{R}_X$ be defined as in Definition 4.2.

If $V \in D\text{-SEP}(X, Y, \mathcal{R}_X) \cap \text{possibleDe}(X, \mathcal{G})$, then $V \in \text{an}(Y, \mathcal{R}_X)$.

**Proof.** Let $V \in D\text{-SEP}(X, Y, \mathcal{R}_X) \cap \text{possibleDe}(X, \mathcal{G})$. This means that there is a collider path $p_1$ between $X$ and $V$ in $\mathcal{R}_X$ such that every vertex on the path is an ancestor of $X$ or $Y$ in $\mathcal{R}_X$. In particular, $V \in \text{an}(\{X, Y\}, \mathcal{R}_X)$.

We first show that $V \in \text{pa}(X, \mathcal{R}_X)$ leads to a contradiction. Thus, suppose there is a directed path $X \leftarrow V$ in $\mathcal{R}_X$. By construction of $\mathcal{R}_X$, $\mathcal{G}$ then contains an edge of the form $X \leftarrow V$ or $X \leftrightarrow V$. But this forms a contradiction together with $V \in \text{possibleDe}(X, \mathcal{G})$, by Lemma 7.5.

We now show that $V \in \text{an}(X, \mathcal{R}_X) \setminus \text{pa}(X, \mathcal{R}_X)$ leads to a contradiction. Thus, suppose there is a directed path from $V$ to $X$ in $\mathcal{R}_X$ of the form $\langle V, \ldots, W, X \rangle$, where $V \neq W$ and $W \neq X$. By construction of $\mathcal{R}_X$, the edge $W \rightarrow X$ must also be into $X$ in $\mathcal{G}$, so that $\mathcal{G}$ contains $W \rightarrow X$ or $W \leftrightarrow X$. Since $V \in \text{possibleDe}(X, \mathcal{G})$, there is a possibly directed path $p_{XV}$ from $X$ to $V$ in $\mathcal{G}$. Since $\mathcal{R}_X$ contains a directed path from $V$ to $W$, $\mathcal{G}$ must also contain a possibly directed path $p_{WV}$ from $V$ to $W$. Now consider $p_{XV} \oplus p_{WV}$, and remove any possible loops. Then this is a possibly directed path from $X$ to $W$ in $\mathcal{G}$, so that $W \in \text{possibleDe}(X, \mathcal{G})$. But this forms a contradiction with $W \rightarrow X$ or $W \leftrightarrow X$ in $\mathcal{G}$, by Lemma 7.5.

Hence, we must have $V \in \text{an}(Y, \mathcal{R}_X)$. \hfill $\square$

We can now prove the results in Section 4.

**Proof of Theorem 4.1.** Suppose that $Y \in \text{adj}(X, \mathcal{R}_X)$. Then there is a definite status backdoor path of length 1 in $\mathcal{G}$ that cannot be blocked. Hence, condition (B-ii) cannot be satisfied.

Next, suppose that there exists some vertex $V \in D\text{-SEP}(X, Y, \mathcal{R}_X) \cap \text{possibleDe}(X, \mathcal{G}) \neq \emptyset$. Then there is a collider path $p_1$ between $X$ and $V$ in $\mathcal{R}_X$ such that every vertex on the path is in $\text{an}(\{X, Y\}, \mathcal{R}_X)$. Moreover, by Lemma 7.7 there is a directed path $p_2$ from $V$ to $Y$ in $\mathcal{R}_X$. Now consider $p_1 \oplus p_2$ and remove possible loops. Call the resulting path $p$. Note that all non-endpoint vertices on $p$ that are not on $p_2$ are colliders on $p$ and in $\text{an}(\{X, Y\}, \mathcal{R}_X)$. The remaining non-endpoint vertices on $p$ are non-colliders and in $\text{possibleDe}(X, \mathcal{G})$ (since $V \in \text{possibleDe}(X, \mathcal{G})$), so that taking them into the conditioning set would violate condition (B-i). It then follows by Lemma 7.3 that for any subset $Z$ satisfying condition (B-i), there exists an $m$-connecting path between $X$ and $Y$ given $Z$ in $\mathcal{R}_X$. By Lemma 7.4, this
means that we cannot block all definite status backdoor paths from $X$ to $Y$ in $G$ without violating condition (B-i).

Now suppose that $Y \notin \text{adj}(X, R_X)$ and $\text{D-SEP}(X, Y, R_X) \cap \text{possibleDe}(X, G) = \emptyset$. By Lemma 4.1 it then follows that $\text{D-SEP}(X, Y, R_X)$ m-separates $X$ and $Y$ in $R_X$. By Lemma 7.4 this implies that $\text{D-SEP}(X, Y, R_X)$ blocks all definite status backdoor paths from $X$ to $Y$ in $G$, so that $\text{D-SEP}(X, Y, R_X)$ satisfies condition (B-ii). Finally, $\text{D-SEP}(X, Y, R_X)$ trivially satisfies condition (B-i), since $\text{D-SEP}(X, Y, R_X) \cap \text{possibleDe}(X, G) = \emptyset$.

Proof of Corollary 4.1. Although this result for DAGs is well-known, we show how one can derive this from Theorem 4.1. Note that $D_X$ is the graph resulting from removing all directed edges out of $X$ from $D$. Moreover, $\text{D-SEP}(X, Y, D_X) = \text{pa}(X, D)$ and $\text{possibleDe}(X, D) = \text{de}(X, D)$. Now the condition $\text{D-SEP}(X, Y, D_X) \cap \text{possibleDe}(X, D) = \emptyset$ reduces to $\text{pa}(X, D) \cap \text{de}(X, D) = \emptyset$, and this is fulfilled automatically by the acyclicity of $D$. Hence, Theorem 4.1 reduces to the given statement.

Proof of Corollary 4.2. Let $D$ be a DAG in the Markov equivalence class represented by $C$, constructed without orienting additional edges into $X$. Let $D_X$ be the graph resulting from removing from $D$ all directed edges out of $X$ that were directed out of $X$ in $C$.

We first show that the conditions in Corollary 4.2 imply the conditions in Theorem 4.1. Suppose that $Y \in \text{pa}(X, C)$. Then $Y \in \text{adj}(X, D_X)$. Next, suppose that $Y \in \text{possibleDe}(X, C_X)$. Then there is a possibly directed definite status path from $X$ to $Y$ in $C_X$, by Lemma 7.2. All non-endpoint vertices on this path must be definite non-colliders. By construction of $C_X$, the first edge on this path must be non-directed in $C_X$, and by construction of $D_X$, this edge must be oriented out of $X$ in $D_X$. The latter implies that the entire path must be directed from $X$ to $Y$ in $D_X$, since all non-endpoint vertices are non-colliders. Let $V$ be the vertex adjacent to $X$ on the path. Then $V \in \text{D-SEP}(X, Y, D_X)$. Moreover, $V \in \text{possibleDe}(X, C_X) \subseteq \text{possibleDe}(X, C)$. Hence, $\text{D-SEP}(X, Y, D_X) \cap \text{possibleDe}(X, C) \neq \emptyset$.

We now show that the conditions in Theorem 4.1 imply the conditions in Corollary 4.2. Suppose that $Y \in \text{adj}(X, D_X)$. Then either $X \leftarrow Y$ or $X \rightarrow Y$ in $C$. This implies that $Y \in \text{pa}(X, C)$ or $Y \in \text{possibleDe}(X, C_X)$. Next, suppose that there exists a vertex $V \in \text{D-SEP}(X, Y, D_X) \cap \text{possibleDe}(X, C)$. Note that $V \in \text{D-SEP}(X, Y, D_X)$ implies (i) $V \in \text{pa}(X, D_X)$ or (ii) $V \in \text{ch}(X, D_X) \cap \text{an}(Y, D_X)$ or (iii) $V \in \text{pa}(\text{ch}(X, D_X) \cap \text{an}(Y, D_X))$. Case (i) is not possible since $V \in \text{possibleDe}(X, C)$, by Lemma 7.3. In case (ii), we have $X \rightarrow V$ and a directed path from $V$ to $Y$ in $D_X$, so that $Y \in \text{de}(X, D_X)$. 28
Similarly, we can obtain $Y \in \text{de}(X, \overline{D_X})$ in case (iii). This implies $Y \in \text{possibleDe}(X, \overline{C_X})$ in cases (ii) and (iii).

The above shows the following: If $Y \in \text{pa}(X, \mathcal{C})$ or $Y \in \text{possibleDe}(X, \overline{C_X})$, then $f(y|do(x))$ cannot be identified via the generalized backdoor criterion. Otherwise, $\text{D-SEP}(X, Y, \overline{D_X})$ satisfies the generalized backdoor criterion relative to $(X, Y)$ and $\mathcal{C}$. It is left to show that in the latter case, we can replace $\text{D-SEP}(X, Y, \overline{D_X})$ by $\text{pa}(X, \mathcal{C})$. Since $\text{pa}(X, \mathcal{C}) \subseteq \text{D-SEP}(X, Y, \overline{D_X})$, it is clear that $\text{pa}(X, \mathcal{C})$ satisfies condition (B-i) of Definition 3.7. We will now show that it also satisfies condition (B-ii).

Thus, suppose that $Y \notin \text{pa}(X, \mathcal{C})$ and $Y \notin \text{possibleDe}(X, \overline{C_X})$. Consider a definite status backdoor path $p = \langle X = U_1, \ldots, U_k = Y \rangle$ from $X$ to $Y$ in $\mathcal{C}$. Since $p$ is a backdoor path, it must start with $X \leftarrow$ or $X \vartriangleleft$. In the former case, it is clear that $\text{pa}(X, \mathcal{C})$ blocks $p$. In the latter case, $p$ cannot have a sub-path of the form $U_{i-1} \vartriangleleft U_i \leftarrow U_{i+1}$, $i \in \{2, \ldots, k-1\}$, because $U_i$ is of a definite status. Moreover, $p$ cannot be possibly directed, because then $Y \in \text{possibleDe}(X, \overline{C_X})$. Hence, there must be at least one collider on $p$. Let $Q$ be the collider on $p$ that is closest to $X$. Then the sub-path $p(X, Q)$ is a possibly directed path from $X$ to $Q$ in $\mathcal{C}$. Suppose that $Q$ is an ancestor of $W \in \text{pa}(X, \mathcal{C})$. Then there is a possibly directed path from $X$ to $W$ in $\mathcal{C}$, as well as an edge $W \rightarrow X$. But this is impossible by Lemma 7.5. Hence, $Q$ cannot be an ancestor of a member of $\text{pa}(X, \mathcal{C})$. This implies that $p$ is blocked by $\text{pa}(X, \mathcal{C})$.

**Acknowledgements**

We thank Markus Kalisch and Thomas Richardson for their valuable comments.

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