ALTERNATIVE POLARIZATIONS OF BOREL FIXED IDEALS,
ELIAHOU-KERVAIRE TYPE RESOLUTION AND
DISCRETE MORSE THEORY

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Abstract. We construct an Eliahou-Kervaire-like minimal free resolution of the
alternative polarization $b$-$pol(I)$ of a Borel fixed ideal $I$. It yields new descriptions
of the minimal free resolutions of $I$ itself and $I^s$, where $(-)^s$ is the squarefree
operation in the shifting theory. These resolutions are cellular, and the (common)
supporting cell complex is given by discrete Morse theory. If $I$ is generated in
one degree, our description is equivalent to that of Nagel and Reiner.

1. Introduction

Let $S := k[x_1, \ldots, x_n]$ be a polynomial ring over a field $k$. For a monomial ideal
$I \subset S$, $G(I)$ denotes the set of minimal (monomial) generators of $I$. We say a
monomial ideal $I \subset S$ is Borel fixed (or strongly stable), if $m \in G(I)$, $x_i|m$ and
$j < i$ imply $(x_j/x_i) \cdot m \in I$. Borel fixed ideals are important, since they appear as
the generic initial ideals of homogeneous ideals (if char($k$) = 0).

As shown in [1], a squarefree analog of a Borel fixed ideal is also import ant
in combinatorial commutative algebra. We say a squarefree monomial ideal
$I$ is squarefree strongly stable, if $m \in G(I)$, $x_i|m$, $x_j \not|m$ and $j < i$ imply $(x_j/x_i) \cdot m \in I$.

Any monomial $m \in S$ with $\deg(m) = e$ has a unique expression

$$m = \prod_{i=1}^{e} x_{\alpha_i} \quad \text{with} \quad 1 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_e \leq n.$$  

Now we can consider the squarefree monomial

$$m^s = \prod_{i=1}^{e} x_{\alpha_i+i-1}$$

in the “larger” polynomial ring $T = k[x_1, \ldots, x_N]$ with $N \gg 0$. If $I \subset S$ is Borel
fixed, then

$$I^s := \{ m^s \mid m \in G(I) \} \subset T$$

is squarefree strongly stable. Moreover, for a Borel fixed ideal $I$ and all $i, j$, we have

$$\beta_{i,j}^S(I) = \beta_{i,j}^T(I^s).$$

See [1] for further information.

A minimal free resolution of a Borel fixed ideal $I$ has been constructed by Eliahou
and Kervaire [6]. While the minimal free resolution is unique up to isomorphism,
its “description” depends on the choice of a free basis, and further analysis of the
minimal free resolution is still an interesting problem. See, for example, [2, 8, 9,
10, 12. In this paper, we will give a new approach which is applicable to both $I$ and $I^{sa}$. Our main tool is the “alternative” polarization $b\text{-pol}(I)$ of $I$.

Let $\tilde{S} := k[x_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq d]$ be the polynomial ring, and set

$$\Theta := \{x_{i,1} - x_{i,j} \mid 1 \leq i \leq n, 2 \leq j \leq d\} \subset \tilde{S}.$$ 

Then there is an isomorphism $\tilde{S}/(\Theta) \cong S$ induced by $\tilde{S} \ni x_{i,j} \mapsto x_i \in S$. Throughout this paper, $\tilde{S}$ and $\Theta$ are used in this meaning.

Assume that $m \in G(I)$ has the expression (1.1). If $\deg(m) (= e) \leq d$, we set

$$b\text{-pol}(m) = \prod_{i=1}^{e} x_{a_{i,j}} \in \tilde{S}.$$ 

Note that $b\text{-pol}(m)$ is a squarefree monomial. If there is no danger of confusion, $b\text{-pol}(m)$ is denoted by $\tilde{m}$. If $m = \prod_{i=1}^{n} x_{i}^{a_{i}}$, then we have

$$\tilde{m} (= b\text{-pol}(m)) = \prod_{1 \leq i \leq n} x_{i,j} \in \tilde{S}, \text{ where } b_{i} := \sum_{i=1}^{n} a_{i}.$$ 

If $\deg(m) \leq d$ for all $m \in G(I)$, we set

$$b\text{-pol}(I) := \{b\text{-pol}(m) \mid m \in G(I)\} \subset \tilde{S}.$$ 

In [14], we have seen that if $I$ is Borel fixed, then $\tilde{I} := b\text{-pol}(I)$ is a polarization of $I$, that is, $\Theta$ forms an $\tilde{S}/\tilde{I}$-regular sequence with the natural isomorphism

$$\tilde{S}/(\tilde{I} + (\Theta)) \cong S/I.$$ 

(In the present paper, we give a new proof of this fact. See Corollary 4.1.) Note that the construction of $b\text{-pol}(\cdot)$ is different from the standard polarization. In fact, it does not give a polarization for a general monomial ideal.

Moreover,

$$\Theta' = \{x_{i,j} - x_{i+1,j-1} \mid 1 \leq i < n, 1 < j \leq d\} \subset \tilde{S}$$

forms an $\tilde{S}/\tilde{I}$-regular sequence too, and we have

$$\tilde{S}/(\tilde{I} + (\Theta')) \cong T/I^{sa}$$

through $\tilde{S} \ni x_{i,j} \mapsto x_{i+j-1} \in T$ (if we adjust the value of $N = \dim T$). The equation $\beta_{i,j}^a(I) = \beta_{i,j}^T(I^{sa})$ mentioned above easily follows from this observation.

In §2, we will construct a minimal $\tilde{S}$-free resolution $\tilde{P}_\bullet$ of $\tilde{S}/\tilde{I}$, which is analogous to the Eliahou-Kervaire resolution of $S/I$. However, their description can not be lifted to $\tilde{I}$, and we need modification. Roughly speaking, for $m \in G(I)$ and $i$ with $i < \max\{l \mid x_l \text{ divides } m\} =: \nu(m)$, we use the operation $m \mapsto (x_i/x_k) \cdot m$ (and take $b\text{-pol}(\cdot)$ of both sides), where $k = \min\{l > i \mid x_l \text{ divides } m\}$. Recall that Eliahou and Kervaire use the operation $m \mapsto (x_i/x_{\nu(m)}) \cdot m$. Clearly, $\tilde{P}_\bullet \otimes_{\tilde{S}} \tilde{S}/(\Theta)$ and $\tilde{P}_\bullet \otimes_{\tilde{S}} \tilde{S}/(\Theta')$ give the minimal free resolutions of $S/I$ and $T/I^{sa}$ respectively. We also remark that the method of Herzog and Takayama ([8, Theorem 1.12]) is not applicable to our case, while $\tilde{I}$ is a variant of the ideals treated there.
Under the assumption that a Borel fixed ideal $I$ is generated in one degree (i.e., all elements of $G(I)$ have the same degree), Nagel and Reiner [12] constructed the alternative polarization $\tilde{I} = b-pol(I)$ of $I$, and described a minimal $\tilde{S}$-free resolution of $\tilde{I}$ explicitly (and induced minimal free resolution of $I$ itself and $I^{\mathbb{N}}$). More precisely, they gave a polytopal CW complex supporting a minimal free resolution of $\tilde{I}$. In §4, we will show that their resolution is equivalent to our description. In this sense, our results are generalizations of those for [12].

Batzies and Welker ([2]) developed a strong theory which tries to construct minimal free resolutions of monomial ideals using Forman’s discrete Morse theory ([7]). If a monomial ideal $J$ is shellable in the sense of [2] (i.e., has linear quotients, in the sense of [8]), their method is applicable to $J$, and we can get a Batzies-Welker type minimal free resolution. Their resolution depends on the choice of an acyclic matching on $\mathcal{P}^{G(J)}$, and the matchings are far from unique in general. For most matchings, it is almost impossible to compute the differential map explicitly.

A Borel fixed ideal $I$ and the polarization $\tilde{I} = b-pol(I)$ are shellable. In §5, we will show that our resolution $\tilde{P}_* \to \tilde{S}/\tilde{I}$ and the induced resolutions of $S/I$ and $T/I^{\mathbb{N}}$ are Batzies-Welker type. In particular, these resolutions are cellular. As far as the authors know, an explicit description of a Batzies-Welker type resolution of a general Borel fixed ideal has never been obtained before.

Theoretically, §3, in which we show directly that $\tilde{P}_*$ is a resolution, is unnecessary, and §5 is enough. However the technique developed in [2] is quite different from the usual one in this area. If we give only the proof based on [2], the paper might become unreadable to novice readers. Therefore we keep §3.

If $I$ is generated in one degree, the CW complex supporting our resolution $\tilde{P}_*$ is regular. In fact, it coincides with the CW complex of Nagel and Reiner. We strongly believe that our CW complex is regular in general, but there is no way to prove it now.

2. The Eliahou-Kervaire type resolution of $\tilde{S}/b-pol(I)$

Throughout the rest of the paper, $I$ is a Borel fixed monomial ideal with $\deg m \leq d$ for all $m \in G(I)$. For the definitions of the alternative polarization $b-pol(I)$ of $I$ and related concepts, consult the previous section. For a monomial $m = \prod_{i=1}^{n} x_i^{a_i} \in S$, set $\mu(m) := \min \{ i \mid a_i > 0 \}$ and $\nu(m) := \max \{ i \mid a_i > 0 \}$. In [6], it is shown that any monomial $m \in I$ has a unique expression $m = m_1 \cdot m_2$ with $\nu(m_1) \leq \mu(m_2)$ and $m_1 \in G(I)$. Following [6], we set $g(m) := m_1$.

For a monomial $m \in S$ and $i$ with $i < \nu(m)$, set

$$b_i(m) = (x_i/x_k) \cdot m,$$

where $k := \min \{ j \mid a_j > 0, j > i \}$.

Since $I$ is Borel fixed, $m \in I$ implies $b_i(m) \in I$.

**Definition 2.1.** For a finite subset $\mathcal{F} = \{ (i_1, j_1), (i_2, j_2), \ldots, (i_q, j_q) \}$ of $\mathbb{N} \times \mathbb{N}$ and a monomial $m = \prod_{i=1}^{r} x_{a_i} = \prod_{i=1}^{n} x_i^{a_i} \in G(I)$ with $1 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_r \leq n$, we say the pair $(\mathcal{F}, \bar{m})$ is admissible (for $b-pol(I)$), if the following conditions are satisfied:

(a) $1 \leq i_1 < i_2 < \cdots < i_q < \nu(m)$,

(b) $j_r = \max \{ i \mid \alpha_i \leq i_r \} + 1$ (equivalently, $j_r = 1 + \sum_{l=1}^{i_r} a_l$) for all $r$. 


For \( m \in G(I) \), the pair \((0, \widetilde{m})\) is also admissible.

The following lemma is easy, and we omit the proof.

**Lemma 2.2.** Let \((\widetilde{F}, \widetilde{m})\) be an admissible pair with \( \widetilde{F} = \{(i_1, j_1), \ldots, (i_q, j_q)\} \) and \( m = \prod x_i^{a_i} \in G(I) \). Then we have the following.

1. \( j_1 \leq j_2 \leq \cdots \leq j_q \).
2. \( x_{k,j} \cdot b_{i,j} (m) = x_{i,j} \cdot b_{i,j} (m) \), where \( k = \min \{ l | l > i_1, a_l > 0 \} \).

For \( m \in G(I) \) and an integer \( i \) with \( 1 \leq i < \nu(m) \), set \( m_{(i)} := g(b_i(m)) \) and \( \widetilde{m}_{(i)} := b_{i,j} (m_{(i)}) \). If \( i \geq \nu(m) \), we set \( m_{(i)} := m \) for the convenience.

In the situation of Lemma 2.2, \( \widetilde{m}_{(i)} \) divides \( x_{i,j} \cdot \widetilde{m} \) for all \( 1 \leq r \leq q \).

**Lemma 2.3.** With the above notation, \( m_{(i)} \) and \( b_i(m) \) have the same exponents in the variables \( x_k \) with \( k \leq i \). In particular, \( \nu(m_{(i)}) \geq i \).

**Proof.** If the assertion does not hold, then \( m_{(i)} \) divides \( m \), and it contradicts the assumption that \( m \in G(I) \).

For \( \widetilde{F} = \{(i_1, j_1), \ldots, (i_q, j_q)\} \) and \( r \) with \( 1 \leq r \leq q \), set
\[
\widetilde{F}_r := \widetilde{F} \setminus \{(i_r, j_r)\}.
\]

If \((\widetilde{F}, m)\) is an admissible pair for \( b_{i,j} (I) \), we set
\[
B(\widetilde{F}, \widetilde{m}) := \{ r | (\widetilde{F}_r, \widetilde{m}_{(i_r)}) \text{ is admissible} \}.
\]

**Example 2.4.** Let \( I \subset S = \mathbb{k}[x_1, x_2, x_3, x_4] \) be the smallest Borel fixed ideal containing \( m = (x_1)^2 x_3 x_4 \). In this case, \( m_{(i)} = g(b_i(m')) \) for all \( m' \in G(I) \). Hence, we have \( m_{(1)} = (x_1)^3 x_4 \), \( m_{(2)} = (x_1)^2 x_2 x_4 \) and \( m_{(3)} = (x_1)^2 (x_3)^2 \).

Clearly,
\[
(\widetilde{F}, \widetilde{m}) = \{(1, 3), (2, 3), (3, 4)\}, x_{1,1} x_{1,2} x_{3,3} x_{4,4}
\]
is admissible. (For this \( \widetilde{F} \), \( i_r = r \) holds and the reader should be careful.) Now
\[
(\widetilde{F}_2, \widetilde{m}_{(2)}) = \{(1, 3), (3, 4)\}, x_{1,1} x_{1,2} x_{2,3} x_{4,4}
\]
and
\[
(\widetilde{F}_3, \widetilde{m}_{(3)}) = \{(1, 3), (2, 3)\}, x_{1,1} x_{1,2} x_{3,3} x_{4,4}
\]
are admissible. However, \((\widetilde{F}_1, \widetilde{m}_{(1)}) = \{(2, 3), (3, 4)\}, x_{1,1} x_{1,2} x_{1,3} x_{4,4}\) does not satisfy the condition (b) of Definition 2.1. Hence \( B(\widetilde{F}, \widetilde{m}) = \{2, 3\} \).

Next let \( I' \) be the smallest Borel fixed ideal containing \( m = (x_1)^2 x_3 x_4 \) and \( (x_1)^2 x_2 \).

For \( \tilde{F} = \{(1, 3), (2, 3), (3, 4)\} \), \( (\tilde{F}, \tilde{m}) \) is admissible again. However \( \tilde{m}_{(2)} = (x_1)^2 x_2 \) in this time, and \( (\tilde{F}_2, \tilde{m}_{(2)}) = \{(1, 3), (3, 4)\}, x_{1,1} x_{1,2} x_{2,3}\) is no longer admissible.

Hence \( B(\tilde{F}, \tilde{m}) = \{3\} \) for \( b_{i,j} (I') \).

**Lemma 2.5.** Let \((\tilde{F}, \tilde{m})\) be as in Lemma 2.2.

1. For all \( r \) with \( 1 \leq r \leq q \), \((\tilde{F}_r, \tilde{m})\) is admissible.
2. We always have \( q \in B(\tilde{F}, \tilde{m}) \).
3. Assume that \((\tilde{F}_r, \tilde{m}_{(i_r)})\) satisfies the condition (a) of Definition 2.1. Then \( r \in B(\tilde{F}, \tilde{m}) \) if and only if either \( j_r < j_{r+1} \) or \( r = q \).
(iv) For \( r, s \) with \( 1 \leq r < s \leq q \) and \( j_r < j_s \), we have \( b_{i_r}(b_{i_s}(m)) = b_{i_s}(b_{i_r}(m)) \)
and hence \((\tilde{m}_{(i_r)})_{(i_s)} = (\tilde{m}_{(i_s)})_{(i_r)}\).

(v) For \( r, s \) with \( 1 \leq r < s \leq q \) and \( j_r = j_s \), we have \( b_{i_r}(m) = b_{i_s}(b_{i_r}(m)) \)
and hence \( \tilde{m}_{(i_r)} = (\tilde{m}_{(i_s)})_{(i_r)} \).

Proof. (i) Clear.

(ii) It suffices to show that \((\tilde{F}_{q}, \tilde{m}_{(i_q)})\) satisfies the condition (a) of Definition 2.1.
However, this is clear since \( \nu(m_{(i_q)}) \geq i_q > i_{q-1} \) by Lemma 2.3.

(iii) By (ii), we may assume that \( r < q \). It suffices to consider the condition (b) of Definition 2.1.
Set \( k := \min\{ l \mid l > i_r, a_l > 0 \} \) and \( \prod x_l^{b_l} := b_{i_r}(m) = (x_{i_r}/x_k) \cdot m \).
Hence the exponents of the variables \( x_l \) in \( m_{(i_r)} \) equals \( b_l \) for all \( l \leq i_q \). Clearly, \( j_r = j_{r+1} \), if and only if \( a_l = 0 \) for all \( l \) with \( i_r < l < i_{r+1} \), if and only if \( k > i_{r+1} \).
Hence \( j_r = j_{r+1} \) implies \( \sum_{l \leq i_r} b_l = 1 + \sum_{l \leq i_r} a_l = j_r = j_{r+1} \), and \((\tilde{F}_{r}, \tilde{m}_{(i_r)})\) does not satisfy the condition (b). Next we assume that \( j_r < j_{r+1} \). Then we have \( \sum_{l \leq i_r} b_l = \sum_{l \leq i_r} a_l = j_s - 1 \) for all \( s \leq q \) with \( s \neq r \), and \((\tilde{F}_{r}, \tilde{m}_{(i_r)})\) satisfies (b).

(iv) Note that if \( j_r < j_s \) then \( a_l = 0 \) for some \( l \) with \( i_r < l < i_s \). Assume that \( l \) is the smallest among these integers, and set \( k := \min\{ m \mid m > i_s, a_m > 0 \} \). Then both \( b_{i_r}(b_{i_s}(m)) \) and \( b_{i_s}(b_{i_r}(m)) \) are equal to \( (x_{i_r}/x_k) \cdot m \).

(v) Since \( a_l = 0 \) for all \( l \) with \( i_r < l \leq i_s \) by the assumption, we have
\[
b_{i_r}(b_{i_s}(m)) = b_{i_r}(x_{i_r}^{1} / x_k) \cdot m = (x_{i_r}^{1}/x_k) \cdot (x_{i_s} / x_k) \cdot m = (x_{i_r} / x_k) \cdot m = b_{i_r}(m),
\]
where \( k = \min\{ l \mid l > i_r, a_l > 0 \} \) (note that \( i_s < k \) now).

For \( F = \{ i_1, \ldots, i_q \} \subset \mathbb{N} \) with \( i_1 < i_2 < \cdots < i_q \) and \( m \in I \), Eliahou-Kervaire [6]
call the pair \((F, m)\) admissible for \( I \), if \( i_q < \nu(m) \). Clearly, there is a unique sequence \( j_1, \ldots, j_q \) such that \((\tilde{F}, \tilde{m})\) is admissible for \( \tilde{I} \), where \( \tilde{F} = \{ (i_1, j_q), \ldots, (i_q, j_q) \} \). In this way, there is a one-to-one correspondence between the admissible pairs for \( I \) and those of \( \tilde{I} \). As the free summands of the Eliahou-Kervaire resolution of \( I \) are indexed by the admissible pairs for \( I \), the free summands of our resolution of \( \tilde{I} \) are indexed by the admissible pairs for \( \tilde{I} \).

We will define a \( \mathbb{Z}^{n \times d} \)-graded chain complex \( \tilde{P}_\bullet \) of free \( \tilde{S} \)-modules as follows. First, set \( \tilde{P}_0 := \tilde{S} \). For each \( q \geq 1 \), we set
\[
A_q := \text{the set of admissible pairs } (\tilde{F}, \tilde{m}) \text{ for } b \text{-pol}(I) \text{ with } \# \tilde{F} = q,
\]
and
\[
\tilde{P}_q := \bigoplus_{(\tilde{F}, \tilde{m}) \in A_q-1} \tilde{S} e(\tilde{F}, \tilde{m}),
\]
where \( e(\tilde{F}, \tilde{m}) \) is a basis element with
\[
\deg \left( e(\tilde{F}, \tilde{m}) \right) = \deg \left( \prod_{(i_r, j_r) \in \tilde{F}} x_{i_r, j_r} \right) \in \mathbb{Z}^{n \times d}.
\]
We define the \( \widetilde{S} \)-homomorphism \( \partial : \widetilde{P}_q \rightarrow \widetilde{P}_{q-1} \) for \( q \geq 2 \) so that \( e(\widetilde{F}, \widetilde{m}) \) with \( \widetilde{F} = \{(i_1, j_1), \ldots, (i_q, j_q)\} \) is sent to
\[
\sum_{1 \leq r \leq q} (-1)^r \cdot x_{i_r, j_r} \cdot e(\widetilde{F}_{r}, \widetilde{m}) - \sum_{r \in \mathcal{B}(\widetilde{F}, \widetilde{m})} (-1)^r \cdot \frac{x_{i_r, j_r}}{m_{(i_r)}} \cdot e(\widetilde{F}_r, \widetilde{m}_{(i_r)}),
\]
and \( \partial : \widetilde{P}_1 \rightarrow \widetilde{P}_0 \) by \( e(\emptyset, \widetilde{m}) \mapsto \widetilde{m} \in \widetilde{S} = \widetilde{P}_0 \). Clearly, \( \partial \) is a \( \mathbb{Z}^{n \times d} \)-graded homomorphism.

Set
\[
\widetilde{P}_r : \cdots \rightarrow \widetilde{P}_i \rightarrow \cdots \rightarrow \widetilde{P}_1 \rightarrow \widetilde{P}_0 \rightarrow 0.
\]
Then we have the following.

**Theorem 2.6.** The complex \( \widetilde{P}_* \) is a \( \mathbb{Z}^{n \times d} \)-graded minimal \( \widetilde{S} \)-free resolution for \( \widetilde{S}/ \text{b-pol}(I) \).

3. THE PROOF OF THEOREM 2.6

This section is devoted to the proof of Theorem 2.6, but it is not easy to see that \( \widetilde{P}_* \) is even a chain complex.

**Proposition 3.1.** With the above notation, we have \( \partial \circ \partial = 0 \), that is, \( \widetilde{P}_* \) is a chain complex.

**Proof.** It suffices to prove \( \partial \circ \partial (e(\widetilde{F}, \widetilde{m})) = 0 \) for each admissible pair \( (\widetilde{F}, \widetilde{m}) \). If \( \# \widetilde{F} = 1 \) (i.e., \( \widetilde{F} = \{(i, j)\} \)), then the assertion is easy. In fact,
\[
\partial \circ \partial (e(\{(i, j)\}, \widetilde{m})) = \partial \left( -x_{i,j} \cdot e(\emptyset, \widetilde{m}) + \frac{x_{i,j}}{m_{(i)}} \cdot e(\emptyset, \widetilde{m}_{(i)}) \right)
= -x_{i,j} \cdot \widetilde{m} + \frac{x_{i,j} \cdot \widetilde{m}}{m_{(i)}} \cdot \widetilde{m}_{(i)}
= 0.
\]

So we may assume that \( q := \# \widetilde{F} > 1 \). For \( r, s \) with \( 0 \leq r, s \leq q \) and \( r \neq s \), set \( \widetilde{F}_{r,s} := \widetilde{F} \setminus \{i_r, j_r\} \cup \{i_s, j_s\} \). We have a unique expression
\[
\partial \circ \partial (e(\widetilde{F}, \widetilde{m})) = C_{r,s} + C,
\]
where \( C_{r,s} \) (resp. \( C \)) is an \( \widetilde{S} \)-linear combination of the basis elements of the form \( e(\widetilde{F}_{r,s}, -) \) (resp. \( e(\widetilde{F}_{t,u}, -) \) with \( \{t, u\} \neq \{r, s\} \)). To show \( \partial \circ \partial (e(\widetilde{F}, \widetilde{m})) = 0 \), it suffices to check that \( C_{r,s} = 0 \) for each \( r, s \). Set
\[
\partial (e(\widetilde{F}, \widetilde{m})) = C_r + C_s + C',
\]
where \( C_r \) (resp. \( C_s \)) is an \( \widetilde{S} \)-linear combination of the basis elements of the form \( e(\widetilde{F}_r, -) \) (resp. \( e(\widetilde{F}_s, -) \)), and \( C' \) is an \( \widetilde{S} \)-linear combination of \( e(\widetilde{F}_t, -) \) with \( t \neq r, s \). Then it is easy to see that \( C_{r,s} \) is the “\( \widetilde{F}_{r,s} \)-part” of \( \partial (C_r + C_s) \).

We will show \( C_{r,s} = 0 \) dividing the arguments into several cases.

First, consider the case \( r, s \in B(\widetilde{F}, \widetilde{m}) \). Then it is clear that \( r \in B(\widetilde{F}_s, \widetilde{m}) \) and \( s \in B(\widetilde{F}_r, \widetilde{m}) \). Since \( r, s \in B(\widetilde{F}, \widetilde{m}) \), we have \( j_r \neq j_s \) by Lemma 2.5 (iii), and hence
by Lemma 2.5 (iv). Therefore $r \in B(\tilde{F}_s, \tilde{m}_{(i_r)})$ if and only if $s \in B(\tilde{F}_r, \tilde{m}_{(i_r)})$. We have

$$\partial(e(\tilde{F}, \tilde{m})) = (-1)^r X \cdot e(\tilde{F}_r, \tilde{m}) - (-1)^r X \cdot e(\tilde{F}_r, \tilde{m}_{(i_r)})$$

$$+ (-1)^s X \cdot e(\tilde{F}_s, \tilde{m}) - (-1)^s X \cdot e(\tilde{F}_s, \tilde{m}_{(i_r)}) + C',$$

where $C'$ is an $\tilde{S}$-linear combination of the basis elements $e(\tilde{F}_t, -)$ with $t \neq r, s$, and $X$'s are certain monomials in $\tilde{S}$. Of course, each $X$ are not the same. Since $\partial$ is $\mathbb{Z}^{n \times d}$-graded, the explicit form of $X$ is not important. Anyway, we will use $X$ in this meaning in the rest of the proof. Without loss of generality, we may assume that $r < s$ in the following computation.

Assume that $r \in B(\tilde{F}_s, \tilde{m}_{(i_r)})$ (equivalently, $s \in B(\tilde{F}_r, \tilde{m}_{(i_r)})$). Then

$$\partial \circ \partial(e(\tilde{F}, \tilde{m})) = (-1)^{r+s} X \cdot e(\tilde{F}_{r,s}, \tilde{m}) + (-1)^{r+s} X \cdot e(\tilde{F}_{r,s}, \tilde{m}_{(i_r)})$$

$$+ (-1)^{r+s} X \cdot e(\tilde{F}_{r,s}, \tilde{m}_{(i_r)}) - (-1)^{r+s} X \cdot e(\tilde{F}_{r,s}, \tilde{m}_{(i_r)})$$

$$+ (-1)^{r+s} X \cdot e(\tilde{F}_{r,s}, \tilde{m}_{(i_r)}) - (-1)^{r+s} X \cdot e(\tilde{F}_{r,s}, \tilde{m}_{(i_r)}) + (-1)^{r+s} X \cdot e(\tilde{F}_{r,s}, \tilde{m}_{(i_r)}) + C$$

where $C$ is an $\tilde{S}$-linear combination of the basis elements $e(\tilde{F}_t, -)$ with $t \neq r, s$. Since $(\tilde{m}_{(i_r)})_{(i_r)} = (\tilde{m}_{(i_r)})_{(i_r)}$, now, $\partial \circ \partial(e(\tilde{F}, \tilde{m})) = C = 0$.

If $r \not\in B(\tilde{F}_s, \tilde{m}_{(i_r)})$ (equivalently, $s \not\in B(\tilde{F}_r, \tilde{m}_{(i_r)})$), then both

$$-(-1)^{r+s} X \cdot e(\tilde{F}_{r,s}, (\tilde{m}_{(i_r)}))_{(i_r)}$$

and

$$-(-1)^{r+s} X \cdot e(\tilde{F}_{r,s}, (\tilde{m}_{(i_r)}))_{(i_r)}$$

do not appear in the above expansion of $\partial \circ \partial(e(\tilde{F}, \tilde{m}))$. Hence $\partial \circ \partial(e(\tilde{F}, \tilde{m})) - C = 0$ remains true.

**Lemma 3.2.** If $r, s \not\in B(\tilde{F}, \tilde{m})$, then we have $r \not\in B(\tilde{F}_s, \tilde{m})$ and $s \not\in B(\tilde{F}_r, \tilde{m})$.

**Proof.** It suffices to prove $r \not\in B(\tilde{F}_s, \tilde{m})$. For the contrary, assume that $(\tilde{F}_{r,s}, \tilde{m}_{(i_r)})$ is admissible, in particular, it satisfies the condition (a) of Definition 2.1. Since $s \neq q$ by Lemma 2.5 (ii), $(\tilde{F}_r, \tilde{m}_{(i_r)})$ also satisfies the condition (a). Since $r \not\in B(\tilde{F}, \tilde{m})$ now, we have $j_r = j_{r+1}$ by Lemma 2.5 (iii). Hence the assumption $r \in B(\tilde{F}_s, \tilde{m})$ implies that $s = r + 1$ and $(j_r =) j_{r+1} < j_{r+2}$. If $(\tilde{F}_s, \tilde{m}_{(i_r)})$ satisfies the condition (a) of Definition 2.1, then we have $s \in B(\tilde{F}, \tilde{m})$ and it contradicts the assumption. Hence $(\tilde{F}_s, \tilde{m}_{(i_r)})$ does not satisfy (a), that is, $\nu(\tilde{m}_{(i_r)}) \leq i_q$. Since $\tilde{m}_{(i_r)} = (m_{(i_r)})_{(i_r)}$ by Lemma 2.5 (v), we have $\nu(m_{(i_r)}) \leq \nu(\tilde{m}_{(i_r)}) \leq i_q$ and $(\tilde{F}_{r,s}, \tilde{m}_{(i_r)})$ does not satisfy (a). This is a contradiction. \[\square\]

Hence, when $r, s \not\in B(\tilde{F}, \tilde{m})$, $\partial \circ \partial(e(\tilde{F}, \tilde{m})) - C = 0$ is easy. So it remains to prove the case where $r \not\in B(\tilde{F}, \tilde{m})$ but $s \in B(\tilde{F}, \tilde{m})$.

**Lemma 3.3.** Assume that $r \not\in B(\tilde{F}, \tilde{m})$ and $s \in B(\tilde{F}, \tilde{m})$. Then we always have $s \in B(\tilde{F}_r, \tilde{m})$. Moreover, $r \in B(\tilde{F}_s, \tilde{m}_{(i_r)})$ if and only if $r \in B(\tilde{F}_s, \tilde{m})$. If this is the case, we have $m_{(i_r)} = (\tilde{m}_{(i_r)})_{(i_r)}$.
Proof. The first assertion is clear, and we will show the second and the third (at the same time). Since \( \nu(v(m_{(i_1)})) \leq \nu(v(m_{(i_2)})), r \in B(F_r, \tilde{m}), \) \( r \in B(F_r, \tilde{m}) \) implies \( r \in B(F_r, \tilde{m}) \).
So, assuming that \( r \in B(F_r, \tilde{m}) \), we will show that \( m_{(i_1)} = (m_{(i_2)}), \) which implies \( r \in B(F_r, \tilde{m}) \). Since \( r \not\in B(F_r, \tilde{m}), \) \( (F_r, \tilde{m}), \) does not satisfy the condition (a) or (b) of Definition 2.1.

Case 1. Assume that \( (F_r, \tilde{m}_{(i_1)}) \) fails (a), that is, \( \nu(v(m_{(i_1)})) \leq i_q. \) Since \( r \in B(F_r, \tilde{m}) \) now, we have \( s = q \) and \( (i_r) \in (i_q, \nu(S)) \). Since \( i_r < \nu(v(m_{(i_1)})) \), \( j_r < j_q, \) and \( b_{i_r}(b_{i_q}(m)) = b_{i_q}(b_{i_r}(m)). \) Therefore, \( (\tilde{m}(i_1)) = (m_{(i_1)}), \) where the first equality follows from Lemma 2.5 (v), and the second follows from \( \nu(v(m_{(i_1)})) \leq i_q. \)

Case 2. Assume that \( (F_r, \tilde{m}_{(i_1)}) \) fails (b) of Definition 2.1. By Lemma 2.5 (iii), we have \( s = r + 1 \) and \( j_r = j_s. \) Hence \( m_{(i_1)} = (m_{(i_1)}), \) by Lemma 2.5 (v).

The continuation of the proof of Proposition 3.1. Finally, consider the case \( r \not\in B(F_r, \tilde{m}) \) and \( s \in B(F_r, \tilde{m}). \) Then we have
\[
\partial(e(F_r, \tilde{m})) = (-1)^s X \cdot e(F_r, \tilde{m}) + (-1)^s X \cdot e(F_r, \tilde{m}) - (-1)^s X \cdot e(F_r, \tilde{m}) + C',
\]
where \( C' \) is an \( S \)-linear combination of the elements \( e(F_r, -) \) with \( t \not= r, s. \)

Assume that \( r \in B(F_r, \tilde{m}) \) (equivalently, \( r \in B(F_r, \tilde{m}) \)). Then we have \( r < s \) as shown in the proof of Lemma 3.3, and
\[
\partial \circ \partial(e(F_r, \tilde{m})) = (-1)^{r+s} X \cdot e(F_r, \tilde{m}) + (-1)^{r+s} X \cdot e(F_r, \tilde{m}) + (-1)^{r+s} X \cdot e(F_r, \tilde{m}) + C'
\]
where \( C \) is an \( S \)-linear combination of the basis elements \( e(F_r, -) \) with \( \{t, u\} \not= \{r, s\}. \) Since \( m_{(i_r)} = (m_{(i_1)}), \) by Lemma 3.3, we have \( \partial \circ \partial(e(F_r, \tilde{m})) - C = 0. \)

Next, assume that \( r \not\in B(F_r, \tilde{m}), \) (equivalently, \( r \not\in B(F_r, \tilde{m}) \)). Then both
\[
(-1)^{r+s}X e(F_r, \tilde{m}_{(i_1)}) \quad \text{and} \quad (-1)^{r+s}X e(F_r, \tilde{m}_{(i_1)})(i_1)
\]
do not appear in the above expansion of \( \partial \circ \partial(e(F_r, \tilde{m})). \) Hence \( \partial \circ \partial(e(F_r, \tilde{m})) = C = 0 \) remains true.

Next, we will show that \( \tilde{P} \) is acyclic. To do so, we need some preparation.

Let \( \triangleright \) be the lexicographic order on the monomials of \( S \) with \( x_1 \triangleright x_2 \triangleright \cdots \triangleright x_n. \)
In the rest of this section, \( \triangleright \) means this order. In the sequel, the minimal monomial generators of the Borel fixed ideal \( I \) are denoted by \( m_1, \ldots, m_t \) with \( m_1 \triangleright m_2 \triangleright \cdots \triangleright m_t. \)

Let \( \tilde{I} \) denote the initial ideal generated by \( m_1, \ldots, m_r. \)

**Lemma 3.4.** Each \( \tilde{I} \) is also Borel fixed.

**Proof.** We use induction on \( r. \) It suffices to show that for \( i, j \) with \( 1 \leq i < j, x_j \mid m_r \) implies \( (x_i/x_j)m_r \in \tilde{I}. \) Set \( m'_r := (x_i/x_j)m_r. \) Since \( I \) is Borel fixed, \( m'_r \in I. \) By the similar argument as in the proof of Lemma 2.3, we see that for \( k \leq i \) the exponent of \( x_k \) in \( g(m'_r) \) coincides with that in \( m'_r. \) Thus the exponent of \( x_k \) in \( g(m'_r) \) is equal to that in \( m_r \) if \( k < i \) and is more than if \( k = i. \) This implies that \( g(m'_r) \triangleright m_r \) and hence \( g(m'_r) \in \tilde{I}. \) Therefore \( m'_r \in \tilde{I}. \)

\( \square \)
For simplicity, we set $\tilde{I}_r := b-pol(I_r) (\subset \tilde{S})$.

**Lemma 3.5.** Let $m_r = \prod_{k=1}^{n} x_k^{a_k}$ and set $b_i := 1 + \sum_{k=1}^{i} a_k$. Then it follows that $(\tilde{I}_{r-1} : \tilde{m}_r) = (x_{l_1 b_i}, \ldots, x_{\lambda b_i})$, where $\lambda := \nu(m_r) - 1$.

**Proof.** Since $\tilde{m}_r$ is a monomial and $\tilde{I}_{r-1}$ is a monomial ideal, we have

$$(\tilde{I}_{r-1} : \tilde{m}_r) = \left( \frac{\text{lcm}(\tilde{m}_1, m_r), \ldots, \text{lcm}(\tilde{m}_{r-1}, m_r)}{m_r} \right).$$

Let $s$ be an integer with $1 \leq s \leq r - 1$. Since $m_s > m_r$, there exists an integer $l$ such that the exponent of $x_k$ in $m_s$ is equal to that in $m_r$ for $k < l$ and is more than if $k = l$. Thus $x_{l b_i} | \tilde{m}_s$ and $x_{l b_i} \notin \tilde{m}_r$. Hence $x_{l b_i} | (\text{lcm}(\tilde{m}_s : \tilde{m}_r)/m_r)$. Since $m_s, m_r \in G(I)$, $l$ must be less than $\nu(m_r)$; otherwise $m_r | m_s$, which is a contradiction. Hence it follows that $\text{lcm}(\tilde{m}_s, \tilde{m}_r)/m_r \in (x_{l b_i}, \ldots, x_{\lambda b_i})$, and therefore $(\tilde{I}_{r-1} : \tilde{m}_r) \subset (x_{l b_i}, \ldots, x_{\lambda b_i})$.

Now let us consider the inverse inclusion. Let $s$ be the integer with $1 \leq s \leq \lambda$. We can take $k := \min \{ j | a_j > 0, j > s \}$. By Lemma 2.2 (ii), we have $(\tilde{m}_r)_{(s)} | x_{s b_i} \tilde{m}_r$. On the other hand, Lemma 2.3 implies $(m_r)_{(s)} \succ m_r$, and hence $(\tilde{m}_r)_{(s)} \subset \tilde{I}_{r-1}$. It follows that $x_{s b_i} \tilde{m}_r \in \tilde{I}_{r-1}$, or equivalently $x_{s b_i} \in (\tilde{I}_{r-1} : \tilde{m}_r)$. Therefore we conclude that $(\tilde{I}_{r-1} : m_r) \supseteq (x_{l b_i}, \ldots, x_{\lambda b_i})$.

The differential $\partial$ of the complex $\tilde{P}_r$, constructed in the above, consists of the following 2 maps. Let $\delta, \delta' : \tilde{P}_r \to \tilde{P}_q$ be the family of $\tilde{S}$-homomorphism $\delta_q, \delta'_q : \tilde{P}_q \to \tilde{P}_{q-1}$ which sends $e(\tilde{F}, \tilde{m})$ with $\tilde{F} = \{(i_1, j_1), \ldots, (i_q, j_q)\}$ and $q \geq 2$ to

$$\sum_{1 \leq r \leq q} (-1)^r \cdot x_{i_r, j_r} \cdot e(\tilde{F}_r, \tilde{m}),$$

$$\sum_{r \in B(\tilde{F}, \tilde{m})} (-1)^r \cdot \frac{x_{i_r, j_r} \cdot \tilde{m}}{\tilde{m}(i_r)} \cdot e(\tilde{F}_r, \tilde{m}(i_r)),$$

respectively. In the case $q = 1$, we set $\delta(e(\emptyset, \tilde{m})) = 0$ and $\delta'(e(\emptyset, \tilde{m})) = -\tilde{m}$. Then it follows that $\partial = \delta - \delta'$.

To show that $\tilde{P}_r$ is acyclic, we make use of the technique by means of mapping cones as the one by Peeva and Stillman in [13]. Let $(U_\bullet, \partial^U), (V_\bullet, \partial^V)$ be a complex and $f : U_\bullet \to V_\bullet$ be a homomorphism of complexes. The mapping cone $\text{Con}_p(f)$ of $f$ is a complex such that $\text{Con}_q(f) := V_q \oplus U_{q-1}$ for all $q$ and the differential $\partial^\text{Con}(f)$ are defined by $\partial^\text{Con}(f)(v, u) = (\partial^V(v) + f(u), -\partial^U(u))$ for $(v, u) \in \text{Con}_q(f)$. It is noteworthy that $V_\bullet$ is then a subcomplex of $\text{Con}_p(f)$.

**The proof of Theorem 2.6.** We use induction on $t = \sharp G(I)$. The case $t = 1$ is trivial. Assume $t > 1$. Let $\tilde{I}, \tilde{I}_r, \tilde{m}_r$, where $1 \leq r \leq \sharp G(I)$, be as above. There is the following exact sequence

$$0 \to \left( \tilde{S}/(\tilde{I}_{r-1} : \tilde{m}_r) \right) (-\tilde{a}) \tilde{m}_r \tilde{S}/\tilde{I}_{r-1} \to \tilde{S}/\tilde{I} \to 0,$$

where $\tilde{a}$ denotes the multi-degree of $\tilde{m}_r$. Set $\lambda := \nu(m_r) - 1$, $m_r = \prod_{k=1}^{n} x_k^{a_k}$, and $b_i := 1 + \sum_{k=1}^{i} a_k$ for $i$ with $1 \leq i \leq n$. It follows from Lemma 3.5 that
of $x_{i_1,j_1}, \ldots, x_{i_q,j_q}$ of $K_q$ of complexes which is a lifting of the map $\tilde{x}$ of $\tilde{\lambda}$ that in $H$ is injective. It is clear that $-\tilde{x}$ induced by the one is minimal, and is acyclic since there exists a long exact sequence of homologies $\tilde{x}$ to $\tilde{\lambda}$ is a minimal free resolution $\tilde{x}$ to $\tilde{\lambda}$ is a complex, we have $\partial^2 = 0$, and hence $\delta^2 - \delta^2 = 0$. Therefore we have $-\delta + \delta^2 = 0$ over $K_\bullet$, as desired.

Let $\text{Con}_\bullet(-\delta')$ be the mapping cone of $-\delta'$. It coincides with $\tilde{P}$, regarding $\tilde{P}$ as a subcomplex of $\tilde{P}$, and each $K_q$ as a direct summand of $\tilde{P}$. Moreover $\text{Con}_\bullet(-\delta')$ is minimal, and is acyclic since there exists a long exact sequence of homologies

$\cdots \rightarrow H_q(\tilde{Q}) \rightarrow H_q(\text{Con}_\bullet(-\delta')) \rightarrow H_{q-1}(\tilde{K}) \xrightarrow{\tilde{m}} H_{q-1}(\tilde{Q}) \rightarrow \cdots$,

induced by the one

$0 \rightarrow \tilde{Q} \rightarrow \text{Con}_\bullet(-\delta') \rightarrow \tilde{K}_\bullet[1] \rightarrow 0$,

and since the map

$H_0(\tilde{K}) = (\tilde{S}/(\tilde{I} : \tilde{m})) \xrightarrow{(-\tilde{a})} \tilde{S}/\tilde{I}_{t-1} = H_0(\tilde{Q})$

is injective. It is clear that $H_0(\text{Con}_\bullet(-\delta')) \cong \tilde{S}/\tilde{I}_t$. Thus $\tilde{P}$ is a minimal $\mathbb{Z}^{n \times d}$ graded free resolution of $\tilde{S}/b \cdot \text{pol}(I)$.

\begin{remark}
In their paper [8], Herzog and Takayama explicitly gave a minimal free resolution of a monomial ideal with linear quotients admitting a regular decomposition function (see [8] for the definitions). A Borel fixed ideal $I$ is a typical example with this property. However, while our $\tilde{I}$ has linear quotients, the decomposition function $\tilde{g}$ can not be regular in general. For example, if $I = (x^2, y, x^2 y, x^3 y, y^3)$, then $\tilde{I} = (x_1 x_2 x_3, x_1 x_2 y_3, x_1 y_2 y_3, y_1 y_2 y_3)$ has linear quotients in several total orders. Checking each case, we see that $\tilde{g}$ is not regular for every order. Here we check this in two typical cases. First, consider the order $x_1 x_2 x_3 > x_1 x_2 y_3 > x_1 y_2 y_3 > y_1 y_2 y_3$ which gives linear quotients. We have $(x_1 x_2 x_3, x_1 x_2 y_3) : (x_1 y_2 y_3) = (x_2) = (1)$. It means that $\tilde{g}$ is not regular in this case. Next, consider the order $x_1 x_2 y_3 > x_1 x_2 x_3 > x_1 y_2 y_3 > y_1 y_2 y_3$ which also gives linear quotients. $(x_1 x_2 x_3, x_1 x_2 y_3, x_1 y_2 y_3) : (y_1 y_2 y_3) = (x_1) = (1)$. Hence $\tilde{g}$ is not regular again.
\end{remark}
Anyway, we can not directly apply [8] to our case.

4. Applications and Remarks

Let \( I \subset S \) be a Borel fixed ideal, and \( \Theta \subset \tilde{S} \) the sequence defined in Introduction. In [14], the second author has shown that \( \tilde{I} := b-pol(I) \) gives a polarization of \( I \), that is, \( \Theta \) forms an \( (\tilde{S}/\tilde{I}) \)-regular sequence. The proof there is rather direct, and uses the sequentially Cohen-Macaulay property of \( \tilde{S}/\tilde{I} \). However we can give a new proof now.

Recall that Eliahou and Kervaire [6] constructed a minimal \( S \)-free resolution of \( I \), and free summands of their resolution are indexed by the admissible pairs for \( I \). Here an admissible pair for \( I \) is a pair \( (F,m) \) such that \( m \in G(I) \) and \( F = \{i_1, \ldots, i_q \} \subset \mathbb{N} \) with \( 1 \leq i_1 < \cdots < i_q < \nu(m) \). As remarked after the proof of Lemma 2.5, there is a one-to-one correspondence between the admissible pairs for \( \tilde{I} \) and those for \( I \), and if \( (\tilde{F},\tilde{m}) \) corresponds to \( (F,m) \) then \( \#\tilde{F} = \#F \). Hence we have

\[
(4.1) \quad \beta^\tilde{S}_{i,j}(\tilde{I}) = \beta^S_{i,j}(I)
\]

for all \( i,j \), where \( S \) and \( \tilde{S} \) are considered to be \( \mathbb{Z} \)-graded by setting the degree of all variables to be 1.

Of course, this equation is clear, if we know the fact that \( \tilde{I} \) is a polarization of \( I \) ([14, Theorem 3.4]). Conversely, this equation induces the theorem.

**Corollary 4.1** ([14, Theorem 3.4]). The ideal \( \tilde{I} \) is a polarization of \( I \).

**Proof.** Follows from the equation (4.1) and [12, Lemma 6.9] (see also [14, Lemma 2.2]). \( \square \)

The next result also follows from [12, Lemma 6.9].

**Corollary 4.2.** \( \tilde{P}_* \otimes_\tilde{S} \tilde{S}/(\Theta) \) is a minimal \( S \)-free resolution of \( S/I \).

**Remark 4.3.** (1) The correspondence between the admissible pairs for \( I \) and those for \( \tilde{I} \), does not give a chain map between the Eliahou-Kervaire resolution and our \( \tilde{P}_* \otimes_\tilde{S} \tilde{S}/(\Theta) \). In this sense, two resolutions are not the same. See (the figures) in Remark 6.2 below.

(2) Eliahou and Kervaire ([6]) constructed minimal free resolutions of stable monomial ideals, which form a wider class than Borel fixed ideals. As shown in [14, Example 2.3 (2)], \( b-pol(J) \) is not a polarization for a stable monomial ideal \( J \) in general, and the construction of \( \tilde{P}_* \otimes_\tilde{S} \tilde{S}/(\Theta) \) does not work for \( J \).

(3) By the equation (4.1) and Corollary 4.1, one might expect that \( b-pol(-) \) preserves lcm-lattices. But it does not in general. Recall that the lcm-lattice of a monomial ideal \( J \) is the set \( \text{LCM}(J) := \{\text{lcm}\{m \mid m \in \sigma \} \mid \sigma \subset G(J)\} \) with the order given by divisibility. Clearly, \( \text{LCM}(J) \) forms a lattice. Let \( \vee \) denote the join in the lcm-lattice. For the Borel fixed ideal \( I = (x^2, xy, xz, y^2, yz) \), we have \( xy \vee xz = xy \vee yz = xz \vee yz \) in \( \text{LCM}(I) \). On the other hand, \( \tilde{xy} \vee \tilde{xz} = x_1y_2z_2 \), \( \tilde{xy} \vee \tilde{yz} = x_1y_1y_2z_2 \) and \( \tilde{xz} \vee \tilde{yz} = x_1y_1z_2 \) are all distinct in \( \text{LCM}(\tilde{I}) \).
Let \( a = \{a_0, a_1, a_2, \ldots \} \) be a non-decreasing sequence of non-negative integers with \( a_0 = 0 \), and \( T = k[x_1, \ldots, x_N] \) a polynomial ring with \( N \gg 0 \). In his paper \([11]\), Murai defined an operator \((-)^{(a)}\) acting on monomials and monomial ideals of \( S \). For a monomial \( m \in S \) with the expression \( m = \prod_{i=1}^{e} x_{a_i} \), as \((1.1)\), set
\[
m^{\gamma(a)} := \prod_{i=1}^{e} x_{a_i+a_i-1} \in T,
\]
and for a monomial ideal \( I \subset S \),
\[
I^{\gamma(a)} := (m^{\gamma(a)} \mid m \in G(I)) \subset T.
\]
If \( a_{i+1} > a_i \) for all \( i \), then \( I^{\gamma(a)} \) is a squarefree monomial ideal. Particularly, in the case \( a_i = i \) for all \( i \), the operator \((-)^{(a)}\) is just the squarefree operator \((-)^{sq}\) due to Kalai, which plays an important role in the construction of the symmetric shifting of a simplicial complex (see \([1]\)).

The operator \((-)^{(a)}\) also can be described by \( b-pol(-) \) as is shown by the second author (\([14]\)). Let \( L_a \) be the \( k \)-subspace of \( \tilde{S} \) spanned by \( \{x_{i,j} - x_{i',j'} \mid i + a_{j-1} = i' + a_{j'-1}\} \), and \( \Theta \) a basis of \( L_a \). For example, we can take
\[
\{ x_{i,j} - x_{i+1,j-1} \mid 1 \leq i < n, \ 1 < j \leq d \}
\]
as \( \Theta_a \) in the case \( a_i = i \) for all \( i \). With a suitable choice of the number \( N \), the ring homomorphism \( \tilde{S} \to T \) with \( x_{i,j} \mapsto x_{i+a_{j-1}} \) induces the isomorphism \( \tilde{S}/(\Theta_a) \cong T \).

**Proposition 4.4** (\([14, \text{Proposition 4.1}]\)). With the above notation, \( \Theta_a \) forms an \( \tilde{S}/I \)-regular sequence, and we have \((\tilde{S}/(\Theta_a)) \otimes_{\tilde{S}} (\tilde{S}/I) \cong T/I^{\gamma(a)} \) through the isomorphism \( \tilde{S}/(\Theta_a) \cong T \).

Applying Proposition 4.4 and \([3, \text{Proposition 1.1.5}]\), we have the following.

**Corollary 4.5.** The complex \( \tilde{P}_i \otimes_{\tilde{S}} \tilde{S}/(\Theta_a) \) is a minimal \( T \)-free resolution of \( T/I^{\gamma(a)} \). In particular, a minimal free resolution of \( T/I^{sq} \) is given in this way.

To draw a “diagram” of an admissible pair \((\tilde{F}, \tilde{m})\), we put a white square in the \((i,j)\)-th position if \( (i,j) \in \tilde{F} \) and a black square there if \( x_{i,j} \) divides \( \tilde{m} \). If \( \tilde{F} \) is maximal among \( \tilde{F}' \) such that \((\tilde{F}', \tilde{m})\) is admissible, then the diagram of \((\tilde{F}, \tilde{m})\) forms a “right side down stairs”. For a non-maximal \( \tilde{F} \), some white squares are removed from the diagram for the maximal case. Figure 1 (resp. Figure 2) below is a diagram of \((\tilde{F}, \tilde{m})\) with \( \tilde{m} = x_{1,1}x_{2,2}x_{3,3}x_{6,4}x_{6,5} \) and \( \tilde{F} = \{(1,3),(2,4),(3,4),(4,4),(5,4)\} \) (resp. \( \tilde{F} = \{(1,3),(3,4),(4,4)\} \)). Clearly, the former is a maximal case.

The lower right end of the stairs must be a black square. For each \( j \) with \( 1 \leq j \leq \deg \tilde{m} \), there is a unique black square in the \( j \)-th column. The black square in the \( j \)-th column is the “lowest” of the squares in this column.

**Lemma 4.6.** Assume that \( I \) is generated in one degree (i.e., all element of \( G(I) \) have the same degree). Let \((\tilde{F}, \tilde{m})\) with \( \tilde{F} = \{(i_1,j_1),\ldots,(i_q,j_q)\} \) be an admissible pair for \( b-pol(I) \). Then, \( r \in B(\tilde{F}, \tilde{m}) \) if and only if \( j_r < j_{r+1} \) or \( r = q \).

**Proof.** By Lemma 2.5 (iii), it suffices to show that \( r < q \) and \( j_r < j_{r+1} \) imply \( i_q < \nu(m_{(i_r)}) \). Under this assumption, we have \( k := \min\{ l > i_r \mid x_l \text{ divides } m \} \leq i_{r+1} <
Since $I$ is generated in one degree, we have $m_{(i_r)} = b_{i_r}(m) = (x_{i_r}/x_k) \cdot m$ and $v(m_{(i_r)}) = v(m) > i_q$. □

In the rest of this section, we assume that $I$ is generated in one degree unless otherwise specified. Let us go back to the diagram of $(\tilde{F}, \tilde{m})$. By Lemma 4.6, $r \in B(\tilde{F}, \tilde{m})$ if and only if the white square in the $(i_r, j_r)$-th position is the “lowest” one among white squares in the $j_r$-th column. In this case, we can get the diagram of the admissible pair $(\tilde{F}_r, \tilde{m}_{(i_r)})$ from that of $(\tilde{F}, \tilde{m})$ by the following procedure.

(i) Remove the (sole) black square in the $j_r$-th column.

(ii) Replace the white square in the $(i_r, j_r)$-th position by a black one.

After minor modification, the above mentioned relation between the diagram of $(\tilde{F}, \tilde{m})$ and that of $(\tilde{F}_r, \tilde{m}_{(i_r)})$ remains true for a general Borel fixed ideal. In general, we might have $m_{(i_r)} \neq b_{i_r}(m)$. If this is the case, we have to remove (possibly several) black squares from the right end of the diagram of $(\tilde{F}, \tilde{m})$.

For an admissible pair $(\tilde{F}, \tilde{m})$ with $\tilde{F} = \{(i_1, j_1), \ldots, (i_q, j_q)\}$, set

$$x(\tilde{F}, \tilde{m}) := \tilde{m} \times \prod_{r=1}^q x_{i_r, j_r} \in \tilde{S}.$$ 

Clearly, the “support” of $x(\tilde{F}, \tilde{m})$ coincides with the all squares (i.e., black and white squares) in the diagram of $(\tilde{F}, \tilde{m})$. We also set

$$\text{rm}(\tilde{F}, \tilde{m}, j) := \begin{cases} \{(i, j) \mid x_{i,j} \text{ divides } x(\tilde{F}, \tilde{m})\} & \text{if } j = j_r \text{ for some } r, \\ \emptyset & \text{otherwise,} \end{cases}$$

and

$$\text{rm}(\tilde{F}, \tilde{m}) := \bigcup_{j \in \mathbb{N}} \text{rm}(\tilde{F}, \tilde{m}, j)$$

(here, “rm” stands for “removable”). The diagram of $(\tilde{F}, \tilde{m})$ introduced above is helpful to understand $\text{rm}(\tilde{F}, \tilde{m})$. If there is a white square in the $j$-th column, the elements in $\text{rm}(\tilde{F}, \tilde{m}, j)$ correspond to the all squares in the $j$-th column.

We can take $r_1, \ldots, r_k$ so that

$$\text{rm}(\tilde{F}, \tilde{m}) = \bigcup_{l=1}^k R_l \text{ with } R_l := \text{rm}(\tilde{F}, \tilde{m}, j_{r_l}).$$
For $j := j_{ri}$, set $s := \min\{ r \mid j_r = j \}$, $t := \max\{ r \mid j_r = j \}$ and $u := \min\{ l \mid i_l \mid x_l \text{ divides } m \}$. Then, we have

$$R_t = \{ (i_r, j) \in \tilde{F} \mid s \leq r \leq t \} \cup \{ (u, j) \}.$$ 

In particular, we have $\#R_t \geq 2$, and $(i_r, j) \in \tilde{F}$ with $s \leq r \leq t$ (resp. $(u, j)$) corresponds to a white (resp. black) square in the $j$-th column of the diagram of $(\tilde{F}, \tilde{m})$. For $r$ with $s \leq r \leq t$, $r \in B(\tilde{F}, \tilde{m})$ if and only if $r = t$ by Lemma 4.6.

**Lemma 4.7.** With the above notation (in particular, $j = j_{ri}$), for $r$ with $s \leq r \leq t$, we have

$$x(\tilde{F}_r, \tilde{m}) = x(\tilde{F}_r, \tilde{m})/x_{i_r,j}$$

and

$$\text{rm}(\tilde{F}_r, \tilde{m}) = \begin{cases} \text{rm}(\tilde{F}, \tilde{m}) \setminus \{x_{i_r,j}\} & #R_t > 2, \\ \text{rm}(\tilde{F}, \tilde{m}) \setminus R_t & #R_t = 2. \end{cases}$$

Similarly, we have

$$x(\tilde{F}_t, \tilde{m}(b_i)) = x(\tilde{F}, \tilde{m})/x_{i_r,j}$$

and

$$\text{rm}(\tilde{F}_t, \tilde{m}(b_i)) = \begin{cases} \text{rm}(\tilde{F}, \tilde{m}) \setminus \{x_{u,j}\} & #R_t > 2, \\ \text{rm}(\tilde{F}, \tilde{m}) \setminus R_t & #R_t = 2. \end{cases}$$

**Proof.** Easily follows from the above observation. \hfill \Box

Let $A_f$ be the set of all admissible pairs for $I$. We make $A_f$ a poset (i.e., partially ordered set) as follows; $(\tilde{F}, \tilde{m})$ covers $(\tilde{F}', \tilde{m}')$ if and only if $e(\tilde{F}', \tilde{m}')$ appears in $\partial(e(\tilde{F}, \tilde{m}))$, where $\partial$ is the differential of $\tilde{P}$. By Lemma 4.7, $(\tilde{F}, \tilde{m})$ covers $(\tilde{F}', \tilde{m}')$ if and only if $x(\tilde{F}', \tilde{m}') = x(\tilde{F}, \tilde{m})/x_{i_r,j}$ for some $(i, j) \in \text{rm}(\tilde{F}, \tilde{m})$.

**Proposition 4.8.** With the above situation, let $(\tilde{F}, \tilde{m})$ and $(\tilde{F}', \tilde{m}')$ be admissible pairs of $\tilde{I}$, and consider the decomposition $\text{rm}(\tilde{F}, \tilde{m}) = \bigsqcup_{l=1}^k R_l$ as in (4.2). Then $(\tilde{F}, \tilde{m}) \succeq (\tilde{F}', \tilde{m}')$ with respect to the order defined above if and only if there is a subset $R \subset \text{rm}(\tilde{F}, \tilde{m})$ such that $R_l \not\subset R$ for all $l$ and

$$x(\tilde{F}', \tilde{m}') = x(\tilde{F}, \tilde{m})/\prod_{(i,j) \in R} x_{i,j}.$$ 

Let $(\tilde{F}, \tilde{m})$ be an element of $A_f$, and $R_1, \ldots, R_k$ as in Proposition 4.8. By the proposition, it is easy to see that the order ideal $\{ (\tilde{F}', \tilde{m}') \mid (\tilde{F}', \tilde{m}') \succeq (\tilde{F}, \tilde{m}) \}$ of $A_f$ is isomorphic to the Cartesian product

$$(2^{R_1} \setminus \{\emptyset\}) \times (2^{R_2} \setminus \{\emptyset\}) \times \cdots \times (2^{R_k} \setminus \{\emptyset\}).$$

Under the assumption that $I$ is generated in one degree, Nagel and Reiner [12] constructed a CW complex which supports a minimal free resolution of $\tilde{I}$ (or $I$, $I^\alpha$). See [12, Theorem 3.13]. If we regard their CW complex as a poset in the natural way, then it is isomorphic to our $A_f$ by Proposition 4.8. Hence we have the following. (Since their CW complex is regular, the choice of an incidence function does not cause a problem.)
Proposition 4.9. Let $I$ be a Borel fixed ideal generated in one degree. Then Nagel-Reiner description of a minimal free resolution of $\tilde{I}$ coincides with our $\tilde{P}$ (more precisely, the truncation $\tilde{P}_{\geq 1}$, since $\tilde{P}$ is a resolution of $\tilde{S}/I$, not $\tilde{I}$ itself).

In the next section, we will show that our resolution $\tilde{P}$ is cellular even if $I$ is not generated in one degree.

5. Relation to Batzies-Welker Theory

We use the same notation as in the previous sections. In particular, $I$ is a Borel fixed ideal, and $\tilde{I} := b_{\text{pol}}(I)$ is the alternative polarization.

In [2], Batzies and Welker connected the theory of cellular resolutions of monomial ideals with Forman’s discrete Morse theory ([7]). In this section, we will show that our resolution $\tilde{P}$ of $\tilde{S}/I$ can also be obtained by their construction.

Definition 5.1. A monomial ideal $J$ is called shellable if there is a total order $\sqsupseteq$ on $G(J)$ satisfying the following condition.

$\star$ For any $m, m' \in G(J)$ with $m \sqsupseteq m'$, there is an $m'' \in G(J)$ such that $m \sqsupseteq m''$, $\text{deg} \left( \frac{\text{lcm}(m, m'')}{m} \right) = 1$ and $\text{lcm}(m, m'')$ divides $\text{lcm}(m, m')$.

Remark 5.2. One can show that $J$ is shellable in the above sense if and only if $I$ has linear quotients in the sense of [8].

For a shellable monomial ideal, we have a Batzies-Welker type minimal free resolution. Since a Borel fixed ideal is shellable, we can apply their method. However, their description is not as explicit as that of Eliahou and Kervaire (see, for example, [9, §6.2]). It makes our argument in this section complicated.

Let $\sqsubseteq$ be the total order on $G(\tilde{I}) = \{ \tilde{m} \mid m \in G(I) \}$ such that $\tilde{m}' \sqsubseteq \tilde{m}$ if and only if $m' \triangleright m$ in the lexicographic order of $S$ with $x_1 \triangleright x_2 \triangleright \cdots \triangleright x_n$. In the rest of this section, $\sqsubseteq$ means this order.

Lemma 5.3. The order $\sqsubseteq$ makes $\tilde{I}$ shellable.

Proof. If we set $l := \min \{ i \mid x_i \text{ divides } \text{lcm}(m, m')/m \}$, then $\tilde{m}'' = \tilde{m}_{(l)}$ satisfies the expected property (see also the proof of Lemma 3.5). 

Hence we have a Batzies-Welker type minimal free resolution of $\tilde{I}$. This resolution is cellular, and the corresponding CW complex $X_\lambda$ is obtained from the simplex $X$ (essentially, the power set $2^{G(I)}$, which supports the Taylor resolution of $I$) using discrete Morse theory.

The following construction is taken from [2, Theorems 3.2 and 4.3]. For the background of their theory, the reader is recommended to consult the original paper.

For $\emptyset \neq \sigma \subset G(\tilde{I})$, let $\tilde{m}_\sigma$ denote the largest element of $\sigma$ with respect to the order $\sqsubseteq$, and set $\text{lcm}(\sigma) := \text{lcm}\{ \tilde{m} \mid \tilde{m} \in \sigma \}$.

Definition 5.4. We define a total order $\prec_\sigma$ on $G(\tilde{I})$ as follows.

- Set $N_\sigma := \{ (\tilde{m}_\sigma)_{(i)} \mid 1 \leq i < \nu(m_\sigma), (\tilde{m}_\sigma)_{(i)} \text{ divides } \text{lcm}(\sigma) \}$.
- For all $\tilde{m} \in N_\sigma$ and $\tilde{m}' \in G(\tilde{I}) \setminus N_\sigma$, we have $\tilde{m} \prec_\sigma \tilde{m}'$. 

• The restriction of \( \prec_{\sigma} \) to \( N_{\sigma} \) coincides with \( \sqsubseteq \), and the same is true for the
restriction to \( G(\tilde{I}) \setminus N_{\sigma} \).

**Remark 5.5.** In the above definition, \( N_{\sigma} \) plays the same role as \( \{ n_j^m \mid j \in J_m \} \)
in the proof of [2, Proposition 4.3]. For a general shellable monomial ideal (e.g., a
Borel fixed ideal), there is an ambiguity in the choice of \( \{ n_j^m \mid j \in J_m \} \), but \( N_{\sigma} \) is
the unique one in our case. In this sense, \( I \) is simpler than \( I \) itself.

Let \( X \) be the \((\#G(\tilde{I}) - 1)\)-simplex associated with \( 2^{G(\tilde{I})} \) (more precisely, \( 2^{G(\tilde{I})} \setminus \{ \emptyset \} \)). Hence we freely identify \( \sigma \subset G(\tilde{I}) \) with the corresponding cell of the simplex \( X \). Let \( G_X \) be the directed graph defined as follows.

• The vertex set of \( G_X \) is \( 2^{G(\tilde{I})} \setminus \{ \emptyset \} \).

• For \( \emptyset \neq \sigma, \sigma' \subset G(\tilde{I}) \), there is an arrow \( \sigma \rightarrow \sigma' \) if and only if \( \sigma \supset \sigma' \) and
\( \# \sigma = \# \sigma' + 1 \).

For \( \sigma = \{ \tilde{m}_1, \tilde{m}_2, \ldots, \tilde{m}_k \} \) with \( \tilde{m}_1 \prec_{\sigma} \tilde{m}_2 \prec_{\sigma} \cdots \prec_{\sigma} \tilde{m}_k = \tilde{m}_{\sigma} \) and \( l \in \mathbb{N} \) with
\( 1 \leq l < k \), set \( \sigma_l := \{ \tilde{m}_{k-l}, \tilde{m}_{k-l+1}, \ldots, \tilde{m}_k \} \) and
\( u(\sigma) := \sup \{ l \mid \exists \tilde{m} \in G(I) \text{ s.t. } \tilde{m} \prec_{\sigma} \tilde{m}_{k-l} \text{ and } \tilde{m} \text{ divides lcm}(\sigma_l) \} \).

If \( u := u(\sigma) \neq -\infty \), we can define
\( \tilde{n}_{\sigma} := \min_{\prec_{\sigma}} \{ \tilde{m} \mid \tilde{m} \text{ divides lcm}(\sigma_u) \} \).

Let \( E_X \) be the set of edges of \( G_X \). We define a subset \( A \) of \( E_X \) by
\( A := \{ \sigma \cup \{ \tilde{n}_{\sigma} \} \rightarrow \sigma \mid u(\sigma) \neq -\infty, \tilde{n}_{\sigma} \notin \sigma \} \).

It is easy to see that \( A \) is a matching, that is, every \( \sigma \) occurs in at most one edges
of \( A \). We say \( \emptyset \neq \sigma \subset G(\tilde{I}) \) is critical, if it does not occurs in any edge of \( A \).

We have the directed graph \( G_X^A \) with the vertex set \( 2^{G(\tilde{I})} \setminus \{ \emptyset \} \) (i.e., same as \( G_X \))
and the set of edges
\( (E_X \setminus A) \cup \{ \sigma \rightarrow \tau \mid (\tau \rightarrow \sigma) \in A \} \).

In the proof of [2, Theorem 3.2], it is shown that the matching \( A \) is acyclic, that is, \( G_X^A \) has no directed cycle. A directed path in \( G_X^A \) is called a gradient path.

Forman’s discrete Morse theory [7] (see also [4]) guarantees the existence of a
CW complex \( X_A \) with the following conditions.

• There is a one-to-one correspondence between the \( i \)-cells of \( X_A \) and the
critical \( i \)-cells of \( X \).

• \( X_A \) is contractible, that is, homotopy equivalent to \( X \).

The cell of \( X_A \) corresponding to a critical cell \( \sigma \) of \( X \) is denoted by \( \sigma_A \). By [2,
Proposition 7.3], the closure of \( \sigma_A \) contains \( \tau_A \) if and only if there is a gradient path
from \( \sigma \) to \( \tau \). See also Proposition 5.8 below and the argument before it.

The following is a direct consequence of [2, Theorem 4.3].

**Proposition 5.6** (Batzies-Welker, [2]). With the above notation, the CW complex \( X_A \) supports a minimal free resolution \( F_{X_A} \) of \( I \).

While the explicit form of \( F_{X_A} \) will be introduced before Theorem 5.11, we remark
now that the free summands of \( F_{X_A} \) are indexed by the cells of \( X_A \) (equivalently,
the critical subsets of \( G(\tilde{I}) \)).
The purpose of this section is to show that the Batzies-Welker resolution $\mathcal{F}_{X_A}$ is “same” as our $\tilde{P}_\bullet$, that is, there is the isomorphism $\mathcal{F}_{X_A} \cong \tilde{P}_\bullet$ given by a correspondence between the basis elements of $\mathcal{F}_{X_A}$ and those of $\tilde{P}_\bullet$. For consistent notation, set $\tilde{Q}_\bullet := \mathcal{F}_{X_A}$. The following (long) discussion is necessary to compute the differential map of $\tilde{Q}_\bullet$.

Lemma 5.7. Let $\sigma$ and $\tau$ be (not necessarily critical) subsets of $G(\tilde{I})$ admitting a gradient path from $\sigma$ to $\tau$. Then $\text{lcm}(\tau)$ divides $\text{lcm}(\sigma)$.

Proof. It suffices to consider the case there is an arrow $\sigma \to \tau$ in $G_A^A$. If $\sigma \supset \tau$, the assumption is clear. If $\tau \supset \sigma$, then $\tau = \sigma \cup \{\tilde{n}_\sigma\}$ and the assumption follows from the fact that $\tilde{n}_\sigma$ divides $\text{lcm}(\sigma)$. \hfill \Box

Assume that $\emptyset \neq \sigma \subset G(\tilde{I})$ is critical. Recall that $\tilde{m}_\sigma$ denotes the largest element of $\sigma$ with respect to $\sqsubset$. Take $m_\sigma = \prod_{d=1}^{\nu} x_{st}^d \in G(I)$ with $\tilde{m}_\sigma = \text{b-pol}(m_\sigma)$, and set $q := \#\sigma - 1$. Then there are integers $1 \leq i_1 < \ldots < i_q < \nu(m_\sigma)$ and (5.1) $\sigma = \{(\tilde{m}_\sigma)_{(i_r)} | 1 \leq r \leq q\} \cup \{\tilde{m}_\sigma\}$ (see the proof of [2, Proposition 4.3]). Equivalently, we have $\sigma = N_\sigma \cup \{\tilde{m}_\sigma\}$ in the notation of Definition 5.4. Set $j_r := 1 + \sum_{i=1}^{q} a_{i}$ for each $1 \leq r \leq q$, and $F_\sigma := \{(i_1,j_1),\ldots,(i_q,j_q)\}$. Then $(F_\sigma,\tilde{m}_\sigma)$ is an admissible pair for $\tilde{I}$. Conversely, any admissible pair comes from a critical cell $\sigma \subset G(\tilde{I})$ in this way. Hence there is a one-to-one correspondence between critical cells and admissible pairs. Note that (5.2) $\text{lcm}(\sigma) = \tilde{m}_\sigma \times \prod_{r=1}^{q} x_{ir,jr}$ is the “support” of the diagram of the admissible pair $(\tilde{F}_\sigma,\tilde{m}_\sigma)$ introduced in §4.

From now on, we study gradient paths starting from a critical subset $\sigma$ of the form (5.1). Since $\sigma$ is critical, the first step of any path must be $\sigma \to \sigma \setminus \{\tilde{m}\}$ for some $\tilde{m} \in \sigma$. Let $\tau \subset G(\tilde{I})$ be a critical subset with $\#\tau = \#\sigma - 1 = q$, and assume that there is a gradient path $\sigma \to \sigma \setminus \{\tilde{m}\} = \sigma_0 \to \sigma_1 \to \ldots \to \sigma_l = \tau$. Since $\tau$ is critical, we have $\#\sigma_1 - 1 = \#\tau + 1 = q + 1$. Hence $\#\sigma_i = q$ or $q + 1$ for each $i$, and $\sigma_i$ is not critical for all $0 \leq i < l$. Since $\sigma \setminus \{(\tilde{m}_{\sigma})_{(i_r)}\}$ is critical for $1 \leq r \leq q$, it remains to consider a gradient path starting from $\sigma \to \sigma \setminus \{\tilde{m}_\sigma\}$.

Proposition 5.8. Let $\sigma, \tau \subset G(\tilde{I})$ be critical subsets with $\#\sigma = \#\tau + 1$, and $(\tilde{F}_\sigma,\tilde{m}_\sigma)$ and $(\tilde{F}_\tau,\tilde{m}_\tau)$ the admissible pairs corresponding to $\sigma$ and $\tau$ respectively. Set $F_\sigma = \{(i_1,j_1),\ldots,(i_q,j_q)\}$ with $1 \leq i_1 < \ldots < i_q$. There is a gradient path $(\sigma \setminus \{\tilde{m}_\sigma\}) = \sigma_0 \to \sigma_1 \to \ldots \to \sigma_l = \tau$ if and only if there is some $r \in B(\tilde{F}_\sigma,\tilde{m}_\sigma)$ with $(\tilde{F}_\tau,\tilde{m}_\tau) = (\tilde{F}_\sigma)_r \setminus \{(\tilde{m}_\sigma)_{(i_r)}\}$.

Proof. First, we assume that there is a gradient path from $\sigma \setminus \tilde{m}_\sigma$ to $\tau$. By Lemma 5.7 and (5.2), the diagram of $(\tilde{F}_\tau,\tilde{m}_\tau)$ is a “subset” of that of $(\tilde{F}_\sigma,\tilde{m}_\sigma)$. By the definition of gradient paths, we have $\tilde{m} \sqsubseteq \max_{\leq A}(\sigma \setminus \{\tilde{m}_{\sigma}\}) \sqsubseteq \tilde{m}_\sigma$ for all $\tilde{m} \in \tau$. It follows that $\tilde{m}_\tau \neq \tilde{m}_\sigma$, and the set of the black squares in the diagram of $(\tilde{F}_\sigma,\tilde{m}_\sigma)$ is different from that of $(\tilde{F}_\tau,\tilde{m}_\tau)$. By the assumption $\#\sigma = \#\tau + 1$ (equivalently, $\#\tilde{F}_\sigma = \#\tilde{F}_\tau + 1$), the number of the white square in the diagram of
$(F_r, \bar{m}_r)$ is one smaller than that of $(F'_r, \bar{m}_r)$, by the shapes of the diagrams of admissible pairs, we have $(F_r, \bar{m}_r) = (F'_r, (\bar{m}_r)_{(i,s)})$ for some $r \in B(\bar{F}, \bar{m}_r)$.

Next, assuming $\bar{F} = (\bar{F}_r)$, and $\bar{m}_r = (\bar{m}_r)_{(i,s)}$ for some $r \in B(\bar{F}_r, \bar{m}_r)$, we will construct a gradient path from $\sigma \setminus \{\bar{m}_r\}$ to $\tau$. For short notation, set

$$\bar{m}_{[s]} := (\bar{m}_r)_{(i,s)} \quad \text{and} \quad \bar{m}_{[s,t]} := ((\bar{m}_r)_{(i,s)})_{(i,t)}.$$ 

By (5.1), we have $\sigma_0 := (\sigma \setminus \{\bar{m}_r\}) = \{\bar{m}_{[s]} | 1 \leq s \leq q\}$ and

$$\tau = \{ (\bar{m}_r)_{(i,s)} | 1 \leq s \leq q, s \neq r \} \cup \{\bar{m}_r\} = \{ \bar{m}_{[s,r]} | 1 \leq s \leq q, s \neq r \} \cup \{\bar{m}_r\}.$$ 

**Case 1.** First consider the case $r = q$. If $q = 1$, then $\sigma_0 = \tau$. So we may assume that $q \geq 2$. Then $\bar{m}_r = \bar{m}_{[q]}$ is the largest element of $\sigma_0$ with respect to the order $\subset$. (The same is true for $\sigma_1, \ldots, \sigma_{2q-1}$ constructed below in this paragraph.)

We might have $\bar{m}_{[s]} = \bar{m}_{[q,s]}$ for some $s < q$. For instance, if $j_1 = j_2$, then the equality holds for all $s$. Even if $j_s < j_q$ we can not ignore the effect of $g(-)$. However, we always have $\bar{m}_{[q,s]} \mid \text{lcm}\{\bar{m}_{[s]}, \bar{m}_{[q]}\}$. 

First, we assume that $\bar{m}_{[s]} \neq \bar{m}_{[q,s]}$ for all $s < q$. Note that $\bar{m}_{[q]}$ divides $\bar{n} := \text{lcm}(\sigma_0)$, and it is the smallest element in $G(\bar{F})$ with respect to the order $\prec_{\sigma_0}$ defined above. By the present assumption that $\bar{m}_{[1]} \neq \bar{m}_{[q]}$, we have $\bar{m}_{[q]} \notin \sigma_0$. Hence if we set $\sigma_1 := \sigma_0 \cup \{\bar{m}_{[q]}\}$, then $\sigma_0 \to \sigma_1$ is an arrow in $G_0$. Clearly, we have an arrow $\sigma_1 \to \sigma_2$, where $\sigma_2 := \sigma_1 \setminus \{\bar{m}_{[1]}\}$. If $q = 2$, then $\sigma_2 = \tau$, and we are done. So assume that $q > 2$. While $x_{i_1j_1}$ divides $\bar{m}_{[1]}$, it does not divide $\bar{n}' := \text{lcm}(\sigma_2 \setminus \{\bar{m}_{[1]}\})$. Hence $\bar{m}_{[1]}$ does not divide $\bar{n}'$, and $\bar{m}_{[q,2]}$ is the smallest elements of $\{ \bar{m} \in G(\bar{F}) | \bar{m} \text{ divides } \bar{n}' \}$ with respect to $\prec_{\sigma_2}$. If we set $\sigma_3 := \sigma_2 \cup \{\bar{m}_{[q,2]}\}$ and $\sigma_4 := \sigma_3 \setminus \{\bar{m}_{[2]}\}$, then $\sigma_2 \to \sigma_3 \to \sigma_4$ is a gradient path. If $q = 3$, then $\sigma_4 = \tau$. If $q > 3$, we repeat this procedure till we get $\sigma_{2(q-1)}$. Then

$$\sigma_{2(q-1)} = (\sigma_0 \setminus \{\bar{m}_{[1]}, \ldots, \bar{m}_{[q-1]}\}) \cup \{\bar{m}_{[q]}, \ldots, \bar{m}_{[q,q-1]}\} = \tau,$$

and we are done.

Next we consider the case $\bar{m}_{[s]} = \bar{m}_{[q,s]}$ for some $s < q$. Then, in the above construction, we skip the part $\sigma_k \to \sigma_k \cup \{\bar{m}_{[q]}\} \to (\sigma_k \cup \{\bar{m}_{[q,s]}\}) \setminus \{\bar{m}_{[s]}\}$ (these are “loops” now), and just change its “name” from $\bar{m}_{[s]}$ to $\bar{m}_{[q,s]}$.

**Case 2, Step 1.** Assume that $r < q$. We will use monomials $\bar{m}_{[s,t]}$ for $s,t$ with $t \leq r < s$. Clearly, $x_{i_sj_s}$ divides $\bar{m}_{[s,t]}$. However, by the effect of $g(-)$, $x_{i_sj_s}$ need not divide $\bar{m}_{[s,t]}$, and we might have $\bar{m}_{[q,t]} = \bar{m}_{[q]}$ and $\bar{m}_{[s,t]} = \bar{m}_{[s+1,t]}$.

For the simplicity, we assume that $\bar{m}_{[q,t]} \neq \bar{m}_{[q]}$ and $\bar{m}_{[s,t]} \neq \bar{m}_{[s+1,t]}$ for all $t \leq r < s$ in the following argument. If we have $\bar{m}_{[q,t]} = \bar{m}_{[q]}$ or $\bar{m}_{[s,t]} = \bar{m}_{[s+1,t]}$, we can avoid the problem as in Case 1. Since $r \in B(\bar{F}_r, \bar{m}_r)$, we have $\nu(\bar{m}_{[r]}) > i_q$ and $x_{i_sj_s}|\bar{m}_{[r,s]}$ for all $r < s \leq q$. Hence $\bar{m}_{[r,s]}$ is irrelevant to the above problem.

We inductively define $\sigma_1, \ldots, \sigma_{2r-1}$ by

$$\sigma_{2s-1} := \sigma_{2s-2} \cup \{\bar{m}_{[q,s]}\} = \{\bar{m}_{[q,1]}, \bar{m}_{[q,2]}, \ldots, \bar{m}_{[q,s]}, \bar{m}_{[s]}, \bar{m}_{[s+1]}, \ldots, \bar{m}_{[q]}\}$$

for each $1 \leq s \leq r$, and $\sigma_{2s} := \sigma_{2s-1} \setminus \{\bar{m}_{[s]}\}$ for each $1 \leq s < r$. These are same as those in Case 1, and we have a gradient path $\sigma_0 \to \cdots \to \sigma_{2r-1}$. Next we set

$$\sigma_{2r} := \sigma_{2r-1} \setminus \{\bar{m}_{[q]}\} = \{\bar{m}_{[q,1]}, \bar{m}_{[q,2]}, \ldots, \bar{m}_{[q,r]}, \bar{m}_{[r]}, \bar{m}_{[r+1]}, \ldots, \bar{m}_{[q-1]}\}.$$
Of course, there is an arrow $\sigma_{2r-1} \to \sigma_{2r}$. If $r = q - 1$, we move to Step 2 below. Hence we assume that $r < q - 1$.

Now $\tilde{m}_{[q-1]}$ is the largest element of $\sigma_{2r}$ with respect to $\square$. We inductively define $\sigma_{2r+1}, \ldots, \sigma_{4r-1}$ by

\[
\sigma_{2r+2s-1} = \sigma_{2r+2s-2} \cup \{ \tilde{m}_{[q-1,s]} \}
\]

\[
= \{ \tilde{m}_{[q-1,t]} | 1 \leq t \leq s \} \cup \{ \tilde{m}_{[q,t]} | s \leq t \leq r \} \cup \{ \tilde{m}_{[t]} | r \leq t \leq q - 1 \}.
\]

for $1 \leq s \leq r$, and $\sigma_{2r+2s} = \sigma_{2r+2s-1} \setminus \{ \tilde{m}_{[q,s]} \}$ for $1 \leq s < r$. Then we have a gradient path $\sigma_{2r} \to \sigma_{2r+1} \to \cdots \to \sigma_{4r-1}$ as before. Set

\[
\sigma_{4r} := \sigma_{4r-1} \setminus \{ \tilde{m}_{[q-1]} \}
\]

\[
= \{ \tilde{m}_{[q-1,1], \tilde{m}_{[q-1,2]}, \ldots}, \tilde{m}_{[q-1,r]}, \tilde{m}_{[q,r]}, \tilde{m}_{[r,1]}, \tilde{m}_{[r,2]} \}.
\]

If $r = q - 2$, we move to Step 2. If $r < q - 2$, then we set $\sigma_{4r+1} := \sigma_{4r} \cup \{ \tilde{m}_{[q-2,1]} \}$, $\sigma_{4r+2} := \sigma_{4r+1} \setminus \{ \tilde{m}_{[q-1,1]} \}$, $\ldots, \sigma_{6r+1} := \sigma_{6r-1} \setminus \{ \tilde{m}_{[q-2]} \}$. We repeat this procedure till we get $\sigma(2q-2)$, which equals

\[
\{ \tilde{m}_{[r+1,1]} | 1 \leq s < r \} \cup \{ \tilde{m}_{[r]} \} \cup \{ \tilde{m}_{[r,s]} | r < s \leq q \}
\]

(since $r \in B(\tilde{F}_{\sigma}, \tilde{m}_{\sigma})$, we have $\tilde{m}_{[s,r]} = \tilde{m}_{[r,s]}$ for all $s \geq r$ by Lemma 2.5 (iv)). Now we move to Step 2.

Case 2, Step 2. Set $\sigma' := \sigma_{2q-2}$ for short. The construction in this step is similar to Case 1. While we might have $\tilde{m}_{[s]} = \tilde{m}_{[r,s]}$ for some $s < r$, we can avoid the problem by the same way as Case 1. So we assume that $\tilde{m}_{[s]} \neq \tilde{m}_{[r,s]}$ for all $s < r$.

Anyway, $\tilde{m}_{[r]}$ is the largest element of $\sigma'$ with respect to $\square$. By the same argument as in Step 1, we have a gradient path $\sigma' = \sigma'_0 \to \cdots \to \sigma'_{2(r-1)}$ with $\sigma'_{2s-1} = \sigma'_{2s-2} \cup \{ \tilde{m}_{[r,s]} \}$ and $\sigma'_{2s} = \sigma'_{2s-1} \setminus \{ \tilde{m}_{[r+1,s]} \}$ for $1 \leq s \leq r - 1$. We have

\[
\sigma'_{2(r-1)} = (\sigma' \setminus \{ \tilde{m}_{[r+1,s]} | 1 \leq s < r \}) \cup \{ \tilde{m}_{[r,s]} | 1 \leq s < r \} = \tau.
\]

We also need the following.

**Proposition 5.9.** Let $\sigma, \tau \in G(\tilde{I})$ be as in Proposition 5.8. Then the gradient path constructed in the proof of Proposition 5.8 is the unique one connecting $\sigma \setminus \{ \tilde{m}_{\sigma} \}$ with $\tau$.

For the proof of the proposition, the next lemma is useful.

**Lemma 5.10.** Let $\sigma$ and $\tau$ be (not necessarily critical) subsets of $G(\tilde{I})$ admitting a gradient path from $\sigma$ to $\tau$. If a variable $x_{i,j}$ divides an element $\tilde{m} \in \tau$, then there is some $\tilde{m}' \in \sigma$ with $\tilde{m}' \supseteq \tilde{m}$ and $x_{i,j} | \tilde{m}'$.

**Proof.** We use induction on the length $l$ of a gradient path $\sigma = \sigma_0 \to \cdots \to \sigma_l = \tau$. If $\tilde{m} \in \sigma_{l-1}$, the assertion is clear from the induction hypothesis. Hence we may assume that $\tau = \sigma_{l-1} \cup \{ \tilde{m} \}$. By the definition of gradient paths, there is a subset $v \subset \sigma_{l-1}$ with $\tilde{m} \mid \text{lcm}(v)$ and $\tilde{m} \supseteq \tilde{n}$ for all $\tilde{n} \in v$. Since $x_{i,j}$ divides $\tilde{m}$, it also divides lcm$(v)$, and there is $\tilde{n}' \in v \subset \sigma_{l-1}$ with $x_{i,j} | \tilde{n}'$. By the induction hypothesis, there is $\tilde{m}' \in \sigma$ with $\tilde{m}' \supseteq \tilde{n}'$ and $x_{i,j} | \tilde{m}'$. Since $\tilde{m}' \supseteq \tilde{m}$, it is a desired element. □
The proof of Proposition 5.9. Let $P : (σ \setminus \{\tilde{m}_s\}) = σ_0 \rightarrow σ_1 \rightarrow \cdots \rightarrow σ_l = τ$ be the path constructed in the proof of Proposition 5.8. We will show that if we once leave from $P$ then we can not arrive $τ$ anymore. Since $G^A_X$ is acyclic, there is no (non-trivial) path from $τ$ to $τ$ itself. Hence it completes the proof of the uniqueness.

Recall that $#σ_{2s+1} = #σ_{2s} + 1 = q + 1$ for all $s$. If we take an arrow $σ_{2s} → σ'$ with $σ' \neq σ_{2s+1}$, then we have $#σ' = q - 1$, and there is no gradient path from $σ'$ to $τ$. Hence it suffices to consider the step $σ_{2r-1} → σ'$. In this case, $σ' = σ_{2s-1} \setminus \{\tilde{m}\}$ for some $\tilde{m} ∈ σ_{2s-1}$. We follow the framework of the proof of Proposition 5.8.

Case 1. First consider the case $r = q$. For the simplicity, we assume that $\tilde{m}_{[q,s]} \neq \tilde{m}_{[s]}$ for all $s < q$. The following argument also works without this assumption after minor modification.

Recall that
\[
σ_{2s-1} = \{ \tilde{m}_{[q,1]}, \tilde{m}_{[q,2]}, \ldots, \tilde{m}_{[q,s]}, \tilde{m}_s, \tilde{m}_{s+1}, \ldots, \tilde{m}_{[q]} \}
\]
and $σ_{2s} = σ_{2s-1} \setminus \{\tilde{m}_s\}$ for each $1 ≤ s < q$. Since $σ_{2s-1} = σ_{2s-2} \cup \{\tilde{m}_{[q,s]}\}$, we can not remove $\tilde{m}_{[q,s]}$ from $σ_{2s-1}$. Hence it suffices to show that there is no gradient path from $σ' := σ_{2s-1} \setminus \{\tilde{m}\}$ to $τ$ for all $\tilde{m} ∈ σ_{2s-1} \setminus \{\tilde{m}_s, \tilde{m}_{[q,s]}\}$. If $\tilde{m} = \tilde{m}_{[q,t]}$ for some $1 ≤ t < s$, then $x_{\tilde{m}_{[q,t]}}, / lcm(σ')$, and hence $lcm(τ) / lcm(σ')$. By Lemma 5.7, there is no path from $σ'$ to $τ$. If $\tilde{m} = \tilde{m}_{[q]}$ for $t > s$, the same argument also works.

Case 2, Step 1. Next consider the case $r < q$. For the simplicity, we assume that $\tilde{m}_{[q,t]} \neq \tilde{m}_q$ and $\tilde{m}_{[q,t]} \neq \tilde{m}_{[s+1,t]}$ for all $t < r < s$. The following argument also works without this assumption after minor modification.

Recall that the first $(2r-1)$-steps $σ_0 → \cdots → σ_{2r-1}$ of $P$ coincide with those in Case 1. In this part of the path, the proof in Case 1 also works. So we consider the next step $σ_{2r} → σ'$. Note that
\[
σ_{2r-1} = \{ \tilde{m}_{[q,1]}, \tilde{m}_{[q,2]}, \ldots, \tilde{m}_{[q,r]}, \tilde{m}_r, \tilde{m}_{r+1}, \ldots, \tilde{m}_{[q]} \}
\]
and $σ_{2r} = σ \setminus \{\tilde{m}_{[q]}\}$. We can not remove $\tilde{m}_{[q,r]}$ from $σ_{2r-1}$, since it is the new comer of this set. If $σ' = σ \setminus \{\tilde{m}\}$ with $\tilde{m} = \tilde{m}_{[q,s]}$ for some $s < r$ or $\tilde{m} = \tilde{m}_{[s]}$ for some $s > r$, then $x_{\tilde{m}_{[q,s]}}, / lcm(σ')$, and hence $lcm(τ) / lcm(σ')$. So we may assume that $σ' = σ_{2r-1} \setminus \{\tilde{m}_r\}$. Note that $\tilde{m}_r \in τ$, and $x_{\tilde{m}_r} \tilde{m}_r$. On the other hand, $\tilde{m}_{[q,r]}$ is the only element in $σ'$ which can be divided by $x_{\tilde{m}_r} \tilde{m}_r$. Since $\tilde{m}_{[q,r]} \subseteq \tilde{m}_r$, there is no gradient path from $σ'$ to $τ$ by Lemma 5.10.

A similar argument works for the step $σ_{2l-1} → σ_{2l}$ for $2 ≤ l ≤ q - r$. Note that
\[
σ_{2l-1} = \{ \tilde{m}_{[s,1]}, \tilde{m}_{[s,2]}, \ldots, \tilde{m}_{[s,r]}, \tilde{m}_r, \tilde{m}_{r+1}, \ldots, \tilde{m}_s, \tilde{m}_{s+1}, \tilde{m}_{r,s+1}, \tilde{m}_{[r,s+2]}, \ldots, \tilde{m}_{[r,q]} \}
\]
and $σ_{2l} = σ \setminus \{\tilde{m}_{[q]}\}$, where $s = q - l + 1$. Since $\tilde{m}_{[s,r]}$ is the new comer, we can not remove it. If $σ' = σ_{2l-1} \setminus \{\tilde{m}_{[s]}\}$ for $1 ≤ t < r$, there is no gradient path from $σ'$ to $τ$. In fact, the variable $x_{\tilde{m}_{[s,t]}}$ matters. The same is true for $\tilde{m}_{[q]}$ for $r < s < q$. If $σ' = σ_{2l-1} \setminus \{\tilde{m}_{[r]}\}$, the element in $σ'$ which can be divided by $x_{\tilde{m}_{[r,t]}}$ are $\tilde{m}_{[r+1,t]}, \ldots, \tilde{m}_{[r,q]}$. Since all of them are strictly smaller than $\tilde{m}_{[r]} \in τ$ with respect to $\subseteq$, there is no gradient path from $σ'$ to $τ$ by Lemma 5.10.

The remaining case of this step is $σ_{2(l+r-1)} → σ'$ for $2 ≤ l ≤ q - r$ and $1 ≤ t < r$. Note that
\[
σ_{2(l+r-1)} = \{ \tilde{m}_{[u]} \mid 1 ≤ u ≤ l \} \cup \{ \tilde{m}_{[s+1,u]} \mid t ≤ u < r \}
\]
\[
\cup \{ \tilde{m}_{[u]} \mid r ≤ u ≤ s \} \cup \{ \tilde{m}_{[v,u]} \mid s + 1 ≤ u ≤ q \},
\]
\(\sigma_{2(t+1)} = \sigma_{2(t+1)-1} \setminus \{\tilde{m}_{[s+1,t]}\}\), and \(\tilde{m}_{[s,t]}\) is the new comer of \(\sigma_{2(t+1)-1}\). The above argument is also applicable to this case. In fact, for \(\tilde{m}_{[s,t]}\), we can use Lemma 5.10 again. For the other elements, the variable \(x_{i,j}\) matters.

**Case 2, Step 2.** Now we connect \(\sigma_{2(q-r)}\) with \(\tau\). Since the construction is quite similar to Case 1, we can prove the assertion by a similar way to that case. \(\square\)

Applying the construction of Batzies and Welker (see [2, Remark 4.4] for its most explicit form), we get the following resolution \(\tilde{Q}_\bullet\) of \(\tilde{I} = b-pol(I)\). For a critical cell \(\sigma \subset G(\tilde{I})\), \(\varepsilon(\sigma)\) denotes a basis element with degree \(\deg(\text{lcm}(\sigma))\) \(\in \mathbb{Z}^{n \times d}\). Set

\[
\tilde{Q}_q = \bigoplus_{\sigma: \text{critical} \atop \# \sigma = q+1} \tilde{S}\varepsilon(\sigma) \quad (q \geq 0).
\]

For a critical cell \(\sigma\) of the form (5.1), the differential map sends \(\varepsilon(\sigma)\) to

\[
(5.3) \quad \sum_{r=1}^{q} (-1)^r \ x_{i_r,j_r} \cdot \varepsilon(\sigma \setminus \{\tilde{m}_\sigma(i_r)\}) - (-1)^q \sum_{\tau: \text{critical} \atop \# \tau = \# \sigma - 1} m(\mathcal{P}) \cdot \frac{\text{lcm}(\sigma)}{\text{lcm}(\tau)} \cdot \varepsilon(\tau).
\]

The definition of \(m(\mathcal{P})\) is found in the proof of Theorem 5.11 below (the original definition is in [2, p.166]).

**Theorem 5.11.** Our description of the resolution \(\tilde{P}_\bullet\) coincides the Batzies-Welker type resolution \(\tilde{Q}_\bullet\) (more precisely, the truncation \(\tilde{P}_{\geq 1}\) of \(\tilde{P}_\bullet\) coincides with \(\tilde{Q}_\bullet\)).

**Proof.** Recall that there is the one-to-one correspondence between the critical cells \(\sigma \subset G(\tilde{I})\) and the admissible pairs \((\tilde{F}_\sigma, \tilde{m}_\sigma)\). Hence, for each \(q\), we have the isomorphism \(\tilde{Q}_q \to \tilde{P}_q\) induced by \(e(\sigma) \mapsto e(\tilde{F}_\sigma, \tilde{m}_\sigma)\). We will show that they induce a chain isomorphism \(\tilde{Q}_\bullet \cong \tilde{P}_\bullet\).

Note that \(\sigma \setminus \{\tilde{m}_\sigma(i_r)\}\) corresponds to \((\tilde{F}_\sigma)_r, \tilde{m}_\sigma\), and \(\tau \subset G(\tilde{I})\) appears in the second \(\sum\) of (5.3) if and only if \((\tilde{F}_\tau, \tilde{m}_\tau) = ((\tilde{F}_\sigma)_r, (\tilde{m}_\sigma)(i_r))\) for some \(r \in B(\tilde{F}_\sigma, \tilde{m}_\sigma)\) by Proposition 5.8. Hence, if we forget “coefficients”, the differential map of \(\tilde{Q}_\bullet\) and that of \(\tilde{P}_\bullet\) are compatible with the maps \(e(\sigma) \mapsto e(\tilde{F}_\sigma, \tilde{m}_\sigma)\). So it remains to check the equality of the coefficients.

To define the coefficient \(m(\mathcal{P})\) \(\in \mathbb{Z}\) used in (5.3), we fix an orientation of the simplex \(X\). Identifying \(X\) with the power set \(2^{G(\tilde{I})}\), we set \([\sigma : \sigma'] = (-1)^r\) for \(\sigma = \{\tilde{m}_1, \ldots, \tilde{m}_{q+1}\}\) and \(\sigma' = \sigma \setminus \{\tilde{m}_r\}\) with \(\tilde{m}_1 \sqsubseteq \tilde{m}_2 \sqsubseteq \cdots \sqsubseteq \tilde{m}_{q+1}\). Then \(m(\mathcal{P})\) \(\in \mathbb{Z}\) for a gradient path \(\mathcal{P} = \sigma_0 \to \sigma_1 \to \cdots \to \sigma_1\) is defined by

\[
m(\mathcal{P}) = \prod_{1 \leq i \leq t \atop (i_r \to \sigma_{i-1}) \in A} [\sigma_{i-1} : \sigma_i] \times \prod_{1 \leq i \leq t \atop (i_r \to \sigma_{i-1}) \in A} (-[\sigma_i : \sigma_{i-1}]).
\]

Let \(\sigma, \tau \subset G(\tilde{I})\) be critical cells with \((\tilde{F}_\tau, \tilde{m}_\tau) = ((\tilde{F}_\sigma)_r, (\tilde{m}_\sigma)(i_r))\) for some \(r \in B(\tilde{F}_\sigma, \tilde{m}_\sigma)\), and \(\mathcal{P} : \sigma \setminus \{\tilde{m}_\sigma\} \to \cdots \to \tau\) the gradient path constructed in the proof
of Propositions 5.8. By Proposition 5.9, $\mathcal{P}$ is the unique one connecting $\sigma \setminus \{\tilde{m}_\sigma\}$ with $\tau$. It suffices to show that $(-1)^q m(\mathcal{P}) = (-1)^r$.

The path $\mathcal{P}$ is a succession of short paths

\begin{equation}
(5.4) \quad \sigma^{(1)} \to \sigma^{(2)} \to \sigma^{(3)}
\end{equation}

such that $\#\sigma^{(1)} = \#\sigma^{(3)} = q$ and $\#\sigma^{(2)} = q + 1$. For convenience, we refer to a path of type (5.4) as a peak. According to the construction of $\mathcal{P}$, the peaks $\sigma^{(1)} \to \sigma^{(2)} \to \sigma^{(3)}$ appearing in $\mathcal{P}$ are classified into the following two types $A$ and $B$. The type $A$ is such that $\sigma^{(1)} \to \sigma^{(2)}$ is given by insertion an element into the $s$-th position of $\sigma^{(1)}$ and $\sigma^{(2)} \to \sigma^{(3)}$ by deletion of the $(s + 1)$-th element of $\sigma^{(2)}$, for some $1 \leq s \leq r - 1$. To be more explicit, set $\sigma^{(1)} := \{\tilde{n}_1, \ldots, \tilde{n}_q\}$ with $\tilde{n}_1 \subset \tilde{n}_2 \subset \cdots \subset \tilde{n}_q$. A peak of type $A$ is of the following form

$$\{\tilde{n}_1, \ldots, \tilde{n}_q\} \to \{\tilde{n}_1, \ldots, \tilde{n}_{s-1}, \tilde{n}_s, \tilde{n}_s, \ldots, \tilde{n}_q\}$$

$$\to \{\tilde{n}_1, \ldots, \tilde{n}_{s-1}, \tilde{n}_s, \tilde{n}_{s+1}, \ldots, \tilde{n}_q\},$$

where $\tilde{n}_{s-1} \subset \tilde{n}_s \subset \tilde{n}_s$. The type $B$ is such that $\sigma^{(1)} \to \sigma^{(2)}$ is given by insertion an element into the $r$-th position of $\sigma^{(1)}$ and $\sigma^{(2)} \to \sigma^{(3)}$ by deletion of the $(q + 1)$-th element of $\sigma^{(2)}$. That is,

$$\{\tilde{n}_1, \ldots, \tilde{n}_q\} \to \{\tilde{n}_1, \ldots, \tilde{n}_{r-1}, \tilde{n}_r, \tilde{n}_r, \ldots, \tilde{n}_q\}$$

$$\to \{\tilde{n}_1, \ldots, \tilde{n}_{r-1}, \tilde{n}_r, \tilde{n}_r, \cdots, \tilde{n}_{q-1}\},$$

where $\tilde{n}_{r-1} \subset \tilde{n}_r \subset \tilde{n}_r$.

Let $\mathcal{P}' = (\sigma^{(1)} \to \sigma^{(2)} \to \sigma^{(3)})$ be a peak of type $A$ or $B$. By the choice of orientation of $X$,

$$m(\mathcal{P}') = -[\sigma^{(2)} : \sigma^{(1)}] [\sigma^{(2)} : \sigma^{(3)}] = \begin{cases} 1 & \text{if } \mathcal{P}' \text{ is of type } A \\ (-1)^{q-r} & \text{if } \mathcal{P}' \text{ is of type } B. \end{cases}$$

Hence it follows that $m(\mathcal{P}) = (-1)\ell(q-r)$, where $\ell$ is the number of the peaks of type $B$ which appears in $\mathcal{P}$.

To count the number of the peaks of type $B$, we use the framework of the proof of Proposition 5.8. It is clear that the paths constructed in Case 1 and Step 2 of Case 2 are successions of peaks of type $A$. In Step 1 of Case 2, an easy observation shows that the path is given by $(q - r)$-times repetition of construction of a path

$$\mathcal{P}_1 \cdots \mathcal{P}_{s-1} \mathcal{P}_s,$$

for some $s$ with $1 \leq s \leq r$, where $\mathcal{P}_i$ is a peak of type $A$ for $i < s$, and $\mathcal{P}_s$ is a peak of type $B$. Summing up these observation, we have $\ell = q - r$ (Recall that $q = r$ in Case 1). Therefore it follows that

$$(-1)^q m(\mathcal{P}) = (-1)^q \cdot (-1)^{q-r} = (-1)^{q+(q-r)} = (-1)^r,$$

as desired. \qed

The following is easy.

**Corollary 5.12.** The free resolution $\widetilde{P}_\bullet \otimes_S \widetilde{S}/(\Theta)$ (resp. $\widetilde{P}_\bullet \otimes_{\widetilde{S}} \widetilde{S}/(\Theta_0)$) of $S/I$ (resp. $T/I^{(a)}$) is also a cellular resolution supported by $X_A$. In particular, these resolutions are Batzies-Welker type.
Final Remarks

For the formal definition of the regularity of a (finite) CW complex, consult suitable textbooks. For our purpose, a regular CW complex is a CW complex such that for all $i \geq 0$ the closure \( \overline{\sigma} \) of any $i$-cell $\sigma$ is homeomorphic to an $i$-dimensional closed ball, and $\overline{\sigma} \setminus \sigma$ is the closure of the union of some $(i-1)$-cells. Recently, Mermin [10] (see also [5]) showed that the Eliahou-Kervaire resolution of a Borel fixed ideal (more generally, a stable monomial ideal) is cellular, and the supporting CW complex is regular. In the previous section, we showed that our resolution $\tilde{P}_\bullet$ is cellular. However, the regularity of the supporting complex $X_A$ constructed by discrete Morse theory is not clear, while we have the following.

**Proposition 6.1.** Let $X_A$ be the CW complex constructed in the previous section (recall that there is a one-to-one correspondence between critical subsets $\sigma \subset G(\tilde{I})$ and cells $\sigma_A$ of $X_A$). Then the following hold.

1. If the closure of an $(i+1)$-cell $\sigma_A$ contains an $(i-1)$-cell $\tau_A$, there are exactly two cells between them.
2. The incidence number $[\sigma_A : \sigma'_A]$ is 1, $-1$, or 0 for all $\sigma_A, \sigma'_A$.

If $X_A$ is regular, the conditions of the above proposition hold obviously.

**Proof.** (1) Let $\sigma'_A$ be an $i$-cell of $X_A$. As shown in the previous section, the closure of $\sigma_A$ contains $\sigma'_A$ if and only if $e(\tilde{F}_{\sigma'}, \tilde{m}_{\sigma'})$ appears in $\partial(e(\tilde{F}_{\sigma}, \tilde{m}_{\sigma}))$, where $\partial$ is the differential of $\tilde{P}_\bullet$. Hence the assertion follows from the analysis of $\partial$ given in §3.

(2) Since $[\sigma_A : \sigma'_A]$ coincides with the coefficient in $\partial$, the assertion is also clear. In other words, the assertion is a consequence of Proposition 5.9. \( \square \)

Recall that if a Borel fixed ideal $I$ is generated in one degree then our resolution $\tilde{P}_\bullet$ of $b\text{-pol}(I)$ is equivalent to the resolution of Nagel and Reiner [12]. Their resolution is cellular, and the supporting CW complex is polytopal, hence is regular. Since their CW complex is contractible as shown in the proof of [12, Theorem 3.13], it can be taken as $X_A$. Anyway, if $I$ is generated in one degree, the Nagel-Reiner resolution of $I$ is Batzies-Welker type, and the CW complex $X_A$ is regular.

**Example 6.2.** Consider the Borel fixed ideal $I = (x^2, xy^2, xyz, xyw, xz^2, xzw)$. Then $b\text{-pol}(I) = (x_1x_2, x_1y_2y_3, x_1y_2z_3, x_1y_2w_3, x_1z_2z_3, x_1z_3w_3)$, and easy computation shows that the CW complex $X_A$, which supports our resolutions $\tilde{P}_\bullet$ of $\tilde{S}/\tilde{I}$ and $\tilde{P}_\bullet \otimes \tilde{S}/(\Theta)$ of $S/I$, is the one illustrated in Figure 3. The complex consists of a square pyramid and a tetrahedron glued along trigonal faces of each. For a Borel fixed ideal generated in one degree, any face of the Nagel-Reiner CW complex is a product of several simplices. Hence a square pyramid can not appear in the case of Nagel and Reiner.

We remark that the Eliahou-Kervaire resolution of $I$ is supported by the CW complex illustrated in Figure 4. This complex consists of two tetrahedrons glued along edges of each. These figures show visually that the description of the Eliahou-Kervaire resolution and that of ours are really different.

**Question 6.3.** Is the CW complex $X_A$ is regular for a general Borel fixed ideal?
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