Geometric global quantum discord

Jianwei Xu

Beijing Computational Science Research Center, Beijing 100084, People’s Republic of China
Key Laboratory for Radiation Physics and Technology, Institute of Nuclear Science and Technology, Sichuan University, Chengdu 610065, People’s Republic of China

E-mail: xxujianwei@yahoo.cn

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Abstract
Geometric quantum discord, proposed by Dakic et al (2010 Phys. Rev. Lett. 105 190502), is an important measure for bipartite correlations. In this paper, we generalize it to multipartite states, we call the generalized version geometric global quantum discord (GGQD). We characterize GGQD in different ways, give a lower bound for GGQD and provide some special states which allow analytical GGQD.

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1. Introduction
Quantum correlation is one of the most striking features in quantum theory. Entanglement is by far the most famous and best-studied kind of quantum correlation and leads to powerful applications [1]. Another kind of quantum correlation, called quantum discord, captures more correlations than entanglement in the sense that even separable states may possess nonzero quantum discord. Quantum discord has attracted much attention in recent years, due to its theoretical interest in quantum theory, and also due to its potential applications (see, e.g., a recent review [2]). Up to now, the studies on quantum correlations, such as entanglement and quantum discord, are mainly focused on the bipartite case.

Quantifying multipartite correlations is a fundamental and difficult question. The direct idea is that we can properly generalize the quantifiers of bipartite correlations to the case of multipartite correlations [3–7]. Recently, generalizing the quantum discord of bipartite states to multipartite states has been discussed in different ways [14–17]. As an important measure of bipartite correlations, the geometric quantum discord, proposed in [18], has been extensively studied [2]. Although some authors question the general validity of geometric quantum discord as a conceptually meaningful measure of quantumness [8], even as a useful measure, geometric quantum discord has found many interesting applications [9–13]. In this paper, we generalize the geometric quantum discord to multipartite states.
This paper is organized as follows. In section 2, we review the definition of geometric quantum discord for bipartite states. In section 3, we give the definition of geometric global quantum discord (GGQD) for multipartite states and give two equivalent expressions for GGQD. In section 4, we provide a lower bound for GGQD by using the high-order singular value decomposition of tensors. In section 5, we obtain the analytical expressions of GGQD for three classes of states. Section 6 is a brief summary.

2. Geometric quantum discord of bipartite states

The original quantum discord was defined for bipartite systems over all projective measurements performed on one subsystem [19, 20]. That is, the quantum discord (with respect to A) of a bipartite state $\rho_{AB}$ of the composite system $AB$ (we suppose $\dim A = n_A < \infty$, $\dim B = n_B < \infty$) was defined as

$$D_A(\rho_{AB}) = S(\rho_A) - S(\rho_{AB}) + \min_{\Pi_A} [S(\Pi_A(\rho_{AB})) - S(\Pi_A(\rho_A))]. \tag{1}$$

In equation (1), $S(\cdot)$ is the von Neumann entropy, $\rho_A = \text{tr}_B \rho_{AB}$. $\Pi_A$ is a projective measurement on $A$, $\Pi_A(\rho_{AB})$ is the abbreviation of $(\Pi_A \otimes I_B)(\rho_{AB})$, where $I_B$ is the identity operator of system $B$. Note that $\Pi_A(\text{tr}_B(\rho_{AB})) = \text{tr}_B(\Pi_A(\rho_{AB}))$, that is, taking partial trace and performing local projective measurement can be interchanged.

It can be proved that

$$D_A(\rho_{AB}) \geq 0, \tag{2}$$

$$D_A(\rho_{AB}) = 0 \iff \rho_{AB} = \sum_{i=1}^{n_A} p_i |i\rangle \langle i| \otimes \rho_i^B, \tag{3}$$

where, $n_A = \dim A$, $\{|i\rangle\}_{i=1}^{n_A}$ is any orthonormal basis of system $A$, $\{|\rho_i^B\rangle\}_{i=1}^{n_A}$ are density operators of system $B$, $p_i \geq 0$, $\sum_{i=1}^{n_A} p_i = 1$.

The original definition of quantum discord in equation (1) is hard to calculate, by far we only know a small class of states which allow analytical expressions [2, 21, 22]. Dakic et al. proposed the geometric quantum discord, as [18]

$$D^G_A(\rho_{AB}) = \min_{\sigma_{AB}} [\text{tr}((\rho_{AB} - \sigma_{AB})^2)] : D_A(\sigma_{AB}) = 0. \tag{4}$$

Equations (1) and (4) are two very different measures of quantum correlations, but it holds that

$$D^G_A(\rho_{AB}) = 0 \iff D_A(\rho_{AB}) = 0. \tag{5}$$

We give a brief proof for equation (5). Suppose $D_A(\rho_{AB}) = 0$, then from equation (4), we can see $D^G_A(\rho_{AB}) = 0$. Conversely, if $D^G_A(\rho_{AB}) = 0$, then there must exist $\sigma_{AB}$ such that $D_A(\sigma_{AB}) = 0$ and $\text{tr}[(\rho_{AB} - \sigma_{AB})^2] = 0$. Note that $\rho_{AB} - \sigma_{AB}$ is a Hermitian operator, so it allows an eigendecomposition as $\rho_{AB} - \sigma_{AB} = \sum r_i |i\rangle \langle i|$ with $r_i$ being the real numbers. Consequently, $\text{tr}[(\rho_{AB} - \sigma_{AB})^2] = \sum r_i^2 = 0$ implies $\rho_{AB} = \sigma_{AB}$, so $D_A(\rho_{AB}) = D_A(\sigma_{AB}) = 0$.

For many cases, $D^G_A(\rho_{AB})$ is much easier to calculate than $D_A(\rho_{AB})$, since $D^G_A(\rho_{AB})$ avoids the complicated entropy function. For instance, $D^G_A(\rho_{AB})$ allows analytical expressions for all $2 \times d$ ($2 \leq d < \infty$) states [18, 22].

3. Geometric global quantum discord

In [17], the authors generalized the original definition of quantum discord to multipartite states, called global quantum discord (GQD). Consider an $N$-partite ($N \geq 2$) system, with
each subsystem $A_k$ ($1 \leq k \leq N$) having Hilbert space $H_k$ with $\dim H_k = n_k$ (we suppose $n_k < \infty$). The GQD of an $N$-partite state $\rho_{A_1A_2...A_N}$ is defined as (here we use an equivalent expression for GQD [23])

$$D(\rho_{A_1A_2...A_N}) = \sum_{k=1}^{N} S(\rho_k) - S(\rho_{A_k}^{A_2...A_N}) - \max_{\Pi} \left[ \sum_{k=1}^{N} S(\Pi_k(\rho_k)) - S(\Pi(\rho_{A_1A_2...A_N})) \right],$$

(6)

where $\Pi = \Pi_{A_1A_2...A_N}$ is a locally projective measurement on $A_1A_2...A_N$.

Similar to equations (2) and (3), we have lemma 1 below.

**Lemma 1.**

$$D(\rho_{A_1A_2...A_N}) \geq 0,$$

(7)

$$D(\rho_{A_1A_2...A_N}) = 0 \iff \rho_{A_1A_2...A_N} = \sum_{\{i_1,i_2,...,i_N\}} p_{\{i_1,i_2,...,i_N\}} |i_1\rangle \langle i_1| \otimes |i_2\rangle \langle i_2| \otimes ... \otimes |i_N\rangle \langle i_N|,$$

(8)

where $\{i_k\}_{k=1}^{N}$ is any orthonormal basis of $H_k$, $k = 1, 2, ... N$, $p_{\{i_1,i_2,...,i_N\}} \geq 0$, $\sum_{\{i_1,i_2,...,i_N\}} p_{\{i_1,i_2,...,i_N\}} = 1$.

**Proof.** Note that $\Pi_{A_1A_2...A_N}(\rho_{A_2...A_N}) = \Pi_{A_1}(\Pi_{A_2}(\Pi_{A_3}(... \Pi_{A_N}(\rho_{A_2...A_N})...)))$, and $\Pi_{A_k}(\rho_{A_k}) = \rho_{A_k}$ for $i \neq j$. On the right-hand side of equation (6),

$$\left[ \sum_{k=1}^{N} S(\rho_k) - S(\rho_{A_1A_2...A_N}) \right] - \left[ \sum_{k=1}^{N} S(\Pi_k(\rho_k)) - S(\Pi(\rho_{A_1A_2...A_N})) \right]$$

$$= \left[ \sum_{k=1}^{N} S(\rho_k) - S(\rho_{A_1A_2...A_N}) \right] - \left[ \sum_{k=1}^{N} S(\Pi_k(\rho_k)) - S(\Pi_k(\rho_{A_1A_2...A_N})) \right]$$

$$+ \left[ \sum_{k=1}^{N} S(\Pi_{A_1}(\rho_{A_k})) - S(\Pi_{A_1}(\rho_{A_1A_2...A_N})) \right] - \left[ \sum_{k=1}^{N} S(\Pi_{A_1A_2}(\rho_{A_k})) - S(\Pi_{A_1A_2}(\rho_{A_1A_2...A_N})) \right]$$

$$+ \cdots + \left[ \sum_{k=1}^{N} S(\Pi_{A_1A_2...A_{N-1}}(\rho_{A_k})) - S(\Pi_{A_1A_2...A_{N-1}}(\rho_{A_1A_2...A_N})) \right]$$

$$- \left[ \sum_{k=1}^{N} S(\Pi_{A_1A_2...A_N}(\rho_{A_k})) - S(\Pi_{A_1A_2...A_N}(\rho_{A_1A_2...A_N})) \right].$$

Then, by equations (1)–(3) and induction, with some direct calculations, equations (7) and (8) can be proved. We remark that equation (7) is also proved in [17].

With lemma 1, in the same spirit of defining geometric quantum discord for bipartite states in equation (4), we now define the GGQD below.

**Definition 1.** The GGQD of state $\rho_{A_1A_2...A_N}$ is defined as

$$D^G(\rho_{A_1A_2...A_N}) = \min_{\sigma_{A_1A_2...A_N}} \{ \text{tr} [\rho_{A_1A_2...A_N} - \sigma_{A_1A_2...A_N}]^2 : D(\sigma_{A_1A_2...A_N}) = 0 \}.$$  

(9)

With this definition, it is obvious that

$$D^G(\rho_{A_1A_2...A_N}) = 0 \iff D(\rho_{A_1A_2...A_N}) = 0.$$  

(10)

In [24], two equivalent expressions for equation (4) were given (theorems 1 and 2 in [24]), and they are very useful for simplifying the calculation of equation (4) and yielding the lower bound of equation (4) [24–26]. Inspired by this observation, we now derive the corresponding versions of these two equivalent expressions for GGQD. These are theorems 1 and 2.
Theorem 1. With $D^G(\rho_{A_1A_2...A_N})$ as defined as in equation (9), then

$$D^G(\rho_{A_1A_2...A_N}) = \min_{\Pi} \{\text{tr}[\rho_{A_1A_2...A_N} - \Pi(\rho_{A_1A_2...A_N})]^2\} = \text{tr}[\rho_{A_1A_2...A_N}]^2 - \max_{\Pi} \{\text{tr}[\Pi(\rho_{A_1A_2...A_N})]^2\}, \quad (11)$$

where $\Pi$ is any locally projective measurement performed on $A_1A_2...A_N$.

Proof. In equation (9), for any $\sigma_{A_1A_2...A_N}$ satisfying $D(\sigma_{A_1A_2...A_N}) = 0$, $\sigma_{A_1A_2...A_N}$ can be expressed in the form

$$\rho_{A_1A_2...A_N} = \sum_{i_1,i_2,...,i_N} \rho_{i_1i_2...i_N} |i_1\rangle \langle i_1| \otimes |i_2\rangle \langle i_2| \otimes \cdots \otimes |i_N\rangle \langle i_N|,$$ \quad (12)

where $\{|i_k\rangle\}_{k=1}^N$ is any orthonormal basis of $H_k$, $k = 1, 2, \ldots, N$. $P_{i_1i_2...i_N} \geq 0$, $\sum_{i_1,i_2,...,i_N} P_{i_1i_2...i_N} = 1$. We now expand $\rho_{A_1A_2...A_N}$ by the bases $\{|i_k\rangle\}_{k=1}^N = \{|j_k\rangle\}_{k=1}^N$, $k = 1, 2, \ldots, N$. Then

$$\rho_{A_1A_2...A_N} = \sum_{i_1,j_1,i_2,j_2,...,i_N,j_N} \rho_{i_1j_1i_2j_2...i_Nj_N} |i_1\rangle \langle j_1| \otimes |i_2\rangle \langle j_2| \otimes \cdots \otimes |i_N\rangle \langle j_N|,$$ \quad (13)

$$\text{tr}[\rho_{A_1A_2...A_N} - \sigma_{A_1A_2...A_N}]^2 = \text{tr}[(\rho_{A_1A_2...A_N})^2] + \sum_{i_1i_2...i_N} (P_{i_1i_2...i_N})^2$$

$$- 2 \sum_{i_1i_2...i_N} \rho_{i_1i_2...i_N} P_{i_1i_2...i_N}$$

$$= \text{tr}[(\rho_{A_1A_2...A_N})^2] + \sum_{i_1i_2...i_N} (\rho_{i_1i_2...i_N} - P_{i_1i_2...i_N})^2$$

$$- \sum_{i_1i_2...i_N} (\rho_{i_1i_2...i_N})^2.$$ \quad (14)

Hence, it is simple to see that when $\rho_{i_1i_2...i_N} = P_{i_1i_2...i_N}$ for all $i_1, i_2, \ldots, i_N$, equation (14) achieves its minimum.$\square$

Theorem 2. If $D^G(\rho_{A_1A_2...A_N})$ is defined as in equation (9), then

$$D^G(\rho_{A_1A_2...A_N}) = \sum_{\alpha_1,\alpha_2,...,\alpha_N} (C_{\alpha_1\alpha_2...\alpha_N})^2 - \max_{\Pi} \sum_{i_1i_2...i_N} \left( \sum_{\alpha_1,\alpha_2,...,\alpha_N} A_{\alpha_1i_1A_{\alpha_2i_2}...A_{\alpha_Ni_N}} C_{\alpha_1\alpha_2...\alpha_N} \right)^2,$$ \quad (15)

where $C_{\alpha_1\alpha_2...\alpha_N}$ and $A_{\alpha_1i_1}$ are all real numbers specified as follows. For any $k$, $1 \leq k \leq N$, let $L(H_k)$ be the real Hilbert space consisting of all Hermitian operators on $H_k$, with the inner product $(X|X') = \text{tr}(XX')$ of $X, X' \in L(H_k)$. For all $k$, for given orthonormal basis $\{X_{\alpha_k}\}_{\alpha_k=1}^{n_k}$ of $H_k$ (there indeed exists such a basis, see [27]) and orthonormal basis $\{|i_k\rangle\}_{k=1}^{n_k}$ of $H_k$, $C_{\alpha_1\alpha_2...\alpha_N}$ and $A_{\alpha_1i_1}$ are determined by

$$\rho_{A_1A_2...A_N} = \sum_{\alpha_1,\alpha_2,...,\alpha_N} C_{\alpha_1\alpha_2...\alpha_N} X_{\alpha_1} \otimes X_{\alpha_2} \otimes \cdots \otimes X_{\alpha_N},$$ \quad (16)

$$A_{\alpha_1i_1} = \langle i_1|X_{\alpha_1}|i_1\rangle.$$ \quad (17)
Proof. According to equation (11), and by equations (16) and (17), we have
\[
D^G(\rho_{A_1A_2\ldots A_N}) = \text{tr}[\rho_{A_1A_2\ldots A_N}]^2 - \max_{\Pi} \{\text{tr}[\Pi(\rho_{A_1A_2\ldots A_N})]^2\}
\]
\[
= \sum_{a_1a_2\ldots a_N} \langle C_{a_1a_2\ldots a_N} \rangle^2 - \max_{i_1i_2\ldots i_N} \left\{ \text{tr} \left[ \sum_{a_1a_2\ldots a_N} C_{a_1a_2\ldots a_N} (i_1|X_{a_1}|i_1) (i_2|X_{a_2}|i_2) \right. \right.
\]
\[
\left. \left. \cdots (i_N|X_{a_N}|i_N)(j_1| \otimes |i_2| \otimes \cdots \otimes |i_N|)(j_N) \right] \right)^2 \right\}
\]
\[
= \sum_{a_1a_2\ldots a_N} \langle C_{a_1a_2\ldots a_N} \rangle^2 - \max_{i_1i_2\ldots i_N} \left( \sum_{a_1a_2\ldots a_N} A_{a_1i_1}A_{a_2i_2}\ldots A_{a_Ni_N}C_{a_1a_2\ldots a_N} \right)^2.
\]

(18)

\[\square\]

4. A lower bound of GGQD

With the help of theorem 2, we now provide a lower bound for GGQD.

If we regard \(\rho_{A_1A_2\ldots A_N}\) as a bipartite state in the partition \(\{A_k, A_1\ldots A_N-A_k\}\), then the original quantum discord and geometric quantum discord of \(\rho_{A_1A_2\ldots A_N}\) with respect to the subsystem \(A_k\) can be defined according to equations (1) and (4). We denote them by \(D_{A_k}(\rho_{A_1A_2\ldots A_N})\) and \(D_{A_k}^G(\rho_{A_1A_2\ldots A_N})\). Comparing equation (3) with (8), it is easy to find that

\[D^G(\rho_{A_1A_2\ldots A_N}) = 0 \implies D_{A_k}^G(\rho_{A_1A_2\ldots A_N}) = 0.\]  

(19)

Consequently, comparing equations (4) and (9), we obtain

\[D^G(\rho_{A_1A_2\ldots A_N}) \geq D_{A_k}^G(\rho_{A_1A_2\ldots A_N}).\]  

(20)

To proceed further, we need a mathematical fact, called high-order singular value decomposition for tensors. We state it as lemma 2.

Lemma 2 [28]. High-order singular value decomposition for tensors. For any tensor \(T = \{T_{\beta_1\beta_2\ldots \beta_N} : \beta_k \in \{1, 2, \ldots, m_k\}, k = 1, 2, \ldots, N\}\), there exist unitary matrices \(U^{(k)}_{\beta_k\gamma_k} : \gamma_k \in \{1, 2, \ldots, m_k\}\), such that

\[T_{\beta_1\beta_2\ldots \beta_N} = \sum_{\gamma_1\gamma_2\ldots \gamma_N} U^{(1)}_{\beta_1\gamma_1} U^{(2)}_{\beta_2\gamma_2} \cdots U^{(N)}_{\beta_N\gamma_N} \Lambda_{\gamma_1\gamma_2\ldots \gamma_N},\]  

(21)

\[\sum_{\gamma_1\gamma_2\ldots \gamma_N} \Lambda_{\gamma_1\gamma_2\ldots \gamma_N}^* \Lambda_{\gamma_1'\gamma_2'\ldots \gamma_N'} = s^{(k)}_{\gamma_k\gamma_k'} \delta_{\gamma_k\gamma_k'},\]  

(22)

\[s^{(k)}_{\gamma_1} \geq s^{(k)}_{\gamma_2} \geq \cdots \geq s^{(k)}_{\gamma_{m_k}} \geq 0.\]  

(23)

Combining lemma 2, equation (20) and the lower bound of \(D_{A_k}^G(\rho_{A_1A_2\ldots A_N})\) in [24], we can readily obtain a lower bound of \(D^G(\rho_{A_1A_2\ldots A_N})\).

Theorem 3. \(D^G(\rho_{A_1A_2\ldots A_N})\) is defined as in equation (9), then a lower bound of \(D^G(\rho_{A_1A_2\ldots A_N})\) is

\[\text{tr}[\rho_{A_1A_2\ldots A_N}]^2 - \min \left\{ \sum_{\gamma_k=1}^{m_k} s^{(k)}_{\gamma_k} : k = 1, 2, \ldots, N \right\},\]  

(24)

where \(s^{(k)}_{\gamma_k}\) are obtained by lemma 2 in which let \(T = \{C_{a_1a_2a_N} : a_k \in \{1, 2, \ldots, m_k^2\}, k = 1, 2, \ldots, N\}, C_{a_1a_2a_N}\) are defined in theorem 2.
Proof. Since $D^G(\rho_{A_1A_2...A_N})$ and $D^G(\rho_{A_1A_2...A_N})$ remain invariant under locally unitary transformation, the state $\rho_{A_1A_2...A_N}$ in equation (16) and the state

$$\Lambda_{A_1, A_2, ..., A_N} = \sum_{a_1, a_2, ..., a_N} A_{a_1a_2...a_N} X_{a_1} \otimes X_{a_2} \otimes \cdots \otimes X_{a_N},$$

have the same GGQD, and

$$D^G(\rho_{A_1A_2...A_N}) = D^G(\Lambda_{A_1A_2...A_N}).$$

From equation (20), we have

$$D^G(\Lambda_{A_1A_2...A_N}) \geq D^G(\rho_{A_1A_2...A_N}).$$

From the definition of $D^G(\Lambda_{A_1A_2...A_N})$, lemma 2 and theorem 1 in [24], we have

$$D^G(\Lambda_{A_1A_2...A_N}) = \text{tr}[\rho_{A_1A_2...A_N}^2] - \max_{\Pi_{i_k}} \sum_{k} \left( \sum_{a_i} A_{a_i}^{a_k} s(a_i) \right)^2$$

$$\geq \text{tr}[\rho_{A_1A_2...A_N}^2] - \sum_{a_k=1}^{m} s(a_k).$$

We then attain theorem 3. □

5. Examples

We provide some special states which possess analytical GGQD.

Example 1. For the $N$-qubit ($N \geq 2$) Werner–GHZ state

$$\rho = (1 - \mu) I^{\otimes N} + \mu |\psi\rangle \langle \psi|,$$

the GGQD of $\rho$ is

$$D^G(\rho) = \mu^2/2.$$ (30)

In equation (29), $I$ is the $2 \times 2$ identity operator, $\mu \in [0, 1]$, $|\psi\rangle$ is the $N$-qubit GHZ state $|\psi\rangle = (|00...0\rangle + |11...1\rangle)/\sqrt{2}$.

Proof. We prove equation (30) according to equation (11).

$$\text{tr}(\rho^2)$$ can be directly calculated, that is,

$$\text{tr}(\rho^2) = \left( \frac{1 - \mu}{2N} + \mu \right)^2 + (2N - 1) \left( \frac{1 - \mu}{2N} \right)^2.$$

$$\max_{\Pi} \{\text{tr}[\Pi(\rho)]^2\}$$ can be obtained by the similar calculations of theorem 4 in [23], the only difference is that the monotonicity of the entropy function under the majorization relation (lemma 4 in [23]) will be replaced by the case of the function

$$f(p_1, p_2, ..., p_n) = -\sum_{i=1}^{n} p_i^2.$$ (33)

That is, $\max_{\Pi} \{\text{tr}[\Pi(\rho)]^2\}$ can be achieved by the eigenvalues

$$\left\{ \frac{1 - \mu}{2N}, \frac{\mu}{2}, \frac{1 - \mu}{2N}, \frac{1 - \mu}{2N}, \cdots, \frac{1 - \mu}{2N} \right\}.$$ (34)
Thus,
\[
\max_{\Pi} \{ \text{tr}[\Pi(\rho)]^2 \} = 2 \left( \frac{1 - \mu}{2^N} + \frac{\mu}{2} \right)^2 + (2^N - 2) \left( \frac{1 - \mu}{2^N} \right)^2. 
\] (35)

Combining equations (32) and (35), we then proved equation (30).

Example 2. For the \(N\)-qubit state
\[
\rho = \frac{1}{2^N} \left( I \otimes I^N + c_1 \sigma_x^N + c_2 \sigma_y^N + c_3 \sigma_z^N \right), 
\] (36)
the GGQD of \(\rho\) is
\[
D^G(\rho) = c_1^2 + c_2^2 + c_3^2 - \max \left\{ |c_1|, |c_2|, |c_3| \right\} \frac{2^N}{2^N}. 
\] (37)

In equation (36), \(I\) is the \(2 \times 2\) identity operator, \(\{c_1, c_2, c_3\}\) are the real numbers constrained by the condition that the eigenvalues of \(\rho\) must lie in \([0, 1]\).

Proof. We prove equation (37) by using equation (11).
\[
\text{tr}(\rho^2) \text{ can be directly found, that is,} \quad \text{tr}(\rho^2) = c_1^2 + c_2^2 + c_3^2. 
\] (38)

\(\max_{\Pi} \{ \text{tr}[\Pi(\rho)]^2 \}\) can again be obtained similarly to theorem 4 in [23], the only difference is that the monotonicity of the entropy function under the majorization relation (lemma 4 in [23]) will be replaced by the case of the function (equation (33)).

Similar reduction shows that \(\max_{\Pi} \{ \text{tr}[\Pi(\rho)]^2 \}\) can be achieved by \(\left\{ 1 \pm c \right\} \frac{2^N}{2^N}\), each of them has multiplicity \(2^N - 1\), where \(c = \max\{|c_1|, |c_2|, |c_3|\}\). Therefore,
\[
\max_{\Pi} \{ \text{tr}[\Pi(\rho)]^2 \} = 1 + \frac{c^2}{2^N}. 
\] (39)

Combining equations (38) and (39), we then obtain equation (37).

Example 3. For the \(N\)-isotropic state
\[
\rho = (1 - s) I^N + s |\phi\rangle \langle \phi|, 
\] (40)
the GGQD of \(\rho\) is
\[
D^G(\rho) = s^2 \left( 1 - \frac{1}{d} \right). 
\] (41)

where \(d = \text{dim}H_1, H_1 = H_2 = \ldots = H_N, I\) is the \(d \times d\) identity operator, \(s \in [0, 1]\), \(|\phi\rangle = \frac{1}{\sqrt{d}} \sum_l^d |l\rangle \langle l|\), \(|l\rangle\) is a fixed orthonormal basis of \(H_1\).

Proof. We prove equation (41) according to equation (9). Any locally projective measurement \(\Pi\) corresponds to \(N\) orthonormal bases of \(H_1\) denoted by \(\{|i_k\rangle\}_{k=1}^d\), \(k = 1, 2, \ldots, N\). Let \(\{|l\rangle\}_{l=1}^d = \{|m\rangle\}_{m=1}^d\), then
\[
\Pi(|\phi\rangle \langle \phi|) = \frac{1}{d} \sum_{i_k, i_k', l} |i_k\rangle \langle i_k| \langle m| \langle m|i_k'\rangle \langle i_k'| |l\rangle \langle l| \langle i_k'| \otimes \cdots \otimes |i_N\rangle \langle i_N|. 
\] (42)
Here, let the state equation (36) undergo a locally phase-flip channel with Kraus operators $s$; sudden transition phenomenon and the frozen-discord phenomenon, see e.g., a recent review [30]. We then proved equation (41).

We make some remarks. For states in equation (29) and states in equation (36), the GQD can also be analytically obtained [17, 23]; then we can compare the GQD and GGQD for these two classes of states. For states in equation (36) and states in equation (40), when $N = 2$, the GGQD in equations (37) and (41) recover the corresponding results in [29].

After this channel, the state $\rho$ in equation (36) becomes

$$\rho(p) = \frac{1}{2^N} (D^{\otimes N} + c_1(p)\sigma_x^{\otimes N} + c_2(p)\sigma_y^{\otimes N} + c_3(p)\sigma_z^{\otimes N}),$$

and the minimum can be achieved by taking $\langle i_k|l\rangle = \delta_{i_k,l}$, $\langle m|i_k\rangle = \delta_{m,i_k}$. We then proved equation (41).

So, $D^{\otimes}(\rho(p))$ can be calculated by equation (37). Therefore, similar to the bipartite case discussed in [30], it can be found that, when $0 < |c_3| = \max[|c_1|, |c_2|]$ the sudden transition occurs, when $0 < |c_3| < \max[|c_1|, |c_2|]$ and $c_1c_2 = 0$, frozen GGQD occurs.

We also remark that GGQD can increase as well as decrease under local measurements. We give such an example (for simplicity, we only consider the case $N = 2$). Consider the example in [31], the initial state

$$\tau_{ini} = \frac{1}{2}(|+\rangle\langle+| \otimes |0\rangle\langle0| + |-\rangle\langle-| \otimes |1\rangle\langle1|)$$

can create quantum discord under a local amplitude damping channel, where

$$|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle).$$

Note that the GGQD of equation (47) is zero, and quantum discord is not zero implies GGQD is not zero. That is to say, the zero-GGQD state equation (47) after a local amplitude channel becomes a nonzero-GGQD state. So, GGQD can increase under local measurements.

6. Conclusion

In summary, we generalized the geometric quantum discord of bipartite states to multipartite states, which we call geometric global quantum discord (GGQD). We gave different characterizations of GGQD which provided new insights for calculating GGQD. As demonstrations, we provided a lower bound for GGQD by using the high-order singular value decomposition of tensors, and obtained the analytical expressions of GGQD for three classes of multipartite states. We also pointed out that GGQD can also manifest the sudden transition phenomenon and the frozen-GGQD phenomenon.

Understanding and quantifying multipartite correlations is a very challenging question; we hope that the GGQD proposed in this paper may provide a useful attempt for this issue.
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