Synchronization, Consensus of Complex Networks and Lyapunov Function Approach

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Abstract—In this paper, we focus on the topic synchronization and consensus of Complex Networks and their relationships. It is revealed that two topics are closely related to each other and all results given in [1], [2] and many other papers can be obtained by the results in [10]. It is pointed out that QUAD condition plays important role in discussing synchronization and consensus.

Index Terms—Consensus, Synchronization, Synchronization Manifold.

Recently, Consensus Protocols for Linear Multi-Agent Systems has attracted some researchers’ attention see [1], [2]. Based on controllable and detectable theory for linear systems, in [1], [2], authors discussed following consensus of multiagent systems and synchronization of complex networks

\[ \dot{x}_i(t) = Ax_i(t) + Bu_i(t), \quad y_i(t) = Cx_i(t) \]  

where \( x_i(t) \in \mathbb{R}^n \) is the state, \( u_i(t) \in \mathbb{R}^p \) is the control input, and \( y_i(t) \in \mathbb{R}^q \) is the measured output. \( A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times p}, C \in \mathbb{R}^{q \times n} \). It is assumed that is stabilizable and detectable.

An observer-type consensus protocol is proposed, which can be written as

\[ \begin{align*} 
\dot{v}_i(t) &= (A + BK)v_i(t) + \sum_{j=1}^{N} F C \delta_{ij}(v_j(t) - x_j(t)) \\
\dot{x}_i(t) &= Ax_i(t) + BKv_i(t) 
\end{align*} \]  

where \( K \in \mathbb{R}^{p \times n} \), \( F \in \mathbb{R}^{n \times q} \), and \( F \).

Let \( e_i(t) = v_i(t) - x_i(t) \), one can transfer (2) to

\[ \begin{align*} 
\dot{e}_i(t) &= (A + BK)e_i(t) + FC \sum_{j=1}^{N} l_{ij} e_j(t) \\
\dot{x}_i(t) &= (A + BK)x_i(t) + BK e_i(t) 
\end{align*} \]  

In case \( A + BK \) is controllable, the synchronization of the system (2) transfers to the synchronization of the system

\[ \dot{e}_i(t) = Ae_i(t) + FC \sum_{j=1}^{N} l_{ij} e_j(t) \]  

In [10], following model was discussed

\[ \frac{dx_i(t)}{dt} = f(x_i(t)) + c \sum_{j=1}^{N} l_{ij} \Gamma x_j(t), \quad i = 1, \ldots, N \]  

where \( x_i(t) \in \mathbb{R}^n \) is the state variable of the \( i \)-th node, \( t \in [0, +\infty) \) is a continuous time, \( f : \mathbb{R} \times [0, +\infty) \to \mathbb{R}^n \) is continuous map, \( L = (l_{ij}) \in \mathbb{R}^{N \times N} \) is the coupling matrix with zero-sum rows and \( l_{ij} \geq 0 \), for \( i \neq j \), which is determined by the topological structure of the LC0DEs, and \( \Gamma \in \mathbb{R}^{n \times n} \) is an inner coupling matrix. Some time, picking \( \Gamma = \text{diag}\{\gamma_1, \gamma_2, \ldots, \gamma_n\} \) with \( \gamma_i \geq 0 \), for \( i = 1, \ldots, n \).

Let \( \Gamma = FC \). Then system (4) is a special case of (3). In fact, consensus of multiagent systems of complex networks can be viewed as special cases of the synchronization of nonlinear systems.

Therefore, main results in [1] can be obtained from the results given in [10].

In this note, we show how to apply Lyapunov function approach to the consensus of multi-agents.

I. SOME BASIC CONCEPTS AND BACKGROUND

Let’s recall some basic concepts.

Following Lemma can be found in [10] (see Lemma 1 in [10]).

Lemma 1. If \( L \) is a coupling matrix with \( \text{Rank}(L) = N - 1 \), then the following items are valid:

1) If \( \lambda \) is an eigenvalue of \( L \) and \( \lambda \neq 0 \), then \( \text{Re}(\lambda) < 0 \);  
2) \( L \) has an eigenvalue 0 with multiplicity 1 and the right eigenvector \([1, 1, \ldots, 1]^T\);  
3) Suppose \( \xi = [\xi_1, \xi_2, \ldots, \xi_m]^T \in \mathbb{R}^m \) (without loss of generality, assume \( \sum_{i=1}^{m} \xi_i = 1 \)) is the left eigenvector of \( A \) corresponding to eigenvalue 0. Then, \( \xi_i \geq 0 \) holds for all \( i = 1, \ldots, m \); more precisely, \( \xi_i > 0 \) holds for all \( i = 1, \ldots, m \);  
4) \( L \) is irreducible if and only if \( \xi > 0 \) holds for all \( i = 1, \ldots, m \);  
5) \( L \) is reducible if and only if for some \( i, \xi_i = 0 \). In such case, by suitable rearrangement, assume that \( \xi^T = [\xi_1, \xi_2, \ldots, \xi_p]^T \in \mathbb{R}^p \), with all \( \xi_i > 0 \), \( i = 1, \ldots, p \), and \( \xi_0 = [\xi_{p+1}, \xi_{p+2}, \ldots, \xi_N]^T \in \mathbb{R}^{N-p} \) with all \( \xi_i \geq 0 \), \( p + 1 \leq j \leq N \). Then, \( L \) can be rewritten as \( L = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \) where \( L_{11} \in \mathbb{R}^{p \times p} \) is irreducible and \( L_{12} = 0 \).

Definition 1. Transverse space \( \mathcal{L} = \{ x = [x_1^T, x_2^T, \ldots, x_m^T]^T : x_i \in \mathbb{R}^n, i = 1, \ldots, m, \text{ and } \sum_{k=1}^{m} \xi_k x_k = 0 \} \). For the case of \( n = 1 \) define \( L = \{ (z_1, \ldots, z_m)^T : z_i \in \mathbb{R}, \sum_{i=1}^{m} \xi_i z_i = 0 \} \).

For the convenience of later use, introduce the following notations:

\( \dot{x}(t) = \sum_{i=1}^{m} \xi_i x_i(t), \)

\( \dot{X}(t) = [\dot{x}(t), \dot{x}(t), \ldots, \dot{x}(t)]^T \in \mathcal{S} \), which can be regarded as a projection of \( x(t) = [x_1(t), x_2(t), \ldots, x_m(t)]^T \) on the synchronization manifold \( \mathcal{S} \) (generally, nonorthogonal).
Denote $\delta x(t) = \begin{bmatrix} \delta x_1(t) \quad \cdots \quad \delta x_m(t) \end{bmatrix}^T$, where $\delta x_i(t) = x_i(t) - \bar{x}(t)$, $i = 1, \cdots, m$. It is easy to see that $\sum_{i=1}^{m} \xi_i \delta x_i(t) = 0$.

Thus, we have following result.

**Proposition 1.** For any $x = (x_1^T, \cdots, x_m^T) \in \mathbb{R}^{mn}$, we have $x = \bar{x} + \delta x$, where $\bar{x}$ and $\delta x$ are defined as above, and it holds that $\bar{x} \in S$ and $\delta x \in \mathcal{L}$.

With this decomposition, the stability of the synchronization manifold $S$ for the model (5) is equivalent to $\delta x(t) = 0$. Equivalently, the dynamical flow in the $(m - 1) \times n$ dimensional subspace $\mathcal{L}$ converges to zero. In the sequel, instead of investigating $x_i(t)$, we investigate dynamical behaviors of $\delta x_i(t)$ directly.

Following function class also plays key role in discussing synchronization and consensus with Lyapunov functions.

**Definition 2.** Function class QUAD($\Delta, P$): let $P = \text{diag}(p_1, \cdots, p_n)$ is a positive definite diagonal matrix and $\Delta = \text{diag}(\delta_1, \cdots, \delta_n)$ is a diagonal matrix. QUAD($\Delta, P$) denotes a class of continuous functions $f(x, t) : \mathbb{R}^n \times [0, +\infty) \rightarrow \mathbb{R}^n$ satisfying

$$
(x - y)^T P \left\{ f(x, t) - f(y, t) - \Delta [x - y] \right\} \leq -\epsilon (x - y)^T (x - y)
$$

(6)

for some $\epsilon > 0$, all $x, y \in \mathbb{R}^n$ and $t > 0$.

**Remark 1.** The matrix $P$ and $\Delta$ can be replaced by any other suitable matrices. In general,

$$
(x - y)^T P \left\{ f(x, t) - f(y, t) - \Gamma [x - y] \right\} \leq -\epsilon (x - y)^T (x - y)
$$

(7)

where $P$ and $\Gamma = BB^T$ are positive definite matrices. In some cases, for example, $P$ and $\Gamma$ are commutable, by a suitable coordinate transform, it can be seen that both (6) and (8) are equivalent.

II. Synchronization Analysis of Complex Networks with Lyapunov Functions

Based on the synchronization state $\bar{x}(t)$, decomposition $\delta x(t) = x(t) - \bar{x}(t)$, and QUAD condition, synchronization problem of Complex Networks can be solved easily with Lyapunov function, which was first proposed in [10].

Since $\sum_{j=1}^{m} l_{ij} = 0$, it is clear that

$$
\sum_{j=1}^{m} l_{ij} \Gamma x_j(t) = \sum_{j=1}^{m} l_{ij} \Gamma \delta x_j(t)
$$

Therefore,

$$
\frac{d\delta x_i(t)}{dt} = f(x_i(t)) - f(\bar{x}(t)) + \sum_{j=1}^{m} l_{ij} \Gamma \delta x_j(t) + J
$$

where $J = f(\bar{x}(t)) - \sum_{k=1}^{m} \xi_k f(x_k(t))$ is independent of any index $i$.

Define a Lyapunov function as first proposed in [10],

$$
V(\delta x) = \frac{1}{2} \delta x^T \Xi P \delta x = \frac{1}{2} \sum_{i=1}^{m} \xi_i \delta x_i^T P \delta x_i
$$

where $\Xi = \Xi \otimes I_m$ and $P = I_n \otimes P$.

Denote $\delta y(t) = B^T \delta x(t)$, and differentiating $V(\delta x)$ (noticing that $\sum_{i=1}^{m} \xi_i \delta x_i^T J = 0$ and $f(x) \in QUAD$ (8)), we have

$$
\frac{dV(\delta x)}{dt} = \sum_{i=1}^{m} \xi_i \delta x_i^T (t) P \frac{d\delta x_i(t)}{dt}
$$

$$
\leq -\epsilon \delta x^T (t) \Xi \delta x(t) + \delta y^T (t) \left\{ [\Xi + c(\Xi L)^s] \otimes I_m \right\} \delta y(t)
$$

Let $0 = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_m$ be the eigenvalues of the matrix $(\Xi A)^s$. Noting

$$
\delta y(t)^T [\Xi \otimes I_m] \delta y(t) \leq \max_{i=1, \cdots, m} \{ \xi_i \} \delta y(t)^T \delta y(t)
$$

(9)

and

$$
\delta y(t)^T [(\Xi L)^s \otimes I_m] \delta y(t) \leq \frac{\lambda_2}{m \sum_{i=1}^{m} \xi_i} \delta y(t)^T \delta y(t)
$$

Then, in case that

$$
c > \max_{i=1, \cdots, m} \{ \xi_i \} \left( -\frac{\lambda_2}{m \sum_{i=1}^{m} \xi_i} \right)^{-1}
$$

we have

$$
\delta y(t)^T [\Xi \otimes I_m] \delta y(t) + c \delta y(t)^T [(\Xi L)^s \otimes I_m] \delta y(t) \leq 0
$$

and there exists a constant $c_1 > 0$, such that

$$
\frac{dV(\delta x)}{dt} \leq -\epsilon \delta x^T (t) \Xi \delta x(t) < -c_1 V(\delta x)
$$

and $V(t)$ converges to zero exponentially.

Now, we can give following

**Proposition 1.** Under the QUAD condition (8), the system (5) can reach synchronization if the coupling strength $c$ is large enough.

**Remark 2.** In the abstract of [10], it is said that a general framework is presented for analyzing the synchronization stability of Linearly Coupled Ordinary Differential Equations (LCODEs). The uncoupled dynamical behavior at each node is general, which can be chaotic or others; the coupling configuration is also general, with the coupling matrix not assumed to be symmetric or irreducible. On the basis of geometrical analysis of the synchronization manifold, a new approach is proposed for investigating the stability of the synchronization manifold of coupled oscillators. In this way, criteria are obtained for both local and global synchronization. These criteria indicate that the left and right eigenvectors corresponding to eigenvalue zero of the coupling matrix play key roles in the stability analysis of the synchronization manifold. Furthermore, the roles of the uncoupled dynamical behavior on each node and the coupling configuration in the synchronization process are also studied.
A. Pinning Control Synchronization of Complex Networks

Let $s(t)$ be a solution of $\dot{s}(t) = g(s(t))$. Consider the following pinning control model

$$\begin{aligned}
\frac{dx(t)}{dt} &= f(x(t)) + c \sum_{j=1}^{m} \ell_{ij} \Gamma x_j(t), \\
-\epsilon x_1(t) &= s(t), \\
\frac{dx_1(t)}{dt} &= g(x_1(t)) + c \sum_{j=1}^{m} \ell_{ij} \Gamma x_j(t),
\end{aligned}$$

(10)

As addressed in [11], following proposition plays a key role.

**Proposition 2.** (see [11]) Under the QUAD condition, the system (8) can synchronize all $x_i(t)$, let $\ell_{ij}$ be the eigenvalues of $L$ with eigenvalue $\lambda$. Then, $L$ is a non-singular M-matrix: all the eigenvalues of $L$ have negative real part, and all the eigenvalues of the matrix $(\Xi L)^{T}$ satisfy $\text{rank}(L) = m-1$, satisfying $\ell_{ij} \geq 0$, if $i \neq j$, and $\sum_{j=1}^{m} \ell_{ij} = 0$, for $i = 1, 2, \ldots, m$.

Let

$$\tilde{L} = \begin{pmatrix}
\ell_{11} - \epsilon & \ell_{12} & \cdots & \ell_{1m} \\
\ell_{21} & \ell_{22} & \cdots & \ell_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
\ell_{m1} & \ell_{m2} & \cdots & \ell_{mm}
\end{pmatrix}$$

Proposition 3. Under the QUAD condition, the system (11) can synchronize all $x_i$ to $s(t)$, if the coupling strength is large enough.

**Remark 3.** In some paper, authors discussed leader-follower system. In fact, it is just a special case of pinning control system. It was clearly addressed in [11].

B. Adaptive Algorithms

In previous parts, we revealed that we can always synchronize pinning a coupled complex network if the coupling strength is large enough. However, in practice, it is not allowed that the coupling strength is arbitrarily large. For synchronization, it was pointed out in [12] that theoretical value of the coupling strength is much larger than needed in practice. Therefore, the following question was arisen in [12]: Can we find the sharp bound $c_{\min}$? Similarly, in pinning process, it is also important to make the coupling strength as small as possible. It is clear that theoretical value of strength given in previous theorems are highly based on the QUAD condition, which is too strong. Therefore, it is possible to lessen coupling strength dramatically.

For this purpose, consider following adaptive algorithm

$$\begin{aligned}
\frac{dx_i(t)}{dt} &= f(x_i(t)) + c(t) \sum_{j=1}^{m} \ell_{ij} \Gamma x_j(t), \\
\dot{c}(t) &= \frac{m}{2} \sum_{i=1}^{m} \delta x_i^T(t)BB^T \delta x_i(t)
\end{aligned}$$

(12)

$$\begin{aligned}
\frac{dx_1(t)}{dt} &= f(x_1(t)) + c(t) \sum_{j=1}^{m} \ell_{ij} \Gamma x_j(t), \\
\dot{c}(t) &= \frac{m}{2} \sum_{i=1}^{m} \delta x_i^T(t)BB^T \delta x_i(t)
\end{aligned}$$

(13)

$$\delta x_i(t) = x_i(t) - \bar{x}(t)$$

for synchronization/consensus without control.

$$\begin{aligned}
\frac{dx_i(t)}{dt} &= f(x_i(t)) + c(t) \sum_{j=1}^{m} \ell_{ij} \Gamma x_j(t), \\
\dot{c}(t) &= \frac{m}{2} \sum_{i=1}^{m} \delta x_i^T(t)BB^T \delta x_i(t)
\end{aligned}$$

Here, we just give a brief proof for the adaptive algorithm (13), which can be traced to [11].

**Proof:** Pick a constant $\alpha > 0$. Define a Lyapunov function

$$V_2(\delta x(t)) = \frac{1}{2} \sum_{i=1}^{m} \xi_i \delta x_i^T(t)P \delta x_i(t) + \frac{\beta}{\alpha} (c - c(t))^2;$$

(14)

where constants $c$ and $\beta$ will be decided later.

Differentiating $V_2(\delta x)$ (noticing that $\sum_{i=1}^{m} \xi_i \delta x_i^T J = 0$ and $f(x) \in QUAD$), we have

$$\begin{aligned}
\frac{dV_2(\delta x(t))}{dt} &= \frac{m}{2} \sum_{i=1}^{m} \xi_i \delta x_i^T(t)P \frac{d\delta x_i(t)}{dt} \\
&\leq -\epsilon \delta x^T(t)\Xi \delta x(t) + \frac{\beta}{\alpha} (c - c(t))^2
\end{aligned}$$

and $V_1(t)$ converges to zero exponentially.

**Proposition 4.** Under the QUAD condition, the system (11) can synchronize all $x_i$ to $s(t)$, if the coupling strength is large enough.
Then, pick $c$ sufficiently large such that $\beta c > 1$, we have

$$
\frac{dV(\delta x)}{dt} \leq -c \sum_{i=1}^{m} \xi_i \delta x_i^T(t) \delta x_i(t)
$$

Therefore,

$$
\epsilon \int_{0}^{\infty} \sum_{i=1}^{m} \xi_i \delta x_i(t)^T \delta x_i(t) dt < V_2(0) - V_2(\infty)
$$

which implies $\delta x_i(t) \rightarrow 0$ and $c(t) \rightarrow c_0$, where $c_0$ is a positive constant. The proof is completed.

Therefore, we have

**Proposition 4.** If $L$ is a connecting coupling matrix and QUAD condition (8) is satisfied, then the adaptive algorithm (12) can reach synchronization.

Now, consider so called distributive adaptive algorithms

$$
\begin{aligned}
\frac{dx_i(t)}{dt} &= f(x_i(t)) + \sum_{j=1}^{m} c_{ij}(t) l_{ij} P^{-1} B^T B c_j(t) x_j(t), \\
\dot{c}_{ij} &= \frac{2}{\delta} \delta x_i^T(t) BB^T \delta x_i(t)
\end{aligned}
$$

(15)

proposed in [14] for more general cluster synchronization. And in case we do not know the structure of the coupled system, a simple approach is to adapt the coupling weight.

Following adaptive algorithm

$$
\begin{aligned}
\frac{dx_i(t)}{dt} &= f(x_i(t)) + c \sum_{j=1}^{m} \omega_{ij}(t) \Gamma(x_j(t) - x_i(t)), \\
\dot{\omega}_{ij} &= \rho_{ij} \xi_i [x_i(t) - \bar{x}(t)]^T \Gamma [x_i(t) - x_j(t)]
\end{aligned}
$$

(16)

proposed in [13], which adapts all weights for more general cluster synchronization.

$$
\dot{x}_i(t) = f(x_i(t)) + c \sum_{j=1}^{m} l_{ij} \Gamma x_j(t)
$$

(17)

Here, we just give simple derivations for the algorithm (16), as for the algorithm (15), readers can refer to [14].

Define Lyapunov function

$$
V_3(t) = \frac{1}{2} \sum_{i=1}^{m} \xi_i (x_i(t) - \bar{x}(t))^T P(x_i(t) - \bar{x}(t)) + \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \frac{1}{\rho_{ij}} (\omega_{ij}(t) - cl_{ij})^2
$$

(18)

$$
V_3(t) = \sum_{i=1}^{m} \xi_i (x_i(t) - \bar{x}(t))^T P(f(x_i(t)) - f(\bar{x}(t)))
$$

$$
+ \sum_{i,j=1}^{m} \xi_i (x_i(t) - \bar{x}(t))^T w_{ij}(t) P \Gamma(x_j(t) - x_i(t))
$$

$$
+ \sum_{i,j=1}^{m} (w_{ij}(t) - cl_{ij}) \xi_i (x_i(t) - \bar{x}(t))^T \Gamma(x_i(t) - x_j(t))
$$

$$
= \sum_{i=1}^{m} \xi_i (x_i(t) - \bar{x}(t))^T P(f(x_i(t)) - f(\bar{x}(t)))
$$

$$
+ \sum_{i,j=1}^{m} \xi_i (x_i(t) - \bar{x}(t))^T l_{ij} \Gamma(x_j(t) - x_i(t))
$$

$$
= \sum_{i=1}^{m} \xi_i (x_i(t) - \bar{x}(t))^T P(f(x_i(t)) - f(\bar{x}(t)))
$$

$$
+ c \sum_{i,j=1}^{m} \xi_i (x_i(t) - \bar{x}(t))^T l_{ij} \Gamma(x_j(t) - x_i(t))
$$

$$
+ c \sum_{i,j=1}^{m} \xi_i (x_i(t) - \bar{x}(t))^T l_{ij} \Gamma(x_j(t) - \bar{x}(t))
$$

(19)

By similar arguments, for sufficient large $c$, we have

$$
\dot{V_3}(t) < -c \sum_{i=1}^{m} \xi_i \delta x_i(t)^T \delta x_i(t)
$$

and

$$
\epsilon \int_{0}^{\infty} \sum_{i=1}^{m} \xi_i \delta x_i(t)^T \delta x_i(t) dt < V_2(0) - V_2(\infty)
$$

which implies $\delta x(t) \rightarrow 0$.

Therefore, we can give

**Proposition 5.** If $L$ is a connecting coupling matrix and QUAD condition (8) is satisfied, then the adaptive algorithms (15) or (16) can reach synchronization.

Instead, in [8]

$$
\begin{aligned}
\frac{dx_i(t)}{dt} &= Ax_i(t) + c \sum_{j=1}^{m} \omega_{ij}(t) \Gamma(x_j(t) - x_i(t)), \\
\dot{\omega}_{ij} &= \rho_{ij} \xi_i [x_i(t) - x_j(t)]^T \Gamma [x_i(t) - x_j(t)]
\end{aligned}
$$

(20)

for the system

$$
\dot{x}_i(t) = Ax_i(t) + c \sum_{j=1}^{m} l_{ij} \Gamma x_j(t)
$$

(21)

where $L$ corresponds to an indirected graph.

In this case,

$$
[x_i(t) - \bar{x}(t)]^T \Gamma [x_i(t) - x_j(t)]
$$

$$
= - [x_j(t) - \bar{x}(t)]^T \Gamma [x_i(t) - x_j(t)]
$$

$$
= [x_i(t) - x_j(t)]^T \Gamma [x_i(t) - x_j(t)]
$$

By previous proposition, we have

**Proposition 6.** If $L$ is a symmetric connecting coupling matrix and QUAD condition (8) is satisfied, then the adaptive algorithm (20) can reach synchronization.

Therefore, the model (20) discussed in [3] is a special case of the model (16) discussed in [13].
III. CONSENSUS OF MULTIAGENT SYSTEMS OF COMPLEX NETWORKS WITH LYAPUNOV FUNCTIONS

In model (5), let \( f(x_i(t)) = Ax_i(t) \), we obtain

\[
\frac{dx_i(t)}{dt} = Ax_i(t) + c \sum_{j=1}^{N} l_{ij} \Gamma x_j(t), \quad i = 1, \ldots, N \tag{22}
\]

Then, all results obtained in previous section can apply to the consensus.

First of all, we discuss the case \((A, C)\) is detectable. In this case, for some fixed \( \bar{t} \),

\[
P = 2 \int_0^{\bar{t}} e^{-At} C^T C e^{-At} dt > 0
\]

\[
PA + A^T P = -2 \int_0^{\bar{t}} \frac{d}{dt} [e^{-At} C^T C e^{-At}] dt
\]

\[
= 2C^T C - 2e^{-At} C^T C e^{-At}
\]

Therefore, there exists \( \epsilon > 0 \) such that

\[
PA + A^T P - 2CTC < -e^{-At} C^T C e^{-At} < -\epsilon I_n
\]

which is equivalent to the QUAD condition

\[
(x-y)^T P \left\{ A(x-y) - P^{-1} BB^T (x-y) \right\} \leq -\epsilon (x-y)^T (x-y)
\]

Proposition 7. Under the QUAD condition (24), or \((A, C)\) is detectable, the system

\[
\dot{x}_i(t) = Ax_i(t) + c P^{-1} BT Bc(t) x_j(t)
\]

(25)

can reach synchronization if the coupling strength \( c \) is large enough.

Proposition 8. If \((A, B)\) is controllable, then

\[
\dot{x}_i(t) = A\delta x_i(t) + c P^{-1} BB^T \sum_{j=1}^{N} l_{ij} x_j(t)
\]

where the QUAD condition

\[
(x-y)^T P \left\{ A(x-y) - P^{-1} BB^T (x-y) \right\} \leq -\epsilon (x-y)^T (x-y)
\]

is satisfied, can reach consensus for sufficient large constant \( \epsilon \).

As direct consequences of Proposition 3, we have

Proposition 9. If \((A, B)\) is controllable, then

\[
\begin{align*}
\frac{dx_1(t)}{dt} &= Ax_1(t) + c \sum_{j=1}^{m} l_{1j} P^{-1} BB^T x_j(t), \quad i = 1, \ldots, N \\
\end{align*}
\]

and the matrix \( P \) satisfies the QUAD condition

\[
(x-y)^T P \left\{ A(x-y) - P^{-1} BB^T (x-y) \right\} \leq -\epsilon (x-y)^T (x-y)
\]

(30)

(31)

can reach consensus to the trajectory \( \dot{s}(t) = s(t) \) for sufficient large constant \( \epsilon \).

In [10], fully distributed consensus protocols for linear multi-agent systems were discussed. In fact, we should consider following systems

\[
\begin{align*}
\frac{dx_1(t)}{dt} &= Ax_1(t) + \sum_{j=1}^{m} c_i(t) l_{1j} P^{-1} BB^T x_j(t) \\
\frac{dx_2(t)}{dt} &= Ax_2(t) + \sum_{j=1}^{m} a_{ij} P^{-1} B B^T x_j(t), \\
\end{align*}
\]

(32)

which can be rewritten as

\[
\begin{align*}
\frac{dx_1(t)}{dt} &= Ax_1(t) + \sum_{j=1}^{m} c_i(t) l_{1j} P^{-1} BB^T x_j(t), \\
\frac{dx_2(t)}{dt} &= Ax_2(t) + \sum_{j=1}^{m} a_{ij} P^{-1} B B^T x_j(t), \\
\end{align*}
\]

(33)

where \( c(0) \geq 0 \) and \( \alpha > 0 \), can synchronize the coupled system to the given trajectory \( s(t) \).

For proof, what we need to do is to replace \( V_2(t) \) by following

\[
V_4(t) = \frac{1}{2} \sum_{i=1}^{m} \xi_i \delta x_i(t) P \delta x_i(t) + \sum_{i=1}^{m} \frac{\beta}{\alpha} (c_i(t))^2;
\]

(34)

and proceed the same way as before. The details are omitted.

IV. COMPARISIONS

In [10], time varying synchronization state \( \bar{x}(t) \) as a non-orthogonal projection in synchronization manifold was first introduced and a distance between the state and the synchronization manifold \( \delta(x(t) = x(t) - \bar{x}(t) \) was used to discuss synchronization was proposed and played key role.

It is clear that \( \delta(t) \) used in [11, 22, 24, 25] and other papers, \( \delta(t) \) is nothing new other than the \( \delta(x(t) = x(t) - \bar{x}(t) \), though the authors did not mention this fact and cite [10].

In [10], following results were given, too.

Theorem 1. see [10]

Let \( \lambda_2, \lambda_3, \ldots, \lambda_m \) be the non-zero eigenvalues of the coupling matrix \( L \). If either one condition is satisfied

1) all variational equations

\[
\frac{dz(t)}{dt} = [Df(x(t), t) + \lambda_k \Gamma] z(t), \quad k = 2, 3, \ldots, m
\]

(35)

are exponentially stable,

2) or there exist a positive definite matrix \( P \) and a constant \( \epsilon > 0 \), such that

\[
(P(D(t) + \lambda_k \Gamma))^s < -\epsilon I_n, \quad k = 2, 3, \ldots, m
\]

(36)
where \( D(t) = (D_{ij}(t)) \) denotes the Jacobian matrix \( Df(\tilde{x}(t), t) \), \( H^* = (H^* + H)/2 \), \( H^* \) is Hermite conjugate of \( H \), and \( I_n \in \mathbb{R}^{n \times n} \) is identity matrix, then the synchronization manifold \( S \) is locally exponentially stable for the coupled system

\[
\dot{x}_i(t) = f(x_i(t)) + \sum_{j=1}^{m} l_{ij} \Gamma x_j(t)
\]  

(37)

Similarly, noting that \( \tilde{L} \) is a non-singular M-matrix see [11], we have

**Theorem 2.** Let \( 0 > \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \) be the eigenvalues of the coupling matrix \( \tilde{L} \). If either one condition is satisfied

1) all variational equations

\[
\frac{dz(t)}{dt} = [Df(s(t)) + \lambda_k \Gamma]z(t), \quad k = 1, 2, \cdots, m
\]

(38) are exponentially stable,

2) or there exist a positive definite matrix \( P \) and a constant \( \epsilon > 0 \), such that

\[
\left\{ P(Df(s(t)) + \lambda_k \Gamma) \right\}^* < -\epsilon I_n, \quad k = 1, 2, \cdots, m
\]

(39) then \( s(t) \) satisfying \( \dot{s}(t) = f(s(t)) \) is locally exponentially stable for the coupled system

\[
\dot{x}_i(t) = f(x_i(t)) + \sum_{j=1}^{m} \tilde{l}_{ij} \Gamma (x_j(t) - s(t))
\]

(40)

**A. On the paper Consensus of Multiagent Systems and Synchronization of Complex Networks: A Unified Viewpoint [11]**

Large number of results concerning with the consensus of multi-agents can be derived from above Theorem 1 and Theorem 2 as special cases.

In fact, let \( f(x) = Ax \), synchronization model becomes consensus of multi-agents

\[
\dot{x}_i(t) = Ax_i(t) + c \sum_{j=1}^{m} l_{ij} \Gamma x_j(t)
\]

(41) and the local consensus and global consensus are equivalent. Then, as special cases of Theorem 1 and Theorem 2, we have two following Theorems.

**Theorem 3.** Let \( \lambda_2, \lambda_3, \cdots, \lambda_m \) be the non-zero eigenvalues of the coupling matrix \( L \). If either one condition is satisfied

1) all variational equations

\[
\frac{dz(t)}{dt} = [A + \lambda_k \Gamma]z(t), \quad k = 2, 3, \cdots, m
\]

(42) are exponentially stable,

2) or there exist a positive definite matrix \( P \) and a constant \( \epsilon > 0 \), such that

\[
\left\{ P(A + \lambda_k \Gamma) \right\}^* < -\epsilon I_n, \quad k = 2, 3, \cdots, m
\]

(43) then the synchronization manifold \( S \) is globally exponentially stable for the coupled system

\[
\dot{x}_i(t) = Ax + \sum_{j=1}^{m} l_{ij} \Gamma x_j(t)
\]

(44)

**Theorem 4.** Let \( 0 > \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \) be the eigenvalues of the coupling matrix \( L \). If either one condition is satisfied

1) all variational equations

\[
\frac{dz(t)}{dt} = [A + \lambda_k \Gamma]z(t), \quad k = 1, 2, \cdots, m
\]

(45) are exponentially stable,

2) or there exist a positive definite matrix \( P \) and a constant \( \epsilon > 0 \), such that

\[
\left\{ P(A + \lambda_k \Gamma) \right\}^* < -\epsilon I_n, \quad k = 1, 2, \cdots, m
\]

(46) then \( s(t) \) satisfying \( \dot{s}(t) = As(t) \) is globally exponentially stable for the coupled system

\[
\dot{x}_i(t) = Ax_i(t) + \sum_{j=1}^{m} \tilde{l}_{ij} \Gamma (x_j(t) - s(t))
\]

(47)

We will show that the results given in the following observer-type consensus protocol of multi-agents and synchronization of complex networks discussed in [11, 2] and some other papers

\[
\begin{cases}
\dot{x}_i(t) = Ax_i(t) + u_i(t), \\
\dot{y}_i(t) = Cx_i(t), \\
\dot{v}_i(t) = (A + BK)v_i(t) + cF \sum_{j=1}^{N} C_{ij}(v_j(t) - x_j(t)) \\
\dot{u}_i = Kv_i
\end{cases}
\]

(48) can be easily derived from the results in [10].

By routine technique used in linear system theory, let \( e_i(t) = v_i(t) - x_i(t) \), then (48) becomes

\[
\begin{cases}
\dot{x}_i(t) = Ax_i(t) + BK \dot{v}_i(t), \\
\dot{e}_i(t) = A \dot{e}_i(t) + cF \sum_{j=1}^{N} C_{ij} \dot{e}_j(t)
\end{cases}
\]

(49)

In case \((A, B)\) is controllable, the synchronization of the system (48) transfers to the synchronization of the system

\[
\dot{e}_i(t) = A \dot{e}_i(t) + cF \sum_{j=1}^{N} l_{ij} \dot{e}_j(t)
\]

(50) By previous results, we have

**Theorem 5.** Let \( \lambda_2, \lambda_3, \cdots, \lambda_m \) be the non-zero eigenvalues of the coupling matrix \( L \). If \((A, B)\) is controllable, either one condition is satisfied

1) all variational equations

\[
\frac{dz(t)}{dt} = [A + c \lambda_k FC]z(t), \quad k = 2, 3, \cdots, m
\]

(51) are exponentially stable,

2) or there exist a positive definite matrix \( P \) and a constant \( \epsilon > 0 \), such that

\[
\left\{ P(A + c \lambda_k FC) \right\}^* < -\epsilon I_n, \quad k = 2, \cdots, m
\]

(52)
then the synchronization manifold $S$ is globally exponentially stable for the coupled system (48).

**Theorem 6.** Let $0 > \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$ be the eigenvalues of the coupling matrix $\hat{L}$. If either one condition is satisfied

1) all variational equations

$$\frac{dz(t)}{dt} = [A + c_\lambda FC]z(t), \quad k = 1, 2, \cdots, m \quad (53)$$

are exponentially stable,

2) or there exist a positive definite matrix $P$ and a constant $\epsilon > 0$, such that

$$\left\{ P(A + c_\lambda FC) \right\}^s < -\epsilon I_n, \quad k = 1, 2, \cdots, m \quad (54)$$

then 0 is exponentially stable for the coupled system

$$\dot{x}_i(t) = Ax_i(t) + c \sum_{j=1}^m \hat{L}_{ij} FCx_j(t) \quad (55)$$

Therefore, Theorem 1 in [11] is just item 1 in Theorem 5 given above, and is a simple consequence of the results given in [10]. Corollary 1 in [7] is just item 1 in Theorem 5 given above, and is simple consequence of the results given in [10].

In case $(A, C)$ is detectable, it can be proved that there exists a positive definite matrix $P$ such that

$$PA + A^TP - C^TC < -\epsilon I_n \quad (56)$$

which was reported in [11].

In this case, let $\Gamma = P^{-1}C^TC$, and by (56), if $c > \frac{1}{|\text{Re} x_2|}$, we have

$$\left\{ P(A + \lambda_k \Gamma) \right\}^s = PA + A^TP + c\text{Re}((\lambda_k)C^TC < -\epsilon I_n \quad (57)$$

Therefore, by the item 2 in Theorem 3, we have

**Corollary 1.** If $(A, C)$ is detectable, and $c > \frac{1}{|\text{Re} x_2|}$, the system

$$\dot{x}_i(t) = Ax_i(t) + c \sum_{j=1}^m \hat{L}_{ij} P^{-1}C^TCx_j(t) \quad (58)$$

is exponentially stable.

Many results given in [2] can also be obtained as direct consequences of those given in [10].

**B. Consensus of Multi-Agent Systems With General Linear and Lipschitz Nonlinear Dynamics [3]**

Similarly, in case $(A, B)$ is controllable, then there exists a positive definite matrix $P$ such that

$$PA + A^TP = BB^T < -\epsilon I_n \quad (59)$$

**Corollary 2.** If $L$ is symmetric, $(A, B)$ is controllable, and $c > \frac{1}{|\lambda_2|}$, the system

$$\dot{x}_i(t) = Ax_i(t) + c \sum_{j=1}^m \hat{L}_{ij} P^{-1}BB^T x_j(t) \quad (60)$$

is exponentially stable.

It means that Lemma 1 in [3] is a direct consequence of the Corollary given in [10].

As pointed out above that the adaptive algorithm

$$\begin{align*}
\frac{dx_i(t)}{dt} &= Ax_i(t) + c \sum_{j=1}^m w_{ij}(t)\Gamma(x_j(t) - x_i(t)), \\
\dot{w}_{ij}(t) &= \rho_{ij}\xi_i[x_i(t) - x_j(t)]^T \Gamma[x_i(t) - x_j(t)]
\end{align*} \quad (61)$$

for the system model

$$\dot{x}_i(t) = Ax_i(t) + c \sum_{j=1}^m l_{ij}\Gamma x_j(t) \quad (62)$$

discussed in [3, 5], where $L$ corresponds to an indirected graph, is a special case of the adaptive algorithm

$$\begin{align*}
\frac{dx_i(t)}{dt} &= f(x_i(t))) + c \sum_{j=1}^m w_{ij}(t)\Gamma(x_j(t) - x_i(t)), \\
\dot{w}_{ij}(t) &= \rho_{ij}\xi_i[x_i(t) - \bar{x}(t)]^T \Gamma[x_i(t) - x_j(t)]
\end{align*} \quad (63)$$

for the system

$$\dot{x}_i(t) = f(x_i(t))) + c \sum_{j=1}^m l_{ij}\Gamma x_j(t) \quad (64)$$

where $L$ corresponds to a directed graph discussed in [13] for adaptive cluster synchronization algorithm

Moreover, Theorem 1 in [3] is a direct consequence of the corresponding adaptive cluster synchronization algorithm in [13].

In [3], authors considered following mixed model

$$\dot{x}_i(t) = Ax_i(t) + f(x_i(t)) + \sum_{j=1}^m l_{ij}\Gamma x_j(t) \quad (65)$$

In fact, let $g(x) = Ax + f(x)$, which is a special case of the model (5).

$$\dot{x}_i(t) = g(x_i(t)) + \sum_{j=1}^m l_{ij}\Gamma x_j(t) \quad (66)$$

discussed in [10].

Notice $(x - y)^T P[f(x) - f(y)] \leq (x - y)^T (PP + \gamma^2 I)(x - y)$, under the Theorem 2' assumptions, we have

$$\begin{align*}
(x - y)^T P \left\{ A(x - y) + [f(x) - f(y)] - P^{-1}\Gamma(x - y) \right\} \\
\leq -\epsilon(x - y)^T (x - y)
\end{align*} \quad (67)$$

with $\Gamma = BB^T$, which is equivalent to

$$\begin{pmatrix}
PA + A^TP - BB^T - \gamma^2 I & P \\
P & -I
\end{pmatrix} > 0 \quad (68)$$

Therefore, Theorem 2 in [3] is a direct consequence of the results given in [10].
C. Designing Fully Distributed Consensus Protocols for Linear Multi-Agent Systems With Directed Graphs

In [4], authors discussed leader-follower consensus problem for the agent. In fact, it is nothing new other than Pinning Complex Networks by a Single Controller discussed in [11].

In [4], following Lemma was given

Lemma 4 There exists a positive diagonal matrix G such that 
GL + L^T G > 0, where G > 0. One such G is given by diag(q_1,\ldots,q_N), where q = [q_1,\ldots,q_N]^T = (L^T)^{-1}L.

In fact, it has been pointed out many years ago in [11], where it was revealed that all the eigenvalues of the matrix \{\Xi L\}^x = \frac{1}{2}[\Xi L + L^T \Xi] are negative.

In [2], [5], fully distributed consensus protocols for linear multi-agent systems were discussed. In fact, we should consider following systems

\[
\begin{align*}
\frac{dx_1(t)}{dt} &= Ax_1(t) + \sum_{j=1}^{m} c_i(t) l_{ij} P^{-1} B^T B c_j(t) x_j(t) \\
&\quad - c_i(t) (x_1(t) - s(t)), \\
\frac{dx_2(t)}{dt} &= Ax_2(t) + \sum_{j=1}^{m} a_{ij} P^{-1} B^T B c_j(t) x_j(t), \\
&\quad \dot{c}_i(t) = -\frac{4}{3} \delta x_i^T(t) BB^T \delta x_i(t)
\end{align*}
\]  
(69)

which can be rewritten as

\[
\begin{align*}
\frac{dx_3(t)}{dt} &= Ax_3(t) + \sum_{j=1}^{m} c_i(t) l_{ij} P^{-1} B^T B c_j(t) x_j(t), \\
&\quad \dot{c}_i(t) = -\frac{4}{3} \delta x_i^T(t) BB^T \delta x_i(t)
\end{align*}
\]  
(70)

where c(0) ≥ 0 and α > 0, can synchronize the coupled system to the given trajectory s(t).

For proof, what we need to do is to replace V_2(t) by following

\[
V_4(t) = \frac{1}{2} \sum_{i=1}^{m} \xi_i \delta x_i(t) P \delta x_i^T(t) + \sum_{i=1}^{m} \frac{\beta}{\alpha} (c - c_i(t))^2;
\]  
(71)

and proceed the same way as before. The details are omitted. Readers can also refer to [14].

V. Conclusions

In this note, we revisit synchronization and consensus of multi-agents. As pointed out in [10], synchronization relates two main points, one is connection structure, and the other is the intrinsic property of the uncoupled system.

- In 2006, time varying synchronization state \( \tilde{x}(t) \) as a non-orthogonal projection in synchronization manifold was first introduced and a distance between the state and the synchronization manifold \( \delta(x(t)) = x(t) - \tilde{x}(t) \) was used to discuss synchronization was proposed and played key role. It describe the connection structure.
- It is clear that \( \delta(t) \) used in [11], [2], [3], [4], [5] and other papers, is nothing new other than the \( \delta(x(t)) = x(t) - \tilde{x}(t) \), though the authors did not mention this fact and cited [10].
- In 10, 2006, QUAD condition is introduced, which describes intrinsic property of the uncoupled system.
- Based on non-orthogonal projection, \( \delta(x(t)) \) and QUAD condition, conditions to ensure synchronization are given.

- It is clear that

\[
\frac{dx_i(t)}{dt} = Ax_i(t) + c \sum_{j=1}^{N} l_{ij} \Gamma x_j(t), \quad i = 1, \ldots, N
\]  
(72)

is a special case of

\[
\frac{dx_i(t)}{dt} = f(x_i(t)) + c \sum_{j=1}^{N} l_{ij} \Gamma x_j(t), \quad i = 1, \ldots, N
\]  
(73)

Therefore, all the results on synchronization model (74) can apply to consensus of multi-agents model (72). All the results given in [1], [2], [3], [4], [5] can be given as applications of the [10].

- In fact, all papers on consensus focus on the QUAD condition

\[
PA + A^T P - BB^T < -\epsilon I_n
\]  
(74)

As we point out that it is a natural consequence of the controllability, and is just another expression of QUAD condition.

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