Non–transversal colliding singularities in F–theory

Silvia Penati, Alberto Santambrogio and Daniela Zanon

Dipartimento di Fisica dell’Università di Milano and
INFN, Sezione di Milano, via Celoria 16, I-20133 Milano, Italy

ABSTRACT

This is a short introduction to the study of compactifications of F–theory on elliptic Calabi–Yau threefolds near colliding singularities. In particular we consider the case of nontransversal intersections of the singular fibers.

*To appear in the proceedings of Quantum aspects of gauge theories, supersymmetry and unification, Neuchatel University, 18–23 September 1997
1 Introduction

We consider F-theory compactified to six dimensions on elliptic Calabi–Yau threefolds and study the case in which the loci of the singular fibers intersect each other non transversally. We show how these singularities can be resolved by suitable blow–ups of the base of the elliptic Calabi–Yau and explore new branches of the moduli space where the gauge group is enhanced and the charged matter content allows cancellation of all the anomalies. Since these new branches are characterized by the presence of more than one tensor multiplet, they do not admit a dual description in terms of heterotic strings.

The talk is organized as follows: we start by reviewing some basic concepts about F-theory compactifications to eight and six dimensions. We briefly discuss the conjectured F-theory/heterotic duality and, in particular, we examine the correspondence between singularities of the compactification manifold on the F-theory side, and gauge group enhancements on the heterotic side. Then we turn our attention to the resolution of nontransversal colliding singularities by means of a suitable blow–up procedure on the base of the elliptic Calabi–Yau manifold. In this way we go to new branches of the moduli space that do not correspond anymore to a dual perturbative heterotic theory. Finally, in these new branches, we illustrate with an example how to determine physical properties, namely the gauge group enhancement and the matter content.

2 F-theory and its compactifications

F-theory is a 12–dimensional theory that, when compactified on a torus, gives rise to type IIB strings in 10 dimensions [1]. It allows to interpret in a natural geometric way the $SL(2, Z)$ symmetry of the type IIB theory: the complex field constructed with the R-R and NS-NS scalars (axion $\tilde{\phi}$ and dilaton $\phi$) of type IIB, $\tau = \tilde{\phi} + i e^{-\phi}$, can be identified with the complex modulus of the torus over which F-theory is compactified, and the $SL(2, Z)$ symmetry is then interpreted as the modular invariance of this torus.

New compactifications of type IIB theory are obtained starting from F-theory and then compactifying it on manifolds that are elliptic fibrations [1, 2]. We consider in particular compactifications to 8 dimensions on elliptic $K3$, and to 6 dimensions on elliptic Calabi–Yau threefolds.
2.1 Compactification to eight dimensions

We start by considering a compactification over a $K3$ which admits an elliptic fibration. In this simple case, the equation for the elliptic fiber (torus) is

$$y^2 = x^3 + f_8(z)x + g_{12}(z)$$  \hspace{1cm} (2.1)

where the two functions $f$ and $g$, polynomials of degree 8 and 12 respectively, vary over the base $\mathbb{P}^1$ parametrized by $z$ \[1\].

The torus is degenerate when the discriminant of the cubic vanishes

$$\Delta = 4(f_8)^3 + 27(g_{12})^2 = 0$$  \hspace{1cm} (2.2)

that is, with this choice for the degrees of $f$ and $g$, on 24 points over the sphere $\mathbb{P}^1$. In fact 24 singular points are necessary in order to satisfy the Calabi–Yau condition for the $K3$ manifold \[4\]. On these 24 points the modulus $\tau$ of the torus becomes singular near $z = 0$, $\tau(z) \sim \frac{1}{2\pi} \log z$. As we go around $z = 0$, $\tau \rightarrow \tau + 1$, i.e. $\tilde{\phi} \rightarrow \tilde{\phi} + 1$. This signals the presence of a magnetically charged 7–brane in $z = 0$, filling the uncompactified space–time, with unitary magnetic charge

$$Q_{magn} = \int_{S^1} d\tilde{\phi} = \tilde{\phi}(e^{2\pi i} z) - \tilde{\phi}(z) = 1$$  \hspace{1cm} (2.3)

The corresponding 8 dimensional theory is conjectured to be dual to the heterotic string compactified on a torus \[1\].

2.2 Compactification to six dimensions

Further compactifications to six dimensions can be obtained from a one parameter family of the above 8 dimensional dual theories, parametrized by $\mathbb{P}^1$. In this way we have the following $N = 1$ dual theories in 6 dimensions \[2\]: $\underline{F_{\text{elliptic CY}}} \sim \frac{het}{K3}$.

In this case the equation for the elliptically fibered CY threefold can be written in the form

$$y^2 = x^3 + xf(z_1, z_2) + g(z_1, z_2)$$  \hspace{1cm} (2.4)

where $(z_1, z_2)$ parametrize the two-dimensional base, a $\mathbb{P}^1$ bundle over $\mathbb{P}^1$. These bundles are classified by an integer $n$ and are called Hirzebruch surfaces $F_n$ (roughly speaking $n$ is the first Chern class of the bundle). The trivial case corresponds to $F_0 = \mathbb{P}^1 \times \mathbb{P}^1$. 

2
The condition (2.2) of degeneration of the torus gives rise to a surface, contained in the base \( F_n \), which can be interpreted as the intersection of the world-volume of magnetically charged 7–branes and the base. In general, \( \Delta = 0 \) describes objects of complex codimension one in the base (divisors).

Let us consider now the heterotic string theory compactified on \( K3 \): the Bianchi identity \( dH = trF \wedge F - trR \wedge R \) imposes that the total instanton number is equal to 24. For the \( E_8 \times E_8 \) case we can distribute the instantons among the two \( E_8 \)'s as \( (12 + n, 12 - n) \). The heterotic theory with such a configuration is conjectured to be dual to the F-theory compactified on a \( CY \) with \( F_n \) as base \[\text{[2]}\]. The \( SO(32) \) case is assumed to be dual to the \( n = 4 \) case.

To test these dualities it is crucial to assume that if the heterotic string has a gauge symmetry \( G \) then on the F-theory side the elliptic fibration must have a singularity of type \( G \). The types of singularities have been classified by Kodaira. (For a complete list of singularities see \[3\].) From the Kodaira table we can read the type of singularity of the \( CY \) manifold simply by looking at the order of zero of the polynomials \( f \) and \( g \) and of the discriminant \( \Delta \).

The groups obtained in this way are always simply–laced (ADE groups). In some cases the singularity does not correspond to these simply laced groups but to a quotient of them. This happens when there is a monodromy action on the singularity which is an outer automorphism of the root lattice \[3, 4\]. The groups so obtained are nonsimply–laced (CBFG groups). In this case the singularity is called non–split (it is called split when the entire group survives).

We now present an explicit example in which we check the duality between the F-theory and the heterotic string with \( E_8 \times E_8 \) as gauge group \[3\].

### 2.2.1 Heterotic side

We consider the case of heterotic string theory compactified on \( K3 \) with \((12 + n)\) instantons in \( E_8 \). In general \( E_8 \) is completely broken on the hypermultiplet moduli space \( \mathcal{H} \) of the gauge bundle \((\dim \mathcal{H} = 30n + 112)\). However, if we restrict the instantons to sit in \( SU(2) \subset E_8 \) we obtain a theory with an unbroken \( E_7 \) as gauge group \((E_7 \times SU(2) \subset E_8 \) is a maximal subgroup).

Standard index theorems give the matter content

\[
\text{neutral hypers} = 2n + 21 \quad (\equiv \dim \text{ of subspace of } \mathcal{H} \text{ with } E_7 \text{ enhancement})
\]

\[
\text{charged hypers} = \frac{1}{2}(n + 8)\text{in the } 56\text{ of } E_7 \quad \text{(2.5)}
\]
where the $\frac{1}{2}$ comes from the pseudoreality of the representation $56$ of $E_7$.

### 2.2.2 F-theory side

From the Kodaira singularity table one can see that the moduli of the $E_7$ enhancement are the degrees of freedom of the two polynomials $f_{8+n}(z_2)$ and $g_{12+n}(z_2)$ in the equation of the elliptic fibration, that is

$$(9 + n) + (13 + n) - 1 = 2n + 21 \quad (2.6)$$

(the $-1$ comes from the rescaling of $z_1$) in agreement with the number of neutral hypermultiplets in the heterotic calculation (2.5).

Where can we read the charged matter content on the F-theory side from? The form of the discriminant near the $E_7$ locus $\{z_1 = 0\}$ is $\Delta = z_1^9 \left( 4f_{8+n}^3(z_2) + o(z_1) \right)$. This tells us that there are $(n + 8)$ extra zeroes, corresponding to $(n + 8)$ 7–branes intersecting the one corresponding to $E_7$ (note, however, that the types of singularity over these extra branes do not necessarily correspond to an extra gauge group enhancement). It is then natural to conclude that each charged hypermultiplet in the $56$ is localized at the points of collision of the divisors. This fact was already been conjectured before the introduction of F-theory [5]. A purely F-theory derivation (without the use of duality with heterotic theory) of the charged matter content was given in [6].

### 3 Colliding singularities

In the last section we have described how F–theory compactified on elliptic Calabi–Yau threefolds with an Hirzebruch surface as base is dual to a perturbative heterotic string theory compactified on a $K3$. Now we are going to explore new situations on the F–theory side that, as we will see at the end of this section, cannot be dual to a perturbative heterotic theory.

By generalizing what we have seen in the last example, if two divisors corresponding to gauge groups $G$ and $G'$ meet each other, then one expects to have a theory with gauge group $G \times G'$ with some matter content, either neutral or charged. However it turns out that in many cases it is not possible to satisfy the anomaly cancellation conditions for the F-theory [7]. These conditions restrict severely the nonanomalous matter content and depend crucially upon the intersection numbers.
among the divisors $D$ (topological conditions)

$$\sum_{(R_a,R_b)} n_{(R_a,R_b)} Ind(R_a) Ind(R_b) = (D_a \cdot D_b)$$

$$Ind(Ad_a) - \sum_{R_a} Ind(R_a) n_{R_a} = 6(K \cdot D_a)$$

$$y_{Ad_a} - \sum_{R_a} y_{R_a} n_{R_a} = -3(D_a \cdot D_a)$$

$$x_{Ad_a} - \sum_{R_a} x_{R_a} n_{R_a} = 0 \quad (3.1)$$

where $K$ is the canonical divisor of the base. The index of a representation $R_a$ is defined by $tr(T_i^a T_j^a) = Ind(R_a) \delta_{ij}$ and the coefficients $x$ and $y$ are defined through the decomposition $tr_{R_a} F^4 = x_{R_a} tr F^4 + y_{R_a} (tr F^2)^2$ (‘tr’ is a trace in a preferred representation, usually the fundamental). Anomaly cancellation imposes also the relation $n_H - n_V = 273 - 29n_T$ among the number of vectors, tensors and hypermultiplets.

As mentioned above, in many cases the (3.1) cannot be satisfied. For example, if $D \cdot D' = 1$ (transversal collision), the case $SO(n) \times SO(m)(n,m \geq 7)$ cannot satisfy the first of them (all $Ind \geq 2$). In these cases one can resolve the singularity and satisfy the (3.1) by making a blow up of the base [8]: the intersection point is replaced by a whole $\mathbb{P}^1$ (called the exceptional divisor $E$). After the blow-up, the two divisors corresponding to the gauge groups do not intersect each other anymore and one can hope that the (3.1) can be satisfied

$$D \cdot D' = 1 \quad \text{blow-up} \quad \hat{D} \cdot \hat{D}' = 0 \quad \hat{D} \cdot E = 1 \quad \hat{D}' \cdot E = 1 \quad (3.2)$$

In many cases $E$ itself becomes a component of the discriminant locus corresponding to a new gauge group $H$.

After the blow-up one has to check that the blown-up surface still satisfies the Calabi–Yau condition. This leads to the condition

$$a(D) + a(D') - a(E) = 1 \quad (3.3)$$

where the coefficients $a$ depend on the singularity type on the corresponding divisor.

So, in order to determine the new gauge group $H$, one looks for a coefficient $a(E)$ in Table 1 such that the (3.3) is satisfied. If $a(E) \neq 0$ then $E$ is a component of the discriminant locus and after the blow-up the gauge group is $G \times H \times G'$. 

5
Table 1: Coefficients appearing in the Calabi–Yau condition

|    | none | $I_n$ | $II$ | $III$ | $IV$ | $I_n^*$ | $IV^*$ | $III^*$ | $II^*$ |
|----|------|-------|------|-------|------|--------|-------|--------|-------|
| 0  | $\frac{n}{12}$ | $\frac{1}{6}$ | $\frac{1}{4}$ | $\frac{1}{3}$ | $\frac{1}{2}$ | $\frac{n}{12}$ | $\frac{2}{3}$ | $\frac{3}{4}$ | $\frac{5}{6}$ |

Now either the (3.1) can be satisfied with some matter content, or else other blow-ups might be needed.

The degree of freedom corresponding to the radius of the blown-up sphere is parametrized by the scalar of a new tensor multiplet. Indeed, the number of tensor multiplets is given by the number of Kähler classes of the base, $h^{1,1}(B)$, as \[ n_T = h^{1,1}(B) - 1 \] (note that $h^{1,1} = 2$ for Hirzebruch surfaces) and the blow-up introduces one more Kähler modulus. We are then in a Coulomb branch of the moduli space, and the theory has more than one tensor multiplet. Note that the heterotic string has $n_T = 1$ and so after the phase transition our theory does not have a dual heterotic description. The gauge group enhancement is of non-perturbative kind.

The case of transversal collisions has been studied in [8]. Let us now look at the non-transversal situation.

### 4 The non-transversal case

For non-transversal collisions ($D \cdot D' > 1$) we need to perform more than one blow-up to resolve the singularity, in such a way that at the end the two divisors do not intersect each other anymore. Since every blow-up introduces one new tensor multiplet, the phase transitions we are now going to explore are characterized by $\delta n_T > 1$.

For example, let us consider the case $D \cdot D' = 2$. After a first blow-up $D$ and $D'$ intersect again each other ($\hat{D} \cdot \hat{D}' = \hat{D} \cdot \hat{E}_1 = \hat{D}' \cdot \hat{E}_1 = 1$). Only after a second blow-up, with the introduction of a second exceptional divisor $E_2$, the two divisors do not intersect

\[ \hat{D} \cdot \hat{D}' = \hat{D} \cdot \hat{E}_1 = \hat{D}' \cdot \hat{E}_1 = 0 \quad \hat{D} \cdot E_2 = \hat{D}' \cdot E_2 = \hat{E}_1 \cdot E_2 = 1 \] (4.1)

More generally, if $D \cdot D' = p$ we need to perform $p$ blow-ups, with the introduction of $p$ exceptional divisors.
There are now $p$ Calabi–Yau conditions, one for every blow–up. For example, in the $p = 2$ case they are
\[
\begin{align*}
a(D) + a(D') - a(E_1) &= 1 \\
a(D) + a(D') + a(E_1) - a(E_2) &= 1
\end{align*}
\] (4.2)
Again, if $E_1$ and $E_2$ belong to the discriminant locus, then we obtain the non–perturbative gauge group enhancement to $G \times H_1 \times H_2 \times G'$ (with $p H$ factors if $D \cdot D' = p$).

As an example we may consider the case of the collision between the fibers $I^n$ and $I^m$, corresponding to the gauge groups $SO(2n + 8) \times SO(2m + 8)$, with $D \cdot D' = 2$. It is easy to see that the (3.1) cannot be verified. As described above, we need to perform at least two blow–ups to resolve the singularity. Then we use (4.2) to find the gauge group enhancement and (3.1) to determine the matter content.

### 4.1 Determining the gauge group enhancement

From Table 1 we know that $a(D) = \frac{1}{2} + \frac{n}{12}$, $a(D') = \frac{1}{2} + \frac{m}{12}$ and then, to solve the (4.2), we have to take
\[
a(E_1) = \frac{n + m}{12} \quad \quad a(E_2) = \frac{n + m}{6}
\] (4.3)
corresponding to the fibers $I_{n+m}$ and $I_{2(n+m)}$ respectively. Note that the singular fibers $I_{n+m}$ correspond either to $SU(n + m)$, in the split case, or to $Sp\left(\frac{n+m}{2}\right)$, in the non split case. The gauge group enhancement is then
\[
SO(2n + 8) \times SU(n + m) \times SU(2(n + m)) \times Sp\left(\frac{n+m}{2}\right) \times SO(2m + 8)
\] (4.4)
where the upper choices correspond to the split cases and the lower choices to the non split ones.

### 4.2 Determining the matter content

Let us study for simplicity the case $n = m = 1$, corresponding to the enhancement
\[
SO(10) \times SO(10) \rightarrow SO(10) \times SU(2) \times SU(4) \times SO(10)
\] (4.5)
We determine the matter content by solving (3.1). The first of them gives the content of matter in mixed representations of the gauge group. Note in particular
that \( \hat{D} \cdot \hat{D}' = \hat{D} \cdot \hat{E}_1 = \hat{D}' \cdot \hat{E}_1 = 0 \) and so there is no matter in mixed representations of the corresponding pairs of groups. We have instead \( \hat{D} \cdot E_2 = 1 \), that means that there is matter in the mixed representation of the first \( SO(10) \) and of \( SU(4) \) (or \( Sp(2) \)). We make a minimal choice for the possible representations of these gauge groups: the fundamental, the adjoint, the spinorial and the antisymmetric.

By imposing the first of (3.1) we fix the gauge group to be \( SO(10) \times SU(2) \times Sp(2) \times SO(10) \) (picking out the non–split case) and we determine the matter content \( \frac{1}{2}(10, 1, 4, 1) \).

Analogously, for the other intersections we find \( \frac{1}{2}(1, 1, 4, 10) \) and \( 2 \times \frac{1}{2}(1, 2, 4, 1) \) or \( \frac{1}{2}(1, 2, 5, 1) \).

The other equations in (3.1) give conditions that allow us to determine the matter content in pure representations of the gauge groups. Note that for the two divisors \( \hat{D} \) and \( \hat{D}' \), the right hand sides of the second and of the third equation in (3.1) depend on the particular choice of the base of the compactification manifold and on the choice of the two divisors. Thus the matter content in pure representations of \( G \) and \( G' \) is not universal, while the mixed–matter content is universal depending only on the local structure of the colliding singularities.

Looking for a realization of \( D \) and \( D' \), one can make again a minimal choice by imposing that there are no hypermultiplets in the adjoint. This requires \( \hat{D} \) that the genus of the divisors is zero and that

\[
K \cdot D = -D \cdot D - 2
\]  

Very often the most convenient choice for the base of the elliptic CY is the Hirzebruch surface \( F_n \).

We then find a universal matter content

\[
\frac{1}{2}(10, 1, 4, 1) + \frac{1}{2}(1, 1, 4, 10) + 2 \times \frac{1}{2}(1, 2, 4, 1)
\]  

and a matter part depending on our particular choice for \( D \) and \( D' \)

\[
(n_1 + 2) [(10, 1, 1, 1) + (16, 1, 1, 1)] + (n_2 + 2) [(1, 1, 1, 10) + (1, 1, 1, 16)]
\]  

where \( n_1 = D \cdot D \) and \( n_2 = D' \cdot D' \).

5 Conclusions

We have shown with an example how non–transversal colliding singularities can be resolved for the \( SO \times SO \) case by using a blow–up procedure on the base of the
fibration. The analysis for general $G \times G'$ and general intersection numbers is in progress.

We have seen that the new branches of the moduli space found in the resolution of non–transversal colliding singularities cannot correspond to a perturbative heterotic description. Their interpretation, maybe in terms of instantons shrinking to zero size \[\text{[9]},\] certainly deserves further study.

**Acknowledgements.** We thank M. Bershadsky and A. Johansen for useful conversations and suggestions.

This work has been supported by the European Commission TMR programme ERBFMRX-CT96-0045, in which the authors are associated to Torino.

**References**

[1] C. Vafa, Nucl. Phys. **B469** (1996), 403.

[2] D.R. Morrison, C.Vafa Nucl. Phys. **B473** (1996), 74; Nucl. Phys. **B476** (1996), 437.

[3] M. Bershadsky, K. Intriligator, S. Kachru, D.R. Morrison, V. Sadov, C. Vafa, Nucl. Phys. **B481** (1996), 215.

[4] P. Aspinwall, M. Gross, Phys. Lett. **B387** (1996), 735.

[5] M. Bershadsky, V. Sadov, C. Vafa Nucl. Phys. **B463** (1996), 398.

[6] S. Katz, C. Vafa, Nucl. Phys. **B497** (1997), 146.

[7] V. Sadov, [hep-th/9606008](http://arxiv.org/abs/hep-th/9606008).

[8] M. Bershadsky, A. Johansen, Nucl. Phys. **B489** (1997), 122.

[9] E. Witten, Nucl. Phys. **B460** (1996), 541.