Teleparallel Version of the Stationary Axisymmetric Solutions and their Energy Contents

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Abstract

This work contains the teleparallel version of the stationary axisymmetric solutions. We obtain the tetrad and the torsion fields representing these solutions. The tensor, vector and axial-vector parts of the torsion tensor are evaluated. It is found that the axial-vector has component only along $\rho$ and $z$ directions. The three possibilities of the axial vector depending on the metric function $B$ are discussed. The vector related with spin has also been evaluated and the corresponding extra Hamiltonian is furnished. Further, we use the teleparallel version of Möller prescription to find the energy-momentum distribution of the solutions. It is interesting to note that (for $\lambda = 1$) energy and momentum densities in teleparallel theory are equal to the corresponding quantities in GR plus an additional quantity in each, which may become equal under certain conditions. Finally, we discuss the two special cases of the stationary axisymmetric solutions.

Keywords: Teleparallel Theory, Axial-Vector, Energy.

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1 Introduction

The attempts made by Einstein and his followers to unify gravitation with other interactions led to the investigation of structures of gravitation other than the metric tensor. These structures yield the metric tensor as a by product. Tetrad field is one of these structures which leads to the theory of teleparallel gravity (TPG) [1,2]. TPG is an alternative theory of gravity which corresponds to a gauge theory of translation group [3,4] based on Weitzenböck geometry [5]. This theory is characterized by the vanishing of curvature identically while the torsion is taken to be non-zero. In TPG, gravitation is attributed to torsion [4] which plays a role of force [6]. In General Relativity (GR), gravitation geometrizes the underlying spacetime. The translational gauge potentials appear as a non-trivial part of the tetrad field and induce a teleparallel (TP) structure on spacetime which is directly related to the presence of a gravitational field. In some other theories [3-8], torsion is only relevant when spins are important [9]. This point of view indicates that torsion might represent additional degrees of freedom as compared to curvature. As a result, some new physics may be associated with it. Teleparallelism is naturally formulated by gauging external (spacetime) translations which are closely related to the group of general coordinate transformations underlying GR. Thus the energy-momentum tensor represents the matter source in the field equations of tetradic theories of gravity like in GR.

There is a large literature available [10] about the study of TP versions of the exact solutions of GR. Recently, Pereira, et al. [11] obtained the TP versions of the Schwarzschild and the stationary axisymmetric Kerr solutions of GR. They proved that the axial-vector torsion plays the role of the gravitomagnetic component of the gravitational field in the case of slow rotation and weak field approximations. In a previous paper [12], we have found the TP versions of the Friedmann models and of the Lewis-Papapetrou spacetimes, and also discussed their axial-vectors.

There has been a longstanding, controversial and still unresolved problem of the localization of energy (i.e., to express it as a unique tensor quantity) in GR [13]. Einstein [14] introduced the energy-momentum pseudo-tensor and then Landau-Lifshitz [15], Papapetrou [16], Bergmann [17], Tolman [18] and Weinberg [19] proposed their own prescriptions to resolve this issue. All these prescriptions can provide meaningful results only in Cartesian coordinates. But Möller [20] introduced a coordinate-independent prescription. The idea of coordinate-independent quasi-local mass was introduced [21] by associ-
ating a Hamiltonian term to each gravitational energy-momentum pseudo-tensor. Later, a Hamiltonian approach in the frame of Schwinger condition [22] was developed, followed by the construction of a Lagrangian density of TP equivalent to GR [4,6,23,24]. Many authors explored several examples in the framework of GR and found that different energy-momentum complexes can give either the same [25] or different [26] results for a given spacetime.

Mikhail et al. [27] defined the superpotential in the Moller’s tetrad theory which has been used to find the energy in TPG. Vargas [28] defined the TP version of Bergman, Einstein and Landau-Lifshitz prescriptions and found that the total energy of the closed Friedman-Robinson-Walker universe is zero by using the last two prescriptions. This agrees with the results of GR available in literature [29]. Later, many authors [30] used TP version of these prescriptions and showed that energy may be localized in TPG. Keeping these points in mind, this paper is addressed to the following two problems: We obtain TP version of the stationary axisymmetric solutions and then calculate the axial-vector part of the torsion tensor. The energy-momentum distribution of the solutions is explored by using the TP version of Møller prescription.

The scheme adopted in this paper is as follows. In section 2, we shall review the basic concepts of TP theory. Section 3 contains the TP version of the stationary axisymmetric solutions and the tensor, vector and axial-vector parts of the torsion tensor. Section 4 is devoted to evaluate the energy-momentum distribution for this family of solutions using the TP version of Møller prescription. In section 5, we present two special solutions for this class of metrics and investigate the corresponding quantities. The last section contains a summary and a discussion of the results obtained.

2 An Overview of the Teleparallel Theory

In teleparallel theory, the connection is a Weitzenböck connection given as [31]

$$\Gamma^a_{\mu\nu} = h^a_\theta \partial_\nu h^\theta_\mu,$$

where $h^\mu_\nu$ is a non-trivial tetrad. Its inverse field is denoted by $h^a_\mu$ and satisfy the relations

$$h^a_\mu h^\mu_\nu = \delta^a_\nu; \quad h^a_\mu h^\mu_b = \delta^a_b.$$
In this paper, the Latin alphabet ($a, b, c, \ldots = 0, 1, 2, 3$) will be used to denote tangent space indices and the Greek alphabet ($\mu, \nu, \rho, \ldots = 0, 1, 2, 3$) to denote spacetime indices. The Riemannian metric in TP theory arises as a by product [4] of the tetrad field given by

$$g_{\mu\nu} = \eta_{ab} h^a_{\mu} h^b_{\nu},$$

where $\eta_{ab}$ is the Minkowski metric $\eta_{ab} = \text{diag}(+1, -1, -1, -1)$. For the Weitzenböck spacetime, the torsion is defined as [2]

$$T^\theta_{\mu\nu} = \Gamma^\theta_{\nu\mu} - \Gamma^\theta_{\mu\nu}$$

which is antisymmetric w.r.t. its last two indices. Due to the requirement of absolute parallelism, the curvature of the Weitzenböck connection vanishes identically. The Weitzenböck connection also satisfies the relation

$$\Gamma^{0\theta}_{\mu\nu} = \Gamma^\theta_{\mu\nu} - K^\theta_{\mu\nu},$$

where

$$K^\theta_{\mu\nu} = \frac{1}{2} [T^\theta_{\mu\nu} + T^\theta_{\nu\mu} - T^\theta_{\theta\mu\nu}]$$

is the contortion tensor and $\Gamma^{0\theta}_{\mu\nu}$ are the Christoffel symbols in GR. The torsion tensor of the Weitzenböck connection can be decomposed into three irreducible parts under the group of global Lorentz transformations [4]: the tensor part

$$t^\theta_{\lambda\mu\nu} = \frac{1}{2} (T^\theta_{\lambda\mu\nu} + T^\theta_{\mu\lambda\nu}) + \frac{1}{6} (g_{\nu\lambda} V_{\mu} + g_{\nu\mu} V_{\lambda}) - \frac{1}{3} g_{\lambda\mu} V_{\nu},$$

the vector part

$$V_{\mu} = T^{\nu}_{\nu\mu}$$

and the axial-vector part

$$A^\mu = \frac{1}{6} \epsilon^{\mu\nu\rho\sigma} T_{\nu\rho\sigma}.$$
where

\[ \epsilon^{\lambda\mu\nu\rho} = \frac{1}{\sqrt{-g}} \delta^{\lambda\mu\nu\rho}. \]  

(11)

Here \[ \delta = \{ \delta^{\lambda\mu\nu\rho} \} \] and \[ \delta^* = \{ \delta^{\lambda\mu\nu\rho} \} \] are completely skew symmetric tensor densities of weight -1 and +1 respectively [4]. TP theory provides an alternate description of the Einstein’s field equations which is given by the teleparallel equivalent of GR [24,31].

Mikhail et al. [27] defined the super-potential (which is antisymmetric in its last two indices) of the Möller tetrad theory as

\[ U_{\mu}{}^{\nu\beta} = \frac{\sqrt{-g}}{2\kappa} P_{\chi\rho\sigma}^{\tau\nu\beta} [V^{\rho} g^{\sigma\chi} g_{\mu\tau} - \lambda g_{\tau\mu} K^{\chi\rho\sigma} - (1 - 2\lambda) g_{\tau\mu} K^{\rho\sigma\chi}], \]

(12)

where

\[ P_{\chi\rho\sigma}^{\tau\nu\beta} = \delta_{\chi}^{\tau} g_{\rho\sigma}^{\nu\beta} + \delta_{\rho}^{\tau} g_{\sigma\chi}^{\nu\beta} - \delta_{\sigma}^{\tau} g_{\chi\rho}^{\nu\beta} \]

(13)

and \[ g_{\rho\sigma}^{\nu\beta} \] is a tensor quantity defined by

\[ g_{\rho\sigma}^{\nu\beta} = \delta_{\rho}^{\nu} \delta_{\sigma}^{\beta} - \delta_{\sigma}^{\nu} \delta_{\rho}^{\beta}. \]

(14)

\[ K^{\rho\sigma\chi} \] is the contortion tensor given by Eq.(6), \[ g \] is the determinant of the metric tensor \[ g_{\mu\nu} \], \[ \lambda \] is the free dimensionless coupling constant of TPG, \[ \kappa \] is the Einstein constant and \[ V_{\mu} \] is the basic vector field given by Eq.(8). The energy-momentum density is defined as

\[ \Xi_{\mu}^{\nu} = U_{\mu}^{\nu\rho}, \]

(15)

where comma means ordinary differentiation. The momentum 4-vector of Möller prescription can be expressed as

\[ P_{\mu} = \int_{\Sigma} \Xi_{\mu}^{0} dx dy dz, \]

(16)

where \[ P_{0} \] gives the energy and \[ P_{1}, P_{2} \] and \[ P_{3} \] are the momentum components while the integration is taken over the hypersurface element \[ \Sigma \] described by \[ x^{0} = t = constant. \] The energy may be given in the form of surface integral [20] as

\[ E = \lim_{r \to \infty} \int_{r=constant} U_{0}^{0\rho} u_{\rho} dS, \]

(17)

where \[ u_{\rho} \] is the unit three-vector normal to the surface element \[ dS. \]
3 Teleparallel Version of the Stationary Axisymmetric Solutions

Tupper [32] found five classes of non-null electromagnetic field plus perfect fluid solutions in which the electromagnetic field does not inherit one of the symmetries of the spacetime. The metric representing the stationary axisymmetric solutions is given by [32]

\[ ds^2 = dt^2 - e^{2K}d\rho^2 - (F^2 - B^2)d\phi^2 - e^{2K}dz^2 + 2Bdt d\phi, \]

(18)

where \( B = B(\rho, z) \), \( K = K(\rho, z) \) and \( F = F(\rho) \) are such functions which satisfy the following relations

\[ \dot{B} = FW', \quad B' = -\frac{1}{4}aF(W^2 - W'^2), \]

\[ K' = -\frac{1}{2}aFWW', \quad \ddot{W} + \dot{F}F^{-1}\dot{W} + W'' = 0, \]

(19)

dot and prime denoting the derivatives w.r.t. \( \rho \) and \( z \) respectively. Here \( a \) is constant and \( W \) is an arbitrary function of \( \rho \) and \( z \), in general. In McIntosh’s solution, \( W \) is taken to be \(-2bz\) while McLenaghan et. al. solution is obtained by substituting \( W = 2 \ln \rho \) [33]. The metric given by Eq.(18) represents five classes of non-null electromagnetic field and perfect fluid solutions which possesses a metric symmetry not inherited by the electromagnetic field and admits a homothetic vector field. Two of these classes contain electrovac solutions as special cases, while the other three necessarily contain some fluid. The generalization of this metric is given in [34].

Using the procedure adopted in the papers [11,12], the tetrad components of the above metric can be written as

\[ h_{a\mu} = \begin{bmatrix} 1 & 0 & B & 0 \\ 0 & e^K \cos \phi & -F \sin \phi & 0 \\ 0 & e^K \sin \phi & F \cos \phi & 0 \\ 0 & 0 & 0 & e^K \end{bmatrix} \]

(20)

with its inverse

\[ h^{a\mu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{B}{F} \sin \phi & e^{-K} \cos \phi & -\frac{1}{F} \sin \phi & 0 \\ \frac{B}{F} \cos \phi & e^{-K} \sin \phi & \frac{1}{F} \cos \phi & 0 \\ 0 & 0 & 0 & e^{-K} \end{bmatrix}. \]

(21)
The non-vanishing components of the torsion tensor are
\[ T^0_{12} = \dot{B} + \frac{B}{F}(e^K - \dot{F}), \quad T^0_{32} = B', \]
\[ T^1_{13} = -K', \quad T^2_{12} = \frac{1}{F}(\dot{F} - e^K), \quad T^3_{31} = -\dot{K}. \] (22)

Using these expressions in Eqs.(7)-(9), we obtain the following non-zero components of the tensor part
\[ t^0_{01} = \frac{1}{3}[\dot{K} + \frac{1}{F}(\dot{F} - e^K)], \quad t^0_{03} = \frac{1}{3}K', \]
\[ t_{010} = \frac{1}{6}\left\{ \frac{1}{F}(e^K - \dot{F}) - \dot{K} \right\} = t_{100}, \quad t_{030} = -\frac{1}{6}K' = t_{300}, \]
\[ t_{012} = \frac{1}{2}\dot{B} + \frac{B}{6}\left\{ \frac{1}{F}(e^K - \dot{F}) - \dot{K} \right\} = t_{102}, \]
\[ t_{021} = -\frac{1}{2}\dot{B} - \frac{B}{3}\left\{ \frac{1}{F}(e^K - \dot{F}) - \dot{K} \right\} = t_{201}, \]
\[ t_{023} = -\frac{1}{2}B' + \frac{1}{3}BK' = t_{203}, \quad t_{032} = \frac{1}{2}B' - \frac{1}{6}BK' = t_{302}, \]
\[ t_{122} = \frac{1}{2}\left\{ F(e^K - \dot{F}) + B\dot{B} \right\} + \frac{1}{6}(B^2 - F^2)\left\{ \frac{1}{F}(e^K - \dot{F}) - \dot{K} \right\} = t_{212}, \]
\[ t_{120} = \frac{B}{6}\left\{ \frac{1}{F}(e^K - \dot{F}) - \dot{K} \right\} = t_{210}, \]
\[ t_{131} = \frac{2K'}{3}e^{2K}, \quad t_{131} = -\frac{K'}{3}e^{2K} = t_{311}, \]
\[ t_{133} = -\frac{e^{2K}}{6}\left\{ \frac{1}{F}(e^K - \dot{F}) + 2\dot{K} \right\} = t_{313}, \]
\[ t_{221} = -F(e^K - \dot{F}) - B\dot{B} - \frac{1}{3}(B^2 - F^2)\left\{ \frac{1}{F}(e^K - \dot{F}) - \dot{K} \right\}, \]
\[ t_{223} = -BB' + \frac{K'}{3}(B^2 - F^2), \quad t_{331} = \frac{e^{2K}}{3}\left\{ \frac{1}{F}(e^K - \dot{F}) + 2\dot{K} \right\}, \]
\[ t_{322} = \frac{1}{2}BB' - \frac{K'}{6}(B^2 - F^2) = t_{232}, \quad t_{320} = -\frac{1}{6}BK' = t_{232}, \] (23)

the vector part
\[ V_1 = -\frac{1}{F}(e^K - \dot{F}) - \dot{K}, \] (24)
\[ V_3 = -K', \] (25)
and the axial-vector part

\begin{align}
A^1 &= \frac{B^\prime}{3F}e^{-2K}, \\
A^3 &= \frac{\dot{B}}{3F}e^{-2K},
\end{align}

respectively. The axial-vector component along the \(\phi\)-direction vanishes and hence the spacelike axial-vector can be written as

\[ A = \sqrt{-g_{11}}A^1 \hat{e}_\rho + \sqrt{-g_{33}}A^3 \hat{e}_z, \]

where \(\hat{e}_\rho\) and \(\hat{e}_z\) are unit vectors along the radial and \(z\)-directions respectively. Substituting \(A^1, A^3, g_{11}\) and \(g_{33}\) in Eq.(28), it follows that

\[ A = \frac{e^{-K}}{3F}(B^\prime \hat{e}_\rho + \dot{B} \hat{e}_z). \]

This shows that the axial-vector lies along radial direction if \(B = B(z)\), along \(z\)-direction if \(B = B(\rho)\) and vanishes identically if \(B\) is constant. As the axial-vector torsion represents the deviation of axial symmetry from cylindrical symmetry, the symmetry of the underlying spacetime will not be affected even for \(B\) constant. Also, the torsion plays the role of the gravitational force in TP theory, hence a spinless particle will obey the force equation [11,24]

\[ \frac{du_\rho}{ds} - \Gamma^\mu_{\rho\nu}u^\mu u^\nu = T^\mu_{\rho\nu}u^\mu u^\nu. \]

The left hand side of this equation is the Weitzenböck covariant derivative of \(u_\rho\) along the particle world-line. The appearance of the torsion tensor on its right hand side indicates that the torsion plays the role of an external force in TPG. It has been shown, both in GR and TP theories, by many authors [4,35] that the spin precession of a Dirac particle in torsion gravity is related to the torsion axial-vector by

\[ \frac{dS}{dt} = -b \times S, \]

where \(S\) is the spin vector of a Dirac particle and \(b = \frac{3}{2}A\), with \(A\) the spacelike part of the torsion axial-vector. Thus

\[ b = \frac{e^{-K}}{2F}\{B^\prime \hat{e}_\rho + \dot{B} \hat{e}_z\}. \]
The corresponding extra Hamiltonian \([36]\) is given by
\[
\delta H = -b \sigma, \tag{33}
\]
where \(\sigma\) is the spin of the particle \([35]\). Using Eq.\((32)\), this takes the form
\[
\delta H = -\frac{e^{-K}}{2F}(B\dot{\epsilon}_\rho + \dot{B}\dot{\epsilon}_x)\sigma. \tag{34}
\]

4 Teleparallel Energy of the Stationary Axisymmetric Solutions

In this section we evaluate the component of energy-momentum densities by using the teleparallel version of Möller prescription. Multiplying Eqs.\((24)\) and \((25)\) by \(g^{11}\) and \(g^{33}\) respectively, it follows that
\[
V^1 = \dot{K}e^{-2K} + \frac{e^{-2K}}{F} (\dot{F} - e^K), \tag{35}
\]
\[
V^3 = K'e^{-2K}. \tag{36}
\]

In view of Eqs.\((6)\) and \((22)\), the non-vanishing components of the contorsion tensor are
\[
K^{100} = -e^{-2K} \left\{ \frac{B^2}{F^3}(e^K - \dot{F}) + \frac{B\dot{B}}{F^2} \right\} = -K^{010},
\]
\[
K^{300} = -\frac{BB'}{F^2}e^{-2K} = -K^{030}, \quad K^{122} = -\frac{e^{-2K}}{F^3}(e^K - \dot{F}) = -K^{212},
\]
\[
K^{133} = -\dot{K}e^{-4K} = -K^{313}, \quad K^{311} = K'e^{-4K} = -K^{131},
\]
\[
K^{102} = K^{120} = e^{-2K} \left\{ \frac{B}{F^3}(e^K - \dot{F}) + \frac{\dot{B}}{2F^2} \right\} = -K^{012} = -K^{210},
\]
\[
K^{302} = K^{320} = K^{023} = \frac{B'}{2F^2}e^{-2K} = -K^{032} = -K^{230} = -K^{203},
\]
\[
K^{021} = \frac{\dot{B}}{2F^2}e^{-2K} = -K^{201}. \tag{37}
\]

It should be mentioned here that the contorsion tensor is antisymmetric w.r.t. its first two indices. Making use of Eqs.\((35)-(37)\) in Eq.\((12)\), we obtain the
required independent non-vanishing components of the superpotential in Møller’s tetrad theory as

\begin{align*}
U_{01}^0 &= \frac{1}{\kappa} [e^K - \dot{F} - F\dot{K} + \frac{1}{2}(1 + \lambda)\frac{B\dot{B}}{F}] = -U_{01}^{10}, \\
U_{03}^0 &= \frac{1}{\kappa} [-FK' + \frac{1}{2}(1 + \lambda)\frac{BB'}{F}] = -U_{03}^{30}, \\
U_{21}^0 &= -\frac{1}{2\kappa}(1 + \lambda)\frac{\dot{B}}{F} = -U_{01}^{12}, \quad U_{03}^{23} = -\frac{1}{2\kappa}(1 + \lambda)\frac{B'}{F} = -U_{03}^{32}, \\
U_{01}^2 &= \frac{1}{\kappa} [B(e^K - \dot{F}) + \frac{1}{2}(1 + \lambda)\frac{B^2\dot{B}}{F} + \frac{1}{2}(1 - \lambda)\dot{B}F] = -U_{21}^{10}, \\
U_{03}^2 &= \frac{1}{\kappa} \left[\frac{1}{2}(1 + \lambda)\frac{B^2B'}{F} + \frac{1}{2}(1 - \lambda)B'F\right] = -U_{23}^{30}, \\
U_{12}^1 &= \frac{1}{2\kappa F}(\lambda - 1)\dot{B}e^{2K} = -U_{12}^{20}, \\
U_{32}^3 &= \frac{1}{2\kappa F}(\lambda - 1)B'e^{2K} = -U_{32}^{30}.
\end{align*}

(38)

It is worth mentioning here that the superpotential is skew symmetric w.r.t. its last two indices. When we make use of Eqs. (15), (37), (38) and take \(\lambda = 1\), the energy density turns out to be

\[\Xi_0^0 = \frac{1}{\kappa} \left[\ddot{K}e^K - \ddot{F} - \dot{F}\dot{K} - F(\ddot{K} + K'\ddot{F}) + \frac{1}{F^2}\{BF(\dot{B} + E''\}\right.\]
\[\quad + \left. (\dot{B}^2 + B'^2)F - B\dot{B}\dot{F}\right].\]

(39)

This implies that

\[E_{TPT}^d = E_{GR}^d + \frac{1}{\kappa} \left[\ddot{K}e^K - \ddot{F} - \dot{F}\dot{K} - F(\ddot{K} + K'\ddot{F})\right],\]

(40)

where \(E^d\) stands for energy density. The only non-zero component of momentum density is along \(\phi\)-direction and (for \(\lambda = 1\)) it takes the form

\[\Xi_0^2 = \frac{1}{\kappa F^2} \left\{F^3(\ddot{B} + B'\ddot{F} + B''F + 2BF(\dot{B}^2 + B^2) - B\dot{F}(B^2 + F^2)\right\}
\[\quad + \frac{1}{\kappa}\left\{\dot{B}e^K + B(\ddot{K}e^K - \ddot{F}) - F(\ddot{B} + B'\ddot{F})\right\},\]

(41)

that is,

\[M_{TPT}^d = M_{GR}^d + \frac{1}{\kappa} \left\{\dot{B}e^K + B(\ddot{K}e^K - \ddot{F}) - F(\ddot{B} + B'\ddot{F})\right\},\]

(42)

where \(M^d\) stands for momentum density.
5 Special Solutions of the Non-Null Einstein Maxwell Solutions

In this section, we evaluate the above quantities for some special cases of the non-null Einstein Maxwell solutions.

5.1 Electromagnetic Generalization of the Gödel Solution

A special case of the non-null Einstein-Maxwell solutions can be obtained by choosing

\[ B = \frac{m}{n} e^{n \rho}, \quad F = e^{n \rho}, \quad K = 0. \]  

(43)

This is known as electromagnetic generalization of the Gödel solution [32]. When we make use of Eq.(43) in Eqs.(23)-(27), (29), (32), (34) and (39)-(42), the corresponding results reduce to

\[ t_{001} = \frac{1}{3} (n - e^{-n \rho}), \quad t_{010} = \frac{1}{6} (e^{-n \rho} - n) = t_{100}, \]

\[ t_{012} = \frac{m}{6n} (1 + 2ne^{n \rho}) = t_{102}, \quad t_{021} = -\frac{m}{3n} (1 + 2ne^{n \rho}) = t_{201}, \]

\[ t_{122} = \frac{e^{n \rho}}{6n^2} \{ m^2 + 2n^2 + 2n(m^2 - n^2)e^{n \rho} \} = t_{212}, \]

\[ t_{120} = \frac{m}{6n} (1 - ne^{n \rho}) = t_{210}, \quad t_{133} = \frac{1}{6} (n - e^{-n \rho}) = t_{313}, \]

\[ t_{221} = -\frac{e^{n \rho}}{3n^2} \{ m^2 + 2n^2 + 2n(m^2 - n^2)e^{n \rho} \}, \]

\[ t_{331} = -\frac{1}{3} (n - e^{-n \rho}), \]  

(44)

\[ V_1 = e^{-n \rho} - n, \quad V_3 = 0, \]

\[ A^1 = 0, \quad A^3 = \frac{m}{3}, \]  

(45)

(46)

\[ A = \frac{m}{3} \hat{e}_z, \quad b = \frac{m}{2} \hat{e}_z, \]

\[ \delta H = \frac{m}{2} \hat{e}_z \sigma, \]

\[ \Xi_0^0 = \frac{1}{\kappa} (m^2 - n^2)e^{n \rho}, \]  

(47)

(48)

(49)

\[ E_{TPT}^d = E_{GR}^d - \frac{n^2}{\kappa} e^{n \rho}, \]  

(50)
\[ \Xi_2^0 = \frac{1}{\kappa} \left( \frac{2m^2}{n} \right) + \frac{m}{\kappa} (1 - 2ne^{\alpha \rho}) e^{\alpha \rho}, \]  
(51)
\[ M_{TPT}^d = M_{GR}^d + \frac{m}{\kappa} (1 - 2ne^{\alpha \rho}) e^{\alpha \rho}. \]  
(52)

The metric (43) reduces to the usual perfect fluid solution when \( m = \sqrt{2} n \) \[ 32 \], i.e., \( B = \sqrt{2} e^{\alpha \rho} \). The corresponding energy and momentum densities take the form as
\[ E_{TPT}^d = E_{GR}^d - \frac{n^2}{\kappa} e^{\alpha \rho}, \]  
(53)\[ M_{TPT}^d = M_{GR}^d + \frac{\sqrt{2} n}{\kappa} (1 - 2ne^{\alpha \rho}) e^{\alpha \rho}. \]  
(54)

5.2 The Gödel Metric

When we choose \( B = e^{\alpha \rho}, F = e^{\alpha \rho} \sqrt{2} \) and \( K = 0 \), the metric given by Eq.(18) reduces to the Gödel metric \[ 32 \]. The results corresponding to Eqs.(23)-(27), (29), (32), (34) and (39)-(42) take the following form
\[ t_{001} = \frac{1}{3} (a - \sqrt{2} e^{-\alpha \rho}), \quad t_{010} = -\frac{1}{6} (a - \sqrt{2} e^{-\alpha \rho}) = t_{100}, \]  
(55)\[ t_{012} = \frac{1}{6} (\sqrt{2} + 2ae^{\alpha \rho}) = t_{102}, \quad t_{021} = -\frac{1}{6} (\sqrt{2} + ae^{\alpha \rho}) = t_{102}, \]  
(55)\[ t_{122} = \frac{1}{6} (2\sqrt{2} + ae^{\alpha \rho}) = t_{212}, \quad t_{120} = \frac{1}{6} (\sqrt{2} - ae^{\alpha \rho}) = t_{210}, \]  
(55)\[ t_{133} = \frac{1}{6} (a - \sqrt{2} e^{-\alpha \rho}) = t_{313}, \quad t_{221} = -\frac{e^{\alpha \rho}}{3} (2\sqrt{2} + ae^{\alpha \rho}), \]  
(55)\[ t_{331} = \frac{1}{3} (a - \sqrt{2} e^{-\alpha \rho}), \]  
(55)\[ V_1 = \sqrt{2} e^{-\alpha \rho} - a, \quad V_3 = 0, \]  
(57)\[ A^1 = 0, \quad A^3 = \frac{\sqrt{2} a}{3}, \]  
(58)\[ A = \frac{\sqrt{2} a}{3} \hat{e}_z, \quad b = \frac{a}{\sqrt{2}} \hat{e}_z, \]  
(59)\[ \delta H = \frac{a}{\sqrt{2}} \hat{e}_z \sigma, \]  
(60)\[ \Xi_0^0 = \frac{\sqrt{2}}{\kappa} a^2 e^{\alpha \rho} - \frac{a^2}{\kappa \sqrt{2}} e^{\alpha \rho}, \]  
(61)
\begin{align*}
E^d_{TPT} &= E^d_{GR} - \frac{a^2}{\kappa \sqrt{2}} e^{a\rho} \\
\Xi_2^0 &= \frac{a^2}{\kappa \sqrt{2}} e^{2a\rho} + \frac{a}{\kappa} (1 - \sqrt{2}ae^{a\rho}) e^{a\rho}, \\
M^d_{TPT} &= M^d_{GR} + \frac{a}{\kappa} (1 - \sqrt{2}ae^{a\rho}) e^{a\rho}.
\end{align*}

6 Summary and Discussion

The purpose of this paper is twofold: Firstly, we have found the TP version of the non-null Einstein Maxwell solutions. This provides some interesting features about the axial vector and the corresponding quantities. Secondly, we have used the TP version of Möller prescription to evaluate the energy-momentum distribution of the solutions. The axial-vector torsion of these solutions has been evaluated. The only non-vanishing components of the vector part are along the radial and the \(z\)-directions due to the cross term \(dx^0dx^2\) involving in the metric. This corresponds to the case of Kerr metric \([11]\), which involves the cross term \(dx^0dx^3\). We also find the vector \(b\) which is related to the spin vector \([4]\) as given by Eq.(32). The axial-vector torsion lies in the \(\rho z\)-plane, as its component along the \(\phi\)-direction vanishes everywhere. The non-inertial force on the Dirac particle can be represented as a rotation induced torsion of spacetime.

There arise three possibilities for the axial-vector, depending upon the metric function \(B(\rho, z)\). When \(B\) is a function of \(z\) only, the axial-vector lies only along the radial direction. When \(B\) is a function of \(\rho\) only, the axial-vector will lie along \(z\)-direction. The axial-vector vanishes identically for \(B\) to be constant. As the axial-vector represents the deviation from the symmetry of the underlying spacetime corresponding to an inertial field with respect to the Dirac particle, the symmetry of the spacetime will not be affected in the third possibility. Consequently there exists no inertial field with respect to the Dirac particle and the spin vector of the Dirac particle becomes constant. The corresponding extra Hamiltonian is expressed in terms of the vector \(b\) which vanishes when the metric function \(B\) is constant, i.e., when the axial-vector becomes zero.

The energy-momentum distribution of the non-null Einstein-Maxwell solutions has been explored by using the TP version of Möller prescription. It is found that energy in the TP theory is equal to the energy in GR (as found...
by Sharif and Fatima [37]) plus some additional part. If, for a particular case, we have $\dot{K} = 0$ and $K'$, $\dot{F} = constant$ (or if $\dot{F}$, $\dot{K} = 0$ and $K'$ = constant), then

$$E_{TPG}^d = E_{GR}^d.$$  \hspace{1cm} (65)

On the other hand, the only non-vanishing component of the momentum density lies along $\phi$-direction, similar to the case of Kerr metric [11], due to the cross term appearing in both the metrics. When we choose $\lambda = 1$, it becomes equal to be the momentum in GR [37] plus an additional quantity. If $\ddot{F}$, $\ddot{B}$, $B''$, $\dot{K}$ all vanish, then

$$M_{TPG}^d = M_{GR}^d.$$  \hspace{1cm} (66)

By taking particular values of $E$, $F$ and $K$, we obtain the electromagnetic generalization of the Gödel solution and the Gödel metric as two special cases. The corresponding results for both the special cases are obtained. It is shown that, for the electromagnetic generalization of the Gödel solution, Eq.(65) does not hold, while Eq.(66) holds when $m = 0$. However, for the perfect fluid case, i.e., when $m = \sqrt{2n}$, both Eqs.(65) and (66) hold by taking $n = 0$. In the case of the Gödel metric, these equations hold if we choose the arbitrary constant $a = 0$. For the special solutions, the vector part lies along the radial direction while the axial-vector part along $z$-direction.

We would like to re-iterate here that the tetrad formalism itself has some advantages which comes mainly from its independence from the equivalence principle and consequent suitability to the discussion of quantum issues. In TPG, an energy-momentum gauge current $j_i^\mu$ for the gravitational field can be defined. This is covariant under a spacetime general coordinate transformation and transforms covariantly under a global tangent space Lorentz transformation [38]. It, then, follows that $j_i^\mu$ is a true spacetime tensor but not a tangent space tensor. When we re-write the gauge field equations in a purely spacetime form, they lead to the Einstein field equations and the gauge current $j_i^\mu$ reduces to the canonical energy-momentum pseudo-tensor of the gravitational field. Thus TPG seems to provide a more appropriate environment to deal with the energy problem.

Finally, it is pointed out that we are not claiming that this paper has resolved the problems of GR using the TPG. This is an attempt to touch some issues in TPG with the hope that this alternative may provide more feasible results. Also, it is always an interesting and enriching to look at things from another point of view. This endeavor is in itself commendable.
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