Information Spreading in Stationary Markovian Evolving Graphs

Andrea Clementi† Angelo Monti‡ Francesco Pasquale§ Riccardo Silvestri‡

Abstract

Markovian evolving graphs are dynamic-graph models where the links among a fixed set of nodes change during time according to an arbitrary Markovian rule. They are extremely general and they can well describe important dynamic-network scenarios.

We study the speed of information spreading in the stationary phase by analyzing the completion time of the flooding mechanism. We prove a general theorem that establishes an upper bound on flooding time in any stationary Markovian evolving graph in terms of its node-expansion properties.

We apply our theorem in two natural and relevant cases of such dynamic graphs. Geometric Markovian evolving graphs where the Markovian behaviour is yielded by $n$ mobile radio stations, with fixed transmission radius, that perform independent random walks over a square region of the plane. Edge-Markovian evolving graphs where the probability of existence of any edge at time $t$ depends on the existence (or not) of the same edge at time $t-1$.

In both cases, the obtained upper bounds hold with high probability and they are nearly tight. In fact, they turn out to be tight for a large range of the values of the input parameters. As for geometric Markovian evolving graphs, our result represents the first analytical upper bound for flooding time on a class of concrete mobile networks.

1 Introduction

Markovian evolving graphs and Flooding. Graphs that evolve over time are currently a very hot topic in computer science. They arise from several areas such as mobile networks, networks of users exchanging e-mail or instant messages, citation networks and hyperlinks networks, peer-to-peer networks, social networks (who-trust-whom, who-talks-to-whom, etc.), and many other more [2, 15, 10, 29, 21, 30, 31].

Markovian evolving graphs are a natural and very general class of models for evolving graphs introduced in [2]. In these models, the set of nodes is fixed and the edge set at time $t$ stochastically depends on the edge set at time $t-1$: so, we have an infinite sequence of graphs that is a Markov chain. It is important to observe that, on one hand, these models make the underlying mechanism of how the graph evolves explicit; on the other hand, they are very general since, by a suitable choice of the matrix transition probability yielding the graph Markovian process, it is possible to model several important network scenarios such as faulty-networks and geometric-mobile networks (such scenarios will be described later).

In [2], the hitting time and cover time of random walks in some specific cases of Markovian evolving graphs have been analytically studied. We instead investigate the speed of information spreading on general Markovian evolving graphs. Reaching all nodes from a given source/initiator node is typically required to disseminate or retrieve information: this task is performed via suitable protocols that aim to achieve low delay and message overhead. However, when the network topology is highly dynamic and unknown, (e.g. unstructured peer-to-peer networks, faulty/mobile networks, etc), it is very hard to design efficient protocols and, as a result, the flooding mechanism is often adopted [8, 16, 17, 26]. In the flooding mechanism, any informed node (i.e. any node that has the source message) always sends the source message to all its neighbors. So, the source is informed since the beginning and, clearly, any other...
node gets informed at time step \( t \) iff any of its neighbors (w.r.t. the edge set at time \( t \)) is informed at time step \( t - 1 \).

The completion time of the flooding mechanism (shortly flooding time) is the first time step in which all nodes of the network are informed.

It is important to observe that flooding time of a dynamic network may largely differ from its diameter: for instance, it is easy to construct an \( n \)-node mobile network over a finite square that has, at every time, diameter \( D = 3 \) while its flooding time is \( \Theta(n) \). In general, any diameter bound for a given dynamic network implies nothing about its flooding time but the fact that the latter is finite. Flooding time in fact represents the “natural” lower bound for broadcast protocols in dynamic networks. For this reason, flooding is often used in order to evaluate the relative efficiency of alternative protocols, especially in networks with unknown dynamic topology \([8, 16, 29]\).

**Our results.** We study flooding time in stationary Markovian evolving graphs, i.e., when the initial graph is random with a stationary distribution of the underlying Markov chain \([1]\). In network mobility simulation, this corresponds to the important concept of perfect simulation (see \([25, 3]\)).

We prove an upper bound on flooding time in any stationary Markovian evolving graph. This upper bound is expressed in terms of the parameterized node-expansion properties satisfied by the stationary graphs. As far as we know, this is the first analytical result on the speed of information spreading in so general dynamic models.

We then show the tightness (so the “goodness”) of this bound in two relevant concrete network scenarios: geometric Markovian evolving graphs (in short, geometric-MEG) and edge-Markovian evolving graphs (in short, edge-MEG).

**Geometric Markovian evolving graphs.** We consider a model of evolving graphs that is based on node mobility. It is the discrete version of the well-known random-walk model \([19, 6, 13, 22]\). In this model, denoted here as geometric-MEG, nodes (i.e. radio stations) move over a region of the plane (typically a square region) and each node performs, independently from the others, a sort of Brownian motion. At any time there is an edge (i.e. a bidirectional connection link) between two nodes if they are at distance at most \( R \) (typically \( R \) represents the transmission range). We make time discrete and consider a square grid of arbitrary resolution as a node support-space (see Section 3 for details). This model can also be viewed as the walkers model \([13]\) on the square grid.

It is important to observe that geometric-MEG yield stochastic dependency among the dynamic edges, i.e., the probability of an edge depends on the existence of other edges.

The impact of mobility in information spreading has been the subject of several papers over the last years. However, only few analytical results are currently available. In \([18]\), some bounds on the network capacity (i.e. the number of received packets) has been analyzed on a mobility model that is not explicitly defined. In \([24]\), the authors analyze the broadcast time over a restricted mobility model. In this restricted model, at every time step, the position of each node is selected independently at random inside a disk that is fixed at the starting time. Observe that, in this restriction, there is no stochastic dependence between two consecutive node positions: the model is significantly far from the random walk model. Then, the same work provides some experimental results for the random walk model. Finally, in \([22]\), the speed of data communication between two nodes is studied over a class of Random-Direction models yielding uniform stationary node distributions (including the random walk model with reflection). They provide an upper bound on this speed that can be interpreted as a lower bound on flooding (routing) time when the mobile network is very sparse and disconnected (so, differently from our result, they consider geometric-MEG under the connectivity threshold). Their adopted technique based on Laplacian transform of independent journeys strongly departs from ours and it cannot be extended to provide any upper bound on flooding time. Further related analytical results that have been obtained after the conference version of our work are discussed in Section 5.

We first prove that stationary geometric MEG, yielding connected graphs, satisfy certain parameterized node-expansion properties. We then apply our general result and achieve an upper bound on flooding time. The obtained bound is shown to be tight whenever flooding time is \( \Omega(\log \log n) \). Informally speaking, this happens whenever (i) the transmission radius is not “almost” equal to the diameter of the square region and (ii) the maximal node speed is less than the message-transmission speed. Both assumptions are satisfied by most of real mobile networks. In general, our upper bound is thus at most an \( O(\log \log n) \) additive factor larger than the optimum.

**Further mobility models.** The node-expansion properties of geometric-MEG are mainly due to the fact
that the stationary distribution of node positions is almost uniform. In this paper, we provide formal results and proofs only for flooding in geometric-MEG. However, our expansion technique can be applied to any mobility model yielding a uniform or almost uniform stationary distribution of node positions. Several variants of the random waypoint model, one of the most commonly used mobility models [23, 6, 25], enjoy this uniformity property. Among the others, we mention the random-direction model with reflection (also called the billiard model) [3, 25, 28], the random waypoint on a torus [19, 20, 25, 28] and the random waypoint on a sphere [25]. Furthermore, the uniformity property is also satisfied by the walkers model on a toroidal grid [14].

To the best of our knowledge, our results are the first analytical bounds on flooding time for natural and concrete models of mobile networks.

We finally remark that our flooding analysis does not take care of the interference problem in message transmissions: this is typically managed at the MAC layer of a wireless network architecture [3, 10]. The impact of message interferences in geometric-MEG is a further interesting issue which is out of the scope of our work focusing instead on dynamic-topological properties of MEG.

**Edge-Markovian evolving graphs.** In several network scenarios, there is a strong dependence between the existence (or the absence) of a link between two nodes at a given time step and the existence (or the absence) of the same link at the previous time step. Important examples of this behavior arise in faulty communication networks, peer-to-peer networks, and social networks.

We thus consider edge-MEG, special Markovian evolving graphs, recently studied in [9, 4], which are a time-discrete version of the reciprocity graph model introduced in the context of evolving social networks [32]. At every time step, every edge changes its state (existing or not) according to a two-state Markovian process with probabilities \( p(n) \) and \( q(n) \) where \( n \) is the number of nodes. If an edge exists at time \( t \) then at time \( t+1 \) it dies with probability \( q(n) \) (i.e. death-rate). If instead the edge does not exist at time \( t \), then it will come into existence at time \( t+1 \) with probability \( p(n) \) (i.e. birth-rate). For brevity’s sake, functions \( p(n) \) and \( q(n) \) will be simply denoted as \( p \) and \( q \), respectively. Observe that setting \( q = 1 - p \) yields (time-independent) dynamic random graphs studied in [10] to model dynamic radio networks and in [5] to model epidemic biological processes; here links, at every time, are chosen independently at random. So, edge-MEG are (in turn) a wider and more realistic class of dynamic random graphs. Observe that when \( 0 < p, q < 1 \), the stationary distribution is unique.

Similarly to the case of geometric-MEG, we first prove that stationary edge-MEG, yielding connected graphs, satisfy certain parameterized node-expansion properties. Thanks to these properties, we can apply our general result and achieve an upper bound on flooding time. The obtained bound is shown to be tight whenever flooding time is \( \Omega(\log \log n) \): this includes, for instance, the relevant case where the expected node-degree is \( O(\text{polylog } n) \). In general, our upper bound for edge-MEG is thus at most an \( O(\log \log n) \) additive factor larger than the optimum.

In [9], the maximal flooding time has been studied in edge-MEG with respect to any initial probability distribution. In that paper, in fact, almost tight bounds for the worst-case flooding time have been derived. However, those results do not say whether flooding can be (significantly) faster in stationary edge-MEG. Interestingly enough, our stationary bound implies that, whenever the birth-rate \( p \) is \( O(1/n^{1+\epsilon}) \) and the death-rate \( q \) is \( O(np/\log n) \), there is an exponential gap between the stationary case and the worst-case. An exponential gap also holds whenever \( p = O(\log n/n) \) and \( q = O(p\sqrt{n}) \) (for instance, set \( q = \text{polylog } n/n \)).

**Organization of the paper.** In Section 2 we prove our upper bound for flooding time in general Markovian evolving graphs. The results for geometric-MEG and edge-MEG are described in Sections 3 and 4 respectively. Finally, further related analytical results (obtained after the conference version of our work) and some open questions are discussed in Section 5.

---

1 Notice that, in some of these settings, there is an underlying physical network that supports the abstraction of point-to-point communication.

2 Hence, any inequality \( p \leq (\geq) b(n) \) means that \( p(n) \) is eventually not larger (not smaller) than \( b(n) \). The same holds for \( q = q(n) \).
2 Markovian evolving graphs: the general theorem

Through this paper, the set $[n] = \{1, \ldots, n\}$ will represent the set of $n$ nodes. Let $G = ([n], E)$ be a graph and $I \subseteq [n]$ be a subset of nodes. We denote by $N(I)$ the out-neighborhood of $I$, i.e.

$$N(I) = \{v \in [n] \setminus I : \{u, v\} \in E, \text{ for some } u \in I\}$$

Given a source node $s \in [n]$, the flooding process can be represented by the sequence $\{I_t \subseteq [n] : t \in \mathbb{N}\}$ where $I_t$ is the subset of informed nodes defined recursively as follows

$$\begin{cases}
I_0 &= \{s\} \\
I_{t+1} &= I_t \cup N(I_t)
\end{cases}$$

Notice that the subset $N(I_t)$ refers to the graph at time step $t$. Let $T(s)$ be the first time step such that all nodes are informed. The flooding time is the maximum $T(s)$ over all possible choices of source $s$.

**Definition 2.1 (Markovian evolving graph)** Let $G$ be a family of graphs with the same node set $[n]$. A Markovian evolving graph $\mathcal{M} = \{G_t : t \in \mathbb{N}\}$ is a Markov chain with state space $G$.

A stationary Markovian evolving graph is a Markovian evolving graph $\mathcal{M} = \{G_t : t \in \mathbb{N}\}$ such that $G_0$ is random with a stationary distribution of $\mathcal{M}$.

The following definition concerns a sort of parameterized node-expansion. This is a key-ingredient, in our analysis of flooding in Markovian evolving graphs, to cope with the difficulties due to the stochastic dependence.

**Definition 2.2 (Expander)** A graph $G = ([n], E)$ is a $(h, k)$-expander if, for every set of nodes $I \subseteq [n]$ with $|I| \leq h$, it holds that $|N(I)| \geq k|I|$.

The above definition naturally extends to random variables and their probability distributions.

**Definition 2.3 (Expander II)** Let $X$ be a random variable with values in a family of graphs with the same node set $[n]$. Then $X$ is a $(h, k)$-expander with probability $p$ if

$$\mathbb{P}(X \text{ is a } (h, k)\text{-expander}) \geq p$$

In this case, we also say that the probability distribution of $X$ yields an $(h, k)$-expander with probability $p$.

We are now able to provide our main result for general stationary Markovian evolving graphs. We first show a lemma that connects the parameterized expansion of a (deterministic) evolving graph with its flooding time, then we use it to prove our theorem on the flooding time of stationary Markovian evolving graphs. In what follows, all logarithms are in base $e$.

**Lemma 2.4 (Flooding and Expansion: Deterministic case)** Let $\mathcal{G} = \{G_t : t \in \mathbb{N}\}$ be an evolving graph (i.e. a sequence of graphs with the same node set $[n]$). Suppose an increasing sequence $1 = h_0 \leq h_1 < \cdots < h_s = n/2$ and a non-increasing sequence $k_1 \geq \cdots \geq k_s$ of positive real numbers exist such that, for every $t \in \mathbb{N}$, graph $G_t$ is a $(h_i, k_i)$-expander for every $i = 1, \ldots, s$. Then the flooding time of $\mathcal{G}$ is

$$O\left(\sum_{i=1}^{s} \frac{\log(h_i/h_{i-1})}{\log(1+k_i)}\right)$$

**Proof.** Let $m_t$ be the number of informed nodes at time step $t$, at the beginning $m_0 = 1$. For $i = 1, \ldots, s$ let $T_i$ be the first time step such that the number of informed nodes is larger than $h_i$,

$$T_i = \min\{t \in \mathbb{N} : m_t > h_i\}$$

If $t$ is a time step such that $h_{i-1} < m_t < h_i$ for some $i = 1, \ldots, s$, then, since the graph $G_t$ is a $(h_i, k_i)$-expander, it holds that the number of informed nodes at the next time step is

$$m_{t+1} \geq (1 + k_i)m_t$$
Hence, after $t'$ of such time steps it holds that
\[ m_{t+t'} \geq (1 + k_i)^t m_t \geq (1 + k_i)^{t'} h_{i-1} \]
So the number of time steps between $T_{i-1}$ and $T_i$ is at most
\[ T_i - T_{i-1} \leq \left\lceil \frac{\log(h_i/h_{i-1})}{\log(1 + k_i)} \right \rceil \]
If \( \frac{\log(h_i/h_{i-1})}{\log(1 + k_i)} = \Omega(1) \) then $T_i - T_{i-1} = O \left( \frac{\log(h_i/h_{i-1})}{\log(1 + k_i)} \right)$. If \( \frac{\log(h_i/h_{i-1})}{\log(1 + k_i)} = o(1) \) then $(1 + k_i)h_{i-1} \gg h_i$, that is in just one time step the number of informed nodes jumps from $m_t \leq h_i$ to $m_{t+1}$ much larger than $h_i$. Let $j$ be the index such that $h_j < (1 + k_i)h_{i-1} \leq h_{j+1}$ (if none of such index exists it means $m_{t+1} > n/2$, so $T_s - T_{i-1} = 1$), then it holds that
\[ 1 \leq \frac{\log(h_{j+1}/h_i)}{\log(1 + k_i)} = \sum_{\ell = i}^{j+1} \frac{\log(h_{\ell}/h_{\ell-1})}{\log(1 + k_i)} \leq \sum_{\ell = i}^{j+1} \frac{\log(h_{\ell}/h_{\ell-1})}{\log(1 + k_i)} \]
In the last inequality we used that $k_\ell \leq k_i$ for $\ell \geq i$. Hence we can bound
\[ T_j - T_{i-1} \leq \left\lceil \frac{\log(h_{j+1}/h_i)}{\log(1 + k_i)} \right \rceil = O \left( \sum_{\ell = i}^{j+1} \frac{\log(h_{\ell}/h_{\ell-1})}{\log(1 + k_i)} \right) \]
Finally, by summing up the contributions of all the considered time intervals we have
\[ T_s = O \left( \sum_{i = 1}^{s} \frac{\log(h_i/h_{i-1})}{\log(1 + k_i)} \right) \]
Once we have at least $n/2$ informed nodes, then a symmetric argument holds. Indeed, consider the number $\tilde{m}_t$ of non-informed nodes at time step $t$. Observe that the neighbors, in graph $G_t$, of such nodes were not informed at the previous time step $t-1$ (otherwise at time step $t$ they would have informed their neighbors in graph $G_t$). Let $i \in \{0, 1, \ldots, s\}$ be the index such that $h_{i-1} < \tilde{m}_t \leq h_i$, since $G_t$ is a $(h_i, k_i)$-expander the number of such neighbors is at least $k_i \tilde{m}_t$, so the number of non-informed nodes at the previous time step $t-1$ were at least
\[ \tilde{m}_{t-1} \geq (1 + k_i)\tilde{m}_t \]
In other words, the number of non-informed nodes follows the same growth rate, when there are at least $n/2$ informed nodes and we look at the system going backward in time, of the number of informed nodes, when there are less than $n/2$ informed nodes and the time moves forward. Hence, to go from $n/2$ informed nodes to $n$ informed nodes it takes further $O \left( \sum_{i = 1}^{s} \frac{\log(h_i/h_{i-1})}{\log(1 + k_i)} \right)$ time steps.

As usual, we say that an event $(E)(n)$ holds with high probability (for short w.h.p.) if $P \left( (E)(n) \right) \geq 1 - 1/n$.

**Theorem 2.5** Let $\mathcal{M} = \{ G_t : t \in \mathbb{N} \}$ be a stationary Markovian evolving graph. Assume an increasing sequence $1 = h_0 \leq h_1 < \cdots < h_s = n/2$ and a non-increasing sequence $k_1 \geq \cdots \geq k_s$ (for any $s \leq n/2$) of positive real numbers exist such that, with probability $1 - 1/n^2$, for every $i = 1, \ldots, s$ the stationary distribution of $\mathcal{M}$ yields an $(h_i, k_i)$-expander. Then the flooding time of $\mathcal{M}$ is w.h.p.
\[ O \left( \sum_{i = 1}^{s} \frac{\log(h_i/h_{i-1})}{\log(1 + k_i)} \right) \]

**Proof.** For $t = 0, 1, \ldots$ define the event
\[ \mathcal{E}_t = \{ G_t \text{ is a (} h_i, k_i \text{)-expander for every } i = 1, \ldots, s \} \]
By stationarity hypothesis we have that $P \left( \mathcal{E}_t \right) \geq 1 - 1/n^2$ for every $t$. Now consider the event
\[ \mathcal{F} = \left\{ \text{The flooding time of } \mathcal{M} \text{ is } O \left( \sum_{i = 1}^{s} \frac{\log(h_i/h_{i-1})}{\log(1 + k_i)} \right) \right\} \]
and observe that from Lemma \ref{lemma:property} it follows that \( \bigcap_{t=0}^{n} E_t \subseteq \mathcal{F} \), hence \( \mathcal{F} \subseteq \bigcup_{t=0}^{n} \overline{E}_t \) and we have that
\[
\mathbb{P}(\mathcal{F}) \leq \mathbb{P}\left(\bigcup_{t=0}^{n} \overline{E}_t\right) \leq n \mathbb{P}(\overline{E}_0) \leq \frac{1}{n}
\]

An easy consequence of Theorem \ref{thm:main} is the following

**Corollary 2.6** Let \( M = \{G_t : t \in \mathbb{N}\} \) be a stationary Markovian evolving graph. Assume a non-decreasing sequence \( k_1 \geq \cdots \geq k_{n/2} \) of positive real numbers exists such that, with probability \( 1 - \frac{1}{n^0} \), for every \( i = 1, \ldots, n/2 \) the stationary distribution of \( M \) yields an \((i, k_i)\)-expander. Then the flooding time of \( M \) is w.h.p.
\[
O\left(\sum_{i=1}^{n/2} \frac{1}{i \log(1 + k_i)}\right)
\]

\section{Geometric Markovian evolving graphs}

We introduce a model of dynamic graphs that is a discrete version of the random walk mobility model for radio networks \cite{6}. In the latter model, nodes (i.e. radio stations) move on a bounded region of the plane (typically a square region) and each node performs, independently from the others, a sort of Brownian motion. At any time there is an edge (i.e. a bidirectional connection link) between two nodes if they are at distance at most \( R \) (typically \( R \) represents the transmission range). In our model we discretize time and space. We choose to keep constant the density (i.e. the ratio between the number of nodes and the area) as the number \( n \) of nodes grows. The node region is a square of side \( \sqrt{n} \) and the density equals to 1. This choice is only for the sake of simplicity and all the results can be scaled to any density \( \delta(n) \) (see Observation \ref{obs:scaling}). The nodes can assume positions whose coordinates are integer multiple of a sufficiently small resolution coefficient \( \epsilon > 0 \); in the sequel, we always assume that \( \epsilon \leq 1 \) and \( \epsilon < R \).

Formally, nodes move on the following set of points
\[
L_{n, \epsilon} = \{(i \epsilon, j \epsilon) \mid i, j \in \mathbb{N} \land i, j \leq \frac{\sqrt{n}}{\epsilon}\}
\]

At any time step, a node can move to one of the positions of \( L_{n, \epsilon} \) within distance \( r \) from the previous position. The positive real number \( r \) is a fixed parameter that we call move radius. It can be interpreted as the maximum velocity of a node\footnote{Indeed, a node can run through a distance of at most \( r \) in a unit of time.}. Formally, we introduce the move graph \( M_{n, r, \epsilon} = (L_{n, \epsilon}, E_{n, r, \epsilon}) \), where
\[
E_{n, r, \epsilon} = \{(x, y) \mid x, y \in L_{n, \epsilon}, \ d(x, y) \leq r\}
\]

and \( d(\cdot, \cdot) \) is the Euclidean distance. A node in position \( x \), in one time step, can move in any position in \( \Gamma(x) \), where \( \Gamma(x) = \{y \mid (x, y) \in E_{n, r, \epsilon}\} \). The nodes are identified by the first \( n \) positive integers \( [n] \). The time-evolution of the movement of a single node \( i \) is represented by a Markov chain \( \{P_{i, t} : t \in \mathbb{N}\} \) where \( P_{i, t} \) are random variables whose state-space is \( L_{n, \epsilon} \) and
\[
\mathbb{P}(P_{i, t+1} = x) = \begin{cases} \frac{1}{|\Gamma(P_{i, t})|} & \text{if } x \in \Gamma(P_{i, t}) \\ 0 & \text{otherwise} \end{cases}
\]

In other words, \( P_{i, t} \) is the position of node \( i \) at time \( t \). Thus, the time-evolution of the movements of all the nodes is represented by a Markov chain \( \mathcal{P}(n, r, \epsilon) = \{P_t : t \in \mathbb{N}\} \) whose state-space is \( L_{n, \epsilon} \times L_{n, \epsilon} \times \cdots \times L_{n, \epsilon} \) \( (n \text{ times}) \) and
\[
P_t = (P_{1, t}, P_{2, t}, \ldots, P_{n, t})
\]

Let us fix a transmission radius \( R > 0 \). A geometric-MEG is a sequence of random variables \( G(n, r, \epsilon) = \{G_t : t \in \mathbb{N}\} \) such that \( G_t = ([n], E_t) \) with
\[
E_t = \{(i, j) \mid d(P_{i, t}, P_{j, t}) \leq R\}
\]

From a formal point of view, geometric-MEGs are not Markovian evolving graphs according to Definition \ref{def:markovian}. In order to include them, we need a slight generalization of the definition
Definition 3.1 (Markovian Evolving Graph II) Let $G$ be a family of graphs with the same node set $[n]$. A Markovian evolving graph $G = \{G_t : t \in \mathbb{N}\}$ is a sequence of random variables with state space $G$ and such that there exist both a Markov chain $X = \{X_t : t \in \mathbb{N}\}$ and a function $f$ so that $G_t = f(X_t)$. A stationary Markovian evolving graph is a Markovian evolving graph $G = \{G_t : t \in \mathbb{N}\}$ such that $G_0$ is random with a stationary distribution of $X$ translated by $f$.

Theorem 2.5 easily extends to the above generalized definition by straightforward arguments. As for the stationary case, standard results of Markov chain theory (see [1]) easily imply that the (unique) stationary distribution $\pi_i$ of Markov chain $\{P_{t,t} : t \in \mathbb{N}\}$ is

$$\pi_i(x) = \frac{|\Gamma(x)|}{\sum_{y \in L_n} |\Gamma(y)|}$$

Notice that $\pi_i$ is almost uniform since, for any two positions $x$ and $y$, the values $\pi_i(x)$ and $\pi_i(y)$ can differ by at most a constant factor. Moreover, the stationary distribution of $P(n,r,\epsilon)$ is the product of the independent distributions $\pi_i$ for all $i \in [n]$. We say that a geometric-MEG $G(n,r,R,\epsilon) = \{G_t : t \in \mathbb{N}\}$ is a stationary geometric-MEG if the underlying $P_0$ is random with the stationary distribution of the Markov chain $P(n,r,\epsilon) = \{P_t : t \in \mathbb{N}\}$. Notice that if $G(n,r,R,\epsilon) = \{G_t : t \in \mathbb{N}\}$ is a stationary geometric-MEG then all random variables $G_t$ are random with the same probability distribution that we call stationary distribution of $G(n,r,R,\epsilon)$.

Stationary geometric-MEG enjoy of the following expansion properties.

Theorem 3.2 If $r \geq 0$ and $c \sqrt{\log n} \leq R \leq \sqrt{n}$ for a sufficiently large constant $c$, then constants $\alpha, \beta > 0$ exist such that, with probability $1 - \frac{1}{n^2}$, the stationary distribution of $G(n,r,R,\epsilon)$ yields:

- $A(h, \alpha \frac{R^2}{\sqrt{n}})$-expander for $1 \leq h \leq \alpha R^2$;
- $A(h, \beta \frac{R^2}{\sqrt{n}})$-expander for $\alpha R^2 \leq h \leq n/2$.

Observation 3.3 For general node density $\delta(n)$, Theorem 3.2 holds under the scaled assumption $R \geq c \sqrt{\log n/\delta(n)}$.

Proof. Let $m = \lceil \sqrt{n}/R \rceil$. Consider the partition of the square $\sqrt{n} \times \sqrt{n}$ into $m \times m$ congruent sub-squares, called cells. Every cell can be identified by the pair of indices $(i,j)$, for $1 \leq i, j \leq m$, such that $i$ is the index of row and $j$ is the index of column of the cell. Let $c_{i,j}$ be the subset of the points of $L_{n,\epsilon}$ that fall into the cell $(i,j)$. Notice that the side length $\ell$ of a cell satisfies

$$R/(\sqrt{5} + 1) \leq \ell \leq R/\sqrt{5}$$

Thus, any point of a cell is at distance less than $R$ from any point of a side-by-side adjacent cell. Through the following, we assume that the positions of the nodes are random with the stationary distribution of the Markov chain $P(n,r,\epsilon)$. Moreover, we say that a node belongs to a cell whenever its position belongs to the cell. Let $N_{i,j}$ be the random variable counting the number of nodes in cell $c_{i,j}$. Now, we prove a simple but crucial claim.

Claim 1 If $\epsilon \leq 1$ and $R \geq c \sqrt{\log n}$ for a sufficiently large constant $c$, then a constant $\lambda \geq 1$ exists such that, with probability $1 - \frac{1}{n^2}$, it holds that, for every $1 \leq i, j \leq m$,

$$\frac{R^2}{\lambda} \leq N_{i,j} \leq \lambda R^2$$

Proof. Firstly, consider a fixed cell $(i,j)$. For every $u \in [n]$, let $X_u$ be the $(0,1)$ random variable that is 1 if node $u$ is in the cell $c_{i,j}$. Clearly, these are independent random variables and it holds that $N_{i,j} = \sum_{u \in [n]} X_u$. As for the probability distribution of $X_u$, we have that

$$\mathbb{P}(X_u = 1) = \sum_{x \in c_{i,j}} \pi_u(x)$$
Since 
\[ \pi_u(x) = \frac{|\Gamma(x)|}{\sum_{y \in L_{n,c}} |\Gamma(y)|} \]
it is easy to see that, if \( \epsilon \) is sufficiently small (say \( \epsilon \leq 1 \)) then there is a constant \( \gamma \geq 1 \) such that, for every \( x \in L_{n,c} \), it holds that 
\[ \frac{1}{\gamma|L_{n,c}|} \leq \pi_u(x) \leq \frac{\gamma}{|L_{n,c}|} \]
This implies that 
\[ \frac{|c_{i,j}|}{\gamma|L_{n,c}|} \leq P(X_u = 1) \leq \frac{\gamma|c_{i,j}|}{|L_{n,c}|} \]
By taking into account the side length of the cells, it is easy to verify that 
\[ \frac{R^2}{10n} \leq \frac{|c_{i,j}|}{|L_{n,c}|} \leq \frac{2R^2}{5n} \]
Now, in virtue of the Chernoff’s bound \[27\], if \( R \geq c\sqrt{\log n} \), for a sufficiently large constant \( c \), then a constant \( \lambda \geq 1 \) exists such that 
\[ \frac{R^2}{\lambda} \leq \frac{N_{i,j}}{L_{n,c}} \leq \lambda R^2 \]
with probability at least \( 1 - \frac{1}{n^3} \). Since the number of cells is less than \( n \), a simple application of the union bound proves the thesis of the claim. \( \square \)

Let \( B \) be the event that occurs when, for every \( 1 \leq i, j \leq m \), 
\[ \frac{R^2}{\lambda} \leq \frac{N_{i,j}}{L_{n,c}} \leq \lambda R^2 \]
where \( \lambda \) is the constant of Claim 1. We now prove event \( B \) implies the expansion properties stated in the thesis of the theorem.

Claim 2 If event \( B \) holds then the graph induced by \( R \) and by the positions of the nodes is a \((h, \alpha \frac{R^2}{n})\)-expander for \( 1 \leq h \leq \alpha R^2 \), where \( \alpha = 1/(2\lambda) \).

Proof. Let \( I \subseteq [n] \) be such that \( |I| \leq \alpha R^2 \). Consider a node \( u \in I \) and let \( c_{i,j} \) be the cell that contains \( u \). Since \( B \) holds, \( N_{i,j} \geq \frac{n^2}{R^2} \). All the nodes in \( c_{i,j} \) are adjacent to \( u \). Thus, there are at least \( N_{i,j} - |I| \) nodes that are adjacent to \( u \) and that are not in \( I \). It follows that 
\[ |N(I)| \geq N_{i,j} - |I| \geq \frac{R^2}{\lambda} - \alpha R^2 \geq \frac{R^2}{2\lambda} = \alpha R^2 \]
In other terms, \( |N(I)| \geq \alpha \frac{R^2}{n} |I| \).

Claim 3 If event \( B \) holds then the graph induced by \( R \leq \sqrt{n} \) and by the positions of the nodes is a \((h, \beta \frac{R^2}{\sqrt{n}})\)-expander for \( \alpha R^2 \leq h \leq n/2 \), where \( \beta = \frac{1}{2\lambda^2} \).

Proof. Let \( I \subseteq [n] \) be any subset of nodes with \( |I| \leq n/2 \). We say that a cell is black if it contains at least one node in \( I \). Let \( B \) be the random variable that is the set of black cells. Let \( J \) be the random variable defined as follows
\[ J = \{ u \in [n] | u \not\in I \land \exists c \in B : \text{node } u \text{ belongs to } c \} \]
Now, two cases are possible: either $|J| \geq \beta R\sqrt{|I|}$ or not. Firstly, suppose that $|J| \geq \beta R\sqrt{|I|}$. Since every node in $J$ is in a black cell, it holds that $J \subseteq N(I)$, and thus

$$|N(I)| \geq |J| \geq \beta R\sqrt{|I|}$$

In other terms, $N(I) \geq \beta \frac{R}{\sqrt{|I|}}|I|$ and the expansion property is proved.

Consider now the case $|J| < \beta R\sqrt{|I|}$. We say that a row (column) of cells is black if all the cells of the row (column) are black. Similarly, we say that a row (column) is white if all the cells of the row (column) are white. A row (column) that is neither black nor white is said to be gray. Notice that a gray row (column) contains at least two adjacent cells such that one is non-black and the other is black. Let $B_r$ and $B_c$ be, respectively, the number of black rows and the number of black columns. Three cases may arise.

$[B_r \geq 1]$: Observe that in this case all the columns are either black or gray. Let $Y$ be the number of gray columns. It holds that

$$Y \geq m - B_c \geq m - \frac{|B|}{m}$$

Since event $B$ holds, the number of nodes in non-black cells is bounded by $\lambda R^2 (m^2 - |B|)$ and thus

$$\lambda R^2 (m^2 - |B|) \geq n - |I| - |J| \geq n - |I| - \beta R\sqrt{|I|}$$

It follows that

$$|B| \leq m^2 - \frac{n - |I| - \beta R\sqrt{|I|}}{\lambda R^2}$$

By combining this bound with the previous bound on $Y$ we obtain

$$Y \geq \frac{n - |I| - \beta R\sqrt{|I|}}{\lambda R^2 m} \geq \frac{n - |I| - \beta R\sqrt{|I|}}{\lambda 2\sqrt{5}nR}$$

where the last inequality follows from $m = \lceil \sqrt{5n}/R \rceil$ and $R \leq \sqrt{n}$.

Observe that every gray column contains at least one non-black cell that is adjacent to a black cell. So, all the nodes belonging to those non-black cells are included in $N(I)$. Since event $B$ holds, it follows that

$$|N(I)| \geq Y R^2 \lambda \geq R \left( \frac{n - |I| - \beta R\sqrt{|I|}}{\lambda 2\sqrt{5}n} \right)$$

Now, recalling that $\beta = \frac{1}{\alpha R}$, $|I| \leq n/2$, and $R \leq \sqrt{n}$, it is easy to verify that

$$\frac{n - |I| - \beta R\sqrt{|I|}}{\lambda 2\sqrt{5}n} \geq \beta \sqrt{|I|}$$

It follows that $|N(I)| \geq \beta R\sqrt{|I|}$ and the expansion property holds.

$[B_r \geq 1 \text{ and } B_c = 0]$: This case is symmetric to the previous one.

$[B_r = 0 \text{ and } B_c = 0]$: In this case, all the rows and columns are either gray or white. Let $Y_r$ and $Y_c$ be the number of gray rows and the number of gray columns, respectively. Since there are neither black rows nor black columns, it must be the case that every black cell belongs to both a gray row and a gray column. As a consequence it holds that $Y_r \cdot Y_c \geq |B|$. Without loss of generality, assume that $Y_r \geq Y_c$. Then $Y_r^2 \geq |B|$ and thus $Y_r \geq \sqrt{|B|}$. Since event $B$ holds and every gray row contains a non-black cell adjacent to a black one, it holds that

$$|N(I)| \geq Y_r R^2 \lambda \geq \sqrt{|B|} R^2 \sqrt{\frac{\lambda}{\lambda}}$$
By using again the fact (implied by event $B$) that every cell contains at most $\lambda R^2$ nodes, we have that $|B|\lambda R^2 \geq |I|$ and thus $\sqrt{|B|} \geq \frac{\sqrt{|I|}}{\lambda R^2}$. It follows that

$$|N(I)| \geq R\sqrt{\frac{|I|}{\lambda}} \geq \beta R \sqrt{|I|}$$

and the expansion property holds.

Since, by Claim 1, event $B$ occurs with probability at least $1 - \frac{1}{n}$, Claims 2 and 3 imply that also the expansion properties will hold with probability $1 - \frac{1}{n}$.

Thanks to the general bound given by Corollary 2.6, the above expansion properties can be exploited in order to bound the flooding time in stationary geometric-MEG.

**Theorem 3.4** Let $G(n, r, R, \epsilon)$ be a stationary geometric-MEG. If $r \geq 0$ and $c \sqrt{\log n} \leq R \leq \sqrt{n}$ for a sufficiently large constant $c$, then the flooding time of $G(n, r, R, \epsilon)$ is w.h.p. $O\left(\frac{\sqrt{n}}{R} + \log \log R\right)$

**Proof.** From Theorem 3.2, the stationary geometric-MEG $G(n, r, R, \epsilon)$ enjoys, with probability $1 - \frac{1}{n}$, the following expansion properties:

- $(h, \alpha R^2)$-expander for $1 \leq h \leq \alpha R^2$
- $(h, \beta R^{1/2})$-expander for $\alpha R^2 \leq h \leq n/2$.

Thus, by applying Corollary 2.6 we obtain that flooding time is w.h.p.

$$O\left(\sum_{h=1}^{\alpha R^2} \frac{1}{h \log(1 + \frac{\alpha R^2}{h})} + \sum_{h=\alpha R^2}^{n/2} \frac{1}{h \log(1 + \beta R^{1/2})}\right)$$

We now evaluate the above two sums separately. For the sake of convenience, set $T = \alpha R^2$. It holds that

$$\sum_{h=1}^{T} \frac{1}{h \log(1 + \frac{T}{h})} \leq 2 \sum_{h=1}^{T} \frac{1}{h \log(1 + \frac{T}{h})} \frac{T}{(T+h)}$$

This holds since $\frac{T}{(T+h)} \geq 1/2$ for $h \leq T$. Moreover,

$$\sum_{h=1}^{T} \frac{1}{h \log(1 + \frac{T}{h})} \frac{T}{(T+h)} = \frac{T}{(T+1) \log(T+1)} + \sum_{h=2}^{T} \frac{1}{h \log(1 + \frac{T}{h})} \frac{T}{(T+h)}$$

$$\leq 1 + \int_{1}^{T} \frac{T}{x \log(1 + \frac{T}{x})} \log(1 + \frac{T}{x}) dx$$

$$= 1 + \left[\log \log(1 + \frac{T}{x})\right]_{1}^{T} = \log \log(T) + c$$

where $c$ is a constant. Therefore we have shown that

$$\sum_{h=1}^{\alpha R^2} \frac{1}{h \log(1 + \frac{\alpha R^2}{h})} = O(\log \log R)$$

Now consider the second sum. By using the inequality $\log(1 + x) \geq \frac{x}{1+x}$, we have that

$$\sum_{h=\alpha R^2}^{n/2} \frac{1}{h \log(1 + \beta R^{1/2})} \leq \sum_{h=\alpha R^2}^{n/2} \frac{\sqrt{h} + \beta R}{h \beta R} \leq \frac{1 + \frac{\sqrt{h}}{\beta R}}{\beta R} \sum_{h=\alpha R^2}^{n/2} \frac{1}{\sqrt{h}}.$$
where the last inequality comes from inequality
\[ \sqrt{h} + \beta R \leq (1 + \frac{\beta}{\sqrt{\alpha}}) \sqrt{h} \]
for \( h \geq \alpha R^2 \). Moreover, it holds that
\[
\sum_{h=\alpha R^2}^{n/2} \frac{1}{\sqrt{h}} \leq \int_{\alpha R^2}^{n/2} \frac{dx}{\sqrt{x}} \leq 2\sqrt{n}
\]
By combining the above inequalities we obtain
\[
\sum_{h=\alpha R^2}^{n/2} \frac{1}{h \log(1 + \beta R \sqrt{h})} \leq 2 \frac{1 + \beta R}{\sqrt{n}}
\]
that is,
\[
\sum_{h=\alpha R^2}^{n/2} \frac{1}{h \log(1 + \beta R \sqrt{h})} = O\left( \frac{\sqrt{n}}{R} \right)
\]
We remark that the proof of the expansion properties of Theorem 3.2 only relies on the fact that the stationary distribution of node positions is almost uniform. In fact we can get the same expansion properties for any mobility model yielding a stationary distribution of node position that is uniform or almost uniform. As mentioned in the Introduction, several relevant mobility models enjoy this uniformity property. So, thanks to our Theorem 2.5, we can get an upper bound on flooding time similar to that of Theorem 3.4.

Next theorem shows a lower bound on flooding time in stationary geometric-MEG.

**Theorem 3.5** Let \( G(n, r, R, \epsilon) \) be a stationary geometric-MEG. If \( r \geq 0 \), then its flooding time is w.h.p.

\[ \Omega\left( \frac{\sqrt{n}}{R + r} \right) \]

**Proof.** Since the geometric-MEG is stationary, it is not hard to see that, w.h.p., at time 0 there exist at least two nodes \( u \) and \( v \) that are at distance greater than \( \sqrt{n}/2 \). Consider the flooding process with source node \( v \). Let \( x_0 \) be the position of \( v \) at time 0. For any \( t \), let \( d_t \) be the minimum distance from \( x_0 \) that node \( u \) has ever reached during the first \( t \) time steps. It is immediate to see that \( d_{t+1} \geq d_t - r \). Since \( d_0 \geq \sqrt{n}/2 \), it holds that \( d_t \geq \sqrt{n}/2 - r \cdot t \).

Let \( D_t \) be the maximal distance from \( x_0 \) that any informed node has ever reached during the first \( t \) time steps. It is easy to see that \( D_{t+1} \leq D_t + R + r \cdot t \).

Let \( \tau \) be the time step in which node \( u \) gets informed. It must be the case that \( D_\tau \geq d_\tau \). It follows that
\[ (R + r)\tau \geq D_\tau \geq d_\tau \geq \sqrt{n}/2 - r \cdot \tau. \]

It follows that \( \tau \geq \sqrt{n}/(2(R + 2r)) \). Therefore, the flooding cannot be completed in less than \( \Omega\left( \frac{\sqrt{n}}{R + r} \right) \) time steps.

By comparing Theorem 3.4 and Theorem 3.5 we obtain the following

**Corollary 3.6** Let \( G(n, r, R, \epsilon) \) be a stationary geometric-MEG. If \( r = \Omega(R) \), and \( c\sqrt{\log n} \leq R \leq \frac{\sqrt{n}}{\log \log n} \) for a sufficiently large constant \( c \), then the flooding time of \( G(n, r, R, \epsilon) \) is w.h.p.

\[ \Theta\left( \frac{\sqrt{n}}{R} \right) \]

Under the very reasonable conditions of the above corollary, the general bound on flooding time in Markovian evolving graphs thus turns out to be asymptotically tight for stationary geometric-MEG.
4 Edge-Markovian evolving graphs

We recall the model introduced in [12, 32]. An edge-MEG \( \mathcal{M}(n, p, q) = \{ G_t : t \in \mathbb{N} \} \) is a Markov chain such that \( G_t = ([n], E_t) \) with

\[
E_t = \left\{ e \in \binom{[n]}{2} : X_t(e) = 1 \right\}
\]

where \( \{ X_t(e) : e \in \binom{[n]}{2} \} \) are independent Markov chains with transition matrix

\[
M = \begin{pmatrix}
0 & 1 & \frac{1}{p} \\
1 - p & p & \frac{1}{q} \\
q & 1 - q & 0
\end{pmatrix}
\]

Remind that \( p \) is the birth-rate and \( q \) is the death-rate and notice that an edge-MEG is a Markovian evolving graph according to Definition 2.1. Observe that if \( 0 < p, q, < N \) then by Lemma 4.2 we here provide a detailed proof for our specific setting.

The proof of the theorem is a simple consequence of the expansion properties of the Markov chains \( \{ X_t(e) : t \in \mathbb{N} \} \) are irreducible and aperiodic; so there is a unique stationary distribution

\[
\pi_e = \left( \frac{q}{p + q}, \frac{p}{p + q} \right)
\]

Hence, the stationary distribution of \( \mathcal{M}(n, p, q) \) is \( G_{n, \hat{p}} \) (i.e. Erdős-Rényi distribution in which each possible edge occurs independently with probability \( \hat{p} \)) where here and in the sequel

\[
\hat{p} = \frac{p}{p + q}
\]

Stationary edge-MEG enjoy the following node-expansion properties.

**Theorem 4.1** Let \( \mathcal{M}(n, p, q) \) be an edge-MEG such that \( \hat{p} \geq c \log n \) for a sufficiently large constant \( c \). Then, the stationary distribution of \( \mathcal{M}(n, p, q) \) yields, with probability at least \( 1 - \frac{1}{n^2} \), a \((h, \frac{2\hat{p}}{c})\)-expander for \( 1 \leq h \leq \frac{1}{p} \) and a \((h, \frac{\hat{p}}{c})\)-expander for \( \frac{1}{p} \leq h \leq \frac{c}{p} \).

The proof of the theorem is a simple consequence of the expansion properties of the \( G_{n, \hat{p}} \) model (see for instance [12]). We here provide a detailed proof for our specific setting.

**Lemma 4.2** Let \( \hat{p} \geq c \log n \) for a sufficiently large constant \( c \). With probability \( 1 - \frac{1}{n^2} \) for \( G_{n, \hat{p}} \) it holds that for any \( I \subseteq [n] \) with \( |I| \leq \frac{n}{2} \),

\[
|N(I)| \geq \min \left\{ \frac{|I|m\hat{p}}{c} : \frac{n}{c} \right\}
\]

**Proof.** We first consider the case when \( |I| \leq \frac{1}{p} \) and prove that, with probability at least \( 1 - \frac{1}{n^2} \), it holds \( |N(I)| \geq |I|m\hat{p} \). Then we consider the case \( \frac{1}{p} \leq |I| \leq \frac{n}{2} \) and prove that, with probability at least \( 1 - \frac{1}{n^2} \), it holds \( |N(I)| \geq \frac{n}{2} \).

Let \( m = |I| \leq \frac{1}{p} \). For any \( u \in [n] \setminus I \) consider the random variable \( X_u \) so that \( X_u = 1 \) if \( u \in N(I) \) and \( X_u = 0 \) otherwise. Since \( \mathbb{P}(X_v = 1) \geq m\hat{p} \) we have

\[
\mathbb{E}[|N(I)|] = \sum_{u \in [n]\setminus I} \mathbb{E}[X_u] = (n - m)m\hat{p} \geq \left( n - \frac{1}{\hat{p}} \right)m\hat{p} \geq \frac{1}{2}nm\hat{p}
\]

From Chernoff’s bound we get

\[
\mathbb{P}
\left(|N(I)| \leq \frac{1}{c}nm\hat{p}\right) \leq e^{-\frac{1}{4}nm\hat{p}(\frac{c}{2m})^2} \leq e^{-\frac{1}{4}m\log n \frac{(n-1)^2}{c^2}} \leq n^{-\frac{c}{4}m}
\]
Therefore
\[
P \left( \exists I \subseteq [n], 1 \leq |I| \leq 1/\hat{p} : |N(I)| \leq \frac{1}{e} |I| \hat{p} \right) \leq \sum_{I \subseteq [n]} P \left( |N(I)| \leq \frac{1}{e} |I| \hat{p} \right)
\]
\[
\leq \sum_{|I| = 1/\hat{p}} \left( \frac{n}{m} \right) n^{-\frac{e-4}{2}}
\]
\[
\leq \sum_{m=1}^{[1/\hat{p}]} n^{m} n^{-\frac{e-4}{2}} \leq \frac{1}{\hat{p}} n^{-\frac{e-4}{2}} \leq n^{-\frac{e-12}{2}}
\]

And the thesis follows if we choose \( c \geq 20 \).

Now consider the case where \( \frac{1}{\hat{p}} \leq |I| = m \leq \frac{n}{c} \). Notice that \( |N(I)| \leq \frac{n}{c} \) if and only if there exists a set \( A \subseteq [n] \setminus (I \cup N(I)) \) such that \( |A| \geq n - m - \frac{n}{c} \). Hence
\[
P \left( \exists I \subseteq [n], |I| = m : |N(I)| \leq \frac{n}{c} \right) = \left( \frac{n}{m} \right) (n - m \frac{n}{|A|})(1 - \hat{p})^{m \frac{n}{|A|}}
\]

From the following inequalities
- \( \left( \frac{n}{m} \right) \leq n^m \leq e^{\frac{mn}{c}} \)
- \( \left( \frac{n-m}{|A|} \right) = \left( \frac{n-m}{n-m-|A|} \right) \leq \left( \frac{n}{2} \right) \leq (ec) \hat{p} = e^{\hat{p} \log(ec)} \leq e^{mn\hat{p} \left( \frac{1}{2} + \frac{\log c}{2} \right)} \)
- \( (1 - \hat{p})^{m \frac{n}{|A|}} \leq e^{mn\hat{p} \left( \frac{1}{2} - \frac{\log c}{2} \right)} \)

by choosing \( c \) sufficiently large, we get
\[
P \left( \exists I \subseteq [n], |I| = m : |N(I)| \leq \frac{n}{c} \right) \leq e^{mn\hat{p} \left( \frac{1}{2} - \frac{\log c}{2} \right)} \leq e^{-\frac{n}{c}}
\]

Hence
\[
P \left( \exists I \subseteq [n], \frac{1}{\hat{p}} \leq |I| \leq \frac{n}{2} : |N(I)| \leq \frac{n}{c} \right) \leq \sum_{m=1/\hat{p}}^{[n/2]} P \left( \exists I \subseteq [n], |I| = m : |N(I)| \leq \frac{n}{c} \right)
\]
\[
\leq \sum_{m=1/\hat{p}}^{[n/2]} e^{-\frac{n}{c}} \leq ne^{-\frac{n}{c}} \leq n^{-2}
\]

where the last inequality holds for sufficiently large \( n \).

The expansion properties of stationary edge-MEG, stated in Theorem 4.1, allow us to apply Corollary 2.6 and thus get the following

**Theorem 4.3** Let \( \mathcal{M}(n, p, q) \) be a stationary edge-MEG such that \( \hat{p} \geq c \frac{\log n}{n} \) for a sufficiently large constant \( c \). Then flooding time in \( \mathcal{M}(n, p, q) \) is w.h.p.

\[
O \left( \frac{\log n}{\log(np)} + \log \log(np) \right)
\]

**Proof.** Thanks to Theorem 4.1 we can apply Corollary 2.6 with sequence

\[
k_i = \begin{cases} \frac{np}{c} & \text{for } 1 \leq i \leq \left\lfloor \frac{1}{\hat{p}} \right\rfloor \\ \frac{1}{c} & \text{for } \left\lfloor \frac{1}{\hat{p}} \right\rfloor < i \leq \frac{1}{\hat{p}} \end{cases}
\]

Thus we obtain that the order of flooding time is w.h.p. bounded by
\[
\sum_{i=1}^{[1/\hat{p}]} \frac{1}{i \log (1 + \frac{np}{c})} + \sum_{i=\lfloor 1/\hat{p} \rfloor + 1}^{\lfloor n/c \rfloor} \frac{1}{i \log (1 + \frac{1}{c})} + \sum_{i=\lfloor n/c \rfloor}^{n/2} \frac{1}{i \log (1 + \frac{1}{c})}
\]

13
We now evaluate the above sums separately. For the first sum, by using \( \sum_{i=1}^{m} \frac{1}{i} \leq \log m + 1 \), we have

\[
\sum_{i=1}^{\lceil \log n \rceil} \frac{1}{i \log (1 + \frac{n}{c})} = \frac{\log \frac{1}{\hat{\Delta}} + 1}{\log (1 + \frac{n}{c})} = O\left( \frac{\log n}{\log(n\hat{\rho})} \right)
\]

For the second sum, by using \( \log(1 + x) \geq \log x \), we have

\[
\sum_{i=\lceil \frac{n}{c} \rceil}^{\lceil \frac{n}{c} \rceil - 1} \frac{1}{i \log (1 + \frac{n}{c})} \leq \int_{\lceil \frac{n}{c} \rceil}^{\lceil \frac{n}{c} \rceil - 1} \frac{1}{x \log (1 + \frac{n}{c})} dx = -\log\log \frac{n}{\hat{c}x} \leq O(\log(n\hat{\rho}))
\]

For the third sum, we apply \( \log(1 + x) \geq x/(1 + x) \) for \( x < 1 \) and get

\[
\sum_{i=\lceil \frac{n}{c} \rceil}^{\lceil \frac{n}{c} \rceil} \frac{1}{i \log (1 + \frac{n}{c})} \leq \sum_{i=\lceil \frac{n}{c} \rceil}^{\lceil \frac{n}{c} \rceil} \frac{1 + \frac{n}{c}}{i \frac{n}{c}} = \sum_{i=\lceil \frac{n}{c} \rceil}^{\lceil \frac{n}{c} \rceil} \frac{1}{i} \leq O(1)
\]

Next theorem gives a lower bound on flooding time in stationary edge-MEG.

**Theorem 4.4** Let \( M(n, p, q) \) be a stationary edge-MEG such that \( \hat{\rho} \geq c \frac{\log n}{n} \) for a sufficiently large constant \( c \). Then the flooding time of \( M(n, p, q) \) is w.h.p.

\[ \Omega \left( \frac{\log n}{\log(n\hat{\rho})} \right) \]

**Proof.** Let \( \Delta_t \) be the random variable indicating the maximal node degree of \( G_t \). Since the marginal distribution of \( G_t \) is Erdős-Rényi \( G_{n, \hat{\rho}} \) then, for a sufficiently large \( c \) (say \( c = 4 \)), it holds that \( \mathbb{P} (\Delta_t > 2n\hat{\rho}) < 1/n^2 \). By applying the union bound the probability that a time step \( t < n \) exists such that \( \Delta_t > 2n\hat{\rho} \) is at most \( 1/n \). Thus, the number of informed nodes at time step \( t < n \) is at most \( (2n\hat{\rho})^t \) w.h.p., and this number is less than \( n/2 \) for \( t < \frac{\log(n/2)}{\log(2n\hat{\rho})} \).

By comparing the upper bound of Theorem 4.3 and the lower bound of Theorem 4.4 we obtain the following.

**Corollary 4.5** Let \( M(n, p, q) \) be a stationary edge-MEG such that \( c \frac{\log n}{n} \leq \hat{\rho} \leq c \frac{\log n}{n} \), for a sufficiently large constant \( c \). Then flooding time in \( M(n, p, q) \) is w.h.p.

\[ \Theta \left( \frac{\log n}{\log(n\hat{\rho})} \right) \]

**5 Conclusions**

We showed that in geometric-MEG, under some conditions on the maximal node speed and transmission radius, node mobility has an almost negligible impact on flooding time: the latter turns out to be equivalent to the diameter of the static stationary graph. A similar fact holds for edge-MEG as well.

After the conference version of this paper, an improved, dynamic version of our expansion technique has been derived in [11] in order to obtain almost tight bounds for the flooding time of highly-sparse and disconnected geometric-MEG when the maximal node speed is larger than the transmission radius. So, in this case, mobility significantly speeds-up flooding time with respect to the static case.

An important open issue is to provide analytical bounds for the flooding time of evolving graphs that are somewhat non homogeneous. Interesting examples are the evolving graphs yielded by node performing random walks over highly-irregular support graphs and those yielded by nodes moving according the random-waypoint model over a non-convex, irregular region.
References

[1] D. Aldous and J. Fill. *Reversible Markov Chains and Random Walks on Graphs*. http://stat-www.berkeley.edu/users/aldous/RWG/book.html, 2002.

[2] C. Avin, M. Koucky, and Z. Lotker. How to explore a fast-changing world. In *Proc. of 35th International Colloquium on Automata, Languages and Programming (ICALP’08)*, volume 5125 of LNCS, pages 121–132. Springer, 2008.

[3] N. Bansal and Z. Liu. Capacity, delay, and mobility in wireless ad-hoc networks. In *Proc. of 22nd IEEE INFOCOM*, 2003.

[4] H. Baumann, P. Crescenzi, and P. Fraigniaud. Parsimonious flooding in dynamic graphs. In *Proc. of 28th ACM PODC*, pages 260–269, 2009.

[5] F. Brauer, P. van den Driessche, and J. Wu (Eds). *Mathematical Epidemiology*. Lecture Notes in Mathematics, subseries in Math. Biosciences, 2008.

[6] T. Camp, J. Boleng, and V. Davies. A survey of mobility models for ad hoc network research. *Wireless Communication and Mobile Computing*, 2(5):483–502, 2002.

[7] T. Camp, W. Navidi, and N. Bauer. Improving the accuracy of random waypoint simulations through steady-state initialization. In *Proc. of 15th Int. Conf. on Modelling and Simulation*, pages 319–326, 2004.

[8] N.B. Chang and M. Liu. Optimal controlled flooding search in a large wireless network. In *Proc. of 3rd International Symposium on Modeling and Optimization in Mobile, Ad Hoc and Wireless Networks (WIOPT’05)*, pages 229–237, 2005.

[9] A. Clementi, C. Macci, A. Monti, F. Pasquale, and R. Silvestri. Flooding time in edge-markovian dynamic graphs. In *Proc. of 27th Annual ACM SIGACT-SIGOPS Symposium on Principles of Distributed Computing (PODC’08)*, pages 213–222. ACM Press, 2008.

[10] A. Clementi, A. Monti, F. Pasquale, and R. Silvestri. Communication in dynamic radio networks. In *Proc. of 26th Annual ACM SIGACT-SIGOPS Symposium on Principles of Distributed Computing (PODC’07)*, pages 205–214. ACM Press, 2007.

[11] A. Clementi, F. Pasquale, and R. Silvestri. Manets: High mobility can make up for low transmission power. In *Proc. of the 36th International Colloquium on Automata, Languages and Programming (ICALP’09)*, volume 5556, pages 387–398. Springer, LNCS, 2009.

[12] C. Cooper and A. Frieze. The cover time of sparse random graphs. 2008.

[13] J. Diaz, D. Mitsche, and X. Perez-Gimenez. On the connectivity of dynamic random geometric graphs. In *Proc. of 19th annual ACM-SIAM symposium on Discrete algorithms (SODA’08)*, pages 601–610, 2008.

[14] J. Diaz, X. Perez, M.J. Serna, and N.C. Wormald. Walkers on the cycle and the grid. *SIAM J. Discrete Math.*, 22(2):747–775, 2008.

[15] S.N. Dorogovtsev and J.F.F. Mendes. *Evolution of Networks*. Oxford University Press, 2003.

[16] M.M.C. Gkantsidis and A. Saberi. Hybrid search schemes for unstructured peer-to-peer networks. In *Proc. of 24th INFOCOM*, pages 1526–1537. IEEE Computer Society, 2005.

[17] Gnutella. Gnutella rfc. http://rfc-gnutella.sourceforge.net, 2002.

[18] M. Grossglauser and N.C. Tse. Mobility increases the capacity of ad-hoc wireless networks. *IEEE/ACM Trans. on Networking*, 10(4), 2002.

[19] R.A. Guerin. Channel occupancy time distribution in a cellular radio system. *IEEE Trans. on Vehicular Technology*, 36(3):89–99, 1987.
[20] Z.J. Haas. The routing algorithm for the reconfigurable wireless networks. In *Proc. of the ICUPC*, 1997.

[21] S.M. Hedetniemi, S.T. Hedetniemi, and A.L. Liestman. A survey of gossiping and broadcasting in communication networks. *Networks*, 18(4):319–349, 1988.

[22] P. Jacquet, B. Mans, and G. Rodolakis. Information propagation speed in mobile and delay tolerant networks. 2009.

[23] D.B. Johnson and D.A. Maltz. Dynamic source routing in ad-hoc wireless networks. *Mobile Computing*, pages 153–181, 1996.

[24] Z. Kong and E. M. Yeh. On the latency for information dissemination in mobile wireless networks. pages 139–148, 2008.

[25] J.-Y. Leboudec and M. Vojnovic. Perfect simulation and the stationarity of a class of mobility models. In *Proc. of 24th IEEE INFOCOM'05*, pages 2743–2754, 2005.

[26] Q. Lv, P. Cao, E. Cohen, K. Li, and S. Shenker. Search and replication in unstructured peer-to-peer networks. In *Proc. of 16th International Conference on Supercomputing (ICS’02)*, pages 84–95, 2002.

[27] M. Mitzenmacher and E. Upfal. *Probability and Computing*. Cambridge University Press, 2005.

[28] P. Naine, D.F. Towsley, B. Liu, and Z. Liu. Properties of random direction models. In *Proc. of 24th IEEE INFOCOM*, pages 1896–1907, 2005.

[29] K. Oikonomou and I. Stavrakakis. Performance analysis of probabilistic flooding using random graphs. In *Proc. of 1st IEEE WoWMoM Workshop on Autonomic and Opportunistic Communications (AOC)*, pages 1–6. IEEE Computer Society, 2007.

[30] B. Pittel. On spreading a rumor. *SIAM Journal on Applied Mathematics*, 47(1):213–223, 1987.

[31] C. Scheideler. Models and techniques for communication in dynamic networks. In *Proc. of 19th Symposium on Theoretical Aspects of Computer Science (STACS'02)*, volume 2285 of *LNCS*, pages 27–49. Springer, 2002.

[32] S. Wasserman. Analyzing social networks as stochastic processes. *Journal of the American Statistical Association*, 75:280–294, 1980.