A. Geometric derivations of the elliptical region

Here we present the analytic form of the centre $c$, semi-major axis $y$, and semi-minor axis $z$ of the elliptical region $L$ (see Sec. 3.1 in the main text) following the method in [1, 2] (subscript $i$ and explicit dependency on $B$ are omitted for simplicity). See Fig. 2a in the main text for a visual representation of the aforementioned geometric entities.

1. Calculate direction of the cone-beam
\[ \hat{u} = \frac{R(t; \omega_c) \hat{n}}{\|R(t; \omega_c) \hat{n}\|_2}, \] (1)
its radius
\[ r = \sin \alpha(B), \] (2)
and the norm vector to the image plane \( \hat{n} = [0 \ 0 \ 1]^T \).

2. Calculate the semi-major axis direction within the cone-beam
\[ \hat{y} = \frac{\hat{u} \times (\hat{u} \times \hat{n})}{\|\hat{u} \times (\hat{u} \times \hat{n})\|} \] (3)
and semi-minor axis direction
\[ \hat{z} = \frac{\hat{y} \times \hat{u}}{\|\hat{y} \times \hat{u}\|}. \] (4)

3. Calculate the intersecting points between the ray with the direction of the semi-major axis and the cone-beam
\[ y^{(a)} = \hat{u} - r \hat{y}, \] (5)
\[ y^{(b)} = \hat{u} + r \hat{y}, \]
and the analogous points for the semi-minor axis
\[ z^{(a)} = \hat{u} - r \hat{z}, \] (6)
\[ z^{(b)} = \hat{u} + r \hat{z}. \]

4. Obtain $y^{(a)}$, $y^{(b)}$, $z^{(a)}$ and $z^{(b)}$ as the projection of (5) and (6) into the image plane with the intrinsic matrix $K$.

5. Calculate $c = 0.5(y^{(a)} + y^{(b)})$, $y = \|y^{(a)} - y^{(b)}\|_2$, and $z = \|z^{(a)} - z^{(b)}\|_2$.

B. Proofs

We state our integer quadratic problem again.
\[ \mathcal{S}_d^*(B) = \max_{\omega \in \mathbb{B}} \sum_{i=1}^{K} \left( \sum_{j=1}^{N} Z_{i,j} \right)^2 \]
\[ \text{s.t.} \ Z_{i,j} \leq M_{i,k}, \ \forall i, k, \quad \text{(IQP)} \]

B.1. Proof of Lemma 1 in the main text

Lemma 1.
\[ \overline{\mathcal{P}}_c(x_j; B) \geq \max_{\omega \in \mathbb{B}} H_c(x_j; \omega) \] (7)
with equality achieved if $B$ is singleton, i.e., $B = \{\omega\}$.

Proof. This lemma can be demonstrated by contradiction. Let $\omega^*$ be the optimiser for the RHS of (7). If
\[ H_c(x_j; \omega^*) > \overline{\mathcal{P}}_c(x_j; B), \] (8)
it follows from the definition of pixel intensity (Eq. (1) in the main text) and its upper bound (Eq. (23) in the main text) that
\[ \|x_j - f(u_i, t_i, \omega^*)\| < \max \left( \|x_j - c_i(B)\| - \|y_i(B)\|, 0 \right) \] (9)
for at least one $i = 1, \ldots, N$.

In words, the shortest distance between $x_j$ and the disc $D_i(B)$ is greater than the distance between $x_j$ and the optimal position $f(u_i, t_i, \omega^*)$. However, $f(u_i, t_i, \omega^*)$ is always inside the disc $D_i(B)$, and hence Eq. (9) cannot hold. If $B = \{\omega\}$, then from definition (23) in the main text $\overline{\mathcal{P}}_c(x_j; B) = H_c(x_j; \omega)$. \qed

B.2. Proof of Lemma 2 in the main text

Lemma 2.
\[ \mathcal{S}_d^*(B) \geq \max_{\omega \in \mathbb{B}} \sum_{j=1}^{P} H_d(x_j; \omega)^2, \] (10)
with equality achieved if $B$ is singleton, i.e., $B = \{\omega\}$.

Proof. We pixel-wisely reformulate IQP:

$$\overline{\mathcal{S}}_d(B) = \max_{Q \in \{0,1\}^{N \times P}} \sum_{j=1}^{P} \left( \sum_{i=1}^{N} Q_{i,j} \right)^2$$

s.t. $Q_{i,j} \leq T_{i,j}$, $\forall i, j$, (P-IQP)

and we express the RHS of (10) as a mixed integer quadratic program:

$$\max_{\omega \in B, Q \in \{0,1\}^{N \times P}} \sum_{j=1}^{P} \left( \sum_{i=1}^{N} Q_{i,j} \right)^2$$

s.t. $Q_{i,j} = I(f(u_i, t_i; \omega) \in x_j)$, $\forall i, j$. (MIQP)

Problem P-IQP is a relaxed version of MIQP - hence (10) holds - as for every $e_i$, the feasible pixel $x_j$ is in $D_i(B)$; whereas for MIQP, the feasible pixel is dictated by a single $\omega \in B$. If $B$ collapses into $\omega$, every event $e_i$ can intersect only one pixel $x_j$, hence $T_{i,j} = I(f(u_i, t_i; \omega) \in x_j)$, $\forall i, j$; $\sum_{j=1}^{P} T_{i,j} = 1$, $\forall i$; and $\sum_{j=1}^{P} Q_{i,j} = 1 \Rightarrow Q_{i,j} = T_{i,j}$, $\forall i$; therefore, MIQP is equivalent to P-IQP if $B = \{\omega\}$.

B.3. Proof of Lemma 3 in the main text

Lemma 3. Problem IQP has the same solution if $M$ is replaced with $M'$.

Proof. We show that removing an arbitrary non-dominant column from $M$ does not change the solution of IQP. Without loss of generality, assume the last column of $M$ is non-dominant. Equivalent to solving IQP on $M$ without its last column is the following IQP reformulation:

$$\overline{\mathcal{S}}'_d(B) = \max_{Z \in \{0,1\}^{N \times K'}} \sum_{k=1}^{K'} \left( \sum_{i=1}^{N} Z_{i,k} M_{i,k} \right)^2$$

s.t. $Z_{i,k} \leq M_{i,k}$, $\forall i, k$, (11c)

$$\sum_{k=1}^{K'} Z_{i,k} = 1$, $\forall i$, (11d)

$Z_{i,K} = 0$, $\forall i$, (11e)

which is same as IQP but with additional constraint (11e). Since $M_{i,K}$ is non-dominant, it must exists a dominant column $M_{i,\eta}$ such that

$$M_{i,K} \leq M_{i,\eta}$, $\forall i.$ (12)

Hence, if $M_{i,K} = 1$, then $M_{i,\eta} = 1$ must holds $\forall i$. Let $Z^*$ be the optimiser of IQP with $Z_{i',k,K}^* = \ldots = Z_{i',K,K}^* = 1$. Let define $Z'^*$ same as $Z^*$ but with $Z_{i',K}^* = 0$ and $Z_{i,\eta,\eta}^* = 1$. In words, we “move” the 1 values from the last column to its dominant one. We show that $Z'^*$ is an equivalent solution (same objective value than $Z^*$). $Z'^*$ is feasible since (12) ensures condition (11c), (11d) is not affected by “moving ones” in the same row, and (11e) is true for the definition of $Z'^*$. Finally we show that

$$\sum_{i=1}^{N} Z_{i,K}^* M_{i,K} = \sum_{i=1}^{N} Z_{i,\eta}^* M_{i,\eta}$$ (13)

dependence $Z'^*$ produces same objective value than IQP. We prove (13) by contradiction. Assume exists at least one $i' \notin \{i_1, \ldots, i_b\}$ such that $Z_{i',\eta}^* = 1 \Rightarrow Z_{i',\eta}^* = 1$. Then, $Z'^*$ produces a larger objective value than $Z^*$ which is a contradiction since problem (11) is most restricted than IQP. Thus, removing any arbitrary non-dominant column will not change the solution which implies this is also true if we remove all non-dominant columns (i.e., if we replace $M$ with $M'$).

\Box

B.4. Proof of Lemma 4 in the main text

Lemma 4.

$$\overline{\mathcal{S}}_d(B) \geq \overline{\mathcal{S}}'_d(B)$$ (14)

with equality achieved if $B$ is singleton, i.e., $B = \{\omega\}$.

Proof. To prove (14), it is enough to show

$$\overline{\mathcal{S}}_d(B) \max_{Z \in \{0,1\}^{N \times K'}} \sum_{k=1}^{K'} \left( \sum_{i=1}^{N} Z_{i,k} M'_{i,k} \right)^2$$

s.t. $Z_{i,k} \leq M'_{i,k}$, $\forall i, k$, (R-IQP)

$$\sum_{k=1}^{K'} \sum_{i=1}^{N} Z_{i,k} = N$$

is a valid relaxation of IQP. This is true as the constraint $\sum_{k=1}^{K'} \sum_{i=1}^{N} Z_{i,k} = N$ in R-IQP is a necessary but not sufficient condition for the constraints $\sum_{k=1}^{K'} Z_{i,k} = 1$, $\forall i$ in IQP. If $B$ collapse into $\omega$, every event $e_i$ can intersect only one CC $G_k \Rightarrow \sum_{k=1}^{K'} Z_{i,k} = 1$; hence, R-IQP is equivalent to IQP.

\Box
B.5. Proof of lower bound (39) in the main text

Lemma 5.

\[ H_c(x_j; B) \leq \min_{\omega \in B} H_c(x_j; \omega) \]  
(15)

with equality achieved if \( B \) is singleton, i.e., \( B = \{\omega\} \).

Proof. Analogous to Lemma 1, we prove this Lemma by contradiction. Let \( \omega^* \) be the optimiser for the RHS of (15). If

\[ H_c(x_j; \omega^*) < H_c(x_j; B), \]  
(16)

it follows from the definition of pixel intensity (Eq. (1) in the main text) and its lower bound (Eq. (39) in the main text) that

\[ \|x_j - f(u_i, t_i; \omega^*)\| > \|x_j - c_i(B)\| + \|y_i(B)\|, \]  
(17)

for at least one \( i = 1, \ldots, N \).

In words, the longest distance between \( x_j \) and the disc \( D_i(B) \) is less than the distance between \( x_j \) and the optimal position \( f(u_i, t_i; \omega^*) \). However, \( f(u_i, t_i; \omega^*) \) is always inside the disc \( D_i(B) \), and hence Eq. (17) cannot hold. If \( B = \{\omega\} \), then from definition (39) in the main text \( H_c(x_j; B) = H_c(x_j; \omega) \).

B.6. Proof of lower bound (41) in the main text

Lemma 6.

\[ \mu_d(B) \leq \min_{\omega \in B} \frac{1}{P} \sum_{j=1}^{P} H_d(x_j; \omega), \]  
(18)

with equality achieved if \( B \) is singleton, i.e., \( B = \{\omega\} \).

Proof. This lemma can be demonstrated by contradiction. Let \( \omega^* \) be the optimiser of the RHS of (18). If

\[ \frac{1}{P} \sum_{j=1}^{P} H_d(x_j; \omega^*) < \mu_d(B), \]  
(19)

after replacing the pixel intensity and the lower bound pixel value with their definitions (Eqs. (3) and (41) in the main text) in (19), it leads to

\[ \sum_{i=1}^{N} \sum_{j=1}^{P} I(f(u_i, t_i; \omega^*) \text{ lies in pixel } x_j) \]  
(20a)

\[ < \sum_{i=1}^{N} I(D_i \text{ fully lie in the image plane}). \]  
(20b)

In words, for every warped event \( f(u_i, t_i; \omega^*) \in D_i \) that lies in any pixel \( x_j \in X \) of the image plane, the discs \( D_i \) must fully lie in the image plane. Since (20a) is a less restricted problem than (20b), (19) cannot hold. If \( B = \{\omega\} \), \( D_i = f(u_i, t_i; \omega) \); therefore, the two sides in (18) are equivalent.

C. Additional qualitative results

Figs. 1, 2 and 3 show additional motion compensation results (Sec. 4.2 in the main text) for subsequences from boxes, dynamic and poster.

References

[1] Rolf Clackdoyle and Catherine Mennessier. Centers and centroids of the cone-beam projection of a ball. Physics in Medicine & Biology, 56(23):7371, 2011.

[2] Yinlong Liu, Yuan Dong, Zhijian Song, and Manning Wang. 2d-3d point set registration based on global rotation search. IEEE Transactions on Image Processing, 28(5):2599–2613, 2018.
Figure 1. Qualitative results (motion compensated event images) for boxes.
Figure 2. Qualitative results (motion compensated event images) for dynamic.
Figure 3. Qualitative results (motion compensated event images) for *poster*. 