Separation of variables for bi-Hamiltonian systems

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Abstract

We address the problem of the separation of variables for the Hamilton-Jacobi equation within the theoretical scheme of bi-Hamiltonian geometry. We use the properties of a special class of bi-Hamiltonian manifolds, called $\omega N$ manifolds, to give intrinsic tests of separability (and Stäckel separability) for Hamiltonian systems. The separation variables are naturally associated with the geometrical structures of the $\omega N$ manifold itself. We apply these results to bi-Hamiltonian systems of the Gel’fand-Zakharevich type and we give explicit procedures to find the separated coordinates and the separation relations.

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1 Introduction

The technique of additive separation of variables for solving by quadratures the Hamilton-Jacobi (HJ) equation is a very important tool in analytical mechanics, initiated by Jacobi and others back in the nineteenth century (see, e.g., [35, 34]). Following these classical works, an \( n \)-tuple \((H_1, \ldots, H_n)\) of functionally independent Hamiltonians will be said to be separable in a set of canonical coordinates \((q_1, \ldots, q_n, p_1, \ldots, p_n)\) if there exist \( n \) relations, called separation relations, of the form

\[
\phi_i(q_i, p_i, H_1, \ldots, H_n) = 0, \quad i = 1, \ldots, n, \quad \text{with } \det \left[ \frac{\partial \phi_i}{\partial H_j} \right] \neq 0. \tag{1.1}
\]

The reason for this definition is that the stationary Hamilton-Jacobi equations for the Hamiltonians \( H_i \) can be collectively solved by the additively separated complete integral

\[
W(q_1, \ldots, q_n; \alpha_1, \ldots, \alpha_n) = \sum_{i=1}^{n} W_i(q; \alpha_1, \ldots, \alpha_n), \tag{1.2}
\]

where the \( W_i \) are found by quadratures as the solutions of ordinary differential equations.

One of the first systematic results was found by Levi-Civita, who provided, in 1904, a test for the separability of a given Hamiltonian in a given system of canonical coordinates. Stäckel and Eisenhart concentrated on Hamiltonians quadratic in the momenta and orthogonal separation variables. In particular, Stäckel considered the Hamiltonian

\[
H(q, p) = \frac{1}{2} \sum g^{ii}(q) p_i^2 + V(q)
\]
and showed that $H$ is separable in the coordinates $(q, p)$ if there exist an invertible matrix $S(q)$ and a column vector $U(q)$ such that the $i$-th rows of $S$ and $U$ depend only on the coordinate $q_i$, and $H$ is among the solutions $(H_1, \ldots, H_n)$ of the linear system

$$\sum_{j=1}^{n} S_{ij}(q_i) H_j = \frac{1}{2} p_i^2 - U_i(q_i).$$

These equations provide the separation relations for the (commuting) Hamiltonians $(H_1, \ldots, H_n)$.

With the works of Eisenhart, the theory of separation of variables was inserted in the context of global Riemannian geometry, and this still represents an active area of research, where the notions of Killing tensor and Killing web play a key role (see, e.g., [43, 24, 4]).

Starting from the study of algebraic-geometric solutions of (stationary reductions of) soliton equations and the introduction of the concept of algebraic completely integrable system [3, 8, 33], separation of variables has received a renewed attention (see, e.g., [16, 1, 20, 22, 39]). This research activity, also connected with the theory of quantum integrable systems, deals with Hamiltonian systems admitting a Lax representation with spectral parameter and an $r$-matrix formulation. In this case, the separation relations are provided by the spectral curve

$$\det(\mu I - L(\lambda)) = 0$$

associated with the Lax matrix $L(\lambda)$. Indeed, one can often find canonical coordinates $(\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n)$ on the phase space such that every pair $(\lambda_i, \mu_i)$ belongs to the spectral curve. Since the Hamiltonians are defined by the spectral curve, they are separable in these coordinates.

The two classes of separable systems briefly recalled above strongly suggest that a “theory of separability” should start from the following data,

1. A class of symplectic manifolds $M$;
2. A class of canonical coordinates on $M$;
3. A class of Hamiltonian functions on $M$,

and should provide
a) separability test(s) to ascertain whether the HJ equations associated with
the selected Hamiltonians admit a complete integral which is additively
separated in the chosen coordinates;

b) algorithms to compute the separation coordinates and to exhibit the sepa-
ration relations, so that the HJ equations can be explicitly solved.

In the context of Riemannian geometry, the manifolds are cotangent bundles
of Riemannian manifolds, the coordinates are (fibered) orthogonal coordinates,
and the Hamiltonians are quadratic in the momenta. For Lax systems, roughly
speaking, the manifolds are suitable coadjoint orbits in loop algebras, the coor-
dinates are the so-called spectral Darboux coordinates [1], possibly to be found
using the “Sklyanin magic recipe” [39], and the separable Hamiltonians are the
spectral invariants.

The point of view herewith presented is the following. The class of mani-
folds we will consider are particular bi-Hamiltonian manifolds, to be termed \( \omega N \)
manifolds, where one of the two Poisson brackets is nondegenerate and thus de-
fines a symplectic form \( \omega \) and, together with the other one, a recursion operator
\( N \). The class of coordinates, called Darboux-Nijenhuis (DN) coordinates,
are canonical with respect to \( \omega \) and diagonalize \( N \).

The first result is that an \( n \)-tuple \( (H_1, \ldots, H_n) \) of Hamiltonians on \( M \) (where
\( n = \frac{1}{2} \dim M \)) is separable in DN coordinates if and only if they are in involution
with respect to both Poisson brackets. This condition is clearly intrinsic, i.e.,
it can be checked in any coordinate system. A second result of the present
paper is that examples of separable systems on \( \omega N \) manifolds are provided by
suitable reductions of bi-Hamiltonian hierarchies, called Gel’fand-Zakharevich
systems. They are bi-Hamiltonian systems defined on a bi-Hamiltonian manifold
\((M, \{\cdot, \cdot\}, \{\cdot, \cdot\}')\) by the coefficients of the Casimir functions of the Poisson pencil
\( \{\cdot, \cdot\}_\lambda := \{\cdot, \cdot\}' - \lambda \{\cdot, \cdot\} \). Such coefficients are in involution with respect to both
Poisson brackets, and are supposed to be enough to define integrable systems
on the symplectic leaves of \( \{\cdot, \cdot\} \). If there exists a foliation of \( M \), transversal to
these symplectic leaves and compatible with the Poisson pencil (in a suitable
sense), then every symplectic leaf of \( \{\cdot, \cdot\} \) becomes an \( \omega N \) manifold, and the
(restrictions of the) GZ systems naturally fall in the class of systems which are
separable in DN coordinates. For this reason, we can say that the Poisson pencil
separates its Casimirs.

The third result concerns the Stäckel separability. With a slight extension
of the classical notion, we say that \( (H_1, \ldots, H_n) \) are Stäckel separable if the
separation relations (1.1) are affine in the $H_i$:

$$\sum_{j=1}^{n} S_{ij}(q_i, p_i) H_j - U_i(q_i, p_i) = 0, \quad i = 1, \ldots, n.$$  \hspace{1cm} (1.3)

In this case, the collection $(H_1, \ldots, H_n)$ is called a Stäckel basis. We give an intrinsic test for the Stäckel separability in DN coordinates, which has a straightforward application to GZ systems. This goes as follows. We notice that if $(H_1, \ldots, H_n)$ are in involution with respect to both Poisson brackets (and therefore separable in DN coordinates), then there exists a matrix $F$ (depending on the choice of the $H_i$) such that

$$N^* dH_i = \sum_{j=1}^{n} F_{ij} dH_j.$$  

We prove that $(H_1, \ldots, H_n)$ is a Stäckel basis if and only if

$$N^* dF_{ij} = \sum_{k=1}^{n} F_{ik} dF_{kj}.$$  

The geometric theory of separability we present in this paper may be, in our opinion, regarded as an effective bridge between the “classical” and the “modern” aspects of the theory of separability. More evidence of this claim will be given in [10], where we will also show how to frame Eisenhart’s theory within our approach, and discuss the problem of associating a Lax representation to GZ systems.

This paper is organized as follows. The first part (Section 2 to 5) is devoted to the geometry of separability on $\omega N$ manifolds. In Section 2 we will introduce the notion of $\omega N$ manifold and we will study the DN coordinates. Section 3 contains the main results about separability on $\omega N$ manifolds, whereas in Section 4 the Stäckel separability is considered. In Section 5 we will come back to DN coordinates, pointing out some algorithms for their explicit computation.

In the second part of the paper we will turn our attention to GZ systems. Section 6 deals with the particular case where there is only one Casimir of the Poisson pencil (i.e., one bi-Hamiltonian hierarchy), and contains the example of the 3-particle open Toda lattice. This section is intended for an introduction to Section 7, where the general case is treated. We will give conditions under which a bi-Hamiltonian manifold is foliated in $\omega N$ manifolds, and we will show that the GZ systems are separable in DN coordinates. Subsection 7.3 is devoted to the Stäckel separability of such systems. In Section 8 we will show an efficient...
way to determine, in the Stäckel separable case, the separation relations for GZ systems. Finally, we present an example in the loop algebra of $\mathfrak{sl}(3)$.

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## 2 $\omega N$ manifolds

In this section we describe the manifolds where our (separable) systems will be defined. They are called $\omega N$ manifolds, since they are Poisson-Nijenhuis (PN) manifolds [25, 27, 29] such that the first Poisson structure is nondegenerate, and therefore defines a symplectic form. In turn, PN manifolds are particular instances of bi-Hamiltonian manifolds, i.e., smooth (or complex) manifolds $M$ endowed with a pair of of compatible Poisson brackets, $\{\cdot, \cdot\}$ and $\{\cdot, \cdot\}'$. This means that every linear combination of them is still a Poisson bracket.

**Definition 2.1** An $\omega N$ manifold is a bi-Hamiltonian manifold $(M, \{\cdot, \cdot\}, \{\cdot, \cdot\}')$ in which one of the Poisson brackets (say, $\{\cdot, \cdot\}$) is nondegenerate.

Therefore, $M$ is endowed with a symplectic form $\omega$ defined by

$$\{f, g\} = \omega(X_f, X_g),$$

(2.1)

where $X_f$ is the Hamiltonian vector field associated with $f$ by means of $\{\cdot, \cdot\}$. In terms of the Poisson tensor $P$ corresponding to $\{\cdot, \cdot\}$, viewed as a section of $\text{Hom}(T^*M, TM)$, this simply means that $P$ is invertible and $\omega$ is its inverse. Using also the Poisson tensor $P'$ associated with $\{\cdot, \cdot\}'$, one can construct the tensor field $N := P'P^{-1}$, of type $(1,1)$, to be termed recursion operator of the $\omega N$ manifold $M$.

**Proposition 2.2** The Nijenhuis torsion of $N$,

$$T(N)(X, Y) := [NX, NY] - N([NX, Y] + [X, NY] - N[X, Y]) ,$$

(2.2)

vanishes as a consequence of the compatibility between $P$ and $P'$. 

A proof of this well known fact can be found in [27].

There are two main sources of examples of \(\omega_N\) manifold. The first one comes from classical mechanics. Let \(Q\) be an \(n\)-dimensional manifold endowed with a \((1, 1)\) tensor field \(L\) with vanishing Nijenhuis torsion, and let us consider its cotangent bundle \(T^*Q\) with the canonical Poisson bracket \(\{\cdot, \cdot\}\). As shown in [23], the vanishing of the Nijenhuis torsion of \(L\) entails that one can use it to define a second Poisson bracket \(\{\cdot, \cdot\}'\) on \(T^*Q\) as

\[
\{q_i, q_j\}' = 0, \quad \{q_i, p_j\}' = -L_j^i, \quad \{p_i, p_j\}' = \left(\frac{\partial L_k^i}{\partial q_i} - \frac{\partial L_k^j}{\partial q_j}\right) p_k,
\]

where \((p_i, q_i)\) are fibered coordinates. This Poisson bracket is compatible with \(\{\cdot, \cdot\}\), so that the phase space \(T^*Q\) becomes an \(\omega_N\) manifold, whose recursion operator \(N\) is the complete lifting [44] of \(L\).

The second class of examples of \(\omega_N\) manifolds can be obtained by reduction from a bi-Hamiltonian manifold \((M, P, P')\) where both Poisson tensors are degenerate (see, e.g., [14]).

This happens, in particular, in the following situation. Suppose that \(P\) has constant corank \(k\), that \(\dim M = 2n + k\), and that one can find a \(k\)-dimensional foliation \(Z\) of \(M\) with the properties:

1. The foliation \(Z\) is transversal to the symplectic foliation of \(P\);
2. The functions which are constant along \(Z\) form a Poisson subalgebra of \((C^\infty(M), \{\cdot, \cdot\})\) and of \((C^\infty(M), \{\cdot, \cdot\}')\), i.e., if \(f\) and \(g\) are constant along \(Z\), then the same is true for \(\{f, g\}\) and \(\{f, g\}'\).

Then any symplectic leaf \(S\) of \(\{\cdot, \cdot\}\) inherits a bi-Hamiltonian structure from \(M\). Moreover, the reduction of the first Poisson structure coincides with the symplectic form of \(S\), so that \(S\) is an \(\omega_N\) manifold. Such a procedure is one of the main topics of the paper, and will be fully discussed in Section [4], where we will also show that bi-Hamiltonian systems on \(M\) give rise to separable systems on \(S\). The corresponding variables of separation are going to be introduced in the next subsection.

### 2.1 Darboux–Nijenhuis coordinates

In this subsection we will describe a class of canonical coordinates on \(\omega_N\) manifolds, called Darboux–Nijenhuis coordinates. They will play the important role of variables of separation for (suitable) systems on \(\omega_N\) manifolds.
Definition 2.3 A set of local coordinates \((x_i, y_i)\) on an \(\omega N\) manifold is called a set of Darboux–Nijenhuis (DN) coordinates if they are canonical with respect to the symplectic form \(\omega\),

\[
\omega = \sum_{i=1}^{n} dy_i \wedge dx_i ,
\]

and put the recursion operator \(N\) in diagonal form,

\[
N = \sum_{i=1}^{n} \lambda_i \left( \frac{\partial}{\partial x_i} \otimes dx_i + \frac{\partial}{\partial y_i} \otimes dy_i \right) .
\]

(2.3)

This means that the only nonzero Poisson brackets are

\[
\{x_i, y_j\} = \delta_{ij} , \quad \{x_i, y_j\}' = \lambda_i \delta_{ij} .
\]

The assumption, contained in (2.3), that the eigenvalues \(\lambda_i\) of \(N\) are (at least) double is not restrictive, since its eigenspaces have even dimension, equal to the dimension of the kernel of \(P' - \lambda_i P\). For the \(\omega N\) manifold \(T^*Q\) described in the previous section, it is easy to check that the eigenvalues of \(L\) (if they are independent) and their conjugate momenta are DN coordinates. In order to ensure the existence of DN coordinates on more general \(\omega N\) manifolds, we give the following

Definition 2.4 A 2n-dimensional \(\omega N\) manifold \(M\) is said to be semisimple if its recursion operator \(N\) has, at every point, \(n\) distinct eigenvalues \(\lambda_1, \ldots, \lambda_n\). It is called regular if the eigenvalues of \(N\) are functionally independent on \(M\).

It can be shown [18, 29, 11] that every point of a semisimple \(\omega N\) manifold has a neighborhood where DN coordinates can be found, and that, if the \(\omega N\) manifold \(M\) is regular, one half of these coordinates are “canonically” provided by the recursion operator. Indeed, as a consequence of the vanishing of the Nijenhuis torsion of \(N\), the eigenvalues \(\lambda_i\) always satisfy

\[
N^* d\lambda_i = \lambda_i d\lambda_i ,
\]

where \(N^*\) is the adjoint of \(N\), and one has

Proposition 2.5 In a neighborhood of a point of a regular \(\omega N\) manifold where the eigenvalues of \(N\) are distinct it is possible to find by quadratures \(n\) functions \(\mu_1, \ldots, \mu_n\) that, along with the eigenvalues \(\lambda_1, \ldots, \lambda_n\), are DN coordinates.
Such coordinates will be called a set of *special Darboux–Nijenhuis (sDN) coordinates*. They will often be used in the sequel, because the \( \lambda_i \) are simply the roots of the minimal polynomial of \( N \). Proposition 2.3 means also that every regular \( \omega N \) manifold is locally equal to the “lifted” \( \omega N \) manifold \( T^* Q \) we have seen in Section 2.

The distinguishing property of the pairs of DN coordinates \((x_i, y_i)\), and, a fortiori, of the “special” pairs \((\lambda_i, \mu_i)\), is that their differentials span an eigenspace of \( N^* \), that is, satisfy the equations

\[
N^* dx_i = \lambda_i dx_i, \quad N^* dy_i = \lambda_i dy_i, \quad i = 1, \ldots, n.
\]

(2.4)

This leads us to the following

**Definition 2.6** A function \( f \) on an \( \omega N \) manifold is said to be a Stäckel function (relative to the eigenvalue \( \lambda_i \) of \( N \)) if

\[
N^* df = \lambda_i df.
\]

(2.5)

The following property of Stäckel functions, which also explains their name, will be used many times in the rest of the paper.

**Proposition 2.7** Let \( M \) be a semisimple \( \omega N \) manifold. A function \( f \) on \( M \) is a Stäckel function relative to \( \lambda_i \) if and only if, in any (some) system \((x_1, \ldots, y_n)\) of DN coordinates, \( f \) depends only on \( x_i \) and \( y_i \).

**Proof.** It is obvious that if \( f = f(x_i, y_i) \) then \( N^* df = \lambda_i df \). Conversely, if (2.3) holds, then \( df \) belongs to the \( \lambda_i \)-eigenspace of \( N^* \), so that \( df \) is a linear combination of \( dx_i \) and \( dy_i \) and therefore \( f \) depends only on \( x_i \) and \( y_i \).

\( QED \)

### 3 Separability on \( \omega N \) manifolds

In Section 2 we have introduced a class of (symplectic) manifolds and we have selected a class of (canonical) coordinates on such manifolds. Now we are going to characterize, from a geometric point of view, those integrable Hamiltonian systems on \( \omega N \) manifolds which are separable in DN coordinates. In the next section we will consider the same problem for Stäckel separability.
We recall that an $n$-tuple $(H_1, \ldots, H_n)$ of functionally independent Hamiltonians on an $\omega N$ manifold $M$ is said to be separable in the DN coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ if there exist relations of the form
\[
\phi_i(x_i, y_i, H_1, \ldots, H_n) = 0, \quad i = 1, \ldots, n, \quad \text{with } \det \left[ \frac{\partial \phi_i}{\partial H_j} \right] \neq 0. \tag{3.1}
\]
It can be easily shown (e.g., via the Hamilton-Jacobi method) that this entails the involutivity of the $H_i$. Obviously enough, the separability property is not peculiar of the specific choice of the functions $H_i$. If $K_i = K_i(H_1, \ldots, H_n)$ are functions of the $H_i$, they are also separable according to (3.1). So we see that the property (3.1) concerns the geometrical features of an integrable system, i.e., is to be regarded as a property of the Lagrangian distribution defined by the mutually commuting functions $H_i$. Thus one can say that the $H_i$ define a separable foliation of $M$. According to the following theorem, that will be proved during this section, the separability property can be formulated in terms of the geometric objects $\omega$ and $N$, or $\{\cdot, \cdot\}$ and $\{\cdot, \cdot\}'$, of the $\omega N$ manifold $M$.

**Theorem 3.1** Let $M$ be a semisimple $\omega N$ manifold and let $(H_1, \ldots, H_n)$ be a set of $n$ functionally independent Hamiltonians on $M$. Then the following statements are equivalent:

a) The foliation defined by $(H_1, \ldots, H_n)$ is separable in DN coordinates (and therefore Lagrangian with respect to $\omega$);

b) The distribution tangent to the foliation defined by $(H_1, \ldots, H_n)$ is Lagrangian with respect to $\omega$ and invariant with respect to $N$;

c) The functions $(H_1, \ldots, H_n)$ are in bi-involution, i.e., $\{H_i, H_j\} = 0$ and $\{H_i, H_j\}' = 0$ for all $i, j$.

We will often refer to property c) by saying that the foliation defined by the $H_i$ is bi-Lagrangian. This is a fundamental property in our approach to separability, and will be exploited especially in Sections 3 and 4. Incidentally, we notice that bi-Lagrangian foliations play an important role in the study of special Kähler manifolds [21].

Throughout the rest of the section $M$ will be a semisimple $\omega N$ manifold, $(\lambda_1, \ldots, \lambda_n)$ the eigenvalues of the recursion operator $N$, and $(x_i, y_i)$ DN coordinates on $M$. We begin with showing that the invariance with respect to $N$ is a necessary condition for separability.

**Proposition 3.2** Let $(H_1, \ldots, H_n)$ be functions on $M$ that are separable in DN coordinates. Then the subspace spanned by $(dH_1, \ldots, dH_n)$ is invariant with
respect to $N^*$. More precisely, there exists a (simple) matrix $F$ with eigenvalues $(\lambda_1, \ldots, \lambda_n)$ such that

$$N^*dH_i = \sum_{j=1}^{n} F_{ij}dH_j, \quad i = 1, \ldots, n. \quad (3.2)$$

Consequently, the Lagrangian distribution defined by $(H_1, \ldots, H_n)$, which is spanned by the Hamiltonian vector fields $X_{H_i}$, is invariant with respect to $N$.

**Proof.** Differentiate the relations (3.1),

$$\frac{\partial \phi_i}{\partial x_i} dx_i + \frac{\partial \phi_i}{\partial y_i} dy_i + \sum_{j=1}^{n} \frac{\partial \phi_i}{\partial H_j} dH_j = 0, \quad (3.3)$$

then apply $N^*$ to obtain

$$\frac{\partial \phi_i}{\partial x_i} \lambda_i dx_i + \frac{\partial \phi_i}{\partial y_i} \lambda_i dy_i + \sum_{j=1}^{n} \frac{\partial \phi_i}{\partial H_j} N^*dH_j = 0. \quad (3.4)$$

It follows that

$$\sum_{j=1}^{n} \frac{\partial \phi_i}{\partial H_j} N^*dH_j = -\lambda_i \left( \frac{\partial \phi_i}{\partial x_i} dx_i + \frac{\partial \phi_i}{\partial y_i} dy_i \right) = \lambda_i \sum_{j=1}^{n} \frac{\partial \phi_i}{\partial H_j} dH_j, \quad (3.5)$$

that is, in matrix form,

$$JN^*dH = \Lambda JdH, \quad (3.6)$$

where $J_{ij} = \frac{\partial \phi_i}{\partial H_j}$, $dH = (dH_1, \ldots, dH_n)^T$, $N^*dH = (N^*dH_1, \ldots, N^*dH_n)^T$, and $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$. Therefore (3.2) is satisfied with $F = J^{-1}\Lambda J$, and the eigenvalues of $F$ are $(\lambda_1, \ldots, \lambda_n)$. The final assertion easily follows. \[QED\]

The matrix $F$ will be called the *control matrix*, with respect to the basis $(H_1, \ldots, H_n)$, of the separable foliation.

**Proposition 3.3** If $(H_1, \ldots, H_n)$ define a distribution which is invariant with respect to $N$, that is,

$$N^*dH_i = \sum_{j=1}^{n} F_{ij}dH_j, \quad i = 1, \ldots, n, \quad (3.7)$$

and the eigenvalues of $F$ are distinct, then the $H_i$ are separable in $DN$ coordinates.
Proof. Since the eigenvalues of $F$ are distinct, they are the eigenvalues $(\lambda_1, \ldots, \lambda_n)$ of $N$, so that there exists a matrix $S$ such that $F = S^{-1} \Lambda S$. With $S$ we define the 1-forms $\theta_i := \sum_{j=1}^{n} S_{ij} dH_j$, for $i = 1, \ldots, n$. They are eigenvectors of $N^*$, since

$$N^* \theta_i = \sum_{j=1}^{n} S_{ij} N^* dH_j = \sum_{j} S_{ij} F_{jk} dH_k = \sum_{k=1}^{n} \lambda_i S_{ik} dH_k = \lambda_i \theta_i . \quad (3.8)$$

Then there exist functions $L_i$ and $M_i$ such that $\theta_i = L_i dx_i + M_i dy_i$, that is,

$$\sum_{j=1}^{n} S_{ij} dH_j - L_i dx_i - M_i dy_i = 0 . \quad (3.9)$$

This means that $\dim \langle dH_1, \ldots, dH_n, dx_i, dy_i \rangle \leq n + 1$, so that there exists a relation of the form (3.1), i.e., the functions $(H_1, \ldots, H_n)$ are separable in DN coordinates.

QED

In order to complete the proof of the equivalence between statements a) and b) of Theorem 3.1, we need the following:

Lemma 3.4 If $(H_1, \ldots, H_n)$ are independent functions in involution with respect to $\omega$ such that (3.2) holds, then the eigenvalues of $F$ are distinct.

Proof. Suppose that $N^* dH_i = \sum_{j=1}^{n} F_{ij} dH_j$, with $\{H_i, H_j\} = 0$ for all $i, j$. Since $F$ represents the restriction of $N^*$ to $\langle dH_1, \ldots, dH_n \rangle$, it is diagonalizable. Thus, if $\lambda_i$ would be a double eigenvalue of $F$, the span $\langle dH_1, \ldots, dH_n \rangle$ would contain the 2-dimensional eigenspace spanned by $dx_i$ and $dy_i$. But the involutivity of the $H_i$ would entail that $\{x_i, y_i\} = 0$, which is false.

QED

Relations (3.2) may be called generalized Lenard relations (and the functions $H_i$ fulfilling them a Nijenhuis chain, as in [13]), as enlightened by the following example.

Example 3.5 If $H_k := \frac{1}{2k} \text{tr} N^k = \sum_{j=1}^{n} \lambda_j^k$, then $N^* dH_k = dH_{k+1}$ for $k = 1, \ldots, n - 1$, that is, the Lenard relations $P^* dH_k = P dH_{k+1}$ hold. Moreover, $dH_{n+1} = \sum_{j=1}^{n} c_j dH_{n+1-j}$, where $\lambda^n - \sum_{j=0}^{n-1} c_{n-j} \lambda^j$ is the minimal polynomial
of $N$. Therefore, condition (3.2) is satisfied with

$$
F = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & 1 \\
0 & 0 & \cdots & \cdots & 1 \\
0 & 0 & \cdots & \cdots & 1 \\
\end{bmatrix} \quad (3.10)
$$

**Remark 3.6** It is well known that functions $H_i$ satisfying the Lenard relations are in involution with respect to both Poisson brackets, and so they provide a first instance of correspondence between invariant distributions and bi-involutivity, which is at the same time trivial and paradigmatic.

Indeed, it is trivial from the point of view of the theory of separation of variables, since such Hamiltonians are easily seen to depend only on $(\lambda_1, \ldots, \lambda_n)$ if the $\omega N$ manifold $M$ is regular and semisimple. Then the Hamilton–Jacobi equations associated with the $H_i$ are trivially separable in the sDN coordinates $(\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n)$. Nevertheless, it is paradigmatic with respect to the issues of this paper. Indeed, the $H_i$ (that is, the $\lambda_i$) define a distinguished bi-Lagrangian foliation, called principal foliation, which coincides with the canonical fibration $\pi : T^*Q \to Q$ of classical phase spaces when $T^*Q$ is the $\omega N$ manifold considered in Section 2. However, there are in general bi-Lagrangian foliations which are different from the principal one, as we will explicitly see in Section 3. We are going to show that such foliations are characterized by the invariance with respect to $N$, so that they give rise to separable systems. This means that our theory deals with cases in which the Hamiltonians are not simply the traces of the recursion operator. In other words, we will deal with cases in which the control matrix $F$ of equation (3.2) need not be a companion matrix of the form (3.10). Accordingly, the separable vector fields we will consider are tangent to a bi-Lagrangian foliation, but they are not, in general, bi-Hamiltonian.

**Proposition 3.7** Let $(H_1, \ldots, H_n)$ be independent functions on $M$. Then (3.2) holds, with a matrix $F$ with distinct eigenvalues, if and only if the functions $H_i$ are in bi-involution:

$$
\{H_i, H_j\} = \{H_i, H_j\}' = 0 \quad \text{for all } i, j = 1, \ldots, n. \quad (3.11)
$$
Proof. We know from Proposition 3.3 that condition (3.2), with a simple matrix $F$, implies separability and therefore involutivity with respect to $\{\cdot,\cdot\}$. Moreover,

$$\{H_i, H_j\}' = \langle dH_i, P'dH_j \rangle = \langle dH_i, NPdH_j \rangle = \langle N^*dH_i, PdH_j \rangle = \sum_{k=1}^n F_{ik}\{H_k, H_j\}.$$  

(3.12)

showing that $\{H_i, H_j\}'$ vanishes as well.

Conversely, suppose that $\{H_i, H_j\} = \{H_i, H_j\}' = 0$ for all $i, j$. Then the foliation $\mathcal{H}$ defined by the $H_i$ is Lagrangian with respect to $\{\cdot,\cdot\}$, and

$$\langle N^*dH_i, PdH_j \rangle = \langle dH_i, NPdH_j \rangle = \langle dH_i, P'dH_j \rangle = 0.$$

Thus, $N^*dH_i$ belongs, for every $i$, to the annihilator of $\langle PdH_1, \ldots, PdH_n \rangle$, which is tangent to $\mathcal{H}$, since $\mathcal{H}$ is Lagrangian. This shows that (3.2) holds, and Lemma 3.4 entails that $F$ has distinct eigenvalues.

$QED$

Thus we have proved also the equivalence between b) and c) of Theorem 3.1.

Remark 3.8 One could also prove that a function $F$ is separable in DN coordinates if and only if its Hamiltonian vector field $X_F$ is tangent to a bi-Lagrangian foliation $\mathcal{H}$. The “if” part of this statement is a simple corollary of Theorem 3.1. Indeed, let $\mathcal{H}$ be defined by the functions $(H_1, \ldots, H_n)$. Then $F$ is a function of the $H_i$, since the distribution is Lagrangian, and one can find other $(n - 1)$ functions $K_2, \ldots, K_n$ such that $\mathcal{H}$ is defined by $(F, K_2, \ldots, K_n)$. The “only if” part of this statement is deeper, and essentially gives rise to the intrinsic picture of the Levi-Civita conditions for separability, to be fully discussed in [10].

Summing up, we have proved a criterion for the separability in DN coordinates, which can be tested without knowing explicitly these coordinates. Indeed, the statement (3.11) can be checked in any coordinate system. An important application of this criterion will be given in Section 7, where we will show that the bi-Hamiltonian hierarchies on a bi-Hamiltonian manifold admitting a transversal distribution with the properties mentioned at the end of Section 2 give rise to separable Hamiltonian vector fields on the reduced $\omega N$ manifolds.

4 Stäckel separability on $\omega N$ manifolds

The separability criteria of the previous section do not give explicit information on the form of the separating relations (3.1). For this reason, in this section
we will concentrate on the more stringent notion of Stäckel separability. Recall that \((H_1, \ldots, H_n)\), independent functions on an \(\omega N\) manifold, were defined to be \textit{Stäckel separable} in the DN coordinates \((x_1, \ldots, y_n)\) if there exist relations of the form \((3.1)\), given by affine equations in the \(H_j\), that is,

\[
\sum_{j=1}^{n} S_{ij}(x_i, y_i) H_j - U_i(x_i, y_i) = 0, \quad i = 1, \ldots, n,
\]

(4.1)

with \(S\) an invertible matrix. In this case, we say that the \(H_i\) are a \textit{Stäckel basis} of the (separable) foliation. The entries \(S_{ij}\) and \(U_i\) depend only on \(x_i\) and \(y_i\), i.e., they are Stäckel functions according to Proposition 2.7. Usually, \(S\) is called a \textit{Stäckel matrix}, and \(U\) a \textit{Stäckel vector}. Notice that the definition of Stäckel separability depends on the choice of the \(H_i\) defining the Lagrangian distribution. Indeed, if \((H_1, \ldots, H_n)\) are Stäckel-separable, then \(K_i = K_i(H_1, \ldots, H_n)\), for \(i = 1, \ldots, n\), will not, in general, fulfill relations of the form \((4.1)\). A natural problem, that will not be discussed in this paper, is to give a geometrical characterization of the Lagrangian foliations admitting a set of defining functions for which Stäckel separability holds. Some results in this direction will be presented in [10].

Now we will give a necessary and sufficient condition for the Stäckel separability in DN coordinates of a given \(n\)-tuple \((H_1, \ldots, H_n)\) of functions on an \(\omega N\) manifold. We will also show that in this case one can explicitly find the relations \((3.1)\) and has useful information to algebraically determine the separation variables.

Suppose \((H_1, \ldots, H_n)\) to be independent functions on a regular semisimple \(\omega N\) manifold that are Stäckel separable in the DN coordinates. Then we know from Proposition 3.2 that there exists a control matrix \(F\), with eigenvalues \((\lambda_1, \ldots, \lambda_n)\), such that \(N^*dH = FdH\). Since Proposition 2.7 entails that \(N^*dS = \Lambda dS\) and \(N^*dU = \Lambda dU\), we can show:

\textbf{Proposition 4.1} In the above-mentioned hypotheses, the matrix \(F\) satisfies

\[
N^*dF = FdF, \quad \text{that is,} \quad N^*dF_{ij} = \sum_{k=1}^{n} F_{ik}dF_{kj} \quad \forall \ i, j = 1, \ldots, n.
\]

(4.2)

\textbf{Proof.} First we show that \(F = S^{-1}\Lambda S\). Indeed,

\[
SFdH = SN^*dH = N^*[d(SH) - (dS)H] = N^*dU - (N^*dS)H
\]

\[
= \Lambda dU - \Lambda dS H = \Lambda SdH.
\]

(4.3)
Then we have
\[ N^*dF = N^*(-S^{-1}dSS^{-1}\Lambda S + S^{-1}d\Lambda S + S^{-1}\Lambda dS) \]
\[ = -S^{-1}\Lambda dSS^{-1}\Lambda S + S^{-1}\Lambda d\Lambda S + S^{-1}\Lambda^2dS \]  
\[ = S^{-1}\Lambda S(-S^{-1}dSS^{-1}\Lambda S + S^{-1}d\Lambda S + S^{-1}\Lambda dS) = FdF, \]  
and the proof is complete.  
\[ QED \]
Condition (4.2) is also sufficient for the Stäckel separability, as shown in the following:

**Theorem 4.2** Let \((H_1, \ldots, H_n)\) be independent functions, defining a bi-Lagrangian foliation on a regular semisimple \(\omega N\) manifold. If the control matrix \(F\) fulfills (4.2), then:

1. The left eigenvectors of \(F\), if suitably normalized, form a Stäckel matrix. More precisely, if \(S\) is a matrix such that \(F = S^{-1}\Lambda S\), and such that in every row of \(S\) there is an entry equal to 1, then \(S\) is a Stäckel matrix in DN coordinates \((x_1, \ldots, y_n)\);

2. The functions \((H_1, \ldots, H_n)\) are Stäckel separable in DN coordinates.

**Proof.** From (4.2) we have that
\[ N^*(-S^{-1}dSS^{-1}\Lambda S + S^{-1}d\Lambda S + S^{-1}\Lambda dS) \]
\[ = S^{-1}\Lambda S(-S^{-1}dSS^{-1}\Lambda S + S^{-1}d\Lambda S + S^{-1}\Lambda dS), \]  
that is,
\[ N^*(-dSS^{-1}\Lambda S + \Lambda dS) = \Lambda(-dSS^{-1}\Lambda S + \Lambda dS), \]  
or \((-N^*dS + \Lambda dS)F = \Lambda(-N^*dS + \Lambda dS).\) Hence the \(j\)-th row of \((-N^*dS + \Lambda dS)\) is a left eigenvector of \(F\), relative to \(\lambda_j\). This entails that it is proportional to the \(j\)-th row of \(S\), i.e., there exists a 1-form \(\alpha_j\) such that
\[ e_j(-N^*dS + \Lambda dS) = \alpha_j e_j S, \]  
where \(e_j\) is the \(j\)-th row vector of the standard basis. Multiplying equation (4.7) by \(e_k^T\), where \(S_{jk} = e_j^T e_k = 1\), we obtain \(\alpha_j = 0\), so that
\[ N^*dS = \Lambda dS. \]
In components, this reads $N^*dS_{jk} = \lambda_j dS_{jk}$, which implies (see Proposition 2.7) that $S_{jk}$ depends only on $x_j$ and $y_j$, i.e., $S$ is a Stäckel matrix. Finally, the fact that $U := SH$ is a Stäckel vector follows from

$$N^*dU = N^*(dS H + SdH) = \Lambda dS H + SFdH = \Lambda(dS H + SdH) = \Lambda dU .$$

(4.9)

This completes the proof.

$QED$

The results obtained so far can be summarized in the following statements. An $n$-tuple of functions $(H_1, \ldots, H_n)$ in involution is separable (in DN coordinates) if and only if the span of their differentials is invariant for $N^*$. Let $F$ be the matrix which represents (the restriction of) $N^*$ on such a span. Then equation (4.2) represents a test for the Stäckel separability of the $H_i$. Once this test is passed, the Stäckel matrix is easily constructed as a (suitably normalized) matrix that diagonalize $F$, and the separation procedure can be quite explicitly performed. Therefore, in our setting the Hamiltonians provide their Stäckel matrix as well as the separation relations (4.1).

We end this section with the following comment on the intrinsic meaning of the Stäckel separability conditions (4.2). It is known [17] that, as a consequence of the vanishing of the Nijenhuis torsion of $N$, the de Rham complex of $M$ is endowed with a second derivation $d_N$, which is defined to be the unique (anti)derivation with respect to the wedge product extending

$$d_Nf(X) = df(NX) = N^*df(X)$$

$$d_N\theta(X,Y) = X(\theta(NY)) - Y(\theta(NX)) - \theta([X,Y]_N) ,$$

where $f$ is a function, $\theta$ is a 1-form, $X$, $Y$ are vector fields on $M$, and

$$[X,Y]_N = [NX,Y] + [X, NY] - N[X,Y] .$$

This differential is a cohomology operator ($d_N^2 = 0$) and anticommutes with the usual exterior derivative $d$. One notices that the invariance condition (3.2) can be equally be written, in matrix notation, as

$$d_N H = F dH .$$

(4.11)

Imposing the condition $d_N^2 = 0$ on this equation, taking into account the anticommutativity of $d$ and $d_N$, and translating back $d_N f = N^* df$ if $f$ is a function on $M$, one gets

$$(N^*dF - FdF) \wedge dH = 0 .$$

(4.12)
So we see that the Stäckel separability conditions (4.2) are nothing but a “strong” solution of the equations imposed on the control matrix $F$ by the cohomological condition $d_N^2 = 0$.

5 Special DN coordinates

In this section we will discuss the problem of explicitly finding sets of special DN coordinates on an $\omega N$ manifold $M$. We assume that $M$ be regular and complex, so that the eigenvalues $(\lambda_1, \ldots, \lambda_n)$ of $N$ can be used as (half of the) coordinates on $M$. We know that in a neighborhood of a point where the $\lambda_i$ are distinct there exist functions $(\mu_1, \ldots, \mu_n)$ forming with the eigenvalues a system of DN coordinates, and that the $\mu_i$ can be computed by quadratures. However, they can often be found in an algebraic way, as we will see below. We divide our argument in three main points.

We start remarking that there are simple conditions to be checked the $\mu_i$ in order to ensure that they form with the $\lambda_i$ a set of DN coordinates. To this aim, we observe that the $\mu_i$ must fulfill two kinds of requirements:

1. They have to be Stäckel functions, that is, they must satisfy $N^* d\mu_i = \lambda_i d\mu_i$;

2. They have to fulfill the canonical commutation relations with respect to the first Poisson bracket: $\{\lambda_i, \mu_j\} = \delta_{ij}$, $\{\mu_i, \mu_j\} = 0$.

In principle, these conditions require the computation of the $\lambda_i$. We will show that this can be avoided, and that a smaller number of equations must be checked. The first step is to notice that, once conditions 1 are satisfied, conditions 2 can be replaced with the $n$ equations

$$\{\lambda_1 + \cdots + \lambda_n, \mu_i\} = 1,$$

(5.1)

which do not require the computation of the $\lambda_i$, but only of their sum, that is, $c_1 := \frac{1}{2} \text{tr} N$ and, consequently, of the Hamiltonian vector field $Y := -P dc_1 = \sum_{i=1}^n \frac{\partial}{\partial \mu_i}$. Indeed, suppose that $\mu_j$ be a Stäckel function, and observe that

$$\lambda_i \{\lambda_i, \mu_j\} = \lambda_i \langle d\lambda_i, P d\mu_j \rangle = \langle \lambda_i d\lambda_i, P d\mu_j \rangle = \langle N^* d\lambda_i, P d\mu_j \rangle = \langle d\lambda_i, P N^* d\mu_j \rangle = \langle d\lambda_i, \lambda_j P d\mu_j \rangle = \{\lambda_i, \lambda_j \} P d\mu_j = \{\lambda_i, \mu_j \},$$

(5.2)
so that \( \{ \lambda_i, \mu_j \} = 0 \) if \( i \neq j \). Then equation \( \text{(5.1)} \) becomes \( \{ \lambda_i, \mu_i \} = 1 \). In the same way one shows that \( \{ \mu_i, \mu_j \} = 0 \). Hence, in order to find the \( \mu_i \) coordinate we have to look for a Stäckel function (relative to \( \lambda_i \)) such that \( \text{(5.1)} \) holds.

The second point starts from the following idea, which will be extensively used in the part of the paper dealing with Gel’fand–Zakharevich systems. Let us consider the minimal polynomial

\[
\Delta (\lambda) = \lambda^n - (c_1 \lambda^{n-1} + c_2 \lambda^{n-2} + \cdots + c_n)
\]

(5.3)

of \( N \). Using the Newton formulas relating the traces of the powers of \( N \) and the coefficients \( c_i \) of \( \Delta (\lambda) \), one easily verifies that the latter satisfy

\[
\begin{align*}
N^*dc_i &= dc_{i+1} + c_i dc_1, & i &= 1, \ldots, n-1 \\
N^*dc_{n-1} &= dc_n.
\end{align*}
\]

(5.4)

These relations are equivalent to the following equation for the polynomial \( \Delta (\lambda) \),

\[
N^*d\Delta (\lambda) = \lambda d\Delta (\lambda) + \Delta (\lambda) dc_1.
\]

(5.5)

Relations of this kind are very interesting for our purposes. For instance, it holds:

**Proposition 5.1** Let \( f(x; \lambda) \) be a function defined on \( M \), depending on an additional parameter \( \lambda \). Suppose that there exists a 1-form \( \alpha_f \) such that

\[
N^*d(f(x; \lambda)) = \lambda d(f(x; \lambda)) + \Delta (\lambda) \alpha_f.
\]

(5.6)

Then, the function \( f_i \) defined by \( f_i(x) := f(x; \lambda_i(x)) \), i.e., the evaluation of \( f(x; \lambda) \) on \( \lambda = \lambda_i \), is a Stäckel function relative to \( \lambda_i \).

**Proof.** The differential of \( f_i \) equals

\[
df_i(x) = df(x; \lambda) \big|_{\lambda=\lambda_i} + \frac{\partial f(x; \lambda)}{\partial \lambda} \big|_{\lambda=\lambda_i} d\lambda_i,
\]

(5.7)

where, in the term \( d(f(x; \lambda)) \big|_{\lambda=\lambda_j} \), one treats \( \lambda \) as a parameter. Applying the adjoint of the recursion operator we get

\[
N^*d(f(x; \lambda_i)) = N^*d(f(x; \lambda)) \big|_{\lambda=\lambda_i} + \lambda_i \frac{\partial f(x; \lambda)}{\partial \lambda} \big|_{\lambda=\lambda_i} d\lambda_i,
\]

(5.8)

whence the assertion, since \( \Delta (\lambda_i) = 0 \).

\( QED \)
Definition 5.2 We will call a function on $M$ depending on the additional parameter $\lambda$ a Stäckel function generator if it satisfies (5.4) with a suitable 1-form $\alpha_f$.

Lemma 5.3 The space of Stäckel function generators is closed under sum and multiplication, and is invariant with respect to the action of $Y$. If $f$ is a Stäckel function generator and $g$ is a function of one variable, then $g \circ f$ is a Stäckel function generator.

Proof. The only assertion whose proof is not straightforward is the invariance with respect to $Y$. This follows from the fact, already noticed in Example 3.5, that $Y$ is a bi-Hamiltonian vector field; hence $L_Y(N^*) = L_Y(P^{-1}P') = 0$ and, consequently, $L_Y(\lambda_i) = 0$.

QED

It is clear that if $f_i$ is a Stäckel function relative to $\lambda_i$ for $i = 1, \ldots, n$, then there exists a Stäckel function generator $f(x; \lambda)$ such that $f_i = f(x; \lambda_i)$, e.g., the interpolating polynomial. In terms of the generator, condition (5.1) can be written as

$$Y(f(x; \lambda)) = 1 \quad \text{for} \; \lambda = \lambda_i, \; i = 1, \ldots, n.$$  \hfill (5.9)

For further use, we state and prove the following:

Proposition 5.4 Let $f$ be a Stäckel function generator, and suppose that for $n \geq 1$ the action of $Y$ on $f$ closes, that is, the relation

$$Y^n(f) = \sum_{j=0}^{n-1} a_j Y^j(f),$$  \hfill (5.10)

with $Y(a_j) = 0$, holds. Then equation (5.9) can be algebraically solved.

Proof. There are two cases: in a first instance, suppose that, actually, $Y$ is nilpotent, that is, $Y^n(f) = 0$ is satisfied for some $n \geq 1$ (whilst $Y^{n-1}(f) \neq 0$). Then it is easily seen that $Y^{n-2}(f)/Y^{n-1}(f)$ is a Stäckel function generator fulfilling (5.9).

On the contrary, if $(a_0, \ldots, a_{n-1}) \neq (0, \ldots, 0)$, then the matrix $A$ representing the action of $Y$ on $\Phi := (f, Y(f), \ldots, Y^{n-1}(f))^T$ has at least one nonzero eigenvalue $\nu$, which is a solution of $\nu^n = \sum_{i=0}^{n-1} a_j \nu^j$. Let $w = (w_0, \ldots, w_{n-1})$
be a (left) eigenvector of $A$ relative to $\nu$, e.g., the one given by $w_{n-1} = 1$ and $w_k = \nu^{n-k-1} - \sum_{j=0}^{n-k-2} a_{k+j+1} \nu^j$ for $k = 0, \ldots, n-2$. Then

$$Y \left( \frac{1}{\nu} \log \sum_{j=0}^{n-1} w_j Y^j(f) \right) = 1 . \quad (5.11)$$

Indeed, $\sum_{j=0}^{n-1} w_j Y^j(f) = w\Phi$ and

$$Y(w\Phi) = wA\Phi = \nu w\Phi ,$$

implying (5.11).

$QED$

These arguments reveal a further important aspect of Stäckel separability within our approach to separation of variables. Indeed, the condition of Stäckel separability, whose intrinsic form is given by equation (4.2), entails that the matrix of the (suitable normalized) eigenvectors of the control matrix $F$ is a Stäckel matrix, that is, its columns are Stäckel functions of $N^*$. Since we have shown that a way to algebraically find the $\mu_i$ coordinates is to find Stäckel functions (or Stäckel function generators) and to combine them in order to fulfill equation (5.9), we see that, in the Stäckel case, the Hamiltonians themselves may algebraically provide the coordinates in which the corresponding flows can be separated.

6 Separability on odd-dimensional bi-Hamiltonian manifolds

This section starts the second (and more applicative) part of the paper, in which we will use the results of Sections 3 and 4 to discuss the separability of a specific family of integrable systems. They are defined on a class of bi-Hamiltonian manifolds, known in the literature as complete torsionless bi-Hamiltonian manifolds of pure Kronecker type (see [19, 34] and the references quoted therein).

In this section we will consider the simplest case, corresponding to generic odd-dimensional bi-Hamiltonian manifolds (while in Section 3 we studied the case of regular $\omega N$ manifolds, which are generic even-dimensional bi-Hamiltonian manifolds). Their Poisson tensors have maximal rank. The more general case will be treated (with detailed proofs) in the next section.
Let \((M, P, P')\) be a \((2n + 1)\)-dimensional bi-Hamiltonian manifold, and let the rank of \(P\) be equal to \(2n\). Suppose that the Poisson pencil \(P_\lambda := P' - \lambda P\) has a polynomial Casimir function

\[ H(\lambda) = \sum_{i=0}^{n} H_i \lambda^{n-i} . \]

This amounts to saying that the functions \( (H_0, \ldots, H_n)\), which we assume to be functionally independent, form a bi-Hamiltonian hierarchy, starting from a Casimir \(H_0\) of \(P\) and terminating with a Casimir of \(P'\),

\[
P \, dH_0 = 0 \ , \quad P \, dH_{i+1} = P' \, dH_i \ , \quad P' \, dH_n = 0 . \tag{6.1}
\]

In particular, they are in involution with respect to \(\{\cdot, \cdot\}\) and \(\{\cdot, \cdot\}'\). If \(dH_0 \neq 0\) at every point of \(M\), then the symplectic foliation of \(P\) is simply given by the level surfaces of \(H_0\). The restrictions of \( (H_1, \ldots, H_n)\) to a symplectic leaf \(S\) of \(P\) form an integrable system (in the Arnold-Liouville sense). The corresponding Hamiltonian vector fields are the restrictions to \(S\) of \(X_i := P \, dH_i\), where \(i = 1, \ldots, n\).

At this point it is natural to wonder whether the bi-Hamiltonian structure of \(M\) can give information on the separability of the (restrictions of the) Hamiltonians \( (H_1, \ldots, H_n)\). More concretely, one can try to induce an \(\omega N\) structure on \(S\) in order to apply the separability theorems of Sections 3 and 4. As anticipated in Section 2, this can be done if there exists a vector field \(Z\) which is transversal to the symplectic foliation of \(P\) and fulfills the following condition:

C) if \(F, G\) are functions on \(M\) which are invariant for \(Z\), that is, \(Z(F) = Z(G) = 0\), then \(\{F, G\}\) and \(\{F, G\}'\) are also invariant.

In this case, any symplectic leaf of \(P\) inherits a bi-Hamiltonian structure from \(M\). Clearly, the first reduced bracket is the one associated with the symplectic form of \(S\), so that \(S\) is an \(\omega N\) manifold.

In the following section we will prove that, if \(Z\) is normalized in such a way that \(Z(H_0) = 1\), condition C) takes the infinitesimal form

\[
L_Z P = 0 \ , \quad L_Z P' = Y \wedge Z , \tag{6.2}
\]

for a suitable vector field \(Y\). In this case there is a useful form of the reduced Poisson brackets \(\{\cdot, \cdot\}_S\) and \(\{\cdot, \cdot\}'_S\) on the symplectic leaf \(S\). If \(f\) and \(g\) are functions on \(S\), by definition \(\{f, g\}_S\) and \(\{f, g\}'_S\) should be computed by taking local extensions of \(f\) and \(g\) which are invariant along \(Z\). But one can avoid the
use of invariant functions and consider arbitrary extensions $F, G$. Then

$$\{f, g\}_S = \{F, G\}$$

$$\{f, g\}'_S = \{F, G\}' + X'(F)Z(G) - X'(G)Z(F),$$

where $X' := P'dH_0 = PdH_1$ and the right-hand sides of the previous equations are implicitly understood to be restricted to $S$. These equations show that the restrictions of $(H_1, \ldots, H_n)$ to $S$ are in bi-involution, and then separable in DN coordinates because of Theorem 3.1. We are going to show that they are even Stäckel separable, by computing their control matrix $F$ and checking that it satisfies the condition $N^*dF = FdF$.

To this purpose, we notice that the Lenard relations (6.1) on $M$ give rise to the equations

$$N^*d\hat{H}_i = d\hat{H}_{i+1} - \hat{Z}(H_i)d\hat{H}_1, \quad i = 1, \ldots, n-1,$$

$$N^*d\hat{H}_n = -\hat{Z}(H_n)d\hat{H}_1,$$

where $N$ is the recursion operator of the $\omega N$ manifold $S$ and $^\wedge$ denotes the restriction to $S$. Therefore, the control matrix of $(\hat{H}_1, \ldots, \hat{H}_n)$ is given by a single Frobenius block:

$$F = \begin{bmatrix}
-Z(H_1) & 1 & 0 & \cdots & 0 \\
-Z(H_2) & 0 & 1 \\
\vdots & \ddots & \ddots & \ddots \\
\vdots & & \ddots & 1 \\
-Z(H_n) & 0 & & & \\
\end{bmatrix}.$$  \hspace{1cm} (6.5)

So we see that the (restriction to the symplectic leaf $S$) of the functions $c_i = -Z(H_i)$ are the coefficients of the characteristic polynomial of the matrix $F$, that is, the coefficients of the minimal polynomial of the recursion operator $N$, $\Delta(\lambda) = \lambda^n - (c_1\lambda^{n-1} + \cdots + c_n)$. Recalling that the coefficients of the minimal polynomial of $N$ satisfy

$$N^*dc_i = dc_{i+1} + c_idc_1, \quad N^*dc_n = c_ndc_n,$$

we see that the condition $N^*dF = FdF$ for the Stäckel separability of the Hamiltonians is automatically verified. Hence we have proven
Theorem 6.1 The Hamiltonians of a corank-1 torsionless GZ system are Stäckel separable in DN coordinates.

It is worthwhile to notice that the examples previously considered in the literature within the theory of quasi-bi-Hamiltonian systems (see, e.g., [3, 32, 45]) fall into this class. The link with the classical Stäckel-Eisenhart theory of separation of variables is discussed in [23].

We remark that the vector field $Y$ appearing in (6.2) can be chosen to be tangent to $S$. In this case, $Y = P d(Z(H_1)) = -P d c_1$, so that its restriction to $S$ is the vector field we used in the previous section to determine the $\mu_i$ coordinates. (This explains why we made use of the same notation).

Now we will write the separation equations for the GZ Hamiltonians. The Stäckel matrix $S$, being the (normalized) matrix of the left eigenvectors of $F$, is easily seen to be the Vandermonde-like matrix

$$S = \begin{bmatrix}
\lambda_1^{n-1} & \cdots & \lambda_1 & 1 \\
\vdots & \cdots & \vdots & \vdots \\
\lambda_n^{n-1} & \cdots & \lambda_n & 1
\end{bmatrix},$$

where the $\lambda_i$ are the eigenvalues of $N$, i.e., the roots of $\Delta(\lambda)$. Therefore, the separation relations take the form

$$\hat{H}_1 \lambda_i^{n-1} + \hat{H}_2 \lambda_i^{n-2} + \cdots + \hat{H}_n = U_i(\lambda_i, \mu_i),$$

(6.7)

where $(\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n)$ are special DN coordinates on $S$ and the $U_i$ are the entries of the Stäckel vector. Such entries can be explicitly computed once we have the map sending the DN coordinates to the corresponding point of $S$, as we will check in the example of the 3-particle nonperiodic Toda lattice.

Another way to arrive at the separation equations is to multiply (6.3) by $\lambda_i^{n-i}$ and then to add to (6.4). The result is

$${\lambda^n}^* d\hat{H}(\lambda) = \lambda d\hat{H}(\lambda) - \Delta(\lambda)d\hat{H}_1,$$

meaning that $\hat{H}(\lambda) := \sum_{i=1}^{n} \hat{H}_i \lambda^{n-i}$ is a Stäckel function generator according to Proposition 5.1. Thus, in DN coordinates, $\hat{H}(\lambda_i) = U_i(\lambda_i, \mu_i)$, which coincides with (6.7). We stress that $\hat{H}(\lambda)$, being a Stäckel function generator, can be in some cases used to determine the $\mu_i$ coordinates. Instances of this situation are provided by the Toda lattice, as discussed in [12], and by the stationary reductions of the KdV hierarchy [11]. Here we will present the example of the 3-particle nonperiodic Toda lattice.

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Example 6.2 The Hamiltonian of the system is
\[ H_{\text{Toda}} = \frac{1}{2} \sum_{i=1}^{3} p_i^2 + \sum_{i=1}^{2} \exp(q^i - q^{i+1}) \, . \] (6.8)

As usual (see, e.g., [13], and [16] for the separability), one introduces the “Flaschka-Manakov coordinates” \((a_1, a_2, b_1, b_2, b_3)\), where
\[ b_i = p_i \, , \quad a_i = -\exp(q^i - q^{i+1}) \, , \]
and consider the manifold \(M = (\mathbb{C}^*)^2 \times \mathbb{C}^3\), or \(M = \mathbb{R}_{>0}^2 \times \mathbb{R}^3\). We endow it with the Poisson pencil \(P_{\lambda} = P' - \lambda P\) given by (see, e.g., [31] and references cited therein)
\[
P_{\lambda} = \begin{bmatrix}
0 & -a_1 a_2 & (b_1 - \lambda) a_1 & (\lambda - b_2) a_1 & 0 \\
a_1 a_2 & 0 & 0 & (b_2 - \lambda) a_2 & (\lambda - b_3) a_2 \\
(\lambda - b_1) a_1 & 0 & 0 & a_1 & 0 \\
(b_2 - \lambda) a_1 & (\lambda - b_2) a_2 & -a_1 & 0 & a_2 \\
0 & (b_3 - \lambda) a_2 & 0 & -a_2 & 0
\end{bmatrix} \, . \] (6.9)

It has a polynomial Casimir \(H(\lambda) = H_0 \lambda^2 + H_1 \lambda + H_2\), where
\[
H_0 = b_1 + b_2 + b_3 \\
H_1 = -(b_1 b_2 + b_2 b_3 + b_3 b_1 + a_1 + a_2) \\
H_2 = b_1 b_2 b_3 + a_1 b_3 + a_2 b_1 \, .
\]
The Hamiltonian (6.8) is related to the coefficients of \(H(\lambda)\) by \(H_{\text{Toda}} = H_1 + \frac{1}{2} H_0\). There are two nontrivial flows, given by:
\[
X_1 = P_0 dH_1 = a_1(b_1 - b_2) \frac{\partial}{\partial a_1} + a_2(b_2 - b_3) \frac{\partial}{\partial a_2} + a_1 \frac{\partial}{\partial b_1} + (a_2 - a_1) \frac{\partial}{\partial b_2} - a_2 \frac{\partial}{\partial b_3} \\
X_2 = P_0 dH_2 = a_1[a_2 + b_3(b_2 - b_1)] \frac{\partial}{\partial a_1} + a_2[a_1 + b_1(b_3 - b_2)] \frac{\partial}{\partial a_2} - a_1 b_3 \frac{\partial}{\partial b_1} \\
+ (a_1 b_3 - a_2 b_1) \frac{\partial}{\partial b_2} + a_2 b_1 \frac{\partial}{\partial b_3} \, .
\]
The symplectic leaves of \(P\) are the level surfaces of \(H_0\), so that they can be parametrized by \((a_1, a_2, b_1, b_2)\). A possible choice for the normalized transversal vector field is \(Z = \frac{\partial}{\partial a_3}\), because \(Z(H_0) = 1\) and
\[
L_Z P = 0 \, , \quad L_Z P' = Y \wedge Z \, ,
\]

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with \( Y = a_2 \frac{\partial}{\partial a_2} \). Since \( Y(H_0) = 0 \), we know that \( Y = P d(Z(H_1)) = -P \, d c_1 \). If \( S \) is a symplectic leaf of \( P \), the reduced bi-Hamiltonian structure on \( S \) is simply obtained by removing the last row and the last column of \( P_\lambda \):

\[
P_S = \begin{bmatrix}
0 & 0 & a_1 & -a_1 \\
0 & 0 & 0 & a_2 \\
-a_1 & 0 & 0 & 0 \\
a_1 & -a_2 & 0 & 0
\end{bmatrix}, \quad P'_S = \begin{bmatrix}
0 & -a_1 a_2 & a_1 b_1 & -a_1 b_2 \\
a_1 a_2 & 0 & 0 & a_2 b_2 \\
-a_1 b_1 & 0 & 0 & a_1 \\
a_1 b_2 & -a_2 b_2 & -a_1 & 0
\end{bmatrix}.
\]

For completeness, we display recursion operator,

\[
N = P'_S P_S^{-1} = \begin{bmatrix}
b_1 & a_1 (b_1 - b_2) / a_2 & a_1 & a_1 \\
0 & b_2 & -a_2 & 0 \\
0 & a_1 / a_2 & b_1 & 0 \\
-1 & -a_1 / a_2 & 0 & b_2
\end{bmatrix},
\]

whose minimal polynomial is

\[
\Delta(\lambda) = \lambda^2 + Z(H_1) \lambda + Z(H_2) = \lambda^2 - (b_1 + b_2) \lambda + a_1 + b_1 b_2.
\]

The coordinates \( \lambda_1, \lambda_2 \) are its roots.

The restrictions of \( H_1 \) and \( H_2 \) to the symplectic leaf \( H_0 = c \) are

\[
\hat{H}_1 = -c(b_1 + b_2) + b_1^2 + b_2^2 + b_1 b_2 - a_1 - a_2 \\
\hat{H}_2 = c(a_1 + b_1 b_2) - (a_1 + b_1 b_2)(b_1 + b_2) + a_2 b_1.
\]

We know that \( \hat{H}(\lambda) := \hat{H}_1 \lambda + \hat{H}_2 \) is a Stäckel function generator, and that the separation equations are \( \hat{H}(\lambda_i) = U(\lambda_i, \mu_i), \) for \( i = 1, 2 \). To write them explicitly, we need the form of the \( \mu_i \). They can be found using Proposition 5.4 and the fact that

\[
Y^2(\hat{H}(\lambda)) = Y(\hat{H}(\lambda)).
\]

This entails that \( f(\lambda) := \log Y(\hat{H}(\lambda)) \) satisfies \( Y(f(\lambda)) = 1 \), so that, according to the results of Section 3,

\[
\mu_i = \log Y(\hat{H}(\lambda_i)) = \log(a_2 b_1 - a_2 \lambda_i), \quad i = 1, 2,
\]

form with the eigenvalues \( \lambda_1 \) and \( \lambda_2 \) of \( N \) a set of (special) DN coordinates.
Finally, using the expression of \((a_1, a_2, b_1, b_2)\) in terms of the DN coordinates one can easily find the separation relations

\[
\hat{H}(\lambda_i) = \lambda_i^3 + \exp \mu_i - c\lambda_i^2, \quad i = 1, 2,
\]

leading to the solution by quadratures of the Hamilton-Jacobi equations for \(\hat{H}_1\) and \(\hat{H}_2\).

We notice that the “change of variables” \((a_i, b_i) \mapsto (\lambda_i, \mu_i)\) is not the lift of a point trasformation on the configuration space; thus, there is no contradiction with the results of [3], stating that it is impossible to separate the 3-particle Toda lattice with point tranformations.

7 Separability of Gel’fand–Zakharevich systems

In this section we will generalize (and give proofs of) the results of the previous section to the case of corank \(k\). As we will see, the picture outlined in the previous section still holds good. The only relevant difference concerns the Stäckel separability, which is no longer valid in general, but requires an additional assumption on the Hamiltonians.

We consider a bi-Hamiltonian manifold \((M, P, P')\) admitting \(k\) polynomial Casimir functions of the Poisson pencil \(P_\lambda = P' - \lambda P\),

\[
H^{(a)}_i(\lambda) = \sum_{i=0}^{n_a} H^{(a)}_i \lambda^{n_a-i}, \quad a = 1, \ldots, k,
\]

such that \(n_1 + n_2 + \cdots + n_k = n\), with \(\dim M = 2n + k\), and such that the differentials of the coefficients \(H^{(a)}_i\) are linearly independent on \(M\). The \(H^{(a)}_i\), for a fixed \(a\), form a bi-Hamiltonian hierarchy and, in particular, \(H^{(a)}_0\) (resp. \(H^{(a)}_{n_a}\)) is a Casimir of \(P\) (resp. \(P'\)). We assume that the corank of \(P\) is exactly \(k\), so that the \(H^{(a)}_0\), for \(a = 1, \ldots, k\), are a maximal set of independent Casimirs of \(P\). The collection of the \(n\) bi-Hamiltonian vector fields

\[
X^{(a)}_i = P \, dH^{(a)}_i = P' \, dH^{(a)}_{i-1}, \quad i = 1, \ldots, n_a, \quad k = 1, \ldots, a,
\]

associated with the Lenard sequences defined by the Casimirs \(H^{(a)}_i\) is called the Gel’fand–Zakharevich (GZ) system, or the axis, of the bi-Hamiltonian manifold \(M\). Since standard arguments from the theory of Lenard–Magri chains show that all the coefficients \(H^{(a)}_i\) pairwise commute with respect to both \(\{\cdot, \cdot\}\) and \(\{\cdot, \cdot\}'\), we have
Proposition 7.1 Let $S$ be a symplectic leaf of $P$, that is, a $2n$-dimensional submanifold defined by $H_0^{(1)} = c_1, \ldots, H_0^{(k)} = c_k$. Then the vector fields $X_i^{(a)}$ of the Lenard sequences associated with the polynomial Casimirs (7.1) of $\{\cdot, \cdot\}$ on $M$ define a completely integrable Hamiltonian system on $S$.

We call the family $\{\hat{H}_i^{(a)} | i = 1, \ldots, n_a, k = 1, \ldots, a\}$ of the restrictions to $S$ of the coefficients of the $H^{(a)}$ the GZ basis of the symplectic leaf $S$. The lagrangian foliation defined by the GZ basis will be referred to as the GZ foliation of $S$.

In the following subsection we will give sufficient conditions so that a symplectic leaf $S$ of $P$ inherits an $\omega N$ structure from the bi-Hamiltonian structure of $M$. Then we will come back to the integrable system described in the previous proposition and we will discuss its separability in DN coordinates.

7.1 The induced $\omega N$ structure

Our strategy to induce on a symplectic leaf $S$ of $P$ a second Poisson bracket which is compatible with the "canonical" one is based on the geometrical considerations already mentioned at the end of Section 2. We suppose that there exists a $k$-dimensional foliation $\mathcal{Z}$ of $M$ such that

C1) the foliation $\mathcal{Z}$ is transversal to the symplectic foliation of $P$;

C2) the functions that are constant on $\mathcal{Z}$ form a Poisson subalgebra with respect to both $\{\cdot, \cdot\}$ and $\{\cdot, \cdot\}'$.

Thus $S$ has a (projected) bi-Hamiltonian structure. The projection of $\{\cdot, \cdot\}$ coincides with the symplectic structure $\{\cdot, \cdot\}_S$ of $S$, while the projection of $\{\cdot, \cdot\}'$ defines a second Poisson bracket $\{\cdot, \cdot\}'_S$ on $S$. Since the compatibility between $\{\cdot, \cdot\}_S$ and $\{\cdot, \cdot\}'_S$ is guaranteed by the fact that the whole pencil $\{\cdot, \cdot\}_\lambda$ is projectable on $S$, we have endowed $S$ with an $\omega N$ structure. We will suppose it to be a regular $\omega N$ manifold, in order to apply (in the open dense set where the eigenvalues of $N$ are distinct) the results of Section 3 and 4, leaving the discussion of the problem of finding the conditions on $(M, P, P')$ and $\mathcal{Z}$ ensuring the regularity of $S$ for future work.

Let $(Z_1, \ldots, Z_k)$ be local vector fields spanning the distribution tangent to $\mathcal{Z}$. Because of the transversality condition, we can always normalize these vector fields with respect to the Casimirs $H_0^{(a)}$ of $P$:

$$Z_b(H_0^{(a)}) = \delta_b^a.$$  \hfill (7.3)
In terms of these generators, the projectability condition takes a very concise form, as shown in

**Proposition 7.2**

1. The normalized vector fields $Z_a$ locally generating $\mathcal{Z}$ are symmetries of $P$,

$$L_{Z_a}(P) = 0,$$

and satisfy

$$L_{Z_a}P' = \sum_b Y^b_a \wedge Z_b,$$  \hspace{1cm} (7.5)

where $Y^b_a = P(\partial(Z_a(H^b_1))) = [Z_a, P' dH^b_1] = [Z_a, X^b_1]$.

2. Viceversa, suppose that there exists a $k$-dimensional integrable distribution on $M$ which is transversal to the symplectic leaves of $P$ and such that (7.4) and (7.5) hold for a suitable local basis $(Z_1, \ldots, Z_k)$ of the distribution (and for suitable vector fields $Y^b_a$). Then the integral foliation of the distribution satisfies the projectability requirements $C1)$ and $C2)$, so that every symplectic leaf of $P$ becomes an $\omega N$ manifold. Moreover, if the $Z_a$ are normalized, then they commute.

**Proof.** First of all, we recall ([12], p. 54) that the condition that the functions constant along $\mathcal{Z}$ form a Poisson subalgebra with respect to $\{\cdot, \cdot\}$ is equivalent to the assertion that the following equations hold,

$$L_{Z_a}P = \sum_{b=1}^{k} W^b_a \wedge Z_b,$$  \hspace{1cm} (7.6)

for some vector fields $W^b_a$. This entails the validity of assertion 2, except the commutativity of the vector fields $Z_a$, normalized according to (7.3), that can be proved as follows. The integrability of the distribution implies that there are functions $\phi^c_{ab}$ such that

$$[Z_a, Z_b] = \sum_{c=1}^{k} \phi^c_{ab} Z_c,$$

and evaluating this relation on the Casimirs $H^{(d)}_0$ of $P$ we easily see that $\phi^c_{ab} = 0$.

In order to prove assertion 1, we notice that the vector fields $W^b_a$ are not uniquely defined, and can be taken to be tangent to the symplectic leaves of $P$. This is accomplished by changing

$$W^b_a \rightarrow W^b_a - \sum_{c=1}^{k} W^b_a(H^{(c)}_0)Z_c.$$

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Indeed,
\[ \sum_{b=1}^{k} (W^b_a - \sum_{c=1}^{k} W^b_a (H^{(c)}_0) Z_c) \land Z_b = \sum_{b=1}^{k} W^b_a \land Z_b - \sum_{b,c=1}^{k} W^b_a (H^{(c)}_0) Z_c \land Z_b = \sum_{b=1}^{k} W^b_a \land Z_b \]

since \( L_{Z_a} \langle dH^{(c)}_0, P dH^{(d)}_0 \rangle = 0 \) and \( (7.6) \) implies that
\[ W^d_b (H^{(c)}_0) = W^c_b (H^{(d)}_0) . \]

Thus the vector fields \( W^b_a \) in \( (7.3) \) can be chosen in such a way that \( W^b_a (H^{(c)}_0) = 0 \). Now, deriving the relation \( P dH^{(c)}_0 = 0 \) along \( Z_a \) one obtains that the normalized vector fields \( W^b_a \) vanish, so that, indeed, the vector fields \( Z_a \) are symmetries of \( P \).

As far as the second Poisson tensor \( P' \) is concerned, in the same way we can show that there exist vector fields \( Y^b_a \) tangent to the symplectic leaves of \( P \) such that
\[ L_{Z_a} P' = \sum_{b=1}^{k} Y^b_a \land Z_b . \quad (7.7) \]

By deriving the relation \( P' dH^{(c)}_0 = X^{(c)}_1 \) with respect to \( Z_a \), one has that
\[ Y^c_a = [Z_a, X^{(c)}_1] = L_{Z_a} (P dH^{(c)}_1) = P d(Z_a (H^{(c)}_1)) . \]

This completes the proof.

\[ QED \]

In the sequel we will always suppose that the normalization conditions \( (7.4) \) on the transversal vector fields \( Z_a \) and the tangency conditions on the \( Y^b_a \) are satisfied. For the sake of simplicity, we will also assume that the \( Z_a \) are defined on the whole manifold \( M \), or at least in a tubular neighborhood of \( S \). Next we give a useful formula for the (second) reduced Poisson bracket on \( S \).

**Proposition 7.3** Let \( f, g \) be functions on a symplectic leaf \( S \) of \( P \), and \( F, G \) arbitrary extensions of \( f, g \) to \( M \). Then
\[ \{f, g\}_S = \{F, G\} \]
\[ \{f, g\}'_S = \{F, G\}' + \sum_{a=1}^{k} \left( X^{(a)}_1 (F) Z_a (G) - X^{(a)}_1 (G) Z_a (F) \right) , \]
where \( X^{(a)}_1 = P' dH^{(a)}_0 = \{H^{(a)}_0, \cdot\}' \).
Proof. The symplectic leaf $S$ is given by the equations $H_0^{(a)} = c^a$, for $a = 1, \ldots, k$, where the $c^a$ are suitable constants. The first formula simply says that $\{\cdot, \cdot\}_S$ corresponds to the symplectic structure of $S$. The second formula follows from the remark that $\tilde{F} := F - \sum_{a=1}^{k} Z_a(F) \left( H_0^{(a)} - c^a \right)$ coincides with $F$ and fulfills $Z_b(\tilde{F}) = 0$ on $S$. Hence it can be used to compute $\{f, g\}_S'$, giving (7.9).

$QED$

Remark 7.4 The projectability conditions we have imposed in order to endow a fixed symplectic leaf $S$ with an $\omega N$ structure can be weakened in the following way. We can consider a distribution transversal to $TS$ and defined only at the points of $S$, generated by a family of vector fields $(Z_1, \ldots, Z_k)$, normalized as $Z_a(H_0^{(b)}) = \langle dH_0^{(b)}, Z_a \rangle = \delta_a^b$. Then we introduce, according to (7.9), a composition law $\{\cdot, \cdot\}'_S$ on $C^\infty(S)$ and we look for conditions ensuring that it is a Poisson bracket, compatible with $\{\cdot, \cdot\}_S$. One can show [14] that $\{\cdot, \cdot\}'_S$ is a Poisson bracket if and only if

$$\sum_{a=1}^{k} X_1^{(a)} \wedge \left( L_{Z_a}(P') + \sum_{b=1}^{k} [Z_a, X_b'] \wedge Z_b \right) + \frac{1}{2} \sum_{a,b=1}^{k} X_1^{(a)} \wedge X_1^{(b)} \wedge [Z_a, Z_b] = 0$$

(7.10)

at the points of $S$. In this case, the two Poisson brackets are compatible if and only if

$$\sum_{a=1}^{k} X_1^{(a)} \wedge L_{Z_a}(P) = 0$$

(7.11)

at the points of $S$. Hence, the requirements (7.4) and (7.5), on the whole manifold $M$, are very “strong” solutions for (7.10) and (7.11). Finally, we mention that the reduction process presented in this remark does not fit in the Marsden-Ratiu scheme [30], whereas the one based on C1) and C2) clearly does.

7.2 Separability and the control matrix

After endowing any symplectic leaf $S$ of $P$ with an $\omega N$ structure, we can reconsider the GZ foliation of $S$ and prove its separability in DN coordinates. Notice that (see also below) the restrictions to $S$ of the bi-Hamiltonian vector fields $X_i^{(a)}$ are not bi-Hamiltonian with respect to the $\omega N$ structure of $S$. This is due to the fact that this structure is obtained by means of a projection, while the Hamiltonian are restricted to $S$. 
We suppose that \((Z_1, \ldots, Z_k)\) are vector fields on \(M\), fulfilling the hypotheses of part 2 of Proposition \(7.2\) and normalized, i.e., \(Z_a(H_b^{(a)}) = \delta^b_a\). Then the expressions \((7.8)\) and \((7.9)\) of the reduced Poisson brackets immediately show that the restrictions of \(H_i^{(a)}\) to \(S\) are in bi-involution. Therefore, they are separable in DN coordinates.

**Theorem 7.5** The GZ foliation of \(S\) is separable in DN coordinates.

Using once more Theorem \(3.1\), we can conclude that the distribution tangent to the GZ foliation is invariant with respect to the recursion operator \(N\). We are going to describe the form of the associated control matrix, which will be needed to discuss the Stäckel separability of the GZ basis.

Let \(g\) be any function on \(S\) and let \(G\) be an extension of \(g\) to \(M\). Using \((7.9)\) and the Lenard relations on the \(H_i^{(a)}\), we have

\[
\{\hat{H}_i^{(a)}, g\}'_S = \{H_i^{(a)}, G\}' + \sum_{b=1}^k \left( X_1^{(b)}(H_i^{(a)})Z_b(G) - X_1^{(b)}(G)Z_b(H_i^{(a)}) \right) = \{H_{i+1}^{(a)}, G\} - \sum_{b=1}^k Z_b(H_i^{(a)})\{H_1^{(b)}, G\},
\]

where we have put \(H_{n+1}^{(a)} := 0\). Therefore, for all \(g \in C^\infty(S)\),

\[
\{\hat{H}_i^{(a)}, g\}'_S = \{\hat{H}_i^{(a)}, g\}_S - \sum_{b=1}^k \{Z_b(H_i^{(a)}), \{\hat{H}_1^{(b)}, g\}_S\},
\]

or, in terms of the (reduced) Poisson tensors \(P_S\) and \(P'_S\),

\[
P'_S d\hat{H}_i^{(a)} = P_S d\hat{H}_i^{(a+1)} - \sum_{b=1}^k \{Z_b(H_i^{(a)}), P_S d\hat{H}_1^{(b)}\}.
\]

Hence, we can conclude that

\[
N^* d\hat{H}_i^{(a)} = d\hat{H}_i^{(a+1)} - \sum_{b=1}^k Z_b(H_i^{(a)}),
\]

and read the form of the control matrix \(F\) associated with the GZ basis. Indeed, if we order the \(n\) functions of the GZ basis as

\[
\hat{H}_1^{(1)}, \hat{H}_2^{(1)}, \ldots, \hat{H}_{n_1}^{(1)}, \hat{H}_1^{(2)}, \ldots, \hat{H}_{n_k}^{(k)},
\]

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then we realize that $F$ has a $k \times k$ block form,

$$F = \begin{bmatrix}
F_1 & C_{1,2} & \cdots & C_{1,k} \\
C_{2,1} & F_2 & \cdots & C_{2,k} \\
\vdots & \vdots & \ddots & \vdots \\
C_{k,1} & F_k & \cdots & C_{k,k}
\end{bmatrix},$$

(7.17)

with $F_a$ an $n_a \times n_a$ square matrix of Frobenius type of the form

$$F_a = \begin{bmatrix}
-Z_a(H_1^{(a)}) & 1 & 0 & \cdots & 0 \\
-Z_a(H_2^{(a)}) & 0 & 1 & \cdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
-Z_a(H_n^{(a)}) & 0 & \cdots & \cdots & 1
\end{bmatrix},$$

(7.18)

and $C_{a,b}$ a rectangular matrix with $n_a$ rows and $n_b$ columns where only the first column is nonzero:

$$C_{a,b} = \begin{bmatrix}
-Z_b(H_1^{(a)}) & 0 & \cdots & 0 \\
-Z_b(H_2^{(a)}) & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-Z_b(H_n^{(a)}) & 0 & \cdots & 0
\end{bmatrix}.$$

(7.19)

**Remark 7.6** The vector field $Y$, defined in Section 5 as the Hamiltonian vector field associated with $-\frac{1}{2} \text{tr} N$ by the first Poisson structure, can be obtained in the present setting by restricting to $S$ the vector field $\sum_{a=1}^{k} Y_a^a$. Indeed,

$$\sum_{a=1}^{k} Y_a^a = P d \left( \sum_{a=1}^{k} Z_a(H_1^{(a)}) \right),$$

and using (7.17) we have $\sum_{a=1}^{k} Z_a(H_1^{(a)}) = -\text{tr} F = -\frac{1}{2} \text{tr} N$.

Thus, we have seen that GZ systems on bi-Hamiltonian manifolds admitting a suitable transversal foliation provide examples of non trivial (but still somewhat special) Hamiltonian systems for which the separability condition in DN coordinates holds, that is, they provide interesting examples of control matrices, discussed in Section 4. Such matrices were introduced (in the specific example of a stationary reduction of the Boussinesq equation) in [13].

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7.3 Stäckel separability of GZ systems

Let us now consider the Stäckel (i.e., linear) separability of GZ systems. We have seen that the invariance with respect to $N$ of the Lagrangian distribution defined by the restricted Hamiltonians $\hat{H}^{(a)}_i$ is a consequence of the Lenard recursion relations on $M$, and that the nontrivial coefficients in $F$ are given by the deformations of the polynomial Casimirs along the normalized generators $Z_a$ of the foliation $\mathcal{Z}$. On the other hand, in Section 6 we have proved that, in the corank 1 case, the control matrix $F$ automatically satisfies the condition for Stäckel separability, $N^*dF = FdF$. The next proposition shows that, in order to ensure this condition in the general case, one has to require that the Hamiltonians $H^{(a)}_i$ be affine with respect to the vector fields $Z_a$.

**Proposition 7.7** The GZ basis, formed by the $\hat{H}^{(a)}_i$, is a Stäckel basis (i.e., it is Stäckel separable in DN coordinates) if and only if

$$Z_b(Z_c(H^{(d)}_j)) = 0$$

on $S$, for all $b, c, d = 1, \ldots, k$ and for all $j = 1, \ldots, n_d$.

**Proof.** Stäckel separability is equivalent to $N^*dF = FdF$, where $F$ is the control matrix (7.17). Since $dF$ has nonvanishing entries only in the columns $1, n_1 + 1, n_2 + 1, \ldots, n_{k-1} + 1$, this condition takes the form

$$N^*d(Z_a(H^{(b)}_i)) = d(Z_a(H^{(b)}_{i+1})) - \sum_{c=1}^k Z_c(H^{(b)}_i)d(Z_a(H^{(c)}_1)), \quad (7.20)$$

where, as usual, we have put $H^{(b)}_{n_k+1} := 0$. In order to compute the left-hand side of (7.20), we observe that (7.9) implies

$$P'_S df = \left( P' + \sum_{c=1}^k Z_c \wedge X^{(c)}_1 \right) dF,$$

where $f \in C^\infty(S)$ and $F$ is any extension of $f$. Moreover, we have that

$$L_{Z_a} \left( P' + \sum_{c=1}^k Z_c \wedge X^{(c)}_1 \right) = \sum_{c=1}^k [Z_a, Z_c] \wedge X^{(c)}_1 = 0,$$

(7.21)
since the $Z_b$ commute. Hence,

$$P' S (\hat{Z}_a) = \left( P' + \sum_{c=1}^{k} Z_c \land X_1^{(c)} \right) d(\hat{H}_i^{(b)})$$

$$= L_{Z_a} \left[ \left( P' + \sum_{c=1}^{k} Z_c \land X_1^{(c)} \right) dH_i^{(b)} \right] = L_{Z_a} \left( P dH_i^{(b)} - \sum_{c=1}^{k} Z_c (H_i^{(b)}) P dH_1^{(c)} \right)$$

$$= P d(\hat{Z}_a (H_i^{(b)})) - \sum_{c=1}^{k} Z_c (H_i^{(b)}) P d(\hat{Z}_a (H_1^{(c)})) - \sum_{c=1}^{k} Z_a (Z_c (H_i^{(b)})) P dH_1^{(c)}.$$  \hspace{1cm} (7.22)

so that

$$N^* d(\hat{Z}_a (H_i^{(b)})) = d(\hat{Z}_a (H_i^{(b)})) - \sum_{c=1}^{k} Z_c (H_i^{(b)}) d(\hat{Z}_a (H_1^{(c)}))$$

$$- \sum_{c=1}^{k} Z_a (Z_c (H_i^{(b)})) d\hat{H}_1^{(c)}.$$ \hspace{1cm} (7.23)

A comparison with (7.20) completes the proof.

\hspace{1cm} \hspace{1cm} \hspace{1cm} QED

Thus, the GZ basis is Stäckel separable if (and only if) the second derivatives of the Hamiltonians along the transversal vector fields vanish. This condition is automatically verified in the case of corank $k = 1$. This “discrepancy” between the generic and the rank 1 case can be understood as follows. Since, by assumption, the transversal distribution $\mathcal{Z}$ is integrable, the tubular neighborhood in which it is defined is equipped with a fibered structure, in which the fibers are the symplectic leaves of $P$. The conditions

$$L_{Z_a} (P) = 0; \quad L_{Z_a} \left( P' + \sum_{c=1}^{k} Z_c \land X_1^{(c)} \right) = 0$$

of equations (7.4) and (7.21) imply that the recursion operator (to be seen, in this picture, as an endomorphism of the vertical tangent bundle to the local fibration) is invariant along all the $Z_a$. So its eigenvalues and hence its minimal polynomial are invariant with respect to the $Z_a$. In the case $k = 1$, as we have seen in Section 6, the coefficients of the minimal polynomials are the derivatives of the Casimir with respect to the (single) transversal vector field $Z$, but this is not necessarily true in the higher corank case. Notice that, whenever the second derivatives of the Casimirs vanish, our separated variables are “invariant” with respect to the Casimirs, as the one considered in [40].
Still under the assumptions of the above proposition, the results of Section 4 tell us how to construct the Stäckel matrix and, in principle, the separation relations. We also know that the entries of the Stäckel matrix and of the Stäckel vector can be used (under additional hypotheses) to explicitly find the separation coordinates, i.e., the DN coordinates. In the next section we will exploit the special properties of the GZ foliation in order to determine the separation relations and, eventually, the DN coordinates without computing the Stäckel matrix.

8 Separation relations for GZ systems

Let us consider the GZ foliation (on the symplectic leaf $S$) studied in Subsection 7.2. The aim of this section is to write, in the Stäckel separable case, the separation relations for the Hamiltonians of the GZ basis. To simplify the notations, we will not use anymore the symbol $\hat{}$ to denote the restriction to $S$.

First of all, we notice that the relevant information contained in the $n \times n$ control matrix $F$ is actually encoded in the $k \times k$ polynomial matrix $F(\lambda)$, which is the Jacobian matrix of the Casimirs $H^{(a)}(\lambda)$ with respect to the transversal (normalized) vector fields $Z_b$, that is, the matrix

$$F(\lambda) = \begin{bmatrix}
Z_1(H^{(1)}(\lambda)) & \cdots & Z_k(H^{(1)}(\lambda)) \\
\vdots & & \vdots \\
Z_1(H^{(k)}(\lambda)) & \cdots & Z_k(H^{(k)}(\lambda))
\end{bmatrix}. \quad (8.1)$$

We can translate the results about separability and Stäckel separability of GZ systems, based on the $n \times n$ matrix equations

$$N^*dH = FdH \quad (8.2)$$
$$N^*dF = FdF, \quad (8.3)$$

into corresponding equations for the polynomial matrix $F(\lambda)$. To this end we denote by $\underline{H}(\lambda) = (H^{(1)}(\lambda), H^{(2)}(\lambda), \ldots, H^{(k)}(\lambda))^T$ the $k$-component vector of the polynomial Casimir functions, and by $\underline{H}_1 = (H_1^{(1)}, H_1^{(2)}, \ldots, H_1^{(k)})^T$ and $F_1 = \begin{bmatrix} Z_b(H_1^{(a)}) \end{bmatrix}$ the analogs of the vector $\underline{H}(\lambda)$ and of the matrix $F(\lambda)$, constructed by using the coefficients $H_1^{(a)}$ instead of the full Casimir functions $H^{(a)}(\lambda)$. 

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Lemma 8.1  The polynomial control matrix $F(\lambda)$ satisfies the equation

$$(N^* - \lambda) dH(\lambda) = -F(\lambda) dH_1,$$  \hspace{1cm} (8.4)

which is the counterpart of the matrix equation (8.2).

Proof. The $\lambda^n a^{-i}$-coefficient of the $a$-th row of (8.4) is exactly (7.15). \hfill QED

In complete analogy, we obtain the “polynomial form” of the Stäckel separability condition (8.3).

Lemma 8.2  The GZ basis is a Stäckel basis iff $F(\lambda)$ satisfies the condition

$$(N^* - \lambda) dF(\lambda) = -F(\lambda) dF_1.$$  \hspace{1cm} (8.5)

Proof. The simplest way to prove this lemma is to expand both sides in powers of $\lambda$. We first write (8.5) in componentwise form as

$$N^* dF_a^b(\lambda) = \lambda dF_a^b(\lambda) - \sum_{c=1}^k F_c^b(\lambda) d(F_1)^c_a,$$

and then expand in powers of $\lambda$, getting

$$N^* d(Z_a(H_i^{(b)}) = d(Z_a(H_i^{(b)}) - \sum_{c=1}^k Z_c(H_i^{(b)}) d(Z_a(H_1^{(c)})),$$

which are exactly the Stäckel conditions (7.20) for the GZ basis. \hfill QED

The following lemma shows that the eigenvalues of $N$ can be easily obtained from the matrix $F(\lambda)$.

Lemma 8.3  The determinant of $F(\lambda)$ is the characteristic polynomial of $F$. In particular, it coincides with the minimal polynomial $\Delta(\lambda)$ of the recursion operator $N$, that is,

$$\det F(\lambda) = \det(\lambda I - F) = \Delta(\lambda).$$  \hspace{1cm} (8.6)

Proof. Let $\lambda_i$ be an eigenvalue of $F$. Then one can check that the relative (left) eigenvectors have the form

$$v_i = (\sigma_1^i \lambda_{i_1}^{n_1-1}, \sigma_1^i \lambda_{i_1}^{n_1-2}, \ldots, \sigma_1^i, \sigma_2^i \lambda_{i_2}^{n_2-1}, \ldots, \sigma_k^i \lambda_{i_k}^{n_k-1}, \ldots, \sigma_k^i),$$
where $\sigma_i := (\sigma^1_i, \ldots, \sigma^k_i)$ is a nonzero vector such that $\sigma_i F(\lambda_i) = 0$. This shows that $\det F(\lambda_i) = 0$. Since $\det F(\lambda)$ is a monic degree $n$ polynomial and the $\lambda_i$ are distinct, we can conclude that (8.4) holds.

\[QED\]

The next step is to introduce the adjoint (or cofactor) matrix $F^\vee(\lambda)$, satisfying the equation

$$F^\vee(\lambda) F(\lambda) = F(\lambda) F^\vee(\lambda) = \det F(\lambda) I .$$

(8.7)

We will show that the rows of $F^\vee(\lambda)$, after a suitable normalization, provide Stäckel function generators and play the role of the Stäckel matrix. If $\sigma(\lambda) := e_k F^\vee(\lambda)$ is a row of the adjoint matrix, then, obviously,

$$\sigma(\lambda) F(\lambda) = \Delta(\lambda) e_k .$$

(8.8)

Let $\sigma_j(\lambda)$ be a nonvanishing entry of $\sigma(\lambda)$ and let us consider the normalized row

$$\rho(\lambda) = \frac{1}{\sigma_j(\lambda)} \sigma ,$$

(8.9)

which satisfies the equation

$$\rho(\lambda) F(\lambda) = \frac{\Delta(\lambda)}{\sigma_j(\lambda)} e_k .$$

(8.10)

**Proposition 8.4** Suppose that the component $\rho_a(\lambda)$ of $\rho(\lambda)$ is defined for $\lambda = \lambda_i, i = 1, \ldots, n$. Then it is a Stäckel function generator, that is, it verifies the equation

$$(N^* - \lambda) d\rho_a(\lambda) = 0, \quad \text{for } \lambda = \lambda_i, i = 1, \ldots, n .$$

(8.11)

**Proof.** It is convenient to consider the full vector $\rho(\lambda)$. From equation (8.10) we have

$$(N^* - \lambda) d\rho(\lambda) \cdot F(\lambda) + \rho(\lambda) \cdot (N^* - \lambda) dF(\lambda) = 0 \quad \text{for } \lambda = \lambda_i .$$

(8.12)

Using Lemma 8.2 we can write the second summand in this equation as

$$\rho(\lambda) \cdot (N^* - \lambda) dF(\lambda) = -\rho(\lambda) F(\lambda) dF_1 \quad \text{for } \lambda = \lambda_i ,$$

(8.13)
so that we finally obtain
\[(N^* - \lambda_i)d\rho(\lambda_i) \cdot F(\lambda_i) = 0 \quad \text{for } i = 1, \ldots, n.\] (8.14)

But the kernel of \(F(\lambda_i)\) is 1-dimensional, due to the fact that the \(\lambda_i\) are distinct.

Indeed, from (8.7) we have that
\[
\det F^\vee(\lambda) = (\det F(\lambda))^{k-1} = \prod_{i=1}^{n} (\lambda - \lambda_i)^{k-1}.
\] (8.15)

If \(\dim \ker F(\lambda_i) \geq 2\) for some \(i\), then the rank of \(F(\lambda_i)\) would be less than \(k-1\), so that \(F^\vee(\lambda_i) = 0\), and therefore \(F^\vee(\lambda) = (\lambda - \lambda_i)^{k-1}\det \tilde{F}(\lambda)\), contradicting (8.15).

Coming back to (8.14), we can assert that there exist 1-forms \(\nu_i\) such that
\[(N^* - \lambda_i)d\rho(\lambda_i) = \nu_i\rho(\lambda_i) \quad \text{for } i = 1, \ldots, n.\] (8.16)

Since the \(j\)-th component of \(\rho(\lambda_i)\) is 1, we have that the \(\nu_i\) vanish, and this closes the proof.

\[QED\]

Now we are ready to show how to compute the separation equations for GZ systems.

**Proposition 8.5** Let the \(\rho_a(\lambda)\) be as in the previous proposition and suppose that they are defined for \(\lambda = \lambda_i\). Then \(\sum_{a=1}^{k} \sigma_a(\lambda)H^{(a)}(\lambda)\) is a Stäckel function generator.

**Proof.** Let us write compactly \(\sum_{a=1}^{k} \rho(\lambda)H^{(a)}(\lambda) = \rho(\lambda) \cdot \underline{H}(\lambda)\) and compute
\[(N^* - \lambda)d(\rho(\lambda) \cdot \underline{H}(\lambda)) = (N^* - \lambda)d\rho(\lambda) \cdot \underline{H}(\lambda) + \rho(\lambda)(N^* - \lambda)d\underline{H}(\lambda)\]. (8.17)

For \(\lambda = \lambda_i\) the first summand vanishes thanks to Proposition 8.4, while the second equals (according to Lemma 8.1)
\[-\rho(\lambda_i)F(\lambda_i) d\underline{H}_1,\]
and so vanishes as well.

\[QED\]

Therefore we have shown that the separation relations of the GZ basis (in the Stäckel case) are given by
\[
\sum_{a=1}^{k} \rho_a(\lambda_i)H^{(a)}(\lambda_i) = \Phi_i(\lambda_i, \mu_i), \quad i = 1, \ldots, n,
\] (8.18)
that in the corank 1 case boils down to equation (6.7).

We end this section with the following remark. Let us suppose that the multipliers \( \rho_a(\lambda) \) and the coordinates \( \mu_1, \ldots, \mu_n \) be related by a “simple” algebraic expression, e.g., that there exist integer numbers \( p_1 = 0, p_2, \ldots, p_k \) such that

\[
\mu_i^{p_a} = \rho_a(\lambda_i), \quad i = 1, \ldots, n \text{ and } a = 1, \ldots, k.
\]

This means, according to the results of Section 5, that the \( p_a \)-th root \( \rho_a(\lambda) \) is a Stäckel function generator satisfying the equation (5.9), i.e., \( Y(\sqrt[p]{\rho_a(\lambda)}) = 1 \), for \( \lambda = \lambda_i \). Then the separation relations (8.18) “degenerate” to a single one, that is, they can be read as the vanishing of the two-variable function

\[
\sum_{a=1}^{k} \mu_i^{p_a} H^{(a)}(\lambda) - \Phi(\lambda, \mu)
\]

evaluated at the points \((\lambda_i, \mu_i)\), for \( i = 1, \ldots, n \). Hence, in such an instance, we can associated with the GZ system a “spectral curve” over which the separation coordinates lie.

This is an indication, which is verified in several concrete examples, that the theory herewith presented may provide an effective bridge between the classical theory of the Hamilton-Jacobi equation and its modern outspings, related to algebraic integrability. In this respect, several questions naturally arise, namely,

1. Can the degeneration property of the separation relations be characterized in terms of the bi-Hamiltonian structure?

2. In this case, what can one say about the algebraicity of the separation relation?

We will further address these problems in [10]. In this paper we limit ourselves to give an example related to loop algebras, where all these features are present.

9  An example related to \( \mathfrak{sl}(3) \)

Applications of the scheme we have presented in this paper have already appeared in the literature. Namely, in [13] a preliminary picture of these ideas has been applied to the \( t_5 \)-stationary reduction of the Boussinesq hierarchy. Subsequently, in [11] we have shown how to frame all stationary reductions of the KdV theory inside this picture, and in [12] the classical \( A_n \)-Toda lattices have been considered (see also [36] for the Neumann system, and [8]). In this
final section we will illustrate how our theoretical scheme concretely works in an example, which is related to the $t_5$–stationary Boussinesq system, in the sense that the latter can be obtained via a reduction from the one we will present. Even if, for the sake of brevity, we will stick to such a particular example, we claim that the same arguments hold for a wide class of integrable systems on finite-dimensional orbits of loop algebras, studied, e.g., in [1]. In these cases, the DN (separation) coordinates turn out to be the so-called spectral Darboux coordinates [1, 2].

The system we are going to study is defined on the space $\mathfrak{sl}(3) \times \mathfrak{sl}(3)$ of pairs $(X_0, X_1)$ of $3 \times 3$ traceless matrices. The cotangent (and the tangent) space at a point is identified with the manifold itself via the pairing

$$\langle (V_0, V_1), (W_0, W_1) \rangle = \text{tr} (V_0 W_0 + V_1 W_1),$$

so that the differential of a scalar function $F$ is represented by a pair of matrices,

$$dF = \left( \frac{\partial F}{\partial X_0}, \frac{\partial F}{\partial X_1} \right).$$

We introduce the two compatible Poisson tensors defined, at the point $(X_0, X_1)$, by

$$P : \begin{bmatrix} V_0 \\ V_1 \end{bmatrix} \mapsto \begin{bmatrix} [X_1, V_0] + [A, V_1] \\ [A, V_0] \end{bmatrix},$$

$$P' : \begin{bmatrix} V_0 \\ V_1 \end{bmatrix} \mapsto \begin{bmatrix} -[X_0, V_0] \\ [A, V_1] \end{bmatrix},$$

where

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

One can easily see that the functions

$$C_1(X_0, X_1) = \text{tr} (AX_1) = (X_1)_{12} + (X_1)_{23}$$

$$C_2(X_0, X_1) = \text{tr} (A^2 X_1) = (X_1)_{13}$$
are common Casimirs of $P$ and $P'$. Thus the bi-Hamiltonian structure can be trivially restricted to

$$M = \{(X_0, X_1) \in \mathfrak{sl}(3) \times \mathfrak{sl}(3) \mid (X_1)_{12} + (X_1)_{23} = 0, (X_1)_{13} = 1\},$$

which is the 14-dimensional manifold where our GZ system will be defined. Indeed, it can be directly shown (see also [37]) that, if

$$L(\lambda) = \lambda^2 A + \lambda X_1 + X_0,$$

then

$$H^{(1)} = \frac{1}{2} \text{tr} L(\lambda)^2 \quad \text{and} \quad H^{(2)} = \frac{1}{3} \text{tr} L(\lambda)^3 \quad (9.1)$$

are Casimir functions of the Poisson pencil $P_\lambda = P' - \lambda P$. One finds that

$$H^{(1)} = \lambda^3 + H_0^{(1)} \lambda^2 + H_1^{(1)} \lambda + H_2^{(1)}$$

$$H^{(2)} = \lambda^5 + H_0^{(2)} \lambda^4 + H_1^{(2)} \lambda^3 + H_2^{(2)} \lambda^2 + H_3^{(2)} \lambda + H_4^{(2)}, \quad (9.2)$$

where

$$H_0^{(1)} = \text{tr} (AX_0 + \frac{1}{2} X_1^2) \quad H_1^{(1)} = \text{tr} (X_0 X_1) \quad H_2^{(1)} = \frac{1}{2} \text{tr} X_0^2$$

$$H_0^{(2)} = \text{tr} (A^2 X_0 + AX_1^2) \quad H_1^{(2)} = \text{tr} \left( \frac{1}{2} X_1^3 + AX_0 X_1 + AX_1 X_0 \right)$$

$$H_2^{(2)} = \text{tr} (X_1^2 X_0 + AX_0^2) \quad H_3^{(2)} = \text{tr} (X_1 X_0^2) \quad H_4^{(2)} = \frac{1}{3} \text{tr} X_0^3.$$

Obviously, $H_0^{(1)}$ and $H_0^{(2)}$ are Casimirs of $P$, whereas $H_2^{(1)}$ and $H_4^{(2)}$ are Casimirs of $P'$. Since the differentials of the functions $H_i^{(a)}$ are linearly independent on a dense open subset of $M$, and the corank of $P$ and $P'$ is 2, we can conclude that the hypotheses of Section 7 are verified, with $k = 2$, $n_1 = 2$, and $n_2 = 4$. The GZ system on $M$ is given by the 6 bi-Hamiltonian vector fields associated with the coefficients of the Casimirs (9.2). The first vector fields of the two bi-Hamiltonian hierarchy are, respectively,

$$X_1^{(1)} = ([A, X_0], [A, X_1]), \quad X_1^{(2)} = ([A^2, X_0], [A^2, X_1]). \quad (9.3)$$

Let us fix a symplectic leaf $S$ of $P$, defined by the constraints $H_0^{(1)} = c_1$, $H_0^{(2)} = c_2$. According to Proposition 7.1, the 6 remaining Hamiltonians define a Lagrangian foliation, called the GZ foliation, on the 12-dimensional symplectic manifold $S$. The results of Section 7 entail that, in order to separate the GZ system, we need a distribution which is transversal to the symplectic leaves.
of $P$. More precisely, let $\{\cdot, \cdot\}$ and $\{\cdot, \cdot\}'$ be the Poisson brackets associated with $P$ and $P'$. Then we must find a pair of vector fields $(Z_1, Z_2)$, spanning a 2-dimensional integrable distribution on $M$, such that

$$Z_a(H_0^{(b)}) = \delta_a^b$$

(9.4)

and such that the functions invariant along the distribution form a Poisson subalgebra with respect to both $\{\cdot, \cdot\}$ and $\{\cdot, \cdot\}'$. It is not difficult to show that these requirements are fulfilled by

$$Z_1 : \begin{align*} \dot{X}_0 &= E_{23} , \quad \dot{X}_1 = 0 \\
\end{align*}$$

$$Z_2 : \begin{align*} \dot{X}_0 &= E_{13} , \quad \dot{X}_1 = 0 \\
\end{align*}$$

where $E_{ij}$ is the matrix with 1 in the $(i, j)$ entry and 0 elsewhere. In fact, a function $F \in C^\infty(M)$ is invariant with respect to both $Z_1$ and $Z_2$ if and only if

$$\left( \frac{\partial F}{\partial X_0} \right)_{31} = \left( \frac{\partial F}{\partial X_0} \right)_{32} = 0 ,$$

(9.5)

and such functions form a Poisson subalgebra, because the matrices fulfilling (9.3) are a Lie subalgebra of $\mathfrak{sl}(3)$. Moreover, $Z_1(H_0^{(1)}) = \text{tr}(AE_{23}) = 1$ and $Z_1(H_0^{(2)}) = \text{tr}(A^2E_{23}) = 0$, and the analogous equations for $Z_2$ hold, so that we have the normalization (9.4). Then Proposition 7.2 implies that

$$L_{Z_a} P = 0 , \quad L_{Z_a} P' = \sum_{b=1}^2 Y_a^b \wedge Z_b ,$$

with $Y_a^b = [Z_a, X_1^{(b)}]$. The vector field (see Remark 7.6)

$$Y = Y_1^1 + Y_2^2 = [Z_1, X_1^{(1)}] + [Z_2, X_1^{(2)}]$$

is given, on account of (9.3), by

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Hence, the symplectic leaf $S$ has an $\omega N$ structure and Theorem 7.4 tells us that the above-defined GZ foliation is separable in DN coordinates.

Now we will use the results of Section 8 and 9 to discuss the Stäckel separability and the separation relations of the GZ basis. Indeed, $H^{(1)}(\lambda)$ and $H^{(2)}(\lambda)$ are easily seen to be affine with respect to the transversal vector fields,

$$Z_a(Z_b(H^{(1)}(\lambda))) = Z_a(Z_b(H^{(2)}(\lambda))) = 0 \quad \text{for all } a, b = 1, 2,$$
meaning that the GZ basis is Stäckel separable.

A set of special DN coordinates \((\lambda_i, \mu_i)_{i=1,...,6}\) on \(S\) is determined as follows. We write the polynomial matrix

\[
F(\lambda) = \begin{bmatrix}
Z_1(H^{(1)}(\lambda)) & Z_2(H^{(1)}(\lambda)) \\
Z_1(H^{(2)}(\lambda)) & Z_2(H^{(2)}(\lambda))
\end{bmatrix} = \begin{bmatrix}
L(\lambda)_{32} & L(\lambda)_{31} \\
(L(\lambda)^2)_{32} & (L(\lambda)^2)_{31}
\end{bmatrix},
\]

whose determinant gives the minimal polynomial of the recursion operator \(N\) of \(S\):

\[
\det F(\lambda) = \lambda^6 - \sum_{i=1}^{6} c_i \lambda^{6-i} . \tag{9.6}
\]

Its roots are the eigenvalues \((\lambda_1, \ldots, \lambda_6)\) of \(N\). The \(\mu_i\) coordinates can be found with the strategy described in Section 5, which consists in looking for a Stäckel function generator \(f(\lambda)\) such that \(Y(f(\lambda)) = 1\). In Section 8 we saw that the normalized rows of the adjoint matrix \(F^\vee(\lambda)\) of \(F(\lambda)\) provide Stäckel function generators. We have

\[
F^\vee(\lambda) = \begin{bmatrix}
(L(\lambda)^2)_{31} & -L(\lambda)_{31} \\
-(L(\lambda)^2)_{32} & L(\lambda)_{32}
\end{bmatrix},
\]

so that \(f(\lambda) := -(L(\lambda)^2)_{31}/L(\lambda)_{31}\) is a Stäckel function generator. Since

\[
Y((L(\lambda)^2)_{31}) = -L(\lambda)_{31} \text{ and } Y(L(\lambda)_{31}) = 0 ,
\]

we obtain \(Y(f(\lambda)) = 1\), and therefore

\[
\mu_i = f(\lambda_i) = -(L(\lambda_i)^2)_{31}/L(\lambda_i)_{31} , \quad i = 1, \ldots, 6 , \tag{9.7}
\]

form with the \(\lambda_i\) a set of special DN coordinates.

At this point we could, in principle, use (9.6) and (9.7) to explicitly write the point \((X_0, X_1)\) of \(S\) in terms of \((\lambda_i, \mu_i)_{i=1,...,6}\), and we could compute the functions \(\Phi_i\) in (8.18) in order to obtain the separation relations for the GZ basis:

\[
\rho_1(\lambda_i)H^{(1)}(\lambda_i) + \rho_2(\lambda_i)H^{(2)}(\lambda_i) = \Phi_i(\lambda_i, \mu_i) ,
\]

with \(\rho_1(\lambda) = f(\lambda)\) and \(\rho_2(\lambda) = 1\). Thus we have

\[
\mu_iH^{(1)}(\lambda_i) + H^{(2)}(\lambda_i) = \Phi_i(\lambda_i, \mu_i) . \tag{9.8}
\]
However, we can directly show that these separation relations coincide with the ones given by the spectral curves, i.e.,

$$\det(\mu I - L(\lambda)) = 0.$$ 

Since \(\det(\mu I - L(\lambda)) = \mu^3 - \frac{1}{2}\text{tr} \ (L(\lambda)^2)\mu - \frac{1}{3}\text{tr} \ (L(\lambda)^3)\), the points \((\lambda_i, \mu_i)_{i=1,...,6}\) given by (9.6) and (9.7) belong to the spectral curve if and only if

$$- \left( \frac{L(\lambda_i)^2}{L(\lambda_i)^3} \right)^3 + \frac{1}{2}\text{tr} \ (L(\lambda_i)^2) \left( \frac{L(\lambda_i)^2}{L(\lambda_i)^3} \right) - \frac{1}{3}\text{tr} \ (L(\lambda_i)^3) = 0 \quad (9.9)$$

for all \(\lambda_i\) such that

$$L(\lambda_i)^3 - L(\lambda_i)^3 - L(\lambda_i)^3 = 0. \quad (9.10)$$

Since it can be checked that equation (9.9) holds for every traceless \(3 \times 3\) matrix fulfilling (9.10), we have indeed shown that the separation relations (9.8) are given by the spectral curve.

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