Some Multiplicity Results to the Existence of Three Solutions for a Dirichlet Boundary Value Problem Involving the $p$-Laplacian

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Abstract. In this paper we prove the existence of two intervals of positive real parameters $\lambda$ for a Dirichlet boundary value problem involving the $p$-Laplacian which admit three weak solutions, whose norms are uniformly bounded with respect to $\lambda$ belonging to one of the two intervals. Our main tool is a three critical points theorem due to G. Bonanno [A critical points theorem and nonlinear differential problems, J. Global Optim., 28:249–258, 2004].

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1 Introduction

The purpose of this paper is to establish the existence of two intervals of positive real parameters $\lambda$ for which the problem

$$
\begin{align*}
\Delta_p u + \lambda f(x,u) &= 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega
\end{align*}
$$

(1.1)

where $\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)$ is the $p$-Laplacian operator, $\Omega \subset R^N$ ($N \geq 1$) is a non-empty bounded open set with smooth boundary $\partial \Omega$, $p > N$, $\lambda$ is a positive parameter and $f : \Omega \times R \rightarrow R$ is an $L^1$- Carathéodory function,

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admits three weak solutions, whose norms are uniformly bounded in respect to \( \lambda \) belonging to one of the two intervals.

We recall that a function \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) is said to be \( L^1 \)-Carathéodory if

1. \( x \to f(x, t) \) is measurable for every \( t \in \mathbb{R} \);
2. \( t \to f(x, t) \) is continuous for almost every \( x \in \Omega \);
3. for every \( \varrho > 0 \) there exists a function \( l_\varrho \in L^1(\Omega) \) such that
   \[
   \sup_{|t| \leq \varrho} |f(x, t)| \leq l_\varrho(x)
   \]
   for almost every \( x \in \Omega \).

We say that \( u \) is a weak solution to the problem (1.1) if \( u \in W^{1,p}_0(\Omega) \) and

\[
\int_\Omega |\nabla u(x)|^{p-2} \nabla u(x) \nabla v(x) \, dx - \lambda \int_\Omega f(x, u(x)) v(x) \, dx = 0
\]

for every \( v \in W^{1,p}_0(\Omega) \).

In recent years, many publications [1, 7, 8, 9, 10, 11, 12, 14] have appeared about elliptic problems with Dirichlet boundary conditions which have been used in a great variety of application. For example, Ramaswamy and Shivaji in [14] established the existence of three positive solutions for classes of non-decreasing, \( p \)-sublinear functions \( f \) belonging to \( C^1([0, +\infty)) \) for a \( p \)-Laplacian version of [3], i.e., the problem

\[
\begin{cases}
-\Delta_p u = \lambda f(u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( p > 1, \lambda > 0 \) is a parameter and \( \Omega \) is a bounded domain in \( \mathbb{R}^N; N \geq 2 \) with \( \partial \Omega \) of class \( C^2 \) and connected. uniqueness of positive solutions to the problem (1.2) when \( p > 1 \) and \( f(u)/u^{p-1} \) is decreasing on \((0, +\infty)\) was obtained in Guo and Webb [11] and Drabek and Hernandez [9]. A natural question is that, whether uniqueness holds under the weaker condition than \( f(u)/u^{p-1} \) is decreasing for large \( u \). When \( \Omega \) is a ball, Hai and Shivaji [12] showed that the answer is affirmative. However, the approach used in [12] depends on ordinary differential equations techniques and cannot be applied to the case of a general domain. In [7], Ricceri’s three critical points theorem [15] has been successfully used to obtain existence of at least three weak solutions to the problem (1.1) in \( W^{1,p}_0(\Omega) \). In [1], based on Ricceri’s three critical points theorem [15] we obtained the existence of an interval \( \Lambda \subseteq [0, +\infty] \) and a positive real number \( q \) such that for each \( \lambda \in \Lambda \) problem

\[
\begin{cases}
\Delta_p u + \lambda f(x, u) = a(x)|u|^{p-2}u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( \Omega \subset \mathbb{R}^N (N \geq 2) \) is non-empty bounded open set with smooth boundary \( \partial \Omega \), \( p > N, \lambda > 0, f : \Omega \times \mathbb{R} \to \mathbb{R} \) is a continuous function and positive weight function \( a(x) \in C(\overline{\Omega}) \), admits at least three weak solutions whose norms in

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\[ W^{1, p}_0(\Omega) \] are less than \( q \) that we extended the main result of [4] by using of the results of [7] to the general case. In [8], the authors employing Ricceri’s three critical points theorem [16] obtained multiple weak solutions for the following BVP

\[
\begin{aligned}
-\Delta_p u &= \lambda f(x, u) + \mu g(x, u) & \text{in } \Omega, \\
 u &= 0 & \text{on } \partial \Omega,
\end{aligned}
\]

where \( \Omega \subset \mathbb{R}^N \) is a non-empty bounded open set with smooth boundary \( \partial \Omega \), \( p > N, f, g : \Omega \times \mathbb{R} \to \mathbb{R} \) are two Carathéodory functions and \( \lambda, \mu \) are two positive parameters.

Bonanno in [6] established the existence of two intervals of positive real parameters \( \lambda \) for which the functional \( \Phi + \lambda \Psi \) has three critical points, whose norms are uniformly bounded with respect to \( \lambda \) belonging to one of the two intervals. He illustrated the result for a two point boundary value problem, and here we are interested to illustrate this result to the problem (1.1). Our main result is Theorem 1 that ensures the existence of two intervals \( \Lambda_1^{'} \) and \( \Lambda_2^{'} \) such that, for each \( \lambda \in \Lambda_1^{'} \cup \Lambda_2^{'} \), the problem (1.1) admits at least three weak solutions whose norms are uniformly bounded with respect to \( \lambda \in \Lambda_2^{'} \). The technique used in our proof has been introduced in [7].

As an immediate consequences of Theorem 1, we obtain Corollary 1, in which the function \( f \) has separated variables. The applicability of the result is illustrated by Example 1. Finally, we present the application of Theorem 1 in the ordinary case with \( p = 2 \), that Example 2 illustrates the result.

2 Main Results

First we recall for the reader’s convenience Theorem 3.1 of [6] (see also [2, 5, 13, 15, 16] for related results) to transfer the existence of three solutions of the problem (1.1) into the existence of critical points of the Euler functional:

**Theorem A ([6, Theorem 3.1])** Let \( X \) be a separable and reflexive real Banach space; \( \Phi : X \to \mathbb{R} \) a nonnegative continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on \( X^* \); \( J : X \to \mathbb{R} \) a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that there exists \( x_0 \in X \) such that \( \Phi(x_0) = J(x_0) = 0 \) and that

(i) \( \lim_{\|x\| \to +\infty} (\Phi(x) - \lambda J(x)) = +\infty \) for all \( \lambda \in [0, +\infty[ \).

Further, assume that there are \( r > 0, x_1 \in X \) such that:

(ii) \( r < \Phi(x_1) \),

(iii) \( \sup_{x \in \Phi^{-1}([-\infty, r])^w} J(x) < \frac{r}{r + \Phi(x_1)} J(x_1) \).

Then, for each

\[
\lambda \in \Lambda_1' = \left[ \frac{\Phi(x_1)}{J(x_1) - \sup_{x \in \Phi^{-1}([-\infty, r])^w} J(x)} \right] \cdot \left[ \frac{r}{\sup_{x \in \Phi^{-1}([-\infty, r])^w} J(x)} \right],
\]

we have at least three weak solutions of the problem (1.1).
the equation \( \Phi'(u) - \lambda J'(u) = 0 \) has at least three solutions in \( X \) and, moreover, for each \( h > 1 \), there exist an open interval

\[
\Lambda_2 \subseteq \left[ 0, hr/(rJ(x_1)/\Phi(x_1) - \sup_{x \in \Phi^{-1}(-\infty,r]} J(x)) \right]
\]

and a positive real number \( \sigma \) such that, for each \( \lambda \in \Lambda_2 \), the equation given above has at least three solutions in \( X \) whose norms are less than \( \sigma \).

Here and in the sequel, \( X \) will denote the Sobolev space \( W^{1,p}_0(\Omega) \) with the norm

\[
\|u\| = \left( \int_{\Omega} |\nabla u(x)|^p \, dx \right)^{1/p}.
\]

Put \( F(x,t) = \int_0^t f(x,\xi) \, d\xi \) for each \((x,t) \in \Omega \times R\), and

\[
c = \sup_{u \in X \setminus \{0\}} \frac{\max_{x \in \overline{\Omega}} |u(x)|}{\|u\|}.
\]

Since \( p > N \), \( X \) is compactly embedded in \( C^0(\overline{\Omega}) \), one has \( c < +\infty \). In addition, it is known [18, formula (6b)] that

\[
c \leq \frac{N^{-1/p}}{\sqrt{\pi}} \left[ \Gamma \left( 1 + \frac{N}{2} \right) \right]^{1/N} \left( \frac{p-1}{p-N} \right)^{1-1/p} [m(\Omega)]^{1/N-1/p},
\]

where \( \Gamma \) denotes the Gamma function and \( m(\Omega) \) is the Lebesgue measure of the set \( \Omega \), and equality occurs when \( \Omega \) is a ball.

Now, fix \( x^0 \in \Omega \) and pick \( r_1, r_2 \) with \( 0 < r_1 < r_2 \) such that

\[
S(x^0, r_1) \subset S(x^0, r_2) \subseteq \Omega
\]

where \( S(x^0, r_i) \) denotes the ball with center at \( x^0 \) and radius of \( r_i \) for \( i = 1, 2 \).

Put

\[
k_i = k_1(N,p,r_1, r_2) = \frac{c}{r_2 - r_1} \left( \frac{r_2^N - r_1^N}{\pi^{N/2}} \right)^{1/p} \left( 1 + \frac{N}{2} \right)^{1/N}.
\]

We formulate our main result as follows:

**Theorem 1.** Let \( f : \Omega \times R \to R \) be an \( L^1 \)-Carathéodory function, and denote \( F(x,t) = \int_0^t f(x,\xi) \, d\xi \) for each \((x,t) \in \Omega \times R\). Assume that there exist three positive constants \( \theta, \tau \) and \( \gamma \) with \( k_1 \tau > \theta, \gamma < p \) and a function \( \mu \in L^1(\Omega) \) such that

\[
\begin{align*}
(\alpha_1) \ & F(x,t) \geq 0 \text{ for each } (x,t) \in (\Omega \setminus S(x^0, r_1)) \times [0, \tau], \\
(\alpha_2) \ & \int_{\Omega} \sup_{t \in [-\theta,\theta]} F(x,t) \, dx < \frac{1}{2} \left( \frac{\theta}{k_1 \tau} \right)^p \int_{S(x^0, r_1)} F(x, \tau) \, dx, \\
(\alpha_3) \ & F(x,t) \leq \mu(x)(1 + |t|^\gamma) \text{ for almost every } x \in \Omega \text{ and for all } t \in R,
\end{align*}
\]

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where $k_1$ is given in (2.1). Then, for each

$$\lambda \in A'_1 = \left[ \frac{\frac{1}{p} \left( \frac{\theta_1}{c} \right)^p}{\int_{S(x^0, r_1)} F(x, \tau) \, dx - \int_{\Omega} \sup_{t \in [-\theta, \theta]} F(x, t) \, dx}, \frac{\frac{1}{p} \left( \frac{\theta_1}{c} \right)^p}{\int_{\Omega} \sup_{t \in [-\theta, \theta]} F(x, t) \, dx} \right],$$

the problem (1.1) admits at least three weak solutions in $X$ and, moreover, for each $h > 1$, there exist an open interval

$$A'_2 \subseteq \left[ 0, \frac{\frac{h}{p} \left( \frac{\theta_1}{c} \right)^p}{\int_{S(x^0, r_1)} F(x, \tau) \, dx - \int_{\Omega} \sup_{t \in [-\theta, \theta]} F(x, t) \, dx} \right]$$

and a positive real number $\sigma$ such that, for each $\lambda \in A'_2$, the problem (1.1) admits at least three weak solutions in $X$ whose norms are less than $\sigma$.

**Proof.** In order to apply Theorem A, we begin by setting

$$\Phi(u) = \frac{\|u\|^p}{p}, \quad J(u) = \int_{\Omega} F(x, u(x)) \, dx$$

for each $u \in X$. It is well known that $J$ is a continuously Gâteaux differentiable functional whose Gâteaux derivative at the point $u \in X$ is the functional $J'(u) \in X^*$, given by

$$J'(u)(v) = \int_{\Omega} f(x, u(x))v(x) \, dx$$

for every $v \in X$. We claim that $J' : X \to X^*$ is a compact operator. To this end, it is enough to show that $J'$ is strongly continuous on $X$. For this, for fixed $u \in X$ let $u_n \to u$ weakly in $X$ as $n \to +\infty$, then we have $u_n$ converges uniformly to $u$ on $\Omega$ as $n \to +\infty$ (see [17]). Since $F(x, \cdot)$ is $C^1$ in $R$ for every $x \in \Omega$, so it is continuous in $R$ for every $x \in \Omega$, and we get that $F(x, u_n) \to F(x, u)$ strongly as $n \to +\infty$ which follows $J'(u_n) \to J'(u)$ strongly as $n \to +\infty$. Thus we proved that $J'$ is strongly continuous on $X$, which implies that $J'$ is a compact operator by Proposition 26.2 of [19]. Hence the claim is true.

Moreover, the functional $\Phi$ is a continuously Gâteaux differentiable whose Gâteaux derivative at the point $u \in X$ is the functional $\Phi'(u) \in X^*$, given by

$$\Phi'(u)(v) = \int_{\Omega} |\nabla u(x)|^{p-2}\nabla u(x) \nabla v(x) \, dx.$$ 

$\Phi'$ admits a continuous inverse on $X^*$. Indeed, owing to (2.2) of [17], for every $u, v \in X$ there exists a positive constant $c_p$ such that

$$\langle |\nabla u(x)|^{p-2}\nabla u(x) - |\nabla v(x)|^{p-2}\nabla v(x), \nabla u(x) - \nabla v(x) \rangle \geq c_p |\nabla u(x) - \nabla v(x)|^p$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $R$. So, we have

$$\langle \Phi'(u) - \Phi'(v) \rangle (u - v) \geq c_p \|u - v\|^p$$
for every \( u, v \in X \), namely \( \Phi' \) is an uniformly monotone operator in \( X \), and since \( \Phi \) is coercive and hemicontinuous in \( X \), by applying Theorem 26.A. [19], we have that \( \Phi' \) admits a continuous inverse on \( X^* \). Using again that \( \Phi' \) is monotone, we obtain that \( \Phi \) is sequentially weakly lower semi continuous (see [19, Proposition 25.20]).

Thanks to \((\alpha_3)\), for each \( \lambda > 0 \) one has that
\[
\lim_{\|u\| \to +\infty} (\Phi(u) - \lambda J(u)) = +\infty.
\]
Now, set
\[
u^*(x) = \begin{cases} 0, & x \in \Omega \setminus S(x^0, r_2) \\ \frac{\tau}{r_2 - r_1} \left[ r_2 - \sqrt{\sum_{i=1}^N (x_i - x_i^0)^2} \right], & x \in S(x^0, r_2) \setminus S(x^0, r_1) \\ \tau, & x \in S(x^0, r_1) \end{cases}
\]
and \( r = \frac{1}{p} \left( \frac{\theta}{c} \right)^p \). It is easy to see that \( u^* \in X \) and, in particular, one has
\[
\Phi(u^*) = \frac{1}{p} \left( r_2 - r_1 \right) \frac{\pi^{N/2}}{\Gamma(1 + N/2)} \left( \frac{\tau}{r_2 - r_1} \right)^p.
\]
So, since \( k_1 \tau > \theta \), we have \( \Phi(u^*) > r \). Moreover, since
\[
\sup_{x \in \Omega} |u(x)| \leq c \|u\|
\]
for each \( u \in X \), one has
\[
\sup_{u \in \Phi^{-1}(\Omega \setminus S(x^0, r_2))} J(u) = \sup_{u \in \Phi^{-1}(\Omega \setminus S(x^0, r_1))} J(u) \leq \int_{\Omega} \sup_{t \in [-\theta, \theta]} F(x, t) \, dx,
\]
and since \( 0 \leq u^*(x) \leq \tau \) for each \( x \in \Omega \), the condition \((\alpha_1)\) ensures that
\[
\int_{\Omega \setminus S(x^0, r_2)} F(x, u^*(x)) \, dx + \int_{S(x^0, r_2) \setminus S(x^0, r_1)} F(x, u^*(x)) \, dx \geq 0.
\]
Therefore, owing to our assumptions, we have
\[
\sup_{u \in \Phi^{-1}(\Omega \setminus S(x^0, r_1))} J(u) = \sup_{\|u\| \leq pr} \int_{\Omega} F(x, u(x)) \, dx \leq \int_{\Omega} \sup_{|t| \leq \theta} F(x, t) \, dx < \frac{1}{2} \left( \frac{\theta}{k_1 \tau} \right)^p \int_{S(x^0, r_1)} F(x, \tau) \, dx \leq \frac{1}{p} \left( \frac{\theta}{c} \right)^p + \frac{k_1 \tau}{p} \int_{S(x^0, r_1)} F(x, \tau) \, dx \leq \frac{r}{r + \Phi(u^*)} J(u^*).
\]
Now, we can apply Theorem A. Taking into account that
\[
\Phi(u^*) / \left( J(u^*) - \sup_{x \in \Phi^{-1}(\Omega \setminus S(x^0, r_1))} J(u^*) \right)
\]
\[
\leq \frac{1}{p} \left( \frac{k_1 \tau}{c} \right)^p \int_{S(x^0, r_1)} F(x, \tau) \, dx - \int_{\Omega} \sup_{t \in [-\theta, \theta]} F(x, t) \, dx;
\]
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\[
\sup_{u \in \Phi^{-1}([0,1])} J(u) > \frac{1}{p^2} \left( \frac{\theta}{c} \right)^p \int_{\Omega} \sup_{\theta \in [-\theta, \theta]} F(x, t) \, dx \; ;
\]

and with \( r = \frac{J(u^*) - \sup_{u \in \Phi^{-1}([0,1])} J(u)}{r \Phi(u^*)} \), we have

\[
hr \leq \left( \frac{\theta}{k_1 \tau} \right)^p \int_{S(x^0, r_1)} F(x, \tau) \, dx - \int_{\Omega} \sup_{\theta \in [-\theta, \theta]} F(x, t) \, dx = \rho;
\]

and with \( x_0 = 0, \ x_1 = u^* \), and see \( A'_1 \subset A_1, \ A_2 \subset A'_2 \), and also taking into account that the weak solutions of the problem (1.1) are exactly the solutions of the equation

\[
\Phi'(u) - \lambda J'(u) = 0,
\]

from Theorem A it follows that, for each \( \lambda \in A'_1 \), the problem (1.1) admits at least three weak solutions, and there exist an open interval \( A'_2 \subset [0, \rho] \) and a real positive number \( \sigma \) such that, for each \( \lambda \in A'_2 \), the problem (1.1) admits at least three weak solutions that whose norms in \( X \) are less than \( \sigma \). Hence, we have the conclusion. \( \square \)

**Remark 1.** In Theorem 1,

\[
\frac{1}{p^2} \left( \frac{k_1 \tau}{c} \right)^p \int_{S(x^0, r_1)} F(x, \tau) \, dx - \int_{\Omega} \sup_{\theta \in [-\theta, \theta]} F(x, t) \, dx < \frac{1}{p^2} \left( \frac{\theta}{c} \right)^p \int_{\Omega} \sup_{\theta \in [-\theta, \theta]} F(x, t) \, dx.
\]

Because, from (\( \alpha_2 \)) we have

\[
2(k_1 \tau)^p \int_{\Omega} \sup_{\theta \in [-\theta, \theta]} F(x, t) \, dx < \theta^p \int_{S(x^0, r_1)} F(x, \tau) \, dx,
\]

and since \( k_1 \tau > \theta \), we get

\[
(\theta^p + (k_1 \tau)^p) \int_{\Omega} \sup_{\theta \in [-\theta, \theta]} F(x, t) \, dx < \theta^p \int_{S(x^0, r_1)} F(x, \tau) \, dx,
\]

and so

\[
(k_1 \tau)^p \int_{\Omega} \sup_{\theta \in [-\theta, \theta]} F(x, t) \, dx < \theta^p \left( \int_{S(x^0, r_1)} F(x, \tau) \, dx - \int_{\Omega} \sup_{\theta \in [-\theta, \theta]} F(x, t) \, dx \right).
\]

Hence, multiplying by \( \frac{1}{pc^p} \) we obtain

\[
\frac{1}{p} \left( \frac{k_1 \tau}{c} \right)^p \int_{\Omega} \sup_{\theta \in [-\theta, \theta]} F(x, t) \, dx < \frac{1}{p} \left( \frac{\theta}{c} \right)^p \left( \int_{S(x^0, r_1)} F(x, \tau) \, dx - \int_{\Omega} \sup_{\theta \in [-\theta, \theta]} F(x, t) \, dx \right),
\]
which follows
\[
\frac{1}{p}\left(\frac{k_1\tau}{c}\right)^p \frac{1}{p}\left(\frac{\tilde{\gamma}}{\tilde{c}}\right)^p \int_{S(x^0, r_1)} F(x, \tau) \, dx - \int_{\Omega} \sup_{t \in [-\theta, \theta]} F(x, t) \, dx < \frac{1}{p}\left(\frac{k_1\tau}{c}\right)^p \frac{1}{p}\left(\frac{\tilde{\gamma}}{\tilde{c}}\right)^p \int_{\Omega} \sup_{t \in [-\theta, \theta]} F(x, t) \, dx.
\]

**Remark 2.** In applying Theorem 1, it is enough to know as explicit upper bound of the constant \(c\). To be precise, we can use formula (2.1) as constant \(c\) the right-hand term of the formula in page 393, so that the constant \(k_1\) in Theorem 1 is numerically well determined.

We now present a particular case of Theorem 1, in which the function \(f\) has separated variables.

**Corollary 1.** Let \(f_1 \in L^1(\Omega)\) and \(f_2 \in C(R)\) be two functions. Put \(\tilde{F}(t) = \int_0^t f_2(\xi) \, d\xi\) for all \(t \in R\), and assume that there exist four positive constants \(\theta, \tau, \eta\) and \(\gamma\) with \(k_1 \tau > \theta, \gamma < p\) such that

- \((\alpha_1')\) \(f_1(x) \geq 0\) for each \(x \in \Omega \setminus S(x^0, r_1)\) and \(f_2(t) \geq 0\) for each \(t \in [0, \tau]\),
- \((\alpha_2')\) \(\max_{t \in [-\theta, \theta]} \tilde{F}(t) (\int_{\Omega} f_1(x) \, dx) < \frac{\tilde{F}(\tau)}{2} \max_{t \leq \theta} \tilde{F}(t) (\int_{\Omega} f_1(x) \, dx),
- \((\alpha_3')\) \(|\tilde{F}(t)| \leq \eta(1 + |t|\gamma)\) for all \(t \in R\),

where \(k_1\) is given in (2.1). Then, for each

\[
\lambda \in \Lambda_1' = \left\{ \lambda : \frac{1}{p}\left(\frac{k_1\tau}{c}\right)^p \int_{S(x^0, r_1)} F(x, \tau) \, dx - \max_{t \leq \theta} \tilde{F}(t) (\int_{\Omega} f_1(x) \, dx) \right\},
\]

the problem

\[
\begin{align*}
\Delta_p u + \lambda f_1(x) f_2(u) &= 0 & \text{in } \Omega, \\
u &= 0 & \text{on } \partial\Omega,
\end{align*}
\]

admits at least three weak solutions in \(X\) and, moreover, for each \(h > 1\), there exists an open interval

\[
\Lambda_2' = \left[ 0, \frac{h}{p}\left(\frac{\tilde{\gamma}}{\tilde{c}}\right)^p \frac{1}{p}\left(\frac{k_1\tau}{c}\right)^p \int_{S(x^0, r_1)} F(x, \tau) \, dx - \max_{t \leq \theta} \tilde{F}(t) (\int_{\Omega} f_1(x) \, dx) \right]
\]

and a positive real number \(\sigma\) such that, for each \(\lambda \in \Lambda_2'\), the problem (2.2) admits at least three weak solutions in \(X\) whose norms are less than \(\sigma\).

**Proof.** Set \(f(x, u) = f_1(x) f_2(u)\) for each \((x, u) \in \Omega \times R\). Since

\[
F(x, t) = f_1(x) \tilde{F}(t),
\]

from \((\alpha_1')\) and \((\alpha_2')\) we obtain \((\alpha_1)\) and \((\alpha_2)\), respectively. From (2.3) and \((\alpha_3')\) we have

\[
F(x, t) \leq |f_1(x) \tilde{F}(t)| \leq \eta |f_1(x)| (1 + |t|\gamma)
\]

for each \((x, t) \in \Omega \times R\), so condition \((\alpha_3)\) follows with \(\mu(x) = \eta |f_1(x)|\). Then, Theorem 1 yields the conclusion. \(\square\)
Example 1. Consider the problem

\[
\begin{cases}
\text{div}(\nabla u|\nabla u) + \lambda(e^{-u}u^{10}(11 - u)) = 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( \Omega = \{(x, y) \in R^2; x^2 + y^2 < 9\} \). Taking into account \( c = \sqrt[3]{36/\pi^2} \), choosing \( x^0 = (0, 0), r_1 = 1, r_2 = 2, f_1(x) = 1 \) for all \( x \in \Omega \) and

\[ f_2(u) = e^{-u}u^{10}(11 - u) \]

for each \( u \in R \), so that \( k_1 = \sqrt{324} \), all the assumptions of Corollary 1, with \( p = 3 \), are satisfied by choosing, for instance \( \theta = 1, \tau = 3, \gamma = 2 \) and \( \eta \) sufficiently large. So for each \( \lambda \in [3\sqrt{e} - \frac{3}{\pi}, \frac{4}{102}] \), the problem (2.4) admits at least three non-trivial weak solutions in \( W_{0}^{1,3}(\Omega) \) and, moreover, for each \( h > 1 \), there exist an open interval \( \Lambda \subseteq [0, \frac{2h\theta}{729r_0 - 108}] \) and a positive real number \( \sigma \) such that, for each \( \lambda \in \Lambda \), the problem (2.4) admits at least three weak solutions in \( W_{0}^{1,3}(\Omega) \) whose norms are less than \( \sigma \).

Finally, we want to point out a simple consequence of Theorem 1 in the ordinary case with \( p = 2 \), and then we present an example of application.

For simplicity, we fix \( \Omega = (a, b) \) for \( a, b \in R \) and \( x^0 \in \Omega \). Taking into account that, in this situation, \( c = \frac{(b-a)^2}{2}, k_1 = \frac{(b-a)}{2(r_2 - r_1)} \frac{r_2}{2}, \) and \( k_2 = \frac{1}{2} \frac{r_2 - r_1}{r_1} (b-a) \frac{r_2}{2} \), we have the following result:

**Corollary 2.** Let \( f : [a, b] \times R \rightarrow R \) be a continuous function and put \( F(x, t) = \int_{a}^{t} f(x, \xi) d\xi \) for each \((x, t) \in [a, b] \times R \). Assume that there exist three positive constants \( \theta, \tau \) and \( \gamma \) with \( \frac{b-a}{2(r_2 - r_1)} \frac{r_2}{2} > \theta, \tau < 2 \) and a function \( \mu \in L^{1}([a, b]) \) such that

\[
\begin{align*}
&\text{(a'_1)} \quad F(x, t) \geq 0 \text{ for each } (x, t) \in ((a, b) \setminus (x^0 - r_1, x^0 + r_1)) \times [0, \tau], \\
&\text{(a'_2)} \quad \int_{a}^{b} \sup_{t \in [-\theta, \theta]} F(x, t) dx < \frac{r_2 - r_1}{b-a} \left( \frac{\theta}{2} \right)^2 \int_{x^0 - r_1}^{x_0 + r_1} F(x, \tau) dx, \\
&\text{(a'_3)} \quad F(x, t) \leq \mu(x)(1 + |t|^{\gamma}) \text{ for almost every } x \in (a, b) \text{ and for all } t \in R.
\end{align*}
\]

Then, for each

\[
\lambda \in \Lambda_1' = \left[ \frac{\tau^2/(r_2 - r_1)}{\int_{x^0 - r_1}^{x_0 + r_1} F(x, \tau) dx - \int_{a}^{b} \sup_{t \in [-\theta, \theta]} F(x, t) dx}, \frac{2\theta^2}{(b-a) \int_{a}^{b} \sup_{t \in [-\theta, \theta]} F(x, t) dx} \right],
\]

the problem

\[
\begin{cases}
u'' + \lambda f(x, u) = 0 & \text{in } (a, b), \\
u(a) = u(b) = 0,
\end{cases}
\]

admits at least three weak solutions in \( X \) and, moreover, for each \( h > 1 \), there exists an open interval

\[
\Lambda_2' \subseteq \left[ 0, \frac{2h\theta^2}{2(r_2 - r_1)(b-a) \int_{a}^{b} \sup_{t \in [-\theta, \theta]} F(x, t) dx} \right]
\]
and a positive real number $\sigma$ such that, for each $\lambda \in \Lambda'_2$, the problem (2.5) admits at least three classical solutions in $X$ whose norms are less than $\sigma$.

**Example 2.** Put

$$f(x, u) = e^{-(x+u)}u^6(7 - u)$$

for each $(x, u) \in (-3, 3) \times R$, and choose $x^0 = 0$, $r_1 = 1$, $r_2 = 2$. It is easy to verify that with $\theta = 1$, $\tau = 3$, $\gamma = 1$ and $\mu(x)$ for each $x \in (-3, 3)$ sufficiently large, all the assumptions of Corollary 2, are satisfied. So for each $\lambda \in \bigl[\frac{9}{2187(e^{-2} - e^{-4}) + e^{-4} - e^2}, \frac{1}{3(e^2 - e^{-4})}\bigr]$, the problem

$$\begin{cases} u'' + \lambda(e^{-(x+u)}u^6(7 - u)) = 0 & \text{in } (-3, 3), \\
 u(-3) = u(3) = 0 \end{cases}$$

(2.6)

admits at least three non-trivial classical solutions in $W^{1,2}_0([-3, 3])$ and, moreover, for each $h > 1$, there exist an open interval $\Lambda \subseteq \bigl[0, \frac{h}{32(e^{-2} - e^{-4}) - 3(e^2 - e^{-4})}\bigr]$ and a positive real number $\sigma$ such that, for each $\lambda \in \Lambda$, the problem (2.6) admits at least three classical solutions in $W^{1,2}_0([-3, 3])$ whose norms are less than $\sigma$.

**Remark 3.** The weak solutions of the problem (1.1) where $f$ is a continuous function, in the ordinary case with $\Omega = (a, b)$, $a, b \in R$ and $p = 2$, by using standard methods, belong to $C^2([a, b])$ and are classical solutions for the problem (1.1). Namely, in this case, the classical and the weak solutions of the problem (1.1) coincide.

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