Schematic interactions with many degeneracies

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Abstract

In previous works we examined the spectra for systems of 2 protons and 2 neutrons, in a single j shell calculation, by obtaining matrix elements from experiment. More recently we considered schematic interactions in the same model space. We continue in this vein here. The present work and the former can be regarded as 2 bookends on a bookshelf.

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1 Introduction

In 1963 and 1964, calculations were performed to obtain wave functions and energy levels in the $f_{7/2}$ shell by Bayman et al. [1], McCullen et al. [2] and Ginocchio and French [3]. At that time, the $T=1$ two-body matrix elements were well known but not so the $T=0$. In 1985, the $T=0$ matrix elements were better known and the calculations were repeated by Escuderos et al. [4]. The two-particle matrix elements, obtained mainly from the spectrum of $^{42}$Sc, are shown in Ref [4]. Note not only is $J=0$ $T=1$ low lying, but $J=1$ $T=0$ and $J=7$ $T=0$.

We then considered [5] a schematic interaction $E(J)=0$ for odd $J$ ($T=0$) and $E(J)=J$ for even $J$ ($T=1$). For convenience, we here change from $J$ to $J/2$ as was done in ref. [6]. We also note work by K. Neergaard ref [7], which was important in getting analytical results in ref [5]. We call this the 0123 interaction. In more detail this can be defined by the input SET1 $\{0,0,1,0,2,0,3,0\}$ Our initial motivation was to find a 2-particle interaction which yielded an equally spaced spectrum for even $I$ states of a system of 2 protons and 2 neutrons e.g. $^{44}$Ti. We were partly successful—the lowest few states were not equally spaced, but then there was a critical angular momentum beyond which we found equal spacing.

In this work we will consider another extreme. Rather than an equally spaced spectrum for high angular momentum we get a collapse with many degenerate
levels. These 2 models will then be like the bookends between which there will be less extreme cases.

As a physical motivation we note that this work puts greater emphasis on the top of the spectrum rather than the more studied at the beginning. As an interesting example compare the I=10\(^+\) and 12\(^+\) states in \(^{44}\)Ti and \(^{52}\)Fe. In the single j shell model (f\(_{7/2}\)) with a fixed interaction, the spectra of the 2 nuclei should be identical. In \(^{44}\)Ti the 12\(^+\) state is at 8039.9 keV and the 10\(^+\) state is at 7671.1 keV, i.e. 368 keV below the 12\(^+\) state. In \(^{52}\)Fe the 10\(^+\) state is at 7381.9 keV whilst the 12\(^+\) state is 423.9 keV below at 6958.8 keV. Thus the 12\(^+\) state cannot decay by quadrupole radiation and is strongly isomeric. In \(^{44}\)Ti the half-life of the 12\(^+\) state is 2.1 ns (weakly isomeric) whilst for \(^{52}\)Fe the corresponding value is 45.9 s. All this is discussed in Ref. [8].

In this work we get an exaggeration of this behavior, but hopefully we gain some insight into the workings of complex shell model results.

2 A new schematic interaction

As a counterpoint to the 0123 interaction we consider here a new schematic interaction 0122, or in more detail SET2 \(\{0,0,1,0,2,0,2,0\}\). That is, we have \(E(J) = 0\) for odd \(J\) whilst for even \(J\) \(E(J)\) is still equal to \(J/2\) for \(J \leq (2j-3)\) but \(E(2j-1) = E(2j-3)\). In the g\(_{9/2}\) shell we have the 0123 interaction and in h\(_{11/2}\) we have the 01234 interaction. We will later also consider the interaction \(\{0,0,1,0,2,0,C,0\}\). By choosing \(C=3\) we retrieve the results of ref [4]. With \(C=2\) we get the new results and with \(C\) in between we see how the changes evolve.

3 Spectra (MeV) of 2 protons and 2 neutrons in the \(f_{7/2}\)shell \(^{44}\)Ti.

We show in Table I the spectra of 2 protons and 2 neutrons in the \(f_{7/2}\) shell. First we have the MBZE interaction, in which the 2 body matrix elements were taken from experiment, more specifically from the spectrum of \(^{42}\)Sc. Then we show results for the previously considered 0123 interaction and then new results for the 0122 interaction.

| \(J\) | MBZE | 0123 | 0122 |
|-----|-----|-----|-----|
| 0   | 0.0000 | 0.0000 | 0.0000 |
| 2   | 1.1631 | 0.7552 | 0.8242 |
| 4   | 2.7900 | 1.8330 | 1.9051 |
| 6   | 4.0618 | 3.1498 | 2.1497 |
| 8   | 6.0842 | 4.6498 | 3.5221 |
| 10  | 7.3839 | 6.1498 | 4.7840 |
| 12  | 7.7022 | 7.6498 | 4.7840 |
Note that with the MBZE interaction, the gap from $I=12$ to $I=10$ is much smaller than from $I=10$ to $I=8$. With the 0123 interaction, however, from $I=6$ on we get equally spaced spectra with 1.5 MeV gaps. That is to say, the 12-10 splitting is the same as the 10-8 splitting is the same as the 8-6 splitting. This was discussed extensively in Ref [5].

We now consider the new result in the last column. In contrast to what happens with the 0123 interactions, we find with 0122 there is a collapse at the top, with $I=12^+$ and $I=10^+$ degenerate. Thus, the 2 interactions form the extremes, or two “bookends” with equal spacing at one end and collapse at the other.

We note that the excitation energy 4.7840 MeV occurs many times, and sometimes we have degenerate doublets.

To show what is happening we list all the wave functions and energies of $I=8^+$, $I=5^+$, and $I=10^+$ states for 2 protons and 2 neutrons in the $f_{7/2}$ shell with the 0122 interaction in Table II, IV, and V respectively; and in Table III we show a special state of $I=4^+$.

| $J_p$ | $J_n$ | $E=3.5221$ | $E=4.7840$ | $E=4.7840$ | $E=5.6691$ | $E=6.4942$ | $E=10.3078$ |
|------|------|----------|----------|----------|----------|----------|----------|
| 2.0  | 6.0  | 0.6486   | 0.0000   | -0.0000  | -0.6927  | 0.1421   | -0.2817  |
| 4.0  | 4.0  | 0.1652   | 0.9087   | 0.0488   | 0.0000   | 0.0000   | 0.3803   |
| 4.0  | 6.0  | 0.1594   | -0.2117  | 0.5433   | -0.1421  | -0.6927  | 0.3669   |
| 6.0  | 2.0  | 0.6486   | -0.0000  | -0.0000  | 0.6927   | -0.1421  | -0.2817  |
| 6.0  | 4.0  | 0.1594   | -0.2117  | 0.5433   | 0.1421   | 0.6927   | 0.3669   |
| 6.0  | 6.0  | -0.2840  | 0.2910   | 0.6381   | 0.0000   | -0.0000  | -0.6538  |

Table III: Special State

| $J_p$ | $J_n$ | $E=4.7840$ |
|------|------|------------|
| 0.0  | 4.0  | 0.0000     |
| 2.0  | 2.0  | 0.0000     |
| 4.0  | 6.0  | 0.0000     |
| 4.0  | 4.0  | -0.8870    |
| 4.0  | 6.0  | -1.735     |
| 6.0  | 2.0  | 0.0000     |
| 6.0  | 4.0  | -1.735     |
| 6.0  | 6.0  | 0.3912     |
Table IV: Wave functions and energies I=5⁺

| J_p J_n | E=3.2840 | E=3.8554 | E=4.4950 | E=4.7528 |
|---------|----------|----------|----------|----------|
| 2.0 4.0 | -0.4707  | 0.4151   | -0.3356  | 0.4850   |
| 2.0 6.0 | 0.5276   | 0.3704   | 0.5492   | 0.3708   |
| 4.0 2.0 | 0.4707   | -0.4151  | -0.3356  | 0.4850   |
| 4.0 4.0 | -0.0000  | 0.0000   | 0.1239   | 0.0902   |
| 4.0 6.0 | -0.0000  | 0.4364   | 0.1922   | 0.1098   |
| 6.0 2.0 | -0.5276  | -0.3704  | 0.5492   | 0.3708   |
| 6.0 4.0 | 0.0000   | -0.4364  | 0.1922   | 0.1098   |
| 6.0 6.0 | 0.0000   | 0.0000   | -0.2868  | 0.4714   |

| J_p J_n | E=5.8703 | E=6.1990 | E=6.6387 | E=9.6411 |
|---------|----------|----------|----------|----------|
| 2.0 4.0 | -0.3513  | -0.1334  | -0.1047  | -0.3257  |
| 2.0 6.0 | 0.0966   | 0.1878   | 0.1274   | 0.2905   |
| 4.0 2.0 | 0.3513   | -0.1334  | -0.1047  | 0.3257   |
| 4.0 4.0 | -0.1488  | 0.8221   | -0.5278  | 0.0000   |
| 4.0 6.0 | -0.2927  | 0.2572   | 0.5470   | 0.5563   |
| 6.0 2.0 | 0.0966   | -0.1878  | -0.1274  | 0.2905   |
| 6.0 4.0 | -0.2927  | 0.2572   | 0.5470   | -0.5563  |
| 6.0 6.0 | 0.7356   | 0.2929   | 0.2620   | 0.0000   |

Table V: Wave functions and energies I=10⁺

| J_p J_n | E=4.7840 | E=4.7840 | E=6.7840 |
|---------|----------|----------|----------|
| 4.0 6.0 | 0.6704   | -0.2249  | 0.7071   |
| 6.0 4.0 | 0.6704   | -0.2249  | -0.7071  |
| 6.0 6.0 | 0.3180   | 0.9481   | -0.0000  |

Note that the amplitude D(J_p J_n) is either plus or minus D(J_n J_p). This is due to charge symmetry of the 2-body interaction. In general, we have for the N=Z nucleus:

\[ D^{IT}(J_p J_n) = (-1)^{(I+T)} D^{IT}(J_n J_p) \]

where I is the total angular momentum and T is the isospin.

Clearly then the 5.6091 and 6.4942 MeV states are T=1 states and we will defer discussing them until later. Less obvious is that the 10.3078 state has isospin T=2. It is the double analog of the unique T=2 J=8⁺ state in ⁴⁴Ca.

### 4 Degeneracies and 2 particle fractional parentage coefficients

We now discuss this in a more systematic way. There are many degeneracies in the new interaction, they are listed in the tables IX, X, and XI for the f⁷/₂, g⁹/₂, and h¹¹/₂ shells respectively. We can correlate these with the number of
T=2 states in the last columns of tables VI, VII and VIII. These can easily be obtained from the work of Bayman and Lande ref [9].

Table VI: Occurrence of special energies: $f_{7/2}$

| $I$ | $3.2840$ | $4.7840$ | $6.7840$ | # of $T=2$ |
|-----|----------|----------|----------|-------------|
| 0   | 1        |          |          | 1           |
| 1   |          |          |          |             |
| 2   | 1        |          |          | 2           |
| 3   | 1        | 1        |          |             |
| 4   | 1        |          |          | 2           |
| 5   | 1        |          |          | 1           |
| 6   | 1        | 2        |          | 1           |
| 7   | 1        | 1        | 1        |             |
| 8   | 2        |          |          | 1           |
| 9   | 1        | 1        |          |             |
| 10  | 2        | 1        |          |             |
| 11  |          |          |          |             |
| 12  |          |          |          | 1           |

Table VII: Occurrence of Special Energies $g_{9/2}$

| $I$ | $4.3644$ | $5.8644$ | $7.3644$ | $10.3644$ | # of $T=2$ |
|-----|----------|----------|----------|-----------|-------------|
| 0   |          |          |          | 2         |             |
| 1   |          |          |          |           |             |
| 2   |          | 1        |          |           |             |
| 3   |          |          |          | 1         |             |
| 4   |          |          |          | 3         |             |
| 5   | 1        |          |          | 1         |             |
| 6   |          |          |          | 3         |             |
| 7   | 1        | 1        |          | 1         |             |
| 8   | 1        | 1        |          | 2         |             |
| 9   | 1        |          |          | 1         |             |
| 10  | 1        | 2        |          | 1         |             |
| 11  | 1        | 1        | 1        |           |             |
| 12  | 2        |          |          | 1         |             |
| 13  |          |          |          | 1         |             |
| 14  |          | 2        | 1        |           |             |
| 15  |          |          |          | 1         |             |
| 16  |          |          |          | 1         |             |
Let us now focus on the 2-fold degenerate doublet at 4.7840 MeV for I=8$^+$. We know that the non-vanishing amplitudes involve angular momenta 4 and 6. There are 4 non-vanishing amplitudes, but only 3 are independent D(44), D(46) (which is equal to D(64)) and D(66). Why this strange behavior? Since for the 0122 interaction E(4)=E(6) we see that the interaction is effectively a constant in the limited J=4 and 6 state. But there is a constraint: for these 2 states to be T=0 states, they must be orthogonal to the single T=2 state. There is one more constraint: normalization.

$$D(44)^2 + 2D(46)^2 + D(66)^2 = 1.$$  

It is easy to see that there are 2 solutions since we have one more parameter than there are constraints. Thus we have a degenerate doublet.

In contrast for I=4$^+$ we have only one special state. This is because there are 2 isospin T=2 states for I=4$^+$, thus 3 parameters and 3 constraints: only one solution.

Let us now consider odd I states and focus on I=5$^+$. We now have a quite different behavior. There is only one special state and it is at 3.2840 MeV, 1.5 MeV lower than the special I=8$^+$ states. This is because for odd I, T=0 states, D(44)=D(66)=0 so there is only one parameter D(46) to play with. We are unable to construct a state orthogonal to the lone T=2 state with these limited configurations. However, with the configurations 24 and 26 we have 2 independent parameters and 2 constraints: normalization and orthogonality to

| I  | 5.5137 | 7.0137 | 8.5137 | 10.0137 | 14.0137 | # of T=2 |
|----|--------|--------|--------|---------|---------|---------|
| 0  | 2      |        |        |         |         |         |
| 1  |        |        |        |         |         |         |
| 2  |        |        |        |         |         | 3       |
| 3  |        |        |        |         |         | 1       |
| 4  |        |        |        |         |         | 4       |
| 5  |        |        |        |         |         | 2       |
| 6  |        |        |        |         |         | 4       |
| 7  |        |        |        |         |         | 2       |
| 8  |        |        |        |         |         | 4       |
| 9  | 1      |        |        |         |         | 2       |
| 10 |        |        |        |         |         | 3       |
| 11 | 1      | 1      |        |         |         | 1       |
| 12 | 1      | 1      |        |         |         | 2       |
| 13 |        | 1      |        |         |         | 1       |
| 14 |        | 1      | 2      |         |         | 1       |
| 15 | 1      | 1      | 1      |         |         |         |
| 16 | 2      |        |        |         |         | 1       |
| 17 |        | 1      | 1      |         |         |         |
| 18 |        | 2      | 1      |         |         |         |
| 19 |        |        | 1      |         |         |         |
| 20 |        |        |        |         |         | 1       |

Table VIII: Occurrence of Special Energies $h_{11/2}$
the lone $I=5^+ T=2$ state. Thus, there is only one possible solution.

Note that all the "special sates" with $T=0$ all have the same energy: 4.784 MeV. Things are perhaps clearer if we make the lowest $I=0, T=0$ energy to be -4.784 MeV. Then all the special states have zero energy. We would get this result with a Hamiltonian $H=0$, and this is the Hamiltonian that this subclass of states sees. Note that all the $T=0$ special states are linear combinations of basis states with $(J_p,J_n)$ $(4,4)$, $(4,6+6,4)$ and $(6,6)$. The coefficients have been obtained by making them orthogonal to $T=2$ states. It can be shown that they are eigenstates of the Hamiltonian by noting one cannot add a component like say $(2,6+6,2)$ to the special wave function. This new state would not be orthogonal to a $T=2$ state and hence would be a mixture of $T=0$ and $T=2$. Some more details can be found in Rule 2 of the previous publication [5].

5 From old to new

In previously published work [5,6] with the 0123, 01234, and 012345 interactions for the $f_{7/2}$, $g_{9/2}$ and $h_{11/2}$ shells respectively, we find critical angular momentum beyond which we get equally spaced spectra. The values are respectively 6, 8, and 10 with an obvious generalization to higher shells.

| $J_p + J_n$ | E (MeV) | $I$   |
|------------|---------|-------|
| 6          | 3.15    | 3, 6  |
| 8          | 4.65    | 6, 7, 8|
| 10         | 6.15    | 3, 7, 9, 10|
| 12         | 7.65    | 10, 12|

Table IX: Special states in the $f_{7/2}$ shell (0123 interaction)

| $J_p + J_n$ | E (MeV) | $I$   |
|------------|---------|-------|
| 8          | 4.29    | 8     |
| 10         | 5.79    | 7, 9, 10|
| 12         | 7.29    | 10, 11, 12|
| 14         | 8.79    | 11, 13, 14|
| 16         | 10.29   | 14, 16|

Table X: Special states in the $g_{9/2}$ shell (01234 interaction)
Table XI: Special states in the h_{11/2} shell (012345 interaction))

| J_p + J_n | E (MeV) | I     |
|-----------|---------|-------|
| 10        | 5.46    | 10    |
| 12        | 6.96    | 11, 12|
| 14        | 8.46    | 11, 13, 14 |
| 16        | 9.96    | 14, 15, 16 |
| 18        | 11.46   | 15, 17, 18 |
| 20        | 12.96   | 18, 20 |

Tables IX, X and XI show the equally spaced spectra (1.5 MeV gaps) in the f_{7/2}, g_{9/2} and h_{11/2} shells respectively, as well as the angular momenta that belong to these states.

Let us focus on the I=12^+–I=10^+ splitting in the f_{7/2} shell. With the old interaction, SET1, this splitting is 1.5 MeV while with the current interaction, SET2, the states are degenerate. With an interaction SET3={0,0,1,0,2,0,C,0} we find the splitting is 1.5C. That is to say, the splitting is linear in C. This behavior is also found in higher shells. Specifically in the g_{9/2} shell with the interaction {0,0,1,0,2,0,3,0,C,0}, the J=16 and J=14 splitting is also proportional to C.

Some of the results can be explained by work in of Robinson and Zamick [8,9]. This pertains to states for which (J_p,J_n) are good quantum numbers, i.e. T=0 states with angular momenta which do not occur for T=2 states. In the f_{7/2} shell, these angular momenta are 3, 7, 9, 10, and 12. Let us look at I=10^+. The absence of the coupling between (4,6) and (6,6) in both SET1 and SET2 was explained in the early work [8,10,11] and is shown by the vanishing of the unitary 9j coefficient (\(\binom{7/2,7/2}{6} \binom{7/2,7/2}{6} \binom{7/2,7/2}{4}\))^{10}.

The special states of SET1 go beyond this and are characterized by having wavefunctions such that (J_p+J_n) is a constant. This is discussed in ref[5].

6 Level Inversions

Although it is somewhat out of the scope of the model we are discussing here, an interesting phenomenon is level inversion at the high end of the spectrum. One of us has previously discussed this in [7] and [14] so our discussion here will be brief. For example in ^{44}Ti, the J=10^+ state is lower in energy than J=12^+ but they are sufficiently close so that the J=12^+ state is isomeric. However, in ^{52}Fe there is an inversion with J=12^+ below J=10^+. Since no B(E2) is possible, this 12^+ state is strongly isomeric.

In the single j shell model with the same interaction these 2 nuclei would have identical spectra since one has 2 protons and 2 neutrons and the other the proton holes and 2 neutron holes. To get changes one has to use different interactions. In ref [14] table 7 we see that for ^{44}Ti the J=J_{max}=7 2 body matrix element is 0.6163 MeV above the J=0 2 body matrix element, whereas for ^{52}Fe it is 0.1999 MeV. Lowering J=J_{max} helps to create level inversion.
Indeed, investigating the spectrum of the 2 hole system $^{54}\text{Co}$, one sees that this splitting is smaller than in $^{44}\text{Ti}$.

Other nuclei are also considered, e.g. the $^{44}\text{Sc}$ and $^{52}\text{Mn}$; and in the g9/2 shell $^{96}\text{Cd}$ and $^{96}\text{Ag}$

We now briefly consider how the splitting $V(6)-V(4)$ for the 2 particle system affects the splitting $E(12)-E(10)$ for the 4 particle system. With the interaction INTa relevant for the $^{44}\text{Ti}$ calculation we have $V(6)=3.242$ MeV and $V(4)=2.815$ MeV, hence $V(6)-V(4)=0.427$ MeV. The splitting in $^{44}\text{Ti}$ is $E(12)-E(10)=0.282$ MeV. The corresponding numbers for the INTb interaction relevant to $^{52}\text{Fe}$ are $2.960$ MeV, $2.645$ MeV, $0.325$ MeV and $E(12)-E(10)= -0.122$ MeV. If we now keep the $V(J)$ in INTb as is, except we modify $V(6)$ by making $V(6)-V(4)=0.427$. That is we assume the INTa gap. This makes $V(6)=3.072$ MeV. We now find $E(12)-E(10)=+0.038$ MeV.

Experimentally $E(12)-E(10)$ is negative so we see that lowering the gap $V(6)-V(4)$ in going from INTa to INTb is important for obtaining the spin reversal.

| Table XII: $E(12)-E(10)$ MeV Splitting |
|--------------------------------------|
| V(4)  | V(6)  | V(6)-V(4)  | E(12)-E(10) |
| INTa($^{44}\text{Ti}$) 2.815 | 3.242 | 0.427 | +0.282 |
| INTb($^{52}\text{Fe}$) 2.645 | 2.960 | 0.325 | -0.122 |
| Mod 2.645 | 3.072 | 0.427 | +0.038 |

7 Closing remarks

There are schematic interactions with many more degeneracies than what we have found here. For example we have the $J=0 \ T=1$ pairing interaction of Flowers and Edmonds [12,13] which were recently used by one of us (L.Z.) to explain the "gaps in nuclear spectra as traces of seniority changes." [15]. In [12,13] seniority is a good quantum number (as well as isospin and reduced isospin). The word "traces" in the title of [15] suggests how to properly make use of schematic interactions. In the realistic case seniority is not a good quantum number but remnants of it have some effects on the nuclear spectra. We hope the models that we have presented here will also help to cast some insight into the behaviours of complex nuclear spectra.

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