Parametrized geometric cobordism and smooth Thom stacks

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Abstract

We develop a theory of parametrized geometric cobordism by introducing smooth Thom stacks. This requires identifying and constructing a smooth representative of the Thom functor acting on vector bundles equipped with extra geometric data, leading to a geometric refinement of the the Pontrjagin-Thom construction in stacks. We demonstrate that the resulting theory generalizes the parametrized cobordism of Galatius-Madsen-Tillman-Weiss. The theory has the feature of being both versatile and general, allowing for the inclusion of families of various geometric data, such as metrics on manifolds and connections on vector bundles, as in recent work of Cohen-Galatius-Kitchloo and Ayala.

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1 Introduction and overview

Bordism was introduced by Pontrjagin in order to understand manifolds from a homotopy-theoretic point of view (see [Po59]). Thom showed that cobordism groups could be computed by means of homotopy theory, via the Thom complex construction [Th54]. Excellent surveys can be found in [Ad74][Pe68][Ru08][St68]. Being originally a geometric construction (see also [Ko93] for more on this approach), it is then natural to aim to tie back homotopic constructions with differential geometric ones.

Conner and Floyd [CP64, Section I.9] define differentiable bordism groups $D_n(X^k)$ of degree $n$ of a $k$-dimensional differentiable manifold without boundary as follows. Consider pairs $(M^n, f)$ consisting of a closed unoriented manifold $M^n$ and a differentiable map $f : M^n \to X^k$. Such a pair bords if and only if there is a compact manifold $B^{n+1}$ with $\partial B^{n+1} = M^n$ and a differentiable map $g : B^{n+1} \to X^k$, with $g|_{\partial B^{n+1}} = f$. In order for the bordism relation to be transitive, it is required further that there exist an open set $U \supset \partial B^{n+1}$ and a diffeomorphism $h : \partial B^{n+1} \times [0, 1) \to U$ with $h(x, 0) = x$ and $g(h(x, t)) = f(x)$ for all $0 \leq t < 1$ and $x \in \partial B^{n+1}$. The resulting group $D_n(X^k)$ admits a natural homomorphism to the unoriented cobordism group $D_n(X^k) \to \Omega_n(X^k)$, taking the class $[(M^n, f)]$ to the class $[(M^n, f)]$, and which is an isomorphism when $X^k$ is a differentiable manifold. The upshot of this is that one can always take the map to be differentiable but then this by itself does not give anything new.

Cobordism can be viewed as a generalized cohomology theory [At61][Mi62]. One formulation which captures geometric and topological aspects of manifolds is differential cohomology, within which developing cobordism is therefore desirable. There has been some very foundational recent work in this direction. Hopkins and Singer gave a geometric definition of differential bordism in the context of differential function spectra in [HS05, Section 4.9]. In [BSSW09] a differential extension $\mathcal{M}U$ of complex bordism $MU$ was constructed. The geometry is encoded in cycles, requires transversality, hence heavy differential topology, as well as harmonic differential characters from $\mathcal{H}Z$.

We provide a general approach to differential cobordism via stacks. We initially wanted to focus on studying differential refinement of unoriented cobordism [Th54][Wa60][Li62][Mi82]. This is the cobordism theory of all compact differentiable manifolds and, as such, is perhaps the most general cobordism theory, as highlighted by Stong [St68, Chapter VI]. From a homotopy point of view, being associated to a $BO$-structure makes it more tractable for some purposes than having an additional structure. However, our formulation via simplicial sheaves turned out to be general enough to allow for differential refinements of cobordisms with $BG$-structures in the sense of [St68][Ko93].

Our initial motivation for considering cobordism was to study differential refinements of cohomology operations. We have constructed primary and secondary differential cohomology operations in [GST16][GS15], and have utilized them in constructing spectral sequences in differential cohomology in [GS15b]. Cohomology operations are intimately related to characteristic classes and to
Thom classes of vector bundles. The way this article has evolved, made it clear that it is better to defer the treatment of characteristic classes to a forthcoming article and focus here on development of the smooth cobordism theory itself.

Thom spaces \([\text{Th}54]\) provide a convenient setting which involves a smooth structure and bundles on \(M\). Note that Hopkins and Singer \([\text{HS05, Section 4.2}]\) have defined Thom complexes geometrically in the context of differential function spectra, and Bunke \([\text{Bu12, Def 4.181}]\) has defined differential Thom classes via a slightly different approach. The Thom homomorphism has also been considered in other generalized differential cohomology theories, for instance in differential K-theory by Freed and Lott \([\text{FL10}]\).

We seek a smooth analog of a Thom space \(\text{Th}(E)\) of a vector bundle \(p : E \to B\). When \(p\) is a smooth map between manifolds, this is essentially accomplished by the usual Thom space construction, but with the crucial difference that the quotient is taken not in the ambient category of topological spaces, but rather smooth stacks. As a consequence of this, there are some subtleties which one does not encounter in spaces. For example, recall that in the category of topological spaces \(\text{Top}\), the Thom space of a rank \(n\) trivial bundle \(p : E \to B\) is equivalent to the \(n\)-fold suspension \(\Sigma^n B\). It is an important point for us that this is not the case in the category of smooth stacks. Essentially, this is because we are remembering the manifold structure on both \(\text{Th}(E)\) and \(X\), while (the \(n\)-fold) suspension is a homotopy theoretic construction. We can, nevertheless, still be able to move forward by remembering a little more of the geometry. The idea is the following. We observe that if \(p : E \to X\) is a trivial vector bundle of rank \(n\), defining the Thom stack as the quotient of the smooth disc bundle modulo the smooth sphere bundle gives an identification

\[
\text{Th}(E) \simeq D^n / \partial D^n \wedge B_+.
\]

This indicates that we should be taking a smooth analogue of the suspension in the category of smooth stacks: the quotient stack \(D^n / \partial D^n\). We establish this in Section 3.1.

These considerations seem to point toward an alternative to the usual definition of spectrum in the category of smooth stacks, in which the “smooth circle” \(D^1 / \partial D^1\) serves as a model for smooth suspension. It turns out that for stacks which are \(\mathbb{R}^1\)-invariant (i.e. usual homotopy-invariant) theories, nothing new is gained. However, more geometrically refined cohomology theories, e.g. differential cohomology, are often not homotopy invariant (see e.g. \([\text{Bu12, Sc13}]\)) and thus are able to distinguish between the simplicial and smooth spheres. In order to maintain consistency with the goal of this paper, we will not explore the full theory of such objects (which we call \(D^1 / \partial D^1\)-spectra) in full generality and save these fundamentals for development elsewhere. Here, we will only provide the definition and focus more on the particular example of differential cobordism in

\(^1\)We started studying differential cobordism as a differential cohomology theory, which led us to smooth motivic Thom spectrum with connection. Generalizing these led us to general smooth motivic Thom spectra (different \(\mathcal{F}\)’s). While working on this we learned about Madsen-Tillman spectra (as also indicated in acknowledgements) and we started working on the stacky cobordism category and then we linked our smooth motivic construction with that stack via this project.
Section 3.2

In their seminal works [MT01][MW07][GMTW09], Galatius, Madsen, Tillman, and Weiss studied the moduli space of embeddings $j : M \to \mathbb{R}^\infty$ and connected this space with the classifying space of the category of cobordisms and the infinite loop space of the Madsen-Tillman spectrum $\text{MT}(d)$, with $d$ indexing the dimension of the bordisms. The homotopy groups of the cobordism category are given a geometric interpretation in [BS14]. In [GRW10] Galatius and Randal-Williams investigated subcategories with classifying spaces homotopy equivalent to that of the GMTW category $C_\theta$ of closed smooth $(d - 1)$-manifolds and smooth $d$-dimensional cobordisms, equipped with a $\theta$-structure, proving a result similar to that of [GMTW09]. The homotopy type of the cobordism category with objects $(d - 1)$-dimensional submanifolds of a fixed background manifold $M$ is identified in [RW11].

Our goals in this paper are the following.

1. First, to develop a smooth refinement of the Madsen-Tillman spectrum in stacks, which carries with it geometric data coming from the Thom-space construction. [MT01][GMTW09].

2. We will then discuss a refined classifying space construction for a smooth category which takes place entirely in the context of smooth stacks. We apply this construction to the smooth cobordism category defined in [GMTW09].

3. Finally, we will describe a refinement of the Pontrjagin-Thom construction in this setting and prove an equivalence of two smooth stacks which geometrically realizes to the equivalence proved in [GMTW09]. Thus, our construction here can be regarded as a generalization of the main theorem in [GMTW09].

4. We will also show how our construction can be used to include connections and geometric data, thus making contact with the setting of Cohen, Galatius and Kitchloo [CGK09] and Ayala [Ay08].

We will work in the context of smooth $\infty$-bundles, as presented in [NSS15a][NSS15b]. While it might seem overly abstract at first sight, this theory has the desirable advantage of being both conceptually simple and unifying, as it recovers various disconnected concepts in the 1-categorical setting as being manifested by one single concept in the $\infty$-category context [Lu09a]. However, we will aim to keep abstraction to a minimum, just enough to allows us to present our constructions and results. Furthermore, the $\infty$ setting is particularly efficient in keeping track of automorphisms, such as diffeomorphisms or gauge transformations, in a systematic way (see for instance [FSS15] for an illustration). Techniques from $\infty$-categories have been used recently in [FSV15] to describe higher extensions of diffeomorphism groups as group stacks of automorphisms of manifolds, equipped with certain topological structures. Stacks have also been used in the cobordism context earlier. For instance, in [EG11] the classical construction of Pontrjagin-Thom maps is extended to the category of differentiable local quotient stacks and used to detect torsion in the homology of the moduli stack of stable curves.

We present constructions and machinery in stacks in Section 2. Most of these have been studied extensively, and from various points of view (see, for instance, [Lu09a][DH10][Ja15][Sc13]).
Although these constructions may seem overly abstract to the reader who is not well versed in abstract homotopy theory, we will see that wading through this level of abstraction has many advantages. Indeed, one of the points of this paper is to show that the classical theorem of [GMTW09] is fairly systematic to prove by making use of these techniques. In particular, one does not need the rather complicated transversality theorems used there. These differential topology techniques are even more particularly pronounced in other extensions of the cobordism category, for instance to manifolds with corners [Ge12].

Note that whenever we speak of a smooth stack we will mean a simplicial presheaf on the site of Cartesian spaces with smooth functions between them, i.e., an object in (what we called) the category of smooth stacks $\mathcal{S}h_{\mathcal{C}}(\text{CartSp})$ (see Definition 2.4). This might be a possible source of confusion, as a smooth stack is usually thought of as a sheaf of 1-categories on a fixed smooth manifold $M$. The reason we have chosen this name in favor of the alternative, stems from the fact that our “stacks” are in fact generalizations of the above familiar 1-categorical concept to $(\infty,1)$-categories. Given a 1-truncated stack (in our sense), if one restricts to covers by convex subsets of a smooth manifold $M \in \mathcal{S}h_{\mathcal{C}}(\text{CartSp})$, one recovers the usual notion of a smooth stack on $M$. We have also chosen stack, instead of sheaf, since our objects do not satisfy the strict gluing condition, but only satisfy descent in general.

In Section 2 we will first review some of the theory and techniques from derived geometry and then use these to study bundles and their classifications in stacks. To help dissolve some of these conceptual difficulties, we provide some intuition as to how to bridge the gap between familiar smooth objects and smooth stacks. Let us consider the familiar example of the category of smooth manifolds $\text{Man}$. It is well known that this category is very poorly behaved under category theoretic constructions (see [St11] for a detailed discussion). For example, pullbacks and pushouts of smooth maps between manifolds are very seldom smooth manifolds themselves. However, there is a fully faithful embedding $\text{Man} \hookrightarrow \mathcal{S}h_{\mathcal{C}}(\text{CartSp})$ into the category of smooth sheaves. In contrast, the latter category is very well-behaved under these constructions; in fact, it is complete and cocomplete (i.e. has all limits and colimits). Thus we see that, by identifying the category of manifolds with a subcategory of sheaves, we can perform categorical constructions and identify the results with a *smooth sheaf*. It is worth emphasizing that the fact that the category of smooth manifolds embeds fully faithfully into sheaves means that any construction which uses only smooth manifolds and maps between these holds equally well in the category of smooth sheaves.

The category of smooth stacks $\mathcal{S}h_{\mathcal{C}}(\text{CartSp})$ is a higher categorical analogue of the category of smooth sheaves. Thus, we can view smooth sheaves as being a subcategory of smooth stacks in the same way that sets, viewed as discrete topological spaces, are a subcategory of topological spaces. So we see that smooth manifolds can really be thought of as certain “discrete” smooth stacks. At first glance this might be confusing since, as a space, a smooth manifold is far from being discrete and it may have highly nontrivial higher homotopy groups. Here we come to an interesting

\[ \pi_i(M) = \pi_i(M/\Gamma) \text{ for } i \geq 2 \text{ while } \pi_1(M/\Gamma) \hookrightarrow \pi_1(M) \text{ is an injection. Indeed, with } \Gamma \text{ viewed as a discrete space, the} \]

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2 Note that this is analogous to the classical situation of a free action of a finite group $\Gamma$ on a space $M$, where $\pi_i(M) = \pi_i(M/\Gamma)$ for $i \geq 2$ while $\pi_1(M/\Gamma) \hookrightarrow \pi_1(M)$ is an injection. Indeed, with $\Gamma$ viewed as a discrete space, the
phenomenon that occurs in smooth stacks. The higher homotopy data is absorbed by the geometry of the manifold. More precisely, smooth stacks have a way of separating out the homotopy theoretic data arising via the smooth structure from that which came from more combinatorial constructions. The smooth data is absorbed by the simplicially discrete sheaves, while the combinatorial data is encoded in the simplicial homotopy groups of the stack. So, in a sense that can be made precise, a smooth stack is a level-wise smooth object whose levels are glued together according to simplicial rules. As an example of a smooth stack with nontrivial combinatorial data, the smooth stack $B^{n-1}U(1)$, which is a smooth version of the Eilenberg-MacLane space $K(\mathbb{Z}, n)$, has nontrivial simplicial homotopy group in degree $n$ (see [FSS12] [SSS12]).

The Madsen-Tillman construction [MT01] involves an interplay between the tangent bundle $TM$ and the normal bundle $T^\perp M$ of an embedding $j : M \to \mathbb{R}^{d+N}$ of a manifold $M$ (of dimension $d$) into a Euclidean spaces of large dimension $(N \to \infty)$. The two bundles combine together to provide a decomposition of vector bundles $TM \oplus T^\perp M \cong j^*T\mathbb{R}^{d+N}$, exhibiting the two original bundles as bundles of vector spaces which are orthogonal complements in a large vector space. For an abstract manifold, these relationships implicitly involve a number of choices. In particular, we have a cover of $M$ by coordinate patches and a trivialization of both bundles over the local patches. Smooth stacks provide a way for making all this information manifest. At first glance, keeping track of such data might seem to be irrelevant, if not pedantic. Nevertheless, these choices turn out to be crucial in the Pontrjagin-Thom construction in smooth stacks. The reason essentially boils down to the following observation. If one is given a submanifold $M \subset \mathbb{R}^{d+N}$ of Euclidean space, there is a canonical map to the Grassmannian classifying the tangent bundle. This map has nothing to do with a choice of cover: it simply maps a point in the manifold to the tangent space (viewed as a subspace of $\mathbb{R}^{d+N}$). If we view $M$ as an abstract manifold, there is no canonical choice of map and the data of local trivializing patches needs to be taken into account. The key observation in the proof of our main theorem is that if we unravel the local trivializing information, then the inverse to the Pontryagin-Thom map can be calculated locally and glued together to give a submanifold. Moreover, the local problem turns out to be nothing more than of finding a regular value of a smooth map $f : D^{N+d} \to D^N$ and witnessing the local inverse as the corresponding Pontryagin submanifold of $D^{d+N}$.

In Section 2.3 we develop the classification of vectors bundles and their complements in stacks. Hence we consider stacky generalizations of the classifying space as $BO(n)$, of the universal bundle as $\mathcal{U}(d,N)$, of its orthogonal complement as $\mathcal{U}^\perp(d,N)$, of Grassmannians as $Gr(d,N)$, and of Stiefel manifolds as $V(d,N)$. The main point will be that, in order to compute the inverse map, we need various models for the corresponding smooth manifolds in stacks. With the exception of $BO(d)$, these stacks will all be equivalent to their classical counterparts. We hope that these

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fibration $\mathcal{F} \to M_\mathcal{F} \to M$ is similar to $\Gamma \to M/\Gamma \to M$ in that the only effect of the fiber is on $\pi_0$. Therefore, a sheaf in the world of stacks is, in the above sense, the analog of a discrete space.

Note again that being discrete in spaces and in stacks have totally different meanings. See the discussion preceding footnote $2$.

Here we mean, equivalent in the category of smooth stacks.
constructions will be of use in their own right. These are summarized in Table 1, which we hope would also be useful for notation.

The techniques developed by Galatius, Madsen, Tillman and Weiss [MT01, MW07, GMTW09], which led to a proof [MW07] of the Mumford conjecture [Mu83], have seen a number of fascinating applications in recent years (see [Co10] for an exposition). Notably, their work has led to a deeper understanding of topological quantum field theories (TQFT’s), which has since been far generalized by the work of Lurie on the cobordism hypothesis [Lu09b] (see [Fr13] for an exposition). One facet, which has not been so extensively studied in the literature, is the application of their techniques to quantum field theories (QFT’s) which have additional geometric data (such as connections, Riemannian metrics, etc.). These types of QFT’s are inherently not topological, but at the same time are really the field theories which one needs to understand in physical applications. Nevertheless, there has been some work in this direction, for example Cohen, Galatius and Kitchloo [CGK09] have applied these techniques to moduli spaces of flat connections and Ayala [Ay08] has used sheaf-theoretic techniques to study some of the effects of adding other geometric data to the mix. However, the end result in both cases is an identification of the geometric realization of certain cobordism categories with a space. As such, the homotopy type of these realizations often lose a large amount of the geometric data. We plan for a future study of field theories in our current context.

Adding connections to the moduli spaces, when viewed from a topological point of view might seem like nothing has happened, as the space of connections is affine, i.e. contractible. However, we certainly would like for the connections to have a nontrivial effect. The way to improve upon this situation is to describe these categories not just as topological categories, but a smooth sheaves of topological categories, or smooth stacks. For our purposes, it will be better to work with simplicial sheaves as a model for smooth stacks, rather than topological valued sheaves and we will make the transition back to topological spaces by geometrically realizing.

As an example of how this occurs, consider the sheaf of Lie-algebra valued 1-forms $\Omega^1(\cdot; g)$. This sheaf encodes globally defined connections on $G$-principal bundles: Given a smooth manifold $M$, the sheaf condition provides us with an identification

$$\pi_0 \text{Map}(M, \Omega^1(\cdot; g)) \simeq \Omega^1(M; g).$$

This example highlights that homotopy theory of smooth stacks captures the geometric data encoded by global connections. However, under the geometric realization functor

$$|\Pi| : \text{Sh}_{\infty}(\text{CartSp}) \to \text{sSet} \to \text{Top},$$

the space $|\Pi \Omega^1(\cdot; g)|$ can be identified with the affine space of $g$-valued 1-forms on $\mathbb{R}^\infty$ (understood as the colimit of 1-forms on $\mathbb{R}^N$ as $N \to \infty$). Being a vector space, this space is contractible and

$${}^5$$We thank Ralph Cohen for very useful discussions in that direction.
| Stack                                                                 | Description                                                                                                                                 |
|----------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------|
| Classifying stack of orthogonal bundles $BO(n)$                     | Homotopy orbit stack of the smooth sheaf $O(n)$ acting on the point $d$                                                                     |
| Universal bundle $\mathcal{U}(d) \to BO(d)$                        | $\mathcal{U}(d) \simeq \mathbb{R}^d/O(d)$                                                                                                  |
| Real Grassmannian stack $\text{Gr}(d, N)$                           | $	ext{Gr}(d, N) := O(d + N)/O(d) \times O(N)$ $\simeq O(d + N)/O(d) \times O(N)$ $=: \text{Gr}(d, N)$                                        |
| Real Stiefel stack $V(d, N)$                                        | $V(d, N) := O(N)/O(N - d)$ $\simeq O(N)/O(N - d) =: V(d, N)$                                                                                     |
| Universal vector bundle $\mathcal{U}(d, N) \to \text{Gr}(d, N)$     | $(\mathbb{R}^d \times V(d, N))/O(d)$                                                                                                                                                             |
| Universal complement bundle $\mathcal{U}^\bot(d, N) \to \text{Gr}(d, N)$ | $(\mathbb{R}^d \bot V(N, d))/O(N)$                                                                                                                                                             |
| Thom stack $\text{Th}(\eta)$ of a vector bundle $\eta \to X$       | $\text{Th}(\eta) := D(\eta)/S(\eta)$ $\simeq D(\eta)/S(\eta) =: \text{Th}(\eta)$                                                      |
| Stacky sphere $D^n/\partial D^n$                                    | $D(V)/S(V)$, $\dim(V) = n$                                                                                                                   |
| Thom stack of the universal bundle $\mathcal{U}(d) \to BO(d)$       | $\text{Th}(\mathcal{U}(d)) \simeq (D^d/\partial D^d)/O(d)$                                                                               |
| Thom stack of the universal bundle $\mathcal{U}(d, N) \to \text{Gr}(d, N)$ | $\text{Th}(\mathcal{U}(d, N)) \simeq (D^d/\partial D^d \times V(d, N))/O(d)$                                                               |
| Thom stack of the universal bundle $\mathcal{U}^\bot(d, N) \to \text{Gr}(d, N)$ | $\text{Th}(\mathcal{U}^\bot(d, N)) \simeq (D^d/N/\partial D^N \times V(N, d))/O(N)$                                                         |
| Stacky Madsen-Tillman spectrum $\text{MT}(d)$                      | $\text{MT}(d) := \text{Th}(\mathcal{U}^\bot(d, N))$ $\simeq (D^d/\partial D^d \times V(N, d))/O(d)$                                              |
| Smooth motivic model for the Thom spectrum $MO$                     | $MO := \text{Th}(\mathcal{U}(d)) \simeq (D^d/\partial D^d)/O(d)$                                                                               |
| $D^1/\partial D^1$-suspension spectrum $\Sigma_{D^1/\partial D^1}^\infty X$ | Successively smashing with the stacky circle $D^1/\partial D^1$                                                                           |
| $D^1/\partial D^1$-infinite loop stack $\Omega_{D^1/\partial D_1}^\infty X(n)$ | $\Omega_{D^1/\partial D_1}^\infty X(n) := \lim_{\to_n} \text{Map}_+(D^n/\partial D^n, X(n))$                                      |
| Smooth concordance category with ordering on $[0, 1]$: $\text{Conc}^> (X)$ | Diagram on the full substack of collared and ordered maps                                                                                   |
| $\text{BConc}^>(X)$                                                 | Realization of the nerve of groupoid $\text{Conc}^> (X)$                                                                                   |
| Smooth cobordism category $\text{Cob}_d$                           | Quotient of of embeddings and diffeomorphisms                                                                                               |
| Smooth cobordism stack $\text{B Cob}_d$                            | Stacky realization of $\text{Cob}_d$                                                                                                       |
| Smooth stack of bundle maps $\text{Bun}(T^*M, \mathcal{U}(d, N))$      | Stackification of $\text{Map}(T^*M, \mathcal{U}(d, N))$ on those maps which are bundle maps                                                |
| Smooth stack of embeddings with local trivializations $\text{Emb}_{loc}(M, \mathbb{R}^{d+N-1})$ | Homotopy pullback of $\text{Bun}(T^*M, \mathcal{U}^\bot(d, N))$ and $\text{Emb}(M, \mathbb{R}^{d+N-1})$ |

Table 1: Various stacks and categories used in the paper. As we show, most of these stacks end up being zero-truncated.
therefore the space of maps \( \text{Map}(M, \Pi \Omega^1(-, \mathfrak{g})) \) is contractible. Thus, we see that the homotopy type of the realization loses the geometric information encoded by the differential forms.

Starting in Section 4.1, we refine the classifying space of the cobordism category to smooth stacks. It turns out that in the category of smooth stacks, the corresponding classifying stack is zero-truncated and is therefore equivalent to its sheaf of connected components. This will allow us to only define the abstract Pontryagin-Thom collapse map at only at the level of connected components and we can mimic the construction in spaces to produce the map.

In Section 4.2 we provide a smooth refinement of the Pontryagin-Thom construction to the category of smooth stacks. Upon geometrically realizing our stacks, we recover the usual collapse map. This will be a morphism of sheaves

\[
\text{PT} : \tilde{\pi}_0(B\text{Cob}_d) \longrightarrow \tilde{\pi}_0(B\text{Conc}^\prec(\Omega^{x-1}_{D^1/\partial D^1} \text{MT}(d)))
\]

where the source stack is the stacky analogue of the classifying space of the cobordism category defined in Section 4.1 (see Definition 4.2) and the target category is defined in Section 3.2 (see Corollary 3.18).

**Theorem 1 (Stacky Pontrjagin-Thom equivalence).** The map PT induces a weak equivalence of smooth stacks

\[
\text{PT} : B\text{Cob}_d \longrightarrow B\text{Conc}^\prec(\Omega^{x-1}_{D^1/\partial D^1} \text{MT}(d))
\]

This is Theorem 4.13 in Section 4.2. Consequently, the result we will prove in Proposition 2.19 leads to a weak equivalence \( |B\text{Cob}_d| \rightarrow B|\text{Cob}_d| \rightarrow B\text{Cob}_d \). Similarly, Proposition 3.14 gives a weak equivalence \( |B\text{Conc}^\prec(\Omega^{x-1}\text{MT}(\theta))| \rightarrow \Omega^{x-1}\text{MT}(\theta) \). By the theorem, PT defines a weak equivalence of smooth stacks and since the geometric realization functor sends weak equivalences to weak equivalences we have the following.

**Corollary 2 (Galatius-Madsen-Tillman-Weiss).** The map PT induces a weak equivalence

\[
\text{PT} : B\text{Cob}_\theta \longrightarrow \Omega^{x-1}\text{MT}(\theta)
\]

For a tangential structure \( \theta \), which is induced by a faithful representation \( \theta : G \hookrightarrow O(d) \), we have a generalization of the first theorem. The following is Theorem 5.5 in Section 5.1.

**Theorem 3 (Stacky Pontrjagin-Thom equivalence with \( \theta \)-structure).** The map

\[
\text{PT}_\theta : B\text{Cob}_\theta \longrightarrow B\text{Conc}^\prec(\Omega^{x-1}_{D^1/\partial D^1} \text{MT}(\theta))
\]

is a weak equivalence of smooth stacks.

Going beyond G-structure, we can also add more refined geometric structure in our setting (Section 5.2). We can consider, for example, the cobordism category with objects smooth manifolds.
whose tangent bundles are equipped with a connection and whose morphisms are bordisms with tangent bundles equipped with connections extending those of the bounding manifolds. We can also consider manifolds equipped with Riemannian structure, symplectic structure, complex structure and so on.

**Theorem 4 (Stacky Pontrjagin-Thom equivalence with extra geometric structure).** The map

$$\PT^F : \mathcal{B}\text{Cob}_d^F \longrightarrow \mathcal{B}\text{Conc}^>(\Omega^{x-1}_{D^1/\mathbb{R}^1}, \text{MT}(d)_F)$$

is a weak equivalence of smooth stacks.

This is Theorem 5.9 in Section 5.2. In [Ay08], the topological category of cobordisms with $F$-structure (where $F$ is a sheaf on the site of smooth manifolds with values in $\text{Top}$) is defined. This category has space of objects equivalent to the coproduct of homotopy orbit spaces $\mathcal{F}(M)/\text{Diff}(M)$ where $M$ ranges through diffeomorphism classes of manifolds of fixed dimension. Similarly, the morphisms are identified with coproducts of $\mathcal{F}(W)/\text{Diff}(W)$, where $W$ is a bordism between manifolds. Ayala proves that there is a weak homotopy equivalence

$$\mathcal{B}\text{Cob}_d^F \simeq \Omega^{x-1}\text{MT}(d)_F,$$

where $\text{MT}(d)_F$ is the spectrum which at level $n$ is given by $\text{Th}(p_F^\# \mathcal{U}^N(d, N))$ with the map $p_F : \text{Gr}(d, N)_F(\mathbb{R}^d) \to \text{Gr}(d, N)$, where $\text{Gr}(d, N)_F(\mathbb{R}^d)$ is defined analogous to our definition in smooth stacks, but in the category of spaces. Theorem 5.9 then achieves a similar goal as Ayala upon geometric realization.

We can also consider mixed situations, i.e. when we have both a tangential structure, as a $\theta$-structure, and geometric data, as an $F$-structure.

- These structures might a priori not be directly related, such as a Riemannian metric and a Spin structure. In this case we would have an action of the Spin group on the space $\text{Riem}(n) \times \Omega^1(-; \mathfrak{so}(n))$ via the surjection $\text{Spin}(n) \to \text{O}(n)$.
- We could also consider situations when the $\theta$-structure and the $F$-structure are required to be compatible. For instance, we could consider a Spin structure together with a compatible connection.

In general, we describe such combinations via $(\theta, F)$-structures, in which case we have yet another generalization.

**Theorem 5 (Stacky Pontrjagin-Thom equivalence with geometric and $\theta$-structures).** The map

$$\PT^F_\theta : \mathcal{B}\text{Cob}_d^F \longrightarrow \mathcal{B}\text{Conc}^>(\Omega^{x-1}_{D^1/\mathbb{R}^1}, \text{MT}(\theta)_F)$$

is a weak equivalence of smooth stacks.
In [RS16], a different notion of parametrized cobordism is introduced, where the parametrization is with respect to the \(\theta\)-structure, while ours is more about geometrically refining the cobordism category in [GMTW09].

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# 2 Vector bundles and their classification in derived geometry

## 2.1 Recollection and techniques from smooth stacks

In this section, we provide a quick review of smooth stacks and the basic properties used frequently throughout this paper. The idea of the proofs of the main theorems are more or less easy to understand, at least conceptually, without having to fully grasp all the technical details involved. Thus, the reader who is not interested in the technical details of the proofs might wish to only have a quick glance at this section.

In essence, smooth stacks provide us with a convenient place to do parametrized homotopy theory on objects which glue with respect to some local data (e.g. manifolds, algebraic varieties, etc.). To begin, we start with a category equipped with the notion of covering families (i.e. a site, see for example [MM94] for details). This category provides us with our parametrizing spaces and the coverings tell us how to glue along local data. Since we are concerned with smooth stacks, we will restrict our attention to prestacks on the following site.

**Definition 2.1** (The category of Cartesian spaces and the category of smooth prestacks). (i) Let \(\text{CartSp}\) be the site with objects the convex subspaces of Euclidean space and morphisms the smooth maps between them.\(^6\) We topologize \(\text{CartSp}\) by taking good open covers (i.e. covers with contractible finite intersections) of convex subspaces.

(ii) Let \(\mathcal{Psh}_{\text{op}}(\text{CartSp}) := [\text{CartSp}^{\text{op}}, s\mathbb{Set}]\) denote the category of contravariant functors

\[
X : \text{CartSp}^{\text{op}} \longrightarrow s\mathbb{Set},
\]

with natural transformations between them. We call this category, the category of smooth prestacks.

\(^6\)By smooth map, we mean in the sense of manifolds with corners. Since all our objects are subsets of Euclidean spaces, this is equivalent to requiring the existence of a smooth extension on an open neighborhood.
Essentially all the tools that one has in the category of simplicial sets can be applied to prestacks. For instance, we have the following (see [Ja15]).

**Definition 2.2 (Presheaf of connected components).** The presheaf of connected components of a prestack $X$ is defined as the presheaf

$$\pi_0(X) : \text{CartSp}^{op} \to \text{Set},$$

which assigns each convex subspace $U \in \text{CartSp}$ to the set of simplicial connected components $\pi_0(X(U))$.

Note that $\pi_0$ is functorial and $X$ is functorial in $U$, which indeed defines a presheaf. In a similar way, we have the following (see [Ja15]).

**Definition 2.3 (Presheaf of homotopy groups).** Given a basepoint of a prestack $x : * \to X$, the presheaf of homotopy groups can be defined, for $m > 0$,

$$\pi_m(X, x) : \text{CartSp}^{op} \to \text{Set},$$

as the functor which assigns to each $U \in \text{CartSp}$ the simplicial homotopy group $\pi_m(X(U), x_U)$.

Here the basepoint $x_U$ is prescribed by the natural transformation $x : * \to X$. Given that these are presheaves on the site $\text{CartSp}$, equipped with the notion of covering families (i.e. the good open covers of convex subspaces), we can sheafify the resulting presheaves with respect to the coverage (see [Ja15] for details).

These observations lead us to a natural definition for a model structure on prestacks. With the injective model structure, this was first defined by Jardine in [Ja15]. The projective model structure is considered in other places (for instance, in [DH10], [FSS12], [Sc13]). For us, it will turn out that the projective model structure is slightly more convenient for calculations and hence we will adopt it as our model structure.

**Definition 2.4 (The category of smooth stacks).** We define the $\infty$-category of smooth stacks

$$\text{Sh}_x(\text{CartSp}) := \text{fib}([\text{CartSp}, \text{sSet}]^{\text{proj},\text{loc}})$$

as the full subcategory on fibrant objects of the smooth prestack category $[\text{CartSp}, \text{sSet}]^{\text{proj},\text{loc}}$, equipped with the projective model structure on functors (i.e. the fibrations are objectwise Kan fibrations) and localized at the morphisms of prestacks $X \to Y$ which induce isomorphisms on all sheaves of homotopy groups.

We will often need mappings between stacks.

---

1. As in the case of topological spaces, $\pi_0$ does not admit a group structure in general.
2. If the stack has more than one connected component, this means that the map must induce an isomorphism on sheaves of homotopy groups, with basepoint taken in each connected component.
Definition 2.5 (Mapping spaces of stacks). The mapping spaces between two smooth stacks is given via the simplicial model structure on the localization. More precisely, given two smooth stacks $X$ and $Y$, the mapping space is defined as $\text{Map}(X,Y) := \text{Map}(Q(X),Y)$, where $Q(X)$ denotes some cofibrant replacement of $X$.

Later we will also encounter mapping stacks which are not spaces (see Proposition 2.14 and Definition 3.12).

Note that if a map $X \rightarrow Y$ induces an isomorphism at the level of presheaves of homotopy groups, this immediately implies that the corresponding sheafifications are isomorphic. The converse is not necessarily true, however. Thus, the class of weak equivalence in $\text{Sh}_\infty(\text{CartSp})$ is strictly larger than the objectwise weak equivalences, i.e. an equivalence of simplicial sets for each $U$.

Remark 2.6 (Other models for smooth stacks). There are many possible presentations for the $\infty$-category defined above. For example, one could take localization hammocks on the presheaf category with local weak equivalences (see [DHKS04]). One could also consider smooth stacks on the site of smooth manifolds, which would lead to an equivalent $\infty$-category (see [Sc13]). We have chosen the above model since it seems to simplify some calculations.

At first glance, it might seem awkward that we have defined the category of smooth stacks as the fibrant objects with respect to some model structure. Indeed, if we are to follow traditional literature, we would like for a stack to be characterized by some sort of descent condition. Thus it would seem that we have departed from what has been traditionally done in the literature, namely characterize a stack via descent condition. However, happily the two notions agree. Indeed, Dugger, Hollander and Isakson show in [DHI04], that the fibrant objects are precisely the objectwise Kan complexes which satisfy descent with respect to all hypercovers. Put another way this means that, for a functor $F : \text{CartSp}^{\text{op}} \rightarrow s\text{Set}$ which takes values in Kan complexes, we have an equivalence

$$F(X) \simeq \text{holim} \left\{ \ldots \xrightarrow{=} F(Y_2) \xrightarrow{=} F(Y_1) \xrightarrow{=} F(Y_0) \right\},$$

for all hypercovers $Y_\bullet \rightarrow X$, if and only if $F$ is an object in $\text{Sh}_\infty(\text{CartSp})$. Using the basic properties of the mapping space in a simplicial model category, it is easy to show (see again [DHI04]) that this is equivalent to saying that $F$ is fibrant in Bousfield localization taken at the hypercovers $Y_\bullet \rightarrow U$.

Remark 2.7 (Hypercover vs. Čech nerve). In general, hypercovers form a strictly larger class than the ordinary Čech nerves of covers. However, over the site of Cartesian spaces, the fiber products of cover inclusions are again Cartesian spaces. Thus, every hypercover is actually a Čech nerve and we can safely restrict to this case.

Notice that if we fix a functorial fibrant replacement functor $L$ on the smooth prestack model category $[\text{CartSp}^{\text{op}},s\text{Set}]^{\text{proj},\text{loc}}$, such a functor gives us a way to turn a prestack into a smooth stack. Clearly $L^2 \simeq L$ and $L$ defines a left $\infty$-adjoint to the inclusion. Moreover, it can be shown that $L$ is also left exact (i.e. preserves finite limits) [Lu09a]. Summarizing, there is an $\infty$-adjunction
between the category of stacks and that of prestacks

\[ \text{Sh}_\mathcal{X}(\mathcal{C}art\mathcal{S}p) \xrightarrow{L} \text{PSh}_\mathcal{X}(\mathcal{C}art\mathcal{S}p), \]

with \( L \) preserving finite \( \infty \)-limits. Any such functor \( L \) is called a \textit{stackification functor}. This functor is not unique, but is unique up to canonical equivalence.

**Example 2.8** (Smooth manifold as a smooth stack). For a smooth manifold \( M \) (possibly with boundary, or even corners), let \( C^\infty(-, M) \) be the smooth sheaf given by sending a convex space \( U \) to \( C^\infty(U, M) \), where smoothness is defined as the usual notion of a smooth map between manifolds with corners. By extending this sheaf to be constant in the simplicial direction, we get an object of \( \text{Sh}_\mathcal{X}(\mathcal{C}art\mathcal{S}p) \). This assignment gives rise to a fully faithful embedding

\[ \text{Man} \hookrightarrow \text{Sh}_\mathcal{X}(\mathcal{C}art\mathcal{S}p). \]

Henceforth, as we mentioned earlier, whenever we speak of a smooth manifold we will mean the corresponding object in \( \text{Sh}_\mathcal{X}(\mathcal{C}art\mathcal{S}p) \).

For a general smooth prestack, and not just an object in the site \( \mathcal{C}art\mathcal{S}p \), there is a more general notion of covering which reduces to the usual notion of Čech nerve when restricted to convex open subsets of \( \mathbb{R}^n \). A \textit{local epimorphism} is a morphism of smooth prestacks \( f : X \to Y \), such that for each \( U \in \mathcal{C}art\mathcal{S}p \) and each map \( i : U \to X \), the iterated fiber product

\[
\cdots \xrightarrow{f} X \times_Y X \times_Y X \xrightarrow{i} U, \]

is a Čech nerve of a cover \( \coprod U_\alpha \to U \). The following example illustrates how useful this more general notion can be when working over the small site of Cartesian spaces.

**Example 2.9** (Cover for smooth manifolds). Let \( M \) be a smooth manifold and let \( \{U_\alpha\} \) be an open cover of \( M \) with contractible finite intersections (i.e. a good open cover). Then we can form the Čech nerve of the cover and regard it as a simplicial diagram in prestacks,

\[
\cdots \xrightarrow{f} \coprod U_{\alpha\beta} \xrightarrow{i} \coprod U_\alpha \xrightarrow{i} M .
\]

Since \( M \) itself is not representable, this Čech nerve cannot define a covering. However, it is clear that this object defines a local epimorphism of prestacks, since given a smooth map \( f : U \to M \), the iterated fiber products is just the nerve of the cover \( \coprod U_\alpha f^{-1}(U_\alpha) \).

Every local epimorphism \( f : X \to Y \) comes equipped with a map

\[ \text{hocolim}_{\Delta^{op}} C(f) \to Y, \]

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and in the category of prestacks there is not too much one can say about this map. However, as a consequence of the characterization of fibrant objects via descent, one sees that fibrantly replacing (or stackifying) this map gives an equivalence of smooth stacks. On the other hand, the stackification functor $L$ is both left exact and a left adjoint (in the $(\infty, 1)$-sense). As a consequence, we have an equivalence

$$\text{hocolim}_{\Delta^\mathrm{op}} \hat{\mathcal{C}}(L(f)) \xrightarrow{\simeq} L(Y).$$

The morphisms of stacks $f: X \to Y$, which induces an equivalences out of their Čech nerves are called effective epimorphisms. The above argument shows that effective epimorphisms are precisely the stackification of local epimorphisms.

**Example 2.10** (Cover for homotopy quotients). Let $G$ be a Lie group acting on a smooth stack $X$. Let $q: X \to X/G$ be the canonical map to the homotopy quotient by the $G$-action. The Čech nerve of the quotient takes the form

$$\ldots \rightsquigarrow X \times G \times G \rightsquigarrow X \times G \rightsquigarrow X \to X/G.$$ 

There is an internal characterization of $(\infty, 1)$-topos due to Rezk \cite{Re10} and Lurie \cite{Lu09a} and one of the characterizing axioms asserts that quotients by $G$-actions are always effective. In other words, the above diagram is a homotopy colimiting diagram. Thus, the morphism $q$ in prestacks must be a local epimorphism and can be thought of as a cover for the homotopy quotient.

In the above example, if $X$ is a smooth sheaf and $G$ acts freely on $X$, then the homotopy quotient $X/G$ is modeled by the strict quotient sheaf $X/G$. So we arrive at the identification

$$\text{hocolim}_{\Delta^\mathrm{op}} \left\{ \ldots \rightsquigarrow X \times G \times G \rightsquigarrow X \times G \rightsquigarrow X \right\} \simeq X/G. $$

On the other hand the homotopy colimit on the right can be calculated by stackifying the objectwise homotopy colimit. This, in turn, can be computed via the diagonal of the relevant bisimplicial complex. In this case, both $G$ and $X$ are zero-truncated and the bisimplicial complex is completely degenerate in one direction. Applying the stackification functor $L$ gives the identification

$$L \left\{ \ldots \rightsquigarrow X \times G \times G \rightsquigarrow X \times G \rightsquigarrow X \right\} \simeq X/G, \quad (2.1)$$

where we are now viewing the simplicial object on the left as a single smooth stack, with presheaves at each level.

**Remark 2.11** (Descent for quotient sheaves). The equivalence (2.1) can be regarded as a descent condition for certain well-behaved quotient sheaves. In the proof of our main theorem, we will crucially use this identification to explicitly calculate a certain derived pullback which defines the inverse to the usual Pontrjagin-Thom construction.

There are several adjoint functors which relate the category $\text{Sh}_{\mathcal{E}}(\mathcal{C}art\mathcal{S}p)$ to the category $s\mathcal{S}et$.
which will be useful for us throughout the paper. We now review these functors and the corresponding basic properties that we will need.

**Remark 2.12** (Adjunctions for the model category of prestacks). The ∞-category of smooth stacks participates in a quadruple adjunction (Π → Γ → disc → codisc), in the set-up and notation of Schreiber [Sc13],

\[
\begin{array}{ccc}
\text{Sh}_\infty(\text{CartSp}) & \xrightarrow{\Pi} & \infty \text{-Spd} \\
\xleftarrow{\Gamma} & & \\
\xleftarrow{\text{disc}} & & \xleftarrow{\text{codisc}}
\end{array}
\]

where

- Γ is the global sections functor, which evaluates a smooth stack on the point manifold.
- disc is the ‘discrete’ functor which assigns each ∞-groupoid to the constant prestack and then stackifies the result.
- Π is the geometric realization functor (which we will describe in further detail below).

The functor in this adjunction of which we will make the most use is the geometric realization functor Π. Notice that it makes sense to take the locally constant stack associated to the simplicial set Π(X). In fact, this operation is functorial and one gets a natural transformation

\[\eta : \text{id} \longrightarrow L(\text{const}) \circ \Pi =: \Pi,\]  

(2.2)
where L denotes the stackification functor already encountered above. The natural transformation Π, sometimes called the shape map, can be regarded the map which sends a smooth stack to its associated homotopy invariant stack [Sc13] [Lu09a]. This map will appear in our construction of the smooth cobordism category. It turns out that on smooth manifolds, this functor has a very convenient presentation.

**Remark 2.13** (Singular realization). The following result, due to Pavlov [Pa17], will be useful for us. Let \(\text{Sing} : \text{Sh}_\infty(\text{Man}) \to \mathcal{PSh}_\infty(\text{Man})\) be the functor given by sending a smooth stack \(F\), on the large site of smooth manifolds, to the homotopy colimit of smooth stacks (taken in the category of prestacks)

\[\text{Sing}(F) := \text{hocolim}_{n \in \Delta^\text{op}} \text{Map}(\Delta^n, F),\]

where \(\Delta^n\) is the smooth \(n\)-simplex, viewed as a manifold with corners. Then \(\text{Sing}\) factors through \(\text{Sh}_\infty(\text{Man})\) and the canonical map \(\text{Sing} \to \Pi\) is an equivalence [Pa17].

We use the above result to establish the following useful equivalences. This entails working with the category of compactly generated weak Hausdorff spaces (CGWH) [Mc69]. This category is well-behaved for mapping spaces and is closed under pushouts which involve inclusion of a closed subspace (see [Sh]).

**Proposition 2.14** (Equivalences of realizations). For every smooth manifold \(M\), and smooth stack \(X\) in \(\text{Sh}_\infty(\text{CartSp})\), we have an equivalence of simplicial realizations
Moreover, we have an equivalence of topological realizations
\[ |\Pi(\text{Map}(M, X))| \simeq \text{Map}_{\text{CGWH}}(M, |\Pi(X)|) \, . \]

**Proof.** Since $\text{CartSp} \subset \text{Man}$ is a dense subsite, the category $\text{Sh}_{\times}(\text{CartSp})$ is equivalent to the category of smooth stacks on manifolds and this equivalence is exhibited by restriction to cartesian spaces. Thus, the claim will follow provided that it holds on the larger site of smooth manifolds. By Remark 2.14, we can compute
\[ \Pi(\text{Map}(M, X)) \simeq \Gamma \circ \text{Sing}(\text{Map}(M, X)) \, . \]
Calculating the homotopy colimit in simplicial sets via the diagonal of the relevant bisimplicial set, we see that the simplicial set on the right has simplices at level $k$ is given by the set of maps $M \times \Delta^k \to X_k$, where $X_k$ is the presheaf at level $k$ of $X$. The face maps induced by $\Delta^k$ and those of $X$. Using the closed Cartesian structure on ordinary presheaves, these maps are in bijective correspondence with the set of maps of the form $M \to \text{hom}(\Delta_k, X_k)$, and this correspondence clearly respects the face and degeneracy maps. This set is, in turn, immediately identified with the $k$-simplices of $\text{Map}(M, \text{Sing}(X))$, taken in prestacks. Since $\text{Sing}(X)$ satisfies descent, this gives an equivalence
\[ \Gamma \circ \text{Sing}(\text{Map}(M, X)) \simeq \text{Map}(M, \text{Sing}(X)) \simeq \text{Map}(M, \Pi(X)) \, . \]
But then, by the adjunction $\Pi \dashv \text{disc}$, the space on the right is $\text{Map}(\Pi(M), \Pi(X))$. The second claim is well-known and follows from the Quillen equivalence between CGWH spaces and simplicial sets [Qu67]. \[ \Box \]

**Remark 2.15 (Simplicial realization).** If $X$ is a smooth manifold, thought of as a smooth stack, remark 2.14 implies that we can identify the geometric realization as the simplicial set of $X$, which at level $n$ is given by the set $C^X(\Delta^n, X)$. Thus, we see that under the further geometric realization $| \cdot | : s\text{Set} \to \text{Top}$, we recover the geometric realization of the smooth singular simplicial set. By the theorem of Milnor [Mi57], this is indeed equivalent to $X$ itself.

The previous discussion gives the notion of geometric realization of a smooth stack, but there is also a notion of realization which is intrinsic to the category of smooth stacks. Indeed, we can form the stacky realization of a simplicial diagram of stacks via the left Kan extension of the above inclusion along the Yoneda embedding $\Delta^{op} \hookrightarrow [\Delta^{op}, \text{PSh}_{\times}(\text{CartSp})]$. We can compute this Kan extension via the local formula which gives the classifying stack of a simplicial object.

\[ B(X.) := \int^{[n] \in \Delta} X_n \times \Delta[n] \, , \]
where $X_n$ is stack in the $n$-th level of $X_\bullet$, and the underline indicates that we are taking the constant stack associated to the corresponding simplicial set. More explicitly, the coend can be computed as the coequalizer of the diagram

$$
\coprod_{d : [k] \to [m]} X_k \times \Delta[m] \xrightarrow{id \times d} \coprod_{d : [n]} X_n \times \Delta[n],
$$

where $d : [k] \to [m]$ is any map in $\Delta$. In general, the result of this construction is not a smooth stack. Thus, we need to stackify the result so that we stay in the right category. In the local model structure on prestacks, these two objects are equivalent and we will often denote both the the prestack $B\mathcal{P}X\mathcal{Q}^\bullet$ and its stackification by the same symbol.

**Proposition 2.16 (Stacky realization as homotopy colimit).** The stacky realization $B\mathcal{P}X\mathcal{Q}^\bullet$ computes the homotopy colimit over $X_\bullet$ in $\mathbf{Sh}_{\mathbf{X}}(\mathbf{Cart}\mathbf{Sp})$.

**Proof.** First observe that, before stackifying, the prestack $B\mathcal{P}X\mathcal{Q}^\bullet$ is objectwise the realization of the resulting bisimplicial set. Since the realization computes the homotopy colimit in $\mathbf{sSet}$ and homotopy colimits in prestacks are computed objectwise, we see that $B\mathcal{P}X\mathcal{Q}^\bullet$ is a model for the homotopy colimit in prestacks. Homotopy colimits of diagrams of stacks, taken in prestacks, are always computed by stackifying the result. The claim follows immediately.

**Remark 2.17 (Stackifying homotopy colimits).** Proposition 2.16 invokes a trick that will be frequently used throughout this paper. Namely, that if we are given a diagram of stacks $F : J \to \mathbf{Sh}_{\mathbf{X}}(\mathbf{Cart}\mathbf{Sp})$, the homotopy colimit can be computed by first computing the homotopy colimit in prestacks, and then the stackifying the result. This immediately follows from the fact that the stackification functor $L$ is left adjoint to the inclusion $i : \mathbf{Sh}_{\mathbf{X}}(\mathbf{Cart}\mathbf{Sp}) \hookrightarrow \mathcal{P}\mathbf{Sh}_{\mathbf{X}}(\mathbf{Cart}\mathbf{Sp})$.

The following example will be of particular interest.

**Example 2.18 (Classifying stack of a smooth category).** Let $\mathcal{C}$ be an internal category of smooth stacks (i.e. a smooth category). This is, by definition, a diagram of the form $\xymatrix{X \ar[r] & Y}$ along with a map

$$
m : X \times_Y X \longrightarrow X,
$$

which gives a composition law, and which makes the relevant diagrams commute. Taking the Čech nerve of this diagram gives a simplicial diagram of the form

$$
\xymatrix{X \times_Y X \times_Y X \ar[r] & X \times_Y X \ar[r] & X \times_Y X \ar[r]^m & X \ar[r] & Y.}
$$

We will denote the stacky realization of the nerve by $B\mathcal{C}$ and call it the classifying stack of the smooth category $\mathcal{C}$.

We now show that the geometric realization of the classifying stack of a simplicial object coincides with the usual notion of classifying space of the underlying topological space (viewed simplicially). Thus the geometric realization of the stacky realization of a smooth category is the
topological realization of geometric realization of the category. Note that the notion of a classifying space of a topological category is described in [Sc74][Mo95][We05].

**Proposition 2.19** (Commuting realizations). Let \( X_\bullet \) be a simplicial object in \( \mathbf{Sh}_X(\mathbf{CartSp}) \). We have a homotopy equivalence of CW-complexes

\[
|B(X_\bullet)| \cong B(|X_\bullet|),
\]

where \( |\cdot| : \mathbf{Sh}_X(\mathbf{CartSp}) \to \mathbf{sSet} \to \mathbf{Top} \) denotes the composite of geometric realization functors and \( |X_\bullet| \) is the simplicial object in \( \mathbf{Top} \) given by level-wise geometrically realizing.

**Proof.** First note that \( \Pi \) and \( |\cdot| \) both preserve (even strict) colimits and products.\(^9\) Thus, we have a homeomorphism

\[
|B(X_\bullet)| \cong \text{coeq} \left\{ \coprod_{d: [k] \to [m]} |X_k| \times |\Delta[m]| \xrightarrow{id \times d} |X_n| \times |\Delta[n]| \right\}.
\]

Since \( \Delta[k] \) is locally constant, we have an equality of simplicial sets \( \Pi(\Delta[k]) = \Delta[n] \), where \( \Delta[n] \) is the simplicial \( n \)-simplex. The geometric realization of \( \Delta[n] \) is equal to the standard topological \( n \)-simplex \( \Delta^n \) (as follows from [Mi57 Theorem 1]). Thus, the right hand side of the above homeomorphism is exactly the topological realization of the complex \( |X_\bullet| \). Finally, since the stackification morphism \( L : B(X_\bullet) \to L(B(X_\bullet)) \) is a weak equivalence in the local model structure on prestacks, and geometric realization preserves these equivalences, we have a weak homotopy equivalence

\[
|L : B(X_\bullet)| \longrightarrow |L(B(X_\bullet))|.
\]

Since \( |\cdot| \) takes values in CW-complexes, the claim follows from Whitehead’s Theorem. \( \square \)

### 2.2 Classification of vector bundles in smooth stacks

In this section we will define a stacky operation on vector bundles which is analogous to the Thom space construction in topological spaces. We begin by describing a proper notion of a vector bundle in the setting of smooth stacks (see [NSS15a]). This just places a familiar concept in a new, more general, setting.

**Definition 2.20** (Vector bundle over a stack). Let \( X \) be a smooth stack. A vector bundle with fiber \( V \) is a fibration \( \pi : \eta \to X \) between two smooth stacks such that for there is an effective epimorphism \( p : Y \to X \), so that we have a homotopy Cartesian square

\[
\begin{array}{ccc}
V \times Y & \longrightarrow & \eta \\
\downarrow \text{proj} & & \downarrow \pi \\
Y & \longrightarrow & X.
\end{array}
\]

\(^9\)The functor \( \Pi \) is presented by the colimit operation on prestacks, which is left adjoint to the constant prestack functor.
Definition 2.20 recovers the usual definition of a locally trivial vector bundle over a smooth manifold. Indeed, if $X$ is a smooth manifold then any good open cover $\{U_\alpha\}$ of $X$ defines an effective epimorphism $\coprod_\alpha U_\alpha \to X$. Taking the full nerve of the cover, we are led to the diagram

$$\cdots \longrightarrow \coprod_{\alpha \beta} U_{\alpha \beta} \times V \longrightarrow \coprod_\alpha U_\alpha \times V \longrightarrow \eta \quad \quad \cdots \longrightarrow \coprod_{\alpha \beta} U_{\alpha \beta} \longrightarrow \coprod_\alpha U_\alpha \longrightarrow X,$$

(2.3)

where the first square is homotopy Cartesian. By the Pasting Lemma, the first square also implies that the homotopy fiber of $\eta \to X$ is isomorphic to $V$.

The above diagram may seem redundant, since the first pullback square seems to give all the information about the local triviality of the bundle. Note, however, that the higher stages in the simplicial diagram in the top tell us how to glue together the total space of the bundle. The maps in this simplicial diagram are not just determined by the combinatorial data needed to glue together $X$, but also the transition functions of the bundle. The axiom of descent for an $\infty$-topos (see Rezk [Re10] and Lurie [Lu09a]) implies that whenever we have an iterated pullback diagram, such as (2.3), then the first square is homotopy Cartesian if and only if the top right arrow is a homotopy colimiting cocone. Thus, we must have

$$\text{hocolim}\left\{ \cdots \longrightarrow \coprod_{\alpha \beta} U_{\alpha \beta} \times V \longrightarrow \coprod_\alpha U_\alpha \times V \right\} \longrightarrow \eta.$$

which allows us to recover $\eta$ from local data. In the case where $\eta$ is zero-truncated, like a smooth manifold, this reduces to the usual construction for the total space of the bundle

$$\text{colim}\left\{ \coprod_{\alpha \beta} U_{\alpha \beta} \times V \longrightarrow \coprod_\alpha U_\alpha \times V \right\} \longrightarrow \eta.$$

Remark 2.21 (General fibers and morphisms). (i) Definition 2.20 works just as well for bundles with any fibers, not just vector spaces. Thus, one could consider the more general theory of $\infty$-bundles, where the fibers are allowed to have higher simplicial data in smooth stacks. For us, however, we will only be concerned with the zero-truncated fibers as our main interest is structures on the tangent bundle of a smooth manifold.

(ii) One can also define a morphism of vector bundles $\eta \to \xi$ over $M$ in the obvious way: namely a morphism of smooth stacks which induces a linear isomorphism on the fibers. Since linear isomorphisms are in particular smooth, this condition is well-defined in smooth stacks.

Let us first recall that there is a smooth stack which classifies locally trivial principal $O(d)$-bundles, denoted $BO(d)$. Up to equivalence, this stack can be defined simply as the homotopy orbit stack of the smooth sheaf $O(d)$ acting on the point manifold $\ast$. However, we will need an explicit model of this stack with which to work in practice (see [FSS12] [SSS12]).

Definition 2.22 (Classifying stack of orthogonal bundles). We define the classifying stack of
smooth, locally trivial, principal O(d)-bundles as the stack given level-wise by

\[ \text{BO}(d) := \left\{ \cdots \longrightarrow \text{O}(d) \times \text{O}(d) \longrightarrow \text{O}(d) \longrightarrow \ast \right\}, \tag{2.4} \]

where the face maps \( d_k : \text{O}(d)^n \to \text{O}(d)^{n-1} \) are given by the projections for \( k \leq n \) and \( d_{n+1} : \text{O}(d)^n \to \text{O}(d)^{n-1} \) sends \((g_1, g_2, \ldots, g_n) \mapsto (1, \ldots, 1, g_1 g_2 \ldots g_n)\), for \( g_i \in C^\infty(-, \text{O}(d))\).

For any Lie group \( G \), one can define \( \text{BG} \) analogously. Note that, since we are working over the small site of Cartesian spaces, the simplicial presheaf defined by \( (2.4) \) does in fact satisfy descent (see [FSS12] for details) and is, therefore, a well-defined object in \( \text{Sh}_X(\text{CartSp})\).

This stack indeed classifies locally trivial, smooth principal O(d)-bundles. In fact, in [FSS12] it was shown that at the level of connected components we have a natural bijection

\[ \pi_0 \text{Map}(\hat{\mathcal{C}}(\{U_\alpha\}), \text{BO}(d)) \cong \check{H}^1(\{U_\alpha\}, \text{O}(d)), \]

where on the left we have replaced the smooth manifold \( M \) by the homotopy colimit over the Čech nerve of some good open cover \( \mathcal{U} \) and on the right we have the group of nonabelian Čech cohomology with respect to the cover. Such a class forms precisely the data needed to construct an isomorphism class of principal O(d)-bundles, and each representative of the class corresponds to a choice of transition functions of the bundle. Even more generally, it was shown in [FSS12] [SSS12] [Sc13] that this isomorphism lifts to an equivalence of \( \infty \)-groupoids

\[ \text{Map}(M, \text{BO}(d)) \congto \text{Bun}(M), \tag{2.5} \]

where on the right we have the \( \infty \)-groupoid of locally trivial principal O(d)-bundles and on the left we have the derived mapping space between \( M \) and \( \text{BO}(d) \).

In the proof of the main theorem, we will need to have explicit cocycle representations for maps out of a smooth manifold to certain classifying stacks. We now illustrate how to derive the Čech cocycle data in this case. Let us take the model for \( \hat{\mathcal{C}}(\{U_\alpha\}) \) given by the realization over the nerve (see [FSS12] for more on this argument). The resulting stack takes the form

\[ \cdots \longrightarrow \coprod_{\alpha \beta \gamma} U_{\alpha \beta \gamma} \times \Delta[2] \longrightarrow \coprod_{\alpha \beta} U_{\alpha \beta} \times \Delta[1] \longrightarrow \coprod_{\alpha} U_{\alpha} \times \Delta[0], \]

where the face maps are induced by the inclusion and the usual face maps of \( \Delta[k] \). Since \( \text{BO}(d) \) is 1-truncated, the Yoneda Lemma along with descent implies that a map from this stack to \( \text{BO}(d) \)

\(^{10}\text{From the point of view of model category theory, these replacements are cofibrant replacements in the local projective model structure [DH04].}\)
is equivalently given by a commutative diagram

\[
\begin{array}{ccc}
\Delta[2] & \xrightarrow{\Delta[1]} & \prod_{\alpha\beta} \text{BO}(d)(U_{\alpha\beta}) \\
\uparrow & & \uparrow \\
\Delta[1] & \xrightarrow{\Delta[1]} & \prod_{\alpha} \text{BO}(d)(U_{\alpha}) \\
\end{array}
\]

where the vertical maps on the right are given by restriction and those on the left are given by the face inclusions. From the definition of BO(d), one sees that such a diagram determines – and is uniquely determined – by a choice of smooth O(d)-valued function \(g_{\alpha\beta} : U_{\alpha\beta} \to O(d)\) on intersections satisfying the cocycle condition \(g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\delta} = 1\) on 3-fold intersections \(U_{\alpha\beta\gamma}\). It follows immediately that we have an isomorphism

\[
\text{hom}(\tilde{C}([U_\alpha]), \text{BO}(d)) \cong \tilde{C}^1([U_\alpha], O(d))
\]

where the right hand side is the set of degree 1 nonabelian Čech cocycles with values in O(d).

In the same way that locally trivial O(d)-bundles are classified by maps to BO(d), there is a universal bundle \(U(d) \to BO(d)\) which classifies vector bundles over \(M\), via pullback along a classifying map. Indeed, let \(U(d)\) be the smooth stack defined by the homotopy quotient \(\mathbb{R}^d//O(d)\), where O(d) acts in the usual way. Lemma 2.24 provides us with an identification

\[
U(d) \simeq \left\{ \ldots \mathbb{R}^d \times O(d) \times O(d) \xrightarrow{\times} \mathbb{R}^d \times O(d) \xrightarrow{\times} \mathbb{R}^d \right\}, \tag{2.6}
\]

where the face maps \(d_k : O(d)^n \times \mathbb{R}^d \to O(d)^{n-1} \times \mathbb{R}^d\) are given by the projections for \(k \leq n\) and \(d_{n+1} : O(d)^n \times \mathbb{R}^d \to O(d)^{n-1} \times \mathbb{R}^d\) sends \((g_1, g_2 \ldots, g_n, v) \mapsto (1, \ldots, 1, g_1g_2 \ldots g_nv)\). If we are given a map \(f : M \to BO(d)\) classifying an O(d)-bundle on \(M\), then the pullback

\[
\begin{array}{ccc}
\eta & \xrightarrow{g} & U(d) \\
\pi \downarrow & & \downarrow \pi \\
M & \xrightarrow{f} & \text{BO}(d)
\end{array}
\]

is a locally trivial vector bundle over \(M\). The trivialization comes from the fact that when we replace \(M\) by a Čech resolution \(\tilde{C}([U_\alpha])\). One such pullback is explicitly computed as \(\tilde{C}([U_\alpha \times \mathbb{R}^d])\), where the face maps in the Čech nerve diagram depend on the transition functions determined by \(f\). Since any two pullbacks are canonically isomorphic, this resolution defines a local trivialization of the bundle.

Notice that immediately one obtains that \(U(d)\) defines a vector bundle in the sense of

**Remark 2.23 (Choice of Čech resolution).** Note that it is not quite right to say that a map \(M \to BO(d)\) determines a locally trivial principal O(d)-bundle on \(M\). Strictly speaking, we need to first choose a Čech resolution of \(M\) and then take a map from the resolution to BO(d) to get such
a bundle. In general, the local structure depends on the choice of resolution, but the full mapping
spaces out of different resolutions will be equivalent (this follows from general properties of simplicial
model categories and the characterization of cofibrant objects via [DH104]). Henceforth, whenever
we write a map out of a smooth manifold $M$, we will implicitly assume a choice of Čech resolution.
If a specific choice is needed, we will make it clear.

The picture is even more clear when one looks at the Čech cocycles obtained by mapping into $\mathcal{U}(d)$. Computing the cocycles in the same way as before, we see that for a good open cover $\{U_\alpha\}$ of $M$, a map to $\mathcal{U}(d)$ determines and is uniquely determined by a commutative diagram

$$
\begin{array}{c}
\Delta[2] \\
\Delta[1] \\
\Delta[0]
\end{array} \xrightarrow{\mathcal{U}(d)} \begin{array}{c}
\prod_{\alpha\beta\gamma} \mathcal{U}(d)(U_{\alpha\beta\gamma}) \\
\prod_{\alpha\beta} \mathcal{U}(d)(U_{\alpha\beta}) \\
\prod_{\alpha} \mathcal{U}(d)(U_{\alpha})
\end{array}
$$

By the definition of $\mathcal{U}(d)$, such a diagram is equivalently a choice of local section $s_\alpha : U_\alpha \to \mathbb{R}^d$ and transition functions $g_{\alpha\beta} : U_{\alpha\beta} \to \text{O}(d)$ such that $s_\alpha = g_{\alpha\beta} s_\beta$ on intersections and $g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\delta} = 1$ on 3-fold intersections. This is precisely the data needed to construct a global section of the bundle with transition functions $g_{\alpha\beta}$. At an abstract level, this is clear since the universal property of the pullback implies that if we are given a map $f : M \to \mathcal{U}(d)$ which fills the diagram \(2.7\), then we have an induced map $s : M \to \eta$ such that $\pi s = \text{id}$, i.e., a global section.

2.3 The Grassmanian and its canonical bundles in the stacky setting

As in the Madsen-Tillman construction [MT01], we now turn our attention to embedded subman-
ifolds and their tangent and normal bundles. Let us recall the Grassmannian manifold, whose
underlying set is defined as the collection of $d$-planes in $\mathbb{R}^{d+N}$. This set is given smooth structure
via the quotient

$$
\text{Gr}(d, N) = \text{O}(d+N)/\text{O}(d) \times \text{O}(N),
$$

where $\text{O}(d) \times \text{O}(N)$ acts via matrix multiplication by block matrices. Similarly, we recall the
definition of the real Steifel manifold $V(d, N)$ as the set of orthonormal $d$-frames in $\mathbb{R}^N$:

$$
V(d, N) = \text{O}(d+N)/\text{O}(d).
$$

There are two canonical bundles over the Grassmannian: the tautological bundle, which maps the
points on a $d$-plane to the corresponding $d$-plane in $\mathbb{R}^{d+N}$, and the orthogonal complement bundle,
given by mapping the points on the complement of a $d$-plane to the underlying $d$-plane. Both
bundles admit the structure of smooth manifolds and are identified, respectively, by the quotients

$$
\mathcal{U}(d, N) := \mathbb{R}^d \times_{\text{O}(d)} V(d, N) \quad \text{and} \quad \mathcal{U}^\perp(d, N) := \mathbb{R}^N \times_{\text{O}(N)} V(N, d).
$$

23
Here we think of $d$ as indexing the tangent direction and $N$ as indexing the normal direction, although clearly we could reverse the roles of $d$ and $N$, which would lead to the canonical identification $U(d, N) \cong U^\perp(N, d)$. In each example, we can take the colimit over the canonical inclusion maps as $N \to \infty$. This leads to the corresponding infinite-dimensional manifolds $\text{Gr}(d, \infty)$, $\text{V}(d)$, $\mathcal{U}(d)$ and $\mathcal{U}^\perp(d)$.

The purpose of this section is to provide convenient descent data for each of these smooth manifolds, viewed as living in the category of smooth stacks. Notice that each of these smooth manifolds arise from quotients of Lie group actions and we can, therefore, use the identification (2.1) for each of these objects. Since we will use this identification frequently, we will spell it out in detail.

**Lemma 2.24** (Homotopy orbit stack of a Lie group action). Let $G$ be a Lie group, regarded as a zero-truncated smooth stack (i.e. a smooth sheaf). Suppose $G$ acts on a freely on a sheaf $X$ and suppose moreover that the corresponding quotient presheaf $X/G$ satisfies descent (i.e. is a sheaf). Then the prestack

$$Y := \left\{ \ldots X \times G \times G \xrightarrow{\text{diag}} X \times G \xrightarrow{\text{diag}} X \right\},$$

(2.12)

satisfies descent and we have an equivalence of smooth stacks

$$q : Y \xrightarrow{\simeq} X/G,$$

where $q$ is the strict quotient map.

**Proof.** Homotopy colimits of prestacks can be calculated objectwise (this follows, for example, from [Lu09a, Theorem 4.2.4.1]). Therefore, the homotopy colimit can be calculated as the presheaf which assigns to each $U \in \text{CartSp}$, the diagonal of the bisimplicial set given by considering (2.12) as a simplicial object in prestacks and evaluating at $U$. Since this diagram is constant in one simplicial direction, the prestack given by (2.12) is a model for the homotopy orbits, taken in prestacks. Now we use the trick discussed in Remark 2.17 to conclude that the stackification models the homotopy quotient in stacks.

On the other hand, since $G$ acts freely on $X$, the stabilizer subgroups of $G$ must be trivial. But these subgroups are precisely the presheaves $\pi_1(Y, g)$, where $g : * \to G$ is a section. Since the higher simplicial homotopy groups vanish ($G$ is zero-truncated), we see that the map $q : Y \to X/G$, induced by the unit of the adjunction $\pi_0 \dashv \text{sk}_0$, induces an isomorphism on all presheaves of homotopy groups. Since $X/G$ already satisfies descent and is equivalent to $Y$ at the level of prestacks, we immediately deduce that $Y$ satisfies descent and that $q$ is an equivalence of smooth stacks. □

**Remark 2.25.** In what follows we will be working with various models for certain smooth stacks. In particular, we will occasionally want to distinguish between strict sheaves and smooth stacks which are equivalent to these strict sheaves. Whenever we are working only up to equivalence, or choosing models for smooth sheaves which have higher simplicial data, we will denote the corresponding stack
by bold characters (for example, $\text{Gr}(d, N)$). When we are identifying the stack with a strict smooth sheaf, we will use upright roman characters (for example, $\text{Gr}(d, N)$). This convention will be used throughout the remainder of the paper.

Notice that in each of the above examples in expressions (2.9), (2.10) and (2.11), the given action is free. Thus, Lemma 2.24 applies and gives us the convenient descent data we have been looking for in each example. This now motivates us to define various stacks which will replace their classical counterparts. Lemma 2.24 will imply that each of these objects is equivalent to the strict quotient and serves as another model for the homotopy quotient in stacks.

**Definition 2.26 (Smooth Grassmannian stack).** We define the smooth Grassmannian stack to be the smooth stack given level-wise by

$$
\text{Gr}(d, N) := \left\{ \ldots \boxtimes (O(d) \times O(N))^2 \times O(d + N) \boxtimes (O(d) \times O(N)) \times O(d + N) \boxtimes O(d + N) \right\}
$$

where the face maps $d_k$ are projections for $k \leq n$ and $d_{n+1}$ is given by the action of $(O(d) \times O(N))$ on $O(d + N)$ via the canonical inclusion $O(d) \times O(N) \hookrightarrow O(d + N)$ as block matrices for $k = n + 1$.

We can also define the stacky analogue of the real Stiefel manifold.

**Definition 2.27 (Smooth Stiefel stack).** Define the smooth Stiefel stack via

$$
\text{V}(d, N) := \left\{ \ldots \boxtimes O(N)^2 \times O(d + N) \boxtimes O(N) \times O(d + N) \boxtimes O(d + N) \right\},
$$

where for $k \leq n$, the face maps $d_k$ are again projections and where $d_{n+1}$ is given by the action by matrix multiplication after the inclusion $i^N_{d+N} : O(N) \hookrightarrow O(d + N)$, which sends an orthogonal $(N \times N)$-matrix $Q$ to the block matrix with $Q$ in the upper left corner, the $d \times d$ identity matrix $1_d$ in the lower right corner and zeros everywhere else.

As in the case of the Grassmannian, this prestack satisfies descent, is zero-truncated and is equivalent to its classical counterpart $\text{V}(d, N)$. Notice also that the smooth Stiefel stack admits an action of $O(d)$, given by the inclusion $i_{d,d+N} : O(d) \hookrightarrow O(d + N)$, which includes as a block matrix in the lower right corner.

**Remark 2.28 (Non-contractibility of the Stiefel Stack in the limit).** Notice that $\text{V}(d, N)$ is clearly not contractible as a smooth stack as $N \to \infty$. Indeed, if it were then $\lim_{N \to \infty} \text{V}(d, N)$ would have to be diffeomorphic to the point manifold $*$ as $N \to \infty$. \footnote{We will use the notation $i^p_n$ to denote the embedding of $p \times p$ matrices as the upper left block in $n \times n$ matrices ($n \geq p$), and similarly $i_{q,n}$ for the embedding of $q \times q$ matrices as the lower right block in $n \times n$ matrices ($n \geq q$).} This is obviously not the case. For example, we can choose a sequence $Q_N$ of orthogonal matrices of the form

$$
Q_N = \begin{pmatrix}
1 & 1 \\
\ast & \ast
\end{pmatrix}
$$

where $Q$ is a zero-truncated smooth manifold.
with \( 1 \neq Q \in O(d) \). Clearly each \( Q_N \) is a non identity element in \( O(N+d)/O(N) \), and \( Q_N \rightarrow Q_{N+1} \), under the induced map \( V(d,N) \rightarrow V(d,N+1) \). Hence, in the colimit \( N \rightarrow \infty \), \( Q_N \) converges to a nonidentity element \( Q_\infty \in O/O \), since for no value of \( N \) is \( Q_N = 1 \). For this reason, we see that the stacks \( \text{Gr}(d,N) \) cannot approximate \( BO(d) \) as \( N \rightarrow \infty \), which is of course different than the case in topological spaces.

We can now define the universal orthogonal complement bundle over the Grassmannian. Indeed, through the map \( i^N_{d+1}: O(N) \hookrightarrow O(d+N) \), the group \( O(N) \) acts on the orthogonal complement of \( \mathbb{R}^d \) in \( \mathbb{R}^{d+N} \) and this leads to the following definition.

**Definition 2.29** (Universal vector bundle and it complement over Grassmannian stacks). (i) We define the universal vector bundle \( \pi: \mathcal{U}(d,N) \rightarrow \text{Gr}(d,N) \) over the smooth Grassmannian, by setting

\[
\mathcal{U}(d,N) := (\mathbb{R}^d \times V(d,N))/O(d) ,
\]

where the action of \( O(d) \) on \( \mathbb{R}^d \) is given by the usual action by linear maps, and on the product diagonally (with the action on the second factor as above).

(ii) Similarly, we define the universal orthogonal complement bundle \( \pi^\perp: \mathcal{U}^\perp(d,N) \rightarrow \text{Gr}(d,N) \) over the Grassmannian, by setting

\[
\mathcal{U}^\perp(d,N) := (\mathbb{R}^\perp \times V(d,N))/O(N) ,
\]

where the orthogonal complement \( \mathbb{R}^\perp \) of \( \mathbb{R}^d \) is taken in \( \mathbb{R}^{d+N} \) (and can, therefore, be identified with \( \mathbb{R}^N \)) and the action of \( O(N) \) is given by the diagonal action. The bundle map is induced by projecting out the first factor \( \mathbb{R}^\perp \).

Since each of the above stacks is equivalent to its classical analogue, we immediately deduce the following.

**Proposition 2.30** (Classical correspondence). The geometric realization of all the above stacks are weak homotopy equivalent to their classical counterparts.

As in the case of the usual Grassmannian, we have obvious maps \( \text{Gr}(d,N) \hookrightarrow \text{Gr}(d,N+1) \) which are induced by the inclusion \( i^N_{1+N}: O(N) \hookrightarrow O(N+1) \). In accord with the classical case, we have the following splitting.

**Proposition 2.31** (Splitting of the universal orthogonal complement bundle). The homotopy pullback of \( \mathcal{U}^\perp(d,N+1) \) to \( \text{Gr}(d,N) \) can be identified with the sum \( \mathcal{U}^\perp(d,N) \oplus 1 \).

**Proof.** This amounts to a direct calculation of the pullback of the bundle. The homotopy pullback can be computed degreewise in presheaves, and we see that in degree \( k \) we are led to the homotopy

\[13\] In fact, we already knew this, since \( \text{Gr}(d,N) \) is zero-truncated.
Note that the right vertical maps project out the Euclidean spaces, and we have used the fact that the map on the right is an objectwise Kan fibration and all objects are objectwise fibrant in order to compute the homotopy pullback as the strict pullback. \[^{14}\] Since the action of O(N) does not act on the factor \( \mathbb{R} \) in the upper left corner, we see that the resulting prestack is simply \( \text{Gr}_\mathbb{R} d, Nq \times \mathbb{R} q ' \). Finally, since the stackification functor is left exact, it preserves homotopy Cartesian squares and products. We compute the pullback in stacks via stackification, which leads to the result.

We will similarly need the colimit of the orthogonal complement bundle.

**Definition 2.32** (Universal orthogonal complement bundle). We define the universal orthogonal complement bundle \( \mathcal{U}^\perp(d) \to \text{Gr}(d, \infty) \) of the classifying stack of orthogonal bundles as the universal map induced by passing to colimits in Definition 2.29, i.e. \( \mathcal{U}^\perp(d) := \lim_k \mathcal{U}^\perp(d, N) \).

Note that one can obtain Čech cocycle data for the bundles \( \mathcal{U}^\perp(d) \) and \( \mathcal{U}^\perp(d, N) \) in a completely analogous way to that of \( \mathcal{U}(d) \), discussed in Section 2.2.

**Remark 2.33** (Stacks vs. spaces). We could have deduced many of the propositions from classical results. We have chosen to illustrate how to work with these smooth stacks, since we will be using similar techniques in the proof of the main theorem.

### 3 Thom stacks and their smooth motivic spectra

#### 3.1 Thom stacks

Having defined vector bundles in the context of stacks, now we consider the stacky analogue of the corresponding Thom spaces. We begin with a classical discussion. Suppose \( V \) is finite-dimensional vector space and equip \( V \) with an inner product. Let \( \pi : \eta \to M \) be a vector bundle with fiber \( V \) over a smooth manifold \( M \). Then \( \eta \) inherits a metric from \( V \) via locally trivializing patches and the transition functions of the bundle can be chosen to be orthogonal, via the metric. Let \( D(V) \) denote the closed unit disc in \( V \) with respect to this inner product. Consider the diagram

\[
\begin{align*}
\prod_{\alpha\beta} U_{\alpha\beta} \times D(V) & \xrightarrow{\prod_{\alpha} U_{\alpha} \times D(V)} D(\eta) \\
\prod_{\alpha\beta} U_{\alpha\beta} \times V & \xrightarrow{\prod_{\alpha} U_{\alpha} \times V} \eta,
\end{align*}
\]

\[^{14}\text{Note that the orthogonal complements are taken in different dimensions.}\]
where the top maps are induced by the transition functions of the bundle on intersections and $D(\eta)$ denotes the unit disc bundle. All the squares to the left of the right most square are Cartesian and by the axiom of descent for the category of sheaves, the top diagram is a coequalizer precisely when the first square is Cartesian. This ensures that we do have a well-defined disc bundle with trivializing charts which are compatible with the ones for $\eta$. We also have a similar picture for the sphere bundle $S(\eta)$ and we have a canonical map $S(\eta) \rightarrow D(\eta)$.

This discussion works well for total spaces which are sheaves, but generally we will need to consider total spaces which are smooth stacks. The same construction works in this case too, but we need to extend the diagram on the left via the entire hypercover

\[
\begin{array}{cccccc}
\ldots & \equiv & \coprod_{\alpha\beta} U_{\alpha\beta} \times D(V) & \rightarrow & \coprod_{\alpha} U_{\alpha} \times D(V) & \rightarrow & D(\eta) \\
& & \downarrow & & \downarrow & & \\
\ldots & \equiv & \coprod_{\alpha\beta} U_{\alpha\beta} \times V & \rightarrow & \coprod_{\alpha} U_{\alpha} \times V & \rightarrow & \eta.
\end{array}
\] (3.1)

Then $\eta$ will be the homotopy colimit over the bottom simplicial diagram and $D(\eta)$ will be the homotopy colimit over the top simplicial diagram. Again the axiom of descent implies that all squares are homotopy Cartesian and the top and bottom maps are homotopy quotient maps. This way, we have a well-defined notion of a unit disc and sphere bundle for vector bundles which have smooth stacks as total spaces.

In fact, we can continue the discussion and consider the situation in even greater generality. Indeed, we can perform the same construction for vector bundles with total spaces given by smooth stacks and base spaces given by smooth stacks. We simply replace our cover of the manifold $M$ by effective epimorphisms $Y \rightarrow X$.

**Definition 3.1 (Thom stack).** Let $V$ be a finite-dimensional vector space, equipped with inner product. For any smooth stack $X$, we define the Thom stack of a $V$-bundle, $\pi : \eta \rightarrow X$ to be the homotopy quotient stack

\[\text{Th}(\eta) := D(\eta)//S(\eta).\]

If the map $S(\eta) \rightarrow D(\eta)$ is an objectwise cofibration (i.e. monomorphisms of simplicial sets), then the homotopy quotient in prestacks $D(\eta)//S(\eta)$ can be modeled by the strict quotient. Thus, the stackification of the strict quotient is a model for the Thom stack in this case. The resulting stack is, in fact, a sheaf up to equivalence. It is the diffeological space whose plots are smooth functions to the $n$-disk bundle, where all plots which land entirely in the boundary of the $n$-disk bundle get identified.

**Remark 3.2 (‘Stacky sphere’).** If $\dim(V) = n$, the quotient stack $D(V)/S(V)$ will serve as a model for the $n$-dimensional sphere and we will denote this stack accordingly as $D^n/\partial D^n$. However, it is important to note that this is not the stack represented by the sheaf of smooth plots of the unit $n$-sphere in $V \oplus \mathbb{R}$, nor is it the simplicial sphere $S^n := \Delta[n]/\partial \Delta[n]$.

We immediately have the following.
Proposition 3.3 (Thom stack of a trivial bundle). Let \( n : \mathbb{R}^n \times X \to X \) be the trivial rank \( n \) bundle on a smooth stack \( X \). We have an equivalence

\[
\text{Th}(n) \simeq D^n/\partial D^n \land X_+.
\]

**Proof.** This is immediate from the definition of the Thom stack. \( \square \)

As in the classical case, we would like the Thom stacks to behave well with respect to the smash product of pointed stacks. With an appropriate understanding of the \( n \)-sphere, we see that this is indeed the case.

Proposition 3.4 (Properties of Thom stacks). Let \( \xi \to X \) and \( \eta \to Y \) be vector bundles of rank \( n \) and \( m \), respectively. The Thom stack operation \( \text{Th} \) satisfies the following properties.

(i) (Functorial) Given a morphism of bundles \( \phi : \xi \to \eta \) which restricts to an isometric injective linear map on the fibers, there is an induced morphism of Thom stacks \( \text{Th}(\phi) : \text{Th}(\xi) \to \text{Th}(\eta) \).

(ii) (Multiplicative) We have an equivalence of smooth stacks \( \text{Th}(\xi \circ \eta) \simeq \text{Th}(\xi) \land \text{Th}(\eta) \), where \( \xi \circ \eta \to X \times Y \) is the external sum bundle.

**Proof.** Part (i) follows directly from the commutative diagram

\[
\begin{array}{ccc}
S(\xi) & \xrightarrow{\phi} & S(\eta) \\
\downarrow & & \downarrow \\
D(\xi) & \xrightarrow{\phi} & D(\eta).
\end{array}
\]

To prove part (ii), note that we have a canonical map

\[
\text{Th}(\eta \circ \xi) \longrightarrow \text{Th}(\eta) \land \text{Th}(\xi) \tag{3.2}
\]

induced by the composition \( D(\xi \times \eta) \hookrightarrow D(\xi) \times D(\eta) \to \text{Th}(\xi) \land \text{Th}(\eta) \), which descends to the quotient. This map is an equivalence fiberwise since we have an isomorphism of pointed sheaves\(^{15}\)

\[
D^{n+m}/\partial D^{n+m} \cong D^n/\partial D^n \land D^m/\partial D^m.
\]

In \( \text{Sh}_{\mathcal{X}}(\mathcal{C}artSp) \), maps which are surjective of sheaves of connected components and which induce fiberwise equivalences are equivalences of smooth stacks (this follows, for example, from the long exact sequence on sheaves of homotopy groups). It follows that the map (3.2) is an equivalence. \( \square \)

The next result allows us to identify the Thom stack of the various universal bundles over the Grassmannian with particularly nice smooth stacks.

Proposition 3.5 (Thom stacks of universal bundles). The total stacks of the Thom stacks of the

\(^{15}\)This is essentially by inspection, along with the fact that the the interior of \( D^{n+m} \) is diffeomorphic to the product of the interiors of \( D^n \) and \( D^m \).
universal bundles $\mathcal{U}(d) \to \text{BO}(d)$, $\mathcal{U}(d, N) \to \text{Gr}(d, N)$ and $\mathcal{U}^\perp(d, N) \to \text{Gr}(d, N)$ can be identified as

\[
\text{Th}(\mathcal{U}(d)) \simeq (D^d/\partial D^d)/\text{O}(d), \\
\text{Th}(\mathcal{U}(d, N)) \simeq (D^d/\partial D^d \wedge V(d, N)_+)/\text{O}(d), \\
\text{Th}(\mathcal{U}^\perp(d, N)) \simeq (D^N/\partial D^N \wedge V(N, d)_+)/\text{O}(N),
\]

where, in each example, the action on $D^d/\partial D^d$ is inherited from the action of $O(d)$ on $\mathbb{R}^d$. \[\text{[16]}\]

**Proof.** We will prove the claim for the first Thom stack and the others are proved analogously. We first make the identification in prestacks and then stackify the result. In prestacks, the map

\[
\partial D^d/\text{O}(d) \to D^d/\text{O}(d)
\]

is an objectwise monomorphisms. Thus, the homotopy quotient can be computed as the strict quotient of prestacks. This, in turn, is immediately identified with $(D^d/\partial D^d)/\text{O}(d)$. Using our standard trick for computing homotopy quotients in smooth stacks gives the identification. \[\square\]

Notice that for the smooth stacks $\text{Th}(\mathcal{U}(d, N))$ and $\text{Th}(\mathcal{U}^\perp(d, N))$, we again have equivalences

\[
\text{Th}(\mathcal{U}(d, N)) \simeq \text{Th}(\mathcal{U}(d, N)) \quad \text{and} \quad \text{Th}(\mathcal{U}^\perp(d, N)) \simeq \text{Th}(\mathcal{U}^\perp(d, N)),
\]

where on the right we have the corresponding classical Thom stacks given by applying the usual Thom space construction to the smooth manifolds $\mathcal{U}(d, N)$ and $\mathcal{U}^\perp(d, N)$ in the ambient category of smooth stacks. All of these constructions geometrically realize to give the underlying Thom spaces.

**Proposition 3.6** (Classical correspondence). *The geometric realization of the Thom stacks $\text{Th}(\mathcal{U}(d))$, $\text{Th}(\mathcal{U}(d, N))$ and $\text{Th}(\mathcal{U}^\perp(d, N))$ agree (up to equivalence) with their classical counterparts.*

**Proof.** The only nontrivial case is $\text{Th}(\mathcal{U}(d))$. But this follows immediately from the fact that the geometric realization preserves homotopy quotients along with naturality of the unit of the Quillen equivalence $\text{sing} \cdot | \cdot$. \[\square\]

### 3.2 A smooth motivic spectrum model for Thom stacks

In this section, we introduce the notion of an $D^1/\partial D^1$-spectrum and show that our geometrically refined cobordism stacks fit nicely into this setting. We also discuss some of the basic properties of these objects, leaving a more comprehensive discussion for a separate treatment.

\[\text{[16]}\text{Note that the unit disc and unit sphere are fixed under the action of O(d). Thus, we indeed get a well-defined action on the quotient.}\]
Definition 3.7 (Motivic $D^1/\partial D^1$-spectra). A $D^1/\partial D^1$-spectrum $X$ is a sequence of pointed smooth stacks $X(n)$, equipped with structure maps

$$\sigma : D^1/\partial D^1 \wedge X(n) \longrightarrow X(n + 1).$$

A morphism of $D^1/\partial D^1$-spectra is a sequence of level-wise maps $X(n) \rightarrow Y(n)$ commuting with the structure maps.

The smash product and wedge product of pointed stacks are defined analogously to those of pointed spaces. That is, for two pointed stacks $X$ and $Y$, we define the wedge product $X \wedge Y$ as the pushout

$$\begin{array}{ccc}
* & \longrightarrow & X \\
\downarrow & & \downarrow \\
Y & \longrightarrow & X \wedge Y
\end{array}$$

and the smash product as the quotient stack $X^\wedge Y$:

$$X^\wedge Y := X \times Y / (X \vee Y).$$

The following examples will be of main interest.

Example 3.8 (Madsen-Tillman spectrum). Notice that the ordinary Madsen-Tillman spectrum $\text{MT}(d)$ already defines a $D^1/\partial D^1$-spectrum via the usual maps

$$D^1/\partial D^1 \wedge \text{Th}(U^\perp(d, N)) \longrightarrow \text{Th}(U^\perp(d, N)),$$

with the subtle difference that we are taking quotients in a different ambient category, namely, that of smooth stacks instead of topological spaces.

Notice that since we have an equivalence $q : U^\perp(d, N) \rightarrow U^\perp(d, N)$, and this equivalence commutes with the structure maps (3.3), we can define the equivalent $D^1/\partial D^1$-spectrum as follows.

Example 3.9 (Stacky Madsen-Tillman spectrum). Let $\text{Th}(U^\perp(d, N))$ be the Thom stack of the universal orthogonal complement bundle over $\text{Gr}(d, N)$. By Proposition 2.37, the pullback of the bundle $U^\perp(d, N + 1) \rightarrow \text{Gr}(d, N + 1)$ by the inclusion $\text{Gr}(d, N) \hookrightarrow \text{Gr}(d, N + 1)$ decomposes as $U^\perp(d, N) \oplus 1$. By Proposition 3.4, the resulting pullback map $U^\perp(d, N) \oplus 1 \rightarrow U^\perp(d, N + 1)$ induces a map of Thom stacks

$$D^1/\partial D^1 \wedge \text{Th}(U^\perp(d, N)) \longrightarrow \text{Th}(U^\perp(d, N + 1)).$$

With these maps, $\text{Th}(U^\perp(d, N))$ becomes a smooth $D^1/\partial D^1$-spectrum with $d$ fixed and $N$ indexing the levels of the spectrum. We denote this spectrum by $\text{MT}(d)$.

Example 3.10 (Smooth motivic Thom spectrum). Let $U(d)$ be the universal bundle over the classifying stack $\text{BO}(d)$. The pullback of $U(d + 1)$ to $\text{BO}(d)$ decomposes as $U(d) \oplus 1$ and we have an induced map at the level of Thom stacks

$$D^1/\partial D^1 \wedge \text{Th}(U(d)) \longrightarrow \text{Th}(U(d + 1)).$$
Thus, Th(∪(d)) forms a smooth spectrum as d varies. We denote this spectrum by MO. It is the smooth motivic model for the Thom spectrum.

**Remark 3.11.** [Higher simplicial data] (i) We emphasize that there is no higher nondegenerate simplicial data in Example 3.9 and that we have an levelwise equivalence of $D^1/\partial D^1$-spectra

$$\text{MT}(d) \simeq \text{MT}(d).$$

(ii) In contrast, Example 3.10 has nondegenerate simplicial data in degree 1. We could also consider the usual cobordism spectrum MO as a $D^1/\partial D^1$-spectrum (similar to the Madsen-Tillman spectrum), but this spectrum would not be equivalent to MO.

**Definition 3.12** ($D^1/\partial D^1$ suspension- and infinite loop stacks). (i) Given a pointed stack $X$, we can define its $D^1/\partial D^1$-suspension spectrum by successively smashing with $D^1/\partial D^1$, with structure maps being identity. We denote this $D^1/\partial D^1$-spectrum by $\Sigma^\infty_{D^1/\partial D^1} X$.

(ii) Conversely, if we are given a $D^1/\partial D^1$-spectrum $X$, we can define its $D^1/\partial D^1$-infinite loop stack as the colimit

$$\Omega^\infty_{D^1/\partial D^1} X(n) := \lim_{\to n} \text{Map}_+(D^n/\partial D^n, X(n)),$$

where the right hand side is the pointed mapping stack, defined in the obvious way.

**Remark 3.13** (Motivic sphere vs. smooth sphere). Note that the sheaf $D^n/\partial D^n$ is not isomorphic to the smooth sphere $S^n$, viewed as a smooth manifold of dimension $n$. Indeed, if this were true, then an isomorphism would induce an isomorphism

$$\text{hom}(D^n/\partial D^n, \mathbb{R}) \simeq \text{hom}(S^n, \mathbb{R}).$$

Since the hom functor sends colimits to limits in the first variable, the Yoneda Lemma implies that the left side is in bijection the set of smooth functions on the closed disc (viewed as a manifold with boundary) $f : D^n \to \mathbb{R}$, which restrict to a constant function on the boundary. Again by the Yoneda Lemma, the right side is in bijection with all smooth functions $f : S^n \to \mathbb{R}$. But there are more elements in the former set, since any radially symmetric smooth function on the closed disc, with non-vanishing derivative at the boundary, will not define a smooth function on the sphere.

**Proposition 3.14** (Geometric realization of $D^1/\partial D^1$-spectra). Let $X(n)$ be a $D^1/\partial D^1$-spectrum. The geometric realization of $X(n)$ is an ordinary (sequential, pre-) spectrum. Moreover, the $D^1/\partial D^1$-suspensions and infinite loop stacks geometrically realize to the usual suspension spectrum and infinite loop space.

**Proof.** We first observe that since geometric realization preserves homotopy quotients (and in fact, strict quotients of sheaves), naturality of the unit of the adjunction $\cdot | \cdot$ sing implies that $|D^1/\partial D^1| \simeq |D^1|/|\partial D^1| \simeq S^1$. As a consequence, the geometric realization of the maps
σ : (D^1/∂D^1) ∩ X(n) → X(n + 1) induce maps

|σ| : S^1 ∩ |Π(X(n + 1))| → |Π(X(n))|,

where S^1 is the topological circle. This proves the first claim. For the geometric realization of the D^1/∂D^1-infinite loop space, the claim follows immediately from Proposition 2.14.

Proposition 3.14 implies that for the D^1/∂D^1-spectra MT(d), the D^1/∂D^1-infinite loop space (zero-truncated in this case) geometrically realizes to recover the topological infinite loop space of MT(d). The first of our two main theorems in the next section is devoted to refining the equivalence

α : BČob ≃ Ω^{∞-1}MT(d), (3.4)
defined in [GMTW09], to the smooth setting.

Remark 3.15 (Identifying the right stacky infinite loop space of the MT-spectrum). A natural choice for smooth analogue of the right hand side of (3.4) would be the smooth stack Ω^{∞-1}_{D^1/∂D^1}MT(d) and, indeed, its geometric realization recovers the usual space on the right side of the above equivalence. However, it turns out that this is not quite right, and for crucial reasons:

(i) The Pontrjagin-Thom construction translates cobordisms between smooth manifolds into smooth paths between maps. The stack Ω^{∞-1}_{D^1/∂D^1}MT(d) does not contain the data of smooth paths between maps

D^{d-1+N}/∂D^{d-1+N} → Th(U^\perp(d, N)).

This is extra data which needs to be accounted for.

(ii) Moreover, if we are to think of bordisms between smooth manifolds as time evolution of a smooth manifold (as motivated by physics and reflected by the definition in [GMTW09]), then the ordering on the interval [0, 1] (which indexes time) plays a crucial role.

All this information needs to be taken into account and motivates the following construction.

Time-ordering of maps. Fix a real number 0 < ε << 1. Let X be a smooth stack and let (t_0 - ε, t_1 + ε) ⊂ R be the open interval containing the closed interval [t_0, t_1] ⊂ R. One entity of main interest will be Map^{\text{col}}((ε - t_0, t_1 + ε), X), the full substack of Map((ε - t_0, t_1 + ε), X) on the maps f : (t_0 - ε, t_1 + ε) → X which make the following diagram commutative

\begin{tikzcd}
(t_0 - ε, t_0) \\

\arrow{r}

(t_0 - ε, t_1 + ε) \\
\arrow{r}

(t_0 - ε, t_1 + ε) \\
\arrow{r}

[t_1, t_1 + ε] \\
\arrow{rd}

\text{ev}_0(f) \\
\arrow{rd}

f \\
\arrow{r}

\text{ev}_1(f) \\
\arrow{r}

X
\end{tikzcd}
This is just a fancy way of saying that restricting \( f \) to the \( \epsilon \)-collars gives a constant map. For each \( \epsilon \), we have a well-defined composition law

\[
m_\epsilon : \operatorname{Map}^{\text{col}}((\epsilon - t_0, t_1 + \epsilon), X) \times_X \operatorname{Map}^{\text{col}}((\epsilon - t_1, t_2 + \epsilon), X) \to \operatorname{Map}^{\text{col}}((\epsilon - t_0, t_2 + \epsilon), X),
\]

which on vertices concatenates smooth paths using the constancy on collars to glue. We get a directed system with respect to the ordering \((\epsilon, <)\) and the maps \(m_\epsilon\) are compatible with the ordering. Thus, letting \(\epsilon \to 0\) gives a composition in the limit. By abuse of notation, we will denote the limit \(\operatorname{Map}^{\text{col}}([t_0, t_1], X) := \lim_{\epsilon \to 0} \operatorname{Map}^{\text{col}}((t_0 - \epsilon, t_1 + \epsilon), X)\) and the limiting composition map by \(m := \lim_\epsilon m_\epsilon\). With this composition map, we have an internal category given by the diagram

\[
\mathbb{R} \times X \sqcup \mathbb{R}^2_+ \times \operatorname{Map}^{\text{col}}([0, 1], X) \longrightarrow \mathbb{R} \times X,
\]

where \(\mathbb{R}^2_+ = C^\infty(\_ ; \mathbb{R}^2_+)\) is the sheaf of pairs of smooth functions \(t_0\) and \(t_1\) such that \(t_1 > t_0\) for all points in the domain. The source map evaluates at \(\{0\}\) and picks out the map \(t_0\) in the pair \((t_0, t_1)\). The target map evaluates at \(\{1\}\) and picks out the function \(t_1\). In particular, for the point Cartesian space \(\ast\), we identify the factor \(\{t_1, t_0\} \times \operatorname{Map}^{\text{col}}([0, 1], X) \cong \operatorname{Map}^{\text{col}}([t_0, t_1], X)\), where the identification precomposes with the affine diffeomorphism \(\varphi : (-\epsilon, 1 + \epsilon) \to (t_0 - \epsilon, t_1 + \epsilon)\). The composition map in (3.5) is a parametrized version of this composition. The source and target maps are induced by the usual evaluation maps.

**Definition 3.16** (Smooth concordance category). Let \(X\) be a smooth stack. We define the smooth concordance category with ordering \(\operatorname{Conc}^>(X)\) to be the diagram (3.5) with composition map \(m\) defined by the limiting operation on collarings above.\(^{17}\) We denote the realization of the nerve of this groupoid as \(\mathcal{B}\operatorname{Conc}^>(X)\).

Note that this new construct still geometrically realizes to the right object.

**Proposition 3.17** (Topological correspondence). Let \(X \in \mathcal{S}h_{\infty}(\mathcal{C}art\mathcal{S}p)\) be a smooth stack. We have a homotopy equivalence of geometric realizations

\[|\mathcal{B}\operatorname{Conc}^>(X)| \simeq |X| .\]

**Proof.** By the result of [Pa17] in Remark [2.13] we have that the geometric realization of the stack \(\operatorname{Map}^{\text{col}}([0, 1], X)\) can be computed as the simplicial set with \(n\)-vertices given by the colimit, as \(\epsilon \to 0\), of collared maps \((-\epsilon, 1 + \epsilon) \times \Delta^n \to X\). Using the Cartesian closed structure on presheaves, we see that this simplicial set can be identified with

\[
i : \operatorname{Map}^{\text{col}}((-\epsilon, 1 + \epsilon), \mathbf{sing}(X)) \longrightarrow \operatorname{Map}((-\epsilon, 1 + \epsilon), \mathbf{sing}(X)),
\]

where the inclusion is that of a full sub-\(\infty\)-groupoid on the collared maps. Any such inclusion will induce an injection on connected components. By homotopy invariance of \(\mathbf{sing}\), the projection

---

\(^{17}\)The superscript \(>\) is used in our notation since the definition of this smooth category depends heavily on the standard ordering of \([0, 1]\).
(-\epsilon, 1+\epsilon) \to *$ induces an equivalence $j: \text{sing}(X) \to \text{Map}((-\epsilon, 1+\epsilon), \text{sing}(X))$. Moreover, it is clear that the restriction of the map $i$ to the components of the constant maps $(-\epsilon, 1+\epsilon) \to * \to \text{sing}(X)$ factors through $j$ and is surjective. Thus, $i$ induces an isomorphism on connected components. By definition, $i$ induces an isomorphism on higher homotopy groups and defines an equivalence of Kan complexes. Since $\epsilon > 0$ was arbitrary, the homotopy colimit as $\epsilon \to 0$ is the homotopy colimit over a constant diagram and is therefore equivalent to $\text{sing}(X)$. Since strict filtered colimits in $\text{Sh}_{\mathcal{X}}(\text{CartSp})$ model their homotopy colimits, we conclude that

$$\text{Map}^{\text{col}}([0, 1], \text{sing}(X)) \simeq \text{sing}(X) \simeq \Pi(X).$$

From this calculation we see that geometrically realizing the nerve of the internal category \((\ref{eq:nerve})\) gives the simplicial diagram in CGWH spaces

$$\left\{ \ldots \right\} \cong \mathcal{N}(\mathcal{C}) \times |X|,$$

where $\mathcal{N}(\mathcal{C})$ is the nerve of the category with a single object and two morphisms $1$ and $a$ satisfying $a^2 = a$. \footnote{This identification is obtained by projecting out both $\mathbb{R}$ and $\mathbb{R}_+^2$, which is levelwise a weak equivalence commuting with the face maps.} But the Kan fibrant replacement of $\mathcal{N}(\mathcal{C})$ is clearly contractible. Therefore, the geometric realization of the simplicial space on the right is equivalent to $|X|$. Finally, the claim follows from Proposition 2.19.

From Proposition 3.14, we immediately get the following crucial characterization.

**Corollary 3.18** (Correspondence of infinite loop spaces). The geometric realization of the smooth stack $B\text{Conc} \to \Omega^{x-1}D_1 \{B D_1 \text{MT}(d)\}$ is equivalent to the infinite loop space $\Omega^{x-1}\text{MT}(d)$.

In the abstract Pontrjagin-Thom construction (Section 4.2), we will need to make use of the stacky homotopy type of the smooth stack $B\text{Conc} \to \Omega^{x-1}D_1 \{B D_1 \text{MT}(d)\}$. In high degrees, this turns out to be trivial.

**Proposition 3.19** (Sheaves of homotopy groups). For each $t \in \mathbb{R}$, indexing a map $t : * \to \mathbb{R}$, we have a canonical inclusion

$$t : X \hookrightarrow B\text{Conc}^\geq(X),$$

corresponding to the inclusion $t : X \hookrightarrow \mathbb{R} \times X$. For each $t$, this inclusion induces an isomorphism on $\tilde{\pi}_k$, based in the $t$-component, for all $k \geq 1$.

**Proof.** Consider the prestack which models $B\text{Conc}^\geq(X)$. From the definition of the realization, we see that this prestack can be modeled by the diagonal of the bisimplicial diagram given by taking the nerve of \((\ref{eq:nerve})\). The first few levels look like

$$B\text{Conc}^\geq(X) = \left\{ \ldots \right\} \cong \text{hom}^{\text{col}}([0, 1], X_1) \cong \mathbb{R} \times X_0 \right\} \quad (\ref{eq:prestack})$$

From Proposition 3.14 we immediately get the following crucial characterization.
where \( \text{hom} \) denotes the presheaf of maps. Consider the prestack given by

\[
Y = \left\{ \ldots \Rightarrow R \sqcup R^2 \Rightarrow R \right\}.
\]  

(3.7)

Then the projection map \( p : B\text{Conc} \to Y \) is an objectwise Kan fibration. Moreover for any fixed map \( x : * \to U \), the induced map \( \text{ev}_x : Y(U) \to Y(*) \simeq \mathcal{N}(\mathbb{R}^\delta, \leq) \) is the Kan fibration which simply evaluates the smooth functions at a fixed point. Thus for each \( U \in \text{CartSp} \), we have a Kan fibration

\[
\text{ev}_x \circ p : B\text{Conc} \to \mathcal{N}(\mathbb{R}^\delta, \leq),
\]

and the fiber at \( r \in \mathbb{R} \) can be identified with the subspace \( X(U) \times S(U) \subset X(U) \times C^\infty(U, \mathbb{R}) \) on those smooth functions \( f : U \to \mathbb{R} \) which satisfy \( f(x) = r \). Since \( C^\infty(U; \mathbb{R}) \) is zero-truncated, the inclusion at the constant map \( r : U \to \mathbb{R} \), which maps every point in the domain to \( r \) induces an isomorphism

\[
\pi_k(X(U)) \xrightarrow{\sim} \pi_k(X(U) \times S(U)).
\]

for \( k \geq 1 \). Since \( \mathcal{N}(\mathbb{R}^\delta, \leq) \) is contractible, we see that \( \text{ev}_x \circ p \) must induce an isomorphism on \( \pi_k \) for \( k \geq 1 \).

Since the stack \( \Omega_{D^1/\mathbb{R}^1}^{\infty-1} \text{MT}(d) \) is zero-truncated, we immediately deduce the following.

**Proposition 3.20.** The smooth stack \( B\text{Conc} \) \( \Omega_{D^1/\mathbb{R}^1}^{\infty-1} \text{MT}(d) \) is zero-truncated.

### 4 A stacky perspective on the cobordism category

#### 4.1 The smooth cobordism category of Galatius-Madsen-Tillman-Weiss

In this section we describe a variant of the topological cobordism category, where we regard both the space of objects and morphisms as smooth objects. Our definition will be closely related to the sheaf of categories describing the parametrized cobordisms introduced in \[\text{GMTW09}\]. In fact, if we work instead over the site of smooth manifolds, we will recover exactly the sheaf of categories described there.

**Definition 4.1** (Cobordism category \[\text{GMTW09}\]). The \( d \)-dimensional cobordism category \( \text{Cob}_d \) is the category with objects given by pairs \((M, t)\), with \( M \subset \mathbb{R}^{\infty-1} = \lim_{N \to \infty} \mathbb{R}^{N-1} \) a smooth submanifold and \( t \in \mathbb{R} \). The morphisms are given by triples \((W, t_0, t_1) : (M_0, t_0) \to (M_1, t_1)\), such that

(i) (Ordering) \( t_0 < t_1 \).

(ii) (Embedding) \( W \subset \mathbb{R}^{\infty-1} \times [t_0, t_1] \).

(iii) (Collared neighborhood) There is \( \epsilon > 0 \) such that \( W \) restricted to \([t_0, t_0 + \epsilon)\) is \( M_0 \times [t_0, t_0 + \epsilon)\) and the restriction of \( W \) to \((t_1 - \epsilon, t_1]\) is \( M_1 \times (t_1 - \epsilon, t_1]\).

(iv) (Boundary condition) \( W \cap \{t_0, t_1\} \times \mathbb{R}^{\infty-1} = \partial W \).

This definition contains quite a bit of data. However, the complexity of the definition is necessary to ensure that one has well-defined compositions of morphisms.
Figure 1: A composition of cobordisms with collared neighborhoods.

This category admits a smooth structure in a natural way, i.e. it can be made into an internal category in smooth sheaves $\mathcal{S}_{\omega}(\text{CartSp})$. To see this, let us recall that if $\text{Emb}(M, \mathbb{R}^{d-1})$ denote the set of embeddings of closed $(d-1)$-dimensional manifolds and $\text{Diff}(M)$ denote the diffeomorphism group of $M$, then both of these spaces admit infinite-dimensional, smooth, Fréchet structure (see for instance [KM97]). Moreover, the quotient space $\text{Emb}(M, \mathbb{R}^{d-1+\infty})/\text{Diff}(M)$, where diffeomorphisms act freely by precomposition, also admits the structure of a smooth infinite-dimensional manifold. In [BF81][KM97], it is shown that $\text{Emb}(M, \mathbb{R}^{d-1+\infty})/\text{Diff}(M)$ is the base space of a smooth fiber bundle

$$\pi : \text{Emb}(M, \mathbb{R}^{d-1+\infty}) \times_{\text{Diff}(M)} M \longrightarrow \text{Emb}(M, \mathbb{R}^{d-1+\infty})/\text{Diff}(M),$$

and that this bundle is a universal bundle which describes smooth manifolds parametrized over a base manifold $B$. More precisely, given a map

$$f : B \longrightarrow \text{Emb}(M, \mathbb{R}^{d-1+\infty})/\text{Diff}(M),$$

the pullback of $\pi$ by $f$ gives a submanifold $E \subset B \times \mathbb{R}^{d-1+\infty}$ and the projection to $B$ defines a smooth fiber bundle with fiber $M$.

Given that we have a smooth structure available, we can regard each of these objects as a smooth sheaf on the site of Cartesian spaces, by embedding them via the sheaf of smooth plots. For bordisms between manifolds, we can consider the analogous smooth object constructed as follows. Let $W$ be a compact $d$-manifold with collared boundary of width $\epsilon > 0$ (as depicted in Figure 1 above) and let $\text{Emb}_{\epsilon}(W, [0, 1] \times \mathbb{R}^{d-1+\infty})$ be the smooth space of embeddings of $W$. Define the infinite-dimensional smooth manifold

$$\text{Emb}(W, [0, 1] \times \mathbb{R}^{d+\infty}) := \lim_{\epsilon \to 0} \text{Emb}_{\epsilon}(W, [0, 1] \times \mathbb{R}^{d+\infty}),$$

where the colimit is taken over the obvious maps as $\epsilon \to 0$. Let $\text{Diff}_{\epsilon}(W, \partial_{\text{in}}, \partial_{\text{out}})$ denote the subgroup of the diffeomorphism group of $W$ which restricts on the collars to diffeomorphisms of
the form $\phi \times \text{id}$, with $\phi$ a diffeomorphism of the boundary. Similar to the embeddings, we set

$$\text{Diff}(W) := \lim_{\epsilon \to 0} \text{Diff}_\epsilon(W, \partial_{\text{in}}, \partial_{\text{out}}).$$

Again, we can divide out the $\text{Diff}(W)$-action on the embeddings and we get a smooth fiber bundle which classifies parametrized bordisms. This leads us to consider the following.

**Definition 4.2 (Smooth cobordism category).** The smooth cobordism category $\text{Cob}_d$ is the category with sheaf of objects and stack of morphisms given, respectively, by

$$\text{Ob}(\text{Cob}_d) := \bigsqcup_{M} \mathbb{R} \times \text{Emb}(M, \mathbb{R}^{d+\infty-1})/\text{Diff}(M),$$

$$\text{Mor}(\text{Cob}_d) := \text{Ob}(\text{Cob}_d) \sqcup \bigsqcup_{W} \mathbb{R}_+^2 \times \text{Emb}(W, [0,1] \times \mathbb{R}^{d+\infty-1})/\text{Diff}(W).$$

Under the geometric realization $|\cdot| \circ \Pi : \text{Sh}_{\infty}(\text{Cart}\mathcal{S}) \xrightarrow{\Pi} \text{sSet} \xrightarrow{|\cdot|} \text{Top}$, this smooth category recovers the topological cobordism category. This follows from two facts: First, that the geometric realization is a left adjoint at the level of 1-categories and hence preserves quotients; second, that the geometric realization of a smooth manifold is equivalent to the underlying topological space of the manifold.

**Remark 4.3 (Strict vs. homotopy quotient).** Note that we have used a strict quotient in our definition of the smooth cobordism category. In fact, since $\text{Diff}(M)$ acts freely and transitively on $\text{Emb}(M, \mathbb{R}^{d+\infty-1})$, the homotopy quotient is modeled by the strict quotient. Thus, we could have (more generally) defined the smooth category via the homotopy quotient. However, the particular model we chose will be more useful in calculation.

Note that, being an internal category in the category of smooth sheaves, we can regard $\text{Cob}_d$ as a sheaf of categories on the site of smooth manifolds, equipped with the usual topology of open covers, by sending each smooth manifold $M \in \text{Man} \to \text{Sh}_{\infty}(\text{Cart}\mathcal{S})$ to the category with objects $\text{hom}(M, \text{Ob}(\text{Cob}_d))$ and morphisms $\text{hom}(M, \text{Mor}(\text{Cob}_d))$. This sheaf of categories was considered in [GMTW09].

How can we think of the objects of the new category? Notice that, for each $U \in \text{Cart}\mathcal{S}$ an element in the set of objects $\text{Ob}(\text{Cob}_d)(U)$ is simply a trivializable bundle $N \cong M \times U \to U$ with fiber a $(d-1)$-dimensional $M$. Moreover, each $N \subset U \times \mathbb{R}^{d-1+\infty}$ and the projection onto $U$ gives the bundle map. The bundle $N$ comes equipped with a smooth function $t : U \to \mathbb{R}$, indexing the position of the fibers in time. The triviality is a consequence of the fact that we are working over the small site of convex subsets of Euclidean spaces, where all smooth fiber bundles trivialize. Similarly, one gets an identification of the set of morphisms (after evaluating at $U$) as trivial bundles $Z \cong W \times U \to U$, with boundary $\partial Z \simeq N \sqcup N'$ and a pair of smooth functions $t, t' : U \to \mathbb{R}^2$, for which $t'(u) > t(u)$ for all $u \in U$. In [GMTW09] a slightly different sheaf of categories, isomorphic to $\text{Cob}_d$, is introduced and this category is defined on the site of smooth manifolds. Since the resulting bundles may be nontrivial, their sheaf of categories involves a slightly more complicated definition.
Definition 4.4 (Classifying stack of the cobordism category). We define the smooth cobordism stack $\mathcal{B}\text{Cob}_d$ as the stacky realization of the smooth cobordism category $\text{Cob}_d$ (from Definition 4.2).

Under geometric realization, this stack indeed recovers the usual cobordism category.

Proposition 4.5 (Classical correspondence). The geometric realization of $\mathcal{B}\text{Cob}_d$ is weak equivalent to the classifying space of the topological bordism category $B\text{Cob}_d$.

Proof. This follows immediately from Proposition 2.19.

As in the case of the concordance category for the Madsen-Tillman spectrum, we also have the following.

Proposition 4.6 (Stacky triviality). The smooth stack $\mathcal{B}\text{Cob}_d$ is zero-truncated.

Proof. The same proof as that of Proposition 3.19 works here as well.

4.2 An abstract Pontrjagin-Thom construction

In this section, we prove the first of our main theorems via a smooth refinement of the Pontrjagin-Thom construction. This construction will take place entirely in the category of smooth stacks. It will also refine the usual Pontrjagin-Thom collapse map in the sense that, if we geometrically realize our stacks, we will recover the usual collapse map.

Before presenting the construction of the Pontrjagin-Thom map and the first of our main theorems, we will need a few lemmas which will be used in the construction and the proof. The first gives us a nice characterization of the hypercovers of the sphere $D^{d+N}/\partial D^{d+N}$, the second will provide a smooth family of diffeomorphisms, while the third connects these to regular values at zero.

Lemma 4.7 (Covers for $D^n/\partial D^n$). Consider the smooth unit sphere $S^{d+N} \subset \mathbb{R}^{d+N+1}$, equipped with a choice of basepoint $\infty : * \to S^{d+N}$. Let $\{U_\beta\}$ be a good open cover of $S^{d+N} - \{\infty\}$ and let $V$ is a sufficiently small geodesically convex open neighborhood of $\{\infty\}$, so that $Y := \bigsqcup_\beta U_\beta \sqcup V$ is a good open cover of $S^{d+N}$. Let $u : D^{d+N}/\partial D^{d+N} \to S^{d+N}$ be the map induced by the universal property of the quotient. Then

(i) the pullback $Y' := u^*Y$ defines an effective epimorphism $Y \to D^{d+N}/\partial D^{d+N}$.

(ii) Moreover, the pullback is of the form $Y' = \bigsqcup_\beta W_\beta \sqcup Z$, with $\{W_\beta\}$ a cover of the interior $(D^{d+N})^\circ$ (in the traditional sense) and $Z$ the pullback of $V$.

Proof. The first claim is immediate, since effective epimorphisms are stable under pullback. The second claim follows from the fact that coproducts commute with pullbacks in any topos (i.e. coproducts are universal).
Lemma 4.8 (A smooth family of diffeomorphisms). Fix $1 > \epsilon > 0$. Then, for each real number $0 < \delta < \epsilon$, and for any point $y \in D^N_\delta \subset D^N$, there is a smooth family of diffeomorphisms $\varphi^y_t : D^N \to D^N$, $t \in [0,1]$, such that

(i) $\varphi^y_0 = \text{id}$, $\varphi^y_1(y) = 0 \in D^N$, and

(ii) for each pair of equidistant points $y, y' \in D^N_\delta$, there is fixed orthogonal transformation $Q^{y,y'} : D^N \to D^N$ satisfying $\varphi^y_t \circ Q^{y,y'} = Q^{y,y'} \circ \varphi^y_t$, for all $t$.

Proof. Let $0 < \delta < \epsilon$. Fix a radially symmetric smooth bump function $\rho_\epsilon$ on $D^N$ which has support in the interior of $D^N_\delta$ and which is 1 on $D^N_\delta$. Let $y \in D^N_\delta$ and consider the smooth family of diffeomorphisms

$$\varphi^y_t(x) := x - \rho_\epsilon(x)ty.$$ 

Then $\varphi^y_0(x) = x$, $\varphi^y_1(y) = y - \rho_\epsilon(y)y = 0$. Moreover if $y$ and $y'$ are equidistant, we can choose an orthogonal transformation $Q^{y,y'}$ taking $y$ to $y'$. Then

$$Q^{y,y'} \circ \varphi^y_t = Q^{y,y'}(x) + \rho_\epsilon ty = \varphi^y_t(Q^{y,y'}(x)).$$ 

Lemma 4.9 (Regular values on the disc). Fix $1 > \delta > 0$. Let $\{f_k\}_{k=1}^\infty : D^N \to D^N$ be any countable collection of smooth functions. Then there is disc of radius $0 \leq \gamma \leq \delta$ whose boundary contains regular values of all the $f_k$’s.\footnote{We use the convention that the boundary of the trivial disc $\{0\}$ is again the point $\{0\}$.}

Proof. Suppose that is not the case. Then for all discs of radius $D^N_\gamma \subset D^N_\delta$, there is a function $f_k \in \{f_k\}_{k=1}^\infty$ such that the boundary contains no regular values of $f_k$. Let $S_k$ be the set of all discs (including the degenerate case $\{0\}$) with no regular values of $f_k$ on the boundary. Let $E_k \subset D^N_\delta$ be the union of all the boundaries of the discs in $S_k$. Since $D^N_\delta$ is the union of all $N$-spheres centered at zero of arbitrary radius $0 \leq \gamma < \delta$, by hypothesis, we can write

$$D^N_\delta = \bigcup_{k=1}^\infty E_k.$$ 

Let $C_k$ denote the set of all critical values of $f_k$. By Sard’s theorem, each $C_k$ has measure zero in $D^N$. If each $E_k \subset C_k$ was measurable, then the right hand side would be a countable union of measure zero sets, and hence would have measure zero. Clearly the left hand side has positive measure and we would arrive at the desired contradiction.

It remains to prove that each $E_k$ is measurable and to this end, it suffices to prove that each $E_k$ is closed. Indeed, let $\{x_m\}$ be a sequence in $E_k$ and suppose $x_m \to x_\infty \in D^N_\delta$. Since the set of critical values $C_k$ of $f_k$ is closed, and every point on $E_k$ is a critical point, we have that $x_\infty \in C_k$. We claim that actually $x_\infty \in E_k$. Fix any point on the sphere of radius $|x_\infty| \geq 0$ and an orthogonal transformation $Q$ sending $x_\infty$ to $y$. Since $Q$ is an orthogonal transformation and $x_m \in E_k$ for all
m, we must have $Q(x_m) \in E_k$ for all $m$. Since $Q$ is continuous on the disc, we have

$$\lim_{m \to \infty} Q(x_m) = Q(x_\infty) = y.$$ 

Since $C_k$ is closed we, therefore, have $y \in C_k$. Since $y$ was an arbitrary point on the sphere of radius $|x_\infty|$, we must have $x_\infty \in E_k$. \qed

**The collapse map.** This map will be a morphism of smooth sheaves

$$\text{PT} : \tilde{\pi}_0(\text{Bcob}_d) \longrightarrow \tilde{\pi}_0(\text{BConc}^{\geq}(\Omega^{x-1}_{D^1/\partial D^1} \text{MT}(d))) \ , \ (4.1)$$

where the source and target categories are defined in Section 4.1 (see Definition 4.2) and Section 3.2 (see Definition 3.16 Corollary 3.18 and Proposition 3.19), respectively. Recall moreover that since $\Omega^{x-1}_{D^1/\partial D^1} \text{MT}(d)$ is zero-truncated as a smooth stack and since $\text{MT}(d)$ is equivalence to the $D^1/\partial D^1$-spectra of strict sheaves $\text{MT}(d)$ (see Remark 3.11), up to equivalence the smooth category $\text{Conc}^{\geq}(\Omega^{x-1}_{D^1/\partial D^1} \text{MT}(d))$ has sheaf of objects and morphisms given, respectively, by

$$\text{Ob} := \lim_{N} \mathbb{R} \times \text{hom}_+(D^{d+N-1}/\partial D^{d+N-1}, \text{Th}(U^\perp(d, N))),$$

$$\text{Mor} := \lim_{N} \mathbb{R}^2 \times \text{hom}_+\left(\left([0,1]_+ \wedge D^{d+N-1}/\partial D^{d+N-1}, \text{Th}(U^\perp(d, N))\right) \sqcup \text{Ob},

$$

where $\text{hom}_+(-, -)$ denotes the sheaf of pointed maps.

Fix a Cartesian space $U \in \text{CartSp}$. After evaluating the sheaf of morphisms at $U$, an element of the resulting set is a choice of submanifold $N \subset U \times \mathbb{R}^{d+N-1}$, such that the map $\pi : N \to U$ is a trivializable bundle, with fiber $M$. We, therefore, consider the diagram

$$T^\perp N \cong T^\perp M \times U \xrightarrow{\gamma} U^\perp(d, N) \longrightarrow U^\perp(d) \ (4.2)$$

$$\text{id} \times \pi \downarrow \downarrow \downarrow \downarrow$$

$$N \cong M \times U \xrightarrow{T^\pi M} \text{Gr}(d, N) \longrightarrow \text{Gr}(d, \infty),$$

where $T^\pi M$ is the fiberwise Gauss map, sending a pair $(x, u) \in M \times U$ to $T_x M \times \mathbb{R} \subset \mathbb{R}^{d+N}$. The map $\gamma$ defined fiberwise as the canonical map which sends a pair $(v, u) \in T^\perp M \times U$, $v \in T_x^\perp M$, to $v \in \mathbb{R}^N$. Taking a tubular neighborhood of $M$ in some disc with sufficiently large radius $D^{d+N-1}$ and collapsing the complement \footnote{Note that we are collapsing in smooth stacks and not in spaces.} gives a map

$$(D^{d+N-1}/\partial D^{d+N-1}) \wedge U^+ \longrightarrow \text{Th}(T^\perp M) \wedge U^+ \longrightarrow \text{Th}(U^\perp(d, N)) ,$$

which is independent of the choices made up to (collared) concordance. Similarly, we can define the map at the level of bordisms, using the Gauss map sending $(y, u) \mapsto T_y W \subset \mathbb{R}^{d+N}$, to get a
(collared) concordance

\[(D^{d+N-1}/\partial D^{d+N-1}) \land [0, 1]_+ \land U_+ \longrightarrow \text{Th}(T^1 \mathcal{W}) \land U_+ \longrightarrow \text{Th}(\mathcal{U}^1(d, N))\,

which depends on a choice of tubular neighborhood and the radius of the disc $D^{d+N-1}$. This shows that the map $\PT$ is well defined and natural in $U$.

We now claim that the collapse map defined above induces a weak equivalence of smooth stacks. In the introduction we claimed that the proof of our main theorem has the advantage of avoiding some rather delicate differential topology theorems (in particular, Phillips’ submersion theorem [Ph67]) used in the proof of [GMTW09] (see [Fr12] for an exposition). We will still need a bit of local transversality, and the price we pay for simple local conditions is complicated gluing conditions. We now proceed with this setup.

**Remark 4.10 (Transversality).** We could probably have avoided all transversality issues had we chosen to work with derived manifolds and derived cobordisms between them. However, since we wanted to directly connect the cobordism category of [GMTW09] with our smooth motivic motivic model for the Madsen-Tillman spectrum, we chose to work with smooth manifolds instead.

Considering now the stacky model $\mathcal{U}^1(d, N)$ for $U\mathcal{K}p_{d, N}$, recall that a morphism $f : M \to \text{Th}(\mathcal{U}^1(d, N))$ is secretly defined as a map out of the Čech nerve $\check{C}\{\{U_\alpha\}\}$ of some good open cover of $M$. Unravelling the resulting cocycle data, we see that such a map in particular gives an assignment morphisms $f_\alpha : U_\alpha \to D^N/\partial D^N$ on open sets, which on intersections obeys the compatibility relation $f_\beta = g_{\alpha\beta} f_\alpha$, for $g_{\alpha\beta} : U_{\alpha\beta} \to O(N)$. We will say that the map $f$ is regular if for all $\alpha$ such that $f_\alpha : U_\alpha \to D^N/\partial D^N$ factor through $D^N$, $0 \in D^N$ is a regular value of the smooth function $f_\alpha$.

**Proposition 4.11 (A homotopy for the classifying space of concordance).** Choose a point $f : * \to \mathcal{B}\text{Conc}^>(\Omega_{{D^1/\partial D^1}}^X \mathcal{M}\text{T}(d))(U)$. Such a point is equivalently a choice of map

\[f : D^{d-1+N}/\partial D^{d-1+N} \land U_+ \longrightarrow \text{Th}(\mathcal{U}^1(d, N))\]

and a smooth function $a : U \to \mathbb{R}$. Then there exists an edge

\[H : \Delta[1] \longrightarrow \mathcal{B}\text{Conc}^>(\Omega_{{D^1/\partial D^1}}^{X-1} \mathcal{M}\text{T}(d))(U)\ , \quad (4.3)\]

such that $d_0 H = (f, a)$ and $d_1 H = (g, a')$, with $a' > a$ and $g$ regular. Moreover, $H$ can be chosen so that the preimages $N_\beta = g_\beta^{-1}(0) \cong M_\beta \times U$, with $g_\beta$ the restriction of $g$ to a specified cover of $D^{d-1+N}/\partial D^{d-1+N}$ and $M_\beta \subset D^{d-1+N}$ a submanifold.

---

\[21\text{By local, we mean a condition that holds on local patches and is appropriately compatible on intersections.}\]

\[22\text{Here } d_0 \text{ and } d_1 \text{ denote the evaluations at the respective endpoints of the edge.}\]
Proof. An edge \([4.3]\) is a choice of collared maps

\[
H : [0, 1]_+ \wedge D^{d-1+N}/\partial D^{d-1+N} \wedge \Delta[1]_+ \wedge U_+ \rightarrow \text{Th}(U^\perp(d, N)),
\]
and a pair of smooth functions \(a < a' : U \rightarrow \mathbb{R}\). The face maps are given by precomposing with the composition of coface maps \(\delta^0, \delta^1 : \{\ast\} \hookrightarrow [0, 1]\) and \(d^0, d^1 : \Delta[0] \rightarrow \Delta[1]\). The face maps send the ordered pair \((a, a')\) to \(a\) and \(a'\), respectively. In particular, we can take the collared map to be of the form

\[
H : [0, 1]_+ \wedge D^{d-1+N}/\partial D^{d-1+N} \wedge U_+ \rightarrow \text{Th}(U^\perp(d, N)),
\]
by considering it as degenerate in the simplicial direction.

We need to identify what such an edge like this looks like in terms of cocycle data. To this end, recall that \(\text{Th}(U^\perp(d, N))\) can be identified with the smooth stack given level-wise by

\[
\text{Th}(U^\perp(d, N)) = \left\{ \ldots \left\langle (O(N) \times V(N, d))_+ \wedge D^N/\partial D^N, V(N, d)_+ \wedge D^N/\partial D^N \right\rangle \right\}.
\]

Let \(\{W_\beta\} \cup \{Z\}\) be a generalized cover of \(D^N/\partial D^N\) of the form presented in Lemma \(4.7\). After passing to this cover and working out the cocycle data, we see that a map \(H\) is uniquely determined by the following maps

\[
\begin{align*}
(a, a') : U &\rightarrow \mathbb{R}^2_+ \\
H^\beta : [0, 1] \times U \times W_\beta &\rightarrow D^N/\partial D^N, \\
H^Z : [0, 1]_+ \wedge U_+ \wedge Z &\rightarrow D^N/\partial D^N,
\end{align*}
\]

\[
\begin{align*}
g^{\alpha\beta} : [0, 1] \times U \times W_{\alpha\beta} &\rightarrow O(N) \times V(N, d), \\
g^{Z\beta} : [0, 1] \times U \times W_\beta \cap Z &\rightarrow O(N) \times V(N, d),
\end{align*}
\]

which satisfy the usual compatibility condition on the intersection. \(\text{[23]}\) Moreover, the last two maps must interpolate between the transition data for the endpoints. First observe that since \(U\) is a convex open subset of \(\mathbb{R}^k\), for some \(k\), \(U\) smoothly deformation retracts to any fixed point \(u \in U\). Choose such a retraction \(r_1 : U \rightarrow U\) which is constant on the collars of the interval. Precomposing all the cocycle maps determining \(f\) with this deformation retraction gives an edge connecting \(f\) to a map which is constant on the factor \(U\). By composing edges, we can therefore assume that \(f\) is constant on \(U\).

Let \(f^\beta : U \times W_\beta \rightarrow D^N/\partial D^N\) be the the cocycle maps determined by \(d_0H = f\). Let \(\{f^\gamma\}\) be the subset of all \(\{f^\beta\}\) factoring through \(D^N\). Since the chosen cover is countable, in fact finite, Lemma \(4.9\) implies that we can find a small sphere in \(D^N\) containing regular values \(y_\gamma\) for each \(f^\gamma\). For each \(\gamma\) in this finite set, let \(H^\gamma\) be post-composition with the smooth family of diffeomorphisms guaranteed by Lemma \(4.8\). Choose \(g^{\alpha\beta}\) and \(g^{Z\beta}\) to be the transition maps of \(f\), constant in the direction of the interval \([0, 1]\). For all the \(f^\beta\)'s which do not factor through \(D^N\), choose \(H^\beta = f^\beta\) and \(H^Z = f^Z\), constant in the direction of \([0, 1]\). The edge corresponding to these choices has the desired property.

\(\text{[23]}\) Note that we are abusing notation by writing the interval \([0, 1]\) as closed. These maps are colimits of collared maps on open intervals \((0 - \epsilon, 1 + \epsilon)\) as \(\epsilon \rightarrow 0\).
Finally, it is clear from the definition that if \( f \) is constant on \( U \), then \( g = d_1H \) is constant on \( U \) and for each \( \beta \), the projection of the preimage \( N_\beta = g^{-1}(0) \to U \) is a trivial bundle \( N_\beta \cong M_\beta \times U \) with \( M_\beta = (g \circ u)^{-1}(0) \) and \( u : W_\beta \to U \times W_\beta \) induced by a point \( u : * \to U \).

With all the above setting up, we are now ready for our first main theorem.

**Proposition 4.12 (Sheaves of connected components).** The map (4.1) is an isomorphism.

**Proof.** We construct the inverse map to \( PT \) objectwise. Suppose we are given a map

\[
f' : D^{d-N-1}/\partial D^{d-N-1} \times U \to \text{Th}(\mathcal{U}^\perp(d, N))
\]

representing an element in the right hand side of (4.1). We want to construct a convenient representative for the homotopy pullback

\[
\begin{array}{ccc}
D^{d-N-1}/\partial D^{d-N-1} \times U & \xrightarrow{f'} & \text{Th}(\mathcal{U}^\perp(d, N)) \\
\downarrow & & \downarrow \text{zero section} \\
N & \xrightarrow{\text{zero section}} & \text{Gr}(d, N)
\end{array}
\]

Since \( \text{Th}(\mathcal{U}^\perp(d, N)) \cong \text{Th}(\mathcal{U}^\perp(d, N)) \) and \( \text{Gr}(d, N) \cong \text{Gr}(d, N) \), it suffices to consider instead the homotopy pullback diagram

\[
\begin{array}{ccc}
D^{d-N-1}/\partial D^{d-N-1} \times U & \xrightarrow{f'} & \text{Th}(\mathcal{U}^\perp(d, N)) \\
\downarrow & & \downarrow \text{zero section} \\
N & \xrightarrow{\text{zero section}} & \text{Gr}(d, N)
\end{array}
\]

Let \( \{W_\beta\} \cup \{Z\} \) be a cover of \( D^{d-N}/\partial D^{d-N} \), as in Lemma 4.7. We will define the inverse map locally on the covering and show that we can "glue" the results together to get the desired object. For the sake of readability, we will break the proof up into steps.

**Step 1 (The local calculation)** For each \( W_\beta \), let \( N_\beta \) be defined by the homotopy pullback

\[
\begin{array}{ccc}
W_\beta \times U & \xrightarrow{f'_\beta} & \text{Th}(\mathcal{U}^\perp(d, N)) \\
\downarrow & & \downarrow \text{zero section} \\
N_\beta & \xrightarrow{\text{zero section}} & \text{Gr}(d, N)
\end{array}
\]

where \( f'_\beta \) is the restriction to \( W_\beta \). Since \( W_\beta \) is zero-truncated, we can use the explicit presentations of the stacks on the right discussed in Section 2.3 to compute this homotopy pullback explicitly as the strict pullback

\[
\begin{array}{ccc}
W_\beta \times U & \xrightarrow{f'_\beta} & D^N/\partial D^N \\
\downarrow & & \downarrow 0 \\
N_\beta & \xrightarrow{0} & *
\end{array}
\]

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where the vertical map is induced by the map \( 0 : * \to D^N \to D^N/\partial D^N \) which picks out zero in the closed unit disc. Since \( W_\beta \) is representable, the definition of \( D^N/\partial D^N \) implies that \( f_\beta' \) can be identified with an equivalence class smooth functions \( f_\beta' : W_\beta \times U \to D^N \). This equivalence identifies all smooth functions which factor through the boundary. Notice that if \( f_\beta' \) factors through the boundary, then since the boundary is far away from zero, \( N_\beta = \emptyset \) and this is true for any representative of the class. Thus, we can assume that \( f_\beta' \) factors as \( f_\beta' : W_\beta \times U \to D^N \to D^N/\partial D^N \) and we are reduced to the pullback square of smooth manifolds

\[
\begin{array}{ccc}
W_\beta \times U & \xrightarrow{f_\beta'} & D^N \\
\downarrow & & \downarrow 0 \\
N_\beta & \xrightarrow{\sim} & * \\
\end{array}
\]  

(4.6)

If \( f_\beta' \) has zero as a regular value, then \( N_\beta \) will be a smooth submanifold of \( W_\beta \). Here is where we will use the fact that the inverse need only be well-defined at the level of \( \tilde{r}_0 \), and our transversality from Proposition 4.11 implies that we can assume this is the case. Moreover (by the same proposition), we can assume that \( f_\beta \) is constant in the \( U \) direction so that \( N_\beta \cong M_\beta \times U \).

**Step 2. (Understanding the construction as a “local” inverse)** Each \( N_\beta \) comes as the preimage of a regular value of \( 0 \in D^N \). Hence the kernel of \( df_\beta \) is identified with the tangent bundle \( TN_\beta \cong TM_\beta \times \mathbb{R}^{\dim(U)} \), where the second factor is the trivial bundle of rank \( \dim(U) \) over \( U \). In addition, \( N_\beta \) inherits a local framing of the normal bundle from a choice of basis of orthonormal basis \( \{v_i\} \), with \( v_i \in D^N \). Summarizing in our case, the normal bundle is identified with \( T^\perp N_\beta \cong T^\perp M_\beta \times U \) where the \( T^\perp M_\beta \) is normal to \( TM_\beta \) in \( D^{d+N-1} \) and \( f_\beta' \) sends each framing of \( T^\perp M_\beta \times U \) to a basis \( \{v_i\} \).

The map \( f_\beta' \) appearing in (4.6) is implicitly the restriction of the original map \( f' \) to the factor \( D^N/\partial D^N \). Consequently, each \( f_\beta' \) also determined a map to \( V(N,d) \), which chooses a \( d \)-plane in \( \mathbb{R}^{d+N} \), up to some ambiguity parametrized by a choice of orthonormal basis of the complement. The discussion above shows that each framing of \( T^\perp M_\beta \) gives a basis of the complement of this \( d \)-plane. After dividing out by the action of \( O(N) \), we thus see that \( f_\beta' \) gives exactly the map \( \gamma \) in (4.2), but restricted to some local patch of \( M \). Thus, provided we can glue the \( N_\beta \cong M_\beta \times U \) together, we will immediately see that this construction gives a two-sided inverse.

**Step 3. (Gluing)** It remains to show that the manifolds \( N_\beta \) glue properly. Recall that, since \( \{W_\beta \times U\} \cup \{Z \cap U_+\} \) is a cover of \( D^{d+N-1}/\partial D^{d+N-1} \cap U_+ \), a map of the form (4.4) is uniquely determined by the data of a collection of maps \( f_\beta' : W_\beta \times U \to D^N/\partial D^N \times V(N,d) \) on open sets and maps \( g_{\beta\gamma} : W_{\beta\gamma} \times U \to O(N) \times V(N,d) \) on intersections satisfying some cocycle conditions. We will only be concerned with the condition that affects the base level. That is, we have \( f_\beta' = g_{\beta\gamma} f_\gamma' \) on intersections, where the juxtaposition on the right is given by the \( O(N) \)-action. Since \( 0 \in D^N/\partial D^N \) is fixed by the \( O(N) \)-action, we have

\[
(W_\beta \times U) \cap N_\gamma = f_\gamma^{-1}_{|W_{\beta\gamma}}(0) = (g_{\beta\gamma} f_\beta')^{-1}_{|W_{\beta\gamma}}(0) .
\]
Since the $g_{\beta,\gamma}$ act by invertible linear maps, we immediately get

$$(g_{\beta,\gamma} f_\beta)^{-1}_{W_{\beta,\gamma}}(0) = (f_\beta)^{-1}_{W_{\beta,\gamma}}(0) = (W_\gamma \times U) \cap N_\beta.$$  

Then the union $\bigcup_\beta N_\beta = N$ defines a smooth submanifold of $(D^{d+N-1})^0 \times U \subset D^{d+N-1}/\partial D^{d+N-1} \times U_+$. The projection $N \to U$ is a fiber bundle with typical fiber $M = \bigcup_\beta M_\beta$ and we have determined a submanifold $N \subset D^{d+N-1} \times U \subset \mathbb{R}^{d+N-1}$, which is diffeomorphic to the product $M \times U$ and this defines an object in $\mathcal{B}\text{Cob}_d(U)$. By the local explanation above (Step 2), we immediately see that this map is a two-sided inverse. $\square$

Finally, we observe that since both $\mathcal{B}\text{Cob}_d$ and $\mathcal{B}\text{Conc}^{>}(\Omega^{\infty-1}_{D^1}; \text{MT}(d))$ are zero-truncated, they are equivalent to their sheaves of connected components. This immediately gives the first of our main theorems

**Theorem 4.13.** The map (4.1) induces an equivalence

$$\text{PT} : \mathcal{B}\text{Cob}_d \xrightarrow{\simeq} \mathcal{B}\text{Conc}^{>}(\Omega^{\infty-1}_{D^1}; \text{MT}(d)).$$

## 5 Higher smooth tangential structures

### 5.1 Adding $G$-structure

In this section, we describe how to add smooth tangential structures to the picture. Much of the machinery established in the previous section can be used to include these various structures on the tangent bundle via pullback. In general, if we are given a Lie group $G$ and a faithful representation $G \to O(d)$, then delooping this map gives rise to a map

$$\theta : BG \to BO(d).$$

For example, $\theta$ could be induced by a representation of $G$. Note that although it is not true that $\text{Gr}(d, N) \to BO(d)$ as $N \to \infty$, there is still a map $\text{Gr}(d, \infty) \to BO(d)$ which is induced by taking the model $\text{Gr}(d, \infty) = V(d, \infty)/O(d)$ and projecting out $V(d, \infty)$. These observations lead to the following definition, in analogy to the tangential structures considered in [GMTW09].

**Definition 5.1** (Smooth Grassmannian stack and universal bundle with $\theta$-structure). (i) We define the smooth Grassmanian stack with $\theta$-structure as the homotopy pullback

$$
\begin{array}{ccc}
\text{Gr}(\theta, N) & \rightarrow & BG \\
\downarrow \theta_N & & \downarrow \theta \\
\text{Gr}(d, N) & \rightarrow & BO(d).
\end{array}
$$
We define the universal bundle $U_p^\theta, N \rightarrow \text{Gr}(\theta, N)$ as the homotopy pullback

\[
\begin{array}{c}
\text{Gr}(\theta, N) \\
\downarrow \theta_N \\
\text{Gr}(d, N) \\
\end{array}
\quad
\begin{array}{c}
\text{Gr}(\theta, N) \\
\downarrow \theta_N \\
\text{Gr}(d, N) \\
\end{array}
\]

We define the universal orthogonal complement bundle $U_p^\perp, N \rightarrow \text{Gr}(\theta, N)$ as the homotopy pullback

\[
\begin{array}{c}
\text{Gr}(\theta, N) \\
\downarrow \theta_N \\
\text{Gr}(d, N) \\
\end{array}
\quad
\begin{array}{c}
\text{Gr}(\theta, N) \\
\downarrow \theta_N \\
\text{Gr}(d, N) \\
\end{array}
\]

Since these stacks are given by pullbacks, they enjoy much of the same properties as their counterparts without $\theta$-structure. In particular, we have an extension of the splitting result in Proposition [2.31] to include $\theta$-structures.

**Proposition 5.2.** The pullback of $U_p^\perp, N \rightarrow \text{Gr}(\theta, N)$ decomposes as the sum $U_p^\perp, N \oplus 1$.

**Proof.** Let $i : \text{Gr}(\theta, N) \hookrightarrow \text{Gr}(\theta, N + 1)$ be the map induced by the inclusion on Grassmannian stacks. By Proposition [2.31], we have an isomorphism of bundles

\[
i^\ast \theta_{N+1}^\ast U_p^\perp(d, N + 1) \cong \theta_{N+1}^\ast i^\ast U_p^\perp(d, N + 1)
\cong \theta_{N+1}^\ast (U_p^\perp(d, N) \oplus 1)
\cong (\theta_N^\ast U_p^\perp(d, N)) \oplus 1.
\]

As in the case without $\theta$-structure, Proposition [5.2] gives rise to maps

\[
D^1/\partial D^1 \wedge \text{Th}(U_p^\perp(\theta, N)) \longrightarrow \text{Th}(U_p^\perp(\theta, N + 1)),
\]

which turn $\text{Th}(U_p^\perp(\theta, N))$ into a $D^1/\partial D^1$-spectrum as $N$ varies. We denote this spectrum accordingly as $\text{MT}(\theta)$. The homotopy pullbacks in Proposition [5.2] are easily seen to be zero-truncated. Indeed, fix a basepoint, say $x : * \rightarrow \text{Gr}(\theta, N)$. Then we have a corresponding homotopy fiber sequence

\[
\text{Gr}(\theta, N) \longrightarrow \text{B}G \times \text{Gr}(d, N) \longrightarrow \text{BO}(d).
\]

Since $G \cong \tilde{\pi}_1(BG) \hookrightarrow O(d) \cong \tilde{\pi}_1(BO(d))$, the long exact sequence on sheaves of verifies the claim. It follows that each of the above homotopy pullbacks must be equivalent to its sheaf of components, which we denote by $\text{Gr}(\theta, N)$, $U_p(\theta, N)$ and $U_p^\perp(\theta, N)$. The latter two objects are still vector bundles over $\text{Gr}(\theta, N)$. We can, therefore, define the sheaf of bundle maps $\text{Bun}(TM, U_p(\theta))$ as the sheaf which on each Cartesian space $U$ assigns the set of bundle maps

\[
\text{Bun}(TM, U_p(\theta))(U) := \{ p : TM \times U \rightarrow U_p(\theta) : p \text{ is a fiberwise bundle map} \}.
\]
We can make the same definition using the stack $U(\theta)$, and given that the canonical equivalence $\varphi : U(\theta) \to U(\theta)$ defines a bundle map, we have an equivalence $\text{Bun}(TM, U(\theta)) \simeq \text{Bun}(TM, U(\theta))$.

We define the smooth sheaf $\text{Emb}_\theta(M, \mathbb{R}^{d-1+\infty})$ as the pullback of sheaves

$$
\begin{align*}
\text{Emb}_\theta(M, \mathbb{R}^{d-1+\infty}) & \to \text{Bun}(TM, U(\theta)) \\
\text{Emb}(M, \mathbb{R}^{d-1+\infty}) & \to \text{Bun}(TM, U(d))
\end{align*}
$$

where $TM$ denotes the canonical bundle map lifting the Gauss map $TM : M \times \mathbb{R} \to \text{Gr}(d, \infty)$.

**Remark 5.3** (Space vs. smooth sheaves of bundle maps). *It is well known that the space of bundle maps $\text{Bun}(TM, U(d))$ is contractible (see for example [GMTW09, Lemma 5.1]). However, this is of course not the case as in smooth sheaves. Nevertheless, the geometric realization of $\text{Emb}_\theta(M, \mathbb{R}^{d-1+\infty})$ agrees with the corresponding space defined in [GMTW09].*

In the same spirit as the previous section, we can form the smooth cobordism category with smooth $G$-structure (hence using classifying stacks $BG$ rather than classifying spaces $BG$). We make the following definition.

**Definition 5.4** (Smooth cobordism category with $G$-structure). The cobordism category with $G$-structure has as sheaf of objects and morphisms, respectively,

$$
\begin{align*}
\text{Ob}(\text{Cob}_G) & := \bigsqcup_{[M]} \mathbb{R} \times \text{Emb}_\theta(M, \mathbb{R}^{d-1+\infty})/\text{Diff}(M) \\
\text{Mor}(\text{Cob}_G) & := \bigsqcup_{[W]} \mathbb{R}^2 \times \text{Emb}_\theta(W, [0,1] \times \mathbb{R}^{d-1+\infty})/\text{Diff}(W)
\end{align*}
$$

The sections of the sheaf of objects can be identified with a triple $(N, t, l)$, where $N \cong M \times U \subset \mathbb{R}^{d+\infty-1}$ is a bundle of $(d-1)$-dimensional manifolds $M$, $t : U \to \mathbb{R}$ is a smooth function and $l : N \cong M \times U \to BG$ is a lift of $TM : N \cong M \times U \to \text{Gr}(d, \infty) \cong \text{Gr}(d, \infty) \to BO(d)$. The morphisms are identified similarly, with lifts $l_{t,t'} : Z \cong W \times U \to BG$ which are required to restrict to the maps $l_t : N \to BG$ and $l_{t'} : N' \to BG$ on collars.

Notice that the proof of the theorem with $G$-structure is almost a direct extension of the proof without $G$-structure (Theorem 4.13), given by replacing $U^\downarrow(d)$ with $U^\downarrow(\theta)$. Indeed, we still have a well-defined collapse map

$$
\text{PT} : \tilde{\pi}_0(B\text{Cob}_G) \to \tilde{\pi}_0(B\text{Conc}^\gamma(\Omega_{D^1/D^1}^{\infty-1} \times MT(\theta)))
$$

defined via the universal property of the pullback, and the same proof of Proposition 3.19 applies. We will simply sketch how to construct the inverse map in this case.

\[24\text{Here we really mean a lift of any map in the connected component that TM : N \to Gr}(d, \infty) \text{ corresponds to in Map}(N, \text{Gr}(d, \infty)).\]
Theorem 5.5 (Stacky Pontrjagin-Thom equivalence with \( \theta \)-structure). The map \( (5.2) \) induces an equivalence

\[
\text{PT} : \text{B} \text{Cob}_\theta \longrightarrow \text{B} \text{Conc} \prec \left( \Omega_{D^1/\partial D^1}^{\infty-1} \text{MT}(\theta) \right)
\]

(5.3)

Proof. As discussed above, we will sketch the construction of the inverse to the map \( (5.2) \). Fix a map

\[
f' : D^{d+N-1}/\partial D^{d+N-1} \wedge U_+ \longrightarrow \text{Th}(\mathcal{U}^\perp(\theta, N))
\]

and cover \( D^{d+N-1}/\partial D^{d+N-1} \) as in the constructions in Section 4.2. We are again led to the homotopy pullback diagram

\[
\begin{array}{ccc}
D^{d+N-1}/\partial D^{d+N-1} \wedge U_+ & \rightarrow & \text{Th}(\mathcal{U}^\perp(\theta, N)) \\
\downarrow & & \downarrow \\
M & \rightarrow & \text{Gr}(\theta, N)
\end{array}
\]

(5.4)

where \( M \) has yet to be identified. By definition, the stacks \( \text{Th}(\mathcal{U}^\perp(\theta, N)) \) and \( \text{Gr}(\theta, N) \) fit into a pullback diagram

\[
\begin{array}{ccc}
\text{Th}(\mathcal{U}^\perp(\theta, N)) & \rightarrow & \text{Th}(\mathcal{U}^\perp(d, N)) \\
\downarrow & & \downarrow \\
\text{Gr}(\theta, N) & \rightarrow & \text{Gr}(d, N)
\end{array}
\]

Precomposing both vertical maps with the maps induced from the zero section, we get a diagram

\[
\begin{array}{ccc}
\text{Gr}(\theta, N) & \rightarrow & \text{Th}(\mathcal{U}^\perp(\theta, N)) \\
\downarrow & & \downarrow \\
\text{Gr}(d, N) & \rightarrow & \text{Th}(\mathcal{U}^\perp(d, N))
\end{array}
\]

The outer diagram has top and bottom horizontal maps as identities and is, therefore, homotopy Cartesian. The right diagram is homotopy Cartesian by definition and so the Pasting Lemma implies that the left square is homotopy Cartesian. Combining the left square above with diagram \( (5.4) \), we see that the pasting law for homotopy pullbacks allows us to compute \( M \) as the homotopy pullback of the resulting outer square, which was already computed in Theorem 4.13. This presents \( M \) manifestly as a smooth manifold, equipped with all the necessary data to identify it with an object in \( \text{Cob}_\theta \). \( \square \)

This then gives a Pontrjagin-Thom construction at the level of \( G \)-bundles with data of gauge transformations. The group \( G \) can be taken to be a subgroup of \( O(n) \), such as \( U(m) \), \( \text{Sp}(m) \), or even exceptional groups if \( n \) is large enough.

5.2 Adding geometric data

We now explain how to add geometric data, such as metrics and connections. Let us fix a sheaf \( \mathcal{F} \) which is equipped with an \( O(d) \)-action. Then we can add \( \mathcal{F} \)-structure by considering the relevant
pullbacks along the forgetful maps $O(d)/\mathcal{F} =: \mathbf{BO}(d)\mathcal{F} \to \mathbf{BO}(d)$, induced by the projection $\mathcal{F} \to \ast$. We name the resulting smooth stacks $\mathbf{Gr}(d,N)\mathcal{F}$, $\mathbf{U}(d,N)\mathcal{F}$ and $\mathbf{U}^\perp(d,N)\mathcal{F}$ accordingly. Again each smooth stack is zero-truncated and equivalent to a smooth sheaf $\mathbf{Gr}(d,N)\mathcal{F}$, $\mathbf{U}(d,N)\mathcal{F}$, and $\mathbf{U}^\perp(d,N)\mathcal{F}$ and all the properties that hold for the stacks without $\mathcal{F}$-structure still hold for their $\mathcal{F}$ counterparts, In particular, we have maps (cf. maps (3.1))

$$\begin{array}{rcl}
D^1/\partial D^1 \wedge \text{Th}(\mathbf{U}^\perp(d,N)\mathcal{F}) & \to & \text{Th}(\mathbf{U}^\perp(d,N+1)\mathcal{F}) ,
\end{array}$$

and therefore a $D^1/\partial D^1$-spectrum $\mathbf{MT}(d)\mathcal{F}$. Similarly, we have maps

$$\begin{array}{rcl}
D^1/\partial D^1 \wedge \text{Th}(\mathbf{U}^\perp(d,N)\mathcal{F}) & \to & \text{Th}(\mathbf{U}^\perp(d,N+1)\mathcal{F}) ,
\end{array}$$

which gives an equivalent $D^1/\partial D^1$-spectrum $\mathbf{MT}(d)\mathcal{F}$. On the bordism side, we can define the sheaf of bundle maps $\text{Bun}(M,\mathbf{U}(d)\mathcal{F})$ which lift $TM$ as in the case of $G$-structure. We define

$$\begin{array}{rcl}
\text{Emb}^\mathcal{F}(M,\mathbb{R}^{d-1+\infty}) & \to & \text{Bun}(TM,\mathbf{U}(d)\mathcal{F}) \\
\text{Emb}(M,\mathbb{R}^{d-1+\infty}) & \to & \text{Bun}(TM,\mathbf{U}(d)) ,
\end{array}$$

and define the smooth cobordism category $\mathbf{Cob}^\mathcal{F}_d$ accordingly.

**Definition 5.6 (Smooth cobordism category with geometric data).** Define the smooth cobordism category with geometric data $\mathbf{Cob}^\mathcal{F}_d$ by

$$\begin{array}{rcl}
\text{Ob}(\mathbf{Cob}^\mathcal{F}_d) & := & \bigsqcup_{[M]} \text{Emb}^\mathcal{F}(M,\mathbb{R}^{d-1+\infty})/\text{Diff}(M) , \\
\text{Mor}(\mathbf{Cob}^\mathcal{F}_d) & := & \bigsqcup_{[W]} \text{Emb}^\mathcal{F}(W,\mathbb{R}^{d-1+\infty})/\text{Diff}(W) .
\end{array}$$

In this case, the sections of the sheaf of objects of the resulting smooth cobordism category are triples $(N,t,l)$, with $N \to U$ a bundle of $(d-1)$-dimensional manifolds, a smooth function $t \in C^\infty(U;\mathbb{R})$ and a lift $l : N \cong M \times U \to \mathbf{BO}(d)\mathcal{F}$ of $TM$, which is natural in $U$. The bordisms are identified similarly with the usual compatibility on collars. The definition and construction are very versatile and allow for geometric data such as the following.

**Example 5.7 (Connections).** Let $\mathcal{F} = \nabla := \Omega^1(-;\mathfrak{so}(d))$ and let $O(d)$ act on $\nabla$ via gauge transformations. Then the stack $\mathbf{BO}(d)\nabla$ is the moduli stack of connections on $M$. The corresponding bordism category is the category with objects manifolds equipped with connections on the tangent bundle. The morphisms are bordisms equipped with connections on the tangent bundle which extend those of the bounding manifolds up to gauge transformation.

**Example 5.8 (Metrics).** Let $\mathcal{F} = \mathcal{M}(\mathbb{R}^d) := C^\infty(-,\text{Sym}_d \cap \text{GL}_d)$ be the sheaf of metrics on $\mathbb{R}^d$. For each Cartesian space $U$, $\mathcal{M}(\mathbb{R}^d)(U) = \{\text{Metrics on the bundle } p : \mathbb{R}^d \times U \to U\}$. Then $O(d)$
acts on $\mathcal{M}(\mathbb{R}^d)$ via conjugation of symmetric matrices. The stack $\text{BO}(d)_{\mathcal{M}(\mathbb{R}^d)}$ is the moduli stack of metrics on $M$. The corresponding bordism category has objects – smooth manifolds equipped with Riemannian metrics, and morphisms – bordisms equipped with metrics which extend those of the bounding manifolds, up to local change of basis.

Just as in the case of $G$-structure, the collapse map is defined using information in the modified cobordism category and one observes that the introduction of the sheaf $\mathcal{F}$ causes no difficulty. In fact, the same proof of Theorem 5.5 works in this case. This leads us to the following theorem.

**Theorem 5.9 (The Pontrjagin-Thom equivalence with geometric structure).** We have an equivalence of smooth stacks

$$\text{PT} : \text{BConc}^\mathcal{F}(\Omega_{\mathcal{D}^1/\partial \mathcal{D}^1}^1\text{MT}(d)_{\mathcal{F}}) \cong \text{BConc}^\mathcal{F}(\Omega_{\mathcal{D}^1/\partial \mathcal{D}^1}^1\text{MT}(d)_{\mathcal{F}}).$$

We now quickly produce a nontrivial example already in the case of cobordism of zero-dimensional manifolds.

**Example 5.10 (An application for $d = 1$).** Let $\mathcal{F} := \Omega_{\mathcal{cl}}^1$ be the sheaf of closed 1-forms on Cartesian spaces. The $O(1) = \mathbb{Z}/2$ acts on this sheaf via gauge transformations, which in this case is trivial. We readily calculate

$$U^1(1, N)_{\mathcal{F}} \cong \Omega_{\mathcal{cl}}^1 \times U^1(1, N),$$

so the corresponding Thom stack simplifies to $\text{Th}(U^1(1, N)_{\mathcal{F}}) \cong (\Omega_{\mathcal{cl}}^1)_+ \wedge \text{Th}(U^1(1, N))$. The geometric realization of the smooth sheaf $\Omega_{\mathcal{cl}}^1$ can be identified with $K(\mathbb{R}, 1)$ [Pa17]. Then, we see that if we geometrically realize the smooth stack $\text{BConc}^\mathcal{F}(\Omega_{\mathcal{D}^1/\partial \mathcal{D}^1}^1\text{MT}(0)_{\mathcal{F}})$, we get the space

$$|\text{BConc}^\mathcal{F}(\Omega_{\mathcal{D}^1/\partial \mathcal{D}^1}^1\text{MT}(0)_{\mathcal{F}})| \cong |\Omega_{\mathcal{D}^1/\partial \mathcal{D}^1}^1\text{MT}(0)_{\mathcal{F}}| \cong \Omega_{\mathcal{D}^1/\partial \mathcal{D}^1}^1(K(\mathbb{R}, 1)_+ \wedge (\mathbb{R}P^\infty)^{-L}),$$

where $L \to \mathbb{R}P^\infty$ is the canonical line bundle with virtual inverse $-L$, and $(\mathbb{R}P^\infty)^{-L}$ denotes the corresponding Thom spectrum. Notice, however, that this is the suspension of the sphere spectrum $(\mathbb{R}P^\infty)^{-L} \cong \Sigma S$. Indeed, the unit disc bundle $D \to \mathbb{R}P^{n+1}$ can be obtained by intersecting each hyperplane with the unit disc $D^{n+1} \subset \mathbb{R}^{n+1}$. So the Thom space is readily identified with $D^{n+1}/\partial D^{n+1} \cong S^{n+1}$.

On the bordism side, we see that the sheaf of objects, evaluated at a cartesian space $U$, are zero-dimensional smooth manifolds $M = \{x_i\}_{i=1}^n \times U \subset \mathbb{R}^{n-1} \times U$, equipped with maps $M \times U \times \mathbb{R} \to \Omega_{\mathcal{cl}}^1 \times \mathbb{BZ}/2$, lifting

$$T^\infty M : M \times U \times \mathbb{R} \to \text{Gr}(1, \infty) \simeq \mathbb{R}P^\infty \simeq S^\infty//\mathbb{Z}/2 \to *//\mathbb{Z}/2.$$ 

Such lifts are in bijective correspondence with closed 1-forms, thought of as flat connections on the line bundle $M \times U \times \mathbb{R} \to M \times U$. The morphisms are embedded 1-dimensional manifolds $W \times U \subset \mathbb{R}^\infty \times U$, where $W$ is a collection of smooth paths with boundary the disjoint union.
$M_0 \sqcup M_1, M_0 \cong M_1$. Graphically, we have

![Graph](image)

We also have to consider maps $W \times U \to \Omega^{1}_{\text{cl}} \times B\mathbb{Z}/2$, which restrict to the given forms on collared neighborhoods of the boundary. A closed 1-form on $W$ is equivalent to associating a nonzero real number on each strand, multiplication by which gives the notion of parallel transport along the strand. The group generated under composition is then seen to be the group generated by elementary matrices: $\text{GL}_n(\mathbb{R})$. Since this group is discrete, the lifts which restrict fiberwise to lifts of $T^\infty M$ are constant in the direction of $U$. Therefore, after geometrically realizing, we get

$$|\mathcal{B}	ext{Cob}_d^\mathcal{F}| \cong B\left( \prod_{n \geq 0} \text{BGL}_n(\mathbb{R}) \right).$$

After looping, we get the group completion (see [Se74] [Ad78]) and this gives the identification

$$\mathbb{Z} \times \text{BGL}_{\infty}(\mathbb{R})^+ \cong \Omega^\infty \Sigma^\infty K(\mathbb{R}, 1)^+.$$

This technique of identifying certain infinite loops spaces via geometric realization of our equivalence deserves a fuller development and indeed we will revisit it elsewhere.

Lastly, we can combine Theorem 5.5 ($\theta$-structures) and Theorem 5.9 ($\mathcal{F}$-structures) to give Theorem 5 in the Introduction.

References

[Ad74] J. F. Adams, Stable homotopy and generalised cohomology, The Univ. of Chicago Press, Chicago 1974.

[Ad78] J. F. Adams, Infinite Loop Spaces, Princeton University Press, Princeton, NJ, 1978.

[At61] M. F. Atiyah, Bordism and cobordism, Proc. Camb. Phil. Soc. 57 (1961), 200–208.

[Ay08] D. Ayala, Geometric cobordism categories, arXiv:0811.2280 [math.AT].

[BF81] E. Binz and H. R. Fischer, The manifold of embeddings of a closed manifold, Differential Geometric Methods in Mathematical Physics (Pro. Internat. Conf., Tech. Univ. Clausthal, Clausthal-Zellerfeld, 1978), Lecture Notes in Phys., vol. 139, 310–329, Springer, Berlin, 1981.

[BS14] M. Bökstedt and A. M. Svane, A geometric interpretation of the homotopy groups of the cobordism category, Algebr. Geom. Topol. 14 (2014), no. 3, 1649-1676.
[BK72] A. Bousfield and D. Kan, *Homotopy limits, completions and localizations*, Springer-Verlag, Berlin, 1972.

[BSSW09] U. Bunke, T. Schick, I. Schröder, and M. Wiethaup, *Landweber exact formal group laws and smooth cohomology theories*, Algebr. Geom. Topol. **9** (2009), 1751-1790.

[Bu12] U. Bunke, *Differential cohomology*, [arXiv:1208.3961] [math.AT].

[Co10] R. L. Cohen, *Stability phenomena in the topology of moduli spaces*, Surveys in Differential Geometry XIV, L. Ji, S. Wolpert, and S-T Yau (eds.), International Press (2010), 23-56, [arXiv:0908.1938].

[CGK09] R. L. Cohen, S. Galatius, and N. Kitchloo, *Universal moduli spaces of surfaces with flat bundles and cobordism theory*, Adv. Math. **221** (2009), no. 4, 1227-1246.

[CF64] P. E. Conner and E. E. Floyd, *Differentiable Periodic Maps*, Springer-Verlag, Berlin, 1964.

[DHI04] D. Dugger, S. Hollander, and D. C. Isaksen, *Hypercovers and simplicial presheaves*, Math. Proc. Cambridge Philos. Soc. **136** (2004), no. 1, 9-51.

[DHKS04] W. Dwyer, P. Hirschhorn, D. Kan, and J. Smith, *Homotopy Limit Functors on Model Categories and Homotopical Categories*, Amer. Math. Soc., Providence, RI, 2004.

[EG11] J. Ebert and J. Giansiracusa, *Pontryagin-Thom maps and the homology of the moduli stack of stable curves*, Math. Ann. **349** (2011) 543-575.

[FSS15] D. Fiorenza, H. Sati, and U. Schreiber, *A Higher stacky perspective on Chern-Simons theory*, Mathematical Aspects of Quantum Field Theories (Damien Calaque and Thomas Strobl eds.), Springer, Berlin, 2015, [arXiv:1301.2580].

[FSS12] D. Fiorenza, U. Schreiber, and J. Stasheff, *Čech cocycles for differential characteristic classes: an ∞-Lie theoretic construction*, Adv. Theor. Math. Phys. **16** (2012), no. 1, 149-250, [arXiv:1011.4735] [math.AT].

[FSV15] D. Fiorenza, U. Schreiber, and A. Valentino, *Central extensions of mapping class groups from characteristic classes*, [arXiv:1503.00888] [math.AT].

[Fr12] D. S. Freed, Course notes, Bordism: old and new, 2012, Lecture 22: Remarks on the proof of GMTW, [https://www.ma.utexas.edu/users/dafr/M392C-2012/Notes/lecture22.pdf].

[Fr13] D. S. Freed, *The Cobordism hypothesis*, Bull. Amer. Math. Soc. **50** (2013), 57-92.

[FL10] D. S. Freed and J. Lott, *An index theorem in differential K-theory*, Geom. & Topol. **14** (2010), 903-966.

[GMTW09] S. Galatius, I. Madsen, U. Tillmann, and M. Weiss, *The homotopy type of the cobordism category*, Acta Math. **202** (2) (2009), 195-239.

53
[GRW10] S. Galatius and O. Randal-Williams, *Monoids of moduli spaces of manifolds*, Geom. Topol. **14** (2010) 1243-1302, [arXiv:0905.2855] [math.AT].

[Ge12] J. Genauer, *Cobordism categories of manifolds with corners*, Trans. Amer. Math. Soc. **364** (2012), no. 1, 519-550.

[GJ99] P. G. Goerss and J. F. Jardine, *Simplicial homotopy theory*, Progress in Mathematics 174, Birkhäuser Verlag, Basel, 1999.

[GS15] D. Grady and H. Sati, *Massey products in differential cohomology via stacks*, to appear in J. Homotopy Relat. Struct. [arXiv:1510.06366] [math.AT].

[GS16a] D. Grady and H. Sati, *Primary operations in differential cohomology*, [arXiv:1604.05988] [math.AT].

[GS16b] D. Grady and H. Sati, *Spectral sequences in smooth generalized cohomology*, to appear in Algeb. Geom. Top., [arXiv:1605.03444] [math.AT].

[HS05] M. J. Hopkins and I. M. Singer, *Quadratic functions in geometry, topology, and M-theory*, J. Differential Geom. **70** (3) (2005), 329-452.

[Ja15] J. F. Jardine, *Local homotopy theory*, Springer Monographs in Mathematics. Springer, New York, 2015.

[Ko93] A. A. Kosinski, *Differential manifolds*, Academic Press, Inc., Boston, MA, 1993.

[KM97] A. Kriegl and P. W. Michor, *The convenient setting of global analysis*, Amer. Math. Soc., Providence, RI, 1997.

[Li62] A. L. Liulevicius, *A proof of Thom’s theorem*, Comment. Math. Helv. **37** (1962/1963), 121-131.

[Lu09a] J. Lurie, *Higher topos theory*, Princeton University Press, Princeton, NJ, 2009.

[Lu09b] J. Lurie, *On the classification of topological field theories*, Current Developments in Mathematics 2008 (2009), 129-280, International Press, Boston, [arXiv:0905.0465] [math.CT].

[MM94] S. MacLane and I. Moerdijk, *Sheaves in Geometry and Logic: A first introduction to topos theory*, Universitext, Springer-Verlag, New York, 1994.

[MT01] I. Madsen and U. Tillmann, *The stable mapping class group and Q(CP∞)*, Invent. Math. **145** (2001), 509-544.

[MW07] I. Madsen and M. Weiss, *The stable moduli space of Riemann surfaces: Mumford’s conjecture*, Ann. of Math. (2) **165** (2007), no. 3, 843-941.

[Mc69] M. C. McCord, *Classifying spaces and infinite symmetric products*, Trans. Amer. Math. Soc. **146** (1969), 273-298.
[Mi57] J. Milnor, *The geometric realization of a semi-simplicial complex*, Ann. of Math. (2) **65** (1957), 357-362.

[Mi62] J. Milnor, *A survey of cobordism theory*, Enseignement Math. (2) **8** (1962), 16-23.

[Mi82] S. A. Mitchell, *Power series methods in unoriented cobordism*, Proceedings of the Northwestern Homotopy Theory Conference (Evanston, Ill., 1982), 247–254, Contemp. Math. 19, Amer. Math. Soc., Providence, RI, 1983.

[Mo95] I. Moerdijk, *Classifying spaces and classifying topos*, Lecture Notes in Math. 1616, Springer-Verlag, New York, 1995.

[Mu83] D. Mumford, *Towards an enumerative geometry of the moduli space of curves*, in Arithmetic and Geometry, Vol. II, Progr. Math. 36, Birkhäuser Boston, Boston, MA (1983), 271-328.

[NSS15a] T. Nikolaus, U. Schreiber, and D. Stevenson, *Principal $\infty$-bundles: general theory*, J. Homotopy Relat. Struct. **10** (2015), no. 4, 749-801.

[NSS15b] T. Nikolaus, U. Schreiber, and D. Stevenson, *Principal $\infty$-bundles: presentations*, J. Homotopy Relat. Struct. **10** (2015), no. 3, 565-622.

[Pa17] D. Pavlov, *Brown representability via concordance*, preliminary draft, available at [https://dmitripavlov.org/concordance.pdf](https://dmitripavlov.org/concordance.pdf).

[Pe68] F. P. Peterson, *Lectures on cobordism theory*, Kyoto University Press, Kyoto, 1968.

[Ph67] A. Phillips, *Submersions of open manifolds*, Topology **6** (1967), 171-206.

[Po59] L. Pontryagin, *Smooth manifolds and their applications in homotopy theory*, Amer. Math. Soc. Translations, Ser. 2, Vol. 11, pp. 1-114, 1959.

[Qu67] D. Quillen, *Homotopical Algebra*, Lecture Notes Math **43**, Springer, Berlin, 1967.

[RW11] O. Randal-Williams, *Embedded cobordism categories and spaces of manifolds*, Int. Math. Res. Not. **2011**, no. 3, 572-608, [arXiv:0912.2505](https://arxiv.org/abs/0912.2505) [math.AT].

[RS16] G. Raptis and W. Steimle, *Parametrized cobordism categories and the Dwyer-Weiss-Williams index theorem*, to appear in J. Topology, [arXiv:1606.07925](https://arxiv.org/abs/1606.07925) [math.AT].

[Re10] C. Rezk, *Toposes and homotopy toposes*, draft available at [http://www.math.uiuc.edu/~rezk/homotopy-topos-sketch.pdf](http://www.math.uiuc.edu/~rezk/homotopy-topos-sketch.pdf).

[Ru08] Yu. B. Rudyak, *On Thom spectra, orientability, and (co)bordism*, Springer, Berlin, 2008.

[SSS09] H. Sati, U. Schreiber, and J. Stasheff, *Fivebrane structures*, Rev. Math. Phys. **21** (2009) 1-44, [arXiv:0805.0564](https://arxiv.org/abs/0805.0564) [math.AT].

55
[SSS12] H. Sati, U. Schreiber, and J. Stasheff, *Differential twisted String- and Fivebrane structures*, Commun. Math. Phys. 315 (2012), 169-213, [arXiv:0910.4001] [math.AT].

[Sc13] U. Schreiber, *Differential cohomology in a cohesive infinity-topos*, [arXiv:1310.7930] [math-ph].

[Se74] G. B. Segal, *Categories and cohomology theories*, Topology 13 (1974), 293-312.

[St11] A. Stacey, *Comparative smootheology*, Theory Appl. Categ. 25 (2011), No. 4, 64-117, [arXiv:0802.2225] [math.DG].

[St68] R.E. Stong, *Notes on Cobordism Theory*, Princeton University Press, Princeton, NJ, 1968.

[St] N. Strickland, *The category of CGWH spaces*, homotopy theory course notes, [http://neil-strickland.staff.shef.ac.uk/courses/homotopy/cgwh.pdf](http://neil-strickland.staff.shef.ac.uk/courses/homotopy/cgwh.pdf).

[Th54] R. Thom, *Quelques propriétés globales des variétés différentiables*, Comm. Math. Helv. 28 (1954), 17-86.

[Wa60] C. T. C. Wall, *Determination of the cobordism ring*, Ann. of Math. (2) 72 (1960), 292-311.

[We05] M. Weiss, *What does the classifying space of a category classify?*, Homology Homotopy Appl. 7 (2005), 185-195.