A contribution to the Zarankiewicz problem

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Abstract

Given positive integers \(m, n, s, t\), let \(z(m, n, s, t)\) be the maximum number of ones in a \((0, 1)\) matrix of size \(m \times n\) that does not contain an all ones submatrix of size \(s \times t\). We show that if \(s \geq 2\) and \(t \geq 2\), then for every \(k = 0, \ldots, s - 2\),

\[
z(m, n, s, t) \leq (s - k - 1)^{1/2} \frac{nm^{1-1/t}}{2} + kn + (t - 1) m^{1+k/t}.
\]

This generic bound implies the known bounds of Kövari, Sós and Turán, and of Füredi. As a consequence, we also obtain the following results:

Let \(G\) be a graph of \(n\) vertices and \(e(G)\) edges, and let \(\mu\) be the spectral radius of its adjacency matrix. If \(G\) does not contain a complete bipartite subgraph \(K_{s,t}\), then the following bounds hold

\[
\mu \leq (s - t + 1)^{1/2} n^{1-1/t} + (t - 1) n^{1-2/t} + t - 2,
\]

and

\[
e(G) < \frac{1}{2} (s - t + 1)^{1/2} n^{2-1/t} + \frac{1}{2} (t - 1) n^{2-2/t} + \frac{1}{2} (t - 2) n.
\]

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Introduction

How large can be the spectral radius \(\mu\) of a graph order \(n\) that does not contain a complete bipartite subgraph \(K_{s,t}\)? This is a spectral version of the famous Zarankiewicz problem: how many edges can have a graph of order \(n\) if it does not contain \(K_{s,t}\)? Except for few cases, no satisfactory solution to either of these problems is known. In an unpublished pioneering work, Babai and Guiduli (see, e.g., [7]) have shown that

\[
\mu \leq \left( (s - 1)^{1/2} + o(1) \right) n^{1-1/t}.
\]

Using a different method, here we improve this result as follows:
Theorem 1. Let $s \geq t \geq 2$, and let $G$ be a $K_{s,t}$-free graph of order $n$ and spectral radius $\mu$. If $t = 2$, then
\[ \mu \leq 1/2 + \sqrt{(s - 1) (n - 1) + 1/4}. \] (1)
If $t \geq 3$, then
\[ \mu \leq (s - t + 1)^{1/t} n^{1-1/t} + (t - 1) n^{1-2/t} + t - 2. \] (2)

Below we show that the bounds (1) and (2) are tight for some values of $s$ and $t$. On the other hand, in view of the inequality $2e(G) \leq \mu n$, we see that if $G$ is a $K_{s,t}$-free graph of order $n$, then
\[ e(G) \leq \frac{1}{2} (s - t + 1)^{1/t} n^{2-1/t} + \frac{1}{2} (t - 1) n^{2-2/t} + \frac{1}{2} (t - 2) n. \] (3)

This is a slight improvement of a result of Füredi [5].

To prove Theorem 1, we first find a family of new upper bounds for the matrix Zarankiewicz problem, thereby extending some previous results.

The matrix Zarankiewicz problem

Let $J_{s,t}$ denote the all ones matrix of size $s \times t$. Given positive integers $m, n, s, t$, let $z(m, n, s, t)$ be the maximum number of ones in a $(0, 1)$ matrix of size $m \times n$ that does not contain $J_{s,t}$ as a submatrix.

Here is an equivalent definition: $z(m, n, s, t)$ is the maximum number of edges in a bipartite graph $G$ with vertex classes $A$ of size $n$ and $B$ of size $m$ such that $G$ does not contain a copy of $K_{s,t}$ with vertex class of size $s$ in $A$ and vertex class of size $t$ in $B$.

The problem of finding $z(m, n, s, t)$ is known as the general Zarankiewicz problem. In [8], Kövari, Sós and Turán gave one of the earliest bounds on $z(m, n, s, t)$, which in simplified form reads as
\[ z(m, n, s, t) \leq (s - 1)^{1/t} nm^{1-1/t} + (t - 1) m. \] (4)

Later, Füredi [5] improved this bound showing that if $s \geq t$, then
\[ z(m, n, s, t) \leq (s - t + 1)^{1/t} nm^{1-1/t} + tm^{2-2/t} + tn. \] (5)

The proof of Füredi, although rather involved, is based on double counting as in [8]. Using a different approach, we show that, in fact, (5) and (4) are particular cases of a whole sequence of subtler bounds on $z(m, n, s, t)$. Instead of using double counting, we start with (4) and deduce by induction a number of inequalities, one of which implies (5). The following theorem gives the precise statement.

Theorem 2. If $s \geq 2$ and $t \geq 2$, then for every $k = 0, \ldots, s - 2$,
\[ z(m, n, s, t) \leq (s - k - 1)^{1/t} nm^{1-1/t} + (t - 1) m^{1+k/t} + kn. \] (6)
Given (6), letting \( k = 0 \), we obtain the bound of Kövari, Sós and Turán (4). Also, if \( s \geq t \), letting \( k = t - 2 \), we obtain

\[
z(m,n,s,t) < (s - t + 1)^{1/t} nm^{1-1/t} + (t - 1) m^{2-2/t} + (t - 2) n,
\]

which is a slight improvement of Füredi’s bound (5).

At first glance it is unclear whether the parameter \( k \) is really useful in inequality (6). Indeed, for \( n = m \), setting \( k = \min \{ s, t \} - 2 \) gives the best inequality for \( n \) large enough. However, for arbitrary \( n \) and \( m \), the parameter \( k \) can give additional improvement, as shown in the following proposition, whose proof is omitted.

**Proposition 3** Let \( s \geq 3 \) and \( t \geq 3 \), and \( 0 \leq k \leq s - 2 \). There exist \( A = A(s,t,k) > 0 \) and \( B = B(s,t,k) > 0 \) such that for all sufficiently large \( n \) and \( m \) satisfying

\[
Am^{(k+1)/t} \leq n \leq Bm^{(k+2)/t},
\]

we have

\[
(s - k - 1)^{1/t} nm^{1-1/t} + (t - 1) m^{1+k/t} + kn < (s - i - 1)^{1/t} nm^{1-1/t} + (t - 1) m^{1+i/t} + in
\]

for all \( i \in [0, s - 2] \setminus \{k\} \).

**Tightness of the bounds (1) and (2)**

For some values of \( s \) and \( t \) the bounds given by (1) and (2) are tight.

**The case \( t = 2 \)**

For \( s = t = 2 \) inequality (1) gives that every \( K_{2,2} \)-free graph \( G \) of order \( n \) satisfies

\[
\mu(G) \leq 1/2 + \sqrt{n - 3/4}.
\]

This bound is tight: equality holds for the friendship graph. Note that letting \( q \) be a prime power, the Erdős-Renyi polarity graph is a \( K_{2,2} \)-free graph of order \( n = q^2 + q + 1 \) and \( q(q + 1)^2/2 \) edges. Thus, its spectral radius \( \mu(ER_q) \) satisfies

\[
\mu(ER_q) \geq \frac{q^3 + 2q^2 + q}{q^2 + q + 1} > q + 1 - \frac{1}{q} = 1/2 + \sqrt{n - 3/4} - \frac{1}{\sqrt{n - 1}},
\]

which is also close to the upper bound.

For \( s > 2 \), equality in (1) is attained when \( G \) is a strongly regular graph in which every two vertices have exactly \( s - 1 \) common neighbors. There are examples of strongly regular graphs of this type; here is a small selection from Gordon Royle’s webpage:

| \( s \) | \( n \) | \( \mu(G) \) |
|-------|-------|-------|
| 3     | 45    | 12    |
| 4     | 96    | 20    |
| 5     | 175   | 30    |
| 6     | 36    | 15    |
We are not aware whether there are infinitely many strongly regular graphs in which every two vertices have the same number of common neighbors. However, Füredi [6] has shown that for any $n$ there exist $K_{s,t}$-free graph $G_n$ of order $n$ such that

$$e(G_n) \geq \frac{1}{2}n\sqrt{s}n + O(n^{4/3}) ,$$

and so,

$$\mu(G_n) \geq \sqrt{s}n + O(n^{1/3}) ;$$

thus (1) is tight up to low order terms.

The case $s = t = 3$

The bound (2) implies that if $G$ is a $K_{3,3}$-free graph of order $n$, then

$$\mu(G) \leq n^{2/3} + 2n^{1/3} + 1. $$

On the other hand, a construction due to Alon, Rónyai and Szabó [1] implies that for all $n = q^3 - q^2$, where $q$ is a prime power, there exists a $K_{3,3}$-free graph $G_n$ of order $n$ with

$$\mu(G_n) \geq n^{2/3} + \frac{2}{3}n^{1/3} + C$$

for some constant $C > 0$. Thus, the bound (2) is asymptotically tight for $s = t = 3$. The same conclusion can be obtained from Brown’s construction of $K_{3,3}$-free graphs [3].

The general case

As proved in [1], there exists $c > 0$ such that for all $t \geq 2$ and $s \geq (t - 1)! + 1$, there is a $K_{s,t}$-free graph $G_n$ of order $n$ with

$$e(G_n) \geq \frac{1}{2}n^{2-1/t} + O(n^{2-1/t-c}) .$$

Hence, for such $s$ and $t$ we have

$$\mu(G) \geq n^{1-1/t} + O(n^{1-1/t-c}) ;$$

thus, the bound (2) and the earlier bound of Babai and Guiduli give the correct order of the main term.

Proof of Theorem 2

Some matrix notation Let $|X|$ denote the cardinality of a finite set $X$. Let $A = (a_{ij})$ be a $(0,1)$-matrix, and let the rows and columns of $A$ be indexed by the elements of two disjoint sets $R(A)$ and $C(A)$. Then:

- for any $i \in R$, we let $C_i = \{j : j \in C(A) , a_{ij} = 1\}$ and set $r_i = |C_i| ;$
- for any \( j \in C \), we let \( R_j = \{ i : i \in R(A), \ a_{ij} = 1 \} \) and set \( c_j = |R_j| \);
- \( \|A\| \) stands for the sum of the entries of \( A \);
- given nonempty sets \( I \subset R(A) \), \( J \subset C(A) \), we write \( A[I, J] \) for the submatrix of the entries \( a_{ij} \) satisfying \( i \in I, \ j \in J \).

**Proof of Theorem 2** We shall use induction on \( k \). For \( k = 0 \), the assertion is given by (1). Suppose \( k \geq 1 \) and assume the assertion true for all \( k' < k \). Let \( A = (a_{ij}) \) be a \((0, 1)\)-matrix of size \( m \times n \), and let \( R = R(A) \), \( C = C(A) \). Suppose that \( A \) does not contain \( J_{s,t} \) as a submatrix and that \( k \leq s - 2 \). Our goal is to prove that

\[
\|A\| \leq (s - k - 1)^{1/t} nm^{1-1/t} + (t - 1) m^{1+1/t} + kn.
\]

Select \( i \in R \) and define the sets

\[
U = R \setminus \{ i \}, \quad W = C_i.
\]

Note that the matrix \( A[U, W] \) does not contain \( J_{s-1,t} \) as a submatrix since the \( i \)'th row of \( A[R, W] \) consists of all ones and we would have a \( J_{s,t} \) in \( A \). Therefore,

\[
\|A[U, W]\| \leq z(|U|, |W|, s - 1, t),
\]

and by the induction assumption applied for \( s - 1 \) and \( k - 1 \), we have

\[
\|A[U, W]\| \leq (s - k - 1)^{1/t} |U| |W|^{1-1/t} + (t - 1) |U|^{1+(k-1)/t} + (k - 1) |W| \leq (s - k - 1)^{1/t} r_i m^{1-1/t} + (t - 1) m^{1+(k-1)/t} + (k - 1) r_i. \quad (7)
\]

A closer look at \( A[U, W] \) shows that

\[
\|A[U, W]\| = \sum_{j \in C_i} \sum_{k \in R \setminus \{ i \}} a_{kj} = \sum_{j \in C} \sum_{k \in R \setminus \{ i \}} a_{kj} = \sum_{j \in C} \sum_{k \in R} a_{ij} a_{kj} - \sum_{j \in C} a_{ij} = \sum_{j \in C} \sum_{k \in R} a_{ij} a_{kj} - r_i.
\]

Substituting the value of \( \|A[U, W]\| \) in (7), we see that

\[
\sum_{j \in C} \sum_{k \in R} a_{ij} a_{kj} - \left( (s - k - 1)^{1/t} m^{1-1/t} + k \right) r_i + (t - 1) m^{1+(k-1)/t} \leq 0.
\]

Summing this inequality for all \( i \in R \), we get

\[
\sum_{i \in R} \sum_{j \in C} \sum_{k \in R} a_{ij} a_{kj} - \left( (s - k - 1)^{1/t} m^{1-1/t} + k \right) \|A\| + (t - 1) m^{1+(k-1)/t} \leq 0.
\]

Now note that

\[
\sum_{i \in R} \sum_{j \in C} \sum_{k \in R} a_{ij} a_{kj} = \sum_{j \in C} \sum_{i \in R} \sum_{k \in R} a_{ij} a_{kj} = \sum_{j \in C} r_j^2 \geq \frac{1}{n} \|A\|^2,
\]

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and so,
\[ \frac{1}{n} \|A\|^2 - \left( (s - k - 1)^{1/t} m^{1-1/t} + k \right) \|A\| - (t - 1) m^{2+(k-1)/t} \leq 0. \]

Solving this inequality, we find that
\[ \|A\| \leq \left( 1 + \sqrt{1 + \frac{4 (t - 1) m^{2+(k-1)/t}}{n \left( (s - k - 1)^{1/t} m^{1-1/t} + k \right)^2}} \right) \frac{(s - k - 1)^{1/t} m^{1-1/t} + k) n}{2} \]
and bounding the radical by the Bernoulli inequality, we obtain
\[ \|A\| \leq \left( 1 + 1 + \frac{2 (t - 1) m^{2+(k-1)/t}}{n \left( (s - k - 1)^{1/t} m^{1-1/t} + k \right)^2} \right) \frac{(s - k - 1)^{1/t} m^{1-1/t} + k) n}{2} \]
\[ = (s - k - 1)^{1/t} n m^{1-1/t} + kn + \frac{(t - 1) m^{2+(k-1)/t}}{(s - k - 1)^{1/t} m^{1-1/t} + k} \]
\[ \leq (s - k - 1)^{1/t} n m^{1-1/t} + (t - 1) m^{1+k/t} + kn. \]

This completes the induction step and the proof of Theorem 2.

Stated in terms of bipartite graphs, Theorem 2 is equivalent to the following one:

**Theorem 4** Let \( s \geq 2, t \geq 2, 0 \leq k \leq s - 2, \) and let \( G(A, B) \) be a bipartite graph with parts \( A \) and \( B. \) Suppose that \( G \) contains no copy of \( K_{s,t} \) with a vertex class of size \( s \) in \( A \) and a vertex class of size \( t \) in \( B. \) Then \( G(A, B) \) has at most
\[ (s - k - 1)^{1/t} |B| |A|^{1-1/t} + (t - 1) |A|^{1+k/t} + k |B| \]
edges.

**The proof of Theorem 1**

**Some graph notation** Our graph notation follows [2]; in particular, given a graph \( G \) and a vertex \( u \) of \( G, \) we write:
- \( V(G) \) for the vertex set of \( G; \)
- \( E(G) \) for the edge set of \( G \) and \( e(G) \) for \( |E(G)|; \)
- \( G - u \) for the graph obtained from \( G \) by removing the vertex \( u. \)
- \( \Gamma(u) \) for the set of neighbors of \( u \) and \( d(u) \) for \( |\Gamma(u)|. \)

**Proof of Theorem 1** Inequality (1) has been proved in [9], so we shall assume that \( s \geq 3 \) and \( t \geq 3. \) Let \( u \in V(G) \) be any vertex of \( G, \) let \( U \) and \( W \) be disjoint sets satisfying \( |U| = d(u) \) and \( |W| = n - 1, \) and let \( \varphi_U \) and \( \varphi_W \) be bijections
\[ \varphi_U : U \to \Gamma(u), \quad \varphi_W : W \to V(G) \setminus \{u\}. \]
Define a bipartite graph $H$ with vertex classes $U$ and $W$ by joining $v \in U$ and $w \in W$ whenever $\{\varphi_U(v), \varphi_W(w)\} \in E(G)$.

We claim that $H$ does not contain a copy of $K_{t,s-1}$ with $s-1$ vertices in $W$ and $t$ vertices in $U$. Indeed, the map $\psi : V(H) \to V(G)$ defined as

$$
\psi(x) = \begin{cases} 
\varphi_U(x) & \text{if } x \in U \\
\varphi_W(x) & \text{if } x \in W
\end{cases}
$$

is a homomorphism of $H$ into $G - v$. Assume for a contradiction that $F \subset H$ is a copy of $K_{t,s-1}$ with a set $S$ of $s-1$ vertices in $W$ and a set $T$ of $t$ vertices in $U$. Clearly $S$ and $T$ are the vertex classes of $F$. Note that $\psi(F)$ is a copy of $K_{t,s-1}$ in $G - u$, and $\psi(T) = \varphi_U(T) \subset \Gamma_G(u)$ is the vertex class of $\psi(F)$ of size $t$; now, adding $u$ to $\psi(F)$, we see that $G$ contains a $K_{t,s}$, a contradiction proving the claim.

Suppose that $0 \leq k \leq \min \{s,t\} - 2$. Setting $k' = k - 1$, $s' = s - 1$, $t' = t$, $A = W$, $B = U$, Theorem 4 implies that

$$
e(H) \leq (s - k - 1)^{1/t} |U||W|^{1-1/t} + (k - 1) |U| + (t - 1) |W|^{1+(k-1)/t}$$

$$\leq (s - k - 1)^{1/t} d(u) n^{1-1/t} + (k - 1) d(u) + (t - 1) n^{1+(k-1)/t}.$$ 

On the other hand, we see that

$$e(H) = \sum_{v \in \Gamma(u)} d(v) - d(u),$$

and so,

$$\sum_{v \in \Gamma(u)} d(v) \leq \left((s - k - 1)^{1/t} n^{1-1/t} + k\right) d(u) + (t - 1) n^{1+(k-1)/t}. \tag{8}$$

Letting $A$ be the adjacency matrix of $G$, note that the $u$’th row sum of the matrix

$$C = A^2 - \left((s - k - 1)^{1/t} n^{1-1/t} + k\right) A$$

is equal to

$$\sum_{v \in \Gamma(u)} d(v) - \left((s - k - 1)^{1/t} n^{1-1/t} + k\right) d(u);$$

consequently, the maximum row sum $r_{\text{max}}$ of $C$ satisfies

$$r_{\text{max}} \leq (t - 1) n^{1+(k-1)/t}.$$ 

Letting $x$ be an eigenvector of $A$ to $\mu$, we see that the value

$$\lambda = \mu^2 - \left((s - k - 1)^{1/t} n^{1-1/t} + k\right) \mu$$

is an eigenvalue of $C$ with eigenvector $x$. Therefore,

$$\mu^2 - \left((s - k - 1)^{1/t} n^{1-1/t} + k\right) \mu = \lambda \leq r_{\text{max}} \leq (t - 1) n^{1+(k-1)/t}.$$
Solving this inequality we obtain
\[
\mu \leq \left(1 + \frac{1 + \frac{4(t-1)n^{1+(k-1)/t}}{(s-k-1)^{1/t}n^{1-1/t}+k}}{2}\right)\frac{(s-k-1)^{1/t}n^{1-1/t}+k}{2} \\
\leq \left(1 + 1 + \frac{2(t-1)n^{1+(k-1)/t}}{(s-k-1)^{1/t}n^{1-1/t}+k}\right)\frac{(s-k-1)^{1/t}n^{1-1/t}+k}{2} \\
\leq (s-k-1)^{1/t}n^{1-1/t} + (t-1)n^{k/t} + k.
\]

Now, if \( s \geq t \geq 3 \), setting \( k = t - 2 \), we obtain inequality [2], completing the proof of Theorem 1.

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