The group law on the tropical Hesse pencil

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Abstract

We show that the addition of points on the tropical Hesse curve can be realized via the intersection with a tropical line. Then the addition formula for the tropical Hesse curve is reduced from those for the level-three theta functions through the ultradiscretization procedure. A tropical analogue of the Hessian group $G_{216}$, the group of linear automorphisms acting on the Hesse pencil, is also investigated; it is shown that the dihedral group $D_3$ of degree three is the group of linear automorphisms acting on the tropical Hesse pencil.

Key words: Hesse pencil, Hessian group, Integrable system, Theta function, Tropical curve, Ultradiscretization

1 Introduction

The Hesse pencil is a canonical elliptic pencil consisting of the Hesse cubic curves to which all nonsingular cubic curves are projectively equivalent [4,5]. The Hessian curve $\text{He}(E)$ of the Hesse cubic curve is the curve itself, and therefore the nine base points of the Hesse pencil are its inflection points [1]. Moreover, the pencil is parametrized with the level-three theta functions, and hence it has the level-three structure [1]. In addition to these remarkable properties, the Hesse pencil also plays an important role from the view point of tropical geometry; the tropical analogue of the Hesse pencil (the tropical Hesse pencil) is a pencil of tropical cubic curves of genus one and has a group structure in analogy to the Hesse pencil [7,11]. This property contrasts well with the fact that the Weierstraß canonical elliptic curve loses its group structure in the tropical limit. The group structure of the tropical Hesse pencil confirms the

$^1$ The Hessian curve $\text{He}(E)$ of a nonsingular cubic curve $E$ is the plane cubic curve defined by the equation $\text{He}(F) = 0$, where $\text{He}(F)$ is the determinant of the matrix of the second partial derivatives of the defining polynomial $F$ of $E$. 

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existence of tropical analogues of the level-three theta functions parametrizing it.

It is well known that once we find a group structure of a plane curve then we can construct two kinds of dynamical systems realized as the evolutions of points on the curve; one is integrable and the other is solvable but chaotic \[ [14,18,8,7,13] \]. Actually, we can construct such dynamical systems on the Hesse pencil as follows. Let \( P_0 \) be an arbitrary point on the plane \( \mathbb{C}^2 \). Then there exists a unique curve in the Hesse pencil passing through \( P \). We denote the curve by \( E_{\lambda} \). Fix a point \( T \) on \( E_{\lambda} \). Let us consider the addition map

\[
P_0 \mapsto P_1 = P_0 + T. \tag{1}
\]

Then \( P_1 \) is a point on \( E_{\lambda} \) because both \( P_0 \) and \( T \) are on \( E_{\lambda} \). By applying the map repeatedly, we obtain the sequence of points \( P_0, P_1, \ldots \) on \( E_{\lambda} \). This dynamical system is nothing but a member of the celebrated QRT system, a family of paradigmatic two-dimensional integrable maps \[ [14,18,10,13] \]. The parameter \( \lambda \) of the pencil is the conserved quantity of the dynamical system, and the general solution to the dynamical system is concretely constructed by using the level-three theta functions \[ [10] \].

On the other hand, if we consider the duplication map

\[
P_0 \mapsto P_1 = P_0 + P_0 = 2P_0
\]

instead of \[ (1) \] then we obtain the sequence of points \( P_0, P_1, \ldots \) on \( E_{\lambda} \) so called the solvable chaotic system \[ [17,19,8,7] \]. This dynamical system behaves chaotically, nevertheless we can construct its general solution by using the level-three theta functions as in the case of the QRT system \[ [8,7] \].

In this paper, we review the group structure of the tropical Hesse pencil on which the above-mentioned types of piecewise linear map dynamical systems are constructed. This group structure is geometrically realized as the intersection of the tropical Hesse curve with a tropical line. It is also realized analytically as the ultradiscrete limit of the addition formulae for the level-three theta functions. In addition, we show that the group of linear automorphisms acting on the Hesse pencil (the Hessian group) survives as the dihedral group of degree three acting on the tropical Hesse pencil in the tropical limit \[ [12] \].
2 Tropical Hesse pencil

2.1 Hesse pencil

The Hesse pencil is a one-dimensional linear system of plane cubic curves in \( \mathbb{P}^2(\mathbb{C}) \) given by the equation

\[
f(x_0, x_1, x_2; t_0, t_1) := t_0 \left( x_0^3 + x_1^3 + x_2^3 \right) + t_1 x_0 x_1 x_2 = 0,
\]

where \((x_0, x_1, x_2)\) is the homogeneous coordinate of \( \mathbb{P}^2(\mathbb{C}) \) and the parameter \((t_0, t_1)\) ranges over \( \mathbb{P}^1(\mathbb{C}) \) \([1]\). Each curve composing the pencil is called the Hesse cubic curve and is denoted by \( E_{t_0,t_1} \). It is well known that every nonsingular cubic curve can be transformed projectively into a member of the Hesse pencil \( \{E_{t_0,t_1}\}_{(t_0,t_1)\in\mathbb{P}^1(\mathbb{C})} \) \([4,5]\).

The nine base points \( p_0, p_1 \ldots, p_8 \) of the pencil are given as follows

\[
\begin{align*}
p_0 &= (0, 1, -1), & p_1 &= (0, 1, -\zeta_3), & p_2 &= (0, 1, -\zeta_3^2), \\
p_3 &= (1, 0, -1), & p_4 &= (1, 0, -\zeta_3^2), & p_5 &= (1, 0, -\zeta_3), \\
p_6 &= (1, -1, 0), & p_7 &= (1, -\zeta_3, 0), & p_8 &= (1, -\zeta_3^2, 0),
\end{align*}
\]

where \( \zeta_3 \) denotes the primitive third root of 1. Any smooth curve in the pencil has these nine base points as its inflection points, and hence they are in the Hesse configuration \([1,16]\).

2.2 Tropicalization of the Hesse pencil

Let us tropicalize the Hesse pencil. For the defining polynomial of the Hesse cubic curve we apply the procedure of tropicalization \([15,2,6]\). Replacing the operations + and \( \times \) with \( \max \) and + respectively, the defining polynomial \( f(x_0, x_1, x_2; t_0, t_1) \) reduces to the tropical one

\[
\tilde{f}(\tilde{x}_0, \tilde{x}_1, \tilde{x}_2; \tilde{t}_0, \tilde{t}_1) := \max \left( \tilde{t}_0 + 3\tilde{x}_0, \tilde{t}_0 + 3\tilde{x}_1, \tilde{t}_0 + 3\tilde{x}_2, \tilde{t}_1 + \tilde{x}_0 + \tilde{x}_1 + \tilde{x}_2 \right).
\]

To distinguish the tropical variables form the original ones, they are ornamented with “\( \tilde{\cdot} \).”

Let \((\tilde{t}_0, \tilde{t}_1)\) be a point in \( \mathbb{T} \mathbb{P}^1 \), the tropical projective line. Then \( \tilde{f} \) can be regarded as a function \( \tilde{f} : \mathbb{T} \mathbb{P}^2 \to \mathbb{T} \), where \( \mathbb{T} \mathbb{P}^2 \) is the tropical projective plane and \( \mathbb{T} := \mathbb{R} \cup \{ -\infty \} \) is the tropical semi-field \([7]\). The tropicalization of the Hesse cubic curve is defined to be the set of points such that the function \( \tilde{f} \) is not differentiable with respect to \( \tilde{x}_1, \tilde{x}_2, \) or \( \tilde{x}_3 \). We call the curve thus
obtained the tropical Hesse curve and denote it by $C_{\tilde{t}_0, \tilde{t}_1}$. Upon introduction of the inhomogeneous coordinate $(X, Y) := (\tilde{x}_1 - \tilde{x}_0, \tilde{x}_2 - \tilde{x}_0) \in \mathbb{TP}^2$ and $\kappa := \tilde{t}_1 - \tilde{t}_0 \in \mathbb{TP}^1$ the tropical Hesse curve is given by the inhomogeneous tropical polynomial of degree three

$$F(X, Y; \kappa) := \max (3X, 3Y, 0, \kappa + X + Y)$$

and is denoted by $C_\kappa$ (see figure 1).

The one-dimensional linear system $\{C_\kappa\}_{\kappa \in \mathbb{TP}^1}$ consisting of the tropical Hesse curves is called the tropical Hesse pencil. The complement of the unbounded edges of $C_\kappa$ is denoted by $\overline{C}_\kappa := C_\kappa \setminus \{\text{unbound edges}\}$. We denote the vertices whose coordinates are $(\kappa, \kappa), (-\kappa, 0), \text{ and } (0, -\kappa)$ by $V_1, V_2, \text{ and } V_3$, respectively. The three base points of the tropical Hesse pencil $\{C_\kappa\}_{\kappa \in \mathbb{TP}^1}$ are the end points of the unbounded edges emanating from $V_1, V_2, \text{ and } V_3$, respectively.

We define the tropical Jacobian $J(C_\kappa)$ of the tropical Hesse curve $C_\kappa$ to be $J(C_\kappa) := \mathbb{R}/3\kappa\mathbb{Z} = \{u \in \mathbb{R} \mid 0 \leq u < 3\kappa\}$.

### 3 Addition of points on the tropical Hesse curve

#### 3.1 Addition of points on the Hesse cubic curve

Fix the parameter $(t_0, t_1) \in \mathbb{P}^1(\mathbb{C})$ so that the Hesse cubic curve $E_{t_0, t_1}$ is nonsingular, or equivalently

$$(t_0, t_1) \neq (0, 1), (1, -3), (1, -3\zeta_3), (1, -3\zeta_3^2).$$
Let $K(E_{t_0,t_1})$ be the set of rational functions on the Hesse cubic curve $E_{t_0,t_1}$. Let us consider the linear system with respect to the divisor $3p_0$ of degree three on $E_{t_0,t_1}$

$$L(3p_0) := \{ h \in K(E_{t_0,t_1}) \mid (h) + 3p_0 > 0 \},$$

where $(h)$ stands for the principal divisor of the rational function $h$ and $D > 0$ means that $D$ is an effective divisor.

Consider the rational function $X_i$ ($i = 0, 1, 2$) on $E_{t_0,t_1}$ such that $X_i(Q) = q_i$ for $Q = (q_0, q_1, q_2) \in E_{t_0,t_1}$. Since $E_{t_0,t_1}$ is projectively equivalent to the Weierstraß canonical form, the linear system $L(3p_0)$ is the three dimensional linear space spanned by the rational functions $X_0$, $X_1$, and $X_2$:

$$L(3p_0) = \langle X_0, X_1, X_2 \rangle.$$

Let $\mathcal{D}(E_{t_0,t_1})$ be the divisor group on $E_{t_0,t_1}$. Let $\mathcal{D}_0(E_{t_0,t_1})$ be the subgroup of $\mathcal{D}(E_{t_0,t_1})$ generated by the divisors of degree 0. Also let $\mathcal{D}_l(E_{t_0,t_1})$ be the group generated by the principal divisors on $E_{t_0,t_1}$. Since $\deg(h) = 0$ for any $h \in K(E_{t_0,t_1})$, we have $\mathcal{D}_l(E_{t_0,t_1}) \subset \mathcal{D}_0(E_{t_0,t_1})$. There exists a bijection $\tilde{\phi}$ from $E_{t_0,t_1}$ to the Jacobian $J(E_{t_0,t_1}) \simeq \text{Pic}^0(E_{t_0,t_1}) := \mathcal{D}_0(E_{t_0,t_1})/\mathcal{D}_l(E_{t_0,t_1})$ of $E_{t_0,t_1}$. We can choose the bijection $\tilde{\phi} : E_{t_0,t_1} \to J(E_{t_0,t_1})$ to be

$$\tilde{\phi}(P) \equiv P - p_0 \pmod{\mathcal{D}_l(E_{t_0,t_1})}$$

for $P \in E_{t_0,t_1}$. The additive group structure of $E_{t_0,t_1}$ is induced from $J(E_{t_0,t_1})$ by $\tilde{\phi}$. Actually, the addition $P + Q$ of points $P, Q \in E_{t_0,t_1}$ is defined to be

$$P + Q := \tilde{\phi}(P) + \tilde{\phi}(Q).$$

Since $\tilde{\phi}(p_0) = 0$, the inflection point $p_0$ is the unit of addition.

Let $P$ and $Q$ be points on $E_{t_0,t_1}$. If $P$ and $Q$ are in generic position then there exists a unique line $l_1$ passing through both $P$ and $Q$. The line $l_1$ is the locus of all points $P$ such that $h(P) = 0$ for $h \in L(3p_0)$. Let the third intersection point of $E_{t_0,t_1}$ and $l_1$ be $\bar{M}$. Then $P$, $Q$, and $\bar{M}$ satisfy $h(P) = h(Q) = h(\bar{M}) = 0$, and hence $(h) = P + Q + \bar{M} - 3p_0$ holds. Thus we obtain

$$P + Q + \bar{M} = \tilde{\phi}(P) + \tilde{\phi}(Q) + \tilde{\phi}(\bar{M})$$

$$\equiv P + Q + \bar{M} - 3p_0 \pmod{\mathcal{D}_l(E_{t_0,t_1})}$$

$$\equiv 0 \pmod{\mathcal{D}_l(E_{t_0,t_1})}$$

$$= p_0.$$

Successively let $l_2$ be the line passing through both $p_0$ and $\bar{M}$. Let the third intersection point of $E_{t_0,t_1}$ and $l_2$ be $M$. Then we have

$$\bar{M} + M = p_0,$$

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and hence we obtain
\[ P + Q = M. \]  

(2)

Thus the addition of points on \( E_{a,t_1} \) is realized via the intersection with lines \( l_1 \) and \( l_2 \).

### 3.2 Theta functions

Next we show that the addition of points on the Hesse cubic curve can also be realized via the addition formulae for the level-three theta functions.

The level-three theta functions \( \theta_0(z, \tau), \theta_1(z, \tau) \), and \( \theta_2(z, \tau) \) are defined to be the theta functions with characteristics:

\[
\theta_k(z, \tau) := \vartheta_{\left( \frac{k}{6}, \frac{1}{6}, \frac{1}{6} \right)}(3z, 3\tau) = \sum_{n \in \mathbb{Z}} e^{3\pi i (n + \frac{k}{3} - \frac{1}{3}) \tau} e^{\pi i (n + \frac{k}{3} - \frac{1}{3})(z + \frac{1}{3})}
\]

for \( k = 0, 1, 2 \), where \( z \in \mathbb{C} \) and \( \tau \in \mathbb{H} := \{ \tau \in \mathbb{C} \mid \text{Im} \tau > 0 \} \). Fixing \( \tau \in \mathbb{H} \), we abbreviate \( \theta_k(z, \tau) \) and \( \theta_k(0, \tau) \) as \( \theta_k(z) \) and \( \theta_k \), respectively. We can easily see that the following holds

\[ \theta_0 = -\theta_1, \quad \theta_2 = 0. \]

(3)

Let \( L_\tau := (-\tau)\mathbb{Z} + (3\tau + 1)\mathbb{Z} \) be the lattice in \( \mathbb{C} \). Consider the map \( \varphi : \mathbb{C} \to \mathbb{P}^2(\mathbb{C}) \),

\[
\varphi : z \mapsto (\theta_2(z), \theta_0(z), \theta_1(z)).
\]

This induces an isomorphism from the complex torus \( \mathbb{C}/L_\tau \) to the Hesse cubic curve \( E_{\theta_2,\theta_0} \). It also induces the additive group structure on \( E_{\theta_2,\theta_0} \) from \( \mathbb{C}/L_\tau \) through the addition formulae for the level-three theta functions. Note that the relation (3) implies the unit of addition on \( E_{\theta_2,\theta_0} \) to be \( p_0 \):

\[
\varphi : 0 \mapsto (\theta_2, \theta_0, \theta_1) = (0, 1, -1) = p_0.
\]

**Theorem 1** (See [7]) Let the unit of addition on the Hesse cubic curve \( E_{\theta_2,\theta_0} \) be \( p_0 \). Let \( P = (x_0, x_1, x_2) \) and \( Q = (x'_0, x'_1, x'_2) \) be points on \( E_{\theta_2,\theta_0} \). Then the addition \( P + Q \) of the points is given as follows

\[
P + Q = (x_1x_2x'_0 - x'_0x_0x'_1, x_0x_1x'_2 - x_2x_0x'_2, x_0x_2x'_0 - x'_2x_1x'_1)
\]

\[
= (x_0x_1x'_0 - x'_0x_0x'_1, x_0x_1x'_2 - x_2x_0x'_2, x_1x_2x'_0 - x'_2x_1x'_1)
\]

\[
= (x_0x_2x'_1 - x'_1x_0x'_2, x_1x_2x'_0 - x'_0x_1x'_2 - x_2x_0x'_1).
\]

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These formulae give the coordinate-wise expression of \( [2] \).

### 3.3 Tropicalization of addition

Fix the parameter \( \kappa \in \mathbb{T} \mathbb{P}^1 \) so that the tropical Hesse curve \( C_\kappa \) is nonsingular, or equivalently

\[
0 < \kappa < \infty.
\]

Let \( K(C_\kappa) \) be the set of rational functions on \( C_\kappa \). A rational function on a tropical curve \( E \) is a continuous function \( h : E \to \mathbb{T} \mathbb{P}^1 \) such that the restriction of \( h \) to any edge of \( E \) is a piecewise linear integral function [3].

The order \( \text{ord}_P h \) of a rational function \( h \) at a point \( P \) on a tropical curve \( E \) is defined to be the sum of the outgoing slopes of all segments of \( E \) emanating from \( P \). If \( \text{ord}_P h > 0 \) then the point \( P \) is called the zero of \( h \) of order \( \text{ord}_P h \). If \( \text{ord}_P h < 0 \) then \( P \) is called the pole of \( h \) of order \( |\text{ord}_P h| \). We define the principal divisor of a rational function \( h \) on \( E \) to be

\[
(h) := \sum_{P \in E} (\text{ord}_P h) P.
\]

Let us consider the set \( K(\bar{C}_\kappa) \) of rational functions on \( \bar{C}_\kappa \) (not \( C_\kappa \)). We define the linear system \( R(3V_1) \) with respect to the divisor \( 3V_1 \) of degree three on \( \bar{C}_\kappa \) as in the case of a non-tropical curve:

\[
R(3V_1) := \{ h \in K(\bar{C}_\kappa) \mid (h) + 3V_1 > 0 \}.
\]

Consider the rational function \( \tilde{X}_i \) \( (i = 1, 2) \) on \( \bar{C}_\kappa \) such that \( \tilde{X}_i(Q) = q_i \) for \( Q = (q_1, q_2) \in \bar{C}_\kappa \). Then the principal divisors of \( \tilde{X}_i \) are given as follows

\[
(\tilde{X}_1) = 3V_2 - 3V_1, \quad (\tilde{X}_2) = 3V_3 - 3V_1.
\]

Therefore we have

\[
(\tilde{X}_1) + 3V_1 = 3V_2 > 0, \quad (\tilde{X}_2) + 3V_1 = 3V_1 > 0,
\]

and hence \( \tilde{X}_1, \tilde{X}_2 \in R(3V_1) \). Consider the tropical module \( \mathcal{M} \) generated by the rational functions 0, \( \tilde{X}_1 \), and \( \tilde{X}_2 \) [9]

\[
\mathcal{M} := \{ h \in K(\bar{C}_\kappa) \mid h = \max\{a, b + \tilde{X}_1, c + \tilde{X}_2\}, \ a, b, c \in \mathbb{T} \}.
\]

We see that the following holds

\[
\mathcal{M} = R(3V_1).
\]
Let $\mathcal{D}(\bar{C}_\kappa)$ be the divisor group on $\bar{C}_\kappa$. Let $\mathcal{D}_0(\bar{C}_\kappa)$ be the subgroup of $\mathcal{D}(\bar{C}_\kappa)$ generated by the divisors of degree 0. Also let $\mathcal{D}_l(\bar{C}_\kappa)$ be the group generated by the principal divisors on $\bar{C}_\kappa$. Since $\deg(h) = 0$ for any $h \in \mathbb{K}(\bar{C}_\kappa)$, we have $\mathcal{D}_l(\bar{C}_\kappa) \subseteq \mathcal{D}_0(\bar{C}_\kappa)$. There exists a bijection $\tilde{\phi}$ from $\bar{C}_\kappa$ (not $C_\kappa$) to the Jacobian $J(C_\kappa) \cong \text{Pic}^0(\bar{C}_\kappa) = \mathcal{D}_0(\bar{C}_\kappa)/\mathcal{D}_l(\bar{C}_\kappa)$ of $C_\kappa$ [20]. We can choose the bijection $\tilde{\phi} : \bar{C}_\kappa \rightarrow J(C_\kappa)$ to be $\tilde{\phi}(P) \equiv P - V_1 \pmod{\mathcal{D}_l(\bar{C}_\kappa)}$ for $P \in \bar{C}_\kappa$. Thus the additive group structure of $\bar{C}_\kappa$ is induced from $J(C_\kappa)$ by $\tilde{\phi}$. Actually, the addition $P + Q$ of points $P, Q \in \bar{C}_\kappa$ is defined to be $P + Q := \tilde{\phi}(P) + \tilde{\phi}(Q)$. Since $\tilde{\phi}(V_1) = 0$, the vertex $V_1$ is the unit of addition.

**Remark 1** If a point $P$ on $\bar{C}_\kappa$ is the zero of a rational function $h \in R(3V_1)$ then $h$ is not smooth at $P$ by definition. Therefore the curve $B$ defined by $h$ passes through the point $P$. Conversely, if $h$ passes through a point $P$ on $\bar{C}_\kappa$ then $P$ must be the zero of $h$ because $h$ is not smooth at $P$ and convex on the plane. Thus the zero $P$ of $h$ is the intersection point of $\bar{C}_\kappa$ and $B$.

Let $P$ and $Q$ be points on $\bar{C}_\kappa$. If $P$ and $Q$ are in generic position then there exists a unique line $r_1$ passing through both $P$ and $Q$. The line $r_1$ is the locus of all points $P$ such that $h \in \mathcal{M} = R(3V_1)$ is not smooth at $P$. Let the third intersection point of $\bar{C}_\kappa$ and $r_1$ be $\bar{M}$. Then $P, Q,$ and $\bar{M}$ satisfy $h(P) = h(Q) = h(\bar{M}) = 0$, and hence $(h) = P + Q + \bar{M} - 3V_0$ holds. Thus we obtain $P + Q + \bar{M} = \tilde{\phi}(P) + \tilde{\phi}(Q) + \tilde{\phi}(\bar{M}) \equiv P + Q + \bar{M} - 3V_1 \pmod{\mathcal{D}_l(\bar{C}_\kappa)} \equiv 0 \pmod{\mathcal{D}_l(\bar{C}_\kappa)} = V_1.$

Successively let $r_2$ be the line passing through both $V_1$ and $\bar{M}$. Let the third intersection point of $\bar{C}_\kappa$ and $r_2$ be $M$. Then we have $\bar{M} + M = V_1,$ and hence we obtain $P + Q = M.$

Thus the addition of points on $\bar{C}_\kappa$ is realized via the intersection with lines $r_1$ and $r_2$. 

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3.4 Ultradiscrete theta functions

Next we show that the addition of points on the tropical Hesse curve can be derived from the addition formulae for the ultradiscrete theta functions. Since we choose the inhomogeneous coordinate of $\mathbb{TP}^1$ to describe the addition on the tropical Hesse curve, we rather consider the elliptic functions $c(z) := \theta_0(z, \tau)/\theta_2(z, \tau)$ and $s(z) := \theta_1(z, \tau)/\theta_2(z, \tau)$ and their ultradiscretizations denoted by $\tilde{c}(u)$ and $\tilde{s}(u)$.

Let $\kappa$ and $\varepsilon$ be positive numbers. Let us fix the parameter $\tau$ in $\theta_k(z; \tau)$:

$$\tau = -\frac{3\kappa}{9\kappa + 2\pi i \varepsilon}.$$  

Then the complex torus $\mathbb{C}/L_\tau$ converges into $J(C_\kappa)$ in the limit $\varepsilon \to 0$ with respect to the Hausdorff metric [11].

In order to ultradiscretize the level-three theta functions, we further assume

$$z = \frac{(1 - i\xi) u}{9\kappa} \quad \text{and} \quad u \in \mathbb{R},$$

where $\xi = 2\pi\varepsilon/9\kappa$, and take the limit $\varepsilon \to 0$. Then we have [7,11]

$$\lim_{\varepsilon \to 0} \varepsilon \log c(z) = \tilde{c}(u), \quad \lim_{\varepsilon \to 0} \varepsilon \log s(z) = \tilde{s}(u),$$

where we define

$$\tilde{c}(u) := -\frac{9\kappa}{2} \left\{ \left( \frac{u - 2\kappa}{3\kappa} - \frac{1}{2} \right) \right\}^2 + \frac{9\kappa}{2} \left\{ \left( \frac{u - 3\kappa}{3\kappa} - \frac{1}{2} \right) \right\}^2,$$

$$\tilde{s}(u) := -\frac{9\kappa}{2} \left\{ \left( \frac{u - 2\kappa}{3\kappa} - \frac{1}{2} \right) \right\}^2 + \frac{9\kappa}{2} \left\{ \left( \frac{u - 3\kappa}{3\kappa} - \frac{1}{2} \right) \right\}^2,$$

and $((u)) := u - \text{Floor}(u)$. These piecewise linear functions $\tilde{c}(u)$ and $\tilde{s}(u)$ are periodic with period $3\kappa$.

Now take the following representatives $z_{0k}, z_{k1}, z_{k2}$ of the zeros of the level-three theta functions $\theta_k(z)$ in $\mathbb{C}/L_\tau$ for $k = 0, 1, 2$

$$
\begin{pmatrix}
z_{20} & z_{21} & z_{22} \\
z_{00} & z_{01} & z_{02} \\
z_{10} & z_{11} & z_{12}
\end{pmatrix}
= \begin{pmatrix}
0 & \tau + \frac{1}{3} & 2\tau + \frac{2}{3} \\
-\frac{\tau}{3} & \frac{2\tau}{3} + \frac{1}{3} & \frac{5\tau}{3} + \frac{2}{3} \\
-\frac{2\tau}{3} & \frac{\tau}{3} + \frac{1}{3} & \frac{4\tau}{3} + \frac{2}{3}
\end{pmatrix}.
$$

These nine zeros of the level-three theta functions are mapped into the nine
Inflection points on $E_{\theta_2,6\theta_0'}$ by $\varphi$, respectively:

$$
\begin{align*}
z_{20} & \quad z_{21} & \quad z_{22} & \quad p_0 & \quad p_1 & \quad p_2 \\
\varphi : & \quad z_{00} & \quad z_{01} & \quad z_{02} & \quad p_3 & \quad p_4 & \quad p_5 \\
z_{10} & \quad z_{11} & \quad z_{12} & \quad p_6 & \quad p_7 & \quad p_8
\end{align*}
$$

In terms of the variable $u$, we put the limit of zeros $z_{kj}$ ($k, j = 0, 1, 2$) of the level-three theta functions as follows

$$
\begin{align*}
u_2 & := \lim_{\varepsilon \to 0} 9\kappa z_{20} = \lim_{\varepsilon \to 0} 9\kappa z_{21} = \lim_{\varepsilon \to 0} 9\kappa z_{22} = 0, \\
u_0 & := \lim_{\varepsilon \to 0} 9\kappa z_{00} = \lim_{\varepsilon \to 0} 9\kappa z_{01} = \lim_{\varepsilon \to 0} 9\kappa z_{02} = \kappa, \\
u_1 & := \lim_{\varepsilon \to 0} 9\kappa z_{10} = \lim_{\varepsilon \to 0} 9\kappa z_{11} = \lim_{\varepsilon \to 0} 9\kappa z_{12} = 2\kappa,
\end{align*}
$$

where it should be noted that $\tau \to -1/3$ in the limit $\varepsilon \to 0$.

Let us introduce the map $\tilde{\varphi} : J(C_\kappa) \to \mathbb{R}^2 \subset \mathbb{T}^2$

$$
\tilde{\varphi} : \quad u \mapsto (\tilde{c}(u), \tilde{s}(u)).
$$

The map $\tilde{\varphi}$ induces an isomorphism $\tilde{C}_\kappa \simeq J(C_\kappa)$. Thus $\tilde{\varphi}$ induces the additive group structure on $\tilde{C}_\kappa$ equipped with the unit of addition $V_1 = \tilde{\varphi}(0)$ from $J(C_\kappa)$.

Consider the map $\eta : E_{\theta_2,6\theta_0'} \to \tilde{C}_\kappa$ so defined that the diagram commutes

$$
\begin{array}{ccc}
\mathbb{C}/L_{\tau} & \xrightarrow{\varepsilon \to 0} & J(C_\kappa) \\
\varphi \downarrow & & \downarrow \tilde{\varphi} \\
E_{\theta_2,6\theta_0'} & \xrightarrow{\eta} & \tilde{C}_\kappa.
\end{array}
$$

The inflection points of $E_{\theta_2,6\theta_0'}$ are mapped into the vertices of $\tilde{C}_\kappa$ by $\eta$:

$$
\begin{align*}
\eta : & \quad p_0, \quad p_1, \quad p_2 & \xleftarrow{\varepsilon \to 0} & \quad z_{20}, \quad z_{21}, \quad z_{22} & \xleftarrow{\varepsilon \to 0} & \quad u_2 & \xmapsto{\tilde{\varphi}} & \quad V_1, \\
\eta : & \quad p_3, \quad p_4, \quad p_5 & \xleftarrow{\varepsilon \to 0} & \quad z_{00}, \quad z_{01}, \quad z_{02} & \xleftarrow{\varepsilon \to 0} & \quad u_0 & \xmapsto{\tilde{\varphi}} & \quad V_2, \\
\eta : & \quad p_6, \quad p_7, \quad p_8 & \xleftarrow{\varepsilon \to 0} & \quad z_{10}, \quad z_{11}, \quad z_{12} & \xleftarrow{\varepsilon \to 0} & \quad u_1 & \xmapsto{\tilde{\varphi}} & \quad V_3.
\end{align*}
$$

Note that the unit $p_0$ of addition on the Hesse cubic curve is mapped into $V_0$, the unit of addition on the tropical Hesse curve, by $\eta$.

Define the open subsets $D_1, D_2,$ and $D_3$ of $J(C_\kappa)$ to be

$$
D_j := \{ u \in J(C_\kappa) \mid (j-1)\kappa < u < j\kappa \} \quad (j = 1, 2, 3).
$$
Then we have $J(C_\kappa) = \bigcup_{j=0}^3 (D_{j+1} \cup u_j)$. The addition formula for $C_\kappa$ is explicitly given as follows. In this theorem we denote $\max(\ )$ simply by $(\ )$.

**Theorem 2** (See [11]) Assume $u \in D_j$ for a fixed $j = 1, 2, 3$, where $D_j$ is the closure of $D_j$. Then the ultradiscrete elliptic functions $\tilde{c}$ and $\tilde{s}$ satisfy the following addition formulae

$$
\tilde{c}(u + v) = (\tilde{s}(u), 2\tilde{c}(u) + \tilde{c}(v) + \tilde{s}(v)) - (\tilde{c}(u) + 2\tilde{c}(v), 2\tilde{s}(u) + \tilde{s}(v)),
$$

$$
\tilde{s}(u + v) = (\tilde{c}(u) + \tilde{s}(u) + 2\tilde{s}(v), \tilde{c}(v)) - (\tilde{c}(u) + 2\tilde{c}(v), 2\tilde{s}(u) + \tilde{s}(v)),
$$

if and only if $v \in D_j \cup D_{j+1}$,

$$
\tilde{c}(u + v) = (\tilde{c}(u) + \tilde{s}(u) + 2\tilde{c}(v), \tilde{s}(v)) - (\tilde{s}(u) + 2\tilde{s}(v), 2\tilde{c}(u) + \tilde{c}(v)),
$$

$$
\tilde{s}(u + v) = (\tilde{c}(u), 2\tilde{s}(u) + \tilde{c}(v) + \tilde{s}(v)) - (\tilde{s}(u) + 2\tilde{s}(v), 2\tilde{c}(u) + \tilde{c}(v)),
$$

if and only if $v \in D_j \cup D_{j+2}$, or

$$
\tilde{c}(u + v) = (\tilde{c}(u) + 2\tilde{s}(v), 2\tilde{c}(u) + \tilde{c}(v)) - (\tilde{c}(u) + \tilde{s}(u), \tilde{c}(v) + \tilde{s}(v)),
$$

$$
\tilde{s}(u + v) = (\tilde{s}(u) + 2\tilde{c}(v), 2\tilde{c}(u) + \tilde{s}(v)) - (\tilde{c}(u) + \tilde{s}(u), \tilde{c}(v) + \tilde{s}(v)),
$$

if and only if $v \in D_{j+1} \cup D_{j+2}$, where the subscripts are reduced modulo 3.

It immediately follows the addition formula for the points on the tropical Hesse curve $C_\kappa$ from theorem 2. Let the edge of $C_\kappa$ connecting the vertex $V_i$ with $V_{i+1}$ be $E_i$ for $i = 1, 2, 3$, where we assume $V_4 = V_1$. Then we have the following corollary of theorem 2.

**Corollary 1** Let $P = (X, Y)$ be a point on an edge $E_j$ of the tropical Hesse curve $C_\kappa$ for a fixed $j = 1, 2, 3$. Then the point $P + Q = (X' + X, Y' + Y)$ is given by the following addition formulae

$$
X + X' = \max(Y, 2X + X' + Y') - \max(X + 2X', 2Y + Y'),
$$

$$
Y + Y' = \max(X + Y + 2Y', X') - \max(X + 2X', 2Y + Y'),
$$

if and only if $Q = (X', Y') \in E_j \cup E_{j+1}$,

$$
X + X' = \max(X + Y + 2X', Y') - \max(Y + 2Y', 2X + X'),
$$

$$
Y + Y' = \max(X, 2Y + X' + Y') - \max(Y + 2Y', 2X + X'),
$$

if and only if $Q \in E_j \cup E_{j+2}$, or

$$
X + X' = \max(X + 2Y', 2Y + X') - \max(X + Y, X' + Y'),
$$

$$
Y + Y' = \max(Y + 2X', 2X + Y') - \max(X + Y, X' + Y'),
$$

if and only if $Q \in E_{j+1} \cup E_{j+2}$, where the subscripts are reduced modulo 3.
4 A tropical analogue of the Hessian group

4.1 Hessian group

The Hessian group $G_{216} \simeq \Gamma \times SL(2, \mathbb{F}_3)$ is a subgroup of $PGL(3, \mathbb{C})$, where $\Gamma = (\mathbb{Z}/3\mathbb{Z})^2$ and $SL(2, \mathbb{F}_3)$ is the special linear group over the finite field $\mathbb{F}_3$ of characteristic three. The Hessian group is generated by the following four linear transformations

$$g_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3^2 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \zeta_3 & \zeta_3^2 \\ 1 & \zeta_3^2 & \zeta_3 \end{pmatrix}, \quad g_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3 \end{pmatrix}.$$ 

The name, “Hessian” group, comes from the fact that $G_{216}$ is the group of linear automorphisms acting on the Hesse pencil [III].

The group of three torsion points on $E_{t_0, t_1}$, denoted by $E_{t_0, t_1}[3]$, consists of its nine inflection points $p_0, p_1, \ldots, p_8$. The map

$$p_1 \mapsto (1, 0) \quad p_3 \mapsto (0, 1)$$

induces a group isomorphism from $E_{t_0, t_1}[3]$ to the normal subgroup $\Gamma = \langle g_1, g_2 \rangle$ of $G_{216}$. Therefore, the action of $\Gamma$ fixes the parameter $(t_0, t_1)$ of the pencil.

Let $\alpha : G_{216} \to PGL(2, \mathbb{C})$ be the map given by

$$\alpha(g) : (t_0, t_1) = (x_0x_1x_2, x_0^3 + x_1^3 + x_2^3) \mapsto (t'_0, t'_1) = (x'_0x'_1x'_2, x'_0^3 + x'_1^3 + x'_2^3),$$

where $g \in G_{216}$ and $g : (x_0, x_1, x_2) \mapsto (x'_0, x'_1, x'_2)$. Then we have Ker($\alpha$) $\supseteq \Gamma$, while $\alpha(g_3)$ and $\alpha(g_4)$ act effectively on $\mathbb{P}^1(\mathbb{C})$:

$$\alpha(g_3) : (t_0, t_1) \mapsto (t'_0, t'_1) = (3t_0 + t_1, 18t_0 - 3t_1)$$
$$\alpha(g_4) : (t_0, t_1) \mapsto (t'_0, t'_1) = (t_0, \zeta_3^2t_1).$$

Thus $g_3$ and $g_4$ induce the action on the Hesse pencil independent of its additive group structure on each curve. We can easily see that $\alpha(g_3)^2 = \alpha(g_4)^3 = 1$ holds. It follows that we have

$$\alpha(G_{216}) = \langle \alpha(g_3), \alpha(g_4) \rangle \simeq \mathcal{T},$$

where $\mathcal{T}$ is the tetrahedral group. Thus the group $\alpha(G_{216})$ acts on $\mathbb{P}^1(\mathbb{C})$ as
the permutations among the following 12 points

\[
\lambda := \frac{t_1}{t_0}, \zeta_3 \lambda, \zeta_3^2 \lambda, \frac{18 - 3 \lambda}{3 + \lambda}, \frac{18 \zeta_3 - 3 \zeta_3 \lambda}{3 + \lambda}, \frac{18 \zeta_3^2 - 3 \zeta_3^2 \lambda}{3 + \lambda}, \frac{18 \zeta_3 - 3 \zeta_3 \lambda}{3 + \lambda}, \frac{18 \zeta_3^2 - 3 \zeta_3^2 \lambda}{3 + \lambda}, \frac{18 \zeta_3 - 3 \zeta_3 \lambda}{3 + \lambda}.
\]

The Hesse pencil contains four singular members each of which is with multiplicity three. These singular members correspond to the following four points in \(\mathbb{P}^1(\mathbb{C})\), respectively [1]

\[(t_0, t_1) = (0, 1), (1, -3), (1, -3\zeta_3^2), (1, -3\zeta_3)\].

Denote these points by \(s_i\) \((i = 1, 2, 3, 4)\) in order. These \(s_i\)'s are permuted by \(\alpha(G_{216})\) as follows

\[\alpha(g_3): s_1 \leftrightarrow s_2, \quad s_3 \leftrightarrow s_4\]

\[\alpha(g_4): s_2 \rightarrow s_3 \rightarrow s_4 \rightarrow s_2 \quad (s_1 \text{ is fixed.})\]

Let

\[g_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.\]

Then we have

\[g_3^2 = (g_4g_0)^3 = g_0.\]

Therefore, we obtain

\[G_{216}/\Gamma = \langle g_3, g_4 \rangle \simeq \tilde{T},\]

where \(\tilde{T}\) is the binary tetrahedral group. Since \(\tilde{T}\) is isomorphic to \(SL(2, \mathbb{F}_3)\), we obtain the semi-direct product decomposition \(G_{216} \simeq \Gamma \rtimes SL(2, \mathbb{F}_3)\).

Note that the actions of \(g_1\) and \(g_2\) on \(E_{9',60'}\) can be realized as the additions with \(p_6\) and \(p_1\), respectively

\[(x_0, x_1, x_2) \xrightarrow{g_1} (x_1, x_2, x_0) = (x_0, x_1, x_2) + p_6,
(x_0, x_1, x_2) \xrightarrow{g_2} (x_0, \zeta_3 x_1, \zeta_3^2 x_2) = (x_0, x_1, x_2) + p_1.\]
4.2 Tropicalization of the Hessian group

Now we investigate the tropical analogue of the Hessian group $G_{216} \simeq \Gamma \times SL(2, \mathbb{F}_3)$. At first we consider the normal subgroup $\Gamma \simeq (\mathbb{Z}/3\mathbb{Z})^2$. Note that $\Gamma = \langle g_1, g_2 \rangle$ and the actions of $g_1$ and $g_2$ on $E_{t_0, t_1}$ is realized as the additions with $p_6$ and $p_1$, respectively.

The correspondence (1), (5), and (6) in terms of $\eta$ tell us that the addition with $p_6$ corresponds to that with $V_3$ on $\bar{C}_\kappa$, while that with $p_1$ vanishes in the limit $\varepsilon \to 0$. (Note that $V_1$ is the unit of addition on $\bar{C}_\kappa$.) Since the addition with $p_3$ (resp. $V_2$) is equivalent to that with $2p_6$ (resp. $2V_3$), the tropical analogue of $\Gamma$ consists of the addition with $V_3$. Actually, it is the group $\bar{C}_\kappa[3] = \langle V_3 \rangle \simeq \mathbb{Z}/3\mathbb{Z}$ of three torsion points on $\bar{C}_\kappa$. Denote the tropical analogue of a group $G$ by $\text{trop}(G)$. Then we have

$$\text{trop}(\Gamma) \simeq \mathbb{Z}/3\mathbb{Z}.$$ 

The addition of points $(X, Y)$ on $\bar{C}_\kappa$ with $V_3$ is explicitly computed as follows

$$(X, Y) + V_3 = (X, Y) + (0, -\kappa) = (Y - X, -X),$$

where we apply the addition formula in corollary [1].

The group $\text{trop}(\Gamma)$ can also be obtained by applying the procedure of ultradiscretization directly to $g_1$ and $g_2$. Let us consider the inhomogeneous coordinate $(x := x_1/x_0, y := x_2/x_0)$ of $\mathbb{P}^2(\mathbb{C})$. Let $g_1 : (x, y) \mapsto (x', y')$ and $g_2 : (x, y) \mapsto (x'', y'')$. Then we have

$$(|x'|, |y'|) = \left( \frac{|y|}{|x|}, \frac{1}{|x|} \right), \quad (|x''|, |y''|) = (|x|, |y|).$$

Replace $|x|$ and $|y|$ with $e^{X/\varepsilon}$ and $e^{Y/\varepsilon}$ for $X, Y \in \mathbb{R}$ and $\varepsilon > 0$, respectively. If we take the limit $\varepsilon \to 0$ then we obtain

$$(X, Y) \xrightarrow{\tilde{g}_1} (Y - X, -X) = (X, Y) + V_3, \quad (X, Y) \xrightarrow{\tilde{g}_2} (X, Y),$$

where we denote the action on $\mathbb{T}\mathbb{P}^2$ induced form $g_1$ and $g_2$ by $\tilde{g}_1$ and $\tilde{g}_2$, respectively.

Next we consider $\alpha(G_{216}) \simeq \mathcal{T}$. Consider the singular members of the tropical Hesse pencil $C_\infty$ and $C_0$, each of which is the tropicalization of the singular members $E_{s_1}$ or $E_{s_i}$, $(i = 2, 3, 4)$ of the Hesse pencil [11]. Then the action of $\alpha(g_3)$, which permutes $s_1$ and $s_3$ with $s_2$ and $s_4$ respectively, must vanish; while the action of $\alpha(g_4)$, which fixes $s_1$ and permutes $s_2$, $s_3$, and $s_4$ cyclically,
reduces to the action fixing both $C_0$ and $C_\infty$. Therefore, we conclude

$$trop(\alpha(G_{216})) \simeq trop(\mathcal{T}) \simeq \langle 1 \rangle.$$  

Thus the tropical analogue of the Hessian group fixes each member of the tropical Hesse pencil.

Furthermore, we consider the tropicalization of the element $g_0 = g_3^2$ of $G_{216}/\Gamma$. We ultradiscretize $g_0$ directly as well as $g_1$ and $g_2$. In the inhomogeneous coordinate, the action of $g_0$ on $\mathbb{P}^2(\mathbb{C})$ is simply

$$(x, y) \mapsto g_0 \cdot (y, x).$$

It follows that we have

$$(X, Y) \mapsto \tilde{g}_0 \cdot (Y, X),$$

where $(X, Y)$ is the inhomogeneous coordinate of $\mathbb{T}\mathbb{P}^2$ and $\tilde{g}_0$ is the action on $\mathbb{T}\mathbb{P}^2$ induced from $g_0$ by applying the procedure of ultradiscretization. Thus we conclude that the tropical analogue of $\tilde{T} \simeq \langle g_3, g_4 \rangle = G_{216}/\Gamma$ is the group of order two generated by $\tilde{g}_0$:

$$trop(\tilde{T}) \simeq \langle \tilde{g}_0 \rangle \simeq \left\langle \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right\rangle \subset SL(2, \mathbb{F}_3).$$

We finally obtain the following theorem concerning the tropical analogue of the Hessian group $G_{216}$.

**Theorem 3** The dihedral group $D_3$ of degree three,

$$D_3 = \langle \tilde{g}_0, \tilde{g}_1 \rangle \simeq (\mathbb{Z}/3\mathbb{Z}) \rtimes \left\langle \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right\rangle$$

where $\tilde{g}_0, \tilde{g}_1 \in PGL(3, \mathbb{T})$ satisfy $\tilde{g}_0^2 = \tilde{g}_1^3 = (\tilde{g}_0 \tilde{g}_1)^2 = 1$, is the group of linear automorphisms acting on the tropical Hesse pencil. The action of $\tilde{g}_0$ on each curve of the pencil is realized as the reflection with respect to the line $Y = X$ passing through the vertex $V_1$; and the action of $\tilde{g}_1$ on each curve is realized as the addition with $V_3$.

The semi-direct product in $D_3$ is defined as follows. Let

$$\gamma_1, \gamma_2 \in \mathbb{Z}/3\mathbb{Z} \quad \text{and} \quad m_1, m_2 \in \left\langle \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right\rangle.$$
Define the multiplication of \((\gamma_1, m_1), (\gamma_2, m_2) \in \mathcal{D}_3\) to be
\[
(\gamma_1, m_1) \cdot (\gamma_2, m_2) = (\gamma_1 + m_1 \gamma_2, m_1 m_2),
\]
where we identify \(\gamma_i\) with \((\gamma_i, 0) \in (\mathbb{Z}/3\mathbb{Z})^2\).

5 Conclusion

We gave two realizations of the addition of points on the tropical Hesse curve. The one realized as the intersection of the tropical Hesse curve with a tropical line is reduced from the divisor class group and the linear system on the curve; the other realized as the addition formula for the ultradiscrete theta functions is reduced as the ultradiscrete limit of the addition formulae for the level-three theta functions. We also showed that the Hessian group, the group of linear automorphisms acting on the Hesse pencil, reduced to the dihedral group of degree three acting automorphically on the tropical Hesse pencil. By using the group structure on the tropical Hesse pencil, we can construct both integrable dynamical systems and solvable chaotic systems each of which is given by a piecewise linear map on the curve.

Although we only considered the linear automorphisms acting on the Hesse pencil in this paper, there exists the group of a wide class of automorphisms acting on the pencil. To investigate a tropical analogue of the Cremona group, the group of birational automorphisms acting on the Hesse pencil, is a further problem.

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