Optimal Record and Replay under Causal Consistency

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Abstract

We investigate the minimum record needed to replay executions of processes that share causally consistent memory. For a version of causal consistency, we identify optimal records under both offline and online recording setting. Under the offline setting, a central authority has information about every process’ view of the execution and can decide what information to record for each process. Under the online setting, each process has to decide on the record at runtime as the operations are observed.

1 Introduction

In this paper, we explore Record and Replay (RnR) of multi-process applications where processes communicate via shared memory. Record and replay (RnR) mechanisms aim to allow parallel program debugging to proceed as follows. The programmer runs the program, and potentially observes incorrect behavior. The programmer then re-runs the program, while more closely watching the program state, and attempts to discover where a program bug may have occurred. However, even when a parallel program is re-executed with the same input, different executions of the program may proceed differently, due to non-determinism introduced by the uncertainty in the delays incurred in performing various operations. Thus, the observed bug may not re-occur during re-run, making it quite difficult to discover the cause of the original problem. Record and Replay (RnR) aims to solve this problem by creating a record during the original execution, and using it.

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during replay to guarantee that the re-run produces the same outcomes as the original execution. In other words, while the original execution may be non-deterministic, the replay using the record eliminates the non-determinism as desired.

There can be many sources of non-determinism in parallel programs. For example user inputs, readings from sensors, random coin flips, etc. However, in this paper we focus specifically on the non-determinism allowed by the shared memory consistency models in the read-write memory model. For a given program, the shared memory consistency model defines a space of allowed executions possible when the program is run. By creating a record during an execution and enforcing it in the replay, this space is further restricted hence reducing the inherent non-determinism. The goal is to record enough from the original execution so as to reproduce the same outcomes in the replay.

The work in this paper is motivated by the trade-off between the consistency model for shared memory and the amount of information that must be recorded to facilitate a replay. A stronger consistency model imposes more constraints on the execution, resulting in a smaller space of allowed executions. Intuitively, a stronger consistency model should require a smaller record to resolve the non-determinism during replay. In Section 5.3 we present an example execution to illustrate that this intuition is indeed correct. In prior work, Netzer [14] identified the minimum record necessary for RnR under the sequential consistency model [10]. The computer architecture research community has also investigated RnR systems under various consistency models, for example [5], [6], and [11]. See also a survey by Chen et. al. [3]. However, to the best of our knowledge, only Netzer’s work [14] has addressed identification of minimum record for RnR under read-write memory model.

This paper builds on Netzer’s work to address the minimum record for correct replay under causal consistency. Whether a certain record is necessary and sufficient for replay depends on several factors, as discussed next. Lee et. al. [11] have also discussed a classification of RnR strategies and Chen et. al. [3] provide a taxonomy of deterministic replay schemes.

1) How faithful should the replay be to the original execution? To understand the different scenarios that are plausible, let us consider an implementation of shared memory. Suppose that each process maintains a local replica of the shared variables. When a process writes to a shared variable, the new value is propagated to other processes via update messages. The new value is eventually written at each replica, while ensuring that the consistency model is obeyed. Figure 1(a) illustrates an execution of two processes that implement sequential consistency. In this case, in the original execution, \( x \) is updated to equal 1 due to the write operation \( w_1(x = 1) \) by process 1, and then \( y \) is updated to 2 due to the write operation \( w_2(y = 2) \) by process 2. Subsequently, process 1 reads \( y \) as 2 with the read operation \( r_1(y = 2) \). Figures 1(b) and (c) show two possible replays of the execution in Figure 1(a). Observe that, while the read returns the same value in both replays, the order in which the variables are updated is different in the replay in Figure 1(b) than the original execution. On the other hand, the replay in Figure 1(c) performs the updates in an identical order as in the original execution.

Depending on whether we must reproduce the replay as in Figure 1(b), or allow a replay as in Figure 1(c), the minimum record necessary will be different. As one may expect, the record required for replay in Figure 1(b) is smaller, since the replay is not as faithful as that in Figure 1(c). Netzer’s minimum record [14] for sequential consistency allows the replay in Figure 1(b), which ensures that all the reads and writes to the same variable occur in the same order during replay as in the original execution. However, the updates to different variables may not necessarily
occur in the same order during replay as in the original execution.

At a minimum, the read operations in the replay must return the same values as the corresponding read operations in the original execution. This ensures that the program state for each process, and so the output, in the replay is the same as the one in the original execution (i.e., the same branches are taken in both the executions as the next step to be performed by a process depends on the current program state and the values read from shared memory) and so the replay is indistinguishable to the high-level user from the original execution. We discuss the exact formal model for this work in Section 4.

2) At what level of abstraction is the RnR system implemented? The abstraction level where the RnR system operates influences what can and needs to be recorded. For instance, if the shared memory is implemented via message passing, then, for the purpose of RnR, we may treat this as a message-passing system and record messages rather than shared memory operations. In this case, the RnR system can be viewed as residing below the shared memory implementation.

Alternatively, the RnR system may operate at the library level where the low level details, including interactions with the shared memory, are abstracted via the provided libraries. The RnR system is only allowed to record interactions with the APIs of the given libraries. We refer the reader to Section 4.3 for a more detailed explanation of different abstract levels.

In this paper, our focus is on RnR for the shared memory. In our model, the RnR system resides on top of the shared memory layer so that the inner workings of the shared memory are abstracted while the interactions with the shared memory, via the read and write operations on shared variables, is exposed. In this case, we assume that the RnR module may observe, at each process, the reads of that process and the writes of all the processes.

3) Offline versus online recording. In the offline setting, the RnR module is provided with a completed execution in its entirety, and can use this information to obtain a record that suffices
for a correct replay. In the online setting, each process has its own RnR module that observes the execution incrementally, and must decide incrementally what information must be recorded. The online record can be useful when, for example, the replay proceeds in tandem with the original execution for redundancy purposes. Netzer’s result [14] applies to both the offline and online setting for sequential consistency. In this paper, we consider both offline and online settings in the context of causal consistency.

A summary of our contributions is presented in Table 1. In this work, we present the optimal record for a version of causal consistency which we call strong causal consistency. This is formally defined in Section 3 and is followed by many practical implementations of causal consistency. We consider both the RnR model for replay as in Figure 1(b) and as in Figure 1(c). These are defined formally in Section 4. Sequential consistency was considered by Netzer [14]. We consider the first RnR model in Section 5. In Sections 5.1 and 5.2 we present the optimal records for strong causal consistency for the offline and online scenarios respectively. The question of optimal record for causal consistency is still open and we discuss this in Section 5.3. We consider the second RnR model in Section 6 with optimal record for the offline case of strong causal consistency given in Section 6.1 and the one for causal consistency discussed in Section 6.2. We finish the paper with a discussion in Section 7 along with some open problems.

2 Preliminaries

A relation \( R \) on a set \( O \) is a set of tuples \((a,b)\) such that \( a, b \in O \). We use the notation \( a <_R b \) if \((a,b) \in R \). We denote \( a \leq_R b \) if either \( a <_R b \) or \( a = b \). An irreflexive, antisymmetric, and transitive relation is called a partial order. A partial order \( R \) on a set \( O \) is a total order if for any \( a, b \in O \), either \( a <_R b \) or \( b <_R a \). A partial order can be represented by a directed acyclic graph which is closed under transitivity. For two relations \( A \) and \( B \) on a set \( O \), we say that \( A \) respects \( B \) if \( B \subseteq A \). We use the notation \( A \mid O' \) to restrict the relation \( A \) on set \( O \) to a subset \( O' \subseteq O \). \( \hat{A} \) denotes the (unique) transitive reduction of the partial order \( A \) and \( a <_A b \) denotes \((a,b) \in \hat{A} \). We use \( A \cup B \) to denote the union, with the transitive closure, of relations \( A \) and \( B \), and \( A \cdot \cup B \) to denote the disjoint union of \( A \) and \( B \). For example, consider two partial orders \( A \) and \( B \) on the set \{a, b\}, given by \( A = \{(a,b)\} \) and \( B = \{(b,a)\} \). Then, \( A \cup B = \{(a,b),(b,a),(a,a),(b,b)\} \) while \( A \cdot \cup B = \{(a,b),(b,a)\} \). Observe that union and disjoint union of two partial orders may
not be a partial order, as the previous example shows.

We borrow some notation by Steinke and Nutt \[15\] for shared memory formalism. The shared memory consists of a set of variables $X$ and supports two operations, read and write. We use $w$ for writes, $r$ for reads, and $o$ when the operation can be either read or write. We use a subscript for process identifier or leave it blank if it is unspecified. If the variable and the corresponding value read/written is relevant, we specify it in parenthesis. For example, $w_i(x = 1)$ denotes a write of value 1 to variable $x$ performed by process $i$ and $o_j(y)$ denotes an operation performed by process $j$ that can either be a read or write to variable $y$. Formally, an operation is a 4-tuple $(op, i, x, id)$ where $op$ is $r$ for read and $w$ for write, $i$ is the unique identifier of the process that performed the operation, $x$ is the (shared) variable on which the operation was performed, and $id$ is the unique identifier of the operation. This notation allows for wild-card entries, e.g. $(w, i, *, *)$ is the set of all writes executed by process $i$. Observe that we do not specify the values in the notation. We assume that each write operation writes a unique value $1$. The values read by read operations may vary between executions, but each read operation reads a value written by some write.

All operations in $(*, i, *, *)$ are totally ordered. We denote this total order by $PO(i)$. The disjoint union of these is the program order given by $PO = \cup_i PO(i)$. This is the order on operations implied by the program text. In figures representing total orders, we draw operations from left to right as they appear in the total order. For example, Figure 2(a) draws the program order for two processes, 1 and 2. The two total orders, $PO(1)$ and $PO(2)$, corresponding to processes 1 and 2 respectively, are drawn from left to right.

We model the distributed system as a network of processes that communicate with each other via reads and writes to the shared memory. Each process comes with a program that specifies the operations to be executed and the order in which they should be executed. Formally, a shared memory system is a set of processes $P$, a set of operations $O$, a program order $PO$ on $O$, a set of shared variables $X$, and a shared memory $\Pi$. An execution is the result of processes running their programs on a shared memory system where each read operation returns a value written by some write operation.

**Definition 2.1** (Writes-to). Given an execution, a write operation $w$ writes-to a read operation $r$, denoted $w \mapsto r$, if $w$ and $r$ are on the same variable and $r$ returns the value written by $w$.

We reason about executions as a collection of read and write operations on shared variables. We do not distinguish any operation as special, e.g. synchronization operation, but view all operations to the shared memory uniformly. This is the same as Netzer’s model \[14\].

**Assumptions about Programs**

In general, programs are dynamic where the next operation to be executed depends on the current program state. Our model requires reproducing the execution faithfully; at the very least all read operations must return the same values. Since we consider deterministic programs, that read the same values from the shared memory via the corresponding read operations, therefore we claim, without proof, that program at each process will execute the same operations in the same order.

\[\text{Since the unique write values have a one-to-one correspondence with the unique identifiers of the respective write operations, therefore formally specifying the write values is redundant.}\]
in both the original execution and the replay. A similar result is shown in [13] for a different setting. So we assume that the program order $PO$ is fixed.

One standard practice for writing concurrent programs is to ensure that they are properly synchronized such that they are data race free [1]. This guarantees sequential semantics for such programs under most concurrent languages and multiprocessors. We do not make any such assumptions since

1) we do not distinguish any operation as special, e.g. synchronization operations,

2) one of the aims of this work is to replay programs for debugging purposes, so assuming that the programmer has written the program correctly is a dangerous assumption, and

3) the guarantee of sequential semantics for data race free programs is for a different consistency model (cache consistency) and it does not hold for causal consistency.

3 Shared Memory Consistency

For an execution, a view $V$ on a set of operations $O' \subseteq O$ is a total order on $O'$ such that each read $r \in O'$ returns the last value written to the corresponding variable in $V$. For a view $V$, the data-race order is given by $DRO(V) = \bigcup_{x \in X} V | (*,*,x,*)$. Reasoning about allowed executions under a shared memory consistency model relies on existence of some collection of views $V$ that satisfy some properties, depending on the shared memory consistency model. We say that $V$ explains the execution under the consistency model. For example, causal consistency [2] requires existence of per-process views that satisfy causality, which is the union (with the transitive closure) of the writes-to relation and the program order. Formally, we use the definition by Steinke and Nutt [15].

**Definition 3.1** (Write-read-write Order [Steinke and Nutt [15]]). Given an execution with a writes-to relation $\rightarrow$, two writes, $w^1 \in (w,*,*,*)$ and $w^2 \in (w,*,*,*)$, are ordered by write-read-write order, $(w^1, w^2) \in WO$, if there exists a read operation $r \in (r,*,*,*)$ such that $w^1 \rightarrow r <_{PO} w^2$.

**Definition 3.2** (Causal Consistency [Steinke and Nutt [15]]). An execution is causally consistent if there exists a set of views $V = \{V_i\}_{i \in P}$ such that, for every process $i$,

- $V_i$ is a view on the set of operations $(*,i,*,*) \cup (w,*,*,*)$, and
- $V_i$ respects $WO \cup (PO)((*,i,*,*) \cup (w,*,*,*))$.

A shared memory $\Pi$ is causally consistent if every execution run on $\Pi$ is causally consistent.

Note that, by definition, each view $V_i$ already respects the writes-to relation restricted to $(*,i,*,*) \cup (w,*,*,*)$ since, by definition of a view, each read returns the last value written to the corresponding variable in $V_i$. Note also that read operations are only observed by the processes that perform them while write operations are observed by every process. We work with a version of causal consistency which we call strong causal consistency. This model is motivated by an implementation of causal consistency via lazy replication [9]. Ladin et. al. [9] use vector timestamps to ensure that a write operation $w_i$ from process $i$ is only committed locally when all write operations in $w_i$’s history, as summarized by $w_i$’s vector timestamp, have been observed.
Many practical systems use vector timestamps to determine order of operations and detect conflicts in systems with weak consistency guarantees (e.g. Dynamo [4], COPS [12], and Bayou [16]) although these systems have conflict resolution schemes which make their actual consistency guarantees stronger than strong causal consistency (see also Section 7). Formally, we define strong causal consistency as follows.

**Definition 3.3 (Strong Causal Order).** Given a set of views $V = \{V_i\}_{i \in P}$, two writes, $w^1 \in (w, *, *, *, *)$ and $w^2_i \in (w, i, *, *, *)$, are ordered by strong causal order, $(w^1, w^2_i) \in SCO(V)$, if $(w^1, w^2_i) \in V_i$.

This is stronger than the write-read-write order $WO$ since two writes $w^1 \in (w, *, *, *, *)$ and $w^2_i \in (w, i, *, *, *)$ are ordered by $WO$ if and only if $w^1$ has been read by process $i$ before it performs $w^2_i$. However, $w^1$ has to be merely observed by process $i$ for the two operations to be ordered by strong causal order. Intuitively, this corresponds to causality when each write operation observed is immediately read.

**Definition 3.4 (Strong Causal Consistency).** An execution is strongly causal consistent if there exists a set of views $V = \{V_i\}_{i \in P}$ such that, for every process $i$,

- $V_i$ is a view on the set of operations $(*, i, *, *) \cup (w, *, *, *)$, and
- $V_i$ respects $SCO(V) \cup (PO(\{*, i, *, *\} \cup (w, *, *, *)))$.

A shared memory $\Pi$ is strongly causal consistent if every execution run on $\Pi$ is strongly causal consistent.

Observe that strong causal consistency does not violate the write-read-write order and thus it is at least as strong as causal consistency. In fact, it is strictly stronger than causal consistency.
Figure 2 shows a causally consistent execution of a two process program. The read and write values are given in Figure 2(a). Figure 2(b) gives a set of views that explains this execution under causal consistency. The values of read and write operations have been omitted with the dotted edges giving the writes-to relation. Some obvious PO edges have also been omitted to avoid clutter. We reason that no set of views can explain the execution under strong causal consistency.

Observe that ordering \((w_2(x), w_1(x))\) \(\in V_1\) implies an \(SCO(V)\) edge that must be respected by \(V_2\). Therefore, any set of views that explain the execution under strong causal consistency must have either \((w_2(x), w_1(x))\) \(\in V_2\) or \((w_1(x), w_2(x))\) \(\in V_1\). We show that none of these is possible.

For the first case, note that \(w_1(x) \prec PO w_1(y) \rightarrow r_2(y) \prec PO r_2^2(x)\). Therefore \(w_1(x)\) can not be placed after \(r_2^2(x)\) in \(V_2\). Now if \(w_1(x)\) is placed after \(w_2(x)\) in \(V_2\), then \(r_2^2(x)\) does not return the last value written to \(x\) in \(V_2\). This violates the definition of a view.

For the second case, we have that \(w_2(x) \prec PO w_2(y) \prec WO w_1(y) \prec PO r_1^2(x)\). Therefore \(w_2(x)\) can not be placed after \(r_1^2(x)\) in \(V_1\). Now if \(w_2(x)\) is placed after \(w_1(x)\) in \(V_1\), then \(r_1^2(x)\) does not return the last value written to \(x\) in \(V_1\). Again, this violates the definition of a view.

Compiler and Hardware Optimizations

In real world systems, many optimizations are applied to the provided program by both the compiler at compile time and the hardware at runtime. The shared memory consistency model ensures that these optimizations are such that the guarantees provided are still maintained by these optimizations. For example, consider a uniprocessor and a shared memory consistency model that guarantees a view consistent with the program order implied by the written program. The compiler and hardware optimizations may result in operations being executed out of order in apparent violation of the program order constraints. However, the resulting execution can still be explained by the existence of a view (or views) where the operations are executed exactly as specified by the program order. Using view based definitions of shared memory consistency models allows us to abstract these implementation details. Therefore we allow all optimizations to be applied to the given program as long as the relevant shared memory consistency guarantees are satisfied.

4 RnR Model

For replaying executions, we assume that the per-process views are provided to the RnR system. The RnR system uses the views to determine the record. In case of online recording, the views are provided to the RnR system incrementally, as and when new operations occur that affect the views. Now let us illustrate how this requirement may be implemented in practice. Consider a shared memory implementation wherein each process has a copy of the shared variable and the shared memory is implemented via message passing. Then the shared memory adds a write operation to process \(i\)'s view when the local copy of the corresponding variable is updated at process \(i\). Similarly a read by process \(i\) is added to process \(i\)'s view when the local copy is read.

The RnR system will record some edges from each view (i.e. on each process) and the replay execution is only allowed views that enforce these records. Note that we do not place any restriction on how the record is enforced. We assume that any set of views can explain the replay as long as it extends the record and is consistent under the shared memory consistency model.
Formally, we define two RnR models with different fidelities. Under the first model, the RnR system is allowed to record any edge from each view and we require that the replay reproduces the per-process views exactly as in the original execution. Under the second model, the RnR system is only allowed to record data races from each view and we only require that the data races are resolved identically in the replay.

**RnR Model 1:** Given a set of views $V = \{V_i\}_{i \in P}$, $R = \{R_i\}_{i \in P}$ is a record of $V$ if each $R_i \subseteq V_i$. An execution is a replay of $R$ if there exists a set of views $V' = \{V'_i\}_{i \in P}$ that explain the execution under the consistency model and each $V'_i$ respects $R_i$. We say that $V'$ certifies the replay to be valid for $R$. A record $R$ of a set of views $V$ is good if, for any replay of $R$, under the same consistency model, any set of views $V' = \{V'_i\}_{i \in P}$ that certifies the replay to be valid for $R$ must have $V'_i = V_i$ for all $i \in P$ (i.e. only $V$ certifies the replay to be valid for $R$).

**RnR Model 2:** Given a set of views $V = \{V_i\}_{i \in P}$, $R = \{R_i\}_{i \in P}$ is a record of $V$ if each $R_i \subseteq DRO(V_i)$. An execution is a replay of $R$ if there exists a set of views $V' = \{V'_i\}_{i \in P}$ that explain the execution under the consistency model and each $V'_i$ respects $R_i$. We say that $V'$ certifies the replay to be valid for $R$. A record $R$ of a set of views $V$ is good if, for any replay of $R$, under the same consistency model, any set of views $V' = \{V'_i\}_{i \in P}$ that certifies the replay to be valid for $R$ must have $DRO(V'_i) = DRO(V_i)$ for all $i \in P$.

The second replay model is the same as the one considered by Netzer [14]. Observe that for each record, there exists at least one replay, specifically the original execution. Note that RnR Model 1 forces all writes to appear in the same order for a process’ view as they did in the original execution, which is different than Netzer’s model in [14]. This may seem expensive since reordering writes to different variables can result in performance optimizations while still returning the same values for reads and allowing the program state in the replay to progress the same as in the original execution. RnR Model 2 allows writes to different variables to be executed in different order, which is the same as Netzer’s model in [14]. But for RnR Model 1 we require that each process’ point of view with respect to the order of events must be indistinguishable between the original execution and the replay.

In contrast to the discussion at the end of Section 3, the optimizations for the replay execution may be more restrictive than those for the original execution. Exactly what optimizations are allowed in the replay execution versus the original execution depends on the shared memory consistency model as well as the replay system implementation. In this work, we do not discuss replay systems, their implementations, or how they may enforce the provided record. So we do not discuss the optimizations during the replay.

## 5 Optimal Records for RnR Model 1

### 5.1 Offline Record for Strong Causal Consistency

In this section we consider offline record for strong causal consistency. In this case the entire set of per-process views $V = \{V_i\}_{i \in P}$ is made available to the RnR system. The RnR system determines the record that must be saved. If the RnR system decides to record the entire views $V_i$ for every process $i$, then this would be sufficient to reproduce the original execution exactly. However, this is wasteful since the transitive reduction $\hat{V}_i$ for each process $i$ would also achieve the same result.

We first give intuition on what edges from each $\hat{V}_i$ do not need to be recorded before formalizing
Fix a process $i$. Since $PO$ is fixed and independent of executions the RnR system does not have to record these edges in $V_i$ as they are guaranteed by the consistency model. Now consider two write operations $w^1 \in (w,*,*,*)$ and $w^2_j \in (w,j,*,*)$, for $j \neq i$, such that $(w^1, w^2_j) \in SCO(V)$. If process $j$ correctly orders the two operations $(w^1, w^2_j)$ in the replay, then this edge will be guaranteed by the consistency model, due to strong causal order, and process $i$ does not need to record it. Such edges are captured by the following definition.

**Definition 5.1.** Given a set of views $V = \{V_i\}_{i \in P}$, the relation $SCO_i(V)$, for a process $i \in P$, is defined as follows. Two writes, $w^1 \in (w,*,*,*)$ and $w^2_j \in (w,j,*,*)$, are ordered $(w^1, w^2_j) \in SCO_i(V)$, if $(w^1, w^2_j) \in SCO(V)$ and $j \neq i$.

Observe that the subscript distinguishes the relation $SCO_i(V)$ from $SCO(V)$ (Definition 5.2) which is a partial order for strongly causal executions. We now present an example to illustrate another set of edges that do not need to be recorded, although they are not directly guaranteed by the consistency model. Consider the following execution on three processes and a set of views that explains it under strong causal consistency (Figure 3). Process 1 performs the write $w_1 \in (w,1,*,*)$, process 2 performs $w_2 \in (w,2,*,*)$, and process 3 does not perform any operations. Now process 1 orders $w_1 < V_1 w_2$, process 2 orders $w_2 < V_2 w_1$, and process 3 orders $w_1 < V_3 w_2$. It can be easily verified that this set of views satisfies Definition 5.1 of strong causal consistency where both $PO$ and $SCO(V)$ are empty. Now note that if process 3 records $w_1 < R_3 w_2$, process 1 does not need to record its order of the two operations. The reason is that any possible set of views $V' = \{V_i'\}_{i \in P}$, that certify a replay to be valid for $R_i$, will have $V'_3 \setminus w_1 < V'_3 w_2$. So if process 1 orders $w_2 < V'_1 w_1$, this will create an $SCO(V')$ edge $w_2 < SCO(V') w_1$. Since $V'_3$ respects $SCO(V')$, therefore process 3 will order $w_2 < V'_3 w_1$. This conflicts with the recorded edge $w_1 < R_3 w_2$. Thus, such a set of views can not certify a replay execution to be valid for $R$. The set of such edges is captured by the following relation.

**Definition 5.2.** Given a set of views $V = \{V_i\}_{i \in P}$, the relation $B_i(V)$, for a process $i \in P$, is defined as follows. Two writes, $w^1_i \in (w,i,*,*)$ and $w^2_j \in (w,j,*,*)$ such that $i \neq j$, are ordered $(w^1_i, w^2_j) \in B_i(V)$ if $(w^1_i, w^2_j) \in V_i$ and there exists a process $k \neq i,j$ such that $(w^1_i, w^2_j) \in V_k$.

Informally, in any set of views $V'$ that explain a replay of $R$, setting $(w^2_j, w^1_i) \in V'_i$ will create an $SCO(V')$ edge $(w^2_j, w^1_i)$ which will conflict with $V'_i$. The following theorem states that for every process $i$ it suffices to record all edges in $\hat{V}_i$, except those in $SCO_i(V)$, $PO$, or $B_i(V)$.

**Theorem 5.3.** Consider a set of views $V = \{V_i\}_{i \in P}$ that explain a strongly causal consistent execution. For each process $i \in P$, let $R_i = \hat{V}_i \setminus (SCO_i(V) \cup PO \cup B_i(V))$. Then, $R = \{R_i\}_{i \in P}$ is a good record of $V$.

The formal proof of the theorem is given in Appendix A. We first show that the strong causal order and the $B_i$’s are preserved in the replay (Lemma 5.1). The proof then proceeds by arguing that, for every process $i$, each path in $\hat{V}_i$ is reproduced correctly in the replay. We refer the reader to Appendix A for the details. The following theorem states that, for every process $i$, each edge in $\hat{V}_i \setminus (SCO_i(V) \cup PO \cup B_i(V))$ is necessary for a good record under strong causal consistency.

**Theorem 5.4.** Consider a set of views $V = \{V_i\}_{i \in P}$ that explain a strongly causal consistent execution. For any good record $R = \{R_i\}_{i \in P}$ of $V$, for any process $i \in P$ and any two operations $o^1, o^2 \in (*,i,*,*) \cup (w,*,*,*)$, if $(o^1, o^2) \in \hat{V}_i \setminus (PO \cup SCO_i(V) \cup B_i(V))$, then $(o^1, o^2) \in R_i$. 

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Figure 3: \(\{V_i\}_{i=1}^3\) explains a strongly causal execution and \(\{V'_i\}_{i=1}^3\) explains an invalid replay. Process 1 orders \(w_1 < V_1 w_2\) in the replay which would force process 3 to violate the record.

The formal proof of the theorem is presented in Appendix A. We show that if any two operations \(o^1, o^2\) are such that, for some process \(i\), \((o^1, o^2) \in \hat{V}_i \setminus (PO \cup SCO_i(\mathcal{V}) \cup B_i(\mathcal{V}))\) but \((o^1, o^2)\) is not recorded, then we can swap the two operations during the replay without violating consistency or replay constraints. This violates the definition of a good record. Theorems 5.3 and 5.4 show that the record \(R = \{R_i\}_{i \in P}\) such that \(R_i = \hat{V}_i \setminus (SCO_i(\mathcal{V}) \cup PO \cup B_i(\mathcal{V}))\) is both sufficient and necessary for a correct replay under strong causal consistency.

5.2 Online Record for Strong Causal Consistency

We now look at the optimal record in an online setting. Consider the following implementation of shared memory. Each process keeps a copy of every shared variable in \(X\). Processes exchange messages to propagate their writes to shared variables. Based on the received messages, each process updates the current value of its copy of the shared variables. At any point in the execution, a read on variable \(x\) at process \(i\) returns the current value of \(x\) stored at \(i\). We abstract this perspective of shared memory as follows. Each process has a fixed set of read and write operations \((\ast, i, \ast, \ast)\) that it executes in their local order \(PO(i)\) by communicating with the shared memory. Executing an operation may take arbitrarily long and the process may spend arbitrarily long time to execute the next operation but each process only executes one operation at a time. Via the shared memory, a process \(i\) observes its own operations and write operations from other processes one at a time. The order in which these operations are observed give rise to the view \(V_i\). More formally, the execution proceeds in time steps. At each time step in the execution, a unique\(^2\) process \(i\) observes an operation from \((\ast, i, \ast, \ast) \cup (w, \ast, \ast, \ast)\) and adds it to its view \(V_i\).

The online record algorithm proceeds as follows. Suppose process \(i\) wants to record \((o^1, o^2) \in V_i\). Then, process \(i\) must record \((o^1, o^2)\) at the time when it observes \(o^2\). In the online setting, process

\(\text{SCO}(\mathcal{V})\)

\(\text{SCO}(\mathcal{V})\)

\(\text{SCO}(\mathcal{V})\)

\(\text{SCO}(\mathcal{V})\)

\(\text{SCO}(\mathcal{V})\)
i has limited information about views of other processes at any given time in the execution. How much does process i know? We assume that, at most, process i has access to the history of other processes brought with the observed operation. More precisely, at any time in the execution, if process i is aware that \((o^1, o^2) \in V_j\), for some process \(j \neq i\), then process i must have already observed \(o^3_j \in (\text{*}, j, \text{*}, \text{*})\) such that \(o^1 \prec_{V_j} o^2 \leq_{V_j} o^3\). As discussed in Sections 1 and 3, the recording proceeds without information about the internal workings of the shared memory. However, we assume that the RnR system is aware of the shared memory guarantees. More precisely, for strong causal consistency, we assume that any process i can check if \((o^1, o^2) \in SCO(V)\) and also if \((o^1, o^2) \in PO\). For a given execution \(V = \{V_i\}_{i \in P}\), we say that a record \(R = \{R_i\}_{i \in P}\) is an online record of \(V\) if \(R\) can be recorded in this manner.

Recall from Theorems 5.3 and 5.4 that for any process \(i\), \(R_i = \widehat{V_i} \setminus (SCO_i(V) \cup PO \cup B_i(V))\) is both sufficient and necessary in the offline setting. Therefore, if the recording unit can detect, for an edge \((o^1, o^2) \in \widehat{V_i}\), if it is one of \(SCO_i(V)\), \(PO\), or \(B_i(V)\), then the optimal record in the online setting would match exactly the one in the offline scenario. However, it turns out that the membership of \((o^1, o^2)\) in \(B_i(V)\) cannot be checked by the recording unit online. This is formalized in Theorems 5.5 and 5.6 which state that for each process \(i\), \(R_i = \widehat{V_i} \setminus (SCO_i(V) \cup PO)\) is both sufficient and necessary in the online setting. The formal proofs are presented in Appendix A.

**Theorem 5.5.** Consider a set of views \(V = \{V_i\}_{i \in P}\) that explain a strongly causal consistent execution. For each process \(i \in P\), let \(R_i = \widehat{V_i} \setminus (SCO_i(V) \cup PO)\). Then, \(R = \{R_i\}_{i \in P}\) is a good online record of \(V\).

**Theorem 5.6.** Consider a set of views \(V = \{V_i\}_{i \in P}\) that explain a strongly causal consistent execution. For any good online record \(R = \{R_i\}_{i \in P}\) of \(V\), for any process \(i \in P\) and any two operations \(o^1, o^2 \in (\text{*}, i, \text{*}, \text{*}) \cup (w, \text{*}, \text{*}, \text{*})\), if \((o^1, o^2) \in \widehat{V_i} \setminus PO \cup SCO_i(V)\), then \((o^1, o^2) \in R_i\).

### 5.3 Causal Consistency

Causal consistency (Definition 3.2) imposes less restrictions on views that can explain an execution as compared to strong causal consistency. As discussed in Section 1, we expect a smaller record for strong causal consistency than causal consistency. Indeed consider a simple execution on two processes and two operations where process 1 performs \(w_1\) and process 2 performs \(w_2\). Consider the set of views given in Figure 4 that explains this execution under both causal and strong causal consistency. Under strong causal consistency, only process 1 has to record \((w_2, w_1)\). However, since causal consistency imposes no restrictions in this particular example, a good record for causal consistency will require process 2 to record \((w_2, w_1)\) as well.

The question of what is the optimal record for causal consistency is still open. We give a simple counterexample that shows that the natural strategy following the scheme of strong causal consistency does not work. More concretely, consider a set of views \(V = \{V_i\}_{i \in P}\) that explain a causally consistent execution. For each process \(i\), let \(R_i = \widehat{V_i} \setminus (WO \cup PO)\). We give a simple four process example that shows that \(R = \{R_i\}_{i \in P}\) is not a good record of \(V\). The program for this example is given in Figures 5 and 6. Figure 5 gives the writes-to relation in bold edges, for the original execution of the program, as well as a set of views \(V\) that explains the execution. The red edges represent the recorded edges, as specified above. Figure 6 gives one possible replay where the reads return the default values for the variables (so that the writes-to relation is empty), as well as a set of views \(V'\) that certifies the replay to be valid for the given record.
Figure 4: A simple example where the required record is smaller for strong causal consistency. \( \{V'_1, V'_2\} \) can certify a replay under causal consistency but not under strong causal consistency.

\[ \begin{align*}
V_1: & \quad w_2 \xrightarrow{R_1} w_1 \\
V_2: & \quad w_2 \xrightarrow{SCO_2(V)} w_1
\end{align*} \]

\[ \begin{align*}
V'_1: & \quad w_2 \xrightarrow{R_1} w_1 \\
V'_2: & \quad w_1 \xrightarrow{} w_2
\end{align*} \]

Figure 5: A 4 process program where the bold edges represent the writes-to relation for a possible execution. The set of views \( \{V_i\}_{i=1}^4 \) explains this execution. The recorded edges are given in red.

\[ \begin{align*}
\text{Process 1: } w_1(x) & \quad \downarrow \\
\text{Process 2: } r_2(x) & \xrightarrow{PO} w_2(x) \\
\text{Process 3: } w_3(y) & \quad \downarrow \\
\text{Process 4: } r_4(y) & \xrightarrow{PO} w_4(y)
\end{align*} \]

\[ \begin{align*}
V_1: & \quad w_1(x) \xrightarrow{R_1} w_3(y) \xrightarrow{WO} w_4(y) \xrightarrow{R_1} w_2(x) \\
V_2: & \quad w_1(x) \xrightarrow{R_2} w_3(y) \xrightarrow{WO} w_4(y) \xrightarrow{R_2} r_2(x) \xrightarrow{PO} w_2(x) \\
V_3: & \quad w_3(y) \xrightarrow{R_3} w_1(x) \xrightarrow{WO} w_2(x) \xrightarrow{R_3} w_4(y) \\
V_4: & \quad w_3(y) \xrightarrow{R_4} w_1(x) \xrightarrow{WO} w_2(x) \xrightarrow{R_4} r_4(y) \xrightarrow{PO} w_4(y)
\end{align*} \]

Figure 6: A possible replay of the execution in Figure 5 where the reads return the default values. The set of views \( \{V'_i\}_{i=1}^4 \) certify that this replay is valid for the record from Figure 4.

\[ \begin{align*}
\text{Process 1: } w_1(x) & \quad \downarrow \\
\text{Process 2: } r_2(x) & \xrightarrow{PO} w_2(x) \\
\text{Process 3: } w_3(y) & \quad \downarrow \\
\text{Process 4: } r_4(y) & \xrightarrow{PO} w_4(y)
\end{align*} \]

\[ \begin{align*}
V'_1: & \quad w_4(y) \xrightarrow{R_1} w_2(x) \xrightarrow{} w_1(x) \xrightarrow{R_1} w_3(y) \\
V'_2: & \quad w_4(y) \xrightarrow{R_2} r_2(x) \xrightarrow{PO} w_2(x) \xrightarrow{} w_1(x) \xrightarrow{R_2} w_3(y) \\
V'_3: & \quad w_2(x) \xrightarrow{R_3} w_4(y) \xrightarrow{} w_3(y) \xrightarrow{R_3} w_1(x) \\
V'_4: & \quad w_2(x) \xrightarrow{R_4} r_4(y) \xrightarrow{PO} w_4(y) \xrightarrow{} w_3(y) \xrightarrow{R_4} w_1(x)
\end{align*} \]
Observe that $\mathcal{V}' \neq \mathcal{V}$. There are two WO edges $(w_1, w_2)$ and $(w_3, w_4)$ in the original execution while $WO'$, the write-read-write order for the replay, is empty. Note that, in this example, not only do the views differ, but the reads return the wrong values in the replay as well.

The example replay execution is causally consistent, but it has the strange property that processes do not commit their writes locally before informing other processes. For example, consider $w_2$ and $w_4$. We have $(w_4, w_2) \in V_2$ but $(w_2, w_4) \in V_4$. Both process 2 and 4 observed the other process’s write before they saw their own; one of these processes distributed it’s write to the other, then observed the other process’s write, then committed it’s own write. This does not violate causality because neither process had read the other process’ write (note, however, that this does violate strong causality). Consider the setting where each process keeps a copy of each variable and the shared memory is implemented via message passing. Then either process 2 or process 4 sends messages for its write before writing the local copy of the corresponding variable. Such an execution would not be possible if each process always wrote to their local copy of the variable first and then sent the relevant messages to other processes.

6 Optimal Records for RnR Model 2

6.1 Offline Record for Strong Causal Consistency

In this section we consider offline record for strong causal consistency. In this case, as in Section 5.1, the entire set of per-process views $\mathcal{V} = \{V_i\}_{i \in P}$ are made available to the RnR system which then determines the record that must be saved. We define strong write order inductively as below. It will be important for the optimal record for this RnR model.

**Definition 6.1 (Strong Write Order).** Given a set of views $\mathcal{V} = \{V_i\}_{i \in P}$, two writes, $w^1 \in (w, *, *, *)$ and $w^2 \in (w, i, *, *)$, are ordered

1. $(w^1, w^2) \in SWO^1(\mathcal{V})$ if $(w^1, w^2) \in DRO(V_i) \cup (PO)((*, i, *, *) \cup (w, *, *, *))$,
2. $(w^1, w^2) \in SWO^k(\mathcal{V})$ if $(w^1, w^2) \in DRO(V_i) \cup SWO^{k-1}(\mathcal{V}) \cup (PO)((*, i, *, *) \cup (w, *, *, *))$.

We say that $w^1$ and $w^2$ are ordered by strong write order, $(w^1, w^2) \in SWO(\mathcal{V})$, if $(w^1, w^2) \in SWO^k(\mathcal{V})$ for some $k$. Furthermore, if $(w^1, w^2) \in SWO(\mathcal{V})$, then for every process $j \neq i$, we say that $(w^1, w^2) \in SWO_j(\mathcal{V})$.

Note that for strongly causal consistent executions the strong write order is a subset of strong causal order. Hence strong write order is a partial order for strongly causal consistent executions. In contrast with RnR Model 1, we are only allowed to record DRO edges. Intuitively, SWO captures those SCO edges that can be used to influence the views of other processes under this model. The base case captures those edges that will be forced on every process if process $i$ reproduces $DRO(V_i)$ faithfully. The inductive case captures those edges that would be forced on every process if the previous level is forced and if process $i$ reproduces $DRO(V_i)$ faithfully. However note that, in contrast with the RnR Model 1, SWO may influence some relations than cannot be recorded.

The following definition will be useful in presenting the optimal records.

**Definition 6.2.** Given a set of views $\mathcal{V} = \{V_i\}_{i \in P}$, the relation $A_i(\mathcal{V})$, for a process $i \in P$, is defined as $A_i(\mathcal{V}) = DRO(V_i) \cup SWO_i(\mathcal{V}) \cup (PO)((*, i, *, *) \cup (w, *, *, *))$. Furthermore $A(\mathcal{V}) = \{A_i(\mathcal{V})\}_{i \in P}$.
Observation 6.3. Consider a set of views $V = \{V_i\}_{i \in P}$ that explain a strongly causal execution and two writes, $w^1 \in (w,*,*,*)$ and $w^2_2 \in (w,i,*,*)$. Then $(w^1, w^2_2) \in A_i(V)$ if and only if $(w^1, w^2_2) \in SWO(V)$.

Note that this implies that $A_i(V) \supseteq SWO(V)$, for all $i \in P$, as follows. Each edge in $SWO(V)$ is either a $SWO_i(V)$ edge or a $SWO(V) \setminus SWO_i(V)$ edge. Observation 6.3 implies $(SWO(V) \setminus SWO_i(V)) \subseteq A_i(V)$ and $SWO_i(V) \subseteq A_i(V)$ by Definition 6.2.

Proof:

$\Rightarrow$ Suppose $(w^1, w^2_2) \in A_i(V)$. Then $(w^1, w^2_2) \in SWO(V)$ by Definition 6.1.

$\Leftarrow$ Suppose $(w^1, w^2_2) \in SWO^k(V)$ for some $k > 0$. We proceed by induction on $k$. For the base case, we have that $(w^1, w^2_2) \in DRO(V_i) \cup (PO([*,i,*,*]) \cup (w,*,*,*))$ and so $(w^1, w^2_2) \in A_i(V)$. For the inductive step, we have that $(w^1, w^2_2) \in DRO(V_i) \cup SWO^{k-1}(V) \cup (PO([*,i,*,*]) \cup (w,*,*,*))$. Now $SWO^{k-1}(V) = SWO^{k-1}_i(V) \cup (SWO^{k-1}(V) \setminus SWO^{k-1}_i(V))$.

Observe that $(SWO^{k-1}(V) \setminus SWO^{k-1}_i(V)) \subseteq A_i(V)$ by the inductive hypothesis. Furthermore $SWO^{k-1}_i(V) \subseteq A_i(V)$ by Definition 6.2. Since $A_i(V)$ is closed under transitivity, the result follows.

Similar to the record for RnR Model 1 in Section 5.1, we wish to capture the effect of reordering two operations on the SWO that violates the views of some other process. More specifically, for two operations $o^1 \in ([*,*,*,*])$ and $o^2 \in ([*,*,*,*])$ such that $o^1 <_{DRO(V_i)} o^2$ for some process $i$, reordering them as $o^2 <_{DRO(V_i)} o^1$, may introduce some $SWO(V)$ edges that violate some other process’s view. The following two definitions capture this notion.

Definition 6.4. Given a set of views $V = \{V_i\}_{i \in P}$, a process $i \in P$, and two operations $o^1 \in ([*,*,*,*])$ and $o^2 \in ([*,*,*,*])$, the relation $C_i(V, o^1, o^2)$ is defined inductively as follows.

1. Two write operations $w^3 \in (w,*,*,*)$ and $w^4_1 \in (w,i,*,*)$ are ordered $(w^3, w^4_1) \in C_i^1(V, o^1, o^2)$ if
   
   (a) $o^1 \leq_{A_i(V)} w^4_1$, and
   (b) $w^3 \leq_{A_i(V)} w^2_2$.

2. Two write operations $w^3 \in (w,*,*,*)$ and $w^4_1 \in (w,i',*,*)$ are ordered $(w^3, w^4_1) \in C_i^k(V, o^1, o^2)$ if there exist two write operations $w^5 \in (w,*,*,*)$ and $w^6 \in (w,*,*,*)$ such that
   
   (a) $(w^5, w^6) \in C_i^{k-1}(V, o^1, o^2)$,
   (b) $w^3 \leq_{A_i(V) \cup C_i^{k-1}(V, o^1, o^2)} w^5$, and
   (c) $w^6 \leq_{A_i(V)} w^4_1$.

Two write operations $w^3 \in (w,*,*,*)$ and $w^4 \in (w,*,*,*)$ are ordered $(w^3, w^4) \in C_i(V, o^1, o^2)$ if $(w^3, w^4) \in C_i^k(V, o^1, o^2)$ for some $k \geq 1$.

Definition 6.5. Given a set of views $V = \{V_i\}_{i \in P}$, the relation $B_i(V)$, for a process $i \in P$, is defined as follows. Two operations on the same variable $x$, $o^1 \in ([*,*,x,*)$ and $o^2 \in ([w,*,x,*])$, are ordered $(o^1, o^2) \in B_i(V)$ if

1. $(o^1, o^2) \in DRO(V_i)$, and
2. there exists a process $m \in P$ such that either
Informally, in any set of views $\mathcal{V}'$ that explain a replay of $\mathcal{R}$, setting $(w^2, o^1) \in DRO(V_i')$ will create a SWO($\mathcal{V}'$) edge which will conflict with $A_m(\mathcal{V}')$. The Appendix B contains some useful observations which are needed in the proofs later on. The following theorem states that for every process $i$ it suffices to record all edges in $\hat{A}_i(\mathcal{V})$, except those in SWO($\mathcal{V}$), PO, or $B_i(\mathcal{V})$.

**Theorem 6.6.** Consider a set of views $\mathcal{V} = \{V_i\}_{i \in P}$ that explain a strongly causal consistent execution. For each process $i \in P$, let $R_i = \hat{A}_i(\mathcal{V}) \setminus (SWO_i(\mathcal{V}) \cup PO \cup B_i(\mathcal{V}))$. Then, $\mathcal{R} = \{R_i\}_{i \in P}$ is a good record of $\mathcal{V}$.

The formal proof of the theorem is given in Appendix C. It proceeds similarly to the proof of Theorem 5.3 but is significantly more complicated. The following theorem states that, for every process $i$, each edge in $\hat{A}_i(\mathcal{V}) \setminus (SWO_i(\mathcal{V}) \cup PO \cup B_i(\mathcal{V}))$ is necessary for a good record under strong causal consistency.

**Theorem 6.7.** Consider a set of views $\mathcal{V} = \{V_i\}_{i \in P}$ that explain a strongly causal consistent execution. For any good record $\mathcal{R} = \{R_i\}_{i \in P}$ of $\mathcal{V}$, for any process $i \in P$ and any two operations $o^1, o^2 \in (*, i, *, *) \cup (w, *, *, *)$, if $(o^1, o^2) \in \hat{A}_i(\mathcal{V}) \setminus (PO \cup SWO_i(\mathcal{V}) \cup B_i(\mathcal{V}))$, then $(o^1, o^2) \in R_i$.

The formal proof of the theorem is presented in Appendix C. We follow the same strategy as proof of Theorem 5.3 and show that if any two operations $o^1, o^2$ are such that, for some process $i$, $(o^1, o^2) \in \hat{A}_i \setminus PO \cup SWO_i(\mathcal{V}) \cup B_i(\mathcal{V})$ but $(o^1, o^2)$ is not recorded, then we can swap the two operations during the replay without violating consistency or replay constraints. This violates the definition of a good record. Theorems 6.6 and 6.7 show that the record $\mathcal{R} = \{R_i\}_{i \in P}$ such that $R_i = \hat{A}_i \setminus (SWO_i(\mathcal{V}) \cup PO \cup B_i(\mathcal{V}))$ is both sufficient and necessary for a correct replay under strong causal consistency.

### 6.2 Causal Consistency

The question of what is the optimal record for causal consistency is still open for RnR Model 2 as well. Similar to Section 5.3, we give a counterexample that shows that the natural strategy following the scheme of strong causal consistency does not work. More concretely, consider a set of views $\mathcal{V} = \{V_i\}_{i \in P}$ that explain a causally consistent execution. For each process $i$, let $A_i = DRO(V_i) \cup WO \cup (PO(*, i, *, *) \cup (w, *, *, *))$ and $R_i = \hat{A}_i \setminus (WO \cup PO)$. We give a simple four process example that shows that $\mathcal{R} = \{R_i\}_{i \in P}$ is not a good record of $\mathcal{V}$. The program for this example is given in Figures 7 and 8. Figure 7 also gives the writes-to relation in bold edges, for the original execution. The writes-to relation is empty for the replay, as shown in Figure 8. Figure 9 gives a set of views $\mathcal{V}$ that explains the original execution. The red edges represent the recorded edges. Figure 10 gives one possible replay where the reads return the default values for the variables (so that the writes-to relation is empty), as well as a set of views $\mathcal{V}'$ that certifies the replay to be valid for the given record.

There are two $WO$ edges $(w_1, w_2)$ and $(w_3, w_4)$ in the original execution while $WO'$, the write-read-write order for the replay, is empty. Note that, in this example, not only do the views differ, but the reads return the wrong values in the replay as well.
Figure 7: A 4 process program where the bold edges represent the writes-to relation for a possible execution.

Figure 8: A possible replay of the execution in Figure 7 where the reads return the default values.

7 Discussion and Open Problems

In this work we have looked at the optimal record for RnR under strong causal consistency, a strengthened version of causal consistency followed by practical implementations of causally consistent shared memory [4], [9], [12], [16]. Table 1 provides a summary of RnR results. The optimal record for causal consistency is still an open problem. In Section 5.3 and Section 6.2 we showed that a simple strategy following the scheme of strong causal consistency does not work for either RnR Model 1 or RnR Model 2.

As discussed in Section 1 to the best of our knowledge, only one other work by Netzer [14] looks at optimal record for RnR. However, Netzer considered sequential consistency and his setting is the same as RnR Model 2 where the objective is to record only data races so that all data races are resolved. Another interesting setting is if the RnR system is allowed to record any edge in the views but the objective is to resolve all data races. We have not yet looked at this setting, which we leave open to investigate in a future work.

We have not discussed how the record is enforced during replay. For example, a simple strategy could be to simply wait for an operation until all its dependencies in the record have been observed. This may not work with every record since the replay may be forced to choose between a record constraint and a consistency constraint. We leave addressing this question to a future work.

Another problem of interest is to look at optimal RnR for weaker models. Cache consistency is

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3Two operations form a data race if they are on the same variable and at least one of them is a write.
\[ V_1: \begin{align*} &w_1(x) \ w_1(y) \ w_3(y) \ w_4(z) \ w_4(\alpha) \ w_2(\alpha) \ w_2(z) \ w_3(x) \\
\end{align*} \]

\[ P1: \begin{align*} &w_1(x) \xrightarrow{PO} w_1(y) \\
\end{align*} \]

\[ \hat{A}_1(\mathcal{V}): \]

\[ P2: \]

\[ \xrightarrow{PO} w_3(y) \xrightarrow{WO} w_3(x) \]

\[ \xrightarrow{R_1} \]

\[ P3: \]

\[ \xrightarrow{PO} w_4(z) \xrightarrow{PO} w_4(\alpha) \]

\[ P4: \]

\[ \xrightarrow{PO} w_2(\alpha) \xrightarrow{PO} w_2(z) \]

\[ \hat{A}_2(\mathcal{V}): \]

\[ P1: \]

\[ \xrightarrow{PO} w_1(x) \xrightarrow{PO} w_1(y) \]

\[ \hat{A}_3(\mathcal{V}): \]

\[ P2: \]

\[ \xrightarrow{PO} w_2(\alpha) \xrightarrow{WO} w_2(z) \]

\[ \xrightarrow{R_3} \]

\[ P3: \]

\[ \xrightarrow{PO} w_3(y) \xrightarrow{PO} w_3(x) \]

\[ \xrightarrow{R_2} \]

\[ P4: \]

\[ \xrightarrow{R_2} \xrightarrow{PO} w_3(x) \]

\[ \xrightarrow{PO} w_4(z) \xrightarrow{PO} w_4(\alpha) \]

\[ \hat{A}_4(\mathcal{V}): \]

\[ P1: \]

\[ \xrightarrow{PO} w_1(x) \xrightarrow{PO} w_1(y) \]

\[ \hat{A}_i(\mathcal{V}) \]

\[ P2: \]

\[ \xrightarrow{WO} w_2(\alpha) \xrightarrow{PO} w_2(z) \]

\[ \xrightarrow{R_4} \]

\[ P3: \]

\[ \xrightarrow{PO} w_3(y) \xrightarrow{PO} w_3(x) \]

\[ \xrightarrow{R_4} \xrightarrow{PO} w_3(x) \]

\[ P4: \]

\[ \xrightarrow{R_4} \xrightarrow{PO} r_4(y) \xrightarrow{PO} w_4(\alpha) \]

Figure 9: The set of views \( \{V_i\}_{i=1}^4 \) explains the execution in Figure 7. \( \hat{A}_i(\mathcal{V}) \) for \( i = 1, 2, 3, 4 \) are also given with the recorded edges drawn in red.
Figure 10: The set of views \( \{V'_i\}_{i=1}^4 \) certifies that the replay in Figure 8 is valid for the record from Figure 9. \( \hat{A}_i(V') \) for \( i = 1, 2, 3, 4 \) are also given with the recorded edges drawn in red.
defined as sequential consistency on a per variable basis.

**Definition 7.1.** An execution is cache consistent if there exists a set of views \( V = \{ V_x \}_{x \in X} \) such that, for every variable \( x \),

- \( V_x \) is a view on the set of operations \( (\ast, \ast, x, \ast) \), and
- \( V_x \) respects \( (PO|(\ast, \ast, x, \ast)) \).

A shared memory \( \Pi \) is cache consistent if every execution run on \( \Pi \) is cache consistent.

For this definition, the optimal record follows from Netzer’s result on sequential consistency [14]. However, this assumes that per variable views are available to be recorded. From the per process perspective, Steinke and Nutt [15] have an alternate equivalent definition which sees cache consistency as providing per process views. We refer the reader to Theorem B.8 in [15] for this alternate definition of cache consistency. What does the optimal record look like in this setting? Cache consistency is implemented by virtually all commercial multiprocessors.

Cache consistency is incomparable to causal consistency. What does the optimal record look like for a system that ensures both cache and causal consistency? With the per process view of cache consistency it is easy to define cache+causal consistency by combining the restrictions on per process views. In causal consistency views for two different processes may diverge so that after all operations have been observed, the two processes may have different values for the same shared variable. Real world distributed systems provide some sort of conflict resolution on top of causal consistency to alleviate this problem [4], [12], [16]. This results in “eventual” consistency where the different processes are eventually in agreement on the value of the shared variables, if all updates are stopped. When this is implemented via a simple last writer wins rule, this is equivalent to all processes agreeing on the per variable ordering of write operations [7], i.e. cache consistency.

It would be interesting to experimentally evaluate how the theoretically optimum record performs on real systems, as opposed to the naive solution. We leave that investigation open to a future work.

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A Proofs for Section 5

Lemma A.1. Consider a set of views $\mathcal{V} = \{V_i\}_{i \in P}$ that explain a strongly causal consistent execution. For each process $i \in P$, let $R_i = \hat{V}_i \setminus (SCO_i(\mathcal{V}) \cup PO \cup B_i(\mathcal{V}))$. Then, for any set of views $\mathcal{V}' = \{V'_i\}_{i \in P}$ that certify a strongly causal consistent replay to be valid under $\mathcal{R} = \{R_i\}_{i \in P}$, we have that

(a) $SCO(\mathcal{V}') \supseteq SCO(\mathcal{V})$, and
(b) \( V'_i \supseteq B_i(V) \) for every process \( i \in P \).

**Proof of Lemma A.1(a):** Consider any arbitrary set of views \( V' = \{ V'_i \}_{i \in P} \) that certify a strongly causal consistent replay to be valid for \( R \). We will call a write operation \( w \) bad if there exists a write operation \( w' \) such that \( (w', w) \in SCO(V) \) but \( (w', w) \not\in SCO(V') \). Recall from Definitions 3.3 and 3.4 that \( SCO(V) \) orders only write operations and is a partial order for strongly causal consistent executions. Consider any bad write operation, WLOG executed on process 1, \( w'_1 \in \{ w, 1, *, * \} \), which is minimal with respect to \( SCO(V) \); i.e. for every write operation \( w' <_{SCO(V)} w'_2 \), we have that \( w' \) is not bad. We proceed via contradiction.

Since \( w'_1 \) is a bad write operation, so there exists a write operation \( w^1 \in \{ w, *, *, * \} \) such that \( (w^1, w'_1) \in SCO(V) \) and \( (w^1, w'_1) \not\in SCO(V') \). Consider a path \( \rho \) from \( w^1 \) to \( w'_1 \) in \( V'_1 \) (such a path must exist since \( w^1 <_{SCO(V)} w'_1 \)). Then \( \rho \) is minimal with respect to \( w'_1 \). If \( (\rho, \rho^j, \rho^j+1) \in V' \) for every \( j \in [0, k-1] \), then \( (w^1, w'_1) \in V'_1 \) and so \( (w^1, w'_1) \in SCO(V') \) by Definition 3.3, which is a contradiction. So there exists a \( j \in [0, k-1] \) such that \( (\rho, \rho^j, \rho^j+1) \not\in V'_1 \).

Consider the smallest \( j \in [0, k-1] \) such that \( (\rho^j, \rho^j+1) \not\in V'_1 \). Therefore \( w^1 \leq_{V'_1} \rho^j \). There are 4 cases to consider.

**Case 1:** \( (\rho^j, \rho^j+1) \in V'_1 \). Then \( \rho^j \) respects \( R \), so \( (w^1, w'_1) \in V'_1 \), a contradiction.

**Case 2:** \( (\rho, \rho^j+1) \in PO \). Then \( V'_1 \) respects \( PO \) due to consistency and \( PO \) is independent of executions. Thus \( (\rho, \rho^j+1) \in V'_1 \), a contradiction.

**Case 3:** \( (\rho^j, \rho^j+1) \in SCO(1)(V) \). Then both \( \rho^j \) and \( \rho^j+1 \) must be write operations. There are now two cases to consider.

**Case i:** \( j < k - 1 \). Then \( \rho^j+1 \neq w'_1 \). \( \rho^j+1 \) is not a bad write. Thus \( (\rho^j, \rho^j+1) \in SCO(V') \). Since \( V'_1 \) respects \( SCO(V') \), therefore \( (\rho^j, \rho^j+1) \in V'_1 \), a contradiction.

**Case ii:** \( j = k - 1 \). So \( \rho^j+1 = w'_1 \) and \( (\rho^j, w'_1) \in SCO(1)(V) \). From Definition 5.2 we have that \( w'_1 \) is not executed on process 1, a contradiction to the initial assumption that \( w'_1 \in \{ w, 1, *, * \} \).

**Case 4:** \( (\rho, \rho^j+1) \in B(1)(V) \). Then by Definition 5.2 \( \rho \in \{ w, 1, *, * \} \) is a write operation on process 1. Therefore \( (\rho, w'_1) \in PO \) and we get that \( w^1 \leq_{V'_1} \rho^j <_{V'_1} w'_1 \), a contradiction.

In all cases, we get the desired contradiction. \( \square \)

**Proof of Lemma A.1(b):** Consider any arbitrary set of views \( V' = \{ V'_i \}_{i \in P} \) that certify a strongly causal consistent replay to be valid for \( R \). We will call a write operation \( w \) bad if there exists a write operation \( w' \) such that \( (w', w) \in SCO(V) \) but \( (w', w) \not\in SCO(V') \). Recall that \( SCO(V) \) is a partial order for strongly causal consistent executions. Consider any bad write operation \( w^1 \in \{ w, i, *, * \} \) which is maximal with respect to \( SCO(V) \); i.e. for every write operation \( w' >_{SCO(V)} w^1 \), we have that \( w' \) is not bad. We proceed via contradiction.

Since \( w^1 \) is a bad write operation, so there exists a write operation \( w^2 \in \{ w, *, *, * \} \) such that \( (w^1, w^2) \in B_i(V) \) and \( (w^2, w^1) \in V'_i \). Therefore, \( (w^2, w^1) \in SCO(V') \). By Definition 5.2, there
exists a process, WLOG process 1 \( \neq i \), such that \((w_1^1, w_2^1) \in V_1\). If \((w_1^2, w_2^2) \in V_1'\) then \(V_1'\) does not respect \(SCO(V')\), a contradiction since \(V'\) explains a strongly causal consistent execution. Therefore \((w_1^1, w_2^2) \notin V_1'\). Consider a path \(\rho\) from \(w_1^1\) to \(w_2^2\) in \(\tilde{V}_1\) given by \(w_1^1 = o_0^1 \prec_{V_1} o_1^1 \prec_{V_1} o_{0,2} \prec_{V_1} \cdots \prec_{V_1} o_{0,k} = w_2^2\). If \((o^{i,j}, o^{i,j+1}) \in V_1'\) for every \(j \in [0, k-1]\), then \((w_1^1, w_2^2) \in V_1'\) which is a contradiction. So there exists a \(j \in [0, k-1]\) such that \((o^{i,j}, o^{i,j+1}) \notin V_1'\).

Consider the smallest \(j \in [0, k-1]\) such that \((o^{i,j}, o^{i,j+1}) \notin V_1'\). Therefore \(w_1^1 \leq V_1' o^{i,j}\). There are 4 cases to consider.

**Case 1:** \((o^{i,j}, o^{i,j+1}) \in V_1 \setminus \left(SOCO(V) \cup \left(PO \cup B_1(V)\right)\right)\). Then \((o^{i,j}, o^{i,j+1}) \in R_1\) and \(V_1'\) respects \(R_1\) since \(V'\) certifies a replay to be valid for \(\mathcal{R}\). Thus \((o^{i,j}, o^{i,j+1}) \in V_1'\), a contradiction.

**Case 2:** \((o^{i,j}, o^{i,j+1}) \in PO\). Then \(V_1'\) respects \(PO\) due to consistency and \(PO\) is independent of executions. Thus \((o^{i,j}, o^{i,j+1}) \in V_1'\), a contradiction.

**Case 3:** \((o^{i,j}, o^{i,j+1}) \in SOCO(V)\). Then \(V_1'\) respects \(SOCO(V')\) due to consistency and \(SOCO(V') \supseteq SOCO(V)\) by Lemma A.1(a). Thus \((o^{i,j}, o^{i,j+1}) \in V_1'\), a contradiction.

**Case 4:** \((o^{i,j}, o^{i,j+1}) \in B_1(V)\). By Definition 5.2 \(o^{i,j} \in (w, 1, *, *)\) is a write operation on process 1. Recall that \(w_1^1 \leq V_1' o^{i,j}\). If \(w_1^1 = o^{i,j}\), then \(i = 1\), which contradicts the initial assumption that \(i \neq 1\). Thus \(w_1^1 \neq o^{i,j}\) and \((w_1^1, o^{i,j}) \in V_1'\) so that by Definition 3.3 \((w_1^1, o^{i,j}) \in SOCO(V')\). Therefore, by the maximality of \(w_1^1\), we have that \(o^{i,j}\) is not a bad write. Thus \((o^{i,j}, o^{i,j+1}) \in V_1'\), a contradiction.

In all cases, we get the desired contradiction. So we have that \((w_1^1, w_2^2) \in V_1'\) but \((w_2^2, w_1^1) \in SOCO(V')\), which is a contradiction since \(V_1'\) respects \(SOCO(V')\).

**Proof of Theorem 5.3:** Consider any arbitrary set of views \(V' = \{V_1'\}_{i \in P}\) that certify a strongly causal consistent replay to be valid for \(\mathcal{R}\). We show that \(V' = V\). More precisely, we show that for any process \(i\) and any two operations \(o^1, o^2 \in (\ast, \ast, \ast, \ast)\) such that \((o^1, o^2) \in V_i\) we must have \((o^1, o^2) \in V_i'\). Consider any arbitrary process \(i\). We have that

- \(V_i'\) respects \(R_i\), since \(V'\) certifies a replay to be valid for \(\mathcal{R}\).
- \(V_i'\) respects \(SOCO_i(V) \cup (PO \cup (\ast, i, \ast, \ast) \cup (w, \ast, \ast, \ast)) \cup B_i(V)\) due to consistency and Lemma A.1.

Consider a \(o^1 o^2\)-path \(\rho\) in \(\tilde{V}_i\) given by \(o^1 = o_0^1 \prec_{V_i} o_{0,1} \prec_{V_i} o_{0,2} \prec_{V_i} \cdots \prec_{V_i} o_{0,k} = o^2\). By construction of \(\tilde{V}_i\), each edge is either a \(R_i\) edge or a \(PO\) edge or a \(SOCO_i(V)\) edge or a \(B_i(V)\) edge. Thus \(o^1 = o_0^1 \prec_{V_i} o_{0,1} \prec_{V_i} o_{0,2} \prec_{V_i} \cdots \prec_{V_i} o_{0,k} = o^2\) and \((o^1, o^2) \in V_i'\), as required.

**Proof of Theorem 5.4:** Assume for the sake of contradiction that there exists a good record \(\mathcal{R}\) of \(V\), a process, WLOG process 1, and two operations \(o^1, o^2 \in (\ast, 1, \ast, \ast) \cup (w, \ast, \ast, \ast)\) such that \((o^1, o^2) \in \tilde{V}_i \setminus (PO \cup SOC0_i(V) \cup B_i(V))\) and \((o^1, o^2) \notin R_i\). Then, we construct a set of views \(V'\), that differs from \(V\), but certifies a strongly causal replay to be valid for \(\mathcal{R}\), i.e. \(V'\) explains a strongly causal execution and extends the record \(\mathcal{R}\). This violates the definition of a good record (see Section 3). We construct \(V'\) from \(V\) as follows. Let \(V'_1 := (V_1 \setminus \{(o^1, o^2)\}) \cup \{(o^2, o^1)\}\). For each \(i > 1\), set \(V'_i = V_i\). There are two things to be shown:

1) each \(V'_i\) is a total order (so that it is indeed a view), and
2) \( \mathcal{V}' \) certifies a strongly causal replay to be valid for \( \mathcal{R} \), i.e., satisfies properties for both strong causal consistency and replay.

We first show that for each \( i \in P \), \( V'_i \) is a total order. Since \( V'_i = V_i \) for \( i > 1 \), we focus on \( V'_1 \). Suppose \( V'_1 \) is not a total order. \( V'_1 \) orders all operations in \((w, *, *, *) \cup (w, *, *, *)\) by construction. So we must have introduced a cycle in \( V'_1 \). This implies that there is a \( o^1, o^2 \)-path in \( V'_1 \). Let this \( o^1, o^2 \)-path \( \rho \) be given by \( o^1 = o_0 \prec V_i o^1 \prec V_i \cdots \prec V_i o^{i \prec k} = o^2 \). Note that since \( V_1 \) preserves all paths in \( V_i \), so there must be a \( o^1 \prec o^1 \prec j+1 \)-path \( P_j \) in \( V_1 \) for every \( j \in [0, k-1] \). Note also that these paths do not include the edge \((1, o^2)\) because \( V_1 \) is acyclic. So there is an \( o^1, o^2 \)-path in \( V_1 \) that does not use the edge \((o^1, o^2)\) given by \( \bigcup_{j=0}^{k-1} P_j \). Hence, the edge \((o^1, o^2)\) can be removed from \( V_1 \) while preserving all paths in \( V_i \). This contradicts the fact that \( V_1 \) is the (unique) transitive reduction of \( V_1 \).

We now show that \( \mathcal{V}' \) certifies a strongly causal replay to be valid for \( \mathcal{R} \). More precisely, we show that, for each process \( i \),

1) \( V'_i \) respects \( R_i \), and
2) \( V'_i \) respects \( SCO(V') \cup \{(w, *, *, *) \cup (w, *, *, *)\} \).

Observe that for each \( i > 1 \), \( V'_i = V_i \) and so \( V'_i \) respects \( R_i \subseteq V_i \) and \( PO | \{(w, *, *, *) \cup (w, *, *, *)\} \). For \( i = 1 \), recall that \((o^1, o^2) \notin R_1 \) and \((o^1, o^2) \notin PO \), both of which are independent of \( \mathcal{V}' \). Therefore \( V'_1 \) respect \( R_1 \) and \( PS | \{(w, *, *, *) \cup (w, *, *, *)\} \) as well. So it is left to show that each \( V'_i \) respects \( SCO(V') \). There are 4 cases to consider.

Case 1: Either \( o^1 \in (r, 1, *, *) \) or \( o^2 \in (r, 1, *, *) \). Since strong causal order only orders write operations (Definition 3.3) so \( SCO(V') = SCO(V) \). Therefore, for each \( i \in P \), \( V'_i \) respects \( SCO(V') \).

Case 2: \( o^2 \in (w, 1, *, *) \) and \( o^1 \notin (w, 1, *, *) \). Then \( (o^1, o^2) \notin SCO(V) \) and therefore \( SCO(V') = SCO(V) \backslash \{(o^1, o^2)\} \). Since \( SCO(V) \supset SCO(V') \), therefore, for every \( i > 1 \), \( V'_i \) respects \( SCO(V') \). \( V'_1 \) respects \( SCO(V) \backslash \{(o^1, o^2)\} \) by construction.

Case 3: \( o^1 \in (w, 1, *, *) \) and \( o^2 \notin (w, 1, *, *) \). Then \( SCO(V') = SCO(V) \cup \{(o^1, o^2)\} \) by Definition 3.3 WLOG \( o^2 \in (w, 2, *, *) \). Since \((o^1, o^2) \notin SCO(V) \), therefore \((o^2, o^1) \notin V_2 \). We have that \( V'_2 = V_2 \) respects \( SCO(V) \cup \{(o^2, o^1)\} \). Since \((o^1, o^2) \notin B_1(V) \), therefore for all \( i > 2 \), \((o^2, o^1) \notin V_i \) and so \( V'_i = V_i \) respects \( SCO(V) \cup \{(o^2, o^1)\} = SCO(V') \). Now \( V'_1 \) respects \( SCO(V) \cup \{(o^2, o^1)\} \) by construction.

Case 4: \( o^1 \), \( o^2 \) are writes and \( o^1 \notin (w, 1, *, *) \). Then \( SCO(V) = SCO(V') \) and for each \( i \in P \), \( V'_i \) respects \( SCO(V') \) since \((o^1, o^2) \notin SCO(V) \).

So we have shown that \( \mathcal{V}' \) certifies a strongly causal replay to be valid for \( \mathcal{R} \). Since \((o^1, o^2) \in V_1 \) and \((o^2, o^1) \in V'_1 \), thus \( V'_1 \neq V_1 \). This contradicts the initial assumption that \( \mathcal{R} \) is a good record. □

**Proof of Theorem 5.3:** By Theorem 5.3 it follows that \( \mathcal{R} \) is a good record of \( \mathcal{V} \), so we show that \( \mathcal{R} \) can be recorded online. Fix a process \( i \) and consider an arbitrary time step in the execution when process \( i \) observes an operation say \( o^2 \). Let \( o^1 \in (*, *, *, *) \) be the last operation in \( V_i \). Process \( i \) can check if \((o^1, o^2) \in PO \) and also if \((o^1, o^2) \in SCO(V) \). To check if \((o^1, o^2) \in SCO(V) \), process \( i \) follows the following procedure. If \( o^2 \) was executed by process \( i \),
then the edge cannot be in \( SCO_1(V) \). If \( o^2 \) was not executed by process \( i \), then \((o^1, o^2) \in SCO_1(V)\) if and only if \((o^1, o^2) \notin SCO(V)\). Process \( i \) records \((o^1, o^2)\) if \((o^1, o^2) \notin SCO(V) \cup PO\).

Observe that \((o^1, o^2) \in \hat{V}_i\) if and only if, when \( o^2 \) is observed by process \( i \), \( o^1 \) is the last operation in \( V_i \). Therefore, the above procedure records \( \hat{V}_i \setminus (SCO(V) \cup PO) \) at process \( i \).

\[ \square \]

**Proof of Theorem 5.6:** By Theorem 5.4, it follows that for any process \( i \), \( \hat{V}_i \setminus (SCO(V) \cup PO \cup B_i(V)) \) is necessary to record even in the offline setting. We show that an arbitrary process, WLOG process 1, can not detect an edge in \( \hat{V}_1 \setminus PO \cup SCO(V) \) is also in \( B_1(V) \) in an online setting. Recall from Definition 5.2 that \( B_1(V) \) orders only write operations. Suppose, at a given time step in the execution, that process 1 observes \( w_2 \in (w, 2, *, *), \) and \( w_1 \in (w, 1, *, *) \) is the last operation in \( V_1 \) so that \((w_1, w_2) \in B_1(V) \cap \hat{V}_1 \). Assume further that \((w_1, w_2) \notin PO \cup SCO(V) \).

Let \( \{\hat{V}_i\}_{i \geq 2} \) be the (parts of) views of processes \( i \geq 2 \) that process 1 is aware of. Observe that for each \( i \geq 2 \), the last operation in \( \hat{V}_i \) was executed by process \( i \). Let this last operation be \( w_i \). Note that each \( w_i \) has already been observed by process 1. For \( i > 2 \), \((w_2, w_i) \notin \hat{V}_i \), since otherwise \((w_2, w_i) \in SCO(V)\), which contradicts the fact that \( w_2 \) is the last operation observed by process 1. Similarly, for \( i > 2 \), \((w_1, w_i) \notin \hat{V}_i \). Therefore, as far as process 1 is aware, no process \( i > 2 \) has observed either \( w_1 \) or \( w_2 \). Thus, for each \( i > 2 \), both \( \hat{V}_1 \cup \{(w_1, w_2)\} \) and \( \hat{V}_1 \cup \{(w_2, w_1)\} \) are valid for future observation by process \( i \). So process 1 cannot decide whether \((w_1, w_2) \in B_1(V) \) or not (see Definition 5.2).

\[ \square \]

**B Some Observations for Section 6**

**Observation B.1.** Consider a set of views \( V = \{V_i\}_{i \in P} \) that explain a strongly causal consistent execution, an arbitrary process \( i \), and two operations \( o^1 \in (*, *, *, *) \) and \( o^2 \in (w, *, *, *) \) such that \( C_i(V, o^1, o^2) \) is non-empty. Let \( w_i^{\text{min}} \in (w, i, *, *) \) be the minimal (with respect to \( PO \)) write on process \( i \) such that \( o^1 \leq A_i(V) w_i^{\text{min}} \). Then \( w_i^{\text{min}} \) exists and

1. \( C_i^k(V, o^1, o^2) = C_i^k(V, w_i^{\text{min}}, o^2) \) for any \( k \), and
2. for any two write operations \( o^3 \in (w, *, *, *) \) and \( o^4 \in (w, *, *, *) \), if \((o^3, o^4) \in C_i^1(V, o^1, o^2)\), then \((o^3, w_i^{\text{min}}) \in C_i^1(V, o^1, o^2)\).

**Proof:** The existence of \( w_i^{\text{min}} \) follows from the assumption that \( C_i(V, o^1, o^2) \), and so \( C_i^1(V, o^1, o^2) \), is non-empty. Therefore, by Definition 6.3, there exists at least one write \( w_i \in (w, i, *, *) \) on process \( i \) such that \( o^1 \leq A_i(V) w_i \).

We proceed via induction on \( k \). The inductive step for \( k > 1 \) follows from Definition 6.4 by applying the inductive hypothesis \( C_i^{k-1}(V, o^1, o^2) = C_i^{k-1}(V, w_i^{\text{min}}, o^2) \). For the base case, we show the equality for \( k = 1 \).

- \( C_i^1(V, o^1, o^2) \subseteq C_i^1(V, w_i^{\text{min}}, o^2) \). Consider any two operations \( o^3 \in (w, *, *, *) \) and \( o^4 \in (w, *, *, *) \) such that \((o^3, o^4) \in C_i^1(V, o^1, o^2)\). Then, \( w_i^{\text{min}} \leq PO w_i^{\text{min}} \), by the minimality of \( w_i^{\text{min}} \), and \( w_i^{\text{min}} \leq PO w_i^{\text{min}} \), by Definition 6.4.

Therefore, by Definition 6.3, \((o^3, w_i^{\text{min}}) \in C_i^1(V, w_i^{\text{min}}, o^2)\).
Consider any two operations \( w^3 \in (w,*,* ,*) \) and \( w^4 \in (w,i,* ,*) \) such that \( (w^3, w^4) \in C_i^1(\mathcal{V}, w_i^{\text{min}}, w_2) \). Then,

\[
\begin{align*}
- w^3 & \leq_{A_i(\mathcal{V})} w^2, \text{ by Definition } 6.4 \\
- o^1 & \leq_{A_i(\mathcal{V})} w_i^{\text{min}}, \text{ by the definition of } w_i^{\text{min}}, \text{ and} \\
- w_i^{\text{min}} & \leq_{PO} w^4, \text{ by the minimality of } w_i^{\text{min}}.
\end{align*}
\]

Therefore, \( o^1 \leq_{A_i(\mathcal{V})} w^4 \), by Definition 6.2 and so \((w^3, w^4) \in C_i^1(\mathcal{V}, o^1, w_2) \) by Definition 6.4.

2. Consider any two operations \( w^3 \in (w,*,* ,*) \) and \( w^4 \in (w,i,* ,*) \) such that \((w^3, w^4) \in C_i^1(\mathcal{V}, o^1, w_2) \). Then,

\[
\begin{align*}
- o^1 & \leq_{A_i(\mathcal{V})} w_i^{\text{min}}, \text{ by the definition of } w_i^{\text{min}}, \text{ and} \\
- w^3 & \leq_{A_i(\mathcal{V})} w^2, \text{ by Definition } 6.4.
\end{align*}
\]

Therefore, by Definition 6.4 \((w^3, w_i^{\text{min}}) \in C_i^1(\mathcal{V}, o^1, w_2) \).

\(\square\)

Observation B.2. Consider a set of views \( \mathcal{V} = \{V_i\}_{i \in P} \) that explain a strongly causal consistent execution, an arbitrary process \( i \), and two operations \( o^1 \in (\ast, \ast, \ast, \ast) \) and \( w^2 \in (w, \ast, \ast, \ast) \). We have that if \( C_i^1(\mathcal{V}, o^1, w_2) \subseteq SWO(\mathcal{V}) \), then

1. \( C_i(\mathcal{V}, o^1, w^2) \subseteq SWO(\mathcal{V}) \), and

2. \((o^1, w^2) \not\in B_i(\mathcal{V}) \).

Proof:

1. By induction on \( k \), we show that for every positive integer \( k \), \( C_i^k(\mathcal{V}, o^1, w^2) \subseteq SWO(\mathcal{V}) \). The base case follows by assumption. For the inductive step, for \( k > 1 \), consider any two operations \( w^3 \in (w,*,* ,*) \) and \( w^4 \in (w,i,* ,*) \) such that \((w^3, w^4) \in C_i^{k-1}(\mathcal{V}, o^1, w_2) \). Then, by Definition 6.4, there exist two write operations \( w^5, w^6 \in (w,*,* ,*) \), such that

\[
\begin{align*}
(a) & \ (w^5, w^6) \in C_i^{k-1}(\mathcal{V}, o^1, w^2), \\
(b) & \ w^3 \leq_{A_i(\mathcal{V}) \cup C_i^{k-1}(\mathcal{V}, o^1, w_2)} w^5, \text{ and} \\
(c) & \ w^6 \leq_{A_i(\mathcal{V})} w^4.
\end{align*}
\]

By the inductive hypothesis, we have that \( C_i^{k-1}(\mathcal{V}, o^1, w_2) \subseteq SWO(\mathcal{V}) \). Therefore \( w^3 \leq_{A_i(\mathcal{V}) \cup SWO(\mathcal{V})} w^5 \), which implies \((w^3, w^4) \in SWO(\mathcal{V}) \) by Observation 6.3.

2. Since \( C_i(\mathcal{V}, o^1, w^2) \subseteq SWO(\mathcal{V}) \) and, for each process \( m \in P, A_m(\mathcal{V}) \supseteq SWO(\mathcal{V}) \), thus

\[
\begin{align*}
(a) & \ \text{if } m \neq i, \text{ then } A_m(\mathcal{V}) \cup C_i(\mathcal{V}, o^1, w^2) = A_m(\mathcal{V}) \text{ which is acyclic, and} \\
(b) & \ \text{if } m = i, \text{ then } (A_m(\mathcal{V}) \setminus \{(o^1, w^2)\}) \cup C_i(\mathcal{V}, o^1, w^2) \subseteq A_m(\mathcal{V}) \text{ which is acyclic.}
\end{align*}
\]

Therefore \((o^1, w^2) \not\in B_i(\mathcal{V}) \) by Definition 6.5.

\(\square\)

Observation B.3. Consider a set of views \( \mathcal{V} = \{V_i\}_{i \in P} \) that explain a strongly causal consistent execution, an arbitrary process \( i \), and two write operations \( w^1, w^2, w^3 \in (w,*,* ,*) \), and \( w^4 \in (w,i',* ,*) \). We have that if \((w^3, w^4) \in C_i^k(\mathcal{V}, w^1, w^2) \) for some \( k \geq 0 \), then \( w^1 \leq_{SWO(\mathcal{V})} w^4 \).
Proof: We proceed by induction on $k$. Base case, for $k = 1$, we have $w^1 \leq_{A_i(V)} w^4_i$ and $i' = i$, by Definition 6.3. Therefore $w^1 \leq_{SWO(V)} w^4_i$. For the inductive step, for $k > 1$, by Definition 6.3 there exist $(w^5, w^6) \in C_i(k-1)(V, w^1, w^2)$ such that $w^6 \leq_{A_i'(V)} w^4_i$. By the inductive hypothesis, we have that $w^1 \leq_{SWO(V)} w^6$. Therefore $w^1 \leq_{SWO(V)} w^6 \leq_{A_i'(V)} w^4_i$, and so $w^1 \leq_{SWO(V)} w^4_i$. □

C Proofs for Section 6

Lemma C.1. Consider a set of views $V = \{V_i\}_{i \in P}$ that explain a strongly causal consistent execution. For each process $i \in P$, let $R_i = A_i(V) \setminus (SWO_i(V) \cup PO \cup B_i(V))$. Then, for any set of views $V'$ that certify a strongly causal consistent replay to be valid under $R = \{R_i\}_{i \in P}$, we have that

(a) $SWO(V') \supseteq SWO(V)$, and

(b) $V'_i \supseteq B_i(V)$ for every process $i \in P$.

Proof of Lemma [C.1(a)]: Consider any arbitrary set of views $V' = \{V'_i\}_{i \in P}$ that certify a strongly causal consistent replay to be valid for $R$. We will call a write operation $w$ bad if there exists a write operation $w'$ such that $(w', w) \in SWO(V)$ but $(w', w) \not\in SWO(V')$. Recall from Definition 6.1 that $SWO(V)$ orders only write operations and is a partial order for strongly causal consistent executions. Consider any bad write operation, WLOG executed on process 1, $w^1_2 \in (w, 1, *, *)$, which is minimal with respect to $SWO(V)$; i.e. for every write operation $w' <_{SWO(V)} w^1_2$, we have that $w'$ is not bad. We proceed via contradiction.

Since $w^1_2$ is a bad write operation, so there exists a write operation $w^1$ such that $(w^1, w^1_2) \in SWO(V)$ and $(w^1, w^1_2) \not\in SWO(V')$. Consider a path $\rho$ from $w^1$ to $w^1_2$ in $A_1(V)$ (such a path must exist since $w^1 <_{SWO(V)} w^1_2 \Rightarrow w^1 <_{A_1(V)} w^1_2$ by Observation 6.3) given by $w^1 = o_{\rho,0} <_{A_1(V)} o_{\rho,2} <_{A_1(V)} \cdots <_{A_1(V)} o_{\rho,k} = w^1_2$. Note that each operation in the path is in the view $V_i$ and hence in $V'_i$. If $(o_{\rho,j}, o_{\rho,j+1}) \in A_1(V')$ for every $j \in [0, k-1]$, then $(w^1, w^1_2) \in A_1(V')$ and so $(w^1, w^1_2) \in SWO(V')$ by Observation 6.3 which is a contradiction. So there exists a $j \in [0, k-1]$ such that $(o_{\rho,j}, o_{\rho,j+1}) \not\in A_1(V')$.

Consider the smallest $j \in [0, k-1]$ such that $(o_{\rho,j}, o_{\rho,j+1}) \not\in A_1(V')$. Therefore $w^1 \leq_{A_1(V')} o_{\rho,j}$. There are 4 cases to consider.

Case 1: $(o_{\rho,j}, o_{\rho,j+1}) \in A_1(V') \setminus (SWO(V) \cup PO \cup B_1(V))$. Then $(o_{\rho,j}, o_{\rho,j+1}) \in R_1$ and $V'_i$ respects $R_1$ since $V'$ is a replay of $R$. Thus $(o_{\rho,j}, o_{\rho,j+1}) \in DRO(V'_i)$ and so $(o_{\rho,j}, o_{\rho,j+1}) \in A_1(V')$, a contradiction.

Case 2: $(o_{\rho,j}, o_{\rho,j+1}) \in PO$. Then $V'_i$ respects PO due to consistency and PO is independent of executions. Thus $(o_{\rho,j}, o_{\rho,j+1}) \in A_1(V')$, a contradiction.

Case 3: $(o_{\rho,j}, o_{\rho,j+1}) \in SWO_1(V)$. Then both $o_{\rho,j}$ and $o_{\rho,j+1}$ must be write operations. There are now two cases to consider.

Case i: $j < k - 1$. Then $o_{\rho,j+1} \not\in w^1_2$. Observe that $(o_{\rho,j+1}, w^1_2) \in A_1(V)$ and so $(o_{\rho,j+1}, w^1_2) \in SWO(V)$. Therefore, by the minimality of $w^1_2$, we have that $o_{\rho,j+1}$ is not a bad write. Thus $(o_{\rho,j}, o_{\rho,j+1}) \in SWO(V')$ and so $(o_{\rho,j}, o_{\rho,j+1}) \in A_1(V')$, a contradiction.

Case ii: $j = k - 1$. So $o_{\rho,j+1} = w^1_2$ and $(o_{\rho,j}, w^1_2) \in SWO_1(V)$. From Definition 6.3 we have that $w^1_2$ is not executed on process 1, a contradiction to the initial assumption that $w^1_2 \in (*, 1, *, *)$. 27
In all cases, we get the desired contradiction.

\[ \square \]
**Proof of Lemma C.1(b):** Consider any arbitrary set of views \( V' = \{ V'_i \}_{i \in P} \) that certify a strongly causal consistent replay to be valid for \( \mathcal{R} \). We will call a pair of write operations \((w^1_i, w^2_i)\), \(w^1_i \in (w, i, *, *)\) and \(w^2_i \in (w, *, *, *), \) bad if

1. \((w^2_i, w^1_i) \in A_i(V')\), and 
2. there exists a process \( m \in P \) such that either 
   a. \( m \neq i \) and \( A_m(V) \cup C_i(V, w^1_i, w^2_i) \) has a cycle, or 
   b. \( m = i \) and \( (A_m(V) \setminus \{(w^1_i, w^2_i)\}) \cup C_i(V, w^1_i, w^2_i) \) has a cycle.

First note that if there is no bad write pair, then this implies the result as follows. We show the contrapositive that if \( V'_i \not\supseteq B_i(V) \), then there exists a bad write pair. Suppose there exist two distinct operations \( o \in (\ast, \ast, \ast, \ast) \) and \( w' \in (w, \ast, \ast, \ast) \) such that \((o, w') \in B_i(V)\) but \((w'_i, o) \in V'_i\), for some process \( i \in P \). Recall from Definition 6.5 that \( B_i(V') \subseteq DRO(V'_i)\). Therefore \((w'_i, o) \in DRO(V'_i)\). Then, from Observation 13.1 there exists a write operation \( w^\min_i \in (w, i, \ast, \ast)\) such that \( C_i(V, o, w') = C_i(V, w^\min_i, w') \) and \( o \leq A_i(V) w^\min_i \). We show that \((w^\min_i, w')\) is a bad write pair. Condition 2 follows from the fact that \( C_i(V, o, w') = C_i(V, w^\min_i, w') \) and \((o, w') \in B_i(V)\) by interchanging \( C_i(V, o, w')\) with \( C_i(V, w^\min_i, w')\) in Definition 6.5. So it is left to show that \((w'_i, w^\min_i) \in A_i(V')\). If \( o \) is a read operation then \( o \in (r, i, \ast, \ast) \) and so \( o \leq_P w^\min_i \). If \( o \) is a write operation then \( o \leq_{SWO(V)} w^\min_i \) (recall that \( o \leq_{A_i(V)} w^\min_i \)) and, by Lemma C.1(a), \( o \leq_{SWO(V')} w^\min_i \). Therefore \( w'_i <_{DRO(V')} o \leq_{A_i(V')} w^\min_i \) and so \((w'_i, w^\min_i) \in A_i(V')\) by Definition 6.2.

We now proceed via contradiction to show that there are no bad write pairs. Suppose that there exists at least one bad write pair. Recall from Definition 6.1 that \( SWO(V) \) orders only write operations and is a partial order for strongly causal consistent executions. Consider any bad write pair \((w^1_i, w^2_i)\) such that \( w^1_i \in (w, i, \ast, \ast)\) is maximal with respect to \( SWO(V)\); i.e. for every write operation \( w' >_{SWO(V)} w^1_i \), we have that there is no \( w'' \) such that \((w', w'')\) is a bad write pair. We also assume that \( w^2_i \) is minimal with respect to \( V_i\); i.e. for every write operation \( w' <_{V_i} w^2_i \) we have that \((w^1_i, w')\) is not a bad write pair. Since \((w^1_i, w^2_i) \in B_i(V)\), by Definition 6.3 there exists a process, WLOG process 1, such that either

1. \( i \neq 1 \) and \( A_1(V) \cup C_i(V, w^1_i, w^2_i) \) has a cycle, or
2. \( i = 1 \) and \( (A_1(V) \setminus \{(w^1_i, w^2_i)\}) \cup C_i(V, w^1_i, w^2_i) \) has a cycle.

We will show that

1. \( C_i(V, w^1_i, w^2_i) \subseteq SWO(V')\), and
2. \( A_1(V') \cup C_i(V, w_1^i, w_2^i) \) has a cycle.

This implies that \( V_1' \supseteq A_1(V') \) does not respect \( C_i(V, w_1^i, w_2^i) \subseteq SWO(V') \), hence giving us the desired contradiction.

**Claim C.2.** \( C_i(V, w_1^i, w_2^i) \subseteq SWO(V') \).

**Proof:** Suppose, for the sake of contradiction, that \( C_i(V, w_1^i, w_2^i) \not\subseteq SWO(V') \). Consider the smallest \( \ell \geq 1 \) such that \( C_i^{\ell}(V, w_1^i, w_2^i) \not\subseteq SWO(V') \). Consider any two writes \( w^3 \in (w, *, *, *) \) and \( w^4 \in (w, i', *, *) \) such that \((w^3, w^4) \in C_i^{\ell}(V, w_1^i, w_2^i) \) but \((w_3, w_4^i) \not\subseteq SWO(V') \). Using \( C_i^{\ell}(V, w_1^i, w_2^i) = \{ (w^2, w_1^i) \} \), by Definition 6.4, we have that there exists a \( w^3 w^4 \)-path \( \psi \) in \( A_1(V') \cup C_i^{\ell-1}(V, w_1^i, w_2^i) \) given by \( w^3 = w^{\psi,0} \leq_A V (V') w^{\psi,1} <_{C_i^{\ell-1}(V, w_1^i, w_2^i)} w^{\psi,2} \leq_A V (V') w^{\psi,3} \cdots <_{C_i^{\ell-1}(V, w_1^i, w_2^i)} w^{\psi,k-1} \leq_A V (V') w^{\psi,k} = w_1^i \). By the choice of \( \ell \), we have that \( C_i^{\ell-1}(V, w_1^i, w_2^i) \subseteq SWO(V') \). WLOG, we can assume that \( w^{\psi,j} \leq_{SWO(V')} w^4 \) for every \( j \in [1, k-1] \), since otherwise we can consider \((w^{\psi,j}, w^4) \in C_i^{\ell}(V, w_1^i, w_2^i) \) instead of \((w^3, w^4) \). Therefore it is sufficient to show that \( w^3 = w^{\psi,0} \leq_A V (V') w^{\psi,1} \), since this implies a \( w^3 w^4 \)-path in \( A_1(V') \cup SWO(V') \) which is a contradiction since \((w^3, w_1^i) \not\subseteq SWO(V') \).

If \( w^3 = w^{\psi,1} \), then we are done. Suppose \( w^3 <_{A_1(V')} w^{\psi,1} \). Consider a path \( \rho \) from \( w^3 \) to \( w^{\psi,1} \) in \( \hat{A}_1(V') \) given by \( w^3 = o^{\rho,0} <_{A_1(V')} o^{\rho,1} <_{A_1(V')} o^{\rho,2} <_{A_1(V')} \cdots <_{A_1(V')} o^{\rho,k'} = w^{\psi,1} \). Note that each operation in the path is in the view \( V_{i'} \) and hence in \( V_{i'}' \). If \((o^{\rho,j'}, o^{\rho,j'+1}) \in A_1(V') \) for every \( j' \in [0, k'-1] \), then \((w^3, w^{\psi,1}) \subseteq SWO(V') \) which is a contradiction. So there exists a \( j' \in [0, k'-1] \) such that \((o^{\rho,j'}, o^{\rho,j'+1}) \not\subseteq A_1(V') \).

Consider any \( j' \in [0, k'-1] \) such that \((o^{\rho,j'}, o^{\rho,j'+1}) \not\subseteq A_1(V') \). There are 4 cases to consider.

**Case 1:** \((o^{\rho,j'}, o^{\rho,j'+1}) \in \hat{A}_1(V') \setminus (SWO(V') \cup PO \cup B_2'(V')) \). Then \((o^{\rho,j'}, o^{\rho,j'+1}) \in R_{i'} \) and \( V_{i'} \) respects \( R_{i'} \) since \( V' \) is a replay of \( R \). Thus \((o^{\rho,j'}, o^{\rho,j'+1}) \in DRO(V_{i'}') \) and so \((o^{\rho,j'}, o^{\rho,j'+1}) \in A_1(V') \), a contradiction.

**Case 2:** \((o^{\rho,j'}, o^{\rho,j'+1}) \in PO \). Then \( A_1(V') \) respects PO due to consistency and PO is independent of executions. Thus \((o^{\rho,j'}, o^{\rho,j'+1}) \in A_1(V') \), a contradiction.

**Case 3:** \((o^{\rho,j'}, o^{\rho,j'+1}) \in SWO_i(V) \). Then \( A_1(V') \subseteq SWO(V') \) by Definition 6.2 and \( SWO(V') \supseteq SWO(V) \) by Lemma C.1(a). Thus \((o^{\rho,j'}, o^{\rho,j'+1}) \in A_1(V') \), a contradiction.

**Case 4:** \((o^{\rho,j'}, o^{\rho,j'+1}) \in B_2'(V') \). Then by Definition 6.5, we have that \((o^{\rho,j'}, o^{\rho,j'+1}) \in DRO(V_{i'}) \) and that \( C_{i'}(V, o^{\rho,j'}, o^{\rho,j'+1}) \) is non-empty. Thus, from Observation B.1, there exists \( w_{i'}^{\psi,\min} \in (w, i', *, *) \) such that \( C_{i'}(V, o^{\rho,j'}, o^{\rho,j'+1}) = C_{i'}(V, w_{i'}^{\psi,\min}, o^{\rho,j'+1}) \). Since \( w^4_i \in (w, i', *, *) \), therefore either \( w_{i'}^{\psi,\min} \leq_{PO} w_i^4 \) or \( w_{i'}^{\psi,\min} <_{PO} w_i^4 \). We consider both cases.

**Case i:** \( w_{i'}^{\psi,\min} \leq_{PO} w_i^4 \). Then \( w^3 \leq_A V (V') o^{\rho,j'} \leq_{A_1(V')} w_{i'}^{\psi,\min} \leq_{PO} w_i^4 \). This implies that \((w^3, w_i^4) \in SWO(V) \), and therefore \((w^3, w_i^4) \in SWO(V') \) by Lemma C.1(a), a contradiction to the initial assumption that \((w^3, w_i^4) \not\subseteq SWO(V') \).

**Case ii:** \( w_i^4 <_{PO} w_{i'}^{\psi,\min} \). Now \( o^{\rho,j'} \leq_{A_1(V')} w_{i'}^{\psi,\min} \), by Observation B.1, \( o^{\rho,j'} \) is either a read or a write operation. If \( o^{\rho,j'} \) is a read operation, then \( o^{\rho,j'} \in (r, i', *, *) \) and so \((o^{\rho,j'}, w_{i'}^{\psi,\min}) \in PO \). If \( o^{\rho,j'} \) is a write operation, then \((o^{\rho,j'}, w_{i'}^{\psi,\min}) \in PO \). Therefore, in either case \( o^{\rho,j'} \leq_{SWO(V')} w_{i'}^{\psi,\min} \) and by Lemma C.1(a) \( o^{\rho,j'} \leq_{SWO(V')} w_{i'}^{\psi,\min} \). Therefore, in either case \( o^{\rho,j'} \leq_{A_1(V')} w_{i'}^{\psi,\min} \). Now since \((o^{\rho,j'}, o^{\rho,j'+1}) \in DRO(V') \) and \((o^{\rho,j'}, o^{\rho,j'+1}) \not\subseteq SWO(V') \).
Claim C.3. \( A_1(V') \cup C_i(V, w^1_i, w^2) \) has a cycle.

**Proof:** Since \((w^1_i, w^2)\) is a bad write pair, therefore either

1. \( i \neq 1 \) and \( A_1(V) \cup C_i(V, w^1_i, w^2) \) has a cycle, or
2. \( i = 1 \) and \( (A_1(V) \setminus \{(w^1_i, w^2)\}) \cup C_i(V, w^1_i, w^2) \) has a cycle.

Consider one such cycle \( \psi \) given by \( w^{\psi,0} \leq A_1(V) w^{\psi,1} < C_i(V, w^1_i, w^2) w^{\psi,2} \leq A_1(V) \cdots \leq A_1(V) w^{\psi,k-1} < C_i(V, w^1_i, w^2) w^{\psi,k} = w^{\psi,0} \). If \( i \neq 1 \), then we let \( \psi \) be any cycle. However, if \( i = 1 \), then we select \( \psi \) to be a cycle with some particular properties. If there exists a cycle \( \psi \) such that there is no even \( j \) with \( w^{\psi,j} = w^1_i \), then we select that cycle. Otherwise, we select \( \psi \) as follows. Since we can rotate cycles, we assume WLOG that
\[ w^{\psi,k} = w^{\psi,0} = w^1. \] We say that \( \psi \) has level \( \ell \) if \( \ell \) is the smallest integer such that \((w^{\psi,k-1}, w^1) \in C^j_i(V, w^1, w^2)\). We select \( \psi \) such that it has the lowest level \( \ell \). The reason behind this choice will become clearer in case 1 below.

**Case 1:** \( i = 1 \) and there exists an even \( j \in [0, k - 1] \) such that \( w^{\psi,j} = w^1 \). WLOG we can assume that \( w^{\psi,k} = w^{\psi,0} = w^1 \) since we can rotate the cycle \( \psi \). We first show that the choice of \( \psi \) implies that \((w^{\psi,k-1}, w^1) \in C^j_i(V, w^1, w^2)\) and \((w^{\psi,j}, w^1) \in C^j_i(V, w^1, w^2)\). Suppose for the sake of contradiction that the level of \( \psi \) is \( \ell > 1 \) so that \( \ell \) is the smallest integer such that \((w^{\psi,k-1}, w^1) \in C^j_i(V, w^1, w^2)\). By Definition 6.4 there exists a \( w^{\psi,k-1} \)-path \( \rho \) in \( A_1(V) \cup C^j_i(V, w^1, w^2) \). Then either \( \psi \cup \rho \) is a cycle or \( \rho \) intersects with \( \psi \) other than at endpoints. In the first case we have found a cycle with level smaller than \( \psi \) and in the second case \( \psi \cup \rho \) has a cycle that does not use \( w^1 \). In either case we have a contradiction with the choice of \( \psi \).

We now show that there exists a path from \( w^1 \) to \( w^2 \) in \((A_1(V) \setminus \{(w^1, w^2)\})\). Since \((w^{\psi,k-1}, w^1) \in C^j_i(V, w^1, w^2)\) we have that \( w^{\psi,k-1} \leq A_1(V) w^2 \) by Definition 6.4. If \( k > 2 \), then \((w^{\psi,k-3}, w^{\psi,k-2}) \in C^j_i(V, w^1, w^2)\) and \( w^1 \leq SWO(V, w^{\psi,k-2}) \) by Observation 6.3. If \( k = 2 \), then \( w^1 = w^{\psi,k-2} \). In either case we get that \( w^1 \leq SWO(V) w^{\psi,k-2} \leq A_1(V) w^2 \).

Note that \( w^1 \neq w^2 \). There are 3 cases to consider

**Case i:** \( w^{\psi,k-2} = w^{\psi,k-1} = w^2 \). Then \((w^1, w^2) \in SWO(V) \) and by Lemma C.1(a) \((w^1, w^2) \in SWO(V') \). This contradicts with the assumption that \((w^1, w^2) \) is a bad write pair (which implies \((w^2, w^1) \in A_1(V') \)).

**Case ii:** \( w^{\psi,k-2} = w^{\psi,k-1} \neq w^2 \). Since \((w^{\psi,k-1}, w^1) \in C^j_i(V, w^1, w^2)\), by Definition 6.4 we have that \( w^1 \neq w^{\psi,k-1} \). Therefore \( w^1 <swop(V) w^{\psi,k-2} = w^{\psi,k-1} < A_1(V) w^2 \) is a \( w^1w^2 \)-path in \((A_1(V) \setminus \{(w^1, w^2)\})\). Hence there is a \( w^1w^2 \)-path in \((A_1(V) \setminus \{(w^1, w^2)\})\).

**Case iii:** \( w^{\psi,k-2} \neq w^{\psi,k-1} \). Since, by construction of \( \psi \), there is a \( w^{\psi,k-2}w^{\psi,k-1} \)-path in \((A_1(V) \setminus \{(w^1, w^2)\})\), therefore there is a \( w^1w^2 \)-path in \((A_1(V) \setminus \{(w^1, w^2)\})\). Therefore, there exists a path from \( w^1 \) to \( w^2 \) in \((A_1(V) \setminus \{(w^1, w^2)\})\). We now show that \((w^1, w^2) \in A_1(V') \) which contradicts with the assumption that \((w^2, w^1) \) is a bad write pair (which implies \((w^2, w^1) \in A_1(V') \)). Since \( A_1(V) \) preserves all paths, we can consider the corresponding \( w^1w^2 \)-path \( \rho \) in \( A_1(V) \) given by \( w^1 = \rho^{o,0} \leq A_1(V) \rho^{o,1} \leq A_1(V) \rho^{o,2} \leq A_1(V) \cdots \leq A_1(V) \rho^{o,k'}. \) Observe that \( \rho \) does not use the \((w^1, w^2) \) edge (property of transitive reduction). If \((\rho^{o,j'}, \rho^{o,j'+1}) \in A_1(V') \) for every \( j' \in [0, k' - 1] \), then \((w^1, w^2) \in A_1(V') \) which is a contradiction. So there exists a \( j' \in [0, k' - 1] \) such that \((\rho^{o,j'}, \rho^{o,j'+1}) \notin A_1(V') \).

Consider the minimum \( j' \in [0, k' - 1] \) such that \((\rho^{o,j'}, \rho^{o,j'+1}) \notin A_1(V') \). Therefore \( w^1 \leq A_1(V') \rho^{o,j'} \). Similar to proof of Claim C.2, the interesting case is when \((\rho^{o,j'}, \rho^{o,j'+1}) \in B_1(V) \). Then by Definition 6.5 we have that \((\rho^{o,j'}, \rho^{o,j'+1}) \in DRO(V_1) \) and that \( C_1(\rho^{o,j'}, \rho^{o,j'+1}) \) is non-empty. Thus, from Observation B.1, there exists \( w^1_{min} \in (w, 1, *, *) \) such that \((C_1(\rho^{o,j'}, \rho^{o,j'+1}) = C_1(\rho^{o,j'}, w^1_{min}, \rho^{o,j'+1}) \). Now \( \rho^{o,j'} \leq A_1(V) w^1_{min} \), by Observation B.1 \( \rho^{o,j'} \) is either a read or a write operation. If \( \rho^{o,j'} \) is a read operation, then \( \rho^{o,j'} \in (r, 1, *, *) \) and so \((\rho^{o,j'}, w^1_{min}) \in PO \). If \( \rho^{o,j'} \) is a write operation, then \( \rho^{o,j'} \in (w, *, *, *) \) and so \( \rho^{o,j'} \leq SWO(V) w^1_{min} \) and by Lemma C.1(a) \( \rho^{o,j'} \leq SWO(V) w^1_{min} \). Therefore, in either case \( \rho^{o,j'} \leq A_1(V') w^1_{min} \). Now since \((\rho^{o,j'}, \rho^{o,j'+1}) \in DRO(V_1) \) and \((\rho^{o,j'}, \rho^{o,j'+1}) \notin A_1(V') \), thus \((\rho^{o,j'+1}, \rho^{o,j'}) \in DRO(V') \) and \((\rho^{o,j'+1}, \rho^{o,j'}) \in A_1(V') \). It follows that \( \rho^{o,j'+1} < A_1(V') \rho^{o,j'} \leq A_1(V') w^1_{min} \). Since
$A_1(V)$

$B_1(V)$

$\rho, j'$

$\rho, j' + 1$

$w^2$

$w_1^1$

$w_1^m$

$\rho, j' + 1$ is a write operation (by Definition 6.5), thus $(w_1^m, \rho, j' + 1)$ is a bad write pair (recall that $(\rho, j', \rho, j' + 1) \in B_1(V)$ and $C_1(V, \rho, j', \rho, j' + 1) = C_1(V, w_1^m, \rho, j' + 1)$).

Now,

- $w_1^1 \leq A_1(V) \rho, j'$ by choice of $j'$, and
- $\rho, j' \leq A_1(V) w_1^m$ by Observation B.1.

Therefore we have that $w_1^1 \leq \text{SWO}(V) w_1^m$. There are two cases to consider.

Case i: $(w_1^1, w_1^m) \in \text{SWO}(V)$. This contradicts the maximality of $w_1^1$ since both $(w_1^1, \rho, j' + 1)$ and $(w_1^1, w_2)$ are bad write pairs.

Case ii: $w_1^1 = \rho, j' = w_1^m$. Since $\rho$ is a path in $(A_1(V) \setminus \{(w_1^1, w_2^1)\})$ and $w_1^1 = \rho, j'$, thus $\rho, j' + 1 \neq w_2$ and by the minimality of $w_2$, we have that $(w_1^m, \rho, j' + 1)$ is not a bad write pair, a contradiction.

Case 2: Either $i \neq 1$ or there does not exist an even $j \in [0, k - 1]$ such that $w_1^j = w_1^1$.

We show that for every even $j \in [0, k - 1]$, we have that $w_1^j \leq A_1(V) w_1^j + 1$. It follows that $A_1(V) \cup C_1(V, w_1^1, w_2)$ has a cycle and we are done.

Consider any even $j \in [0, k - 1]$. WLOG $j = 0$ since we can rotate the cycle. If $w_1^j = w_1^1$, then we are done. So assume $w_1^j < A_1(V) w_1^1$. Suppose for the sake of contradiction that $w_1^j \not\leq A_1(V) w_1^j + 1$. Consider a path $\rho$ from $w_1^j$ to $w_1^j + 1$ in $A$ given by $w_1^j = \rho, 0 \preceq A_1(V) \rho, 1 \preceq A_1(V) \rho, 2 \preceq A_1(V) \cdots \preceq A_1(V) \rho, k = w_1^j + 1$. Note that each operation in the path is in the view $V_j$ and hence in $V_j'$. If $(\rho, j', \rho, j' + 1) \in A_1(V)$ for every $j' \in [0, k - 1]$, then $(w_1^j, w_1^j + 1) \in A_1(V)$ which is a contradiction. So there exists a $j' \in [0, k - 1]$ such that $(\rho, j', \rho, j' + 1) \not\in A_1(V)$.

Consider the minimum $j' \in [0, k' - 1]$ such that $(\rho, j', \rho, j' + 1) \not\in A_1(V)$. Therefore $w_1^j \leq A_1(V) \rho, j'$. Similar to proof of Claim C.2 the interesting case is when $(\rho, j', \rho, j' + 1) \in B_1(V)$. Then by Definition 6.3, we have that $(\rho, j', \rho, j' + 1) \in \text{DRO}(V_j)$ and that $C_1(V, \rho, j', \rho, j' + 1)$ is non-empty. Thus, from Observation B.1 there exists $w_1^m \in (w_1, *, *, *)$ such that $C_1(V, \rho, j', \rho, j' + 1) = C_1(V, w_1^m, \rho, j' + 1)$. Similar to Case 1, we get that $(w_1^m, \rho, j' + 1)$ is a bad write pair.

Now,

- $w_1^1 \leq \text{SWO}(V) w_1^0$, by Observation B.3 since $w_1^j = w_1^0$ and $(w_1^j, w_1^k) \in C_1(V, w_1^j, w_1^k)$.
- $w_1^j \leq A_1(V) w_1^j$ by choice of $j'$, and
- $\rho, j' \leq A_1(V) w_1^m$ by Observation B.1.

Therefore we have that $w_1^1 \leq \text{SWO}(V) w_1^m$. There are two cases to consider.
At each step $i$. Given a set of partial orders $\mathcal{U}$, we have that each $w_i^1 <_{\text{SWO}(\mathcal{V})} w_i^{\min}$. This contradicts the maximality of $w_i^1$ since both $(w_i^{\min}, o_0, j')$ and $(w_i^1, w_2^1)$ are bad write pairs.

**Case ii:** $w_i^1 = w_i^{\psi, 0} = o_0, j' = w_i^{\min}$. Then $i = 1$ and $w_i^{\psi, 0} = w_i^{\min}$. This contradicts the assumption that either $i \neq 1$ or there does not exist an even $j \in [0, k - 1]$ such that $w_i^{\psi, j} = w_i^1$.

In both cases, we get a contradiction. Therefore for every even $j \in [0, k - 1]$ we have that $w_i^{\psi, j} \leq_{A_i(\mathcal{V})} w_i^{\psi, j + 1}$ and so $A_i(\mathcal{V}) \cup C_i(\mathcal{V}, w_i^1, w_2^1)$ has a cycle, as required.

\[\Box\]

**Proof of Theorem 6.6:** Consider any arbitrary set of views $\mathcal{V}'$ that certify a strongly causal consistent replay to be valid for $\mathcal{R}$. We show that for any process $i$ and any two operations $o_i^1, o_i^2 \in \{\ast, \ast, \ast, \ast\}$ such that $(o_i^1, o_i^2) \in \text{DRO}(V_i)$ we must have that $(o_i^1, o_i^2) \in V_i'$. Consider any arbitrary process $i$. We have that

- $V_i'$ respects $R_i$, since $V_i'$ certifies a replay to be valid for $\mathcal{R}$;
- $V_i'$ respects $\text{SWO}_i(\mathcal{V}) \cup (\text{PO}(\ast, i, \ast, \ast) \cup (w, \ast, \ast, \ast)) \cup B_i(\mathcal{V})$ due to consistency and Lemma C.1

Consider the $o_i^1 o_i^2$-path $\rho$ in $\hat{A}_i$ given by $o_i^1 = o_i^{\rho, 0} \prec_{A_i(\mathcal{V})} o_i^{\rho, 1} \prec_{A_i(\mathcal{V})} o_i^{\rho, 2} \prec_{A_i(\mathcal{V})} \cdots \prec_{A_i(\mathcal{V})} o_i^{\rho, k} = o_i^2$. By construction of $\hat{A}_i$, each edge is either a $R_i$ edge or a $\text{PO}$ edge or a $\text{SWO}_i(\mathcal{V})$ edge or a $B_i(\mathcal{V})$ edge. Thus $o_i^1 = o_i^{\rho, 0} <_{V_i'} o_i^{\rho, 1} <_{V_i'} o_i^{\rho, 2} <_{V_i'} \cdots <_{V_i'} o_i^{\rho, k} = o_i^2$ and $(o_i^1, o_i^2) \in V_i'$, as required.

We extend Definition 3.3 of strong causal order to be applicable to a set of partial orders as follows.

**Definition C.4.** Given a set of partial orders $\mathcal{U} = \{U_i\}_{i \in P}$, two writes, $w_i^1 \in (w, \ast, \ast, \ast)$ and $w_i^2 \in (w, i, \ast, \ast)$, are ordered $(w_i^1, w_i^2) \in \text{SCO}(U_i)$, if $(w_i^1, w_i^2) \in U_i$. Furthermore, $\text{SCO}(\mathcal{U}) = \bigcup_{i \in P} \text{SCO}(U_i)$.

**Lemma C.5.** Given a set of partial orders $\mathcal{U} = \{U_i\}_{i \in P}$ such that for each process $i \in P$, $U_i$ is a partial order on $(\ast, i, \ast, \ast) \cup (w, \ast, \ast, \ast)$ that satisfies transitivity and respects $\text{SCO}(\mathcal{U}) \cup \text{PO}(\ast, i, \ast, \ast) \cup (w, \ast, \ast, \ast))$. Then there exists a strongly causal consistent execution $\mathcal{V} = \{V_i\}_{i \in P}$ such that each $V_i \supseteq U_i$.

**Proof:** We extend $\mathcal{U}$ to $\mathcal{V}$ iteratively. Let $U_i^t$ be the partial order after $t$ steps. Initially, $U_i^0 = U_i$. After some finite number of steps, $U_i^t$ will be a total order and we set $V_i = U_i^t$ at that step. We first order all the write operations for each process $i$ and then add edges for reads appropriately. At each step $t$, we consider two write operations $w_1 \in (w, 1, \ast, \ast)$ and $w_2 \in (w, 2, \ast, \ast)$.

1. If $w_1, w_2$ are not related in $U_i^{t-1}$, then we set $U_i^t = U_i^{t-1} \cup \{(w_1, w_2)\}$.
2. If $w_1, w_2$ are not related in $U_i^{t-1}$, then we set $U_i^t = U_i^{t-1} \cup \{(w_2, w_1)\}$.

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3. For every process $k \neq 1, 2$, if $w_1^k, w_2^k$ are not related in $U_k^{t-1}$, then we do the following. If $SCO(U_k^{t-1}) \cup \{(w_1^k, w_2^k)\} = SCO(U_k^{t-1})$, then we set $U_k^t = U_k^{t-1} \cup \{(w_1^k, w_2^k)\}$. Otherwise, we set $U_k^t = U_k^{t-1} \cup \{(w_2^k, w_1^k)\}$.

After processing all pairs of write operations, we add edges for read operations as follows. If both operations are reads, then they are already related by $PO$. For each read $r \in (r, i, *, *)$ and write $w \in (w, *, *, *)$ such that $r, w$ are not related in $U_i^{t-1}$, set $U_i^t = U_i^{t-1} \cup \{(w, r)\}$. At the end we set $V_i = U_i^t$ for each process $i$.

We now show the correctness of the above procedure. First observe that we add two type of edges 1) for operations that are not already related, and 2) the edges implied by transitivity. Therefore, each $V_i$ is acyclic. Thus, each $V_i$ is a total order on $(*, i, *, *) \cup (w, *, *, *)$ by construction. Now note that each $V_i \supseteq U_i^0 \supseteq (PO((*, i, *, *) \cup (w, *, *, *)))$ by construction. So we show that each $V_i$ respects $SCO(V)$. We proceed via induction and show that at each step $t$, each $U_i^t$ respects $SCO(U^t)$ by construction. For the base case $U_i^0$ respects $SCO(U^0)$ by construction. For the inductive step, we show that $SCO(U_i^t) = SCO(U_i^{t-1})$. This implies the result since each $V_i \supseteq U_i^{t-1}$ and $U_i^{t-1}$ respects $SCO(U_i^{t-1})$ by the inductive hypothesis.

If at step $t$ we considered two write operations, then we have 3 cases to consider.

1. $w_1^i \in (w, 1, *, *)$ and $w_2^i \in (w, 2, *, *)$ are not related in $U_i^{t-1}$ and we set $U_i^t = U_i^{t-1} \cup \{(w_1^i, w_2^i)\}$. We show, via contradiction, that $SCO(U_i^t) \setminus SCO(U_i^{t-1})$ is empty and so there are no new $SCO$ edges in this case. Suppose $(w', w'') \in SCO(U_i^t) \setminus SCO(U_i^{t-1})$. Then $w' \leq_{U_1^{t-1}} w_1^i, w_2^i \leq_{U_1^{t-1}} w''$, and $w'' \in (w, 1, *, *)$. Therefore, $w_1^i$ and $w''$ are related by $(PO((*, i, *, *) \cup (w, *, *, *))$. If $w'' \leq_{PO} w_1^i$, then $w'' \leq_{U_i^{t-1}} w_1^i$ and so $w_2^i \leq_{U_i^{t-1}} w'' \leq_{U_i^{t-1}} w_1^i$, which contradicts the initial assumption that $w_1^i$ and $w_2^i$ are not related in $U_i^{t-1}$. Thus $w_1^i \leq_{U_i^{t-1}} w''$. This implies that $w_1^i \leq_{U_i^{t-1}} w''$ and so $w' \leq_{U_i^{t-1}} w_1^i \leq_{U_i^{t-1}} w''$. Thus $(w', w'') \leq_{SCO(U_i^{t-1})}$, which contradicts the initial assumption that $(w', w'') \leq_{SCO(U_i^{t-1})}$.

2. $w_1^i \in (w, 1, *, *)$ and $w_2^i \in (w, 2, *, *)$ are not related in $U_i^{t-1}$ and we set $U_i^t = U_i^{t-1} \cup \{(w_2^i, w_1^i)\}$. This is the same as Case 1.

3. For every process $k \neq 1, 2$ such that $w_1^k, w_2^k$ are not related in $U_k^{t-1}$, we do the following. If $SCO(U_k^{t-1}) \cup \{(w_1^k, w_2^k)\} = SCO(U_k^{t-1})$, then we set $U_k^t = U_k^{t-1} \cup \{(w_1^k, w_2^k)\}$. Otherwise, we set $U_k^t = U_k^{t-1} \cup \{(w_2^k, w_1^k)\}$. We proceed via contradiction to show that either $SCO(U_k^{t-1}) \cup \{(w_1^k, w_2^k)\} \setminus SCO(U_k^{t-1})$ or $SCO(U_k^{t-1}) \cup \{(w_2^k, w_1^k)\} \setminus SCO(U_k^{t-1})$ is empty and so there are no new $SCO$ edges in this case. Suppose $(w_3^k, w_4^k) \in SCO(U_k^{t-1}) \cup \{(w_1^k, w_2^k)\} \setminus SCO(U_k^{t-1})$ and $(w_5^k, w_6^k) \in SCO(U_k^{t-1}) \cup \{(w_2^k, w_1^k)\} \setminus SCO(U_k^{t-1})$. It follows that $w_3^k, w_6^k \in (w, k, *, *)$ and therefore, are related by $(PO((*, k, *, *) \cup (w, *, *, *))$. There are two cases to consider.

i) $w_6 \leq_{PO} w_4$. Since $(w_3^k, w_4^k) \in SCO(U_k^{t-1}) \cup \{(w_1^k, w_2^k)\} \setminus SCO(U_k^{t-1})$, so $w_3 \leq_{U_k^{t-1}} w_1^k$ and $w_2^k \leq_{U_k^{t-1}} w_4^k$. Since $(w_5^k, w_6^k) \in SCO(U_k^{t-1}) \cup \{(w_2^k, w_1^k)\} \setminus SCO(U_k^{t-1})$, so $w_5 \leq_{U_k^{t-1}} w_2^k$ and $w_1^k \leq_{U_k^{t-1}} w_6^k$. Therefore $w_3 \leq_{U_k^{t-1}} w_1^k \leq_{U_k^{t-1}} w_6^k \leq_{PO} w_4 \leq_{U_k^{t-1}} w_4^k$ and thus $(w_3^k, w_4^k) \in SCO(U_k^{t-1})$. This contradicts the initial assumption that $(w_3^k, w_4^k) \in SCO(U_k^{t-1}) \cup \{(w_1^k, w_2^k)\} \setminus SCO(U_k^{t-1})$.

ii) $w_4 \leq_{PO} w_6$. This is the same as Case i with the role of $w_4$ and $w_6$ switched.
Now, if step $t$ considered read operations, then all write operations have already been ordered by each $V_i^{t-1}$ and therefore $SCO(V^t) = SCO(V^{t-1})$. This completes the proof that $SCO(V^t) = SCO(V^{t-1})$.

Proof of Theorem 6.7: Assume for the sake of contradiction that there exists a good record $R$ of $V$ such that there exists a process, WLOG process 1, and two operations $o^1, o^2 \in (\ast, 1, \ast, \ast) \cup (\ast, w, \ast, \ast)$ such that $(o^1, o^2) \in \hat{A}_1(V) \setminus PO \cup SWO_1(V) \cup B_1(V)$ and $(o^1, o^2) \notin R_1$. Then, we construct a set of views $V'$, such that $DRO(V'_1) \neq DRO(V_1)$ but $V'$ satisfies the strongly causal consistent replay to be valid for $R$, i.e. $V'$ explains a strongly causal execution and extends the record $R$. This violates the definition of a good record. We use Lemma C.5 and construct, for each process $i$, a partial order $U_i \supseteq R_i \cup SCO(V) \cup (PO(\ast, i, \ast, \ast) \cup (w, \ast, \ast, \ast))$ such that $(o^2, o^1) \in U_1$. From Lemma C.5 it follows that there exists a strongly causal execution $V'$ such that for each process $i$, $V_i \supseteq U_i \supseteq R_i$ (and therefore a replay of $R$) and $(o^2, o^1) \in V_i$. Observe that since $(o^1, o^2) \in A_1(V) \setminus (PO \cup SWO_1(V) \cup B_1(V))$, therefore $(o^1, o^2) \in DRO(V_1)$, $(o^2, o^1) \in DRO(V'_1)$, and so $DRO(V_1') \neq DRO(V_1)$.

We construct $U$ from $A(V)$ as follows. We slightly abuse notation to set $C_1(V, o^1, o^2) = \emptyset$ if $o^2$ is a read operation (recall that $C_1(V, o^1, o^2)$ is only defined when $o^2$ is a write operation in Definition 6.4). Let $U_1 := (A_1(V) \setminus \{o^1, o^2\}) \cup \{(o^2, o^1)\} \cup C_1(V, o^1, o^2)$. For each $i > 1$, set $U_i = A_i(V) \cup C_1(V, o^1, o^2)$ (see Definition 6.4).

For correctness we have to show that each $U_i$

1. is a partial order, and
2. respects $SCO(U)$.

We first consider the case when $o^2$ is a read operation. We claim that $SCO(U) \setminus SCO(A(V))$ is empty. If not, then there exist two write operation $w^3 \in (w, \ast, \ast, \ast)$ and $w^4 \in (w, 1, \ast, \ast)$ such that $w^3 \leq_{A_1(V)} o^2$ and $o^1 \leq_{A_1(V)} w^4$. Since $o^2 \in (r, 1, \ast, \ast)$, therefore either $(o^2, w^4) \in PO$ or $(w^4, o^2) \in PO$. In the first case $(o^2, w^4) \in SCO(A(V))$. In the second case if $o^1 = w^4$, then $(w^1, o^2) \in PO$. So $o^1 <_{A_1(V)} w^4 <_{A_1(V)} o^2$ which contradicts the fact that $(o^1, o^2) \in \hat{A}_1(V)$. In either case, we have the desired contradiction. Therefore, each $U_i$ is a partial order that respects $SCO(U)$.

We now consider the case when $o^2$ is a write operation. Since $(o^1, o^2) \notin B_i(V)$, therefore each $U_i$ is a partial order by Definition 6.5. So we show that for any process $i$, $SCO(U_i) \setminus SCO(A_i(V)) \subseteq C_1(V, o^1, o^2)$. Consider any $(w^3, w^4) \in SCO(U_i) \setminus SCO(A_i(V))$. Then there exists a $w^3 w^4$-path $\rho$ in $A_1 \cup C_1(V, o^1, o^2)$, given by $w^3 = o^0, o^1 <_{A_1(V)} o^{0,1} <_{C_1(V, o^1, o^2)} o^{0,2} <_{A_i(V)} \cdots <_{A_i(V)} o^{0,k-1} \leq_{A_i(V)} o^{0,k} = w^4$ with $k > 2$. It follows by Definition 6.4 that $(w^3, w^4) \in C_1(V, o^1, o^2)$.

So we have shown that $U$ meets the conditions of Lemma C.5 and therefore, we can find a strongly causal consistent replay $V'$ of $R$ such that $DRO(V'_1) \neq DRO(V_1)$. This contradicts the initial assumption that $R$ is a good record. □