Charges in nonlinear higher-spin theory

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Abstract
Nonlinear higher-spin equations in four dimensions admit a closed two-form that defines a
gauge-invariant global charge as an integral over a two-dimensional cycle. In this paper we
argue that this charge gives rise to partitions depending on various lower- and higher-spin
chemical potentials identified with modules of topological fields in the theory. The vacuum
contribution to the partition is calculated to the first nontrivial order for a solution to
higher-spin equations that generalizes AdS⁴ Kerr black hole of General Relativity. The
resulting partition is non-zero being in parametric agreement with the ADM-like behavior
of a rotating source. The linear response of chemical potentials to the partition function
is also extracted. The explicit unfolded form of 4d GR black holes is given. An explicit
formula relating asymptotic higher-spin charges expressed in terms of the generalized
higher-spin Weyl tensor with those expressed in terms of Fronsdal fields is obtained.

1 Introduction

Higher-spin (HS) gauge theories [1, 2] have recently attracted much of attention in the context
of AdS/CFT correspondence (see e.g. [3] and references therein). The conjecture of Klebanov
and Polyakov [4, 5] associates HS theories with vectorial models on the flat boundary and
provides an example of the weak-weak type duality that can be verifiable in practice. The main
obstacle that however prevents one from solid tests of the correspondence is the absence of the
full nonlinear extension of the free Fronsdal action principle on the HS side [6] (see however
[7]). The available tests favoring to Klebanov-Polyakov conjecture were completed either at the
level of equations of motion [8, 9] or by implying symmetry arguments [10, 11]. Recently an
on-shell invariant functional has been proposed in [12], conjectured to produce the generating
functional for boundary correlation functions. As such, it may provide a proper substitute of
action principle for the AdS/CFT HS problem. In d = 4 the invariant functional is a closed
space-time four-form which arises upon an extension of the original HS equations [13] by adding certain auxiliary extra fields.

Apart from the four-form anticipated to be relevant to the AdS/CFT dictionary, there is also a closed two-form defined on the equations of motion. Being integrated over a two-dimensional cycle it provides a conserved charge on solutions of HS equations. Moreover, this construction allows a straightforward inclusion of HS topological fields introduced in [13], which can be identified with various chemical potentials.

In this paper we analyze the topological moduli contribution at first order in the linearized approximation. We show how the obtained formulae reproduce asymptotic charges associated with moduli parameters of arbitrary rank. In general this results in some higher-derivative expression for a closed two-form which is hard to derive using standard General Relativity (GR) methods. We also consider the example of gravity in which case one reproduces asymptotic charge of a kind considered in [14].

To explain the idea of our approach we first review the generating construction of GR BHs on AdS along with their HS generalization in terms of a single AdS global symmetry parameter [15]. We put emphasis on particular cases of most physical relevance that are Kerr, Schwarzschild, planar and topological BHs. We show that the charge calculated this way for the Kerr case parametrically agrees with the standard asymptotic ADM-like behavior.

The paper is organized as follows. In section 2 we review 4d HS equations and define the invariant two-form functional. In section 3 we introduce the notion of asymptotic symmetries in HS theory and introduce charges and partitions with the emphasis on HS topological sector. Then in section 3.1 the simplest vacuum partition is further derived at free level. In section 3.2 the topological contribution is found to the lowest order and the examples of spin two and spin four are considered. In section 4 we review how AdS BHs and their HS cousins originate from AdS global symmetry parameter and consider various BH examples in detail. Finally in section 4.3 the charge for the Kerr case is explicitly computed.

2 Structure of HS equations

HS equations in four dimensions have the following standard form [13]

\begin{align}
    dW + W * W &= 0, \quad \text{(2.1)} \\
    dS + [W, S]_* &= 0, \quad \text{(2.2)} \\
    dB + [W, B]_* &= 0, \quad \text{(2.3)} \\
    S * S &= -i\theta_\alpha \wedge \theta^\alpha (1 + F_*(B) * k \mathbf{r}) - i\bar{\theta}_\dot{\alpha} \wedge \bar{\theta}^{\dot{\alpha}} (1 + \bar{F}_*(B) \bar{\mathbf{k}} \bar{\mathbf{r}}), \quad \text{(2.4)} \\
    [S, B]_* &= 0. \quad \text{(2.5)}
\end{align}

The conventions are as follows. W, S and B in (2.1)-(2.5) denote space-time dependent generating functions of commuting twistor-like variables \( \mathbf{Y} = (y_\alpha, \bar{y}_{\dot{\alpha}}) \) and \( \mathbf{Z} = (z_\alpha, \bar{z}_{\dot{\alpha}}) \), where spinor indices \( \alpha, \beta, ... \) range two values. The associative star-product operation acts on functions of \((\mathbf{Y}, \mathbf{Z})\)-space

\[
(f * g)(\mathbf{Y}, \mathbf{Z}) = \frac{1}{(2\pi)^4} \int dUdV f(\mathbf{Y} + U, \mathbf{Z} + V)g(\mathbf{Y} + V, \mathbf{Z} - U)e^{iU_{A}V^{A}}, \quad \text{(2.6)}
\]

where \( U_{A}V^{A} := U_{A}V_{B}\epsilon^{AB} \) with some \( sp(4) \)-invariant form \( \epsilon_{AB} = -\epsilon_{BA} \). Indices are raised and lowered with the aid of \( \epsilon_{AB} \) as follows, \( X^A = \epsilon^{AB}X_B \) and \( X_A = \epsilon_{BA}X^B \). The integration
measure is chosen in such a way that
\[
1 \ast F(Y, Z) = F(Y, Z) \ast 1 = F(Y, Z) .
\] (2.7)

The star product induces the following commutation relations
\[
[Y_A, Y_B]_\ast = - [Z_A, Z_B]_\ast = 2i\epsilon_{AB} ,
\]
\[
[Y_A, Z_B]_\ast = 0 .
\] (2.8)

Important elements of the star-product algebra entering (2.4) are the inner Klein operators \(\kappa\) and \(\bar{\kappa}\)
\[
\kappa = \exp iz_\alpha y^\alpha ,
\]
\[
\bar{\kappa} = \exp i\bar{z}_\dot{\alpha}\bar{y}^{\dot{\alpha}} .
\] (2.9)

Using (2.6) it is easy to check their characteristic properties
\[
\{\kappa, y_\alpha\}_\ast = \{\kappa, z_\alpha\}_\ast = 0 ,
\]
\[
\kappa \ast \kappa = 1 ,
\] (2.10)

analogously in the antiholomorphic sector for \(\bar{\kappa}\).

There are also extra Clifford variables \(K = (k, \bar{k})\) which enter system (2.1)-(2.5). \(k\) anticommutes with the holomorphic variables \(y_\alpha, z_\alpha, \theta_\alpha\) and commutes with every antiholomorphic one, while \(\bar{k}\) does the opposite. In addition
\[
k^2 = \bar{k}^2 = 1 .
\] (2.11)

Note that along with (2.10) this makes elements \(k\kappa\) and \(\bar{k}\bar{\kappa}\) central in the star-product algebra.

A space-time one-form \(W(Y, Z; K|x) = W_\mu dx^\mu\) parameterizes on-shell HS potentials and their derivatives. The 0-form \(B(Y, Z; K|x)\) encodes spin \(s = 0, s = 1/2\) matter fields and all HS curvatures. The \(S(Y, Z; K|x) = S_A\theta^A\) field, where \(\theta_A = (\theta_\alpha, \bar{\theta}_{\dot{\alpha}})\), representing a one-form with respect to spinorial differentials, is purely auxiliary on-shell carrying no local degrees of freedom. All differentials anticommute
\[
\{dx_\mu, dx_\nu\} = \{dx_\mu, \theta_A\} = \{\theta_A, \theta_B\} = 0 .
\] (2.13)

By virtue of (2.11) the dependence of \(W, B\) and \(S\) on \(K\) is at most bilinear,
\[
W = \sum_{i,j=0,1} W_{i,j} k^i \bar{k}^j ,
\]
\[
B = \sum_{i,j=0,1} B_{i,j} k^i \bar{k}^j .
\] (2.14)

Such dependence is known to split the field content of the theory into physical and topological (non-propagating) sectors [13]. In the \(AdS\) background fields \(W_{i,i}\) and \(B_{i,1-i}\) are propagating, while \(W_{i,1-i}\) and \(B_{i,i}\) on the contrary are non-dynamical (topological). The fact that topological sector contains no propagating fields follows from the inspection of the zero-form sector at free level. Indeed, within the unfolded formulation physical degrees are stored in zero-forms. (For more detail see e.g. [1, 2]). Upon resolution for the dependence on the coordinates \(Z^A\) the free level analysis over \(AdS\) background implies in the topological sector
\[
D_\Omega C_{i,i}(Y|x) := dC_{i,i} + [\Omega, C_{i,i}]_\ast = 0 ,
\] (2.15)
which is the adjoint flatness condition. Here $\Omega$ is the $AdS$-flat connection spanned by $Y$-bilinears

$$\Omega = \frac{i}{4} \Omega_{AB} Y^A Y^B.$$  \hfill (2.16)

The adjoint covariant derivative acts on the finite-dimensional modules, spanned by various homogeneous polynomials in $Y$, and therefore the respective equations describe an infinite set of topological systems, each describing at most a finite number of degrees of freedom. In this respect, topological sector is different from the dynamical sector in which zero-forms are valued in the twisted-adjoint module, obeying the equations

$$dC_{i,1-i} + \Omega \ast C_{i,1-i} - C_{i,1-i} \ast \pi(\Omega) = 0,$$  \hfill (2.17)

where the antiautomorphism $\pi$ is defined to flip the sign of undotted oscillators

$$\pi(y, z, \bar{y}, \bar{z}) = (-y, -z, \bar{y}, \bar{z}).$$  \hfill (2.18)

The twisted-adjoint module is infinite-dimensional being appropriate for the description of relativistic fields carrying an infinite number of degrees of freedom.

Note that the total numbers of degrees of freedom in the topological and dynamical systems are the same, being described by a pair of unrestricted functions of $Y^A$. This is possible because the HS theory contains infinitely many topological fields.

In principle, topological fields can be truncated away from 4$d$ HS theory by requiring $B_{ii} = W_{i,i+1} = S_{i,i+1} = 0$. However, this leads to enormous reduction of the space of HS theories since, as was recently emphasized in [17], the topological fields are in fact the modules distinguishing between different theories. These fields will also play a central role in this paper being associated with chemical potentials in the partitions.

We apply the bosonic truncation in this paper that allows one to stay with a single copy of every integer spin in the spectrum. This is achieved by setting $kk = 1$ which is only consistent if $W$ and $B$ are bosons obeying $(W, B)(-y, \bar{y}, -z, \bar{z}) = (W, B)(y, -\bar{y}, z, -\bar{z})$ and $S(-y, \bar{y}, -z, \bar{z}) = -S(y, -\bar{y}, z, -\bar{z})$.

Apart from topological fields, the only freedom in (2.1)-(2.5) is an arbitrary complex function

$$F_\ast(B) = \eta B + O(B^2),$$  \hfill (2.19)

where $\eta$ is a constant phase factor, $|\eta| = 1$. When all higher powers of $B$ in (2.19) are discarded $\eta$ breaks down parity of the theory unless $\eta = 1$ or $\eta = i$ in which cases one is left with the so-called A and B HS models correspondingly [18].

A convenient way to look at (2.1)-(2.5) that makes issues of gauge invariance and consistency check straightforward is the unification of all one-forms into a single field

$$\mathcal{W} = d + W + S$$  \hfill (2.20)

which reduces (2.1)-(2.5) to

$$\mathcal{W} \ast \mathcal{W} = -i \theta_\alpha \wedge \theta^\alpha (1 + F_\ast(B) \ast \mathcal{K}) - i \bar{\theta}_\dot{\alpha} \wedge \bar{\theta}^{\dot{\alpha}} (1 + \bar{F}_\ast(B) \ast \bar{\mathcal{K}}),$$  \hfill (2.21)

$$[\mathcal{W}, B]_\ast = 0.$$  \hfill (2.22)

\footnote{Note that no such truncation is available in $d = 3$ theory of [16].}
As was shown recently in [12], written this way, equations (2.21)-(2.22) admit straightforward generalization allowing to define the on-shell closed two- and four-forms. The four-form is related presumably to on-shell HS action thus being anticipated to play important role in the AdS/CFT analysis. The two-form $L^2$, which is of most interest in this paper, gives rise to a surface charge of a particular solution of HS equations. Namely, following [12] we deform (2.21)-(2.22) as follows

\[ W^*W = -i\theta_\alpha^\alpha \wedge \bar{\theta}^\dot{\alpha}(1 + F_* (B) * k \kappa) - i\bar{\theta}_\dot{\alpha} \wedge \theta^\alpha (1 + \bar{F}_* (B) * \bar{k} \bar{\kappa}) + L^2, \]  

(2.23)

\[ dL^2 = 0, \]  

(2.24)

\[ [W, B]_* = 0, \]  

(2.25)

where $L^2(x)$ is a space-time two-form, independent of $Z$ and $Y$ variables. According to equations (2.23), $L^2$ is responsible for the following gauge transformations

\[ \delta L^2(x) = d\epsilon(x), \quad \delta W(Y, Z|x) = \epsilon(x), \]  

(2.26)

where gauge parameter $\epsilon(x)$ represents a space-time one-form.

Thus, by virtue of (2.24) and (2.26) $L^2$ provides a gauge-invariant surface charge via integration over a two-cycle $\Sigma$

\[ Q = \int_\Sigma L^2. \]  

(2.27)

Here $L^2$ is expressed in terms of HS fields $W$ and $B$ through (2.23). By construction $L^2$ is $Y$ and $Z$ – independent. At free level this implies that spin-1 field only can contribute to the conserved charge, while all higher spins affect through interaction. This fact reveals essential difference between (2.27) and standard canonical constructions for gravity and HS fields based on the action principle which either reproduce asymptotic charge (equivalently exact charge of any spin for a free theory) or imply existence of a global symmetry. Let us stress that (2.27) is exactly conserved irrespectively of any leftover global symmetry of a particular solution.

It might seem that the simplest free HS charges for Fronsdal fields escape from (2.27). We will see that those can be still obtained using the topological sector of (2.23)-(2.25). In this case, the topological fields play a role of a book-keeping device for contracting indices to combine all closed forms into a single spin-one form that however depends on the Killing tensors associated with the topological fields whose field equations are just those of the Killing tensors. In this respect it is not necessary to take into account the back reaction of the the topological fields on the matter ones to define asymptotic charges. Instead, as shown below, it is enough to plug the Killing tensors into the formula for topological fields which thus lead to asymptotic charges defined in the usual HS theory free of the topological fields.

However, in order to use proposed construction for the globally closed forms, the effect of back reaction of the propagating fields on the topological ones should be fully taken into account. This means that it is not clear how to define closed forms of spins greater than one if the topological fields are not included into the model. As speculated in the end of Section 5, this phenomenon may have important implications for the resolution of the information paradox.

Denoting $\omega(y, \bar{y}|x) = W(Y, 0|x)$ and $C(y, \bar{y}|x) = B(Y, 0|x)$, within perturbation theory around $AdS_4$ vacuum one arrives at the following schematic form of equations resulting from (2.23), (2.25)

\[ R(Y|x) := d\omega + \omega * \omega = L^2 - \Upsilon(\omega, \omega, C) - \Upsilon(\omega, \omega, C, C) - \ldots, \]  

(2.28)

\[ dC + [\omega, C]_* = -\Upsilon(\omega, C) - \Upsilon(\omega, C, C) - \ldots, \]  

(2.29)
where $\Upsilon$ denotes interaction vertices that can be found order by order from the perturbative expansion. A powerful method of extracting such vertices has been recently proposed in [19].

The second term on the l.h.s. of (2.28) is identically zero at $Y = 0$ because it can be represented as $tr(\omega * \omega) = 0$ (recall that $\omega$ is a one-form). Indeed, the star product admits the supertrace operation which in the bosonic case has the property of usual trace

$$tr(f(Y) := f(0), \quad tr(f(Y) * g(Y)) = tr(g(Y) * f(Y)).$$

So one has the following equation determining $L^2$

$$(2.30)$$

Next, using gauge symmetry (2.26) one can impose the canonical gauge $\omega(0| x) = 0$ [12]. Then a two-form $L^2$ can be expressed as

$$(2.31)$$

Being closed, $L^2$ may or may not be exact. In the former case it results in zero charge (2.27). This happens if (2.31), considered as an equation for $\omega(0| x)$, admits solutions with $L^2 = 0$ and $\omega(0| x)$ regular on the integration cycle $\Sigma$. On the contrary, if $\omega(0| x)$ at $L^2 = 0$ gets singular on $\Sigma$, then $L^2$ may be nontrivial.

This situation is analogous to Dirac monopole problem. There, due to Dirac string, vector-potential is ill-defined on any surface surrounding a monopole. However, magnetic field strength remains regular and provides a conserved magnetic charge via integration over the surface. Here we have ill-defined Abelian spin-1 potential $\omega(0| x)$ and well-defined ‘field strength’ $\left(\Upsilon(\omega, \omega, C) + \ldots\right)_{Y = 0}$. Imposing the canonical gauge allows one to work entirely in terms of the latter, though in general it is not obligatory: alternatively, as in the monopole problem, one can use the fiber bundle picture, covering the spacetime with several charts. Nontriviality of $L^2$ then results from patching conditions. Since this analysis is somewhat more complicated, we prefer to work in the canonical gauge.

A related issue that we would like to emphasize is that it may be tempting to solve eq. (2.1) in a pure gauge form suggesting that any flat $W(Z, Y| x)$ is gauge equivalent to $W = 0$. Again, this immediately entails $Q = 0$. This manipulation should not be expected to be globally well-defined, however. This argument resurrects an old problem of admissible gauge transformations and classes of functions in general in HS theories, see [16, 20, 21, 22] for a recent account.

The main new point stressed in this paper is that the construction of the invariant functionals associated with $L^2$ admits a natural extension to various chemical potentials $\xi$ associated with modules of topological fields in the HS theory introduced in [13]. The respective partition

$$Z = \exp - \int_{\Sigma} L^2(\xi)$$

turns out to be independent of the variations of the integration two-cycle $\Sigma$. In the asymptotic limit where $\Sigma$ tends to infinity and the theory becomes nearly free, this construction is argued to reproduce usual asymptotic charges in GR [23, 24, 25, 26, 27], as well as their HS extension. For surface charges in the HS context see [28] and [29, 30, 31, 32]. While it may not be immediately straightforward comparing our charge with the canonical asymptotic charges based on the action
principle and the metric-like approach, there is a simple argument in favor of its equivalence. Our conclusions are in full agreement with those of [27] since asymptotically conserved charges are shown to be represented by on-shell closed two-forms expressed in terms of dynamical fields of the theory equivalent to the Fronsdal fields in the metric-like approach. The complete set of these two-forms expressed via (generalized) Weyl tensors is the same in the metric-like and frame-like formalism. Hence they necessarily coincide.

3 Charges and partitions in HS equations

3.1 Charges

In Einstein case (Ricci=0) with some Killing one-form $\xi = \xi_\mu dx^\mu$ there exists a conserved charge given by the Komar integral [33]

$$Q = \int_\Sigma K, \quad (3.1)$$

where $K$ is a two-form defined via Killing one-form $\xi$

$$K = \ast d\xi, \quad (3.2)$$

which is closed on the equations of motion. For non-zero cosmological constant $\Lambda \neq 0$ Komar expression (3.1) no longer works and one needs either subtracting infinities due to the volume proportional to $\Lambda$ in (3.1) [34], or use other refined methods, see e.g. [35, 26, 27]. This way one normally produces BH charges such as mass and angular momentum using appropriate exact isometries. The resulting mass is of course linear in BH mass parameter $m$.

At the linearized level the analog of GR isometries in HS theory are global symmetries, that is the redundant symmetries $\epsilon(Y|x)$ of a vacuum solution such that $\delta_\epsilon \omega = \delta_\epsilon C = 0$. In [14], the two-form that generates asymptotic symmetries in GR was expressed in terms of the Weyl tensor contracted with vector fields representing asymptotic symmetries. HS theory admits a natural generalization of this construction. Indeed, consider the following exact two-form

$$d\text{tr} (\xi \ast \tilde{n}\omega_1). \quad (3.3)$$

Here $\xi(Y|x)$ is an AdS Killing tensor, i.e.

$$D_\Omega \xi(Y|x) = 0, \quad D_\Omega = d + [\frac{i}{4} \Omega_{AB} Y^A Y^B, \ast], \quad (3.4)$$

and $\omega_1(Y|x)$ is a linear fluctuation of the HS connection $\omega$, $\Omega_{AB}$ is the background AdS$_4$ connection (more details in the following section) and the operator $\tilde{n}$ is defined via

$$\tilde{n} F(Y) = \begin{cases} F(Y), & N_y > \tilde{N}_\tilde{y}, \\ 0, & N_y = \tilde{N}_\tilde{y}, \\ -F(Y), & N_y < \tilde{N}_\tilde{y}, \end{cases} \quad (3.5)$$

where $N_y$ and $\tilde{N}_\tilde{y}$ count the powers of $y$ and $\tilde{y}$ respectively

$$y^\alpha \frac{\partial}{\partial y^\alpha} F(Y) = N_y F(Y), \quad \tilde{y}^\dot{\alpha} \frac{\partial}{\partial \tilde{y}^\dot{\alpha}} F(Y) = \tilde{N}_{\dot{y}} F(Y). \quad (3.6)$$
Using (3.4)-(3.6), one finds that
\[
dtr(\xi \ast \hat{n}\omega_1) = tr\left\{ \xi \ast e^{\alpha\beta} \bar{y}_\beta \frac{\partial}{\partial y^\alpha} (\bar{c}_2 + 2\bar{c}_1 + \bar{c}_0)\omega_1 - \xi \ast e^{\alpha\beta} y_\alpha \frac{\partial}{\partial \bar{y}^\beta} (\bar{c}_{-2} + 2\bar{c}_{-1} + \bar{c}_0)\omega_1 + \xi \ast \hat{n}D_\Omega\omega_1 \right\},
\]
where \(\hat{c}_n\) is the projector to the proper (sub)diagonal in the \((y, \bar{y})\)-space.
\[
\hat{c}_n F(Y) := \begin{cases} F(Y), & N_y - \bar{N}_{\bar{y}} = n, \\ 0, & N_y - \bar{N}_{\bar{y}} \neq n. \end{cases}
\]
\(\omega_1\) satisfies what is known as the First on-mass-shell theorem
\[
D_\Omega\omega_1(y, \bar{y}|x) = -\frac{i}{4} \epsilon^g \wedge \epsilon^\beta \frac{\partial^2}{\partial y^\alpha \partial \bar{y}^\beta} C(0, \bar{y}|x) - \frac{i}{4} \epsilon^g \wedge \epsilon^\beta \frac{\partial^2}{\partial y^\alpha \partial \bar{y}^\beta} C(y, 0|x),
\]
following from linearization of the nonlinear HS equations. Each term on the r.h.s. of (3.9) represents an independent \(D_\Omega\)-cohomology.

Components of the 1-form \(\omega_1(Y|x)\) with \(N_y = \bar{N}_{\bar{y}}\) describe free bosonic Fronsdal fields (in terms of base vector indices \(\omega_1^{(n), 1}(x)\) corresponds to double-traceless rank-\((n+1)\) tensor \(\phi^{(n+1)}(x)\)), while components with \(N_y = \bar{N}_{\bar{y}} \pm 1\) describe fermionic ones. That is \(\hat{c}_0\omega_1(Y|x)\) projects out bosonic Fronsdal fields. Similarly, by virtue of (3.9), \(\hat{c}_{\pm 1}\omega_1\) extracts \(n\)-th derivatives thereof. For fermions this is done by \(\hat{c}_{\pm 1}\omega_1\) and \(\hat{c}_{\pm (2n+1)}\omega_1\), respectively (for more detail see [1] and references therein).

Now we define closed non-exact two-form, corresponding to the last term from the r.h.s. of (3.7)
\[
\mathcal{R}_\xi := tr(\xi \ast \hat{n}D_\Omega\omega_1),
\]
which according to (3.9) equals
\[
\mathcal{R}_\xi = tr\left\{ \xi \ast \left( \frac{i}{4} \epsilon^g \wedge \epsilon^\beta \frac{\partial^2}{\partial y^\alpha \partial \bar{y}^\beta} C(0, \bar{y}|x) - \frac{i}{4} \epsilon^g \wedge \epsilon^\beta \frac{\partial^2}{\partial y^\alpha \partial \bar{y}^\beta} C(y, 0|x) \right) \right\}.
\]
That \(\mathcal{R}_\xi\) is closed follows from the linearized equations of motion on \(C(Y|x)\) (2.29), which encode Bianchi identities and Maxwell equations. \(\mathcal{R}_\xi\) generates asymptotic charge via integration over two-dimensional cycle \(\Sigma^2\),
\[
Q_\xi = \int_{\Sigma^2} \mathcal{R}_\xi.
\]
Charge (3.12) provides a HS generalization of conserved quantities in asymptotic AdS defined in [14]. The analogy is straightforward since charges of [14] arise after integration of a two-form made of Weyl tensor and a symmetry parameter. In our approach HS Weyl tensors naturally appear on the r.h.s of (2.28) in perturbative expansion around AdS background as explained in more detail in the next section.

Charge (3.12) is built of HS Weyl tensors \(C(y, 0|x), C(0, \bar{y}|x)\) which correspond to \(s\)-th derivatives of spin-\(s\) Fronsdal fields. At the same time there are canonical one-derivative asymptotic Fronsdal charges, obtained in [28]. The relation between two constructions is provided by Eq. (3.7) implying that
\[
\mathcal{R}_\xi = -tr\left\{ \xi \ast e^{\alpha\beta} \bar{y}_\beta \frac{\partial}{\partial y^\alpha} (\bar{c}_2 + 2\bar{c}_1 + \bar{c}_0) - e^{\alpha\beta} y_\alpha \frac{\partial}{\partial \bar{y}^\beta} (\bar{c}_{-2} + 2\bar{c}_{-1} + \bar{c}_0)\omega_1 \right\} + dtr(\xi \ast \hat{n}\omega_1).
\]
Since d-exact term does not contribute to the charge, (3.12) can be equivalently expressed in terms of Fronsdal fields and their first derivatives. This also explains a peculiarity of the spin-1 sector. Spin-1 is described by Y-independent connection $\omega_1(x)$, so it drops out from (3.10). However, nonzero spin-1 contribution can be embedded into (3.11) with $C_{AB}Y^AY^B(x)$. The reason is that first derivatives of $\omega_1(x)$ are already described by gauge-invariant $C_{AB}Y^AY^B(x)$, which is the Faraday tensor, so there is no representation of spin-1 current in the form (3.13), while (3.7) becomes an identity 0=0.

An asymptotically covariantly constant parameter $\xi(Y|x)$, $D\xi_{x\to0} = 0$ generates the asymptotically d-closed $\mathcal{R}_\xi$ and therefore asymptotically conserved charge (3.12). Here $\xi$ plays an auxiliary role of the parameter that has nothing to do with a given HS solution. This simple construction suggests an interesting possibility for the study of BH charges in HS theory.

The appearance of the parameter $\xi$ that approaches $AdS$ (HS) global symmetry parameter is natural in the HS system (2.1)-(2.5) with the topological sector. The key observation is that the form of the linearized equations in the topological sector precisely coincides with that of the asymptotic symmetry parameters in (3.4). This suggests an idea that the topological fields can play a role of generalized chemical potentials $\xi$ conjugated to HS charges.

At the nonlinear level the two sectors become entangled with topological fields sourcing the physical ones. The two-form $\mathcal{L}^2$ defined as (2.32) at $Y = 0$ is still closed in presence of topological fields. This allows us to introduce an invariant partition (2.33) that depends on the modules of a chosen solution like the BH mass as well as on the chemical potentials $\xi$ identified with the topological fields. In this setup, the partition is insensitive to local variations of the integration cycle giving the same result upon integration at infinity, where it should reduce to the asymptotic charges, and over any other cycle in the same homotopy class including the horizon.

Naively, the existence of the closed form $\mathcal{L}^2$ in the nonlinear theory and the surface-independent representation for the partition (2.33) may look surprising and even impossible. However one should take into account the following two facts. First of all, the functional $\mathcal{L}^2(\xi)$ is not a local object in terms of dynamical and topological fields, containing an infinite expansion in powers of derivatives of the fields in $AdS_4$. Secondly, nonzero topological fields $\xi$ will affect the form of the original (e.g. BH) solution at $\xi = 0$. So, to find a full partition (2.33) for certain chemical potentials $\xi$ one has to find the respective solution at $\xi \neq 0$.

In this paper we compute the vacuum value of $\mathcal{L}^2$ at $\xi = 0$ to illustrate how the computation goes. It is simple to find the contribution to $\mathcal{L}^2(\xi)$ linear in $\xi$, associated in particular with asymptotic charges and we present the explicit formulae for that case as well. Computation of the nontrivial charges to higher orders will be given elsewhere.

### 3.2 Vacuum partition

To extract $\mathcal{L}^2$ to the lowest order one truncates away the topological sector and starts with the proper vacuum solution of (2.1)-(2.5) that reads

$$
B^{(0)} = 0, \quad S_A^{(0)} = Z_A, \quad W^{(0)} = \Omega = i\frac{1}{4} \Omega^{\alpha\beta} Y^\alpha Y^\beta, \quad (3.14)
$$

where the $AdS_4$ flat vacuum connection is given by

$$
\Omega = i\frac{1}{4}(\omega_\alpha^\beta y^\alpha y^\beta + \bar{\omega}_\dot{\alpha}\dot{\beta} \bar{y}^\dot{\alpha}\dot{y}^\dot{\beta} + 2\lambda e_\alpha\dot{\alpha} y^\alpha \bar{y}^\dot{\alpha}) \quad (3.15)
$$
that gives as a consequence of (2.1) equations for the background $AdS_4$ space
\begin{align}
    de_{\alpha \dot{\alpha}} + \omega_{\alpha}^{\gamma} \wedge e_{\gamma \dot{\alpha}} + \bar{\omega}_{\alpha}^{\dot{\gamma}} \wedge e_{\alpha \dot{\gamma}} &= 0, \\
    d\omega_{\alpha \beta} + \omega_{\alpha}^{\gamma} \wedge \omega_{\gamma \beta} + \lambda^2 e_{\alpha}^{\dot{\gamma}} \wedge e_{\beta \dot{\gamma}} &= 0
\end{align}
and complex conjugate, where $\lambda$ is the $AdS_4$ cosmological parameter. The following free-level analysis is straightforward (see e.g. [1]) and results in $B^1(Y, Z|x) = C(Y|x)$ that fulfills the twisted adjoint covariant constancy condition (2.17).

As mentioned before, the HS potentials resided in $\omega(Y|x) = W^1(Y, Z|x)_{Z=0}$ obey First on-mass-shell Theorem
\begin{align}
    D_{Q}\omega(y, \bar{y}|x) = -\frac{i\eta}{4} e_{\gamma}^{\dot{\gamma}} \wedge e_{\gamma}\frac{\partial^2}{\partial y^\alpha \partial y^\beta} C(0, \bar{y}|x) - \frac{i\bar{\eta}}{4} e_{\dot{\gamma}}^{\gamma} \wedge e_{\dot{\gamma}}\frac{\partial^2}{\partial \bar{y}^\alpha \partial \bar{y}^\beta} C(y, 0|x).
\end{align}
It is clear that the left-hand side of (3.18) is a closed two-form at $Y_A = 0$. Therefore, the vacuum contribution to $\mathcal{L}^2$ at the free level is
\begin{align}
    \mathcal{L}^2 = -\frac{i}{4} \left( \eta e_{\gamma}^{\dot{\gamma}} \wedge e_{\gamma}\frac{\partial^2}{\partial y^\alpha \partial y^\beta} C(0, \bar{y}|x) + \bar{\eta} e_{\dot{\gamma}}^{\gamma} \wedge e_{\dot{\gamma}}\frac{\partial^2}{\partial \bar{y}^\alpha \partial \bar{y}^\beta} C(y, 0|x) \right)_{Y=0}.
\end{align}
The field $C(y, \bar{y}|x)$ contains matter fields and gauge-invariant information of all free HS fields. The spin-$s$ contribution is captured by the following components
\begin{align}
    \left( y^\alpha \frac{\partial}{\partial y^\alpha} - \bar{y}^{\dot\alpha} \frac{\partial}{\partial \bar{y}^{\dot\alpha}} \right) C(y, \bar{y}|x) = \pm 2s C(y, \bar{y}|x).
\end{align}
In other words, the components $C_{\alpha(2s+m),\dot{\alpha}(m)}$ and $C_{\alpha(m),\dot{\alpha}(m+2s)}$ for any $m$ in
\begin{align}
    C(y, \bar{y}|x) = \sum_{m,n=0}^{\infty} \frac{1}{m!n!} C_{\alpha(m),\dot{\alpha}(n)} (y^{\alpha})^m (\bar{y}^{\dot\alpha})^n
\end{align}
correspond to a spin-$s$ field and all its on-shell derivatives. Particularly, purely (anti)holomorphic components stored in $C(0, 0|x)$ ($C(0, \bar{y}|x)$) are the so-called generalized HS Weyl tensors $C_{\alpha(2s)}$ and $\bar{C}_{\dot{\alpha}(2s)}$. For example, for $s = 2$, $C_{\alpha_1...\alpha_4}$ and $C_{\dot{\alpha}_1...\dot{\alpha}_4}$ are the spinor components of the usual gravity Weyl tensor. Indeed, being totally symmetric the spin-tensors $C_{\alpha_1...\alpha_4}$ and $\bar{C}_{\dot{\alpha}_1...\dot{\alpha}_4}$ are equivalent to totally traceless window-like Young tensor $C_{ab,cd}$ in the vector language. For $s = 1$, one has $C_{\alpha\dot{\alpha}}$ and $\bar{C}_{\dot{\alpha}\alpha}$ being the spinor version of the antisymmetric Maxwell tensor $C_{ab} = -C_{ba}$. For integer $s > 2$, both $C_{\alpha(2s)}$ and $\bar{C}_{\dot{\alpha}(2s)}$ encode totally traceless two-row rectangular Young diagram in the vector language describing the spin-$s$ Weyl tensors. These components are constrained by the generalized Bianchi identities for any $s$ arising from equation of motion (2.17). All the rest mixed components $C_{\alpha(m),\dot{\alpha}(n)}$ can be expressed via the derivatives of HS Weyl tensors through (2.17).

Therefore, the only nonzero contribution to $\mathcal{L}^2$ at the free level comes from the Maxwell tensor $C_{\alpha\beta}$ and $\bar{C}_{\dot{\alpha}\dot{\beta}}$ of the Abelian spin-one field
\begin{align}
    \mathcal{L}^2 = -\frac{i\eta}{4} e_{\gamma}^{\dot{\gamma}} \wedge e_{\gamma}\bar{C}_{\dot{\alpha}\dot{\beta}} - \frac{i\bar{\eta}}{4} e_{\dot{\gamma}}^{\gamma} \wedge e_{\dot{\gamma}}\bar{C}_{\alpha\beta}.
\end{align}

The space-time contribution in (3.21) can be split into local and nonlocal part. Indeed, slicing four-dimensional space time into boundary coordinates $\vec{x}$ and bulk direction $z$ as $x^a =$
(z, x̄), the two-form e ∧ e has the following schematic components e_z ∧ e_− → x being nonlocal and e_− → e_− → x being local from the boundary perspective. More precisely, each contribution can be easily singled out using Poincaré coordinates

\[ ds^2 = \frac{1}{z^2}(4\lambda^2 dz^2 + dx_i dx^i). \]  

A convenient choice for the background fields is

\[ \omega_{\alpha\beta} = \frac{i\lambda}{2z} dx_{\alpha\beta}, \quad \bar{\omega}_{\dot{a}\dot{b}} = -\frac{i\lambda}{2z} dx_{\dot{a}\dot{b}}, \quad e_{\alpha\dot{b}} = \frac{1}{2z}(-dx_{\alpha\beta}\delta_{\dot{a}}^\beta + i\lambda^{-1}\epsilon_{\alpha\beta}dz), \]  

where we have introduced the boundary coordinates x_{\alpha\beta} = x_{\beta\alpha} and the antisymmetric \( \epsilon_{\alpha\dot{a}} = -\epsilon_{\dot{a}\alpha} \) that explicitly breaks Lorentz symmetry \( X_{\alpha\dot{a}} = (iz\epsilon_{\alpha\dot{a}}, x_{\alpha\beta}\delta_{\dot{a}}^\beta) \). Substituting the vierbein from (3.23) into (3.21) one finds

\[ \mathcal{L}^2 = \mathcal{L}_{loc}^2 + \mathcal{L}_{nloc}^2, \]  

where

\[ \mathcal{L}_{loc}^2 = -\frac{i}{16z^2} dx_\gamma^\alpha \wedge dx_\gamma^\beta(\eta\bar{C}_{\alpha\beta} + \bar{\eta}C_{\alpha\beta}), \]  

\[ \mathcal{L}_{nloc}^2 = \frac{\lambda^{-1}}{8z^2} dx^{\alpha\beta} \wedge dz(\eta\bar{C}_{\alpha\beta} - \bar{\eta}C_{\alpha\beta}). \]  

Note that, as pointed out in [12], the nonlocal contribution vanishes in particular in parity invariant A model with \( \eta = 1 \). Indeed, in this case \( \mathcal{L}_{nloc}^2 \) remains invariant under parity transform, while the right-hand side of (3.26) changes its sign, therefore \( \mathcal{L}_{nloc}^2 = 0 \) for A model. A way out to have nonlocal contribution to the vacuum charges in this important case is to keep the phase \( \eta = \exp i\phi \) arbitrary and then define

\[ \mathcal{L}_{nloc}^2 A = \frac{1}{2} \frac{\partial \mathcal{L}^2}{\partial \phi} |_{\phi=0}. \]  

This procedure is to large extent analogous to considering a variation over topological fields hence indicating that the full partition function is likely to be nontrivial even in cases with vanishing vacuum partitions. Note that the effect of this differentiation is the same as of the insertion of the operator \( \hat{n} \) into (3.7).

### 3.3 Topological sector contribution

Let us look at the effect that topological sector of HS equations brings into the partition. The most general bosonic theory is given by (2.21)-(2.22) with some arbitrary complex function

\[ F_*(B) = \eta B + \mu B^\ast B + \ldots, \]  

where \( \eta \) and \( \mu \) are arbitrary complex parameters and we assume

\[ B = B_{ph}(k + \bar{k}) + B_{top}(1 + k\bar{k}), \quad W = W_{ph}(1 + k\bar{k}) + W_{top}(k + \bar{k}). \]  

In this paper we consider theories with essentially nonlinear \( F_*(B) \). As pointed out in [12], though terms nonlinear in \( B \) in (3.28) taken in the conformal frame may lead to boundary
divergencies at conformal infinity, this does not imply any inconsistency of the theory in the bulk. Hence we do not expect it to affect the consideration of this paper. To fully clarify this issue it is necessary to analyze in detail nonlinear corrections to the conserved charge which analysis is beyond the scope of this paper.

The closed two-form as defined in (2.32)

\[ \mathcal{L}^2 = \frac{\partial^2}{\partial k \partial \bar{k}} (W \ast W)_{(y, z = 0)} \]  

depends now not only on physical HS fields \( \omega(y, \bar{y} | x) \) and \( C(y, \bar{y} | x) \) but also on topological modules \( \xi \) stored in \( C^{\text{top}}(y, \bar{y} | x) \). At the free level the topological sector is decoupled and therefore does not contribute to (3.30). Indeed, linearized equations are

\[ dC + \Omega \ast C - C \ast \pi(\Omega) = 0, \]  
\[ dC^{\text{top}} + [\Omega, C^{\text{top}}] \ast = 0. \]

Starting from second order the topological contribution is nontrivial and intertwined with the HS one.

Terms bilinear in \( C \) and \( C^{\text{top}} \) appear in a simple way

\[
\mathcal{L}^2(\xi) = -\frac{i}{4} \left( \mu e_\gamma \hat{\alpha} \wedge e^{\gamma \hat{\beta}} \frac{\partial^2}{\partial y^\alpha \partial \bar{y}^\beta} (C^{\text{top}}(\xi) \ast C + C \ast \pi(C^{\text{top}}(\xi))) + c.c. \right),
\]

coming from the second-order piece in \( F_*(B) \). More precisely, taking \( F_*(B) \) in the form (3.28) and recalling that \( B \sim C \oplus C^{\text{top}} \) one notes the First on-shell theorem (3.18) in this case accumulates the cross \( C \times C^{\text{top}} \) piece proportional to \( \mu \). Eq. (3.33) provides a perturbative contribution to the conserved charge bilinear in \( C, C^{\text{top}} \). It therefore can be viewed as an asymptotic two-form generated by Killing parameters \( \xi \) stored in \( C^{\text{top}} \) for HS theory without topological backreaction. Indeed, what eq. (3.33) says is that for free (equivalently asymptotic) HS equations there is a two-form which is d-closed on free HS equations.

Let us show now how one can reproduce this way an asymptotic charge for gravity analogous to the one of [14]. In the construction of [14] a closed \((d - 2)\)-form was built out of rescaled Weyl tensor and a conformal Killing vector. For that let us choose \( C^{\text{top}} \) in the following form

\[ C^{\text{top}} = \xi_{AB} Y^A Y^B, \]

where \( D_B \xi_{AB} = 0 \) so that \( C^{\text{top}} \) verifies (3.32). Though for gravity case the Weyl module should be restricted to spin \( s = 2 \) sector, let us keep all spins excited to see which of those will contribute to (3.33). Simple star-product calculation of (3.33) with \( C^{\text{top}} \) from (3.34) results in

\[
\mathcal{L}^2(\xi) = \mathcal{L}^2_{s=0}(\xi) + \mathcal{L}^2_{s=2}(\xi),
\]

where

\[
\mathcal{L}^2_{s=0}(\xi) = -i \mu e_\gamma \hat{\alpha} \wedge e^{\gamma \hat{\beta}} \left( \xi_{\hat{\alpha}\hat{\beta}} C(x) - 2i \xi^{\beta \hat{\alpha}} C_{\beta \hat{\alpha}}(x) - \frac{1}{2} \xi^{\beta \hat{\beta}} C^{\hat{\beta}, \hat{\alpha}}(x) \right) + c.c.,
\]
\[
\mathcal{L}^2_{s=2}(\xi) = -\frac{i \mu}{2} e_\gamma \hat{\alpha} \wedge e^{\gamma \hat{\beta}} \xi^{\hat{\alpha} \hat{\beta}} C_{\hat{\alpha}}(4) + c.c.
\]
We see that in this case the two-form picks up only spin-zero and spin-two contributions. Spin one gets cancelled due to twisting in (3.33). There are no HS contribution as well because the generating parameter $\xi_{AB}$ carries only two indices. Had one picked some higher-rank spin tensors $\xi_{A_1...A_n}$ there would be nonzero HS contribution to two-form (3.33). Expression (3.37) reproduces the result of [14] with the difference that the Weyl tensor in (3.37) is free of conformal scaling and a conformal Killing vector is replaced with a Killing parameter $\xi_{\alpha\beta}$. Let us stress that our construction is general allowing to reproduce higher-derivative contribution for higher-rank parameters $\xi$ in a straightforward and uniform fashion. Example of spin $s = 4$ conserved charge arises for parameter $C^{top} = \xi_{A_1...A_6} Y^{A_1} \ldots Y^{A_6}$ and can be straightforwardly calculated using (3.33)

$$\mathcal{L}_{s=4}^2(\xi) = -\frac{i\mu}{2} e^{\gamma^\alpha} \wedge e^{\gamma^\bar{\alpha}} \bar{\xi}^{(6)} \tilde{C}^{(8)} + c.c. \quad (3.38)$$

One comment now is in order. It may look odd that to reproduce charge for a free Fronsdal field one extends minimal HS system by adding topological sector. In fact there are many ways in obtaining asymptotic charges. One of the simplest is given by tracing HS curvature with external parameters like in (3.3), (3.12) which does not require any extension of the HS equations. These external parameters can be made a part of moduli space of the theory being included via topological extension which paves a way to chemical potentials. On a formal side such an extension allows one to generate linearized charges of a minimal theory where $\mu$ in (3.28) plays a role of a coupling-like parameter which however does not have an independent meaning. Namely, its product with the topological field has to be identified with the generalized asymptotic Killing tensor that appears in the usual approach to asymptotic symmetries, which can be defined in the minimal HS theory, in which case only a linear response to the $\mu$-dependent terms is needed. However, beyond the linearized approximation, these terms allow us to make the Killing tensor-dependent two-form closed everywhere which is indeed impossible in the genuine minimal HS model.

Let us stress again that system (2.23)-(2.25) admits the unique totally conserved charge (2.27) rather than an asymptotic one.

Let us now proceed with the calculation of vacuum contribution to conserved charge (2.27). We start with a HS BH solution corresponding to a rotating source in the first nontrivial order.

## 4 Black holes in GR and HS theory

Remarkable property of any 4d Einstein BH including those with nonzero cosmological constant $\Lambda$ is its ‘linearized’ nature. Namely, the BH Weyl tensor satisfies linearized and full nonlinear equations simultaneously. This is manifested by the famous Kerr-Schild decomposition

$$g_{mn} = g_{0mn} + \frac{M}{U} k_m k_n \quad (4.1)$$

where $g_{0mn}$ is the background $(A)dS$ metric, $M$ is the parameter of mass, $U$ is some function and $k_m$ is the Kerr-Schild vector that brings Einstein equations to the linearized Fierz-Pauli.

At the level of curvature tensor such BHs and their Petrov D-type analogs as well as the HS generalization originate from a single $(A)dS$ global symmetry parameter [15]. To recall the
basic construction consider \(d\)-dimensional \((A)dS\) space described by the structure equations

\[
dw_{ab} + w_a^c \wedge w_{cb} = \Lambda e_a \wedge e_b, \tag{4.2}
\]
\[
De_a = de_a + w_a^b \wedge e_b = 0, \tag{4.3}
\]

where \(w_{ab} = -w_{ba}\) and \(e_a\) are one-forms of the Lorentz connection and the vielbein, respectively. A nice feature of Cartan form (4.2)-(4.3) for \((A)\) geometry is its explicit local gauge invariance

\[
\delta w_{ab} = D\kappa_{ab} + \Lambda(v_a e_b - v_b e_a), \quad \delta e_a = Dv_a - \kappa_{ab} e^b, \tag{4.4}
\]

with arbitrary zero-forms \(\kappa_{ab} = -\kappa_{ba}\) and \(v_a\), allowing one to fix \((A)dS\) global symmetries requiring \(\delta w_{ab} = \delta e_a = 0\),

\[
Dv_a = \kappa_{ab} e^b, \tag{4.5}
\]
\[
D\kappa_{ab} = -\Lambda(v_a e_b - v_b e_a). \tag{4.6}
\]

Here \(v_a\) is just a Killing vector since from (4.5) it follows that \(D_a v_b + D_b v_a = 0\), while \(\kappa_{ab} = -\kappa_{ba}\) is its covariant derivative. \(\kappa_{ab}\) and \(v_a\) can be packed into \((d + 1) \times (d + 1)\) - dimensional matrix, forming an element of \(o(d-1,2)\) for \(\Lambda < 0\) or \(o(d,1)\) for \(\Lambda > 0\). This matrix will be called the \((A)dS\) global symmetry parameter. It turns out that system (4.5)-(4.6) generates exact solutions to Einstein and HS equations in terms of \(\kappa_{ab}\) and \(v_a\) fields including all GR BHs. Now we consider the case of \(AdS_4\) in more detail following [15].

### 4.1 Four dimensions

For \(AdS_4\), the well-known isomorphism \(o(3,2) \sim sp(4)\) allows us to use the two-component spinor language that facilitates analysis a lot and suits the HS needs. Introducing \(\kappa_{a\dot{\beta}} = \kappa_{\dot{\beta}a}\), \(\kappa_{\dot{a}\dot{\beta}} = \kappa_{\dot{\beta}\dot{a}}\) and \(v_{a\dot{a}}\) as spinor counterparts for \(\kappa_{ab}\) and \(v_a\) fields, the spinor analog of (4.5)-(4.6) reads

\[
\begin{align*}
D^L \kappa_{a\dot{\beta}} &= -\lambda^2(e_{a\dot{\gamma}} \wedge v_{{\dot{\beta}}\dot{\gamma}} + e_{{\dot{\beta}}\dot{\gamma}} \wedge v_{a\dot{\gamma}}), \tag{4.7} \\
D^L v_{a\dot{a}} &= -e^\gamma_{\dot{a}} \kappa_{{\gamma}a} - e_{a\dot{\gamma}} \kappa_{\dot{\gamma}a}, \tag{4.8}
\end{align*}
\]

where \(\lambda = -4\lambda^2\) and \(D^L A_a = dA_a + \omega_\alpha^\beta A_{a\dot{\alpha}} + \bar{\omega}_{\dot{\alpha}}^\beta A_a\dot{\beta}\). Introducing then

\[
\Omega_{AB} = \begin{pmatrix} \omega_{a\beta} & \lambda e_{a\dot{\beta}} \\ \lambda e_{\dot{a}\beta} & \bar{\omega}_{\dot{a}\dot{\beta}} \end{pmatrix}, \quad K_{AB} = \begin{pmatrix} \kappa_{a\dot{\beta}} & \lambda v_{a\dot{\beta}} \\ \lambda v_{\dot{a}\dot{\beta}} & \bar{\kappa}_{\dot{a}\dot{\beta}} \end{pmatrix} \tag{4.9}
\]

\(A, B = 1, \ldots, 4\) one rewrites (3.16)-(3.17) and (4.7)-(4.8) as

\[
\begin{align*}
d\Omega_{AB} + \Omega_A^C \wedge \Omega_{CB} &= 0, \tag{4.10} \\
dK_{AB} + \Omega_A^C K_{CB} + \Omega_B^C K_{AC} &= 0. \tag{4.11}
\end{align*}
\]

Equation (4.10) is the \(AdS_4\) flatness condition and \(K_{AB}\) that satisfies (4.11) is an \(AdS_4\) global symmetry parameter. In accordance with the fact that \(\text{rank}(sp(4)) = 2\) there are two \(sp(4)\) invariants

\[
\begin{align*}
C_2 &= Tr K^2, \quad C_4 = 4 Tr K^4. \tag{4.12}
\end{align*}
\]
An important observation [36] is that (4.7)-(4.8) generates a tower of solutions for free massless equations of all spins \( s = 0, 1, 2, \ldots \), with the following generalized HS Weyl tensors

\[
C_{\alpha (2s)} = \frac{m_s \lambda^{-2s}(\kappa_{\alpha\alpha})^{s}}{2^{s}s!q^{2s+1}}, \quad \bar{C}_{\dot{\alpha}(2s)} = \frac{\bar{m}_s \lambda^{-2s}(\bar{\kappa}_{\dot{\alpha}\dot{\alpha}})^{s}}{2^{s}s!\bar{q}^{2s+1}},
\]

where

\[
q = \frac{1}{2\lambda^2} \sqrt{-\kappa_{\alpha\beta}\kappa^{\alpha\beta}/2}, \quad \bar{q} = \frac{1}{2\lambda^2} \sqrt{-\bar{\kappa}_{\dot{\alpha}\dot{\beta}}\bar{\kappa}^{\dot{\alpha}\dot{\beta}}/2}
\]

(4.14) and \( m_s \) are arbitrary constants.

To show how solutions (4.13) can be reproduced from (2.17), it is useful to rewrite equation (4.11) in terms of star product as

\[
D_{\Omega}(K_{AB}(x)Y^{A}Y^{B}) = 0.
\]

(4.15)

Since \( D_{\Omega} \) is a first-order differential operator both in \( x \) and in \( Y \), any function of \( \xi = K_{AB}(x)Y^{A}Y^{B} \) enjoys covariant constancy condition as well

\[
D_{\Omega}f(\xi) = 0.
\]

(4.16)

The solution to twisted-adjoint covariant condition (2.17) can be obtained by Fourier transform

\[
C(y, \bar{y}|x) = f(\xi) * 2\pi\delta^2(y).
\]

(4.17)

Generally, (4.17) does not meet the reality condition, still it enables one to extract HS Weyl tensors that do satisfy proper reality condition as

\[
C(y, 0|x) = (f(\xi) * 2\pi\delta^2(y))_{y=0}, \quad C(0, \bar{y}|x) = (f(\xi) * 2\pi\delta^2(\bar{y}))_{\bar{y}=0}.
\]

(4.18)

Straightforward Gaussian integration

\[
F(y, \bar{y}) * 2\pi\delta^2(y) = \int d^2 u F(u, \bar{y})e^{-iu\alpha y^\alpha}
\]

(4.19)

gives (4.13), where particular values of \( m_s \) depend on \( f(\xi) \).

The solutions resulting from the construction are of generalized Petrov type D in a sense that all the fields \( s \geq 1 \) are made of two principal spinors which are the null directions of \( \kappa_{\alpha\beta} \).

The class of inequivalent solutions is represented by the conjugacy classes with respect to \( Sp(4) \) adjoint action

\[
K_{AB} \sim (U^{-1}KU)_{AB},
\]

(4.20)

where \( U_{A}^{B} \in Sp(4) \). Particularly, different values of (4.12) correspond to different \( Sp(4) \) orbits. Note, however, that by normalizing \( K_{AB} \) one can always set one of the invariants, say \( C_2 \) to \( \pm 1 \) or 0. There are nine conjugacy classes of \( o(3,2) \) classified in e.g. [38]. Finally, although not obvious (see [15]) for \( s = 2 \) a class of thus obtained solutions is covered by Carter-Plebanski metric which captures all type-D solutions of General Relativity including all BHs except for accelerated metrics. We can therefore refer to HS generalization as to HS Carter-Plebanski solutions. An interesting parallelism can be drawn with the BTZ BH, where its type is driven by a conjugacy class of \( o(2,2) \) parameter [39]. The analogy is not direct though, since \( d = 4 \) solutions in question are not topological.
4.2 Examples

To be more specific let us review some examples of physical significance including the most symmetric BHs, the Kerr BH and the Carter-Plebanski solution.

4.2.1 Schwarzschild, planar and hyperbolic

The examples below fall out from the general scheme of [15]. These were not analyzed explicitly in [15], so we do it here. Consider the \(AdS_4\) metric in the following coordinates

\[
ds^2 = -f_\epsilon dt^2 + f_\epsilon^{-1} dr^2 - \Lambda r^2 d\Sigma_\epsilon^2 ,
\]

where \(\epsilon = \pm 1, 0\),

\[
f_\epsilon = \epsilon - r^2 \Lambda
\]

and

\[
d\Sigma_\epsilon^2 = \begin{cases} -\Lambda^{-1}(d\theta^2 + \sin^2 \theta d\phi^2) & \text{for } \epsilon = 1 \\
 dx^2 + dy^2 & \text{for } \epsilon = 0 \\
 -\Lambda^{-1}(d\psi^2 + \sinh^2 \psi d\theta^2) & \text{for } \epsilon = -1 \end{cases}.
\]

Recall, that \(\Lambda = -4\lambda^2\). Each value of \(\epsilon\) is designed to reproduce the Schwarzschild (\(\epsilon = 1\)), the planar (\(\epsilon = 0\)) and the hyperbolic (\(\epsilon = -1\)) BHs. The hyperbolic solution is often referred to as topological due to possibility for quotienting over a discrete subgroup of the hyperbolic horizon to yield arbitrary genus Riemann surfaces [30].

In each case we take the following Killing vector

\[
v^m = (1, 0, 0, 0) = \frac{\partial}{\partial t}
\]

and the vierbein

\[
e^0 = f_\epsilon^{1/2} dt , \quad e^1 = f_\epsilon^{-1/2} dr , \quad e^2 = \begin{cases} \sqrt{-\Lambda^{-1}} d\theta & \epsilon = 1 \\
 dx & \epsilon = 0 \\
 \sqrt{-\Lambda^{-1}} d\psi & \epsilon = -1 \end{cases} , \quad e^3 = \begin{cases} \sqrt{-\Lambda^{-1}} \sin \theta d\phi & \epsilon = 1 \\
 dy & \epsilon = 0 \\
 \sqrt{-\Lambda^{-1}} \sinh \psi d\theta & \epsilon = -1 \end{cases}.
\]

which results in the following spinorial form for global symmetry parameter components

\[
\kappa_{\alpha\beta} = 2\lambda^2 r \begin{pmatrix} 1 & 0 \\
 0 & -1 \end{pmatrix} , \quad v_{\alpha\beta} = -f_\epsilon^{1/2} \begin{pmatrix} 1 & 0 \\
 0 & 1 \end{pmatrix}.
\]

The corresponding \(K_{AB}\) from (4.9) has a distinguishing \(Sp(4)\) invariant property

\[
K_A^C K_C^B = -\lambda^2 \epsilon \delta_A^B
\]

that along with the choice of the Killing vector (4.24) singles out the most symmetric BH solutions. The eigenvalues are

\[
K_A^B \xi_B = \tilde{\lambda} \xi_A , \quad \tilde{\lambda} = \pm i \lambda \sqrt{\epsilon}.
\]

The Weyl tensors are given by (4.13) with

\[
q_\epsilon = \bar{q}_\epsilon = r .
\]
Let us note that solutions of invariant condition (4.27) correspond to most symmetric GR BHs. In other words, should $K_{AB}$ be more generic, so that (4.27) no longer holds the resulting BHs would be less symmetric. Clearly (4.27) respects the $Sp(4)$ conjugacy transformation (4.20). Still it provides more than one inequivalent global symmetry parameter (4.21). Expression (4.24) corresponds to a Killing vector which is time-like, space-like or null and fixes algebraical type of $K_{AB}$ unambiguously. Let us also note that the centralizer of $K_{AB}$ in $sp(4)$ generates all isometries of the corresponding BH solution.

4.2.2 Kerr

To single out the Kerr case we take the $AdS_4$ Boyer-Lindquist metric suitably adopted to account for rotation

$$ds^2 = -\frac{\Delta_r}{\rho^2}(dt - \frac{a}{\Xi} \sin^2 \theta d\phi)^2 + \frac{\rho^2}{\Delta_r} dr^2 + \frac{\rho^2}{\Delta_\theta} d\theta^2 + \frac{\sin^2 \theta \Delta_\theta}{\rho^2} (adt - \frac{r^2 + a^2}{\Xi} d\phi)^2, \quad (4.30)$$

where

$$\rho^2 = r^2 + a^2 \cos^2 \theta, \quad \Delta_r = (r^2 + a^2)(1 - \Lambda r^2), \quad \Delta_\theta = 1 + \Lambda a^2 \cos^2 \theta, \quad \Xi = 1 + \Lambda a^2. \quad (4.31-4.34)$$

A convenient form of the vierbein field reads

$$e^0 = \frac{\sqrt{\Delta_r}}{\rho} (dt - \frac{a \sin^2 \theta}{\Xi} d\phi), \quad e^1 = \frac{\rho}{\sqrt{\Delta_r}} dr, \quad (4.35)$$

$$e^2 = \frac{\rho}{\sqrt{\Delta_\theta}} d\theta, \quad e^3 = \frac{\sqrt{\Delta_\theta} \sin \theta}{\rho} (adt - \frac{r^2 + a^2}{\Xi} d\phi). \quad (4.36)$$

Taking the following Killing vector

$$v^m = \frac{\partial}{\partial t} = (1, 0, 0, 0) \quad (4.37)$$

one finds the $sp(4)$ global symmetry parameter components to be

$$\gamma_{\alpha \beta} = 2 \lambda^2 (r - ia \cos \theta) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad v_{\alpha \beta} = \frac{1}{\rho} \begin{pmatrix} -\sqrt{\Delta_r} + a \sqrt{\Delta_\theta} \sin \theta & 0 \\ 0 & -\sqrt{\Delta_r} - a \sqrt{\Delta_\theta} \sin \theta \end{pmatrix}. \quad (4.38)$$

Note now that Killing vector (4.37) is time-like in $AdS$ case and not sign-definite in $dS$. Indeed, its norm is given by

$$v_t \cdot v_t = -1 + \Lambda (r^2 + a^2 \sin^2 \theta), \quad (4.39)$$

which is sign-definite at $\Lambda \leq 0$. The $sp(4)$ invariants (4.12) and the eigenvalues (4.28) are

$$C_2 = \Lambda (1 - a^2 \Lambda), \quad C_4 - C_2^2 = -4a^2 \Lambda^3, \quad (4.40)$$

$$\tilde{\lambda}_{1,2} = \pm i (1 + 2a \lambda), \quad \tilde{\lambda}_{3,4} = \pm i (1 - 2a \lambda). \quad (4.41)$$
Again, for \( s = 2 \) we reproduce Kerr BH Weyl tensor in (4.13) with
\[
q = r - ia \cos \theta, \quad \bar{q} = r + ia \cos \theta.
\]
(4.42)

In the Kerr case eq. (4.27) no longer holds and the solution in \( AdS \) is fixed by generic time-like symmetry parameter (4.37). The global symmetry invariants (4.40) are expressed in terms of the rotation parameter \( a \).

### 4.2.3 Carter-Plebanski

To describe the most general case that captures all above cases as different limits it is convenient to introduce two-parameter \( AdS_4 \) metric in the Carter-Plebanski form, [15]

\[
ds^2 = -\frac{\Delta_r}{r^2 + y^2}(d\tau + y^2 d\psi)^2 + \frac{\Delta_y}{r^2 + y^2}(d\tau - r^2 d\psi)^2 + \frac{r^2 + y^2}{\Delta_r} dr^2 + \frac{r^2 + y^2}{\Delta_y} dy^2,
\]
where
\[
\Delta_r = r^2(-\Lambda r^2 + \epsilon) + a^2, \quad \Delta_y = y^2(-\Lambda y^2 - \epsilon) + a^2.
\]
(4.44)

From that metric the Carter-Plebanski solution is generated via \( \frac{\partial}{\partial \tau} \) Killing vector for which the norm equals
\[
v_{\tau} \cdot v_{\tau} = -\epsilon + \Lambda (r^2 - y^2).
\]
(4.45)

Choosing the vierbein as
\[
e^0 = \sqrt{\frac{\Delta_r}{r^2 + y^2}}(d\tau + y^2 d\psi), \quad e^1 = \sqrt{\frac{\Delta_y}{r^2 + y^2}}(d\tau - r^2 d\psi), \quad e^2 = \sqrt{\frac{r^2 + y^2}{\Delta_r}} dr, \quad e^3 = \sqrt{\frac{r^2 + y^2}{\Delta_y}} dy
\]
(4.46)

we find the generating \( AdS_4 \) global symmetry parameter in the form
\[
\kappa_{\alpha \beta} = 2\lambda^2 \begin{pmatrix} y - ir & 0 \\ 0 & y - ir \end{pmatrix}, \quad v_{\alpha \beta} = \frac{1}{\sqrt{r^2 + y^2}} \begin{pmatrix} \sqrt{\Delta_r} & \sqrt{\Delta_y} \\ \sqrt{\Delta_y} & \sqrt{\Delta_r} \end{pmatrix}
\]
(4.47)

that corresponds to
\[
C_2 = \Lambda \epsilon, \quad C_4 - C_2^2 = -4a^2\Lambda^2,
\]
\[
\tilde{\lambda}_{1,2} = \pm i\lambda\sqrt{\epsilon + 4a\Lambda}, \quad \tilde{\lambda}_{3,4} = \pm i\lambda\sqrt{\epsilon - 4a\lambda}.
\]
(4.48)

and
\[
q = y - ir, \quad \bar{q} = y + ir.
\]
(4.50)

Note that Kerr solution is reproduced at \( \epsilon = 1 - \Lambda a^2 \).

### 4.3 Charges of Kerr BH

Now we are in a position to obtain explicit expressions for conserved charges (2.27) of Kerr-like HS BH.

First, let us consider a vacuum contribution at the free level. It is generated by (3.21) for Kerr-type Maxwell tensor (4.13)
\[
C_{\alpha \beta} = \frac{m_1 \kappa_{\alpha \beta}}{2\lambda^2 q^3},
\]
(4.51)
where $\kappa_{\alpha\beta}$ and $q$ were calculated in (4.38) and (4.42). Straightforward calculation puts (3.21) in the following simple form

$$L^2 = -i\left(\frac{m_1 \bar{\eta}}{q^2} + \frac{\bar{m}_1 \eta}{q^2}\right)e^0 \wedge e^1 + \left(\frac{m_1 \bar{\eta}}{q^2} - \frac{\bar{m}_1 \eta}{q^2}\right)e^3 \wedge e^2.$$ (4.52)

Corresponding charge (2.27) can be easily evaluated via integration of (4.52) around $t = \text{const}$, $r \to \infty$ and is eventually given by

$$Q = 4\pi \frac{m_1 \bar{\eta} - \bar{m}_1 \eta}{1 + \Lambda a^2}.$$ (4.53)

As anticipated, the so obtained vacuum charge is zero for $\eta = \bar{\eta}$ for the Kerr-case with $m_1 = \bar{m}_1$. However it is non-zero for parity violating theories. Let us also note, that in this case $Q$ agrees up to a numerical normalization with the standard $s = 1$ charge for AdS-Kerr-Newman BH, see e.g. [41]. In the parity preserving case one uses (3.27) to reproduce the charge.

Next we compute the lowest nontrivial topological contribution to the spin-2 charge, provided by (3.37). To this end we identify Killing parameter $\xi_{\alpha\beta}$ in (3.37) with $\kappa_{\alpha\beta}$ from (4.38) and substitute there $s = 2$ Weyl tensor (4.13) built of the same $\kappa_{\alpha\beta}$

$$C_{\alpha\beta\gamma\delta} = \frac{m_2}{8\lambda^2 q^3} \kappa_{(\alpha\beta} \kappa_{\gamma\delta)}.$$ (4.54)

This yields

$$L^2_{s=2} = 3i\lambda^2 \left(\frac{m_2 \bar{\mu}}{q^2} + \frac{\bar{m}_2 \mu}{q^2}\right)e^0 \wedge e^1 - 3\lambda^2 \left(\frac{m_2 \bar{\mu}}{q^2} - \frac{\bar{m}_2 \mu}{q^2}\right)e^3 \wedge e^2.$$ (4.55)

Once again, to obtain a charge we integrate this around $t = \text{const}$, $r \to \infty$, arriving at

$$Q_{s=2} = 3\pi \Lambda \frac{m_2 \bar{\mu} - \bar{m}_2 \mu}{1 + \Lambda a^2}.$$ (4.56)

Now we can adjust the parameters $\mu$ and $\bar{\mu}$ to reproduce the proper BH charge.

5 Conclusion

In this paper we conjecture that the partition associated with solutions of HS theory is determined by the closed two-form associated with the full HS theory that contains the topological fields which from the thermodynamical perspective represent chemical potentials conjugated to various higher-spin and lower-spin charges. This construction not only properly reproduces the asymptotic charges but also allows a proper deformation into the bulk. Specifically, in this paper we compute the vacuum contributions to the partition at zero values of the chemical potentials at free level and also the first order contribution of the chemical potentials which allows us to extract at lowest order the BH conserved charge for the proper choice of moduli parameters.

It is demonstrated that the invariant functional two-form defined on-shell provides nontrivial vacuum partition. The reason why it is not trivial is cohomological. Whenever solutions are globally well-defined the corresponding functional is trivial. However, in case of global obstruction it may be non-zero. That this is indeed the case was illustrated with an example of...
the HS analog of $AdS_4$ Kerr BH at the linearized level. The corresponding global charge turned out to be nonzero and agreed with the asymptotic ADM behaviour. Let us stress that those solutions that are not everywhere well-defined are often of most physical interest including BHs, Coulomb potential, Dirac string and its gravity cousin, the NUT solution.

We have also analyzed leading contribution of the chemical potentials to asymptotic charge which are the parameters stored in the topological sector of the theory. Their effect can be easily captured to the first order in physical fields and gives a simple formula that reproduces known result [14] for the case of gravity. Yet the obtained result is uniformly applicable to the asymptotic charge of any spin associated with an arbitrary rank parameter. Therefore we propose an efficient tool for extracting HS asymptotic charges. It would be interesting to study how the proposed approach can be applied to other Lagrangian HS systems like those of [7].

We believe that the proposed approach will make it possible not only evaluating nontrivial charges for the HS BH solutions of [37], [42] starting from the linear response to the various chemical potentials but eventually can shed light on the deep issue of the BH information paradox. The most important ingredient of the developed formalism is that the charges are independent of the integration cycle and hence may be integrated equally well both at infinity and at the horizon. Particularly this paves a way to HS analogs of thermodynamical Smarr formulas. Another possibly relevant issue is that the physical interpretation of various fields in the theory such as dynamical and topological fields may depend on the behaviour of the vacuum solution that can change the interpretation of the fields in different regions of space-time. All this makes HS theory extremely interesting for the study of BH physics. We hope to discuss these intriguing issues in the future.

As a byproduct of our construction we obtained an explicit formula (3.13) relating asymptotic charges expressed in terms of the generalized HS Weyl tensor with those expressed in terms of Fronsdal fields.

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