Rogue waves in nonlinear science

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Abstract. Rogue waves, as a special type of solitary waves, play an important role in nonlinear optics, Bose-Einstein condensates, ocean, atmosphere, and even finance. In this report, we mainly review on the history of the rogue wave phenomenon and recent development of rogue wave solutions in some nonlinear physical models arising in the fields of nonlinear science.

Introduction. Rogue waves (RWs), similar to solitary waves, have no universal definition, and display the basic feature that RWs ‘appear from nowhere and disappear without a trace’. In general, the common criteria is available for RWs in the ocean, i.e., the height of a RW (vertical distance from trough to crest) is two or more times greater than the significant wave height (the average wave height among one third of the highest waves in a time series, e.g., usually of length \(10 - 30\) min) \([1-6]\). The term RW (or freak wave) was first introduced in the scientific community due to Draper in 1964 \([7]\). RWs are also known as freak waves, monster waves, killer waves, abnormal waves, steep waves, giant waves, or extreme waves. More recently, the RW (or freak wave) was also coined ‘rogue’ (or ‘freakon’) if they reappear virtually unaffected in size or shape shortly after their interactions \([8]\).

RWs in the ocean have led to many marine misfortunes. For example, the oceanic RWs have made more than 22 supercarriers lose between 1969 and 1994 \([1-5]\). The New Year’s wave or Draupner wave was regarded as the first rogue wave recorded by scientific measurement in North Sea (see Fig. 1(a)(b)). RWs have been paid much attention in order to understand better their physical mechanisms in the past decades \([1-5]\). There exist many reasons generating RWs in the ocean such as nonlinear wave modulation, the diffractive focusing, the wave refraction, the wind waves, and etc. (see e.g. \([1,9]\)).

Except for the oceanic RWs, the RWs have found to exist other fields such as nonlinear optics \([8,10,11]\), Bose-Einstein condensates \([12,13]\), atmospherics \([14]\), and even finance \([15,16]\). In particular, Solli et al. \([10]\) first observed the optical rogue waves in an optical fibre and found that they could be used to stimulate supercontinuum generation (see Fig. 1(c)(d)).

Figure 1. (color online). (a) The Draupner S and E oil platform (front) are located in the North Sea, midway between Norway and Scotland (Photo: Statoil); (b) A rogue wave recorded at Draupner oil platform on Jan. 1, 1995; (c) Schematic of experimental apparatus for optical rogue waves; (d) Time-wavelength profile of an optical rogue wave \([10]\).

Single rogue wave. Many rogue wave phenomena have been described by using the nonlinear Schrödinger equation and its extensions or deformations (see, e.g. \([1,8,10,12–17,24,29]\)).
basic solution (rogue waves or rogons) was first presented by Peregrine [18] to describe the rogue wave phenomenon, which was known as by Peregrine soliton (or breather), i.e., the one-dimensional forcussing or attractive NLS equation
\[ i\psi_t = -(1/2)\psi_{xx} - |\psi|^2\psi, \]
possessed the rogue wave solution (see Fig. 2(a))
\[ \psi(x, t) = [1 - 4(1 + 2it)/(1 + 4x^2 + 4t^2)]e^{it} \]
which was obtained from the limit case of Ma soliton [19] which is periodic in time. In fact, the Peregrine soliton can be also found by taking the limit case of Akhmediev soliton [20] which is periodic in space (see Ref. [21]).

N-rogon solutions —. The NLS equation (1) is a condition of compatibility of the two following linear equations (i.e., Lax pair) [23]:
\[ \Phi_x = U\Phi, \quad \Phi_t = V\Phi, \quad U = \begin{bmatrix} i\lambda & i\psi^* \\ i\psi & -i\lambda \end{bmatrix}, \quad V = \begin{bmatrix} i\lambda^2 - (i/2)|\psi|^2 & i\lambda\psi^* + (1/2)\psi_x^* \\ i\lambda\psi^* - (1/2)\psi_x & -i\lambda^2 + (i/2)|\psi|^2 \end{bmatrix}, \]
where \( \Phi = (\phi_1(x, t), \phi_2(x, t))^T \), and \( \lambda \) being a complex eigenvalue. The \( N \)-soliton solutions have been constructed via the inverse scattering transformation [22] and Darboux transformation [23].

Recently, based on the Lax pair (3), the \( N \)-rogon solutions were also constructed via the deformed Darboux transformation approach [24, 25], in which the first one is just the Peregrine soliton (2) and the second-order rogon solution is given by (see Fig. 2(b))
\[ \psi(x, t) = [1 - (G + itH)/D]e^{it}, \]
where \( G = (x^2 + t^2 + 3/4)(x^2 + 5t^2 + 3/4) - 3/4, \quad H = (2(x^2 + t^2)^2 - 3x^2 + t^2 - 15/8, \quad \text{and} \quad D = (1/3)(x^2 + t^2)^3 + (1/4)(x^2 - 3t^2)^2 + (3/64)(12x^2 + 44t^2 + 1). \)

**Figure 2.** (color online). (a) First-order rogue wave; (b) Second-order rogue wave [25].

Moreover, the NLS equation with perturbing terms (i.e, three-order NLS equation)
\[ i\psi_t = -(1/2)\psi_{xx} - |\psi|^2\psi - is\left[\alpha\psi(|\psi|^2)x + \beta(|\psi|^2)x + \gamma|x^2| \right] \]
and its special case (i.e., Hirota equation) were also shown to admit exact one- and two-rogue wave solutions [26], where \( s, \alpha, \beta, \) and \( \gamma \) are real constants.

*Nonautonomous rogon solutions —. Recently, the rogue wave (rogon) solutions of the NLS equation have been extended to nonlinear wave equations with variable coefficients, in which the rogue-like wave (rogon-like) solutions were called nonautonomous rogue wave (rogon) solutions. The NLS equation with variable coefficients
\[ i\psi_t = -(1/2)\beta(t)\psi_{xx} - v(x, t)\psi - g(t)|\psi|^2\psi + i\gamma(t)\psi, \]
has been shown to admit multi-rogon solutions in which the first two examples are listed below:
\[ \psi_1(x, t) = a_0\sqrt{|\alpha(t)|} \left\{ 1 - 4 + 8ir(t) \right\} / \left[ 1 + 2(\alpha(t)x + \delta(t))^2 + 4r^2(t) \right] e^{i|x(t) + r(t)| + f_0^s \gamma(s)ds}, \]
\[ \psi_2(x, t) = a_0\sqrt{|\alpha(t)|} \left\{ 1 + P_2(\eta, \tau) + iQ_2(\eta, \tau) / A_2(\eta, \tau) \right\} e^{i|x(t) + r(t)| + f_0^s \gamma(s)ds}, \]
where \( \eta(x,t) = \alpha(t)x + \delta(t) \), \( \chi(x,t) = \frac{1}{2\alpha(t)} \), \( \psi(x,t) = \chi(t) + (1/2)\beta(t)\chi^2 \), \( \tau(t) = (1/2)\int_0^t \alpha^2(s)\beta(s)ds \), \( g(t) = -\alpha(t)\beta(t)/(2\alpha^2(t)\beta(t)) \), and \( P_2 = -(1/2)\eta^4 - 6\eta^2 + 10\tau^4 - (3/2)\eta^2 - 9\tau^4 + 3/8 \). \( Q_2 = -\tau(\eta^4 + 4\eta^2 + 4\tau^4 - 3\eta^2 + 2\tau^4 - 15/4) \).

The matter rogue waves of the one-dimensional generalized Gross-Pitaevskii (GP) equation with dissipative term

\[
i \psi_t = -\psi_{xx} + \sigma|\psi|^2 \psi - ig|\psi|^4 \psi,
\]

where \( \sigma = \text{arg}(a_s) \), \( \alpha \) is a parameter of three-body interaction, and \( g \) is a parameter of three-body interaction, were realized by using the numerical simulation [12]. Moreover, exact three-dimensional nonautonomous rogue waves and their interaction were also found for the three-dimensional GP equation with varying coefficients [13]

\[
i \Psi_t = -(1/2)\nabla^2 \Psi + v(\mathbf{r},t)\Psi + g(t)|\Psi|^2 \Psi + i\gamma(t)\Psi,
\]

where the physical field \( \Psi \equiv \Psi(\mathbf{r},t) \), \( \mathbf{r} \in \mathbb{R}^3 \), \( \nabla \equiv (\partial_x, \partial_y, \partial_z) \) with \( \partial_\xi \equiv \partial/\partial_x \), the external potential \( v(\mathbf{r},t) \) is a real-valued function of time and spatial coordinates, the nonlinear coefficient \( g(t) \) and gain/loss coefficient \( \gamma(t) \) are real-valued functions of time.

**Rogue waves in atmosphere**—. When the scalar nonlinearity was considered, the slowly varying amplitude \( A(x,t) \) of the linearized pressure perturbation in atmosphere can be described by the NLS equation: \( 2iKA \chi + fA_{\tau\tau} - 2KA^2n_1/n_0|A|^2A = 0 \), where \( \tau = t - x/v_0 \) with \( v_0 \) being the linear group velocity, \( K = \omega_0/c \) denotes the wavenumber, \( f = K/v_0\omega_0 \) is the group velocity dispersion, and \( n_0 \) is the linear refractive. This model was also shown to possess multi-rogon solutions [14].

**Financial rogue waves**—. Recently, Ivancevic proposed a novel nonlinear option pricing model (called the Ivancevic option pricing model) [27]

\[
i \psi_t(S,t) = -(1/2)\sigma \psi_{SS}(S,t) - \beta|\psi(S,t)|^2 \psi(S,t),
\]

in order to satisfy efficient and behavioral markets, and their essential nonlinear complexity, where \( \psi(S,t) \) denotes the option-price wave function, the dispersion frequency coefficient \( \sigma \) is the volatility (which can be either a constant or stochastic process itself), the Landau coefficient \( \beta = \beta(r,w) \) represents the adaptive market potential. The nonlinear model can be regarded as the nonlinear extension of the linear Black-Schole model [28].

The financial rogue wave solutions and their interactions are constructed for the nonlinear financial model (11) with the constant parameters \( \sigma \) and \( \beta \) [15]. Moreover the vector financial rogue wave solutions are also presented for the coupled financial models [16]. These financial rogue waves may further excite the possibility of relative researches and potential applications for the financial rogue wave phenomenon in the financial markets and related fields (e.g., 1997 Asian financial crisis/storm and the current global financial crisis/storm).

**Discrete rogue waves**—. Recently, the integrable discrete Hirota lattice

\[
i \psi_{n,t} = -[(a + ib)\psi_{n+1} + (a - ib)\psi_{n-1}](1 + |\psi_n|^2) + 2\psi_n \quad (a, b \in \mathbb{R})
\]
has been shown to possess discrete one- and two-RW solutions based on the limit cases of its multi-soliton solutions [29]. Eq. (12) with \((a, b) = (1, 0)\) or \((0, 1)\) reduces to the Ablowitz-Ladik or discrete mKdV lattice, respectively. More recently, we presented nonautonomous discrete multi-soliton solutions [28]. Eq. (12) with \((a, b)\) has been shown to possess discrete one- and two-RW solutions based on the limit cases of its multi-soliton solutions [29].

In addition, the rogue waves of the non-integrable discrete NLS lattice: \[i\psi_{n+1} + \psi_{n-1} - 2\psi_n + \sigma|\psi_n|^2\psi_n = 0\] have been considered by using the numerical simulations [31].

**Conclusions**— In brief, we have reviewed on rogue wave solutions in some nonlinear physical models which contains the continuous and discrete cases with constant coefficients and variable coefficients. These methods may also be extended to other nonlinear physical models. Due to the limit of the paper length, we impossibly list all other documents about rogue wave phenomenon (see e.g. Refs. [32–34]).

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