THE MODULI SPACE OF 2-DIMENSIONAL ASSOCIATIVE ALGEBRAS

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ABSTRACT. In this paper, we study moduli spaces of 2-dimensional complex associative algebras. We give a complete calculation of the cohomology of every element in the moduli space, as well as compute their versal deformations.

1. Introduction

A constructive approach to the computation of versal deformations of a large class of algebraic algebras, including infinity algebras, was developed in [3], and a procedure for computing examples was applied in [4] to construct some new examples of infinity algebras, as well as to construct their miniversal deformations. Even though infinity algebras have recently emerged as an important concept in the area of mathematical physics called string theory, few finite dimensional examples of such algebras have been studied, and their versal deformations have not been constructed. The examples in this paper are the first step in the study of moduli spaces of low dimensional \(A_\infty\) algebras.

In [6, 17], the moduli space of 3-dimensional Lie algebras was studied, and the miniversal deformations of the Lie algebras played an important role in understanding how the moduli space was glued together. In [7], the moduli space of 4-dimensional Lie algebras was studied in the same manner. The new perspective, using miniversal deformations to study the moduli spaces of Lie algebras, led to a more complete picture of these moduli spaces. The intention in this paper is to undertake a similar study for 2-dimensional associative algebras.

One method of constructing the moduli space of ordinary associative algebras in dimension 2 is to consider extensions of a 1-dimensional associative algebra by a 1-dimensional associative algebra. This is possible because there are no simple 2-dimensional associative algebras, by a theorem of Wedderburn, so all such algebras have an ideal, and therefore arise as extensions.

In this paper, we will give a complete description of the moduli space of 2-dimensional complex associative algebras, including a computation of a miniversal deformation of each of these algebras. From the miniversal deformations, a decomposition of the moduli space into strata is obtained, with the only connections between strata given by jump deformations. In the 2-dimensional case, the description is simple, because each of the strata consists of a single point, so the only interesting information is given by the jump deformations.

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The versal deformation of an associative algebra depends only on the second and third Hochschild cohomology groups. However, we give a complete calculation of the cohomology for each of the algebras. What makes the study of associative algebras of low dimension much more complicated than the corresponding study of low dimensional Lie algebras is that while for a Lie algebra, the $n$-th cohomology group $H^n$ vanishes for $n$ larger than the dimension of the vector space, in general, for an associative algebra $H^n$ does not vanish. Thus we had to develop arguments on a case by case basis for each of the six distinct algebras. In particular, one of these algebras has an unusual pattern for the cohomology, which made its computation rather nontrivial.

The main result of this paper is the complete description of the Hochschild cohomology for all 2-dimensional associative algebras. It turns out that the calculation of cohomology even for low dimensional associative algebras is a nontrivial problem. To construct extensions of associative algebras to $A_\infty$ algebras, it is necessary to have a complete description of the cohomology in all degrees, not just $H^2$ and $H^3$, which are needed for the deformation theory of these algebras as associative algebras. What we compute in this paper is the first step in constructing 2-dimensional $A_\infty$ algebras. These results may be of interest on their own, especially as an indication of the difficulty which occurs in computing the deformation theory of associative algebras, even in low dimension.

2. Preliminaries

Suppose that $V$ is a 2-dimensional vector space, defined over a field $K$ whose characteristic is not 2 or 3, equipped with an associative multiplication structure $m: V \otimes V \to V$. The associativity relation can be given in the form

$$m \circ (m \otimes 1) = m \circ (1 \otimes m).$$

When the space $V$ is $\mathbb{Z}_2$-graded, there is no difference in the relation of associativity, but only even maps $m$ are allowed, so the set of associative algebra structures depends on the $\mathbb{Z}_2$-grading in this way.

The notion of equivalence of associative algebra structures is given as follows. If $g$ is a linear automorphism of $V$, then define

$$g^*(m) = g^{-1} \circ m \circ (g \otimes g).$$

Two algebra structures $m$ and $m'$ are equivalent if there is an automorphism $g$ such that $m' = g^*(m)$. The set of equivalence classes of algebra structures on $V$ is called the moduli space of associative algebras on $V$.

When $V$ is $\mathbb{Z}_2$-graded, we require that $g$ be an even map. Thus the set of equivalence classes of $\mathbb{Z}_2$-graded associative algebra structures will be different than the set of equivalence classes of associative algebra structures on the same space, ignoring the grading. Because the set of equivalences is more restricted in the $\mathbb{Z}_2$-graded case, two algebra structures which are equivalent as ungraded algebra structures may not be equivalent as $\mathbb{Z}_2$-graded algebra structures. There is a map between the moduli space of $\mathbb{Z}_2$-graded algebra structures on $V$ and the space of all algebra structures on $V$. In general, this map will be neither injective nor surjective.

Hochschild cohomology was introduced in [15], and used to classify infinitesimal deformations of associative algebras. Suppose that

$$m_t = m + t\varphi,$$
is an infinitesimal deformation of $m$. By this we mean that the structure $m_t$ is associative up to first order. From an algebraic point of view, this means that we assume that $t^2 = 0$, and then check whether associativity holds. It is not difficult to show that this is equivalent to the following.

$$a\varphi(b, c) - \varphi(ab, c) + \varphi(a, bc) - \varphi(a, b)c = 0,$$

where, for simplicity, we denote $m(a, b) = ab$. Moreover, if we let

$$g_t = I + t\lambda$$

be an infinitesimal automorphism of $V$, where $\lambda \in \text{Hom}(V, V)$, then it is easily checked that

$$g_t^*(m)(a, b) = ab + t(a\lambda(b) - \lambda(ab) + \lambda(a)b).$$

This naturally leads to a definition of the Hochschild coboundary operator $D$ on $\text{Hom}(T(V), V)$ by

$$D(\varphi)(a_0, \cdots, a_n) = a_0\varphi(a_1, \cdots, a_n) + (-1)^{n+1}\varphi(a_0, \cdots, a_{n-1})a_n$$

$$+ \sum_{i=0}^{n-1} (-1)^{n+1}\varphi(a_0, \cdots, a_{i-1}, a_i a_{i+1}, a_{i+2}, \cdots, a_n).$$

If we set $C^n(V) = \text{Hom}(V^n, V)$, then $D : C^n(V) \to C^{n+1}(V)$. One obtains the following classification theorem for infinitesimal deformations.

**Theorem 2.1.** The equivalence classes of infinitesimal deformations $m_t$ of an associative algebra structure $m$ under the action of the group of infinitesimal automorphisms on the set of infinitesimal deformations are classified by the Hochschild cohomology group

$$H^2(m) = \ker(D : C^2(V) \to C^3(V))/\text{Im}(D : C^1(V) \to C^2(V)).$$

When $V$ is $\mathbb{Z}_2$-graded, the only modifications that are necessary are that $\varphi$ and $\lambda$ are required to be even maps, so we obtain that the classification is given by $H^2_2(V)$, the even part of the Hochschild cohomology group.

We wish to transform this classical viewpoint into the more modern viewpoint of associative algebras as being given by codifferentials on a certain coalgebra. To do this, we first introduce the parity reversal $\Pi V$ of a $\mathbb{Z}_2$-graded vector space $V$. If $V = V_\epsilon \oplus V_\sigma$ is the decomposition of $V$ into its even and odd parts, then $W = \Pi V$ is the $\mathbb{Z}_2$-graded vector space given by $W_\epsilon = V_\sigma$ and $W_\sigma = V_\epsilon$. In other words, $W$ is just the space $V$ with the parity of elements reversed.

Denote the tensor (co)-algebra of $W$ by $T(W) = \bigoplus_{k=0}^{\infty} W^k$, where $W^k$ is the $k$-th tensor power of $W$ and $W^0 = K$. For brevity, the element in $W^k$ given by the tensor product of the elements $w_i$ in $W$ will be denoted by $w_1 \cdots w_k$. The coalgebra structure on $T(W)$ is given by

$$\Delta(w_1 \cdots w_n) = \sum_{i=0}^{n} w_1 \cdots w_i \otimes w_{i+1} \cdots w_n.$$  

Define $d : W^2 \to W$ by $d = \pi \circ m \circ (\pi^{-1} \otimes \pi^{-1})$, where $\pi : V \to W$ is the identity map, which is odd, because it reverses the parity of elements. Note that $d$ is an odd map. The space $C(W) = \text{Hom}(T(W), W)$ is naturally identifiable with the
space of coderivations of $T(W)$. In fact, if $\varphi \in C^k(W) = \text{Hom}(W^k, W)$, then $\varphi$ is extended to a coderivation of $T(W)$ by

$$\varphi(w_1 \cdots w_n) = \sum_{i=0}^{n-k} (-1)^{\sum_j w_j} w_1 \cdots w_i \varphi(w_{i+1} \cdots w_{i+k}) w_{i+k+1} \cdots w_n.$$  

The space of coderivations of $T(W)$ is equipped with a $\mathbb{Z}_2$-graded Lie algebra structure given by

$$[\varphi, \psi] = \varphi \circ \psi - (-1)^{\varphi \psi} \psi \circ \varphi.$$  

The reason that it is more convenient to work with the structure $d$ on $W$ rather than $m$ on $V$ is that the condition of associativity for $m$ translates into the codifferential property $[d, d] = 0$. Moreover, the Hochschild coboundary operation translates into the codifferential operator $D$ on $C(W)$, given by

$$D(\varphi) = [d, \varphi].$$  

This point of view on Hochschild cohomology first appeared in [21]. The fact that the space of Hochschild cochains is equipped with a graded Lie algebra structure was noticed much earlier [10, 11, 12, 13, 14].

For notational purposes, we introduce a basis of $C^n(W)$ as follows. Suppose that $W = \langle w_1, \cdots, w_m \rangle$. Then if $I = (i_1, \cdots, i_n)$ is a multi-index, where $1 \leq i_k \leq m$, denote $w_I = w_{i_1} \cdots w_{i_n}$. Define $\varphi^I_t \in C^n(W)$ by

$$\varphi^I_t(w_I) = \delta^I_t w_I,$$

where $\delta^I_t$ is the Kronecker delta symbol. In order to emphasize the parity of the element, we will denote $\varphi^I_t$ by $\psi^I_t$ when it is an odd coderivation.

For a multi-index $I = (i_1, \cdots, i_k)$, denote its length by $\ell(I) = k$. If $K$ and $L$ are multi-indices, then denote $KL = (k_1, \cdots, k_{|K|}, l_1, \cdots, l_{|L|})$. Then

$$(\varphi^I_t \circ \varphi^J_s)(w_K) = \sum_{k_1, k_2, k_3 = K} (-1)^{w_{k_1} \varphi^I_t} \varphi^I_t(w_{k_1}, \varphi^J_s(w_{k_2}), w_{k_3})$$

$$= \sum_{k_1, k_2, k_3 = K} (-1)^{w_{k_1} \varphi^I_t} \delta_{k_1}^J \delta_{k_2}^J w_{k_3},$$

from which it follows that

$$\varphi^I_t \circ \varphi^J_s = \sum_{k=1}^{\ell(I)} (-1)^{\sum_{l=1}^{k-1} w_{i_l}} \varphi^I_t \delta^J_{k_1} \delta^J_{k_2} \varphi^J_s(I, J, k),$$

where $(I, J, k)$ is given by inserting $J$ into $I$ in place of the $k$-th element of $I$; i.e.,

$$(I, J, k) = (i_1, \cdots, i_{k-1}, j_1, \cdots, j_{|J|}, i_{k+1}, \cdots, i_{\ell(I)}).$$

Let us recast the notion of an infinitesimal deformation in terms of the language of coderivations. We say that

$$d_t = d + t\psi$$

is a deformation of the codifferential $d$ precisely when $[d_t, d_t] = 0 \mod t^2$. This condition immediately reduces to the cocycle condition $D(\psi) = 0$. Note that we require $d_t$ to be odd, so that $\psi$ must be an odd coderivation. One can introduce a more general idea of parameters, allowing both even and odd parameters, in which case even coderivations play an equal role, but we will not adopt that point of view in this paper.
For associative algebras, we require that \(d\) and \(\psi\) lie in \(\text{Hom}(W^2, W)\). This notion naturally generalizes to considering \(d\) simply to be an arbitrary odd coderivative, in which case we would obtain an \(A_\infty\) algebra, a natural generalization of an associative algebra.

3. Associative algebra structures on an \(0|2\) vector space

Suppose that \(W = \langle 1, 2 \rangle\), where both 1, 2 are odd elements. Then \(C^n = \langle \varphi^1_1, \ell(I) = n \rangle\) has dimension \(\dim C^n = 2^{n+1}\). For later convenience, we decompose \(C^n\) as follows. Let

\[
C^n_1 = \langle \varphi^1_1, \ell(I) = n \rangle \\
C^n_2 = \langle \varphi^1_2, \ell(I) = n \rangle.
\]

Then \(C^n = C^n_1 \oplus C^n_2\). Moreover \(\dim C^n_1 = \dim C^n_2 = 2^n\).

The matrix of a generic odd coderivation is

\[
A = \begin{pmatrix}
a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\
a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4}
\end{pmatrix}
\]

The codifferential condition \([d, d] = 0\) gives 8 solutions,

| Number | Expression | Conditions |
|--------|------------|------------|
| 1      | \(a_{1,2} = a_{2,3} = a_{2,1} = a_{1,4} = a_{1,1} = a_{2,2} = 0, a_{1,3} = a_{2,4}\) | \(a_{1,2} = a_{2,3} = a_{2,1} = a_{1,4} = a_{1,1} = a_{2,2} = 0, a_{1,3} = a_{2,4}\) |
| 2      | \(a_{1,3} = a_{2,3} = a_{2,1} = a_{1,4} = a_{1,1} = a_{2,2} = 0, a_{1,2} = a_{2,4}\) | \(a_{1,3} = a_{2,3} = a_{2,1} = a_{1,4} = a_{1,1} = a_{2,2}\) |
| 3      | \(a_{1,2} = a_{1,3} = a_{2,3} = a_{2,1} = a_{1,4} = a_{2,2} = 0, a_{1,1} = a_{2,4}\) | \(a_{1,2} = a_{1,3} = a_{2,3} = a_{2,1} = a_{1,4} = a_{2,2}\) |
| 4      | \(a_{2,3} = a_{1,4} = a_{2,2} = 0, a_{1,3} = a_{2,4} = a_{1,2}\) | \(a_{2,3} = a_{1,4} = a_{2,2} = 0, a_{1,3} = a_{2,4} = a_{1,2}\) |
| 5      | \(a_{1,3} = a_{2,1} = a_{1,4} = a_{2,2} = 0, a_{2,4} = a_{1,2}, a_{1,1} = a_{2,3}\) | \(a_{1,3} = a_{2,1} = a_{1,4} = a_{2,2} = 0, a_{2,4} = a_{1,2}, a_{1,1} = a_{2,3}\) |
| 6      | \(a_{1,2} = a_{2,3} = a_{2,1} = a_{1,4} = a_{1,1} = a_{2,4} = a_{1,3}, a_{1,1} = a_{2,2}\) | \(a_{1,2} = a_{2,3} = a_{2,1} = a_{1,4} = a_{1,1} = a_{2,4} = a_{1,3}, a_{1,1} = a_{2,2}\) |
| 7      | \(a_{1,2} = a_{1,3} = a_{1,4} = a_{2,2} = a_{2,3}, a_{1,1} = - \frac{a_{2,1}^2a_{2,4}^2 - a_{2,3}^2}{a_{2,3}}\) | \(a_{1,2} = a_{1,3} = a_{1,4} = a_{2,2} = a_{2,3}, a_{1,1} = - \frac{a_{2,1}^2a_{2,4}^2 - a_{2,3}^2}{a_{2,3}}\) |
| 8      | \(a_{2,2} = a_{2,3}, a_{1,2} = a_{1,3}, a_{2,1} = \frac{a_{1,1}a_{2,3}}{a_{1,4}}, a_{1,1} = \frac{a_{1,1}^2 - a_{1,3}a_{2,4}^2 + a_{1,4}a_{2,3}}{a_{1,4}}\) | \(a_{2,2} = a_{2,3}, a_{1,2} = a_{1,3}, a_{2,1} = \frac{a_{1,1}a_{2,3}}{a_{1,4}}, a_{1,1} = \frac{a_{1,1}^2 - a_{1,3}a_{2,4}^2 + a_{1,4}a_{2,3}}{a_{1,4}}\) |

Each of these solutions corresponds to at least one nonequivalent codifferential listed below in the following manner:

- Number (1) corresponds to \(d_4\). This correspondence can be seen by setting \(a_{2,4} = 1\) and applying an automorphism that interchanges \(w_1\) with \(w_2\).
- Number (2) corresponds to \(d_3\), for reasons identical to those above.
- Number (3) corresponds to \(d_1\). This correspondence can be seen by setting \(a_{2,4} = 1\).
- Number (4) corresponds to \(d_1, d_2\), and \(d^6\). For \(d_2\) we set \(a_{2,4} = 0\). For \(d_6\) we set \(a_{2,2} = 0\) and \(a_{2,4} = 1\) and then apply an automorphism similar to that mentioned with regards to number (1). Though the arguments are similar for \(d_1\), the automorphisms are slightly more complicated.
- Number (5) also corresponds to \(d_3\). This correspondence can be seen by setting \(a_{1,1} = 1\) and \(a_{2,4} = 0\).
- Number (6) corresponds to \(d_4\). This correspondence is identical to that above.
- Number (7) also corresponds to \(d_1, d_2\), and \(d_6\). For \(d_6\), we set \(a_{2,3} = 1\) and everything else to 0. The other two correspondences have slightly more complicated automorphisms.
- Number (8) has the least restrictions and so corresponds to \(d_1, d_2, d_5\), and \(d_6\).
Nonequivalent Codifferentials:

\[ d_1 = \psi_{22}^{22} + \psi_{11}^{11} \]
\[ d_2 = \psi_{22}^{22} \]
\[ d_3 = \psi_{22}^{22} + \psi_{12}^{12} \]
\[ d_4 = \psi_{22}^{22} + \psi_{21}^{21} \]
\[ d_5 = \psi_{22}^{22} + \psi_{12}^{12} + \psi_{21}^{21} \]
\[ d_6 = \psi_{22}^{22}. \]

Note that if we define \( D(\varphi) = [d^*, \varphi] \), where \( d^* \) is one of the above codifferentials, then \( D^2 = 0 \), so the coboundary operator \( D \) determines a differential on \( C(W) \).

Since \( d^* \in C^2 \), \( D(C^k) \subseteq C^{k+1} \), and we can define the \( k \)-th cohomology \( H^k(d^*) \) of \( d^* \) by

\[ H^k(d^*) = \ker(d^* : C^k \rightarrow C^{k+1}) / \text{Im}(d^* : C^{k-1} \rightarrow C^k). \]

The cohomology of these codifferentials is given in Table 3 below.

| Codifferential | \( H^0 \) | \( H^2 \) | \( H^1 \) | \( H^3 \) | \( H^4 \) |
|----------------|--------|--------|--------|--------|--------|
| \( d_1 = \psi_{11}^{11} + \psi_{22}^{22} \) | 2 | 0 | 0 | 0 | 0 |
| \( d_2 = \psi_{22}^{22} \) | 2 | 1 | 1 | 1 | 1 |
| \( d_3 = \psi_{22}^{22} + \psi_{12}^{12} \) | 0 | 0 | 0 | 0 | 0 |
| \( d_4 = \psi_{22}^{22} + \psi_{21}^{21} \) | 0 | 0 | 0 | 0 | 0 |
| \( d_5 = \psi_{22}^{22} + \psi_{12}^{12} + \psi_{21}^{21} \) | 2 | 1 | 1 | 1 | 1 |
| \( d_6 = \psi_{22}^{22} \) | 2 | 2 | 2 | 2 | 2 |

Table 1. Cohomology of the six codifferentials on a 0|1-dimensional space.

4. Elements of the Moduli Space

In this section we give a complete description of both the cohomology and the multiplication structure generated by each codifferential. For a complete proof of the cohomological structure see the next section. Let us suppose that \( V = \{x, \theta\} \), where \( x, \theta \) are even, and that \( W = \Pi V = \{1, 2\} \), where \( \pi(x) = 2 \) and \( \pi(\theta) = 1 \). Let \( m = \pi \circ d \circ (\pi^{-1} \otimes \pi^{-1}) \). Then \( m \) is an associative algebra structure on \( V \), corresponding to the codifferential \( d \). For each of the codifferentials, we give the multiplication structure \( m \) on \( V \).

\[
\begin{align*}
d_1 & : x^2 = x & x\theta = 0 & \theta x = 0 & \theta^2 = \theta \\
d_2 & : x^2 = 0 & x\theta = 0 & \theta x = 0 & \theta^2 = \theta \\
d_3 & : x^2 = 0 & x\theta = x & \theta x = 0 & \theta^2 = \theta \\
d_4 & : x^2 = 0 & x\theta = 0 & \theta x = x & \theta^2 = \theta \\
d_5 & : x^2 = 0 & x\theta = x & \theta x = x & \theta^2 = \theta \\
d_6 & : x^2 = 0 & x\theta = 0 & \theta x = 0 & \theta^2 = x
\end{align*}
\]

Of these algebras, \( d_1 \), \( d_2 \), \( d_5 \) and \( d_6 \) are commutative; and \( d_1 \) and \( d_5 \) are unital, with unit \( \theta \). In the algebras \( d_1 \) and \( d_5 \), \( \theta \) generates a nontrivial proper ideal, while \( x \) generates a nontrivial proper ideal in every algebra. The algebra \( d_1 \) is the unique
semisimple 2-dimensional algebra, which is the direct sum of two copies of the 1-dimensional simple algebra \( \mathbb{C} \).

The algebras \( d_2, d_3, d_4 \) and \( d_5 \) are all extensions of the simple 1-dimensional associative algebra (whose structure is just the associative algebra structure of \( \mathbb{C} \)). In fact, they fit a certain pattern of extensions. The algebras \( d_3 \) and \( d_4 \) are opposite algebras, and they are rigid in the cohomological sense. These two rigid algebras are just the first in a sequence of rigid extensions of the 1-dimensional simple algebra.

The algebra \( d_5 \) is the unique extension of the simple 1-dimensional algebra by the trivial 1-dimensional algebra as a unital algebra. The algebra \( d_2 \) is just the direct sum of the trivial 1-dimensional algebra and the simple 1-dimensional algebra.

Finally, the algebra \( d_6 \) is an extension of the trivial 1-dimensional algebra by the trivial 1-dimensional algebra, and as a consequence, it is a nilpotent algebra. By nilpotent algebra, we mean an algebra such that a power of the algebra vanishes, which in the finite dimensional case is equivalent to the fact that every element in this algebra is nilpotent.

We did not use the method of extensions in calculating the nonequivalent codifferentials. In this simple case, it is easy to solve the codifferential property \([d, d] = 0\), which gives a system of quadratic coefficients, and study the action of the group of linear automorphisms of the underlying vector space, to arrive at the six codifferentials. However, calculating this space by extensions reveals more of its properties, and also gives a natural manner of organizing the codifferentials.

5. Calculating the cohomology

The cohomology of the codifferentials is given in Table 3 above. With the exception of \( d_6 \), the pattern of cohomology is easily deduced from the information in the table.

For later use, we define the following operator on \( C(W) \). If \( I \) is a multi-index with \( i_k \in \{1, 2\} \), with \( \ell(I) = m \), then define \( \lambda^I : C^k \to C^{k+m} \) by \( \lambda^I \varphi_j^I = \varphi_j^{I+1} \). Note that the parity of \( \lambda^I \) is the same as the parity of \( I \). We abbreviate \( \lambda^{(1)} = \lambda^1 \).

We give a computation of the cohomology of the codifferentials on a case by case basis.

**Theorem 5.1.** Suppose that a coboundary operator \( D : C^n \to C^{n+1} \) decomposes as \( D = D' + D'' \), given by the following diagram

\[
\begin{array}{cccccccc}
C^n_a & \xrightarrow{D'} & C^{n+1}_a & \xrightarrow{D''} & C^{n+1}_{a+1} \\
\downarrow & & \downarrow & & \downarrow \\
C^{n+2}_a & \xrightarrow{D'} & C^{n+2}_{a+1} & = & C^{n+2}_{a+1} \\
\downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow \\
C^n_a & \xrightarrow{D'} & C^{n+1}_a & \xrightarrow{D''} & C^{n+1}_a \\
\downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots \\
\end{array}
\]

for \( a \leq k \leq n \) where \( C^n_a = C^n_a \oplus \cdots \oplus C^n_a \oplus \cdots \oplus C^n_n \), such that \( D'' \) is injective when \( k = a \), and \( H(D'') = 0 \). Then \( H(D) = 0 \) on the subcomplex \( C^k_a \) for \( k \geq a \).

This result is well known. A proof is given in [2].
5.1. \( d_1 = \psi_2^{22} + \psi_1^{11} \). We begin by computing the coboundary operator with representatives from the \( C_n^m \) and \( C_1^n \) spaces,

\[
D(\phi_1^I) = \phi_1^{11} + (1)^{I+1} \phi_1^{11} + (1)^I \phi_2^{22} + (1)^{I+1} \phi_2^{11} \\
D(\phi_2^I) = \phi_2^{11} + (1)^{I+1} \phi_2^{11} + (1)^I \phi_2^{22} + (1)^{I+1} \phi_2^{11}
\]

We decompose the \( C_1^n \) and \( C_2^n \) spaces as follows.

\[
C_{1,k}^n = \langle \phi_1^{1+2I} | \ell(I) = n - k - 1 \rangle \quad P_1^n = \langle \phi_1^n \rangle \\
C_{2,k}^n = \langle \phi_2^{2+2I} | \ell(I) = n - k - 1 \rangle \quad P_2^n = \langle \phi_2^n \rangle
\]

We also decompose the coboundary operator \( D \) as follows.

\[
D = D_1' + D_1'' : C_{1,k}^n \to C_{1,k+1}^{n+1} \\
D = D_2' + D_2'' : C_{2,k}^n \to C_{2,k+1}^{n+1} \\
D : P_a^n \to P_a^{n+1}.
\]

By computation we see,

\[
D_2''(\phi_2^{2+I}) = \begin{cases} 
\phi_2^{2k+1+I}, & \text{k is even;} \\
0, & \text{k is odd}
\end{cases} \\
D_2''(\phi_1^{1+2I}) = \begin{cases} 
\phi_1^{1k+2+I}, & \text{k is even;} \\
0, & \text{k is odd}
\end{cases}
\]

\[
D_2'' = \begin{cases} 
0, & n \text{ is even;} \\
\phi_2^{n+1}, & n \text{ is odd}
\end{cases} \\
D_1'' = \begin{cases} 
0, & n \text{ is even;} \\
\phi_1^{n+1}, & n \text{ is odd}
\end{cases}
\]

Using Theorem 5.1 and a direct computation of \( H^0(d_1) \), we obtain that

\[
H^n(d_1) = \begin{cases} 
\langle \psi_2, \psi_1 \rangle, & n = 0; \\
0, & n \geq 1.
\end{cases}
\]

5.2. \( d_2 = \psi_2^{22} \). We will begin by computing the coboundary operator with representatives from the \( C_2^n \) and \( C_1^n \) spaces:

\[
D(\phi_2^I) = \phi_2^{22} + (1)^{I+1} \phi_2^{22} + (1)^I \phi_2^{22} \\
D(\phi_2^I) = (1)^I \phi_2^{22}.
\]

We decompose our spaces as follows,

\[
C_{a,k}^n = \langle \phi_a^{2+2I} | \ell(I) = n - k - 1 \rangle \quad P_a^n = \langle \phi_a^n \rangle
\]

Using this decomposition we have the following maps,

\[
D = D_a' + D_a'' : C_{a,k}^n \to C_{a,k+1}^{n+1} \\
D : P_a^n \to P_a^{n+1}.
\]

By computation we see,

\[
D_2''(\phi_2^{2+I}) = \begin{cases} 
\phi_2^{2k+1+I}, & \text{k is even;} \\
0, & \text{k is odd}
\end{cases} \\
D_2''(\phi_1^{1+2I}) = \begin{cases} 
0, & \text{k is even;} \\
\phi_1^{1k+2+I}, & \text{k is odd}
\end{cases}
\]

\[
D_2'' = \begin{cases} 
0, & n \text{ is even;} \\
\phi_2^{n+1}, & n \text{ is odd}
\end{cases} \\
D_1'' = \begin{cases} 
0, & n \text{ is even;} \\
\phi_1^{n+1}, & n \text{ is odd}
\end{cases}
\]
Thus using Theorem 5.1 we see that $H(D) = 0$ on the subcomplexes $C^n_{2,k}$ for $k \geq 0$, $C^n_{1,k}$ for $k \geq 1$ and $P^n_{α}$ for $n \geq 1$. If we let $D = D_1 + D''_1$, then we see that $λ^1 D_1 = D_1 λ^1$ which implies the following diagram,

$$
\begin{align*}
H^n(C_1) & \longrightarrow H^n(C_{1,0}) \\
\downarrow^D & \downarrow^D \\
H^{n+1}(C_{1,0}) & 
\end{align*}
$$

This is true for $n \geq 0$ if we define $C^n_{1,0} = C_0^n = P^n_1 = \langle ϕ_1 \rangle$. Therefore we conclude,

$$
H^n(d_2) = \begin{cases}
\langle ψ_2, ψ_1 \rangle, & n = 0; \\
\langle φ^n_1 \rangle, & n \geq 1.
\end{cases}
$$

5.3. $d_3 = ψ^{22}_2 + ψ^{12}_1$. We begin by computing the coboundary operator with representatives from the $C^n_2$ and $C^n_1$ spaces,

$$
D(ϕ^n_2) = ϕ^{22}_2 + (-1)^{ℓ+1} ϕ^{22}_2 + (-1)^{ℓ} ϕ^n_2 ψ^{22}_2 + (-1)^{ℓ+1} ϕ^{22}_1 + (-1)^{ℓ+1} ϕ^{12}_1 \\
D(ϕ^n_1) = ϕ^n_1 + (-1)^{ℓ} ϕ^n_1 ψ^{22}_2 + (-1)^{ℓ} ϕ^n_1 ψ^{12}_2.
$$

Now we decompose $C^n_{1,k}$ as follows,

$C^n_{1,k} = \langle ϕ^{2+k}_1 | ℓ(I) = n - k - 1 \rangle, \quad P^n_{1} = \langle ϕ^n_1 \rangle$

Using this decomposition we have,

$D = D_2 + D_1 : C^n_2 \rightarrow C^{n+1}_2 \oplus C^{n+1}_{1,0}$

$D = D + D'' : C^n_{1,k} \rightarrow C^{n+1}_{k} \oplus C^{n+1}_{1,k+1}$

$D : P^n_{1} \rightarrow P^{n+1}_{1}$

Furthermore we see that $D : C^n_{1,0} \rightarrow C^{n+1}_{1,0}$. Now we check how $D''$ acts

$D''(ϕ^{2+k}_1) = \begin{cases}
0 & k \text{ is even}; \\
ϕ^{2+k+1}_1 & k \text{ is odd}
\end{cases}$

Thus by Theorem 5.1 we see that the cohomology vanishes on the subcomplex $C^n_{1,k}$, for $k \geq 1$. Notice that $D_1 : C^n_2 \rightarrow C^{n+1}_1$ and that

$$
DD_1(ϕ^n_1) = ϕ^{12}_1 + (-1)^{ℓ+1} ϕ^{12}_1 ψ^{22}_2 + (-1)^{ℓ+1} ϕ^{12}_1 ψ^{12}_1 + (-1)^{ℓ} λ^1 ϕ^{12}_1 ψ^{12}_1 \\
= (-1)^{n+1} D_1 D_2(ϕ^n_1)
$$

Let $a φ \in C^n_2$ for $a \in \mathbb{C}$ and $ξ \in C^n_{1,0}$ be such that $a φ + ξ \in \ker D$, then

$D_1(a φ) + D(ξ) = 0 \quad \quad D_2(a φ) = 0.$

However the second equation follows from the first,

$$
0 = DD_1(a φ) + D^1(ξ) = (-1)^{n+1} D_1 D_2(a φ)
$$
and since $D_1$ is injective then $D_2(a\phi) = 0$. Since $D_1$ is an isomorphism then
\[
D = D_1(a\phi) + D(\xi) = D_1(a\phi) + DD_1(\eta) = D_1(a\phi + (-1)^{n+1}D_2(\eta)).
\]
This implies that $a\phi = (-1)^nD_2(\eta)$. Therefore we have shown $(-1)^n a\phi + \xi = D(\eta)$
since $a$ is arbitrary we can absorb the $(-1)^n$, and we have
\[
a\phi + \xi = D(\eta)
\]
Next we consider is $P^n_1$, but
\[
D(\varphi^n_1) = \begin{cases} 0, & n \text{ odd} \\ \varphi^{n+1}_1, & n \text{ even} \end{cases}
\]
We’ve now shown for all spaces except $C^n_0$, but for this we calculate
\[
D(\varphi_2) = -\varphi^n_1 \neq 0
\]
So we discover
\[
H^n(d_4) = 0 \quad \text{for all } n
\]
5.4. $d_4 = \psi_2^{22} + \psi_1^{21}$. This is analogous to section 5.3. We define the spaces as follows,
\[
C^n_{2,k} = \langle \varphi^{I^{12}k}_2 \mid \ell(I) = n - k - 1 \rangle, \quad P^n_2 = \langle \varphi^{2n}_2 \rangle
\]
\[
C^n_{1,k} = \langle \varphi^{I^{12}k}_1 \mid \ell(I) = n - k - 1 \rangle, \quad P^n_1 = \langle \varphi^{n}_1 \rangle
\]
The cohomology is given by,
\[
H^n(d_4) = 0 \quad \text{for all } n
\]
5.5. $d_5 = \psi_2^{22} + \psi_1^{21} + \psi_1^{12}$. We will begin by computing the coboundary operator with representatives from the $C^n_2$ and $C^n_1$ spaces:
\[
D(\varphi^n_2) = \varphi^n_2 + (-1)^{l+1}\varphi^{1l}_2 + \varphi^{I}_1 + (-1)^{l+1}\varphi^{1l}_1 + (-1)^{l}\varphi^{I}_2\varphi^{22}_2
\]
\[
+ (-1)^{l}\varphi^{I}_2\varphi^{21}_1 + (-1)^{l}\varphi^{I}_2\varphi^{12}_1
\]
\[
D(\varphi^n_1) = (-1)^{l+1}\varphi^{2l}_1 + \varphi^{I}_1 + (-1)^{l}\varphi^{I}_1\varphi^{22}_2 + (-1)^{l}\varphi^{I}_1\varphi^{21}_1
\]
\[
+ (-1)^{l}\varphi^{I}_1\varphi^{12}_1
\]
We decompose our spaces in the following manner,
\[
C^n_{a,k} = \langle \varphi^{a l+1}_a \mid \ell(I) = n - k - 1 \rangle, \quad P^n_a = \langle \varphi^{an}_a \rangle
\]
\[
C^n_{a,0,k} = \langle \varphi^{a l+2l}_a \mid \ell(I) = n - k - 1 \rangle, \quad S^n_a = \langle \varphi^{an}_a \rangle
\]
With these definitions there is ambiguity when $n = 0$, so we define $C^n_0 = S^n_0$. Note
that we can write $C^n_{a,0}$ as
\[
C^n_{a,0} = S^n_a \oplus \bigoplus_{k=1}^{n-1} C^n_{a,0,k}
\]
We now decompose the $D$ operator as follows,

$$D = D' + D'' + D_k + D_0 : C_{2,k} \to C_{2,k+1} \oplus C_{2,k+1} \oplus C_{1,k} \oplus C_{1,0}$$

$$D = D'_1 + D''_1 : C_{1,k} \to C_{1,k+1} \oplus C_{1,k+1}$$

For convenience we denote $D_1 = D'_1 + D''_1$ and $D_2 = D'_2 + D''_2$. With this decomposition we obtain the following relations,

$$(D'_1)^2 = 0 \quad (D''_1)^2 = 0 \quad D_1 D'_1 = -D'_1 D_2$$

$$D_2 D'_2 = -D_2 D''_2 \quad D'_1 D''_1 = -D''_1 D'_1 \quad D_1 D''_1 = -D''_1 D_2$$

$$(D''_2)^2 = 0 \quad (D'_2)^2 = 0$$

Some sort of transitional sentence that will start the proving process.

$$D''_n(\varphi^{2n} I) = \begin{cases} 0, & k \text{ even} \\ \pm \varphi^{2n+1} I, & k \text{ odd} \end{cases}$$

$$D(\varphi^{2n}) = \begin{cases} 0, & n \text{ even} \\ \pm \varphi^{2n+1}, & n \text{ odd} \end{cases}$$

Let $\varphi \in C_{2}^{n}$ and $\xi \in C_{1}^{n}$ satisfy $\varphi + \xi \in \ker D$. We can write $\varphi$ and $\xi$ as

$$\varphi = \varphi_0 + \cdots + \varphi_{n-1} + \varphi_P$$

$$\xi = \xi_0 + \cdots + \xi_{n-1} + \xi_P.$$  

Where $\xi_k \in C_{1,k}$ and $\xi_P \in P_{1}^{n}$. Since $D(\varphi + \xi) = 0$ we see,

$$D_2(\varphi) = 0$$

(2) \hspace{1cm} D'_1(\varphi_0) + D''_1(\varphi) + D_1(\xi_0) = 0, \text{ if } k = 0$$

(3) \hspace{1cm} D'_2(\varphi_k) + D''_2(\xi_k) + D_1(\xi_{k-1}) = 0, \text{ if } k \geq 1.$$  

Our goal is to find $\alpha \in C_{2}^{n-1}$ and $\eta \in C_{1}^{n-1}$ such that $D(\alpha + \eta) = \varphi + \xi$. Note that we can write $\alpha$ and $\eta$ as a sum of terms as we did with $\varphi$ and $\xi$. Using THEOREM on $D_2$ we can find $\alpha_{k+1} \in C_{2,k+1}^{n-1}$ such that $D_2(\alpha_{k+1}) = \varphi_{k+1}$. Now we find $\eta_{k+1}$ to satisfy (3). For $k = n + 1$ we have

$$D''_1(\xi_n) = 0$$

and since $H(D'_1) = 0$ on the subcomplex $C_{1}^{n-1}$, we can find an $\eta_{n-1} \in C_{1}^{n-1}$ such that $D''_1(\eta_{n-1}) = \xi_n$. Assume we have shown for $k + 1$, or assume we have shown shown

$$\xi_{k+1} = D'_1(\alpha_{k+1}) + D'_1(\eta_{k+1}) + D''_1(\eta.k).$$

Then we have

$$0 = D''_1(\xi_{k+1}) + D'_1(\xi_{k+1}) + D''_1(\xi_k)$$

$$= D'_1(\alpha_{k+1}) + D'_1(\eta_{k+1}) + D''_1(\xi_{k+1})$$

Thus $\xi_k = D'_1(\alpha_k) + D'_1(\eta_k) + D''_1(\eta_{k-1})$ for some $\eta_{k-1} \in C_{1,k-1}^{n-1}$. This holds until $k = 1$ in which case we show

$$0 = D''_1(\varphi_2) + D'_1(\xi_2) + D''_1(\xi_1)$$

$$= D'_1(\alpha_1) + D'_1(\eta_1) + D''_1(\xi_1)$$

$$= D''_1(\xi_1 - D''_1(\alpha_1) - D'_1(\eta_1)).$$

However, $D''_1$ is injective when $k = 1$ so $\xi_1 = D'_1(\alpha_1) + D'_1(\eta_1)$.
When \( k = 0 \) a separate technique is required. We begin by showing that \( \lambda^1 \) commutes with the \( D_a \) operators.

\[
D_2(\lambda^1 \varphi^1_2) = \varphi^1_2 + (-1)^i \varphi^2_2 + (-1)^i \varphi^3_2 + (-1)^i \varphi^4_2 + (-1)^i \varphi^5_2 + (-1)^i \varphi^6_2 + (-1)^i \varphi^7_2 + (-1)^i \varphi^8_2 + (-1)^i \varphi^9_2 + (-1)^i \varphi^{11}_2 + (-1)^i \varphi^{12}_2 = \lambda^1 D_2(\varphi^1_2)
\]

The same proof will hold for \( D_1 \lambda^1 = \lambda^1 D_1 \). This implies that \( \lambda^k \) commutes with \( D_a \) and notice that

\[
\lambda^k : C^0_{a} \rightarrow C^0_{a,0,k}
\]

Similar to before we can write \( \varphi_0 \) and \( \xi_0 \) as,

\[
\varphi_0 = \varphi_{0,2} + \cdots + \varphi_{0,n-1} + \varphi_{0,S} \quad \xi_0 = \xi_{0,2} + \cdots + \xi_{0,n-1} + \xi_{0,S}
\]

Let \( \varphi'_k \in C^m_{2,k} \) be such that \( \lambda^k \varphi'_k = \varphi_{0,k} \). Now we rewrite our condition on \( \varphi_0 \) as,

\[
D_2(\varphi_0) = 0 \Rightarrow 0 = D_2(\varphi_{0,k}) = \max_k \lambda^k D_2(\varphi'_k).
\]

Since we pulled out the maximum number of 1’s, then if \( \varphi'_k \not\in C^m_{2,k} \) then \( \varphi'_k \) has a leading 2, and since \( H(D_2) = 0 \) on \( C^m_{2,k} \geq 1 \) and \( P^m_{2} \) we can conclude that \( \varphi'_k = D_2(\alpha'_k) \) for some \( \alpha'_k \in C^m_{2,k} \) and thus,

\[
\varphi_0 = D(\alpha_0).
\]

This means we are left only with the term \( \varphi_{0,S} \) from \( \varphi \).

Now consider \( \xi_0 \), define \( \xi'_k \in C^m_{1,k} \) by \( \xi_{0,k} = \lambda^k \xi'_k \). We want to find \( \eta_{0,k} \in C^m_{1,0,k} \) such that

\[
\xi_{0,1} = D^0_1(\alpha_{0,k+1}) + D_1(\eta_{0,1}) + D^k_1(\alpha_{0,1})
\]

\[
\xi_{0,k+2} = D^0_1(\alpha_{0,k+1}) + D_1(\eta_{0,k}) + D^k_1(\alpha_{0,k})
\]

However, we’ve already chosen our \( \alpha \)’s, so we really want

\[
D_1(\eta_{0,1}) = \xi_{0,1} - D^0_1(\alpha_{0,k+1}) - D^k_1(\alpha_{0,1})
\]

\[
D_1(\eta_{0,k}) = \xi_{0,k+2} - D^0_1(\alpha_{0,k+1}) - D^k_1(\alpha_{0,k})
\]

For \( k \geq 2 \) we have

\[
0 = D^0_1(\varphi_{0,k-1}) + D^k_1(\varphi_{0,k}) + D_1(\xi_{0,k})
\]

\[
= D^0_1 D_2(\alpha_{0,k-1}) + D_1(\xi_{0,k}) + D^k_1(\alpha_{0,k})
\]

\[
= D_1(\xi_{0,k} - D^0_1(\alpha_{0,k-1}) - D^k_1(\alpha_{0,k}))
\]

\[
= \max_k \lambda^k D_1(\xi_{0,k} - D^0_1(\alpha_{0,k-1}) - D^k_1(\alpha_{0,k})).
\]

By an argument similar to the \( \varphi_0 \), if \( \xi'_k \not\in C^0_1 \) then we can find an \( \eta'_k \in C^m_{1,k} \) such that \( \xi'_k = D^0_1(\alpha_{0,k-1}) + D^k_1(\alpha_{0,k}) + D_1(\eta'_k) \), more specifically if \( k \geq 2 \) we can find \( \eta_{0,k} \) such that

\[
\xi_{0,k+2} = D^0_1(\alpha_{0,k-1}) + D^k_1(\alpha_{0,k}) + D_1(\eta_{0,k})
\]
If \( k = 1 \) then we have

\[
0 = D_k^1(\varphi_{0,1}) + D_0^1(\varphi_{k \geq 2}) + D_1(\xi_{0,1})
\]

\[
= D_1^kD_2(\alpha_{0,1}) + D_0^kD_2(\alpha_{k \geq 2}) + D_1(\xi_{0,1})
\]

\[
= - D_1D_1(\alpha_{0,1}) - D_1D_0^k(\alpha_{k \geq 2}) + D_1(\xi_{0,1}) = D_1(\xi_{0,1} - D_1D_0^k(\alpha_{k \geq 2}))
\]

Thus we can write

\[
\xi_{0,1} = D_1^0(\alpha_{0,k \geq 2}) + D_1(\eta_{0,1}) + D_1(\alpha_{0,1})
\]

for some \( \eta_{0,1} \in \mathbb{C}_{2,0,1}^{n-1} \).

Finally consider the spaces \( S_2^0 \) and \( S_1^0 \). To begin we need to be more specific about the action of the coboundary operator on these spaces.

\[
D(\varphi_2^n) = \begin{cases} 2\varphi_1^{n+1} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}
\]

When \( n \) is odd \( D : S_2^0 \rightarrow S_1^{n+1} \). Thus we obtain the diagram in Figure 1. The spaces which are boxed are isolated spaces that go to zero, or contribute cohomology. Thus we see the cohomology is given by

\[
H^n = \begin{cases} \psi_2, \psi_1, & n = 0 \\ \phi_1^n, & n \text{ odd} \\ \phi_2^n, & n \neq 0 \text{ and } n \text{ even} \end{cases}
\]

Figure 1.

5.6. \( d_6 = \psi_1^{22} \). For case six, the methods employed in the previous cases completely fail, and as a result, a different method is necessary. Once again, we begin by computing the bracket of \( d \) with a general element in \( C_2^2 \) and \( C_1^1 \).

\[
D(\varphi_2^l) = \varphi_1^{l2} + (-1)^{l+1}\varphi_1^{2l} + (-1)^l\varphi_2^l\varphi_1^{22}
\]

\[
D(\varphi_1^l) = (-1)^l\varphi_1^l\varphi_1^{22}
\]
Therefore, we have decompositions
\[ D = D_2 + D_1 : C^n_2 \to C^{n+1}_2 \oplus C^{n+1}_1 \]
\[ D : C^n_1 \to C^{n+1}_1 \]

Note that \( D_1 \) is injective and that
\[ D_2^2 = 0, \quad D_1D_2 = -DD_1. \]

Thus \( D_2 \) is a coboundary operator on \( C^n_2 \), giving a cohomology \( H^n_2 = H^n(D_2) \).

First, let us say that \( \text{Ch} \) is essentially the same as \( \text{Ch} \) restricted only to the \( F \) space. We shall discuss later when it is a nontrivial cocycle on the whole space \( C^n = C^n_1 \oplus C^n_2 \).

Let \( B^n \) be the space of \( D_3 \) \( n \)-coboundaries, \( Z^n \) be the \( n \)-cocycles, \( z_n = \dim(Z^n) \), \( b_n = \dim(B^n) \) and \( h_n = \dim H^n \). Then \( h_n = z_n - b_n \) and \( z_n + b_{n+1} = 2^n \). Because there is a nontrivial Decleene cocycle in each degree, we know that \( h_n \geq 1 \). We wish to show that \( h_n = 1 \).

Thus, by induction, the formula holds for all \( n \). Using this formula, we obtain
\[ 1 \leq h_n = z_n - b_n = 2^n - b_{n+1} - b_n = 2^n - (2^n - 1) = 2. \]

First, let us say that \( \text{Ch} \) extends to a \( D \)-cocycle if there is some \( \eta \in C^n_1 \) such that \( \text{Ch} + \eta \) is a \( D \)-cocycle. If \( \text{Ch} \) extends, then let \( \text{Ch} = \text{Ch} + \eta \) be some arbitrary extension of \( \text{Ch} \).

Suppose that \( D(\phi + \xi) = 0 \) for some \( \phi \in C^n_2 \) and \( \xi \in C^n_1 \). Then \( D_1(\phi) + D(\xi) = 0 \) and \( D_2(\phi) = 0 \). In fact, the second equation follows from the first one. For, suppose the first equality holds. Then
\[ D_1D_2(\phi) = -DD_1(\phi) = D^1(\xi) = 0. \]
Using the fact that $D_1$ is injective, we see that $D_2(\varphi) = 0$. Now we can write $\varphi = a \text{Ch}_2^n + D_2(\alpha)$ for some $\alpha \in \mathbb{C}^{n-1}_2$, because we know that $h_n = 1$.

Note that if $\text{Ch}_2^n$ does not extend to a $D$-cocycle, then $a = 0$. This is because

$$0 = D_1(\varphi) + D(\xi) = D_1(a \text{Ch}_2^n) + D_1D_2(\alpha) + D(\xi) = D_1(a \text{Ch}_2^n) - D\text{D}_1(\alpha) + D(\xi) = D_1(a \text{Ch}_2^n) - D(D_1(\alpha) - \xi),$$

so that if $a \neq 0$ we have $D_1(\text{Ch}_2^n) = D(\eta)$, where $\eta = \frac{1}{2}(D_1(\alpha) - \xi)$. Next, we claim that $\varphi + \xi = b \text{Ch}_1^n + D(\alpha + \beta)$ for some $\beta \in \mathbb{C}^{n-1}_1$. To see this, first suppose that $a = 0$. Then

$$0 = D(\varphi + \xi) = D_1D_2(\alpha) + D(\xi) = -D(D_1(\alpha) + D(\xi) = D(\xi - D_1(\alpha)).$$

Thus $\xi - D_1(\alpha)$ is a $D$-cocycle lying in $\mathbb{C}^{n}_1$, which means it can be written in the form $\xi - D_1(\alpha) = b \text{Ch}_1^n + D(\beta)$ for some $\beta \in \mathbb{C}^{n-1}_1$. But this means $\varphi + \xi = b \text{Ch}_1^n + D(\alpha + \beta)$, as desired.

On the other hand, if $a \neq 0$ then $a \text{Ch}_2^n = a \text{Ch}_1^n + a\eta$, where $\eta \in \mathbb{C}^{n-1}_1$, so $\varphi = a \text{Ch}_1^n + a\eta + D_2(\alpha)$, and then

$$0 = D(\varphi + \xi) = D(a\eta) + D_1D_2(\alpha) + D(\xi) = D(\xi + a\eta - D_1(\alpha)).$$

Thus in this case, we can express $\xi + a\eta - D_1(\alpha) = b \text{Ch}_1^n + D(\beta)$, so we obtain

$$\varphi + \xi = a \text{Ch}_1^n + a\eta + \xi = a \text{Ch}_1^n + b \text{Ch}_2^n + D(\alpha + \beta).$$

From the equation above, it follows that the dimension of $H^n$ is at most 1, depending on whether $\text{Ch}_2^n$ extends to a $D$-cocycle and whether $\text{Ch}_1^n$ is a nontrivial cocycle.

Now we show that the non triviality of $\text{Ch}_1^n$ is linked to the whether or not we can extend $\text{Ch}_2^{n-1}$.

Suppose $\text{Ch}_1^n$ is trivial, ie. $\text{Ch}_1^n = D(\varphi + \xi)$ for some $\varphi \in \mathbb{C}^{n-1}_2$, $\xi \in \mathbb{C}^{n-1}_2$. $D_2(\varphi) = 0$ so $\varphi = a \text{Ch}_2^{n-1} + D_2(\alpha)$ for some $\alpha \in \mathbb{C}^{n-1}_2$. If $\text{Ch}_2^{n-1}$ extends, so $\text{Ch}_2^{n-1} = \text{Ch}_1^{n-1} + \eta$ then

$$\text{Ch}_1^n = D_1(\varphi) + D(\xi) = aD(\eta) + D_1D_2(\alpha) + D(\xi) = D(\eta) - D\text{D}_1(\alpha) + D(\xi) = D(\xi - D_1(\alpha)).$$

But then $\text{Ch}_1^n$ is a coboundary in the $C^n_1$ space, which is impossible. Thus if $\text{Ch}_1^n$ is trivial, $\text{Ch}_2^{n-1}$ does not extend to a $D$-cocycle.

On the other hand, suppose $\text{Ch}_2^{n-1}$ does not extend to a $D$-cocycle. Then $D_1(\text{Ch}_2^{n-1})$ is a $D$-cocycle, lying in $C^n_1$, which is nontrivial in terms of the $D$-cohomology restricted to $\mathbb{C}^{n}_1$, so we must have $D_1(\text{Ch}_2^{n-1}) = a \text{Ch}_1^n + D(\beta)$ for some $\beta \in \mathbb{C}^{n-1}_1$, where $a \neq 0$. But then $\text{Ch}_1^n = D(a \text{Ch}_2^{n-1} - \beta)$, and therefore $\text{Ch}_1^n$ is trivial.

We now show that the $\text{Ch}_2^n$ is non-trivial when $n$ is both even and odd. First when $n = 2 \mod 1$ suppose that $\text{Ch}_1^n = D(\varphi + \xi)$, where $\varphi \in \mathbb{C}^{n-1}_2$ and $\xi \in \mathbb{C}^{n-1}_2$. Since we also want $\varphi$ to be a $D_2$ cocycle, we conclude that $\varphi = a \text{Ch}_2^{n-1} + D_2(\alpha)$ for some $\alpha \in \mathbb{C}^{n-1}_2$. Since $\text{Ch}_1^n$ contains terms of the form $\varphi_1^{(21)} \frac{n-1}{2}$ then, by assumption, we must have this term appear in $D_1(\varphi) = aD_1(\text{Ch}_2^{n-1}) + D_1D_2(\alpha)$. The moduli space of complex two dimensional associative algebras 15
Clearly this term won’t appear in $D_1 D_2(\alpha)$, and $D_1 (\text{Ch}_2^{-n})$ will produce two terms of the proper form but of opposite sign. So $\text{Ch}_1^n$ must be non-trivial.

Now assume $n = 0 \mod 2$. Following the same lines as the previous argument, we see $\text{Ch}_1^n$ will contain terms of the form $\varphi_1^{(21)}$ and $\varphi_1^{(12)}$. But $\text{Ch}_2^{-n-1}$, will have no terms with a 1 on the edge and no double 2. Thus $\text{Ch}_1^n$ must be non-trivial.

We are able to conclude that $h^n(d) = 2$ for all $n$.

6. Infinitesimal Deformations

To compute the infinitesimal deformations we only need consider cohomology in degree two. That leaves only $d_2$, $d_5$, and $d_6$; each of which will be considered separately.

6.1. $d_2$. The cohomology of $d_2$ is given by $\psi_1^{11}$. We determine that

$$d_\varepsilon = \psi_2^{22} + t\psi_1^{11}$$

is isomorphic to $d_1$ when $t \neq 0$.

6.2. $d_5$. The cohomology of $d_5$ is given by $\psi_2^{11}$. We determine that

$$d_\varepsilon = \psi_2^{22} + \psi_2^{11} + \psi_4^{12} + t\psi_2^{11}$$

is isomorphic to $d_1$ when $t \neq 0$.

6.3. $d_6$. The cohomology of $d_6$ is given by $\psi_2^{21} + \psi_2^{12} + \psi_1^{11}$ and $\psi_1^{21} + \psi_1^{12}$. We determine that

$$d_\varepsilon = \psi_1^{22} + t_2(\psi_2^{21} + \psi_2^{12} + \psi_1^{11}) + t_1(\psi_1^{21} + \psi_1^{12})$$

is isomorphic to $d_2$ when $t_1 = t_2$, to $d_2$ when $t_1 = \frac{3t_2}{4}$, and when $t_2$ and $t_1$ are not both zero to $d_1$.

7. Versal Deformations

In this case the versal deformations coincide exactly with the infinitesimal deformations. This can be seen by taking the bracket of the infinitesimal deformation with itself. If the result is zero then the deformations coincide.

7.1. Diagram of Deformations. For a visual of the deformations we provide Figure 2

![Figure 2](image-url)

Figure 2. Infinitesimal and Versal deformations of an 0|2-dimensional vector space.
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