On residually finite groups satisfying an Engel type identity

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Abstract
Let \( n, q \) be positive integers. We show that if \( G \) is a finitely generated residually finite group satisfying the identity \( [x, n y^q] \equiv 1 \), then there exists a function \( f(n) \) such that \( G \) has a nilpotent subgroup of finite index of class at most \( f(n) \). We also extend this result to locally graded groups.

Keywords Engel element · Engel groups · Residually finite groups · Locally graded groups · Lie algebras

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1 Introduction
Let \( n \) be a positive integer. We say that a group \( G \) is (left) \( n \)-Engel if it satisfies the identity \( [y, n x] \equiv 1 \), where the word \([x, n y]\) is defined inductively by the rules

\[
[x, 1 y] = x^{-1} y^{-1} x y, \quad [x, n y] = [[[x, n-1 y], y] \quad \text{for all } n \geq 2.
\]

A important theorem of Wilson [13, Theorem 2] says that finitely generated residually finite \( n \)-Engel groups are nilpotent. More specific properties of residually finite \( n \)-Engel groups can be found for example in a theorem of Burns and Medvedev (quoted below as Theorem 5) stating that there exist functions \( c(n) \) and \( e(n) \) such that any residually finite \( n \)-Engel group \( G \) has a nilpotent normal subgroup \( N \) of class at most \( c(n) \) such that the quotient group \( G/N \) has exponent dividing \( e(n) \). The interested reader is referred to the survey [12] and references therein for further results on finite and

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Dedicated to Pavel Shumyatsky on the occasion of his 60th birthday.

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residually finite Engel groups. The purpose of the present article is to provide the proof for the following theorem.

**Theorem 1** Let $G$ be a finitely generated residually finite group satisfying the identity $[x, y^n] \equiv 1$. Then there exists a function $f(n)$ such that $G$ has a nilpotent subgroup of finite index of class at most $f(n)$.

A group is called locally graded if every non-trivial finitely generated subgroup has a proper subgroup of finite index. The class of locally graded groups contains locally (soluble-by-finite) groups as well as residually finite groups. We can extend the Theorem 1 to the class of locally graded groups.

**Corollary 1** Let $G$ be a finitely generated locally graded group satisfying the identity $[x, y^n] \equiv 1$. Then there exists a function $f(n)$ such that $G$ has a nilpotent subgroup of finite index of class at most $f(n)$.

In the next section we describe the Lie-theoretic machinery that will be used in the proof of Theorem 1. The proof of the theorem and of the corollary is given in Sect. 3.

### 2 About Lie algebras

Let $L$ be a Lie algebra over a field $K$ and $X$ a subset of $L$. By a commutator in elements of $X$ we mean any element of $L$ that can be obtained as a Lie product of elements of $X$ with some system of brackets. If $x_1, \ldots, x_k, x, y$ are elements of $L$, we define inductively

$$[x_1] = x_1; [x_1, \ldots, x_k] = [[x_1, \ldots, x_{k-1}], x_k]$$

and $[x_0, y] = x; [x, m y] = [[x, m-1 y], y]$, for all positive integers $k, m$. As usual, we say that an element $a \in L$ is ad-nilpotent if there exists a positive integer $n$ such that $[x, a^n] = 0$ for all $x \in L$. Denote by $F$ the free Lie algebra over $K$ on countably many free generators $x_1, x_2, \ldots$. Let $f = f(x_1, x_2, \ldots, x_n)$ be a non-zero element of $F$. The algebra $L$ is said to satisfy the identity $f \equiv 0$ if $f(l_1, l_2, \ldots, l_n) = 0$ for any $l_1, l_2, \ldots, l_n \in L$.

The next theorem represents the most general form of the Lie-theoretical part of the solution of the Restricted Burnside Problem [15,17,18]. It was announced by Zelmanov [15]. A detailed proof can be found in [18].

**Theorem 2** Let $L$ be a Lie algebra over a field and suppose that $L$ satisfies a polynomial identity. If $L$ can be generated by a finite set $X$ such that every commutator in elements of $X$ is ad-nilpotent, then $L$ is nilpotent.

### 2.1 Associating a Lie ring to a group

Let $G$ be a group. A series of subgroups

$$G = G_1 \geq G_2 \geq \ldots$$

(*)
is called an $N$-series if it satisfies $[G_i, G_j] \leq G_{i+j}$ for all $i, j \geq 1$. Obviously any $N$-series is central, i.e. $G_i/G_{i+1} \leq Z(G/G_{i+1})$ for any $i$. Let $p$ be a prime. An $N$-series is called $N_p$-series if $G^p_i \leq G_{pi}$ for all $i$. Given an $N$-series ($\ast$), let $L^*(G)$ be the direct sum of the abelian groups $L^*_i = G_i/G_{i+1}$, written additively. Commutation in $G$ induces a binary operation $[\cdot, \cdot]$ in $L^*(G)$. For homogeneous elements $xG_{i+1} \in L^*_i$, $yG_{j+1} \in L^*_j$ the operation is defined by

$$[xG_{i+1}, yG_{j+1}] = [x, y]G_{i+j+1} \in L^*_{i+j}$$

and extended to arbitrary elements of $L^*(G)$ by linearity. It is easy to check that the operation is well-defined and that $L^*(G)$ with the operations $+$ and $[\cdot, \cdot]$ is a Lie ring. If all quotients $G_i/G_{i+1}$ of an $N$-series ($\ast$) have prime exponent $p$ then $L^*(G)$ can be viewed as a Lie algebra over the field with $p$ elements. In the important case where the series ($\ast$) is the $p$-dimension central series (also known under the name of Zassenhaus-Jennings-Lazard series) of $G$ we write $D_i = D_i(G) = \prod_{j \geq i} G_j(G)^{p^k}$ for the $i$-th term of the series of $G$, $L(G)$ for the corresponding associated Lie algebra over the field with $p$ elements and $L_p(G)$ for the subalgebra generated by the first homogeneous component $D_1/D_2$ in $L(G)$. Observe that the $p$-dimension central series is an $N_p$-series (see [5, p. 250] for details).

The nilpotency of $L_p(G)$ has strong influence in the structure of a finitely generated pro-$p$ group $G$. The proof of the following theorem can be found in [4, 1.(k) and 1.(o) in Interlude A].

**Theorem 3** Let $G$ be a finitely generated pro-$p$ group. If $L_p(G)$ is nilpotent, then $G$ is $p$-adic analytic.

Let $x \in G$ and let $i = i(x)$ be the largest positive integer such that $x \in D_i$ (here, $D_i$ is a term of the $p$-dimensional central series to $G$). We denote by $\tilde{x}$ the element $xD_{i+1} \in L(G)$. We now quote two results providing sufficient conditions for $\tilde{x}$ to be ad-nilpotent. The following lemma was established in [6, p. 131].

**Lemma 1** For any $x \in G$ we have $(ad \tilde{x})^p = ad (\tilde{x}^p)$.

**Corollary 2** Let $x$ be an element of a group $G$ for which there exists a positive integer $m$ such that $x^m$ is $n$-Engel. Then $\tilde{x}$ is ad-nilpotent.

The following theorem is a particular case of a result that was established by Wilson and Zelmanov in [14].

**Theorem 4** Let $G$ be a group satisfying an identity. Then for each prime number $p$ the Lie algebra $L_p(G)$ satisfies a polynomial identity.

### 3 Proof of the main theorem

The following useful result is a consequence of [13, Lemma 2.1] (see also [11, Lemma 3.5] for details).
Lemma 2 Let $G$ be a finitely generated residually finite-nilpotent group. For each prime $p$ let $J_p$ be the intersection of all normal subgroups of $G$ of finite $p$-power index. If $G/J_p$ has a nilpotent subgroup of finite index of class at most $c$ for each $p$, then $G$ also has a nilpotent subgroup of finite index of class at most $c$.

Proof It follows from proof of [11, Lemma 3.5] that there exists a finite set of primes $\pi$ such that $G$ embeds in the direct product \( \prod_{p \in \pi} G/J_p \). We will identify $G$ with its images in direct product. By hypothesis, for any $p \in \pi$, $G/J_p$ contains a nilpotent subgroup of finite index $H_p$ with class at most $c$. Set $H = \cap_{p \in \pi} H_p$. Thus, $G \cap H$ has finite index in $G$ and has nilpotency class at most $c$, which completes the proof.

Recall that a group is locally graded if every non-trivial finitely generated subgroup has a proper subgroup of finite index. Note that the quotient of a locally graded group need not be locally graded, since free groups are locally graded (see [10, 6.1.9]), but no finitely generated infinite simple group is locally graded. However, the following results give a sufficient conditions for a quotient to be locally graded (see [7] for details).

Lemma 3 Let $G$ be a locally graded group and $N$ a normal locally nilpotent subgroup of $G$. Then $G/N$ is locally graded.

Let $p$ be a prime and $q$ be a positive integer. A finite $p$-group $G$ is said to be powerful if and only if $[G, G] \leq G^p$ for $p \neq 2$ (or $[G, G] \leq G^4$ for $p = 2$), where $G^q$ denotes the subgroup of $G$ generated by all $q$th powers. While considering a pro-$p$ group $G$ we shall be interested only in closed subgroups. So by the commutator subgroup $G^q = [G, G]$ we mean the closed commutator subgroup, $G^q$ means the closed subgroup generated by the $q$th powers. Similarly to powerful finite $p$-groups, we may define the powerful pro-$p$ groups. For more details we refer the reader to [7, Chapters 2 and 3]. In [1] the following useful result for powerful finite $n$-Engel $p$-group was established.

Lemma 4 There exists a function $s(n)$ such that any powerful finite $n$-Engel $p$-group is nilpotent of class at most $s(n)$.

The proof of Theorem 1 will requires the following lemma.

Lemma 5 Let $s(n)$ be as in Lemma 4. If $G$ is a finitely generated powerful pro-$p$ group satisfying the identity $[x, n y^q] = 1$, then $G^q$ has nilpotency class at most $s(n)$.

Proof Since $G$ satisfies the identity $[x, n y^q] = 1$, we can deduce from [4, Corollary 3.5] that $H = G^q = \{x^q \mid x \in G\}$ is a powerful $n$-Engel pro-$p$ group. According to [4, Corollary 3.3], $H$ is the inverse limit of an inverse system of powerful finite $p$-groups $H_\lambda$. Lemma 4 implies that any group $H_\lambda$ has class at most $s(n)$, and so, $H$ has class at most $s(n)$ as well. Finally, by a result due to Zelmanov [16, Theorem 1] saying that any torsion profinite group is locally finite we get that the quotient group $G/H$ is finite. This completes the proof.\[\square\]

The proof of Theorem 1 will also require the following result, due to Burns and Medvedev [3].
Theorem 5 There exist functions $c(n)$ and $e(n)$ such that any residually finite $n$-Engel group $G$ has a nilpotent normal subgroup $N$ of class at most $c(n)$ such that $G/N$ has exponent dividing $e(n)$. □

We are now ready to embark on the proof of our main result.

Proof of Theorem 1 For any positive integer $n$ let $s(n)$ and $c(n)$ be as in Lemma 4 and Theorem 5, respectively. Set $f(n) = \max\{s(n), c(n)\}$. Since $G$ satisfies the identity $[x, y^n] \equiv 1$ we can deduce from [2, Theorem A] that $H = G^q$ is locally nilpotent. According to Lemma 3, $G/H$ is locally graded. By Zelmanov’s solution of the Restricted Burnside Problem [15,17,18], locally graded groups of finite exponent are locally finite (see for example [8, Theorem 1]), and so $G/H$ is finite. Thus $H$ is finitely generated and so it is nilpotent.

By Lemma 2, we can assume that $H$ is residually (finite $p$-group) for some prime $p$. If $p$ does not divides $q$, then $H$ is finitely generated residually finite $n$-Engel group. By Theorem 5, $H$ contains a nilpotent normal subgroup $N$ of class at most $f(n)$ such that the quotient group $G/N$ has exponent dividing $e(n)$. Thus, we can see that $G/N$ is finite. Thereby, in what follows we can assume that $H$ is residually (finite $p$-group), where $p$ divides $q$.

Set $H = \langle h_1, \ldots, h_t \rangle$. Let $L = L_p(H)$ be the Lie algebra associated with the $p$-dimensional central series of $H$. Then $L$ is generated by $h_i \in h_i D_2, i = 1, 2, \ldots, t$. Let $h$ be any Lie-commutator in $\tilde{h}_i$ and $h$ be the group-commutator in $h_i$ having the same system of brackets as $h$. Since for any group commutator $h$ in $h_1, \ldots, h_t$ we have that $h^q$ is $n$-Engel, Corollary 2 shows that any Lie commutator in $\tilde{h}_1, \ldots, \tilde{h}_t$ is ad-nilpotent. Since $H$ satisfies the identity $[x, y^n] \equiv 1$, by Theorem 4, $L$ satisfies some non-trivial polynomial identity. According to Theorem 2 $L$ is nilpotent.

Let $\hat{H}$ be the pro-$p$ completion of $H$, that is, the inverse limit of all quotients of $H$ which are finite $p$-groups. Notice that $\hat{H}$ is finitely generated, being $H$ finitely generated.

Since the finite $p$-quotients of $H$ are the same as the finite $p$-quotients of $\hat{H}$ by (a) and (d) of [9, Proposition 3.2.2], we get that $L_p(\hat{H}) = L$. Hence, $L_p(\hat{H})$ is nilpotent and so, $\hat{H}$ is $p$-adic analytic group by Theorem 3.

By [4, 1.(a) and 1.(o) in Interlude A], $\hat{H}$ is virtually powerful, that is, $\hat{H}$ has a powerful subgroup $K$ of finite index. By Lemma 5, $K^q$ has class at most $f(n)$. Furthermore, it follows from [16, Theorem 1] that group $K^{\hat{K}}$ is finite. Finally, since $H$ is residually-$p$, it embeds in $\hat{H}$. Thus, $H \cap K^q$ is a nilpotent subgroup of finite index in $G$ of class at most $f(n)$. This completes the proof. □

Proof of Corollary 1 Let $f(n)$ be as in Theorem 1. It follows from [2, Theorem C] that $H = G^q$ is locally nilpotent. By Lemma 3, $G/H$ is a locally graded group. By Zelmanov’s solution of the Restricted Burnside Problem, locally graded groups of finite exponent are locally finite. Thus, $G/H$ is finite and so, $H$ is a finitely generated nilpotent group. Since polycyclic groups are residually finite [10, 5.4.17], we can deduce from Theorem 1 that $H$ contains a subgroup of finite index and of class at most $f(n)$. The proof is complete.

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