GKZ Systems, Gröbner Fans and Moduli Spaces of Calabi-Yau Hypersurfaces

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Abstract

We present a detailed analysis of the GKZ (Gel’fand, Kapranov and Zelevinski) hypergeometric systems in the context of mirror symmetry of Calabi-Yau hypersurfaces in toric varieties. As an application we will derive a concise formula for the prepotential about large complex structure limits.

1. Introduction

Mirror symmetry of Calabi-Yau manifolds has been playing a central role in revealing non-perturbative aspects of the type II string vacua, i.e., the moduli spaces for a family of Calabi-Yau manifolds. Since the success in determining the quantum geometry on the IIA moduli space made by Candelas et al. [CdGP] in 1991, there have been many progresses and a lot of communications between physics and mathematics on this topic [CY].

In this article, we will be concerned with the mirror symmetry of Calabi-Yau hypersurfaces in toric varieties. In this case, the mirror symmetry may be traced to a rather combinatorial properties of the reflexive polytopes which determines the ambient toric varieties due to ref. [Bat1]. Furthermore since the period integrals of Calabi-Yau hypersurfaces turn out to satisfy the hypergeometric differential equation, $A$-hypergeometric system, introduced by Gel’fand, Kapranov and Zelevinski (GKZ), we can study in great detail the moduli spaces of Calabi-Yau hypersurfaces. Based on the analysis of GKZ-hypergeometric system in our context, we will derive a closed formula for the prepotential, which defines the special Kähler geometry on the moduli spaces.

In section 2, we will review the mirror symmetry of Calabi-Yau hypersurfaces in toric varieties. This is meant to fix our notations as well as to introduce the mirror symmetry due to Batyrev. In section 3, we will introduce GKZ-hypergeometric system ($\Delta^*$-hypergeometric system) as an infinite set of differential equations satisfied by period integrals and summarize basic results following [GKZ]. We also define the extended $\Delta^*$-hypergeometric system incorporating the automorphisms of the toric
varieties. We will remark that the $\Delta^*$-hypergeometric system in our context is resonant in general. In section 4, we will review basic properties of the toric ideal and the Gröbner fan as an equivalence classes of the term orders in the toric ideal. We will use the Gröbner fan to compactify the space of the variables in the $\Delta^*$-hypergeometric system, and propose it as a natural compactification of the corresponding family of Calabi-Yau hypersurfaces. In section 5, we will prove general existence of the so-called large complex structure limits, at which the monodromy becomes maximally unipotent [Mor]. We will also present a general formula for the local solutions about these points. In the final section, we will derive a closed formula for the prepotential, which is valid about a large complex structure limit for arbitrary Calabi-Yau hypersurfaces in toric varieties. Our formula determines the special Kähler geometry about a large complex structure limit as well as the quantum corrected Yukawa coupling. Claim 5.8, Claim 5.11, and Claim 6.8 in the last two sections are meant to state those results that are verified in explicit calculations by many examples without general proofs.

All the results except Prop.6.7 for the prepotential in the final section have already reported in refs. [HKTY1] [HKTY2] [HLY1] [HLY2].

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2. Mirror Symmetry of Calabi-Yau Hypersurfaces

In this section, we will summarize mirror symmetry of Calabi-Yau hypersurfaces in toric varieties due to Batyrev. We refer the paper [Bat1] for details.

Let $M \cong \mathbb{Z}^d$ be a lattice of rank $d$ and $N$ be its dual. We denote the dual pairing $M \times N \to \mathbb{Z}$ by $\langle , \rangle$. A (convex) polytope $\Delta$ is a convex hull of a finite set of points in $M_{\mathbb{R}} := M \otimes \mathbb{R}$. In the following, we assume $\Delta$ contains the origin in its interior. The polar dual $\Delta^* \subset N_{\mathbb{R}}$ is defined by

$$\Delta^* = \{ x \in N_{\mathbb{R}} | \langle x, y \rangle \geq -1, \ y \in \Delta \} . \quad (1)$$

Definition 2.1. A polytope $\Delta$ is called reflexive if it is a convex hull of
a finite set of integral points in $M_\mathbb{R}$ and contains only the origin in its interior.

**Proposition 2.2.** When a polytope $\Delta$ is reflexive, its dual $\Delta^*$ is also reflexive.

Since $(\Delta^*)^* = \Delta$, reflexive polytopes come with a pair $(\Delta, \Delta^*)$. The following descriptions about $\Delta^*$ with $N$ equally apply to $\Delta$ with $M$ by symmetry.

**Definition 2.3.** A maximal triangulation $T_o$ of $\Delta^*$ is a simplicial decomposition of $\Delta^*$ with properties; 1) every $d$-simplex contains the origin as its vertex, 2) 0-simplices consist of all integral points of $\Delta^*$.

For a maximal triangulation $T_o$ of $\Delta^*$, we consider a complete fan $\Sigma(\Delta^*, T_o)$ over the triangulation $T_o$ in $N_\mathbb{R}$. Associated to the data $(\Sigma(\Delta^*, T_o), N)$ we consider a toric variety $P_{\Sigma(\Delta^*, T_o)}$ [Oda][Ful]. Due to the property that $\Delta^*$ is reflexive, we have

**Proposition 2.4.** (Prop.2.2.19 in [Bat1]) $P_{\Sigma(\Delta^*, T_o)}$ is a projective variety for at least one maximal triangulation with its anti-canonical class $-K = \sum_{\rho \in N \cap \Delta^* \setminus \{0\}} D_\rho$ ample.

**Note.** In [Bat1], the maximal triangulations with the property in this proposition are called projective. In case of $d \leq 4$, we can observe widely that every maximal triangulation is projective. More generally we observe that every triangulation of a reflexive polytope is regular which generalize projectivity (see right after eq.(11) for the definition). For a restricted class of reflexive polytopes (the type I or II in the following classification), it has been proved (Th.4.10 in [HLY2]) that every nonsingular maximal triangulation is projective, see also Remark after Th.2.5. In the following, we will write a projective toric variety by $P_{\Sigma(\Delta^*, T_o)}$ choosing a projective maximal triangulation $T_o$ of $\Delta^*$.

Let us fix a basis $\{n_1, \cdots, n_d\}$ of $N$ and denote its dual basis by $\{m_1, \cdots, m_d\}$. With respect to this basis, we denote the coordinate ring of the torus $T_N := \text{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*) \subset P_{\Sigma(\Delta^*, T_o)}$ by $\mathbb{C}[Y_1^{\pm 1}, \cdots, Y_d^{\pm 1}]$ with the generators $Y_k = e(m_k) : T_N \to \mathbb{C}^*$ defined by $e(m_k)(t) = t(m_k)$. Consider a Laurent polynomial $f_\Delta = \sum_{\nu \in \Delta \cap M} c_\nu Y^\nu$ with complex coefficients $c_\nu$. We denote by $X_\Delta$ the Zariski closure of the zero locus ($f_\Delta = 0$) in $P_{\Sigma(\Delta^*, T_o)}$ for generic $c_\nu$'s. Similarly, we consider a projective toric variety $P_{\Sigma(\Delta^*, T_o)}$ associated to a projective maximal triangulation $T_o$ of $\Delta$, and denote the coordinate ring of $T_M := \text{Hom}_{\mathbb{Z}}(N, \mathbb{C}^*) \subset P_{\Sigma(\Delta^*, T_o)}$ by $\mathbb{C}[X_1^{\pm 1}, \cdots, X_d^{\pm 1}]$ with $X_k = e(n_k)$. 
Theorem 2.5. (Th.4.2.2, Corollary 4.2.3, Th.4.4.3 in [Bat1]) Let $(\Delta, \Delta^*)$ be a pair of reflexive polytopes in dimensions $d \leq 4$ (in $M_\mathbb{R}$ and $N_\mathbb{R}$, respectively). Then; 1) generic hypersurfaces $X_\Delta \subset \mathbb{P}_{\Sigma(\Delta^*, T_0)}$ and $X_{\Delta^*} \subset \mathbb{P}_{\Sigma(\Delta, T_0)}$ define smooth Calabi-Yau manifolds, 2) these two hypersurfaces are mirror symmetric in their Hodge numbers, i.e., $h^{1,1}(X_\Delta) = h^{d-2,1}(X_{\Delta^*})$; $h^{d-2,1}(X_\Delta) = h^{1,1}(X_{\Delta^*})$.

Remark. Depending on the toric data of the reflexive polytopes, the ambient spaces have (Gorenstein) singularities (Prop.2.2.2 in [Bat1]) in general. We call a maximal triangulation is nonsingular if its simplices of maximal dimensions consists of unit simplices, i.e., simplices with unit volume. It is easy to deduce that the toric variety is nonsingular if the maximal triangulation is so. Now we classify the reflexive polytopes into the following three types:

- type I; the polytope has no integral point in the interior of all codimension one faces, and has a nonsingular maximal triangulation,
- type II; the polytope has at least one integral point in the interior of some codimension one face, and has a nonsingular maximal triangulation,
- type III; the polytope does not have a nonsingular maximal triangulation.

In the following, we always consider a nonsingular maximal triangulation $T_o$ for the polytopes of type I and II. Then the toric varieties $\mathbb{P}_{\Sigma(\Delta^*, T_o)}$ are projective and nonsingular for both the polytopes $\Delta^*$ of type I and II (Th.4.10 in [HLY2]), however we distinguish these two because of the difference in the root system for their dual polytopes $\Delta$;

$$R(\Delta, M) = \{ \alpha \in \Delta^* \cap N \mid \exists m_\alpha \in \Delta \cap M \text{ s.t. } \langle m_\alpha, \alpha \rangle = -1 \text{ and } \langle m, \alpha \rangle \geq 0 (m \neq m_\alpha \in \Delta \cap M) \}$$

The root system determines the automorphisms of the toric variety $\mathbb{P}_{\Sigma(\Delta, T_o)}$ infinitesimally due to the following result, which we will utilize in the next section;

Proposition 2.6. (Prop.3.13 in [Oda]) For a nonsingular toric variety $\mathbb{P}_{\Sigma(\Delta, T_o)}$, we have a direct sum decomposition via the root system $R(\Delta, M)$;

$$\text{Lie}(\text{Aut}(\mathbb{P}_{\Sigma(\Delta, T_o)})) = H^0(\mathbb{P}_{\Sigma(\Delta, T_o)}, \Theta_{\mathbb{P}_{\Sigma(\Delta, T_o)}}) = \text{Lie}(T_M) \oplus (\oplus_{\alpha \in R(\Delta, M)} \mathbb{C} e(\alpha) \delta_{m_\alpha})$$

where $\delta_m (m \in M)$ is the derivation on $T_M$ defined by $\delta_m e(n) := \langle m, n \rangle e(n)$. 


3. Resonance in GKZ Hypergeometric System

We consider a family of Calabi-Yau hypersurfaces $X_{\Delta^*}(a) \subset \mathbb{P}_{\Sigma(\Delta^*, T_0)}$ varying the coefficients $a_{\nu^*}$ in the defining equation $f_{\Delta^*}(a) = \sum_{\nu^* \in \Delta^* \cap N} a_{\nu^*} X_{\nu^*}$. By this polynomial deformation, we describe the complex structure deformation of $X_{\Delta^*}$. This deformation space is mapped to that of (complexified) Kähler class of $X_{\Delta} \subset \mathbb{P}_{\bar{\Sigma}(\Delta^*, T_0)}$ under the mirror symmetry. According to the local Torelli theorem [BG], we can introduce a local coordinate on the moduli space in terms of period integrals. In case of hypersurfaces in toric varieties, we have one canonical period integral [Bat2][BC]

$$\Pi(a) = \frac{1}{(2\pi i)^d} \int_{C_0} \frac{1}{f_{\Delta^*}(X, a)} \prod_{i=1}^d \frac{dX_i}{X_i},$$

with the cycle $C_0 = \{|X_1| = \cdots = |X_d| = 1\}$ in $T_M$. Here we study the differential equation satisfied by (4).

(3-1) Extended GKZ hypergeometric system Let $A = \{\bar{\chi}_0, \cdots, \bar{\chi}_p\}$ be a finite set of integral points in $\{1\} \times \mathbb{R}^n \subset \mathbb{R}^{n+1}$. We assume the vectors $\bar{\chi}_0, \cdots, \bar{\chi}_p$ span $\mathbb{R}^{n+1}$.

**Definition 3.1.** Consider the lattice of relations among the set $A$,

$$L = \{(l_0, \cdots, l_p) \in \mathbb{Z}^{p+1} | \sum_{i=0}^p l_i \bar{\chi}_{i,j} = 0, (j = 1, \cdots, n+1)\} ,$$

where $\bar{\chi}_{i,j}$ represents the $j$-th component of the vector $\bar{\chi}_i$. $A$-hypergeometric system with exponent $\beta \in \mathbb{C}^{n+1}$ is a system of differential equations for a complex function $\Psi(a)$ on $\mathbb{C}^A$,

$$D_l \Psi(a) = \left\{ \prod_{l_i > 0} \left( \frac{\partial}{\partial a_i} \right)^{l_i} - \prod_{l_i < 0} \left( \frac{\partial}{\partial a_i} \right)^{-l_i} \right\} \Psi(a) = 0 \ (l \in L)$$

$$Z \Psi(a) = \{\sum_{i=0}^p \bar{\chi}_i a_i \frac{\partial}{\partial a_i} - \beta\} \Psi(a) = 0 .$$

**Proposition 3.2.** ([Bat2]) The period integral (4) satisfies the $A$-hypergeometric system with $A = \{1\} \times (\Delta^* \cap N)$ and $\beta = (-1) \times \bar{0}$. We call this hypergeometric system as $\Delta^*$-hypergeometric system.

By direct evaluation of the action of $D_l$ and $Z$ on the period integral (4), we obtain this proposition. Here we consider the meaning of the linear operator $Z$. The first component of this vector operator is exactly
the Euler operator, and just says that the period integral has homogeneous
degree $-1$ as a function of $a_i$’s. For the other components, it is easy to
deduce that these come from the invariance of the period integral under the
torus actions, which act infinitesimally on the coordinate $X_k = e(n_k)$ by
delta_m X_k = \langle m, n_k \rangle X_k$. It is now clear that these actions should be considered
for all elements in Lie(Aut ($P_{\Sigma(\Delta, T_o)}$)). Then we may write the invariance of
the period integral under the infinitesimal action of $\xi \in$ Lie(Aut ($P_{\Sigma(\Delta, T_o)}$))
by a linear differential operator $Z_\xi$ acting on $\Pi(a)$ through
$$Z_\xi \Pi(a) = \int_{C_0} \xi \left( \frac{1}{f_{\Delta^*}(X, a)} \right) \prod_{i=1}^{d} \frac{dX_i}{X_i} = 0 .$$
(8)

For explicit forms of the operators $Z_\xi$, we refer to the examples given in
p.541 of [HLY1].

Proposition 3.3. ((2.13) in [HLY1]) The period integral $\Pi(a)$ satisfies
$$D_l \Pi(a) = 0 \ (l \in L) ,$$
$$Z_E \Pi(a) = 0 , \ Z_\xi \Pi(a) = 0 \ \ (\xi \in$ Lie(Aut ($P_{\Sigma(\Delta, T_o)}$))$),$$
where we denote the Euler operator by $Z_E = \sum_{i=0}^{p} a_i \frac{\partial}{\partial a_i} + 1$.

We call this system as extended GKZ-hypergeometric system or
extended $\Delta^*$-hypergeometric system. By Prop.2.6, it is clear that this ex-
tended system reduces to the GKZ system if the polytope $\Delta^*$ is of type I. In
the following, we take an approach to study mainly the $\Delta^*$-hypergeometric
system because the set of the solutions of the extended $\Delta^*$-hypergeometric
system can be found in that of the $\Delta^*$-hypergeometric system.

(3-2) Convergent series solutions Here we summarize general results
in [GKZ1] about the convergent series solution of the $\mathcal{A}$-hypergeometric
system with exponent $\beta$. This is to introduce the notion of the secondary
fan as well as to fix our conventions and notations. Since our interest
is in the period integrals, we assume $\mathcal{A} = \{1\} \times (\Delta^* \cap N)$ and $\beta =
(-1) \times \vec{0}$. Hereafter we write the integral points explicitly by $\Delta^* \cap N =
\{\nu_0, \ldots, \nu_p\} (\nu_0^* \equiv \vec{0})$ and $\vec{\nu}_i^* := 1 \times \nu_i^* \ (i = 0, \ldots, p)$.

We start with a formal solution of the $\mathcal{A}$-hypergeometric system with
exponent $\beta$ given by
$$\Pi(a, \gamma) = \sum_{\alpha \in \mathcal{A}} \frac{1}{\prod_{0 \leq i \leq p} \Gamma(l_i + \gamma_i + 1)} d^{l+\gamma} ,$$
(10)
where $\beta = \sum_1 \gamma_i \vec{\nu}_i^*$. Now define an affine space $\Phi(\beta) := \{ \gamma \in \mathbb{R}^{p+1} | \beta =
\sum_1 \gamma_i \vec{\nu}_i^* \}$. A subset $I \subset \{0, \ldots, p\}$ is called a base if $\vec{\nu}_I^* := \{\vec{\nu}_i^* | i \in I\}$ form
a basis of $\mathbb{R}^{d+1}$. Given a base $I$, we may solve $\sum_{j \in I} \gamma_j \bar{v}_j = \beta - \sum_{j \notin I} \gamma_j \bar{v}_j$ for $\gamma_j$ ($j \in I$) and define $\Phi_{\mathbb{Z}}(\beta, I) = \{ \gamma \in \Phi(\beta) \mid \gamma_j \in \mathbb{Z} \text{ for } j \notin I \}$. We choose an integral basis $A = \{ l^{(1)}, \ldots, l^{(p-d)} \}$ of the lattice $L$, and define $\Phi_{\mathbb{Z}}^A(\beta, I) = \{ \gamma \in \Phi_{\mathbb{Z}}(\beta, I) \mid \gamma_j = \sum_{k=1}^{p-d} \lambda_k I_j^k (0 \leq \lambda_k < 1, j \notin I) \}$. Then $\Phi_{\mathbb{Z}}^A(\beta, I)$ provides a set of representatives of the quotient $\Phi_{\mathbb{Z}}(\beta, I)/L$ and kills the invariance $\gamma \rightarrow \gamma + v (v \in L)$ in the formal solution $(\mathbb{I})$.

**Definition 3.4.** For a base $I$, define a cone in $L_\mathbb{R} = L \otimes \mathbb{R}$ by $\mathcal{K}(\mathcal{A}, I) = \{ l \in L_\mathbb{R} \mid l_i \geq 0 \text{ for } i \notin I \}$. A $\mathbb{Z}$-basis $A \subset L$ is said compatible with a base $I$ if it generates a cone that contains $\mathcal{K}(\mathcal{A}, I)$.

**Theorem 3.5.** (Prop.1 in [GKZ1]) Fix a base $I$ and choose a $\mathbb{Z}$-basis $A = \{ l^{(1)}, \ldots, l^{(p-d)} \}$ compatible with it. Then the formal solution $(\mathbb{I})$ takes the form $\Pi(\alpha, \gamma) = a^\gamma \sum_{m \in \mathbb{Z}^{p-d}} c_m(\gamma) x^m$ for each $\gamma \in \Phi_{\mathbb{Z}}^A(\beta, I)$ with $x_k = a^{l_j^k}$. This powerseries converges for sufficiently small $|x_k|$.

**Remark.** The coefficient $c_m(\gamma)$ is given explicitly by

$$c_m(\gamma) = \frac{1}{\prod_{i=0}^{p} \Gamma(\sum_k m_k I_i^k + \gamma_i + 1)}.$$ 

For some index $i$ of the base $I$ in the above theorem it can happen that $\sum m_k I_i^k + \gamma_i + 1$ is non-positive for all $m \in \mathbb{Z}_{\geq 0}^{p-d}$, which means we have the trivial solution $\Pi(\alpha, \gamma) \equiv 0$. In this case, we multiply an infinite number $\Gamma(\gamma_i + 1)$ to obtain nonzero powerseries, i.e., $\frac{\Gamma(\gamma_i + 1)}{\Gamma(\sum m_k I_i^k + \gamma_i + 1)} := \lim_{\varepsilon \rightarrow 0} \frac{\Gamma(\gamma_i + 1 + \varepsilon)}{\Gamma(\sum m_k I_i^k + \gamma_i + 1 + \varepsilon)}$.

**Definition 3.6.** Consider $P = \text{Conv.} \left( \{0, \bar{v}_0^*, \ldots, \bar{v}_p^*\} \right)$ in $\mathbb{R}^{d+1}$. A collection of bases $T = \{ I \}$ is called a triangulation of $P$ if $\bigcup_{I \in T} \{ \bar{v}_I^* \} = P$ for simplices $\langle \bar{v}_I^* \rangle = \text{Conv.} \left( \{0\} \cup \bar{v}_I^* \right)$ ($I \in T$), and $\langle \bar{v}_{I_1}^* \rangle \cap \langle \bar{v}_{I_2}^* \rangle$ ($I_1, I_2 \in T$) is a lower dimensional common face.

**Note.** Since $(d+1)$-simplices in $P$ are in one-to-one correspondence to $d$-simplices in $\Delta^*$, we identify a triangulation of $T$ of $P$ with its corresponding triangulation of $\Delta^*$. We call a triangulation $T$ of $P$ is maximal if it corresponds to a maximal triangulation of $\Delta^*(\text{Def.2.3})$.

For a base $I$ and a point $\eta \in \mathbb{R}^{p+1}$, we consider a linear function $h_{I, \eta}$ on $\mathbb{R}^{p+1}$ by $h_{I, \eta}(\bar{v}_I^*) = \eta$ ($i \in I$). We define $\mathcal{C}(\mathcal{A}, I) = \{ \eta \in \mathbb{R}^{p+1} \mid h_{I, \eta}(\bar{v}_I^*) \leq \eta_i \text{ for } i \notin I \}$ and $\mathcal{C}(\mathcal{A}, T) := \cap_{I \in T} \mathcal{C}(\mathcal{A}, I)$ for a triangulation $T$. Then it is easy to see that $\mathcal{C}(\mathcal{A}, T)$ consists of $\eta \in \mathbb{R}^{p+1}$ for which we have a convex piecewise linear function $h_{T, \eta}$ on $T$ determined by $h_{T, \eta}(\bar{v}_I^*) = h_{I, \eta}(I \in T)$
and satisfies \( h_{T,\eta}(\bar{v}_i^*) \leq \eta_i \) for \( \bar{v}_i^* \) not a vertex of \( T \), i.e.,

\[
\mathcal{C}(A, T) = \{ \eta \in \mathbb{R}^{p+1} \mid h_{I_1,\eta}(v) \leq h_{I_2,\eta}(v) \ (v \in \langle \bar{v}_i^* \rangle, \ I_1, I_2 \in T), \\
h_{T,\eta}(\bar{v}_i^*) \leq \eta_i \ (\bar{v}_i^* \text{ is not a vertex of } T) \} \tag{11}
\]

A triangulation is called regular if \( \mathcal{C}(A, T) \) has interior points. We say a \( \mathbb{Z} \)-basis \( A \subset L \) is compatible with a triangulation \( T \) if it is compatible with all bases \( I \) in \( T \).

**Proposition 3.7.** (Prop.5 in [GKZ1]) For every regular triangulation \( T \), there exists (infinitely many) \( \mathbb{Z} \)-basis of \( L \) compatible with \( T \).

The exponent \( \beta \) is called \( T \)-nonresonant if the set \( \Phi_\mathbb{Z}(\beta, I) \ (I \in T) \) are pairwise disjoint. We set \( \Phi_\mathbb{Z}(\beta, T) := \cup_{I \in T} \Phi_\mathbb{Z}(\beta, I) \). We normalize the volume of the standard \((d+1)\)-simplex to 1.

**Theorem 3.8.** (Th.3 in [GKZ1]) For a regular triangulation \( T \) of the polytope \( P \), and a \( \mathbb{Z} \)-basis \( A = \{l^{(1)}, \ldots, l^{(p-d)}\} \) of \( L \) compatible with \( T \), there are integral powerseries in the variables \( x_k = a^{(k)} \) for \( a^{-\gamma}\Pi(a, \gamma) \ (\gamma \in \Phi_\mathbb{Z}(\beta, T)) \), which converge for sufficiently small \( |x_k| \). If the exponents \( \beta \) is \( T \)-nonresonant, these series constitute \( \text{vol}(P) \) linearly independent solutions.

**Remark.** In our case of \( \Delta^* \)-hypergeometric system with \( \beta = (-1) \times \bar{0} \), we have one special element \( \gamma = (-1, 0, \ldots, 0) \) in the set \( \Phi_\mathbb{Z}(\beta, I) \) for any base \( I \). If the polytope \( \Delta^* \) is of type I or II in our classification, a nonsingular maximal triangulation \( T \) of \( P \) consists of those bases \( I \) for which \( |\det(\bar{v}_i^*)|_{1 \leq i \leq d+1, j \in I} = 1 \). Because of this unimodularity, we have

\[
\Phi_\mathbb{Z}(\beta, I) = (-1, 0, \ldots, 0) + L ,
\tag{12}
\]

and \( \Phi_\mathbb{Z}(\beta, I) = \{(-1, 0, \ldots, 0)\} \) for every base \( I \) of the maximal triangulation and any \( \mathbb{Z} \)-basis \( A \) compatible with it. Thus we encounter a “maximally \( T \)-resonant” situation.

**Definition 3.9.** For a regular triangulation \( T \) and a \( \mathbb{Z} \)-basis \( A = \{l^{(1)}, \ldots, l^{(p-d)}\} \) compatible with \( T \), we define a power series \( w_0(x, \rho; A) = a_0\Pi(a, \gamma) \) (with \( \gamma = \sum_{k=1}^{p-d} \rho_k l^{(k)} + (-1, 0, \ldots, 0) \)) by

\[
w_0(x, \rho; A) = \sum_{m \in \mathbb{Z}^{p-d}} \frac{\Gamma(-\sum_k(m_k + \rho_k)l_0^{(k)} + 1)}{\prod_{1 \leq i \leq p} \Gamma(\sum_k(m_k + \rho_k)l_i^{(k)} + 1)} x^{m+\rho} , \tag{13}
\]

where \( x_k = (-1)^{l_0} a^{l^{(k)}} \).
Remark. Here we have applied our recipe of multiplying the constant $\Gamma(\gamma_0 + 1)$ to the formal solution $\Pi(a, \gamma)$. We adopt this definition because for a maximal triangulation $T_o$, we encounter the situation $\Pi(a, \gamma) \equiv 0$, namely, $\sum_{m \in \mathbb{Z}_{\geq 0}} (m \in \mathbb{Z}_{\geq 0})$ for a $\mathbb{Z}$-basis $A = \{l^{(1)}, \ldots, l^{(p-n)}\}$ compatible with $T_o$. (See Prop.4.8, Prop.4.9 and Th.4.10 in [HLY2]). In general a basis $A$ compatible with a regular triangulation $T$ contains both the bases vectors $l^{(k)}$ with positive 0-th component and nonpositive 0-th component. Taking a value $\rho_0 \in \mathbb{Z}_{p-n} \geq 0$ under this situation cause infinity for some $m$ in the numerator of the coefficients of (13). In this case we understand in our definition (13) to take a limit $\rho \to \rho_0$ in a generic way. (If we encounter infinities under this limit in some coefficient, we go back to the original definition (10). For a maximal triangulation $T_o$, we observe that this limit exists for all coefficients in (13), see Claim5.8.)

(3-3) Secondary fan It is known that the set $\mathcal{C}(A, T)$ is a closed polyhedral cone and that these cones cover $\mathbb{R}^{p+1}$ when we vary the triangulations. Thus the set of these cones and their lower dimensional faces all together define a complete, polyhedral fan $\mathcal{F}(A)$ called the secondary fan [BFS][OP]. Let $M = \mathbb{Z} \times M$ and $N = \mathbb{Z} \times N$. We extend naturally the pairing $\langle , \rangle$ to that of $M$ and $N$. Consider the lattice $\mathcal{M} := \oplus_{\nu^* \in \Delta \cap N} \mathbb{Z} e_{\nu^*} = \oplus_{i=0}^p \mathbb{Z} e_{\nu^*_i}$ and its dual $\mathcal{N} = \text{Hom}_{\mathbb{Z}}(\mathcal{M}, \mathbb{Z})$. Then we have the following exact sequences

$$0 \to \mathcal{M} \overset{A}{\to} \mathcal{M} \overset{B}{\to} \Xi(\mathcal{M}) \to 0, \quad \mathcal{N} \overset{A^*}{\leftarrow} \mathcal{N} \overset{B^*}{\leftarrow} \Xi(\mathcal{N}) \leftarrow 0,$$

where $A(m) = \sum_{i=0}^p \langle m, \nu^*_i \rangle e_{\nu^*_i}$ ($m \in \mathcal{M}$) and $B$ is the quotient. The dual is given by $A^*(\mu) = \sum_{i=0}^p \mu(\nu^*_i) \nu^*_i$ ($\mu \in \mathcal{N}$). The pair $\{B, \Xi(\mathcal{M})\}$ with $B := \{B(e_{\nu^*_i}), \ldots, B(e_{\nu^*_p})\}$ is called Gale transform of a pair $\{A, \mathcal{N}\}$. Under this general setting, let us consider a polyhedral cone in $\Xi(\mathcal{M})_{\mathbb{R}}$

$$\mathcal{C}'(A, T) = \cap_{I \in T} \left( \sum_{i \notin I} \mathbb{R}_{\geq 0} B(e_{\nu^*_i}) \right).$$

**Proposition 3.10.** (Lemma 4.2 in [BFS]) The map $B$ induces the following decomposition

$$\mathcal{C}(A, T) = \text{Ker}(B) \oplus \mathcal{C}'(A, T).$$

By definition, the cone $\mathcal{C}'(A, T)$ is strongly convex. Using the above decomposition, we redefine the secondary fan to be

$$\mathcal{F}(A) = \{\mathcal{C}'(A, T) | T : \text{regular triangulation}\}.$$
Now the secondary fan consists of strongly convex, polyhedral cones. If the polytope \( \Delta^* \) is of type I or II, then the quotient \( \Xi(\mathcal{M}) \) is torsion free and thus \( (\mathcal{F}(\mathcal{A}), \Xi(\mathcal{M})) \) defines a toric variety. Even in the case of type III, we may consider the corresponding toric variety by simply projecting out the torsion part of \( \Xi(\mathcal{M}) \). We will use this toric variety for the compactification of the moduli space in the next section.

Now let us note that \( \Xi(N) \cong \text{Ker}(A^*) \) is identified with the lattice \( L \), and thus \( K(A, T) \subseteq \Xi(N) \). By definition of \( C'(A, T) \), we may deduce \( K(A, T) = C'(A, T)^\vee \). (18)

Since for a regular triangulation \( T \), \( C'(A, T) \) is a strongly convex polyhedral cone with interior points, the dual cone \( K(A, T) \) is also strongly convex polyhedral cone. Therefore we see that there are infinitely many \( \mathbb{Z} \)-basis of the lattice \( L \) compatible with \( T \) (Prop.3.7).

4. Toric Ideal and Gröbner Fan

In this section we will reduce the infinite set of operators \( D_l \) (\( l \in L \)) in our \( \Delta^* \)-hypergeometric system to a finite set. This will be related to the compactification problem of the moduli spaces.

(4-1) Toric ideal and Gröbner fan We write the operators \( D_l \) in (6) simply by \( D_l = (\frac{\partial}{\partial a})^{l_+} - (\frac{\partial}{\partial a})^{l_-} \) with \( l = l_+ - l_- \). Keeping this form in mind we define toric ideal in a polynomial ring:

**Definition 4.1.** Consider a polynomial ring \( \mathbb{C}[y] := \mathbb{C}[y_0, \ldots, y_p] \). Toric ideal \( \mathcal{I}_A \) is defined to be an ideal generated by binomials \( y^{l_+} - y^{l_-} \) (\( l \in L \)),

\[
\mathcal{I}_A = \left\langle y^{l_+} - y^{l_-} \mid l \in L \right\rangle. \tag{19}
\]

Toric ideal has been extensively studied in ref.[Stu1][GKZ2]. Here we summarize relevant results for our purpose.

A term order (monomial ordering) on \( \mathbb{C}[y] \) is a total order \( \prec \) on the set of monomials \( \{ y^\alpha \mid \alpha \in \mathbb{Z}_{\geq 0}^{p+1} \} \) satisfying, 1) \( y^\alpha \prec y^\beta \) implies \( y^{\alpha + \gamma} \prec y^{\beta + \gamma} \) and 2) 1 is the unique minimal element. When we have an ideal \( \mathcal{I} \subseteq \mathbb{C}[y] \) and fix a term order, we can speak of the leading term \( LT_{\prec}(f) \) for every non-zero polynomial in \( \mathcal{I} \). Then we define initial ideal of \( \mathcal{I} \) by

\[
\langle LT_{\prec}(\mathcal{I}) \rangle = \left\langle LT_{\prec}(f) \mid f \in \mathcal{I}, f \neq 0 \right\rangle. \tag{20}
\]

A finite set \( \mathcal{G} \subseteq \mathcal{I} \) is called Gröbner basis with respect to a term order \( \prec \) if it generates the initial ideal:

\[
\langle LT_{\prec}(\mathcal{I}) \rangle = \langle LT_{\prec}(g) \mid g \in \mathcal{G} \rangle. \tag{21}
\]
Theorem 4.2. (Th.1.2 in [Stu2]) For every ideal $\mathcal{I} \subset \mathbb{C}[y]$, there are only finitely many distinct initial ideals.

We consider representing the term orders by weight vectors $\omega = (w_0, \cdots, w_p) \in \mathbb{R}^{p+1}$. For a polynomial $f = \sum \alpha c_\alpha y^\alpha$, we define its leading terms $LT_\omega(f)$ to be a sum of terms $c_\alpha y^\alpha$ whose weight $t_\omega(y^\alpha) := \omega_0\alpha_0 + \cdots + \omega_p\alpha_p$ is maximal. It is obvious that if the components of $\omega \in \mathbb{R}^{p+1}$ are rationally independent, the weight determines a term order on $\mathbb{C}[y]$. When we fix an ideal $\mathcal{I} \subset \mathbb{C}[y]$, we may relax the condition for the weight $\omega$ to be a term order; we say a weight $\omega \in \mathbb{R}^{p+1}$ defines a term order of $\mathcal{I}$ if $\langle LT_\omega(\mathcal{I}) \rangle = \langle LT_<(\mathcal{I}) \rangle$ for some term order $\prec$. The following proposition provides a 'converse' statement.

Proposition 4.3. (Prop.1.11 in [Stu2]) For any term order $\prec$, there exists a weight $\omega \in \mathbb{R}_{\geq 0}^{p+1}$ such that $\langle LT_\omega(\mathcal{I}) \rangle = \langle LT_<(\mathcal{I}) \rangle$.

Now Gröbner region is defined to be a set

$$GR(\mathcal{I}) = \{ \omega \in \mathbb{R}^{p+1} | \langle LT_\omega(\mathcal{I}) \rangle = \langle LT_<(\mathcal{I}) \rangle \text{ for some } \omega' \in \mathbb{R}_{\geq 0}^{p+1} \}. \quad (22)$$

Proposition 4.4. (Prop.1.12 in [Stu2]) If an ideal $\mathcal{I} \subset \mathbb{C}[y]$ is a homogeneous ideal with some grading $deg(y_i) = d_i > 0$, then $GR(\mathcal{I}) = \mathbb{R}^{p+1}$.

Since the toric ideal $\mathcal{I}_A$ is homogeneous ideal with $deg(y_0) = \cdots = deg(y_p) = 1$, we see $GR(\mathcal{I}_A) = \mathbb{R}^{p+1}$. For a term order $\omega$ of $\mathcal{I}_A$, we define

$$C(\mathcal{I}_A, \omega) = \{ \omega' \in \mathbb{R}^{p+1} | \langle LT_\omega(\mathcal{I}_A) \rangle = \langle LT_{\omega'}(\mathcal{I}_A) \rangle \}.$$  \hspace{1cm} (23)

It is known that this set constitutes an open, convex, polyhedral cone in $\mathbb{R}^{p+1}$ (Prop.2.1 in [Stu1]). In the following, we mean by $C(\mathcal{I}_A, \omega)$ the closure of the set \((23)\). Then due to Th.4.2 and Prop.4.4, the collection $\{C(\mathcal{I}_A, \omega)\}$ is finite and defines a complete polyhedral fan $\mathcal{F}(\mathcal{I}_A)$ in $\mathbb{R}^{p+1}$, called the Gröbner fan.

(4-2) Indicial ideal and compactification of $\operatorname{Hom}_Z(L, \mathbb{C}^*)$ In the previous section, we called a triangulation $T$ of the polytope $P$ regular if the cone $C(A, T)$ has interior points. Here we characterize the regular triangulation in a geometrical way. To this aim let us first consider a polytope $P_\omega := \operatorname{Conv.} \left\{ (\omega_0 \times \nu_0^*, \cdots, \omega_p \times \nu_p^*) \right\}$ in $\mathbb{R}^{d+1}$ for a weight $\omega \in \mathbb{R}^{p+1}$. If we project a polytope $P_\omega$ to $1 \times \mathbb{R}^d$, then we have the polytope $1 \times \Delta^*$. Thus we may regard the weight $\omega$ giving a height to each vertex of $1 \times \Delta^*$. For generic weight $\omega$, the 'lower' faces of the polytope $P_\omega$ consist of simplices and define, under the projection, a simplicial decomposition of $\Delta^*$ and thus induce a triangulation $T_\omega$. The regular triangulation of
the polytope $P$ is a triangulation $T_\omega$ obtained for some weight $\omega$ in this way (see Def.5.3 of [Zie] for more details). It is not difficult to see the relation of the polytope $P_\omega$ to the piecewise linear function $h_{T,\eta}$ in (11) with $\eta = \omega$.

Given a (regular) triangulation $T$ of the polytope $P$, the Stanley-Reisner ideal $\text{SR}_T$ in $\mathbb{C}[y]$ is defined to be the ideal generated by all monomials $y_{i_1} \cdots y_{i_k}$ for which the vertices $\bar{v}_{i_1}, \ldots, \bar{v}_{i_k}$ do not make a simplex in $T$. The following theorem is due to Sturmfels:

**Theorem 4.5.** (Thm. 3.1 in [Stu1]) If a weight $\omega$ defines a term order of the toric ideal $I_A$, then it induces a regular triangulation $T_\omega$. Moreover the Stanley-Reisner ideal $\text{SR}_{T_\omega}$ is equal to the radical of the initial ideal $\langle LT_\omega(I_A) \rangle$.

As an immediate corollary to this theorem, we see that the Gröbner fan is a refinement of the fan $\{C(A, T_\omega)\}$. Since the cone $C(A, T_\omega)$ decomposes according to (11), we have similar decomposition of $C(I_A, \omega)$ to $C'(I_A, \omega)$. In the following we call the collection $\{C'(I_A, \omega)\}$ as the Gröbner fan $\mathcal{F}(I_A)$.

Now we determine a finite set of operators $D_l$ which characterize the power series $w_0(x, \rho; A)$ for each regular triangulation and a $\mathbb{Z}$-basis $A$ compatible with it. This provides us a way to analyze our resonant GKZ hypergeometric system.

Let us consider a term order $\omega$ of $I_A$. According to Th.4.5, the term order $\omega$ determines a regular triangulation $T_\omega$ and also a cone $C'(I_A, \omega) \subset C'(A, T_\omega)$. If the cone $C'(I_A, \omega)$ is simplicial and regular, i.e., the integral generators of its one-dimensional boundary cones generate the lattice points $C'(I_A, \omega) \cap \Xi(M)$, we simply make its dual cone $C'(I_A, \omega)^\vee$ and take the integral generators of this cone as a canonical $\mathbb{Z}$-basis $A$ of $L$ which is compatible with $T_\omega$. If not, we subdivide the cone $C'(I_A, \omega)$ into simplicial, regular cones and reduce the problem to the former case. More generally, we may take a $\mathbb{Z}$-basis $A_\tau = \{l^{(1)}_\tau, \ldots, l^{(p-d)}_\tau\}$ of $L$ compatible with $T_\omega$ considering any simplicial, regular cone $\tau$ contained in $C'(I_A, \omega)$ and making its dual $\tau^\vee$.

Associated to $\omega$, we have a Gröbner basis $B_\omega \subset I_A$. By Büchberger’s algorithms to construct the (reduced) Gröbner basis, we see that every generator $g \in B_\omega$ is a binomial of the form $y^{l^+} - y^{l^-}$ with some $l \in L$. In the following, we assume $B_\omega$ to be the reduced Gröbner basis which is determined uniquely for a term order $\omega$ (see Chapter 2 of [CLO] for the properties of the reduced Gröbner basis). Translating this to differential operator, we write the Gröbner basis $B_\omega = \{D_{l_1}, \ldots, D_{l_s}\}$ ($1 \leq s < \infty$).
Now, for each generator, we define

\[ J_l(\rho; A_\tau) := a_0 x_\tau^{-\rho} a_l^\pm \left( \frac{\partial}{\partial a} \right) ^{l_\pm} a_0^{-1} x_\tau^\rho, \quad (24) \]

where the choice in \( l_\pm \) is made respectively by \( \omega \cdot l_+ - \omega \cdot l_- > 0 \ (\ < 0) \). (The factor \( a_0 \) originate from the definition \( w_0(x, \rho; A) := a_0 \Pi(a, \gamma) \) in Def.3.9.)

**Definition 4.6.** For a term order \( \omega \) of \( \mathcal{I}_A \) and an arbitrary regular cone \( \tau \) contained in \( \mathcal{C}'(\mathcal{I}_A, \omega) \), we define, through the Gröbner basis \( \mathcal{B}_\omega = \{ \mathcal{D}_{l_1}, \ldots, \mathcal{D}_{l_\rho} \} \), an **indicial ideal** in \( \mathbb{C}[\rho_1, \ldots, \rho_{p-d}] \):

\[ \text{Ind}_\omega(\tau) = \langle J_{l_1}(\rho, A_\tau), \ldots, J_{l_\rho}(\rho, A_\tau) \rangle. \quad (25) \]

Similarly to the indicial equations of the differential equations of Fuchs type, we also consider the **indicial equations** for our \( \Delta^* \)-hypergeometric system as algebraic equations for \( \rho \) coming from the leading terms of the operators \( D_l \ (l \in L) \). (Note that the leading term of an operator \( D_l \) varies in general when a term order \( \omega \) varies. Here we consider for the indicial equations all possible leading terms when \( \omega \) varies inside \( \tau \).)

**Proposition 4.7.** In the notation above, the indicial ideal \( \text{Ind}_\omega(\tau) \) coincides with the ideal generated by the indicial equations for the indices \( \rho \) of the power series \( w_0(x, \rho; A_\tau) \).

**(Proof)** Consider an operator \( \mathcal{D}_l \in \mathcal{B}_\omega \). If \( \omega \cdot l_+ - \omega \cdot l_- > 0 \), we multiply \( a^l_+ \) to obtain

\[ a^l_+ \mathcal{D}_l = a^l_+ \left( \frac{\partial}{\partial a} \right) ^{l_+} - a^{l_+ - l_-} a^l_- \left( \frac{\partial}{\partial a} \right) ^{l_-}. \quad (26) \]

Since the initial ideal \( \langle LT_\omega(\mathcal{I}_A) \rangle \) and thus the reduced Gröbner basis \( \mathcal{B}_\omega \) does not change for \( \omega \) in the interior of \( \tau \), \( \text{Int}(\tau) \), we have \( \omega \cdot (l_+ - l_-) > 0 \) for all \( \omega \in \text{Int}(\tau) \), i.e., \( l_+ - l_- \in \tau^\vee \cap L \). Since we have chosen the \( \mathbb{Z} \)-basis \( A_\tau = \{ l_1^{(1)}, \ldots, l_{p-d}^{(p-d)} \} \) so that it generates all integral points in \( \tau^\vee \cap L \), \( a^{l_+ - l_-} \) is a monomial of \( x_\tau \), which vanish in the limit \( x_\tau \rightarrow 0 \). The same argument applies to the case \( \omega \cdot l_+ - \omega \cdot l_- < 0 \). Therefore the indicial equations arising from the operators \( \mathcal{D}_l \in \mathcal{B}_\omega \) exactly coincide with the generators of the indicial ideal \( \langle \mathcal{P} \rangle \). For general operators \( \mathcal{D}_l \ (l \in L) \), depending on the weight \( \omega \in \text{Int}(\tau) \), we have two possible leading terms. However for both of them, owing to the defining property of the Gröbner basis, we have \( LT_\omega(\mathcal{D}_l) = \left( \frac{\partial}{\partial a} \right)^{\mu} LT_\omega(\mathcal{D}_{l_k}) \) for some \( k \) and \( \mu \). Multiplying a monomial \( a^{\mu + l_k} \), we obtain

\[ a_0 x_\tau^{-\rho} a^{\mu + l_k} LT_\omega(\mathcal{D}_l) a_0^{-1} x_\tau^\rho = F(\rho) J_{l_k}(\rho; A_\tau), \quad (27) \]
with some polynomial $F(\rho)$. Thus we see all polynomial relations of $\rho$ coming from the leading terms are in $\text{Ind}_{\omega}(\tau)$.

Conversely, since all generators of the ideal $\text{Ind}_{\omega}(\tau)$ give the indi-
cial equations related to $\mathcal{B}_\omega$, the ideal $\text{Ind}_{\omega}(\tau)$ is contained in the other. Therefore the two ideals are the same. □

Now based on Prop.4.7, we may claim the following:

**Proposition 4.8.** Consider a compact toric variety $\mathbb{P}_{\mathcal{F}(\mathcal{I}_A)}$ associated to the Gröbner fan $(\mathcal{F}(\mathcal{I}_A),\Xi(\mathcal{M}))$. Then for any resolution $\mathbb{P}_{\tilde{\mathcal{F}}(\mathcal{I}_A)} \to \mathbb{P}_{\mathcal{F}(\mathcal{I}_A)}$ associated to a refinement $(\tilde{\mathcal{F}}(\mathcal{I}_A),\Xi(\mathcal{M})) \to (\mathcal{F}(\mathcal{I}_A),\Xi(\mathcal{M}))$, we have integral powerseries of the form $w_0(x,\rho;A_\tau)$ ($\rho \in V(\text{Ind}_{\omega}(\tau))$) at each boundary point given by the normal crossing toric divisors, namely at the origin of $\text{Hom}_{s.g.}(\tau^\vee \cap L,\mathbb{C})$. We will call this compactification Gröbner compactification.

**Remark.** Since Prop.4.7 provides us only a necessary condition for the indices $\rho$ to give a powerseries solution $w_0(x,\rho;A_\tau)$, we do not claim by Prop.4.8, although we expect, that all $\rho \in V(\text{Ind}_{\omega}(\tau))$ form the powerseries solutions of our $\Delta^*$-hypergeometric system.

**(4-3) Resonance of $\Delta^*$-hypergeometric system** When the polytope $\Delta^*$ is of type I or II, we have seen in the Remark right after Th.3.8 that the $\Delta^*$-hypergeometric system becomes “maximally resonant” for a maximal triangulation $T_o$. Here we study this resonance in detail restricting our attention to the polytopes of type I or II. We also comment about the case of type III.

We call a collection of vertices $\mathcal{P} = \{\nu_{i_1}^*, \cdots, \nu_{i_a}^*\}$ primitive if $\mathcal{P}$ does not form a simplex in $T_o$ but $\mathcal{P} \setminus \{\nu_{i_s}^*\}$ does for any $\nu_{i_s}^* \in \mathcal{P}$. By definition of the Stanley-Reisner ideal, it is easy to deduce that the monomials that corresponds to primitive collections generate the ideal $SR_{T_o}$.

Let us denote by $\Sigma(1 \times \Delta^*,T_o)$ the fan in $\overline{\Delta}_{\mathbb{R}}$ that is naturally associated to the triangulation $T_o$ of $P$. Since the volumes of all $d+1$ simplices in $T_o$ are unimodular for the polytope $\Delta^*$ of type I or II, the fan $\Sigma(1 \times \Delta^*,T_o)$ consists of regular cones. Therefore if we have a primitive collection $\mathcal{P} = \{\nu_{i_1}^*, \cdots, \nu_{i_a}^*\}$, we obtain

$$\nu_{i_1}^* + \cdots + \nu_{i_a}^* = \sum_k c_k \nu_{j_k}^* \quad (c_k \in \mathbb{Z}_{\geq 0})$$

(28)

where $\{\nu_{j_k}^* | c_k \neq 0\}$ generates a cone that contains the vector in the left hand side. Writing (28) as $\nu_{i_1}^* + \cdots + \nu_{i_a}^* - \sum c_k \nu_{j_k}^* = 0$, we read the corresponding primitive relation $l(\mathcal{P}) \in L$.
Lemma 4.9. Every primitive collection of a maximal triangulation $T_o$ does not contain the point $\bar{\nu}^*_0 = 1 \times \bar{0}$.

(Proof) Suppose a primitive collection is given by $\mathcal{P} = \{\bar{\nu}^*_{i_1}, \bar{\nu}^*_{i_2}, \ldots, \bar{\nu}^*_{i_a}\}$ $(1 \leq i_1, \ldots, i_a \leq p)$. Since it is primitive, the simplex $\langle \bar{\nu}^*_{i_1}, \ldots, \bar{\nu}^*_{i_a}\rangle$ must be a simplex in the triangulation $T_o$, which means that this simplex is a face of some maximal dimensional simplex in $T_o$. Since $T_o$ is a maximal triangulation in which every maximal dimensional simplex contains the vertex $\bar{\nu}^*_0$, we see the simplex $\langle \bar{\nu}^*_{i_1}, \bar{\nu}^*_{i_2}, \ldots, \bar{\nu}^*_{i_a}\rangle$ must be a simplex in $T_o$, which is a contradiction. \qed

Proposition 4.10. For a term order $\omega$ such that $T_\omega$ is a maximal triangulation, the initial ideal $\langle LT_\omega(\mathcal{I}_A) \rangle$ is radical and $\langle LT_\omega(\mathcal{I}_A) \rangle = SR_{T_\omega}$.

(Proof) Consider the primitive collections for the triangulation $T_\omega$, which generate the Stanley-Reisner ideal $SR_{T_\omega}$. Write a primitive collection $\mathcal{P} = \{\bar{\nu}^*_{i_1}, \ldots, \bar{\nu}^*_{i_a}\}$ and the corresponding primitive relation as $l(\mathcal{P})$ considering the relation (28). For a term order $\omega$, the regular triangulation $T_\omega$ is induced from the lower faces of the polytope $P_\omega = \text{Conv.}(\{\bar{\nu}^*_0, \ldots, \bar{\nu}^*_p \mid \bar{\nu}^*_k = \omega_k \times \nu^*_k \ (k = 0, \ldots, p)\})$. Then the convex hull $\text{Conv.}(\{\bar{\nu}^*_i \mid \bar{\nu}^*_i \notin \mathcal{P}\})$ is not a simplex that corresponds to a lower face of $P_\omega$. Therefore we have a “height” inequality $(\bar{\nu}^*_{i_1} + \cdots + \bar{\nu}^*_{i_a})_1 > (\sum_k c_k \bar{\nu}^*_{j_k})_1$, namely,

$$\omega_{i_1} + \cdots + \omega_{i_a} > \sum_k c_k \omega_{j_k}.$$ (29)

This means that $LT_\omega(y^{l(\mathcal{P})+} - y^{l(\mathcal{P})-}) = y_{i_1} \cdots y_{i_a}$, which is one of the generators of the ideal $SR_{T_\omega}$. Since this argument applies to all primitive collections, we conclude $SR_{T_\omega} \subseteq \langle LT_\omega(\mathcal{I}_A) \rangle$. Since the opposite inclusion follows from $SR_{T_\omega} = \sqrt{\langle LT_\omega(\mathcal{I}_A) \rangle}$ (Th.4.5), we conclude $SR_{T_\omega} = \langle LT_\omega(\mathcal{I}_A) \rangle$, which proves the initial ideal is radical. \qed

Corollary 4.11. Under the hypothesis in the previous proposition, the set of all possible primitive collections $\{\mathcal{P}_1, \ldots, \mathcal{P}_s\}$ of $T_\omega$ determines the Gröbner basis by $\mathcal{B}_\omega = \{\mathcal{D}_l(\mathcal{P}_1), \ldots, \mathcal{D}_l(\mathcal{P}_s)\}$. And the indicial ideal $\text{Ind}_\omega(\tau)$ is homogeneous for an arbitrary regular cone $\tau$ contained in $\mathcal{C}(\mathcal{I}_A, \omega)$.

(Proof) By definition, $SR_{T_\omega}$ is generated by the monomials corresponding to primitive collections. From the argument in the proof of Prop.4.10, we know $SR_{T_\omega} = \langle LT_\omega(\mathcal{D}_l(\mathcal{P}_1)), \ldots, LT_\omega(\mathcal{D}_l(\mathcal{P}_s))\rangle$. This combined with $\langle LT_\omega(\mathcal{I}_A) \rangle = SR_{T_\omega}$ establishes that $\mathcal{B}_\omega$ is the Gröbner basis. For the rest, we note that any primitive collection $\mathcal{P} = \{\bar{\nu}^*_{i_1}, \ldots, \bar{\nu}^*_{i_a}\}$ does not contain $\bar{\nu}^*_0$ according to Lemma 4.9. Now we have

$$J_l(\mathcal{P})(\rho; A_\tau) = a_0 x^\rho a^{l(\mathcal{P})+} \left(\frac{\partial}{\partial a}\right)^{l(\mathcal{P})} a_0^{-1} x^\rho \theta_{a_1} \cdots \theta_{a_{i_a}} x^\rho, \quad (30)$$
where \( \theta_a = a \frac{\partial}{\partial a} \), which proves that the generator \( J_l(P)(\rho; A_\tau) \) is homogeneous in \( \rho \).

**Remark.** If we combine a general result that the GKZ system is holonomic \([GKZ1]\), i.e., its solution space is finite dimensional, with our Corol. 4.11, we may conclude that the zero is the only solution for the indices \( \rho \). This is the maximal \( T \)-resonance in our approach. We will give following \([HLY2]\) an independent proof about this in the next section.

As we remarked before, our \( \Delta^* \)-hypergeometric system for the polytope \( \Delta^* \) of type III does not share this property. Here we can explain the difference. We first note that the primitive collections generate the Stanley-Reisner ideal and has the property in Lemma 4.9 irrespective to the type of polytopes. The only change in the above arguments is in the definition of the primitive relation. Namely, since all cones are not regular in type III case, for some primitive collection the equation (28) should be replaced by

\[
\lambda_{i_1} \vec{P}_{i_1}^* + \cdots + \lambda_{i_a} \vec{P}_{i_a}^* = \sum_k c_k \vec{P}_{j_k}^*,
\]

with some positive integers \( \lambda_{i_1}, \cdots, \lambda_{i_a} \) not all equal to one. Accordingly the leading term \( LT_\omega(y^{l(P)_+} - y^{l(P)_-}) \) will be replaced by \( (y_{i_1})^{\lambda_{i_1}} \cdots (y_{i_a})^{\lambda_{i_a}} \). This indicates that the initial ideal \( LT_\omega(I_A) \) is no longer radical, and therefore the generators \( J_l(\rho; A_\tau) \) become inhomogeneous. When translating the monomial \( y^{l(P)_+} \) to the differential operator \( a^{l(P)_+} \left( \frac{\partial}{\partial a} \right)^{l(P)_+} \), each \( \lambda \)-fold degeneration to zero ‘splits’ to simple zeros. Thus every index does not degenerate to zero, although we still have a simple zero.

5. Large complex Structure Limit

Here we will study in detail the maximal resonance of the \( \Delta^* \)-hypergeometric system. We will identify this resonance with the large complex structure limit (LCSL), i.e., a celebrated boundary point in the moduli space of Calabi-Yau manifolds\([Mor]\).

(5-1) **Maximal degeneration** In this subsection, we will restrict our arguments to the polytopes of type I or II. In these two cases, we have a nonsingular projective toric variety \( \mathbb{P}_{\Sigma(\Delta^*, T_\circ)} \) for a maximal triangulation. We focus on the Chow ring of this toric variety. The Chow ring of a variety is a free abelian group generated by irreducible closed subvarieties, modulo rational equivalence, which is endowed with the ring structure via the intersection products. In case of non-singular compact toric varieties \( \mathbb{P}_{\Sigma(\Delta^*, T_\circ)} \), it has a simple description in terms of the (toric) divisors;
Proposition 5.1. (sect.3.3 of [Oda], sect.5.2 of [Ful]) The Chow ring $A^*(\mathbb{P}_{\Sigma(\Delta^*, T_o)})$ is isomorphic to the cohomology ring $H^{2*}(\mathbb{P}_{\Sigma(\Delta^*, T_o)}, \mathbb{Z})$ and is given by

$$A^*(\mathbb{P}_{\Sigma(\Delta^*, T_o)}) = \mathbb{Z}[D_0, \ldots, D_p]/(SR_{T_o} + \mathcal{R}),$$

where $D_k$ ($k > 0$) represents the toric-divisor determined by the integral point $\nu_k^*$. $SR_{T_o}$ is the Stanley-Reisner ideal and $\mathcal{R}$ is the ideal generated by linear relations $\sum_{k=0}^p (1 \times u, \nu_k^*)D_k = 0$ ($u \in \mathcal{M}$).

Note. Owing to lemma 4.9, we can take the generators of $SR_{T_o}$ that do not contain $D_0$. Therefore the generator $D_0$ plays only a dummy role, although it makes sense as a divisor if we consider a toric variety defined by the (non-complete) fan $\Sigma(1 \times \Delta^*, T_o) \subset \mathcal{N}_R$.

Now consider a term order $\omega$ of the toric ideal $\mathcal{I}_A$ and denote the Gröbner basis by $\mathcal{B}_\omega = \{\mathcal{D}_{l_1}, \ldots, \mathcal{D}_{l_s}\}$. We define

$$I_l(\theta_a) := a_0 a^{l_\pm} \left( \frac{\partial}{\partial a} \right)^{l_\pm} a_0^{-1}$$

for each $LT_{\omega}(\mathcal{D}_l) = (\frac{\partial}{\partial a})^{l_\pm}$ in a similar way to $J_l(\rho; A_r)$. Obviously these two are related by $J_l(\rho; A_r) = x_r^{-\rho} I_l(\theta_a)x_r^\rho$. We consider the following ideals in $\mathbb{C}[\theta_{a_0}, \ldots, \theta_{a_p}]$,

$$I_\omega := \langle I_{l_1}(\theta_a), \ldots, I_{l_s}(\theta_a) \rangle, \quad \tilde{R}_a := \langle \sum_{i=0}^p (1 \times u, \nu_i^*)a_{i_a} \mid u \in \mathcal{M} \rangle. \quad (34)$$

Proposition 5.2. For a term order $\omega$ of $\mathcal{I}_A$ and an arbitrary regular cone $\tau$ contained in $\mathcal{C}(\mathcal{I}_A, \omega)$, we have

$$\mathbb{C}[\rho]/\text{Ind}_{\omega}(\tau) \cong \mathbb{C}[\theta_a]/(I_\omega + \tilde{R}_a). \quad (35)$$

(Proof) When we take the $\mathbb{Z}$-basis $A_\tau = \{l^{(1)}_\tau, \ldots, l^{(p-d)}_\tau\}$, we have $\theta_{a_i} = \sum_{k=1}^{p-d}(l^{(k)}_\tau)_i \theta_{x^{(k)}_r}$. Then the homomorphism $\phi : \mathbb{C}[\theta_a] \to \mathbb{C}[\theta_{x_r}] \cong \mathbb{C}[\rho]$ induced by this relation is surjective, since rank($L$) = $p - d$, and satisfies $\text{Ker} \phi = \tilde{R}_a$ and $\phi(I_\omega) = \text{Ind}_{\omega}(\tau)$. This proves the assertion. □

Proposition 5.3. Consider a term order $\omega$ of $\mathcal{I}_A$ with $T_\omega$ a maximal triangulation $T_o$. Then for any regular cone $\tau$ contained in $\mathcal{C}(\mathcal{I}_A, \omega)$, the variety associated to the indicial ideal $\text{Ind}_{\omega}(\tau)$ consists only one point, i.e.,

$$V(\text{Ind}_{\omega}(\tau)) = \{0\}. \quad (36)$$
By Corol. 4.11, we know the indicial ideal is homogeneous for a
term order $\omega$ of the given property. Moreover the ideal $I_\omega$
coincides with the Stanley-Reisner ideal $SR_{T_\omega}$. Therefore we have
\[
\mathbb{C}[\rho]/\text{Ind}_\omega(\tau) \cong \mathbb{C}[\theta_a]/(I_\omega + \bar{R}_\alpha) \cong A^*(\mathbb{P}_{\Sigma(\Delta^*,T_\omega)}) \otimes \mathbb{C}, \tag{37}
\]
which is finite dimensional. Since the ideal is homogeneous, the claim
follows. \qed

We write our series (13) for generic $\rho$ by
\[
w_0(x, \rho; A) = \sum_{m \in \mathbb{Z}_{p-n} \geq 0} c(m + \rho)x^m + \rho. \tag{38}
\]

**Theorem 5.4.** (Th.5.2 in [HLY2]) For a term order $\omega$ with $T_\omega$ a maximal
triangulation and any regular cone $\tau$ contained in $C^\prime(I_A, \omega)$, the series
\[
w_0(x, 0; A_{\tau}) \geq 0
\]
is the only power series solution of the $\Delta^*$-hypergeometric
system about the origin of $U_{\tau} = \text{Hom}_{s.g.}(\tau^\vee \cap L, \mathbb{C})$.

To prove this theorem, we prepare the following lemma;

**Lemma 5.5.** Consider a subset $S \neq \{\phi\}$ that is contained in $\tau^\vee \cap L$.
There exist an element $\delta \in S$ and a simplicial, regular cone $C^\prime_\delta \subset L_\mathbb{R}$
such that $C^\prime_\delta$ contains both the subset $S - \delta$ and the cone $\tau^\vee$.

**Proof** Consider a hyperplane $H(v; z_0)$ with a normal vector $v \in \tau$
and passing through a point $z_0$ in $\tau^\vee$. When we consider a parallel transport
$H(v, t z_0)$ $(t \geq 0)$ of the hyperplane, we may find the minimal $t_0$ such that
$H(v, t_0 z_0) \cap S \neq \{\phi\}$ while $H(v, t z_0) \cap S = \{\phi\}$ $(t < t_0)$. Changing
the normal vector $v$ slightly, if necessary, we may assume the intersection
$H(v, t_0 z_0) \cap S$ occurs at a point $\delta$. Now for this $\delta$, we see that the union
$U := (\tau^\vee \cap L) \cup (S - \delta) \setminus \{0\}$ is contained in the half space $H_{>}(v, 0)$.
Therefore the normal cone to the set $U$ at the origin is strongly convex,
polyhedral cone. Since a strongly convex, polyhedral cone can be inside a
simplicial, regular cone, the assertion follows. \qed

**Proof of Th.5.4.** To prove the theorem, we write the series $w_0(x, 0; A_{\tau})$ in terms of $a_0, \cdots, a_p$ by
\[
w_0(a, 0, \tau) = \sum_{l \in \tau^\vee \cap L} c_l a^l, \tag{39}
\]
with $c_0 = 1$. Now suppose we have two different series of this form. Then the difference of the two may be written by $r(a, 0, S) = \sum_{l \in S} d_l a^l$ with a subset $S \subset \tau^\vee \cap L \setminus \{0\}$. Using the result in the lemma 5.5, we may write this series via nonzero $\delta$ as

$$r(a, 0, C_\delta) = a^\delta \sum_{l \in C_\delta} d_l a^l,$$

(40)

or $r(x_\tau, 0, A_{C_\delta}) = x_\tau^{\rho(\delta)} \sum_{n \in \mathbb{Z}d_{-d}} d(n) x_\tau^n$ with $\rho(\delta) \neq 0, d(0) \neq 0$ and $C_\delta \subset \tau$. This is a contradiction to Prop.5.3.

**Remark.** By direct evaluation of the period integral \([\text{Bat}2]\), we can verify that $a_0 \Pi(a)$ exactly coincides with the powerseries in Th.5.4 when expressed in terms of the $\mathbb{Z}$-basis $A_{\tau}$ (Prop.5.15 [HLY2]).

(5-2) All solutions about maximal degeneration points

Here we determine other solutions about maximal degeneration points, all of which contains logarithmic singularities. As in the previous subsection, our arguments are restricted to the polytopes of type I or II.

Let us note that the first degree elements of the Chow ring, $A^1(\mathbb{P}_{\Sigma(\Delta^*, T_o)})$, describe the Picard group of the toric variety $\mathbb{P}_{\Sigma(\Delta^*, T_o)}$ and may be expressed by

$$A^1(\mathbb{P}_{\Sigma(\Delta^*, T_o)}) = \mathbb{Z}D_0 \oplus \cdots \oplus \mathbb{Z}D_p / \bar{R} \cong \Xi(M).$$

(41)

From this we see a dual pairing between the Picard group and the lattice $L \cong \Xi(N)$;

$$A^1(\mathbb{P}_{\Sigma(\Delta^*, T_o)}) \times L \rightarrow \mathbb{Z}. \quad (42)$$

**Definition 5.6.** For a $\mathbb{Z}$-basis $A_\tau = \{l^{(1)}_\tau, \cdots, l^{(p-d)}_\tau\}$ of $L$ determined from a term order $\omega$ with $T_\omega$ equal to a maximal triangulation $T_o$, we denote its dual by $A_\tau^\vee = \{J_{\tau,1}, \cdots, J_{\tau,p-d}\}$ or simply by $\{J_1 \cdots, J_{p-d}\}$ when its dependence on $\tau$ is obvious.

**Note.** By construction, the basis $A_\tau^\vee$ consists of the integral generators of the simplicial, regular cone $\tau$ contained in $\mathcal{C}'(\mathcal{I}_A, \omega) = \mathcal{C}'(A, T_o)$ by Prop.4.10. $\mathcal{C}'(A, T_o)$ consists of convex functions on $T_o$ which may be identified with the convex functions on the fan $\Sigma(\Delta^*, T_o)$. Since the set of all convex functions on the fan $\Sigma(\Delta^*, T_o)$ determines the closure of the Kähler cone of $\mathbb{P}_{\Sigma(\Delta^*, T_o)}$ (see Corol.2.15 in [Oda]), the bases $J_{\tau,1}, \cdots, J_{\tau,p-d}$ generate a simplicial, regular cone contained in this closure of the Kähler cone.
**Definition 5.7.** For the power series \( w_0(x_r, \rho; A_r) = \sum_{n \in \mathbb{Z}_{\geq 0}} c(n+\rho)x_r^{n+\rho} \) in (I3), we define
\[
 w_0(x_r, J; A_r) := \sum_{n \in \mathbb{Z}_{\geq 0}} c \left( n + \frac{J}{2\pi i} \right) x_r^{n+\frac{J}{2\pi i}},
\]
as the Taylor series expansion of \( w_0(x_r, \rho; A_r) \) about \( \rho = 0 \) followed by the substitution \( \rho = \frac{J}{2\pi i} \), where \( J \)'s are defined in Def.5.6.

In refs. [HKTY1] [HKTY2] [HLY1] [HLY2], it is widely verified

**Claim 5.8.** The expansion (43) exists as an element in \( A^*(\mathbb{P}_{\Sigma(\Delta^*, T_0)}) \otimes \mathbb{C}\{x_r\}[\log x_r] \), and the coefficient series constitute a complete set of the local solutions about the maximal degeneration points. Especially the limit \( w_0(x_r, \rho; A_r)|_{\rho \rightarrow 0} \) coincides with \( w_0(x_r, 0; A_r) \geq 0 \) in Th.5.4.

**Remark.** We comment about the case of the polytopes of type III. In this case, since the toric variety \( \mathbb{P}_{\Sigma(\Delta^*, T_0)} \) is singular, the Chow ring should be considered over \( \mathbb{Q} \). Under this modification the expansion (I3) makes sense in \( A^*(\mathbb{P}_{\Sigma(\Delta^*, T_0)})_{\mathbb{Q}} \otimes \mathbb{C}\{x_r\}[\log x_r] \). Then the coefficient series should be in a subspace of the whole solution space of the \( \Delta^* \)-hypergeometric system. More precisely, as we see in Remark after Corol. 4.11, the initial ideal \( LT_\omega(\mathcal{I}_A) \) is no longer radical but we have a strict inclusion \( LT_\omega(\mathcal{I}_A) \subset \sqrt{LT_\omega(\mathcal{I}_A)} \). As discussed there, we have \( LT_\omega(\mathcal{D}_I) = (\frac{\partial}{\partial a_i})^{\lambda_1} \cdots (\frac{\partial}{\partial a_i})^{\lambda_\omega} \) for some element of the Gröbner basis \( \mathcal{B}_\omega = \{\mathcal{D}_I, \cdots, \mathcal{D}_{I_s}\} \). If we define \( \text{rad}(LT_\omega(\mathcal{D}_I)) := \frac{\partial}{\partial a_i_1} \cdots \frac{\partial}{\partial a_i_n} \), then the radical may be expressed by \( \sqrt{LT_\omega(\mathcal{I}_A)} = (\text{rad}(LT_\omega(\mathcal{D}_{I_1})), \cdots, \text{rad}(LT_\omega(\mathcal{D}_{I_s}))) \). Correspondingly, if we define \( \tilde{I}_\omega(\theta_a) := a_0a_i_1 \cdots a_i_n \text{rad}(LT_\omega(\mathcal{D}_I))a_0^{-1} \), we naturally come to the “radical” of the indicial ideal \( \tilde{\text{Ind}}_\omega(\tau) := (\tilde{J}_{I_{1}}(\rho; A_r), \cdots, \tilde{J}_{I_{s}}(\rho; A_r)) \) with \( \tilde{J}_i(\rho; A_r) := x_r^{-\rho} \tilde{I}_\omega(\theta_a)x_r^{\epsilon} \). By definition, we have strict inclusions \( \tilde{\text{Ind}}_\omega(\tau) \subset \text{Ind}_\omega(\tau) \) and \( \text{V}(\text{Ind}_\omega(\tau)) \supset \text{V}(\tilde{\text{Ind}}_\omega(\tau)) \). As is clear now, our Prop.5.2 and Prop.5.3 apply to the “radical” \( \tilde{\text{Ind}}_\omega(\tau) \) under the replacements \( \tilde{I}_\omega \) by \( \tilde{I}_\omega \) and the Chow ring by that over \( \mathbb{Q} \). The expansion (I3) gives all logarithmic solutions which arise from the degeneration \( \text{V}(\tilde{\text{Ind}}_\omega(\tau)) = \{0\} \).

**5.3 LCSL of Calabi-Yau hypersurfaces** So far we have been concerned with the \( \Delta^* \)-hypergeometric system. Since the period integral (I1) of Calabi-Yau hypersurface \( X_{\Delta^*} \) satisfies the (extended) \( \Delta^* \)-hypergeometric system, a complete set of the period integrals of \( X_{\Delta^*} \) should be found in the set of solutions of the \( \Delta^* \)-hypergeometric system. We will find that the expansion (I3) contains the period integrals in a natural way from the mirror symmetry.
Before going into this topic, we need to discuss about the compactification of the moduli space $\mathcal{M}(X_{\Delta^*}(a))$ of the polynomial deformation of the Calabi-Yau hypersurface $X_{\Delta^*}$. Through a detailed analysis of the local solutions of the $\Delta^*$-hypergeometric system, we have arrived at a natural compactification, the Gröbner compactification $\mathbb{P}_{F(I_A)}$ in Prop.4.8. Now it is natural to adopt this compactification as that of the moduli space $\mathcal{M}(X_{\Delta^*}(a))$. However, one problem arises when the hypersurface (precisely its ambient space,) has non-trivial automorphisms. We need to mod out the space $\mathbb{P}_{F(I_A)}$ by the induced actions from the automorphisms, whose infinitesimal forms are described in (9). Here to avoid getting involved in the problems related to the actions of the automorphisms, we take a “gauge choice” that sets to zero all polynomial deformations corresponding to integral points on codimension-one faces of $\Delta^*$. Note that, in view of Prop.2.6, the degree of the freedom associated to the non-trivial automorphisms would be fixed by this gauge choice. In the following, we use the subscript $s$ (s of simply-minded!) to indicate this naive choice of the “gauge”; for example $\Delta^s$-hypergeometric system, the toric ideal $I_A$ etc. Note that all the polytopes of type II will be treated as the polytopes of type III under this prescription.

**Definition 5.9.** As an compactification of $\mathcal{M}(X_{\Delta^*}(a))$, we define

$$\overline{\mathcal{M}}(X_{\Delta^*}(a)) = \mathbb{P}_{F(I_A_s)}.$$  \hspace{1cm} (44)

Now we consider the toric part of the Chow ring of the Calabi-Yau hypersurface $X_{\Delta^s}$, which comes from the ambient space by restriction. Since we have $[X_{\Delta^s}] = D_1 + \cdots + D_p$ for the divisor of the Calabi-Yau hypersurface, the restriction may be attained by the quotient as follow;

**Definition 5.10.**

$$A^*(X_{\Delta^s})_{toric} = A^*(\mathbb{P}_{\Sigma(\Delta^s,T^0)})_Q/Ann(D_1 + \cdots + D_p),$$  \hspace{1cm} (45)

where $Ann(x)$ is defined by $Ann(x) = \{ y \in \mathcal{R} \mid xy = 0 \}$ for a ring $\mathcal{R}$.

**Claim 5.11.** Period integrals about a LCSL of Calabi-Yau hypersurface $X_{\Delta^s}$ are extracted from the series $w_0(x_{\tau}, J; A_{\tau})$ (43) expanded in $A^*(X_{\Delta^s})_{toric} \otimes \mathbb{C}\{x_{\tau}\}[\log x_{\tau}]$.

**Remark.** In general, the period integrals of Calabi-Yau hypersurfaces satisfy the differential equations of Fuchs type, so-called the Picard-Fuchs equations [10]. Picard-Fuchs equations determines the period integrals as its solutions. Our Claim 5.11 says that our $\Delta^*$-hypergeometric system is
reducible in general and contains the Picard-Fuchs equation as a component of it. In refs. [HKTY], it is verified in several examples of all types of the polytopes that Picard-Fuchs equations are derived from the (extended) $\Delta^*$-hypergeometric system after a factorization of the operator $\theta a_1 + \cdots + \theta a_p$ from the left, which we may identify with the quotient by $\text{Ann}(D_1 + \cdots + D_p)$ in (45).

6. Prepotential

In this section, we study so-called the prepotential [Saito] near a LCSL in detail. Under the mirror map, a LCSL is mapped to a large radius limit in which the instanton corrections to the prepotential are suppressed exponentially, and has important applications to the enumerative geometry. Also the prepotential determines the special Kähler geometry on the moduli space $\mathcal{M}(X_{\Delta_*})$ and that on the complexified Kähler moduli space of the mirror $X_{\Delta_*}$.

In this section, we fix a term order $\omega$ for which $T_\omega$ is a maximal triangulation of $\Delta_*$ and take the $\mathbb{Z}$-basis $A_\tau$ choosing a regular cone $\tau$ in $C'(I_{A_\tau}, \omega)$. Based on Claim 5.11, we expand the series $w_0(x_\tau, J; A_\tau)$ defined in (43) (see also [ST]) as;

$$w_0(x_\tau, J; A_\tau) = w^{(0)}(x_\tau, J) + w^{(1)}(x_\tau, J) + \frac{1}{2!}w^{(2)}(x_\tau, J) + \frac{1}{3!}w^{(3)}(x_\tau, J), \quad (46)$$

where the superscripts indicate the degree in the Chow ring $A^*(X_{\Delta_*})_{\text{toric}} \otimes \mathbb{C}\{x_\tau\}[\log x_\tau]$.

**Definition 6.1.** The special coordinate $(t_1, \cdots, t_{p-d})$ of the special Kähler geometry is defined by the ratios of the period integrals;

$$t_\cdot J = \frac{w^{(1)}(x_\tau, J)}{w^{(0)}(x_\tau, J)} \quad , \quad (47)$$

where we abuse the letters $J_1, \cdots, J_{p-d}$ to represent the images of the $J$'s under the quotient [45]. The inverse series of this relation will be called the mirror map.

**Note.** Since $w^{(1)}(x_\tau, J)$ is linear in $x_\tau$, the mirror map takes the form $x_\tau(q) := x_\tau(q_1, \cdots, q_{p-d})$ with $q_k := e^{2\pi i k}$. It is easy to see that $x^{(k)}(q) = q_k(1 + O(q))$.

**Definition 6.2.** We define the prepotential in the special coordinate by

$$F(t) := \int_{X_{\Delta_*}} \mathcal{F}(x_\tau(q), J) \quad , \quad (48)$$
with the *invariant* density

\[
\mathcal{F}(x_\tau(q), J) = \frac{1}{2} \left( \frac{1}{w^{(0)}} \right)^2 \left\{ w^{(0)} \left( -\frac{1}{3!} w^{(3)} - \frac{c_2(X_{\Delta_s})}{12} w^{(1)} \right) + w^{(1)} \left( \frac{1}{2!} w^{(2)} \right) \right\}.
\]

(49)

The integration symbol \( \int_{X_{\Delta_s}} := \int_{\mathbb{P}(\Sigma(\Delta_s).T_{\mathbb{R}})} [X_{\Delta_s}] \) is meant to take the coefficient of the 'volume form' in the Chow ring \( A^*(\mathbb{P}(\Sigma(\Delta_s).T_{\mathbb{R}}))_{\mathbb{Q}} \) normalized by \( \int_{\mathbb{P}(\Sigma(\Delta_s).T_{\mathbb{R}})} [X_{\Delta_s} c(\mathbb{P}(\Sigma(\Delta_s).T_{\mathbb{R}})) / (1 + [X_{\Delta_s}]) = \chi(X_{\Delta_s}) \). (For the normalization when \( \chi(X_{\Delta_s}) = 0 \), see ref. \[\text{HLY1}\].)

**Note.** It would be instructive to summarize the general description \[\text{Str}\] of the special Kähler geometry on the complex structure moduli space of Calabi-Yau threefolds. Let us denote the holomorphic 3-form of a family of Calabi-Yau threefolds \( W \) by \( \Omega(\psi) \). We take a symplectic basis \( \{A_a, B_b\} \) \((a, b = 0, \ldots, h^{2,1}(W)) \) of \( H_3(W, \mathbb{Z}) \) and construct the period integrals \( z_a(\psi) = \int_{A_a} \Omega(\psi) \) and \( G_b(\psi) = \int_{B_b} \Omega(\psi) \). Then the holomorphic 3-form may be written by \( \Omega(\psi) = \sum_a z_a(\psi) \alpha_a + \sum_b G_b(\psi) \beta_b \) in terms of the dual bases \( \alpha_a \) and \( \beta_b \) in \( H^3(W, \mathbb{Z}) \). Locally we can introduce on the moduli space a Kähler metric, so-called the Weil-Petersen metric \[\text{Tian}\], through the Kähler potential \( K(\psi, \tilde{\psi}) = -\log i \int_{M} \Omega(\psi) \wedge \bar{\Omega}(\psi) = -\log i \sum_a (z_a(\psi) \overline{G_a(\psi)} - G_a(\psi) \bar{z}_a(\psi)) \). It is shown in ref. \[\text{Str}\] that the prepotential \( F(\psi) = \frac{1}{2} \sum_a z_a(\psi) G_a(\psi) \) describes the potential \( K(\psi, \tilde{\psi}) \) by

\[
K(\psi, \tilde{\psi}) = -\log i \sum_a \left( z_a(\psi) \frac{\partial F(\psi)}{\partial z_a} - z_a(\psi) \frac{\partial F(\psi)}{\partial z_a} \right),
\]

(50)

and defines the *special Kähler geometry* on the moduli space.

Our definition \[\text{H13}\] of the prepotential, up to the prefactor \((w^{(0)})^{-2}\) which makes the prepotential invariant under \( \Omega(\psi) \mapsto f(\psi) \Omega(\psi) \), implicitly contains a claim that \((w^{(0)}, w^{(1)}, \frac{1}{2!} w^{(2)}, -\frac{1}{3!} w^{(3)} - \frac{c_2(X_{\Delta_s})}{12} w^{(1)}) \) form the period integrals for a symplectic basis of \( H_3(X_{\Delta_s}, \mathbb{Z}) \). Several evidences for this claim are reported in \[\text{HLY3}\]. In the following, we will restrict our attention to the form of the prepotential near a LCSL assuming its application to the enumerative geometry (the instanton counting).

Now consider the following expansion in the Chow ring associated to the series \( w_0(x_\tau, 0, A_\tau) = \sum_{n \in \mathbb{Z}_{\geq 0}} c(n) x^n_\tau \):

\[
\sum_{n \in \mathbb{Z}_{\geq 0}} c \left( n + \frac{J}{2\pi i} \right) x^n_\tau = \bar{w}^{(0)}(x_\tau, J) + \bar{w}^{(1)}(x_\tau, J) + \frac{1}{2!} \bar{w}^{(2)}(x_\tau, J) + \frac{1}{3!} \bar{w}^{(3)}(x_\tau, J).
\]

(51)
Lemma 6.3. The two definitions of the series (44) and (51) are related by
\[
\begin{align*}
w^{(0)} &= \tilde{w}^{(0)}, \\
w^{(1)} &= (\log x\cdot \tilde{J})w^{(0)} + \tilde{w}^{(1)}, \\
w^{(2)} &= (\log x\cdot \tilde{J})^2w^{(0)} + 2(\log x\cdot \tilde{J})\tilde{w}^{(1)} + \tilde{w}^{(2)}, \\
w^{(3)} &= (\log x\cdot \tilde{J})^3w^{(0)} + 3(\log x\cdot \tilde{J})^2\tilde{w}^{(1)} + 3(\log x\cdot \tilde{J})\tilde{w}^{(2)} + \tilde{w}^{(3)},
\end{align*}
\]
where we have introduced an abbreviation \(\log x\cdot \tilde{J} = \sum_{a=1}^{p-d}(\log x_a)\frac{1}{2\pi i} J_a\).

Lemma 6.4. The series \(\tilde{w}^{(d)}(x_{\tau}, J)\) (\(d = 1, 2, 3\)) in (57) have the form
\[
\begin{align*}
\tilde{w}^{(1)}(x_{\tau}, J) &= \sum_n c(n)\Psi^{(1)}(n)x^n_{\tau}, \\
\tilde{w}^{(2)}(x_{\tau}, J) &= \sum_n c(n)\{(\Psi^{(1)}(n))^2 + \Psi^{(2)}(n)x^n_{\tau}, \\
\tilde{w}^{(3)}(x_{\tau}, J) &= \sum_n c(n)\{(\Psi^{(1)}(n))^3 + 3\Psi^{(1)}(n)\Psi^{(2)}(n) + \Psi^{(3)}(n)x^n_{\tau},
\end{align*}
\]
where \(\Psi^{(k)}(n)\)'s are elements in the Chow ring of degree \(k\) defined by
\[
\begin{align*}
\Psi^{(1)}(n) &= -(\tilde{J}_0)\psi(1 - n\tilde{I}_0) - \sum_{i=1}^{p}(\tilde{J}_i)\psi(1 + n\tilde{I}_i), \\
\Psi^{(2)}(n) &= (\tilde{J}_0)^2\psi'(1 - n\tilde{I}_0) - \sum_{i=1}^{p}(\tilde{J}_i)^2\psi'(1 + n\tilde{I}_i), \\
\Psi^{(3)}(n) &= -(\tilde{J}_0)^3\psi''(1 - n\tilde{I}_0) - \sum_{i=1}^{p}(\tilde{J}_i)^3\psi''(1 + n\tilde{I}_i),
\end{align*}
\]
with \(\tilde{J}_0 = \sum_{a=1}^{p-d} \frac{1}{2\pi i} l_k^{(a)} (k = 0, \cdots, p)\), \(\psi(z) = \frac{d}{dz}\log \Gamma(z)\) and the derivatives of \(\psi(z)\).

Lemma 6.5.
\[
\Psi^{(1)}(0) = 0, \quad \Psi^{(2)}(0) = -\frac{c_2(X_{\Delta})}{12}, \quad \Psi^{(3)}(0) = -\frac{6\zeta(3)}{(2\pi i)^3}c_3(X_{\Delta}).
\]

(Proof) These constant terms originate from those of the \(\psi\)-functions; \(\psi(1) = -\gamma\), \(\psi'(1) = \frac{-\pi^2}{6}\), \(\psi''(1) = -2\zeta(3)\). These values of the \(\psi\)-functions combined with the adjunction formula for the total Chern class, with \(D_i = J\tilde{I}_i\) under the rational equivalence in the Chow ring,
\[
c(X_{\Delta}) = \frac{\prod_{i=1}^{p}(1 + D_i)}{1 + [X_{\Delta}]} = \frac{\prod_{i=1}^{p}(1 + J\tilde{I}_i)}{1 - J\tilde{I}_0},
\]
result in our claim for the leading terms. (Note that \(c_1(X_{\Delta}) = 0\) for \(\Psi^{(1)}(0)\).) \(\square\)

Remark. We can subtract these constant terms \(\Psi^{(k)}(0)\) in a systematic way modifying the expansion (51) slightly as follows;
\[
\sum_{n \in \mathbb{Z}_{\geq 0}} \frac{c(n + \frac{j}{2\pi i})}{c(j/2\pi i)} x^n_{\tau} = w^{(0)}(x_{\tau}) + \tilde{w}^{(1)}(x_{\tau}, J) + \frac{1}{2} \tilde{w}^{(2)}(x_{\tau}, J) + \frac{1}{3!} \tilde{w}^{(3)}(x_{\tau}, J).
\]
This is because this change of normalization in the series \( w_0 \) simply results in the replacement \( \Psi^{(k)}(n) \) with \( \Psi^{(k)}_r(n) := \Psi^{(k)}(n) - \Psi^{(k)}(0) \) in (53).

Now it is immediate from Lemmas 6.4 and 6.5 to obtain

**Lemma 6.6.**

\[
\tilde{w}^{(1)} = \tilde{w}^{(1)}_r , \quad \tilde{w}^{(2)} = - \frac{c_2(X_{\Delta_s})}{12} w^{(0)} + \tilde{w}^{(2)}_r , \\
\tilde{w}^{(3)} = - \frac{6\zeta(3)}{(2\pi i)^3} c_3(X_{\Delta_s}) w^{(0)} - \frac{c_2(X_{\Delta_s})}{4} \tilde{w}^{(1)} + \tilde{w}^{(3)}_r .
\]  

(58)

Now using the results in Lemmas 6.3-6.6, we may arrive at our final form of the prepotential, see also [Sti], modulo the kernel of the integration \( \int_{X_{\Delta_s}} \) in Def.6.2;

**Proposition 6.7.** The invariant form of the prepotential \( F(x, J) \) may be expressed by

\[
F(t, J) = \frac{1}{6} (t \cdot J)^3 - \frac{c_2(X_{\Delta_s})}{24} (t \cdot J) + \frac{\zeta(3)}{2(2\pi i)^3} c_3(X_{\Delta_s}) \\
- \frac{1}{2} \log \left( \sum_{n \in \mathbb{Z}^{p-d}_{\geq 0}} \frac{c(n + \frac{J}{2\pi i})}{c(J/2\pi i)} x^\tau_n \right) ,
\]

(59)

with the mirror map \( x_\tau = x_\tau(q) \).

**Claim 6.8.** Three times derivatives of the prepotential give the instanton corrected Yukawa couplings;

\[
K_{t_a t_b t_c}(t) = \frac{\partial^3}{\partial t_a \partial t_b \partial t_c} F(t) \\
= \int_{X_{\Delta_s}} J_a J_b J_c + \sum_{\Gamma \in H_2(X_{\Delta_s}, \mathbb{Z}) \setminus \Gamma \neq 0} (\Gamma \cdot J_a)(\Gamma \cdot J_b)(\Gamma \cdot J_c) N(\Gamma) \frac{e^{2\pi i \Gamma \cdot (t \cdot J)}}{1 - e^{2\pi i \Gamma \cdot (t \cdot J)}},
\]

(60)

where \( \Gamma \cdot J := \int_{\Gamma} J \) and \( N(\Gamma) \) counts the number of the rational curves of class \( \Gamma \) on the Calabi-Yau manifolds \( X_{\Delta_s} \).

**Note.** Since the mirror map has the \( q \)-expansion \( x^{(k)}_\tau(q) = q^k (1 + \mathcal{O}(q)) \), it is immediate to deduce that the number of lines \( N(\Gamma) \) in \( X_{\Delta_s} \) is counted by

\[
N(\Gamma) = \int_{X_{\Delta_s}} \frac{-1}{2} \frac{c((\Gamma \cdot J) + J)}{c(J)} .
\]  

(61)
We see that the famous number 2785 for the quintic in $\mathbb{P}^4$ [CdGP] is counted by this formula as

$$N(1) = -\frac{1}{2} \int_{\mathbb{P}^4} 5J(5 + 5J)(4 + 5J)(3 + 5J)(2 + 5J)(1 + 5J) \left(1 + 5J\right)^5. \quad (62)$$

The invariant form of the prepotential $\mathcal{F}(x, J)$ may have significant applications to extracting the predicted numbers of the rational curves $N(\Gamma)$. In a recent work [HSS], this form has been utilized efficiently to verify that the numbers $N(\Gamma)$ of a certain Calabi-Yau manifold (Schoen’s Calabi-Yau manifold) are related to the modular forms, the theta function of the $E_8$ lattice and Dedekind’s eta function.

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