A study on bipolar valued multi I-fuzzy subhemirings of a hemiring

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Abstract
In this paper, bipolar valued multi I-fuzzy subhemiring of a hemiring is introduced and some properties are discussed. Bipolar valued multi I-fuzzy subhemiring of a hemiring is a generalized form of bipolar valued multi fuzzy subhemiring of a hemiring. The paper will be useful to further research.

Keywords
Interval valued fuzzy subset, bipolar valued fuzzy subset, bipolar valued multi fuzzy subset, bipolar valued I-fuzzy subset, bipolar valued multi I-fuzzy subhemiring, union, intersection, product, strongest.

AMS Subject Classification
03E72.

1. Introduction

In 1965, Zadeh [13] introduced the notion of a fuzzy subset of a crisp set, fuzzy subsets are a kind of useful mathematical structure to represent a collection of objects whose boundary is vague. Multi fuzzy set was introduced by Sabu Sebastian, T.V. Ramakrishnan [9]. Lee [5] introduced the notion of bipolar valued fuzzy sets. Bipolar valued fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval $[0, 1]$ to $[-1, 1]$. In a bipolar valued fuzzy set, the membership degree 0 means that elements are irrelevant to the corresponding property, the membership degree $(0, 1]$ indicates that elements somewhat satisfy the property and the membership degree $[-1, 0)$ indicates that elements somewhat satisfy the implicit counter property. Bipolar valued fuzzy sets and intuitionistic fuzzy sets look similar each other. However, they are different each other [5, 6]. Fuzzy subgroup was introduced by Azriel Rosenfeld [2]. K. Murugalingam and K. Arjunan [7] have discussed about interval valued fuzzy subsemiring of a semiring. A study on interval valued fuzzy, anti fuzzy, intuitionistic fuzzy subrings of a ring by Somasunda Moorthy [11], the thesis was useful to write the paper. Anitha et.al. [1] defined as bipolar valued fuzzy subgroups of a group and and Balasubramanian et.al. [3] introduced about bipolar interval valued fuzzy subgroups of a group. The papers [4] and [10] was useful to work this field. After that bipolar valued multi fuzzy subsemirings of a semiring have been introduced by Yasodara and KE. Sathappan [12]. Muthukumaran & Anandh [8] defined the bipolar valued multi fuzzy subnearring of a nearing. Here, the concept of bipolar valued multi I-fuzzy subhemiring of a hemiring is introduced and established some results.

2. Preliminaries

Definition 2.1. Let $X$ be any nonempty set. A mapping $[M]: X \rightarrow D[0, 1]$ is called an interval valued fuzzy subset (I-fuzzy subset) of $X$, where $D[0, 1]$ denotes the family of all closed subintervals of $[0, 1]$ and $[M](x) = [M^-(x), M^+(x)]$, for all $x$ in $X$, where $M^-$ and $M^+$ are fuzzy subsets of $X$ such that $M^-(x) \leq M^+(x)$, for all $x$ in $X$. Thus $M^-(x)$ is an interval (a closed subset of $[0, 1]$) and not a number from the interval $[0, 1]$ as in the case of fuzzy subset. Note that $[0] = [0, 0]$ and $[1] = [1, 1]$.

Definition 2.2 ([5]). A bipolar valued fuzzy set (BVFS) $A$ in
$X$ is defined as an object of the form

$$A = \{ < x, A^+(x), A^-(x) > / x \in X \},$$

where $A^+: X \to [0, 1]$ and $A^-: X \to [-1, 0]$. The positive membership degree $A^+(x)$ denotes the satisfaction degree of an element $x$ to the property corresponding to a bipolar valued fuzzy set $A$ and the negative membership degree $A^-(x)$ denotes the satisfaction degree of an element $x$ to some implicit counter-property corresponding to a bipolar valued fuzzy set $A$.

Example 2.3. $A = \{ < a, 0.7, -0.3 >, < b, 0.6, -0.5 >, < c, 0.2, -0.8 > \}$ is a bipolar valued fuzzy subset of $X = \{ a, b, c \}$.

**Definition 2.4** ([12]). A bipolar valued multi fuzzy set (BVMFS) $A$ in $X$ is defined as an object of the form

$$A = \{ < x, A^+_1(x), A^+_2(x), \ldots, A^+_n(x), A^-_1(x), A^-_2(x), \ldots, A^-_n(x) > / x \in X \},$$

where $A^+_i: X \to [0, 1]$ and $A^-_i: X \to [-1, 0]$ for all $i = 1, 2, \ldots, n$. The positive membership degrees $A^+_i(x)$ denote the satisfaction degrees of an element $x$ to the property corresponding to a bipolar valued multi fuzzy set $A$ and the negative membership degrees $A^-_i(x)$ denote the satisfaction degrees of an element $x$ to some implicit counter-property corresponding to a bipolar valued multi fuzzy set $A$.

Example 2.5. $A = \{ < a, 0.7, 0.3, 0.7, -0.4, -0.5, -0.9 >, < b, 0.6, 0.7, 0.2, -0.8, -0.1, -0.4 >, < c, 0.5, 0.9, 0.8, -0.2, -0.6, -0.9 > \}$ is a bipolar valued multi fuzzy subset of $X = \{ a, b, c \}$.

**Definition 2.6.** A bipolar interval valued fuzzy subset (bipolar valued I-fuzzy subset) $[A]$ in $X$ is defined as an object of the form $[A] = \{ < x, [A]_1^+(x), [A]_1^-(x) > / x \in X \}$, where $[A]_1^+: X \to [0, 1]$ and $[A]_1^- : X \to [-1, 0]$, where $[D, 0, 1]$ denotes the family of all closed subintervals of $[0, 1]$ and $D[-1, 0]$ denotes the family of all closed subintervals of $[-1, 0]$. The positive interval membership degrees $[A]_1^+(x)$ denotes the satisfaction degrees of an element $x$ to the property corresponding to a bipolar valued multi I-fuzzy subset $[A]$ and the negative interval membership degrees $[A]_1^-(x)$ denotes the satisfaction degrees of an element $x$ to some implicit counter-property corresponding to a bipolar valued multi I-fuzzy subset $[A]$. Note that

$$[0] = ([0, 0], [0, 0], \ldots, [0, 0]), [1] = ([1, 1], [1, 1], \ldots, [1, 1])$$

and

$$[-1] = ((-1, -1), (-1, -1), \ldots, (-1, -1)).$$

It is denoted as

$$[A] = ([A]_1^+, [A]_2^+, \ldots, [A]_n^+, [A]_1^-, [A]_2^-, \ldots, [A]_n^-).$$

Example 2.9. $[A] = \{ < a, [0.3, 0.6], [0.2, 0.5], [0.6, 0.8], [-0.7, -0.2], [-0.9, -0.1], [-0.5, -0.2] >, < b, [0.2, 0.4], [0.2, 0.6], [0.3, 0.6], [-0.6, -0.3], [-0.7, -0.22, -0.5, -0.2] >, < c, [0.2, 0.6], [0.5, 0.7], [0.6, 0.8], [-0.4, -0.2], [-0.6, -0.3], [-0.4, -0.2] > \}$ is a bipolar valued multi I-fuzzy subset of $X = \{ a, b, c \}$.

**Definition 2.10.** Let $[A] = ([A]_1^+, [A]_2^+, \ldots, [A]_n^+, [A]_1^-, [A]_2^-, \ldots, [A]_n^-)$ and $[B] = ([B]_1^+, [B]_2^+, \ldots, [B]_n^+, [B]_1^-, [B]_2^-, \ldots, [B]_n^-)$ be two bipolar valued multi I-fuzzy subsets of a set $X$. We define the following relations and operations:

(i) $[A] \subseteq [B]$ if and only if for each $i$, $[A]_i^+(u) \leq [B]_i^+(u)$ and $[A]_i^-(u) \geq [B]_i^-(u), \forall u \in X$

(ii) $[A] = [B]$ if and only if for each $i$, $[A]_i^+(u) = [B]_i^+(u)$ and $[A]_i^-(u) = [B]_i^-(u), \forall u \in X$

(iii) $[A] \cap [B] = \{ < u, \min ([A]_1^+(u), [B]_1^+(u)), \ldots, \min ([A]_n^+(u), [B]_n^+(u)), \max ([A]_1^-(u), [B]_1^-(u)), \ldots, \max ([A]_n^-(u), [B]_n^-(u)) > / u \in X \}$

(iv) $[A] \cup [B] = \{ < u, \max ([A]_1^+(u), [B]_1^+(u)), \ldots, \max ([A]_n^+(u), [B]_n^+(u)), \min ([A]_1^-(u), [B]_1^-(u)), \ldots, \min ([A]_n^-(u), [B]_n^-(u)) > / u \in X \}$

**Definition 2.11.** Let $R$ be a hemiring. A bipolar valued multi I-fuzzy subset $[A] = ([A]_1^+, [A]_2^+, \ldots, [A]_n^+, [A]_1^-, [A]_2^-, \ldots, [A]_n^-)$ of $R$ is said to be a bipolar valued multi I-fuzzy subhemiring (BVMIFSHR) if the following conditions are satisfied for each $i$,

(i) $[A]_i^+(x + y) \geq \min \{ [A]_i^+(x), [A]_i^+(y) \}$

(ii) $[A]_i^+(xy) \geq \min \{ [A]_i^+(x), [A]_i^+(y) \}$

(iii) $[A]_i^-(x + y) \leq \max \{ [A]_i^-(x), [A]_i^-(y) \}$
Theorem 2.15. Let \( R = Z_3 = \{0, 1, 2\} \) be a hemiring with respect to the ordinary addition and multiplication. Then

\[
[A] = \{< 0, [0.55, 0.6], [0.65, 0.7], [0.75, 0.8], [-0.6, -0.5] \\
[-0.7, -0.6], [-0.8, -0.75], < 1, [0.41, 0.5], [0.51, 0.6], [0.61, 0.7], [-0.51, -0.4], [-0.61, -0.5], [-0.71, -0.6], < 2, [0.41, 0.5], [0.51, 0.6], [0.61, 0.7], [-0.51, -0.4], [-0.61, -0.5], [-0.71, -0.6] \}
\]

is a bipolar valued multi I-fuzzy subhemiring of \( R \).

Example 2.12. Let \([A] = \langle [A_1^+, [A_2^+, \ldots, [A_n^+, [A_1^-], [A_2^-], \ldots, [A_n^-] \rangle \) and \([B] = \langle [B_1^+, [B_2^+, \ldots, [B_n^+, [B_1^-], [B_2^-], \ldots, [B_n^-] \rangle \) be any two bipolar valued multi I-fuzzy subsets of \( G \) and \( H \), respectively. The product of \([A] \) and \([B] \), denoted by \([A] \times [B] \), is defined as

\[
[A] \times [B] = \{ ([x, y], ([A_1] \times [B_1])^+, ([x_2, y_2])^+, \ldots, ([A_n] \times [B_n])^+, (x, y), ([A_1] \times [B_1])^+, (x, y) \} = \{ ([A_2] \times [B_2])^+, (x, y), \ldots, ([A_n] \times [B_n])^+, (x, y) \} \}
\]

where each \( i \), \([A_i] \times [B_i])^+ (x, y) = \text{rmin} \{ [A_i]^+ (x), [B_i]^+ (y) \} \) and \([A_i] \times [B_i])^- (x, y) = \text{rmax} \{ [A_i]^+ (x), [B_i]^+ (y) \} \) for all \( x \) in \( G \) and \( y \) in \( H \).

Definition 2.13. Let \([A] = \langle [A_1^+, [A_2^+, \ldots, [A_n^+, [A_1^-], [A_2^-], \ldots, [A_n^-] \rangle \) and \([B] = \langle [B_1^+, [B_2^+, \ldots, [B_n^+, [B_1^-], [B_2^-], \ldots, [B_n^-] \rangle \) be any two bipolar valued multi I-fuzzy subsets of \( S \), the strongest bipolar valued multi I-fuzzy relation on \( S \), that is a bipolar valued multi I-fuzzy relation on \([A] \) is

\[
[V] = \{ ([x, y], [V_1]^+, (x, y), [V_2]^+, (x, y), \ldots, [V_n]^+, (x, y)) \} \times \text{yin} \}
\]

given by \([V_1]^+, (x, y) = \text{rmin} \{ [A_1]^+ (x), [A_1]^+ (y) \} \) and \([V_1]^+, (x, y) = \text{rmax} \{ [A_1]^+ (x), [A_1]^+ (y) \} \) for all \( i \), \( j \) and \( y \) in \( S \).

Theorem 2.16. Let \([A] = \langle [A_1^+, [A_2^+, \ldots, [A_n^+, [A_1^-], [A_2^-], \ldots, [A_n^-] \rangle \) be a bipolar valued multi I-fuzzy subhemiring of a hemiring \( R \).

(i) For each \( i \), if \( [A_i]^+ (x+y) = [A_i]^+ (0) \) then either \( [A_i]^+ (x) = [A_i]^+ (0) \) or \( [A_i]^+ (y) = [A_i]^+ (0) \) for \( x, y \) in \( R \).

(ii) For each \( i \), if \( [A_i]^+ (x) = [A_i]^+ (0) \) then either \( [A_i]^+ (y) = [A_i]^+ (0) \) for \( x, y \) in \( R \).

(iii) For each \( i \), if \( [A_i]^+ (x+y) = [A_i]^+ (0) \) then either \( [A_i]^+ (x) = [A_i]^+ (0) \)

(iv) For each \( i \), if \( [A_i]^+ (y) = [A_i]^+ (0) \) then either \( [A_i]^+ (x) = [A_i]^+ (0) \) for \( x, y \) in \( R \).

Proof. Let \( x, y \) in \( R \).

(i) For each \( i \), if \( [A_i]^+ (x+y) \geq [A_i]_1 (x) \) \( [A_i]_1^+ (y) \) which implies that \( 0 \geq [A_i]_1 (x) \) \( [A_i]_1^+ (y) \).

(ii) For each \( i \), if \( [A_i]^+ (x) \geq [A_i]_1 (0) \) then either \( [A_i]^+ (y) \geq [A_i]_1 (0) \).

Theorem 2.17. If \([A] = \langle [A_1^+, [A_2^+, \ldots, [A_n^+, [A_1^-], [A_2^-], \ldots, [A_n^-] \rangle \) and \([B] = \langle [B_1^+, [B_2^+, \ldots, [B_n^+, [B_1^-], [B_2^-], \ldots, [B_n^-] \rangle \) are two bipolar valued multi I-fuzzy subhemirings of a hemiring \( R \), then their intersection \([A] \cap [B] \) is also a bipolar valued multi I-fuzzy subhemiring of \( R \).

Proof. Let \([C] = [A] \cap [B] \) and let \( x, y \) in \( R \). For each \( i \),

\[
[C_i]^+ (x+y) = \text{rmin} \{ [A_i]^+ (x+y), [B_i]^+ (x+y) \}
\]

\[
\geq \text{rmin} \{ [A_i]^+ (x), [B_i]^+ (y) \} \]
Therefore $|C|^+_r(x+y) \geq rmin\{ |C|^+_r(x), |C|^+_r(y) \}$, for all $x, y$ in $R$ and

\[
|C|^+_r(xy) = rmin\{ |A|^+_r(xy), |B|^+_r(xy) \}
\]

\[
\geq rmin\{ rmax\{ |A|^+_r(x), |A|^+_r(y) \}, rmin\{ |B|^+_r(x), |B|^+_r(y) \} \}
\]

\[
\geq rmin\{ rmax\{ |A|^+_r(x), |B|^+_r(x) \}, rmax\{ |A|^+_r(y), |B|^+_r(y) \} \}
\]

\[
= rmin\{ |C|^+_r(x), |C|^+_r(y) \}.
\]

Therefore $|C|^+_r(xy) \geq rmin\{ |C|^+_r(x), |C|^+_r(y) \}$, for all $x, y$ in $R$. Also

\[
|C|^+_r(x+y) = rmax\{ |A|^+_r(x+y), |B|^+_r(x+y) \}
\]

\[
\leq rmax\{ rmax\{ |A|^+_r(x), |A|^+_r(y) \}, rmax\{ |B|^+_r(x), |B|^+_r(y) \} \}
\]

\[
\leq rmax\{ rmax\{ |A|^+_r(x), |B|^+_r(x) \}, rmax\{ |A|^+_r(y), |B|^+_r(y) \} \}
\]

\[
= rmax\{ |C|^+_r(x), |C|^+_r(y) \}.
\]

Therefore $|C|^+_r(xy) \leq rmax\{ |C|^+_r(x), |C|^+_r(y) \}$, for all $x, y$ in $R$. Hence $|A| \cap |B|$ is a bipolar valued multi $I$-fuzzy subhemiring of $R$.

Theorem 2.18. The intersection of a family of bipolar valued multi $I$-fuzzy subhemirings of a hemiring $R$ is a bipolar valued multi $I$-fuzzy subhemiring of $R$.

Proof. The proof follows from the Theorem 2.17.

Theorem 2.19. If $|A| = \langle |A|_1^+, |A|_2^+, \ldots, |A|_n^+, |A|_1^-, |A|_2^-, \ldots, |A|_n^- \rangle$ is a bipolar valued multi $I$-fuzzy subhemiring of a hemiring $R$, then

\[
H = \{ x \in R \mid |A|_i^+(x) = [1], |A|_i^-(x) = [-1] \ for \ all \ i \}
\]

is either empty or a subhemiring of $R$.

Proof. If no element satisfies this condition then $H$ is empty. If $x$ and $y$ in $H$ then for all $i$,

\[
|A|_i^+(x+y) \geq rmin\{ |A|_i^+(x), |A|_i^+(y) \} = rmin\{[1], [1] \} = [1].
\]

Therefore $|A|_i^+(x+y) = [1]$ and

\[
|A|_i^-(xy) \geq rmin\{ |A|_i^-(x), |A|_i^-(y) \} = rmin\{[1], [1] \} = [1].
\]

Therefore $|A|_i^+(xy) = [1]$. Also

\[
|A|_i^-(x+y) \leq rmax\{ |A|_i^-(x), |A|_i^-(y) \}
\]

\[
= rmax\{[-1], [-1] \} = [-1].
\]

Therefore $|A|_i^-(x+y) = [-1]$. And

\[
|A|_i^+(xy) \leq rmax\{ |A|_i^+(x), |A|_i^+(y) \}
\]

\[
= rmax\{[-1], [-1] \} = [-1].
\]

Therefore $|A|_i^-(xy) = [-1]$. That is $x + y \in H$ and $xy \in H$. Hence $H$ is a subhemiring of $R$. Hence $H$ is either empty or a subhemiring of $R$.

Theorem 2.20. If $|A| = \langle |A|_1^+, |A|_2^+, \ldots, |A|_n^+, |A|_1^-, |A|_2^-, \ldots, |A|_n^- \rangle$ is a bipolar valued multi $I$-fuzzy subhemiring of a hemiring $R$, then

\[
H = \{ x \in R \mid |A|_i^+(x) = |A|_i^+(0) & |A|_i^-(x) = |A|_i^-(0), \forall i \}
\]

is a subhemiring of $R$.

Proof. Here $H = \{ x \in R \mid |A|_i^+(x) = |A|_i^+(e) & |A|_i^-(x) = |A|_i^-(e) \}$ by Theorem 2.15. For each $i$,

\[
|A|^+_i(x+y) \geq rmin\{ |A|^+_i(x), |A|^+_i(y) \}
\]

\[
= rmin\{ |A|^+_i(0), |A|^+_i(0) \} = |A|^+_i(0).
\]

Hence $|A|_i^+(0) = |A|^+_i(x+y)$. And

\[
|A|_i^-(xy) \geq rmin\{ |A|_i^-(x), |A|_i^-(y) \}
\]

\[
= rmin\{ |A|_i^-(0), |A|_i^-(0) \} = |A|_i^-(0).
\]

Hence $|A|_i^+(0) = |A|^+_i(xy)$. Also

\[
|A|_i^-(x+y) \leq rmax\{ |A|_i^-(x), |A|_i^-(y) \}
\]

\[
= rmax\{ |A|_i^-(0), |A|_i^-(0) \} = |A|_i^-(0).
\]

Therefore $|A|_i^-(0) = |A|^-_i(xy)$. Therefore $x + y$ and $xy$ are in $H$. Hence $H$ is a subhemiring of $R$.

Theorem 2.21. If $|A| = \langle |A|_1^+, |A|_2^+, \ldots, |A|_n^+, |A|_1^-, |A|_2^-, \ldots, |A|_n^- \rangle$ and $|B| = \langle |B|_1^+, |B|_2^+, \ldots, |B|_n^+, |B|_1^-, |B|_2^-, \ldots, |B|_n^- \rangle$ are any two bipolar valued multi $I$-fuzzy subhemirings of the hemirings $R_1$ and $R_2$ respectively, then

\[
|A| \times |B| = \langle (|A|_1 \times |B|_1)^+, (|A|_2 \times |B|_2)^+, \ldots, (|A|_n \times |B|_n)^+, (|A|_1 \times |B|_1)^-, (|A|_2 \times |B|_2)^-, \ldots, (|A|_n \times |B|_n)^- \rangle
\]

is a bipolar valued multi $I$-fuzzy subhemiring of $R_1 \times R_2$.

Proof. Let $|A|$ and $|B|$ be two bipolar valued multi $I$-fuzzy subhemirings of the hemirings $R_1$ and $R_2$ respectively. Let
\[ x_1, x_2 \text{ be in } R_1, y_1 \text{ and } y_2 \text{ be in } R_2. \text{ Then } (x_1, y_1) \text{ and } (x_2, y_2) \text{ are in } R_1 \times R_2. \text{ For each } i, \]

\[
([A_i]_i \times [B_i])^+ [(x_1, y_1) + (x_2, y_2)]
\]

\[
= ([A_i]_i \times [B_i])^+ (x_1 + x_2, y_1 + y_2)
\]

\[
= \min \{([A_i]_i^+ (x_1 + x_2), [B_i]_i^+ (y_1 + y_2))
\]

\[
\geq \min \{\min \{([A_i]_i^+ (x_1), [A_i]_i^+ (x_2)),
\min \{([B_i]_i^+ (y_1), [B_i]_i^+ (y_2))\}\},
\min \{([A_i]_i^+ (x_1), [B_i]_i^+ (y_1)),
\min \{([A_i]_i^+ (x_2), [B_i]_i^+ (y_2))\}\}\}
\]

\[
= \min \{\min \{([A_i]_i \times [B_i])^+ (x_1, y_1) + (x_2, y_2),
\min \{([A_i]_i \times [B_i])^+ (x_1, y_1) + (x_2, y_2)\}\}\}
\]

\[
\text{Therefore}
\]

\[
([A_i]_i \times [B_i])^+ [(x_1, y_1) + (x_2, y_2)]
\]

\[
\geq \min \{([A_i]_i \times [B_i])^+ (x_1, y_1) + (x_2, y_2)\}.
\]

And

\[
([A_i]_i \times [B_i])^+ [(x_1, y_1) + (x_2, y_2)]
\]

\[
= \min \{([A_i]_i^+ (x_1 + x_2), [B_i]_i^+ (y_1 + y_2))
\]

\[
\geq \min \{\min \{([A_i]_i^+ (x_1), [A_i]_i^+ (x_2)),
\min \{([B_i]_i^+ (y_1), [B_i]_i^+ (y_2))\}\},
\min \{([A_i]_i^+ (x_2), [B_i]_i^+ (y_2))\}\}
\]

\[
= \min \{\min \{([A_i]_i \times [B_i])^+ (x_1, y_1) + (x_2, y_2),
\min \{([A_i]_i \times [B_i])^+ (x_1, y_1) + (x_2, y_2)\}\}\}
\]

\[
\text{Therefore}
\]

\[
([A_i]_i \times [B_i])^+ [(x_1, y_1) + (x_2, y_2)]
\]

\[
\geq \min \{([A_i]_i \times [B_i])^+ (x_1, y_1) + (x_2, y_2)\}.
\]

Also

\[
([A_i]_i \times [B_i])^- [(x_1, y_1) + (x_2, y_2)]
\]

\[
= ([A_i]_i \times [B_i])^- (x_1 + x_2, y_1 + y_2)
\]

\[
= \max \{([A_i]_i^- (x_1 + x_2), [B_i]_i^- (y_1 + y_2))
\]

\[
\leq \max \{\max \{([A_i]_i^- (x_1), [A_i]_i^- (x_2)),
\max \{([B_i]_i^- (y_1), [B_i]_i^- (y_2))\}\},
\max \{([A_i]_i^- (x_2), [B_i]_i^- (y_2))\}\}
\]

\[
= \max \{\max \{([A_i]_i \times [B_i])^- (x_1, y_1) + (x_2, y_2),
\max \{([A_i]_i \times [B_i])^- (x_1, y_1) + (x_2, y_2)\}\}\}
\]

\[
\text{Therefore}
\]

\[
([A_i]_i \times [B_i])^- [(x_1, y_1) + (x_2, y_2)]
\]

\[
\leq \max \{([A_i]_i \times [B_i])^- (x_1, y_1) + (x_2, y_2)\}.
\]
Therefore \( V_i^+(x + y) \geq \min \{ V_i^+(x), V_i^+(y) \} \) for all \( x \) and \( y \) in \( R \times R \). Also we have

\[
\begin{align*}
[V_i^-(x + y)] &= [V_i^-(x_1 + y_1, x_2 + y_2)] \\
&= [V_i^-(x_1 + x_2, y_1 + y_2)] \\
&= \max \{ [A_i^-(x_1 + y_1), [A_i^-(x_2 + y_2)] \} \\
&\leq \max \{ \max \{ [A_i^-(x_1), [A_i^-(y_1)] \}, \max \{ [A_i^-(x_2), [A_i^-(y_2)] \} \} \\
&= \max \{ [V_i^-(x_1, x_2), [V_i^-(y_1, y_2)] \} \\
&= \max \{ [V_i^-(x), [V_i^-(y)] \}.
\end{align*}
\]

Therefore \( V_i^-(x + y) \leq \max \{ [V_i^-(x), [V_i^-(y) \} \) for all \( x, y \) in \( R \times R \). This proves that \( V \) is a bipolar valued multi-I-fuzzy subhemiring of \( R \times R \).

Conversely assume that \( V \) is a bipolar valued multi-I-fuzzy subhemiring of \( R \times R \), then for any \( x = (x_1, x_2) \) and \( y = (y_1, y_2) \) are in \( R \times R \), for all \( i \),

\[
\begin{align*}
\min \{ [A_i^+(x_1 + y_1), [A_i^+(x_2 + y_2)] \} &= [V_i^+(x_1 + x_2, y_1 + y_2)] \\
&= [V_i^+(x_1, x_2) + (y_1, y_2)] \\
&= [V_i^+(x + y)] \leq \min \{ [V_i^+(x), [V_i^+(y)] \} \\
&= \min \{ [V_i^+(x_1, x_2)] [V_i^+(y_1, y_2)] \} \\
&= \min \{ \min \{ [A_i^+(x_1), [A_i^+(x_2)] \}, \min \{ [A_i^+(y_1), [A_i^+(y_2)] \} \}.
\end{align*}
\]

If \( x_2 = y_2 = 0 \), we get

\[
[A_i^+(x_1 + y_1)] \geq \min \{ [A_i^+(x_1), [A_i^+(y_1)] \}
\]

for all \( x_1 \) and \( y_1 \) in \( R \) and

\[
\begin{align*}
\min \{ [A_i^+(x_1y_1), [A_i^+(x_2y_2)] \} &= [V_i^+(x_1y_1, x_2y_2)] \\
&= [V_i^+(x_1, x_2)] (y_1, y_2) \} = [V_i^+(x_1y_1)] \\
&\leq \min \{ [V_i^+(x), [V_i^+(y))] \} \\
&= \min \{ \min \{ [A_i^+(x_1), [A_i^+(x_2)] \}, \min \{ [A_i^+(y_1), [A_i^+(y_2)] \} \}.
\end{align*}
\]

If \( x_2 = y_2 = 0 \), we get \( [A_i^+(x_1y_1)] \leq \max \{ [A_i^+(x_1), [A_i^+(y_1)] \} \) for all \( x_1 \) and \( y_1 \) in \( R \). Hence \( [A_i] \) is a bipolar valued multi-I-fuzzy subhemiring of \( R \).

**Theorem 2.23.** Let \( [C] \) be a bipolar valued multi-I-fuzzy subset of a hemiring \( R \). Then \([C] \) is a bipolar valued multi-I-fuzzy subhemiring of \( R \) if and only if \( -C \) and \( +C \) are bipolar valued multi-I-fuzzy subhemiring of \( R \), where \( -C \) is the lower limits of the closed intervals and \( +C \) is the upper limits of the closed intervals.

**Proof.** Let \( a, b \in R \). Suppose \([C] \) is a bipolar valued multi-I-fuzzy subhemiring of \( R \), for each \( i \), \([C_i^+(a)] = [-C_i^-(a), +C_i^+(a)] \) and \([C_i^-(a)] = [-C_i^+(a), -C_i^-(a)] \). So,

\[
[C_i^+(a + b)] \geq \min \{ [-C_i^+(a), -C_i^-(b)] \} = \min \{ [-C_i^+(a), -C_i^-(b)] \}.
\]
Thus

\[
- C_i^+(a+b), + C_i^-(a+b) \geq \min \{ - C_i^+(a), - C_i^-(b) \}, \min \{ + C_i^+(a), - C_i^-(b) \}
\]
and

\[
+ C_i^+(a+b) \geq \min \{ + C_i^+(a), + C_i^-(b) \}.
\]

And

\[
[C_i^+(ab)] \geq \min \{ [C_i^+(a)]^+, [C_i^-(b)]^- \} = \min \{ [C_i^- (a)]^+, [C_i^- (b)]^- \}.
\]

which implies that

\[
[C_i^+(ab)] \leq \max \{ [C_i^-(a)]^+, [C_i^-(b)]^- \} = \max \{ [C_i^- (a)]^+, [C_i^- (b)]^- \}.
\]

References

[1] G. Adomian and G. E. Adomian, Cellular systems and aging models, Comput. Math. Appl. 11(1985), 283–291.

[2] M.S. Anitha, Muruganantha Prasad and K. Arjunan, Notes on bipolar valued fuzzy subgroups of a group, Bulletin of Society for Mathematical Services and Standards, 2(3), (2013), 52–59.

[3] Azriel Rosenfeld, fuzzy groups, Journal of mathematical analysis and applications, 35(1971), 512–517.

[4] A. Balasubramanian, K.L. Muruganantha Prasad and K. Arjunan, Properties of Bipolar interval valued fuzzy subgroups of a group, International Journal of Scientific Research, 4(4), (2015), 262–268.

[5] Grattan Guiness, Fuzzy membership mapped onto interval and many valued quantities, Z. Math. Logik. Grundl. Math 22(1975), 149–160.

[6] K.M. Lee, bipolar valued fuzzy sets and their operations, Proc. Int. Conf. on Intelligent Technologies, Bangkok, Thailand, (2000), 307–312.

[7] K.M. Lee, Comparison of interval valued fuzzy sets, intuitionistic fuzzy sets and bipolar valued fuzzy sets, J. fuzzy Logic Intelligent Systems, 14(2), (2004), 125–129.

[8] K. Murugalingam and K. Arjunan, A study on interval valued fuzzy subsemiring of a semiring, International Journal of Applied Mathematics Modeling, 1(5), (2013), 1–6.

[9] S. Muthukumaran and B. Anandh, Some theorems in bipolar valued multi fuzzy subsemiring of a nearing, Infokara, 8(11), (2019).

[10] Sabu Sebastian, T.V. Ramakrishnan, Multi fuzzy sets, International Mathematical Forum, 5(50), (2010), 2471–2476.

[11] V.K Santhi and K. Anbarasi, Bipolar valued multi fuzzy subsemirings of a hemiring, Advances in Fuzzy Mathematics, 10(1), (2015), 55–62.
M.G. Somasundra Moorthy, *A study on interval valued fuzzy, anti fuzzy, intuitionistic fuzzy subrings of a ring*, Ph.D Thesis, Bharathidasan University, Trichy, Tamilnadu, India (2014).

B. Yasodara and KE. Sathappan, *Bipolar-valued multi fuzzy subsemirings of a semiring*, *International Journal of Mathematical Archive*, 6(9), (2015), 75–80.

L.A. Zadeh, *fuzzy sets*, *Inform. And Control*, 8(1965), 338–353.

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