BiHom-Lie superalgebra structures

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ABSTRACT

The aim of this paper is to introduce the notion of BiHom-Lie superalgebras. This class of algebras is a generalization of both BiHom-Lie algebras and Hom-Lie superalgebras. In this article, we first present two ways to construct BiHom-Lie superalgebras from BiHom-associative superalgebras and Hom-Lie superalgebras by Yau’s twist principle. Also, we explore some general classes of BiHom-Lie admissible superalgebras and describe all these classes via $G$-BiHom-associative superalgebras, where $G$ is a subgroup of the symmetric group $S_3$. Finally, we discuss the concept of $\beta^k$-derivation of BiHom-Lie superalgebras and prove that the set of all $\beta^k$-derivation has a natural BiHom-Lie superalgebra structure.

Key words: BiHom-Lie superalgebra; BiHom-associative superalgebra; BiHom-Lie admissible superalgebra; derivation

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INTRODUCTION

As generalizations of Lie algebras, Hom-Lie algebras were introduced motivated by applications to physics and to deformations of Lie algebras, especially Lie algebras of vector fields. The notion of Hom-Lie algebras was firstly introduced by Hartwig, Larsson and Silvestrov to describe the structure of certain q-deformations of the Witt and the Virasoro algebras, see [1, 6, 11, 12]. More precisely, a Hom-Lie algebras are different from Lie algebras as the Jacobi identity is replaced by a twisted form using a morphism. This twisted Jacobi identity is called Hom-Jacobi identity given by

$$[\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] = 0.$$
The twisting of parts of the defining identities was transferred to other algebraic structures. In [13, 14, 15], Makhlouf and Silvestrov introduced the notions of Hom-associative algebras, Hom-coassociative coalgebras, Hom-bialgebras and Hom-Hopf algebras. The original definition of a Hom-bialgebra involved two linear maps, one twisting the associativity condition and the other one twisting the coassociativity condition. Later, two directions of study on Hom-bialgebras were developed, one in which the two maps coincide (these are still called Hom-bialgebras) and another one, started in [4], where the two maps are assumed to be inverse to each other (these are called monoidal Hom-bialgebras).

The main tool for constructing examples of Hom-type algebras is the so-called twisting principle introduced by Yau for Hom-associative algebras and extended afterwards to other types of Hom-algebras, see [20, 21]. Later, Yau [22] proposed the definition of quasitriangular Hom-Hopf algebras and showed that each quasitriangular Hom-Hopf algebra yields a solution of Hom-Yang-Baxter equations. Meanwhile, Yau [23] defined the classical Hom-Yang-Baxter equation in the same manner and studied Hom-Lie bialgebras. In fact, the quasi-element of quasitriangular Hom-Lie bialgebras is a solution of classical Hom-Yang-Baxter equation.

A categorical interpretation of Hom-associative algebras has been given by Caenepeel and Goyvaerts in [4]. To any monoidal category $\mathcal{C}$, they associate a new monoidal category $\tilde{\mathcal{H}}(\mathcal{C})$ and call it a Hom-category. They proved that a Hom-associative algebra is just an algebra in $\tilde{\mathcal{H}}(\mathcal{M}_k)$, where $\mathcal{M}_k$ is the category of linear spaces over a base field $k$. The similar results holds for Hom-coassociative coalgebras and Hom-bialgebras. Later, Chen et al. [7] studied the quasitriangular structures of monoidal Hom-Hopf algebras and gave an equivalent description via a braided monoidal category of Hom-modules. Many more properties and structures of Hom-Hopf algebras have been developed, see [8, 9, 16, 18, 19] and references cited therein.

In [10], Graziani et al. studied Hom-bialgebras and Hom-Lie algebras in a so-called group Hom-category and called them BiHom-bialgebras and BiHom-Lie algebras. They defined BiHom-bialgebras using two commuting multiplicative linear maps $\alpha, \beta$, which unify Hom-bialgebras and monoidal Hom-bialgebras by setting $\alpha = \beta$ and $\alpha = \beta^{-1}$ respectively. Also they extended the enveloping algebras and representations of Hom-Lie algebras to BiHom-Lie algebras.

In [2], Ammar and Makhlouf introduced the notion of Hom-Lie superalgebras, they gave a classification of Hom-Lie admissible superalgebras and proved a graded version of Hartwig-Larsson-Silvestrov Theorem. Later, Ammar, Makhlouf and Saadaoui [3] studied the representation and the cohomology of Hom-Lie superalgebras, and calculated the derivations and the second cohomology group of $q$-deformed Witt superalgebra. In [5], Cao and Luo studied Hom-Lie superalgebra structures on finite-dimensional simple Lie superalgebras, while Yuan, Sun and Liu considered Hom-Lie superalgebra structures on
infinite-dimensional simple Lie superalgebras in [24].

Motivated by these results, we generalize the notion of Hom-Lie superalgebras and BiHom-Lie algebras to BiHom-Lie superalgebras and study the structures of BiHom-Lie superalgebras and BiHom-Lie admissible superalgebras. This paper is organized as follows.

In Section 1, we recall some basic definitions and facts related with BiHom-associative algebras and BiHom-Lie superalgebras.

In Section 2, we introduce the notion of BiHom-Lie superalgebras and show that any BiHom-associative algebra gives rise to a BiHom-Lie superalgebra (see Theorem 2.6). Meanwhile, we show a method to construct BiHom-Lie superalgebras from Hom-Lie superalgebras by Yau’s twist principle (see Theorem 2.7).

In Section 3, we introduce BiHom-Lie admissible superalgebras and more general $G$-BiHom-associative superalgebras, where $G$ is a subgroup of the symmetric group $S_3$. We show that BiHom-Lie admissible superalgebras are $G$-BiHom-associative superalgebras (see Propositions 3.7). As a corollary, we obtain a classification of BiHom-Lie admissible superalgebras using the symmetric group $S_3$.

In Section 4, we study the $\beta^k$-derivation of a BiHom-Lie superalgebra and prove that the set of all $\beta^k$-derivation of a BiHom-Lie superalgebra forms a BiHom-Lie superalgebra (see Propositions 4.4). As an application, we prove that the inner derivation is a $\beta^{k+1}$-derivation (see Propositions 4.5).

1 Preliminaries

In this section we recall some basic definitions and results related to our paper. Throughout the paper, all algebraic systems are supposed to be over a field $k$. Any unexplained definitions and notations can be found in [10] and [17].

Definition 2.1 ([10]) A BiHom-associative algebra is a 4-tuple $(A, \mu, \alpha, \beta)$, where $A$ is a $k$-linear space, $\alpha : A \to A$, $\beta : A \to A$ and $\mu : A \otimes A \to A$ are linear maps, with notation $\mu(a \otimes b) = ab$, satisfying the following conditions, for all $a, a', a'' \in A$:

$$\begin{align*}
\alpha \circ \beta & = \beta \circ \alpha, \\
\alpha(aa') & = \alpha(a)\alpha(a'), \beta(aa') = \beta(a)\beta(a'), \\
\alpha(a)(a'a'') & = (aa')\beta(a'').
\end{align*}$$

And the maps $\alpha, \beta$ are called the structure maps of $A$.

Clearly, a Hom-associative algebra $(A, \mu, \alpha)$ can be regarded as the BiHom-associative algebra $(A, \mu, \alpha, \alpha)$.

Definition 2.2 ([10]) A BiHom-Lie algebra is a 4-tuple $(L, [, ], \alpha, \beta)$, where $L$ is a $k$-linear space, $\alpha : L \to L$, $\beta : L \to L$ and $[., .] : L \otimes L \to L$ are linear maps, satisfying the
following conditions, for all $a, a', a'' \in A$:

\[
\begin{align*}
\alpha \circ \beta &= \beta \circ \alpha, \\
\alpha[a, a'] &= [\alpha(a), \alpha(a')], \beta[a, a'] = [\beta(a), \beta(a')], \\
[\beta(a), \alpha(a')] &= -[\beta(a'), \alpha(a)]. \\
[\beta^2(a), [\beta(a'), \alpha(a'')]] + [\beta^2(a'), [\beta(a''), \alpha(a)]] + [\beta^2(a''), [\beta(a), \alpha(a')]] &= 0.
\end{align*}
\]

Obviously, a Hom-Lie algebra $(L, [\cdot, \cdot], \alpha)$ is a particular case of a BiHom-Lie algebra, namely $(L, [\cdot, \cdot], \alpha, \alpha)$. Conversely, a BiHom-Lie algebra $(L, [\cdot, \cdot], \alpha, \alpha)$ with bijective $\alpha$ is the Hom-Lie algebra $(L, [\cdot, \cdot], \alpha)$.

2 BiHom-associative superalgebras and BiHom-Lie superalgebras

In this section, we will present the notions of BiHom-associative superalgebras and BiHom-Lie superalgebras, and construct BiHom-Lie superalgebras from BiHom-associative superalgebras and Hom-Lie superalgebras, as a generalization of results in [2] and [10].

Now, let $V$ be a linear superspace over $k$ that is a $\mathbb{Z}_2$-graded linear space with a direct sum $V = V_0 \oplus V_1$. The elements of $V_j, j = 0, 1$, are said to be homogenous and of parity $j$. The parity of a homogeneous element $x$ is denoted by $|x|$. 

**Definition 2.1.** A BiHom-associative superalgebra is a 4-tuple $(A, \mu, \alpha, \beta)$, where $A$ is a superspace, $\alpha : A \to A$ and $\beta : A \to A$ are even homomorphisms, $\mu : A \otimes A \to A$ is an even bilinear map, with notation $\mu(a \otimes b) = ab$ satisfying

\[
\begin{align*}
\alpha \circ \beta &= \beta \circ \alpha, \\
\alpha(ab) &= \alpha(a)\alpha(b), \beta(ab) = \beta(a)\beta(b), \\
\alpha(a)(bc) &= (ab)\beta(c),
\end{align*}
\]

for all homogeneous elements $a, b, c \in A$.

Let $(A, \mu_A, \alpha_A, \beta_A)$ and $(B, \mu_B, \alpha_B, \beta_B)$ be two BiHom-associative superalgebras, an even homomorphism $f : A \to B$ is said to be a morphism of BiHom-associative superalgebras if $\alpha_B \circ f = f \circ \alpha_A$, $\beta_B \circ f = f \circ \beta_A$ and $f \circ \mu_A = \mu_B \circ (f \otimes f)$.

**Remark 2.2.** Assume that $\beta = \alpha$ in Definition 2.1, then the BiHom-associative superalgebra $(A, \mu, \alpha, \beta)$ is the Hom-associative superalgebra in [2]. If the part of parity one in $(A, \mu, \alpha, \beta)$ is trivial, then it is just the BiHom-associative algebra in [10].

**Definition 2.3.** A BiHom-Lie superalgebra is a 4-tuple $(L, [\cdot, \cdot], \alpha, \beta)$, where $L$ is a superspace, $\alpha : L \to L$ and $\beta : L \to L$ are even homomorphisms, $[\cdot, \cdot] : L \otimes L \to L$ is an even
bilinear map satisfying
\[
\alpha \circ \beta = \beta \circ \alpha, \tag{2.4}
\]
\[
\alpha[x, y] = [\alpha(x), \alpha(y)], \beta[x, y] = [\beta(y), \beta(y)], \tag{2.5}
\]
\[
[\beta(x), \alpha(y)] = -(\alpha, \beta), \tag{2.6}
\]
\[
\circ_{\alpha, \beta} \cdot (-1)^{|x||y|}[\beta^{2}(x), [\beta(y), \alpha(z)]] = 0, \tag{2.7}
\]
for all homogeneous elements \(x, y, z \in L\).

Let \((L, [\cdot, \cdot], \alpha, \beta)\) and \((L', [\cdot, \cdot]', \alpha', \beta')\) be two BiHom-Lie superalgebras, an even homomorphism \(f : L \rightarrow L'\) is said to be a morphism of BiHom-Lie superalgebras if \(\alpha' \circ f = f \circ \alpha, \beta' \circ f = f \circ \beta\) and \(f \circ [\cdot, \cdot] = [\cdot, \cdot'] \circ (f \otimes f)\).

**Example 2.4.** Let \(L = L_{0} \oplus L_{1}\) be a 2-dimensional superspace, \(L_{0}\) is generated by \(x\) and \(L_{1}\) is generated by \(y\) such that \([x, y] = 0\). Then for any commutative even homomorphism \(\alpha, \beta : L \rightarrow L\), \((L, [\cdot, \cdot], \alpha, \beta)\) is a BiHom-Lie superalgebra.

**Example 2.5.** Let \(L = L_{0} \oplus L_{1}\) be a 3-dimensional superspace, \(L_{0}\) is generated by \(e_{1}, e_{2}\) and \(L_{1}\) is generated by \(e_{3}\). Define a bracket product \([\cdot, \cdot]\) on \(L\) by
\[
[e_{1}, e_{2}] = e_{1}, [e_{1}, e_{3}] = [e_{2}, e_{3}] = [e_{3}, e_{3}] = 0.
\]
Let \(\lambda, \mu\) be two nonzero scalars in \(k\). Consider the maps \(\alpha, \beta : L \rightarrow L\) defined on the basis elements by
\[
\alpha(e_{1}) = \mu(e_{1}), \alpha(e_{2}) = e_{2}, \alpha(e_{3}) = \lambda e_{3},
\]
\[
\beta(e_{1}) = \mu(e_{1}), \beta(e_{2}) = e_{2}, \beta(e_{3}) = -\lambda e_{3}.
\]
It is straightforward to check that \(\alpha, \beta\) defines two BiHom-Lie superalgebra homomorphisms and \(\alpha \circ \beta = \beta \circ \alpha\). Also one may check that the bracket product \([\cdot, \cdot]\) and the structure maps \(\alpha, \beta\) satisfy Eq. (2.6) and Eq. (2.7), then \((L, [\cdot, \cdot], \alpha, \beta)\) is a BiHom-Lie superalgebra.

**Theorem 2.6.** Let \((A, \mu, \alpha, \beta)\) be a BiHom-associative superalgebra with bijective homomorphisms \(\alpha\) and \(\beta\). One can define the supercommutator on homogeneous elements by
\[
[x, y] = xy - (\alpha, \beta), \tag{2.8}
\]
and then extending by linearity to all elements. Then \((A, [\cdot, \cdot], \alpha, \beta)\) is a BiHom-Lie superalgebra.

**Proof** First we check that the bracket product \([\cdot, \cdot]\) is compatible with the structure maps \(\alpha\) and \(\beta\). For any homogeneous elements \(x, y \in A\), we have
\[
[\alpha(x), \alpha(y)] = \alpha(x)\alpha(y) - (-1)^{|x||y|}[\alpha(x)], \tag{2.9}
\]
\[
\beta^{2}(x), [\beta(\alpha(y))]\alpha(\beta^{-1}(\alpha(x)))
\]
\[
\alpha(x)\alpha(y) - (-1)^{|x||y|}[\beta(y)\alpha^{2}(\beta^{-1}(x))
\]
\[
= \alpha(x, y).
\]
The second equality holds since $\alpha$ is even and $\alpha \circ \beta = \beta \circ \alpha$. Similarly, one can prove that $\beta[x,y] = [\beta(x), \beta(y)]$.

To verify the skew-supersymmetry, let $x, y \in A$. Then

$$\begin{align*}
[\beta(x), \alpha(y)] &= \beta(x)\alpha(y) - (-1)^{|\beta(x)||\alpha(y)|} \alpha^{-1}(\beta(\alpha(y)))\alpha(\beta^{-1}(\beta(x))) \\
&= \beta(x)\alpha(y) - (-1)^{|x||y|} \beta(y)\alpha(x).
\end{align*}$$

Similarly, $[\beta(x), \alpha(y)] = \beta(y)\alpha(x) - (-1)^{|y||x|} \beta(x)\alpha(y) = -(-1)^{|y||x|}[\beta(x), \alpha(y)]$. So Eq. (2.6) holds.

Now we prove the Eq. (2.7). For any $x, y, z \in A$, we have

$$\begin{align*}
(1-1)^{|x||z|[\beta^2(x), [\beta(y), \alpha(z)]]} \\
= (-1)^{|x||z|} |\beta^2(x), \beta(y)\alpha(z) - (-1)^{|y||z|} |\alpha^{-1}(\beta(\alpha(z)))\alpha(\beta^{-1}(\beta(y))) \\
= (-1)^{|x||z|} |\beta^2(x), \beta(y)\alpha(z) - (-1)^{|y||z|} |\beta(z)\alpha(y)] \\
= (-1)^{|x||z|} |\beta^2(x)(\beta(y)\alpha(z)) - (-1)^{|x||y|} |\alpha^{-1}(\beta(\beta(y)))\beta(z)\alpha(\beta(x)) \\
&- (-1)^{|x||z|+|y||z|} |\beta^2(x)(\beta(z)\alpha(y)) + (-1)^{|z||y|+|z||y|} |\alpha^{-1}(\beta(\beta(y)))\beta(z)\alpha(\beta(x)).
\end{align*}$$

Similarly, we have

$$\begin{align*}
(1-1)^{|y||x|[\beta^2(y), [\beta(x), \alpha(z)]]} \\
= (-1)^{|y||x|} |\beta^2(y), \beta(z)\alpha(x) - (-1)^{|z||x|} |\alpha^{-1}(\beta(\beta(z)))\beta(x)\alpha(\beta(y)) \\
&- (-1)^{|y||x|+|z||x|} |\beta^2(y)(\beta(x)\alpha(z)) + (-1)^{|z||x|+|y||x|} |\alpha^{-1}(\beta(\beta(x)))\beta(z)\alpha(\beta(y)), \\
(1-1)^{|z||y|[\beta^2(z), [\beta(x), \alpha(y)]]} \\
= (-1)^{|z||y|} |\beta^2(z), \beta(x)\alpha(y) - (-1)^{|z||x|} |\alpha^{-1}(\beta(\beta(x)))\beta(y)\alpha(\beta(z)) \\
&- (-1)^{|z||y|+|z||y|} |\beta^2(z)(\beta(y)\alpha(x)) + (-1)^{|z||y|+|z||y|} |\alpha^{-1}(\beta(\beta(y)))\beta(x)\alpha(\beta(z)).
\end{align*}$$

By the associativity Eq. (2.3), it is not hard to check that

$$\bigcirc_{x,y,z} (1-1)^{|x||z|[\beta^2(x), [\beta(y), \alpha(z)]]} = 0,$$

as desired. And this finishes the proof.

**Theorem 2.7.** Let $(L, [\cdot, \cdot])$ be a Lie superalgebra. Assume that $\alpha, \beta$ are two even commuting algebra homomorphisms of $L$. Then $(L, [\cdot, \cdot]_{\alpha, \beta}, \alpha, \beta)$, where $[x,y]_{\alpha, \beta} = [\alpha(x), \beta(y)]$, is a BiHom-Lie superalgebra.

**Proof** For any $x, y \in L$, we have

$$\begin{align*}
[\beta(x), \alpha(y)]_{\alpha, \beta} &= [\alpha \beta(x), \beta \alpha(y)] = \alpha \beta([x, y]), \\
[\beta(y), \alpha(x)]_{\alpha, \beta} &= [\alpha \beta(y), \beta \alpha(x)] = \alpha \beta([y, x]) = -(-1)^{|z||y|} \alpha \beta([x, y]).
\end{align*}$$

So $[\beta(x), \alpha(y)]_{\alpha, \beta} = (-1)^{|z||y|}[\beta(y), \alpha(x)]_{\alpha, \beta}$, that is, Eq. (2.6) holds.
For Eq. (2.7), we have
\[
\circ_{x,y,z} (-1)^{|x||z|[\beta^2(x), \beta(y), \alpha(z)_{\alpha,\beta}]} \alpha_{\alpha,\beta} \\
= \circ_{x,y,z} (-1)^{|x||z|[\beta^2(x), \alpha\beta(y), \alpha\beta(z)]_{\alpha,\beta}} \\
= \circ_{x,y,z} (-1)^{|x||z|[\alpha\beta^2(x), \alpha\beta^2(y), \alpha\beta^2(z)]} = 0.
\]
The last equality holds since \((L, [\cdot, \cdot])\) is a Lie superalgebra. Thus \((L, [\cdot, \cdot]_{\alpha,\beta,\alpha,\beta})\) is a BiHom-Lie superalgebra.

\section{3 BiHom-Lie admissible superalgebras}

In this section, we introduce the notion of BiHom-Lie admissible superalgebras and provide a classification of BiHom-Lie admissible superalgebras using the symmetric group \(S_3\). In this section, we always assume that the structure maps \(\alpha\) and \(\beta\) are bijective.

A BiHom-superalgebra is a 4-tuple \((V, \mu, \alpha, \beta)\), where \(V\) is a superspace, \(\alpha : V \rightarrow V\) and \(\beta : V \rightarrow V\) are even homomorphism, \(\mu : V \otimes V \rightarrow V\) is an even bilinear map satisfying
\[
\alpha \circ \beta = \beta \circ \alpha, \alpha \circ \mu = \mu (\alpha \otimes \alpha), \beta \circ \mu = \mu (\beta \otimes \beta).
\]

**Definition 3.1.** Let \(A = (V, \mu, \alpha, \beta)\) be a BiHom-superalgebra. Then \(A\) is said to be a BiHom-Lie admissible superalgebra over \(V\) if the bracket defined by
\[
[x, y] = \mu(x \otimes y) - (-1)^{|x||y|}\mu(\alpha^{-1}(\beta(y)) \otimes \alpha^{-1}(\beta(y))) \quad (3.1)
\]
satisfies the BiHom-superJacobi identity (2.7), for all homogeneous elements \(x, y \in V\).

**Remark 3.2.** By Theorem 2.5, any BiHom-associative superalgebra is a BiHom-Lie admissible superalgebra.

Let \((L, [\cdot, \cdot], \alpha, \beta)\) be a BiHom-Lie superalgebra. Define a new supercommutator bracket \([\cdot, \cdot]'\) on \(L\) by
\[
[x, y]' = [x, y] - (-1)^{|x||y|}[\alpha^{-1}(\beta(y)), \alpha^{-1}(\beta(x))].
\]

It is easy to see that the bracket \([\cdot, \cdot]'\) satisfies Eq. (2.6). Moreover, we have
\[
(-1)^{|x||z|[\beta^2(x), \beta(y), \alpha(z)]}]' \\
= (-1)^{|x||z||[\beta^2(x), \beta(y), \alpha(z)]} - (-1)^{|y||z|[\alpha^{-1}(\beta(\alpha(z)))}, \alpha(\beta(\alpha(y)))]'} \\
= (-1)^{|y||z|[\beta^2(x), \beta(y), \alpha(z)]} - (-1)^{|y||z|[\beta(z), \alpha(y)]} \\
= (-1)^{|x||z|[\beta^2(x), \beta(y), \alpha(z)]} \\
= 2(-1)^{|x||z|[\beta^2(x), \beta(y), \alpha(z)]} - 2(-1)^{|x||y|[\alpha^{-1}\beta([\beta(y), \alpha(z)]}, \alpha(\beta(x)))] \\
= 4(-1)^{|x||z|[\beta^2(x), \beta(y), \alpha(z)]}.
\]
where supercommutator. Then Lemma 3.4.

Then we have the following lemmas:

**Proposition 3.3.** Let $A = (V, \mu, \alpha, \beta)$ be a BiHom-superalgebra. The $(\alpha, \beta)$-associator of the multiplication $\mu$ is a trilinear map $\text{as}_{\alpha, \beta} \in V$ defined by

$$\text{as}_{\alpha, \beta}(x_1, x_2, x_3) = \mu(\alpha(x_1), \mu(x_2, x_3)) - \mu(\mu(x_1, x_2), \beta(x_3)),$$

where $x_1, x_2, x_3$ are homogeneous elements in $V$.

Now let us introduce the notation:

$$S(x, y, z) := \langle x, y, z \rangle = (1)_{\|x\| |y| |z|} \text{as}_{\alpha, \beta}(\alpha^{-1} \beta^2(x), \beta(y), \alpha(z)).$$

Then we have the following lemmas:

**Lemma 3.4.** Let $A = (V, \mu, \alpha, \beta)$ be a BiHom-superalgebra and $[\cdot, \cdot]$ the associated supercommutator. Then

$$\langle x, y, z \rangle \langle -1 \|x\| |y| |z| \rangle \text{as}_{\alpha, \beta}(\alpha^{-1} \beta^2(x), \beta(y), \alpha(z)) = 0.$$
Proposition 3.5. Let $A = (V, \mu, \alpha, \beta)$ be a BiHom-superalgebra. Then $A$ is a BiHom-Lie admissible superalgebra if and only if it satisfies

$$S(x, y, z) = (-1)^{|x||z|+|z||x|+|x||y|}S(x, z, y),$$

for all homogeneous elements $x, y, z \in V$.

Proof For any homogeneous elements $x, y, z \in V$, it is easy to check that

$$(-1)^{|x||z|+|z||x|+|x||y|}S(x, z, y) = (-1)^{|x||z|+|z||x|+|x||y|}S(x, z, y)$$

Therefore, by Lemma 3.4, we have

$$S(x, y, z) = (-1)^{|x||z|+|z||x|+|x||y|}S(x, z, y)$$

The proof is completed.

In the following, we will provide a classification of BiHom-Lie admissible superalgebras using the symmetric group $S_3$, whereas it was classified in [2] [13] [25] for Hom-Lie admissible algebras, Hom-Lie admissible superalgebras and Hom-Lie color admissible algebras, respectively.

Let $S_3$ be the symmetric group generated by $\sigma_1 = (12), \sigma_2 = (23)$ and $A = (V, \mu, \alpha, \beta)$ a BiHom-superalgebra. Suppose that $S_3$ acts on $V \times 3$ in the usual way, i.e., $\sigma(x_1, x_2, x_3) = (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})$.

For convenience, define the parity of the transposition $\sigma_i$ with $i \in \{1, 2\}$ as follows:

$$|\sigma_i(x_1, x_2, x_3)| = |x_i||x_{i+1}|.$$

It is natural to assume that the parity of the identity is 0 and for the composition $\sigma_i\sigma_j$, it is defined by

$$|\sigma_i\sigma_j(x_1, x_2, x_3)| = |\sigma_j(x_1, x_2, x_3)| + |\sigma_i(\sigma_j(x_1, x_2, x_3))|$$

$$= |\sigma_j(x_1, x_2, x_3)| + |\sigma_i(x_{\sigma_j(1)}, x_{\sigma_j(2)}, x_{\sigma_j(3)})|.$$
One can define by induction the parity for any composition as follows:

\[
|\text{id}(x_1, x_2, x_3)| = 0, \\
|\sigma_1(x_1, x_2, x_3)| = |x_1||x_2|, \\
|\sigma_2(x_1, x_2, x_3)| = |x_2||x_3|, \\
|\sigma_1\sigma_2(x_1, x_2, x_3)| = |x_2||x_3| + |x_1||x_3|, \\
|\sigma_2\sigma_1(x_1, x_2, x_3)| = |x_1||x_2| + |x_1||x_3|, \\
|\sigma_2\sigma_1\sigma_2(x_1, x_2, x_3)| = |x_2||x_3| + |x_1||x_3| + |x_1||x_2|,
\]

where \(x_1, x_2, x_3\) are homogeneous elements in \(V\).

**Lemma 3.6.** A BiHom-superalgebra \(A = (V, \mu, \alpha, \beta)\) is a BiHom-Lie admissible superalgebra if and only if the following condition holds

\[
\sum_{\sigma \in S_3} (-1)^{\varepsilon(\sigma)} (-1)^{\sigma(x_1, x_2, x_3)} \mathbf{as}_{\alpha, \beta} \circ \sigma(\alpha^{-1}\beta^2(x_1), \beta(x_2), \alpha(x_3)) = 0,
\]

for all homogeneous elements \(x_1, x_2, x_3 \in V\), where \((-1)^{\varepsilon(\sigma)}\) is the signature of \(\sigma\).

**Proof** It is sufficient to verify the BiHom-superJacobi identity (2.7). By Lemma 3.4,

\[
\sum_{\sigma \in S_3} (-1)^{\varepsilon(\sigma)} (-1)^{\sigma(x_1, x_2, x_3)} \mathbf{as}_{\alpha, \beta} \circ \sigma(\alpha^{-1}\beta^2(x_1), \beta(x_2), \alpha(x_3)) = 0,
\]

since \(\alpha, \beta\) are even homomorphism. \(\square\)

Let \(G\) be a subgroup of \(S_3\), any BiHom-superalgebra \((V, \mu, \alpha, \beta)\) is said to be \(G\)-BiHom-associative if the following equation holds:

\[
\sum_{\sigma \in G} (-1)^{\varepsilon(\sigma)} (-1)^{\sigma(x_1, x_2, x_3)} \mathbf{as}_{\alpha, \beta} \circ \sigma(\alpha^{-1}\beta^2(x_1), \beta(x_2), \alpha(x_3)) = 0,
\]

for all homogeneous elements \(x_1, x_2, x_3 \in V\).

**Proposition 3.7.** Let \(G\) be a subgroup of the symmetric group \(S_3\). Then any \(G\)-BiHom-associative superalgebra \((V, \mu, \alpha, \beta)\) is BiHom-Lie admissible.

**Proof** The BiHom-supersymmetry (2.6) follows straightaway from the definition. Assume that \(G\) is a subgroup of \(S_3\). Then \(S_3\) can be written as the disjoint union of the left cosets of \(G\). Say \(S_3 = \bigcup_{I \in \mathcal{I}} I\), with \(I \subseteq S_3\), and for any \(\sigma, \sigma' \in I, \sigma \neq \sigma' \in I \Rightarrow \sigma G \cap \sigma' G = \emptyset\). It follows that

\[
\sum_{\sigma \in S_3} (-1)^{\varepsilon(\sigma)} (-1)^{\sigma(x_1, x_2, x_3)} \mathbf{as}_{\alpha, \beta} \circ \sigma(\alpha^{-1}\beta^2(x_1), \beta(x_2), \alpha(x_3))
\]

\[= \sum_{\tau \in I} \sum_{\sigma \in \tau G} (-1)^{\varepsilon(\sigma)} (-1)^{\sigma(x_1, x_2, x_3)} \mathbf{as}_{\alpha, \beta} \circ \sigma(\alpha^{-1}\beta^2(x_1), \beta(x_2), \alpha(x_3)) = 0,
\]
for all homogeneous elements $x_1, x_2, x_3 \in V$. By Lemma 3.6, $(V, \mu, \alpha, \beta)$ is a BiHom-Lie admisssible superalgebra. The proof is completed. \qed

Now we provide a classification of the BiHom-Lie admissisble superalgebras via $G$-

BiHom-associative superalgebras. The subgroups of $S_3$ are

$$G_1 = \{\text{id}\}, \quad G_2 = \{\text{id}, \sigma_1\}, \quad G_3 = \{\text{id}, \sigma_2\},$$

$$G_4 = \{\text{id}, \sigma_2 \sigma_1 = (13)\}, \quad G_5 = A_3, \quad G_6 = S_3,$$

where $A_3$ is the alternating subgroup of $S_3$.

1. The $G_1$-BiHom-associative superalgebras are the BiHom-associative superalgebras defined in Definition 2.1.

2. The $G_2$-BiHom-associative superalgebras satisfy the condition:

$$\mu(\beta^2(x), \mu(\beta(y), \alpha(z))) - \mu(\mu(\alpha^{-1} \beta^2(x), \beta(y)), \alpha\beta(z))$$

$$= (-1)^{|x||y|} \{\mu(\alpha\beta(y), \mu(\alpha^{-1} \beta^2(x), \alpha(z))) - \mu(\mu(\beta(y), \alpha^{-1} \beta^2(x)), \alpha\beta(z))\}.$$ 

3. The $G_3$-BiHom-associative superalgebras satisfy the condition:

$$\mu(\beta^2(x), \mu(\beta(y), \alpha(z))) - \mu(\mu(\alpha^{-1} \beta^2(x), \beta(y)), \alpha\beta(z))$$

$$= (-1)^{|y| |z|} \{\mu(\beta^2(x), \mu(\alpha(z), \alpha(y))) - \mu(\mu(\alpha^{-1} \beta^2(x), \alpha(z)), \beta^2(y))\}.$$ 

4. The $G_4$-BiHom-associative superalgebras satisfy the condition:

$$\mu(\beta^2(x), \mu(\beta(y), \alpha(z))) - \mu(\mu(\alpha^{-1} \beta^2(x), \beta(y)), \alpha\beta(z))$$

$$= (-1)^{|x||y| + |x||z| + |y||z|} \{\mu(\alpha^2(z), \mu(\beta(y), \alpha^{-1} \beta^2(x))) - \mu(\mu(\alpha(z), \beta(y)), \alpha^{-1} \beta^3(x))\}.$$ 

5. The $G_5$-BiHom-associative superalgebras satisfy the condition:

$$\mu(\beta^2(x), \mu(\beta(y), \alpha(z))) - \mu(\mu(\alpha^{-1} \beta^2(x), \beta(y)), \alpha\beta(z))$$

$$= -(-1)^{|x||y| + |x||z| + |y||z|} \{\mu(\alpha\beta(y), \mu(\alpha(z), \alpha^{-1} \beta^2(x))) - \mu(\mu(\beta(y), \alpha(z)), \alpha^{-1} \beta^3(x))\}$$

$$- (-1)^{|y||z| + |x||z|} \{\mu(\alpha^2(z), \mu(\alpha^{-1} \beta^2(x), \beta(y))) - \mu(\mu(\alpha(z), \alpha^{-1} \beta^2(x)), \beta^2(y))\}.$$ 

6. The $G_6$-BiHom-associative superalgebras are the BiHom-Lie admissible superalgebras.

4 Derivations of BiHom-Lie superalgebras

In this section, we provide the notion of derivations of a BiHom-Lie superalgebra $L$ and prove that the set of all derivations of $L$ has a natural BiHom-Lie superalgebra structure.
Let \( L = (L, [\cdot, \cdot], \alpha, \beta) \) be a BiHom-Lie superalgebra. For any nonnegative integer \( k \), denote by \( \alpha^k \) the \( k \)-times composition of \( \alpha \), i.e.

\[
\alpha^k = \alpha \circ \cdots \circ \alpha \text{ (\( k \)-times)}.
\]

In particular, \( \alpha^{-1} = 0, \alpha^0 = id \) and \( \alpha^1 = \alpha \). And similarly for the notion \( \beta^k \).

**Definition 4.1.** For any integer \( k \geq -1 \), a homogeneous linear map \( D : L \to L \) of degree \(|D|\) is called a \( \beta^k \)-derivation of the BiHom-Lie superalgebra \((L, [\cdot, \cdot], \alpha, \beta)\) if it satisfies

\[
D \circ \alpha = \alpha \circ D, \quad D \circ \beta = \beta \circ D,
\]

(4.1)

\[
D[x, y] = [D(x), \beta^k(y)] + (-1)^{|x||D|}[\beta^k(x), D(y)],
\]

(4.2)

for all homogeneous elements \( x, y \in L \).

We denote by \( \text{Der}_{\beta^k}(L) = (\text{Der}_{\beta^k}(L))_0 \oplus (\text{Der}_{\beta^k}(L))_1 \) the set of \( \beta^k \)-derivation of the BiHom-Lie superalgebra \((L, [\cdot, \cdot], \alpha, \beta)\), and \( \text{Der}(L) = \bigoplus_{k \geq -1} \text{Der}_{\beta^k}(L) \). Define the endomorphisms \( \tilde{\alpha}, \tilde{\beta} \) on \( \text{Der}(L) \) by

\[
\tilde{\alpha}(D) = \alpha \circ D, \quad \tilde{\beta}(D) = \beta \circ D.
\]

For any \( D, D' \in \text{Der}(L) \), define their commutator \([D, D']\) as follows:

\[
[D, D'] = D \circ D' - (-1)^{|D||D'|}[D' \circ D, D'].
\]

**Lemma 4.2.** Let \((L, [\cdot, \cdot], \alpha, \beta)\) be a BiHom-Lie superalgebra. For any \( D \in (\text{Der}_{\beta^k}(L))_i, D' \in (\text{Der}_{\beta^s}(L))_j \), where \( k + s \geq -1 \) and \((i, j) \in \mathbb{Z}_2^2\), then \([D, D'] \in (\text{Der}_{\beta^{k+s}}(L))_1\).

**Proof** For any \( x, y \in L \), we have

\[
[D, D']([x, y])
= (D \circ D' - (-1)^{|D||D'|}[D' \circ D])([x, y])
= D([D'(x), \beta^s(y)] + (-1)^{|x||D|}[\beta^s(x), D'(y)])
\]

\[
- (-1)^{|D||D'|}[D'(D(x), \beta^k(y)) + (-1)^{|x||D|}[\beta^k(x), D(y)])
\]

\[
= [DD'(x), \beta^{s+k}(y)] + (-1)^{|D||D'|}[D'(\beta^k(x)), D(\beta^s(y)]
\]

\[
+ (-1)^{|x||D'|}([D'(\beta^s(x)), D(\beta^k(y)]) + (-1)^{|x||D|}[\beta^{s+k}(x), DD'(y)])
\]

\[
- (-1)^{|D||D'|}[D'(DD(x), \beta^{s+k}(y)] + (-1)^{|D'||D'(x)]}[D(\beta^s(x)), D'(\beta^k(y)])
\]

\[
- (-1)^{|D||D'|}[D'(D(x), \beta^{s+k}(y)] + (-1)^{|D'|}[\beta^{s+k}(x), D'D'(y)])
\]

\[
= [DD'(x) - (-1)^{|D||D'|}D'D(x), \beta^{s+k}(y)]
\]

\[
+ (-1)^{|x||D|+|D'|}[\beta^{s+k}(x), (DD' - (-1)^{|D'||D'|}D'D)(y)]
\]

\[
= ([D, D'](x), \beta^{s+k}(y)] + (-1)^{|x||[D, D']|}[\beta^{s+k}(x), [D, D'](y)]).
\]
It is easy to check that \([D, D'] \circ \alpha = \alpha \circ [D, D'], [D, D'] \circ \beta = \beta \circ [D, D']\), which leads to
\([D, D'] \in \text{Der}_{\alpha, k+1}(L)\). □

**Remark 4.3.** Obviously, we have

\[
\text{Der}_{\beta^{-1}}(L) = \{D \in \text{End}(L) | D \circ \alpha = \alpha \circ D, D \circ \beta = \beta \circ D, D[x, y] = 0, \forall x, y \in L\}.
\]

Thus for any \(D, D' \in \text{Der}_{\beta^{-1}}(L)\), we have \([D, D'] \in \text{Der}_{\beta^{-1}}(L)\).

**Proposition 4.4.** Let \((L, [\cdot, \cdot], \alpha, \beta)\) be a BiHom-Lie superalgebra. Then \((\text{Der}(L), [\cdot, \cdot], \tilde{\alpha}, \tilde{\beta})\) is a BiHom-Lie superalgebra.

**Proof.** We prove that the bracket \([\cdot, \cdot]\) on \(\text{Der}(L)\) satisfies the conditions in Definition 2.3. Let \(D \in (\text{Der}_{\alpha}(L))_i\), \(D' \in (\text{Der}_{\alpha^*}(L))_j\), \(D'' \in (\text{Der}_{\alpha^*}(L))_l\) and \(x \in L\), we have

\[
(\tilde{\alpha} \circ \tilde{\beta})(D) = D \circ \alpha \circ \beta = D \circ \beta \circ \alpha = (\tilde{\beta} \circ \tilde{\alpha})(D).
\]

So Eq. (2.4) holds and similarly for Eq. (2.5). For Eq. (2.6), we have

\[
[\tilde{\beta}(D), \tilde{\alpha}(D')] = [D \circ \beta, D' \circ \alpha]
\]

\[
= (D \circ \beta) \circ (D' \circ \alpha) - (-1)^{|D||D'|}(D' \circ \alpha) \circ (D \circ \beta)
\]

\[
= (D \circ D' - (-1)^{|D||D'|}D' \circ D) \circ (\alpha \beta)
\]

\[
= -(-1)^{|D||D'|}(D' \circ D - (-1)^{|D||D'|}D \circ D') \circ (\alpha \beta)
\]

\[
= -(-1)^{|D||D'|}[\tilde{\beta}(D'), \tilde{\alpha}(D)]
\]

For Eq. (2.7), we calculate

\[
(-1)^{|D||D''|}[\tilde{\beta}^2(D), [\tilde{\beta}(D'), \tilde{\alpha}(D'')]] = (-1)^{|D||D''|}[D \circ \beta^2, [D' \circ \beta, D'' \circ \alpha]]
\]

\[
= (-1)^{|D||D''|}[D \circ \beta^2, (D' \circ D'') \circ (\beta \alpha)] - (-1)^{|D||D''|}(D'' \circ D') \circ (\beta \alpha)
\]

\[
= (-1)^{|D||D''|}[D \circ (D' \circ D'')] - (-1)^{|D||D''|}((D' \circ D'') \circ D) \circ (\beta^3 \alpha)
\]

\[
-(-1)^{|D''|(|D|+|D'|)}((D' \circ D') \circ D) \circ (\beta^3 \alpha).
\]

Therefore, one can check that \(\circ_{D, D', D''} (-1)^{|D||D''|}[\tilde{\beta}^2(D), [\tilde{\beta}(D'), \tilde{\alpha}(D'')]] = 0\), as desired. And this finishes the proof. □

For any homogeneous elements \(a \in L\) satisfying \(\alpha(a) = a = \beta(a)\), define \(ad_k(a) \in \text{End}(L)\) by

\[
ad_k(a)(x) = [a, \beta^k(x)], \forall x \in L.
\]

**Proposition 4.5.** Let \((L, [\cdot, \cdot], \alpha, \beta)\) be a BiHom-Lie superalgebra and \(a\) an homogeneous element in \(L\). Assume that the structure maps \(\alpha\) and \(\beta\) are bijective, then \(ad_k(a)\) is an \(\beta^{k+1}\)-derivation, which we call inner \(\beta^{k+1}\)-derivation.
Proof For any homogeneous elements $x, y \in L$, on the one hand we have

$$ad_k(a)[x, y] = [a, \beta^k[x, y]] = [\beta^2(a), [\beta^k(x), \beta^k(y)]]$$

$$= - (-1)^{|a||y|}(-1)^{|x||a|} [[\beta^{k+1}(x), [\beta^{k+1}\alpha^{-1}(y), \alpha(a)]]$$

$$= - (-1)^{|a||y|}(-1)^{|x||a|} [[\beta^{k+2}\alpha^{-1}(y), [\beta(a), \alpha\beta^{k-1}(x)]]$$

On the other hand, we have

$$[ad_k(a)(x), \beta^{k+1}(y)] = [[a, \beta^k(x)], \beta^{k+1}(y)] = [\beta[a, \beta^{k-1}(x)], \beta^{k+1}(y)]$$

$$= - (-1)^{(l|a|+|x|)|y|} [\beta^{k-1}\alpha^{-1}(y), \alpha[a, \beta^{k-1}(x)]]$$

and

$$[\beta^{k+1}(x), ad_k(a)(y)] = [\beta^{k+1}(x), [a, \beta^k(y)]]$$

$$= [\beta^{k+1}(x), [\beta(a), \alpha\beta^k\alpha^{-1}(y)]]$$

$$= - (-1)^{|a||y|} [\beta^{k+1}(x), [\beta(y), \alpha(a)]]$$

$$= - (-1)^{|a||y|} [\beta^{k+2}\alpha^{-1}(y), [\alpha\beta^{k-1}(x), \alpha(a)]]$$

It follows that

$$ad_k(a)[x, y] = [ad_k(a)(x), \beta^{k+1}(y)] + (-1)^{|x||a|}[\beta^{k+1}(x), ad_k(a)(y)]$$

as desired. And this finishes the proof. □

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