One and two side generalisations of the log-Normal distribution 
by means of a new product definition

Sílvio M. Duarte Queirós Ⓟ

Unilever R&D Port Sunlight
Quarry Road East, CH63 3JW, UK

(Dated: The 28th August 2009)

In this manuscript we introduce a generalisation of the log-Normal distribution
that is inspired by a modification of the Kaypten multiplicative process using the
$q$-product of Borges [Physica A 340, 95 (2004)]. Depending on the value of $q$ the
distribution increases the tail for small (when $q < 1$) or large (when $q > 1$) values
of the variable upon analysis. The usual log-Normal distribution is retrieved when
$q = 1$. The main statistical features of this distribution are presented as well as a re-
lated random number generators and tables of quantiles of the Kolmogorov-Smirnov.
Lastly, we illustrate the application of this distribution studying the adjustment of
a set of variables of biological and financial origin.

Keywords: generalized log Normal, q-product, shadow prices

I. INTRODUCTION

The two-parameter log-Normal distribution, with probability density function,

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma x} \exp \left[ -\frac{(\ln x - \mu)^2}{2\sigma^2} \right], \quad x > 0,$$  

has played a major role in the statistical characterisation of many data sets for several
decades (empirical fitting) and has been an inspiration for theoretical studies as well. The
form of Eq. (1) has been derived in several ways with particularly emphasis to the works of
Kapteyn [Kapteyn, 1903], the Gibrat’s law of proportionate effect [Gibrat, 1930], the Theory
of Breakage introduced by Kolmogorov [Kolmogorov, 1941] or more recently in the theory
of chemical reactions [Fa, 2003]. Concomitantly, Eq. (1) has been systematically modified
to cope with different sets of data. Of those generalisations the most famous of them is the
truncated log-normal distribution,

\[ p(x) = \frac{1}{\sqrt{2\pi}\sigma (x - \gamma)} \exp \left[-\frac{(\ln(x - \gamma) - \mu)^2}{2\sigma^2}\right], \quad 0 < \gamma < x, \quad (2) \]

which has become a mathematical object of study in itself.

In this manuscript we introduce an alternative generalisation of the log-normal distribution which we will term the \( q \)-log Normal distribution for historical reasons. This purported probability density function emerges from changing the traditional algebra in the Kaypten dynamics by a modified multiplication operation independently introduced by Borges \[\text{[Borges, 2004]}\] and Nivanen \textit{et al.} \[\text{[Nivanen \textit{et al.}, 2003]}\]. This algebra has got direct consequences on the emergence of asymptotic scale-free behaviour. Specifically, in this manuscript we survey a new family of probability density functions based on,

\[ p_q(x) = \frac{1}{Z_q x^q} \exp \left[-\frac{(\ln_q x - \mu)^2}{2\sigma^2}\right], \quad (x \geq 0), \quad (3) \]

where \( \ln_q(x) \) represents a generalisation of the logarithm of base \( e \) with the normalisation,

\[ Z_q = \begin{cases} \sqrt{\frac{\pi}{2}} \text{erfc} \left[-\frac{1}{\sqrt{2}\sigma} \left(\frac{1}{1-q} + \mu\right)\right] & \text{if } q < 1 \\ \sqrt{\frac{\pi}{2}} \text{erfc} \left[\frac{1}{\sqrt{2}\sigma} \left(\frac{1}{1-q} + \mu\right)\right] & \text{if } q > 1. \end{cases} \quad (4) \]

where \( \text{erfc}(x) \equiv 2\Phi(\sqrt{2}x) - 1, \)

and which in the limit \( q \to 1^{\pm} \) exactly gives the traditional log-Normal distribution. The aim of the present work is to introduce the functional form of the distribution, its dynamical origins and statistical features as well as applying it to data of biological and financial origin. The manuscript is organised as follows. In Sec. \( \text{[II]} \) we give a historical and mathematical introduction of the underlying algebra; In Sec. \( \text{[III]} \) we reinterpret the Kaypten scenario for the emergence of the log-normal, but using the \( q \)-algebra formalism which lead to the \( q < 1, q > 1, \) and double \( q \)-log- Normal probability density function. In Secs. \( \text{[IV]} \) and \( \text{[V]} \) we analyse their statistical properties and generate random variables according to the distribution. Finally, in Sec. \( \text{[VI]} \) we introduce some real examples to which the new distribution is shown to be a worthy candidate for modelling the data.
II. PRELIMINARIES: THE $q$-PRODUCT

The $q$-product, $\otimes_q$, has its origins in the endeavour to extend the subject of statistical mechanics to systems exhibiting anomalous behaviour when compared to systems described at Boltzmann-Gibbs equilibrium, i.e., to deal with systems presenting long-lasting correlations, ageing phenomena, non-exponential sensitivity to initial conditions, and scale-invariance occupancy of the allowed phase space (for detailed explanation of these concepts see [Tsallis, 2009, Abe et al., 2007]). The proposed extension of statistical mechanics theory is grounded on the entropic functional

$$S_q \equiv \frac{1 - \int [p(x)]^q \, dx}{q - 1}, \quad (q \in \mathbb{R})$$

(in its continuous and one-dimensional version) usually called Tsallis entropy as well [Tsallis, 1988]. This entropic form recovers the celebrated Boltzmann-Gibbs-Shannon information measure,

$$S = -\int p(x) \ln (x) \, dx,$$  

in the limit that the entropic parameter $q$ approaches 1. The interpretation of Eq. (5) as a $q$ generalisation of Eq. (6) induced the introduction of analogue functions of the exponential and the logarithm, namely, the $q$-exponential

$$\exp_q (x) \equiv [1 + (1-q) x]^{\frac{1}{1-q}}, \quad (x, q \in \mathbb{R}),$$

($\exp_q (x) = 0$ if $1 + (1-q) x \leq 0$) and its inverse the $q$-logarithm [Tsallis, 1994],

$$\ln_q (x) \equiv \frac{x^{1-q} - 1}{1-q}, \quad (x > 0, q \in \mathbb{R}).$$

A functional form that generalises the mathematical identity,

$$\exp [\ln x + \ln y] = x \times y, \quad (x, y > 0),$$

for the $q$-product is,

$$x \otimes_q y \equiv \exp_q [\ln_q x + \ln_q y].$$

For $q \to 1$, Eq. (10) recovers the usual property,

$$\ln (x \times y) = \ln x + \ln y$$
$(x, y > 0)$, with $x \times y \equiv x \otimes_1 y$. Its inverse operation, the $q$-division, $x \odot_q y$, satisfies the following equality $(x \otimes_q y) \odot_q y = x$.

Bearing in mind that the $q$-exponential is a non-negative function, the $q$-product must be restricted to the values of $x$ and $y$ that respect the condition,

$$|x|^{1-q} + |y|^{1-q} - 1 \geq 0.$$  \hspace{1cm} (11)

Moreover, we can extend the domain of the $q$-product to negative values of $x$ and $y$ writing it as,

$$x \otimes_q y \equiv \text{sign } (xy) \exp_q [\ln_q |x| + \ln_q |y|].$$  \hspace{1cm} (12)

Regarding some key properties of the $q$-product we mention:

1. $x \otimes_1 y = xy$;
2. $x \otimes_q y = y \otimes_q x$;
3. $(x \otimes_q y) \otimes_q z = x \otimes_q (y \otimes_q z) = [x^{1-q} + y^{1-q} - 2]^{\frac{1}{1-q}}$;
4. $(x \otimes_q 1) = x$;
5. $\ln_q [x \otimes_q y] \equiv \ln_q x + \ln_q y$;
6. $\ln_q (xy) = \ln_q (x) + \ln_q (y) + (1-q) \ln_q (x) \ln_q (y)$;
7. $(x \otimes_q y)^{-1} = x^{-1} \otimes_{2-q} y^{-1}$;
8. $(x \otimes_0 0) = \begin{cases} 0 & \text{if } (q \geq 1 \text{ and } x \geq 0) \text{ or } (q < 1 \text{ and } 0 \leq x \leq 1) \\ (x^{1-q} - 1)^{\frac{1}{1-q}} & \text{otherwise} \end{cases}$

For particular values of $q$, e.g., $q = 1/2$, the $q$-product provides non-negative values at points for which the inequality $|x|^{1-q} + |y|^{1-q} - 1 < 0$ is verified. According to the cut-off of the $q$-exponential, a value of zero for $x \otimes_q y$ is set down in these cases. Restraining our analysis of Eq. (11) to the sub-space $x, y > 0$, we can observe that for $q \to -\infty$ the region $\{0 \leq x \leq 1, 0 \leq y \leq 1\}$ is not defined. As the value of $q$ increases, the forbidden region decreases its area, and when $q = 0$, we have the limiting line given by $x + y = 1$, for which $x \otimes_0 y = 0$. Only for $q = 1$, the entire set of $x$ and $y$ real values of has a defined value for
the $q$-product. For $q > 1$, the condition (11) implies a region, $|x|^{1-q} + |y|^{1-q} = 1$ for which the $q$-product diverges. This undefined region augments its area as $q$ goes to infinity. When $q = \infty$, the $q$-product is only defined in $\{x \geq 0, 0 < y < 1\} \cup \{0 < x \leq 1, y > 1\}$. Illustrative plots are presented in Fig. (1) of [Duarte Queirós and Tsallis, 2007].

From the properties presented above we ascertain that the $q$-product has got a neutral element and opposite and inverse elements under restrictions. However, distributive property is not held and this fact thwarts the $q$-product of having commutative ring or field structures. Nevertheless, it does not diminish the importance of this algebraic structure as other algebras like the tropical algebra [Clay, 1992] do not present all the standard algebra properties and because the $q$-product represents a quite rare case of a both-side non-distributive structure [Green, 1948].

Besides its inherent exquisiteness, this generalisation has found its own field of applicability in the definition of the $q$-Fourier transform [Umarov and Tsallis, 2008] which plays a key part in non-linear generalisations of the $q$-Central Limit Theorem [Umarov et al., 2006], the definition of a modified characteristics methods which allows the full analytical solution of the porous medium equations [Umarov and Duarte Queirós, 2008] and the structure of Pascal-Leibniz triangles [Petit Lobão et al., 2009].

III. MULTIPLICATIVE PROCESSES AS GENERATORS OF DISTRIBUTIONS

Multiplicative processes, particularly stochastic multiplicative processes, have been the source of plentiful models applied in several fields of science and knowledge. In this context, we can name the study of fluid turbulence [Frisch, 1997], fractals [Feder, 1988], finance [Mandelbrot, 1997], linguistics [Stauffer et al., 2006], etc. Specifically, multiplicative processes play a very important role in the emergence of the log-Normal distribution as a natural and ubiquitous distribution. With regard to the dynamical origins of the log-Normal distribution, we have mentioned in Sec. I the most celebrated examples. Now, we shall give a brief account of the Kapteyn’s process. To that, let us consider a variable $\tilde{Z}$ obtained from a multiplicative random process,

$$\tilde{Z} = \prod_{i=1}^{N} \tilde{\zeta}_i,$$  \hspace{1cm} (13)
where \( \tilde{\zeta}_i \) are nonnegative microscopic variables associated with a distribution \( f'(\tilde{\zeta}) \). If we consider the following change of variables \( Z \equiv \ln \tilde{Z} \), then we have,

\[
Z = \sum_{i=1}^{N} \zeta_i,
\]

with \( \zeta \equiv \ln \tilde{\zeta} \). Assume now that \( \zeta \) has a distribution \( f(\zeta) \) with mean \( \mu \) and variance \( \sigma^2 \). Then, \( Z \) converges to the Gaussian distribution in the limit of \( N \) going to infinity as entailed by the Central Limit Theorem [Araujo and Guiné, 1980]. Explicitly, considering that the variables \( \zeta \) are independently and identically distributed, the Fourier Transform of \( p(Z') \) is given by,

\[
\mathcal{F}[p(Z')] (k) = \left[ \int_{-\infty}^{+\infty} e^{ik \tilde{\zeta}} f(\zeta) \, d\zeta \right]^N, \quad (14)
\]

where \( Z' = N^{-1}Z \). For all \( N \), the integrand can be expanded as,

\[
\mathcal{F}[p(Z')] (k) = \left[ \sum_{n=0}^{\infty} \frac{(ik)^n (\zeta^n)}{n!} \right]^N, \quad (15)
\]

\[
\mathcal{F}[p(Z')] (k) = \exp \left\{ N \ln \left[ 1 + ik \frac{\langle \zeta \rangle}{N} - \frac{1}{2} k^2 \frac{\langle \zeta^2 \rangle}{N^2} + O(N^{-3}) \right] \right\},
\]

where \( \langle \zeta^n \rangle \) represents the \( n \)th order raw moment of \( \zeta \). Expanding the logarithm,

\[
\mathcal{F}[P(Z')] (k) \approx \exp \left[ ik \mu - \frac{1}{2N} k^2 \sigma^2 \right]. \quad (16)
\]

Applying the inverse Fourier Transform, and reverting the \( Z' \) change of variables we finally obtain,

\[
p(Z) = \frac{1}{\sqrt{2\pi N\sigma}} \exp \left[ -\frac{(Z - N\mu)^2}{2\sigma^2 N} \right]. \quad (17)
\]

We can define the attracting distribution in terms of the original multiplicative random process which yields the usual log-Normal distribution [Crow and Shimizu, 1988],

\[
p(\tilde{Z}) = \frac{1}{\sqrt{2\pi N\sigma \tilde{Z}}} \exp \left[ -\frac{(\ln \tilde{Z} - N\mu)^2}{2\sigma^2 N} \right]. \quad (18)
\]

Although this distribution with two parameters, \( \mu \) and \( \sigma \), is able to appropriately describe a large variety of data sets, there are cases for which the log-Normal distribution fails statistical testing [Crow and Shimizu, 1988]. In some of these cases, such a failure
has been overcome by introducing different statistical distributions (e.g., Weibull distributions [Fréchet, 1927, Rosin and Rammler, 1933, Weibull, 1951]) or by changing the 2-parameter log-Normal distribution into a 3-parameter log-Normal distribution [Yuang, 1933, Finney, 1941],

\[
p(x) = \frac{1}{\sqrt{2\pi} \sigma} \exp \left[ -\frac{(\ln (x - \theta) - \mu)^2}{2\sigma^2} \right],
\]

(19)

which is very well characterised in the current scientific literature [Cohen, 1988].

A. One side generalisations

Moving ahead, we now present our alternative procedure to generalise the distribution in Eq. (1). The motivation for this proposal comes from changing the \( N \) products in Eq. (13) by \( N q \)-products,

\[
\tilde{Z} = \prod_{i=1}^{N} \tilde{\zeta}_i \equiv \tilde{\zeta}_1 \otimes_q \tilde{\zeta}_2 \otimes_q \ldots \otimes_q \tilde{\zeta}_N.
\]

(20)

Applying the \( q \)-logarithm we have a sum of \( N \) terms. If every term is identically and independently distributed, then for variables \( \zeta_i = \ln_q \tilde{\zeta}_i \) with finite variables we have a Gaussian has stable distribution [1], i.e., a Gaussian distribution in the \( q \)-logarithm variable. From this scenario we can obtain our \( q \)-log Normal probability density function,

\[
p_q(x) \equiv \frac{1}{Z_q x^q} \exp \left[ -\frac{(\ln_q x - \mu)^2}{2\sigma^2} \right], \quad (x \geq 0),
\]

(21)

with the normalisation,

\[
Z_q \equiv \begin{cases} 
\sqrt{\frac{\pi}{2}} \text{erfc} \left[ -\frac{1}{\sqrt{2\sigma}} \left( \frac{1}{1-q} + \mu \right) \right] \sigma & \text{if } q < 1 \\
\sqrt{\frac{\pi}{2}} \text{erfc} \left[ \frac{1}{\sqrt{2\sigma}} \left( \frac{1}{1-q} + \mu \right) \right] \sigma & \text{if } q > 1.
\end{cases}
\]

(22)

In the limit of \( q \) equal to 1, \( \ln_{q-1} x = \ln x \) and \( Z_{q-1} = \sqrt{2\pi} \sigma \) and the usual log-Normal is recovered. The cumulative distribution,

\[
P(x) \equiv \int_0^x p(z) \, dz,
\]

is given by the following expressions,

\[
P_{q>1}(x) = \frac{1 + \text{erf} \left[ \frac{\ln_q(x) - \mu}{\sqrt{2\sigma}} \right]}{1 + \text{erf} \left[ \frac{1}{\sqrt{2\sigma}} \right]},
\]

(21)
and,
\[ P_{q<1}(x) = \frac{\text{erf}\left[ \frac{m_q(x) - \mu}{\sqrt{2}\sigma} \right] - \text{erf}\left[ -\frac{1}{\sqrt{2}\sigma} \left( \frac{1}{1-q} + \mu \right) \right]}{1 + \text{erfc}\left[ -\frac{1}{\sqrt{2}\sigma} \left( \frac{1}{1-q} + \mu \right) \right]}, \]

Typical plots for cases with \( q = \frac{4}{5}, q = 1, q = \frac{5}{4} \) are depicted in Fig. 1. It can be seen that for \( q \) greater than one the likelihood of events round the peak as well as large values is greater than that for the log-Normal case whereas the case \( q < 1 \) favours events of small value and the intermediate regime between the peak and the tail.

The raw statistical moments,
\[ \langle x^n \rangle \equiv \int_0^\infty x^n p(x) \, dx, \quad (23) \]
can be analytically computed for \( q < 1 \) giving [Gradshteyn and Ryzhik, 1980],
\[ \langle x^n \rangle = \frac{\Gamma[\nu] \exp\left[-\frac{\gamma^2}{2\beta}\right] D_{-\nu} \left[ \frac{\gamma}{\sqrt{2\beta}} \right]}{\sqrt{3^{\nu} \pi} \sigma (1-q) \text{erfc}\left[ -\frac{1}{\sqrt{2}\sigma} \left( \frac{1}{1-q} + \mu \right) \right]}, \quad (24) \]
with
\[
\beta = \frac{1}{2\sigma^2 (1-q)^2}; \quad \gamma = -\frac{1+\mu (1-q)}{(1-q)^2 \sigma^2}; \quad \nu = 1 + \frac{n}{1-q},
\]
(25)
where \( D_{-\alpha} [z] \) is the parabolic cylinder function [Wolfram, 2001]. Equation (24) allows us to write the Fourier Transform or the generating function as,
\[
\varphi (k) = \int p(x) e^{ikx} dx
\]
\[
= \sum_{n=0}^{\infty} \frac{\Gamma[\nu_n]}{\sqrt{2\pi} \pi \sigma (1-q) \text{erfc}\left[\frac{\gamma}{\sqrt{2\sigma}} \sqrt{1-q} + \mu\right]} (ik)^n.
\]
(26)

For \( q > 1 \), the raw moments are given by an expression quite similar to the Eq. (24) with the argument of the \text{erfc} replaced by
\[
\frac{1}{\sqrt{2\sigma}} \left( \frac{1}{1-q} + \mu \right).
\]
However, the finiteness of the raw moments is not guaranteed for every \( q > 1 \) for two reasons. First, according to the definition of \( D_{-\nu} [z] \), \( \nu \) must be greater than 0. Second, the core of the probability density function (21),
\[
\exp \left[ -\frac{(\ln_q x - \mu)^2}{2\sigma^2} \right],
\]
do not vanish in the limit of \( x \) going to infinity,
\[
\lim_{x \to \infty} \exp \left[ -\frac{(\ln_q x - \mu)^2}{2\sigma^2} \right] = \exp \left[ -\frac{\gamma^2}{2} \right].
\]
(27)
This means that the limit \( p(x \to \infty) = 0 \) is introduced by the normalisation factor \( x^{-q} \), which comes from redefining the Normal distribution of variables,
\[
y \equiv \ln_q x,
\]
(28)
as the probability density function of variables \( x \). Because of that, if the moment exceeds the value of \( q \), then the integral (23) diverges. This has got severe repercussion in the adjustment procedures that can be applied.

B. Two side generalisation

As visible from Fig. (1), our generalisation modifies the tail behaviour for small and large values of the variable depending on the value of \( q \) which describes the dependence between
the variable that is transferred into the \( q \)-product. It is well-known that many processes are actually defined by a mixture of different laws of formation, some simpler than others. Within this context, dual relations namely,

\[
q' = 2 - q, \\
q' = \frac{1}{2 - q},
\]

wherefrom property 7 of the \( q \)-product (see Sec. II) emerges, are very inviting in the way that they represent the mapping of a certain rule onto another which seems to be different at first but for which there is actually a univocal transformation. Accordingly, we can imagine a scenario in which variables follow two distinct paths either \( q \)-multiplying or \((2 - q)\)-multiplying (corresponding to the inverse of the \( q \)-product) according to some proportions \( f \) and \( f' = 1 - f \). This proposal is in fact quite plausible if we bear in mind few of the rife examples of mixing in dynamical processes. From that, we establish the law,

\[
p_{q,2-q}(x) = f p_q(x) + f' p_{2-q}(x), \tag{29}
\]

for which we hold that \( f = f' = \frac{1}{2} \) is the most paradigmatic case.

### C. Alternative interpretation

The \( q \)-log-Normal distribution can introduce another clear advantage. Namely, it provides us with an natural and dynamical interpretation of the truncated Normal distribution [Johnson and Lotz, 1970]. In other words, we can look at the left(right) truncated Normal distribution,

\[
\mathcal{G}_b(y) = \sqrt{\frac{2}{\pi \sigma}} \text{erfc} \left[ \text{sgn}(b) \frac{1}{\sqrt{2\sigma}} (\mu - b) \right]^{-1} \exp \left[ -\frac{(y - \mu)^2}{2 \sigma^2} \right], \tag{30}
\]

in which the truncation factor,

\[
b = \frac{1}{q - 1}, \tag{31}
\]

and \( y = \ln_q x \) are intimately related to the value of \( q \) which controls the \( q \)-product part of the dynamical process. In this case the Fourier Transform can be analytically determined. For left truncations we obtain,

\[
\mathcal{F}[\mathcal{G}_b](y) = \frac{1 + \text{erf} \left[ -\frac{i \kappa \sigma}{\sqrt{2}} - B \right]}{\text{erfc}[B] \exp \left[ -\frac{1}{2} i (2 \mu + k \sigma^2) \right]}, \tag{32}
\]
where \( \text{erf}(x) \equiv \Phi(x) - 1 \). For right-truncations,

\[
\mathcal{F}[G_b](y) = \frac{\text{erfc}\left[-i \frac{k}{\sqrt{2}} - B\right]}{\text{erfc}\left[-B\right]} \exp\left[-\frac{1}{2} k \left(2i \mu + k \sigma^2\right)\right],
\]

with

\[
B = \frac{b - \mu}{\sqrt{2} \sigma}.
\]

IV. EXAMPLES OF CASCADE GENERATORS

In this section, we discuss the upshot of two simple cases in which the dynamical process described in the previous section is applied. We are going to verify that the value of \( q \) influences the nature of the attractor in probability space.

A. Compact distribution \([0, b]\)

First, let us consider a compact distribution for identically and independently distributed variables \( x \) within the interval 0 and \( b \). Following what we have described in the preceding section, we can transform our generalised multiplicative process into a simple additive process of \( y_i \) variables which are now distributed in conformity with the distribution,

\[
p'(y) = \frac{1}{b} [1 + (1 - q) y]^{\frac{-2}{1-q}},
\]

with \( y \) defined between \( \frac{1}{q-1} \) and \( \frac{b^{1-q} - 1}{1-q} \) if \( q < 1 \), whereas \( y \) ranges over the interval between \( -\infty \) and \( \frac{b^{1-q} - 1}{1-q} \) when \( q > 1 \). Some curves for the special case \( b = 2 \) are plotted in Fig. 2.

If we look at the variance of this independent variable,

\[
\sigma_y^2 = \langle y^2 \rangle - \langle \mu_y \rangle^2,
\]

which is the moment whose finiteness plays the leading role in the Central Limit Theory, we verify that for \( q > \frac{3}{2} \), we obtain a divergent value,

\[
\sigma_y^2 = \frac{b^{2-2q}}{(3 - 2q)(2-q)^2}.
\]

Hence, if \( q < \frac{3}{2} \), we can apply the Lyapunov’s central Limit theorem and our attractor in the probability space is the Gaussian distribution. On the other hand, if \( q > \frac{3}{2} \), the Lévy-Gnedenko’s version of the central limit theorem [Lévy, 1954] asserts that the attracting
distribution is a Lévy distribution with a tail exponent,
\[ \alpha = \frac{1}{q - 1}. \]  
Furthermore, it is simple to verify that the interval \((\frac{3}{2}, \infty)\) of \(q\) values maps onto the interval \((0, 2)\) of \(\alpha\) values, which is precisely the interval of validity of the Lévy class of distributions that is defined by its Fourier transform,
\[ \mathcal{F} [L_{\alpha}] (k) = \exp \left[ -a |k|^\alpha \right]. \]
In Fig. 3 we depict some sets generated by this process for different values of \(q\).

**B. \(q\)-log Normal distribution**

In this example, we consider the case of generalised multiplicative processes in which the variables follow a \(q\)-log Normal distribution. In agreement with what we have referred to in Sec. III, the outcome strongly depends on the value of \(q\). Consequently, in the associated \(x\) space, if we apply the generalised process to \(N\) variables \(y = \ln_q x\ (x \in [0, \infty))\) which follows a Gaussian-like functional form \([2]\) with average \(\mu\) and finite standard deviation \(\sigma\), i.e., \(\forall q<1\) or \(q > 3\) in Eq.\([21]\), the resulting distribution in the limit of \(N\) going to infinity corresponds to the probability density function \([21]\) with \(\mu \rightarrow N\mu\) and \(\sigma^2 \rightarrow N\sigma^2\). In respect of the conditions of \(q\) we have just mentioned here above, the \(q\)-log normal can be seen as an asymptotic attractor, a stable attractor for \(q = 1\), and an unstable distribution for
FIG. 3: Sets of random variables generated from the process (20) with $N = 100$ and $q = -\frac{1}{2}$ (green), 0 (red), $\frac{1}{2}$ (blue), 1 (black), $\frac{5}{4}$ (magenta) in linear (upper panel) and log scales (lower panel). The generating variable is uniformly distributed within the interval $[0, 1]$ as is the same for all of the cases that we present. As visible, the value of $q$ deeply affects the values of $X_N = \tilde{Z}$.

the remaining cases with the resulting attracting distribution being computed by applying the convolution operation.

V. RANDOM NUMBER GENERATION AND TESTING

The generation of random numbers is by itself a subject of study and depending on distribution different kinds of strategies can be used which start from the shrewd von Neumann-Buffon acceptance-rejection method [von Neumann, 1951] and go to more sophisticated techniques [Gentle, 2004]. Since we aim to introduce a global portrait of the distribution we have not tried to develop bespoken algorithms but applied the robust method
of the Smirnov transformation (or inverse transformation sampling). From a classic robust
generator of uniform numbers, \( \{z\} \), between \(-1\) and \(1\) and considering the probability con-
servation when \( z \) is transformed into \( y \) where \( y \) is associated with a truncated log Normal
distribution with parameters \( \mu \) and \( \sigma \) and \( b \) given by Eq. (31). For \( q < 1 \), i.e., for \( y = \ln_q x \)
between \( b \) and \(+\infty\) we must use,

\[
y = \mu + \sqrt{2\sigma} \operatorname{erf}^{-1} \left[ \infty, \frac{1}{2} (z - 1) \operatorname{erfc} [B] \right],
\]

whereas for \( q > 1 \), i.e., for \( y = \ln_q x \) between \(-\infty\) and \( b \)

\[
y = \mu + \sqrt{2\sigma} \operatorname{erf}^{-1} \left[ 0, \frac{1}{2} \left\{ z (\operatorname{erf} [B] + 1) + \operatorname{erf} [B] - 1 \right\} \right].
\]

From these formulae we have defined the Kolmogorov-Smirnov distance tables that we
present for typical cases \( q = 4/5 \) and \( q = 5/4 \) with \( \mu = 0 \) and \( \sigma = 1 \). For each case
\( 10^6 \) samples have been considered.

VI. EXAMPLES OF APPLICABILITY

In the following examples parameter estimation has been made using traditional max-
imum log-likelihood methods. In spite of using Brent’s method for optimisation [Brent, 1973]
of the log-likelihood function, the following set of equations can be solved if a differential
method is preferred:

\[
\begin{align*}
\sum_{i=1}^{n} \frac{d}{dq} \ln P(x_i) &= 0 \\
\sum_{i=1}^{n} \frac{d}{d\mu} \ln P(x_i) &= 0 \\
\sum_{i=1}^{n} \frac{d}{d\sigma} \ln P(x_i) &= 0.
\end{align*}
\]

(42)

The specific equations can be obtained after straightforward (and tedious) calculus.

A. Shadow prices in metabolic networks

The representation of metabolic networks is often related to linear programming ap-
proaches [Palsson, 2006] for which there is a dual optimisation procedure. In other words,
the maximisation of the reaction fluxes of a metabolic network with a given stoichiometry
matrix has as its dual solution the minimisation of a certain function defined by quantities
TABLE I: Quantiles of the Kolmogorov-Smirnov statistics of a q-log-Normal distribution with $q = 4/5$, $\mu = 0$ and $\sigma = 1$.

| $n \setminus P$ | 0.80 | 0.85 | 0.90 | 0.95 | 0.99 |
|-----------------|------|------|------|------|------|
| 5               | 0.442| 0.471| 0.504| 0.558| 0.663|
| 10              | 0.318| 0.339| 0.362| 0.404| 0.485|
| 15              | 0.261| 0.277| 0.296| 0.333| 0.399|
| 20              | 0.211| 0.225| 0.245| 0.274| 0.334|
| 25              | 0.192| 0.205| 0.222| 0.249| 0.302|
| 30              | 0.176| 0.188| 0.204| 0.228| 0.278|
| 35              | 0.164| 0.175| 0.190| 0.212| 0.257|
| 40              | 0.154| 0.165| 0.178| 0.200| 0.242|
| 45              | 0.146| 0.156| 0.169| 0.189| 0.230|
| 50              | 0.140| 0.149| 0.161| 0.18 | 0.219|
| 60              | 0.128| 0.139| 0.148| 0.165| 0.199|
| 70              | 0.119| 0.127| 0.138| 0.154| 0.186|
| 80              | 0.112| 0.121| 0.1301| 0.144| 0.175|
| 90              | 0.106| 0.113| 0.122| 0.135| 0.164|
| 100             | 0.101| 0.108| 0.115| 0.129| 0.156|

$n > 100 \quad 1.02 \ n^{-1/2} \quad 1.17 \ n^{-1/2} \quad 1.30 \ n^{-1/2} \quad 1.42 \ n^{-1/2} \quad 1.56 \ n^{-1/2}$

traditionally called shadow prices, $\Pi$, which for this case correspond to the chemical potential [Warren and Jones, 2006]. In a previous study the shape of the distribution of the shadow prices has been analysed. From the set of tested PDFs the log-normal has proven to be the better description.

Our first example is composed of shadow prices of the genome-scale model for *E. coli* (iJR 904) growing on a D-glucose substrate [Reed *et al.*, 2003, Kummel *et al.*, 2006, Reed and Palsson, 2007]. The number of shadow prices is 649. Minimisation of the log-likelihood function we obtained $\mu = -0.432$, $\sigma = 0.838$ and $q = 1.21$ in comparison with $\mu = -0.454$ and $\sigma = 0.741$ for the log-Normal. The corresponding Kolmogorov-Smirnov distances are 0.072 and 0.123, respectively, as we depict in Fig. 4. Other qualitatively similar
TABLE II: Quantiles of the Kolmogorov-Smirnov statistics of a q-log-Normal distribution with \( q = 5/4 \), \( \mu = 0 \) and \( \sigma = 1 \).

| \( n \backslash P \) | 0.80 | 0.85 | 0.90 | 0.95 | 0.99 |
|-----------------|------|------|------|------|------|
| 5               | 0.382| 0.413| 0.454| 0.513| 0.627|
| 10              | 0.286| 0.307| 0.334| 0.377| 0.461|
| 15              | 0.246| 0.262| 0.282| 0.317| 0.384|
| 20              | 0.204| 0.218| 0.237| 0.277| 0.327|
| 25              | 0.189| 0.202| 0.218| 0.246| 0.299|
| 30              | 0.174| 0.186| 0.202| 0.225| 0.276|
| 35              | 0.161| 0.174| 0.188| 0.213| 0.256|
| 40              | 0.155| 0.165| 0.178| 0.204| 0.242|
| 45              | 0.146| 0.156| 0.169| 0.189| 0.229|
| 50              | 0.137| 0.148| 0.162| 0.183| 0.217|
| 60              | 0.128| 0.138| 0.148| 0.165| 0.201|
| 70              | 0.118| 0.127| 0.138| 0.154| 0.186|
| 80              | 0.111| 0.121| 0.130| 0.143| 0.175|
| 90              | 0.107| 0.113| 0.122| 0.135| 0.164|
| 100             | 0.099| 0.107| 0.115| 0.128| 0.155|
| \( n > 100 \)   | 1.01 | 1.15 | 1.28 | 1.41 | 1.57 |

results, \( q > 1 \), are found for the shadow prices of models growing upon aerobic conditions.

A different kind of distribution was obtained when a metabolic network like the *M. barkeri* (iAF 692 model) evolving in a Hydrogen medium was considered. In this case 517 metabolites are taken into account. The values of the best fit obtained were \( \mu = -0.633 \), \( \sigma = 1.24 \) and \( q = 0.822 \) in comparison with \( \mu = -0.454 \) and \( \sigma = 0.741 \) for the log-Normal.

These parameters yield the following Kolmogorov-Smirnov distances 0.049 and 0.080 which represent a blunt improvement. Moreover, the introduction of the extra parameter \( q \) is completely justified when we calculate the Akeike information criterion (AIC) [Akeike, 1974],

\[
\text{AIC} = 2k + n \ln \left( \frac{RSS}{n} \right),
\]

where \( k \) is the number of parameters, \( n \) is the number of metabolites of the metabolic network.
FIG. 4: Cumulative density function of the shadow prices vs shadow price of the metabolic network of the E. coli (iJR 904) growing on a D-glucose substrate. The symbols are obtained from the data and the lines the best fits with the q-log-Normal distribution and log-Normal. The values of the parameters and error are mentioned in the text.

and RSS is the residual sum of squares. The values of AIC per metabolite are $-6.288$ and $-5.607$ for the q-log-Normal and the log-Normal in the case of the E. coli, respectively. For the M. barkeri the values are $-7.474$ and $-6.682$. This is clearly adduces that the q-log-Normal outperforms the log-Normal distribution which had given the best results. From a biological perspective is even more appealing that metabolic networks developed in a aerobic environment have presented a value of $q > 1$ and networks related to anaerobic environments yield $q$ values smaller than 1. Whence, we can infer that $q$ value can be possibly used as a signature of aerobic and anaerobic growing.

B. Volatility in financial markets

One of the keystone elements of mathematical finance is the volatility. Despite appearing in every theory of financial markets the truth is that volatility still lacks a precise definition [Engle, 1995]. Nonetheless, it is customarily associated with average of squared
FIG. 5: Cumulative density function of the shadow prices vs shadow price of the metabolic network of the M. barkeri (iAF 692 model) growing in Hydrogen medium [Feist, 2006]. The symbols are obtained from the data and the lines the best fits with the q-log-Normal distribution and log-Normal. The values of the parameters and error are mentioned in the text.

Fluctuations,

$$ r_\Delta (t) \equiv \ln S (t + \Delta) - \ln S (t), $$

of the (log-)price (or index) $S$ over some window $T$ ($\Delta$ is lag). It is well-known for a long time that price fluctuations are nicely fitted by the Student’s $t$-distribution. An explanation for that relies on the local Gaussianity of the price distributions but with a time dependent variance as it has been hold by heteroskedastic processes [Engle, 1995]. In that sense, we can consider a variable $B (t) = v (t)^{-1}$, where

$$ v (t) = \frac{1}{T} \sum_{i=1}^{T} r^2 (t - i). \quad (43) $$

Accordingly, the distribution of price fluctuations following a Bayesian approach

$$ p (r) = \int P (\beta) p (r|B) dB. \quad (44) $$

Assuming the Student’s $t$-distribution hypothesis for $p (r)$ and

$$ p (r|B) = \frac{B}{2\pi} \exp [-B r^2], \quad (45) $$
the distribution of $\mathcal{B}$ must be,

$$P_\Gamma(\mathcal{B}) = \frac{\theta^{-1-\kappa}}{\Gamma[1+\kappa]} \mathcal{B}^\kappa \exp[-\frac{\mathcal{B}}{\theta}], \quad (46)$$

In this case, we have analysed the distribution of $\mathcal{B}$ according to the definition given in Eq. (43) using the daily fluctuations of the SP500 index from the 3rd January 1950 to the 3rd April 2009 and considering 5-business days windows with data obtained from http://finance.yahoo.com (see Fig. 6).

```
FIG. 6: Evolution of the five-day volatility of the SP500 index as defined in the text after normalisation by its average value. Value above the dashed red line can be considered extreme events.
```

Regarding $P(\mathcal{B})$, we have actually noticed that the distribution is poorly described by Eq. (46). Conversely, we verified a very good agreement with Eq. (29) as we show in Fig. 7. The values obtained for $P_\Gamma(\mathcal{B})$ are $\kappa = 0.314$ and $\theta = 1.41$ and for $p_{q,2-q}(\mathcal{B})$ we have $\mu = 0.391$, $\sigma = 1.15$ and $q = 1.22$. The values of the Kolmogorov-Smirnov distances yield 0.0959 and 0.0126 which has passed the statistical test for $\alpha = 2\%$. A representation of the probability density function adjustment is shown in Fig. 7. Although not shown here a log-Normal adjustment which yielded $\mu = 0.379$ and $\sigma = 1.121$ and a Kolmogorov-Smirnov distance of 0.0177 which is 40% greater than the Kolmogorov-Smirnov distance of the $q$-Log-Normal. The utilisation of two values for $q$ (although they relate one another) can
be understood if we accept tested hypotheses that the volatility runs over two mechanisms (short and long scale) \cite{Bouchaud and Potters, 2000}. Nevertheless, it is worth mentioning that applying Eqs. \eqref{eq:44}-\eqref{eq:46} with the values we have determined brings about a Student-$t$ distribution with $\nu = 2.64$ that is in accordance with the value measured for the tail exponent of that distribution (see e.g. \cite{Duarte Queirós, 2005}).

![Graph showing probability density function of the 5-day volatility vs $B$. The symbols are obtained from the data and the lines the best fits with the Gamma distribution and the double sided q-log-Normal. The values of the parameters and error are mentioned in the text.](image.png)

FIG. 7: Probability density function of the 5-day volatility vs $B$. The symbols are obtained from the data and the lines the best fits with the Gamma distribution and the double sided q-log-Normal. The values of the parameters and error are mentioned in the text.

VII. FINAL REMARKS

In this manuscript we have introduced a new kind of generalisation of the log-Normal distribution which stem from a modification of the multiplicative cascade using a new type of algebra recently introduced in a physical context. This modification of the $q$-product, as shown in other cases, represents a way of describing a type of dependence between the variables. Accordingly, the new distribution differs from the classical log-Normal by a single parameter $q$ which favours the right-side tail for $q > 1$, the left-side tail if $q < 1$ and recovers the traditional form when $q = 1$. We have made an extensive description of the distribution namely by defining the moments, the Fourier Transform we have purported a generator...
of random numbers as well which yield the distributions we mentioned here. Using these random number generators we have depicted the construction of a $P$-value table for the Kolmogorov-Smirnov distance when the $q$-log-Normal distribution is assumed. Moreover, we have tested the distribution against real data of biological and financial origin. Both results have shown its usefulness and all the cases we have studied curiously present values of $q$ or $2 - q$ close to $5/4$. Concerning future work we can mention the modification of the two branched distribution to accommodate equal weights for $f$ and $f'$ as we have considered here, using different dual relations for the $q$ parameters or parameters that are not related by any dual relation as well. This last approach corresponds to accepting different mixtures of dynamical or structural processes. It is obvious that such modifications augment the number of the parameters which might be plainly justified by usual statistical criteria.

SMDQ acknowledges P.B. Warren for having provided the shadow prices data, T. Cox for several comments on the subject matter and the critical reading of the manuscript and M.A. Naenini for preliminary discussions. This work benefited from financial support from the Marie Curie Fellowship programme (European Union).

* Electronic address: sdqueiro@gmail.com

[Abe et al., 2007] (2007) Complexity, Metastability, and Nonextensivity: An International Conference (eds S. Abe, H. Herrmann, P. Quarati, A. Rapisarda, C. Tsallis), AIP Conf. Proc., 965.

[Akeike, 1974] Akeike H. (1974) A new look at the statistical model identification. IEEE Trans. Aut. Control, 19, 716.

[Araujo and Guiné, 1980] Araujo A. and Guiné E. (1980) The Central Limit Theorem for Real and Banach Valued Random Variables. New York: John Wiley & Sons.

[Beck et al., 2005] Beck C., Cohen E.G.D., and Swinney H.L. (2005) From time series to super-statistics. Phys. Rev. E, 72, 056133.

[Borges, 2004] Borges E. P. (2004) A possible deformed algebra and calculus inspired in nonextensive thermostatistics. Physica A, 340, 95.

[Bouchaud and Potters, 2000] Bouchaud J. P. and Potters M. (2000) Theory of financial risk and derivative pricing: from statistical physics to risk management. Cambridge: Cambridge Uni-
[Brent, 1973] Brent R. P. (1973) *Algorithms for Minimization Without Derivatives*. Englewood Cliffs - NJ: Prentice & Hall.

[Cohen, 1988] A.C. Cohen (1988) Three-parameter estimation in Lognormal Distributions: Theory and Applications. In *Lognormal Distributions: Theory and Applications* (eds E.L. Crow and K. Shimizu). New York: CRC Pess.

[Clay, 1992] Clay J. R. (1992) *Nearrings: Genesis and Applications*. Oxford: Oxford University Press.

[Crow and Shimizu, 1988] (1988) *Lognormal Distributions: Theory and Applications* (eds E.L. Crow and K. Shimizu). New York: CRC Pess.

[Duarte Queirós, 2005] Duarte Queirós S. M. (2005) On non-Gaussianity and dependence in financial time series: a nonextensive approach. *Quant. Finance* **5**, 475.

[Duarte Queirós and Tsallis, 2007] Duarte Queirós S. M. and Tsallis C. (2007) Nonextensive statistical mechanics and central limit theorems I - Convolution of independent random variables and the q-product. In *Complexity, Metastability, and Nonextensivity: An International Conference* (eds S. Abe, H. Herrmann, P. Quarati, A. Rapisarda, C. Tsallis), AIP Conf. Proc. **965**, 8.

[Engle, 1995] (1995) *ARCH - Selected readings* (ed R.F. Engle). Oxford: Oxford University Press.

[Fa, 2003] Fa K. S. (2003) Linear Langevin equation with time-dependent drift and multiplicative noise term: exact study. *Chem. Phys.*, **287**, 1.

[Feder, 1988] Feder J. (1988) *Fractals*. New York: Plenum.

[Feist, 2006] Feist A.M., Scholten J.C.M., Palsson B.O., Brockman F.J. and Ideker T. (2006) Modeling methanogenesis with a genome-scale metabolic reconstruction of M. barkeri. *Mol Syst Biol*, **2**, 4.

[Finney, 1941] Finney D. J. (1941) On the distribution of a variate whose logarithm is normally distributed. *J. Roy. Statist. Soc. B*, **7**, 155.

[Fréchet, 1927] Fréchet M. (1927) Sur la loi de probabilité de l’écart maximum. *Ann. Soc. Pol. Math.*, **6**, 93.

[Frisch, 1997] Frisch U. (1997) *Turbulence: The Legacy of A. Kolmogorov*. Cambridge: Cambridge University Press.

[Gentle, 2004] Gentle J. E. (2004) *Random Number Generation and Monte Carlo Methods (Statistics and Computing)*. Berlin: Springer.
[Gibrat, 1930] Gibrat R. (1930) Une loi des répartitions économiques. Bull. Statist. Gén. Fr., 19, 469.

[Gradshteyn and Ryzhik, 1980] Gradshteyn I. S. and Ryzhik I. M. (1980) Table of Integrals, Series, and Products. New York: Academic Press. 3.462.1.

[Green, 1948] Green L. C. (1948) Maximum Uncertainty as a Simple Example of a Non-Distributive Algebra. Amer. Math Monthly, 55, 363.

[Johnson and Lotz, 1970] Johnson N. L. and Lotz S. (1970) Continuous univariate distributions. New York: John Wiley & Sons.

[Kapteyn, 1903] Kapteyn J. C. (1903) Skew Frequency Curves in Biology and Statistics. Groningen: Astronomical Laboratory, Noordhoff.

[Kolmogorov, 1941] Kolmogorov A. N. (1941) On the logarithmic normal distribution law of particles with dimensions of fragmentation. Dok. Acad. Nauk SSSR, 31, 99.

[Kummel et al., 2006] Kümmel A., Panke S. and Heinemann M. (2006) Systematic assignment of thermodynamic constraints in metabolic network models. BMC Bioinformatics, 7, 512.

[Lévy, 1954] Lévy P. (1954) Théorie de l’addition des variables aléatoires. Paris:Gauthier-Villars.

[Mandelbrot, 1997] Mandelbrot B. B. (1997) Fractals and Scaling in Finance. New York: Springer.

[Nivanen et al., 2003] Nivanen L., Le Mehaute A. and Wang Q. A. (2003) Generalized algebra within a nonextensive statistics. Rep. Math. Phys., 52, 437.

[Palsson, 2006] Palsson B. O. (2006) Systems Biology: Properties of reconstructed networks. Cambridge: Cambridge University Press.

[Petit Lobão et al., 2009] T.C. Petit Lobão, P.G.S. Cardoso, S.T.R. Pinho and E.P. Borges (2009) Some properties of deformed $q$-numbers. arXiv:0901.4501v1[math-ph]. Preprint.

[Reed et al., 2003] Reed J.L., Vo T.D., Schilling C.H. and B. O. Palsson (2003) An expanded genome-scale model of E. coli K-12 (iJR904 GSM/GPR). Genome Biology, 4,R54.1.

[Reed and Palsson, 2007] Reed J.L., Palsson B.O. (2007) Genome-Scale in silico models of E. coli have multiple equivalent phenotypic states: Assessment of correlated reaction subsets that comprise network states. Genome Res., 14, 1797.

[Rosin and Rammler, 1933] Rosin P. and Rammler E. (1933) The laws governing the finiteness of Powdered Coal. J. Inst. Fuel, 7, 29.

[Stauffer et al., 2006] D. Stauffer, S.M. Moss de Oliveira, P.M.C. de Oliveira and J.M. de Sá
Martins (2006) *Biology, Sociology, Geology by Computational Physicists*, vol. 1. Amsterdam: Elsevier.

[Tsallis, 1988] Tsallis C. (1988) Possible generalization of Boltzmann–Gibbs statistics. *J. Stat. Phys.*, **52**, 479.

[Tsallis, 1994] Tsallis C. (1994) What are the numbers that experiments provide? *Química Nova*, **17**, 468.

[Tsallis, 2009] Tsallis C. (2009) *Introduction to Nonextensive Statistical Mechanics: Approaching a Complex World*. Berlin: Springer.

[Umarov et al., 2006] Umarov S., Tsallis C., Gell-Mann M. and Steinberg S., Symmetric \((q, \alpha)\)-Stable Distributions. Part I: First Representation. arXiv:cond-mat/0606038[cond-mat.stat-mech]. Preprint. and Symmetric \((q, \alpha)\)-Stable Distributions. Part II: Second Representation. arXiv:cond-mat/0606040[cond-mat.stat-mech]. Preprint.

[Umarov and Duarte Queirós, 2008] Umarov S. and Duarte Queirós S. M. (2008) Functional-differential equations for \(F_q\)-transforms of \(q\)-Gaussians. arXiv:0711.2550[cond-mat.stat-mech]. Preprint.

[Umarov and Tsallis, 2008] Umarov S. and Tsallis C. (2008), On a representation of the inverse \(F_q\)-transform. *Phys. Lett. A* **372**, 4874.

[von Neumann, 1951] von Neumann J. (1951) Various techniques used in connection with random digits. Monte-Carlo methods. *Nat. Bureau Standards*, **12**, 36.

[Warren and Jones, 2006] Warren P. B. and Jones J. L. (2006) Duality, Thermodynamics, and the Linear Programming Problem in Constraint-Based Models of Metabolism. *Phys. Rev. Lett.*, **99**, 108101.

[Weibull, 1951] Weibull W. (1951) A statistical distribution function of wide applicability. *J. Appl. Mech. - Trans. ASME*, **18**, 293.

[Wolfram, 2001] http://functions.wolfram.com/HypergeometricFunctions/ParabolicCylinderD/.

[Yuang, 1933] Yuan P. T. (1933) On the logarithmic frequency distributions and the semilogarithmic correlation surface. *Ann. Math. Statist.*, **4**, 30.

[1] Stable in the sense that if we consider the addition of two variables with that distribution the outcome of the convolution of the probability density functions is a probability density function with exactly the same functional form.
Strictly speaking, we cannot use the term Gaussian distribution because it is not defined in the interval \((−\infty, \infty)\). The limitations in the domain do affect the Fourier transform and thus the result of the convolution of the probability density function.