Novel solutions to the tetrahedron equation

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March 1989

Abstract

This is the English translation of the short note where the first non-trivial tetrahedron relation (solution of the Zamolodchikov tetrahedron equation) with variables on the edges was presented.

1 Formulation of the results

Let $V_1, V_2, V_3, V_4$ be two-dimensional complex linear spaces with fixed bases, and $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ be complex numbers. Consider operators

$$R^0_{ij}(\lambda_i, \lambda_j) = \begin{pmatrix} a & d \\ b & c \\ c & b \\ d & a \end{pmatrix},$$

$$R^1_{ij}(\lambda_i, \lambda_j) = \begin{pmatrix} -a' & d' \\ -b' & c' \\ -c' & b' \\ -d' & a' \end{pmatrix}$$

acting in $V_i \otimes V_j$, $1 \leq i < j \leq 4$. Here

$$a = \text{cn}(\lambda_i - \lambda_j), \quad b = \text{sn}(\lambda_i - \lambda_j) \text{dn}(\lambda_i - \lambda_j),$$

$$c = \text{dn}(\lambda_i - \lambda_j), \quad d = k \text{sn}(\lambda_i - \lambda_j) \text{cn}(\lambda_i - \lambda_j),$$

$$a' = \text{cn}(\lambda_i + \lambda_j), \quad b' = \text{sn}(\lambda_i + \lambda_j) \text{dn}(\lambda_i + \lambda_j),$$

$$c' = \text{dn}(\lambda_i + \lambda_j), \quad d' = k \text{sn}(\lambda_i + \lambda_j) \text{cn}(\lambda_i + \lambda_j);$$

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1 With only this abstract and all the footnotes added in 2013.
2 I.G. Korepanov. Novel solutions to the tetrahedron equation. Chelyabinsk: Chelyabinsk Polytechnical Institute, 1989. Deposited at VINITI (http://viniti.ru) scientific database, no. 1751-V89 (Russian).
3 And the only nontrivial tetrahedron relation known by that time was due to Zamolodchikov himself.
all elliptic functions are of modulus $k$.

As we show in Section 2, there exist $8 \times 8$ matrices $S_{123}, S_{124}, S_{134}, S_{234}$, whose components will be denoted as, for instance, $(S_{123})^{abc}_{def}$, $a, \ldots, f = 0$ or 1, such that the relations of tetrahedral Zamolodchikov algebra [1] hold:

$$R^a_{12}R^b_{13}R^c_{23} = \sum_{d,e,f=0}^{1} (S_{123})^{abc}_{def} R^f_{23}R^e_{13}R^d_{12},$$

(1)

and similarly for the other $S$-matrices. Arguments $\lambda_i$ are omitted in formula (1); it is understood also that $S_{123}$ depends on $\lambda_1, \lambda_2$ and $\lambda_3$, and so on.

In contrast to the special case $k = 0$ of paper [1], the $S$-matrices are determined uniquely in the general case (making $k$ tend to zero, we can get unique matrices for $k = 0$ as well). Introduce two-dimensional spaces $E_{12}, E_{13}, E_{14}, E_{23}, E_{24}$ and $E_{34}$, and consider $S_{123}$ as an operator in $E_{12} \otimes E_{13} \otimes E_{23}$, $S_{124}$ as an operator in $E_{12} \otimes E_{14} \otimes E_{24}$, and so on. Does the tetrahedron equation

$$S_{123}S_{124}S_{134}S_{234} = S_{234}S_{134}S_{124}S_{123}$$

hold?

The author has performed direct calculations [1] for $k = 0$, and they gave the positive answer. Below we write out matrix $S_{123}$ explicitly, in terms of values $\tau_i = \tan \lambda_i$ (the rest of $S$-matrices are obtained by the obvious changes of subscripts). We introduce the function

$$f(\rho, \sigma) = \frac{1 + \rho \sigma}{\rho + \sigma}.$$
So, all elements of matrix $S_{123}$ are zeros except the following:

$$S^{000}_{000} = S^{011}_{011} = S^{101}_{101} = S^{110}_{110} = 1,$$

$$S^{010}_{010} = f(\tau_1, \tau_3)f(\tau_2^{-1}, \tau_3),$$

$$S^{001}_{100} = f(\tau_1, -\tau_2^{-1})f(\tau_2^{-1}, \tau_3),$$

$$S^{001}_{111} = f(\tau_1, -\tau_2^{-1})f(\tau_1, \tau_3),$$

$$S^{010}_{001} = f(\tau_2^{-1}, -\tau_3)f(\tau_1, -\tau_3),$$

$$S^{010}_{100} = f(\tau_1, -\tau_2^{-1})f(\tau_1, -\tau_3),$$

$$S^{010}_{111} = f(\tau_1, -\tau_2^{-1})f(\tau_2^{-1}, -\tau_3),$$

$$S^{100}_{001} = -f(\tau_2^{-1}, -\tau_3)f(\tau_1, \tau_2^{-1}),$$

$$S^{100}_{010} = f(\tau_1, \tau_3)f(\tau_1, \tau_2^{-1}),$$

$$S^{110}_{111} = -f(\tau_1, \tau_3)f(\tau_2^{-1}, -\tau_3),$$

$$S^{110}_{001} = f(\tau_1, \tau_2^{-1})f(\tau_1, -\tau_3),$$

$$S^{110}_{010} = -f(\tau_1, \tau_2^{-1})f(\tau_2^{-1}, \tau_3),$$

$$S^{110}_{100} = -f(\tau_1, -\tau_3)f(\tau_2^{-1}, \tau_3).$$

Recall that a model on two two-dimensional layers, related to the $S$-matrices constructed here, and whose “Boltzmann weights” can be made positive, has been constructed in [1].

2 Possible generalizations

Consider a Yang–Baxter equation

$$R_{12}L_{01}M_{02} = M_{02}L_{01}R_{12},$$

where the subscripts show the numbers of two-dimensional spaces in whose tensor product a given operator acts. Let $L_{01}$ and $M_{02}$ be $L$-matrices of Felderhof [2] type. In particular, they are symmetric with respect to the transposition $^T$. Then, as is known (see, for example, [3]), there exists, besides a symmetric matrix $R_{12} = R_{12}^0$, also a non-symmetric $R_{12}^1$, for which

$$(R_{12}^1)^TL_{01}M_{02} = M_{02}L_{01}R_{12}^1.$$

It can be shown, by developing ideas of paper [3], that the space of operators $\mathcal{R}_{123}$ performing the following permutation of $L$-matrices:

$$(\mathcal{R}_{123})^TL_{01}M_{02}N_{03} = N_{03}M_{02}L_{01}\mathcal{R}_{123},$$
Введем функцию

\[ f(\rho, \varepsilon) = \frac{1 + \rho \varepsilon}{\rho + \varepsilon} \]

Итак, все элементы матрицы \( S_{23} \) равны нулю, кроме следующих:

- \( S_{000} = S_{011} = S_{101} = S_{110} = 1 \)
- \( S_{010} = f(\tau_1, \tau_3) f(\tau_2^{-t}, \tau_3) \)
- \( S_{001} = f(\tau_1, -\tau_2^{-t}) f(\tau_2^{-t}, \tau_3) \)
- \( S_{101} = f(\tau_1, -\tau_2^{-t}) f(\tau_1, \tau_3) \)
- \( S_{010} = f(\tau_2^{-t}, \tau_3) f(\tau_1, -\tau_3) \)
- \( S_{100} = f(\tau_1, -\tau_2^{-t}) f(\tau_1, -\tau_3) \)
- \( S_{110} = f(\tau_1, -\tau_2^{-t}) f(\tau_2^{-t}, -\tau_3) \)
- \( S_{111} = f(\tau_1, -\tau_2^{-t}) f(\tau_2^{-t}, -\tau_3) \)
is eight-dimensional. The author hopes to explain this in more detail in another paper. Exactly 8 such operators $R_{123}$ are obtained, on one hand, in the form

$$R_{12}^a R_{13}^b R_{23}^c,$$

and on another hand — in the form

$$R_{23}^f R_{13}^e R_{12}^d,$$

and this is what leads to linear dependencies of type [1]. Checking the validity of the tetrahedron equation for thus obtained $S$-matrices is, however, extremely difficult and has not been done as yet.

References

[1] Korepanov I.G. *Tetrahedral analogue of Zamolodchikov algebra and a two-layer model of two-dimensional statistical physics*. Chelyabinsk Polytechnical Institute, Chelyabinsk, 1988, 9 p., Deposited at VINITI 06 June 1988, no. 4433-V88 (Russian).

[2] Felderhof B.U. *Diagonalization of the transfer matrix of the free fermion model*. Physica, 1973, vol. 66, no. 2, pp. 279–298.

[3] Krichever I.M. *Baxter equations and algebraic geometry*. Funct. Anal. and Apps., 1981, vol. 15, issue 2, pp. 92–103.