Connecting the von Neumann and Rényi entropies for fermions

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Abstract
We explore the relation between the von Neumann entropy and the Rényi entropies of integer orders for shift-invariant quasi-free fermionic lattice systems. We investigate the approximation of the von Neumann entropy by a combination of integer-order Rényi entropies and give an estimate for the quality of such an approximation.

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(Some figures may appear in colour only in the online journal)

1. Introduction
The motivation for this paper is to better understand the relation between the average von Neumann entropy and the average Rényi entropies of integer order. Entropies are non-local characteristics of a state and are, e.g., an essential input in maximal entropy principles such as the variational principle for thermal equilibrium. Restricting the variational principle to specific classes of states leads to well-known approximations like mean-field or Hartree–Fock [13]. More refined approximation schemes like using matrix product states for computing ground states of quantum spin chains turned out to be quite effective [11, 15]. Extensions to higher dimensional quantum spin lattices are currently being investigated [8] and one might wonder about using general finitely correlated states for thermal states. Computing the mean von Neumann entropy for a general, shift-invariant, finitely correlated state is, however, still an open problem [9].

In [17], Rényi introduced a generalized entropy of order \( \alpha \) defined as

\[
S_\rho(\alpha) := -\frac{1}{\alpha - 1} \log \text{Tr} \rho^\alpha,
\]

where \( \rho \) is a density matrix, \( \alpha \neq 1 \) and \( \alpha \geq 0 \). The von Neumann entropy is obtained as the limit

\[
S_\rho = \lim_{\alpha \to 1} S_\rho(\alpha) = -\text{Tr} \rho \log \rho.
\]
Rényi entropies of integer order $\alpha \in \{2, 3, \ldots\}$ are easily computed: one just needs matrix multiplication and to take the trace. For fractional orders or for the von Neumann entropy, one should, in principle, diagonalize the density matrix which is a much harder problem.

For integer orders, it is often convenient to express the entropy in terms of several copies of the system. This is called the replica trick:

$$\text{Tr} \rho^n = \text{Tr} \rho \otimes \rho \otimes \cdots \otimes \rho T_n,$$

where $T_n$ is the cyclic shift on the $n$-fold tensor power of the original system:

$$T_n(\varphi_1 \otimes \varphi_2 \otimes \cdots \otimes \varphi_n) = \varphi_2 \otimes \cdots \otimes \varphi_n \otimes \varphi_1.$$

Formula (3) can be checked by evaluating its right-hand side in a tensor basis:

$$\text{Tr} \rho \otimes \rho \otimes \cdots \otimes \rho T_n = \sum_{i_1, \ldots, i_n} \langle e_{i_1} \otimes \cdots \otimes e_{i_n}, \rho \otimes \cdots \otimes \rho T_n e_{i_1} \otimes \cdots \otimes e_{i_n} \rangle$$

$$= \sum_{i_1, \ldots, i_n} \langle e_{i_1} \otimes \cdots \otimes e_{i_n}, \rho \otimes \cdots \otimes \rho e_{i_2} \otimes \cdots \otimes e_{i_n} \rangle$$

$$= \sum_{i_1, \ldots, i_n} \langle e_{i_1}, \rho e_{i_2} \rangle \langle e_{i_2}, \rho e_{i_3} \rangle \cdots \langle e_{i_n}, \rho e_{i_1} \rangle$$

$$= \text{Tr} \rho^n.$$

For an example of this replica trick applied to a spin glass, see [5] and for an example of the replica trick applied to the computation of an average Rényi entropy, see [9]. Apart from the obvious application in statistical mechanics, Rényi entropies are also used in the analysis of multiparticle states produced in high-energy collisions, see e.g., [1] and [2]. Rényi entropies are also of interest in the context of catalysis in quantum information theory. The question that inspired this paper is: whether it is possible to reconstruct the von Neumann entropy given the Rényi entropies of integer order $2, 3, \ldots$, and, if so, how to do this?

In fact, the relevant quantity for shift-invariant states is the average Rényi entropy. Unfortunately, these densities pose several serious problems with regard to existence and continuity. In general, they simply do not exist and they are also not affine on convex subsets of shift-invariant states. Their use is therefore limited to states with strong clustering. Moreover, it is completely unclear whether, in general, the knowledge of integer-order average Rényi entropies uniquely determines the average von Neumann entropy.

A number of papers have considered relations between the integer-order Rényi and von Neumann entropies [12, 10, 13]. These relations do not always scale properly with the system size and therefore do not necessarily survive on the level of densities. There is certainly no general procedure, even under strong assumptions on clustering, for passing from integer-order Rényi to von Neumann entropies. Further suggested reading on the subject can be found in [3, 14, 16].

In this paper, we consider the case of shift-invariant quasi-free fermionic states on a lattice. We show that such a reconstruction procedure exists in this case and obtain some simple approximations in terms of the first few Rényi densities.

The paper is organised as follows. In section 2, we remind the reader of the description of fermions on a lattice and introduce the notation. Section 3 gives the expression for the average Rényi entropies of quasi-free states. In section 4, we introduce a completely monotonic entropy function which is then used in section 5 to reconstruct the von Neumann entropy. Finally, we provide an explicit approximation scheme in section 6.
2. Fermions on a lattice

We consider a system of fermions living on some Bravais lattice $\mathcal{L}$ in $\mathbb{R}^d$:

$$\mathcal{L} = \{ n_1 \mathbf{e}_1 + n_2 \mathbf{e}_2 + \cdots + n_d \mathbf{e}_d \mid n_1, n_2, \ldots, n_d \in \mathbb{Z} \} = \{ \mathbf{n} \cdot \mathbf{e} \mid \mathbf{n} \in \mathbb{Z}^d \},$$

where $\{ \mathbf{e}_j \}$ are the primitive vectors of the lattice. For our purposes, we can identify $\mathbf{n} \cdot \mathbf{e}$ with $\mathbf{n}$.

Fermions on the lattice are described by smeared out creation and annihilation operators $c^\dagger$ and $c$ obeying the canonical anticommutation relations. The smearing is done by square summable sequences on the lattice:

$$\mathcal{L}^2(\mathbb{Z}^d) \ni \varphi \mapsto c^\dagger(\varphi)$$

is $\mathbb{C}$-linear

and the anticommutation relations are

$$\{c(\varphi), c(\psi)\} = 0 \quad \text{and} \quad \{c(\varphi), c^\dagger(\psi)\} = \langle \varphi, \psi \rangle \mathbb{I}.\quad (8)$$

The creation and annihilation operators generate the C*-algebra $\mathcal{A}(\mathbb{Z}^d)$ of canonical anticommutation relations on $\mathbb{Z}^d$ (CAR).

Lattice translations induce *-automorphisms on $\mathcal{A}(\mathbb{Z}^d)$ by extending

$$c(\varphi) \mapsto c(U(\mathbf{n})\varphi).\quad (9)$$

Here, $U(\mathbf{n})$ is the unitary shift on $\mathcal{L}^2(\mathbb{Z}^d)$ by $\mathbf{n} \in \mathbb{Z}^d$

$$U(\mathbf{n})\varphi(\mathbf{k}) = \varphi(\mathbf{k} - \mathbf{n}).\quad (10)$$

We also need quasi-free states $\omega_Q$, sometimes called Gaussian fermionic states. They are uniquely determined by their two-point correlation functions:

$$\langle \varphi, \psi \rangle \mapsto \omega(c^\dagger(\varphi)c(\psi)), \quad \varphi, \psi \in \mathcal{L}^2(\mathbb{Z}^d).\quad (11)$$

Due to the complex linearity of $c^\dagger$, and the conjugate linearity of $c$,

$$\langle \varphi, \psi \rangle \mapsto \omega(c^\dagger(\varphi)c(\psi))$$

is a sesquilinear form on $\mathcal{L}^2(\mathbb{Z}^d)$. One can show that the necessary and sufficient condition for (12) to define a state that is the form given in equation (13) corresponds to a linear operator $Q$ on $\mathcal{L}^2(\mathbb{Z}^d)$ such that $0 \leq Q \leq 1$:

$$\omega(c^\dagger(\varphi)c(\psi)) = \langle \psi, Q \varphi \rangle.\quad (14)$$

The operator $Q$, called the symbol, defines the state $\omega_Q$, where the notation stresses the dependence of the state $\omega$ on $Q$.

A quasi-free state $\omega_Q$ is shift-invariant if and only if $Q$ commutes with the lattice-shift unitaries $U(\mathbf{n})$ defined in (10). In the standard basis $\{ \mathbf{e}_j \}$ of $\mathcal{L}^2(\mathbb{Z}^d)$, this is equivalent to

$$\langle \mathbf{e}_j, Q \mathbf{e}_k \rangle = \langle \mathbf{e}_{j+n}, Q \mathbf{e}_{k+n} \rangle, \quad j, k, n \in \mathbb{Z}^d.\quad (15)$$

If $F$ denotes the unitary Fourier transformation

$$(F \varphi)(\mathbf{x}) := \sum_{\mathbf{n} \in \mathbb{Z}^d} \varphi(\mathbf{n}) e^{2\pi i \mathbf{n} \cdot \mathbf{x}}, \quad \mathbf{x} \in [0, 1]^d,\quad (16)$$

then this is equivalent to

$$F \ Q = q \ F,\quad (17)$$

where $q$ is the multiplication operator with the function

$$q(\mathbf{x}) = \sum_{\mathbf{j} \in \mathbb{Z}^d} \langle \mathbf{e}_j, Q \mathbf{e}_j \rangle e^{2\pi i \mathbf{j} \cdot \mathbf{x}}\quad (18)$$

on $\mathcal{L}^2([0, 1]^d)$. As $0 \leq Q \leq 1$, $q$ takes values in $[0, 1]$. For more details on these matters, we refer to [4, 6, 7].
3. Rényi entropies

Let $\Lambda$ be a finite subset of $\mathbb{Z}^d$. The local algebra $\mathcal{A}(\Lambda)$ is generated by the creation and annihilation operators with smearing functions supported in $\Lambda$. This algebra is isomorphic to the algebra of matrices of dimension $2^{\#(\Lambda)}$ and has therefore, up to unitary equivalence, a unique irreducible representation. This allows one to assign to any state $\omega$ on $\mathcal{A}(\Lambda)$ the Rényi entropies:

$$S_\omega(\alpha) := -\frac{1}{\alpha - 1} \log \text{Tr} \rho^\alpha, \quad \alpha > 0. \quad (19)$$

Here, $\rho$ is the density matrix defining $\omega$ in an irreducible representation of $\mathcal{A}(\Lambda)$. For $\alpha = 1$, the expression in (19) has to be replaced by its limit, the von Neumann entropy:

$$S_\omega := S_\omega(1) = -\text{Tr}(\rho \log \rho). \quad (20)$$

The local Rényi entropies of quasi-free states can be readily expressed in terms of the local restrictions of their corresponding symbols $Q$. Moreover, one can show that for shift-invariant quasi-free states, the limiting average Rényi entropies in the sense of growing boxes exist and one obtains an explicit expression in terms of the Fourier transform $q$:

$$s_q(\alpha) := -\frac{1}{\alpha - 1} \int_{[0,1]^d} \text{d}x \log(q^\alpha + (1 - q)^\alpha). \quad (21)$$

In the case of the von Neumann entropy density, this expression must again be understood as

$$s_q(\alpha) = -\int_{[0,1]^d} \text{d}x [q \log q + (1 - q) \log(1 - q)]. \quad (22)$$

4. A completely monotonic entropy

It is known that for a general density matrix $\rho$ in a matrix algebra,

$$\alpha \in [0, \infty[ \mapsto -\frac{1}{\alpha - 1} \log \text{Tr} \rho^\alpha \quad (23)$$

is a monotonically decreasing function, in particular

$$S_\rho \geq S_\rho(2) \geq S_\rho(3) \geq \ldots. \quad (24)$$

This ordering then extends to densities, provided they exist. This is certainly the case for shift-invariant quasi-free states, where it is not hard to compute the asymptotic value:

$$s_q(\alpha) = \lim_{\alpha \to \infty} s_q(\alpha) = -\int_{[0,1]^d} \text{d}x \log[\max\{q, (1 - q)\}]. \quad (25)$$

We now introduce a modified entropy-like function:

$$g_q(\alpha) := s_q(\infty) - \frac{\alpha - 1}{\alpha} s_q(\alpha), \quad \alpha > 0. \quad (26)$$

To avoid technical complications, we assume from now on that $q = 0$ or $q = 1$ on a set of measure zero. This is, e.g., true for thermal states of quadratic interactions at non-zero temperature. The general case can be handled by removing the union of the kernels of $q$ and $1 - q$ from $[0, 1]^d$. One now easily verifies that

$$g_q(\alpha) = \frac{1}{\alpha} \int_{[0,1]^d} \text{d}x \log[1 + \exp(-\alpha h)], \quad (27)$$

where we have introduced the function:

$$h : [0, 1]^d \to \mathbb{R}^+ : h := -\log\left(\min\left\{\frac{q}{1 - q}, \frac{1 - q}{q}\right\}\right). \quad (28)$$
A function \( f : ]0, \infty[ \to \mathbb{R} \) is called completely monotonic if its derivatives to all orders exist and if

\[
(-1)^n f^{(n)} \geq 0, \quad n = 0, 1, 2, \ldots
\]  

Bernstein’s theorem [18] characterises completely monotonic functions as the Laplace transforms of positive measures that do not grow too fast at infinity. We show here that, for a given \( q \), \( \alpha \in ]0, \infty[ \mapsto g_q(\alpha) \) is completely monotonic by computing its inverse Laplace transform.

First, consider the function \( k \) on \( \mathbb{R}^+ \):

\[
k(t) = \sum_{j \in \mathbb{N}_0} (-1)^{j+1} j \quad t \geq 0.
\]  

This function is piecewise constant, non-negative, continuous from the right and tends to \( \log 2 \), see figure 1. As \( k \) is of exponential order 0, we may compute its Laplace transform for all \( s \in \mathbb{C} \) with \( \Re(s) > 0 \) by evaluating the integral:

\[
\mathcal{L}(k)(s) = \int_{\mathbb{R}^+} dt k(t) e^{-st} = \sum_{\ell=1}^{\infty} \sum_{j=1}^{\ell} \frac{(-1)^{j+1}}{j} \int_{\ell}^{\ell+1} dt e^{-st}
\]

\[
= s \sum_{\ell=1}^{\infty} \frac{1}{s} \frac{e^{-\ell s} - e^{-(\ell+1)s}}{e^{-(\ell+1)s}} \left( \sum_{j=1}^{\ell} \frac{(-1)^{j+1}}{j} \right)
\]

\[
= \frac{1}{s} \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell+1}}{\ell} e^{-\ell s} = \frac{1}{s} \log(1 + \exp(-s)).
\]  

This shows that

\[
s > 0 \mapsto \frac{1}{s} \log(1 + \exp(-s))
\]  

is completely monotonic.
As complete monotonicity is preserved by rescaling the argument, rescaling the function, and addition, we conclude from (27) that
\[ \alpha \in ]0, \infty[ \mapsto g_q(\alpha) \] (34)
is completely monotonic. We have, in fact, a rather explicit expression for the inverse Laplace transform of \( g_q \):
\[ L^{-1}(g_q)(t) = \int_{[0,1]^d} \frac{t}{h} \, dx. \] (35)

5. Reconstructing the von Neumann entropy

In general, one cannot hope to uniquely reconstruct a function \( g \) that is analytic in \( \{ z \in \mathbb{C} \mid \Re(z) > 0 \} \) given the values of \( g \) on \( \mathbb{N}_0 \). This is nevertheless often attempted in statistical mechanics of disordered systems where one tries to reconstruct the von Neumann entropy given the Rényi entropies of order 2, 3, ...which can be computed using the replica trick. Suppose, however, that there exists a non-negative measurable function \( G \) of exponential order 0 on \( \mathbb{R}^+ \) such that
\[ g(\alpha) = \int_0^\infty dt \, G(t) \, e^{-\alpha t}, \quad \alpha > 0, \] (36)
where \( g \) is related to the entropy densities \( s \) as in (26). We then have
\[ s(\alpha) = \frac{\alpha}{\alpha - 1} \int_0^\infty dt \, G(t) (e^{-t} - e^{-\alpha t}), \quad \alpha \neq 1 \] (37)
and
\[ s = s(1) = \int_0^\infty dt \, G(t) \, t \, e^{-t}. \] (38)

Suppose that we know the integer-order entropies \( s(n) \) for \( n = 2, 3, \ldots \) and that we wish to reconstruct the von Neumann entropy \( s \) given by (37) and (38). The vector space generated by the functions
\[ f_n : t \in \mathbb{R}^+ \mapsto \frac{n}{n-1} (e^{-t} - e^{-nt}), \quad n = 2, 3, \ldots \] (39)
is actually an algebra of continuous functions that vanish at 0 and at \( \infty \). It is closed under complex conjugation and it separates the points in \( \mathbb{R}^+ \). We can therefore approximate
\[ t \in \mathbb{R}^+ \mapsto t \, e^{-t} \] (40)
uniformly to arbitrary precision by a linear combination of the \( f_n \) using the Stone–Weierstrass theorem.

Given a \( d \)-dimensional density matrix \( \rho \) and \( d - 1 \) Rényi entropies of integer order in \( \{2, 3, \ldots\} \), we can uniquely reconstruct the ordered eigenvalues of \( \rho \). This means that two density matrices with the same integer Rényi entropies are related by a unitary transformation.

The mean Rényi entropies of a shift-invariant quasi-free state are of the form \( \int_{[0,1]^d} dx \, f(q) \), where \( f \) is a bounded measurable function on \( [0, 1] \). We can associate a distribution function to a symbol \( q \) in the Fourier space as follows:
\[ \gamma_q : y \in [0, 1] \mapsto \int_{q(x) \leq y} \, dx. \] (41)

We can then write
\[ \int_{[0,1]^d} dx \, f(q) = \int_0^1 dy \gamma_q(y) \, f(y). \] (42)
Obviously, different symbols may yield the same distribution function and therefore the same mean Rényi entropies. Knowledge of the integer mean entropies with \( \alpha \in \{2, 3, \ldots\} \) actually determines the distribution function except for contributions at 0 or 1 which do not effectively contribute to any mean entropy for \( \alpha > 0 \).

Suppose that \( t \) is a Lebesgue measure preserving rearrangement of \([0, 1]^d\), i.e. a transformation of \([0, 1]^d\) such that

\[
\int_{x \in \Lambda} dx = \int_{t(x) \in \Lambda} dx
\]

for each measurable set \( \Lambda \). The mapping

\[
U_t : \varphi \mapsto \varphi \circ t
\]

is a unitary transformation of \( L^2([0, 1]^d) \) such that for a multiplication operator \( q \),

\[
U_t q = (q \circ t) U_t.
\]

The Fourier transformed symbols \( q \) and \( q \circ t \) define different shift-invariant quasi-free states on \( \mathcal{A}(\mathbb{Z}^d) \) which are related by the automorphism defined through \( U_t \). The distribution functions (41) of these states coincide and therefore have the same mean entropies. This situation can be seen as a partial quasi-free analogue of the density matrix case discussed above. There are, however, plenty of symbols that yield the same mean entropies and that are not related by a rearrangement.

6. Explicit approximations

Let us extend the notation in (39) by defining

\[
f_1(t) = t e^{-t}.
\]

A simple-minded \( n \)-term approximation of \( f_1 \) consists in finding the minimizers of

\[
(\gamma_1, \ldots, \gamma_n) \mapsto \|f_1 - \gamma_1 f_2 - \cdots - \gamma_n f_{n+1}\|_\infty
\]

by demanding that all extrema of the above function are equal in magnitude. The first few optimal approximations are

\[
\begin{align*}
\|f_1 - 0.800 f_2\|_\infty &= 0.09 \\
\|f_1 - 2.219 f_2 + 1.314 f_3\|_\infty &= 0.04 \\
\|f_1 - 4.233 f_2 + 6.133 f_3 - 2.850 f_4\|_\infty &= 0.02 \\
\|f_1 - 6.833 f_2 + 17.498 f_3 - 18.780 f_4 + 7.148 f_5\|_\infty &= 0.01.
\end{align*}
\]

In comparison, \( \|f_1\|_\infty \approx 0.368 \).

This corresponds to the approximations of the mean von Neumann entropy by the first few mean Rényi entropies:

\[
s_q \approx 0.800 s_q(2) \\
\approx 2.219 s_q(2) - 1.314 s_q(3) \\
\approx 4.233 s_q(2) - 6.133 s_q(3) + 2.850 s_q(4) \\
\approx 6.833 s_q(2) - 17.498 s_q(3) + 18.780 s_q(4) - 7.148 s_q(5)
\]

The single-term approximation is certainly terrible because we know from the monotonicity of the Rényi densities that

\[
s_q \geq s_q(2).
\]
The multi-term approximations are, however, no longer direct consequences of the monotonicity of the Rényi densities. The quality of (49) is not easily assessed because the measure $G(t)\,dt$ is generally unbounded.

Under our assumption on $q$, the function $G$ in (37) tends to $\log 2$ at infinity. As the supremum norm is attained close to the origin, we cannot expect to obtain a very good approximation this way. A better approach, especially at high temperatures, is to subtract the asymptotic value $\log 2$ from $G$ and then use (48). This yields:

$$s_q \approx 0.200 \log 2 + 0.800 s_q(2)$$
$$\approx 0.095 \log 2 + 2.192 s_q(2) - 1.314 s_q(3)$$
$$\approx 0.050 \log 2 + 4.233 s_q(2) - 6.133 s_q(3) + 2.850 s_q(4)$$
$$\approx 0.033 \log 2 + 6.833 s_q(2) - 17.498 s_q(3) + 18.785 s_q(4) - 7.148 s_q(5),$$

as illustrated in figure 2. One can, however, still not hope that $G - \log 2$ is integrable.

A controllable approximation scheme can be set up by using Rényi densities of order less than 1 and using the uniform bound

$$s_q(\alpha) = \frac{\alpha}{\alpha - 1} \int_0^\infty dt \, G(t)(e^{-t} - e^{-\alpha t}) \leq \log 2.$$  

Extending the definition of $f_n$ in (39) to general $\alpha \in \mathbb{R}^+$, we write

$$|s_q - \gamma_1 s_q(2) - \cdots - \gamma_n s_q(n + 1)|$$
$$= \left| \int_0^\infty dt G(t) (f_1(t) - \gamma_1 f_2(t) - \cdots - \gamma_n f_{n+1}(t)) \right|$$
$$= \left| \int_0^\infty dt G(t) f_\alpha(t) \left( \frac{f_1(t)}{f_\alpha(t)} - \gamma_1 \frac{f_2(t)}{f_\alpha(t)} - \cdots - \gamma_n \frac{f_{n+1}(t)}{f_\alpha(t)} \right) \right|$$
$$\leq \left\| \frac{f_1}{f_\alpha} - \gamma_1 \frac{f_2}{f_\alpha} - \cdots - \gamma_n \frac{f_{n+1}}{f_\alpha} \right\|_\infty \log 2.$$
Figure 3. Difference between $f_1/f_\alpha$ and its controlled one-, two-, three- and four-term approximations re-scaled by $f_\alpha$, given by equations (53).

It now remains to minimize the bound (53) for a given $n$ with respect to $\gamma_1, \ldots, \gamma_n$ and $\alpha \in [0, 1]$. This leads to

\[
|s_q - 0.666 s_q(2)| \leq 0.35
\]
\[
|s_q - 1.938 s_q(2) + 1.005 s_q(3)| \leq 0.19
\]
\[
|s_q - 3.892 s_q(2) + 4.967 s_q(3) - 2.048 s_q(4)| \leq 0.12
\]
\[
|s_q - 6.556 s_q(2) + 15.064 s_q(3) - 14.413 s_q(4) + 4.923 s_q(5)| \leq 0.08
\]
\[
\vdots
\]
\[
|s_q - 37.181 s_q(2) + 529.415 s_q(3) - 3846.261 s_q(4) + 16301.725 s_q(5)
- 43168.833 s_q(6) + 73647.855 s_q(7) - 80999.681 s_q(8) + 55517.489 s_q(9)
- 21580.373 s_q(10) + 3634.848 s_q(11)| \leq 0.03,
\]

as illustrated in figure 3.

The corresponding values for $\alpha$ are: 0.661, 0.515, 0.435, 0.384 and 0.261, respectively.

7. Conclusion

In this paper, we consider the reconstruction of the average von Neumann entropy in terms of average Rényi entropies of integer order, which, for some classes of states, are computed rather easily. Obtaining general estimates and relations for these quantities is a long-standing problem.

We have restricted our attention to shift-invariant quasi-free fermionic states on a regular lattice and obtained two results. Firstly, we proved that the mean von Neumann entropy is reconstructible in terms of mean Rényi entropies of integer order. Next, we set up an explicit approximation scheme. This scheme, including the controlled approximations (54), relies only on the validity of representation (37) and is therefore applicable to general systems for which (37) holds and is not only limited to the shift-invariant quasi-free states considered here. Of course, the log 2 asymptotic limit given in the controlled approximation scheme should be
adjusted according to the dimensionality of the states in question. It should be noted that the coefficients in (48) and (54) are not easily expressible in a general analytical form. Moreover, the obtained approximations are generally neither upper nor lower bounds.

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