Note on the Sum of Powers of Signless Laplacian Eigenvalues of Graphs

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Abstract

For a simple graph $G$ and a real number $\alpha$ ($\alpha \neq 0, 1$) the graph invariant $s_\alpha (G)$ is equal to the sum of powers of signless Laplacian eigenvalues of $G$. In this note, we present some new bounds on $s_\alpha (G)$. As a result of these bounds, we also give some results on incidence energy.

1 Introduction

Let $G$ be a simple graph with $n$ vertices and $m$ edges. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ be the set of vertices of $G$. For $v_i \in V(G)$, the degree of the vertex $v_i$, denoted by $d_i$, is equal to the number of vertices adjacent to $v_i$. Throughout this paper, the maximum, the second maximum and the minimum vertex degrees of $G$ will be denoted by $\Delta_1$, $\Delta_2$ and $\delta$, respectively.

Let $A(G)$ be the $(0,1)$-adjacency matrix of a graph $G$. The eigenvalues of $G$ are the eigenvalues of $A(G)$ [6] and denoted by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then the energy of a graph $G$ is defined by [17]

$$E = E(G) = \sum_{i=1}^{n} |\lambda_i| .$$

There is an extensive literature on this topic. For more details see [18, 28] and the references cited therein.

The concept of graph energy was extended to energy of any matrix in the following manner [36]. Recall that the singular values of any (real) matrix $M$ are equal to the square roots of the eigenvalues of $MM^T$, where $M^T$ is the transpose of $M$. Then
the energy of the matrix $M$ is defined as the sum of its singular values. Clearly, $E(A(G)) = E(G)$.

Let $D(G)$ be the diagonal matrix of vertex degrees of $G$. Then the Laplacian matrix of $G$ is $L(G) = D(G) - A(G)$ and the signless Laplacian matrix of $G$ is $Q(G) = D(G) + A(G)$. As well known in spectral graph theory, both $L(G)$ and $Q(G)$ are real symmetric and positive semidefinite matrices, so their eigenvalues are non-negative real numbers. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$ be the eigenvalues of $L(G)$ and let $q_1 \geq q_2 \geq \cdots \geq q_n$ be the eigenvalues of $Q(G)$. These eigenvalues are called Laplacian and signless Laplacian eigenvalues of $G$, respectively. For details on Laplacian and signless Laplacian eigenvalues, see [7–10, 33, 34].

The incidence matrix $I(G)$ of a graph $G$ with the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$ is the matrix whose $(i, j)$-entry is 1 if the vertex $v_i$ is incident with the edge $e_j$, and 0 otherwise. In [24], Jooyandeh et al. motivated the idea in [36] and defined the incidence energy of $G$, denoted by $IE(G)$, as the sum of singular values of $I(G)$. Since $Q(G) = I(G)I(G)^T$, it was later proved that [20]

$$IE = IE(G) = \sum_{i=1}^{n} \sqrt{q_i}.$$ 

For the basic properties of $IE$ involving also its lower and upper bounds, see [3,4,13,20,21,24,32,38,42,43].

In [30] Liu and Lu introduced a new graph invariant based on Laplacian eigenvalues

$$LEL = LEL(G) = \sum_{i=1}^{n-1} \sqrt{\mu_i}$$

and called it Laplacian energy like invariant. At first it was considered that [30] $LEL$ shares similar properties with Laplacian energy [22]. Then it was shown that it is much more similar to the ordinary graph energy [23]. For survey and details on $LEL$, see [29].

For a graph $G$ with $n$ vertices and a real number $\alpha$, to avoid trivialities it may be required that $\alpha \neq 0, 1$, the sum of the $\alpha$th powers of the non-zero Laplacian eigenvalues is defined as [41]

$$\sigma_\alpha = \sigma_\alpha(G) = \sum_{i=1}^{n-1} \mu_i^\alpha.$$ 

The cases $\alpha = 0$ and $\alpha = 1$ are trivial as $\sigma_0 = n - 1$ and $\sigma_1 = 2m$, where $m$ is the number of edges of $G$. Note that $\sigma_{1/2}$ is equal to $LEL$. It is worth noting that $n\sigma_{-1}$ is also equal to the Kirchhoff index of $G$ (one can refer to the papers [2,19,37] for its definition and extensive applications in the theory of electric circuits, probabilistic theory and chemistry). Recently, various properties and the estimates of $\sigma_\alpha$ have been well studied in the literature. For details, see [14,31,39,41,43].

Motivating the definitions of $IE$, $LEL$ and $\sigma_\alpha$, Akbari et al. [1] introduced the sum of the $\alpha$th powers of the signless Laplacian eigenvalues of $G$ as

$$s_\alpha = s_\alpha(G) = \sum_{i=1}^{n} q_i^\alpha.$$
and they also gave some relations between $\sigma_\alpha$ and $s_\alpha$. In this sum, the cases $\alpha = 0$ and $\alpha = 1$ are trivial as $s_0 = n$ and $s_1 = 2m$. Note that $s_{1/2}$ is equal to the incidence energy $IE$. Note further that Laplacian eigenvalues and signless Laplacian eigenvalues of bipartite graphs coincide \[7\] [33, 34]. Therefore, for bipartite graphs $\sigma_\alpha$ is equal to $s_\alpha$ [3] and $\text{LEL}$ is equal to $IE$ [20]. Recently some properties and the lower and upper bounds of $s_\alpha$ have been established in [1, 3, 27, 32].

In this paper, we obtain some new bounds on $s_\alpha$ of bipartite graphs which improve the some bounds in [14]. In addition to this, we extend these bounds to non-bipartite graphs. As a result of these bounds, we also present some results on incidence energy.

2 Lemmas

Let $t = t(G)$ denotes the number of spanning trees of $G$. Let $\overline{G}$ be the complement of $G$ and let $G_1 \times G_2$ be the Cartesian product of the graphs $G_1$ and $G_2$ [6]. Now, we give two auxiliary quantities for a graph $G$ as

$$t_1 = t_1(G) = \frac{2t(G \times K_2)}{t(G)} \quad \text{and} \quad T = T(G) = \frac{1}{2} \left[ \Delta_1 + \delta + \sqrt{(\Delta_1 - \delta)^2 + 4\Delta_1} \right]$$

where $\Delta_1$ and $\delta$ are the maximum and the minimum vertex degrees of $G$, respectively.

Lemma 2.1. \[25\] Let $G$ be a graph with $n$ vertices and $m$ edges. Then

$$\sum_{i=1}^{n} d_i^2 \leq m \left( \frac{2m}{n-1} + n - 2 \right)$$

Moreover, if $G$ is connected, then the equality holds in (2) if and only if $G$ is either a star $K_{1,n-1}$ or a complete graph $K_n$.

Lemma 2.2. \[7, 33, 34\] The spectra of $L(G)$ and $Q(G)$ coincide if and only if the graph $G$ is bipartite.

Lemma 2.3. \[9\] If $G$ is a connected bipartite graph of order $n$, then $\prod_{i=1}^{n-1} q_i = \prod_{i=1}^{n-1} \mu_i = nt(G)$. If $G$ is a connected non-bipartite graph of order $n$, then $\prod_{i=1}^{n} q_i = t_1(G)$.

Lemma 2.4. \[5, 33\] Let $G$ be a connected graph with $n \geq 3$ vertices and maximum vertex degree $\Delta_1$. Then

$$q_1 \geq T \geq \Delta_1 + 1$$

with either equalities if and only if $G$ is a star graph $K_{1,n-1}$.

Lemma 2.5. \[11\] Let $G$ be a graph with second maximum vertex degree $\Delta_2$. Then

$$q_2 \geq \Delta_2 - 1.$$
Lemma 2.6. [11] Let $G$ be a connected graph with $n$ vertices and minimum vertex degree $\delta$. Then
\[ q_n < \delta. \]

Lemma 2.7. [8] Let $G$ be a connected graph with diameter $d(G)$. If $G$ has exactly $k$ distinct signless Laplacian eigenvalues, then $d(G) + 1 \leq k$.

Lemma 2.8. [20] Let $G$ be a connected graph with $n \geq 3$ vertices and second maximum vertex degree $\Delta_2$. Then
\[ \mu_2 \geq \Delta_2 \]
with equality if $G$ is a complete bipartite graph $K_{p,q}$ or a tree with degree sequence $\pi(T_n) = (n/2, n/2, 1, 1, \ldots, 1)$, where $n \geq 4$ is even.

Lemma 2.9. [13] Let $G$ be a graph with $n$ vertices, different from $K_n$ and let $\delta$ be the minimum vertex degree of $G$. Then
\[ \mu_{n-1} \leq \delta \]

Lemma 2.10. [12, 41] Let $G$ be a simple graph with $n$ vertices. Then $\mu_1 = \mu_2 = \cdots = \mu_{n-1}$ if and only if $G \cong K_n$ or $G \cong \overline{K}_n$.

Lemma 2.11. [16] For $a_1, a_2, \ldots, a_n \geq 0$ and $p_1, p_2, \ldots, p_n \geq 0$ such that $\sum_{i=1}^n p_i = 1$
\[ \sum_{i=1}^n p_i a_i - \prod_{i=1}^n a_i^{p_i} \geq n \lambda \left( \frac{1}{n} \sum_{i=1}^n a_i - \prod_{i=1}^n a_i^{1/n} \right) \]
(3)
where $\lambda = \min \{p_1, p_2, \ldots, p_n\}$. Moreover, equality holds in (3) if and only if $a_1 = a_2 = \cdots = a_n$.

Lemma 2.12. [35] Let $a_i > 0$, $i = 1, 2, \ldots, p$ be the $p$ real numbers. Then
\[ p (A_p - G_p) \geq (p - 1) (A_{p-1} - G_{p-1}), \]
where
\[ A_p = \frac{\sum_{i=1}^p a_i}{p} \text{ and } G_p = \left( \prod_{i=1}^p a_i \right)^{1/p}. \]

3 Main Results

In this section, we give the main results of the paper. First, we need the following lemma. For a graph $G$ with signless Laplacian eigenvalues $q_1 \geq q_2 \geq \cdots \geq q_n$, let
\[ M_k = M_k(G) = \sum_{i=1}^k q_i \]
for $1 \leq k \leq n - 1$. Then, we have:
Lemma 3.1. Let $G$ be a connected graph with $n$ vertices and $m$ edges.

i) If $G$ is bipartite, then for $1 \leq k \leq n-2$

$$M_k(G) \leq \frac{2mk + \sqrt{mk(n-k-1)(n^2-n-2m)}}{n-1}$$

with equality holding in (4) if and only if $G$ is either a star $K_{1,n-1}$ or a complete graph $K_n$ when $k = 1$ and $G$ is a complete graph $K_n$ when $2 \leq k \leq n-2$.

ii) If $G$ is non-bipartite, then for $1 \leq k \leq n-1$

$$M_k(G) \leq \frac{2mk + \sqrt{mk(n-k)(n^2+2mn/n-1-4m)}}{n}$$

with equality holding in (5) if and only if $G \cong K_n$ when $k = 1$.

Proof. The inequality (4) was established in [40]. So we omit its proof here. Now we only prove the inequality (5). Let $M_k = M_k(G)$. It is clear that

$$q_1 + q_2 + \cdots + q_n = 2m$$

and

$$q_1^2 + q_2^2 + \cdots + q_n^2 = 2m + \sum_{i=1}^{n} d_i^2$$

Then, using Cauchy-Schwarz inequality, we get

$$(2m - M_k)^2 = (q_{k+1} + \cdots + q_n)^2$$

$$\leq (n-k)(q_{k+1}^2 + \cdots + q_n^2)$$

$$= (n-k) \left( 2m + \sum_{i=1}^{n} d_i^2 - (q_1^2 + \cdots + q_k^2) \right)$$

$$\leq (n-k) \left( 2m + \sum_{i=1}^{n} d_i^2 - \frac{1}{k}M_k^2 \right).$$

Therefore

$$M_k \leq \left\{ 2mk + \left[ k(n-k) \left( n \left( 2m + \sum_{i=1}^{n} d_i^2 \right) - 4m^2 \right) \right]^{1/2} \right\} / n.$$ (6)

From the inequality (6) and Lemma 2.1, the inequality (5) holds. Now we suppose that the equality holds in (5). Then, by Cauchy-Schwarz inequality we have $q_1 = \cdots = q_k$ and $q_{k+1} = \cdots = q_n$. Since $G$ is connected non-bipartite graph, by Lemma 2.1 and Lemma 2.7, we conclude that $G \cong K_n$ when $k = 1$. \hfill $\Box$

The following result can be found in [14].

Theorem 3.2. [14] Let $G$ be a bipartite graph with $n \geq 2$ vertices, $m$ edges and positive integer $k$ ($1 \leq k \leq n-2$).

(i) If $0 < \alpha < 1$, then

$$s_\alpha (G) = \sigma_\alpha (G) \leq k^{1-\alpha} \left( \frac{2mk}{n-1} \right)^{\alpha} + (n-k-1)^{1-\alpha} \left( \frac{2m - 2mk}{n-1} \right)^{\alpha}$$

(7)
with equality holding in (7) if and only if $G \cong K_n$ or $G \cong \overline{K}_n$.

(ii) If $\alpha > 1$, then
\[
s_\alpha(G) = \sigma_\alpha(G) \geq k^{1-\alpha} \left( 2mk \right)^{\alpha} + (n-k-1)^{1-\alpha} \left( 2m - \frac{2mk}{n-1} \right)^{\alpha}
\]
with equality holding in (8) if and only if $G \cong K_n$ or $G \cong \overline{K}_n$.

(iii) If $G$ is connected and $\alpha < 0$, then
\[
s_\alpha(G) = \sigma_\alpha(G) \leq \min_{1 \leq k \leq n-1} \left\{ k^{1-\alpha} \left[ \frac{2mk + \sqrt{mk(n-k-1)(n^2 - 2m)}}{n-1} \right]^\alpha 
+ (n-k-1)^{1-\alpha} \left[ \frac{2m(n-k-1) - \sqrt{mk(n-k-1)(n^2 - 2m)}}{n-1} \right]^\alpha \right\}
\]
with equality holding in (9) if and only if $G \cong K_{1,n-1}$ ($k = 1$) and $G \cong K_n$ ($2 \leq k \leq n-2$).

We now extend the above result to non-bipartite graphs.

**Theorem 3.3.** Let $G$ be a non-bipartite graph with $n \geq 2$ vertices, $m$ edges and positive integer $k$ ($1 \leq k \leq n-1$).

(i) If $0 < \alpha < 1$, then
\[
s_\alpha(G) \leq k^{1-\alpha} \left( \frac{2mk}{n} \right)^{\alpha} + (n-k)^{1-\alpha} \left( 2m - \frac{2mk}{n} \right)^{\alpha}
\]
with equality holding in (10) if and only if $G \cong \overline{K}_n$.

(ii) If $\alpha > 1$, then
\[
s_\alpha(G) \geq k^{1-\alpha} \left( \frac{2mk}{n} \right)^{\alpha} + (n-k)^{1-\alpha} \left( 2m - \frac{2mk}{n} \right)^{\alpha}
\]
with equality holding in (11) if and only if $G \cong \overline{K}_n$.

(iii) If $G$ is connected and $\alpha < 0$, then
\[
s_\alpha(G) \leq \min_{1 \leq k \leq n-1} \left\{ k^{1-\alpha} \left[ \frac{2mk + \sqrt{mk(n-k)(n^2 - 2m)}}{n} \right]^\alpha 
+ (n-k)^{1-\alpha} \left[ \frac{2m(n-k) - \sqrt{mk(n-k)(n^2 - 2m)}}{n} \right]^\alpha \right\}
\]
with equality holding in (12) if and only if $G \cong K_n$ when $k = 1$.

**Proof.** Using power mean inequality, we get
\[
\sum_{i=1}^{k} q_i^\alpha \leq k^{1-\alpha} \left( \sum_{i=1}^{k} q_i \right)^{\alpha}, \text{ as } 0 < \alpha < 1
\]
with equality holding in (13) if and only if $q_1 = q_2 = \cdots = q_k$. 


Considering the above manner, we also get
\[
\sum_{i=k+1}^{n} q_i^\alpha \leq (n-k)^{1-\alpha} \left( 2m - \sum_{i=1}^{k} q_i \right)^\alpha, \text{ as } \sum_{i=1}^{n} q_i = 2m \tag{14}
\]
with equality holding in (14) if and only if \( q_{k+1} = q_{k+2} = \cdots = q_n \). Since \( q_1 \geq q_2 \geq \cdots \geq q_n \), we have
\[
\frac{\sum_{i=1}^{k} q_i}{k} \geq \frac{\sum_{i=k+1}^{n} q_i}{n-k} = \frac{2m - \sum_{i=1}^{k} q_i}{n-k}.
\]
Therefore, we get
\[
\sum_{i=1}^{k} q_i \geq \frac{2mk}{n}. \tag{15}
\]
By Eqs. (13) and (14), we obtain
\[
s_\alpha (G) = \sum_{i=1}^{n} q_i^\alpha = \sum_{i=1}^{k} q_i^\alpha + \sum_{i=k+1}^{n} q_i^\alpha \leq k^{1-\alpha} \left( \sum_{i=1}^{k} q_i \right)^\alpha + (n-k)^{1-\alpha} \left( 2m - \sum_{i=1}^{k} q_i \right)^\alpha.
\]
Now consider the following function
\[
f (x) = k^{1-\alpha}x^\alpha + (n-k)^{1-\alpha} (2m-x)^\alpha
\]
for \( x \geq \frac{2mk}{n} \). Then it is easy to see that
\[
f' (x) = \alpha \left[ \left( \frac{x}{k} \right)^{\alpha-1} - \left( \frac{2m-x}{n-k} \right)^{\alpha-1} \right] \leq 0, \text{ as } 0 < \alpha < 1.
\]
Thus, by (15), we get
\[
f (x) \leq f \left( \frac{2mk}{n} \right) = k^{1-\alpha} \left( \frac{2mk}{n} \right)^\alpha + (n-k)^{1-\alpha} \left( 2m - \frac{2mk}{n} \right)^\alpha.
\]
Hence we get the the inequality (10). Now we suppose that the equality holds in (10). Then, from (13) and (14) we have \( q_1 = q_2 = \cdots = q_k \) and \( q_{k+1} = q_{k+2} = \cdots = q_n \), respectively. Furthermore from (15), we have
\[
\sum_{i=1}^{k} q_i = \frac{2mk}{n}.
\]
Therefore
\[
q_1 = q_2 = \cdots = q_n = \frac{2m}{n}.
\]
Then, we conclude that \( G \cong \overline{K}_n \).
Conversely, one can easily show that the equality holds in (10) for the complement of the complete graph $\overline{K}_n$.

(ii) Using power mean inequality, from (i), we obtain

$$s_\alpha (G) \geq k^{1-\alpha} \left( \sum_{i=1}^{k} q_i \right)^{\alpha} + (n-k)^{1-\alpha} \left( 2m - \sum_{i=1}^{k} q_i \right)^{\alpha}, \text{ as } \alpha > 1.$$ 

Note that $f(x)$ is increasing function for $x \geq \frac{2mk}{n}$ as $\alpha > 1$. Then, similar to the proof of (i), we get the inequality (11). Furthermore, the equality holds in (11) if and only if $G \cong \overline{K}_n$.

(iii) From Lemma 3.1, we have

$$\sum_{i=1}^{k} q_i \leq \frac{2mk + \sqrt{mk (n-k) \left( n^2 + \frac{2mn}{n-1} - 4m \right)}}{n}.$$ 

As $\alpha < 0$, from (i), we obtain that $f(x)$ is increasing function for

$$\frac{2mk}{n} \leq x \leq \frac{1}{n} \left[ 2mk + \sqrt{mk (n-k) \left( n^2 + \frac{2mn}{n-1} - 4m \right)} \right].$$ 

Therefore

$$f(x) \leq k^{1-\alpha} \left( \frac{2mk + \sqrt{mk (n-k) \left( n^2 + \frac{2mn}{n-1} - 4m \right)}}{n} \right)^{\alpha} + (n-k)^{1-\alpha} \times \left( \frac{2m (n-k) - \sqrt{mk (n-k) \left( n^2 + \frac{2mn}{n-1} - 4m \right)}}{n} \right)^{\alpha}.$$ 

Hence the inequality (12) holds. Now we suppose that the equality holds in (12). Therefore we get that

$q_1 = q_2 = \cdots = q_k, q_{k+1} = q_{k+2} = \cdots = q_n$

and

$$\sum_{i=1}^{k} q_i = \frac{2mk + \sqrt{mk (n-k) \left( n^2 + \frac{2mn}{n-1} - 4m \right)}}{n}.$$ 

Then, from Lemma 3.1, we conclude that $G \cong K_n$ when $k = 1$.

Conversely, let $G$ be isomorphic to the complete graph $K_n$ when $k = 1$. Thus

$$k^{1-\alpha} \left[ \frac{2mk + \sqrt{mk (n-k) \left( n^2 + \frac{2mn}{n-1} - 4m \right)}}{n} \right]^{\alpha} + (n-k)^{1-\alpha} \times \left( \frac{2m (n-k) - \sqrt{mk (n-k) \left( n^2 + \frac{2mn}{n-1} - 4m \right)}}{n} \right)^{\alpha} = \left( 2(n-1) \right)^{\alpha} + (n-1)(n-2)^{\alpha}, \text{ as } k = 1, m = n(n-1)/2$$ 

$$= s_\alpha (G), \text{ since } q_1 = 2(n-1), q_2 = \cdots = q_n = n-2.$$
This completes the proof of theorem.

**Theorem 3.4.** Let $\alpha$ be a real number with $\alpha \neq 0, 1$ and let $G$ be a connected graph with $n \geq 3$ vertices and $t$ spanning trees and also let $t_1$ and $T$ be given by (1). For any real number $k \geq 0$,

i) if $G$ is bipartite, then

$$s_\alpha (G) = \sigma_\alpha (G) > (n - 2) (nt)^{\alpha/(n-1)} \left[ \frac{(k + 1)(nt)^{\alpha/[(k+1)(n-1)n]} - k}{T^{\alpha/[(k+1)(n-1)]}} \right] + T^\alpha. \quad (16)$$

ii) If $G$ is non-bipartite, then

$$s_\alpha (G) > (n - 1) (t_1)^{\alpha/n} \left[ \frac{(k + 1)(t_1)^{\alpha/[(k+1)n(n-1)]} - k}{T^{\alpha/[(k+1)n(n-1)]}} \right] + T^\alpha. \quad (17)$$

**Proof.** By Lemmas 2.2–2.4, 2.10 and 2.11, the inequality (16) can be proved using similar method of Theorem 3.4 in [14]. We now only prove the inequality (17).

Setting in Lemma 2.11 $a_i = q_{i1}^\alpha$, $i = 1, 2, \ldots, n$ and $p_1 = \frac{k}{(k+1)n}$, $p_i = \frac{(k+1)n - k}{(k+1)n(n-1)}$, $i = 2, 3, \ldots, n$

we obtain

$$kq_1^\alpha \left( \frac{k}{(k+1)n} \right) + \frac{(k+1)n - k}{(k+1)n(n-1)} \sum_{i=2}^n q_i^\alpha - q_1 \prod_{i=2}^n \frac{(k+1)n - k}{(k+1)n(n-1)} q_i^\alpha$$

$$\geq \frac{k}{(k+1)n} \sum_{i=1}^n q_i^\alpha - \frac{k}{k + 1} \prod_{i=1}^n q_i^{\alpha/n}.$$ 

Then, by Lemma 2.3, we have

$$q_1 \left( \frac{k}{(k+1)n} \right) + \frac{(k+1)n - k}{(k+1)n(n-1)} (s_\alpha (G) - q_1^\alpha) - q_1 \prod_{i=2}^n \frac{(k+1)n - k}{(k+1)n(n-1)} q_i^\alpha$$

$$\geq \frac{k}{(k+1)n} s_\alpha (G) - \frac{k}{k + 1} (t_1)^{\alpha/n},$$

i.e.,

$$s_\alpha (G) \geq (n - 1) \left[ \frac{(k + 1)(t_1)^{\alpha/[(k+1)n(n-1)]}}{q_1^{\alpha/[(k+1)n(n-1)]}} + \frac{q_1^\alpha}{n - 1} - k (t_1)^{\alpha/n} \right]. \quad (18)$$

Let us consider the auxiliary function

$$f(x) = \frac{(k + 1)(t_1)^{\alpha/[(k+1)n(n-1)]}}{x^{\alpha/[(k+1)n(n-1)]}} + \frac{x^\alpha}{n - 1}.$$ 

It is easy to see that $f(x)$ is increasing for $x > (t_1)^{1/n}$ whether $\alpha > 0$ or $\alpha < 0$. By Lemmas 2.3, 2.4 and Theorem 3.3 in [4], we have

$$q_1 \geq T \geq \Delta_1 + 1 > \Delta_1 \geq \frac{2m}{n} \geq (t_1)^{1/n}$$

where $\Delta_1$ is the degree sequence of graph $G$.
Therefore
\[ f(x) \geq f(T) = \frac{(k+1)\left(t_1\right)^{\frac{k(1+k)n-k}{k+1(n-k)}}}{T^{\frac{1}{k+1(n-k-1)}}} + \frac{T^\alpha}{n-1}. \]

Combining this with (18) we get the inequality (17). Now we assume that the equality holds in (17). Then all inequalities in the above arguments must be equalities. Thus \( q_1 = T \) and \( q_1 = q_2 = \cdots = q_n = \frac{2m}{n} \). Thus we have that \( q_1 = \frac{2m}{n} \leq \Delta_1 < \Delta_1 + 1 \leq T \) which contradicts with the result in Lemma 2.4 \[4\]. Hence (17) cannot become an equality.

\[ \square \]

**Remark 3.5.** By Lemmas 2.2 and 2.4, we have that \( \mu_1 = q_1 \geq T \geq \Delta_1 + 1 \) for bipartite graphs. Then from the proof of Theorem 3.4 in \[14\], one can arrive at the bound (16) improves the bound of Theorem 3.4 in \[14\] for bipartite graphs.

Taking \( k = 1 \) in Theorem 3.4, we have the following result.

**Corollary 3.6.** Let \( \alpha \) be a real number with \( \alpha \neq 0, 1 \) and let \( G \) be a connected graph with \( n \geq 3 \) vertices and \( t \) spanning trees and also let \( t_1 \) and \( T \) be given by (1).

i) if \( G \) is bipartite, then
\[ s_\alpha(G) = \sigma_\alpha(G) > (n-2)\left(\frac{2(n-1)}{T^{\frac{1}{2}(n-1)}}\right) + T^\alpha. \]

ii) If \( G \) is non-bipartite, then
\[ s_\alpha(G) > (n-1)\left(\frac{2(t_1)^{\frac{1}{2(n-1)}}}{T^{\frac{1}{2(n-1)}}}\right) + T^\alpha. \]

As in Remark 3.5, one can easily conclude that the bound (19) of Corollary 3.6 improves Corollary 3.5 in \[14\]. Moreover, taking \( \alpha = 1/2 \) in Corollary 3.6, we have the following result.

**Corollary 3.7.** \[4\] Let \( G \) be a connected graph with \( n \geq 3 \) vertices and \( t \) spanning trees and also let \( t_1 \) and \( T \) be given by (1).

i) if \( G \) is bipartite, then
\[ IE(G) = LEL(G) > \sqrt{T} + (n-2)\left(\frac{2(n-1)}{T^{1/4(n-1)}}\right) - 1. \]

ii) if \( G \) is non-bipartite, then
\[ IE(G) > \sqrt{T} + (n-1)\left(\frac{2(t_1)^{1/2(n-1)}}{T^{1/4(n-1)}}\right) - 1. \]

**Theorem 3.8.** Let \( \alpha \) be a real number with \( \alpha \neq 0, 1 \) and let \( G \) be a connected graph with \( n \geq 3 \) vertices and \( t \) spanning trees and also \( t_1 \) and \( T \) be given by (1).
i) if $G$ is bipartite, then
\[ s_\alpha(G) = \sigma_\alpha(G) \geq T^\alpha + (n-2) \left( \frac{nt}{T} \right)^{\alpha/(n-2)} + \left( \frac{\Delta_2}{2} - \frac{\delta}{2} \right)^2. \] (23)

ii) if $G$ is non-bipartite, then
\[ s_\alpha(G) > T^\alpha + (n-1) \left( \frac{t_1}{T} \right)^{\alpha/(n-1)} + \left( (\Delta_2 - 1)^{\alpha/2} - \frac{\delta}{2} \right)^2. \] (24)

where $\Delta_2$ and $\delta$ are the second maximum and the minimum vertex degrees of the graph $G$, respectively.

Proof. Using Lemmas 2.2–2.4, 2.8, 2.9 and 2.12, one can prove inequality (23) similar to the proof of Theorem 3.9 in [14]. Here we only prove the inequality (24).

By Lemma 2.12, we have
\[ p(A_p - G_p) \geq (p-1)(A_{p-1} - G_{p-1}) \geq \cdots \geq 2(A_2 - G_2) \]
i.e.,
\[ A_p \geq G_p + \frac{2}{p} \left( \frac{a_1 + a_2}{2} - \sqrt{a_1 a_2} \right) = G_p + \frac{1}{p} (\sqrt{a_1} - \sqrt{a_2})^2 \] (25)
see, [14]. Setting $p = n-1$, $(a_1, a_2, \ldots, a_{n-1}) = (q_2^\alpha, q_3^\alpha, \ldots, q_n^\alpha)$ and $a_1 = q_2^\alpha, a_2 = q_n^\alpha$ in (25), we obtain
\[ s_\alpha(G) = \sum_{i=1}^{n} q_i^\alpha \geq q_1^\alpha + (n-1) \left( \prod_{i=2}^{n} q_i \right)^{\alpha/(n-1)} + \left( q_2^{\alpha/2} - q_n^{\alpha/2} \right)^2. \]

Considering Lemmas 2.3, 2.5 and 2.6, we have
\[ s_\alpha(G) = \sum_{i=1}^{n} q_i^\alpha \geq q_1^\alpha + (n-1) \left( \frac{t_1}{q_1} \right)^{\alpha/(n-1)} + \left( (\Delta_2 - 1)^{\alpha/2} - \frac{\delta}{2} \right)^2. \] (26)

Let us consider the auxiliary function
\[ f(x) = x^\alpha + (n-1) \left( \frac{t_1}{x} \right)^{\alpha/(n-1)} \].

Note that $f(x)$ is increasing for $x > (t_1)^{1/n}$ for both $\alpha > 0$ and $\alpha < 0$ [3]. Then by Lemmas 2.3, 2.4 and Theorem 4.9 in [3], we have
\[ f(x) \geq f(T) = T^\alpha + (n-1) \left( \frac{t_1}{T} \right)^{\alpha/(n-1)}. \]

Combining this with Eq. (26), we get the inequality (24).

Remark 3.9. By Lemmas 2.2 and 2.4, we have that $\mu_1 = q_1 \geq T \geq \Delta_1 + 1$ for bipartite graphs. Then, from the proof of Theorem 3.9 in [14], one can arrive at the bound (23) improves the bound of Theorem 3.9 in [14] for bipartite graphs. Moreover, it is clear that the results of Theorem 3.8 are better than the results of Theorem 4.9 in [3].
Taking $\alpha = 1/2$ in Theorem 3.8, we get the following result on $IE$.

**Corollary 3.10.** Let $G$ be a connected graph with $n \geq 3$ vertices and $t$ spanning trees and also let $t_1$ and $T$ be given by (1).

i) if $G$ is bipartite, then

$$IE(G) = LEL(G) \geq \sqrt{T} + (n - 2) \left(\frac{nt}{T}\right)^{1/2(n-2)} + \left(\Delta_2^{1/4} - \delta^{1/4}\right)^2. \quad (27)$$

ii) if $G$ is non-bipartite, then

$$IE(G) > \sqrt{T} + (n - 1) \left(\frac{t_1}{T}\right)^{1/2(n-1)} + \left((\Delta_2 - 1)^{1/4} - \delta^{1/4}\right)^2. \quad (28)$$

where $\Delta_2$ and $\delta$ are the second maximum and the minimum vertex degrees of the graph $G$, respectively.

**Remark 3.11.** It is clear that the results of Corollary 3.10 improve the results of Theorem 4.8 in [3].

**Remark 3.12.** We finally note that, if we can establish a new lower bound such that $q_1 \geq \beta \geq T$, then we can improve the results in Theorems 3.4 and 3.8.

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