Abstract. The generalized quantum group $\mathcal{U}(\epsilon)$ of type $A$ is an affine analogue of quantum group associated to a general linear Lie superalgebra $\mathfrak{gl}_{M|N}$. We prove that there exists a unique $R$ matrix on tensor product of fundamental type representations of $\mathcal{U}(\epsilon)$ for arbitrary parameter sequence $\epsilon$ corresponding to a non-conjugate Borel subalgebra of $\mathfrak{gl}_{M|N}$. We give an explicit description of its spectral decomposition, and then as an application, construct a family of finite-dimensional irreducible $\mathcal{U}(\epsilon)$-modules which have subspaces isomorphic to the Kirillov-Reshetikhin modules of usual affine type $A_{M-1}^{(1)}$ or $A_{N-1}^{(1)}$.

Contents

1. Introduction 1
2. Generalized quantum group $\mathcal{U}(\epsilon)$ of type $A$ 3
3. Schur-Weyl duality and polynomial representations of $\mathcal{U}(\epsilon)$ 8
4. $R$ matrix for finite-dimensional $\mathcal{U}(\epsilon)$-modules 14
5. Kirillov-Reshetikhin modules 22
References 26

1. Introduction

A generalized quantum group $\mathcal{U}(\epsilon)$ associated to $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ with $\epsilon_i \in \{0, 1\}$ is a Hopf algebra introduced in [17], which appears in the study of solutions to the tetrahedron equation or the three-dimensional Yang-Baxter equation.

The generalized quantum group $\mathcal{U}(\epsilon)$ of type $A$ is equal to the usual quantum affine algebra of type $A_{n-1}^{(1)}$, when $\epsilon$ is homogeneous, that is, $\epsilon_i = \epsilon_j$ for all $i \neq j$. But it becomes a more interesting object when $\epsilon$ is non-homogeneous, which is closely related to the quantized enveloping algebra associated to an affine Lie superalgebra [22], or which can be viewed as an affine analogue of the quantized enveloping algebra of the general linear Lie superalgebra $\mathfrak{gl}_{M|N}$ [21], where $M$ and $N$ are the numbers of 0 and 1 in $\epsilon$, respectively. We remark that the subalgebra $\hat{\mathcal{U}}(\epsilon)$ of $\mathcal{U}(\epsilon)$ associated to the Lie superalgebra $\mathfrak{gl}_{M|N}$ was also introduced in [21].

1991 Mathematics Subject Classification. 17B37,17B10.

Key words and phrases. quantum group, crystal base, Lie superalgebra.

J.-H.K. was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIT) (No. 2019R1A2C108483311).
in [17] independently, as symmetries appearing in the study of wave functions of quantum mechanical systems [23].

When the parameter $\epsilon$ is standard, that is, $\epsilon_{M|N} = (0^M,1^N)$, it is shown in [17] that there exists a unique $R$ matrix on the tensor product of finite-dimensional $\mathcal{U}(\epsilon_{M|N})$-modules $W_{s,\epsilon}(x)$, which correspond to fundamental representations of type $A_{N-1}^{(1)}$ with spectral parameter $x$ when $N \geq 3$. Indeed, the $R$ matrix is obtained by reducing the solution of the tetrahedron equation, and the uniqueness follows from the irreducibility of tensor product $W_{l,\epsilon}(x) \otimes W_{m,\epsilon}(y)$ for generic $x$ and $y$. An explicit spectral decomposition of the associated $R$ matrix is obtained by analyzing the maximal vectors with respect to $\mathcal{U}(\epsilon_{M|N})$.

By applying fusion construction using the $R$ matrix in [17], a family of irreducible $\mathcal{U}(\epsilon_{M|N})$-modules is constructed in [16], which are parametrized by rectangular partitions inside an $(M|N)$-hook. Moreover the existence of their crystal base is proved together with a combinatorial description of the associated crystal graphs. It can be viewed as a natural super-analogue of Kirillov-Reshetikhin modules (simply KR modules) of type $A^{(1)}_1$, which is a most important family of finite-dimensional irreducible modules of quantum affine algebras (cf. [3] [13]).

The results in [17] and [16] suggests that there is a close connection between finite-dimensional representations of $\mathcal{U}(\epsilon_{M|N})$ and $U_q(A^{(1)}_{n-1})$. The purpose of this paper is to extend the results in [17] and [16] to arbitrary parameter sequence $\epsilon$, and find a more concrete connection between the finite-dimensional representations of $\mathcal{U}(\epsilon)$ and $U_q(A^{(1)}_1)$. From a viewpoint of representations of $\mathfrak{g}_{M|N}$, the sequence $\epsilon$ represents the type of Borel subalgebras of $\mathfrak{g}_{M|N}$, which are not conjugate to each other. It is not obvious whether the representation theory of $\mathcal{U}(\epsilon)$ is the same under a different choice of permutations of $\epsilon_{M|N}$. For example, if we change the Borel in the generalized quantum group, then the defining relations and the crystal structure associated to $\mathcal{U}(\epsilon)$-modules become much different from the ones with respect to $\epsilon_{M|N}$ as $\epsilon$ gets far from $\epsilon_{M|N}$ (cf. [2] [15]).

We first show that there exists a unique $R$ matrix on the tensor product of finite-dimensional $\mathcal{U}(\epsilon)$-modules $W_{s,\epsilon}(x)$ of fundamental type (Theorem 4.10). Since the existence of $R$ matrix for arbitrary $\epsilon$ was shown in [17], it suffices to prove the irreducibility of tensor product $W_{l,\epsilon}(x) \otimes W_{m,\epsilon}(y)$ for generic $x$ and $y$. We use a method completely different from [17]. Indeed, motivated by the work [6], we introduce a functor called truncation, and show that it sends any $\mathcal{U}(\epsilon)$-module with polynomial weights to a $\mathcal{U}(\epsilon')$-module, preserving the comultiplications in tensor product, where $\epsilon'$ is a subsequence of $\epsilon$. This in particular enables us to define an oriented graph structure on $W_{l,\epsilon}(x) \otimes W_{m,\epsilon}(y)$ when $x = y = 1$ with additional arrows other than the ones associated to $\mathcal{U}(\epsilon)$. With this structure, we prove the connectedness of the crystal (Theorem 4.18), and hence the irreducibility for generic $x$ and $y$.

Next, we prove that the truncation functor is compatible with the $R$ matrix. This immediately implies that the spectral decomposition of the $R$ matrix for $\mathcal{U}(\epsilon)$ is the same as that of type $A^{(1)}_1$ (Theorem 5.2) and hence does not depend on the choice of $\epsilon$. As an application, we construct a family of irreducible $\mathcal{U}(\epsilon)$-modules $W_{s,\epsilon}^{(r)}$ which yields the usual KR modules.
under truncation (Theorem 5.3). We conjecture that \( W_{\epsilon, \ell}^{(r)} \) has a crystal base as in the case of \( \epsilon = \epsilon_{M|N} \). We expect that the compatibility of truncation with the \( R \) matrix will also play a crucial role in understanding arbitrary finite-dimensional \( \mathcal{U}(\epsilon) \)-modules in connection with those of type \( A_{\ell}^{(1)} \).

There are other recent works on the finite-dimensional representations of quantum affine superalgebra associated to \( \mathfrak{gl}_{M|N} \) [24, 25, 26]. It would be interesting to compare with these results.

The paper is organized as follows. In Section 2 we review basic materials for a generalized quantum group and its crystal base. In Section 3 we present the classical Schur-Weyl duality for \( \mathcal{U}(\epsilon) \) and then realize the irreducible polynomial representation of \( \mathcal{U}(\epsilon) \). In Section 4 we prove the main theorem on the existence of the \( R \) matrix. In Section 5 we construct KR type modules of \( \mathcal{U}(\epsilon) \) using the \( R \) matrix.

Acknowledgement The authors would like to thank Euiyong Park and Masato Okado for helpful discussions and thank Shin-Myung Lee for careful reading of the manuscript and comments.

2. GENERALIZED QUANTUM GROUP \( \mathcal{U}(\epsilon) \) OF TYPE A

2.1. Generalized quantum group. We fix a positive integer \( n \geq 4 \). Let \( \epsilon = (\epsilon_1, \cdots, \epsilon_n) \) be a sequence with \( \epsilon_i \in \{0, 1\} \) for \( 1 \leq i \leq n \). We denote by \( \mathbb{I} \) the linearly ordered set \( \{1 < 2 < \cdots < n\} \) with \( \mathbb{Z}_2 \)-grading given by \( I_0 = \{ i \mid \epsilon_i = 0 \} \) and \( I_1 = \{ i \mid \epsilon_i = 1 \} \). We assume that \( M \) is the number of \( i \) with \( \epsilon_i = 0 \) and \( N \) is the number of \( i \) with \( \epsilon_i = 1 \) in \( \epsilon \). We denote by \( \epsilon_{M|N} \) the sequence when \( \epsilon_1 = \cdots = \epsilon_M = 0 \) and \( \epsilon_{M+1} = \cdots = \epsilon_n = 1 \).

Let \( P = \bigoplus_{i \in \mathbb{I}} \mathbb{Z} \delta_i \) be the free abelian group generated by \( \delta_i \) with a symmetric bilinear form \( \langle \cdot | \cdot \rangle \) given by \( \langle \delta_i | \delta_j \rangle = (-1)^{\epsilon_i \epsilon_j} \delta_{ij} \) for \( i, j \in \mathbb{I} \). Let \( \{ \delta_i^\vee | i \in \mathbb{I} \} \subset P^\vee := \text{Hom}_\mathbb{Z}(P, \mathbb{Z}) \) be the dual basis such that \( \langle \delta_i, \delta_j^\vee \rangle = \delta_{ij} \) for \( i, j \in \mathbb{I} \).

Let \( I = \{ 0, 1, \ldots, n-1 \} \) and

\[
\alpha_i = \delta_i - \delta_{i+1}, \quad \alpha_i^\vee = \delta_i^\vee - (-1)^{\epsilon_i+\epsilon_{i+1}} \delta_{i+1}^\vee \quad (i \in I).
\]

Throughout the paper, we understand the subscript \( i \in I \) modulo \( n \). When \( \epsilon = \epsilon_{M|N} \), the Dynkin diagram associated to the Cartan matrix \( (\langle \alpha_j, \alpha_i^\vee \rangle)_{0 \leq i, j \leq n} \) is

![Dynkin diagram](image)

where \( \otimes \) denotes an isotropic simple root.
Let $q$ be an indeterminate. We put $I_{\text{even}} = \{ i \in I \mid (\alpha_i | \alpha_i) = \pm 2 \}$ and $I_{\text{odd}} = \{ i \in I \mid (\alpha_i | \alpha_i) = 0 \}$, and set

$$ q_i = (-1)^{\epsilon_i} q^{-\epsilon_i} = \begin{cases} q & \text{if } \epsilon_i = 0, \\ q^{-1} & \text{if } \epsilon_i = 1, \end{cases} \quad (i \in I). $$

**Definition 2.1.** We define $\mathcal{U}(\epsilon)$ to be the associative $\mathbb{Q}(q)$-algebra with 1 generated by $q^h, e_i, f_i$ for $h \in P^\vee$ and $i \in I$ satisfying

\begin{align*}
(2.1) & \quad q^0 = 1, \quad q^{h+h'} = q^h q^{h'} \quad (h, h' \in P^\vee), \\
(2.2) & \quad \omega_j e_i \omega_j^{-1} = q^{(\alpha_i, \delta_j)} e_i, \quad \omega_j f_i \omega_j^{-1} = q^{-(\alpha_i, \delta_j)} f_i, \\
(2.3) & \quad e_i f_j - f_j e_i = \delta_{ij} \frac{\omega_i \omega_{i+1} - \omega_i^{-1} \omega_{i+1}}{q - q^{-1}}, \\
(2.4) & \quad e_i^2 = f_i^2 = 0 \quad (i \in I_{\text{odd}}),
\end{align*}

where $\omega_j = q^{(-1)^{\epsilon_j} \delta_j^\vee} \quad (j \in I)$, and the Serre-type relations

\begin{align*}
(2.5) & \quad e_i e_j - e_j e_i = f_i f_j - f_j f_i = 0, \quad (|i - j| > 1), \\
& \quad e_i^2 e_j - (-1)^{\epsilon_i} [2] e_i e_j e_i + e_j e_i^2 = 0, \quad e_i f_j - f_j e_i = \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}}, \\
& \quad f_i^2 f_j - (-1)^{\epsilon_i} [2] f_i f_j f_i + f_j f_i^2 = 0, \quad (i \in I_{\text{even}} \text{ and } |i - j| = 1),
\end{align*}

and

\begin{align*}
(2.6) & \quad e_i e_{i+1} e_i + e_i e_i e_{i+1} e_i - e_i e_i e_{i+1} e_i + e_i e_{i+1} e_i = 0, \quad (i \in I_{\text{odd}}), \\
& \quad f_i f_{i-1} f_i f_{i+1} - f_i f_{i-1} f_i f_{i+1} + f_{i+1} f_i f_{i-1} f_i - f_{i-1} f_i f_{i+1} f_i = 0.
\end{align*}

We call $\mathcal{U}(\epsilon)$ the generalized quantum group of affine type $A$ associated to $\epsilon$ (see [17]).

Put $k_i = \omega_i \omega_{i+1}^{-1}$ for $i \in I$. Then we have for $i, j \in I$

$$ k_i e_j k_i^{-1} = D_{ij} e_j, \quad k_i f_j k_i^{-1} = D_{ij}^{-1} f_j, \quad e_i f_j - f_j e_i = \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}}, $$

where $D_{ij} = q^{(\alpha_j, \delta_i^\vee)} q_i^{-1} q_{i+1}^{-(\alpha_j, \delta_i^\vee)}$. There is a Hopf algebra structure on $\mathcal{U}(\epsilon)$, where the comultiplication $\Delta$, the antipode $S$, and the counit $\varepsilon$ are given by

\begin{align*}
\Delta(q^h) & = q^h \otimes q^h, \\
\Delta(e_i) & = e_i \otimes 1 + k_i^{-1} \otimes e_i, \\
\Delta(f_i) & = f_i \otimes k_i + 1 \otimes f_i, \\
S(q^h) & = q^{-h}, \quad S(e_i) = -e_i k_i^{-1}, \quad S(f_i) = -k_i f_i, \\
\varepsilon(q^h) & = 1, \quad \varepsilon(e_i) = \varepsilon(f_i) = 0,
\end{align*}

for $h \in P^\vee$ and $i \in I$. Let $\eta$ be the anti-automorphism on $\mathcal{U}(\epsilon)$ defined by

$$ \eta(q^h) = q^h, \quad \eta(e_i) = q_i f_i k_i^{-1}, \quad \eta(f_i) = q_i^{-1} k_i e_i, $$

where $q_i = (-1)^{\epsilon_i} q^{-\epsilon_i}$. Let $\eta$ be the anti-automorphism on $\mathcal{U}(\epsilon)$ defined by
for \( h \in P^\vee \) and \( i \in I \). It satisfies \( \eta^2 = id \) and
\[
\Delta \circ \eta = (\eta \otimes \eta) \circ \Delta.
\]

We have an isomorphism between \( \mathcal{U}(\epsilon) \) and \( \mathcal{U}(\bar{\epsilon}) \) where \( \bar{\epsilon} \) is obtained from \( \epsilon \) by permutation of \( \epsilon_i \)'s, which is not an isomorphism of Hopf algebras [20, Theorem 2.7] (cf. [19, 37.1]).

**Theorem 2.2.** For \( 1 \leq i \leq n - 1 \), let \( \overline{\epsilon} = (\overline{\epsilon}_1, \ldots, \overline{\epsilon}_n) \) be the sequence obtained by exchanging \( \epsilon_i \) and \( \epsilon_{i+1} \) in \( \epsilon \). Then there exists an isomorphism of algebras \( \tau_i : \mathcal{U}(\epsilon) \to \mathcal{U}(\overline{\epsilon}) \) given by
\[
\tau_i(k_i) = k_i^{-1}, \quad \tau_i(e_i) = -f_i k_i, \quad \tau_i(f_i) = -k_i^{-1} e_i,
\]
\[
\tau_i(k_j) = k_i k_j, \quad \tau_i(e_j) = [e_i, e_j]_{D_{ij}}, \quad \tau_i(f_j) = [f_i, f_j]_{D_{ij}^{-1}} \quad (|i - j| = 1),
\]
\[
\tau_i(k_j) = k_j, \quad \tau_i(e_j) = e_j, \quad \tau_i(f_j) = f_j \quad (|i - j| > 1),
\]
where the inverse map is given by
\[
\tau_i^{-1}(k_i) = k_i^{-1}, \quad \tau_i^{-1}(e_i) = -k_i^{-1} f_i, \quad \tau_i^{-1}(f_i) = -e_i k_i,
\]
\[
\tau_i^{-1}(k_j) = k_i k_j, \quad \tau_i^{-1}(e_j) = [e_i, e_j]_{D_{ij}}, \quad \tau_i^{-1}(f_j) = [f_i, f_j]_{D_{ij}^{-1}} \quad (|i - j| = 1),
\]
\[
\tau_i^{-1}(k_j) = k_j, \quad \tau_i^{-1}(e_j) = e_j, \quad \tau_i^{-1}(f_j) = f_j \quad (|i - j| > 1).
\]

\[\square\]

### 2.2. Crystal base of \( \mathcal{U}(\epsilon) \)-modules

For a \( \mathcal{U}(\epsilon) \)-module \( V \) and \( \mu = \sum_i \mu_i \delta_i \in P \), let
\[
V_\mu = \{ u \in V | \omega_i u = q_i^{\mu_i} u \ (i \in I) \}
\]
be the \( \mu \)-weight space of \( V \). For a non-zero vector \( u \in V_\mu \), we denote by \( \operatorname{wt}(u) = \mu \) the weight of \( u \). Let \( P_{\geq 0} = \sum_{i \in I} \mathbb{Z}_{\geq 0} \delta_i \) and let \( \mathcal{O}_{\geq 0} \) be the category of \( \mathcal{U}(\epsilon) \)-modules with objects \( V \) such that
\[
(2.8) \quad V = \bigoplus_{\mu \in P_{\geq 0}} V_\mu \text{ with } \dim V_\mu < \infty.
\]
which is closed under taking submodules, quotients and tensor products.

**Remark 2.3.** There is another comultiplication on \( \mathcal{U}(\epsilon) \) given by
\[
\Delta_+(q^h) = q^h \otimes q^h,
\]
\[
\Delta_+(e_i) = 1 \otimes e_i + e_i \otimes k_i,
\]
\[
\Delta_+(f_i) = k_i^{-1} \otimes f_i + f_i \otimes 1,
\]
(while \( \Delta_+^{\mathrm{op}} \) is used in [16]). Let \( \otimes \) and \( \otimes_+ \) denote the tensor product with respect to \( \Delta \) and \( \Delta_+ \), respectively. For \( \mathcal{U}(\epsilon) \)-modules \( M \) and \( N \), we have a \( \mathcal{U}(\epsilon) \)-linear isomorphism \( \psi : M \otimes N \to M \otimes_+ N \) given by
\[
(2.10) \quad \psi(u \otimes v) = \left( \prod_{i \in I} q_i^{\mu_i \nu_i} \right) u \otimes v,
\]
for \( u \in M_\mu \) and \( v \in N_\nu \) with \( \mu = \sum_i \mu_i \delta_i \) and \( \nu = \sum_i \nu_i \delta_i \).
Let us recall the notion of crystal base for \( V \in \mathcal{O}_{\geq 0} \) (cf. [2]). The Kashiwara operators \( \tilde{e}_i \) and \( \tilde{f}_i \) on \( V \) for \( i \in I \) are defined as follows. Suppose that \( u \in V_\mu \) is given.

**Case 1.** Suppose that \( i \in I_{\text{odd}} \) and \((\epsilon_i, \epsilon_{i+1}) = (0, 1)\). We define

\[
\tilde{e}_i u = \eta(f_i)u = q_i^{-1}k_i e_i u, \quad \tilde{f}_i u = f_i u.
\]

**Case 2.** Suppose that \( i \in I_{\text{odd}} \) and \((\epsilon_i, \epsilon_{i+1}) = (1, 0)\). We define

\[
\tilde{e}_i u = e_i u, \quad \tilde{f}_i u = \eta(e_i)u = q_i f_i k_i^{-1} u.
\]

**Case 3.** Suppose that \( i \in I_{\text{even}} \) and \((\epsilon_i, \epsilon_{i+1}) = (0, 0)\). Let \( \zeta : U_q(\mathfrak{sl}_2) \to U(\mathfrak{sl}_2) \) be the \( \mathbb{Q}(q) \)-algebra isomorphism given by \( \zeta(e) = e_i, \zeta(f) = f_i \) and \( \zeta(k) = k_i \), where \( U_q(\mathfrak{sl}_2) = \langle e, f, k^{\pm 1} \rangle \) is the usual quantum group for \( \mathfrak{sl}_2 \) with relation \( k e k^{-1} = q^2 e, k f k^{-1} = q^{-2} f \), \( e f - f e = \frac{k - k^{-1}}{q - q^{-1}} \). The induced comultiplication \( \Delta^\zeta := (\zeta^{-1} \otimes \zeta^{-1}) \circ \Delta \circ \zeta \) on \( U_q(\mathfrak{sl}_2) \) is

\[
\Delta^\zeta(k^{\pm 1}) = k^{\pm 1} \otimes k^{\pm 1}, \quad \Delta^\zeta(e) = k^{-1} \otimes e + e \otimes 1, \quad \Delta^\zeta(f) = 1 \otimes f + f \otimes k.
\]

So we define \( \tilde{e}_i \) and \( \tilde{f}_i \) on \( V \) to be the usual Kashiwara operators on the lower crystal base of \( U_q(\mathfrak{sl}_2) \)-module induced from \( \zeta \). In other words, if \( u = \sum_{k \geq 0} f_i^{(k)} u_k \), where \( f_i^{(k)} = f_i^k/[k]! \) and \( e_i u_k = 0 \) for \( k \geq 0 \), then we define

\[
\tilde{e}_i u = \sum_{k \geq 1} f_i^{(k-1)} u_k, \quad \tilde{f}_i u = \sum_{k \geq 0} f_i^{(k+1)} u_k.
\]

**Case 4.** Suppose that \( i \in I_{\text{even}} \) and \((\epsilon_i, \epsilon_{i+1}) = (1, 1)\). Let \( \xi : U_q(\mathfrak{sl}_2) \to U(e) \) be the \( \mathbb{Q}(q) \)-algebra homomorphism given by \( \xi(e) = -e_i, \xi(f) = f_i \) and \( \xi(k) = k_i^{-1} \). Then the induced comultiplication \( \Delta^\xi \) on \( U_q(\mathfrak{sl}_2) \) is

\[
\Delta^\xi(k^{\pm 1}) = k^{\pm 1} \otimes k^{\pm 1}, \quad \Delta^\xi(e) = k \otimes e + e \otimes 1, \quad \Delta^\xi(f) = 1 \otimes f + f \otimes k^{-1}.
\]

So we define \( \tilde{e}_i \) and \( \tilde{f}_i \) on \( V \) to be the Kashiwara operators on the upper crystal base of \( U_q(\mathfrak{sl}_2) \)-module induced from \( \xi \). In other words, if \( u = \sum_{k \geq 0} f_i^{(k)} u_k \), where \( e_i u_k = 0 \) for \( k \geq 0 \) and \( l_k = (\text{wt}(u_k), \alpha_i^\vee) \), then we define

\[
\tilde{e}_i u = \sum_{k \geq 1} q^{-l_k+2k-1} f_i^{(k-1)} u_k, \quad \tilde{f}_i u = \sum_{k \geq 0} q^{l_k-2k-1} f_i^{(k+1)} u_k.
\]

Let \( A_0 \) be the subring of \( \mathbb{Q}(q) \) consisting of \( f(q)/g(q) \) with \( f(q), g(q) \in \mathbb{Q}[q] \) and \( g(0) \neq 0 \).

**Definition 2.4.** Let \( V \in \mathcal{O}_{\geq 0} \) be given. A pair \((L, B)\) is a crystal base of \( V \) if it satisfies the following conditions:

1. \( L \) is an \( A_0 \)-lattice of \( V \) and \( L = \bigoplus_{\mu \in \mathbb{P}_{\geq 0}} L_\mu \), where \( L_\mu = L \cap V_\mu \),
2. \( B \) is a signed basis of \( L/qL \), that is \( B = B \cup -B \) where \( B \) is a \( \mathbb{Q} \)-basis of \( L/qL \),
3. \( B = \bigcup_{\mu \in \mathbb{P}_{\geq 0}} B_\mu \) where \( B_\mu \subset (L/qL)_\mu \),
4. \( \tilde{e}_i L \subset L, \tilde{f}_i L \subset L \) and \( \tilde{e}_i B \subset B \cup \{0\}, \tilde{f}_i B \subset B \cup \{0\} \) for \( i \in I \),
Proof. The proof is almost the same as in [16 Proposition 3.4], where the order of tensor product is reversed due to a different convention of comultiplication.

Remark 2.6. Let \( V = \bigoplus_{i \in I} \mathbb{Q}(q)v_i \) denote the \( \mathcal{U}(e) \)-module, where

\[
\omega_i v_j = q_i^{\delta_{ij}} v_j, \quad e_k v_j = \delta_{kj-1} v_{j-1}, \quad f_k v_j = \delta_{kj} v_{j+1},
\]
for $i, j \in I$ and $k \in I$. It is clear that the pair $\mathcal{L} = \bigoplus_{i \in I} A_i v_i$ and $\mathcal{B} = \{ \pm v_i \pmod{q\mathcal{L}} \mid i \in I \}$ is a crystal base of $\mathcal{V}$. The crystal structure on $B^{\otimes \ell}/\{ \pm 1 \}$ for $\ell \geq 1$ can be described explicitly by Proposition 2.5, which is the same as in [2] or [16] except that the tensor product order is reversed.

3. Schur-Weyl duality and polynomial representations of $\tilde{U}(\epsilon)$

3.1. Schur-Weyl duality. Put $\bar{I} = I \setminus \{0\}$. Let $\tilde{U}(\epsilon)$ be the $\mathbb{Q}(q)$-subalgebra of $U(\epsilon)$ generated by $q^h$ and $e_i, f_i$ for $h \in P^\vee$ and $i \in \bar{I}$.

Let us consider $V = \bigoplus_{i \in \bar{I}} \mathbb{Q}(q)v_i$ in (2.14) as a $\tilde{U}(\epsilon)$-module. Fix $\ell \geq 2$. Let $\Phi_\ell : \tilde{U}(\epsilon) \to \text{End}_{\mathbb{Q}(q)}(V^{\otimes \ell})$ denote the action of $\tilde{U}(\epsilon)$ on $V^{\otimes \ell}$ via (2.7). Note that $V^{\otimes \ell}$ is semisimple (see [16] Corollary 4.1)).

Assume that $\epsilon_1 = 0$. We have a $\tilde{U}(\epsilon)$-linear map $R : V^{\otimes 2} \to V^{\otimes 2}$ given by

$$R(v_i \otimes v_j) = \begin{cases} q^{-1}q_i^{-1}v_i \otimes v_j, & \text{if } i = j, \\ q^{-1}v_j \otimes v_i, & \text{if } i > j, \\ (q^{-2} - 1)v_i \otimes v_j + q^{-1}v_j \otimes v_i, & \text{if } i < j, \end{cases}$$

satisfying the Yang-Baxter equation;

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23},$$

where $R_{ij}$ denotes the map acting as $R$ on the $i$-th and the $j$-th component and the identity elsewhere on $V^{\otimes 3}$ (cf. [11]).

Let $\mathcal{H}_\ell(q^{-2})$ be the Iwahori-Hecke algebra of type $A$ over $\mathbb{Q}(q)$ generated by $h_i$ for $i \in \{1, \ldots, \ell - 1\}$ subject to the relations;

$$(h_i - q^{-2})(h_i + 1) = 0,$$

$h_i h_j = h_j h_i$, \quad (|i - j| > 1),

$h_i h_j h_i = h_i h_j h_j$, \quad (|i - j| = 1),

for $i, j \in \{1, \ldots, \ell - 1\}$. Let $W$ be the symmetric group on $\{1, \ldots, \ell\}$ and $s_i = (i, i + 1)$ be the transposition for $1 \leq i \leq \ell - 1$. For $w \in W$, $\ell(w)$ denote the length of $w$ and let $h(w)$ be the element in $H_\ell(q^{-2})$ associated to $w$ such that $h(s_i) = h_i$ for $1 \leq i \leq \ell - 1$.

We can check that there exists a well-defined action of $\mathcal{H}_\ell(q^{-2})$ on $V^{\otimes \ell}$, say, $\Psi_\ell : \mathcal{H}_\ell(q^{-2}) \to \text{End}_{\mathbb{Q}(q)}(V^{\otimes \ell})$, where $\Psi_\ell(h_i)$ acts as $R$ on the $i$-th and $(i + 1)$-th component and the identity elsewhere. Then we have an analogue of Schur-Weyl duality for $\tilde{U}(\epsilon)$ (cf. [11]) as follows. The proof is similar to the case when $\epsilon_i = 0$ for all $i$.

**Theorem 3.1.** We have

$$\text{End}_{\mathcal{H}_\ell(q^{-2})}(V^{\otimes \ell}) = \Phi_\ell(\tilde{U}(\epsilon)), \quad \text{End}_{\tilde{U}(\epsilon)}(V^{\otimes \ell}) = \Psi_\ell(\mathcal{H}_\ell(q^{-2})).$$

$\square$
3.2. Polynomial representations of $\mathcal{U}(\epsilon)$. Recall that $M$ is the number of $i$'s with $\epsilon_i = 0$ and $N$ is the number of $i$'s with $\epsilon_i = 1$ in $\epsilon$.

Let $\mathcal{P}$ be the set of all partitions. A partition $\lambda = (\lambda_i)_{i \geq 1} \in \mathcal{P}$ is called an $(M|N)$-hook partition if $\lambda_{M+1} \leq N$ (cf. [1]). We denote the set of all $(M|N)$-hook partitions by $\mathcal{P}_{M|N}$. For a Young diagram $\lambda$, a tableau $T$ obtained by filling $\lambda$ with letters in $I$ is called semistandard if (1) the letters in each row (resp. column) are weakly increasing from left to right (resp. from top to bottom), (2) the letters in $I_0$ (resp. $I_1$) are strictly increasing in each column (resp. row). Let $SST_{\epsilon}(\lambda)$ be the set of all semistandard tableaux of shape $\lambda$. Then $SST_{\epsilon}(\lambda)$ is non-empty if and only if $\lambda \in \mathcal{P}_{M|N}$. For $T \in SST_{\epsilon}(\lambda)$, let $w(T)$ be the word given by reading the entries in $T$ column by column from left to right, and from bottom to top in each column.

For $T \in SST_{\epsilon}(\lambda^r)$ with $r \geq 1$, let $d(T) = \sum_{u < v} d_u d_v$, where $d_u$ is the number of occurrences of $u$ in $T$ for $u \in I$. In general, for a column-semistandard tableau $T$, that is, each column of $T$ is semistandard, we define $d(T) = \sum_{k \geq 1} d(T_k)$, where $T_k$ is the $k$-th column from the left.

We fix $\ell \geq 2$, and let $W$ denote the symmetric group on $\{1, \ldots, \ell\}$. Suppose that $\lambda \in \mathcal{P}$ is given with $\sum_{i \geq 1} \lambda_i = \ell$. Let $T^0_\ell$ be the standard tableau obtained by filling $\lambda$ with $\{1, \ldots, \ell\}$ row by row from top to bottom and from left to right in each row, and let $T^\lambda_\ell$ be the tableau obtained by filling $\lambda$ with $\{1, \ldots, \ell\}$ column by column from left to right and from bottom to top in each column.

Let $w_\lambda \in W$ be such that $w_\lambda(T^0_\ell) = T^\lambda_\ell$, where $w_\lambda(T^0_\ell)$ is the tableau obtained by acting $w_\lambda$ on the letters in $T^0_\ell$. Let $W^\lambda_\ell$ (resp. $W^\lambda_\ell$) be the Young subgroup of $W$ stabilizing the rows (resp. columns) of $T^\lambda_\ell$ (resp. $T^\lambda_\ell$). Then the $q$-deformed Young symmetrizer is given by

$$Y^\lambda(q) = h(w_\lambda^{-1}) e^\lambda_+ h(w_\lambda) e^\lambda_-, \quad (3.2)$$

where

$$e^\lambda_+ = \sum_{w \in W_\lambda^+} h(w), \quad e^\lambda_- = \sum_{w \in W_\lambda^-} (-q^2)^{-\ell(w)} h(w).$$

For $1 \leq u < v \leq \ell$, let $W_{uv} = \langle s_i \rangle | u \leq i \leq v - 1 \rangle$. Suppose that $a$ is a letter in $T^\lambda_\ell$ such that $a + 1$ is located in the same column. We put $C_a = 1 + h_a$. Then we have

$$Y^\lambda(q) C_a = 0, \quad (3.3)$$

Next, suppose that $a$ is a letter in $T^\lambda_\ell$, where there is another letter $d$ to the right. Let $b$ be the letter at the bottom of column where $a$ is placed, and $c = b + 1$ the letter at the top of the column where $d$ is placed. Let $G^\lambda_a$ be the set of minimal length right coset representatives of $W_{ab} \times W_{ad}$ in $W_{ad}$. We define the Garnir element at $a$ to be

$$G^\lambda_a = \sum_{w \in G^\lambda_a} (-q^2)^{\ell(w)} h(w). \quad (3.4)$$
The collection of boxes in the Young diagram \( \lambda \) corresponding to the letters from \( a \) to \( d \) in \( T_\lambda \) is called a Garnir belt at \( a \). Then we have the following relations \([9](15)\);

\[
Y_\lambda(q)G_\lambda^a = 0.
\]

Let \( T \) be a tableau of shape \( \lambda \) with letters in \( \mathbb{I} \), and let \( T(i) \) be the letter in \( T \) at the position corresponding to \( i \) in \( T_\lambda \) for \( 1 \leq i \leq \ell \). Let

\[
v_T = Y_\lambda(q) \left( v_{T(1)} \otimes \ldots \otimes v_{T(\ell)} \right).
\]

For \( \sigma \in W \), let \( T^\sigma \) be the tableau given by replacing \( T(i) \) with \( T(\sigma(i)) \) for \( 1 \leq i \leq \ell \).

Let \( a \) be a letter in \( T_\lambda \) with \( d \) to the right in the same row and with \( b, c \) as above. Let \( w_0 \) be the longest element in \( W_{ab} \times W_{cd} \), and let \( G_\lambda^a = w_0G_\lambda^a w_0 \). Let \( u_1, \ldots, u_s \) and \( u_{s+1}, \ldots, u_{r+s} \) be the letters in \( T \) corresponding to \( c, \ldots, d \) and \( a, \ldots, b \) in \( T_\lambda \), respectively. Then we may identify \( \sigma \in G_\lambda^a \) with a permutation on \( \{1, \ldots, r+s\} \) satisfying \( \sigma(1) < \cdots < \sigma(s) \) and \( \sigma(s+1) < \cdots < \sigma(s+r) \) so that \( T^\sigma \) is the tableau obtained from \( T \) by replacing \( u_i \)'s with \( u_{\sigma(i)} \)'s for \( 1 \leq i \leq r+s \). With this identification, we let \( \ell(\sigma) \) be the length of \( \sigma \) as a permutation on \( \{1, \ldots, r+s\} \), and put

\[
X_\sigma = \{i \mid 1 \leq i \leq s, \ s+1 \leq \sigma^{-1}(i) \leq s+r\}, \quad Y_\sigma = \{j \mid s+1 \leq j \leq s+r, \ 1 \leq \sigma^{-1}(j) \leq r\}.
\]

**Lemma 3.2.** Suppose that \( T \) is column-semistandard such that either \( T(a) = T(d) \in \mathbb{I}_1 \) or \( T(a) > T(d) \). Then under the above hypothesis, we have

\[
v_T = - \sum_{\sigma \in G_\lambda^a, \sigma \neq 1} (-q)^{\ell(\sigma)+m(\sigma,T)} v_{T^\sigma},
\]

where

\[
m(\sigma,T) = - \left| \{ (i,j) \mid 1 \leq i < j \leq s, \ i \notin X_\sigma, \ j \notin X_\sigma, \ u_i = u_j \} \right|
\]

\[
- \left| \{ (k,l) \mid s+1 \leq k < l \leq s+r, \ k \notin Y_\sigma, \ l \notin Y_\sigma, \ u_k = u_l \} \right|
\]

\[
+ \left| \{ (x,y) \mid 1 \leq x \leq s, \ s+1 \leq y \leq s+r, \ x \in X_\sigma \ or \ y \in Y_\sigma, \ u_x = u_y \} \right|.
\]

**Proof.** We have \( v_T = Y_\lambda(q)v \), where \( v = (v_{T(1)} \otimes \ldots \otimes v_{T(\ell)}) \). Following the above notations, we have \( v = v' \otimes v_{u_1+s} \otimes \cdots \otimes v_{u_{r+s}} \otimes v_{u_1} \otimes \cdots \otimes v_{u_s} \otimes v'' \). Note that

\[
v_\omega = Y_\lambda(q) \left( v' \otimes v_{u_{r+s}} \otimes \cdots \otimes v_{u_{1+s}} \otimes v_{u_1} \otimes \cdots \otimes v_{u_s} \otimes v'' \right),
\]

where \( u_{r+s} \geq \cdots \geq u_1 = T(a) \geq u_s = T(d) \geq \cdots \geq u_1 \).

For \( w \in G_\lambda^a \), we have by (3.1) and (3.3)

\[
h(w) \left( v' \otimes v_{u_{r+s}} \otimes \cdots \otimes v_{u_{1+s}} \otimes v_{u_1} \otimes \cdots \otimes v_{u_s} \otimes v'' \right)
\]

\[
= q^{-\ell(w)}(-q)^{m(\sigma,T_\omega)} \left( v' \otimes v_{u_{\sigma(r+s)}} \otimes \cdots \otimes v_{u_{\sigma(1+s)}} \otimes v_{u_{\sigma(1)}} \otimes \cdots \otimes v_{u_1} \otimes v'' \right),
\]

where \( \sigma \) is the permutation on \( \{1, \ldots, r+s\} \) corresponding to \( w_0w_0 \) and

\[
m(\sigma,T_\omega) = \left| \{ (i,j) \mid i < j, \ \sigma^{-1}(i) < \sigma^{-1}(j), \ u_i = u_j \} \right|.
\]
Hence it follows from (3.5) and that (3.6)

\[ 0 = Y^\lambda(q)G_\alpha^\lambda(q) \left( \psi' \otimes v_{u_{r+s}} \otimes \cdots \otimes v_{u_{l+1+s}} \otimes v_{u_1} \otimes \psi'' \right) \]

\[ = Y^\lambda(q) \sum_{w \in \mathcal{G}_\alpha^\lambda} (-q^2)^{\ell(w)} h(w) \left( \psi' \otimes v_{u_{r+s}} \otimes \cdots \otimes v_{u_{l+1+s}} \otimes v_{u_1} \otimes \psi'' \right) \]

\[ = Y^\lambda(q) \sum_{w \in \mathcal{G}_\alpha^\lambda} (-q)^{\ell(w)+m(\sigma,T^{w_0})} \left( \psi' \otimes v_{u_{\sigma(r+s)}} \otimes \cdots \otimes v_{u_{\sigma(l+1+s)}} \otimes v_{u_{\sigma(1)}} \otimes \psi'' \right) \]

\[ = \sum_{\sigma \in \mathcal{G}_\alpha^\lambda} (-q)^{\ell(\sigma)+m(\sigma,T^{w_0})} v_{T^{w_0}w_0} = \sum_{\sigma \in \mathcal{G}_\alpha^\lambda} (-q)^{\ell(\sigma)+m(\sigma,T^{w_0})} v_{T^{w_0}w_0}. \]

We have

\[ \sum_{\sigma \in \mathcal{G}_\alpha^\lambda} (-q)^{\ell(\sigma)+m(\sigma,T^{w_0})} v_{(T^\sigma)^{w_0}} = 0. \]

For \( \sigma \in \mathcal{G}_\alpha^\lambda \), let \( U^\sigma \) be the subtableau of \( T^\sigma \) corresponding to the Garnir belt at \( a \), where \( U = U^{id} \). We define \( d_\alpha(T^\sigma) \) in the same way as in \( d(T) \) only by using the letters in \( U^\sigma \). Let \( l_p > \cdots > l_1 \geq r_q > \cdots > r_1 \) be the distinct letters appearing in \( U \), where \( l_i \) and \( r_j \) are located in the left and right columns of \( U \), respectively.

Let \( m_i \) (resp. \( n_j \)) be the number of occurrences of \( l_i \)'s (resp. \( r_j \)'s) in \( U \), which remain in the same column after applying \( \sigma \). Let \( m_i' \) (resp. \( n_j' \)) be the number of \( l_i \)'s (resp. \( r_j \)'s) which are placed on the right (resp. left) column of \( U^\sigma \) after applying \( \sigma \) to \( U \). Note that \( \sum_i m_i' = \sum_j n_j' \).

Case 1. Suppose that \( l_1 \neq r_q \). We have

\[ d_\alpha(T) = \sum_{1 \leq i < j \leq p} (m_i + m_i')(m_j + m_j') + \sum_{1 \leq k < l \leq q} (n_k + n_k')(n_l + n_l'), \]

while

\[ d_\alpha(T^\sigma) = \sum_{1 \leq i < j \leq p} (m_i m_j + m_i m_j') + \sum_{i,k} m_i n_k + \sum_{1 \leq k < l \leq q} (n_k n_l + n_k n_l') + \sum_{j,l} m_j n_l \]

\[ = \sum_{1 \leq i < j \leq p} (m_i m_j + m_i m_j') + \sum_{i} m_i \sum_{k} n_k' + \sum_{1 \leq k < l \leq q} (n_k n_l + n_k n_l') + \sum_{j} m_j' \sum_{l} n_l. \]

Since we have

\[ m(\sigma,T^{w_0}) = \left| \{(i,j) \mid 1 \leq i < j \leq s, \ i \in X_\sigma, j \notin X_\sigma, \ u_i = u_j \} \right| \]

\[ + \left| \{(k,l) \mid s + 1 \leq k < l \leq s + r, \ k \notin Y_\sigma, l \in Y_\sigma, \ u_k = u_l \} \right|, \]

\[ m(\sigma,T) = - \left| \{(i,j) \mid 1 \leq i < j \leq s, \ i \notin X_\sigma, j \in X_\sigma, \ u_i = u_j \} \right| \]

\[ - \left| \{(k,l) \mid s + 1 \leq k < l \leq s + r, \ k \in Y_\sigma, l \notin Y_\sigma, \ u_k = u_l \} \right|, \]

one can check easily that

\[ m(\sigma,T^{w_0}) - m(\sigma,T) = \sum_{1 \leq i \leq p} m_i m_i' + \sum_{1 \leq j \leq q} n_j n_j'. \]

By (3.8), (3.9), and (3.10), we have

\[ d_\alpha(T) - d_\alpha(T^\sigma) = m(\sigma,T) - m(\sigma,T^{w_0}). \]
By (3.3), (3.7) and (3.11), we have
\[
0 = \sum_{\sigma \in \mathfrak{D}_\lambda} (-q)^{\tilde{\ell}(\sigma) + m(\sigma,T^{w_0})} v_{(T^{w_0})} = \sum_{\sigma \in \mathfrak{D}_\lambda} (-q)^{\tilde{\ell}(\sigma) + m(\sigma,T^{w_0}) - d_\sigma(T^{w_0})} v_{T^{w_0}} = \sum_{\sigma \in \mathfrak{D}_\lambda} (-q)^{\tilde{\ell}(\sigma) + m(\sigma,T)} v_{T^{w_0}} = (-q)^{-d_\sigma(T)} \sum_{\sigma \in \mathfrak{D}_\lambda} (-q)^{\tilde{\ell}(\sigma) + m(\sigma,T)} v_{T^{w_0}}.
\]

This proves the identity in the lemma.

Case 2. Suppose that \(l_1 = r_q\). In this case, \(d_\sigma(T)\) is the same as in Case 1, and
\[
d_\sigma(T) - d_\sigma(T^{w_0}) = - \sum_{1 \leq i \leq p} m_i m_i' - \sum_{1 \leq j < q} n_j n_j' + m_p n_1' + m_1 n_1.
\]

Note that
\[
m(\sigma,T^{w_0}) = |\{(i,j) | 1 \leq i < j \leq s, i \in X_\sigma, j \notin X_\sigma, u_i = u_j\}|
+ |\{(k,l) | s + 1 \leq k < l \leq s + r, k \notin Y_\sigma, l \in Y_\sigma, u_k = u_l\}|
+ |\{(x,y) | 1 \leq x \leq s, s + 1 \leq y \leq s + r, x \in X_\sigma, y \in Y_\sigma, u_x = u_y\}|
\]

where the last summand is equal to \(m_i' n_i'\). By similar arguments as in (3.10), we have
\[
d_\sigma(T) - d_\sigma(T^{w_0}) = m(\sigma,T) - m(\sigma,T^{w_0}).
\]

This also proves the identity in the lemma as in (3.11). \(\square\)

For \(\lambda \in \mathcal{P}_{M|N}\) with \(\sum_\lambda \lambda_i = \ell\), let
\[
V_\epsilon(\lambda) = \sum_{T \in SST_\epsilon(\lambda)} Q(q) v_T.
\]

Let \(H_\lambda\) be the tableau in \(SST_\epsilon(\lambda)\), which is defined inductively as follows:

1. Fill the first row (resp. column) of \(\lambda\) with 1 if \(\epsilon_1 = 0\) (resp. \(\epsilon_1 = 1\)).
2. Suppose that we have filled a subdiagram of \(\lambda\) from 1 to \(i\). Then fill the first row (resp. column) of the remaining diagram with \(i + 1\) if \(\epsilon_{i+1} = 0\) (resp. \(\epsilon_{i+1} = 1\)).

Example 3.3. Suppose that \(n = 5\), \(\epsilon = (0, 1, 1, 0, 0)\) and \(\lambda = (6, 5, 4, 2, 1)\). In this case, we have

\[
H_\lambda = \begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
2 & 3 & 1 & 3 & 1 & 1 \\
2 & 3 & 1 & 3 & 1 & 1 \\
2 & 3 & 1 & 3 & 1 & 1 \\
2 & 3 & 1 & 3 & 1 & 1 \\n\end{array}
\]

Proposition 3.4. We have the following.

1. \(V_\epsilon(\lambda)\) is a \(\mathcal{U}(\epsilon)\)-submodule of \(V^{\otimes \ell}\).
2. \(V_\epsilon(\lambda)\) is an irreducible \(\mathcal{U}(\epsilon)\)-module with basis \(\{ v_T | T \in SST_\epsilon(\lambda) \}\).
3. \(v_{H_\lambda}\) is a highest weight vector in \(V_\epsilon(\lambda)\).

Proof. (1) It is clear that \(V_\epsilon(\lambda)\) is invariant under \(q^h\) for \(h \in P^\vee\). It suffices to check \(f_i V_\epsilon(\lambda) \subset V_\epsilon(\lambda)\) for \(i \in \check{I}\) since the proof for \(e_i\) is the same. The proof is similar to the case when \(\epsilon = (0, \ldots, 0)\) (cf. [9]). For column-semistandard tableaux \(U\) and \(V\) of shape \(\lambda\),
we define $U < V$ if there exists $1 \leq k \leq \ell$ such that $U(k) < V(k)$ and $U(k') = V(k')$ for $k < k' \leq \ell$.

Suppose that $T \in \text{SST}_\epsilon(\lambda)$ is given. By (2.7), $f_i v_T$ is a $\mathbb{Q}(q)$-linear combination of $v_{T'}$’s, where we may assume that $T$ is column-semistandard by (3.13). If such $T'$ is not semistandard, then we may apply Lemma 3.2 to $T'$ so that $v_{T'}$ is a linear combination of $T''$’s which is column-semistandard and $T' < T''$. Repeating this process finitely many times, we conclude that $f_i T$ is a linear combination of $v_S$’s for some $S \in \text{SST}_\epsilon(\lambda)$. Therefore, have $f_i V_\epsilon(\lambda) \subset V_\epsilon(\lambda)$.

(2) Since $V_\epsilon(\lambda) = Y^\lambda(q) V^\otimes \ell$ and $Y^\lambda(q)$ is a primitive idempotent up to scalar multiplication [10], it follows from Theorem 3.4 that $V_\epsilon(\lambda)$ is a crystal base of $\mathcal{U}(\mathfrak{gl}_N)$, which depends only on $\epsilon$. Recall that the dimension of the irreducible $\mathcal{H}_\ell(q^{-2})$-module $S_\lambda$ generated by $Y^\lambda(q)$ is the number of standard tableaux of shape $\lambda$. We may have an analogue of the Robinson-Schensted type correspondence, which is a bijection from the set of words of length $\ell$ with letters in $I$ to the set of pair of standard tableau and semistandard tableau of shape $\lambda$ (cf. [4]). Comparing the dimensions of $V^\otimes \ell$ and its decomposition into $\mathcal{H}(q^{-2}) \otimes \mathcal{U}(\epsilon)$-module $S_\lambda \otimes V_\epsilon(\lambda)$, we conclude that dim$\mathbb{Q}(q) V_\epsilon(\lambda)$ is equal to $|\text{SST}_\epsilon(\lambda)|$, and hence $\{v_T \mid T \in \text{SST}_\epsilon(\lambda)\}$ is a linear basis of $V_\epsilon(\lambda)$.

(3) The character of $V_\epsilon(\lambda)$ is equal to that of polynomial representations of the general linear Lie superalgebra $\mathfrak{gl}_{M|N}$ corresponding to $\lambda \in \mathcal{P}_{M|N}$, and wt($v_{H_\lambda}$) is maximal [5, Theorem 2.55]. This implies that $e_i v_{H_\lambda} = 0$ for all $i \in \tilde{I}$ and hence $v_{H_\lambda}$ is a highest weight vector. □

**Remark 3.5.** The character of $V_\epsilon(\lambda)$ for $\lambda \in \mathcal{P}_{M|N}$ is called a hook Schur polynomial [4], which depends only on $\epsilon$ up to permutations. The tensor product of two polynomial representations is completely reducible and the multiplicity of each irreducible component is given by usual Littlewood-Richardson coefficient.

### 3.3. Crystal base of $V_\epsilon(\lambda)$

Let $\lambda \in \mathcal{P}_{M|N}$ be given. We may define an $\tilde{I}$-colored oriented graph structure by identifying $T$ with $w(T)^{\text{rev}}$, the reverse word of $w(T)$.

Let

$$L_\epsilon(\lambda) = \bigoplus_{T \in \text{SST}_\epsilon(\lambda)} A_{0} v_T^*, \quad B_\epsilon(\lambda) = \{ \pm v_T^* \ (\text{mod} \ q L_\epsilon(\lambda)) \mid T \in \text{SST}_\epsilon(\lambda) \},$$

where $v_T^* = q^{-d(T)} v_T$ for $T \in \text{SST}_\epsilon(\lambda)$.

**Lemma 3.6.** When $\lambda = (1^r)$ or $(r)$ for $r \geq 1$, $(L_\epsilon(\lambda), B_\epsilon(\lambda))$ is a crystal base of $V_\epsilon(\lambda)$, and the crystal $B_\epsilon(\lambda)/\{\pm 1\}$ is isomorphic to $\text{SST}_\epsilon(\lambda)$.

**Proof.** The proof is similar to that of [16, Proposition 3.3]. □

**Proposition 3.7.** Suppose that $\epsilon = \epsilon_{M|N}$. For $\lambda \in \mathcal{P}_{M|N}$, $(L_\epsilon(\lambda), B_\epsilon(\lambda))$ is a crystal base of $V_\epsilon(\lambda)$. 
Proof. The proof is similar to that of [18, Theorem 4.4]. Let \((L(\lambda), B(\lambda))\) be given by

\[
L(\lambda) = \sum_{r \geq 0, i_1, \ldots, i_r \in I} A_0 \bar{x}_{i_1} \cdots \bar{x}_{i_r} v_\lambda,
\]

\[
B(\lambda) = \{ \pm \bar{x}_{i_1} \cdots \bar{x}_{i_r} v_\lambda \mod qL(\lambda) \mid r \geq 0, i_1, \ldots, i_r \in I \} \setminus \{0\},
\]

where \(v_\lambda\) is a highest weight vector in \(V_\epsilon(\lambda)\) and \(x = e, f\) for each \(i_k\). Following the same arguments in [2], it is shown in [16] that \((L(\lambda), B(\lambda))\) is a crystal base of \(V_\epsilon(\lambda)\). The crystal \(B(\lambda)/\{\pm 1\}\) is equal to \(SST_\epsilon(\lambda)\) which is connected.

Let \(\mu = (\mu_1, \ldots, \mu_r) = \lambda'\) be the conjugate partition of \(\lambda\), and

\[
V_\epsilon^\mu = V_\epsilon((1^{\mu_1})) \otimes \cdots \otimes V_\epsilon((1^{\mu_r})�).
\]

Let \(I_\mu^\epsilon\) be the subspace of \(V_\epsilon^\mu\) spanned by the vectors induced from the relation (3.5), which includes the relations in Lemma 3.2. Since \(I_\mu^\epsilon\) is a \(\mathcal{U}(\epsilon)\)-submodule, the quotient \(V_\epsilon^\mu/I_\mu^\epsilon\) is isomorphic to \(V_\epsilon(\lambda)\) by Proposition 3.4. So we have a well-defined \(\mathcal{U}(\epsilon)\)-linear map

\[
\pi^\mu : V_\epsilon^\mu \longrightarrow V_\epsilon(\lambda)
\]

given by \(\pi^\mu(v_{T_1} \otimes \cdots \otimes v_{T_r}) = v_\lambda\) where \(T\) is the column semistandard tableau whose \(i\)-th column (from the left) is \(T_i\) for \(1 \leq i \leq r\). Since the decomposition of \(V^\mu\) is equal to the usual Pieri rule of Schur functions, it has exactly one component isomorphic to \(V_\epsilon(\lambda)\). Hence \(\pi^\mu\) is equal to the projection onto \(V_\epsilon(\lambda)\) up to scalar multiplication.

Let \(L^\mu_\epsilon = L_\epsilon((1^{\mu_1})) \otimes \cdots \otimes L_\epsilon((1^{\mu_r}))\) be the crystal lattice of \(V^\mu_\epsilon\). By [16, Theorem 4.14], \(\pi^\mu(L^\mu_\epsilon)\) is a crystal lattice of \(V_\epsilon(\lambda)\) whose \(\text{wt}(H_\lambda)\)-weight space is equal to \(A_0 v_\lambda H_\lambda\). Since the crystal of \(V_\epsilon(\lambda)\) is connected, we conclude that \(\{ v_\lambda T \mid T \in SST_\epsilon(\lambda) \}\) is an \(A_0\)-basis of \(\pi^\mu(L^\mu_\epsilon)\) which is equal to \(L_\epsilon(\lambda)\).

Remark 3.8. For arbitrary \(\epsilon\), the \(I\)-colored oriented graph \(SST_\epsilon(\lambda)\) is not in general connected (see [15] for more details). Furthermore, it is not known yet whether \(V_\epsilon(\lambda)\) has a crystal base for any \(\lambda \in \mathcal{P}_{M|N}\). We expect that \((L_\epsilon(\lambda), B_\epsilon(\lambda))\) in (3.13) is a crystal base of \(V_\epsilon(\lambda)\).

4. \(R\) matrix for finite-dimensional \(\mathcal{U}(\epsilon)\)-modules

4.1. Finite-dimensional \(\mathcal{U}(\epsilon)\)-modules of fundamental type. Let \(Z_+\) be the set of non-negative integers. Let

\[
Z_+^\epsilon(\mu) = \{ \mathbf{m} = (m_1, \ldots, m_n) \mid m_i \in Z_+ \text{ if } \epsilon_i = 0, m_i \in \{0, 1\} \text{ if } \epsilon_i = 1, (i \in I) \}.
\]

For \(\mathbf{m} \in Z_+^\epsilon(\mu)\), let \(|\mathbf{m}| = m_1 + \cdots + m_n\). For \(i \in I\), put \(e_i = (0, \ldots, 1, \cdots, 0)\) where 1 appears only in the \(i\)-th component.

For \(s \in Z_+\), let

\[
W_{s, \epsilon} = \bigoplus_{\mathbf{m} \in Z_+^\epsilon(\mu), |\mathbf{m}| = s} \mathbb{Q}(q)|\mathbf{m}|
\]

be the \(\mathbb{Q}(q)\)-vector space spanned by \(|\mathbf{m}|\) for \(\mathbf{m} \in Z_+^\epsilon(\mu)\) with \(|\mathbf{m}| = s\).
For a parameter $x \in \mathbb{Q}(q)$, we denote by $W_{s,\epsilon}(x)$ a $\mathcal{U}(\epsilon)$-module $V$, where $V = W_{s,\epsilon}$ as a $\mathbb{Q}(q)$-space and the actions of $e_i, f_i, \omega_j$ are given by

$$
e_i(m) = \begin{cases} x^{\delta_i,0}q^{m_{i+1}-m_i}[m_{i+1}]m + e_i - e_{i+1}, & \text{if } m + e_i - e_{i+1} \in \mathbb{Z}_+^n(\epsilon), \\ 0, & \text{otherwise}, \end{cases}$$

$$f_i(m) = \begin{cases} -x^{-\delta_i,0}q^{m_i}[m] - e_i + e_{i+1}, & \text{if } m - e_i + e_{i+1} \in \mathbb{Z}_+^n(\epsilon), \\ 0, & \text{otherwise}, \end{cases}$$

$$\omega_j(m) = q^{m_j}m,$$

for $i \in I$, $j \in \mathbb{I}$, and $m = (m_1, \ldots, m_n) \in \mathbb{Z}_+^n(\epsilon)$. Here we understand $e_0 = e_n$.

**Remark 4.1.** We may identify $W_{s,\epsilon}(x)$ with $V_t((s))$ as a $\mathcal{U}(\epsilon)$-module, where $|m\rangle$ corresponds to $v_T$, where $T$ is the tableau of shape $(s)$ with $m_i$ the number of occurrences of $i$ in $T$ ($i \in \mathbb{I}$). Also the map

$$(4.1) \quad \phi(|m\rangle) = q^{-\sum_{i<j} m_i m_j}|m\rangle$$

gives an isomorphism of $\mathcal{U}(\epsilon)$-modules from $W_{s,\epsilon}(x)$ to itself with another $\mathcal{U}(\epsilon)$-action defined in [10], (2.15)).

Let us regard $W_{s,\epsilon} = W_{s,\epsilon}(1)$ and set

$$(4.2) \quad \mathcal{L}_{s,\epsilon} = \bigoplus_{m \in \mathbb{Z}_+^n(\epsilon), |m| = s} A_0|m\rangle, \quad B_{s,\epsilon} = \{ \pm |m\rangle \mod q\mathcal{L}_{s,\epsilon} | m \in \mathbb{Z}_+^n(\epsilon), |m| = s \}.$$

**Proposition 4.2.** For $s \in \mathbb{Z}_+$, the pair $(\mathcal{L}_{s,\epsilon}, B_{s,\epsilon})$ is a crystal base of $W_{s,\epsilon}$, where the crystal $B_{s,\epsilon}/\{\pm 1\}$ is connected.

**Proof.** It follows from the same arguments as in Lemma 3.4 that $(\mathcal{L}_{s,\epsilon}, B_{s,\epsilon})$ is a crystal base of $W_{s,\epsilon}$. The crystal $SST_t((s))$ is connected with highest element $H_t(s)$. Since the crystal $B_{s,\epsilon}/\{\pm 1\}$ of $W_{s,\epsilon}$ is equal to $SST_t((s))$ as an $I$-colored graph, $B_{s,\epsilon}/\{\pm 1\}$ is connected as an $I$-colored oriented graph. \hfill \Box

4.2. **Subalgebra $\mathcal{U}(\epsilon')$.** Suppose that $n \geq 4$ and let $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ be given. Let $\epsilon' = (\epsilon'_1, \ldots, \epsilon'_{n-1})$ be the sequence obtained from $\epsilon$ by removing $\epsilon_i$ for some $i \in \mathbb{I}$. We further assume that $\epsilon'$ is homogeneous when $n = 4$, that is, $\epsilon' = (000)$ or $(111)$.

Put $I' = \{0, 1, \ldots, n-2\}$. Let us denote by $\omega'_j, e'_j$, and $f'_j$ the generators of $\mathcal{U}(\epsilon')$ for $1 \leq l \leq n-1$ and $j \in I'$, where $k'_j = \omega'_j(\omega'_{j+1})^{-1}$. Let us define $K_j, E_j, F_j$ for $j \in I'$ as follows:
Case 1. Assume that $2 \leq i \leq n - 1$. For $j \in I'$, put

$$K_j = \begin{cases} k_j, & \text{if } j \leq i - 2, \\ k_{i-1}k_i, & \text{if } j = i - 1, \\ k_{j+1}, & \text{if } j \geq i, \end{cases}$$

(4.3)

$$E_j = \begin{cases} e_j, & \text{if } j \leq i - 2, \\ [e_{i-1}, e_i]_{D_{i-1}}, & \text{if } j = i - 1, \\ e_{j+1}, & \text{if } j \geq i, \end{cases}$$

and

$$F_j = \begin{cases} f_j, & \text{if } j \leq i - 2, \\ [f_i, f_{i-1}]_{D_{i-1}}, & \text{if } j = i - 1, \\ f_{j+1}, & \text{if } j \geq i. \end{cases}$$

Case 2. Assume that $i = n$. For $j \in I'$, put

$$K_j = \begin{cases} k_j, & \text{if } j \neq 0, \\ k_{n-1}k_0, & \text{if } j = 0, \end{cases}$$

(4.4)

$$E_j = \begin{cases} e_j, & \text{if } j \neq 0, \\ [e_{n-1}, e_0]_{D_{n-1}}, & \text{if } j = 0, \end{cases}$$

and

$$F_j = \begin{cases} f_j, & \text{if } j \neq 0, \\ [f_0, f_{n-1}]_{D_{n-1}}^{-1}, & \text{if } j = 0. \end{cases}$$

Case 3. Assume that $i = 1$. For $j \in I'$, put

$$K_j = \begin{cases} k_0k_1, & \text{if } j = 0, \\ k_{j+1}, & \text{if } j \neq 0, \end{cases}$$

(4.5)

$$E_j = \begin{cases} [e_0, e_1]_{D_0}, & \text{if } j = 0, \\ e_{j+1}, & \text{if } j \neq 0, \end{cases}$$

and

$$F_j = \begin{cases} [f_1, f_0]_{D_0}^{-1}, & \text{if } j = 0, \\ f_{j+1}, & \text{if } j \neq 0. \end{cases}$$

**Theorem 4.3.** There exists a homomorphism of $\mathbb{Q}(q)$-algebras $\phi: \mathcal{U}(\epsilon') \rightarrow \mathcal{U}(\epsilon)$ such that

$$\phi(k_j') = K_j, \quad \phi(e_j') = E_j, \quad \phi(f_j') = F_j \quad (j \in I').$$

**Proof.** Let us prove Case 1 since the the proof of the other cases are similar. Let $\bar{\epsilon} = (\bar{\epsilon}_1, \ldots, \bar{\epsilon}_n)$ be the sequence obtained from $\epsilon$ by exchanging $\epsilon_i$ and $\epsilon_{i+1}$, and let $\tau_i: \mathcal{U}(\epsilon) \rightarrow \mathcal{U}(\bar{\epsilon})$ be the isomorphism in Theorem 2.2

Put $\Omega_j = \omega_j$ for $1 \leq j \leq i - 1$ and $\omega_{j+1}$ for $i \leq j \leq n - 1$, and let $\phi(\omega_j') = \Omega_j, \phi(e_j') = E_j$, and $\phi(f_j') = F_j$ for $j = 1, \ldots, n - 1$. Let us check that $\Omega_j, E_j, F_j$ satisfy the relations in Definition 2.1. Note that $D_{i-1} = q_i^{-1}$.

First, the relations (2.1) and (2.2) are trivial. Let us check that (2.3) holds. Let $E_j$ and $F_l$ be given for $j, l \in I'$. If $j \neq l$ or $j = l \neq i - 1$, then it is clear. When $j = l = i - 1$, we have $\tau_{i-1}(e_{i-1}) = [e_{i-1}, e_i] = E_{i-1}$, $\tau_{i-1}(f_{i-1}) = F_{i-1}$, and $\tau_{i-1}(k_{i-1}) = K_{i-1}$. Hence (2.3) holds. We can check the relation (2.4) by the same argument.

Next, consider the relations (2.5). The first one is immediate. So it is enough to show the second one. We may only consider four non-trivial cases when the pair of relevant indices in $I'$ are $(i - 2, i - 1), (i - 1, i - 2), (i - 1, i), (i, i - 1)$ with the first index in the pair in $I'_{\text{even}}$. 
In case of \((i - 2, i - 1)\), we have
\[
E_{i-2}^2E_{i-1} - (-1)^{i-2}[2]E_{i-2}E_{i-1}E_{i-2} + E_{i-1}E_{i-2}^2
\]
\[
= e_{i-2}^2e_{i-1}e_i - q_i^{-1}e_{i-2}e_{i-1}e_i - (q_i^{-1} + q_i^{-1})e_{i-2}e_{i-1}e_i - 2e_{i-1}e_i - q_i^{-1}e_{i-1}e_i^2 - q_i^{-1}e_i e_{i-1}e_i^2.
\]
which is zero, since \(e_{i-2}^2e_{i-1} + e_{i-1}e_i^2 = (q_i^{-1} + q_i^{-1})e_{i-2}e_{i-1}e_i\) and hence
\[
e_{i-2}^2e_{i-1}e_i - (q_i^{-1} + q_i^{-1})e_{i-2}e_{i-1}e_i + e_{i-1}e_i^2 = 0,
\]
\[-q_i^{-1}e_{i-2}e_i - q_i^{-1}q_i^{-1}e_{i-2}e_{i-1}e_i - 2q_i^{-1}e_{i-1}e_i - q_i^{-1}e_i e_{i-1}e_i^2 = 0.
\]
The proof for \((i, i-1)\) is the same. In case of \((i-1, i-2)\) and \((i-1, i)\), the proof reduces to the case of \((i-2, i-1)\) or \((i-1, i)\) by applying \(\tau_i\) to \(E_l\)'s for \(l = i - 2, i - 1, i\).

Finally let us check the relation \((2.8)\). We may only consider the cases when the relevant triple of indices in \(I'\) are \((i-3, i-2, i-1), (i-2, i-1, i), (i-1, i, i+1)\) with the index in the middle in \(I'_{odd}\). In case of \((i-1, i, i+1)\) and \(i \in I'_{odd}\), we have
\[
E_i E_{i-1} E_{i+1} E_i - E_i E_{i+1} E_i E_{i-1} + E_{i+1} E_i E_{i-1} E_i
\]
\[
- E_{i-1} E_i E_{i+1} E_i + (-1)^i[2]E_i E_{i-1} E_{i+1} E_i
\]
\[
= e_{i+1}^2e_{i-1}e_i - q_i^{-1}e_{i-1}e_i e_{i+1} - e_{i+1}e_{i-1}e_i e_{i+1} + (q_i^{-1}e_{i-1} + q_i^{-1}e_{i-1})e_{i-1}e_i e_{i+1}
\]
\[
+ e_{i+1}^2e_{i-1}e_i - q_i^{-1}e_{i-1}e_i e_{i+1} - (q_i^{-1}e_{i-1} + q_i^{-1}e_{i-1})e_{i-1}e_i e_{i+1}
\]
\[
+ (-1)^i[2]e_{i+1}^2e_{i-1}e_i - q_i^{-1}e_{i-1}e_i e_{i+1} - e_{i+1}e_{i-1}e_i e_{i+1},
\]
which is zero by \((2.8)\) for \(U(\epsilon)\) with respect to \(i+1 \in I_{odd}\). The proof for \((i-3, i-2, i-1)\) is the same. The proof for \((i-2, i-1, i)\) reduces to the previous cases by applying \(\tau_i\) to \(E_l\) for \(l = i-2, i-1, i\). We leave the proof for \(F_j\)'s to the reader.  

4.3. **Truncation to \(U(\epsilon')\)-modules.** Let \(\epsilon'\) be as in Section 1.2. Suppose that \(M'\) is the number of \(j\)'s with \(\epsilon'_j = 0\) and \(N'\) is the number of \(j\)'s with \(\epsilon'_j = 1\) in \(\epsilon'\).

For a submodule \(V\) of \(\mathcal{V}^\otimes \ell\) \((\ell \geq 1)\), we define
\[
\text{tr}_{\epsilon'}^V(V) = \bigoplus_{\mu \in \text{wt}(V) \atop (\mu|\lambda_i) = 0} V_{\mu},
\]
where \(\text{wt}(V)\) is the set of weights of \(V\). For any submodules \(V, W\) of a tensor power of \(\mathcal{V}\), it is clear that
\[
\text{tr}_{\epsilon'}^V(V \otimes W) = \text{tr}_{\epsilon'}^V(V) \otimes \text{tr}_{\epsilon'}^V(W),
\]
as a vector space.

**Lemma 4.4.** Let \(\mathcal{V}' = \text{tr}_{\epsilon'}^V(V)\). Then

1. \(\mathcal{V}'\) is isomorphic to the natural representation of \(U(\epsilon')\) given in \((2.1)\),
2. \(\text{tr}_{\epsilon'}^V(V^\otimes \ell)\) is isomorphic to \(V^\otimes \ell\) as a \(U(\epsilon')\)-module.
Proof. (1) Let us assume that $2 \leq i \leq n - 2$ since the proof for the other cases is similar. Let $j \in (\check{J} \setminus \{i\}$ given. It is clear from (4.3) that

$$E_j v_k = \begin{cases} v_j, & \text{if } k = j + 1, \\ 0, & \text{if } k \neq j + 1, \end{cases} \quad (j \leq i - 2),$$

$$E_j v_k = \begin{cases} v_{j+1}, & \text{if } k = j + 2, \\ 0, & \text{if } k \neq j + 2, \end{cases} \quad (j \geq i).$$

When $j = i - 1$, we have $E_{i-1} = e_i e_i - q^{-1}_i e_i e_i - 1$, and

$$E_{i-1} v_k = \begin{cases} v_{i-1}, & \text{if } k = i + 1, \\ 0, & \text{if } k \neq i + 1. \end{cases}$$

We have similar formulas for $F_j$ for $j \in (\check{I} \setminus \{i\}$. Hence $V'$ is invariant under the action of $\check{U}(\epsilon')$. In fact, $V'$ is isomorphic to the natural representation of $\check{U}(\epsilon')$ (2.15).

(2) We see that the actions of $E_j, F_j, K_j (j \in (\check{I} \setminus \{i\})$ on $V' \otimes V'$ are equal to those of

$$K_j^{-1} \otimes E_j + E_j \otimes 1, \quad 1 \otimes F_j + F_j \otimes K_j, \quad K_j \otimes K_j,$$

respectively. This implies that $V' \otimes V'$ and hence $(V')^\otimes \ell$ are invariant under the action of $\check{U}(\epsilon')$. For example, in case of $E_{i-1} = e_i e_i - q^{-1}_i e_i e_i - 1$, we have

$$\Delta(E_{i-1}) = \Delta(e_{i-1})\Delta(e_i) - q^{-1}_i \Delta(e_i)\Delta(e_{i-1})$$

$$= k^{-1}_i k^{-1}_i e_i e_i - q^{-1}_i k^{-1}_i e_i e_i + k^{-1}_i e_i e_i + 1 \otimes e_i + 1 \otimes e_i$$

Then the action of $\Delta(E_{i-1})$ on $V' \otimes V'$ is equal to $k^{-1}_i k^{-1}_i e_i e_i + 1 \otimes e_i + 1 \otimes e_i$, and hence $K^{-1}_i \otimes E_{i-1} + E_{i-1} \otimes 1.$ \hfill \Box

Proposition 4.5. Let $\lambda \in \mathcal{P}_{M|N}$ be given.

(1) $\text{tr}_\epsilon(V_\epsilon(\lambda))$ is a $\check{U}(\epsilon')$-submodule of $V_\epsilon(\lambda)$ via $\phi$.

(2) $\text{tr}_\epsilon(V_\epsilon(\lambda))$ is non-zero if and only if $\lambda \in \mathcal{P}_{M|N'}$. In this case, we have

$$\text{tr}_\epsilon(V_\epsilon(\lambda)) \cong V_\epsilon(\lambda),$$

as a $\check{U}(\epsilon')$-module.

Proof. (1) It follows from Lemma 4.4 and

$$(4.8) \quad \text{tr}_\epsilon(V_\epsilon(\lambda)) = \text{tr}_\epsilon(Y^\lambda(q)\mathcal{V}^{\otimes \ell}) = Y^\lambda(q)\text{tr}_\epsilon(Y^\lambda(q)\otimes \ell) = Y^\lambda(q)(Y^\lambda(q))^{\otimes \ell}.$$

(2) Note that $SST_\epsilon(\lambda) \subset SST_\epsilon(\lambda)$. By Proposition 3.4 and (4.8), we see that $\text{tr}_\epsilon(V_\epsilon(\lambda))$ is a $\mathbb{Q}(q)$-span of $\{v_T \mid T' \in SST_\epsilon(\lambda)\}$, which in fact forms a basis. This implies that $\text{tr}_\epsilon(V_\epsilon(\lambda))$ is non-zero if and only if $\lambda \in \mathcal{P}_{M_{-1}\setminus N}$ when $\epsilon_i = 0$, and $\lambda \in \mathcal{P}_{M_{-1}\setminus N}$ when $\epsilon_i = 1$. Hence, $\text{tr}_\epsilon(V_\epsilon(\lambda))$ is isomorphic to $V_\epsilon(\lambda)$ when it is non-zero by (4.7) and Proposition 3.4 \hfill \Box

Corollary 4.6. Let $V, W$ be submodules of a tensor power of $V$. Then

(1) $\text{tr}_\epsilon(V), \text{tr}_\epsilon(W)$, and $\text{tr}_\epsilon(V \otimes W)$ are $\check{U}(\epsilon')$-modules via $\phi$.

(2) $\text{tr}_\epsilon(V \otimes W) \cong \text{tr}_\epsilon(V) \otimes \text{tr}_\epsilon(W)$ as $\check{U}(\epsilon')$-modules.
Proof. Since $V^\otimes t$ is completely reducible, it follows from Proposition 4.5 and 4.7. □

We may define $\text{tr}^e$, and have similar results for $U(\epsilon)$-modules in $O_{\geq 0}$.

Proposition 4.7.

(1) For $s \in \mathbb{Z}_+$ and $x \in \mathbb{Q}(q)$, $\text{tr}^e(x')$ is a $U(\epsilon')$-submodule of $W_s, e(x)$ via $\phi$, and

$$
\text{tr}^e(x') \cong W_s, e(x).
$$

Moreover, $(\text{tr}^e(L_{s, e}), \text{tr}^e(B_{s, e}))$ is a crystal base of $\text{tr}^e(W_{s, e})$ isomorphic to $(L_{s, e'}, B_{s, e'})$.

(2) For $l, m \in \mathbb{Z}_+$ and $x, y \in \mathbb{Q}(q)$, $\text{tr}^e(W_{l, e}(x) \otimes W_{m, e}(y))$ is a $U(\epsilon')$-module via $\phi$, and

$$
\text{tr}^e(W_{l, e}(x) \otimes W_{m, e}(y)) \cong \text{tr}^e(W_{l, e}(x)) \otimes \text{tr}^e(W_{m, e}(y)),
$$

as $U(\epsilon')$-modules.

Proof. The proof is the same as in Proposition 4.5. □

4.4. Irreducibility of $W_{l, e}(x) \otimes W_{m, e}(y)$. Let us show that $W_{l, e}(x) \otimes W_{m, e}(y)$ is irreducible for $l, m \in \mathbb{Z}_+$ and generic $x, y \in \mathbb{Q}(q)$. When $\epsilon = \epsilon_{M+N}$, the irreducibility is shown in 17. In this paper, we give a different proof of it, which is also available for arbitrary $\epsilon$.

Theorem 4.8. For $l, m \in \mathbb{Z}_+$, $W_{l, e} \otimes W_{m, e}$ is irreducible.

Proof. Let us assume without loss of generality that $M, N \geq 1$ with $e_1 = 0$.

Let $(L_{s, e}, B_{s, e})$ be the crystal base of $W_{s, e}$ in (4.2) for $s = l, m$. By Proposition 2.5, $(L_{l, e} \otimes L_{m, e}, B_{l, e} \otimes B_{m, e})$ is a crystal base of $W_{l, e} \otimes W_{m, e}$. If $M = 1$, then it is proved in 10 that $B_{l, e} \otimes B_{m, e}/\{\pm 1\}$ is connected. Since $\dim_{\mathbb{Q}(q)}(W_{l, e} \otimes W_{m, e}) = 1$ and $B_{l, e} \otimes B_{m, e}/\{\pm 1\}$ is connected, it follows from 2 Lemma 2.7 that $W_{l, e} \otimes W_{m, e}$ is irreducible.

We assume that $M \geq 2$. Let $\epsilon' = \epsilon_{M+1}$, which is the subsequence of $\epsilon$ obtained by removing all $e_i = 1$'s. Note that the length of $\epsilon'$ may be less than 4 so that $U(\epsilon')$ is not well-defined, but $\text{tr}^e$ can be defined in the same way as in (4.6). We put

$$
W_{s, \epsilon'} := \text{tr}^e(W_{s, e}), \quad L_{s, \epsilon'} := \text{tr}^e(L_{s, e}) \subset L_{s, e}, \quad B_{s, \epsilon'} := \text{tr}^e(B_{s, e}) \subset B_{s, e}.
$$

Let $1 \leq j_1 < \cdots < j_M \leq n$ be such that $e_{j_k} = 0$ for $1 \leq k \leq M$. By Theorem 1.3 we have a $U_q(\mathfrak{sl}_2)$-action on $W_{l, \epsilon'} \otimes W_{m, \epsilon'}$ corresponding to the pair $(\epsilon_{j_k}, \epsilon_{j_{k+1}})$ or $(\epsilon_{j_M}, \epsilon_{j_1})$.

For $0 \leq k \leq M-1$, let us denote by $\tilde{e}_{k'}$ and $\tilde{f}_{k'}$ the Kashiwara operators corresponding to $(\epsilon_{j_k}, \epsilon_{j_{k+1}})$ when $k \neq 0$ and to $(\epsilon_{j_M}, \epsilon_{j_1})$ when $k = 0$.

If we put $I' = \{k' \mid k = 0, \ldots, M-1\}$, then $L_{l, \epsilon'} \otimes L_{m, \epsilon'}$ is invariant under $\tilde{e}_{k'}$ and $\tilde{f}_{k'}$ for $k' \in I'$, and hence $B_{l, \epsilon'} \otimes B_{m, \epsilon'}/\{\pm 1\}$ is an $I'$-colored oriented graph. Since $L_{l, \epsilon'} \otimes L_{m, \epsilon'} \subset L_{l, e'} \otimes L_{m, e'}$ and $B_{l, \epsilon'} \otimes B_{m, \epsilon'} \subset B_{l, e'} \otimes B_{m, e'}$, we may regard $B_{l, \epsilon'} \otimes B_{m, \epsilon'}/\{\pm 1\}$ as an $(I \cup I')$-colored oriented graph.

Let $b = |m_1\rangle \otimes |m_2\rangle \in B_{l, \epsilon} \otimes B_{m, \epsilon}$ be given. We will show that $b$ is connected to $|e_1\rangle \otimes |m_{e_1}\rangle$, which implies that $B_{l, \epsilon} \otimes B_{m, \epsilon}/\{\pm 1\}$ is connected as an $(I \cup I')$-colored oriented graph. Let us write $m_i = (m_{i_1}, \ldots, m_{i_n})$ for $i = 1, 2$.

We first claim that there exists a sequence $i_1, \ldots, i_r \in I$ such that $(\epsilon_{i_k}, \epsilon_{i_{k+1}}) \neq (0, 0)$ for $1 \leq k \leq r$ and

$$
b' := \tilde{x}_{i_1} \cdots \tilde{x}_{i_r} b \equiv |m_1\rangle \otimes |m_2\rangle \mod qL_{l, \epsilon'} \otimes L_{m, \epsilon'}.
$$
Hence we may apply the induction hypothesis to conclude (4.9).

Suppose that there exists k with \( \epsilon_k = 1 \) such that \( m_{1k} = 1 \) or \( m_{2k} = 1 \). Let \( i \) and \( j \) be the maximal and minimal indices respectively such that \( i < k < j \) and \( \epsilon_i = \epsilon_j = 0 \). If there is no such \((i,j)\), then we have \( \epsilon = \epsilon_M/\epsilon_N \) and identify this case with the one of \( \epsilon = (0^M,1^N,0) \).

Since we will choose \( i_1, \ldots, i_r \) in \( \{i,i+1,\ldots,j-1\} \), we may assume for simplicity that \( m_{ab} = 0 \) for \( a = 1,2 \) and \( b \notin \{i,1,\ldots,j\} \).

Let us use induction on \( L = |m_1| + |m_2| \). Suppose that \( L = 1 \). If \( m_{1k} = 1 \), then \( \tilde{f}_{j-1} \tilde{f}_{j-2} \ldots \tilde{f}_i b \) satisfies (4.9). If \( m_{2k} = 1 \), then \( \tilde{e}_i \tilde{e}_{i+1} \ldots \tilde{e}_{k-1} b \) satisfies (4.9).

Suppose that \( L > 1 \). We may assume that \( \tilde{f}_{i+1} b = \tilde{f}_{i+2} b = \cdots = \tilde{f}_{j-1} b = 0 \). Then by tensor product rule in Proposition 2.5 we have

\[
\begin{align*}
m_1 &= m_{1i} e_i + \sum_{x \leq u \leq y} e_u + \sum_{z \leq v \leq j-1} e_v + m_{1j} e_j, \\
\end{align*}
\]

(4.10)

\[
\begin{align*}
m_2 &= m_{2i} e_i + \sum_{y+1 \leq u \leq j-1} e_u + m_{2j} e_j, \\
\end{align*}
\]

for some \( i < x < y < z < j \). Here we assume that \( \sum_{z \leq v \leq j-1} e_v \) in \( m_1 \) is empty if there is no such \( z \). Now we take the following steps to construct \( b' \) in (4.9).

Step 1. If there exists \( z \) such that \( y < z < j \) and \( m_{1z} \cdots = m_{1j-1} = 1 \), then by applying \( \tilde{f}_z \tilde{f}_{z-1} \ldots \tilde{f}_{j-1} \) to \( b, m_1 \) in (4.10) is replaced by

\[
\begin{align*}
m_{1i} e_i + \sum_{x \leq u \leq y} e_u + \sum_{z+1 \leq v \leq j-1} e_v + (m_{1j} + 1) e_j. \\
\end{align*}
\]

(4.11)

Repeating this step, (4.11) is replaced by

\[
\begin{align*}
m_{1i} e_i + \sum_{x \leq u \leq y} e_u + (m_{1j} + j - z) e_j. \\
\end{align*}
\]

Hence we may assume that \( m_1 \) in (4.10) is of the form \( m_{1i} e_i + \sum_{x \leq u \leq y+1} e_u + m_{1j} e_j \).

Step 2. If \( m_{1j} = 0 \), then we have

\[
\tilde{f}_{j-1} b = |m_1| \otimes |m_2 - e_{j-1} + e_j|. 
\]

Hence we may apply the induction hypothesis to conclude (4.9).

Step 3. If \( m_{ij} \neq 0 \), then by applying \( \tilde{e}_i \tilde{e}_{i+1} \ldots \tilde{e}_{j-2} \tilde{e}_{j-1} \) to \( b, m_1 \) and \( m_2 \) are replaced by

\[
\begin{align*}
m_{1i} e_i + \sum_{x \leq u \leq y+1} e_u + (m_{1j} - 1) e_j, \\
(m_{2i} + 1) e_i + \sum_{y+2 \leq v \leq j-1} e_v + m_{2j} e_j, \\
\end{align*}
\]

respectively. Repeating this step \( d \) times such that \( m_{1j} - d \geq 0 \) and \( y + d + 1 \leq j \), \( m_1 \) and \( m_2 \) are replaced by

\[
\begin{align*}
m_{1i} e_i + \sum_{x \leq u \leq y+d} e_u + (m_{1j} - d) e_j, \\
(m_{2i} + d) e_i + \sum_{y+d+1 \leq v \leq j-1} e_v + m_{2j} e_j, \\
\end{align*}
\]
respectively. We may keep this process until $m_{1j} - d = 0$, which belongs to the case in Step 2, or $\sum_{y+d+1 \leq x \leq j-1} e_v$ is empty. In the latter case, $m_1$ is replaced by $m_1 e_i + \sum_{x \leq u \leq j-1} e_u + (m_{1j} - d) e_j$, so that we may apply $\tilde{f}_{j-1}$ and use induction hypothesis to have $b'$. This proves the claim.

By construction of $b'$ and its weight, we have

$$b' - |m_1' \otimes m_2' \otimes L_{\alpha} \otimes L_{m,\epsilon'} \cap (qL_{\alpha} \otimes L_{m,\epsilon}) = qL_{\alpha} \otimes L_{m,\epsilon},$$

and hence $b' \in (L_{\alpha} \otimes L_{m,\epsilon})/qL_{\alpha} \otimes L_{m,\epsilon}) \subset (L_{\alpha} \otimes L_{m,\epsilon}) (L_{\alpha} \otimes L_{m,\epsilon})$. If $M = 2$, then it is easy to show that $b' = (m_1' \otimes m_2' \otimes B_{\alpha} \otimes B_{m,\epsilon})$ is connected to $|e_1 \otimes me_1\rangle$ under $e_k$ for $k = 0, 1$. If $M \geq 3$, then we can also show that $b' = (m_1' \otimes m_2' \otimes B_{\alpha} \otimes B_{m,\epsilon})$ is connected to $|e_1 \otimes me_1\rangle$ by using the fact that $B_{\alpha} \otimes B_{m,\epsilon}/\{\pm 1\}$ is a connected crystal of type $A_{M-1}^{(1)}$ (cf. [11]).

Finally, since $\dim_{\mathbb{Q}(q)}(W) = 1$ and $B_{\alpha} \otimes B_{m,\epsilon}/\{\pm 1\}$ is connected, it follows from [2] Lemma 2.7 that $W \otimes W$ is irreducible. This completes the proof. □

**Corollary 4.9.** For $l, m \in \mathbb{Z}_+$ and generic $x, y \in \mathbb{Q}(q)$, $W_{\alpha}(x) \otimes W_{\alpha}(y)$ is irreducible.

**Proof.** It follows from [12] Lemma 3.4.2. □

### 4.5. Existence of $R$ matrix

For $l, m \in \mathbb{Z}_+$ and generic $x, y \in \mathbb{Q}(q)$, consider a non-zero $\mathbb{Q}(q)$-linear map $R$ on $W_{\alpha}(x) \otimes W_{\alpha}(y)$ such that

$$\Delta^{op}(g) \circ R = R \circ \Delta(g),$$

for $g \in U(\alpha)$, where $\Delta^{op}$ is the opposite coproduct of $\Delta$ in [21], that is, $\Delta^{op}(g) = P \circ \Delta(g) \circ P$ and $P(a \otimes b) = b \otimes a$. We denote it by $R(z)$, where $z = x/y$, since $R$ depends only on $z$.

We say that $R(z)$ satisfies the Yang-Baxter equation if we have

$$R_{12}(u)R_{13}(uv)R_{23}(v) = R_{23}(v)R_{13}(uv)R_{12}(u),$$

on $W_{x_1}(x_1) \otimes W_{x_2}(x_2) \otimes W_{x_3}(x_3)$ with $u = x_1/x_2$ and $v = x_2/x_3$ for $(s_1), (s_2), (s_3) \in \mathcal{P}_{M+N}$. Here $R_{ij}(z)$ denotes the map which acts as $R(z)$ on the $i$-th and the $j$-th component and the identity elsewhere. We call $R(z)$ the (quantum) $R$ matrix.

**Theorem 4.10.** Let $l, m \in \mathbb{Z}_+$ given with $(l), (m) \in \mathcal{P}_{M+N}$. Suppose that $\epsilon_1 = 0$. There exists a unique non-zero linear map $R(z) \in \mathrm{End}_{\mathbb{Q}(q)}(W_{\alpha}(x) \otimes W_{\alpha}(y))$ satisfying (4.12) and (4.13), and $R(z)(|e_1\rangle \otimes |me_1\rangle) = |e_1\rangle \otimes |me_1\rangle$ for generic $x, y \in \mathbb{Q}(q)$.

**Proof.** The existence of such a map for arbitrary $\epsilon$ is proved in [17] Theorem 5.1 with respect to $\Delta_+$. Let $R_+$ be defined in (2.3), say $R_+$. Let

$$\chi = \psi \circ (\phi \otimes \phi),$$

where $\psi$ and $\phi$ are given in (2.10) and (4.11), respectively. Then

$$R := \chi^{-1} \circ R_+ \circ \chi$$

satisfies the conditions (4.10) and (4.13), and $R(z)(|e_1\rangle \otimes |me_1\rangle) = |e_1\rangle \otimes |me_1\rangle$ with respect to $\Delta$. The uniqueness follows from the irreducibility in Corollary 4.9 and normalization by $R(z)(|e_1\rangle \otimes |me_1\rangle) = |e_1\rangle \otimes |me_1\rangle$. □
We also define Lemma 5.1. For $W$:

We have as a $\mathcal{U}(\mathfrak{g})$-module.

5. Kirillov-Reshetikhin modules

5.1. Spectral decomposition. Suppose that $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ is given with $n \geq 4$. Recall that $M$ is the number of $\iota$'s with $\epsilon_i = 0$ and $N$ is the number of $\iota$'s with $\epsilon_i = 1$ in $\epsilon$.

Let $l, m \in \mathbb{Z}_+$ be given. Let $R_\epsilon(z)$ be the $R$ matrix on $W_{l,\epsilon}(x) \otimes W_{m,\epsilon}(y)$ in Theorem 4.10. We have as a $\mathcal{U}(\mathfrak{g})$-module,

$$W_{l,\epsilon}(x) \otimes W_{m,\epsilon}(y) \cong \bigoplus_{t \in H(l, m)} V_t((l + m - t, t)),$$

where $H(l, m) = \{ t | 0 \leq t \leq \min\{l, m\}, (l + m - t, t) \in \mathcal{P}_{M,N} \}$.

Let us take a sequence $\epsilon'' = (\epsilon_1'', \ldots, \epsilon_n'')$ of $0, 1$'s with $n'' \gg n$ satisfying the following:

1. $\epsilon$ is a subsequence of $\epsilon''$,
2. we have as a $\mathcal{U}(\mathfrak{g})$-module

$$W_{l,\epsilon''}(x) \otimes W_{m,\epsilon''}(y) \cong \bigoplus_{0 \leq t \leq \min\{l, m\}} V_t''((l + m - t, t)),$$

3. if $\epsilon' = \epsilon_{M'' \leq 0}$ with $M'' = |\{ i | \epsilon_i'' = 0 \}| > 0$, then we have as a $\mathcal{U}(\mathfrak{g})$-module

$$W_{l,\epsilon'}(x) \otimes W_{m,\epsilon'}(y) \cong \bigoplus_{0 \leq t \leq \min\{l, m\}} V_t'((l + m - t, t)),$$

Let $R_{\epsilon'}(z)$ and $R'_\epsilon(z)$ denote the $R$ matrices on $W_{l,\epsilon''}(x) \otimes W_{m,\epsilon''}(y)$ and $W_{l,\epsilon'}(x) \otimes W_{m,\epsilon'}(y)$, respectively.

Lemma 5.1. For $\epsilon = \epsilon$ or $\epsilon'$, we have the following commutative diagram:

$$\begin{align*}
W_{l,\epsilon''}(x) \otimes W_{m,\epsilon''}(y) & \xrightarrow{PR_{\epsilon''}(z)} W_{m,\epsilon''}(y) \otimes W_{l,\epsilon''}(x) \\
W_{l,\epsilon}(x) \otimes W_{m,\epsilon}(y) & \xrightarrow{PR_{\epsilon}(z)} W_{m,\epsilon}(y) \otimes W_{l,\epsilon}(x)
\end{align*}$$

Proof. For $\epsilon = \epsilon$ or $\epsilon'$, the restriction of $PR_{\epsilon''}(z)$ on $\text{tr}''(W_{l,\epsilon}(x) \otimes W_{m,\epsilon}(y))$, which gives a well-defined $\mathcal{U}(\mathfrak{g})$-linear endomorphism. By Proposition 4.7 and Theorem 4.10 the restricted $R$ matrix is the quantum $R$ matrix on $W_{l,\epsilon}(x) \otimes W_{m,\epsilon}(y)$, which proves the commutativity of the diagram.

For $0 \leq t \leq \min\{l, m\}$, let $v'(l, m, t)$ be the highest weight vectors of $V_{\epsilon'}((l + m - t, t))$ in $W_{l,\epsilon'}(x) \otimes W_{m,\epsilon'}(y)$ such that

$$v'(l, m, t) \in \mathcal{L}_{l,\epsilon'} \otimes \mathcal{L}_{m,\epsilon'},$$

$$v'(l, m, t) \equiv |le_i\rangle \otimes |(m - t)e_1 + te_2\rangle \pmod{q\mathcal{L}_{l,\epsilon'} \otimes \mathcal{L}_{m,\epsilon'}}.$$
as a \(\mathbb{Q}(q)\)-space, and let \(P_t^{l,m} : \mathcal{W}_{l',e'}(x) \otimes \mathcal{W}_{m,e'}(y) \to \mathcal{W}_{m,e'}(y) \otimes \mathcal{W}_{l',e'}(x)\) be a \(\hat{U}(\epsilon')\)-linear map given by
\[
P_t^{l,m}(v'(l, m, t')) = \delta_{t,t'}v'(m, l, t')\]
Then we have the following spectral decomposition of \(PR_{e'}(z)\)
\[
PR_{e'}(z) = \sum_{0 \leq t \leq \min\{l, m\}} \rho_t(z)P_t^{l,m},
\]
for some \(\rho_t(z) \in \mathbb{Q}(q)\). By Proposition 17.5 and Lemma 5.1 we also have the following spectral decomposition of \(PR_{e}(z)\)
\[
PR_{e}(z) = \sum_{0 \leq t \leq \min\{l, m\}} \rho_t(z)P_t^{l,m},
\]
(5.1)
where we understand \(P_t^{l,m}\) as defined on \(\mathcal{W}_{l,e}(x) \otimes \mathcal{W}_{m,e}(y)\). Then we have the following explicit description of \(PR_{e}(z)\), which is proved in case of \(\epsilon = \epsilon_{M|N}\) [17].

**Theorem 5.2.** We have

\[
PR_{e}(z) = \sum_{t=\max\{l+m-n, 0\}}^{\min\{l+m, n\}} \left( \prod_{i=1}^{t} \frac{z-q^{l+m-2i+2}z}{1-q^{l+m-2i+2}z} \right) P_t^{l,m} \quad (M = 0),
\]
(5.2)
\[
PR_{e}(z) = \sum_{t=0}^{\min\{l+m, n\}} \left( \prod_{i=1}^{t} \frac{1-q^{l+m-2i+2}z}{z-q^{l+m-2i+2}z} \right) P_t^{l,m} \quad (M = 1),
\]
(5.3)
\[
PR_{e}(z) = \sum_{t=0}^{\min\{l+m, n\}} \left( \prod_{i=1}^{t} \frac{1-q^{l+m-2i+2}}{z-q^{l+m-2i+2}} \right) P_t^{l,m} \quad (2 \leq M \leq n),
\]
(5.4)
where we assume that \(\rho_{\min\{l,m\}}(z) = 1\) in (5.2) and \(\rho_0(z) = 1\) in (5.3) and (5.4).

**Proof.** We may consider the case of \(1 \leq M \leq n\) only since the case when \(M = 0\) is known (see [16] (5.6) or [17] (6.10)). It is well-known that \(PR_{e'}(z)\) for \(e' = \epsilon_{N'|0}\) has the following spectral decomposition
\[
PR_{e'}(z) = \sum_{0 \leq t \leq \min\{l, m\}} \rho_t(z)P_t^{l,m},
\]
where
\[
\rho_0(z) = 1, \quad \rho_t(z) = \prod_{i=1}^{t} \frac{1-q^{l+m-2i+2}z}{z-q^{l+m-2i+2}} \quad (1 \leq t \leq \min\{l, m\}),
\]
(cf. [16] (5.8) or [17] (6.16)). We remark that \(\chi(v'(l, m, t))\) and \(\chi(v'(m, l, l))\) for \(0 \leq t \leq \min\{l, m\}\) are the same scalar multiplications of the highest weight vectors in [17] (6.14), where \(\chi\) is as in [14]. Hence it follows from (5.1) that
\[
\rho_t(z) = \rho_t'(z) \quad (t \in H(l, m)),
\]
which completes the proof. \(\square\)
5.2. Kirillov-Reshetikhin modules. As an application of Theorem 5.2, let us construct a family of irreducible $U(\epsilon)$-modules in $O_{\geq 0}$ which corresponds to usual Kirillov-Reshetikhin modules under truncation. Let us assume that $1 \leq M \leq n - 1$ since the results when $M \in \{0, n\}$ are well-known [13].

Fix $s \geq 1$ and put $V_x = W_{s,x}(x)$ for $x \in \mathbb{Q}(q)$. We take a normalization

$$\tilde{R}(z) = \left( \prod_{i=1}^{s} \frac{z - q^{2s-2l+2}}{1 - q^{2s-2l+2}z} \right) PR(z),$$

where $R(z)$ is the $R$ matrix on $V_x \otimes V_y$. Since $(s^2) \notin \mathcal{P}_{M|N}$ if and only if $M = 1$ and $s > n - 1$, we have

$$\tilde{R}(z) = \left\{ \begin{array}{ll}
\sum_{t=0}^{n-1} \left( \prod_{i=t+1}^{s} \frac{z - q^{2s-2l+2}}{1 - q^{2s-2l+2}z} \right) P_{s,t}^{s,s}, & \text{if } (s^2) \notin \mathcal{P}_{M|N}, \\
\sum_{t=0}^{s-1} \left( \prod_{i=t+1}^{s} \frac{z - q^{2s-2l+2}}{1 - q^{2s-2l+2}z} \right) P_{s,t}^{s,s}, & \text{if } (s^2) \in \mathcal{P}_{M|N}.
\end{array} \right.$$

For $r \geq 2$, let $W$ denote the group of permutations on $r$ letters generated by $s_i = (i, i+1)$ for $1 \leq i \leq r - 1$. By Theorem 4.10, we have $U(\epsilon)$-linear maps

$$\hat{R}_w(x_1, \ldots, x_r) : V_{x_1} \otimes \cdots \otimes V_{x_r} \rightarrow V_{x_{w(1)}} \otimes \cdots \otimes V_{x_{w(r)}}$$

for $w \in W$ and generic $x_1, \ldots, x_r$ satisfying the following:

$$\hat{R}_1(x_1, \ldots, x_r) = \text{id}_{V_{x_1}} \otimes \cdots \otimes \text{id}_{V_{x_r}},$$

$$\hat{R}_{s_i}(x_1, \ldots, x_r) = \left( \otimes_{j < i} \text{id}_{V_{x_j}} \right) \otimes \hat{R}(x_i/x_{i+1}) \otimes \left( \otimes_{j > i+1} \text{id}_{V_{x_j}} \right),$$

$$\hat{R}_{ww'}(x_1, \ldots, x_r) = \hat{R}_w(x_{w(1)}, \ldots, x_{w(r)}) \hat{R}_w(x_1, \ldots, x_r),$$

for $w, w' \in W$ with $\ell(ww') = \ell(w) + \ell(w')$. Let $w_0$ denote the longest element in $W$. By Theorem 5.2, $\hat{R}_{w_0}(x_1, \ldots, x_r)$ does not have a pole at $q^{2k}$ for $k \in \mathbb{Z}_+$ as a function in $x_1, \ldots, x_r$. Hence we have a $U(\epsilon)$-linear map

$$\hat{R}_r := \hat{R}_{w_0}(q^{-1}, q^{-3}, \ldots, q^{1-r}) : V_{q^{-r}} \otimes \cdots \otimes V_{q^{1-r}} \rightarrow V_{q^{1-r}} \otimes \cdots \otimes V_{q^{1-r}}.$$

Then we define a $U(\epsilon)$-module

$$(5.5) \quad W_{s,\epsilon}^{(r)} := \text{Im} \hat{R}_r.$$

It is proved in [16] that $W_{s,\epsilon}^{(r)}$ is irreducible when $\epsilon = \epsilon_{M|N}$, where the proof uses the crystal base of polynomial representation of $U_{M|N}(\epsilon)$. Now we give another proof of the irreducibility of $W_{s,\epsilon}^{(r)}$, which is available for arbitrary $\epsilon$.

**Theorem 5.3.** Let $r, s \geq 1$ be given. Then $W_{s,\epsilon}^{(r)}$ is non-zero if and only if $(s^r) \in \mathcal{P}_{M|N}$. In this case, $W_{s,\epsilon}^{(r)}$ is irreducible, and it is isomorphic to $V_{\epsilon}(s^r)$ as a $U(\epsilon)$-module.

**Proof.** Let us take a sequence $\epsilon'' = (\epsilon''_1, \ldots, \epsilon''_n)$ of 0, 1’s satisfying the following:

1. $\epsilon$ is a subsequence of $\epsilon''$. 


(2) we have as a $\tilde{U}(\epsilon'')$-module

\begin{equation}
V_\epsilon'(\langle s \rangle)_{\otimes r} \cong \bigoplus_{\lambda \in \mathcal{G}} V_\epsilon'(\lambda)^{\otimes K_\lambda(r)},
\end{equation}

where $K_\lambda(r)$ is the Kostka number associated to $\lambda$ and $(r)$ (cf. Remark 3.3).

(3) if $\epsilon' = \epsilon_{M''0}$ with $M'' = \{ i | \epsilon''_i = 0 \}$, then we have as a $\tilde{U}(\epsilon')$-module

\begin{equation}
V_\epsilon'(\langle s \rangle)_{\otimes r} \cong \bigoplus_{\lambda \in \mathcal{G}} V_\epsilon'(\lambda)^{\otimes K_\lambda(r)}.
\end{equation}

Let us define a $\tilde{U}(\epsilon'')$-module $\mathcal{W}^{(r)}_{s,\epsilon''}$ by the same way as in (5.5), where $\tilde{R}''_r$ and $V''_r$ denote the corresponding ones. We define $\mathcal{W}^{(r)}_{s,\epsilon''}$, $\tilde{R}''_r$ and $V''_r$ similarly.

By Lemma 5.1 we have the following commutative diagram:

\begin{align*}
V''_{q_{r-1}} \otimes \cdots \otimes V''_{q_{1-r}} & \hspace{1cm} \tilde{R}''_r \hspace{1cm} V''_{q_{r-1}} \otimes \cdots \otimes V''_{q_{1-r}} \\
\downarrow \text{tr}''_{r'} & \hspace{1cm} \downarrow \text{tr}''_{r'} \\
V'_{q_{r-1}} \otimes \cdots \otimes V'_{q_{1-r}} & \hspace{1cm} \tilde{R}'_r \hspace{1cm} V'_{q_{r-1}} \otimes \cdots \otimes V'_{q_{1-r}}
\end{align*}

By (5.6), (5.7) and Proposition 4.5 the decomposition of $\mathcal{W}^{(r)}_{s,\epsilon''}$ into polynomial $\tilde{U}(\epsilon'')$-modules is the same as that of $\mathcal{W}^{(r)}_{s,\epsilon'}$ into polynomial $\tilde{U}(\epsilon')$-modules. It is well-known that $\mathcal{W}^{(r)}_{s,\epsilon'}$ is irreducible and isomorphic to $V_\epsilon'((s'))$ as a $\tilde{U}(\epsilon')$-module since $\tilde{U}(\epsilon'') \cong U_q(A^{(1)}_{1-r-1})$.

Therefore, $\mathcal{W}^{(r)}_{s,\epsilon'}$ is irreducible and isomorphic to $V_\epsilon'((s'))$ as a $\tilde{U}(\epsilon'')$-module.

Again by Lemma 5.1 we have the following commutative diagram:

\begin{align*}
V''_{q_{r-1}} \otimes \cdots \otimes V''_{q_{1-r}} & \hspace{1cm} \tilde{R}''_r \hspace{1cm} V''_{q_{r-1}} \otimes \cdots \otimes V''_{q_{1-r}} \\
\downarrow \text{tr}''_{r'} & \hspace{1cm} \downarrow \text{tr}''_{r'} \\
V'_{q_{r-1}} \otimes \cdots \otimes V'_{q_{1-r}} & \hspace{1cm} \tilde{R}'_r \hspace{1cm} V'_{q_{r-1}} \otimes \cdots \otimes V'_{q_{1-r}}
\end{align*}

Since $\text{tr}''_{r'}(V_\epsilon'((s'))) \neq 0$ if and only if $(s') \in \mathcal{P}_{M,N}$, which is equal to $V_\epsilon'((s'))$ in this case, it follows that $\mathcal{W}^{(r)}_{s,\epsilon'}$ is non-zero if and only if $(s') \in \mathcal{P}_{M,N}$. This implies in this case that $\mathcal{W}^{(r)}_{s,\epsilon'}$ is irreducible, and it is isomorphic to $V_\epsilon'((s'))$ as a $\tilde{U}(\epsilon')$-module.

The following can be proved by similar arguments.

**Corollary 5.4.** Suppose that $(s') \in \mathcal{P}_{M,N}$ is given.

1. If $r \leq M$ and $M \geq 3$, then $\text{tr}''_{r'}(\mathcal{W}^{(r)}_{s,\epsilon'})$ is the Kirillov-Reshetikhin module of type $A^{(1)}_{M-1}$ corresponding to the partition $(s')$, where $\epsilon' = \epsilon_{M0}$.

2. If $s \leq N$ and $N \geq 3$, then $\text{tr}''_{r'}(\mathcal{W}^{(r)}_{s,\epsilon'})$ is the Kirillov-Reshetikhin module of type $A^{(1)}_{N-1}$ corresponding to the partition $(r^s)$, where $\epsilon' = \epsilon_{0N}$.

**Remark 5.5.** As in case of $\epsilon = \epsilon_{M,N}$ [16], we also expect that $\mathcal{W}^{(r)}_{s,\epsilon}$ has a crystal base for arbitrary $\epsilon$ (cf. Remark 3.3).
One may use a similar argument as in the proof of Theorem 5.3 to prove the irreducibility of a tensor product of $W_{l,\epsilon}(x)$’s and its image under $R$ matrix in some special cases. Let $l_1, \ldots, l_r \in \mathbb{Z}_+$ and $x_1, \ldots, x_r \in \mathbb{Q}(q)$ be given and let $\epsilon' = \epsilon_{M|0}$.

**Proposition 5.6.** If $M$ is sufficiently large and $W_{l,\epsilon'}(x_1) \otimes \cdots \otimes W_{l,\epsilon'}(x_r)$ is irreducible, then $W_{l,\epsilon'}(x_1) \otimes \cdots \otimes W_{l,\epsilon'}(x_r)$ is also irreducible.

**Proof.** Suppose that $W_{l,\epsilon'}(x_1) \otimes \cdots \otimes W_{l,\epsilon'}(x_r)$ is not irreducible and let $W$ be a proper non-trivial submodule. Since $M$ is sufficiently large, the multiplicity of $V(\lambda)$ for $\lambda \in \mathcal{P}$ in $W_{l,\epsilon'}(x_1) \otimes \cdots \otimes W_{l,\epsilon'}(x_r)$ is equal to that of $V(\lambda')$ for $\lambda' \in \mathcal{P}$ in $W_{l,\epsilon'}(x_1) \otimes \cdots \otimes W_{l,\epsilon'}(x_r)$ (cf. Remark 5.5). This also holds for $W$ and $\mathcal{R}_{\epsilon'}(W)$, which implies that $\mathcal{R}_{\epsilon'}(W)$ is a proper non-zero subspace of $\mathcal{R}_{\epsilon'}(W_{l,\epsilon'}(x_1) \otimes \cdots \otimes W_{l,\epsilon'}(x_r))$. Since $\mathcal{R}_{\epsilon'}(W) = W \cap \mathcal{R}_{\epsilon'}(W_{l,\epsilon'}(x_1) \otimes \cdots \otimes W_{l,\epsilon'}(x_r))$, it follows that $\mathcal{R}_{\epsilon'}(W)$ is a proper non-zero $\mathcal{U}(\epsilon')$-submodule, which is a contradiction. □

**Remark 5.7.** Proposition 5.6 together with the irreducibility of $W_{l,\epsilon'} \otimes W_{m,\epsilon'}$ also implies Theorem 1.8 when $M \geq 3$. But we do not know whether it holds for $M = 2$. We also would like to point out that the proof of Theorem 1.8 has its own interest since it describes a new connected crystal graph structure on $\mathcal{B}_{l,\epsilon} \otimes \mathcal{B}_{m,\epsilon}/\{\pm 1\}$.

**Proposition 5.8.** Suppose that $x_i/x_{i+1} \not\in \mathbb{Q}^{2\mathbb{Z}_+}$ for $1 \leq i \leq r - 1$. If $M$ is sufficiently large and the image of

$$
\hat{R}_{w_0}(x_1, \ldots, x_r) : W_{l,\epsilon'}(x_1) \otimes \cdots \otimes W_{l,\epsilon'}(x_r) \rightarrow W_{l,\epsilon'}(x_r) \otimes \cdots \otimes W_{l,\epsilon'}(x_1)
$$

is irreducible, then the image of

$$
\hat{R}_{w_0}(x_1, \ldots, x_r) : W_{l,\epsilon'}(x_1) \otimes \cdots \otimes W_{l,\epsilon'}(x_r) \rightarrow W_{l,\epsilon'}(x_r) \otimes \cdots \otimes W_{l,\epsilon'}(x_1)
$$

is also irreducible, where $\hat{R}_{w_0}(x_1, \ldots, x_r)$ is the restriction of $\hat{R}_{w_0}(x_1, \ldots, x_r)$ on $W_{l,\epsilon'}(x_1) \otimes \cdots \otimes W_{l,\epsilon'}(x_r)$.

**Proof.** It follows from Lemma 5.1 and the same argument as in Proposition 5.6. □

**References**

[1] T. Akasaka, M. Kashiwara, Finite-dimensional representations of quantum affine algebras, Publ. Res. Inst. Math. Sci. 33 (1997) 839–867.

[2] G. Benkart, S.-J. Kang, M. Kashiwara, Crystal bases for the quantum superalgebra $U_q(\mathfrak{gl}(m,n))$, J. Amer. Math. Soc. 13 (2000) 295–331.

[3] Č. Burdík, R. C. King, T. A. Welsh, The explicit construction of irreducible representations of the quantum algebras $U_q(\mathfrak{sl}(n))$, AIP Conference Proceedings 589 (2001), 158–169.

[4] A. Berele, A. Regev, Hook Young diagrams with applications to combinatorics and to representations of Lie superalgebras, Adv. Math. 64 (1987) 118–175.

[5] V. Chari, D. Hernandez, Beyond Kirillov-Reshetikhin modules, Contemp. Math. 506 (2010) 49–81.

[6] S.-J. Cheng, N. Lam, Irreducible characters of general linear superalgebra and super duality, Comm. Math. Phys. 298 (2010) 645–672.

[7] J. Cheng, Y. Wang, R.B. Zhang, Degenerate quantum general linear groups, preprint (2018) arXiv:1806.07191.
Department of Mathematical Sciences and Research Institute of Mathematics, Seoul National University, Seoul 08826, Korea
E-mail address: jaehoonku@snu.ac.kr

Department of Mathematical Sciences, Seoul National University, Seoul 08826, Korea
E-mail address: ycw453@snu.ac.kr