FLUCTUATIONS OF THE LONGEST COMMON SUBSEQUENCE IN THE ASYMMETRIC CASE OF 2- AND 3-LETTER ALPHABETS

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October 10, 2018

Abstract

We investigate the asymptotic standard deviation of the Longest Common Subsequence (LCS) of two independent i.i.d. sequences of length $n$. The first sequence is drawn from a three letter alphabet $\{0, 1, a\}$, whilst the second sequence is binary. The main result of this article is that in this asymmetric case, the standard deviation of the length of the LCS is of order $\sqrt{n}$. This confirms Waterman’s conjecture [22] for this special case. Our result seems to indicate that in many other situations the order of the standard deviation is also $\sqrt{n}$.

1 Introduction

In computational genetics and computational linguistics one of the basic problem is to find an optimal alignment between two given sequences $X := X_1 \ldots X_n$ and $Y := Y_1 \ldots Y_n$. This requires a scoring system which can rank the alignments. Typically a substitution matrix gives the score for each possible pair of letters. The total score of an alignment is the sum of terms for each aligned pair of residues, plus a usually negative term for each gap (gap penalty).

Let us look at an example. Take the sequences $X$ and $Y$ to be binary sequences. Let the substitution matrix be equal to:

$$
\begin{array}{c|cc}
0 & 0 & 1 \\
2 & 2 & 1 \\
1 & 1 & 3 \\
\end{array}
$$

With the above matrix we get the following scores for pairs of letter:

\[ s(0, 0) = 2, s(0, 1) = s(1, 0) = 1, s(1, 1) = 3. \]

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(Here, \( s(a, b) \) designates the score when we align letter \( a \) with letter \( b \).) Take \( X = 0101 \) and \( Y = 1100 \) with the above substitution matrix and a zero gap penalty. The optimal alignment is:

\[
\begin{array}{cccccc}
0 & 1 & 0 & 1 & - & - \\
- & 1 & - & 1 & 0 & 0 \\
\end{array}
\]

The above alignment gives the score \( s(1, 1) + s(1, 1) = 3 + 3 = 6 \). This is the alignment with maximal score. We denote by \( L_n \) the maximal alignment-score. In our example \( L_n = 6 \).

Let \( \{X_i\}_{i \in \mathbb{N}} \) and \( \{Y_i\}_{i \in \mathbb{N}} \) be two ergodic processes independent of each other. Let \( L_n \) denote the optimal alignment score of the two finite sequences \( X := X_1 \ldots X_n \) and \( Y := Y_1 \ldots Y_n \). Waterman \[22\] conjectured that in many cases \( \text{VAR}[L_n] \) is of order \( n \).

Throughout this paper the substitution matrix is equal to the identity and there is no gap penalty. In this case, the optimal score is equal to the length of the Longest Common Subsequence (LCS) of \( X \) and \( Y \). (A common subsequence of \( X \) and \( Y \) is a sequence which is a subsequence of \( X \) as well as of \( Y \).)

We take the two sequences \( X \) and \( Y \) to be i.i.d. sequences. The letters of \( X \) are drawn from the three letter alphabet \( \{0, 1, a\} \) and \( Y \) is a binary sequence. The main result of this article is that \( \text{VAR}[L_n] \) is of order \( n \).

Let \( X = a11a1000 \) and \( Y = 00110011 \). We remove the \( a \)'s from \( X \) and obtain the binary sequence \( X' = 111000 \). The length of the LCS of \( X \) and \( Y \) is equal to the length of the LCS of \( X' \) and \( Y \), since no \( a \)'s appear in \( Y \). In this example a LCS is 1100 and corresponds to the following alignment:

\[
\begin{array}{cccccc}
1 & - & - & 1 & 1 & 0 & 0 & - & - \\
- & 0 & 0 & 1 & 1 & 0 & 0 & - & 1 & 1 \\
\end{array}
\]

(The letters which appear stacked one on top of the other correspond to the letters of the common subsequence 1100.)

The reader might wonder why the case considered in the present article is relevant. Three letters in one sequence and two in the other might seem an unrealistic example. Our motivation is the following: in any i.i.d. sequence there are finite patterns (i.e. finite words) which tend to have below-average expected matching scores. The number of times any given finite pattern occurs in \( X = X_1 \ldots X_n \) is roughly a binomial variable with variance proportional to \( n \). Hence, the number of times we observe a given pattern in \( Y \) behaves roughly like the number of \( a \)'s in \( Y \). The number of \( a \)'s in \( Y \), decreases the optimal score linearly. For a given finite pattern with low average matching score the same should be true.

Bonetto and Matzinger \[10\] simulated the situation where both sequences \( X \) and \( Y \) are binary i.i.d.. When the zeros and one's are equally likely (and for \( n \) around 10000), surprisingly the standard deviation of \( L_n \) is of order \( o(n^{1/3}) \). This is similar to the behavior of the Longest Increasing Sequence (LIS) as studied by Baik-Deift-Johansson \[9\] and Aldous-Diaconis \[1\]. On the other hand, Durringer, Lember and Matzinger \[18\] investigate the case where the sequence \( Y \) is periodic with a short period. They also find the standard deviation of \( L_n \) to be of order \( o(\sqrt{n}) \). Hence a small change in the parameters of the model can change the asymptotic behaviour of the random variable \( L_n \) completely.
Let us mention a little bit of the history of these problems: Using a sub additivity argument, Chvátal-Sankoff [11] prove that the limit
\[ \gamma := \lim_{n \to \infty} \frac{E[L_n]}{n} \]
exists. The exact value of \( \gamma \) remains however unknown. Chvátal-Sankoff [11] derive upper and lower bounds for \( \gamma \), and similar upper bounds were found by Baeza-Yates, Gavaldà, Navarro and Scheihing [8] using an entropy argument. These bounds have been improved by Deken [14], and subsequently by Dancik-Paterson [13, 20]. In [16], Hauser, Martinez and Matzinger developed a Monte Carlo and large deviation-based method which allows to further improve the upper bounds on \( \gamma \). Their approach can be seen as a generalization of the method of Dancik-Paterson.

For sequence with many letters, Kiwi, Loebl and Matousek, [17] have the following interesting result:
when both sequences \( X \) and \( Y \) are drawn from the alphabet \( \{1, 2, \ldots, k\} \) and the letters are equiprobable, then \( \gamma \to 2/\sqrt{k} \) as \( k \to \infty \).

Waterman-Arratia [7] derive a law of large deviation for \( L_n \) for fluctuations on scales larger than \( \sqrt{n} \). The order of magnitude of the deviation from the mean of \( L_n \) is unknown, and in fact it is not even known if these deviations are larger than a power of \( n \). However, using first passage percolation methods, Alexander [2] proves that \( E[L_n]/n \) converges at a rate of order at least \( \sqrt{\log n/n} \).

Waterman [22] studies the statistical significance of the results produced by sequence alignment methods. An important problem that was open for decades concerns the longest increasing subsequence (LIS) of random permutations and appears to be related to the LCS-problem. However, it is an open question to know if solutions of the LIS-problem can be used to study the LCS problem, see Baik-Deift-Johansson [9] and Aldous-Diaconis [1].

Another problem related to the LCS-problem is that of comparing sequences \( X \) and \( Y \) by looking for longest common words that appear both in \( X \) and \( Y \), and generalizations of this problem where the word does not need to appear in exactly the same form in the two sequences. The distributions that appear in this context have been studied by Arratia-Gordon-Goldstein-Waterman [3] and Neuhauser [19]. A crucial role is played by the Chen-Stein Method for the Poisson-Approximation. Arratia-Gordon-Waterman [4, 5] shed some light on the relation between the Erdős-Rényi law for random coin tossing and the above mentioned problem. In [6] the same authors also developed an extreme value theory for this problem.

For a general discussion of the relevance of string comparison for biology and of other similar problem in computational biology the reader can refer to the standard texts [15, 21, 12].

2 Main result

Throughout this paper \( \{X_i\}_{i \in \mathbb{N}} \) and \( \{Y_i\}_{i \in \mathbb{N}} \) are two i.i.d. sequences which are independent of each other and which satisfy all of the following three conditions:
1. The variables $X_i, i \in \mathbb{N}$, have state space $\{0, 1, a\}$.

2. There exists $p, 0 < p < 1$ such that

$$P(X_1 = a) = p, \quad P(X_1 = 0) = P(X_1 = 1) = \frac{1-p}{2}. \quad (2.1)$$

3. The variables $Y_i, i \in \mathbb{N}$, are Bernoulli variables with parameter $1/2$.

When all the three conditions above are satisfied, we say we are in case I. The main result of this paper is:

**Theorem 2.1** When we are in case I, there exists $k > 0$ not depending on $n$, such that for all $n \in \mathbb{N}$, we have

$$\text{VAR}[L_n] \geq k \cdot n. \quad (2.2)$$

There is also an upper bound for the variance

$$\text{VAR}[L_n] \leq K \cdot n$$

where $K > 0$ is a constant not depending on $n$. This upper bound follows directly from the large deviation result for LCS of Waterman-Arratia [7]. Let us give this result:

**Lemma 2.1** Assume that we are in case I, then: there exists a constant $c > 0$ (not depending on $n$ and $\Delta$) such that for all $n$ large enough and all $\Delta > 0$, we have that:

$$P(|L_n - E[L_n]| \geq n\Delta) \leq e^{-cn\Delta^2} \quad (2.3)$$

Theorem 2.1 and lemma 2.1 together imply that the typical size of $L_n - E[L_n]$ is $o(\sqrt{n})$. More precisely, let $D_n := (L_n - E[L_n])/\sqrt{n}$ denote the rescaled fluctuation of $L_n$. Then:

**Theorem 2.2** The sequence $\{D_n\}$ is tight. Moreover, the limit of any weakly convergent subsequence of $\{D_n\}$ is not a Dirac measure.

Theorem 2.2 is a rather direct consequence of theorem 2.1 and lemma 2.1. We refer the reader to [18] for the proof.

### 3 Proof of main theorem

Let $N^a$ designate the numbers of $a$’s in the sequence $X = X_1X_2\ldots X_n$. Let $X^{01}$ designate the subsequence of $X$ consisting of all the 0’s and 1’s contained in $X$. In other words, $X^{01}$ is obtained by removing the $a$’s from the finite sequence $X$. Thus, $X^{01}$ is a finite sequence of i.i.d. Bernoulli variables with parameter $1/2$ with random length. The length of the random binary string $X^{01}$ is equal to $(n - N^a)$.

Let us illustrate this with a practical example. For $n = 6$, assume that $X = 011a0a$ and $Y = 101011$. In this case $N^a = 2$ and $X^{01} = 0110$. Obviously the $a$’s from sequence $X$ can not be matched since $Y$ does not contain any $a$’s. Hence, The length $L_6$ of the LCS of
X and Y is equal to the length of the LCS between $X^{01}$ and $Y$. The length of the LCS is $L_6 = 3$. There are actual three longest common subsequences: 011, 010 and 110.

The main idea why $L_n$ fluctuates on the scale $\sqrt{n}$ is the following: The binomial variable $N^a$ has variance of order $o(n)$. The variable $L_n$ tends to decrease linearly with an increase of $N^a$ (since the $a$’s are not matched and thus constitute losses). Hence $L_n$ should also fluctuate on the scale $\sqrt{n}$.

To prove this rigorously, we simulate the variable $L_n$ in a special way. We first simulate a variable with same distribution as $N^a$. (We can call it $N^{a'}$.) Then we generate $X^{01}$ by using a drop-scheme of random bits. Instead of flipping a coin independently $n - N^a$ times in a row we generate a sequence $Z_1, Z_2, \ldots$ of binary strings where $Z_k$ has length $k$. $Z^{k+1}$ is obtained by adding to $Z^k$ a random bit at a random location.

For example, assume that we have the binary string $Z^6 = 00010$. There are four possible positions where the next bit could come:

| position 1 | position 2 | position 3 | position 4 |
|------------|------------|------------|------------|
| 0x0010     | 00x010     | 000x10     | 0001x0     |

where $x$ designates the possible position of the next bit. We assign the same probability to each of the four above possibilities and draw one of them at random. We flip a fair coin, and fill the previously chosen position with the number obtained from the fair coin. If the position chosen is the second one and the fair coin gives us a 1, then we obtain $Z^7 = 001010$.

We apply this scheme recursively on $k$ and obtain a sequence of random binary strings $Z^1, Z^2, \ldots, Z^n$. Let $Z_i^k$ designate that $i$-th bit of the $k$-th string. With that notation:

$$Z^k = Z_1^k Z_2^k \ldots Z_k^k.$$ 

Hence, $\{Z_i^k\}_{i\leq k\leq n}$ is a triangular array of Bernoulli variables. Let us next define the $Z^k$’s in a formal way: let $V_k, k \in \mathbb{N}$ be a sequence of i.i.d. Bernoulli variables with parameter 1/2. Let $T_k, k \in \mathbb{N}$ be a sequence of independent integer variables, so that $\{V_k\}_{k \in \mathbb{N}}$ is independent of $\{T_k\}_{k \in \mathbb{N}}$. Furthermore, for $k \in \mathbb{N}$, let the distribution of $T_{k+1}$ be the uniform distribution on the set $\{2, \ldots, k\}$, (i.e. for all $s \in \{1, \ldots, k\}$, we have that $P(T_k = s) = 1/(k-1)$.) We define $Z^k$ recursively in $k$:

- Let $Z^2 := V_1 V_2$.
- Given the binary string $Z^k = Z_1^k Z_2^k \ldots Z_k^k$, we define $Z^{k+1}$:
  - For all $j < T_{k+1}$, let $Z_j^{k+1} := Z_j^k$.
  - For $j = T_{k+1}$, let $Z_j^{k+1} = V_{k+1}$.
  - For $j$, such that $T_{k+1} < j \leq k + 1$, let $Z_j^{k+1} := Z_{j-1}^k$.

(Thus $V_k$ designates the $k$-th bit added and $T_k$ designates the position where it gets added.)
To prove the main result of this paper, we generate a variable having same distribution as $L_n$ using the bit-drop-scheme. Instead of generating the sequence $X$, we generate the triangular array $\{Z^k_i\}_{i \leq k \leq n}$ and, independently, a random number $N^a$ with binomial distribution with parameters $p$ and $n$. Then, we look for the longest common subsequence of $Y$ and $Z^k$ with $k = n - N^a$.

More precisely, let $L^a_n(k)$ designate the length of the Longest Common Subsequence of $Z^k$ and $Y = Y_1Y_2\ldots Y_n$. Then:

**Lemma 3.1** Assume that case I holds and $Z^k$ is generated independently of $Y$ and $N^a$, according to the mechanism described above. Then, $L_n$ has same distribution as $L^a_n(n - N^a)$.

**Proof.** For every $l, k \geq 0$ we have that $P(L_n = l|N^a = k) = P(L^a_n(n - k) = l)$. This gives the thesis. □

We can now explain the main idea behind the proof of Theorem 2.1: assume $f$ is a map with bounded slope so that $f'(x) \geq c > 0$ for all $x \in \mathbb{R}$. Let $B$ be any random variable. Lemma 3.2 tells us, that in this case, the variance of $f(B)$ is bounded below by $c^2 \cdot \text{VAR}[B]$. On the other hand, the map $k \mapsto L^a_n(k)$ is very likely to increase above a linear rate larger than a constant $k_1 > 0$. Hence $\text{VAR}[L_n] = \text{VAR}[L^a_n(n = N^a)]$ should be larger then $k_1^2 \text{VAR}[N^a]$. The most difficult part in the proof is showing that with high probability the slope of $k \mapsto L^a_n(k)$ is "everywhere" bounded below by a positive constant. This problem is solved in the next section. Let us look at the details of the proof of Theorem 2.1.

**Lemma 3.2** Let $c > 0$. Assume that $f : \mathbb{R} \to \mathbb{R}$ is a map which is everywhere differentiable and such that for all $x \in \mathbb{R}$:

$$\frac{df}{dx} \geq c. \quad (3.1)$$

Let $B$ be a random variable such that $E[|f(B)|] \lt +\infty$. Then:

$$\text{VAR}[f(B)] \geq c^2 \cdot \text{VAR}[B]. \quad (3.2)$$

**Proof.** We have that $E[B]$ and $E[f(B)]$ are finite. Observe that $\lim_{x \to \pm\infty} f(x) = \pm\infty$ and $f(x)$ is strictly increasing so that there exists $x_0 \in \mathbb{R}$ such that

$$f(x_0) = E[f(B)]. \quad (3.3)$$

By the mean value theorem, we know that there exists a map $\delta : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$ we have

$$f(x) = f(x_0) + f'(\delta(x))(x - x_0). \quad (3.4)$$

By definition of variance and eqs. (3.3) (3.4) we have:

$$\text{VAR}[f(B)] = E[(f(B) - f(x_0))^2] = E[f'(\delta(B))^2 (B - x_0)^2] \quad (3.5)$$

\[6\]
Using eq. (3.1) we get:
\[ \text{VAR}[f(B)] \geq c^2 E[(B - x_0)^2]. \] (3.6)

Observe that
\[ E[(B - x_0)^2] \geq \min_y E[(B - y)^2] = \text{VAR}[B] \] (3.7)

where we used a well known minimizing property of the variance. This immediately gives
\[ \text{VAR}[f(B)] \geq c^2 \text{VAR}[B] \] (3.8)

which finishes this proof. ■

Typically, the (random) map \( k \mapsto L^a(k) \) does not strictly increase for every \( k \in [0, n] \). But it is likely that every order \( o(\ln n) \) points, it increases by a linear quantity. Next we define an event which guarantees that the map \( k \mapsto L^a(k) \) increases linearly on the scale \( o(\ln n) \):

**Definition 3.1** Let \( E_{\text{slope}}^n \) designate the event that \( \forall i, j, \) such that \( 0 < i < j \leq n \) and \( i + k_2 \ln n \leq j \), we have:
\[ L^a(j) - L^a(i) \geq k_1 |i - j|. \] (3.9)

Here \( k_1, k_2 > 0 \) designate constants which do not depend on \( n \) and which will be fixed in the proofs in sects. 4,5.

The above definition gives the discrete equivalent of condition (3.1) in the case of a discrete function. Before proceeding we need a discrete version of Lemma 3.2.

**Lemma 3.3** Let \( c, m > 0 \) be two constants. Let \( f : \mathbb{Z} \to \mathbb{Z} \) be a non decreasing map such that:
- for all \( i < j \):
  \[ f(j) - f(i) \leq (j - i) \] (3.10)
- for all \( i, j \) such that \( i + m \leq j \):
  \[ f(j) - f(i) \geq c \cdot (j - i). \] (3.11)

Let \( B \) be an integer random variable such that \( E[|f(B)|] \leq +\infty \). Then:
\[ \text{VAR}[f(B)] \geq c^2 \left( 1 - \frac{2m}{c \sqrt{\text{VAR}[B]}} \right) \text{VAR}[B]. \] (3.12)

**Proof.** Because of conditions (3.10) and (3.11), we can find a continuously differentiable map \( g : \mathbb{R} \to \mathbb{R} \) satisfying the following conditions:
- \( g \) agrees with \( f \) on every integer which is a multiple of \( m \).
- \( \forall x \in \mathbb{R} \), we have that \( c \leq g'(x) \leq 1. \) (3.13)
Thus, we can apply lemma 3.2 to \( g(B) \) and find:

\[
\text{VAR}[g(B)] \geq c^2 \cdot \text{VAR}[B].
\] (3.14)

The random variable \( g(B) \) approximates \( f(B) \):

\[
|f(B) - g(B)| \leq (1 - c) \cdot m
\] (3.15)

Hence,

\[
\text{VAR}[f(B) - g(B)] \leq m^2
\] (3.16)

Since, \( f(B) = g(B) + (f(B) - g(B)) \), we can apply the triangular inequality and find:

\[
\sqrt{\text{VAR}[f(B) - g(B)]} \geq \sqrt{\text{VAR}[f(B)]} - \sqrt{\text{VAR}[g(B)]} - \sqrt{\text{VAR}[f(B) - g(B)]}
\] (3.17)

Hence:

\[
\text{VAR}[f(B)] \geq \text{VAR}[g(B)] - 2\sqrt{\text{VAR}[f(B)]} \cdot \sqrt{\text{VAR}[f(B) - g(B)]} =
\text{VAR}[g(B)] \left(1 - \frac{2\sqrt{\text{VAR}[f(B) - g(B)]}}{\sqrt{\text{VAR}[g(B)]}}\right).
\]

Applying the inequalities (3.14) and (3.16) to the last inequality above, yields

\[
\text{VAR}[f(B)] \geq c^2 \text{VAR}[B] \left(1 - \frac{2m}{c\sqrt{\text{VAR}[B]}}\right)
\] (3.18)

which finishes this proof. 

Let \( \sigma_Z \) designate the \( \sigma \)-algebra of the triangular array \( Z^k_i \) and \( \sigma_{YZ} \) the \( \sigma \)-algebra of the triangular array \( Z^k_i \) and of the \( Y_i \). Thus:

\[
\sigma_Z := \sigma(Z^k_i|i \leq k \leq n) \quad \sigma_{YZ} := \sigma(Z^k_i, Y_j|i \leq k \leq n, j \leq n).
\]

We are now ready for the proof of the main theorem 2.1 of this article.

**Proof of theorem 2.1** By Lemma 3.1 it is enough to prove that there exits \( k > 0 \) not depending on \( n \), such that:

\[
\text{VAR}[L^n(a(n - N^n))] \geq kn.
\] (3.19)

Note that for any random variable \( D \) and any \( \sigma \)-field \( \sigma \), we have

\[
\text{VAR}[D] = \text{VAR}[E[D|\sigma]] + E[\text{VAR}[D|\sigma]].
\] (3.20)

Thus, since the variance is never negative, we find that

\[
\text{VAR}[D] \geq E[\text{VAR}[D|\sigma]].
\] (3.21)

Taking \( L^n(a(n - N^n)) \) for \( D \) and \( \sigma_{YZ} \) for \( \sigma \), we find:

\[
\text{VAR}[L^n(a(n - N^n))] \geq E[\text{VAR}[L^n(a(n - N^n))|\sigma_{YZ}]].
\] (3.22)
Note that the map $L^a(\cdot)$ is $\sigma_{YZ}$-measurable. Thus, conditional on $\sigma_{YZ}$, $L^a(\cdot)$ becomes a non-random increasing map. The event $E_{\text{slope}}^n$ is $\sigma_{YZ}$-measurable. When $E_{\text{slope}}^n$ holds, then the hypotheses of Lemma 3.3 holds for $f = L^a(\cdot)$ with $c = k_1$ and $m = k_2 \ln n$. This implies that

$$\text{VAR}[L^a(n - N^a)|\sigma_{YZ}] \geq (k_1)^2 \left( 1 - \frac{2k_2 \ln n}{k_1 \sqrt{\text{VAR}[N^a|\sigma_{YZ}]}} \right) \text{VAR}[N^a|\sigma_{YZ}] \quad (3.23)$$

Since $N^a$ is a binomial variable with parameter $p$ and $n$ and is independent from $\sigma_{YZ}$, we have that

$$\text{VAR}[N^a] = \text{VAR}[N^a|\sigma_{YZ}] = np(1 - p). \quad (3.24)$$

Using the last equality with inequality (3.23), we obtain:

$$\text{VAR}[L^a(n - N^a)|\sigma_{YZ}] \geq np(1 - p) \left( k_1 \right)^2 \left( 1 - \frac{2k_2 \ln n}{k_1 \sqrt{p(1 - p)n}} \right) \quad (3.25)$$

Since, $\text{VAR}[L^a(n - N^a)|\sigma_{YZ}]$ is never negative and since inequality (3.25) holds, whenever $E_{\text{slope}}^n$ holds, we find

$$\text{VAR}[L^n] \geq E[\text{VAR}[L^a(n - N^a)|\sigma_{YZ}]] \geq n \cdot P(E_{\text{slope}}^n) \cdot \left[ p(1 - p) \left( k_1 \right)^2 \left( 1 - \frac{2k_2 \ln n}{k_1 \sqrt{p(1 - p)n}} \right) \right]. \quad (3.26)$$

The expression on the right side of inequality (3.26) divided by $n$ converges to

$$P(E_{\text{slope}}^n)p(1 - p) \left( k_1 \right)^2.$$  

We will show in Lemma 4.1 below that $P(E_{\text{slope}}^n) \to 1$ as $n \to \infty$. Hence, for all $n$ big enough, $\text{VAR}[L^n]$ is larger than $np(1 - p) \left( k_1 \right)^2 / 2 > 0$. This finishes the proof of theorem 2.1.

### 4 Slope of $L^a(\cdot)$

This section is dedicated to the proof of the following lemma:

**Lemma 4.1** We have that:

$$P(E_{\text{slope}}^n) \to 1 \quad (4.1)$$

as $n \to \infty$.

We first need a few definitions. A common subsequence of length $m$ of the two sequences $Z^k$ and $Y$, can be viewed as a pair of strictly increasing functions:

$$(\pi, \eta)$$

such that $\pi : [1, m] \to [1, k]$, $\eta : [1, m] \to [1, n]$ and

$$\forall i \in [1, m], Z^k_{\pi(i)} = Y_{\eta(i)}. \quad (4.2)$$
Definitions:

1. Let \( \pi : [1, m] \rightarrow [1, k] \) and \( \eta : [1, m] \rightarrow [1, n] \) be two increasing functions. The pair of \((\pi, \eta)\) is called a pair of matching subsequences of \(Z^k\) and \(Y\) iff it satisfies condition (4.2).

2. Let \(M^k_1\) designate the set of all pairs of matching subsequences of \(Z^k\) and \(Y\).

3. Let \(M^k_2\) designate the set of all pairs of matching subsequences of \(Z^k\) and \(Y\) of maximal length, (i.e. of maximal length in the set \(M^k_1\).)

4. Let \(\leq\) indicate the natural partial order relation between increasing functions \(\pi : [1, m] \rightarrow \mathbb{N}\), i.e. \(\pi_1 \leq \pi_2\) iff, for every \(i \in [1, m]\), \(\pi_1(i) \leq \pi_2(i)\). With a slight abuse of notation we will indicate with \(\leq\) also the partial order induced on the pairs of increasing function \((\pi, \eta)\), i.e. \((\pi_1, \eta_1) \leq (\pi_2, \eta_2)\) iff \(\pi_1 \leq \pi_2\) and \(\eta_1 \leq \eta_2\).

5. Let \(M^k \subseteq M^k_2\) designate the set of all \((\pi, \eta) \in M^k_2\) which are minimal according to the relation \(\leq\), (i.e. minimal in the set \(M^k_2\)).

6. Let \((\pi, \eta)\) be a pair of matching subsequences of length \(m\) and let \(i \in [0, m - 1]\). We call the quadruple

\[
(\pi(i), \pi(i + 1), \eta(i), \eta(i + 1)),
\]

(4.3)
a match of \((\pi, \eta)\). If \(\eta(i) + 2 \leq \eta(i + 1)\), we call the match a non-empty match. If there exists \(j\), such that \(\eta(i) < j < \eta(i + 1)\) and \(Y_j = 1\), resp. \(Y_j = 0\), we say that the match contains a 1, resp. a 0. We also say that the match contains the point \(j\) and call the bit \(Y_j\) a free bit of the match \((\pi(i), \pi(i + 1), \eta(i), \eta(i + 1))\).

Sometimes we identify the match \((\pi(i), \pi(i + 1), \eta(i), \eta(i + 1))\) with the couple of binary words:

\[
\left( Z^k_{\pi(i)}Z^k_{\pi(i)+1}\cdots Z^k_{\pi(i+1)}, Y_{\eta(i)}Y_{\eta(i)+1}\cdots Y_{\eta(i+1)} \right).
\]

7. Let \(0 < s < t \leq n\). We call the integer interval \([s, t] = \{s, s + 1, \ldots, t\}\) a block of \(Y\), if for all \(r \in [s, t]\) we have \(Y_r = Y_s\) but \(Y_{s-1} \neq Y_s\) and \(Y_t \neq Y_{t+1}\). The cardinality \(|[s, t]| = s - t + 1\) is called length of the block \([s, t]\).

Let us give an illustrative example. Take \(Z^6 = 101011\), \(n = 9\) and \(Y = 111000111\). Let \((\pi, \eta)\) be defined as follows:

\[
\pi(1) = 1, \pi(2) = 3, \pi(3) = 4, \pi(4) = 5, \pi(5) = 6
\]

and

\[
\eta(1) = 1, \eta(2) = 2, \eta(3) = 4, \eta(4) = 7, \eta(5) = 8.
\]

Then, \((\pi, \eta)\) is a pair of matching subsequences of \(Z^6\) and \(Y\). The common subsequence associated with it is:

\[
Z^6_1Z^6_3Z^6_4Z^6_5Z^6_6 = Y_1Y_3Y_4Y_7Y_8 = 11011.
\]

We represent the pair of matching subsequences \((\pi, \eta)\) using an alignment of \(Z^6\) and \(Y\):

\[
\begin{array}{cccccccccc}
1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & \\
1 & - & 1 & 1 & 0 & 0 & 1 & 1 & 1
\end{array}
\]

In this example \((\pi, \eta)\) contains the four following matches:
The first match above is empty. The second match contains a one. Here, \( Y_3 \) is a free bit of the second match. The third match contains two zero’s: \( Y_5 \) and \( Y_6 \) are free bits of the third match. The forth match is empty. The common subsequence 11011 is of maximal length (among all the common subsequences of \( Z^6 \) and \( Y \)). So, we have that \( L^a(6) = 5 \). Hence, \( L^a(7) \) can only be equal to 5 or 6.

What is the probability that \( L^a(7) \) is larger by one than \( L^a(6) \)? When we generate \( Z^7 \) by dropping the bit \( V_7 \) on \( Z^6 \), then there are five positions where it can fall:

| position 1 | position 2 | position 3 | position 4 | position 5 |
|------------|------------|------------|------------|------------|
| 1x01011    | 10x1011    | 101x011    | 1010x11    | 10101x1    |

where \( x \) designates the possible positions of the bit \( V^7 \). Each of these positions has same probability. Positions 1 and 2 correspond to the first match. Position 3 corresponds to the second match. Position 4 correspond to the third match and position 5 corresponds to match number four.

If \( V_7 = 1 \) and the bit drops on the match which contains a one (that is match number two corresponding to position three, i.e. \( T_7 = 3 \)), then \( L^a(7) = L^a(6) + 1 \). The reason is that the bit \( V^7 \) can then get matched with the free 1-bit in match two and increase the score \( L^a(6) \) by one. Similarly, if \( V_7 = 0 \) and the bit \( V^7 \) drops on match number three, the score gets increased by one, since then \( V^7 \) gets matched with the “free” zero contained in match number three. Hence, when \( V^7 \) drops on match number three, the result is: \( L^a(7) = L^a(6) + 1 \). In general \( L^a(k + 1) = L^a(k) + 1 \), if the bit \( V_{k+1} \) drops on a match which contains a bit of the same color as to \( V_{k+1} \). (By color, we mean 0 or 1.)

From the idea of the previous example, we can get a lower bound for the probability that the score \( L^a(k) \) increases by one. The bit \( V_{k+1} \) is equally likely to be equal to one or equal to zero. So, when it drops on a nonempty match, the score has at least 50% probability to increase. Each nonempty match corresponds to at least one position. The bit \( V^{k+1} \) has \( k - 1 \) equally likely positions. It follows: for any pair \((\pi, \eta)\) of matching subsequences of \( Z^k \) and \( Y \):

\[
P \left( L^a(k + 1) = L^a(k) + 1 \mid Z^k, Y \right) \geq \frac{1}{2} \frac{\# \text{ of nonempty matches of } (\pi, \eta)}{k} \tag{4.4}
\]

if \((\pi, \eta)\) is of maximal length.
Let us explain at this stage the main ideas for the proof of lemma 4.1. We distinguish two cases depending on the value of $k$.

We first deal with the case $k < 0.45n$. In this case it easy to show that with large probability all the bits in $Z^k$ are matched. Let $E_{1k}^n$ be the event:

$$E_{1k}^n := \{ L_n^a(k) = k \} \quad (4.5)$$

and

$$E_1^n := \bigcap_{k=1}^{0.45n} E_{1,k}^n. \quad (4.6)$$

Observe that we have

$$E_1^n = \{ L_n^a(k + 1) - L_n^a(k) = 1, \forall k < 0.45n \} \quad (4.7)$$

i.e. the slope of $L_n^a(k)$ is equal to 1 for all $k < 0.45n$ if $E_1^n$ holds. In the next section we prove the following lemma:

**Lemma 4.2** We have

$$\lim_{n \to \infty} P(E_1^n) = 1. \quad (4.8)$$

Assume that instead of looking for a LCS, we want to know if one sequence is contained in another. For example for given $l \in \mathbb{N}$, we may be interested in finding out if the sequence $Z^k$ is a subsequence of $Y_1Y_2 \ldots Y_l$. For this let $\nu(i)$ be the smallest $l$ such that $Z^k_i$ is a subsequence of $Y_1Y_2 \ldots Y_l$. Then, $\nu(1), \nu(2), \nu(3), \ldots$ defines a renewal process. The interarrival times $I_i = \nu(i+1) - \nu(i)$ have geometric distribution and expectation $E[I_i] = 2$. Thus, $E[\nu(i)] = 2i$ and $\text{VAR}[n] = o(n)$. From this it follows that if we want $Z^k$ to be with high probability a subsequence of $Y_1Y_2 \ldots Y_l$, we need to take $l$ somewhat above $2k$. Let us give a numerical example. Take $Z^3 = 001$ and $Y = 10101000111$. Then, $\nu(1)$ denotes the indices of the first $Y_i$ equal to zero. In this case, $\nu(1) = 2$. Similarly, $\nu(2)$ is the smallest $i \geq \nu(1)$ such that $Y_i = Z_3^3 = 0$. Here: $\nu(2) = 4$. Finally, $\nu(3)$ is the smallest $i \geq \nu(2)$, such that $Y_3 = 1$, hence $\nu(3) = 5$.

Let us next give the main ideas, why with high probability, the slope of $k \mapsto L^a(k)$ is increasing linearly on the domain $[0.45n, n]$. We use the bit-drop scheme to prove this: we show that typically the random map $k \mapsto L^a(k)$ has a positive drift $\gamma > 0$. We define:

$$E_{2k}^n := \{ \nu(\pi, \eta) \in M^k, \# \text{ of nonempty matches of } (\pi, \eta) \text{ is larger than } \gamma n \} \quad (4.9)$$

When $E_{2k}^n$ holds, every pair $(\pi, \eta) \in M^k$ has at least $\gamma n$ non-empty matches. The proportion of non-empty matches to $k$ hence is larger or equal to $\gamma$. Using inequality 4.3 it follows that

$$P \left( L_n^a(k + 1) = L_n^a(k) + 1 \mid Z^k, Y \right) \geq 0.5 \cdot \gamma \quad (4.10)$$
when $E_{2k}^n$ holds. Let $E_2^n$ be the event:

$$E_2^n := \bigcap_{k=0.45n}^n E_{2k}^n.$$  \hspace{1cm} (4.11)

Inequality 4.10 implies, that when $E_{2k}^n$ holds, the map $k \mapsto L_a(k)$ has positive drift $0.5\gamma > 0$ for $k \in [0.45n, n]$. By large deviation it follows, that with high probability $k \mapsto L_a(k)$ has positive slope on $[0.45n, n]$ as soon as $E_2^n$ holds. (See lemma 4.9.)

It remains to explain why $E_2^n$ holds with high probability.

Let us first summarize the general idea:

We proceed by contradiction. Assume all the matches of $(\pi, \eta) \in M_k^k$ were empty. Then all of the following would hold:

- $(\eta(1), \eta(2), \eta(3), \ldots, \eta(m)) = (\eta(1), \eta(1) + 1, \eta(1) + 2, \ldots, \eta(1) + m)$
  where $m$ is the length of the LCS of $Z^{k}$ and $Y$: $m = L_a(k)$.
- The sequence $Y_{\eta(1)}Y_{\eta(2)}\ldots Y_{\eta(m)} = Y_{\eta(1)}Y_{\eta(1)+1}\ldots Y_{\eta(1)+m}$
  is a subsequence of $Z^k_{\pi(1)}Z^k_{\pi(1)+1}\ldots Z^k_{\pi(m)}$.

Hence we would have two independent i.i.d. sequences of Bernoulli variables with parameter 1/2, where one is contained in the other as subsequence. This implies that the sequence containing the other must be approximately twice as long. Hence $k$ is approximately at least twice as large as $m = L_a(k)$. Thus, the ratio $L_a(k)/k$ is close to 50% or below. This is very unlikely, since it is known that the $L_a(k)/k$ is typically above 80%. This is our contradiction.

From the previous argument it follows that with high probability any $(\pi, \eta) \in M_k^k$ contains a non-vanishing proportion $\epsilon > 0$ of free bits. (Hence, $L_a^k(k)/\eta(L_a^k(k)) \geq \epsilon$.)

We need to show that this proportion $\epsilon$ of free bits generates sufficiently many non-empty matches: the free bits should not be concentrated in a too small number of matches.

Let us go back to the numerical example on page 9 to illustrate how we count the proportion of bits that are free. In that example, the first match of $(\pi, \eta)$ contains no free bit. The second match contains one free bit which is a one. The third match contains two free bits which are zero’s. The forth match contains no free bit. The sequence $Y$ contains a total of 8 bits which are involved in a match of $(\pi, \eta)$. (Note that the last bit $Y_9$ of $Y$ is not counted since it is not involved in a match of $(\pi, \eta)$.) We have a proportion of free bits to bits involved in matches equal to:

$$3/8 = (8-5)/5 = \frac{\eta(L_a^k(k)) - L_a^k(k)}{\eta(L_a^k(k))} = \frac{\eta(5) - 5}{\eta(5)}.$$  

The 3 free bits generate two non-empty matches.

To prove that there are more than $\gamma n$ nonempty matches two arguments are used:

\[ \]
• Any pair of matching subsequence \((\pi, \eta)\) which is minimal according to our partial order for pairs of matches satisfies:
  every match of \((\pi, \eta)\) can contain zero’s or one’s but not both at the same time. Hence, each match of \((\pi, \eta) \in M^k\) contains free bits from at most one block of \(Y\).

• With high probability, the total number of integer points in \([0, n]\) contained in blocks of \(Y\) of length \(\geq D\) is very small. (By choosing \(D\) large, we make the total number of points contained in blocks longer than \(D\), much smaller than the number of free bits.)

From the two points above, it follows that for \((\pi, \eta) \in M^k\), the majority of free bits are at most \(D\) per match. This ensures that the proportion \(\epsilon\) of free bits, generates a proportion of at least order \(\epsilon/D\) non-empty matches.

Let us look at an example of a pair \((\pi, \eta)\) which is of maximal length but not minimal according to our order relation on \(M^2_k\). Take \(Z^7 = 0101101\) and \(Y = 00110010111\). Define the pair of matching subsequences \((\pi, \eta)\) as follows:

\[
\begin{align*}
\pi(1) &= 1, \pi(2) = 2, \pi(3) = 3, \pi(4) = 4, \pi(5) = 5, \pi(6) = 7 \\
\eta(1) &= 1, \eta(2) = 7, \eta(3) = 8, \eta(4) = 9, \eta(5) = 10, \eta(6) = 11.
\end{align*}
\]

Let us represent this pair of matching subsequences by an alignment:

\[
\begin{array}{cccccccccccc}
0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & - & 1
\end{array}
\]

This gives the common subsequence 010111. The pair \((\pi, \eta)\) is of maximal length, but it is not minimal for our order relation on \(M^2_k\). Instead of \(\eta(2) = 7\), take \(\eta^*(2) = 3\). Let otherwise \(\eta^*\) be equal to \(\eta\). Then \((\pi, \eta^*)\) is strictly below \((\pi, \eta)\). To construct \(\eta^*\) we used the fact that a match of \((\pi, \eta)\) contained both zero’s and one’s. It is always possible to find a strictly smaller pair \((\pi, \eta^*) \in M^k_2\) when a match of \((\pi, \eta)\) contains hero’s and one’s at the same time.

Note that \((\pi, \eta)\) contains 5 free bits, but only one non-empty match. All the free bits of \((\pi, \eta)\) are concentrated in one match. The match containing all the free bits contains several blocks. By taking a minimal pair of matching subsequences, this kind of situation is avoided.

Let us look at the details of the proof of lemma 4.1. Let \(L^q_l(k)\) denote the length of the LCS of \(Z^k\) and the sequence \(Y^l := Y_1Y_2\ldots Y_l\). For \(Y^l\) to be entirely contained as a subsequence in \(Z^k\), one needs \(k\) to be approximately twice as long as \(l\). (We have that \(Y^l\) is a subsequence of \(Z^k\) iff \(L^q_l(k) = l\).) Hence, it is unlikely that that \(Y^l\) is a subsequence of \(Z^k\), when \(k = 2l(1 - \delta)\). (Here \(\delta > 0\) is a constant not depending on \(l\).) In other words, it is unlikely that:

\[
L^q_l(2l(1 - \delta)) \geq l.
\]

Similarly, it is unlikely, that \(Y^l\) is “close to being a subsequence of \(Z^{kn}\), when \(k = 2l(1 - \delta)\):
Lemma 4.3 There exists a function $\delta : \mathbb{R} \to \mathbb{R}$ such that $\lim_{\epsilon \to 0} \delta(\epsilon) = 0$ and

$$P \left( L_n^a \left( 2l(1 - \delta(\epsilon)) \right) > l(1 - \epsilon) \right) \leq Ce^{-cl} \quad (4.12)$$

for all $l > 0$ and suitable constants $c > 0$ and $C > 0$ not depending on $l$. (Note that the constants $c > 0$ and $C > 0$ may depend on $\epsilon$.)

We can now define:

$$E_{3l}^n = \{ L_n^a \left( 2l(1 - \delta(\epsilon)) \right) \leq (1 - \epsilon)l \} \quad (4.13)$$

and

$$E_3^n := \bigcap_{k=0.2n}^{n} E_{3k}^n \quad (4.14)$$

where $\epsilon$ is a suitable number, to be fixed in the following, and $\delta(\epsilon)$ is given by Lemma 4.3. It follows that:

**Corollary 4.1** If $\delta(\epsilon)$ in the definition of $E_3^n$ is given by lemma 4.3, we have

$$\lim_{n \to \infty} P(E_3^n) = 1. \quad (4.15)$$

Typically, $L_n^a(k)$ is above $80\% \cdot k$. However, to make things easier, we prove only that it is above $65\% \cdot k$. We define:

$$E_{4k}^n := \{ L_n^a(k) \geq 0.65k \} \quad (4.16)$$

and

$$E_4^n := \bigcap_{k=0.45n}^{n} E_{4,k}^n. \quad (4.17)$$

The next lemma is proven in the next section:

**Lemma 4.4** We have

$$\lim_{n \to \infty} P(E_4^n) = 1. \quad (4.18)$$

Let us define the event $E_{6k}^n$: 

$$E_{6k}^n := \left\{ L_n^a(k) \leq (1 - \epsilon)\eta(L_n^a(k)), \forall (\pi, \eta) \in M^k \right\} \quad (4.19)$$

and

$$E_6^n := \bigcap_{k=0.45n}^{n} E_{6k}^n. \quad (4.20)$$

The event $E_{6k}^n$ says that any pair of matching subsequences $(\pi, \eta) \in M^k$ has a proportion of at least $\epsilon$ free bits. (Note that $\eta(L_n^a(k))$ is the number of the last bit of $Y$ involved in a match of $(\pi, \eta)$. Furthermore, $L_n^a(k)$ represents the number of bits that are “matched” by $(\pi, \eta)$. Hence, $\eta(L_n^a(k)) - L_n^a(k)$ is the number of “free” bits.)
Lemma 4.5 Take $\epsilon > 0$ small enough, so that

$$\frac{50\%}{1 - \delta(\epsilon)} < 65\%.$$ (4.21)

Then, we have that, for all $k > 0.45n$,

$$E_3^n \cap E_4^n \subset E_6^n.$$ (4.22)

Thus

$$E_3^n \cap E_4^n \subset E_6^n.$$ (4.23)

Proof. Let $k \in [0.45n, n]$. We show that if $E_6^n$ does not hold and $E_3^n$ holds, then $E_4^n$ can not hold. This in terms implies 4.22.

Let $(\pi, \eta) \in M$. If $E_6^n$ does not hold, than the proportion of "free" bits of $(\pi, \eta)$ is below $\epsilon$. In other words:

$$\frac{L\eta(k)}{l} \geq 1 - \epsilon$$

where $l := \eta(L\eta(k))$. (Note that $L\eta_1(k) = L\eta_n(k)$, since $(\pi, \eta)$ is of maximal length.) It follows that

$$L\eta(k) \geq l(1 - \epsilon).$$ (4.24)

Now, when $E_3^n$ holds, then

$$L\eta(2l(1 - \delta(\epsilon))) \leq l(1 - \epsilon).$$ (4.25)

Comparing inequality 4.24 with 4.25 and noting that the (random) map $x \mapsto L\eta(x)$ is increasing, yields:

$$k \geq 2l(1 - \delta(\epsilon))$$

and hence

$$k \geq 2\eta(L\eta_n(k))(1 - \delta(\epsilon)) \geq 2L\eta_n(k)(1 - \delta(\epsilon)).$$

From this it follows, that:

$$\frac{L\eta(k)}{k} \leq \frac{50\%}{1 - \delta(\epsilon)} < 65\%$$ (4.26)

where the 65%-bound is obtained from inequality 4.21. Inequality 4.26 contradicts $E_4^n$. ■

To obtain $E_2^n$ we must be sure that the free bits of $Y$ do not concentrate in a small amount of of matches of $(\pi, \eta) \in M$. As explained in the example on page 12, any match of $(\pi, \eta) \in M$ can contain 0’s or 1’s, (or nothing) but not 0’s and 1’s at the same time. This is due to the minimality respect to the ordering $\prec$. In fact if $(\pi(i), \pi(i+1), \eta(i), \eta(i+1))$ is a non empty match we must have that $Y_1 \neq Y_{\eta(i+1)}$ for all $\eta(i) < l < \eta(i+1)$. Otherwise, we could match the bit $Z_{\pi(i+1)}$ with $Y_1$ instead of $Y_{\eta(i+1)}$. This modification would yield a pair of matching subsequences of same length but strictly smaller according to our order relation on $M_k$. Thus, all the free bits of a match of $(\pi, \eta) \in M$ are contained in only one block of $Y$. 

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It is useful to see how many bits are contained in long blocks. Let $BLOCK^D$ designate the set of all blocks $[i, j] \subset [0, n]$ of $Y$ of length at least $D$. (For the definition of blocks see the definitions at the beginning of this section.) Let $N^D$ denote the total number of points in the sequence $Y$ which are contained in a block of length at least $D$:

$$N^D := | \{ s \in [1, n] | \exists[i, j] \in BLOCK^D, s \in [i, j] \} |.$$  \hspace{1cm} (4.27)

Let $E^n_5$ designate the event:

$$E^n_5 := \{ N^D \leq \epsilon n/4 \}$$ \hspace{1cm} (4.28)

We will show in sec. 6 that:

**Lemma 4.6** For every $\epsilon$ there exists $D$ such that

$$\lim_{n \to \infty} P(E^n_5) = 1.$$  \hspace{1cm} (4.29)

We then have the following combinatorial fact:

**Lemma 4.7** We have that, for all $k > 0.45n$:

$$E^n_4 \cap E^n_5 \cap E^n_{6k} \subset E^n_{2k}$$ \hspace{1cm} (4.30)

with $\gamma = \frac{0.0425\epsilon}{D-1}$. Thus also:

$$E^n_4 \cap E^n_5 \cap E^n_6 \subset E^n_2.$$ \hspace{1cm} (4.31)

**Proof.** We prove \[4.30\] The event $E^n_{6k}$ implies that for each $(\pi, \eta) \in M^k$ there are at least $\epsilon \eta(L^a_n(k))$ free bits. We have:

$$\eta(L^a_n(k)) \geq L^a_n(k).$$ \hspace{1cm} (4.32)

When $E^n_4$ holds, we have that:

$$L^a_n(k) \geq 0.65k.$$ \hspace{1cm} (4.33)

Since we take $k \geq 0.45n$, inequalities \[4.32\] and \[4.33\] together imply that the number of free bits of $(\pi, \eta) \in M^k$ is at least

$$\epsilon \cdot 0.65 \cdot 0.45n = \epsilon \cdot 0.2925n.$$ 

By $E^n_5$, there are at most $0.25\epsilon n$ bits contained in blocks of length $\geq D$. Thus, there are at least $0.0425\epsilon \cdot n$ free bits contained in blocks of length $< D$. Recall that every match of $(\pi, \eta) \in M^k$ contains free bits from only one block. Hence, every match of $(\pi, \eta) \in M^k$ can contain at most $D - 1$ free bits from blocks of length $< D$. Hence, these $\epsilon 0.0425n$ free bits which are not in $N^D$, must fill at least $\epsilon 0.0425n/(D - 1)$ matches of $(\pi, \eta) \in M^k$. It follows that $(\pi, \eta) \in M^k$ has at least $0.0425\epsilon \cdot n/(D - 1)$ non-empty matches. \[\square\]
Lemmas 4.5 and 4.7 jointly imply that $E_{n}^{3} \cap E_{n}^{4} \cap E_{n}^{5} \subset E_{n}^{2}$. Hence:

$$P(E_{n}^{nc}) \leq P(E_{n}^{nc} \cap (E_{n}^{2} \cap E_{n}^{1})) + P(E_{n}^{nc}) + P(E_{n}^{nc})$$  \hfill (4.34)

where $E_{x}^{nc}$ denotes the complement of $E_{x}^{n}$. We have that $P(E_{n}^{nc})$, $P(E_{n}^{nc} \cap E_{n}^{5})$ and $P(E_{n}^{nc})$ all converge to zero when $n \to \infty$. (This follows from Lemmas 4.1, 4.4 and 4.6.) Hence, we have that:

$$\lim_{n \to \infty} P(E_{n}^{2}) = 1. \quad \hfill (4.35)$$

Let $\sigma_{k}$ denote the $\sigma$-algebra:

$$\sigma_{k} := \sigma(Z_{i}^{k}, Y_{j}^{k}|i \leq k, j \leq n).$$

It is easy to check that $E_{n}^{nc}$ is $\sigma_{k}$-measurable. Note that $L_{a}(k+1) - L_{a}(k)$ is always equal to one or zero.

**Lemma 4.8** When $E_{2k}^{n}$ holds, then

$$P(L_{a}(k+1) - L_{a}(k) = 1|\sigma_{k}) \geq 0.5\gamma. \quad \hfill (4.36)$$

**Proof.** This has already been explained. (See inequality 4.10). ■

We finally observe that

$$P(E_{slope}^{nc}) \leq P(E_{slope}^{nc} \cap (E_{n}^{2} \cap E_{n}^{1})) + P(E_{2}^{nc}) + P(E_{1}^{nc}). \quad \hfill (4.37)$$

Since $P(E_{slope}^{nc})$ and $P(E_{2}^{nc})$ both go to zero as $n$ goes to infinity, we only need to prove that

$$P(E_{slope}^{nc} \cap (E_{n}^{2} \cap E_{n}^{1})) \to 0 \text{ for } n \to \infty; \quad \hfill (4.38)$$

to establish lemma 4.1

**Lemma 4.9** We have that

$$P(E_{slope}^{nc} \cap (E_{n}^{2} \cap E_{n}^{1})) \to 0$$

as $n \to \infty$.

**Proof.** We can assume that $\gamma < 1$. Define $k_{1} := 0.4\gamma$, so that $k_{1} \leq 0.4$. Let

$$\Delta(k) := L_{n}(k+1) - L_{n}(k)$$

when $E_{2k}^{n}$ holds, and $\Delta(k) := 1$ otherwise. From eq. (4.36), it follows that:

$$P(\Delta(k) = 1|\sigma_{k}) \geq 0.5\gamma. \quad \hfill (4.39)$$

Furthermore, $\Delta(k)$ is equal to zero or one and $\sigma_{k}$-measurable. For $k \in [0.45n, n]$, let

$$\tilde{L}_{n}(k) = L_{n}(0.45n) + \sum_{i=0.45n}^{k-1} \Delta(i).$$

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For $k \in [0, 0.45n]$, let $\tilde{L}^a_n(k) := L^a_n(k)$. Note that when $E^n_2$ holds, then

$$L^a_n(k) = \tilde{L}^a_n(k) \quad (4.40)$$

for all $k \in [0, n-1]$. Introduce the event $\tilde{E}^n_{\text{slope}}$ to be the event such that $\forall i, j$, with $0.45n < i < j \leq n$ and $i + k_2 \ln n \leq j$, we have:

$$\tilde{L}^a_n(j) - \tilde{L}^a_n(i) \geq k_1|i - j|. \quad (4.41)$$

When $E^n_1$ holds, then $L^a_n(k)$ has a slope of one on the domain $[0, 0.45]$. Hence, the slope condition of $E^n_{\text{slope}}$ holds on the domain $[0, 0.45n]$, since we have $k_1 \leq 0.4$. When $E^n_2$ holds, then $L^a_n(k)$ and $\tilde{L}^a_n(k)$ are equal. It follows that when $E^n_2$ and $\tilde{E}^n_{\text{slope}}$ both hold, then the slope condition of $E^n_{\text{slope}}$ is verified on the domain $[0.45n, n]$. Hence

$$E^n_1 \cap E^n_2 \cap \tilde{E}^n_{\text{slope}} = E^n_1 \cap E^n_2 \cap E^n_{\text{slope}}. \quad (4.42)$$

Thus

$$P(E^n_{\text{slope}} \cap E^n_1 \cap E^n_2) = P(\tilde{E}^n_{\text{slope}} \cap E^n_1 \cap E^n_2) \leq P(\tilde{E}^n_{\text{slope}}).$$

It only remains to prove that $P(\tilde{E}^n_{\text{slope}})$ goes to zero as $n \to \infty$. For this we can use large deviation. Let $\tilde{E}^n_{i,j}$ be the event that

$$\tilde{L}^a_n(j) - \tilde{L}^a_n(i) \geq k_1|i - j|$$

Then

$$\tilde{E}^n_{\text{slope}} = \bigcap_{i,j} \tilde{E}^n_{i,j}$$

where the intersection in the last equation above is taken over all $i, j \in [0.45n, n]$ such that $i + k_2 \ln n \leq j$. It follows that

$$P(\tilde{E}^n_{\text{slope}}) \leq \sum_{i,j} P(\tilde{E}^n_{i,j}) \quad (4.43)$$

where the last sum is taken over all $i, j \in [0.45n, n]$ such that $i + k_2 \ln n \leq j$. Since we took $k_1 = 0.4 \gamma$ and because of 4.39 large deviation tells us that there exists constants $c, C > 0$ such that

$$P(\tilde{E}^n_{i,j}) \leq Ce^{-c|i-j|} \quad (4.44)$$

for all $i, j \in \mathbb{N}$. (The constants $C, c$ do not depend on $i, j$.) Take $k_2 := 3/c$. With this choice, 4.44 becomes:

$$P(\tilde{E}^n_{i,j}) \leq Cn^{-3} \quad (4.45)$$

when $k_2 \ln n \leq |i - j|$. Note that there are less than $n^2$ terms in the sum in inequality 4.43. By 4.45, each term in the sum in inequality 4.43 is less or equal to $Cn^{-3}$. Thus inequality 4.43 and 4.45 together imply that

$$P(\tilde{E}^n_{\text{slope}}) \leq \frac{C}{n}.$$
5 Bounds for the probabilities.

We report in this section several proofs of the lemmas used in sec. 4.

**Lemma 5.1** for every \( n \) and \( \nu < 0.5 \) we have

\[
P(L^n_\nu(\nu n) = \nu n) \geq 1 - e^{c(0.5 -\nu)^2 n}
\]  

(5.1)

**Proof.** We can build a pair of matching subsequences as follows: start from \( Z_k \) and match it with the first \( Y_i = Z_k \), then match \( Z_k \) with the first \( Y_i = Z_k \) such that \( i_2 > i_1 \). We can proceed as before until we reach the end of the \( Z_k \) or the \( Y \). More precisely we can define a matching \( (\pi, \eta) \) such that \( \pi(i) = i \) and \( \nu(i) = \inf_{l > \nu(i-1)} \{ Y_l = Z_k \} \) (see remark after Lemma 4.2 for an explicit example). Given \( Z_k \) and \( Y \) we call \( T_j \) the sequence of random variables defined by

\[
T_j = \nu(j) - \nu(j-1).
\]

Observe that the \( T_j \) is a sequence of independent random variables all with geometric distribution of parameter \( \frac{1}{2} \). It follows that

\[
P(L^n_\nu(\nu n) = \nu n) \geq P \left( \sum_{i=0}^{\nu n} T_i < n \right) = P \left( \sum_{i=0}^{\nu n} T_i - \frac{1}{\nu} < 0 \right)
\]

but

\[
P \left( \sum_{i=0}^{\nu n} T_i - \frac{1}{\nu} > 0 \right) \leq \inf_{s > 0} E \left( e^{s \sum_{i=0}^{\nu n} T_i - \frac{1}{\nu}} \right)
\]

(5.3)

Due to the independence of the \( T_i \) we have

\[
E \left( e^{s \sum_{i=0}^{\nu n} T_i - \frac{1}{\nu}} \right) = E \left( e^{s (T_0 - \frac{1}{\nu})} \right)^{\nu n} = \left( \frac{e^s}{2 - e^s} \right)^{\nu n} e^{-ns}
\]

(5.4)

It is easy to check that

\[
\inf_{s > 0} \left( \frac{e^s}{2 - e^s} \right)^{\nu} e^{-s} \leq e^{c(0.5 -\nu)^2}
\]

(5.5)

for a suitable constant \( c \), so that we get

\[
P(L^n_\nu(\nu n) = \nu n) \geq 1 - e^{c(\nu - 0.5)^2 n}.
\]

(5.6)

Proof of lemma 4.2. It follows immediately from the above lemma.

**Lemma 5.2** For every \( k \)

\[
P(L^n_k(2(1 - \delta)k) = k) \leq Ce^{c^2k}
\]

(5.7)
**Proof.** Observe that the only possibility for $L^n_a(k) = k$ is that the pair of matching subsequences constructed at the beginning of the proof of lemma 5.1 has length $k$. Using the notation of that proof we have that

$$P\left( L^n_a(2-\delta)k(k) = k \right) = P\left( \sum_{i=0}^{k} T_i \leq (2-\delta)k \right)$$

This quantity can be evaluated as in the previous proof to obtain the lemma.

We can now estimate the probability of $E_{3k}^n$. **Proof of Lemma 4.3.** Consider a subset of $S \subset [0, l]$ containing $(1-\epsilon)l$ points. There are $\left(\binom{l}{(1-\epsilon)l} \right)$ such subset. We can fix the sequence $Y$ on the subset $S$. We have $2^l$ $Y$'s that agree on $S$. Calling $\delta(\epsilon) = \epsilon + \delta'(\epsilon)$ we have, due to Lemma 5.2, that the probability of matching all $Y$ in $S$ is bounded by $e^{-\delta'(\epsilon)^2l}$. Collecting the above estimates we get that

$$P\left( L^n_a(2l(1-\delta(\epsilon))) > l(1-\epsilon) \right) \leq 2^l \left( \frac{l}{l(1-\epsilon)} \right) e^{-\delta'(\epsilon)^2l} \leq C e^{[\epsilon \ln 2 + \ln \epsilon + (1-\epsilon) \ln(1-\epsilon) - \delta'(\epsilon)^2]l}$$

where we have used Stirling’s formula. Thus it is enough to chose

$$\delta'(\epsilon) = \sqrt{\frac{2}{c} [\epsilon \ln 2 + \ln \epsilon + (1-\epsilon) \ln(1-\epsilon)]}$$

(5.10)

to obtain the lemma.

**Proof of lemma 4.4.** We can divide the sequences $Z^k$ and $Y$ is subsequences of length 10 and write $L^n_a(k) < \sum_{i=1}^{k/10} L_i$ where $L_i$ is the longest common subsequence between $Y_{10(i-1)+1} \ldots Y_{10i}$ and $Z^k_{10(i-1)+1} \ldots Z^k_{10i}$. From Chvatal we know that $E(L_i) = 6.97844$. From a standard large deviation argument we get

$$P\left( \sum_{i=1}^{k/10} L_i < k \left( \frac{E(L_i)}{10} - \delta \right) \right) < \left( \inf_{s<0} E \left( e^{s(L_0-(0.09-\delta))} \right) \right)^{\frac{k}{10}}$$

(5.11)

Calling $p(s, \delta) = E \left( e^{s(L_0-(0.09-\delta))} \right)$ it easy to see that $p(s, \delta)$ is smooth in $s$, $p(0, \delta) = 1$ and $\partial_s p(0, \delta) < 0$ for every $\delta > 0$. This implies that

$$\inf_{s<0} p(s, \delta) < e^{-c(\delta)}$$

(5.12)

for suitable $c(\delta) > 0$. This immediately give the thesis of the Lemma.

Finally we prove the lemma 4.6. **Proof of lemma 4.6.** Let $\tilde{N}^D$ be the number of integer points in $[0, n-D]$ which are followed by at least $D$ times the same color in the sequence $Y$. Thus, $\tilde{N}^D$ is the number of integer points $s \in [0, n-D]$ so that

$$Y_s = Y_{s+1} = \ldots = Y_{s+D}. \tag{5.13}$$
It is easy to check that
\[ N^D \leq D \tilde{N}^D. \] (5.14)
Let now \( \tilde{Y}_s, \ s \in [0, n - D], \) be equal to 1 iff \( 5.13 \) holds, and 0 otherwise. We find:
\[ \sum_{s=1}^{n} \tilde{Y}_s = \tilde{N}^D. \] (5.15)
To estimate the sum \( 5.15 \) we can decompose it into \( D \) sub sums \( \Sigma_1, \Sigma_2, \ldots, \Sigma_D \) where
\[ \Sigma_i = \sum_{\substack{s=1, \ldots, n \atop s \mod D = i}} \tilde{Y}_s \] (5.16)
so that
\[ \tilde{N}^D = \sum_{i=1}^{D} \Sigma_i \] (5.17)
It is easy to see that
\[ P \left( N^D > \frac{c}{4} n \right) \leq P \left( \tilde{N}^D > \frac{c}{4D} n \right) \leq D \cdot P \left( \Sigma_0 > \frac{c}{4D^2} n \right) \] (5.18)
where the last inequality follows from the fact that at least one of the addends in \( 5.17 \) has to be larger than \( \frac{c}{4D^2} n \). Now, the \( Y_s \) appearing in the sub sum \( \Sigma_0 \) are i.i.d. Bernoulli random variable with \( P(Y_s = 1) = 2^{-D} \). We can apply a large deviation argument analogous to the one used in the previous proof and obtain
\[ P \left( \Sigma_0 > \left( 2^{-D} + \delta \right) \frac{n}{D} \right) \leq e^{-c(\delta) \frac{n}{D}}. \] (5.19)
with \( c(\delta) > 0 \) for \( \delta > 0 \). Thus it is enough to choose \( D \) such that \( D 2^{-D} < \frac{c}{4} \). 

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