Statistical Models with a Line of Defect

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Abstract

The factorization condition for the scattering amplitudes of an integrable model with a line of defect gives rise to a set of Reflection-Transmission equations. The solutions of these equations in the case of diagonal $S$-matrix in the bulk are only those with $S = \pm 1$. The choice $S = -1$ corresponds to the Ising model. We compute the transmission and reflection amplitudes relative to the interaction of the Majorana fermion with the defect and we discuss their relevant features.
1 Introduction

The theory of two-dimensional integrable statistical models with a finite correlation length can be elegantly formulated in terms of an ensemble of particle excitations in bootstrap interaction. Originally proposed by Zamolodchikov for systems defined in the infinite volume [1] (see also [2-6]), the bootstrap approach has been recently generalized to the case of integrable systems with boundary [7-13]. Here, in addition to the usual scattering matrices of the particles in the bulk, one needs to introduce the reflection amplitudes relative to the interaction with the boundary.

In this letter we will discuss the properties of integrable 2-d statistical models in the presence of a line of defect. These systems may interpolate between a bulk or a boundary statistical behaviour and their theoretical understanding has stimulated quite a large literature (see for instance [14-22]). We initially derive the Reflection-Transmission (RT) equations for these models. In the case of diagonal $S$-matrix in the bulk, one of the solutions of these equations corresponds to the Ising model with a line of defect. We compute the reflection and transmission amplitudes for the interaction of the Majorana fermion with the defect in two different ways, i.e. resumming the perturbative series and implementing the boundary conditions of the equations of motion.

2 Reflection-Transmission Equations

Consider an integrable 2-d model in the bulk, defined by an action

$$\mathcal{A}_B = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \mathcal{L} (\partial_\mu \phi_i; \phi_i) ,$$

(2.1)

where $\phi_i$ are local fields of the theory. Let us assume we have completely solved the dynamics of the corresponding theory in the Minkowski space and as result, we know the mass spectrum $\{m_a\}$ and the bulk scattering amplitudes $S_{cd}^{ab}(\beta_{ab})$ [1]. The presence of a linear defect line can be described by adding to the action of the bulk an interaction localized, say, along the $y$-axes,

$$\mathcal{A} = \mathcal{A}_B + \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \delta(x) \mathcal{L}_D \left( \phi_i, \frac{d\phi_i}{dy} \right) .$$

(2.2)

Suppose that the introduction of this additional term does not affect the integrability of the original model in such a way that the action (2.2) is still supported by an infinite number of conserved charges in involution. Since the system is invariant along the $y$-direction (which we choose to identify with the time axis in the Minkowski space), the

$$\beta_{ab} = \beta_a - \beta_b,$$

where $\beta_i$ is the rapidity variable of the particle $A_i$. It is related to the momenta by $p_0^{(i)} = m_i \cosh \beta_i$, $p_1^{(i)} = m_i \sinh \beta_i$. 



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energy is one of the integrals of motions. On the other hand, the momentum will not be generally conserved and we may have processes with an exchange of momentum on the defect line, compatible though with the conservation of the energy.

The conservation of the energy and of the other higher charges assures the complete elasticity of the scattering processes which take place on the defect line. In particular, this means that a particle which hits the defect line with rapidity $\beta$ can only proceed forward with the same rapidity or reverses its motion acquiring a rapidity of $-\beta$. A further effect of the interaction with the defect line may be a change of the label of the particle inside its multiplet of degeneracy. The interactions of the particles $|\beta; i>$ with the defect line will then be described in terms of the transmission and reflection amplitudes, denoted respectively by $T_{ij}(\beta)$ and $R_{ij}(\beta)$ (fig. 1). It is convenient to introduce the operator $D$ associated to the defect. This can be considered as an additional particle of the theory with zero rapidity in the entire time evolution of the system. Its commutation relations with the creation operators $A_{i}^{\dagger}(\beta)$ relative to the asymptotic particles are given by

$$A_{i}^{\dagger}(\beta) D = R_{ij}(\beta) A_{j}^{\dagger}(\beta) D + T_{ij}(\beta) D A_{j}^{\dagger}(\beta) ;$$

$$D A_{i}^{\dagger}(\beta) = R_{ij}(\beta) D A_{j}^{\dagger}(\beta) + T_{ij}(\beta) D A_{j}^{\dagger}(\beta) .$$

The consistency condition of this algebra requires the unitarity equations

$$R_{ij}(\beta) R_{jk}(\beta) + T_{ij}(\beta) T_{jk}(\beta) = \delta_{ik} ;$$

$$R_{ij}^{\dagger}(\beta) T_{jk}^{\dagger}(\beta) + T_{ij}^{\dagger}(\beta) R_{jk}^{\dagger}(\beta) = 0 .$$

Additional constraints emerge from the crossing relations

$$R_{ij}(\beta) = S_{ij}^{kl}(2\beta) R_{kl}(i\pi - \beta) ;$$

$$T_{ij}(\beta) = T_{ij}(i\pi - \beta) .$$

The first of (2.5) is obtained according to the argument proposed in [9, 10], and we assume the validity of the second equation as in the bulk.

Usually the presence of an infinite number of integrals of motion implies not only the elasticity of all scattering processes but also their complete factorization, i.e. an $n$-particle scattering amplitude can be expressed in terms of the elementary two-body interactions [23]. A crucial step for proving the factorization property of the total S-matrix is to impose the associativity condition of the algebra (2.3). In this case it gives rise to a set of Reflection-Transmission (RT) equations, those which are relevant to the following considerations shown in fig. 2. The first of them (fig. 2.a) coincides with the well-known boundary equations analysed in [4, 8],

$$S_{ac}^{ef}(\beta_{1} - \beta_{2}) R_{fg}(\beta_{2}) S_{ge}^{dh}(\beta_{1} + \beta_{2}) R_{gb}(\beta_{1}) = R_{ah}(\beta_{2}) S_{ch}^{fe}(\beta_{2} + \beta_{1}) R_{fg}(\beta_{2}) S_{eg}^{bd}(\beta_{1} - \beta_{2}) .$$

\[2\] For simplicity we consider the case of a defect without internal degrees of freedom.
The RT equations associated to the configurations of figs. (2.b), (2.c) and (2.d) are given respectively by

\[
S_{ac}^{lm}(\beta_1 - \beta_2) T_{lb}(\beta_1) T_{md}(\beta_2) = S_{md}^{bd}(\beta_1 - \beta_2) T_{cm}(\beta_2) T_{ad}(\beta_1) ;
\]
\[
S_{ac}^{fe}(\beta_1 - \beta_2) T_{fb}(\beta_1) R_{ed}(\beta_2) = R_{ce}(\beta_2) S_{ac}^{fd}(\beta_1 + \beta_2) T_{fb}(\beta_1) ;
\]
\[
S_{ac}^{fe}(\beta_1 - \beta_2) R_{fg}(\beta_2) S_{dh}^{ge}(\beta_1 + \beta_2) T_{hb}(\beta_1) = T_{ab}(\beta_1) R_{ed}(\beta_2) .
\]

The RT equations become very restrictive once applied to models with diagonal S-matrix in the bulk. In fact, whereas eq. (2.6) and the first in (2.7) are identically satisfied, the last two equations in (2.7) become

\[
S_{ab}(\beta_a + \beta_b) = S_{ab}(\beta_b - \beta_a) ,
\]
\[
S_{ab}(\beta_a + \beta_b) S_{ab}(\beta_a - \beta_b) = 1 ,
\]

whose solutions are only \( S_{ab}(\beta) = \pm 1 \). Hence, we arrive to the conclusion that the only integrable QFT with diagonal S-matrix in the bulk and factorizable scattering in the presence of the defect line are those associated to generalized-free theory. It is an open problem whether or not the Reflection-Transmission equations admit a non-trivial solution in the case of non-diagonal bulk S-matrix.

### 3 Ising Model with a Line of Defect

A particularly interesting theory is that associated to \( S = -1 \). It corresponds to the Ising model, which is described in the bulk by the free Majorana fermion with Lagrangian given by \[ 24, 25 \]

\[
L_M = \overline{\Psi}(\rho) (i\gamma^\mu \partial_\mu - m) \Psi(\rho) .
\]

In the Majorana representation, given by \( \gamma^0 = \sigma_2, \gamma^1 = -i\sigma_1 \), the fermionic field \( \Psi(\rho) \) is real, i.e. \( \Psi^\dagger(\rho) = \Psi(\rho) \). The Ising model with a line of defect is obtained by adding to the Lagrangian (3.1) the interaction \[ 16-22 \]

\[
L_D = -g \delta(x) \overline{\Psi}(\rho) \Psi(\rho) ,
\]

where \( g \) is a dimensionless coupling constant. The resulting Lagrangian is still quadratic in the fermionic field and the beta-function associated to the coupling constant \( g \) is identically zero. As a consequence, the theory presents a non-universal ultraviolet behaviour and the critical exponent of the magnetization operators depends continuously on the parameter \( g \). \[ 16, 17, 18, 19 \].

We are interested in determining the reflection \( R(\beta) \) and transmission \( T(\beta) \) amplitudes for the scattering of the fermion with the defect line, i.e. the S-matrix elements between 

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3We denote with \( \rho \) the couple of coordinate \((x, t)\).
initial and final states $u(p_i)$ and $\pi(p_f)$ with $p_f = \pm p_i$. To this aim, let us compute the Green-function of the fermion field using the Feynman rules given by

\[
\frac{\not{p} - \not{p}'}{\not{p}^2 - m^2 + i\varepsilon} = i(2\pi)^2 \delta^2(p - p') \frac{\not{p} + m}{\not{p}^2 - m^2 + i\varepsilon}, \quad \frac{\not{p} - \not{p}'}{\not{p}^2 - m^2 + i\varepsilon} = -ig 2\pi \delta(p_0 - p'_0)
\]

The self-energy entering the exact propagator is given by

\[
\Sigma = \cdots + \Sigma_2 + \Sigma_3 + \Sigma_4 + \cdots
\]

where we have to integrate on the spatial component of the momentum running in the internal lines. Using

\[
\Sigma(p_0) = 2\pi i \delta(p_0 - p_0') \sin \chi \frac{\omega - i \frac{g}{1 + \frac{g^2}{4}}(p_0 - m)}{\omega - i m \sin \chi},
\]

the geometric series for $\Sigma$ can be resummed as\footnote{In comparison with [21], notice that in the massless limit our result implies a different coupling constant dependance of the Green functions of the fermionic fields.}

\[
\omega = \sqrt{p_0^2 - m^2}, \quad \sin \chi = -\frac{g}{1 + \frac{g^2}{4}}.
\]

Hence, for the transmission and reflection amplitudes defined by

\[
< \beta' | \beta > = 2\pi \delta(\beta - \beta') T(\beta, g) + 2\pi \delta(\beta + \beta') R(\beta, g)
\]

we have

\[
T(\beta, g) = \frac{\cos \chi \sinh \beta}{\sinh \beta - i \sin \chi},
\]

\[
R(\beta, g) = i \frac{\sin \chi \cosh \beta}{\sinh \beta - i \sin \chi}.
\]

The transmission amplitude also contains the disconnected part relative to the free motion.

An alternative derivation of the transmission and reflection amplitudes is obtained by implementing the algebra (2.3) on the creation operators of the fermion field. Let $\Psi_{\pm}(x, t)$ be the solutions of the free Dirac equation in the two intervals $x > 0$ and $x < 0$, i.e.

\[
\Psi(\rho) = \theta(x) \Psi_+(x, t) + \theta(-x) \Psi_-(x, t),
\]
with the value at the origin given by $\Psi(0,t) = \frac{1}{2}(\Psi_+(0,t) + \Psi_-(0,t))$. The mode expansion of the two components of the fields $\Psi_{\pm}(\rho)$ is expressed as

$$
\psi_{(\pm)}^{(1)}(x,t) = \int \frac{d\beta}{2\pi} \left[ \omega e^{\frac{\beta}{2}} A_{(\pm)}(\beta) e^{-im(t \cosh \beta - x \sinh \beta)} + \omega e^{\frac{\beta}{2}} A_{(\pm)}^\dagger(\beta) e^{im(t \cosh \beta - x \sinh \beta)} \right] \quad (3.6)
$$

$$
\psi_{(\pm)}^{(2)}(x,t) = -\int \frac{d\beta}{2\pi} \left[ \omega e^{-\frac{\beta}{2}} A_{(\pm)}(\beta) e^{-im(t \cosh \beta - x \sinh \beta)} + \omega e^{-\frac{\beta}{2}} A_{(\pm)}^\dagger(\beta) e^{im(t \cosh \beta - x \sinh \beta)} \right],
$$

with $\omega = \exp(i\pi/4)$, $\omega = \exp(-i\pi/4)$. The operators $A_{\pm}(\beta)$ and $A_{\pm}^\dagger(\beta)$ satisfy the usual anti-commutation relations. They are not independent since they are related to each other by the conditions at $x = 0$ which arise from applying the eqs. of motion to (3.3)

$$
\begin{align*}
(p_{(\pm)}^{(2)} - p_{(\pm)}^{(1)})(0,t) &= \frac{g}{2}(p_{(\pm)}^{(1)} + p_{(\pm)}^{(2)})(0,t) ; \\
(p_{(1)}^{(1)} - p_{(1)}^{(2)})(0,t) &= \frac{g}{2}(p_{(2)}^{(1)} + p_{(2)}^{(2)})(0,t),
\end{align*}
$$

i.e.

$$
M \begin{pmatrix} A_{(\beta)}^\dagger(\beta) \\ A_{(\gamma)}^\dagger(-\beta) \end{pmatrix} = N \begin{pmatrix} A_{(-\beta)}^\dagger(\beta) \\ A_{(\beta)}^\dagger(\beta) \end{pmatrix},
$$

(3.8)

where

$$
M = \begin{pmatrix} \omega e^{\frac{\beta}{2}} + \frac{g}{2} \omega e^{\frac{\beta}{2}} & -\omega e^{\frac{\beta}{2}} + \frac{g}{2} \omega e^{\frac{\beta}{2}} \\ \omega e^{\frac{\beta}{2}} + \frac{g}{2} \omega e^{\frac{\beta}{2}} & \omega e^{\frac{\beta}{2}} - \frac{g}{2} \omega e^{\frac{\beta}{2}} \end{pmatrix};
$$

$$
N = \begin{pmatrix} -\omega e^{\frac{\beta}{2}} - \frac{g}{2} \omega e^{\frac{\beta}{2}} & \omega e^{\frac{\beta}{2}} - \frac{g}{2} \omega e^{\frac{\beta}{2}} \\ -\omega e^{\frac{\beta}{2}} - \frac{g}{2} \omega e^{\frac{\beta}{2}} & -\omega e^{\frac{\beta}{2}} + \frac{g}{2} \omega e^{\frac{\beta}{2}} \end{pmatrix}.
$$

Hence,

$$
\begin{pmatrix} A_{(\beta)}^\dagger(\beta) \\ A_{(\gamma)}^\dagger(-\beta) \end{pmatrix} = M^{-1} N \begin{pmatrix} A_{(-\beta)}^\dagger(\beta) \\ A_{(\beta)}^\dagger(\beta) \end{pmatrix} = \begin{pmatrix} R(\beta, g) & T(\beta, g) \\ T(\beta, g) & R(\beta, g) \end{pmatrix} \begin{pmatrix} A_{(-\beta)}^\dagger(\beta) \\ A_{(\beta)}^\dagger(\beta) \end{pmatrix},
$$

(3.9)

with $R(\beta, g)$ and $T(\beta, g)$ given in (3.4).

It is easy to see that the amplitudes (3.4) satisfy the unitarity and crossing equations (2.4) and (2.3). For negative values of $g$ the interaction with the defect line is attractive and the theory presents a bound state with binding energy $e_b = m \cos \chi$. Notice that simple expressions are obtained for the partial-wave phase shifts

$$
e^{2i\delta_0} \equiv T(\beta, g) + R(\beta, g) = \frac{\sinh \frac{1}{2}(\beta + i\chi)}{\sinh \frac{1}{2}(\beta - i\chi)};
$$

$$
e^{2i\delta_1} \equiv T(\beta, g) - R(\beta, g) = \frac{\cosh \frac{1}{2}(\beta - i\chi)}{\cosh \frac{1}{2}(\beta + i\chi)},
$$

where $\delta_0$ and $\delta_1$ are crossed functions of each other. Quite interesting is also the strong-weak duality presented by the scattering amplitudes, i.e.

$$
T\left(\beta, \frac{4}{g}\right) = -T(\beta, g), \quad R\left(\beta, \frac{4}{g}\right) = R(\beta, g).
$$

(3.10)
At the self-dual points $g^2 = 4$ the transmission amplitude vanishes and the reflection amplitudes $R(\beta, \pm 2)$ reduce to those of the Ising model with free (−) and fixed (+) boundary conditions, as determined in [9]. This can also be seen directly by analysing the resulting boundary conditions (3.7).

4 Conclusion

Integrable models with a line of defect are described, in addition to the usual scattering processes in the bulk, by a set of reflection and transmission amplitudes. The factorization condition of the $n$-particle amplitude gives rise to the Reflection-Transmission equations which put severe constraints on the possible realization of those models. In the case of diagonal $S$-matrix in the bulk, there are only two solutions of the RT equations, one of them corresponding to the Ising model with a line of defect. In this paper we have determined the reflection and transmission amplitudes relative to the interaction of the Majorana fermion with the defect line. This formulation of the model can be very useful to compute its correlation functions by using the form factor approach. We hope to come back to this problem in a future publication.

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Figure Captions

**Figure 1**. Reflection and Transmission Amplitudes.

**Figure 2**. Reflection-Transmission Equations.
Figure 1
Figure 2.b
Figure 2.c
Figure 2.d