No point-localized photon states

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The aim of this paper is to critically examine claims that it is possible to construct point-localized state vectors for the photon. We supply a brief proof of the impossibility of this. Then it is found that the authors making these claims use a non-standard scalar product, not equal to the quantum-mechanical one. This alternative scalar product is found to be proportional to a Dirac delta function in position for the state vectors they use, but the remaining elements of the proof, namely satisfying all three Newton-Wigner criteria, are lacking.

I. INTRODUCTION

Newton and Wigner \[1\] defined three physical criteria that must be satisfied by the point-localized state vectors of a massive or massless particle. The first is that a nonzero spatial translation of a localized state vector must produce another localized state vector that is orthogonal to the original. Then a rotation of a state vector localized at the spatial origin at any time, \(t\), must produce another state vector localized at the origin. Such a state vector can be labelled by another quantum number that must carry an irreducible representation of rotations, with state vectors for different values of this quantum number being mutually orthogonal. Lastly, the scalar product of a localized state vector and any boost of this state vector must be a continuous function of the boost velocity.

It is not possible to satisfy all three criteria for the photon. The reason is not the masslessness of the photon but because of its limited helicity spectrum, as we will confirm below. This point was noted by Wightman \[2\]. Within this limitation, it is possible to define measures of partial (not point) localization for the photon \[3\].

Given the impossibility result, it was surprising to learn that some authors \[4–7\] claim to have constructed point-localized states for the photon. Perhaps the result was not derived in sufficient detail, so we remedy that situation in Section II. Then, in Section III, we closely examine some of the claims of point-localized photons and attempt to reproduce their results. Conclusions follow in Section IV.

II. IMPOSSIBILITY OF POINT-LOCALIZED PHOTON STATE VECTORS

As stated above, the helicity spectrum of a massless particle determines whether it can be point-localized. A hypothetical particle of zero mass and zero helicity could be localized at a point, with state vectors

\[
|x, 0\rangle = \frac{1}{(2\pi)^\frac{3}{2}} \int \frac{d^3k}{\sqrt{\omega}} \left| k, 0 \right> e^{ik \cdot x},
\]

where \(\omega = |k|\) and the covariant normalization of the momentum-helicity eigenvectors (for general helicity, \(\lambda\)),

\[
\langle k_1, \lambda_1 | k_2, \lambda_2 \rangle = \delta_{\lambda_1,\lambda_2} \omega_1 \omega_2 \delta^3(k_1 - k_2),
\]

will be used throughout this paper. Also, in this paper, we use Heaviside-Lorentz units, in which \(\hbar = c = \epsilon_0 = \mu_0 = 1\). Only positive energies are used in this superposition. These state vectors have the equal-time scalar product

\[
\langle (t, x_1), 0 | (t, x_2), 0 \rangle = \delta^3(x_1 - x_2)
\]

and rotate and translate as expected for localized state vectors. They boost continuously.

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The photon has only two physical helicities, \( \lambda = \pm 1 \). If there were a \( \lambda = 0 \) photon in addition to these two, we could construct three state vectors (for \( \sigma = -1, 0, 1 \))

\[
|x, \sigma, 3 \rangle = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \frac{d^3 k}{\sqrt{\omega}} \sum_{\lambda=1,0,1} |k, \lambda\rangle \mathcal{R}_{\lambda\sigma}^{(1)}(\hat{k}) e^{ik\cdot x}.
\]  

(4)

Here \( \mathcal{R}_{\lambda\mu}^{(1)}(\hat{k}) \) are \( j = 1 \) matrix elements of the inverse of the standard rotation

\[
R_0[\hat{k}] = R_z(\varphi)R_y(\theta)R_z(-\varphi)
\]

(5)

takes \( \hat{z} \) into the momentum direction \( \hat{\kappa} = (\theta, \varphi) \).

The basis vectors rotate as

\[
U(R)|k, \lambda\rangle = |Rk, \lambda\rangle e^{-i\lambda \omega(R, \hat{k})},
\]

(6)

with a Wigner rotation, a rotation about the momentum direction, \( \hat{\kappa} \), specified by \( \hat{\kappa} \).

Then we have (no sum over \( \lambda \))

\[
U(R)|k, \lambda\rangle \mathcal{R}_{\lambda\sigma}^{(1)}(\hat{k}) = \sum_{\sigma' = -1}^1 |Rk, \lambda\rangle \mathcal{R}_{\lambda\sigma'}^{(1)}(R\hat{k}) D_{\sigma' \sigma}^{(1)}(R).
\]

(7)

This is an important result, as it shows that each helicity rotates with the same transformation matrix.

This leads to the required rotation behaviour

\[
U(R)|(t, x), \sigma, 3\rangle = \sum_{\mu' = -1}^1 |(t, Rx), \sigma', 3\rangle D_{\sigma' \sigma}^{(1)}(R),
\]

(9)

according to the unitary irreducible rotation representation with angular momentum quantum number \( j = 1 \). It is clear that this construction (Eq. 4) is the unique solution (up to phase changes dependent on helicity) of the requirement in Eq. 9.

The unitary transformation

\[
\langle \sigma | i \rangle = \sqrt{\frac{4\pi}{3}} Y_{1\sigma}^i(i) \quad \text{for } i = 1, 2, 3,
\]

(10)

produces three state vectors for every \( x \),

\[
|x, i, 3\rangle = \sum_{\sigma = -1}^1 |x, \sigma, 3\rangle \langle \sigma | i \rangle \quad \text{for } i = 1, 2, 3.
\]

(11)

We note that the coefficients in the superposition can be identified as the components of complex conjugate polarization vectors in a particular gauge,

\[
\epsilon^*_i(\hat{k}, \lambda) = \sum_{\sigma = -1}^1 \mathcal{R}_{\lambda\sigma}^{(1)}(\hat{k}) \langle \sigma | i \rangle = \sum_{\sigma = -1}^1 \sum_{j = 1}^3 \sum_{k = 1}^3 \langle \lambda | j \rangle R_{0j}^{-1}(\hat{k}) \langle k | \sigma \rangle \langle \sigma | i \rangle = \sum_{j = 1}^3 R_{0ij}(\hat{k}) \langle \lambda | j \rangle.
\]

(12)

They are obtained by a rotation from their \( \hat{x} = \hat{z} \) values

\[
\epsilon^*(\hat{z}, +1) = -\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}, \quad \epsilon^*(\hat{z}, 0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \epsilon^*(\hat{z}, -1) = +\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}.
\]

(13)

So we can write

\[
|x, i, 3\rangle = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \frac{d^3 k}{\sqrt{\omega}} \sum_{\lambda=1,0,1} |k, \lambda\rangle \epsilon^*_i(\hat{k}, \lambda) e^{ik\cdot x}.
\]

(14)
The three state vectors in Eq. (14) rotate according to the three-vector representation of rotations

\[ U(R) | (t, x), i, 3 \rangle = \sum_{i' = -1}^{1} | (t, R x), i', 3 \rangle \sum_{\sigma', \sigma = -1}^{1} \langle i' | \sigma' \rangle \mathcal{D}^{(1)}_{\sigma' \sigma}(R) \langle \sigma | i \rangle = \sum_{i' = -1}^{1} | (t, R x), i', 3 \rangle R_{\epsilon i}^{\epsilon i}. \]  

(15)

Both sets of state vectors translate correctly and satisfy the equal-time orthonormality relations

\[ \langle (t, x_1), \sigma_1, 3 | (t, x_2), \sigma_2, 3 \rangle = \frac{1}{(2\pi)^3} \int d^3k \sum_{\lambda = -1, 1} \mathcal{R}^{(1)}_{\sigma_1 \lambda}(k) \mathcal{R}^{(1)}_{\lambda \sigma_2}(\hat{k}) e^{ik(x_1 - x_2)} = \delta_{\sigma_1 \sigma_2} \delta^3(x_1 - x_2). \]  

(16)

and

\[ \langle (t, x_1), i_1, 3 | (t, x_2), i_2, 3 \rangle = \delta_{i_1 i_2} \delta^3(x_1 - x_2), \]  

(17)

as required for sets of three point-localized state vectors.

Now, for the physical photon, we must remove the zero helicity in the new definition

\[ | x, \sigma, \gamma \rangle = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \frac{d^3k}{\sqrt{\omega}} \sum_{\lambda = -1, 1} | k, \lambda \rangle \mathcal{R}^{(1)}_{\lambda \sigma}(\hat{k}) e^{ikx} \quad \text{for } \sigma = -1, 0, 1. \]  

(18)

We continue to use the \( j = 1 \) rotation representation, as it is expected to come closest to the desired result. From the result of Eq. (15), we see that these three still rotate in the desired way. However, the equal-time overlap becomes

\[ \langle (t, x_1), \sigma_1, \gamma | (t, x_2), \sigma_2, \gamma \rangle = \frac{1}{(2\pi)^3} \int d^3k \sum_{\lambda = -1, 1} \mathcal{R}^{(1)}_{\sigma_1 \lambda}(k) \mathcal{R}^{(1)}_{\lambda \sigma_2}(\hat{k}) e^{ik(x_1 - x_2)}. \]  

(19)

The sum over helicities is

\[ \sum_{\lambda = -1, 1} \mathcal{R}^{(1)}_{\sigma_1 \lambda}(k) \mathcal{R}^{(1)}_{\lambda \sigma_2}(\hat{k}) = \delta_{\sigma_1 \sigma_2} - \mathcal{R}^{(1)}_{\sigma_1 0}(\hat{k}) \mathcal{R}^{(1)}_{0 \sigma_2}(\hat{k}) = \delta_{\sigma_1 \sigma_2} - \mathcal{R}^{(1)}_{\sigma_1 0}(\hat{k}) \mathcal{R}^{(1)}_{0 \sigma_2}(\hat{k}) = \delta_{\sigma_1 \sigma_2} \left( \begin{array}{ccc} \frac{1}{2} \sin^2 \theta & - \frac{i e^{-i\varphi}}{\sqrt{2}} \sin \theta \cos \theta & - \frac{i e^{-i2\varphi}}{\sqrt{2}} \sin^2 \theta \\ - \frac{i e^{i\varphi}}{\sqrt{2}} \sin \theta \cos \theta & \cos^2 \theta & \frac{i e^{i2\varphi}}{\sqrt{2}} \sin \theta \cos \theta \\ - \frac{1}{2} e^{i2\varphi} \sin^2 \theta & \frac{i e^{i2\varphi}}{\sqrt{2}} \sin \theta \cos \theta & \frac{1}{2} \sin^2 \theta \end{array} \right). \]  

(20)

So

\[ \langle (t, x_1), \sigma_1, \gamma | (t, x_2), \sigma_2, \gamma \rangle = \delta_{\sigma_1 \sigma_2} \delta^3(x_1 - x_2) - \frac{1}{(2\pi)^3} \int d^3k \, M_{\sigma_1 \sigma_2}(\hat{k}) e^{ik(x_1 - x_2)}, \]  

(21)

where \( M_{\sigma_1 \sigma_2}(\hat{k}) \) is the matrix appearing in the second term of Eq. (20).

For the three-vector states

\[ \langle (t, x_1), i_1, \gamma | (t, x_2), i_2, \gamma \rangle = \frac{1}{(2\pi)^3} \int d^3k \, (\delta_{i_1 i_2} - \sum_{\lambda = -1, 1} \langle i_1 | \sigma_1 \rangle \mathcal{R}^{(1)}_{\sigma_1 0}(\hat{k}) \mathcal{R}^{(1)}_{0 \sigma_2}(\hat{k}) \langle \sigma_2 | i_2 \rangle) e^{ik(x_1 - x_2)}. \]  

(22)

with summation over \( \mu_1 \) and \( \mu_2 \) implied. We find the result

\[ \langle (t, x_1), i_1, \gamma | (t, x_2), i_2, \gamma \rangle = \delta_{i_1 i_2} \delta^3(x_1 - x_2) - \frac{1}{(2\pi)^3} \int d^3k \, \hat{i}_1 \cdot \hat{k} \hat{i}_2 e^{ik(x_1 - x_2)}, \]  

(23)

illustrating the impossibility of point-localized photon state vectors, at least using the \( j = 1 \) rotation representation.

Clearly using \( j = 0 \) gives the wrong rotation properties for the state vectors. Using \( j \geq 2 \), there will be unwanted terms inside the momentum integral for the scalar products, given by

\[ T^{(j)}_{\sigma_1 \sigma_2}(\hat{k}) = - \sum_{\lambda} \mathcal{R}^{(j)}_{\sigma_1 \lambda}(\hat{k}) \mathcal{R}^{(j)-1}_{\lambda \sigma_2}(\hat{k}), \]  

(24)

a sum over all helicities other than \( \lambda = \pm 1 \).
III. ANALYSIS OF CLAIMED POINT-LOCALIZED STATES

Hawton’s (her Eq. (34), summed over \( \epsilon \) (the sign of the energy) and \( \lambda \)) constructs state vectors (using our normalization conventions)

\[
|A^\mu(x)\rangle = \frac{1}{(2\pi)^2} \int \frac{d^3k}{\omega} \sum_{\lambda = \pm 1} |k, \lambda\rangle \epsilon^{\mu(\hat{k}, \lambda)} e^{+ik\cdot x}.
\]  

Here \( \epsilon^{\mu(\hat{k}, \lambda)} \) is now a four-component polarization vector, with \( \mu = 0, 1, 2, 3 \).

Negative frequency contributions, \( |A^{(-)^\mu}(x)\rangle \), with the phase factor \( \exp(-ik\cdot x) \), are also included, but they don’t translate correctly:

\[
U(T(a)) |A^{(-)^\mu}(x)\rangle = |A^{(-)^\mu}(x-a)\rangle
\]  

instead of

\[
U(T(a)) |A^{(-)^\mu}(x)\rangle = |A^{(-)^\mu}(x+a)\rangle.
\]  

It seems that these negative frequencies are not essential to their result, so we omit them here.

The first issue is gauge variation. The four-component polarization vectors appear in the electromagnetic field strength operators, \( F^{\mu\nu}(x) \), and in the four-component gauge field operator, \( A^\mu(x) \). A gauge transformation,

\[
\epsilon^{\mu(\hat{k}, \lambda)} \rightarrow \epsilon^{\mu(\hat{k}, \lambda) + g(|k|) k^\mu},
\]  

for arbitrary \( g \), leaves the observable electromagnetic field strengths unchanged, but changes the gauge field components. We note that such a gauge transformation preserves the Lorentz condition,

\[
k \cdot \epsilon(\hat{k}, \lambda) \equiv 0,
\]  

which is essential for the gauge fields to satisfy the Maxwell equations.

Unless a particular gauge is chosen, Eq. (24) does not give a unique definition. We choose to consider the radiation gauge, with polarization vectors \( \epsilon^{\mu R}(\hat{k}, \lambda) \), in which \( \epsilon^{\mu R}(\hat{k}, \lambda) \equiv 0 \) and the spatial parts are given by Eqs. (12,13). This is part of the Lorentz family of gauges since \( k \cdot \epsilon_R(\hat{k}, \lambda) = \hat{z} \cdot \epsilon_R(\hat{z}, \lambda) \equiv 0 \).

Then we arrive at three state vectors for every \( x \),

\[
|x, i, H\rangle = \frac{1}{(2\pi)^2} \int \frac{d^3k}{\omega} \sum_{\lambda = \pm 1} |k, \lambda\rangle \epsilon^{i}_R(\hat{k}, \lambda) e^{+ik\cdot x} \quad \text{for } i = 1, 2, 3.
\]  

Since a different factor of the rotational invariant, \( \omega \), in the superposition will not change the rotation properties, we see that these state vectors rotate according to the irreducible representation. Their equal-time quantum-mechanical scalar products are

\[
\langle (t, x_1), i_1, H | (t, x_2), i_2, H\rangle = \frac{1}{(2\pi)^3} \int \frac{d^3k}{\omega} \sum_{\lambda = \pm 1} \epsilon^{i_1R}_R(\hat{k}, \lambda)\epsilon^{i_2R}_R(\hat{k}, \lambda) e^{ik\cdot (x_1-x_2)}
\]

\[
= \frac{1}{(2\pi)^3} \int \frac{d^3k}{\omega} \delta(i_1 - i_2) \hat{k} \cdot \hat{k} e^{ik\cdot (x_1-x_2)},
\]

clearly not giving the required orthogonality result.

It is remarkable that these authors [4–7] define an alternative scalar product, not equal to the quantum-mechanical scalar product. If a state vector is defined as a superposition of basis vectors with known scalar products, then the quantum-mechanical scalar product of two such state vectors is completely defined and not subject to arbitrary redefinition. The orthogonality properties of the basis vectors, in this case, follow from the fact that they are eigenvectors of Hermitian observables. State vectors can be defined in different ways, but then their scalar products are fixed.

The alternative scalar product is defined by

\[
\langle A_1(t, x_1), A_2(t, x_2) \rangle = \frac{1}{(2\pi)^3} \int \frac{d^3k}{\omega} \sum_{\lambda = \pm 1} \sum_{i=1}^3 \epsilon^{i}_R(\hat{k}, \lambda)\epsilon^{i}_R(\hat{k}, \lambda) e^{ik\cdot (x_1-x_2)}.
\]  

\[i \]
To verify the rotation criterion of Newton and Wigner, nine scalar products for each \( t, x_1, x_2 \) are needed, one for each pair \( i_1, i_2 \). But only one is given here, with the index \( i \) summed over. If there were no sum over \( i \), we would again have

\[
\sum_{\lambda = \pm 1} e_R^i(\hat{k}, \lambda) e_R^i(\hat{k}, \lambda) = \delta_{i_1 i_2} - \hat{i}_1 \cdot \hat{k} \hat{k} \cdot \hat{i}_2.
\]  

(32)

Using

\[
\epsilon_R(\hat{k}, \lambda) \cdot \epsilon^*_R(\hat{k}, \lambda) = \epsilon_R(\hat{z}, \lambda) \cdot \epsilon^*_R(\hat{z}, \lambda) = 1
\]  

(33)
gives their result

\[
(A_1(t, x_1), A_2(t, x_2)) = 2 \delta^3(x_1 - x_2).
\]  

(34)

This result does not, as claimed, prove that localized photon state vectors have been constructed.

IV. CONCLUSIONS

We have confirmed that no point-localized state vectors can be constructed for the photon that satisfy all three of the criteria of Newton and Wigner [1]. The reason is the limited helicity spectrum of the photon. If there were a zero helicity state in addition to \( \lambda = \pm 1 \), such localized states, three for every position and time, could be constructed.

We considered the claims of other authors [4–7] that they could construct point-localized photon states satisfying the three criteria of Newton and Wigner. It was found that they employ an alternative definition of scalar product, not equal to the well-defined quantum-mechanical one. It is only by using this alternative scalar product that they are able to find a Dirac delta function in position. But the argument is incomplete. The rotation of their state vectors involves a subspace of dimension 3. Orthogonality of state vectors with different values of the index would be required to complete the proof, but no such result is given and would clearly not be possible.

We note that these authors also claim to have constructed a positive-definite position probability density for the photon (the zero component of a locally conserved four-current). If this were the case, it would have to take the form

\[
\hat{\rho}(x) = \sum_\sigma |x, \sigma\rangle\langle x, \sigma|.
\]

The basis vectors would have to rotate according to an irreducible representation and satisfy mutual orthogonality between different values of the index, \( \sigma \). Since we know this is not possible, the claim is baseless.

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