DYSON DIFFUSION ON A CURVED CONTOUR

A. V. Zabrodin∗†‡

We define the Dyson diffusion process on a curved smooth closed contour in the plane and derive the Fokker–Planck equation for the probability density. Its stationary solution is shown to be the Boltzmann weight for the logarithmic gas confined on the contour.

Keywords: diffusion process, logarithmic gas, partition function

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1. Introduction

In the pioneering paper [1], Dyson defined a diffusion process for eigenvalues of a unitary matrix confined to the unit circle. In this paper, we introduce a more general diffusion process for $N$ interacting particles in an external potential confined to a smooth closed contour of arbitrary shape in the plane. This is done using $N$ copies of the standard Brownian motion $B_i(t)$.

The time evolution of the probability density is described by the Fokker–Planck equation. We derive this equation and find its stationary solution, which is the Boltzmann weight for the logarithmic gas in an external potential. The logarithmic gases were introduced by Dyson in [2], where it was shown that eigenvalues of random matrices can be represented as a statistical ensemble of charged particles with the 2D Coulomb (logarithmic) interaction. A comprehensive presentation of the theory of logarithmic gases can be found in [3]. In our case, we obtain the Boltzmann weight for the logarithmic gas on a curved contour at the inverse temperature $β$ studied in [4]. In Sec. 4, we present the results for the large-$N$ expansion of the free energy of this gas.

2. The diffusion process on a curved contour

Let $Γ$ be a smooth non-self-intersecting closed contour in the plane. We consider a system of $N$ interacting particles with complex coordinates $z_i ∈ Γ$ moving along the contour and subject to Gaussian random forces.

To describe the diffusion process, we introduce $N$ copies of the standard Brownian motion $B_i(t)$ with $B_i(0) = 0$ under the conditions

$$\langle B_i(t) \rangle = 0, \quad \langle (B_i(t) - B_i(t'))(B_j(t) - B_j(t')) \rangle = \delta_{ij}|t - t'|,$$

or

$$\langle \dot{B}_i(t) \dot{B}_j(t') \rangle = \delta_{ij}\delta(t - t').$$

∗Skolkovo Institute of Science and Technology, Moscow, Russia, e-mail: zabrodin@itep.ru.
†National Research University Higher School of Economics, Moscow, Russia.
‡Alikhanov Institute for Theoretical and Experimental Physics, National Research Center “Kurchatov Institute,” Moscow, Russia.

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For $dB_i(t) = B_i(t + dt) - B_i(t)$, we have

$$
\langle dB_i(t) \rangle = 0, \quad \langle dB_i(t)dB_j(t) \rangle = 0, \quad i \neq j, \quad \langle (dB_i(t))^2 \rangle = dt,
$$

and we should therefore consider $(dB_i)^2$ as a first-order quantity in $dt$.

Below, we use the standard notions and facts from stochastic calculus and the theory of stochastic differential equations [5], [6]. We can write

$$
dz_i(t) = z_i(t + dt) - z_i(t),
$$

where both points $z_i(t)$ and $z_i(t + dt)$ belong to $\Gamma$. For $z \in \Gamma$, let $\tau(z)$ and $\nu(z)$ be respectively the unit tangential and normal vectors to $\Gamma$ at $z$, represented as complex numbers. We assume that the tangential vector is oriented counterclockwise and the normal vector is directed to the exterior of the contour; we then have $\tau = i\nu$. For $z = z_i$, we write $\tau(z_i) = \tau_i$ and $\nu(z_i) = \nu_i$ for brevity. Then $\tau_i|dz_i|$ is the tangent vector at $z_i$ with length $|dz_i|$. However, for the needs of stochastic calculus we, cannot simply write $dz_i = \tau_i|dz_i|$ because it is necessary to expand through the second order. It is easy to see that the second-order correction is $-\nu_i|dz_i|d\theta_i/2$, where $\theta(z) = \arg \tau(z)$. By definition, $k(z) = d\theta(z)/|dz|$ is the curvature of $\Gamma$ at the point $z$, and we can therefore write

$$
dz_i = \tau_i|dz_i| - \frac{1}{2} \nu_i k_i |dz_i|^2.
$$

We define the diffusion process by setting

$$
|dz_i| = \partial_e E dt + \sqrt{\kappa} dB_i.
$$

Here,

$$
E = 2 \sum_{i<j} \ln |z_i - z_j| + \sum_i W(z_i),
$$

$W(z)$ is an external potential, $\partial_e = \tau \partial_z + \bar{\tau} \partial_{\bar{z}}$ is the tangential derivative, and $\kappa$ is a parameter characterizing the strength of the random force. The real-valued function $W$ is defined in a strip-like neighborhood of the contour and depends on both $z$ and $\bar{z}$ (we write $W = W(z)$ just for simplicity of notation). Explicitly, we have

$$
\partial_e E = \tau_i \sum_{j \neq i} \frac{1}{z_i - z_j} + \tau_i \partial_z W(z_i) + \text{c.c.}
$$

Substituting (4) in Eq. (3), we define the diffusion process on $\Gamma$ as the stochastic differential equation

$$
dz_i = \tau_i \partial_e E dt - \frac{\kappa}{2} \nu_i k_i (dB_i)^2 + \sqrt{\kappa} \tau_i dB_i.
$$

In the right-hand side, we keep only terms of the first order in $dt$.

We note that Eq. (7) agrees with the familiar diffusion process on the unit circle, where $z_j = e^{i\theta_j}$ (see [1]). In terms of the $\theta_j$, the stochastic differential equation (the Langevin equation) is of the standard form

$$
d\theta_j = \partial_{\theta_j} E dt + \sqrt{\kappa} dB_j.
$$

Applying the Itô formula

$$
df(\theta_j) = \frac{\partial f}{\partial \theta_j} d\theta_j + \frac{\kappa}{2} \frac{\partial^2 f}{\partial \theta_j^2} (dB_j)^2
$$

to the function $f(\theta) = e^{i\theta}$, we can rewrite Eq. (8) in the form

$$
dz_j = iz_j \partial_{\theta_j} E dt - \frac{\kappa}{2} z_j (dB_j)^2 + iz_j \sqrt{\kappa} dB_j,
$$

which agrees with (7) because $\tau_j = iz_j$ and $k_j = 1$ for the unit circle. Equation (7) is a generalization of (9) for curved contours of arbitrary shape.
We now derive a stochastic differential equation for an arbitrary (sufficiently smooth) function \( f = f(z_1, \ldots, z_N) \). As is customary in the Itô calculus, the differential \( df \) contains not only the first-order but also the second-order terms in \( dz_i \):

\[
df = \sum_i \left( \frac{\partial f}{\partial z_i} \, dz_i + \frac{\partial f}{\partial z_i} \, d\bar{z}_i \right) + \frac{1}{2} \sum_i \left( \frac{\partial^2 f}{\partial z_i^2} (dz_i)^2 + \frac{\partial^2 f}{\partial z_i \partial \bar{z}_i} (dz_i \, d\bar{z}_i) + \frac{\partial^2 f}{\partial \bar{z}_i^2} (d\bar{z}_i)^2 \right). \tag{10}
\]

Using (7) and taking into account that \( \nu_i \partial_z f + \bar{\nu}_i \partial_{\bar{z}} f = \partial_n f \) is the normal derivative of \( f \) at the point \( z_i \in \Gamma \) (the normal vector directed to the exterior of \( \Gamma \)), we can see from (10) that

\[
df = \sum_i \partial_{s_i} E \partial_{s_i} f \, dt + \frac{\kappa}{2} \sum_i (-k_i \partial_n f + \tau_i^2 \partial_z^2 f + \tau_i^2 \partial_{\bar{z}}^2 f + 2 \partial_z \partial_{\bar{z}} f) \, (d\bar{B}_i)^2 + \sqrt{\kappa} \sum_i \partial_{s_i} f \, dB_i. \tag{11}
\]

According to the Itô calculus rules, we should put \((d\bar{B}_i)^2 = dt\). In this way, we obtain the stochastic differential equation

\[
df = \sum_i \left( \partial_{s_i} E \partial_{s_i} f + \frac{\kappa}{2} (-k_i \partial_n f + \tau_i^2 \partial_z^2 f + \tau_i^2 \partial_{\bar{z}}^2 f + 2 \partial_z \partial_{\bar{z}} f) \right) dt + \sqrt{\kappa} \sum_i \partial_{s_i} f \, dB_i. \tag{12}
\]

In order to simplify the right-hand side, for \( f = f(z) \), we write

\[
\partial^2 f = \partial_k (\partial_z f) + \partial_{\bar{z}} (\partial_{\bar{z}} f) = \partial_k \tau \partial_z f + \partial_{\bar{z}} (\partial_{\bar{z}} f) + \text{c.c.}
\]

Taking into account that \( \partial_k \tau = i \kappa k \), we have

\[
\partial^2 f = -k (\nu \partial_z f + \bar{\nu} \partial_{\bar{z}} f) + \tau (\tau \partial_z^2 f + \bar{\tau} \partial_{\bar{z}}^2 f + 2 \partial_z \partial_{\bar{z}} f)
\]

or, finally,

\[
\partial^2 f = -k \partial_n f + \tau^2 \partial_z^2 f + \bar{\tau}^2 \partial_{\bar{z}}^2 f + 2 \partial_z \partial_{\bar{z}} f. \tag{13}
\]

Substituting this in the right-hand side of (12), we obtain the stochastic differential equation in the simple form

\[
df = \sum_i \left( \frac{\kappa}{2} \partial^2_{s_i} f + \partial_{s_i} E \partial_{s_i} f \right) dt + \sqrt{\kappa} \sum_i \partial_{s_i} f \, dB_i. \tag{14}
\]

with the derivatives in the tangential direction only.

3. The Fokker–Planck equation and its stationary solution

Let \( \hat{A} \) be the differential operator

\[
\hat{A} = \sum_i \left( \frac{\kappa}{2} \partial^2_{s_i} + \partial_{s_i} E \partial_{s_i} \right). \tag{15}
\]

Equation (14) can then be written as

\[
df = \hat{A} f + \sqrt{\kappa} \sum_i \partial_{s_i} f \, dB_i.
\]

As is known, the Fokker–Planck equation for the time evolution of the probability density \( P = P(z_1, \ldots, z_N) \) is

\[
\partial_t P = \hat{A}^* P,
\]

where \( \hat{A}^* \) is the operator adjoint to \( \hat{A} \). Explicitly, in our case the Fokker–Planck equation is

\[
\partial_t P = \sum_i \left( \frac{\kappa}{2} \partial^2_{s_i} P - \partial_{s_i} E \partial_{s_i} P - \partial^2_{s_i} EP \right). \tag{16}
\]
The similarity transformation of the operator \( \hat{A}^* \) with the function \( e^{E/\kappa} \) eliminates the first-order derivatives. With the help of this procedure, we obtain the “Hamiltonian”

\[
\hat{H} = e^{-E/\kappa} \hat{A}^* e^{E/\kappa} = \frac{1}{2} \sum_i \left( \kappa \partial_{\theta_i}^2 - \kappa^{-1} \left( \partial_{\theta_i} E \right)^2 - \partial_{\theta_i}^2 E \right).
\]

We note that if the contour \( \Gamma \) is the unit circle and \( W = 0 \), then \( \hat{H} \) is the Hamiltonian of the Calogero–Sutherland model:

\[
\hat{H} = \frac{\kappa}{2} \left( \sum_i \partial_{\theta_i}^2 - \frac{2}{\kappa} \left( \frac{2}{\kappa} - 1 \right) \sum_{i \neq j} \frac{1}{4 \sin^2((\theta_i - \theta_j)/2)} \right) + \text{const.}
\]

The stationary solution \( P_0 \) of (16) such that \( \partial_\beta P_0 = 0 \) is the zero mode of the operator \( \hat{A}^* \): \( \hat{A}^* P_0 = 0 \). It is easy to verify that it has the form

\[
P_0 = e^{2E/\kappa}.
\]

Setting \( \beta = 2/\kappa \) and substituting Eq. (5) for \( E \), we have

\[
P_0(z_1, \ldots, z_N) = \exp \left( 2\beta \sum_{i<j} \ln |z_i - z_j| + \beta \sum_i W(z_i) \right) = \prod_{i<j} |z_i - z_j|^{2\beta} \prod_k e^{\beta W(z_k)}.
\]

This is the Boltzmann weight for the logarithmic gas in the external potential \( W \) at the inverse temperature \( \beta \). The Boltzmann weight (20) defines the \( \beta \)-ensemble of \( N \) particles on a simple closed contour \( \Gamma \) in the plane [4]. An early reference on this problem is [7] (section 7). A similar problem with the support in the complex plane, the 2D Dyson gas, was considered in [8]–[11].

### 4. Free energy of the logarithmic gas: large-\( N \) asymptotics

For completeness, we present the results for the large-\( N \) asymptotics of the free energy of the logarithmic gas obtained in [4] (see also [12] for the rigorous proof for \( \beta = 1 \)). We also give a new interpretation of the \( O(1) \)-contribution.

The partition function is the \( N \)-fold integral

\[
Z_N = \oint_{\Gamma} \cdots \oint_{\Gamma} P_0(z_1, \ldots, z_N) \, ds_1 \cdots ds_N
\]

with \( P_0 \) given by (20), where \( ds_i = |dz_i| \) is the line element along the contour. As \( N \to \infty \), the partition function is known to behave as

\[
Z_N = N! N^{(\beta - 1)N} e^{F(N)},
\]

where the free energy \( F^{(N)} \) has the expansion in integer powers of \( N \),

\[
F^{(N)} = N^2 F_0 + NF_1 + F_2 + O\left( \frac{1}{N} \right).
\]

The nonvanishing contributions as \( N \to \infty \), \( F_0, F_1, \) and \( F_2 \), are of primary interest.

The curve \( \Gamma \) divides the plane into the interior domain \( \mathbb{D}_{\text{int}} \) and the exterior domain \( \mathbb{D}_{\text{ext}} \), which contains \( \infty \). Without loss of generality, we assume that \( 0 \in \mathbb{D}_{\text{int}} \). The results for \( F_0, F_1, \) and \( F_2 \) are respectively expressed in terms of conformal maps \( w_{\text{int}}(z) \) and \( w_{\text{ext}}(z) \) of the domains \( \mathbb{D}_{\text{int}} \) and \( \mathbb{D}_{\text{ext}} \) to the interior and exterior of the unit circle. We normalize the conformal maps as

\[
\begin{align*}
  w_{\text{int}}(0) &= 0, \\
  w_{\text{int}}'(0) &= 1, \\
  w_{\text{ext}}(\infty) &= \infty, \\
  w_{\text{ext}}'(\infty) &= \frac{1}{r} > 0.
\end{align*}
\]

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The quantity $r$ is called the conformal radius of the domain $\mathbb{D}_{\text{ext}}$. It is also convenient to use the notation
\begin{equation}
\psi_{\text{int}}(z) = \ln |w'_{\text{int}}(z)|, \quad \psi_{\text{ext}}(z) = \ln |w'_{\text{ext}}(z)|. \tag{24}
\end{equation}
We note that $\psi_{\text{int}}$ and $\psi_{\text{ext}}$ are harmonic functions in the respective domains $\mathbb{D}_{\text{int}}$ and $\mathbb{D}_{\text{ext}}$.

It was shown in [4] that the leading and subleading contributions to the free energy are given by
\begin{equation}
F_0 = \beta \ln r,
F_1 = \frac{\beta}{2\pi} \int_{\Gamma} |w'_{\text{ext}}| W \, ds - (\beta - 1) \ln \frac{er}{2\pi\beta} + \ln \frac{2\pi}{\Gamma(\beta)}. \tag{25}
\end{equation}

The result for $F_2$ is most interesting and nontrivial. The contribution $F_2$ can be naturally divided into two parts, “classical” and “quantum”: $F_2 = F_2^{(\text{cl})} + F_2^{(\text{q})}$. The “classical” part has the electrostatic nature while the “quantum” part is due to the fluctuations of particles. The result for the “classical” part is expressed through the Neumann jump operator $\hat{N}$ associated with the contour. This operator takes a function $f$ on $\Gamma$ to the jump of the normal derivative of its harmonic continuations $f_H$ to $\mathbb{D}_{\text{int}}$ and $f^H$ to $\mathbb{D}_{\text{ext}}$:
\begin{equation}
\hat{N} f = \partial_n f_H - \partial_n f^H,
\end{equation}
where the normal vector is directed to the exterior of $\Gamma$ in both cases. The result for $F_2^{(\text{cl})}$ is
\begin{equation}
F_2^{(\text{cl})} = \frac{\beta}{8\pi} \int_{\Gamma} ((1 - \beta^{-1})\psi_{\text{ext}} + W)\hat{N}((1 - \beta^{-1})\psi_{\text{ext}} + W) \, ds. \tag{26}
\end{equation}
The result for $F_2^{(\text{q})}$ is
\begin{equation}
F_2^{(\text{q})} = \frac{1}{24\pi} \int_{\Gamma} (\psi_{\text{int}} \partial_n \psi_{\text{int}} - \psi_{\text{ext}} \partial_n \psi_{\text{ext}}) \, ds + \frac{1}{6} (\psi_{\text{ext}}(\infty) - \psi_{\text{int}}(0)) + \ln \sqrt{\beta}. \tag{27}
\end{equation}
Remarkably, the quantity in the right-hand side of (27) is $1/24$ times the Loewner energy $I^L(\Gamma)$ of $\Gamma$,
\begin{equation}
F_2^{(\text{q})} = \frac{1}{24} I^L(\Gamma) + \ln \sqrt{\beta}, \tag{28}
\end{equation}
where
\begin{equation}
I^L(\Gamma) = \frac{1}{\pi} \int_{\mathbb{D}_{\text{int}}} |\nabla \psi_{\text{int}}|^2 \, d^2 z + \frac{1}{\pi} \int_{\mathbb{D}_{\text{ext}}} |\nabla \psi_{\text{ext}}|^2 \, d^2 z + 4(\psi_{\text{ext}}(\infty) - \psi_{\text{int}}(0)) \tag{29}
\end{equation}
(see [13]). The notion of Loewner energy was recently actively discussed in the literature in the context of the Schramm–Loewner evolution (see, e.g., survey [14] and the references therein). The quantity in (29) is also known as the universal Liouville action [15].

5. Concluding remarks

We have defined the diffusion process for $N$ interacting particles on a smooth closed contour $\Gamma$ in the plane. Each particle is subject to the action of a random force represented by the Brownian process $B(t)$ such that $\langle dB(t)^2 \rangle = \kappa \, dt$, where the coefficient $\kappa$ characterizes the strength of the random force. We have derived the Fokker–Planck equation for the corresponding probability density. The stationary solution of the Fokker–Planck equation was shown to be given by the Boltzmann weight for the $N$ particles confined on the contour and interacting via the 2D Coulomb (logarithmic) potential. The temperature of this logarithmic gas is $\kappa/2$. This model was studied in [4], where the nonvanishing contributions to the free energy as $N \to \infty$ were found. The leading term is of the order $O(N^2)$ and it has the electrostatic nature. The most interesting quantity is the “quantum” part (i.e., the part that is entirely due to fluctuations) of the $O(1)$-contribution to the free energy. Although the result in (27) was already obtained in [4], it was not observed there that it is actually $1/24$ times the Loewner energy $I^L(\Gamma)$ of the contour $\Gamma$. We make this explicit in this paper.
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