On the slowdown of random walk in random environment with bounded jumps

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Abstract

In this paper we prove that under certain assumptions the transient random walk in random environment with bounded jumps (in \( \mathbb{Z} \)) grows much slower than the speed \( \frac{1}{n} \). Precisely, there is \( 0 < s < 1 \), such that although \( X_n \to \infty \) we have \( \frac{X_n}{n^s} \to 0 \) for \( 0 < s < s' \) almost surely.

Keywords: random walk in random environment, slowdown, trap.

AMS Subject Classification: Primary 60F15; secondary 60F10.

1. Introduction

Slowdown property is one of the most important features for the random walk in random environment (RWRE in short) in \( \mathbb{Z} \). More precisely, although \( X_n \to \infty \) we have that \( \frac{X_n}{n} \to 0 \) almost surely ([4]). However, this phenomena is impossible for random walk in non-random environment, since the law of large numbers implies that the walk grows with a positive speed as long as it is transient. Intuitively, because of the random environment, there are “many” environments formulated “traps” in which the random walk spend “much” time. For the nearest RWRE (i.e., the walk which goes to right and left for only one unit in one step), even a much slower speed has been revealed, i.e., under certain assumptions there is \( 0 < s < 1 \), such that although \( X_n \to \infty \) we have \( \frac{X_n}{n^s} \to 0 \) for \( 0 < s < s' \) almost surely ([3]). In this paper, we will prove this property for the random walk in random environment with bounded jumps. One should note that the RWRE with bounded jumps makes the situation more complicated than the nearest RWRE ([2]).

Let us recall the RWRE with bounded jumps firstly. We will adopt the notations in [2]. \( \Lambda = \{-L, \ldots, 1\} \), \( \Sigma \) is the simplex in \( \mathbb{R}^{L+2} \), and \( \Omega := \Sigma^\mathbb{Z} \). Let \( \mu \) be a measure on \( \Sigma \) and \( \omega_0 = (\omega_0(z))_{z \in \Lambda} \) be a \( \Sigma \)-valued random vector with distribution \( \mu \), satisfying \( \sum_{z \in \Lambda} \omega_0(z) = 1 \), and \( \mu(\omega_0(z)/\omega_0(1) > \kappa, z \in \Lambda, z \neq 0) = 1 \) for some \( \kappa > 0 \). Let \( \mathbb{P} = \mu^\otimes \mathbb{Z} \) on \( \Omega \) making \( \omega_x, x \in \mathbb{Z} \) i.i.d.. The random

*The project is partially supported by National Nature Science Foundation of China(Grant No.11226199).
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walk in random environment $\omega$ with bounded jumps is the Markov chain defined by $X_0 = x$ and the transition probabilities
\[ P_{x,\omega}(X_{n+1} = y + z|X_n = y) = \omega_y(z), \forall y, z \in \Lambda. \] (1)

In the sequel we refer to $P_{x,\omega}(\cdot)$ as the “quenched” law. One also defines the “annealed” law on $\Omega \times \mathbb{Z}^N$ by:
\[ P_x(\cdot) = \int P_{x,\omega}(\cdot) \mathbb{P}(d\omega) \quad \text{for } x \in \mathbb{Z}. \] (2)

In the rest of the paper, we use $E$ corresponding to $\mathbb{P}$, $E_{x,\omega}$ corresponding to $P_{x,\omega}$ and $E_x$ corresponding to $P_x$ to denote the expectation respectively. Define the shift $T$ on $\Omega$ by relation $(T\omega)_i = \omega_{i+1}$. Let
\[ a_i = \frac{\omega_0(-i) + \cdots + \omega_0(-L)}{\omega_0(1)}, 1 \leq i \leq L, \]
\[ A := A(0) = \begin{pmatrix} a_1 & \cdots & a_{L-1} & a_L \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}. \]

For $k \geq l$ set $A(k, l) = A(k) \cdots A(l)$, $A(k) := T^k A$, and for $l \in \mathbb{Z}$ set
\[ \delta(l, l + 1) = 1 \quad \text{and} \quad \forall k \geq l, \delta(k, l) = \langle e_1, A(k) \cdots A(l)e_1 \rangle. \]

Note that all $\delta(k, l)$ defined above are strictly positive.

Define the norm of matrix $A$ by
\[ \| A \| = < e_1, A e_1 >. \]

We have $\delta(k, l) = \| A_k \cdots A_l \|$. It is easy to verify that for $n \geq L$, $A_n A_{n-1} \cdots A_0 \gg 0$, where for a matrix $A$, $A \gg 0$ means that all entries of $A$ are strictly positive. Then one follows from Frobenious theory of positive matrices that there exists a number $\lambda_0$ such that $Ax = \lambda_0 x$ for some $x \in \mathbb{R}^L$ and $|\lambda| < \lambda_0$ for all other eigenvalues of $A$. Consequently, $A$ is contracting. Next, suppose $V$ is linear subspace of $\mathbb{R}^L$ with dimension $1 \leq d \leq L$. Then for any $v \in V$, $Av \in \mathbb{R}^{d+1}$. Therefore the set $\{ A(\omega) : \omega \in \text{supp}\mu \}$ is strongly irreducible.

For the definition of contracting set and strongly irreducible set, see [1]. Let
\[ l(A) = \sup \{ \log^+ \| A \|, \log - \| A^{-1} \| \}. \]

By the elliptic condition of $\mu$, we have that for all $u \in \mathbb{R}$,
\[ E(e^{ul(A)}) < \infty. \]
Then we have the following facts which could be found in [1].

**Facts:**

1. The limit
   \[
   \gamma_L := \lim_{n \to \infty} \frac{1}{n} \mathbb{E}(\log \|A_{n-1} \cdots A_0\|)
   \]  
   exists;

2. For \( u \in \mathbb{R} \) the limit
   \[
   F(u) := \lim_{n \to \infty} \frac{1}{n} \log \mathbb{E}(\|A_{n-1} \cdots A_0\|^u)
   \]  
   exists and the function \( F(\cdot) \) is analytic;

3. Consequently, \( \{\frac{1}{n} \log \|A_{n-1} \cdots A_0\|\}_{n \geq 1} \) satisfies a large deviation principle. Precisely, for any \( \epsilon > 0 \),
   \[
   \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\log \|A_{n-1} \cdots A_0\| > \epsilon) = -I(\epsilon),
   \]
   \[
   \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\log \|A_{n-1} \cdots A_0\| < -\epsilon) = -I(-\epsilon),
   \]
   where the rate function
   \[
   I(x) = \sup_{u \in \mathbb{R}} \{ux - F(u)\}.
   \]

The number \( \gamma_L \) is called the greatest Liapounov exponent of \( A \). It serves as a criteria for RWRE with bounded jumps. The following results can be found in Brémont [2], (see page 1271, lemma 4 in page 1272, theorem 2.4 in page 1275, theorem 3.5 in page 1284 respectively).

**Theorem A.** (Brémont. [2]) For the RWRE with bounded jumps \( X_n \), we have

1. There exists a unique unit random vector \( V \) with strictly positive components and a unique random variable \( \lambda \) such that \( AV = \lambda TV \).

2. \( \gamma_L = \mathbb{E}(\log \lambda) \).

3. There exists a constant \( C > 0 \) such that for \( k > l \)
   \[
   \frac{1}{(C)(T^k \lambda \cdots T^l \lambda)} \leq \delta(k,l) \leq C(T^k \lambda \cdots T^l \lambda). \]

4. If \( \gamma_L < 0 \), then \( X_n \to \infty \) \( P_0 \)-a.s.. If \( \gamma_L > 0 \), then \( X_n \to -\infty \) \( P_0 \)-a.s.. If \( \gamma_L = 0 \), the walk is recurrent almost surely.

5. If \( \mathbb{E}(\sum_{n=1}^{\infty} T^{n-1} \lambda \cdots \lambda) < \infty \), then \( \frac{X_n}{n} \to c > 0 \) \( P_0 \)-a.s.. If \( \mathbb{E}(\sum_{n=1}^{\infty} (T^{n-1} \lambda \cdots \lambda)^{-1}) < \infty \), then \( \frac{X_n}{n} \to c < 0 \) \( P_0 \)-a.s.. If \( \mathbb{E}(\sum_{n=1}^{\infty} T^{n-1} \lambda \cdots \lambda) = \infty \) and \( \mathbb{E}(\sum_{n=1}^{\infty} (T^{n-1} \lambda \cdots \lambda)^{-1}) < \infty \), then \( \frac{X_n}{n} \to c = 0 \) \( P_0 \)-a.s.. \( \square \)

We have from (4) that
\[
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{E}((T^{n-1} \lambda \cdots \lambda)^u) = \lim_{n \to \infty} \frac{1}{n} \log \mathbb{E}(\|A_{n-1} \cdots A_0\|^u) = F(u).
\]

Consequently \( \mathbb{E}(\sum_{n=1}^{\infty} T^{n-1} \lambda \cdots \lambda) < \infty \) if and only if \( F(1) < 0 \), while \( \mathbb{E}(\sum_{n=1}^{\infty} (T^{n-1} \lambda \cdots \lambda)^{-1}) < \infty \) if and only if \( F(-1) < 0 \).
In this point of view, we have that

\[ F(1) < 0 \Rightarrow \frac{X}{n} \to c > 0; \]
\[ F(-1) < 0 \Rightarrow \frac{X}{n} \to c < 0; \]
\[ F(1) \geq 0, F(-1) \geq 0 \Rightarrow \frac{X}{n} \to c = 0. \]

**Remark 1.1**

1. By the convexity of function \( x^{-1} \), we have \( F(-1) \geq -F(1) \). As a consequence, there is one and only one case in 5 of Theorem A happens.

2. Note that the function \( \log x \) is concave, we have \( \gamma_L \leq F(1) \). If \( \gamma_L < 0 \) (by 4 of theorem A, \( X_n \to \infty \), a.s.) it is possible that \( F(1) > 0 \). Then by 5 of theorem A we have \( \frac{X_n}{n} \to 0 \) a.s. In these two situations, the slowdown properties occur and we show in the following main theorem that \( X_n \) grows with only sub-linear speed.

**2. Main result and proofs**

**Theorem 2.1** For the RWRE with bounded jumps \( X_n \), one of the following conditions holds

(1) \( \gamma_L := \lim_{n \to \infty} \frac{1}{n} \mathbb{E}(\log \| A_{n-1} \cdots A_0 \|) < 0 \) and \( F(1) := \lim_{n \to \infty} \frac{1}{n} \log \mathbb{E}(\| A_{n-1} \cdots A_0 \|) > 0; \)

(2) \( \gamma_L := \lim_{n \to \infty} \frac{1}{n} \mathbb{E}(\log \| A_{n-1} \cdots A_0 \|) > 0 \) and \( F(-1) := \lim_{n \to \infty} \frac{1}{n} \log \mathbb{E}(\| A_{n-1} \cdots A_0 \|^{-1}) > 0. \)

Then there exists \( s \in (0, 1) \) such that \( \frac{X_n}{n^{s'}} \to 0, P_0\text{-a.s., for all } s' > s. \)

**Remark 2.1**

1. By the discussion in Remark 1.1, \( X_n \to \infty \) but \( \frac{X_n}{n} \to 0, P_0\text{-a.s., while case (1) of the theorem happens; and similarly } X_n \to -\infty \) but \( \frac{X_n}{n} \to 0, P_0\text{-a.s., while case (2) of the theorem happens.} \)

2. For a fixed environment \( \omega \), we call \([-K \log n, K \log n]\) a “trap” for the walk if the walk with positive (quenched) probability spends more than \( n \) steps in \([-K \log n, K \log n]\), i.e., \( P_0,\omega [T_{[-K \log n, K \log n]} > n] \geq \varepsilon > 0; \) and we say the fixed environment \( \omega \) formulated a trap for the walk, where \( T_{[-K \log n, K \log n]} \) is the first exit time of the walk from \([-K \log n, K \log n].\)

3. The key step in the proof of the Theorem is to show that there are “many” environments formulated traps for the walk in the sense of the following (16) under the conditions of the
Theorem. For this purpose we need some estimations for the (quenched) exit probabilities and the environment factors.

Proof of Theorem 2.1:
Consider integers \((a, b, k)\) with \(a < b\) and define

\[ P_{k,\omega}\{a, b, -\} := P_{k,\omega}\{\text{the walk reaches } (-\infty, a] \text{ before } [b, \infty)\} \]

and similarly

\[ P_{k,\omega}\{a, b, +\} := P_{k,\omega}\{\text{the walk reaches } [b, \infty) \text{ before } (-\infty, a]\}. \]

We have the following lemma which was proved in Brémont [2].

Lemma 2.1 If \(a < k < b\), then

\[ P_{k,\omega}\{a, b, -\} = \sum_{j=a}^{b-1} \delta(j, a + 1) = \sum_{j=a}^{b-1} \delta(j, a + 1). \]

With this Lemma in hands, defining for \(z \in \mathbb{Z}\)

\[ H_1^z := \min\{n > 0 : X_n \leq z\}, \quad H_r^z := \min\{n > 0 : X_n \geq z\}, \]

for \(M > 1, N > L, 1 \leq k \leq L\), we have

\[ P_{1,\omega}\{H_0^k < H_{M+1}^k\} = P_{1,\omega}\{0, M + 1, -\} = \sum_{j=0}^{M} \frac{\delta(j, 1)}{\sum_{j=0}^{M} \delta(j, 1)} = 1 - \frac{\delta(0, 1)}{\sum_{j=0}^{M} \delta(j, 1)}, \]

and

\[ P_{-k,\omega}\{H_0^k < H_{-(N+1)}^k\} = P_{-k,\omega}\{-(N + 1), 0, +\} = 1 - P_{-k,\omega}\{-(N + 1), 0, -\} \]

\[ = 1 - \frac{\sum_{j=-k}^{-1} \delta(j, -N)}{\sum_{j=-N}^{-1} \delta(j, -N)} = 1 - \frac{\sum_{j=-k}^{-1} \delta(j, -N)}{1 + \sum_{j=-N}^{-1} \delta(j, -N)} \]

\[ \geq 1 - \sum_{j=-k}^{-1} \delta(j, -N) \geq 1 - \sum_{j=-L}^{-1} \delta(j, -N) \]

\[ = 1 - e^{\log \sum_{j=-L}^{-1} \delta(j, -N)}. \]

Let \(R_M = \frac{1}{M} \log \delta(M, 1)\) and let \(R_N = \frac{1}{N} \log \sum_{j=-L}^{-1} \delta(j, -N)\). Then we have

\[ P_{1,\omega}\{H_0^k < H_{M+1}^k\} \geq (1 - e^{-MR_M})^+, \]

and

\[ P_{-k,\omega}\{H_0^k < H_{-(N+1)}^k\} \geq (1 - e^{NR_N})^+. \]
From (3) one follows that \(\mathbb{P}\)-a.s.,

\[
\lim_{M \to \infty} R_M = \lim_{N \to \infty} R_n = \gamma_L.
\]

**Case 1.** Suppose that \(\gamma_L < 0\) but \(F(1) > 0\). Not that \(F\) is a strictly convex function satisfying \(F(0) = 0, F(1) > 0\) and \(F'(0) < 0\). Therefore, there exists a unique \(s \in (0, 1)\) such that \(F(s) = 0\).
We fix such \(s\) in the remainder of the proof.

We now set, for \(U = [-N, M]\), \(\gamma(U) = 1 \wedge \max[e^{NR^-}, e^{-MR^+}]\), where \(R^- = R_N\) and \(R^+ = R_M\). Define \(T_U := \inf\{k, X_k \in U^c\}\). Note that \(T_U\) is the exit time of the walk from the set \(U\). Then we have from the strong Markov property that

\[
P_{0,\omega}(T_U > n) \geq P_{0,\omega}[\#\{1 < k \leq T_U, X_{k-1}X_k \leq 0\} > n] \geq (1 - \gamma(U))^n. \quad (8)
\]

The first inequality of the last expression follows immediately. For the second one, we define

\[
\tilde{H}_0 := \inf\{k > 1, X_kX_{k-1} \leq 0\}.
\]

Note that \(\tilde{H}_0\) can be explained as the first time the walk crosses 0 after time 1. By decomposing the event \(\{\tilde{H}_0 < T_U\}\) according to the value of \(X_1\), we have

\[
P_{0,\omega}(\tilde{H}_0 < T_U) = P_{0,\omega}(H_0^1 < H_{M+1}^l) P_{0,\omega}(X_1 = 1) + \sum_{j=1}^L P_{-j,\omega}(H_0^j < H_{-(N+1)}^l) P_{0,\omega}(X_1 = -j)
\]

by the estimation in (6) and (7)

\[
\geq P_{0,\omega}(X_1 = 1) \left(1 - e^{-MR^+}\right) + \sum_{j=1}^L P_{0,\omega}(X_1 = -j) \left(1 - e^{NR^-}\right)
\]

\[
\geq P_{0,\omega}(X_1 = 1) (1 - \gamma(U)) + \sum_{j=1}^L P_{0,\omega}(X_1 = -j) (1 - \gamma(U)) = 1 - \gamma(U).
\]

Then (8) follows. The remainder of the proof is similar as Sznitman [3], to make the proof complete we still give the details here. In particular, if \(\gamma_L(U) \leq \frac{1}{n}\), we have

\[
P_{0,\omega}(T_U > n) \geq (1 - \gamma(U))^n \geq \left(1 - \frac{1}{n}\right)^n \to e^{-1}.
\]

Hence for \(n\) large enough, \(P_{0,\omega}(T_U > n) \geq \epsilon = e^{-2} > 0\). Note that for \(N \geq \frac{2}{|\gamma_L|} \log n, M, \epsilon\) with \(M \epsilon \geq \log n\), by independence of \(R^+\) and \(R^-\) under \(\mathbb{P}\),

\[
\mathbb{P}\left(\gamma(U) \leq \frac{1}{n}\right) \geq \mathbb{P}\left(R^- \leq \frac{\gamma_L}{2}, R^+ \geq \epsilon\right) = \mathbb{P}\left(R^- \leq \frac{\gamma_L}{2}\right) \mathbb{P}\left(R^+ \geq \epsilon\right). \quad (9)
\]

Then for \(n\) large enough and \(\eta > 0\) small, by the large deviations, we have that

\[
\mathbb{P}\left(\gamma(U) \leq \frac{1}{n}\right) \geq \frac{1}{2 \mathbb{P}\left(R^+ \geq \epsilon\right)} \mathbb{P}\left(R^+ \geq \epsilon\right) \geq \frac{1}{2} \exp\{-I(\epsilon)M(1 + \eta)\}.
\]
Now we optimize $\epsilon, M$ by looking at

$$\inf[I(\epsilon)M, M\epsilon \geq \log n, I(\epsilon) = \inf_{\epsilon > 0} \left[ \frac{I(\epsilon)}{\epsilon} \log n \right],$$

and recall that $F(u) = \sup_{x} [e^{u} - I(x)]$. Let $\alpha := \inf_{\epsilon > 0} \left[ \frac{I(\epsilon)}{\epsilon} \right]$. By a duality argument (see [?] lemma 4.5.8), we see that $F(\alpha) = 0$. In the other words $\inf_{\epsilon > 0} \left[ \frac{I(\epsilon)}{\epsilon} \right] = s$, recalling that $s \in (0, 1)$ is the unique positive zero of the function $F(\cdot)$. Therefore, choosing $K > 0, \eta > 0$ properly, for large $n$, from the discussion above we have

$$\mathbb{P} \left( P_{0, \omega} \left[ T_{[K \log n, K \log n]} > n \right] \geq e^{-2} \right) \geq n^{s(1+\eta)}. \quad (10)$$

With (10) we have created a trap of size $2K \log n$ which retains the walk for $n$ units of time with large probability. If $s' > s$, choosing $\eta$ small in (10) there will be many such traps in $[0, n^{s'}]$ which will prevent the walk from moving to distance $n^{s'}$ from the origin before time $n$. Precisely, for large $n$, with $M$ the number of traps in $[0, n^{s'}]$, which is of order $\frac{n^{s'}}{\log n}$, and with $T_{i}$ the time to exit the $i$-th trap after reaching its center, for $\lambda > 0$, we have

$$P_{0} \left( X_{n} > n^{s'} \right) \leq P_{0} \left( T_{1} + \cdots + T_{M} < n, X_{n} \text{ reaches the center of the } i \text{-th trap, } i = 1, \ldots, M \right)$$

$$\leq e^{\lambda n} \mathbb{E} \left( e^{-\lambda (T_{1} + \cdots + T_{M})}, X_{n} \text{ reaches the center of the } i \text{-th trap, } i = 1, \ldots, M \right)$$

(using Markov property under $P_{0, \omega}$, the independence under $P$ of the environments in different traps and the stationarity)

$$= e^{\lambda n} \mathbb{E} \left( e^{-\lambda T_{[K \log n, K \log n]}} \right)^{M}$$

$$\leq e^{\lambda n} \left( 1 - \frac{e^{-2}}{n^{s(1+\eta)}} + \frac{e^{-2\lambda n}}{n^{s(1+\eta)}} \right)^{M} \text{, using the fact } 1 - x \leq e^{-x}$$

$$\leq e^{\lambda n - M} \frac{e^{-2}}{n^{(s+(1+\eta))}} \left( 1 - e^{-\lambda n} \right).$$

Since $M \sim \text{const} \frac{n^{s'}}{\log n}$, if we now choose $\eta$ small, $\lambda = \frac{1}{2} n^{s' - s(1+2\eta) - 1}$, for $n$ large, we have

$$P_{0} \left( X_{n} > n^{s'} \right) \leq e^{\frac{1}{2} n^{s' - s(1+2\eta) - 1}} = e^{-\frac{1}{2} n^{s' - s(1+2\eta)}}.$$

Then it follows from Borel-Cantelli lemma that,

$$P_{0}\text{-a.s.}, \lim_{n \to \infty} \frac{X_{n}}{n^{s'}} = 0, \text{ for all } s' > s.$$

**Case 2:** By assumption $\gamma > 0, F(-1) > 0$. Then $F$ is a strictly convex function satisfying $F(0) = 0, F(-1) > 0$ and $F'(0) > 0$. Therefore, there exists a unique $s \in (-1, 0)$ such that $F(s) = \log \mathbb{E} (\lambda^{s}) = 0$, that is, $\mathbb{E} (\lambda^{s}) = 1$. Fix such $s$. The proof moves on as that of Case 1. Using large deviation and changing he role of $R^{+}$ and $R^{-}$ in (9), we can get an estimation as (10), i.e.,

$$\mathbb{P} \left( P_{0, \omega} \left[ T_{[K \log n, K \log n]} > n \right] \geq e^{-2} \right) \geq n^{s(1+\eta)}, \quad (11)$$

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for $n$ large and $\eta > 0$ small. Recall that $s \in (-1, 0)$ in this case. Using a similar argument of Case 1 below (10), we can get

$$P_0 \text{-a.s., } \lim_{n \to \infty} \frac{X_n}{n^{-s'}} = 0, \text{ for all } s' < s,$$

which completes the proof. □

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