We study color entanglement effect for T-odd cases in collinear twist-3 factorization. As an example, we compute the transverse single spin asymmetry for direct photon production in pp collisions in pure collinear twist-3 approach. By analyzing the gauge link structure of the collinear gluon distribution on unpolarized target side, we demonstrate how color entanglement effect arises in the presence of the additional gluon attachment from polarized projectile. The result is consistent with that obtained from a hybrid approach calculation.

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I. INTRODUCTION

One of the key steps involved in proving QCD factorization theorems is to decouple longitudinal gluon exchange between active partons and the remanents of projectile/target nucleons in a high energy scattering process. By invoking the Ward identity argument, longitudinal gluon exchange to all orders can be absorbed into gauge link that ensures gauge invariance of the operator definitions of parton distribution functions. In the context of transverse momentum dependent(TMD) factorization framework [1], depending on color flow in a hard scattering, TMD parton distributions could possess quite complicated gauge link structure [2–5]. A generalized TMD factorization was proposed [3, 4] to express cross sections in terms of process dependent TMDs when computing physical observables that are sensitive to incoming parton transverse momenta.

However, a further investigation [6] reveals that it is not possible to disentangle simultaneous longitudinal gluon attachments from both nucleon sides in a nucleon-nucleon collisions provided that color flow is nontrivial in final state. This phenomena, commonly refereed to as color entanglement, originates from the non-Abelian feature of QCD as a gauge theory. It prevents us from describing parton transverse momentum with separate correlation functions for each external nucleon, and thus leads to the breakdown of generalized TMD factorization. The phenomenological implications of color entanglement effect have been explored from both theoretical and experimental sides [7–10].

On the other hand, it is not yet entirely clear whether color entanglement effect shows up in nucleon-nucleon collisions in collinear factorization calculation. Though it is usually believed to be absent in leading twist collinear factorization, at twist-3 level, one has to take into account an additional gluon re-scattering which makes color flow more complicate, and could potentially give rise to color entanglement effect. In fact, the transverse single spin asymmetry(SA) for prompt photon production in pp collisions computed in genuine collinear twist-3 factorization [11, 12] differs from that obtained in the hybrid approach [13] by terms proportional to a new gluon distribution $G_4$, whose emergence can be attributed to color entanglement effect. To some extend, the hybrid approach is a more complete method in the sense that gluon re-scattering on the unpolarized target side has been summed to all orders. One thus has good reason to believe that the color entanglement effect related to the $G_4$ term contribution are missing in the conventional collinear twist-3 calculation.

The objective of this paper is to explicitly work out color structure for the diagrams with simultaneous longitudinal gluon attachments from both incoming nucleon sides within the pure collinear twist-3 factorization framework. To the best of our knowledge, the gauge link structure of leading twist collinear parton distributions on unpolarized target side has never been carefully examined in the presence of additional gluon attachment from polarized projectile. To identify color entanglement effect and compare with the hybrid approach, it is sufficient to take into account one longitudinal gluon attachment from unpolarized target on each side of cut, since this is the lowest nontrivial order where $G_4$ receives nonvanishing contribution. As discussed in Sec. III, the gluon distribution $G_4$ indeed enters into the spin dependent cross section in pure collinear twist-3 factorization when going beyond one gluon exchange approximation. As expected, we verified that the hybrid approach and the collinear twist-3 approach yield the same result in the collinear limit at the order under consideration.

The paper is structured as follows. In the next section, we briefly review the conventional collinear twist-3 calculations for the SSA in direct photon production and the derivation of gauge link of the collinear gluon distribution in leading twist collinear factorization. In sec. III, we present our analysis for the gauge link structure at twist-3 level in details, and recover the hybrid approach result. In the end, we comment on more general cases and discuss possible extensions of the current work in sec. IV.
II. BRIEF REVIEW OF CONVENTIONAL CALCULATIONS

The dominant production mechanism for prompt photons in high energy collisions is Compton scattering $gq \rightarrow \gamma q$ as shown in Fig.1. We start by introducing the relevant kinematical variables and assign 4-momenta to the particles according to

$$g(x'\bar{P}) + q(xP) \longrightarrow \gamma(l_\gamma) + q(l_q)$$

where $\bar{P}^\mu = P^\mu - n^\mu$ and $P^\mu = P^+ p^\mu$ with $n^\mu$ and $p^\mu$ being the commonly defined light cone vectors, normalized according to $p \cdot n = 1$. The corresponding unpolarized Born cross section reads,

$$\frac{d^3\sigma}{d^2l_{\gamma\perp} dz} = \frac{\alpha_s\alpha_{em}}{N_c} \frac{z^2}{l_{\gamma\perp}^2} \sum_q c_q^2 \int_{x_{min}}^1 dx f_q(x) x G(x')$$

where $z \equiv l_{\gamma} \cdot n/(xP \cdot n)$ is the fraction of the incoming quark momentum $xP$ carried by the outgoing photon, and $l_{\gamma\perp}$ is the photon transverse momentum. The meaning of the other coefficients should be self-evident. Note that $x' = \frac{xP - l_q}{xP}$ is a function of $x$, and $x_{min}$ is given by $x_{min} = \frac{P l_{\gamma}}{xP - P l_q}$. In the above formula, $f_q(x)$ and $G(x')$ are the usual integrated quark and gluon distributions, respectively.

The operator definition of the collinear gluon distribution is given by [1]

$$x'G(x') = \int \frac{d\xi^+}{2\pi P} e^{-ix'\vec{P}\cdot\vec{\xi}^+} \langle P|F_{a^-}^\mu(\xi^+)\tilde{L}_{ac}F_{c^-}(0)|P\rangle$$

where $F_{a^-}^\mu$ is the gauge field strength tensor and $\tilde{L}_{ac} = \mathbb{P} \exp\{-g \int_0^{\xi^+} dz^+ f^{bac} A_b^-(z)\}$ is the gauge link in the adjoint representation. This gluon distribution can also be defined in the fundamental representation [14],

$$x'G(x') = 2\int \frac{d\xi^+}{2\pi P} e^{-ix'\vec{P}\cdot\vec{\xi}^+} \langle P|\text{Tr} \left[ F_{a^-}^\mu(\xi^+)T^a \mathcal{L} F_{c^-}(0)T^c \mathcal{L}^{-1} \right] |P\rangle,$$

where the gauge link takes form $\mathcal{L} = \mathbb{P} \exp\{-ig \int_0^{\xi^+} dz^+ T^b A_b^-(z)\}$. The above gauge link is built up by summing longitudinal gluon ($A^-$) attachment to all orders. As a warm up exercise, we first rederive the gauge link at lowest nontrivial order by computing the diagrams illustrated in Fig.2. The gluon pole and the color structure associated with the initial state interaction in the amplitude is given by,

$$\frac{1}{x'_{\gamma} + i\epsilon} \text{Tr} \left[ T^aqT^cT^b \right]$$

which yields the following contribution to the first order expansion of the gauge link,

$$\langle P|\text{Tr} \left[ F_{a^-}^\mu(\xi^+)T^a F_{c^-}(0)T^c \left( -ig \int_{-\infty}^0 dz^+ T^b A_b^-(z) \right) \right] |P\rangle$$
And similarly, for the final state interaction in the amplitude, one has,
\[
\frac{1}{-x'_{g} - i\epsilon} \text{Tr} \left[ T^{a} T^{b} T^{c} \right] \Rightarrow \langle P | \text{Tr} \left[ F_{a}^{-\mu}(\xi^{+}) T^{a} \left( -ig \int_{0}^{\infty} dz^{+} T^{b} A_{b}^{-}(z) \right) F_{c}^{-\mu}(0) T^{c} \right] | P \rangle
\] (7)

For Fig.2(c), one has,
\[
\frac{1}{-x'_{g} - i\epsilon} \text{Tr} \left[ T^{b} T^{a} T^{c} \right] \Rightarrow \langle P | \text{Tr} \left[ \left( ig \int_{-\infty}^{\xi^{+}} dz^{+} T^{b} A_{b}^{-}(z) \right) F_{a}^{-\mu}(\xi^{+}) T^{a} F_{c}^{-\mu}(0) T^{c} \right] | P \rangle
\] (8)

and Fig.2(d),
\[
\frac{1}{-x'_{g} - i\epsilon} \text{Tr} \left[ T^{a} T^{b} T^{c} \right] \Rightarrow \langle P | \text{Tr} \left[ F_{a}^{-\mu}(\xi^{+}) T^{a} \left( ig \int_{-\infty}^{\xi^{+}} dz^{+} T^{b} A_{b}^{-}(z) \right) F_{c}^{-\mu}(0) T^{c} \right] | P \rangle
\] (9)

Summing up all terms gives rise to the first nontrivial order expansion of the gauge link. It is easy to generalize the above derivation and absorb longitudinal gluon exchange into the gauge link to all orders. Meanwhile, the gauge link of the integrated quark distribution from the projectile nucleon is build up by summing longitudinal gluon exchange in order to generate an imaginary phase necessary for the nonvanishing SSA. One doesn’t expect that these two procedures (summing A⁻ and A⁺ gluon exchanges) interfere with each other at leading twist level.

We now turn to review the conventional twist-3 calculation for the SSA in prompt photon production process [11, 12]. In a covariant gauge calculation, as shown in Fig.3 an additional A⁺ gluon which carries small transverse momentum \( p_{\perp} \) must be exchanged in order to generate an imaginary phase necessary for the nonvanishing SSA. One can isolate imaginary part by picking up gluon poles generated via the initial state interactions or the final state interactions as shown in Fig.3. However, as well known, there is a complete cancelation between the contributions from the different cut diagrams with an longitudinal gluon attaching to the unobserved final state produced particle. This can be best seen by explicitly writing down the gluon pole contribution and the on shell condition from Fig.3(c),
\[
\frac{1}{(l_{q} - x g P - k_{\perp})^{2} + i\epsilon} |_{\text{pole}} \delta(l_{q}^{2}) = -i\pi \delta \left( (l_{q} - x g P - k_{\perp})^{2} \right) \delta(l_{q}^{2})
\] (10)

and from the left cut diagram Fig.3(d),
\[
\frac{1}{l_{q}^{2} - i\epsilon} |_{\text{pole}} \delta \left( (l_{q} - x g P - k_{\perp})^{2} \right) = +i\pi \delta(l_{q}^{2}) \delta \left( (l_{q} - x g P - k_{\perp})^{2} \right)
\] (11)

where \( l_{q} = x P + x' \bar{P} + x g P + p_{\perp} - l_{\gamma} \). Obviously, they cancel out each other as the rest part of the amplitude squared represented by Fig.3(c) and Fig.3(d) are exactly same. The absence of the final state interaction contributions to the SSA in the current case is actually the key observation which leads us to conclude that color entanglement effect also shows up in collinear twist-3 factorization. We will explain the reasoning in details in the next section.

To isolate the twist-3 effect, one proceeds by expanding the amplitudes squared and the on shell condition in terms of \( p_{\perp} \),
\[
\mathcal{H}(p_{\perp}) \delta(l_{q}^{2}) = \mathcal{H}(p_{\perp}) \delta(l_{q}^{2}) |_{p_{\perp}=0} + \frac{\partial \mathcal{H}(p_{\perp}) \delta(l_{q}^{2})}{\partial p_{\perp}^{2}} |_{p_{\perp}=0} p_{\perp}^{2} + ...
\] (12)

The twist-3 spin dependent part is the term linear in \( p_{\perp} \). In this work, we only take into account the derivative term contribution, for which case our analysis can be greatly simplified without losing generality. For the derivative term contribution, one can
simply neglect $p_\perp$ in the hard part $\mathcal{H}$,

$$ \frac{-p_\perp^0}{q_\perp} \frac{d^4 F}{d^4 \mathcal{H}}(p_\perp = 0) \left[ \frac{\partial \delta(t_f^2)}{\partial x} \right]_{p_\perp = 0} p_{\perp, \rho} $$

(13)

One then can carry out integration over $p_\perp$ after converting $A^+$ into the gauge invariant form $F^{\rho+}$ by partial integration. The corresponding three parton correlation function can be cast into the form of the ETQS function defined as [15, 16],

$$ T_{F,q}(x_1, x_2) = \int \frac{dy^-_1 dy^-_2}{4\pi} e^{ix_1 P^+ y^-_1 + i(x_2 - x_1) P^+ y^-_2} \langle P, S_\perp \rangle \bar{\psi}_q(0) \gamma^+ g_{\rho np} F_{\rho}^+(y^+_2) \psi_q(y^-_1) \langle P, S_\perp \rangle $$

(14)

where we have suppressed gauge links. $S_\perp$ denotes the proton transverse spin vector. Note that our definition of the ETQS functions differs by a factor $g$ from the convention used in Ref. [12].

Making use of the ingredients described above, the calculation is straightforward. The derivative term contribution to the spin dependent cross section is given in [11, 12],

$$ \frac{d^3 \sigma}{d^2 \mathbf{q}_\perp dz} = \frac{\alpha_s \alpha_em N_c}{N_c^2 - 1} \left( \frac{z^2 - z}{1 + (z - 1)^2} \right) \frac{1}{L_{T,\perp}} \sum_q e_q^2 \int_{x_{min}}^1 dx \, x' G(x') \left[ -x \frac{d}{dx} T_{F,q}(x, x) \right] $$

(15)

The same observable has also been computed in a hybrid approach [13]. It has been found that two approaches don’t produce the same result. To be more explicit, $x' G(x')$ in the above formula is replaced with the combination $x' G(x') - x' G_4(x')$ in the hybrid approach calculation in the collinear limit. Numerically, the impact of $G_4$ on the SSA for photon production is limited as $G_4$ is $1/N_c^2$ suppressed compared to the normal integrated gluon distribution $G$ [13, 17, 18]. However, a recent work [19] shows that the SSA for the forward inclusive jet production in pp/pA collisions is proportional to the combination $x' G(x') - N_c^2 x' G_4(x')$, which drastically deviates from the prediction of the conventional collinear twist-3 calculations. Therefore, from both theoretical and phenomenological point of view, it is important to pin down the source of the inconsistency between two approaches. We address this issue in the next section.

![FIG. 3: Soft gluon pole contributions to the single spin asymmetry for direct photon production in pp collisions. Final state interaction contributions (c) and (d) to the spin asymmetry cancel out. Note that $A^+$ gluon carries small transverse momentum in the twist-3 calculation.](image)

III. GAUGE LINK STRUCTURE IN COLLINEAR TWIST-3 FACTORIZATION

The new gluon distribution $G_4$ appears in the hybrid approach calculation results from color entanglement which arises when considering $A^+$ and $A^-$ gluon exchanges from each side simultaneously. The operator definition of the integrated $G_4$ reads [17],

$$ x' G_4(x') = \frac{2}{N_c} \int \frac{d\xi^+}{2\pi F^-} e^{-ix' F^- - \xi^+} \langle P | Tr \left[ \mathcal{L}_\xi F^- \mu(\xi^+ \rangle \right] Tr \left[ \mathcal{L}_0 F^- \mu(0) \right] | P \rangle $$

(16)

where the gauge link is given by,

$$ \mathcal{L}_\xi = \mathcal{P} \exp \left[ ig \int_{\xi^+}^{+\infty} dz^+ A^-(z^+, 0_\perp) \cdot T \right] \mathcal{P} \exp \left[ ig \int_{0}^{+\infty} dz A_\perp (+\infty^+, z_\perp) \cdot T \right] \mathcal{P} \exp \left[ ig \int_{-\infty}^{\xi^+} dz A^-(z^+, 0_\perp) \cdot T \right] $$

(17)
with transverse gauge links [20, 21] being included. We are now aiming at recovering the above novel color structure within the pure collinear twist-3 formalism. Before making an all order analysis, it is instructive to carry out an explicit calculation at lowest nontrivial order. To this end, one has to compute diagrams with one $A^-$ gluon attachments on each side of cut due to $\text{Tr}[T^a] = 0$. The relevant diagrams are shown in Fig.5.

As we focus on the derivative term contribution, we can neglect transverse momentum $p_{\perp}$ carried by $A^+$ gluon in the hard parts except for $p_{\perp}$ dependence in the on shell condition $\delta(l_2^2)$. The hard parts computed from different diagrams are then proportional to the twist-2 spin independent Born diagram contribution, but with different gluon pole and color structure. Therefore, for the current purpose, it is sufficient to only present the associated gluon pole and color structure for each diagram shown in Fig.5.

To simplify the calculation of diagrams with three gluon vertex, it is convenient to organize $A^+$ and $A^-$ gluon fusion amplitude into two terms,

$$
\frac{-igf^{abc}}{(x_gP + x_g'P)^2 + i\epsilon} \left[ g^{\mu\nu} (x_g'P - x_gP)^{\rho} + g^{\nu\rho} (2x_gP + x_g'P)_{\mu} + g^{\rho\mu} (-2x_g'P - x_gP)^{\nu} \right] n^\mu p^\nu
$$

$$
= \frac{-igf^{abc}}{(x_gP + x_g'P)^2 + i\epsilon} \left[ x_gP + x_g'P \right]_{\rho} - \frac{-igf^{abc} P^\rho}{x_gP \cdot P + i\epsilon}
$$

(18)

where the first term in the second line doesn’t contribute to the final result due to the Ward identity. For example, the contribution from the first term is canceled out when summing up diagrams Fig.5(a), Fig.5(d) and Fig.5(i). One can use the second term as the effective Feynman rule for the $A^+$ and $A^-$ gluon fusion vertex in the following calculation. It is important to notice that the associated gluon pole $\frac{1}{x_gP \cdot P + i\epsilon}$ corresponds to the initial state interaction contribution.

Using the above trick, it is easy to verify that the summation of the hard parts Fig.5(i), Fig.5(j) and Fig.5(k) vanishes,

$$
\mathcal{H}_i + \mathcal{H}_j + \mathcal{H}_k = 0
$$

(19)

And in an exactly analogous fashion, one has,

$$
\mathcal{H}_t + \mathcal{H}_u + \mathcal{H}_v = 0
$$

(20)

Meanwhile, we also notice that,

$$
\mathcal{H}_h = 0, \quad \mathcal{H}_s = 0
$$

(21)

We continue with the evaluation of Fig.5(a) and Fig.5(b),

$$
\mathcal{H}_a \propto \frac{1}{x_g + i\epsilon} \frac{1}{x_g' - x_g'_{1} - i\epsilon} \frac{1}{x'_g + i\epsilon} \text{Tr} \left[ T^a T^b T^c T^d T^e \right] i_f^{def}
$$

(22)

$$
\mathcal{H}_b \propto \frac{1}{x_g + i\epsilon} \frac{1}{x_g' - x_g'_{1} - i\epsilon} \frac{1}{x'_g + i\epsilon} \text{Tr} \left[ T^a T^b T^d T^e \right] i_f^{ce\bar{f}}
$$

(23)

Summing up them, one obtains,

$$
\mathcal{H}_{a+b} \propto \frac{1}{x_g + i\epsilon} \frac{1}{x_g' - x_g'_{1} - i\epsilon} \frac{1}{x'_g + i\epsilon} \left\{ C_F \text{Tr} \left[ T^a T^b T^c T^d T^e \right] - \text{Tr} \left[ T^a T^b T^e T^c T^d \right] \right\}
$$

(24)
which leads to the following gauge link structure,

\[
\mathcal{H}_{a+b} \Rightarrow C_F \langle P | \text{Tr} \left[ F_{a}^{-\mu}(\xi^+) T^a \left( i g \int_{\xi^+}^{\infty} dz^+ T^b A_{\mu}^-(z) \right) F_{c}^{-\mu}(0) T^c \left( -i g \int_{-\infty}^{0} dz^+ T^d A_{\mu}^-(z) \right) \right] | P \rangle - \langle P | \text{Tr} \left[ F_{a}^{-\mu}(\xi^+) T^a \left( i g \int_{\xi^+}^{\infty} dz^+ T^b A_{\mu}^-(z) \right) T^e F_{c}^{-\mu}(0) T^c \left( -i g \int_{-\infty}^{0} dz^+ T^d A_{\mu}^-(z) \right) \right] | P \rangle
\]  

(25)

The gluon poles and color structures associated with Fig.5(c) to Fig.5(g) are listed in the following.

\[
\mathcal{H}_c \propto \frac{1}{x_2 + i \epsilon} \frac{1}{x_3 + i \epsilon} \frac{1}{x_4 + i \epsilon} \text{Tr} \left[ T^a T^b T^d T^e T^f T^g \right] i f^{cde}
\]  

(26)

\[
\mathcal{H}_d \propto \frac{1}{x_2 + i \epsilon} \frac{1}{x_3 + i \epsilon} \frac{2l_q \cdot \bar{P}}{(l_q - x'_g P - x_g) P + i \epsilon} \text{Tr} \left[ T^a T^b T^e T^d T^c T^f \right] i f^{cde}
\]  

(27)

\[
\mathcal{H}_e \propto \frac{-1}{x_2 + i \epsilon} \frac{1}{x_3 + i \epsilon} \frac{1}{x_4 + i \epsilon} \text{Tr} \left[ T^a T^b T^e T^d T^c \right] i f^{cde}
\]  

(28)

\[
\mathcal{H}_f \propto \frac{-1}{x_2 + i \epsilon} \frac{1}{x_3 + i \epsilon} \frac{2l_q \cdot \bar{P}}{(l_q - x'_g P - x_g) P + i \epsilon} \text{Tr} \left[ T^a T^b T^d T^c T^e \right] i f^{cde}
\]  

(29)

\[
\mathcal{H}_g \propto \frac{-1}{x_2 + i \epsilon} \frac{1}{x_3 + i \epsilon} \frac{2l_q \cdot \bar{P}}{(l_q - x'_g P - x_g) P + i \epsilon} \text{Tr} \left[ T^a T^d T^c T^e \right] i f^{cde}
\]  

(30)

We further decompose the hard parts \(\mathcal{H}_d\) and \(\mathcal{H}_f\) into two terms \(\mathcal{H}_d = \mathcal{H}_{d1} + \mathcal{H}_{d2}\) and \(\mathcal{H}_f = \mathcal{H}_{f1} + \mathcal{H}_{f2}\), where \(\mathcal{H}_{d1}, \mathcal{H}_{f1}\) denote contributions by picking up the gluon poles: \(\frac{1}{x_2 + i \epsilon}\) and \(\frac{1}{x_3 + i \epsilon}\), while the imaginary phase of \(\mathcal{H}_{d2}, \mathcal{H}_{f2}\) is only generated from the gluon pole \(\frac{1}{(l_q - x'_g P - x_g) P + i \epsilon}\), an internal quark propagator is effectively put on shell. The Ward identity then implies that

\[
\mathcal{H}_{d2} + \mathcal{H}_{f2} + \mathcal{H}_g = 0
\]  

(31)

This relation also can be readily verified by explicit calculation. The hard parts \(\mathcal{H}_{f1}\) and \(\mathcal{H}_c\) represent the final state interaction contributions, which are canceled out by the corresponding left cut diagrams, as explained in the previous section. We are now only left with

\[
\mathcal{H}_{c+d1} \propto \frac{1}{x_2 + i \epsilon} \frac{1}{x_3 + i \epsilon} \frac{1}{x_4 + i \epsilon} \left\{ C_F \langle P | \text{Tr} \left[ T^a T^b T^d T^e \right] \text{Tr} \left[ T^a T^b T^e T^d T^c T^f \right] \right\}
\]  

(32)

which results in,

\[
\mathcal{H}_{c+d1} \Rightarrow C_F \langle P | \text{Tr} \left[ F_{a}^{-\mu}(\xi^+) T^a \left( i g \int_{\xi^+}^{\infty} dz^+ T^b A_{\mu}^-(z) \right) \left( -i g \int_{-\infty}^{0} dz^+ T^d A_{\mu}^-(z) \right) F_{c}^{-\mu}(0) T^c \right] | P \rangle - \langle P | \text{Tr} \left[ F_{a}^{-\mu}(\xi^+) T^a \left( i g \int_{\xi^+}^{\infty} dz^+ T^b A_{\mu}^-(z) \right) T^e \left( -i g \int_{-\infty}^{0} dz^+ T^d A_{\mu}^-(z) \right) F_{c}^{-\mu}(0) T^c \right] | P \rangle
\]  

(33)

The similar analysis applies to the rest of diagrams. One finds that

\[
\mathcal{H}_{d2} + \mathcal{H}_{f2} + \mathcal{H}_g = 0
\]  

(34)

and the contributions from \(\mathcal{H}_{q1}\) and \(\mathcal{H}_g\) drop out once combining with the corresponding left cut diagrams. One eventually has,

\[
\mathcal{H}_{t+m} \Rightarrow C_F \langle P | \text{Tr} \left[ \left( i g \int_{-\infty}^{\xi^+} dz^+ T^b A_{\mu}^-(z) \right) F_{a}^{-\mu}(\xi^+) T^a F_{c}^{-\mu}(0) T^c \left( -i g \int_{-\infty}^{0} dz^+ T^d A_{\mu}^-(z) \right) \right] | P \rangle - \langle P | \text{Tr} \left[ \left( i g \int_{-\infty}^{\xi^+} dz^+ T^b A_{\mu}^-(z) \right) F_{a}^{-\mu}(\xi^+) T^a T^e F_{c}^{-\mu}(0) T^c \left( -i g \int_{-\infty}^{0} dz^+ T^d A_{\mu}^-(z) \right) \right] | P \rangle
\]  

(35)

and

\[
\mathcal{H}_{n+o1} \Rightarrow C_F \langle P | \text{Tr} \left[ \left( i g \int_{-\infty}^{\xi^+} dz^+ T^b A_{\mu}^-(z) \right) F_{a}^{-\mu}(\xi^+) T^a \left( -i g \int_{-\infty}^{\infty} dz^+ T^d A_{\mu}^-(z) \right) F_{c}^{-\mu}(0) T^c \right] | P \rangle - \langle P | \text{Tr} \left[ \left( i g \int_{-\infty}^{\xi^+} dz^+ T^b A_{\mu}^-(z) \right) F_{a}^{-\mu}(\xi^+) T^a T^e \left( -i g \int_{-\infty}^{\infty} dz^+ T^d A_{\mu}^-(z) \right) F_{c}^{-\mu}(0) T^c \right] | P \rangle
\]  

(36)
Collecting all pieces $\mathcal{H}_{a+b}$, $\mathcal{H}_{c+d}$, $\mathcal{H}_{t+m}$, $\mathcal{H}_{n+o}$ together, it is easy to see that the summation of them gives rise to the lowest nontrivial order expansion of the following gauge link structure,

$$
C_F \langle P | \text{Tr} \left[ F_a^{-\mu}(\xi^+) T^a \mathcal{L}(\xi^+, 0) F_{c,-\mu}(0) T^c \mathcal{L}^i(\xi^+, 0) \right] | P \rangle 
$$

$$
- \langle P | \text{Tr} \left[ \mathcal{L}^i(\infty, \xi^+) F_a^{-\mu}(\xi^+) T^a \mathcal{L}(\xi^+, \infty) T^c \mathcal{L}(\infty, 0) F_{c,-\mu}(0) T^c \mathcal{L}(0, -\infty) T^i \right] | P \rangle 
$$

(37)

where the term in the second line has quite peculiar color structure. Note that the final state interaction $\mathcal{H}_{q1}$, $\mathcal{H}_{p1}$, $\mathcal{H}_{f1}$ and $\mathcal{H}_{e}$ contribute to the twist-2 unpolarized cross section. The principal value part of the gluon poles $\frac{1}{x_{g+1} e}$ and $\frac{1}{x_{g+1} e}$ is canceled out when adding up these contributions with the second term in the above formula. Correspondingly, $A^+$ gluon attachment is decoupled from the hard part and can be absorbed into the gauge link of the collinear quark distribution. This leads us to conclude that the collinear twist-2 factorization is not affected by color entanglement effect at the order under consideration. Meanwhile, it becomes clear that the emergence of color entanglement effect in the collinear twist-3 factorization essentially can be attributed to the fact that the final state interaction doesn’t contribution to the SSA. With this observation, one can generalize the above analysis to all orders. Considering a diagram with arbitrary number of $A^-$ gluon attachments and one $A^+$ attachment from the opposite side, $A^+$ gluon can be first decoupled from the hard part by invoking the Ward identity argument once we sum over all possible $A^+$ insertion points. After having done this, multiple $A^-$ exchange can be incorporated into the normal gauge link as shown in the first line of Eq. 37. However, one has to keep in mind that only initial state interaction contributes to the SSA in the current case. The gauge link structure associated with final state interaction given in the second line of Eq. 37 has to be subtracted. The overall color structure associated with initial state interaction is thus given by Eq. 37. Note that the similar argument also applies to the SSA in other channels, for instance, inclusive hadron production in pp collisions, because initial state interaction and final state interaction don’t generate contributions with equal weight.

Eq. 37 can be further cast into the following form using the Fierz identity,

$$
\frac{N_c}{2} \langle P | \text{Tr} \left[ F_a^{-\mu}(\xi^+) T^a \mathcal{L}(\xi^+, 0) F_{c,-\mu}(0) T^c \mathcal{L}^i(\xi^+, 0) \right] | P \rangle
$$

$$
- \frac{1}{2} \langle P | \text{Tr} \left[ \mathcal{L}^i(\infty, \xi^+) F_a^{-\mu}(\xi^+) T^a \mathcal{L}(\xi^+, \infty) \right] \text{Tr} \left[ \mathcal{L}(\infty, 0) F_{c,-\mu}(0) T^c \mathcal{L}(0, -\infty) \right] | P \rangle
$$

(38)

One immediately recognizes this operator structure as the one that gives rise to the combination $x' G(x') - x' G_4(x')$. The
corresponding spin dependent cross section,
\[
\frac{d^3\Delta\sigma}{d^2l_\perp dz} = \frac{\alpha_s\alpha_{em}N_c}{N_c^2 - 1} \frac{(z^2 - z)[1 + (1 - z)^2]}{l_{\gamma\perp}^4} e^{i\cdot S_{\perp mp}} \int_1^2 \sum_q e_q^2 x_{\text{min}}^1 dx \left[ x'G(x') - x'G_4(x') \right] \left[ -x \frac{d}{dx} T_{F,q}(x, x) \right]
\]  
(39)
which is in full agreement with that obtained by extrapolating the hybrid approach result to the collinear limit [13]. As expected, the pure collinear twist-3 formalism and the hybrid approach are consistent with each other in the collinear limit after properly taking into account color entanglement effect in collinear twist-3 formalism.

IV. SUMMARY

In summary, we have shown that color entanglement effect plays a role in contributing to T-odd observables within the genuine collinear twist-3 factorization framework. As an example, we compute the SSA for direct photon production in pp collisions using the collinear twist-3 approach. Our calculation differs from the conventional collinear twist-3 treatment by deriving the gauge link structure on unpolarized target side explicitly. We found that the existence of the longitudinal gluon \(G\) summarized into the novel gluon distribution \(G_4\) yield the same result in the overlap region where they both apply.

In the present work, we only focus on the derivative term contribution. But we anticipate that the non-derivative term contribution and the soft fermion pole contribution [22] are also affected by color entanglement effect. Moreover, it has been found in the hybrid approach calculations [17–19] that color entanglement effect also contributes to the SSAs in other processes in pp/pA collisions. It is of great interest to confirm these results within pure collinear twist-3 approach, particularly in view of that color entanglement effect plays a crucial role in solving the sign mismatch problem [19]. In a word, all previous collinear twist-3 calculations for T-odd observables (including these related to the Boer-Mulders effect [23–26]) should be thoroughly re-examined by analyzing the gauge link structure of the collinear twist-2 parton distributions on target side.

There are also some other theoretical issues remain to be addressed in future study. First of all, one has to investigate if all pure gauge gluons \(A^+\) attachments other than the one carried small transverse momentum can be decoupled from hard parts and absorbed into the gauge link in the ETQS function. If true, relevant cross sections are factorizable in collinear twist-3 factorization. Otherwise, collinear twist-3 factorization breaks down for T-odd observables in pp collisions. Second, it is worthwhile to make effort to study the properties of the gluon distribution \(G_4\) at moderate or large \(x\), such as its scale evolution, behavior in some models. Finally, though we tend to believe that the conventional collinear twist-3 formulation of T-even observables [27–31] and fragmentation effects [32–39] are not affected by color entanglement effect, it would be nice to verify this point by explicit calculations.

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