SEGREGATED SOLUTIONS FOR SOME NON-LINEAR SCHRÖDINGER SYSTEMS WITH CRITICAL GROWTH

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ABSTRACT. We find infinitely many positive non-radial solutions for a system of Schrödinger equations with critical growth in a fully attractive or repulsive regime in presence of an external radial trapping potential.

1. Introduction

The well-known Gross-Pitaevskii system

$$-i\partial_t \phi_i = \Delta \phi_i - V_i(x)\phi_i + \mu_i |\phi_i|^2 \phi_i + \sum_{j=1, j\neq i}^m \beta_{ij} |\phi_j|^2 \phi_i, \quad i = 1, \ldots, m$$ (1)

has been proposed as a mathematical model for multispecies Bose-Einstein condensation in m different states. We refer to [7, 8, 9, 12] for a detailed physical motivation. Here the complex valued functions $\phi_i$’s are the wave functions of the $i$-th condensate, $|\phi_i|$ is the amplitude of the $i$-th density, $\mu_i$ describes the interaction between particles of the same component and $\beta_{ij}$, $i \neq j$, describes the interaction between particles of different components, which can be attractive if $\beta_{ij} > 0$ or repulsive if $\beta_{ij} < 0$. To obtain solitary wave solutions of the Gross-Pitaevskii system (1) one sets $\phi_i(t,x) = e^{-i\lambda_i t}u_i(x)$ and the real functions $u_i$’s solve the system

$$-\Delta u_i + \lambda_i u_i + V_i(x)u_i = \mu_i u_i^3 + u_i \sum_{j=1, j\neq i}^m \beta_{ij} u_j^2 \text{ in } \mathbb{R}^n, \quad i = 1, \ldots, m$$ (2)

where $\mu_i > 0$, $\lambda_i > 0$, $\beta_{ij} = \beta_{ji} \in \mathbb{R}$, $V_i \in C^0(\mathbb{R}^n)$ and $n \geq 2$.

We are interested in the so-called vector solutions, i.e. $u_i \not\equiv 0$ for any $i = 1, \ldots, m$. Indeed, if one component $u_i$ identically vanishes, the system (2) is reduced to a system

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with $m - 1$ components.

In the last decades, the nonlinear Schrödinger system (2) has been widely studied. Most of the work has been done when the spacial dimension is $n = 2$ or $n = 3$ (in this case the cubic non-linearity has sub-critical growth) and in absence of potentials (i.e. all the $V_i$’s are zero). We refer the reader to a couple of recent papers [1, 20] where the authors provide an exhaustive list of references. On the other hand the non-autonomous case is much less studied. It has been treated firstly by Peng and Wang [13] and more recently by Pistoia and Vaira [15]. In particular, they considered radial trapping potentials $V_i$’s and built (via a careful Ljapunov-Schmidt procedure) unbounded sequences of non-radial positive vector solutions in a fully repulsive (i.e. all the $\beta_{ij}$’s are negative) or attractive (i.e. all the $\beta_{ij}$’s are negative) regime.

In dimension $n = 4$ the cubic growth is critical and the existence of solutions to (2) is a much more difficult issue. The autonomous case has been studied by Clapp and Pistoia [4] and Clapp and Szulkin [5], who proved the existence of vector non-radial solutions to (2) in a fully repulsive regime using an interesting variational approach. Recently, Chen, Medina and Pistoia [3] built (via a sophisticated Ljapunov-Schmidt procedure) a new type of non-radial solutions in a weak repulsive regime (i.e. some $\beta_{ij}$’s are equal, negative and small). In the present paper, we will focus on the existence of solutions in the non-autonomous case which is by far quite unexplored.

We assume that all the coupling parameters $\beta_{ij}$’s are equal to a real number $\beta$ and all the trapping potentials $V_i$’s coincide with a positive and radially symmetric function $V \in L^2(\mathbb{R}^4) \cap C^2(\mathbb{R}^4)$. Therefore the system (2) reduces to the system

$$-\Delta u_i + V(x)u_i = u_i^3 + \beta \sum_{j \neq i} u_i u_j^2 \text{ in } \mathbb{R}^4, \quad i = 1, \ldots, m. \quad (3)$$

We are going to build infinitely many non-radial solutions to (3), whose building blocks are the so-called bubbles

$$U_{\delta,\xi}(x) = \frac{1}{\delta} U \left( \frac{x - \xi}{\delta} \right) \text{ with } U(x) = \frac{c}{1 + |x|^2}, \quad c = 2\sqrt{2}, \quad (4)$$

which are the positive solutions to the critical equation

$$-\Delta u = u^3 \text{ in } \mathbb{R}^4.$$

We assume that

$$r_0 > 0 \text{ is a non-degenerate critical point of the function } r \rightarrow r^2V(r). \quad (5)$$

Now, we can state our main result.

**Theorem 1.1.** There exists $k_0 > 0$ such that for any even integer $k \geq k_0$, there exists a solution $(u_1, \ldots, u_m) \in [D^{1,2}(\mathbb{R}^4)]^m$ to problem (3) of the form

$$u_q \sim \sum_{i=1}^k U_{\delta,\xi_i}^{q_i} \text{ for any } q = 1, \ldots, m.$$
where the bubbles $U_{\delta,\xi}$ are defined in (4).
All the bubbles have the same blow-up rate which satisfies
\[
\delta = e^{-d_k k^2} \quad \text{and} \quad d_k \sim \frac{3}{r_0^2 V(r_0)} > 0 \quad \text{as} \quad k \to \infty
\]
for some positive constant $\delta$.
The blow-up points $\xi^q_i$ (see (14)) of the component $u_q$ lie on a circle $\Gamma_q$ whose distance from the origin approaches $r_0$ as $k \to \infty$. Moreover, $\Gamma_p$ with $\Gamma_q$ is an Hopf link if $p \neq q$.

Remark 1.2. The solutions we find are of segregated type in the sense that the components tend to segregate from each other leading to phase separation. This phenomena has been widely studied by Terracini and her collaborators in a series of papers (see for example [6, 11, 16, 17, 18] and references therein) and naturally appears in the strongly repulsive case (i.e. $\beta_{ij} \to -\infty$). In our case the existence of this kind of solutions is not affected at all by the presence of the coupling parameter $\beta$: they do exist in both fully repulsive (i.e. $\beta < 0$) and attractive regime (i.e. $\beta > 0$). This is due to the fact that the regime of the system is entirely encrypted in the non-local term of (7) (see below) which does not appear in the reduced problem (i.e. it is an higher order term in (32) and (36)). It would be extremely interesting to exhibit examples of potential $V$ (maybe changing-sign) and/or different configurations of bubbles for which the existence of solutions of (7) strongly depends on the sign of the parameter $\beta$.

Remark 1.3. Let us describe the strategy of the proof. First, we look for solutions having a subtle symmetry which allows to reduce the system (3) to a single non-local equation. We follow an idea introduced by Chen, Medina and Pistoia in [3]. More precisely, if
\[
\mathcal{J}_q := \begin{pmatrix}
\cos \left(\frac{(q-1)\pi}{m}\right) & -\sin \left(\frac{(q-1)\pi}{m}\right) & 0 & 0 \\
\sin \left(\frac{(q-1)\pi}{m}\right) & \cos \left(\frac{(q-1)\pi}{m}\right) & 0 & 0 \\
0 & 0 & \cos \left(\frac{(q-1)\pi}{m}\right) & \sin \left(\frac{(q-1)\pi}{m}\right) \\
0 & 0 & -\sin \left(\frac{(q-1)\pi}{m}\right) & \cos \left(\frac{(q-1)\pi}{m}\right)
\end{pmatrix}
\]
for any $q = 1, \ldots, m$. (6)
and $u$ is an even function (i.e. $u(x) = u(-x)$) which solves the non-local equation
\[
-\Delta u + V(x)u = u^3 + \beta u \sum_{q=2}^{m} u^2(\mathcal{J}_q x), \quad \text{in} \quad \mathbb{R}^4,
\]
(7)
then the functions defined via (6)
\[
u_q(x) = u(\mathcal{J}_q x) \quad \text{for any} \quad q = 1, \ldots, m
\]
(8)
solve the system (3).

Next, we build a solution to (7) whose shape resembles $k$ copies of bubbles whose peaks are located at the vertices of a regular polygon placed in a great circles $\Gamma$ with radius $r$. As a result the peaks of the function $u_q$ which solves the system (3) and is defined via the symmetry (8), lie on the great circle $\Gamma_q = \mathcal{J}_q(\Gamma)$ and different components concentrate along linked great circles (i.e. $\Gamma_q$ with $\Gamma_p$ is an Hopf link if $q \neq p$). In order to carry
out the construction of the solution to (7) we are inspired by Peng, Wang and Yan [14], who build positive solutions to the Schrödinger equation

$$-\Delta u + V(x)u = u^{\frac{4+2}{n-2}}, \text{ in } \mathbb{R}^n \ (n \geq 5)$$

combining the classical Ljapunov-Schmidt procedure together with a clever use of local Pohozaev identities to find the algebraic equations which determine the location of the peaks. Here we follow the same general approach to study the non-local Schrödinger equation (7). However a substantial difference arises when we write the local Pohozaev identities, due to the presence of the non-local term. To overcome the problem we use in a delicate and clever way the subtle symmetries owned by the solutions we aim to build (see Proposition 2.10).

**Remark 1.4.** If we replace $\mathcal{S}_q$ in (6) by

$$\mathcal{S}_q := \begin{pmatrix} \cos \left(\frac{(q-1)\pi}{m}\right) & -\sin \left(\frac{(q-1)\pi}{m}\right) & 0 & 0 \\ \sin \left(\frac{(q-1)\pi}{m}\right) & \cos \left(\frac{(q-1)\pi}{m}\right) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

for any $q = 1, \ldots, m$.

a result similar to Theorem 1.1 can be proved once we choose the peaks as $\xi_i^q = \mathcal{S}_q^{-1} \xi_i$ where $\xi_i := \rho \left(\cos \frac{2\pi(i-1)}{k}, \sin \frac{2\pi(i-1)}{k}, 0, 0\right)$. In this case we can also relax the assumption on the potential $V$ only requiring that it is radially symmetric in the first two variables in the spirit of [14].

**Remark 1.5.** It is also worthwhile to point out that we introduce the good weighted spaces where the reduction procedure in dimension $n = 4$ can be carried out (as far as we know only the case $n \geq 5$ has been treated in the literature). This choice is a delicate issue, since as it is usual we can not use the standard Sobolev spaces because of the arbitrary large number of bubbles in the solution and the presence of the linear term in the critical equation (7) in 4D is not an innocent matter due the slow decay of the bubbles.

**Notation.** In what follows we agree that $f \lesssim g$ or $f = O(g)$ means $|f| \leq c|g|$ for some positive constant $c$ independent of $k$ and $f \sim g$ means $f = g + o(g)$.

2. Proof of Theorem 1.1

2.1. Rewriting the non-local equation via the finite dimensional reduction method. We will find solutions of (7) in the space of symmetric functions

$$X := \{ u \in D^{1,2}(\mathbb{R}^4) : u \text{ satisfies } (9), \ (10) \text{ and } (11) \},$$
i.e.
\[
\begin{align*}
    u(x_1, x_2, x_3, x_4) &= u(x_1, -x_2, x_3, -x_4), \\
    u(x_1, x_2, x_3, x_4) &= u(x_3, x_4, x_1, x_2) \\
    u(x) &= u(\mathcal{R}_i x) \text{ for any } i = 1, ..., k,
\end{align*}
\]

where \( k \) is an even integer and
\[
\mathcal{R}_i = \begin{pmatrix}
    \cos \frac{2\pi(i-1)}{k} & \sin \frac{2\pi(i-1)}{k} & 0 & 0 \\
    -\sin \frac{2\pi(i-1)}{k} & \cos \frac{2\pi(i-1)}{k} & 0 & 0 \\
    0 & 0 & \cos \frac{2\pi(i-1)}{k} & \sin \frac{2\pi(i-1)}{k} \\
    0 & 0 & -\sin \frac{2\pi(i-1)}{k} & \cos \frac{2\pi(i-1)}{k}
\end{pmatrix}.
\]

In particular, we are going to build a solution to (7) as
\[
u = \sum_{i=1}^{k} \chi U_{\delta, \xi_i} + \phi \text{ as } k \to +\infty,
\]

where the bubbles \( U_{\delta, \xi_i} \) are defined in (4) whose blow-up points are
\[
\xi_i := \frac{\rho}{\sqrt{2}} \left( \cos \frac{2\pi(i-1)}{k}, \sin \frac{2\pi(i-1)}{k}, \cos \frac{2\pi(i-1)}{k}, \sin \frac{2\pi(i-1)}{k} \right), \text{ with } |\rho - r_0| \leq \vartheta
\]

for some \( \vartheta > 0 \) small and blow-up rate satisfies \( \delta = e^{-dk^2} \) with \( d \in [d_0, d_1] \) for some \( d_1 > d_0 > 0 \). Here \( r_0 \) is given in (5). Moreover, \( \chi(x) = \chi(|x|) \) is a radial cut-off function whose support is close to the sphere \( \{|x| = r_0\} \), namely
\[
\chi = 1 \text{ in } |r - r_0| \leq \sigma \text{ and } \chi = 0 \text{ in } |r - r_0| > 2\sigma \text{ for some } \sigma > 0 \text{ small.}
\]

Finally, as usual, \( \phi \) is an higher order term.

It is worthwhile to point out that the function \( u_q \) given in (8) blows-up at the points
\[
\xi_i^q := \mathcal{S}_q^{-1} \xi_i = \frac{\rho}{\sqrt{2}} \left( \cos \left( \frac{(q-1)\pi}{m} + \frac{2\pi(i-1)}{k} \right), \sin \left( \frac{(q-1)\pi}{m} + \frac{2\pi(i-1)}{k} \right), \cos \left( \frac{(q-1)\pi}{m} + \frac{2\pi(i-1)}{k} \right), \sin \left( \frac{(q-1)\pi}{m} + \frac{2\pi(i-1)}{k} \right) \right),
\]

Plugging \( u = W + \phi \) into the non-local equation (7), it can be rewritten as
\[
\mathcal{L}(\phi) = \mathcal{E} + \mathcal{N}(\phi), \text{ in } \mathbb{R}^4
\]
where \( \mathcal{L}(\phi), \mathcal{E}, \mathcal{N}(\phi) \) are defined as

\[
\mathcal{L}(\phi) := -\Delta \phi + V(x)\phi - 3W^2\phi - \beta \phi \sum_{q=2}^{m} W^2(\mathcal{S}_q x),
\]

\[
\mathcal{E} := W^3 + \Delta W - V(x)W + \beta W \sum_{q=2}^{m} W^2(\mathcal{S}_q x),
\]

\[
\mathcal{N}(\phi) := \phi^3 + 3W\phi^3 + \beta \phi \sum_{q=2}^{m} \phi^2(\mathcal{S}_q x) + 2\beta \phi \sum_{q=2}^{m} W(\mathcal{S}_q x)\phi(\mathcal{S}_q x)
\]

\[
+ \beta W \sum_{q=2}^{m} \phi^2(\mathcal{S}_q x) + 2\beta W \sum_{q=2}^{m} W(\mathcal{S}_q x)\phi(\mathcal{S}_q x).
\]

As it is usual, to solve \((15)\) we will follow the classical steps of the Ljapunov-Schmidt procedure:

(i) we show there exists \( \phi \in X \) solution to the problem

\[
\begin{cases}
\mathcal{L}(\phi) = \mathcal{E} + \mathcal{N}(\phi) + \sum_{l=0}^{1} \alpha_l(\delta, \rho) \sum_{i=1}^{k} (\chi U_{\delta, \xi_i})^2 Z_{\delta, \xi_i}^l \\
\int_{\mathbb{R}^4} \sum_{i=1}^{k} (\chi U_{\delta, \xi_i})^2 Z_{\delta, \xi_i}^l \phi \, dx = 0, l = 0, 1
\end{cases}
\]

where

\[
Z_{\delta, \xi_i}^0 = \frac{\partial (\chi U_{\delta, \xi_i})}{\partial \delta} \quad \text{and} \quad Z_{\delta, \xi_i}^1 = \frac{\partial (\chi U_{\delta, \xi_i})}{\partial \rho}
\]

(ii) we find \( \delta > 0 \) and \( \rho \) close to \( r_0 \) such that \( \alpha_0(\delta, \rho) = \alpha_1(\delta, \rho) = 0 \).

2.2. Preliminaries. Let us introduce the norms

\[
\|\phi\|_* = \sup_{x \in \mathbb{R}^4} \left( \sum_{q=1}^{m} \sum_{i=1}^{k} \frac{1}{\delta + |x - \mathcal{S}_q^{-1} \xi_i|} \right)^{-1} |\phi(x)|,
\]

\[
\|h\|_{**} = \sup_{x \in \mathbb{R}^4} \left( \sum_{q=1}^{m} \sum_{i=1}^{k} \frac{1}{|\delta + |x - \mathcal{S}_q^{-1} \xi_i||^3} \right)^{-1} |h(x)|.
\]

The estimates of the norm of varies quantities rely on a couple of important results whose proofs can be found in Appendix B of [19].

**Lemma 2.1.** For any \( 0 < \alpha \leq \min\{\alpha_1, \alpha_2\}, i \neq j \), it holds

\[
\frac{1}{(1 + |x - x_i|)^{\alpha_1}} \frac{1}{(1 + |x - x_j|)^{\alpha_2}} \lesssim \frac{1}{|x_i - x_j|^{\alpha}} \left( \frac{1}{(1 + |x - x_i|)^{\alpha_1 + \alpha_2 - \alpha}} + \frac{1}{(1 + |x - x_j|)^{\alpha_1 + \alpha_2 - \alpha}} \right)
\]
Lemma 2.2. For any constant $0 < \alpha < 2$, there is a constant $C > 0$, such that
\[
\int_{\mathbb{R}^4} \frac{1}{|z - x|^2} \frac{1}{(1 + |z|)^{2+\alpha}} \, dz \leq \frac{C}{(1 + |x|)^\alpha}.
\]

Finally, we also state the following result which will be used throughout the paper.

Lemma 2.3. There exists a positive constant $c, \zeta, C_\alpha$ such that
\[
|\xi_p^i - \xi_q^j| \begin{cases} 
\geq c > 0 & \text{if } p \neq q, \\
\geq \frac{\zeta}{k} & \text{if } p = q \text{ and } i \neq j.
\end{cases}
\]

and
\[
\sum_{(q,i) \neq (p,j)} \frac{1}{|\xi_p^i - \xi_q^j|^\alpha} \begin{cases} 
\sim C_\alpha \frac{k^\alpha}{\rho^2} & \text{if } \alpha > 1, \\
\lesssim \zeta \ln k & \text{if } \alpha = 1, \\
\lesssim k^{1+\alpha} & \text{if } \alpha < 1.
\end{cases}
\]

Proof. It is enough to remark that
\[
|\xi_p^i - \xi_q^j|^2 = 2\rho^2 \left(1 - \cos \frac{(p-q)\pi}{m} \cos \frac{(i-j)2\pi}{k}\right)
\]
and
\[
\sum_{(q,i) \neq (p,j)} \frac{1}{|\xi_p^i - \xi_q^j|^\alpha} = \sum_q \sum_{i \neq j} \frac{1}{|\xi_q^j - \xi_i^i|^\alpha} + \sum_{p \neq q} \sum_{i,j} \frac{1}{|\xi_p^p - \xi_q^q|^\alpha}
\]
\[
= \sum_q \sum_{i \neq j} \frac{1}{|\xi_q^j - \xi_i^i|^\alpha} + \mathcal{O}(k)
\]
\[
\sim A_\alpha \frac{k^\alpha}{\rho^2} \sum_{i=2}^{k} \frac{1}{i^\alpha}
\]
for some constant $A_\alpha$ depending on $\alpha$.

In the following we set $\eta_q^q = \frac{\xi_q^q}{\delta}$ and $\eta_i = \frac{\xi_i}{\delta}$ if $i, j = 1, \ldots, k$ and $q = 1, \ldots, m$.

2.3. The size of the error term $\mathcal{E}$.

Proposition 2.4. Let $\mathcal{E}$ be defined as in (17). Then
\[
\|\mathcal{E}\|_{**} \lesssim \delta.
\]

Proof. We have
\[
\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 + \beta W \sum_{q=2}^{m} W^2(\mathcal{S}_q(x)) = \mathcal{E}_3.
\]
First

\[
\mathcal{E}_1 := W^3 + \Delta W = \left( \sum_{i=1}^{k} \chi U_{\delta,\xi_i} \right)^3 - \chi \sum_{i=1}^{k} U_{\delta,\xi_i}^3 + \Delta \chi \sum_{i=1}^{k} U_{\delta,\xi_i} + 2\Delta \chi \sum_{i=1}^{k} \nabla U_{\delta,\xi_i}.
\]

Now,

\[
|I_1| \lesssim \left( \sum_{j \neq i} U_{\delta,\xi_j}^2 U_{\delta,\xi_i} + (\chi^3 - \chi) \sum_{i=1}^{k} U_{\delta,\xi_i}^3 \right),
\]

with

\[
(\chi^3 - \chi) \sum_{i=1}^{k} U_{\delta,\xi_i}^3 \lesssim \delta^3 \sum_{i=1}^{k} \frac{1}{(\delta + |x - \xi_i|)^3}
\]

and for any \( x \in \mathbb{R}^4 \), denote \( \omega = \frac{x}{\delta} \), by Lemma 2.1 and (21)

\[
U_{\delta,\xi_i}^2 \sum_{j \neq 1} U_{\delta,\xi_j} = \frac{1}{\delta^3} \frac{1}{(1 + |\omega - \eta_1|^2)^\alpha} \sum_{j \neq 1} \frac{1}{1 + |\omega - \eta_j|^2}
\]

\[
\lesssim \frac{1}{\delta^3} \sum_{j \neq 1} \frac{1}{|\eta_1 - \eta_j|^\alpha} \left( \frac{1}{(1 + |\omega - \eta_1|)^{6-\alpha}} + \frac{1}{1 + |\omega - \eta_j|^{6-\alpha}} \right)
\]

\[
\lesssim \delta^\alpha k \sum_{j=1}^{k} \frac{1}{(\delta + |x - \xi_j|)^3} \text{ for any } 1 < \alpha \leq 2,
\]

and so using the rotation transform

\[
\sum_{i=1}^{k} \sum_{j \neq i} U_{\delta,\xi_i}^2 U_{\delta,\xi_j} \lesssim k \cdot \delta^\alpha k \sum_{j=1}^{k} \frac{1}{(\delta + |x - \xi_j|)^3},
\]

i.e.,

\[
|I_1| \leq \sum_{i=1}^{k} \frac{1}{(\delta + |x - \xi_i|)^3} \delta^\alpha k_2 \text{ for any } 1 < \alpha \leq 2.
\]

Moreover,

\[
|I_2| \lesssim \left[ \delta \sum_{i=1}^{k} \frac{1}{(\delta + |x - \xi_i|)^3} \cdot (\delta + |x - \xi_i|) |\Delta \chi| 
\right.
\]

\[
+ \delta \sum_{i=1}^{k} \frac{1}{(\delta + |x - \xi_i|)^3} \cdot \frac{|x - \xi_i|}{\delta + |x - \xi_i|} |\nabla \chi| \right]
\]

\[
\lesssim \delta \sum_{i=1}^{k} \frac{1}{(\delta + |x - \xi_i|)^3}.
\]

Therefore

\[
\|\mathcal{E}_1\|_* \leq \|I_1\|_* + \|I_2\|_* \lesssim \delta.
\]
Next
\[ |E_2| \lesssim \sum_{i=1}^{k} \frac{\delta}{\delta^2 + |x - \xi_i|^2} |\chi| \lesssim \delta \sum_{i=1}^{k} \frac{1}{(\delta + |x - \xi_i|)^\beta}. \]

Finally by (20) we get
\[ |E_3| \lesssim \sum_{q=1}^{m} \sum_{i=1}^{k} \frac{1}{(\delta + |x - \xi_i|^\alpha)} \delta^\alpha k^2 \text{ for any } 1 < \alpha \leq 2. \]

Collecting all the previous estimates, it follows
\[ \|E\|_* \leq \|E_1\|_* + \|E_2\|_* + \|E_3\|_* \lesssim \delta. \]

\[ \square \]

2.4. Solving a linear problem. Denote by \((\Lambda_i)_{i \in \mathbb{N}}\) the sequences of eigenvalues of the problem
\[-\Delta Z = \Lambda_i U^2 Z \text{ in } \mathbb{R}^4, \quad Z \in \mathcal{D}^{1,2}(\mathbb{R}^4).\]

It is well known that the first eigenvalue is \(\Lambda_1 = 1\) and the associated eigenspace is generated by the function \(U\). The second eigenvalue is \(\Lambda_2 = 3\) and the associated eigenspace is generated by the function
\[ Z_0(x) = \frac{1 - |x|^2}{(1 + |x|^2)^2}, \quad Z_1(x) = \frac{x_1}{(1 + |x|^2)^2}, \quad \ldots, \quad Z_4(x) = \frac{x_4}{(1 + |x|^2)^2}. \]

**Proposition 2.5.** Let \(L(\phi)\) be defined as in (16). Assume \(\beta \notin (\Lambda_i)_{i \in \mathbb{N}}\). There exist \(k_0 > 0\) and a constant \(C > 0\) independent of \(k\) such that for any even \(k \geq k_0\), \(\delta = e^{-dk^2}\) with \(d \in [d_0, d_1]\) for some \(d_1 > d_0 > 0\), \(\rho \in (r_0 - \vartheta, r_0 + \vartheta)\) with \(\vartheta > 0\) and for any \(h \in L^4(\mathbb{R}^4)\) satisfying (9)-(11), the linear problem
\[ \begin{cases} 
L(\phi) = h + \sum_{l=0}^{1} \sum_{i=1}^{k} \zeta_i (\chi U_{\delta_i})^2 Z_{\delta_i \xi_i}^l, & \text{in } \mathbb{R}^4, \\
\int_{\mathbb{R}^4} \sum_{i=1}^{k} (\chi U_{\delta_i})^2 Z_{\delta_i \xi_i}^l \phi \, dx = 0, \quad l = 0, 1.
\end{cases} \quad (22) \]

admits a unique solution \(\phi \in X\) satisfying
\[ \|\phi\|_* \lesssim \|h\|_{**} \text{ and } |\zeta_i| \lesssim \delta \|h\|_{**}, \quad l = 0, 1. \]

(23)

for some \(\zeta_0, \zeta_1\).

**Proof.** Step 1: Assume first that (23) holds. Define
\[ X_0 := \{ \phi \in X, \|\phi\|_* < +\infty, \int_{\mathbb{R}^4} \sum_{i=1}^{k} (\chi U_{\delta_i})^2 Z_{\delta_i \xi_i}^l \phi \, dx = 0, \quad l = 0, 1 \}. \]
The first equation in (22) can be rewritten as

$$
\phi + (-\Delta)^{-1}(V(x)\phi - 3W^2\phi - \beta\phi \sum_{q=2}^m W^2(\mathcal{Q}_q x)) = (-\Delta)^{-1}(h + \sum_{l=0}^1 \sum_{i=1}^k (\chi U_{\delta,\xi_i})^2 Z_{\delta,\xi_i}^l) := K\phi
$$

where $K : X_0 \to X_0$ is a compact operator for each fixed $k$ since $W^2, W^2(\mathcal{Q}_q x), V(x) \in L^2(\mathbb{R}^4)$ (see Lemma 2.3 in [2]) and there exist $c_0, c_1$ such that $f \in X_0$. Thus there is a unique $\phi \in X_0$ such that $\phi + K\phi = f$ with $f \in X_0$ by Fredholm-alternative theorem.

Now, let us prove (23). Assume by contradiction that there exist a sequence $\phi_n$ satisfying (22) with $h = h_n$, and $\|\phi_n\|_* = 1, \|h_n\|_{**} \to 0$. For sake of simplicity, we will drop the subscript $n$.

**Step 2:** We claim that

$$
c_l = O(\delta^2 k^{1+\tau}\|\phi\|_*) + O(\delta\|h\|_{**}), \ l = 0, 1 \text{ for some small } \tau > 0. \quad (24)
$$

Testing the first equation in (22) by $Z_{\delta,\xi_i}^l, j = 0, 1$, we get

$$
\sum_{l=0}^1 c_l \int_{\mathbb{R}^4} \sum_{i=1}^k (\chi U_{\delta,\xi_i})^2 Z_{\delta,\xi_i}^l Z_{\delta,\xi_i}^j dx = \int_{\mathbb{R}^4} \left(-\Delta \phi + V(x)\phi - 3W^2\phi - \beta\phi \sum_{q=2}^m W^2(\mathcal{Q}_q x) - h\right) Z_{\delta,\xi_i}^l dx.
$$

Concerning the L.H.S. it is immediate to check that there exist $B_l \neq 0, l = 0, 1$ such that

$$
\int_{\mathbb{R}^4} \sum_{i=1}^k (\chi U_{\delta,\xi_i})^2 Z_{\delta,\xi_i}^l Z_{\delta,\xi_i}^j dx = \begin{cases} B_l \delta^{-2}(1 + o(1)), & l = j, \\ o(\delta^{-2}), & l \neq j. \end{cases}
$$

Now, let us look at the R.H.S.. First, by Lemma 2.1, (21) and the fact that $|Z_{\delta,\xi_i}^l| \lesssim \frac{1}{\delta^{1+\tau}|x - \xi_i|^\tau}$, $j = 0, 1$

$$
\int_{\mathbb{R}^4} h Z_{\delta,\xi_i}^l dx \lesssim \|h\|_{**} \int_{\mathbb{R}^4} \sum_{q=1}^m \sum_{i=1}^k \frac{1}{(\delta + |x - \xi_q^i|)^3} \frac{1}{\delta^2 + |x - \xi_1|^2} dx
$$

$$
\lesssim \delta^{-1}\|h\|_{**} \int_{\mathbb{R}^4} \sum_{q=1}^m \sum_{i=1}^k \frac{1}{(1 + |\omega - \eta_q^i|)^3} \frac{1}{1 + |\omega - \eta_1|^2} d\omega
$$

$$
\lesssim \delta^{-1}\|h\|_{**} \int_{\mathbb{R}^4} \frac{1}{(1 + |\omega - \eta_1|)^3} + \sum_{(q,l) \neq (1,1)} \sum_{i=2}^k \frac{1}{|\eta_i - \eta_l^i|^{3-\alpha}}
$$

$$
\cdot \left( \frac{1}{(1 + |\omega - \eta_1|)^{5-\alpha}} + \frac{1}{(1 + |\omega - \eta_l^i|)^{5-\alpha}} \right) d\omega
$$

$$
\lesssim \delta^{-1}\|h\|_{**} \text{ for any } 0 < \alpha < 1,
$$
\[
\int_{\mathbb{R}^4} V(x) \phi Z^j_{\delta, \xi_1} \, dx
\]
\[
\lesssim \| \phi \|_* \int_{|r-r_0| \leq 2\sigma} \sum_{q=1}^m \sum_{i=1}^k \frac{1}{\delta + |x - \xi^q_i|} \frac{1}{\delta^2 + |x - \xi_1|^2} \, dx
\]
\[
\lesssim \| \phi \|_* \int_{|r-r_0| \leq 2\sigma} \frac{1}{(\delta + |x - \xi_1|)^3} \, dx
\]
\[
+ \sum_{(q,i) \neq (1,1)} \frac{\delta^\alpha}{\xi^q_i - \xi_1} \frac{\delta^4}{\alpha} \int_{|r-r_0| \leq \frac{2\sigma}{\delta}} \left( \frac{1}{(1 + |x - \xi^q_i|^{3-\alpha})} + \frac{1}{(1 + |x - \xi_1|^{3-\alpha})} \right) \, dx
\]
\[
\lesssim k^{1+\tau} \| \phi \|_* \quad \text{where } \tau > 0 \text{ is small for any } 0 < \alpha \leq 1.
\] (25)

and

\[
\int_{\mathbb{R}^4} \phi \sum_{q=2}^m W^2(\mathcal{S}^q x) Z^j_{\delta, \xi_1} \, dx
\]
\[
\lesssim \| \phi \|_* \int_{\mathbb{R}^4} \sum_{q=1}^m \sum_{i=1}^k \frac{1}{\delta + |x - \xi^q_i|} \sum_{p=2}^m \left( \sum_{j=1}^k \frac{\delta}{\delta^2 + |x - \xi^p_j|^2} \right)^2 \frac{1}{\delta^2 + |x - \xi_1|^2} \, dx
\]
\[
\lesssim \| \phi \|_* \int_{\mathbb{R}^4} \frac{1}{(\delta + |x - \xi_1|)^3} \sum_{p=2}^m \left( \sum_{j=1}^k \frac{\delta}{\delta^2 + |x - \xi^p_j|^2} \right)^2 \, dx
\]
\[
\lesssim \delta^{-1} k \| \phi \|_* \int_{\mathbb{R}^4} \sum_{q=1}^m \sum_{j=1}^k \frac{1}{|\eta_1 - \eta^q_j|^\alpha} \left( \frac{1}{(1 + |\omega - \eta_1|)^{7-\alpha}} + \frac{1}{(1 + |\omega - \eta^q_j|)^{7-\alpha}} \right) \, d\omega
\]
\[
\lesssim \delta^{\alpha-1} k^{1+\alpha} \| \phi \|_* \quad \text{for any } 1 < \alpha < 3.
\]

Next, let us consider the term

\[
\int_{\mathbb{R}^4} (-\Delta \phi - 3W^2 \phi) Z^j_{\delta, \xi_1} \, dx, \quad j = 0, 1.
\]

If \( j = 0 \) (the case \( j = 1 \) is similar), arguing as in (25) for \( \tau > 0 \) small enough

\[
\int_{\mathbb{R}^4} (-\Delta \chi) \frac{\partial U_{\delta, \xi_1}}{\partial \delta} \phi \, dx = O(k^{1+\tau} \| \phi \|_*)
\]
and then

\[
\int_{\mathbb{R}^4} (-\Delta \phi \ldots 3W^2_\phi) Z^0_{\delta, \xi_l} \, dx
\]

\[
= \int_{\mathbb{R}^4} \phi \left( \frac{\partial (-\Delta U_{\delta, \xi_l} \ldots 2\nabla \nabla U_{\delta, \xi_l} + U^{3}_{\delta, \xi_l})}{\partial \delta} - 3W^2 Z^1_{\delta, \xi_l} \right) \]

\[
= \int_{\mathbb{R}^4} \left( (3U^2_{\delta, \xi_l} \frac{\partial U_{\delta, \xi_l}}{\partial \delta} - 3\chi^3 U^2_{\delta, \xi_l} \frac{\partial U_{\delta, \xi_l}}{\partial \delta}) \phi \right) + \left[ 3(\chi^2 U^2_{\delta, \xi_l} \ldots W^2) \chi \frac{\partial U_{\delta, \xi_l}}{\partial \delta} \phi \right] \, dx
\]

\[
+ \int_{\mathbb{R}^4} (-\Delta \chi) \frac{\partial U_{\delta, \xi_l}}{\partial \delta} \phi \, dx - \int_{\mathbb{R}^4} 2\nabla \chi \nabla \frac{\partial U_{\delta, \xi_l}}{\partial \delta} \phi \, dx
\]

\[
= o(||\phi||_*) + O(k^{1+\tau}||\phi||_*) + O(k||\phi||_*) = O(k^{1+\tau}||\phi||_*). \tag{26}
\]

Finally (24) follows by all the above estimates.

**Step 3:** We claim that if \(\tau > 0\) is small enough

\[
|\phi(x)| \leq (o(||\phi||_*) + O(||h||_*)) \sum_{q=1}^m \sum_{i=1}^k \frac{1}{\delta + |x - \xi^q_i|} + O\left( \sum_{q=1}^m \sum_{i=1}^k \frac{\delta^r}{\delta + |x - \xi^q_i|^{1+\tau}} \right). \tag{27}
\]

Indeed, since \(V\) is positive, by [10, Lemma C.1],

\[
|\phi(x)| \lesssim \left| \int_{\mathbb{R}^4} \frac{1}{|x - y|^2} \left( 3W^2 \phi + \beta \phi \sum_{q=2}^m W^2(\mathcal{S}_q x) \right) + h + \sum_{l=0}^1 \sum_{i=1}^k (\chi U_{\delta, \xi_l})^2 Z^l_{\delta, \xi_l} \right| \, dy
\]

\[
:= II_1 + II_2 + II_3.
\]

Since for small \(\tau > 0\), for any \(x \in \mathbb{R}^4\), using (21) and Lemma 2.1,

\[
\sum_{q=1}^m \left( \sum_{i=1}^k \frac{\delta^2 + |y - \xi^q_i|^2}{\delta + |y - \xi^q_i|} \right)^2 \sum_{p=1}^m \sum_{j=1}^k \frac{1}{\delta + |y - \xi^p_j|}
\]

\[
\lesssim \sum_{q=1}^m \sum_{i=1}^k \left( \frac{\delta^2 + |y - \xi^q_i|^2}{\delta + |y - \xi^q_i|} \right)^2 \sum_{p=1}^m \sum_{j=1}^k \frac{1}{\delta + |y - \xi^p_j|}
\]

\[
\lesssim \frac{1}{\delta^3} \left( \sum_{q=1}^m \sum_{i=1}^k \frac{1}{(1 + |\omega - \eta^q_i|)^5} + \sum_{(p, j) \neq (q, i)}^k \frac{1}{|\eta^q_i - \eta^p_j|^{3+\alpha}} \right)
\]

\[
\lesssim \frac{1}{\delta^3} \sum_{q=1}^m \sum_{i=1}^k \frac{1}{(1 + |\omega - \eta^q_i|)^{5+\tau}} (1 + \delta^{\alpha} k^{3+\alpha}) \text{ for some } 0 < \alpha < 1,
\]
2.2. \[
II_1 \lesssim \|\phi\|_* \left( \int_{\mathbb{R}^4} \frac{1}{|x-y|^2} \sum_{q=1}^{m} \left( \sum_{i=1}^{k} \frac{\delta^2 + |y-\xi_i|^2}{\delta + |y-\xi_i|^2} \sum_{j=1}^{k} \right) dy \right) \\
\lesssim \|\phi\|_* \sum_{q=1}^{m} \sum_{i=1}^{k} \frac{\delta^\tau}{(\delta + |x-\xi_i|)^{1+\tau}} \text{ for small } \tau > 0
\]
and
\[
II_2 \lesssim \|h\|_* \left( \int_{\mathbb{R}^4} \frac{1}{|x-y|^2} \sum_{q=1}^{m} \sum_{i=1}^{k} \frac{1}{\delta + |y-\xi_i|^2} dy \right) \\
\lesssim \|h\|_* \sum_{q=1}^{m} \sum_{i=1}^{k} \frac{1}{\delta + |x-\xi_i|}.
\]
Next, taking into account that \[|Z_l(\delta,\xi)| \lesssim \frac{1}{\delta^{1+|x-\xi|} l}, \text{ } l = 0, 1,\] we get
\[
II_3 \lesssim \sum_{l=0}^{1} \frac{1}{|x-\xi|} \left( \int_{\mathbb{R}^4} \frac{1}{|x-y|^2} \sum_{i=1}^{k} \frac{\delta^2}{(\delta^2 + |y-\xi_i|^2)} dy \right) \\
\lesssim \sum_{l=0}^{1} \frac{|\xi|}{|x-\xi|} \left( \int_{\mathbb{R}^4} \frac{1}{|x-y|^2} \sum_{i=1}^{k} \frac{1}{(1 + |\omega-\eta_i|)^6} d\omega \right) \\
\lesssim \sum_{l=0}^{1} \frac{|\xi|}{|x-\xi|} \left( \int_{\mathbb{R}^4} \frac{1}{|x-y|^2} \sum_{i=1}^{k} \frac{1}{(1 + |\omega-\eta_i|)^{3+\tau}} d\omega \right) \\
\lesssim \sum_{l=0}^{1} \frac{|\xi|}{|x-\xi|} \left( \int_{\mathbb{R}^4} \frac{1}{|x-y|^2} \sum_{i=1}^{k} \frac{\delta^{-1+\tau}}{(1 + |\delta - \eta_i|)^{1+\tau}} \right) \\
\lesssim \sum_{l=0}^{1} \frac{|\xi|}{|x-\xi|} \left( \int_{\mathbb{R}^4} \frac{1}{|x-y|^2} \sum_{i=1}^{k} \frac{\delta^{-1+\tau}}{(\delta + |x-\xi_i|)^{1+\tau}} \right).
\]
Finally, (27) follows by all the above estimates.

**Step 4:** We claim that there exist \(c, R > 0\) and \(\xi_1^{q_0}\) for some \(q_0\) such that
\[
\|\delta \phi(y + \xi_1^{q_0})\|_{L^\infty(B_R(0))} \geq c > 0.
\]
Indeed, from (27), we have
\[
\left( \sum_{q=1}^{m} \sum_{i=1}^{k} \frac{1}{\delta + |x-\xi_i|} \right) \left( \sum_{q=1}^{m} \sum_{i=1}^{k} \frac{\delta^\tau}{(\delta + |x-\xi_i|^2)^{1+\tau}} \right) \geq c > 0.
\]
Let \(\zeta > 0\) be a small constant. We have
\[
\sup_{x \in \bigcup_{q=1}^{m} \bigcup_{i=1}^{k} B(x, \zeta)} \left( \sum_{q=1}^{m} \sum_{i=1}^{k} \frac{1}{\delta + |x-\xi_i|^2} \right) \geq c > 0.
\]
On the other side, since φ satisfies (11) by (20),
\[
\sup_{x \in \cup_{q=1}^{m}B(\xi_{0}^{q}, \frac{\epsilon}{2})} \left( \sum_{q=1}^{m} \sum_{i=1}^{k} \frac{1}{\delta + |x - \xi_{i}^{q}|} \right)^{-1} |\phi(x)|
\]
\[
= \sup_{x \in \cup_{q=1}^{m}B(\xi_{0}^{q}, \frac{\epsilon}{2})} \left( \sum_{q=1}^{m} \sum_{i=1}^{k} \frac{1}{\delta + |x - \xi_{i}^{q}|} \right)^{-1} |\phi(x)|
\]
\[
\lesssim \sup_{x \in \cup_{q=1}^{m}B(\xi_{0}^{q}, \frac{\epsilon}{2})} \sum_{q=1}^{m} \sum_{i=1}^{k} \frac{\delta^{\tau}}{|\delta + |x - \xi_{i}^{q}||^{\tau}}
\]
\[
\lesssim \sup_{x \in \cup_{q=1}^{m}B(\xi_{0}^{q}, \frac{\epsilon}{2})} \sum_{q=1}^{m} \sum_{i=1}^{k} \frac{\delta^{\tau}}{|\delta + |x - \xi_{i}^{q}||^{\tau}} + \delta^{\tau} k^{1+\tau},
\]
where the last inequality we used \( \frac{1}{k} \lesssim |\xi_{i}^{q} - \xi_{i_{0}}^{q}| - |x - \xi_{i_{0}}^{q}| \lesssim |x - \xi_{i}^{q}| \) for \( x \in \cup_{q=1}^{m}B(\xi_{0}^{q}, \frac{\epsilon}{2}) \).

It gives from \( \|\psi\|_{*} = 1 \)
\[
1 \lesssim \sup_{x \in \cup_{q=1}^{m}B(\xi_{0}^{q}, \frac{\epsilon}{2})} \sum_{q=1}^{m} \frac{\delta^{\tau}}{|\delta + |x - \xi_{i}^{q}||^{\tau}} \text{ for any } i_{0} \in \{1, ..., k\}
\]
which implies that there exist \( x_{*}, \xi_{0}^{0} \) and \( R > 0 \) such that \( |x_{*} - \xi_{0}^{0}| \leq \delta R \). Then, by (27)
\[
1 \lesssim \sup_{B_{3R}(\xi_{0}^{0})} \left( \sum_{q=1}^{m} \sum_{i=1}^{k} \frac{1}{\delta + |x - \xi_{i}^{0}|} \right)^{-1} |\phi(x)|
\]
\[
\lesssim \sup_{B_{3R}(\xi_{0}^{0})} \left( \frac{1}{\delta + |x - \xi_{0}^{0}|} \right)^{-1} |\phi(x)|
\]
\[
\lesssim (1 + R) \|\delta \phi\|_{L^{\infty}(B_{3R}(\xi_{0}^{0}))},
\]
which proves the claim.

**Step 5:** Let us prove that \( \delta \phi(\delta y + \xi_{0}^{0}) \to 0 \) in \( L^{\infty}(B_{R}(0)) \) and a contradiction arises.

Indeed, let us consider the function \( \tilde{\phi}(y) = \delta \phi(\delta y + \xi_{0}^{0}) \). Since \( \|\phi\|_{*} = 1 \), by the standard regularity theories and Arzelà-Ascoli theorem, we can assume that, up to a subsequence, \( \phi \to \phi^{*} \) in \( C^{1}(B_{R}(0)) \) for any \( R > 0 \) and \( \phi^{*} \) solves either
\[
-\Delta \phi^{*} - 3U^{2}\phi^{*} = 0, \text{ in } \mathbb{R}^{4}, \text{ if } \xi_{0}^{0} = \xi_{1}
\]
(28)
or
\[
-\Delta \phi^{*} - \beta U^{2}\phi^{*} = 0, \text{ in } \mathbb{R}^{4}, \text{ if } \xi_{0}^{0} \neq \xi_{1}.
\]
(29)
We claim that \( \phi^{*} = 0 \) and a contradiction arises because of what we proved in Step 4. In case (29), since \( \beta \not\in (\Lambda_{i})_{i \in \mathbb{N}} \), we immediately get that \( \phi^{*} = 0 \). In case (28), first we
observe that $\phi$ and also $\phi^*$ satisfy (9), (10) and so
\[
\int_{\mathbb{R}^4} \frac{1}{1 + |y|^2} \frac{y_2}{(1 + |y|^2)} \phi^* dy = 0 = \int_{\mathbb{R}^4} \frac{1}{1 + |y|^2} \frac{y_4}{(1 + |y|^2)} \phi^* dy,
\]
\[
\int_{\mathbb{R}^4} \frac{1}{1 + |y|^2} \frac{y_1}{(1 + |y|^2)} \phi^* dy = \int_{\mathbb{R}^4} \frac{1}{1 + |y|^2} \frac{y_3}{(1 + |y|^2)} \phi^* dy.
\]
Moreover, since $\int_{\mathbb{R}^4} \phi \sum_{i=1}^k (\chi U_{\delta,\xi_i})^2 Z^l_{\delta,\xi_i} dy = 0$, $l = 0, 1$, we have
\[
\int_{\mathbb{R}^4} \frac{1}{1 + |y|^2} \frac{y_1 + y_3}{(1 + |y|^2)} \phi^* dy = 0,
\]
and so $\int_{\mathbb{R}^4} 3U^2Z_l \phi^* dy = 0$, $l = 0, 1, 3$. Therefore $\phi^* = 0$. $\square$

2.5. Solving the non-linear problem. We are going to solve the non-linear problem (19) by using the standard fixed point theorem.

Proposition 2.6. Assume $\beta \notin (A_i)_{i \in \mathbb{N}}$. There exist $k_0 > 0$ and a constant $C > 0$ independent of $k$ such that for any even $k \geq k_0$, $\delta = e^{-\alpha k^2}$ with $d \in [d_0, d_1]$ and $\rho \in (r_0 - \vartheta, r_0 + \vartheta)$ with $\vartheta > 0$ the problem (19) has a unique solution $\phi \in X$ satisfying
\[
\|\phi\|_* \lesssim \delta, \quad |q| \lesssim \delta^2, \quad l = 0, 1.
\]

Proof. The proof relies on a standard contraction mapping argument together with Proposition 2.4 and Lemma 2.7 below. $\square$

Lemma 2.7. Let $N(\phi)$ be defined as in (18). Then
\[
\|N(\phi)\|_{**} \lesssim (\|\phi\|_*^2 + k^2\|\phi\|_*^3 + \delta^2k^4\|\phi\|_*).
\]

Proof. First of all, since $\|\phi(\mathcal{S}_q x)\|_* = \|\phi\|_*$ and
\[
\sum_{l=1}^m \sum_{i=1}^k \frac{1}{\delta + |x - \xi_l|} = \sum_{p=1}^m \sum_{j=1}^k \frac{1}{\delta + |\mathcal{S}_q x - \xi_j|}, q = 2, \ldots, m,
\]
we get
\[
|N(\phi)| \lesssim \left[ \|\phi\|_*^3 \left( \sum_{q=1}^m \sum_{i=1}^k \frac{1}{\delta + |x - \xi_q|} \right)^3 + \|\phi\|_*^2 \sum_{q=1}^m W(\mathcal{S}_q x) \left( \sum_{q=1}^m \sum_{i=1}^k \frac{1}{\delta + |x - \xi_q|} \right)^2 \right] + \|\phi\|_* W \sum_{q=2}^m W(\mathcal{S}_q x) \left( \sum_{q=1}^m \sum_{i=1}^k \frac{1}{\delta + |x - \mathcal{S}_q^{-1} \xi_q|} \right) \right].
\]
By Lemma 2.1 and (21) again, for any \( x \in \mathbb{R}^4 \), let \( \omega = \frac{c}{2} \),
\[
\sum_{q=1}^{m} \sum_{i=1}^{k} \frac{\delta}{\alpha^2 + |x - \xi_i^q|^2} \left( \sum_{p=1}^{m} \sum_{j=1}^{k} \frac{1}{\delta + |x - \xi_j^p|} \right)^2 \lesssim \sum_{q=1}^{m} \sum_{i=1}^{k} \frac{\delta}{\alpha^2 + |x - \xi_i^q|^2} \left[ \frac{1}{(\delta + |x - \xi_i^q|)^2} + \left( \sum_{(p,j) \neq (q,i)} \frac{1}{\delta + |x - \xi_j^p|} \right)^2 \right] \lesssim \sum_{q=1}^{m} \sum_{i=1}^{k} \left[ \frac{\delta}{(\delta + |x - \xi_i^q|)^{4-\alpha}} + \frac{\delta}{(\delta + |x - \xi_j^p|)^{4-\alpha}} \right] \lesssim \sum_{q=1}^{m} \sum_{i=1}^{k} (\delta + |x - \xi_i^q|)^3 (1 + \delta^\alpha k^{2+\alpha}) \text{ for any } 1 < \alpha \leq 2,
\]
and similarly from (20)
\[
W \sum_{q=2}^{m} W(\mathcal{S}_q x) \left( \sum_{q=1}^{m} \sum_{i=1}^{k} \frac{1}{\delta + |x - \mathcal{S}_q^{-1} \xi_i^q|} \right) \lesssim \frac{k}{\delta^3} \left[ \sum_{j=1}^{k} \frac{1}{(1 + |\omega - \eta_j|)^3} \sum_{q=2}^{m} \sum_{i=1}^{k} \frac{1}{(1 + |\omega - \eta_i^q|)^2} + \sum_{j=1}^{k} \frac{1}{(1 + |\omega - \eta_j|)^2} \sum_{q=2}^{m} \sum_{i=1}^{k} \frac{1}{(1 + |\omega - \eta_i^q|)^2} \right] \lesssim \frac{k}{\delta^3} \cdot \delta^\alpha \cdot k \sum_{q=1}^{m} \sum_{i=1}^{k} \frac{1}{(1 + |\omega - \eta_i^q|)^{3-\alpha}} \lesssim \delta^\alpha k^2 \sum_{q=1}^{m} \sum_{i=1}^{k} \frac{1}{(\delta + |x - \xi_i^q|)^3} \text{ for any } 1 < \alpha \leq 2.
\]
Last but not least,
\[
\left( \sum_{q=1}^{m} \sum_{i=1}^{k} \frac{1}{\delta + |x - \xi_i^q|} \right)^3 \lesssim k^2 \sum_{q=1}^{m} \sum_{i=1}^{k} \frac{1}{(\delta + |x - \xi_i^q|)^3}.
\]
Collecting all the previous estimates the proof ends.

2.6. **Solving the reduced problem.** We need to find \((\delta, \rho)\) such that \(c_0 = c_1 = 0\) in the equations (19). We will follow the same strategy of [14]. In particular, we will find out a local Pohozaev’s identity for the non-local equation (12). This is the most
technical part of the paper.

First of all, arguing as in \[14\), Proposition 3.1, we prove that

**Proposition 2.8.** Let \( D_{\varepsilon} := \{ x : |r - r_0| \leq \varepsilon, \varepsilon \in (2\sigma, 5\sigma) \} \) for some \( \sigma > 0 \) (see (13)). Suppose that \((\delta, \rho)\) satisfies

\[
\int_{\mathbb{R}^4} (\mathcal{L}(\phi) - \mathcal{E} - \mathcal{N}(\phi)) \frac{\partial W}{\partial \delta} dx = 0, \tag{30}
\]

\[
\int_{D_{\varepsilon}} (\mathcal{L}(\phi) - \mathcal{E} - \mathcal{N}(\phi))(x, \nabla u) dx = 0, \tag{31}
\]

then \( c_0 = c_1 = 0 \).

Next, we compute the first order term in the expansion of the L.H.S. of (30).

**Proposition 2.9.** There exist positive constants \( a \) and \( b \) such that

\[
\int_{\mathbb{R}^4} (\mathcal{L}(\phi) - \mathcal{E} - \mathcal{N}(\phi)) \frac{\partial W}{\partial \delta} = k(-a\delta \ln \delta V(\rho) - b\frac{k^2}{\rho^2}) + o(\delta k^3). \tag{32}
\]

**Proof.** We have

\[
\int_{\mathbb{R}^4} (\mathcal{L}(\phi) - \mathcal{E} - \mathcal{N}(\phi)) \frac{\partial W}{\partial \delta} dx \]

\[
= \int_{\mathbb{R}^4} V(|x|)W \frac{\partial W}{\partial \delta} dx - \int_{\mathbb{R}^4} (W^3 + \Delta W) \frac{\partial W}{\partial \delta} dx \]

\[
+ \int_{\mathbb{R}^4} \left( -\Delta \phi - 3W^2 \phi + V(x)\phi - \beta(W + \phi) \sum_{q=2}^{m} (W(\mathcal{F}_q x) + \phi(\mathcal{F}_q x))^2 - \phi^3 - 3W \phi^2 \right) \frac{\partial W}{\partial \delta} dx.
\]

First we prove that

\[
\mathcal{M}_3 = o(\delta k^3). \tag{34}
\]
By (20) and Lemma 2.2 and the fact that \(|\frac{\partial W}{\partial \delta}| \lesssim \sum_{p=1}^{k} \frac{1}{\delta^2 + |x - \xi_p|^2}
abla W| \lesssim \sum_{p=1}^{k} \frac{1}{\delta^2 + |x - \xi_p|^2}
abla W| dx

\leq k^2 \int_{\mathbb{R}^4} \sum_{j=1}^{k} (\delta^2 + |x - \xi_j|^2)^2 \sum_{q=2}^{m} \frac{\delta^2}{(\delta^2 + |x - \xi_q|^2)^2} \sum_{i=1}^{l} \frac{1}{|\eta_j - \eta_i|^{\alpha}} \left( \frac{1}{(1 + |\omega - \eta_j|)^{8-\alpha}} + \frac{1}{(1 + |\omega - \eta_i|)^{8-\alpha}} \right) d\omega

\leq \delta^{-1} k^2 \int_{\mathbb{R}^4} \sum_{j=1}^{k} \sum_{q=2}^{m} \sum_{i=1}^{l} \frac{1}{|\eta_j - \eta_i|^{\alpha}} \left( \frac{1}{(1 + |\omega - \eta_j|)^{8-\alpha}} + \frac{1}{(1 + |\omega - \eta_i|)^{8-\alpha}} \right) d\omega

= o(\delta k^3) \text{ if we choose } \alpha \in (1, 4),

and from \(\|\phi\|_* = \|\phi(\mathcal{S}_q x)\|_*\), Proposition 2.6

\int_{\mathbb{R}^4} W(\phi^2 + \sum_{q=2}^{m} \phi^2(\mathcal{S}_q x)) \frac{\partial W}{\partial \delta} dx

\lesssim \|\phi\|_*^2 \int_{\mathbb{R}^4} \sum_{i=1}^{k} \frac{\delta^2}{\delta^2 + |x - \xi_i|^2} \sum_{q=1}^{m} \left( \sum_{p=1}^{k} \sum_{j=1}^{l} \frac{1}{\delta^2 + |x - \mathcal{S}_q^{-1} \xi_j|^2} \right)^2 \sum_{p=1}^{k} \frac{1}{\delta^2 + |x - \xi_p|^2}

\lesssim k \|\phi\|_*^2 \int_{\mathbb{R}^4} \frac{\delta}{(\delta^2 + |x - \xi_1|)^6}

= O(\delta^2 k^4),

and

\int_{\mathbb{R}^4} \left( \phi^3 + \phi \sum_{q=2}^{m} \phi^2(\mathcal{S}_q x) \right) \frac{\partial W}{\partial \delta} dx \lesssim k^4 \|\phi\|_*^3 \int_{\mathbb{R}^4} \frac{1}{(\delta^2 + |x - \xi_1|)^6} = O(\delta^2 k^4).

Moreover since \(V, W, \phi\) satisfy (11), by (25)-(26) and Proposition 2.6,

\int_{\mathbb{R}^4} (-\Delta \phi - 3W^2 \phi + V(x) \phi + \phi \sum_{q=2}^{m} W^2(\mathcal{S}_q x) \frac{\partial W}{\partial \delta}) dx

= k \int_{\mathbb{R}^4} (-\Delta \phi - 3W^2 \phi + V(x) \phi + \phi \sum_{q=2}^{m} W^2(\mathcal{S}_q x) Z_{q,\xi_1}^0) dx

= O(\delta k^{2+\tau}) + o(\delta) = O(\delta k^{2+\tau}) \text{ for some small } \tau > 0.

By all the above estimates (34) follows.
Next, let us estimate $M_1$.

$$M_1 = k \left[ \int_{\mathbb{R}^4} V(x) \chi^2 U_{\delta, \xi_1} \frac{\partial U_{\delta, \xi_1}}{\partial \delta} dx + \int_{\mathbb{R}^4} V(x) \chi^2 U_{\delta, \xi_1} \sum_{i=2}^{k} \frac{\partial U_{\delta, \xi_i}}{\partial \delta} dx \right]$$

$$\sim k V(\rho) \int_{|x - r_0| \leq \sigma} U_{\delta, \xi_1} \frac{\partial U_{\delta, \xi_1}}{\partial \delta} dx + \mathcal{O}(\delta^2 \ln k)$$

$$\sim k V(\rho) \delta \int_{B(0, \frac{\delta}{2})} -U(\langle y, \nabla U \rangle + U) dx + \mathcal{O}(\delta^2 \ln k)$$

$$\sim -a k V(\rho) \delta \ln \delta$$

where $a := c^2 > 0$ (see (4)). Indeed, taking $x = |\xi_1 - \xi_j| y + \xi_1$, then there exists $R > 0$ such that $|y| \leq Rk$ if $x \in \{ x : |x - r_0| \leq \sigma \}$ since $|\xi_1 - \xi_i| \geq \frac{\zeta}{k}$ for $i \neq 1$, and then

$$\int_{\mathbb{R}^4} V(x) \chi^2 U_{\delta, \xi_1} \sum_{i=2}^{k} \frac{\partial U_{\delta, \xi_i}}{\partial \delta} dx \lesssim \delta k \int_{\{ x : |x - r_0| \leq \sigma \}} \frac{1}{|x - \xi_1|^2} \frac{1}{|x - \xi_1|^2} dx$$

$$\lesssim \delta k \int_{B(0, Rk)} \frac{1}{|y|^2} \frac{1}{|y + \frac{\xi_1 - \xi_i}{|\xi_1 - \xi_i|}|} dy$$

$$\lesssim \delta k \left[ \int_{|y| \leq 2} \frac{1}{|y|^2} \frac{1}{|y + \frac{\xi_1 - \xi_i}{|\xi_1 - \xi_i|}|} dy + \int_{2 \leq |y| \leq Rk} \frac{1}{|y|^4} dy \right]$$

$$= \delta k (\mathcal{O}(1) + \mathcal{O}(\ln k)) = \mathcal{O}(\delta k \ln k).$$

Let us estimate $M_2$. by Taylor’s expansion for fixed small $\zeta > 0$ and for any $y \in B(0, \frac{\zeta}{k})$,

$$\sum_{i=2}^{k} U_{\delta, \xi_i}(\delta y + \xi_1)$$

$$= c \sum_{i=2}^{k} \frac{\delta}{|\xi_1 - \xi_i|^2} \left[ 1 - \frac{\delta^2 + \delta^2 |y|^2 - 2\delta \langle y, \xi_1 - \xi_i \rangle}{|\xi_1 - \xi_i|^2} \right]$$

$$+ \mathcal{O}\left( \left( \frac{\delta^2 + \delta^2 |y|^2 - 2\delta \langle y, \xi_1 - \xi_i \rangle}{|\xi_1 - \xi_i|^2} \right)^2 \right).$$
\[ M_2 \sim \int_{\text{supp}(\chi)} \left( \left( \sum_{i=1}^{k} U_{\delta, \xi_i} \right)^3 - \sum_{i=1}^{k} \frac{\partial U_{\delta, \xi_i}}{\partial \delta} \right) \sum_{j=1}^{k} \frac{\partial U_{\delta, \xi_j}}{\partial \delta} \, dx \]

\[ \sim k \int_{\text{supp}(\chi)} \left( \left( \sum_{i=1}^{k} U_{\delta, \xi_i} \right)^3 - \sum_{i=1}^{k} \frac{\partial U_{\delta, \xi_i}}{\partial \delta} \right) \, dx \]

\[ \sim 3k \int_{\text{supp}(\chi) \cap \mathbb{R}^4} \sum_{i=2}^{k} U_{\delta, \xi_i}^2 U_{\delta, \xi_i} \frac{\partial U_{\delta, \xi_i}}{\partial \delta} + \sum_{i=2}^{k} U_{\delta, \xi_i} U_{\delta, \xi_i}^2 \frac{\partial U_{\delta, \xi_i}}{\partial \delta} + \sum_{i \neq j, i \neq 1} U_{\delta, \xi_i}^2 U_{\delta, \xi_j} \frac{\partial U_{\delta, \xi_j}}{\partial \delta} \, dx \]

\[ \sim 3\delta k \sum_{i=2}^{k} \frac{1}{|\xi_1 - \xi_i|^2} \int_{B(0, \frac{\xi}{\delta k})} -U^2(\langle y, \nabla U \rangle + U) \, dy + O(\delta^3 k^7) \]

\[ \sim 3\delta k \sum_{i=2}^{k} \frac{1}{|\xi_1 - \xi_i|^2} \int_{B(0, \frac{\xi}{\delta k})} \frac{1}{3} U^3 \, dy \]

\[ \sim b_0 \frac{k^3}{\rho^2} \quad (35) \]

where \( b := C_2 \int_{\mathbb{R}^4} U^3 \, dx > 0 \) (see (21)). Indeed, we have

\[ \sum_{i=2}^{k} \left[ \int_{B(\xi_i, \frac{\xi}{\delta k})} U_{\delta, \xi_i}^2 \frac{\partial U_{\delta, \xi_i}}{\partial \delta} + \int_{(B(\xi_i, \frac{\xi}{\delta k}) \cup B(\xi_1, \xi))^{c}} U_{\delta, \xi_i}^2 \frac{\partial U_{\delta, \xi_i}}{\partial \delta} \right] \]

\[ \lesssim \delta^5 k^7 \int_{B(0, \frac{\xi}{\delta k})} \frac{1}{1 + |y|^2} \, dy + \delta k^3 \int_{B^c(0, \frac{\xi}{\delta k})} \frac{1}{1 + |y|^6} \, dy \]

\[ \lesssim \delta^3 k^5 \]

and using Lemma 2.1, it holds

\[ \int_{\text{supp}(\chi) \cap \mathbb{R}^4} \sum_{i=2}^{k} U_{\delta, \xi_i} U_{\delta, \xi_i}^2 \frac{\partial U_{\delta, \xi_i}}{\partial \delta} \, dx \]

\[ \lesssim \frac{1}{\delta} \sum_{i=2}^{k} \int_{|r - r_0| \leq \frac{\xi}{\delta}} \frac{1}{(1 + |\omega - \eta_i|)^4} \left( \frac{1}{(1 + |\omega - \eta_i|)^4} \right) \, d\omega \]

\[ \lesssim \frac{1}{\delta} \sum_{i=2}^{k} \int_{|r - r_0| \leq \frac{\xi}{\delta}} \frac{1}{|\eta_i - \eta_i|^4} \left( \frac{1}{(1 + |\omega - \eta_i|)^4} + \frac{1}{(1 + |\omega - \eta_i|)^4} \right) \, d\omega \]

\[ \lesssim \delta^3 k^4 |\ln \delta|, \]
\[ \int_{|r-r_0| \leq \sigma} U_{\delta,\xi_1}^2 U_{\delta,\xi_j} \frac{\partial U_{\delta,\xi_j}}{\partial \delta} dx \]
\[ \lesssim \frac{1}{\delta} \sum_{i \neq j, i,j \neq 1} \int_{|r-r_0| \leq \frac{\sigma}{\delta}} \frac{1}{|\eta_i - \eta_j|^2} \left( \frac{1}{1 + |\omega - \eta_i|^4} + \frac{1}{1 + |\omega - \eta_j|^4} \right) (1 + |\omega - \eta_1|^4)^2 d\omega \]
\[ \lesssim \delta^3 k^4 \int_{|r-r_0| \leq \frac{\sigma}{\delta}} \frac{1}{(1 + |\omega - \eta_i|^4)^4 + (1 + |\omega - \eta_j|^4)^4 + (1 + |\omega - \eta_1|^4)^4} d\omega \]
\[ \lesssim \delta^3 k^4 \ln \delta. \]

Finally, (32) follows by (33)-(35).

□

Now, let us compute the first order term in the expansion of the L.H.S. of (31).

**Proposition 2.10.** There exists a positive constant \( c \) such that

\[ \int_{D_\epsilon} (\mathcal{L}(\phi) - \mathcal{E} - \mathcal{N}(\phi)) \langle x, \nabla u \rangle dx = \frac{1}{2\rho} \frac{\partial (\rho^2 V(\rho))}{\partial \rho} \epsilon k^2 \ln \delta + o(\epsilon^2 k^3). \]  

(36)

**Proof.** Denote \( u_q(x) := u(\mathcal{S}_q x), q = 1, \ldots, m \) where \( u_1(x) = u(x) \). Recalling that

\[ \mathcal{L}(\phi) - \mathcal{E} - \mathcal{N}(\phi) = -\Delta u + V(x)u - u^3 - \beta u \sum_{q=2}^m u^2(\mathcal{S}_q x). \]  

(37)

For the sake of clarity, we divide the proof into several steps.

**Step 1:** Let us prove that for any \( q = 2, \ldots, m \),

\[ \int_{D_\epsilon} \beta u \sum_{q=2}^m u^2(\mathcal{S}_q x) \langle x, \nabla u \rangle dx = \int_{D_\epsilon} \beta u_q \sum_{p \neq q} u_p^2(x, \nabla u_q) dx, \]  

(38)

First of all we point out that for any \( q = 2, \ldots, m \),

\[ u_q(x) = u(\mathcal{S}_q x) \]

\[ = u(\cos \frac{\pi(q-1)}{mk} x_1 - \sin \frac{\pi(q-1)}{mk} x_2, \sin \frac{\pi(q-1)}{mk} x_1 + \cos \frac{\pi(q-1)}{mk} x_2, \cos \frac{\pi(q-1)}{mk} x_3 + \sin \frac{\pi(q-1)}{mk} x_4, -\sin \frac{\pi(q-1)}{mk} x_3 + \cos \frac{\pi(q-1)}{mk} x_4) \]

\[ := u(y_1, y_2, y_3, y_4) \text{ where } y = \mathcal{S}_q x \]

and

\[ u(\mathcal{S}_{m+1} x) = u(\mathcal{S}_{k+1} x) = u(x), V(x) = V(|x|) = V(\mathcal{S}_q x), q = 2, \ldots, m. \]
Next, since
\[
\nabla_x u_q = \nabla_y u \cdot \mathcal{J}_q
\]
\[
\frac{\partial^2 u_q}{\partial x_1^2} = \cos^2 \frac{\pi(q-1)}{mk} \frac{\partial^2 u}{\partial y_1^2} + \cos \frac{\pi(q-1)}{mk} \sin \frac{\pi(q-1)}{mk} \frac{\partial^2 u}{\partial y_1 \partial y_2} + \sin \frac{\pi(q-1)}{mk} \cos \frac{\pi(q-1)}{mk} \frac{\partial^2 u}{\partial y_2^2} + \sin^2 \frac{\pi(q-1)}{mk} \frac{\partial^2 u_1}{\partial y_2^2} + \sin \frac{\pi(q-1)}{mk} \cos \frac{\pi(q-1)}{mk} \frac{\partial^2 u_1}{\partial y_2^2}
\]
\[
\frac{\partial^2 u_q}{\partial x_2^2} = \sin^2 \frac{\pi(q-1)}{mk} \frac{\partial u}{\partial y_1^2} - \cos \frac{\pi(q-1)}{mk} \sin \frac{\pi(q-1)}{mk} \frac{\partial u}{\partial y_1 \partial y_2} + \cos \frac{\pi(q-1)}{mk} \frac{\partial u}{\partial y_2^2} + \cos^2 \frac{\pi(q-1)}{mk} \frac{\partial u_1}{\partial y_2^2} + \cos \frac{\pi(q-1)}{mk} \frac{\partial u_1}{\partial y_2^2}
\]
\[
\frac{\partial^2 u_q}{\partial x_3^2} = \cos^2 \frac{\pi(q-1)}{mk} \frac{\partial u}{\partial y_3^2} - \cos \frac{\pi(q-1)}{mk} \sin \frac{\pi(q-1)}{mk} \frac{\partial u}{\partial y_3 \partial y_1} + \cos \frac{\pi(q-1)}{mk} \frac{\partial u}{\partial y_3 \partial y_2} + \cos^2 \frac{\pi(q-1)}{mk} \frac{\partial u_1}{\partial y_3 \partial y_2} + \cos \frac{\pi(q-1)}{mk} \frac{\partial u_1}{\partial y_3 \partial y_2}
\]
\[
\frac{\partial^2 u_q}{\partial x_4^2} = \sin^2 \frac{\pi(q-1)}{mk} \frac{\partial u}{\partial y_3^2} + \cos \frac{\pi(q-1)}{mk} \sin \frac{\pi(q-1)}{mk} \frac{\partial u}{\partial y_3 \partial y_1} + \cos \frac{\pi(q-1)}{mk} \frac{\partial u}{\partial y_3 \partial y_2} + \cos^2 \frac{\pi(q-1)}{mk} \frac{\partial u_1}{\partial y_3 \partial y_2} + \cos \frac{\pi(q-1)}{mk} \frac{\partial u_1}{\partial y_3 \partial y_2}
\]
we have
\[
-\Delta u(y) = -\Delta u_q(x),
\]
and
\[
\langle x, \nabla_x u_q \rangle = \langle x, \nabla_y u \cdot \mathcal{J}_q \rangle = \langle \mathcal{J}_q^{-1} y, \nabla_y u \cdot \mathcal{J}_q \rangle = \langle y, \nabla_y u \rangle
\]
Finally
\[
\int_{D_x} \beta u_q \sum_{p \neq q} u^2(x, \nabla u_q) dx = \int_{D_x} \beta u \sum_{q=2}^m u^2(\mathcal{J}_q y)(y, \nabla u) dy
\]
\[
= \int_{D_x} \beta u \sum_{q=2}^m u^2(\mathcal{J}_q x)(x, \nabla u) dx
\]
and the claim is proved.

**Step 2:** Let us prove that
\[
\int_{D_x} (-\Delta u + V(x)u - u^3 - \beta u \sum_{q=2}^m u^2(\mathcal{J}_q x))(x, \nabla u) dx
\]
\[
= \int_{D_x} -|\nabla u|^2 - V(x)u^2 - (V(x) + \frac{1}{2} \nabla V, x)u^2 + u^4 + \beta u^2 \sum_{q=2}^m u^2(\mathcal{J}_q x) dx
\]
\[
+ \mathcal{O} \left( \int_{\partial D_x} |\nabla \phi|^2 + |\phi|^2 + |\phi|^4 \right).
\]  
(39)
Indeed, for any $q = 1, \ldots, m$

\[
\int_{D_\epsilon} \beta u_q \sum_{p \neq q} u_{p}^2(x, \nabla u_q) \, dx
\]

\[
= \int_{D_\epsilon} \beta u_q \sum_{p \neq q} u_{p}^2 \sum_{l=1}^{4} x_l \frac{\partial u_q}{\partial x_l} \, dx = \int_{D_\epsilon} \beta \sum_{p \neq q} u_{p}^2 \sum_{l=1}^{4} x_l \frac{\partial (u_{q}^2)}{\partial x_l} \, dx
\]

\[
= \int_{\partial D_\epsilon} \frac{1}{2} \beta u_q \sum_{p \neq q} u_{p}^2(x, \nabla u_q) \, dS - \int_{D_\epsilon} \frac{1}{2} \beta u_q^2 \sum_{p \neq q} \left[ \sum_{p \neq q} u_{p}^2 + x_l \sum_{p \neq q} \frac{\partial u_p}{\partial x_l} \right] \, dx
\]

\[
= \int_{\partial D_\epsilon} \frac{1}{2} \beta \phi_q^2 \sum_{p \neq q} \phi_p^2(x, \nabla u_q) \, dS - \int_{D_\epsilon} 2 \beta u_q^2 \sum_{p \neq q} u_{p}^2 + \beta \sum_{q=1}^{m} \sum_{p \neq q} u_{p}^2 \sum_{q=1}^{m} \sum_{p \neq q} u_{p} \nabla u_p \, dx
\]

\[
= \int_{\partial D_\epsilon} \frac{1}{4} \beta \sum_{q=1}^{m} \phi_q^2 \sum_{p \neq q} \phi_{p}^2(x, \nabla u_q) \, dS - \int_{D_\epsilon} \beta \sum_{q=1}^{m} \sum_{p \neq q} u_{p}^2 \, dx
\]

since

\[
\int_{D_\epsilon} \beta \sum_{q=1}^{m} \sum_{p \neq q} u_{p}^2(x, \nabla u_q) \, dx = \int_{D_\epsilon} \beta \sum_{q=1}^{m} \sum_{p \neq q} u_{p} \nabla u_p \, dx.
\]

Then by (38)

\[
\int_{D_\epsilon} \beta u \sum_{q=2}^{m} u^2(\mathcal{F}_q x)(x, \nabla u_q) \, dx = \frac{1}{m} \int_{D_\epsilon} \beta \sum_{q=1}^{m} \sum_{p \neq q} u_{p}^2(x, \nabla u_q) \, dx
\]

\[
= \frac{1}{m} \left[ \int_{\partial D_\epsilon} \frac{1}{4} \beta \sum_{q=1}^{m} \phi_q^2 \sum_{p \neq q} \phi_p^2(x, \nabla u_q) \, dS - \int_{D_\epsilon} \beta \sum_{q=1}^{m} \sum_{p \neq q} u_{p}^2 \, dx \right]
\]

\[
= \frac{1}{4} \int_{\partial D_\epsilon} \beta \phi^2 \sum_{q=2}^{m} \phi^2(\mathcal{F}_q x)(x, \nabla u_q) \, dS - \int_{D_\epsilon} \beta u^2 \sum_{q=2}^{m} u^2_{q} \, dx. \quad (40)
\]

By standard computations taking into account that $u = \phi$ on $\partial D_\epsilon$ (due to the presence of the cut-off function (13)),

\[
\int_{D_\epsilon} (-\Delta u + V(x)u - u^3)(x, \nabla u) \, dx
\]

\[
= \int_{D_\epsilon} \left( -|\nabla u|^2 - \frac{1}{2} (4V(x) + \langle \nabla V, x \rangle) u^2 + u^4 \right) \, dx
\]

\[
+ O \left( \int_{\partial D_\epsilon} (|\nabla \phi|^2 + |\phi|^2 + |\phi|^4) \, dS \right) \quad (41)
\]

Finally, (39) follows by (40) and (41).
Step 3: Let us prove that
\[
\int_{D_\varepsilon} \left( |\nabla u|^2 + V(x)u^2 \right) \, dx = \int_{D_\varepsilon} \left( u^4 + \beta u^2 \sum_{q=2}^{m} u^2(\mathcal{I}_q x) \right) \, dx + O \left( \int_{\partial D_\varepsilon} (|\nabla \phi|^2 + |\phi|^2) \, dx \right) + o(\delta^2 k^3). \tag{42}
\]

First of all, testing (19) by \( u \) and using (37), we get
\[
\int_{D_\varepsilon} \left( |\nabla u|^2 + V(x)u^2 \right) \, dx = \int_{D_\varepsilon} \left( u^4 + \beta u^2 \sum_{q=2}^{m} u^2(\mathcal{I}_q x) \right) \, dx + \sum_{l=0}^{k} \sum_{i=1}^{k} c_l (\chi U_{\delta, \xi_i})^2 Z_{\delta, \xi_i}^l \, u \, dx \\
+ O \left( \int_{\partial D_\varepsilon} (|\nabla \phi|^2 + |\phi|^2) \, dx \right).
\]

Now we claim
\[
\int_{D_\varepsilon} \sum_{l=0}^{k} c_l (\chi U_{\delta, \xi_i})^2 Z_{\delta, \xi_i}^l \, W \, dx = o(\delta^2 k^3), \quad l = 0, 1.
\]

Indeed, it holds
\[
\int_{D_\varepsilon} \sum_{l=1}^{k} (\chi U_{\delta, \xi_i})^2 Z_{\delta, \xi_i}^0 \, W \, dx \sim k \int_{D_\varepsilon} (\chi U_{\delta, \xi_i})^3 Z_{\delta, \xi_i}^0 \, dx \sim k \int_{D_\varepsilon} \frac{1}{4} \frac{\partial (\chi U_{\delta, \xi_i})^4}{\partial \delta} \, dx \\
\sim k \left[ \int_{D_\varepsilon} \frac{1}{4} \frac{\partial U_{\delta, \xi_i}^4}{\partial \delta} \, dx + \int_{D_\varepsilon} \frac{1}{4} (\chi^4 - 1) \frac{\partial U_{\delta, \xi_i}^4}{\partial \delta} \, dx \right] \\
\sim k \frac{1}{4} \int_{D_\varepsilon} \frac{\partial}{\partial \delta} \sum_{l=1}^{k} U_{\delta, \xi_i}^l \, dx + \int_{D_\varepsilon} \frac{1}{4} (\chi^4 - 1) \frac{\partial U_{\delta, \xi_i}^4}{\partial \delta} \, dx \\
= O(\delta^3 k)
\]

and
\[
\int_{D_\varepsilon} \sum_{l=1}^{k} (\chi U_{\delta, \xi_i})^2 Z_{\delta, \xi_i}^1 \, W \, dx \sim k \int_{D_\varepsilon} \frac{1}{4} \frac{\partial (\chi U_{\delta, \xi_i})^4}{\partial \rho} \, dx \\
\sim k \left[ \int_{D_\varepsilon} \frac{1}{4} \frac{\partial U_{\delta, \xi_i}^4}{\partial \rho} \, dx + \int_{D_\varepsilon} \frac{1}{4} (\chi^4 - 1) \frac{\partial U_{\delta, \xi_i}^4}{\partial \rho} \, dx \right] \\
\sim k \int_{D_\varepsilon} \frac{1}{4} \frac{\partial U_{\delta, \xi_i}^4}{\partial \rho} \, dx + O(\delta^3 k) \\
= O(\delta^3 k)
\]

The claim follows since \( c_l = O(\delta^2) \), \( l = 0, 1 \) (see Proposition 2.6) and the orthogonality condition on \( \phi \).
Step 4: By (39) and (42)

\[
\int_{D_ε} (-\Delta u + V(x)u - u^3 - \beta u \sum_{q=2}^{m} u^2 (\mathcal{H}_q x))(x, \nabla u) dx
\]

\[
= \int_{D_ε} -(V(x) + \frac{1}{2} \nabla V(x), x)u^2 dx + O \left( \int_{\partial D_ε} (|\nabla \phi|^2 + |\phi|^2 + |\phi|^4) dS \right) + o(\delta^2 k^3)
\]

\[
= \int_{D_ε} \frac{1}{2} \frac{\partial (r^2 V(r))}{\partial r} u^2 dx + O \left( \int_{\partial D_ε} (|\nabla \phi|^2 + |\phi|^2 + |\phi|^4) dS \right) + o(\delta^2 k^3).
\]

Step 5: Let us prove that

\[
\int_{\partial D_ε} (|\nabla \phi|^2 + \phi^2 + \phi^4) dS = O(\delta^2 k^2).
\]

Indeed, for any bounded domain \( D \subset \mathbb{R}^4 \),

\[
\int_D \phi^4 dx \leq \|\phi\|^4 \int_D \left( \sum_{q=1}^{m} \sum_{i=1}^{k} \frac{1}{\delta + |x - \xi_q^i|} \right)^4 dx 
\]

\[
\lesssim \delta^4 \left[ \int_D \sum_{q=1}^{m} \sum_{i=1}^{k} \left( \frac{1}{\delta + |x - \xi_q^i|} \right)^4 dx + \sum_{(p,j) \neq (q,i)} \left( \frac{1}{\delta + |x - \xi_q^i|} \right)^{\alpha_1} \left( \frac{1}{\delta + |x - \xi_j^p|} \right)^{\alpha_2} dx \right] 
\]

\[
\lesssim \delta^4 k \int_{B(0,\frac{3}{4})} \frac{1}{(1 + |y|)^4} dy + \int_D \frac{1}{\delta^4} \sum_{(p,j) \neq (q,i)} \left( \frac{\delta^\alpha}{|\xi_j^p - \xi_q^i|^\alpha} \right) dx 
\]

\[
\cdot \left( \frac{1}{1 + \frac{x - \xi_j^p}{\delta |4 - \alpha|}} + \frac{1}{1 + \frac{x - \xi_q^i}{\delta |4 - \alpha|}} \right) dx 
\]

\[
= O(\delta^4 \ln \delta |k|) \text{ where we choose } 0 < \alpha \ll 1 \text{ for any } (\alpha_1, \alpha_2) \text{ with } \alpha_1 + \alpha_2 = 4,
\]

and

\[
\int_D \phi^2 dx \leq \|\phi\|^2 \int_D \left( \sum_{q=1}^{m} \sum_{i=1}^{k} \frac{1}{\delta + |x - \xi_q^i|} \right)^2 dx 
\]

\[
\lesssim k^2 \delta^2 \int_D \left( \frac{1}{\delta + |x - \xi_q^i|} \right)^2 dx 
\]

\[
= O(\delta^2 k^2).
\]

Now, we multiply (19) by \( \phi \) and integrate over \( D_{4\sigma} \setminus D_{3\sigma} \) and taking into account that \( W = 0 \) in \( D_{5\sigma} \setminus D_{2\sigma} \) (because of the choice of the cut-off function (13)) we get

\[
\int_{D_{4\sigma} \setminus D_{3\sigma}} |\nabla \phi|^2 dx \lesssim \int_{D_{5\sigma} \setminus D_{2\sigma}} (\phi^2 + \phi^4) dx = O(\delta^2 k^2).
\]
So there exists \( \varepsilon \in (3\sigma, 4\sigma) \) such that
\[
\int_{\partial D_\varepsilon} |\nabla \phi|^2 dS = O(\delta^2 k^2)
\]
and \( \int_{\partial D_\varepsilon} \phi^2 + \phi^4 dS = O(\delta^2 k^2) \)
and the claim follows.

**Step 6:** There is a constant \( c > 0 \) such that
\[
\int_{D_\varepsilon} -\frac{1}{2r} \frac{\partial (r^2 V(r))}{\partial r} u^2 dx = k \left( \frac{1}{2\rho} \frac{\partial (\rho^2 V(\rho))}{\partial \rho} \right) c \delta^2 \ln \delta + o(\delta^2 |\ln \delta|)).
\] (44)

Indeed, by Lemma 2.1 and (21)
\[
\int_{D_\varepsilon} \sum_{i \neq j} U_{\delta, \xi_i} \cdot U_{\delta, \xi_j} dx
\]
\[
= k \int_{D_\varepsilon} \sum_{j \neq 1} U_{\delta, \xi_j} \cdot U_{\delta, \xi_j} dx
\]
\[
\lesssim k \delta^2 \int_{\mathbb{R}} \sum_{j \neq 1} \frac{1}{(1 + |\omega - \eta_1|^2 (1 + |\omega - \eta_j|^2 d\omega
\]
\[
\lesssim k \delta^2 \int_{\mathbb{R}} \sum_{j \neq 1} \left( \frac{1}{(1 + |\eta_1 - \eta_j|^4 + (1 + |\omega - \eta_j|^4 \right) d\omega
\]
\[
\lesssim \delta^2 k^{2+\tau} \text{ for small } \tau > 0
\]
and then
\[
\int_{D_\varepsilon} -\frac{1}{2r} \frac{\partial (r^2 V(r))}{\partial r} W^2 dx
\]
\[
= \int_{D_\varepsilon} -\frac{1}{2r} \frac{\partial (r^2 V(r))}{\partial r} \chi \sum_{i=1}^k (\chi U_{\delta, \xi_i})^2 dx + \int_{D_\varepsilon} -\frac{1}{2r} \frac{\partial (r^2 V(r))}{\partial r} \sum_{i \neq j} \chi U_{\delta, \xi_i} \cdot \chi U_{\delta, \xi_j} dx
\]
\[
= -\frac{1}{2\rho} \frac{\partial (\rho^2 V(\rho))}{\partial \rho} k \int_{D_\varepsilon} \chi^2 U_{\delta, \xi_i}^2 dx
\]
\[
+ k \int_{D_\varepsilon} \left( \frac{1}{2\rho} \frac{\partial (\rho^2 V(\rho))}{\partial \rho} - \frac{1}{2r} \frac{\partial (r^2 V(r))}{\partial r} \right) (\chi U_{\delta, \xi_i})^2 dx + O(\delta^2 k^{2+\tau})
\]
\[
= -\frac{1}{2\rho} \frac{\partial (\rho^2 V(\rho))}{\partial \rho} k \left[ \int_{D_\varepsilon} U_{\delta, \xi_i}^2 dx + \int_{D_\varepsilon} (\chi^2 - 1) U_{\delta, \xi_i}^2 dx \right]
\]
\[
+ O \left( k \int_{D_\varepsilon} |x| - \rho |U_{\delta, \xi_i}^2 dx \right) + O(\delta^2 k^{2+\tau})
\]
\[
= -\frac{1}{2\rho} \frac{\partial (\rho^2 V(\rho))}{\partial \rho} c \delta^2 \ln \delta + O(\delta^2 k) + O(\delta^2 k^{2+\tau}),
\]
where \( c := c^2 > 0 \) (see (4)).
Next by Proposition 2.6 and (43)
\[ \left| \int_{D_\varepsilon} \frac{1}{2r} \frac{\partial (r^2 V(r))}{\partial r} \phi^2 - \frac{1}{r} \frac{\partial (r^2 V(r))}{\partial r} W \phi dx \right| \]
\[ \lesssim \int_{D_\varepsilon} \phi^2 dx + \| \phi \|_* \int_{D_\varepsilon} \sum_{i=1}^k \chi_{U_\delta,\xi_i} \sum_{q=1}^m \sum_{j=1}^k \frac{1}{\delta + |x - \xi_q|} dx \]
\[ \lesssim \delta^2 k^2. \]
Finally (44) follows and the proof of (36) is complete. □

2.7. Proof of Theorem 1.1: completed. By (32), (36) and Proposition 2.8, the problem reduces to finding \((\delta, \rho)\) such that
\[ \begin{align*}
  -a \delta \ln V(\rho) - b \delta^2 \rho^2 &= o(\delta^3), \\
k \delta \rho^2 \ln \rho \frac{\partial (\rho^2 V(\rho))}{\partial \rho} &= o(\delta^2 k^2),
\end{align*} \]
which is equivalent to finding \(d = d_k > 0\) (remember that \(\delta = e^{-dk^2}\) for some \(d > 0\)) and \(\rho = \rho_k\) solutions of
\[ \begin{align*}
  adV(\rho) - \frac{b}{\rho^2} &= o(1), \\
  \frac{\partial (\rho^2 V(\rho))}{\partial \rho} &= o(1).
\end{align*} \]
Eventually, since \(r_0\) is a non-degenerate critical point of the function \(r^2 V(r)\) with \(V(r_0) > 0\), this last problem has a solution \(d_k \sim \frac{b}{ar_0 V(r_0)}\) and \(\rho_k \sim r_0\) as \(k\) is large enough. That concludes the proof. □

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