ON THE JENSEN FUNCTIONAL AND SUPERQUADRATICITY

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ABSTRACT. In this note we give a recipe which describes upper and lower bounds for the Jensen functional under superquadraticity conditions. Some results involve the Chebychev functional. We give a more general definition of these functionals and establish the analogue results.

1. Introduction

The object of this paper is to derive some results related to the Jensen functional in the framework of superquadratic functions. We are interested in finding bounds both for the discrete and continuous case.

For the reader’s convenience, let us briefly state known facts regarding the principal tools, the superquadraticity and the Jensen functional. See S. Abramovich and S. S. Dragomir [1] for details and proofs.

Definition 1 ([2, Definition 2.1]). A function $f$ defined on an interval $I = [0, a]$ or $[0, \infty)$ is superquadratic if for each $x$ in $I$ there exists a real number $C(x)$ such that
\begin{equation}
    f(y) - f(x) \geq f(|y - x|) + C(x)(y - x)
\end{equation}
for all $y \in I$.

We say that $f$ is a subquadratic function if $-f$ is superquadratic. The set of superquadratic functions is closed under addition and positive scalar multiplication.

Example 1 ([3]). The function $f(x) = x^p$, $p \geq 2$ is superquadratic with $C(x) = f'(x) = px^{p-1}$. Similarly, $g(x) = -(1 + x^{1/p})^p$, $p > 0$ is superquadratic with $C(x) = 0$. Also $h(x) = x^2 \log x$ with $C(x) = h'(x) = x(2 \log x + 1)$ is a superquadratic function (but not monotonic and not convex).

Example 2 ([11]). Some elementary functions are not superquadratic, such as $f(x) = x$ and $f(x) = \exp x$.

Lemma 1 ([2 Lemma 2.2]). Let $f$ be a superquadratic function with $C(x)$ defined as above.

(i) Then $f(0) \leq 0$.
(ii) If $f(0) = f'(0) = 0$, then $C(x) = f'(x)$, whenever $f$ is differentiable at $x > 0$.
(iii) If $f \geq 0$, then $f$ is convex and $f(0) = f'(0) = 0$.

Theorem 1 ([2 Theorem 2.3]). The inequality
\begin{equation}
    f \left( \int \varphi d\mu \right) \leq \int f(\varphi(s)) - f \left( \varphi(s) - \int \varphi d\mu \right) d\mu(s)
\end{equation}

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holds for all probability measures $\mu$ and all nonnegative, $\mu$-integrable functions $\varphi$ if and only if $f$ is superquadratic.

**Definition 2** ([1]). Let $f$ be a real valued function defined on an interval $I$, $x_1, ..., x_n \in I$ and $p_1, ..., p_n \in (0,1)$ such that $\sum_{i=1}^{n} p_i = 1$. The Jensen functional is defined by

\[ (1.3) \quad J(f, p, x) = \sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) \]

and the Chebychev functional is defined by

\[ (1.4) \quad T(f, p, x) = \sum_{i=1}^{n} p_i \left( x_i - \sum_{j=1}^{n} p_j x_j \right) f(x_i) . \]

See [6] and [10].

The discrete form of Theorem 1 is as follows.

**Proposition 1** ([1 Lemma 2]). Let $x_i \geq 0$, $i = 1, ..., n$, and $p_i > 0$, $i = 1, ..., n$, with $\sum_{i=1}^{n} p_i = 1$. If $f$ is superquadratic, then

\[ J(f, p, x) \geq \sum_{i=1}^{n} p_i f\left( x_i - \sum_{j=1}^{n} p_j x_j \right) . \]

**Theorem 2** ([1 Theorem 4]). Assume that $x_i \in I$, $i = 1, ..., n$, $p_i > 0$ are such that $\sum_{i=1}^{n} p_i = 1$ and $r_i > 0$ are such that $\sum_{i=1}^{n} r_i = 1$. We denote

\[ m = \min_{i=1, \ldots, n} \left\{ \frac{p_i}{r_i} \right\} , \quad M = \max_{i=1, \ldots, n} \left\{ \frac{p_i}{r_i} \right\} . \]

Then

\[ J(f, p, x) - mJ(f, r, x) \geq mf \left( \sum_{i=1}^{n} (r_i - p_i) x_i \right) + \sum_{i=1}^{n} (p_i - m r_i) f\left( x_i - \sum_{j=1}^{n} p_j x_j \right) \]

and

\[ M J(f, r, x) - J(f, p, x) \geq f \left( \sum_{i=1}^{n} (r_i - p_i) x_i \right) + \sum_{i=1}^{n} (Mr_i - p_i) f\left( x_i - \sum_{j=1}^{n} r_j x_j \right) \]

for all superquadratic functions $f$.

**Definition 3.** Assume that $x = (x_1, ..., x_n) \in I^n$, $p = (p_1, ..., p_n)$ are such that $p_i > 0$, $\sum_{i=1}^{n} p_i = 1$, $q = (q_1, ..., q_k)$ are such that $q_i > 0$, $\sum_{i=1}^{k} q_i = 1$ ($1 \leq k \leq n$). We define

\[ J_k(f, p, q, x) := \sum_{i_1, \ldots, i_k=1}^{n} p_{i_1} \ldots p_{i_k} f\left( \sum_{j=1}^{k} q_j x_{i_j} \right) - f\left( \sum_{i=1}^{n} p_i x_i \right) . \]

Obviously $J_1(f, p, q, x) = J(f, p, x)$. We quote now some results that we refine in the following section.
Proposition 2 ([1] Theorem 6). Let $x_i \geq 0$, $i = 1, \ldots, n$, and $p_i > 0$, $i = 1, \ldots, n$, such that $\sum_{i=1}^{n} p_i = 1$ and $q_i > 0$, $i = 1, \ldots, k$, $\sum_{i=1}^{k} q_i = 1$ ($1 \leq k \leq n$). If $f$ is superquadratic, then

$$\mathcal{J}_k (f, p, q, x) \geq \sum_{i_1, \ldots, i_k = 1}^{n} p_{i_1} \cdots p_{i_k} f \left( \sum_{j=1}^{k} q_{i_j} x_{i_j} - \sum_{j=1}^{n} p_j x_j \right).$$

Notice that the Proposition 2 is a slightly more general assertion than Proposition 1 above.

Theorem 3 ([1] Theorem 7). Assume that $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in I^n$, $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ are such that $p_i > 0$, $\sum_{i=1}^{n} p_i = 1$, $\mathbf{q} = (q_1, q_2, \ldots, q_k)$ are such that $q_i > 0$, $\sum_{i=1}^{k} q_i = 1$ ($1 \leq k \leq n$) and $\mathbf{r} = (r_1, r_2, \ldots, r_n)$ are such that $r_i > 0$, $\sum_{i=1}^{n} r_i = 1$. We denote

$$m = \min_{1 \leq i_1, \ldots, i_k \leq n} \left\{ \frac{p_{i_1} \cdots p_{i_k}}{r_{i_1} \cdots r_{i_k}} \right\} \quad \text{and} \quad M = \max_{1 \leq i_1, \ldots, i_k \leq n} \left\{ \frac{p_{i_1} \cdots p_{i_k}}{r_{i_1} \cdots r_{i_k}} \right\}.$$

If $f$ is superquadratic then

$$\mathcal{J}_k (f, \mathbf{p}, \mathbf{q}, \mathbf{x}) - m \mathcal{J}_k (f, \mathbf{r}, \mathbf{q}, \mathbf{x}) \geq mf \left( \sum_{i=1}^{n} (r_i - p_i) x_i \right)$$

$$+ \sum_{i_1, \ldots, i_k = 1}^{n} (p_{i_1} \cdots p_{i_k} - m r_{i_1} \cdots r_{i_k}) f \left( \sum_{j=1}^{k} q_{i_j} x_{i_j} - \sum_{j=1}^{n} p_j x_j \right)$$

and

$$M \mathcal{J}_k (f, \mathbf{r}, \mathbf{q}, \mathbf{x}) - \mathcal{J}_k (f, \mathbf{p}, \mathbf{q}, \mathbf{x}) \geq f \left( \sum_{i=1}^{n} (r_i - p_i) x_i \right)$$

$$+ \sum_{i_1, \ldots, i_k = 1}^{n} (Mr_{i_1} \cdots r_{i_k} - p_{i_1} \cdots p_{i_k}) f \left( \sum_{j=1}^{k} q_{i_j} x_{i_j} - \sum_{j=1}^{n} r_j x_j \right).$$

Results involving Jensen’s and Chebychev’s inequalities are sometimes stated in terms of probability measures rather than summation or Lebesgue integration. Then some of our results can be derived from the ones above applied to a product measure, as Gord Sinnamon pointed out during some useful discussions.

Section 3 contains a definition of such functionals and analogue results. Their study is done for the discrete and integral case, not in probabilistic terms. The distinction between summation and integration is not artificial, but useful for different areas of study as information theory and transport theory.

For convex, strong convex and superquadratic functions the interested reader can also find relevant results in [8] and [9].
2. Main results

2.1. More on the Jensen functional.

**Theorem 4.** Let $f$ be a superquadratic function defined on an interval $I = [0,a]$ or $[0,\infty)$, $x_1, x_2, \ldots, x_n \in I$ and $p_1, p_2, \ldots, p_n \in (0,1)$ such that $\sum_{i=1}^{n} p_i = 1$, $\lambda \in [0,1]$. Then

\[
\sum_{i=1}^{n} p_i f \left( (1 - \lambda) \sum_{j=1}^{n} p_j x_j + \lambda x_i \right) - f \left( \sum_{j=1}^{n} p_j x_j \right) \geq \left| \sum_{i=1}^{n} p_i f \left( \lambda \left| x_i - \sum_{j=1}^{n} p_j x_j \right| \right) \right|.
\]

**Proof.** Let $f$ be a superquadratic function with $C(x)$ defined as above. Then replacing $y$ by $(1 - \lambda) x + \lambda y$ in (2.1) we deduce the inequality

\[
f((1 - \lambda) x + \lambda y) - f(x) \geq f(\lambda |y - x|) + \lambda C(x)(y - x).
\]

In this inequality we put $x = \sum_{j=1}^{n} p_j x_j$ and $y = x_i$. Multiplying by $p_i$ and summing over $i$ we get the conclusion. \hfill \Box

For $\lambda = 1$ we recover the result of Proposition 1.

As an immediate consequence of this result, due to the convexity of positive superquadratic functions, we get the following lower bound of interest:

**Corollary 1.** Let $f \geq 0$ be a superquadratic function defined on an interval $I = [0,a]$ or $[0,\infty)$, $x_1, x_2, \ldots, x_n \in I$ and $p_1, p_2, \ldots, p_n \in (0,1)$ such that $\sum_{i=1}^{n} p_i = 1$, $\lambda \in [0,1]$. Then

\[
\mathcal{J}(f, p, x) \geq 2 \sum_{i=1}^{n} p_i f \left( \frac{1}{2} \left| x_i - \sum_{j=1}^{n} p_j x_j \right| \right).
\]

**Proof.** In (2.1) we consider $\lambda = \frac{1}{2}$:

\[
\sum_{i=1}^{n} p_i f \left( \frac{\sum_{j=1}^{n} p_j x_j + x_i}{2} \right) - f \left( \sum_{j=1}^{n} p_j x_j \right) \geq \sum_{i=1}^{n} p_i f \left( \frac{1}{2} \left| x_i - \sum_{j=1}^{n} p_j x_j \right| \right).
\]

Since by Jensen’s inequality one has

\[
\frac{1}{2} \left[ f \left( \sum_{j=1}^{n} p_j x_j \right) + f(x_i) \right] \geq f \left( \frac{\sum_{j=1}^{n} p_j x_j + x_i}{2} \right),
\]

we get

\[
\frac{1}{2} \sum_{i=1}^{n} p_i \left[ f \left( \sum_{j=1}^{n} p_j x_j \right) + f(x_i) \right] - f \left( \sum_{j=1}^{n} p_j x_j \right) \geq \sum_{i=1}^{n} p_i f \left( \frac{1}{2} \left| x_i - \sum_{j=1}^{n} p_j x_j \right| \right).
\]

This completes the proof. \hfill \Box

The interested reader can refine our last result by using refinements of Jensen’s inequality instead of the classic result.
2.2. The discrete case. Motivated by the above results, introduce in a natural other functionals.

Definition 4. Assume that we have a real valued function \( f \) defined on an interval \( I \), the real numbers \( p_{ij} \), \( i = 1, \ldots, k \) and \( j = 1, \ldots, n_i \) such that \( p_{ij} > 0 \), \( \sum_{j=1}^{n_i} p_{ij} = 1 \) for all \( i = 1, \ldots, k \) (we denote \( p_i = (p_{i1}, p_{i2}, \ldots, p_{in_i}) \)), \( x_i = (x_{i1}, x_{i2}, \ldots, x_{in_i}) \in I^{n_i} \) for all \( i = 1, \ldots, k \) and \( q = (q_1, q_2, \ldots, q_k) \), \( q_i > 0 \) such that \( \sum_{i=1}^{k} q_i = 1 \). We define the generalized Jensen functional by

\[
J_k (f, p_1, \ldots, p_k, q, x_1, \ldots, x_k) : = \sum_{j_1, \ldots, j_k = 1}^{n_1, \ldots, n_k} p_{1j_1} \cdots p_{kj_k} f \left( \sum_{i=1}^{k} q_i x_{ij_i} \right)
\]

and the generalized Chebychev functional by:

\[
T_k (f, p_1, \ldots, p_k, q, x_1, \ldots, x_k) = \sum_{j_1, \ldots, j_k = 1}^{n_1, \ldots, n_k} p_{1j_1} \cdots p_{kj_k} \sum_{i=1}^{k} q_i \left( x_{ij_i} - \sum_{j=1}^{n_i} p_{ij} x_{ij} \right) f \left( \sum_{i=1}^{k} q_i x_{ij_i} \right).
\]

We easily notice also that for \( k = 1 \) this definition reduces to Definition 2. In the following estimation is obtained:

Remark 1. If \( f \) is a convex function then we have

\[
\min_{1 \leq j_1 \leq n_1} \min_{1 \leq j_k \leq n_k} \left\{ \frac{p_{1j_1} \cdots p_{kj_k}}{r_{1j_1} \cdots r_{kj_k}} \right\} J_k (f, r_1, \ldots, r_k, q, x_1, \ldots, x_k)
\]

\[
\leq J_k (f, p_1, \ldots, p_k, q, x_1, \ldots, x_k)
\]

\[
\leq \max_{1 \leq j_1 \leq n_1} \max_{1 \leq j_k \leq n_k} \left\{ \frac{p_{1j_1} \cdots p_{kj_k}}{r_{1j_1} \cdots r_{kj_k}} \right\} J_k (f, r_1, \ldots, r_k, q, x_1, \ldots, x_k).
\]

In this paper, we investigate upper and lower bounds that we have if the function \( f \) is superquadratic.

Now we extend the earlier results. The following lemma describes the behaviour of the functional under the superquadraticity condition:

Lemma 2. Let \( p_i, x_i, q \) be as in Definition 4. If \( f \) is superquadratic then we have

\[
J_k (f, p_1, \ldots, p_k, q, x_1, \ldots, x_k) \geq \sum_{j_1, \ldots, j_k = 1}^{n_1, \ldots, n_k} p_{1j_1} \cdots p_{kj_k} f \left( \sum_{i=1}^{k} q_i x_{ij_i} - \bar{x} \right),
\]

where \( \bar{x} = \sum_{i=1}^{k} q_i \sum_{j=1}^{n_i} p_{ij} x_{ij} \) (we will keep this notation throughout this subsection).

Proof. Straightforward from the definition of superquadratic functions. \( \square \)

Using the same recipe as in the proof of Corollary 1 we get:
Corollary 2. Let $p_i$, $x_i$, $q$ be as in Definition 4. Let $f \geq 0$ be a superquadratic function defined on an interval $I = [0, a]$ or $[0, \infty)$, $\lambda \in [0, 1]$. Then

$$
\mathcal{J}_k (f, p_1, \ldots, p_k, q, x_1, \ldots, x_k) \geq 2 \sum_{j_1, \ldots, j_k} p_{ij} \cdots p_{kji} f \left( \frac{1}{2} \sum_{i=1}^k q_i x_{ij} - \bar{x} \right).
$$

The next result of the paper can be expressed as:

Theorem 5. Let $f$, $p_i$, $x_i$, $q$ be as in Definition 4 and the positive real numbers $r_{ij}$, $i = 1, \ldots, k$ and $j = 1, \ldots, n_i$ such that $\sum_{j=1}^{n_i} r_{ij} = 1$ for all $i = 1, \ldots, k$. We denote

$$
r_i = (r_{i1}, r_{i2}, \ldots, r_{in_i}) \quad \text{for all } i = 1, \ldots, k,
$$

$$
m = \min_{1 \leq j_1 \leq n_1} \min_{1 \leq j_k \leq n_k} \frac{p_{ij_1} \cdots p_{kji_k}}{r_{ij_1} \cdots r_{kji_k}},
$$

$$
M = \max_{1 \leq j_1 \leq n_1} \max_{1 \leq j_k \leq n_k} \frac{p_{ij_1} \cdots p_{kji_k}}{r_{ij_1} \cdots r_{kji_k}}.
$$

If $f$ is a superquadratic function, then:

$$
\mathcal{J}_k (f, p_1, \ldots, p_k, q, x_1, \ldots, x_k) = m \mathcal{J}_k (f, r_1, \ldots, r_k, q, x_1, \ldots, x_k)
$$

$$
\geq mf \left( \sum_{i=1}^k q_i \sum_{j=1}^{n_i} (r_{ij} - p_{ij}) x_{ij} \left| \sum_{i=1}^k q_i x_{ij} - \bar{x} \right| \right)
$$

$$
+ \sum_{j_1, \ldots, j_k=1}^{n_1, \ldots, n_k} (p_{ij_1} \cdots p_{kji_k} - m r_{ij_1} \cdots r_{kji_k}) \int \left( \sum_{i=1}^k q_i x_{ij} - \bar{x} \right).
$$

and

$$
M \mathcal{J}_k (f, r_1, \ldots, r_k, q, x_1, \ldots, x_k) - \mathcal{J}_k (f, p_1, \ldots, p_k, q, x_1, \ldots, x_k)
$$

$$
\geq f \left( \sum_{i=1}^k q_i \sum_{j=1}^{n_i} (r_{ij} - p_{ij}) x_{ij} \left| \sum_{i=1}^k q_i x_{ij} - \bar{x} \right| \right)
$$

$$
+ \sum_{j_1, \ldots, j_k=1}^{n_1, \ldots, n_k} (M r_{ij_1} \cdots r_{kji_k} - p_{ij_1} \cdots p_{kji_k}) \int \left( \sum_{i=1}^k q_i x_{ij} - \bar{x} \right).\]

Proof. We prove only the first inequality. Obviously

$$
\mathcal{J}_k (f, p_1, \ldots, p_k, q, x_1, \ldots, x_k) - m \mathcal{J}_k (f, r_1, \ldots, r_k, q, x_1, \ldots, x_k)
$$

$$
= \sum_{j_1, \ldots, j_k=1}^{n_1, \ldots, n_k} (p_{ij_1} \cdots p_{kji_k} - m r_{ij_1} \cdots r_{kji_k}) \int \left( \sum_{i=1}^k q_i x_{ij} \right)
$$

$$
+ mf \left( \sum_{i=1}^k q_i \sum_{j=1}^{n_i} r_{ij} x_{ij} \right) - f \left( \bar{x} \right).
$$

Since

$$
\bar{x} = \sum_{j_1, \ldots, j_k=1}^{n_1, \ldots, n_k} (p_{ij_1} \cdots p_{kji_k} - m r_{ij_1} \cdots r_{kji_k}) \sum_{i=1}^k q_i x_{ij} + m \sum_{i=1}^k q_i \sum_{j=1}^{n_i} r_{ij} x_{ij},
$$
we conclude by Lemma 2 that
\[
\mathcal{J}_k (f, p_1, \ldots, p_k, q, x_1, \ldots, x_k) - m \mathcal{J}_k (f, r_1, \ldots, r_k, q, x_1, \ldots, x_k) \\
\geq \sum_{j_1, \ldots, j_k=1}^{n_1, \ldots, n_k} (p_{1j_1} \ldots p_{kj_k} - m r_{1j_1} \ldots r_{kj_k}) f \left( \sum_{i=1}^{k} q_i x_{ij_i} - \bar{x} \right)
\]
\[+ m f \left( \left| \sum_{i=1}^{k} q_i \sum_{j=1}^{n} r_{ij} x_{ij} - \bar{x} \right| \right)
\]
\[= \sum_{j_1, \ldots, j_k=1}^{n_1, \ldots, n_k} (p_{1j_1} \ldots p_{kj_k} - m r_{1j_1} \ldots r_{kj_k}) f \left( \sum_{i=1}^{k} q_i x_{ij_i} - \bar{x} \right)
\]
\[+ m f \left( \left| \sum_{i=1}^{k} q_i \sum_{j=1}^{n} (r_{ij} - p_{ij}) x_{ij} \right| \right).
\]

The proof of the other inequality goes likewise and we omit the details. □

The following particular case is of interest.

**Remark 2.** Let \( p_1 = \ldots = p_k = p \) and \( x_1 = \ldots = x_k = x \). In this case we see that Lemma 4 and Theorem 6 are recovering Proposition 2, respectively Theorem 5. Also for \( k = 1 \) Lemma 5 and Theorem 6 recapture Proposition 1, respectively Theorem 4.

According to [2, Lemma 2.2], if a superquadratic function is also nonnegative then it is convex. We may conclude that in this particular case Theorem 6 is a refinement of the result stated in Remark 1.

Different results are obtained by using the Chebychev functional:

**Lemma 3.** Let \( f : [0, \infty) \to \mathbb{R} \). If there exist real numbers \( \bar{m}, \bar{M} \) such that \( \bar{m} \leq f \left( \sum_{i=1}^{k} q_i x_{ij_i} \right) \leq \bar{M} \), for all \( j_i \in 1, \ldots, n_i, i = 1, \ldots, k \), then

\[
(2.4) \quad |\mathcal{T}_k (f, p_1, \ldots, p_k, q, x_1, \ldots, x_k)| \leq \frac{\bar{M} - \bar{m}}{2} \sum_{j_1, \ldots, j_k=1}^{n_1, \ldots, n_k} p_{1j_1} \ldots p_{kj_k} \left| \sum_{i=1}^{k} q_i x_{ij_i} - \bar{x} \right|.
\]

This lemma is a discrete version of a result due to P. Cerone and S. S. Dragomir [4, Theorem 2]. See also [3, Lemma 5.58].

**Proof.** Notice that
\[
\left| f \left( \sum_{i=1}^{k} q_i x_{ij_i} \right) - \frac{\bar{M} + \bar{m}}{2} \right| \leq \frac{\bar{M} - \bar{m}}{2},
\]
for all \( j_i \in 1, \ldots, n_i, i = 1, \ldots, k \). Since
\[
\sum_{j_1, \ldots, j_k=1}^{n_1, \ldots, n_k} p_{1j_1} \ldots p_{kj_k} \left( \sum_{i=1}^{k} q_i x_{ij_i} - \bar{x} \right) = 0,
\]

we have

$$
\mathcal{T}_k (f, p_1, ..., p_k, q, x_1, ..., x_k)
= \sum_{j_1, ..., j_k = 1}^{n_1, ..., n_k} p_{j_1} ... p_{j_k} \left( \sum_{i=1}^{k} q_i x_{ij_i} - \bar{x} \right) \left( f \left( \sum_{i=1}^{k} q_i x_{ij_i} \right) - \frac{M + \hat{m}}{2} \right),
$$

whence it follows that

$$
\mathcal{T}_k (f, p_1, ..., p_k, q, x_1, ..., x_k)
\leq \sum_{j_1, ..., j_k = 1}^{n_1, ..., n_k} p_{j_1} ... p_{j_k} \left| \sum_{i=1}^{k} q_i x_{ij_i} - \bar{x} \right|
\leq \frac{M - \hat{m}}{2} \sum_{j_1, ..., j_k = 1}^{n_1, ..., n_k} p_{j_1} ... p_{j_k} \left| \sum_{i=1}^{k} q_i x_{ij_i} - \bar{x} \right|.
$$

Proof. We apply

$$
\mathcal{J}_k (f, p_1, ..., p_k, q, x_1, ..., x_k)
\leq \mathcal{T}_k (C, p_1, ..., p_k, q, x_1, ..., x_k)
- \sum_{j_1, ..., j_k = 1}^{n_1, ..., n_k} p_{j_1} ... p_{j_k} \left( \sum_{i=1}^{k} q_i x_{ij_i} - \bar{x} \right).
$$

(2.5)

and the inequality [24] in order to get the claimed result.

The proof of (2.5) can be found in [9] Theorem 3].

We close this subsection with a proposition that gives us an upper bound for the Jensen functional under the superquadraticity condition, via the above result on the Chebyshev functional.

**Proposition 3.** Let \( f : [0, \infty) \to \mathbb{R} \) be a superquadratic function. If for \( C(x) \) there exist real numbers \( \hat{m}, \hat{M} \) such that \( \hat{m} \leq C \left( \sum_{i=1}^{k} q_i x_{ij_i} \right) \leq \hat{M} \), for all \( j_1 \in 1, ..., n_i, i = 1, ..., k \), then we have:

$$
\mathcal{J}_k (f, p_1, ..., p_k, q, x_1, ..., x_k)
\leq \mathcal{T}_k (C, p_1, ..., p_k, q, x_1, ..., x_k)
- \sum_{j_1, ..., j_k = 1}^{n_1, ..., n_k} p_{j_1} ... p_{j_k} \left( \sum_{i=1}^{k} q_i x_{ij_i} - \bar{x} \right).
$$

This proposition extends a result due to S. Abramovich and S. S. Dragomir [11 Theorem 9]. The inequality (2.5) is establishing a connection between the Jensen functional and the Chebyshev functional and is interesting in itself.

### 2.3. The integral case.

In what follows we shall concentrate on the integral analogue of some of the results from the previous section. Let \( p_i (x) \) dx and \( r_i (x) \) dx, \( i = 1, ..., k \) be absolutely continuous measures, where \( p_i, r_i : [a, b] \subset (0, \infty) \to (0, \infty) \) are such that \( \int_{a}^{b} p_i (x) \) dx = 1, \( \int_{a}^{b} r_i (x) \) dx = 1. We also consider \( q = (q_1, q_2, ..., q_k) \), \( q_i > 0 \) with \( \sum_{i=1}^{k} q_i = 1 \). We define

$$
\mathcal{J}_k (f, p_1, ..., p_k, q) := \int_{[a,b]^k} f \left( \sum_{i=1}^{k} q_i x_i \right) \prod_{i=1}^{k} (p_i (x_i) \) dx_i \) - f \left( \sum_{i=1}^{k} q_i \int_{a}^{b} x p_i (x) \right) dx_i \right)
$$
and
\[ T_k(f, p_1, \ldots, p_k, q) = \int_{[a,b]^k} \sum_{i=1}^k q_i \left( x_i - \int_a^b x p_i(x) \, dx \right) f \left( \sum_{i=1}^k q_i x_i \right) \prod_{i=1}^k (p_i(x_i) \, dx_i) \]
for all positive integers \( k \).

Before we prove the main result, we need the following lemma providing an inequality that is interesting in itself as well. For the case of superquadratic non-negative functions (hence convex) this result is a refinement of Jensen’s inequality.

**Lemma 4** (the integral analogue of Lemma [2]. Assume that \( f \) is superquadratic. Then
\[ \mathcal{J}_k(f, p_1, \ldots, p_k, q) \geq \int_{[a,b]^k} f \left( \sum_{i=1}^k q_i x_i - \bar{x} \right) \prod_{i=1}^k (p_i(x_i) \, dx_i), \]
where
\[ \bar{x} = \sum_{i=1}^k q_i \int_a^b x p_i(x) \, dx \]
(we will keep this notation in this subsection).

**Proof.** Straightforward from the definition of superquadratic functions. \( \square \)

**Example 3.** A particularly interesting case is pointed out by assuming, for simplicity, that \( p_i(x) \, dx = dx/(b-a) \), \( i = 1, ..., k \), when
\[ \frac{1}{(b-a)^k} \int_{[a,b]^k} f \left( \sum_{i=1}^k q_i x_i \right) \prod_{i=1}^k (p_i(x_i) \, dx_i) \geq \frac{1}{(b-a)^k} \int_{[a,b]^k} f \left( \sum_{i=1}^k q_i x_i - \frac{a+b}{2} \right) \prod_{i=1}^k (p_i(x_i) \, dx_i) \]

**Remark 3.** For the case \( k = 1 \) the lemma gives us the following inequality
\[ \int_a^b f(x) \, dx \geq \int_a^b f \left( \int_a^b x p(x) \, dx \right) + \int_a^b f \left( \int_a^b x p(x) \, dx \right) p(x) \, dx \]
for every \( f \) superquadratic. This is the integral counterpart of Proposition [4] an example of [2] Theorem 2.3.

We derive the following result.

**Theorem 6** (the integral analogue of Theorem [3]. We denote
\[ m = \inf_{t,s \in [a,b], t \neq s} \left\{ \int_{[t,s]^k} \prod_{i=1}^k p_i(x_i) \, dx_i \right\} \]
and
\[ M = \sup_{t,s \in [a,b], t \neq s} \left\{ \int_{[t,s]^k} \prod_{i=1}^k r_i(x_i) \, dx_i \right\}. \]
If $f$ is superquadratic then

$$J_k (f, p_1, ..., p_k, q) - m J_k (f, r_1, ..., r_k, q)$$

$$\geq m f \left( \sum_{i=1}^{k} q_i \int_{a}^{b} x (p_i (x) - r_i (x)) \, dx \right)$$

$$+ \int_{[a, b]^k} f \left( \sum_{i=1}^{k} q_i x_i - \bar{x} \right) \prod_{i=1}^{k} ((p_i (x_i) - m r_i (x_i)) \, dx_i)$$

and

$$M J_k (f, r_1, ..., r_k, q) - J_k (f, p_1, ..., p_k, q)$$

$$\geq f \left( \sum_{i=1}^{k} q_i \int_{a}^{b} x (p_i (x) - r_i (x)) \, dx \right)$$

$$+ \int_{[a, b]^k} f \left( \sum_{i=1}^{k} q_i x_i - \bar{x} \right) \prod_{i=1}^{k} ((M r_i (x_i) - p_i (x_i)) \, dx_i).$$

Proof. We will prove the first inequality. Lemma 4 implies that

$$J_k (f, p_1, ..., p_k, q) - m J_k (f, r_1, ..., r_k, q)$$

$$= \int_{[a, b]^k} f \left( \sum_{i=1}^{k} q_i x_i \right) \prod_{i=1}^{k} ((p_i (x_i) - m r_i (x_i)) \, dx_i)$$

$$+ m f \left( \sum_{i=1}^{k} q_i \int_{a}^{b} x r_i (x) \, dx \right) - f \left( \sum_{i=1}^{k} q_i \int_{a}^{b} x p_i (x) \, dx \right)$$

$$\geq \int_{[a, b]^k} f \left( \sum_{i=1}^{k} q_i x_i - \bar{x} \right) \prod_{i=1}^{k} ((p_i (x_i) - m r_i (x_i)) \, dx_i)$$

$$+ m f \left( \sum_{i=1}^{k} q_i \int_{a}^{b} x (p_i (x) - r_i (x)) \, dx \right).$$

The proof of the second inequality goes likewise and has been omitted. \hfill \Box

Now we turn our attention to the Chebychev functional. By an essentially similar method as in the discrete case already discussed above, one can prove the following lemma.

**Lemma 5.** We consider a superquadratic function $f : [0, \infty) \to \mathbb{R}$. If there exist real numbers $\bar{m}, \bar{M}$ such that $\bar{m} \leq f(x) \leq \bar{M}$, for all $x \geq 0$, then we get

$$|T_k (f, p_1, ..., p_k, q)| \leq \frac{\bar{M} - \bar{m}}{2} \int_{[a, b]^k} \left| \sum_{i=1}^{k} q_i x_i - \bar{x} \right| \prod_{i=1}^{k} (p_i (x_i) \, dx_i).$$

This lemma can be used to point out our last result.

**Proposition 4** (the integral analogue of Proposition 3). Using the above notations, we also consider a superquadratic function $f : [0, \infty) \to \mathbb{R}$. If there exist real
numbers $\tilde{m}$, $\tilde{M}$ such that $\tilde{m} \leq C(x) \leq \tilde{M}$, for all $x \geq 0$, then we have:

$$J_k \left( f, p_1, \ldots, p_k, q \right) \leq \int_{[a,b]^k} \left( \frac{\tilde{M} - \tilde{m}}{2} \sum_{i=1}^{k} q_i x_i - \bar{x} \right) - f \left( \left| \sum_{i=1}^{k} q_i x_i - \bar{x} \right| \right) \prod_{i=1}^{k} \left( p_i(x_i) dx_i \right).$$

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