Universal R-matrix formalism for the spin Calogero-Moser system and its difference counterpart

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Abstract. The expression of the quantum Ruijsenaars-Schneider Hamiltonian is obtained in the framework of the dynamical $R$-matrix formalism. This generalizes to the case of $U_q(sl_n)$ the result obtained in [1] for $U_q(sl_2)$ which is the higher difference Lame operator. The same result was obtained in [2] using the “fusion” procedure. The distinctive starting point of the present paper is the universal dynamical $R$-matrix.

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1 Introduction

The connection between the finite gap integrability and the R-matrix formalism comes from the famous work \[3\] and the claim therein that the Lax representation

$$\dot{L} = [L, M]$$

is equivalent to the existence of a classical r-matrix satisfying the following fundamental property:

$$\{L_1, L_2\} = [r_{12}, L_1] - [r_{21}, L_2];$$

$r$ lies in the tensor product $\text{Mat}(n)^{\otimes 2}$, where $L_1 = L \otimes \text{id}$. This sets the path to quantization. For example, in \[4\], it was shown that in the case of the spin Calogero-Moser model one can treat the dynamical Yang-Baxter equation (Gervais-Neveu-Felder equation \[5\]) as the quantized r-matrix representation for this model. There the GNF equation in the form of Gervais-Neveu writes:

$$R_{12}(x)R_{13}(xq^{H_2})R_{23}(x) = R_{23}(xq^{H_1})R_{13}(x)R_{12}(xq^{H_3})$$

where $R(x)$ lies universally in $U_q(sl_n) \otimes U_q(sl_n)$ or in the tensor product of the representations. Also is known the L-variant form of this equation:

$$R_{12}(xq^{-H_3/2})L_{13}(x)L_{23}(x) = L_{23}(x)L_{13}(x)R_{12}(xq^{H_3/2}).$$

The formula above provides a machinery for the construction of the quantum integrable system, as was done in \[4\]. The main claim of this procedure is that the operators $I_n$ on some space of functions with values in some fixed representation of $U_q(sl_n)$ constructed by the formula

$$I_n = Tr_{1...n}[L_1(x)...L_n(x)\hat{R}_{12}(xq^{h(3,n)})...\hat{R}_{n-1,n}(x)]$$

where $\hat{R} = PR$, $P$ is the permutation operator in the tensor product and $h^{(k,l)} = \sum_{i=k}^{l} h^{(i)}$, leave the subspace of zero weight in the representation invariant. The restrictions of the operators $I_n$ to this subspace form a set of commuting operators.

The crucial ideas to obtain the integrable system are:

1. take the solution of the standard Yang-Baxter equation as the universal Drinfeld R-matrix \[3\] for $U_q(sl_n)$;
2. use the quasi-Hopf shift $F_{12}(x)$ \[7\] to obtain the quasi-Hopf structure on this algebra and the solution of the GNF equation in the form $F_{21}^{-1}(x)R_{12}F_{12}(x)$;
3. take as the L-operator the expression

$$L = q^{-(H_1+H_2/2)p}Rq^{H_2p/2}, \quad (1)$$

where $p = x\frac{\partial}{\partial x}$ in the tensor product of the representation $\rho \otimes \rho_q$, where $\rho$ is the matrix representation and $\rho_q$ is some quantum representation;
4. the set of commuting \( I_n \) operators form an integrable system.

This scheme was successively investigated in [1] for the case \( U_q(sl_2) \). For the spin-\( j \) representation on the zero-weight subspace the element \( I_1 \) was shown to be the Hamiltonian generalizing, in the case of spin-\( j \), the one-particle Ruijsenaars-Schneider system. The quantum problem was solved using the existence of a spin-shift operator. The constructed quantum Hamiltonians were shown to be, in the limit \( q \to 1 \), those of the Calogero-Moser system.

The main goal of the present paper is to generalize this construction for the case \( U_q(sl_n) \). This result can be treated as an exercise but the evident advantage of this approach is that it allows one to investigate the full representation, full algebra and not only the zero-weight space. It is likely to clarify the original Poisson structure of the spin RS system.

In the second part, I explain the method to construct the Drinfeld R-matrix, emphasizing the generalization to \( U_q(sl_n) \).

In the third part I explicitly calculate the dynamical \( R \)-matrix by using the linear equation satisfied by the quasi-Hopf shift.

In the fourth and last part I construct the Hamiltonian of the system.

2 Universal R-matrix.

The construction of the universal R-matrix arises from the notion of Drinfeld double for quasitriangular Hopf algebras. It is the case for \( U_q(sl_n) \), described by the set of generators \( f_{\alpha}, e_{\alpha}, h_{\alpha} \); for primitive roots \( \alpha_i \) we shall write them for simplicity \( f_i, e_i, h_i, i = 1, ..., n-1 \). They satisfy the following relations:

\[
[h_i, h_j] = \delta_{ij}, \quad [e_i, f_j] = \delta_{ij} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}} \\
[h_i, e_j] = (\delta_{ij} 2 - \delta_{i,j+1})e_j, \quad [h_i, f_j] = (-\delta_{ij} 2 + \delta_{i,j+1})f_j.
\]

The Serre’s relations in this case read:

\[
[e_i, e_j] = 0 \quad if |i - j| > 1 \quad [f_i, f_j] = 0 \quad if |i - j| > 1 \\
e_i e_{i+1} + e_{i+1} e_i = (q + q^{-1})e_i e_{i+1} e_i,
\]

and the same for the \( f_i \)'s.

For a fixed order of the positive roots one introduces canonical generators by requiring that, for \( \alpha, \beta, \alpha + \beta \) - positive roots such that \( \alpha < \alpha + \beta < \beta \), we have

\[
e_{\alpha+\beta} = e_{\alpha} e_{\beta} - q e_{\beta} e_{\alpha} \quad f_{\alpha+\beta} = f_{\beta} f_{\alpha} - q^{-1} f_{\alpha} f_{\beta}.
\]
We associate the elements $e_{i,i+1}, f_{i,i+1}$ to the primitive generators $e_i, f_i$ respectively, and extend this by induction as

$$
e_{ij} = e_{ik}e_{kj} - qe_{kje_{ik}}, \quad f_{ij} = f_{kj}f_{ik} - q^{-1}f_{ik}f_{kj} \quad \text{for } i < k < j.$$

Due to the Serre’s relations it does not depend on the choice of $k$. The order here is lexicographical, like $(12), (13), (1, n), (2, 3), ..., (n-1, n)$.

A Poincare-Birkhoff-Witt basis of $U_q(N_+)$ (resp.$U_q(N_-)$) is formed here by:

$$e^p = \prod_{\alpha \in \Phi^+} (e_{ij})^{p_{ij}} \quad \text{(resp.} f^p = \prod_{\alpha \in \Phi^+} (f_{ij})^{p_{ij}}\text{).}$$

(2)

The R-matrix can be expressed as:

$$R = K\hat{R}, \quad K = \prod_{j=1}^{n-1} q^{h_i\otimes h_i}, \quad \hat{R} = \prod_{\alpha \in \Phi^+, >} \hat{R}_\alpha,$$

where $\hat{R}_\alpha = \exp_q((q - q^{-1})e_\alpha \otimes f_\alpha)$. The q-exponential is defined as follows:

$$\exp_q(z) = \sum_{n=0}^{\infty} \frac{1}{[n]!} z^n,$$

where $[n] = (q^n - q^{-n})/(q - q^{-1})$.

We will be interested in calculating the R-matrix when the first or the second space is the space of matrix representation of $U_q(sl_n)$; then, in the series for the q-exponential, there is no terms with degrees of generators but 0 and 1, i.e. the formula for the R-matrix reads:

$$\hat{R} = \prod_{ij, >} (1 + (q - q^{-1})e_{ij} \otimes f_{ij}).$$

We will separately treat two cases:

1. matrix representation on the first space: $e_{ij}$ is just the matrix element $\delta_{ij}$ for $i < j$. Looking at the order in the product and noting that $\delta_{ij}\delta_{km} = 0$ if $(ij) > (km)$ we have

$$\hat{R} = 1 + (q - q^{-1}) \sum_{i < j} e_{ij} \otimes f_{ij};$$

2. matrix representation on the second space: $f_{ij}$ is then the matrix element $\delta_{ji}$ for $i < j$ and we have for the R-matrix:

$$\hat{R} = 1 + \sum_{i < j} (\prod_{m=1} c_{i_m-1i_m}...c_{i_1i_2} (q - q^{-1})^{m-1}) \otimes f_{ij}.$$

The inner summation is taken on all partitions $i = i_1 < i_2 < ... < i_m = j$. 

3
3 Quasi-Hopf shift.

The main representation of $U_q(sl_n)$ on the quantum space is a highest weight representation $(n,0,...,0)$. In the latter the weight spaces have the fundamental property of being one-dimensional. For example on the zero-weight space all the elements $f_{i_{m-1}i_m}...f_{i_1i_2}$ with the same $i_1,i_m$ are proportional. Looking for the representations of $U_q(sl_2)$ subalgebras corresponding to all positive roots we find

$$f_{i_{m-1}i_m}...f_{i_1i_2}v_0 = ([l+1]q^{-l})(m-1-i_m+i_1)f_{i_1i_m}^*v_0$$

where $f_{i_1}^* = f_{j-1,j}f_{j-2,j-1}...f_{i,i+1}$ and $v_0$ is the zero-weight vector. Introducing the element $e_{ij}^* = e_{j-1,j}e_{j-2,j-1}...e_{i,i+1}$ we have on the space generated by $f_{i_1i_m}^*v_0$

$$e_{i_{m-1}i_m}...e_{i_1i_2}v_0 = ([l]q^l)(m-1-i_m+i_1)e_{i_1i_m}^*v_0.$$ 

It is worth saying that those relations are preserved through respectively right and left multiplication by a positive root generator $e_{i_mj}$ and $f_{imj}$. It will be important for further calculations.

This structure of representation leads for example to the property that, in the case of matrix representation on the second space, the $R$-matrix reads:

$$\hat{R}_{21} = 1 + \sum_{i<j} f_{ij} \otimes e_{ij}^*(q^l)^{-1}j^{-i-1}. \quad (3)$$

The last fact needed from the representation is

$$e_{ij}^*f_{ij}^*v_0 = ([l][l+1])j^{-i}v_0. \quad (4)$$

The construction of the quasi-Hopf $F_{12}(x)$ shift is based on the main proposition of [6], which claims that it satisfies the linear equation:

$$F_{12}(x)B_2(x) = \hat{R}_{12}^{-1}B_2(x)F_{12}(x)$$

or its equivalent form:

$$\hat{R}_{12}F_{12}(x) = B_2(x)F_{12}(x)B_2(x)^{-1} \quad (5)$$

where $B(x) = q^{\sum_{j=1}^{n-1}(h_jh_j'-x_jh_j')}$ The element $F_{12}(x)$ must be of the form:

$$F(x) = \sum_{p,r} e^p \otimes f^r \phi_{p,r}(x)$$

where the summation is done over the Poincare-Birkhoff-Witt basis elements (2) with the same weight. Introducing the variables $y_j$ such that $x_i = y_{i+1} - y_i$ we have $Bf_{ij}B^{-1} = f_{ij}q^{y_j-y_{i-2}h_{ij}+2}$ where $h_{ij} = \sum_{k=i}^{<j} h_k$. Then, taking $F$ of the form:

$$F_{12}(x) = 1 + \sum_{i<j} e_{ij} \otimes f_{ij}^* \Phi_{ij}(x)$$
one finds
\[ B_2 F_{12} B_2^{-1} = 1 + \sum_{i<j} e_{ij} \otimes f_{ij}^* \Phi_{ij}(x) q^{y_j-y_i+2}, \]
where we have restricted the right-hand-side of the equation above to the zero-weight subspace. Treating alike its left-hand-side one obtains:
\[ \hat{R}_{12} F_{12}(x) = 1 + \sum_{i<j} e_{ij} \otimes (f_{ij}^* \Phi_{ij} + f_{ij}(q - q^{-1}) + (q^2 - 1) \sum_{k,i<k<j} f_{j-1,j} \cdots f_{k+1,k+2} (f_{k,k+1} f_{ik} - f_{i,k+1}) \Phi_{kj}). \]
Reducing the proportional terms we have:
\[ \hat{R}_{12} F_{12}(x) = 1 + \sum_{i<j} e_{ij} \otimes (f_{ij}^* (\Phi_{ij} + q_{i-j-1} (q - q^{-1}) + (q^2 - 1) \sum_{k,i<k<j} q_{i-k} (q_l - 1) \Phi_{kj})). \]
where \( q_l = [l+1] q^{-l} \). Comparing the two sides of (5) that we just evaluated yields an equation to be satisfied by the functions \( \Phi_{ij} \):
\[ \Phi_{ij}(q^{y_j-y_i+2} - 1) = q_{i-j-1} (q - q^{-1}) + \sum_{k,i<k<j} (q^2 - 1)(q_l - 1) q_{i-k} \Phi_{kj}. \]
It leads to the relations:
\[ \Phi_{ij} = \Phi_{i+1,j} q_l^{-1} \frac{q^{y_j-y_i+2} - q^{-2l}}{q^{y_j-y_i+2} - 1}, \quad \Phi_{j-1,j} = \frac{q - q^{-1}}{q^{y_j-y_i+2} - 1}. \]
To evaluate the element \( F_{21}^{-1} \) we rewrite (3) as follows:
\[ F_{21}^{-1} = B_1 F_{21}^{-1} B_1 \hat{R}_{21}. \]
Taking then \( F_{21}^{-1} \) in the form:
\[ F_{21}^{-1} = 1 + \sum_{i<j} \Psi_{ij} f_{ij} \otimes e_{ij}^* \]
we get
\[ B_1 F_{21}^{-1} B_1^{-1} = 1 + \sum_{i<j} \Psi_{ij} q^{y_j-y_i} f_{ij} \otimes e_{ij}^*, \]
and, plugging the expression (3) for \( R_{21} \) in (7) gives
\[ B_1 F_{21}^{-1} B_1^{-1} \hat{R}_{21} = 1 + \sum_{i<j} f_{ij} \otimes e_{ij}^* (\Psi_{ij} q^{y_j-y_i} + (q - q^{-1})(q^l[l]^{-1})^{j-i-1}
+ \sum_{k,i<k<j} \Psi_{kj} q^{y_j-y_i} (q - q^{-1})(q^l[l]^{-1})^{k-i-1}). \]
Identifying the two sides of (4) gives a system of equations for the functions $\Psi_{ij}$:

$$
\Psi_{ij}(1 - q^{y_j-y_i}) = (q - q^{-1})(q^{l}[l]^{-1})^{j-i-1} + \sum_{k,i<k<j} \Psi_{kj}q^{y_j-y_k}(q - q^{-1})(q^{l}[l]^{-1})^{k-i-1}.
$$

Solving this system we get:

$$
\Psi_{ij} = q^{-l}[l]^{-1} \frac{q^{y_j-y_i+1} - q^{2l}}{q^{y_j-y_i} - 1} \Psi_{i+1,j}, \quad \Psi_{j-1,j} = \frac{q - q^{-1}}{1 - q^{y_j-y_{j-1}}}. \quad (8)
$$

4 The Hamiltonian.

In this last step all calculations are performed in the matrix representation on the first space and in the zero weight subspace of the $(n,0,...,0)$-highest weight representation on the quantum space. We are going to calculate the product:

$$
F_{21}^{-1}K\hat{R}_{12}F_{12}
$$

where

$$
K\hat{R}_{12}F_{12}(x) = 1 + \sum_{i < j} e_{ij} \otimes f_{ij}^* \Phi_{ij}(y)q^{y_j-y_i+1}.
$$

Because we will take traces, only the diagonal elements of $R_{12}(x)$ matter. They can be expressed as:

$$
R_{12}(y)_{jj} = 1 + \sum_{i < j} e_{ij}^* f_{ij}^* \Phi_{ij}(y)\Psi_{ij}(y)q^{y_j-y_i+1},
$$

which can be simplified by using equation (4) into:

$$
R_{12}(y)_{jj} = 1 + \sum_{i < j} \Phi_{ij}(y)\Psi_{ij}(y)q^{y_j-y_i+1}([l][l+1])^{j-i}.
$$

The reduced form of this sum is

$$
R_{12}(y)_{jj}^k = 1 + \sum_{i < j} \Phi_{ij}(y)\Psi_{ij}(y)q^{y_j-y_i+1}([l][l+1])^{j-i}.
$$

Lemma 1. For $\tilde{R}_{jj}^k = R_{12}(y)_{jj}^k$ the following property is fulfilled:

$$
\tilde{R}_{jj}^k = \prod_{i=k}^{i<j} (1 - [l][l+1]\xi(y_j-y_i))
$$

where

$$
\xi(y) = \frac{(q - q^{-1})^2q^{y+1}}{(q^y - 1)(q^{y+2} - 1)}.
$$
Proof. For \( k = j - 1 \) it is true. We then proceed by induction. Supposing the claim valid for \( k < m \) leads to

\[
\tilde{R}_j^k = \tilde{R}_j^{k+1}(1 - [l][l + 1]\xi(y_j - y_k)),
\]
and it is then sufficient to prove that

\[
\Phi_{kj}\Psi_{kj}q^{y_j - y_k + 1}([l][l + 1])^{j-k} = -[l][l + 1]\xi(y_j - y_k) \prod_{i=k+1}^{l} (1 - [l][l + 1]\xi(y_j - y_i)).
\]

(9)

Carrying out the induction for (9) requires proving

\[
\Phi_{k-1,j}\Psi_{k-1,j}q^{y_k - y_{k-1}} = \frac{\xi(y_j - y_{k-1})}{\xi(y_j - y_k)}(1 - [l][l + 1]\xi(y_j - y_k)),
\]

which is straightforwardly achieved by plugging in it the expressions for the \( \Phi \)'s and \( \Psi \)'s obtained in (6) and (8). This proves lemma 1.

Lemma 2. The Hamiltonian reads:

\[
H = I_1 = \sum_{i=1}^{n} e^{p_i} \prod_{k<i} (1 - [l][l + 1]\xi(y_i - y_k))
\]

where \( p_i = 2 \frac{\partial}{\partial y_i} \).

Proof. The main thing to prove is that \( q^H = q^{\sum \tilde{p}_i h^i} \) of (9) in the matrix representation is just the diagonal matrix with elements \( e^{p_i}, p_i \) as above. The variables used in (9) are \( \tilde{x}_i = q^{-\frac{y_i + 1 - y_l}{2}} \). There, the authors introduce the operator \( \tilde{p}_i = \tilde{x}_i \frac{\partial}{\partial \tilde{x}_i} \). The link between the \( p_i \)'s of this paper and the \( \tilde{p}_i \)'s is \( q^{\tilde{p}_i} = e^{p_{i+1} - p_i} \). In this representation the elements \( h^{i-1} - h^i, h^0 = h^n = 0 \), are just the diagonal \( \delta_{ii} \)'s. So

\[
q^{\sum \tilde{p}_i h^i} = e^{\sum h^i(p_{i+1} - p_i)} = e^{\sum p_i(h^{i-1} - h^i)} = \sum \delta_{ii} e^{p_i}.
\]

Introducing the variables \( t_i = y_i/2, t = y/2 \) we have

\[
\xi(y) = \frac{(q - q^{-1})^2}{(q^t - q^{-t})(q^{t+1} - q^{t-1})}.
\]

Conjugating the Hamiltonian

\[
\hat{H}_f = f H f^{-1}
\]

where the function \( f(t_1, ..., t_n) = \prod_{i<k} g(t_i - t_k) \) and \( g(t) = ((q^t - q^{-t})(q^{t+1} - q^{t-1}))^{-1} \) one obtains

\[
\hat{H}_f = \sum_i e^{\frac{\tilde{p}_i}{2}} \prod_{k \neq i} \frac{q^{t_i - t_k + l} - q^{-t_i + t_k - l}}{q^{t_i - t_k} - q^{-t_i + t_k}}.
\]
The elliptic version of this Hamiltonian was obtained in \cite{2}. The similar Hamiltonian
\[ H_\pm = \sum_i (e^{i\theta_i} \pm e^{-i\theta_i}) \prod_{j \neq i} f(q_{ij}), \]
where
\[ f^2(q) = (1 - \frac{\sin^2(\pi \nu/kN)}{\sin^2(\pi q/k)}), \]
was obtained in \cite{8} in the framework of hamiltonian reduction for the group $G$ on the space $T^*\hat{G}$ where $G$ is the suitable affine Lie group.

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