Generalization Bounds for Stochastic Saddle Point Problems

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Abstract

This paper studies the generalization bounds for the empirical saddle point (ESP) solution to stochastic saddle point (SSP) problems. For SSP with Lipschitz continuous and strongly convex strongly concave objective functions, we establish an \(O(1/n)\) generalization bound by using a probabilistic stability argument. We also provide generalization bounds under a variety of assumptions, including the cases without strong convexity and without bounded domains. We illustrate our results in three examples: batch policy learning in Markov decision process, stochastic composite optimization problem, and mixed strategy Nash equilibrium estimation for stochastic games. In each of these examples, we show that a regularized ESP solution enjoys a near-optimal sample complexity. To the best of our knowledge, this is the first set of results on the generalization theory of ESP.

1 Introduction

Consider the stochastic saddle point (SSP) problem

\[
\text{(SSP)} \quad \min_{x \in X} \max_{y \in Y} \Phi(x, y) := \mathbb{E}[\Phi_\xi(x, y)], \tag{1}
\]

where \(X\) and \(Y\) are compact and convex sets, and \(\xi\) is a random variable. We denote the optimal solution to \(\text{(1)}\) as \((x^*, y^*)\). SSP problem finds a wide range of applications in machine learning, reinforcement learning, operations research and game theory. Many stochastic approximation (SA) algorithms have been proposed for approximating the SSP solution based on samples of \(\xi\), see e.g. [Natele et al. (2018); Nemirovski et al. (2009); Shalev-Shwartz and Zhang (2013); Xiao et al. (2019); Yan et al. (2019); Zhang and Xiao (2017); Zhao (2019)]. Most of the algorithms make primal-dual updates and enjoy convergence guarantees with appropriately chosen stepsizes.

In this paper, we provide understanding of \(\text{(1)}\) from a finite sample perspective. Consider the empirical counterpart of the SSP, which we refer to as the empirical saddle point (ESP) problem

\[
\text{(ESP)} \quad \min_{x \in X} \max_{y \in Y} \Phi_\xi(x, y) := \frac{1}{n} \sum_{i=1}^n \Phi_{\xi_i}(x, y), \tag{2}
\]

where \(\Gamma := \{\xi_1, ..., \xi_n\}\) is a collection of \(n\) i.i.d. samples of \(\xi\)’s. A natural way of estimating the optimal solution \((x^*, y^*)\) to \(\text{(1)}\) is to solve instead its empirical approximation given by \(\text{(2)}\). This approach is also known as sample average approximation (SAA), see [Shapiro et al. (2014)]. We denote by \((\hat{x}, \hat{y})\) the empirical saddle point (ESP) solution to problem \(\text{(2)}\). Based on a given set of samples, one can compute the ESP solution using any convergent algorithm for minimax optimization. In parallel to the generalization theory for empirical risk minimization [Vapnik (1992, 2006, 2013)], we aim to analyze the empirical saddle point and establish finite-sample generalization bounds.

1.1 Motivating Examples

Stochastic saddle points \(\text{(1)}\) are very common in machine learning, game theory, and operations research. A generalization theory for ESP would be useful for establishing generalization bounds for a number of machine learning tasks that are beyond empirical risk minimization. We will study three examples in this paper.

One example is batch policy learning for Markov Decision Process (MDP). For the infinite-horizon average-reward MDP, the policy optimization problem is equivalent to an SSP, known as the Bellman saddle point [Puterman (2014); Wang (2017)], given by

\[
\min_{x \in X} \max_{y \in Y} \Phi(x, y) := \langle y, r \rangle + \sum_{a \in \mathcal{A}} y_a^T (P_a - I)x, \tag{3}
\]

where \(x\) is the value function, \(y = \{y_a\}_{a \in \mathcal{A}}\) is the state-action occupancy measure, \(a\) and \(\mathcal{A}\) denote the action and action space respectively, \(P_a\) denotes the transition probability matrix under \(a\), \(r\) is the reward function (see Section 3 for...
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We will study the generalization properties of the empirical matrix game, where the expectations are taken over the sample set \(X\) and \(Y\). Another example is the stochastic composite optimization, which finds applications in off-policy evaluation and risk-averse optimization, of the form

\[
\min_{x \in X} \max_{y \in Y} r(x) + E\left[ A_x x - b_x \right],
\]

(4)

where \(A_x\) is a random matrix, \(b_x\) is a random vector, \(r\) is a convex loss, \(r\) is some regularizer; see [Nesterov 2007; Wang et al. 2017; Zhang and Xiao 2019] and references therein. By using the convex conjugate \(f^*\) of \(f\), we can reformulate the problem as an SSP:

\[
\min_{x \in X} \max_{y \in Y} r(x) + E\left[ A_x x - b_x \right] y - f^*(y).
\]

Thus a generalization theory for this SSP would lead to generalization bounds for the original composite optimization problem. The third example is the two-person stochastic matrix game

\[
\min_{x \in X} \max_{y \in Y} x^T E[A_x y]
\]

(5)

where the constraint sets are defined as \(\Delta_N \coloneqq \{ z \in \mathbb{R}^N : z \geq 0, \mathbf{1}^T z = 1 \}\), \(x, y\) are the mixed strategies of the two players, and \(A_x \in \mathbb{R}^{N \times N}\) is a stochastic payoff matrix. Based on sample payoffs from past plays, we can estimate the mixed strategy Nash equilibrium by solving a regularized version of the empirical matrix game. Our generalization bound will be used to evaluate the quality of the empirical Nash equilibrium learned from data.

1.2 Weak and Strong Generalization Measures

We will study the generalization properties of the empirical saddle point (ESP) solution \((\hat{x}, \hat{y})\) via two metrics. The first metric, referred to as the weak generalization measure \(^\wedge\), is defined as

\[
\Delta^w(\hat{x}, \hat{y}) := \max_{y \in Y} E[\Phi(\hat{x}, y)] - \min_{x \in X} E[\Phi(x, \hat{y})],
\]

(6)

where the expectations are taken over the sample set \(\{\xi_1, \ldots, \xi_n\}\). In some applications, one desires a stronger metric of optimality, which we refer to as the strong generalization measure \((SGM)\), given by

\[
\Delta^s(\hat{x}, \hat{y}) := E\left[ \max_{y \in Y} \Phi(\hat{x}, y) - \min_{x \in X} \Phi(x, \hat{y}) \right].
\]

(7)

The SGM is often referred to as the expected duality gap in the optimization literature. A third commonly used metric is \(d^2(\hat{x}, \hat{y}) := ||\hat{x} - x^*||^2 + ||\hat{y} - y^*||^2\), which is the squared distance between the ESP solution to the true saddle point solution. Note that here we allow the use of two different norms \(\| \cdot \|\) and \(\| \cdot \|\) to measure the distances in \(X\) and \(Y\), respectively. Suppose that \(\Phi(\cdot, \cdot)\) is \(\mu_x\)-strongly convex and \(\mu_y\)-strongly concave. Then

\[
\Delta^w(\hat{x}, \hat{y}) \geq \frac{\mu_x}{2} ||\hat{x} - x^*||^2 + \frac{\mu_y}{2} ||\hat{y} - y^*||^2
\]

\[
\geq \min\{\mu_x, \mu_y\} \cdot E\left[ d^2(\hat{x}, \hat{y}) \right].
\]

That is, we have \(\Delta^w(\hat{x}, \hat{y}) \geq \Omega\left(E[d^2(\hat{x}, \hat{y})]\right)\). Due to the Jensen’s inequality, we also have \(\Delta^w(\hat{x}, \hat{y}) \leq \Delta^s(\hat{x}, \hat{y})\). Therefore the SGM is strongest among the three, WGM is the second and \(d^2(\hat{x}, \hat{y})\) is the weakest, i.e.,

\[
\Omega\left(E[d^2(\hat{x}, \hat{y})]\right) \leq \Delta^w(\hat{x}, \hat{y}) \leq \Delta^s(\hat{x}, \hat{y})
\]

1.3 Main Results

In this paper, we establish the generalization bounds for solving SSP using the empirical saddle point solution under various assumptions. See Table 1 for an overview of our technical results. Contributions of the paper are three-folded:

- We establish a uniform stability argument for the ESP solution by extending the technique of Shalev-Shwartz et al. (2009). For SSP over compact domains that are Lipschitz continuous and strongly convex-strongly concave (SC-SC), we provide an \(O(1/n)\) bound for the WGM metric. With an additional assumption on gradient Lipschitz continuity, we also establish an \(O(1/n)\) bound for the SGM metric. Especially, the WGM bound has matched the state-of-the-art result in [Yan et al. 2020], though with slightly different measures and assumptions\(^\dagger\). Especially, when \(\mu_x \ll \mu_y\) and \(\ell_x \ll \ell_y\), our result provides a tighter bound.

- Further, we extend the generalization theory to SSP problems with unbounded domains or without the SC-SC property. By using a different analysis, we establish an \(O(1/n)\) generalization bound for the ESP solution even if the feasible regions are unbounded. We also provide a generalization bound for the regularized ESP problem.

- We illustrate the applications of our theory in three important learning tasks: the batch policy learning in MDP, the stochastic composite optimization, and the Nash equilibrium estimation. In each of these tasks, the empirical saddle point provides a conceptually simple estimator. Further, we use the generalization theory to show these estimators have

\[^\dagger\]Yan et al. (2020) proved result under a slightly stronger measure, yet it also requires a stronger assumption on the tails of the distribution.
Yan et al. (2019) designed a stochastic gradient method with Xiao et al. (2019); Zhang and Xiao (2017). The interested readers are also there exist a rich body of literatures on the generalization theory for solving stochastic convex optimization (SCO) by empirical risk minimization (ERM), namely, Du and Hu (2018); Shalev-Shwartz and Zhang (2013); Mehta (2016) derived a risk bound of $O(1/n)$. Under certain stronger conditions, a tighter $O(1/n^2)$ risk bound has been shown (Zhang et al. 2017).

In a weakly related work Dikkala et al. (2020), the authors introduce a min-max approach for estimating models with conditional moment restrictions. Under a different setting, Dikkala et al. (2020) studies the statistical properties of the ESP solutions in terms of some distance between the empirical and true solutions in the minimization side, which is analogous to $E[||\hat{x} - x^\star||]$ in our setting. To the authors’ best knowledge, there is no existing generalization bound for stochastic saddle point problems.

### 2 Generalization Bounds for Empirical Saddle Points

#### 2.1 Assumptions

In most of our analysis, we require that $\Phi$ is strongly convex and strongly concave (SC-SC).

**Assumption 1** (SC-SC objective function). $\exists \mu_x, \mu_y \geq 0$, s.t. for almost every $\xi$, $\Phi_{\xi}(x, y)$ is $\mu_x$-strongly convex under norm $\| \cdot \|$ and $\Phi_{\xi}(x, \cdot)$ is $\mu_y$-strongly concave under the norm $\| \cdot \|$. Namely, denote $\partial_x \Phi_{\xi}(x, y)$ and $\partial_y \Phi_{\xi}(x, \cdot)$ the subgradients and supergradients respectively, then for $\forall x, x' \in X, \forall y, y' \in Y$, $u \in \partial_x \Phi_{\xi}(x, y)$ and $v \in \partial_y \Phi_{\xi}(x, y)$.

| Table 1: Generalization bounds for stochastic saddle points (results from this paper) |
|---------------------------------|-----------------|-----------------|-----------------|
| $\Delta^w(\hat{x}, \hat{y})$    | $O\left( \frac{1}{\mu_x \mu_y} \cdot \left( \frac{\ell^2_x}{\mu_x} + \frac{\ell^2_y}{\mu_y} \right) \right)$ | Same as left | $O\left( \frac{\kappa^4}{\mu_x \mu_y} \cdot \left( \frac{\ell^2_x}{\mu_x} + \frac{\ell^2_y}{\mu_y} \right) \right)$ |
| $\Delta^x(\hat{x}, \hat{y})$    | $O\left( \frac{1}{\mu_x \mu_y} \cdot \left( \frac{\ell^2_x}{\mu_x} + \frac{\ell^2_y}{\mu_y} \right) \right)$ | Same as left | $O\left( \frac{\kappa^4}{\mu_x \mu_y} \cdot \left( \frac{\ell^2_x}{\mu_x} + \frac{\ell^2_y}{\mu_y} \right) \right)$ |

**The definition of these parameters can be found in the corresponding sections.**
it holds that
\[
\begin{cases}
    \Phi_\xi(x', y) \geq \Phi_\xi(x, y) + \langle u, x' - x \rangle + \frac{\nu_x}{2} \|x' - x\|^2, \\
    \Phi_\xi(x, y') \leq \Phi_\xi(x, y) + \langle v, y' - y \rangle - \frac{\nu_y}{2} \|y' - y\|^2.
\end{cases}
\]  
(9)

For convex analysis of strongly convex function under non-Euclidean norms, see [Shalev-Shwartz and Singer, 2007; Kakade et al., 2012] and references therein. We further assume that the feasible regions $X, Y$ are bounded and the objective function is Lipschitz continuous.

**Assumption 2 (Function Lipschitz continuity).** The feasible regions $X$ and $Y$ are compact convex sets. For almost every $\xi$, there exist constants $\ell_x(\xi)$ and $\ell_y(\xi, x)$ s.t.
\[
\begin{align*}
    |\Phi_\xi(x', y) - \Phi_\xi(x, y)| & \leq \ell_x(\xi, y') \|x' - x\|, \\
    |\Phi_\xi(x, y') - \Phi_\xi(x', y)| & \leq \ell_y(\xi, x) \|y' - y\|,
\end{align*}
\]  
(10)

for all $x, x' \in X$ and $y, y' \in Y$. To bound the WGM, we need to assume
\[
\begin{align*}
    (\ell_x^n)^2 & := \sup_{y \in Y} E_\xi [\ell_x^2(\xi, y)] < +\infty, \\
    (\ell_y^n)^2 & := \sup_{x \in X} E_\xi [\ell_y^2(\xi, x)] < +\infty.
\end{align*}
\]  
(11)

To bound the SGM, we assume
\[
\begin{align*}
    (\ell_x^m)^2 & := E_\xi [\sup_{y \in Y} \ell_x^2(\xi, y)] < +\infty, \\
    (\ell_y^m)^2 & := E_\xi [\sup_{x \in X} \ell_y^2(\xi, x)] < +\infty.
\end{align*}
\]  
(12)

Due to Jensen's inequality, $\ell_x^m \leq \ell_x^n$ and $\ell_y^m \leq \ell_y^n$ always hold.

In our analysis, Assumptions 1 and 2 only guarantee the $O(1/n)$ bound for the WGM metric. In order to prove an $O(1/n)$ bound for the stronger metric SGM, we will require additional smoothness of $\Phi$.

**Assumption 3 (Gradient Lipschitz continuity).** There exist constants $L_x$, $L_y$ and $L_{xy}$ s.t. for all $x, x' \in X$, $y, y' \in Y$, it holds that
\[
\begin{align*}
    \|\nabla_x \Phi_\xi(x, y) - \nabla_x \Phi_\xi(x', y)\|_x & \leq L_x \|x - x'\|, \\
    \|\nabla_y \Phi_\xi(x, y) - \nabla_y \Phi_\xi(x, y')\|_y & \leq L_y \|y - y'\|, \\
    \|\nabla_x \Phi_\xi(x, y) - \nabla_y \Phi_\xi(x, y')\|_x & \leq L_{xy} \|x - x'\|, \\
    \|\nabla_y \Phi_\xi(x, y) - \nabla_y \Phi_\xi(x', y)\|_y & \leq L_{xy} \|y - y'\|.
\end{align*}
\]

where $\|\cdot\|$ and $\|\cdot\|_a$ stand for the dual norms of $\|\cdot\|$ and $\|\cdot\|_a$ respectively.

Finally, we also study the case where $X$ and $Y$ are unbounded. Such unboundedness would invalidate Assumption 2 as well as the stability argument. To remedy this issue, we would replace Assumption 2 with the following assumption about the true optimal solution $(x^*, y^*)$.

**Assumption 4.** $\exists C > 0$ s.t. $E_\xi [\|\nabla \Phi_\xi(x^*, y^*)\|^2_2] \leq C$.

### 2.2 Main Results

We use the leave-one-out technique in [Shalev-Shwartz et al., 2009] to analyze the stability of the ESP solutions. Let $\Gamma := \{\xi_1, \ldots, \xi_n\}$ be a set of $n$ i.i.d. samples, and let $\xi'_i$ be another independent sample. We then define the perturbed sample set $\Gamma(i) = \Gamma \cup \{\xi'_i\} \setminus \{\xi_i\}$. That is, $\Gamma(i)$ is constructed by replacing just the $i$-th sample $\Gamma$ with another i.i.d. sample $\xi'_i$. For the sake of generality, instead of the ESP solution, we will establish the stability property of the regularized ESP (R-ESP) solution:

\[
\begin{align*}
    (R-ESP) \quad \min_{x \in X} \max_{y \in Y} & \Phi_\Gamma(x, y) + \Psi(x, y), \\
    \text{s.t.} & \quad \xi_i(x, y) = \Psi \left( \xi_i, \hat{\xi}_i \right), \quad \forall i \in \{1, \ldots, n\}.
\end{align*}
\]  
(13)

where $\Psi$ is a regularization function that is SC-SC and can be specified by the user. In particular, if we set $\Psi \equiv 0$, the R-ESP problem (13) is reduced to the ESP problem (2) where no regularization is applied.

**Lemma 1 (Stability property for R-ESP solution).** Suppose the regularization function $\Psi(\cdot, y)$ is $\nu_y$-strongly convex under norm $\|\cdot\|$ and $\Psi(x, \cdot)$ is $\nu_x$-strongly concave under norm $\|\cdot\|_a$, and there exists $R > 0$ s.t. $\|\Psi(x, y)\| \leq R$, $\forall (x, y) \in X \times Y$. Under the Assumptions 2, let $(\hat{x}, \hat{y})$ and $(\hat{x}_i, \hat{y}_i)$ be the solution to the R-ESP problem (13) with sample set $\Gamma$ and $\Gamma(i)$ respectively, and suppose $\mu_x + \nu_x > 0$, $\mu_y + \nu_y > 0$, then

\[
\begin{align*}
    \max \left\{ \frac{\mu_x + \nu_x}{\mu_x} \|\hat{x} - \hat{x}_i\|^2, \frac{\mu_y + \nu_y}{\mu_y} \|\hat{y} - \hat{y}_i\|^2 \right\} \\
    \leq \frac{1}{n} \left( \ell_x(\xi, \hat{y}_i) + \ell_y(\xi, \hat{y}_i) \right)^2 + \frac{\ell_x(\xi, \hat{x}_i) + \ell_y(\xi, \hat{x}_i)}{\mu_x + \nu_x} \|\hat{x} - \hat{x}_i\|^2 + \frac{\ell_y(\xi, \hat{y}_i)}{\mu_y + \nu_y} \|\hat{y} - \hat{y}_i\|^2
\end{align*}
\]

**Proof.** First, in parallel to $\hat{\Phi}_n$, we define
\[
\hat{\Phi}_{n,i}(x, y) = \frac{1}{n} \sum_{\xi \in \Gamma(i)} \Phi_\xi(x, y).
\]

Then we have
\[
\begin{align*}
    \Phi_n(\hat{x}_i, \hat{y}_i) + \Psi(\hat{x}_i, \hat{y}_i) - \Phi_n(\hat{x}_i, \hat{y}_i) - \Psi(\hat{x}_i, \hat{y}_i) \\
    = \frac{1}{n} \sum_{j=1}^n \left( \Phi_{\xi_j}(\hat{x}_j, \hat{y}_j) - \Phi_{\xi_j}(\hat{x}_j, \hat{y}_j) \right) + \Psi(\hat{x}_i, \hat{y}_i) - \Psi(\hat{x}_i, \hat{y}_i) \\
    = \frac{1}{n} \left( \Phi_{\xi_j}(\hat{x}_j, \hat{y}_j) + \Phi_{\xi_j}(\hat{x}_j, \hat{y}_j) \right) + \sum_{j=1}^n \left( \Phi_{\xi_j}(\hat{x}_j, \hat{y}_j) - \Phi_{\xi_j}(\hat{x}_j, \hat{y}_j) \right) \\
    = \frac{1}{n} \left( \Phi_{\xi_j}(\hat{x}_j, \hat{y}_j) - \Phi_{\xi_j}(\hat{x}_j, \hat{y}_j) + \Phi_{\xi_j}(\hat{x}_j, \hat{y}_j) - \Phi_{\xi_j}(\hat{x}_j, \hat{y}_j) \right) \\
    + \Psi(\hat{x}_i, \hat{y}_i) - \Psi(\hat{x}_i, \hat{y}_i) \\
    \leq \frac{\ell_x(\xi, \hat{y}_i) + \ell_y(\xi, \hat{y}_i)}{n} \|\hat{x} - \hat{x}_i\|^2 + \frac{\ell_x(\xi, \hat{y}_i) + \ell_y(\xi, \hat{y}_i)}{n} \|\hat{y} - \hat{y}_i\|^2
\end{align*}
\]
For the ESP problem, it is worth noting that by the McDiarmid’s inequality \cite{McDiarmid1989, McDiarmid1998}, the stability argument of Corollary 1 immediately results in an $O(1/\sqrt{n})$ generalization bound for SC-SC problems, which, however, is not tight. In Theorem 1 we establish a tighter $O(1/n)$ bound for SC-SC problems by using a more careful analysis.

**Lemma 2** (Generalization bound for R-ESP). Under the settings of Lemma 1, the R-ESP solution $(\hat{x}, \hat{y})$ satisfies

\[
\Delta^n(\hat{x}, \hat{y}) \leq 2\sqrt{\frac{2}{n} \left( \left( \frac{\ell_x^w}{\mu_x} \right)^2 + \left( \frac{\ell_y^w}{\mu_y} \right)^2 \right) + 2R},
\]

where the WGM $\Delta^n(\cdot, \cdot)$ is defined for the original unregularized SSP problem.

**Proof.** By the function Lipschitz continuity of Assumption 2 for any $1 \leq i \leq n$, and for any $x \in \mathcal{X}$ and $y \in \mathcal{Y}$,

\[
\Phi_{\xi_i}(\hat{x}_{i|y}), \Phi_{\xi_i}(x, \hat{y}_{i|y}) \leq \Phi_{\xi_i}(\hat{x}, \hat{y}) - \Phi_{\xi_i}(x, y) + \ell_x(\hat{x}, y) + \ell_y(x, \hat{y}) \tag{16}
\]

As a result, we have

\[
\frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \Phi(\hat{x}_{i|y}, y) - \Phi(x, \hat{y}_{i|y}) \right] + \mathbb{E} \left[ \Psi(\hat{x}, y) - \Psi(x, \hat{y}) \right] \tag{17}
\]

The step (a) is due to the fact that $(\hat{x}_{i|y}, \hat{y}_{i|y})$ is independent from $\xi_i$, and hence one can take the expectation over the $\xi_i$'s first. And the step (b) is due to (16). Then, because the distribution of $(\hat{x}_{i|y}, \hat{y}_{i|y})$ are the same as that of $(\hat{x}, \hat{y})$ for any $1 \leq i \leq n$. Therefore, the expectation term on the LHS of (17) can be simplified to

\[
\frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \Phi(\hat{x}_{i|y}, y) - \Phi(x, \hat{y}_{i|y}) \right] = \mathbb{E} \left[ \Phi(\hat{x}, y) - \Phi(x, \hat{y}) \right].
\]

Second, the first term on the RHS of (17) is actually

\[
\mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} \left( \Phi_{\xi_i}(\hat{x}_{i|y}, y) - \Phi_{\xi_i}(x, \hat{y}_{i|y}) \right) \right] + \mathbb{E} \left[ \Psi(\hat{x}, y) - \Psi(x, \hat{y}) \right] \leq 0,
\]

for $\Psi(x, y) \in \mathcal{X}$ and $y \in \mathcal{Y}$, which is because $(\hat{x}, \hat{y})$ solves the R-ESP problem (13). Third, because the distributions of $T(i)$'s are the same, for the second term on the RHS of (17),
we have

\[\frac{1}{n} \sum_{i=1}^{n} E[T(i)]\]

\[\leq E \left[ \left( \ell_x(\xi_i, y) \right)^2 \right] \] (20)

\[\leq \sqrt{\mu_x + v_x + \mu_y + v_y} \left( \left\| \hat{x} - \xi(i) \right\|^2 + (\mu_y + v_y) \left\| \hat{y} - \xi(i) \right\|^2 \right)\]

The step (a) uses the vector Cauchy-Schwartz inequality \( \alpha^T \beta \leq \|a\|_2 \cdot \|b\|_2 \) for some vectors \( a \) and \( b \). The step (b) uses the expectation version of Cauchy-Schwartz inequality \( E[ab] \leq \sqrt{E[a^2]} \cdot \sqrt{E[b^2]} \) for some random variables \( a \) and \( b \). The step (c) is due to Lemma [1]. And the step (d) is due to Assumption [2] and the fact that \( (\hat{x}_y, \hat{y}_i) \) is independent of \( \xi \) and \( (\hat{x}, \hat{y}) \) is independent from \( \xi^t \). Finally, substituting [18], [19], and [20] into [17] provides the following result:

\[E[\Phi(\hat{x}, y)] - E[\Phi(x, y)] + \Psi(\hat{x}, y) - \Psi(x, y)] \]

\[\leq \frac{2\sqrt{n}}{n} \left( \frac{(\ell_{\xi_x}^2 + \ell_{\xi_y}^2)}{\mu_x + v_x + \mu_y + v_y} \right) \]

Due to the bound the regularizer, we know \( |\Psi(\hat{x}, y) - \Psi(x, y)| \leq 2R \). Note that the above inequality is true for any \( x \) and \( y \). Therefore, we prove the overall result that

\[\max_{x \in X, y \in Y} E[\Phi(\hat{x}, y)] - \min_{x \in X} E[\Phi(x, y)] \]

\[\leq \frac{2\sqrt{n}}{n} \left( \frac{(\ell_{\xi_x}^2 + \ell_{\xi_y}^2)}{\mu_x + v_x + \mu_y + v_y} \right) + 2R \]

This completes the proof.

Theorem 1. (Upper bound on WGM) For the SSP problem (1), let \((\hat{x}, \hat{y})\) be the solution to the ESP problem (2). And let \( n = |\Gamma| \) be sample size. Under Assumptions [1,2] and suppose \( \mu_x, \mu_y > 0 \), we have

\[\left\{ E \left[ d^2(\hat{x}, \hat{y}) \right] \right\} \leq \frac{2\Delta_w(\hat{x}, \hat{y})}{\min(\mu_x, \mu_y)} \]

\[\Delta_w(\hat{x}, \hat{y}) \leq \frac{2\sqrt{n}}{n} \left( \ell_{\xi_x}^2 + \ell_{\xi_y}^2 \right) \]

The first inequality of (21) is directly due to [6]. The second inequality of (21) can be derived from Lemma [2] by setting the regularizer \( \Psi \) to be 0. Namely, \( R = v_x = v_y = 0 \).

The generalization bounds given in Theorem 1 have tight dependence on the sample size \( n \), as well as the problem’s parameters \( \ell_{\xi_x}, \ell_{\xi_y} \) and \( \mu_x, \mu_y \). To see this, we can simply consider the special case of SCO. When \( \Phi_{\xi}(x, \cdot) = f_{\xi}(x) \), i.e., the objective function is constant in \( y \), the SSP and ESP reduce to the classical SCO and ERM respectively. In this case, the differences between \( \ell_{\xi}^w \) and \( \ell_{\xi}^v \) vanish, and we denote them as \( \ell_x := \ell_{\xi}^w = \ell_{\xi}^v \). The WGM also reduces to

\[\Delta_w(\hat{x}, \hat{y}) = \max_{y \in Y} E[f(\hat{x})] - \min_{x \in X} E[f(x)] \]

\[= \left. E[f(x)] \right|_{x \in X} \]

The generalization bound (21) becomes \( O\left( \frac{\ell_x^2}{\sqrt{n}} \right) \) and matches the generalization bound for ERM [Shalev-Shwartz et al. 2009].

Note that Theorem 1 requires \( \mu_x, \mu_y > 0 \), i.e., the problem is strongly convex and strongly concave. When the problem is general convex and concave with \( \mu_x = \mu_y = 0 \), we will need to choose the regularizer optimally and establish a generalization bound that depends only on the diameters of \( X \) and \( Y \) and constants of Lipschitz continuity of \( \Phi \).

Theorem 2. Suppose the \( \Phi_{\xi} \)'s are convex concave but not SC-SC. Namely, \( \mu_x = \mu_y = 0 \). Suppose the \( \Phi_{\xi} \)'s satisfy Assumption 2 under the L2 norm. Then we can set the regularizer to be \( \Psi(x, y) = \frac{\alpha_x}{2} \|x\|_2^2 - \frac{\alpha_y}{2} \|y\|_2^2 \). Consequently, \( R = \frac{\alpha_x}{2} D_x^2 + \frac{\alpha_y}{2} D_y^2 \), where \( D_x \) and \( D_y \) denote the diameters of \( X \) and \( Y \) under L2-norm respectively. If we set \( \alpha_x = \frac{\ell_x^w}{\sqrt{n}D_x} \) and \( \alpha_y = \frac{\ell_y^w}{\sqrt{n}D_y} \), Lemma 2 implies

\[\Delta_w(\hat{x}, \hat{y}) \leq O\left( \frac{\ell_x^w D_x + \ell_y^w D_y}{\sqrt{n}} \right) \]

Similar to Theorem 1, Theorem 2 is also tight for the special case of general convex stochastic optimization problem.

The above two theorems characterize the generalization bounds under the measure of weak generalization measure. By utilizing additional smoothness of the objective function, we provide an \( O(1/n) \) bound on SGM, for the SC-SC case where \( \mu_x, \mu_y > 0 \). We provide the proof in Appendix A.1.
Theorem 1. (Upper bound on SGM) Under the settings of Theorem 2 if Assumption 3 holds in addition, we have
\[ \Delta^2(\hat{x}, \hat{y}) \leq 2\sqrt{\frac{L_{x,y}}{\mu_x \mu_y}} + 1 \cdot \left( \frac{\sigma_y^2}{\mu_y} + \frac{\sigma_y^2}{\mu_y} \right). \] (22)

We remark that the bound (22) has an additional multiplicative \( L_{x,y}/\sqrt{\mu_x \mu_y} \) factor compared to bound (21). It remains open whether this dependence can be improved, as a question for future work. It is worth noting that for SA type method, [Yan et al. (2020)] has derived a similar bound for SGM of \( O(\max\{\frac{L_{x,y}}{\mu_x \mu_y}, 1\}) \), yet with stronger requirements on tails of the stochastic gradients. When the parameters are unbalanced, i.e. \( \mu_x \ll \mu_y \) and \( \ell_x^* \ll \ell_y^* \), our result can be better.

Note that Theorem 1, 2 and 3 are all based on the stability argument in Lemma 1 which relies heavily on the Lipschitz continuity of the objective function. Next we study SSP problem over unbounded domains. In this case, the SC-SC property and function Lipschitz continuity are mutually exclusive. In the next theorem, we provide a generalization bound for this class of problem.

Theorem 4 (Generalization error for unbounded problems). Let Assumptions 2, 3 and 4 hold, and let the SSP be unconstrained. Let \( \| \cdot \| = \| \cdot \|_\infty = \| \cdot \|_2 \). Then
\[ \mathbb{E} \left[ d^2 (\hat{x}, \hat{y}) \right] \leq O \left( \frac{\kappa^2}{n} \right), \]
\[ \Delta^2(\hat{x}, \hat{y}) \leq O \left( \frac{\kappa^2}{n} \right), \] (23)
where \( \kappa = \min\{\mu_x, \mu_y\} \) and \( \kappa = \frac{\max\{L_{x,y}, L_{x,y}\}}{\min\{\mu_x, \mu_y\}} \) is the condition number.

3 Application to Batch Policy Learning for MDP

3.1 Saddle Point Formulation of MDP

Consider the policy learning problem for an infinite-horizon average-reward Markov Decision Process (MDP). The MDP instance is specified by \( M = (S, A, P, r) \) where \( S \) is a finite state space. \( A \) is a finite action space, \( P = \{P_a\} \) are state transition matrices with \( P_a(s, s') = \text{Prob}(s_{t+1} = s' | s_t = s, a_t = a) \), for \( s, s' \in S \) and \( a \in A \). \( r \) is the reward function with \( r_{sa} \in [0, 1] \) being the reward received after taking action \( a \) at state \( s \). A policy \( \pi: S \mapsto \Delta_A \) maps a state \( s \) to a distribution over the action space \( A \), where we denote the probability of taking action \( a \) at state \( s \) as \( \pi(a|s) \).

The objective is to maximize the long-term average reward, defined as
\[ \hat{\pi}^* := \max_{\pi} \lim_{T \to \infty} \mathbb{E} \left[ \frac{1}{T} \sum_{t=0}^{T-1} r_{s_{t+1}a_{t+1}} \right], \] (24)

The optimal Bellman equation has an equivalent saddle point formulation [Puterman (2014); Chen and Wang (2016)]
\[ \min_{x} \max_{y} \Phi(x, y) := \langle y, r \rangle + \sum_{a \in A} y_a^T (P_a - I)x, \]
where \( x \in \mathbb{R}^{|S|} \) is the difference-of-value vector, \( y \in \mathbb{R}^{|S|\times|A|} \) stands for the stationary state-action distribution under certain policy \( \pi \). \( y_a = (y_{1,a}, \ldots, y_{|S|,a})^T \) is the \( a \)-th column of \( y \). Under the assumption of fast mixing time and uniform ergodicity (Assumptions 2.3 of Chen et al. (2018a)), there exists constant \( t_{mix} \) and \( \tau \) such that one can set the feasible regions \( \mathcal{X} \) and \( \mathcal{Y} \) as
\[ \mathcal{X} := \{ x \in \mathbb{R}^{|S|} : \| x \|_\infty \leq 2t_{mix} \}, \] (25)
\[ \mathcal{Y} := \{ y \in \mathbb{R}^{|S|\times|A|} : y \geq 0, \| y \|_1 = 1, \frac{1}{\sqrt{|S|}} \leq \sum_{a \in A} y_a \leq \sqrt{\frac{1}{|S|}} \}, \] (26)
(see Appendix B.1 for details). In the policy learning setting, we do not know either \( P \) or \( r \). Instead, we want to estimate the optimal policy \( \pi^* \) based on sample transitions.

We construct an unbiased sample of \( P = \{P_a\} \) by generating one sample transition from every \( (s, a) \), i.e., \( \xi := \{ (s, a, s', r_{sa}) : s \in S, a \in A, s' \sim P_a(s) \} \). In other words, each \( \xi \) consists of \( |S|\times|A| \) sample transitions. Thus we obtain a sample transition matrix \( P_{\xi} = \{P_a\} \) where \( P_{\xi,a}(s, s') = 1 \) if \( s' \) is sampled and \( P_{\xi,a}(s, s') = 0 \) otherwise. Thus, we can define a stochastic sample of the objective function of (3) as
\[ \Phi_{\xi}(x, y) = \sum_{s \in S} \sum_{a \in A} y_{sa} r_{sa} - \sum_{a \in A} y_a^T (P_{\xi,a} - I)x. \] (27)

It is easy to see that \( \Phi(x, y) = \mathbb{E}_{\xi} \Phi_{\xi}(x, y) \).

3.2 Efficiency of the Empirical Optimal Policy

To handle the bilinear objective function, we consider the regularized empirical saddle point (R-ESP) problem, given by
\[ \min_{x \in \mathcal{X}, y \in \mathcal{Y}} \max_{\alpha_x, \alpha_y} \frac{\alpha_x}{2} \| x \|_2^2 + \Phi_{\xi}(x, y) - \alpha_y \sum_{s,a} y_{sa} \log(y_{sa}), \] (28)
where \( \Phi_{\xi}(x, y) := \frac{1}{|S|} \sum_{\xi \in \xi} \Phi_{\xi}(x, y) \) is the empirical objective, \( \alpha_x, \alpha_y \) are to be chosen later. Let \( (\hat{x}, \hat{y}) \) be the solution to the R-ESP problem (28). Then we obtain the empirical optimal policy \( \hat{\pi} \) given by
\[ \hat{\pi}(a|s) := \frac{\hat{y}_{sa}}{\sum_{a' \in A} \hat{y}_{sa'}}, \quad \forall s \in S, a \in A. \]

The entropy regularizer \( \sum_{s,a} y_{sa} \log(y_{sa}) \) plays an important role in the analysis. It is 1-strongly convex in \( L_1 \)-norm due to the Pinsker’s inequality. To analyze the efficiency of \( \hat{\pi} \), we will apply the generalization theory for SSP problem by choosing the norms as \( \| \cdot \| := \| \cdot \|_2 \) and \( \| \cdot \| := \| \cdot \|_1 \), respectively.
Generalization Bounds for Stochastic Saddle Point Problems

**Theorem 5** (Sample Efficiency of Empirical Optimal Policy). Let $\alpha_x = \frac{1}{\sqrt{\log(|S||A|)}}$, $\alpha_y = \frac{t_{mix}}{\sqrt{n} \log(|S||A|)}$. Then the empirical optimal policy $\bar{\pi}$ satisfies

$$E \left[ \hat{v}^* - v^* \right] \leq O \left( \frac{t_{mix}}{\sqrt{n}} \left( \tau^{1.5} + \log(|S||A|) \right) \right).$$

Consequently, to guarantee that $E \left[ \hat{v}^* - v^* \right] \leq \epsilon$, we need $n \geq \Omega \left( \frac{t_{mix}}{\epsilon} \left( \tau^5 + \tau^3 \log(|S||A|) \right) \right)$. Since each $\xi \in \{ |S||A| \}$ samples of state transitions, the total sample complexity will be $|S||A| \cdot n = O \left( \frac{t_{mix}^2 |S||A|}{\epsilon^2} \right)$. The $|S||A|/\epsilon^2$ dependence in this bound is optimal.

Theorem 5 has several implications:

- The regularized empirical optimal policy $\bar{\pi}$ achieves a near-optimal sample complexity, which matches known upper/lower bounds in their dependences on $|S|, |A| \in [\text{Chen and Wang} (2017)]$. This result is somewhat surprising: it means that one can simply compute an empirical MDP and solve it for estimating the optimal policy. This approach is conceptually simple, yet has satisfying error bound.

- Also note that the transition matrix $P$ contains $|S|^2 |A|$ unknown variables, but the policy error of $\bar{\pi}$ scales with $|S||A|$ which is significantly smaller. Namely, one does not need to estimate the full matrix $P$ but can still get a good policy estimator by solving the R-ESP.

- The proof of Theorem 5 is nontrivial because we want to evaluate $v^*$, which is the average reward of the state-transition process if $\bar{\pi}$ is implemented. The first step of the proof is to apply the result of Lemma 2 to the R-ESP 28, so that we obtain a WGM upper bound as

$$\Delta^w(x, \bar{y}) \leq O \left( t_{mix} \cdot \left( \tau^{1.5} + \sqrt{\log(|S||A|)} \right) \cdot n^{-\frac{1}{2}} \right) \quad (29)$$

Then, we exploit the stationarity condition of the MDP 2 and prove that

$$\hat{v}^* - E \left[ \sum_{a \in A} \hat{y}_a^* ((P_a - I) x^* + r_a) \right] = E \left[ \Phi(x, y^*) - \Phi(x^*, \bar{y}) \right] \leq \Delta^w(x, \bar{y}) \quad (30)$$

Finally, we use the uniform ergodicity property of the MDP to show that $E [ \hat{v}^* - v^* ] \leq \tau E [ \hat{v}^* - \sum_{a \in A} \hat{y}_a^* ((P_a - I) x^* + r_a) ] \leq \Delta^w(x, \bar{y})$, which further leads to our theorem. Proofs of (29), (30) are given in Appendices [B.2, B.3].

**4 Application to Stochastic Composite Optimization**

Another example is stochastic composite optimization of the general form

$$\min_{x \in X} F(x) := f \left[ E_{\xi} [ g_{\xi}(x) ] \right] + r(x).$$

Such problems have been studied in many literatures, see [Nesterov (2007), Wang et al. (2017), Zhang and Xiao (2019)] and references therein. Let us consider the case where $f$ is convex and has Lipschitz continuous gradient and $g_{\xi}(x) = A_{\xi} x - b_{\xi}$ is linear, where $A_{\xi}$ is a random matrix and $b_{\xi}$ is a random vector. This case applies to inverse problems and policy evaluation with function approximation [Wang et al. (2017)]. This problem is equivalent to an SSP problem

$$\min_{x \in X} \max_{y \in Y} \Phi(x, y) = r(x) + E_{\xi} [ y^T (A_{\xi} x - b_{\xi}) ] - f^*(y), \quad (31)$$

where $f^*(y) = \sup_{x \in X} y^T x - f(x)$ is the convex conjugate of $f$. Suppose $X$ is a convex compact set and both $f$ and $r$ are convex functions. To guarantee the SC-SC property of $\Phi$, we assume the function $r$ is $\mu$-strongly convex and $f^*$ is smooth and has $L$-Lipschitz continuous gradient. We also assume that the $L_2$ norms of the sample $A_{\xi}$ and $b_{\xi}$ are almost surely bounded.

Note that in general, we should choose $\mathcal{Y}$ to be the whole space. However, in (31), because we assume $f$ is convex and $L$-smooth in the whole space, $f^*$ is strongly convex. Since $X, A_{\xi}, b_{\xi}$ are all required to be bounded, $\argmax_y y^T x - f(x)$ is bounded for all $x \in X$. Consequently, we assume the domain $\mathcal{Y}$ is convex and compact without hurting the optimality.

Due to the compactness of $\mathcal{Y}$, we also assume that $f^*$ is $\ell_{f^*}$-Lipschitz continuous in $\mathcal{Y}$. Since $\mu$ is often very small, for simplify the result, we assume that $L_{\mu} = O(1)$.

Let $(\hat{x}, \bar{y})$ solve the ESP version of (31). Under these conditions, if directly applying Theorem 3, we would be able to prove

$$E [ F(\hat{x}) ] - \min_{x \in X} F(x) \leq O \left( \frac{1}{n \cdot \mu^{-1.5}} \right), \quad (32)$$

which corresponds to an $O \left( \frac{1}{\sqrt{n} \cdot \mu} \right)$ sample complexity (see its proof in Appendix C.1). In the next theorem, we show that the ESP solution is actually more sample efficient and has smaller error. The proof is given in Appendix C.2.

**Theorem 6.** Let $(\hat{x}, \bar{y})$ be the ESP solution to (31) based on $n$ i.i.d. samples. Then

$$E [ F(\hat{x}) ] - \min_{x \in X} F(x) \leq \tilde{\Delta}(\hat{x}, \bar{y}) \leq O \left( \frac{L_{f^*} + \ell_{f^*}}{\mu \cdot n} \right) = O \left( \frac{1}{\mu \cdot n} \right),$$

where $\tilde{\Delta}(\hat{x}, \bar{y}) = E \left[ \max_{y \in \mathcal{Y}} \Phi(\hat{x}, y) \right] - \min_{x \in X} E \left[ \Phi(x, \bar{y}) \right]$ is a hybrid duality gap measure between WGM and SGM.

In other words, the ESP solution leads to an $\epsilon$-optimal solution to the original composite problem, when the sample size satisfies $n \geq O \left( \frac{1}{\mu \cdot \epsilon} \right)$. In terms of the dependence on $n$ and $\mu$, this result matches the best known sample complexity for this problem given by [Zhao (2019)]. However, we should also point out that the dependence on $L$ and $\sigma_A$ is not tight yet. How the dependence on these constants can be improved remains a future task.
5 Application to Stochastic Games

Consider the two-player stochastic matrix game problem [5]:

$$\min_{x \in \Delta_{N_1}} \max_{y \in \Delta_{N_2}} x^T \mathbb{E}_{\xi}[A_{\xi}] y,$$

with $\Delta_{N_i} := \{ z \in \mathbb{R}^{N_i} : z \succeq 0, 1^T z = 1 \}$, $i = 1, 2$, and $x, y$ denote the mixed strategies of players 1 and 2, respectively. Based on $n$ i.i.d. samples of the payoff matrix (with $nN_1N_2$ individual sample payoffs), we estimate the Nash equilibrium $(x^*, y^*)$ by constructing the following R-ESP

$$\min_{x \in \Delta_{N_1}} \max_{y \in \Delta_{N_2}} \sum_{i} \frac{x_i \log x_i}{n \log N_1} + x^T \left( \frac{1}{n} \sum_{i=1}^{n} A_{\xi} \right) y - \frac{\sum_j y_j \log y_j}{n \log N_2}.$$

Let $(\bar{x}, \bar{y})$ be the solution to the preceding R-ESP, which is referred to as the empirical Nash equilibrium. Then the following theorem holds.

Theorem 7. Assume $\max_{i,j} |A_{\xi}(i,j)| \leq 1$ almost surely. Therefore,

$$E[x^T A_{\xi}] - O\left(\sqrt{\log(N_1N_2)/n}\right) \leq E[\bar{x}^T A_{\xi}] \leq E[\bar{x}^T A_{\xi}] + O\left(\sqrt{\log(N_1N_2)/n}\right),$$

for $\forall x \in \Delta_{N_1}, y \in \Delta_{N_2}$.

The theorem means that the empirical strategy $(\bar{x}, \bar{y})$ is an $\epsilon$-Nash equilibrium with high probability, as long as the total number of sample payoffs is greater than $O(N_1N_2\epsilon^{-2})$. This sample complexity is statistically optimal.

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