MULTIPLIERS OF DISPOSITION $p$-GROUPS

MAHBUBE ALIZADEH SANATI

Abstract. Let $p$ be a prime number and $c, d$ natural numbers. Up to isomorphism, there is a unique $p$-group $G_{c}^{d}$ of least order with rank $d$ and nilpotency class $c$ named disposition group. This group plays an important role in the construction of Galois extensions over number fields with given $p$-group as Galois group. Also, it has a central series with all factors being elementary. Since $G_{1}^{1}$ is abelian we consider $d \geq 2$. In this article, first, we determine the order of all its subgroups of lower central series and $n$-th center subgroups of $G_{c}^{d}$, $(n \in \mathbb{N})$. Then we deduce these groups are $n$-capable. Also, the structure of the $m$-nilpotent multiplier of $G_{c}^{d}$ is determined in two cases $m \geq c$ and $m \leq c$. Finally, polynilpotent multiplier of disposition group of class row $(m_{1}, m_{2}, \ldots, m_{t})$, when $m_{1} \leq c$ is calculated.

1. Introduction

Several papers from the beginning of the twentieth century tried to find some structures for the notion of the Schur-multiplier. Undoubtedly, Karpilovsky’s book [8] is concluded with comprehensive information on this notion. Some results about its varietal generalization, Baer-invariant, of some well-known groups can be found in [11], [10]. Since the $p$-part of the multiplier of $G$ is embedded into the multiplier of its Sylow $p$-subgroup, it is of interest to study the multiplier of $p$-groups. Also, by Schur’s literatures, one can use the Schur multiplier of a $p$-group for classifying $p$-groups.

Since the 1950s it has been known the Bake-Campbell-Hausdorff formula gives an isomorphism between the category of nilpotent Lie ring with order $p^{n}$ and nilpotency class $c$ and the category of finite $p$-groups with order $p^{n}$ and nilpotency class $c$, provided $p > c$. This is known as the Lazard correspondence [9].

Among all finite $p$-groups of class $c$ with $d$ generators, our interest is disposition group $G_{c}^{d}$. A group $G$ has Frattini class $m$ if $m$ is the length of a shortest central series of $G$ with all factors being elementary abelian. There is up to isomorphism a unique largest $p$-group $G_{c}^{d}$ with $d$ generators and Frattini class $c$, and $G$ is an epimorphic image of $G_{c}^{d}$.

Let $F_{\infty}$ be a free group on the infinite many countable set $X = \{x_{1}, x_{2}, \ldots \}$. Every element $v$ has the form $x_{i_{1}}^{\alpha_{1}} \cdots x_{i_{k}}^{\alpha_{k}}$ in which, $\alpha_{j} = \pm 1$ for each $1 \leq j \leq k$, $k \in \mathbb{N}$ and $x_{i_{j}}$ s are distinct elements of $X$ is called as word.

Now, suppose $V$ is a set of words, $G$ is a group, $v = x_{i_{1}}^{\alpha_{1}} \cdots x_{i_{k}}^{\alpha_{k}}$ a word in $V$ and $g_{1}, \ldots, g_{k}$ are arbitrary elements of $G$. The value of $v$ with respect to $(g_{1}, \ldots, g_{k})$ is denoted by $v(g_{1}, \ldots, g_{k})$ and defined by $v(g_{1}, \ldots, g_{k}) = g_{1}^{\alpha_{1}} \cdots g_{k}^{\alpha_{k}}$. The subgroup generated by all values of the words $V$ in $G$ is called the verbal subgroup...
of $G$ with respect to the set of words $V$ and is denoted by $V(G)$. i.e. $V(G) = \langle v(g_1, \ldots, g_k) \mid v \in V, g_i \in G, 1 \leq i \leq k, k \in \mathbb{N} \rangle$. Let $N$ be a normal subgroup of $G$. Then $V(N)$ is defined to be the subgroup of $G$ generated by the following set \( \{ v(g_1, \ldots, g_n, \ldots, g_k) \mid v \in V; g_1, \ldots, g_k \in G; n \in \mathbb{N} \} \). The \textit{marginal} subgroup of $G$ with respect to the set of words $V$, $V^*(G)$, is defined as
\[
\{ a \in G \mid v(g_1, \ldots, g_a, \ldots, g_k) = v(g_1, \ldots, g_k) \mid v \in V; g_j \in G, 1 \leq i, j \leq k, k \in \mathbb{N} \}.
\]

It is shown this set is a characteristic subgroup of $G$. A subgroup $N$ of $G$ is called $\mathcal{V}$-marginal, if $N \subseteq V^*(G)$. A class of all groups $G$ such that $V(G) = 1$ is called the \textit{variety} $\mathcal{V}$ determined by $V$ and we say $V$ is a \textit{set of laws} for the variety $\mathcal{V}$.

For two subsets $X_1, X_2$ of $G$, we let $[X_1, X_2] = \langle [x_1, x_2] \mid x_1 \in X_1, x_2 \in X_2 \rangle$ in which $[x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2$. Also, we consider expanding the commutator on the left hand side, $[x_1, \ldots, x_n, x_{n+1}] = [[x_1, \ldots, x_n], x_{n+1}], (n \geq 2)$. Hence the subgroups of the lower central series of $G$ are defined recursively, $\gamma_1(G) = G$ and $\gamma_{n+1}(G) = [\gamma_n(G), G]$. Usually we write $\gamma_n(G) = G^n$. The subgroups of the upper central series are defined by $Z_1(G) = Z(G)$ and $Z_{n+1}(G)/Z_n(G) = Z(G/Z_n(G))$.

If $V = \{ [x_1, x_2] \}$, then $V(G) = G'$, $V^*(G) = Z(G)$ and $V(N, G) = [N, G]$. Also, $\mathcal{V} = \mathcal{A}$ is the variety of abelian groups. More generally, for each natural number $m, V = \{ [x_1, \ldots, x_{m+1}] \}$ implies $V(G) = \gamma_{m+1}(G)$, $V^*(G) = Z_m(G)$ and $V(N, G) = [N_m G]$. In this case, $\mathcal{V} = \mathcal{N}_m$ is the variety of nilpotent groups of class at most $m$. If $V = \{ [x_{c_1}, \ldots, x_{c_1+c_2+1}], \ldots, [x_{c_1+c_2+1}, \ldots, x_{c_1+c_2+1+c_2}] \}$, for some natural numbers $c_1, c_2$, then it is proved that $V(G) = \gamma_{c_1+c_2} (\gamma_{c_1+1}(G))$ and $V(N, G) = [N_{c_1, c_2} G, \gamma_{c_1+c_1+1}(G)]$. The related variety is denoted by $\mathcal{N}_{c_1, c_2}$.

The following lemma gives basically a summary of the known properties of the verbal and the marginal subgroups of a group $G$ with respect to the variety $\mathcal{V}$, which is useful in our investigation, see [6].

\textbf{Lemma 1.1.} Let $\mathcal{V}$ be a variety of groups and $N$ be a normal subgroup of a group $G$. Then the following statements hold.

(i) $V(V^*(G)) = 1, V^*(G/V(G)) = G/V(G)$.

(ii) $V(G) = 1 \iff V^*(G) = G \iff G \in \mathcal{V}$.

(iii) $V(N, G) = 1 \iff N \subseteq V^*(G)$.

(iv) $V(G/N) = V(G)N/N, V^*(G)N/N \subseteq V^*(G/N)$.

(v) $[N, V(G)] \subseteq V(N, G) \subseteq N \cap V(G), V(G, G) = V(G)$.

(vi) If $N \cap V(G) = 1$, then $N \subseteq V^*(G)$ and $V^*(G/N) = V^*(G)/N$.

(vii) $V(N, G)$ is the smallest normal subgroup $T$ of $G$ contained in $N$, such that $N/T \subseteq V^*(G/T)$.

(viii) If $H$ and $K$ are subgroups of $G$, then $V(HK, G) = V(H, G)V(K, G)$.

A group $G$ is said to be $\mathcal{V}$-\textit{nilpotent} if it has a normal series,
\[
1 = G_0 \leq G_1 \leq \cdots \leq G_n = G,
\]
such that each factor is marginal, i.e. $G_{i+1}/G_i \subseteq V^*(G/G_i)$ for all $0 \leq i \leq n-1$. Such a series is called a $\mathcal{V}$-\textit{marginal series}. The least integer $c$ for such series, is called the $\mathcal{V}$-\textit{nilpotency class} of $G$.

It is obvious that each $\mathcal{A}$-nilpotent group is the usual nilpotent group. In the following we introduce a $\mathcal{V}$-marginal series.

\textbf{Definition 1.2.} Let $\mathcal{V}$ be a variety of groups defined by a set of words $V$. The lower $\mathcal{V}$-marginal series of a group $G$ is defined as
\[
G = V_0(G) \supseteq V_1(G) = V(G) \supseteq V_2(G) \supseteq \cdots \supseteq V_n(G) \supseteq \cdots,
\]
such that \( V_n(G) = V(V_{n-1}(G), G) \), for each \( n \in \mathbb{N} \).

Note that by Lemma 1.1(viii), we have \( V_i(G)/V_{i+1}(G) \leq V^*(G/V_{i+1}(G)) \), i.e. the above series is \( V \)-marginal.

If \( V = \{x_1, x_2\} \), then \( V_n(G) = \{V_{n-1}(G), G\} \) and so \( V_n(G) = \gamma_{n+1}(G), (n \in \mathbb{N}) \).
Thus the above series coincides with the lower central series of the group,
\[
G = \gamma_1(G) \triangleright \gamma_2(G) = G' \triangleright \gamma_3(G) \triangleright \cdots \triangleright \gamma_{n+1}(G) \triangleright \cdots
\]
The following theorem is vital in our main result.

**Theorem 1.3.** ([4]) Let \( F = \langle x_1, \cdots, x_d \rangle \) be a free group, then
\[
\frac{\gamma_i(F)}{\gamma_{i+1}(F)}
\]
is the free abelian group freely generated by \( \chi_n(d) \) elements is given by Witt’s formula
\[
\chi_n(d) = \frac{1}{n} \sum_{m|n} \mu(m)q^{n/m},
\]
where \( \mu(m) \) is the Mobiuous function and defined to be
\[
\mu(m) = \begin{cases} 
1 & \text{if } m = 1, \\
0 & \text{if } m = p_1^{\alpha_1} \cdots p_k^{\alpha_k}, \exists \alpha_i > 1. \\
(-1)^s & \text{if } m = p_1 \cdots p_s,
\end{cases}
\]
Fix a prime number \( p \) and a group \( G \) in what follows. Write
\[
G^p^n = \langle g^p^n | g \in G \rangle \quad (n \in \mathbb{N}).
\]
For an arbitrary group \( G \), define
\[
\Lambda_n(G) = V_0(G)p^{n-1}V_1(G)p^{n-2} \cdots V_{n-1}(G).
\]
Trivially, \( \Lambda_1(G) = G \), \( \Lambda_2(G) = G^pV(G) \), \( \Lambda_3(G) = G^{p^2}V(G)pV_2(G) \) and so on. Also \( \Lambda_n(G) \) is a characteristic subgroup of \( G \) and we have
\[
G = \Lambda_1(G) \triangleright \Lambda_2(G) \triangleright \cdots \triangleright \Lambda_n(G) \triangleright \cdots
\]
Observe that if \( H \) and \( K \) are two subgroups of \( G \) with \( H \leq K \), then \( \Lambda_n(H) \leq \Lambda_n(K) \). In the case of normality of \( H \), we have \( \Lambda_n(G/H) \leq \Lambda_n(G)H/H \).

Also, by Lemma 1.1(viii)
\[
V(\Lambda_n(G), G) = V(V_0(G)p^{n-1}V_1(G)p^{n-2} \cdots V_{n-1}(G), G)
\]
\[
= V(V_0(G)p^{n-1}, G)V(V_1(G)p^{n-2}, G) \cdots V(V_{n-1}(G), G)
\]
\[
\subseteq V(V_0(G), G)p^{n-1}V_1(G, G)p^{n-2} \cdots V_{n-1}(G) = V_1(G)p^{n-1}V_2(G)p^{n-2} \cdots V_{n-1}(G)
\]
\[
\subseteq V_0(G)p^{n}V_1(G)p^{n-1}V_2(G)p^{n-2} \cdots V_{n-1}(G) = \Lambda_{n+1}(G).
\]
Hence \( \Lambda_n(G)/\Lambda_{n+1}(G) \) is a \( V \)-marginal group.

If \( V = A \) is the variety of abelian groups, then the above series is one of the many introduced in the fundamental paper of Lazard (1954) as follows, [9]
\[
\lambda_n(G) = \gamma_1(G)p^{n-1} \gamma_2(G)p^{n-2} \cdots \gamma_n(G).
\]
In 1979, Blackburn and Evens showed \( \lambda_n(G) = [\lambda_{n-1}(G), G], \lambda_{n-1}(G)p \) and \( G/\lambda_n(G) \) is a finite \( p \)-group if \( G \) is finitely generated, see [3] for more details. If \( G \neq 1 \) is a finite \( p \)-group, then \( \lambda_2(G) = \phi(G) \) is the Frattini subgroup. Blackburn and Evens proved the following useful lemma which will be required in the proof of our main theorems.

**Theorem 1.4.** ([3], Theorems 2.4. and 2.7.) Let \( G \) be an arbitrary group. Then for all \( c \geq 1 \)
(i) \( [\gamma_{c-j+1}(G)p^{c-1} \cdots \gamma_c(G), G] = \gamma_{c-j+2}(G)p^{c-1} \cdots \gamma_{c+1}(G) \) (\( j \in \{1, 2, \cdots, c\} \)).
(ii) \( [\lambda_c(G), G] = \gamma_2(G)p^{c-1} \cdots \gamma_{c+1}(G) \).
(iii) \( \lambda_c(G) \cap \gamma_j(G) = \gamma_j(G)p^{c-j} \cdots \gamma_c(G) \) (\( j \in \{1, 2, \cdots, c\} \)).

As a corollary of Theorem 1.4, we can inductively prove
\[ [\lambda_c(G), m G] = \gamma_{m+c}(G)p^{c-1} \cdots \gamma_{c+m}(G) \] for each \( m \geq 2 \).

**Lemma 1.5.** ([3], Lemma 2.9) For each free group \( F \) and natural number \( n \) the quotient group
\[ H_k = \frac{\gamma_k(F)p^{n-k} \cdots \gamma_n(F)}{\gamma_k(F)p^{n-k+1} \cdots \gamma_n+1(F)} = \frac{\lambda_n(F) \cap \gamma_k(F)}{\lambda_{n+1}(F) \cap \gamma_k(F)}, \quad (1 \leq k \leq n) \]
is elementary abelian of order \( p^s_k \), where \( s_k = \chi_k(d) + \cdots + \chi_n(d) \).

2. **Polynilpotent Multipliers of disposition groups**

Let \( 1 \to R \to F \to G \to 1 \) be a free presentation for \( G \), in which \( F \) is a free group. In 1945, R. Baer [2] defined the notion of **Baer- invariant** as \( \mathcal{V}M(G) = R \cap V(F) \) and proved that this quotient group is abelian and independent from the choice of the free presentation of \( G \).

In the variety of abelian groups, the Baer- invariant of \( G \) will be \( R \cap F' \) which is called the **Schur-multiplier** of \( G \), and was defined by I. Schur [13] in 1904, for a finite group.

Also \( \mathcal{N}_m M(G) = \frac{R \cap \gamma_{m+1}(F)}{[R, F]} \), the **\( m \)-nilpotent multiplier** of \( G \), is the Baer-invariant of \( G \) with respect to the variety of nilpotent group of class at most \( m \).

**Two nilpotent multiplier** of \( G \) of class row \( (m_1, m_2) \) is the Baer-invariant of \( G \) with respect to the word \( \{[[x_1, \cdots, x_{m_1+1}], \cdots, [x_{m_2+1}, \cdots, x_{m_1+m_2+1}]]\} \) and is denoted by \( \mathcal{N}_{m_1,m_2} M(G) \). A generalization of it, for \( t \geq 2 \), is the **polynilpotent multiplier** of \( G \) of class row \( (m_1, \cdots, m_t) \) and denoted by \( \mathcal{N}_{m_1,\cdots,m_t} M(G) \) which Hekseter in [6] proved it is

\[ \mathcal{N}_{m_1,\cdots,m_t} M(G) = \frac{R \cap \gamma_{m_t+1}(\cdots(\gamma_{m_{t+1}}(F))\cdots)}{[R, F_{m_1} \cap \gamma_{m_t+1}(\cdots(\gamma_{m_{t+1}}(F))\cdots)]}. \]

(for more details see [8]).

In 1973, M.R. Jones [7] by applying the exact sequence \( 1 \to \frac{\gamma_{c+1}(F)}{[R, F] \cap \gamma_{c+1}(F)} \to M(G) \to M(G/\gamma_c(G)) \to \gamma_c(G) \to 1 \) for a nilpotent group \( G \) of class \( c \), gave inequalities for the order, number of generators and exponent of \( M(G) \). He concluded if \( G \) is a \( p \)-group of class \( c \) generated by \( d \) elements, \( d(M(G)) \leq \sum_{i=1}^{c} \chi_{i+1}(d) \). In 1979 Blackburn and Evans, by calculating the Schur-multiplier of disposition groups, proved that this bound is best possible.

Let \( F \) be a free group of rank \( d \geq 2 \). In 2016, P. Schmid called the group
\[ G_d^c = F/\lambda_{c+1}(F) \] (\( c \geq 1 \)).
as **disposition group**. Trivially \( G_d^c \) is a finite \( p \)-group having Frattini class \( c \) and rank \( d \), nilpotency class \( c \) and exponent \( p^c \) having the center \( Z(G_d^c) = \lambda_c(G_d^c) \), for \( c \geq 2 \). Every \( p \)-group \( G \) with Frattini class at most \( c \) and rank \( d(G) \leq d \) is an
epimorphism image of $G^c_d$. Now, we present the upper central series and the order of each subgroup of lower its central series. For all $1 \leq i \leq c$, by the Lemma 1.5

$$
\gamma_i(G^c_d) = \gamma_i \left( \frac{F}{\lambda_{c+1}(F)} \right) = \frac{\gamma_i(F)\lambda_{c+1}(F)}{\lambda_{c+1}(F)} \simeq \frac{\gamma_i(F)}{\lambda_{c+1}(F) \cap \gamma_i(F)}.
$$

so

$$
|\gamma_i(G^c_d)| = \left| \frac{\lambda_i(F) \cap \gamma_i(F)}{\lambda_{i+1}(F) \cap \gamma_i(F)} \right| \cdot \left| \frac{\lambda_{i+1}(F) \cap \gamma_i(F)}{\lambda_{i+2}(F) \cap \gamma_i(F)} \right| \cdot \cdots \cdot \left| \frac{\lambda_{c}(F) \cap \gamma_i(F)}{\lambda_{c+1}(F) \cap \gamma_i(F)} \right| \cdot \lambda_i(F) \cap \gamma_i(F)
$$

where $i = 1, \ldots, c$. In particular,

$$
|\gamma_i(G^c_d)| = \prod_{i=1}^{c} \lambda_i(F) \cap \gamma_i(F) \simeq \prod_{i=1}^{c} \lambda_i(F) \cap \gamma_i(F).
$$

In particular, $G^1_d = \lambda_1(G^c_d)$. Hence the desired assertion is established in all cases.

**Proposition 2.1.** The upper central series of $G^c_d$ is as follows ($c \geq 2$).

$$
1 = Z_0(G^c_d) \subseteq Z_1(G^c_d) = \lambda_c(G^c_d) \subseteq \cdots \subseteq Z_i(G^c_d) = \lambda_{c-i+1}(G^c_d) \subseteq \cdots \subseteq Z_c(G^c_d) = \lambda_1(G^c_d) = G^c_d.
$$

**Proof.** Schmid in [13] proved that $Z(G^c_d) = \lambda_c(G^c_d)$. By the definition we have always $\lambda_i(G^c_d) = \lambda_i(F)/\lambda_{c+1}(F)$. Suppose $1 \leq i < c$. Inductively, we can see

$$
\frac{Z_{i+1}(G^c_d)}{Z_i(G^c_d)} = Z \left( \frac{G^c_d}{\lambda_{c-i+1}(G^c_d)} \right) \simeq Z \left( \frac{\lambda_{c-i+1}(F) / \lambda_{c+1}(F)}{\lambda_{c-i+1}(F) / \lambda_{c+1}(F)} \right) = Z \left( \frac{\lambda_{c-i+1}(F)}{\lambda_{c-i+1}(F)} \right) = Z(G^c_{d-i}).
$$

Hence the desired assertion is established in all cases.

Baer in 1938 concentrated in his study on a group $G$ which there is a group $H$ such that $H/Z(H) \simeq G$, [1]. Hall and Senior are called this group **capable**, [5]. A generalization of this notion, $n$-capability, was simultaneously introduced by Burns and Ellis and also by Moghaddam and Kayvanfar, [12]; A group $G$ is called **$n$-capable** if there is a group $H$ such that $H/Z_n(H) \simeq G$. Trivially, 1-capability implies capability and also $n$-capability implies 1-capability for a group. The capability of abelian groups has been described by Baer [1] to be direct sums of cyclic groups. An interesting application of the Proposition 2.1 is the following fact which is the generalization of the main theorem of [13].

**Corollary 2.2.** Disposition $p$-groups are $n$-capable, for each $n \in \mathbb{N}$.

**Proof.** For each $c \geq 1$, by the Theorem 2.1, we have

$$
\frac{G^n_{d+c} \lambda_{c+1}(G^{n+c}_{d})}{Z_n(G^{n+c}_{d})} = \frac{\lambda_{c+1}(G^{n+c}_{d})}{Z_n(G^{n+c}_{d})} = \frac{F/\lambda_{n+c+1}(F)}{\lambda_{c+1}(F)/\lambda_{n+c+1}(F)} \simeq G^c_d.
$$

In the sequel, we compute the $m$–nilpotent multiplier of the disposition group.

**Theorem 2.3.** With the above notations, the $m$-nilpotent multiplier $\mathcal{N}_m(M(G^c_d))$ is elementary abelian of order $p^s$ where

(i) $s = m \sum_{i=m}^{c} \chi_{i+1}(d) + m \sum_{i=1}^{m} (m-i+1) \chi_{c-i+1}(d)$, if $m \leq c$

(ii) $s = \sum_{i=1}^{(c-i+1) \chi_{m+i}(d)}$, if $m \geq c$. 
Proof. By the definition of the $m$-nilpotent multiplier, $\mathcal{N}_m M(G_d^c) \simeq \frac{\lambda_{c+1}(F) \cap \gamma_{m+1}(F)}{\lambda_{c+1}(F) \cap \gamma_{m+1}(F)}$. We know $[\lambda_{c+1}(F),_m F] = \gamma_{m+1}(F)p^{c} \cdots \gamma_{c+m+1}(F) = \lambda_{c+m+1}(F) \cap \gamma_{m+1}(F)$. Hence

$$\mathcal{N}_m M(G_d^c) \simeq \frac{\lambda_{c+1}(F) \cap \gamma_{m+1}(F)}{\lambda_{c+m+1}(F) \cap \gamma_{m+1}(F)}.$$

(i) If $m \leq c$, by invoking Lemma 1.5

$$|\mathcal{N}_m M(G_d^c)| = |\frac{\lambda_{m+1}(F) \cap \gamma_{m+1}(F)}{\lambda_{m+1}(F) \cap \gamma_{m+1}(F)| = p^{t_1}p^{t_2} \cdots p^{t_m},$$

where, for each $i \in \{1, \ldots, m\}$, $t_i = \chi_{n+1}(d) + \cdots + \chi_{c+1}(d)$.

(ii) If $m \geq c$, then $\gamma_{m+1}(F) \subseteq \gamma_{c+1}(F) \subseteq \lambda_{c+1}(F)$ and so by Lemma 1.5 we can write

$$|\mathcal{N}_m M(G_d^c)| = |\frac{\gamma_{m+1}(F)}{\lambda_{m+1}(F) \cap \gamma_{m+1}(F)}| = p^{t_{m+1}(d)}p^{t_{m+1}(d) + \chi_{n+1}(d)} \cdots p^{t_{m+1}(d) + \chi_{n+1}(d)},$$

as required.

Note that in the case of $m = c$, the above values coincide.

An immediate result of the first part of the theorem when $m = 1$ is the following statement.

**Corollary 2.4.** ([3], Theorem 2.10) Let $F$ be a free group of rank $d$ and $c$ a natural number. Then $\mathcal{M}(F/\lambda_{c+1}(F))$ is an elementary abelian group of order $p^{s}$, where $s = \sum_{i=1}^{c} \chi_{i+1}(d)$.

In 2019, Niroomand, Johari and Parvizi have proved that if $G$ is a finite $p$-group of order $p^{n}$ with $G' = p^{k}$ and $m \geq 2$ then

$$|\mathcal{N}_m M(G)| \leq p^{x_{m+1}(n-k)+x_{m+1}(2)+

But the order of $m$-nilpotent multiplier of $G_d^c$ is very less than of the above bound. For example $|G_2^1| = p^{18}$, $|(G_2^2)'| = p^{10}$ and $|\mathcal{N}_2 M(G_2^c)| = p^{12}$ whereas the bound in (s) is $p^{4608}$.

Also, Burns and Ellis in 1998 proved if $G$ is a $d$-generator $p$-group and $||\phi(G),_i-1 G|| = p^{k_{i}}$ ($i \geq 1$) then

$$|\mathcal{N}_m M(G)||\gamma_{m+1}(F)| \leq p^{x_{m+1}(d)+k_m d+k_{m-1} d^2+d \cdots + k_1 d^m}.$$

In disposition group by Theorem 1.4 (ii) we have

$$||\phi(G',_i-1 G')|| = |\lambda_{2}(G_d^c),_i-1 G_d^c| = |\gamma_{i}(G_d^c)'| = p^{(c-i)\chi_{i}(d)} \cdots \chi_{i+1}(d) \cdots \chi_{c}(d),$$

because Schmid in [14] proved that $|\gamma_{i}(G_d^c)'| = p^{c-i} \chi_{i}(d)$ and we know $|\gamma_{i}(G_d^c)'| = p^{\sum_{i=1}^{c-i} (c-i) \chi_{i}(d)}$. This shows our bound is very less than their result.

Mashayekhy, Homaibadi and Mohammazadeh in [11] proved that if $G$ is a nilpotent group of class $e$, then polynilpotent multiplier of $G$ of class row $(m_1, m_2, \cdots, m_t)$ with condition $m_t \leq c$ satisfies in the following relation.

$$N_{m_1, m_2, \cdots, m_t} M(G) \simeq N_{m_1} M \left( \cdots N_{m_2} M \left( N_{m_1} M(G) \right) \cdots \right)$$
On the other hand as a corollary of the main result of [10] we have

\[ N_{m_1,m_2,\ldots,m_t} M(\bigoplus_{i=1}^{k\text{-times}} \mathbb{Z}_p) \simeq \mathbb{Z}_p^{(f_k)} \]

in which \( f_k = \chi_{m_1+1}(\chi_{m_{t-1}+1}(\ldots(\chi_{m_2+1}(\chi_{m_1+1}(k)))\ldots)) \). Now we can conclude some of polynilpotent multiplier of \( G_d^c \).

**Theorem 2.5.** The polynilpotent multiplier of \( G_d^c \) of class row \( (m_1,m_2,\ldots,m_t) \), when \( m_1 \leq c \), is as follows:

\[ N_{m_1,m_2,\ldots,m_t} M(G_d^c) \simeq \mathbb{Z}_p^{(g_s)} \]

where \( g_s = \chi_{m_1+1}(\chi_{m_{t-1}+1}(\ldots(\chi_{m_2+1}(\chi_{m_1+1}(s)))\ldots)) \) and \( s = m_1 \sum_{i=m_1}^{c-1} \chi_{i+1}(d) + \sum_{i=1}^{m_1} (m_1 - i + 1) \chi_{c+i}(d) \).

**References**

[1] R. Baer, Groups with preassigned central and central quotient group, Trans. Amer. Math. Soc. 44, 387-412 (1938).
[2] R. Baer, Representations of groups as quotient groups I-II-III, Trans. Amer. Math. Soc. 58, 295-419 (1945).
[3] N. Blackburn and L. Evens, Schur multipliers of \( p \)-groups, J. Reine Angew. Math. 309, 100-113 (1979).
[4] M. Hall, The theory of groups, Mac Millan Company, New York, (1959).
[5] M. Hall and J. K. Senior, The groups of order \( 2^n \) \((n \leq 6)\) (1964) (Macmillan: New York)
[6] N. S. Hekster, Varieties of groups and isologism, J. Aust. Math. Soc. (series A), 46, 22-60 (1998).
[7] M. R. Jones, Some inequalities for the multiplicator of a finite group. Proc. Amer. Math. Soc. (3) 39, 450-456 (1973). doi:10.2307/2039572
[8] G. Karpilovsky, The Schur Multiplier, London Math. Soc. Monographs, New Series no. 2, (1987).
[9] M. Lazard, Sur les groupes nilpotents et les anneaux de Lie, Ann. Ecole Normale Sup. 71, 101-190 (1954).
[10] B. Mashayekhy and M. Parvizi, Polynilpotent multiplier of finitely generated abelian groups. Int. J. Math. Game Theory Algebra 16(1), 93-102 (2006).
[11] B. Mashayekhy, A. Hokmabadi and F. Mohammadzadeh, Polynilpotent Multiplier of Some Nilpotent Products of Cyclic Groups. Int. J. Math. Game Theory Algebra, 17(5-6), 279-287 (2009). doi:10.1007/s13369-011-0041-0
[12] M. R. Moghaddam and S. Kayvanfar, A new notion derived from varieties of groups, Algebra Colloq. 4, 1-11 (1997).
[13] P. Schmid, On a class of finite capable \( p \)-groups. Arch. Math. 106, 301-304 (2016). doi:10.1007/s00013-016-0872-8.
[14] P. Schmid, Disposition \( p \)-groups. Arch. Math. 108, 113-121 (2017). doi:10.1007/S00013-016-0987-Y
[15] I. Schur, Uber die darstellung der endlichen gruppen durch gebrochene lineare substitutionen. J. Fur Math. 127, 20-50 (1904).