Hamilton-Jacobi Equations for Nonholonomic Magnetic Hamiltonian Systems

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In Memory of Great Geometer Shiing Shen Chern

Abstract. In order to describe the impact of the different geometric structures and the constraints for the dynamics of a Hamiltonian system, in this paper, for a magnetic Hamiltonian system defined by a magnetic symplectic form, we drive precisely the geometric constraint conditions of the magnetic symplectic form for the magnetic Hamiltonian vector field, which are called the Type I and Type II Hamilton-Jacobi equations. Second, for the magnetic Hamiltonian system with a nonholonomic constraint, we can define a distributional magnetic Hamiltonian system, then derive its two types of Hamilton-Jacobi equations. Moreover, we generalize the above results to nonholonomic reducible magnetic Hamiltonian system with symmetry, we define a nonholonomic reduced distributional magnetic Hamiltonian system, and prove the two types of Hamilton-Jacobi theorems. These research reveal the deeply internal relationships of the magnetic symplectic structure, the nonholonomic constraint, the distributional two-form, and the dynamical vector field of the nonholonomic magnetic Hamiltonian system.

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1 Introduction

It is well-known that Hamilton-Jacobi theory is an important research subject in

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mathematics and analytical mechanics (see Abraham and Marsden [1], Arnold [2] and Marsden and Ratiu [19]), and the Hamilton-Jacobi equation is also fundamental in the study of the quantum-classical relationship in quantization, and it plays an important role in the study of stochastic dynamical systems (see Woodhouse [32], Ge and Marsden [10], and Lázaro-Camí and Ortega [12]). For these reasons, the equation is described as a useful tool in the study of Hamiltonian system theory, which has been extensively developed in recent years and become one of the most active subjects in the study of modern applied mathematics and analytical mechanics.

The Hamilton-Jacobi theory, from the variational point of view, was originally developed by Jacobi in 1866, and it states that the integral of the Lagrangian of a mechanical system along the solution of its Euler-Lagrange equation satisfies the Hamilton-Jacobi equation. The classical description of this problem, from the generating function and the geometrical point of view, was given by Abraham and Marsden in [1] as follows: letting $Q$ be a smooth manifold and $TQ$ the tangent bundle, $T^*Q$ is the cotangent bundle with a canonical symplectic form $\omega$, and the projection $\pi_Q:T^*Q \to Q$ induces the map $T\pi_Q:TT^*Q \to TQ$.

**Theorem 1.1.** Assume that the triple $(T^*Q,\omega,H)$ is a Hamiltonian system with Hamiltonian vector field $X_H$, and $W:Q \to \mathbb{R}$ is a given generating function. Then the following two assertions are equivalent:

(i) For every curve $\sigma: \mathbb{R} \to Q$ satisfying that $\dot{\sigma}(t) = T\pi_Q(X_H(dW(\sigma(t))))$, $\forall t \in \mathbb{R}$, then $dW \cdot \sigma$ is an integral curve of the Hamiltonian vector field $X_H$.

(ii) $W$ satisfies the Hamilton-Jacobi equation $H(q^i, \frac{\partial W}{\partial q^i}) = E$, where $E$ is a constant.

From the proof of the above theorem given in Abraham and Marsden [1], we know that the assertion (i), equivalent to Hamilton-Jacobi equation (ii) by the generating function, gives a geometric constraint condition of the canonical symplectic form on the cotangent bundle $T^*Q$ for the Hamiltonian vector field of the system. Thus, the Hamilton-Jacobi equation reveals the deeply internal relationships of the generating function, the canonical symplectic form, and the dynamical vector field of a Hamiltonian system.

Now, it is a natural problem how to generalize Theorem 1.1 to fit the nonholonomic systems and their reduced systems. Note that if we take that $\gamma = dW$ in Theorem 1.1, then $\gamma$ is a closed one-form on $Q$, and the equation $d(H \cdot dW) = 0$ is equivalent to the Hamilton-Jacobi equation $H(q^i, \frac{\partial W}{\partial q^i}) = E$, where $E$ is a constant, which was called the classical Hamilton-Jacobi equation. This result was used the formulation of a geometric version of the Hamilton-Jacobi theorem for
Hamiltonian systems, see Cariñena et al. [5, 6]. Moreover, noting that Theorem 1.1 was also generalized in the context of a time-dependent Hamiltonian system by Marsden and Ratiu in [19], and the Hamilton-Jacobi equation may be regarded as a nonlinear partial differential equation for some generating function $S$. Thus, the problem becomes how to choose a time-dependent canonical transformation $\Psi : T^*Q \times \mathbb{R} \to T^*Q \times \mathbb{R}$ which transforms the dynamical vector field of a time-dependent Hamiltonian system to equilibrium such that the generating function $S$ of $\Psi$ satisfies the time-dependent Hamilton-Jacobi equation. In particular, for the time-independent Hamiltonian system, one may look for a symplectic map as the canonical transformation. This work offers an important idea: that one can use the dynamical vector field of a Hamiltonian system to describe the Hamilton-Jacobi equation. As a consequence, if we assume that $\gamma : Q \to T^*Q$ is a closed one-form on $Q$, and define that $X^\gamma_H = T\pi_Q \cdot X_H \cdot \gamma$, where $X_H$ is the dynamical vector field of the Hamiltonian system $(T^*Q, \omega, H)$, then we have that $X^\gamma_H$ and $X_H$ are $\gamma$-related; that is, $T\gamma \cdot X^\gamma_H = X_H \cdot \gamma$, is equivalent to that $d(H \cdot \gamma) = 0$, which was given in Cariñena et al. [5, 6]. Motivated by the above research, Wang in [27] proved an important lemma, which is a modification of the corresponding result of Abraham and Marsden in [1], such that we can derive precisely the geometric constraint conditions of the regular reduced symplectic forms for the dynamical vector fields of a regular reducible Hamiltonian system on the cotangent bundle of a configuration manifold; these are called the Type I and Type II Hamilton-Jacobi equations, because they are the development of the classical Hamilton-Jacobi equation given by Theorem 1.1 (see Abraham and Marsden [1] and Wang [27]). Moreover, León and Wang in [14] generalized the above results to the nonholonomic Hamiltonian system and the nonholonomic reducible Hamiltonian system on a cotangent bundle, by using the distributional Hamiltonian system and the reduced distributional Hamiltonian system.

In order to describe the impact of the different geometric structures and the constraints for the dynamics of a Hamiltonian system, in the following we shall consider the magnetic Hamiltonian system. Define that a magnetic symplectic form is $\omega^B = \omega - \pi_Q^*B$, where the $\pi_Q^*B$ is (called) a magnetic term on $T^*Q$, and $\omega$ is the usual canonical symplectic form on $T^*Q$, and $B$ is the closed two-form on $Q$, and the map $\pi_Q^*: T^*Q \to T^*T^*Q$. A magnetic Hamiltonian system is a Hamiltonian system defined by the magnetic symplectic form, which is a canonical Hamiltonian system coupling the action of the magnetic field $B$. Under the impact of the magnetic term $\pi_Q^*B$, the magnetic symplectic form $\omega^B$, in general, is not the canonical symplectic form on $T^*Q$, we cannot prove the Hamilton-Jacobi theorem for the magnetic Hamiltonian system the same as in Theorem 1.1. We have to look for a new way. In this paper, we drive precisely the geometric constraint condi-
tions of the magnetic symplectic form for the magnetic Hamiltonian vector field. These conditions are called the Type I and Type II Hamilton-Jacobi equations, which are the development of the Type I and Type II Hamilton-Jacobi equations for a Hamiltonian system given in Wang [27].

Second, we consider the magnetic Hamiltonian system with a nonholonomic constraint, which is called the nonholonomic magnetic Hamiltonian system. In mechanics, it happens very often that systems have constraints, and usually, under the restrictions of the nonholonomic constraints, in general, the dynamical vector field of a nonholonomic magnetic Hamiltonian system may not be Hamiltonian. Thus, we cannot describe the Hamilton-Jacobi equations for the nonholonomic magnetic Hamiltonian system from the viewpoint of a generating function as in the classical Hamiltonian case; that is, we cannot prove the Hamilton-Jacobi theorems for the nonholonomic magnetic Hamiltonian system the same as in Theorem 1.1. In this paper, by analyzing carefully the structures of the nonholonomic dynamical vector fields, we give a geometric formulation of the distributional magnetic Hamiltonian system for the nonholonomic magnetic Hamiltonian system, and the system is determined by a non-degenerate distributional two-form induced from the magnetic symplectic form. Note that the distributional magnetic Hamiltonian system is not Hamiltonian, however, it is a dynamical system closely related to a magnetic Hamiltonian system. Thus, we can drive precisely the two types of Hamilton-Jacobi equations for the distributional magnetic Hamiltonian system, which are the development of the Type I and Type II Hamilton-Jacobi equations for a distributional Hamiltonian system given in León and Wang [14].

Third, a natural problem is to consider the nonholonomic magnetic Hamiltonian system with symmetry. In this paper, we shall generalize the above results to the nonholonomic reducible magnetic Hamiltonian system with symmetry. Using the method of nonholonomic reduction given in Bates and Śniatycki [3], and analyzing carefully the structures of the nonholonomic reduced dynamical vector fields, we can give a geometric formulation of the nonholonomic reduced distributional magnetic Hamiltonian system. Since the nonholonomic reduced distributional magnetic Hamiltonian system is not Hamiltonian, and it is a dynamical system closely related to a magnetic Hamiltonian system, then we can derive precisely the geometric constraint conditions of the non-degenerate, and nonholonomic reduced distributional two-form for the nonholonomic reducible dynamical vector field; that is, the two types of Hamilton-Jacobi equations for the nonholonomic reduced distributional magnetic Hamiltonian system. These research reveal the deeply internal relationships of the magnetic symplectic structure, the nonholonomic constraint, the induced (resp. reduced) distributional
two-form, and the dynamical vector field of the nonholonomic magnetic Hamiltonian system.

The paper is organized as follows. In Section 2 we give some definitions and basic facts on the magnetic Hamiltonian system, the nonholonomic constraint, the nonholonomic magnetic Hamiltonian system and the distributional magnetic Hamiltonian system, which will be used in subsequent sections. In Section 3, for a magnetic Hamiltonian system defined by a magnetic symplectic form, we drive precisely the geometric constraint conditions of the magnetic symplectic form for the magnetic Hamiltonian vector field, which are called the Type I and Type II Hamilton-Jacobi equations. In Section 4, we derive the two types of Hamilton-Jacobi equations for a distributional magnetic Hamiltonian system, by the analysis and calculations in detail. The nonholonomic reducible magnetic Hamiltonian system with symmetry is considered in Section 5, and we derive precisely the geometric constraint conditions of the non-degenerate, and nonholonomic reduced distributional two-form for the nonholonomic reducible dynamical vector field; that is, the two types of Hamilton-Jacobi equations for the nonholonomic reduced distributional magnetic Hamiltonian system. These research develop the Hamilton-Jacobi theory for the nonholonomic magnetic Hamiltonian system, as well as with symmetry, and make us have much deeper understanding and recognition for the structures of the nonholonomic magnetic Hamiltonian systems.

2 Nonholonomic magnetic Hamiltonian system

In this section we first give some definitions and the basic facts on the magnetic Hamiltonian system, the nonholonomic constraint and the nonholonomic magnetic Hamiltonian system. Moreover, by analyzing carefully the structures for the nonholonomic dynamical vector fields, we can give a geometric formulation of a distributional magnetic Hamiltonian system, which is determined by a non-degenerate distributional two-form induced from the magnetic symplectic form. All of them will be used in subsequent sections.

Let $Q$ be an $n$-dimensional smooth manifold, $TQ$ be the tangent bundle, and $T^*Q$ be the cotangent bundle with a canonical symplectic form $\omega$. The projection $\pi_Q:T^*Q \to Q$ induces the map $\pi_Q^*:T^*Q \to T^*T^*Q$. We consider the magnetic symplectic form $\omega^B = \omega - \pi_Q^*B$, where $\omega$ is the canonical symplectic form on $T^*Q$, and $B$ is a closed two-form on $Q$, and $\pi_Q^*B$ is called a magnetic term on $T^*Q$. A magnetic Hamiltonian system is a triple $(T^*Q, \omega^B, H)$, which is a Hamiltonian system defined by the magnetic symplectic form $\omega^B$; that is, a canonical Hamiltonian
system coupling the action of the magnetic field $B$. For a given Hamiltonian $H$, the dynamical vector field is $X^B_H$, which is called the magnetic Hamiltonian vector field, and it satisfies the magnetic Hamilton’s equation; that is, $i_{X^B_H} \omega^B = dH$. In canonical cotangent bundle coordinates, for any $q \in Q$, $(q, p) \in T^*Q$, we have that

$$\omega = \sum_{i=1}^n dq^i \wedge dp_i, \quad B = \sum_{i,j=1}^n B_{ij} dq^i \wedge dq^j, \quad dB = 0,$$

$$\omega^B = \omega - \pi_B^* B = \sum_{i=1}^n dq^i \wedge dp_i - \sum_{i,j=1}^n B_{ij} dq^i \wedge dq^j,$$

and the magnetic Hamiltonian vector field $X^B_H$ can be expressed that

$$X^B_H = \sum_{i=1}^n \left( \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i} \right) - \sum_{i,j=1}^n B_{ij} \frac{\partial H}{\partial p_j} \frac{\partial}{\partial p_i} \left(\text{see Marsden et al. [17]}\right).$$

In order to describe the nonholonomic magnetic Hamiltonian system, we first give the completeness and the regularity conditions of the nonholonomic constraints of a mechanical system (see León and Wang [14]). In fact, we note that the set of Hamiltonian vector fields forms a Lie algebra with respect to its Lie bracket, since $[X^f, X^g]$. However, the Lie bracket operator, in general, may not be closed on the restrictions of the nonholonomic constraints. Thus, we have to give the completeness condition of nonholonomic constraints of a system.

**D-completeness:** Let $Q$ be a smooth manifold and $TQ$ be the tangent bundle. A distribution $D \subset TQ$ is said to be completely nonholonomic (or bracket-generating) if $D$ along with all of its iterated Lie brackets $[D, D], [D, [D, D]], \ldots$, spans the tangent bundle $TQ$. Moreover, we consider a nonholonomic mechanical system on $Q$, which is given by a Lagrangian function $L : TQ \to \mathbb{R}$ subject to constraints determined by a nonholonomic distribution $D \subset TQ$ on the configuration manifold $Q$. Then the nonholonomic system is said to be completely nonholonomic if the distribution $D \subset TQ$ determined by the nonholonomic constraints is completely nonholonomic.

**D-regularity:** In the following we always assume that $Q$ is a smooth manifold with coordinates $(q^i)$, and that $TQ$ is the tangent bundle with coordinates $(q^i, \dot{q}^i)$, and that $T^*Q$ is the cotangent bundle with coordinates $(q^i, p_j)$ that are the canonical cotangent coordinates of $T^*Q$, and that $\omega = dq^i \wedge dp_j$ is canonical symplectic form on $T^*Q$. If the Lagrangian $L : TQ \to \mathbb{R}$ is hyperregular; that is, the
Hessian matrix \( (\partial^2 L / \partial \dot{q}^i \partial \dot{q}^j) \) is nondegenerate everywhere, then the Legendre transformation \( FL : TQ \to T^*Q \) is a diffeomorphism. In this case the Hamiltonian \( H : T^*Q \to \mathbb{R} \) is given by \( H(q,p) = \dot{q} \cdot p - L(q, \dot{q}) \) with Hamiltonian vector field \( X_H \) that is defined by the Hamilton’s equation \( i_{X_H} \omega = dH \), and \( M = FL(\mathcal{D}) \) is a constraint submanifold in \( T^*Q \). In particular, for the nonholonomic constraint \( \mathcal{D} \subset TQ \), the Lagrangian \( L \) is said to be \( \mathcal{D} \)-regular, if the restriction of Hessian matrix \( (\partial^2 L / \partial \dot{q}^i \partial \dot{q}^j) \) on \( \mathcal{D} \) is nondegenerate everywhere. Moreover, a nonholonomic system is said to be \( \mathcal{D} \)-regular if its Lagrangian \( L \) is \( \mathcal{D} \)-regular. Note that the restriction of a positive definite symmetric bilinear form to a subspace is also positive definite, and hence, is nondegenerate. Thus, for a simple nonholonomic mechanical system; that is, whose Lagrangian is the total kinetic energy minus potential energy, is \( \mathcal{D} \)-regular automatically.

A nonholonomic magnetic Hamiltonian system is a 4-tuple \( (T^*Q, \omega^B, \mathcal{D}, H) \), which is a magnetic Hamiltonian system with a \( \mathcal{D} \)-completely and \( \mathcal{D} \)-regularly nonholonomic constraint \( \mathcal{D} \subset TQ \). Under the restrictions given by the constraints, in general, the dynamical vector field of the nonholonomic magnetic Hamiltonian system may not be magnetic Hamiltonian. However, the system is a dynamical system closely related to a magnetic Hamiltonian system. In the following we shall derive a distributional magnetic Hamiltonian system of the nonholonomic magnetic Hamiltonian system \( (T^*Q, \omega^B, \mathcal{D}, H) \), by analyzing carefully the structures for the nonholonomic dynamical vector fields using the method similar to that given in León and Wang [14]. It is worth noting that the leading distributional Hamiltonian system is also called a semi-Hamiltonian system in Patrick [23].

We consider that the constraint submanifold \( \mathcal{M} = FL(\mathcal{D}) \subset T^*Q \) and that \( i_{\mathcal{M}} : \mathcal{M} \to T^*Q \) is the inclusion, and that the symplectic form \( \omega^B_{\mathcal{M}} = i_{\mathcal{M}}^* \omega^B \), is induced from the magnetic symplectic form \( \omega^B \) on \( T^*Q \). We define the distribution \( \mathcal{F} \) as the pre-image of the nonholonomic constraints \( \mathcal{D} \) for the map \( T\pi_Q : TT^*Q \to TQ \); that is, \( \mathcal{F} = (T\pi_Q)^{-1}(\mathcal{D}) \subset TT^*Q \), which is a distribution along \( \mathcal{M} \), and denote that \( \mathcal{F}^\circ := \{ \alpha \in T^*T^*Q | \langle \alpha, v \rangle = 0, \forall v \in TT^*Q \} \) as the annihilator of \( \mathcal{F} \) in \( T^*T^*Q |_{\mathcal{M}} \). We consider that the following nonholonomic constraints condition is given by

\[
(i_X \omega^B - dH) \in \mathcal{F}^\circ, \quad X \in TM.
\]  

(2.1)

From Cantrijn et al. [4], we know that there exists a unique nonholonomic vector field \( X_n \) satisfying the above condition (2.1), if the admissibility condition \( \dim \mathcal{M} = \text{rank} \mathcal{F} \) and the compatibility condition \( TM \cap \mathcal{F}^\perp = \{0\} \) hold, where \( \mathcal{F}^\perp \) denotes the magnetic symplectic orthogonal of \( \mathcal{F} \) with respect to the magnetic symplectic form \( \omega^B \) on \( T^*Q \). In particular, if we consider the Whitney
sum decomposition that $T(T^*Q)|_M = TM \oplus F^\perp$ and the canonical projection that $P: T(T^*Q)|_M \to TM$, then we have that $X_n = P(X_n^B)$.

From the condition (2.1), we know that the nonholonomic vector field, in general, may not be magnetic Hamiltonian, because of the restrictions of nonholonomic constraints. However, we hope to study the dynamical vector field of the nonholonomic magnetic Hamiltonian system using the method similar to study the magnetic Hamiltonian vector field. From León and Wang [14] and Bates and Śniatycki [3], we can define the distributions that $K = F \cap TM$ and $K^\perp = F^\perp \cap TM$, where $K^\perp$ denotes the magnetic symplectic orthogonal of $K$ with respect to the magnetic symplectic form $\omega^B$. Assume that the admissibility condition $\dim M = \text{rank } F$ and the compatibility condition $TM \cap F^\perp = \{0\}$ hold, then we know that the restriction of the symplectic form $\omega^B$ on $TM$ fiberwise to the distribution $K$; that is, $\omega^B_K = \tau_K \cdot \omega^B_M$ is non-degenerate, where $\tau_K$ is the restriction map to distribution $K$. It is worth noting that $\omega^B_K$ is not a true two-form on a manifold, so it does not make sense to speak that it is closed. We regard $\omega^B_K$ as a distributional two-form to avoid any confusion. Since $\omega^B_K$ is non-degenerate as a bilinear form on each fibre of $K$, there exists a vector field $X^B_K$ on $M$, which takes values in the constraint distribution $K$ such that the distributional magnetic Hamiltonian equation holds, that is,

$$i_{X^B_K} \omega^B_K = dH_K,$$  \hspace{1cm} (2.2)  

where $dH_K$ is the restriction of $dH_M$ to $K$, and the function $H_K$ satisfies that $dH_K = \tau_K \cdot dH_M$, and that $H_M = \tau_M \cdot H$ is the restriction of $H$ to $M$. Moreover, from the distributional magnetic Hamiltonian equation (2.2), we have that $X^B_K = \tau_K \cdot X^B_M$. Then the triple ($K, \omega^B_K, H_K$) is a distributional magnetic Hamiltonian system of the nonholonomic magnetic Hamiltonian system $(T^*Q, \omega^B, D, H)$. Thus, the geometric formulation of the distributional magnetic Hamiltonian system may be summarized as follows:

**Definition 2.1 (Distributional magnetic Hamiltonian system).** Assume that the 4-tuple $(T^*Q, \omega^B, D, H)$ is a $D$-completely and $D$-regularly nonholonomic magnetic Hamiltonian system, where the magnetic symplectic form is $\omega^B = \omega - \pi_Q^* B$ on $T^*Q$, and $\omega$ is the canonical symplectic form on $T^*Q$ and $B$ is a closed two-form on $Q$, and $D \subset TQ$ is a $D$-completely and $D$-regularly nonholonomic constraint of the system. If there exist a distribution $K$, an associated non-degenerate distributional two-form $\omega^B_K$ which is induced by the magnetic symplectic form $\omega^B$ and a vector field $X^B_K$ on the constraint submanifold $M = F(L(D) \subset T^*Q$, such that the distributional magnetic Hamiltonian equation holds; that is, $i_{X^B_K} \omega^B_K = dH_K$, where $dH_K$ is the restriction of $dH_M$ to $K$ and the
function $H_K$ satisfies that $dH_K = \tau_K \cdot dH_M$ as defined above, then, the triple $(K, \omega^B_K, H_K)$ is called a distributional magnetic Hamiltonian system of the nonholonomic magnetic Hamiltonian system $(T^*Q, \omega^B, D, H)$, and $X^B_K$ is called a nonholonomic dynamical vector field of the distributional magnetic Hamiltonian system $(K, \omega^B_K, H_K)$. Under the above circumstances, we refer to $(T^*Q, \omega^B, D, H)$ as a nonholonomic magnetic Hamiltonian system with the associated distributional magnetic Hamiltonian system $(K, \omega^B_K, H_K)$.

Moreover, in Section 5, we consider the nonholonomic magnetic Hamiltonian system with symmetry, and using the method similar to that of the nonholonomic reduction given in León and Wang [14] and in Bates and Śniatycki [3], and analyzing carefully the structures of the nonholonomic reduced dynamical vector fields, we can also give a geometric formulation of the nonholonomic reduced distributional magnetic Hamiltonian system.

### 3 Hamilton-Jacobi equations of magnetic Hamiltonian system

In order to describe the impact of the different geometric structures and the constraints for the Hamilton-Jacobi theory, in this paper, we shall give two types of Hamilton-Jacobi equations for the magnetic Hamiltonian system, the distributional magnetic Hamiltonian system and the nonholonomic reduced distributional magnetic Hamiltonian system.

In this section, we first derive precisely the geometric constraint conditions of the magnetic symplectic form for the dynamical vector field of a magnetic Hamiltonian system, that is, the Type I and Type II Hamilton-Jacobi equations for the magnetic Hamiltonian system. In order to do this, in the following we first give some important notions and prove a key lemma, which is an important tool for the proofs of two types of Hamilton-Jacobi theorems for the magnetic Hamiltonian system.

Denote $\Omega^i(Q)$ as the set of all $i$-forms on $Q$, $i = 1, 2$. For any $\gamma \in \Omega^1(Q)$, $q \in Q$, we have that $\gamma(q) \in T_q^*Q$, and that we define a map $\gamma : Q \rightarrow T^*Q$, $q \rightarrow (q, \gamma(q))$. Hence, we say often that the map $\gamma : Q \rightarrow T^*Q$ is a one-form on $Q$. If the one-form $\gamma$ is closed, then $d\gamma(x,y) = 0, \forall x,y \in TQ$. Noting that, for any $v,w \in TT^*Q$, $d\gamma(T\pi_Q(v), T\pi_Q(w)) = \pi^*(d\gamma)(v,w)$ is a two-form on the cotangent bundle $T^*Q$, where $\pi^* : T^*Q \rightarrow T^*T^*Q$. Thus, in the following we can give a weaker notion.

**Definition 3.1.** The one-form $\gamma$ is called to be closed with respect to $T\pi_Q : TT^*Q \rightarrow TQ$, if for any $v,w \in TT^*Q$, we have that $d\gamma(T\pi_Q(v), T\pi_Q(w)) = 0$. 
For the one-form $\gamma: Q \to T^*Q$, $d\gamma$ is a two-form on $Q$. Assume that $B$ is a closed two-form on $Q$, then we say that the $\gamma$ satisfies condition $d\gamma = -B$ if for any $x, y \in TQ$, we have that $(d\gamma + B)(x, y) = 0$. In the following we can give a new notion.

**Definition 3.2.** Assume that $\gamma: Q \to T^*Q$ is a one-form on $Q$, we say that the $\gamma$ satisfies condition that $d\gamma = -B$ with respect to $T\pi_Q: TT^*Q \to TQ$ if for any $v, w \in TT^*Q$, we have that $(d\gamma + B)(T\pi_Q(v), T\pi_Q(w)) = 0$.

From the above definition we know that if $\gamma$ satisfies condition $d\gamma = -B$, then it must satisfy condition $d\gamma = -B$ with respect to $T\pi_Q: TT^*Q \to TQ$. Conversely, if $\gamma$ satisfies condition $d\gamma = -B$ with respect to $T\pi_Q: TT^*Q \to TQ$, then it may not satisfy condition $d\gamma = -B$. We can prove a general result as follows, which states that the notion that $\gamma$ satisfies condition $d\gamma = -B$ with respect to $T\pi_Q: TT^*Q \to TQ$, is not equivalent to the notion that $\gamma$ satisfies condition $d\gamma = -B$.

**Proposition 3.1.** Assume that $\gamma: Q \to T^*Q$ is a one-form on $Q$ and that it does not satisfy condition $d\gamma = -B$. We define the set $N$, which is a subset of $TQ$ such that the one-form $\gamma$ on $Q$ satisfies the condition that, for any $x, y \in N$, $(d\gamma + B)(x, y) \neq 0$. Denote that $Ker(T\pi_Q) = \{u \in TT^*Q | T\pi_Q(u) = 0\}$, and that $T\gamma: TQ \to TT^*Q$ is the tangent map of $\gamma: Q \to T^*Q$. If $T\gamma(N) \subseteq Ker(T\pi_Q)$, then $\gamma$ satisfies condition $d\gamma = -B$ with respect to $T\pi_Q: TT^*Q \to TQ$.

**Proof.** For any $v, w \in TT^*Q$, if $T\pi_Q(v) \notin N$, or $T\pi_Q(w)) \notin N$, from the definition of $N$, we know that $(d\gamma + B)(T\pi_Q(v), T\pi_Q(w)) = 0$. If $T\pi_Q(v) \in N$, and $T\pi_Q(w) \in N$, from the condition $T\gamma(N) \subseteq Ker(T\pi_Q)$, we know that $T\pi_Q \cdot T\gamma \cdot T\pi_Q(v) = T\pi_Q(v) = 0$, and that $T\pi_Q \cdot T\gamma \cdot T\pi_Q(w) = T\pi_Q(w) = 0$, where we have used the relation $\pi_Q \cdot \gamma \cdot \pi_Q = \pi_Q$, and hence, $(d\gamma + B)(T\pi_Q(v), T\pi_Q(w)) = 0$. Thus, for any $v, w \in TT^*Q$, we have always that $(d\gamma + B)(T\pi_Q(v), T\pi_Q(w)) = 0$; that is, $\gamma$ satisfies condition $d\gamma = -B$ with respect to $T\pi_Q: TT^*Q \to TQ$. 

From the Definitions 3.1 and 3.2, we know that when $B = 0$, the notion that $\gamma$ satisfies condition $d\gamma = -B$ with respect to $T\pi_Q: TT^*Q \to TQ$ become the notion that $\gamma$ is closed with respect to $T\pi_Q: TT^*Q \to TQ$. Now, we can prove the following lemma, which is a generalization of the corresponding to lemma given by Wang [27], and the lemma is a very important tool for our research.

**Lemma 3.1.** Assume that $\gamma: Q \to T^*Q$ is a one-form on $Q$, and that $\lambda = \gamma \cdot \pi_Q: T^*Q \to T^*Q$. For the magnetic symplectic form $\omega^B = \omega - \pi^*_QB$ on $T^*Q$, where $\omega$ is the canonical symplectic form on $T^*Q$, and $B$ is a closed two-form on $Q$, then we have that the following two assertions hold:
(i) For any \(v, w \in TT^*Q\), we have that 
\[
\lambda^*\omega^B(v, w) = -(d\lambda + B)(T\pi_Q(v), T\pi_Q(w)).
\]

(ii) For any \(v, w \in TT^*Q\), we have that 
\[
\omega^B(T\lambda \cdot v, w) = \omega^B(v, w - T\lambda \cdot w) - (d\gamma + B)(T\pi_Q(v), T\pi_Q(w)).
\]

**Proof.** We first prove assertion (i). Since \(\omega\) is the canonical symplectic form on \(T^*Q\), there is an unique canonical one-form \(\theta\) such that \(\omega = -d\theta\). From the [1, Proposition 3.2.11], we have that for the one-form \(\gamma: Q \to T^*Q, \gamma^*\theta = \gamma\). Then we can obtain that, for any \(x, y \in TQ\),
\[
\gamma^*(x, y) = \gamma^*(-d\theta)(x, y) = -d(\gamma^*(\theta))(x, y) = -d\gamma(x, y).
\]

Note that \(\lambda = \gamma \cdot \pi_Q: T^*Q \to T^*Q\), and that \(\lambda^* = \pi_Q^* \cdot \gamma^* : T^*T^*Q \to T^*T^*Q\), then we have that, for any \(v, w \in TT^*Q\),
\[
\lambda^*\omega(v, w) = \lambda^*(-d\theta)(v, w) = -d(\lambda^*(\theta))(v, w) = -d(\pi_Q^* \cdot \gamma^*)(v, w)
\]
\[
= -d(\pi_Q^* \cdot \gamma)(v, w) = -d\gamma(T\pi_Q(v), T\pi_Q(w)).
\]

Hence, we have that
\[
\lambda^*\omega^B(v, w) = \lambda^*\omega(v, w) - \lambda^* \cdot \pi_Q^* B(v, w)
\]
\[
= -d\gamma(T\pi_Q(v), T\pi_Q(w)) - (\pi_Q \cdot \gamma \cdot \pi_Q)^* B(v, w)
\]
\[
= -d\gamma(T\pi_Q(v), T\pi_Q(w)) - \pi_Q^* B(v, w)
\]
\[
= -(d\gamma + B)(T\pi_Q(v), T\pi_Q(w)),
\]

where we have used the relation \(\pi_Q \cdot \gamma \cdot \pi_Q = \pi_Q\). It follows that the assertion (i) holds.

Next, we prove assertion (ii). For any \(v, w \in TT^*Q\), noting that \(v - T(\gamma \cdot \pi_Q) \cdot v\) is vertical, because
\[
T\pi_Q(v - T(\gamma \cdot \pi_Q) \cdot v) = T\pi_Q(v) - T(\pi_Q \cdot \gamma \cdot \pi_Q) \cdot v = T\pi_Q(v) - T\pi_Q(v) = 0.
\]

Thus, \(\omega(v - T(\gamma \cdot \pi_Q) \cdot v, w - T(\gamma \cdot \pi_Q) \cdot w) = 0\), and hence,
\[
\omega(T(\gamma \cdot \pi_Q) \cdot v, w) = \omega(v, w - T(\gamma \cdot \pi_Q) \cdot w) + \omega(T(\gamma \cdot \pi_Q) \cdot v, T(\gamma \cdot \pi_Q) \cdot w).
\]

However, the second term on the right-hand side is given by
\[
\omega(T(\gamma \cdot \pi_Q) \cdot v, T(\gamma \cdot \pi_Q) \cdot w)
\]
\[
= \gamma^*\omega(T\pi_Q(v), T\pi_Q(w)) = -d\gamma(T\pi_Q(v), T\pi_Q(w)).
\]
It follows that
\[
\omega(T\lambda \cdot v, w) = \omega(T(\gamma \cdot \pi_Q) \cdot v, w) = \omega(v, w - T(\gamma \cdot \pi_Q) \cdot w) - d\gamma(T\pi_Q(v), T\pi_Q(w)) = \omega(v, w - T\lambda \cdot w) - d\gamma(T\pi_Q(v), T\pi_Q(w)).
\]

Hence, we have that
\[
\omega^B(T\lambda \cdot v, w) = \omega(T\lambda \cdot v, w) - \pi_Q^* B(T\lambda \cdot v, w) = \omega(v, w - T\lambda \cdot w) - d\gamma(T\pi_Q(v), T\pi_Q(w)) - B(T\pi_Q(T\lambda \cdot v), T\pi_Q(v)) = \omega^B(v, w - T\lambda \cdot w) - d\gamma(T\pi_Q(v), T\pi_Q(w)) - B(T\pi_Q(T\lambda \cdot v), T\pi_Q(v)) = \omega^B(v, w - T\lambda \cdot w) - d\gamma(T\pi_Q(v), T\pi_Q(w)) - B(T\pi_Q(T\lambda \cdot v), T\pi_Q(v)) = \omega^B(v, w - T\lambda \cdot w) - (d\gamma + B)(T\pi_Q(v), T\pi_Q(w)) = \omega^B(v, w - T\lambda \cdot w) - (d\gamma + B)(T\pi_Q(v), T\pi_Q(w)).
\]

Thus, the assertion (ii) holds. \(\square\)

Since a magnetic Hamiltonian system is also a Hamiltonian system defined by the magnetic symplectic form, and it is a canonical Hamiltonian system coupling the action of a magnetic field \(B\). Usually, under the impact of magnetic term \(\pi_Q^* B\), the magnetic symplectic form \(\omega^B = \omega - \pi_Q^* B\), in general, is not the canonical symplectic form \(\omega\) on \(T^* Q\), we cannot prove the Hamilton-Jacobi theorem for a magnetic Hamiltonian system the same as in Theorem 1.1. However, we can give precisely the geometric constraint conditions of a magnetic symplectic form for the dynamical vector field of a magnetic Hamiltonian system; that is, the Type I and Type II Hamilton-Jacobi equations for the magnetic Hamiltonian system. First, for a given magnetic Hamiltonian system \((T^* Q, \omega^B, H)\) on \(T^* Q\), using the Lemma 3.1, the magnetic symplectic form \(\omega^B\) and the magnetic Hamiltonian vector field \(X^B_H\), we can derive precisely the Type I Hamilton-Jacobi equation for the magnetic Hamiltonian system \((T^* Q, \omega^B, H)\).
Theorem 3.1 (Type I Hamilton-Jacobi theorem for a magnetic Hamiltonian system). For a given magnetic Hamiltonian system \( (T^*Q, \omega^B, H) \) with a magnetic symplectic form \( \omega^B = \omega - \pi_Q^*B \) on \( T^*Q \), where \( \omega \) is the canonical symplectic form on \( T^*Q \) and \( B \) is a closed two-form on \( Q \), assume that \( \gamma : Q \to T^*Q \) is a one-form on \( Q \), and that \( X^\gamma = T\pi_Q \cdot X^B_H \cdot \gamma \), where \( X^B_H \) is the dynamical vector field of the magnetic Hamiltonian system \( (T^*Q, \omega^B, H) \); that is, the magnetic Hamiltonian vector field. If the one-form \( \gamma : Q \to T^*Q \) satisfies the equation \( d\gamma = -B \) with respect to \( T\pi_Q : T^*Q \to TQ; \) then \( \gamma \) is a solution of the equation \( T\gamma \cdot X^\gamma = X^B_H \cdot \gamma \), and the equation is called the Type I Hamilton-Jacobi equation for the magnetic Hamiltonian system \( (T^*Q, \omega^B, H) \). Here the maps involved in the theorem are shown in Fig. 1.

\[
\begin{align*}
T^*Q & \xrightarrow{\pi_Q} Q \xrightarrow{T} T^*Q \\
T(T^*Q) & \xrightarrow{T\gamma} TQ \xrightarrow{T\pi_Q} T(T^*Q)
\end{align*}
\]

Figure 1: Diagram-1.

Proof. We take that \( v = X^B_H \cdot \gamma \in TT^*Q \), and that, for any \( w \in TT^*Q, T\pi_Q(w) \neq 0 \), from Lemma 3.1(ii) and the condition that \( d\gamma = -B \) with respect to \( T\pi_Q : TT^*Q \to TQ \); that is, \( (d\gamma + B)(T\pi_Q \cdot X^B_H \cdot \gamma, T\pi_Q \cdot w) = 0 \), we have that

\[
\omega^B \left(T\gamma \cdot X^\gamma, w\right) = \omega^B \left(T\gamma \cdot T\pi_Q \cdot X^B_H \cdot \gamma, w\right) = \omega^B \left(T(\gamma \cdot \pi_Q) \cdot X^B_H \cdot \gamma, w\right)
\]

\[
= \omega^B \left(X^B_H \cdot \gamma, w - T(\gamma \cdot \pi_Q) \cdot w\right) - (d\gamma + B) \left(T\pi_Q \cdot X^B_H \cdot \gamma, T\pi_Q \cdot w\right)
\]

\[
= \omega^B \left(X^B_H \cdot \gamma, w\right) - \omega^B \left(X^B_H \cdot \gamma, T\lambda \cdot w\right).
\]

Hence, we have that

\[
\omega^B \left(T\gamma \cdot X^\gamma, w\right) - \omega^B \left(X^B_H \cdot \gamma, w\right) = -\omega^B \left(X^B_H \cdot \gamma, T\lambda \cdot w\right). \tag{3.1}
\]

If \( \gamma \) satisfies the equation \( T\gamma \cdot X^\gamma = X^B_H \cdot \gamma \), from Lemma 3.1(i), we know that the right side of the (3.1) becomes that

\[
\omega^B \left(X^B_H \cdot \gamma, T\lambda \cdot w\right) = \omega^B \left(T\gamma \cdot X^\gamma, T\lambda \cdot w\right) = \omega^B \left(T\gamma \cdot T\pi_Q \cdot X^B_H \cdot \gamma, T\lambda \cdot w\right)
\]

\[
= \omega^B \left(T\lambda \cdot X^B_H \cdot \gamma, T\lambda \cdot w\right) = \lambda^* \omega^B \left(X^B_H \cdot \gamma, w\right)
\]

\[
= -(d\gamma + B) \left(T\pi_Q \cdot X^B_H \cdot \gamma, T\pi_Q \cdot w\right) = 0,
\]
since $\gamma: Q \to T^*Q$ satisfies the condition that $d\gamma = -B$ with respect to $T\pi_Q: TT^*Q \to TQ$. However, because the magnetic symplectic form $\omega^B$ is non-degenerate, the left side of (3.1) equals zero, only when $\gamma$ satisfies the equation $T\gamma \cdot X^\gamma = X^B_H \cdot \gamma$. Thus, if the one-form $\gamma: Q \to T^*Q$ satisfies the condition that $d\gamma = -B$ with respect to $T\pi_Q: TT^*Q \to TQ$, then $\gamma$ must be a solution of the Type I Hamilton-Jacobi equation $T\gamma \cdot X^\gamma = X^B_H \cdot \gamma$ for the magnetic Hamiltonian system $(T^*Q, \omega^B, H)$.

It is worth noting that, when $B = 0$, the magnetic symplectic form $\omega^B$ is just the canonical symplectic form $\omega$ on $T^*Q$, and in this case, the magnetic Hamiltonian system $(T^*Q, \omega^B, H)$ becomes the Hamiltonian system $(T^*Q, \omega, H)$ with the canonical symplectic form $\omega$, and the condition that the one-form $\gamma: Q \to T^*Q$ satisfies the condition, $d\gamma = -B$ with respect to $T\pi_Q: TT^*Q \to TQ$, becomes the condition that $\gamma$ is closed with respect to $T\pi_Q: TT^*Q \to TQ$. Thus, from Theorem 3.1, we can obtain Theorem 2.5 (given in Wang [27]); that is the Type I Hamilton-Jacobi theorem for a Hamiltonian system. On the other hand, from the proof of [27, Theorem 2.5], we know that if an one-form $\gamma: Q \to T^*Q$ is not closed on $Q$ with respect to $T\pi_Q: TT^*Q \to TQ$, then $\gamma$ is not a solution of the Type I Hamilton-Jacobi equation $T\gamma \cdot X^\gamma = X^B_H \cdot \gamma$. However, noting that if $\gamma: Q \to T^*Q$ is not closed on $Q$ with respect to $T\pi_Q: TT^*Q \to TQ$; that is, there exist $v, w \in TT^*Q$ such that $d\gamma(T\pi_Q(v), T\pi_Q(w)) \neq 0$, and hence, $\gamma$ is not yet closed on $Q$. But, because $d \cdot d\gamma = d^2\gamma = 0$, and hence, the $d\gamma$ is a closed two-form on $Q$. Thus, we can construct a magnetic symplectic form on $T^*Q$; that is, $\omega^B = \omega + \pi^*_Q(d\gamma) = \omega - \pi^*_Q B$, where $B = -d\gamma$, and $\omega$ is the canonical symplectic form on $T^*Q$, and $\pi^*_Q: T^*Q \to T^*T^*Q$. Moreover, we hope to look for a new magnetic Hamiltonian system such that $\gamma$ is a solution of the Type I Hamilton-Jacobi equation for the new magnetic Hamiltonian system. In fact, for a given Hamiltonian system $(T^*Q, \omega, H)$ with the canonical symplectic form $\omega$ on $T^*Q$, and assume that $\gamma: Q \to T^*Q$ is a one-form on $Q$, and that it is not closed with respect to $T\pi_Q: TT^*Q \to TQ$. Then we can construct a magnetic symplectic form on $T^*Q$; that is, $\omega^B = \omega + \pi^*_Q(d\gamma)$, where $B = -d\gamma$, and a magnetic Hamiltonian system $(T^*Q, \omega^B, H)$, its dynamical vector field is given by $X^B_H$, which satisfies the magnetic Hamiltonian equation; that is, $i_{X^B_H} \omega^B = dH$. In this case, for any $x, y \in TQ$, we have that $(d\gamma + B)(x, y) = 0$, and hence that, for any $v, w \in TT^*Q$, we have $(d\gamma + B)(T\pi_Q(v), T\pi_Q(w)) = 0$; that is, the one-form $\gamma: Q \to T^*Q$ satisfies the condition that $d\gamma = -B$ with respect to $T\pi_Q: TT^*Q \to TQ$. Thus, using Lemma 3.1 and the magnetic Hamiltonian vector field $X^B_H$, from Theorem 3.1, we can obtain the Theorem 3.2 as follows:

**Theorem 3.2.** For a given Hamiltonian system $(T^*Q, \omega, H)$ with the canonical symplec-
tic form $\omega$ on $T^*Q$, and assume that the one-form $\gamma: Q \to T^*Q$ is not closed with respect to $T\pi_Q: TT^*Q \to TQ$, then one can construct a magnetic symplectic form on $T^*Q$; that is, $\omega^B = \omega + \pi_Q^*(d\gamma)$, where $B = -d\gamma$, and a magnetic Hamiltonian system $(T^*Q, \omega^B, H)$. Denote that $X^\gamma = T\pi_Q \cdot X^B_H \cdot \gamma$, where $X^B_H$ is the dynamical vector field of the magnetic Hamiltonian system $(T^*Q, \omega^B, H)$. Then $\gamma$ is a solution of the Type I Hamilton-Jacobi equation $T\gamma \cdot X^\gamma = X^B_H \cdot \gamma$, for the magnetic Hamiltonian system $(T^*Q, \omega^B, H)$.

Next, for any symplectic map $\epsilon: T^*Q \to T^*Q$ with respect to the magnetic symplectic form $\omega^B$, we can derive precisely the Type II Hamilton-Jacobi equation for the magnetic Hamiltonian system $(T^*Q, \omega^B, H)$. For convenience, the maps involved in the theorem and its proof are shown in Fig. 2.

![Diagram 2](image)

**Theorem 3.3** (Type II Hamilton-Jacobi theorem for a magnetic Hamiltonian system). For the magnetic Hamiltonian system $(T^*Q, \omega^B, H)$ with a magnetic symplectic form $\omega^B = \omega - \pi_Q^*B$ on $T^*Q$, where $\omega$ is the canonical symplectic form on $T^*Q$ and $B$ is a closed two-form on $Q$, assume that $\gamma: Q \to T^*Q$ is a one-form on $Q$, and that $\lambda = \gamma \cdot \pi_Q: T^*Q \to T^*Q$, and that for any symplectic map $\epsilon: T^*Q \to T^*Q$ with respect to $\omega^B$, denote that $X^\epsilon = T\pi_Q \cdot X^B_H \cdot \epsilon$, where $X^B_H$ is the dynamical vector field of the magnetic Hamiltonian system $(T^*Q, \omega^B, H)$; that is, the magnetic Hamiltonian vector field. Then $\epsilon$ is a solution of the equation $T\epsilon \cdot X^B_H \cdot \epsilon = T\lambda \cdot X^B_H \cdot \epsilon$ if and only if it is a solution of the equation $T\gamma \cdot X^\epsilon = X^B_H \cdot \epsilon$, where $X^B_H \in TT^*Q$ is the magnetic Hamiltonian vector field of the function $H \cdot \epsilon: T^*Q \to \mathbb{R}$. The equation $T\gamma \cdot X^\epsilon = X^B_H \cdot \epsilon$, is called the Type II Hamilton-Jacobi equation for the magnetic Hamiltonian system $(T^*Q, \omega^B, H)$.

**Proof.** We take that $v = X^B_H \cdot \epsilon \in TT^*Q$, and that for any $w \in TT^*Q$, $T\lambda(w) \neq 0$, from Lemma 3.1, we have that

\[
\omega^B(T\gamma \cdot X^\epsilon, w) = \omega^B(T\gamma \cdot T\pi_Q \cdot X^B_H \cdot \epsilon, w) = \omega^B(T(\gamma \cdot \pi_Q) \cdot X^B_H \cdot \epsilon, w)
\]

\[
= \omega^B(X^B_H \cdot \epsilon, w - T(\gamma \cdot \pi_Q) \cdot w) - (d\gamma + B)(T\pi_Q(X^B_H \cdot \epsilon), T\pi_Q(w))
\]

\[
= \omega^B(X^B_H \cdot \epsilon, w) - \omega^B(X^B_H \cdot \epsilon, T\lambda \cdot w) + \lambda^* \omega^B(X^B_H \cdot \epsilon, w)
\]

\[
= \omega^B(X^B_H \cdot \epsilon, w) - \omega^B(X^B_H \cdot \epsilon, T\lambda \cdot w) + \omega^B(T\lambda \cdot X^B_H \cdot \epsilon, T\lambda \cdot w).
\]
Because $\varepsilon: T^*Q \to T^*Q$ is symplectic with respect to $\omega^B$, and hence, $X^B_{H\cdot \varepsilon} = T\varepsilon \cdot X^B_{H\cdot \varepsilon}$ along $\varepsilon$. From the above arguments, we can obtain that

\[
\omega^B( T\gamma \cdot X^\varepsilon, \omega) - \omega^B( X^B_{H\cdot \varepsilon}, \omega) = -\omega^B( T\varepsilon \cdot X^B_{H\cdot \varepsilon}, T\lambda \cdot w ) + \omega^B( T\lambda \cdot X^B_{H\cdot \varepsilon}, T\lambda \cdot w )
\]

\[
= \omega^B( T\lambda \cdot X^B_{H\cdot \varepsilon} - T\varepsilon \cdot X^B_{H\cdot \varepsilon}, T\lambda \cdot w ).
\]

Since the magnetic symplectic form $\omega^B$ is non-degenerate, it follows that $T\gamma \cdot X^\varepsilon = X^B_{H\cdot \varepsilon}$ is equivalent to $T\varepsilon \cdot X^B_{H\cdot \varepsilon} = T\lambda \cdot X^B_{H\cdot \varepsilon}$. Thus, $\varepsilon$ is a solution of the equation $T\varepsilon \cdot X^B_{H\cdot \varepsilon} = T\lambda \cdot X^B_{H\cdot \varepsilon}$ if and only if it is a solution of the Type II Hamilton-Jacobi equation $T\gamma \cdot X^\varepsilon = X^B_{H\cdot \varepsilon}$.

\[\square\]

**Remark 3.1.** It is worth noting that the Type I Hamilton-Jacobi equation $T\gamma \cdot X^\gamma = X^B_{H\cdot \gamma}$ is the equation of the differential one-form $\gamma$, and that the Type II Hamilton-Jacobi equation $T\gamma \cdot X^\varepsilon = X^B_{H\cdot \varepsilon}$ is the equation of the symplectic diffeomorphism map $\varepsilon$. When $B = 0$, the magnetic symplectic form $\omega^B$ is just the canonical symplectic form $\omega$ on $T^*Q$, and the magnetic Hamiltonian system is just the canonical Hamiltonian system itself. From the Type I and Type II Hamilton-Jacobi theorems; that is, Theorems 3.1 and 3.3, we can get the Theorems 2.5 and 2.6 given in Wang [27]. It shows that Theorems 3.1 and 3.3 can be regarded as an extension of the two types of Hamilton-Jacobi theorems for a Hamiltonian system (given in [27]) to that for the magnetic Hamiltonian system.

### 4 Hamilton-Jacobi equations for distributional magnetic Hamiltonian system

In this section we shall derive precisely the geometric constraint conditions of the induced distributional two-form for the nonholonomic dynamical vector field of a distributional magnetic Hamiltonian system; that is, the two types of Hamilton-Jacobi equations for a distributional magnetic Hamiltonian system. In order to do this, in the following we first give some important notions and prove a key lemma, which is an important tool for the proofs of the two types of Hamilton-Jacobi theorems for the distributional magnetic Hamiltonian system.

Assuming that $\mathcal{D} \subset TQ$ is a $\mathcal{D}$-regularly nonholonomic constraint, the constraint submanifold $\mathcal{M} = \mathcal{F}L(\mathcal{D}) \subset T^*Q$, the distribution $\mathcal{F} = (T\pi_Q)^{-1}(\mathcal{D}) \subset TT^*Q$, and
\( \gamma : Q \to T^*Q \) is a one-form on \( Q \), and \( B \) is a closed two-form on \( Q \), we introduce two weaker notions.

**Definition 4.1.** (i) The one-form \( \gamma \) is called to be closed on \( D \) with respect to \( T\pi_Q : TT^*Q \to TQ \) if, for any \( v,w \in TT^*Q \) and \( T\pi_Q(v),T\pi_Q(w) \in D \), we have that 
\[
d\gamma(T\pi_Q(v),T\pi_Q(w)) = 0.
\]

(ii) The one-form \( \gamma : Q \to T^*Q \) is called satisfying the condition that \( d\gamma = -B \) on \( D \) with respect to \( T\pi_Q : TT^*Q \to TQ \) if, for any \( v,w \in TT^*Q \) and \( T\pi_Q(v),T\pi_Q(w) \in D \), we have that 
\[
(d\gamma + B)(T\pi_Q(v),T\pi_Q(w)) = 0.
\]

From the Definition 4.1, we know that when \( B = 0 \), the notion that \( \gamma \) satisfies the condition that \( d\gamma = -B \) on \( D \) with respect to \( T\pi_Q : TT^*Q \to TQ \) becomes the notion that \( \gamma \) is closed on \( D \) with respect to \( T\pi_Q : TT^*Q \to TQ \). On the other hand, noting that \( \gamma \) satisfies the condition that \( d\gamma = -B \) on \( D \) with respect to \( T\pi_Q : TT^*Q \to TQ \) is weaker than that \( \gamma \) satisfies the condition \( d\gamma = -B \) on \( D \); that is, \( (d\gamma + B)(x,y) = 0, \forall x,y \in D \). In fact, if \( \gamma \) satisfies the condition that \( d\gamma = -B \) on \( D \), then it must satisfy the condition that \( d\gamma = -B \) on \( D \) with respect to \( T\pi_Q : TT^*Q \to TQ \). Conversely, if \( \gamma \) satisfies the condition that \( d\gamma = -B \) on \( D \) with respect to \( T\pi_Q : TT^*Q \to TQ \), then it may not satisfy the condition \( d\gamma = -B \) on \( D \).

We can prove a general result as follows; which states that the notion that \( \gamma \) satisfies the condition that \( d\gamma = -B \) on \( D \) with respect to \( T\pi_Q : TT^*Q \to TQ \) is not equivalent to the notion that \( \gamma \) satisfies the condition \( d\gamma = -B \) on \( D \).

**Proposition 4.1.** Assume that \( \gamma : Q \to T^*Q \) is a one-form on \( Q \) and that it does not satisfy the condition \( d\gamma = -B \) on \( D \). We define the set \( N \), which is a subset of \( TQ \), such that the one-form \( \gamma \) on \( N \) satisfies the condition that for any \( x,y \in N \), \( (d\gamma + B)(x,y) \neq 0 \). Denote that \( \text{Ker}(T\pi_Q) = \{ u \in TT^*Q \mid T\pi_Q(u) = 0 \} \), and that \( T\gamma : TQ \to TT^*Q \). If \( T\gamma(N) \subset \text{Ker}(T\pi_Q) \), then \( \gamma \) satisfies the condition that \( d\gamma = -B \) with respect to \( T\pi_Q : TT^*Q \to TQ \) and hence, \( \gamma \) satisfies the condition that \( d\gamma = -B \) on \( D \) with respect to \( T\pi_Q : TT^*Q \to TQ \).

**Proof.** In fact, if the \( \gamma : Q \to T^*Q \) does not satisfy the condition that \( d\gamma = -B \) on \( D \), then it does not yet satisfy the condition that \( d\gamma = -B \). From the proof of Proposition 4.1, for any \( v,w \in TT^*Q \), we have always that \( (d\gamma + B)(T\pi_Q(v),T\pi_Q(w)) = 0 \). In particular, for any \( v,w \in TT^*Q \), and \( T\pi_Q(v),T\pi_Q(w) \in D \), we have that \( (d\gamma + B)(T\pi_Q(v),T\pi_Q(w)) = 0 \); that is, \( \gamma \) satisfies the condition that \( d\gamma = -B \) on \( D \) with respect to \( T\pi_Q : TT^*Q \to TQ \).

Now, we prove the Lemma 4.1. It is worth noting that this lemma and Lemma 3.1 given in Section 3 are the important tools for the proofs of the two types
of Hamilton-Jacobi theorems for the distributional magnetic Hamiltonian system and the nonholonomic reduced distributional magnetic Hamiltonian system.

**Lemma 4.1.** Assume that $\gamma: Q \rightarrow T^* Q$ is a one-form on $Q$, and that $\lambda = \gamma \cdot \pi_Q: T^* Q \rightarrow T^* Q$, and that $\omega$ is the canonical symplectic form on $T^* Q$, and that $\omega^B = \omega - \pi^*_Q B$ is the magnetic symplectic form on $T^* Q$. If the Lagrangian $L$ is $D$-regular, and assume that $\text{Im}(\gamma) \subset M$, where $M = \mathcal{F}L(D)$, then we have that $X^B_H \cdot \gamma \in \mathcal{F}$ along $\gamma$, and that $X^B_H \cdot \lambda \in \mathcal{F}$ along $\lambda$; that is, $T\pi_Q(X^B_H \cdot \gamma(q)) \in \mathcal{D}_q, \forall q \in Q$ and $T\pi_Q(X^B_H \cdot \lambda(q,p)) \in \mathcal{D}_q, \forall q \in Q, (q,p) \in T^* Q$. Moreover, if a symplectic map $\varepsilon: T^* Q \rightarrow T^* Q$ with respect to the magnetic symplectic form $\omega^B$ satisfies the condition that $\varepsilon(M) \subset M$, then we have that $X^B_H \cdot \varepsilon \in \mathcal{F}$ along $\varepsilon$.

**Proof.** Under the canonical cotangent bundle coordinates, for any $q \in Q, (q,p) \in T^* Q$, we have that

$$X^B_H \cdot \gamma(q) = \left( \sum_{i=1}^{n} \left( \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i} \right) - \sum_{i,j=1}^{n} B_{ij} \frac{\partial H}{\partial p_j} \frac{\partial}{\partial p_i} \right) \gamma(q)$$

and that

$$X^B_H \cdot \lambda(q,p) = \left( \sum_{i=1}^{n} \left( \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i} \right) - \sum_{i,j=1}^{n} B_{ij} \frac{\partial H}{\partial p_j} \frac{\partial}{\partial p_i} \right) \gamma \cdot \pi_Q(q,p).$$

Thus, we have that

$$T\pi_Q \left( X^B_H \cdot \gamma(q) \right) = T\pi_Q \left( X^B_H \cdot \lambda(q,p) \right) = \sum_{i=1}^{n} \left( \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} \right) \gamma(q) \in T_q Q.$$ 

Since $\text{Im}(\gamma) \subset M$, and $\gamma(q) \in M_{(q,p)} = \mathcal{F}L(D_q)$, from the Lagrangian $L$ is $D$-regular, and $\mathcal{F}L$ is a diffeomorphism, then there exists a point $(q,v_q) \in D_q$ such that $\mathcal{F}L(q,v_q) = \gamma(q)$. Thus,

$$T\pi_Q \left( X^B_H \cdot \gamma(q) \right) = T\pi_Q \left( X^B_H \cdot \lambda(q,p) \right) = \mathcal{F}L(q,v_q) \sum_{i=1}^{n} \left( \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} \right) \in D_q.$$ 

It follows that $X^B_H \cdot \gamma \in \mathcal{F}$ along $\gamma$, and that $X^B_H \cdot \lambda \in \mathcal{F}$ along $\lambda$. Moreover, for the symplectic map $\varepsilon: T^* Q \rightarrow T^* Q$ with respect to the magnetic symplectic form $\omega^B$, we have that

$$X^B_H \cdot \varepsilon(q,p) = \left( \sum_{i=1}^{n} \left( \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i} \right) - \sum_{i,j=1}^{n} B_{ij} \frac{\partial H}{\partial p_j} \frac{\partial}{\partial p_i} \right) \varepsilon(q,p).$$
If \( \varepsilon \) satisfies the condition that \( \varepsilon(M) \subset M \), then, for any \((q,p) \in M_{(q,p)}\), we have that \( \varepsilon(q,p) \in M_{(q,p)} \), and that there exists a point \((q,v_q) \in D_q \) such that \( FL(q,v_q) = \varepsilon(q,p) \). Thus,

\[
T\pi_Q \left(X_H^B \cdot \varepsilon(q,p) \right) = \sum_{i=1}^{n} \left( \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} \right) \varepsilon(q,p) = FL(q,v_q) \sum_{i=1}^{n} \left( \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} \right) \in D_q.
\]

It follows that \( X_H^B \cdot \varepsilon \in \mathcal{F} \) along \( \varepsilon \).

We note that for a nonholonomic magnetic Hamiltonian system, under the restriction of the nonholonomic constraint, in general, the dynamical vector field of the nonholonomic magnetic Hamiltonian system may not be Hamiltonian. On the other hand, since the distributional magnetic Hamiltonian system is determined by a non-degenerate distributional two-form, which is induced from the magnetic symplectic form, but, the non-degenerate distributional two-form is not a "true two-form" on a manifold, and hence, the leading distributional magnetic Hamiltonian system cannot be Hamiltonian. Thus, we cannot describe the Hamilton-Jacobi equations for the nonholonomic magnetic Hamiltonian system from the viewpoint of a generating function just as in the classical Hamiltonian case; that is, we cannot prove the Hamilton-Jacobi theorem for the nonholonomic magnetic Hamiltonian system the same as that in Theorem 1.1. Since the distributional magnetic Hamiltonian system is a dynamical system closely related to a magnetic Hamiltonian system, using Lemmas 3.1 and 4.1, and the non-degenerate distributional two-form \( \omega_K^B \) and the nonholonomic dynamical vector field \( X_K^B \) given in Section 2 for the distributional magnetic Hamiltonian system, we can derive precisely the geometric constraint conditions of the non-degenerate distributional two-form \( \omega_K^B \) for the nonholonomic dynamical vector field \( X_K^B \); that is, the two types of Hamilton-Jacobi equations for the distributional magnetic Hamiltonian system \((K,\omega_K^B,H_K)\). First, we prove the Type I Hamilton-Jacobi theorem for the distributional magnetic Hamiltonian system.

**Theorem 4.1** (Type I Hamilton-Jacobi theorem for distributional magnetic Hamiltonian system). For the nonholonomic magnetic Hamiltonian system \((T^*Q,\omega^B,D,H)\) with the associated distributional magnetic Hamiltonian system \((K,\omega_K^B,H_K)\), assume that \( \gamma : Q \to T^*Q \) is a one-form on \( Q \), and that \( X^\gamma = T\pi_Q \cdot X_H^B \cdot \gamma \), where \( X_H^B \) is the magnetic Hamiltonian vector field of the associated unconstrained magnetic Hamiltonian system \((T^*Q,\omega^B,H)\). Moreover, assume that \( \text{Im}(\gamma) \subset M (= FL(D)) \) and \( \text{Im}(T\gamma) \subset C \). If the one-form \( \gamma : Q \to T^*Q \) satisfies the condition that \( d\gamma = -B \) on \( D \) with respect to \( T\pi_Q : TT^*Q \to TQ \), then \( \gamma \) is a solution of the equation \( T\gamma \cdot X^\gamma = X_K^B \cdot \gamma \). Here \( X_K^B \) is the nonholonomic dynamical vector field of the distributional magnetic Hamiltonian system.
The equation $T\gamma \cdot X^\gamma = X^B_K \cdot \gamma$ is called the Type I Hamilton-Jacobi equation for the distributional magnetic Hamiltonian system $(K, \omega^B_K, H_K)$. Here the maps involved in the theorem are shown in Fig. 3.

![Diagram 3](image-url)

**Proof.** We note that $\text{Im}(\gamma) \subset M$ and $\text{Im}(T\gamma) \subset K$, in this case,

$$\omega^B_K \cdot \tau_K = \tau_K \cdot \omega^B_M = \tau_K \cdot i^*_M \cdot \omega^B$$

along $\text{Im}(T\gamma)$. Moreover, from the distributional magnetic Hamiltonian equation (2.2), we have that $X^B_K = \tau_K \cdot X^B_H$, and that $\tau_K \cdot X^B_H \cdot \gamma = X^B_K \cdot \gamma$. Thus, using the non-degenerate distributional two-form $\omega^B_K$, from Lemmas 3.1(ii) and 4.1, we take that $v = X^B_H \cdot \gamma \in F$, and that for any $w \in F$, $T\lambda(w) \neq 0$ and $\tau_K \cdot w \neq 0$, then we have that

$$\omega^B_K(T\gamma \cdot X^\gamma, \tau_K \cdot w) = \omega^B_K(\tau_K \cdot T\gamma \cdot X^\gamma, \tau_K \cdot w)$$

$$= \tau_K \cdot i^*_M \cdot \omega^B(T \gamma \cdot T\pi_Q \cdot X^B_H \cdot \gamma, w) = \tau_K \cdot i^*_M \cdot \omega^B(T(\gamma \cdot \pi_Q) \cdot X^B_H \cdot \gamma, w)$$

$$= \tau_K \cdot i^*_M \cdot \omega^B(X^B_H \cdot \gamma, w) - \tau_K \cdot i^*_M \cdot \omega^B(X^B_H \cdot \gamma, T(\gamma \cdot \pi_Q) \cdot w)$$

$$= \omega^B_K(T \gamma \cdot X^B_H \cdot \gamma, \tau_K \cdot w) - \omega^B_K(T \gamma \cdot X^B_H \cdot \gamma, T(\gamma \cdot \pi_Q) \cdot w)$$

$$= \omega^B_K(T \gamma \cdot X^B_H \cdot \gamma, \tau_K \cdot w) - \omega^B_K(T \gamma \cdot X^B_H \cdot \gamma, T(\gamma \cdot \pi_Q) \cdot w)$$

$$= \omega^B_K(T \gamma \cdot X^B_H \cdot \gamma, \tau_K \cdot w) - \omega^B_K(T \gamma \cdot X^B_H \cdot \gamma, T(\gamma \cdot \pi_Q) \cdot w)$$

$$= \omega^B_K(T \gamma \cdot X^B_H \cdot \gamma, \tau_K \cdot w) - \omega^B_K(T \gamma \cdot X^B_H \cdot \gamma, T(\gamma \cdot \pi_Q) \cdot w)$$

where we have used that $\tau_K \cdot T\gamma = T\gamma$, since $\text{Im}(T\gamma) \subset K$ and $\tau_K \cdot X^B_H \cdot \gamma = X^B_K \cdot \gamma \in K$. Note that $X^B_K \cdot \gamma, w \in F$, and that $T\pi_Q(X^B_H \cdot \gamma), T\pi_Q(w) \in D$. If the one-form $\gamma : Q \rightarrow$...
from the proofs of the Theorem 3.5 given in León and Wang [14], we know that the Type I Hamiltonian system (\(T^*Q\)) is closed on \(\gamma\). On the other hand, if \(\gamma\) satisfies the condition that \(d\gamma = -B\) on \(D\) with respect to \(T\pi_Q: TT^*Q \to TQ\), then \((d\gamma + B)(T\pi_Q \cdot X^B_H \cdot \gamma, T\pi_Q \cdot w) = 0\), and hence,

\[
\tau_K \cdot i_M^* \cdot (d\gamma + B) \left( T\pi_Q (X^B_H \cdot \gamma), T\pi_Q (w) \right) = 0.
\]

Thus, we have that

\[
\omega^K_B (T\gamma \cdot X^\gamma, \tau_K \cdot w) - \omega^K_B (X^B_K \cdot \gamma, \tau_K \cdot w) = -\omega^K_B (X^B_K \cdot \gamma, \tau_K \cdot T\gamma \cdot T\pi_Q (w)). \tag{4.1}
\]

If \(\gamma\) satisfies the equation \(T\gamma \cdot X^\gamma = X^B_K \cdot \gamma\), from Lemma 3.1 (i), we know that the right side of the (4.1) becomes that

\[
-\omega^K_B \left( X^B_K \cdot \gamma, \tau_K \cdot T\gamma \cdot T\pi_Q (w) \right)
= -\omega^K_B (T\gamma \cdot X^\gamma, \tau_K \cdot T\gamma \cdot T\pi_Q (w))
= -\omega^K_B (\tau_K \cdot T\gamma \cdot X^\gamma, \tau_K \cdot T\gamma \cdot T\pi_Q (w))
= -\tau_K \cdot i_M^* \cdot \omega^K_B (T\gamma \cdot T\pi_Q (X^B_H \cdot \gamma), T\gamma \cdot T\pi_Q (w))
= -\tau_K \cdot i_M^* \cdot \lambda^* \omega^K_B (X^B_H \cdot \gamma, w)
= \tau_K \cdot i_M^* \cdot (d\gamma + B) \left( T\pi_Q \cdot X^B_H \cdot \gamma, T\pi_Q \cdot w \right) = 0.
\]

Since the distributional two-form \(\omega^K_B\) is non-degenerate, the left side of the (4.1) equals zero, only when \(\gamma\) satisfies the equation \(T\gamma \cdot X^\gamma = X^B_K \cdot \gamma\). Thus, if the one-form \(\gamma: Q \to T^*Q\) satisfies the condition that \(d\gamma = -B\) on \(D\) with respect to \(T\pi_Q: TT^*Q \to TQ\), then \(\gamma\) must be a solution of the Type I Hamilton-Jacobi equation \(T\gamma \cdot X^\gamma = X^B_K \cdot \gamma\) for the distributional magnetic Hamiltonian system \((K, \omega^K_B, H_K)\).

It is worth noting that when \(B = 0\), the magnetic symplectic form \(\omega^K_B\) is just the canonical symplectic form \(\omega\) on \(T^*Q\), and that the nonholonomic magnetic Hamiltonian system \((T^*Q, \omega^K_B, D, H)\) becomes the nonholonomic Hamiltonian system \((T^*Q, \omega, D, H)\) with the canonical symplectic form \(\omega\), and that the distributional magnetic Hamiltonian system \((K, \omega^K_B, H_K)\) becomes the distributional Hamiltonian system \((K, \omega_K, H_K)\), and that the one-form \(\gamma: Q \to T^*Q\) satisfies the condition that \(d\gamma = -B\) on \(D\) with respect to \(T\pi_Q: TT^*Q \to TQ\), becomes that \(\gamma\) is closed on \(D\) with respect to \(T\pi_Q: TT^*Q \to TQ\). Thus, from Theorem 4.1, we can obtain the Theorem 3.5 (given in León and Wang [14]); that is, the Type I Hamilton-Jacobi theorem for the distributional Hamiltonian system. On the other hand, from the proofs of the Theorem 3.5 given in León and Wang [14], we know that...
if the one-form $\gamma: Q \to T^*Q$ is not closed on $\mathcal{D}$ with respect to $T\pi_Q: TT^*Q \to TQ$, then the $\gamma$ is not yet closed on $\mathcal{D}$; that is, $d\gamma(x,y) \neq 0, \forall x,y \in \mathcal{D}$, and hence, $\gamma$ is not yet closed on $Q$. However, in this case, we note that $d \cdot d\gamma = d^2\gamma = 0$, and that the $d\gamma$ is a closed two-form on $Q$. Thus, we can construct a magnetic symplectic form on $T^*Q$; that is, $\omega^B = \omega + \pi_Q^*(d\gamma)$, where $B = -d\gamma$. Moreover, we can construct a nonholonomic magnetic Hamiltonian system $(T^*Q, \omega^B, \mathcal{D}, H)$ with the associated distributional magnetic Hamiltonian system $(\mathcal{K}, \omega^B_K, H_K)$, which satisfies the distributional magnetic Hamiltonian equation (2.2); that is, $i_{X^B_H} \omega^B_K = dH_K$.

In this case, noting that the one-form $\gamma: Q \to T^*Q$ also satisfies the condition that $d\gamma = -B$ on $\mathcal{D}$ with respect to $T\pi_Q: TT^*Q \to TQ$, using Lemmas 3.1 and 4.1, and the magnetic Hamiltonian vector field $X^B_H$, from Theorem 4.1, we can obtain the Theorem 4.2 as follows:

**Theorem 4.2.** For a given nonholonomic Hamiltonian system $(T^*Q, \omega, \mathcal{D}, H)$ with the canonical symplectic form $\omega$ on $T^*Q$ and the $\mathcal{D}$-completely and $\mathcal{D}$-regularly nonholonomic constraint $\mathcal{D} \subset TQ$, and assume that the one-form $\gamma: Q \to T^*Q$ is not closed on $\mathcal{D}$ with respect to $T\pi_Q: TT^*Q \to TQ$, then one can construct a magnetic symplectic form on $T^*Q$; that is, $\omega^B = \omega + \pi_Q^*(d\gamma)$, where $B = -d\gamma$, and a nonholonomic magnetic Hamiltonian system $(T^*Q, \omega^B, \mathcal{D}, H)$ with the associated distributional magnetic Hamiltonian system $(\mathcal{K}, \omega^B_K, H_K)$. Denote that $X^\gamma = T\pi_Q \cdot X^B_H \cdot \gamma$, where $X^B_H$ is the dynamical vector field of the magnetic Hamiltonian system $(T^*Q, \omega^B, H)$. Then $\gamma$ is a solution of the Type I Hamilton-Jacobi equation $T\gamma \cdot X^\gamma = X^B_H \cdot \gamma$ for the distributional magnetic Hamiltonian system $(\mathcal{K}, \omega^B_K, H_K)$.

Next, for any symplectic map $\varepsilon: T^*Q \to T^*Q$ with respect to the magnetic symplectic form $\omega^B$, we can prove the Type II Hamilton-Jacobi theorem for the distributional magnetic Hamiltonian system. For convenience, the maps involved in the theorem and its proof are shown in Fig. 4.

**Theorem 4.3** (Type II Hamilton-Jacobi theorem for distributional magnetic Hamiltonian system). Let $(T^*Q, \omega^B, \mathcal{D}, H)$ be a nonholonomic magnetic Hamiltonian system with the associated distributional magnetic Hamiltonian system $(\mathcal{K}, \omega^B_K, H_K)$, assume that $\gamma: Q \to T^*Q$ is a one-form on $Q$, and that $\lambda = \gamma \cdot \pi_Q: T^*Q \to T^*Q$, and that for
any symplectic map $\varepsilon:T^*Q \to T^*Q$ with respect to the magnetic symplectic form $\omega^B$, denote that $X^\varepsilon = T\pi_Q X^B_H \varepsilon$, where $X^B_H$ is the dynamical vector field of the magnetic Hamiltonian system $(T^*Q, \omega^B, H)$. Moreover, assume that $\text{Im}(\gamma) \subset M = F(L(D))$, and that $\varepsilon(M) \subset M$, and that $\text{Im}(T\gamma) \subset \mathcal{K}$. Then $\varepsilon$ is a solution of the equation $T\varepsilon(X^B_H \varepsilon) = TL \cdot X^B_H \varepsilon$ if and only if it is a solution of the equation $T\gamma \cdot X^\varepsilon = X^B_{K \varepsilon} \cdot \varepsilon$. Here $X^B_{H \varepsilon}$ is the magnetic Hamiltonian vector field of the function $H \cdot \varepsilon: T^*Q \to \mathbb{R}$ and $X^B_K$ is the dynamical vector field of the distributional magnetic Hamiltonian system $(\mathcal{K}, \omega^B_K, H_K)$. The equation $T\gamma \cdot X^\varepsilon = X^B_{K \varepsilon} \cdot \varepsilon$ is called the Type II Hamilton-Jacobi equation for the distributional magnetic Hamiltonian system $(\mathcal{K}, \omega^B_K, H_K)$.

Proof. In the same way, we note that $\text{Im}(\gamma) \subset M$ and $\text{Im}(T\gamma) \subset \mathcal{K}$, in this case, we have that $\omega^B_K \cdot \tau_K = \tau_K^* \cdot \omega^B_M = \tau_K^* \cdot i_M \cdot \omega^B$ along $\text{Im}(T\gamma)$. Moreover, from the distributional magnetic Hamiltonian equation (2.2), we have that $X^B_K = \tau_K \cdot X^B_H$, and that $\tau_K \cdot X^B_H \varepsilon = X^B_K \varepsilon$. Note that $\varepsilon(M) \subset M$, and that $T\pi_Q (X^B_H \varepsilon(q, p)) \in D_q, \forall q \in Q, (q, p) \in M (\subset T^*Q)$, and hence, $X^B_K \varepsilon \in F$ along $\varepsilon$. Thus, using the non-degenerate distributional two-form $\omega^B_K$, from Lemmas 3.1 and 4.1, we take that $v = X^B_H \varepsilon \in F$, and that for any $w \in F$, $T\lambda(w) \neq 0$ and $\tau_K \cdot w \neq 0$, then we have that

$$
\omega^B_K (T\gamma \cdot X^\varepsilon, \tau_K \cdot w) = \omega^B_K (\tau_K \cdot T\gamma \cdot X^\varepsilon, \tau_K \cdot w)
$$

$$
\tau_K \cdot i_M \cdot \omega^B \left( T\gamma \cdot T\pi_Q \cdot X^B_H \varepsilon, w \right) = \tau_K \cdot i_M \cdot \omega^B \left( T(\gamma \cdot \pi_Q) \cdot X^B_H \varepsilon, w \right)
$$

$$
\tau_K \cdot i_M \cdot \omega^B \left( X^B_H \varepsilon, w \right) = \tau_K \cdot i_M \cdot \omega^B \left( X^B_H \varepsilon, T(\gamma \cdot \pi_Q) \cdot w \right)
$$

$$
\tau_K \cdot i_M \cdot \omega^B \left( X^B_H \varepsilon, w \right) = \omega^B_K \left( \tau_K \cdot X^B_H \varepsilon, \tau_K \cdot w \right) - \tau_K \cdot i_M \cdot \omega^B \left( X^B_H \varepsilon, T\lambda \cdot w \right)
$$

$$
= \omega^B_K \left( \tau_K \cdot X^B_H \varepsilon, \tau_K \cdot w \right) - \omega^B_K \left( \tau_K \cdot X^B_H \varepsilon, T\lambda \cdot w \right) + \tau_K \cdot i_M \cdot \lambda^* \omega^B \left( X^B_H \varepsilon, w \right)
$$

where we have used that $\tau_K \cdot T\gamma = T\gamma \cdot \tau_K \cdot T\lambda = T\lambda$ and $\tau_K \cdot X^B_H \varepsilon = X^B_K \varepsilon$, since $\text{Im}(T\gamma) \subset \mathcal{K}$. Note that $\varepsilon: T^*Q \to T^*Q$ is symplectic with respect to the magnetic symplectic form $\omega^B$, and that $X^B_H \varepsilon = T\varepsilon \cdot X^B_H \varepsilon$ along $\varepsilon$, and hence, $\tau_K \cdot X^B_H \varepsilon = \tau_K \cdot T\varepsilon \cdot X^B_H \varepsilon$ along $\varepsilon$. Then we have that

$$
\omega^B_K \left( T\gamma \cdot X^\varepsilon, \tau_K \cdot w \right) - \omega^B_K \left( X^B_K \varepsilon, \tau_K \cdot w \right)
$$

$$
= -\omega^B_K \left( \tau_K \cdot X^B_H \varepsilon, T\lambda \cdot w \right) + \omega^B_K \left( T\lambda \cdot X^B_H \varepsilon, T\lambda \cdot w \right)
$$

$$
= \omega^B_K \left( T\lambda \cdot X^B_H \varepsilon - \tau_K \cdot T\varepsilon \cdot X^B_H \varepsilon, T\lambda \cdot w \right).
$$
Since the induced distributional two-form $\omega^B_K$ is non-degenerate, it follows that the equation $T\gamma \cdot X^\varepsilon = X^B_K \cdot \varepsilon$ is equivalent to the equation $\tau^*_K \cdot T\varepsilon \cdot X^B_H \cdot \varepsilon = T\lambda \cdot X^B_H \cdot \varepsilon$. Thus, $\varepsilon$ is a solution of the equation $\tau^*_K \cdot T\varepsilon \cdot X^B_H \cdot \varepsilon = T\lambda \cdot X^B_H \cdot \varepsilon$ if and only if it is a solution of the Type II Hamilton-Jacobi equation $T\gamma \cdot X^\varepsilon = X^B_K \cdot \varepsilon$.

Remark 4.1. It is worth noting that the Type I Hamilton-Jacobi equation $T\gamma \cdot X^\gamma = X^B_K \cdot \gamma$ is the equation of the differential one-form $\gamma$, and that the Type II Hamilton-Jacobi equation $T\gamma \cdot X^\varepsilon = X^B_K \cdot \varepsilon$ is the equation of the symplectic diffeomorphism map $\varepsilon$. If the nonholonomic magnetic Hamiltonian system, we considered, has not any constrains, the distributional magnetic Hamiltonian system is just the magnetic Hamiltonian system itself. From the Type I and Type II Hamilton-Jacobi theorems; that is, Theorems 4.1 and 4.3, we can get the Theorems 3.1 and 3.3. It has shown that Theorems 4.1 and 4.3 can be regarded as an extension of the two types of Hamilton-Jacobi theorems for the magnetic Hamiltonian system to the system with nonholonomic context. On the other hand, when $B = 0$, the magnetic symplectic form $\omega^B$ is just the canonical symplectic form $\omega$ on $T^*Q$, and the distributional magnetic Hamiltonian system is just the distributional Hamiltonian system itself. From the Type I and Type II Hamilton-Jacobi theorems; that is, Theorems 4.1 and 4.3, we can get the Theorems 3.5 and 3.6 given in León and Wang [14]. It has shown that Theorems 4.1 and 4.3 can be regarded as an extension of the two types of Hamilton-Jacobi theorems for the distributional Hamiltonian system to that for the distributional magnetic Hamiltonian system.

5 Nonholonomic reduced distributional magnetic Hamiltonian system

It is well-known that the reduction theory for the mechanical system with symmetry is an important subject and that it is widely studied in the theory of mathematics and mechanics, as well as applications; see Abraham and Marsden [1], Arnold [2], Marsden and Ratiu [19], Marsden et al. [17, 18], Libermann and Marle [15], Marsden [16], Marsden and Weinstein [21], and Ortega and Ratiu [22] and so on., for more details and development. In particular, the reduction of the nonholonomically constrained mechanical systems is also very important subject in the study of the geometric mechanics, which is regarded as a useful tool for simplifying and studying concrete nonholonomic systems; see León and Wang [14], Bates and Śniatycki [3], Cantrijn et al. [4], Cendra et al. [7], Cushman et al. [8, 9], Koiller [11], León and Rodrigues [13], and so on.
In this section, we shall consider the nonholonomic reduction and the Hamilton-Jacobi theory of a nonholonomic magnetic Hamiltonian system with symmetry. First, we give the definition of a nonholonomic magnetic Hamiltonian system with symmetry, then, using the method similar to that given in León and Wang [14] and Bates and Śniatycki [3] and analyzing carefully the dynamics and structures of the nonholonomic magnetic Hamiltonian system with symmetry, we give a geometric formulation of the nonholonomic reduced distributional magnetic Hamiltonian system. Moreover, we derive precisely the geometric constraint conditions of the non-degenerate, and nonholonomic reduced distributional two-form for the nonholonomic reducible dynamical vector field; that is, the two types of Hamilton-Jacobi equations for the nonholonomic reduced distributional magnetic Hamiltonian system, which are an extension of the two types of Hamilton-Jacobi equation for the distributional magnetic Hamiltonian system given in Section 4 under nonholonomic reduction.

Assume that the Lie group $G$ acts smoothly on the manifold $Q$ by the left, and that we also consider the natural lifted actions on $TQ$ and $\mathcal{T}^*Q$, and that the cotangent lifted action on $\mathcal{T}^*Q$ is free, proper and symplectic with respect to the magnetic symplectic form $\omega^B = \omega - \pi_Q^*B$ on $\mathcal{T}^*Q$, where $\omega$ is the canonical symplectic form on $\mathcal{T}^*Q$ and $B$ is a closed two-form on $Q$. Then the orbit space $\mathcal{T}^*Q/G$ is a smooth manifold and the canonical projection $\pi: \mathcal{T}^*Q \rightarrow \mathcal{T}^*Q/G$ is a surjective submersion. Assume that $H: \mathcal{T}^*Q \rightarrow \mathbb{R}$ is a $G$-invariant Hamiltonian, and that the $D$-completely and $D$-regularly nonholonomic constraint $D \subset TQ$ is a $G$-invariant distribution; that is, the tangent of the group action maps $D_q$ to $D_{gq}$ for any $q \in Q$. A nonholonomic magnetic Hamiltonian system with symmetry is a 5-tuple $(\mathcal{T}^*Q, G, \omega^B, D, H)$, which is a magnetic Hamiltonian system with symmetry and with the $G$-invariant nonholonomic constraint $D$.

In the following we consider the nonholonomic reduction of a nonholonomic magnetic Hamiltonian system with symmetry $(\mathcal{T}^*Q, G, \omega^B, D, H)$. Note that the Legendre transformation $FL: TQ \rightarrow T^*Q$ is a fiber-preserving map, and that $D \subset TQ$ is $G$-invariant for the tangent lifted left action $\Phi^T: G \times TQ \rightarrow TQ$, then the constraint submanifold $M = FL(D) \subset \mathcal{T}^*Q$ is $G$-invariant for the cotangent lifted left action $\Phi^{T^*}: G \times \mathcal{T}^*Q \rightarrow \mathcal{T}^*Q$. For the nonholonomic magnetic Hamiltonian system with symmetry $(\mathcal{T}^*Q, G, \omega^B, D, H)$, in the same way, we define the distribution $\mathcal{F}$, which is the pre-image of the nonholonomic constraints $D$ for the map $T\pi_Q: \mathcal{T}^*Q \rightarrow TQ$; that is, $\mathcal{F} = (T\pi_Q)^{-1}(D)$, and the distribution $\mathcal{K} = \mathcal{F} \cap TM$. Moreover, we can also define the distributional magnetic two-form $\omega^{B}_{\mathcal{K}}$, which is induced from the magnetic symplectic form $\omega^B$ on $\mathcal{T}^*Q$; that is, $\omega^{B}_{\mathcal{K}} = \tau_{\mathcal{K}} \cdot \omega^B_{\mathcal{M}}$, where $\omega^B_{\mathcal{M}} = i_{\mathcal{M}}^* \omega^B$. If the admissibility condition that $\dim M = \text{rank} \mathcal{F}$ and the compatibility condition that $TM \cap \mathcal{F}^\perp = \{0\}$ hold, then $\omega^{B}_{\mathcal{K}}$ is non-degenerate as
a bilinear form on each fibre of $K$, and there exists a vector field $X^K_B$ on $M$, which takes values in the constraint distribution $K$, such that for the function $H_K$, the following distributional magnetic Hamiltonian equation holds, that is,

$$\mathbf{i}_X^K \omega^K_B = \mathbf{d}H_K, \quad (5.1)$$

where the function $H_K$ satisfies that $\mathbf{d}H_K = \tau_K \cdot \mathbf{d}H_M$, and that $H_M = \tau_M \cdot H$ is the restriction of $H$ to $M$. From Eq. (5.1), we have that $X^K_B = \tau_K \cdot X^B_H$.

In the following we define that the quotient space $\tilde{M} = M/G$ of the $G$-orbit in $M$ is a smooth manifold with projection $\pi_G : M \rightarrow \tilde{M}(\subset T^*Q/G)$, which is a surjective submersion. The reduced magnetic symplectic form $\omega^{B}_M = \pi_G^* \cdot \omega^{B}_M$ on $\tilde{M}$ is induced from the magnetic symplectic form $\omega^{B}_M = i^K_M \cdot \omega^{B}_M$ on $M$. Since $G$ is the symmetry group of the system $(T^*Q, G, \omega^B, D, H)$, all intrinsically defined vector fields and distributions are pushed down to $\tilde{M}$. In particular, the vector field $X^K_B$ on $M$ is pushed down to the vector field $X^K_B = T\pi_G \cdot X^K_B$, and the distribution $K$ is pushed down to the distribution $T\pi_G \cdot K$ on $\tilde{M}$, and the Hamiltonian $H$ is pushed down to $h_M$ such that $h_M \cdot \pi_G = \tau_M \cdot H$. However, $\omega^K_B$ need not to be pushed down to a distributional two-form defined on $T\pi_G \cdot K$, despite of we have the fact that $\omega^K_B$ is $G$-invariant. This is because there may be infinitesimal symmetry $\eta_K$ that lies in $M$, such that $\mathbf{i}_{\eta_K} \omega^K_B \neq 0$. From Bates and Śniatycki [3], we know that in order to eliminate this difficulty, $\omega^K_B$ is restricted to a sub-distribution $\mathcal{U}$ of $K$ defined by

$$\mathcal{U} := \left\{ u \in K \mid \omega^K_B(u,v) = 0, \forall v \in \mathcal{V} \cap K \right\},$$

where $\mathcal{V}$ is the distribution on $M$ tangent to the orbits of $G$ in $M$ and it is spanned by the infinitesimal symmetries. Clearly, $\mathcal{U}$ and $\mathcal{V}$ are both $G$-invariant, project down to $\tilde{M}$ and $T\pi_G \cdot \mathcal{V} = 0$, and we can define the distribution $\tilde{K}$ by $\tilde{K} = T\pi_G \cdot \mathcal{U}$. Moreover, we take that $\omega^K_B = \pi_G \cdot \omega^{B}_M$ is the restriction of the induced magnetic symplectic form $\omega^{B}_M$ on $T^*M$ fiberwise to the distribution $\mathcal{U}$, where $\pi_G$ is the restriction map to distribution $\mathcal{U}$, and the $\omega^K_B$ is pushed down to the distributional magnetic two-form $\omega^K_B$ on $\tilde{K}$, such that $\pi_G \cdot \omega^K_B = \omega^K_B$. It is worth noting that the distributional magnetic two-form $\omega^K_B$ is not a "true two-form" on a manifold, so it does not make sense to speak that it is closed. Thus, it is called the nonholonomic reduced distributional magnetic two-form to avoid any confusion.

From the above construction we know that if the admissibility condition that $\dim \tilde{M} = \text{rank} \tilde{F}$ and the compatibility condition that $T\tilde{M} \cap \tilde{F}^\perp = \{0\}$ hold, where $\tilde{F}^\perp$ denotes the symplectic orthogonal of $\tilde{F}$ with respect to the reduced magnetic symplectic form $\omega^{B}_{M'}$, then the nonholonomic reduced distributional magnetic
two-form $\omega^B_K$ is non-degenerate as a bilinear form on each fibre of $\mathcal{K}$, and hence, there exists a vector field $X^B_K$ on $\mathcal{M}$, which takes values in the constraint distribution $\mathcal{K}$, such that the nonholonomic reduced distributional magnetic Hamiltonian equation holds, that is

$$i_{X^B_K} \omega^B_K = dh_K,$$

(5.2)

where $dh_K$ is the restriction of $dh_{\mathcal{M}}$ to $\mathcal{K}$ and the function $h_K$ satisfies that $dh_K = \tau_K \cdot dh_{\mathcal{M}}$ and $h_{\mathcal{M}} \cdot \pi_G = H_M$, where $H_M$ is the restriction of the Hamiltonian function $H$ to $\mathcal{M}$. In addition, from the distributional magnetic Hamiltonian equation (5.1); that is, $i_{X^B_K} \omega^B_K = dh_K$, we have that $X^B_K = \tau_K \cdot X^B_{\mathcal{M}}$. From the nonholonomic reduced distributional magnetic Hamiltonian equation (5.2); that is, $i_{X^B_K} \omega^B_K = dh_K$, we have that $X^B_K = \tau_K \cdot X^B_{\mathcal{M}}$, where $X^B_{\mathcal{M}}$ is the magnetic Hamiltonian vector field of the function $h_K$ with respect to the reduced magnetic symplectic form $\omega^B_{\mathcal{M}}$. Moreover, the vector fields $X^B_K$ and $X^B_{\mathcal{M}}$ are $\pi_G$-related; that is, $X^B_K \cdot \pi_G = T_{\pi_G} \cdot X^B_{\mathcal{M}}$. Thus, the geometrical formulation of a nonholonomic reduced distributional magnetic Hamiltonian system may be summarized as follows:

**Definition 5.1** (Nonholonomic reduced distributional magnetic Hamiltonian system). Assume that the 5-tuple $(T^*Q, G, \omega^B, \mathcal{D}, H)$ is a nonholonomic magnetic Hamiltonian system with symmetry, where $\omega^B$ is the magnetic symplectic form on $T^*Q$, and $\mathcal{D} \subset TQ$ is a $\mathcal{D}$-completely and $\mathcal{D}$-regularly nonholonomic constraint of the system, and $\mathcal{D}$ and $H$ are both $G$-invariant. If there exists a nonholonomic reduced distribution $\bar{\mathcal{K}}$, an associated non-degenerate and nonholonomic reduced distributional two-form $\omega^B_{\bar{\mathcal{K}}}$ and a vector field $X^B_{\bar{\mathcal{K}}}$ on the reduced constraint submanifold $\bar{\mathcal{M}} = \mathcal{M} / G$, where $\mathcal{M} = \mathcal{F}(\mathcal{D}) \subset T^*Q$, such that the nonholonomic reduced distributional magnetic Hamiltonian equation holds; that is, $i_{X^B_{\bar{\mathcal{K}}}} \omega^B_{\bar{\mathcal{K}}} = dh_{\bar{\mathcal{K}}}$, where $dh_{\bar{\mathcal{K}}}$ is the restriction of $dh_{\mathcal{M}}$ to $\bar{\mathcal{K}}$ and the function $h_{\bar{\mathcal{K}}}$ satisfies that $dh_{\bar{\mathcal{K}}} = \tau_{\bar{\mathcal{K}}} \cdot dh_{\mathcal{M}}$ and $h_{\mathcal{M}} \cdot \pi_G = H_M$ as defined above. Then the triple $(\bar{\mathcal{K}}, \omega^B_{\bar{\mathcal{K}}}, h_{\bar{\mathcal{K}}})$ is called the nonholonomic reduced distributional magnetic Hamiltonian system of the nonholonomic magnetic Hamiltonian system with symmetry $(T^*Q, G, \omega^B, \mathcal{D}, H)$, and $X^B_{\bar{\mathcal{K}}}$ is called a nonholonomic reduced dynamical vector field of the system $(\bar{\mathcal{K}}, \omega^B_{\bar{\mathcal{K}}}, h_{\bar{\mathcal{K}}})$. Under the above circumstances, we refer to $(T^*Q, G, \omega^B, \mathcal{D}, H)$ as a nonholonomic reducible magnetic Hamiltonian system with the associated distributional magnetic Hamiltonian system $(\bar{\mathcal{K}}, \omega^B_{\bar{\mathcal{K}}}, h_{\bar{\mathcal{K}}})$ and nonholonomic reduced distributional magnetic Hamiltonian system $(\bar{\mathcal{K}}, \omega^B_{\bar{\mathcal{K}}}, h_{\bar{\mathcal{K}}})$.

Since the non-degenerate and nonholonomic reduced distributional two-form $\omega^B_{\bar{\mathcal{K}}}$ is not a “true two-form” on a manifold, and it is not symplectic, and hence, the
nonholonomic reduced distributional magnetic Hamiltonian system \((\mathcal{K}, \omega_{\mathcal{K}}^B, h_{\mathcal{K}})\) may not be a Hamiltonian system, and may have no generating function, and hence, we can not describe the Hamilton-Jacobi equation for the nonholonomic reduced distributional magnetic Hamiltonian system the same as in Theorem 1.1. However, the nonholonomic reduced distributional magnetic Hamiltonian system is a dynamical system closely related to a magnetic Hamiltonian system. Let \((T^*Q, G, \omega^B, \mathcal{D}, H)\) be a nonholonomic reducible magnetic Hamiltonian system with associated distributional magnetic Hamiltonian system \((\mathcal{K}, \omega_{\mathcal{K}}^B, H_{\mathcal{K}})\) and nonholonomic reduced distributional magnetic Hamiltonian system \((\mathcal{K}, \omega_{\mathcal{K}}^B, h_{\mathcal{K}})\). Using Lemmas 3.1 and 4.1, we can derive precisely the geometric constraint conditions of the nonholonomic reduced distributional two-form \(\omega_{\mathcal{K}}^B\) for the dynamical vector field \(X_{\mathcal{K}}^B\); that is, the two types of Hamilton-Jacobi equations for the nonholonomic reduced distributional magnetic Hamiltonian system \((\mathcal{K}, \omega_{\mathcal{K}}^B, h_{\mathcal{K}})\).

First, using the fact that the one-form \(\gamma: Q \to T^*Q\) satisfies the condition \(d\gamma = -B\) on \(\mathcal{D}\) with respect to \(T\pi_Q : TT^*Q \to TQ\), and that \(\text{Im}(\gamma) \subset \mathcal{M}\), and that \(\gamma\) is \(G\)-invariant, as well as that \(\text{Im}(T\gamma) \subset \mathcal{K}\), we can prove the Type I Hamilton-Jacobi theorem for the nonholonomic reduced distributional magnetic Hamiltonian system. For convenience, the maps involved in the theorem and its proof are shown in Fig. 5.

\[
\begin{array}{cccccccc}
\mathcal{M} & \xrightarrow{i_M} & T^*Q & \xrightarrow{\pi_Q} & Q & \xrightarrow{\gamma} & T^*Q & \xrightarrow{\pi/G} & T^*Q/G & \xrightarrow{i_M} & \hat{\mathcal{M}} \\
X_{\mathcal{K}}^B & \downarrow \tau_{\mathcal{K}} & X_{H}^B & \downarrow X^\gamma & T(T^*Q) & \xrightarrow{T\pi_Q} & TQ & \xrightarrow{T\pi_Q} & T(T^*Q/G) & \xrightarrow{T\pi_Q} & \hat{\mathcal{K}} \\
\mathcal{K} & \xrightarrow{T(T^*Q)} & TQ & \xrightarrow{T\pi_Q} & T(T^*Q/G) & \xrightarrow{T\pi_Q} & \hat{\mathcal{K}} \\
\end{array}
\]

Figure 5: Diagram-5.

**Theorem 5.1** (Type I Hamilton-Jacobi theorem for a nonholonomic reduced distributional magnetic Hamiltonian system). For a given nonholonomic reducible magnetic Hamiltonian system \((T^*Q, G, \omega^B, \mathcal{D}, H)\) with the associated distributional magnetic Hamiltonian system \((\mathcal{K}, \omega_{\mathcal{K}}^B, H_{\mathcal{K}})\) and the nonholonomic reduced distributional magnetic Hamiltonian system \((\mathcal{K}, \omega_{\mathcal{K}}^B, h_{\mathcal{K}})\), assume that \(\gamma: Q \to T^*Q\) is a one-form on \(Q\), and that \(X^\gamma = T\pi_Q \cdot X_{H}^B \cdot \gamma\), where \(X_{H}^B\) is the magnetic Hamiltonian vector field of the corresponding unconstrained magnetic Hamiltonian system with symmetry \((T^*Q, G, \omega^B, H)\). Moreover, assume that \(\text{Im}(\gamma) \subset \mathcal{M}\), and that \(\gamma\) is \(G\)-invariant, and that \(\text{Im}(T\gamma) \subset \mathcal{K}\), and that \(\tilde{\gamma} = \pi/G(\gamma): Q \to T^*Q/G\). If the one-form \(\gamma: Q \to T^*Q\) satisfies the condition that \(d\gamma = -B\) on \(\mathcal{D}\) with respect to \(T\pi_Q : TT^*Q \to TQ\), then \(\tilde{\gamma}\) is a solution of the equation \(T\tilde{\gamma} \cdot X^\gamma = X_{\mathcal{K}}^B \cdot \tilde{\gamma}\). Here \(X_{\mathcal{K}}^B\) is the dynamical vector field of the nonholonomic reduced
distributional magnetic Hamiltonian system \((\mathcal{K}, \omega^B_\mathcal{K}, h_\mathcal{K})\). The equation \(T\vec{\gamma} \cdot X^\gamma = X^B_\mathcal{K} \cdot \vec{\gamma}\) is called the Type I Hamilton-Jacobi equation for the nonholonomic reduced distributional magnetic Hamiltonian system \((\mathcal{K}, \omega^B_\mathcal{K}, h_\mathcal{K})\).

**Proof.** First, from Theorem 4.1, we know that \(\gamma\) is a solution of the Type I Hamilton-Jacobi equation \(T\gamma \cdot X^\gamma = X^B_\mathcal{K} \cdot \gamma\). Next, we note that \(\text{Im}(\gamma) \subset \mathcal{M}\), and that \(\gamma\) is \(G\)-invariant, and that \(\text{Im}(T\gamma) \subset \mathcal{K}\), and hence, \(\text{Im}(T\gamma) \subset \mathcal{K}\), in this case, we have that \(\pi^*_G \cdot \omega^B_\mathcal{K} \cdot \gamma = \tau_U \cdot \omega^B_M = \pi_U \cdot i^*_M \cdot \omega^B\) along \(\text{Im}(T\gamma)\). From the distributional magnetic Hamiltonian equation (5.1), we have that \(X^B_\mathcal{K} = \tau_\mathcal{K} \cdot X^B_H\), and that \(\tau_\mathcal{K} \cdot X^B_H \cdot \gamma = X^B_\mathcal{K} \cdot \gamma\). Because the vector fields \(X^B_\mathcal{K}\) and \(X^B_H\) are \(\pi/G\)-related, \(T\pi/G(X^B_\mathcal{K}) = X^B_\mathcal{K} \cdot \pi/G\), and hence,

\[
\tau_\mathcal{K} \cdot T\pi/G \left( X^B_\mathcal{K} \cdot \gamma \right) = \tau_\mathcal{K} \cdot \left( T\pi/G \left( X^B_\mathcal{K} \right) \right) \cdot (\gamma) \\
= \tau_\mathcal{K} \cdot \left( X^B_\mathcal{K} \cdot \pi/G \right) \cdot (\gamma) = \tau_\mathcal{K} \cdot X^B_\mathcal{K} \cdot \pi/G(\gamma) = X^B_\mathcal{K} \cdot \gamma.
\]

Thus, using the non-degenerate and nonholonomic reduced distributional two-form \(\omega^B_\mathcal{K}\), from Lemmas 3.1 (ii) and 4.1, we take that \(v = X^B_H \cdot \gamma \in \mathcal{F}\), and that for any \(w \in \mathcal{F}\), \(T\pi/G \cdot \gamma \neq 0\) and \(\tau_\mathcal{K} \cdot T\pi/G \cdot w \neq 0\), then we have that

\[
\omega^B_\mathcal{K}(T\vec{\gamma} \cdot X^\gamma, \tau_\mathcal{K} \cdot T\pi/G \cdot w) = \omega^B_\mathcal{K}(\tau_\mathcal{K} \cdot T(\pi/G \cdot \gamma) \cdot X^\gamma, \tau_\mathcal{K} \cdot T\pi/G \cdot w) \\
= \pi^*_G(\omega^B_\mathcal{K} \cdot \gamma \cdot X^\gamma, \tau_\mathcal{K} \cdot T(\pi/G \cdot \gamma)) = \pi_U \cdot i^*_M \cdot \omega^B \left( T\gamma \cdot T\pi_Q \cdot X^B_H \cdot \gamma, w \right) \\
= \pi_U \cdot i^*_M \cdot \omega^B \left( T\gamma \cdot T\pi_Q \cdot X^B_H \cdot \gamma, w \right) \\
= \pi_U \cdot i^*_M \cdot \omega^B \left( X^B_H \cdot \gamma, w - T(\gamma \cdot \pi_Q) \cdot w \right) - (d \gamma + B) \left( T\pi_Q \left( X^B_H \cdot \gamma \right), T\pi_Q(\gamma) \right) \\
= \pi_U \cdot i^*_M \cdot \omega^B \left( X^B_H \cdot \gamma, w - T(\gamma \cdot \pi_Q) \cdot w \right) \\
= \pi_U \cdot i^*_M \cdot \omega^B \left( X^B_H \cdot \gamma, T(\gamma \cdot \pi_Q) \cdot w \right) \\
= \omega^B_\mathcal{K} \left( \tau_\mathcal{K} \cdot T\pi/G(X^B_H \cdot \gamma), \tau_\mathcal{K} \cdot T\pi/G \cdot w \right) \\
= \omega^B_\mathcal{K} \left( \tau_\mathcal{K} \cdot T\pi/G(X^B_H \cdot \gamma), \tau_\mathcal{K} \cdot T\pi/G(\gamma) \cdot T\pi_Q(\gamma) \right) \\
= \tau_\mathcal{K} \cdot T\pi/G(X^B_H \cdot \gamma), \tau_\mathcal{K} \cdot T\pi/G \cdot w \right) \\
= \omega^B_\mathcal{K} \left( \tau_\mathcal{K} \cdot T\pi/G(X^B_H \cdot \gamma), \tau_\mathcal{K} \cdot T\pi/G(\gamma) \cdot T\pi_Q(\gamma) \right) \\
= \omega^B_\mathcal{K} \left( \tau_\mathcal{K} \cdot T\pi/G(X^B_H \cdot \gamma), \tau_\mathcal{K} \cdot T\pi/G \cdot w \right) \]
If \( \bar{\gamma} \) hence, thus, we have that \( T \) the one-form reduced distributional two-form equals zero, only when \( \bar{\gamma} \) where we have used that \( \tau \), \( \gamma \) satisfies the equation \( \gamma = -B \) on \( D \) with respect to \( T\pi_Q : T^*Q \rightarrow TQ \), then we have that \( (d\gamma + B)(T\pi_Q(X^B, \gamma), T\pi_Q(\omega)) = 0 \), and hence,

\[
\tau U \cdot i^*_M \cdot (d\gamma + B) \left( T\pi_Q(X^B, \gamma), T\pi_Q(\omega) \right) = 0.
\]

Thus, we have that

\[
\omega^K_B \left( T\bar{\gamma} \cdot X^\gamma, \tau K \cdot T\pi_G \cdot w \right) - \omega^K_B \left( X^K_B \cdot \bar{\gamma}, \tau K \cdot T\pi_G \cdot w \right) = -\omega^K_B \left( X^K_B \cdot \bar{\gamma}, T\bar{\gamma} \cdot T\pi_Q(\omega) \right) = 0.
\]

(5.3)

If \( \bar{\gamma} \) satisfies the equation \( T\bar{\gamma} \cdot X^\gamma = X^K_B \cdot \bar{\gamma} \), from Lemma 3.1 (i), we know that the right side of the (5.3) becomes that

\[
-\omega^K_B \left( X^K_B \cdot \bar{\gamma}, T\bar{\gamma} \cdot T\pi_Q(\omega) \right) = -\omega^K_B \cdot \tau K \left( T\bar{\gamma} \cdot X^\gamma, T\bar{\gamma} \cdot T\pi_Q(\omega) \right) = -\bar{\gamma}^* \cdot \omega^K_B \cdot \tau K \left( T\pi_Q \cdot X^K_B \cdot \gamma, T\pi_Q(\omega) \right) = -\gamma^* \cdot \pi^*_G \cdot \omega^K_B \cdot \tau K \left( T\pi_Q \cdot X^K_B \cdot \gamma, T\pi_Q(\omega) \right) = -\gamma^* \cdot \tau U \cdot i^*_M \cdot \omega^B \left( T\pi_Q \left( X^K_B \cdot \gamma \right), T\pi_Q(\omega) \right) = -\tau U \cdot i^*_M \cdot \gamma^* \omega^B \left( T\pi_Q \left( X^K_B \cdot \gamma \right), T\pi_Q(\omega) \right) = 0,
\]

where \( \gamma^* \cdot \tau U \cdot i^*_M \cdot \omega^B = \tau U \cdot i^*_M \cdot \gamma^* \cdot \omega^B \), because \( \text{Im}(\gamma) \subset M \). Since the nonholonomic reduced distributional two-form \( \omega^K_B \) is non-degenerate, the left side of the (5.3) equals zero, only when \( \bar{\gamma} \) satisfies the equation \( T\bar{\gamma} \cdot X^\gamma = X^K_B \cdot \bar{\gamma} \). Thus, if the one-form \( \gamma : Q \rightarrow T^*Q \) satisfies the condition that \( d\gamma = -B \) on \( D \) with respect to \( T\pi_Q :
$T^*Q \to TQ$, then $\dot{\gamma}$ must be a solution of the Type I Hamilton-Jacobi equation $T\dot{\gamma} \cdot X = X^B \cdot \dot{\gamma}$.

Next, for any $G$-invariant symplectic map $\varepsilon: T^*Q \to T^*Q$ with respect to $\omega^B$, we can prove the Type II Hamilton-Jacobi theorem for the nonholonomic reduced distributional magnetic Hamiltonian system. For convenience, the maps involved in the theorem and its proof are shown in Fig. 6.

\[ \begin{array}{cccccccc}
\mathcal{M} & \xrightarrow{i_M} & T^*Q & \xrightarrow{\pi_Q} & Q & \xrightarrow{\gamma} & T^*Q & \xrightarrow{\pi/G} & T^*Q/G \\
X^K & \xrightarrow{X^B} & T(T^*Q) & \xrightarrow{T\pi_Q} & TQ & \xrightarrow{T\pi/G} & T(T^*Q/G) & \xrightarrow{\tau^K} & \mathcal{K}
\end{array} \]

Figure 6: Diagram-6.

**Theorem 5.2** (Type II Hamilton-Jacobi theorem for a nonholonomic reduced distributional magnetic Hamiltonian system). For a given nonholonomic reducible magnetic Hamiltonian system $(T^*Q, G, \omega^B, \mathcal{D}, H)$ with the associated distributional magnetic Hamiltonian system $(\mathcal{K}, \omega^K, H^K)$ and the nonholonomic reduced distributional magnetic Hamiltonian system $(\mathcal{K}, \omega^K, h^K)$, assume that $\gamma: Q \to T^*Q$ is a one-form on $Q$, and that $\lambda = \gamma \cdot \pi_Q: T^*Q \to T^*Q$, and that for any $G$-invariant symplectic map $\varepsilon: T^*Q \to T^*Q$ with respect to $\omega^B$, denote that $X^\varepsilon = T\pi_Q \cdot X^B \cdot \varepsilon$, where $X^B$ is the magnetic Hamiltonian vector field of the corresponding unconstrained magnetic Hamiltonian system with symmetry $(T^*Q, G, \omega^B, H)$. Moreover, assume that $\operatorname{Im}(\gamma) \subset \mathcal{M}$, and that $\gamma$ is $G$-invariant, and that $\varepsilon(\mathcal{M}) \subset \mathcal{M}$, and that $\operatorname{Im}(T\gamma) \subset \mathcal{K}$, and that $\dot{\gamma} = \pi_G(\gamma): Q \to T^*Q/G$, and that $\dot{\lambda} = \pi_G(\lambda): T^*Q \to T^*Q/G$, and that $\dot{\varepsilon} = \pi_G(\varepsilon): T^*Q \to T^*Q/G$. Then $\varepsilon$ and $\dot{\varepsilon}$ satisfy the equation $T\dot{\gamma} \cdot X^\varepsilon = X^K \cdot \dot{\varepsilon}$. Here $X^K \cdot \dot{\varepsilon}$ is the magnetic Hamiltonian vector field of the function $h^K \cdot \dot{\varepsilon}: T^*Q \to \mathcal{R}$ and $X^K$ is the dynamical vector field of the nonholonomic reduced distributional magnetic Hamiltonian system $(\mathcal{K}, \omega^K, h^K)$. The equation $T\dot{\gamma} \cdot X^\varepsilon = X^K \cdot \dot{\varepsilon}$ is called the Type II Hamilton-Jacobi equation for the nonholonomic reduced distributional magnetic Hamiltonian system $(\mathcal{K}, \omega^K, h^K)$.

**Proof.** In the same way, we note that $\operatorname{Im}(\gamma) \subset \mathcal{M}$, and that $\gamma$ is $G$-invariant, and that $\operatorname{Im}(T\gamma) \subset \mathcal{K}$, and hence, $\operatorname{Im}(T\dot{\gamma}) \subset \mathcal{K}$, in this case, we have that $\pi_G \cdot \omega^K \cdot \tau^K = \tau_M \cdot \omega_M = \tau_M \cdot i_M \cdot \omega^B$ along $\operatorname{Im}(T\dot{\gamma})$. Moreover, from the distributional magnetic Hamiltonian equation (5.1), we have that $X^K = \tau_M \cdot X^B$. Note that $\varepsilon(\mathcal{M}) \subset \mathcal{M}$, and that $T\pi_Q(X^B \cdot \varepsilon(q, p)) \in \mathcal{D}_q$, $\forall q \in Q$, $(q, p) \in \mathcal{M}(\subset T^*Q)$, and hence, $X^B \cdot \varepsilon \in \mathcal{F}$.
along \( \varepsilon \). Because the vector fields \( X^B_K \) and \( X^B_K \) are \( \pi_{/G} \)-related; that is, \( T\pi_{/G}(X^B_K) = X^B_K \cdot \pi_{/G} \), and hence,

\[
\tau_K \cdot T\pi_{/G}(X^B_K \cdot \varepsilon) = \tau_K \cdot \left( T\pi_{/G}(X^B_K) \right) \cdot (\varepsilon) = \tau_K \cdot (X^B_K \cdot \pi_{/G})(\varepsilon) = X^B_K \cdot \xi.
\]

Thus, using the non-degenerate and nonholonomic reduced distributional two-form \( \omega^B_K \), from Lemmas 3.1 and 4.1, we take that \( \varepsilon = X^B_H \cdot \varepsilon \in \mathcal{F} \), and that for any \( w \in \mathcal{F}, T\lambda(w) \neq 0 \) and \( \tau_K \cdot T\pi_{/G} \cdot w \neq 0 \), then we have that

\[
\omega^B_K(T\gamma \cdot X^e, \tau_K \cdot T\pi_{/G} \cdot w) = \omega^B_K(\tau_K \cdot T(\pi_{/G} \cdot \gamma) \cdot X^e, \tau_K \cdot T\pi_{/G} \cdot w)
\]

\[
= \pi^*_{/G} \cdot \omega^B_K(T\gamma \cdot X^e, w) = \tau_K \cdot i^*_M \cdot \omega^B_K(T\gamma \cdot X^e, w)
\]

\[
= \tau_K \cdot i^*_M \cdot \omega^B_K(T(\gamma \cdot \pi_Q) \cdot X^e, w)
\]

\[
= \tau_K \cdot i^*_M \cdot \omega^B_K(\omega^B_K(\gamma \cdot X^e, w - T(\gamma \cdot \pi_Q) \cdot w) - (d\gamma + B)(T\pi_Q(X^B_H \cdot \varepsilon), T\pi_Q(w)))
\]

\[
= \tau_K \cdot i^*_M \cdot \omega^B_K(X^B_H \cdot \varepsilon, w) - \tau_K \cdot i^*_M \cdot \omega^B_K(X^B_H \cdot \varepsilon, T\lambda \cdot w)
\]

\[
- \tau_K \cdot i^*_M \cdot (d\gamma + B)(T\pi_Q(X^B_H \cdot \varepsilon), T\pi_Q(w))
\]

\[
= \pi^*_{/G} \cdot \omega^B_K(\tau_K \cdot T\pi_{/G} \cdot X^B_H \cdot \varepsilon, \tau_K \cdot T\pi_{/G} \cdot w) - \pi^*_{/G} \cdot \omega^B_K(\tau_K \cdot T\pi_{/G} \cdot X^B_H \cdot \varepsilon, T\lambda \cdot w) + \tau_K \cdot i^*_M \cdot \lambda^* \omega^B_K(X^B_H \cdot \varepsilon, w)
\]

\[
= \omega^B_K(\tau_K \cdot T\pi_{/G} \cdot X^B_H \cdot \varepsilon, \tau_K \cdot T\pi_{/G} \cdot w) - \omega^B_K(\tau_K \cdot T\pi_{/G} \cdot X^B_H \cdot \varepsilon, \tau_K \cdot T(\pi_{/G} \cdot \lambda) \cdot w)
\]

\[
+ \pi^*_{/G} \cdot \omega^B_K(\tau_K \cdot T\lambda \cdot X^B_H \cdot \varepsilon, T\lambda \cdot w)
\]

\[
= \omega^B_K(\tau_K \cdot T\pi_{/G} \cdot X^B_H \cdot \varepsilon, \tau_K \cdot T\pi_{/G} \cdot w)
\]

\[
- \omega^B_K(\tau_K \cdot T\pi_{/G} \cdot X^B_H \cdot \varepsilon, \tau_K \cdot T\lambda \cdot w)
\]

\[
+ \omega^B_K(\tau_K \cdot T\pi_{/G} \cdot T\lambda \cdot X^B_H \cdot \varepsilon, \tau_K \cdot T\pi_{/G} \cdot T\lambda \cdot w)
\]

\[
= \omega^B_K(X^B_K \cdot \xi, \tau_K \cdot T\pi_{/G} \cdot w) - \omega^B_K(X^B_K \cdot \xi, T\lambda \cdot w) + \omega^B_K(T\lambda \cdot X^B_K \cdot \xi, T\lambda \cdot w),
\]

where we used the relations

\[
\tau_K \cdot T\pi_{/G}(X^B_K \cdot \varepsilon) = \tau_K \left( X^B_K \right) \cdot \xi = X^B_K \cdot \xi, \quad \tau_K \cdot T\pi_{/G} \cdot T\lambda = T\lambda
\]

since \( \text{Im}(T\gamma) \subset \mathcal{K} \). From the nonholonomic reduced distributional magnetic Hamiltonian equation (5.2); that is, \( i_{\gamma_K}^B \omega^B_K = d\xi_k \), we have that \( X^B_K = \tau_K \cdot X^B_K \), where
$X^B_{h_K}$ is the magnetic Hamiltonian vector field of the function $h_K: \tilde{M}(\subset T^*Q/G) \to \mathbb{R}$. Note that $\varepsilon: T^*Q \to T^*Q$ is symplectic with respect to $\omega^B$, and that $\tilde{\varepsilon} = \pi/G(\varepsilon): T^*Q \to T^*Q/G$ is also symplectic along $\tilde{\varepsilon}$, and hence, $X^B_{h_K} \cdot \tilde{\varepsilon} = T\tilde{\varepsilon} \cdot X^B_{h_K}$ along $\tilde{\varepsilon}$, and $X^B_{\tilde{\varepsilon}} = \tau_{\tilde{\varepsilon}} \cdot X^B_{h_K}$ along $\tilde{\varepsilon}$. Then we have that

$$\omega^B_K(T\tilde{\gamma} \cdot X^e, \tau_{\tilde{\varepsilon}} \cdot T\pi/G \cdot w) - \omega^B_K \left( X^B_{h_K} \cdot \tilde{\varepsilon}, \tau_{\tilde{\varepsilon}} \cdot T\pi/G \cdot w \right) = -\omega^B_K \left( X^B_{h_K} \cdot \tilde{\varepsilon}, T\tilde{\lambda} \cdot w \right) + \omega^B_K \left( T\tilde{\lambda} \cdot X^B_{h_K} \cdot \varepsilon, T\tilde{\lambda} \cdot w \right) = \omega^B_K \left( T\tilde{\lambda} \cdot X^B_{h_K} \cdot \varepsilon - \tau_{\tilde{\varepsilon}} \cdot T\tilde{\varepsilon} \cdot X^B_{h_K} \cdot \varepsilon, T\tilde{\lambda} \cdot w \right).$$

Since the nonholonomic reduced distributional two-form $\omega^B_K$ is non-degenerate, it follows that the equation $T\tilde{\gamma} \cdot X^e = X^B_{h_K} \cdot \tilde{\varepsilon}$ is equivalent to the equation $T\tilde{\lambda} \cdot X^B_{h_K} \cdot \varepsilon = \tau_{\tilde{\varepsilon}} \cdot T\tilde{\varepsilon} \cdot X^B_{h_K} \cdot \varepsilon$. Thus, $\varepsilon$ and $\tilde{\varepsilon}$ satisfy the equation $T\tilde{\lambda} \cdot X^B_{h_K} \cdot \varepsilon = \tau_{\tilde{\varepsilon}} \cdot T\tilde{\varepsilon} \cdot X^B_{h_K} \cdot \varepsilon$ if and only if they satisfy the Type II Hamilton-Jacobi equation $T\tilde{\gamma} \cdot X^e = X^B_{h_K} \cdot \tilde{\varepsilon}$.\hfill $\square$

Let $(T^*Q, G, \omega^B, D, H)$ be nonholonomic reducible magnetic Hamiltonian system with associated distributional magnetic Hamiltonian system $(K, \omega^B_K, H_K)$ and nonholonomic reduced distributional magnetic Hamiltonian system $(\tilde{K}, \omega^B_{\tilde{K}}, h_{\tilde{K}})$. We know that the nonholonomic dynamical vector field $X^B_K$ and the nonholonomic reduced dynamical vector field $X^B_{\tilde{K}}$ are $\pi/G$-related, i.e., $X^B_K \cdot \pi/G = T\pi/G \cdot X^B_{\tilde{K}}$. Then we can prove the Theorem 5.3 to state the relationship between the solutions of Type II Hamilton-Jacobi equations and nonholonomic reduction.

**Theorem 5.3.** Let $(T^*Q, G, \omega^B, D, H)$ be nonholonomic reducible magnetic Hamiltonian system with the associated distributional magnetic Hamiltonian system $(K, \omega^B_K, H_K)$ and the nonholonomic reduced distributional magnetic Hamiltonian system $(\tilde{K}, \omega^B_{\tilde{K}}, h_{\tilde{K}})$, assume that $\gamma: Q \to T^*Q$ is a one-form on Q, and that $\lambda = \gamma \cdot \pi_Q: T^*Q \to T^*Q$, and that $\varepsilon: T^*Q \to T^*Q$ is a $G$-invariant symplectic map with respect to $\omega^B$. Moreover, assume that $\text{Im}(\gamma) \subset M$, and that $\gamma$ is $G$-invariant, and that $\varepsilon(M) \subset M$, and that $\text{Im}(T\gamma) \subset K$, and that $\tilde{\gamma} = \pi/G(\gamma): Q \to T^*Q/G$, and that $\tilde{\lambda} = \pi/G(\lambda): T^*Q \to T^*Q/G$, and that $\tilde{\varepsilon} = \pi/G(\varepsilon): T^*Q \to T^*Q/G$. Then $\varepsilon$ is a solution of the Type II Hamilton-Jacobi equation $T\tilde{\gamma} \cdot X^e = X^B_{\tilde{K}} \cdot \tilde{\varepsilon}$ for the distributional magnetic Hamiltonian system $(K, \omega^B_K, H_K)$ if and only if $\varepsilon$ and $\tilde{\varepsilon}$ satisfy the Type II Hamilton-Jacobi equation $T\gamma \cdot X^e = X^B_K \cdot \varepsilon$ for the nonholonomic reduced distributional magnetic Hamiltonian system $(\tilde{K}, \omega^B_{\tilde{K}}, h_{\tilde{K}})$.

**Proof.** Note that $\text{Im}(\gamma) \subset M$, and that $\gamma$ is $G$-invariant, and that $\text{Im}(T\gamma) \subset K$, and hence, $\text{Im}(T\tilde{\gamma}) \subset \tilde{K}$, in this case, we have that $\pi^*_G \cdot \omega^B_{\tilde{K}} \cdot \tau_{\tilde{K}} = \tau_{\tilde{U}} \cdot \omega^B_M = \tau_{\tilde{U}} \cdot \pi^*_M \cdot \omega^B$
along \( \text{Im}(T\dot{\gamma}) \), and that \( \tau_K \cdot T\dot{\gamma} = T\dot{\gamma}, \tau_K \cdot X^B_K = X^B_K \). Since the nonholonomic vector field \( X^B_K \) and the vector field \( X^B_K \) are \( \pi/G \)-related; that is, \( X^B_K \cdot \pi/G = T\pi/G \cdot X^B_K \), using the non-degenerate and nonholonomic reduced distributional two-form \( \omega^B_K \), we have that
\[
\omega^B_K \left( T\dot{\gamma} \cdot X^\varepsilon - X^B_K \cdot \varepsilon, \tau_K \cdot T\pi/G \cdot w \right) = \omega^B_K \left( T\dot{\gamma} \cdot X^\varepsilon, \tau_K \cdot T\pi/G \cdot w \right) - \omega^B_K \left( X^B_K \cdot \varepsilon, \tau_K \cdot T\pi/G \cdot w \right) = \omega^B_K \left( \tau_K \cdot T\dot{\gamma} \cdot X^\varepsilon, \tau_K \cdot T\pi/G \cdot w \right) - \omega^B_K \left( \tau_K \cdot X^B_K \cdot \pi/G \cdot \varepsilon, \tau_K \cdot T\pi/G \cdot w \right) = \omega^B_K \cdot \tau_K \left( T\pi/G \cdot T\gamma \cdot X^\varepsilon, T\pi/G \cdot w \right) - \omega^B_K \cdot \tau_K \left( T\pi/G \cdot X^B_K \cdot \varepsilon, T\pi/G \cdot w \right) = \pi^*_G \cdot \omega^B_K \cdot \tau_K \left( T\gamma \cdot X^\varepsilon, w \right) - \pi^*_G \cdot \omega^B_K \cdot \tau_K \left( X^B_K \cdot \varepsilon, w \right) = \tau_U \cdot \iota_M^* \cdot \omega^B \left( T\gamma \cdot X^\varepsilon, w \right) - \pi_U \cdot \iota_M^* \cdot \omega^B \left( X^B_K \cdot \varepsilon, w \right).
\]

In the case we considered that \( \tau_U \cdot \iota_M^* \cdot \omega^B = \tau_K \cdot \iota_M^* \cdot \omega^B = \omega^B_K \cdot \tau_K \), and that \( \tau_K \cdot T\gamma = T\gamma, \tau_K \cdot X^B_K = X^B_K \), since \( \text{Im}(\gamma) \subset \mathcal{M} \), and \( \text{Im}(T\gamma) \subset \mathcal{K} \). Thus, we have that
\[
\omega^B_K \left( T\dot{\gamma} \cdot X^\varepsilon - X^B_K \cdot \varepsilon, \tau_K \cdot T\pi/G \cdot w \right) = \omega^B_K \cdot \tau_K \left( T\gamma \cdot X^\varepsilon, w \right) - \omega^B_K \left( X^B_K \cdot \varepsilon, w \right) = \omega^B_K \left( \tau_K \cdot T\gamma \cdot X^\varepsilon, \tau_K \cdot w \right) - \omega^B_K \left( \tau_K \cdot X^B_K \cdot \varepsilon, \tau_K \cdot w \right) = \omega^B_K \left( T\gamma \cdot X^\varepsilon - X^B_K \cdot \varepsilon, \tau_K \cdot w \right) = \omega^B \left( T\gamma \cdot X^\varepsilon - X^B_K \cdot \varepsilon, \tau_K \cdot w \right).
\]

Since the distributional two-form \( \omega^B_K \) and the nonholonomic reduced distributional two-form \( \omega^B_K \) are both non-degenerate, it follows that the equation \( T\dot{\gamma} \cdot X^\varepsilon = X^B_K \cdot \varepsilon \) is equivalent to the equation \( T\gamma \cdot X^\varepsilon = X^B_K \cdot \varepsilon \). Thus, \( \varepsilon \) is a solution of the Type II Hamilton-Jacobi equation \( T\gamma \cdot X^\varepsilon = X^B_K \cdot \varepsilon \) for the distributional magnetic Hamiltonian system \( (\mathcal{K}, \omega^B_K, H_K) \) if and only if \( \varepsilon \) and \( \bar{\varepsilon} \) satisfy the Type II Hamilton-Jacobi equation \( T\dot{\gamma} \cdot X^\varepsilon = X^B_K \cdot \varepsilon \) for the nonholonomic reduced distributional magnetic Hamiltonian system \( (\bar{\mathcal{K}}, \omega^B_K, H_{\bar{K}}) \).

**Remark 5.1.** It is worth noting that the Type I Hamilton-Jacobi equation \( T\dot{\gamma} \cdot X^\varepsilon = X^B_K \cdot \varepsilon \) is the equation of the nonholonomic reduced differential one-form \( \gamma \), and that the Type II Hamilton-Jacobi equation \( T\dot{\gamma} \cdot X^\varepsilon = X^B_K \cdot \varepsilon \) is the equation of the symplectic diffeomorphism map \( \varepsilon \) and the nonholonomic reduced symplectic diffeomorphism map \( \bar{\varepsilon} \). When \( B = 0 \), the magnetic symplectic form \( \omega^B \) is just the
canonical symplectic form $\omega$ on $T^*Q$, and the nonholonomic reducible magnetic Hamiltonian system is just the nonholonomic reducible Hamiltonian system itself, and the nonholonomic reduced distributional magnetic Hamiltonian system is just the nonholonomic reduced distributional Hamiltonian system. From the Type I and Type II Hamilton-Jacobi theorems; that is, Theorems 5.1 and 5.2, we can get the Theorems 4.2 and 4.3 given in León and Wang [14]. It has shown that Theorems 5.1 and 5.2 can be regarded as an extension of the two types of Hamilton-Jacobi theorems for the nonholonomic reduced distributional Hamiltonian system to that for the nonholonomic reduced distributional magnetic Hamiltonian system.

6 Conclusion

In order to describe the impact of the different geometric structures and the constraints for the dynamics of a Hamiltonian system, in this paper, we study the Hamilton-Jacobi theory for the magnetic Hamiltonian system, the nonholonomic magnetic Hamiltonian system and the nonholonomic reducible magnetic Hamiltonian system on a cotangent bundle, by using the distributional magnetic Hamiltonian system and the nonholonomic reduced distributional magnetic Hamiltonian system, which are the developments of the Hamilton-Jacobi theory for the nonholonomic Hamiltonian system and the nonholonomic reducible Hamiltonian system given in León and Wang [14]. These research reveal, from the geometrical point of view, the internal relationships of the magnetic symplectic forms, the nonholonomic constraints, the non-degenerate distributional two forms and dynamical vector fields of a nonholonomic magnetic Hamiltonian system and the nonholonomic reducible magnetic Hamiltonian system. It is worth noting that Marsden et al. in [20] set up the regular reduction theory of the regular controlled Hamiltonian systems on a symplectic fiber bundle, by using the momentum maps and the associated reduced symplectic forms, and from the viewpoint of completeness of the Marsden-Weinstein symplectic reduction. Some developments around the above work are given in Wang and Zhang [31], Ratiu and Wang [24], and Wang [25, 27, 28]. Since the Hamilton-Jacobi theory is developed based on the Hamiltonian picture of dynamics, a natural idea is to extend the Hamilton-Jacobi theory to the (regular) controlled (magnetic) Hamiltonian systems and their a variety of reduced systems, and it is also possible to describe the application and the relationship between the RCH-equivalence for the controlled Hamiltonian systems and the solutions of the corresponding Hamilton-Jacobi equations, see Wang [26, 29, 30] for more details. Thus, our next topic is
how to set up and develop the nonholonomic reduction and the Hamilton-Jacobi theory for the nonholonomic controlled (magnetic) Hamiltonian systems and the distributional controlled (magnetic) Hamiltonian systems, by analyzing carefully the geometrical and topological structures of the phase spaces of these systems. It is the key thought of the researches of geometrical mechanics of Professor Jerrold E. Marsden to explore and reveal the deeply internal relationship between the geometrical structure of phase space and the dynamical vector field of a mechanical system. It is also our goal of pursuing and inheriting. In addition, we note also that there have been a lot of beautiful results of the reduction theory of Hamiltonian systems in celestial mechanics, hydrodynamics and plasma physics. Thus, it is an important topic to study the application of the reduction theory and the Hamilton-Jacobi theory of the systems in celestial mechanics, hydrodynamics and plasma physics. These are our goals in future research.

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