Duality properties for quantum groups

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Abstract : Some duality properties for induced representations of enveloping algebras involve the character $\text{Trad}_g$. We extend them to deformation Hopf algebras $A_h$ of a noetherian Hopf $k$-algebra $A_0$ satisfying $\text{Ext}^i_{A_0}(k, A_0) = \{0\}$ except for $i = d$ where it is isomorphic to $k$. These duality properties involve the character of $A_h$ defined by right multiplication on the one dimensional free $k[[h]]$-module $\text{Ext}^d_{A_h}(k[[h]], A_h)$. In the case of quantized enveloping algebras, this character lifts the character $\text{Trad}_g$. We also prove Poincaré duality for such deformation Hopf algebras in the case where $A_0$ is of finite homological dimension. We explain the relation of our construction with quantum duality.

1. Introduction

In this article $k$ will be a field of characteristic 0 and we set $K = k[[h]]$.

Let $A_0$ be a noetherian algebra. We assume moreover that $k$ has a left $A_0$-module structure such that there exists an integer $d$ satisfying

\[
\begin{cases}
    \text{Ext}^i_{A_0}(k, A_0) = \{0\} \text{ if } i \neq d \\
    \text{Ext}^d_{A_0}(k, A_0) \simeq k.
\end{cases}
\]

It follows from Poincaré duality that any finite dimensional Lie algebra $\mathfrak{g}$ verifies these assumptions. In this case $d = \dim \mathfrak{g}$ and the character defined by the right representation of $U(\mathfrak{g})$ on $\text{Ext}^d_{U(\mathfrak{g})}(k, U(\mathfrak{g}))$ is $\text{Trad}_\mathfrak{g}$ ([Cl]). The algebra of regular functions on an affine algebraic Poisson group and algebra of formal power series also satisfy these hypothesis. Let $A_h$ be a deformation algebra of $A_0$. Assume that there exists an $A_h$-module structure on $K$ that reduces modulo $h$ to the $A_0$-module structure we started with. The following theorem constructs a new character of $A_h$, which will be denoted by $\theta_{A_h}$.

**Theorem 5.0.7**

*With the assumptions made above, one has :

a) $\text{Ext}^i_{A_h}(K, A_h) = \{0\}$ is zero if $i \neq d$

b) $\text{Ext}^d_{A_h}(K, A_h)$ is a free $K$-module of dimension one. The right $A_h$-module structure given by right multiplication lifts that of $A_0$ on $\text{Ext}^d_{A_0}(k, A_0)$.***
The right $A_h$-module $\text{Ext}^d_{A_h}(K,A_h)$ will be denoted by $\Omega_{A_h}$. If there is an ambiguity, the integer $d$ will be written $d_{A_h}$.

Theorem 5.0.7 applies to universal quantum enveloping algebras, quantization of affine algebraic Poisson groups and to quantum formal series Hopf algebras.

Let $g$ be a Lie bialgebra. Denote by $F[[g]]$ the formal series Poisson algebra $U(g^*)$. If $U_h(g^*)$ is a quantum enveloping algebra such that $U_h(g^*)/hU_h(g^*)$ is isomorphic to $U(g^*)$ as a coPoisson Hopf algebra, we show that one may construct a resolution of the trivial $U_h(g^*)$-module $k[[h]]$ that lifts the Koszul resolution of the trivial $U(g^*)$-module $k$. If $F_h[g]$ is a quantum formal series algebras such that $F_h[g]/hF_h[g]$ is isomorphic to $F[g]$ as a Poisson Hopf algebra, we construct a resolution of the trivial $F_h[g]$-module that lifts the Koszul resolution of the trivial $F[g]$-module $k$ and that respects quantum duality ([Dr], [Ga]). This construction is not explicit but it allows to show that, if $F_h[g]$ and $U_h(g^*)$ are linked by quantum duality, the following equality holds $\theta_{F_h[g]} = h\theta_{U_h(g^*)}$.

As an application of theorem 5.0.7, we show Poincaré duality:

**Theorem 8.1.1**

We make the same assumptions as above. Let $M$ be an $A_h$-module. Assume that $K$ is an $A_h$-module of finite projective dimension. One has an isomorphism of $K$-modules for all integer $i$:

$$\text{Ext}^i_{A_h}(K,M) \cong \text{Tor}^{A_h}_{d_{A_h}-i}(\Omega_{A_h},M).$$

From now on, we assume that $A_h$ is a deformation Hopf algebra.

Brown and Levasseur ([B-L]) and Kempf ([Ke]) had shown that, in the semi-simple context, the Ext-dual of a Verma module is a Verma module. In [C1], we have extended this result to the Ext-dual of an induced representation of any Lie superalgebra. In this article, we show that this result can be generalized to quantum groups provided that the quantization is functorial. Such a functorial quantization has been constructed by Etingof and Kazdhan ([E-K1], [E-K2], [E-K3], [E-S]). As the result holds for quantized universal enveloping algebras, for quantized functions algebras and for quantum formal series Hopf algebras, we state it in the more general setting of Hopf algebras.

**Corollary 8.2.2**

Let $A_h$ (respectively $B_h$) be a topological Hopf deformation of $A_0$ (respectively $B_0$). We assume that there exists a morphism of Hopf algebras from $B_h$ to $A_h$ such that $A_h$ is a flat $B_h^{op}$-module. We also assume that $B_h$ satisfies the condition of the theorem 5.0.7. Let $V$ be a $B_h$-module which is a free finite dimensional $K$-module. Then

a) $\text{Ext}^i_{A_h}(A_h \otimes_{B_h} V, A_h)$ is $\{0\}$ if $i$ is different from $d_{B_h}$.
In this case one has Lie subalgebras of $W_C$ of this theorem (namely the case where the semi-simple hypothesis) but in a slightly different context.

Collingwood and B. Shelton ([C-S]) also proved a duality of this type (still under assumptions that $B$ where $Ω_B$ that $morphism of Hopf algebras from $n$.

Then, for all integer $V = \Omega_B⊗ V^*$ one has an isomorphism

$$\text{Ext}^{n+d_B}_{A_h} \left( A_h ⊗ V, A_h ⊗ W \right) \simeq \text{Ext}^{n+d_C}_{A_h} \left( (Ω_{C_h} ⊗ W^*) ⊗ A_h, (Ω_{B_h} ⊗ V^*) ⊗ A_h \right).$$

The right $B_h$ (respectively $C_h$)-module structure on $Ω_{B_h} ⊗ V^*$ (respectively $Ω_{C_h} ⊗ W^*$ ) are as in Corollary 8.2.2.

**Remark :**

Proposition 8.2.3 is already known in the case where $g$ is a Lie algebra, $h$ and $k$ are Lie subalgebras of $g$, $A_h$, $B_h$ and $C_h$ are their corresponding enveloping algebras.

In this case one has $d_{B_h} = dim h$ and $d_{C_h} = dim k$. More precisely :

Generalizing a result of G. Zuckerman ([B-C]), A. Gyoja ([G]) proved a part of this theorem (namely the case where $h = g$ and $n = dim h = dim k$) under the assumptions that $g$ is split semi-simple and $h$ is a parabolic subalgebra of $g$. D.H Collingwood and B. Shelton ([C-S]) also proved a duality of this type (still under the semi-simple hypothesis) but in a slightly different context.

M. Duflo ([Du2]) proved proposition 8.2.3 for a general Lie algebra, $h = k$, $V = W^*$ being one dimensional representations.

Proposition 8.2.3 is proved in full generality in the context of Lie superalgebras in [C1].

Wet set $A_h^k = A_h ⊗ A_h^k$. Using the properties as a Hopf algebra (as in [C2]), we show that all the $\text{Ext}^{n+d_{B_h}}_{A_h} (A_h, A_h ⊗ A_h)$’s are zero except one. More precisely :

**Proposition 8.3.1** Assume that $A_h$ satisfies the conditions of the theorem 5.0.7. Assume moreover that $A_0 ⊗ A_0^{op}$ is noetherian. Consider $A_h ⊗ A_h$ with the following $A_h^k$-module structure :

$$\forall (α, β, x, y) ∈ A_h, \quad α · (x ⊗ y) · β = αx ⊗ yβ.$$
a) $HH_{A_h}(A_h \otimes_{k[[h]]} A_h)$ is zero if $i \neq d_{A_h}$.

b) The $\widehat{A}_h$-module $HH^{d_{A_h}}_{A_h}(A_h \otimes_{k[[h]]} A_h)$ is isomorphic to $\Omega_{A_h} \otimes A_h$ with the following $\widehat{A}_h$-module structure:

$$\forall (\alpha, \beta, x) \in A_h, \quad \alpha \cdot (\omega \otimes x) \cdot \beta = \omega \theta_{A_h}(\beta'') x S(\beta') x S^{-1}(\alpha)$$

where $\alpha = \sum \alpha'_i \otimes \alpha''_i$ (to be taken in the topological sense)

This result has already been obtained in [D-E] for a deformation of the algebra of regular functions on a smooth algebraic affine variety. From this, as in [VdB], we deduce a relation between Hochschild homology and Hochschild cohomology for the ring $A_h$.

We start the article by a study of algebras endowed with a decreasing filtration and filtered modules over such algebras. Our study relies on the use of the associated graded algebra and graded module and on the use of the topology defined by a decreasing filtration. We apply this study to deformation algebras endowed with the $h$-adic filtration and filtered modules over such algebras. In [K-S], a study of the derived category of $A_h$-modules is carried out using the right derived functor of the functor $M \mapsto \frac{M}{h^n M}$.

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2. Graded linear algebra

In this section, we fix notation about graded linear algebra. A graded $k$-algebra $GA$ is the data of a $k$-algebra with unit and a family of $k$-vector spaces $(G_tA)_{t \in \mathbb{Z}}$ of $A$ satisfying:

a. $A = \bigoplus_{t \in \mathbb{Z}} G_tA$

b. $1 \in G_0A$

c. $G_tA \cdot G_lA \subset G_{t+l}A$.

We will also assume that $G_tA = 0$ for $t < 0$.

A graded $GA$-module $GM$ is the data of a $GA$-module and a family of $k$-vector space $(G_tM)_{t \in \mathbb{Z}}$ of $GM$ such that

$$GM = \bigoplus_{t \in \mathbb{Z}} G_tM$$

$$G_tA \cdot G_lM \subset G_{t+l}M$$

We will always also assume that $G_tM = 0$ if $t << 0$.

Let $GM$ and $GN$ be two graded $GA$-modules. A morphism of graded $GA$-modules from $GM$ to $GN$ is a morphism of $GA$-modules $f : GM \to GN$ such that $f(G_tM) \subset G_tN$. The group of morphisms of graded $GA$-modules from $GM$ to $GN$ will be denoted $\text{Hom}_{GA}(GM, GN)$. With this notion of morphisms, the category of graded $GA$-modules is abelian. Thus it is suitable for homological algebra.

For $r \in \mathbb{Z}$ and any graded $GA$-module $GM$, we define the shifted graded $GA$-module $GM(r)$ to be the $GA$-module $GM$ endowed with the grading defined by

$$\forall t \in \mathbb{Z}, \quad G_tM(r) = G_{t+r}M.$$
Let us denote $\text{Hom}_{GA}(GM, GN)$ the graded group defined by setting
\[ G_t \text{Hom}_{GA}(GM, GN) = \text{Hom}_{GA}(GM, GN(t)). \]
The $i^{\text{th}}$ right derived functor of the functor $\text{Hom}_{GA}(-, N)$ will be denoted $\text{Ext}_{GA}^i(-, N)$.

A graded $GA$-module $GL$ is finite free if there are integers $d_1, d_2, \ldots, d_n$ such that
\[ GL \cong \bigoplus_{i=1}^{n} GA(-d_i). \]
A graded $GA$-module $GM$ is of finite type if there exists a finite free graded $GA$-module $GL$ and an exact sequence in the category of graded $GA$-modules
\[ GL \to GM \to 0. \]
This means that there are homogeneous elements $m_1 \in G_{d_1}M, \ldots, m_n \in G_{d_n}M$ such that any $m \in G_dM$ may be written as
\[ m = \sum_{i=1}^{n} a_{d-d_i}m_d, \]
where $a_{d-d_i} \in G_{d-d_i}A$.

A graded ring $GA$ is noetherian if any graded $GA$-submodule of a graded $GA$-module of finite type is of finite type.

In the sequel, all the $GA$-modules we will consider will be graded so that we will say "$GA$-module" for "graded $GA$-module".

3. Decreasing filtrations

In this section, we give results about decreasing filtrations. These results are proved in [Schn] in the framework of increasing filtrations. For the sake of completeness, we give detailed proofs of the results even if most of our proofs are obtained by adjusting those of Schneider.

We will consider a $k$-algebra endowed with a decreasing filtration $\ldots F_{t+1}A \subset F_tA \subset \cdots \subset F_1A \subset F_0A = A$. The order of an element $a$, $o(a)$, is the biggest $t$ such that $a \in F_tA$. The principal symbol of $a$ is the image of $a$ in $F_{o(a)}/F_{o(a)+1}$. It will be denoted by $[a]$.

A filtered module over $FA$ is the data of an $A$-module $M$ and a family $(F_tM)_{t \in \mathbb{Z}}$ of $k$-subspaces such that
- $\bigcup_{t \in \mathbb{Z}} F_tM = M$
- $F_{t+1}M \subset F_tM$
- $F_tA \cdot F_tM \subset F_{t+1}M$

We will assume that $F_tM = M$ for $t << 0$. We have the notion of principal symbol. We endow such a module with the topology for which a basis a neighborhoods is $(F_tM)_{t \in \mathbb{Z}}$. The topological space $M$ is Hausdorff if and only if $\cap_{t \in \mathbb{Z}} F_tM = \{0\}$. If $M$ is Hausdorff, the topology defined by the filtration is defined by the following metric
\[ \forall (x, y) \in FM, \ d(x, y) = || x - y || \text{ with} \]  
\[ || x - y || = 2^{-t} \text{ where } t = \text{Sup} \{ j \in \mathbb{Z} \mid x - y \in F_jM \} \]
Note that $M$ is Hausdorff if and only if the natural map from $M$ to $\lim_{t \in \mathbb{Z}} \frac{M}{F_t M}$ is injective. The metric space $(M, d)$ is complete if and only if the natural map from $M$ to $\lim_{t \in \mathbb{Z}} \frac{M}{F_t M}$ is an isomorphism.

**Example:**
Let $k$ be a field and set $K = k[[h]]$. If $V$ is a $K$-module, it is endowed with the following decreasing filtration $\cdots \subset h^nV \subset h^{n-1}V \subset \cdots \subset hV \subset V$.

The topology induced by this filtration is the $h$-adic topology.

Recall the following result:

**Lemma 3.0.1.** Let $N$ be a Hausdorff filtered module. Let $P$ be a submodule of $N$ which is closed in $N$. Let $p$ the canonical projection from $N$ to $N/P$.

a) The topology defined by the filtration $p(F_t N)$ on $N/P$ is the quotient topology.

b) If $N$ is complete, then $N/P$ is complete for the quotient topology.

**Proof of the lemma:**
a) As $P$ is closed in $N$, then 0 is closed in $N/P$. Thus, its complement in $N/P$, $U$, is open. Let $\bar{x}$ an element of $N/P$ different from 0. As $U$ is open, there exists $n \in \mathbb{N}$ such that $\bar{x} \in p(F_n N) + \bar{x} \subset U$. Hence $\bar{x} \notin p(F_n N)$ and we have proved that $\bigcap_{n \in \mathbb{N}} p(F_n N) = \{0\}$. Hence $N/P$ is Hausdorff. It is easy to check that the open ball of center 0 and radius $2^{-t}$ in $N/P$ for the distance defined is $p(F_{t+1} N)$.

b) We refer to [Schw] p 245. □.

Let $FM$ and $FN$ be two filtered $FA$-modules. A filtered morphism $Fu : FM \rightarrow FN$ is a morphism $u : M \rightarrow N$ of the underlying $A$-modules such that $u(F_t M) \subset F_t N$. It is continuous if we endow $M$ and $N$ with the topology defined by the filtrations. Denote by $F_t u$ the morphism $u|_{F_t M} : F_t M \rightarrow F_t N$. Denote by $Hom_{F,A}(FM, FN)$ the group of filtered morphisms from $FM$ to $FN$. The kernel of $Fu$ is the kernel of $u$ filtered by the family $Ker Fu \cap F_t M$. If $M$ is complete and $N$ is Hausdorff, then $Ker Fu$, endowed with the induced topology is complete.

To a filtered ring $FA$ is associated a graded ring $GA$ defined by

$$GA = \bigoplus_{t \in \mathbb{Z}} G_t A \text{ with } G_t A = F_t A/F_{t+1} A$$

the multiplication being induced by that of $FA$. To a filtered $FA$-module $FM$ is associated a graded $GA$-module $GM$ defined by setting

$$GM = \bigoplus_{t \in \mathbb{Z}} G_t M \text{ with } G_t M = F_t M/F_{t+1} M$$

the action of $GA$ on $GM$ being induced by that of $FA$. If $x$ is in $F_t M$, we will write $\sigma_t(x)$ for the class of $x$ in $F_t M/F_{t+1} M$. A filtered morphism of $FA$-modules $Fu : FM \rightarrow FN$ induces a morphism of abelian groups $G_t u : G_t M \rightarrow G_t N$ and a morphism of $GA$-modules $Gu : GM \rightarrow GN$. 
An arrow $Fv : FM \to FN$ is strict if it satisfies $u(FtM) = u(M) \cap FtN$.

An exact sequence of $FA$-modules is a sequence

$$FM \xrightarrow{Fu} FN \xrightarrow{Fv} FP$$

such that $Ker Fv = Im Fu$. It follows from this definition that $Fu$ is strict. Moreover $Fv$ is strict, we say that it is a strict exact sequence.

**Proposition 3.0.2.** a) Consider $Fu : FM \to FN$ and $Fv : FN \to FP$ two filtered $FA$-morphisms such that $Fv \circ Fu = 0$. If the sequence

$$FM \xrightarrow{Fu} FN \xrightarrow{Fv} FP$$

is strict exact, then

$$GM \xrightarrow{Gu} GN \xrightarrow{Gv} GP$$

is exact.

b) Conversely, assume that $FM$ is complete for the topology defined by the filtration and that $FN$ is Hausdorff for the topology defined by the filtration. If the sequence

$$GM \xrightarrow{Gu} GN \xrightarrow{Gv} GP$$

is exact, then the sequence

$$FM \xrightarrow{Fu} FN \xrightarrow{Fv} FP$$

is strict exact.

**Proof of the proposition:**

a) Let $n_t \in GtN$ be such that $Gtv(n_t) = 0$. There is $n'_t \in FtN$ such that $n_t = \sigma_t(n'_t)$. Hence $v(n'_t) \in Ft+1P$. Since $Fv$ is strict, we find $n''_{t+1} \in Ft+1N$ such that $v(n''_{t+1}) = v(n'_t)$. Then $v(n'_t - n''_{t+1}) = 0$ and there is $m_t \in FtM$ such that $u(m_t) = n'_t - n''_{t+1}$. This shows that

$$Gtv(\sigma_t(m_t)) = \sigma_t(n'_t) = n_t.$$

b) Let us prove that $Fv$ is strict. Assume that $p_t \in FtP \cap Imv$. Let $l$ be the biggest integer such that $p_t = v(n_l)$ with $n_l \in FtN$. We need to show that $l \geq t$. Assume that $l < t$. One has

$$Gtv(\sigma_t(n_l)) = \sigma_t(v(n_l)) = \sigma_t(p_t) = 0.$$

Hence $\exists m_t \in FtM$ such that $Gtv(\sigma_t(m_t)) = \sigma_t(n_l)$. Thus we have

$$n_l - u(m_t) \in Ft+1M \text{ and } v(n_l - u(m_t)) = p_t$$

which contradicts the definition of $l$.

Let us prove that $Ker Fv = Im Fu$. Let $n_t \in Ker Ftv$. One has: $Gt(v)(\sigma_t(n_t)) = 0$. Hence there exists $m_t$ in $FtM$ such that

$$\sigma_t(n_t) = Gt(u)(\sigma_t(m_t)).$$

Hence $n_t - u(m_t) \in Ker Ftv \cap Ft+1N$. We can reproduce the previous reasoning to $n_t - u(m_t)$ and produce an element $m_{t+1}$ in $Ft+1M$ such that $n_t - u(m_t + m_{t+1}) \in KgFtv \cap KgFtvN$. Since $KgFtvN$ is closed at $KgFtvN$, there is $m_{t+1}$ in $Ft+1M$ such that $n_t - u(m_t + m_{t+1}) \in KgFtvN$. Hence $n_t = u(m_t + m_{t+1})$.
Corollary 3.0.3. Let $FA$ be a filtered $k$-algebra and let $FM$ and $FN$ two $FA$-modules. Let $Fu : FM \to FN$ be a morphism of $FA$-modules. Then $G \ker Fu \subset \ker GFu$ and $\operatorname{Im} GFu \subset \operatorname{Im} Fu$. Assume moreover that $FM$ is complete and $FN$ is Hausdorff, then the following conditions are equivalent:

(a) $Fu$ is strict

(b) $G \ker Fu = \ker GFu$

(c) $\operatorname{Im} GFu = \operatorname{Im} Fu$.

Proof:

One has:

\[
\begin{align*}
F_t \ker u &= \ker u \cap F_t M \\
F_t \operatorname{Im} u &= \operatorname{Im} u \cap F_t N \\
G_t \ker u &= F_t M \cap \ker u \\
G_t \operatorname{Im} u &= \frac{F_{t+1} M \cap \ker u}{F_t M \cap \ker u} \\
\ker G_t u &= \frac{F_{t+1} M \cap u^{-1}(F_{t+1} N)}{u(F_t M)} \\
\operatorname{Im} G_t u &= \frac{F_{t+1} N \cap u(F_t M)}{\operatorname{Im} u \cap F_t N} \\
G_t \operatorname{Im} u &= \frac{F_{t+1} N \cap u(F_t M)}{\operatorname{Im} u \cap F_{t+1} M}
\end{align*}
\]

The second part of the corollary follows from applying the previous proposition to the strict exact sequence $FM \to \operatorname{Im} u \to 0$.

Indeed $Fu$ is strict if and only the following sequence

\[
FM \xrightarrow{Fu} \operatorname{Im} u \to 0
\]

is a strict exact sequence of $FA$-modules when $\operatorname{Im} u$ is endowed with the induced topology. Then we apply 3.0.2.

Let us recall this well known result about complexes of filtered modules.

Proposition 3.0.4. Let $(M^\bullet, d^\bullet)$ be a complex of complete $FA$-modules. $H^i(M^\bullet)$ is filtered as follows $F_t H^i(M^\bullet) = \frac{\ker d_i \cap F_t M^i + \operatorname{Im} d_{i-1}}{\operatorname{Im} d_{i-1}} \simeq \frac{\ker d_i \cap F_t M^i}{\operatorname{Im} d_{i-1} \cap F_t M^{i-1}}$.

If $d_i$ and $d_{i-1}$ are strict, then $G H^i(M^\bullet)$ is isomorphic to $H^i(GM^\bullet)$.

Proof of the proposition 3.0.4:

We consider the following exact sequence

\[
0 \to \operatorname{Im} d_{i-1} \to \ker d_i \xrightarrow{p_i} H^i M^\bullet \to 0.
\]
we endow \( \text{Kerd}_i \) and \( \text{Imd}_{i-1} \) with the induced filtration. One has
\[
\begin{align*}
F_i \text{Kerd}_i &= \text{Kerd}_i \cap F_i M^i \\
F_i \text{Imd}_{i-1} &= \text{Imd}_{i-1} \cap F_i M^i \\
p(F_i \text{Kerd}^i) &= \frac{\text{Kerd}_i \cap F_i M^i + \text{Imd}_i}{\text{Imd}_i} = F_i H^1(M^i).
\end{align*}
\]
The exact sequence above is strict exact. It stays exact if one takes the graded modules. Thus, we have the following exact sequence of \( GA \)-modules.

\[
0 \to \text{GImd}_{i-1} \to \text{GKerd}_i \xrightarrow{p} \text{GH}^1(M^i) \to 0.
\]

Then \( \text{GH}^1(M^i) \equiv \frac{\text{GKerd}_i}{\text{GImd}_{i-1}} \equiv \frac{\text{KerGd}_i}{\text{ImGd}_{i-1}} \equiv H^1(\text{GM}^i) \). This finishes the proof of the proposition. □

**Remark :**

The isomorphism from \( G_t H^i(M^i) \) to \( H^i(G_t M^i) \) is given by
\[
\begin{align*}
G_t H^i(M^i) &\to H^i(G_t M^i) \\
\sigma_t \text{cl}(x) &\mapsto \text{cl}(\sigma_t(x)).
\end{align*}
\]

For any \( r \in \mathbb{Z} \) and for any \( FA \)-module \( FM \), we define the shifted module \( FM(r) \) as the module \( M \) endowed with the filtration \( (F_{t+r}M)_{t \in \mathbb{Z}} \).

An \( FA \)-module module is finite free if it is isomorphic to an \( FA \)-module of the type \( \oplus_{i=1}^{p} FA(−d_i) \) where \( d_1, \ldots, d_p \) are integers. An \( FA \)-module \( FM \) is of finite type if there exists a strict epimorphism \( FL \to FM \), where \( FL \) is a finite free \( FA \)-module. This means that we can find \( m_1 \in F_{d_1}M, \ldots, m_p \in F_{d_p}M \) such that any \( m \in F_d M \) may be written as
\[
m = \sum_{i=1}^{p} a_{d − d_i} m_i
\]

where \( a_{d − d_i} \in F_{d − d_i}A \).

**Proposition 3.0.5.** Let \( FA \) be a filtered \( k \)-algebra and \( FM \) be an \( FA \)-module.

a) If \( FM \) is an \( FA \)-module of finite type generated by \( (s_1, \ldots, s_r) \) then \( GM \) is a \( GA \)-module of finite type generated by \( ([s_1], \ldots, [s_r]) \). Conversely, assume that \( FA \) is complete for the topology given by the filtration and that \( FM \) is an \( FA \)-module which is Hausdorff for the topology defined by the filtration. If \( GM \) is a \( GA \)-module of finite type generated by \( ([s_1], \ldots, [s_r]) \), then \( FM \) is an \( FA \)-module of finite type generated by \( (s_1, \ldots, s_r) \).

b) If \( FM \) is a finite free \( FA \)-module, then \( GM \) is a finite free \( GA \)-module. Conversely, assume that \( FA \) is complete for the topology given by the filtration and \( FM \) is a \( FA \)-module Hausdorff for the topology defined by the filtration. If \( GM \) is a finite free \( GA \)-module, then \( FM \) is a finite free \( FA \)-module.

**Proof of the proposition :**

a) If \( FM \) is an \( FA \)-module of finite type, then there is a strict exact sequence \( \oplus_{i=1}^{N} FA(−d_i) \to FM \to 0 \). If we apply proposition 3.0.2, we see that \( GM \) is a \( GA \)-module of finite type. Conversely, assume that \( GM \) is a \( GA \)-module of finite
type generated by \( \sigma_1 = [s_1], \ldots, \sigma_r = [s_r] \). Assume that \( s_i \in F_{d_i}M - F_{d_i-1}M \). Let \( x \) in \( F_nM \). There exists \( a_{i,0} \in G_{n-d_i}A \) such that

\[
\sigma_n(x) = \sum_{i=1}^r a_{i,0} \sigma_i.
\]

Let \( \alpha_{i,0} \in F_{n-d_i}A \) such that \( \sigma_{n-d_i}(\alpha_{i,0}) = a_{i,0} \). We have

\[
x - \sum_{i=1}^r \alpha_{i,0} s_i \in F_{n+1}M.
\]

Reasoning in the same way, one can construct \( \alpha_{i,1} \in F_{n-d_i+1}A \) such that

\[
x - \sum_{i=1}^r (\alpha_{i,0} + \alpha_{i,1}) s_i \in F_{n+2}M.
\]

Going on that way, we construct an element \( \sum_{j=1}^\infty \alpha_{i,j} \) in \( F_{n-d_i}A \) such that

\[
x = \sum_{i=1}^r \left( \sum_{j=1}^\infty \alpha_{i,j} \right) s_i.
\]

Hence \( FM \) is a finite type \( FA \)-module.

b) apply proposition 3.0.2. □

**Definition 3.0.6.** A filtered \( k \)-algebra is said to be (filtered) noetherian if it satisfies one of the following equivalent conditions:

- Any filtered submodule (not necessarily a strict submodule) of a finite type \( FA \)-module is of finite type
- Any filtered ideal (not necessarily a strict ideal) of \( FA \) is of finite type.

**Proposition 3.0.7.** Let \( FA \) be a filtered complete \( k \)-algebra and denote by \( GA \) its associated graded algebra. If \( GA \) is graded noetherian, then \( FA \) is filtered noetherian.

**Proof of the proposition:**

We assume that \( GA \) is a noetherian algebra. We need to prove that a filtered submodule \( FM' \) of a finitely generated \( FA \)-module \( FM \) is finitely generated.

First we assume that \( FM \) is Hausdorff. For this case, we reproduce the proof of [Sch].

If \( FM' \) is strict, then the associated \( GA \)-module \( GM' \) is a submodule of the \( GA \)-module \( GM \) associated to \( FM \). Since \( GA \) is noetherian and \( GM \) is finitely generated so is \( GM' \) and the conclusion follows.

To prove the general case, we may assume that the image of the inclusion \( FM' \rightarrow FM \) is equal to \( FM \). In this case, using a finite systems of generators of \( FM \), it is easy to find an integer \( l \) such that

\[
F_lM' \subset F_lM \subset F_{l-1}M'.
\]

We will prove the result by induction on \( l \).
For \( l = 1 \), let us introduce the auxiliary \( GA \)-modules
\[
G K_0 = \oplus_{t \in \mathbb{Z}} F_t M' / F_{t+1} M \\
G K_1 = \oplus_{t \in \mathbb{Z}} F_t M / F_t M'
\]
These modules satisfy the exact sequences
\[
0 \rightarrow G K_0 \rightarrow G M \rightarrow G K_1 \rightarrow 0 \\
0 \rightarrow G K_1(1) \rightarrow G M' \rightarrow G K_0 \rightarrow 0
\]
Since \( G M \) is a finite type \( GA \)-module, so are \( G K_0 \) and \( G K_1 \). Hence \( G M' \) is also finitely generated and the conclusion follows.

For \( l > 1 \), we define the auxiliary \( FA \)-module \( F M'' \) by setting
\[
F_t M'' = F_{t+1} M + F_t M'
\]
Since we have
\[
F_t M'' \subset F_t M \subset F_{t-1} M'
\]
the preceding discussion shows that \( F M'' \) is finitely generated. Moreover
\[
F_t M' \subset F_t M'' \subset F_{t-(l-1)} M'
\]
and the conclusion follows from the induction hypothesis.

We no longer assume that \( F M \) is Hausdorff
As \( F M \) is a finite type \( FA \)-module, there exists a strict exact sequence
\[
F L = \bigoplus_{i=1}^n F A(-d_i) \xrightarrow{p} F M \rightarrow 0.
\]
We will denote by \( p_t \) the map from \( F_t L \) to \( F_t M \) induced by \( p \). As \( p \) is strict, the map \( p_t \) is surjective. Let \( F M' \) be a submodule (not necessarily strict) of \( F M \). Then \( p^{-1}(F M') \) is an \( FA \)-submodule of \( F L \) if we endow it with the filtration
\[
F_i [p^{-1}(M')] = p_t^{-1}(F_t M') = p^{-1}(F_t M') \cap F_t L.
\]
As \( F L \) is Hausdorff, we know from the first part of the proof that the \( FA \)-module \( p^{-1}M' \) is finite type. Hence there exist \( \alpha_1 \in F_{d_1} [p^{-1}M'], \ldots, \alpha_p \in F_{d_p} [p^{-1}M'] \) such that any \( x \) of \( F_d [p^{-1}M'] \) can be written
\[
x = \sum_{i=1}^p a_{d-d_i} \alpha_i \text{ with } a_{d-d_i} \in F_{d-d_i} A.
\]
Let \( y \) in \( F_d M' \). As \( p \) is strict, there exist \( x \in F_d [p^{-1}M'] \) such that \( y = p(x) \). Then \( y \) can be written
\[
y = \sum_{i=1}^p a_{d-d_i} p(\alpha_i) \text{ with } a_{d-d_i} \in F_{d-d_i} A.
\]
We have proved that \( F M' \) is a finite type \( FA \)-module \( \square \).

**Proposition 3.0.8.** Assume that \( FA \) is noetherian for the topology given by the filtration. Any \( FA \)-module of finite type has an infinite resolution by finite free \( FA \)-modules i.e there is an exact sequence
\[
\cdots \rightarrow F L_s \rightarrow F L_{s-1} \rightarrow \cdots \rightarrow F L_0 \rightarrow F M \rightarrow 0
\]
where each \( F L_s \) is a finite free \( FA \)-module.
Remark:
For such a resolution of $FM$, the sequence
\[ \cdots \to GL_s \to GL_{s-1} \to \cdots \to GL_0 \to GM \to 0 \]
is a resolution of the $GA$-module $GM$.

**Proposition 3.0.9.** Assume $FA$ is noetherian and complete. If $GA$ is of finite (left) global homological dimension, so is $A$.

**Proof:** We adjust the proof of [Schn] proposition 10.3.5. to decreasing filtrations. Let us start by a lemma.

**Lemma 3.0.10.** If $FN$ is a finite type $FA$-module, then it is complete.

First we assume that $FN$ is Hausdorff. Let $FN$ be a finite type Hausdorff $FA$-module. We have a strict exact sequence
\[ FL = \oplus_{i=1}^{n} FA(−d_i) \to FN \to 0. \]
The filtration on $FN$ is given by $p(F_iL)$. Let us endow the kernel $K$ of $p$ with the induced topology. We have a strict exact sequence
\[ 0 \to FK \to FL \to FN \to 0. \]
As $N$ is Hausdorff, $K = p^{-1}(\{0\})$ is closed in $FL$. The filtered $FA$-module $FN$ is isomorphic to $FL/K$, endowed with the quotient topology. Hence, $FN$ is complete (see lemma 3.0.1).

We no longer assume that $FN$ is Hausdorff. From the first case, $FK$, endowed with the induced topology is complete and hence closed in $FL$. As $FN \cong FL/K$, the $FA$-module $FN$ is Hausdorff.

**Lemma 3.0.11.** Assume that $FA$ is noetherian and complete. Then, for any $FA$-module of finite type $FM$ and any complete $FA$-module $FN$,
\[ \text{Ext}^j_{GA}(GM,GN) = 0 \implies \text{Ext}^j_A(M,N) = 0. \]

Let
\[ \cdots \to FL_n \to FL_{n-1} \to \cdots \to FL_0 \to FM \to 0 \]
be a filtered resolution of $FN$ by finite free $FA$-modules. Applying the graduation functor, we get a resolution
\[ \cdots \to GL_n \to GL_{n-1} \to \cdots \to GL_0 \to GM \to 0. \]
Assuming $\text{Ext}^j_{GA}(GM,GN) = \{0\}$ means that the sequence
\[ Hom_{GA}(GL_{j-1},GN) \to Hom_{GA}(GL_{j},GN) \to Hom_{GA}(GL_{j+1},GN) \]
is an exact sequence of $GA$-modules. When $FL = \bigoplus_{i=1}^{n} FA(−d_i)$ is finite free, the $FA$-module $FHom(FL,FN) = \bigoplus_{i=1}^{n} FN(d_i)$ is complete and the natural map
\[ GHom_{FA}(FL,FN) \to Hom_{GA}(GL,GN) \]
is an isomorphism. Hence the sequence
\[ FHom_{FA}(FL_{j-1},FN) \to FHom_{FA}(FL_{j},FN) \to FHom_{FA}(FL_{j+1},FN) \]
is a strict exact sequence of $FA$-modules (proposition 3.0.2). When $FL$ is finite free, the underlying module of $\text{FHom}_{FA}(FL, FN)$ is $\text{Hom}_A(L, N)$. This finishes the proof of the lemma.

Denote by $d_{GA}$ the (left) global homological dimension of $GA$. Let $M$ be a finite type $A$-module. One has an epimorphism $A^n \xrightarrow{p} M \rightarrow 0$. We endow $M$ with the filtration $p(A^n)$. Similarly, we endow $N$ with a filtration $FN$ such that $FN$ is a finite $FA$-module. From the two previous lemmas, we deduce that for any finite type $A$ modules $M$ and $N$,

$$\text{Ext}^j_A(M, N) = 0 \text{ if } j \geq d_{GA} + 1.$$

Let now $N$ be any $A$-module. We have $N = \lim_{\rightarrow} N'$ where $N'$ runs over all finitely generated submodules of $N$. Let $L^\bullet$ be a resolution of $M$ by finitely generated free $A$-modules. We have for all $j \geq d_{GA} + 1$

$$\text{Ext}^j_A(M, N) = \lim_{\rightarrow} \text{Ext}^j_A(M, N').$$

where, in the equality before the last equality, we used the fact that the functor $\lim_{\rightarrow}$ is exact because the set of finitely generated submodules of $M$ is a directed set ([Ro] proposition 5.33). Thus we have proved : if $M$ is a finitely generated $A$-module and $N$ is any $A$-module, then

$$\text{Ext}^j_A(M, N) = \{0\} \text{ if } j \geq d_{GA} + 1.$$

From this, we deduce ([Ro] theorem 8.16), that the global (left) dimension of $A$ is finite and inferior or equal to $d_{GA}$. □

Remark : The lemma 3.0.10 is proved in [K-S] in the case an $A_h$-module ($A_h$ being a deformation algebra) endowed with the $h$-adic filtration.

4. Deformation algebras

4.1. Definition and properties. In this section $k$ will be a field of characteristic 0 and we will set $K = k[[h]]$.

Definition 4.1.1. A topologically free $K$-algebra $A_h$ is a topologically free $K$-module together with a $K$-bilinear (multiplication) map $A_h \times A_h \rightarrow A_h$ making $A_h$ into an associative algebra.

Let $A_0$ be an associative $k$-algebra. A deformation of $A_0$ is topologically free $K$-algebra $A_h$ such that $A_0 \simeq A_h/hA_h$ as algebras.

Remark : If $A_h$ is a deformation algebra of $A_0$, we may endow it with the $h$-adic filtration. We then have $GA_h = \bigoplus_{i \in \mathbb{N}} h^i A_h \simeq A_0[h]$ as $k[h]$-algebra.
From proposition 3.0.6, we deduce that a deformation algebra of a noetherian algebra is noetherian.

**Examples ([C-P]):**

Before giving a list of examples, let us recall the following definition:

**Definition 4.1.2.** A deformation of a Hopf algebra \((A, \iota, \mu, \epsilon, \Delta, S)\) over a field \(k\) is a topological Hopf algebra \((A_h, \iota_h, \mu_h, \epsilon_h, \Delta_h, S_h)\) over the ring \(k[[h]]\) such that

i) \(A_h\) is isomorphic to \(A_0[[h]]\) as a \(k[[h]]\)-module

ii) \(A_h/hA_h\) is isomorphic to \(A_0\) as Hopf algebra.

**Example 1:** Quantized universal enveloping algebras (QUEA)

**Definition 4.1.3.** Let \(g\) be a Lie bialgebra. A Hopf algebra deformation of \(U(g)\), \(U_h(g)\), such that \(U_h(g)/hU_h(g)\) is isomorphic to \(U(g)\) as a coPoisson Hopf algebra is called a quantization of \(U(g)\).

Quantizations of Lie bialgebras have been constructed in \([E-K]\).

**Example 2:** Quantization of affine algebraic Poisson groups

**Definition 4.1.4.** A quantization of an affine algebraic Poisson group \((G, \{,\})\) is a Hopf algebra deformation \(F_h(G)\) of the Hopf algebra \(F(G)\) of regular functions on \(G\), such that \(F_h(G)/hF_h(G)\) is isomorphic to \((F(G), \{,\})\) as Poisson Hopf algebra.

Quantization of affine algebraic Poisson groups have been constructed by Etingof and Kazhdan ([E-S], see also [C-P] for the case where \(G\) is simple).

**Examples 3:** Quantum formal series Hopf algebras (QFSHA)

The vector space dual \(U(g)^*\) of the universal enveloping algebra \(U(g)\) of a Lie algebra can be identified with an algebra of formal power series and it has a natural Hopf algebra structure, provided we interpret the tensor product \(U(g)^* \otimes U(g)^*\) in a suitable completed sense. If \(g\) is a Lie bialgebra, \(U(g)^*\) is a Hopf Poisson algebra.

**Definition 4.1.5.** A quantum formal series Hopf algebra is a topological Hopf algebra \(B_h\) over \(k\) such that \(B_h/hB_h\) is isomorphic to \(U(g)^*\) as a topological Poisson Hopf algebra for some finite dimensional Lie bialgebra.

The following proposition is proved in [K-S] (theorem 2.6)

**Proposition 4.1.6.** Let \(A_h\) be a deformation algebra of \(A_0\) and let \(M\) be an \(A_h\)-module. Assume that

(i) \(M\) has no \(h\)-torsion

(ii) \(M/hM\) is a flat \(A_0\)-module

(iii) \(M = \lim_{n} M/h^nM\)

then \(M\) is a flat \(A_h\)-module.
5. A Quantization of the Character $\text{trad}$

**Theorem 5.0.7.** Let $A_0$ be a noetherian $k$-algebra and let $A_h$ be a deformation of $A_0$. Assume that $k$ has a left $A_0$-module structure such that there exists an integer $d$ such that

\[
\begin{align*}
\text{Ext}^i_{A_0}(k, A_0) &= \{0\} \text{ if } i \neq d \\
\text{Ext}^d_{A_0}(k, A_0) &\simeq k
\end{align*}
\]

Assume that $K$ is endowed with a $A_h$-module structure which reduces modulo $h$ to the $A_0$-module structure on $k$ we started with. Then

a) $\text{Ext}^i_{A_h}(K, A_h)$ is zero if $i \neq d$.

b) $\text{Ext}^d_{A_h}(K, A_h)$ is a free $K$-module of dimension 1. By right multiplication, it is a right $A_h$-module. It is a lift of the right $A_0$-module structure (given by right multiplication) on $\text{Ext}^d_{A_0}(k, A_0)$.

**Notation:** The right $A_h$-module $\text{Ext}^d_{A_h}(k, A_h)$ will be denoted $\Omega_{A_h}$ and the character defined by this action $\theta_{A_h}$.

**Remark:** In [K-S] (paragraph 6), Kashiwara and Schapira make a similar construction in the set up of $DQ$-algebroids. In [C2], it is shown that a result similar to theorem 5.0.7 holds for $U_q(\mathfrak{g})$ ($\mathfrak{g}$ semi-simple).

**Example 1:** Quantized universal enveloping algebras

Poincaré duality gives us the following result for any finite dimensional Lie algebra.

\[
\begin{align*}
\text{Ext}^i_{U(\mathfrak{g})}(k, U(\mathfrak{g})) &= \{0\} \text{ if } i \neq 0 \\
\text{Ext}^d_{U(\mathfrak{g})}(k, U(\mathfrak{g})) &\simeq \Lambda^{\text{dim}\mathfrak{g}}(\mathfrak{g}^*)
\end{align*}
\]

The character defined by the right action of $U(\mathfrak{g})$ on $\text{Ext}^d_{U(\mathfrak{g})}(k, U(\mathfrak{g}))$ is $\text{trad}_\mathfrak{g}$ ([C1]). Thus, the character defined by the theorem 5.0.7 is a quantization of the character $\text{trad}_\mathfrak{g}$.

- If $\mathfrak{g}$ is a complex semi-simple algebra, as $H^1(\mathfrak{g}, k) = \{0\}$ ([H-S] p 247), there exists a unique lift of the trivial representation of $U_h(\mathfrak{g})$, hence the representation $\Omega_{U_h}(\mathfrak{g})$ is the trivial representation.

- Let $\mathfrak{a}$ be a $k$-Lie algebra. Denote by $a_h$ the Lie algebra obtained from $\mathfrak{a}$ by multiplying the bracket of $\mathfrak{a}$ by $h$. Thus, for any elements $X$ and $Y$ of $a_h$ $\simeq \mathfrak{a}$,

\[
[X, Y]_{a_h} = h[X, Y]_\mathfrak{a}.
\]

Denote by $\widehat{U(a_h)}$ the $h$-adic completion of $U(a_h)$. Then $\widehat{U(a_h)}$ is a Hopf deformation of $(\mathfrak{a}^h, \delta = 0)$. The character $\theta_{\mathfrak{a}, X}$ defined by the theorem in this case is given by

\[
\forall X \in \mathfrak{a}, \quad \theta_{\mathfrak{a}, X}(X) = \text{htrad}_\mathfrak{a}(X).
\]

Thus, even if $\mathfrak{g}$ is unimodular, the character defined by the right action of $U_h(\mathfrak{g})$ on $\Omega_{U_h}(\mathfrak{g}) \simeq \Lambda^{\text{dim}\mathfrak{g}}(\mathfrak{g}^*)[[h]]$ might not be trivial.

- We consider the following Lie algebra: $\mathfrak{a} = \bigoplus_{i=1}^5 ke_i$ with non zero bracket $[e_2, e_4] = e_1$. Consider $k[[h]]$-Lie algebra structure on $\mathfrak{a}[[h]]$ defined by the following non zero brackets

\[
[e_3, e_5] = he_3 \\
[e_2, e_4] = 2e_1
\]
GA
[0x0]U
[127x715]16
[127x639]If
[127x639]G
[127x163]A
[127x163]GA
[127x286]Hom
[127x286]Recall that there is a natural filtration on $F$ is a resolution of the bras.
[127x572]where
[127x338]so that the graded complex $\Ext^{\bullet}_{\mathcal{F}(G)}(k, \mathcal{F}(G))$ acts trivially on $\Ext^{\bullet}_{\mathcal{F}^\bullet(k^\bullet)}$ whereas the action of $\mathcal{F}_h(k^\bullet)$ is not trivial.

**Example 2**

The theorem 5.0.7 also applies to quantization of affine algebraic Poisson groups. If $G$ is an affine algebraic Poisson group with neutral element $e$, we take $k$ to be given by the counit of the Hopf algebra $\mathcal{F}(G)$. One has [A-K]

$$\Ext^{\bullet}_{\mathcal{F}(G)}(k, \mathcal{F}(G)) = \{0\} \text{ if } i \neq \dim G$$

$$\Ext^{\dim G}_{\mathcal{F}(G)}(k, \mathcal{F}(G)) \simeq \wedge^{\dim G} \left( \mathcal{M}_e / \mathcal{M}_e^2 \right)^{\ast}$$

where

$$\mathcal{M}_e = \{ f \in \mathcal{F}(G) \mid f(e) = 0 \}.$$  

Let $\mathfrak{g}$ be a real Lie algebra. The algebra of regular functions on $\mathfrak{g}^\bullet$, $\mathcal{F}(\mathfrak{g}^\bullet)$, is isomorphic to $S(\mathfrak{g})$ and is naturally equipped with a Poisson structure given by:

$$\forall X, Y \in \mathfrak{g}, [X, Y] = [X, Y].$$

In the example above, $\hat{U}(\mathfrak{g}_h)$ is a quantization of the Poisson algebra $\mathcal{F}(\mathfrak{g}^\bullet)$. The algebra of regular functions $\hat{U}(\mathfrak{g}_h)$ is naturally equipped with a Poisson structure given by:

$$\forall X, Y \in \mathfrak{g}, [X, Y] = [X, Y].$$

**Example 3**: The theorem 5.0.7 also applies to quantum formal series Hopf algebras.

**Proof of the theorem 5.0.7:**

Let us consider a resolution of the $A_h$-module $K$ by filtered finite free $A_h$-modules

$$\cdots \rightarrow FL^{i+1} \xrightarrow{\partial_{i+1}} FL^i \xrightarrow{\partial_i} \cdots \xrightarrow{\partial_2} FL^1 \xrightarrow{\partial_1} FL^0 \rightarrow K \rightarrow \{0\}$$

so that the graded complex

$$\cdots \rightarrow GL^{i+1} \xrightarrow{G\partial_{i+1}} GL^i \xrightarrow{G\partial_i} \cdots \xrightarrow{G\partial_2} GL^1 \xrightarrow{G\partial_1} GL^0 \rightarrow k[h] \rightarrow \{0\}$$

is a resolution of the $A_0[h]$-module $k[h]$. Consider the complex $M^\bullet = (\Hom_{A_h}(L^\bullet, A_h), \partial^\bullet)$. Recall that there is a natural filtration on $\Hom_{A_h}(L^\bullet, A_h)$ defined by

$$F_i \Hom_{A_h}(L^\bullet, A_h) = \{ \lambda \in \Hom_{A_h}(L^\bullet, A_h) \mid \lambda (F_p L^\bullet) \subset F_{i+p} A_h \}.$$  

One has an isomorphism of right $FA$-modules

$$\Hom_{A_h}(L^\bullet, A_h) \simeq \bigoplus_{j=1}^{r} FA(m_{j,i})$$

Hence

$$\GFHom_{A_h}(L^\bullet, A_h) \simeq \Hom_{GA_h}(GL^\bullet, GA_h)$$

and the complex $\Hom_{GA_h}(GL^\bullet, GA_h)$ computes $\Ext^{\bullet}_{GA_h}(k[h], GA_h)$. We have the following isomorphisms of right $A_0[h]$-modules.

$$\Ext^{\bullet}_{GA_h}(k[h], GA_h) \simeq \Ext^{\bullet}_{A_0[h]}(k[h], A_0[h]) \simeq \Ext^{\bullet}_{A_h}(k[h], A_0[h]).$$
Duality properties for quantum groups

If $i \neq d$, then $\Ext^i_{\mathcal{A}_h}(k[h], GA_h) = \{0\}$. This means that the sequence

$$\Hom_{G\mathcal{A}}(GL_{i-1}, GA_h) \overset{\partial_h}{\longrightarrow} \Hom_{G\mathcal{A}}(GL_i, GA_h) \overset{\partial_{i+1}}{\longrightarrow} \Hom_{G\mathcal{A}}(GL_{i+1}, GA_h)$$

is an exact sequence of $G\mathcal{A}_h$-modules. Hence, applying 3.0.2 the sequence

$$\text{FHom}_{\mathcal{A}_h}(FL_{i-1}, FN) \overset{\partial_h}{\longrightarrow} \text{FHom}_{\mathcal{A}_h}(FL_i, FN) \overset{\partial_{i+1}}{\longrightarrow} \text{FHom}_{\mathcal{A}_h}(FL_{i+1}, FN)$$

is strict exact. As $FL_i$ is finite free, the underlying module of $\text{FHom}_{\mathcal{A}_h}(FL_i, FN)$ is $\text{Hom}_{\mathcal{A}_h}(L_i, N)$, hence we have proved that $\Ext^i_{\mathcal{A}_h}(K, A_h) = \{0\}$ if $i \neq d$.

We have also proved that all the maps $\partial_h$ are strict. Hence, by proposition 3.0.4, we have for all integer $i$

$$G\Ext^i_{\mathcal{A}_h}(k[[h]], A_h) \simeq \Ext^i_{\mathcal{A}_h}(k[h], A_0[h]) \simeq \Ext^i_{\mathcal{A}_h}(k, A_0)[h]$$

The $\mathcal{A}_h$-modules $\Ext^i_{\mathcal{A}_h}(K, A_h)$ are finite type $\mathcal{A}_h$-modules, hence they are Hausdorff and even complete (see lemma 3.0.10).

As $\Ext^d_{\mathcal{A}_h}(K, A_h)$ is Hausdorff and $G\Ext^d_{\mathcal{A}_h}(k[[h]], A_h) \simeq \Ext^d_{\mathcal{A}_h}(k, A_0)[h]$, the $k[[h]]$-module $\Ext^d_{\mathcal{A}_h}(K, A_h)$ is a one dimensional.

This finishes the proof of the theorem 5.0.7. □

*From now on, we assume that $A_h$ is a topological Hopf algebra and that its action on $K$ is given by the counit. The antipode of $A_h$ will be denoted $S_h$. *

If $V$ is a left $\mathcal{A}_h$-module, we set $V^\vee$ (respectively $V^\rho$) the right $\mathcal{A}_h$-module defined by

$$\forall a \in A_h, \forall v \in V, v \cdot S_h a = S_h(a) \cdot v \quad (\text{respectively} \quad v \cdot S_h^{-1} a = S_h^{-1}(a) \cdot v).$$

Similarly, if $W$ is a right $\mathcal{A}_h$-module, we set $W^\vee$ (respectively $W^\lambda$) the left $\mathcal{A}_h$-module defined by

$$\forall a \in A_h, \forall w \in W, a \cdot S_h w = w \cdot S_h(a) \quad (\text{respectively} \quad a \cdot S_h^{-1} w = w \cdot S_h^{-1}(a)).$$

One has $(V^\vee)^\lambda = V$, $(V^\rho)^\lambda = V$, $(W^\vee)^\rho = W$ and $(W^\lambda)^\vee = W$. Thus, we have defined two (in the case where $S_h^2 \neq id$) equivalences of categories between the category of left $\mathcal{A}_h$-modules and the category of right $\mathcal{A}_h$-modules, that is to say left $A_h^\text{op}$-modules.

Let $\text{Mod}(A_h)$ be the abelian category of left $A_h$-modules and $D(\text{Mod}(A_h))$ be the derived category of the abelian category $\text{Mod}(A_h)$. We may consider $A_h$ as an $A_h \otimes A_h^\text{op}$-module. Introduce the functor $D_{A_h}$ from $D(\text{Mod}(A_h))$ to $D(\text{Mod}(A_h^\text{op}))$

$$\forall M^\bullet \in D(A_h), \quad D_{A_h}(M^\bullet) = R\text{Hom}_{A_h}(M^\bullet, A_h).$$

If $M$ is a finitely generated module, the canonical arrow $M \to D_{A_h^\text{op}} \circ D_{A_h}(M)$ is an isomorphism.

Let $V$ be a left $\mathcal{A}_h$-module, then, by transposition, $V^* = \text{Hom}_K(V, K)$ is naturally endowed with a right $\mathcal{A}_h$-module structure. Using the antipode, we can also see it as a left module structure. Thus, one has :

$$\forall u \in A_h, \forall f \in V^*, \quad u \cdot f = f \cdot S_h(u).$$
We endow $\Omega_{A_h} \otimes V^*$ with the following right $A_h$-module structure:
\[
\forall u \in A_h, \forall f \in V^*, \forall \omega \in \Omega_{A_h},
(\omega \otimes w) \cdot u = \lim_{n \to +\infty} \sum_j \theta_{A_h}(u'_{j,n}) \omega \otimes f \cdot S_h^2(u''_{j,n})
\]
where $\Delta(u) = \lim_{n \to +\infty} \sum_j u'_{j,n} \otimes u''_{j,n}$.

**Theorem 5.0.8.** Let $V$ be an $A_h$-module free of finite type as a $k[[h]]$-module. Then $D_{A_h}(V)$ and $V^* \otimes \Omega_{A_h}$ are isomorphic in $D(A_h^{op})$.

**Proof of the theorem:**
In the proof of this theorem, we will make use of the following lemma (see [Du1], [C1]).

**Lemma 5.0.9.** Let $W$ be a left $A_h$-module. $A_h \widehat{\otimes} W$ is endowed with two different structures of $A_h \otimes A_h^{op}$-modules. The first one denoted $(A_h \widehat{\otimes} W)_1$ is described as follows: Let $w$ be an element of $W$ and let $u, a$ be two elements of $A_h$. We set $\Delta(a) = \lim_{n \to +\infty} \sum_i a'_{i,n} \otimes a''_{i,n}$. Then
\[
(u \otimes w) \cdot a = ua \otimes w
\]
\[
a \cdot (u \otimes w) = \lim_{n \to +\infty} \sum_i a'_{i,n} u \otimes a''_{i,n} \cdot w
\]
The second one denoted $(A_h \widehat{\otimes} W)_2$ is described as follows: Then
\[
a \cdot (u \otimes w) = au \otimes w
\]
\[
(u \otimes w) \cdot a = \lim_{n \to +\infty} \sum_i u a'_{i,n} \otimes S_h(a''_{i,n}) \cdot w
\]
The $A_h \otimes A_h^{op}$-modules $(A_h \widehat{\otimes} W)_1$ and $(A_h \widehat{\otimes} W)_2$ are isomorphic.

**Proof of the lemma:**
The map
\[
\Psi : (A_h \widehat{\otimes} W)_2 \to (A_h \widehat{\otimes} W)_1
\]
\[
u \otimes w \mapsto \lim_{n \to +\infty} \sum_i u'_{i,n} \otimes u''_{i,n} \cdot w
\]
where $\Delta(u) = \lim_{n \to +\infty} \sum_i u'_{i,n} \otimes u''_{i,n}$ is an isomorphism of $A_h \otimes A_h^{op}$-modules from $(A_h \widehat{\otimes} W)_2$ to $(A_h \widehat{\otimes} W)_1$. Moreover
\[
\Psi^{-1}(u \otimes w) = \sum u'_{i,n} \otimes S_h(a''_{i,n}) \cdot w.
\]
This finishes the proof of the lemma.

Let $L^\bullet$ be a resolution of $K$ by free $A_h$-modules. We endow $L^i \otimes V$ with the following left $A_h$-module structure:
\[
a \cdot (l \otimes v) = \lim_{n \to +\infty} \sum_i a'_{i,n} \cdot l \otimes a''_{i,n} \cdot v
\]
Then $L^\bullet \otimes V$ is a resolution of $V$ by free $A_h$-modules. Using the relation
\[
a \cdot l \otimes v = \lim_{n \to +\infty} \sum_i a'_{i,n} \left[l \otimes S_h(a''_{i,n}) \cdot v \right]
\]
one shows the following sequence of $A_h$-isomorphisms

$$D_{A_h}(V) \cong \text{Hom}_{A_h}(L \otimes V, A_h) \cong \text{Hom}_{A_h}(L, (A_h \otimes V^*)_1) \cong \text{Hom}_{A_h}(L, (A_h \otimes V^*)_2) \cong \text{RHom}_{A_h}(K, A_h) \otimes V^*.$$

6. Link with quantum duality

6.1. Recollection on the quantum dual principle. The quantum dual principle ([Dr], see [G] for a detailed treatment) states that there exist two functors, namely $(^{\vee}) : QESA → QFSA$ and $(^\vee) : QFSA → QUEA$ which are inverse of each other. If $U_h(g)$ is a quantization of $U(g)$ and $F_h[[g]]$ is a quantization of $F[[g]] = U(g)^*$, then $U_h(g^\vee)$ is a quantization of $F[[g^\vee]]$ and $F_h[[g]]^\vee$ is a quantization of $U(g^\vee)$.

Let’s recall the construction of the functor $(^\vee) : QFSA → QUEA$ which is the one we will need. Let $g$ be a Lie bialgebra and $F_h[[g]]$ a quantization of $F[[g]] = U(g)^*$. For simplicity we will write $F_h$ instead of $F_h[[g]]$. If $\epsilon_h$ denotes the counit of $F_h$, set $I := \epsilon^{-1}_h(hk[h])$ and $J = \text{Ker} \epsilon_h$. Let

$$F_h^\vee := \sum_{n \geq 0} h^{-n} I^n = \sum_{n \geq 0} (h^{-1})^n = \bigcup_{n \geq 0} (h^{-1})^n$$

be the $k[[h]]$-subalgebra of $k((h)) \otimes F_h$ generated by $h^{-1}I$. As $I = J + hF_h$, one has $F_h^\vee = \sum_{n \geq 0} h^{-n} J^n$. Define $F_h^\vee$ to be the $h$-adic completion of the $k[[h]]$-module $F_h^\vee$. The coproduct (respectively counit, antipode) on $F_h$ provides a coproduct (respectively counit, antipode) on $F_h^\vee$ and $F_h^\vee$ is endowed with a Hopf algebra structure. A precise description of $F_h^\vee$ is given in [G]. Let us recall it as we will need it for our computations. The algebras $F_h/hF_h$ and $k[[x_1, \ldots, x_n]]$ are isomorphic. We denote $\pi : F_h → F_h/hF_h$ be the natural projection. We may choose $x_j \in \pi^{-1}(x_j)$ for any $j$ such that $\epsilon_h(x_j) = 0$, then $F_h$ and $k[[x_1, \ldots, x_n]]$ are isomorphic as $k[[h]]$-topological module and $J$ is the set of formal series $f$ whose degree in the $x_j$, $\partial x_i(f)$ (that is the degree of the lowest degree monomials occurring in the series with non zero coefficients) is strictly positive. As $F_h/hF_h$ is commutative, one has

$$x_i x_j - x_j x_i = h \chi_{i,j}$$

with $\chi_{i,j} \in F_h$. As $\chi_{i,j}$ is in $J$, it can be written as follows :

$$\chi_{i,j} = \sum_{a=1}^{n} c_a(h) x_a + f_{i,j}(x_1, \ldots, x_n, h)$$

with $\partial x_i(f_{i,j}) > 1$. If $\tilde{x}_i = h^{-1} x_j$, then

$$F_h^\vee = \{ f = \sum_{r \in \mathbb{N}} P^r(\tilde{x}_1, \ldots, \tilde{x}_n) h^r \mid P^r(X_1, \ldots, X_n) \in k[X_1, \ldots, X_n] \}.$$

Thus $F_h^\vee$ and $k[\tilde{x}_1, \ldots, \tilde{x}_n][[h]]$ are isomorphic as a topological $k[[h]]$-modules. One has

$$\tilde{x}_i \tilde{x}_j - \tilde{x}_j \tilde{x}_i = \sum_{a=1}^{n} c_a(h) \tilde{x}_a + h^{-1} f_{i,j}(\tilde{x}_1, \ldots, \tilde{x}_n, h)$$
where \( \bar{f}_{i,j}(\bar{x}_1, \ldots, \bar{x}_n, h) \) is obtained from \( f_{i,j}(x_1, \ldots, x_n) \) by writing \( x_j = h \bar{x}_j \). The element \( h^{-1} \bar{f}_{i,j}(\bar{x}_1, \ldots, \bar{x}_n, h) \) is in \( h\bar{k}[\bar{x}_1, \ldots, \bar{x}_n][[h]] \). The \( k \)-span of the set of cosets \( \{ e_i = \bar{x}_i \mod hF_h^\vee \} \) is a Lie algebra isomorphic to \( g^\ast \). The map \( \Psi : F_h^\vee \to U(g^\ast)[[h]] \) defined by

\[
\Psi \left( \sum_{r \in \mathbb{N}} P_r(\bar{x}_1, \ldots, \bar{x}_n)h^r \right) = \sum_{r \in \mathbb{N}} P_r(e_1, \ldots, e_n)h^r
\]

is an isomorphism of topological \( k[[h]] \)-modules. The algebra \( \frac{F_h^\vee}{hF_h^\vee} \) is isomorphic to \( U(g^\ast) \) and \( F_h(g)^\vee \) is a quantization of the coPoisson Hopf algebra \( U(g^\ast) \). Denote by \( \cdot_h \) multiplication on \( F_h \) and its transposition to \( U(g^\ast)[[h]] \) by \( \Psi \). To compute \( e_1^{b_1} \cdots e_n^{b_n} \cdot_h e_1^{b_1} \cdots e_n^{b_n} \) we proceed as follows: we compute \( \bar{x}_1^{a_1} \cdots \bar{x}_n^{a_n} \cdot_h \bar{x}_1^{b_1} \cdots \bar{x}_n^{b_n} \) in \( F_h^\vee \) and write it under the form \( \sum_{r \in \mathbb{N}} P_r(\bar{x}_1, \ldots, \bar{x}_n)h^r \). Then

\[
e_1^{a_1} \cdots e_n^{a_n} \cdot_h e_1^{b_1} \cdots e_n^{b_n} = \sum_{r \in \mathbb{N}} P_r(e_1, \ldots, e_n)h^r.
\]

If \( u \) and \( v \) are in \( U(g^\ast)[[h]] \), one writes \( u \ast v = \sum_{r \in \mathbb{N}} h^r \mu_r(u, v) \). One knows that the first non zero \( \mu_r \) is a 1-cocycle of the Hochschild cohomology.

If \( P \) in \( k[X_1, \ldots, X_n] \) can be written \( P = \sum_{i_1, \ldots, i_n} a_{i_1, \ldots, i_n} X_1^{i_1} \cdots X_n^{i_n} \), one sets

\[
P^{\otimes}(e_1, \ldots, e_n) = \sum_{i_1, \ldots, i_n} a_{i_1, \ldots, i_n} e_1^{\otimes i_1} \cdots e_n^{\otimes i_n}
\]

and if \( g \in k[X_1, \ldots, X_n][[h]] \) can be written \( g = \sum_{i=1}^r P_r(X_1, \ldots, X_r)h^r \), then one sets:

\[
g^{\otimes}(e_1, \ldots, e_n) = \sum_{i=1}^r P_r^{\otimes}(e_1, \ldots, e_r)h^r.
\]

**Fact:**

\((F_h)^{\vee}\) is isomorphic as an algebra to

\[
U_h(g^\ast) \simeq \frac{T_{k[[h]]} \left( \bigoplus_{i=1}^n k[[h]] e_i \right)}{I}
\]

where \( I \) is the closure (in the \( h \)-adic topology) of the two sided ideal generated by the relations

\[
e_i \otimes e_j - e_j \otimes e_i = \sum_{k=1}^n c_k(h) e_k + h^{-1} f_{i,j}^{\otimes}(e_1, \ldots, e_n, h).
\]

Let us prove this fact. Let \( \Omega : T_{k[[h]]} \left( \bigoplus_{i=1}^n k[[h]] e_i \right) \to F_h^\vee \) that sends \( e_i \) to \( \bar{x}_i \). One has \( I \subset \text{Ker} \Omega \) and we need to prove that \( \text{Ker} \Omega \subset I \). Let \( \mathcal{R} \) be in \( T_{k[[h]]} \left( \bigoplus_{i=1}^n k[[h]] e_i \right) \) be such that \( \Omega(\mathcal{R}) = 0 \). Then, modulo \( h \), we get \( \bar{\Omega}(\mathcal{R}) = 0 \).
Hence there exist \((u^0_{i,j})\) and \((v^0_{i,j})\) in \(T_k(\bigoplus_{i=1}^n \mathfrak{e}_i)\) such that
\[
\mathcal{R} = \sum_{i,j} u^0_{i,j} \otimes (e_i \otimes e_j - e_j \otimes e_i - [e_i, e_j]) \otimes v^0_{i,j}
\]
and \(\mathcal{R} - \sum_{i,j} u^0_{i,j} \otimes \left(e_i \otimes e_j - e_j \otimes e_i - \sum_{a=1}^n c_a(h) e_a - h^{-1} f^0_{i,j}(e_1, \ldots, e_n, h)\right) \otimes v^0_{i,j} \in h \ker \Omega.\) Hence there exist \(\mathcal{R}_1 \in \ker \Omega\) be such that
\[
\mathcal{R} - \sum_{i,j} u^0_{i,j} \otimes \left(e_i \otimes e_j - e_j \otimes e_i - \sum_{a=1}^n c_a(h) e_a - h^{-1} f^0_{i,j}(e_1, \ldots, e_n, h)\right) \otimes v^0_{i,j} = h \mathcal{R}_1.
\]
Reproducing the same reasoning, we find \((u^1_{i,j})\) and \((v^1_{i,j})\) in \(T_k(\bigoplus_{i=1}^n \mathfrak{e}_i)\) such that
\[
\mathcal{R}_1 - \sum_{i,j} u^1_{i,j} \otimes \left(e_i \otimes e_j - e_j \otimes e_i - \sum_{a=1}^n c_a(h) e_a - h^{-1} f^1_{i,j}(e_1, \ldots, e_n, h)\right) \otimes v^1_{i,j} = h \mathcal{R}_2
\]
and going on like this, we show that \(\mathcal{R}\) is in \(I.\)

### 6.2. Deformation of the Koszul complex

Let \(\mathfrak{a}\) be a \(k\)-Lie algebra. There is a well-known resolution of the trivial \(U(\mathfrak{a})\)-module, namely the Koszul resolution \(K = (U(\mathfrak{a}) \otimes \wedge \mathfrak{a}, \partial)\) where
\[
\partial \left( u \otimes X_1 \wedge \cdots \wedge X_n \right) = \sum_{i=1}^n (-1)^{i-1} u X_i \otimes X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_n
\]
\[
\sum_{i<j} (-1)^{i+j} u \otimes [X_i, X_j] \wedge X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge \hat{X}_j \wedge \cdots \wedge X_n.
\]

We will now show that the Koszul resolution can be deformed.

**Theorem 6.2.1.** Let \(\mathfrak{a}\) be a Lie algebra and let \((e_1, \ldots, e_n)\) be a basis of \(\mathfrak{a}.\) Denote by \(C^a_{i,j}\) the structure constants of \(\mathfrak{a}\) with respect to the basis \((e_1, \ldots, e_n)\) so that we have \([e_i, e_j] = \sum_{k=1}^n C^a_{i,j} e_a.\) Consider \(U_k(\mathfrak{a})\) a deformation of \(U(\mathfrak{a})\) given under the form
\[
U_k(\mathfrak{a}) \simeq \frac{T_k[[h]] \left( \bigoplus_{i=1}^n k[[h]] e_i \right)}{I}
\]
where \(I\) is the closure (in the \(h\)-adic topology) of the two-sided ideal generated by the relations
\[
e_i \otimes e_j - e_j \otimes e_i - g^0_{i,j}(e_1, \ldots, e_n, h)
\]
where \(g_{i,j}\) satisfies the following:
\[
g_{i,j} \in k[X_1, \ldots, X_n][[h]] \text{ and } \partial X(g_{i,j}) \geq 1
\]
\[
g^0_{i,j}(e_1, \ldots, e_n) = \sum_{a=1}^n C^a_{i,j} e_a \mod h T_k[[h]] \left( \bigoplus_{i=1}^n k[[h]] e_i \right).
\]
$k[[h]]$ is an $U_h(\mathfrak{a})$-module (called the trivial $U_h(\mathfrak{a})$-module) if we let the $e_i$’s act trivially. There exists a resolution of the trivial $U_h(\mathfrak{a})$-module $k[[h]]$, $K_h = (U_h(\mathfrak{a}) \otimes_k \wedge \mathfrak{a}, \partial_d^*),$ such that $\text{Gr}K_h$ is the resolution of the trivial $U(\mathfrak{a})[h]$-module $k[h].$

**Remarks:**

1) Any quantized universal enveloping algebra, $U_h(\mathfrak{a})$, has a presentation as in the theorem because we might write it $U_h(\mathfrak{a}) = (U_h(\mathfrak{a})')^\wedge.$

2) The proof of the theorem gives an algorithm to construct the resolution $K_h.$

3) By theorem 6.2.1, we even get a filtered resolution of the $FU_h(\mathfrak{g})$-module $k[[h]].$

**Proof of the theorem 6.2.1:**

We will prove by induction that on $q$ that one can construct $\partial_0^h, \ldots, \partial_q^h$ morphisms of $U_h(\mathfrak{a})$-modules such that:

- $\forall r \in [1, q], \partial_r^h \partial_{r-1}^h = 0.$

- $\partial_1^h (1 \otimes e_{p_1} \wedge \cdots \wedge e_{p_r}) = \sum_{i=1}^r (-1)^{i-1} e_{p_i} \otimes e_{p_1} \wedge \cdots \wedge \widehat{e_{p_i}} \wedge \cdots \wedge e_{p_r}$

$+ \sum_{k<l} (-1)^{k+l} C_{p_k,p_l}^a \otimes e_{p_1} \wedge \cdots \wedge \widehat{e_{p_k}} \wedge \cdots \wedge \widehat{e_{p_l}} \wedge \cdots \wedge e_{p_r} + \sum_{p_1, \ldots, p_r} \in hU_h(\mathfrak{a}) \otimes \wedge^{r-1}(\mathfrak{a})$ so that $G\partial_r^h$ is the $q$th differential of the Koszul complex of the trivial $U(\mathfrak{a})[h]$-module $k[h]$. From proposition 3.0.2, this implies that $\ker\partial_{q-1}^h = \text{Im}\partial_q^h.$

We take $\partial_0^h : U_h(\mathfrak{a}) \to k[[h]]$ to be the algebra morphism determined by $\partial_0^h (e_i) = 0.$

We take $\partial_1^h : U_h(\mathfrak{a}) \otimes_k \mathfrak{a} \to U_h(\mathfrak{a})$ to be the morphism of $U_h(\mathfrak{a})$-modules determined by $\partial_1^h (u \otimes e_i) = u e_i.$

One writes

$$g_{i,j}(e_1, \ldots, e_n, h) = \sum_{a=1}^n P_{i,j}^a e_a + \sum_{a=1}^n C_{i,j}^a e_a$$

where the $P_{i,j}^a$’s are in $hU_h(\mathfrak{a}).$

We look for a morphism of $U_h(\mathfrak{a})$-modules, $\partial_2^h : U_h(\mathfrak{a}) \otimes_k \wedge^2 \mathfrak{a} \to U_h(\mathfrak{a}) \otimes_k \mathfrak{a}$ under the form

$$\partial_2^h (1 \otimes e_i \wedge e_j) = e_i \otimes e_j - e_j \otimes e_i - \sum_a C_{i,j}^a 1 \otimes e_a - \alpha_{i,j}$$

where $\alpha_{i,j}$ is in $hU_h(\mathfrak{a}) \otimes \mathfrak{a}.$

One has

$$\partial_2^h \left( e_i \otimes e_j - e_j \otimes e_i - \sum_a C_{i,j}^a 1 \otimes e_a \right) = \sum_{a=1}^n P_{i,j}^a e_a = \partial_1^h \left( \sum_{a=1}^n P_{i,j}^a \otimes e_a \right).$$

We might take

$$\partial_2^h (1 \otimes e_i \wedge e_j) = e_i \otimes e_j - e_j \otimes e_i - \sum_{a=1}^n C_{i,j}^a 1 \otimes e_a - \sum_{a=1}^n P_{i,j}^a \otimes e_a.$$

We have $\partial_0^h \circ \partial_2^h = \partial_1^h$.

Let $q \geq 2$. Assume that $\partial_0^h, \partial_1^h, \ldots, \partial_q^h$ are constructed and let us construct $\partial_{q+1}^h$ as required.
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We look for \( \partial_{q+1}^h(1 \otimes e_{p_1} \wedge \cdots \wedge e_{p_{q+1}}) \) under the form

\[
\partial_{q+1}^h(1 \otimes e_{p_1} \wedge \cdots \wedge e_{p_{q+1}}) = \sum_{i=1}^{q+1} (-1)^{i-1} e_{p_i} \otimes e_{p_1} \wedge \cdots \wedge \widehat{e_{p_i}} \wedge \cdots \wedge e_{p_{q+1}} + \sum_{r<s} (-1)^{r+s} c_{p_r,p_s}^a 1 \otimes e_a \wedge \cdots \wedge \widehat{e_{p_r}} \wedge \cdots \wedge \widehat{e_{p_s}} \wedge \cdots \wedge e_{p_{q+1}} + \alpha_{p_1,\ldots,p_{q+1}}
\]

where \( \alpha_{p_1,\ldots,p_{q+1}} \) is in \( hU_h(a) \otimes \wedge^q(a) \).

The term

\[
\partial_{q}^h \left( \sum_{i=1}^{q+1} (-1)^{i-1} e_{p_i} \otimes e_{p_1} \wedge \cdots \wedge \widehat{e_{p_i}} \wedge \cdots \wedge e_{p_{q+1}} \right) + \partial_{q}^h \left( \sum_{r<s} (-1)^{r+s} c_{p_r,p_s}^a 1 \otimes e_a \wedge \cdots \wedge \widehat{e_{p_r}} \wedge \cdots \wedge \widehat{e_{p_s}} \wedge \cdots \wedge e_{p_{q+1}} \right)
\]

equals 0 modulo \( h \). Hence it is in \( hU_h(a) \otimes \wedge^{q-1}(a) \).

As \( \partial_{q-1}^h \partial_q^h = 0 \), it is in \( hKer\partial_{q-1}^h = hIm\partial_q^h \). The existence of \( \alpha_{p_1,\ldots,p_{q+1}} \) follows. Hence we have constructed \( \partial_{q+1}^h \) as required. The complex \( K_h = \left( U_h(a) \otimes \wedge^*a, \partial_q^h \right) \) is a resolution of the trivial \( U_h(a) \)-module \( k[[h]] \).

6.3. Quantum duality and deformation of the Koszul complex. We may construct resolutions of the trivial \( F_h[\mathfrak{g}] \) and \( F_h[\mathfrak{g}]^* \)-modules that respects the quantum duality.

**Theorem 6.3.1.** Let \( \mathfrak{g} \) be a Lie bialgebra, \( F_h[\mathfrak{g}] \) a QFSHA such that \( \frac{F_h[\mathfrak{g}]}{hK_F[\mathfrak{g}]} \) is isomorphic to \( F[\mathfrak{g}] \) as a topological Poisson Hopf algebra and \( F_h[\mathfrak{g}]^* = U_h(\mathfrak{g}^*) \) the quantization of \( U(\mathfrak{g}^*) \) constructed from \( F_h[\mathfrak{g}] \) by the quantum duality principle. Let \( x_1, \ldots, x_n \) be elements of \( F_h[\mathfrak{g}] \) such that \( F_h[\mathfrak{g}] \simeq k[[x_1, \ldots, x_n]] \). Choose \( x_1, \ldots, x_n \) elements of \( F_h[\mathfrak{g}] \) such that \( x_i = \tilde{x}_i \mod h \) and \( \epsilon_h(x_i) = 0 \). Then \( U_h(\mathfrak{g}^*) \simeq k[[x_1, \ldots, x_n]][[h]] \) with \( \tilde{x}_i = h^{-1} x_i \). Let \( (\epsilon_1, \ldots, \epsilon_n) \) be a basis of \( \mathfrak{g}^* \) and let \( c_{i,j}^a \) the structural constants of \( \mathfrak{g}^* \) with respect to this basis. We can construct a resolution of the trivial \( F_h[\mathfrak{g}] \)-module \( K_h = (F_h[\mathfrak{g}] \otimes \wedge^*\mathfrak{g}^*, \partial_q^h) \) of the form

\[
\partial_q^h (1 \otimes \epsilon_{p_1} \wedge \cdots \wedge \epsilon_{p_q}) = \sum_{i=1}^{q} (-1)^{i-1} x_i \otimes \epsilon_{p_1} \wedge \cdots \wedge \widehat{\epsilon_{p_i}} \wedge \cdots \wedge \epsilon_{p_q} + \sum_{r<s} (-1)^{r+s} h c_{p_r,p_s}^a 1 \otimes \epsilon_a \wedge \cdots \wedge \widehat{\epsilon_{p_r}} \wedge \cdots \wedge \widehat{\epsilon_{p_s}} \wedge \cdots \wedge \epsilon_{p_q} + \sum_{t_1, \ldots, t_{q-1}} \alpha_{t_1,\ldots,t_{q-1}} \otimes \epsilon_{t_1} \wedge \cdots \wedge \epsilon_{t_{q-1}}
\]

such that \( \alpha_{t_1,\ldots,t_{q-1}} \in I = \epsilon_{h}^{-1}(hk[[h]]) \). Set

\[
\alpha_{t_1,\ldots,t_{q-1}}(x_1, \ldots, \tilde{x}_n) = \alpha_{t_1,\ldots,t_{q-1}}(x_1, \ldots, x_n).
\]
Then $U$ take $\partial + \sum_{r<s} \partial p = \sum_{r<s} \partial p h 1 i \times \cdots \times p j \times \cdots \times p q$. One sets $\alpha i,j, p h a, p h 1 i, p h 2$ is in $hk[\hat{x}_1, \ldots, \hat{x}_n][[h]]$. Define the morphism of $U_h(g^*)$-modules $\partial^h_q : U_h(g^*) \otimes \wedge^q(g^*) \rightarrow U_h(g^*) \otimes \wedge^{q-1}(g^*)$ by

\[
\partial^h_q (1 \otimes \epsilon p_1 \wedge \cdots \wedge \epsilon p_q) = \sum_{i=1}^{n} (\alpha i, p h a, \epsilon p_1 \wedge \cdots \wedge \epsilon p_i \wedge \cdots \wedge \epsilon p_q)
\]
\[
+ \sum_{r<s} \sum_{a} (-1)^{r+s} C^{a}_{p r, p_s} 1 \otimes \epsilon a \wedge \epsilon p_1 \wedge \cdots \wedge \epsilon p_r \wedge \cdots \wedge \epsilon p_s \wedge \cdots \wedge \epsilon p_q
\]
\[
+ \sum_{r<s} \sum_{t_1, \ldots, t_{q-1}} \alpha^{t_1, \ldots, t_{q-1}} \otimes \epsilon t_1 \wedge \cdots \wedge \epsilon t_{q-1}.
\]

Then $K^*_h = (U_h(g^*) \otimes \wedge^* g^*, \partial^h_q)$ is a resolution of the trivial $U_h(g^*)$-module.

Proof of the theorem:

One sets $x, x_j - x_j x_i = \sum_{a=1}^{n} hC^{a}_{i,j} x_a + h\omega^a_{i,j} x_a$. We know that $\omega^a_{i,j}$ is in $I$. We take

\[
\partial^h_0 = \epsilon_h
\]
\[
\partial^h_1 (1 \otimes \epsilon_i) = x_i
\]

We have $\partial^h_0 \circ \partial^h_q = 0$ and we may choose $\omega^a_{i,j} = \omega^a_{i,j}$.

Assume that $\partial^h_0, \partial^h_1, \ldots, \partial^h_q$ have been constructed such that

- $\forall r \in [1, q] \partial^h_{r-1} \partial^h_r = 0$
- $\forall r \in [1, q] \text{Im} \partial^h_r = \text{Ker} \partial^h_{r-1}$ and satisfying the required relation.
- $\alpha i, p h a, p h 1 i, p h 2 \in I$.

and let us show that we can construct $\partial^h_{q+1}$ satisfying these three conditions.

The computation below is in [Kn] p 173.

\[
\partial^h_q \left( \sum_{i=1}^{q+1} (-1)^{i-1} x p_i \otimes \epsilon p_1 \wedge \cdots \wedge \epsilon p_i \wedge \cdots \wedge \epsilon p_{q+1} \right) +
\]
\[
+ \partial^h_q \left( \sum_{r<s} \sum_{a} hC^{a}_{p r, p_s} (-1)^{r+s} 1 \otimes \epsilon a \wedge \epsilon p_1 \wedge \cdots \wedge \epsilon p_r \wedge \cdots \wedge \epsilon p_s \wedge \cdots \wedge \epsilon p_{q+1} \right)
\]
\[
= \sum_{i=1}^{q+1} (-1)^{i} \sum_{r<s} hC^{a}_{p r, p_s} x p_i \otimes \epsilon a \wedge \epsilon p_1 \wedge \cdots \wedge \epsilon p_r \wedge \cdots \wedge \epsilon p_s \wedge \cdots \wedge \epsilon p_{q+1}
\]
\[
+ \sum_{r<s} \sum_{a} \sum_{p<s} \epsilon a \wedge \epsilon p_1 \wedge \cdots \wedge \epsilon p_s \wedge \cdots \wedge \epsilon p_{q+1}
\]
\[
+ \sum_{r<s} \sum_{a} \epsilon a \wedge \epsilon p_1 \wedge \cdots \wedge \epsilon p_{q+1}
\]
\[
+ \sum_{r<s} \sum_{k<l, s \neq k} hC^{a}_{p r, p_s} \epsilon a \wedge \epsilon p_1 \wedge \cdots \wedge \epsilon p_s \wedge \cdots \wedge \epsilon p_{q+1}
\]
\[
+ \sum_{r<s} \sum_{a} \epsilon a \wedge \epsilon p_1 \wedge \cdots \wedge \epsilon p_{q+1}
\]
\[
+ \sum_{i=1}^{q+1} (-1)^{i} x p_i h \alpha p_{i-1} \cdots \alpha p_{q+1} + \sum_{r<s} \sum_{a} (-1)^{r+s} hC^{a}_{p r, p_s} h \alpha p_{i-1} \cdots \alpha p_{q+1}
\]
where
\[ \delta = 1 \text{ if } r < i < s \text{ and } \delta = 0 \text{ otherwise} \]
\[ \sigma = 1 \text{ if exactly one of } k \text{ and } 1 \text{ is between } r \text{ and } s \]
\[ \tau = 1 \text{ if } r < j < s \text{ and } \tau = 0 \text{ otherwise} \]

The fifth term and the sixth term cancel. The second term and fourth term cancel
with each other so that we have
\[
\partial_q^h \left( \sum_{i=1}^{q+1} \epsilon_{p_i} \otimes \epsilon_{p_1} \wedge \cdots \wedge \epsilon_{p_i} \wedge \cdots \wedge \epsilon_{p_{q+1}} \right) +
\]
\[
+ \partial_q^h \left( \sum_{k<l} hC_{p_k,p_l}^a \otimes \epsilon_a \wedge \epsilon_{p_1} \wedge \cdots \wedge \epsilon_{p_k} \wedge \cdots \wedge \epsilon_{p_{q+1}} \right)
\]
\[ = \sum_{j<i} (-1)^{i+j} \left( x_{p_j}x_{p_j} - x_{p_j}x_{p_i} \right) \otimes \epsilon_1 \wedge \cdots \wedge \epsilon_{p_j} \wedge \cdots \wedge \epsilon_{p_i} \wedge \cdots \wedge \epsilon_{p_{q+1}}
\]
\[ + \sum_{i} (-1)^{i-1} h^i \alpha_{p_1, \ldots, p_i} + \sum_{r<s} (-1)^{r+s} h^2 C_{p_r,p_s}^{a_{p_1, \ldots, p_r}, \ldots, p_s} \alpha_{p_1, \ldots, p_r} \alpha_{p_r, \ldots, p_s} \alpha_{p_1, \ldots, p_s}.
\]

As \( \partial_q^h \otimes \partial_q^h = 0 \), the term
\[
\partial_q^h \left( \sum_{i=1}^{q+1} (-1)^{i-1} \epsilon_{p_i} \otimes \epsilon_{p_1} \wedge \cdots \wedge \epsilon_{p_i} \wedge \cdots \wedge \epsilon_{p_{q+1}} \right) +
\]
\[ + \partial_q^h \left( \sum_{k<l} (-1)^{k+l} C_{p_k,p_l}^a \otimes \epsilon_a \wedge \epsilon_{p_1} \wedge \cdots \wedge \epsilon_{p_k} \wedge \cdots \wedge \epsilon_{p_{q+1}} \right)
\]
is in \( h\text{Ker}\partial_q^h = h\text{Im}\partial_q^h \). We can choose \( \alpha_{p_1, \ldots, p_{q+1}} \) in \( F_h[\mathfrak{g}] \) so that the expression
above equals \( -\partial_q^h \left( h^2 \alpha_{p_1, \ldots, p_{q+1}} \right) \).

Let us now prove that \( \alpha_{p_1, \ldots, p_{q+1}} \) is in \( I \). It is easy to see that
\( -\partial_q^h \left( h^2 \alpha_{p_1, \ldots, p_{q+1}} \otimes \epsilon_1 \wedge \cdots \wedge \epsilon_{q} \right) \) is element of \( I^3 \otimes \wedge^{q+1} \mathfrak{g}^* \). Note that \( \partial_q^h \) sends
\( I^r \otimes \wedge^q \mathfrak{g}^* \) to \( I^{r+1} \otimes \wedge^q \mathfrak{g}^* \). Let us write
\[ \alpha_{p_1, \ldots, p_{q+1}} = \sum_{i_1, \ldots, i_n} \left( \alpha_{p_1, \ldots, p_{q+1}} \right)_{i_1, \ldots, i_n} x_1^{i_1} \cdots x_n^{i_n}
\]
with \( \left( \alpha_{p_1, \ldots, p_{q+1}} \right)_{i_1, \ldots, i_n} \) in \( k[[h]] \). From the remarks we have just made, we see that
\[ \partial_q^h \left( h \sum_{i_1, \ldots, i_n} \left( \alpha_{p_1, \ldots, p_{q+1}} \right)_{0, \ldots, 0} \epsilon_{i_1} \wedge \cdots \wedge \epsilon_{i_n} \right) \) is in \( I^3 \otimes \wedge^q \mathfrak{g}^* \). Hence \( \left( \alpha_{p_1, \ldots, p_{q+1}} \right)_{0, \ldots, 0} \) is in \( h\text{Ker}\partial_q^h \).

As \( \text{Im}\partial_q^h = \text{Ker}\partial_q^h \), one has \( \text{Im}\partial_q^h = \text{Ker}\partial_q^h \).

Set
\[ \alpha_{p_1, \ldots, p_{q+1}} = \sum_{i_1, \ldots, i_n} \left( \alpha_{p_1, \ldots, p_{q+1}} \right)_{i_1, \ldots, i_n} x_1^{i_1} \cdots x_n^{i_n}.
\]
\[ \frac{\partial h}{\partial \epsilon} = \epsilon \]
\[ \frac{\partial_1}{\partial \epsilon_i} = \delta_i \]
\[ \frac{\partial_2}{\partial \epsilon_j} = \delta_j \]
\[ \frac{\partial_{q+1}}{\partial \epsilon_{p_1}} \cdots \frac{\partial_{q+1}}{\partial \epsilon_{p_q}} = \sum_{i=1}^{q+1} (-1)^{i-1} \delta_i \]
\[ + \sum_{s < \delta} \sum_{a} \left( -1 \right)^{r+s} c_{a}^{q} \frac{\partial_{p_1}}{\partial \epsilon_{p_1}} \cdots \frac{\partial_{p_q}}{\partial \epsilon_{p_q}} \]
\[ If P is in \mathcal{F}_h, one has \]
\[ \frac{\partial_q}{\partial \epsilon_{p_1}} \cdots \frac{\partial_q}{\partial \epsilon_{p_q}} = h \frac{\partial_q}{\partial \epsilon_{p_1}} \cdots \frac{\partial_q}{\partial \epsilon_{p_q}} \]

The relation \( \frac{\partial_q}{\partial \epsilon_{p_1}} \cdots \frac{\partial_q}{\partial \epsilon_{p_q}} = 0 \) is obtained by multiplying the relation \( \frac{\partial_q}{\partial \epsilon_{p_1}} \cdots \frac{\partial_q}{\partial \epsilon_{p_q}} = 0 \) by \( h^{-2} \).

As \( G\delta_q^h \) is the differential of the Koszul complex of the trivial \( U(\mathfrak{g}^*)[\hbar] \)-module, the complex \( \tilde{K}^* = (U_0(\mathfrak{g}^*) \otimes \Lambda^* \mathfrak{g}^*, \delta_n^h) \) is a resolution of the trivial \( U_h(\mathfrak{g}^*) \)-module. \( \square \)

### 6.4. A link between \( \theta_{\mathcal{F}_h} \) and \( \theta_{\mathcal{F}^*_h} \)

**Theorem 6.4.1.** One has \( \theta_{\mathcal{F}_h} = h \theta_{\mathcal{F}^*_h} \)

**Proof of the theorem:**

We keep the notation of the previous proposition and we will use the proof of the theorem 5.0.7.

The complex \( (\Lambda^* \mathfrak{g}^* \otimes \mathcal{F}_h, \delta_n) \) computes the \( \mathbb{k}[\hbar] \)-modules \( \text{Ext}_{\mathcal{F}_h}^1(\mathbb{k}[\hbar], \mathcal{F}_h) \).

The cohomology class \( \text{cl} \left( \begin{array}{c} 1 \\ \epsilon_1^* \end{array} \right) \wedge \cdots \wedge \epsilon_n^* \) is a basis of \( \text{Ext}_{\mathcal{F}_h}^n(\mathbb{k}[\hbar], \mathcal{F}_h) \cong \text{GExt}_{\mathcal{F}_h}^n(\mathbb{k}[\hbar], \mathcal{F}_h) \). Hence there exists \( \sigma = 1 + h \sigma_1 + \cdots \in \text{Ker} \delta_n^h \) such that \( [\text{cl} \left( \begin{array}{c} \sigma \\ \epsilon_1^* \end{array} \right) \wedge \cdots \wedge \epsilon_n^*] \) is a basis of \( \text{GExt}_{\mathcal{F}_h}^n(\mathbb{k}[\hbar], \mathcal{F}_h) \). As the filtration on \( \text{Ext}_{\mathcal{F}_h}^p(\mathbb{k}[\hbar], \mathcal{F}_h) \) is Hausdorff, the cohomology class \( \text{cl} \left( \begin{array}{c} \sigma \\ \epsilon_1^* \end{array} \right) \wedge \cdots \wedge \epsilon_n^* \) is a basis of \( \text{Ext}_{\mathcal{F}_h}^n(\mathbb{k}[\hbar], \mathcal{F}_h) \).

Define \( \tilde{\sigma} \) by

\[ \tilde{\sigma}(x_1, \ldots, x_n) = \sigma(x_1, \ldots, x_n) \]

One has \( \delta_n = h^t \tilde{\sigma}_n \) and it is easy to check that \( \tilde{\sigma} \otimes \epsilon_1^* \wedge \cdots \wedge \epsilon_n^* \) is in \( \text{Ker} \delta_n^h - 1 \). If we had

\[ \tilde{\sigma} \otimes \epsilon_1^* \wedge \cdots \wedge \epsilon_n^* = \delta_n^h \left( \sum_{i=1}^{n} \tilde{\sigma}_i \otimes \epsilon_1^* \wedge \cdots \wedge \epsilon_i^* \wedge \cdots \wedge \epsilon_n^* \right) \]

then, reducing modulo \( h \), we would get

\[ \tilde{\sigma} \otimes \epsilon_1^* \wedge \cdots \wedge \epsilon_n^* = \frac{\partial_n^h}{\partial \epsilon_{p_1}} \left( \sum_{i=1}^{n} \tilde{\sigma}_i \otimes \epsilon_1^* \wedge \cdots \wedge \epsilon_i^* \wedge \cdots \wedge \epsilon_n^* \right) \]

This would implies that \( \text{cl} \left( \begin{array}{c} 1 \\ \epsilon_1^* \end{array} \right) \wedge \cdots \wedge \epsilon_n^* \) is 0 in \( \text{Ext}_{\mathcal{F}_h}^n(\mathbb{k}[\hbar], U_0^*(\mathfrak{g}^*)) \), which is impossible because \( \text{cl} \left( \begin{array}{c} 1 \\ \epsilon_1^* \end{array} \right) \wedge \cdots \wedge \epsilon_n^* \) is a basis of \( \text{Ext}_{\mathcal{F}_h}^n(\mathbb{k}[\hbar], U_0^*(\mathfrak{g}^*)) \). Thus \( \text{cl} \left( \begin{array}{c} \tilde{\sigma} \otimes \epsilon_1^* \end{array} \right) \wedge \cdots \wedge \epsilon_n^* \) is a non zero element of \( \text{Ext}_{U_0^*(\mathfrak{g}^*)}^n(\mathbb{k}[\hbar], U_0^*(\mathfrak{g}^*)) \). For all \( i \) in \( [1, n] \), one has the relation

\[ \text{cl} \left( \begin{array}{c} \sigma x_i \otimes \epsilon_1^* \end{array} \right) \wedge \cdots \wedge \epsilon_n^* = \theta_{\mathcal{F}_h}(x_i) \text{cl} \left( \begin{array}{c} \sigma \\ \epsilon_1^* \end{array} \right) \wedge \cdots \wedge \epsilon_n^* + \frac{\partial_n^h}{\partial \epsilon_{p_1}}(\mu) \]
Let us write
\[ \mu = \sum_i \mu_i \otimes \epsilon^*_i \wedge \cdots \wedge \epsilon^*_n \]
with \( \mu_i \in F_h[g] \). We set \( \tilde{\mu}_i(x_1, \ldots, x_n) = \mu_i(x_1, \ldots, x_n) \) and
\[ \tilde{\mu} = \sum_i \tilde{\mu}_i \otimes \epsilon^*_i \wedge \cdots \wedge \epsilon^*_n. \]

Then we have
\[ h\delta \tilde{x}_i \otimes \epsilon^*_i \wedge \cdots \wedge \epsilon^*_n = \theta F_h(x_i) \delta \otimes \epsilon^*_i \wedge \cdots \wedge \epsilon^*_n + h' \delta_h^i(\tilde{\mu}). \]

This finishes the proof of the theorem 6.4.1. □

7. Study of an example

We will now study explicitly an example suggested by B. Enriquez. Chloup ([Chl]) introduced the triangular Lie bialgebra \((g = kX_1 \oplus kX_2 \oplus kX_3 \oplus kX_4 \oplus kX_5, r = 4(X_2 \wedge X_3))\) where the non zero brackets are given by
\[ [X_1, X_2] = X_3, \quad [X_1, X_3] = X_4, \quad [X_1, X_4] = X_5 \]
and the cobracket \( \delta_g \) is the following:
\[ \forall X \in g, \quad \delta(X) = X \cdot 4(X_2 \wedge X_3). \]

The dual Lie bialgebra of \( g \) will be denoted \((a = ke_1 \oplus ke_2 \oplus ke_3 \oplus ke_4 \oplus ke_5, \delta)\). The only non zero Lie bracket of \( a \) is \([e_2, e_4] = 2e_1 \) and its cobracket \( \delta \) is non zero on the basis vectors \( e_3, e_4, e_5 \):
\[ \delta(e_3) = e_1 \otimes e_2 - e_2 \otimes e_1 = 2e_1 \wedge e_2, \quad \delta(e_4) = 2e_1 \wedge e_3, \quad \delta(e_5) = 2e_1 \wedge e_4. \]

The invertible element of \( U(g)[[h]] \otimes U(g)[[h]], \quad R = \exp(h(X_2 \otimes X_3 - X_3 \otimes X_2)), \)
satisfies the equations
\[ R^{12}(\Delta \otimes 1)(R) = R^{23}(1 \otimes \Delta)(R), \]
\[ (\epsilon \otimes id)(R) = 1 = (id \otimes \epsilon)(R). \]

Thus, we may twist the trivial deformation of \( (U(g)[[h]], \mu_0, \Delta_0, \iota_0, \epsilon_0, S_0) \) by \( R \) ([C-P] p. 130). The topological Hopf algebra obtained has the same multiplication, antipode, unit and counit but its coproduct is \( \Delta^R = R^{-1} \Delta_0 R \). It is a quantization of \( (g, r) \). We will denote it by \( U_h(g) \). The Hopf algebra \( U_h(g)^* \) is a QFSHA and \((U_h(g)^*)^\vee \) is a quantization of \((a, \delta_a)\). We will compute it explicitly.

**Proposition 7.0.2.** a) \((U(g)^*)^\vee \) is isomorphic as a topological Hopf algebra to the topological \( k[[h]]\)-algebra \( T_k[[h]](ke_1 \oplus k[[h]]e_2 \oplus k[[h]]e_3 \oplus k[[h]]e_4 \oplus k[[h]]e_5) / I \)
where \( I \) is the closure of the two-sided ideal generated by
\[ e_2 \otimes e_4 - e_4 \otimes e_2 - 2e_1, \]
\[ e_3 \otimes e_5 - e_5 \otimes e_3 - \frac{2}{3} h^2 e_1 \otimes e_1 \otimes e_1, \]
\[ e_4 \otimes e_5 - e_5 \otimes e_4 - \frac{1}{6} h^3 e_1 \otimes e_1 \otimes e_1 \otimes e_1, \]
\[ e_2 \otimes e_5 - e_5 \otimes e_2 + he_1 \otimes e_1, \]
\[ e_3 \otimes e_4 - e_4 \otimes e_3 + he_1 \otimes e_1, \]
\[ e_1 \otimes e_j - e_j \otimes e_1 \text{ if } \{i, j\} \neq \{2, 4\}, \{3, 5\}, \{4, 5\}, \{2, 5\}, \{3, 4\} \]
with the coproduct $\Delta_h$, counit $\epsilon_h$ and antipode $S$ defined as follows:

$$
\Delta_h(e_1) = e_1 \otimes 1 + 1 \otimes e_1 \\
\Delta_h(e_2) = e_2 \otimes 1 + 1 \otimes e_2 \\
\Delta_h(e_3) = e_3 \otimes 1 + 1 \otimes e_3 - he_2 \otimes e_1 \\
\Delta_h(e_4) = e_4 \otimes 1 + 1 \otimes e_4 - he_3 \otimes e_1 + \frac{h^2}{2} e_2 \otimes e_1 \\
\Delta_h(e_5) = e_5 \otimes 1 + 1 \otimes e_5 - he_4 \otimes e_1 + \frac{h^2}{2} e_3 \otimes e_1 - \frac{h^3}{6} e_2 \otimes e_1.
$$

$\forall i \in \{1, 5\}$, $\epsilon_h(e_i) = 0$

$\forall i \in \{1, 5\}$, $S(e_i) = -e_i$

h) $(U(g)^*)^\vee$ is not isomorphic to the trivial deformation of $U(a)$, $U(a)[[h]]$, as algebra.

Proof of the proposition

Let $\xi_1$ be the element of $U(g)^*$ defined by

$$
<\xi_1, X_1^{a_1}X_2^{a_2}X_3^{a_3}X_4^{a_4}X_5^{a_5} > = \delta_{a_1,0} \ldots \delta_{a_1,1} \ldots \delta_{a_5,0}.
$$

The algebras $U(g)^*$ and $k[[\xi_1, \ldots, \xi_n]]$ are isomorphic. The topological Hopf algebra $(U_h(g)^*, \Delta_h^0 = \gamma) = (U_h, \mu, \eta, \epsilon_h, \delta_h, S_h)$ is a QFSHA. Remark that $U_h(g)^*$ and $k[[\xi_1, \ldots, \xi_n, h]]$ are isomorphic as $k[[h]]$-modules. The elements $\xi_1, \ldots, \xi_n$ generate topologically the $k[[h]]$- algebra $U_h(g)^*$ and satisfy $\epsilon_h(\xi_1) = 0$.

$<\xi_2 \otimes \xi_4 - \xi_4 \otimes \xi_2, \Delta^R(X_1^{a_1} \ldots X_5^{a_5}) > \neq 0 \Leftrightarrow (a_1, a_2, a_3, a_4, a_5) = (1, 0, 0, 0, 0)$

and $<\xi_2 \otimes \xi_4 - \xi_4 \otimes \xi_2, \Delta^R(\xi_1) > = 2h$. Hence $\xi_2 \cdot h \xi_4 - \xi_4 \cdot h \xi_2 = 2h\xi_1$.

$<\xi_3 \otimes \xi_5 - \xi_5 \otimes \xi_3, \Delta^R(X_1^{a_1} \ldots X_5^{a_5}) > \neq 0 \Leftrightarrow (a_1, a_2, a_3, a_4, a_5) = (3, 0, 0, 0, 0)$

and $<\xi_3 \otimes \xi_5 - \xi_5 \otimes \xi_3, \Delta(\xi_1^2) > = 4h$. Hence $\xi_3 \cdot h \xi_5 - \xi_5 \cdot h \xi_3 = 2h\xi_1$.

$<\xi_3 \otimes \xi_5 - \xi_5 \otimes \xi_3, \Delta^R(X_1^{a_1} \ldots X_5^{a_5}) > \neq 0 \Leftrightarrow (a_1, a_2, a_3, a_4, a_5) = (4, 0, 0, 0, 0)$

and $<\xi_3 \otimes \xi_5 - \xi_5 \otimes \xi_3, \Delta^R(\xi_1^2) > = 4h$. Hence $\xi_3 \cdot h \xi_5 - \xi_5 \cdot h \xi_3 = 2h\xi_1$.

$<\xi_2 \otimes \xi_5 - \xi_5 \otimes \xi_2, \Delta^R(X_1^{a_1} \ldots X_5^{a_5}) > \neq 0 \Leftrightarrow (a_1, a_2, a_3, a_4, a_5) = (2, 0, 0, 0, 0)$

and $<\xi_2 \otimes \xi_5 - \xi_5 \otimes \xi_2, \Delta^R(\xi_1^2) > = 2h$. Hence $\xi_2 \cdot h \xi_5 - \xi_5 \cdot h \xi_2 = 2h\xi_1$.

$<\xi_i \otimes \xi_4 - \xi_4 \otimes \xi_i, \Delta^R(X_1^{a_1} \ldots X_5^{a_5}) > \neq 0 \Leftrightarrow (a_1, a_2, a_3, a_4, a_5) = (2, 0, 0, 0, 0)$

Let us now compute the coproduct $\Delta_h$ of $U_h(g)^*$.

$<\Delta_h(\xi_3), X_1^{a_1}X_2^{a_2}X_3^{a_3}X_4^{a_4}X_5^{a_5} \otimes X_1^{b_1}X_2^{b_2}X_3^{b_3}X_4^{b_4}X_5^{b_5} > \neq 0 \Leftrightarrow (a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, b_5) = (0, 0, 1, 0, 0, 0, 0, 0, 0, 0)$ or $(0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$

and $<\Delta_h(\xi_3), X_2X_1 > = -1$. Hence

$$
\Delta_h(\xi_3) = \xi_3 \otimes 1 + 1 \otimes \xi_3 - \xi_2 \otimes \xi_1.
$$

$<\Delta_h(\xi_4), X_1^{a_1}X_2^{a_2}X_3^{a_3}X_4^{a_4}X_5^{a_5} \otimes X_1^{b_1}X_2^{b_2}X_3^{b_3}X_4^{b_4}X_5^{b_5} > \neq 0 \Leftrightarrow (a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, b_5) = (0, 0, 1, 0, 0, 0, 0, 0, 0, 0)$ or $(0, 0, 0, 0, 0, 0, 0, 0, 1, 0)$

or $(0, 0, 1, 0, 0, 1, 0, 0, 0, 0)$ or $(0, 1, 0, 0, 0, 0, 0, 0, 0, 0)$.
Moreover
\[ <\Delta_h(\xi_4), X_3 \otimes X_1> = -1 \quad \text{and} \quad <\Delta_h(\xi_4), X_2 \otimes X_1^2> = 1.\]

Hence
\[ \Delta_h(\xi_4) = \xi_4 \otimes 1 + 1 \otimes \xi_4 - \xi_3 \otimes \xi_1 + \frac{1}{2} \xi_2 \otimes \xi_1 \otimes \xi_1.\]

\[ <\Delta_h(\xi_5), X_1^{a_1} X_2^{a_2} X_3^{a_3} X_4^{a_4} X_5^{a_5} \otimes X_1^{b_1} X_2^{b_2} X_3^{b_3} X_4^{b_4} X_5^{b_5} > \neq 0 \iff\]
\[ (a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, b_5) = (0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0) \quad \text{or} \quad (0, 0, 0, 0, 0, 0, 0, 0, 0, 1)
\]
\[ \text{or} \quad (0, 0, 0, 0, 0, 0, 0, 0, 0) \quad \text{or} \quad (0, 0, 1, 0, 0, 2, 0, 0, 0) \quad \text{or} \quad (0, 0, 0, 3, 0, 0, 0, 0).\]

Moreover
\[ <\Delta_h(\xi_5), X_4 \otimes X_1> = -1, \quad <\Delta_h(\xi_4), X_3 \otimes X_1^2> = 1, \quad <\Delta_h(\xi_4), X_2 \otimes X_1^3> = -1.\]

Hence
\[ \Delta_h(\xi_5) = \xi_5 \otimes 1 + 1 \otimes \xi_5 - \xi_4 \otimes \xi_1 + \frac{1}{2} \xi_3 \otimes \xi_1 \otimes \xi_1 - \frac{1}{6} \xi_2 \otimes \xi_1 \otimes \xi_1 \otimes \xi_1.\]

We set \( \xi_i = b^{-1} \xi_i \) and \( e_i = \xi_i \mod h \). Let \( \chi : (U(g))^\vee \to U(a)[[h]] \) be the isomorphism of topologically \( k[[h]] \)-modules defined by
\[ \chi \left( \sum_{r \in \mathbb{N}} P_r(\xi_1, \ldots, \xi_n) h^r \right) = \sum_{r \in \mathbb{N}} P_r(e_1, \ldots, e_n) h^r. \]

From what we have reviewed in the first paragraph of this section, the first part of this theorem is proved.

If \( u \) and \( v \) are in \( U(a) \), one sets
\[ u \cdot_h v = uv + \sum_{r=1}^{\infty} h^r \mu_r(u, v). \]

one has
\[ \mu_1(e_3, e_4) = 0, \quad \mu_1(e_4, e_3) = e_1^2, \quad \mu_1(e_2, e_5) = 0, \quad \mu_1(e_5, e_2) = e_1^2. \]

Let us show now that \( \mu_1 \) is a coboundary in the Hochschild cohomology. The Hochschild cohomology \( HH^*(U(a), U(a)) \) is computed by the complex \( (\text{Hom}(U(a)^{\otimes n+1}, U(a)), b) \) where : if \( f \in \text{Hom}(U(a)^{\otimes n+1}, U(a)) \), then
\[ b(f)(a_0, \ldots, a_n) = a_0 f(a_1, \ldots, a_n) + \sum_{i=1}^{n} (-1)^i f(a_0, \ldots, a_{i-1} a_i, \ldots, a_n) + f(a_0, \ldots, a_{n-1}) a_n (-1)^n. \]

The Lie algebra cohomology of \( a \) with coefficients in \( U(a) \) (with the adjoint action), \( H^* (a, U(a)^{ad}) \), is computed by the Chevalley-Eilenberg complex \( (\text{Hom}(\wedge a, U(a)), d) \) where : if \( f \in \text{Hom}(\wedge^{n+1} a, U(a)) \)
\[ d(f)(z_1, \ldots, z_{n+1}) = \sum_{i=1}^{n} (-1)^{i-1} z_1 \cdot f(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_{n+1}) + \sum_{i<j} (-1)^{i+j} f([z_i, z_j], \ldots, z_{i-1}, z_{i+1}, z_{j-1}, z_{j+1}, z_{n+1}). \]
The map ([L] lemma 3.3.3) $\Psi^* : (Hom (U(a)^{\otimes}, U(a)), b) \to (Hom (\wedge a, U(a)^{ad}), d)$ defined by antisymmetrization
\[
\Psi^*(f) (z_1, \ldots, z_n) = f \left( \sum_{\sigma \in S_n} \epsilon(\sigma) z_{\sigma(1)} \otimes \cdots \otimes z_{\sigma(n)} \right)
\]
is a morphism of complexes. One checks easily that
\[
\Psi^*(\mu_1) = d \left( -\frac{1}{2} e_1 e_2 \otimes e_3 - \frac{1}{2} e_1 e_4 \otimes e_5 \right).
\]
There exists $\alpha \in Hom (U(a), U(a))$ such that $\mu_1 = b(\alpha)$. The map $\alpha$ is determined by
\[
\alpha|_n = -\frac{1}{2} e_1 e_2 \otimes e_3 - \frac{1}{2} e_1 e_4 \otimes e_5\quad \forall (u, v) \in U(a), \quad \mu_1(u, v) = u\alpha(v) - \alpha(uv) + u\alpha(v)
\]
We set $\beta_h = id - h\alpha$. Then one has $\beta_h^{-1} = \sum_{i=0}^{\infty} h^i \alpha^i$. If $u$ and $v$ are elements of $U(a)$, we put
\[
u = u \cdot_h v = \beta_h^{-1} (\beta_h(u) \cdot_h \beta_h(v)).
\]
Let’s compute $e_i \cdot_h e_j - e_j \cdot_h e_i$. If $i$ and $j$ are different from 3 and 5, then $e_i \cdot_h e_j = e_i \cdot_h e_j$.
\[
e_1 \cdot_h e_3 - e_3 \cdot_h e_1 = f_h^{-1} \left[ e_1 \cdot_h \left( e_3 + \frac{h e_1 e_2}{2} \right) - \left( e_3 + \frac{h e_1 e_2}{2} \right) \cdot_h e_1 \right]
= f_h^{-1} \left[ e_1 \cdot_h \frac{h e_1 e_2}{2} + \frac{h e_1 e_2}{2} \cdot_h e_1 \right] = 0
\]
Similarly, the following relations hold
\[
e_1 \cdot_h e_5 = e_5 \cdot_h e_1, \quad e_2 \cdot_h e_3 = e_3 \cdot_h e_2, \quad e_2 \cdot_h e_5 = e_5 \cdot_h e_2, \quad e_3 \cdot_h e_4 = e_4 \cdot_h e_3.
\]
Let us now compute $e_3 \cdot_h e_5 - e_5 \cdot_h e_3$. Easy computations lead to the following equalities: one has
\[
e_1 e_2 \cdot_h e_5 - e_5 \cdot_h e_1 e_2 = e_1^3
\]
\[
e_3 \cdot_h e_4 - e_4 \cdot_h e_3 = -e_1^3
\]
\[
e_1 e_2 \cdot_h e_4 - e_4 \cdot_h e_1 e_2 = 2e_1^3
\]
One deduces easily from this that
\[
e_3 \cdot_h e_5 - e_5 \cdot_h e_3 = \frac{1}{6} h^2 e_1^3.
\]
Similarly, one has
\[
e_4 \cdot_h e_5 - e_5 \cdot_h e_4 = -\frac{h^2}{6} e_1^3.
\]
The topological algebras $[U(a)[[h]], \cdot_h]$ and $[U(a)[[h]], \cdot_h']$ are isomorphic, hence their centers are isomorphic. Let us compute the center of $[U(a)[[h]], \cdot_h']$. Let $z$ be an element of the center $Z [U(a)[[h]], \cdot_h']$. One writes $z$ under the form $\sum_{r \geq 0} P_r (e_1, e_2, e_3, e_4, e_5) h^r$ (where the multiplications in $P_r (e_1, e_2, e_3, e_4, e_5)$ are $\cdot_h'$). One has
\[
e_2 \cdot_h z - z \cdot_h e_2 = \sum_{r \in \mathbb{N}} 2h^r \frac{\partial P_r}{\partial X_4} (e_1, e_2, e_3, e_4, e_5).
\]
Hence the polynomials $P_r$ don’t depend on $X_4$ and $z$ can be written $z = \sum_{n \geq 0} P_r(e_1, e_2, e_3, e_5)h^r$.

\[
e_3', z - z'e_3 = \sum_{r \in \mathbb{N}} \frac{1}{6} h^{r+2} \left( X_1^3 \frac{\partial P_r}{\partial X_3} \right) (e_1, e_2, e_3, e_5).
\]

Hence the polynomials $P_r$ don’t depend on $X_5$ and $z$ can be written $z = \sum_{n \geq 0} P_r(e_1, e_2, e_3)h^r$.

\[
e_4', z - z'e_4 = \sum_{r \in \mathbb{N}} -2h^{r+2} \left( X_1^3 \frac{\partial P_r}{\partial X_2} \right) (e_1, e_2, e_3).
\]

Hence the polynomials $P_r$ don’t depend on $X_2$ and $z$ can be written $z = \sum_{n \geq 0} P_r(e_1, e_3)h^r$.

\[
e_5', z - z'e_5 = \sum_{r \in \mathbb{N}} -\frac{1}{6} h^{r+2} \left( X_1^3 \frac{\partial P_r}{\partial X_3} \right) (e_1, e_3).
\]

Hence the polynomials $P_r$ don’t depend on $X_3$ and $z$ can be written $z = \sum_{n \geq 0} P_r(e_1)h^r$.

Hence

\[
Z \{ U(a)[[h]], \mu' \} = \left\{ \sum_{n \geq 0} P_r(e_1)h^r \mid P_r \in k[X_1] \right\}.
\]

But, the center of the trivial deformation of $U(a)$ is

\[
Z \{ U(a)[[h]], \mu_0 \} = \left\{ \sum_{n \geq 0} P_r(e_1, e_3)h^r \mid P_r \in k[X_1, X_3, X_5] \right\}.
\]

The algebras $\{ U(a)[[h]], \mu' \}$ and $\{ U(a)[[h]], \mu_0 \}$ are not isomorphic as their center are not isomorphic. □

**Proposition 7.0.3.** We consider the quantized enveloping algebra of the proposition 7.0.2 We write the relations defining the ideal $I$ as follows

\[
e_i \otimes e_j - e_j \otimes e_i = \sum a C_{i,j}^a e_a - P_{i,j}.
\]

As all the $P_{i,j}'s$ are monomials in $e_1$’s, the notation $\frac{P_{i,j}}{e_1}$ makes sense. The complex

\[
0 \rightarrow U_h(a) \otimes \wedge^a a \xrightarrow{\partial_k^h} U_h(a) \otimes \wedge^4 a \xrightarrow{\partial_k^h} \cdots \xrightarrow{\partial_k^h} U_h(a) \otimes a \xrightarrow{\partial_k^h} U_h(a) \xrightarrow{\partial_k^h} k[[h]] \rightarrow 0
\]

where the morphisms of $U_h(a)$, $\partial_k^h$, are described below is a resolution of the trivial $U_h(a)$-module $k[[h]]$. We set

\[
\partial_k(1 \otimes \wedge \cdots \wedge \wedge e_p) = \sum_{i=1}^n (-1)^{i-1} e_p \otimes e_p \wedge \cdots \wedge \wedge e_p \wedge \cdots \wedge e_p
\]

\[
+ \sum_{k<l} (-1)^{k+l} \sum a C_{k,l}^a 1 \otimes e_a \wedge e_p \wedge \cdots \wedge e_p \wedge \cdots \wedge e_p.
\]
Then
\[ \partial_0^h = e_h \]
\[ \partial_1^h (1 \otimes e_i) = e_i \]
\[ \partial_2^h (1 \otimes e_i \wedge e_j) = \partial_2 (1 \otimes e_i \wedge e_j) - \frac{P_{i,j}}{e_1} \otimes e_i \]
\[ \partial_3^h (1 \otimes e_i \wedge e_j \wedge e_k) = \partial_4 (1 \otimes e_i \wedge e_j \wedge e_k) - \frac{P_{i,j}}{e_1} \otimes e_1 \wedge e_k + \frac{P_{i,k}}{e_1} \otimes e_1 \wedge e_j - \frac{P_{j,k}}{e_1} \otimes e_1 \wedge e_i \]
\[ \partial_4^h \]

The character defined by the right multiplication of \( h \) is zero.

**Proof of the proposition:** The resolution of \( k[[h]] \) constructed in the proposition is obtained by applying the proof of theorem 6.2.1. Moreover, one has

\[ \partial_0^h (1 \otimes e_1 \wedge e_2 \wedge e_3 \wedge e_4 \wedge e_5) = e_1 \otimes e_3 \wedge e_4 \wedge e_5 + e_1 \otimes e_4 \wedge e_5 + e_1 \otimes e_5 \]
\[ \partial_2^h (1 \otimes e_2 \wedge e_3 \wedge e_4 \wedge e_5) = \partial_3 (1 \otimes e_2 \wedge e_3 \wedge e_4 \wedge e_5) + \frac{P_3}{e_1} \otimes e_1 \wedge e_2 \wedge e_4 \]
\[ - \frac{P_2}{e_1} \otimes e_1 \wedge e_2 \wedge e_3 - \frac{P_2}{e_1} \otimes e_1 \wedge e_3 \wedge e_4 - \frac{P_2}{e_1} \otimes e_1 \wedge e_4 \wedge e_5 \]
\[ \partial_4^h (1 \otimes e_2 \wedge e_3 \wedge e_4 \wedge e_5) = \partial_5 (1 \otimes e_2 \wedge e_3 \wedge e_4 \wedge e_5) \]

These equalities show that the character defined by the right multiplication of \( U_h(a) \) on \( Ext^5_{U_h(a)} (k[[h]], U_h(a)) \) is zero.

8. **Applications**

8.1. **Poincaré duality.** Let \( M \) be an \( A_h^{op} \)-module and \( N \) an \( A_h \)-module. The right exact functor \( M \otimes - \) has a left derived functor. We set \( Tor^A_{ab} (M,N) = L^i \left( M \otimes - \right) (N) \).

**Theorem 8.1.1.** Let \( A_h \) be a deformation algebra of \( A_0 \) satisfying the hypothesis of theorem 5.0.7. Assume moreover that the \( A_h \)-module \( K \) is of finite projective dimension. Let \( M \) be an \( A_h \)-module. One has an isomorphism of \( K \)-modules

\[ \text{Ext}^i_{A_h} (K, M) \simeq Tor^A_{d A_h} (\Omega_{A_h}, M) \]

**Remark:** Theorem 8.1.1 generalizes classical Poincaré duality ([K7]).

**Proof of the theorem**

As the \( A_h \)-module \( K \) admits a finite length resolution by finitely generated projective \( A_h \)-modules, \( P^* \rightarrow K \), the canonical arrow

\[ RHom_{A_h} (K, A_h) \otimes \rightarrow RHom_{A_h} (K, M) \]

is an isomorphism in \( D(\text{Mod} A_h) \). Indeed the canonical arrow

\[ Hom_{A_h} (P^*, A_h) \otimes M \rightarrow Hom_{A_h} (P^*, M) \]
8.2. Duality property for induced representations of quantum groups.

From now on, we assume that $A_h$ is a topological Hopf algebra.

In this section, we keep the notation of theorem 5.0.8. Let $V$ be a left $A_h$-module, then, by transposition, $V^* = \text{Hom}_K(V, K)$ is naturally endowed with a right $A_h$-module structure. Using the antipode, we can also see $V^*$ as a left module structure. Thus, one has:

\[ \forall u \in A_h \forall f \in V^*, \quad u \cdot f = f \cdot S(u). \]

We endow $\Omega_{A_h} \otimes V^*$ with the following right $A_h$-module structure:

\[ \forall u \in A_h \forall f \in V^*, \forall \omega \in \Omega_{A_h}, \quad (\omega \otimes f) \cdot u = \lim_{n \to +\infty} \sum_j \theta_{A_h}(u'_j,n) \omega \otimes f \cdot S_{A_h}^2(u''_j,n) \]

where $\Delta(u) = \lim_{n \to +\infty} \sum_j u'_j,n \otimes u''_j,n$.

Let $A_h$ be a topological Hopf deformation of $A_0$ and $B_h$ be a topological Hopf deformation of $B_0$. We assume moreover that there exists a morphism of Hopf algebras from $B_h$ to $A_h$ and that $A_h$ is a flat $B_{op}^h$-module (by proposition 4.1.6 this is verified if the induced $B_0$-module structure on $A_0$ is flat). If $V$ is an $A_h$-module, we can define the induced representation from $V$ as follows:

\[ \text{Ind}_{B_h}^{A_h}(V) = A_h \otimes_{B_h} V \]

on which $A_h$ acts by left multiplication.

**Proposition 8.2.1.** Let $A_h$ be a topological Hopf deformation of $A_0$ and $B_h$ be a topological deformation of $B_0$. We assume that there exists a morphism of Hopf algebras from $B_h$ to $A_h$ such that $A_h$ is a flat $B_{op}^h$-module. We also assume that $B_h$ satisfies the hypothesis of theorem 5.0.7. Let $V$ be an $B_h$-module which is a free finite dimensional $K$-module. Then $D_{B_h}\left(\text{Ind}_{B_h}^{A_h}(V)\right)$ is isomorphic to $(\Omega_{B_h} \otimes V^*) \otimes A_h[-d_{B_h}]$ in $D(\text{Mod}B_{op}^h)$.

**Corollary 8.2.2.** Let $A_h$ be a topological Hopf deformation of $A_0$ and $B_h$ be a topological deformation of $B_0$. We assume that there exists a morphism of Hopf algebras from $B_h$ to $A_h$ such that $A_h$ is a flat $B_{op}^h$-module. We also assume that $B_h$ satisfies the condition of the theorem 5.0.7. Let $V$ be a $B_h$-module which is a free finite dimensional $K$-module. Then

a) $\text{Ext}_{A_h}^i\left(A_h \otimes_{B_h} V, A_h\right)$ is reduced to 0 if $i$ is different from $d_{B_h}$.

b) The right $A_h$-module $\text{Ext}_{A_h}^{d_{B_h}}\left(A_h \otimes_{B_h} V, A_h\right)$ is isomorphic to $(\Omega_{B_h} \otimes V^*) \otimes A_h$.

**Remarks:**

Proposition 8.2.1 is already known in the case where $\mathfrak{g}$ is a Lie algebra, $\mathfrak{h}$ is a Lie subalgebras of $\mathfrak{g}$, $A$ and $B$ are the corresponding enveloping algebras.
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In this case one has $d_B = \dim \mathfrak{b}$ and $d_C = \dim \mathfrak{t}$. More precisely: It was proved by Brown and Levasseur ([B-L] p. 410) and [Ke] in the case where $\mathfrak{g}$ is a finite dimensional semi-simple Lie algebra and $\text{Ind}_{U(h)}^{U(g)}(V)$ is a Verma-module. Proposition 8.2.3 is proved in full generality for Lie superalgebras in [C1].

Here are some examples of situations where we can apply the proposition 8.2.1:

**Example 1:**
Let $k$ be a field of characteristic 0. We set $K = k[[h]]$. Etingof and Kazhdan have constructed a functor $Q$ from the category $LB_k$ of Lie bialgebras over $k$ to the category $HA(K)$ of topological Hopf algebras over $K$. If $(\mathfrak{g}, \delta)$ is a Lie bialgebra, its image by $Q$ will be denoted $U_h(\mathfrak{g})$.

**Example 2:** If $\mathfrak{g}$ is a Lie bialgebra, we will denote by $F(\mathfrak{g})$ the formal group attached to it and $F_h(\mathfrak{g})$ its Etingof Kazhdan quantization. Let $\mathfrak{g}$ and $\mathfrak{h}$ be two Lie algebras and assume that there exists a surjective morphism of Lie bialgebras from $\mathfrak{g}$ to $\mathfrak{h}$. Then $F_h(\mathfrak{g})$ is a flat $F_h(\mathfrak{h})$-module and $A_h = F_h(\mathfrak{g})$ and $B_h = F_h(\mathfrak{h})$ satisfies the hypothesis of the theorem.

**Example 3:** If $G$ is an affine algebraic Poisson group, we will denote by $F(G)$ the algebra of regular functions on $G$ and $F_h(G)$ its Etingof Kazhdan quantization. Let $G$ and $H$ be affine algebraic Poisson groups. Assume that there is a Poisson group map $G \rightarrow H$ such that $F(G)$ is a flat $F(H)^{op}$-module. By functoriality of Etingof Kazhdan quantization, $A_h = F_h(G)$ and $B_h = F_h(H)$ satisfies the hypothesis of the theorem.

**Proof of the proposition 8.2.1:**
We proceed as in [C1]. Let $L^\bullet \rightarrow V$ be a resolution of $V$ by finite free $B_h$-modules. As $A_h$ is a flat $B_h^{op}$-module, $A_h \otimes_{B_h} L^\bullet \rightarrow A_h \otimes_{B_h} V$ is a resolution of the $A_h$-module $A_h \otimes_{B_h} V$ by finite free $A_h$-modules.

We have the following sequence of isomorphisms in $D(\text{Mod}A_h)$

$$
\text{RHom}_{A_h}(A_h \otimes_{B_h} V, A_h) \cong \text{Hom}_{A_h}(A_h \otimes_{B_h} L^\bullet, A_h) \\
\cong \text{Hom}_{B_h}(L^\bullet, B_h \otimes_{B_h} A_h) \\
\cong (\Omega_{B_h} \otimes V^*) \otimes_{B_h} A_h[-d_{B_h}], \square
$$

We now extend to Hopf algebras another duality property for induced representations of Lie algebras ([C1]).

**Proposition 8.2.3.** Let $A_h$ be a Hopf deformation of $A_0$, $B_h$ be a Hopf deformation of $B_0$ and $C_h$ be a Hopf deformation of $C_0$. We assume that there exists a
morphism of Hopf algebras from \(B_h\) to \(A_h\) and a morphism of Hopf algebras from \(C_h\) to \(A_h\) such that \(A_h\) is a flat \(B_h^{\text{op}}\)-module and a flat \(C_h^{\text{op}}\)-module. We also assume that \(B_h\) and \(C_h\) satisfies the hypothesis of theorem 5.0.7. Let \(V\) (respectively \(W\)) be an \(B_h\)-module (respectively \(C_h\)-module) which is a free finite dimensional \(K\)-module. Then, for all integer \(n\), one has an isomorphism

\[
\text{Ext}^{n+d_{B_h}}_{A_h} \left( A_h \otimes V, A_h \otimes W \right) = \text{Ext}^{n+d_{C_h}}_{A_h^{\text{op}}} \left( \Omega_{C_h} \otimes W^*, \Omega_{B_h} \otimes V^* \otimes A_h \right)
\]

**Remarks:**

Proposition 8.2.3 is already known in the case where \(g\) is a Lie algebra, \(\mathfrak{h}\) and \(\mathfrak{t}\) are Lie subalgebras of \(g\), \(A\), \(B\) and \(C\) are the corresponding enveloping algebras. In this case one has \(d_{B_h} = \text{dim}\mathfrak{h}\) and \(d_{C_h} = \text{dim}\mathfrak{t}\). More precisely:

Generalizing a result of G. Zuckerman ([B-C]), A. Gyoja ([G]) proved a part of this theorem (namely the case where \(\mathfrak{h} = g\) and \(n = \text{dim}\mathfrak{h} = \text{dim}\mathfrak{t}\)) under the assumptions that \(g\) is split semi-simple and \(\mathfrak{h}\) is a parabolic subalgebra of \(g\). D.H Collingwood and B. Shelton ([C-S]) also proved a duality of this type (still under the semi-simple hypothesis) but in a slightly different context.

M. Duflo [Du2] proved proposition 8.2.3 for a \(g\) general Lie algebra, \(\mathfrak{h} = \mathfrak{t}\), \(V = W^*\) being one dimensional representations.

Proposition 8.2.3 is proved in full generality in the context of Lie superalgebras in [C1].

**Proof of the proposition 8.2.3:**

We will proceed as in [C2]. As \(D_{A_h^{\text{op}}} \circ D_{A_h} \left( A_h \otimes V \right) = A_h \otimes V\), we have the following isomorphism

\[
\text{Hom}_{D(A_h)} \left( A_h \otimes V, A_h \otimes W \right) = \text{Hom}_{D(A_h^{\text{op}})} \left[ D_{A_h} \left( A_h \otimes W \right), D_{A_h} \left( A_h \otimes V \right) \right]
\]

the corollary follows now from proposition 8.2.1.

8.3. **Hochschild cohomology.** In this subsection, \(A_h\) is a topological Hopf algebra. We set \(A_h^\wedge = A_h \otimes A_h^{\text{op}}_{k[[h]]}\) and \(A_h^\wedge = A_h \otimes A_h^{\text{op}}_{k[[h]]}\). If \(M\) is an \(A_h^\wedge\)-module, we set

\[
\HH^*_A(M) = \text{Ext}^{*}_{A_h^\wedge} \left( A_h, M \right)
\]

\[
\HH^*_A(M) = \text{Tor}^{*}_{A_h^\wedge} \left( A_h, M \right)
\]

**Proposition 8.3.1.** Assume that \(A_h\) satisfies the condition of the theorem 5.0.7. Assume moreover that \(A_0 \otimes A_0^{\text{op}}_{k[[h]]}\) is noetherian. Consider \(A_h \otimes A_h\) with the following \(A_h^\wedge\)-module structure :

\[
\forall (\alpha, \beta, x, y) \in A_h, \quad \alpha \cdot (x \otimes y) \cdot \beta = \alpha x \otimes y \beta.
\]

a) \(\HH^*_A(A_h \otimes A_h)\) is zero if \(i \neq d_{A_h}\).
b) The $\widehat{A}_h^k$-module $H^d_{A_k}(A_h \otimes A_h)$ is isomorphic to $\Omega_{A_h} \otimes A_h$ with the following $\widehat{A}_h^k$-module structure:

$$\forall (\alpha, \beta, x) \in A_h, \quad \alpha \cdot (\omega \otimes x) \cdot \beta = \omega \theta_{A_h}(\beta') \otimes S(\beta'') x S^{-1}(\alpha)$$

where $\alpha = \sum \alpha'_i \otimes \alpha''_i$ (to be taken in the topological sense).

This result was obtained in [D-E] for a deformation of the algebra of regular functions on a smooth algebraic affine variety.

Proof of the theorem:
The proof is analogous to that of [C2] (theorem 3.3.2).

Using the antipode $S_h$ of $A_h$, we have the following isomorphism in $D\left(\text{Mod}\widehat{A}_h^k\right)$,

$$\text{RHom}_{\widehat{A}_h^k}(A_h, A_h \otimes A_h) \simeq \text{RHom}_{A_h}(\Omega_{A_h} \otimes A_h, (A_h \otimes A_h)^\#).$$

where the structures on $(A_h)^\#$ and $(A_h \otimes A_h)^\#$ are given by:

$$\forall (\alpha, \beta, u, v) \in A_h$$

$$(\alpha \otimes \beta) \cdot u = \alpha u S_h(\beta)$$

$$(\alpha \otimes \beta) \cdot (u \otimes v) = \alpha u \otimes v S_h(\beta)$$

$$(u \otimes v) \cdot \alpha \otimes \beta = u \alpha \otimes v \beta.$$ 

Using the version of lemma 5.0.9 for right modules (see [C2] lemma 1;1), one sees that $(A_h)^\#$ is isomorphic to $(A_h \otimes A_h) \otimes K$ as an $A_h \otimes A_h$-module. we get

$$\text{RHom}_{\widehat{A}_h^k}(A_h, A_h \otimes A_h) \simeq \text{RHom}_{A_h}(\Omega_{A_h} \otimes A_h, (A_h \otimes A_h)^\#) \simeq \text{RHom}_{A_h}(K, (A_h \otimes A_h)^\#) \simeq \text{RHom}_{A_h}(K, A_h) \otimes_{A_h} (A_h \otimes A_h)^\# \simeq \Omega_{A_h} \otimes_{A_h} (A_h \otimes A_h)^\#$

The isomorphism $id \otimes S_h^{-1}$ transforms $(A_h \otimes A_h)^\#$ into the natural $(A_h \otimes A_h) \otimes (A_h \otimes A_h)^\#_{\text{nat}}$-module $(A_h \otimes A_h)^\#_{\text{nat}}$:

$$\forall (\alpha, \beta, u, v) \in A_h$$

$$(\alpha \otimes \beta) \cdot (u \otimes v) = \alpha u \otimes v \beta$$

$$(u \otimes v) \cdot \alpha \otimes \beta = u \alpha \otimes v \beta.$$ 

Then, using the lemma 5.0.9, one sees that $\Omega_{A_h} \otimes_{A_h} (A_h \otimes A_h)^\#_{\text{nat}}$ is isomorphic to $\Omega_h \otimes A_h$ endowed with the following $(A_h \otimes A_h)^\#_{\text{nat}}$-module structure:

$$\forall (\alpha, \beta) \in A_h,$$ 

$$(u \otimes v) \cdot \alpha \otimes \beta = \sum_i u \theta_{A_h}(\alpha'_i) \otimes S(\alpha''_i) v \beta.$$ 

This finishes the proof of the proposition. $\square$

We are in the case where $\text{Ext}_{A_h}^{i, A_h}(A_h, \widehat{A}_h^k)$ is 0 unless when $i = d_{A_h}$, so we have a duality between Hochschild homology and Hochschild cohomology ([VdB]).
Corollary 8.3.2. Let $A_h$ be a $k$-algebra satisfying the hypothesis of theorem 5.0.7. Assume moreover that $A_0^0 = A_0 \otimes A_0^{op}$ is noetherian and that the $A_h$-module $A_h$ is finite projective dimension. Let $M$ be an $A_h$-module. One has
\[ HH^i(M) \simeq HH_{d_{A_h}}(U \otimes M). \]

Proof of the corollary: The proof of the corollary is similar to that of [vdB].

First case: $M$ is a finite type $A_h$-module. Let $P^* \to A_h \to 0$ be a finite length finite type projective resolution of the $A_h$-module $A_h$ and let $Q^* \to M \to 0$ be a finite type projective resolution of the $A_h$-module $M$. As $Q^i$ and $U \otimes Q^i$ are complete, one has the following sequence of isomorphisms:
\[
HH^i_{A_h}(M) \simeq H^i \left( \text{Hom}_{A_h}(P^*, M) \right) \simeq H^i \left( \text{Hom}_{A_h}(P^*, A_h) \otimes M \right)
\]
\[
\simeq H^i (U[-d]: M) \simeq H^{i-d_{A_h}} (U \otimes Q^*) \simeq H^{i-d_{A_h}} \left( (A_h \otimes U) \otimes Q^* \right)
\]
\[
\simeq H^{i-d_{A_h}} \left( A_h \otimes (U \otimes Q^*) \right) \simeq HH_{d_{A_h}}(U \otimes M).
\]

General case: We no longer assume that $M$ is a finite type $A_h$-module. We have $M = \lim_{\to} M'$ where $M'$ runs over all finitely generated submodules of $M$.

\[
\text{Ext}_{A_h}^i(A_h, M) = \text{Ext}_{A_h}^i(A_h, \lim_{\to} M') \simeq \lim_{\to} \text{Ext}_{A_h}^i(A_h, M') \simeq \lim_{\to} \text{Tor}_{d_{A_h}}^{A_h}(A_h, U \otimes M')
\]
\[
\simeq \text{Tor}_{d_{A_h}}^{A_h}(A_h, \lim_{\to} U \otimes M') \simeq \text{Tor}_{d_{A_h}}^{A_h}(A_h, U \otimes M)
\]
where we used the fact that the functor lim is exact because the set of finitely generated submodules of $M$ is a directed set ([Ro] proposition 5.33)

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