Composably secure device-independent encryption with certified deletion

Srijita Kundu *          Ernest Tan †

November 26, 2020

Abstract

We study the task of encryption with certified deletion (ECD) introduced by Broadbent and Islam [BI19], but in a device-independent setting: we show that it is possible to achieve this task even when the honest parties do not trust their quantum devices. Moreover, we define security for the ECD task in a composable manner and show that our ECD protocol satisfies conditions that lead to composable security. Our protocol is based on device-independent quantum key distribution (DIQKD), and in particular the parallel DIQKD protocol based on the magic square non-local game, given by Jain, Miller and Shi [JMS20]. To achieve certified deletion, we use a property of the magic square game observed by Fu and Miller [FM18], namely that a two-round variant of the game can be used to certify deletion of a single random bit. In order to achieve certified deletion security for arbitrarily long messages from this property, we prove a parallel repetition theorem for two-round non-local games, which may be of independent interest.

1 Introduction

Consider the following scenario: Alice wants to send a message to Bob that is secret from any third party. She may do this by sending Bob a ciphertext which contains the message encrypted with a secret key, such that when the key is revealed to Bob he may learn the message. Now suppose after sending the ciphertext Alice decides that she does not want Bob to learn the message after all, but she cannot prevent the secret key from eventually being revealed to him. So Alice wants to encrypt the message in such a way that she can ask Bob for a deletion certificate if she changes her mind. If Bob sends a valid deletion certificate, Alice can be convinced that Bob has indeed deleted his ciphertext and cannot hereafter learn the message even if the secret key is revealed to him. In this scenario Alice is not actually forcing Bob to delete the ciphertext, but she is making sure that he cannot simultaneously convince her that he has deleted the ciphertext, and also learn the message.

An encryption scheme for the above scenario is called encryption with certified deletion (ECD) and was introduced by Broadbent and Islam [BI19]. It is easy to see that ECD cannot be achieved

*Centre for Quantum Technologies, National University of Singapore, Singapore. srijita.kundu@u.nus.edu
†Institute for Theoretical Physics, ETH Zürich, Switzerland. ernestt@ethz.ch
with a classical ciphertext: since classical information can always be copied, any deletion certificate Bob sends to Alice can only convince her that he has deleted one copy of it – he may have kept another copy to decrypt from, when he learns the key. However, quantum states cannot in general be copied, and are disturbed by measurements. So if Bob has a quantum ciphertext that he cannot copy, and needs to perform a measurement on it to produce a deletion certificate, the state may be disturbed to such an extent that it is no longer possible to recover the message from it, even with the key.

The no-cloning property and the fact that measurements disturb quantum states have been useful for various cryptographic tasks, such as quantum key distribution (QKD) [BB84] and unforgeable quantum money [Wie83]. The concept of revocable timed-release encryption — a task which has some similarities to encryption with certified deletion — was studied by Unruh [Unr14], who showed it can be achieved with quantum encodings. Another related task of tamper-evident delegated quantum storage — here Alice wants to store data that she encrypts using a short key on a remote server, so that she can retrieve it later and also detect if the server has tampered with it — was studied by van der Vecht, Coiteaux-Roy and Škorić [vCSR20]. Lütkenhaus, Marwah and Touchette [LMT20] studied a different form of delegated storage, where Alice commits to a single random bit that Bob can learn at some fixed time, or she can erase, using a temporarily trusted third party. Finally, the ECD task itself, as mentioned before, was introduced by Broadbent and Islam, who achieved it using Wiesner’s encoding scheme [Wie83].

All of the works mentioned above are in the device-dependent setting, where the honest parties trust either the quantum states that are being used in the protocol, or the measurement devices, or both. However, in general a sufficiently powerful dishonest party may make the quantum state preparation and measurement devices used in a protocol behave however they want. As it turns out, with some mild assumptions it is possible to achieve certain cryptographic tasks even in this scenario. There is a long line of works studying the device-independent security of QKD [PAB+09, AFDF+18, JMS20]. Device-independent protocols for two-party cryptographic tasks such as coin flipping [ACK+14], bit commitment [SAC+11, AMPS16] and XOR oblivious transfer [KST20] have also been shown. Fu and Miller [FM18] studied the task of a sharing between two parties a single random bit, which can be certifiably deleted, in the device-independent setting.

A desirable property of cryptographic protocols is that they should be composable, meaning that if a protocol is used as part of a larger protocol to achieve some more complex task, then security of the larger protocol should follow from the security of its constituent protocols. While it is possible to achieve composable security of various cryptographic tasks such as QKD [BOHL+05, PR14], this is in general not so easy to achieve for many examples, such as the others mentioned above.

1.1 Our contributions

Informally stated, our main contributions in this work are:

1. We define the ECD task and its security in a composable manner.

2. We give a quantum protocol for the ECD task which satisfies certain properties of correctness, completeness and soundness, even when the honest parties do not trust their own quantum devices.

3. We show how to prove that a protocol that satisfies the above properties achieves the ECD task in a compositely secure manner.
The reason we don’t combine items 2 and 3 above to make the claim that we give a protocol that achieves the ECD task in a composably secure manner is because the notion of device-independence itself has not been precisely formalized in a composable manner yet. So item 3 in the device-independent setting requires an additional conjecture that we shall soon explain (though our proof of that point is already valid in the existing formulations for the device-dependent setting at least). In contrast, our proof that our protocol satisfies the security properties in item 2 holds under standard device-independent conditions, without additional conjectures.

Our composable security definition uses the framework of abstract cryptography introduced by Maurer and Renner [MR11]. In the abstract cryptography framework, a resource is an abstract system with an interface available to each party involved, to and from which they can supply some inputs and receive some outputs. A protocol uses some resources (meaning it interacts with the outer interfaces of such resources) in order to construct new resources. The protocol is said to construct the new resource in a composably secure manner if it is not possible to tell the ideal resource apart from the protocol acting on the resources it uses, under certain conditions. As such, a composable security definition for a cryptographic task would be the description of a reasonable (in the sense of being potentially achievable by actual protocols) ideal functionality or resource corresponding to that task, and a composable security proof for a protocol for this task would show that the constructed resource and ideal resource are indistinguishable.

We describe the notion of a device-independence resource in the abstract cryptography framework as a resource which supplies some black boxes representing quantum states to the honest parties, and the honest parties may press some buttons on these boxes to perform certain measurements. However, the resource allows the boxes themselves to be chosen by a dishonest third party Eve, and they may implement whatever states and measurements Eve wants.

Strictly speaking, to avoid the memory attack in the device-independent setting described in [BCK13], some additional constraints need to be placed on the registers that the measurements act on. Namely, one has to impose the condition that the measurements cannot access any registers storing private information from previous (potentially unrelated) protocols. Such a condition is implicitly imposed, albeit not always obvious, in the standard (device-dependent) framework for abstract cryptography [MR11, PR14], where the measurements are assumed to be fully characterized. However, the question of precisely formalizing this condition in the device-independent setting has not been completely resolved, and is currently a topic of active research. For the purposes of this work, we consider the technical treatment of this subject to be beyond our scope, and for ease of presentation we shall proceed under the assumption that it will be possible to find an appropriate such formulation in the device-independent setting. That is, we shall assume the following conjecture and prove our main theorem assuming the quantum boxes satisfy the conditions in it.

Conjecture 1. There is a set of conditions that can be imposed on quantum boxes, regarding their behaviour in potential past and future protocols, that leads to a compositely secure device-independent model in the abstract cryptography framework. In particular, there should be a constraint placed on the registers accessible to the parties’ boxes, that prevents the boxes from releasing private information from past protocols.

The abstract cryptography framework also lets us formalize the notion of Alice not being able to prevent the secret key from leaking to Bob in the ECD setting. We model this as follows: Alice has access to a trusted temporarily private randomness source — meaning it supplies random variables with any requested distribution, but it will make public any randomness used by Alice for encryption after some fixed time. We constrain Alice to have no private source of randomness apart from
this, thereby formalizing the notion that all randomness she uses is eventually leaked. Overall, our protocol constructs the ECD resource using only the above, untrusted quantum boxes, and an *authenticated classical channel* — which are all fairly weak.

Aside from device-independence and composability, our ECD security definition and protocol construction has some advantages over the ones in [BI19]:

- We consider the possibility of Bob being honest and consider security against a third party eavesdropper Eve when this is the case. This helps motivate the encryption aspect of the ECD task: if Alice did not need to conceal the message from Eve, she could have waited until she was sure whether she trusts Bob or not, and then sent the message as plaintext. Security against Eve is not considered in [BI19], and indeed in their protocol Eve may be able to learn the message whenever Bob does.

- We let all the randomness used by Alice for encryption be revealed to Bob (and Eve). [BI19] instead consider randomness in the form of a decryption key, which Bob can use to decrypt, and an auxiliary key, which Alice can use to check the validity of Bob’s deletion certificate, and they only have certified deletion security if the decryption key but not the auxiliary key is revealed to Bob after Alice has received a valid deletion certificate. In contrast, we do not consider the randomness used by Alice as two separate categories, and prove security when all of the randomness is eventually made public. We do note however that we still require that the outputs of the quantum devices used in the protocol are not made public.

### 1.2 Our techniques

#### 1.2.1 Constructing the DI ECD protocol

All device-independent security proofs are based on non-local games. One approach towards constructing such proofs is to use the property known as self-testing or rigidity displayed by certain non-local games. Specifically, suppose we play a non-local game with boxes implementing some unknown state and measurements, and in fact even the dimension of the systems are unspecified. If these state and measurements regardless achieve a winning probability for the game that is close to its optimal winning probability, then self-testing tells us that the state and measurements are close to the ideal state and measurements for that game, up to trivial isometries. For DIQKD, this means in particular that the measurement outputs of the devices given the inputs are random, i.e., they cannot be predicted by a third party even if they have access to the inputs used. This lets us use the outputs of the devices to produce a secret key.

**Parallel DIQKD protocol.** We make use of the parallel DIQKD protocol given by Jain, Miller and Shi [JMS20], and its subsequent simplification given by Vidick [Vid17], based on the magic square non-local game. In the magic square game, henceforth deonoted by MS, Alice and Bob respectively receive trits $x$ and $y$, and they are required to output 3-bit strings $a$ and $b$, which respectively have even parity and odd parity, and satisfy $a[y] = b[x]$. The classical winning probability of MS is 8/9, whereas the quantum winning probability is 1. The [JMS20] protocol works as follows: Alice and Bob have boxes which can play $l$ many instances of MS. Using trusted private randomness, Alice and Bob generate i.i.d. inputs $x, y$ for each of their boxes and obtain outputs $a, b$ (which are not necessarily i.i.d.). The inputs $x, y$ are then publicly communicated. Alice and Bob select a small subset of instances on which to communicate their outputs and test if the MS
winning condition is satisfied (up to some error tolerance) on those instances. If this test passes, then they go ahead and select their common bits \( a[y] = b[x] \) from all the instances — they can do this since Alice has \( a \), Bob has \( b \), and they both have \( x, y \) — as their raw secret key (some privacy amplification of the raw key is required in order to get the final key). Otherwise, the protocol aborts.

If the MS winning condition is satisfied on the test instances with high probability, then self-testing says that the states and measurements are close to the ideal ones for MS; but this property is not directly used in the security proof. In the version of the security proof given by [Vid17], instead a guessing game variant MSE of MS, involving three players Alice, Bob and Eve, is considered. MSE is the same as MS on Alice and Bob’s parts, and additionally, Eve also gets Alice and Bob’s inputs and has to guess Alice and Bob’s common bit. It can be shown that MSE cannot be won with probability 1 by all three players, and in particular if Alice and Bob’s winning condition is satisfied, then Eve cannot guess their common bit with high probability. Now making use of a parallel repetition theorem for the MSE game, which first requires a small transformation called anchoring in order to make the parallel repetition proof work, we can say that Eve’s guessing probability for the shared bit in \( l \) many instances of MSE, is exponentially small in \( l \). Since Alice and Bob’s winning condition is satisfied on a random subset of instances, we can say it is satisfied on all instances with high probability by making use of a sampling lemma. Hence Eve’s guessing probability for the raw key is exponentially small in \( l \). Now using the operational interpretation of min-entropy, this means that the min-entropy of the raw key conditioned on Eve’s quantum system is linear in \( l \), and we can use privacy amplification to get a final key that looks almost uniformly random to Eve.

**Using DIQKD for ECD.** It is easy to see how to use DIQKD to achieve the encryption aspect of ECD — Alice and Bob can perform the DIQKD protocol to share a secret key, and then Alice can encrypt the message by one-time padding it with the key, and send it to Bob. This certainly achieves security against Eve if Bob is honest. However, in the ECD scenario, Bob may not be honest, and hence the QKD security proof may not apply, since it requires him to honestly follow the protocol. Moreover, it is not clear how to achieve the certified deletion aspect of ECD this way. Instead, we do the following for our ECD protocol: we make Alice obtain the inputs \( x, y \) for MS from the trusted temporarily private randomness source, and obtain the raw key by herself using her boxes and the inputs, but she does not reveal the inputs to Bob (who hence does not have the raw key). She then one-time pads the message with the final key obtained from the raw key and sends the resulting ciphertext to Bob. Bob cannot decrypt the message at this time since he does not have the raw key, but he can get the key from his boxes and decrypt as soon as the temporarily private randomness source reveals \( x, y \) to him. Hence in order to achieve certified deletion security, Alice needs to make Bob do some operation on his boxes which destroys his ability to learn the raw key even if he gets \( x, y \).

We also note that for technical reasons, Alice actually needs to one-time pad the message with an extra uniformly random string \( u \) that she gets from the randomness source, in addition to the final key. This makes no difference to Bob’s ability to decrypt the message when all the randomness is revealed by the source or in certified deletion (since \( u \) is revealed at the end), but it does potentially make a difference at intermediate stages in the protocol. Such an extra one-time pad is also used by [BI19] in their protocol.

**Achieving certified deletion security.** Fu and Miller [FM18] made the following observation about the magic square game: suppose Alice does the measurement corresponding to \( x \) and Bob does the measurement corresponding to \( y' \) on the MS shared state, then if Bob is later given \( x \) he
can guess \( a[y'] \) perfectly as \( b[x] \) from his output. But if Bob has indeed performed the \( y' \) measurement, then he cannot guess the value of \( a[y] \) for some \( y \neq y' \), even given \( x \). In fact this property holds in a device-independent manner, i.e., if Alice and Bob have boxes implementing some unknown state and measurements which are compatible with MS, and Alice and Bob input \( x \) and \( y' \) into their boxes and get outputs that satisfy the MS winning condition with probability close to 1, then Bob cannot later perfectly guess \( a[y] \) given \( x, y' \). Now consider a 2-round variant of MS, which we shall call MSB: in the first round, Alice and Bob are given \( x, y' \) and are required to output \( a, b' \) that satisfy the MS winning condition; in the second round, Bob is given \( x, y \) such that \( y \neq y' \) and he is required to produce a bit equal to \( a[y] \). We note that Bob can use his first round input, his first round measurement outcome, and his half of the post-measured shared state in order to produce the second round output. The [FM18] observation implies that the winning probability of MSB is less than 1.

Using the same anchoring trick as in MSE, we can prove a parallel repetition theorem for the 2-round MSB game. Now to achieve certified deletion, Alice gets i.i.d. \( y' \neq y \) for all \( l \) instances from the randomness source, and if she wants Bob to delete his ciphertext, she sends Bob \( y' \) and asks for \( b' \) as a deletion certificate. If the \( b' \) sent by Bob satisfies \( a[y'] = b'[x] \) (up to some error tolerance) then Alice accepts his deletion certificate. Due to the parallel repetition theorem for MSB, if Bob’s deletion certificate has been accepted, then his guessing probability for \( a[y] \), i.e., the raw key, given \( x, y \), is exponentially small in \( l \). Due to privacy amplification, the final key looks uniformly random to Bob, and thus the message is secret from him. We note that certified deletion security should be against Bob and Eve combined rather than just Bob, as a dishonest Bob could collude with Eve in order to try and guess the message. This is fine for our security proof approach, as we can consider Bob and Eve combined as a single party for the purposes of the MSB game.

**Remark 1.** Most security proofs for DIQKD work in the sequential rather than parallel setting. In the sequential setting, Alice and Bob provide inputs to and get outputs from each instance of their boxes sequentially, which limits the kinds of correlations that are possible between the whole string of their inputs and outputs. While this is easy to justify for DIQKD when Alice and Bob are both honest, justification is harder for the ECD scenario where Bob may be dishonest and need not use his boxes sequentially, so that more general correlations between his inputs and outputs are possible. Hence a parallel rather than sequential security proof is crucial for us.

### 1.2.2 Proving composable security

To prove composable security for our protocol, we need to show that a *distinguisher* cannot tell the protocol apart from the ideal ECD functionality, when it is constrained to interact with the ideal functionality via a *simulator* acting on the dishonest parties’ interfaces. That is, for any possible behaviour of the dishonest parties in the real protocol, we need to construct a simulator such that the above is true. This needs to be done for all possible combinations of honest/dishonest parties involved, and here we only describe the idea for the case when Bob is dishonest and the dishonest third party Eve is present.

Our simulator construction is inspired by the composable security proofs of QKD [BOHL+05, PR14]: it internally simulates the real protocol using whatever outputs it gets from the ideal functionality, so that the distinguisher is able to get states on the dishonest parties’ side similar to what it would have gotten in the real protocol. However, the ideal functionality is only supposed to reveal the message \( m \) to Bob at the end (if either Alice did not ask for a deletion certificate, or
Alice asked for a deletion certificate and Bob did not produce a valid one), and since the simulator needs to simulate the real protocol before this time, it has to instead release a dummy ciphertext that does not depend on the message. Hence we require that the states on the dishonest parties’ side corresponding to \( m \) and the dummy ciphertext in the protocol be indistinguishable, if the message has not been revealed. This is related to the security notions of ciphertext indistinguishability and certified deletion security considered in [BI19]. But these properties hold only as long as the protocol does not actually reveal \( m \). If \( m \) is actually revealed at the end, the simulator needs to fool the distinguisher into believing it originally released the ciphertext corresponding to \( m \). This is where the extra one-time pad \( u \) we use comes in handy: the simulator can edit the value on the one-time pad register to a value compatible with the true message \( m \).

Overall, our security proof is fairly “modular”: our simulator construction for dishonest Bob and Eve works for any protocol in which the extra \( u \) OTP is used and which satisfies the ciphertext indistinguishability and certified deletion security properties (jointly called soundness). For other combinations of honest/dishonest parties, the proof works for any protocol that satisfies notions of completeness and correctness, even for devices with some small noise. Completeness here means that if all parties are honest then the protocol aborts with small probability, and Bob’s deletion certificate is accepted by Alice with high probability; correctness means that an honest Bob can recover the correct message from the quantum ciphertext with high probability.

### 1.2.3 Proving parallel repetition for 2-round games

As far as we are aware, our proof of the parallel repetition theorem for the MSB game is the first parallel repetition result for 2-round games, which may be of independent interest. First we clarify what we mean by a 2-round game: in the literature, boxes that play multiple instances of a game, whether sequentially or in parallel, are sometimes referred to as multi-round boxes, and certainly the nomenclature makes sense in the sequential setting. However, the two rounds for us are not two instances of the same game — they both constitute a single game and in particular, the outputs of the second round are required to satisfy a winning condition that depend on the inputs and outputs of the first round. Alice and Bob share a single entangled state at the beginning of the game, and the second round outputs are gotten by performing a measurement that can depend on the first round inputs and outputs in addition to the second round inputs, on the post-measured state from the first round.

We actually prove a parallel repetition theorem for a wider class of 2-round games than just the anchored MSB game, what we call product-anchored games. This captures elements of both product games and anchored games, whose parallel repetition has been studied for 1-round games [JY14, BVY15, BVY17, JK20] (although we consider only a specific form of anchoring which is true of the MSB game — anchored distributions can be more general), and our proof is inspired by techniques from proving parallel repetition for both product and anchored 1-round games. We call a 2-round game product-anchored if the first round inputs \( x, y \) are from a product distribution, and in the second round, only Bob gets an input \( z \) which takes a special value \( \bot \) with constant probability such that the distribution of \( x, y \) conditioned on \( z = \bot \) is the same as their marginal distribution, and otherwise \( z = (x, y') \) (where \( y' \) may be arbitrarily correlated with \( x, y \)). The first and second round outputs are \( (a, b) \) and \( b' \) respectively.\(^1\)

\(^1\)In this notation we switch around the roles of \( y, y', b, b' \) as compared to our definition of MSB. We do this in order to make our notation more compatible with standard parallel repetition theorems. As this definition refers to a wider class of games than just MSB, we hope this will not cause any confusion.
We use the information theoretic framework for parallel repetition established by [Raz95, Hol07]: we consider a strategy $S$ for the $l$ instances of the game $G$, condition on the event $E$ of the winning condition being satisfied on some $C \subseteq [l]$ instances, and show that if $\Pr[E]$ is not already small, then we can find another coordinate in $i \in \hat{C}$ where the winning probability conditioned on $E$ is bounded away from 1. For 1-round games (where there is no $z_i$), this is done in the following way: Alice and Bob’s overall state in $S$ conditioned on $E$ is considered; this state depends on Alice and Bob’s inputs — suppose it is $|\phi\rangle_{x_i,y_i}$ when Alice and Bob’s inputs in the $i$-th coordinate are $(x_i,y_i)$. It is then argued that there exists some coordinate $i$ and unitaries $\{U_{x_i}\}_{x_i}, \{V_{y_i}\}_{y_i}$ acting on Alice and Bob’s registers respectively, such that $U_{x_i} \otimes V_{y_i}$ takes some shared initial state (in the product case, the state $|\phi\rangle$, which is the superposition over $x_i$ and $y_i$ according to their respective distributions, of $|\phi\rangle_{x_i,y_i}$) close to $|\phi\rangle_{x_i,y_i}$. Hence, unless the winning probability in the $i$-th coordinate is bounded away from 1, Alice and Bob can play a single instance of $G$ by sharing this initial state, performing $U_{x_i}, V_{y_i}$ on it on inputs $(x_i,y_i)$, and giving the measurement outcome corresponding to the $i$-th coordinate on the resulting state; the winning probability of this strategy would then be higher than the optimal winning probability of $G$ — a contradiction.

For 2-round games, the state conditioned on success depends on all three inputs $x_i,y_i,z_i$, and Alice and Bob obviously cannot perform unitaries $U_{x_i}$ and $V_{y_i,z_i}$ in order to produce their first round outputs, since Bob has not received $z_i$ yet. However, we observe that Alice and Bob don’t actually need the full $|\phi\rangle_{x_i,y_i,z_i}$ state in order to produce their first round outputs — they only need a state whose $A_iB_i$ registers, containing their first round outputs, are close to those of $|\phi\rangle_{x_i,y_i,z_i}$. We observe that $|\phi\rangle_{x_i,y_i,z_i}$ is indeed such a state. In fact, in the unconditioned state, given $x_i,y_i$, all of Alice’s registers as well as all of $B_i$ are independent of $z_i$, as the second round unitary depending on $z_i$ does not act on these registers (the second round unitary may use $B_i$ as a control register, but that does not affect the reduced state of $B_i$). Conditioning on the high probability event $E$ does not disturb the state too much, and by chain rule of mutual information, we can argue that there exists an $i$ such that Alice’s registers and $B_i$ in $\varphi_{x_i,y_i,z_i}$ are close to those in $\varphi_{x_i,y_i}$ (i.e., averaged over $z_i$). Since $z_i = \perp$ with constant probability, this means that these registers are indeed close in $\varphi_{x_i,y_i,z_i}$ and $\varphi_{x_i,y_i,\perp}$.

Conditioned on $z_i = \perp$, the situation in the first round is identical to the product case; we can argue the same way as in the product parallel repetition proof by [JPY14] that there exist unitaries $\{U_{x_i}\}_{x_i}, \{V_{y_i}\}_{y_i}$ such that $U_{x_i} \otimes V_{y_i}$ takes $|\phi\rangle_{x_i,y_i,\perp}$ close to $|\phi\rangle_{x_i,y_i,\perp}$. Now we use the fact that Alice’s registers and $B_i$ are close in $\varphi_{x_i,y_i,z_i}$ and $\varphi_{x_i,y_i,\perp}$ once again to argue that there exist unitaries $\{W_{x_i,y_i,z_i}\}_{x_i,y_i,z_i}$ acting on Bob’s registers except $B_i$ that take $|\phi\rangle_{x_i,y_i,\perp}$ to $|\phi\rangle_{x_i,y_i,z_i}$. We notice that $W_{x_i,y_i,z_i}$ is in fact just $W_{y_i,z_i}$, because either $z_i$ contains $x_i$ or it can just be the identity, which means Bob can use $W_{y_i,z_i}$ as his second round unitary. Moreover, these unitaries commute with the measurement operator on the $A_iB_i$ registers, hence $W_{y_i,z_i}$ acting on the post-measured $|\phi\rangle_{x_i,y_i,\perp}$ also takes it to the post-measured $|\phi\rangle_{x_i,y_i,z_i}$. Thus the distribution Bob would get by measuring $B_i$ after applying $W_{y_i,z_i}$ on his post $A_iB_i$ measurement state is close to the correct distribution of $B_i$ conditioned on any values $(a_i,b_i)$ obtained in the first round measurement. This gives a strategy $S'$ for a single instance of $G$, where $U_{x_i}, V_{y_i}$ are the first round unitaries and $W_{y_i,z_i}$ is Bob’s second round unitary.

### 1.3 Organization of the paper

In Section 2 we describe in detail the ideal ECD functionality and formally state our result regarding it. In Section 3 we provide definitions and known results for the quantities used in our...
proofs. In Section 4, we describe the variants of the magic square non-local games and state the parallel repetition theorems for them that we use. In Section 5 we give our real ECD protocol and prove various intermediate results that help establish its composable security, which is done in Section 6. Finally, in Section 7 we provide the proofs for the parallel repetition theorems stated in Section 4.

2 Composable security definition for ECD

2.1 Abstract cryptography

We give a very brief overview of the abstract cryptography framework here, with more detailed or pedagogical explanations being available in [MR11, PR14, VPdR19].

In this framework, a resource is an abstract system with an interface available to each party, to and from which they can supply some inputs and receive some outputs. We also have the notion of converters which can interact with a resource to produce a new resource. A converter is an abstract system with an inner and outer interface, with the inner interface being connected to the resource interfaces, and the outer interface becoming the new interface of the resulting resource. If \( P \) is a subset of the parties and we have a converter \( \chi^P \) that connects to their interfaces in a resource \( \mathcal{F} \), we shall denote this as \( \chi^P \mathcal{F} \) or \( \mathcal{F} \chi^P \) (the ordering has no significance except for readability).

As an important basic example, a protocol is essentially a tuple \( \mathcal{P} = (\Pi^A, \Pi^B, \ldots) \) of converters, one for each party. Each converter describes how that party interacts with its interfaces in \( \mathcal{F} \), producing a new set of inputs and outputs “externally” (i.e. at the outer interface). If we have (for instance) a protocol with converters \( \Pi^A \) and \( \Pi^B \) for parties A and B, for brevity we shall use \( \Pi^{AB} \) to denote the converter obtained by attaching both the converters \( \Pi^A \) and \( \Pi^B \).

Given two resources \( \mathcal{F} \) and \( \mathcal{F}' \), a distinguisher is a system that interacts with the interfaces of these resources, and then produces a single bit \( G \) (which can be interpreted as a guess of which resource it is interacting with). For a given distinguisher, let \( P_{G|\mathcal{F}} \) be the probability distribution it produces on \( G \) when supplied with \( \mathcal{F} \), and analogously for \( \mathcal{F}' \). Its distinguishing advantage \( \lambda \) between these two resources is defined to be

\[
\lambda = \left| P_{G|\mathcal{F}}(0) - P_{G|\mathcal{F}'}(0) \right| = \frac{1}{2} \left\| P_{G|\mathcal{F}} - P_{G|\mathcal{F}'} \right\|_1.
\]

We can now discuss the security definitions in this framework. In a situation where there is some set \( Q \) of potentially dishonest parties, we model this as a tuple of resources \( (\mathcal{F}_P)_{P \subseteq Q} \), where \( \mathcal{F}_P \) denotes the resources available when parties \( P \) are dishonest (which presumably have more functionalities than when they are honest). Suppose we have such a resource tuple \( (\mathcal{F}^\text{real}_P)_{P \subseteq Q} \) describing the “real” functionalities available to the various parties, and a protocol \( \mathcal{P} \) which connects to the interfaces of \( \mathcal{F}^\text{real} \), with the (informal) goal of constructing a more idealized resource tuple \( (\mathcal{F}^\text{ideal}_P)_{P \subseteq Q} \). We shall formalize this as follows:

**Definition 1.** For a scenario in which there is some set \( Q \) of potentially dishonest parties, we say that \( \mathcal{P} \) constructs \( (\mathcal{F}^\text{ideal}_P)_{P \subseteq Q} \) from \( (\mathcal{F}^\text{real}_P)_{P \subseteq Q} \) within distance \( \lambda \) iff the following holds: for every \( P \subseteq Q \), there exists a converter \( \Sigma^P \) which connects to their interfaces, such that for every distinguisher, the distinguishing advantage between \( \Pi^P \mathcal{F}^\text{real}_P \) and \( \mathcal{F}^\text{ideal}_P \Sigma^P \) is at most \( \lambda \). The converters \( \Sigma^P \) shall be referred to as simulators.
We have stated the above definition slightly differently from [MR11], in which an individual simulator is required for each dishonest party. If necessary, one could convert a security proof satisfying Definition 1 into one satisfying the [MR11] definition by explicitly including quantum channels between the dishonest parties in \((\mathcal{F}_{\text{ideal}}^P)_{P \subseteq Q}\), which would allow for individual simulators that communicate using these quantum channels in order to effectively implement the simulator \(\Sigma^P\) in Definition 1. From the perspective of [MR11], this would basically reflect the inability of a protocol to guarantee that the dishonest parties cannot communicate with each other. For subsequent ease of describing the simulators, in this work we will follow Definition 1 as stated, instead of the exact definition in [MR11].

Definition 1 has an important operational interpretation, regarding the effects of composing protocols with each other.\(^2\) Namely, suppose we have a larger protocol that uses \((\mathcal{F}_{\text{ideal}}^P)_{P \subseteq Q}\) as a resource, and take any event that might be considered a “failure” in the larger protocol (we impose no restrictions on the nature of a failure, except that it be a well-defined event). Suppose we also have a proof that for any strategy by the dishonest parties, the probability of this failure event in the larger protocol is upper-bounded by some \(p_0\) when using \((\mathcal{F}_{\text{ideal}}^P)_{P \subseteq Q}\). In that case, one implication of Definition 1 being satisfied is that the probability of this failure event is upper-bounded by \(p_0 + \lambda\) if \((\mathcal{F}_{\text{ideal}}^P)_{P \subseteq Q}\) is replaced by the protocol \(\mathcal{P}\) applied to \((\mathcal{F}_{\text{real}}^P)_{P \subseteq Q'}\). This follows from the following observations, taking an arbitrary \(P \subseteq Q\): firstly, since the bound \(p_0\) for the functionality \(\mathcal{F}_{\text{ideal}}^P\) holds for any strategy by the dishonest parties, it must in particular hold when they apply the simulator \(\Sigma^P\), i.e. it holds if they are using \((\mathcal{F}_{\text{ideal}}^P)_{P \subseteq Q}\) instead of \((\mathcal{F}_{\text{ideal}}^P)_{P \subseteq Q}\). Secondly, since the distinguishing advantage between \(\mathcal{F}_{\text{ideal}}^P\Sigma^P\) and \(\Pi^\beta \mathcal{F}_{\text{real}}^P\) is at most \(\lambda\) when Definition 1 is satisfied, the probability of the failure event cannot differ by more than \(\lambda\) between them (otherwise the event would serve as a way to distinguish the two cases).

There is a technicality in the above operational interpretation, namely that in order for the argument to be valid, the bound \(p_0\) (for the larger protocol using \(\mathcal{F}_{\text{ideal}}^P\)) must be derived for a class of dishonest-party strategies that includes the simulator \(\Sigma^P\), in order for the bound to hold for \(\mathcal{F}_{\text{ideal}}^P\Sigma^P\) as well. This means that if a more “powerful” simulator is used in Definition 1, then the bound \(p_0\) must be proved against a more “powerful” class of strategies. In particular, for instance the simulators \(\Sigma^P\) we construct in this work assume that the dishonest parties \(P\) cooperate to some extent (when there are multiple dishonest parties), which means that to apply the above operational interpretation, the bound \(p_0\) for the larger protocol must be valid against cooperating dishonest parties. However, this is more of a consideration for the larger protocol rather than the protocol \(\mathcal{P}\) satisfying Definition 1, and we do not consider it further in this work.

### 2.2 Ideal ECD functionality

We work in a setting with three parties: Alice who is always honest, and Bob and Eve who may independently be honest or dishonest. The inputs for Alice and honest Bob into the functionality are:

(i) Message \(M \in \{0, 1\}^n\) from Alice at time \(t_2\)

(ii) Deletion decision \(D \in \{0, 1\}\) from Alice at time \(t_3\)

\(^2\)A more abstract compositability notion given by this definition is that if several protocols satisfying this definition are composed, the “error” \(\lambda\) of the resulting larger protocol can be bounded by simply by adding those of the sub-protocols [MR11].
and their outputs are:

(i) Abort decision $O \in \{\top, \bot\}$ to Alice and Bob at time $t_1$

(ii) Deletion decision $D$ to Bob at time $\tilde{t}_3$

(iii) Deletion flag $F \in \{\checkmark, \times\}$ to Alice at time $t_4$

(iv) $\tilde{M} = \begin{cases} M & \text{if } D \land F = 0 \\ 0^n & \text{if } D \land F = 1 \end{cases}$ to Bob at time $t_5$

where for the purposes of applying the AND function to the binary symbols $\{\times, \checkmark\}$, $\times$ is interpreted as 0.

The times corresponding to the inputs and outputs must satisfy $t_1 \leq t_2 \leq t_3 \leq \tilde{t}_3 \leq t_4 < t_5$. In particular, we shall call a functionality an ideal ECD functionality if it produces the above inputs and outputs at any points in time satisfying the above constraints. We have strict inequality only between $t_4$ and $t_5$ because this is necessary in any real protocol for achieving the functionality.

We now describe how the honest inputs and outputs are to be interpreted. Alice and Bob’s output $O$ is to detect interference by Eve. If $O = \bot$ then the protocol stops and no further inputs are fed in or outputs are received. Alice’s input $M$ is self-explanatory: this is the secret message that she potentially wants Bob to learn. Alice’s decision $D$ is her later decision about whether she wants Bob to learn $M$: some time after inputting the message but strictly before the time $t_5$ when the message is supposed to be revealed, Alice inputs $D = 1$ if she does not want Bob to learn $M$; otherwise she inputs $D = 0$. $D$ is directly output to Bob some time after Alice inputs it. The output $F$ to Alice is only produced if $D = 1$, and this indicates whether Bob has produced a valid deletion certificate (although the deletion certificate itself is not part of the ideal functionality). If Bob is honest then he always produces a valid certificate if Alice asks him to, and $F$ is always $\checkmark$. Finally, the output $M$ to Bob is a function of $M$, $D$ and $F$: if $D = 0$, i.e., Alice wanted him to learn the message, or $F = \times$, i.e., he did not produce a valid deletion certificate, then $\tilde{M} = M$; otherwise it is the dummy string $0^n$.

Now we come to the inputs and outputs of dishonest parties, which are the following:

(i) Abort decisions $O^B$ and $O^E \in \{\top, \bot\}$ from Bob and Eve at times $t'_1, t''_1$ respectively

(ii) Deletion decision $D \in \{0, 1\}$ to Eve at time $\tilde{t}_3$

(iii) Deletion flag $F \in \{\checkmark, \times\}$ from Bob at time $t'_4$.

The times corresponding to these inputs and outputs must satisfy the following ordering with respect to the previously specified times: $t'_1, t''_1 \leq t_1, t_3 \leq \tilde{t}_3$ and $\tilde{t}_3 \leq t'_4 \leq t_4$. We remark that we are indifferent about the relative ordering of $t'_1, t''_1$ and $\tilde{t}_3, \tilde{t}_3$.

Eve’s input $O^E$ is similar to what she has in the ideal key distribution functionality that is achieved by quantum key distribution (see e.g. [PR14]). She has the ability to interfere in a way that makes the honest parties abort the protocol. If Bob is honest then Eve’s choice of $O^E$ directly gets output to Alice and Bob as $O$ and the protocol stops if $O^E = \bot$. However, if $O^E = \top$, then the protocol continues and Eve gets nothing other than $D$ as further outputs, and in particular she is not able to learn the message. We include $D$ as an output for Eve because we cannot prevent her
from learning this in our actual protocol. Dishonest Bob also has an input $O^B$ that he can use to make the protocol abort: Alice and Bob’s output $O$ is $\bot$ if either one of Bob and Eve inputs $\bot$. We include this input for Bob because in the real protocol we cannot prevent Bob from deliberately sabotaging whatever test Alice and Bob are supposed to perform in order to detect interference from Eve, so that the output is $O = \bot$. Finally, Bob’s input $F$ indicates whether he has decided to produce a valid deletion certificate and hence lose his ability to learn the message or not, and this is directly output to Alice. Honest Bob does not have this functionality as he always deletes his ability to learn the message if Alice asks.

The final ECD$_n$ functionality, parametrized by the message length $n$, is depicted in Figure 1 in the four possible combinations of honest and dishonest Bob and Eve.

2.3 Achievability result

Before stating the result about our protocol constructing the ideal ECD functionality, we clarify what resources are used by the protocol, and what assumptions are needed on said resources.

Resources used.

(i) Untrusted boxes $(B^1_1, B^2_1)$ compatible with (several instances of) the magic square non-local game, supplied by Eve and held by Alice and Bob;\(^4\)

(ii) A trusted temporarily private randomness source $R$, which if used by Alice by time $t_1$, makes public all the randomness it supplied at some time $t_4 < t'_5 \leq t_5$.

(iii) An authenticated classical channel $C$ between Alice and Bob, which faithfully transmits all classical messages sent between them, but also supplies copies of the messages to Eve.

Assumptions about quantum boxes. We make the following standard assumptions about the boxes $(B^1, B^2)$ for device-independent settings:

(i) The boxes do not broadcast the inputs supplied to them and outputs obtained;

(ii) The measurements performed by $B^1$ and $B^2$ are in tensor product.

We remark that the first assumption here is rather necessary to do any cryptography at all, and the second can be ensured by spatially separating Alice and Bob’s boxes.

In addition we assume that the boxes are of the form $(B^1_1 \ldots B^1_l, B^2_1 \ldots B^2_l)$, where each $(B^1_i, B^2_j)$ is compatible with one instance of the magic square game, MS. Note that this gives Alice and Bob the ability to supply inputs to and get outputs to the boxes corresponding to any $S \subseteq [l]$, which are compatible with $\text{MS}^{[S]}$, i.e., $|S|$ parallel instances of MS, and later supply inputs and get outputs for any subset of $[l] \setminus S$. Further, we assume if Alice and Bob supply inputs to and get outputs for $S_1$ and then $S_2$, the output distribution is the same as they would have gotten if they had supplied inputs to $S_1 \cup S_2$ at once. This means that we allow signalling between all of Alice’s

\(^3\)Certainly from the point of view of the ECD functionality such an action by Bob seems pointless, but we cannot exclude this possibility for any composable security proof, since we do not know Bob’s motivations in some hypothetical larger protocol in which the real ECD protocol is used, and it may be useful for Bob there if the protocol outputs $\bot$.

\(^4\)This describes the resource behaviour in the cases where Eve is dishonest. In the case where only Bob is dishonest, we shall consider Bob to be the party choosing the box behaviour.
Figure 1: Ideal ECDₙ functionality in four cases. The times at which various events occur satisfy $t'_1, t''_1 \leq t_1 \leq t_2 \leq t_3 \leq i_3 \leq t'_4 \leq t_4 < t_5$. All inputs and outputs after $O$ are only provided if $O = \top$; the $F$ input and output are only provided if $D = 1$. $\bot$ and $\times$ are interpreted as 0 for the AND function.

boxes and all of Bob’s boxes, even if the inputs and outputs were obtained at different times (in particular, earlier outputs are allowed to depend on later inputs and outputs). This is somewhat more structure on the boxes than the truly parallel setting where $(B^1, B^2)$ only receive inputs and give outputs for MS⁻ⁿ, but proof techniques for parallel boxes still apply.
When Eve and Bob are honest, we assume that the boxes \((B_1 \ldots B_1') \cup (B_1' \ldots B_2')\) play MS with an i.i.d. strategy, although they may do so \(\varepsilon/2\)-noisily. That is, each box independently wins MS with probability \(1 - \varepsilon/2\) instead of 1, for some \(\varepsilon > 0\).

Finally, we shall assume that the boxes satisfy the conditions in Conjecture 1. We shall use \((B_1' \ldots B_1', B_1 \ldots B_2')\) to refer to boxes with the above properties.

**Theorem 2.** Assuming Conjecture 1, there exists a universal constant \(\varepsilon_0 \in (0, 1)\) such that for any \(\varepsilon \in (0, \varepsilon_0], \lambda \in (0, 1] \text{ and } n \in \mathbb{N}\), there is a protocol that constructs the ECD functionality depicted in Figure 1, within distance \(\lambda\), using only the resources \(R, C\) and \((B_1' \ldots B_1', B_1 \ldots B_2')\) for some \(l = l(\lambda, \varepsilon, n)\).

We reiterate here from section 1.1 that in the above theorem, Conjecture 1 is not required to show intermediate security properties (see Lemmas 24–26), but only to show that these properties lead to composable security.

**Remark 2.** The resources we use put some constraints on the timings achievable in the ECD functionality. For example, if \(R\) makes the randomness used by Alice public at time \(t_5\), then \(t_4 \text{ and } t_5\) must satisfy \(t_4 < t_5\). Similarly, the delay between \(t_3 \text{ and } t'_3\), \(t'_4\) and \(t_4\) depend on the time taken to transmit information between Alice and Bob using \(C\).

## 3 Preliminaries

### 3.1 Probability theory

We shall denote the probability distribution of a random variable \(X\) on some set \(\mathcal{X}\) by \(P_X\). For any event \(\mathcal{E}\) on \(\mathcal{X}\), the distribution of \(X\) conditioned on \(\mathcal{E}\) will be denoted by \(P_{X|\mathcal{E}}\). For joint random variables \(XY\), \(P_{X|Y}(x)\) is the conditional distribution of \(X\) given \(Y = y\); when it is clear from context which variable’s value is being conditioned on, we shall often shorten this to \(P_{X|Y}\).

We shall use \(P_{XY}P_{Z|X}\) to refer to the distribution

\[
(P_{XY}P_{Z|X})(x, y, z) = P_{XY}(x, y) \cdot P_{Z|X=x}(z).
\]

For two distributions \(P_X\) and \(P_{X'}\) on the same set \(\mathcal{X}\), the \(\ell_1\) distance between them is defined as

\[
||P_X - P_{X'}||_1 = \sum_{x \in \mathcal{X}} |P_X(x) - P_{X'}(x)|.
\]

**Fact 3.** For joint distributions \(P_{XY}\) and \(P_{X'Y'}\) on the same sets,

\[
||P_X - P_{X'}||_1 \leq ||P_{XY} - P_{X'Y'}||_1.
\]

**Fact 4.** For two distributions \(P_X\) and \(P_{X'}\) on the same set and an event \(\mathcal{E}\) on the set,

\[
|P_X(\mathcal{E}) - P_{X'}(\mathcal{E})| \leq \frac{1}{2} ||P_X - P_{X'}||_1.
\]

The following result is a consequence of the well-known Serfling bound.

**Fact 5 ([TL17]).** Let \(Z = Z_1 \ldots Z_l\) be \(l\) binary random variables with an arbitrary joint distribution, and let \(T\) be a random subset of size \(\gamma l\) for \(0 \leq \gamma \leq 1\), picked uniformly among all such subsets of \([l]\) and independently of \(Z\). Then,

\[
\text{Pr} \left[ \left( \sum_{i \in T} Z_i \geq (1 - \varepsilon)\gamma l \right) \land \left( \sum_{i \notin [l] \setminus T} Z_i < (1 - 2\varepsilon)(1 - \gamma)l \right) \right] \leq 2^{-2\varepsilon^2 \gamma l}.
\]
### 3.2 Quantum information

The $\ell_1$ distance between two quantum states $\rho$ and $\sigma$ is given by

$$\|\rho - \sigma\|_1 = \text{Tr} \sqrt{(\rho - \sigma)^\dagger (\rho - \sigma)} = \text{Tr} |\rho - \sigma|.$$  

The fidelity between two quantum states is given by

$$F(\rho, \sigma) = \|\sqrt{\rho}\sqrt{\sigma}\|_1.$$  

$\ell_1$ distance and fidelity are related in the following way.

**Fact 6** (Fuchs-van de Graaf inequality). For any pair of quantum states $\rho$ and $\sigma$,

$$2(1 - F(\rho, \sigma)) \leq \|\rho - \sigma\|_1 \leq 2 \sqrt{1 - F(\rho, \sigma)}^2.$$  

For two pure states $|\psi\rangle$ and $|\phi\rangle$, we have

$$\| |\psi\rangle\langle\psi| - |\phi\rangle\langle\phi| \|_1 = \sqrt{1 - F(|\psi\rangle\langle\psi|, |\phi\rangle\langle\phi|)^2} = \sqrt{1 - |\langle\psi, \phi\rangle|^2}.$$  

**Fact 7** (Uhlmann’s theorem). Suppose $\rho$ and $\sigma$ are mixed states on register $X$ which are purified to $|\rho\rangle$ and $|\sigma\rangle$ on registers $XY$, then it holds that

$$F(\rho, \sigma) = \max_U |\langle\rho|1_X \otimes U|\sigma\rangle|$$

where the maximization is over unitaries acting only on register $Y$. Due to the Fuchs-van de Graaf inequality, this implies that there exists a unitary $U$ such that

$$\left\| (1_X \otimes U) |\rho\rangle\langle\rho| (1_X \otimes U^\dagger) - |\sigma\rangle\langle\sigma| \right\|_1 \leq 2 \sqrt{\|\rho - \sigma\|_1}.$$  

**Fact 8.** For a quantum channel $\mathcal{E}$ and states $\rho$ and $\sigma$,

$$\|\mathcal{E}(\rho) - \mathcal{E}(\sigma)\|_1 \leq \|\rho - \sigma\|_1 \quad F(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \geq F(\rho, \sigma).$$  

The entropy of a quantum state $\rho$ on a register $Z$ is given by

$$H(\rho) = -\text{Tr}(\rho \log \rho).$$

We shall also denote this by $H(Z)_\rho$. For a state $\rho_{YZ}$ on registers $YZ$, the entropy of $Y$ conditioned on $Z$ is given by

$$H(Y|Z)_\rho = H(YZ)_\rho - H(Z)_\rho$$

where $H(Z)_\rho$ is calculated w.r.t. the reduced state $\rho_Z$. The conditional min-entropy of $Y$ given $Z$ is defined as

$$H_\infty(Y|Z)_\rho = \inf \{ \lambda : \exists \sigma_Z \text{ s.t. } \rho_{YZ} \leq 2^{-\lambda} 1_Y \otimes \sigma_Z \}.$$  

The conditional Hartley entropy of $Y$ given $Z$ is defined as

$$H_0(Y|Z)_\rho = \log \left( \sup_{\sigma_Z} \text{Tr}(\text{supp}(\rho_{YZ})(1_Y \otimes \sigma_Z)) \right).$$
where $\text{supp}(\rho_{YZ})$ is the projector on to the support of $\rho_{YZ}$. For a classical distribution $P_{YZ}$, this reduces to

$$H_0(Y|Z)_{P_{YZ}} = \log \left( \sup_z \{|y: P_{YZ}(y,z) > 0|\} \right).$$

For $0 \leq \delta \leq 2$, the $\delta$-smoothed versions of the above entropies are defined as

$$H_{\infty}^\delta(Y|Z)_\rho = \sup_{\rho^{'}} H_{\infty}(Y|Z)_{\rho'}, \quad H_{0}^\delta(Y|Z)_{P_{YZ}} = \inf_{\rho^{'}} \inf H_0(Y|Z)_{\rho'}.$$

**Fact 9.** For any state $\rho_{XYZ}$ and any $0 \leq \delta \leq 2$,

$$H_{\infty}^\delta(Y|XZ)_\rho \geq H_{\infty}^\delta(Y|Z)_\rho - \log |X|.$$

The relative entropy between two states $\rho$ and $\sigma$ of the same dimensions is given by

$$D(\rho||\sigma) = \text{Tr}(\rho \log \rho) - \text{Tr}(\rho \log \sigma).$$

**Fact 10** (Pinsker's Inequality). For any two states $\rho$ and $\sigma$, $\|\rho - \sigma\|_1 \leq \sqrt{D(\rho||\sigma)}$.

The mutual information between $Y$ and $Z$ with respect to a state $\rho$ on $YZ$ can be defined in the following equivalent ways:

$$I(Y : Z)_\rho = D(\rho_{YZ}||\rho_Y \otimes \rho_Z) = H(Y)_{\rho} - H(Y|Z)_{\rho} = H(Z)_{\rho} - H(Z|Y)_{\rho}.$$

The conditional mutual information between $Y$ and $Z$ conditioned on $X$ is defined as

$$I(Y : Z|X)_\rho = H(Y|X)_{\rho} - H(Y|XZ)_{\rho} = H(Z|X)_{\rho} - H(Z|XY)_{\rho}.$$

Mutual information can be seen to satisfy the chain rule

$$I(XY : Z)_\rho = I(X : Z)_\rho + I(Y : Z|X)_\rho.$$

A state of the form

$$\rho_{XY} = \sum_x P_X(x) |x\rangle \langle x|_X \otimes \rho_{Y|x}$$

is called a CQ (classical-quantum) state, with $X$ being the classical register and $Y$ being quantum. We shall use $X$ to refer to both the classical register and the classical random variable with the associated distribution. As in the classical case, here we are using $\rho_{Y|x}$ to denote the state of the register $Y$ conditioned on $X = x$, or in other words the state of the register $Y$ when a measurement is done on the $X$ register and the outcome is $x$. Hence $\rho_{XY|x} = |x\rangle \langle x|_X \otimes \rho_{Y|x}$. When the registers are clear from context we shall often write simply $\rho_x$.

**Fact 11.** For a CQ state $\rho_{XY}$ where $X$ is the classical register, $H_{\infty}(X|Y)_{\rho}$ is equal to the negative logarithm of the maximum probability of guessing $X$ from the quantum system $\rho_{Y|x}$, i.e.,

$$H_{\infty}(X|B)_\rho = - \log \left( \sup_{\{M_x\}_x} \sum_x P_X(x) \text{Tr}(M_x \rho_{Y|x}) \right)$$

where the maximization is over the set of POVMs with elements indexed by $x$. 

16
For CQ states, the expression for relative entropy for $\rho_{XY}$ and $\sigma_{XY}$ given by
\[
\rho_{XY} = \sum_x P_X(x) |x\rangle \langle x|_X \otimes \rho_Y|x,
\sigma_{XY} = \sum_x P_X'(x) |x\rangle \langle x|_X \otimes \sigma_Y|x,
\]
reduces to
\[
S(\rho_{XY} \parallel \sigma_{XY}) = S(P_X \parallel P_{X'}^Y) + \mathbb{E}_{P_X} S(\rho_Y|x \parallel \sigma_Y|x).
\]
Accordingly, the conditional mutual information between $Y$ and $Z$ conditioned on a classical register $X$, is simply
\[
I(Y:Z|X) = \mathbb{E}_{P_X} I(Y:Z|\rho_x).
\]

3.3 2-universal hashing

Definition 2. A family $\mathcal{H}$ of functions from $X$ to $Z$ is called a 2-universal family of hash functions iff
\[
\forall x \neq x' \quad \Pr[h(x) = h(x')] = \frac{1}{|Z|}
\]
where the probability is taken over the choice of $h$ uniformly over $\mathcal{H}$.

2-universal hash function families always exist if $|X|$ and $|Z|$ are powers of 2, i.e., bit strings of some fixed length (see e.g. [CW79]). We shall denote a family of 2-universal hash functions from $\{0,1\}^s$ to $\{0,1\}^n$ by $\mathcal{H}(s,n)$.

2-universal hash functions are used for privacy amplification in cryptography. For privacy amplification against an adversary with quantum side information, the following lemma is used.

Fact 12 (Leftover Hashing Lemma, [Ren05]). The CQ state $\rho_{\text{CKHE}}$, where $C$ is an $n$-bit classical register, $K$ is an $s$-bit classical register, and $H$ is a classical register of dimension $|\mathcal{H}(s,n)|$, is defined as
\[
\rho_{\text{CKHE}} = \sum_{k \in \{0,1\}^s} \sum_{h \in \mathcal{H}(s,n)} \frac{1}{|\mathcal{H}(s,n)|} P_k(k) |h(k),k,h\rangle \langle h(k),k,h|_{\text{CKH}} \otimes \rho_{\text{E}|k}.
\]
Then for any $\varepsilon \in [0,1)$,
\[
\left\| \rho_{\text{CHE}} - \frac{1}{2^n} \otimes \rho_{\text{HE}} \right\|_1 \leq 2^{-\frac{1}{2}(H_{\text{SE}(K)\cdot E}) - n} + 4\varepsilon.
\]

4 The magic square game & its parallel repetition

In a 2-player $k$-round game $G$, Alice and Bob share an entangled state at the beginning of the game. In the $j$-th round, they receive inputs $(x^j, y^j)$ and produce outputs $(a^j, b^j)$ respectively. They can do this by performing measurements that depend on the inputs and outputs of all rounds up to the $j$-th, on the post-measured state from the previous round. Each round has an associated

\[\text{Strictly speaking, in our applications of this concept we will need to use bitstrings of variable length (up to some upper bound $l$) as inputs to the hash functions. This can be handled simply by noting that there are } 2^{l+1} - 1 \text{ bitstrings of length less than or equal to } l. \text{ Hence there is an injective mapping from such bitstrings to bitstrings of length } l + 1, \text{ and we can then apply 2-universal hash families designed for the latter.}\]
predicate \( \mathcal{V}_j \) which is a function of all inputs and outputs up to the \( j \)-th round. Alice and Bob win the game iff

\[
\bigwedge_{j=1}^k \mathcal{V}_j(x^1 \ldots x^j, y^1 \ldots y^j, a^1 \ldots a^j, b^1 \ldots b^j) = 1.
\]

For a \( k \)-round game \( G \), let \( G^l \) denote the \( l \)-fold parallel repetition of it, and let \( G^{t/l} \) denote the following game:

- For \( j = 1 \) to \( k \), in the \( j \)-th round, Alice and Bob receive \( x^1_j, \ldots, x^l_j \) and \( y^1_j, \ldots, y^l_j \) as inputs.
- For \( j = 1 \) to \( k \), in the \( j \)-th round, Alice and Bob output \( a^1_j, \ldots, a^l_j \) and \( y^1_j, \ldots, y^l_j \).
- Alice and Bob win the game iff \((x^1_1 \ldots x^1_k, y^1_1 \ldots y^1_k, a^1_1 \ldots a^1_k, b^1_1 \ldots b^1_k)\) win \( G \) for at least \( t \) many \( i \)-s.

A parallel repetition threshold theorem gives an upper bound on the winning probability of \( G^{t/l} \) which is exponentially small in \( l \), for sufficiently high values of \( t \).

In the magic square game,

- Alice and Bob receive respective inputs \( x \in \{0, 1, 2\} \) and \( y \in \{0, 1, 2\} \) independently and uniformly at random.
- Alice outputs \( a \in \{0, 1\}^3 \) such that \( a[0] \oplus a[1] \oplus a[2] = 0 \), and Bob outputs \( b \in \{0, 1\}^3 \) such that \( b[0] \oplus b[1] \oplus b[2] = 1 \).
- Alice and Bob win the game if \( a[y] = b[x] \).

The classical value of the magic square game is \( \omega(\text{MS}) = 8/9 \), whereas the quantum value is \( \omega^*(\text{MS}) = 1 \).

We introduce two variants of the magic square game: a 3-player 1-round version where Alice and Bob’s distributions are product and are anchored w.r.t. the third player Eve’s input, and a 2-player 2-round version where Alice and Bob’s first round inputs are product and anchored w.r.t. Bob’s second round input. We shall make use of parallel repetition threshold theorems for both these kinds of games.

### 4.1 2-player 2-round MSB

MSB is defined as follows:

- In the first round, Alice and Bob receive inputs \( x \in \{0, 1, 2\} \) and \( y' \in \{0, 1, 2\}^2 \) independently and uniformly at random.
- Alice outputs \( a \in \{0, 1\}^3 \) such that \( a[0] \oplus a[1] \oplus a[2] = 0 \), and Bob outputs \( b' \in \{0, 1\}^3 \) such that \( b'[0] \oplus b'[1] \oplus b'[2] = 1 \).
- In the second round, Bob gets input \( z = \perp \) (indicating no input) with probability \( \alpha \), and otherwise \( z = (x, y) \) where \( y \) is uniformly distributed on \( \{0, 1, 2\} \setminus y' \). (Alice has no input.)
• Bob outputs $c \in \{0, 1\}$. (Alice has no output.)

• Alice and Bob win the game if $a[y'] = b'[x]$, and either Bob gets input $\perp$ or $a[y] = c$.

**Lemma 13.** There exists a constant $0 < c^B < 1$ such that $\omega^*(\text{MSB}_a) = 1 - c^B(1 - \alpha)$.

In order to prove Lemma 13, we shall make use of a result due to [FM18].

**Fact 14 ([FM18]).** Suppose Alice and Bob have a state and measurements that can win the MS game with probability $1 - \delta$. Consider any $x, y, y' \in \{0, 1, 2\}$ such that $y' \neq y$, and suppose Alice and Bob perform the aforementioned measurements for inputs $x, y'$, receiving outputs $a, b'$. Then if Bob is subsequently given $y$ and Alice’s input $x$, the probability that he can guess $a[y]$ is at most $\frac{1}{2} + 9\sqrt{\delta}$.

**Proof of Lemma 13.** Since $z = (x, y)$ with probability $1 - \alpha > 0$, it suffices to show that the probability of winning the game for $z = (x, y)$ is at most $1 - c^B$ for some $c^B > 0$. If $z = \perp$, the game being played is just the standard magic square game, which can be won with probability 1.

The probability of winning the game if $z = (x, y)$ can be written as $\Pr[(a[y'] = b'[x]) \land (a[y] = c) | z = (x, y)]$, which is upper bounded by

$$\min\{\Pr[a[y'] = b'[x] | z = (x, y)], \Pr[a[y] = c | z = (x, y)]\}.$$

Denote $\Pr[a[y'] = b'[x] | z = (x, y)] = 1 - \delta$. Then by Fact 14 we have $\Pr[a[y] = c | z = (x, y)] \leq \frac{1}{2} + 9\sqrt{\delta}$, hence the above expression is upper bounded by the maximum value of $\min\{1 - \delta, \frac{1}{2} + 9\sqrt{\delta}\}$ over all possible $\delta$. Since $\frac{1}{2} + 9\sqrt{\delta}$ is continuous in $\delta$ and has value less than 1 at $\delta = 0$, this maximum must be less than 1. In fact the maximum is obtained at the intersection of $1 - \delta$ and $\frac{1}{2} + 9\sqrt{\delta}$ for $\delta \in [0, 1]$, where the value is $\sim 0.997$.

**Theorem 15.** For $c^B_\alpha = c^B(1 - \alpha)$ from Lemma 13, $\delta$ such that $t = (1 - c^B_\alpha + \delta)\dagger \in ((1 - c^B_\alpha)\dagger, \dagger]$, there exists $d^B > 0$ such that

$$\omega^*(\text{MSB}^t_{\dagger}) \leq 2^{-d^B_\delta x^2}. $$

We prove more a general version of Theorem 15 in Section 7.

### 4.2 3-player MSE$_\alpha$ game

MSE$_\alpha$ is defined as follows:

• Alice and Bob receive inputs $x \in \{0, 1, 2\}$ and $y \in \{0, 1, 2\}^2$ independently and uniformly at random.

• Eve receives an input $z = \perp$ (indicating no input) with probability $\alpha$, and $z = (x, y)$ with probability $1 - \alpha$.

• Alice outputs $a \in \{0, 1\}^3$ such that $a[0] \oplus a[1] \oplus a[2] = 0$, Bob outputs $b \in \{0, 1\}^3$ such that $b[0] \oplus b[1] \oplus b[2] = 1$ and Eve outputs $c \in \{0, 1\}$.

• Alice, Bob and Eve win the game if $a[y] = b[x]$, and either Eve gets input $\perp$ or $c$ is equal to both of these.
Fact 16 ([KKM+08], and modification described in [Vid17]). There exists a constant $0 < c^E < 1$ such that $\omega^*(\text{MSE}_\alpha) = 1 - c^E(1 - \alpha)$.

**Theorem 17.** For $c^E_\alpha = c^E(1 - \alpha)$ from Fact 16, $\delta$ such that $t = (1 - c^E_\alpha + \delta)l \in ((1 - c^E_\alpha)l, l]$, there exists a constant $d^E > 0$ such that

$$
\omega^*(\text{MSE}_t^l) \leq 2 - d^E \delta \alpha^2 l.
$$

We prove a more general version of Theorem 17 in Section 7. Note that it is possible to use the result of [BVY15, BVY17] for $k$-player anchored games to get a version of Theorem 17 with worse parameters; we provide a different proof in order to improve the parameters.

## 5 ECD protocol

Our ECD protocol, which uses the resources $C, R$ and $(B_1 \ldots B_l, B_2 \ldots B_l)_E$, is given in Protocol 1. To be specific, Protocol 1 describes the steps performed by Alice and honest Bob; we have highlighted the steps that a dishonest Bob need not perform in red. We have also indicated in blue steps that occur at specific times corresponding to the ideal functionality. The parameters $l, \alpha, \gamma$ in the protocol need to satisfy conditions specified in Sec. 5.4. The function $\text{syn}$ used in the protocol is specified by Fact 19 later.

### 5.1 Notation

We shall introduce some notation that will be used in the rest of the section and the composable security proof. Firstly, note that even though for ease of presentation in the protocol, we have indicated Alice getting $R$ step by step from $R$, in reality she could have gotten it all in step 3 and here we shall consider her having done so.

Consider the following state shared by Alice Bob and Eve after step 8 of the protocol, when Alice has produced the deletion decision $O$ but has not sent it to Bob yet:

$$
\varphi_{\text{CForkA}\tilde{A}\tilde{B}B_T\tilde{E}} = |0^n\hat{x}angle|0^n\hat{x}\rangle_{\text{CF}} \otimes \sum_{\text{ork}^A} P_{\text{ork}^A}(\text{ork}^A) |\text{ork}^A\rangle|\text{ork}^A\rangle_{\text{ORK}^A} \otimes \varphi_{\tilde{A}\tilde{B}B_T\tilde{E}|\text{ork}^A}.
$$

Here the ciphertext register $C = C_1C_2$ and the flag register $F$ — which are initialized to default values — are with Alice, as is the randomness $R$ received from $R$. The answer $B_T$ Alice got from Bob is with both Alice and Eve, but for the sake of brevity we only explicitly specify the copy with Eve. $\tilde{A}, \tilde{B}, \tilde{E}$ are the quantum registers held by Alice, Bob and Eve. We shall assume $\tilde{B}\tilde{E}$ includes $\text{TY}_T$ that Bob (and Eve) got from Alice, and $\tilde{A}\tilde{B}$ include $B_T$ Alice and Bob’s copies of $B_T$. Finally, we shall assume $\tilde{B}$ contains the register $K^B$ on which Bob would obtain his raw key, if he were honest. Further states in the protocol are obtained from $\varphi$ by passing some registers from Alice to Bob (and Eve) and local operations on the registers possessed by Alice or jointly Bob and Eve.

At times $t_2$ and $t_3$ the message $M = m$ and the deletion decision $D = 0/1$ enter the protocol, and we shall specify these parameters when talking about states from these points on — although the message dependence is only on the $C$ register, so we may drop the $M$ dependence when talking about other registers. We use the following notation to denote states at various times in the protocol conditioned on various events (all the states are conditioned on outputting $\top$ at time
Protocol 1 ECD protocol

Phase 1: Encryption

1. Alice and Bob receive $B_1^i \ldots B_l^i$ and $B_1^T \ldots B_l^T$ respectively from Eve.
2. Alice gets $S \subseteq [l]$ obtained by choosing each index independently with probability $(1 - \alpha)$, and $T \subseteq S$ (or $S \subseteq [l]$) if $|S| \leq \gamma l$ uniformly at random, from $\mathcal{R}$.
3. Alice gets $x_S$, $y_S$ and $y_T$ uniformly at random such that $y_i \neq y'_i$ for each $i$, from $\mathcal{R}$.
4. Alice inputs $x_S$ into her boxes corresponding to $S$ and gets output $a_S$.
5. Alice sends $(T, y_T)$ to Bob using $C$.
6. Bob inputs $y_T$ into his boxes corresponding to $T$ and gets output $b_T$.
7. Bob sends $b_T$ to Alice using $C$.
8. Alice tests if $|S| > \gamma l$ and $a_i[y_i] = b_i[x_i]$ for at least $(1 - \epsilon)|T|$ many $i$-s in $T$.
9. if the test passes then
10. Alice sends $\top$ to Bob.
11. At time $t_1$, Alice and Bob output $\top$.
12. Alice sets $K^A = (a_i[y_i])_{i \in S}$.
13. Alice gets $h \in \mathcal{H}(l + 1, n)$, $U_1 \in \{0, 1\}^n$, $U_2 \in \{0, 1\}^{\text{syn}(K^A)}$ uniformly at random from $\mathcal{R}$.
14. At time $t_2$, Alice selects input $M \in \{0, 1\}^n$.
15. Alice sends $C = (C_1, C_2) = (M \oplus h(K^A) \oplus U_1, \text{syn}(K^A) \oplus U_2)$ to Bob using $C$.
16. else
17. Alice sends $\bot$ to Bob.
18. At time $t_1$, Alice and Bob output $\bot$ and the protocol ends.

Phase 2: Certified Deletion

19. At time $t_3$, Alice selects input $D \in \{0, 1\}$.
20. if $D = 0$ then
21. Alice sends $0$ to Bob using $C$.
22. else
23. Alice sends $(1, y'_T)$ to Bob using $C$.
24. Bob inputs $y'_T$ into his boxes corresponding to $\hat{T}$ and gets output $b'_T$.
25. Bob sends $b'_T$ to Alice using $C$.
26. Alice tests if $a_i[y'_i] = b'_i[x_i]$ for at least $(1 - 2\epsilon)|S \setminus T|$ many $i$-s in $S \setminus T$.
27. if the test passes then
28. At time $t_4$, Alice outputs $\checkmark$.
29. else
30. At time $t_4$, Alice outputs $\times$.

Phase 3: Decryption

31. $\mathcal{R}$ reveals $R = (x_S, y_S, y'_T, S, T, h, u_1, u_2)$.
32. if $D = 0$ then
33. Bob inputs $y_{S \cap T}$ into his boxes corresponding to $S \cap \hat{T}$ and gets output $b_{S \cap T}$.
34. Bob sets $K^B = (b_i[x_i])_{i \in S}$.
35. Bob uses $K^B$ and syn($K^A$) = $C_2 \oplus u_2$ to compute a guess $\tilde{K}^A$ for $K^A$.
36. At time $t_5$, Bob outputs $\tilde{M} = C_1 \oplus h(\tilde{K}^A) \oplus u_1$.
37. else
38. At time $t_5$, Bob outputs $\tilde{M} = 0^n$. 

21
$t_1$, though we only mention this in the first one, since the protocol only continues after $t_1$ under this condition):

\[
\begin{align*}
\rho_{CFORK^AABB,F}^1 &: \phi_{CFORK^AABB,F} \text{ conditioned on } O = \top \\
\rho_{CFORK^AABB,F}^2(m, 0) &: \text{ state after honest Bob’s measurement in step 33} \\
\rho_{CFORK^AABB,F}^2(m, 1) &: \text{ state at time } t_4 \text{ when Alice has produced the flag } F \\
\sigma_{CFORK^AABB,F}^2(m, 1) &: \rho_{CFORK^AABB,F}^2(m, 1) \text{ conditioned on } F = \checkmark
\end{align*}
\]

We shall use $p^\top$ to denote the probability of outputting $\top$ at time $t_1$, which is clearly the probability of $\rho$ within $\phi$. Let $p_{\checkmark}^\top$ denote the probability of Alice outputting $\checkmark$ at time $t_4$ conditioned on outputting $\top$ at time $t_1$, for message $M = m$ and $D = 1$, i.e., the probability of $\sigma(m, 1)$ within $\rho^2(m, 1)$. This probability is independent of $m$, as we shall argue in Lemma 23.

### 5.2 Completeness and correctness

**Lemma 18.** Suppose $\alpha, \gamma < \frac{1}{2}$ and $l \geq \frac{4}{(1-2\gamma^2)}$. If Bob and Eve are honest (so Alice and Bob’s boxes are $\epsilon/2$-noisy, i.e., able to win each instance of MS with probability $1 - \epsilon/2$) then

\[
\begin{align*}
p^\top &\geq \left(1 - 2^{-\gamma l/8}\right)\left(1 - 2^{-\epsilon l/8}\right) \geq 1 - 2^{-2\gamma^2 l/8} - 2^{-\epsilon l/8}.
\end{align*}
\]

Moreover, if Bob is honest, then regardless of Eve, we have

\[
\begin{align*}
p^\top(1 - p_{\checkmark}^\top) \leq 2 \cdot 2^{-2\gamma^2 l}.
\end{align*}
\]

**Proof.** Since each element of $[l]$ is included in $S$ independently with probability $(1 - \alpha)$, we see from the Chernoff bound that the probability of outputting $\bot$ due to $|S| \leq \gamma l$ is bounded by

\[
\begin{align*}
\Pr[|S| \leq \gamma l] \leq 2^{-\frac{(1-\gamma)^2 l}{2(1-\alpha)}} \leq 2^{-\frac{(1-2\gamma^2)^l}{8}},
\end{align*}
\]

for the choice of $\alpha$. Moreover, for our choice of $l$ the above quantity is at most $\frac{1}{4}$. Conditioned on $|S| > \gamma l$, the behaviour of the boxes on $T$ is independent of $S$. For any $i \in [l]$, let $W_i$ denote the indicator variable for the event $a_i[y_i] = b_i[x_i]$. Since each instance of MS is won with probability at least $1 - \epsilon/2$ by honest boxes, the probability of aborting due to $a_i[y_i] \neq b_i[x_i]$ in at least $\epsilon |T|$ boxes in $T$, i.e., $\sum_{i \in T} W_i \leq (1 - \epsilon) |T|$, is

\[
\begin{align*}
\Pr\left[\sum_{i \in T} W_i < (1 - \epsilon) |T| \bigg| |S| > \gamma l\right] \leq 2^{-\frac{\epsilon^2 |T|}{4}} = 2^{-\frac{\epsilon^2 l}{16}}.
\end{align*}
\]

Hence overall,

\[
\begin{align*}
p^\top \geq\left(1 - 2^{-\gamma^2 l/8}\right)\left(1 - 2^{-\epsilon^2 l/8}\right).
\end{align*}
\]

$p_{\checkmark}^\top$ is independent of $m$ in general (see Lemma 23 below), but it is easy to see why this is so for honest Bob: his behaviour in Phase 2 is entirely independent of $m$. To lower bound $p_{\checkmark}^\top$, let $W_i$ be defined as before, and let $W'_i$ be the indicator variable for the event that when Bob inputs $y'_i$ into his box and gets output $b'_i$, they satisfy $a_i[y'_i] = b'_i[x_i]$. As the marginal distributions of $y_i$ and $y'_i$ are exactly the same, $W'_i$ and $W_i$ are identically distributed.
Recall that $W_i$ is the same variable regardless of when the inputs are provided and the outputs obtained, so we can consider doing the $y_T'$ measurement on $\varphi$. $p_T$ is the probability of the event

$$\left( |S| > \gamma l \right) \land \left( \sum_{i \in T} W_i \geq (1 - \varepsilon)|T| \right)$$

when all the measurements are done on $\varphi$. Let $p_\varphi$ denote the probability of

$$\left( |S| > \gamma l \right) \land \left( \sum_{i \in T} W_i \geq (1 - \varepsilon)|T| \right) \land \left( \sum_{i \notin S \setminus T} W_i' \geq (1 - 2\varepsilon)|S \setminus T| \right).$$

Since the distribution of $S$ is independent of the $W_i$-s and $W_i'$-s, from Lemma 5,

$$\begin{align*}
\Pr &\left[ \left( \sum_{i \in T} W_i \geq (1 - \varepsilon)|T| \right) \land \left( \sum_{i \notin S \setminus T} W_i' \geq (1 - 2\varepsilon)|S \setminus T| \right) \right] \\
= &\Pr \left[ \left( \sum_{i \in T} W_i \geq (1 - \varepsilon)|T| \right) \land \left( \sum_{i \notin S \setminus T} W_i \leq (1 - 2\varepsilon)|S \setminus T| \right) \right] \\
\leq &2^{-2^{2^\frac{2\gamma}{l}}|S|} = 2^{-2^{2\gamma l}}.
\end{align*}$$

This gives us $p_\varphi \geq p_T - 2^{-2^{2\gamma l}} / (1 - 2^{-\frac{(1-2\varepsilon)^2}{8}l}) \geq p_T - 2 \cdot 2^{-2^{2\gamma l}}$.

Now clearly, $p_\varphi = p_T p_{\varphi | T}$, which gives us the required result. Note that here we required that upon receiving $y_T'$, Bob produces $b_T'$ by the same procedure by which he produced $b_T$ upon receiving $y_T$, which is his honest behaviour (even though the boxes themselves are untrusted), but we did not assume that the procedure actually implements anything close to the ideal MS measurements. A dishonest Bob, on the other hand, may produce $b_T'$ by some different procedure, and hence this bound does not apply to him. \hfill \square

Further analysis will be done assuming $a, \gamma, l$ satisfy the conditions of Lemma 18, though we shall not state it explicitly in each case.

The correctness of Protocol 1, i.e., the fact that Bob is able to produce the correct message if $D = 0$ and he is honest, uses the following fact.

**Fact 19** ([Ren05], Lemma 6.3.4). Suppose Alice and Bob respectively hold random variables $K^A, K^B \in \{0, 1\}^s$. Then for $0 < \delta \leq 1$, there exists a protocol in which Alice communicates a single message $\text{syn}(K^A)$ of at most $H_E^0(K^A | K^B) + \log(1/\lambda_{\text{EC}})$ bits to Bob, after which Bob can produce a guess $\tilde{K}^A$ that is equal to $K^A$ with probability at least $1 - (\delta + \lambda_{\text{EC}})/2$.

**Lemma 20.** There is a choice of $C_2 = \text{syn}(K^A)$ of length $h_2(2\varepsilon)l + \log(1/\lambda_{\text{EC}})$ bits, such that $\tilde{K}^A$ produced by honest Bob in step 35 of Phase 3 of the protocol is equal to $K^A$ with probability at least $1 - (4 \cdot 2^{-2^2\gamma l} / p_T + \lambda_{\text{EC}}/2)$, where $h_2$ is the binary entropy function.

**Proof.** Let $\rho^1$ be the state conditioned on

$$\left( |S| > \gamma l \right) \land \left( \sum_{i \in T} W_i \geq (1 - \varepsilon)|T| \right) \land \left( \sum_{i \notin S \setminus T} W_i \geq (1 - 2\varepsilon)|S \setminus T| \right).$$


when all the measurements are done on \( \varphi \), with \( W_i \) defined as in the proof of Lemma 18. \( \rho^1 \) is the state conditioned on \((|S| > \gamma l) \land (\sum_{i \in T} W_i \geq (1 - \varepsilon)|T|)\) when the measurements are done on \( \varphi \) (we have defined \( \rho^1 \) to be the state where the measurements on \( T \) are done first, then the event in \( T \) conditioned on, and then the measurements on \( T \) done; but since it does not matter in what order the measurements are done, clearly we can do all the measurements first and then condition on the event in \( T \)). By similar arguments as in the proof of Lemma 18, \( \| \rho^1 - \rho^1 \|_1 \leq 4 \cdot 2^{-2\varepsilon \gamma l} / p_T \).

In \( \rho^1 \), the \( K^B \) thus obtained differs from \( K^A \) in at most \( 2\varepsilon |S| \) many indices. The number of \(|S|\)-bit binary strings that can disagree with \( K^B \) in at most \( 2\varepsilon |S| \) places is at most \( 2^{h_2(2\varepsilon)l} \). Hence,

\[
H_0(K^A|K^B)_{\rho^1} \leq h_2(2\varepsilon)l
\]

and this implies that the \( 4 \cdot 2^{-2\varepsilon \gamma l} / p_T \)-smoothed entropy of \( \rho^1 \) is at most \( h_2(2\varepsilon)l \). Hence by Fact 19, we get the required result.6

5.3 Soundness

We prove two lower bounds for the (smoothed) min-entropy of \( K^A \) in the states \( \rho \) and \( \sigma \), conditioned on Bob and Eve’s side information and the randomness \( R \). These guarantee soundness of the protocol via the Leftover Hashing Lemma, as we shall later show.

**Lemma 21.** If Bob plays honestly, the state \( \rho^1_{C_2 RK^A \overline{BB_T E}} \) satisfies

\[
H^\rho_0(K^A|C_2 RB_T \overline{E})_{\rho^1} \geq d^E(c^E_0 - 2\varepsilon)^2\alpha^2 l - \log(1 / p_T) - 2\gamma l - h_2(2\varepsilon)l - \log(1 / \lambda_{EC}),
\]

where \( c^E_0, d^E \) are the constants from Theorem 17, and \( \delta_T = 4 \cdot 2^{-2\varepsilon \gamma l} / p_T \).

**Proof.** We follow the proof approach of [Vid17] for the protocol in [JMS20]. First we shall bound \( H^\rho_0(K^A|SXSY \overline{E})_{\rho^1} \). Consider the MSE\(^{t/}\) for \( t = (1 - 2\varepsilon)l \) game being played on the shared state between Alice, Bob and Eve. Let \( W_i \) denote the indicator variable for the event \( a_i[y_i] = b_i[x_i] \) and \( V_i \) denote the indicator variable for the event \( a_i[y_i] = c_i \) (Eve’s guess for the \( i \)-th bit). The winning condition for the \( i \)-th game is \( W_i \land V_i = 1 \).

Define \( \tilde{\rho} \) as in the proof of Lemma 20, and we have, \( \| \rho^1 - \tilde{\rho}^1 \|_1 \leq \delta_T \). Now consider the winning probability of MSE\(^{t/}\) on the state \( \tilde{\rho}^1 \). Due to Theorem 17, the winning probability of MSE\(^{t/}\) on the original state shared by Alice, Bob and Eve is \( 2^{-d^p(c^p_0 - 2\varepsilon)^p \alpha^2 l} \). Since we obtain \( \tilde{\rho}^1 \) from this original state by conditioning on an event of probability \( \tilde{\rho} \), we have

\[
\Pr_{\tilde{\rho}^1}[\text{Win MSE}\,^{t/}] \leq \frac{2^{-d^p(c^p_0 - 2\varepsilon)^p \alpha^2 l}}{\tilde{\rho}}.
\]

Note that since there is some subset of size at least \((1 - 2\varepsilon)|S|\) on which \( W_i = 1 \) for each \( i \) in \( \tilde{\rho}^1 \). MSE\(^{t/}\) is won on \( \tilde{\rho} \) if \( V_i = 1 \) on this subset. Hence,

\[
\Pr_{\tilde{\rho}^1}\left[\sum_{i \in S} V_i = |S|\right] \leq \Pr_{\tilde{\rho}}[\text{Win MSE}\,^{t/}].
\]

---

6Strictly speaking, in order to actually implement the [Ren05] protocol (Fact 19), it is not sufficient to only have the upper bound \( H_0(K^A|K^B) \leq h_2(2\varepsilon)l \). Rather, for each value of \( K^B \) Bob needs to know the set of \( K^A \) values such that \( \Pr[K^A | K^B] > 0 \), where the probability is with respect to some distribution in the \( \delta \)-ball that attains \( H_0(K^A|K^B) = h_2(2\varepsilon)l \). The proof we give here indeed characterizes this set, namely the set of \( K^A \) that differ from \( K^B \) in at most \( 2\varepsilon \) indices, so it is possible to apply that protocol.
But the probability of \( V_i = 1 \) for all \( i \in S \) is the probability that Eve is able to guess \( a_i[y_i] \) given \( x_i, y_i \) for all \( i \in S \). Hence from Fact 11,

\[
H_{\infty}(K^A | S X_S Y_S \bar{E})_{\rho^1} = \log \left( \frac{1}{\Pr_{\rho^1} [ \sum_{i \in S} V_i = |S| ]} \right) \\
\geq d^E (c_a^E - 2\varepsilon)^{2}l \log (1/p^1) \\
\geq d^E (c_a^E - 2\varepsilon)^{2}l \log (1/p^\top).
\]

Using the fact that \( \|\rho^1 - \tilde{\rho}^1\|_1 \leq \delta^\top, \) we get the smoothed result for \( \rho^1. \)

Now the other parts of \( R \) besides \( S X_S Y_S \) are \( T \) — which Eve already has in \( \bar{E}, Y'_T, H \) and \( U_1 U_2. \) But \( K^A \) is independent of \( Y'_T \) given \( S Y_S, \) and \( H \) and \( U_1 U_2 \) are independent of everything else. Hence giving Eve the extra registers in \( R \) makes no difference. \( C_2 \) and \( B_T \) are correlated with \( K^A, \) and hence giving Eve these can increase her guessing probability for it. Since \( C_2 \) is at most \( h_2(2\varepsilon)l + \log(1/\lambda_{EC}) \) bits and \( B_T \) is at most \( 2\gamma l \) bits, by Fact 9 we get the desired result. \( \square \)

For proving the next bound, we will need the following fact, which is easily proven by a summation relabelling:

**Fact 22.** Consider a CQ state \( \rho_{ZQ} \) where \( Z \) is an \( s \)-bit classical register. If we select an independent uniformly random \( U \in \{0,1\}^s \) and generate a register \( C = Z \oplus U, \) then the resulting global state,

\[
\sum_{z \in \{0,1\}^s} \sum_{u \in \{0,1\}^s} \frac{1}{2^s} P_Z(z) |z \oplus u \rangle \langle z \oplus u|_C \otimes |u \rangle_U \otimes |z \rangle_Z \otimes \rho_{Q|z}
\]

is equal to

\[
\sum_{z \in \{0,1\}^s} \sum_{u \in \{0,1\}^s} \frac{1}{2^s} P_Z(z) |u \rangle_U \otimes |z \oplus u \rangle_C \otimes |z \oplus u \rangle_U \otimes |z \rangle_Z \otimes \rho_{Q|z}.
\]

When applying this fact, we will take \( U \) to correspond to \((U_1, U_2)\) in Protocol 5, which is basically a one-time pad. Intuitively, Fact 22 expresses a symmetry between the “ciphertext” and the “padding string” when applying a one-time pad — while we usually think of the ciphertext as taking the value \( Z \oplus U \) and the padding string as taking the independent uniform value \( U, \) this fact implies that we have an exactly equivalent situation by thinking of the ciphertext as taking the value \( U \) and the padding string as taking the value \( Z \oplus U. \) We use this to prove the following lemma:

**Lemma 23.** The probability \( p_{\bar{V} | \bar{T}} \) is independent of the message \( m. \) Furthermore, letting \( R' \) denote all the registers in \( R \) except \( U_1, \) the state \( \sigma_{K^A CR' B_T \bar{E}} \) satisfies

\[
H_{\infty}(K^A | CR' B_T \bar{E})_{\sigma} \geq d^B (c_a^B - \varepsilon)^{2}(1 - \gamma)l - \log(1/p^\top p_{\bar{V} | \bar{T}}) - \gamma l - h_2(2\varepsilon)l - \log(1/\lambda_{EC}),
\]

where \( c_a^B, d^B \) are the constants from Theorem 17.

---

For the purposes of our proof, we technically do not need such an exact symmetry — it would suffice to have functions \( f \) and \( g \) such that (1) and (2) are equal when register \( C \) in (1) is set to \( f(z, u) \) and register \( U \) in (2) is set to \( g(z, u). \) However, our proof is easier to describe using the formulation shown. Furthermore, in principle our proofs also hold using a relaxed version of this statement in which (1) and (2) are only \( \lambda_{OTP} \)-close in \( \ell_1 \) distance, at the cost of increasing our composable security parameter by \( O(\lambda_{OTP}). \)
Proof. Recall that $U$ was initially generated as a uniformly random value independent of all the other registers, and that it is not revealed to Bob and Eve until the final time $t_5$. Also, by Fact 22, we know that at the point at which $C$ is generated, the global state remains the same if we swap the roles of the registers $C$ and $U$. This means that it is perfectly equivalent to instead consider the following “virtual” process: Bob and Eve generate an independent uniformly random value in the register $C$, and this is used to generate a register $U = (M \oplus h(K^A) \oplus C_1, \text{syn}(K^A) \oplus C_2)$ which is given to Alice only (until time $t_5$). We stress that this virtual process does not correspond to a physical procedure which is actually performed, but it produces exactly the same state as the original protocol, so it is valid to study it in place of the original protocol.

With this process in mind, it is clear that $p_{\sigma | T}$ is independent of $m$, since the only register that depends on $m$ at that point is always with Alice and not acted upon. We shall now prove

$$H_\infty(K^A | CR''BBT\tilde{E})_{\sigma} \geq d^B(c_\alpha^B - \epsilon)^5 \alpha^2 (1 - \gamma)l - \log(1/p_{\sigma | T}) - \gamma l,$$

where $R''$ denotes all the registers in $R$ except $U_1U_2$. From there the desired bound would follow by subtracting the number of bits in $U_2$, via Fact 9.

Under the virtual process, the register $C$ is locally generated from the joint system of Bob and Eve, without access to any of Alice’s registers. Hence we can proceed as in the proof of Lemma 21, this time by considering the game $\text{MSB}_{1/(1-\gamma)}^t$ for $t = (1 - \epsilon)(1 - \gamma)l$ on the set $T = [l] \setminus T$ between Alice and the joint system of Bob and Eve.

However, even apart from the $|S| > \gamma l$ conditioning (whose probability is included in $p_{\sigma | T}$), the input distribution in $T$ is not quite right for $\text{MSB}_{1/(1-\gamma)}^l$. In particular the distribution of the subset $S \setminus T$ on which $(X_{S \setminus T}, Y_{S \setminus T})$ is revealed to Bob in second round is not right, since the elements of this were not selected independently at random. Consider the following random process of selecting a pair $(T', S')$ of which the first is a subset of $[l]$ on which $\text{MSE}^t_{T'/|T'|}$ is played, and the second is a subset of the former on which Alice’s inputs are revealed to Bob in the game. First a subset $S$ is chosen by selecting whether to include each element of $[l]$ independently at random, then a subset $T'$ of size $\gamma l$ of the entirety of $[l]$ is chosen uniformly at random and removed, so that our final selection is $(T' = [l] \setminus T', S' = S \setminus (S \cap T'))$. Clearly, the distribution of $S'$ is correct for $\text{MS}^t_{1/(1-\gamma)}$ on $T'$. But our actual distribution of $T$ and $S \setminus T$ is gotten by conditioning this correct distribution on the event that each element of $T'$ is in $S$, which happens with probability $(1 - \alpha)^{\gamma l}$, and we shall get an additional factor due to this conditioning.

The state $\sigma$ is conditioned on the first round of $\text{MSB}_{1/(1-\gamma)}^t$ winning, as well as the initial conditioning of outputting $T$. The probability of winning both the first and second rounds on an unconditioned state with an unconditioned input distribution is at most $2^{-d^B(c_\alpha^B - \epsilon)^5 \alpha^2 (1 - \gamma)l}$. Hence,

$$H_\infty(K^A | R''BBT\tilde{E})_{\sigma} = \log \left( \frac{1}{\Pr_{\sigma} \left[ \text{Win } \text{MSB}_{1/(1-\gamma)}^t \right]} \right) \geq \log \left( \frac{2^{d^B(c_\alpha^B - \epsilon)^5 \alpha^2 (1 - \gamma)l}}{(1 - \alpha)^{\gamma l} p_{\sigma | T}} \right).$$

We get the min-entropy instead of smoothed min-entropy here because the first round is checked on the entire $T$ instead of a test subset.

\[ \Box \]

5.4 Parameter choices

Take any values of $\alpha \in (0, \frac{1}{2})$ and $\epsilon \in (0, 1)$ satisfying

$$\min \{ d^E(c_\alpha^E - 2\epsilon)^5 \alpha^2, \ d^B(c_\alpha^B - \epsilon)^5 \alpha^2 \} > h_2(2\epsilon). \quad (3)$$
We remark that more specifically, we could focus on a fixed choice of $\alpha$ (say, $\alpha = 0.4$), in which case there clearly exists $\varepsilon_0$ such that $\min\{d^E(c^E - 2\varepsilon_0)^5 \alpha^2, d^B(c^B - \varepsilon_0)^5 \lambda^2\} > h_2(2\varepsilon_0)$ (by noting the behaviour of both sides as $\varepsilon_0 \to 0$). This value is then a valid choice of $\varepsilon_0$ in Theorem 2, since (3) would then be satisfied for any $\varepsilon \in (0, \varepsilon_0]$ (using that fixed choice of $\alpha$).

Now take some $\lambda_{\text{com}}, \lambda_{\text{CI}}, \lambda_{\text{EC}} \in (0, 1]$ and some desired message length $n$, and choose $l$ large enough such that when setting

$$\gamma = \frac{1}{\varepsilon l} \max \left\{ 8 \log \frac{2}{\lambda_{\text{com}}}, \frac{1}{2} \log \frac{32}{\lambda_{\text{CI}}}, \frac{1}{2} \log \frac{8}{\lambda_{\text{EC}}} \right\},$$

(4)

the following conditions are satisfied: firstly, $\gamma < \frac{1}{2}$, secondly,

$$2^{-(1-2\gamma)^2 l/8} \leq \frac{\lambda_{\text{com}}}{2},$$

(5)

and lastly,

$$n \leq d^E(c^E - 2\varepsilon)^5 \alpha^2 l - 2\gamma l - h_2(2\varepsilon) l - \log(1/\lambda_{\text{EC}}) - 2 \log(2/\lambda_{\text{CI}}),$$

$$n \leq d^B(c^B - \varepsilon)^5 \alpha^2 (1 - \gamma) l - \gamma l - h_2(2\varepsilon) l - \log(1/\lambda_{\text{EC}}) - 2 \log(2/\lambda_{\text{CI}}).$$

(6)

The conditions $\gamma < \frac{1}{2}$ and (5) ensure that the conditions on $\gamma, l$ for Lemma 18 are satisfied. Since the choice of $\gamma$ in (4) satisfies $\gamma \to 0$ as $l \to \infty$, these conditions can all be satisfied by taking sufficiently large $l$. Furthermore, given that (3) holds, for any $n$ the conditions (6) will be satisfied at sufficiently large $l$, because all the $\gamma l$ terms are independent of $l$ for this choice of $\gamma$.

For these parameter choices, together with a choice of $\text{syn}$ satisfying Lemma 20 for the specified $\lambda_{\text{EC}}$ value, the described ECD protocol satisfies the following security properties (which hold independently of Conjecture 1).

**Lemma 24.** Given parameter choices satisfying (3)–(6), if Bob and Eve are honest, then

$$1 - p_\top \leq \lambda_{\text{com}}.$$

Also, if Bob is honest, then

$$p_\top (1 - p_{\mathcal{V} | \top}) \leq \lambda_{\text{com}}.$$

**Proof.** This follows immediately from Lemma 18, recalling that we chose parameters such that $2^{-(1-2\gamma)^2 l/8} \leq \lambda_{\text{com}}/2$ and $\gamma \geq 8 \log(2/\lambda_{\text{com}})/(\varepsilon^2 l) \geq \log(2/\lambda_{\text{com}})/(2\varepsilon^2 l)$.

**Lemma 25.** Given parameter choices satisfying (3)–(6), if Bob is honest, then conditioned on the outcome $\top$ in step 10 of the protocol, for any specific message value $M = m$ we have

$$p_\top \sum_{\bar{m} \neq m} \Pr[\bar{M} = \bar{m}] \leq \lambda_{\text{EC}}.$$

**Proof.** This follows immediately from Lemma 20, recalling that we chose parameters such that $\gamma \geq \log(8/\lambda_{\text{EC}})/(2\varepsilon^2 l)$.

In the following lemma, the first bound essentially corresponds to the notion of certified deletion security considered in [BI19], while the second bound describes ciphertext indistinguishability against Eve if Bob is honest, which is not considered in [BI19]. In our context, these are intermediate results that we use to prove composable security.
Lemma 26. Given parameter choices satisfying (3)–(6), we have for any specific message value \( M = m \):

\[
p^\top p^\top \left\| \sigma_{\text{CRBB}_T \tilde{E}}(m, 1) - \sigma_{\text{CRBB}_T \tilde{E}}(0^n, 1) \right\|_1 \leq 2 \lambda_{\text{CI}},
\]

and if Bob plays honestly,

\[
p^\top \left\| \rho_{\text{CRBB}_T \tilde{E}}^1(m, 0) - \rho_{\text{CRBB}_T \tilde{E}}^1(0^n, 0) \right\|_1 \leq 2 \lambda_{\text{CI}}.
\]

Proof. We first prove the expression for \( \rho^1 \), which is under the assumption that Bob plays honestly. Putting together Lemma 21 with the first of the bounds on \( n \), we have

\[
\frac{1}{2} \left( H_{\infty}^{\tau} \left( KA | C_2 RB_T \tilde{E} \right) \rho^1 - n \right) \geq \log(2 / \lambda_{\text{CI}}) - \frac{1}{2} \log(1 / p^\top) \geq \log(2 / \lambda_{\text{CI}}) - \log(1 / p^\top).
\]

Let \( S \) be a register storing the value of the hash \( h(KA) \). Recalling that we chose \( \gamma \geq \log(32 / \lambda_{\text{CI}}) / (2 e^2 I) \), the Leftover Hashing Lemma then implies

\[
\left\| \rho_{\text{SC}_2 RB_T \tilde{E}}^1 - \frac{1}{2^n} \otimes \rho_{\text{SC}_2 RB_T \tilde{E}}^1 \right\|_1 \leq 2^{-\log(2 / \lambda_{\text{CI}}) + \log(1 / p^\top)} + 4 \cdot 2^{\frac{1}{2} \cdot 2e^2 \gamma m} \leq \frac{\lambda_{\text{CI}}}{p^\top}.
\]

The state on these registers is independent of the value of \( M \). Now for any message value \( m \), let \( \mathcal{E}^m \) denote the map that generates the ciphertext register \( C_1 = m \oplus s \) by reading \( s \) off the register \( S \) and then tracing it out. By the properties of the one-time pad, we know that

\[
\mathcal{E}^m \left( \frac{1}{2^n} \otimes \rho_{\text{SC}_2 RB_T \tilde{E}}^1 \right) = \mathcal{E}^{0^n} \left( \frac{1}{2^n} \otimes \rho_{\text{SC}_2 RB_T \tilde{E}}^1 \right).
\]

This yields the desired result:

\[
\left\| \rho_{\text{CRBB}_T \tilde{E}}^1(m, 0) - \rho_{\text{CRBB}_T \tilde{E}}^1(0^n, 1) \right\|_1 \leq \frac{2 \lambda_{\text{CI}}}{p^\top},
\]

using Fact 8 in the last line.

For \( \sigma \), it is again easier to analyze the situation by using Fact 22 to switch to the virtual process of \( C \) being a uniformly random value and \( U \) being set to \( U = (M \oplus h(KA) \oplus C_1, \text{syn}(KA) \oplus C_2) \). We then follow a similar argument as above: by Lemma 23 and the second bound on \( n \), we have

\[
\frac{1}{2} \left( H_{\infty} \left( K_A | CR' \tilde{B} B_T \tilde{E} \right) \sigma - n \right) \geq \log(2 / \lambda_{\text{CI}}) - \frac{1}{2} \log(1 / p^\top p^\top) \geq \log(2 / \lambda_{\text{CI}}) - \log(1 / p^\top p^\top),
\]

where \( R' \) denotes all the registers in \( R \) except \( U_1 \). Defining \( \hat{\mathcal{E}}^m \) the same way as \( \mathcal{E}^m \) above, except with the output register being \( U_1 \) instead of \( C_1 \), we follow the same line of reasoning and obtain

\[
\left\| \sigma_{\text{CRBB}_T \tilde{E}}(m, 1) - \sigma_{\text{CRBB}_T \tilde{E}}(0^n, 1) \right\|_1 = \left\| \hat{\mathcal{E}}^m \left( \sigma_{\text{CR'BB}_T \tilde{E}} \right) - \hat{\mathcal{E}}^{0^n} \left( \sigma_{\text{CR'BB}_T \tilde{E}} \right) \right\|_1 \leq \frac{2 \lambda_{\text{CI}}}{p^\top p^\top}.
\]

(In fact, a tighter bound of \( \lambda_{\text{CI}} / (p^\top p^\top) \) holds here since the min-entropy bound for \( \sigma \) is not smoothed, but we will not track this detail.)
6 Composable security proof

In this section, we prove our main security result, which implies Theorem 2. The argument essentially only depends on Fact 22 and Lemmas 24–26, without requiring the details of the analysis leading up to those lemmas.

**Theorem 27.** Assuming Conjecture 1, there exists a universal constant $\epsilon_0 \in (0, 1)$ such that for any $\epsilon \in (0, \epsilon_0]$, $\lambda_{\text{com}}, \lambda_{\text{CI}}, \lambda_{\text{EC}} \in (0, 1]$ and $n \in \mathbb{N}$, there exist parameter choices for Protocol 1 such that it constructs the $ECD_n$ functionality from the resources $R$, $C$ and $(B_1^1 \ldots B_1^1, B_2^2 \ldots B_2^2)\epsilon$, within distance $\lambda = 2\lambda_{\text{com}} + \lambda_{\text{CI}} + \lambda_{\text{EC}}$.

As noted in the previous section, using the value of $\epsilon_0$ specified there allows us to choose parameters such that (3)–(6) are satisfied, in which case Lemmas 24–26 hold and we can use them in our subsequent proof. To prove composable security according to Definition 1, we need to consider the four possible combinations of honest/dishonest Bob and Eve’s behaviours, and for each case bound the distinguishing probability between the real functionality with the honest parties performing the honest protocol, versus the ideal functionality with some simulator attached to the dishonest parties’ interfaces. We shall construct appropriate simulators and argue that for a distinguisher interacting with either scenario, the states held by the distinguisher in the two scenarios differ in $\ell_1$ distance by at most $2\lambda = 4\lambda_{\text{com}} + 2\lambda_{\text{CI}} + 2\lambda_{\text{EC}}$ at all times. This implies the distinguishing advantage is bounded by $\lambda$ via Fact 8, since the process of the distinguisher producing a value on the guess register $G$ can be viewed as a channel applied to the states it holds.

Note that it suffices to consider only the points where output registers are released to the distinguisher, since by Fact 8, any operations the distinguisher performs between these points cannot increase the $\ell_1$ distance. Furthermore, we observe that for classical inputs, it is not necessary to bound the distinguishability for all possible input distributions that the distinguisher could supply — it suffices to find a bound that holds for all specific values that could be supplied as input, since by convexity of the $\ell_1$ norm, the same bound would hold when considering arbitrary distributions over those input values. In particular, for the subsequent arguments we shall assume the distinguisher supplies a specific value $m$ for the input $M$, and we shall split the analysis into different cases for the two possible values for the input $D$.

**Remark 3.** In the following proofs, we shall construct simulators by explicitly using Fact 22, but an alternative approach appears possible, which we sketch out here. First, observe that the use of the one-time pad $U$ in Protocol 1 is in fact a composable secure realization of a functionality we could call a trusted-sender channel with delay, which is defined in exactly the same way as the channel with delay in [VPdR19], except that only the recipient is potentially dishonest.\(^8\) If we now view Protocol 1 as sending the value $(M \oplus h(K^A), \text{syn}(K^A))$ through a compositely secure implementation of a channel with delay, we can safely assume that the $C$ register gives no information to the dishonest parties about Alice's outputs until the final step, which may be a helpful perspective to keep in mind when considering the proofs below. Essentially, our approach below has the simulator in the composable security proof for the trusted-sender channel with delay “built into” the argument directly, by repeated use of Fact 22.

\(^8\)Proving this would be fairly simple: simply follow the argument in the composable security proof for the one-time pad [PR14], except with appropriate changes in timing.
6.1 Dishonest Bob and Eve

As mentioned in the introduction, intuitively speaking the simulator here runs the honest protocol internally with a simulated Alice. Since it does not initially have access to the true message $m$, it first releases a dummy ciphertext $C_1 = u_1$. If it later receives $m$ from the ideal functionality, it uses $m$ to set the value on the register $U_1$ to a value that is compatible with the message $m$, before releasing it as part of $R$.

Figure 2: Simulator $\Sigma^\mathrm{BE}$ acting on the ideal functionality $F_{\mathrm{BE}}^{\mathrm{ideal}}$ with dishonest Bob and Eve. As before in the honest functionality, the $F$ input and output is provided only if $D = 1$, and the simulator only sends $y'_T$ if $D = 1$ as well. The version of $R$ the simulator releases has $U_1$ set to $\bar{m} \oplus h(k^A) \oplus u_1$.

We now describe the simulator in detail, with a schematic depiction in Figure 2. Furthermore, after each step in the description, we derive bounds on the distinguishability of the real and ideal functionalities up to that point.

- The simulator accepts the input states from Eve at the outer interface, and follows the ECD protocol with an internal simulated Alice until step 10. The inner interface of the simulator then feeds the output $O$ of that step as the values $O^B$ and $O^E$ to the ideal functionality, which

\footnote{Note that the simulator is not constrained to use the actual resources of the ECD protocol. In particular, it does not have to use a temporarily private randomness source, which is why in some of the cases we describe, the value of $R$ the simulator reveals at the outer interface does not describe the randomness used by the simulated Alice.}
releases the same value $O$ to the distinguisher (and also the simulator, though the simulator does not need it).

If $O = \perp$, the simulator stops here, apart from releasing its register $R$ at the end. Otherwise, it continues on with the ECD protocol, except that the simulated Alice instead prepares $C_1$ by generating an independent and uniformly random $u_1$ and setting $C_1 = u_1$. Furthermore, the simulator does not initialize a register $U_1$ yet — this is valid because after generating $C_1$, the register $U_1$ is not needed at any point in the ECD protocol until the last step. The simulator then proceeds until it receives $D$ from the ideal functionality at the inner interface.

By Fact 22, it is easily seen that the states produced by $\Pi^A F^{\text{real}}_{BE}$ and $F^{\text{ideal}}_{\Sigma BE}$ remain perfectly indistinguishable throughout these steps: we can equivalently consider the virtual process where $\Pi^A F^{\text{real}}_{BE}$ initializes the register $C_1$ with the independent uniform value $u_1$, exactly as $F^{\text{ideal}}_{\Sigma BE}$ did. (The distinguisher does not yet have access to $U_1$, the only register which differs between $\Pi^A F^{\text{real}}_{BE}$ and $F^{\text{ideal}}_{\Sigma BE}$ under this virtual process.)

If $D = 0$, the simulator receives the message $m$ at the inner interface and sets $U_1 = m \oplus h(k^A) \oplus u_1$, then it outputs the register $R$ at the outer interfaces.

Through this process, the distinguisher only receives $D$ followed by $R$. Since it already knows $D$, the former is trivial, and we only need to bound the distinguishability after receiving $R$. At this point, the state produced by $\Pi^A F^{\text{real}}_{BE}$ is such that $U_1$ was initialized with the independent uniform value $u_1$, and $C_1$ with the value $m \oplus h(k^A) \oplus u_1$. In comparison, the state produced by $F^{\text{ideal}}_{\Sigma BE}$ is such that $C_1$ was initialized with the independent uniform value $u_1$ and $U_1$ was initialized with the value $m \oplus h(k^A) \oplus u_1$. Applying Fact 22, the situations for $\Pi^A F^{\text{real}}_{BE}$ and $F^{\text{ideal}}_{\Sigma BE}$ are hence exactly equivalent.

If $D = 1$, the simulator releases $y'_{Fair}$ at the outer interfaces, then receives an input $b'_{Fair}$. Using this value, it runs step 26, and feeds the output $F$ of that step to the ideal functionality. Depending on the value of $F$, it performs one of the following actions:

- If $F = \xmark$, the simulator does the same as in the $D = 0$ case: it receives the message $m$ at the inner interface and sets $U_1 = m \oplus h(k^A) \oplus u_1$, then it outputs the register $R$ at the outer interfaces.

- If $F = \checkmark$, the simulator sets $U_1 = 0^n \oplus h(k^A) \oplus u_1$, then it outputs the register $R$ at the outer interfaces.

Through this process, the distinguisher receives $(D, y'_{Fair})$, supplies an input $b'_{Fair}$, then receives $F$ followed by $R$. By Fact 22, it is again easily seen that the states produced by $\Pi^A F^{\text{real}}_{BE}$ and $F^{\text{ideal}}_{\Sigma BE}$ remain perfectly indistinguishable up until $R$ is released, because as long as the distinguisher does not have access to $R$ (and hence $U_1$), we can consider the virtual process where both $\Pi^A F^{\text{real}}_{BE}$ and $F^{\text{ideal}}_{\Sigma BE}$ initialized $C_1$ with the independent uniform value $u_1$.

After $R$ is released, we note that the conditional states for the $O = \perp$ component are
perfectly indistinguishable, because in that component all the registers are independent of the message (possibly by being set to “blank” values). Also, the conditional states for \( F = \text{x} \) are perfectly indistinguishable, by the same argument as in the \( D = 0 \) case above. As for the conditional states for \( F = \text{✓} \), the states produced by \( \Pi^A \mathcal{F}_{BE}^{\text{real}} \) and \( \mathcal{F}_{BE}^{\text{ideal} \Sigma^{BE}} \) are \( \sigma_{\text{CRBBBE}}(m, 1) \) and \( \sigma_{\text{CRBBBE}}(0^n, 1) \) respectively — the former holds by definition, while the latter can be understood by noting the simulator set the values \( C_1 = u_1 \) and \( U_1 = 0^n \oplus h(k^\Lambda) \oplus u_1 \), but by Fact 22 the values on \( C_1 \) and \( U_1 \) can be swapped, which would then result in the state \( \sigma_{\text{CRBBBE}}(0^n, 1) \).

Overall, the distinguisher’s states produced by \( \Pi^A \mathcal{F}_{BE}^{\text{real}} \) and \( \mathcal{F}_{BE}^{\text{ideal} \Sigma^{BE}} \) at this point are respectively of the form

\[
(1 - p_T) |\bot\rangle \langle \bot|_O \otimes \omega_{\text{CRBBBE}} + p_T |\top\rangle \langle \top|_O \otimes \left( (1 - p_T) |\bot\rangle \langle \bot|_F \otimes \psi_{\text{CRBBBE}} + p_T |\top\rangle \langle \top|_F \otimes \sigma_{\text{CRBBBE}}(m, 1) \right),
\]

\[
(1 - p_T) |\bot\rangle \langle \bot|_O \otimes \omega_{\text{CRBBBE}} + p_T |\top\rangle \langle \top|_O \otimes \left( (1 - p_T) |\bot\rangle \langle \bot|_F \otimes \psi_{\text{CRBBBE}} + p_T |\top\rangle \langle \top|_F \otimes \sigma_{\text{CRBBBE}}(0^n, 1) \right),
\]

where \( \omega \) and \( \psi \) are appropriate conditional states for \( O = \bot \) and \( F = \text{x} \) (as argued above, these states are the same in the two scenarios), and \( p_T \) is the same in both scenarios (by Lemma 23). The only components that differ in the two scenarios are the \( \sigma \) terms, hence by Lemma 26 we see that the \( \ell_1 \) distance between the states is bounded by \( 2\lambda_{CI} \).

6.2 Dishonest Bob and honest Eve

Since Eve has no inputs to the protocol after the initial step, the argument for this case is essentially the same as the preceding section, just with \( E \) traced out.

6.3 Honest Bob and dishonest Eve

- The simulator accepts the input states from Eve at the outer interface, and follows the ECD protocol with an internal simulated Alice and Bob until step 10. The inner interface of the simulator then feeds the output \( O \) of that step as the value \( O^E \) to the ideal functionality, which releases the same value \( O \) to the distinguisher (and also the simulator, though the simulator does not need it).

\( \Pi^{AB} \mathcal{F}_{BE}^{\text{real}} \) and \( \mathcal{F}_{BE}^{\text{ideal} \Sigma^{BE}} \) are perfectly indistinguishable throughout this process, since no message has been chosen yet, and hence the states produced by \( \Pi^{AB} \mathcal{F}_{BE}^{\text{real}} \) and \( \mathcal{F}_{BE}^{\text{ideal} \Sigma^{BE}} \) are identical.

- If \( O = \bot \), the simulator stops here, apart from releasing its register \( R \) at the end. Otherwise, it continues on with the ECD protocol, except that the simulated Alice instead prepares \( C_1 \) by generating an independent and uniformly random \( u_1 \) and setting \( C_1 = u_1 \). Furthermore, the simulator does not initialize a register \( U_1 \) yet. The simulator then proceeds until it receives \( D \) from the ideal functionality at the inner interface.
By Fact 22, it is easily seen that the states produced by $\Pi^{AB}J^\text{real}_E$ and $J^\text{ideal}_E\Sigma^E$ remain perfectly indistinguishable throughout these steps: we can equivalently consider the virtual process where $\Pi^{AB}J^\text{real}_E$ initializes the register $C_1$ with the independent uniform value $u_1$, exactly as $J^\text{ideal}_E\Sigma^E$ did. (The distinguisher does not yet have access to $U_1$, the only register which differs between $\Pi^{AB}J^\text{real}_E$ and $J^\text{ideal}_E\Sigma^E$ under this virtual process.)

- If $D = 0$, the simulator’s internal versions of Alice and Bob proceed with the ECD protocol until it finishes, upon which the simulator sets $U_1 = 0^n \oplus h(k^A) \oplus u_1$ and releases $R$ at the outer interface.

Through this process, the distinguisher receives $D$ followed by $R\tilde{M}$. (There is no $F$ output for $D = 0$.) The distinguisher supplies no inputs, so we can suppose without loss of generality that it applies no operations on its systems through this process, and we only need to bound the distinguishability after $R\tilde{M}$ is released.

We note that the conditional states for the $O = \bot$ component at this point are perfectly indistinguishable, because in that component all the registers are independent of the message (possibly by being set to “blank” values). For the $O = \top$ component, the conditional states produced by $\Pi^{AB}J^\text{real}_E$ and $J^\text{ideal}_E\Sigma^E$ are $\rho_{\text{MCRB}_E}(m,0)$ and $\rho_{\text{MCRB}_E}(0^n,0)$ respectively, where the latter can be understood by again using Fact 22 to swap the values on the $C_1$ and $U_1$ registers.

Overall, the distance between the states from $\Pi^{AB}J^\text{real}_E$ and $J^\text{ideal}_E\Sigma^E$ at this point is

$$p_\top \left\| \rho_{\text{MCRB}_E}(m,0) - |m\rangle\langle m|_M \otimes \rho_{\text{MCRB}_E}(0^n,0) \right\|_1$$

$$\leq p_\top \left\| \rho_{\text{MCRB}_E}(m,0) - |m\rangle\langle m|_M \otimes \rho_{\text{MCRB}_E}(m,0) \right\|_1$$

$$+ p_\top \left\| |m\rangle\langle m|_M \otimes \rho_{\text{MCRB}_E}(m,0) - |m\rangle\langle m|_M \otimes \rho_{\text{MCRB}_E}(0^n,0) \right\|_1.$$  

By Lemma 26, the second term is bounded by $2\lambda_{\text{CI}}$. As for the first term, we have

$$p_\top \left\| \rho_{\text{MCRB}_E}(m,0) - |m\rangle\langle m|_M \otimes \rho_{\text{MCRB}_E}(m,0) \right\|_1$$

$$\leq p_\top \sum_{\tilde{M}} \Pr[\tilde{M} = \tilde{m}] \left\| |\tilde{m}\rangle\langle \tilde{m}|_M \otimes \rho_{\text{MCRB}_E}(\tilde{m},0) - |m\rangle\langle m|_M \otimes \rho_{\text{MCRB}_E}(m,0) \right\|_1$$

$$\leq p_\top \sum_{\tilde{m} \neq m} 2\Pr[\tilde{M} = \tilde{m}] \leq 2\lambda_{\text{EC}},$$

applying Lemma 25 in the last line. Adding the two bounds, we arrive at a final bound of $2\lambda_{\text{CI}} + 2\lambda_{\text{EC}}$.

- If $D = 1$, the simulator’s internal versions of Alice and Bob proceed with the ECD protocol: the simulator releases $y'_T$ and $b'_T$, then sets $U_1 = 0^n \oplus h(k^A) \oplus u_1$ and finally releases $R$ at the outer interface.

Through this process, the distinguisher receives $(D, y'_T)$ (at $t_3$), then $b'_T$ followed by $F$ (at $t_4$), and finally $R\tilde{M}$ (at $t_5$). The distinguisher supplies no inputs, so we can suppose without loss of generality that it applies no operations on its systems through this pro-
cess, and we only need to bound the distinguishability after \( R \bar{M} \) is released. Also, \( D \) is trivial since the distinguisher chose it, and so is \( \bar{M} \) since here it is always set to 0\(^n\), so we shall ignore these registers.

We note that the conditional states for the \( O = \perp \) component at this point are perfectly indistinguishable, because in that component all the registers are independent of the message (possibly by being set to “blank” values). Also, for the \( F = \checkmark \) component the conditional states produced by \( \Pi^A \mathcal{F}_{BE}^{\text{real}} \) and \( \mathcal{F}_{BE}^{\text{ideal}} \) are \( \sigma_{\text{RCY}_i'B_i'B_i\bar{E}}(m, 1) \) and \( \sigma_{\text{RCY}_i'B_i'B_i\bar{E}}(0^n, 1) \) respectively, where the latter can be understood by again using Fact 22 to swap the values on the \( C_1 \) and \( U_1 \) registers.

Overall, the states produced by \( \Pi^{AB} \mathcal{F}_{E}^{\text{real}} \) and \( \mathcal{F}_{E}^{\text{ideal}} \) at this point are respectively of the form

\[
(1 - p_\top) \left| \perp \right\rangle \left\langle \perp \right|_O \otimes \omega_{\text{RCY}_i'B_i'B_i\bar{E}}(m, 1), \\
(1 - p_\top) \left| \perp \right\rangle \left\langle \perp \right|_O \otimes \omega_{\text{RCY}_i'B_i'B_i\bar{E}}(0^n, 1),
\]

where \( \omega \) is an appropriate conditional state (as argued above, it is the same in both scenarios), and

\[
\rho_{\text{RCY}_i'B_i'B_i\bar{E}}^2(m, 1) = (1 - p_\perp) \left| \checkmark \right\rangle \left\langle \checkmark \right|_F \otimes \psi_{\text{RCY}_i'B_i'B_i\bar{E}}(m, 1) \\
+ p_\perp \left| \top \right\rangle \left\langle \top \right|_F \otimes \sigma_{\text{RCY}_i'B_i'B_i\bar{E}}(m, 1), \\
\left| \checkmark \right\rangle \left\langle \checkmark \right|_F \otimes \rho_{\text{RCY}_i'B_i'B_i\bar{E}}^2(0^n, 1) = \left| \checkmark \right\rangle \left\langle \checkmark \right|_F \otimes \left( (1 - p_\perp) \psi_{\text{RCY}_i'B_i'B_i\bar{E}}(0^n, 1) \\
+ p_\perp \sigma_{\text{RCY}_i'B_i'B_i\bar{E}}(0^n, 1) \right).
\]

where \( \psi \) are appropriate conditional states, and \( p_\perp \left| \top \right\rangle \) is the same in both scenarios (by Lemma 23). The \( \ell_1 \) distance between the two expressions is bounded by

\[
2p_\top (1 - p_\perp) + p_\top p_\perp \left\| \sigma_{\text{RCY}_i'B_i'B_i\bar{E}}(m, 1) - \sigma_{\text{RCY}_i'B_i'B_i\bar{E}}(0^n, 1) \right\| \leq 2\lambda_{\text{com}} + 2\lambda_{\text{CI}},
\]

where we have applied Lemmas 24 and 26 (for the latter we use the fact that \( \bar{B} \) can contain a copy of \( Y_i'B_i' \), and apply Fact 8).

### 6.4 Honest Bob and Eve

In this case there are no dishonest parties, so the simulator is trivial and our task is simply to bound the distinguishability between \( \Pi^{AB} \mathcal{F}_{BE}^{\text{real}} \) and \( \mathcal{F}_{E}^{\text{ideal}} \).

- We first consider the situation up until \( D \) is supplied.

Through this process, the distinguisher releases \( O \), then supplies \( M \) and \( D \). When \( O \) is released, the states produced by \( \Pi^{AB} \mathcal{F}_{BE}^{\text{real}} \) and \( \mathcal{F}_{E}^{\text{ideal}} \) are \( (1 - p_\top) \left| \perp \right\rangle \left\langle \perp \right|_O + p_\top \left| \top \right\rangle \left\langle \top \right|_O \) and \( \left| \top \right\rangle \left\langle \top \right|_O \) respectively, where \( p_\top \) is computed with respect to the honest behaviour in the ECD protocol. Then Lemma 24 implies the \( \ell_1 \) distance between them is bounded.
by

\[ 2(1 - p^\top) \leq 2\lambda_{\text{com}}. \]

After that, \( \Pi^{\text{ABE \ real}} \) and \( \mathcal{F}^{\text{ideal}} \) do not release any outputs during the steps described here, hence the distance between the states cannot increase.

- If \( D = 0 \):

The distinguisher receives \( D \) followed by \( \bar{M} \). (There is no \( F \) output for \( D = 0 \).) \( D \) is trivial since the distinguisher chose it, so we only need to bound the distinguishability after \( \bar{M} \) is released. The states produced by \( \Pi^{\text{ABE \ real}} \) and \( \mathcal{F}^{\text{ideal}} \) at this point are respectively (filling in the register \( \bar{M} \) with a “blank value” \( \phi \) in the case where \( O = \bot \) for \( \Pi^{\text{ABE \ real}} \)):

\[
(1 - p^\top) |\bot\rangle\langle \bot|_O \otimes |\phi\rangle_{\bar{M}} + p^\top |T\rangle\langle T|_O \otimes \sum_m \text{Pr}[\bar{M} = \bar{m}] |\bar{m}\rangle\langle \bar{m}|_{\bar{M}},
\]

and the \( \ell_1 \) distance between them is upper bounded by

\[
2(1 - p^\top) + p^\top \left\| |T\rangle\langle T|_O \otimes \sum_{\bar{m}} \text{Pr}[\bar{M} = \bar{m}] |\bar{m}\rangle\langle \bar{m}|_{\bar{M}} - |T\rangle\langle T|_O \otimes |m\rangle\langle m|_{\bar{M}} \right\|_1
\]

\[
\leq 2(1 - p^\top) + p^\top \sum_{\bar{m} \neq m} 2\text{Pr}[\bar{M} = \bar{m}] \leq 2\lambda_{\text{com}} + 2\lambda_{\text{EC}},
\]

applying Lemmas 24 and 25 in the last line.

- If \( D = 1 \):

The distinguisher receives \( D \), followed by \( F \) and \( \bar{M} \). \( D \) is trivial, and no inputs occur between \( F \) and \( \bar{M} \), so we only need to bound the distinguishability after \( \bar{M} \) is released. The states produced by \( \Pi^{\text{ABE \ real}} \) and \( \mathcal{F}^{\text{ideal}} \) at this point are respectively (filling in the registers \( M \) with a “blank value” \( \phi \) in the case where \( O = \bot \) for \( \Pi^{\text{ABE \ real}} \)):

\[
(1 - p^\top) |\bot\rangle\langle \bot|_O \otimes |\phi\rangle_E \otimes |\phi\rangle_{\bar{M}} + p^\top |T\rangle\langle T|_O \otimes \rho_F^2 \otimes |0^n\rangle\langle 0^n|_{\bar{M}},
\]

where

\[
\rho_F^2 = (1 - p_\chi^{|T\rangle}) \chi \chi |F\rangle + p_\chi^{|T\rangle} |\checkmark\rangle \langle \checkmark|_F. \]

The \( \ell_1 \) distance between them is upper bounded by

\[
2(1 - p^\top) + 2p^\top (1 - p_\chi^{|T\rangle}) \leq 4\lambda_{\text{com}},
\]

applying Lemma 24.
7 Parallel repetition theorems

7.1 Parallel repetition theorem for 2-round 2-player product-anchored game

**Definition 3.** A 2-round 2-player non-local game is called a product-anchored game with anchoring probability $\alpha$ iff

- Alice and Bob get $(x, y) \in X \times Y$ from a product distribution as their first round inputs.
- Alice and Bob produce $(a, b) \in A \times B$ as their first round outputs.
- Bob gets $z = \perp$ with probability $\alpha$ and $z = (x, y')$ with probability $1 - \alpha$, as his second round input, such the distribution of $(x, y)$ conditioned on $z = \perp$ is the same as the marginal distribution of $(x, y)$. (Alice has no input.)
- Bob produces $b'$ as his second round output. (Alice has no output.)
- Alice and Bob win the game iff $V(x, y, a, b)$ and $V'(x, y, z, a, b, b')$ are both satisfied.

**Theorem 28.** Let $G$ be a 2-round 2-player non-local product-anchored game with satisfying the conditions above with parameter $\alpha$. Then for $\delta > 0$ and $t = \left( \omega^*(G) + \delta \right) l$,

$$
\omega^*(G^l) = \left( 1 - (1 - \omega^*(G))^{5} \right)^{\Omega \left( \frac{\epsilon^2}{\log(|A| \cdot |B| \cdot |B'|)} \right)}
$$

$$
\omega^*(G'^{l/2}) = \left( 1 - \delta^{5} \right)^{\Omega \left( \frac{\epsilon^2}{\log(|A| \cdot |B| \cdot |B'|)} \right)}.
$$

We shall use the following results in order to prove the theorem.

**Fact 29 ([Hol07]).** Let $P_{TVU_1 \ldots U_l V} = P_T P_{U_1 | T} P_{U_2 | T} \ldots P_{U_l | T} P_{V | T U_1 \ldots U_l}$ be a probability distribution over $T \times U^l \times V$, and let $E$ be any event. Then,

$$
\sum_{i=1}^{l} \left\| P_{TVU_i V | E} - P_{TV | E} P_{U_i | T} \right\|_1 \leq \sqrt{l \left( \log(|V|) + \log \left( \frac{1}{\Pr[E]} \right) \right)}.
$$

**Fact 30 ([BVY15], Lemma 16).** Suppose $TVW$ are random variables satisfying $P_{VW}(v, w^*) = \alpha \cdot P_V(v)$ for all $v$. Then,

$$
\left\| P_{TVW} - P_{VW} P_T | V, w^* \right\|_1 \leq 2 \alpha \left\| P_{TVW} - P_{VW} P_T | V \right\|_1.
$$

**Fact 31 ([JPY14], Lemma III.1).** Suppose $\rho$ and $\sigma$ are CQ states satisfying $\rho = \epsilon \sigma + (1 - \epsilon) \sigma'$ for some other state $\sigma'$. Suppose $Z$ is a classical register of size $|Z|$ in $\sigma$ and $\sigma$ such that the distribution on $Z$ in $\sigma$ is $P_Z$, then

$$
\mathbb{E}_{P_Z} D(\sigma_z \| \rho) \leq \log(1/\epsilon) + \log |Z|.
$$

**Fact 32 (Quantum Raz’s Lemma, [BVY15]).** Let $\rho_{XY}$ and $\sigma_{XY}$ be two CQ states with $X = X_1 \ldots X_l$ being classical, and $\sigma$ being product across all registers. Then,

$$
\sum_{i=1}^{l} I(X_i : Y)\rho \leq D(\rho_{XY} \| \sigma_{XY}).
$$
Fact 33 ([JPy14], Lemma II.15). Let the following pure state be shared between Alice and Bob, with Alice holding registers $XXA$ and Bob holding registers $YB$:

$$|\varphi\rangle_{XXY\tilde{Y}AB} = \sum_{x,y} \sqrt{P_{XY}(x,y)} |xx\rangle_{XX} |yy\rangle_{YY} |\varphi\rangle_{AB|xy}.$$ 

If $I(X:Y\tilde{Y}B) \leq \epsilon$ and $I(Y:X\tilde{X}A) \leq \epsilon$, then there exist unitary operators $\{U_x\}_x$ on $X\tilde{X}A$ and $\{V_y\}_y$ $Y\tilde{Y}B$ such that

$$\mathbb{E}_{P_{XY}} \left| \langle \varphi | \varphi \rangle_{XXY\tilde{Y}AB|xy} - (U_x \otimes V_y) |\varphi\rangle_{XXY\tilde{Y}AB}(U_x^\dagger \otimes V_y^\dagger) \right|_1 \leq 8\sqrt{\epsilon} + \| P_{XY} - P_XP_Y \|_1.$$ 

Proof of Theorem 28. Consider a strategy $S$ for $l$ copies of $G$ (it may correspond to $G^l$ or $G^{l/l}$ — it doesn’t really matter): before the game starts, Alice and Bob share an entangled state on registers $A\tilde{A}BB\tilde{B}B'E^AEB$. Here $A, B, B'$ will be the registers in which the outputs are measured in the computational basis, and $\tilde{A}, \tilde{B}, \tilde{B}'$ are registers onto which the contents of $A, B, B'$ are copied — we can always assume the outputs are copied since they are classical. Alice and Bob apply unitaries based on their first round inputs $XY$ to their respective halves of this entangled state and measure in the computational basis to obtain their first round outputs. We define the following pure state to represent the inputs, outputs and other registers in the protocol at this stage:

$$|\rho\rangle_{X\tilde{Y}Z\tilde{Z}X\tilde{A}BB\tilde{B}B'E^AEB} = \sum_{x,y,z} \sqrt{P_{XYZ}(x,y,z)} |xx\rangle_{XX} |yy\rangle_{YY} |zz\rangle_{ZZ} \sum_{a,b} \sqrt{P_{AB|xy}(ab)} |aa\rangle_{A\tilde{A}} |bb\rangle_{B\tilde{B}} |\rho\rangle_{B'B'E^AEB|x_{yab}}$$

where we have used $Z$ to denote Bob’s second round input, which is either $\perp$ or $(x, y')$. We have included the $ZZ$ registers in this state even though Bob has not received the $z$ input yet; the state in the entangled registers has no dependence on $z$ however. Here $P_{AB|xy}(a, b)$ is the probability of Alice and Bob obtaining outputs $(a, b)$ on inputs $(x, y)$ in the first round.

In the actual protocol, the $AB$ registers are measured on $|\rho\rangle$, and the subsequent unitary Bob applies on the $B'B'E^B$ registers can depend on his first round output, as well as both his inputs. We represent the state of the protocol at this state by:

$$|\sigma\rangle_{X\tilde{Y}Z\tilde{Z}X\tilde{A}BB\tilde{B}B'E^AEB} = \sum_{x,y,z} \sqrt{P_{XYZ}(x,y,z)} |xx\rangle_{XX} |yy\rangle_{YY} |zz\rangle_{ZZ} \sum_{a,b} \sqrt{P_{AB|xy}(ab)} |aa\rangle_{A\tilde{A}} |bb\rangle_{B\tilde{B}} \otimes \sum_{b'} \sqrt{P_{B'|xyzab}(b')} |b'b'\rangle_{B'B'} |\sigma\rangle_{E^AEB|x_{yabb'}}.$$ 

Note that $|\sigma\rangle$ is related to $|\rho\rangle$ by a unitary on the $B'B'E^B$ registers that is controlled on the registers $YZB$, which is why the marginal distribution of $AB$ is the same in $|\rho\rangle$ and $|\sigma\rangle$.

For each $i \in [l]$, we define the correlation-breaking random variables $D_i, G_i$ as follows: $D_i$ is a uniformly random bit, and $G_i$ takes value $X_iY_i$ or $Z_i$ respectively depending on whether $D_i$ is 0 or 1. Clearly $XYZ$ are independent conditioned on $DG$. We can consider the states $|\rho\rangle$ and $|\sigma\rangle$ conditioned on $DG = dg$, and this simply means that the distribution of $XYZ$ used is conditioned on $dg$.

We shall prove the following lemma, from which, using standard techniques (see e.g. [Rao08]), the theorem follows. \hfill $\Box$
Lemma 34. Let \( W_i \) denote the indicator variable for the event that using \( S \) Alice and Bob win \( G \) in the coordinate \( i \), and \( E \) denote the event \( \prod_{i \in C} W_i = 1 \) in \( C \subseteq [ l ] \) such that \( |C| \leq l/2 \). Then there exists an \( i \in C \) such that

\[
\Pr[W_i = 1 | E] \leq \omega^*(G) + \frac{542\delta^{1/4}}{\alpha^{1/2}}
\]

where

\[
\delta = \frac{|C| \log(|A| \cdot |B| \cdot |B'|)}{l} + \log \left( \frac{1}{\Pr[E]} \right).
\]

Proof. We define the following state, which is \( |\sigma\rangle_{d g} \) conditioned on success in \( C \):

\[
|\phi\rangle_{X Y Z A B B' B'' E A E B|^d g} = \frac{1}{\sqrt{T_{d g}}} \sum_{x, y} \sqrt{P_{X Y Z A B B' B'' E A E B}|x y z\rangle_{X Y Z} |x y\rangle_{Y} |z z\rangle_{Z Z} \sum_{a, b, b': (x, y, z, c, a, b, b')} \sqrt{P_{A B B' | x y z} (a b)} |a a\rangle_{A A} |b b\rangle_{B B} \otimes (x, y, z, c, a, b, b') \varepsilon_{x y z}
\]

where \( \varepsilon_{x y z} \) is the probability of winning in \( C \) conditioned on \( D G = d g \), in \( S \). It is easy to see that \( P_{X Y Z A B B' | x y z} \) is the distribution on the registers \( X Y Z A B B' \) in \( |\phi\rangle_{d g} \) and \( \mathbb{E}_{P_{D G}^l} P_{X Y Z A B B' | x y z} \) is \( P_{X Y Z A B B' | x y z} \).

Applying Fact 29 with \( T, V \) being trivial and \( U_i = X_i Y_i Z_i \), we get,

\[
\mathbb{E}_{i \in C} \| P_{X_i Y_i Z_i | E} - P_{X_i Y_i Z_i} \|_1 \leq \frac{1}{l - |C|} \sqrt{l \cdot \log (1/ \Pr[E])} \leq \sqrt{2\delta}.
\]

(7)

In particular by Facts 3 and 4 this means, \( \mathbb{E}_{i \in C} P_{Z_i | E} (\perp) \geq \alpha - \sqrt{\delta/2} \geq \alpha/2 \) (for all \( \delta \) for which the lemma statement is nontrivial). We also note that \( P_{X_i Y_i | E, \perp} = P_{X_i Y_i | E} \). Applying Fact 29 again with \( U_i \) the same, \( T = X_i Y_i Z_i C D G \) and \( V = A_i B_i C_i B'_i \), and using \( R_i \) to denote the random variable \( X_i Y_i Z_i C_i A_i B_i C_i B'_i D_i \cdot \perp \cdot C_i \cdot \perp \cdot G \),

\[
\sqrt{2\delta} \geq \mathbb{E}_{i \in C} \| P_{X_i Y_i Z_i D_i G_i R_i | E} - P_{D_i G_i R_i | E} P_{X_i Y_i Z_i D_i G_i} \|_1 = \frac{1}{2} \mathbb{E}_{i \in C} \left( \| P_{X_i Y_i Z_i R_i | E} - P_{X_i Y_i R_i | E} P_{Z_i | X_i Y_i} \|_1 + \| P_{X_i Y_i Z_i R_i | E} - P_{Z_i R_i | E} P_{X_i Y_i | Z_i} \|_1 \right)
\]

(8)

From the first term in (8) we get,

\[
\mathbb{E}_{i \in C} \| P_{X_i Y_i Z_i R_i | E} - P_{X_i Y_i Z_i | E} P_{R_i | E, X_i Y_i} \|_1 \leq \mathbb{E}_{i \in C} \left( \| P_{X_i Y_i Z_i R_i | E} - P_{X_i Y_i R_i | E} P_{Z_i | X_i Y_i} \|_1 + \| P_{X_i Y_i Z_i | E} - P_{X_i Y_i Z_i} \|_1 \right) \leq 2 \sqrt{2\delta}.
\]

Applying Fact 30 we then have,

\[
\mathbb{E}_{i \in C} \| P_{X_i Y_i Z_i R_i | E} - P_{X_i Y_i Z_i | E} P_{R_i | E, X_i Y_i, \perp} \|_1 = \mathbb{E}_{i \in C} \| P_{X_i Y_i Z_i R_i | E} - P_{X_i Y_i R_i | E} P_{Z_i | E, X_i Y_i} \|_1 \leq \frac{8 \sqrt{2\delta}}{\alpha}.
\]

(9)

From the second term in (8) we get,

\[
2 \sqrt{2\delta} \geq \mathbb{E}_{i \in C} \frac{\alpha}{2} \| P_{X_i Y_i R_i | E, \perp} - P_{X_i Y_i | \perp} P_{R_i | E, \perp} \|_1.
\]

(10)
Now,

\[
\mathbb{E}_{i \in \mathcal{C}} \| P_{X_i Y_i Z_i | \epsilon} P_{R_i | \epsilon, X_i Y_i, \bot} - P_{X_i Y_i Z_i | \epsilon, X_i Y_i, \bot} \|_1
\]

\[
\leq \mathbb{E}_{i \in \mathcal{C}} \left( \| (P_{X_i Y_i | \epsilon} P_{R_i | \epsilon, X_i Y_i, \bot} - P_{X_i Y_i | \epsilon} P_{R_i | \epsilon, \bot}) P_{Z_i | \epsilon, X_i Y_i} \|_1 + \| (P_{X_i Y_i Z_i | \epsilon} - P_{X_i Y_i Z_i | \epsilon}) P_{R_i | \epsilon, \bot} \|_1 \right)
\]

\[
= \mathbb{E}_{i \in \mathcal{C}} \left( \| (P_{X_i Y_i | \epsilon, \bot} - P_{X_i Y_i | \epsilon} P_{R_i | \epsilon, \bot}) P_{Z_i | \epsilon, X_i Y_i} \|_1 + \| (P_{X_i Y_i Z_i | \epsilon} - P_{X_i Y_i Z_i | \epsilon}) P_{R_i | \epsilon, \bot} \|_1 \right)
\]

\[
\leq \mathbb{E}_{i \in \mathcal{C}} \left( \| P_{X_i Y_i | \epsilon, \bot} - P_{X_i Y_i | \epsilon} P_{R_i | \epsilon, \bot} \|_1 + \| (P_{X_i Y_i Z_i | \epsilon} - P_{X_i Y_i Z_i | \epsilon}) P_{R_i | \epsilon, \bot} \|_1 \right)
\]

\[
= \mathbb{E}_{i \in \mathcal{C}} \left( \| P_{X_i Y_i | \epsilon, \bot} - P_{X_i Y_i | \epsilon} P_{R_i | \epsilon, \bot} \|_1 + \| P_{X_i Y_i | \epsilon} - P_{X_i Y_i Z_i | \epsilon} P_{R_i | \epsilon, \bot} \|_1 \right)
\]

\[
\leq \frac{6\sqrt{2\delta}}{\alpha}
\]

where we have used (7) and (10) in the last line. Combining this with (9) we get,

\[
\mathbb{E}_{i \in \mathcal{C}} \| P_{X_i Y_i Z_i | \epsilon} - P_{X_i Y_i Z_i | \epsilon, \bot} \|_1 \leq \frac{14\sqrt{2\delta}}{\alpha}.
\] (11)

Finally,

\[
\mathbb{E}_{i \in \mathcal{C}} \| P_{X_i Y_i | \epsilon, \bot} - P_{X_i Y_i Z_i | \epsilon, \bot} P_{R_i | \epsilon, \bot} \|_1
\]

\[
\leq \mathbb{E}_{i \in \mathcal{C}} \left( \| P_{X_i Y_i | \epsilon, \bot} - P_{X_i Y_i | \epsilon} P_{R_i | \epsilon, \bot} \|_1 + \| P_{X_i Y_i | \epsilon} - P_{X_i Y_i Z_i | \epsilon} \|_1 \right)
\]

\[
\leq \frac{4\sqrt{2\delta}}{\alpha} + \mathbb{E}_{i \in \mathcal{C}} \left( \| (P_{X_i Y_i | \epsilon} P_{R_i | \epsilon, \bot} - P_{X_i Y_i | \epsilon} P_{R_i | \epsilon, \bot}) P_{Y_i | \bot} \|_1 + \| P_{X_i Y_i | \epsilon} \|_1 \right)
\]

\[
\leq \frac{4\sqrt{2\delta}}{\alpha} + \mathbb{E}_{i \in \mathcal{C}} \left( \| P_{X_i Y_i | \epsilon} \|_1 + \| P_{Y_i \bot} \|_1 \right)
\]

\[
\leq \frac{4\sqrt{2\delta}}{\alpha} + 2 \mathbb{E}_{i \in \mathcal{C}} \| P_{X_i Y_i | \epsilon, \bot} - P_{X_i Y_i | \epsilon} \|_1 \leq \frac{12\sqrt{2\delta}}{\alpha}.
\] (12)

Note that \( \sigma_{X_i Y_i Z_i} \) is product across \( X_i \) and the rest of the registers, since \( d_g \) is being conditioned on. Hence using Fact 31 and Quantum Raz’s Lemma,

\[
\delta l \geq \mathcal{E}_{X_i Y_i Z_i C_i A_i B_i \bot} D \left( \varphi_{X_i Y_i Z_i C_i A_i B_i \bot} || \varphi_{X_i Y_i Z_i C_i A_i B_i \bot} \right)
\]

\[
\geq \sum_{i \in \mathcal{C}} I(X_i : Y \tilde{C} C Y \tilde{C} C B \tilde{B} C B \tilde{B} C E) |X_i Y_i C A_i C B_i \bot \delta_g | d_g \varphi_{X_i Y_i Z_i C_i A_i B_i \bot}
\]

\[
\geq \frac{1}{2} \mathcal{E}_{i \in \mathcal{C}} P_{D_i | \epsilon, \bot} I(X_i : Y \tilde{C} C Y \tilde{C} C B \tilde{B} C B \tilde{B} C E) \delta_{\epsilon, \bot}
\]

\[
\geq \frac{1}{2} \frac{\alpha}{4} \mathcal{E}_{i \in \mathcal{C}} P_{R_i | \epsilon, \bot} I(X_i : Y \tilde{C} C Y \tilde{C} C B \tilde{B} C B \tilde{B} C E) \delta_{\epsilon, \bot}.
\] (13)

Similarly,

\[
\frac{8\delta}{\alpha} \geq \mathbb{E}_{i \in \mathcal{C}} P_{R_i | \epsilon, \bot} I(Y_i : X \tilde{C} A \tilde{C} A \tilde{C} E \delta G) \delta_{\epsilon, \bot},
\] (14)
Finally, note that $\sigma_{Z_cX_c\bar{X}_cA_c\bar{A}_cB_c\bar{B}_cE^A|X_cY_cZ_c\cdots}$ is product across $Z_\bar{c}$ and the other registers, since they were product in $\rho$, and $\sigma$ is obtained from $\rho$ by a unitary on other registers, which only acts on $Z_\bar{c}B_\bar{c}$ as a control\(^{10}\). Hence,

$$
\delta l \geq \frac{1}{2} \sum_{\ell} E \sum_{i \in C} P_{\ell,i} \left( I(Z_i : X_\bar{c} \bar{X}_\bar{c} A_\bar{c} \bar{A}_\bar{c} B_\bar{c} \bar{B}_\bar{c}) \right)
$$

$$
\geq \frac{1}{2} \frac{1}{2} \sum_{\ell} E \sum_{i \in C} P_{\ell,i} \left( I(Z_i : X_\bar{c} \bar{X}_\bar{c} A_\bar{c} \bar{A}_\bar{c} B_\bar{c} \bar{B}_\bar{c}) \right)
$$

$$
= \frac{1}{2} \frac{1}{2} \sum_{\ell} E \sum_{i \in C} P_{\ell,i} \left( I(Z_i : X_\bar{c} \bar{X}_\bar{c} A_\bar{c} \bar{A}_\bar{c} B_\bar{c} \bar{B}_\bar{c}) \right)
$$

$$
\geq \frac{1}{4} \sum_{\ell} E \sum_{i \in C} P_{\ell,i} \left( I(Z_i : X_\bar{c} \bar{X}_\bar{c} A_\bar{c} \bar{A}_\bar{c} B_\bar{c} \bar{B}_\bar{c}) \right)
$$

$$
\geq \frac{1}{4} \frac{1}{2} \sum_{\ell} E \sum_{i \in C} P_{\ell,i} \left( I(Z_i : X_\bar{c} \bar{X}_\bar{c} A_\bar{c} \bar{A}_\bar{c} B_\bar{c} \bar{B}_\bar{c}) \right)
$$

where we have used Pinsker’s inequality in (15). Using (9) to change the distribution over which the expectation is taken to $P_{X,Y,Z,R_i|\bar{c}}$ from $P_{X,Y,Z,R_i|\bar{c},\perp} P_{\bar{c},\bar{c}}$, and using Jensen’s inequality on (16) gives us

$$
\mathbb{E} \sum_{i \in C} P_{X_i,Y_i,Z_i,R_i|\bar{c}} \left( I(X_i : X_\bar{c} \bar{X}_\bar{c} A_\bar{c} \bar{A}_\bar{c} B_\bar{c} \bar{B}_\bar{c}) \right)
$$

$$
\leq \sqrt{\frac{8\delta}{\alpha}} + \frac{4\sqrt{2\delta}}{\alpha}
$$

and using triangle inequality on the above equation and (15),

$$
\mathbb{E} \sum_{i \in C} P_{X_i,Y_i,Z_i,R_i|\bar{c}} \left( I(X_i : X_\bar{c} \bar{X}_\bar{c} A_\bar{c} \bar{A}_\bar{c} B_\bar{c} \bar{B}_\bar{c}) \right)
$$

$$
\leq \sqrt{\frac{8\delta}{\alpha}} + \frac{4\sqrt{2\delta}}{\alpha} + \sqrt{4\delta} \leq \frac{11\sqrt{\delta}}{\alpha}
$$

Using Markov’s inequality on (7), (11), (12), (13), (14) and (17) we get that there exists an $i \in \bar{c}$ such that

$$
\left\| P_{X_i,Y_i,Z_i|\bar{c}} - P_{X_i,Y_i,Z_i} \right\|_1 \leq 7\sqrt{2\delta}
$$

$$
\left\| P_{X_i,Y_i,Z_i,R_i|\bar{c}} - P_{X_i,Y_i,Z_i,R_i|\perp} \right\|_1 \leq 98\sqrt{2\delta}
$$

$$
\left\| P_{X_i,Y_i,R_i|\bar{c},\perp} - P_{X_i,Y_i,R_i|\perp} \right\|_1 \leq 84\sqrt{2\delta}
$$

$$
\left\| P_{X_i,Y_i,R_i|\bar{c},\perp} - P_{X_i,Y_i,R_i|\perp} \right\|_1 \leq \frac{56\delta}{\alpha}
$$

$$
\left\| P_{X_i,Y_i,R_i|\bar{c},\perp} - P_{X_i,Y_i,R_i|\perp} \right\|_1 \leq \frac{56\delta}{\alpha}
$$

\(^{10}\)This would also have been true if we included $Y_c\bar{Y}_c$ along with $X_c\bar{X}_cA_c\bar{A}_cB_c\bar{B}_cE^A$, but we don’t need to include these.
Let \( N \) new registers. Note that \( O \)

Using the Fuchs-van de Graaf inequality and Uhlmann’s theorem on (24)

where the pure states \( |\varphi\rangle \) \( X \)

Note that if the \( A_iB_i \)

Using the Fuchs-van de Graaf inequality and Uhlmann’s theorem on (23), there exist unitaries \( \{ \mathcal{W}_{x_iy_i,\perp} \} \)

Let \( \mathcal{O}_{A_iB_i} \)

Now, suppose \( \Pr[W_i = 1|\mathcal{E}] > \omega^*(G) + 542\delta^{1/4} + \frac{\alpha}{\alpha^{1/2}} \), i.e.,

\[
\mathbb{E}_{P_{x_iy_i,\perp}} \left| \mathcal{O}_{A_iB_i}(|\varphi\rangle_{xy_{i,z_i}} - (1 \otimes \mathcal{W}_{y_{i,z_i}}) |\varphi\rangle_{xy_{i,z_i}} (1 \otimes \mathcal{W}_{y_{i,z_i}}^\dagger) \right| \leq 3 \left( \frac{77\sqrt{\delta}}{\alpha} \right)^{1/2} .
\]
With this assumption, we shall exhibit a quantum $S'$ for $G$ which has winning probability $> \omega^*(G)$, which is a contradiction; hence $\Pr[W_i = 1|\mathcal{E}]$ must be $\leq \omega^*(G) + \frac{542\delta^{1/4}}{\alpha^{1/2}}$. The strategy $S'$ works as follows:

- Alice and Bob share $r_i$ according to $P_{R_i|\mathcal{E},\perp}$ as randomness, and for each $R_i = r_i$, the state $|\varphi\rangle_{\perp,r_i}$ as shared entanglement, with Alice holding registers $X_{\tilde{c}}\tilde{X}_{\tilde{c}}A_{\tilde{c}}\tilde{A}_{\tilde{c}}E_{\tilde{A}}$, and Bob holding registers $Y_{\tilde{c}}\tilde{Y}_{\tilde{c}}Z_{\tilde{c}}B_{\tilde{c}}\tilde{B}_{\tilde{c}}E_{\tilde{B}}$.

- On inputs $(x_i, y_i)$ in the first round, Alice and Bob sample $r_i$ from $P_{R_i|\mathcal{E},\perp}$, then Alice applies $U_{s,r_i}$ to her half of $|\varphi\rangle_{\perp,r_i}$ and Bob applies $V_{y,r_i}$ on his half of it, and they measure the $A_i, B_i$ registers of the resulting state in the computational basis to give their first round outputs.

- On input $z_i$ in the second round, Bob applies the unitary $W_{y,z_i,r_i}$ to his half of the post-measurement state and measures the $B_i'$ register of the resulting state in the computational basis to give his second round output.

To analyze the success probability of this strategy, let us first assume Alice and Bob have $X_iY_iZ_iR_i$ from $P_{X_iY_iZ_iR_i|\mathcal{E}}$ instead of $P_{X_iY_iZ_i|\mathcal{E},\perp}$. Let us denote the conditional distribution of $(a_i, b_i)$ obtained by Alice and Bob in $S'$ after the first round by $P_{A_i,B_i|X_iY_iZ_i,R_i}$, and the conditional distribution of $b_i'$ obtained by Bob in the second round by $P_{B_i'|X_iY_iZ_i,R_i,A_i,B_i}$. By (24) and (25),

$$\|P_{X_iY_iZ_iR_i|\mathcal{E}}(P_{A_i,B_i|X_iY_iZ_i,R_i} - P_{A_i,B_i|X_iY_iZ_i})\|_1 \leq \frac{227\sqrt{\delta}}{\alpha}.$$

and by (26),

$$\|P_{X_iY_iZ_iR_iA_iB_i|\mathcal{E}}(P_{B_i'|X_iY_iZ_i,R_i,A_i,B_i} - P_{B_i'|X_iY_iZ_i,A_i,B_i})\|_1 \leq 3 \left(\frac{77\sqrt{\delta}}{\alpha}\right)^{1/2}.$$

Overall then we have,

$$\|P_{X_iY_iZ_iR_i|\mathcal{E}}(P_{A_i,B_i,B_i'|X_iY_iZ_i,R_i} - P_{A_i,B_i|X_iY_iZ_i})\|_1 \leq \frac{403\delta^{1/4}}{\alpha^{1/2}}.$$

Finally, from (19),

$$\|P_{X_iY_iZ_iR_i|\mathcal{E},\perp}(P_{A_i,B_i,B_i'|X_iY_iZ_i,R_i} - P_{X_iY_iZ_i,R_iA_i,B_i,B_i'}\|\|_1 \leq \frac{403\delta^{1/4}}{\alpha^{1/2}} + \frac{98\sqrt{2\delta}}{\alpha} \leq \frac{542\delta^{1/4}}{\alpha^{1/2}}.$$

Hence, if $\Pr[W_i = 1|\mathcal{E}] > \omega^*(G) + \frac{542\delta^{1/4}}{\alpha^{1/2}}$, then the probability that $S'$ wins $G$ is $> \omega^*(G)$. \qed

### 7.2 Parallel repetition theorem for 1-round 3-player product-anchored game

A 1-round 3-player game is called product-anchored iff Alice and Bob’s marginal input distribution is a product distribution, and there is an input $z = \perp$ to Eve such that $p(\perp) = \alpha$, and $p(x, y, \perp) = \alpha \cdot p(x, y)$. 
Theorem 35. Let $G$ be a 1-round 3-player non-local product-anchored game with parameter $\alpha$. Then for $\delta > 0$ and $t = (\omega^*(G) + \delta)l$,

$$\omega^*(G^l) = (1 - (1 - \omega^*(G))^5) \Omega\left(\frac{\delta^2}{\log(|A| |B| |C| |E|)}\right)$$
$$\omega^*(G^{l/l}) = (1 - \delta^5) \Omega\left(\frac{\delta^2}{\log(|A| |B| |C| |E|)}\right).$$

Proof sketch. The proof goes via a lemma analogous to Lemma 34. The correlation-breaking variables are defined similar to [JK20] as follows: for each $i \in [l]$, $D_i$ is a uniform bit as usual, and depending on the value of $D_i$, $G_i X_i Y_i Z_i$ are correlated in the following way:

$$G_i = \begin{cases} (x, y) \text{ w.p. } p(x, y) & \text{if } D_i = 0 \\ \perp \text{ w.p. } 1 - (1 - \alpha)^2/3 & \text{if } D_i = 1 \\ z \text{ w.p. } (1 - \alpha)^2/3 \cdot p(z|z \neq \perp) & \text{if } D_i = 1 (z \neq \perp) \end{cases}$$

$$X_i Y_i Z_i = \begin{cases} (x, y, \perp) \text{ w.p. } \alpha & \text{if } D_i = 0, G_i = (x, y) \\ (x, y, z) \text{ w.p. } (1 - \alpha) \cdot p(x, y|z) & \text{if } D_i = 0, G_i = (x, y) \\ (x, y, \perp) \text{ w.p. } p(x, y) & \text{if } D_i = 1, G_i = \perp \\ (x, y, \perp) \text{ w.p. } (1 - (1 - \alpha)^{1/3}) \cdot p(x, y|z) & \text{if } D_i = 1, G_i = z \\ (x, y, z) \text{ w.p. } (1 - \alpha)^{1/3} \cdot p(x, y|z) & \text{if } D_i = 1, G_i = z. \end{cases}$$

Let $C$ be the subset where we condition on success and $E$ denote the success event on $C$, and $|\varphi\rangle$ the state of the protocol conditioned on $E$ as before. Defining $R_i$ analogously, and using further techniques from [JK20] we can show that there exists an $i \in \hat{C}$ and unitaries $\{U_{x_i r_i}\}_{x_i r_i}, \{V_{y_i r_i}\}_{y_i r_i}, \{W_{z_i r_i}\}_{z_i r_i}$ on Alice Bob and Eve’s systems such that:

$$\mathbb{E}_{P_{X_i Y_i Z_i R_i | E, \perp}} \| P_{X_i Y_i Z_i R_i | E, \perp} - P_{X_i Y_i Z_i} P_{R_i | E, \perp} \|_1 = O\left(\frac{\delta^{1/2}}{\alpha}\right)$$

and

$$\mathbb{E}_{P_{X_i Y_i Z_i R_i | E, \perp}} \| |\varphi\rangle_{x_i y_i r_i} - (U_{x_i r_i} \otimes V_{y_i r_i} \otimes 1) |\varphi\rangle_{z_i r_i} (U_{x_i r_i}^+ \otimes V_{y_i r_i}^+ \otimes 1) \|_1 = O\left(\frac{\delta^{1/2}}{\alpha}\right)$$

By the usual commuting argument, this means that

$$\mathbb{E}_{P_{X_i Y_i Z_i R_i | E}} \| |\varphi\rangle_{x_i y_i z_i r_i} - (U_{x_i r_i} \otimes V_{y_i r_i} \otimes W_{z_i r_i}) |\varphi\rangle_{z_i r_i} (U_{x_i r_i}^+ \otimes V_{y_i r_i}^+ \otimes W_{z_i r_i}^+) \|_1 = O\left(\frac{\delta^{1/4}}{\alpha^{1/2}}\right)$$

and Alice, Bob and Eve have a strategy for $G$ in which they share $P_{R_i | E, \perp}$ as randomness, $|\varphi\rangle_{z_i r_i}$ as entanglement, and apply the unitaries $\{U_{x_i r_i}\}_{x_i r_i}, \{V_{y_i r_i}\}_{y_i r_i}, \{W_{z_i r_i}\}_{z_i r_i}$ on it depending on their inputs. \qed

Acknowledgements

We thank Anne Broadbent for discussions on the result in [BI19], as well as Lídia del Rio, Christopher Portmann, Renato Renner and Vilasini Venkatesh for discussions on composable security.
S. K. is supported by the National Research Foundation, including under NRF RF Award No. NRF-NRFF2013-13, the Prime Minister’s Office, Singapore; the Ministry of Education, Singapore, under the Research Centres of Excellence program and by Grant No. MOE2012-T3-1-009; and in part by the NRF2017-NRF-ANR004 VanQuTe Grant.

E. Y.-Z. T. is supported by the Swiss National Science Foundation via the National Center for Competence in Research for Quantum Science and Technology (QSIT), the Air Force Office of Scientific Research (AFOSR) via grant FA9550-19-1-0202, and the QuantERA project eDICT.

References

[ACK+14] Nati Aharon, André Chailloux, Iordanis Kerenidis, Serge Massar, Stefano Pironio, and Jonathan Silman. Weak coin flipping in a device-independent setting. In Theory of Quantum Computation, Communication, and Cryptography, pages 1–12, 2014.

[AFDF+18] Rotem Arnon-Friedman, Frédéric Dupuis, Omar Fawzi, Renato Renner, and Thomas Vidick. Practical device-independent quantum cryptography via entropy accumulation. Nature Communications, 9(1):459, 2018.

[AMPS16] Nati Aharon, Serge Massar, Stefano Pironio, and Jonathan Silman. Device-independent bit commitment based on the CHSH inequality. New Journal of Physics, 18(2):025014, 2016.

[BB84] Charles H. Bennett and Gilles Brassard. Quantum cryptography: Public key distribution and coin tossing. In Proceedings of International Conference on Computers, Systems and Signal Processing, page 175, 1984.

[BCK13] Jonathan Barrett, Roger Colbeck, and Adrian Kent. Memory Attacks on Device-Independent Quantum Cryptography. Physical Review Letters, 110:010503, 2013.

[BI19] Anne Broadbent and Rabib Islam. Quantum encryption with certified deletion. https://arxiv.org/abs/1910.03551, 2019.

[BOHL+05] Michael Ben-Or, Michał Horodecki, Debbie W. Leung, Dominic Mayers, and Jonathan Oppenheim. The universal composable security of quantum key distribution. In Theory of Cryptography, pages 386–406, 2005.

[BVY15] Mohammad Bavarian, Thomas Vidick, and Henry Yuen. Anchoring Games for Parallel Repetition. https://arxiv.org/abs/1509.07466, 2015.

[BVY17] Mohammad Bavarian, Thomas Vidick, and Henry Yuen. Hardness Amplification for Entangled Games via Anchoring. In Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing, STOC '17, page 303–316, 2017.

[CW79] J. Lawrence Carter and Mark N. Wegman. Universal classes of hash functions. Journal of Computer and System Sciences, 18(2):143–154, 1979.

[FM18] Honghao Fu and Carl A. Miller. Local randomness: Examples and application. Physical Review A, 97:032324, 2018.
Thomas Holenstein. Parallel Repetition: Simplifications and the No-Signaling Case. In *Proceedings of the Thirty-Ninth Annual ACM Symposium on Theory of Computing*, STOC ’07, page 411–419, 2007.

Rahul Jain and Srijita Kundu. A Direct Product Theorem for One-Way Quantum Communication. https://arxiv.org/abs/2008.08963, 2020.

Rahul Jain, Carl A. Miller, and Yaoyun Shi. Parallel Device-Independent Quantum Key Distribution. *IEEE Transactions on Information Theory*, 66(9):5567–5584, 2020.

Rahul Jain, Attila Pereszlényi, and Penghui Yao. A Parallel Repetition Theorem for Entangled Two-Player One-Round Games under Product Distributions. In *2014 IEEE 29th Conference on Computational Complexity (CCC ’14)*, pages 209–216, 2014.

Julia Kempe, Hirotada Kobayashi, Keiji Matsumoto, Ben Toner, and Thomas Vidick. Entangled Games are Hard to Approximate. In *2008 49th Annual IEEE Symposium on Foundations of Computer Science*, pages 447–456, 2008.

Srijita Kundu, Jamie Sikora, and Ernest Y.-Z. Tan. A Device-Independent Protocol for XOR Oblivious Transfer. In *15th Conference on the Theory of Quantum Computation, Communication and Cryptography (TQC 2020)*, volume 158 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 12:1–12:15, 2020.

Norbert Lütkenhaus, Ashutosh S. Marwah, and Dave Touchette. Erasable Bit Commitment from Temporary Quantum Trust. https://arxiv.org/abs/1910.13949, 2020.

Ueli Maurer and Renato Renner. Abstract Cryptography. In *The Second Symposium on Innovations in Computer Science*, ICS 2011, pages 1–21. Tsinghua University Press, 2011.

Stefano Pironio, Antonio Acín, Nicolas Brunner, Nicolas Gisin, Serge Massar, and Valerio Scarani. Device-independent quantum key distribution secure against collective attacks. *New Journal of Physics*, 11(4):045021, 2009.

Christopher Portmann and Renato Renner. Cryptographic security of quantum key distribution. https://arxiv.org/abs/1409.3525v1, 2014.

Anup Rao. Parallel repetition in projection games and a concentration bound. In *Proceedings of the Fortieth Annual ACM Symposium on Theory of Computing*, STOC ’08, page 1–10. Association for Computing Machinery, 2008.

Ran Raz. A Parallel Repetition Theorem. In *Proceedings of the Twenty-Seventh Annual ACM Symposium on Theory of Computing*, page 447–456, 1995.

Renato Renner. *Security of Quantum Key Distribution*. PhD thesis, ETH Zürich, 2005.

Jonathan Silman, André Chailloux, Nati Aharon, Iordanis Kerenidis, Stefano Pironio, and Serge Massar. Fully distrustful quantum bit commitment and coin flipping. *Physical Review Letters*, 106:220501, 2011.

Marco Tomamichel and Anthony Leverrier. A largely self-contained and complete security proof for quantum key distribution. *Quantum*, 1:14, 2017.
[Unr14] Dominique Unruh. Revocable quantum timed-release encryption. In *Advances in Cryptology – EUROCRYPT 2014*, pages 129–146, 2014.

[vdVCRŠ20] Bart van der Vecht, Xavier Coiteaux-Roy, and Boris Škorić. Can’t Touch This: unconditional tamper evidence from short keys. https://arxiv.org/abs/2006.02476, 2020.

[Vid17] Thomas Vidick. Parallel DIQKD from parallel repetition. https://arxiv.org/abs/1703.08508, 2017.

[VPdR19] Vilasini Venkatesh, Christopher Portmann, and Lídia del Rio. Composable security in relativistic quantum cryptography. *New Journal of Physics*, 21(4):043057, 2019.

[Wie83] Stephen Wiesner. Conjugate coding. *SIGACT News*, 15(1):78–88, 1983.