Obstacles to State Preparation and Variational Optimization from Symmetry Protection

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Local Hamiltonians with topological quantum order exhibit highly entangled ground states that cannot be prepared by shallow quantum circuits. Here, we show that this property may extend to all low-energy states in the presence of an on-site $\mathbb{Z}_2$ symmetry. This proves a version of the No Low-Energy Trivial States (NLTS) conjecture for a family of local Hamiltonians with symmetry protected topological order. A surprising consequence of this result is that the Goemans-Williamson algorithm outperforms the Quantum Approximate Optimization Algorithm (QAOA) for certain instances of MaxCut, at any constant level. We argue that the locality and symmetry of QAOA severely limits its performance. To overcome these limitations, we propose a non-local version of QAOA, and give numerical evidence that it significantly outperforms standard QAOA for frustrated Ising models on random 3-regular graphs.

Classifying topological phases of matter is amongst the main objectives of modern condensed matter physics [1]. Central to this program is the characterization of entanglement structures that can emerge in ground states of many-body systems. Of particular interest are topologically non-trivial ground states [2]. Such states cannot be generated by a constant-depth quantum circuit starting from a product state. Non-trivial states exhibit complex, non-local entanglement properties and are thus expected to have highly non-classical features. Remarkably, certain gapped local Hamiltonians have non-trivial ground states. For example, preparing a ground state of Kitaev’s toric code [3] from a product state requires a circuit depth growing at least polynomially in the system size using us-

going beyond ground states, a natural next question is whether there are local Hamiltonians with the property that any low-energy state is non-trivial. Formalized by Freedman and Hastings [6], this is known as the No Low-Energy Trivial States (NLTS) conjecture. To state it in summary: the latter posits that there are local Hamiltonians whose ground state energy is QMA-hard to approximate with an extensive error $\epsilon n$ for some constant $\epsilon > 0$.

A proof of the NLTS conjecture is still outstanding. Although many natural families of Hamiltonians provably do not have the NLTS property (see [8] for a comprehen-

sive list), evidence for its validity has been provided by a number of related results. Ref. [6] constructs Hamiltonians satisfying a certain one-sided NLTS property: these have excitations of two kinds (similar to the toric code), and low-energy states with no excitations of the first kind are non-trivial. The construction crucially relies on ex-

pander graphs as even the one-sided NLTS property does not hold for similar constructions on regular lattices [6].

Eldar and Harrow [8] construct families of local Hamiltonians (also based on expander graphs) such that any state whose reduced density operators on a constant fraction of sites coincides with that of a ground state is non-trivial. This feature, called the No Low-Error Trivial States (NLETS) property, is clearly related to robustness of entanglement in the ground state with respect to erasure errors [9][10]. The existence of Hamiltonians with the NLETS property is a necessary condition [4] for the existence of good quantum LDPC codes, another central conjecture in quantum information.

Here we pursue a different approach to the NLTS con-

jecture by imposing an additional symmetry in the ini-

tial state as well as the preparation circuit. This mirrors similar considerations in the classification of topological phases, where the concept of symmetry-protected topolog-

ical (SPT) phases [11] has been extremely fruitful. In-

deed, the study of SPT equivalence classes of states, pio-

neered in [12][13], has led to a complete classification of 1D phases [11][14][15], and also plays an essential role in measurement-based quantum computation [16][17].
For concreteness, we focus on the simplest case of on-site $Z_2$-symmetry. A local Hamiltonian is said to be $Z_2$-symmetric if all interaction terms commute with $X^{\otimes n}$, where $X$ is the single-qubit Pauli-$X$ operator. Likewise, a quantum circuit $U$ acting on $n$ qubits is said to be $Z_2$-symmetric if it obeys

$$UX^{\otimes n} = X^{\otimes n}U.$$  

We do not impose the symmetry on the individual gates of $U$, though this will naturally be the case in many interesting examples, such as the QAOA circuits considered below. Finally, let us say that a state $\Psi$ of $n$ qubits is $Z_2$-symmetric if $X^{\otimes n}\Psi = \pm \Psi$. Our first result is a proof of the NLTS conjecture in the presence of onsite $Z_2$-symmetry:

**Theorem 1.** There exist constants $\epsilon, c > 0$ and a family of $Z_2$-symmetric local Hamiltonians $\{H_n\}_{n \in \mathbb{Z}}$ such that $H_n$ has ground state energy 0 for any $n \in \mathbb{Z}$ while

$$\langle \varphi | U | H_n | U \varphi \rangle > c n$$

for any $Z_2$-symmetric depth-$d$ circuit $U$ composed of two-qubit gates, any $Z_2$-symmetric product state $\varphi$, and any $n \geq 2^{cd}$, $n \in \mathbb{Z}$.

Our starting point to establish Theorem 1 is a fascinating result by Eldar and Harrow stated as Corollary 43 in [5]. It shows that the output distribution of a shallow quantum circuit cannot assign any non-negligible probability to subsets of bit strings that are separated far apart w.r.t. the Hamming distance. More precisely, define the distribution $p(x) = |\langle x | U | \varphi \rangle|^2$, where $x \in \{0, 1\}^n$. Given a subset $S \subseteq \{0, 1\}^n$, let $p(S) = \sum_{x \in S} p(x)$.

**Fact 1 ([5]).** For all subsets $S, S' \subseteq \{0, 1\}^n$ one has

$$\text{dist}(S, S') \leq \frac{4n^{1/2}2^{3d/2}}{\min\{p(S), p(S')\}}.$$  

Here $\text{dist}(S, S')$ is the Hamming distance, i.e. the minimum number of bit flips required to get from $S$ to $S'$. We emphasize that Eq. (2) holds for all depth-$d$ circuits $U$ and all product states $\varphi$ ($Z_2$-symmetry is not needed).

Given a bit string $x$ let $\overline{x}$ be the bit-wise negation of $x$. Note that $p(x) = p(\overline{x})$ since $U\varphi$ is $Z_2$-symmetric. Choose $S$ and $S'$ as the sets of all $n$-bit strings with the Hamming weight $\leq n/3$ and $\geq 2n/3$ respectively. Then $p(S') = p(S)$ and $\text{dist}(S, S') = n/3$. Eq. (2) gives

$$p(S) \leq 12n^{-1/2}2^{3d/2}.$$  

Our strategy is to choose the Hamiltonian $H_n$ such that low-energy states of $H_n$ are concentrated on bit strings with the Hamming weight close to $0$ or $n$ such that $p(S)$ is non-negligible. Then Eq. (3) provides a logarithmic lower bound on the depth $d$ for symmetric low-energy states.

**Suppose $G = (V, E)$ is a graph with $n$ vertices. The Cheeger constant of $G$ is defined as

$$h(G) = \min_{S \subseteq V} \frac{\partial S}{|S|},$$

where $\partial S \subseteq E$ is the subset of edges that have exactly one endpoint in $S$. Families of expander graphs are infinite collections of bounded degree graphs $\{G_n\}_{n \in \mathbb{Z}}$ whose Cheeger constant is lower bounded by a constant, i.e. $h(G_n) \geq h > 0$ for all $n \in \mathbb{Z}$. Explicit constructions of degree-3 expanders can be found in [18]. Fix a family of degree-3 expanders $\{G_n\}_{n \in \mathbb{Z}}$ and define $H_n$ as the ferromagnetic Ising model on the graph $G_n$, i.e.

$$H_n = \frac{1}{2} \sum_{(u,v) \in E} (I - Z_u Z_v).$$

Here $Z_u$ is the Pauli-$Z$ applied to a site $u$ and $I$ is the identity. Clearly, $H_n$ is $Z_2$-symmetric, each term in $H_n$ acts on two qubits and each qubit is involved in three terms. Thus $\{H_n\}_{n \in \mathbb{Z}}$ is a family of local Hamiltonians. $H_n$ has $Z_2$-symmetric ground states $\frac{1}{\sqrt{(0^n) + |1^n|}}$ with zero energy. Given a bit string $x$, let $\text{supp}(x) = \{j \in [n] : x_j = 1\}$ be the support of $x$. From Eqs. (4) and (5), one gets

$$\langle x | H_n | x \rangle = |\text{supp}(x)| \geq h \cdot \min\{|x|, n - |x|\},$$

where $|x|$ is the Hamming weight of $x$. Assume Eq. (1) is false. Then $p(x)$ is a low-energy distribution such that $\sum_x p(x) |\langle x | H_n | x \rangle| \leq cn$. By Markov’s inequality, $p(x)$ has a non-negligible weight on low-energy basis states,

$$p(S_{\text{low}}) \geq 1/2, \quad S_{\text{low}} = \{x : |\langle x | H_n | x \rangle| \leq 2cn\}. \quad (7)$$

By Eq. (4), $\min\{|x|, n - |x|\} \leq 2n \epsilon h^{-1}$ for all $x \in S_{\text{low}}$. Choose $\epsilon = h/6$. Then $S_{\text{low}} \subseteq S \cup S'$ and $p(S_{\text{low}}) \leq p(S) + p(S') = 2p(S)$. Here the last equality uses the symmetry of $p(x)$. By Eq. (7), $p(S) \geq 1/4$. Combining this and Eq. (3) one gets $n \geq 48 \cdot 2^{3d/2}$. We conclude that Eq. (1) holds whenever $n > 48^2 \cdot 8^d$. This proves Theorem 1.

The Hamiltonians Eq. (5) are diagonal in the computational basis and have product ground states $|0^n\rangle$ and $|1^n\rangle$. The presence of the $Z_2$-symmetry is therefore essential: the same family of Hamiltonians do not exhibit NLTS without it. In this sense, the NLTS property here behaves similarly to topological order in 1D systems which only exists under symmetry protection [11,14].

Theorem 1 implies restrictions on the performance of variational quantum algorithms for combinatorial optimization. Recall that the Quantum Approximate Optimization Algorithm (QAOA) [19] seeks to approximate the maximum of a cost function $C : \{0, 1\}^n \rightarrow \mathbb{R}$ by encoding it into a Hamiltonian $H_n = \sum_{x \in \{0, 1\}^n} C(x) |x\rangle \langle x|$. It variationally optimizes the expected energy of $H_n$ over quantum states of the form $U(\beta, \gamma) |+\rangle^n$, where

$$U(\beta, \gamma) = \prod_{k=1}^p e^{i \beta_k B_k} e^{i \gamma_k H_n},$$

with $B_k$ acting on two qubits and each qubit is involved in three terms.

[5] This shows that the output distribution of a shallow quantum circuit cannot assign any non-negligible probability to subsets of bit strings that are separated far apart w.r.t. the Hamming distance.
and where $B = \sum_{j=1}^{n} X_j$. The integer $p \geq 1$ is called the QAOA level. It controls non-locality of the variational circuit.

A paradigmatic test case for QAOA is the MaxCut problem. Given a graph $G_n = (V,E)$ with $n$ vertices, the corresponding MaxCut Hamiltonian is defined by Eq. (5). The maximum energy of $H_n$ coincides with the number of edges in the maximum cut of $G_n$. Crucially, the QAOA circuit $U(\beta, \gamma)$ with the Hamiltonian $H_n$ as well as the initial QAOA state $\ket{+^n}$ obey the $\mathbb{Z}_2$-symmetry property. Furthermore, the circuit $U(\beta, \gamma)$ has depth $O(Dp)$, where $D$ is the maximum vertex degree of $G_n$. Specializing Theorem 1 to bipartite graphs we obtain an upper bound on the approximation ratio achieved by the level-$p$ QAOA circuits for the MaxCut cost function (see Appendix A for a proof):

**Corollary 1.** For every integer $D \geq 3$ there exists an infinite family of bipartite $D$-regular graphs $\{G_n\}_{n \in \mathbb{Z}}$ such that the Hamiltonians $H_n$ defined in Eq. (5) obey

$$\frac{1}{|E|} \langle +^n | U^{-1} H_n U | +^n \rangle \leq \frac{5}{6} + \frac{\sqrt{D-1}}{3D}$$

(8)

for any level-$p$ QAOA circuit $U \equiv U(\beta, \gamma)$ as long as $p \leq (1/3 \log_2 n - 4)D^{-1}$.

Note that any bipartite graph with a set of edges $E$ has maximum cut size $|E|$. In this case, the left-hand side of Eq. (8) coincides with the approximation ratio, i.e., the ratio between the expected value of the MaxCut cost function on the (optimal) level-$p$ variational state and the maximum cut size. Thus Corollary 1 provides an explicit upper bound on the approximation ratio achieved by level-$p$ QAOA. Such bounds were previously known only for $p = 1$ [19]. Statement (8) severely limits the performance of QAOA at any level $p$; rigorously establishing a widely believed conjecture [20]: constant-level QAOA is inferior to the classical Goemans-Williamson algorithm for MaxCut, which achieves an approximation ratio of approximately 0.878 on an arbitrary graph [21]. Indeed, the right-hand side of Eq. (8) is approximately $5/6 \approx 0.833$ for large vertex degree $D$.

QAOA circuits $U(\beta, \gamma)$ possess a form of locality which is stronger than the one assumed in Theorem 1. Indeed, if $p$ and $D$ are constants, the unitary $U(\beta, \gamma)$ can be realized by a constant-depth circuit composed of nearest-neighbor gates, i.e., the circuit is geometrically local.

A natural question is whether more general bounds on the variational energy can be established for states generated by geometrically local $\mathbb{Z}_2$-symmetric circuits. Of particular interest are graphs that lack the expansion property, such as regular lattices, where the arguments used in the proof of Theorem 1 no longer apply. A simple model of this type is the ring of disagrees [12]. It describes the MaxCut problem on the cycle graph $\mathbb{Z}_n$.

Quite recently, Ref. [22] proved that the optimal approximation ratio achieved by level-$p$ QAOA for the ring of disagrees is bounded above by $(2p+1)/(2p+2)$ for all $p$ and conjectured that this bound is tight. Here we prove a version of this conjecture for arbitrary geometrically local $\mathbb{Z}_2$-symmetric circuits. To quantify the notion of geometric locality, let us say that a unitary $U$ acting on $n$ qubits located at vertices of the cycle graph has range $R$ if the operator $U^\dagger Z_j U$ has support on the interval $[j-R, j+R]$ for any qubit $j$. For example, the level-$p$ QAOA circuit associated with the ring of disagrees has range $R = p$.

**Theorem 2.** Let $H_n$ be the ring of disagree Hamiltonian,

$$H_n = \frac{1}{2} \sum_{p \in \mathbb{Z}_n} (I - Z_p Z_{p+1}),$$

where $n$ is even. Let $U$ be a $\mathbb{Z}_2$-symmetric unitary with range $R < n/4$. Then

$$\frac{1}{n} \langle +^n | U^\dagger H_n U | +^n \rangle \leq \frac{2R + 1/2}{2R + 1}.$$  

(9)

This bound is tight whenever $n$ is a multiple of $2R + 1$.

Since one can always round $n$ to the nearest multiple of $2R + 1$, the bound Eq. (9) is tight for all $n$ up to corrections $O(1/n)$, assuming that $R = O(1)$. Let us first prove the upper bound Eq. (9). Define $X = (XI)^{\otimes n}/2$. Then

$$X H_n X + H_n = nI$$

(10)

Let $V = Xu$. Note that $V$ is a $\mathbb{Z}_2$ symmetric circuit with range $R$. Taking the expected value of Eq. (10) on the state $U|+^n\rangle$, one infers that Eq. (9) holds whenever

$$\frac{1}{n} \langle +^n | V^\dagger H_n V | +^n \rangle \geq 1 - \frac{2R + 1/2}{2R + 1} = \frac{1}{2(2R + 1)}.$$  

(11)

Thus it suffices to prove that Eq. (11) holds for any $\mathbb{Z}_2$-symmetric range-$R$ circuit $V$. For each $j, k \in \mathbb{Z}_n$ define

$$\epsilon_{j,k} = \frac{1}{2} \langle +^n | V^\dagger (I - Z_j Z_k) V | +^n \rangle.$$  

Let dist$(j, k)$ be the distance between $j$ and $k$ with respect to the cycle graph $\mathbb{Z}_n$. We claim that

$$\epsilon_{j,k} = 1/2 \text{ if dist}(j, k) > 2R.$$  

(12)

Indeed, $\langle +^n | V^\dagger Z_i Z_0 V | +^n \rangle = 0$ for any qubit $i$ since $V|+^n\rangle$ and $Z_i V | +^n \rangle$ are eigenvectors of $X^{\otimes n}$ with eigenvalues $1$ and $-1$. Such eigenvectors have to be orthogonal. From dist$(j,k) > 2R$ one infers that $V^\dagger Z_j V$ and $V^\dagger Z_k V$ have disjoint support. Thus

$$\langle +^n | V^\dagger Z_j Z_k V | +^n \rangle = \langle +^n | (V^\dagger Z_j V)(V^\dagger Z_k V) | +^n \rangle$$

$$= \langle +^n | V^\dagger Z_j V | +^n \rangle \cdot \langle +^n | V^\dagger Z_k V | +^n \rangle = 0.$$
This proves Eq. (12). Suppose one prepares the state $V|+^n\rangle$ and measures a pair of qubits $j < k$ in the standard basis. Then $\epsilon_{j,k}$ is the probability that the measured values on qubits $j$ and $k$ disagree. By the union bound, 

$$\epsilon_{j,k} \leq \sum_{i=j}^{k-1} \epsilon_{i,i+1}. \quad (13)$$

Indeed, if qubits $j$ and $k$ disagree, at least one pair of consecutive qubits located in the interval $[j,k]$ must disagree. Set $k = j + 2R + 1$. Then $\epsilon_{j,k} = 1/2$ by Eq. (12). Take the expected value of Eq. (13) with respect to random uniform $j \in \mathbb{Z}_n$. This gives

$$\frac{1}{2} \leq \frac{2R+1}{n} \sum_{i \in \mathbb{Z}_n} \epsilon_{i,i+1} = \frac{2R+1}{n} (|V\rangle H_n |V\rangle)$$

proving Eq. (11). In Appendix B we construct a $\mathbb{Z}_2$-symmetric range-$R$ circuit $U$ such that $U|+^n\rangle$ is a tensor product of GHZ-like states on consecutive segments of $2R+1$ qubits. We show that such circuit saturates the upper bound Eq. (9). This completes the proof of Theorem 2.

Concerns about limitations of QAOA have previously been voiced by Hastings [20], who showed analytically that certain local classical algorithms match the performance of level-1 QAOA for Ising-like cost functions with multi-spin interactions. Hastings also gave numerical evidence for the same phenomenon for MaxCut with $p = 1$, and argued that this should extend to $p > 1$ [20].

Motivated by these limitations, we propose a non-local modification of QAOA which we call the recursive quantum approximate optimization algorithm (RQAOA). To sketch the main ideas behind RQAOA consider an Ising-like Hamiltonian

$$H_n = \sum_{(p,q) \in E} J_{p,q} Z_p Z_q \quad (14)$$

defined on a graph $G_n = (V,E)$ with $n$ vertices. Here $J_{p,q}$ are arbitrary real coefficients. RQAOA aims to approximate the maximum energy $\max_z \langle z | H_n | z \rangle$, where $z \in \{1,-1\}^n$. It consists of the following steps.

First, run the standard QAOA to maximize the expected value of $H_n$ on the state $|\psi\rangle = U(\beta, \gamma) |+^n\rangle$. For every edge $(j,k) \in E$ compute $M_{j,k} = \langle \psi^* | Z_j Z_k | \psi \rangle$, where $\psi^*$ is the optimal variational state.

Next, find a pair of qubits $(i,j) \in E$ with the largest magnitude of $M_{i,j}$ (breaking ties arbitrarily). The corresponding variables $Z_i$ and $Z_j$ are correlated if $M_{i,j} > 0$ and anti-correlated if $M_{i,j} < 0$. Impose the constraint

$$Z_j = \text{sgn}(M_{i,j}) Z_i \quad (15)$$

and substitute it into the Hamiltonian $H_n$ to eliminate the variable $Z_j$. For example, a term $Z_i Z_k$ with $k \notin \{i,j\}$ gets mapped to $\text{sgn}(M_{i,j}) Z_i Z_k$. The term $J_{i,j} Z_i Z_j$ gets mapped to a constant energy shift $J_{i,j} \text{sgn}(M_{i,j})$. All other terms remain unchanged. This yields a new Ising Hamiltonian $H_{n-1}$ that depends on $n-1$ variables. By construction, the maximum energy of $H_{n-1}$ coincides with the maximum energy of $H_n$ over the subset of assignments satisfying the constraint Eq. (15).

Finally, call RQAOA recursively to maximize the expected value of $H_{n-1}$. Each recursion step eliminates one variable from the cost function. The recursion stops when the number of variables reaches some specified threshold value $n_c \ll n$. The remaining instance of the problem with $n_c$ variables is then solved by a purely classical algorithm (for example, by a brute force method). Thus the value of $n_c$ controls how the workload is distributed between quantum and classical computers. We describe a generalization of RQAOA applicable to Ising-like cost functions with multi-spin interactions in Appendix C.

Impose a constraint of the form (14) can be viewed as rounding correlations among the variables $Z_i$ and $Z_j$. Indeed, the constraint demands that these variables must be perfectly correlated or anti-correlated. This is analogous to rounding fractional solutions obtained by solving linear programming relaxations of combinatorial optimization problems. We note that reducing the size of a problem to the point that it can be solved optimally by brute force is a widely used and effective approach in combinatorial optimization.

We compare the performance of the standard QAOA, RQAOA, and local classical algorithms by considering the Ising Hamiltonians in Eq. (14) with couplings $J_{p,q} = \pm 1$ defined on the cycle graph. In Appendix D we prove:

**Theorem 3.** For each integer $n$ divisible by 6 there is a family of $2^{n/3}$ Ising Hamiltonians of the form $H_n = \sum_{k \in \mathbb{Z}_n} J_k Z_k Z_{k+1}$ with $J_k \in \{1,-1\}$ such that the following holds for all Hamiltonians in the family:

(i) There is a local classical algorithm which achieves the approximation ratio 1.

(ii) Level-$p$ QAOA achieves an approximation ratio of at most $p/(p+1)$.

(iii) Level-1 RQAOA achieves the approximation ratio 1.

Our definition of local classical algorithms follows [20]. We also show that the level-1 RQAOA achieves the optimal approximation ratio for any 1D Ising model with coupling coefficients $J_k \in \{1,-1\}$.

Finally, we numerically compare level-1 versions of QAOA and RQAOA. We consider Ising-type Hamiltonians Eq. (14) with random couplings $J_{p,q} = \pm 1$ defined on random 3-regular graphs. For each problem instance we compute the optimal approximation ratios achieved by the two algorithms, see Figure 1. The level-1 RQAOA significantly outperforms the standard level-1 QAOA for these problem instances. The details of this simulation and further discussion of RQAOA can be found in Appendix C.4.

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FIG. 1: Approximation ratios achieved by the level-1 QAOA (blue) and RQAOA (red) for the Ising-type Hamiltonian $H_n$ defined Eq. (14) with $n = 32$ (left) and $n = 100$ (right). We consider 16 problem instances with random 3-regular graphs and random couplings $J_{p,q} = \pm 1$. The cutoff value for variable elimination was chosen as $n_c = 8$ (left) and $n_c = 30$ (right). The approximation ratio achieved by a given algorithm is defined as $\langle z|H_n|z \rangle / E_{\text{max}}$, where $z \in \{1,-1\}^n$ is the algorithm’s output and $E_{\text{max}} = \max_z \langle z|H_n|z \rangle$.

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Appendix A: Proof of Corollary 1

In this appendix, we give a proof of Corollary 1 in the main text. Here and below, we will denote the expected approximation ratio achieved by the QAOA with Hamiltonian $H$ as

$$\text{QAOA}_p(H) = \left( \max_{\beta, \gamma \in \mathbb{R}^p} \langle \Psi_H(\beta, \gamma)|H|\Psi_H(\beta, \gamma) \rangle \right) \cdot \left( \max_{x \in \{0,1\}^n} \langle x|H|x \rangle \right)^{-1},$$

where

$$|\Psi_H(\beta, \gamma) \rangle = U_H(\beta, \gamma)|+^n \rangle \quad \text{and} \quad U_H(\beta, \gamma) = \prod_{m=1}^p (e^{i\beta_m B} e^{i\gamma_m H})$$

(A1)

for $\beta, \gamma \in \mathbb{R}^p$ and where $B = \sum_{j=1}^n X_j$. Let us first record a few general features of the QAOA for later use.

Let $G = (V,E)$ be a graph, $n = |V|$, $m = |E|$, and let $J = (J_e)_{e \in E} \in \mathbb{R}^E$ be an assignment of edge weights on $G$. Let us define the Hamiltonian $H_G(J)$ as

$$H_G(J) = \sum_{\{u,v\} \in E} J_{\{u,v\}} Z_u Z_v.$$

(A2)

It will be useful for later to also define

$$H_G = \sum_{\{u,v\} \in E} Z_u Z_v, \quad \text{and} \quad H_G^{\text{MaxCut}} = \frac{1}{2} (mI - H_G),$$

where $H_G^{\text{MaxCut}}$ is the Hamiltonian used in QAOA for the Maximum Cut problem on the graph $G$. We will use the following bound on the circuit depth of a QAOA unitary.
Lemma A.1. Let $U = U_H(\beta, \gamma)$ with $\beta, \gamma \in \mathbb{R}^p$ be a level-$p$ QAOA unitary (cf. Eq. (A1)) for a Hamiltonian $H = H_G(J)$ on a graph $G$ (cf. (A2)). Let $D$ be the maximum degree of $G$. Then $U$ can be realized by a circuit of depth $d \leq p(D + 1)$ consisting of 2-qubit gates.

If $G$ is $D$-regular and bipartite, then the circuit depth of $U$ can be bounded by $d \leq pD$.

Proof. By Vizing’s theorem [23] there is an edge coloring of $G$ with at most $D + 1$ colors. Taking such a coloring $E = E_1 \cup \cdots \cup E_{D+1}$, we may apply each level $e^{i\beta E_v}e^{i\gamma H}$ of $U$ in depth $D + 1$ by applying $(\prod_{v \in V} e^{i\beta X_v})\prod_{c=1}^{D+1} V_c(\gamma)$, where each $V_c(\gamma) = (\prod_{(u,v) \in E_c} e^{i\gamma I_{(u,v)}Z_uZ_v})$ is a depth-1-circuit of two-local gates.

If $G$ is $D$-regular and bipartite, we may reduce the chromatic number upper bound from $D + 1$ to $D$ since all bipartite graphs are $D$-edge-colorable by König’s line coloring theorem.

The expected QAOA approximation ratios of suitably related instances are identical:

Lemma A.2. Let $\mathcal{L} \subset V$ be an arbitrary subset of vertices and $\partial \mathcal{L}$ be the set of edges that have exactly one endpoint in $\mathcal{L}$. Let $J = (J_e)_{e \in E} \in \mathbb{R}^E$ be arbitrary edge weights. Define $\tilde{J} = (\tilde{J}_e)_{e \in E} \in \mathbb{R}^E$ by

$$\tilde{J}_e = \begin{cases} -J_e & \text{if } e \in \partial \mathcal{L} \\ J_e & \text{otherwise} \end{cases}.$$

Then expected QAOA approximation ratios satisfy

$$\text{QAOA}_p(H_G(J)) = \text{QAOA}_p(H_G(\tilde{J})).$$

Proof. Let us write $H = H_G(J)$ and $\tilde{H} = H_G(\tilde{J})$ for brevity. Let $\tilde{X} = X|\mathcal{L}|$ be a tensor product of Pauli-$X$ operators acting on every qubit in $\mathcal{L} \subset V$. Then $\tilde{H} = \tilde{X}H\tilde{X}$, which implies that

$$\max_{x \in \{0,1\}^n} \langle x|\tilde{H}|x \rangle = \max_{x \in \{0,1\}^n} \langle x|H|x \rangle.$$

(A3)

Let $\beta, \gamma \in \mathbb{R}^p$ be arbitrary. Then we also have

$$\tilde{X}|\Psi_H(\beta, \gamma)) = \prod_{m=1}^{p} (X_{e^{i\beta_m B}}e^{i\alpha_m \tilde{H}}|X|^{|\alpha_m\rangle} = \prod_{m=1}^{p} (e^{i\beta_m B}e^{i\gamma_m H})|\alpha_m\rangle = |\Psi_H(\beta, \gamma)),$$

where identities in the middle follow since $|\alpha_m\rangle$ is stabilized by $X$, and since $[X, B] = 0$. Therefore we have

$$\langle \Psi_H(\beta, \gamma)|\tilde{H}|\Psi_H(\beta, \gamma)) = \langle \Psi_H(\beta, \gamma)|XH\tilde{X}|\Psi_H(\beta, \gamma)) = \langle \Psi_H(\beta, \gamma)|H|\Psi_H(\beta, \gamma)).$$

Combined with (A3), this implies the claim.

In particular, if $G = (V, E)$ is a bipartite graph, then Lemma A.2 implies that

$$\text{QAOA}_p(H_G) = \text{QAOA}_p(-H_G)$$

and

$$\text{QAOA}_p(H_G^{\text{MaxCut}}) = \frac{1}{2}(1 + \text{QAOA}_p(H_G)).$$

(A4)

We now prove Corollary 1. It is a direct consequence of Theorem 1, which we restate here for convenience in the notation of this appendix:

Theorem 1. Consider a family $\{G_n = ([n], E_n)\}_{n \in \mathcal{I}}$ of graphs with Cheeger constant lower bounded as $h(G_n) \geq h > 0$ for all $n \in \mathcal{I}$. Then

$$\langle \varphi|U^\dagger H_{G_n} U|\varphi \rangle < |E_n| - \frac{hn}{3}$$

for any $\mathbb{Z}_2$-symmetric depth-$d$ circuit $U$ composed of two-qubit gates, any $\mathbb{Z}_2$-symmetric product state $\varphi$, and any $n > 48^2 8^d$, $n \in \mathcal{I}$.

Then we have the following:
Corollary 1. For every integer \( D \geq 3 \) there exists an infinite family of bipartite \( D \)-regular graphs \( \{ G_n \}_{n \in \mathcal{I}} \) such that
\[
\text{QAOA}_p(H_{G_n}^{\text{MaxCut}}) \leq \frac{5}{6} + \frac{\sqrt{D-1}}{3D}
\]
as long as
\[
p < (1/3 \log_2 n - 4)D^{-1}.
\] (A5)

Proof. Fix some \( D \geq 3 \). By the results of \cite{24, 25}, there exists an infinite family \( \{ G_n \}_{n \in \mathcal{I}} \) of bipartite \( D \)-regular Ramanujan graph with \( n \) vertices for every \( n \in \mathcal{I} \). Consider a fixed \( n \in \mathcal{I} \) and let \( p = p(n) \) be the associated QAOA level. Let \( U_n = U_{H_{G_n}}(\beta, \gamma^*) \) be a level-\( p \) QAOA unitary for the Hamiltonian \( H_{G_n} \) on \( G_n \), and assume that \( \beta^*, \gamma^* \in \mathbb{R}^p \) are such that the expectation of \( H_{G_n} \) is maximized. Because \( G_n \) is \( D \)-regular, the circuit depth of \( U_n \) can be bounded from above by \( pD \) according to Lemma \ref{A.1}. Condition (A5) implies that \( n > 48^2 8^p D \), thus
\[
\text{QAOA}_p(H_{G_n}^{\text{MaxCut}}) < 1 - \frac{h}{3D}.
\]
by Theorem 1, where we have used that \( |E_n| = nD/2 \). With (A4) (using that \( G_n \) is bipartite) we conclude that
\[
\text{QAOA}_p(H_{G_n}^{\text{MaxCut}}) < 1 - \frac{h}{3D}.
\]
The claim then follows from the bound \( h/D \geq (D - 2\sqrt{D-1})/(2D) \), valid for all Ramanujan graphs. \( \square \)

Appendix B: Optimal variational circuit for the ring of disagrees
In this section we prove that the upper bound of Theorem 2 in the main text is tight whenever \( n \) is a multiple of \( 2R + 1 \). Let
\[
|\text{GHZ}_n\rangle = 2^{-1/2}(|0^n\rangle + |1^n\rangle)
\]
be the GHZ state of \( n \) qubits.

Lemma B.1. Suppose \( n = 2p + 1 \) for some integer \( p \). There exists a \( \mathbb{Z}_2 \)-symmetric range-\( p \) quantum circuit \( V \) such that
\[
|\text{GHZ}_n\rangle = V|+n\rangle.
\] (B1)

Proof. We shall write \( \text{CX}_{c,t} \) for the CNOT gate with a control qubit \( c \) and a target qubit \( t \). Let \( c = p + 1 \) be the central qubit. One can easily check that
\[
|\text{GHZ}_n\rangle = \left( \prod_{j=1}^{p} \text{CX}_{c,c-j} \text{CX}_{c,c+j} \right) H_c |0^n\rangle.
\]
All CX gates in the product pairwise commute, so the order does not matter. Inserting a pair of Hadamards on every qubit \( j \in [n] \setminus \{c\} \) before and after the respective CX gate and using the identity \((I \otimes H)\text{CX}(I \otimes H) = \text{CZ} \) one gets
\[
|\text{GHZ}_n\rangle = \left( \prod_{j \in [n] \setminus \{c\}} H_j \right) \left( \prod_{j=1}^{p} \text{CZ}_{c,c-j} \text{CZ}_{c,c+j} \right) |+n\rangle.
\] (B2)
Let \( S = \exp [i(\pi/4)Z] \) be the phase-shift gate. Define the two-qubit Clifford gate
\[
\text{RZ} = (S \otimes S)^{-1} \text{CZ} = \exp(-i\pi/4) \exp[-i(\pi/4) (Z \otimes Z)].
\]
Expressing CZ in terms of RZ and \( S \) in Eq. (B2) one gets
\[
|\text{GHZ}_n\rangle = S_c^{2p} \left( \prod_{j \in [n] \setminus \{c\}} H_j S_j \right) \left( \prod_{j=1}^{p} \text{RZ}_{c,c-j} \text{RZ}_{c,c+j} \right) |+n\rangle.
\] (B3)
Multiply both sides of Eq. (B3) on the left by a product of $S$ gates over qubits $j \in [n] \setminus \{c\}$. Noting that
$$SHS = i \exp \left[ -i(\pi/4)X \right]$$
one gets (ignoring an overall phase factor)
$$\prod_{j \in [n] \setminus \{c\}} S_j |\text{GHZ}_n\rangle = S_c^{2p} \left( \prod_{j \in [n] \setminus \{c\}} \exp \left[ -i(\pi/4)X_j \right] \right) \left( \prod_{j=1}^{p} \text{RZ}_{c,c-j}\text{RZ}_{c,c+j} \right) |+^n\rangle. \quad (B4)$$

Using the identity
$$\prod_{j \in [n] \setminus \{c\}} S_j |\text{GHZ}_n\rangle = S_c^{2p} |\text{GHZ}_n\rangle.$$
one can cancel $S_c^{2p}$ that appears in both sides of Eq. (B4). We arrive at Eq. (B1) with
$$V = \left( \prod_{j \in [n] \setminus \{c\}} \exp \left[ -i(\pi/4)X_j \right] \right) \left( \prod_{j=1}^{p} \text{RZ}_{c,c-j}\text{RZ}_{c,c+j} \right).$$

Obviously, $V$ is $\mathbb{Z}_2$-symmetric since any individual gate commutes with $X^\otimes n$. Let us check that $V$ has range-$p$. Consider any single-qubit observable $O_q$ acting on the $q$-th qubit. Consider three cases. Case 1: $q = c$. Then $V^\dag O_q V$ may be supported on all $n$ qubits. However, $|c-p,c+p\rangle = |1,n\rangle$, so the $p$-range condition is satisfied trivially. Case 2: $1 \leq q < c$. Then all gates $\text{RZ}_{c,c-j}$ in $V$ cancel the corresponding gates in $V^\dag$, so that $V^\dag O_q V$ has support in the interval $[1,c] \subseteq [q-p,q+p]$. Thus the $p$-range condition is satisfied. Case 3: $c < q \leq n$. This case is equivalent to Case 2 by symmetry.

Recall that we consider the ring of disagrees Hamiltonian
$$H_n = \frac{1}{2} \sum_{p \in \mathbb{Z}_n} (I - Z_p Z_{p+1}).$$

**Lemma B.2.** Consider any integers $n,p$ such that $n$ is even and $n$ is a multiple of $2p + 1$. Then there exists a $\mathbb{Z}_2$-symmetric range-$p$ circuit $U$ such that
$$\langle +^n | U^\dagger H_n U | +^n \rangle = \frac{2p + 1/2}{2p + 1}.$$

**Proof.** Let $V$ be the $\mathbb{Z}_2$-symmetric range-$p$ unitary operator preparing the GHZ state on $2p + 1$ qubits starting from $|+^{2p+1}\rangle$, see Lemma B.1. Suppose $n = m(2p + 1)$ for some integer $m$. Define
$$U = U_1 U_2,$$
where
$$U_1 = (X \otimes I)^{\otimes n/2} \quad \text{and} \quad U_2 = V^\otimes m.$$

Since each copy of $V$ acts on a consecutive interval of qubits and has range $p$, one infers that $U$ has range $p$. We have
$$U_1^\dagger H_n U_1 = \sum_{p \in \mathbb{Z}_n} G_p, \quad G_p = \frac{1}{2} (I + Z_p Z_{p+1}).$$
The state $U_2|+^n\rangle$ is a tensor product of GHZ states supported on consecutive tuples of $2p + 1$ qubits. The expected value of $G_p$ on the state $U_2|+^n\rangle$ equals 1 if $G_p$ is supported on one of the GHZ state. Otherwise, if $G_p$ crosses the boundary between two GHZ states, the expected value of $G_p$ on the state $U_2|+^n\rangle$ equals 1/2. Thus
$$\langle +^n | U^\dagger H_n U | +^n \rangle = \sum_{p \in \mathbb{Z}_n} \langle +^n | U_2^\dagger G_p U_2 | +^n \rangle = m(2p + 1/2) = \frac{2p + 1/2}{2p + 1}.$$

□
Appendix C: Recursive QAOA

In this appendix, we outline the Recursive QAOA algorithm (RQAOA) for general cost functions.

C.1. Variable Elimination

Let $G = (V, E)$ be a hypergraph with $|V| = n$ vertices. Suppose a variable $x_v \in \{1, -1\}$ is associated with each vertex $v \in V$. Let $\{1, -1\}^n = \{1, -1\}^V$ be the set of all possible variable assignments. Let $J : E \to \mathbb{R}$ be a function which assigns a real weight $J_e$ to every hyperedge $e$. Given a subset $f \subset V$ and an assignment $x \in \{1, -1\}^V$, let us write

$$x(f) = \prod_{v \in f} x_v.$$  

Let us agree that $x(\emptyset) = 1$. We consider the problem of maximizing cost functions of the form

$$C(x) = \sum_{e \in E} J_e x(e)$$

over $x \in \{1, -1\}^V$. Fix some vertex $v \in V$. As a motivation, we first describe how a single variable $x_v$ can be eliminated when a suitably constrained problem is considered. Namely, suppose that instead of trying to approximate $\max_{x \in \{1, -1\}^V} C(x)$, we restrict to $x \in \{1, -1\}^V$ satisfying

$$x(f) = \sigma,$$  

where $f \subset V$ is some fixed subset of vertices containing $v$, and $\sigma \in \{1, -1\}$ is a constant. If $x \in \{1, -1\}^V$ satisfies the constraint (C1) then

$$J_e x(e) = J_e x(e)(f) \sigma = \sigma J_e x(e \Delta f).$$

Here and below $A \triangle B$ denotes the symmetric difference of sets $A$ and $B$. We arrive at

$$C(x) = \sum_{e \in E} J_e x(e) + \sum_{e \in E : v \notin e} \sigma J_e x(e \triangle f).$$  

(C2)

Note that $C(x)$ does not depend on $x_v$. Expression (C2) can be written as a sum over the hyperedges of a hypergraph $G' = (V', E')$ with vertex set $V' = V \setminus \{v\}$ and hyperedges

$$E' = E_0' \cup E_1',$$  

(C3)

where

$$E_0' = \{e \in E : v \notin e\} \quad \text{and} \quad E_1' = \{e \triangle f : e \in E, \ v \in e\}.$$  

(C4)

Note that $G'$ no longer contains the vertex $v$. Define a function $J' : E' \to \mathbb{R}$ such that

$$J'_e = J_e \quad \text{if } e \in E_0',$$  

(C5)

and

$$J'_e = \sigma J_e \Delta f \quad \text{if } e \in E_1'.$$  

(C6)

By construction, the maximum of

$$C'(x) = \sum_{e \in E'} J'_e x(e)$$

over all assignments $x \in \{1, -1\}^{V'}$ coincides with the maximum of $C(x)$ over all $x \in \{1, -1\}^V$ satisfying the constraint Eq. (C1). Furthermore, any maximum $x^*$ of $C'$ can directly be translated to a corresponding maximum of $C$ over
the restricted set defined by the constraint (C1) by setting \( x_v^* = \sigma \cdot x(f \setminus \{v\}) \). That is, we have \( x^* = \xi(x) \) for the function \( \xi : \{1, -1\}^{V'} \to \{1, -1\}^V \) defined by

\[
\xi(x)_w = \begin{cases} \sigma \cdot x(f \setminus \{v\}) & \text{for } v = w \\ x_w & \text{otherwise} \end{cases}
\]  

(C7)

for all \( w \in V \).

In summary, we have reduced the problem of maximizing \( C(x) \) over \( n \) variables \( x \in \{1, -1\}^V \) satisfying (C1) to the problem of maximizing \( C'(x) \) over \( n-1 \) variables \( x \in \{1, -1\}^{V'} \). If a global maximum \( x \) of \( C(x) \) happens to satisfy (C1), the new reduced problem yields a solution to the original problem.

### C.2. Correlation rounding

To construct an approximation algorithm, we simply impose a constraint of the form (C1) by choosing \( f \subset V, v \in f \) and \( \sigma \in \{1, -1\} \) appropriately. To make the latter choice, we use the standard QAOA\(_p\) algorithm with \( p = O(1) \). That is, let us set

\[
H_G(J) = \sum_{e \in E} J_e Z(e) \quad \text{where} \quad Z(e) = \prod_{v \in e} Z_v
\]

(C8)

and write \( H = H_G(J) \). We first use the standard QAOA\(_p\)(H) algorithm to find an optimal state

\[
\Psi = \Psi_{H_G}(\beta_*, \gamma_*) \in (\mathbb{C}^2)^\otimes |V|
\]

maximizing the energy of \( H_G \). The expected value

\[
M_e = \langle \Psi | Z(e) | \Psi \rangle
\]

can be efficiently approximated on a quantum computer for any \( e \in E \).

Suppose the state \( \Psi \) is measured in the computational basis giving a string \( x \in \{1, -1\}^V \). Clearly, if \( |M_f| \) is close to 1, then the variables \( \{x_v\}_{v \in f} \) satisfy a constraint of the form (C1) with high probability with \( \sigma = \text{sign}(M_f) \in \{1, -1\} \) and any \( v \in f \). Thus it is natural to choose \( f \) such that \( |M_f| \) is maximal. Combined with the procedure for eliminating the corresponding variable \( x_v \) described in Section C.2, we obtain a subroutine for reducing the problem size by one variable. Pseudocode for this routine is given below.

Imposing a constraint of the form (C1) can be viewed as rounding correlations among the variables \( \{x_w\}_{w \in f} \): indeed, the constraint demands that for \( v \in f \), the variable \( x_v \) and \( x(f \setminus \{v\}) \) must be perfectly correlated or anti-correlated.

1: \textbf{function} \text{eliminateVariable}(G = (V, E), J)
2: \textbf{Input:} A hypergraph \( G = (V, E) \) and a weight function \( J : E \to \mathbb{R} \)
3: \textbf{Output:} A hypergraph \( G' = (V', E'), J' : E' \to \mathbb{R} \) and a function \( \xi : \{1, -1\}^{V'} \to \{1, -1\}^V \).
4: \textbf{end function}

### C.3. The recursive QAOA (RQAOA) algorithm

The recursive QAOA algorithm (RQAOA) we propose here proceeds simply by iterating the process of eliminating one variable at a time until the number of variables reaches some specified threshold value \( n_c \ll n \). The remaining instance of the problem with \( n_c \) variables is solved by a purely classical algorithm (for example, by the brute force method). Thus the value of \( n_c \) controls how the workload is distributed between the quantum and the classical computers. Pseudocode for the RQAOA algorithm is given in Fig. 2.
1: function RQAOA(G = (V, E), J)
2: Input: A hypergraph G = (V, E) with n = |V| and a weight function J : E → R defining a Hamiltonian H_G(J), see Eq. [CS].
3: Output: A variable assignment x ∈ {−1, 1}^V
4: 6: Let ξ^{(0)} : {1, −1}^V → {1, −1}^V be the identity map.
5: 7: for k = 1 to n − n_c do
6: (G, J, ξ) ← eliminateVARIABLE(G, J).
7: ξ^{(k)} ← ξ^{(k−1)} o ξ.
8: end for
9: 11: Let G = (V, E) be the final hypergraph with |V| = n_c vertices.
10: 12: Find x* = arg max_{x ∈ {−1, 1}^V} (x|H_G(J)|x).
11: return ξ(x*)
12: end function

FIG. 2: Pseudocode for the recursive QAOA algorithm.

C.4. Classical simulability of level-1 RQAOA for Ising models

Suppose J is a real symmetric matrix of size n. Here we consider Ising-like cost functions such that the corresponding Hamiltonian is

\[ H = \sum_{1 \leq p < q \leq n} J_{p,q} Z_p Z_q. \]

The mean values of a Pauli operator Z_p Z_q on the level-1 QAOA state

\[ |\Psi_H(\beta, \gamma)\rangle = e^{i\beta B} e^{i\gamma H} |+^n\rangle \]

can be computed in time O(n) using an an explicit analytic formula. Such a formula was derived for the Max-Cut cost function by Wang et al. [26, Theorem 1]. Below we provide a generalization to general Ising Hamiltonians. Since the total number of terms in the cost function is O(n^2), simulating each step of RQAOA takes time at most O(n^3). Assuming that n_c = O(1), the number of steps is roughly n so that the full simulation cost is O(n^4). Crucially, the simulation cost of this method does not depend on the depth of the variational circuit. This is important because RQAOA may potentially increase the depth from O(1) to O(n) since it adds many new terms to the cost function.

Lemma C.1. Fix a pair of qubits 1 ≤ u < v ≤ n. Let c = cos(2\beta) and s = sin(2\beta). Then

\[ \langle \Psi_H(\beta, 1)|Z_u Z_v|\Psi_H(\beta, 1)\rangle = (s^2/2) \prod_{p \neq u,v} \cos(2J_{u,p} - 2J_{v,p}) = (s^2/2) \prod_{p \neq u,v} \cos(2J_{u,p} + 2J_{v,p}) \]

\[ + cs \cdot \sin(2J_{u,v}) \left[ \prod_{p \neq u,v} \cos(2J_{u,p}) + \prod_{p \neq u,v} \cos(2J_{v,p}) \right]. \]

Here we only consider the case γ = 1 since γ can be absorbed into the definition of J.

Proof. Given a 2-qubit observable O define the mean value

\[ \mu(O) = \langle \Psi_H(\beta, 1)|O_{u,v}|\Psi_H(\beta, 1)\rangle. \]

We are interested in the observable O = ZZ ≡ Z ⊗ Z.

We note that all terms in H and B that act trivially on \{u, v\} do not contribute to \mu(O). Such terms can be set to zero. Given a 2-qubit observable O, define a mean value

\[ \mu'(O) = \langle +^n|e^{iH'} O_{u,v} e^{-iH'}|+^n\rangle, \] \hspace{1cm} \text{where} \hspace{1cm} H' = \sum_{p \neq u,v} (J_{u,p} Z_u + J_{v,p} Z_v) Z_p. \hspace{1cm} (C10)
and noting that

\[ \mu O \]

Next consider the case

\[ b | H \]

Using the explicit form of \( \mu Eqs. (C14),(C15) \) one infers that

\[ b \]

Finally, consider the case

\[ O \]

From Eqs. (C14),(C15) one can easily compute the mean value \( \mu \)\((Z \)I\) + \( \mu \)\((IX))\].

Using the explicit form of \( H' \) one gets

\[ e^{-iH'} + n = \frac{1}{2} \sum_{a,b=0,1} |a, b, u, v \rangle \langle \Phi(a, b) \rangle \]

where \( \Phi(a, b) \) is a tensor product state of \( n - 2 \) qubits defined by

\[ \langle \Phi(a, b) \rangle = 0 \]

Combining Eqs. (C10),(C13) one gets

\[ \mu'(O) = (1/4) \sum_{a,b,a',b'=0,1} \langle a', b'|O|a, b \rangle \cdot \langle \Phi(a', b')|\Phi(a, b) \rangle \]

Using the tensor product form of the states \( \Phi(a, b) \) and the identity \( \langle \theta' \theta \rangle = \cos(\theta - \theta') \) gives

\[ \langle \Phi(a', b')|\Phi(a, b) \rangle = \prod_{p \neq u, v} \cos[J_{u,p}(-1)^a - J_{v,p}(-1)^b - J_{v,p}(-1)^b'] \]

From Eqs. (C14),(C15) one can easily compute the mean value \( \mu'(O) \) for any 2-qubit observable.

Consider first the case \( O = YY \). Then the only terms contributing to Eq. (C14) are those with \( a' = a \oplus 1 \) and \( b' = b \oplus 1 \). The identity \( \langle a' \oplus 1|Y|a \rangle = -i(-1)^a \) gives

\[ \mu'(YY) = (1/4) \sum_{a,b=0,1} (-1)^{a+b} \prod_{p \neq u, v} \cos[2J_{u,p}(-1)^a + 2J_{v,p}(-1)^b] \]

that is,

\[ \mu'(YY) = (1/2) \prod_{p \neq u, v} \cos[2J_{u,p} - 2J_{v,p}] - (1/2) \prod_{p \neq u, v} \cos[2J_{u,p} + 2J_{v,p}] \]

Next consider the case \( O = YZ \). Note that the matrix elements \( \langle a', b'|O|a, b \rangle \) have zero real part. From Eqs. (C14),(C15) one infers that \( \mu'(YZ) \) has zero real part. This implies

\[ \mu'(YZ) = \mu'(ZY) = 0 \]

Finally, consider the case \( O = XI \). Then the only terms that contribute to Eq. (C14) are those with \( a' = a \oplus 1 \) and \( b' = b \). We get

\[ \mu'(XI) = \prod_{p \neq u, v} \cos(2J_{u,p}) \]

Here we noted that the inner product Eq. (C15) with \( a' = a \oplus 1 \) and \( b' = b \) does not depend on \( a, b \). By the same argument,

\[ \mu'(IX) = \prod_{p \neq u, v} \cos(2J_{v,p}) \]

Combining Eq. (C12) and Eqs. (C16),(C17),(C18),(C19) one arrives at Eq. (C9).
For more general cost functions including interactions among three or more variables, there are two complications: First, unlike in the Ising case, the variable elimination process will typically increase the degree of non-locality of interactions. Second, mean values of Pauli operators on the QAOA state $\Psi_H(\beta, \gamma)$ lack a simple analytic formula (as far as we know). However, one can approximately compute the mean values using the Monte Carlo method due to Van den Nest [27]. A specialization of this method to simulation of the level-1 QAOA is described in [28]. The Monte Carlo simulator has runtime scaling polynomially with the number of qubits, number of terms in the cost function, and the inverse error tolerance, see [28] for details. This method also requires no restrictions on the depth of the variational circuit.

An important distinction between QAOA and RQAOA lies in the measurement step. QAOA requires few-qubit measurements to estimate the variational energy as well as the final $n$-qubit measurement that assigns a value to each individual variable. This last step is what makes QAOA hard to simulate classically and may lead to a quantum advantage [29]. In contrast, RQAOA only needs few-qubit measurements to estimate mean values of individual terms in the cost function. The $n$-qubit measurement step is replaced by the correlation rounding that eliminates variables one by one. One may ask whether the lack of multi-qubit measurements also precludes a quantum advantage. Indeed, in the special case of level-1 variational circuits and the Ising-like cost function RQAOA can be efficiently simulated classically, see above. However, level-$p$ RQAOA with $p > 1$ as well as level-1 RQAOA with more general cost functions are not known to be classically simulable in polynomial time, leaving room for a quantum advantage.

Appendix D: Comparison of QAOA, RQAOA, and Classical Algorithms

D.1. QAOA versus Classical Local Algorithms

In this section, we discuss another limitation of QAOA which results from its locality and the covariance condition discussed in Lemma [A.2]: we compare QAOA to a certain very simple classical local algorithm (see Lemma [D.1] below). We show that there is an exponential number of problem instances for which the classical local algorithm outperforms QAOA.

Let us briefly sketch the notion of a local classical algorithm. We envision that the tuple $(J_e)_{e \in E}$ is given as input. Here we are interested in algorithms which are local with respect to the underlying graph $G$. For $r \in \mathbb{N}$ and $v \in V$, define

$$E_r(v) = \bigcup_{\ell=1}^r \bigcup_{\text{path with } v \in e_1} \{e_1, \ldots, e_{\ell}\}$$

to be the set of edges that belong to a path starting at $v$ of length bounded by $r$. Consider a classical algorithm $A$ which on input $\{J_e\}_{e \in E}$ outputs $x = \{x_1, \ldots, x_n\} \in \{0, 1\}^n$. We say that $A$ is $r$-local if there is a family of functions $\{g_v : \mathbb{R}^{E_r(v)} \to \{0, 1\}\}_{v \in V}$ such that the following holds for every problem instance $(J_e)_{e \in E} \in \mathbb{R}^E$:

$$x_v = g_v \left(\{J_e\}_{e \in E_r(v)}\right) \quad \text{for every } v \in V. \quad \text{(35)}$$

In other words, in an $r$-local classical algorithm, every output bit $x_v$ only depends on edge weights $J_e$ belonging to paths of length bounded by $r$ starting at $v$. We note that this definition can easily be generalized to the probabilistic case (e.g., by including local random bits). For the purposes of this section, deterministic functions turn out to be sufficient.

The (choice of) family $\{g_v\}_{v \in V}$ can be considered as a set of variational parameters for the classical algorithm. To keep the number of variational parameters constant, we consider vertex-transitive graphs $G$. Fix $v_0 \in V$. For every $v \in V$, fix an automorphism $\pi_v$ of $G$ such that $\pi_v(v_0) = v$. Then the sets $E_r(v)$ for different $v \in V$ can be identified via $E_r(v) = \pi_v(E_r(v_0))$. We say that an $r$-local classical algorithm is uniform if (after this identification) $g_v \equiv g$ for all $v \in V$, i.e., if there is a single function $g : \mathbb{R}^{E_r(v_0)} \to \{0, 1\}$ specifying the behavior of the algorithm. To obtain general-purpose algorithms (applicable to any instance), the function $g : \mathbb{R}^{E_r(v_0)} \to \{0, 1\}$ should be chosen adaptively (i.e., potentially depending on the instance). The definition of local classical algorithm sketched here includes e.g., the algorithms considered in Ref. [20], though it is slightly more general as the local functions can be arbitrary.

Let $n = 6r$ be a multiple of 6. Consider $n$-qubit Hamiltonians (cf. [A.2]) of the form

$$H(J) = \sum_{k \in \mathbb{Z}_n} J_k Z_k Z_{k+1}$$

where $J = (J_0, \ldots, J_{n-1}) \in \{1, -1\}^n$. 

To define locality and uniformity for the cycle graph \( \mathbb{Z}_n \), let \( \pi_v(w) = v + w \ (\text{mod} \ n) \) be chosen as translation modulo \( n \) for \( v \in \mathbb{Z}_n \). We show the following:

**Lemma D.1.** There is a subset \( S \subset \{1,-1\}^n \) of \( 2^n/3 \) problem instances such that the following holds:

(i) QAOA\(_p(H(J)) \leq p/(p+1) \) for every \( p \in \mathbb{N} \) and every \( J \in S \).

(ii) There is a 1-local uniform classical algorithm such that for every \( J \in S \), the algorithm outputs \( x \in \{0,1\}^n \) such that \( \langle x|H(J)|x \rangle = 1 \).

(iii) Level-1 RQAOA achieves the approximation ratio 1.

**Proof.** For every \( s = (s_0, \ldots, s_{2r-1}) \in \{0,1\}^{2r} \) define \( J = J(s) \in \{1,-1\}^n \) by

\[
J_{3a} = J_{3a+1} = (-1)^{s_a}, \quad \text{and} \quad J_{3a+2} = 1,
\]

for all \( a = 0, 1, \ldots, 2r-1 \). We claim that the set \( S = \{ J(s) \mid s \in \{0,1\}^{2r} \} \) has the required properties. Consider an instance \( H(J(s)) \) with \( s \in S \). Define

\[
\overline{X}(s) = \prod_{a=0}^{2r-1} X_{3a+1}.
\]

Then \( H(J(s)) \) is related to \( H_{\mathbb{Z}_n} = \sum_{j \in \mathbb{Z}_n} Z_j Z_{j+1} \) by the gauge transformation

\[
H(J(s)) = \overline{X}(s) H_{\mathbb{Z}_n} \overline{X}(s)^{-1}.
\]

Since the QAOA algorithm is invariant under such gauge transformation (see Lemma A.2), we obtain

\[
\text{QAOA}_p(H(J(s))) = \text{QAOA}_p(H_{\mathbb{Z}_n}) \leq \frac{p}{p+1}
\]

where we use the bound

\[
\text{QAOA}_p(H_{\mathbb{Z}_n}^{\text{MaxCut}}) \leq \frac{2p+1}{2p+2},
\]

proven in [22] for even \( n \), in combination with Lemma A.2. This shows (i).

For the proof of (ii), consider the classical algorithm \( A \) which on input \( J = (J_0, \ldots, J_{n-1}) \) outputs

\[
x_v = g(J_{v-1}, J_v) \quad \text{for every} \ v \in \mathbb{Z}_n,
\]

where

\[
g(J, J') = \begin{cases} 1 & \text{if } (J, J') = (-1, -1) \\ 0 & \text{otherwise} \end{cases}.
\]

Clearly, the algorithm \( A \) is uniform and 1-local, and it is easy to check that the output satisfies \( \langle x|H(J)|x \rangle = 1 \).

The proof of (iii) is given as a part of Lemma D.2. \( \square \)

**D.2. RQAOA on the Ising ring**

Here we prove that level-1 RQAOA achieves approximation ratio 1 on the ring of disagrees, in sharp contrast to (arbitrary) level-\( p \) QAOA (see Lemma D.1). More generally, level-1 RQAOA produces \( x \in \{0,1\}^n \) which maximizes the cost function (that is, achieves approximation ratio 1) for any 1D Ising model where the coupling coefficients are either +1 or -1.

**Lemma D.2.** Consider a cost function of the form

\[
C_n(x) = \sum_{k \in \mathbb{Z}_n} J_k x_k x_{k+1} \quad \text{for} \ x \in \{1,-1\}^n,
\]

where \( J_k \in \{1,-1\} \) for all \( k \in \mathbb{Z}_n \). Then the level-1 RQAOA produces \( x^* \in \{1,-1\}^n \) such that \( C_n(x^*) = \max_{x \in \{1,-1\}^n} C_n(x) \).
It would be interesting to see additional, more general families of examples where approximation ratios achieved by RQAOA can be computed or lower bounded analytically.

Proof. Let

\[ H = \sum_{k \in \mathbb{Z}_n} J_k Z_k Z_{k+1}. \]  

(D1)

Observe first that \( \langle \Psi_H(\beta, \gamma) | Z_i Z_j | \Psi_H(\beta, \gamma) \rangle = 0 \) if \( |i - j| > 2 \) since in this case the operators \( U^{-1} Z_i U \) and \( U^{-1} Z_j U \) have disjoint support. Lemma \[ C.1 \] shows that a QAOA-state \( \Psi_H(\beta, \gamma) \) has expectation values

\[ \langle \Psi_H(\beta, \gamma) | Z_i Z_j | \Psi_H(\beta, \gamma) \rangle = \begin{cases} \frac{1}{2} J_i \sin(4\beta) \sin(4\gamma) & \text{if } j = i + 1 \\ \frac{1}{2} J_i J_{i+1} \sin^2(2\beta) \sin^2(4\gamma) & \text{if } j = i + 2 \\ 0 & \text{otherwise} \end{cases} \]  

(D2)

when \( J_k \in \{1, -1\} \) for every \( k \in \mathbb{Z}_n \). Thus

\[ |\langle \Psi(\beta, \gamma) | Z_i Z_{i+2} | \Psi(\beta, \gamma) \rangle| \leq 1/4 \quad \text{for all } (\beta, \gamma). \]  

(D3)

Assume \( (\beta^*, \gamma^*) \) are such that \( (\beta^*, \gamma^*) = \arg \max_{(\beta, \gamma)} |\langle \Psi_H(\beta^*, \gamma^*) | H | \Psi_H(\beta^*, \gamma^*) \rangle| \). Then we can infer from (D2) that

\[ \langle \Psi(\beta^*, \gamma^*) | Z_i Z_{i+1} | \Psi(\beta^*, \gamma^*) \rangle = J_i / 2. \]  

(D4)

Combined with (D3) and (D2) we conclude that

\[ \arg \max_{(i,j):j < i} |\langle \Psi_H(\beta^*, \gamma^*) | Z_i Z_j | \Psi_H(\beta^*, \gamma^*) \rangle| = (i^*, i^* + 1) \]  

(D5)

for some \( i^* \in \mathbb{Z}_n \). Without loss of generality, assume that \( i^* = n - 2 \). Then, according to (D5), the RQAOA algorithm eliminates the variable \( x_{n-1} \) (i.e., \( v = n - 1, f = \{n-2, n-1\} \)). By (D4), this is achieved by imposing the constraint

\[ x_{n-1} = x_{n-2} J_{n-2} \]  

(D6)

i.e., \( \sigma = J_{n-2} \). The resulting reduced graph \( G' = (V', E') \) has vertex set \( V' = V \setminus \{n-1\} = \mathbb{Z}_{n-1} \) and edges

\[ E' = \{\{i, i + 1\} \mid i \in \mathbb{Z}_n \setminus \{n - 2\}\} \cup \{\{n - 2, 0\}\} \]

\[ = \{\{i, i + 1\} \mid i \in \mathbb{Z}_{n-2}\}, \]

and it is easy to check that the new cost function takes the form

\[ C'(x) = 1 + \sum_{k \in \mathbb{Z}_{n-1}} J'_k x_k x_{k+1} \]  

(D7)

with

\[ J'_i = \begin{cases} J_i & \text{when } i \neq n - 2 \\ J_{n-2} J_{n-1} & \text{when } i = n - 2 \end{cases} \]  

(D8)

We note that the transformation (D8) preserves the parity of the couplings in the sense that

\[ \prod_{k \in \mathbb{Z}_n} J_k = \prod_{k \in \mathbb{Z}_{n-1}} J'_k. \]  

(D9)

Inductively, the RQAOA thus eliminates variables \( x_{n-1}, x_{n-2}, \ldots, x_{n_c} \) while imposing the constraints (cf. (D6))

\[ x_{n-1} = x_{n-2} J_{n-2} \]

\[ x_{n-2} = x_{n-3} J'_{n-3} \]

\[ \vdots \]
arriving at the cost function $C_{n_c}(x)$ associated with an Ising chain of length $n_c$ having couplings belonging to $\{1, -1\}$. Because of (D9) and because (D1) is frustrated if and only if $\prod_{k \in \mathbb{Z}^n} J_k = -1$, we conclude that any maximum $x^* \in \{1, -1\}^{n_c}$ of $C_{n_c}(x)$ satisfies

$$
C_{n_c}(x^*) = \begin{cases} 
n_c + 1 & \text{if } \prod_{k \in \mathbb{Z}^n} J_k = 1 
n_c - 2 & \text{otherwise} \end{cases}.
$$

Because the cost function acquires a constant energy shift in every variable elimination step (cf. (D7)), the output $x = \xi(x^*)$ of the RQAOA algorithm satisfies

$$
C(x) = n - n_c + C_{n_c}(x^*) = \begin{cases} 
n + 1 & \text{if } \prod_{k \in \mathbb{Z}^n} J_k = 1 
n - 2 & \text{otherwise} \end{cases}.
$$

This implies the claim. \hfill \Box

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