MINIMAL BRICKS

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Abstract. A brick is a 3-connected graph such that the graph obtained from it by deleting any two distinct vertices has a perfect matching. A brick is minimal if for every edge $e$ the deletion of $e$ results in a graph that is not a brick. We prove a generation theorem for minimal bricks and two corollaries: (1) for $n \geq 5$, every minimal brick on $2n$ vertices has at most $5n - 7$ edges, and (2) every minimal brick has at least three vertices of degree three.

1. Introduction

All the graphs considered in this paper are finite and simple. A brick is a 3-connected graph such that the graph obtained from it by deleting any two distinct vertices has a perfect matching. The importance of bricks stems from the fact that they are building blocks of the matching decomposition procedure of Kotzig, and Lovász and Plummer [5]. In particular, many matching problems of interest (such as, for example, computing the dimension of the linear hull [2] or lattice [4] of incidence vectors of perfect matchings, or characterizing graphs that admit a “Pfaffian orientation” [7]) can be reduced to bricks.

In an earlier paper we proved a generation theorem for bricks. The precise statement requires a large number of definitions, and is given in Theorem 2.3 below. Let us describe the result informally first. Let $G$ be a graph, and let $v_0$ be a vertex of $G$ of degree two incident with the edges $e_1 = v_0v_1$ and $e_2 = v_0v_2$. Let $H$ be obtained from $G$ by contracting both $e_1$ and $e_2$ and deleting all resulting parallel edges. We say that $H$ was obtained from $G$ by bicontracting or bicontracting the vertex $v_0$, and write $H = G/v_0$. A subgraph $J$ of a graph $G$ is central if $G \setminus V(J)$ has a perfect matching. We say that a graph $H$ is a matching minor of a graph $G$ if $H$ can be obtained from a central subgraph of $G$ by repeatedly bicontracting vertices of degree two. We denote the fact that $H$ is isomorphic to a matching minor of $G$ by writing $H \hookrightarrow G$. Our generation theorem of [6] asserts that, except for a few well-described exceptions, if $H \hookrightarrow G$, then a graph isomorphic to $H$ can be obtained from $G$ by repeatedly applying a certain operation in such a way that all the intermediate graphs are bricks and no parallel edges are produced. The operation is as follows: first delete an edge, and for every vertex of degree two that results contract both edges incident with it. The theorem improves a recent result of de Carvalho, Lucchesi and Murty [1], but in this paper we seem to need our result.

We found our theorem useful for generating interesting examples of bricks and testing various conjectures, but even more useful was a variant for minimal bricks, which we prove...
in this paper. A brick $G$ is minimal if $G \setminus e$ is not a brick for every edge $e \in E(G)$. (We use \ for deletion.) The theorem asserts that every minimal brick other than the Petersen graph can be obtained from $K_4$ or the prism (the complement of a cycle of length six) by taking “strict extensions” in such a way that all the intermediate graphs are minimal bricks not isomorphic to the Petersen graph. The theorem is formally stated as Theorem 3.2. We postpone the definition of strict extensions until they are needed.

The paper is organized as follows. In the next section we introduce the results from [6] that we need. In Section 3 we state and prove our generation theorem for minimal bricks; we deduce it from the more general Theorem 3.1. In Section 4 we prove that, except for four graphs on at most eight vertices, every minimal brick on $2n$ vertices has at most $5n - 7$ edges. Finally, in Section 5 we prove that every minimal brick has at least three vertices of degree three.

2. The tools

In this section we state the results of [6] that we need, but let us start with the following theorem of Lovász [3]; see also [5, Theorem 5.4.11].

**Theorem 2.1.** Every brick has a matching minor isomorphic to $K_4$ or the prism.

The theorem of de Carvalho, Lucchesi and Murty [1] mentioned in the introduction uses $K_4$ and the prism as the starting graphs of their generation procedure. We use a more restricted set of operations, and the price we pay for that is that the starting set has to be expanded. We now introduce the relevant classes of graphs.

Let $C_1$ and $C_2$ be two vertex-disjoint cycles of length $n \geq 3$ with vertex-sets $\{u_1, u_2, \ldots, u_n\}$ and $\{v_1, v_2, \ldots, v_n\}$ (in order), respectively, and let $G_1$ be the graph obtained from the union of $C_1$ and $C_2$ by adding an edge joining $u_i$ and $v_i$ for each $i = 1, 2, \ldots, n$. We say that $G_1$ is a planar ladder. Let $G_2$ be the graph consisting of a cycle $C$ with vertex-set $\{u_1, u_2, \ldots, u_{2n}\}$ (in order), where $n \geq 2$ is an integer, and $n$ edges with ends $u_i$ and $u_{n+i}$ for $i = 1, 2, \ldots, n$. We say that $G_2$ is a Möbius ladder. A ladder is a planar ladder or a Möbius ladder. Let $G_1$ be a planar ladder as above on at least six vertices, and let $G_3$ be obtained from $G_1$ by deleting the edge $u_1u_2$ and contracting the edges $u_1v_1$ and $u_2v_2$. We say that $G_3$ is a staircase. Let $t \geq 2$ be an integer, and let $P$ be a path with vertices $v_1, v_2, \ldots, v_t$ in order. Let $G_4$ be obtained from $P$ by adding two distinct vertices $x, y$ and edges $xv_i$ and $yv_j$ for $i = 1, t$ and all even $i \in \{1, 2, \ldots, t\}$ and $j = 1, t$ and all odd $j \in \{1, 2, \ldots, t\}$. Let $G_5$ be obtained from $G_4$ by adding the edge $xy$. We say that $G_5$ is an upper prismoid, and if $t \geq 4$, then we say that $G_4$ is a lower prismoid. A prismoid is a lower prismoid or an upper prismoid.

We need the following strengthening of Theorem 2.1, proved in [6, Theorem (1.8)].

**Theorem 2.2.** Let $G$ be a brick not isomorphic to $K_4$, the prism or the Petersen graph. Then $G$ has a matching minor isomorphic to one of the following seven graphs: the graph obtained from the prism by adding an edge, the lower prismoid on eight vertices, the staircase
on eight vertices, the staircase on ten vertices, the planar ladder on ten vertices, the wheel on six vertices, and the Möbius ladder on eight vertices.

In the introduction we described our generation theorem by means of operations that reduce the larger graph $G$ to its matching minor $H$. This version is easier to describe concisely, but for both the proof and the applications it is better to proceed the other way, namely to describe how to obtain $G$ from $H$. Thus we reverse the process now and proceed in the other direction. Here are the relevant definitions.

Let $H, G, v_0, v_1, v_2, e_1, e_2$ be as in the definition of bicontraction. Assume that $v_1$, $v_2$ are not adjacent, that they both have degree at least three and that they have no common neighbors except $v_0$; then no parallel edges are produced during the contraction of $e_1$ and $e_2$. Let $v$ be the new vertex that resulted from the contraction. We say that $G$ was obtained from $H$ by bisplitting the vertex $v$. We call $v_0$ the new inner vertex and $v_1$ and $v_2$ the new outer vertices. Let $H$ be a graph. We wish to define a new graph $H''$ and two vertices of $H''$. Either $H'' = H$ and $u, v$ are two nonadjacent vertices of $H$, or $H''$ is obtained from $H$ by bisplitting a vertex, $u$ is the new inner vertex of $H''$ and $v \in V(H'')$ is not adjacent to $u$, or $H''$ is obtained by bisplitting a vertex of a graph obtained from $H$ by bisplitting a vertex, and $u$ and $v$ are the two new inner vertices of $H''$. Finally, let $H'$ be obtained from $H''$ by adding an edge with ends $u, v$. We say that $H'$ is a linear extension of $H$.

Since in the next theorem the graph $H$ need not be a brick we need two more exceptional classes of graphs. Let $C$ be an even cycle with vertex-set $v_1, v_2, \ldots, v_{2t}$ in order, where $t \geq 2$ is an integer and let $G_6$ be obtained from $C$ by adding vertices $v_{2t+1}$ and $v_{2t+2}$ and edges joining $v_{2t+1}$ to the vertices of $C$ with odd indices and $v_{2t+2}$ to the vertices of $C$ with even indices. Let $G_7$ be obtained from $G_6$ by adding an edge $v_{2t+1}v_{2t+2}$. We say that $G_7$ is an upper biwheel, and if $t \geq 3$ we say that $G_6$ is a lower biwheel. A biwheel is a lower biwheel or an upper biwheel. Please note that biwheels are bipartite, and therefore are not bricks.

We are now ready to state a version of our generation theorem [6, Theorem (1.10)]. The version mentioned in the introduction follows easily, because a linear extension of a brick is a brick.

**Theorem 2.3.** Let $G$ be a brick other than the Petersen graph, and let $H$ be a 3-connected matching minor of $G$. Assume that if $H$ is a planar ladder, then there is no strictly larger planar ladder $L$ with $H \leftrightarrow L \leftrightarrow G$, and similarly for Möbius ladders, wheels, lower biwheels, upper biwheels, staircases, lower prismoids and upper prismoids. If $H$ is not isomorphic to $G$, then some matching minor of $G$ is isomorphic to a linear extension of $H$.

### 3. Generation Theorem for Minimal Bricks

In this section we prove a generation theorem for minimal bricks, Theorem 3.2 below. We derive it from the more general Theorem 3.1.

If $H$ is a graph, and $u, v \in V(H)$ are distinct nonadjacent vertices, then $H + (u, v)$ or $H + uv$ denotes the graph obtained from $H$ by adding an edge with ends $u$ and $v$. If $u$ and $v$ are adjacent or equal then $H + uv = H$. Now let $u, v \in V(H)$ be adjacent. By bisubdividing...
the edge $uv$ we mean replacing the edge by a path of length three, say a path with vertices $u, x, y, v$, in order. Let $H'$ be obtained from $H$ by this operation. We say that $x, y$ (in that order) are the new vertices. Thus $y, x$ are the new vertices resulting from subdividing the edge $vu$ (we are conveniently exploiting the notational asymmetry for edges). Now if $w \in V(H) - \{u\}$, then by $H + (w, uv)$ we mean the graph $H' + (w, x)$. Notice that the graphs $H + (w, uv)$ and $H + (w, vu)$ are different.

Let $H$ be a graph, let $u, v \in V(H)$ be distinct, and let $H'$ be obtained from $H + uv$ by bisubdividing $uv$, where the new vertices are $x, y$. Let $x' \in V(H) - \{u\}$ and $y' \in V(H) - \{v\}$ be not necessarily distinct vertices such that not both belong to $\{u, v\}$. In those circumstances we say that $H' + (x, x') + (y, y')$ is a quasiquadratic extension of $H$. We say that it is a quadratic extension of $H$ if $u$ and $v$ are not adjacent in $H$. (Recall our convention that if $u$ and $v$ are adjacent in $H$, then $H + uv = H$.) We say that $uv$ is the base of this quasiquadratic extension.

Now let $u, v, H', x, y$ be as above, and let $a, b \in V(H)$ be not necessarily distinct vertices such that $\{u, v\} \neq \{a, b\}$, and if $a = b$ then $a \notin \{u, v\}$. If $a \neq b$, then let $H''$ be obtained from $H' + ab$ by bisubdividing $ab$, and let $x', y'$ be the new vertices. If $a = b$, then let $H''$ be obtained from $H'$ by adding new vertices $x', y'$ and edges $ax', x'y'$ and $ya$. Then the graph $H'' + (x, x') + (y, y')$ is called a quasiquartic extension of $H$. It is a quartic extension of $H$ if $uv, ab \in E(H)$. We say that $uv, ab$ are the bases of the quasiquartic extension. Quadratic and quartic extensions were used in the proof of Theorem 2.3 in [6]; quasiquadratic and quasiquartic extensions are new.

We need to define two new types of extension. We say that a linear extension $H'$ of a graph $H$ is strict if $|V(H')| > |V(H)|$. Let $u, v, w$ be pairwise distinct vertices of $H$, let $H'$ be obtained from $H$ by bisplitting $u$, and let $u_0$ be the new inner vertex and $u_1$ a new outer vertex. If $u_1v \in E(H')$ and $vw \notin E(H)$ then the graph $H' + (u_0, vu_1) + (y, w)$, where $x, y$ are the new vertices of $H' + (u_0, vu_1)$, is called a bilinear extension of $H$. If $uv \notin E(H)$ then the graph $H' + (u_0, u_1u_0) + (b, w)$, where $a, b$ are the new vertices of $H' + (u_0, u_1u_0)$, is called a pseudolinear extension of $H$. See Figure 1.

Finally, we say that $H'$ is a strict extension of $H$ if $H'$ is a quasiquadratic, quasiquartic, bilinear, pseudolinear or strict linear extension of $H$. It is not hard to see that a strict extension of a brick is a brick.

**Theorem 3.1.** Let $G$ be a brick other than the Petersen graph, and let $H$ be a 3-connected matching minor of $G$ such that $|V(H)| < |V(G)|$. Then some matching minor of $G$ is isomorphic to a strict extension of $H$.

**Proof.** Let a graph $H'$ be chosen so that $H$ is a spanning subgraph of $H'$, $H' \hookrightarrow G$ and $|E(H')|$ is maximal.

Suppose first that $H'$ is a planar ladder and there exists a planar ladder $L$ with $H' \hookrightarrow L \hookrightarrow G$ and $|V(L)| > |V(H')|$. Then clearly $H' = H$, and if we choose $L$ with $|V(L)|$ minimum, then $L$ is a quartic extension of $H$ and therefore the theorem holds. Therefore we can assume that if $H'$ is a planar ladder, then there is no strictly larger planar ladder $L$ with
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Figure 1. (a) Bilinear extension, (b) Pseudolinear extension

$H \rightarrow L \rightarrow G$, and similarly for M"obius ladders, wheels, lower biwheels, upper biwheels, staircases, lower prismoids and upper prismoids. By Theorem 2.3 and the choice of $H'$ there exists a strict linear extension $K$ of $H'$ such that $K \rightarrow G$. We denote $E(H') - E(H)$ by $E'$ and break the analysis into cases depending on the type of strict linear extension.

Suppose first that $K = K' + uv$, where $K'$ is obtained from $H'$ by bisplitting a vertex, $v$ is the new inner vertex of $K'$ and $u \in V(H')$. Let $v_1$ and $v_2$ be the new outer vertices. We have $E(H') \subseteq E(K')$, in the natural way. For $i = 1, 2$ let $d_i$ be the number of edges of $E(H)$ that are incident with $v_i$ in $K'$ (or $K$). We assume without loss of generality that $d_1 \geq d_2$. Note that $d_1 + d_2 \geq 3$, because $v$ has degree at least three in $H$.

If $d_2 \geq 2$ then $K \setminus E'$ is a strict linear extension of $H$. If $d_2 = 1$ let $f \in E'$ be an edge incident with $v_2$; then $K \setminus (E' - \{f\})$ is a quadratic extension of $H$. Finally, if $d_2 = 0$ and $f_1, f_2 \in E'$ are incident with $v_2$ then $K \setminus (E' - \{f_1, f_2\})$ is a quasiquadratic extension of $H$.

Now suppose $K = K' + u_1u_2$, where $K'$ is obtained by bisplitting a vertex of a graph obtained from $H'$ by bisplitting a vertex, and $u_1$ and $u_2$ are the two new inner vertices of $K'$. Let $v_1, v_2$ and $v_3, v_4$, respectively, be the corresponding new outer vertices. Let $d_1, d_2, d_3$ and $d_4$ be defined analogously as above. We start by assuming that $v_1, v_2, v_3$ and $v_4$ are pairwise distinct and without loss of generality assume $d_1 \geq d_2, d_3 \geq d_4 \geq d_2$.

If $d_2 \geq 2$ then $K \setminus E'$ is a strict linear extension of $H$. If $d_2 = 1, d_4 \geq 2$ then $K \setminus E'/v_2$ is isomorphic to a strict linear extension of $H$ unless the edge of $H$ incident with $v_2$ is incident also with one of the vertices $v_3$ and $v_4$. In this case $K \setminus (E' - \{f\})$ is a bilinear extension of
H, for every \( f \in E' \) incident with \( v_2 \). If \( d_2 = d_4 = 1 \) for \( i \in \{1, 2\} \) let \( e_i \) denote the unique edge in \( E(H) \) incident with \( v_{2i} \) and let \( f_i \) denote some edge in \( E' \) incident with \( v_{2i} \). If \( e_1 = e_2 \) then \( K \setminus (E' - \{f_1, f_2\}) \) is a quasiquartic extension of \( H \). (If \( f_1 \) is adjacent to \( f_2 \), then we need the provision of \( a = b \) in the definition of quasiquartic extension.) Otherwise, without loss of generality we assume that \( e_2 \) is not incident with \( v_1 \) and deduce that \( K \setminus (E' - \{f_1\})/v_4 \) is a quadratic extension of \( H \) with base \( e_1 \).

It remains to consider the subcase when \( d_2 = 0 \). Let \( f, f' \in E' \) be incident with \( v_2 \) such that \( f \) has no end in \( \{v_3, v_4\} \). If \( d_4 \geq 2 \) then \( K \setminus (E' - \{f\})/u_1v_1/u_1 \) is a strict linear extension of \( H \). If \( d_4 = 1 \) let \( e \) denote the unique edge in \( E(H) \) incident with \( v_4 \). If \( e \) is not incident with \( v_1 \) then \( K \setminus (E' - \{f, f'\})/v_4 \) is a quasiquadratic extension of \( H \) if \( f' \) is not incident with \( v_4 \) and \( K \setminus (E' - \{f, f'\}) \) is a quasiquartic extension of \( H \) if \( f' \) is incident with \( v_4 \). If on the other hand \( e \) is incident with \( v_1 \) then \( K \setminus (E' - \{f, f''\})/u_1v_1/u_1 \) is a quadratic extension of \( H \), where \( f'' \) is any edge in \( E' \) incident with \( v_4 \). Finally, if \( d_4 = 0 \) let \( f^* \in E' \) be incident with \( v_4 \) and have no end in \( \{v_1, v_2\} \). Then \( K \setminus (E' - \{f, f', f^*\})/u_2v_3/u_2 \) is a quasiquadratic extension of \( H \). This completes the case when \( v_1, v_2, v_3 \) and \( v_4 \) are pairwise distinct.

We now assume without loss of generality that \( v_1 = v_4 \). Then \( v_1, v_2 \) and \( v_3 \) are pairwise distinct and we assume \( d_2 \geq d_3 \), again without loss of generality. Suppose first \( d_1 = 0 \). If \( d_3 \geq 2 \) then \( K \setminus (E' - \{g\})/v_3 \) is a pseudolinear extension of \( H \), where \( g \in E' \) is incident with \( v_1 \); if \( d_3 = 1 \) then \( K \setminus (E' - \{g\})/v_3 \) is a quadratic extension of \( H \) and if \( d_3 = 0 \) then \( K \setminus (E' - \{f, g\})/v_3 \) is a quasiquadratic extension of \( H \), where \( f \) is an edge in \( E' \) incident with \( v_3 \) and not adjacent to \( g \). Therefore we may assume \( d_1 \geq 1 \). If \( d_2 \geq 2 \) and \( d_3 \geq 1 \) then \( K \setminus E' \) or \( K \setminus E'/v_3 \) is a strict linear extension of \( H \). If \( d_2 \geq 2 \) and \( d_3 = 0 \) then \( K \setminus (E'/f)/v_3 \) is a quadratic extension of \( H \), where \( f \) is as above. If, finally, \( d_2 \leq 1 \) then let \( E'' \) be obtained from \( E' \) by deleting \( 2 - d_2 \) edges of \( E' \) incident with \( v_2 \) and \( 1 - d_3 \) edges incident with \( v_3 \); in that case \( K \setminus E'' \setminus v_1u_2/u_2 \) is a quasiquadratic extension of \( H \).

This completes the case analysis. \( \square \)

Theorem 3.1 implies the following generation theorem for minimal bricks.

**Theorem 3.2.** Let \( G \) be a minimal brick other than the Petersen graph. Then \( G \) can be obtained from \( K_4 \) or the prism by taking strict extensions, in such a way that all the intermediate graphs are minimal bricks not isomorphic to the Petersen graph.

**Proof.** Suppose the statement of the theorem is false and let \( G \) be a counterexample with \( |V(G)| \) minimum.

By Theorem 2.1 we may choose a minimal brick \( H \leftrightarrow G \) such that \( H \) can be obtained from \( K_4 \) or the prism by taking strict extensions and, subject to that, \( |V(H)| \) is maximum. If \( |V(H)| = |V(G)| \) then \( H \) is isomorphic to \( G \) by the minimality of \( G \). If, on the other hand, \( |V(H)| < |V(G)| \), then by Theorem 3.1 there exists a strict extension \( H' \leftrightarrow G \) of \( H \). Let \( H'' \leftrightarrow H' \) be a minimal brick with \( |V(H'')| = |V(H')| \); then \( H'' \leftrightarrow G \). It follows that \( H'' \) is not isomorphic to \( G \), for otherwise so is \( H' \), contrary to our assumption that \( G \) is a
counterexample to the theorem. By the minimality of $G$ the graph $H''$ can be obtained from $K_4$ or the prism by taking strict extensions, contrary to the choice of $H$. \hfill \Box

Note that there exist bricks obtained from $K_4$ or the prism by a sequence of strict extensions, that are not minimal. A simple example follows.

Let $G$ be the prism, $V(G) = \{v_1, v_2, v_3, u_1, u_2, u_3\}$, the vertices $v_1, v_2, v_3$ are pairwise adjacent and so are the vertices $u_1, u_2, u_3$, and $u_i$ is adjacent to $v_i$ for $i \in \{1, 2, 3\}$. Let $G' = G + u_4v_2$ and let $G'' = G' + (u_2, u_1v_2) + v_1y$, where $x, y$ are the new vertices of $G' + (u_2, u_1v_2)$. Then $G''$ is a quasiquadratic extension of $G$ and $G'' \setminus u_1v_1$ is a brick, which can be obtained from a prism by a quadratic extension or a sequence of two linear extensions.

4. Edge Bound for Minimal Bricks

The following theorem is [5, Corollary 5.4.16].

**Theorem 4.1.** If $G$ is a minimal bicritical graph with $n \geq 6$ vertices, then $|E(G)| \leq 5(n - 2)/2$.

We use Theorem 3.1 to prove a similar bound for minimal bricks.

**Theorem 4.2.** Let $G$ be a minimal brick on $2n$ vertices. Then $|E(G)| \leq 5n - 7$, unless $G$ is the prism or the wheel on four, six or eight vertices.

**Proof.** The theorem holds for the Petersen graph, so from now on we assume that $G$ is not the Petersen graph, the prism or the wheel on six or eight vertices. Denote the last three graphs by $R_6, W_6$ and $W_8$, respectively.

Note that a strict linear extension increases the number of vertices in a graph by 2 or 4 and the number of edges by 3 or 5, respectively. Similarly, a quasiquadratic extension increases the number of vertices by 2 and the number of edges by at most 5, while quasiquartic, bilinear and pseudolinear extensions increase the number of vertices by 4 and the number of edges by at most 8.

We say that a brick $H$ is sparse if $|E(H)| \leq \frac{5}{2}|V(H)| - 7$ and we say that $H$ is dense otherwise. We claim that any minimal brick that contains a sparse matching minor is sparse. Suppose $G_1$ and $G_2$ are bricks, $G_1 \leftarrow G_2$, $G_1$ is sparse and $G_2$ is minimal. Let a sparse brick $H \leftarrow G_2$ be chosen with $|V(H)|$ maximum. From Theorem 3.1 we deduce that either $|V(H)| = |V(G_2)|$ or some strict extension $H'$ of $H$ is a matching minor of $G_2$. In the latter case, by the calculations above, $H'$ is sparse in contradiction with the choice of $H$. Therefore $|V(H)| = |V(G_2)|$ and $G_2$ is isomorphic to $H$ by the minimality of $G_2$. The claim follows.

Suppose $G$ is dense. By Theorem 2.2 $G$ has a matching minor isomorphic to one of the seven graph mentioned therein, and hence $G$ has a matching minor isomorphic to one of the following four graphs: $R_6, W_6$, the staircase on eight vertices, and the Möbius ladder on eight vertices. Among these graphs only two are dense: $R_6$ and $W_6$.

Assume first that $G$ contains $R_6$ as a matching minor. By Theorem 3.1 there exists a strict extension $H$ of the prism such that $H \leftarrow G$. By the calculations above $H$ is sparse, unless $H$ is a quadratic extension of $R_6 + uv$ with base $uv$, where $uv \notin E(R_6)$. We will show
that there exists $e \in E(H)$ such that $H \setminus e$ is a brick. Note that $H \setminus e$ is sparse. Therefore it follows that any minimal brick containing the prism as a matching minor and not equal to it is sparse. We prove the existence of $e$ by listing all possible quasiquadratic extensions of $R_6$ with 14 edges in Figure 2. An edge $e$ that satisfies the conditions above is indicated by a cross. A spanning bisubdivision or bisplit of $R_6$ or $W_6$ in $H \setminus e$ is indicated by bold lines and allows the reader to easily verify that the claim holds in each of the cases.

Therefore we may assume that $G$ contains $W_6$ as a matching minor and does not contain $R_6$. By Theorem 2.3 $G$ is a wheel or $G$ contains a linear extension of a wheel as a matching
minor. All the wheels on at least ten vertices and all strict linear extensions of $W_6$ and $W_8$ are sparse and therefore $G$ must contain a graph obtained from $W_6$ or $W_8$ by an edge addition. Every graph obtained from $W_6$ by adding an edge has a matching minor isomorphic to a graph obtained from $W_6$ by adding an edge. The latter graph is unique up to isomorphism and contains $R_6$ as a spanning subgraph, in contradiction with our assumptions. □

The bound given in Theorem 4.2 is tight for every $n \geq 4$. An example of a minimal brick $G_n$ on $2n + 4$ vertices with $5n + 3$ edges for $n \geq 2$ follows. Let $V(G_n) = \{x, y, z, t, v_1, u_1, v_2, u_2, \ldots, v_n, u_n\}$. For every $i \in \{1, 2, \ldots, n\}$ let $xt, yt, zt, xu_i, yu_i, yv_i, zv_i$ and $u_iv_i$ be the edges of $G_n$. Then for every $e \in E(G_n)$ the graph $G_n \setminus e$ contains a vertex of degree two, and hence is not a brick. It remains to show that $G_n$ is a brick for every $n$. Note that $G_k$ is a quasiquadratic extension of $G_{k-1}$ for every $k > 2$. Therefore it suffices to show that $G_2$ is a brick. The graph $G_2 \setminus u_1y \setminus v_1y$ is isomorphic to the prism with one of its edges bisubdivided and consequently $G_2$ can be obtained from the prism by a quadratic extension.

5. Three Cubic Vertices

De Carvalho, Lucchesi and Murty [1] proved that every minimal brick has a vertex of degree three. According to them (private communication) it had been conjectured by Lovász. We prove the following strengthening.

Theorem 5.1. Every minimal brick has at least three vertices of degree three.

Proof. Let a minimal brick $G$ that has at most two vertices of degree three be chosen with $|V(G)|$ minimal. By Theorem 3.2 there exists a minimal brick $H \hookrightarrow G$ with at least three vertices of degree three, such that $G$ is isomorphic to a strict extension of $H$.

Note that if a strict linear extension is used to obtain $G$ from $H$ then the degree of at most one vertex of $H$ increased and at least one vertex in $V(G) - V(H)$ has degree three. If a quasiquartic, bilinear or pseudolinear extension is used to obtain $G$ then $V(G) - V(H)$ contains at least three vertices of degree three. Therefore $G$ is isomorphic to a quasiquadratic extension of $H$ that is not quadratic.

We assume without loss of generality that $V(G) - V(H) = \{u_1, u_2\}$ and there exist $v_1, v_2, v_3, v_4 \in V(H)$ such that $E(G) - E(H) = \{u_1v_1, u_1v_2, u_2v_3, u_2v_4, u_1u_2\}$, at least three of the vertices $v_1, v_2, v_3, v_4$ are distinct, $v_1 \neq v_2$ and $v_3 \neq v_4$. Note that the vertices of degree three in $H$ must form a subset of $\{v_1, v_2, v_3, v_4\}$ and that $v_1v_3, v_2v_3, v_2v_4, v_1v_4 \notin E(H)$, for the deletion of such an edge from $G$ results in a quadratic extension of $H$, contrary to the fact that $G$ is a minimal brick.

Since $H$ is a brick, it is not a biwheel. By Theorem 2.3 either $H$ is a ladder, wheel, staircase or prismoid or $H$ is a linear extension of a brick. If $H$ is a ladder, wheel, staircase or prismoid distinct from $K_4$ then $H$ has at least 5 vertices of degree three, and consequently $G$ has at least three vertices of degree three. If $H = K_4$ then $G$ is not minimal, by an observation in the previous paragraph.
Therefore, $H$ is a linear extension of a brick, and hence there exists $e \in E(H)$ such that $H \setminus e$ becomes a brick after possible bicontractions of vertices of degree two in such a way that no parallel edges are created by these bicontractions. Note that $H$ is minimal and therefore at least one end of $e$ is a vertex of degree three in $H$. Assume first that exactly one end of $e$ has degree three in $H$. Without loss of generality this end is $v_1$. The graph $G \setminus e$ is a brick, because it can be obtained by a linear extension (first bisplit to produce $H \setminus e$, then add the edge $v_1v_3$) followed by a quadratic extension with base $v_1v_3$, a contradiction. Recall that $v_1$ is not adjacent to $v_3$ in $H$.

It remains to consider the case when both of the ends of $e$ have degree three in $H$. Without loss of generality we assume that $e = v_1v_2$, and hence $v_1, v_2, v_3$ and $v_4$ are pairwise distinct. It follows that $G \setminus e$ is a strict linear extension of $H + v_1v_3 + v_1v_4$ and is again a brick. This completes the case analysis. □

We conjecture the following strengthening of Theorem 5.1.

**Conjecture 5.2.** There exists $\alpha > 0$ such that every minimal brick $G$ has at least $\alpha |V(G)|$ vertices of degree three.

Even a much weaker strengthening, namely, the conjecture that every brick has at least four vertices of degree three, seems to require new ideas or a substantial refinement of our techniques.

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