ADEQUACY OF NONSINGULAR MATRICES OVER COMMUTATIVE PRINCIPAL IDEAL DOMAINS

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Dedicated to the memory of Professor Bohdan Zabavsky

Abstract. The notion of the adequacy of commutative domains was introduced by Helmer in Bull. Amer.Math. Soc., 49 (1943), 225–236. In the present paper we extend the concept of adequacy to noncommutative Bézout rings. We show that the set of nonsingular second-order matrices over a commutative principal ideal domain is adequate.

1. Introduction and results

Let \( U(R) \) be the group of units of an associative commutative ring \( R \) with \( 1 \neq 0 \). The elements \( a, b \in R \) are called strong associates if there exists \( e \in U(R) \) such that \( a = be \) (see [1, Definition 2.1, p. 441] and [2]). The set of all non strong associate elements of the ring \( R \) is denoted by \( R^\ast \). The matrix \( \text{diag}(d_1, \ldots, d_n) \) means a matrix having \( d_1, \ldots, d_n \in R \) on the main diagonal and zeros elsewhere (by the main diagonal we mean the one beginning at the upper left corner). The set of all matrices of size \( n \times m \) over a ring \( R \) is denoted by \( R^{n \times m} \).

A commutative ring \( R \) is called an elementary divisor ring [7, p. 465], if for each matrix \( A \in R^{n \times m} \) there exist invertible matrices \( P_A \) and \( Q_A \), such that

\[
P_A A Q_A = \text{diag}(\alpha_1, \ldots, \alpha_s) \in R^{n \times m},
\]

in which \( s := \min(n, m) \) and each \( \alpha_i \) is a divisor of \( \alpha_{i+1} \) for \( i = 1, \ldots, s - 1 \). The diagonal matrix \( \text{diag}(\alpha_1, \ldots, \alpha_s) \) is called a Smith form (unique up to strong associates of its diagonal elements). Accordingly, we can always choose \( \alpha_1, \ldots, \alpha_s \in R^\ast \), so the matrix \( \text{diag}(\alpha_1, \ldots, \alpha_s) \) is uniquely defined, and it is called the Smith normal form (SNF) of the matrix \( A \) and is denoted by \( \text{SNF}(A) \). The matrices \( P_A \) and \( Q_A \) (see (1)) are called left and right transformation matrices of the matrix \( A \), respectively. The greatest common divisor and the least common multiple of \( a_1, \ldots, a_t \in R \), which are unique up to strong associates, are denoted by \( (a_1, \ldots, a_t) \) and \( [a_1, \ldots, a_t] \), respectively; and \( a \mid b \) means that \( a \) is a divisor of \( b \).

Let \( R \) be a commutative ring with \( 1 \neq 0 \) in which every finitely generated ideal is principal (Bézout ring). By a relatively prime part of \( b \in R \) with respect to \( a \in R \) written as \( \text{RP}(b, a) \), is a divisor \( t \) of \( b = st \) such that \( (t, a) = 1 \), but \( (s', a) \neq 1 \) for any non-unit divisor \( s' \in R \) of \( s \in R \). The element \( s \) is called an adequate part of \( b \) with respect to \( a \). The commutative ring \( R \) is called adequate [3, p. 225] if \( \text{RP}(b, a) \) exists for all \( a, b \in R \) with \( b \neq 0 \).

This concept is basically a formalization of the properties of the entire analytic functions rings. Each commutative principal ideal domain (PID) is adequate, but the converse is not true, in particular, the chain condition right may not be satisfied. Each adequate ring is an elementary divisor ring [6, Theorem 3, p. 234]. The ring of all continuous real-valued functions
defined on a completely regular (Hausdorff) space $X$ is an example of an adequate ring, which is regular and every prime ideal is maximal \[5\] Corollaries 3.6, 3.8 p. 386. Each local ring as well as each commutative von Neumann regular ring is adequate \[4\] Theorem 11, p. 365. Adequate rings with zero-divisors in their Jacobson radical were investigated by Kaplansky \[7\] Theorem 5.3, p. 473. Note that not every elementary divisor ring is adequate \[5\] Corollary 6.7, p. 386] and in an adequate domain each nonzero prime ideal is contained in a unique maximal ideal \[5\] Corollary 6.6, p. 386]. Bézout rings in which each regular element is adequate were investigated in \[11\]. Moreover, in \[10\] generalized adequate rings were introduced, that is a new family of elementary divisor rings which contains adequate rings.

Gatalevych \[3\] was the first who tried to apply the notion of adequacy to noncommutative rings. He proposed a new concept of adequacy of noncommutative rings and proved that the generalized right adequate (in the sense of Gatalevych) duo Bézout domain is an elementary divisor domain \[3\] Theorem 2, p. 117]. In the present article we propose another definition of adequate rings, which differs from the one proposed by Gatalevych \[3\] Definition 1, p. 116]. Using an example in §4, we highlight certain advantages of our definition.

Our definition of the adequacy of a ring is the following:

Let $K$ be a Bézout (not necessary commutative) ring with $1 \neq 0$ and let $a \in K$. An element $b \in K$ is called a left adequate to $a \in K$ if there exist $s, t \in K \setminus U(K)$ such that $b = st$ and the following conditions hold:

\begin{enumerate}[(i)]
  \item $s'K + aK \neq K$ for each $s' \in K \setminus U(K)$ such that $sK \subset s'K \neq K$;
  \item for each $t' \in K \setminus U(K)$ such that $tK \subset t'K \neq K$ there exists a decomposition $st' = pq$ such that $pK + aK = K$.
\end{enumerate}

The element $s$ is called a left adequate part of $b$ with respect to $a$. The right adequate part of $b$ with respect to $a$ is defined by analogy.

A subset $A \subseteq K$ is called left (right) adequate if each of its elements is left (right) adequate to all elements of $A$. If each element of $A$ is left and right adequate to the rest of elements, then the set $A$ is called adequate.

It is easy to see if $K$ is a commutative PID, then our definition coincides with the definition given by Helmer \[6\] p. 225].

Our first main result is the following:

**Theorem 1.** Let $R$ be a commutative principal ideal domain such that $1 \neq 0$. The set of nonsingular $2 \times 2$ matrices over $R$ is an adequate set.

Let $R$ be a commutative PID with $1 \neq 0$. A subset of $R^*$ consisting of all indecomposable divisors of an element $a \in R$ is called the spectrum of $a$ and is denoted by $\Sigma(a)$. Of course, we always assume $1 \in \Sigma(a)$. The spectrum of a matrix $A \in R^{2 \times 2}$ is the set $\Sigma(A) := \Sigma(\alpha_2)$ (see \[11\]). The matrices $M, N \in R^{2 \times 2}$ are called strong right associates if there is a matrix $U \in \text{GL}_2(R)$ such that $M = NU$.

Let $A, B, C, D, A_1, B_1 \in R^{2 \times 2}$. If $A = BC$, then $A$ is called a right multiple of $B$. If $A = DA_1$ and $B = DB_1$, then $D$ is called a left common divisor of $A$ and $B$. In addition, if $D$ is a right multiple of each left common divisor of $A$ and $B$, then $D$ is called a left greatest common divisor of $A$ and $B$, which we denoted by $D := (A, B)_l$. The left greatest common divisor $D$ is unique up to right strong associates \[5\] Theorem 1.12, p. 39].
Let $A \in R^{2 \times 2}$ with the Smith normal form $\text{SNF}(A) := \text{diag}(\alpha_1, \alpha_2)$. In the sequel we use the following notation for this matrix:

$$A := P_A^{-1} \cdot \text{diag}(\alpha_1, \alpha_2) \cdot Q_A^{-1},$$

(2)

where $P_A, P_B$ are the left and right transforming matrices of $A$. This means that the datum of each matrix determines a triple of matrices: left and right transforming matrices, and its Smith normal form.

Our next main result is the following:

**Theorem 2.** Let $R$ be a commutative principal ideal domain and let

$$A := P_A^{-1} \cdot \text{diag}(\alpha_1, \alpha_2) \cdot Q_A^{-1} \quad \text{and} \quad S := P_S^{-1} \cdot \text{diag}(\sigma_1, \sigma_2) \cdot Q_S^{-1}$$

be nonsingular matrices of the form (2) from $R^{2 \times 2}$.

Each left divisor of the matrix $S$ has a nontrivial left common divisor with the matrix $A$ if and only if $\Sigma(\sigma_i) \subseteq \Sigma(\alpha_i)$ for $i = 1, 2$ and one of the following conditions holds:

(i) if $\Sigma(\sigma_2) \subseteq \Sigma(\alpha_1)$, then there isn’t any restriction on the structure of $P_S$;
(ii) if $\Sigma(\sigma_2) \backslash \Sigma(\alpha_1) = \{q_1, \ldots, q_k\}$, where $k \geq 1$, then

$$P_S = \begin{bmatrix} m_{11} & m_{12} \\ (q_1 \cdots q_k) m_{21} & m_{22} \end{bmatrix} P_A, \quad (m_{ij} \in R).$$

2. Preliminaries, lemmas and proofs

For each pair $A, B \in R^{2 \times 2}$ of the form (2) we define the matrix $[\tau_{ij}] := P_B P_A^{-1}$ and the set

$$L_{\alpha_1, \beta_2} := \left\{ \begin{bmatrix} l_{11} & l_{12} \\ \frac{\beta_2}{(\beta_2, \alpha_1)} l_{12} & l_{22} \end{bmatrix} \in \text{GL}_2(R) \mid l_{ij} \in R \right\}.

(3)

In the sequel we freely use the following facts:

**Fact 1.** [9] Theorem 1, p. 851] Let $R$ be a commutative PID and let $A, B \in R^{2 \times 2}$ of the form (2). The following conditions hold:

(i) the Smith normal form of $(A, B)_1$ has the following form:

$$\text{diag}((\alpha_1, \beta_1), (\alpha_2, \beta_2, [\alpha_1, \beta_1] \tau_{21}));$$

(ii) $A, B$ are left relatively prime (i.e. $(A, B)_1 = I$) if and only if

$$(\alpha_2, \beta_2, [\alpha_1, \beta_1] \tau_{21}) = 1.$$

**Fact 2.** [8] Theorem 4.2, p. 127] Let $R$ be a commutative PID and let $A, B \in R^{2 \times 2}$ of the form (2). The matrix $B$ is a left divisor of $A$ (i.e. $A = BC$) if and only if $\beta_i | \alpha_i$ for $i = 1, 2$ and $P_B = LP_A$, in which $L \in L_{\alpha_1, \beta_2}$ (see (3)).

**Fact 3.** [8] Theorem 4.4, p. 128] Let $R$ be a commutative PID and let $A \in R^{2 \times 2}$ of the form (2). Let $\beta_1, \beta_2 \in R$ such that $\beta_1 | \beta_2$ and $\beta_i | \alpha_i$ for $i = 1, 2$. The set of the form

$$(L_{\alpha_1, \beta_2} P_A)^{-1} \cdot \text{diag}(\beta_1, \beta_2) \cdot \text{GL}_2(R) \subset R^{2 \times 2}$$

is the set of all left divisors $L$ of $A$ for which $\text{SNF}(L) = \text{diag}(\beta_1, \beta_2)$.

**Lemma 1.** Let $R$ be a commutative Bézout domain and let $A, B \in R^{n \times n}$ (with $n \geq 2$) such that $\det(B)$ is indecomposable in $R$. If $(A, B)_1 \neq I$ then $A = BC$. 
Proof. Let \( D := (A, B)_l \neq I \). Clearly \( B = DB_1 \) and \( \det(D) \mid \det(B) \). Thus \( \det(D) \) and \( \det(B) \) are strong associates in \( R, \) i.e. \( \det(B) = \det(D)e \) for some \( e \in U(R) \). Consequently, \( \det(B_1) = e, \) so \( B_1 \in \text{GL}_n(R) \) and \( D = BB^{-1}_1. \) Since \( A = DA_1, \) we have \( A = BB^{-1}_1A_1 = BC, \) where \( C = B_1^{-1}A_1. \) \( \square \)

Proof of the 'if' part of Theorem 2. Let \( 1 \neq \omega \in \Sigma(\sigma_1). \) Thus \( \sigma_1 = \omega \sigma'_1 \) and \( \sigma_2 = \omega \sigma'_2 \) for some \( \sigma'_1, \sigma'_2 \in R. \) If \( M := P^{-1} \cdot \text{diag}(1, \omega) \cdot Q^{-1} \) and

\[
M_1 := (Q \cdot \text{diag}(\omega, 1) \cdot P) \left( P^{-1} \cdot \text{diag}(\sigma'_1, \sigma'_2) \cdot Q^{-1} \right),
\]

in which \( P, Q \) are arbitrary invertible matrices, then

\[
S = P^{-1} \cdot \text{diag}(\sigma_1, \sigma_2) \cdot Q^{-1} = M \cdot M_1.
\]

Taking into account that \( \det(M) \) is indecomposable in \( R \) and \( (A, M)_l \neq I, \) we obtain that \( A = MA_1 \) by Lemma 1. Consequently, all matrices \( L \) with \( \text{SNF}(L) = \text{diag}(1, \omega) \) are left divisors of \( A. \) In accordance with [5] Theorem 5.3 p.152 and Property 4.11 p.147] we have \( \omega|\sigma_1 \) and \( \Sigma(\sigma_1) \subseteq \Sigma(\sigma_2). \)

Let \( \mu \in \Sigma(\sigma_2) \setminus \Sigma(\sigma_1). \) Thus \( \sigma_2 = \mu \cdot \mu_1 \) and \( (\mu, \sigma_2) = 1. \) If \( C := P^{-1} \cdot \text{diag}(1, \mu) \) and \( C_1 := \text{diag}(\sigma_1, \mu_1) \cdot Q^{-1}, \) then \( S = CC_1. \) Since \( (\det(C), \det(A)) = 1, \) we have \( (A, C)_l = I, \) a contradiction. Consequently \( \Sigma(\sigma_2) \subseteq \Sigma(\sigma_1). \)

Let \( \Sigma(\sigma_2) \setminus \Sigma(\sigma_1) = \{ q_1, \ldots, q_k \} \) for \( k \geq 1 \) and let \( i \in \{1, \ldots, k\}. \) Thus \( \sigma_2 = q_i \delta_i \) for some \( \delta_i \in R. \) If \( D := P^{-1} \cdot \text{diag}(q_i) \) and \( D_1 := \text{diag}(\sigma_1, \delta_i) \cdot Q^{-1}, \) then

\[
S = P^{-1} \cdot \text{diag}(\sigma_1, \sigma_2) \cdot Q^{-1} = D \cdot D_1.
\]

All left divisors of \( S \) with the Smith normal form \( \text{diag}(1, q_i) \) (included \( D \)) belong to the set

\[
W = \left\{ (L_{\sigma_1, q_i} P) S^{-1} \cdot \text{diag}(1, q_i) \cdot \text{GL}_2(R) \right\}
\]

by Fact 3

\[
= \left\{ \left( \begin{array}{cc}
 l_{11} & l_{12} \\
 q_i l_{12} & l_{22}
\end{array} \right) P S^{-1} \cdot \text{diag}(1, q_i) \cdot \text{GL}_2(R) \mid (q_i, \sigma_1) = 1 \right\}
\]

Let us fix \( M := P^{-1} \cdot \text{diag}(1, q_i) \cdot Q^{-1} \in W, \) in which \( P := \begin{bmatrix} h_{11} & h_{12} \\
 q_i h_{21} & h_{22} \end{bmatrix} P \) for some \( h_{kl} \in R \) and \( Q := \text{GL}_2(R) \) is fixed. The matrix \( M \) is a left divisor of \( S, \) so \( (A, M)_l \neq I. \) Hence \( d_i := (\alpha_2, q_i, \alpha_1 \tau_2^{(i)}) \neq 1 \) (see Fact 1(ii)), where

\[
\tau_{mn}^{(i)} := P M A^{-1} = \begin{bmatrix} h_{11} & h_{12} \\
 h_{21} & h_{22} \end{bmatrix} \left( P S P A^{-1} \right).
\]

Since \( d_i | q_i \) and \( q_i \) is an indecomposable element, \( d_i \) and \( q_i \) are strong associates (i.e. \( d_i = q_i e_i \) for some \( e_i \in U(R) \)). Therefore

\[
q_i e_i = (\alpha_2, q_i, \alpha_1 \tau_2^{(i)}) = (\alpha_2, (q_i, \alpha_1 \tau_2^{(i)})) = (\alpha_2, q_i, \tau_2^{(i)}),
\]

so \( q_i | \tau_2^{(i)}, \) i.e. \( \tau_2^{(i)} = q_i n_i \) for some \( n_i \in R. \) It is obvious (see [1]) that

\[
P S P A^{-1} = \begin{bmatrix} h_{11} & h_{12} \\
 q_i h_{21} & h_{22} \end{bmatrix}^{-1} \begin{bmatrix} \tau_{11}^{(i)} & \tau_{12}^{(i)} \\
 q_i n_i & \tau_{22}^{(i)} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\
 q_i p_{21} & p_{22} \end{bmatrix}, \quad (p_{mn} \in R).
\]

This yields

\[
P S = \begin{bmatrix} p_{11} & p_{12} \\
 q_i p_{21} & p_{22} \end{bmatrix} P_{\alpha}.
\]
Let us show that (5) holds independently of the choices of $P_S$ and $P_A$. If we choose a different ordered pair $(P_S', P_A') \neq (P_S, P_A)$, then (see [8 Property 2.2 p. 61]) we obtain that

$$P_S' = FP_S \quad \text{and} \quad P_A' = T P_A,$$

where $F := \begin{bmatrix} f_{11} & f_{12} \\ \frac{2\pi}{\sigma} f_{21} & f_{22} \end{bmatrix}$, $T^{-1} := \begin{bmatrix} t_{11} & t_{12} \\ \frac{\alpha}{\sigma} t_{21} & t_{22} \end{bmatrix} \in \text{GL}_2(R)$ and $f_{mn}, t_{mn} \in R$. Thus

$$P_S' (P_A')^{-1} = F (P_S P_A^{-1}) T^{-1} = \begin{bmatrix} f_{11} & f_{12} \\ \frac{2\pi}{\sigma} f_{21} & f_{22} \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ q_{p21} & p_{22} \end{bmatrix} \begin{bmatrix} t_{11} & t_{12} \\ \frac{\alpha}{\sigma} t_{21} & t_{22} \end{bmatrix}.$$

Since $\Sigma(q_i) \subseteq \Sigma\left(\frac{2\pi}{\sigma}\right) \cap \Sigma\left(\frac{\alpha}{\sigma}\right)$, from the last equality we obtain that

$$P_S' = \begin{bmatrix} p_{11}' & p_{12}' \\ q_{p21}' & p_{22}' \end{bmatrix} P_A' \quad (p_{mn}' \in R).$$

Consequently, (5) is independent by choice of $P_S$ and $P_A$.

Now we need to do the same with the rest of the elements of the set $\{q_1, \ldots, q_k\}$. As a result the matrix $P_S$ has the form as in Theorem 2(ii).

Proof of the ‘only if’ part. Let $S = LM$, in which the nontrivial divisor $L := P_L^{-1} \cdot \text{diag}(\lambda_1, \lambda_2) \cdot Q_L^{-1}$ has the form (2).

(i) If $\Sigma(\sigma_2) \subseteq \Sigma(\alpha_1)$, then $\Sigma(\lambda_2) \subseteq \Sigma(\sigma_1)$ by Fact 2. This yields $(\alpha_2, \lambda_2, \alpha_1) \neq 1$, so $(A, L)i \neq I$ by Fact 1(ii) for arbitrary $P_S \in \text{GL}_2(R)$.

(ii) Let $\Sigma(\sigma_2) = \Sigma(\alpha_1) \cup \{q_1, \ldots, q_k\}$ for $k \geq 1$ and each $q_i \notin \Sigma(\alpha_1)$. If $1 \neq \gamma \in \Sigma(\lambda_2) \cap \Sigma(\alpha_1)$, then $L = L_1 L_2$, where

$$L_1 := P_L^{-1} \cdot \text{diag}(1, \gamma) \quad \text{and} \quad L_2 := \text{diag} \left( \frac{\alpha}{\gamma} \right) \cdot Q_L^{-1}.$$ 

Using $(\alpha_2, \gamma, \alpha_1) \neq 1$ and Fact 1(ii) we have $(A, L_1)i \neq I$. The element $\det(L_1)$ is indecomposable in $R$, so $A = L_1 A_1$ by Lemma 1 and $(A, L_1)i \neq I$.

Suppose $\delta \in \{q_1, \ldots, q_k\} \cap \Sigma(\lambda_2)$. It is easy to see that $L = F_1 F_2$, where

$$F_1 := P_L^{-1} \cdot \text{diag}(1, \delta) \quad \text{and} \quad F_2 := \text{diag} \left( \frac{\alpha}{\delta} \right) \cdot Q_L^{-1}.$$ 

The set of all left divisors of $S$ with the Smith normal form $\text{diag}(1, \delta)$ (see Fact 2) is

$$W := \left\{ \left( L_{\alpha_1, \delta} P_S \right)^{-1} \cdot \text{diag}(1, \delta) \cdot \text{GL}_2(R) \right\}.$$ 

Since $(\delta, \sigma_1) = 1$, any matrix $D \in W$ can be written in the form $D = P_D^{-1} \cdot \text{diag}(1, \delta) \cdot Q_D^{-1}$, where $P_D = \begin{bmatrix} l_{11} & l_{12} \\ \delta l_{12} & l_{22} \end{bmatrix}$ $P_S$, and $Q_D \in \text{GL}_2(R)$. Consequently we have

$$P_D P_A^{-1} = \begin{bmatrix} l_{11} & l_{12} \\ \delta l_{12} & l_{22} \end{bmatrix} P_S P_A^{-1} = \begin{bmatrix} l_{11} & l_{12} \\ \delta l_{12} & l_{22} \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} \\ (q_1 \cdots q_k) m_{21} & m_{22} \end{bmatrix} = \begin{bmatrix} l_{11}' & l_{12}' \\ \delta l_{12}' & l_{22}' \end{bmatrix},$$

so $P_D = \begin{bmatrix} l_{11}' & l_{12}' \\ \delta l_{12}' & l_{22}' \end{bmatrix} P_A$. Therefore $A = DA_2$ by Fact 2. It follows that each left divisor $D$ of $S$ with $\text{SNF}(D) = \text{diag}(1, q_i)$ for $i = 1, \ldots, k$ (included $L_1$) is a left divisor of the matrix $A$ too. Consequently $F_1 := P_L^{-1} \cdot \text{diag}(1, \delta)$ is a left divisor of $A$. It means that $(A, L)i \neq I$. \[\square\]
Let $A$ and $B$ be nonsingular matrices. We investigate the properties and structure of the left divisors of the matrix $B$ that have a nontrivial left common divisor with $A$.

**Lemma 2.** Let $R$ be a commutative PID and let $A, B, S, T$ be nonsingular matrices in $R^{2 \times 2}$, such that $B = ST$. If all left divisors of $S$ have a common left divisor with $A$, then

$$\Sigma(S) \subseteq \Sigma((A, B)_i).$$

**Proof.** Let $B := P_B^{-1} \cdot \text{diag}(\beta_1, \beta_2) \cdot Q_B^{-1}$ and $S = P_S^{-1} \cdot \text{diag}(\sigma_1, \sigma_2) \cdot Q_S^{-1}$ have form (2). Let $1 \neq \mu \in \Sigma(S)$. Thus $\sigma_2 = \mu \sigma'_2$ and $S = S_1 S_2$, where

$$S_1 := P_S^{-1} \cdot \text{diag}(1, \mu) \quad \text{and} \quad S_2 := \text{diag}(\sigma_1, \sigma'_2) \cdot Q_S^{-1}.$$ 

By the assumption, $(A, S_1)_i \neq I$. Since $\det(S_1)$ is indecomposable element of $R$ we get that $S_1$ is a left divisor of $A$ by Lemma 1. It follows that $S_1$ is a left common divisor of matrices $A$ and $B = ST$. So $S_1$ is a left divisor of $(A, B)_i$. Consequently, $\Sigma(S_1) \subseteq \Sigma((A, B)_i)$ and $\mu \in \Sigma((A, B)_i)$. It means that $\Sigma(S) \subseteq \Sigma((A, B)_i)$. \square

**Lemma 3.** Let $R$ be a commutative PID and let $A, B, S \in R^{2 \times 2}$ be nonsingular matrices of the form (2):

$$A := P_A^{-1} \cdot \text{diag}(\alpha_1, \alpha_2) \cdot Q_A^{-1}, \quad B := P_B^{-1} \cdot \text{diag}(\beta_1, \beta_2) \cdot Q_B^{-1},$$

$$S := P_S^{-1} \cdot \text{diag}(\sigma_1, \sigma_2) \cdot Q_S^{-1},$$

such that $B = ST$ for some $T \in R^{2 \times 2}$. Set $[\tau_{ij}] := P_B P_A^{-1}$.

Each left divisor of the matrix $S$ has a nontrivial left common divisor with the matrix $A$ if and only if $S$ satisfies the conditions of Theorem 2 and

$$\left( \frac{\sigma_2}{(\sigma_2, \beta_1)}, q_1 \cdots q_k \right) | \tau_{21}, \quad (6)$$

where $\sigma_2 = q_1^{r_1} \cdots q_k^{r_k} d_2$ for $q_1, \ldots, q_k \in \Sigma(\sigma_2) \setminus \Sigma(\alpha_1)$, and $\Sigma(d_2) \subseteq \Sigma(\alpha_1)$.

**Proof of the 'if' part.** Since $S$ is a left divisor of $B$, $\Sigma(\sigma_i) \subseteq \Sigma(\beta_i)$ for $i = 1, 2$ and $P_S = LP_B$, where

$$L := \begin{bmatrix} l_{11} & l_{12} \\ \sigma_2 & l_{21} \end{bmatrix}$$

(for some $l_{ij} \in R$)

by Fact 2. Each left divisor of $S$ has a left common divisor with $A$, so $S$ satisfies the conditions of Theorem 2. Hence $P_S = NP_A$, where

$$N := \begin{bmatrix} n_{11} & n_{12} \\ (q_1 \cdots q_k) n_{21} & n_{22} \end{bmatrix} \quad (\text{for some } n_{ij} \in R).$$

Consequently, $P_S = NP_A = LP_B$. It follows that

$$[\tau_{ij}] = P_B P_A^{-1} = L^{-1} N = \begin{bmatrix} \Sigma_{11}^{l_{11}} & \Sigma_{12}^{l_{12}} \\ \sigma_2 & \Sigma_{21}^{l_{21}} \end{bmatrix} N = \begin{bmatrix} n_{11} & n_{12} \\ (q_1 \cdots q_k) n_{21} & n_{22} \end{bmatrix}$$

$$= \begin{bmatrix} \Sigma_{11}^{l_{11}} & m_{11} \\ \sigma_2 & (q_1 \cdots q_k) m_{21} \end{bmatrix} \begin{bmatrix} m_{12} \\ \Sigma_{22}^{l_{22}} \end{bmatrix} \quad (\text{for some } l_{ij}, m_{ij} \in R).$$

Therefore, the condition (6) is fulfilled.
Proof of the ‘only if’ part. There exist invertible matrices (see [8, Lemma 5.10, p. 193])

\[ C^{-1} := \begin{bmatrix} c_{11} & c_{12} \\ \frac{\sigma_2}{(\sigma_2, \beta_1)} & c_{22} \end{bmatrix} \quad \text{and} \quad D := \begin{bmatrix} d_{11} & d_{12} \\ (q_1 \cdots q_k)d_{21} & d_{22} \end{bmatrix} \]

such that \( PB^{-1} = C^{-1}D \). The matrix \( S := (CP_B)^{-1} \cdot \text{diag}(\sigma_1, \sigma_2) \) is a left divisor of \( B \) by Fact [2]. Moreover, each left divisor of \( S = (DP_A)^{-1} \cdot \text{diag}(\sigma_1, \sigma_2) \) has a nontrivial left common divisor with \( A \) by Theorem [2].

Let us show that [6] holds independently of the choices of \( PB, PA \in GL_2(R) \). Indeed, if we choose a different ordered pair \((PB', PA') \neq (PB, PA)\), then \( PB = HP_B \) and \( PA = TP_A \) by [8, Property 2.2, p. 61], where

\[
H := \begin{bmatrix} h_{11} & h_{12} \\ \frac{\beta_2}{\beta_1}h_{21} & h_{22} \end{bmatrix} \quad \text{and} \quad T^{-1} := \begin{bmatrix} t_{11} & t_{12} \\ \frac{\alpha_2}{\alpha_1}t_{21} & t_{22} \end{bmatrix} \quad \text{(for some} \ h_{ij}, t_{ij} \in R). \]

Thus,

\[
[r_{ij}'] := (PA')^{-1} = HP_BP^{-1}T^{-1} = H[r_{ij}]T^{-1} = \begin{bmatrix} h_{11} & h_{12} \\ \frac{\beta_2}{\beta_1}h_{21} & h_{22} \end{bmatrix} \cdot \begin{bmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{bmatrix} \cdot \begin{bmatrix} t_{11} & t_{12} \\ \frac{\alpha_2}{\alpha_1}t_{21} & t_{22} \end{bmatrix},
\]

and

\[
\tau_{21}' = \tau_{21}(h_{22}t_{11}) + \frac{\beta_2}{\beta_1}(h_{21}\tau_{11}t_{11} + \frac{\alpha_2}{\alpha_1}h_{21}\tau_{12}t_{21}) + \frac{\alpha_2}{\alpha_1}(h_{22}\tau_{21}t_{21}).
\]

Obviously, \( \frac{\beta_2(\sigma_2, \beta_1)}{\beta_1} = \frac{\beta_2(\sigma_2, \beta_1)}{\beta_1} \in R \), so \( \frac{\alpha_2}{\sigma_2, \beta_1} \in R \). Taking into account that \( q_1, \ldots, q_k \in \Sigma(\alpha_2) \) and \( q_1 \cdots q_k, \alpha_1 = 1 \), we obtain that \( (q_1 \cdots q_k)\frac{\alpha_2}{\alpha_1}, \frac{\alpha_2}{\sigma_2, \beta_1}, q_1 \cdots q_k \frac{\beta_2}{\beta_1} \).

Consequently \( \frac{\alpha_2}{\sigma_2, \beta_1}, q_1 \cdots q_k \mid \tau_{21} \).

\[ \square \]

Proof of Theorem [7]. Let \( A := P_A^{-1} \cdot \text{diag}(\alpha_1, \alpha_2) \cdot Q_A^{-1} \) and \( B = P_B^{-1} \cdot \text{diag}(\beta_1, \beta_2) \cdot Q_B^{-1} \) be of the form [2]. Let \( \text{SNF}((A, B)) := \text{diag}(\omega_1, \omega_2) \) and set \( [\tau_{ij}] := P_BP_A^{-1} \). If an adequate part of the matrix \( B \) with respect to \( A \) exists, then we denote it by \( S \).

According to Lemma [2], we have \( \Sigma(S) \subseteq \Sigma((A, B)) \). Since \( \Sigma(\omega_i) \subseteq \Sigma(\alpha_i) \) for \( i = 1, 2 \) (see Fact [II](i)), we set

\[
\Sigma(\omega_1) := \{p_1, \ldots, p_m\}, \quad \Sigma(\omega_2) := \{p_1, \ldots, p_n\} \cup \{q_1, \ldots, q_l\} \cup \{q_{l+1}, \ldots, q_k\},
\]

where \( n \geq m \) and

\[
\{p_1, \ldots, p_n\} \subseteq \Sigma(\alpha_1), \quad \{q_1, \ldots, q_l\} \subseteq \Sigma(\tau_{21}), \quad \{q_{l+1}, \ldots, q_k\} \cap \Sigma(\tau_{21}) = \emptyset.
\]

Moreover \( \omega_1|\beta_i \) for \( i = 1, 2 \) (see Fact [II](i)), so we can write

\[
\beta_1 = \left(\prod_{p_1^r}^{p_m^r} \cdot \prod_{q_1^u}^{q_l^u} \cdot \prod_{q_{l+1}^u}^{q_k^u} \right) \cdot d = \sigma_1 \cdot \beta_1', \quad (7)
\]

\[
\beta_2 = \left(\prod_{p_1^r}^{p_m^r} \cdot \prod_{q_1^u}^{q_l^u} \cdot \prod_{q_{l+1}^u}^{q_k^u} \right) \cdot \beta_2', \quad (8)
\]

\[ = \sigma_2 \cdot \beta_2', \]
where \((d, \alpha_2) = 1\), \((\beta_0, p_1 \cdots p_n \cdot q_1 \cdots q_k) = 1\), \(r'_i \geq r_i\), for \(i = 1, \ldots, m\) and \(u'_j \geq u_j \geq 0\) for \(j = 1, \ldots, l\). It follows that
\[
\frac{\sigma_2}{(\sigma_2, \beta_1)} = \left(\frac{r'_i - r_i}{p_{m+1} \cdots p_n} \cdot (p_{m+1} r_{m+1} \cdots p_n r_n) \cdot (q_i' - u_i) \cdots (q_i - u_i)\right).
\]
Since \(q_1, \ldots, q_l \in \Sigma(\tau_{21})\),
\[
\left(\frac{\sigma_2}{(\sigma_2, \beta_1)} \cdot q_1 \cdots q_k\right) = \left(q_i'^{-1} - u_i \cdots q_i - u_i, q_1 \cdots q_k\right) | \tau_{21}.
\]
According to [8, Lemma 5.10, p. 193], we can write
\[
P_B P_A^{-1} = \left[ \begin{array}{cc} f_{11} & f_{12} \\ f_{21} & f_{22} \end{array} \right]^{-1} P_B = \left[ \begin{array}{cc} l_{11} & l_{12} \\ (q_1 \cdots q_k) l_{21} & l_{22} \end{array} \right] P_A.
\]
Let us consider the matrix
\[
S := \left(\left[ \begin{array}{cc} f_{11} & f_{12} \\ f_{21} & f_{22} \end{array} \right]^{-1} P_B \right)^{-1} \left[ \begin{array}{cc} \sigma_1 & 0 \\ 0 & \sigma_2 \end{array} \right] .
\]
Using Fact 2, \(S\) is a left divisor of \(B\), i.e. \(B = ST\) for some \(T \in R^{2 \times 2}\). From (9) we obtain that
\[
\left[ \begin{array}{cc} f_{11} & f_{12} \\ f_{21} & f_{22} \end{array} \right]^{-1} P_B = \left[ \begin{array}{cc} l_{11} & l_{12} \\ (q_1 \cdots q_k) l_{21} & l_{22} \end{array} \right] P_A.
\]
It follows that the matrix \(S\) can also be written in the following form:
\[
S = \left(\left[ \begin{array}{cc} l_{11} & l_{12} \\ (q_1 \cdots q_k) l_{21} & l_{22} \end{array} \right] P_A \right)^{-1} \left[ \begin{array}{cc} \sigma_1 & 0 \\ 0 & \sigma_2 \end{array} \right] .
\]
Consequently, each left divisor of \(S\) has a nontrivial common left divisor with \(A\) by Theorem 2 so \(S\) satisfies part (i) in the definition of an adequate part of \(B\) with respect to \(A\).

Note that \(B = ST\). Assume that \(T = T_1 T_2\) is a decomposition into a product of its two nontrivial divisors. Let us consider the following two cases:

Case 1. Let \(\Sigma(ST_1) \nsubseteq \Sigma((A, B)_i)\). Hence, there exists \(t \in \Sigma(ST_1) \setminus \Sigma((A, B)_i)\). It means that \(ST_1\) has a left divisor \(L\) with \(\text{SNF}(L) = \text{diag}(1, t)\) such that \((A, L)_l = I\) (by the same trick as the one used in the proof of Lemma 2).

Case 2. Let \(\Sigma(ST_1) \subseteq \Sigma((A, L)_i)\) and \(\text{SNF}(ST_1) = \text{diag}(\mu_1, \mu_2)\) where \(\mu_2 \in \Sigma((A, L)_i)\). In view of (7) and (8), we have \(\det(ST_1)\) has the divisor \(q_i^{u_i + 1}\) in which \(t + 1 \leq i \leq k\).

Case 2a. Let \(q_i | \mu_1\). Any matrix with the Smith normal form \(\text{diag}(q_i, q_i)\) is a left divisor of \(ST_1\) by [8] Theorem 5.3 p. 152 and Property 4.11 p. 147. Let us examine the matrix \(M := P_M^{-1} \text{diag}(q_i, q_i)\), where \(P_M := \left[ \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right] P_A\). It is obvious that \(M = M_1 M_2\), where \(M_1 := P_M^{-1} \text{diag}(1, q_i)\) and \(M_2 := \text{diag}(q_i, 1)\). Since \((\alpha_1, q_i) = 1\) and \(P_M P_A^{-1} = \left[ \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right]\), we have \((A, M_1)_l = I\) by Fact 1(ii). Thus \(ST_1 = MN = M_1 (M_2 N)\) for some \(N\).

Case 2b. Let \((q_i, \mu_1) = 1\). Clearly \(q_i^{u_i + 1} | \mu_2\). It follows that the matrix \(ST_1\) has a left divisor \(K\) with \(\text{SNF}(K) = \text{diag}(1, q_i^{u_i + 1})\). Since \(\frac{q_i^{u_i + 1}}{(q_i^{u_i + 1}, \beta_1)} = q_i\),
\[
K := P_K^{-1} \cdot \text{diag}(1, q_i^{u_i + 1}) \cdot Q_K^{-1},
\]
where $P_K := \begin{bmatrix} k_{11} & k_{12} \\ q_i k_{21} & k_{22} \end{bmatrix}$ $P_B$ by Fact[2] Thus

$$[\tau_{ij}'] := P_K P_A^{-1} = \begin{bmatrix} k_{11} & k_{12} \\ q_i k_{21} & k_{22} \end{bmatrix} P_B P_A^{-1} = \begin{bmatrix} k_{11} & k_{12} \\ q_i k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{bmatrix}$$

$$= \begin{bmatrix} k_{11} & k_{12} \\ q_i k_{21} \tau_{11} + k_{22} \tau_{21} & * \end{bmatrix}.$$ 

It is easy to see that $(q_i, k_{22}) = 1$, because $\begin{bmatrix} k_{11} & k_{12} \\ q_i & k_{22} \end{bmatrix}$ is invertible. By assumption $(q_i, \tau_{21}) = 1$, so $(q_i, \tau_{21}') = 1$. Consequently $A, B$ be pairwise relatively prime indecomposable elements. Let $A, B \in R^{2 \times 2}$ be of the form (2) such that

$$A := \text{diag}(ab, ab^2 cf m), \quad B := \begin{bmatrix} 1 & 0 \\ -f & 1 \end{bmatrix} \text{diag}(b^2 c, ab^2 c^2 fn),$$

$$P_A = I, \quad P_B = \begin{bmatrix} 1 & 0 \\ f & 1 \end{bmatrix}, \quad [\tau_{ij}] := P_B P_A^{-1} = P_B = \begin{bmatrix} 1 & 0 \\ f & 1 \end{bmatrix}.$$ 

Clearly, $\Sigma(A) = \{1, a, b, c, f, m\}, \Sigma(B) = \{1, a, b, c, f, n\}$ and $\text{SNF}(\langle A, B \rangle) = \text{diag}(b, ab^2 cf)$ by Fact[1][ii]. Using the notation of Theorem[1] we have that $q_1 q_2 = cf$. An adequate part of $B$ with respect to $A$ (see Theorem[1]) has the following Smith normal form $\text{diag}(b^2, ab^3 cf) := \text{diag}(\sigma_1, \sigma_2)$. Note that

$$\left(\sigma_2 \sigma_3 \sigma_4^2; q_1 q_2\right) = (ab f, cf) = f | \tau_{21}.$$ 

It is easy to check that

$$P_B P_A^{-1} = \begin{bmatrix} 1 & 0 \\ f & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ abf y & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ cx & 1 \end{bmatrix}$$

in which $cx + aby = 1$. It follows that

$$\begin{bmatrix} 1 & 0 \\ -abf y & 1 \end{bmatrix} P_B = \begin{bmatrix} 1 & 0 \\ cx & 1 \end{bmatrix} P_A.$$ 

Consequently, an adequate part of $B$ with respect to $A$ has the following form:

$$S := \left(\begin{bmatrix} 1 & 0 \\ -abf y & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ f & 1 \end{bmatrix}\right)^{-1} \begin{bmatrix} b^2 & 0 \\ 0 & ab^3 cf \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ f(aby - 1) & 1 \end{bmatrix} \begin{bmatrix} b^2 & 0 \\ 0 & ab^3 cf \end{bmatrix} = \begin{bmatrix} b^2 f(aby - 1) & 0 \\ 0 & ab^3 cf \end{bmatrix}$$

3. Some examples

An algorithm for constructing an adequate part of a matrix from $R^{2 \times 2}$ is presented in the following:

**Example 1.** Let $R$ be a PID and let $a, b, c, f, m, n \in R \setminus \{U(R) \cup \{0\}\}$ be pairwise relatively prime indecomposable elements. Let $A, B \in R^{2 \times 2}$ be of the form (2) such that

$$A := \text{diag}(ab, ab^2 cf m), \quad B := \begin{bmatrix} 1 & 0 \\ -f & 1 \end{bmatrix} \text{diag}(b^2 c, ab^2 c^2 fn),$$

$$P_A = I, \quad P_B = \begin{bmatrix} 1 & 0 \\ f & 1 \end{bmatrix}, \quad [\tau_{ij}] := P_B P_A^{-1} = P_B = \begin{bmatrix} 1 & 0 \\ f & 1 \end{bmatrix}.$$ 

Clearly, $\Sigma(A) = \{1, a, b, c, f, m\}, \Sigma(B) = \{1, a, b, c, f, n\}$ and $\text{SNF}(\langle A, B \rangle) = \text{diag}(b, ab^2 cf)$ by Fact[1][ii]. Using the notation of Theorem[1] we have that $q_1 q_2 = cf$. An adequate part of $B$ with respect to $A$ (see Theorem[1]) has the following Smith normal form $\text{diag}(b^2, ab^3 cf) := \text{diag}(\sigma_1, \sigma_2)$. Note that

$$\left(\sigma_2 \sigma_3 \sigma_4^2; q_1 q_2\right) = (ab f, cf) = f | \tau_{21}.$$ 

It is easy to check that

$$P_B P_A^{-1} = \begin{bmatrix} 1 & 0 \\ f & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ abf y & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ cx & 1 \end{bmatrix}$$

in which $cx + aby = 1$. It follows that

$$\begin{bmatrix} 1 & 0 \\ -abf y & 1 \end{bmatrix} P_B = \begin{bmatrix} 1 & 0 \\ cx & 1 \end{bmatrix} P_A.$$ 

Consequently, an adequate part of $B$ with respect to $A$ has the following form:

$$S := \left(\begin{bmatrix} 1 & 0 \\ -abf y & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ f & 1 \end{bmatrix}\right)^{-1} \begin{bmatrix} b^2 & 0 \\ 0 & ab^3 cf \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ f(aby - 1) & 1 \end{bmatrix} \begin{bmatrix} b^2 & 0 \\ 0 & ab^3 cf \end{bmatrix} = \begin{bmatrix} b^2 f(aby - 1) & 0 \\ 0 & ab^3 cf \end{bmatrix}$$
by Theorem 1. In this case \( B = ST \), where \( T = \begin{bmatrix} c & 0 \\ -y & cn \end{bmatrix} \).

Each commutative PID \( R \) is adequate by the Helmer definition as was noted in the Introduction. Moreover, the adequate and the relatively prime parts of an element \( b \in R \) with respect to \( a \) in \( R \) are defined up to strong associates. This statement is not true in the case of the ring \( R^{2 \times 2} \) as shown in the next example:

**Example 2.** Let \( R = \mathbb{Z} \) be the ring of integers. Let

\[
A := \text{diag}(\alpha_1, \alpha_2) = \text{diag}(2, 2 \cdot 3 \cdot 5 \cdot 7), \quad B := \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \cdot \text{diag}(2 \cdot 3^2 \cdot 5^2, 2^2 \cdot 3^3 \cdot 5^4),
\]

\[
P_A = I, \quad P_B = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}, \quad [\tau_{ij}] := P_B P_A^{-1} = P_B.
\]

Obviously, \( \text{SNF}((A, B)_l) = \text{diag}(\omega_1, \omega_2) = \text{diag}(2, 2 \cdot 3 \cdot 5) \) by Fact 1(i) and

\[
\Sigma(\alpha_1) = \{1, 2\}, \quad \Sigma(\omega_2) = \{1, 2, 3, 5\}, \quad \Sigma(\tau_{21}) = \{1, 3\}, \quad \{5\} \cap \Sigma(\tau_{21}) = \emptyset.
\]

The left adequate part of \( B \) with respect to \( A \) has the following Smith normal form \( \Phi = \text{diag}(2, 2^2 \cdot 3^3 \cdot 5^2) \) (see the proof of Theorem 2). The matrices

\[
S := \begin{bmatrix} 1 & 0 \\ -3 & 5 \end{bmatrix} \cdot \Phi \quad \text{and} \quad S_1 := \begin{bmatrix} 1 & 0 \\ 3 & 5 \end{bmatrix} \cdot \Phi
\]

are left divisors of the matrix

\[
B = S \begin{bmatrix} 3^2 \cdot 5^2 & 0 \\ 2 & 5^2 \end{bmatrix} = S_1 \begin{bmatrix} 3^2 \cdot 5^2 & 0 \\ -3 & 5^2 \end{bmatrix},
\]

and are also adequate parts of \( B \) with respect to \( A \) by Theorem 2. However (see [8, Theorem 4.5, p. 128]) the matrices \( S \) and \( S_1 \) are not right strong associates.

\[ \diamond \]

4. Adequate rings in the sense of Gatalevych

Gatalevych [3, Definition 1, p.116] proposed the following definition for noncommutative Bézout rings which was already indicated in the Introduction.

Let \( K \) be a Bézout ring and let \( a \in K \). An element \( b \in K \) is called left adequate in the sense of Gatalevych to \( a \in K \) if the following conditions hold:

(i) there exist elements \( s, t \in K \) such that \( b = st \) and \( tK + aK = K \);

(ii) \( s'K + aK \neq K \) for each \( s' \in K \setminus U(K) \) such that \( sK \subseteq s'K \neq K \).

The shortcomings of this definition are demonstrated by the next example:

**Example 3.** Let \( R \) be a commutative PID, and let \( a, d, c \in R \setminus \{U(R) \cup \{0\} \} \) be pairwise relatively prime indecomposable elements. Let

\[
A := \text{diag}(a, d^2c), \quad P_A = I, \quad B := \begin{bmatrix} 1 & 0 \\ d & d^2c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} d & d^2c^2 \end{bmatrix}, \quad P_B = \begin{bmatrix} 1 & 0 \\ -d & 1 \end{bmatrix};
\]

\[
A_1 := \text{diag}(a, d^2c), \quad T := \begin{bmatrix} 1 & 0 \\ d^2c & 1 \end{bmatrix} \cdot \text{diag}(1, d^2c^2), \quad S := \text{diag}(1, d).
\]

It is easy to check that \( A = SA_1 \) and \( B = ST \). Since \( (A, T)_l = I \) (see Fact 1(ii)), the decomposition \( B = ST \) satisfies the definition of Gatalevych.
On the other hand, 
\[ B = S_1 T_1 = (P_{S_1}^{-1} \cdot \text{diag}(1, d^3) \cdot Q_{S_1}^{-1}) \cdot (P_{T_1}^{-1} \cdot \text{diag}(1, c^2) \cdot Q_{T_1}^{-1}), \]
where
\[
S_1 = \begin{bmatrix}
1 & 0 \\
d^3 + d & d^3
\end{bmatrix}, \quad P_{S_1} = \begin{bmatrix}
1 & 0 \\
-d & 1
\end{bmatrix}, \quad Q_{S_1} = \begin{bmatrix}
1 & 0 \\
-1 & 1
\end{bmatrix},
\]
\[
T_1 = \begin{bmatrix}
1 & 0 \\
-1 & c^2
\end{bmatrix}, \quad P_{T_1} = \begin{bmatrix}
1 & 0 \\
1 & 1
\end{bmatrix}, \quad Q_{T_1} = I.
\]

Each left divisor of \( S_1 \) has a nontrivial left common divisor with \( A \) by Theorem 2 and \((A, T_1)_l = I\) by Fact 1(ii), so the decomposition \( B = S_1 T_1 \) also satisfies Gatalevych’s definition. However, \( S \) is the left divisor of \( S_1 \), because \( S = S_1 \begin{bmatrix}
1 & 0 \\
1 & d^2
\end{bmatrix}. \)

It should be noted that the decompositions \( B = ST \) and \( B = S_1 T_1 \) also have one more unpleasant property. Let us consider the cosets \( SGL_2(R) \) and \( S_1 GL_2(R) \), i.e. the sets of all right strong associated matrices to the matrices \( S \) and \( S_1 \), respectively. According to Fact 1(ii), each left divisor of the matrices from \( SGL_2(R) \) and \( S_1 GL_2(R) \) has a nontrivial left common divisor with the matrix \( A \). However, if \( U, V \in GL_2(R) \) and \( B = (SU)(U^{-1}T) = (S_1V)(V^{-1}T_1) \) then it does not follow that \((A, U^{-1}T)_l = I \) and \((A, V^{-1}T_1)_l = I \). Indeed, if 
\[
U := \begin{bmatrix}
1 & 0 \\
1 - d & 1
\end{bmatrix} \quad \text{and} \quad V := \begin{bmatrix}
1 & 0 \\
-1 & 1
\end{bmatrix},
\]
then \( T' := U^{-1}T = \begin{bmatrix}
1 & 0 \\
d & 1
\end{bmatrix} \cdot \text{diag}(1, d^2c) \) and \( T'_1 := V^{-1}T_1 = \text{diag}(1, c^2) \). It is easy to see that \((A, T')_l \neq I \) and \((A, T'_1)_l \neq I \).

References

[1] D. D. Anderson and S. Valdes-Leon. Factorization in commutative rings with zero divisors. *Rocky Mountain J. Math.*, 26(2):439–480, 1996.
[2] S. Chun, D. D. Anderson, and S. Valdez-Leon. Reduced factorizations in commutative rings with zero divisors. *Comm. Algebra*, 39(5):1583–1594, 2011.
[3] A. I. Gatalevich. On adequate and generalized adequate duo rings, and duo rings of elementary divisors. *Mat. Stud.*, 9(2):115–119, 223, 1998.
[4] L. Gillman and M. Henriksen. Rings of continuous functions in which every finitely generated ideal is principal. *Trans. Amer. Math. Soc.*, 82:366–391, 1956.
[5] L. Gillman and M. Henriksen. Some remarks about elementary divisor rings. *Trans. Amer. Math. Soc.*, 82:362–365, 1956.
[6] O. Helmer. The elementary divisor theorem for certain rings without chain condition. *Bull. Amer. Math. Soc.*, 49:225–236, 1943.
[7] I. Kaplansky. Elementary divisors and modules. *Trans. Amer. Math. Soc.*, 66:464–491, 1949.
[8] V. Shchedryk. *Arithmetic of matrices over rings*. https://doi.org/10.15407/akademperiodika.430.278. Akademperiodyka, Kyiv, https://www.researchgate.net/publication/353979871_arithmetic_of_matrices, 2021.
[9] V. P. Shchedryk. Bezout rings of stable range 1.5. *Ukrainian Math. J.*, 67(6):960–974, 2015. Translation of *Ukraïn. Mat. Zh.*, 67 (2015), no. 6, 849–860.
[10] B. V. Zabavskii. Generalized adequate rings. *Ukrain. Mat. Zh.*, 48(4):554–557, 1996.
[11] B. V. Zabavsky and A. Gatalevych. Diagonal reduction of matrices over commutative semihereditary Bezout rings. *Comm. Algebra*, 47(4):1785–1795, 2019.
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