Continuity of the maximum-entropy inference and open projections of convex bodies

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Abstract

The maximum-entropy inference assigns to the mean values with respect to a fixed set of observables the unique density matrix, which is consistent with the mean values and which maximizes the von Neumann entropy.

A discontinuity was recently found in this inference method for three-level quantum systems. For arbitrary finite-level quantum systems, we show that these discontinuities are no artefacts. While lying on the boundary of the set of mean values, they influence the inference of nearby mean values. We completely characterize the discontinuities by an openness condition on the linear map that assigns mean values. An example suggests that the openness condition is independent of the inference and can be formulated in terms of the convex geometry of the set of density matrices—this is left as an open problem.

1 Introduction

We consider the \( n \)-dimensional Euclidean vector space \( \mathbb{E}^n \) with inner product \( \langle \cdot, \cdot \rangle \) and norm \( \|x\| \coloneqq \sqrt{\langle x, x \rangle} \). Unless otherwise specified, a subset \( X \subset \mathbb{E}^n \) is endowed with the relative topology induced from \( \mathbb{E}^n \). Another issue is the topology of the affine hull of \( X \), which defines relative interiors and relative boundaries in convex geometry. When topologies of affine hulls are used, this will be explicitly specified.

Let \( X, Y \subset \mathbb{E}^n \). A mapping \( \pi : Y \to X \) is open at \( y \in Y \), if for each neighborhood \( U(y) \) there is a neighborhood \( V(\pi(y)) \) included in \( \pi(U(y)) \). The map \( \pi \) is open if it is open at each \( y \in Y \), or equivalently if every open set \( V \subset Y \) has an open image \( \pi(V) \) in \( X \).

Definition 1. We consider a convex body \( S \subset \mathbb{E}^n \) (compact, convex subset) and refer to it as state space and to its elements as states.

This is motivated by the state space \( S = \{ \rho \in \mathcal{A} \mid \rho \geq 0, \text{tr}(\rho) = 1 \} \) of a real self-adjoint subalgebra \( \mathcal{A} \) of Mat(\( N, \mathbb{C} \)), called \( * \)-subalgebra in [Ko], which is a convex body. It contains all density matrices, i.e. positive semi-definite matrices of trace one. Let \( 1_N \) resp. \( 0_N \) denote the identity resp. zero in Mat(\( N, \mathbb{C} \)). The real vector space of self-adjoint matrices of \( \mathcal{A} \) contains \( S \) and is a Euclidean vector space endowed with the Hilbert-Schmidt scalar product \( \langle a, b \rangle = \text{tr}(ab) \) for self-adjoint \( a, b \in \mathcal{A} \).

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\[2\text{This definition differs from [Pa] where } \pi : Y \to X \text{ is called open at } x \in X \text{ if it is open at every } y \in \pi^{-1}(x). \text{ The present definition suits our main results.}\]

\[3\text{A neighborhood of } y \text{ is an open set containing } y.\]
Example 1. We consider the *-subalgebra $\mathcal{A}$ of $\text{Mat}(3, \mathbb{C})$ defined as the real linear span of the $3 \times 3$-matrices (embedded as block-diagonal matrices)

$$1_2 \oplus 0, \quad \sigma_1 \oplus 0, \quad \sigma_2 \oplus 0, \quad i\sigma_3 \oplus 0, \quad 0_2 \oplus 1$$

with the Pauli $\sigma$-matrices

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad$$

The state space is a three-dimensional cone (conv denoting convex hull)

$$\mathcal{S} = \text{conv}(\{\rho(\alpha) \mid \alpha \in [0, 2\pi]\} \cup \{0_2 \oplus 1\})$$

based on the circle $\rho(\alpha) = \frac{1}{2}(1_2 + \sin(\alpha)\sigma_1 + \cos(\alpha)\sigma_2) \oplus 0$ and with apex $0_2 \oplus 1$. The circle is an equator of the state space of $\text{Mat}(2, \mathbb{C})$, known as Bloch ball.

Definition 2. The second element we introduce is a linear subspace $U \subset \mathbb{E}^n$. We denote the orthogonal projection onto $U$ by

$$\pi_U : \mathbb{E}^n \to U$$

and the projection of the state space by

$$\mathbb{M} := \pi_U(\mathcal{S}).$$

The fiber $(\pi_U|_\mathcal{S})^{-1}(m) = \{\rho \in \mathcal{S} \mid \pi_U(\rho) = m\}$ of $m \in \mathbb{M}$ gives rise to the multi-valued map

$$F : \mathbb{M} \to \mathcal{S}, \quad m \mapsto (m + U^\perp) \cap \mathcal{S}.$$  

We call $\mathbb{M}$ the mean value set and its elements mean values.

The projection is motivated by the real vector space of self-adjoint matrices $U$ of an operator system $[P_n]$, i.e. a complex self-adjoint subspace of $\text{Mat}(N, \mathbb{C})$ containing $1_N$. If $\{u_1, \ldots, u_k\}$ is a spanning set of $U$, then providing the projection $\pi_U(\rho)$ of a density matrix $\rho \in \text{Mat}(N\mathbb{C})$ is equivalent to providing the mean values

$$\langle u_1, \rho \rangle, \ldots, \langle u_k, \rho \rangle.$$

In Jaynes’s maximum-entropy principle [Ja, Wi, St] mean values are commonly used and the self-adjoint matrices $\{u_1, \ldots, u_k\}$ are sometimes called observables. For simplicity and in agreement with [Kn] we use the projection $\pi_U$ in this article.

Definition 3. The third and last element to be introduced is a function

$$\phi : \mathcal{S} \to \mathbb{R},$$

which we refer to as disorderliness measure. The fibers $\{F(m)\}_{m \in \mathbb{M}}$ of $\pi_U|_\mathcal{S}$ are selective (with respect to $\phi$) if for each mean value $m \in \mathbb{M}$ there is a unique $\rho_m \in F(m)$, such that $\phi(\rho_m) = \max\{\phi(\rho) \mid \rho \in F(m)\}$, see [Be]. If the fibers are selective, then the function

$$\Phi : \mathbb{M} \to \mathcal{S}, \quad \Phi(m) = \arg\max\{\phi(\rho) \mid \rho \in F(m)\}$$

is defined. We call $\Phi$ the inference.
We call $\phi$ disorderliness measure in analogy to the “uncertainty” [Ja] or “disorderliness” [Wi] that the von Neumann entropy $-\text{tr}\rho\log(\rho)$ assigns to a density matrix $\rho$. The von Neumann entropy is strictly concave (hence the fibers of $\pi_U|_S$ are selective) and continuous [We].

Our main result is stated in Theorem 2:

**Theorem.** If $\phi : S \to \mathbb{R}$ is a continuous disorderliness measure and the fibers of $\pi_U|_S$ are selective, then the inference $\Phi : \mathbb{M} \to S$ is continuous at $m \in \mathbb{M}$ if and only if the projection $\pi_U|_S : S \to \mathbb{M}$ is open at $\Phi(m)$.

The “if” part of the theorem follows from an argument by Wichmann [Wi] (reformulated in Theorem 1). The “only if” part is trivial and shall be proved now.

**Proof:** We give an indirect proof. If $\pi_U|_S$ is not open at $\Phi(m)$ for some $m \in \mathbb{M}$, then there exists a neighborhood $U(\Phi(m))$ in $S$ and a sequence $m_i \subset \mathbb{M} \setminus \pi_U(U(\Phi(m)))$ such that $m_i \to m$. Since $\Phi(m) \in U(\Phi(m))$, the sequence $\Phi(m_i)$ can not converge to $\Phi(m)$, thus $\Phi$ is not continuous at $m$. \qed

## 2 Discussion and open questions

Physically the openness of $\pi_U|_S$ means that any small ambiguity of the mean value can be realized by a small adjustment of the state. In particular, states of maximal disorderliness are non-local if $\pi_U|_S$ is not open at a state of maximal disorderliness.

A similar physical meaning is attached to the openness of the barycenter map in the description of statistical ensembles of quantum states [Sh2]. This has its origin in the idea of *stability* [Pa] of a convex body $S$, meaning that the *mid-point map* $S \times S \to S$, $(x, y) \mapsto \frac{1}{2}(x + y)$ is open. We are not aware of any literature on the openness of linear projections, except [Ve].

We discuss a discontinuous maximum-entropy inference in an example [Kn].
Example 2. We use the state space $\mathcal{S}$ from Example 1, a three-dimensional (truncated) cone. The linear space $U$ is the real span of $\sigma_1 \oplus 0$ and of $(\sigma_2 \oplus 1) - 1_3/3$. The generating line $[\rho(0), 0_2 \oplus 1]$ of the cone $\mathcal{S}$ is orthogonal to $U$, hence $m := \pi_U(\rho(0))$ has the fiber $F(m) = [\rho(0), 0_2 \oplus 1]$. The states $\rho(0)$ and $0_2 \oplus 1$ being mutually orthogonal rank-one projection operators, the von Neumann entropy is maximized on $F(m)$ at the centroid $c := \frac{1}{2}(\rho(0) + 0_2 \oplus 1)$, see Figure 1a.

It is easy to see that the projection $\pi_U|_{\mathcal{S}}$ is not open at $\Phi(m) = c$. Using $v_z = 0_2 \oplus 1 - \rho(0)$ we consider the neighborhood $U(c) := \{\rho \in \mathcal{S} \mid \langle v_z, \rho \rangle > -\frac{1}{3}\}$. The projection $\pi_U(U(c))$ contains no neighborhood of $m$, see Figure 1b. Hence $\pi_U|_{\mathcal{S}}$ is not open at $\Phi(m)$ and the theorem in the introduction shows that $\Phi(m)$ is not continuous at $m$. The inference $\Phi(m)$ for other points $m$ on the boundary of the mean value set yields a state on the base circle of the cone $\mathcal{S}$, showing discontinuity of $\Phi$.

We might hope that the discontinuity of the inference $\Phi$ in Example 2 could be artificial as it lies on the boundary of the mean value set—at the best, it could be unrelated to the inference $\Phi|_{\text{ri}M}$ for generic mean values. Here $\text{ri}M$ denotes the relative interior of the convex set $M$, i.e. the interior with respect to the affine hull of $M$. However, as is proved in Theorem 2d in [Wi] this is not the case because

$$\Phi(M) \subset \overline{\Phi(\text{ri}M)}.$$  \hspace{1cm} (1)

We will give in Corollary 1.1 a proof of this inclusion for continuous disorderliness measures and selective fibers of the projection.

The meaning of (1) is that $\Phi|_{\text{ri}M}$ can not have a continuous extension to $\text{ri}M \cup \{m\}$ if $\Phi$ is discontinuous at $m \in M$. This is an analytical pathology because $\Phi|_{\text{ri}M}$ itself is continuous, as we shall prove in Theorem 3. For the maximum-entropy inference, where $\Phi|_{\text{ri}M}$ is real analytic and $\Phi(\text{ri}M)$ is known as Gibbsian family, a discontinuity of $\Phi$ at $m \in M$ precludes a continuous extension of that analytic function. The Gibbsian family corresponding to Example 2 is studied in detail in [Kn].

Low convergence rates may arise in empirical inference problems for mean values near to a discontinuity, see [Kn] for a short discussion. Hence, the discontinuities should be taken into account in applications which are sensitive to convergence rates, like e.g. Bayesian model selection [Ra]. The latter is an example to use the maximum-entropy inference in order to test the validity of a physical law (see the introduction in [Ja]), a scientist should look for new laws if a given law has insufficient inference properties for the data provided by nature. This article shows that it may also be worth checking if not the inference itself has insufficient convergence properties due to discontinuities.

Theoretically, a classification of linear spaces $U$ with discontinuous maximum-entropy inference is desirable.

1. Example 2 suggests that the openness condition for discontinuities might be independent of the disorderliness measure, given that the disorderliness measure assumes its maximum in the relative interior of each fiber, like the von Neumann entropy does. In that example, the projection $\pi_U|_{\mathcal{S}}$ is not open at any point of the fiber $[\rho(0), 0_2 \oplus 1]$, the exception $\rho(0)$ lies on the relative boundary of the segment.
Question. Let \( S \) be a stable convex body, let \( m \in \mathbb{M} = \pi_{U}(S) \) and let \( \pi_{U}|S \) be open at a single point \( \rho \) in the relative interior of the fiber \( F(m) \). Is then \( \pi_{U}|S \) open at every point of \( F(m) \)?

If not true for all convex bodies, this question is still interesting for state spaces of density matrices. The standard example

\[
\text{conv} \left[ \{(x, y, 0) \in \mathbb{R}^{3} \mid (x - 1)^{2} + y^{2} \leq 1\} \cup \{(0, 0, 1), (0, 0, -1)\} \right]
\]

shows that a condition like stability has to be assumed. Stability of state spaces of density matrices is proved in Lemma 3 in [Sh1].

2. A convex geometric indicator for discontinuity of the maximum-entropy inference was proposed in [Kn]: \( \Phi \) is discontinuous for linear spaces \( U \) where non-exposed faces are created at the mean value set \( \mathbb{M} = \mathbb{M}(U) \) when \( U \) varies in a Grassmannian manifold of subspaces. From this perspective, it is interesting to have a classification of non-exposed faces of mean value sets. Some references and ideas are collected in [Ws], others may be found in the recent literature on convex algebraic geometry.

3 Upper semi-continuity

Discontinuities of the inference have some structure. Under the assumption of an upper semi-continuous disorderliness measure, we show that the maximal disorderliness (in each fiber) depends upper semi-continuously on mean values. We include this statement for completeness, it should be clear for experts in the field of multi-valued maps.

Let \( X, Y \subset \mathbb{E}^{n} \) be subsets. A multi-valued map

\[
\Gamma : X \rightarrow Y
\]

assigns to each element \( x \in X \) a subset \( \Gamma(x) \subset Y \). If \( \Gamma \) is such that the set \( \Gamma(x) \) always consists of a unique element, then \( \Gamma \) is called single-valued map.

The multi-valued map \( \Gamma : X \rightarrow Y \) is upper semi-continuous at \( x_{0} \in X \) if for each open set \( G \) containing \( \Gamma(x_{0}) \) there exists a neighborhood \( U(x_{0}) \) such that \( x \in U(x_{0}) \) implies \( \Gamma(x) \subset G \). The multi-valued map \( \Gamma : X \rightarrow Y \) is upper semi-continuous, denoted by \( u.s.c. \) if it is upper semi-continuous at each point of \( X \) and if, also, \( \Gamma(x) \) is a compact set for each \( x \). A different concept is used for single-valued maps \( f : X \rightarrow \mathbb{R} \). The map \( f \) is upper semi-continuous at \( x_{0} \in X \) if for each \( \epsilon > 0 \) there exists a neighborhood \( U(x_{0}) \) such that \( x \in U(x_{0}) \) implies \( f(x) < f(x_{0}) + \epsilon \). The multi-valued map \( \Gamma : X \rightarrow Y \) is closed if for all \( x_{0} \in X \) and \( y_{0} \in Y \), \( y_{0} \notin \Gamma(x_{0}) \) there exist neighborhoods \( U(x_{0}) \) and \( V(y_{0}) \) such that \( x \in U(x_{0}) \) implies \( \Gamma(x) \cap V(y_{0}) = \emptyset \).

The multi-valued mapping \( F : \mathbb{M} \rightarrow S \) is closed. For a proof we consider for \( X \subset \mathbb{E}^{n} \), \( x \in \mathbb{E}^{n} \) and \( \epsilon > 0 \) the open ball

\[
B_{\epsilon}^{X}(x) := \{ u \in X \mid \| y - x \| < \epsilon \}.
\]

For \( m \in \mathbb{M} \) and \( \rho \in S \) with \( \rho \notin F(m) \) we can take the positive number \( \epsilon := \frac{1}{2}\| m - \pi_{U}(\rho) \| \) and the neighborhoods \( B_{\epsilon}^{\mathbb{M}}(m) \) and \( B_{\epsilon}^{S}(\rho) \).
In the Corollary to Theorem 7 in Section VI.1 of [Be] it is proved for compact \( Y \) that a closed multi-valued map \( X \to Y \) is u.s.c., this shows
\[
F \text{ is u.s.c.} \tag{3}
\]
Since the fibers \( F(m) \) for \( m \in M \) are compact and \( \phi \) is upper semi-continuous, the maximum \( \max\{\phi(\rho) \mid \rho \in F(m)\} \) exists for all \( m \in M \) (see e.g. Theorem 2 in Section IV.8 of [Be]).

**Lemma 1.** If \( \phi \) is an upper semi-continuous disorderliness measure, then the maximum \( m \mapsto \max\{\phi(\rho) \mid \rho \in F(m)\} \) is upper semi-continuous on \( M \).

**Proof:** This follows from Theorem 2 in Section VI.3 of [Be]. We meet the assumption that \( \tilde{\phi} : M \times S \to \mathbb{R}, \tilde{\phi}(m, \rho) := \phi(\rho), \) is upper semi-continuous on \( M \times S \) and that \( F \) is u.s.c., which we have seen in (3).

Upper-semicontinuity of the maximum is illustrated in Example 2.

**Example 3.** The maximal-entropy inference \( \Phi(\pi_U(\rho(\alpha))) = \rho(\alpha) \) is the rank one state \( \rho(\alpha) \) for angles \( \alpha \neq 0 \), having zero von Neumann entropy. The inference \( \Phi(\pi_U(\rho(0))) = c = \frac{1}{2}(\rho(0) + 0_2 \oplus 1) \) has von Neumann entropy \( \log(2) \). So the maximum is upper semi-continuous on the elliptical boundary of \( M \) but not continuous. It is upper semi-continuous on the whole mean value set according to Lemma 1.

## 4 All discontinuities

We characterize all discontinuities of the inference in terms of open projections of the state space. This will be done using Theorem 1, which is equivalent to an argument in Theorem 2d in [Wi]—except that we have dropped the unnecessary condition of \( (m_i)_{i \in \mathbb{N}} \subset \text{ri} M \). Wichmanns original theorem is given as Corollary 1.1.

**Theorem 1.** Let \( \phi \) be a continuous disorderliness measure and let the fibers of \( \pi_U|S \) be selective. Let \( m_i \subset M \) be a sequence converging to some \( m \in M \) and let \( \rho_i \in F(m_i) \) for all \( i \in \mathbb{N} \). If \( \rho_i \xrightarrow{i \to \infty} \Phi(m) \), then \( \Phi(m_i) \xrightarrow{i \to \infty} \Phi(m) \).

**Proof:** We assume \( \Phi(m) = \lim_{i \to \infty} \rho_i \). Since \( \rho_i \in F(m_i) \) we have \( \phi(\rho_i) \leq \phi(\Phi(m_i)) \) for all \( i \in \mathbb{N} \). As \( S \) is compact, a subsequence of \( \Phi(m_i) \), which we denote \( \Phi(m_i) \) in abuse of notation, must converge. Let \( \lim_{i \to \infty} \Phi(m_i) =: \rho \). As \( \phi \) is continuous on \( S \), we have
\[
\phi(\Phi(m)) = \lim_{i \to \infty} \phi(\rho_i) \leq \lim_{i \to \infty} \phi(\Phi(m_i)) = \phi(\rho).
\]
Since the fibers of \( \pi_U|S \) are selective and since
\[
\pi_U(\Phi(m)) = m = \lim_{i \to \infty} m_i = \lim_{i \to \infty} \pi_U(\Phi(m_i)) = \pi_U(\rho)
\]
this implies \( \rho = \Phi(m) \). The converging subsequence of \( \Phi(m_i) \) was arbitrary, so \( \Phi(m) \) converges to \( \Phi(m) \).

We denote the closure of a subset \( X \subset \mathbb{E}^n \) in the norm topology of \( \mathbb{E}^n \) by \( \overline{X} \).
Corollary 1.1. Let $\phi$ be a continuous disorderliness measure and let the fibers of $\pi_U|_S$ be selective. Then $\Phi(\mathbb{M}) \subset \Phi(\text{ri}\mathbb{M})$.

Proof: Let $m \in \mathbb{M}$ and let $\rho_i \subset \text{ri}S$ be a sequence converging to $\Phi(m) \in S$. Then the mean values $m_i := \pi_U(\rho_i)$ lie in the relative interior of $\mathbb{M}$ for all $i \in \mathbb{N}$, because the relative interior of a convex set $C \subset \mathbb{E}^n$ maps to the relative interior of the image set of $C$ under a linear map (see e.g. Theorem 6.6 in Rockafellar). Theorem 1 shows $\Phi(m_i) \xrightarrow{i \to \infty} \Phi(m)$. □

Theorem 2. Let $\phi$ be a continuous disorderliness measure and let the fibers of $\pi_U|_S$ be selective. For any $m \in \mathbb{M}$ the inference $\Phi$ is continuous at $m$ if and only if $\pi_U|_S$ is open at $\Phi(m)$.

Proof: The “only if” part was proved in the introduction, we assume in this section that $\Phi$ is not continuous at $m \in \mathbb{M}$. Then, there is a sequence $m_i \subset \mathbb{M}$ converging to $m$ such that its inference values $\Phi(m_i)$ do not converge to $\Phi(m)$,

$$\lim_{i \to \infty} \Phi(m_i) \neq \Phi(m).$$

It follows from Theorem 1 that $\Phi(m)$ is not the limit of any sequence $\rho_i \subset S$ that has mean values $\pi_U(\rho_i) = m_i$ for all $i \in \mathbb{N}$. Hence there exists $\epsilon > 0$ and $K \in \mathbb{N}$ such that

$$\inf_{i \geq K} \min_{\rho \in F(m_i)} \|\Phi(m) - \rho\| \geq \epsilon.$$

This proves that the neighborhood of $\Phi(m)$

$$V := \{\rho \in S \mid \|\Phi(m) - \rho\| < \epsilon\}$$

does not intersect any of the fibers $F(m_i)$ for $i \geq K$. So $m_i \notin \pi_U(V)$ for all $i \geq K$. As $m_i \xrightarrow{i \to \infty} m$ this proves that $\pi_U(V)$ contains no neighborhood of $m = \pi_U(\Phi(m))$, so $\pi_U|_S$ is not open at $\Phi(m)$. □

5 Continuity for generic mean values

This section studies a subset—the relative interior—of the mean value set, where the inference is always continuous. A special case is the real analytic maximum-entropy inference, where the relative interior of the mean value set is generic in the sense that it equals the set of mean values of all invertible density matrices [Wi].

The domain of $U$ is the inverse projection of the relative interior of the mean value set $\text{ri}\mathbb{M}$

$$\text{dom}(U) := F(\text{ri}\mathbb{M}) = (\text{ri}\mathbb{M} + U^\perp) \cap S.$$ 

Continuity of the inference on $\text{ri}\mathbb{M}$ translates through Theorem 2 into openness of the projection $\pi_U|_{\text{dom}(U)}$, which we shall prove.
Remark 1. This section could be head-lined "lower semi-continuity" in analogy to Section 3. The u.s.c. property of the fiber function $F|_{\text{dom}(U)}$ has an l.s.c. counterpart, which is equivalent to the openness of the projection $\pi_U|_{\text{dom}(U)}$ on the domain of $U$, see Section VI.1 in [Be]. It seems we can not draw advantage from multi-valued maps in this section.

We make a transformation such that $\text{dom}(U)$ has non-empty interior in the topology of $\mathbb{E}^n$.

Lemma 2. Let $\mathbb{A} \subset \mathbb{E}^n$ be a non-empty affine space with translation vector space $V$. Let $U \subset \mathbb{E}^n$ be a linear subspace and set $W := \pi_V(U)$. Let $\tilde{a} \in \mathbb{A}$ and denote $T(x) := x + \tilde{a} - \pi_W(\tilde{a})$ the translation of $x \in \mathbb{E}^n$ by $\tilde{a} - \pi_W(\tilde{a})$. Then we have for all $a \in \mathbb{A}$

$$\pi_U(a) = \pi_U \circ T \circ \pi_W(a)$$

and the restriction $\pi_U|_{T(W)} : T(W) \to \pi_U(\mathbb{A})$ is an affine isomorphism.

Proof: Let $\tilde{a} \in \mathbb{A}$. We show that $\pi_U \circ T \circ \pi_W(a) = \pi_U(a)$ holds for all $a \in \mathbb{A}$. Let $u \in U$ and $a \in \mathbb{A}$. Then, since $\pi_U, \pi_V, \pi_W$ are self-adjoint, we get

$$\langle u, \pi_U \circ T \circ \pi_W(a) - a \rangle = \langle u, T \circ \pi_W(a) - a \rangle = \langle u, \pi_W(a) + \tilde{a} - \pi_W(\tilde{a}) - a \rangle$$

$$= \langle \pi_V(u), \pi_W(a - \tilde{a}) - (a - \tilde{a}) \rangle = \langle \pi_V(u), a - \tilde{a} - (a - \tilde{a}) \rangle = 0.$$

On the other hand, since $\pi_U|_{\mathbb{A}} : \mathbb{A} \to \pi_U(\mathbb{A})$ is trivially onto, the dimension equality

$$\dim T(W) = \dim W = \dim \pi_V(U) = \dim \pi_U(V) = \dim \pi_U(\mathbb{A})$$

proves that $\pi_U|_{T(W)} : T(W) \to \pi_U(\mathbb{A})$ is an affine isomorphism.

We shall use the neighborhood $B_{\epsilon_1,\epsilon_2}(x) := B_{\epsilon_1}(\pi_U(x)) + B_{\epsilon_2}(\pi_U(x))$ of $x \in \mathbb{E}^n$ in the topology of $\mathbb{E}^n$, for $\epsilon_1 > 0, \epsilon_2 > 0$ and balls defined in (2).

Lemma 3. The projection $\pi_U|_{\text{dom}(U)} : \text{dom}(U) \to \text{ri} \mathbb{M}$ is open.

Proof: We can assume that the state space $\mathcal{S} \subset \mathbb{E}^n$ has full dimension $n$. This follows by using in Lemma 2 the affine hull $\mathbb{A}$ of $\mathcal{S}$ and applying the two homeomorphisms, $T : W \to T(W)$ and $\pi_U|_{T(W)} : T(W) \to \pi_U(\mathbb{A})$. In the topology of $\mathbb{E}^n$, the domain and the state space have the same non-empty interior, which we denote by $\text{dom}(U)^\circ$. Let $\rho \in \text{dom}(U)$. If the convex set

$$G(\rho) := (\rho + U^\perp) \cap \text{dom}(U)^\circ$$

is empty, then by a separation theorem, see Theorem 11.3 in [Ro], there exists a hyperplane $H \supset (\rho + U^\perp)$ which does not intersect $\text{dom}(U)^\circ$. This shows that $\pi_U(\rho)$ lies on the relative boundary of $\text{ri} \mathbb{M}$. The relative boundary of a relatively open set being empty, this is a contradiction.

We can assume that $G(\rho)$ is non-empty. Let $U(\rho)$ be a neighborhood of $\rho$, then there exist $\epsilon_1, \epsilon_2 > 0$ such that the neighborhood of $\rho$

$$V(\rho) := \text{dom}(U) \cap B_{\epsilon_1,\epsilon_2}(\rho)$$
is included in $U(\rho)$. As $G$ a non-empty, there is a state $\sigma \in G \cap V(\rho)$. Indeed, as $\rho \in \text{dom}(U)$, for any state $\tilde{\sigma} \in G$ the state $(1 - \lambda)\tilde{\sigma} + \lambda \rho$ belongs to $\text{dom}(U)^{\circ}$ for any $0 \leq \lambda < 1$ (see Theorem 6.1 in [Ro]). All of these states lie in $G(\rho)$ and for $\lambda$ near one they lie in $V(\rho)$.

As $\sigma$ lies in $\text{dom}(U)^{\circ}$ and in $V(\rho)$, there exist $\tilde{e}_1, \tilde{e}_2 > 0$ such that

$$B_{\tilde{e}_1, \tilde{e}_2}(\sigma) \subset V(\rho).$$

Let us choose $\tilde{e}_1 < \epsilon_1$ and $\tilde{e}_2 < \epsilon_2$. Then the neighborhood of $\rho$

$$W(\rho) := (\text{dom}(U) \cap B_{\tilde{e}_1, \tilde{e}_2}(\rho)) \cup B_{\tilde{e}_1, \tilde{e}_2}(\sigma)$$

is included $V(\rho)$. We have $\pi_U(W(\rho)) = B_{\tilde{e}_1}^{U}(\pi_U(\rho))$. So $\pi_U(W(\rho))$ is a neighborhood of $\pi_U(\rho)$. This neighborhood is by construction included in $\pi_U(U(\rho))$ so $\pi_U|_{\text{dom}(U)}$ is an open map.

**Theorem 3.** Let $\phi$ be a continuous disorderliness measure and let the fibers of $\pi_U|_{\mathcal{S}}$ be selective. Then the restriction $\Phi|_{\mathbb{HM}} : \mathbb{HM} \to \text{dom}(U)$ of the inference is continuous.

**Proof:** This follows from Theorem 2 and Lemma 3. \qed

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