Gravitational lensing in braneworld gravity: formalism and applications

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Received 3 July 2007, in final form 3 November 2007
Published 31 January 2008
Online at stacks.iop.org/CQG/25/045003

Abstract
In this paper, we develop a formalism which is different from the standard lensing scenario and is necessary for understanding lensing by gravitational fields which arise as solutions of the effective Einstein equations on the brane. We obtain general expressions for measurable quantities such as time delay, deflection angle, Einstein ring and magnification. Subsequently, we estimate the deviations (relative to the standard lensing scenario) in the above-mentioned quantities by considering the line elements for clusters and spiral galaxies obtained by solving the effective Einstein equations on the brane. Our analysis reveals that gravitational lensing can be a useful tool for testing braneworld gravity as well as the existence of extra dimensions.

PACS numbers: 04.50.+h, 95.35.+d, 98.62.Sb

1. Introduction

One of the path-breaking successes of Einstein’s general theory of relativity is its prediction of the amount of bending of light by a gravitating object. That a light ray can be deflected by the gravitational field of a massive object was indicated, as early as in 1704, by Newton. It was Einstein, however, who first used the equivalence principle to calculate this ‘Newtonian’ deflection angle [1]. Later on, he obtained the formula [2] based on his general relativistic field equations and found the deflection angle to be exactly twice the Newtonian deflection. This angle, though very small, was found to be in excellent agreement in the solar system, when measured by Eddington and others during a total solar eclipse [3]. Eddington, among others, also pointed out the possibility of having multiple images of a source due to this light
bending [4]. Later on, Chowlson [5] indicated the formation of the Einstein ring by the images for a specific alignment of the source. This effect was also independently shown by Einstein himself [6]. All these properties, resonating with refraction in geometrical optics, led to the conclusion that a gravitating object can behave like a lens—the gravitational lens.

Due to excessively small values for the deflection angle, physicists, including Einstein himself, were not too sure whether these properties could be detected some day. Zwicky, the most enthusiastic of all, calculated the mass of galaxies inside clusters by using gravitational lensing [7] and suggested that the magnification of distant fainter galaxies can make them visible [8]. However, physicists had to wait until 1979 for observational verifications. It was only after the discovery of lensing effects by the quasar QSO 0957+561A,B [9] (that they are in fact double images of a single QSO) when the predictions of Zwicky and others were found to be true. Subsequently, several gravitational lenses have been detected, which have eventually made the subject an attractive and promising field of research today [10–14].

Of late, gravitational lensing has emerged as an important probe of structures and has found several applications in cosmology and astrophysics [15]. To mention a few, most of the lens systems involve sources and lenses at moderate or high redshift, thereby making it possible to study the geometry of the universe by lensing. Thus, the Hubble parameter [16] and the cosmic density can be determined by using multiple-image lens systems and time delay between the different light paths of multiply imaged sources, such as quasars. The quantitative analysis of the multiply imaged sources and Einstein radius can provide detailed information on the mass of the deflector [9], by knowing the angular diameters and redshifts of the source and the lens. Furthermore, the magnification and shear effects due to weak lensing can be used to obtain statistical properties of matter distribution between the observer and the source [10]. So, it can be used to study the properties of dark matter halos surrounding galaxies, and thus, provide a test for its existence. The detection of cosmic shear plays an important role in precision cosmology. The arcs, which result from a very strong distortion of background galaxies, can be used to constrain cosmological parameters [17]. Another interesting application is that it can serve as a crucial test for any modified theory of gravity. In [18] a rigorous, analytical formalism was developed in order to study lensing beyond the weak deflection limit—the motivation there being the search for signatures of modified gravity. This formalism was further investigated in [19] for PPN metrics and then in [20] for metrics that arise in the context of braneworld gravity. Though not entirely a strong lensing study, the analysis in [18–20] goes much beyond the usual weak deflection limit. A nice review of the current status of gravitational lensing beyond the weak field, small angle approximation can be found in [21].

Lensing characteristics are essentially determined by the gravitational potentials. Lensing effects probe the total matter density, no matter whether it is luminous or dark. Gravitational lensing is thus an important tool to test theories of gravity which predict gravitational potentials different from the one in GR.

In [23] it was shown that in order to consider dark matter with pressure in galaxy halos, it is necessary to have two gravitational potentials. In this approach, the weak field equations with the two potentials are first solved to obtain the functional forms of the potentials. The deflection of light due to such a weakly relativistic (but not Newtonian) scenario is then analyzed in the line elements obtained [23]. Subsequent to the work in [23], in [24, 25], we have demonstrated that bulk-induced extra-dimensional effects in braneworld gravity can provide an alternative to particle dark matter. It was claimed that one could re-interpret the standard dark matter scenario as a purely geometric (necessarily extra-dimensional) effect rather than due to some invisible material entity. Along with the Newtonian potential, this theory requires the existence of another potential. These potentials have been found for spiral
galaxies and clusters. One of our aims in this paper is to develop the lensing formalism for a weakly relativistic situation where two gravitational potentials are necessary. This will then be applied to braneworld gravity. To illustrate the formalism, we shall estimate some of the observable quantities for cluster and galaxy metrics. We will also indicate possible links with observational data. It must be mentioned here that there have been some earlier investigations along somewhat similar lines [20, 26, 27, 29–31]. While, in [26], the authors study strong lensing by a braneworld black hole, [27] discusses strong lensing and [31] analyzes certain aspects for a typical galactic metric in braneworlds. In [29], calculations of the bending of light in Garriga–Tanaka and tidal charge metrics have been done, while [20] provides an extensive lensing study with the Garriga–Tanaka metric. Lensing calculations in DGP braneworld models are also around [30]. More recently, in [22], the authors have further explored spherically symmetric line elements (galaxy halos, in particular) in the context of the various existing effective theories on the brane.

2. Bending of light on the brane

Following [23–25], we express a static spherically symmetric metric on the brane in the weak field limit using isotropic coordinates as

$$dS^2 = -\left(1 + \frac{2\Phi(r)}{c^2}\right)c^2 dt^2 + \left(1 - \frac{2\Phi(r) - 2\Psi(r)}{c^2}\right)d\vec{X}^2,$$  \hspace{1cm} (2.1)

where $\Phi(r)$ is the Newtonian potential and $\Psi(r)$—the relativistic potential—adds a non-trivial correction to it, characterizing braneworld gravity (or, in general situations where pressure terms in the energy–momentum tensor are important) and thus, making the theory distinguishable from GR. Note that with the intention of studying optical properties, we have written explicitly included factors of ‘$c$’ in the line element.

Lensing effects in the above spacetime metric can be expressed in terms of an effective refractive index:

$$n = 1 + \frac{|2\Phi - \Psi|}{c^2}. \hspace{1cm} (2.2)$$

Thus the refractive index is greater than 1, confirming that a light ray, analogous to geometrical optics, passes through the lens slower than the speed of light in vacuum. Further, this refractive index is related to the corresponding GR value by

$$n = n_R - \frac{|\Psi|}{c^2}. \hspace{1cm} (2.3)$$

Thus, the lens on the brane acts as an optically rarer medium than a lens in GR. From now on, we shall assume that the absolute value is implicitly written whenever we write the potentials.

Since the light speed is reduced inside the lens, there occurs a delay in the arrival time of a light signal compared to another signal passing far away from the lens with a speed $c$. This leads to the time delay of a photon coming from a distant source (S), propagating through the lens to a distant observer (O):

$$\Delta t = \int_S^O \frac{2\Phi - \Psi}{c^3} dl,$$  \hspace{1cm} (2.4)

where the integral is to be evaluated along the straight-line trajectory between the source and the observer. Hence a light ray passing through the lens on the brane suffers a time delay which is less than its GR value, $\Delta t_R$ (the so-called Shapiro time delay [10]), by an amount

$$\Delta t_R - \Delta t = \frac{1}{c^3} \int_S^O |\Psi| dl.$$  \hspace{1cm} (2.5)
Thus, an accurate measurement of the time delay can discriminate between the two theories of gravity, and thus, can test the scenario from observational ground.

The deflection angle, $\hat{\alpha}$, of a photon in this gravitational field is determined by the integral of the gradient of the effective refractive index perpendicular to the light path. This deflection angle can also be derived by using Fermat’s principle, by extremizing the light travel time from the source to the observer. Thus, we have,

$$\hat{\alpha} = -\int_O^S \hat{\nabla}_\perp n = -\int_O^S \hat{\nabla}_\perp \left(1 - \frac{2\Phi - \Psi}{c^2}\right) dl,$$

(2.6)

where $\hat{\nabla}_\perp$ denotes the derivative in the direction perpendicular to this trajectory. Thus, the deflection angle is related to the GR deflection $\hat{\alpha}_R$ by

$$\hat{\alpha} = \hat{\alpha}_R - \frac{1}{c^2} \int_O^S \hat{\nabla}_\perp \Psi dl = \hat{\alpha}_R - \hat{\alpha}_\Psi,$$

(2.7)

where the term involving $\Psi$ is the braneworld correction (or a correction in a modified theory of gravity) and, for brevity, will be depicted as $\hat{\alpha}_\Psi$ from now on.

What is obvious from the above equation is that a light ray on the brane is deviated by a smaller amount in comparison with its corresponding GR deflection. Consequently, it turns out that measuring the deflection angle can serve as a crucial test while comparing braneworld gravity effects with those of GR.

As a useful illustration, let us consider the thin lens scenario. Most of the spherically symmetric objects can be approximated as a thin lens for which the Schwarzschild radius is much smaller than the impact parameter, so that the lens appears to be thin in comparison with the total extent of the light path.

The GR deflection of such a lens is given by the ‘Einstein angle’ [10]

$$\hat{\alpha}_R = \frac{4GM(\xi)}{c^2 \xi} = \frac{2R_S}{\xi},$$

(2.8)

where $R_S = 2GM/c^2$ is the Schwarzschild radius of the lens (for this reason, this type of lens is also called the Schwarzschild lens) and $M(\xi) = M$ is the constant mass for a point mass source. Note that the general expression for the mass function is given by

$$M(\xi) = \int \frac{\Sigma(\xi')}{(\xi' - \xi)^2} d^2\xi'$$

(2.9)

in terms of a two-dimensional vector $\xi'$ on the lens plane, which is basically the distance from the lens center $\xi' = 0$. This general expression reduces to a constant mass $M(\xi) = M = \text{constant}$ for a point mass source. Hence a thin lens in braneworld gravity deviates a light ray by an amount

$$\hat{\alpha} = \frac{4GM}{c^2 \xi} - \hat{\alpha}_\Psi$$

(2.10)

which can be subject to observational verification.

### 3. Lensing geometry on the brane

Apart from the time delay and the deflection angle, the other observable properties of a gravitational lens are the position of the image and the magnification involving convergence and shear. In order to find out these quantities, it is customary to obtain the lensing geometry in terms of the lens equation. Below is a schematic diagram that shows how a gravitational
lens functions. A light ray, emerging from the source S, is deflected by an angle $\hat{\alpha}$ by the lens L and reaches the observer O, resulting in the image at I. The angular positions of the source and the image with respect to the optical axis of the lens are $\beta$ and $\theta$, respectively. Here, $D_{ds}$, $D_d$, and $D_s$ are the angular diameter distances between source and lens, lens and observer, and source and observer, respectively.

Now, the deflection angle being small, the angular positions bear a simple relation among them. The general lens equation [10] reduces to the following:

$$D_s \beta = D_s \theta - D_{ds} \hat{\alpha}.$$  \hspace{1cm} (3.1)

Thus, in terms of the reduced deflection angle (where $D_d D_{ds} / D_s = D$ measures the effective distance)

$$\alpha = \frac{D_{ds}}{D_s} \hat{\alpha} = \alpha_R - \alpha_\Psi,$$  \hspace{1cm} (3.2)

the vector expression for equation (3.1) on the lens plane can be written as

$$\vec{\beta} = \vec{\theta} - \vec{\alpha}(\theta).$$  \hspace{1cm} (3.3)

This is the desired lens equation—also called the ‘ray-tracing equation’. Note that though this equation resembles the lens equation in GR, quantitatively this is a different equation, since the deflection angle and the angular positions in the braneworld gravity are different from their GR values. This will be revealed from the new results obtained in the subsequent sections.

### 3.1. Image formation and the Einstein ring

Equipped with the lens equation, one can now study the formation of images, which will eventually reveal some interesting facts. A source lying on the optical axis ($\beta = 0$) of a circularly symmetric lens is imaged as a ring, called the Einstein ring [5], and the corresponding radius of the ring is called the Einstein radius.

The lens equation (3.3) with two potentials suggests that the deflection angle $\alpha$ has a modification $\alpha_\Psi$ which is a function of $\theta$. Hence, one needs to know the exact form of $\Psi$ in order to get the correction for a circularly symmetric lens. Of course, for the case $\Psi = 0$ the results are identical to the GR results, but this is not so when $\Psi \neq 0$. Below we shall illustrate the situation with a specific example.

Let us consider the lensing scenario for the Garriga–Tanaka metric which incorporates the effects of extra dimensions in the exterior gravitational field of a spherically symmetric, static massive object living on the brane [28]. The light bending angle in this metric has been calculated in [29]. It is a straightforward exercise to show that this metric can indeed be cast
into the form with two potentials $\Phi$ and $\Psi$ as is being discussed in the present paper. With this new formalism, the net deflection is the same as obtained in [29]. Explicitly,

$$\hat{\alpha} = \frac{4GM}{c^2r} + \frac{4GML^2}{c^2r^3},$$

(3.4)

where the last term on the rhs is the braneworld modification (or, more generally, a modification due to a second potential). For this deflection, we can now rewrite the lens equation (3.3) in the form

$$\theta^4 - \beta \theta^3 - \theta_{ER} \theta^2 - (\theta_l)^2 = 0,$$

(3.5)

where $\theta_l$ is the modification due to the characteristic length scale $l$ of the angular position of the image with respect to the optical axis of the lens.

To obtain the Einstein ring, we put the condition $\beta = 0$ in the lens equation. This results in the following expression for the image position:

$$\theta^2 = \frac{1}{4} \left( \theta_{ER} \pm \sqrt{(\theta_{ER})^2 + 4\theta_l^2} \right).$$

(3.6)

The minus sign is ruled out because it will give imaginary $\theta$. Consequently, with the valid solution with the positive sign, we arrive at the following interesting conclusion: in a theory of gravity with two potentials, the Einstein ring is indeed formed but the radius of the Einstein ring is different from the GR radius. In order to get the full image structure one needs to look at the roots of the quartic equation (3.5), which is not a very trivial exercise. Of course, one can solve the quartic equation and find out the roots depicting the image position for this specific metric and the solutions will definitely give some new results as obvious from equation (3.6) but the results do not always turn out to be tractable. A second independent approach is the perturbative analysis following [20]. However, since the results will vary with the expressions for relativistic potential for different metrics, it is sufficient to realize that the Einstein ring and image position with two potentials will be different from the GR results in general and perform the analysis afresh with the specific potentials under consideration. The situation is applicable to models of dark matter with relativistic stresses, such as [23], as well. Thus, our formalism is quite general irrespective of whether we are studying braneworlds or not.

However, even without the above-mentioned analysis, it is easy to show that the radius of the Einstein ring will be larger if we have some conditions on possible additional terms in the deflection angle. Let us assume that with the additional terms arising out of a modified deflection angle, the condition for the Einstein ring ($\beta = 0$) is of the form

$$\theta = \frac{\theta_{ER}^2}{\theta} + \frac{\theta_{ER}^2}{\theta} \sum_{i=1}^{m} a_{(2n+1)} \theta_{(2n+1)}.$$

(3.7)

where the additional terms are encoded in the second term on the rhs, with arbitrary coefficients $a_{(2n+1)}$. Keeping only the odd-order terms in the summation to make sure that $\beta \to -\beta$ implies $\theta \to -\theta$, one can rearrange the terms of the above equation to give

$$\frac{\theta}{\theta_{ER}^2} - 1 = \sum_{i=1}^{m} \frac{a_{(2n+1)}}{\theta_{2n}}.$$

(3.8)

Obviously, the rhs is positive as long as all the coefficients $a_{(2n+1)}$ are positive. Consequently, wherever such corrections in the deflection angle arise, the Einstein radius will be greater than its value obtained without them.

Thus, following the above analysis, for the Garriga–Tanaka metric, the Einstein ring will be larger than the GR case. This is, in general, true for any such metric with an additional correction term arising due to pressure-like effects in the source. No matter whether it arises...
from relativistic stresses or from braneworld modifications, we will have a similar conclusion as long as the correction varies as inverse powers of $\theta$. This is, indeed, an interesting fact from an observational point of view and is a clear distinction between the two theories.

However, it is worthwhile noting from equation (3.6) that, with the present example, a circularly symmetric lens forms two images of the source, lying on either side. While one image ($\theta^-$) lies inside the Einstein ring, the other one ($\theta^+$) outside. This is how multiple images are formed by a gravitational lens. This situation is identical to GR.

3.2. Singular isothermal sphere

Let us now discuss the image formation by a galaxy modeled as an isothermal sphere. The matter constituents of a galaxy are considered to be in thermal equilibrium, confined by the spherically symmetric gravitational potential of the galaxy, which behaves like a singular isothermal sphere obeying the equation

$$m\sigma_v^2 = kT,$$

where $\sigma_v$ is the line-of-sight velocity dispersion of the stars and HI clouds rotating inside the galaxy. By utilizing the properties of hydrostatic equilibrium and the velocity profile of HI clouds inside galaxies, one can easily derive the relation

$$v_c^2(r) = \frac{GM(r)}{r} = 2\sigma_v^2,$$

which reproduces the observed flat rotation curve. Consequently, under the thin lens approximation, equation (2.10) implies that a light ray on the brane is deflected by an isothermal spherical galaxy by an angle

$$\hat{\alpha} = \frac{4\pi \sigma_v^2}{c^2} - \hat{\alpha}_\Psi.$$

Thus, for $\Psi \neq 0$, there is a non-trivial modification that tends to alter the GR results. Once again the results will differ from GR due to the presence of a nonzero $\alpha_\Psi$ in the above equation. However, as discussed earlier, the quantitative results will depend exclusively on the specific expression for the relativistic potential $\Psi$.

4. Magnification in braneworld gravity

As in geometrical optics, a source not only gets multiply imaged by a gravitational lens but the deflected light rays can also change the shape and size of the image compared to the actual shape and size of the source. This happens due to the distortion of the cross-section of light bundles that changes the solid angle viewed from the location of the observer. However, the surface brightness of the source is not affected by the lens as light neither gets absorbed nor emitted during deflection by the lens.

The quantity representing this change in shape and size of the image with respect to the source is called the magnification which is given as

$$\mu = \det M = \frac{1}{\det A},$$

where $A$ is the Jacobian of the lens-mapping matrix. Below we discuss in detail how to describe and estimate the magnification for metrics in braneworld gravity.
4.1. Lensing potential

The Jacobian matrix can be expressed conveniently in terms of a scalar potential, called the lensing potential, which provides useful physical insight. With a nonzero relativistic potential, the lensing potential is now modified to

\[ V(\theta) = \frac{D_d}{D_D D_s} \int \frac{2 \Phi - \Psi}{c^2} \, dl. \]  

(4.2)

For \( \Psi = 0 \), we get back the Newtonian potential. Hence, in braneworld gravity, the lensing potential is now reduced by an amount

\[ V = \frac{D_d}{D_D D_s} \int \frac{\Psi}{c^2} \, dl. \]  

(4.3)

It is worthwhile to mention two important properties of the lensing potential:

(i) The gradient of \( V \) w.r.t. \( \theta \) is the reduced deflection angle on the brane

\[ \nabla_\theta V = \frac{D_d}{D_D D_s} \int \hat{\nabla}_\perp \left( \frac{2 \Phi - \Psi}{c^2} \right) \, dl = \alpha \]  

(4.4)

which, together with the GR result \( \nabla_\theta V_R = \alpha_R \), implies

\[ \nabla_\theta V = \alpha. \]  

(4.5)

(ii) The Laplacian of \( V \) w.r.t. \( \theta \) is the scaled surface mass density

\[ \nabla_\theta^2 V = \frac{D_d}{D_D D_s} \int \nabla_\perp^2 \left( \frac{2 \Phi - \Psi}{c^2} \right) \, dl = 2 \frac{\Sigma(\theta)}{\Sigma_{cr}}. \]  

(4.6)

where \( \Sigma \) is the surface density as already defined and \( \Sigma_{cr} = (c^2/4\pi G)(D_s/D_D D_{\mu}) \) is its critical value. The scaled surface density, called the convergence \( \kappa \), reveals that \( V \) satisfies the 2D Poisson equation

\[ \nabla_\theta^2 V = 2\kappa. \]  

(4.7)

It is straightforward to verify that equations (4.4) and (4.6) together give the same deflection angle as calculated for a thin lens.

4.2. Convergence and shear

Using the lensing potential, the Jacobian matrix can be written as

\[ A = \delta_{ij} - \frac{\partial^2 (V_R - V)}{\partial \theta_i \partial \theta_j}, \]  

(4.8)

wherefrom the inverse of the magnification tensor turns out to be

\[ M^{-1} = M_R^{-1} + \frac{\partial^2 V}{\partial \theta_i \partial \theta_j}. \]  

(4.9)

and the total magnification is given by

\[ \mu = \det M = \mu_R \left[ 1 + \mu_R \det \left( \frac{\partial^2 V}{\partial \theta_i \partial \theta_j} \right) \right]^{-1}, \]  

(4.10)

where \( \mu_R \) is the magnification calculated from GR. Clearly, the magnification in braneworld gravity is different from the corresponding GR value due to the presence of the additional term inside the square bracket. However, in order to comment conclusively on whether the magnification will be more or less than the GR value, one needs to have a specific expression
for $\Psi$ and check whether the determinant of the potential due to that $\Psi$ has a positive or a negative contribution. In what follows, we shall illustrate this situation in a bit more detail. From now on, we shall use $\partial^2 V / \partial \theta_i \partial \theta_j = V_{ij}$ for brevity.

Two important quantities derived from the linear combinations of the components of the Jacobian matrix provide the real picture of how a source is mapped onto the image. They are

(i) Convergence $\kappa = \frac{1}{2} (V_{11} + V_{22}) = \frac{1}{2} \text{Tr} V_{ij}$.

(ii) Shear $\gamma = \sqrt{\gamma_1^2 + \gamma_2^2}$, where $\gamma_1 = \frac{1}{2} (V_{11} - V_{22}) = \gamma \cos 2\phi$ and $\gamma_2 = V_{12} = V_{21} = \gamma \sin 2\phi$.

The first one depicts the change in the size of the source when imaged, while the latter one gives the change in shape. A combination of the two accounts for the total magnification. In terms of convergence and shear, the Jacobian matrix can be expressed as

\[
A = \begin{pmatrix} 1 - \kappa - \gamma_1 & -\gamma_2 \\ -\gamma_2 & 1 - \kappa + \gamma_1 \end{pmatrix}. \tag{4.11}
\]

The calculation of the convergence and shear can serve as an important tool to distinguish between braneworld gravity and GR. In order to calculate these quantities for a nonzero $\Psi$, we use the spherical symmetry ($\theta_1 = \theta_2 = \theta$) of the lens, which yields

\[
\kappa = \frac{1}{2} \text{Tr} (V_{Rij} - V_{\Psi ij}) = \frac{\partial^2 (V_R - V_{\Psi})}{\partial \theta^2}, \tag{4.12}
\]

\[
\gamma_1 = \frac{1}{2} \left[ (V_{R11} - V_{\Psi 11}) - (V_{R22} - V_{\Psi 22}) \right] = 0, \tag{4.13}
\]

\[
\gamma_2 = V_{R12} - V_{\Psi 12} = V_{R21} - V_{\Psi 21} = \frac{\partial^2 (V_R - V_{\Psi})}{\partial \theta^2}, \tag{4.14}
\]

\[
\gamma = \gamma_2 = \frac{\partial^2 (V_R - V_{\Psi})}{\partial \theta^2}. \tag{4.15}
\]

The results show that both the convergence and the shear are less than the corresponding GR values due to the presence of a nonzero relativistic potential.

We can now construct the Jacobian matrix by using its components as calculated above. Separating the braneworld modifications from the GR values, we finally arrive at

\[
A = \begin{pmatrix} 1 - \kappa_R - \gamma_1 & -\gamma_2 \\ -\gamma_2 & 1 - \kappa_R + \gamma_1 \end{pmatrix} + \begin{pmatrix} \kappa_{\Psi} + \gamma_1 \kappa_{\Psi} \\ \gamma_2 \kappa_{\Psi} \gamma_1 \kappa_{\Psi} \end{pmatrix}. \tag{4.16}
\]

The above equation shows explicitly the role the relativistic potential plays in determining the magnification. The first matrix is the Jacobian in GR, while the second one is the exclusive contribution from a nonzero relativistic potential. This expression clearly reveals that the determinant of the Jacobian with a nonzero $\Psi$ is different from the GR value (where $\Psi = 0$). However, whether this determinant will have a positive or a negative contribution solely depends upon the explicit expression for the relativistic potential one gets on solving the effective Einstein equation. Thus, though the convergence and shear are less than the GR value due to a positive $\Psi$, the expression for the magnification being highly nonlinear, one cannot say a priori whether the magnification is more or less than GR. What one can say at best is that the magnification will be different from GR. It is only when one has a specific expression for $\Psi$, one can calculate this difference (i.e., more or less) conclusively, a fact which resonates with the discussions following equation (4.10). In the following section, we shall calculate these quantities for specific potentials and estimate the difference of the quantities from GR.
Thus, we arrive at the conclusion that finding out the magnification by spherically symmetric lenses by measuring the convergence and shear can help us to test braneworld gravity, and in general, any theory of gravity with two potentials, through observations.

5. Quantitative estimation

Let us now try to make some actual quantitative estimation of lensing effects by clusters and spiral galaxies on the brane and see by how much amount the observable quantities differ from the GR values. To this end, we shall make use of the Newtonian and relativistic potentials obtained in [24, 25].

5.1. Lensing by clusters

For an x-ray cluster on the brane, we employ the Newtonian and relativistic potentials obtained in [24]. Upon scaling with $c$, they read

$$\Phi(r) = \frac{2kT}{\mu m_p} \ln \frac{r}{r_c}, \quad (5.1)$$

$$\Psi(r) = \left[ \frac{kT}{\mu m_p} - 2\pi G \rho_0 r_c^2 \right] \ln \frac{r}{r_c}, \quad (5.2)$$

where $\rho_0, r_c, \mu$ and $T$ are, respectively, the central density, core radius, mean atomic weight of gas particles inside the x-ray cluster and the temperature of the gas.

In the standard GR analysis of x-ray profiles of clusters by using dark matter, $\Psi = 0$ and the deflection angle $\hat{\alpha}_R$ of a photon from a distant source, propagating through the dark matter halo to a distant observer is given by

$$\hat{\alpha}_R = \frac{2}{c^2} \int_0^S \nabla \Phi \cdot ds.$$ \hspace{1cm} (5.3)

Using the above expression for $\Phi$, we find from GR that a photon passing through the halo of a cluster experiences a constant deflection

$$\hat{\alpha}_R = \frac{4\pi kT}{\mu m_p c^2}.$$ \hspace{1cm} (5.4)

In braneworld gravity $\Psi \neq 0$, and the deflection angle is now modified to equation (2.7). For a cluster with the above $\Phi$ and $\Psi$ as calculated from braneworld gravity, this deflection angle turns out to be

$$\hat{\alpha} = \frac{3\pi kT}{\mu m_p c^2} + \frac{2\pi^2 G \rho_0 r_c^2}{c^2} = \hat{\alpha}_R \left[ \frac{3}{4} + \frac{\pi G \rho_0 r_c^2 \mu m_p}{2kT} \right]. \hspace{1cm} (5.5)$$

For a typical x-ray cluster, we use the following representative values for the cluster parameters [32]: $\rho_0 = 5 \times 10^{-24}$ kg m$^{-3}$, $r_c = 0.3$ Mpc, $\mu = 0.6$, $T = 10^8$ k. A good summary of up-to-date cluster data is also available in [33] for further confirmation of these data. Consequently, the deflection angle from braneworld gravity turns out to be around $\sim 80\%$ of the GR value.

As already pointed out, the different observable properties of lensing for a cluster in the brane will also differ significantly from the GR values. In table 1 we mention the estimates for some of the observable quantities, namely convergence and shear, for an x-ray cluster with our choice of parameters.

We find that there is a $\sim 20\%$ difference in the estimation of these observable quantities in lensing in the two different theories. The results can be compared with observations in order to test braneworld gravity using the formalism.
Table 1. A comparative analysis of different observable properties of gravitational lensing by a cluster obtained from braneworld gravity with their GR counterparts.

| Properties     | Estimations                                                                 | Comments                                    |
|----------------|------------------------------------------------------------------------------|---------------------------------------------|
| Image position | $\theta = \theta_{R} \left[ \frac{3}{4} + \frac{\alpha G \rho_0 \gamma}{2 r} \right]$ | Images closer by 20%                        |
| Convergence    | $\kappa = \kappa_{R} \left[ \frac{3}{4} + \frac{\alpha G \rho_0 \gamma}{2 r} \right]$ | 20% less change in the image size           |
| Shear          | $\gamma = \gamma_{2R} \left[ \frac{3}{4} + \frac{\alpha G \rho_0 \gamma}{2 r} \right]$ | $\gamma_2$ less by 20%                     |
|                | $\gamma = \gamma_{2R} \left[ \frac{3}{4} + \frac{\alpha G \rho_0 \gamma}{2 r} \right]$ | $\gamma_2$ unchanged                       |

5.2. Lensing by spiral galaxies

As another interesting situation where we can test braneworld gravity, we intend to estimate the lensing effects for a spiral galaxy on the brane. For explicit calculations, we take up the Newtonian and relativistic potentials found in [25] by scaling with $c$

$$\Phi(r) = \frac{v_c^2}{c^2} \left[ \ln \left( \frac{r}{r_0} \right) - 1 \right], \quad (5.6)$$

$$\Psi(r) = \frac{v_c^2}{2} \left[ \ln \left( \frac{r}{r_0} \right) - 1 \right] - \left[ \frac{4\pi^2 G \rho_0}{\gamma^2} \right] \frac{1}{r}, \quad (5.7)$$

where $v_c$, $r_0$ and $\rho_0$ are, respectively, the rotational velocity in the flat rotation curve region, the impact parameter and the core density.

In the GR analysis of rotation curves of spiral galaxies, the GR deflection angle of a photon is determined by equation (5.3). Consequently, the deflection angle of a photon passing through the galactic halo turns out to be

$$\hat{\alpha}_R = \frac{2\pi v_c^2}{c^2}, \quad (5.8)$$

which is nothing but the deflection angle for a singular isothermal sphere in GR, whereas for the galactic metric obtained from braneworld gravity for a nonzero $\Psi$, the deflection angle is found to be

$$\hat{\alpha} = \frac{3\pi v_c^2}{2c^2} - \frac{8\pi^2 G \rho_0}{\gamma^2 c^2 b} = \hat{\alpha}_R \left[ \frac{3}{4} - \frac{4\pi G \rho_0}{\gamma^2 v_c^2 b} \right], \quad (5.9)$$

where $b$ is the usual impact parameter. For estimation, we use the following values of the parameters for a typical spiral galaxy [34]: $v_c = 220$ km s$^{-1}$, $r_0 = 8$ kpc ($\sim \gamma^{-1} \sim b$), $\rho_0 = 10^{-25}$ kg m$^{-2}$ (note that $\rho_0$ is the surface density). Thus, the deflection angle by a galaxy in the braneworlds turns out to be $\sim 75\%$ of the GR value.

Likewise, the other observable properties for gravitational lensing by a galaxy can also be estimated and compared with their GR counterparts. Table 2 summarizes the results. Here again $\theta_{R\pm}$ is the image position found from GR and $\theta_{\pm}$ the corresponding positions in braneworld gravity.

In a nutshell, the quantities differ by $\sim 25\%$ from GR, which is good enough to distinguish between the two theories. The result can again be subject to observational verification to test braneworld gravity theory.
Table 2. A comparative analysis of different observable properties of weak lensing by a spiral galaxy in braneworlds with their GR counterparts.

| Properties | Estimations | Comments |
|------------|-------------|----------|
| Image position | $\theta_+ = \theta_R + \frac{3}{8} \frac{4\pi G \rho_0}{c^2 b} \gamma_1$ | Image closer by 25% |
| Convergence | $\kappa = \kappa_R + \frac{3}{8} \frac{4\pi G \rho_0}{c^2 b} \gamma_1$ | 25% less change in the image size |
| Shear | $\gamma_1 = 0 = \gamma_{1R}$ | $\gamma_1$ unchanged |
| Shear | $\gamma_2 = \gamma_{2R} \left[ \frac{3}{8} \frac{4\pi G \rho_0}{c^2 b} \right]$ | $\gamma_2$ less by 25% |
| Shear | $\gamma = \gamma_2$ | $\Rightarrow$ change in shape 25% less |

5.3. Present status of observations

We have shown that sufficiently accurate lensing data for clusters and galaxies can be useful to test braneworld gravity. The present observational data [35, 36] reveal that there are significant amount of uncertainties in the galaxy or cluster properties estimated from the lensing data. While a few of them claim that they are consistent [35], some of them [36] indeed show that there are some inconsistencies between the observation and the theory based on dark matter. The uncertainty in these data thus opens up a fair possibility for a modified theory of gravity, e.g., braneworld gravity, to replace GR in explaining those observations. For example, lensing calculations from the nonsymmetric theory of gravity [37] has also shown its possibility to be an alternative to GR in galactic and extragalactic scales.

Using weak lensing data, the best-fit velocity dispersion for a cluster has been found to be $2200 \pm 500$ km s$^{-1}$. Analyzing the change in the background galaxy luminosity function, the cluster mass is obtained in the range $(0.48 \pm 0.16) \times 10^{15} h^{-1} M_\odot$ at a radius $0.25 h^{-1}$ from the cluster core [38]. Further information about the determination of mass can be obtained from [39, 40]. Magnification [41] and shear [42] can also be calculated from the data. For example, [42] estimates the amount of shear for a typical cluster to be $\langle \gamma^2 \rangle^{1/2} = 0.0012 \pm 0.0003$. These results reveal $\sim 25$–$30\%$ uncertainties in determining the precise value of the quantities.

Several properties of galaxy dark matter halos can be derived from weak lensing [43, 44]. Using the galaxy-mass cross-correlation function, it is found that the value of velocity dispersion is $\langle \sigma_v^2 \rangle^{1/2} = 128 \pm 4$ km s$^{-1}$ [43]. But this value is highly sensitive to the selection of the sample of lens galaxies, e.g., with different samples, the value lies in between $118 \pm 4 \pm 2$ km s$^{-1}$ and $140 \pm 4 \pm 3$ km s$^{-1}$. Thus the results are not so precise. A detailed survey of the current status of weak lensing can be found in [45].

To conclude, at the present status of information, both GR and braneworld gravity would fare equally well in explaining those observations. The results showing the present status of weak lensing are thus insufficient for a conclusive remark. A more accurate measurement of those lensing effects will help us to determine conclusively whether or not the braneworld gravity can be accepted as the theory of gravity.

6. Summary and outlook

We have developed a formalism appropriate for understanding gravitational lensing in the line elements which arise in braneworld gravity. Of course, this formalism is general enough for studying lensing in contexts wherever two gravitational potentials are required in order to include relativistic effects. For instance, following earlier work, one may use our general formulae for studying dark matter scenarios where pressure is not negligible [23].
With the intention of studying gravitational lensing in detail, we have obtained, using our formalism, general expressions for the time delay, deflection angle, Einstein ring, image positions, magnification and critical curves. It was noted that significant deviations from the results of weak-field GR was evident in the expressions for each of the above-mentioned quantities.

To illustrate our formalism, we made use of our earlier results on gravitational potentials of clusters and spiral galaxies, as obtained in braneworld gravity (using the relativistic, but weak-field effective Einstein equations on the brane). We estimated quantitatively lensing features for clusters and spiral galaxies by using both the Newtonian and weakly relativistic potentials. The difference between the values of each of the above quantities as compared to those obtained in the standard scenario, is found to be around 20–25%. Analysis of actual data reveals a 25–30% uncertainty in the values of almost all of these quantities. Thus, we conclude that it is only when more precise data become available, the theory can be verified conclusively, using lensing observations.

In this paper, we have primarily focused on weak lensing effects which can act as signatures for a modified theory of gravity. It is surely worthwhile to investigate features of strong lensing as well, which may provide further ways of testing braneworld gravity, or, for that matter, any modified theory of gravity where a two-potential formalism becomes necessary. To this end, we have performed some simplistic calculations of the caustics and critical curves, assuming a spherically symmetric lens considered as a singular isothermal sphere, and have obtained some preliminary results. The critical curves have been found to give qualitatively same but quantitatively different results, though the location of the caustics remain unchanged. Thus, we expect that a detailed survey of strong lensing in braneworld gravity may reveal further interesting and new features. We hope to address such issues related to strong lensing in detail, in future.

In conclusion, it is important to mention a drawback in our formalism. The general results we have obtained are applicable only to lensing by local objects in the sky. We need to include the effects of a background cosmology in order to address more realistic scenarios in an appropriate manner. We hope to return to this and other issues later.

Acknowledgments

We thank S Bharadwaj for discussion and suggestions related to the work reported in this paper. We also acknowledge useful discussions with S Majumdar, R Misra, T Padmanabhan, T D Saini and K Subramanian.

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