ON EQUIVALENCE THEOREMS OF MINKOWSKI SPACES AND APPLICATIONS IN FINSLER GEOMETRY

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Abstract. In the first part of this paper, we establish the correspondence between the Minkowski spaces and hyperovaloids with the centroaffine normalization. This correspondence allow us to prove the equivalence theorem of Minkowski spaces via results in affine geometry. In the second part, we obtain some results related to the Unicorn problem in Finsler geometry. We prove that Landsberg spaces with some additional conditions must be Berwald spaces. These conditions are vanishing mean Chern-Minkowski curvature, vanishing mean Berwald curvature, closeness of Cartan-type form or vanishing S-curvature. As an application, we prove that a specially semi-C-reducible Finsler manifold of dimension \( n \geq 4 \) is a Landsberg manifold if and only if it is a Berwald manifold.

Keywords: Minkowski space, hyperovaloid, Cartan tensor, Cartan form, cubic form, Tchebychev form, Chern-Minkowski curvature, Landsberg curvature, Cartan-type form, S-curvature, nonlinear parallel transport

Contents

Introduction 2
1. Review of centroaffine differential geometry of hypersurfaces 4
1.1. Centroaffine normalization of a nondegenerate hypersurface 4
1.2. Integrability conditions and fundamental theorems 6
1.3. Uniqueness theorem for hyperovaloids 6
2. Equivalence theorems of Minkowski spaces 7
2.1. Minkowski spaces and their indicatrices 7
2.2. Equivalence theorems of Minkowski spaces 12
3. Preliminary of Finsler geometry 15
3.1. Some notations in Finsler geometry 15
3.2. Bott-Chern connection and Bianchi identities 17
3.3. Some properties of Cartan type one form 22
4. Nonlinear parallel transport and special Finsler spaces 25
4.1. Basic properties of nonlinear parallel transport 25
4.2. Characterizations of special Finsler spaces 27
References 31
In this paper, a Minkowski space means an \( n \)-dimensional real vector space \( V \) with a smooth strongly convex norm \( F \). Two Minkowski spaces \((V, F)\) and \((\tilde{V}, \tilde{F})\) are equivalent, if there exists a nondegenerate linear homomorphism \( L : \tilde{V} \to V \), such that \( \tilde{F} = F \circ L \). It is clear that all \( n \)-dimensional vector spaces with Euclidean norms, which are such norms induced by inner products, are equivalent to \((\mathbb{R}^n, \| \cdot \|)\), where \( \| y \| = \sqrt{\sum (y^i)^2} \), for \( y = (y^1, \ldots, y^n) \in \mathbb{R}^n \).

The indicatrix \( I_F \) of a Minkowski norm \( F \) is the set of vectors of unit norm, which is a strongly convex hypersurface with origin in its interior. The indicatrices of two equivalent Minkowski spaces differ by a nondegenerate linear homomorphism, so they have the same centroaffine differential geometric structures. Then it is a natural way to study Minkowski spaces via methods of affine differential geometry.

This idea may appeared firstly in the work of Blaschke [8] and Deicke [12] about the Blaschke-Deicke theorem, which implies the Cartan form \( \eta \) is the sole of the difference between non-Euclidean norms and Euclidean norms. In their works, the equiaffine geometric structure of the indicatrix of a Minkowski norm is investigated. Laugwitz firstly develops the centroaffine differential geometry of the indicatrices of Minkowski norms in [18, 19]. He proves that the pull backs of the fundamental form \( \hat{g} \) and Cartan tensor \( \hat{A} \) of a Minkowski space are exactly the centroaffine invariant metric and cubic form of the indicatrix up to a sign with respect to its centroaffine normalization, respectively. In [9], Bryant also indicates these results. We will reprove these facts in Section 2.1 for our purpose. Using a similar idea, Huang and Mo [25] give some characterization of Randers norms, of which the indicatrices are elliptic quadrics. One also notes that [14] defines a function for each hypersurface with centroaffine normalization, which is exactly the Minkowski norm if the hypersurface is a hyperovaloid with center in its interior. The importance role played by the centroaffine differential geometry of convex hypersurfaces in Finsler geometry is also pointed by Álvarez Paiva in [2].

It is established here an explicit correspondence between Minkowski norms and hyperovaloids with origin in its interior, up to a centroaffine transformation (cf. Lemma 8-11). Under this correspondence, the Cartan tensors and Cartan forms of Minkowski norms are identified with the cubic forms and Tchebychev forms of the centroaffine geometries of hyperovaloids up to a sign. This explicit correspondence theorem allow us to transfer the results from affine differential geometry to the theory of Minkowski spaces. A direct important application is the equivalence problem for Minkowski spaces. In two dimensional case, this problem is solved via a equation of Rund [6]. This special theorem indicates again the importance of the Cartan form. For the cases of dimension greater than 2, Schneider [28] gives remarkable uniqueness theorems for hyperovaloids. His theorems are in fact settled in relative differential geometry, which is a general theory about affine differential geometry of hypersurfaces. For the centroaffine normalizations of hyperovaloids, it is proved that hyperovaloids are determined by their induced affine metrics and Tchebychev forms up to a nondegenerate centroaffine transformation. This result translate to Minkowski spaces solves...
the equivalence problem of Minkowski spaces (cf. Theorem 1, 2). It should be pointed out that the work of Schneider also relates to a class of PDEs which are similar to the equations appeared in the work of Rund in dimension two. As an application, we also obtain the equivalence theorems for specially semi-C-reducible Minkowski spaces of dimension \( n \geq 4 \).

A Finsler manifold \((M, F)\) can be viewed as a family of Minkowski spaces, since that \( F_{T_xM} \) is a Minkowski norm for each \( x \in M \). In Finsler geometry, most of geometric invariants live on the slit tangent bundle \( TM_0 \) or projective sphere bundles \( SM \). The variation of arc length of \( F \) gives the spray vector field \( \mathbf{G} \) and a horizontal splitting of the tangent bundle of \( TM_0 \) and \( SM \). This allows us to define natural Sasaki-type metrics on \( TM_0 \) and \( SM \). Since \( SM \) admits natural foliation structure, the horizontal part of the tangent bundle of \( SM \) has the Bott connection. In \([13]\), we prove that the Bott connection is indeed the Chern connection, and the symmetrization of the Bott-Chern connection is just the Cartan connection. We also introduce the notations Cartan endomorphism \( H \) as the difference between Bott-Chern connection and Cartan connection. The trace \( \eta = \text{tr} H \) is called Cartan-type one form. This two concepts are natural generalization of cubic forms and Tchebychev forms to the family case. The main results of this paper will based on the study of \( H \) and \( \eta \).

The Bott-Chern connection is torsion free and almost metric preserving. Then the curvature of the Bott-Chern connection only have “h-h” part and “h-v” part with respect to the splitting of the tangent bundle of \( SM \). The “h-h” part and “h-v” part are called the Chern-Riemann curvature and Chern-Minkowski curvature, and denoted by \( R \) and \( P \), respectively. A Finsler manifold is called a Berwald manifold if \( P = 0 \). The value of \( P \) along the spray vector field is called the Landsberg curvature and denoted as \( L \). If \( L = 0 \), then the Finsler manifold is called a Landsberg manifold. A famous problems in Finsler geometry is the so called “Unicorn problem” of hunting non-Berwald Landsberg manifolds. So far this problem is still open. The “generalized Unicorn problem” is to hunt non-Berwald weak Landsberg manifolds. For general comment on these problems, one refers to \([1, 5, 23, 37, 38]\). It is proved in \([35]\) that Landsberg \((\alpha, \beta)\) manifolds are Berwald manifolds for dimension \( n \geq 3 \). In \([3, 4]\), some non-regular \((\alpha, \beta)\) metrics are constructed, which are Landsberg but not Berwald metrics. Recently, \([40]\) proves that weakly Landsberg \((\alpha, \beta)\) manifolds are Berwald manifolds for dimension \( n \geq 3 \). We also note that \([22]\) gives some results about this problem for semi-C-reducible Finsler spaces, which may not be regular metrics.

The above investigation of Finsler manifold can be described as “\((x, y)\)-local” viewpoint, for the curvatures appear naturally on \( TM_0 \) or \( SM \). But Ichijyō in \([15, 16]\) introduces another natural “\(x\)-local but \(y\)-global” viewpoint to deal with Finsler geometry. For some problems, it is better to view a Finsler manifold as a family of Minkowski spaces. A natural relation between tangent spaces at different points on \( M \) is the nonlinear parallel transport. One refers to \([1, 5, 11]\) for details about parallel transports. Since each Minkowski space admits a natural Riemannian metric, it is proved (\([5, 11, 16]\)) that \( M \) is a Landsberg manifold if and only if the nonlinear...
parallel transports are all Riemannian isometry. One also proves that for a Berwald
space, the nonlinear parallel transports are in fact linear Riemannian isometry be-
tween tangent spaces. We prove in this paper that the inverse statement is also true
(cf. Theorem 7).

Under the “x-local but y-global” viewpoint, we would better to consider the tangent
space $T_x M$ at a point $x \in M$ globally. By definition, nonlinear parallel transports
give norm preserving mappings between the tangent spaces at different points. Ac-
cording to the equivalence theorem of Minkowski spaces (cf. Theorem 1), we need to
observe the behavior of metric $\hat{g}$ and Cartan-type form $\eta$ under the nonlinear parallel
transports. We prove that $\hat{g}$ is horizontally stable if and only if $L = 0$. And simi-
larly $\eta$ is horizontally stable if and only if mean Chern-Minkowski curvature $\text{tr} P = 0$.
Since Berwald spaces is characterized by $\hat{g}$ and $\eta$ are horizontally stable everywhere
(cf. Theorem 8), then $L = 0$ and $\text{tr} P = 0$ imply that $P = 0$ (cf. Theorem 9). Since
the mean Chern-Minkowski curvature $\text{tr} P$ is just the mean Berwald curvature $E$ if
$L = 0$, then we prove that $L = 0$ and $E = 0$ imply that $P = 0$. This gives a positive
answer of a problem asked by Shen in [34].

This paper will mainly be divided in to two part. The first part contains Section
1 and 2, and the second part contains the remain sections. In the first part, we
will investigate the centroaffine geometry of the indicatix of a Minkowski space. As
applications, we prove equivalence theorems of Minkowski spaces. In the second
part, Finsler geometry will be investigated. The local geometry can be described by
the Bott-Chern connection and its curvatures. But the y-global geometry need the
concept of nonlinear parallel transport. After a review of these two theories, we try
to join them to prove some results relate to the Unicorn problem in Finsler geometry.

In this paper, lower case Latin indices will run from 1 to $n$ and lower case Greek
indices will run from 1 to $n - 1$. We also adopt the summation convention of Einstein.
We will assume in this paper that $n \geq 3$.

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1. Review of centroaffine differential geometry of hypersurfaces

In this section, we would like to review the fundamental equations and some results
of an affine hypersurface with centroaffine normalization. One refers to [20, 26, 30]
for details.

1.1. Centroaffine normalization of a nondegenerate hypersurface.

Let $V$ be a real vector space of dimensional $n$ with a chosen orientation. Let $V^*$ be
the dual space of $V$, and $\langle \cdot, \cdot \rangle : V \times V^* \to \mathbb{R}$ the canonical pairing. $V$ has a smooth
manifold structure and a flat affine connection $\nabla$. 
Let $x : M \rightarrow V$ be an immersed connected oriented smooth manifold $M$ of dimension $n - 1$. Then for each point of $p \in M$, $dx(T_p M)$ is an $n - 1$ dimensional subspace of $T_{x(p)} V$, and defines an one dimensional subspace $C_p M = \{ v^*_p \in V^* | \ker v^*_p = dx(T_p M) \} \subset T_{x(p)}^* V$. The trivial line bundle $CM = \bigcup_p C_p M$ is called the conormal line bundle of $x$.

Let $Y$ be a nowhere vanishing section of $CM$. If $\text{rank}(dY, Y) = n$, then $x$ is called a nondegenerate hypersurface. The nondegenerate property is independent to the choice of the conormal field $Y$. In this paper, we will only discuss nondegenerate hypersurfaces. Let $y : M \rightarrow V$ be the special vector filed $y(p) = -x(p), \forall p \in M$. If $\langle Y, y \rangle = 1$, then the pair $\{Y, y\}$ is called the centroaffine normalization of $x$. We state the structure equations of $x(M)$ with respect to the centroaffine normalization $\{Y, y\}$ as following,

$$\nabla_v y = dy(v) = -dx(v),$$

$$\nabla_v dx(w) = dx(\nabla_v w) + h(v, w)y,$$

$$\nabla^*_v dY(w) = dY(\nabla^*_v w) - h(v, w)Y,$$

where $h$ is a nondegenerate symmetric $(0, 2)$-tensor and called the induced affine metric, $\nabla$ and $\nabla^*$ are torsion free affine connections. These geometric quantities satisfy

$$dh(v_1, v_2) = h(\nabla v_1, v_2) + h(v_1, \nabla^* v_2).$$

Then the triple $\{\nabla, h, \nabla^*\}$ are called conjugate connections. For any triple of conjugate connections $\{\nabla, h, \nabla^*\}$, one can define $C = \frac{1}{2}(\nabla - \nabla^*) \in \Omega^1(M, \text{End}(TM))$. By \[\text{[1.4]}\], the $(0, 3)$-tensor $\hat{C} := h \circ C$ is totally symmetric and called the cubic form of $\{\nabla, h, \nabla^*\}$. One can prove that

$$\hat{C} = -\frac{1}{2} \nabla h.$$

For $\{\nabla, h, \nabla^*\}$, the Tchebychev form $\hat{T}$ is defined as the normalized trace of $C$,

$$\hat{T} = \frac{1}{n-1} \text{tr} C.$$

The Tchebychev field $T$ is the dual of $\hat{T}$ with respect to $h$.

Let $\omega(h)$ be the Riemannian volume of $h$ on $x(M)$ and $\omega$ the induced volume form on $x(M)$ from the orientation of $V$. Then it is proved that

$$\hat{T} = \frac{1}{n-1} d \log \left| \frac{\omega}{\omega(h)} \right|.$$
1.2. Integrability conditions and fundamental theorems.

We have the following equations from the integrability of an affine hypersurface \( x(M) \) with the centroaffine normalization \( \{ Y, y \} \),

(1.8) \[ R^h(U,V)W = -[h(U,W)V - h(V,W)U] + [C(C(U)W)V - C(C(V)W)U], \]
and

(1.9) \[ (\nabla^h_V C)(V)W - (\nabla^h_U C)(U)W = 0, \]

where \( \nabla^h \) and \( R^h \) are the Levi-Civita connection and curvature tensor of \( h \), respectively.

**Lemma 1** ([30], 6.3.3). Let \( V \) be a real vector space of dimension \( n \) and \( M \) a connected, simply-connected, oriented smooth manifold with \( \dim M = n - 1 \). Let \( (M,h) \) be a pseudo-Riemannian manifold. Let \( \hat{C} \) be a totally symmetric \((0,3)\)-tensor field on \( M \). Let \( C \) be the \((1,2)\)-tensor identified to \( \hat{C} \) via \( h \).

If the curvature tensor \( R^h \) of \( h \) and the cubic form \( \hat{C} \) satisfy the conditions (1.8) and (1.9), then there exists a nondegenerate immersion \( x : M \to V \) together with the centroaffine normalization \( \{ Y, y \} \), such that the induced affine metric and cubic form respectively induced by the centroaffine normalization are exactly \( h \) and \( \hat{C} \).

**Lemma 2** ([30], 6.3.3). Let \( x_i : M \to V_i \) be two nondegenerate immersion of \( M \) in \( n \) dimensional real vector spaces \( V_i \) with the centroaffine normalization \( \{ Y_i, y_i \} \), \( i = 1, 2 \). Then \( h_1 = h_2 \), \( \hat{C}_1 = \hat{C}_2 \) if and only if there exists a nondegenerate linear homomorphism \( L \in \text{Hom}(V_1,V_2) \), such that

\[ x_2 = L \circ x_1. \]

1.3. Uniqueness theorem for hyperovaloids.

Compare with the above local uniqueness theorem, there is a remarkable global uniqueness theorems for hyperovaloids. We review some facts about hyperovaloids.

**Lemma 3.** Let \( M \) be an \( n-1 \) dimensional connected closed smooth manifold. Let \( x : M \to V \) be a smooth immersion in an \( n \) dimensional real vector space \( V \). If the centroaffine normalization of \( x(M) \) is nondegenerate, then \( x(M) \) is a hyperovaloid.

**Lemma 4** ([26], Prop.7.3). Assume that \( x : M \to V \) is an \( n \) dimensional hyperovaloid with respect to the centroaffine normalization. Then

(i) \( M \) is diffeomorphic to the \( n-1 \) dimensional standard sphere \( S^{n-1} \);
(ii) \( x \) is an imbedding;
(iii) \( x(M) \) is the boundary of a strongly convex body, in which the origin is contained.

**Remark 1.** In [26], two terms about convexity are used. They are “locally strictly convex” and “globally strictly convex” respectively. But in [6], the convexity “locally strictly convex” is named as “strongly convex”, and “globally strictly convex” as “strictly convex”. It is proved that strongly convexity implies strictly convexity in [6]. Here we follow [6] terminologically.
Now we are going to state a remarkable rigidity theorem about hyperovaloids with centroaffine normalizations, which is due to R. Schneider [28].

**Lemma 5 ([28], Satz 4.2).** Let \( x_1 : M \to V_1 \) be two nondegenerate immersion of \( M \) in an \( n \geq 3 \) dimensional real vector space \( V_i \) with the centroaffine normalization \( \{ Y_i, y_i \} \), \( i = 1, 2 \). Let \( \bar{T}_i = \frac{1}{n-1} \text{tr} C_i \) be the Tchebychev forms of \( x_i \), \( i = 1, 2 \), respectively.

Then \( h_1 = h_2 \) and \( \bar{T}_1 = \bar{T}_2 \) if and only if there exists a nondegenerate linear homomorphism \( L \in \text{Hom}(V_1, V_2) \), such that

\[
x_2 = L \circ x_1.
\]

2. Equivalence theorems of Minkowski spaces

2.1. Minkowski spaces and their indicatrices.

Let \( V \) be an \( n \) dimensional real vector space with a given orientation. Let \( \{ b_1, \ldots, b_n \} \) be an oriented basis. Then the mapping \( \phi : V \to \mathbb{R}^n \) defined by

\[
\phi(y) = (y^1, \ldots, y^n), \quad \forall y = y^i b_i \in V,
\]
gives the standard smooth structure of \( V \). Let \( \{ b_1^*, \ldots, b_n^* \} \) be the dual basis. The orientation of \( V \) is then the \( n \) form \( \omega = b_1^* \wedge \cdots \wedge b_n^* \), which gives a volume form on \( V \).

For any function \( f : V \to \mathbb{R} \), we will denote \( f = f \circ \phi^{-1} \). \( f \) is said to be differentiable on \( V \), if \( f \) is differentiable as a function on \( \mathbb{R}^n \). These definition is independent to the choices of the basis of \( V \).

In the following, we review some basic concepts.

**Definition 1.** Let \( V \) be an \( n \) dimensional real vector space with the standard smooth structure \( (y^i; \phi) \). Let \( \mathbf{F} : V \to [0, +\infty) \) be a function, such that

(i) \( \mathbf{F} \) is continuous on \( V \), and smooth on \( V_0 := V \setminus \{0\} \);

(ii) \( \mathbf{F}(\lambda v) = \lambda \mathbf{F}(v), \quad \forall v \in V, \quad \forall \lambda \in \mathbb{R}^+ \);

(iii) \( \mathbf{F} \) is strongly convex, i.e., the symmetric tensor \( \bar{g} := \bar{\nabla} d \left[ \frac{1}{2} \mathbf{F}^2 \right] \) is positive everywhere. Then \( \mathbf{F} \) is called a Minkowski norm of \( V \). \((V, \mathbf{F})\) is called a Minkowski space.

By definition, \((V_0, \bar{g})\) is a Riemannian manifold related to a Minkowski space \((V, \mathbf{F})\). The Riemannian geometry of \((V_0, \bar{g})\) is fundamental and has many applications in Finsler geometry. But the following concept concentrates almost all geometric information of a Minkowski space.

**Definition 2.** Let \((V, \mathbf{F})\) be a Minkowski space of dimension \( n \). The set

\[
\mathbf{I}_\mathbf{F} := \{ v \in V | \mathbf{F}(v) = 1 \}
\]

is called the indicatrix of the Minkowski space \((V, \mathbf{F})\).

The indicatrix \( \mathbf{I}_\mathbf{F} \) of a Minkowski space \((V, \mathbf{F})\) have been studied as a submanifold of \((V_0, \bar{g})\). For the study of \( \mathbf{I}_\mathbf{F} \) from this viewpoint, one refers to [6] [27] [31]. In the following, we will study the centroaffine geometry of \( \mathbf{I}_\mathbf{F} \). It will be proved that these two kinds of geometry of \( \mathbf{I}_\mathbf{F} \) are in fact the same. But the centroaffine differential geometry are more suitable for the discussion of equivalence problem.
Definition 3. Let \((V, F)\) and \((\tilde{V}, \tilde{F})\) be two Minkowski spaces of dimension \(n\). If there exists a homomorphism \(L \in \text{Hom}(\tilde{V}, V)\), such that

\[
\tilde{F} = F \circ L,
\]

then \((V, F)\) and \((\tilde{V}, \tilde{F})\) are said to be equivalent. We will denote \((V, F) \sim (\tilde{V}, \tilde{F})\), if \((V, F)\) and \((\tilde{V}, \tilde{F})\) are equivalent.

It is clear that “\(\sim\)” is an equivalent relation on the set of \(n\) dimensional Minkowski spaces.

Lemma 6. Let \((V, F)\) and \((\tilde{V}, \tilde{F})\) be two Minkowski spaces of dimension \(n\). Let \(I_F\) and \(\tilde{I}_{\tilde{F}}\) be their indicatrices respectively. Then \((V, F) \sim (\tilde{V}, \tilde{F})\) if and only if

\[
I_F = L(\tilde{I}_{\tilde{F}}),
\]

holds for some \(L \in \text{Hom}(\tilde{V}, V)\).

Proof. We firstly assume that \((V, F) \sim (\tilde{V}, \tilde{F})\). So \(\tilde{F} = F \circ L\) holds for some \(L \in \text{Hom}(\tilde{V}, V)\). For any \(\tilde{v} \in \tilde{I}_{\tilde{F}}\), we have

\[
F(L(\tilde{v})) = (F \circ L)(\tilde{v}) = \tilde{F}(\tilde{v}) = 1.
\]

So \(L(\tilde{v}) \in I_F\).

For any \(v \in I_F\), one chooses that \(\tilde{v} = L^{-1}(v)\). Since

\[
\tilde{F}(\tilde{v}) = (F \circ L)(L^{-1}(v)) = F(v) = 1,
\]

then \(\tilde{v} \in \tilde{F}(\tilde{v})\) and \(v = L(\tilde{v})\). So we have proved that \(I_F = L(\tilde{I}_{\tilde{F}})\).

Now, we assume that \(I_F = L(\tilde{I}_{\tilde{F}})\) for some \(L \in \text{Hom}(\tilde{V}, V)\). For any \(\tilde{v} \in \tilde{V}_0\), it is clear that \(\lambda^{-1}\tilde{v} \in I_F\), where \(\lambda = \tilde{F}(\tilde{v})\). Since \(L(\lambda^{-1}\tilde{v}) \in I_F\), we have

\[
(F \circ L)(\tilde{v}) = F(L(\tilde{v})) = \lambda F(L(\lambda^{-1}\tilde{v})) = \lambda.
\]

It implies that \(\tilde{F} = F \circ L\) for \(L \in \text{Hom}(\tilde{V}, V)\). \(\square\)

Before the study of the centroaffine differential geometry of the indicatrices, We would like to review the definitions of three basic tensors on \((V_0, \bar{g})\).

Definition 4. Let \((V, F)\) be a Minkowski space of dimension \(n\). We define a conformal metric of \(\bar{g}\) as

\[
\hat{g} = \frac{1}{F^2} \bar{g}.
\]

The Cartan tensor is defined by

\[
\hat{A} = \frac{1}{2F^2} \nabla \hat{g}.
\]

Let \(|G| = \left| \frac{\omega(\bar{g})}{\omega} \right|\) be the Radon-Nikodym derivative of the Riemannian measure of \(\bar{g}\) with respect to the measure \(\omega = b_1^* \wedge \cdots \wedge b_n^*\) induced by the orientation of \(V\). The Cartan form is defined by

\[
\eta = d \log |G|.
\]
Remark 2. One notes that tensors $\hat{g}$, $\hat{A}$ and $\eta$ are all homogeneous of degree 0. And $i_{\mathbf{x}} \hat{A} = 0$ and $i_{\mathbf{x}} \eta = 0$, where $\mathbf{x}$ is the position vector field on $V$.

Lemma 7. The tensors $\hat{g}$, $\hat{A}$ and $\eta$ are relative invariants with respect to the equivalence relation of Minkowski spaces.

Proof. Let $(V, F)$ be a Minkowski space. Let $\{b_1, \ldots, b_n\}$ be an oriented basis of $V$. Let $\phi$ be the induced coordinate map from $V$ to $\mathbb{R}^n$. Then we have $dF = \phi^* dF$, and

$$\bar{g} = \phi^* ((\bar{F}_y F_{y^j} + F F_{y^j y^j}) dy^i \otimes dy^j),$$

where $\bar{F} = F \circ \phi^{-1}$.

Let $(\tilde{V}, \tilde{F})$ be a Minkowski space which is equivalent to $(V, F)$. Then there exists $L \in \text{Hom}(\tilde{V}, V)$ such that $\tilde{F} = F \circ L$. Let $\tilde{b}_i = L^{-1}(b_i)$, $i = 1, \ldots, n$. Let $\tilde{\phi}$ be the induced coordinate map from $\tilde{V}$ to $\mathbb{R}^n$. Then we have $\tilde{\phi} = \phi \circ L$. It follows that $\tilde{F} = \tilde{F} \circ \tilde{\phi}^{-1} = (F \circ L) \circ (\phi \circ L)^{-1} = F$. So we obtain

$$\bar{g} = \tilde{\phi}^* ((\bar{F}_y \tilde{F}_{y^j} + \tilde{F} \bar{F}_{y^j y^j}) dy^i \otimes dy^j)
= L^* \circ \phi^* ((F_y F_{y^j} + \bar{F} \bar{F}_{y^j y^j}) dy^i \otimes dy^j)
= L^* \bar{g}.$$

By the same way, we can prove that $\tilde{A} = L^* \hat{A}$ and $\tilde{\eta} = L^* \eta$.

We are going to discuss the centroaffine differential geometry of the indicatrix of a given Minkowski space.

Lemma 8. Let $I_F$ be the indicatrix of a Minkowski space $(V, F)$. Then the identity map $i : I_F \to V$ is an imbedding. For each point $v \in I_F$, if we choose that $y = -v \in T_v V$, $Y = -dF \in T^*_v V$, then the pair $\{Y, y\}$ gives the centroaffine normalization of $I_F$.

Proof. Since the Minkowski norm $F : V_0 \to \mathbb{R}^+$ is a smooth mapping, and

$$F(\lambda v) = \lambda F(v), \quad \forall \lambda > 0,$$

then

$$F_{\ast v} \left( \frac{v}{F(v)} \right) = \frac{\partial}{\partial t} \in T_{F(v)} \mathbb{R}^+, \quad v \in T_v V.$$

So $F$ is a submersion. By the inverse image of a regular value theorem, $I_F = F^{-1}(1)$ is an imbedding submanifold of $V_0$.

For each smooth curve $\gamma : (-\epsilon, \epsilon) \to I_F$ such that $\gamma(0) = v \in I_F$, it is clear $F(\gamma(t)) = 1$. Then

$$0 = \left. \frac{d}{dt} \right|_{t=0} F(\gamma(t)) = \langle dF, \dot{\gamma}(0) \rangle,$$

and $-dF$ is a conormal field of $I_F$. 
By the Euler theorem for homogeneous function,
\[ \langle -dF, -v \rangle|_v = \left. \frac{dF_y, dy^i}{\partial y^j} \right|_{\phi(v)} = F_{y^i} y^i|_{\phi(v)} = 1, \]
where \( F = F \circ \phi^{-1} \). So \( \{ Y = -dF, y = -v \} \) is the centroaffine normalization. □

Let \( \{ Y, y \} \) be the centroaffine normalization of \( I_F \) determined in Lemma 8. Then we have the induced affine metric \( h \) on \( I_F \) by the Gauss equation (1.2).

**Lemma 9.** The induced Riemannian metric \( h \) is given by

\[ (2.2) \quad h = i^* \hat{g} = i^* h, \]

where \( i : I_F \to V \) is the identity map, and \( h := \nabla dF \) is the angular metric. As a consequence, \( h \) is positive definite and \( I_F \) is a hyperovaloid.

**Proof.** The equation (1.2) reads
\[ \nabla_u i_* w = i_*(\nabla_u w) + h(u, w)y, \]
where \( u, w \in \Gamma(T I_F) \) are smooth vector fields on \( I_F \), and \( \nabla \) is the induced affine connection.

Since \( \hat{g} = F^{-2}(dF \otimes dF + F \nabla dF) \), we have
\[ i^* \hat{g}(u, w) = F^{-2}(dF \otimes dF + F \nabla dF)(i_* u, i_* w) \]
\[ = \nabla dF(i_* u, i_* w) \]
\[ = \langle \nabla u dF, i_* w \rangle \]
\[ = \langle -dF, \nabla u i_* w \rangle \]
\[ = h(u, w). \]

So \( h \) is positive definite everywhere and \( I_F \) is a hyperovaloid. □

**Lemma 10.** The cubic form \( \hat{C} \) is given by

\[ (2.3) \quad \hat{C} = -i^* \hat{A}, \]

and the Tchebychev form is

\[ (2.4) \quad \hat{T} = -\frac{1}{n-1} i^* \eta \]

**Proof.** Let \( u, w, z \in \Gamma(T I_F) \) be smooth vector fields on \( I_F \). Since that
\[ A = \frac{1}{2F^2} \nabla(dF \otimes dF + F \nabla dF) \]
\[ = \frac{1}{2F^2} (\nabla dF \otimes dF + 2dF \otimes \nabla dF + F \nabla \nabla dF), \]
and
\[
\bar{\nabla} dF(u, y) = \langle \bar{\nabla} uF, y \rangle \\
= \langle -dF, \bar{\nabla} y \rangle \\
= \langle -dF, -i_uu \rangle \\
= 0.
\]

By (1.5), we have
\[
2i^* A(z, u, w) = \frac{1}{F^2} (\bar{\nabla} dF \otimes dF + 2dF \otimes \bar{\nabla} dF + F\bar{\nabla} dF) (i^* z, i^* u, i^* w)
\]
\[
= \bar{\nabla} \bar{\nabla} dF (i^* z, i^* u, i^* w)
\]
\[
= (\bar{\nabla} z (\bar{\nabla} dF))(i^* u, i^* w)
\]
\[
= z(\bar{\nabla} dF (i^* u, i^* w)) - \bar{\nabla} dF (\bar{\nabla} z (i^* u, i^* w)) - \bar{\nabla} dF (i^* u, \bar{\nabla} z (i^* w))
\]
\[
= z(\bar{\nabla} dF (i^* u, i^* w)) - \bar{\nabla} dF (i^* (\bar{\nabla} z u), i^* w) - \bar{\nabla} dF (i^* u, i^* (\bar{\nabla} z u))
\]
\[
= z(i^* (\bar{\nabla} dF) (u, w)) - i^* (\bar{\nabla} dF) (\bar{\nabla} z u, w) - i^* (\bar{\nabla} dF) (u, \bar{\nabla} z w)
\]
\[
= z(h(u, w)) - h(\bar{\nabla} z u, w) - h(u, \bar{\nabla} z w)
\]
\[
= (\bar{\nabla} h)(u, w)
\]
\[
= -2\hat{C}(z, u, w).
\]
So (2.3) follows. One similarly has (2.4) from (1.7).

The above lemmas show that how to derive the centroaffine differential geometric structures of the indicatrices of Minkowski spaces from the Minkowski norms. Conversely, a hyperovaloid $M$ with origin in its interior can define a Minkowski norm $F$, such that $I_F = M$.

**Lemma 11.** Let $V$ be an $n$ dimensional vector space. Let $M$ be a hyperovaloid in $V$ with origin in its interior. Then there is a Minkowski norm $F$ on $V$ such that $I_F = M$.

**Proof.** Since $M$ is strongly convex, then for each $\tilde{v} \in V_0$, the ray $\{t\tilde{v}|t \geq 0\}$ has a unique intersection point $v$ with $M$. Let $\tilde{v} = \lambda v$. Then we can define a function $F : V_0 \to (0, +\infty)$ as
\[
F(\tilde{v}) = \lambda.
\]

It is clear that $F$ is positive homogeneous degree 1. Let $(U, \psi)$ be a coordinate chart of $M$ with the coordinate map $\psi : U \to \mathbb{R}^{n-1}$. The scalar product $s : \mathbb{R}^+ \times V \to V$ is clearly smooth. Then the restriction of $s$ on $\mathbb{R}^+ \times U$ is also smooth as $M$ is a smooth imbeded submanifold. We will denote that $\mathbb{R}^+ U := s(\mathbb{R}^+ \times U)$. Since the Jacobian of $s$ at $(\lambda, v)$ is $J = \lambda^{n-1}(v, dv) = (-1)^n \lambda^{n-1}(y, dy)$, where $y = -v$ denotes the centroaffine norm of $M$ at $v$. Then $J$ is nonsingular. By inverse function theorem, $s : \mathbb{R}^+ \times U \to \mathbb{R}^+ U$ is a diffeomorphism. Let $\tilde{\phi} = (i \times \psi) \circ s^{-1}$ be a map from $\mathbb{R}^+ U$
to $\mathbb{R} \times \mathbb{R}^{n-1} = \mathbb{R}^n$. Then $\tilde{\phi}$ gives a coordinate map of $\mathbb{R}^+ U$. It is clear that
$$F \circ \tilde{\phi}^{-1}(\lambda, \psi(U)) = \lambda,$$
then $F$ is smooth on $V_0$.

Similar to Lemma \(8\), \{-v, -d F\} is the centroaffine normalization of $M$, where $v \in M$. Moreover, we have
$$s^* \tilde{g} = s^* \left( \nabla d \left[ \frac{1}{2} F^2 \right] \right) = d\lambda \otimes d\lambda + \lambda h,$$
where $h$ is the induced Riemannian metric of $M$ with respect to the centroaffine normalization. So we have proved that $F$ is a Minkowski norm on $V$. $I_F = M$ holds by definition. \(\square\)

2.2. Equivalence theorems of Minkowski spaces.

As a consequence of the discussion in Subsection 2.1, one has the equivalence theorems of Minkowski spaces.

**Theorem 1.** Let $(V_1, F_1)$ and $(V_2, F_2)$ be two Minkowski spaces of dimension $n$, respectively. Let $f : (V_1)_0 \rightarrow (V_2)_0$ be a norm preserving diffeomorphism which satisfies
$$f(tv) = tf(v), \quad \forall v \in (V_1)_0, \forall t > 0.$$
Then $\hat{g}_1 = f^* \hat{g}_2$ and $\eta_1 = f^* \eta_2$ if and only if there exists a nondegenerate linear homomorphism $L \in \text{Hom}(V_1, V_2)$, such that $f = L$ and
$$F_1 = F_2 \circ L.$$

**Proof.** It follows that $f$ induces a diffeomorphism between the indicatrixes $f : I_{F_1} \rightarrow I_{F_2}$. This theorem is a consequence of Lemma 5, 6, 9 and 10. \(\square\)

As a corollary of Theorem 1, one has the following weak equivalence theorem of Minkowski spaces.

**Theorem 2.** Let $(V_1, F_1)$ and $(V_2, F_2)$ be two Minkowski spaces of dimension $n$, respectively. Let $f : (V_1)_0 \rightarrow (V_2)_0$ be a norm preserving diffeomorphism which satisfies
$$f(tv) = tf(v), \quad \forall v \in (V_1)_0, \forall t > 0.$$
Then $\hat{g}_1 = f^* \hat{g}_2$ and $A_1 = f^* A_2$ if and only if there exists a nondegenerate linear homomorphism $L \in \text{Hom}(V_1, V_2)$, such that $f = L$ and
$$F_1 = F_2 \circ L.$$

As an application, we give an equivalent theorem of semi-C-reducible Minkowski norms. We will firstly give the following definition.

**Definition 5.** Let $(V, F)$ be a Minkowski space of dimension $n \geq 3$. For any $q \neq 1-n$, we define a $(0,3)$-tensor on $I_F$ as follows,
\[
M^q = \begin{cases} 
\hat{C} - \frac{n-1}{n+q-1} \left( \frac{q-2}{\|T\|^2} \hat{T} \otimes \hat{T} \otimes \hat{T} \right), & \text{if } \hat{T} \neq 0, \\
\hat{C}, & \text{if } \hat{T} = 0,
\end{cases}
\]
where
\[ B(u, v, w) := [h(u, v)\hat{T}(w) + h(v, w)\hat{T}(u) + h(w, u)\hat{T}(v)] \]
for \( u, v, w \in \Gamma(TI_F) \), and \( \| \cdot \| \) means the norm of tensors with respect to the induced metric \( h \).

If \( M^q \equiv 0 \) for some \( q \neq 1 - n \), then \((V, F)\) is called semi-C-reducible. If \((V, F)\) is semi-C-reducible and \( q \neq 1 \), then \((V, F)\) is called specially semi-C-reducible.

**Remark 3.** The following tensor on \( V_0 \) is defined by Matsumoto in [22] [21],

\[
M^q = \begin{cases} 
\hat{A} - \frac{1}{n + q - 1} \left( B + \frac{q - 2}{\|\eta\|^2} \eta \otimes \eta \right), & \text{if } \eta \neq 0, \\
\hat{A}, & \text{if } \eta = 0,
\end{cases}
\]

where \( B(U, V, W) = [h(U, V)\eta(W) + h(V, W)\eta(U) + h(W, U)\eta(V)] \) for \( U, V, W \in \Gamma(TV_0) \), and \( \| \cdot \| \) means the norm of tensors with respect to \( h \). The tensor \( M^2 \) is in fact the cubic Simon form of \( I_F \) in affine differential geometry. And \( M^2 \) is called Matsumoto tensor in literature.

In fact, \( M^q = -i^*M^q \), where \( i : I_F \to V \) is the identity map. By Remark 2, \( M^q \equiv 0 \) is equivalent to \( M^q \equiv 0 \).

**Theorem 3.** Let \((V, F)\) and \((\tilde{V}, \tilde{F})\) be \( n \geq 4 \) dimensional specially semi-C-reducible Minkowski spaces. Let \( f : V_0 \to \tilde{V}_0 \) be a norm preserving diffeomorphism which satisfies

\[ f(tv) = tf(v), \quad \forall v \in V_0, \ t > 0. \]

If \( \hat{g} = f^*\tilde{g} \), then \( \hat{A} = f^*\tilde{A} \) or \( \hat{A} = -f^*\tilde{A} \). In the first case, \((V, F)\) and \((\tilde{V}, \tilde{F})\) are equivalent.

**Proof.** On the submanifolds \( I_F \) and \( \tilde{I}_F \), we have the centroaffine geometric structure induced by the identity maps. Since that \( f : I_F \to \tilde{I}_F \) is a diffeomorphism, the geometric invariants on \( \tilde{I}_F \) has been pulled back to \( I_F \) via \( f \). Then the induced affine metrics satisfy \( h = f^*\tilde{h} \), because of \( \hat{g} = f^*\tilde{g} \). So the Riemannian curvature tensor of \( h \) and \( f^*\tilde{h} \) coincide. For any \( U, V, W, Z \in \Gamma(TI_F) \), formula \((1.8)\) implies that

\[ h(C(U)W, C(V)Z) = h(C(V)W, C(U)Z) \]

\[ \equiv h(\tilde{C}(U)W, \tilde{C}(V)Z) - h(\tilde{C}(V)W, \tilde{C}(U)Z), \]

where \( C \) and \( \tilde{C} \) are the (1,2)-tensors on \( I_F \) defined by \( C \) and \( f^*\tilde{C} \) with respect to \( h \).

Let \( e_1, \ldots, e_{n-1} \) be an arbitrary local orthonormal frame field on \( I_F \) of the metric \( h \). Then equation \((2.6)\) is equivalent to

\[ \sum_\alpha \left( C^\alpha_{\beta\mu} C^\alpha_{\gamma\nu} - C^\alpha_{\beta\nu} C^\alpha_{\gamma\mu} \right) = \sum_\alpha \left( \tilde{C}^\alpha_{\beta\mu} \tilde{C}^\alpha_{\gamma\nu} - \tilde{C}^\alpha_{\beta\nu} \tilde{C}^\alpha_{\gamma\mu} \right), \]

where \( C(e_\alpha)e_\beta = C^\alpha_{\beta\gamma}e_\gamma \) and \( \tilde{C}(e_\alpha)e_\beta = \tilde{C}^\gamma_{\alpha\beta}e_\gamma \).
Let $T = T^\alpha e_\alpha$ be the dual vector field of $\hat{T}$, then $T^\alpha = \frac{1}{n-1} \sum_\beta C^\alpha_{\beta\beta}$. Then equation (2.7) implies the following two formulas

\begin{equation}
(2.8) \quad (n-1) \sum_\alpha T^\alpha C^\alpha_{\gamma\nu} - \sum_\alpha C^\alpha_{\beta\nu} C^\alpha_{\gamma\beta} = (n-1) \sum_\alpha \hat{T}^\alpha \hat{C}^\alpha_{\gamma\nu} - \sum_\alpha \hat{C}^\alpha_{\beta\nu} \hat{C}^\alpha_{\gamma\beta},
\end{equation}

and

\begin{equation}
(2.9) \quad (n-1)^2 \|T\|^2 - \|C\|^2 = (n-1)^2 \|\hat{T}\|^2 - \|\hat{C}\|^2.
\end{equation}

On another hand, $F$ is specially semi-C-reducible, then (2.5) and $M^q = 0$ implies that

\begin{equation}
(2.10) \quad C_{\alpha\beta\gamma} = \frac{n-1}{n + q - 1} \left[ (\delta_{\alpha\beta} T_\gamma + \delta_{\beta\gamma} T_\alpha + \delta_{\gamma\alpha} T_\beta) + \frac{q - 2}{\|T\|^2} T_\alpha T_\beta T_\gamma \right],
\end{equation}

where $\hat{C}(e_\alpha, e_\beta, e_\gamma) = C_{\alpha\beta\gamma}$, and $\hat{T}(e_\alpha) = T_\alpha$. A similar formula holds for $\hat{C}$ as $\hat{F}$ is also specially semi-C-reducible,

\begin{equation}
(2.11) \quad \hat{C}_{\alpha\beta\gamma} = \frac{n-1}{n + \tilde{q} - 1} \left[ (\delta_{\alpha\beta} \tilde{T}_\gamma + \delta_{\beta\gamma} \tilde{T}_\alpha + \delta_{\gamma\alpha} \tilde{T}_\beta) + \frac{\tilde{q} - 2}{\|\tilde{T}\|^2} \tilde{T}_\alpha \tilde{T}_\beta \tilde{T}_\gamma \right].
\end{equation}

By (2.10) and (2.11), a direct calculation gives

\begin{equation}
(2.12) \quad \|C\|^2 = \frac{(n-1)^2 [3(n-2) + (q+1)^2]}{(n + q - 1)^2} \|T\|^2,
\end{equation}

\begin{equation}
(2.13) \quad \|\hat{C}\|^2 = \frac{(n-1)^2 [3(n-2) + (\tilde{q}+1)^2]}{(n + \tilde{q} - 1)^2} \|\tilde{T}\|^2.
\end{equation}

Combining (2.9) and (2.12), we have

\begin{equation}
(2.14) \quad \frac{n - 3 + 2q}{(n + q - 1)^2} \|T\|^2 = \frac{n - 3 + 2\tilde{q}}{(n + \tilde{q} - 1)^2} \|\tilde{T}\|^2.
\end{equation}

Plugging (2.10) and (2.11) in (2.8), one has

\begin{equation}
(2.15) \quad \frac{(n-3)(q-1)}{(n + q - 1)^2} T_\gamma T_\nu + \frac{n - 3 + q}{(n + q - 1)^2} \delta_{\gamma\nu} \|T\|^2 = \frac{(n-3)(\tilde{q}-1)}{(n + \tilde{q} - 1)^2} \tilde{T}_\gamma \tilde{T}_\nu + \frac{n - 3 + \tilde{q}}{(n + \tilde{q} - 1)^2} \delta_{\gamma\nu} \|\tilde{T}\|^2.
\end{equation}

Now we assume that $q$ and $\tilde{q}$ are both not equal to one. At any point where $T \neq 0$, we can choose the frame field $e_1, \ldots, e_{n-1}$ such that $T_\gamma \neq 0$ for $\gamma = 1, \ldots, n-1$. As $n > 3$, choosing indices $\gamma \neq \nu$ in (2.14) gives

\begin{equation}
(2.16) \quad \frac{q - 1}{(n + q - 1)^2} T_\gamma T_\nu = \frac{\tilde{q} - 1}{(n + \tilde{q} - 1)^2} \tilde{T}_\gamma \tilde{T}_\nu.
\end{equation}

From (2.15), we know that $\tilde{T}_\gamma \neq 0$ for $\gamma = 1, \ldots, n-1$. Then the zero points sets of $T$ and $\tilde{T}$ coincide. Moreover, (2.15) implies that

\begin{equation}
(2.17) \quad \frac{q - 1}{(n + q - 1)^2} T^2_\gamma = \frac{\tilde{q} - 1}{(n + \tilde{q} - 1)^2} \tilde{T}^2_\gamma, \quad \gamma = 1, \ldots, n - 1.
\end{equation}
Then we have
\[
\frac{q - 1}{(n + q - 1)^2} \|T\|^2 = \frac{\bar{q} - 1}{(n + \bar{q} - 1)^2} \|\bar{T}\|^2.
\]
If \(q\) and \(\bar{q}\) are both not equal to \((3 - n)/2\), then by (2.13) and (2.16), one has
\[
\frac{n - 3 + 2q}{q - 1} = \frac{n - 3 + 2\bar{q}}{\bar{q} - 1},
\]
or \(T = \bar{T} = 0\). So we obtain \(q = \bar{q}\). Then (2.15) implies that
\[
T = \pm \bar{T}.
\]
If \(q = (3 - n)/2\), then (2.13) implies \(\bar{q} = (3 - n)/2\) unless \(\bar{T} \equiv 0\). So (2.18) again holds in this case.

By (2.10), (2.11) and (2.18), we obtain
\[
C = \pm \bar{C}.
\]
As \(\hat{A}\) is zero on directions along the rays start from zero, we have
\[
\hat{A} = \pm \bar{\hat{A}}.
\]

\[\square\]

3. Preliminary of Finsler geometry

3.1. Some notations in Finsler geometry.

Let \(M\) be an \(n\) dimensional smooth manifold and \(\pi : TM \to M\) the tangent bundle of \(M\). Let \((U; \phi(x) = (x^1, x^2, \ldots, x^n))\) be a local coordinate system on an open subset \(U\) of \(M\). Then by the standard procedure one gets a local coordinate system \(\psi(x, y) = (x^1, \ldots, x^n, y^1, \ldots, y^n)\) on \(\pi^{-1}(U)\). Set \(TM_0 = TM \setminus 0\), where 0 denotes the zero section of \(TM\). Then \(\psi(x, y)\) with \(y \neq 0\) is a local coordinate system on \(TM_0\).

**Definition 6.** A Finsler structure on \(M\) is a continue function \(F : TM \to \mathbb{R}\), which is smooth on \(TM_0\), such that \(F_{x}^{x}M\) is a Minkowski norm for each \(x \in M\). A manifold \(M\) with a Finsler structure \(F\) is called a Finsler manifold, and denoted by \((M, F)\).

Let \(F = F \circ \psi^{-1}\), then \(F\) is a smooth function of \(2n\) variables. By the definition of Minkowski spaces, the \(n \times n\) matrix
\[
(g_{ij}) := \left(\frac{1}{2}[F^2_{y^iy^j}]\right),
\]
is positive definite everywhere.

Using the Finsler structure \(F\) of a Finsler manifold \((M, F)\), one can compute the energy variation of curves in \(M\). The following important data in Finsler geometry appears naturally in this process:
\[
G^i = \frac{1}{4}g^{ij} \left([F^2_{y^iy^j} y^k - [F^2]_{x^i}]\right),
\]
where \((g^{ij}) = (g_{ij})^{-1}\). It is clear that
\begin{equation}
G^i(x, \lambda y) = \lambda^2 G^i(x, y), \quad \lambda > 0, \ i = 1, \ldots, n.
\end{equation}
The spray \(G\) or the Reeb field of the Finsler manifold \((M, F)\) is defined as a special smooth vector field on \(TM_0\) as follows
\[
G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}.
\]
In literatures, \(G^i\)’s are called the spray coefficients.

Set
\[
\delta = \frac{\partial}{\partial x^i} - \frac{\partial G^j}{\partial y^i} \frac{\partial}{\partial y^j}, \quad \delta = F \frac{\partial}{\partial y^i}.
\]
Clearly, the vectors
\begin{equation}
\left\{ \frac{\delta}{\delta x}, \frac{\delta}{\delta y}, \ldots, \frac{\delta}{\delta x^n}, \frac{\delta}{\delta y^n} \right\}
\end{equation}
form a local tangent frame of \(TM_0\). For another local coordinate system \((U; \tilde{x})\) on \(M\), a routine computation shows that
\begin{equation}
\delta \frac{\delta}{\delta x} = \frac{\partial \tilde{x}^j}{\partial x^i}, \quad \delta \frac{\delta}{\delta y} = \frac{\partial \tilde{y}^j}{\partial x^i}.
\end{equation}
Then \(T(TM_0)\) admits a splitting induced from the Finsler structure \(F\),
\[
T(TM_0) = H(TM_0) \oplus V(TM_0),
\]
where \(H(TM_0) = \text{span}\left\{ \frac{\delta}{\delta x}, \ldots, \frac{\delta}{\delta x^n} \right\}\), and \(V(TM_0) = \text{span}\left\{ \frac{\delta}{\delta y}, \ldots, \frac{\delta}{\delta y^n} \right\}\).

Now by (3.3), one gets a well-defined linear map \(J : T(TM_0) \to T(TM_0)\)
\[
J \left( \frac{\delta}{\delta x} \right) = \frac{\delta}{\delta y}, \quad J \left( \frac{\delta}{\delta y} \right) = -\frac{\delta}{\delta x},
\]
which is in fact an almost complex structure on \(TM_0\). Let
\[
\left\{ \delta x^1, \delta x^2, \ldots, \delta x^n, \delta y^1, \delta y^2, \ldots, \delta y^n \right\}
\]
be the dual frame of (3.2). One has
\[
\delta x^i = dx^i, \quad \delta y^i = \frac{1}{F} \left( dy^i + \frac{\partial G^i}{\partial y^j} dx^j \right),
\]
and
\begin{equation}
J^* (\delta x^i) = -\delta y^i, \quad J^* (\delta y^i) = \delta x^i,
\end{equation}
where \(J^*\) denotes the dual map of \(J\).

Let \(\pi : SM = TM_0/\mathbb{R}^+ \to M\) denote the projective sphere bundle. Now the fundamental tensor \(g = g_{ij} dx^i \otimes dx^j\) defines an Euclidean metric on the pull back
bundle \( \pi^*TM \) over \( SM \). Note that \( \pi^*TM \) admits a distinguished global section \( l : SM \to \pi^*TM \), which is defined by

\[
l(x, [y]) = \left( x, [y], \frac{y}{F(x, y)} \right).
\]

For any local orthonormal frame \( \{e_1, \ldots, e_n\} \) of \((\pi^*TM, g)\) with \( e_n = l \), let \( \{\omega^1, \ldots, \omega^n\} \) be the dual frame. Clearly, \( \omega^i \)'s can be also viewed naturally as (local) one forms on \( SM \) as well as on \( TM_0 \). Here \( \omega^n \), the so called Hilbert form, is a globally defined one form and \( \omega^n = F_y \delta x^i \). Set

\[
\omega^{n+i} = J^*(\omega^i), \quad i = 1, 2, \ldots, n.
\]

The one forms \( \omega^1, \omega^2, \ldots, \omega^{2n-1} \) and \( \omega^{2n} = -F_y \delta y^i = -d \log F \) give rise to a local coframe of \( TM_0 \). The tensor

\[
g^{T(TM_0)} = \sum_{i=1}^{n} \omega^i \otimes \omega^i + \sum_{i=1}^{n} \omega^{n+i} \otimes \omega^{n+i}
\]

gives raise a Riemannian metric on \( TM_0 \).

Moreover, one verifies easily that the forms \( \omega^{n+\alpha}, \alpha = 1, 2, \ldots, n-1 \), are actually the one forms on \( SM \) and the set

\[
\theta = \{\omega^1, \ldots, \omega^n, \omega^{n+1}, \ldots, \omega^{2n-1}\}
\]

forms a local coframe of \( SM \). Let \( \{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2n-1}\} \) denote the dual frame of \( \theta \). By using the local coframe (3.5), the tensor

\[
g^{T(SM)} = \sum_{i=1}^{n} \omega^i \otimes \omega^i + \sum_{\alpha=1}^{n-1} \omega^{n+\alpha} \otimes \omega^{n+\alpha}
\]

gives a well-defined Riemannian metric on \( SM \), which is called the Sasaki-type Riemannian metric on \( SM \).

Write that

\[
\omega^i = v_i^j \delta x^j, \quad \text{and so} \quad \omega^{n+\alpha} = J^*(v_i^\alpha \delta x^i) = -v_i^\alpha \delta y^i.
\]

Then one has

\[
e_i = u_j^i \frac{\delta}{\delta x^j} \quad \text{and} \quad e_{n+\alpha} = -u_j^\alpha \frac{\delta}{\delta y^j},
\]

where \( (u_j^i)^{-1} = (v_i^j)^{-1} \). Here also note that \( v_i^n = F_y \) and \( u_n^i = v_i^n \).

### 3.2. Bott-Chern connection and Bianchi identities.

Let \( H(SM) \) be the horizontal subbundle of \( T(SM) \) spanned by

\[
\{e_1, \ldots, e_n\},
\]

and \( V(SM) \) the vertical subbundle spanned by \( \{e_{n+1}, \ldots, e_{2n-1}\} \). The bundle structure \( \pi : SM \to M \) gives a simple foliation structure on the Riemannian manifold \((SM, g^{T(SM)})\), which is foliated by the vertical bundle \( V(SM) \). Set

\[
\mathcal{F} = V(SM), \quad \mathcal{F}^\perp = H(SM).
\]
Then $F^\perp = H(SM)$ admits the famous Bott connection $\tilde{\nabla}^{F^\perp}$.

By Theorem 1 in [13], the Bott connection $F^\perp = H(SM)$ is exactly the Chern connection, with respect to the canonical identification of $\pi^*TM$ with $H(SM)$. Moreover the Cartan connection is just the symmetrization of the Bott connection. We would like to denote the Bott-Chern connection as

$$\nabla^{BC} : \Omega^*(SM; H(SM)) \to \Omega^{*+1}(SM; H(SM)).$$

The Cartan connection will be denoted by $\hat{\nabla}^{BC}$. Another important quantity appeared here is a $\Omega^1(SM)$-valued endomorphism $H = H_{ij}e_i \otimes \omega^j$ as the difference between $\hat{\nabla}^{BC}$ and $\nabla^{BC}$,

$$H = \hat{\nabla}^{BC} - \nabla^{BC} \in \Omega^1(SM, \text{End}(H(SM))).$$

By Lemma 3 and Lemma 4 in [13], $H_{ij} = H_{ji} = H_{ij\gamma}^{n+\gamma}$ is locally determined with respect to (3.6) by

$$H_{ij\gamma} = -A_{pqk}u_p^iu_q^ju_k^\gamma,$$

where $A_{ijk} = {1\over 4} F[F^2]^{y\gamma}_{y^i y^j y^k}$. In [13], $H$ is called the Cartan endomorphism. The one form

$$\eta = \text{tr}[H] \in \Omega^1(SM)$$

is called the Cartan-type form. Then Cartan-type form has a local formula

$$\eta = \sum_{i=1}^n H_{ii\gamma}^{n+\gamma} =: H_{i\gamma}^{n+\gamma}.$$

**Remark 4.** In literature, Cartan form is locally defined by $I = H_{ij}\omega^j$. This is the reason why we call $\eta$ the Cartan-type form. From the discussion in Section 2, under the identification of projective sphere bundle and indicatrix bundle (cf. [7]), we know that $H$ and $\eta$ are the extension of the cubic form and Tchebychev form to the family case. Some simple calculus show that $\eta$ and $I$ behavior differently. For example, $dI = 0$ means $M$ is a Riemannian manifold. But the properties of $d\eta$ has not been understand well.

Let $\omega = (\omega^i)$ be the connection matrix of the Bott-Chern connection with respect to the orthonormal frame (3.7), i.e.,

$$\nabla^{BC}e_i = \omega^j_i e_j.$$

We list here the structure equations of $\nabla^{BC}$.

**Lemma 12.** The connection matrix $\omega = (\omega^i_j)$ of $\nabla^{BC}$ is determined by the following structure equations,

$$(3.8) \begin{cases} d\vartheta = \vartheta \wedge \omega, \\
\omega + \omega^t = -2H, \end{cases}$$

where $\vartheta = (\omega^1, \ldots, \omega^n)$. Furthermore,

$$\omega^n_\alpha = -\omega^n_\alpha = \omega^{n+\alpha}, \quad \text{and} \quad \omega^n_n = 0.$$
The first equation in (3.8) is described as torsion freeness of Bott-Chern connection. And the second equation in (3.8) implies that Bott-Chern connection is almost preserving metric.

**Remark 5.** Chern connection is constructed originally by Chern in the study of local equivalence problem in Finsler spaces [10]. Since Finsler structure of a Finsler manifold can be viewed as a smooth family of Minkowski norms on the tangent spaces, then we can use the equivalence theorems for Minkowski spaces to deal with local equivalence problem in Finsler spaces. Then Theorem 2 in fact gives the same solution as Chern [10]. But a better solution for local equivalence problem in Finsler spaces can be obtained from Theorem 4. This observation will be noted in a forthcoming paper.

Let $R = \left(\nabla^{BC}\right)^2$ be the curvature of $\nabla^{BC}$. Let $\Omega = \left(\omega^j_i\right)$ be the curvature forms of $R$. Then

$$\Omega^i_j = d\omega^i_j - \omega^k_j \wedge \omega^i_k.$$  

Set

$$\Omega^i_j = \frac{1}{2} R^i_{jkl} \omega^k \wedge \omega^l + P^i_{j k \gamma} \omega^k \wedge \omega^{n+\gamma} + \frac{1}{2} Q^i_{j \alpha \beta} \omega^{\alpha+\gamma} \wedge \omega^{n+\beta},$$

where

$$R^i_{jkl} = -R^i_{jlk}, \quad Q^i_{j \alpha \beta} = -Q^i_{j \beta \alpha}.$$  

Usually the “h-h” part

$$R = \frac{1}{2} R^i_{jkl} \omega^k \wedge \omega^l \otimes \omega^j \otimes e^i$$

of $R$ is called the Chern-Riemann curvature. The “h-v” part

$$P = P^i_{j k \gamma} \omega^k \wedge \omega^{n+\gamma} \otimes \omega^j \otimes e^i$$

is called the Chern-Minkowski curvature. The following Bianchi identities suggest that the “v-v” part of $R$ doesn’t appear.

**Lemma 13.** One has the following Bianchi identities induced from the torsion free property of Bott-Chern connection.

$$R^i_{jkl} + R^i_{k lj} + R^i_{l jk} = 0,$$

$$P^i_{j k \gamma} = P^i_{k j \gamma},$$

and

$$Q^i_{j \alpha \beta} = Q^i_{j \beta \alpha} = 0.$$  

Some more involved Bianchi identities derived from the almost metric preserving properties of Bott-Chern connection.

**Lemma 14.**

$$R^i_{j kl} + R^i_{i kl} + 2H^i_{j \gamma} R^\gamma_{n kl} = 0,$$

$$H^i_{j \gamma \alpha} = H^i_{j \alpha \gamma}.$$
and

\[ P^i_{\ j} k_\gamma + P^j_{\ i} k_\gamma + 2H_{ij\beta}P^n_{\ k} k_\gamma - 2H_{ij\gamma} k = 0. \]  

(3.9)

where we have use the notation

\[ dH_{ij\gamma} - H_{ik\gamma} k^k - H_{jk\gamma} k^j - H_{ij\beta} \omega^\beta_k =: H_{ij\gamma} k^k + H_{ij\gamma} n \omega^{n+\alpha}. \]

As consequences Lemma 13 and Lemma 14 we have the following formulas about the Chern-Minkowski curvature.

**Lemma 15.**

\[ P^i_{\ n} k_\gamma = H_{ki\gamma} n, \]  

(3.10)

and

\[ P^i_{\ j} k_\gamma = -H_{ij\beta} H_{k\beta\gamma} n - H_{ki\beta} H_{j\beta\gamma} n + H_{jk\beta} H_{i\beta\gamma} n + H_{ij\gamma} k + H_{ki\gamma} j - H_{jk\gamma} i. \]

(3.11)

For the proof of the above Bianchi identities, one refers to [24].

**Definition 7.** The Landsberg curvature is defined as

\[ L := P^i_{\ n} k_\gamma (\omega^k \wedge \omega^{n+\gamma}) \otimes e_i, \]

the mean Landsberg curvature is defined by \( J = \text{tr}L. \) If a Finsler manifold satisfies \( P = 0, L = 0 \) or \( J = 0, \) then it is called a Berwald, Landsberg or weak Landsberg manifold, respectively.

The following formulas will be used in the last section of this paper.

**Theorem 4.** We define two tensors on \( TM_0 \) as follows

\[ \hat{g} = \sum_{j=1}^{n} \omega^{n+j} \otimes \omega^{n+j}, \]

and

\[ \hat{A} = H_{\alpha\beta\gamma} \omega^{n+\alpha} \otimes \omega^{n+\beta} \otimes \omega^{n+\gamma}. \]

Then the Lie derivatives of \( \hat{g}, \hat{A} \) and \( \eta \) are given as follows,

\[ \mathcal{L}_e \hat{g} \equiv -2 \sum_{\alpha,\beta=1}^{n-1} P^n_{\ i\alpha} \omega^{n+\alpha} \otimes \omega^{n+\beta} \text{ (mod } \omega^i), \]

(3.12)

\[ \mathcal{L}_e \hat{A} \equiv (H_{\alpha\beta\gamma} - H_{\mu\beta\gamma} P^n_{\ i\alpha} - H_{\alpha\mu\gamma} P^n_{\ i\beta} - H_{\alpha\beta\mu} P^n_{\ i\gamma}) \omega^{n+\alpha} \otimes \omega^{n+\beta} \otimes \omega^{n+\gamma} \text{ (mod } \omega^i), \]

(3.13)

\[ \mathcal{L}_e \eta \equiv P^j_{\ i\gamma} \omega^{n+\gamma} \text{ (mod } \omega^i). \]

(3.14)
Proof. Since that
\[ \Omega_n = d\omega_n^\alpha - \omega_n^\beta \wedge \omega_n^\alpha = -d\omega_n^{n+\alpha} + \omega_n^{n+\beta} \wedge \omega_n^\alpha, \]
then
\[ d\omega_n^{n+\alpha} = -\Omega_n^\alpha + \omega_n^{n+\beta} \wedge \omega_n^\alpha = -\frac{1}{2} R_n^\alpha_{\ jk} \omega_n^j \wedge \omega_n^k - P_n^\alpha_{\ j\beta} \omega_n^j \wedge \omega_n^{n+\beta} + \omega_n^{n+\beta} \wedge \omega_n^\alpha. \]

By Cartan homotopy formula, we have
\[ \mathcal{L}_{e_i} \omega^{n+\alpha} = (i_{e_i} d + di_{e_i}) \omega^{n+\alpha} \]
\[ = i_{e_i} \left[ -\frac{1}{2} R_n^\alpha_{\ jk} \omega_n^j \wedge \omega_n^k - P_n^\alpha_{\ j\beta} \omega_n^j \wedge \omega_n^{n+\beta} + \omega_n^{n+\beta} \wedge \omega_n^\alpha \right] \]
\[ = -R_n^\alpha_{\ ik} \omega_n^k - P_n^\alpha_{\ i\beta} \omega_n^{n+\beta} - \omega_n^\alpha(e_i) \omega_n^{n+\beta}. \]

Using the fact \( e_i(F) = 0 \) and \( \omega^{2n} = -d\log F \), one gets \( \mathcal{L}_{e_i} \omega^{2n} = 0 \). So we have
\[ \mathcal{L}_{e_i} \left( \sum_{j=1}^{n} \omega_n^{n+j} \otimes \omega_n^{n+j} \right) = \mathcal{L}_{e_i} \left( \sum_{\alpha=1}^{n-1} \omega_n^{n+\alpha} \otimes \omega_n^{n+\alpha} \right) \]
\[ = \sum_{\alpha=1}^{n-1} \left[ (\mathcal{L}_{e_i} \omega_n^{n+\alpha}) \otimes \omega_n^{n+\alpha} + \omega_n^{n+\alpha} \otimes (\mathcal{L}_{e_i} \omega_n^{n+\alpha}) \right] \]
\[ = \sum_{\alpha=1}^{n-1} \left[ (-R_n^\alpha_{\ ik} \omega_n^k - P_n^\alpha_{\ i\beta} \omega_n^{n+\beta} - \omega_n^\alpha(e_i) \omega_n^{n+\beta}) \otimes \omega_n^{n+\alpha} \right. \]
\[ + \omega_n^{n+\alpha} \otimes (-R_n^\alpha_{\ ik} \omega_n^k - P_n^\alpha_{\ i\beta} \omega_n^{n+\beta} - \omega_n^\alpha(e_i) \omega_n^{n+\beta}) \]
\[ = -\sum_{\alpha=1}^{n-1} R_n^\alpha_{\ ik} \left( \omega_n^k \otimes \omega_n^{n+\alpha} + \omega_n^{n+\alpha} \otimes \omega_n^k \right) \]
\[ - \sum_{\alpha=1}^{n-1} \left( P_n^\alpha_{\ i\beta} + \omega_n^\alpha(e_i) \right) \left( \omega_n^{n+\beta} \otimes \omega_n^{n+\alpha} + \omega_n^{n+\alpha} \otimes \omega_n^{n+\beta} \right) \]
\[ = -\sum_{\alpha=1}^{n-1} R_n^\alpha_{\ ik} \left( \omega_n^k \otimes \omega_n^{n+\alpha} + \omega_n^{n+\alpha} \otimes \omega_n^k \right) \]
\[ - \sum_{\alpha,\beta=1}^{n-1} \left( P_n^\alpha_{\ i\beta} + P_n^\beta_{\ i\alpha} + \omega_n^\alpha(e_i) + \omega_n^\beta(e_i) \right) \omega_n^{n+\alpha} \otimes \omega_n^{n+\beta} \]
\[ = -\sum_{\alpha=1}^{n-1} R_n^\alpha_{\ ik} \left( \omega_n^k \otimes \omega_n^{n+\alpha} + \omega_n^{n+\alpha} \otimes \omega_n^k \right) - 2 \sum_{\alpha,\beta=1}^{n-1} P_n^\beta_{\ i\alpha} \omega_n^{n+\alpha} \otimes \omega_n^{n+\beta} \]
where we have use the facts
\[ P_n^\alpha_{\ i\beta} = P_n^\beta_{\ i\alpha}, \quad \omega_n^\alpha \wedge \omega_n^\beta = H_{\alpha\beta\gamma} \omega_n^{n+\gamma}. \]
Furthermore, we have

\[ \mathcal{L}_e (H_{\alpha\beta\gamma} \omega^{n+\alpha} \otimes \omega^{n+\beta} \otimes \omega^{n+\gamma}) \]

\[ = e_i (H_{\alpha\beta\gamma}) \omega^{n+\alpha} \otimes \omega^{n+\beta} \otimes \omega^{n+\gamma} + H_{\alpha\beta\gamma} \left( \mathcal{L}_e \omega^{n+\alpha} \right) \otimes \omega^{n+\beta} \otimes \omega^{n+\gamma} \]

\[ + H_{\alpha\beta\gamma} \omega^{n+\alpha} \otimes \left( \mathcal{L}_e \omega^{n+\beta} \right) \otimes \omega^{n+\gamma} + H_{\alpha\beta\gamma} \omega^{n+\alpha} \otimes \omega^{n+\beta} \otimes \left( \mathcal{L}_e \omega^{n+\gamma} \right) \]

\[ \equiv e_i (H_{\alpha\beta\gamma}) \omega^{n+\alpha} \otimes \omega^{n+\beta} \otimes \omega^{n+\gamma} - H_{\alpha\beta\gamma} \left( P_{n \ i} \mu + \omega^\mu (e_i) \right) \omega^{n+\mu} \otimes \omega^{n+\beta} \otimes \omega^{n+\gamma} \]

\[ - H_{\alpha\beta\gamma} \omega^{n+\alpha} \otimes \left( P_{n \ i} \mu + \omega^\mu (e_i) \right) \omega^{n+\mu} \otimes \omega^{n+\gamma} \]

\[ - H_{\alpha\beta\gamma} \omega^{n+\alpha} \otimes \left( P_{n \ i} \mu + \omega^\mu (e_i) \right) \omega^{n+\mu} \otimes \omega^{n+\gamma} \quad (\text{mod} \ \omega^i) \]

\[ \equiv e_i (H_{\alpha\beta\gamma}) \omega^{n+\alpha} \otimes \omega^{n+\beta} \otimes \omega^{n+\gamma} - H_{\alpha\beta\gamma} \omega^\alpha (e_i) \omega^{n+\mu} \otimes \omega^{n+\beta} \otimes \omega^{n+\gamma} \]

\[ - H_{\alpha\beta\gamma} \omega^\alpha (e_i) \omega^{n+\mu} \otimes \omega^{n+\beta} \otimes \omega^{n+\gamma} - H_{\alpha\beta\gamma} \omega^\alpha (e_i) \omega^{n+\mu} \otimes \omega^{n+\beta} \otimes \omega^{n+\gamma} \]

\[ - H_{\alpha\beta\gamma} \omega^\alpha (e_i) \omega^{n+\mu} \otimes \omega^{n+\beta} \otimes \omega^{n+\gamma} \quad (\text{mod} \ \omega^i) \]

\[ \equiv \left( e_i (H_{\alpha\beta\gamma}) - H_{\mu\beta\gamma} \omega^\mu (e_i) - H_{\mu\beta\gamma} \omega^\mu (e_i) - H_{\alpha\beta\gamma} \omega^\alpha (e_i) \right) \omega^{n+\mu} \otimes \omega^{n+\beta} \otimes \omega^{n+\gamma} \]

\[ - \left( H_{\mu\beta\gamma} P_{n \ i} \alpha + H_{\alpha\gamma} P_{n \ i} \beta + H_{\alpha\beta\gamma} P_{n \ i} \gamma \right) \omega^{n+\alpha} \otimes \omega^{n+\beta} \otimes \omega^{n+\gamma} \quad (\text{mod} \ \omega^i) \]

\[ \equiv \left( H_{\alpha\beta\gamma} \omega^\alpha (e_i) - H_{\mu\beta\gamma} P_{n \ i} \alpha - H_{\mu\beta\gamma} P_{n \ i} \beta - H_{\alpha\beta\gamma} P_{n \ i} \gamma \right) \omega^{n+\alpha} \otimes \omega^{n+\beta} \otimes \omega^{n+\gamma} \quad (\text{mod} \ \omega^i). \]

For the Cartan-type form, we have

\[ \mathcal{L}_e \eta = \mathcal{L}_e (H_\gamma \omega^{n+\gamma}) \]

\[ = e_i (H_\gamma) \omega^{n+\gamma} + H_\gamma \mathcal{L}_e \omega^{n+\gamma} \]

\[ \equiv (e_i (H_\gamma) - H_{\mu} \omega^\mu (e_i) - H_{\mu} \omega^\mu (e_i)) \omega^{n+\gamma} \quad (\text{mod} \ \omega^i) \]

\[ \equiv (H_\gamma i - H_{\mu} P_{n \ i} \gamma) \omega^{n+\gamma} \quad (\text{mod} \ \omega^i) \]

\[ \equiv P_{j \ i} \gamma \omega^{n+\gamma} \quad (\text{mod} \ \omega^i). \]

3.3. Some properties of Cartan type one form.

Let \( \mathbf{R} \) and \( \tilde{\mathbf{R}} \) be the curvature of \( \nabla^{BC} \) and \( \tilde{\nabla}^{BC} \), respectively. Then we have

**Lemma 16.** The exterior differentiation of \( \eta \) is given by

\[ (3.15) \quad d\eta = -\text{tr}[\mathbf{R}]. \]

Then \( d\eta \) has the local formula

\[ (3.16) \quad d\eta = d(H_\gamma \omega^{n+\gamma}) = -\frac{1}{2} R_i \ k \ i (\omega^k \wedge \omega^j) - P_i \ k \ i \omega^k \wedge \omega^{n+\gamma}. \]
Proof. It is well known that the Cartan connection is metric-compatible. Then \( \text{tr}[\hat{\mathbf{R}}] = 0 \). So one has

\[
0 = \text{tr}[\hat{\mathbf{R}}] = \text{tr}[(\nabla^{BC})^2]
= \text{tr} \left[ (\nabla^{BC} + H)^2 \right]
= \text{tr} \left[ (\nabla^{BC})^2 + [\nabla^{BC}, H] + [H, H] \right]
= \text{tr} [\mathbf{R}] + \text{tr} \left[ [\nabla^{BC}, H] + [H, H] \right]
\]

where \([\cdot, \cdot]\) denotes the super bracket on \( \Omega^* (SM, \text{End}(H(SM))) \). By the element facts in [39], we have

\[
\text{tr} \left[ [H, H] \right] = 0,
\]

and

\[
d\text{tr}[H] = \text{tr} \left[ [\nabla^{BC}, H] \right].
\]

So the proof is complete. \( \square \)

On a local coordinate chart \((U; x^i)\), let \( d\sigma = \sigma(x) dx^1 \wedge \cdots \wedge dx^n \) be any volume form on \( M \). The following important function on \( SM \) is well defined,

\[
\tau = \ln \sqrt{\frac{\det g_{ij}}{\sigma(x)}}.
\]

\( \tau \) is called the distortion of \((M, F)\). \( \tau \) is a very important invariant of the Finsler manifold, which is firstly introduced by Zhongmin Shen. One refers to [11] for the other discussion about \( \tau \).

Let \( l = \mathbb{R}G \) be the line bundle generated by the Reeb vector field \( G = F e_n \). The quotient subbundle \( H(SM)/l \) will be canonically chosen as the orthogonal complement of \( l \) in \( H(SM) \) with respect to \( g \), i.e.,

\[
l^\perp = H(SM)/l.
\]

**Definition 8.** Using the Levi-Civita connection \( \nabla^{T(SM)} \) of the metric \( g^{T(SM)} \), we introduce the following three operator on \( \Omega^*(SM) \).

\[
\begin{align*}
d^{\perp} &= \omega^\alpha \wedge \nabla^{T(SM)}_{e_n}, \\
d^l &= \omega^n \wedge \nabla^{T(SM)}_{e_n}, \\
d^V &= \omega^{n+\alpha} \wedge \nabla^{T(SM)}_{e_{n+1}},
\end{align*}
\]

where \( \nabla^{T(SM)} \) denotes the Levi-Civita connection on the cotangent bundle \( T^*(SM) \).

It is obvious that the above operators are well defined. So the exterior differential operator \( d \) on \( SM \) splits to three parts

\[
d = d^{\perp} + d^l + d^V.
\]

The following lemma gives another explanation of \( \eta \).

**Lemma 17.**

\[
d^V \tau = \eta.
\]
Proof.

\[ d^V \tau \equiv d\tau \mod(\omega^i) \]
\[ \equiv d\ln \sqrt{\det g_{ij}} - d\ln \sigma(x) \mod(\omega^i) \]
\[ \equiv d\ln \sqrt{\det g_{ij}} \mod(\omega^i) \]

Assume that \( \theta = \{\omega^1, \ldots, \omega^n\} \) has the same orientation with \( M \). Let \( \omega = \omega^1 \land \cdots \land \omega^n \), one has

\[ d\omega = d(\omega^1 \land \cdots \land \omega^n) \]
\[ = \sum_i (-1)^i d\omega^i \land \omega^1 \land \cdots \land \hat{\omega}^i \land \cdots \land \omega^n \]
\[ = \sum_i (-1)^i \omega^i \land \omega^1 \land \cdots \land \hat{\omega}^i \land \cdots \land \omega^n \]
\[ = -(\sum_i \omega^i_j) \land \omega \]
\[ = \eta \land \omega. \]

Since
\[ \omega = \omega^1 \land \cdots \land \omega^n = \sqrt{\det(g_{ij})} dx^1 \land \cdots \land dx^n. \]

Then
\[ d\omega = d\sqrt{\det(g_{ij})} \land dx^1 \land \cdots \land dx^n \]
\[ = d\sqrt{\frac{\det(g_{ij})}{\det(g_{ij})}} \land \omega \]
\[ = d\ln \sqrt{\det g_{ij}} \land \omega. \]

Then one has
\[ d\ln \sqrt{\det g_{ij}} \land \omega = \eta \land \omega. \]
So
\[ \eta \equiv d\ln \sqrt{\det g_{ij}} \mod(\omega^i). \]

Set
\[ S = G(\tau). \]

\( S \) is called the \( S \)-curvature of the Finsler manifold \( (M, F) \), which is also introduced by Zhongmin Shen. For the detail of \( S \)-curvature, one refers to [11, 32, 33, 34].

Set \( \tilde{S} = \tilde{S}_F = e_n(\tau) \). If \( d^V \tilde{S} = 0 \), then \( (M, F) \) is called of isotropic \( S \)-curvature. The following corollary is simple.

**Corollary 1.**

\[ (3.17) \]
\[ d\tau = d^\perp \tau + \tilde{S} \omega^n + \eta. \]

By Lemma [16] and the above corollary, one has a description of \( (M, F) \) of isotropic \( S \)-curvature.
Proposition 1. A Finsler manifold \((M, F)\) is of isotropic \(S\)-curvature if and only if

\[ J^*(d^1 \tau) = J, \]

where \(J^*\) is the mapping given by (3.4).

Proof. Using Lemma 16, the exterior differentiation of (3.17) gives

\[ J = d(e_\alpha(\tau) \omega^\alpha) + d(S \wedge \omega^n + S d\omega^n + d\eta) \]

\[ = d(e_\alpha(\tau)) \wedge \omega^\alpha + e_\alpha(\tau) d\omega^\alpha + dV(S) \wedge \omega^n + dV(S) \wedge \omega^n + \tilde{S} d\omega^n + d\eta \]

\[ = e_\alpha(e_\beta(\tau)) \omega^\alpha \wedge \omega^\beta + e_n(e_\beta(\tau)) \omega^\alpha \wedge \omega^\beta + e_n(e_\beta(\tau)) \omega^n + \tilde{S} \wedge \omega^n + \tilde{S} d\omega^n + d\eta \]

\[ + e_\alpha(\tau) \omega^\beta \wedge \omega^\gamma + d^1 \tilde{S} \wedge \omega^n + d^V \tilde{S} \wedge \omega^n + \tilde{S} \wedge \omega^n + \tilde{S} d\omega^n \]

\[ = \frac{1}{2} R_i^i j k \omega^j \wedge \omega^k - P_i^i j k \omega^j \wedge \omega^n \]

It follows that

\[ d^V \tilde{S} \wedge \omega^n - e_\alpha(\tau) \omega^n + \tilde{S} \wedge \omega^n = 0. \]

So

\[ d^V \tilde{S} = e_\alpha(\tau) \omega^n \wedge \omega^n - P_i^i n o \omega^n \wedge \omega^n = J^*(d^1 \tau) - J. \]

\[ \square \]

Proposition 2. For a Finsler manifold \((M, F)\), let \(\tau\) be the distortion with respect to a given volume element of \(M\). Then \(\eta = d\tau\) if and only if \(S = 0\) and \(J = 0\).

Proof. If \(\eta = d\tau\), then \(d^1 \tau = 0\) and \(S = 0\) directly holds. Since \(S = 0\) obviously implies \(S\) is isotropic, then \(J = 0\). The proof of the converse statement is similarly. \(\square\)

4. Nonlinear parallel transport and special Finsler spaces

4.1. Basic properties of nonlinear parallel transport.

In this subsection, we review some facts of the nonlinear parallel transport. One refers to [1, 5, 11] for details.

Let \((M, F)\) be a Finsler manifold of dimension \(n\). Let \(\sigma : [0, 1] \to M\) be a smooth curve emanates form \(\sigma(0) = p\). We assume that \(\sigma\) is contained in a local coordinate \((U; x^i)\) of \(M\). Let \((\sigma^1(t), \ldots, \sigma^n(t))\) be the local coordinates of \(\sigma(t)\), \(t \in [0, 1]\). The tangent vector of \(\sigma(t)\) is then given by

\[ \dot{\sigma}(t) = \frac{d\sigma^i}{dt} \frac{\partial}{\partial x^i}|_{\sigma(t)}. \]
We have the horizontal lift vector field $\hat{\sigma}$ of $\sigma$ on $\pi^{-1}(\sigma)$ with respect to the decomposition

$$T(TM_0) = H(TM_0) \oplus V(TM_0),$$

where $\pi : TM \to M$ is the canonical projection. At any point $(\sigma(t), z)$, where $z \in T_{\sigma(t)}M \setminus \{0\}$,

$$\hat{\sigma}(\sigma(t), z) = \frac{d\sigma^i}{dt} \frac{\partial}{\partial x^i}|_{(\sigma(t), z)} + \frac{dy^i}{dt} \frac{\partial}{\partial y^i}|_{(\sigma(t), y(t))}.$$

Now we consider the integral curves of the vector field $\hat{\sigma}$ on $\pi^{-1}(\sigma)$. For any $y \in T_pM \setminus \{0\}$, let $\hat{\sigma} : [0, \epsilon) \to \pi^{-1}(\sigma)$ be a curve such that $\hat{\sigma}(0) = (p, y)$, and

$$\hat{\sigma}(t) = (\sigma(t), y(t)), \quad y(t) \in T_{\sigma(t)}M, \quad \forall t \in [0, \epsilon).$$

Then $\hat{\sigma}$ is an integral curve of $\hat{\sigma}$ if and only if

$$\hat{\sigma}(t) = \hat{\sigma}((\sigma(t), y(t))), \quad \forall t \in [0, \epsilon).$$

Under local coordinate, let $y(t) = y^i(t) \frac{\partial}{\partial x^i}|_{\sigma(t)}$, then

$$\hat{\sigma}(t) = \frac{d\sigma^i}{dt} \frac{\partial}{\partial x^i}|_{(\sigma(t), y(t))} + \frac{dy^i}{dt} \frac{\partial}{\partial x^i}|_{(\sigma(t), y(t))}.$$

It follows that $\hat{\sigma}$ is an integral curve of $\hat{\sigma}$ if and only if $\hat{\sigma}(t) = (\sigma(t), y(t))$ is the solution of the following ODE

$$(4.1) \quad \frac{dy^i}{dt} + \frac{d\sigma^i}{dt} \frac{\partial G^i}{\partial y^j}(\sigma(t), y(t)) = 0, \quad i = 1, \ldots, n,$$

with initial value $(\sigma(0), y(0)) = (p, y)$. By the theory of ODE, there exists $\epsilon = \epsilon(y)$, such that $[4.1]$ exist unique solution on $[0, \epsilon(y))$. Moreover, the following homogenous property induced by $[3.1]

$$(4.2) \quad \frac{\partial G^i}{\partial y^j}(\sigma(t), \lambda y(t)) = \lambda \frac{\partial G^i}{\partial y^j}(\sigma(t), y(t)), \quad \forall \lambda > 0,$$

implies that $(\sigma(t), \lambda y(t))$ is the solution for the initial value $(p, \lambda y)$. The compactness of the projective sphere $(T_xM \setminus \{0\})/\mathbb{R}^+$ allow us to uniformly choose an positive number $\epsilon > 0$, such that the solutions of $[4.1]$ exist on $[0, \epsilon)$ for any initial values.

As the curve $\hat{\sigma}(t) = (\sigma(t), y(t))$ is in fact a vector field along $\sigma$, we have the definition of nonlinear parallel vector field.

**Definition 9.** A vector field $\hat{\sigma}(t) = (\sigma(t), y(t))$ along $\sigma$ is called a nonlinear parallel vector filed, if it is a solution of $[4.1]$. The curve $\hat{\sigma}(t) = (\sigma(t), y(t))$ is called the horizontal lift of $\sigma(t)$ start from $y(0) = y$. 


Definition 10. For any \( t_0 \in [0, \varepsilon] \), define \( P_{\sigma, t_0} : T_pM \setminus \{0\} \to T_{\sigma(t_0)}M \setminus \{0\} \), by
\[
(4.3) \quad P_{\sigma, t_0}(y) := y(t_0), \quad \forall \ y \in T_pM \setminus \{0\},
\]
where \( y(t) \) is the nonlinear parallel vector field along \( \sigma \) with \( y(0) = y \). \( P_{\sigma, t_0} \) is called the nonlinear parallel transport along \( \sigma \) from \( 0 \) to \( t_0 \).

Lemma 18 ([1][5][11]). Let \( \sigma : [0, \varepsilon] \to M \) be a smooth curve such that the nonlinear parallel translation \( P_{\sigma, t} \) is defined for \( \forall \ t \in [0, \varepsilon] \). Let \( p = \sigma(0) \), then the map \( P_{\sigma, t} : T_pM \setminus \{0\} \to T_{\sigma(t)}M \setminus \{0\} \) is a norm preserving diffeomorphism, and
\[
(4.4) \quad P_{\sigma, t}(\lambda y) = \lambda P_{\sigma, t}(y), \quad \forall \ \lambda > 0, \forall \ y \in T_pM \setminus \{0\}.
\]

Proof. \( P_{\sigma, t} \) is a diffeomorphism follows form the theory of ODE. As the tangent vector of the curve \( \hat{\sigma}(t) = (\sigma(t), y(t)) \) is horizontal, i.e., \( \dot{\sigma} = \dot{\hat{\sigma}} \), then the norm preserving property follows form the basic fact \( \frac{\delta}{\delta x} F = 0 \). Uniqueness theorem of ODE and (4.2) imply (4.4). \( \square \)

4.2. Characterizations of special Finsler spaces.

In this subsection, we are going to give some characterizations of Landsberg spaces and Berwald spaces via nonlinear parallel transport.

Firstly, we would like to give a description of the differential mapping of the nonlinear parallel transport.

Let \( \sigma : [0, \varepsilon] \to M \) be a curve in \( M \) with \( \sigma(0) = p \). Let \( P_{\sigma, t} \) be the nonlinear parallel transport form \( T_pM \setminus \{0\} \) to \( T_{\sigma(t)}M \setminus \{0\} \), \( t \in [0, \varepsilon] \). Let \( y \in T_pM \setminus \{0\} \) and \( u \in T_y(T_pM \setminus \{0\}) \). Then \( u \) is the tangent vector of the line \( \tau : [-\xi, \xi] \to T_pM \setminus \{0\} \),
\[
\tau(s) = y + su,
\]
through \( y = \tau(0) \). By the nonlinear parallel translation, we obtain a smooth mapping \( H : [-\xi, \xi] \times [0, \varepsilon] \to \pi^{-1}(\sigma) \) defined by
\[
H(s, t) := P_{\sigma, t}(\tau(s)) = P_{\sigma, t}(y + su), \quad \forall \ (s, t) \in [-\xi, \xi] \times [0, \varepsilon].
\]
For \( \forall \ t \in [0, \varepsilon] \), we have
\[
(4.5) \quad H_{s,(0,t)} \left( \frac{\partial}{\partial t} \right) = \frac{\partial}{\partial t} \bigg|_{s=0} P_{\sigma, t}(y + su) = \dot{\hat{\sigma}}(t),
\]
\[
H_{s,(0,t)} \left( \frac{\partial}{\partial s} \right) = \frac{\partial}{\partial s} \bigg|_{s=0} P_{\sigma, t}(y + su) = (P_{\sigma, t})_{\ast,y} u.
\]
By (4.5), we have,
\[
(4.6) \quad \left[ \dot{\hat{\sigma}}(t), (P_{\sigma, t})_{\ast,y} u \right] = H_{s,(0,t)} \left[ \frac{\partial}{\partial t}, \frac{\partial}{\partial s} \right] = 0,
\]
for \( \forall y \in T_pM \setminus \{0\} \), \( \forall u \in T_y(T_pM \setminus \{0\}) \) and \( \forall t \in [0, \varepsilon] \).

Now we consider the well defined tensor on \( TM_0 \),
\[
\hat{g} = \sum_{i=1}^{n} \omega^{n+i} \otimes \omega^{n+i} = g_{ij} \frac{\delta y^i}{F} \otimes \frac{\delta y^j}{F}.
\]
It is obvious that the restriction of \( \hat{g} \) at \( T_p M \setminus \{0\} \), \( p \in M \), gives the Riemannian metric \( \hat{g}_p \) of the Minkowski space \((T_p M \setminus \{0\}, F_{T_p M})\).

**Definition 11.** Let \((M, F)\) be a Finsler manifold, \( p \in M \) and \( y \in T_p M \setminus \{0\} \). For any smooth curve \( \sigma(t) \) start from \( \sigma(0) = p \), let \( \hat{\sigma}(t) \) be the horizontal lift of \( \sigma(t) \) start from \( y \). The Riemannian metric \( \hat{g}_p \) of \((T_p M \setminus \{0\}, F_{T_p M})\) is called horizontally stable at \((p, y)\) along the curve \( \sigma(t) \), if

\[
\frac{d}{dt}

\bigg|_{t=0}

(P_{\sigma(t)})^* \hat{g}_p = 0.
\]

If \( \hat{g}_p \) is horizontally stable at \((p, y)\) along any curve \( \sigma(t) \) start from \( p \), \( \hat{g}_p \) is called horizontally stable at \((p, y)\). If \( \hat{g}_p \) is horizontally stable at \((p, y)\), \( \forall y \in T_p M \setminus \{0\} \), we call \( \hat{g} \) horizontally stable at \( p \).

The following theorem is well known.

**Theorem 5** ([1][5][11]). Let \((M, F)\) be a Finsler manifold, \( p \in M \) and \( y \in T_p M \setminus \{0\} \). Then the Landsberg curvature vanishing at \((p, y)\), i.e., \( L(p, y) = 0 \) if and only if the Riemannian metric \( \hat{g}_p \) of \((T_p M \setminus \{0\}, F_{T_p M})\) is horizontally stable at \((p, y)\).

**Proof.** Let \( \sigma(t) \) be any smooth curve emanated form \( p \). Let \( \hat{\sigma}(t) \) be the horizontal lift of \( \sigma(t) \) start form \( y \). For any \( u, v \in T_y (T_p M \setminus \{0\}) \), we have

\[
\left( (P_{\sigma(t)})^* \hat{g}_p \right) (u, v) = \hat{g}_{(\sigma(t), y(t))} \left( (P_{\sigma(t)})_{*y} u, (P_{\sigma(t)})_{*y} v \right).
\]

By [4][0], one gets that

\[
\frac{d}{dt} \left[ \left( (P_{\sigma(t)})^* \hat{g}_p \right) (u, v) \right]
= \frac{d}{dt} \left[ \hat{g}_{(\sigma(t), y(t))} \left( (P_{\sigma(t)})_{*y} u, (P_{\sigma(t)})_{*y} v \right) \right]
= \hat{\sigma}(t) \left[ \hat{g}_{(\sigma(t), y(t))} \left( (P_{\sigma(t)})_{*y} u, (P_{\sigma(t)})_{*y} v \right) \right]
= \left( L_{\hat{\sigma}(t)} \hat{g}_{(\sigma(t), y(t))} \right) \left( (P_{\sigma(t)})_{*y} u, (P_{\sigma(t)})_{*y} v \right) + \hat{g}_{(\sigma(t), y(t))} \left[ \hat{\sigma}(t), (P_{\sigma(t)})_{*y} u, (P_{\sigma(t)})_{*y} v \right]
= \left( L_{\hat{\sigma}(t)} \hat{g}_{(\sigma(t), y(t))} \right) \left( (P_{\sigma(t)})_{*y} u, (P_{\sigma(t)})_{*y} v \right).
\]

It follows that

\[
\frac{d}{dt} \left|_{t=0} \left[ \left( (P_{\sigma(t)})^* \hat{g}_p \right) (u, v) \right] = \left( L_{\hat{\sigma}(0)} \hat{g} \right) (u, v).
\]

By [3][12], we have

\[
\left( L_{\hat{\sigma}(0)} \hat{g} \right) (u, v) = -2 \sum_{\alpha, \beta=1}^{n-1} P_n^\beta i_\alpha \omega^j (\hat{\sigma}(0)) \omega^{n+\alpha} (u) \omega^{n+\beta} (v).
\]

Since the horizontal lift from \( T_p M \) to \( H_{(p, y)}(TM_0) \) is surjective, and the choice of \( u, v \in T_y (T_p M \setminus \{0\}) \) is arbitrary, the theorem follows. \( \Box \)
As a corollary, we have the well known characterization of Landsberg manifold.

**Corollary 2** ([1], [5], [11]). A connected Finsler manifold $(M, F)$ is a Landsberg manifold if and only if $\hat{g}$ is horizontally stable everywhere, which is equivalent to say that nonlinear parallel transports

$$P_{\sigma,t} : (T_{\sigma(0)}M \setminus \{0\}, \hat{g}_{\sigma(0)}) \to \left(T_{\sigma(t)}M \setminus \{0\}, \hat{g}_{\sigma(t)}\right),$$

are Riemannian isometry, for any piecewise smooth curve $\sigma$ in $M$.

Now we consider the well defined tensor

$$\hat{A} = H_{\alpha\beta\gamma}^{\omega^{n+\alpha}} \otimes \omega^{n+\beta} \otimes \omega^{n+\gamma}.$$ 

The restriction of $\hat{A}$ as well as $\eta = H_{\gamma}^{\omega^{n+\gamma}}$ on $T_p M \setminus \{0\}$, $p \in M$, gives the Cartan tensor and Cartan form of the Minkowski space $(T_p M \setminus \{0\}, F_{T_p M})$ up to a sign, respectively. Then we have the following theorem about Chern-Minkowski curvature.

**Theorem 6.** Let $(M, F)$ be a Finsler manifold, $p \in M$ and $y \in T_p M \setminus \{0\}$. Then Chern-Minkowski curvature $P(p, y) = 0$ if and only if $\hat{g}_p$ and $\hat{A}_p$ of $(T_p M \setminus \{0\}, F_{T_p M})$ are all horizontally stable at $(p, y)$.

**Proof.** Using (3.13) and (4.6), for any smooth curve $\sigma(t)$ with $\sigma(0) = p$ and $u, v, w \in T_y (T_p M \setminus \{0\})$, we have

$$\frac{d}{dt} \bigg|_{t=0} \left[ \left( (P_{\sigma,t})^* \right)_{\sigma(t)} (\hat{A})(u, v, w) \right] = \left( L_{\hat{g}_{\sigma(0)}} \hat{A} \right)(u, v, w)$$

$$= \left( H_{\alpha\beta\gamma}^{\omega^{n+\alpha}} - H_{\gamma \alpha \beta} P_{n \alpha}^{\mu} - H_{\alpha \beta \gamma} P_{n \beta}^{\mu} - H_{\mu \beta \gamma} P_{n \gamma}^{\mu} \right) \omega^i(\hat{\sigma}) \omega^{n+\alpha}(u) \omega^{n+\beta}(v) \omega^{n+\gamma}(w).$$

If $P(p, y) = 0$, then $L(p, y) = 0$ and $\hat{g}_p$ is stable at $(p, y)$. By (3.11) and $P(p, y) = 0$, we have $H_{\alpha\beta\gamma} = 0$. Then $\hat{A}_p$ is stable at $(p, y)$.

Conversely, if $\hat{g}_p$ and $\hat{A}_p$ is stable at $(p, y)$, then $L(p, y) = 0$ and $H_{\alpha\beta\gamma} = 0$. Using (3.11), we conclude that $P(p, y) = 0$. 

As a consequence, we obtain the following characterization of Berwald manifold.

**Theorem 7.** A connected Finsler manifold $(M, F)$ is a Berwald manifold if and only if $\hat{g}$ and $\hat{A}$ are horizontally stable everywhere, which is equivalent to say that nonlinear parallel transports

$$P_{\sigma,t} : \left(T_{\sigma(0)}M \setminus \{0\}, F_{T_{\sigma(0)}M}\right) \to \left(T_{\sigma(t)}M \setminus \{0\}, F_{T_{\sigma(t)}M}\right),$$

are linear mappings which preserving norms, for any piecewise smooth curve $\sigma$ in $M$.

**Proof.** This theorem is a consequence of Theorem 2 and Theorem 6. 

**Remark 6.** It is firstly proved by Ichijyo [15] that the nonlinear parallel transports are necessarily linear mappings for Berwald manifold. This result has many influences in the study of Berwald manifold. Along this way, Szabó [30] gives his local classification of the positive definite Berwald spaces. Our Theorem 7 implies that nonlinear parallel transports are all linear mappings in fact characterizes Berwald manifold.

Now we are able to prove the main theorem of this paper.
Theorem 8. A connected Finsler manifold \((M, F)\) is a Belwald manifold if and only if \(\hat{g}\) and \(\eta\) are horizontally stable everywhere.

Proof. By Theorem 7, we only need to prove the sufficient condition. Assume that \(\hat{g}\) and \(\eta\) are horizontally stable everywhere. For any piecewise smooth curve \(\sigma\) in \(M\), the nonlinear parallel transports

\[
P_{\sigma,t} : \left( T_{\sigma(0)}M \setminus \{0\}, F_{T_{\sigma(0)}M} \right) \to \left( T_{\sigma(t)}M \setminus \{0\}, F_{T_{\sigma(t)}M} \right),
\]

preserving the metric \(\hat{g}\) and Cartan-type form \(\eta\). It follows from Theorem 1 that \(P_{\sigma,t}\) are linear. By Theorem 7, the theorem follows. \(\square\)

Theorem 9. A connected Finsler manifold \((M, F)\) is a Belwald manifold if and only if the Landsberg curvature \(L\) and the mean Chern-Minkowski curvature \(\text{tr}P\) are vanishing.

Proof. Using (3.14), for any smooth curve \(\sigma(t)\) with \(\sigma(0) = p\) and \(u \in T_y(T_pM \setminus \{0\})\), we have

\[
\frac{d}{dt} \bigg|_{t=0} \left[ \left( (P_{\sigma,t})^* \eta \right)(u) \right] = \left( L_{\hat{g}(0)} \eta \right)(u) = P_j^i v^i (\hat{g}) \omega^{j+\gamma}(u).
\]

So \(L = 0\) and \(\text{tr}P = 0\) implies that \(\hat{g}\) and \(\eta\) are horizontally stable everywhere. It follows that \((M, F)\) is a Belwald manifold by Theorem 8. \(\square\)

In literatures, the Berwald curvature \(B\) is a tensor defined by

\[
B_{j}^{i} \omega^{k} = \frac{\partial^3 G^i}{\partial y_j \partial y_i \partial y_k}.
\]

The trace of \(B\) is called the mean Berwald curvature \(E = \text{tr}B\). If \(L = 0\), then it is proved that \(B = P\) and \(E = \text{tr}P\) (cf. [6], p. 67).

So we have another version of Theorem 9.

Theorem 10. A connected Finsler manifold \((M, F)\) is a Belwald manifold if and only if the \(L = 0\) and \(E = 0\).

Remark 7. This theorem gives a positive answer of a problem asked by Shen (cf. [34], p. 322). One also notes that Theorem 9 and Theorem 10 are trivial for 2 dimension cases.

Corollary 3. A connected Finsler manifold \((M, F)\) is a Belwald manifold if and only if the \(L = 0\) and \(d\eta = 0\).

Proof. By Lemma 16, \(d\eta = 0\) implies that \(\text{tr}P = 0\). This corollary follows from Theorem 9. \(\square\)

Corollary 4. A connected Finsler manifold \((M, F)\) is a Belwald manifold if and only if the \(L = 0\) and the \(S\)-curvature \(S = 0\) for the distortion \(\tau\) with respect to certain volume element.

Proof. By Proposition 2, \(L = 0\) and \(S = 0\) imply \(\eta = d\tau\). This corollary follows from Corollary 3. \(\square\)
For Finsler manifold of specially semi-C-reducible Finsler structure, we have the following theorem.

**Theorem 11.** Let \((M, F)\) be a connected Finsler manifold such that the induced Minkowski norms on each tangent spaces are specially semi-C-reducible. Assume that the dimension of \(M\) is \(n \geq 4\). Then \((M, F)\) is a Landsberg manifold if and only if it is a Berwald manifold.

**Proof.** If \((M, F)\) is a Landsberg manifold, then \(\hat{g}\) is horizontally stable everywhere. It follows that the nonlinear parallel transport preserves \(\hat{g}\) along any piecewise smooth curves. By Theorem 3 and the connectedness of \(M\), the nonlinear parallel transport preserves \(\hat{A}\). By Theorem 7, the theorem follows. \(\square\)

**Remark 8.** In the papers [22, 21], the authors have proved that the \((\alpha, \beta)\)-Finsler metrics are semi-C-reducible. But the metrics in [22, 21] may be not regular. We don’t know that whether \((\alpha, \beta)\)-Minkowski norms are regular implies that they are specially semi-C-reducible. If it is true, then Theorem 17 gives another proof of 35 for dimension satisfies \(n \geq 4\).

It is well known that C-reducible regular Minkowski norms must be Randers norms. So it is natural to ask if regular specially semi-C-reducible Minkowski norms are exactly \((\alpha, \beta)\)-norms. Up to now, we don’t know the answer.

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