Non-linear electrodynamics non-minimally coupled to gravity: symmetric-hyperbolicity and causal structure.

Érico Goulart

Federal University of São João d’El Rei, C.A.P. Rod.: MG 443,
KM 7, CEP-36420-000, Ouro Branco, MG, Brazil

Santiago Esteban Perez Bergliaffa

Departamento de Física Teórica, Instituto de Física,
Universidade do Estado de Rio de Janeiro, 20550-013, Rio de Janeiro, Brazil

It is shown here that symmetric hyperbolicity, which guarantees well-posedness, leads to a set of two inequalities for matrices whose elements are determined by a given theory. As a part of the calculation, carried out in a mostly-covariant formalism, the general form for the symmetrizer, valid for a general Lagrangian theory, was obtained. When applied to nonlinear electromagnetism linearly coupled to curvature, the inequalities lead to strong constraints on the relevant quantities, which were illustrated with applications to particular cases. The examples show that non-linearity leads to constraints on the field intensities, and non-minimal coupling imposes restrictions on quantities associated to curvature.

I. INTRODUCTION

Well-posedness of the initial value problem stands out as a basic requirement to be satisfied by any relativistic field theory. Broadly speaking, it implies that solutions for a given problem exist, are unique, and depend continuously on the initial data. Hence, well-posedness is at the roots of physics, for it amounts to the predictability power of a given theory. While it is difficult to decide whether a given non-linear theory has a well-posed initial value problem, a necessary and sufficient condition for well-posedness around a given solution is that all the linearised problems obtained by linearising near such a solution are well-posed, see for instance [1, 2].

In the linearized regime, well-posedness means that the equation of motion is hyperbolic. At least three notions of hyperbolicity can be distinguished [1, 3]: (a) weak hyperbolicity, in which all the roots of the characteristic equation are real, (b) strong hyperbolicity implies that there is an energy estimate that sets a bound for the energy of a solution at a given time in terms of the initial energy, and (c) symmetric hyperbolicity, which is a sufficient condition for well-posedness [4].

1 In fact, strong hyperbolicity is a necessary and sufficient condition for the initial-value problem to be well-posed.
It follows that symmetric hyperbolicity implies strong hyperbolicity, which in turn implies weak hyperbolicity.

There are many examples in which the requirement of some kind of hyperbolicity imposes severe restrictions on the Lagrangian of a given theory. For instance, as shown in [2], the equations of motion of Lovelock’s theory are always weakly hyperbolic for weak fields but not strongly hyperbolic in a generic weak-field background. The well-posedness of Horndeski theory has been analyzed in [2, 5]. As shown in the latter reference, the most general Horndeski theory which is strongly hyperbolic for weak fields in a generalized harmonic gauge is simply \( k\)-essence coupled to Einstein’s gravity (see also [6]). The well-posedness of scalar-tensor effective field theory was studied in [7], where it was shown that the equations of motion are strongly hyperbolic at weak coupling.

We would like to explore here the constraints imposed by the requirement of well-posedness in the case of nonlinear electromagnetic theories coupled to gravity. Nonlinear electromagnetism has been widely studied in several contexts. A non-exhaustive list of applications and references includes black holes [8–12], astrophysics [13–15] and cosmology [16–20]. There are also several articles devoted to different aspects of the propagation of perturbations in nonlinear electromagnetic theories, such as [21–24]. The matter of (symmetric) hyperbolicity for nonlinear electromagnetism minimally coupled to gravity was analyzed in a flat spacetime background in [25], while the hyperbolicity of Maxwell’s equations with a local (and possibly nonlinear) constitutive law in flat spacetime was considered in [26]. Our aim here is to determine the restrictions that follow from the requirement of symmetric hyperbolicity on nonlinear electromagnetism non-minimally coupled to gravity, several aspects of which have been studied in detail in [27, 30]. We shall restrict to couplings linear in the curvature, since higher-order couplings produce higher-than-second order equations for the gravitational field [27].

As mentioned above, a sufficient condition for well-posedness to hold is that the system under study admits a symmetric-hyperbolic representation. The theory of first-order symmetric hyperbolic systems, originally due to Friedrichs [31], has been extensively developed (see for instance [3] and references therein). To study the evolution of a system we shall adopt here the modern geometric approach to the subject outlined by Geroch in [32], in which covariance is kept during most of the calculation, instead of using a 3+1 decomposition of spacetime (as for instance in [26]). We shall see that such approach leads to a general form for the symmetrizer, and to conditions for symmetric hyperbolicity that are easier to evaluate than those for other types of hyperbolicity.

\[^2\] For a list of other approaches to the analysis of the evolution of minimally coupled electromagnetism, see [28].
The structure of the paper is as follows. In Section II the basic notation used in the equations of motion is presented. Symmetric hyperbolicity, the related concept of symmetrizer, and the role of the constraints are analyzed in Section III. The equations defining the characteristic cones will be deduced in Section IV. Our main result, namely the explicit form of the matrices that are needed to investigate the symmetric hyperbolicity of any nonlinear electromagnetic theory non-minimally coupled to gravity can be found in Section V. In Section VI some examples of the restrictions imposed by symmetric hyperbolicity are presented for different theories. Our closing remarks are presented in Section VII.

II. LAGRANGIANS AND EQUATIONS OF MOTION

To begin with, let \( \mathcal{M} \) denote a smooth four-dimensional spacetime with a Lorentzian metric \( g \) of signature \((+,-,-,-)\). For the sake of concreteness we assume \( \mathcal{M} \) to be also oriented and globally hyperbolic i.e. \( \mathcal{M} \cong \mathbb{R} \times \Sigma \), with \( \Sigma \) a codimension-1 hypersurface. Sticking to the conventions \( R_{ab} \equiv R^c_{\ abc} \) and \( [ab] \equiv ab - ba \), the canonical decomposition of the Riemann tensor into its irreducible parts reads

\[
R_{\ ab}^{\ cd} = W_{\ ab}^{\ cd} + \frac{1}{2} \delta_{[c}^{[a} S_{d]}^{b]} + \frac{R}{12} g_{\ ab}^{\ cd},
\]

where \( W_{abcd} \) is the Weyl conformal tensor, \( S_{ab} \) is the traceless part of the Ricci tensor, \( R \) is the scalar curvature and \( g_{abcd} = g_{[a} c d_{b]} \) is the Kulkarni-Nomizu product of the metric with itself. Clearly, each factor in the decomposition has the same algebraic symmetries as the full Riemann tensor, that is

\[
R_{abcd} = -R_{bacd} = -R_{abdc}, \quad R_{abcd} = R_{cdab},
\]

\[
R_{abcd} + R_{acdb} + R_{adbc} = 0.
\]

An arbitrary rank four covariant tensor satisfying Eqs. (2) has 21 independent components and is often referred to as a double symmetric \((2,2)\)-form (the skew pairs can be interchanged). If, in addition, the tensor satisfies Eq. (3), it is called an algebraic curvature tensor (see for instance [33]).

If we are given a double symmetric \((2,2)\)-form on \( \mathcal{M} \), say \( \chi \), we may obtain its left-dual \( \star \chi \) and right-dual \( \chi \star \) by

\[
\star \chi_{abcd} = \frac{1}{2} \epsilon_{ab}^{\ pq} \chi_{pqcd}, \quad \chi_{abcd} \star = \frac{1}{2} \epsilon_{cd}^{\ pq} \chi_{abpq},
\]

\[
\star \chi_{abcd} = \frac{1}{2} g_{\ ab}^{\ pq} \chi_{pqcd}, \quad \chi_{abcd} \star = \frac{1}{2} g_{\ cd}^{\ pq} \chi_{abpq},
\]
where $\varepsilon_{abcd}$ is the Levi-Civita tensor with $\varepsilon_{0123} = \sqrt{-g}$. Notice that the place of the $\star$ indicates the pair of skew indices which are Hodge-dualized. A direct consequence is that

$$\star \star \chi_{abcd} = -\chi_{abcd}, \quad \chi_{abcd} \star \star = -\chi_{abcd}. \quad (5)$$

We aim at constructing Lagrangians describing the coupling of gravity with the electromagnetic field using invariants that are at most linear in the curvature and respecting the $U(1)$ gauge invariance of electromagnetism. Recalling that every algebraic curvature tensor satisfy the Ruse-Lanczos identity, we may construct the following independent rank four tensors (see the Appendix for details)

\begin{align}
(1) \chi^{ab}_{\quad cd} &\equiv g^{ab}_{\quad cd}, \\
(2) \chi^{ab}_{\quad cd} &\equiv \varepsilon^{ab}_{\quad cd}, \\
(3) \chi^{ab}_{\quad cd} &\equiv R g^{ab}_{\quad cd}, \\
(4) \chi^{ab}_{\quad cd} &\equiv R \varepsilon^{ab}_{\quad cd}, \\
(5) \chi^{ab}_{\quad cd} &\equiv W^{ab}_{\quad cd}, \\
(6) \chi^{ab}_{\quad cd} &\equiv \star W^{ab}_{\quad cd}, \\
(7) \chi^{ab}_{\quad cd} &\equiv \delta^{[a}_{[\Gamma} S^{b]}_{\quad cd]}.
\end{align}

Notice that $(\Gamma)\chi_{abcd}$ ($\Gamma = 1, 2, ..., 7$) are double symmetric $(2,2)$-forms by construction and it can be checked that they exhaust the interesting possibilities: the first pair, Eqs. (6), involves only algebraic terms in the metric, while the remaining Eqs. (7)-(9) contain at most linear terms in the curvature.

Let $F_{ab} = \partial_{[a} A_{b]}$ denote a test electromagnetic field propagating on $\mathcal{M}$. We shall focus on the propagation properties of such a field on a fixed gravitational background. The EM field will be described by the gauge-invariant action functional of the form

$$S = \frac{1}{4} \int d^4x \sqrt{-g} \ L(I^1, I^2, ..., I^7), \quad (10)$$

where the factor $1/4$ is introduced for future convenience and the Lagrangian density is taken as an arbitrary smooth function of the following scalars

$$I^\Gamma = (\Gamma) H^{ab} F_{ab}, \quad \text{where} \quad (\Gamma) H^{ab} = \frac{1}{2} (\Gamma) \chi^{ab}_{\quad cd} F^{cd}. \quad (11)$$

\footnote{As discussed in [27], higher-order couplings lead to equations of motion with derivatives of the metric higher than two.}
In analogy with electrodynamics in material media, we call \( (\Gamma) H_{ab} \) the \( \Gamma \)-th \textit{induction tensor} and \( (\Gamma) \chi_{cd}^{ab} \) the \( \Gamma \)-th \textit{constitutive tensor}. Particular instances described by Eq. (10) include Maxwell’s theory, minimally-coupled nonlinear electrodynamics, and the three-parameter non-minimal Einstein–Maxwell model originating from QED vacuum polarization in a background gravitational field (see for instance [34]), among others. With the conventions presented above, the variation of the action with respect to the 4-potential yields a coupled system of first-order quasi-linear PDEs for the fields, given by

\[
\nabla_b H^{ab} = 0, \quad \nabla_b \ast F^{ab} = 0. \tag{12}
\]

Here, \( \nabla_a \) is the covariant derivative compatible with the metric and the \textit{full induction tensor} is defined by

\[
H^{ab} \equiv \sum L_{\Gamma} (\Gamma) H_{ab}, \tag{13}
\]

with \( L_{\Gamma} \equiv \partial L_{\Gamma} / \partial I^I \), for conciseness. Notice that the system defined by Eqs. (12) is composed of eight equations for only six unknowns. In relevant physical situations, the equations must include six dynamical equations (defined with respect to some time coordinate, to be identified later) and two constraints, which are to be imposed on initial data.

### III. SYMMETRIC-HYPERBOLICITY

In order to study whether the equations of motion admit a symmetric-hyperbolic representation, it is convenient to recast the system of equations Eqs. (12) in a unified manner as

\[
K_{\alpha}^m A_{\beta}(x, \Phi) \partial_m \Phi^\beta + J_A(x, \Phi) = 0, \tag{14}
\]

where \( x \in M \), \( K_{\alpha}^m \) is the \textit{principal part} of the PDE and \( J_A(x, \Phi) \) stands for semi-linear contributions (whose explicit form is unnecessary for our discussion). Here capital Latin indices \( (A = 1, \ldots, 8) \) stand for the space of multi-tensorial equations, lowercase Latin indices \( (m = 0, 1, 2, 3) \) stand for space-time indices, and greek indices \( (\beta = 1, \ldots, 6) \) for tensorial unknowns. To start with, we introduce an ordering of the antisymmetric indices to obtain the six possible collective quantities

\[
1 \rightarrow (01) \quad 2 \rightarrow (02) \quad 3 \rightarrow (03) \quad 4 \rightarrow (32) \quad 5 \rightarrow (13) \quad 6 \rightarrow (21). \tag{15}
\]

Making the identification \( \Phi^\beta \rightarrow F^{bc} \) and performing simple manipulations in Eqs. (12), the principal part is then written as

\[
K_{\alpha}^m A_{\beta} = \frac{1}{2} (X_a^m \varepsilon_{abc} \varepsilon_{m}^{abc}), \tag{16}
\]
with

$$X_{abcd} \equiv \sum_{\Gamma} \mathcal{L}_{\Gamma} \, (^{(\Gamma)}X)_{abcd} + 4 \sum_{\Gamma} \sum_{\Lambda} \mathcal{L}_{\Gamma \Lambda} \, (^{(\Gamma)}H_{ab} \, (^{(\Lambda)}H_{cd}).$$

(17)

$X_{abcd}$ consists of a main term involving only first partial derivatives of the Lagrangian density and a nonlinear term including the Hessian matrix of the latter. Clearly, if the Lagrangian is a linear combination of the invariants $I^{\Gamma}$, the last term vanishes and linear equations of motion follow. More importantly, $X_{abcd}$ is always a symmetric double (2,2)-form independently on the specific form of the Lagrangian. This is a direct consequence of the symmetries of the constitutive tensors together with the symmetry of the Hessian matrix, i.e. $\mathcal{L}_{\Gamma \Lambda} = \mathcal{L}_{\Lambda \Gamma}$.

In what follows we shall use the covariant approach for first-order symmetric-hyperbolic systems outlined in [32]: a symmetric hyperbolization of Eqs. (14) means that there exist a smooth symmetrizer $h^{A}_{\alpha}$ and a covector field $n_{m}$, such that:

1. $\hat{K}_{\alpha \beta}^{m} \equiv h^{A}_{\alpha} K_{A \beta}^{m}$ is symmetric in the indices $\alpha, \beta$.

2. The matrix $\tilde{K}_{\alpha \beta}(n) \equiv \hat{K}_{\alpha \beta}^{m} n_{m}$ is positive-definite.

Roughly speaking, the first statement means that it should be possible to construct from Eqs. (14) a new subsystem of first-order quasi-linear PDEs given by

$$\hat{K}_{\alpha \beta}^{m} \partial_{m} \Phi^{\beta} + \hat{J}_{\alpha} = 0$$

(18)

with $\hat{J}_{\alpha} \equiv h^{A}_{\alpha} J_{A}$. Clearly, the new system contains only evolution equations and its dimension is equivalent to the number of unknown fields. The second statement means that the new system can be solved uniquely for any given set of initial conditions on a hypersurface $\Sigma$ with normal covector $n_{m}$. What are the remaining equations the symmetrizer does not capture? For the system to be consistent they should not be of the evolution type. In other words, they must be satisfied automatically once they are satisfied initially, i.e., they should be what we normally call the constraints.

The geometrical meaning of the covectors introduced in the previous paragraph is as follows. At a spacetime point $p \in \mathcal{M}$, the collection of all covectors satisfying condition (2) is denoted by

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4 See references [3, 32, 35] for additional details.
This set defines a nonempty open, convex cone at \( p \) and the tangent vectors \( p^a \in T_p M \) such that \( p^a n_a > 0 \) for all \( n_a \in S_p \) determine the cone of influence of the physical field, i.e., the maximal speed of propagation in any given direction. It turns out that the latter is also a nonempty open, convex cone at \( p \).

In the next subsection, we start by showing that a family of symmetrizers parametrized by a vector field always exists for the system of first order PDEs given by Eqs. (12). It is important to point out that our result is general in the sense that it does not depend on the specific form of the Lagrangian density. We then obtain the conditions that a theory must satisfy for positive-definiteness (see condition (2) above) to hold. This is first obtained for a particular covector and henceforth generalized by inspecting the characteristic varieties (dispersion relations), which are necessarily well-behaved for symmetric hyperbolic systems.

A. Symmetrizer

In order to find a symmetrizer, it is convenient to work with projections. In other words, we seek a multi-tensorial field \( h^A_a \) such that the quantity \( \delta \phi^\alpha (h^A_a K^m_{\quad \beta}) \delta \psi^\beta \) is symmetric in \( \delta \phi \) and \( \delta \psi \). Making the identifications

\[
\delta \phi^\alpha \rightarrow A^{ab}, \quad \delta \psi^\alpha \rightarrow B^{ab},
\]

where \( A \) and \( B \) are generic bivectors, we obtain from Eq. (16), the relation

\[
K^m_{\quad \beta} \delta \psi^\beta = \left( \frac{1}{2} X^m_{a b c} B^{b c} + \star B^m_a \right) t^q.
\]

Since there is no known practical procedure to obtain a symmetrizer for an arbitrary system of PDE’s, hyperbolizations are found, for a sufficiently low number of dimensions, by solving explicitly the algebraic equations inherent to the system defined by Eq. (14) and, in higher dimensions, by guessing. Let us show that the symmetrizer is given by the projection

\[
\delta \phi^\alpha h^A_a \equiv \left( A^a_q, \frac{1}{2} \star X^a_{q r s} A^r s \right) t^q
\]

where \( t^q \) is an auxiliary vector field and \( \star X^a_{q r s} \) is the left Hodge dual as defined before. Indeed, multiplying (21) by (20) one obtains

\[
\delta \phi^\alpha (h^A_a K^m_{\quad \beta}) \delta \psi^\beta = \frac{1}{2} \left( X^m_{a b c} A^a_q B^{b c} + \star X^a_{q r s} A^r s \star B^m_a \right) t^q
\]

Now, defining

\[
Y^a_q \equiv X^a_{q r s} A^r s, \quad \star Y^a_q \equiv \star X^a_{q r s} A^r s,
\]

(23)
and using the well known identity valid for anti-symmetric tensors
\[(\star Y^{aq})(\star B_{am}) = -\frac{1}{2}(Y_{ln}B^{ln})\delta^q_m + Y_{am}B^{aq},\] (24)

it follows that
\[
\delta\phi^\alpha(h^A_\alpha K^m_A\beta)\delta\psi^\beta = \frac{1}{2}\left[ X^m_a_{bc}(A^a_q B^{bc} + A^{bc}B^a_q) - \frac{1}{2}(X_{abcd}B^{ab}A^{cd})\delta^m_q \right]t^q,
\] (25)

which is obviously symmetric in $A$ and $B$ since $X_{abcd} = X_{cdab}$. Therefore, the symmetrizer itself is given by the simple expression
\[
h^A_\alpha = \frac{1}{2}(g^a_{qrz}, \star X^a_{qrz})t^q,
\] (26)

It is remarkable that Eq. (26) is valid independently of the specific content of the tensor field $X_{abcd}$: in particular it does not matter whether the equations of motion contain quasi-linear terms or not.

The application of Eq. (26) to Eq. (16), yields the object
\[\hat{K}^m_{\alpha \beta} = \frac{1}{4}(X^m_a_{bc} + \star X^a_{qrz}\epsilon^m_a_{bc})t^q,
\] (27)

which, after straightforward algebraic manipulations, becomes
\[\hat{K}_m^\alpha \beta = -\frac{1}{4}\left( g_{q[a}X^m_{b]_{cd}} + g_{q[c}X^m_{d]_{ab}} + \delta^m_q X_{abcd} \right) t^q.
\] (28)

Notice that this equation is indeed symmetric in the exchange of antisymmetric indices $ab \Leftrightarrow cd$ and that the auxiliary vector field $t^q$ remains (up to now) arbitrary, so we can use it at our disposal. This concludes the first step of our task.

**B. Positive definiteness**

Let us next investigate whether the above symmetrizer constitutes a true hyperbolization of the equations of motion. This will be achieved if we manage to find a covector field $n_m$ such that the characteristic matrix
\[\hat{K}_{\alpha \beta}(t, n) = -\frac{1}{4}[t_{[a}X^m_{b]_{cd}} + t_{[c}X^m_{d]_{ab}} + t^m X_{abcd}]n_m
\] (29)

is positive definite, i.e.
\[\delta\phi^\alpha \hat{K}_{\alpha \beta}\delta\phi^\beta > 0.
\] (30)
for all nonzero vectors $\delta \phi^\alpha \in \mathbb{R}^6$. A direct calculation gives the equivalent inequality

$$-\left[t_a X_b m_{cd} A^{ab} A^{cd} + \frac{1}{4} t^m X_{abcd} A^{ab} A^{cd}\right] n_m > 0,$$

where $A^{ab}$ is an arbitrary, but nonzero bivector. Notice that the latter is a fully covariant expression in the sense that it does not depend on any particular choice of coordinates. It will restrict, however, the choices of admissible covector and auxiliary vector fields.

In order to proceed, we shall decompose $A^{ab}$ and $X_{abcd}$ with respect to the auxiliary vector field $t^q$. To do so, assuming that $t^q$ is timelike, future-directed and normalized, we write the bivector as

$$A_{ab} = a^a[t_b] + \varepsilon_{ab}^{cd} t_c t_d,$$  

with

$$a_a \equiv A_{ab} t^b, \quad b_a \equiv *A_{ab} t^b, \quad a_a t^a = 0, \quad b_a t^a = 0.$$  

Similarly, for any double $(2, 2)$-form, we have

$$X_{abcd} = \{g_{abpq}(g_{cdrs} P^{pr} + \varepsilon_{cdrs} Q^{pr}) + \varepsilon_{abpq}(g_{cdrs} R^{pr} + \varepsilon_{cdrs} S^{pr})\} t^q t^s,$$

with the $2$-index tensors given by

$$P_{ab} \equiv X_{acbd} t^c t^d, \quad Q_{ab} \equiv -X_{acbd} * t^c t^d, \quad R_{ab} \equiv -*X_{acbd} t^c t^d, \quad S_{ab} \equiv *X_{acbd} * t^c t^d,$$

and orthogonal to $t^q$ by construction. In general, each of the latter has 9 independent components since the number of independent components of $X_{abcd}$ is 36. However, in the particular case in which $X_{abcd} = X_{cdab}$, the following simplified relations are valid:

$$P_{ab} = P_{ba}, \quad S_{ab} = S_{ba}, \quad Q_{ab} = R_{ba},$$

so that $P_{ab}$ and $S_{ab}$ have 6 independent components each, while $Q_{ab}$ (or, equivalently, $R_{ab}$) have the remaining 9 components. Notice that, in the context of electrodynamics in material media, these tensors would be related to the permittivity, permeability and magneto-electric cross terms of the medium.

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5 See for instance [36] for similar decompositions.
Using the above decompositions in Eq. (31), the new covariant inequality
\[
(n_m t^m)[P_{pr} a^p a^r - S_{pr} b^p b^r] > 2(n_m t^m)\varepsilon_{npq} [R_{qr} a^p a^r + S_{qr} a^p b^r]
\] (37)
follows, which is to be satisfied for any 3-vectors \(a^p\) and \(b^q\), not vanishing simultaneously. Furthermore, since our considerations are essentially algebraic we may restrict to an arbitrary point \(p \in M\). In order to complete our symmetric hyperbolization, it suffices to find a specific covector \(n_a \in T_p^* M\) satisfying inequality (37) since, if this is the case, there will be a unique connected, open and convex cone, \(S_p\), containing the initial covector, and this cone will exhaust all possibilities. The natural choice here is the covector \(n_a = g_{ab} t^b\), since the right-hand-side of Eq. (37) will vanish. Now, since \(n_m t^m > 0\) for the latter, we obtain the equivalent inequalities
\[
P_{pr} a^p a^r > 0, \quad S_{pr} b^p b^r < 0,
\]
for all nonzero vectors \(a^p, b^q\). In words, a symmetric hyperbolization is achieved for a covector \(n_a\) if the 2-index tensors \(P_{ab}\) and \(S_{ab}\), obtained from \(X_{abcd}\) and \(*X_{abcd}\) via contractions with \(n^a\), satisfy the above inequalities.

For the sake of completeness, let us repeat the calculations in an adapted frame at \(p\), such that
\[
g_{ab}(p) = \delta_{ab}, \quad t^q = \delta^q_0.
\] (38)
Using Eq. (34), it can be shown that \(X_{abcd}\) has the following block matrix display:
\[
X_{\alpha\beta} = \begin{pmatrix} P & Q \\ Q^T & S \end{pmatrix},
\] (39)
where \(P, Q\) and \(S\) denote the covariant \(3 \times 3\) matrices constructed with the corresponding tensors in the obvious way. In order to compute Eq. (29) in matrix form it is convenient to define the auxiliary \(3 \times 3\) matrices (a similar notation was used in [26])
\[
A^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

Taking into account that the transposition relation \((A^k)^T = -A^k\) holds, a direct calculation gives:
\[
\hat{K}_{\alpha\beta}(n) = \begin{pmatrix} n_0 P - n_k (Q A^k - A^k Q^T) & n_k A^k S \\ -n_k S A^k & -n_0 S \end{pmatrix}.
\] (40)
The above relation may be thought as a linear map from $T^*_x\mathcal{M}$ to the 21-dimensional space of symmetric $6 \times 6$ matrices $\text{Sym}_6$. The set of all symmetric positive definite matrices forms an open convex cone in $\text{Sym}_6$ with apex on the origin. It turns out that the image of the particular covector $n_m = \eta_{mn}t^n$ will lie inside this cone whenever

$$\mathbb{P} > 0, \quad \mathbb{S} < 0.$$ (41)

since $n_m = (1, 0, 0, 0)$ in our frame. Therefore, if Eqs. (41) are satisfied pointwisely, symmetric hyperbolicity is guaranteed for the corresponding covector field. In other words, if the auxiliary vector field $t^a$ is vorticity-free, then initial data given on a hypersurface $\Sigma$ with normal covectors $t_a$ are uniquely evolved away from the hypersurface. As will be shown in Sect. VI, the requirements set by Eqs. (41) may yield severe constraints on the Lagrangian density, the intensity of the curvature tensor or the electromagnetic field.

**C. Constraints**

We now discuss the remaining equations, which are left aside by the symmetrizer. To do so, we recall that, according to Geroch’s formalism, a constraint is a tensor $c^A_n$ such that

$$c^A_n K^m_A \alpha + c^A_m K^n_A \alpha = 0.$$ (42)

It is straightforward to show that, in our case, this tensor is given by

$$c^A_n = (xg^{an}, yg^{an})$$ (43)

where $x, y \in \mathbb{R}$. Indeed, multiplying Eq. (43) by (16) one obtains

$$c^A_n K^m_A \alpha = \frac{1}{2}(xX^{nm} + y\varepsilon^{nm})$$ (44)

which is obviously antisymmetric in $n$ and $m$. Here, the emergence of two real numbers reveals that the vector space of constraints is actually 2-dimensional, as expected.

From the above calculation one concludes that the constraints are complete, in the sense that the number of constraint equations plus the number of evolution equations equals the number of initial equations. The constraints are integrable if the equation

$$c^A_n \nabla_n (K^m_A \beta \nabla_m \Phi^\beta + J_A) = 0,$$ (45)

is identically satisfied solely due the algebraic structure of the principal symbol, independently of the equations (14) of the original system. We leave for the reader to verify that this is indeed the case. This means that the original system of equations is equivalent to a symmetric-hyperbolic one with two additional integrable constraints.
IV. CHARACTERISTIC CONES

Suppose that we manage to find a symmetric hyperbolization with constraints, as described in the previous sections. We have seen that this choice, however, is far from unique. Indeed, by the continuity of Eq. (40), any small deformation of \( t_a \) will be such that condition (2) of symmetric hyperbolization is satisfied. It turns out that the set of all admissible covectors \( S_p \) in \( T_p^*M \) is determined by the unique connected, open, convex, positive cone containing the initial covector \( \gamma \). Its existence is related to the hyperbolicity of the characteristic polynomial, defined by

\[
p(n) \equiv \det \left( \hat{K}_{\alpha\beta}(n) \right)
\]

which is a homogeneous multivariate polynomial of degree 6 in our case. Recall that, at a spacetime point, such a polynomial is called hyperbolic in a direction \( t_a \) if \( p(t_a) > 0 \) and the univariate polynomial \( p(u_a + \lambda t_a) \) only has real roots for all covectors \( u_a \neq t_a \). Now, the vanishing set of the characteristic polynomial will define an algebraic variety: the cone of characteristic conormals (or characteristic cone, for brevity). In general, it will consist of different codimension 1 sheets which may be nested, intersect along lines, or even coincide. Geometrically, hyperbolicity in the direction of \( t_a \) is the requirement that every line parallel to \( t_a \) intersects this algebraic variety at exactly “6” points (counting multiplicities). Clearly, this condition severely constrains the topology of the characteristic cones, thus guaranteeing well-behaved propagation for small wavy excitations. In particular, it was shown by Garding [38] that the closure of the connected component of \( t_a \) in the set \( \{n_a|p(n_a) \neq 0\} \) is necessarily convex: the hyperbolicity cone of the polynomial. That symmetric hyperbolicity implies the hyperbolicity of the characteristic polynomial is direct. To see this one simply observes that the equation

\[
\det(\hat{K}_{\alpha\beta}(a + \lambda t)) = 0
\]

characterizes the eigenvalues of the quadratic form \( \hat{K}_{\alpha\beta}(a_m) \) relative to the metric \( \hat{K}_{\alpha\beta}(t_m) \) - and these eigenvalues have to be real (see [3] for further details).

In order to compute the characteristic polynomial explicitly, we substitute Eq. (40) into Eq. (46) and use Schur’s determinant identity to obtain the product of determinants

\[
p(n) = -\det(S)q(n),
\]

where

\[
q(n) \equiv \det \left( n_0^2P - n_0n_k(QA^k - A^kQ^T) - n_0n_lA^kSA^l \right).
\]
Notice that \( q(t) \) is necessarily positive definite, since \( P > 0 \) and \( S < 0 \) for this particular covector. Interestingly, the determinant in Eq. \((49)\) has been calculated several times in literature, see e.g. \([20,39]\). In particular, it can be shown that it factorizes as

\[
q(n) = (n_m t^n)^2 P(n),
\]

where

\[
P(n) = \frac{1}{24} \varepsilon_{a_1 a_2 a_3 a_4} \varepsilon_{b_1 b_2 b_3 b_4} X^{a_1 a_2 b_1 c_1} X^{a_2 a_3 b_2 c_2} X^{a_3 a_4 b_3 c_3} X^{a_4 a_1 b_4 c_4} n_{c_1} n_{c_2} n_{c_3} n_{c_4}.
\]

In other words, the sixth order polynomial always reduces to a product of a quadratic polynomial and a quartic polynomial. Since the vanishing set of the quadratic polynomial gives a non-compact variety (plane) inconsistent with the constraint equations, the characteristic cone is given by the covectors \( k_a \) that satisfy the fourth-order \textit{Fresnel equation}

\[
P(k) \sim \left( X_{pq}^{ap} X^{bpcs} X^{dq}_{rs} \right) k_a k_b k_c k_d = 0,
\]

The fourth rank tensor defined by the terms between parentheses is called the Kummer tensor, whereas its totally symmetrized version is usually called the Tamm-Rubilar tensor. If symmetric hyperbolicity holds, the above polynomial is necessarily hyperbolic in the direction of \( t_a \), its vanishing set determining the causal structure of the theory up to a conformal factor. In specific situations, where \( X_{abcd} \) is sufficiently simple, Eq. \((52)\) will reduce to the more familiar product of quadratic polynomials \(\textit{\textsuperscript{6}}\)

\[
P(k) \sim \left( g_{(1)}^{ab} k_a k_b \right) \left( g_{(2)}^{cd} k_c k_d \right) = 0.
\]

In these cases, symmetric hyperbolicity guarantees that the rank-2 contravariant tensors \( g_{(1)}^{ab} \) and \( g_{(2)}^{ab} \) are nondegenerate, necessarily of Lorentzian type and with the same signature. Furthermore, if these tensors coincide \(\textit{\textsuperscript{7}}\), then only one of them must be considered in the dispersion relation (reduced polynomial) , i.e.,

\[
P_{\text{red}}(k) \sim g_{(1)}^{ab} k_a k_b = 0.
\]

It is important to point out that the abovementioned result stating that symmetric hyperbolicity leads to a cone strongly constraints the possible shape of the Fresnel surfaces, defined in a convenient 3-space: they must be topologically equivalent to those obtained from the 4-d hyperbolicity cone. Hence, open surfaces are not allowed by symmetric-hyperbolic propagation \(\textit{\textsuperscript{8}}\).

\textit{\textsuperscript{6}} A general \textit{Ansatz} leading to bi-metricity in nonlinear electromagnetism was presented in \([40]\).

\textit{\textsuperscript{7}} Covariant conditions on the Fresnel surface for birefringence to be absent were derived in \([41]\).

\textit{\textsuperscript{8}} For examples of Fresnel surfaces in Maxwell’s theory non-minimally coupled to gravity, see \([42]\).
V. GENERAL RESULT

We have seen that symmetric hyperbolicity imposes restrictions on admissible physical theories. Importantly, they are always expressed in terms of matrix inequalities which are much easier to guarantee than to check the hyperbolicity of the corresponding characteristic polynomial. In order to obtain explicit expressions for the matrix inequalities, it is convenient to introduce the projection tensor

\[ h_{ab} = g_{ab} - t_a t_b, \]

i.e.,

\[ h_{ab} = h_{ba}, \quad h_{ac} h^c_b = h_{ab}, \quad h_{ab} t^b = 0. \]  

(55)

Similarly, we decompose the electromagnetic 2-form, the Weyl tensor and the traceless part of the Ricci tensor as

\[ F_{ab} = E_{[a t b]} + \varepsilon_{abcd} t^c B^d, \]  

(56)

\[ W_{abcd} = \{ g_{abpq} (g_{cdrs} E^{pr} - \varepsilon_{cdrs} B^{pr}) - \varepsilon_{abpq} (g_{cdrs} B^{pr} + \varepsilon_{cdrs} E^{pr}) \} t^q t^s, \]  

(57)

\[ S_{ab} = S t_a t_b + Q_{(a t b)} + N_{ab}, \]  

(58)

with the following definitions

\[ E_a \equiv F_{ab} t^b, \quad B_a \equiv \star F_{ab} t^b, \]  

(59)

\[ \mathcal{E}_{ab} \equiv W_{abcd} t^c t^d, \quad \mathcal{B}_{ab} \equiv \star W_{abcd} t^c t^d, \]  

(60)

\[ S \equiv S_{ab} t^a t^b, \quad Q_a \equiv h^b_a S_{bc} t^c, \quad N_{ab} \equiv h^c_a h^d_b S_{cd}. \]  

(61)

According to the latter, the tensor fields \{E_a, B_a, Q_a, \mathcal{E}_{ab}, \mathcal{B}_{ab}, N_{ab}\} are automatically orthogonal to the auxiliary vector field.

In order to compute \( P_{ab} \) and \( S_{ab} \) from Eq. (17), the following relations involving the constitutive tensors are useful

\[ (1) \chi_{abcd} t^b t^d = h_{ac}, \quad (2) \chi_{abcd} t^b t^d = 0, \]  

(62)

\[ (3) \chi_{abcd} t^b t^d = R h_{ac}, \quad (4) \chi_{abcd} t^b t^d = 0, \]  

(63)
Similarly, for the induction tensors, Eq. (11), one obtains

\begin{align*}
(1) \ H_{ab}^t b^b &= E_a, \\
(2) \ H_{ab}^t b^b &= B_a, \\
(3) \ H_{ab}^t b^b &= R E_a, \\
(4) \ H_{ab}^t b^b &= R B_a, \\
(5) \ H_{ab}^t b^b &= \varepsilon_m a^b E^e - B_{ab} B^b, \\
(6) \ H_{ab}^t b^b &= \varepsilon_m a^b B^e + B_{ab} E^b, \\
(7) \ H_{ab}^t b^b &= S E_a + N_{ab} E^b - \varepsilon_n a b^b Q_{p q r} E^e, \\
\end{align*}

Using the above relations and the Ruse-Lanczos identities (see the Appendix A), it can be checked that the tensors \( P_{ab} \) and \( S_{ab} \) are given by:

\begin{align*}
P_{ab} &= (L_1 + L_3 R) h_{ab} + L_5 E_{ab} + L_6 B_{ab} + L_7 N_{ab} \\
&+ 4 E_a E_b (L_{11} + 2 R L_{13}) \\
&+ 4 E_n (a B_b) [L_{12} + R (L_{14} + L_{23})] \\
&+ 4 B_a B_b (L_{22} + 2 R L_{24}) \\
&+ 4 [L_{15} E_n (a + L_{25} B_{(a)} [E_{b(c)} E^c - B_{b(c)} B^c] \\
&+ 4 [L_{16} E_n (a + L_{26} B_{(a)} [E_{b(c)} B^c + B_{b(c)} E^c] \\
&+ 4 [L_{17} E_n (a + L_{27} B_{(a)} [S E_b + N_{b(c)} E^c - \varepsilon_n b) p q r Q_{p q r} E^e] + ...
\[ S_{ab} = -\left( \mathcal{L}_1 + \mathcal{L}_3 R \right) h_{ab} - \mathcal{L}_5 \mathcal{E}_{ab} - \mathcal{L}_6 \mathcal{B}_{ab} + \mathcal{L}_7 N_{ab} \]  
\[ + 4 B_a B_b (\mathcal{L}_{11} + 2 R \mathcal{L}_{13}) \]
\[ - 4 E_{(a} B_{b)} [\mathcal{L}_{12} + R (\mathcal{L}_{14} + \mathcal{L}_{23})] \]
\[ + 4 E_a E_b (\mathcal{L}_{22} + 2 R \mathcal{L}_{24}) \]
\[ + 4 [\mathcal{L}_{15} B_{(a} - \mathcal{L}_{25} E_{(a]} [\mathcal{E}_{b)} c B^c + \mathcal{B}_{b)} c E^c] \]
\[ - 4 [\mathcal{L}_{16} B_{(a} - \mathcal{L}_{26} E_{(a]} [\mathcal{E}_{b)} c E^c - \mathcal{B}_{b)} c B^c] \]
\[ - 4 [\mathcal{L}_{17} B_{(a} - \mathcal{L}_{27} E_{(a]} [S B_{b)} + N_{b)} c B^c - \varepsilon_{b)pqr} Q^p \varepsilon^q B^r] + ... \]

where the dots stand for possible nonlinear terms in the irreducible parts of the curvature tensor, which we shall not discuss in this work. The tensors \( P_{ab} \) and \( S_{ab} \) given above are the most general ones when dealing with symmetric hyperbolicity in models of nonlinear electromagnetism coupled linearly to curvature. The inequalities given in Eqs. (41), defined in terms of such tensors, lead to drastic restrictions on admissible theories: there must be a strong compromise between the electromagnetic quantities \( \{ E^a, B^a \} \), the spacetime curvature expressed via \( \{ E_{ab}, B_{ab}, Q_a, N_{ab} \} \) and the partial derivatives of the Lagrangian density.

\section{VI. APPLICATIONS}

Let us present next several examples that illustrate the restrictions obtained so far. For the sake of completeness, we display, in each case, the tensor \( X_{abcd} \), the relevant inequalities and the corresponding characteristic polynomial. A generic feature of the latter is non-factorization: in general, the characteristic varieties are described by vanishing sets of fourth-order polynomials which do not split into products of second-order polynomials. Needless to say, this fact makes it considerably difficult to guarantee hyperbolicity straight from the characteristic polynomial. In constrast with this fact, symmetric-hyperbolicity automatically implies that the characteristic varieties are necessarily well-behaved.
A. Maxwell electrodynamics $\mathcal{L} = -I^1$

Our first example is Maxwell’s linear theory, which is the simplest possible case in our setting. Using Eq. (17), it follows that

$X_{abcd} = -g_{abcd}$.  \hfill (72)

Taking the auxiliary vector field $t^a$ as timelike, future-directed and normalized, Eqs. (70) and (71) give the 2-index tensors as

$P_{ab} = -h_{ab}, \quad S_{ab} = h_{ab}$.  \hfill (73)

In a frame such that $g_{ab} = \eta_{ab}$ and $t^a = \delta^a_0$, the corresponding $3 \times 3$ matrices are

$P_{ij} = \delta_{ij}, \quad S_{ij} = -\delta_{ij}, \quad i, j = 1, 2, 3$.  \hfill (74)

Since they satisfy Eqs. (11) identically, symmetric-hyperbolicity is guaranteed for all timelike covector fields $n_m = \eta_{mn}t^n$. A direct calculation using Eq. 52 leads to

$P(k) \sim (g^{ab}k_ak_b)^2$.  \hfill (75)

Only one of the quadratic polynomials is needed to obtain the usual dispersion relation for linear electromagnetic waves in vacuum:

$P_{red}(k) \sim g^{ab}k_ak_b = 0$.  \hfill (75)

Clearly, the cone of hyperbolicity at $p$, $S_p$, is the connected component of $t_a$ i.e., it coincides with the set of future-directed timelike covectors, as expected.

B. Minimally coupled nonlinear electrodynamics

1. Single invariant: $\mathcal{L} = \mathcal{L}(I^1)$

If the Lagrangian density is an arbitrary function of the invariant $I^1$, a direct inspection of Eq. (17) gives

$X_{abcd} = \mathcal{L}_1 g_{abcd} + 4\mathcal{L}_{11} F_{ab} F_{cd}$.  \hfill (76)

From Eqs. (70) and (71), we obtain the projections

$P_{ab} = \mathcal{L}_1 h_{ab} + 4\mathcal{L}_{11} E_a E_b, \quad S_{ab} = -\mathcal{L}_1 h_{ab} + 4\mathcal{L}_{11} B_a B_b$.  \hfill (77)
Simple manipulations using Eq. (41) then yield the matrix inequalities

\[ \mathcal{L}_1 \delta_{ij} - 4 \mathcal{L}_{11} E_i E_j < 0, \]  
\[ \mathcal{L}_1 \delta_{ij} + 4 \mathcal{L}_{11} B_i B_j < 0. \]  

(78)  

(79)

In particular, they imply the condition \( \mathcal{L}_1 < 0 \), since the theory should be well-behaved when either the electric or magnetic field vanishes. However, notice that for field intensities violating the inequalities, the propagation of small wavy excitations about the background field will be, in general, ill-posed.

A straightforward calculation using Eq. (52) gives the well-known bi-metric dispersion relation

\[ P(k) \sim \begin{pmatrix} g_{ab}^{(1)} k_a k_b \\ g_{cd}^{(2)} k_c k_d \end{pmatrix}, \]  

(80)

with the effective metrics given by

\[ g_{(1)}^{ab} \equiv \mathcal{L}_1^2 g^{ab}, \quad \quad g_{(2)}^{ab} \equiv -(\mathcal{L}_1 g^{ab} + 4 \mathcal{L}_{11} \tau^{ab}), \quad \quad \tau^{ab} \equiv F^{ac} F^{b}_{\phantom{b}c}, \]  

(81)

as was shown for instance in [42]. Needless to say, if Eqs. (78) and (79) hold, the effective metrics are well-behaved, and both possess the same Lorentzian signature. However, it should be clear from our discussion that symmetric-hyperbolicity requires more than the simple Lorentzian nature of the effective metrics.

2. Two invariants: \( \mathcal{L}(I^1, I^2) \)

The case of two invariants is naturally more involved. The symmetric double \((2,2)\) form becomes

\[ X_{abcd} = \mathcal{L}_1 g_{abcd} + 4 \{ \mathcal{L}_{11} F_{ab} F_{cd} + \mathcal{L}_{12} (F_{ab} \star F_{cd} + \star F_{ab} F_{cd}) + \mathcal{L}_{22} \star F_{ab} \star F_{cd} \}. \]  

(82)

Apart from minor details on definitions, this tensor coincides with the so-called jump tensor obtained by Obukhov and Rubilar in [43]. Notice also that the term involving \( \mathcal{L}_2 \) in Eq. (17) was discarded, since it is proportional to the Bianchi identity. We then obtain the tensors (see Eqs. (70) and (71))

\[ \mathbb{P}_{ab} = \mathcal{L}_1 h_{ab} + 4 \{ \mathcal{L}_{11} E_a E_b + \mathcal{L}_{12} (E_a B_b + B_a E_b) + \mathcal{L}_{22} B_a B_b \}. \]  

(83)
\[ S_{ab} = -\mathcal{L}_1 h_{ab} + 4\{\mathcal{L}_{11} B_a B_b - \mathcal{L}_{12} (E_a B_b + B_a E_b) + \mathcal{L}_{22} E_a E_b\}, \]  

(84)

which, in the frame described above, lead to the inequalities

\[ \mathcal{L}_1 \delta_{ij} - 4\{\mathcal{L}_{11} E_i E_j + \mathcal{L}_{12} (E_i B_j + B_i E_j) + \mathcal{L}_{22} B_i B_j\} \prec 0, \]  

(85)

\[ \mathcal{L}_1 \delta_{ij} + 4\{\mathcal{L}_{11} B_i B_j - \mathcal{L}_{12} (E_i B_j + B_i E_j) + \mathcal{L}_{22} E_i E_j\} \prec 0, \]  

(86)

A tedious calculation using Eq. (52) yields the effective metrics

\[ g^{ab}_{(1)} = \mathcal{X} g^{ab} + (\mathcal{Y} + \sqrt{\mathcal{Y}^2 - \mathcal{X} \mathcal{Z}}) t^{ab}, \]  

(87)

\[ g^{ab}_{(2)} = \mathcal{X} g^{ab} + (\mathcal{Y} - \sqrt{\mathcal{Y}^2 - \mathcal{X} \mathcal{Z}}) t^{ab}, \]  

(88)

with

\[ \mathcal{X} = \mathcal{L}_1^2 + 2\mathcal{L}_1 (G - \mathcal{L}_{22} F) + (\mathcal{L}_{12} \mathcal{L}_{12} - \mathcal{L}_{11} \mathcal{L}_{22}) G^2, \]

\[ \mathcal{Y} = 2\mathcal{L}_1 (\mathcal{L}_{11} + \mathcal{L}_{22}) + 4(\mathcal{L}_{12} \mathcal{L}_{12} - \mathcal{L}_{11} \mathcal{L}_{22}) F, \]

\[ \mathcal{Z} = (\mathcal{L}_{11} \mathcal{L}_{22} - \mathcal{L}_{12} \mathcal{L}_{12}), \]

and we have used the notation \( I^1 \equiv F, I^2 \equiv G, t^{ab} \equiv F^{ac} F^b_c \) for conciseness. This result coincides with \[43, 44\], with minor modifications of notation.

C. Nonminimally coupled nonlinear electrodynamics.

Since the equations of motion that follow from Eq. (10) are very general, it is convenient to consider some particular cases. Hence, in what follows we assume that the Lagrangian density takes the form

\[ \mathcal{L}(I^1, I^2) + \alpha I^3 + \beta I^5 + \gamma I^7, \]  

(89)

where \( \mathcal{L}(I^1, I^2) \) is an arbitrary function of the usual electromagnetic invariants, and \( \alpha, \beta \) and \( \gamma \) are phenomenological parameters to be determined in principle from experiments. Such a density is still sufficiently general to include most of the relevant important models present in literature.

Let us examine next some particular cases that follow from Eq. (89).
Linear non-minimal model

The corresponding Lagrangian density is given by

\[ \mathcal{L} = -I^1 + \alpha I^3 + \beta I^5 + \gamma I^7, \]  

(90)
since we expect to recover Maxwell’s theory in the flat spacetime regime. This model describes several possible non-minimal couplings of linear electrodynamics with gravity, and encompasses for instance the modifications of Maxwell’s theory due to one-loop vacuum polarization contributions [34]. Clearly, the corresponding equations of motion are linear and, using Eq. (17), one obtains the double symmetric \((2,2)\)-form as

\[ X_{abcd} = (\alpha R - 1)g_{abcd} + \beta W_{abcd} + \gamma (g_{ac}S_{bd} - g_{ad}S_{bc} + g_{bd}S_{ac} - g_{bc}S_{ad}). \]  

(91)

Notice that the irreducible parts of the Riemann tensor contribute in different ways to the equations of motion. Using the above prescription in Eqs. (70) and (71), we obtain the following 2-index tensors

\[ P_{ab} = (\alpha R - 1)h_{ab} + \beta \mathcal{E}_{ab} + \gamma N_{ab} \]  

(92)

\[ S_{ab} = (1 - \alpha R)h_{ab} - \beta \mathcal{E}_{ab} + \gamma N_{ab} \]  

(93)

Let us consider next several sub-cases:

a. Scalar curvature coupling \((\alpha \neq 0, \beta = 0, \gamma = 0)\).

This is by far the simplest type of non-minimal coupling. Indeed, since the double symmetric \((2,2)\)-form in Eq. (91) reduces to

\[ X_{abcd} = (\alpha R - 1)g_{abcd}, \]  

(94)

the relevant 2-index tensors read

\[ P_{ab} = (\alpha R - 1)h_{ab}, \quad S_{ab} = (1 - \alpha R)h_{ab}. \]  

(95)

Hence, the simple matrix inequality

\[ (1 - \alpha R)\delta_{ij} > 0 \]  

(96)
follows. Clearly, symmetric-hyperbolicity requires that $\alpha < 1/R$, which may forbid good propagation for sufficiently high curvature, for a given $\alpha$. Regarding the characteristic cone, Eq. (52) gives

$$P(k) \sim (g^{ab}k_ak_b)^2,$$

(97)

which shows that in this case the dispersion relation is governed by the background metric i.e. the causal structure is not changed by the coupling.

b. Weyl coupling ($\alpha = 0, \beta \neq 0, \gamma = 0$).

It follows from Eqs. (92) and (93) that

$$P_{ab} = -S_{ab} = -h_{ab} + \beta e_{ab}.$$

Although the ensuing inequalities need to be examined on a case by case basis, they will lead to limits on the components of the electric part of the Weyl tensor.

In order to compute the dispersion relation explicitly, we first recall that the Weyl conformal tensor has only one independent Hodge dual i.e. $\ast W_{abcd} = W_{abcd} \ast$. A direct calculation using Eq. (52) gives a quartic equation of the type

$$P(k) \sim G_{abcd} k_ak_bk_ck_d,$$

(98)

with the Kummer tensor given by

$$G_{abcd} \equiv g^{ab}g^{cd} - \frac{\beta^2}{3} \left( W^{apbq}W^{cd}_{pq} + \frac{1}{4} g^{ab}W^{cpqr}W^{rd}_{pq} \right) + \frac{\beta^3}{6} \ast W^{ar}_{pq} W^{bpcs}_{pq} W^{dq}_{rs} \ast.$$

(99)

Using the following identities due to Debever and Lanczos

$$W_{apeq} W_{b}_{ p q} - \ast W_{apeq} \ast W_{b}_{ p q} = A g_{ab} g_{cd},$$

(100)

$$W_{apqr} W^{pq}_{b r} = 2 A g_{ab}$$

(101)

with $A \equiv \frac{1}{8} W_{pqrs} W^{pqrs}$, we obtain the simplified expression

$$G^{abcd} = \left( 1 - \frac{\beta^2}{6} A \right) g^{ab}g^{cd} - \frac{\beta^2}{3} W^{apbq}W^{cd}_{pq} + \frac{\beta^3}{3} W^{ar}_{pq} W^{bpcs}_{pq} W^{dq}_{rs}. $$

(102)

9 Conversely, for a given geometry, the inequalities may furnish $\beta$-dependent limits on the region of space-time where the propagation is symmetric-hyperbolic.
c. **Traceless Ricci couplings** ($\alpha = 0, \beta = 0, \gamma \neq 0$).

The relevant matrices take the form

\[ \mathbb{P}_{ab} = -h_{ab} + \gamma N_{ab}, \]  

(103)

\[ \mathbb{S}_{ab} = h_{ab} + \gamma N_{ab}. \]  

(104)

As in the previous case, the corresponding inequalities will relate the coupling constant $\gamma$ to the curvature quantities described by $N_{ab}$. The Kummer tensor is given by

\[ G^{abcd} = g^{ab} g^{cd} - \gamma g^{ab} \chi_{cpd} p + \frac{\gamma^2}{2} \left( \chi^{apb}_p (7) \chi^{cd}_q q - \chi^{apbq} (7) \chi^{cd}_p q \right) + \frac{\gamma^3}{6} \chi^{apbq} (7) \chi^{crds} (7) \chi_{pqrs}. \]

**d. Non-minimally coupled EM in a cosmological background.**

Another relevant example is that of the propagation in a cosmological background described by the flat Friedman-Lemaître-Robertson-Walker metric, for which the Weyl tensor is null. In such a case, only the terms associated to $L_1$, $L_3$, and $L_7$ survive in Eqs. (70) and (71). Using Einstein's equations, the traceless part of the Ricci tensor is given in terms of the matter by

\[ S_{ab} = T_{ab} - \frac{T}{4} g_{ab}, \]

where $T_{ab}$ is the energy-momentum tensor and $T$, its trace. In a convenient tetrad basis, in which $T_{ab} = \text{diag} (\rho, p, p, p)$, it follows that

\[ \mathbb{P}_{ij} = -\delta_{ij} \left[ 1 + \beta (\rho - 3p) - \frac{1}{2} \gamma (\rho + p) \right], \]  

(105)

\[ \mathbb{S}_{ij} = \delta_{ij} \left[ 1 + \beta (\rho - 3p) + \frac{1}{2} \gamma (\rho + p) \right], \]  

(106)

Hence, the inequalities (11) lead to

\[ 1 + \beta (\rho - 3p) > \frac{1}{2} |\gamma| (\rho + p), \]

which is trivially satisfied in the case of linear EM. The propagation will be symmetric hyperbolic if this inequality is satisfied at all times for which the model is valid.
VII. CONCLUSIONS

Well-posedness is a basic requirement for any field theory, and is guaranteed by symmetric hyperbolicity. We have obtained a general form for the symmetrizer, given in Eq. (26), valid for a general Lagrangian theory. We have shown that symmetric hyperbolicity leads to a set of two inequalities for the matrices $P$ and $S$, whose elements are determined by a given theory. Regarding the constraints, we have verified that they are integrable.

When applied to nonlinear electromagnetism linearly coupled to curvature, the matrices $P$ and $S$ are expressed in terms the fields, the Lagrangian, and its derivatives, and also of the different quantities associated to curvature. They lead to strong constraints on the relevant quantities, which were illustrated with applications to several particular cases. The examples show that while in the linear theory, no constraint arises from symmetric hyperbolic propagation, non-linearity leads to constraints on the field intensities, and non-minimal coupling imposes restrictions on quantities associated to curvature. In the general case, symmetric hyperbolicity relates the electromagnetic quantities $\{E^a, B_a\}$, the spacetime curvature expressed via $\{E_{ab}, B_{ab}, Q_a, N_{ab}\}$ and the partial derivatives of the Lagrangian density.

The ideas presented here can be applied in other settings, such as electromagnetism in material media. We plan to return to this problem in a future publication.

Appendix A: Ruse-Lanczos identity

Let $\chi_{abcd}$ denote an arbitrary double symmetric (2,2) form at a point $p \in \mathcal{M}$. Its double Hodge dual is defined as

$$\star \chi_{abcd} \star = \frac{1}{4} \epsilon_{ab}^{pq} \epsilon_{cd}^{rs} \chi_{pqrs}.$$  \hfill (A1)

Writing the traces of $\chi_{abcd}$ as

$$\chi_{ab} \equiv \chi_{acb}^c, \quad \chi \equiv \chi_a^a,$$  \hfill (A2)

we may construct the symmetric trace-free tensor

$$\psi_{ab} \equiv \chi_{ab} - \frac{1}{4} g_{ab} \chi.$$  \hfill (A3)

Using elementary algebraic manipulations, it can be shown that

$$\star \chi_{abcd} \star + \chi_{abcd} = g_a[c \psi^d b] + g_b[d \psi^c a].$$  \hfill (A4)
which is called the Ruse-Lanczos identity \([45, 46]\). Applying this identity to the constitutive tensors results in the following useful relations

\[
\begin{align*}
\star(1) \chi_{ab}^{cd} + \star(1) \chi_{ab}^{cd} &= 0, \\
\star(2) \chi_{ab}^{cd} + \star(2) \chi_{ab}^{cd} &= 0, \\
\star(3) \chi_{ab}^{cd} + \star(3) \chi_{ab}^{cd} &= 0, \\
\star(4) \chi_{ab}^{cd} + \star(4) \chi_{ab}^{cd} &= 0, \\
\star(5) \chi_{ab}^{cd} + \star(5) \chi_{ab}^{cd} &= 0, \\
\star(6) \chi_{ab}^{cd} + \star(6) \chi_{ab}^{cd} &= 0, \\
\star(7) \chi_{ab}^{cd} - \star(7) \chi_{ab}^{cd} &= 0.
\end{align*}
\]

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