A TOPOLOGICAL INSIGHT INTO THE POLAR INVOLUTION OF CONVEX SETS

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Abstract. Denote by $\mathcal{K}_0^n$ the family of all closed convex sets $A \subset \mathbb{R}^n$ containing the origin $0 \in \mathbb{R}^n$. For $A \in \mathcal{K}_0^n$, its polar set is denoted by $A^\circ$. In this paper, we investigate the topological nature of the polar mapping $A \mapsto A^\circ$ on $(\mathcal{K}_0^n, d_{AW})$, where $d_{AW}$ denotes the Attouch-Wets metric. We prove that $(\mathcal{K}_0^n, d_{AW})$ is homeomorphic to the Hilbert cube $Q = \prod_{i=1}^{\infty} [-1, 1]$ and the polar mapping is topologically conjugate with the standard based-free involution $\sigma : Q \to Q$, defined by $\sigma(x) = -x$ for all $x \in Q$. We also prove that among the inclusion-reversing involutions on $\mathcal{K}_0^n$ (also called dualities), those and only those with a unique fixed point are topologically conjugate with the polar mapping, and they can be characterized as all the maps $f : \mathcal{K}_0^n \to \mathcal{K}_0^n$ of the form $f(A) = T(A^\circ)$, with $T$ a positive-definite linear isomorphism of $\mathbb{R}^n$.

1. Introduction

Let $\mathbb{R}^n$, $n \geq 2$, be the $n$-dimensional Euclidean space endowed with the standard scalar product $\langle \cdot, \cdot \rangle$, and denote by $\mathcal{K}_0^n$ the class of closed convex sets $A \subset \mathbb{R}^n$ such that $0 \in A$. The polar set of a given element $A \in \mathcal{K}_0^n$ is defined as

$$A^\circ := \left\{ x \in \mathbb{R}^n : \sup_{a \in A} \langle a, x \rangle \leq 1 \right\},$$

and it is always an element of $\mathcal{K}_0^n$. A classic result related with the polar set establishes that $\left( A^\circ \right)^\circ = A$ for all $A \in \mathcal{K}_0^n$ (see the Bipolar Theorem in Section 2). This property leads us to the well-known concept of involution. An involution on a topological space $X$ is a...

2020 Mathematics Subject Classification. Primary: 52A20, 52A21, 54B20, 54C10, 54C55. Secondary: 54H15, 57S25.

Key words and phrases. Duality of convex sets, Polar set, Involution, Hilbert cube, Hyperspace, Anderson’s problem.

The first author has been supported by The Post-Doctoral Scholarship Program at UNAM. The second author have been supported by CONACyT grant 252849 (México) and by PAPIIT grant IN101622 (UNAM, México).
continuous map \( \beta : X \to X \) such that \( \beta(\beta(x)) = x \) for every \( x \in X \). If additionally \( \beta \) has a unique fixed point (namely, if there exists a unique point \( x_0 \in X \) such that \( \beta(x_0) = x_0 \)), then \( \beta \) is called based-free (c.f. Definition 2.6).

Let \( X \) and \( Y \) be topological spaces with involutions \( \beta \) and \( \tau \), respectively. A map \( f : X \to Y \) is called equivariant (with respect to \( \beta \) and \( \tau \)), if \( f(\beta(x)) = \tau(f(x)) \) for every \( x \in X \). If we can find an equivariant homeomorphism \( f : X \to Y \), then we say that \( \beta \) and \( \tau \) are conjugate.

An interesting and difficult problem in infinite-dimensional topology, is the characterization of all based-free involutions on the Hilbert cube \( Q = \prod_{i=1}^{\infty} [-1,1] \). In the mid-sixties, R. D. Anderson asked if all based-free involutions on \( Q \) are conjugate (see e.g.36 Problem 930 or 5, Section 3.2). In other words, if we let \( \sigma : Q \to Q \) be the standard involution on \( Q \) given by \( \sigma(x) = -x \), then Anderson’s problem can be stated as:

\[
\text{If } \beta : Q \to Q \text{ is an involution with a unique fixed point, does there exist a homeomorphism } \Psi : Q \to Q \text{ such that } \beta = \Psi \sigma \Psi^{-1} ?
\]

(see 36, p 554)). Despite the many efforts that have been done to answer this question (see for instance 2, 21, 22, 35, 37, 38, 40), Anderson’s problem remains open.

An important tool in this and other topics on infinite-dimensional topology has been the use of different models for the space \( Q \). In this respect, the so called hyperspaces of compact sets have played an important role (see, e.g., 15, 16 and 37). In particular, in the seminal work 23 the authors proved that the family of all compact convex subsets of a convex set \( K \subset \mathbb{R}^n \) is homemorphic to the Hilbert cube, provided that \( K \) is compact and has dimension at least 2.

In this work, we are interested in the topological properties of the polar mapping and its relationship with the so called Anderson’s problem. In this sense, the set \( K_0^n \) can be considered as a metric space with respect to the Attouch-Wets metric \( d_{AW} \) (see Section 2 for the definition). In this case, the polar mapping

\[
\alpha : (K_0^n, d_{AW}) \to (K_0^n, d_{AW})
\]

\[
A \to A^\circ
\]

is in fact a based-free involution for which the Euclidean ball \( B \) is the unique fixed point (see e.g. Proposition 4.2). Additionally, as we will
prove in Section 3 the space $(K^n_0, d_{AW})$ is homeomorphic to the Hilbert cube $Q$. One of our main contributions is the following theorem.

**Theorem 1.** The polar mapping $\alpha : (K^n_0, d_{AW}) \to (K^n_0, d_{AW})$ is topologically conjugate with the standard involution $\sigma : Q \to Q$.

It is worth mentioning that the homeomorphism $\Psi : Q \to K^n_0$ obtained after Theorem 1 does not map the natural order on $Q$ (see Section 7) into the order inclusion on $K^n_0$ (Remark 7.2).

In [6, 7, 11, 31], a deep study on the geometric properties of the polar mapping has been carried out. It is known that this map is a duality (in the sense of [6, Definition 4]) for the class $K^n_0$, and it can be characterized by some of its basic properties. For example, in [31, Corollary 4] (see also [7, Theorem 10]), the following is proved:

**Theorem A.** [31, Corollary 4] Let $n \geq 2$. Let $f : K^n_0 \to K^n_0$ be a mapping satisfying for all $A, K \in K^n_0$ that

$(D1) \ f(f(A)) = A,$

$(D2) \ \text{If } A \subseteq K, \text{ then } f(A) \supseteq f(K).$

Then, there exists a symmetric linear isomorphism $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(A) = T(A^\circ)$ for all $A \in K^n_0$.

An earlier version of this result was proved in [11] for the class $K^n_{(0),b}$ of convex compact sets containing 0 in their interior.

We will notice in Section 6 that all the maps $f : K^n_0 \to K^n_0$ satisfying conditions (D1) and (D2) are always continuous on $(K^n_0, d_{AW})$ (see Remark 6.1). For short, these kinds of maps will be called decreasing involutions. A key observation is that Theorem A translates the whole study of decreasing involutions into the linear category. In this setting, another of our results will be the characterization of the linear isomorphisms on $\mathbb{R}^n$ that induce based-free involutions. More precisely, these mappings can be characterized as follows.

**Theorem 2.** Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a symmetric linear isomorphism and let $f : K^n_0 \to K^n_0$ be defined as $f(A) = T(A^\circ)$. Then, the following statements hold.

1. If $T$ is positive-definite, then $f$ is conjugate with the polar mapping. In particular, $f$ is based-free.
(2) If $T$ is not positive-definite, then $f$ has infinitely many fixed points in $\mathcal{K}^n_{(0),b}$.

Theorem 2 in combination with Theorem 1 and Theorem A, yields the following corollary that could be of interest given its relation with Anderson’s Problem.

**Corollary 3.** Every based-free decreasing involution $f : \mathcal{K}^n_0 \to \mathcal{K}^n_0$ is conjugate with the standard involution on $Q$. Moreover, $f$ is of the form $f(A) = T(A^\circ)$ for some positive-definite linear isomorphism $T : \mathbb{R}^n \to \mathbb{R}^n$.

In terms of [6, Definition 4], one can say that a duality on $\mathcal{K}^n_0$ is topologically equivalent to the polar mapping if and only if it has a unique fixed point.

Let us briefly describe the contents and structure of the paper. In Section 2, we introduce the notation and basic results regarding $\mathcal{K}^n_0$, the Attouch-Wets metric and the polar set. We also introduce some tools from the theory of Hilbert cube manifolds, ANR’s and $G$-spaces that will be used later. Section 3 is dedicated to calculate the topological structure of the space $\mathcal{K}^n_0$, $n \geq 2$. In Theorem 3.4, we show that $(\mathcal{K}^n_0, d_{AW})$ is homeomorphic to $Q$. Additionally, in Corollary 3.5, we prove that the subspace $\mathcal{K}^n_{0,b}$ consisting of all compact convex sets containing 0 is homeomorphic to the Hilbert cube with a point removed.

Section 4 is dedicated to prove Theorem 1. We begin by summarizing the basic properties of the polar mapping $\alpha : \mathcal{K}^n_0 \to \mathcal{K}^n_0$ and the $\mathbb{Z}_2$-action induced by it. After recalling some results from the theory of $G$-spaces, we prove that the $\mathbb{Z}_2$-space $(\mathcal{K}^n_0, \alpha)$ is $\mathbb{Z}_2$-contractible (Theorem 4.7). This allows us to prove that the polar mapping is conjugate to $\sigma$ (Theorem 1), among other results (Corollary 4.8).

In Section 5, we investigate the properties of the maps on $\mathbb{R}^n$ that preserve polarity. A map $f : \mathbb{R}^n \to \mathbb{R}^n$ is called polar preserving map if $f(A^\circ) = f(A)^\circ$ for all $A \in \mathcal{K}^n_0$. Proposition 5.5 is the main result of this section. There, we prove that every injective polar preserving map $f : \mathbb{R}^n \to \mathbb{R}^n$ must be an orthogonal map.

In Section 6, we characterize based-free decreasing involutions on $\mathcal{K}^n_0$. Namely, we prove Theorem 2. We finish the paper with some remarks and questions that could be of general interest (Section 7).
2. Preliminaries

We begin by introducing the notation and basic results we use throughout the work. The Euclidean space $\mathbb{R}^n$, $n \geq 2$, is endowed with the standard scalar product $\langle \cdot, \cdot \rangle$. The corresponding Euclidean norm and closed unit ball are denoted by $\| \cdot \|$ and $B$, respectively. The topology on $\mathbb{R}^n$ is the one determined by $\| \cdot \|$. If $A \subset \mathbb{R}^n$ is an arbitrary subset, the closed convex hull of $A$ is denoted by $\text{conv}(A)$. For every pair of points $a, b \in \mathbb{R}^n$, we denote by $[a, b]$ the segment with end points $a$ and $b$. Namely,

$$[a, b] := \{(1 - t)a + tb : t \in [0, 1]\}.$$ 

By $\mathcal{K}^n$ we denote the family of all nonempty closed and convex subsets of $\mathbb{R}^n$. The family of all elements in $\mathcal{K}^n$ containing the origin $0 \in \mathbb{R}^n$ will be denoted by $\mathcal{K}_0^n$, and the elements of $\mathcal{K}_0^n$ containing the origin in its interior will be denoted by $\mathcal{K}^n_{(0)}$. We also use the symbol $\mathcal{K}_b^n$ to denote the elements of $\mathcal{K}^n$ that are compact, and by following this logic, we define the families

$$\mathcal{K}_{0,b}^n := \mathcal{K}_0^n \cap \mathcal{K}_b^n \quad \text{and} \quad \mathcal{K}_{(0),b}^n := \mathcal{K}_{(0)}^n \cap \mathcal{K}_b^n.$$ 

On $\mathcal{K}^n$ one can consider different topologies (see, e.g. [9]). In this work, we are interested in the Attouch-Wets topology $\tau_{AW}$. This topology is determined by the Attouch-Wets metric $d_{AW}$ which can be defined for $A, K \in \mathcal{K}^n$ as

$$(2.1) \quad d_{AW}(A, K) := \sup_{j \in \mathbb{N}} \left\{ \min \left\{ \frac{1}{j}, \sup_{\|x\| < j} |d(x, A) - d(x, K)| \right\} \right\}. $$

This definition is the one used in [27] and it is convenient to highlight that the formula in (2.1) is equivalent to the Attouch-Wets metric defined in [9, Definition 3.1.2].

Throughout the work we will use several times the following fact, which can be consulted in [9, Theorem 3.1.4 and Exercise 5.1.10(b)] or [28, Remark 1].

**Fact 1.** On $\mathcal{K}^n$, the Attouch-Wets topology, the Fell topology $\tau_F$ and the Wijsman topology $\tau_W$ are the same.

For the convenience of the reader, we recall the definition of $\tau_F$ and $\tau_W$. The Fell topology on $\mathcal{K}^n$ is generated by the sets $U^- := \{A \in \mathcal{K}^n : A \cap U \neq \emptyset\}$ and $(\mathbb{R}^n \setminus C)^+ := \{A \in \mathcal{K}^n : A \subset \mathbb{R}^n \setminus C\}$, where $U \subset \mathbb{R}^n$ is open and $C \subset \mathbb{R}^n$ is compact. This topology is also defined
on $\mathcal{K}_n := \mathcal{K}^n \cup \{\emptyset\}$. On the other hand, the Wijsman topology on $\mathcal{K}^n$ is the one generated by the sets

$$U^-(x,r) := \{ A \in \mathcal{K}^n : d(x,A) < r \}$$

$$U^+(x,r) := \{ A \in \mathcal{K}^n : d(x,A) > r \}$$

where $x \in \mathbb{R}^n$ and $r > 0$.

Recall that the Hausdorff metric between any pair of nonempty sets $A, K \subset \mathbb{R}^n$ is defined by any of the following equivalent expressions

$$d_H(A, K) = \max \left\{ \sup_{x \in A} d(x, K), \sup_{x \in K} d(x, A) \right\}$$

$$= \inf \{ \lambda > 0 : A \subseteq K + \lambda \mathbb{B}, \ K \subseteq A + \lambda \mathbb{B} \}$$

$$= \sup_{x \in \mathbb{R}^n} |d(x, A) - d(x, K)|$$

(see, e.g. [9, §3.2]).

It is important to point out the following fact.

**Fact 2.** ([27, Theorem 3.2]) On the class of compact convex subsets $\mathcal{K}_b^n$, the Attouch-Wets metric and the Hausdorff metric $d_H$ determine the same topology.

In [17, Lemma 2.3], the following useful property of $d_{AW}$ was established: Let $\varepsilon > 0$ be such that $\frac{1}{j+1} < \varepsilon \leq \frac{1}{j}$ for some $j \in \mathbb{N}$, and let $A, K \in \mathcal{K}_b^n$. Then,

$$d_{AW}(A, K) < \varepsilon \text{ if and only if } \sup_{\|x\| < j} |d(x, A) - d(x, K)| < \varepsilon.$$

This relation holds not only for elements of $\mathcal{K}_b^n$, but for any pair of nonempty closed subsets. However, in the context we are working on, we can use the Euclidean structure of $\mathbb{R}^n$ to provide a sharper relationship between $d_{AW}$ and $d_H$ on $\mathcal{K}_b^n$.

**Lemma 2.1.** Let $r > 0$. For any $z \in r\mathbb{B}$ and $A, K \in \mathcal{K}_b^n$, the following equalities hold.

1. $d(z, A) = d(z, A \cap r\mathbb{B})$.
2. $\sup_{\|x\| < r} |d(x, A) - d(x, K)| = \sup_{x \in r\mathbb{B}} |d(x, A) - d(x, K)|$.
3. $d_H(A \cap r\mathbb{B}, K \cap r\mathbb{B}) = \sup_{x \in r\mathbb{B}} |d(x, A) - d(x, K)|$. 
\begin{enumerate}
\item[(4)] \(d_{AW}(A, K) = \sup_{j \in \mathbb{N}} \left\{ \min \left\{ \frac{1}{j}, d_H(A \cap jB, K \cap jB) \right\} \right\}.\)
\item[(5)] For every integer \(j \geq 1\) and every \(\varepsilon \in \left( \frac{1}{j+1}, \frac{1}{j} \right],\)
\[d_{AW}(A, K) < \varepsilon \text{ if and only if } d_H(A \cap jB, K \cap jB) < \varepsilon.\]
\end{enumerate}

\textbf{Proof.} (1) Let \(a \in A\) be the unique element such that \(d(z, A) = \|z-a\|\). Assume that \(\|a\| > r\) and consider the point \(p := \left( \frac{x}{\|a\|^2} a \right).\) By the Cauchy-Schwarz inequality, \(\|p\| \leq \|z\| \leq r\) and therefore \(p\) lies in the closed segment \([-a, a] \cap rB\).

If \(p \in [-a, 0] \cap rB\), then \(\|p\| < \|p-a\|\) and by the Pythagorean theorem we conclude that
\[
\|z\|^2 = \|z-p\|^2 + \|p\|^2 < \|z-p\|^2 + \|p-a\|^2 = \|z-a\|^2 = d(z, A)^2.
\]
This proves that \(0\) is an element of \(A\) closer to \(z\) than \(a\), a contradiction.

On the other hand, if \(p \in [0, a] \cap rB\), clearly \(p \in A \setminus \{a\}\). Using the Pythagorean theorem again, we obtain that \(p\) is an element of \(A\) closer to \(z\) than \(a\). From these contradictions we conclude that \(\|a\| \leq r\) and therefore \(a \in A \cap rB\). This yields to
\[
d(z, A) \leq d(z, A \cap rB) \leq \|z-a\| = d(z, A),
\]
which proves the equality.

(2) Let \(f : \mathbb{R}^n \to \mathbb{R}\) be defined as \(f(x) = |d(x, A) - d(x, K)|\), and consider the set
\[
U := f \left( \text{Int}(rB) \right) = \{ |d(x, A) - d(x, K)| : \|x\| < r \}.
\]
Using the continuity of \(f\) we obtain that
\[
U \subset f(rB) = f \left( \text{Int}(rB) \right) \subset f(\overline{\text{Int}(rB)}) = \overline{U}.
\]
Now, using the fact that \(f(rB)\) is closed, we infer that \(\overline{U} = f(rB)\). Thus
\[
\sup U = \sup \overline{U} = \sup f(rB),
\]
as desired.
(3) Notice that (1) of this proposition and the definition of \( d_H \) imply that
\[
d_H(A \cap rB, K \cap rB) = \sup_{x \in \mathbb{R}^n} |d(x, A \cap rB) - d(x, K \cap rB)|
\geq \sup_{x \in rB} |d(x, A \cap rB) - d(x, K \cap rB)|
= \sup_{x \in rB} |d(x, A) - d(x, K)|.
\]
On the other hand, if \( b \in K \cap rB \) then
\[
d(b, A \cap rB) = |d(b, A \cap rB) - d(b, K \cap rB)|
\leq \sup_{x \in rB} |d(x, A \cap rB) - d(x, K \cap rB)|
= \sup_{x \in rB} |d(x, A) - d(x, K)|.
\]
Similarly, if \( a \in A \cap rB \), then \( d(a, K \cap rB) \leq \sup_{x \in rB} |d(x, A) - d(x, K)| \). Hence, \( d_H(A \cap rB, K \cap rB) \leq \sup_{x \in rB} |d(x, A) - d(x, K)| \). This inequality completes the proof of (3).

(4) Follows directly from (1), (2) and (3) of this proposition.

(5) Follows from (2) and (3) of this proposition, and equivalence \( (22) \). \( \square \)

We recall some basic properties of the polar set \( (1.1) \) that will be used several times along the paper.

(P1) \( A^\circ \in K_0^n \) for every nonempty set \( A \subset \mathbb{R}^n \) and \( A^\circ \in K_{n,0}^n \), provided that \( A \in K_{n,0}^n \). Moreover, if \( A \) and \( A^\circ \) belong to \( K_{(0),b}^n \), then \( A \in K_{(0),b}^n \).

(P2) If \( A \subset B \subset \mathbb{R}^n \), then \( B^\circ \subset A^\circ \).

(P3) \( (\bigcup K_a)^\circ = \bigcap K_a^\circ \).

(P4) \( (\bigcap K_a)^\circ = \overline{\text{conv}} (\bigcup K_a^\circ) \).

(P5) \( A^\circ = A \) if and only if \( A = B \).

(P6) \( (\mathbb{R}^n)^\circ = \{0\} \) and \( \{0\}^\circ = \mathbb{R}^n \).

(P7) For every \( r > 0 \), \( (rA)^\circ = r^{-1}A^\circ \). Particularly, \( (rB)^\circ = \frac{1}{r}B \).

Finally, we recall the Bipolar Theorem which is going to play an important role in this work.

(P8) (The Bipolar Theorem) For every \( A \in K_0^n \), \( (A^\circ)^\circ = A \).
We refer the reader to [8, Chapter IV] and [32, §2.2] for a deeper understanding of the polar map.

In [19], V. Milman and L. Rotem introduced a new operation on $K^n_{(0), b}$. It is called the geometric mean $g(A, K)$ of the compact convex sets $A, K \in K^n_{(0), b}$. Since we will use it in the proof of our main results, we include its definition and basic properties. For $A, K \in K^n_{(0), b}$ and $m \in \mathbb{N}$, let $(A_m)_m$ and $(H_m)_m$ be sequences on $K^n_{(0), b}$ defined as

$$A_0 = A, \quad H_0 = K, \quad A_{m+1} = \frac{A_m + H_m}{2}, \quad H_{m+1} = \left(\frac{A_m^o + H_m^o}{2}\right)^o.$$ 

In [19, Theorem 6], it is proved that $H_m \subseteq A_m$, for all $m \in \mathbb{N}$, and that both sequences converge to the same limit on $(K^n_{(0), b}, d_H)$. This allows to define the geometric mean of $A$ and $K$ as

$$g(A, K) := \lim_{m \to \infty} A_m = \lim_{m \to \infty} H_m,$$

where the limit is calculated with respect to $d_H$. The fundamental properties of $g$ appear in [19] and [25]. Below, we include those that are relevant for this work. Let $A, K \in K^n_{(0), b}$, then $g(A, K) \in K^n_{(0), b}$ and the following hold:

$(\Gamma 1)$ $g(A, A) = A$ and $g(A, K^o) = g(K, A)$.

$(\Gamma 2)$ $g(A^o, K^o) = g(A, K)^o$.

$(\Gamma 3)$ $g(A, A^o) = B$.

$(\Gamma 4)$ $g(A_1, K_1) \subseteq g(A_2, K_2)$ if $A_1 \subseteq A_2$ and $K_1 \subseteq K_2$, with $A_i, K_i \in K^n_{(0), b}$ for $i = 1, 2$.

$(\Gamma 5)$ The map $g : (K^n_{(0), b}, d_H) \times (K^n_{(0), b}, d_H) \rightarrow (K^n_{(0), b}, d_H)$, sending $(A, K)$ to $g(A, K)$ is continuous.

$(\Gamma 6)$ $g(TA, TK) = Tg(A, K)$ for every linear map $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$.

Let us notice that in this work, the words map and mapping do not mean continuous function. When necessary, we will emphasize if a map is continuous or not.

In Sections 3 and 4 we will use several results and notions coming from the theory of retracts, the theory of Q-manifolds and the theory of G-spaces. Let us quickly recall some of their fundamentals that will be used later.
2.1. ANR and AR spaces. We say that a closed subset $Y$ of a topological space $X$ is a retract of $X$ if there exists a continuous map $r : X \to Y$ such that $r(y) = y$ for all $y \in Y$. The map $r$ is then called a retraction. An absolute neighborhood retract (ANR) is a metrizable space $X$ such that for every metrizable space $Z$ containing $X$ as closed subset, there exists a neighborhood $U$ of $X$ in $Z$ and a retraction $r : U \to X$. In this case, we write $X \in \text{ANR}$. If we can always take $U = Z$, we say that $X$ is an absolute retract (AR), and we write $X \in \text{AR}$. Clearly, every AR is a ANR. A simple example of an AR space is any closed convex subset of a Banach space (see, e.g., [20, Theorem 1.5.1]).

Let us summarize some basic properties concerning the ANR spaces.

(R1) Every open subspace of an ANR is an ANR ([18, Chapter 3, Proposition 7.9]).

(R2) If every point of a separable metric space $X$ has a neighborhood which is an ANR, then $X$ is a ANR ([12, Chapter 4, Corollary 10.4]).

(R3) Every AR is contractible ([18, Chapter 3, Theorem 7.1]).

(R4) Every contractible ANR is an AR (see [18, Chapter 3, Proposition 7.2] or [20, Corollary 1.6.7]).

Finally, let us recall that a subset $Y$ of a topological space $X$ is called homotopy dense if there exists a homotopy $H : X \times [0, 1] \to X$ such that $H_0$ is the identity on $X$ and $H_t(X) \subseteq Y$ for all $0 < t \leq 1$. As usual, $H_t$ denotes the map $H_t : X \to X$ given by $H_t(x) := H(x, t)$ for each $x \in X$. The importance of homotopy dense subsets is that they can characterize whether a metrizable space is a ANR, as the following result shows.

(R5) Let $X$ be a metrizable space and $Y$ be a homotopy dense subset of $X$. Then $X$ is an ANR if and only if $Y$ is an ANR ([26, Corollary 6.6.7]).

For more information on the theory of retracts, we refer the reader to [12, 18, 20, 26].

2.2. $Q$-Manifolds. The Hilbert cube, denoted by $Q$, is the topological product $\prod_{i=1}^{\infty} [-1, 1]$. A Hilbert cube manifold ($Q$-manifold) is a separable metrizable space that admits an open cover by sets homeomorphic to open subsets of $Q$. In particular, we have the following.
Fact 3. *Every nonempty open subset of a $Q$-manifold, is a $Q$ manifold too.*

It should be reminded that $Q$ is an AR (see e.g. [20, Corollary 1.5.5]). This, in combination with properties (R1) and (R2), proves the following fact.

Fact 4. *Every $Q$ manifold is an ANR.*

Let us recall some of the most important results from the theory of $Q$-manifolds that will be used later in the paper.

Theorem 2.2. ([14, Theorem 44.1]) *If $X$ in an ANR then $X \times Q$ is a $Q$ manifold.*

Theorem 2.3. ([20, Theorem 7.5.8]) *Every compact contractible $Q$-manifold is homeomorphic to $Q$.*

Theorem 2.4. (Stability Theorem for $Q$-manifolds [14 Theorem 15.1]) *If $M$ is a $Q$-manifold, then $M \times Q$ is homeomorphic to $M$. Moreover, the projection map of $M \times Q$ to $M$ is a near homeomorphism.$^1$

A continuous surjective map $f : X \to Y$ between topological spaces is called proper if $f^{-1}(K)$ is compact for every compact set $K \subseteq Y$. A proper map $f : X \to Y$ is called cell-like (or CE-map) if every fiber $f^{-1}(y)$ has trivial shape. Let us recall that a compact metric space $X$ has trivial shape if for every embedding $f : X \to Z$ where $Z$ is a ANR, the image $f(X)$ is contractible in any of its neighborhoods. In particular, if $X$ is contractible, then it has trivial shape. The reader can find more information on this topic in [20, Subsection 7.1].

The importance of CE-maps is that they often guarantee the existence of a homeomorphism between $Q$-manifolds, as the following Theorem from R. D. Edwards states.

Theorem 2.5. ([14 Theorem 43.1]) *If $M$ is a $Q$-manifold, $X$ is an ANR and $f : M \to X$ is a CE-map, then the map $f \times id_Q : M \times Q \to X \times Q$
is a near homeomorphism. Here, \( \text{id}_Q \) denotes the identity map on \( Q \).

We refer the reader to \([14, 20, 26]\) for a deeper understanding of the theory of \( Q \)-manifolds. However, in Section 3 we still require to introduce some other notions from this theory that have not been mentioned here.

### 2.3. \( G \)-spaces

For the proof of Theorem 1, we will use some important results from the theory of \( G \)-spaces. We refer the reader to \([13]\) for a deeper understanding of this topic. However, in order to make our proof as clear as possible, let us recall some of the basic notions that we will use.

By a \( G \)-space, we mean a topological space \( X \) equipped with a continuous action \( \rho : G \times X \to X \) of a topological group \( G \) on \( X \). As usual, \( hx \) denotes the image under the action of the pair \( (h, x) \), i.e., \( hx := \rho(h, x) \). The set \( G(x) := \{hx \in X : h \in G\} \) is called the orbit of \( x \), and the set of all orbits in \( X \), equipped with the quotient topology, is the orbit space and it is denoted by \( X/G \).

A map \( f : X \to Y \) between \( G \)-spaces is called \( G \)-equivariant (or simply equivariant) if it commutes with the action. Namely, if \( f(hx) = hf(x) \) for every \( x \in X \) and \( h \in G \). If in addition \( f \) is continuous, then we say that \( f \) is a \( G \)-map. Similarly, if \( f \) happens to be a retraction which is also equivariant, then we call it a \( G \)-retraction.

Given a \( G \)-space \( X \), for every \( x \in G \) we denote by \( G_x \) the stabilizer (also called isotropy group) of \( x \), i.e.,

\[
G_x := \{h \in G : hx = x\}.
\]

An action of a group \( G \) on a topological space \( X \) is said to be free, if \( G_x \) is the trivial group for every \( x \in X \). This notion leads us to the following definition.

**Definition 2.6.** A based-free \( G \)-space is a \( G \)-space with a unique point \( x_0 \in X \), such that \( G_{x_0} = G \) and \( G_x \) is trivial for every \( x \in X \setminus \{x_0\} \). Namely, \( x_0 \) is a fixed point and the action of \( G \) on \( X \setminus \{x_0\} \) is free.

Every finite group can define a based-free action on the Hilbert cube. Indeed, for a finite group \( G \), consider its cone

\[
C(G) := G \times [0, 1]/G \times \{0\}.
\]
Notice that $C(G)$ is homeomorphic to a contractible compact non trivial polyhedron and therefore it is an AR\(^2\). Thus, the countable product $\prod_{n \in \mathbb{N}} C(G)$ is homeomorphic to the Hilbert cube (see, e.g. [20, Corollary 8.1.2]). Furthermore, the group $G$ acts continuously on $C(G)$ by $(h_1, [h_2, t]) \rightarrow [h_1 h_2, t]$ and hence we can define an action of $G$ on the Hilbert cube $\prod_{n \in \mathbb{N}} C(G)$ by means of the following rule
\[
(h, ([h_n, t_n])_{n \in \mathbb{N}}) \rightarrow ([hh_n, t_n])_{n \in \mathbb{N}}.
\]
It is not difficult to prove that this action is based-free and the only fixed point is the point with all its coordinates equal to the vertex $\theta := [h, 0]$. We refer the reader to [10, Section 3] for more details on this construction\(^3\). These kinds of actions are called standard based-free $G$-action on a Hilbert cube.

**Remark 2.7.** Notice that if $G = \mathbb{Z}_2$, then $C(\mathbb{Z}_2)$ is homeomorphic with the interval $[-1, 1]$ and therefore the standard based-free $\mathbb{Z}_2$-action on $Q$ corresponds precisely with the one defined by $(h, (x_n)_{n \in \mathbb{N}}) \rightarrow (hx_n)_{n \in \mathbb{N}}$, where $h \in \mathbb{Z}_2 := \{-1, 1\}$.

After this construction, a based-free action $\rho : G \times Q \rightarrow Q$ of a finite group on the Hilbert cube is called standard if it is conjugate with the standard based-free $G$-action on $Q$. Namely, if there exists an equivariant homeomorphism $\varphi : (Q, \rho) \rightarrow (Q, \sigma_G)$, where $\sigma_G$ denotes the standard based-free $G$-action on $Q$.

The easiest way to detect if a based-free action on $Q$ is standard is by means of the following theorem of J. West and R. Y. T. Wong.

**Theorem 2.8.** ([38, Theorem 1]). If $G$ is finite, a based-free action $\rho : G \times Q \rightarrow Q$ is standard if and only if the orbit space $Q/G$ is an AR.

To avoid making this section any longer, all other basic concepts and results needed in specific spots of the paper will be introduced as they are needed.

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\(^2\)We refer the reader to [20, Page 117] for the definition of a polyhedron and more information about the AR-property on these kinds of spaces.

\(^3\)Actually, this construction works for any compact Lie group.
3. Homeomorphism type of $\mathcal{K}_0^n$

From now on, $\mathcal{K}^n$ and all its subspaces are assumed to be metric spaces with respect to the Attouch-Wets metric $d_{AW}$ defined in equation (2.1).

The next lemmas will be used frequently throughout the work.

**Lemma 3.1.** Let $r : (0, 1] \to [0, \infty)$ be a continuous map such that $r(1) = 0$, $\lim_{t \to 0^+} r(t) = \infty$ and $r(t) > 0$ for all $t \in (0, 1)$. Then the map $F : \mathcal{K}_0^n \times [0, 1] \to \mathcal{K}_0^n$ defined by

$$F(K, t) := \begin{cases} K & \text{if } t = 0, \\ K \cap r(t)B & \text{if } t \in (0, 1], \end{cases}$$

is continuous.

**Proof.** Let $(A, s) \in \mathcal{K}_0^n \times [0, 1]$ be fixed. Let us suppose that $s > 0$, and let $j_0 \geq 1$ be an integer such that $r(s) < j_0$. Let $\varepsilon > 0$ be such that $\frac{1}{j_0 + 1} < \varepsilon \leq \frac{1}{j}$, for some $j > j_0 + 1$. By the continuity of $r$, we can pick $\delta \in (0, s)$ such that $|r(t) - r(s)| < \frac{\varepsilon}{4}$, if $|t - s| < \delta$. Now, let $(K, t) \in \mathcal{K}_0^n \times [0, 1]$ be any pair satisfying $d_{AW}(K, A) < \frac{\varepsilon}{4}$ and $|t - s| < \delta$. Observe that, by our choice of $\delta$, $r(t) < j$ and $t > 0$. Therefore,

(3.1) \quad $F_s(A) = A \cap r(s)B = A \cap r(s)B \cap jB \subseteq A \cap jB$

(3.2) \quad $F_t(K) = K \cap r(t)B = K \cap r(t)B \cap jB \subseteq K \cap jB$.

Recall that $F_s(A)$ and $F_t(K)$ stand for $F(A, s)$ and $F(K, t)$, respectively. Also, since $d_{AW}(K, A) < \frac{\varepsilon}{4} < \frac{1}{j}$, we can use Lemma 2.1-(4) to infer that

(3.3) \quad $d_H(K \cap jB, A \cap jB) < \frac{\varepsilon}{4}$.

Now, consider $b \in A \cap r(s)B$. If $b \in r(t)B$, then $d(b, A \cap r(t)B) < \frac{\varepsilon}{4}$. Otherwise, $\|b\| > r(t)$ and $\frac{r(t)}{\|b\|}b \in A \cap r(t)B$. Hence,

$$d(b, A \cap r(t)B) = \left\| b - \frac{r(t)}{\|b\|}b \right\| \leq r(s) - r(t) < \frac{\varepsilon}{4}.$$

Similarly, $d(b', A \cap r(s)B) < \frac{\varepsilon}{4}$ for all $b' \in A \cap r(t)B$. Thus, $d_H(F_s(A), F_t(A)) \leq \frac{\varepsilon}{4}$ which, by the triangle inequality, yields to

(3.4) \quad $d_H(F_s(A), F_t(K)) \leq \frac{\varepsilon}{4} + d_H(F_t(A), F_t(K))$.

We claim that $d_H(F_t(A), F_t(K)) \leq \frac{\varepsilon}{2}$. Indeed, let $a \in A \cap r(t)B \subseteq A \cap jB$. By (3.3), there is $x \in K \cap jB$ such that $\|x-a\| < \frac{\varepsilon}{4}$. If $x \in r(t)B$, then...
then \(d(a, K \cap r(t)\mathbb{B}) < \frac{\varepsilon}{4} \). Otherwise, \(\|x\| > r(t)\) and \(\frac{r(t)}{\|x\|} x \in K\). In this case, \(0 < \|x\| - r(t) < \frac{\varepsilon}{4}\) and
\[
\left\| a - \frac{r(t)}{\|x\|} x \right\| \leq \left\| a - \frac{r(t)}{\|x\|} a \right\| + \left\| \frac{r(t)}{\|x\|} a - \frac{r(t)}{\|x\|} x \right\| \leq \frac{\varepsilon}{4} \left( \frac{\|a\|}{\|x\|} + \frac{r(t)}{\|x\|} \right) < \frac{\varepsilon}{2}.
\]
Thus \(d(a, K \cap r(t)\mathbb{B}) < \frac{\varepsilon}{2}\). In a similar way, we can prove that \(d(a', A \cap r(t)\mathbb{B}) < \frac{\varepsilon}{2}\) for all \(a' \in K \cap r(t)\mathbb{B}\). Therefore \(d_{H}(F_{t}(A), F_{t}(K)) \leq \frac{\varepsilon}{2}\). This, in combination with (3.1), (3.2) and (3.4), implies that \(d_{H}(F_{s}(A) \cap j\mathbb{B}, F_{t}(K) \cap j\mathbb{B}) < \varepsilon\). Hence, by Lemma 2.11(5), \(d_{AW}(F_{s}(A), F_{t}(K)) < \varepsilon\), and therefore \(F\) is continuous at \((A, s)\) whenever \(s \neq 0\).

Now, let us consider the case when \(s = 0\). Let \(\varepsilon > 0\) and assume that \(\varepsilon \in \left(\frac{1}{j+1}, \frac{1}{j}\right]\) for some integer \(j \geq 1\). By our hypothesis, there is \(\delta \in (0, \varepsilon)\) such that \(r(t) > j\) for all \(t \in (0, \delta)\). Let \((K, t) \in K_{0}^{n} \times [0, \delta)\) be such that \(d_{AW}(A, K) < \varepsilon\). Clearly, if \(t = 0\), \(d_{AW}(F(A, 0), F(K, t)) = d_{AW}(A, K) < \varepsilon\). On the other hand, if \(t > 0\), then \(K \cap j\mathbb{B} = F(K, t) \cap j\mathbb{B}\). This, in combination with Lemma 2.11(5) and the fact that \(d_{AW}(A, K) < \varepsilon\), implies that
\[
d_{H}(F(A, 0) \cap j\mathbb{B}, F(K, t) \cap j\mathbb{B}) = d_{H}(A \cap j\mathbb{B}, K \cap j\mathbb{B}) < \varepsilon.
\]
Using Lemma 2.11(5) again, we conclude that \(d_{AW}(F(A, 0), F(K, t)) < \varepsilon\), which proves the continuity of \(F\) at \((A, 0)\).

**Lemma 3.2.** Let \(H : K_{0}^{n} \times [0, 1] \to K_{0}^{n}\) be defined as \(H(K, t) = K + t\mathbb{B}\), for \(K \in K_{0}^{n}\) and \(t \in [0, 1]\). Then \(H\) is continuous.

**Proof.** By Fact 1, it is enough to prove that \(H\) is continuous with respect to the Fell topology on \(K_{0}^{n}\). Let \(U \subset \mathbb{R}^{n}\) be an open set and consider \((K, t) \in K_{0}^{n} \times [0, 1]\) with \(H(K, t) \in U^{-}\). Thus we can find \(a \in K\) and \(b \in \mathbb{B}\) such that \(a + tb \in U\). By the continuity of the scalar product and the sum operation on \(\mathbb{R}^{n}\), we can find open neighborhoods \(V\) and \(W\) of \(a\) and \(t\), respectively, such that
\[
a' + t'b \in U \quad \text{for every } a' \in V \text{ and } t' \in W.
\]
This proves that \(H(K', t') \in U^{-}\) for every \(K' \in V^{-}\) and \(t' \in W\).

On the other hand, if \(M \subset \mathbb{R}^{n}\) is compact and \(K + t\mathbb{B} = H(K, t) \in (\mathbb{R}^{n} \setminus M)^{+}\), we can find \(\delta > 0\) small enough such that
\[
(K + t\mathbb{B} + \delta\mathbb{B}) \cap (M + \delta\mathbb{B}) = \emptyset.
\]
Clearly, \(K \cap (M + (t + \delta)\mathbb{B}) = \emptyset\). Thus, since \(C := (M + (t + \delta)\mathbb{B})\) is compact, \((\mathbb{R}^{n} \setminus C)^{+}\) is an open neighborhood of \(K\). Finally, let
A ∈ (R^n \ C)^+ and s ∈ [0,1] with |s − t| < δ. If a ∈ A, b ∈ B and x ∈ M are arbitrary, then the following inequality holds.

\[ \|a + sb - x\| \geq \|a - x\| - \|sb\| > t + \delta - (t + \delta) > 0 \]

Hence, \( H(A, s) \in (R^n \ M)^+ \), which proves the continuity of \( H \). □

Before proving the main results of this section, let us recall some facts concerning the topology of certain subspaces of \( K^n \).

**Fact 5.** ([27, Proposition 3.5]) The subspace \( K^n_b \subset K^n \) is open.

Let \( cb(n) := \{ A \in K^n_b : \text{Int}(A) \neq \emptyset \} \). The following results from [4] will be used in the proof of the proposition below.

**Fact 6.** ([4, Corollary 3.11]) The space \( (cb(n), d_H) \) is homeomorphic to \( Q \times R^{n+3} \).

**Fact 7.** ([4, Lemma 3.1]) Let \( A, C \in cb(n) \) and let \( x_0 \in A \) be such that \( x_0 + 2\varepsilon B \subseteq A \), for certain \( \varepsilon > 0 \). If \( d_H(A, C) < \varepsilon \), then \( x_0 + \varepsilon B \subseteq C \).

As a direct consequence of Fact 7 we obtain the following.

**Fact 8.** \( (K^n_{(0),b}, d_H) \) is an open set in \( (cb(n), d_H) \).

**Proposition 3.3.** For any integer \( n \geq 2 \), the following statements hold:

(1) \( K^n_{(0)} \) and \( K^n_{(0),b} \) are open subsets of \( K^n \). Furthermore, \( K^n_{(0),b} \) is a \( Q \)-manifold.

(2) \( K^n_0 \) is a contractible space.

(3) \( K^n_0 \) is a compact AR

**Proof.** (1) First, we show that \( K^n_{(0)} \subset K^n \) is open. Let \( A \in K^n_{(0)} \) and pick \( \varepsilon_0 > 0 \) such that \( \varepsilon_0 B \subset A \). We shall prove that every \( K \in K^n \) with \( d_{AW}(A, K) < \varepsilon_0 \) belongs to \( K^n_{(0)} \). Assume, without lost of generality, that \( \frac{1}{j+1} < \varepsilon_0 \leq \frac{1}{j} \) for some integer \( j \geq 1 \). By [2.2],

\[ \sup_{\|x\| < j} |d(x, A) - d(x, K)| < \varepsilon_0. \]
Let $p \in K$ be the closest point to 0. If $p \neq 0$, then $w := \frac{p}{||p||} \in \varepsilon_0B \subset A$ and $d(w, K) = \varepsilon_0 + ||p|| > \varepsilon_0$. Therefore

$$\varepsilon_0 < d(w, K) \leq \sup_{||x|| < j} |d(x, A) - d(x, K)| < \varepsilon_0,$$

a contradiction. This implies that $p = 0$ and hence $K \in \mathcal{K}_0^n$. We can now use Lemma 2.1 to infer that

$$d_H(A \cap jB, K \cap jB) = \sup_{||x|| < j} |d(x, A) - d(x, K)| < \varepsilon_0.$$

Observe that $A \cap jB \in \mathcal{K}^n_{(0),b}$ and $\varepsilon_0B \subset A \cap jB$. Thus, from Fact 7 it follows that $\frac{\varepsilon_0}{2}B \subset K \cap jB \subset K$ and therefore $K \in \mathcal{K}^n_{(0)}$, as desired.

In order to prove that $\mathcal{K}^n_{(0),b}$ is open, let us recall that $\mathcal{K}^n_{0} \subset \mathcal{K}^n$ is an open set (Fact 5). Thus $\mathcal{K}^n_{(0),b} = \mathcal{K}^n_{0} \cap \mathcal{K}^n_{(0)}$ is an intersection of two open sets and therefore it is also open in $\mathcal{K}^n_{0}$.

Recall the metrics $d_H$ and $d_{AW}$ generate the same topology on $\mathcal{K}^n_{(0),b}$ (Fact 2). In consequence, to prove that $\mathcal{K}^n_{(0),b}$ is a $Q$-manifold, it is enough to show that $(\mathcal{K}^n_{(0),b}, d_H)$ is a $Q$-manifold. Indeed, from Fact 8 it follows that $(\mathcal{K}^n_{(0),b}, d_H)$ is an open set in $(cb(n), d_H)$. Since $(cb(n), d_H)$ is a $Q$-manifold (Fact 6 and Theorem 2.2), $(\mathcal{K}^n_{(0),b}, d_H)$ must be a $Q$-manifold too (Fact 3).

(2) Notice that if we let $r(t) = \frac{1-t}{t}$, $t \in (0, 1]$, be the map of Lemma 3.1 then the corresponding map $F : \mathcal{K}^n_0 \times [0, 1] \to \mathcal{K}^n_0$ is a homotopy such that $F(A, 0) = A$ and $F(A, 1) = \{0\}$ for all $A \in \mathcal{K}^n_0$. Hence, $\mathcal{K}^n_0$ is contractible to $\{0\}$.

(3) We will show that $\mathcal{K}^n_0$ is compact with respect to the Fell topology $\tau_F$, then the compactness of $(\mathcal{K}^n_0, d_{AW})$ will follow from the fact that the topologies $\tau_F$ and $\tau_{AW}$ coincide on $\mathcal{K}^n_0$. Since $\mathcal{K}^n_0 = \mathcal{K}^n \cup \{0\}$ endowed with $\tau_F$ is the Alexandroff one-point compactification of $(\mathcal{K}^n, \tau_F)$ ([28, Proposition 1]), it is enough to prove that $\mathcal{K}^n_0$ is a closed subset of $\mathcal{K}^n_0$. But this follows directly from the fact that the complement $\mathcal{K}^n_0 \setminus \mathcal{K}^n_0 = (\mathbb{R}^n \setminus \{0\})^+$ is an open set in $\mathcal{K}^n_0$.

We turn to prove that $\mathcal{K}^n_0$ is an ANR. To do so, observe that by (1) of this proposition, $\mathcal{K}^n_{(0),b}$ is a $Q$-manifold, so it is an ANR (Fact 1). Even more, by the lemmas 3.1 and 3.2 the map $h : \mathcal{K}^n_0 \times [0, 1] \to \mathcal{K}^n_0$, defined by $h(A, t) = A \cap \frac{1-t}{t}B + tB$, $t \neq 0$, and $h(A, 0) = A$, is a homotopy such that $h_0$ is the identity map of $\mathcal{K}^n_0$, and $h(\mathcal{K}^n_0, t) \subseteq \mathcal{K}^n_{(0),b}$ for all $t \in (0, 1]$. Thus, $\mathcal{K}^n_{(0),b}$ is a homotopy dense subset of $\mathcal{K}^n_0$, and therefore,
by Property (R5), $\mathcal{K}_0^n$ is an ANR. Since $\mathcal{K}_0^n$ is a contractible space (see (2) of this proposition), it must be an AR (Property (R4)). □

In order to prove that $\mathcal{K}_n^0$ is homeomorphic to $Q$, let us recall what a $Z$-set is. A closed subset $Y$ of a metric space $(X,d)$ is called a $Z$-set if the set $\{f \in C(Q,X) : f(Q) \cap Y = \emptyset\}$ is dense in $C(Q,X)$. Here, $C(Q,X)$ denotes the space of continuous maps from $Q$ to $X$ endowed with the compact-open topology. In particular, if for every $\varepsilon > 0$, there exists a continuous map $f : X \to X \setminus Y$ such that $d(x, f(x)) < \varepsilon$, then $Y$ is a $Z$-set.

The following fact on $Q$-manifolds will be used in the proof of Theorem 3.4. It shows that any locally compact ANR containing a “big enough” $Q$-manifold as an open set, must be a $Q$-manifold.

**Fact 9.** Let $X$ be a locally compact ANR containing a $Z$-set $Y$. If $X \setminus Y$ is a $Q$-manifold, then $X$ must be a $Q$-manifold too.

The proof of this fact appears in the first paragraph of [33, §3].

**Theorem 3.4.** $\mathcal{K}_0^n$, $n \geq 2$, is homeomorphic to $Q$.

**Proof.** We already proved that $\mathcal{K}_0^n$ is a compact contractible AR (Proposition 3.3). Thus, since $Q$ is the only $Q$-manifold which is both compact and contractible (Theorem 2.3), it is enough to show that $\mathcal{K}_0^n$ is a $Q$-manifold. To this end, observe that $\mathcal{K}_n^0$ is an open set in $\mathcal{K}_0^n$ (Proposition 3.3-(1)). Therefore, $\mathcal{K}_0^n \setminus \mathcal{K}_n^0$ is closed in $\mathcal{K}_0^n$. Even more $\mathcal{K}_0^n \setminus \mathcal{K}_n^0$ is a $Z$-set. To prove this, observe that the map $h$, used in the proof of Proposition 3.3-(3), is a homotopy such that $h_0$ is the identity map of $\mathcal{K}_0^n$, and $h_t(\mathcal{K}_0^n) \subseteq \mathcal{K}_n^0$ for all $t \in (0,1]$. Thus, for every $\varepsilon > 0$, we can pick $\delta \in (0,1)$ such that $d_{AW}(A, h_t(A)) < \varepsilon$ for all $t \in (0, \delta)$. In consequence, the map $h_t : \mathcal{K}_0^n \to \mathcal{K}_n^0$, $t \in (0, \delta)$, is continuous and $d_{AW}(A, h_t(A)) < \varepsilon$ for all $A \in \mathcal{K}_n^0$. Hence, $\mathcal{K}_0^n \setminus \mathcal{K}_n^0$ is a $Z$-set. This, in combination with the fact that $\mathcal{K}_n^0 \subset \mathcal{K}_0^n$ is a $Q$-manifold (Proposition 3.3-(1)), proves that $\mathcal{K}_0^n$ is a $Q$-manifold too (Fact 9).

It is worth noting that, as a consequence of Proposition 3.3 and Theorem 3.4, $\mathcal{K}_n^0 \subset \mathcal{K}_0^n$ is an open set in a Hilbert cube, and therefore it is a $Q$-manifold (Fact 3). Theorem 3.4 also allows us to determine the homeomorphism type of $(\mathcal{K}_0^n, d_{AW})$. In order to do that, it should
be reminded that \( Q \) is a homogeneous space (see e.g. [20, Theorem 6.1.6]) and therefore \( Q \setminus \{ x \} \) and \( Q \setminus \{ y \} \) are homeomorphic for any \( x, y \in Q \). In what follows, \( Q \setminus \{ \ast \} \) denotes the Hilbert cube with a point removed.

**Corollary 3.5.** \( \mathcal{K}^n_{0,b} \) is homeomorphic to \( Q \setminus \{ \ast \} \).

**Proof.** It is a very well-known result that \( Q \setminus \{ \ast \} \) and \( Q \times [0, \infty) \) are homeomorphic spaces (see e.g. the proof of [14, Theorem 12.2]). Thus, in order to achieve our goal, it is enough to prove that \( (\mathcal{K}^n_{0,b}, d_{AW}) \) is homeomorphic to \( Q \times [0, \infty) \). Recall that \( \mathcal{K}^n_p \) is an open set in \( (\mathcal{K}^n, d_{AW}) \) (Fact 3) and \( \mathcal{K}^n_{0,b} = \mathcal{K}^n_0 \cap \mathcal{K}^n_b \). Thus, \( \mathcal{K}^n_{0,b} \) is an open set in \( \mathcal{K}^n \). Hence, by Theorem 3.1, \( \mathcal{K}^n_{0,b} \) is a \( Q \)-manifold. Moreover, since the metrics \( d_{AW} \) and \( d_H \) generate the same topology on \( \mathcal{K}^n_{0,b} \) (Fact 2), both \( (\mathcal{K}^n_{0,b}, d_H) \) and \( (\mathcal{K}^n_{0,b}, d_{AW}) \) are homeomorphic \( Q \)-manifolds. Then, to prove the result, it is enough to show that \( (\mathcal{K}^n_{0,b}, d_H) \) is homeomorphic to \( Q \times [0, \infty) \). Consider the map \( \nu : (\mathcal{K}^n_{0,b}, d_H) \to [0, \infty) \) defined as \( \nu(A) = \max_{a \in A} \| a \| \), for \( A \in \mathcal{K}^n_{0,b} \). In [3, Lemma 4.2], it was shown that \( \nu \) is a continuous map. Since \( \nu(rB) = r \) for every \( r \in [0, \infty) \), we also know that \( \nu \) is surjective. Furthermore, if \( W \subset [0, \infty) \) is a compact set, then \( \nu^{-1}(W) \subset \mathcal{K}^n_{0,b} \) is a closed bounded set. So, by the Blaschke Selection Theorem ([30, Theorem 1.8.6]), \( \nu^{-1}(W) \) must be compact. In addition, \( \nu^{-1}(r) \) is contractible for all \( r \in [0, \infty) \). Indeed, let \( f : \nu^{-1}(r) \times [0, 1] \to \nu^{-1}(r) \) be defined as \( f(A, t) = (1 - t)A + trB \), for \( t \in [0, 1] \) and \( A \in \nu^{-1}(r) \). If \( A \in \nu^{-1}(r) \) and \( a_0 \in A \) is such that \( \nu(A) = \| a_0 \| = r \), then

\[
\begin{align*}
(1 - t)a_0 + \frac{rt}{\| a_0 \|}a_0 &= r.
\end{align*}
\]

Hence, \( f(A, t) \in \nu^{-1}(r) \) for all \( t \in [0, 1] \), which proves that \( f \) is well-defined. The continuity of \( f \) follows directly from [34, Theorem 2.7.5] (see also [30, §1.8]). Clearly, \( f_0 \) is the identity map of \( \nu^{-1}(r) \) and \( f_1(A) = rB \) for all \( A \in \nu^{-1}(r) \). Therefore, \( f \) is a homotopy and \( \nu^{-1}(r) \) is a contractible space. We have thus shown that \( \nu \) is a CE-map. Hence, by Theorem 2.7.5 \( Q \times (\mathcal{K}^n_{0,b}, d_H) \) is homeomorphic with \( Q \times [0, \infty) \). However, since \( (\mathcal{K}^n_{0,b}, d_H) \) and \( Q \times (\mathcal{K}^n_{0,b}, d_H) \) are homeomorphic too (Theorem 2.4), then \( (\mathcal{K}^n_{0,b}, d_H) \) is homeomorphic with \( Q \times [0, \infty) \), as desired. \( \Box \)
4. Topology of the polar involution on $\mathcal{K}_0^n$

In Section 2 we have recalled some fundamental properties of the polar set $A^\circ$. Notice that properties (P1) and (P8) guarantee that the polar mapping

$$\alpha : \mathcal{K}_0^n \to \mathcal{K}_0^n$$
$$\alpha(A) = A^\circ$$

is a well-defined bijective map with the property that $\alpha(\alpha(A)) = A$, for all $A \in \mathcal{K}_0^n$. Moreover, by property (P5), $\alpha$ has a unique fixed point: the Euclidean ball $\mathbb{B}$. Furthermore, in [39, Theorem 7.2], R. Wijsman shows the following:

**Theorem 4.1.** The polar mapping $\alpha : \mathcal{K}_0^n \to \mathcal{K}_0^n$ is continuous with respect to the Wijsman topology $\tau_W$ on $\mathcal{K}_0^n$.

This theorem, in combination with Fact 1 (cf. [9, Theorem 3.1.4]), yields the following.

**Fact 10.** The polar map $\alpha$ is continuous with respect to $d_{AW}$.

These observations are summarized in the next proposition.

**Proposition 4.2.** The polar map $\alpha : (\mathcal{K}_0^n, d_{AW}) \to (\mathcal{K}_0^n, d_{AW})$ is a based-free involution. Moreover, $\mathbb{B} \in \mathcal{K}_0^n$ is the only fixed point of $\alpha$.

After Theorem 3.4, we know that $\mathcal{K}_0^n$ is homeomorphic to the Hilbert cube $Q$. This fact, in combination with Proposition 4.2, shows that the polar mapping determines a based-free involution on a Hilbert cube. Bearing in mind Anderson’s problem on the characterization of all based-free involutions on $Q$, it is then natural to ask for the relation between the involution $\sigma : Q \to Q$, given by $\sigma(x) = -x$, $x \in Q$, and $\alpha : \mathcal{K}_0^n \to \mathcal{K}_0^n$. This section is dedicated to prove Theorem 1, namely, that $\alpha$ and $\sigma$ are conjugate.

To achieve this, we will rely on the theory of $G$-spaces that we introduced in Subsection 2.3. We start by noticing that every involution $\beta : X \to X$ on a topological space $X$, induces a continuous action of the group $\mathbb{Z}_2 = \{-1, 1\}$ as follows

$$(1, x) \to x \quad \text{and} \quad (-1, x) \to \beta(x).$$

Observe that the $\mathbb{Z}_2$-action is based-free if and only if the involution $\beta$ is based-free. In this situation, the orbit space $X/\mathbb{Z}_2$ is precisely the
the quotient space $X/\beta$ induced by the involution. In particular, the involution $\alpha$ turns $K^n_0$ into a based-free $\mathbb{Z}_2$-space (see Subsection 2.3).

In order to prove Theorem 1, by Theorem 2.8, it is enough to show that the quotient space $K^n_0/\alpha$ induced by the polar mapping is an AR. However, by [1, Theorem 8], $K^n_0/\alpha$ is an AR provided that $K^n_0$ is a $\mathbb{Z}_2$-AR. Let us recall what this means.

A set $S \subset X$ is called invariant, if $hx \in S$ for every $(h, x) \in G \times S$. We say that a metrizable $G$-space $X$ is a $G$-equivariant absolute neighborhood retract (denoted by $G$-ANR) if for any metrizable $G$-space $Z$ containing $X$ as an invariant closed subset, there exists an invariant neighborhood $U$ of $X$ in $Z$ and a $G$-retraction $r : U \to X$. If we can always take $U = Z$, then we say $X$ is a $G$-equivariant absolute retract (denoted by $G$-AR).

To prove that $K^n_0$ is a $\mathbb{Z}_2$-AR, we will use the following result of S. Antonyan (see [3, Corollary 3.6]).

**Theorem 4.3.** Let $X \in \text{AR}$ be a metrizable based-free $G$-space, where $G$ is a compact Lie group. If $X$ is $G$-contractible, then $X$ is a $G$-AR.

In the previous theorem, a $G$-space $X$ is called $G$-contractible if there exists a $G$-fixed point $x_0 \in X$ and a homotopy $\varphi : X \times [0, 1] \to X$ such that $\varphi_0$ is the identity on $X$, $\varphi_t$ is a $G$-map for every $t \in [0, 1]$ and $\varphi_1(x) = x_0$ for every $x \in X$.

Notice that $K^n_0$ is indeed a based-free $\mathbb{Z}_2$-space, and $B$ is the unique fixed point. Furthermore, $K^n_0 \in \text{AR}$ (Proposition 3.3). Thus, if we translate Theorem 4.3 to our situation, we obtain the following remark that summarizes all the previous paragraphs.

**Remark 4.4.** $K^n_0$ is $\mathbb{Z}_2$-contractible if there exist a homotopy $\varphi : K^n_0 \times [0, 1] \to K^n_0$ such that $\varphi_0(A) = A$, $\varphi_1(A) = B^n$, and $\varphi_t(A^0) = \varphi_t(A)^0$ for every $A \in K^n_0$ and every $t \in [0, 1]$. Furthermore, if $K^n_0$ is $\mathbb{Z}_2$-contractible, then the following statements hold.

1. $K^n_0$ is a $\mathbb{Z}_2$-AR.
2. $K^n_0/\alpha$ is an AR.
3. Theorem 1 is true.

In the following pages, we will prove that $K^n_0$ is indeed $\mathbb{Z}_2$-contractible.
Throughout the rest of this section we will make extensive use of the map $\psi : K^n_0 \times [0, 1] \to K^n_0$ defined as

$$\psi(A, t) := \begin{cases} A & \text{if } t = 0, \\ (A + tB) \cap \frac{1-t}{t}B & \text{if } t \in (0, 1]. \end{cases}$$

\begin{lemma}
The map $\psi : K^n_0 \times [0, 1] \to K^n_0$ defined in (4.1) is continuous. Moreover, for every $t \in \left(\frac{\sqrt{5} - 1}{2}, 1\right)$ and every $A \in K^n_0$, $\psi(A, t) = \frac{1-t}{t}B$.
\end{lemma}

\textbf{Proof.} First, notice that the continuity of the map $H(A, t) = A + tB$, $A \in K^n_0$ and $t \in [0, 1]$, was proved in Lemma 3.2. Also, observe that if we let $r(t) = \frac{1-t}{t}$ be the map of Lemma 3.1, then the associated map $F$ is such that $\psi(A, t) = F(H(A, t), t)$ for every $A \in K^n_0$ and $t \in [0, 1]$. Hence, the continuity of $\psi$ follows from the continuity of $H$ and $F$.

To show that $\psi(A, t) = \frac{1-t}{t}B$ for $\frac{\sqrt{5} - 1}{2} \leq t < 1$, notice that in this interval we always have that $t \geq \frac{1}{1-t}$. Hence, for every $A \in K^n_0$ and $t \in \left[\frac{\sqrt{5} - 1}{2}, 1\right)$, we conclude that $\frac{1-t}{t}B = \{0\} + \frac{1-t}{t}B \subseteq A + tB$ and therefore $\frac{1-t}{t}B \subseteq \psi(A, t) \subseteq \frac{1-t}{t}B$. \hfill \Box

\begin{lemma}
For each $t \in (0, 1)$ and $A \in K^n_0$, let $\kappa_t(A)$ be the set

$$\kappa_t(A) := \operatorname{conv} \left( \bigcup_{a \in A} \left[ 0, \frac{1}{1+t\|a\|} \right] \cap tB \right).$$

Then $\kappa_t(A) \subseteq \psi_t(A^\circ)$. Recall that $\psi_t$ stands for the map $\psi(\cdot, t)$ and $\psi$ is the map defined in (4.1).
\end{lemma}

\textbf{Proof.} Let $A \in K^n_0$ and observe that $A = \bigcup_{a \in A} [0, a]$. Then, by property (P3), $A^\circ = \bigcap_{a \in A} [0, a]^\circ$ and therefore

$$A^\circ + tB = \bigcap_{a \in A} [0, a]^\circ + tB \subseteq \bigcap_{a \in A} \{0, a\}^\circ + tB.$$
Now we can use some of the basic properties of the polar set to infer that

\[
\psi_t(A^\circ) \supseteq \left( \bigcap_{a \in A} \{ [0, a]^\circ + tB \} \cap \frac{1 - t}{1} B \right)^\circ \\
= \text{conv} \left( \left( \bigcap_{a \in A} \{ [0, a]^\circ + tB \} \right)^\circ \cup \frac{t}{1 - t} B \right) \\
= \text{conv} \left( \bigcup_{a \in A} \{ [0, a]^\circ + tB \}^\circ \cup \frac{t}{1 - t} B \right) \\
= \text{conv} \left( \bigcup_{a \in A} \left[ 0, \frac{1}{1 + t\|a\|} \right] \cup \frac{t}{1 - t} B \right) \\
\supseteq \text{conv} \left( \bigcup_{a \in A} \left[ 0, \frac{1}{1 + t\|a\|} \right] \cup tB \right).
\]

Therefore \( \kappa_t(A) \subseteq \psi_t(A^\circ) \) for all \( A \in \mathcal{K}_0^n \). \( \square \)

**Theorem 4.7.** The space \( \mathcal{K}_0^n, n \geq 2 \), is \( \mathbb{Z}_2 \)-contractible to its fixed point \( B \).

**Proof.** Let \( g \) be the geometric mean on \( \mathcal{K}_{(0),b}^n \times \mathcal{K}_{(0),b}^n \) (see equation (2.3)), and consider the map \( \varphi : \mathcal{K}_0^n \times [0, 1] \to \mathcal{K}_0^n \) defined as

\[
\varphi(A, t) = \begin{cases} 
A & \text{if } t = 0, \\
g(\psi_t(A), \psi_t(A^\circ)) & \text{if } t \in (0, 1), \\
B & \text{if } t = 1,
\end{cases}
\]

for every \((A, t) \in \mathcal{K}_0^n \times [0, 1]\). Observe that \( \varphi_0 \) is the identity on \( \mathcal{K}_0^n \), and \( \varphi_1(A) = B \) for every \( A \in \mathcal{K}_0^n \). In order to complete the proof, it is enough to verify that \( \varphi \) is continuous and that \( \varphi_t(A^\circ) = \varphi_t(A)^\circ \) for every \( t \in [0, 1] \) and every \( A \in \mathcal{K}_0^n \). Clearly \( \varphi_0(A^\circ) = A^\circ = \varphi_0(A)^\circ \) and \( \varphi_1(A^\circ) = B = B^\circ = \varphi_1(A)^\circ \). If \( t \in (0, 1) \), we can use properties (\( \Gamma1 \)) and (\( \Gamma2 \)) to obtain

\[
\varphi_t(A^\circ) = g(\psi_t(A^\circ), \psi_t(A^\circ)) = g(\psi_t(A^\circ), \psi_t(A)^\circ) \\
= g(\psi_t(A)^\circ, \psi_t(A^\circ)) = g(\psi_t(A)^\circ, \psi_t(A)^\circ) \\
= g(\psi_t(A), \psi_t(A)^\circ)^\circ = \varphi_t(A)^\circ.
\]

Hence, \( \varphi_t \) is an equivariant map for every \( t \).
To prove that \( \varphi \) is continuous, take any pair \((K, s) \in K^n_0 \times [0, 1]\). We consider separately the cases \( s \in (0, 1) \), \( s = 1 \) and \( s = 0 \). The continuity of \( \varphi \) on pairs \((K, s) \), \( s \in (0, 1) \) and \( K \in K^n_0 \), follows from the continuity of both \( \psi \) (Lemma 4.5) and \( \alpha \) (Proposition 4.2), together with the continuity of the geometric mean w.r.t. \( d_{AW} \) on \( K^n_0 \). The latter is a consequence of the facts that \( g \) is continuous on \((K^n_0, d_H)\) (see (Г5)), and that the metrics \( d_{AW} \) and \( d_H \) induce the same topology on \( K^n_0 \) (Fact 2).

If \( s = 1 \), recall that \( \psi_t(K) = \psi_t(K^o) = \frac{1 - t}{t} \mathbb{B} \), for all \( K \in K^n_0 \), and \( t \in \left[ \frac{\sqrt{5} - 1}{2}, 1 \right) \) (Lemma 4.3). Hence, by properties (Γ3) and (P7), we know that

\[
\varphi(K, t) = g \left( \frac{1 - t}{t} \mathbb{B}, \frac{t}{1 - t} \mathbb{B} \right) = \mathbb{B}.
\]

Thus, for every \( \delta \in \left[ \frac{\sqrt{5} - 1}{2}, 1 \right) \), the set \( U := K^n_0 \times (\delta, 1] \) is an open neighborhood of the pair \((K, 1)\) such that \( \varphi|_U \) is constant. Therefore \( \varphi \) is continuous on \((K, 1)\), for every \( K \in K^n_0 \).

The case \( s = 0 \) will require the following claims:

**Claim 1.** For any \( t \in \left( 0, \frac{1}{2} \right) \) and \( A \in K^n_0 \) we have that

\[
\varphi_t(A) \subseteq A + \frac{t}{1 - t} \mathbb{B}.
\]

**Proof of Claim 1.** Let \( t \in \left( 0, \frac{1}{2} \right) \) and \( A \in K^n_0 \) be fixed. Since \( \psi_t(A) = (A + t \mathbb{B}) \cap \frac{1 - t}{t} \mathbb{B} \), then \( \psi_t(A) \subseteq A + t \mathbb{B} \subseteq A + \frac{t}{1 - t} \mathbb{B} \), and \( \psi_t(A) \subseteq \frac{1 - t}{t} \mathbb{B} \subseteq \frac{1}{t} \mathbb{B} \). So,

\[
(4.4) \quad \psi_t(A) \subseteq \left( A + \frac{t}{1 - t} \mathbb{B} \right) \cap \frac{1}{t} \mathbb{B}.
\]

On the other hand, we know that \( A^o \subseteq A^o + t \mathbb{B} \), hence \( (A^o + t \mathbb{B})^o \subseteq A^o = A \). Since 0 belongs to both \( (A^o + t \mathbb{B})^o \) and \( \mathbb{B} \), we have that \( (A^o + t \mathbb{B})^o \cup \frac{1}{1 - t} \mathbb{B} \subseteq A + \frac{t}{1 - t} \mathbb{B} \). In consequence, we can use the fact that
\( A + \frac{t}{1-t}B \) is closed and convex to infer that
\[
\psi_t(A^o) = \left( (A^o + tB) \cap \frac{1-t}{t}B \right)^o = \operatorname{conv}(A^o + tB) \cup \frac{t}{1-t}B \] \subseteq A + \frac{t}{1-t}B.
\]

Since \( 0 < t < \frac{1}{2} \), then \( tB \subseteq \frac{1-t}{t}B \). This yields to \( tB \subseteq (A^o + tB) \cap \frac{1-t}{t}B \). Therefore, \( tB \subseteq \psi_t(A^o) \) and \( \psi_t(A^o) \subseteq \frac{1}{t}B \). This, in combination with (4.5), implies that
\[
\psi_t(A^o) \subseteq \left( A + \frac{t}{1-t}B \right) \cap \frac{1}{t}B.
\]

Finally, we can combine the above inclusion together with (4.4), and properties (Γ1) and (Γ4) to obtain
\[
\varphi_t(A) = g(\psi_t(A), \psi_t(A^o)) 
\subseteq g \left( \left( A + \frac{t}{1-t}B \right) \cap \frac{1}{t}B, \left( A + \frac{t}{1-t}B \right) \cap \frac{1}{t}B \right) 
= \left( A + \frac{t}{1-t}B \right) \cap \frac{1}{t}B 
\subseteq A + \frac{t}{1-t}B.
\]

**Claim 2.** For every \( \varepsilon > 0 \), there exists \( \eta > 0 \) such that if \( 0 < t < \eta \), then \( d_{AW}(\varphi_t(A), A) < \varepsilon \) for all \( A \in K^n_0 \).

**Proof of Claim 2.** Let \( A \in K^n_0 \) and let us suppose that \( \frac{1}{j+1} < \varepsilon \leq \frac{1}{j} \) for some integer \( j \geq 1 \). Notice that, by Lemma 2.1.5, \( d_{AW}(\varphi_t(A), A) < \varepsilon \) if and only if \( d_H(\varphi_t(A) \cap jB, A \cap jB) < \varepsilon \). Thus, in order to prove the claim, we will show that \( d_H(\varphi_t(A) \cap jB, A \cap jB) < \varepsilon \) if \( t < \eta < \min\{\frac{1}{2}, \frac{\varepsilon}{\varepsilon+1}, \frac{1}{j+1}, \frac{\varepsilon}{j^2}\} \). Indeed, since \( \varphi_t(A) \subseteq A + \frac{t}{1-t}B \) (Claim 1) and \( \frac{t}{1-t} < \varepsilon \), then \( \varphi_t(A) \subseteq A + \varepsilon B \). In consequence, for every \( x \in \varphi_t(A) \cap jB \), \( d(x, A \cap jB) = d(x, A) < \varepsilon \) (Lemma 2.1.5). So \( \varphi_t(A) \cap jB \subseteq A \cap jB + \varepsilon B \).

To show that \( A \cap jB \subseteq \varphi_t(A) \cap jB + \varepsilon B \), pick an arbitrary point \( a \in A \cap jB \) and define \( b := \frac{1}{1+\|a\|}a \). Notice that \( b \in A \) and \( \|b\| \leq \|a\| \leq j \).
Moreover, since $t < \frac{1}{j+1}$, we also have that $\|b\| \leq j < \frac{1}{t}$. Hence, 

$$b \in (A + tB) \cap \frac{1-t}{t}B = \psi_t(A).$$

On the other hand, since $b \in \left[0, \frac{1}{1+\|a\|} \right] \subseteq \kappa_t(A)$ (where $\kappa_t(A)$ is the set defined in equation (4.2)), then $b \in \psi_t(A^\circ) \circ \psi_t(A^\circ)^\circ$. Thus, if we define $M := \psi_t(A) \cap \psi_t(A^\circ)^\circ$, we can use properties (Γ1) and (Γ4) to conclude that $b \in M = g(M, M) \subseteq g(\psi_t(A), \psi_t(A^\circ)^\circ) = \varphi_t(A)$.

Hence, $b \in \varphi_t(A) \cap jB$ and $\|a - b\| \leq \|a\|t^2 < j^2t < \varepsilon$. This proves that $A \cap jB \subseteq \varphi_t(A) \cap jB + \varepsilon B$ and therefore $d_H(\varphi_t(A) \cap jB, A \cap jB) < \varepsilon$, as desired.

We turn to prove the continuity of $\varphi$ at the point $(K,0) \in K_0^n \times [0,1]$. To this end, let $\rho > 0$ and choose $0 < \eta < \rho$ such that $d_{AW}(\varphi_t(A), A) < \frac{\rho}{3}$, for every $A \in K_0^n$ and $t \in (0,\eta)$ (Claim 2). Finally, pick any pair $(A,t) \in K_0^n \times [0,1]$ with $d_{AW}(K,A) < \frac{\rho}{2}$ and $t < \eta$. If $t = 0$, then $\varphi_t(A) = \varphi_0(A) = A$ and therefore 

$$d_{AW}(\varphi_0(A),\varphi_0(K)) = d_{AW}(A,K) < \frac{\rho}{2}.$$ 

On the other hand, if $t > 0$, by the choice of $\eta$, we conclude that 

$$d_{AW}(\varphi_t(A), K) \leq d_{AW}(\varphi_t(A), A) + d_{AW}(A,K) < \frac{\rho}{2} + \frac{\rho}{2} = \rho.$$ 

Therefore $\varphi$ is continuous at $(K,0)$ and now the proof is completed. \qed 

Finally, we can combine Theorem 4.7 with Remark 4.4 to obtain the following corollary.

**Corollary 4.8.**

1. $K_0^n$ is a $Z_2$-AR.

2. $K_0^n/\alpha$ is an AR.

3. Theorem 7 is true.

### 5. Polar preserving maps

A map $F : K_0^n \to K_0^n$ such that $F(A^\circ) = F(A)^\circ$, for all $A \in K_0^n$, is called polar-equivariant. In the proof of Theorem 4.7, we have constructed a family of polar-equivariant maps $\varphi_t := \varphi(\cdot,t)$ on $K_0^n$, for $t \in [0,1]$. As follows from their definition (4.3), the maps $\varphi_t$ are determined by two “global” operations: the polar mapping and the geometric mean. In contrast, there are polar-equivariant maps determined exclusively by point maps on $\mathbb{R}^n$. This is the case of the orthogonal maps $U \in O(n)$,
since the induced map \( \tilde{U} : K^n_0 \to K^n_0 \), given by \( \tilde{U}(A) := \{ U(a) : a \in A \} \), for \( A \in K^n_0 \), is a polar-equivariant map.

In the following, we investigate the properties of the polar-equivariant maps induced by maps \( f : \mathbb{R}^n \to \mathbb{R}^n \). To do so, we require the following definition.

**Definition 5.1.** A map \( f : \mathbb{R}^n \to \mathbb{R}^n \) is called polar preserving map if \( f(A^o) = f(A)^o \), for all \( A \in K^n_0 \).

The next properties are direct consequences of the above definition:

**Remark 5.2.** Let \( f : \mathbb{R}^n \to \mathbb{R}^n \) be a polar preserving map. Then the following hold:

(1) The induced map \( \tilde{f} : K^n_0 \to K^n_0 \) given by \( \tilde{f}(A) := f(A) \), for \( A \in K^n_0 \), is a well-defined polar-equivariant map.

(2) \( f \) is a surjective map and \( f(0) = 0 \).

(3) \( f(\mathbb{B}) = \mathbb{B} \).

**Proof.** (1) Let \( A \in K^n_0 \). By the Bipolar Theorem (P8), \( f(A) = f(A^o) = f(A)^o \). Since the polar set \( f(A^o) \) always belongs to \( K^n_0 \) (P1), \( f(A) \in K^n_0 \) too. In consequence, the map \( \tilde{f} \) is well-defined and clearly satisfies that \( \tilde{f}(A^o) = f(A^o) = f(A)^o = \tilde{f}(A)^o \) for all \( A \in K^n_0 \).

(2) By (1) of this remark, \( \tilde{f}([0]) = \{ f(0) \} \in K^n_0 \). Since the only singleton in \( K^n_0 \) is \( \{0\} \), we conclude that \( f(0) = 0 \). In addition \( f(\mathbb{R}^n) = f([0]^o) = \{0\}^o = \mathbb{R}^n \). Hence \( f \) is a surjective map.

(3) By (P5), \( f(\mathbb{B}) = f(\mathbb{B}^o) = f(\mathbb{B})^o \), and therefore \( f(\mathbb{B}) = \mathbb{B} \). \( \square \)

**Lemma 5.3.** Let \( f : \mathbb{R}^n \to \mathbb{R}^n \) be a map and \( T \in GL(n) \). If \( f([0,x]) = [0,T(x)] \) for every \( x \in \mathbb{R}^n \), then \( f = T \).

**Proof.** Since \( T(0) = 0 \), we obviously have \( f(0) = 0 = T(0) \). Now, if \( x \in \mathbb{R}^n \setminus \{0\} \), we can use the equality \( f([0,x]) = [0,T(x)] \) to find a \( t \in (0,1] \) such that \( f(tx) = T(x) \). Thus, \( T(x) \in f([0,tx]) = [0,tT(x)] \). But, this is only possible if \( t = 1 \) and therefore \( f(x) = T(x) \), as required. \( \square \)

**Lemma 5.4.** Let \( f : \mathbb{R}^n \to \mathbb{R}^n \) be a surjective map such that for every \( x, y \in \mathbb{R}^n \),

\[
\langle x, y \rangle \leq 1 \text{ if and only if } \langle f(x), f(y) \rangle \leq 1.
\]
Then $f$ is a polar preserving map.

Proof. Let $A \in K_0^n$. If $x \in A^0$, then $\langle a, x \rangle \leq 1$ for every $a \in A$. Hence, by (5.1), $\langle f(a), f(x) \rangle \leq 1$ for every $a \in A$. Thus, $f(x) \in f(A)^o$ and therefore $f(A^o) \subseteq f(A)^o$. Conversely, if $y \in f(A)^o$, by the surjectivity of $f$, there exists $x \in \mathbb{R}^n$ such that $y = f(x)$. In consequence $\langle f(a), f(x) \rangle \leq 1$ for all $a \in A$, and by (5.1), $\langle a, x \rangle \leq 1$ for every $a \in A$. This shows that $x \in A^o$ and therefore $y \in f(A^o)$. Hence, $f(A)^o \subseteq f(A^o)$ and consequently $f(A^o) = f(A)^o$, as desired. \hfill \Box

Below, we shall see that if $n \geq 2$, then the family of injective maps on $\mathbb{R}^n$ which are polar preserving consists only of orthogonal maps. This is not the case if $n = 1$, where we can find maps $f : \mathbb{R} \to \mathbb{R}$ which are bijective, polar-preserving but not orthogonal. For example, all the maps of the form $f_k(x) = x^{2k+1}$ with $k = 1, 2, \ldots$, satisfy condition (5.1) and therefore they are polar-preserving maps.

Recall that for every pair $A, K$ in $K_0^n$ ($K_{(0),b}^n$, resp.), the intersection $A \cap K$ and the closed convex hull $A \vee K := \overline{\text{conv}}(A \cup K)$ belong to $K_0^n$ ($K_{(0),b}^n$, resp.). In fact, these operations endow $K_0^n$ and $K_{(0),b}^n$ with a natural lattice structure (see e.g. [11, 31]).

In the main theorem of [11] and [31] Theorem 2] all endomorphisms of the lattices $(K_{(0),b}^n, \cap, \vee)$ and $(K_0^n, \cap, \vee)$ are characterized, respectively. These theorems are summarized in the following fact that will be used through the section.

**Fact 11.** Let $f : K_0^n \to K_0^n$ ($f : K_{(0),b}^n \to K_{(0),b}^n$, resp.) be a mapping satisfying

(5.2) $f(A \cap K) = f(A) \cap f(K)$ and $f(A \vee K) = f(A) \vee f(K),$

for all $A, K \in K_0^n$ ($A, K \in K_{(0),b}^n$, resp.). Then, either $f$ is constant or there exists a linear map $T \in GL(n)$ such that $f(A) = T(A)$ for all $A \in K_0^n$ (for all $A \in K_{(0),b}^n$, resp.).

**Proposition 5.5.** Let $n \geq 2$. If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a polar preserving injective map, then $f$ is an orthogonal map.

Proof. By Remark 5.2(1), the induced map $\tilde{f} : K_0^n \to K_0^n$ is a polar-equivariant map. Even more, $\tilde{f}$ is not constant, since $f(\{0\}) = \{0\}$ and $\tilde{f}(\mathbb{B}) = \mathbb{B}$ (Remark 5.2(2)-(3)). In addition, by the injectivity of
f, \tilde{f}(A \cap K) = \tilde{f}(A) \cap \tilde{f}(K) \text{ for every } A, K \in \mathcal{K}^n_0. \text{ Hence, by properties (P3), (P4) and (P8),}
\tilde{f}(A \lor K) = \tilde{f}((A \lor K)^\circ) = \tilde{f}((A^\circ \cap K^\circ)^\circ)
= \tilde{f}(A^\circ \cap K^\circ)^\circ = \tilde{f}(A^\circ) \cap \tilde{f}(K^\circ)^\circ
= \tilde{f}(A^\circ) \lor \tilde{f}(K^\circ) = \tilde{f}(A^\circ \lor K^\circ)
= \tilde{f}(A) \lor \tilde{f}(K).

In consequence, the map \tilde{f} is a non-constant endomorphism of the lattice \((\mathcal{K}^n_0, \cap, \lor)\) and, by Fact 11, there exists a linear isomorphism T : \(\mathbb{R}^n \to \mathbb{R}^n\) such that \(\tilde{f}(A) = T(A)\) for all \(A \in \mathcal{K}^n_0\). In particular, \(f([0, x]) = \tilde{f}([0, x]) = T([0, x]) = [0, T(x)]\) for every \(x \in \mathbb{R}^n\), and therefore \(f = T\) (Lemma 5.3). Finally, by Remark 5.2-(3), \(T(\mathbb{B}) = \tilde{f}(\mathbb{B}) = \mathbb{B}\), and hence \(T = f\) is an orthogonal map.

A direct application of Proposition 5.5 and Lemma 5.4 leads to the following characterization of the bijective maps satisfying condition (5.1).

**Remark 5.6.** Let \(f : \mathbb{R}^n \to \mathbb{R}^n, n \geq 2\), be a bijective map such that (5.1) holds. Then \(f\) is an orthogonal map.

We have been interested in polar-equivariant maps \(\tilde{f}\) on \(\mathcal{K}^n_0\) induced by maps \(f\) on \(\mathbb{R}^n\). However, one can consider a wider class of maps by allowing to take closures. The next proposition examines the properties of the maps \(f\) on \(\mathbb{R}^n\) such that \(\tilde{f}(A^\circ) = \tilde{f}(A)^\circ\), for all \(A \in \mathcal{K}^n_0\).

**Proposition 5.7.** Let \(f : \mathbb{R}^n \to \mathbb{R}^n, n \geq 2\), be a continuous injective map such that \(\overline{f(A^\circ)} = \overline{f(A)}^\circ\), for all \(A \in \mathcal{K}^n_0\). Then \(f\) is an orthogonal map.

**Proof.** Let us define \(F(A) := \overline{f(A)}\), for \(A \in \mathcal{K}^n_0\). We will show that the restriction \(F|_{\mathcal{K}^n_0} : \mathcal{K}^n_0 \to \mathcal{K}^n_0\) is a well-defined non-constant map satisfying the hypotheses of Fact 11. This would guarantee the existence of a linear isomorphism \(T : \mathbb{R}^n \to \mathbb{R}^n\) such that \(F(K) = T(K)\) for all \(K \in \mathcal{K}^n_0\). Our work then will consist in proving that \(f = T\) and that \(T\) is an orthogonal map.

Notice that \(F(A) = \overline{f(A^\circ)} = \overline{f(A)}^\circ = F(A^\circ)^\circ\), for all \(A \in \mathcal{K}^n_0\). Hence \(F(A) \in \mathcal{K}^n_0\) and \(F(A^\circ) = F(A)^\circ\) for all \(A \in \mathcal{K}^n_0\). It follows directly that
f(0) = 0, f(R^n) = R^n and f(B) = B. Moreover, by the continuity of f, f(K) is compact for all K ∈ K^n_{(0),b}. Thus, F(K) = f(K) = f(K) for each K ∈ K^n_{(0),b}. Furthermore, if K ∈ K^n_{(0),b}, K^o also belongs to K^n_{(0),b} (property (P1)) and then f(K^o) = f(K)^o is compact. From this, we infer that f(K) ∈ K^n_{(0),b} for all K ∈ K^n_{(0),b}. The previous observations show that F|_{K^n_{(0),b}} : K^n_{(0),b} → K^n_{(0),b} is well-defined, F(B) = B, and F(K) = f(K) for every K ∈ K^n_{(0),b}.

We can now proceed as in the proof of Proposition 5.5 to show that F(A \cap K) = F(A) \cap F(K) and F(A \cup K) = F(A) \cup F(K) for all A, K ∈ K^n_{(0),b}. Hence, by Fact 11, F(K) = T(K) for some linear isomorphism T on R^n and for all K ∈ K^n_{(0),b}. Clearly, T must be an orthogonal map, since F(B) = B. To prove that f = T, let x ∈ R^n and define, for each integer m ≥ 1, the set

X_m := [0, x] + \frac{1}{m}B.

Then f(X_m) = F(X_m) = [0, T(x)] + \frac{1}{m}B. Hence, (f(X_m))_m converges (w.r.t. d_H) to [0, T(x)]. Since (X_m)_m converges (w.r.t. d_H) to [0, x], and f is a continuous map, we have that f([0, x]) = [0, T(x)] for all x ∈ R^n. We can now use Lemma 5.3 to conclude that f = T, as required. □

6. DECREASING INVOLUTIONS ON K^n_0

In Theorem A ([31, Corollary 4]), the maps f : K^n_0 → K^n_0 satisfying conditions (D1) and (D2) are completely described. It is shown that such maps f are of the form f(A) = T(A^o) for some symmetric linear isomorphism T : R^n → R^n and all A ∈ K^n_0. Bearing in mind Theorem 1, it is then natural to ask when does a map f : K^n_0 → K^n_0 satisfying (D1) and (D2) is conjugate with the polar mapping (and therefore with the standard involution on Q). Theorem 2 answers this question, and the purpose of this section is to prove it.

The following remark will be used in the proof of Theorem 2. It establishes the continuity of the maps on K^n_0 induced by linear isomorphisms on R^n.

**Remark 6.1.** Let T : R^n → R^n be a linear isomorphism. Then, the maps T, f : K^n_0 → K^n_0, given by T(A) := T(A) and f(A) := T(A^o), are continuous.

**Proof.** Using that T is a continuous bijection, we can easily check that for every open set U ⊂ R^n and every compact set C ⊂ R^n, the following
equalities hold:
\[ \tilde{T}^{-1}(U^-) = (T^{-1}(U))^\circ \quad \text{and} \quad \tilde{T}^{-1}((\mathbb{R}^n \setminus C)^+) = (\mathbb{R}^n \setminus T^{-1}(C))^+ . \]
Thus, \( \tilde{T}^{-1}(U^-) \) and \( \tilde{T}^{-1}((\mathbb{R}^n \setminus C)^+) \) are open sets with respect to the Fell topology. The continuity of \( \tilde{T} \) then follows immediately from Fact 1. On the other hand, observe that \( f \) is such that \( f = \tilde{T} \alpha \). Therefore, its continuity follows from the continuity of \( \tilde{T} \) and the continuity of the polar mapping (Fact 10). □

We denote by \( M_n(\mathbb{R}) \) the vector space of square-matrices \( R = (r_{ij}), \ i, j = 1, \ldots, n \), with real entries. As usual, \( GL(n) \) and \( O(n) \) denote the set of linear isomorphisms on \( \mathbb{R}^n \) and the set of orthogonal maps on \( \mathbb{R}^n \), respectively. In what follows, we will not distinguish between a linear map on \( \mathbb{R}^n \) and its canonical matrix representation. For every \( T \in M_n(\mathbb{R}) \), we denote by \( T^\top \) the transpose of \( T \). Notice that in the case we are working on, \( T^\top \) coincides with the adjoint operator of \( T \). Hence, for every linear isomorphism \( T : \mathbb{R}^n \to \mathbb{R}^n \) and every \( A \in K_0^n \), the following formula holds:
\[ T(A^\circ) = [(T^\top)^{-1}A]^\circ \]
(see, e.g. [29, Chaper IV §2]).

Finally, we are in condition to prove Theorem 2.

**Proof of Theorem 2.** First, observe that from Remark 6.1, the map \( f : K_0^n \to K_0^n \) is continuous. In particular, it is an involution on \( K_0^n \).

Let us fix an orthogonal diagonalization of \( T \) by some \( U \in O(n) \), and a diagonal matrix \( D \in M_n(\mathbb{R}) \). Then \( T = U^\top DU \) and \( f(A) = U^\top DU(A^\circ) \) for all \( A \in K_0^n \).

(1) We shall show that if \( T \) is positive-definite, then \( f \) is conjugate with the polar mapping and, in particular, it is a based-free involution. Indeed, in this case, all diagonal entries \( d_{ii}, i = 1, \ldots, n, \) of \( D \) are strictly positive. Hence, the diagonal matrix \( R \) with entries \( r_{ii} = \sqrt{d_{ii}}, \ i = 1, \ldots, n, \) is well-defined and \( D = RR \). Furthermore, for every \( A \in K_0^n \),
\[ f(A) = U^\top RRU(A^\circ) = U^\top R \left[ (((RU)^\top)^{-1}A)^\circ \right] \]
\[ = U^\top R \left[ (((U^\top R)^{-1}A)^\circ \right] . \]
By letting \( \Psi = U^\top R \), we have that \( f(A) = \Psi((\Psi^{-1}A)^\circ) \) for all \( A \). The latter shows that \( f = \Psi \alpha \Psi^{-1} \), where \( \Psi : K_0^n \to K_0^n \) and \( \Psi^{-1} : K_0^n \to K_0^n \)
are the maps induced by Ψ and Ψ⁻¹, respectively. Clearly, the maps \( \tilde{Ψ} \) and \( \tilde{Ψ}^{-1} \) are continuous (Remark 6.1), and \( \tilde{Ψ}^{-1} = \tilde{Ψ}^{-1} \). In consequence, \( \tilde{Ψ} \) is a homeomorphism and therefore \( f \) is conjugate with the polar mapping \( α \), as desired.

(2) Let us suppose that \( T \) is not positive-definite. We shall show that \( f \) has infinitely many fixed points lying in \( K^n_0(b) \). Let \( S \in M_n(\mathbb{R}) \) be the diagonal matrix such that \( s_{ii} = \sqrt{|d_{ii}|} \) for \( i = 1, \ldots, n \). Notice that \( W = S^{-1}DS^{-1} \) is a diagonal matrix such that \( \varepsilon_i := w_{ii} \in \{-1, 1\} \) for \( i = 1, \ldots, n \). Moreover, \( \varepsilon_i = w_{ii} = -1 \) iff \( d_{ii} < 0 \). Clearly, \( W \) is an orthogonal matrix and \( W^{-1} = W \).

To exhibit an infinite family of fixed points, let us fix an index \( j \in \{1, \ldots, n\} \) such that \( d_{jj} < 0 \), and define the set

\[
A_j := \left\{ (a_1, \ldots, a_n) \in \mathbb{R}^n : a_j \geq \sqrt{\sum_{i \neq j} a_i^2} \right\}.
\]

The following is a well-known fact about the cone \( A_j \). However, we include its proof for the sake of completeness:

**Claim.** \( W(A_j) = A_j^0 \).

**Proof of the Claim.** Let \( a = (a_1, \ldots, a_n) \) and \( x = (x_1, \ldots, x_n) \) be points in \( A_j \). Then, by the Cauchy-Schwarz inequality,

\[
\langle W(a), x \rangle = \left( \sum_{i \neq j} \varepsilon_i a_i x_i \right) - a_j x_j \leq \left( \sqrt{\sum_{i \neq j} (\varepsilon_i a_i)^2} \right) \left( \sqrt{\sum_{i \neq j} x_i^2} \right) - a_j x_j
\]

Hence, \( \langle W(a), x \rangle \leq 0 \) and therefore \( W(a) \in A_j^0 \). This proves that \( W(A_j) \subseteq A_j^0 \).

On the other hand, if \( y = (y_1, \ldots, y_n) \in A_j^0 \), then for every integer \( m \geq 1 \) and every \( a = (a_1, \ldots, a_n) \in A_j \) such that \( a_j = \sqrt{\sum_{i \neq j} a_i^2} = m \), we have that

\[
\langle y, a \rangle = \left( \sum_{i \neq j} y_i a_i \right) + y_j m \leq 1 \quad \text{and therefore}
\]

\[
\frac{1}{m} \left( \sum_{i \neq j} y_i a_i \right) \leq \frac{1}{m - y_j}.
\]
Since \( m \) is an arbitrary positive integer, and the supremum of \( \frac{1}{m} \sum_{i \neq j} y_i a_i \) over the \((n-1)\)-tuples \((a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n)\) such that \( \sqrt{\sum_{i \neq j} a_i^2} = m \) is \( \sqrt{\sum_{i \neq j} y_i^2} \), we conclude that \( \sqrt{\sum_{i \neq j} y_i^2} \leq -y_j \). As a consequence, the point

\[
y_0 := W(y) = (\varepsilon_1 y_1, \ldots, \varepsilon_j - y_j, \varepsilon_j y_j, \ldots, \varepsilon_n y_n)
\]

belongs to \( A_j \). It then follows that \( y = W(y_0) \in W(A_j) \). We infer that \( A_j^o \subseteq W(A_j) \) and therefore \( A_j^o = W(A_j) \), as required. \( \blacksquare \)

Since \( W^{-1} = W \), then \( A_j = W(A_j^o) \). This, in combination with \((P1)\), implies that \( A_j \in \mathcal{K}_n^o \).

Now, let \( E_t, K_t \in \mathcal{K}_n^{o(0),b} \) with \( t \in (0,1) \), be defined as

\[
E_t := (A_j + t\mathbb{B}) \cap \left(\frac{1-t}{t}\right) \mathbb{B}, \quad \text{and} \quad K_t := \left((A_j^o + t\mathbb{B}) \cap \left(\frac{1-t}{t}\right) \mathbb{B}\right)^o.
\]

Using the claim, it is not difficult to see that \( K_t^o = W(E_t^o) \), and thus \( K_t = W(E_t^o) \). Moreover, the geometric mean \( P_t := g(E_t, K_t) \) belongs to \( \mathcal{K}_n^{o(0),b} \) and, by properties \((\Gamma1)\), \((\Gamma2)\) and \((\Gamma6)\), we have that

\[
P_t^o = g(E_t, K_t)^o = g(E_t^o, W(E_t)) = g(W(K_t), W(E_t)) = W(P_t).
\]

We have just constructed a family of solutions for the equation \( X^o = W(X) \) on \( \mathcal{K}_n^{o(0),b} \). Even more, since \( E_t = \psi_t(A_j), \quad K_t = (\psi_t(A_j^o))^o \) and \( P_t = \varphi_t(A_j) \) (where \( \psi \) and \( \varphi \) are the maps defined in \((4.11)\) and \((4.3)\), resp.), then \( P_t \rightarrow A_j \) as \( t \) tends to 0. Since \( A_j \) is an unbounded set, the family \( \{P_t : t \in (0,1)\} \subset \mathcal{K}_n^{o(0),b} \) must be infinite.

To finish the proof, let \( Y_t \) denote the set \( U^\top S(P_t) \), and notice that \( Y_t^o = (U^\top S(P_t))^o = [(U^\top S)^\top]^{-1}(P_t^o) = U^\top S^{-1}(P_t^o) \). Therefore,

\[
f(Y_t) = U^\top DU(Y_t^o) = U^\top DUU^\top S^{-1}(P_t^o) = U^\top DS^{-1}(P_t^o) = U^\top SW(P_t^o).
\]

Since \( P_t = W(P_t^o) \), then \( f(Y_t) = U^\top S(P_t) = Y_t \) is a fixed point of \( f \). Hence, the family \( \{Y_t : t \in (0,1)\} \subset \mathcal{K}_n^{o(0),b} \) consists only of fixed points of \( f \). In particular, \( f \) has infinite fixed points on \( \mathcal{K}_n^{o(0),b} \). \( \square \)

7. Final remarks and questions

In \cite{[23]}, algebraically inspired results for the class \( \mathcal{K}_n^{o(0),b} \) were obtained. There, by letting the polar set \( A^o \) play the role of the multiplicative inverse of \( A \), different real-equations can be translated to the context
of compact convex sets $K^n_{(0), b}$. In this sense, the equation $X^\circ = -X$ arises from the equation $x^{-1} = -x$ which has no solution over the real numbers. In contrast, as a consequence of Theorem 2-(2), the equation $X^\circ = -X$ has infinite solutions on $K^n_{(0), b}$. This fact follows directly from the remark below.

**Remark 7.1.** Suppose that $T \in GL(n)$ is symmetric and is not positive-definite, then the equation $X^\circ = T(X)$ has infinite solutions on $K^n_{(0), b}$.

The Hilbert cube $Q = \prod_{i=1}^{\infty} [-1, 1]$ has a natural lattice structure. In fact, for all $x = (x_i)_{i \in \mathbb{N}}, y = (y_i)_{i \in \mathbb{N}} \in Q$, the relation $\preceq$, defined by $x \preceq y$ iff $x_i \leq y_i$ for all $i \in \mathbb{N}$ is a partial order for which the operations $\lor$ and $\land$, defined as $x \lor y := (\max\{x_i, y_i\})_i$ and $x \land y := (\min\{x_i, y_i\})_i$, for $x, y \in Q$, provide a lattice structure. In this context, the standard based-free involution $\sigma$ satisfies, additionally, a condition similar to (D2): for $x, y \in Q$, $\sigma(y) \preceq \sigma(x)$ whenever $x \preceq y$. Namely, it is decreasing with respect to $\preceq$. Taking into account that in Theorem 1 we showed that every based-free decreasing involution on $K^n_{(0), b}$ is conjugate with the polar mapping, we propose the following weaker versions of Anderson’s problem:

**Question 1.** Is every based-free decreasing involution $\beta : Q \rightarrow Q$ conjugate with $\sigma$?

**Question 2.** Let $\beta : Q \rightarrow Q$ be a based-free involution with the property that there exist a lattice structure $(Q, \lor', \land')$ with respect to a partial order $\preceq'$ on $Q$, such that $\beta(y) \preceq \beta(x)$ whenever $x \preceq' y$. Is $\beta$ conjugate to $\sigma$?

Notice that the converse of Question 2 is trivially true.

Recall that a homeomorphism $\Phi : Q \rightarrow K^n_0$ is equivariant (with respect to $\sigma$ and $\alpha$) if $\Phi(-x) = \Phi(x)^\circ$ for every $x \in Q$. Since Theorem 1 guarantees the existence of at least one equivariant homomorphism, it is then natural to ask if there exist an equivariant homeomorphism $\Phi : Q \rightarrow K^n_0$ such that $x \preceq y$ if and only if $\Phi(x) \subset \Phi(y)$. However, the answer to this question is negative, as we explain in the following remark.

**Remark 7.2.** There is no homeomorphism $\Phi : Q \rightarrow K^n_0$ such that $x \preceq y$ if and only if $\Phi(x) \subset \Phi(y)$. Indeed, if such a homeomorphism exists, then it is not difficult to show that it would satisfy the following
two equalities

\[ \Phi(x \lor y) = \Phi(x) \lor \phi(y) \quad \text{and} \quad \Phi(x \land y) = \Phi(x) \land \Phi(y). \]

In particular, the following commutative diagram would hold:

\[
\begin{array}{ccc}
Q \times Q & \longrightarrow & Q \\
\Phi^{-1} \times \Phi^{-1} & & \Phi \\
\downarrow & & \downarrow \\
K^n_0 \times K^n_0 & \longrightarrow & K^n_0 \\
\end{array}
\]

Since the operation \( \lor \) is continuous on \( Q \), we would have that the operation \( \lor \) on \( K^n_0 \) is a composition of continuous maps, and therefore it must be continuous too. However, it is not difficult to see that \( \lor \) is not a continuous operation on \( K^n_0 \) and therefore we have a contradiction.

Finally, we want to point out another algebraic similitude between \( Q \) and \( K^n_0 \). For every \( x = (x_i)_{i \in \mathbb{N}} \), define \( \frac{1}{2}x := (\frac{1}{2}x_i)_{i \in \mathbb{N}} \). Clearly \( \frac{1}{2}x \in Q \) for every \( x \in Q \). If we consider the map \( \gamma : Q \times Q \to Q \) given by \( \gamma(x, y) := \frac{1}{2}x + \frac{1}{2}y \), then \( \gamma \) is well-defined and satisfies the following properties.

\((\tilde{\Gamma}1)\) \( \gamma(x, x) = x \) and \( \gamma(x, y) = \gamma(y, x) \).

\((\tilde{\Gamma}2)\) \( \gamma(-x, -y) = -\gamma(x, y) \).

\((\tilde{\Gamma}3)\) \( \gamma(x, -x) = 0 \) (where 0 is the element of \( Q \) with all its coordinates equal to 0).

\((\tilde{\Gamma}4)\) If \( x_1 \preceq x_2 \) and \( y_1 \preceq y_2 \), then \( \gamma(x_1, y_1) \preceq \gamma(x_2, y_2) \).

\((\tilde{\Gamma}5)\) The map \( \gamma \) is continuous.

The reader can notice that if we replace the role of the involution \( \sigma \) by \( \alpha \), and the order \( \preceq \) by the inclusion \( \subseteq \), then the above properties are analogous to properties \((\Gamma1)-(\Gamma5)\) of the geometric mean \( g \). However there is a significant difference: the map \( \gamma \) is defined on the the whole space \( Q \), while \( g \) is only defined on \( K^n_{(0),b} \).

**Remark 7.3.** For any equivariant homeomorphism \( \Phi : Q \to K^n_0 \), we can define the map \( g' : K^n_0 \times K^n_0 \to K^n_0 \) given by

\[ g'(A, K) = \Phi\left( \gamma(\Phi^{-1}(A), \Phi^{-1}(K)) \right). \]

(1) The map \( g' \) satisfies properties \( \Gamma1, \Gamma2, \Gamma3 \).
(2) The map $g'$ also satisfies the following property

$$(\Gamma'5) \text{ The map } g' \text{ is continuous w.r.t. } d_{AW}. $$

After Remark 7.3, we believe that the next final question could be of interest.

**Question 4.** Is it possible to construct explicitly a map $g' : K^n_n \times K^n_0 \to K^n_0$ satisfying properties $(\Gamma1)$, $(\Gamma2)$, $(\Gamma3)$ and $(\Gamma'5)$? And $\Gamma4$?

**Acknowledgments.** We wish to thank the anonymous referee for the constructive comments and recommendations which improved the final version of this paper.

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