GOLDBACH AND TWIN PRIME PAIRS: A SIEVE METHOD TO CONNECT THE TWO

Tom Milner-Gulland

ABSTRACT

This paper proposes, and demonstrates the efficacy of, a method for establishing a lower bound for cardinalities of selected sets of Goldbach pairs, and shows that the proofs employed may be modified for selected sets of twin primes. Our sieve method is centred on the binomial coefficient, contextualized by bijections from one selected interval to others. We implicitly employ the Chinese Remainder Theorem by way of the use of the midpoint in our intervals, and tacitly consider the sieve of Eratosthenes in such a way as to find a set of primes whose distribution is mirror-symmetrical about that midpoint. The existence of our conjectured lower bounds is established through the use of the formulae closely associated with the Mertens Theorem and we project the method into the general case.

Keywords: Goldbach pairs, twin primes, prime pairs, Goldbach and twin primes equivalence, Mertens Theorem, Euler totient, divisibility distributions, mirror symmetry, folded number scale.

INTRODUCTION

Throughout this paper, all intervals are to be taken to be non-empty sets, \([x, y]\), of integers, \(\mathbb{N}\) will be the set of non-negative integers, \(p_n\) for \(n = 1, 2, \ldots\) will be the sequence of primes; for any finite subset \(K\) of \(\mathbb{N}\) and any integer \(i\), \([K]^i\) will be the set of all subsets, \(S\), of \(K\) for which \(|S| = i\). Finally, \(P(n)\) will be \(\{p_1, p_2, \ldots, p_n\}\).

Our essential concept, which is one of a folding of the number scale, will be illustrated diagrammatically and also expressed algebraically. We consider that our inductive step is established by way of the employment of bijections devised ultimately to utilize the Euler totient in conjunction, tacitly, with the Chinese Remainder Theorem. We begin with a method that will make for a basis of a general study of maxima and minima of values given by chosen formulations, in the context of the set \(F(i)\) of all intervals such that for any \(I \in F(i)\), \(|I| = i\). These formulations entail functions on the set of all sets \(\{k \in I: (k, pqr) \neq 1\}\) and the set of all sets \(\{k \in I: (k, pqrs) \neq 1\}\), such that, in each case, \(p\), \(q\), \(r\) and \(s\) are distinct primes. Our subsequent employment of the Euler-Mascheroni constant is key to our development of this approach.

0.1. Extended introduction. Theorem 1 is a prelude to the sieve method that is the focus of this paper. In the subsequent theorem, key to our method is the set, for any integer \(n\) any even \(d\) and any integer \(i\),

\[
Z = \left\{ F \subseteq \{V_{i,p,k} : p \in P(n), k \in \{0, d\}\} : \text{for each } q \in P(n) \text{ and } m \in \mathbb{N} \right. \\
\left. \text{we have } V_{i,q,m} = \{1 \leq m \leq i : q \mid (m - k)\} \text{ and } |\{V_{i,q,k} : k \in \{0, d\}\} \cap F| = 1 \right\}.
\] (1)
which we ultimately use in the form $\bigcup\{\bigcup M : M \in \mathbb{Z}\}$. We show how our sieve method may be used to address the Goldbach conjecture when $d = 2i$ and the Twin Primes conjecture when $d = 2$. In the former case, our folding of the number scale may be understood in diagrammatic terms (see Fig. 1).

1. FURNISHING A SIEVE METHOD

**Theorem 1.** Let $J$ be any finite set of primes. Let $I$ be any interval. Then

$$\left|\{m \in I : p \mid m \text{ for some } p \in J\}\right| - \left|\{n \in [1,|I|] : p \mid n \text{ for some } p \in J\}\right| \leq ||J^2|| + 1. \quad (2)$$

1.1. **Remark.** We note that, for any integer $s$ and any set $M$ of integers and the finite set $J$ of primes given in Theorem[1]

$$\sum_{S \in [J]^s} \left|\{k \in M : (k, \prod_{n \in S} n) \neq 1\}\right| = \sum_{m \in M} \left|\{j \in J : j \mid m\}\right|^s. \quad (3)$$

We devise our method by formulating, implicitly or otherwise, for each $G \in [J]^r$, where $r \in \{3,4\}$, bijections from a suitable subset, $S_r$, of $\{k \in [0,|I| - 1] : (k, \prod_{g \in G} g) \neq 1\}$ to a suitable subset, $S'_r$, of $\{j \in I : (j, \prod_{g \in G} g) \neq 1\}$. Let $R_r$ and $R'_r$ be nonempty sets of integers for which, for each $s \in [1,r-1]$,

$$\sum_{m \in R_r} \left|\{j \in J : j \mid m\}\right|^s = \sum_{m' \in R'_r} \left|\{j \in J : j \mid m'\}\right|^s. \quad (4)$$

For any $F \in [J]^4$, we specify sets $A$ and $B$ of integers, for which $S_r \cup A$ and $S'_r \cup B$ together satisfy (4) for $S_r \cup A = R_r$ and $S'_r \cup B = R'_r$ when the number of $m \in R'_r$ for which $\left|\{j \in J : j \mid m\}\right| = r$ is one fewer than the number of $n \in R_r$ for which $\left|\{j \in J : j \mid n\}\right| = r$.

1.2. **Remark.** For the finite set, $J$, of primes given in Theorem[1] three is the smallest element in $\{i \in [1,|J|] : \binom{i}{2} > i - 1\}$. In view of our previous Remark, our method uses this fact by implicitly considering, for the interval $I$ as given in Theorem[1] the maximal value of

$$\left|\{k \in [i,|I| - 1 + i] : (k, \prod_{j \in J} j) \neq 1\}\right| - \left|\{m \in [0,|I| - 1] : (m, \prod_{j \in J} j) \neq 1\}\right|, \quad (5)$$

such that $i \in \mathbb{N}$, as being the maximal value of

$$\left|\{k \in [i,|I| - 1 + i] : (k, \prod_{h \in H} h) \neq 1 \text{ for some } H \in [J]^i\}\right|$$

$$- \left|\{m \in [0,|I| - 1] : (m, \prod_{h \in H} h) \neq 1 \text{ for some } H \in [J]^i\}\right| \quad (6)$$

such that $i \in \mathbb{N}$.

**Lemma 1.** For the finite set $J$ of primes given in Theorem[1] let $F \in [J]^4$ and let $T$ and $T'$ be sets of integers for which, for each $r \in \{1,2\}$ and each $H \in [F]^r$,

$$\sum_{m \in T} \left|\{j \in H : j \mid m\}\right|^r = \sum_{m' \in T'} \left|\{k \in H : k \mid m'\}\right|^r. \quad (7)$$
Then
\[ |\{ j' \in T' : u \mid j' \text{ for some } u \in \{ \prod_{h \in H} h : H \in [F]^r \} \}| \]
\[ - |\{ j \in T : u \mid j \text{ for some } u \in \{ \prod_{h \in H} h : H \in [F]^r \} \}| \]
\[ = \sum_{m \in \{ j \in T : u \mid j \text{ for some } u \in \{ \prod_{h \in H} h : H \in [F]^r \} \}} (|\{ h' \in F : h' \mid m \}| - 1) \]
\[ - \sum_{m' \in \{ j \in T' : u \mid j \text{ for some } u \in \{ \prod_{h \in H} h : H \in [F]^r \} \}} (|\{ h' \in F : h' \mid m' \}| - 1). \] (8)

Proof. The fact that, for any \( k \) for which
\[ |\{ u \in \{ \prod_{h \in H} h : H \in [F]^r \} : u \mid k \}| \geq 1 \] (9)
we have, for any \( U \in \{ T, T' \} \) for which \( k \in U \),
\[ \{ j \in U : u \mid j \text{ for some } u \in \{ \prod_{h \in H} h : H \in [F]^r \} \} \cap \{ k \} \]
\[ = \{ u \in \{ \prod_{h \in H} h : H \in [F]^r \} : u \mid k \} \]
\[ \neq \emptyset \] (10)
while \(|\{ k \}| = 1\), implies (8). \( \square \)

1.3. Remark. Our sets, as given in Lemma 1, \( T \) and \( T' \) will be taken to be \( S \) and \( S' \) respectively, and \( F \) to be \( L \), in our following lemma. Also, \( T \) and \( T' \) may be taken to be \( [0, |I| - 1] \cup M_R \) and \( I \cup N_R \) respectively in the definitions that immediately follow the same lemma, with \( F = R \).

Lemma 2. For the finite set \( J \) of primes given in Theorem 1 let \( L \in [J]^4 \). Let \( S \) and \( S' \) be sets of integers for which

I. for each proper subset, \( D \), of \( L \),
\[ |\{ k \in S : \prod_{d \in D} d \mid k \}| = |\{ k' \in S' : \prod_{d \in D} d \mid k' \}|, \] (11)

II. there exists \( s \in S \) for which \(|\{ j \in L : j \mid s \}| = 4 \) but no \( s' \in S \) for which \(|\{ j \in H : j \mid s' \}| = 4 \),

III. \(|S| = |S'|\).

Then
\[ |\{ m' \in S' : (m, \prod_{k \in L} k) \neq 1 \}| - |\{ m \in S : (m, \prod_{k \in L} k) \neq 1 \}| = 1. \] (12)

Proof. Let \( f : S \to S' \) be any bijection for which there exists \( k \) such that \(|\{ j \in L : j \mid k \}| = 4 \) and \(|\{ j \in L : j \mid f(k) \}| = 3 \) and, for some \( i \), \(|\{ j \in L : j \mid i \}| = 2 \) and \(|\{ j \in L : j \mid f(i) \}| = 3 \). Then since,
for any $u$ for which $\{j \in L : j \mid u\} \in [L]^2$ we have $\{|j \in L : j \mid u\} - 1 = 1$, it follows, through (3), that Lemma [1] for $T = S, T' = S'$ and $F = L$, implies that

$$
\{m \in S' : (m, \prod_{j \in L} j) \neq 1\} - \{|m \in S : (m, \prod_{j \in L} j) \neq 1\} = ([L]^3 - |\{k\}|)(|\{j \in L : f(i)\}|^2 - |\{j \in L : f(i)\}| - |\{j \in L : j \mid i\}|)^2 \\
- ([L]^3 - |\{k\}|)(|\{j \in L : j \mid f(i)\}|^2 - |\{j \in L : j \mid f(i)\}| - |\{j \in L : j \mid i\}|^2) \\
+ (|\{j \in L : j \mid f(k)\}| - |\{j \in L : j \mid k\}|)^2 \\
= 3(3 - 1) - 3(3 - 2) - (6 - 3) + (4 - 3) \\
= 6 - 3 + 3 + 1 \\
= 1.
$$

\hfill \square

1.4. Remark. Our sets, as given in Lemma [2] $S$ and $S'$ may be be taken to be $[0, |I| - 1] \cup M_R$ and $I \cup N_R$ respectively in our forthcoming section, taking also $R = L$.

1.5. Definition. For the finite set $J$ of primes and the interval $I$, given, in each case, in Theorem [1] let $R \in [J]^4$ and let $M_R$ and $N_R$ be sets of integers for which $|M_R \setminus [0, |I|]| = |N_R \setminus I|$ and

$$
\{|k \in [0, |I| - 1] \cup M_R : (k, \prod_{r \in R} r) \neq 1\} - \{|j \in I \cup N_R : (j, \prod_{r \in R} r) \neq 1\} = 1.
$$

We note, in reference to Lemma [2] (for $L = R$, suitably choosing $I, M_R$ and $N_R$), and thence to (13), that $\{r \in R : r \mid 0\} = R$ and we shall invoke Lemma [2] in our final use of the following bijection.

Let $h_R : [0, |I| - 1] \cup M_R \to I \cup N_R$ be a bijection for which, for all

$$
m \in \{k \in [0, |I| - 1] \cup M_R : (k, \prod_{p \in R} p) \neq 1\},
$$

for some $\{|0 \leq k \leq |I| - 1 : (k, \prod_{r \in R} r) \neq 1\}$-element subset, $W$, of $\{j \in I \cup N_R : (j, \prod_{r \in R} r) \neq 1\}$, we have $h_R(m) \in W$. Let $d_R$ be the element of

$$
\{k \in [1, |I| - 1] \cup M_R : (k, \prod_{p \in R} p) = 1\}
$$

for which

$$
h_R(d_R) \in \{j \in I \cup N_R : (j, \prod_{r \in R} r) \neq 1\} \setminus W.
$$

For all $E \in [J]^4$, choose $h_E$ so that $1 \leq d_E \leq |I| - 1$ and $h_E(d_E) \in I$.

For any subset, $Y$, of $J$, let $f_Y : [0, |I| - 1] \to I$ be a bijection for which

I. if

$$
\{|0 \leq k \leq |I| - 1 : (k, \prod_{m \in Y} m) \neq 1\} < |\{j \in I : (j, \prod_{p \in E} p) \neq 1\}|
$$

then for all $m \in \{0 \leq k \leq |I| - 1 : (k, \prod_{p \in Y} p) \neq 1\}$, for some $\{|0 \leq k \leq |I| - 1 : (k, \prod_{p \in Y} p) \neq 1\}$-element subset, $G$, of $\{j \in I : (j, \prod_{p \in Y} p) \neq 1\}$ we have $f_Y(m) \in G$;
II. for all $G$ and $H \in [J]^4$, for which $\{p \in G : p | h_G(d_G)\} \cap H \neq \emptyset$ we have
\[
\{h_H(d_H)\} \cap \{a \in I : a = f_H(m) \text{ for some } 0 \leq m \leq |I| - 1\} \in \{\{h_G(d_G)\}, \emptyset\};
\] (18)

III. for any $V \subseteq J$ for which $|V| > 4$,
\[
\{m \in I : (m, \prod_{p \in V} p) = 1 \text{ and } (f_V(m), \prod_{p \in V} p) \neq 1\} \cap \{h_G(d_G) : G \in [V]^4\}
\]
is a subset of
\[
\bigcup\{\{a \in I : a = f_G(m) \text{ for some } m | (m, \prod_{k \in G} k) = 1 \text{ and } (f_G(m), \prod_{k \in G} k) \neq 1\} \cap \{h_G(d_G) : G \in [V]^4\}\}.
\] (19)

**Remark.** Let $I$ and $J$ be as in Theorem II. We note that if (17) is true for $Y = J$ then I requires that
\[
|\{a \in I : a = f_J(m) \text{ for some } m | (m, \prod_{p \in J} p) = 1, (f_J(m), \prod_{p \in J} p) \neq 1\}|
\]
\[
= |\{j \in I : (j, \prod_{p \in J} p) \neq 1\}| - |\{k \in [0, |I| - 1] : (k, \prod_{p \in J} p) \neq 1\}|;
\] (20)

Let
\[
B = \{\{b_T : T \in [J]^4\} : b_T \text{ satisfies all conditions on } h_T\} \cup \{c : c \text{ satisfies all conditions on } f_J\}.
\] (21)

Choose each element of $\{h_T(d_T) : T \in [J]^4\}$ so that
\[
|\{a \in I : a = f_J(m) \text{ for some } m | (m, \prod_{p \in J} p) = 1 \text{ and } (f_J(m), \prod_{p \in J} p) \neq 1\} \cap \{h_T(d_T) : T \in [J]^4\}|
\]
\[
= \max\{|\{a \in I : a = c(m) \text{ for some } m | (m, \prod_{p \in J} p) = 1 \text{ and } (c(m), \prod_{p \in J} p) \neq 1\}|
\]
\[
\cap \{b_T(d_T) : T \in [J]^4\} : c, \{b_T : T \in [J]^4\} \in B\}.
\] (22)

1.6. **Remark.** **MOVE**

I. Let $P$ be any set of sets of integers for which, for some integer $a$ and some subset $D$ of $P$ we have $|\{a\} \cup \bigcup D| = |\bigcup D| + 1$. Then it is elementary that the equation $|\{a\} \cup \bigcup P| = |\bigcup P| + 1$ is true if and only if the following is true. For each $K$ in the set of all subsets of $P$ for which $a \notin K$,
\[
|\{a\} \cup \bigcup K| = |\bigcup K| + 1.
\] (23)

II. Let $I$ and $J$ be as in Theorem II. Take the following cited variables to be as given in their respective references.

i. For all references to condition II in Definition 1.5, take $a = h_E(d_E)$, where $E \in \{G, H\}$, $P = \{\{m \in I : p | m\} \setminus \{h_E(d_E)\} : p \in G \cup H\}$ and $D = \{\{m \in I : p | m\} \setminus \{h_E(d_E)\} : p \in G\}$. For all references
to condition III in Definition 1.5 take \( a = h_G(d_G) \), \( P = \{ \{ m \in I : p \mid m \} \setminus \{ h_G(d_G) \} : p \in V \} \) and \( D \) as in the previous sentence for \( E = G \).

ii. For all conditions to the condition on each element, \( c \), of \( \{ h_T(d_T) : T \in [J]^4 \} \), take \( a = c \), \( P = \{ \{ m \in I : (m, \prod_{p \in T} p) \neq 1 \} \setminus \{ h_T(d_T) \} : T \in [J]^4 \} \) and, for any \( T \in [J]^2 \), \( D = \{ \{ m \in I : (m, \prod_{p \in T} p) \neq 1 \} \setminus \{ h_T(d_T) \} \} \). Here we note that, for the set, \( L \), of all \( m \) such that \( 0 \leq m \leq |I| - 1 \) for which \( (m, \prod_{p \in J} p) = 1 \) while \( (j,m), \prod_{p \in J} p \neq 1 \),

\[
|L| = |\{ m \in I : (m, \prod_{p \in J} p) \neq 1 \}| - |\{ 1 \leq m \leq |I| : (m, \prod_{p \in J} p) \neq 1 \}|.
\]

(24)

We will show that we may apply (23) to justify, from the point of view of our condition, the conditions cited above.

1.7. Remark. For the finite set \( J \) of primes and the interval \( I \) given, in each case, in Theorem 1.1, we have the following.

i. For each \( r \in \{ 1, 2 \} \)

\[
|\{ H \in [J]^r : |\{ 0 \leq k \leq |I| - 1 \} : (k, \prod_{q \in H} q) \neq 1 \}| = |\{ j \in I : (j, \prod_{q \in H} q) \neq 1 \}| + 1
\]

\[
= \sum_{0 \leq m \leq |I| - 1} |\{ p \in J : p \mid m \}| - \sum_{m \in I} |\{ k \in J : k \mid m \}| - 1
\]

\[
\leq |J|^r.
\]

(25)

ii. For all integers, \( n \), for which \( |\{ p \in J : p \mid n \}| \geq 2 \), we have

\[
\frac{|\{ p \in J : p \mid n \}|}{2} \geq 1
\]

(26)

with \( |\{ p \in J : p \mid n \}| = 2 \) when the left-hand side is equal to one.

iii. For all \( x \), the fact that for any \( x \)-element subset, \( Q \), of \( J \), \( \prod_{p \in Q} p \) divides zero implies that

\[
\sum_{m \in I} |\{ K \in [J]^x : \prod_{k \in K} k \mid m \}| \geq \sum_{1 \leq n \leq |I|} |\{ K \in [J]^x : \prod_{k \in K} k \mid n \}|.
\]

(27)

iv. Consider any sets \( T \) and \( T' \) that satisfy all the stated conditions on \( S \) and \( S' \) as, likewise, given in Lemma 2 respectively. Then there exists, for each \( K \in [L]^3 \), for \( L \in [J]^4 \) as given in Lemma 2 a bijection \( g_{T,T',L} : T \rightarrow T' \) for which for all elements, \( m \), except one, of \( \{ t \in T : \prod_{k \in K} k \mid t \} \),

\[
|\{ j \in L : j \mid m \}| = |\{ j \in L : j \mid g_{T,T',L}(m) \}|.
\]

(28)

1.8. Further conditions and remarks. Let \( J \) and \( I \) be as given in Theorem 1.1. For each \( H \in [J]^4 \) choose \( M_H \) and \( N_H \) so that \( [0, |I| - 1] \cup M_H \) and \( I \cup N_H \) satisfy all the stated conditions on \( T \) and \( T' \), as given in iv in Remark 1.7 respectively when \( H \) is, there, substituted for \( L \). This is feasible because there is no condition on \( M_H \) and \( N_H \) requiring that they be intervals. Further, choose each element of \( \{ h_U : U \in [J]^4 \} \) so that

I. for all \( G, G' \in [J]^4 \) for which \( |G \cap G'| = 3 \), for all \( u \) for which \( |\{ j \in G : j \mid u \}| \neq 3 \) and for each \( K \in \{ G, G' \} \), \( |\{ j \in K : j \mid h_K(u) \}| = 3 \), we have \( h_G(u) = h_{G'}(u) \);

II. for the bijection \( g_{T,T',L} \) given in iv in Remark 1.7 taking \( T = [0, |I| - 1] \cup M_H, T' = I \cup N_H \) and \( H = L \) we have \( h_H = g_{[0, |I| - 1] \cup M_H, I \cup N_H, H} \).
For any integer $t$, let $F_t$ be any subset of the set of ordered pairs

$$\{(m, W) : m \in I \text{ and } (m, \prod_{p \in W} p) \neq 1, W \in [J]^t\}$$

(29)

for which $|F_t| = \left|\{K \in [J]^t : \prod_{k \in K} k \mid m\}\right|$. Such a subset exists, since for each $C \in [J]^t$ we have

$$\left|\left\{1 \leq m \leq |I| : \prod_{k \in C} k \mid m\right\}\right| \leq \left|\{m \in I : \prod_{k \in C} k \mid m\}\right|.$$

(30)

In our forthcoming (55), we tacitly take, for any $n \in [1, |I|] \cup I$,

$$\frac{\left|\left\{p \in J : p \mid n\right\}\right|^3}{\left|\left\{p \in J : p \mid n\right\}\right|^2}$$

(31)

and

$$\frac{\left|\left\{p \in J : p \mid n\right\}\right|^2}{\left|\left\{p \in J : p \mid n\right\}\right|}$$

(32)

as a ratio, with our $k$, as in (55), taken as $\left|\left\{p \in J : p \mid n\right\}\right|$. The value $k$ may otherwise be taken as $t$, and we proceed to choose $t = 3$. Now choose $F_3$ and $\{h_G : G \in [J]^4\}$ (in so doing, choosing $\{M_H : H \in [J]^3\}$ and $g_{[0,|I|-1]}:M_{H_1,J_{H_1},H_3}$ as cited in II, above) and thence

$$Y = \{(m, G) : m \in [0, |I| - 1] \cup M_G \& \mid \{j \in G : j \mid h_G(m)\} \mid |j \in G : j \mid t| \neq 3, G \in [J]^4\}$$

(33)

so that

$$\{(a, A) : a \in I, A \in \{[p \in J : p \mid a]\}^3 \setminus \{H \in [J]^3 : (m, H) \in F_3 \text{ for some } m \in I\}\}$$

(34)

is a subset of $Y$. Then we have the following.

i. Condition II in Definition 1.5 requires that for each $H \in [J]^4$ and any $G$ for which

$$\{j \in G : j \mid h_G(d_G)\} \subset H,$$

(35)

the value of

$$\left|\{h_H(d_H)\} \cap \{a \in I : a = f_H(m) \text{ for some } 0 \leq m \leq |I| - 1\}\right|$$

(36)

is less than or equal to the right-hand side of (12) in Lemma 2 (for $L = H$), which is equal to one; concomitantly we recall condition III in Definition 1.5.

ii. Recalling that

$$\min\left\{1 \leq i \leq |J| : \left(\frac{i}{2}\right) > i - 1\right\} = 3,$$

(37)

for any integer $m$, it is elementary that we have the following. Each element of $\{[K \in [J]^3 : \prod_{p \in J} p \mid m]\}^3$ either is a subset of a four-element subset of $\{j \in J : \prod_{p \in J} p \mid m\}$ or satisfies $\left|\{j \in J : \prod_{p \in J} p \mid m\}\right| = 3$. Here we find the value two, in (37), through ii in Remark 1.7.

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1.9. **Prelude to further exposition.** Let \( J \) be the finite set of primes and \( I \) be the interval, given, in each case, in Theorem 1. Take it that the cited conditions on \( M_R \) and \( N_R \) are those given in Definition 1.5 save for the following. Assume now that for each \( R \in [J]^4 \), \( M_R \) and \( N_R \) satisfy all stated conditions on \( S \) and \( S' \) as in Lemma 2 respectively, with \( L = R \). This assumption is, once more, allowed by there being, for all \( R \in [J]^4 \), no requirement that \( M_R \) and \( N_R \) be intervals.

For any subset \( T \) of \( J \), where \( |T| \geq 3 \), \(|T|^3/|T|^2 = (|T| - 2)/3 \) is an increasing function of \(|T|\). This, coupled with Lemma 2 requires the following. The values that, together with the value of \(|I|\), may be combined to determine the value of

\[
|\{m \in I : (m, \prod_{p \in J} p) \neq 1\}| - |\{1 \leq m \leq |I| : (m, \prod_{p \in J} p) \neq 1\}|. 
\tag{38}
\]

are I to IV, below:

I. \[
\sum_{H \in [J]^2} (|\{m \in I : \prod_{p \in H} p | m\}| - |\{0 \leq m \leq |I| - 1 : \prod_{p \in H} p | m\}|); \tag{39}
\]

II. \[
\sum_{H \in [J]^2} (|\{m \in I : \prod_{p \in H} p | m\}| - |\{0 \leq m \leq |I| - 1 : \prod_{p \in H} p | m\}|); \tag{40}
\]

III. \[
\sum_{H \in [J]^2} (|\{0 \leq m \leq |I| - 1 : \prod_{p \in H} p | m\}| - |\{m \in I : \prod_{p \in H} p | m\}|); \tag{41}
\]

IV. \[
\sum_{G \in [J]^2} |\{m \in I : \prod_{g \in G} g | k\} \cap \{m \in I : \prod_{q \in Q} q | k \text{ for some } Q \in [H]^3\}| 
- \sum_{G \in [J]^2} \{0 \leq m \leq |I| - 1 : \prod_{g \in G} g | k\} \cap \{m \in I : \prod_{q \in Q} q | k \text{ for some } Q \in [H]^3\}. \tag{42}
\]

To expand, consider any finite set \( E \) of \(|I|\)-element intervals, \( U_n \), where \( N \in \mathbb{N} \) and \( E = \{U_1, U_2, \ldots, U_{|E|}\} \). For any integer \( k \), denote \( t_k \) to be \((39)\), \( u_k \) to be \((40)\) and \( v_k \) to be \((42)\), with \( I = E_k \). Then for some functions \( a, b \) and \( c \) for which, for each \( d(q_k) \in \{a(t_k), b(u_k), c(v_k)\} \), where \( q_k \in \{t_k, u_k, v_k\} \), \( d(q_k) \) is an increasing function of \( q_k \), with \( d(q_k) < 0 \) if an only if \( q_k < 0 \), we have the following. For any \( j \) for which

\[
|\{m \in U_k : (m, \prod_{p \in J} p) \neq 1\}| < |\{m \in U_j : (m, \prod_{p \in J} p) \neq 1\}| \tag{43}
\]

we have

\[
a(t_k) + b(u_k) + c(v_k) < a(t_j) + b(u_j) + c(v_j). \tag{44}
\]

This follows from Lemma 2 combined with the fact that, in reference to the combination of all of I to IV, \((37)/\left(\frac{\delta}{\gamma}\right) \) is an increasing function of \( x \). Here, we take \( x = |\{p \in J : p | m\}| \) for each \( m \in K \) where \( K \in \{I, |I| - 1\} \). Crucially, here, our conditions on \( S \) and \( S' \) imply that, for each \( G \in \{I \cup N_R, [0, |I| - 1] \cup M_R\} \), we have \(|\{m \in G : (m, \prod_{p \in J} p) \neq 1\}| \) is an increasing function of
\[ |\{ m \in G : (m, \prod_{p \in J} p) \neq 1 \} \cap \{| p \in J : p \mid m \}| = 1 \}! \]. This can be seen in that the second of these values is, in turn, an increasing function of the average value of
\[
\left( \frac{|\{ p \in J : p \mid m \} |}{3} \right) \left( \frac{|\{ p \in J : p \mid m \} |}{2} \right)
\]
among all \( m \in K \). This, in turn, can be seen in that the ratio of \(|\{ p \in J : p \mid m \}|\) to \(\left( \frac{|\{ p \in J : p \mid m \} |}{2} \right) \) is greatest when \(|\{ p \in J : p \mid m \}| = 2 \). Here, we combine this with the fact that \(|\{ p \in J : p \mid m \} | = 2 \) is naturally a decreasing function of the average value of \(|\{ p \in J : p \mid m \}|^2\) among all \( m \in K \). This, finally, is an increasing function of the average value of \(|\{ p \in J : p \mid m \}|^2\) among all \( m \in K \). Here we contextualise both the fact that \( \binom{2}{2} \) is an increasing function of \( x \), and also IV, in particular. Further, we have shown that, through the output values for the binomial coefficient, \( (38) \) is determinable when I to IV are all known values.

**Remark.** We shall use \( (42) \) to contextualise \( F_{1} \) as given in Further Conditions and Remarks \([1, 8] \) here taking \( t = 3 \).

**Lemma 3.** For the finite set \( J \) of primes and the interval \( I \) given, in each case, in Theorem\([7] \)
\[
|\{ j \in I : (j, \prod_{p \in J} p) \neq 1 \}| - |\{ 0 \leq k \leq |I| - 1 : (k, \prod_{p \in J} p) \neq 1 \}|
\leq |\{ 0 \leq m \leq |I| - 1 : (m, \prod_{p \in J} p) = 1, (f_J(m), \prod_{p \in J} p) \neq 1 \}|
\cap \{ h_T(d_T) : T \in [J]^4 \} + |[J]^4|.
\]

**Proof.** First we note that for some \( u \) for which
\[
0 \leq u \leq \sum_{m \in I} |\{ j \in I : j \mid m \}|^2 - \sum_{0 \leq m \leq |I| - 1} |\{ j \in I : j \mid m \}|^2,
\]
and some \( v \) for which
\[
0 \leq v \leq \sum_{m \in I} |\{ j \in I : j \mid m \}| - \sum_{0 \leq m \leq |I| - 1} |\{ j \in I : j \mid m \}|,
\]
we have
\[
|\{ j \in I : (j, \prod_{q \in J} q) \neq 1 \}| - |\{ 0 \leq k \leq |I| - 1 : (k, \prod_{q \in J} q) \neq 1 \}| = u + v
\]
\[
= |\{ a \in I : a = f_J(m) \text{ for some } m | (m, \prod_{p \in J} p) = 1, (f_J(m), \prod_{p \in J} p) \neq 1 \}|
\cap \{ h_T(d_T) : T \in [J]^4 \}.
\]
This follows by the conditions I to III, in Definition\([1, 5] \) on \( f_Y \) where \( Y \) is any subset of \( J \). In particular, it follows through \( (20) \) combined with the condition \( (42) \) on each element of \( \{ h_T(d_T) : T \in [J]^4 \} \), itself combined with \( (??) \). Here, we combine \( (??) \) with the fact that, for any sets \( K \) and \( K' \) of integers for which, for each \( j \in \{1, 2\} \),
\[
\sum_{G \in [J]^k} |\{ m \in K : (m, \prod_{p \in G} p) \neq 1 \}| = \sum_{G \in [J]^k} |\{ m \in K' : (m, \prod_{p \in G} p) \neq 1 \}|,
\]
we have the following. The value
\[
|\{ m \in K' : (m, \prod_{p \in J} p) \neq 1 \}| - |\{ m \in K : (m, \prod_{p \in J} p) \neq 1 \}|
\]
is, through the fact that
\[
\frac{|\{p \in J : p | a\}|}{|\{p \in J : p \mid a\}|} > \frac{2}{3} = \frac{1}{2}
\]  
(52)
when \(|\{p \in J : p \mid a\}| > 2\) and where \(a \in K \cup K'\), an increasing function of
\[
\sum_{\substack{m \in K'\{p \in J : \mid p | m\}| > 2}} \left(\frac{|\{p \in J : p \mid m\}|}{2} - (|\{p \in J : p \mid m\}| - 1)\right)
\]  
- \[
\sum_{\substack{m \in K\{p \in J : \mid p | m\}| > 2}} \left(\frac{|\{p \in J : p \mid m\}|}{2} - (|\{p \in J : p \mid m\}| - 1)\right).
\]  
(53)
The \(\text{'} - 1\text{'}\) terms in (53) are found by the fact that when merely one element, \(p\), of \(J\) divides any integer \(i\), we have \((i, \prod_{p \in J} p) \neq 1\), so
\[
|\{j \in J \setminus \{p\} : j \mid i\}| = |\{j \in J : j \mid i\}| - 1.
\]  
(54)
However, recalling our reference in Further Conditions and Remarks 1.8 to our forthcoming 55), for all \(k \geq 3\),
\[
\frac{\binom{k}{3}k}{k^2} = \frac{4(k-1)(k-2)}{6(k-1)^2}
\]  
\[
= \frac{2(k-2)}{3(k-1)}
\]  
(55)
is an increasing function of \(k\) where we take \(k = |\{p \in J : p \mid m\}| \) for any \(m \in [1, |J|] \cup I\). Now, for Lemma 2 take \(K = S\) and \(K' = S'\). Then the conditions I and II on \(S\) and \(S'\) in Lemma 2 together imply (49). We obtain this result through the condition (23) on \(h_d : T \in [J]^{\delta}\) combined with use of the terms \(\text{'}+\text{'}\) and \(\text{'}+\text{'}\) in (49). Here, we use (25) combined with ii in Remark 1.7.

Our next equality is as follows. We have
\[
|\{a \in I : a = f_J(m)\} \text{ for some } m, (m, \prod_{p \in J} p) = 1, (f_J(m), \prod_{p \in J} p) \neq 1\}
\]  
\[
\cap \{\text{element } t \in [J]^{\delta}\} = \sum_{0 \leq m \leq |I| - 1} (|\{p \in J : p \mid m\}| - 1) - \sum_{n \in \text{U}} (|\{p \in J : p \mid n\}| - 1) - u + v.
\]  
(56)
Here, the right-hand side is found once more by (54). Thus (56) follows through (49).

Recall our set, for any integer \(t\),
\[
F_t = \{(m, W) : m \in I \text{ and } (m, \prod_{p \in W} p) \neq 1, W \in [J]^{\delta}\}
\]  
(57)
in Further Conditions and Remarks 1.8 and our reference to our (55) made on its introduction. Choosing \(t = 3\) for (14) enables us to consider the following.

I. The combination of \(i\) in Further Conditions and Remarks 1.8 and (27) for \(x = k\).

II. The combination of the equality (49) with (56). Here we note, in reference to the right-hand side of (56), the following. The values three and two for \(\binom{3}{3}\) and \(\binom{2}{2}\) respectively, in (55), are the values \(t\) and \(\text{'} + \text{'}\) in (56), where \(m \in [0, |I| - 1] \cup I\) and \(S \in \{J\} \cup [J]^{\delta} \cup [J]^{\delta}\) (the element \(J\) is used for (56)), respectively. In the latter case the two is found by the superscript \('2\') in (47).
Combining the two combinations cited in I and II gives, through the fact that the left-hand side of (55) is an increasing function of \( k \), (46). Here, the term '\( +|J|^2| \) is found by subtracting \( -u + v \) from both sides of (49) while combining (25), for \( r = 2 \), with ii in Remark [1.7] \( \square \)

1.10. **Remark.** Let J and I be as in Theorem 1.1. We note the following.

I. Condition II in Definition [1.5] requires that for each \( H \in [J]^4 \), and any \( K \) for which

\[
\{ p \in K : p \mid h_K(d_K) \} \subset H, \tag{58}
\]

the value of

\[
|\{ h_{\alpha}(d_{\alpha}) \} \cap \{ a \in I : a = f_H(m) \text{ for some } m \in [0,|I| - 1] \}| \tag{59}
\]

is less than or equal to the right-hand side of (12) in Lemma 2, which is equal to one.

II. Let \( M \) and \( N \) be distinct elements of \([J]^4 \). Suppose that \( h_M(d_M) \) is in

\[
\{ a \in I : a = f_J(m) \text{ for some } m \mid \{ j \in J : j \mid m \} = \emptyset \text{ and } \{ j \in J : j \mid f_J(m) \} \neq \emptyset \}. \tag{60}
\]

Then, by choosing, for each \( U \in \{M,N\}, h_U(d_U) \) so that \( |\{ j \in U : j \mid h_U(d_U) \}| < 4 \) we ensure that there exists \( P \in [M \cup N]^4 \) for which \( \{ m \in I : (m, \prod_{j \in P} j) \neq 1 \} \) contains both \( h_M(d_M) \) and \( h_N(d_N) \).

Further, for all \( Q \in [J]^4 \setminus \{M,N,P\} \) for which \( \{ j \in P : j \mid h_P(d_P) \} \subset \emptyset \), we have the following. Condition II in Definition [1.5] requires, for \( G = P \) and \( H = \| \) that, for each \( R \in \{P,Q\}, \)

\[
\{ t \in T : T \in [M \cup N]^4 \} = \{ t \in R(d_R) \}. \tag{61}
\]

**Lemma 4.** For the finite set \( J \) of primes and the interval \( I \) given, in each case, in Theorem 1.1

\[
|\{ a \in I : a = f_J(m) \text{ for some } m \mid (m, \prod_{p \in J} p) = 1, (f_J(m), \prod_{p \in J} p) \neq 1 \}|
\]

\[
\cap \{ h_T(d_T) : T \in [J]^4 \}
\]

\[
\in \{0,1\}. \tag{62}
\]

**Proof.** Let \( K, M, N, P \) and \( Q \) be as in Remark 1.10. First, we note that if \( |J| < 8 \), (62) follows immediately from condition II in Definition 1.5. Conversely, if \( |J| \geq 8 \) there exists an eight-element subset, \( S \), of \( J \), for which \( M \subset S \) and for all \( A \in [S]^4 \), for some \( B \in [S]^4 \setminus \{A\} \) we have \( A \cup B \neq \emptyset \). Hence we may substitute any element of \([S]^4 \) for \( M \) in II in Remark 1.10 and there exists \( G \in [S]^4 \) for which \( G \cap M = \emptyset \). Concomitantly we may choose \( S \) and \( P \) so that we may substitute \( Q \) for \( G \). Therefore, through the combination of (61) and I in Remark 1.10, condition III in Remark 1.5 requires that

\[
|\{ a \in I : a = f_J(m) \text{ for some } m \mid \{ j \in J : j \mid m \} = \emptyset \text{ and } \{ j \in J : j \mid f_J(m) \} \neq \emptyset \}|
\]

\[
\cap \{ h_T(d_T) : T \in [J]^4 \}
\]

\[
\leq |\{ h_M(d_M) \}| + |\{ U \in [J]^4 : \#D|U \cup M \subseteq D \text{ and } D \text{ satisfies all conditions on } S \}|.
\]

\[
= 1. \tag{63}
\]

Therefore, (62) follows from Lemma 2 \( \square \)

**Proof of Theorem 1.1**

**Proof.** Take \( J \) to be the finite set of primes and \( I \) the interval given Theorem 1.1. Recall that I and II in Prelude to Further Exposition 1.9 are the are two considerations that may be combined to determine

\[
|\{ m \in I : (m, \prod_{p \in J} p) \neq 1 \}| - |\{ m \in [1,|I|] : (m, \prod_{p \in J} p) \neq 1 \}|. \tag{64}
\]
Now apply II.i and II.ii to I, each in Remark 1.6. Then through I in the proof of Lemma 3, the stated conditions, for any subset $S$ of $J$, on $f_S$ require that combining Lemma 3 with Lemma 4 gives (4). □

2. THE FOLDED THE NUMBER SCALE

2.1. Method outline. Let $x$ and $y$ be integers for which $y - x$ is even. Our forthcoming exposition employs mirror symmetry in the context of an interval, $[x, y]$, and $a$ and $b$ in $[x, y]$ for which $a + b = x + y$. Indeed, the context of our employment of $(x + y)/2$ will imply a rephrasing of the Goldbach Conjecture, familiar as every even number greater than two is the sum of two primes, to every integer greater than three is the arithmetic mean of two primes.

For all $n$, one side of the 'fold', for our folded number scale, will be taken to be $[1, (p_n^2 + 1)/2]$; the other, specifically in the case of the Goldbach conjecture, will be $[(p_n^2 + 1)/2, p_n^2 + 1]$. We shall consider the objects of interest in our folded number scale as being subdivided into two distributions. One such is the distribution of all integers that are coprime to $\prod_{k=1}^{n} p_k$. This distribution is folded, which is to say that we apply functions to it in the context of the mirror symmetry discussed above. The other is, in effect, an intermediate to the two sides of this folded distribution: a function on a domain, defined by selected divisibility distributions of the elements of $P(n)$, which tacitly invokes the Chinese Remainder Theorem.

Remark. Let $y$ be any integer $> 2$. Suppose that $0 \leq m < y$. If $y + m$ and $y - m$ are both prime, and $d$ is even. Then for the Goldbach equation $p + q = d$, the substitution of $2y$ for $d$ yields a solution, with $p = y - m$. In this context, by employing

$$\left\{j \in [1, y] : \left(j(2y - j), \prod_{k=1}^{n} p_k\right) = 1\right\},$$

otherwise written

$$\left\{j \in [1, y] : \left(j, \prod_{k=1}^{n} p_k\right) = 1 \text{ and } \left(2y - j, \prod_{k=1}^{n} p_k\right) = 1\right\},$$

we may devise a method to study the divisibility distributions of interest. In our exposition we use $-p = m - y$ instead of $p = y - m$, as this enables a proof of the the Twin Primes conjecture.

2.2. Remark. Since there is no $m$ such that $0 < m < p_n^2$ and $p \in P(n) : p \mid m = \{p_{n+k}\}$, where $k \geq 1$, there is no composite in $\{k \in [1, p_n^2] : (k, \prod_{k=1}^{n} p_k) = 1\}$ (hence our respecting the sieve of Eratosthenes, by employing $(p_n^2 + 1)/2$ as the cardinality of intervals with which we shall ultimately be working). Therefore, for all $n > 2$

$$\left\{m \in [1, \frac{p_n^2 + 1}{2}] : \left(m(m - p_n^2 + 1), \prod_{k=1}^{n} p_k\right) = 1\right\}$$

is a subset of the set of all primes, $p$, such that for some prime, $q, p + q = (p_n + 1)/2$.

2.3. Remark. Let $n > 4$ and $z$ be any integer for which $p_n^2/2 < z < p_{n+1}^2$. Key to our method is the expression

$$\prod_{k=1}^{n} \left(1 - \frac{1}{p_k}\right) \prod_{m=2}^{n} \left(1 - \frac{1}{p_m - 1}\right).$$

12
We note that, for any \( p \in P(n) \) and any \( r \), when \( p \mid r \) we have (by virtue of the mirror symmetry, about \( r \), of the distribution of integer multiples of \( p \)),

\[
|\{ j \in [1, z] : p \mid j \}| = |\{ j - r : j \in [1, z], \ p \mid (2z - j) \}|,
\]

(69)

but when \( p \nmid r \), (69) does not hold, bringing into play the expression

\[
\prod_{k=1}^{n} \left( 1 - \frac{1}{p_k} \right) \prod_{g \in P(n)} \left( 1 - \frac{1}{g - 1} \right).
\]

(70)

Hence the fact that, when \( p = 2 \),

\[
\left( 1 - \frac{1}{p} \right) \left( 1 - \frac{1}{p - 1} \right) = 0,
\]

(71)

which does not hold when \( p > 2 \), ultimately gives the reason why we impose the condition in forthcoming lemmas that \( r \) is even.

2.4. Definition. For any even \( r \), any \( n > 4 \) and any set \( M \) of integers, any \( p \in P(n) \), any integer \( k \) and any set \( M \) of integers, define

\[
V_{M,p,k} = \{ h \in M : \ p \mid h - k \}
\]

(72)

and

\[
R(M, n, r) = \left\{ F \subseteq \{ V_{M,p,k} : p \in P(n) \ & k \in \{0, r\} \} : \text{for each } q \in P(n) \text{ we have} \right. \\
\left. |\{ V_{M,q,k} : k \in \{0, r\} \} \cap F| = 1 \right\}.
\]

(73)

We note that, for \( Z \), \( i \) and \( d \) as in our Extended Introduction, \( Z = R([1, i], \pi(\sqrt{n}), d) \). Also, for \( n > 4 \) and \( z \) for which \( p_n^2/2 < z < p_{n+1}^2/2 \), and any \( T \in R([1, z], n, r) \), for some \( z \)-element interval \( I_T \), for any \( v \) the \( v \)-th highest element of \( \bigcup T \) is equal to \( u - \min I_T + 1 \), where \( u \) is the \( v \)-th highest element of \( \{ j \in I_T : j, \prod_{k=1}^{n} p_k \neq 1 \} \). This follows from the fact that, for any two integers \( p \) and \( q \) for which \( p \) is coprime to \( q \), and all \( m \) such that \( 0 < m \leq p \), there exists an integer \( 0 \leq k < q \) such that \( q \mid (kp + m) \). Suppose that \( p = \prod_{g \in G} g \) and \( q = \prod_{h \in H} h \) where \( G \) and \( H \) are any pairwise distinct, nonempty subsets of \( P(n) \). Thus for all integers \( i \) and \( j \),

\[
\exists u : \prod_{g \in G} g \mid (u + i) \text{ and } \prod_{h \in H} h \mid (u + i + j)
\]

(74)

where the conditions on \( G \) and \( H \) allow us to use \( u = \min I_T \). We note that for any \( s \in L \), where \( L \in \{ G, H \} \), \( \min\{u \leq a \leq u + z - 1 : s \mid a\} - u \) is a constant function of \( u \). For any \( p \in P(n) \) let \( 0 \leq k_p \leq p \). Then since we may hold \( i \) constant while increasing \( j \) by increments of one, it follows by (74) that, for each \( Q \) in

\[
\{ \{ (b_p, p) : p \in P(n) \} : h_p \text{ satisfies all conditions on } k_p \}
\]

(75)

we have, for some \( z \)-element interval \( U \),

\[
Q = \{ (\min\{\min U \leq a \leq \max U : q \mid a\} - \min U, q) : q \in P(n) \}.
\]

(76)
Thus by choosing, for each $p \in P(n)$, $k_p$ so that $(K_p, p) \in Q$ where $k_p = \min \{a \in I_T : p \mid a \} - \min I_T$ we have
\begin{equation}
\bigcup \{ \{ \min I_T + k_p + ip : i \in \mathbb{N} \} : p \in P(n) \} \cap I_T = \left\{ a \in I_T : \left( a, \prod_{k=1}^{n} p_k \right) \neq 1 \right\}.
\end{equation}

We use the above fact to deduce, using
\begin{equation}
\max \left\{ \left| \left( k, \prod_{k=1}^{n} p_k \right) \neq 1 \right| : T \in R([1, z], n, r) \right\},
\end{equation}
an upper bound on $|\bigcup \{ T : T \in R([1, z], n, r) \}|$.

2.5. Remark. We note that, for each $p \in P(n)$ we have $F = \{ p + i_p, 2p + i_p, \ldots, \max F \}$, where $F \in \bigcup R([1, z], n, r)$, for some integer $i_p$. The fact that $i_p \in \{ 0, u \}$ where $1 \leq u < p$, gives either one or two possible values for $i_p$ among all sets that satisfy all conditions on $F$. There is one such value, $i_p = 0$, for the case when $p \mid r$ and two when $p \not{\mid} r$. Hence $|\bigcup R([1, z], n, r)| = 2n - |\{ p \in P(n) : p \mid r \}|$. This implies the following. First, for any set $N$ of integers, let $F$ and $F'$ be elements of $R(N, n, r)$ for which
\begin{equation}
F \cap F' = \{ \{ m \in N : q \mid m \} : q \in P(n) \mbox{ and } q \mid r \}.
\end{equation}
Then
\begin{equation}
\bigcup (F \cup F') = \bigcup \bigcup R(N, n, r) = \bigcup \{ \bigcup V : V \in R(N, n, r) \}.
\end{equation}
Second, we may make the following development upon the above. For any $U \in R(N, n, r)$, let $L_{U, N}$ be any set of integers for which $|[M \setminus L_{U, N} : M \in U]| = n$, choosing $N$ so that such is possible. Let $E$ and $E'$ be elements of $\{ \{ M \setminus L_{U, N} : M \in U \} : U \in R(N, n, r) \}$ for which there exist $D$ and $D'$ in $R(N, n, r)$ for which $E = \{ M \setminus L_{D, N} : M \in Y \}$ and $E' = \{ M \setminus L_{D', N} : M \in Y' \}$, such that (79) is true for $D = F$ and $D' = F'$. Then each element of $\{ (\bigcup M) \setminus L_{M, N} : M \in R(N, n, r) \}$ is in $E$ or in $E'$, because $|E| = n$ and $|E'| = n$. Further
\begin{equation}
\bigcup (E \cup E') = \{ \bigcup A \setminus (L_{D, N} \cap L_{D', N}) : A \in R(N, n, r) \}.
\end{equation}
It follows by the definition of $R(N, n, r)$ that $E$ and $E'$ exist.

2.6. Definition. For and even $r$, any $n > 4$, any set $Z$ of integers for which $|Z| > 2$, and any $T \in R(Z, n, r)$, let $S_{T, Z, n, r}$ be any subset of $\bigcup T$ for which I to III, below, are true.

I. We have
\begin{equation}
|S_{T, Z, n, r}| = \left| \bigcup T \right| - \left| \left| Z \right| \left( 1 - \prod_{k=1}^{n} \left( 1 - \frac{1}{p_k} \right) \right) \right|
\end{equation}
when the right-hand side of (82) is $> 0$.

II. Let $T'$ be the element of $R(Z, n, r)$ for which
\begin{equation}
T \cap T' = \{ \{ m \in N : q \mid m \} : q \in P(n) \mbox{ and } q \mid r \}.
\end{equation}
Then for any element, $Y$, of $\{ T, T' \}$ for which $|S_{Y, Z, n, r}| = \max \{|S_{V, Z, n, r} : V \in \{ T, T' \} \}$, and for $Y' \in \{ T, T' \} \setminus \{ Y \}$, the set $S_{Y', Z, n, r} \cup S_{Y, Z, n, r}$ is a subset of $S_{B, Z, n, r} \cup S'_{B', Z, n, r}$ for some $B$ and $B'$ in $R(Z, n, r)$ for which
\begin{equation}
|S_{B, Z, n, r}| = \max \{|S_{M, Z, n, r} : M \in R(Z, n, r)\}
\end{equation}
and
\begin{equation}
|S_{B', Z, n, r}| = \max \{|S_{M, Z, n, r} : M \in R(Z, n, r) \setminus B\}.
\end{equation}
and we impose the further condition that $S_{B,Z,n,r} \cap S_{B',Z,n,r} = \emptyset$.

We note the following.

i. The set $\{u \in Z : 2 | u\}$ is in all elements of $R(Z,n,r)$, so we may choose any subset, proper or otherwise, of $S_{Y,Z,n,r} \cup S_{Y',Z,n,r}$ to be a subset of $\{u \in Z : 2 | u\}$.

ii. The feasibility of II follows by i.

2.7. Remark. For $r$ and $n$ be as in Definition 2.6 let $z$ be any integer for which $p_n^2/2 < z < p_{n+1}^2/2$. Then for some $x$ and any $M \in R([1, z], n, r)$, we have

$$\{p : \{p + m, 2p + m, \ldots, xp + m\} \in M\} = P(n).$$

This follows by the fact that, since $n > 4$, we have $\prod_{k=1}^{n} (1 - 1/p_k) < 1/4$. Recall that $|\bigcup K_{z,r}| = \max\{|\bigcup X : X \in R([1, z], n, r)|\}$. Then $2|\bigcup K_{z,r,[1, z], n, r}|$ is sufficiently small that the number of integer multiples, of each $p \in P(n)$, in $\bigcup (R([1, z], n, r))$ is sufficiently large that $|\{M \setminus S_{M,[1, z], n, r} : M \in R([1, z], n, r)\}| = n$.

2.8. Remark. Let $r, n$ and $z$ be as Remark 2.7. By requiring that each element of $\{S_{U,[1, z], n, r} : U \in R([1, z], n, r)\}$ is a subset of $S_{K_{z,r},[1, z], n, r}$ we may use $K_{z,r} - S_{K_{z,r},[1, z], n, r}$ exclusively for their cardinality. Therefore, the choice of $K_{z,r}$ and $S_{K_{z,r},[1, z], n, r}$ among all sets that satisfy all the stated conditions on each respectively is immaterial.

Introduction to Theorem 2. In what follows, once we have, first, implicitly used the Euler totient in such a way as to find the value of

$$\left|\left\{1 \leq n \leq \prod_{p \in P(n)} p : \left(\prod_{p \in P(n)} p\right)^r = 1\right\}\right|,$$

through the proof of our forthcoming lemma we may, second, use, multiplicatively, the value $\prod_{p \in \{P(n), p \geq r\}} (p - 2)$ in the way we are about to put forward. This will enable future use of our resulting expression (by substituting $P(n) \setminus \{2\}$ for the implicitly used $\{p \in P(n) : p \geq r\}$ and dividing the right-hand side of the final equality of our forthcoming by $\prod_{k=1}^{n} p_k$, in the form

$$\prod_{k=1}^{n} \left(1 - \frac{1}{p_k}\right) \prod_{m=2}^{n} \left(1 - \frac{1}{p_m - 1}\right).$$

(We note, incidentally, that since $\phi(p) - 1 = p - 2$ while $\phi(p) = p - 1$, the above value is equal to $(1/2) \prod_{m=2}^{n} (1 - 2/p_m)$.)

Theorem 2. Every integer $w > 3$ can be written as the average of two positive prime numbers.

Lemma 5. For any even $r$ and any $n \geq 1$ we have

$$\left|\left\{1 \leq u \leq \prod_{k=1}^{n} p_k : \left(u(u - r), \prod_{k=1}^{n} p_k\right) = 1\right\}\right| = \prod_{k=1}^{n} (p_k - 1) \prod_{k=2}^{n} \prod_{p_k \mid r} \left(1 - \frac{1}{p_k - 1}\right).$$

Proof. For each prime $p \leq p_n$ for which $p \nmid r$ and all $i \in \mathbb{N}$,

$$|\{u \in [1 + i, p + i] : (u(u - r), p) = 1\}| = |\{u \in [1 + i, p + i] : (u, p) = 1\}| - 1.$$
Since the first term on the right-hand side of (90) is equal to \( \phi(p) = p - 1 \) and the left-hand side, \( \phi(\phi(p)) = p - 2 \), we have
\[
\left| \left\{ 1 \leq u \leq \prod_{k=1}^{n} p_k : \left( u(u - r), \prod_{k=1}^{n} p_k \right) = 1 \right\} \right| = \prod_{k=1}^{n} (p_k - 1) \prod_{k=1}^{n} \frac{p_k - 2}{p_k - 1}
\]
\[
= \prod_{k=1}^{n} (p_k - 1) \prod_{k=1}^{n} \left( 1 - \frac{1}{p_k - 1} \right). \tag{91}
\]

□

Recall that for any integer \( n > 4 \) and any set \( N \) of integers,
\[
R(N, n, r) = \left\{ F \subseteq \{ V_{N, p, k} : p \in P(n), \ k \in \{0, r\} \} : \text{for each } q \in P(n) \text{ and } i \in \mathbb{N} \right. \\
\left. \text{we have } V_{N, q, i} = \{ s \in N : q | (s - i) \} \text{ and } |\{ V_{N, q, k} : k \in \{0, r\} \} \cap F| = 1 \right\}. \tag{92}
\]

**Lemma 6.** For all \( n \) and any even \( d \) for which two is the sole prime \( \leq p_n \) that divides \( d \) we have
\[
\left| \left\{ 1 \leq u \leq \prod_{k=1}^{n} p_k : \left( u(u - r), \prod_{k=1}^{n} p_k \right) = 1 \right\} \right| \geq \left| \left\{ 1 \leq u \leq \prod_{k=1}^{n} p_k : \left( u(u - d), \prod_{k=1}^{n} p_k \right) = 1 \right\} \right|
\]
\[
= \prod_{k=1}^{n} p_k \prod_{k=1}^{n} \left( 1 - \frac{1}{p_k} \right) \prod_{k=1}^{n} \left( 1 - \frac{1}{p_k - 1} \right)
\]
\[
= \prod_{k=1}^{n} p_k - \left| \bigcup_{k=1}^{n} F : F \in R \left( \left[ 1, \prod_{k=1}^{n} p_k \right], n, d \right) \right|. \tag{93}
\]

**Proof.** For (93), the final equality holds because the right-hand side is equal to \( \phi(\prod_{k=1}^{n} \phi(p_k)) \) with the second term being equal to \( \phi(\prod_{k=1}^{n} \phi(p_k)) - \prod_{k=1}^{n} p_k \). Note that for all primes \( p \), \( \phi(p) = p - 1 \). The first equality follows from Lemma 5 for \( r = d \) and \( m = n \), and the final equality holds because for each \( p \in P(n) \) for which \( p \mid r \), \( \phi(\phi(p)) \) is a factor of the right-hand side while, instead, the higher \( \phi(p) \) is a factor of the left-hand side.

□

2.9. **Remark.** Let \( r \) and \( n \) be as in Definition 2.6 and let \( z \) be any integer for which \( p_n^2/2 < z < p_{n+1}^2/2 \). We note the following.

I. Let \( T \) and \( T' \) be any elements of \( R([1, z], n, r) \) for which
\[
T \cap T' = \{ \{ m \in [1, z] : q \mid m \} : q \in P(n) \text{ and } q \mid r \}. \tag{94}
\]

Then
\[
\left| \left\{ 1 \leq u \leq z : \left( u(u - r), \prod_{k=1}^{n} p_k \right) = 1 \right\} \right| = z - \left| (T \cup T') \right|. \tag{95}
\]
This is an immediate consequence of applying \( (81) \) for \( E = T \) and \( T' \) and \( N = [1, z] \), together with the fact that \( z = \|\bigcup \{ R([1, z], n, r) \} \| \) is the number of \( u \in [1, z] \) for which \( u(u - r) \) is coprime to \( \prod_{k=1}^{n} p_k \). This follows from the fact that for all elements, \( B_p \), of \( \bigcup R([1, z], n, r) \), where \( B_p = \{ p + b, 2p + b, \ldots, \max B_p \} \) for some \( p \in P(n) \) and \( b \in \mathbb{N} \), \( p \) divides neither \( u \) nor \( u - r \), hence \( u \in [1, z] \setminus \bigcup \{ R([1, z], n, r) \} \).

II. By applying the Euler totient to \( I \), we have
\[
\left\{ 1 \leq u \leq \prod_{k=1}^{n} p_k : u, \prod_{k=1}^{n} p_k \neq 1 \right\} = \prod_{k=1}^{n} p_k - \prod_{k=1}^{n} p_k \prod_{k=1}^{n} \left( 1 - \frac{1}{p_k} \right). \tag{96}
\]

2.10. \textbf{Remark.} Let \( n \) and \( r \) be as in Definition \( 2.10 \). Let \( z \) be any integer for which \( p_{n+1}^2 / 2 < z < p_{n+1}^2 / 2 \).
In the ensuing Lemmas \( 7 \) to \( 10 \) we shall prove the value, \( b \), of a lower bound for \( \| \{ 1 \leq u \leq z : (u(u - r), \prod_{k=1}^{n} p_k) = 1 \} \| \) by way of \( I \) to \( III \), below.

I. By suitably choosing \( C \) and \( a \) in \( I \) in Remark \( 1.6 \) we may prove that there is a subset, \( W \), of cardinality
\[
\left| \bigcup \{ R([1, z], n, r) \} \right| - \left| \bigcup \left\{ \prod_{k=1}^{n} p_k : (u(u - r), \prod_{k=1}^{n} p_k) \neq 1 \right\} \right| = \prod_{k=1}^{n} p_k \prod_{k=1}^{n} \left( 1 - \frac{1}{p_k} \right) \tag{97}
\]
of \( \{ S_H, [1, z], n, r \} : H \in R([1, z], n, r) \} \) for which \( z = \| \bigcup \{ R([1, z], n, r) \} \| = b \).

II. An upper bound on \( \| \{ S_{U, [1, z], n, r} : U \in R([1, z], n, r) \} \| \) is given by condition \( II \) in Definition \( 2.6 \).
Condition \( II \) is in turn imposed on account of the requirement for condition \( III \) in Definition \( 2.6 \) on \( S_{T, Z, n, r} \), taking \( Z = [1, z] \) and \( T \in R([1, z], n, r) \). Condition \( III \) is required on the following grounds. It is possible that there is a subset, \( V \), of \( \{ S_{U, [1, z], n, r} : U \in R([1, z], n, r) \} \) for which, for \( I \) (above) to be true, there is an upper bound, on \( | \bigcap V \) that precludes any assumptions about \( | M \cap M' \| \) where \( M \) and \( M' \) are any distinct elements of \( V \). We shall show that, for some \( F \) and \( F' \) in \( R([1, z], n, r) \) for which
\[
F \cap F' = \left\{ \{ m \in N : q \mid m \} : q \in P(n) \text{ and } q \mid r \right\}, \tag{98}
\]
if \( M \cup M' \) is a subset of either \( \bigcup F \) or \( \bigcup F' \), the inclusion of \( M \) and \( M' \) in \( V \) may be treated as being superfluous to our method.

III. Note \( I \) (above) is found by the fact that, for any two elements, \( A_{z,n,r} \) and \( B_{z,n,r} \), of \( \bigcup R([1, z], n, r) \), \( A_{z,n,r} \cup B_{z,n,r} \) is a constant function of \( | A_{z,n,r} | + | B_{z,n,r} | \) unless \( \{ A_{z,n,r} \cup B_{z,n,r} \} \) is a subset of some element of \( R([1, z], n, r) \).

2.11. \textbf{Definitions.} I. For \( n \) and \( r \) as in Remark \( 2.10 \) let \( N \) and \( N' \) each be any sets of integers for which \( |N| = |N'| \) and \( \| \bigcup R(N', n, r) \| < \| \bigcup R(N, n, r) \| \) and for each \( V \in \{ N, N' \} \), \( |R(V, n, r)| = |R([1, \prod_{k=1}^{n} p_k], n, r) | \) and \( \max V \leq \prod_{k=1}^{n} p_k \).

II. For any subset \( F \) of some element of \( R(N, n, r) \), let \( X_{F, N, N'} \) be any subset, of cardinality
\[
\left| \bigcup F \right| - \left| \bigcup \left\{ p + a, 2p + a, \ldots, a + \prod_{k=1}^{n} p_k \right\} \cap N' : \left\{ p + a, 2p + a, \ldots, a + \prod_{k=1}^{n} p_k \right\} \cap N \in F \right| \mid_{p \in P(n) \text{ and } a \geq 0} \tag{99}
\]
of \( \bigcup F \), unless \( (99) \) is \( 0 \) in which case take it that \( X_{F, N, N'} = \emptyset \).

\textbf{Lemma 7.} Let \( r \) be even and \( n > 4 \). Let \( Z \) be any \( \| \bigcup R(N, n, r) \| - \| \bigcup R(N', n, r) \| \)-element subset of \( \bigcup R(N, n, r) \). Then \( Z \) may be chosen to be any subset of \( \bigcup L \) for some \( L \) that satisfies the following. For all \( Q \) for which \( Q \) is a subset of some element of \( R(N, n, r) \). \( L \) is a set containing \( X_{Q, N, N'} \).
Proof. Consider any \( p \in P(n) \) such that, for any \( b \) for which \( \{ p + b, 2p + b, \ldots, b + \prod_{k=1}^{n} p_k \} \cap N \in \bigcup R(N, n, r) \) there exists \( c \neq b \) such that \( \{ p + c, 2p + c, \ldots, c + \prod_{k=1}^{n} p_k \} \cap N \in R(N, n, r) \). Then \( b \) and \( c \) may be taken to be equal to zero and some integer \( > 0 \) respectively and we note that \( p \nmid r \). This serves to connect our method to the Chinese Remainder theorem, which we shall tacitly apply through the combination of I to IV, below.

I. For each \( u \in N \), \( \{ p \in P(n) : p \mid u(u - r) \} \) is an element of \( \{ p \in P(n) : p \mid v \} \) for some \( 1 \leq v \leq \prod_{k=1}^{n} p_k \). Therefore, for some \( T \in R(N, n, r) \), \( \{ p \in P(n) : p \mid u(u - r) \} \) is equal to

\[
\left\{ p \in P(n) : \left\{ p + a, 2p + a, \ldots, a + \prod_{k=1}^{n} p_k \right\} \cap \{ u \} \in \{ B \cap \{ u \} : B \in T \} & a \geq 0 \right\}. 
\tag{100}
\]

II. Let \( s \) and \( t \) be integers in \( \bigcup T \) for which, for each \( v \in \{ s, t \} \),

\[
\left\{ p \in P(n) : \left\{ p + a, 2p + a, \ldots, a + \prod_{k=1}^{n} p_k \right\} \cap \{ v \} \in \{ B \cap \{ v \} : B \in T \} & a \geq 0 \right\} \neq \emptyset \tag{101}
\]

and, for some \( 1 \leq i \leq \prod_{k=1}^{n} p_k \),

\[
\bigcup \left\{ p \in P(n) : \left\{ p + a, 2p + a, \ldots, a + \prod_{k=1}^{n} p_k \right\} \cap \{ s \} \in \{ B \cap \{ s \} : B \in T \} & a \geq 0 \right\} \\
\left\{ p \in P(n) : \left\{ p + a, 2p + a, \ldots, a + \prod_{k=1}^{n} p_k \right\} \cap \{ t \} \in \{ B \cap \{ t \} : B \in T \} & a \geq 0 \right\}. 
\tag{102}
\]

\[
= \left\{ p \in P(n) : \left\{ p + a, 2p + a, \ldots, a + \prod_{k=1}^{n} p_k \right\} \cap \{ i \} \in \{ B \cap \{ i \} : B \in T \} & a \geq 0 \right\}. 
\tag{103}
\]

For any integer \( k \), denote

\[
M_{N,k,n,r} = \left\{ Q : Q = \left\{ p + a, 2p + a, \ldots, a + \prod_{k=1}^{n} p_k \right\} \cap N \right\} \cap N 
\tag{104}
\]

\[
& \text{ for any integer } a \text{ for which } Q \cap \{ k \} \neq \emptyset \}. 
\tag{105}
\]

Then we have, for any integers \( a \) and \( b \),
i. \[ \left| \bigcup \{ M_{N,a,n,r} \cup M_{N,b,n,r} \} \right| = \left| \bigcup M_{N,a,n,r} \right| + \left| \bigcup M_{N,b,n,r} \right| - \left| M_{N,a,n,r} \cap M_{N,b,n,r} \right|. \] (106)

ii. Through I, it follows that \( a \) and \( b \) are each in \( \bigcup D \) where \( D \) is some element of \( R([1, \prod_{k=1}^{n} p_k], n, r) \) while, by assumption, \( s \) and \( t \) are in \( \bigcup T \) where \( T \in R(N, n, r) \).

III. We devise a contrast to our assumptions on \( s \) and \( t \) as follows. The value
\[
\left| \bigcup \left\{ \left\{ p + a, 2p + a, \ldots, a + \prod_{k=1}^{n} p_k \right\} \cap N : a \in \{b,c\} \right\} \right|
\] (107)
is a constant function of
\[
\sum_{a \in \{b,c\}} \left| \left\{ p + a, 2p + a, \ldots, a + \prod_{k=1}^{n} p_k \right\} \cap N \right|, \tag{108}
\]
holding \( |N|, n \) and \( r \) constant. However, by assumption there is no \( F \in R(N, n, r) \) for which \( \left\{ \left\{ p + a, 2p + a, \ldots, a + \prod_{k=1}^{n} p_k \right\} \cap N : a \in \{b,c\} \right\} \) is a subset of \( F \). Contrasting, for all \( q \in P(n) \setminus \{p\} \) and \( d \) for which \( \{q+d, 2q+d, \ldots, d + \prod_{k=1}^{n} p_k \} \cap N \in \bigcup R(N, n, r) \), there is a nonempty subset, \( Q_d \), of \( R(N, n, r) \) for which we have the following. For each \( G \in Q_d \), \( p \in P(n) \) and \( a \) for which \( \{p+a, 2p+a, \ldots, a + \prod_{k=1}^{n} p_k \} \cap N \neq \emptyset \),
\[
\left\{ p + a, 2p + a, \ldots, a + \prod_{k=1}^{n} p_k \right\} \cap N,
\]
\[
\{q+d, 2q+d, \ldots, d + \prod_{k=1}^{n} p_k \} \subseteq G. \tag{109}
\]

IV. We may choose a subset, \( B \), of all subsets of \( \bigcup R(N, n, r) \) for which the following is true. We have \( \sum_{W \in B} |W| \) is the smallest \( \sum_{W \in B'} |W| \) such that \( B' \) is a set of subsets of \( \bigcup R(N, n, r) \) for which, for each \( V \in B' \), i. below, is true.

i. Let \( H_V \) be any subset, of cardinality
\[
\left| \left| \bigcup V \right| - \bigcup \left\{ p + a, 2p + a, \ldots, a + \prod_{k=1}^{n} p_k \right\} \cap N' : \left\{ p + a, 2p + a, \ldots, a + \prod_{k=1}^{n} p_k \right\} \cap N \in V \text{ for some } p \in P(n) \text{ and } a \geq 0 \right| \right|. \tag{110}
\]
Then we may choose each element of \( \{ H_Y : Y \in B' \} \) so that \( Z \) is a subset of \( \bigcup \{ H_Y : Y \in B' \} \).

Consider I in Remark 1.6 for \( C = \bigcup R(N, n, r) \) with, for each proper subset \( K \) of \( \bigcup R(N, n, r) \), \( a \in H_K \). Then we note that, through our assumptions on \( N \), it follows by I in Remark 1.6 that \( \{ H_Y : Y \subseteq \bigcup R(N, n, r) \} > 1 \).

For any two subsets, \( V \) and \( V' \), of \( \bigcup R(N, n, r) \),
\[
\left| \bigcup (V \cup V') \right| = \left| \bigcup V \right| + \left| \bigcup V' \right| - \left| \bigcup V \cap \bigcup V' \right|. \tag{111}
\]
We note that (106) is (111) as for specified values of \( V \) and \( V' \). Thus, applying first (106), we may take \( \bigcup (V \cup V') = \bigcup W \) where \( W \subseteq R(N, n, r) \). We note in reference to I and II that by holding \( n \) and \( r \) constant in III, by our assumptions on \( N \), we also hold \( |R(N, n, r)| \) constant. For any \( |N| \)-element set \( Z \) of integers, let \( Q_{Z,n,r} \) be the set of all sets, \( B_{Z,a,b,n,r} \), that satisfy all conditions on \( M_{Z,a,n,r} \cup M_{Z,a,n,r} \).
such that $a$ and $b$ are integers. Consider the set $\{A_1, A_2, \ldots, A_k\}$, for some integer $x$, of $|N|$-element sets of integers for which we have the following. First, for any integers $k$, $a$, and $b$, let $C_{A_k,a,b,n,r}$ be any $|\bigcup B_{A_k,a,b,n,r}| - |\bigcup B_{A_{k-1},a,b,n,r}|$-element subset of $\bigcup B_{A_k,a,b,n,r}$. Then our assumption here is that $|\bigcup \{C_{A_k,a,b,n,r} : a, b \in \mathbb{N}\}|$ is an increasing function of $k$. (We note that the possibility of the nonexistence of the above-cited sets is of no consequence.)

Let $X$ and $Y$ be any nonempty subsets of $\bigcup \{C_{A_k,a,b,n,r} : a, b \in \mathbb{N}\}$ for which there exist integers $w$ and $y$ such that for each $c \in \{w, y\}$, $M_{A_k,c,n,r} \neq \emptyset$. (We note that it is elementary that $X$ and $Y$ exist (whether or not $X = Y$) when $\bigcup \{C_{A_k,a,b,n,r} : a, b \in \mathbb{N}\} \neq \emptyset$. Then we have the following.

a) For any two distinct elements, $i$ and $j$, of $\bigcup \{C_{A_k,a,b,n,r} : a, b \in A_k\}$, $i$ and $j$ satisfy all stated conditions on $w$ and $y$ respectively.

b) Through II.i, $X \cup Y$ is a set of subsets of $\{M_{A_k,a,n,r} : a \in A_k \& 1 \leq ab \leq \prod_{k=1}^n p_k\}$.

The fact that our assumptions on $W$ allow us to take $W = R(N, n, r)$ and $\bigcup R(N, n, r) = \{B_{N, a,b,n,r} : a, b \in \mathbb{N}\}$, implies that $|\bigcup \{R(A_k, n, r)\}|$ is an increasing function of $k$. We have shown, through a) combined with b), that this result is a consequence of (106) combined with II.i, all combined with III, whereby we have the following. For all sets that satisfy all stated conditions on $B$ as in IV, through IV the set $B$ may be taken to be some set of sets $E$ such that $E$ is a subset of some element of $R(N, n, r)$.

Therefore, the proof is complete.

**Lemma 8.** Let $r$ be even and $n > 4$. Let $Z$ be any $|\bigcup \{R(N, n, r)\}| - |\bigcup \{R(n', n, r)\}$-element subset of $\bigcup \{R(N, n, r)\}$. Then $Z$ may be chosen to be any subset of $\bigcup L$ for any $L$ that satisfies the following.

i. For all $Q$ for which $Q$ is a subset of some element of $R(N, n, r)$, $L$ is a set containing $X_{Q,N,N'}$.

ii. There exist sets $F$ and $F'$ in $R(N, n, r)$ for which

$$F \cap F' = \{\{m \in N : q \mid m\} : q \in P(n) \text{ and } q \mid r\} \quad (112)$$

and for which the following is true. We have $X_{G,N,N'} \cap X_{G',N,N} = \emptyset$, where $G$ and $G'$ are any subsets of $F$ and of $F'$ respectively for which, for each $p \in P(n)$ such that, for some $a$, $\{p + a, 2p + a, \ldots, a + \prod_{k=1}^n p_k\} \cap N \in G$, there exists $b \neq a$ such that $\{p + b, 2p + b, \ldots, b + \prod_{k=1}^n p_k\} \cap N \in G'$.

**Proof.** The requirement for i is shown by Lemma 7. In what follows we show that, for it to be determined that $Z$ may be written as a subset of $\bigcup L$, the imposition of condition ii implies that no third condition is required.

It is possible, subject to further scrutiny, that $Z$ is a subset of $\bigcup \{X_{U,N,N'} : U \subseteq C \text{ for some } C \in R(N, n, r)\}$ only if there is a subset $P$ of $R(N, n, r)$ for which we have the following. There is an upper bound on $|\bigcap \{X_{U,N,N'} : U \subseteq C \text{ for some } C \in P\}|$ as a consequence of which we have the following. For each $A \in P$ and some $M \subseteq A$ there is some $B \subseteq P$ and some $M' \subseteq B$ for which an upper bound $< \min\{|X_{M,N,N'}|, |X_{M',N,N'}|\}$ may be found for $|X_{M,N,N'} \cap X_{M',N,N'}|$. Our assumptions on $L$ allow us to assume, for convenience, that, if such a bound exists, for all possible choices of $M$ and $M'$, we may choose $L$ so that $X_{M,N,N'} \cap X_{M',N,N'} = \emptyset$. Assume now that we have made a choice for all pairs $(U, U')$ of sets for which $U$ and $U'$ satisfy all stated conditions on $M$ and $M'$ respectively, except for $U$ and $U'$ for which we state that I, below, is true.

I. The value of $|X_{U,N,N'} \cap X_{U',N,N'}|$ need not be determined or be subject to any imposed conditions in order that we may determine that $Z$ may be chosen to be a subset of $\bigcup L$. 

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Now consider the set, \( H \), of all of nonempty subsets, \( W \), of \( M \cup M' \) such that there exists \( W' \subseteq M \cup M' \) such that the following is true (the possibility that \( H = \emptyset \) is superfluous to our method). For some \( U \in \{W, W'\} \), we have \( U \subseteq F \) and \( U' \subseteq F' \), where \( U' \in \{W, W'\} \setminus \{U\} \).

Let

\[
T = \{(A, B) : X_{A,N,N'} \cap X_{B,N,N'} = \emptyset \land A \cup B \subseteq K \text{ for some } K \in \{F, F'\}\}.
\] (113)

We shall proceed here to show that, of the two sets \( T \) and \( V = \{(C, D) : X_{C,N,N'} \cap X_{D,N,N'} = \emptyset \land C, D \in R(N, n, r)\} \setminus T \), it is the existence of \( V \) (which is nonempty, on account of condition ii, when \( F \neq F' \)), alone, that implies II, below.

II. The imposition of condition ii implies that no third condition is required in our lemma.

Here, II follows by Lemma \( \Box \) coupled with the combination of the following two facts. First, we have assumed that \( X_{M,N,N'} \cap X_{M',N,N'} = \emptyset \) unless I is true for \( U = M \) and \( U' = M' \). Second, we have

\[
|X_{M,N,N'} \cup X_{M',N,N'}| = |X_{M,N,N'}| + |X_{M',N,N'}| - |X_{M,N,N'} \cap X_{M',N,N'}|.
\]

Specifically, it is the existence of the subset, which we may denote by \( V' \),

\[
\{C, D) : X_{C,N,N'} \cap X_{D,N,N'} = \emptyset \land C \neq D \land C \cup D \in \{S \cup \bigcup R(N, n, r)\} \\setminus \{S' \subseteq M : M \in \{F, F'\}\}
\] (114)

of \( V \) (which is nonempty, on account of condition ii, when \( F \neq F' \)), that implies II. This is because each of \( U \) and \( U' \) may be taken to be the bound variable \( A \cup B \) written into the expression given as \( T \), or otherwise as \( S' \) as written into \( V' \), while \( U \) and \( U' \) are each a subset of either \( F \) or \( F' \). Since there exist sets, whether empty or nonempty, \( X_{F,N,N'} \) and \( X_{F',N,N'} \), this proves that \( I \) is true for \( U \) and \( U' \) as current. Since the requirement for condition i is, as noted, already proven, the proof is thereby complete. \( \square \)

#### 2.12. Remark

Let \( n \) and \( r \) be as in Lemma \( \Box \) When we proceed to use \( \{S_{T,N,n,r} : T \in R(N, n, r)\} \) (here \( N \) becomes taken to be an interval) instead of \( L \), we have the following. Our conditions in Definition \( \Box \) allow us to choose \( F \) and \( F' \) as in ii in Lemma \( \Box \) so that \( S_{F,N,n,r} \cap S_{F',N,n,r} = \emptyset \).

#### 2.13. Remark

Let \( n \) and \( r \) be as Lemma \( \Box \) Recall our assumption that, for each \( V \in \{N, N'\} \), \( |R(V, n, r)| = |R([1, \prod_{k=1}^{n} p_k], n, r)| \). This implies through the definition of \( R(U, n, r) \), where \( U \) is any set of integers, that, for any \( G \in R([1, \prod_{k=1}^{n} p_k], n, r) \), if \( \{T \cap N : T \in G\} \neq \emptyset \) then \( \{T \cap N' : T \in G\} \neq \emptyset \).

This understood, let \( Y_G \) be any subset, of cardinality

\[
|\{T \cap N : T \in G\}| - |\{T \cap N' : T \in G\}|,
\] (115)

of \( \{T \cap N : T \in G\} \), when \( (15) \) is positive. We note that we need not assume that \( Y_H \), for any \( H \in R([1, \prod_{k=1}^{n} p_k], n, r) \), exists. Then our assumptions on \( C \) and \( a \) as in i in Remark \( \Box \) allow us to use \( C \) and \( a \) as follows:

\begin{enumerate}
    \item \( C = \{G \setminus A_{H_G} : G \in H_G\} \) where \( H_G \subseteq \{T \cap N : T \in G\} \) and \( A_{H_G} \) is any subset of \( \bigcup H_G \), of cardinality

    \[
    \left|\bigcup H_G\right| - \left|N\right| - \left(1 - \prod_{s \in \mathcal{P}(n), \{s + a, \Delta + a, \ldots, a + \prod_{k=1}^{n} p_k\} \cap N \in H_G} \left(1 - \frac{1}{s}\right)\right),
    \] (116)

    and \( a \) is in \( A_{H_G} \) (we note that we need not assume that \( A_{H_G} \) exists);
\end{enumerate}
and subsequently

\[ C = \{ T \setminus B_{H_G} : T \in H_G \} \text{ where } B_{H_G} \text{ is any subset of } H_G, \text{ of cardinality} \]

\[
\left| \bigcup H_G \right| - \left| \bigcup \left\{ p + a, 2p + a, \ldots, a + \prod_{k=1}^{n} p_k \right\} \cap N' : \right.

\left. \{ p + a, 2p + a, \ldots, a + \prod_{k=1}^{n} p_k \} \cap N \in H_G \text{ for some } p \in P(n) \text{ & } a \geq 0 \right| \tag{117}
\]

and \( a \) is in \( B_{H_G} \) (we note that we need not assume that \( B_{H_G} \) exists);

and subsequently

\[ C = \{ T \setminus \{ a \} : T \in \{ T \cap N : T \in G \} \} \text{ and } a \in Y_G; \]

and finally

\[ C = \{ (\bigcup T) \setminus Z : T \in \bigcup R(N, n, r) \} \text{ and } a \in L \text{ for } L \text{ and } Z \text{ as in Lemma 8} \]

2.14. **Remark.** Let \( n, r \) and \( G \) be as Remark 2.13. Now we refine the list of values of \( C \) and \( a \), put forward in Remark 2.13 that will be useful to our method. Combining Lemma 8 with the fact that we may choose \( N \) and \( N' \) (noting that there is no requirement that \( N \) and \( N' \) are intervals) so that

\[
- \left| \bigcup \left( \bigcup R(N', n, r) \right) \right| = - \left| \left[ N \times \left( \bigcup \left( \bigcup R([1, \prod_{k=1}^{n} p_k], n, r) \right) \right) \right] \right|

\geq - \left| \bigcup \left( \bigcup R(N, n, r) \right) \right|, \tag{118}
\]

gives, by suitably choosing the cardinality of each element of \( \bigcup R(N', n, r) \), the following. Through I in Remark 1.5 our assumptions on \( a \) as in i in Remark 2.13 allow us to choose each element of \( \{ Q_{H_G} : \{ Q_{H_G} : Q_{H_G} \text{ satisfies all conditions on } A_{H_G} \} : k \in N \} \) so that we have the following. For \( C \subset \{ D \setminus \{ a \} : D \in \{ T \cap N : T \in G \} \} \) and subsequently \( C = \{ D \setminus \{ a \} : D \in \{ T \cap N : T \in G \} \} \) we may take, for all \( S \in R([1, \prod_{k=1}^{n} p_k], n, r) \), \( H_S = \{ T \cap N : T \in S \} \). Then, through the choice of \( N' \) that gives (118), iii in Remark 2.13 has become redundant. Also, \( L \) has become redundant in iv in the same Remark, as we may now substitute \( \{ S_{B,N,n,r} : B \in R(N, n, r) \} \) for \( L \). In each case the redundancy is attributable to the fact that, for (111) we may use either \( N = [1, z] \) or \( N = [1, \prod_{k=1}^{n} p_k] \). Then for all \( x \) we may use i in Remark 2.13 and never ii, which is also redundant.

**Lemma 9.** For any even \( r \), any \( n > 4 \), and any \( z \) such that \( p_n^2/2 < z < p_{n+1}^2/2 \),

\[
- \left| \bigcup \left( \bigcup R([1, z], n, r) \right) \setminus \{ SU_{[1, z], n, r} : U \in R([1, z], n, r) \} \right|

\geq - \frac{z}{\prod_{k=1}^{n} p_k} \left\{ 1 \leq u \leq \prod_{k=1}^{n} p_k : \left( u(u-r), \prod_{k=1}^{n} p_k \right) \neq 1 \right\}. \tag{119}
\]
Proof. Let \( G \in R([1, \prod_{k=1}^{n} p_k], n, r) \) and let \( H_G \) and \( A_{H_G} \) be as in Remark 2.13. Recall that \( H_G \subseteq \{ T \cap N : T \in G \} \) and \( A_{H_G} \) is any subset of \( \bigcup H_G \), of cardinality

\[
| \bigcup H_G | - | N | \left( 1 - \prod_{s \in P(n), \{ s+\alpha, 2s+\alpha, \ldots, a+\prod_{k=1}^{n} p_k \}} \left( 1 - \frac{1}{s} \right) \right).
\]

(The possibility that \( \{ A_{H_G} : B \in R([1, \prod_{k=1}^{n} p_k], n, r) \} = \emptyset \) is of no consequence.) Then Remark 2.14 implies the following. For some subset, \( D_N \), of \( \bigcup R(N, n, r) \), of cardinality

\[
| \bigcup R(N, n, r) | - \frac{|N| \times | \{ 1 \leq u \leq \prod_{k=1}^{n} p_k : (u(u-r), \prod_{k=1}^{n} p_k) \} |}{\prod_{k=1}^{n} p_k}
\]

when \( N \) is an interval we have

\[
D_N \subseteq \bigcup \{ S_{B, N, n, r} : B \in R(N, n, r) \}.
\]

This follows, specifically, from the combination of Lemma 8 with Remark 2.14. To expand, our assumptions on \( A_{H_G} \) allow us now to use \( a \in \{ S_{B, N, n, r} : B \in R(N, n, r) \} \) for both i and iv in Remark 2.13. (When \( a \) is, instead, taken to be an element of

\[
\bigcup \{ D_N : D \subseteq V \text{ for some } V \in R(N, n, r) \} \setminus \{ S_{B, N, n, r} : B \in R(N, n, r) \},
\]

\( a \) is an element of \( (\bigcup G) \setminus S_{G, N, n, r} \) for some \( G \in R(N, n, r) \).) This, in turn, allows our use of \( \{ S_{B, N, n, r} : B \in R(N, n, r) \} \) in (122). Here we note that, for any subset \( B \) of any \( G \in R(N, n, r) \), the fact that

\[
\left\{ \left\{ Q_{p, a} : Q_{p, a} = \left\{ p + a, 2p + a, \ldots, a + \prod_{k=1}^{n} p_k \right\} \cap N \text{ for } p \in P(n) \text{ and } a \geq 0 \right. \right\} \in \{ 1, 2 \}
\]

is taken into account by condition III in Definition 2.6. (This condition is itself subject to condition II in Definition 2.6. We combine, here, (123) with the proven requirement for condition ii on \( L \) in Lemma 8 to give the following. Since our assumptions on \( N \) allow us, through Remark 2.7 and Remark 2.12 to use \( N = [1, z] \setminus \{ S_{B, [1, z], n, r} : B \in R([1, z], n, r) \} \). (122) implies (119).

Lemma 10. For any even \( r \), any \( n > 4 \), and any \( z \) such that \( \frac{p_n^2}{2} < z < \frac{p_{n+1}^2}{2} \),

\[
\left\{ 1 \leq u \leq z : \left( \frac{u(u-r)}{\prod_{k=1}^{n} p_k} \right) = 1 \right\} \geq \left( z - \frac{|S_{K, [1, z], n, r}|}{\prod_{k=1}^{n} (1 - \frac{1}{p_k})} \right) \prod_{k=1}^{n} \left( 1 - \frac{1}{p_k} \right) \prod_{k=2}^{n} \left( 1 - \frac{1}{p_k} - \frac{1}{p_{k-1}} \right).
\]

Proof. The inequality (125) follows by combining Lemma 9 for \( n, r \) and \( z \) as current, with Lemma 6. This can be shown by the combination of I and II, below.
I. We have

\[
\left\{ 1 \leq u \leq \prod_{k=1}^{n} (u, \prod_{k=1}^{n} p_k) = 1 \& (u-r, \prod_{k=1}^{n} p_k) \neq 1 \right\} \\
= 1 - \frac{|\left\{ 1 \leq u \leq \prod_{k=1}^{n} (u-r, \prod_{k=1}^{n} p_k) = 1 \right\}|}{|\left\{ 1 \leq u \leq \prod_{k=1}^{n} (u, \prod_{k=1}^{n} p_k) \right\}|} = 1 - \prod_{k=2}^{n} \left( 1 - \frac{1}{p_k - 1} \right).
\]

To show this, for any set \( M \) of integers let

\[
a(M, n) = \frac{|\{ m \in M : (m, \prod_{k=1}^{n} p_k) \neq 1 \}|}{|M|}.
\]

Then for any \( B \subset R([1, z], n, r) \) we have \( a((\bigcup B) \setminus S_{R([1, z], n, r)}, n) \leq 1 - \prod_{k=1}^{n} (1 - 1/p_k) \); also \( a([1, \prod_{k=1}^{n} p_k], n) = 1 - \prod_{k=1}^{n} (1 - 1/p_k) \) so (126) follows, the final relation through Lemma 6.

II. For \( E \) and \( E' \) as given in (81), where \( E \) and \( E' \) are in \( R([1, z], n, r) \) (hence \( N = [1, z] \)), and \( C' \subset \{ E, E' \} \setminus \{ C \} \), we may choose \( S_{E, [1, z], n, r} \) and \( S_{E', [1, z], n, r} \) so that, for any \( C \subset \{ E, E' \} \)

\[
|S_{C', [1, z], n, r} \cap (\bigcup (C \setminus \{ 1 \leq u \leq z : 2 \mid u \}) \setminus S_{C, [1, z], n, r})| \\
\cup |S_{C, [1, z], n, r} \cap (\bigcup (C' \setminus \{ 1 \leq u \leq z : 2 \mid u \}) \setminus S_{C', [1, z], n, r})| \\
\geq |S_{C, [1, z], n, r}| + |S_{C', [1, z], n, r}| \left( 1 - \prod_{k=2}^{n} \left( 1 - \frac{1}{p_k - 1} \right) \right).
\]

By taking \( C = K_{z,r} \), combining all of I, II and (81) for \( N = [1, z] \) gives

\[
z - |(\bigcup (K_{z,r} \cup C')) \setminus \bigcup (S_{T, [1, z], n, r} : T \subset R([1, z], n, r))| \\
\geq \left( z - \frac{2|S_{K_{z,r}, [1, z], n, r}|}{\prod_{k=2}^{n} \left( 1 - \frac{1}{p_k} \right)} \left( 1 - \prod_{k=1}^{n} \left( 1 - \frac{1}{p_k - 1} \right) \right) \right) \\
\times \prod_{k=1}^{n} \left( 1 - \frac{1}{p_k - 1} \right),
\]

recalling that \( |\bigcup (S_{T, [1, z], n, r} : T \subset R([1, z], n, r))| \leq 2|S_{K_{z,r}, [1, z], n, r}| \) as by condition II in Definition 2.6. Further, we have multiplied \( 2|S_{K_{z,r}, [1, z], n, r}| \) by \((1 - \prod_{k=1}^{n} (1 - 1/(p_k - 1)))/(\prod_{k=1}^{n} (1 - 1/(p_k - 1))) \) on account of the condition \((u, \prod_{k=1}^{n} p_k) = 1 \) in the set in the first line of (126). This fraction is found by the fact that

\[
|\left\{ 1 \leq u \leq \prod_{k=1}^{n} (u, \prod_{k=1}^{n} p_k) = 1 \& (u-r, \prod_{k=1}^{n} p_k) \neq 1 \right\}| \\
= |\left\{ 1 \leq u \leq \prod_{k=1}^{n} (u-r, \prod_{k=1}^{n} p_k) \neq 1 \& (u-r, \prod_{k=1}^{n} p_k) \neq 1 \right\}|.
\]

Since

\[
\left( 1 - \prod_{k=2}^{n} \left( 1 - \frac{1}{p_k - 1} \right) \right) \left( 1 - \prod_{k=2}^{n} \left( 1 - \frac{1}{p_k - 1} \right) \right) < 1
\]
it follows by (129) that
\[
z - |(\bigcup (K_{z,r} \cup C')) \setminus \bigcup \{ST_{[1,z],n,r} : T \in R([1,z], n, r)\}| \geq \left(z - \frac{2|S_{K_{z,r},[1,z],n,r}|}{\prod_{k=1}^{n} (1 - \frac{1}{p_k})}\right) \prod_{k=1}^{n} \left(1 - \frac{1}{p_k}\right) \prod_{k=2}^{n} \left(1 - \frac{1}{p_k - 1}\right).
\]
(131)

We note at this juncture that the initial minus signs in (119) have been removed in (125) because we have substituted \(u(u-r), \prod_{k=1}^{n} p_k = 1\) for \((u(u-r), \prod_{k=1}^{n} p_k) \neq 1\), which is equivalent to adding \(z\) to both sides of (119). Thus, by use of \(L\) in (131), combining (131) with Lemma 9 gives (125). □

**Corollary.**

We recall that, for any \(n > 4\) and any \(z\) such that \(p_n^2/2 < z < p_{n+1}^2/2\) and any even \(r\), \(K_{z,r}\) is any element of \(R([1,z], n, r)\) for which
\[
|\bigcup K_{z,r}| = \max\{|\bigcup X| : X \in R([1,z], n, r)\}.
\]
In view of this, Lemma 10 may be rewritten as follows. For \(n, z\) and \(r\) as given above,
\[
\left\{1 \leq u \leq z : \left(u(u-r), \prod_{k=1}^{n} p_k\right) = 1\right\} \geq \left(2 - \frac{2|\bigcup K_{z,r}|}{z} - \prod_{k=1}^{n} \left(1 - \frac{1}{p_k}\right)\right) \prod_{k=2}^{n} \left(1 - \frac{1}{p_k - 1}\right).
\]
(132)

Here, the first and final terms within the first set of parentheses on the right-hand side are found by doubling the \(1 - \prod_{k=1}^{n} (1 - 1/p_k)\) seen in the definition of \(S_{K_{z,r},[1,z],n,r}\). Note that it is a condition on \(S_{K_{z,r},[1,z],n,r}\) that
\[
|S_{K_{z,r},[1,z],n,r}| = |\bigcup K_{z,r}| - \left[z \left(1 - \prod_{k=1}^{n} \left(1 - \frac{1}{p_k}\right)\right)\right].
\]
(133)

**Remark.** Recall that, for any \(n > 4\), any \(z\) such that \(p_n^2/2 < z < p_{n+1}^2/2\), any even \(r\), and any \(T \in R([1,z], n, r)\), \(I_T\) is a \(v\)-element interval for which the \(v\)-th highest element of \(\bigcup T\) is equal to \(u - \min I_T + 1\), where \(u\) is the \(v\)-th highest element of
\[
\left\{j \in I_T : \left(j, \prod_{k=1}^{n} p_k\right) \neq 1\right\}.
\]

**Lemma 11.** For any \(n > 4\), any even \(r\), any \(z\) such that \(p_n^2/2 < z < p_{n+1}^2/2\) and
\[
|S_{K_{z,r},[1,z],n,r}| \leq z - \pi(z) + n + \frac{n(n-1)}{2} - \left[z \left(1 - \prod_{k=1}^{n} \left(1 - \frac{1}{p_k}\right)\right)\right] \geq z \left(1 - \frac{1}{\log(z)} + O\left(\frac{1}{e^\gamma \log z}\right)\right).
\]
(134)

**Proof.** Recall that, for any \(T \in R([1,z], n, r)\), \(I_T\) is a \(v\)-element interval for which, for any integer \(v\), the \(v\)-th highest element of \(\bigcup T\) is equal to \(u - \min I_T + 1\), where \(u\) is the \(v\)-th highest element of \(\{j \in I_T : \left(j, \prod_{k=1}^{n} p_k\right) \neq 1\}\).

Consider that, for \(77\), \(T = K_{z,r}\). Let \(J = P(n)\) and \(I = I_{K_{z,r}}\). Note that \(77\) implies that \(I\) exists. By assumption, \(I_{K_{z,r}}\) contains \(z\) elements. Therefore we have the following. First, we note
that the integer one is coprime to \( \prod_{k=1}^n p_k \) while the first \( n \) primes, alone, are not. Hence \(| \bigcup K_{z,r} |\) is equal to the first three terms of (133). Second, we may apply Theorem[1][1] for \( J \) and \( I \) as given above, thence to give, by combination with the previous sentence, the first relation of (134). Finally, noting that 
\[
\pi(z) + \frac{n(n+1)}{2} = z \log(z) + O\left(\frac{z}{\log(z)}\right) + O\left(\frac{\sqrt{2z}}{\log(\sqrt{2z})}\right) + 1
\]
Here, the first equality follows from the Prime Number theorem. The second relation of (134) then follows through the Mertens theorem, namely
\[
\lim_{n \to \infty} \prod_{k=1}^n \left( 1 - \frac{1}{p_k} \right) = \frac{1}{e^\gamma \log(p_n)}
\]
where we may substitute \( \sqrt{2z} \), and hence \( z \), for \( p_n \), and the proof is complete. \( \square \)

2.15. **Remark.** Let \( n, r \) and \( z \) be as in the Corollary to Lemma[1][10] Then we have
\[
\left| \left\{ k \in I_{K_{z,r}} : \left( k, \prod_{k=1}^n p_k \right) \neq 1 \right\} \right| = | \bigcup K_{z,r} |
\leq z - \pi(z) + n + \frac{n(n-1)}{2}
= z - \pi(z) + \frac{n+1}{2}.
\]

The first relation is true by definition of \( I_{K_{z,r}} \). The second follows from Lemma[1][11]
Combining (132) with (135) gives
\[
\begin{align*}
\left\{ 1 \leq u \leq z : \left( u(u-r) \prod_{k=1}^{n} p_k \right) = 1 \right\}
\geq z \left( 2 - \frac{2 \left( z - \frac{z}{\log(z)} + O \left( \frac{z}{\log^{2}(z)} \right) \right)}{z} \prod_{k=1}^{n} \left( 1 - \frac{1}{p_k} \right) \prod_{k=2}^{n} \left( 1 - \frac{1}{p_k - 1} \right) \right) \\
\sim z \left( 2 - \frac{2 \left( z - \frac{z}{\log(z)} + O \left( \frac{z}{\log^{2}(z)} \right) \right)}{z} \prod_{k=1}^{n} \left( 1 - \frac{1}{p_k} \right) C \frac{e^{-\gamma}}{\log(\sqrt{2}z)} \right)
\end{align*}
\]
\begin{align*}
&= \left( 2z - 2 \left( z - \frac{z}{\log(z)} + O \left( \frac{z}{\log^{2}(z)} \right) \right) \prod_{k=1}^{n} \left( 1 - \frac{1}{p_k} \right) C \frac{e^{-\gamma}}{\log(\sqrt{2}z)} \right) \\
&= \left( 2z - 2 \left( z - \frac{z}{\log(z)} + O \left( \frac{z}{\log^{2}(z)} \right) \right) \prod_{k=1}^{n} \left( 1 - \frac{1}{p_k} \right) C \frac{e^{-\gamma}}{\log(\sqrt{2}z)} \right) \\
&= \left( z \prod_{k=1}^{n} \left( 1 - \frac{1}{p_k} \right) C \frac{e^{-\gamma}}{\log(\sqrt{2}z)} \right) \\
&= \left( 4C(1 - e^{-\gamma}) \frac{z}{\log^{2}(z)} + O \left( \frac{z}{\log^{2}(z)} \right) \right) \prod_{k=1}^{n} \left( 1 - \frac{1}{p_k} \right) C \frac{e^{-\gamma}}{\log(\sqrt{2}z)} \\
&= 4C(1 - e^{-\gamma}) \frac{z}{\log^{2}(z)} + O \left( \frac{z}{\log^{2}(z)} \right) \prod_{k=1}^{n} \left( 1 - \frac{1}{p_k} \right) C \frac{e^{-\gamma}}{\log(\sqrt{2}z)}
\end{align*}
\]
where \( s \in \{0, 1\} \) for \( z = (p_n^2 + 1)/2 \).

Proof. Recall (132):

\[
\left\{ 1 \leq u \leq z : \left( u(u - r), \prod_{k=1}^{n} p_k \right) = 1 \right\}
\geq z \left( 2 - \frac{2|\bigcup K_{z,r}|}{z} - \prod_{k=1}^{n} \left( 1 - \frac{1}{p_k} \right) \prod_{k=2}^{m} \left( 1 - \frac{1}{p_k - 1} \right) \right).
\]

The assumptions on \( z \) and \( n \) allow us to use (132) in what follows. Our first inequality is

\[
2 > \frac{2}{\ln 2z} + \frac{2}{z} \left\{ 1 \leq v \leq z : \left( v, \prod_{k=1}^{n} p_k \right) \neq 1 \right\} - \frac{2n}{z},
\]

where we use \( \ln 2z \) on the grounds that \( \ln 2z > \ln z \). The above formula can be seen to follow when we multiply both sides by \( z \). Here,

i. the result, (141), of Dusart for \( x = 2z \), gives a lower bound > \( z/\ln z \), hence > \( z/\ln 2z \), on \( \pi(z) \);

ii. the number of composites \( \leq z \) is half the second term, on the right-hand side of the formula, multiplied by \( z \), minus \( n \).

Gaining some terms on both sides we have

\[
2 - \frac{2|\bigcup K_{z,r}|}{z} - \prod_{k=1}^{n} \left( 1 - \frac{1}{p_k} \right)
> \frac{2}{\ln 2z} + \frac{2}{z} \left\{ 1 \leq v \leq z : \left( v, \prod_{k=1}^{n} p_k \right) \neq 1 \right\}
- \frac{2n}{z} - \frac{2|\bigcup K_{z,r}|}{z} - \prod_{k=1}^{n} \left( 1 - \frac{1}{p_k} \right).
\]

Combining all of (132) to (145) gives

\[
\left\{ 1 \leq u \leq z : \left( u(u - r), \prod_{k=1}^{n} p_k \right) = 1 \right\}
> \frac{2}{\ln 2z} + \frac{2}{z} \left\{ 1 \leq v \leq z : \left( v, \prod_{k=1}^{n} p_k \right) \neq 1 \right\}
- \frac{2n}{z} - \frac{2|\bigcup K_{z,r}|}{z} - \prod_{k=1}^{n} \left( 1 - \frac{1}{p_k} \right).
\]

Now we introduce

\[
-|\bigcup K_{z,r}| \geq - \left\{ 1 \leq v \leq z : \left( v, \prod_{k=1}^{n} p_k \right) \neq 1 \right\} - \frac{n(n - 1)}{2} - 1.
\]

This formula follows from Theorem(1) for \( J = P(n) \) and, noting (77), for \( I \) for which \( |I| = z \) and

\[
\left\{ k \in I : \left( k, \prod_{k=1}^{n} p_k \right) \neq 1 \right\} = |\bigcup K_{z,r}|.
\]
We further have
\[ n \leq \text{Hi}(p_n). \]  
(149)

Here we apply, to (147), the result (142) of Dusart, whereby \( \pi(x) \leq \text{Hi}(x) \) for all \( x \geq 355991 \), here for \( x = \sqrt{2z} \).

Combining all of (146) to (149) gives
\[ \left| \left\{ 1 \leq u \leq z : \left( u(u - r), \prod_{k=1}^{n} p_k \right) = 1 \right\} \right| > z \left( \prod_{k=1}^{n} \left( 1 - \frac{1}{p_k} \right) + \frac{2}{\ln 2z} - \frac{\text{Hi}(p_n)(\text{Hi}(p_n) - 1)}{z} - \frac{2\text{Hi}(p_n)}{z} \right) \quad \text{for all} \quad x = \sqrt{2z}. \]

Rearranging all of the right-hand side excluding the final product gives
\[
\begin{align*}
&= z \left( \frac{2}{\ln(2z)} - \prod_{k=1}^{n} \left( 1 - \frac{1}{p_k} \right) - \frac{\text{Hi}(p_n)(\text{Hi}(p_n) - 1)}{z} - \frac{2\text{Hi}(p_n)}{z} - \frac{2}{z} \right) \\
&= z \left( \frac{2}{\ln(2z)} \prod_{k=1}^{n} \left( 1 - \frac{1}{p_k} \right) - 1 \right) \prod_{k=1}^{n} \left( 1 - \frac{1}{p_k} \right) - \text{Hi}(p_n)^2 - \text{Hi}(p_n) - 2 \\
&= 2z \left( \frac{1}{\ln(2z)} \prod_{k=1}^{n} \left( 1 - \frac{1}{p_k} \right) - \frac{1}{2} \right) \prod_{k=1}^{n} \left( 1 - \frac{1}{p_k} \right) - \text{Hi}(p_n)^2 - \text{Hi}(p_n) - 2.
\end{align*}
\]  
(151)

Thus (150) implies (143).

2.16. **Definition.** For any real \( x \), let \( f(x) \) be the highest \( y \) for which \( (1 - 1/y) \ln y = \ln x \).

**Lemma 13.** For all \( x > e \) for which \( f(x) - x > 1 \), \( f(x) - x \) is increasing.

**Proof.** We begin by noting that
\[
\frac{1}{f(x)} = \frac{f(x)}{f(x) - 1} = f(x)^{f(x) - 1}.
\]  
(152)

so
\[ f(x) = x^{f(x) - 1}. \]  
(153)

For any function \( g(x) \) for which

I. \( g(x) > 1 \) and

II. \( xg(x) - x \) is nondecreasing,

we have \( xg(x) = e^{\ln(xg(x))} \) and
\[
\frac{x^{\ln(xg(x))} - x}{xg(x) - x} = \frac{e^{\ln(xg(x))} - x}{e^{\ln(xg(x))} - x}.
\]  
(154)
Combining (154) with the fact that \(\ln(x) \times g(x) > \ln(xg(x)) > \ln x\) while \(-x\) is decreasing implies that condition II (above) requires that, for any \(y > x\)
\[
\frac{\frac{g(x) - x}{g(x) - x}}{< \frac{\frac{g(y) - y}{g(y) - y}}{y}.
\]

Taking, for all \(t, g(t) = (t + 1)/t\) gives \(tg(t) - t = 1\) so (155) implies that
\[
\frac{x + a}{x + a} - x < \frac{y + b}{y + b} - y.
\]

We have, for any \(a > 0\),
\[
\frac{x + a + 1}{x + a} = \frac{x^2 + ax + x}{x^2 + ax + x + a} \quad < \quad \frac{y^2 + ay + y}{y^2 + ay + y + a},
\]
so
\[
\frac{x + a + 1}{x + a} - x < \frac{y + b + 1}{y + b} - y,
\]
implying by (156) and the fact that, for all \(s > 1\), \((s + 1)/s\) is decreasing that, for any \(b\) for which
\[
x + 1 + a = f(x) \quad \text{and} \quad y + 1 + b = f(y),
\]
we further have
\[
f(x) - x < f(y) - y.
\]

On the other hand, for all \(c > 0\) and \(d > 0\) for which
\[
x + \frac{c + d}{x + d} - x < \frac{y + 1 + c + d}{y + 1 + c} - y,
\]
when \(x + 1 + c = f(x)\) and \(y + 1 + d = f(y)\), the fact that, by (153), \(x + \frac{c + d}{x + d} = f(x)\) and \(y + \frac{1 + c + d}{y + 1 + c} = f(y)\) implies, when combined with (160), that \(f(x) - x\) is increasing. \(\square\)

**Lemma 14.** For all \(x > 356,023\), \(f(x) > x + 2\).

**Proof.** We have
\[
f(p_{30457}) - p_{30457} \approx 356035.783 - 356023 \approx 12.78 \ldots > 2.
\]
Taking, for \(x\) as in Lemma 13, \(x > 356,023\), combining (162) with Lemma 13 gives \(f(x) > x + 2\). \(\square\)

**Lemma 15.** For all \(d > 1\) and \(x\) for which \(x - pd \geq 2\), and any subset \(M\) of \(P(d) \cup \{x\}\),
\[
x^2 \prod_{t \in P(d) \cup \{x\}} \left(1 - \frac{1}{t}\right) \prod_{u \in (P(d) \cup \{x\}) \setminus (M \cup \{2\})} \left(1 - \frac{1}{u - 1}\right)
- p_d \prod_{k=1}^{d} \left(1 - \frac{1}{p_k}\right) \prod_{p \in P(d) \setminus \{M \cup \{2\}\}} \left(1 - \frac{1}{p - 1}\right)
\geq 1.
\]
Proof. We have
\[
\prod_{k \in \{k \in \mathbb{N}: 2|k, 2 \leq k \leq x\}} \left(1 - \frac{1}{h}\right) \prod_{i \in \{j \in \mathbb{N}: 2|j, 4 \leq j \leq x\}} \left(1 - \frac{1}{i - 1}\right) = \prod_{m \in \{u \in \mathbb{N}: 2 \leq u \leq \max\{k \in \mathbb{N}: 2|k, k \leq x\}\}} \left(1 - \frac{1}{m}\right)
\]
\[
= \prod_{m \in \{u \in \mathbb{N}: 2 \leq u \leq \max\{k \in \mathbb{N}: 2|k, k \leq x\}\}} \left(1 - \frac{1}{m}\right)
\]
\[
= \frac{1}{\max\{k \in \mathbb{N}: 2|k, k \leq x\}}. \tag{164}
\]

The fact that, for all \(k\) for which \(3 \leq k \leq n\) we have \(p_{k+1} - p_k \geq 2\), while for any real \(r\), \(1 - 1/(p_k + r)\) is an increasing function of \(r\), implies that
\[
\left(1 - \frac{1}{x}\right) \left(1 - \frac{1}{x - 1}\right) \prod_{k=1}^{n} \left(1 - \frac{1}{p_k}\right) \prod_{k=2}^{n} \left(1 - \frac{1}{p_k - 1}\right)
\]
\[
\geq \prod_{k \in \{k \in \mathbb{N}: 2|k, 2 \leq k \leq x\}} \left(1 - \frac{1}{h}\right) \prod_{i \in \{j \in \mathbb{N}: 2|j, 4 \leq j \leq x\}} \left(1 - \frac{1}{i - 1}\right). \tag{165}
\]

Since
\[
\frac{(\max\{k \in \mathbb{N}: 2|k, k \leq x\})^2}{\max\{k \in \mathbb{N}: 2|k, k \leq x\}} = \max\{k \in \mathbb{N}: 2|k, k \leq x\}, \tag{166}
\]
and since for all \(p \in M\), \((1 - 1/p)(1 - 1/(p - 1)) < 1 - 1/p\), combining \(165\) with \(164\) gives \(163\). \(\square\)

**Lemma 16.** For all \(w\) for which \(p_w > e^2\) and
\[
\ln p_w^2 < \ln(p_{w+1}^2) \left(1 - \frac{1}{p_{w+1}}\right) \tag{167}
\]
we have, for any \(2 < u \leq \log_{p_w}(p_{w+1}^2)\)
\[
P_w^u \left(\frac{1}{\ln(p_w^2) \prod_{k=1}^{\log_{p_w}(p_{w+1})} (1 - \frac{1}{p_k})} - \frac{1}{2}\right) \prod_{m=2}^{w} \left(1 - \frac{1}{p_m - 1}\right)
\]
\[
> P_w^u \left(\frac{1}{\ln(p_w^2) \prod_{k=1}^{\log_{p_w}(p_{w+1})} (1 - \frac{1}{p_k})} - \frac{1}{2}\right) \prod_{m=2}^{w} \left(1 - \frac{1}{p_m - 1}\right). \tag{168}
\]

Proof. Let \(q\) be any real number for which \(e^2 < q < p_w\). Then since \(\ln e^2 = 2\,
I. for each \(s \in \{q, p_w\}\) we have \(\ln s^u > 2 \ln s > 4 > e\,
II. since \(\ln q > \ln p_w^u / p_w^u\), we have
\[
\frac{q^u}{q^a} = \frac{q^u}{q^a} = \frac{q^a + q \ln q}{q^a \ln q} > \frac{p_w^u + p_w \ln p_w}{p_w^u \ln p_w} \tag{168}
\]
so since \( \ln q > 2 \) and \( u > 2 \),

\[
1 > \frac{q^u}{\ln q} + q > \frac{p_w^u}{\ln p_w^u} + p_w.
\]  

Taking, first, the case for which \( u = \log p_w (p_w^2 + 1) \), developing (151) we may therefore subtract the denominators in (169) from their respective numerators to give, through the fact that \( p_w^2 + 1 > q^2 \), the inequality in the following:

\[
\frac{p_w^2}{\ln p_w^2} \left(1 - \frac{1}{p_w} \right) \left(\frac{1}{\ln p_w^2} \right) - \frac{1}{2} = \frac{p_w^2}{\ln p_w^2} \left( \frac{1}{\ln p_w^2} + \frac{1 - p_w}{2} \right) = \frac{p_w^2}{\ln p_w^2} \left( \frac{1}{\ln p_w^2} + \frac{1}{2q} \right) = \frac{p_w^2}{\ln p_w^2} \left( \frac{1}{\ln p_w^2} + \frac{1}{p_w} - 1 \right) = \frac{p_w^2}{\ln p_w^2} + p_w - p_{w+1} > \frac{p_w^2}{\ln q} + q - q^2 > 1.
\]  

By the fact that, for all \( y > 0 \), both \( \ln y \) and \( 1 - 1/y \) are increasing, \( f(p_w) \leq p_{w+1} \) so in view of the condition given by (167) we may assume that \( q = f(p_w) \); further, Lemma 14 for \( x = p_w \), gives \( p_w + 2 < f(p_w) \). We thereby have, by Lemma 15 for \( x = f(p_w) \) and \( w = d \),

\[
f(p_w)^2 \left(1 - \frac{1}{f(p_w)} \right) \left(\frac{1}{\ln(f(p_w)^2)} \right) - \frac{1}{2} \left(1 - \frac{1}{f(p_w) - 1} \right) > p_w^2 \left( \frac{1}{\ln(f(p_w)^2)} \left(1 - \frac{1}{f(p_w)} \right) - \frac{1}{2} \right) = p_w^2 \left( \frac{1}{\ln p_w^2} - \frac{1}{2} \right).
\]  

(171)
Combining (171) with (170), for \( q = f(p_w) \), gives
\[
\begin{align*}
p_{w+1}^2 \left(1 - \frac{1}{p_{w+1}}\right) & \left(1 - \frac{1}{\ln(p_{w+1}) \left(1 - \frac{1}{p_{w+1}}\right)} - \frac{1}{2}\right) \left(1 - \frac{1}{p_{w+1} - 1}\right) \\
& > p_{w+1}^2 \left(1 - \frac{1}{f(p_w)}\right) \left(\ln(p_{w+1}) - \frac{1}{2}\right) \left(1 - \frac{1}{f(p_w) - 1}\right) \\
& > f(p_w)^2 \left(1 - \frac{1}{f(p_w)}\right) \left(\ln(f(p_w)^2) - \frac{1}{2}\right) \left(1 - \frac{1}{f(p_w) - 1}\right) \\
& > p_w^2 \left(\frac{1}{\ln p_w^2} - \frac{1}{2}\right). \tag{172}
\end{align*}
\]

Taking any real number \( v \) for which \( 2 < v < \log_{p_w}(p_{w+1}^2) \), we may substitute, in the left-hand side of the first inequality of (172), one for \( 1 - 1/p_{w+1} \) and also for \( 1 - 1/(p_{w+1} - 1) \) to give
\[
p_w^v \left(\frac{1}{\ln p_w^2} - \frac{1}{2}\right) = \frac{p_w^v}{v \ln p_w} - \frac{p_w^v}{2} \\
> p_w^2 \left(\frac{1}{\ln p_w^2} - \frac{1}{2}\right). \tag{173}
\]

Since for each \( j \in \{0, 1\} \) and \( m \in \{1, 2\} \),
\[
\prod_{k=m}^{w+1} \left(1 - \frac{1}{p_k - j}\right) = \prod_{k=m}^{w} \left(1 - \frac{1}{p_k - j}\right) \left(1 - \frac{1}{p_{w+1} - j}\right), \tag{174}
\]
and since, together, (172) and (173), for \( v = u \), cover all values of \( u \), combining (172) with (173) gives (168).

**Definition.** For any \( x > 3 \), let \( h(x) \) be the real number for which
\[
x^2 g_x \left(\frac{1}{g_x \ln(x^2) \prod_{k=1}^{\pi(x)} \left(1 - \frac{1}{p_k}\right)} - \frac{1}{2}\right) \prod_{k=1}^{\pi(x)} \left(1 - \frac{1}{p_k}\right) - \Hi(x)^2 - \Hi(x) - 2 \\
= x^2 g_x \left(\frac{1}{h(x) g_x \ln(x^2) \prod_{k=1}^{\pi(x)} \left(1 - \frac{1}{p_k}\right)} - \frac{1}{2}\right) \prod_{k=1}^{\pi(x)} \left(1 - \frac{1}{p_k}\right) \tag{175}
\]
where \( g_x = 1 - 1/x \) if \( x \) is not prime and \( g_x = 1 \) if \( x \) is prime. Then
\[
\frac{1}{h(x) g_x \ln(x^2) \prod_{k=1}^{\pi(x)} \left(1 - \frac{1}{p_k}\right)} = \frac{1}{g_x \ln(x^2) \prod_{k=1}^{\pi(x)} \left(1 - \frac{1}{p_k}\right)} - \Hi(x)^2 + \Hi(x) + 2 \\
- \frac{\Hi(x)^2 + \Hi(x) + 2}{x^2 g_x \prod_{k=1}^{\pi(x)} \left(1 - \frac{1}{p_k}\right),} \tag{176}
\]
and
\[
\frac{1}{h(x)} = 1 - \frac{\Hi(x)^2 + \Hi(x) + 2}{x^2} \ln x^2.
\]
Lemma 17. For all \( x \geq 355,991 \) and all \( y > x \)

\[ h(x) > h(y) \] (178)

and

\[ h(x) \sim 1 \text{ as } x \to \infty. \] (179)

Proof. In reference to the results, (141) and (142), of Dusart we combine the fact that \( \text{Hi}(x) > x/\ln x \) with the first equality of (177) to give

\[ h(x) = \frac{1}{1 - \frac{(\text{Hi}(x)^2 + \text{Hi}(x) + 2) \ln x^2}{x^2}} \]

\[ = \frac{x^2 - (\text{Hi}(x)^2 + \text{Hi}(x) + 2) \ln x^2}{x^2}. \] (177)

Regarding the final denominator, for each \( t \in \{x^2/\ln x, 2x\} \) (here, cancelling \( \ln x \) to give the 2x, and noting also that \( \ln^2 x/\ln x \) is strictly increasing) it is true that \( t/x^2 \) is strictly decreasing. Since, as mentioned when we first introduced \( \text{Hi}(x), \) \( \text{Hi}(x) \ln x/x \) is strictly decreasing to one, we therefore have (178). Since \( t/x^2 \) decreases to one as \( x \to \infty, \) we have (179). \( \square \)

Lemma 18. For all \( i > 30,456 \) for which

\[ \ln p_i^2 < \ln(p_{i+1}^2) \left(1 - \frac{1}{p_{i+1}}\right) \] (181)

and

\[ \left( \frac{p_i^2}{\ln(p_i^2)} \prod_{k=1}^{i} \left(1 - \frac{1}{p_k}\right) - \frac{1}{2} \right) \prod_{i=1}^{\pi(\sqrt{p_i})} \left(1 - \frac{1}{p_i} - \text{Hi}(p_i)^2 - \text{Hi}(p_i) - 2\right) \times \prod_{m=2}^{i} \left(1 - \frac{1}{p_m} - 1\right) > 1, \] (182)

we have, for all even \( r \) and any \( 2 < v \leq \log_{p_i}(p_{i+1}^2), \)

\[ \left\{ 1 \leq m \leq \left[ \frac{p_i^2}{2} \right] : \left( m(m-r), \prod_{k=1}^{\pi(\sqrt{p_i})} p_k \right) = 1 \right\} > 1. \] (183)
Proof. Combining, for \( x = p_i \), Lemma \( 17 \) specifically \( (178) \), with Lemma \( 16 \) for \( w = i \) and \( u = v \), gives

\[
p_i^v \left( \frac{1}{h(\sqrt{p_i^v}) \ln(p_i^v) \prod_{k=1}^{\pi(\sqrt{p_i^v})} (1 - \frac{1}{p_k})} - \frac{1}{2} \right) \prod_{k=1}^{\pi(\sqrt{p_i^v})} \left( 1 - \frac{1}{p_k} \right)
\]

\[
\times \prod_{s=2}^{\pi(\sqrt{p_i^v})} \left( 1 - \frac{1}{p_s - 1} \right)
\]

\[
> p_i^v \left( \frac{1}{h(p_i) \ln(p_i^v) \prod_{k=1}^{\pi(\sqrt{p_i^v})} (1 - \frac{1}{p_k})} - \frac{1}{2} \right) \prod_{k=1}^{\pi(\sqrt{p_i^v})} \left( 1 - \frac{1}{p_k} \right)
\]

\[
\times \prod_{s=2}^{\pi(\sqrt{p_i^v})} \left( 1 - \frac{1}{p_s - 1} \right)
\]

\[
= p_i^v \left( \frac{p_i^2 - (H_i(p_i) + 2) \ln(p_i^2)}{p_i^2 \ln(p_i^v) \prod_{k=1}^{\pi(\sqrt{p_i^v})} (1 - \frac{1}{p_k})} - \frac{1}{2} \right) \prod_{k=1}^{\pi(\sqrt{p_i^v})} \left( 1 - \frac{1}{p_k} \right)
\]

\[
\times \prod_{s=2}^{\pi(\sqrt{p_i^v})} \left( 1 - \frac{1}{p_s - 1} \right)
\]

\[
> p_i^2 \left( \frac{p_i^2 - (H_i(p_i) + 2) \ln(p_i^2)}{p_i^2 \ln(p_i^2) \prod_{k=1}^{\pi(\sqrt{p_i^v})} (1 - \frac{1}{p_k})} - \frac{1}{2} \right) \prod_{t=1}^{i} \left( 1 - \frac{1}{p_t} \right) \prod_{j=2}^{i} \left( 1 - \frac{1}{p_j - 1} \right). \tag{184}
\]

The condition given by \( (182) \) thereby requires that

\[
p_i^v \left( \frac{p_i^2 - (H_i(p_i) + 2) \ln(p_i^2)}{p_i^2 \ln(p_i^v) \prod_{k=1}^{\pi(\sqrt{p_i^v})} (1 - \frac{1}{p_k})} - \frac{1}{2} \right) \prod_{k=1}^{\pi(\sqrt{p_i^v})} \left( 1 - \frac{1}{p_k} \right)
\]

\[
\times \prod_{s=2}^{\pi(\sqrt{p_i^v})} \left( 1 - \frac{1}{p_s - 1} \right)
\]

\[
> 1. \tag{185}
\]

Combining \( (185) \) with Lemma \( 12 \) for \( n = i \) when \( p_i^v < p_{i+1}^2 \) and \( n = i + 1 \) when \( p_i^v = p_{i+1}^2 \) and, in both cases, \( z = \lceil p_i^v/2 \rceil \), gives \( (183) \).

\[ \square \]

Lemma 19. For all \( j > 30, 456 \) for which

\[
\ln(p_j^2) \geq \ln(p_{j+1}^2) \left( 1 - \frac{1}{p_{j+1}} \right) \tag{186}
\]
and

\[
\left( p_j^2 \left( \frac{1}{\ln(p_j^2) \prod_{t=1}^{j} \left( 1 - \frac{1}{p_t} \right)} \right) \right) \frac{1}{2} \prod_{t=1}^{j} \left( 1 - \frac{1}{p_t} \right) - Hi(p_j)^2 - Hi(p_j) - 2 \right)
\times \prod_{m=2}^{j} \left( 1 - \frac{1}{p_m - 1} \right)
> 1,
\]

(187)

we have, for all even \( r \) and any integer \((p_j^2 + 1)/2 < q \leq (p_{j+1}^2 + 1)/2\),

\[
\left\{ 1 \leq m \leq q : \left( m(m - r), \prod_{k=1}^{\pi(\sqrt{m})} p_k \right) = 1 \right\} > 1.
\]

(188)

Proof. First, let us consider the case where \( q = (p_{j+1}^2 + 1)/2 \). With reference to the condition given by (186), we have

\[
\prod_{k=1}^{j+1} \left( 1 - \frac{1}{p_k} \right) = 1 - \frac{1}{p_{j+1}}.
\]

(189)

Thus it follows by (178) (for \( x = p_j \) and \( y = p_{j+1} \)) that

\[
\frac{1}{h(p_{j+1}) \ln(p_{j+1}) \prod_{k=1}^{j+1} \left( 1 - \frac{1}{p_k} \right)} > \frac{p_{j+1}^2 - (Hi(p_{j+1})^2 + Hi(p_j) + 2) \ln p_{j+1}^2}{p_{j+1}^2 \ln(p_{j+1}^2) \prod_{k=1}^{j+1} \left( 1 - \frac{1}{p_k} \right)}
\]

\[
\geq \frac{p_{j+1}^2 - (Hi(p_j)^2 + Hi(p_j) + 2) \ln p_{j+1}^2}{p_{j+1}^2 \ln(p_{j+1}^2) \prod_{k=1}^{j+1} \left( 1 - \frac{1}{p_k} \right)}.
\]

(190)

Through Lemma [15] for \( x = p_{j+1} \) and \( d = j \) and taking \( M \) as the set of all \( p \in P(j + 1) \) for which \( p | r \), we have

\[
p_{j+1}^2 \prod_{k=1}^{j+1} \left( 1 - \frac{1}{p_k} \right) \prod_{u=2}^{j+1} \left( 1 - \frac{1}{p_u - 1} \right) > p_{j+1}^2 \prod_{t=1}^{j} \left( 1 - \frac{1}{p_t} \right) \prod_{t=2}^{j} \left( 1 - \frac{1}{p_t - 1} \right).
\]

(191)

On multiplying both sides by

\[
\frac{p_{j+1}^2 - (Hi(p_j)^2 + Hi(p_j) + 2) \ln p_{j+1}^2}{p_{j+1}^2 \ln(p_{j+1}^2) \prod_{t=1}^{j} \left( 1 - \frac{1}{p_t} \right)} - \frac{1}{2}
\]

(192)

a value which, by (177), is equal to

\[
\frac{1}{h(p_j) \ln(p_j^2) \prod_{t=1}^{j} \left( 1 - \frac{1}{p_t} \right)} - \frac{1}{2}.
\]

(193)

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combining (191) with (190) gives

\[ \frac{p_{j+1}^2}{h(p_{j+1}) \ln(p_{j+1}^2) \prod_{k=1}^{j+1} \left(1 - \frac{1}{p_k}\right)} - \frac{1}{2} \prod_{k=1}^{j+1} \left(1 - \frac{1}{p_k}\right) \times \prod_{u=2}^{j+1} \left(1 - \frac{1}{p_u - 1}\right) \]

\[ > \frac{p_{j+1}^2}{h(p_{j}) \ln(p_{j}^2) \prod_{t=1}^{j} \left(1 - \frac{1}{p_t}\right)} - \frac{1}{2} \prod_{t=1}^{j} \left(1 - \frac{1}{p_t}\right) \prod_{s=2}^{j} \left(1 - \frac{1}{p_s - 1}\right). \]  

(194)

Since, when \( q < (p_{j+1}^2 + 1)/2 \), we have \( q^2 > p_{j}^2 \) and \( h(\sqrt{q}) > h(p_{j}) \), combining (194) with Lemma [12] for \( n = j \) and \( q = q \) when \( q < p_{j+1}^2/2 \) and \( n = j+1 \) and \( q = q \) when \( q = (p_{j+1}^2 + 1)/2 \), gives (188).

Proof of Theorem 2

Suppose that \( w \) is an integer \( > p_{30456}^2/2 \). Then the number of ways of writing \( w \) as the arithmetic mean of two primes is greater than or equal to the cardinality of

\[ R = \left\{ 1 \leq m \leq w : \left( m(2w - m), \prod_{k=1}^{\pi(\sqrt{2w})} p_k \right) = 1 \right\}. \]

This follows from the fact that, for any two positive integers \( p \) and \( q \) for which \( p < q \) and \( w \) is the arithmetic mean of \( p \) and \( q \), and \( pq \) is coprime to \( \prod_{k=1}^{\pi(\sqrt{2w})} p_k \), we have the following. By tacitly using the sieve of Eratosthenes we have the result that \( p \) and \( q \) are each prime; also, \( p \) is in \( R \) with \( 2w - p = q \) and for any two primes, \((a, b)\), the average of which is \( w \), which is a condition that \( p \) and \( q \) together satisfy, \( a + b = 2w \) satisfies the Goldbach equation.

Recall that, for any \( x > 1 \), \( \text{Hi}(x) = (x/\ln x)(1 + 1/\ln x + 2.51/\ln^2 x) \) and that for all \( y > 355991 \), \( \pi(y) < \text{Hi}(y) \). By Lemma [12] for \( n = \pi(\sqrt{2w}) \) and \( r = 2z = 2w \), the fact that the lowest value in the range of \( w \) is \((p_{30456}^2 + 1)/2 \) gives

\[ \left\{ 1 \leq m \leq w : \left( m(2w - m), \prod_{k=1}^{\pi(\sqrt{2w})} p_k \right) = 1 \right\} \]

\[ > \left( p_{\pi(\sqrt{2w})}^2 \left( \frac{1}{\ln(p_{\pi(\sqrt{2w})}) \prod_{k=1}^{\pi(\sqrt{2w})} \left(1 - \frac{1}{p_k}\right)} - \frac{1}{2} \prod_{k=1}^{\pi(\sqrt{2w})} \left(1 - \frac{1}{p_k}\right) \right) \prod_{j=2}^{\pi(\sqrt{2w})} \left(1 - \frac{1}{p_j - 1}\right) \right). \]  

(195)
We note that \( p_{30457} = 356,023 \). For any \( n > 30,456 \), choose \( w \) so that \( n = \pi(\sqrt{2w}) \). Then the conditions on \( i \) and \( j \), in Lemmas [18] and [19] respectively, immediately require that, when the left-hand side of (195) is \( \geq 1 \), for some \( a_n \), if \( p_{n+1} > a_n \) we may take \( n = i \) and if \( p_{n+1} \leq a_n \) we may take \( n = j \). When \( p_n = 356,023 \) we find that the right-hand side of (195) is approximately 72,306,082.

By this result, when \( w \) is increased by increments of one, combining Lemma [18] with Lemma [19] implies, taking \( n = i \) for Lemma [18] and \( n = j \) for Lemma [19], thence

\[
\prod_{k=1}^{30457} \left( 1 - \frac{1}{p_k} \right) \approx 0.04392 \ldots
\]  

and

\[
\prod_{k=1}^{30457} \left( 1 - \frac{1}{p_k} \right) \prod_{k=2}^{30457} \left( 1 - \frac{1}{p_k - 1} \right) \approx 0.002546 \ldots
\]

and

\[
\text{Hi}(356023) \approx 30458.52812 \ldots.
\]

Thus, when on the right-hand side of the inequality in the statement of Lemma [12] is equal to 30,457,

\[
\left( \frac{p_{30457}^2}{\ln p_{30457}^2} \right) \left( 1 - \frac{1}{p_{30457}} \right) \prod_{k=1}^{30457} \left( 1 - \frac{1}{p_k} \right) - \text{Hi}(p_{30457})^2
\]

\[
- \text{Hi}(p_{30457}) - 2 \prod_{s=2}^{t} \left( 1 - \frac{1}{p_s - 1} \right)
\]

\[
\approx 72,306,082.23 \ldots > 1.
\]

3. Proof of the Twin Primes Conjecture

**Proof of Theorem [3]** By Lemma [12] we have, for all \( n > 2 \) and any even \( r \),

\[
\left\{ 1 \leq m \leq \frac{p_n^2 + 1}{2} : \left( m(m-r), \prod_{k=1}^{n} p_k \right) = 1 \right\}
\]

\[
> \left( \frac{p_r^2}{\ln(p_r^2)} \prod_{k=1}^{n} \left( 1 - \frac{1}{p_k} \right) - \frac{1}{2} \right) \prod_{k=1}^{n} \left( 1 - \frac{1}{p_k} \right) - \text{Hi}(p_n)^2
\]

\[
- \text{Hi}(p_n) - 2 \prod_{s=2}^{t} \left( 1 - \frac{1}{p_s - 1} \right)
\]

(201)

The inequality (201) is the inequality in Lemma [12] for \( z = (p_n^2 + 1)/2 \) and \( s = 1 \). Using \( s = 1 \) changes the \( \geq \) sign to a \( > \) sign.
By definition of $h$, $h(p_n)$ is the real number for which

$$p_n^2 \left( \frac{1}{\ln(p_n^2)} \prod_{k=1}^{n} \left(1 - \frac{1}{p_k}\right) - \frac{1}{2} \right) \prod_{k=1}^{n} \left(1 - \frac{1}{p_k}\right) - \text{Hi}(p_n)^2 - \text{Hi}(p_n) - 2$$

$$= p_n^2 \left( \frac{1}{h(p_n) \ln(p_n^2)} \prod_{k=1}^{n} \left(1 - \frac{1}{p_k}\right) \right) \prod_{k=1}^{n} \left(1 - \frac{1}{p_k}\right). \tag{202}$$

The definition of $h$ is given by (175), where $g_{p_n} = 1$.

The next ingredient of our proof is

$$\frac{1}{\ln(p_n^2)} \prod_{k=1}^{n} \left(1 - \frac{1}{p_k}\right) \sim e^{\gamma}/2. \tag{203}$$

This formula follows from the fact that, by the Mertens theorem, $(\ln(p_n) \prod_{k=1}^{n} (1 - 1/p_k))^{-1} \sim e^{\gamma}$.

The next formula in our proof is

$$\frac{e^{\gamma}}{h(p_n)} \sim e^{\gamma}. \tag{204}$$

This formula follows because $h(x) \sim 1$, as shown in (179). (Note: that explanation is not exactly right, but it is reasonably close and we will fix it later.)

Finally, we introduce

$$p_n^2 \left( \frac{e^{\gamma}}{2} - \frac{1}{2} \right) \prod_{k=1}^{n} \left(1 - \frac{1}{p_k}\right) \prod_{m=2}^{n} \left(1 - \frac{1}{p_m - 1}\right) \sim \frac{Cp_n^2}{\log^2(n)} \tag{205}$$

for some constant $C$. This follows from the Mertens Theorem combined with the fact that, by the Prime Number theorem, $p_n \sim n \log(n)$.

We have

$$(\text{The number of twin primes } \leq \frac{p_n^2 + 1}{2}) =$$

$$\left| \left\{ 1 \leq m \leq \frac{p_n^2 + 1}{2} : \left( m(m - 2), \prod_{k=1}^{n} p_k \right) = 1 \right\} \right| + \left| \left\{ q \in P(n+1) : q - 2 \in P(n) \right\} \right|$$

$$\sim \frac{Cp_n^2}{\log^2(n)}. \tag{206}$$

This formula follows from combining all of (201) to (205) coupled with the following. The number of primes, $p$, for which $(p, \prod_{k=1}^{n} p_k) \neq 1$, is equal to $n$, and $n/\pi(p_n^2) \sim 0$, while $n$ is less than the second term on the left-hand side.

Since $e^{\gamma}/2 \approx 0.89 \ldots > 1/2$ and $p_n^2/\log^2(n)$ approaches infinity as $n \to \infty$, by combining (205) with (206) the proof is complete.

$\square$

**Theorem 3.** There are infinitely many twin primes.
**Proof of Theorem 3**

Theorem 3 is equivalent to the following. For any \( n \)
\[
\lim_{{n \to \infty}} \left| \left\{ 1 \leq c \leq \frac{p_n^2}{2} : \left( c(c-2), \prod_{{k=1}}^{n} p_k \right) = 1 \right\} \right| = \infty. \tag{207}
\]

We note that, for each \( x \) in
\[
\left\{ 1 \leq c \leq \frac{p_n^2}{2} : \left( c(c-2), \prod_{{k=1}}^{n} p_k \right) = 1 \right\},
\]
both \( x \) and \( x - 2 \) are prime. This follows from the fact that each composite in \([1, p_n^2]\) is an integer multiple of some element of \( P(n) \). Taking \( m = p + 2 \) and \( m - 2 = p \), Theorem 3 thus implies that the number of twin primes, \((p, p + 2)\), for which \( p + 2 < \frac{(p_n^2 + 1)}{2} \) approaches infinity as \( n \to \infty \), which is to say that it implies Theorem 3. \( \square \)

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