STRUCTURAL STABILITY OF TRANSONIC SHOCK FLOWS WITH AN EXTERNAL FORCE

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Abstract. This paper is devoted to the structural stability of a transonic shock passing through a flat nozzle for two-dimensional steady compressible flows with an external force. We first establish the existence and uniqueness of one dimensional transonic shock solutions to the steady Euler system with an external force by prescribing suitable pressure at the exit of the nozzle when the upstream flow is a uniform supersonic flow. It is shown that the external force helps to stabilize the transonic shock in flat nozzles and the shock position is uniquely determined. Then we are concerned with the structural stability of these transonic shock solutions when the exit pressure is suitably perturbed. One of the new ingredients in our analysis is to use the deformation-curl decomposition to the steady Euler system developed in [27] to deal with the transonic shock problem.

1. Introduction and main results

The studies of transonic shock solutions for inviscid compressible flows in different kinds of nozzles had a long history and had obtained many important new achievements during the past twenty years. Courant and Friedrichs [9] had described the transonic shock phenomena in a de Laval nozzle whose cross section decreases first and then increases. It was observed in experiment that if the upcoming flow becomes supersonic after passing through the throat of the nozzle, to match the prescribed appropriately large exit pressure, a shock front intervenes at some place in the diverging part of the nozzle and the gas is compressed and slowed down to subsonic speed.

People first used the quasi one dimensional model to study the transonic shock problem [1,9,10,14]. The structural stability of multidimensional transonic shocks in flat or diverging nozzles were further investigated in [6,30,31] using the steady potential flows with different kinds of boundary conditions. In particular, [30,31] proved that the stability of transonic shocks for potential flows is usually ill-posed under the perturbations of the exit pressure. Many researchers also considered the transonic shock problem in the flat or almost flat nozzles with the exit pressure satisfying some special constraint, see [3–5,15,29] and the references therein. Recently, there is an interesting progress on the stability and existence of transonic shock solutions to the two dimensional and three dimensional axisymmetric steady compressible Euler system in an almost flat finite nozzle with the receiver pressure prescribed at the exit of the nozzle (see [11,12]), where the shock position was uniquely determined.

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The structural stability of the transonic shock problem in two dimensional divergent nozzles under the perturbations for the exit pressure was first established in [16] when the opening angle of the nozzle is suitably small. Later on, this restriction was removed in [17, 20]. Furthermore, the transonic shock in general two dimensional straight divergent nozzles was shown in [20] to be structurally stable under generic perturbations for both the nozzle shape and the exit pressure. The existence and stability of transonic shock for three dimensional axisymmetric flows without swirl in a conic straight nozzle were established in [18, 19] with respect to small perturbations of the exit pressure. For the structural stability under the axisymmetric perturbation of the nozzle wall, a modified Lagrangian coordinate was introduced in [26] to deal with the corner singularities near the intersection points of the shock surface and nozzle boundary and the artificial singularity near the axis simultaneously. Most recently, the authors in [24, 25] studied radially symmetric transonic flow with/without shock in an annulus. Thanks to the effect of angular velocity, it was found in [24] that besides the well-known supersonic-subsonic shock in a divergent nozzle as in the case without angular velocity, there exists a supersonic-supersonic shock solution, where the downstream state may change smoothly from supersonic to subsonic. Furthermore, there exists a supersonic-sonic shock solution where the shock circle and the sonic circle coincide.

In this paper, we will consider similar transonic shock phenomena occurring in a flat nozzle when the fluid is exerted with an external force. The 2-D steady compressible isentropic Euler system with external force are of the form

\[
\begin{aligned}
\tag{1.1}
\partial_{x_1}(\rho u_1) + \partial_{x_2}(\rho u_2) &= 0, \\
\partial_{x_1}(\rho u_1^2 + P(\rho)) + \partial_{x_2}(\rho u_1 u_2) &= \rho \partial_{x_1} \Phi, \\
\partial_{x_1}(\rho u_1 u_2) + \partial_{x_2}(\rho u_2^2 + P(\rho)) &= \rho \partial_{x_2} \Phi,
\end{aligned}
\]

where \((u_1, u_2) = u : \mathbb{R}^2 \rightarrow \mathbb{R}^2\) is the unknown velocity field and \(\rho : \mathbb{R}^2 \rightarrow \mathbb{R}\) is the density, and \(\Phi(x_1, x_2)\) is a given potential function of external force. For the ideal polytropic gas, the equation of state is given by \(P(\rho) = A\rho^\gamma\), here \(A\) and \(\gamma\) \((1 < \gamma < 3)\) are positive constants. We take \(A = 1\) throughout this paper for the convenience.

To this end, let’s firstly focus on the 1-D steady compressible flow with an external force on an interval \(I = [L_0, L_1]\), which is governed by

\[
\begin{aligned}
\tag{1.2}
& (\bar{\rho} \bar{u})'(x_1) = 0, \\
& \bar{\rho} \bar{u} \bar{u}' + \frac{d}{dx_1} P(\bar{\rho}) = \bar{\rho} \bar{f}(x_1), \\
& \bar{\rho}(L_0) = \rho_0 > 0, \quad \bar{u}(L_0) = u_0 > 0,
\end{aligned}
\]

where we assume that the flow state at the entrance \(x_1 = L_0\) is supersonic, meaning that \(u_0^2 > c^2(\rho_0) = \gamma \rho_0^{\gamma - 1}\).
Denote \( J = \bar{\rho} \bar{u} = \rho_0 u_0 > 0 \), then it follows from (1.2) that

\[
\begin{align*}
\ddot{\rho}(x_1) &= \frac{J}{\bar{u}(x_1)}, \\
((\bar{u})^{\gamma+1} - \gamma J^{\gamma-1}) \ddot{u} &= \bar{u} \ddot{f}.
\end{align*}
\]

Also one has

\[
\ddot{u} = \frac{\bar{u} \ddot{f}}{\bar{u}^2 - c^2(\bar{\rho})}, \quad \ddot{\rho} = -\frac{\bar{\rho} \ddot{f}}{\bar{u}^2 - c^2(\bar{\rho})},
\]

\[
\frac{d}{dx_1} M^2(x_1) = \frac{(\gamma + 1) M^2 \ddot{f}}{M^2 - 1} \frac{\ddot{f}}{c^2(\bar{\rho})},
\]

where \( M(x_1) = \frac{\bar{u}(x_1)}{c(\bar{\rho})} \) is the Mach number.

Since \( M^2(L_0) > 1 \), it follows from (1.5) that if the external force satisfies

\[
f(x_1) > 0, \quad \forall L_0 < x_1 < L_1,
\]

then the problem (1.2) has a global supersonic solution \((\bar{\rho}^-, \bar{u}^-)\) on \([L_0, L_1]\). If one prescribes a large enough end pressure at \( x_1 = L_1 \), a shock will form at some point \( x_1 = L_s \in (L_0, L_1) \) and the gas is compressed and slowed down to subsonic speed, the gas pressure will increase to match the given end pressure. Mathematically, one looks for a shock \( x_1 = L_s \) and smooth functions \((\bar{\rho}^\pm, \bar{u}^\pm, \bar{P}^\pm)\) defined on \( I^+ = [L_s, L_1] \) and \( I^- = [L_0, L_s] \) respectively, which solves (1.3) on \( I^\pm \) with the jump at the shock \( x_1 = L_s \in (L_0, L_1) \) satisfying the physical entropy condition \([\bar{P}(L_s)] = \bar{P}^+(L_s) - \bar{P}^-(L_s) > 0\) and the Rankine-Hugoniot conditions

\[
\begin{align*}
[\bar{\rho} \bar{u}](L_s) &= 0, \\
[\bar{\rho} \bar{u}^2 + \bar{P}(\bar{\rho})](L_s) &= 0.
\end{align*}
\]

and also the boundary conditions

\[
\begin{align*}
\rho(L_0) &= \rho_0, \quad u(L_0) = u_0 > 0, \\
\bar{P}(L_1) &= \bar{P}_e.
\end{align*}
\]

We will show that there is a unique transonic shock solution to the 1-D Euler system when the end pressure \( \bar{P}_e \) lies in a suitable interval. Such a problem will be solved by a shooting method employing the monotonicity relation between the shock position and the end pressure.

**Lemma 1.1.** Suppose that the initial state \((u_0, \rho_0)\) at \( x_1 = L_0 \) is supersonic and the external force \( f \) satisfying (1.6), there exists two positive constants \( P_0, P_1 > 0 \) such that if the end pressure \( \bar{P}_e \in (P_1, P_0) \), there exists a unique transonic shock solution \((\bar{u}^-, \bar{\rho}^-)\) and \((\bar{u}^+, \bar{\rho}^+)\) defined on \( I^- = [L_0, L_s] \) and \( I^+ = (L_s, L_1) \) respectively, with a shock located at \( x_1 = L_s \in (L_0, L_1) \). In addition, the shock position \( x_1 = L_s \) increases as the exit pressure \( \bar{P}_e \) decreases. Furthermore, the shock position \( L_s \) approaches to \( L_1 \) if \( \bar{P}_e \) goes to \( P_1 \) and \( L_s \) tends to \( L_0 \) if \( \bar{P}_e \) goes to \( P_0 \).
Proof. The existence and uniqueness of smooth supersonic flow \((\bar{u}^-, \bar{\rho}^-)\) starting from \((\rho_0, u_0)\) on \([L_0, L_1]\) is trivial. Suppose the shock occurs at \(x_1 = L_s \in (L_0, L_1)\), then it is well-known that there exists a unique subsonic state \((\bar{u}^+(L_s), \bar{\rho}^+(L_s))\) satisfying the Rankine-Hugoniot conditions (1.7) and the entropy condition. With \((\bar{u}^+(L_s), \bar{\rho}^+(L_s))\) as the initial data, the equation (1.2) has a unique smooth solution \((\bar{u}^+, \bar{\rho}^+)\) on \([L_s, L_1]\). Denote \(p_e = (\bar{\rho}^+(L_1))^\gamma\). In the following, we show that the monotonicity between the shock position \(x_1 = L_s\) and the exit pressure \(P_e = (\bar{\rho}^+(L_1))^\gamma\). \(\bar{\rho}^+(L_1)\) is regarded as a function of \(L_s\). Since \((\bar{\rho}^- \bar{u}^+)(L_s) = (\bar{\rho}^- \bar{u}^-(L_s)) = J = \rho_0 u_0 > 0\), then

\[
\bar{u}^-(L_s) + \frac{\gamma - 1}{\gamma} \frac{(\bar{\rho}^- + (L_s))^{\gamma - 1}}{(\bar{\rho}^- (L_s))^{\gamma - 1}} = \bar{u}^+(L_s) + \frac{\gamma - 1}{\gamma} \frac{(\bar{\rho}^+ (L_s))^{\gamma - 1}}{(\bar{\rho}^+ (L_s))^{\gamma - 1}}.
\]

(1.10)

It follows from the second equation in (1.2) that

\[
\frac{1}{2} (\bar{u}^+(L_1))^2 + \frac{\gamma}{\gamma - 1} (\bar{\rho}^+(L_1))^{\gamma - 1} - \Phi(L_1) = \frac{1}{2} (\bar{u}^+(L_s))^2 + \frac{\gamma}{\gamma - 1} (\bar{\rho}^+(L_s))^{\gamma - 1} - \Phi(L_s).
\]

Differentiating with respect to \(L_s\), one deduces that

\[
\left( \frac{\gamma}{\gamma - 1} (\bar{\rho}^+(L_1))^{\gamma - 2} - \frac{J^2}{(\bar{\rho}^+(L_1))^3} \right) \frac{d\bar{\rho}^+(L_1)}{dL_s} = \left( \frac{\gamma}{\gamma - 1} (\bar{\rho}^+(L_s))^{\gamma - 2} - \frac{J^2}{(\bar{\rho}^+(L_s))^3} \right) \frac{d\bar{\rho}^+(L_s)}{dL_s} - \bar{f}(L_s) =: I.
\]

Also (1.10) yields that

\[
\left\{ 1 - \frac{\gamma J^{\gamma - 1}}{(\bar{\rho}^+(L_s))^{\gamma + 1}} \right\} \frac{d\bar{u}^+(L_s)}{dL_s} = \left\{ 1 - \frac{\gamma J^{\gamma - 1}}{(\bar{\rho}^-(L_s))^{\gamma + 1}} \right\} \frac{d\bar{u}^-(L_s)}{dL_s} = \bar{f}(L_s).
\]

Finally, we conclude that

\[
I = - \left\{ \frac{\gamma (\bar{\rho}^+(L_s))^{\gamma - 1} - \frac{J^2}{(\bar{\rho}^+(L_s))^3}}{\bar{u}^+(L_s) \bar{\rho}^+(L_s) - \bar{u}^-(L_s)} \right\} \left\{ 1 - \frac{\gamma J^{\gamma - 1}}{(\bar{\rho}^+(L_s))^{\gamma + 1}} \right\} \frac{d\bar{u}^+(L_s)}{dL_s} - \bar{f}(L_s)
\]

\[
= \frac{\bar{f}(L_s)(\bar{u}^+(L_s) - \bar{u}^-(L_s))}{\bar{u}^-(L_s)} < 0.
\]

Since the coefficients

\[
\gamma (\bar{\rho}^+(L_1))^{\gamma - 2} - \frac{J^2}{(\bar{\rho}^+(L_1))^3} > 0,
\]

then (1.11) implies that the end density \(\bar{\rho}^+(L_1)\) is a strictly decreasing function of the shock position \(x_1 = L_s\). It follows that the end pressure \(P_e = (\bar{\rho}^+(L_1))^\gamma\) is a strictly decreasing and continuous differentiable function on the shock position \(x_1 = L_s\). In particular, when \(L_s = L_0\) and \(L_s = L_1\), there are two different end pressure \(P_1, P_2\) with \(P_0 > P_1\). Hence, by the monotonicity one can obtain a transonic shock for the end pressure \(P_e \in (P_1, P_0)\).

\[\square\]

Remark 1. Lemma 1.1 shows that the external force helps to stabilize the transonic shock in flat nozzles and the shock position is uniquely determined.
The one dimensional transonic shock solution \((\bar{u}^{\pm}, \bar{\rho}^{\pm})\) with a shock occurring at \(x_1 = L_s\) constructed in Lemma 1.1 will be called the background solution in this paper. The extension of the subsonic flow \((\bar{u}^{+}(x_1), \bar{\rho}^{+}(x_1))\) of the background solution to \(L_s - \delta_0 < x_1 < L_1\) for a small positive number \(\delta_0\) will be denoted by \((\hat{u}^{+}(x_1), \hat{\rho}^{+}(x_1))\).

Figure 1. Nozzle

It is natural to focus on the structural stability of this transonic shock flows. For simplicity, we only investigate the structural stability under suitable small perturbations of the end pressure. Therefore, the supersonic incoming flow is unchanged and remains to be \((\bar{u}^{-}(x_1), 0, \bar{\rho}^{-}(x_1))\).

Assume that the possible shock curve \(\Sigma\) and the flow behind the shock are denoted by \(x_1 = \xi(x_2)\) and \((u_1^+, u_2^+, P^+)(x)\) respectively (See Figure 1). Let \(\Omega^+ = \{(x_1, x_2) : \xi(x_2) < x_1 < L_1, -1 < x_2 < 1\}\) denotes the subsonic region of the flow. Then the Rankine-Hugoniot conditions on \(\Sigma\) gives

\[
\begin{align*}
[\rho u_1] - \xi'(x_2)[\rho u_2] &= 0, \\
[\rho u_1^2 + P] - \xi'(x_2)[\rho u_1 u_2] &= 0, \\
[\rho u_1 u_2] - \xi'(x_2)[\rho u_2^2 + P] &= 0.
\end{align*}
\]

(1.12)

In addition, the pressure \(P\) satisfies the physical entropy conditions

\[P^+(x) > P^-(x) \quad \text{on } \Sigma.\]  

(1.13)

Since the flow is tangent to the nozzle walls \(x_2 = \pm 1\), then

\[u_2^+(x_1, \pm 1) = 0.\]  

(1.14)

The end pressure is perturbed by

\[P^+(L_1, x_2) = P_e + \epsilon P_{ex}(x_2), \]

(1.15)

due to some technical reasons, we may readily suppose that \(P_{ex}(x_2) = \bar{P}_{ex}(x_2) \in C^2 Olsen([-1, 1])\) satisfies the compatibility conditions

\[\bar{P}'_{ex}(\pm 1) = 0.\]  

(1.16)
The following theorem gives the main results of this paper.

**Theorem 1.2.** Under the assumptions on the external force and the exit pressure, there exists a constant $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0]$, the system \((1.1), (1.12)-(1.15)\) has a unique transonic shock solution \((u_1^+(x), u_2^+(x), P^+(x); \xi(x_2))\) which admits the following properties:

(i) The shock $x_1 = \xi(x_2) \in C^{3,\alpha}([-1,1])$, and satisfies

\[
\|\xi(x_2) - L_s\|_{C^{3,\alpha}([-1,1])} \leq C \epsilon, \tag{1.17}
\]

where the positive constant $C$ only depends on the background solution, the exit pressure and $\alpha$. 

(ii) The velocity and pressure in subsonic region \((u_1^+, u_2^+, P^+)(x) \in C^{2,\alpha}(\bar{\Omega}^+)\), and there holds

\[
\|(u_1^+, u_2^+, P^+)(x) - (\hat{u}, 0, \hat{P})\|_{C^{2,\alpha}(\bar{\Omega}^+)} \leq C \epsilon, \tag{1.18}
\]

where $\Omega^+ = \{(x_1, x_2) : \xi(x_2) < x_1 < L_1, -1 < x_2 < 1\}$ is the subsonic region and \((\hat{u}, 0, \hat{P}) = (\hat{u}(x_1), 0, \hat{P}(\hat{\rho}(\hat{x}_1)))\) is the extended background solution.

Our proof is influenced by the approach developed in [16, 17, 20], yet the reformulation of the problem is different from there. It is well-known that steady Euler equations are hyperbolic-elliptic coupled in subsonic region. The entropy and Bernoulli’s function are conserved along the particle path, while the pressure and the flow angle satisfy a first order elliptic system in subsonic region. These facts are widely used in the structural stability analysis for the transonic shock problems in flat or divergent nozzles, one may refer to [3, 7, 8, 16, 17, 20, 22, 32, 33] and the reference therein. Here we resort to a different decomposition based on the deformation and curl of the velocity developed in [27, 28] for three dimensional steady Euler and Euler-Poisson systems. The idea in that decomposition is to rewrite the density equation as a Frobenius inner product of a symmetric matrix and the deformation matrix by using the Bernoulli’s law. The vorticity is resolved by an algebraic equation of the Bernoulli’s function and the entropy. We should mention that there are several different decompositions to three dimensional steady Euler system [2, 4, 5, 21, 23, 32] developed by many researchers for different purposes. An interesting issue that deserves further discussion is when using the deformation-curl decomposition to deal with the transonic shock problem, the end pressure boundary condition becomes nonlocal since it involves the information from the shock front (see (3.9)). However, this nonlocal boundary condition reduces to be local after introducing the potential function (see (3.15)).

The rest of this paper will be organized as follows. In Section 2, we reformulate the original 2-D problem \((1.1)-(1.15)\) by deformation-curl decomposition developed in [27, 28] so that one can rewrite the system \((1.1)\) with the velocity and the Bernoulli function. We obtain a $2 \times 2$ first order system for the velocity field, a transport type equation for the Bernoulli function and the first order ordinary differential equation for the shock after linearization. In Section 3, we design an elaborate iteration scheme inspired by the works [17] for the nonlinear system. The investigation of well-posedness and regularity for the linear system are given in the reminder part of this section. In section 4, we prove the main existence and uniqueness theorem.
2. Reformulation of the problem

Different from previous works on transonic shock problems [3, 7, 8, 16, 17, 20], we will use the deformation-curl decomposition developed in [27, 28] for steady Euler system to decompose the original system (1.1) into an equivalent system (2.3), where the hyperbolic quantity $B$ and elliptic quantities $u_1, u_2$ are effectively decoupled in subsonic regions. To this end, define the Bernoulli’s function

$$B = \frac{1}{2}|u|^2 + h(\rho) - \Phi,$$

where $h(s) = \frac{\gamma - 1}{\gamma} s^{\gamma - 1}$ is the enthalpy function. Hence the density can be expressed by the Bernoulli function and velocity field as

$$\rho = H(B, \Phi, |u|^2) = \left[ \frac{\gamma - 1}{\gamma} (B + \Phi - \frac{1}{2} |u|^2) \right]^{\frac{1}{\gamma - 1}}.$$

Consequently, the 2-D Euler system (1.1) with unknown function $(u_1, u_2, P)$ is equivalent to the following system

$$\begin{aligned}
\sum_{i,j=1}^{2} (c^2(H)\delta_{ij} - u_i u_j)\partial_i u_j + u_1 \partial_1 \Phi + u_2 \partial_2 \Phi &= 0, \\
\partial_1 u_2 - \partial_2 u_1 &= -\frac{\partial B}{u_1}, \\
u_1 \partial_1 B + u_2 \partial_2 B &= 0,
\end{aligned}$$

with unknown function $(u_1, u_2, B)$.

The shock curve is determined by

$$\xi'(x_2) = \frac{[\rho u_1 u_2]}{[\rho u_2^2 + P]}(\xi(x_2), x_2), \quad x_2 \in (-1, 1).$$

Furthermore, it follows from the R-H conditions (1.12) that

$$\begin{aligned}
[\rho u_1] &= \left[ \frac{[\rho u_2]}{[\rho u_2^2 + P]} \right][\rho u_1 u_2], \\
[\rho u_2^2 + P(\rho)] &= \left[ \frac{[\rho u_2]}{[\rho u_2^2 + P]} \right]^2.
\end{aligned}$$

A direct computation by using (2.5) shows that on $x_1 = \xi(x_2)$

$$\begin{aligned}
(\rho^+(\xi(x_2), x_2) - \tilde{\rho}^+(L_s), u_1^+(\xi(x_2), x_2) - \tilde{u}^+(L_s)) &= \\
(h_1, h_2)(\rho^-(\xi(x_2)) - \tilde{\rho}^-(L_s), u_1^-(\xi(x_2)) - \tilde{u}^-(L_s), (u_2^+(\xi(x_2), x_2))^2)
\end{aligned}$$
here \( h_i(0, 0, 0) = 0 \) for \( i = 1, 2 \). In addition, we have

\[
\begin{align*}
\frac{\partial h_1}{\partial (\rho^- - \bar{\rho}^-)}|_{(0,0,0)} &= 2\bar{u}^-(L_s)\left(\frac{\bar{u}^+}{\bar{u}^+ + c^2(\bar{\rho}^+)}\right) + 1, \\
\frac{\partial h_1}{\partial (u^- - \bar{u}^-)}|_{(0,0,0)} &= 2\bar{\rho}^-(L_s)\left(\frac{\bar{u}^+ - \bar{u}^-}{\bar{u}^+ + c^2(\bar{\rho}^+)}\right), \\
\frac{\partial h_1}{\partial (u_2^2)}|_{(0,0,0)} &= \frac{(\bar{\rho}^+(L_s)\bar{u}^+(L_s))^2}{\bar{u}^+ + c^2(\bar{\rho}^+)}.
\end{align*}
\]

and

\[
\begin{align*}
\frac{\partial h_2}{\partial (\rho^- - \bar{\rho}^-)}|_{(0,0,0)} &= -(\gamma - 1)\frac{\bar{P}^+(L_s) - \bar{P}^-(L_s)}{\bar{P}^+(L_s) - \bar{P}^-(L_s)} \frac{\bar{u}^+\bar{u}^-}{\bar{u}^+ + c^2(\bar{\rho}^+)} , \\
\frac{\partial h_2}{\partial (u^- - \bar{u}^-)}|_{(0,0,0)} &= \frac{2\bar{\rho}^- (L_s)\bar{u}^+(L_s)}{\rho^+(L_s)\bar{u}^+(L_s)}\frac{\bar{u}^-\bar{u}^+}{\bar{u}^+ + c^2(\bar{\rho}^+)} + \frac{\bar{\rho}^-(L_s)}{\bar{\rho}^+(L_s)}, \\
\frac{\partial h_2}{\partial (u_2^2)}|_{(0,0,0)} &= \frac{\bar{\rho}^+(L_s)\bar{u}^+(L_s)}{\bar{P}^+(L_s) - \bar{P}^-(L_s)}\frac{c^2(\bar{\rho}^+)}{\bar{u}^+ + c^2(\bar{\rho}^+)}. \quad (2.8)
\end{align*}
\]

By substituting (2.6) into (2.1), we conclude that there is a function \( h_3 \) such that

\[
B^+(\xi(x_1), x_2) - \bar{B}^+(L_s) = h_3(\rho^- - \bar{\rho}^-(L_s), u^-(\xi(x_2)) - \bar{u}^-(L_s), u_2^+(\xi(x_2), x_2))^2).
\]

Thus, Theorem 1.2 is established as long as we solve the problem (2.3)-(2.4) with boundary conditions (2.5), (1.14)-(1.15). In order to deal with the free boundary value problem (2.3)-(2.4), we introduce the following transformation to reduce it into a fixed boundary value problem. Setting

\[
y_1 = \frac{x_1 - \xi(x_2)}{L_1 - \xi(x_2)}(L_1 - L_s) + L_s, \quad y_2 = x_2,
\]

then, the domain \( \Omega^+ = \{(x_1, x_2) : \xi(x_2) < x_1 < L_1, -1 < x_2 < 1\} \) is changed into

\[
Q = \{(y_1, y_2) : L_s < y_1 < L_1, -1 < y_2 < 1\}.
\]

The inverse change variable gives

\[
x_1 = \xi(y_2) + \frac{L_1 - \xi(y_2)}{L_1 - L_s}(y_1 - L_s) = y_1 + \frac{L_1 - y_1}{L_1 - L_s}\xi(y_2) - L_s), \quad x_2 = y_2.
\]

We now set for \( y \in Q \)

\[
(\bar{u}_j, \bar{\rho}, \bar{B}, \bar{\Phi})(y_1, y_2) = (u_j, \rho, B, \Phi) \left( \frac{L_1 - \xi(y_2)}{L_1 - L_s}(y_1 - L_s) + \xi(y_2), y_2 \right), \quad j = 1, 2.
\]

The shock equation (2.4) becomes to

\[
\xi'(y_2) = \frac{(\rho u_1 u_2)(\xi(y_2), y_2)}{\bar{P}^+(\rho)(\xi(y_2), y_2) - \bar{P}^-(\xi(y_2)) + \rho(u_2)^2(\xi(y_2), y_2)},
\]

\[
= \frac{\bar{\rho} \bar{u}_2(L_s, y_2)}{\bar{P}^+(\bar{\rho})(L_s, y_2) - \bar{P}^-(\xi(y_2)) + \bar{\rho}(\bar{u}_2)^2(L_s, y_2)}, \quad y_2 \in (-1, 1), \quad (2.12)
\]
and the system (2.3) is changed into

\begin{equation}
(c^2(\bar{\rho}) - \bar{u}_1^2) \frac{L_1 - L_s}{L_1 - \xi(y_2)} \partial_{y_1} \bar{u}_1 + c^2(\bar{\rho}) \partial_{y_2} \bar{u}_2 + \bar{u}_1 \frac{L_1 - L_s}{L_1 - \xi(y_2)} \partial_{y_1} \bar{\Phi} = F_1(\bar{u}, \bar{B}),
\end{equation}

\begin{equation}
\frac{L_1 - L_s}{L_1 - \xi(y_2)} \partial_{y_1} \bar{u}_2 - \partial_{y_2} \bar{u}_1 - \frac{y_1 - L_1}{L_1 - \xi(y_2)} \xi'(y_2) \partial_{y_1} \bar{u}_1 + \frac{y_2 - \bar{B}}{u_1} = F_2(\bar{u}, \bar{B}),
\end{equation}

\begin{equation}
\frac{y_1 - L_1}{L_1 - \xi(y_2)} \partial_{y_1} \bar{B} + \bar{u}_2 \partial_{y_2} \bar{B} + \frac{y_1 - L_1}{L_1 - \xi(y_2)} \xi'(y_2) \bar{u}_2 \partial_{y_1} \bar{B} = 0,
\end{equation}

where

\begin{align*}
F_1(\bar{u}, \bar{B}) &= \bar{u}_2^2 \partial_{y_2} \bar{u}_2 - (c^2(\bar{\rho}) - \bar{u}_2^2) \frac{y_1 - L_1}{L_1 - \xi(y_2)} \xi'(y_2) \partial_{y_1} \bar{u}_2 + \bar{u}_1 \bar{u}_2 \frac{L_1 - L_s}{L_1 - \xi(y_2)} \partial_{y_1} \bar{u}_2 \\
&\quad + \bar{u}_2 \bar{u}_1 \partial_{y_2} \bar{u}_1 + \frac{y_1 - L_1}{L_1 - \xi(y_2)} \xi'(y_2) \partial_{y_1} \bar{u}_1 - \bar{u}_2 \partial_{y_2} \bar{\Phi} + \frac{y_1 - L_1}{L_1 - \xi(y_2)} \xi'(y_2) \partial_{y_1} \bar{\Phi}, \\
F_2(\bar{u}, \bar{B}) &= -\frac{y_1 - L_1}{L_1 - \xi(y_2)} \xi'(y_2) u_1 \partial_{y_1} \bar{B}.
\end{align*}

Consider the perturbed functions \(v_i(y_1, y_2), i = 1, 2, 3, 4\), as

\begin{align*}
v_1(y_1, y_2) &= \bar{u}_1(y_1, y_2) - \bar{u}_1^+(y_1), \quad v_2(y_1, y_2) = \bar{u}_2(y_1, y_2), \\
v_3(y_1, y_2) &= \bar{B}(y_1, y_2) - \bar{B}^+, \quad v_4(y_2) = \xi(y_2) - L_s,
\end{align*}

and define the vector functions

\begin{equation}
V(y_1, y_2) = (v_1(y_1, y_2), v_2(y_1, y_2), v_3(y_1, y_2), v_4(y_2)).
\end{equation}

It follows from (2.12) that the shock satisfies

\begin{equation}
v_4'(y_2) = \frac{\bar{\rho}_v(y_2)}{P^+(\bar{\rho})(L_s, y_2) - P^-(\xi(y_2)) + \bar{\rho}_v(y_2)^2(L_s, y_2)}.
\end{equation}

Through a direct computation, one can derive from (2.9) and (2.14) that the Bernoulli function satisfy a transport type equation

\begin{equation}
[(\bar{u}_1^+ + v_1)(L_1 - L_s) + v_2(y_1 - L_1)v_4'(y_2)] \partial_{y_1} v_3 + v_2(L_1 - v_4 - L_s) \partial_{y_2} v_3 = 0,
\end{equation}

\begin{equation}
v_3(L_s, y_2) = b_3 v_4(y_2) + R_3(y_2),
\end{equation}

where

\begin{equation}
b_3 = \frac{\bar{\rho}_v(L_s) - \bar{\rho}_v(L_s)}{\rho_+(L_s)} f(L_s),
\end{equation}

and \(R_3(y_2) = R_3(V(L_s, y_2)) = O(\{V(L_s, y_2^2)^2\}^2\) is an error term of second order. We may readily drop superscribe + on the background solutions if there is no risk of confusing. And the first order system for \(v_1, v_2\) is given by,

\begin{equation}
\begin{cases}
(c^2(\bar{\rho}_v) - (\bar{u}_1^+)^2) \partial_{y_1} v_1 + c^2(\bar{\rho}_v) \partial_{y_2} v_2 + B_1(y_1) v_1 \\
\quad + B_3(y_1) v_3 + B_4(y_1) v_4(y_2) = F_3(V, \nabla V), \\
\partial_{y_1} v_2 - \partial_{y_2} v_1 + \frac{L_1 - y_1}{L_1 - L_s} \bar{u}_1 v_4' + \frac{y_1 - L_s}{u_1} = F_4(V, \nabla V),
\end{cases}
\end{equation}
where $F_3, F_4$ represent the remainder term of second order with respect to $V$ and $\nabla V$, and

$$B_1(y_1) = \bar{f}(y_1) - (\gamma + 1)\bar{u}\bar{u}' = \frac{\gamma\bar{u}^2 + c^2(\bar{\rho}^+)}{c^2(\bar{\rho}^+) - \bar{u}^2}\bar{f} > 0,$$

$$B_2(y_1) = \bar{f}(y_1)(L_1 - y_1)\bar{u}' - \bar{u}\bar{f}'(y_1)(L_1 - y_1).$$

It’s obvious that the second formula in (2.6) gives the boundary condition of $v_1$ on the entrance $x_1 = L_s$. Meanwhile, the formula (2.2) after changing variable becomes

$$c^2(\bar{\rho})(y) = \gamma\bar{\rho}^{\gamma - 1} = (\gamma - 1)(\bar{B} - \frac{1}{2}|\bar{u}|^2 + \bar{\Phi})$$

$$= (\gamma - 1)(\bar{B}^+ + v_3 - \frac{1}{2}(\bar{u}^+ + v_1)^2 - \frac{1}{2}v_2^2 + \bar{\Phi})$$

$$= c^2(\bar{\rho}^+) + (\gamma - 1)(v_3 - \bar{u}^+ v_1 - \frac{1}{2}|v_1|^2 - \frac{1}{2}|v_2|^2),$$

which together with (1.15) gives the boundary condition of $v_1$ on the exit $x_1 = L_1$. Hence, the boundary conditions to the system (2.19) read as follow

$$v_1(L_s, y_2) = b_2 v_3(y_2) + R_2(y_2),$$

$$v_1(L_1, y_2) = \frac{1}{\bar{\rho}(L_s)}(v_3(L_1, y_2) - \epsilon\hat{P}_{ex}(y_2)) + R_4(y_2),$$

$$v_2(y_1, \pm 1) = 0,$$

where

$$b_2 = \frac{\bar{\rho}^-(L_s)\bar{u}^+(L_s)}{\bar{\rho}^+(L_s)[(\bar{u}^+(L_s))^2 - c^2(\bar{\rho}^+(L_s))]\bar{f}(L_s)} < 0,$$

and $R_2(y_2) = R_2(V(L_s, y_2)) = O(|V(L_s, y_2)|^2), R_4(y_2) = R_4(V(L_1, y_2)) = O(|V(L_1, y_2)|^2)$ are error terms of second order. Based on our reformulation, Theorem 1.2 follows from the following results.

**Theorem 2.1.** Under the same assumptions as in Theorem 1.2 there exists a positive constant $\epsilon_0 > 0$ such that for each $\epsilon \in (0, \epsilon_0]$, the system (2.16)-(2.21) has a unique solution $V \in (C^{2,\alpha}(Q))^3 \times C^{3,\alpha}([-1, 1])$ satisfying the following estimate

$$\sum_{i=1}^{3} \|v_i\|_{C^{2,\alpha}(Q)} + \|v_4\|_{C^{3,\alpha}([-1, 1])} \leq C\epsilon,$$

where the constant $C$ depends only on the background solution, the exit pressure and $\alpha \in (0, 1)$.

### 3. Iteration Scheme and the Linear System

In the first part of this section, we construct an iteration scheme for the nonlinear system, and the problem is reduced to the solvability of corresponding linear systems. Indeed, it turns out that the linear system is a non-local elliptic equation of second order with a free parameter denoting the relative location of the shock position on the wall $x_2 = -1$. Then we study the existence, uniqueness and regularity for this linear system in the remainder part of this section.
3.1. Iteration Scheme. Inspired by [17], we will develop an iteration scheme to prove Theorem 2.1. Consider the Banach space

\[
\mathcal{V}_\delta := \{ V : \sum_{i=1}^{3} \| v_i \|_{C^{2,\alpha}([\bar{Q}])} + |v_4|_{C^{2,\alpha}[-1,1]} \leq \delta; \partial_y v_j(y_1, \pm 1) = 0, \]

\[
j = 1, 3; v_2(y_1, \pm 1) = \partial_{y_2}^2 v_2(y_1, \pm 1) = 0; v_4'(\pm 1) = v_4^{(3)}(\pm 1) = 0, \}

here \( \delta > 0 \) is a small constant to be determined later. For a fix \( \hat{V} \in \mathcal{V}_\delta \), equivalently, we have the following quantity

\[
(\hat{\nu}_1, \hat{\nu}_2, \hat{B}, \hat{\rho}, \hat{P}, \hat{\xi})(y).
\]

We now define the linearized scheme to the problem (2.16)-(2.21) as follows.

Firstly, thanks to (2.16), \( v_4 \) is determined by

\[
v_4'(y_2) = b_0 v_2(L_s, y_2) + F_5(\hat{V})(L_s, y_2),
\]

where

\[
b_0 = \frac{(\hat{\rho}\hat{u})(L_s)}{P^+(\hat{\rho})(L_s) - P^-(L_s)} > 0,
\]

\[
F_5(y_2) = \left\{ \frac{\hat{\rho}\hat{u}_1}{P^+(\hat{\rho})(L_s, y_2) - P^-(\hat{\xi}(y_2)) + \hat{\rho}(\hat{u}_2)^2(L_s, y_2)} - b_0 \right\} v_2(L_s, y_2),
\]

hence, one can express the shock as

\[
v_4(y_2) = v_4(-1) + \int_{-1}^{y_2} b_0 v_2(L_s, \tau)d\tau + R_5(y_2),
\]

where \( R_5(y_2) = \int_{-1}^{y_2} F_5(\tau)d\tau \) is an error term of second order. Due to \( \hat{V} \in \mathcal{V}_\delta \), we have

\[
F_5(\pm 1) = F_5''(\pm 1) = 0, \| F_5 \|_{C^{k,\alpha}[-1,1]} \leq C\delta \| \hat{v}_2 \|_{C^{k,\alpha}(Q)}, \ k = 0, 1, 2.
\]

Secondly, Using (2.17), we get the linearized transport equation for \( v_3 \):

\[
[(\hat{u} + \hat{v}_1)(L_1 - L_s) + \hat{v}_2(y_1 - L_1)\hat{v}_4'(y_2)]\partial_y v_3 + \hat{v}_2(L_1 - \hat{v}_4 - L_s)\partial_{y_2} v_3 = 0 \text{ in } Q,
\]

with initial data

\[
v_3(L_s, y_2) = b_3 v_4(y_2) + R_3(y_2).
\]

Thus, it can be solved by characteristic methods. Let \( y_2(s; \beta) \) be the characteristics going through \((y_1, y_2)\) with \( y_2(L_s) = \beta \), i.e.

\[
\begin{cases}
\frac{dy_2}{ds}(s; \beta) = \frac{\hat{v}_2(L_1 - \hat{v}_4 - L_s)}{(\hat{u} + \hat{v}_1)(L_1 - L_s) + \hat{v}_2(y_1 - L_1)\hat{v}_4'(y_2)}, & L_s < s < L_1, \\
y_2(L_s) = \beta.
\end{cases}
\]

It is noted that \( \beta \) can be also regarded as a function of \( y = (y_1, y_2) \), this is denoted by \( \beta = \beta(y) \), which leads to

\[
v_3(y_1, y_2) = v_3(L_s, \beta(y)) = b_3 v_4(y_2) + F_6(y),
\]
where
\[ F_6(y) = b_3 \int_{y_2}^{\beta(y)} v_4(\tau) d\tau + R_3(\hat{V}(L_s, \beta(y))) \]
is an error term of second order. Furthermore, we have
\[ \partial_y^2 F_6(y_1, \pm 1) = 0, \]
\[ \|F_6\|_{C^{k, \alpha}(\bar{Q})} \leq C\delta \left( \sum_{i=1}^{3} \|\hat{\nu}_i\|_{C^{k, \alpha}(\bar{Q})} + \|\hat{\nu}_4\|_{C^{k+1, \alpha}(\bar{Q})} \right), k = 0, 1, 2. \]

It remains to determine the velocity \( v_1, v_2 \) and the shock position difference \( v_4(-1) \) on the wall \( x_2 = -1 \). Substituting (3.3) and (3.6) into (2.19) and (2.21), we get the following linearized system for \( v_1, v_2 \) with an unknown parameter \( v_4(-1) \):
\[
\begin{cases}
\partial_y v_1 + \frac{1}{1-M_s^2} \partial_{y_2} v_2 + \frac{B_3(y_1)}{c^2(\rho^* \mp (u^*))^2} v_1 \\
\quad + \frac{B_3(y_1) \kappa + B_4(y_1)}{c^2(\rho^* \mp (u^*))^2} (v_4(-1) + \int_{-1}^{y_2} b_0 v_2(L_s, \tau) d\tau) = G_1(y), \\
\partial_y v_2 - \partial_{y_2} v_1 - \lambda(y_1)(v_4(-1) + \int_{-1}^{y_2} b_0 v_2(L_s, \tau) d\tau) = G_2(y),
\end{cases}
\]
and the boundary conditions
\[
\begin{align*}
&v_1(L_s, y_2) = b_2 v_4(-1) + \int_{-1}^{y_2} b_0 v_2(L_s, \tau) d\tau + R_6(y_2), \\
&v_1(L_1, y_2) = \frac{b_2}{a(L_1)} v_4(-1) + \int_{-1}^{y_2} b_0 v_2(L_s, \tau) d\tau - \frac{\partial_y R_5(y_2)}{a(L_1)} + R_7(y_2), \\
&v_2(y_1, \pm 1) = 0,
\end{align*}
\]
where
\[
\begin{align*}
\lambda(y_1) &= \frac{L_1 - y_1}{L_1 - L_s} u' + \frac{b_3}{u}, \\
R_6(y_2) &= b_2 R_5(y_2) + R_2(y_2), \\
R_7(y_2) &= \frac{b_3 R_5(y_2) + F_6(L_1, y_2)}{u(L_1)} + R_4(y_2), \\
G_1(y) &= \frac{F_3(\hat{V}, \nabla \hat{V}) - B_3(y_1) R_5(y_2) - B_3(y_1) F_6(y) - B_4(y_1) R_5(y_2)}{c^2(\rho^* \mp (u^*))^2}, \\
G_2(y) &= F_4(\hat{V}, \nabla \hat{V}) - \frac{\partial_{y_2} F_6(y)}{u} + \lambda(y_1) \partial_{y_2} R_5(y_2),
\end{align*}
\]
are error terms of second order. It follows from (3.4),(3.7) and a simple calculation that
\[
\begin{align*}
\partial_{y_2} G_1(y_1, \pm 1) &= 0, \quad G_2(y_1, \pm 1) = 0, \\
\|G_i\|_{C^{k-1, \alpha}(\bar{Q})} &\leq C\delta \|\hat{V}\|_{C^{k, \alpha}(\bar{Q})}, i, k = 1, 2,
\end{align*}
\]
and
\[
\begin{align*}
R_6'(\pm 1) &= 0, \quad R_7'(\pm 1) = 0, \\
\|\langle R_6, R_7 \rangle\|_{C^{k, \alpha}([-1, 1])} &\leq C\delta \|\hat{V}\|_{C^{k, \alpha}(\bar{Q})}, k = 0, 1, 2.
\end{align*}
\]
The second equation in (3.8) implies that there is a potential function \( \phi(y) \) satisfy

\[
\begin{align*}
\partial_{y_2} \phi &= \nu_2, \quad \phi(L_s, -1) = 0, \\
\partial_{y_1} \phi &= \nu_1 - \lambda(y_1)(\nu_4(-1) + \int_{-1}^{y_2} b_0 \nu_2(L_s, \tau) d\tau) + \int_{-1}^{y_2} G_2(y_1, \tau) d\tau,
\end{align*}
\]

(3.12)

it follows that \( \nu_1, \nu_2 \) can be represented by

\[
\begin{align*}
\nu_2 &= \partial_{y_2} \phi, \\
\nu_1 &= \partial_{y_1} \phi + \lambda(y_1)[\nu_4(-1) + b_0 \phi(L_s, y_2)] - \int_{-1}^{y_2} G_2(y_1, \tau) d\tau.
\end{align*}
\]

(3.13)

Substituting (3.13) into the first equation in (3.8), we conclude that \( \phi \) satisfying the following non-local elliptic equation of second with an unknown constant \( \nu_4(-1) \)

\[
\begin{align*}
\partial_{y_1}^2 \phi + \frac{1}{1 - M^2} \partial_{y_2}^2 \phi + \lambda_1(y_1) \partial_{y_1} \phi + \lambda_0(y_1)b_0(\frac{\nu_4(-1)}{b_0} + \phi(L_s, y_2)) &= G_1(y) + \lambda_1(y_1) \int_{-1}^{y_2} G_2(y_1, \tau) d\tau + \partial_{y_1} \int_{-1}^{y_2} G_2(y_1, \tau) d\tau,
\end{align*}
\]

(3.14)

where

\[
\begin{align*}
\lambda_1(y_1) &= \frac{B_1(y_1)}{\frac{c^2(\rho)}{\bar{u}} - \bar{u}^2} > 0, \\
\lambda_0(y_1) &= \frac{1}{\frac{c^2(\rho)}{\bar{u}} - \bar{u}^2}[(\frac{c^2(\rho)}{\bar{u}} - \bar{u}^2)\lambda' + B_1 \lambda + B_3 b_3 + B_4].
\end{align*}
\]

Similarly, substituting (3.13) into the boundary conditions (3.9), combine with the boundary condition of \( \phi \) in (3.12), we have

\[
\begin{align*}
\partial_{y_1} \phi(L_s, y_2) &= b_0(2 - \lambda(L_s))\frac{\nu_4(-1)}{b_0} + \phi(L_s, y_2) + R_6(y_2) + \int_{-1}^{y_2} G_2(L_s, \tau) d\tau, \\
\partial_{y_2} \phi(L_1, y_2) &= -\frac{\partial P_{\bar{u}}(y_2)}{\bar{u}(L_1)} + R_7(y_2) + \int_{-1}^{y_2} G_2(L_1, \tau) d\tau, \\
\partial_{y_2} \phi(y_1, \pm 1) &= \phi(L_s, -1) = 0.
\end{align*}
\]

(3.15)

So far, we have reduced the problem (3.8)-(3.9) into a non-local elliptic equation of \( \phi \) with an unknown constant \( \nu_4(-1) \). Hence, it is sufficient to establish the solvability and regularity results for the problem (3.14)-(3.15) in order to study the original problem. We are going to do it in the next subsection.

3.2. A non-local elliptic equation with a free constant. In this section, we prove the existence, uniqueness and regularity results for the problem (3.14). To this end, we consider the following more concise form of second order elliptic system with an unknown constant \( \kappa \)

\[
\begin{align*}
\partial_{y_2}^2 \phi + a_2(y_1)\partial_{y_2} \phi + a_1(y_1)\partial_{y_1} \phi - a_0(y_1)(\kappa + \phi(L_s, y_2)) &= \partial_{y_1} f, \quad \text{in } Q, \\
\partial_{y_1} \phi(L_s, y_2) - a_3(\kappa + \phi(L_s, y_2)) &= g_1(y_2), \\
\partial_{y_1} \phi(L_1, y_2) &= g_2(y_2), \\
\partial_{y_2} \phi(y_1, \pm 1) &= \phi(L_s, -1) = 0.
\end{align*}
\]

(3.16)
where the smooth coefficients \( a_i(y_1), i=0,1,2 \) and the constant \( a_3 \) satisfy

\[
(3.17) \quad a_1(y_1) < C_1, \ 0 < C_0 < a_i(y) < C_1, \ i = 0, 2, 3,
\]

and the parameter \( \kappa \) is a constant to be determined with the solution itself.

The first lemma implies that the inhomogeneous problem corresponding to the system (3.16) without the unknown constant has a unique weak solution.

**Lemma 3.1.** Suppose that \( f \in L^2(Q) \) and \( g_i \in L^2(-1,1), i = 1, 2 \), then, there exists a suitable large positive constant \( K \), such that the following inhomogeneous second order elliptic equation

\[
(3.18) \quad \begin{cases}
-\partial_{y_1}^2 \phi - a_2(y_1)\partial_{y_2}^2 \phi - a_1(y_1)\partial_{y_1} \phi + K \phi + a_0(y_1)\phi(L_{s,y_2}) = \partial_{y_1} f, \text{ in } Q, \\
\partial_{y_1} \phi(L_{s,y_2}) - a_3 \phi(L_{s,y_2}) = g_1(y_2), \\
\partial_{y_1} \phi(L_{1,y_2}) = g_2(y_2), \\
\partial_{y_2} \phi(y_1, \pm 1) = 0,
\end{cases}
\]

admits a unique weak solution \( \phi \in H^1(Q) \) satisfying

\[
(3.19) \quad \|\phi\|_{H^1(Q)} \leq C\left(\|(g_1,g_2)\|_{L^2(-1,1)} + \|f\|_{L^2(Q)}\right),
\]

for some positive constant \( C > 0 \).

**Proof.** For \( \phi, \psi \in H^1(Q) \), define the bilinear form

\[
\mathcal{B}[\phi, \psi] = \int_Q \partial_{y_1}\phi \partial_{y_2} \psi dy + \int_Q a_2(y_1)\partial_{y_2} \phi \partial_{y_2} \psi dy - \int_Q a_1(y_1)\psi \partial_{y_1} \phi dy + K\int_Q \phi \psi dy + \int_Q a_0(y_1)\phi(L_{s,y_2}) \psi dy + a_3 \int_{-1}^1 \phi(L_{s,y_2}) \psi(L_{s,y_2}) dy dy_2,
\]

and the linear functional on \( H^1(Q) \)

\[
l(\psi) = \int_{-1}^1 g_2(y_2)\psi(L_{1,y_2}) dy_2 - \int_{-1}^1 g_1(y_2)\psi(L_{s,y_2}) dy_2 - \int_Q \partial_{y_1} f \psi dy.
\]

It’s obviously that the linear functional \( l(\psi) \) on \( H^1(Q) \) is continuous, i.e,

\[
(3.20) \quad |l(\psi)| \leq C\left(\|(g_1,g_2)\|_{L^2(-1,1)} + \|f\|_{L^2(Q)}\right)\|\psi\|_{H^1(Q)},
\]

where we have used the trace theorem. So, what we need to do is just verify that the conditions of Lax-Milgram Theorem is satisfied for the bilinear form \( \mathcal{B} \). The boundedness of \( \mathcal{B}_K \) is trivial, we will show that \( \mathcal{B}_K \) is also coercive. Denote \( \Lambda = \min\{1,C_0\} > 0 \), then

\[
\Lambda \int_Q |\nabla \phi|^2 dy + K \int_Q |\phi|^2 dy \leq \mathcal{B}[\phi, \phi] + \int_Q a_1(y_1)\phi \partial_{y_1} \phi dy - \int_Q a_0(y_1)\phi(L_{s,y_2}) \phi(y_1,y_2) dy - a_3 \int_{-1}^1 |\phi(L_{s,y_2})|^2 dy dy_2,
\]
Cauchy’s inequality gives
\[
\int_Q a_1(y_1) \phi \partial_y \phi dy \leq C_1 \epsilon \int_Q |\nabla \phi|^2 dy + \frac{C_1}{4 \epsilon} \int_Q |\phi|^2 dy,
\]
and
\[
\int_Q a_0(y_1) \phi(L_s, y_2) \phi(y_1, y_2) dy \leq C_L (L_1 - L_s) C_1 \epsilon \int_Q |\nabla \phi|^2 dy + \frac{C_1}{4 \epsilon} \int_Q |\phi|^2 dy.
\]
Then, fix \( \epsilon_0 \) such that \( C_1 \epsilon_0 (1 + (L_1 - L_s) C_L) < \frac{\Lambda}{2} \), and choosing \( K = \max \{ \Lambda, \frac{C_1}{\epsilon_0} \} \), thanks to the positivity of \( a_3 \), we obtain
\[
\mathcal{B}[\phi, \phi] \geq \frac{\Lambda}{2} \| \phi \|_{H^1(Q)},
\]
the unique existence follows from the Lax-Milgram Theorem and (3.20) gives the estimates (3.19). Thus the proof is completed. \( \square \)

The unique existence of regular solution to the non-local system (3.16) is established in the following proposition.

**Proposition 3.2.** For any \( f \in C^{1,\alpha} (\bar{Q}) \), \( g_1 \in C^\alpha (\bar{Q}) \), there is a unique weak solution \((\phi, \kappa)\), such that \( \phi \in H^1(Q) \) and the following estimate holds
\[
\| \phi \|_{H^1(Q)} + |\kappa| \leq C (\| f \|_{C^\alpha(Q)} + |(g_1, g_2)|_{C^{1,\alpha}[1,1]}).
\]
Moreover, if the compatibility conditions
\[
\partial_{y_2} f(y_1, -1) = \partial_{y_2} f(y_1, 1) = 0, \quad g_i'(-1) = g_i'(1) = 0, \quad i = 1, 2,
\]
are fulfilled, then \( \phi \in C^{2,\alpha}(Q) \)
\[
\| \phi \|_{C^{2,\alpha}(Q)} \leq C(|f|_{C^\alpha(Q)} + |(g_1, g_2)|_{C^\alpha[1,1]} + \| \phi \|_{H^1(Q)} + |\kappa|)
\]
and
\[
\| \phi \|_{C^2(Q)} \leq C(|f|_{C^{1,\alpha}(Q)} + |(g_1, g_2)|_{C^{1,\alpha}[1,1]} + \| \phi \|_{H^1(Q)} + |\kappa|)
\]
for some positive constant \( C > 0 \).

**Proof.** The proof is divided into two steps.

Step 1: Regularity of weak solutions. we will use the symmetric extension methods to exclude the possible singularities that may appear at the corner, which implies that the weak solution \( \phi \in H^1(Q) \) to the system (3.16) are essentially more regular. To this end, introduce the notation
\[
Q^* := \{(y_1, y_2) : L_s < y_1 < L_1, \quad -2 < y_2 < 2\}
\]
and define the extended function \( \phi^*(y) \) on \( Q^* \) as
\[
\phi^*(y) = \begin{cases} 
\phi(y_1, 2 - y_2), & 1 < y_2 < 2, \\
\phi(y_1, y_2), & -1 < y_2 < 1, \\
\phi(y_1, -2 - y_2), & -2 < y_2 < -1.
\end{cases}
\]
Then the extended function $\phi^*$ satisfies

$$
\begin{aligned}
\begin{cases}
\partial^2_{y_1} \phi^* + a_2(y_1) \partial^2_{y_2} \phi^* + a_1(y_1) \partial_{y_1} \phi^* - a_0(y_1)(\kappa + \phi^*(L_s, y_2)) = \partial_{y_1} f^*, & \text{in } Q^*, \\
\partial_{y_1} \phi^*(L_s, y_2) - a_3(\kappa + \phi^*(L_s, y_2)) = g_1^*(y_2), \\
\partial_{y_1} \phi^*(L_1, y_2) = g_2^*(y_2), \\
\partial_{y_2} \phi^*(y_1, \pm 1) = \phi^*(L_s, -1) = 0,
\end{cases}
\end{aligned}
$$

(3.26)

Using the standard interior and the boundary estimates for the second order linear elliptic equations in [13], we obtain that $\phi^*(y) \in C^{1,\alpha}(Q^*)$ and

$$
\|\phi^*\|_{C^{1,\alpha}(Q^*)} \leq C(\|f\|_{C^{1,\alpha}(Q^*)} + \|(g_1, g_2)\|_{C^{1,\alpha}([-2,2])} + \|\phi^*(L_s, y_2)\|_{L^2([-2,2])} + |\kappa|),
$$

(3.27)

which implies that $\phi^*(L_s, y_2) \in C^{1,\alpha}([-2,2])$. Use the estimate (3.27) again to conclude that $\phi^*(y) \in C^{2,\alpha}(Q^*)$ and

$$
\|\phi^*\|_{C^{1,\alpha}(Q^*)} \leq C(\|f\|_{C^{1,\alpha}(Q^*)} + \|(g_1, g_2)\|_{C^{1,\alpha}([-2,2])} + \|\phi^*(0, y_2)\|_{L^2([-2,2])} + |\kappa|).
$$

(3.28)

Then (3.23) and (3.24) follows immediately.

Step 2: Existence and Uniqueness of weak solutions. Due to the linearity, any solution $\phi$ to the problem (3.16) can be decomposed as $\phi = \phi_1 + \phi_2$, where $\phi_i, i=1,2,$ satisfy the following equation respectively

$$
\begin{aligned}
\begin{cases}
\partial^2_{y_1} \phi_1 + a_2(y_1) \partial^2_{y_2} \phi_1 + a_1(y_1) \partial_{y_1} \phi_1 - a_0(y_1)\phi_1(L_s, y_2) = \partial_{y_1} f, & \text{in } Q, \\
\partial_{y_1} \phi_1(L_s, y_2) - a_3\phi_1(L_s, y_2) = g_1(y_2), \\
\partial_{y_1} \phi_1(L_1, y_2) = g_2(y_2), \\
\partial_{y_2} \phi_1(y_1, \pm 1) = 0,
\end{cases}
\end{aligned}
$$

(3.29)

and

$$
\begin{aligned}
\begin{cases}
\partial^2_{y_1} \phi_2 + a_2(y_1) \partial^2_{y_2} \phi_2 + a_1(y_1) \partial_{y_1} \phi_2 - a_0(y_1)(\kappa + \phi_2(L_s, y_2)) = 0, & \text{in } Q, \\
\partial_{y_1} \phi_2(L_s, y_2) - a_3(\kappa + \phi_2(L_s, y_2)) = 0, \\
\partial_{y_1} \phi_2(L_1, y_2) = \partial_{y_2} \phi_2(y_1, \pm 1) = 0, \\
\phi_2(L_s, -1) = -\phi_1(L_s, -1).
\end{cases}
\end{aligned}
$$

(3.30)

Combing the Lemma 3.1 with the Fredholm alternative theorem, one can easily derive (3.29) has a unique $H^4(Q)$ solution $\phi_1$ which satisfying (3.21). On the other hand, one can prove that the only weak solution to (3.30) must be $(\phi_2, \kappa) = (-\phi_1(L_s, -1), \phi_1(L_s, -1))$ by applying the maximum principle. For the detailed proof, one can refer to Lemma 4.1 in [17]. Thus the proof is completed.

At this point, we can easily illustrate the well-posedness to the reformulated problem (3.14)-(3.15).
Lemma 3.3. The problem (3.14)-(3.15) has a unique solution \((\phi, \kappa) \in C^{2, \alpha}(\bar{Q}) \times \mathbb{R}\) satisfying

\[
\begin{align*}
||\phi||_{C^{k, \alpha}(\bar{Q})} + |\kappa| & \leq C(||(G_1, G_2)||_{C^{k-1, \alpha}(\bar{Q})} \\
& + ||(R_6(y_2), R_7(y_2))||_{C^{1, \alpha}([-1, 1])} + ||\hat{P}_{ex}(y_2)||_{C^{1, \alpha}([-1, 1])}, \quad k = 1, 2,
\end{align*}
\]

for some positive constant \(C\).

Proof. It suffices to verify the solvability condition (3.17) for the problem (3.14)-(3.15). A direct but tedious computation shows that

\[
\begin{align*}
a_0(y_1) &= -b_0 \lambda_0(y_1) = -\frac{2c^2(\bar{\rho}^+ \bar{f}) b_0 b_3}{u(c^2(\bar{\rho}^+) - \bar{u}^2)} > 0, \\
a_1(y_1) &= \lambda_1(y_1) = \frac{\gamma u^2 + c^2(\bar{\rho}^+) (c^2(\bar{\rho}^+) - \bar{u}^2) \bar{f}}{c^2(\bar{\rho}^+) - \bar{u}^2} > 0, \\
a_2(y_1) &= \frac{1}{1 - M^2} > 0, \\
a_3 &= b_0(b_2 - \lambda(L_s)) = b_0 \frac{c^2(\bar{\rho}^+)(\bar{\rho}^+ - \bar{\rho}^-) \bar{f}(L_s)}{\bar{\rho}^+ u(c^2(\bar{\rho}^+) - \bar{u}^2)} > 0,
\end{align*}
\]

since the background solution is subsonic and smooth, the upper bound is trivial. Hence, Proposition 3.2 implies that there exist a unique weak solution \((\phi, \kappa)\), and the estimate (3.31) follows from (3.23) and (3.24).

In view of the analysis of the problem (3.14)-(3.15), the well-posedness of the equation (3.8)-(3.9) follows.

Lemma 3.4. The problem (3.8)-(3.9) admits a unique solution \((v_1, v_2, v_4(-1)) \in C^{2, \alpha}(\bar{Q}) \times \mathbb{R}\) satisfying

\[
\begin{align*}
||v_1, v_2||_{C^{k, \alpha}(\bar{Q})} + |v_4(-1)| & \leq C(||(G_1, G_2)||_{C^{k-1, \alpha}(\bar{Q})} \\
& + ||(R_6(y_2), R_7(y_2))||_{C^{1, \alpha}([-1, 1])} + ||\hat{P}_{ex}(y_2)||_{C^{1, \alpha}([-1, 1])}, \quad k = 1, 2,
\end{align*}
\]

and the compatibility conditions

\[
\begin{align*}
\partial_{y_2} v_1(y_1, \pm 1) = 0, \quad \partial_{y_2} v_2(y_1, \pm 1) = \partial_{y_2}^2 v_2(y_1, \pm 1) = 0.
\end{align*}
\]

Proof. Combine (3.13) and (3.31), we conclude that there is a unique solution \((v_1, v_2, v_4(-1)) \in C^{1, \alpha} \times \mathbb{R}\) such that

\[
\begin{align*}
||v_1, v_2||_{C^{\alpha}(\bar{Q})} + |v_4(-1)| & \leq C(||(G_1, G_2)||_{C^{\alpha}(\bar{Q})} \\
& + ||(R_6(y_2), R_7(y_2))||_{C^{1, \alpha}([-1, 1])} + ||\hat{P}_{ex}(y_2)||_{C^{1, \alpha}([-1, 1])},
\end{align*}
\]

for some positive constant \(C\).
The similar estimates also holds true for $\|(v_1, v_2)\|_{C^{1,\alpha}(Q)}$, but we can derive an even better estimate by rewriting (3.8) into an elliptic equation of $v_1$. To this end, we firstly rewrite the system (3.8) as

$$
\begin{align*}
\begin{cases}
\partial_{y_1} v_1 + \frac{1}{1-M^2} \partial_{y_2} v_2 + \lambda_1(y_1) v_1 = G_1(y), \\
\partial_{y_1} v_2 - \partial_{y_2} v_1 = G_2(y), \\
v_1(L_s, y_2) = R_6(y_2), \\
v_1(L_1, y_2) = R_7(y_2), \\
v_2(y_1, \pm 1) = 0,
\end{cases}
\end{align*}
$$

(3.35)

where

$$
\begin{align*}
G_1(y) &= G_1(y) - \frac{B_3(y_1)b_3 + B_4(y_1)}{(c^2(\rho^+)-(\bar{u}^+)^2)}(v_4(-1) + \int_{-1}^{y_2} b_0 v_2(L_s, \tau) d\tau), \\
G_2(y) &= G_2(y) + b_0 v_2(L_s, y_2), \\
R_6(y_2) &= b_2(v_4(-1) + \int_{-1}^{y_2} b_0 v_2(L_s, \tau) d\tau) + R_6(y_2), \\
R_7(y_2) &= \frac{b_3}{\bar{u}(L_1)}(v_4(-1) + \int_{-1}^{y_2} b_0 v_2(L_s, \tau) d\tau) - \frac{\epsilon \hat{P}_{ex}(y_2)}{\bar{u}(L_1)} + R_7(y_2),
\end{align*}
$$

and

$$
\partial_{y_2} G_1(y_1, \pm 1) = 0, G_2(y_1, \pm 1) = 0, R_6'(\pm 1) = 0, R_7'(\pm 1) = 0.
$$

(3.36)

Note that (3.13) and the boundary conditions of $\phi$ implies that

$$v_2(y_1, \pm 1) = 0, \partial_{y_2} v_1(y_1, \pm 1) = 0.
$$

A simple cancellation yields to

$$
\begin{align*}
\begin{cases}
\partial_{y_1}((1-M^2)\partial_{y_1} v_1) + \partial_{y_2}^2 v_1 + \partial_{y_1}(\lambda_1(1-M^2) v_1) = \partial_{y_1}((1-M^2)G_1) - \partial_{y_2}G_2, \\
v_1(L_s, y_2) = R_6(y_2), v_1(L_1, y_2) = R_7(y_2), \\
\partial_{y_2} v_1(y_1, \pm 1) = 0,
\end{cases}
\end{align*}
$$

(3.37)

Since the boundary conditions in (3.37) are compatible at the corners, we obtain the following estimates for $v_1$ by using the symmetric extension defined by (3.25)

$$
\begin{align*}
\|v_1\|_{C^{k,\alpha}(Q)} &\leq C(\|v_2\|_{C^{k-1,\alpha}(Q)} + |v_4(-1)| + \|(G_1, G_2)\|_{C^{k-1,\alpha}(Q)} \\
&\quad + \|\hat{P}_{ex}\|_{C^{k,\alpha}[-1,1]} + \|(R_6, R_7)\|_{C^{k,\alpha}[-1,1]}, k = 1, 2.
\end{align*}
$$

(3.38)

This estimates together with (3.35) also implies that

$$
\begin{align*}
\|v_2\|_{C^{k,\alpha}(Q)} &\leq C(\|v_2\|_{C^{k-1,\alpha}(Q)} + |v_4(-1)| + \|(G_1, G_2)\|_{C^{k-1,\alpha}(Q)} \\
&\quad + \|\hat{P}_{ex}\|_{C^{k,\alpha}[-1,1]} + \|(R_6, R_7)\|_{C^{k,\alpha}[-1,1]}, k = 1, 2.
\end{align*}
$$

(3.39)
Hence, (3.32) is follows from (3.34) and (3.38)-(3.39). Finally, differentiating the first equation in (3.35) with respect to \( x_2 \) and combining with \( \partial y_2 \mathcal{G}_1(y_1, \pm 1) = 0 \) and \( \partial y_2 v_1(y_1, \pm 1) = 0 \), we obtain

\[
\partial^2_{y_2} v_2(y_1, \pm 1) = 0,
\]

which gives the compatibility condition (3.33). Thus, the proof is complete. \( \square \)

4. A priori estimates and proofs of main results

In this section, we will use the Banach contraction mapping theorem to prove Theorem 2.1. Given any \( \hat{V} \in \mathcal{V}_\delta \), we could establish some a priori estimates to the linearized problems defined in subsection 3.1, and construct a contractible mapping from \( \mathcal{V}_\delta \) into itself so that there exits a unique fixed point, which is the solutions obtained in Theorem 2.1 and the proof of Theorem 2.1 will be finished.

Lemma 3.4 implies that there is a unique solution \( (v_1, v_2, v_4(-1)) \in C^{2,\alpha}(\mathcal{Q}) \times \mathbb{R} \) to the system (3.8)-(3.9) satisfying

\[
\|(v_1, v_2)\|_{C^{2,\alpha}(\mathcal{Q})} + |v_4(-1)| \leq C(\|(G_1, G_2)\|_{C^{1,\alpha}(\mathcal{Q})}) + \|(R_6(y_2), R_7(y_2))\|_{C^{2,\alpha}[-1, 1]} + \|\epsilon \hat{P}_x(y_2)\|_{C^{2,\alpha}[0, 1]} \leq C(\epsilon + \delta^2),
\]

and the compatibility condition (3.33) holds true.

The shock curve \( v_4 \) is given by (3.3), which satisfies

\[
|v_4|_{C^{3,\alpha}[-1, 1]} \leq C(|v_4(-1)| + \|v_2\|_{C^{2,\alpha}(\mathcal{Q})} + \|F_5\|_{C^{2,\alpha}(\mathcal{Q})}) \leq C(\epsilon + \delta^2).
\]

Moreover, it follows from (3.33) and (3.4) that

\[
v_4'(\pm 1) = 0 = v_4(3)(\pm 1).
\]

It remains to solve \( v_3 \). Due to (3.6), combining with (3.7) and (4.3), we obtain the estimate

\[
|v_3|_{C^{2,\alpha}(\mathcal{Q})} \leq C(|v_4|_{C^{2,\alpha}[-1, 1]} + \|F_5\|_{C^{2,\alpha}(\mathcal{Q})}) \leq C(\epsilon + \delta^2),
\]

and the compatibility condition

\[
\partial y_2 v_3(y_1, \pm 1) = 0.
\]

Taking \( \delta = O(1)\epsilon \), then for any given \( \hat{V} \in \mathcal{V}_\delta \) we can define a continuous mapping \( T : \mathcal{V}_\delta \rightarrow \mathcal{V}_\delta \) as

\[
T \hat{V} = V,
\]

due to the iteration scheme introduced in the previous section and the estimates (4.1)-(4.5). Finally, we show that the mapping is also contractible in the space \( (C^{1,\alpha}(\mathcal{Q}))^3 \times C^{2,\alpha}[-1, 1] \).

For arbitrarily given two states \( \hat{V}_i = (\hat{v}_1^i, \hat{v}_2^i, \hat{v}_3^i, \hat{v}_4^i) \in \mathcal{V}_\delta, i = 1, 2 \) with the corresponding fluid variable \( (\hat{u}_1^i, \hat{u}_2^i, \hat{B}^i, \hat{\xi}^i) \), set

\[
V_i = T \hat{V}_i, i = 1, 2,
\]
where $V_i = (v_i^1, v_i^2, v_i^3, v_i^4)$. For the convenience, we denote $\dot{W} = \dot{V}_1 - \dot{V}_2$ and $W = V_1 - V_2$, or equivalently,

$$\dot{w}_k = \dot{v}_k^1 - \dot{v}_k^2, \quad w_k = v_k^1 - v_k^2, \quad 1 \leq k \leq 4.$$  

The equation (3.2) implies that

$$w'_4 = b_0w_2 + O(\epsilon) \sum_{i=1}^{4} \dot{\hat{w}}_i,$$

which yields that

$$\|w'_4\|_{C^{1,\alpha}[\bar{1},\bar{1}]} \leq C\|w_2\|_{C^{1,\alpha}(\bar{Q})} + C\epsilon \sum_{i=1}^{3} \|\dot{\hat{w}}_i\|_{C^{1,\alpha}(\bar{Q})} + \|\ddot{\hat{w}}_4\|_{C^{2,\alpha}[\bar{1},\bar{1}]}).$$

It follows from (3.6) that

$$w_3 = b_3w_4 - b_3 \int_{\beta_1(y)}^{\beta_2(y)} (\dot{v}_1^1(\tau))^2 d\tau + b_3 \int_{y_2}^{\beta_2(y)} \dot{\hat{w}}'_4(\tau) d\tau + O(\epsilon) \sum_{i=1}^{3} \dot{\hat{w}}_i,$$

where $\beta_i, i = 1, 2$ is the initial position such that the corresponding characteristic $y^i(s, \beta_i)$ going through $(y_1, y_2)$ with $y^i(L_s) = \beta_i$. It is easy to verify that

$$\|\beta_1(y) - \beta_2(y)\|_{C^{1,\alpha}(\bar{Q})} \leq C(\|\dot{\hat{w}}_1\|_{C^{1,\alpha}(\bar{Q})} + \|\dot{\hat{w}}_2\|_{C^{1,\alpha}(\bar{Q})} + \|\ddot{\hat{w}}_4\|_{C^{2,\alpha}[\bar{1},\bar{1}]}),$$

thus,

$$\|w_3\|_{C^{1,\alpha}(\bar{Q})} \leq C\|w'_4\|_{C^{1,\alpha}[\bar{1},\bar{1}]} + C\epsilon \sum_{i=1}^{3} \|\dot{\hat{w}}_i\|_{C^{1,\alpha}(\bar{Q})} + \|\ddot{\hat{w}}_4\|_{C^{2,\alpha}[\bar{1},\bar{1}]}).$$

It is straightforward to show that $w_1, w_2$ satisfies

$$\begin{align*}
\rho_y w_1 + \frac{1}{1-M^2} \partial_y w_2 + \lambda_1(y_1)w_1 + \lambda_2(y_1)(w_4(-1) + \int_{y_2}^{y_3} b_0w_2(L_s, \tau) d\tau) \\
\partial_y w_1 - \partial_y \left[ w_1 - \lambda_1(y_1)(w_4(-1) + \int_{y_2}^{y_3} b_0w_2(L_s, \tau) d\tau) \right] \\
= \sum_{i=1}^{4} O(\epsilon) \dot{\hat{w}}_i + O(\epsilon) \nabla \dot{\hat{w}}_i + O(\epsilon)(\beta_1 - \beta_2) + O(1) \int_{y_2}^{y_3} \ddot{\hat{w}}'_4(\tau) d\tau, \\
\end{align*}$$

and the boundary conditions

$$\begin{align*}
w_1(L_s, y_2) &= b_2(\dot{w}_4(-1) + \int_{y_2}^{y_3} b_0\dot{w}_2(L_s, \tau) d\tau) + \sum_{i=1}^{4} O(\epsilon) \dot{\hat{w}}_i, \\
w_1(L_1, y_2) &= \frac{\dot{\cal{L}}}{\cal{L}}(\dot{w}_4(-1) + \int_{y_2}^{y_3} b_0\dot{w}_2(L_s, \tau) d\tau) + \sum_{i=1}^{4} O(\epsilon) \dot{\hat{w}}_i \\
&+ O(\epsilon)(\beta_1 - \beta_2)(L_1, y_2) \ddot{\hat{w}}'_4(\tau) d\tau, \\
w_2(y_1, \pm 1) &= 0,
\end{align*}$$

where

$$\lambda_2(y_1) = \frac{B_3(y_1)b_3 + B_4(y_1)}{c^2(\bar{\rho}^+) - (\bar{u}^+)^2}.$$
Then, applying the estimate (3.32) to the system (4.12)-(4.13) with $k = 1$ and together with (4.10), we obtain

\[(4.14) \quad \| (w_1, w_2) \|_{C^{1, \alpha}(\bar{Q})} + |w_4(-1)| \leq C \varepsilon \left( \sum_{i=1}^{3} \| \hat{w}_i \|_{C^{1, \alpha}(\bar{Q})} + \| \hat{w}_4 \|_{C^{2, \alpha}[-1,1]} \right).\]

Finally, collecting all these estimates above leads to

\[(4.15) \quad \sum_{i=1}^{3} \| w_i \|_{C^{1, \alpha}(\bar{Q})} + \| w_4 \|_{C^{2, \alpha}[-1,1]} \leq C \varepsilon \left( \sum_{i=1}^{3} \| \hat{w}_i \|_{C^{1, \alpha}(\bar{Q})} + \| \hat{w}_4 \|_{C^{2, \alpha}[-1,1]} \right).\]

By (4.15), there is a small constant $\varepsilon_0$ such that for all $\varepsilon \in (0, \varepsilon_0]$, the mapping $T$ defined by (4.6) is contractible in the Banach space $(C^{1, \alpha}(\bar{Q}))^3 \times C^{2, \alpha}[-1,1]$. Therefore, there exists a unique solution $V$ in $(C^{1, \alpha}(\bar{Q}))^3 \times C^{2, \alpha}[-1,1]$. Due to the Lemma 3.4 and the a priori estimates (4.2),(4.4), we know that $V$ also belongs to $V_\delta$. It follows that $V$ satisfies the estimates (2.23). Thus, the proof of Theorem 2.1 is complete. Theorem 1.2 is a direct inference of Theorem 2.1. We omit the details.

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