Non-perturbative calculations for the effective potential of the \( PT \)
symmetric and non-Hermitian \((-g\phi^4)\) field theoretic model

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Abstract

We investigate the effective potential of the \( PT \) symmetric \((-g\phi^4)\) field theory, perturbatively as well as non-perturbatively. For the perturbative calculations, we first use normal ordering to obtain the first order effective potential from which the predicted vacuum condensate vanishes exponentially as \( G \to G^+ \) in agreement with previous calculations. For the higher orders, we employed the invariance of the bare parameters under the change of the mass scale \( t \) to fix the transformed form totally equivalent to the original theory. The form so obtained up to \( G^3 \) is new and shows that all the 1PI amplitudes are perturbative for both \( G \ll 1 \) and \( G \gg 1 \) regions.

For the intermediate region, we modified the fractal self-similar resummation method to have a unique resummation formula for all \( G \) values. This unique formula is necessary because the effective potential is the generating functional for all the 1PI amplitudes which can be obtained via \( \partial^n E/\partial b^n \) and thus we can obtain an analytic calculation for the 1PI amplitudes. Again, the resummed form of the effective potential is new and interpolates the effective potential between the perturbative regions. Moreover, the resummed effective potential agrees in spirit of previous calculation concerning bound states.

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I. INTRODUCTION

In recent years, it has been established that the $PT$ symmetric and non-Hermitian quantum models have real and discrete spectra [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. This may draw the attention to the reinvestigation of the previously rejected non-Hermitian models especially for the quantum field versions. For those theories, there are some clues concerning the importance of the study of them. For instance, the very recent result concerning the existence of a non-Hermitian representation for Hermitian theory which has simpler calculation [16]. Accordingly, one may aim to describe hadrons in a simple way rather than the complicated QCD model by finding a non-Hermitian representation for QCD. However, this is still a speculation and in this work we concentrate on the study of the effective potential of a simple model but exhibits non-trivial features. A class of simple but non-trivial quantum mechanical models are given by

$$H = p^2 + x^2 (ix) \epsilon, \quad \epsilon > 0.$$  (1)

All such models have real and positive spectra even in the case of $\epsilon = 2$. In fact, all the complex PT symmetric Hamiltonians have real and positive spectra [2].

Unlike the quantum mechanical versions of the $PT$ symmetric models, the quantum field versions of such theories still are not investigated in a sizable way. Moreover, the quantum field Hamiltonians have interesting properties. For instance, a simple model with Lagrangian density like

$$L = \frac{1}{2} ((\partial \phi)^2 - m^2 \phi^2) + \frac{g}{4} \phi^4,$$  (2)

exhibits asymptotic freedom as well as having bound states [17]. Also, for certain range of the coupling values it has two-body bound state (meson-like) and for another range it has three-body bound state (baryon-like). Thus, it is concluded that this simple model has supersymmetric features. This may give us a hope to describe strong interactions with Abelian theories without the need for Glouns. Regardless of these legitimate hopes, the PT symmetric and non-Hermitian theories, like any of the physically acceptable models, deserve the employment of the usual machinery of investigation under different conditions. For example, we need to know how they behave at zero and non-zero temperatures, the presence of external sources, extreme conditions, etc... . Since the effective potential serves as the generating functional for all the one-particle irreducible (1PI) amplitudes, it’s investigation
is the basic stone for all other discussions. Up to the best of our knowledge, the effective potential of the non-Hermitian $\phi_{4+1}^4$ has never been obtained in a form reliable in all regions of the coupling space. This situation is partly due to being a new field of study and partly due to the non-Borel summability of the theory because of the existence of classical soliton solutions. In this paper, we offer a coherent formula for the effective potential of the $PT$ symmetric and non-Hermitian $\phi_{4+1}^4$ theory which is reliable for any coupling value. First, we study the effective potential (at zero temperature) of the model in Eq.(2) in an effective quasi-particle theory which verify perturbation for both $g \ll 1$ and $g \gg 1$. For the intermediate region in the coupling space, the quasi-particle theory ought to be non-perturbative and to resum the perturbation series one has to resort to a resummation technique rather than Borel technique because the theory is not Borel summable. Padé approximation are suggested for the sake of getting reliable results from the input information of perturbation series. However, the knowledge of only a few first terms does not permit one to use these techniques \[18\]. This means that to provide a reasonable accuracy, these techniques need to know tens of first terms of the perturbation theory \[19\]. In fact, going to higher orders in quantum field models is not an easy task as time-ordering of many fields results in many different types of Feynman diagrams and thus one needs (if it is possible to do the calculations) a long time to accomplish the diagrams calculations. To overcome such difficulties, the self-similar method was suggested as a non-perturbative tool for the resummation of divergent series \[18, 19, 20, 21, 22, 23, 24, 25, 26\]. Although this method can give good results even with few terms of perturbation series, sometimes the method is not applicable at all. We will argue, later in this work, it’s applicability and suggestions for modification to render it applicable for the effective potential for any coupling value.

The paper is organized as follows. In Section \[II\] we obtain the first order effective potential by normal ordering of the fields with a field as well as mass shift. Then, the first order calculations is supplemented by perturbative corrections up to $g^3$. In Section \[III\], we briefly review the key points of the fractal self-similar method and introduce more control functions to render it applicable for the effective potential for all the values of the coupling $g$. In Section \[IV\] we present and discuss the results and Conclusion follows in Section \[V\].
II. THE PERTURBATIVE EFFECTIVE POTENTIAL

In low dimensional super-renormalizable theories, it is often enough to work with normal ordering to render the quantum field theory finite. This is because there are only few diagrams that are divergent and these are regulated by normal ordering. The \((\frac{-g^4}{4}\phi^4)_{1+1}\) theory is such an example that has only one divergent diagram in the self-energy amplitude. In that case, one shall start with a Hamiltonian that is normal ordered with respect to the vacuum of mass parameter \(m\).

\[
H = N_m \left( \frac{1}{2} \left( (\nabla \phi)^2 + \pi^2 + m^2 \phi^2 \right) - \frac{g}{4} \phi^4 \right). \tag{3}
\]

We can use the relation \([27]\)

\[
N_m \exp (i\beta \phi) = \exp \left( -\frac{1}{2} \beta^2 \Delta \right) N_{M=\sqrt{t} \cdot m} \exp (i\beta \phi), \tag{4}
\]

to rewrite the Hamiltonian normal ordered with respect to a new mass parameter \(M = \sqrt{t} \cdot m\). In eq.(4), expanding both sides and equating the coefficients of the same power in \(\beta\) yields the result

\[
N_m \phi = N_M \phi, \\
N_m \phi^2 = N_M^2 \phi + \Delta, \\
N_m \phi^3 = N_M \phi^3 + 3\Delta N_M \phi, \tag{5}
\]

\[
N_m \phi^4 = N_M \phi^4 + 6\Delta N_M \phi^2 + 3\Delta^2,
\]

with

\[
\Delta = -\frac{1}{4\pi} \ln t. \tag{6}
\]

Also, it is easy to obtain the result \([28]\)

\[
N_m \left( \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} \pi^2 \right) = N_M \left( \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} \pi^2 \right) + \frac{1}{8\pi} (M^2 - m^2). \tag{7}
\]

The mass shift \(m \to M\) should be accompanied by the canonical transformation \([28]\)

\[
(\phi, \pi) \to (\psi + B, \Pi). \tag{8}
\]
The field $\psi$ has mass $M = \sqrt{t} \cdot m$, $B$ is a constant, the field condensate and $\Pi$ is the conjugate momentum $\psi$. Therefore, the Hamiltonian in Eq.(3) can be written in the form;

$$H = \tilde{H}_0 + \tilde{H}_I + \tilde{H}_1 + E,$$

where

$$\tilde{H}_0 = N_M \left( \frac{1}{2} (\Pi^2 + (\nabla \psi)^2) \right) + \frac{1}{2} N_M (m^2 - 3g (B^2 + \Delta)) \psi^2,$$

$$\tilde{H}_I = -\frac{g}{4} N_M (\psi^4 + 4B\psi^3).$$

$\tilde{H}_1$ can be found as

$$\tilde{H}_1 = N_M (m^2 - g (B^2 + 3\Delta)) B \psi,$$

and the field independent terms can be regrouped as

$$E = \frac{1}{2} \left( m^2 - \frac{12g\Delta}{4} \right) B^2 - \frac{g}{4} B^4 + \frac{1}{8\pi} (M^2 - m^2) - \frac{3g\Delta^2}{4} + \frac{1}{2} m^2 \Delta. \quad (11)$$

Taking $b^2 = 4\pi B^2$ and the dimensionless parameters $t = \frac{M^2}{m^2}$, $G = \frac{g}{2\pi m^2}$, the corresponding vacuum energy density can be written as

$$E(b, t, G) = \frac{m^2}{8\pi} \left( b^2 - G \left( b^4 - 6b^2 \ln t + 3 \ln^2 t \right) + t - 1 - \ln t \right) \quad (12)$$

The renormalization conditions are given by [29]

$$\frac{\partial^n}{\partial b^n} E(b, t, G) = g_n,$$

where $g_n$ is the $\psi^n$ coupling. For instance,

$$\frac{\partial^2 E}{\partial B^2} = M^2,$$

where $g_2 = M^2 = m^2 - 3g (B^2 + \Delta)$. Note that, the renormalization condition $\frac{\partial E}{\partial B} = 0$ enforces $\tilde{H}_1$ to be zero.

The quasi-particle Hamiltonian in Eq.(9) exhibits some interesting properties. For instance, the renormalization conditions predicts an imaginary condensate which turns the Hamiltonian to be non-Hermitian. To show how this comes out, consider the equations

$$\frac{\partial E}{\partial b} = (2 + (-G) (b^2 - 3 \ln t)) b = 0, \quad (15)$$

$$\frac{\partial^2 E}{\partial b^2} = (-G) (3b^2 - 3 \ln t) + 2 = 2t. \quad (16)$$
For $b \neq 0$, they simplify to

\[ 2 - G \left( b^2 - 3 \ln t \right) = 0, \]
\[ -G \left( 3b^2 - 3 \ln t \right) + 2 = 2t. \]  

(17)

Now, Eq.(17) can be parameterized as

\[ G = -\frac{t + 2}{3 \ln t}, \]
\[ b^2 = -\frac{t}{G}. \]

(18)

This parameterization shows that the parameter $t$ should be less than one and $b$ is pure imaginary. This result clearly shows the non-Hermitian property of the theory considering the form of $\bar{H}_I$ in Eq.(9).

The solutions of Eq.(17) are given by

\[ b^2(G) = -3W\left( \frac{1}{3G}e^{-\frac{2}{3G}} \right), \]
\[ t(G) = 3GW\left( \frac{1}{3G}e^{-\frac{2}{3G}} \right), \]

(19)  

(20)

where the Lambert’s $W(x)$ function is defined by $W(x)e^{W(x)} = x$. Also, $W$ has the series expansion \[30]\]

\[ W(z) = \sum_{n \geq 1} \frac{(-n)^{n-1} z^n}{n!}, \]

(21)

which is convergent if $|z| < \frac{1}{e}$. Thus, for $G \to 0^+$, Eq.(19) takes the form

\[ b^2(G) \simeq -3 \left( \frac{1}{G}e^{-\frac{2}{3G}} \right), \]

(22)

\[ b = \pm i\sqrt{3} \left( \frac{1}{\sqrt{G}}e^{-\frac{1}{3G}} \right). \]

(23)

This result predicts pure imaginary condensates which vanishes exponentially as $G \to 0^+$, which agrees with the prediction of Refs. [1, 17]. Another interesting property that the quasi-particle Hamiltonian posses is that it is totally equivalent to the original theory in the sense that setting the parameter $t$ equal to one ($M = m$), the Hamiltonian $H$ (Eq.(9)) reproduces the original form in Eq.(3). In fact, this is very important because, as we will
see later in the work, the direct calculation of higher orders, spoils out this equivalence. Also, since normal ordering can not account for non-cactus Feynman diagrams, we resort to renormalization group invariance to fix the parameters in the theory which automatically turn the quasi-particle Hamiltonian equivalent to the original Hamiltonian. Besides, it is interesting to note that Eq. (17) can be obtained by adding a counter term as well as employing the invariance of the bare couplings on the change of the mass scale \( t \).

In spite of all of the above correct features, the normal ordered effective potential in Eq. (12), as we will discuss, is non-perturbative for intermediate values of the coupling \( G \). In order to improve the representation of the effective potential near the non-perturbative region, we consider the modification of Eq. (12) resulting from the higher order perturbative corrections to the vacuum energy followed by a modified fractal self-similar method (the theory is not Borel summable).

The normal ordered effective potential of \( \frac{g_4}{4} \phi^4 \) theory (Eq. (12)) agrees with GEP results [33] which in turn accounts not only for the leading order diagrams but also for all the non-cactus diagrams [34, 35]. Thus, to go to higher orders we include only non-cactus diagrams (Fig. 1).

![Vacuum Diagrams](image)

**FIG. 1:** The vacuum diagrams (up to \( g^3 \)) of the effective quasi-particle \( (-\frac{g_4}{4} \phi^4 - gB\phi^3) \) theory.

In the equivalent quasi-field theory, the interaction term is \( (-\frac{g_4}{4} \phi^4 - gB\phi^3) \). Up to \( g^3 \), we have the Feynman diagrams (non-cactus) shown in Fig. 1. Accordingly, the perturbation
corrections to the effective potential are

\[
\frac{8\pi E(b, t, G)}{m^2} = t - \ln t + b^2 - 1 - G \left( \frac{1}{4} b^4 + \frac{3}{4} \ln^2 t - \frac{3}{2} b^2 \ln t \right) - G^2 \left( -3.155 \left( \frac{1}{t} - 1 \right) - 3.515 b^2 \left( \frac{1}{t} - 1 \right) \right) - G^3 \left( 4.057 \left( \frac{1}{t^2} - 1 \right) + 9.918 b^2 \left( \frac{1}{t^2} - 1 \right) \right). \tag{24}
\]

In fact, this form of effective potential does not predict the values of \(b\) and \(t\) parameters that makes the quasi-particle theory equivalent to the original theory. To keep the equivalence, we use the fact that the bare parameters are independent of the scale \(t\). Accordingly, we obtain the result

\[
\frac{8\pi E(t, b, G)}{m^2} = t - \ln t + b^2 - 1 - G \left( \frac{1}{4} b^4 + \frac{3}{4} \ln^2 t - \frac{3}{2} b^2 \ln t \right) - G^2 \left( -3.155 \left( \frac{1}{t} - 1 \right) - 3.515 b^2 \left( \frac{1}{t} - 1 \right) \right) - G^3 \left( 4.057 \left( \frac{1}{t^2} - 1 \right) + 9.918 b^2 \left( \frac{1}{t^2} - 1 \right) \right). \tag{25}
\]

We employ the conditions in Eq.(13) to predict the parameters \(b\) and \(t\) which define the vacuum. This leads to the following conditions on the parameters \(b\) and \(t\) (for \(b \neq 0\)):

\[
2 - 7.03 G^2 \left( \frac{1}{t} - 1 \right) - 19.836 G^3 \left( \frac{1}{t^2} - 1 \right) - G \left( b^2 - 3 \ln t \right) = 0 \tag{26}
\]

\[
-19.836 G^3 \left( \frac{1}{t^2} - 1 \right) - 7.03 G^2 \left( \frac{1}{t} - 1 \right) - 3G \left( b^2 - \ln t \right) + 2 = 2t \tag{27}
\]

In all of the \(G^3\) order equations (Eq.(25), Eq.(26) and Eq.(27)) setting \(t = 1\), we have \(b = E = 0\), which when substituted back into Eq.(19) we get the original Hamiltonian form in Eq.(3). Up to the best of our knowledge, such form of the effective potential which conserve the equivalence between original theory and the transformed one has never been obtained before (up to \(g^3\) order).

As in Eq.(18), Eq.(26) and Eq.(27) can be parametrized as

\[
t = -3G \ln t + 7.03 G^2 \left( \frac{1}{t} - 1 \right) - 19.836 (-G)^3 \left( \frac{1}{t^2} - 1 \right) - 2,
\]

\[
b^2 = -\frac{t}{G}. \tag{28}
\]

This parametrization is exactly the same as in Eq.(18) except that the \(t\) parameter takes it’s new form according to the radiative corrections. One may conjecture, relying on this result,
that the prediction of the vacuum condensate as $b^2 = \frac{t}{\bar{G}}$ is correct up to any order of the perturbation series and the higher orders just redefine the parameters due to the radiative corrections.

As we will see in section [IV] the $G^3$ order equations so obtained has an interesting property. These equations are perturbative not only for $G \ll 1$, which is proved in Ref. [17], but also for $G \gg 1$ (see Fig.2, Fig.3 and Fig.4). To obtain an effective potential which is reliable for all the coupling values and noting that the theory is not Borel summable, we resort to the fractal self-similar method [18]. In fact, we will modify it via the introduction of more control functions to make it applicable for all the regions in the coupling space.

III. THE MODIFIED FRACTAL SELF-SIMILAR METHOD

To get finite values of a divergent series, the so-called resummation methods are suggested. The most often used among such techniques are the Borel summation [37, 38] and the construction of Pade approximants [39], including the two-point [40] and multivalued [41, 42] Pade approximants. These techniques have many known limitations. For instance, the Borel method can not be applied in case of the existence of classical soliton solutions [31] like the model we study in this work. Also, Borel and Pade techniques require to have a number of perturbative terms which often are hard to get. Rather than that, the self-similar method can give good approximation with few terms at hand. One of the basic ideas in self-similar approximation theory is the introduction of control functions which govern the evolution of an approximation dynamical system to be close to a fixed point. To introduce control functions into a given asymptotic series, one has to employ a transformation that include trial parameters. This transformation has to simulate the self-similarity property hidden in the given perturbative sequence. For power series, it looks natural to employ the power-law transformations [43]. Since power laws are typical of fractals [44, 45], the power-law transformation can also be called the fractal transformation [43]. Accordingly, it may be more plausible to introduce control functions via a fractal transform because it satisfies the scaling relation

$$\frac{P(\lambda x, s)}{p(\lambda x)} = \lambda^s \frac{P(x, s)}{p(x)},$$

which is a typical of fractals, where $P(x, s)$ and $p(x)$ are the polynomials defined below.
For a function $f(x)$, the fractal transform is given by

$$F(x, s) = x^s f(x), \quad (29)$$

and its inverse transform is:

$$f(x) = x^{-s} F(x, s). \quad (30)$$

Here we follow the work in Ref. [18] to present the key points of the self-similar method. Consider the series given by

$$p_k = \sum_{k=0}^{k} a_n x^n. \quad (31)$$

Applying the fractal transform we get

$$P_k(x, s) = x^s P_k(x) = \sum_{n=0}^{k} a_n x^{s+n}. \quad (32)$$

Define the initial approximation $P_0(x, s) = a_0 x^s = f$. Solving for $x$ we get

$$x(f, s) = \left( \frac{f}{a_0} \right)^\frac{1}{s}. \quad (33)$$

Then

$$y_k(f, s) = P_k(x(f, s), s) = \sum_{n=0}^{k} a_n \left( \frac{f}{a_0} \right)^\frac{n}{s+1}. \quad (34)$$

Note that, self similarity means that

$$y_{k+p}(f, s) = y_k(y_p(f, s), s). \quad (35)$$

Considering the cascade $y_k$ as a dynamical system with the time as $k$, then the cascade velocity is given by

$$y_k(f, s) - y_{k-1}(f, s) = \sum_{n=0}^{k} a_n \left( \frac{f}{a_0} \right)^\frac{n}{s+1} - \sum_{n=0}^{k-1} a_n \left( \frac{f}{a_0} \right)^\frac{n}{s+1}$$

$$= a_k \left( \frac{f}{a_0} \right)^\frac{k}{s+1}. \quad (36)$$

After introducing the control functions, the regime of the self-similar renormalization is to consider the passage from one approximation to another as a motion with respect to the approximation number $k = 0, 1, 2, \ldots$. In fact, the trajectory $y_k(f, s)$ of this dynamical system is bijective, that is, in one-to-one correspondence to the approximation sequence.
$P_k(x, s)$. This dynamical system with discrete time $k$ is called the approximation cascade. The attracting fixed point of the cascade trajectory is, by construction, bijective to the limit of the approximation sequence $P_k(x, s)$, that is, it corresponds to the sought function.

One can deal with continuous time $t$ rather than the discrete time $k$ such that the trajectory passes through the same points when $t = k$. In this case, the flow velocity is governed by a differential equation rather the difference equation in Eq. (36). In other words, the evolution equation for the flow reads

$$\frac{\partial}{\partial t} y(t, f, s) = v(y(t, f, s)).$$

Accordingly, the evolution integral is

$$\int_{P_k}^{P_{k+1}} \frac{df}{v_{k+1}(f, s)} = t^*_k.$$  \hfill (38)

Thus, the self-similar approximation is given by

$$p^*_k = p_{k-1}(x) \left(1 - \frac{ka_k}{sa_0^{1+s}} x^k p_k^* (x) \right)^{-\frac{1}{k}},$$  \hfill (39)

where $t^*_k = 1$ when no restrictions are imposed on the series.

The applicability of the method is governed by the stabilizers

$$\mu_k (f) = \frac{\partial}{\partial f} y_k (f, s),$$  \hfill (40)

or their images

$$m_k (x, s) = \mu_k (P_0 (x, s), s).$$  \hfill (41)

The stability condition is given by

$$|m_k (x, s)| < 1.$$  \hfill (42)

For the series given above we have

$$m_k (x, s) = \sum_{n=0}^{k} \frac{a_n}{a_0} \left(1 + \frac{n}{s}\right) x^n,$$  \hfill (43)

For $k = 3$, the stabilizers are given by

$$m_k (x, s) = \frac{xa_1 + 2x^2a_2 + 3x^3a_3}{sa_0} + \frac{a_0 + xa_1 + x^2a_2 + x^3a_3}{a_0}.$$  \hfill (44)
The most stable approximant is obtained if \( m_k(x, s) = 0 \), or
\[
s = -\frac{xa_1 + 2x^2a_2 + 3x^3a_3}{a_0 + xa_1 + x^2a_2 + x^3a_3}.
\] (45)

If \( m_k(x, s) \) does not have a positive root, then \( m_k(x, s) \) is a monotonic decreasing function of \( s \). Therefore, the minimum is given by
\[
|m_k(x, s)|_{s \to \infty} = \left| \frac{a_0 + xa_1 + x^2a_2 + x^3a_3}{a_0} \right|.
\] (46)

If all \( |m_k(x, s)|_{s \to \infty} \) are less than one, the bootstrap formula given by
\[
p_k^* = a_0 \exp \left( \frac{a_1}{a_0} x \exp \left( \frac{a_2}{a_1} x \exp \left( \frac{a_3}{a_2} x \exp \left( \frac{a_4}{a_3} x \exp \left( \frac{a_5}{a_4} x \ldots \exp \left( \frac{a_k}{a_{k-1}} x \right) \right) \right) \right) \right) \right)
\] (47)
represents the resummed series [26]. However, it is not guaranteed to have all \( |m_k(x, s)| \) less than one for \( s \to \infty \) and for every point of the argument \( x \). In fact, this is the situation in case of applying the fractal self-similar method to the effective potential in Eq. (25) where there are some coupling values for which the bootstrap formula does not exist. So, instead of applying the fractal self-similar method directly to the series in Eq. (25) we introduce another control function via a bijective transformation which has the property of transforming the original series to another one for which the bootstrap formula is applicable. Then, we apply the fractal self-similar method to the transformed series and at the end we apply the inverse of the transformation to get the resummation formula of the original series.

To test the modification we introduced, consider the Lambert \( W \) function defined by
\[
W(x) \exp(W(x)) = x.
\] (48)

The series expansion of \( W(1 + x) \) is
\[
W(1 + x) \approx W(1) + \frac{W(1)}{1 + W(1)} x + \left( -\frac{1}{2} \frac{(W(1))^2}{(1 + W(1))^3} \right) x^2
\] (49)
\[
+ \left( \frac{1}{6} \frac{(W(1))^3}{(1 + W(1))^5} \right) \frac{9 + 8W(1) + 2(W(1))^2}{(1 + W(1))^5} x^3 + O(x^4).
\] (50)

At \( x = 3 \), \( W(1 + x) = 1.2022 \) and the perturbative result (up \( x^3 \)) is 1.9189. The error percent is \( \left| \frac{1.2022 - 1.9189}{1.2022} \right| \% = 59.616 \% \).
Let us apply the transformation $\Upsilon(W(1+x)) = W(1+x) + c$, where $c$ is used as a control function. Apply the fractal self-similar method to $\Upsilon(W(1+x))$ and find $c$ which make all the $|m_k(x,s)|_{s\to\infty}$ less than one and then apply $\Upsilon^{-1}$ to the obtained bootstrap formula we get the result $W(1+x) \approx 1.1798$ with the error percent $\frac{|1.2022-1.1798|}{1.2022} \times 100\% = 2.0697\%$. This result indicates the success of the modification we introduced to the fractal self-self-similar method which can be summarized as: instead of using one control function $s$ we introduce two control functions $s$ and $c$. With $s \to \infty$, we adopt $c$ to obtain stable approximant. In fact, this trick is necessary because it results in a unique formula for the approximant for all the values of the coupling $G$. Since it is well known that the effective potential is the generating functional for all 1PI amplitudes, unique formula for the effective potential makes it easy to obtain the different amplitudes via analytic differentiation.

We applied the modified fractal-self similar method to the perturbation series of the effective potential in Eq.(25). As we will see in the following section, the resummed series fits the perturbative data for regions where the perturbation series is reliable ($G \ll 1$ and $G \gg 1$).

IV. NUMERICAL CALCULATIONS AND DISCUSSIONS

In this section, we present the numerical calculations concerning the perturbative as well as non-perturbative calculations for the vacuum properties of the $(-\frac{g}{4}\phi^4)_{1+1}$ non-Hermitian field theory. The non-Hermiticity of the theory is clear in its quantum field version without the employment of boundary conditions in the complex $x$ space. This can be extracted from Eq.(28), where we realize that the condensation is always pure imaginary and thus the from in Eq.(9) shows that the theory is non-Hermitian as well as PT symmetric.

In Fig.2 the effective potential is plotted as a function of the coupling $G$ for different values of order of perturbation $k$ as well as the modified fractal-self similar resummation formula. Careful analysis of the plot shows that the theory is really perturbative for both $G \ll 1$ and $G \gg 1$. Although we can extract visually from the graph that the non-perturbative calculations coincides for $k = 1, 2$ and $3$ for $G \ll 1$, it is not clear that the theory is perturbative for $G \gg 1$ region. To clarify the perturbative character of the theory for $G \gg 1$, we rearrange Eq.(25) as:
\[
\frac{8\pi E(t, b, G)}{m^2} = E_b + E_0, \quad (51)
\]

where

\[
E_b = b^2 - G \left( \frac{1}{4} b^4 - \frac{3}{2} b^2 \ln t \right) + G^2 \left( -3.515 b^2 \left( \frac{1}{t} - 1 \right) \right) - G^3 \left( 9.918 b^2 \left( \frac{1}{t^2} - 1 \right) \right), \quad (52)
\]

\[
E_0 = t - \ln t - 1 - G \left( \frac{3}{4} \ln^2 t \right) + G^2 \left( -3.155 \left( \frac{1}{t} - 1 \right) \right) - 4.057 G^3 \left( \frac{1}{t^2} - 1 \right). \quad (53)
\]

In fact, the field dependent term \(E_b\) behaves well for both \(G \ll 1\) and \(G \gg 1\). For \(G \ll 1\) it goes to zero as \(G \to 0\) while \(E_0\) goes to infinity. The appearance of infinite effective potential as \(G \to 0\) can not be considered as Infra-Red divergence because it appears only in the vacuum energy which can be rescaled as we can measure only differences in energies. All the 1PI amplitudes are finite as \(G \to 0\) because we can get them by successive differentiation of the effective potential with respect to the condensate \(b\) while \(E_0\) has no contribution. Thus, we can safely consider the effective potential as \(E_b\) only. Also, when \(G \gg 1\) the theory is perturbative (see Fig.3 and Fig.4).

The fractal-self similar resummation for \(E_b\) is plotted in Fig.5 where we can realize that it goes to zero as \(G \to 0\) and decreases as \(G\) be very large. In fact, this is an indication that as we increase the coupling \(G\) the number of bound states decreases as \(G\) increases which is proved before in Ref.\[17\]. To assure this point, we plot the expectation value of the potential term in Fig.6 which shows that the depth of the potential decreases for large \(G\) values and thus it is expected to have no bound states for \(G \gg 1\).

V. CONCLUSION

We investigated the effective potential of the \((-\frac{g}{4}\phi^4)_{1+1}\) in view of the effective potential representation. The normal ordered effective potential predicts a pure imaginary condensate but tends to zero (exponential decrease) as \(g \to 0^+\). The imaginary value of the condensate turns the theory non-Hermitian without a priori postulated boundary conditions which is
necessary in the quantum version of the theory. The calculation of the effective potential is extended perturbatively up to $g^3$ order which in turn predicts the same shape for the vacuum condensate but with higher values. Up to $g^3$, we found that the theory is perturbative for both $g \ll 1$ and $g \gg 1$ regions. Since the theory is ought to be non-perturbative for intermediate regions of the coupling space, we supplemented the perturbation result by the fractal-self similar method in a way that results in a coherent formula to make the calculation of all the 1PI amplitudes accessible analytically.

We believe that these new results concerning the effective potential of the $PT$ symmetric $\phi^4_{1+1}$ theory is a step forward toward the understanding of these novel kind of field theories.

A very interesting note should be mentioned, the simple model we used shows many similar features with $\text{QCD}$. For instance, In Ref. [17], it was shown that the theory has two body bound state similar to $q\bar{q}$ and three body bound state like baryons. Also, our calculations shows that the theory is perturbative for large scales (Fig.4) which may be a clue that the theory may have an Ultra-Violet fixed point like $\text{QCD}$. Finally, the theory may have a symmetry restoration for large $G$ values. All of these expected features will be our task of the next work.

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FIG. 2: The effective potential $8\pi \frac{E}{m^2}$ up to $G^3$ as a function of the coupling $G$.

FIG. 3: The vacuum condensate squared versus the coupling $G$ for first, second and third order in the perturbation series.
FIG. 4: The reciprocal of the 1PI two-point function versus the coupling $G$ for first, second and third order in the perturbation series.

FIG. 5: The resummed vacuum energy $E_b$ as a function of the coupling $G$. 
FIG. 6: The expectation value of the potential term in the Hamiltonian as a function of the coupling $G$. 