TWO-ROWED HECKE ALGEBRA REPRESENTATIONS AT ROOTS OF UNITY

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In this paper, we initiate a study into the explicit construction of irreducible representations of the Hecke algebra $H_n(q)$ of type $A_{n-1}$ in the non-generic case where $q$ is a root of unity. The approach is via the Specht modules of $H_n(q)$ which are irreducible in the generic case, and possess a natural basis indexed by Young tableaux. The general framework in which the irreducible non-generic $H_n(q)$-modules are to be constructed is set up and, in particular, the full set of modules corresponding to two-part partitions is described. Plentiful examples are given.

1 Introduction and notation

The Hecke algebra $H_n(q)$ (of type $A_{n-1}$) is the unital associative algebra over $\mathbb{C}$, generated by $h_i$, $i = 1, 2, \ldots, n-1$, subject to the relations:

$$
\begin{align*}
&h_i h_{i+1} h_i = h_{i+1} h_i h_{i+1}; \\
&h_i h_j = h_j h_i \quad \text{for } |i - j| > 1; \\
&h_i^2 = (q-1) h_i + q. 
\end{align*}
$$

(1)

The parameter $q \in \mathbb{C}$ will be permitted to take any non-zero value. It is said to be generic if $q = 1$ or $q^p \neq 1$ for $p = 2, 3, 4, \ldots$. Otherwise, if $q$ is a primitive $p$th root of unity for $p > 2$, it is said to be non-generic.

When $q = 1$, $H_n(q)$ may be identified with the group algebra $\mathbb{C} S_n$ of the symmetric group on $n$ symbols, through identifying each $h_i$ with the simple transposition $s_i = (i, i+1) \in S_n$.

If $w = s_{i_1} s_{i_2} \cdots s_{i_k}$ and $w \in S_n$ cannot be expressed as a shorter product of the generators $s_i$, then $s_{i_1} s_{i_2} \cdots s_{i_k}$ is said to be a reduced expression for $w$ and the value of $k$ is the length $l(w)$ of $w$. Thereupon, the relations (1) imply that the map $h : \mathbb{C} S_n \to H_n(q)$ for which $h(s_i) = h_i$ and $h(w w') = h(w) h(w')$ for $w, w' \in S_n$ satisfying $l(w w') = l(w) + l(w')$, and extended linearly, is well defined. It follows that if $l(w) = k$ and $w = s_{i_1} s_{i_2} \cdots s_{i_k}$, then $h(w) = h_{i_1} h_{i_2} \cdots h_{i_k}$. Furthermore, a basis of $H_n(q)$ is provided by $\{ h(w) : w \in S_n \}$.
It may be shown that if \( q \) is generic then \( H_n(q) \) is isomorphic to \( \Phi S_n \) [DJ86, Vn88] and the representation theory of \( H_n(q) \) is much the same as that of \( S_n \). In particular, the inequivalent irreducible representations of \( H_n(q) \) are indexed by partitions \( \lambda \) of \( n \). That is, by finite integer sequences \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \) for which \( \lambda_1 + \lambda_2 + \cdots + \lambda_r = n \) and \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0 \). A partition for which no part \( \lambda_i \) is repeated more than \( p - 1 \) times is said to be \( p \)-regular. In Section 2, an explicit construction of the irreducible modules of \( H_n(q) \) with \( q \) generic will be described. This generalisation of the well known Specht module construction (see [JK81]) was first described in [KWy92], and is based on the use of Young diagrams, Young tableaux and \( q \)-analogues of Young symmetrisers. The Young diagram \( F^\lambda \) associated with the partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \) is a left-adjusted, top-adjusted array of square boxes such that the \( i \)th row (counting from the top) contains \( \lambda_i \) boxes. For instance, if \( \lambda = (5, 3, 2, 2) \), then

\[
F^\lambda = \begin{array}{cccc}
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\end{array}
\] (2)

Filling (or replacing) each of the \( n \) boxes of \( F^\lambda \) with distinct elements of \( \{1, 2, \ldots, n\} \) yields what is known as a Young tableau. Of the possible \( n! \) tableaux of a given shape, those for which the entries are increasing across each row and down each column are known as standard tableaux. Examples may be found at (15), (26) and (27). That particular standard tableau of shape \( \lambda \) for which the entries increase down first the leftmost column and then down successive columns taken left to right (e.g. (15)) is denoted \( t^\lambda \). The number of standard tableaux of shape \( \lambda \) is equal to the dimension of the irreducible representation of \( S_n \) (and \( H_n(q) \) with \( q \) generic) labelled by \( \lambda \) (see [JK81]). In fact, the Specht module construction enables a basis to be identified naturally with the set of standard tableaux.

2 The Specht modules

If \( \lambda \) is a partition of \( n \), the Specht module \( S^\lambda \) of \( H_n(q) \) is defined to be the linear span of the vectors \( v_{t^\lambda} \), indexed by Young tableaux \( t^\lambda \) and subject to certain relations (which will be defined below). The natural action of \( H_n(q) \) on these vectors is defined in the following way. We say that the entry \( i \) precedes \( j \) in \( t^\lambda \) if \( i \) occurs before \( j \) on reading the entries of \( t^\lambda \) down the first and then successive columns. If \( x^\lambda \) is identical to \( t^\lambda \) apart from the transposition of \( i \) and \( i+1 \), then \( h_i \) acts on \( v_{t^\lambda} \) as follows:

\[
h_i v_{t^\lambda} = \begin{cases} 
v_{x^\lambda} & \text{if } i \text{ precedes } i+1 \text{ in } t^\lambda; \\
qv_{z^\lambda} + (q-1)v_{t^\lambda} & \text{if } i+1 \text{ precedes } i \text{ in } t^\lambda.
\end{cases}
\] (3)

It is possible to express every \( v_{z^\lambda} \) in terms of standard tableaux, by means of the following two types of relation:

1. **Column relations.** Entries within a column may be transposed, if the corresponding vector is multiplied by \(-1\). Thus if \( x^\lambda \) differs from \( z^\lambda \) only in that
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a single pair of entries within a column are transposed then:
\[ v_{z^\lambda} = -v_{x^\lambda}. \] (4)

For example (denoting $v_{t^\lambda}$ by $t^\lambda$ for typographical reasons),
\[
\begin{array}{cccccc}
1 & 8 & 5 & 10 & 4 & 12 \\
6 & 11 & 3 & 5 & 8 & 12 \\
9 & 2 & 7 & 6 & 11 & 7 \\
13 & 13 & 13 \\
\end{array}
\] =
\[
\begin{array}{cccccc}
1 & 8 & 5 & 10 & 4 & 12 \\
6 & 11 & 3 & 5 & 8 & 12 \\
9 & 11 & 7 & 6 & 8 & 5 \\
9 & 11 & 7 & 6 & 8 & 5 \\
\end{array}
\] . (5)

2. Garnir relations. Assume that $z^\lambda$ is such that its entries increase down each column. If $z^\lambda$ is not standard then an adjacent pair of entries exists with that on the left greater than that on the right. Consider these two entries together with all those below the left one and all those above the right one. For example, we could consider the highlighted entries in:
\[
\begin{array}{cccccc}
1 & 2 & 3 & 10 & 4 & 12 \\
6 & 8 & 5 & 12 & 4 & 12 \\
9 & 11 & 7 & 11 & 7 & 11 \\
13 & 13 & 13 \\
\end{array}
\]
Now form all possible tableaux $t^\lambda$ by permuting these entries in all ways such that the permuted entries are increasing down the portions of each of the two columns being considered. The Garnir relation is then the following expression in which the sum is over all such tableaux:
\[
(-q)^{l(w_{t^\lambda})} \sum_{t^\lambda} (-q)^{-l(w_{t^\lambda})} v_{t^\lambda} = 0, \] (7)
where $w_{t^\lambda} \in S_n$ maps $t^\lambda$ to $t^\lambda$. The above example gives the Garnir relation:
\[
\begin{array}{cccccc}
1 & 2 & 3 & 10 & 4 & 12 \\
6 & 8 & 5 & 12 & 4 & 12 \\
9 & 11 & 7 & 11 & 7 & 11 \\
13 & 13 & 13 \\
\end{array}
\] +
\[
\begin{array}{cccccc}
1 & 2 & 3 & 10 & 4 & 12 \\
6 & 8 & 5 & 12 & 4 & 12 \\
9 & 11 & 7 & 11 & 7 & 11 \\
13 & 13 & 13 \\
\end{array}
\]
\[
\begin{array}{cccccc}
6 & 8 & 5 & 12 & 4 & 12 \\
9 & 8 & 7 & 11 & 7 & 11 \\
13 & 13 & 13 \\
\end{array}
\]
\[
\begin{array}{cccccc}
6 & 8 & 5 & 12 & 4 & 12 \\
9 & 8 & 7 & 11 & 7 & 11 \\
13 & 13 & 13 \\
\end{array}
\]
\[
\begin{array}{cccccc}
6 & 8 & 5 & 12 & 4 & 12 \\
9 & 8 & 7 & 11 & 7 & 11 \\
13 & 13 & 13 \\
\end{array}
\]
\] As in the example above, these relations do not necessarily immediately express an arbitrary $v_{t^\lambda}$ in terms of standard tableaux. However, it may be shown through employing a suitable order on the set of tableaux [JK81], that repeated application of the column and Garnir relations enables any term to be rendered in terms of standard tableaux in a finite number of steps. This completes the construction of the irreducible Specht module $S^\lambda$ of $H_n(q)$ since the number of standard
tableaux is equal to the dimension of the representation of $H_n(q)$ indexed by $\lambda$ and consequently,
\[
\{ v_{t^\lambda} : t^\lambda \text{ is standard} \} \tag{9}
\]
is a basis for $S^\lambda$.

As an example, consider representing $h_1 \in H_5(q)$ in the Specht module $S^{(3,2)}$, by acting on each $v_{t^{(3,2)}}$ for which $t^{(3,2)}$ is standard (once more $v_{t^\lambda}$ is written as $t^\lambda$):
\[
\begin{align*}
    h_1^{135} & = 235 = 135, \\
    h_1^{125} & = 215 = 125 - q^2 135, \\
    h_1^{134} & = 234 = 134, \\
    h_1^{124} & = 214 = 124 - q^2 134, \\
    h_1^{123} & = 213 = 123 - q^2 143 = q 123 - q^3 134 + q^4 135.
\end{align*}
\]

Here, column relations have been used in the first and third calculations, and Garnir relations have been used in the second, fourth and last (twice), to express each result in terms of the standard tableaux. Consequently, in the representation labelled by the partition $(3,2)$, $h_1$ is mapped to the matrix (where zeros are denoted by dots):
\[
\begin{pmatrix}
-1 & -q^2 & . & . & q^4 \\
. & q & . & . & . \\
. & . & -1 & -q^2 & -q^3 \\
. & . & . & q & . \\
. & . & . & . & q
\end{pmatrix}.
\tag{10}
\]

The matrices representing the generators $h_i$ of $H_n(q)$ in each irreducible representation for $n \leq 5$ given in [KWy92] have been produced in a similar way.

### 3 The Young symmetriser and its annihilators

For each entry $a$ of $t^\lambda$ which is not at the bottom of a column, define the column element:
\[
C^\lambda_a = 1 + h_a. \tag{11}
\]

Its action on $v_{t^\lambda}$ gives rise to a Column relation (cf. [W]):
\[
C^\lambda_a v_{t^\lambda} = 0. \tag{12}
\]

The Garnir element $G^\lambda_a$ is defined for each $a$ which is not at the end of a row of $t^\lambda$, through first letting $d$ be the entry to the right of $a$, $b$ be the entry at the bottom of the column containing $a$, and $c ( = b + 1)$ the entry at the top of the column.
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containing $d$ in $t^\lambda$. With $W_{ij}$ the subgroup of $S_n$ permuting only $\{i, i+1, \ldots, j\}$, let $G_a^\lambda$ be a set of left coset representatives for $W_{ab} \times W_{cd}$ in $W_{ad}$ chosen so that each representative is of minimal length in its coset (it is unique). Then let [KWy92]:

$$G_a^\lambda = \sum_{d \in G_a^\lambda} (-q)^{-l(d)} h(d).$$

(13)

Its action on $v_t^\lambda$ gives rise to a Garnir relation (cf. (7)):

$$G_a^\lambda v_t^\lambda = 0,$$

(14)

It is easily shown that the general column and Garnir relations of Section 2 are a consequence of (12) and (14). These properties themselves arise by identifying $v_t^\lambda$ with the $q$-analogue $Y^\lambda(q)$ of the Young symmetriser. $Y^\lambda(q)$ was originally defined in [DJ86, Gy86] and cast in a form suitable for the current purposes in [KWy92, BKW95]. However, as is seen, only its $2n - r - \lambda_1$ annihilators $C_a$ and $G_a$ are required in the construction of the Specht module $S^\lambda$. Thus $S^\lambda$ may be defined as the free module generated by a non-zero vector (say $v_t^\lambda$) subject to (12) and (14). This viewpoint of $S^\lambda$ will be utilised in what follows.

To illustrate it, consider $\lambda = (6, 3, 3, 1)$, for which:

$$t^\lambda = \begin{array}{cccccccc}
1 & 5 & 8 & 11 & 12 & 13 \\
2 & 6 & 9 \\
3 & 7 & 10 \\
4
\end{array}.$$

(15)

Here we have the seven column elements $1 + h_1, 1 + h_2, 1 + h_3, 1 + h_5, 1 + h_6, 1 + h_8$ and $1 + h_9$, each of which annihilates $v_t^\lambda$. There are nine Garnir elements $G_a^\lambda$ for $a = 1, 2, 3, 5, 6, 7, 8, 11, 12$, each of which annihilates $v_t^\lambda$. Typically:

$$G_6^\lambda = q^4 - q^3 h_7 + q^2 h_6 h_7 + q^2 h_8 h_7 - q h_6 h_8 h_7 + h_7 h_6 h_8 h_7;$$
$$G_8^\lambda = q^3 - q^2 h_{10} + q h_9 h_{10} - h_8 h_9 h_{10};$$
$$G_{11}^\lambda = q - h_{11}.$$

(16)

In fact $G_6^\lambda v_t^\lambda = 0$ gives rise to (8).

4 Decomposing $S^\lambda$ at roots of unity

In the generic case when $q$ is not a root of unity, each Specht module $S^\lambda$ of $H_n(q)$ is irreducible. However, this is no longer so if $q$ is a root of unity, although $S^\lambda$ remains well-defined. For $q$ a primitive $p$th root of unity, let $D_p^\lambda$ be the irreducible $H_n(q)$-module obtained by factoring out the maximal proper submodule from $S^\lambda$.

It is shown in [DJS80] that in this case,

$$\{D_p^\lambda : \lambda \text{ is } p\text{-regular}\}$$

(17)
is a complete set of irreducible, irredundant $H_n(q)$-modules. Very little is known about the $D_p^\lambda$ or the composition series of $S^\lambda$ in terms of the $D_p^\mu$ except in a few specific cases (see [Im90] for $n \leq 10$ and [K92] for $n \leq 5$).

The viewpoint developed in the previous section provides a means of tackling these questions in a quite general way. It relies on the fact that $D_p^\mu$ is characterised by the presence of a non-zero vector $v_{t\mu}$ which is annihilated by the set of column and Garnir elements, $C_a^\mu$ and $G_a^\mu$ defined above. This follows because, via (3), $v_{t\mu}$ generates the whole of $S^\mu$, and hence $v_{t\mu}$ cannot be present in any proper submodule. Thus, to determine whether $S^\lambda$ is reducible, it is sufficient to prove the existence of a non-zero $v_{\mu} \in S^\mu$ having the same set of annihilators as $v_{t\mu} \in S^\mu$ for some $p$-regular partition $\mu \neq \lambda$ of $n$. Conversely, the absence of all such $v_{t\mu}$ would prove $S^\lambda$ to be irreducible. (In fact, the results of [DJ86] and [DJ87] considerably restrict the set of $\mu$ for which $D_p^\mu$ may occur as a composition factor of $S^\lambda$.)

As an example, consider $\lambda = (3, 2)$. We will show that if $p = 3$ then

$$v^\mu = (1 + h_4)v_{t(3,2)} = \frac{1 \ 3 \ 5}{2 \ 4} + \frac{1 \ 3 \ 4}{2 \ 5}$$

is annihilated by the column and Garnir elements of $\mu = (4, 1)$, and hence that $S^{(3,2)}$ has a submodule $D_3^{(4,1)}$. Since $t_{(4,1)} = \frac{1 \ 3 \ 4}{2 \ 5}$, the column and Garnir elements of $\mu = (4, 1)$, are:

i) $(1 + h_1)$;

ii) $(q^2 - qh_2 + h_1h_2)$;

iii) $(q - h_3)$;

iv) $(q - h_4)$.

Acting on (18) with each of these, using (3) gives:

\begin{align*}
\text{i)} & \quad (1 + h_1)v^\mu = \frac{1 \ 3 \ 5}{2 \ 4} + \frac{2 \ 3 \ 5}{1 \ 4} + \frac{1 \ 3 \ 4}{2 \ 5} + \frac{2 \ 3 \ 4}{1 \ 5} = 0; \\
\text{ii)} & \quad (q^2 - qh_2 + h_1h_2)v^\mu = q^2\frac{1 \ 3 \ 5}{2 \ 4} - q\frac{1 \ 2 \ 5}{3 \ 4} + \frac{2 \ 1 \ 5}{3 \ 4} \\
& \quad + q^2\frac{1 \ 3 \ 4}{2 \ 5} - q\frac{2 \ 4 \ 5}{3 \ 5} + \frac{2 \ 1 \ 4}{3 \ 5} = 0; \\
\text{iii)} & \quad (q - h_3)v^\mu = q\frac{1 \ 3 \ 5}{2 \ 4} - \frac{1 \ 3 \ 4}{2 \ 5} + q\frac{1 \ 3 \ 4}{2 \ 5} - q\frac{1 \ 3 \ 5}{2 \ 4} - (q - 1)\frac{1 \ 3 \ 4}{2 \ 5} = 0; \\
\text{iv)} & \quad (q - h_4)v^\mu = q\frac{1 \ 3 \ 5}{2 \ 4} - \frac{1 \ 4 \ 5}{2 \ 3} + q\frac{1 \ 3 \ 4}{2 \ 5} - \frac{1 \ 4 \ 3}{2 \ 5} \\
& \quad = (1 + q)\frac{1 \ 3 \ 5}{2 \ 4} + q\frac{1 \ 3 \ 4}{2 \ 5} - q\frac{1 \ 3 \ 4}{2 \ 5} + q^2\frac{1 \ 3 \ 5}{2 \ 4} = (1 + q + q^2)\frac{1 \ 3 \ 5}{2 \ 4} = 0,
\end{align*}

since if $p = 3$, then $1 + q + q^2 = 0$. Therefore, $D_3^{(4,1)}$ is a submodule of $S^{(3,2)}$. It may be shown that the 4-dimensional $S^{(4,1)}$ is irreducible when $p = 3$, so that $D_3^{(4,1)} = S^{(4,1)}$. Hence $D_3^{(3,2)}$ is of dimension $5 - 4 = 1$. It is spanned by $v_{t(3,2)}$. 

In order to express the general structure in the case of two-part partitions, we introduce the notion of a boundary strip of a Young diagram $F^\lambda$. It is a continuous strip of boxes obtained by starting at the rightmost end of a row of $F^\lambda$ and, for a number of steps, recursively passing to the box below if one exists, otherwise passing to the box to the left. It must end at the bottom of a column. The length of the boundary strip is the number of boxes it comprises.

**Theorem 1.** If $\lambda = (\lambda_1, \lambda_2)$ and $q$ is a primitive $p$th root of unity then $S^\lambda$ is reducible if and only if for some integer $k > 0$, $F^\lambda$ has a boundary strip of length $kp$ having at least one, but not more than $p - 1$ boxes in the second row (or equivalently, if there exists an integer $k > 0$ such that $\lambda_1 - \lambda_2 + 2 \leq kp \leq \min\{\lambda_1 + 1, \lambda_1 - \lambda_2 + p\}$).

This theorem is illustrated by the following table, which for various $\lambda = (\lambda_1, \lambda_2)$ and $p$, displays the Young diagram $F^\lambda$ with the appropriate boundary strip indicated, says whether $S^\lambda$ is reducible, and shows its composition series.

| $\lambda$ | $p$ | $F^\lambda$ | $S^\lambda$ | Composition |
|-----------|-----|-------------|-------------|-------------|
| (5, 4)    | 3   | [ ]         | reducible   | $S^{(5,4)} = D^{(6,3)}_3 \supset D^{(5,4)}_3$ |
| (6, 4)    | 3   | [ ]         | irreducible | $S^{(6,4)} = D^{(6,4)}_3$ |
| (7, 4)    | 3   | [ ]         | reducible   | $S^{(7,4)} = D^{(9,2)}_3 \supset D^{(7,4)}_3$ |
| (8, 3)    | 3   | [ ]         | irreducible | $S^{(8,3)} = D^{(8,3)}_3$ |
| (8, 2)    | 5   | [ ]         | irreducible | $S^{(8,2)} = D^{(8,2)}_5$ |
| (9, 3)    | 5   | [ ]         | reducible   | $S^{(9,3)} = D^{(12)}_5 \supset D^{(9,3)}_5$ |

Theorem 1 has the consequence that the character $\tilde{\chi}_p^\lambda$ of the irreducible representation corresponding to $D^\lambda_p$ may be expressed as a finite sum over the characters $\chi^\lambda(q)$ of the generic representations of $H_n(q)$ (which themselves may be calculated using the methods and formulae of [KWy90, KWy92, Rm91, Vj91]).

**Theorem 2.** If $S^{(\lambda_1, \lambda_2)}$ is reducible then, using the notation of Theorem 1,

$$\tilde{\chi}_p^\lambda = \sum_{j=0}^{[\lambda_2/p]} \chi^{(\lambda_1 + jp, \lambda_2 - jp)}(q) - \sum_{j=0}^{[\mu_2/p]} \chi^{(\mu_1 + jp, \mu_2 - jp)}(q),$$

(22)
where \( [x] \) is the largest integer less than or equal to \( x \).

Of course, this Theorem may be used to give the dimension of \( D_p^\lambda \) in terms of the dimensions \( d^\nu \) of the irreducible representations of \( S_n \). For example,

\[
\dim D_3^{(6,5)} = d^{(6,5)} + d^{(9,2)} - d^{(7,4)} - d^{(10,1)} = 132 + 44 - 165 - 10 = 1.
\]

5 Explicit \( D_p^{(\lambda_1, \lambda_2)} \)

When \( q \) is a primitive \( p \)th root of unity, the irreducible \( H_n(q) \)-module \( D_p^{(\lambda_1, \lambda_2)} \) may be constructed along lines similar to the construction of the Specht modules. In this, the column and Garnir relations are retained, and are supplemented by additional relations. These relations will be described elsewhere. A basis for \( D_p^{(\lambda_1, \lambda_2)} \) is defined in terms of a certain subset of the set of standard tableaux. In order to specify this set, let

\[
T^{(\lambda_1, \lambda_2)} = a_1 a_2 a_3 \cdots a_{\lambda_1},
\]

and say that \( T^{\lambda} \) is \( s \)-strip standard at the \( i \)th position if:

\[
b_i < a_{i+s-2} \quad \text{(or if } i > \lambda_1 - s + 2, \text{ when of course } a_{i+s-2} \text{ is undefined}).
\]

**Definition 1.** If \( \lambda = (\lambda_1, \lambda_2) \) and the positive integers \( p \) and \( k \) are such that \( \lambda_1 - \lambda_2 + 2 \leq kp \leq \min\{\lambda_1 + 1, \lambda_1 - \lambda_2 + p\} \), then \( T^{\lambda} \) is said to be \( p \)-root standard if \( T^{\lambda} \) is standard and either:

1. \( T^{\lambda} \) is \( kp \)-strip standard at positions 1, 2, \ldots, \( \lambda_2 \);

or 2. to the right of the rightmost position of a non-standard \( kp \)-strip, there is a position at which \( T^{\lambda} \) is \( ((k-1)p + 2) \)-strip standard.

Note that in the important case of \( k = 1 \), the second condition here can never be satisfied because standardness denies 2-strip standardness. In this case, the tableaux are identical to those defined in [Wn88] for the corresponding representations.

As an example, consider \( \lambda = (7, 4) \) and \( p = 3 \) (so that \( k = 2 \)). In this case, the following are 3-root standard:

\[
\begin{array}{cccccccccccc}
1 & 2 & 3 & 4 & 6 & 8 & 10 \\
5 & 7 & 9 & 11
\end{array}
\quad
\begin{array}{cccccccccccc}
1 & 3 & 4 & 5 & 6 & 7 & 11 \\
2 & 8 & 9 & 10
\end{array}
\quad
\begin{array}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 9 & 10 \\
6 & 7 & 8 & 11
\end{array}
\]

(26)

whereas the following are not 3-root standard:

\[
\begin{array}{cccccccccccc}
1 & 3 & 5 & 6 & 7 & 8 & 9 \\
2 & 4 & 10 & 11
\end{array}
\quad
\begin{array}{cccccccccccc}
1 & 3 & 4 & 5 & 6 & 7 & 10 \\
2 & 8 & 9 & 11
\end{array}
\quad
\begin{array}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 8 & 10 \\
6 & 7 & 9 & 11
\end{array}
\]

(27)

**Theorem 3.** If \( \lambda = (\lambda_1, \lambda_2) \), the dimension of \( D_p^\lambda \) is equal to the number of \( p \)-root standard tableaux of shape \( \lambda \).

The non-generic \( H_n(q) \)-module \( D_p^\lambda \) may then be explicitly constructed with basis:

\[
\{v_{t^\lambda} : t^\lambda \text{ is } p \text{-root standard}\}.
\]

(28)
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