PARAMETRIX FOR WAVE EQUATIONS
ON A ROUGH BACKGROUND
I
REGULARITY OF THE PHASE AT INITIAL TIME
II
CONSTRUCTION AND CONTROL AT INITIAL TIME

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Abstract. — This book is dedicated to the construction and the control of a parametrix to the homogeneous wave equation $\square_g \phi = 0$, where $g$ is a rough metric satisfying the Einstein vacuum equations. Controlling such a parametrix as well as its error term when one only assumes $L^2$ bounds on the curvature tensor $R$ of $g$ is a major step of the proof of the bounded $L^2$ curvature conjecture proposed in [10], and solved jointly with S. Klainerman and I. Rodnianski in [17]. On a more general level, this book deals with the control of the eikonal equation on a rough background, and with the derivation of $L^2$ bounds for Fourier integral operators on manifolds with rough phases and symbols, and as such is also of independent interest.

Abstract. (Parametrix pour l’équation des ondes sur un espace-temps peu régulier : I. Régularité de la phase à l’instant initial. II. Construction et contrôle à l’instant initial) — Cet ouvrage est dédié à la construction et au contrôle d’une paramétrix pour l’équation des ondes homogène $\square_g \phi = 0$, où $g$ est une métrique peu régulière satisfaisant les équations d’Einstein dans le vide. Le contrôle d’une telle paramétrix ainsi que du terme d’erreur associé lorsque l’on suppose seulement des bornes $L^2$ sur le tenseur de courbure $R$ de $g$ est une étape cruciale de la preuve de la conjecture de courbure $L^2$ proposée dans [10], et résolue conjointement avec S. Klainerman et I. Rodnianski dans [17]. Plus généralement, cet ouvrage concerne le contrôle de l’équation eikonale sur un espace-temps peu régulier et la dérivation de bornes $L^2$ pour des opérateurs intégraux de Fourier sur des variétés avec une phase et un symbole peu réguliers, et possède de ce point de vue un intérêt propre.
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