Nonexistence of a universal quantum machine to examine the precision of unknown quantum states

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In this work, we reveal a new type of impossibility discovered in our recent research which forbids comparing the closeness of multiple unknown quantum states with any non-trivial threshold in a perfect or an unambiguous way. This impossibility is distinct from the existing impossibilities in that it is a “collective” impossibility on multiple quantum states while most other “no-go” theorems concern with only one single state each time, i.e., it is an impossibility on a non-local quantum operation. This novel impossibility may provide a new insight into the nature of quantum mechanics and it implies more limitations on quantum information tasks than the existing “no-go” theorems.

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Quantum mechanics has brought many surprises to people with its fantastic features and broad applications for a long time since it was born. In recent decades, it has been applied to the information field and greatly furthered this field by introducing the concepts like nonlocality and entanglement [1] which lead to the emergence of some novel quantum protocols and algorithms such as quantum teleportation [2], quantum dense coding [3], Shor’s factoring algorithm [4], etc, demonstrating higher performance in communication and computation than classical protocols.

On the other hand, quantum mechanics has also laid its distinctive limitations on quantum information tasks due to its linearity and superposition principle, such as the well-known quantum no cloning theorem [5], the quantum no deletion theorem [6], and so on. More recently, some new no-go theorems are found, including the no-broadcasting theorem [7, 8], the impossibility of unconditionally secure quantum bit commitment [9–12], the nonexistence of deterministic purification for two copies of a noisy entangled state [13], and so on. These impossibilities have provided deep insights into the quantum world and have stimulated people to explore more intrinsic nature of the quantum mechanics.

In this article, we are going to study a problem concerning the “closeness” of multiple unknown quantum states and reveal a interesting collective type of impossibility in quantum mechanics.

Consider that when a classical machine produces many copies of a product, people may control the quality of the products by using all kinds of apparatus to find whether the copies of the same product are exactly identical or the difference between them is within a tolerable level. In the quantum world, one may also try to find whether or not several quantum states are the same or the difference between them is within a threshold by quantum measurement to control the quality of the quantum states or test the stability of quantum machines. However, our research in this paper implies that such a task is forbidden in the quantum world. In this work, we shall show that with a sound definition of closeness between multiple quantum states, one cannot determine whether the closeness of the unknown states is within or beyond a given tolerable threshold A.

The problem is defined as follows: suppose there are n unknown states \( |\psi_1\rangle, \ldots, |\psi_n\rangle \) arbitrarily chosen from the Hilbert space \( \mathcal{H} \) of finite dimension \( d \) (any state in \( \mathcal{H} \) can be chosen repeatedly for arbitrary times), and we use the average fidelity between them,

\[
C = \frac{2}{n(n-1)} \sum_{i<j} |\langle \psi_i | \psi_j \rangle|^2 ,
\]

(1)

to measure how close the \( n \) states are, then we ask “is it possible to determine whether or not the average fidelity of the \( n \) unknown states is above a given threshold by quantum measurements?” If we denote the threshold as \( A \), then this task is to test the inequality

\[
C \geq A ,
\]

(2)

by quantum measurements. We shall call \( C \) the closeness of \( |\psi_1\rangle, \ldots, |\psi_n\rangle \) in this paper.

Here, we need to note that there is some natural limit on \( C \). When \( n \leq d \), the minimum of \( C \) is 0 obviously, so \( A \) can take any value between 0 and 1. But when \( n > d \), there are no \( n \) orthogonal states in the Hilbert space \( \mathcal{H} \), so the closeness \( C \) has a non-zero minimum value in this case, and we denote it as \( C_{\text{min}} \). In the appendix, we derive the minimum value of \( C \):

\[
C_{\text{min}} = \frac{n-d}{d(n-1)} , \quad n > d.
\]

(3)

Let \( C_{\text{min}} = 0 \) when \( n \leq d \). We assume \( C_{\text{min}} \leq A \leq 1 \) throughout this paper, whether \( n \leq d \) or not, otherwise no measurement is necessary, and the problem would be trivial.

In our research, we find that the above task is definitely impossible to accomplish by a perfect or an unambiguous measurement, whenever \( C_{\text{min}} < A < 1 \). “Perfect measurement” means that the measurement will deterministically produce a conclusive and correct result, and “unambiguous measurement” means that the measurement may produce an inclusive result with non-zero probability, but if a result is conclusive, it must be error-free. For a review of unambiguous quantum measurement, we refer the readers to [10]. We shall focus on the unambiguous measurement in this article, since the perfect measurement is a special case of the unambiguous measurement.

At first glance, the impossibility of comparing how close several unknown quantum states with a given threshold does not seem strange, since it is widely known that an unknown quantum state cannot be determined generally due to the no cloning theorem, then the average fidelity between the unknown states cannot be determined, either. However, it should be clear that what is actually concerned in our problem is the relation between the unknown states, but not what each state is, and it is not necessary to determine each state first so as to acquire the relation between them. In fact, collective measurement, one of the characteristic operations in quantum mechanics which cannot be decomposed...
to local measurements on single states generally, can be performed on the $n$ unknown states simultaneously, implying more success possibility of discerning the relation between them than that of determining each unknown quantum state.

Our research result is two-folded in detail: i) if $C_{\text{min}} < A \leq 1$, no measurement can unambiguously indicate the case that the average fidelity of the unknown quantum states is above the threshold; ii) if $C_{\text{min}} \leq A < 1$, no measurement can unambiguously indicate the case that the average fidelity of the unknown quantum states is below the threshold.

We use the positive operator-valued measure (POVM) \cite{12} to study the problem in our research. POVM provides a convenient way to describe a general physical process (no matter the process is local or non-local) if only the statistical properties of the process are concerned in the problem. A POVM consists of a set of POVM elements, each of which is a positive operator and corresponds to a possible outcome of the physical process, and the POVM elements sum up to the identity operator on the Hilbert space. In our research, there are three possible outcomes after measuring the closeness of the given states: i) the average fidelity between the states is above the given threshold, ii) the average fidelity between the states is below the given threshold, and iii) the result is inconclusive. We use $R_1$, $R_2$, $R_3$ to denote the three possible result respectively and $M_1$, $M_2$ and $M_3$ to denote the corresponding POVM elements. Each $M_i$ acts on the composite Hilbert space $H^{\otimes n}$ of the $n$ quantum states as simultaneous measurements are allowed on the whole $n$ states in this problem. Quantum mechanics tells that the probability the result $M_i$ occurs is

$$\text{Prob}(R_i) = \langle \psi_1 | \cdots \langle \psi_n | M_i | \psi_1 \rangle \cdots \langle \psi_n | \psi_n \rangle, \quad (4)$$

and the unambiguity of the measurement requires that $\text{Prob}(R_1) = 0$ if the average fidelity of the $n$ states is below the threshold $A$ and $\text{Prob}(R_2) = 0$ if the average fidelity of the $n$ states is above the threshold.

Now we present the proof of our result. Let us divide all product states in the composite Hilbert space $H^{\otimes n}$ into two sets: one set contains all product states whose $n$ factor states satisfy \cite{2}, and the other set contains the remaining product states. We denote the first set by $S_1$ and the second set by $S_2$. Then, the task of examining the closeness of $n$ arbitrary quantum states with a threshold $A$ is equivalent to distinguishing between the sets $S_1$ and $S_2$.

We first prove the first part of our result: that the average fidelity of the unknown quantum states is above the threshold cannot be detected unambiguously if $C_{\text{min}} < A \leq 1$. The core of the proof is to show $S_2$ can span the whole composite Hilbert space $H^{\otimes n}$. We prove this by constructing a spanning set of $H^{\otimes n}$ from the set $S_2$.

Let us arbitrarily select $n$ quantum states $|\psi_1 \rangle, \cdots, |\psi_n \rangle$, of which the average fidelity $C$ \cite{1} is below the threshold from the Hilbert space $H$, then $|\psi_1 \rangle \otimes \cdots \otimes |\psi_n \rangle \in S_2$. Suppose $|\phi_{ij} \rangle$, $j = 1, \cdots, d - 1$, are $d - 1$ orthonormal basis states of the orthogonal complement to $|\psi_i \rangle$ in $H$. Let

$$|\psi_{i,j} \rangle = \frac{1}{\sqrt{1 + |\epsilon|^2}} (|\psi_i \rangle + \epsilon |\phi_{ij} \rangle), \quad j = 1, \cdots, d - 1, \quad (5)$$

and $|\psi_{i,d} \rangle = |\psi_i \rangle$, for each $i = 1, \cdots, n$, it is evident that the $d$ states $|\psi_{i,j} \rangle$, $j = 1, \cdots, d$ are linearly independent. When $\epsilon$ is sufficiently small, for arbitrary $n$ indexes $j_1, \cdots, j_n = 1, \cdots, d$ the average fidelity $C'$ between the states $|\psi_{i,j_1} \rangle, \cdots, |\psi_{i,j_n} \rangle$ becomes

$$C' = \frac{2}{n(n-1)(1+|\epsilon|^2)} \sum_{k<l} |\langle \psi_k | \psi_l \rangle + \epsilon (\langle \psi_k | \phi_{l,j_l} \rangle + (\phi_{l,j_l} | \psi_i \rangle) + c^2 (\phi_{l,j_l} | \phi_{l,j_l} \rangle) |^2. \quad (6)$$

It can be seen that when $\epsilon$ is sufficiently small, $C'$ can be still below the threshold: note that $C'|_{\epsilon = 0} = C$, so when $\epsilon$ is sufficiently small, $|C' - C| \approx \frac{d'C'}{d\epsilon}|_{\epsilon = 0}|\epsilon| < A - C$ if $|\epsilon| < |A - C|(|\frac{d'C'}{d\epsilon}|_{\epsilon = 0})^{-1}$, then $C' \leq C + |C' - C| < C + A - C = A$, thus $|\psi_{i,j_1} \rangle \otimes \cdots \otimes |\psi_{i,j_n} \rangle \in S_2$. Now we show that such $d^n$ states $|\psi_{1,j_1} \rangle \otimes \cdots \otimes |\psi_{n,j_n} \rangle$ forms a spanning set of $H^{\otimes n}$ due to the linear independence of $|\psi_{i,j} \rangle$, $j = 1, \cdots, d$ for each $i$. Let $|\Psi_1 \rangle \otimes \cdots \otimes |\Psi_n \rangle$ be an arbitrary product state in $H^{\otimes n}$. Since the $d$ states $|\psi_{i,j} \rangle$, $j = 1, \cdots, d$ are linearly independent for each $i = 1, \cdots, n$ and the dimension of $H$ is $d$, the state $|\Psi_1 \rangle \otimes \cdots \otimes |\Psi_n \rangle$ can be expanded as

$$|\Psi_1 \rangle \otimes \cdots \otimes |\Psi_n \rangle = \sum_{j_1, \cdots, j_n = 1}^{d} \alpha_{j_1} \cdots \alpha_{j_n} |\psi_{1,j_1} \rangle, \cdots, |\psi_{n,j_n} \rangle. \quad (7)$$

Considering that any state in $H^{\otimes n}$ can be expanded by product states, the whole composite Hilbert space $H^{\otimes n}$ can be spanned by the states $|\psi_{1,j_1} \rangle \otimes \cdots \otimes |\psi_{n,j_n} \rangle$, $j_1, \cdots, j_n = 1, \cdots, d$ then, so the $d^n$ states $|\psi_{1,j_1} \rangle \otimes \cdots \otimes |\psi_{n,j_n} \rangle$, $j_1, \cdots, j_n = 1, \cdots, d$ forms a basis of $H^{\otimes n}$.

In the following, we show that in any case the probability of producing the outcome $R_1$ by an unambiguous measurement must be zero. Since the two operators $M_1$ and $M_2$ are positive, they can be decomposed as

$$M_i = K_i^\dagger K_i, \quad i = 1, 2. \quad (8)$$
The unambiguity of the measurement requires that the outcome $R_1$ should not occur when the average fidelity of the $n$ unknown states is below the threshold, i.e.

$$
\langle \psi_{1,j_1} | \cdots \otimes \langle \psi_{n,j_n}, K_1^\dagger K_1 | \psi_{1,j_1} \rangle \otimes \cdots \otimes | \psi_{n,j_n} \rangle = \| K_1 | \psi_{1,j_1} \rangle \otimes \cdots \otimes | \psi_{n,j_n} \rangle \|^2 = 0, \forall | \psi_{1,j_1} \rangle \otimes \cdots \otimes | \psi_{n,j_n} \rangle \in S_2, \tag{9}
$$

then we have

$$
K_1 | \psi_{1,j_1} \rangle \otimes \cdots \otimes | \psi_{n,j_n} \rangle = 0, \tag{10}
$$

thus

$$
M_1 | \psi_{1,j_1} \rangle \otimes \cdots \otimes | \psi_{n,j_n} \rangle = K_1^\dagger K_1 | \psi_{1,j_1} \rangle \otimes \cdots \otimes | \psi_{n,j_n} \rangle = 0. \tag{11}
$$

For arbitrary $n$ quantum states $| \Psi_1 \rangle, \cdots, | \Psi_n \rangle$, (11) implies that

$$
M_1 | \Psi_1 \rangle \otimes \cdots \otimes | \Psi_n \rangle = 0, \tag{12}
$$

resulting in

$$
M_1 = 0. \tag{13}
$$

Therefore, it can be inferred that the outcome $R_1$, which indicates that the average fidelity of the unknown states is above the threshold, will never be produced in any case if the measurement is unambiguous. Note that the case $A = 0$ is excluded in the first part of our result, because the fidelity between any two quantum states is always non-negative, a trivial case.

The second part of our result, i.e., no quantum measurement can give an unambiguous result when the average fidelity of the $n$ quantum states is below the threshold $A$ if $C_{\min} < A < 1$, can be proved in a similar way as above. We skip the details of the proof here.

Putting the two parts of our result together, we can conclude that neither $M_1$ nor $M_2$ exists in any unambiguous quantum measurement to compare the average fidelity of arbitrary $n$ unknown quantum states with a threshold $A$ if $C_{\min} < A < 1$, and hence perfect or unambiguous examining the closeness of multiple unknown quantum states with a threshold $A$ is definitely prohibited when $C_{\min} < A < 1$.

It is worth mentioning that the case $A = 1$ is excluded here, because when $A = 1$ our problem is equivalent to determining whether $n$ unknown quantum states are exactly identical. This is reduced to the problem of quantum state comparison and it can be shown that $M_2$ exists (but $M_1$ still vanishes) in this situation, and $M_2$ can be chosen as the projector onto the orthogonal complement to the totally symmetric subspace of $H^\otimes n$. In addition, the case $n \leq d$ and $A = 0$ is studied as the problem of determining the orthogonality of multiple quantum states in.

Furthermore, we want to point out that when $A = 1$ although $M_2$ exists for unknown pure quantum states, it still vanishes for unknown mixed quantum states. This is referred to.

It is known that there have been some “no-go” theorems like the famous quantum no cloning theorem and quantum no deletion theorem, and they reveal the limitations in quantum information science due to quantum principles. Compared with these known impossibilities, the impossibility of comparing the average fidelity of multiple unknown quantum states with a threshold in this paper has some interesting features in the following aspect: most existing “no-go” theorems concern with a single quantum system each time and they forbid local quantum operations; however, in our research the comparison operation involves multiple quantum systems simultaneously and the forbidden quantum measurement is non-local indeed, so the impossibility introduced in this article is a “collective” impossibility.

In summary, we have studied the problem of examining the closeness of $n$ arbitrary quantum states with a threshold $A$. We have shown that such a task can never succeed by a perfect or unambiguous quantum measurement if $C_{\min} < A < 1$. This is a new kind of impossibility other than the existing impossibilities, which may pose new challenges in practical situations. For example, it implies that it would be impossible to examine the stability of a deterministic quantum machine by feeding identical quantum states into the machine and comparing how close its outputs are with a threshold. We hope that our research can shed light on further understanding the limitations on quantum information tasks by quantum principles and provide a deeper insight into the quantum world.

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APPENDIX

The aim of this appendix is to obtain the minimum of the closeness when \( n > d \).

Note that

\[
C = \frac{2}{n(n-1)} \sum_{i<j} |\langle \psi_i | \psi_j \rangle|^2 = \frac{1}{n(n-1)} (\text{Tr} G^2 - n),
\]  

(14)

where \( G \) is the Gram matrix of \(|\psi_1\rangle, \ldots , |\psi_n\rangle\), i.e. \( G_{i,j} = \langle \psi_i | \psi_j \rangle \).

Suppose the eigenvalues of \( G \) are \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \). As \( G \) is positive semi-definite and \( \text{rank} G \leq d \), \( \lambda_{d+1} = \cdots = \lambda_n = 0 \). Therefore, we only need to consider the first \( d \) eigenvalues of \( G \). As \( G_{ii} = 1 \), \( \text{Tr}(G) = \lambda_1 + \cdots + \lambda_d = n \).

According to the inequality

\[
\frac{1}{m} \sum_{i=1}^{m} x_i \leq \left( \frac{1}{m} \sum_{i=1}^{m} x_i^2 \right)^{1/2}, \quad x_i \geq 0, \quad i = 1, \ldots , m,
\]

(15)

where “=” holds if \( x_1 = \cdots = x_m \), it can be inferred that

\[
\text{Tr} G^2 = \lambda_1^2 + \cdots + \lambda_d^2 \geq \frac{n^2}{d},
\]

(16)

so

\[
C \geq \frac{n-d}{d(n-1)}.
\]

(17)

The minimum value of \( \text{Tr} G^2 \) can be achieved when \( \lambda_1 = \cdots = \lambda_d = \frac{n}{d} \), and the corresponding \(|\psi_1\rangle, \ldots , |\psi_n\rangle\) can be constructed as follows. Suppose the eigenstates of \( G \) are \(|0\rangle, \ldots , |n-1\rangle\), then \( G = \frac{d}{n} |0\rangle\langle 0| + \cdots + d(d-1)(d-1) \) when \( G \) reaches its minimum. To make all the diagonal elements of \( G \) equal to one, we can change the basis to a mutually unbiased basis of \(|0\rangle, \ldots , |n-1\rangle\) [25, 26, e.g.

\[
|e_j\rangle = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \omega^{jk} |k\rangle, \quad j = 0, \ldots , n-1, \quad \omega = \exp(\frac{2\pi i}{n}).
\]

(18)

Under this basis, \( G_{ii} = \langle e_i | G | e_i \rangle = 1 \), so a feasible choice for \(|\psi_1\rangle, \ldots , |\psi_n\rangle\) with which \( \text{Tr} G^2 \) reaches the minimum is as follows:

\[
|\psi_j\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} \omega^{jk} |k\rangle, \quad j = 0, \ldots , n-1.
\]

(19)

Note that in [19], \( \omega \) is still \( \exp(\frac{2\pi i}{n}) \), but not \( \exp(\frac{2\pi i}{d}) \).

Therefore, the minimum value of \( \text{Tr} G^2 \) is \( \frac{n^2}{d} \), and according to [14] and [17], the minimum value of \( C, C_{\text{min}} \), is

\[
\frac{n-d}{d(n-1)}.
\]

(20)

[1] A. Einstein, B. Podolsky, and N. Rosen, \textit{Phys. Rev.} \textbf{47}, 777 (1935)
[2] C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres and W. K. Wootters, \textit{Phys. Rev. Lett.} \textbf{70}, 1895 (1993)
[3] C. H. Bennett and S. J. Wiesner, \textit{Phys. Rev. Lett.} \textbf{69}, 2881 (1992)
[4] A. Ekert and R. Jozsa, \textit{Rev. Mod. Phys.} \textbf{68}, 733 (1996)
[5] W. K. Wootters and W. H. Zurek, \textit{Nature} \textbf{299}, 802 (1982)
[6] A. K. Pati and S. L. Braunstein, \textit{Nature} \textbf{404}, 164 (2000)
[7] H. Barnum, C. M. Caves, C. A. Fuchs, R. Jozsa, and B. Schumacher, \textit{Phys. Rev. Lett.} \textbf{76}, 2818 (1996)
[8] H. Barnum, J. Barrett, M. Leifer and A. Wilce, \textit{Phys. Rev. Lett.} \textbf{99}, 240501 (2007)
[9] D. Mayers, \textit{Phys. Rev. Lett.} \textbf{78}, 3414 (1997)
[10] H.-K. Lo and H. F. Chau, \textit{Phys. Rev. Lett.} \textbf{78}, 3410 (1997)
[11] L. Magnin, F. Magniez, A. Leverrier, and N. J. Cerf, Phys. Rev. A 81, 010302(R) (2010)
[12] S. Winkler, M. Tomamichel, S. Hengl, R. Renner, Phys. Rev. Lett. 107, 090502 (2011)
[13] L. Lamata, J. León, D. Pérez-García, D. Salgado and E. Solano, Phys. Rev. Lett. 101, 180506 (2008)
[14] H. Saberi, Phys. Rev. A 84, 032323 (2011)
[15] A. J. Short, Phys. Rev. Lett. 102, 180502 (2009)
[16] M. Sedlák, Acta Physica Slovaca 59, 653 (2009)
[17] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information p.90 (Cambridge, 2000)
[18] A. Chefles, E. Andersson and I. Jex, J. Phys. A 37, 7315 (2004)
[19] S. M. Barnett, A. Chefles, and I. Jex, Phys. Lett. A 307, 189 (2003)
[20] I. Jex, E. Andersson and A. Chefles, J. Mod. Opt. 51, 505 (2004)
[21] M. Kleinmann, H. Kampermann, and D. Bruss, Phys. Rev. A 72, 032308 (2005)
[22] M. Sedlák, M. Ziman, V. Bužek, and M. Hillery, Phys. Rev. A 77, 042304 (2008)
[23] S. Pang and S. Wu, Phys. Rev. A 84, 012336 (2011)
[24] S. Pang and S. Wu, Phys. Rev. A 82, 042311 (2010)
[25] I. D. Ivanović, J. Phys. A 14, 3241 (1981)
[26] W. K. Wootters and B. D. Fields, Ann. Phys. N.Y. 191, 363 (1989)