The Spectral Zeta Function for Laplace Operators on Warped Product Manifolds of the type $I \times f N$

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Abstract: In this work we study the spectral zeta function associated with the Laplace operator acting on scalar functions defined on a warped product of manifolds of the type $I \times f N$, where $I$ is an interval of the real line and $N$ is a compact, $d$-dimensional Riemannian manifold either with or without boundary. Starting from an integral representation of the spectral zeta function, we find its analytic continuation by exploiting the WKB asymptotic expansion of the eigenfunctions of the Laplace operator on $M$ for which a detailed analysis is presented. We apply the obtained results to the explicit computation of the zeta regularized functional determinant and the coefficients of the heat kernel asymptotic expansion.

1. Introduction

The spectral zeta function is one of the most widely used tools for the analysis of the spectrum of a class of partial differential operators generally defined on Riemannian manifolds. It is often the case, in both mathematical and physical problems, that particular information needs to be extracted from the spectrum of an elliptic self-adjoint differential operator with positive leading symbol defined on compact Riemannian manifolds with or without boundary. For the Laplace operator, in this situation, the spectrum, $l_n$, is discrete, bounded from below and forms an increasing sequence of numbers tending to infinity with the behavior $l_n \sim n^{2/\dim(M)}$, where $M$ denotes the manifold under consideration [22]. The spectral zeta function is constructed from the eigenvalues $l_n$ as

$$
\xi(s) = \sum_{n=1}^{\infty} l_n^{-s},
$$

where $s$ is a complex variable and each eigenvalue is counted with its (finite) multiplicity. Due to the asymptotic behavior of the eigenvalues mentioned above the representation of the spectral zeta function (1.1) is valid in the halfplane $\Re(s) > \dim(M)/2$. It is possible,
however, to analytically continue $\zeta(s)$ to a meromorphic function in the entire complex plane possessing only simple poles and which is holomorphic at the point $s = 0$ [28].

In physics the spectral zeta function is of pivotal importance because it provides an elegant way of regularizing the divergent quantities that often plague calculations performed in the ambit of quantum field theory in flat or curved spacetimes [10,23]. Zeta function regularization techniques are predominantly used in order to compute, in a variety of situations, the one-loop effective action and the Casimir energy (see for instance [3,4,11,12,20,24]). For these types of applications one needs to evaluate the derivative of the spectral zeta function at $s = 0$ or the value at $s = -1/2$ to obtain the functional determinant, respectively the Casimir energy. Since these points do not belong to the region of convergence of (1.1), methods that provide the analytic continuation of (1.1) to values of $\Re(s) \leq \dim(M)/2$ need to be developed. One of these methods relies on a complex integral representation of the spectral zeta function based on the Cauchy residue theorem [24] and it has been proven to be very useful in problems involving a wide range of geometries. It is this technique that we will employ here in order to find the analytic continuation of the spectral zeta function for Laplace operators on warped products of manifolds.

Warped products of manifolds of the type $I \times f N$ with $I$ being an interval of the real line and $N$ a compact, $d$-dimensional Riemannian manifold play an important role especially in field theoretical models inspired by string theory. In the Randall-Sundrum two-brane model the spacetime is assumed to be five dimensional with only one $S^1/Z_2$ orbifolded extra-dimension. The solution to the five dimensional Einstein equations which preserves Poincare invariance is a warped product manifold with exponential warping function and four dimensional Minkowski branes [29]. Generalized Randall-Sundrum models with higher-dimensional curved branes in an $AdS$ bulk have also been considered (see e.g. [1,8,21] and references therein). In this paper we will mainly be concerned with warped product manifolds possessing an unspecified, strictly positive, warping function and a $d$-dimensional smooth compact manifold $N$.

The Laplace operator acting on scalar functions on a warped product $I \times f N$ is separable and therefore its spectral zeta function is suitably analyzed by using the contour integral method. Its analytic continuation is found by exploiting the asymptotic expansion of the eigenfunctions for large values of a specific parameter. For many geometric configurations considered in the literature the eigenfunctions are Bessel functions and their well known asymptotic properties have been used in order to perform such analytic continuation [24]. Recently, it has been shown how to find the analytic continuation of the spectral zeta function in the ambit of the spherical suspension where the relevant eigenfunctions are associated Legendre functions [17]. The case of a warped product $I \times f N$ is more general because for an arbitrary strictly positive warping function the eigenfunctions are not explicitly known. In this paper we will show that even when the warping function and the manifold $N$ have not been specified, the asymptotic expansion of the eigenfunctions can be found and the process of analytic continuation can be carried to completion. This will allow us to obtain explicit formulas for the zeta regularized functional determinant and for the coefficients of the heat kernel asymptotic expansion of the Laplacian on the warped product manifold.

The outline of the paper is as follows. In the next section we describe the geometry of the warped product $I \times f N$ and present the eigenvalue problem for the Laplacian. In Sects. 3 and 4 we compute the asymptotic expansion of the relevant eigenfunctions and use it in order to explicitly perform the analytic continuation. The analysis needs modifications if the Laplacian on $N$ has zero modes. These are described in Sect. 5. Finally, we use the results of the analytic continuation in order to obtain the regularized functional