Acoustic wave propagation in two-phase heterogeneous porous media

J.I. Osypik\textsuperscript{1}, N.I. Pushkina\textsuperscript{2}, Ya.M. Zhileikin\textsuperscript{3}
M.V.Lomonosov Moscow State University, Research Computing Center, Vorobyovy Gory, Moscow 119991, Russia

\textsuperscript{1}email jam@srcc.msu.ru
\textsuperscript{2}email N.Pushkina@mererand.com
\textsuperscript{3}email jam@srcc.msu.ru
Abstract

The propagation of an acoustic wave through two-phase porous media with spatial variation in porosity is studied. The evolutionary wave equation is derived, and the propagation of an acoustic wave is numerically analyzed in application to marine sediments with different physical parameters.
Introduction

Investigation of acoustic-wave propagation in two-phase porous media, marine sediments in particular, finds an increasing interest in studying physical properties of such media. There has been a significant amount of publications on the propagation of sound in the sea floor. Acoustic-wave propagation in sediments is controlled by intrinsic properties of a sediment, characterized by a number of physical parameters. One of the parameters that significantly influences sound propagation in a sediment is the porosity. The porosity indicates relative amounts of solid and liquid fractions in a sediment and hence it determines the frame bulk and shear moduli and through this the acoustic-wave speed. The variations in the medium properties can arise due to random packing of inhomogeneous sediment grains. In Refs. [1, 2, 3, 4] dispersion and acoustic-wave scattering from randomly varying heterogeneities in the poroelastic medium properties such as the porosity or the frame bulk modulus has been studied experimentally and theoretically. In Ref. [4] perturbation theory is used to derive a poroelastic wave equation which describes the first-order scattering by the heterogeneities of a medium. The scattering of energy from heterogeneities accounts for additional losses of a sound wave propagating through a poroelastic medium. It was shown that random variations in the parameters have more significant effects on the sound propagation through a consolidated medium than through a sand sediment.

In the present paper, we study the influence of porosity variations in space on the propagation of a plain acoustic wave in marine sediments. We shall describe the porosity of a sediment and similarly some other physical parameters as fluctuations about their average values. If the fluctuations are small the sound field is assumed to change slowly at the wave-length scale, and we shall use the method of slowly varying form of the wave (see Ref. [5]) to develop a poroelastic evolutionary wave equation in a heterogeneous medium. This method is similar to the method of a slowly varying amplitude widely used in studying nonlinear wave interactions in media with dispersion, for instance, in studying light-wave interactions in nonlinear optics. To analyze different cases of spatial variation in porosity computer simulation of the obtained evolutionary equation is performed.

1. Derivation of approximate wave equations for finite-amplitude acoustic waves in marine sediments

To derive the approximate evolution wave equation we start from the continuity equations for the densities and momenta of the liquid and solid phases of a sediment composed of a rigid frame and pores filled with water, see Ref. [6]. These equations are equivalent in the main features to the equations developed by Biot [7, 8, 9], but they are presented in a somewhat different form, we shall not list them here. On the basis of these equations, in Ref. [10] the equations for the densities of the liquid and solid phases, $\rho_f$ and $\rho_s$, were derived (in this paper we don’t take diffraction and nonlinearity into account):


\[
\left(1 - \frac{m}{\rho_f c^2 G}\right) \frac{\partial \rho_f}{\partial \tau} - \frac{\nu}{\rho_s c^2 G} \frac{\partial \rho_s}{\partial \tau} = \nabla \cdot \left[ c \left(1 + \frac{m}{\rho_f c^2 G}\right) \frac{\partial \rho_f}{\partial x} - \frac{\nu}{\rho_s c G} \frac{\partial \rho_s}{\partial x}\right],
\]
(1)

\[
\left(1 - m - \frac{k + 4/3 \mu + \nu^2/G}{\rho_s c^2}\right) \frac{\partial \rho_s}{\partial \tau} - \frac{mv}{\rho_f c^2 G} \frac{\partial \rho_f}{\partial \tau} = \nabla \cdot \left[ c \left(1 - m - \frac{k + 4/3 \mu + \nu^2/G}{\rho_s c^2}\right) \frac{\partial \rho_s}{\partial x} - \frac{mv}{\rho_f c G} \frac{\partial \rho_f}{\partial x}\right].
\]
(2)

In these equations the following variables are used,

\[x' = \epsilon \tau\]
(3)

and the moving coordinate

\[\tau = t - x/c.\]
(4)

(In Eqs. (1), (2) and everywhere below the primes for \(x\) are omitted.) In the relation (3) the small parameter \(\epsilon\) is introduced as \(\epsilon \sim v_x/c \sim u_x/c \sim \delta \rho_f/\rho_f \sim \delta \rho_s/\rho_s\),

(5)

here \(c\) is the speed of sound in the sediment and \(v, u\) are the hydrodynamic velocities of the liquid and solid phases, \(\delta \rho_f, \delta \rho_s\) are the deviations from equilibrium values of the densities of the liquid and solid phases. In Eqs. (1), (2) the left-hand sides are of the order of \(\sim \epsilon\), the right-hand side terms are of the order of \(\sim \epsilon^2\). The introduction of the new variables (3), (4) actually signifies the application of the method of slowly varying wave profile. In the equations (1), (2) we wrote \(\rho_n, \rho_s\) instead of \(\delta \rho_n, \delta \rho_s\), \(m\) is the porosity;

\[G = \frac{1 - m}{k_s} + \frac{m}{k_f} - \frac{k}{k_f^2},\]

where \(k_f, k_s\) and \(k\) are the bulk moduli of the fluid, mineral grains constituting the frame, and of the frame itself; \(\mu\) is the shear modulus of the frame; \(\nu = 1 - m - k/k_s\).

Let us now describe the porosity of a sediment as fluctuations about an average value, \(m = m_0 + \Delta m\). Alongside the porosity the bulk and shear moduli of the frame and the speed of sound should similarly fluctuate about their average values, \(k = k_0 - \Delta k, \mu = \mu_0 - \Delta \mu, c = c_0 - \Delta c\). The deviations of the moduli and speed of sound from their equilibrium values have an opposite sign as to the fluctuations of the porosity since with the increase of the porosity the sediment frame becomes softer.

Let us eliminate one of the variables, \(\delta \rho_f\) or \(\delta \rho_s\), from the left-hand sides of Eqs. (1), (2), (let it be, e.g., \(\delta \rho_s\)) by subtracting one equation from the other one. In the right-hand sides the quantity \(\delta \rho_s\) is expressed through \(\delta \rho_f\) with the formula which is valid to an accuracy \(\sim \epsilon\),

\[\delta \rho_s = \left(\frac{\nu}{\rho_s c^2 G}\right)^{-1} \left(1 - \frac{m}{\rho_f c^2 G}\right) \delta \rho_f.\]
(6)
Note, that Eqs. (1), (2) allow two independent longitudinal modes, the so called fast and slow waves. As it is shown in Ref. [11], the slow wave (unlike the fast one) is a strongly attenuated diffusion mode, and it does not contribute significantly to the sound field. In this approximation we arrive at the equation for an acoustic wave in a sediment with parameters, that vary with distance:

\[
2(1 - m) \left[ 1 - \left( \frac{c_f}{c} \right)^2 \right] - \frac{\nu^2 \rho_f}{m \rho_s} \left( \frac{c_f}{c} \right)^2 + \left( 1 - m - \frac{k + 4/3\mu}{\rho_s c^2} \right) \left[ 1 + \left( \frac{c_f}{c} \right)^2 \right] \frac{\partial \rho_f}{\partial x} + \\
\left\{ m + \frac{k + 4/3\mu}{\rho_s c^2} \left( \frac{c_f}{c} \right)^2 \left[ 2\rho_f \nu(1 - 2m) + 2 - 3m - 3\frac{k + 4/3\mu}{\rho_s c^2} \right] \right\} \delta(x) \frac{\partial \rho_f}{\partial \tau} + \\
a_1 D_\tau \rho_f = 0. \tag{7}
\]

To obtain these equations we took into account that in sand sediments the bulk modulus \(k_s\) of quartz grains is much greater than that of the pore water, and in this case \(G\) can be evaluated as \(G \approx m/k_f\), provided \(m\) is not close to zero. In Eq. (7) we have used the quantity \(\delta(x)\) which absolute value would be around 0.1,

\[
\delta(x) \sim \frac{\Delta m}{m_0} \sim \frac{\Delta k}{k_0} \sim \frac{\Delta \mu}{\mu_0} \sim \frac{\Delta c}{c_0}. \tag{8}
\]

In Eq. (7) the term \(a_1 D_\tau \rho_f\) that accounts for dissipation is introduced. \(D_\tau\) is the dissipation linear operator in the variable \(\tau\) which is characterized by the property

\[
D_\tau e^{i\omega \tau} = \alpha(\omega) e^{i\omega \tau}, \tag{9}
\]

where \(\alpha\) is real and positive and it has the meaning of an amplitude attenuation coefficient if the coefficient \(a_1\) is taken equal to the coefficient at \(\partial \rho_f / \partial x\). The relation (9) defines the action of this operator on any function of the variable \(\tau\) which can be represented by a Fourier series or integral. An algebraic expression for \(\alpha(\omega)\) is a combination of physical parameters (complex bulk and shear frame moduli included) of a sediment, and it includes the frequency correction function introduced by Biot [9].

In Introduction it was noted that acoustic wave scattering from randomly varying heterogeneities in the poroelastic medium properties manifests itself in the increase of the sound field attenuation. In this paper, we shall not consider sound scattering from heterogeneities, including it implicitly in the dissipation term.

2. Computer simulation of acoustic wave propagation in heterogeneous absorbing marine sediments

To study numerically the propagation of acoustic waves in absorbing marine sediments with parameters varying with distance we consider Eq. (7) presented in a concise form:

\[
a_1 \frac{\partial \rho_f}{\partial x} + a_2 \frac{\partial \rho_f}{\partial \tau} \delta(x) + a_1 D_\tau \rho_f = 0. \tag{10}
\]

As it is seen from Eq. (7) the coefficients \(a_1, a_2\) are the algebraic combinations of physical parameters of a sediment. It is convenient to divide Eq. (10) by \(a_1\),
\[ \frac{\partial \rho}{\partial x} + \frac{a_2}{a_1} \delta(x) \frac{\partial \rho}{\partial \tau} + D \rho = 0. \]  \hspace{1cm} (11)

Let the density boundary value be

\[ \rho|_{x=0} = A \rho_0, \hspace{0.5cm} A = 10^{-3} - 10^{-5}. \]

Introducing a new variable \( \theta = 10^4 \tau \) we have \( \rho(2\pi 10^4 \tau) = \rho(2\pi \theta) \).

Solving Eq. (11) we are to find the function \( \rho(2\pi \theta) \) periodic in the variable \( \theta \) with the period 1. The boundary condition is taken to be a harmonic function

\[ \rho_0 = -\sin(2\pi \theta). \]

In Eq. (11) it is convenient to normalize the functions and the variables except the variable \( x \) measured in centimeters. We obtain the equation

\[ \frac{\partial \rho}{\partial x} + C \delta'(x) \frac{\partial \rho}{\partial \theta} + D \rho = 0. \]  \hspace{1cm} (12)

\[ \rho|_{x=0} = -\sin(2\pi \theta), \]  \hspace{1cm} (13)

where

\[ C = \frac{\varepsilon b}{a_1 c} 10^4 \delta_0, \hspace{0.5cm} \varepsilon = \pm 1, \hspace{0.5cm} \delta(x) = \delta_0 \delta'(x), \]

\[ \delta'(x) \in [0, 1], \hspace{0.5cm} \delta_0 \in [0.1, 0.2], \]

\[ D \theta = 10^4 D \tau \]

Eq. (12) describes in fact two processes, the change of the wave phase and the wave dissipation. To solve it the so called splitting method [12] is applied.

Consider a simple example,

\[ \frac{du}{dx} = Au + Bu, \hspace{0.5cm} u|_{x=0} = u_0 \]  \hspace{1cm} (14)

and calculate \( u(h) \), were \( h \) is the step in \( x \).

We can divide the problem into two parts

\[ \frac{dw}{dx} = Av, \hspace{0.5cm} v|_{x=0} = u_0, \]

\[ \frac{dw}{dx} = Bw, \hspace{0.5cm} w|_{x=0} = v(h). \]

For smooth solutions the equality \( w(h) = u(h) + O(h^2) \) holds true. So, we can obtain the solution to Eq. (14) solving two more simple problems.

The solution of the equation

\[ \frac{\partial \rho}{\partial x} + C \delta'(x) \frac{\partial \rho}{\partial \theta} = 0, \]  \hspace{1cm} (15)

\[ \rho|_{x=0} = \rho_0, \]
satisfying periodic boundary condition can be received with the difference schemes of the “angle” type. The stencil of this difference scheme is defined by the characteristic equation \( \frac{d\theta}{dx} = C\delta'(x) \). Since \( \delta'(x) \) is positive or equal to zero the direction of the characteristics depends on the sign of the coefficient \( C \). If \( C \geq 0 \) the ”right angle” difference scheme is stable, if \( C \leq 0 \) the ”left angle” difference scheme is also stable. The condition binding the steps in \( x \) and \( \theta \) is of the form \( h \leq \frac{z}{|C|} \) (\( h \) and \( z \) are the steps for \( x \) and \( \theta \) axes respectively).

If \( \rho|_{x=0} \) is represented as the Fourier series \( \rho|_{x=0} = \sum \nu_m(0)e^{2\pi im\theta} \), Eq. (15) can be solved in an explicit form:

\[
\rho(x, \theta) = \sum \nu_m(x)e^{2\pi im\theta}
\]

with

\[
\nu_m(x) = \nu_m(0)e^{-2\pi imC \int_0^x \delta'(\xi)d\xi}.
\]

Since \( m = \pm 1 \) we have \( \rho(x, \theta) = -\sin(2\pi(\theta - \mu)), \mu = C \int_0^x \delta'(\xi)d\xi \). This means that the solution of (15) gives a shift of the phase equal to \( 2\pi\mu \). The phase shift moves to the right if \( C \) is positive and to the left if \( C \) is negative.

Let us consider the equation

\[
\frac{\partial \rho}{\partial x} + D_\tau \rho = 0.
\]

\( D_\tau \) is the linear dissipation operator:

\[
D_\tau \Rightarrow D_\theta = 10^4 \left| \frac{\partial \rho}{\partial \theta} \right| \alpha',
\]

\( 10^4\alpha' = \alpha \) is the attenuation coefficient.

This relation defines the action of this operator on a function of the variable \( \theta \) represented by the Fourier series \( \rho = \sum \nu_m e^{2\pi im\theta} \). As a result we obtain the equation

\[
\frac{d\nu_m}{dx} = -\frac{\alpha'}{2\pi} 10^4|d_m|, \quad d_m \approx 2\pi m,
\]

from which one gets

\[
\nu_m = e^{-\frac{\alpha'}{2\pi} 10^4|d_m|x}.
\]

Since \( m = \pm 1 \), we have

\[
\rho(x, \theta) = -\sin(2\pi\theta)e^{-\alpha'10^4 x}.
\]

The parameter \( L_d = \frac{1}{\alpha'10^4} \) is the propagation distance.

Consider some examples describing the transformation of harmonic acoustic waves propagating along \( x \). It will be seen that the change of the porosity with distance leads to the phase shift of the initial acoustic wave.

In Figures 1 and 2 the graphs of the functions \( \rho|_{x=0} \) and \( \delta'(x) = 0.667 + 0.333 \sin(10^{-1}\pi x) \) are presented.

We take \( C = 0.652 \cdot 10^{-2} \) and \( 10^4\alpha' = \alpha \simeq 0.45 \cdot 10^{-2} cm^{-1} \) (see experimental data in Refs. [13, 14, 15]).

The number of nodes of the variable \( \theta \) in the interval \( [0, 1] \) is 64. This number of nodes is sufficient for approximating a harmonic function. The step of the spatial variable \( x \) is 0.5, that corresponds to the magnitude of \( |C| \) and the stability condition.
The graphs of \( \rho(x_i, \theta) \) at \( x_i = 25, 50, 75, 100 \) are presented in Figs 3–6. A positive value of \( C \) gives the wave phase shift to the right. If \( C \) is negative (with the same module), the wave phase shifts to the left.

In the above examples the porosity has an oscillating character (Fig. 2) and satisfies the inequality \( 0.331 \leq \delta'(x) \leq 1 \). If \( C \) is positive the curve shifts to the right more quickly for larger \( \delta'(x) \) values and slowly for smaller \( \delta'(x) \) values. Negative \( C \) values lead to shifting the initial curve to the left in a similar way. The porosity as an oscillating function has been chosen as an example.

Now we shall consider more realistic cases of an irregular spatial variation in a sediment porosity. Take for example arbitrary continuous \( \delta'(x) \) functions of the form presented in Figs. 7–9.

Fig. 7: let \( C > 0 \). In Figs. 7.1–7.6 the function \( \rho(x_i, \theta) \) is presented for six \( x_i \) values: \( x_i = 30, 40, 50, 60, 70 \) and 80. If \( 0 \leq x_i \leq 40 \) the initial curve \( \rho(x_i, \theta) \) does not shift, that is its phase does not change; if \( 40 < x_i < 60 \) it moves to the right with an increasing speed; if \( 60 \leq x_i \leq 100 \) it moves with a constant high speed.

Fig. 8. The graphs for the initial function \( \rho(x_i, \theta) \) are not listed here, since they are in a sense similar to those in Figs. 7.1–7.6. At \( 0 \leq x_i \leq 40 \) the phase changes so that the function \( \rho(x_i, \theta) \) shifts to the right with a constant speed. At \( x_1 > 40 \) the speed decreases and at \( x_i \geq 60 \) it goes to zero.

Fig. 9: This case qualitatively repeats the cases of Figs. 7–8 where the curves go respectively up or down. The speed of the initial curve shift slows down at \( 0 \leq x_i < 50 \) and at \( x_i > 50 \) the shift moves with an accelerating speed.

In conclusion consider the case of a random pore size distribution (Fig. 10) that can arise, for instance, due to random packing of the sediment nonuniform grains. Such porosity distribution can be presented as a random digital array, and we shall interpolate it with a continuous function, see Fig. 11. This continuous function in its turn can be approximately considered as a series of curves of the types presented in Figs. 7–8. That is, the consideration given above for Figs. 7–8 can be applied to each section of this function.

3. Conclusion

Evolutionary wave equation to describe acoustic wave propagation in a two-phase porous media with spatial variations in porosity is derived. Computer simulation of the obtained equation is performed to analyze diverse cases of the porosity variations that lead to phase shifts in the initial harmonic acoustic wave.
References

[1] Hefner B.T., Jackson D.R., Dispersion and attenuation due to scattering from heterogeneities in the frame bulk modulus of sand sediments J. Acoust. Soc. Am. 2006, 119, N 5, Pt. 2, 3447.

[2] Hefner B.T., Jackson D.R., Dispersion and attenuation due to scattering from heterogeneities in the porosity of sand sediments J. Acoust. Soc. Am. 2006, 120, N 5, Pt. 2, 3098-3099.

[3] Hefner B.T., Jackson D.R., The role of porosity fluctuations in scattering from sand sediments and in propagation losses within the sediment J. Acoust. Soc. Am. 2009, 126, N 4, Pt. 2, 2168.

[4] Hefner B.T., Jackson D.R., Dispersion and attenuation due to scattering from heterogeneities of the frame bulk modulus of a poroelastic medium J. Acoust. Soc. Am. 2010, 127, N 6, 3372-3384.

[5] Zabolotskaya E.A., Khokhlov R.V., Quasi-plane waves in nonlinear acoustics of bounded beams Acust. Zh. 1969, 15, N1, 40-47.

[6] Bykov V.G., Nikolaevskii V.N., Nonlinear geoacoustic waves in sea sediments Acust. Zh. 1990, 36, N 4, 606-610 [Sov. Phys. Acoust. 1990, 36, 342].

[7] Biot M.A., Theory of propagation of elastic waves in a fluid-saturated porous solid. I. Low-frequency range J. Acoust. Soc. Am. 1956, 28, N 2, 168-178.

[8] Biot M.A., Theory of propagation of elastic waves in a fluid-saturated porous solid. II. Higher-frequency range J. Acoust. Soc. Am. 1956, 28, N 2, 179-191.

[9] Biot M.A., Generalized theory of acoustic propagation in porous dissipative media J. Acoust. Soc. Am. 1962, 34, N 9, 1254-1264.

[10] Zhilekin Ya.M., Osypik J.I., Pushkina N.I., Diffracting acoustic beams of finite amplitude in marine sediments Acoustical Physics 2003, 49, N 3, 305-311.

[11] Stoll R.D., Ocean seismo-acoustics, Low-frequency underwater acoustics (Plenum, New York, 1986).

[12] Godunov S.K., Ryaben’kii V.S., Difference Schemes. Nauka, Moscow, 1977.

[13] Chotiros N.P., Biot model of sound propagation in water-saturated sand J. Acoust. Soc. Am. 1995, 97, N 1, 199-214.

[14] Stoll R.D., Comments on ”Biot model of sound propagation in water-saturated sand” [J. Acoust. Soc. Am. 97, 199-214 (1995)] J. Acoust. Soc. Am. 1995, 103, N 5, 2723-2725.

[15] Buchanan J.L., A comparison of broadband models for sand sediments J. Acoust. Soc. Am. 2006, 120, N 6, 3584-3598.
Fig. 1. The function $\rho(\theta)|_{x=0}$

Fig. 2. The function $\delta'(x)$

Fig. 3. The function $\rho(x_i, \theta)$ at $x_i = 25$

Fig. 4. The function $\rho(x_i, \theta)$ at $x_i = 50$

Fig. 5. The function $\rho(x_i, \theta)$ at $x_i = 75$

Fig. 6. The function $\rho(x_i, \theta)$ at $x_i = 100$
Fig. 7.1. The function $\rho(x_i, \theta)$ at $x_i = 30$

Fig. 7.2. The function $\rho(x_i, \theta)$ at $x_i = 40$

Fig. 7.3. The function $\rho(x_i, \theta)$ at $x_i = 50$

Fig. 7.4. The function $\rho(x_i, \theta)$ at $x_i = 60$

Fig. 7.5. The function $\rho(x_i, \theta)$ at $x_i = 70$

Fig. 7.6. The function $\rho(x_i, \theta)$ at $x_i = 80$