Energy formula for Newman-Unti-Tamburino class of black holes

Parthapratim Pradhan

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Abstract
We compute the surface energy ($E^\pm_s$), the rotational energy ($E^\pm_r$) and the electromagnetic energy ($E^\pm_{em}$) for Newman-Unti-Tamburino (NUT) class of black hole having the event horizon ($H^+$) and the Cauchy horizon ($H^-$). Remarkably, we find that the mass parameter can be expressed as sum of three energies i.e. $M = E^\pm_s + E^\pm_r + E^\pm_{em}$. It has been tested for Taub-NUT black hole, Reissner-Nordström-Taub-NUT black hole, Kerr-Taub-NUT black hole and Kerr-Newman-Taub-NUT black hole. In each case of black hole, we find that the sum of these energies is equal to the Komar mass. It is plausible only due to the introduction of new conserved charges i.e. $J_N = M N$ (where $M = m$ is the Komar mass and $N = n$ is the gravitomagnetic charge), which is closely analogue to the Kerr-like angular momentum parameter $J = a M$.

Keywords Surface energy · Rotational energy · Electromagnetic energy · Taub-NUT BH · Kerr-Taub-NUT BH

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Parthapratim Pradhan
pppradhan77@gmail.com

1 Department of Physics, Hiralal Mazumdar Memorial College For Women, Dakshineswar, Kolkata 700035, India
1 Introduction

In recent times there is considerable ongoing interest in the thermodynamics of the NUT class of black holes. Much of the insight gained from the work [1] in which it has been suggested that a generic four dimensional Taub-NUT black hole (BH) should be interpreted as in terms of three or four different types of thermodynamic hairs. They could be defined as the Komar mass \( M = m \), the angular momentum \( J_n = mn \), the gravitomagnetic charge \( N = n \), the dual (magnetic) mass \( \tilde{M} = n \).

Motivated by this formalism, in the present work, we wish to compute the surface energy \( \mathcal{E}_\pm \), the rotational energy \( \mathcal{E}_r^\pm \) and the electromagnetic energy \( \mathcal{E}_{em}^\pm \) for NUT class of BH having two physical horizons i.e., the event horizon \( \mathcal{H}^+ \) and the Cauchy horizon \( \mathcal{H}^- \). Long ago, Smarr [2,3] computed these energies for three-parameter, charged Kerr BH, which is a solution of the Einstein-Maxwell equation. The author calculated the area of the Kerr-Newman (KN) BH, and from this expression he derived the mass parameter as a function of the area, angular momentum, and charge. He also proved the mass differential could be expressed as three physical invariants of the BH horizon. These three physical invariants are effective surface tension \( T \), angular velocity \( \Omega \), and electromagnetic potential \( \Phi \). Moreover, he demonstrated that the mass parameter should be expressed in terms of these physical invariants as a simple bilinear form. This expression could be derived by applying Euler’s theorem on homogeneous functions to the mass parameter.

Furthermore, he showed that the effective surface tension, the angular velocity, and the electromagnetic potentials may be defined and are constant on the horizon for any stationary axisymmetric spacetime. Also, he evaluated three energy components for charged Kerr BH, i.e., the surface energy, the rotational energy, and electromagnetic energy. Further, these integrals are computed using the variational definitions. Then it was easily shown that the sum of three energies is equal to the mass parameter.

There are several important works we should mention here that has been discussed in different aspects of NUT BH [4–11,13–15,18–22]. If we have not considered the formalism developed recently in [1], the following relations do exist [23–25]:

(i) the effective surface tension of \( \mathcal{H}_\pm \) is not proportional to the surface gravity of \( \mathcal{H}_\pm \) i.e.

\[
T_\pm \neq \frac{\kappa_\pm}{8\pi},
\]

where \( \kappa_\pm \) is the surface gravity of \( \mathcal{H}_\pm \).

(ii) the mass parameter can not be expressed in terms of three physical invariants, i.e., the effective surface tension of \( \mathcal{H}_\pm \), the angular velocity of \( \mathcal{H}_\pm \) and the electromagnetic potential of \( \mathcal{H}_\pm \)

\[
M \neq 2T_\pm A_\pm + 2J_N \omega_\pm + \psi_\pm N,
\]

and finally,
(iii) the sum of the surface energy, the rotational energy, and the electromagnetic energy is not equal to the mass parameter i.e.

$$\mathcal{E}_s^\pm + \mathcal{E}_r^\pm, J_N + \mathcal{E}_{em}^\pm, J_N \neq M$$

In the present work, we want to calculate three energy components, i.e., the surface energy of $H^\pm$, the rotational energy of $H^\pm$ and electromagnetic energy of $H^\pm$ for NUT class of BH. Notably, we point out that the sum of these energies is equal to the mass parameter.

Introducing the formalism developed in [1] we are indeed able to prove that

(1) the effective surface tension of $H^\pm$ is proportional to the surface gravity of $H^\pm$ i.e.

$$T^\pm = \frac{\kappa^\pm}{8\pi},$$

(2) the mass parameter can be expressed in terms of three physical invariants i.e.

$$M = 2T^\pm A^\pm + 2J_N \omega^\pm + \psi^\pm N,$$

and finally,

(3) the sum of three energies is equal to the mass parameter i.e.

$$\mathcal{E}_s^\pm + \mathcal{E}_r^\pm, J_N + \mathcal{E}_{em}^\pm, J_N = M$$

We have explicitly examined the result (3) especially for the NUT class of BHs, i.e., for Taub-NUT (TN) BH, Reissner-Nordström Taub-NUT BH, Kerr-Taub-NUT BH and Kerr-Newman-Taub-NUT BH. We show that the sum of three energies is indeed equal to the mass parameter. It has not been studied previously to the best of my knowledge. It is possible only due to the introduction of new conserved charges, i.e., $J_N = MN$ which is analogous to the Kerr like angular momentum parameter $J = aM$. The other reason is that due to multihair features of NUT parameter i.e. both rotation-like and electromagnetic charge like characteristics:

$$\frac{J_n}{m} = n \equiv N = \frac{J_N}{M}.$$

In the next Sect. (2), we have derived the surface energy, the rotational energy, and the electromagnetic energy for Taub-NUT BH. In Sect. (3), we have derived these energies for Reissner-Nordström-Taub-NUT BH and proved that the sum of three energies equal to the mass parameter. A similar analysis has been done for Kerr-Taub-NUT BH in Sect. (4). In Sect. (5), we have done a similar analysis for Kerr-Newman-Taub-NUT BH. Finally, we have given our conclusions in Sect. (6).
2 Energy formula for Taub-NUT BH

First, we consider the Taub-NUT BH. The metric \([5–9]\) in Schwarzschild like coordinates are

\[
\begin{equation}
 ds^2 = -\mathcal{Y}(r) \, (dt + 2n \cos \theta \, d\phi)^2 + \frac{dr^2}{\mathcal{Y}(r)} + \left( r^2 + n^2 \right) \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right),
\end{equation}
\]

where the function \(\mathcal{Y}(r)\) is defined by

\[
\mathcal{Y}(r) = \frac{1}{r^2 + n^2} \left[ r^2 - n^2 - 2mr \right] \quad (2)
\]

Under new procedure \([1]\) for NUT class of BHs, the global conserved charges are defined as

- Komar mass: \(M = m\),
- Gravitomagnetic charge: \(N = n\),
- Angular momentum: \(J_n = mn\),
- Dual (or Magnetic) mass: \(\tilde{M} = n \equiv N\). \quad (3)

(4)

Taking cognizance of Eq. (3) then the metric can be re-written as

\[
\begin{equation}
 ds^2 = -\mathcal{Y}(r) \, (dt + 2N \cos \theta \, d\phi)^2 + \frac{dr^2}{\mathcal{Y}(r)} + \left( r^2 + N^2 \right) \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right),
\end{equation}
\]

where the function \(\mathcal{Y}(r)\) is given by

\[
\mathcal{Y}(r) = \frac{1}{r^2 + N^2} \left( r^2 - N^2 - 2Mr \right) \quad (6)
\]

The Killing horizons are located at

\[
r_{\pm} = M \pm \sqrt{M^2 + N^2}.
\]

\(r_+\) is called EH and \(r_-\) is called CH. Now we know from Hawking’s \([16]\) general theorem which asserted that the surface area of a BH never decrease. Thus for a Taub-NUT class of BH, the area is constant. It is derived to be both for the horizons \([26]\) as

\[
A_\pm = 8\pi \left[ M^2 + N^2 \pm \sqrt{M^4 + J_N^2} \right].
\]

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According to Ramaswamy and Sen [12], it was pointed out that NUT solutions with the mass parameter $M = 0$ i.e. massless dual mass are perfectly well defined. In this case, the horizon radius becomes $r_\pm = \pm N$. Consequently, the area [Eq. (8)] of both the horizons reduced to $A_\pm = 8\pi N^2$. Hence in this circumstances, only the NUT parameter can be expressed as in terms of the area of both the horizons i.e.

$$N = \sqrt{\frac{A_\pm}{8\pi}}. \quad (9)$$

In this situation the mass formula does not exist.

On inverting Eq. (8), one obtains the Komar mass as a function of area, new conserved charges $J_N$ and NUT parameter for both the horizons $\mathcal{H}^\pm$:

$$M(A_\pm, J_N, N) = \sqrt{\frac{A_\pm}{16\pi} + \frac{4\pi J_N^2}{A_\pm} - N^2 + \frac{4\pi N^4}{A_\pm}}, \quad (10)$$

It is interesting to note that the mass parameter can be expressed in terms of both the area of $\mathcal{H}^+$ and $\mathcal{H}^-$. The mass differential is expressed as three physical invariants of both $\mathcal{H}^+$ and $\mathcal{H}^-$

$$dM = T_\pm dA_\pm + \omega_\pm dJ_N + \psi_\pm dN. \quad (11)$$

where

$$T_\pm = \frac{1}{M} \left( \frac{1}{32\pi} - \frac{2\pi J_N^2}{A_\pm^2} - \frac{2\pi N^4}{A_\pm^2} \right). \quad (12)$$

$$\omega_\pm = \frac{4\pi J_N}{MA_\pm}, \quad (13)$$

$$\psi_\pm = -\frac{N}{M} \left( 1 - \frac{8\pi N^2}{A_\pm} \right). \quad (14)$$

and

- $T_\pm =$ Effective surface tension of $\mathcal{H}^+$ and $\mathcal{H}^-$
- $\omega_\pm =$ Angular velocity of $\mathcal{H}^\pm$
- $\psi_\pm =$ NUT potentials of $\mathcal{H}^\pm$

Shortly, we prove that this effective surface tension is proportional to the surface gravity of the horizons, i.e.

$$T_\pm = \frac{1}{M} \left[ \frac{1}{32\pi} - \frac{2\pi J_N^2}{A_\pm^2} - \frac{2\pi N^4}{A_\pm^2} \right] \left. \right|_{\mathcal{H}^\pm} \quad (15)$$
\[
\begin{align*}
&= \frac{1}{32\pi M} \left[ 1 - \frac{64\pi^2 (J_N^2 + N^4)}{A_\pm^2} \right] \\
&= \frac{1}{16\pi M} \left[ 1 - \frac{(M^2 + N^2)^2}{Mr_\pm + N^2} \right] \\
&= \pm \frac{\sqrt{M^2 + N^2}}{8\pi (r_\pm^2 + N^2)} \\
&= \frac{r_\pm - M}{8\pi (r_\pm^2 + N^2)} = \frac{\kappa_\pm}{8\pi},
\end{align*}
\]

where \( \kappa_\pm \) is defined as the surface gravity \([16,17]\) of \( \mathcal{H}_\pm \). The BH temperature of both the horizons is derived via surface gravity on the horizons

\[
T_\pm = \frac{\kappa_\pm}{2\pi} = \frac{r_\pm - M}{2\pi (r_\pm^2 + N^2)} = \frac{1}{4\pi r_\pm}.
\]

The entropy can be easily derived from the area formula via this relation.

\[
S_\pm = \frac{A_\pm}{4}
\]

So, the mass parameter \([1]\) should be expressed in terms of these quantities both for \( \mathcal{H}_\pm \) as a simple bilinear form

\[
M = 2T_\pm A_\pm + 2J_N \omega_\pm + \psi_\pm N.
\]

This striking formula is obtained by applying Euler’s theorem on homogeneous functions to \( M \), which is homogeneous of degree \( \frac{1}{2} \) in \((A_\pm, J_N, N^2)\). This may be rewritten as

\[
M = \frac{\kappa_\pm}{4\pi} A_\pm + 2J_N \omega_\pm + \psi_\pm N.
\]

Therefore the Smarr-Gibbs-Duhem relation of \( \mathcal{H}_\pm \) for Taub-NUT BH is

\[
\frac{M}{2} = T_\pm S_\pm + J_N \omega_\pm + \frac{\psi_\pm N}{2}.
\]

Interestingly, \( T_\pm, \omega_\pm \) and \( \psi_\pm \) should be defined and are constant on the \( \mathcal{H}_+ \) and \( \mathcal{H}_- \) for any stationary, axially symmetric space-time. Since \( dM \) is a perfect differential therefore one can choose freely any path of integration in \((A_\pm, J_N, N)\) space. So the surface energy \( \mathcal{E}_s^\pm \) of \( \mathcal{H}_+ \) \([2,3]\) and \( \mathcal{H}_- \) \([27]\) may be defined by

\[
\mathcal{E}_s^\pm = \int_0^{A_\pm} T_\pm (\tilde{A}_\pm, 0, 0) d\tilde{A}_\pm;
\]
the rotational energy of $\mathcal{H}^+$ [2,3] and $\mathcal{H}^-$ [27] can be defined by

$$E_{r, J_N} = \int_0^{J_N} \omega_\pm (A_\pm, \tilde{J}_N, 0) d\tilde{J}_N, \ A_\pm \text{ fixed}; \quad (22)$$

and the electromagnetic energy of $\mathcal{H}^+$ [2,3] and $\mathcal{H}^-$ [27] can be defined as

$$E_{em, J_N} = \int_0^{N} \psi_\pm (A_\pm, J_N, \tilde{N}) d\tilde{N}, \ A_\pm, J_N \text{ fixed}; \quad (23)$$

These integrals should be directly computed using the variational definitions which is already defined in Eqs. (12–14). First we shall compute the surface energy of $\mathcal{H}^\pm$:

$$E_s^\pm = \int_0^{A_\pm} T_\pm (\tilde{A}_\pm, 0, 0) d\tilde{A}_\pm \quad (24)$$

$$= \sqrt{A_\pm/16\pi} \quad (25)$$

Next we find the rotational energy of $\mathcal{H}^\pm$ as

$$E_r^\pm = \int_0^{J_N} \omega_\pm (A_\pm, \tilde{J}_N, 0) d\tilde{J}_N, \ A_\pm \text{ fixed} \quad (26)$$

$$= \sqrt{A_\pm/16\pi + 4\pi J_N^2/4A_\pm - \sqrt{A_\pm/16\pi}}. \quad (27)$$

and finally we get the electromagnetic energy of $\mathcal{H}^\pm$ as

$$E_{em, J_N} = \int_0^{N} \psi_\pm (A_\pm, J_N, \tilde{N}) d\tilde{N}, \ A_\pm, J_N \text{ fixed} \quad (28)$$

$$= \sqrt{A_\pm/16\pi + 4\pi J_N^2/4A_\pm - N^2 + 4\pi N^4/4A_\pm} - \sqrt{A_\pm/16\pi + 4\pi J_N^2/4A_\pm}. \quad (29)$$

Now we compute the sum of three energies

$$E_s^\pm + E_r^\pm, J_N + E_{em, J_N} = \sqrt{A_\pm/16\pi + 4\pi J_N^2/4A_\pm - N^2 + 4\pi N^4/4A_\pm}. \quad (30)$$

Using Eq. (10), we can rewrite the above equation as
\[ E_s^\pm + E_r, J_N + E_{em, J_N}^\pm = \sqrt{\frac{A_\pm}{16\pi} + \frac{4\pi J_N^2}{A_\pm} - N^2 + \frac{4\pi N^4}{A_\pm}} = M (A_\pm, J_N, N). \] (31)

Remarkably, the Komar mass can be expressed as the sum of three energies, namely the surface energy of \( \mathcal{H}^\pm \), the rotational energy of \( \mathcal{H}^\pm \) and the electromagnetic energy of \( \mathcal{H}^\pm \).

Note that, in the limit \( N = 0 \), we get the energy formula for Schwarzschild BH i.e.

\[ E_s^\pm = \sqrt{\frac{A_\pm}{16\pi}} = M (A_\pm). \] (32)

In the next section, we will do a similar investigation by adding a charge parameter.

### 3 Energy formula for Reissner-Nordström-Taub-NUT BH

Now we want to extend the preceding analysis for Reissner-Nordström-Taub-NUT BH. The metric function modified for this BH as

\[ Y(r) = \frac{1}{r^2 + N^2} \left( r^2 - N^2 - 2Mr + Q^2 \right) \] (33)

where \( Q \) is the purely electric charge. Now the BH horizons are situated at

\[ r_\pm = M \pm \sqrt{M^2 - Q^2 + N^2} \] (34)

Now the area of the BH computed for both the physical horizons are

\[ A_\pm = 4\pi \left[ 2 \left( M^2 + N^2 \right) - Q^2 \pm 2\sqrt{M^4 - M^2 Q^2 + J_N^2} \right]. \] (35)

Now inverting the above expression, one obtains the Komar mass as a function of area, conserved charges \( J_N \), charge parameter \( Q \) and NUT parameter \( N \) for \( \mathcal{H}^\pm \):

\[ M (A_\pm, J_N, N, Q) = \sqrt{\frac{A_\pm}{16\pi} + \frac{4\pi J_N^2}{A_\pm} - N^2 + \frac{4\pi N^4}{A_\pm} + \frac{\pi Q^4}{A_\pm} - \frac{4\pi N^2 Q^2}{A_\pm} + \frac{Q^2}{2}}. \] (36)

So, the mass differential can be expressed as three physical invariants of both \( \mathcal{H}^\pm \)

\[ dM = T_\pm dA_\pm + \omega_\pm dJ_N + \psi_\pm dN + \Phi_\pm dQ. \] (37)
where

\[
T_\pm = \frac{1}{M} \left( \frac{32\pi}{2} - \frac{2\pi J_N^2}{A_\pm^2} - \frac{2\pi N^2 Q^2}{2A_\pm^2} + \frac{2\pi N^2 Q^2}{2A_\pm^2} \right). \tag{38}
\]

\[
\omega_\pm = \frac{4\pi J_N}{MA_\pm}, \tag{39}
\]

\[
\psi_\pm = -\frac{N}{M} \left( 1 - \frac{8\pi N^2 Q^2}{A_\pm} \right). \tag{40}
\]

\[
\Phi_\pm = \frac{1}{M} \left( \frac{2\pi Q^3}{A_\pm} - \frac{4\pi Q^2 N^2}{A_\pm} + \frac{Q^2}{2} \right). \tag{41}
\]

and

\[
T_\pm = \text{Effective surface tension of } \mathcal{H}^+ \text{ and } \mathcal{H}^- \text{ for RN-Taub-NUT BH}
\]

\[
\omega_\pm = \text{Angular velocity of } \mathcal{H}_\pm \text{ for RN-Taub-NUT BH}
\]

\[
\psi_\pm = \text{NUT potentials of } \mathcal{H}_\pm \text{ for RN-Taub-NUT BH}
\]

\[
\Phi_\pm = \text{Electromagnetic potentials of } \mathcal{H}_\pm \text{ for RN-Taub-NUT BH}
\]

Similarly, we show that this effective surface tension is proportional to the surface gravity of the horizons, i.e.

\[
T_\pm = \frac{r_\pm - M}{8\pi \left( r_\pm^2 + N^2 \right)} = \frac{\kappa_\pm}{8\pi}, \tag{42}
\]

where \(\kappa_\pm\) is defined as the surface gravity of \(\mathcal{H}_\pm\) for RN-Taub-NUT BH.

Hence, for RN-Taub-NUT BH, the mass parameter can be expressed in terms of these quantities both for \(\mathcal{H}_\pm\) as a simple bilinear form

\[
M = 2T_\pm A_\pm + 2J_N \omega_\pm + \psi_\pm N + \Phi_\pm Q. \tag{43}
\]

Like Taub-NUT BH, this striking formula could be obtained by applying Euler’s theorem on homogeneous functions to \(M\), which is homogeneous of degree \(\frac{1}{2}\) in \((A_\pm, J_N, N^2, Q^2)\). Thus this can be rewritten as

\[
M = \frac{\kappa_\pm}{4\pi} A_\pm + 2J_N \omega_\pm + \psi_\pm N + \Phi_\pm Q. \tag{44}
\]

Thus the Smarr-Gibbs-Duhem relation of \(\mathcal{H}_\pm\) for RN-Taub-NUT BH is

\[
\frac{M}{2} = T_\pm S_\pm + J_N \omega_\pm + \frac{\psi_\pm N}{2} + \frac{\Phi_\pm Q}{2}. \tag{45}
\]

Interestingly, \(T_\pm, \omega_\pm, \psi_\pm\) and \(\Phi_\pm\) could be defined and are constant on the \(\mathcal{H}_\pm\) for RN-Taub-NUT spacetime.

\[ \text{Springer} \]
As \( dM \) is a perfect differential therefore one can choose freely any path of integration in \((A_\pm, J_N, N, Q)\) space. So the surface energy \( \mathcal{E}_s^\pm \) of \( \mathcal{H}^\pm \) for RN-Taub-NUT BH can be defined by

\[
\mathcal{E}_s^\pm = \int_0^{A_\pm} T_{\pm} (\hat{A}_\pm, 0, 0, 0) \, d\hat{A}_\pm; \tag{46}
\]

The rotational energy of \( \mathcal{H}^\pm \) can be defined by

\[
\mathcal{E}_{r, J_N}^\pm = \int_0^{J_N} \omega_{\pm} (A_\pm, \tilde{J}_N, 0, 0) \, d\tilde{J}_N, \, A_\pm \text{ fixed}; \tag{47}
\]

The electromagnetic energy of \( \mathcal{H}^\pm \) due to NUT parameter \((N)\) can be defined as

\[
\mathcal{E}_{em, J_N}^\pm = \int_0^{N} \psi_{\pm} (A_\pm, J_N, \tilde{N}, 0) \, d\tilde{N}, \, A_\pm, J_N \text{ fixed}; \tag{48}
\]

and the electromagnetic energy of \( \mathcal{H}^\pm \) due to charge parameter \((Q)\) can be defined as

\[
\mathcal{E}_{em, J_N, Q}^\pm = \int_0^{Q} \Phi_{\pm} (A_\pm, J_N, N, \tilde{Q}) \, d\tilde{Q}, \, A_\pm, J_N, N \text{ fixed}; \tag{49}
\]

These integrals should be directly computed using the variational definitions which is already defined in Eqs. (38–41).

Proceeding analogously the surface energy of \( \mathcal{H}^\pm \) for RN-Taub-NUT BH is

\[
\mathcal{E}_s^\pm = \sqrt{\frac{A_\pm}{16\pi}} \tag{50}
\]

Next we compute rotational energy of \( \mathcal{H}^\pm \) for this BH as

\[
\mathcal{E}_{r, J_N}^\pm = \sqrt{\frac{A_\pm}{16\pi} + \frac{4\pi J_N^2}{A_\pm}} - \sqrt{\frac{A_\pm}{16\pi}}. \tag{51}
\]

Due to NUT parameter the electromagnetic energy of \( \mathcal{H}^\pm \) for RN-Taub-NUT BH becomes

\[
\mathcal{E}_{em, J_N}^\pm = \sqrt{\frac{A_\pm}{16\pi} + \frac{4\pi J_N^2}{A_\pm} - N^2 + \frac{4\pi N^4}{A_\pm}} - \sqrt{\frac{A_\pm}{16\pi} + \frac{4\pi J_N^2}{A_\pm}}. \tag{52}
\]

Due to the charge parameter, the electromagnetic energy of \( \mathcal{H}^\pm \) for RN-Taub-NUT BH is

\[
\mathcal{E}_{em, J_N, Q}^\pm = \sqrt{\frac{A_\pm}{16\pi} + \frac{4\pi J_N^2}{A_\pm} - N^2 + \frac{4\pi N^4}{A_\pm} + \frac{\pi Q^4}{A_\pm} - \frac{4\pi N^2 Q^2}{A_\pm} + \frac{Q^2}{2}}.
\]
Next we compute the sum of these energies

$$E_s^\pm + E_r^\pm, J_N + E_{em, J_N}^\pm + E_{em, J_N, Q} = \sqrt{\frac{A_\pm}{16\pi} + \frac{4\pi J_N^2}{A_\pm} - N^2 + \frac{4\pi N^4}{A_\pm} + \frac{\pi Q^4}{A_\pm} - \frac{4\pi N^2 Q^2}{A_\pm} + \frac{Q^2}{2}}.$$  \hspace{1cm} (54)

Using Eq. (36), we can rewrite the above equation as

$$E_s^\pm + E_r^\pm, J_N + (E_{em, J_N}^\pm + E_{em, J_N, Q}) = M (A_\pm, J_N, N, Q).$$  \hspace{1cm} (55)

Remarkably, for RN-Taub-NUT BH, the Komar mass can also be expressed as the sum of these energies, namely the surface energy of $H^\pm$, the rotational energy of $H^\pm$, the electromagnetic energy of $H^\pm$ due to the NUT parameter and the electromagnetic energy of $H^\pm$ due to the charge parameter. Note that, in the limit $N = 0$, we will get the energy formula for RN BH i.e.

$$E_s^\pm + E_{em, Q} = \sqrt{\frac{A_\pm}{16\pi} + \frac{\pi Q^4}{A_\pm} + \frac{Q^2}{2}} = M (A_\pm, Q).$$  \hspace{1cm} (56)

### 4 Energy formula for Kerr-Taub-NUT BH

The metric of Kerr-Taub-NUT BH [10] in Boyer-Lindquist like coordinates $(t, r, \theta, \phi)$ is given by

$$ds^2 = -\frac{\Delta}{\rho^2} [dt - P d\phi]^2 + \frac{\sin^2 \theta}{\rho^2} \left[ (r^2 + a^2 + N^2) d\phi - a dt \right]^2 + \rho^2 \left[ \frac{dr^2}{\Delta} + d\theta^2 \right].$$  \hspace{1cm} (57)

where

$$a \equiv \frac{J}{M}, \rho^2 \equiv r^2 + (N + a \cos \theta)^2$$  \hspace{1cm} (58)

$$\Delta \equiv r^2 - 2Mr + a^2 - N^2$$  \hspace{1cm} (59)

$$P \equiv a \sin^2 \theta - 2N \cos \theta.$$  \hspace{1cm} (60)

where the global conserved charges are the Komar mass $M$, angular momentum $J = aM$ and gravitomagnetic charge or dual mass or NUT parameter $N$. 
The radii of the horizon is evaluated by the function

$$\Delta|_{r=r_\pm} = 0$$

which implies that

$$r_\pm \equiv M \pm \sqrt{M^2 - a^2 + N^2}$$

(61)

Taking cognizance of Eq. (3), the area $[1]$ of $\mathcal{H}_\pm$ is thus

$$A_\pm = 8\pi \left[ (M^2 + N^2) \pm \sqrt{M^4 + J_N^2 - J^2} \right].$$

(62)

Analogously, the Komar mass as a function of area, new conserved charges $J_N$, angular momentum ($J$) and NUT parameter for both the horizons $\mathcal{H}_\pm$:

$$M(A_\pm, J, J_N, N) = \sqrt{\frac{A_\pm}{16\pi} + \frac{4\pi J^2}{A_\pm} + \frac{4\pi J_N^2}{A_\pm} - N^2 + \frac{4\pi N^4}{A_\pm}},$$

(63)

Similarly, the mass differential of $\mathcal{H}_\pm$

$$dM = T_\pm dA_\pm + \Omega_\pm dJ + \omega_\pm dJ_N + \psi_\pm dN.$$ 

(64)

where

$$T_\pm = \frac{1}{M} \left( \frac{1}{32\pi} - \frac{2\pi J^2}{A_\pm^2} - \frac{2\pi J_N^2}{A_\pm^2} - \frac{2\pi N^4}{A_\pm^2} \right),$$

(65)

$$\omega_\pm = \frac{4\pi J_N}{MA_\pm},$$

(66)

$$\Omega_\pm = \frac{4\pi a}{A_\pm},$$

(67)

$$\psi_\pm = -\frac{N}{M} \left( 1 - \frac{8\pi N^2}{A_\pm} \right).$$

(68)

and

$T_\pm =$ Effective surface tension of $\mathcal{H}_\pm$

$\omega_\pm =$ Angular velocity of $\mathcal{H}_\pm$ due to NUT parameter

$\Omega_\pm =$ Angular velocity of $\mathcal{H}_\pm$ due to spin parameter

$\psi_\pm =$ NUT potentials of $\mathcal{H}_\pm$
Similarly, the effective surface tension is proportional to the surface gravity of $\mathcal{H}^\pm$ i.e.

$$T_\pm = \frac{r_\pm - M}{8\pi \left(r_\pm^2 + N^2 + a^2\right)} = \frac{\kappa_\pm}{8\pi},$$

where $\kappa_\pm$ is defined as the surface gravity of $\mathcal{H}^\pm$ for Kerr-Taub-NUT BH. The BH temperature is

$$T_\pm = \frac{\kappa_\pm}{2\pi} = \frac{r_\pm - M}{2\pi \left(r_\pm^2 + N^2 + a^2\right)}.$$  

Similarly, the mass parameter of Kerr-Taub-NUT BH can be expressed in terms of these quantities as a simple bilinear form

$$M = 2T_\pm A_\pm + 2J \Omega_\pm + 2J_N \omega_\pm + \psi_\pm N.$$  

This may be rewritten as

$$M = \frac{\kappa_\pm}{4\pi} A_\pm + 2J \Omega_\pm + 2J_N \omega_\pm + \psi_\pm N.$$  

Therefore the Smarr-Gibbs-Duhem relation of $\mathcal{H}^\pm$ for Kerr-Taub-NUT BH is

$$\frac{M}{2} = T_\pm S_\pm + +J \Omega_\pm + J_N \omega_\pm + \frac{\psi_\pm N}{2}.$$  

As before, $dM$ is a perfect differential therefore one can choose freely any path of integration in $(A_\pm, J, J_N, N)$ space. So the surface energy $\mathcal{E}_s^\pm$ of $\mathcal{H}^\pm$ for Kerr-Taub-NUT BH can be defined by

$$\mathcal{E}_s^\pm = \int_0^{A_\pm} T_\pm (\tilde{A}_\pm, 0, 0, 0) d\tilde{A}_\pm;$$

The rotational energy of $\mathcal{H}^\pm$ due to spin parameter can be defined by

$$\mathcal{E}_{r, J}^\pm = \int_0^{J} \Omega_\pm (A_\pm, \tilde{J}, 0, 0) d\tilde{J}, \ A_\pm \text{ fixed};$$

The rotational energy of $\mathcal{H}^\pm$ due to NUT parameter can be defined by

$$\mathcal{E}_{r, J_N}^\pm = \int_0^{J_N} \omega_\pm (A_\pm, J, \tilde{J}_N, 0) d\tilde{J}_N, \ A_\pm, J \text{ fixed};$$

and the electromagnetic energy of $\mathcal{H}^\pm$ can be defined as

$$\mathcal{E}_{em, J, J_N}^\pm = \int_0^{N} \psi_\pm (A_\pm, J, J_N, \tilde{N}) d\tilde{N}, \ A_\pm, J, J_N \text{ fixed};$$
These integrals should be directly computed using the variational definitions which is already defined in Eqs. (65–68). First, we shall compute the surface energy of $\mathcal{H}^\pm$ for Kerr-Taub-NUT BH

$$\mathcal{E}_s^\pm = \sqrt{\frac{A_\pm}{16\pi}}$$

(78)

Next we find rotational energy of $\mathcal{H}^\pm$ due to spin parameter as

$$\mathcal{E}_r, J = \sqrt{\frac{A_\pm}{16\pi} + \frac{4\pi J^2}{A_\pm}} - \sqrt{\frac{A_\pm}{16\pi}}.$$  

(79)

The rotational energy of $\mathcal{H}^\pm$ due to NUT parameter is

$$\mathcal{E}_r, J, J_N = \sqrt{\frac{A_\pm}{16\pi} + \frac{4\pi J^2}{A_\pm} + \frac{4\pi J_N^2}{A_\pm}} - \sqrt{\frac{A_\pm}{16\pi} + \frac{4\pi J^2}{A_\pm} + \frac{4\pi J_N^2}{A_\pm}}.$$  

(80)

and finally we get the electromagnetic energy of $\mathcal{H}^\pm$ for Kerr-Taub-NUT BH as

$$\mathcal{E}_{em}, J, J_N = \sqrt{\frac{A_\pm}{16\pi} + \frac{4\pi J^2}{A_\pm} + \frac{4\pi J_N^2}{A_\pm} - N^2 + \frac{4\pi N^4}{A_\pm}} - \sqrt{\frac{A_\pm}{16\pi} + \frac{4\pi J^2}{A_\pm} + \frac{4\pi J_N^2}{A_\pm}.}$$

(81)

Now we compute the sum of these energies

$$\mathcal{E}_s^\pm + \mathcal{E}_r, J + \mathcal{E}_r, J, J_N + \mathcal{E}_{em}, J, J_N$$

$$= \sqrt{\frac{A_\pm}{16\pi} + \frac{4\pi J^2}{A_\pm} + \frac{4\pi J_N^2}{A_\pm} - N^2 + \frac{4\pi N^4}{A_\pm}.}$$

(82)

Using Eq. (63), we can rewrite the above equation for Kerr-Taub-NUT BH as

$$\mathcal{E}_s^\pm + \left(\mathcal{E}_r, J + \mathcal{E}_r, J, J_N\right) + \mathcal{E}_{em}, J, J_N = M (A_\pm, J, J_N, N).$$  

(83)

Remarkably for Kerr-Taub-NUT BH, the Komar mass can be expressed as sum of four energies namely the surface energy of $\mathcal{H}^\pm$, the rotational energy of $\mathcal{H}^\pm$ due to spin parameter, the rotational energy of $\mathcal{H}^\pm$ due to NUT parameter and the electromagnetic energy of $\mathcal{H}^\pm$. Note that in the limit $N \to 0$, one obtains the energy formula for Kerr
Energy formula for Newman-Unti-Tamburino class of black holes

BH i.e.

$$\mathcal{E}^\pm_s + \mathcal{E}^\pm_r = \sqrt{\frac{A^\pm}{16\pi}} + \frac{4\pi J^2}{A^\pm} = M (A^\pm, J).$$

(84)

5 Energy formula for Kerr-Newman-Taub-NUT BH

Finally, we consider most general class of BH without cosmological constant. The metric form similar to Eq. (57) and having the horizon function [10] as

$$\Delta \equiv r^2 - 2M r + a^2 + Q^2 - N^2.$$  

(85)

The horizons are located at

$$r^\pm \equiv M \pm \sqrt{M^2 - a^2 - Q^2 + N^2}.$$  

(86)

Analogously, the BH horizon area of $H^\pm$ is

$$A^\pm = 4\pi \left[ 2(M^2 + N^2) - Q^2 \pm 2\sqrt{M^4 + J_N^2 - J^2 - M^2 Q^2} \right].$$

(87)

Analogously the mass parameter of Kerr-Newman-Taub-NUT BH for both the horizons $H^\pm$ are

$$M (A^\pm, J, J_N, N, Q) = \sqrt{\frac{A^\pm}{16\pi} + \frac{4\pi J^2}{A^\pm} + \frac{4\pi J_N^2}{A^\pm} - N^2 + \frac{4\pi N^4}{A^\pm} + \frac{\pi Q^4}{A^\pm} - \frac{4\pi N^2 Q^2}{A^\pm} + \frac{Q^2}{2}},$$

(88)

Hence the mass differential of $H^\pm$ becomes

$$dM = T^\pm dA^\pm + \Omega^\pm dJ + \omega^\pm dJ_N + \psi^\pm dN + \Phi^\pm dQ.$$  

(89)

where

$$T^\pm = \frac{1}{M} \left( \frac{1}{32\pi} - \frac{2\pi J_N^2}{A^\pm} - \frac{2\pi N^4}{A^\pm} - \frac{2\pi J^2}{A^\pm} - \frac{\pi Q^4}{2A^\pm} + \frac{2\pi N^2 Q^2}{A^\pm} \right).$$

(90)

$$\omega^\pm = \frac{4\pi J_N}{MA^\pm},$$

(91)

$$\Omega^\pm = \frac{4\pi a}{A^\pm},$$

(92)

$$\psi^\pm = -\frac{N}{M} \left( 1 - \frac{8\pi N^2}{A^\pm} + \frac{4\pi Q^2}{A^\pm} \right).$$

(93)
Φ_± = \frac{1}{M} \left( \frac{2\pi Q^3}{A_±} - 4\pi Q N^2 A_± + \frac{Q}{2} \right) . \tag{94}

and

\begin{align*}
T_± &= \text{Effective surface tension of } \mathcal{H}_± \text{ for KN-Taub-NUT BH} \\
ω_± &= \text{Angular velocity of } \mathcal{H}_± \text{ due to NUT parameter} \\
Ω_± &= \text{Angular velocity of } \mathcal{H}_± \text{ due to spin parameter} \\
ψ_± &= \text{Electromagnetic potentials of } \mathcal{H}_± \text{ due to NUT parameter} \\
Φ_± &= \text{Electromagnetic potentials of } \mathcal{H}_± \text{ due to spin parameter}
\end{align*}

Similarly, we prove that this effective surface tension is proportional to the surface gravity of the horizons, i.e.

\begin{align*}
T_± &= \frac{r_± - M}{8\pi (2Mr_± + 2N^2 - Q^2)} = \frac{r_± - M}{8\pi (r_±^2 + N^2 + a^2)} = \frac{κ_±}{8\pi} , \tag{95}
\end{align*}

where κ_± is defined as the surface gravity of \( \mathcal{H}_± \) for KN-Taub-NUT BH.

Analogously, the bilinear form of the mass parameter is

\begin{align*}
M &= 2T_± A_± + 2J Ω_± + 2JN ω_± + ψ_± N + Φ_± Q . \tag{96}
\end{align*}

Like Kerr-Taub-NUT BH, this striking formula may be obtained by applying Euler’s theorem on homogeneous functions to \( M \), which is homogeneous of degree \( \frac{1}{2} \) in \( (A_±, J, JN, N^2, Q^2) \). Thus it can be rewritten as

\begin{align*}
M &= \frac{κ_±}{4\pi} A_± + 2J Ω_± + 2JN ω_± + ψ_± N + Φ_± Q . \tag{97}
\end{align*}

So, the Smarr-Gibbs-Duhem relation of \( \mathcal{H}_± \) for Kerr-Newman-Taub-NUT BH is

\begin{align*}
\frac{M}{2} &= T_± S_± + J Ω_± + JN ω_± + \frac{ψ_± N}{2} + \frac{Φ_± Q}{2} . \tag{98}
\end{align*}

Interestingly, \( T_±, ω_±, ψ_± \) and \( Φ_± \) should be defined and are constant on \( \mathcal{H}_± \).

Proceeding similarly, the surface energy \( E_s^± \) of \( \mathcal{H}_± \) for this BH can be defined by

\begin{align*}
E_s^± &= \int_0^{A_±} T_± (\tilde{A}_±, 0, 0, 0) d\tilde{A}_±; \tag{99}
\end{align*}

The rotational energy of \( \mathcal{H}_± \) due to spin parameter can be defined by

\begin{align*}
E_r^± &= \int_0^{J} Ω_± (A_±, \tilde{J}, 0, 0, 0) d\tilde{J}, \ A_± \text{ fixed}; \tag{100}
\end{align*}
The rotational energy of $\mathcal{H}^\pm$ due to NUT parameter can be defined by

$$\mathcal{E}_{r, J, J_N}^\pm = \int_0^{J_N} \omega_\pm (A_\pm, J, \tilde{J}_N, 0, 0) \, d\tilde{J}_N, \, A_\pm, J \text{ fixed}; \quad (101)$$

The electromagnetic energy of $\mathcal{H}^\pm$ due to NUT parameter ($N$) can be defined as

$$\mathcal{E}_{em, J, J_N}^\pm = \int_0^N \psi_\pm (A_\pm, J, J_N, \tilde{N}, 0) \, d\tilde{N}, \, A_\pm, J, J_N \text{ fixed}; \quad (102)$$

and the electromagnetic energy of $\mathcal{H}^\pm$ due to charge parameter ($Q$) can be defined as

$$\mathcal{E}_{em, J, J_N, Q}^\pm = \int_0^Q \Phi_\pm (A_\pm, J, J_N, N, \tilde{Q}) \, d\tilde{Q}, \, A_\pm, J, J_N, N \text{ fixed}; \quad (103)$$

These integrals may be directly computed using the variational definitions which are defined in Eqs. (90–94).

Proceeding similarly, the surface energy of $\mathcal{H}^\pm$ for Kerr-Newman-Taub-NUT BH is

$$\mathcal{E}_s^\pm = \sqrt{\frac{A_\pm}{16\pi}} \quad (104)$$

The rotational energy of $\mathcal{H}^\pm$ due to spin parameter

$$\mathcal{E}_{r, J}^\pm = \sqrt{\frac{A_\pm}{16\pi}} + \frac{4\pi J^2}{A_\pm} \quad (105)$$

The rotational energy of $\mathcal{H}^\pm$ due to NUT parameter is

$$\mathcal{E}_{r, J, J_N}^\pm = \sqrt{\frac{A_\pm}{16\pi}} + \frac{4\pi J^2}{A_\pm} + \frac{4\pi J_N^2}{A_\pm} - \sqrt{\frac{A_\pm}{16\pi}} + \frac{4\pi J^2}{A_\pm} \quad (106)$$

The electromagnetic energy of $\mathcal{H}^\pm$ due to NUT parameter is

$$\mathcal{E}_{em, J, J_N}^\pm = \sqrt{\frac{A_\pm}{16\pi}} + \frac{4\pi J^2}{A_\pm} + \frac{4\pi J_N^2}{A_\pm} - N^2 + \frac{4\pi N^4}{A_\pm}$$

$$- \sqrt{\frac{A_\pm}{16\pi}} + \frac{4\pi J^2}{A_\pm} + \frac{4\pi J_N^2}{A_\pm} \quad (107)$$

The electromagnetic energy of $\mathcal{H}^\pm$ due to the charge parameter is
Next we compute the sum of these energies

\[ \mathcal{E}_{s}^{\pm} + \mathcal{E}_{r}^{\pm}, J, J_{N} + \mathcal{E}_{em}^{\pm}, J, J_{N} + \mathcal{E}_{em}^{\pm}, J, J_{N}, Q = \sqrt{\frac{A_{\pm}}{16\pi} + \frac{4\pi J_{N}^{2}}{A_{\pm}} - N^{2} + \frac{4\pi J^{2}}{A_{\pm}} + \frac{\pi Q^{4}}{A_{\pm}} - \frac{4\pi N^{2} Q^{2}}{A_{\pm}} + \frac{Q^{2}}{2}}. \]  

(108)

Using Eq. (88), we can rewrite the above equation as

\[ \mathcal{E}_{s}^{\pm} + \left( \mathcal{E}_{r}^{\pm}, J + \mathcal{E}_{r}^{\pm}, J, J_{N} \right) + \left( \mathcal{E}_{em}^{\pm}, J, J_{N} + \mathcal{E}_{em}^{\pm}, J, J_{N}, Q \right) = M \left( A_{\pm}, J, J_{N}, N, Q \right). \]  

(109)

Remarkably, for KN-Taub-NUT BH the Komar mass can also be expressed as sum of these energies namely the surface energy of \( \mathcal{H}^{\pm} \), the rotational energy of \( \mathcal{H}^{\pm} \), the electromagnetic energy of \( \mathcal{H}^{\pm} \) due to NUT parameter and the electromagnetic energy of \( \mathcal{H}^{\pm} \) due to charge parameter. It should be noted that in the limit \( N = 0 \), one obtains the energy formula for Kerr-Newman BH [2,3] i.e.

\[ \mathcal{E}_{s}^{\pm} + \mathcal{E}_{r}^{\pm}, J + \mathcal{E}_{em}^{\pm}, J, Q = \sqrt{\frac{A_{\pm}}{16\pi} + \frac{4\pi J^{2}}{A_{\pm}} + \frac{\pi Q^{4}}{A_{\pm}} + \frac{Q^{2}}{2}} = M \left( A_{\pm}, J, Q \right). \]  

(111)

6 Conclusions

It has been suggested that a generic four dimensional Taub-NUT BH [1] should be completely specified in terms of three or four different types of thermodynamic hairs. They are defined as the Komar mass \( (M = m) \), the angular momentum \( (J_{n} = mn) \), the gravitomagnetic charge \( (N = n) \), the dual (magnetic) mass \( (\tilde{M} = n) \). Under this formalism, we derived the surface energy of \( \mathcal{H}^{\pm} \), the rotational energy of \( \mathcal{H}^{\pm} \) and the electromagnetic energy of \( \mathcal{H}^{\pm} \) for NUT class of BHs. Interestingly, we showed that like Kerr-Newman BH [2,3,27] the mass parameter could be expressed as the sum of these three energies i.e. \( M = \mathcal{E}_{s}^{\pm} + \mathcal{E}_{r}^{\pm} + \mathcal{E}_{em}^{\pm} \). We have explicitly examined for Taub-NUT BH, RN-Taub-NUT BH, Kerr-Taub-NUT BH and KN-Taub-NUT BH. In each case of the NUT class of BHs, we showed that the sum of these three energies
is equal to the Komar mass. It has been achieved only due to the introduction of new conserved charges $J_N = MN$, which is closely analogous to Kerr-like angular momentum parameter ($J = aM$).

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