THE FULL HOLONOMY GROUP UNDER THE RICCI FLOW

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Abstract. We give a short, direct proof that the full holonomy group of a solution to the Ricci flow is invariant up to isomorphism using the invariance of the reduced holonomy under the flow.

1. Introduction

It is a classical observation of Hamilton [H2, H3] that the reduced holonomy \( \text{Hol}^0(g(t)) \) of a solution \( g(t) \) to the Ricci flow cannot expand: if it is initially restricted to some subgroup \( G \subset \text{SO}(n) \), then it remains so, provided that the solution is complete and of bounded curvature or otherwise belongs to some class in which the equation is suitably well-posed. This may be proven with entirely general ingredients. After passing to the universal cover and using Berger’s classification, it is only necessary to verify that Einstein, product, and Kähler structures are preserved by the flow. Using the short-time existence theorems of [H1, S1, S2] where needed, one can construct complete Einstein, product, and Kähler solutions to the flow starting from given initial data with those characteristics. The uniqueness results of Hamilton and Chen-Zhu [H1, CZ] then imply that these special solutions are the only solutions within the class with the given initial data.

Under the same assumptions on \( g(t) \), the second author later showed [K2] that \( \text{Hol}^0(g(t)) \) also cannot contract and therefore remains isomorphic to \( \text{Hol}^0(g(0)) \) for all time. In this case, the problem does not reduce in the same way to one of backward uniqueness of solutions to the Ricci flow. Although it is still only necessary to verify that the above three special structures are preserved under the flow, it is not in general possible (except in the Einstein case) to solve the parabolic terminal-value problems needed to obtain solutions with these special structures to compare against the original solution. However, the problem can be framed instead as one of backward uniqueness of the solutions to a related prolonged system which may in turn be treated by the general methods of [K1, K4]. This formulation of the problem as one for a prolonged system also leads to an alternative proof of the non-expansion of \( \text{Hol}^0(g(t)) \).

The purpose of this note is to give a direct and unified proof that the full holonomy \( \text{Hol}(g(t)) \) of a complete solution \( g(t) \) to the Ricci flow of bounded curvature is likewise invariant (up to isomorphism) forward and backward in time. In fact, we will show that the preservation of \( \text{Hol}(g(t)) \) follows from that of \( \text{Hol}^0(g(t)) \).

Theorem 1.1. Let \( g(t) \) be a solution to the Ricci flow on \( M \times [0, T] \) such that \( (M, g(t)) \) is complete for each \( t \in [0, T] \), and \( \sup_{M \times [0, T]} |\text{Rm}((x, t))| < \infty \). Then, for all \( q \in M \) and \( t \in [0, T] \),

\[
\text{Hol}_q(g(t)) = \psi_t \circ \text{Hol}_q(g(0)) \circ \psi_t^{-1},
\]

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where $\psi_t \in O(T_qM, g_q(t))$ is a family of isometries satisfying

$$
\frac{d\psi_t}{dt} = Rc \circ \psi_t, \quad \psi_0 = \text{Id},
$$

for $t \in [0, T]$. Here $Rc = Rc(g_q(t)) : T_qM \to T_qM$.

In particular, Theorem 1.1 implies that if a complete Ricci flow with bounded curvature is Kähler or splits as a product on any time-slice, it will possess these properties at all times. A short additional argument (see Section 4) shows that the complex and product structures will in these cases be independent of time.

Of course, the preservation of Kähler and product structures forward in time is a well-known property of the Ricci flow. We have already sketched one proof of this fact above; one feature of the argument below (when used in conjunction with [K2]) is that it does not make use of the short-time existence and uniqueness of solutions to the equation. In [K3], the second author has also previously shown that the preservation of global Kählerity backward in time follows from the preservation of local Kählerity under the flow. Our proof of Theorem 1.1 is in some sense a generalization of the argument given there. Further results concerning the preservation of the Kähler property for Ricci flows with instantaneously bounded curvature can be found in [HT, LS].

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2. The preservation of reduced holonomy and a reformulation

Let $g(t)$ be a smooth solution to the Ricci flow

$$
\frac{\partial}{\partial t}g = -2 \text{Rc}(g),
$$

on $M \times [0, T]$. The holonomy groups $\text{Hol}^0_p(g(t))$ and $\text{Hol}_p(g(t))$ based at a point $p \in M$ are naturally represented as subgroups of the orthogonal group $O(T_pM, g_p(t))$ relative to the time-varying inner product $g_p(t)$ at $p$. Using Uhlenbeck’s trick, we can transform Theorem 1.1 into an equivalent statement for a family of connections whose holonomy groups are instead realized as subgroups of some fixed representation of the orthogonal group.

2.1. Uhlenbeck’s trick. Fix $t_0 \in [0, T]$ and let $E$ be a vector bundle isomorphic to $TM$ by some fixed isomorphism $\iota_{t_0} : E \to TM$. Using $\iota_{t_0}$, we equip $E$ with the bundle metric $h$ defined by

$$
h_q(V, W) = g_q(t_0)(\iota_{t_0}V, \iota_{t_0}W)
$$

for $q \in M$ and $V, W \in E_q$. Then, we extend $\iota_{t_0}$ forward and backward in time as the solution to the fiber-wise ODE

$$
\frac{\partial \iota_t}{\partial t} = \text{Rc} \circ \iota_t,
$$

where $Rc = Rc(g_q(t))$, to obtain a family $\iota_t : E \to TM$ of bundle isomorphisms for $t \in [0, T]$. With this extension, $\iota_t : (E, h) \to (TM, g(t))$ is in fact a bundle isometry for each $t \in [0, T]$.
Let $\nabla = \nabla^t$ denote the Levi-Civita connection of $g(t)$. We will study the holonomy of $\nabla$ via that of the family of pull-back connections $\overline{\nabla} = \overline{\nabla}^t$ on $E$ defined by
\begin{equation}
(2.3) \quad \overline{\nabla}_X V = \iota_t^{-1}(\nabla^t_X (\iota t V))
\end{equation}
for $X \in TM$ and $V \in C^\infty(E)$. The metric $h$ is compatible with the connection $\overline{\nabla}^t$, and the holonomy groups of $\overline{\nabla}^t$ and $\nabla^t$ are related by
\begin{equation}
(2.4) \quad \Hol_q(\overline{\nabla}^t) = \iota_t^{-1} \circ \Hol_q(\nabla^t) \circ \iota_t.
\end{equation}

2.2. Preservation of reduced holonomy. As discussed in the introduction, the results in [H3], [K2] imply that when $(M, g(t))$ is complete and of uniformly bounded curvature, the reduced holonomy group of $\overline{\nabla}^t$ is isomorphic to the reduced holonomy of $\nabla^0_t$ for any $t$, $t_0 \in [0, T]$. While the analytic arguments used in the verification of this fact for $t < t_0$ are fundamentally different from those used for the case $t_0 < t$, the backward-time and forward-time problems can still be formulated in a unified way in terms of the image of the curvature operator $\text{Rm} = \text{Rm}(g(t))$ of the solution in the bundle of two-forms.

Let $\mathfrak{hol}(\overline{\nabla}^t)$ be the subbundle of $\text{End}(T M)$ whose fiber $\mathfrak{hol}_q(\overline{\nabla}^t) \subset \mathfrak{so}(T_q M, g_q(t))$ at $q$ is the Lie algebra of $\text{Hol}_q^0(\overline{\nabla}^t) \subset \text{O}(T_q M, g_q(t))$. Let $\mathcal{H}(\overline{\nabla}^t) \subset \wedge^2 T^* M$ be the bundle of two-forms isomorphic to $\mathfrak{hol}(\overline{\nabla}^t)$ via the correspondence
\begin{equation*}
A \in \mathfrak{hol}_q(\overline{\nabla}^t) \mapsto g_q(t)(A \cdot, \cdot) \in \mathcal{H}_q(\overline{\nabla}^t),
\end{equation*}
and let $\mathfrak{hol}(\overline{\nabla}) \subset \text{End}(E)$ and $\mathcal{H}(\overline{\nabla}) \subset \wedge^2 E^*$ denote the analogous families of bundles relative to the connection $\overline{\nabla}$.

In Theorem 1.4 and Appendix A of [K2] (compare Theorem 4.1 of [H3]), it is shown that $\mathcal{H}(\overline{\nabla}^t)$ is time-invariant, by showing first that the family of subbundles
\begin{equation*}
H(t) = (\iota_t)_*, \mathcal{H}(\overline{\nabla}^0) \subset \wedge^2 (T^* M)
\end{equation*}
is a $\nabla^t$-parallel subalgebra which contains the image of $\text{Rm}(g(t))$. From this, we deduce that $H(t)$ must coincide with $\mathcal{H}(\overline{\nabla}^t)$. Then, using the definition of $\iota_t$ and the fact that $H(t)$ contains the image of $\text{Rm}(g(t))$, one verifies that $H(t)$ must actually be independent of time. Thus,
\begin{equation*}
\mathcal{H}(\overline{\nabla}^t) = H(t) = H(t_0) = \iota_{t_0}^* \mathcal{H}(\overline{\nabla}^0) = \mathcal{H}(\overline{\nabla}^0).
\end{equation*}
But, $\mathcal{H}(\overline{\nabla}) = \iota_t^* \mathcal{H}(\overline{\nabla}^t)$, so
\begin{equation*}
\mathcal{H}(\overline{\nabla}) = \iota_t^* H(t) = \mathcal{H}(\overline{\nabla}^0),
\end{equation*}
and $\mathcal{H}(\overline{\nabla})$ is also independent of time.

Whereas the fibers of $\mathfrak{hol}(\overline{\nabla}^t)$ are related to the fibers of $\mathcal{H}(\overline{\nabla}^t)$ via the time-dependent isomorphisms $A \mapsto g(t)(A \cdot, \cdot)$, the fibers of $\mathfrak{hol}(\overline{\nabla})$ and $\mathcal{H}(\overline{\nabla})$ are related by the time-independent isomorphism $A \mapsto h(A \cdot, \cdot)$. Thus, the fibers $\mathfrak{hol}_q(\overline{\nabla}) \subset \mathfrak{so}(E_q, h)$ are also independent of time, and it follows that the same is true of $\text{Hol}_q^0(\overline{\nabla}) \subset \text{O}(E_q, h)$. 

Theorem 2.1 (H3, K2). Let $g(t)$, $\nabla^t$, and $\nabla^t$ be as above, and assume that $(M, g(t))$ is complete and of uniformly bounded curvature for $t \in [0, T]$. Then $\mathfrak{hol}_q(\nabla^t) \subset \mathfrak{so}(E_q, h)$ is independent of $t$ for all $q \in M$. Hence, 
\begin{align*}
\text{Hol}_q(\nabla^t) = \text{Hol}_q(\nabla^0), \quad \text{Hol}_q(\nabla^t) = \psi_t \circ \text{Hol}_q(\nabla^{t_0}) \circ \psi_t^{-1},
\end{align*}
for all $q \in M$, $t \in [0, T]$, where $\psi_t = t_t \circ t_0^{-1}$.

Theorem 2.1 can now be restated in terms of the family of connections $\nabla^t$.

Theorem 2.2. Provided $(M, g(t))$ is complete and of uniformly bounded curvature, 
\begin{align*}
\text{Hol}_q(\nabla^t) = \text{Hol}_q(\nabla^0)
\end{align*}
for all $t \in [0, T]$ and $q \in M$. Consequently, 
\begin{align*}
\text{Hol}_q(\nabla^t) = \psi_t \circ \text{Hol}_q(\nabla^{t_0}) \circ \psi_t^{-1},
\end{align*}
where $\psi_t = t_t \circ t_0^{-1}$.

3. Invariance of the full holonomy group

Given a piecewise smooth curve $\gamma : [a, b] \to M$, we will use $D_s = D_s^t$ to denote the covariant derivative along $\gamma$ induced by $\nabla = \nabla^t$. We will temporarily suppress the subscript $t$ on the maps $\alpha = t_\alpha$.

The key to the proof of Theorem 2.2 is the following identity.

Proposition 3.1. Let $\gamma : [a, b] \to M$ denote a smooth curve and $V = V(s, t)$ be a smooth family of smooth sections of $E$ along $\gamma = \gamma(s)$ which is parallel along $\gamma$ with respect to $D_s = D_s^t$ for all $t \in [0, T]$. Then $V$ satisfies 
\begin{align*}
D_s \frac{\partial}{\partial t} V = -i^{-1} \text{div} \text{Rm}(\dot{\gamma})eV.
\end{align*}

Here, div $\text{Rm}$ is the section of $T^*M \otimes \text{End}(TM)$ defined as follows: for any $p \in M$ and $X \in T_pM$, div $\text{Rm}(X) \in \text{End}(T_pM)$ acts on $Y \in T_pM$ by
\begin{align*}
\text{div} \text{Rm}(X)Y = \sum_{i=1}^n \nabla_{e_i} R_p(e_i, X)Y,
\end{align*}
where $\{e_i\}_{i=1}^n$ is a $g(t)$-orthonormal basis of $T_pM$. Note that div $\text{Rm}(X) \in \mathfrak{hol}_p(\nabla^t)$ for any $X \in T_pM$. Indeed, the curvature endomorphisms $R_p \in T^{(3,1)}(T_pM)$ belong to $\mathcal{H}_p(t) \otimes \mathfrak{hol}_p(\nabla^t)$, and so do $\nabla_{X_1} \nabla_{X_2} \cdots \nabla_{X_k} R_p$ for any $X_1, X_2, \ldots, X_k \in T_pM$.

Proof of Proposition 3.1. Let $s_0 \in [a, b]$ be fixed. Choose local frames $(E_\alpha)_{\alpha=1}^n$ for $E$ and $(e_i)_{i=1}^n$ for $TM$ on a neighborhood of $\gamma(s_0)$, and let $\bar{\Gamma}_{i\alpha}^\beta$ be the coefficients of $\nabla$ in terms of these frames, i.e., $\nabla_{e_i} E_\alpha = \bar{\Gamma}_{i\alpha}^\beta E_\beta$.

First, since $V(\cdot, t)$ is parallel for all $t$, 
\begin{align*}
0 = \frac{\partial}{\partial t} D_s V = \frac{\partial^2 V_\alpha}{\partial s \partial t} E_\alpha + \frac{\partial V_\alpha}{\partial t} \nabla_\gamma_\alpha E_\alpha + \gamma^t V_\alpha \frac{\partial \bar{\Gamma}_{i\alpha}^\beta}{\partial t} E_\beta
\end{align*}
at any $(s_0, t)$. On the other hand, for $s$ near $s_0$, 
\begin{align*}
\frac{\partial V}{\partial t}(s, t) = \frac{\partial V_\alpha}{\partial t}(s, t) E_\alpha(\gamma(s)),
\end{align*}
and so
\[ D_s \frac{\partial V}{\partial t} = \frac{\partial^2 V^\alpha}{\partial s \partial t} E_\alpha + \frac{\partial V^\alpha}{\partial t} \nabla_s E_\alpha \]
\[ = -\dot{i}^V \frac{\partial \Gamma^\beta_{is}}{\partial t} E_\beta \]
(3.1)
at \((s_0, t)\).
Now, fix \(\alpha\) and temporarily write \(X = i E_\alpha\). Then we have \(\frac{\partial}{\partial t} X = \text{Rc}(X)\), and, using that
\[ \frac{\partial \Gamma^k_{ij}}{\partial t} = \nabla^k R^l_{ji} - \nabla^i R^l_{jk}, \quad \nabla^k R^l_{ji} - \nabla^j R^l_{ki} = g^{lm} \nabla_l R^k_{mij}, \]
where \(\Gamma^k_{ij}\) denotes the components of \(\nabla\) in terms of \(\{e_i\}_{i=1}^n\), we see that
\[ \frac{\partial}{\partial t} \left( \nabla_{e_i} X \right) = \nabla_{e_i} \left( \frac{\partial}{\partial t} X \right) + \left( \frac{\partial \Gamma^k_{ij}}{\partial t} \right) X^j e_k \]
\[ = \nabla_{e_i} \text{Rc}(X) - (\nabla_{e_i} \text{Rc})(X) - (\nabla X \text{Rc})(e_i) + (\nabla^k \text{Rc})(e_i, X) e_k \]
\[ = \text{Rc}(\nabla_{e_i} X) + \text{div} \text{Rm}(e_i) X. \]
Thus,
\[ \frac{\partial \Gamma^\beta_{is}}{\partial t} E_\beta = \frac{\partial}{\partial t} \left( \nabla_{e_i} E_\alpha \right) \]
\[ = \frac{\partial}{\partial t} \left( i^{-1} \nabla_{e_i} t(E_\alpha) \right) \]
\[ = -i^{-1} \circ \text{Rc}(\nabla_{e_i} t(E_\alpha)) + i^{-1} \left( \frac{\partial}{\partial t} (\nabla_{e_i} t(E_\alpha)) \right) \]
\[ = i^{-1} \text{div} \text{Rm}(e_i) t E_\alpha. \]
Inserting this expression into (3.1) for \(D_s \frac{\partial V}{\partial t}\) completes the proof. \(\square\)

Next we use Proposition 3.1 to determine the evolution of parallel transport along a fixed loop.

**Proposition 3.2.** Let \(q \in M\) and let \(\gamma : [0, 1] \to M\) be a piecewise smooth loop with \(\gamma(0) = \gamma(1) = q\). Let \(P_{s,t} : E_q \to E_{\gamma(s)}\) be parallel translation along \(\gamma\) with respect to \(D_s = D^t_s\). Then
\[ \frac{\partial}{\partial t} P_{1,t} = P_{1,t} B \]
for some \(B = B(t) \in \mathfrak{hol}_q(\nabla)\).

**Proof.** It suffices to show that
\[ P_{1,t}^{-1} \frac{\partial}{\partial t} P_{1,t} \in \mathfrak{hol}_q(\nabla). \]
Let \(0 = a_0 < a_1 < \ldots < a_k = 1\) be such that \(\gamma|[a_{i-1}, a_i]\) is smooth and fix an arbitrary \(W \in E_q\). Applying the previous lemma to \(V = P_{s,t} W\) on any subinterval \([a_{i-1}, a_i]\), we find that
\[ \frac{d}{ds} \left( P_{s,t} \frac{\partial}{\partial t} P_{s,t} W \right) = P_{s,t}^{-1} \left( D_s \frac{\partial}{\partial t} P_{s,t} W \right) \]
\[ = -P_{s,t}^{-1} \left( i^{-1} \text{div} \text{Rm}(\gamma(s)) t(P_{s,t} W) \right). \]
In other words, 
\[ \frac{d}{ds} \left( P_{s,t}^{-1} \frac{\partial}{\partial t} P_{s,t} \right) = -P_{s,t}^{-1} \circ \tau^{-1} \circ \text{div} \, \text{Rm}_{\gamma(s)}(\dot{\gamma}) \circ \tau \circ P_{s,t} \div A(s,t). \]

But \( \text{div} \, \text{Rm}_{\gamma(s)}(\dot{\gamma}) \in \mathfrak{hol}_{\gamma(s)}(\nabla) \) for each \( s \), so \( \tau^{-1} \circ \text{div} \, \text{Rm}_{\gamma(s)}(\dot{\gamma}) \circ \tau \in \mathfrak{hol}_{\gamma(s)}(\nabla) \) for each \( s \). Since \( \mathfrak{hol}(\nabla) \) is invariant under parallel translation, it follows that \( A(s,t) \in \mathfrak{hol}_{\gamma(s)}(\nabla) \) for all \( s \in (a_{i-1}, a_i) \) and \( t \in [0, T] \).

Now let \( \mathfrak{hol}_q^\perp(\nabla) \) denote the orthogonal complement of \( \mathfrak{hol}_q(\nabla) \) in \( \text{End}(E) \) and let \( L \in \mathfrak{hol}_q^\perp(\nabla) \) be arbitrary. Then
\[ F(s) = \left\langle L, P_{s,t}^{-1} \frac{\partial}{\partial t} P_{s,t} \right\rangle_{\mathfrak{hol}_q} = \left\langle P_{s,t} \circ L \circ P_{s,t}^{-1}, \frac{\partial}{\partial t} P_{s,t} \circ P_{s,t}^{-1} \right\rangle_{\mathfrak{hol}_{\gamma(s)}} \]
is continuous on \([0, 1]\) and smooth on each interval \((a_{i-1}, a_i)\). For \( s \) in any such interval,
\[ F'(s) = \left\langle L, A(s,t) \right\rangle_{\mathfrak{hol}_q} = 0. \]
Thus \( F|_{[a_{i-1},a_i]} \) is constant for each \( i \).

But \( P_{0,t} = P_{0,0}^{-1} = \text{Id} \) for all \( t \), so \( P_{0,t}^{-1} \frac{\partial}{\partial t} P_{0,t} = 0 \) and \( F(0) = 0 \). Thus \( F(s) = 0 \) for all \( s \in [0, 1] \). Since \( L \in \mathfrak{hol}_q^\perp(\nabla) \) was arbitrary, it follows that
\[ B(t \div P_{1,t}^{-1} \frac{\partial}{\partial t} P_{1,t} \in \mathfrak{hol}_q(\nabla), \]
completing the proof. \( \square \)

We will use Proposition 3.2 in conjunction with the following simple fact.

**Lemma 3.3.** Suppose \( H \) is a Lie subgroup of the Lie group \( G \), \( B(t) \) is a smooth family of tangent vectors in \( T_eH \subset T_eG \) for \( t \in [0, T] \), and \( X = X(g,t) \) is the left-invariant extension of \( B(t) \) to \( G \) for each \( t \). If \( \alpha : [0, T] \to G \) is an integral curve of \( X \) passing through \( a \in H \) at \( t = t_0 \) then then \( \alpha(t) \in H \) for all \( t \in [0, T] \).

**Proof.** Since \( B(t) \in T_eH \), we may separately form the left-invariant extension \( \bar{X} \) of \( B(t) \) on \( H \) and obtain \( \bar{\alpha} : [0, T] \to H \) solving \( \bar{\alpha}'(t) = \bar{X}(\bar{\alpha}(t), t) \) with \( \bar{\alpha}(t_0) = a \). Then the inclusion \( \iota \circ \bar{\alpha} \) of \( \bar{\alpha} \) into \( G \) will be an integral curve of \( X \) passing through \( a \) at \( t = t_0 \) whose image lies in \( H \subset G \). By uniqueness, it must coincide with \( \alpha \). \( \square \)

Now we put the above pieces together to prove Theorem 2.2.

**Proof of Theorem 2.2.** Fix \( q \in M \) and \( t_0 \in [0, T] \). We will show that \( \text{Hol}_q(\nabla') \subset \text{Hol}_q(\nabla'^0) \) for all \( t \in [0, T] \). Let \( \gamma : [0,1] \to M \) be an arbitrary piecewise-smooth loop based at \( q \) and let \( P(t) = P_{q,t} : E_q \to E_q \) be parallel translation along \( \gamma \) with respect to the covariant derivative \( D_s = D^t_s \) relative to \( \nabla' \). By Proposition 3.2, \( \frac{\partial P}{\partial t} = PB \) for some \( B = B(t) \) in the time-invariant subalgebra \( \mathfrak{hol}_q(\nabla') = \mathfrak{hol}_q(\nabla'^0) \subset \mathfrak{so}(E_q, h) \).

Applying Lemma 3.3 with \( H = \text{Hol}_q(\nabla'^0) \), \( G = O(E_q, h) \), \( \alpha(t) = P(t) \), and \( a = P(t_0) \), we see that \( P(t) \in \text{Hol}_q(\nabla'^0) \) for all \( t \). But \( P(t) \) represents \( \nabla'^0 \)-parallel translation along an arbitrary admissible loop \( \gamma \) based at \( q \), so \( \text{Hol}_q(\nabla') \subset \text{Hol}_q(\nabla'^0) \) for all \( t \) as claimed. \( \square \)
4. Preservation of Parallel Tensors

One consequence of Theorem 1.1 is that if \( g(t) \) is a complete solution to the Ricci flow of uniformly bounded curvature on \( M \times [0, T] \) and \( A_0 \) is a smooth \( \nabla^t \)-parallel tensor for some \( t_0 \in [0, T] \), then there is a smooth family \( A(t) \) of \( \nabla^t \)-parallel tensors on \( M \times [0, T] \) with \( A(t_0) = A_0 \).

**Corollary 4.1.** If the tensor field \( A_0 \in C^\infty(T^{k,l}(M)) \) is \( \nabla^{t_0} \)-parallel for some \( t_0 \in [0, T] \), then \( A(t) = (\iota_t)_* \iota_{t_0}^* A_0 \) is \( \nabla^t \)-parallel for all \( t \).

Indeed, the section \( B_0 = \iota_{t_0}^* A_0 \) of the corresponding tensor product of \( E \) is \( \nabla^{t_0} \)-parallel, and Theorem 2.2 shows that \( \text{Hol}_g(\nabla^t) \) is independent of time for each \( g \). So \( B_0 \) is \( \nabla^t \)-parallel, and \( A(t) = (\iota_t)_* B_0 \) therefore \( \nabla^t \)-parallel, for each \( t \in [0, T] \). The family \( A(t) \) in Corollary 4.1 can be explicitly described as the solution of the fiberwise linear system

\[
\frac{\partial}{\partial t} A^{a_1 \ldots a_l}_{b_1 \ldots b_k} = R^c_{b_1} A^{a_1 \ldots a_l}_{c b_2 \ldots b_k} + \cdots + R^c_{b_k} A^{a_1 \ldots a_l}_{b_1 b_2 \ldots b_{k-1} c} - R^c_{c b_1} A^{a_1 \ldots a_l}_{b_2 \ldots b_k} - \cdots - R^c_{c b_k} A^{a_1 \ldots a_l}_{b_1 \ldots b_{k-1}},
\]

\[
a(t_0) = A_0,
\]

of ordinary differential equations.

In some cases, the extended family of parallel tensors \( A(t) \) will be independent of time. This is true, for example, when the time-slice \( (M, g(t_0)) = (\check{M} \times \check{M}, \check{g} \oplus \check{g}) \) is a Riemannian product, and \( A_0 \) is one of the associated complementary orthogonal projections \( \check{P}, \check{P} \in \text{End}(T\check{M}) \). It is also true when the time-slice \( (M, g(t_0)) \) is Kähler and \( A_0 = J \) is its complex structure.

### 4.0.1. Product structures

Suppose \( M = \check{M} \times \check{M} \) and \( g(t_0) = \check{g} \oplus \check{g} \) is a Riemannian product. Let \( \check{P}_0 \) and \( \check{P}_0 \) denote the orthogonal projections onto the subbundles of \( T\check{M} \) isomorphic to \( T\check{M} \) and \( T\check{M} \), respectively. These sections of \( \text{End}(T\check{M}) \) are parallel at \( t = t_0 \), and therefore, by Corollary 4.1, their extensions defined by

\[
\frac{\partial \check{P}}{\partial t} = \check{P} \circ \text{Re} - \text{Re} \circ \check{P}, \quad \check{P}(t_0) = \check{P}_0,
\]

\[
\frac{\partial \check{P}}{\partial t} = \check{P} \circ \text{Re} - \text{Re} \circ \check{P}, \quad \check{P}(t_0) = \check{P}_0,
\]

are \( \nabla^t \)-parallel for each \( t \). It also follows directly from (4.1) that \( \check{P}(t) \) and \( \check{P}(t) \) will remain complementary \( g(t) \)-orthogonal projections. But these properties imply that \( \check{P} \circ \text{Re} = \text{Re} \circ \check{P} \) and \( \check{P} \circ \text{Re} = \text{Re} \circ \check{P} \) identically on \( M \times [0, T] \). So \( \frac{\partial \check{P}}{\partial t} = \frac{\partial \check{P}}{\partial t} = 0 \), and the product structure these projections define is constant in time.

### 4.0.2. Complex structures

Similarly, suppose \( (M, g(t_0)) \) is Kähler with complex structure \( J_0 \). As above, the family \( J = J(t) \) defined by

\[
\frac{\partial J}{\partial t} = J \circ \text{Re} - \text{Re} \circ J, \quad J(t_0) = J_0,
\]

will be \( \nabla^t \)-parallel and will satisfy \( J^2 = -\text{Id} \) and \( g(J \cdot, J \cdot) = g(\cdot, \cdot) \) for all \( t \). However, these conditions likewise imply that \( \text{Re} \circ J = J \circ \text{Re} \) for each \( t \), and hence that \( \frac{\partial J}{\partial t} = 0 \), so that \( (M, g(t)) \) is, in fact, Kähler relative to the fixed complex structure \( J_0 \) for all \( t \). (Compare Section 3 of [K3].)
References

[CZ] B.-L. Chen and X.-P. Zhu, *Uniqueness of the Ricci flow on complete noncompact manifolds*, J. Diff. Geom. 74 (2006), no. 1, 119–154.

[H1] R. S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Diff. Geom. 17 (1982), no. 2, 255–306.

[H2] R. S. Hamilton, *Four-manifolds with positive curvature operator*, J. Diff. Geom. 24 (1986), no. 2, 153–179.

[H3] R. S. Hamilton, *The formation of singularities in the Ricci flow*, Surveys in differential geometry, Vol. II (Cambridge, MA, 1993), 7–136, Int. Press, Cambridge, MA, 1995.

[HT] S. Huang and L.-F. Tam, *Kähler-Ricci flow with unbounded curvature*, Amer. J. Math. 140 (2018), no. 1, 189–220.

[K1] B. Kotschwar, *Backwards uniqueness of the Ricci flow*, Int. Math. Res. Not. (2010), no. 21, 4064–4097.

[K2] B. Kotschwar, *Ricci flow and the holonomy group*, J. Reine Angew. Math. 690 (2014), 133–161.

[K3] B. Kotschwar, *Kählerity of shrinking gradient Ricci solitons asymptotic to Kähler cones*, J. Geom. Anal., 28 (2018), no. 3, 2609–2623.

[K4] B. Kotschwar, *A short proof of backward uniqueness for some geometric evolution equations*, Int. J. Math. 27 (2016), no. 12, 1650102, 17 pp.

[LS] G. Liu and G. Székelyhidi, *Gromov-Hausdorff limits of Kähler manifolds with Ricci curvature bounded below*, arXiv:1804.08567 [math.DG], (2018).

[S1] W.-X. Shi, *Deforming the metric on complete Riemannian manifolds*, J. Diff. Geom. 30 (1989), no. 1, 223–301.

[S2] W.-X. Shi, *Ricci flow and the uniformization on complete noncompact Kähler manifolds*, J. Diff. Geom. 45 (1997), no. 1, 94–220.

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