Deformed state of viscoelastic bodies in one problem of tidal interaction

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Abstract. In our previous articles we considered the motion of two viscoelastic bodies in the gravitational field of a massive material point. We studied the evolution of the bodies, stationary solutions and their stability. We solved the equations of translational-rotational motion of the bodies together with the equations of their deformed state, using the method proposed by V G Vil’ke for the systems with infinite numbers of degrees of freedom. In our manuscript we continue our research and our aim is to get the global characteristics of the bodies, deformed by tidal forces: their shape, moments of inertia, the tidal lag angle, the relationship between the viscosity coefficient and the lag angle, tidal torques, internal material properties, depending on elastic constant and viscosity coefficient. It is very important for study of an evolutionary motion of the celestial bodies, since tides affect this. In order to achieve this goal we use the tidal potential perturbation and displacements of each point of the bodies. The formulation of the problem, goal set and used methods are the novelty of our research. We applied our theory to the Earth, and it was found that the Earth’s substance (in global) behaves like an auxetic.

1. Introduction
This manuscript is continuation of other our papers. In work [1] we formulated problem of motion of two viscoelastic bodies in the gravitational field of massive material point, solved the equations of motion and applied them for investigating the tidal evolution of the Earth – Moon system in the field of the Sun. In work [2] we studied the stationary solutions of these equations of motions and their stability.

As usually in researching of tides are used the classical methods of celestial mechanics or their modifications, for example, generalisation of the Darwin-Kaula theory for tides in a body librating about a spin-orbit resonance [3]. However, we use relatively “young” approach, developed by V G Vil’ke for system with infinite number of degrees of freedom [4], because we solve the equations of translational and rotational motions of the bodies together with equations of their deformation, the equations of the elasticity theory. This is the novelty of our paper for applying this method for such celestial-mechanical problem.

We find the displacements of each point of the bodies. They depend on the separations of the bodies, their orbital and rotational angular velocities, their inner structure, coefficient of viscosity and so on. These displacements change the surface of the bodies and influence in turn on the motion of the bodies. This is the novelty of our paper in comparison with other works on this subject.

We use the Kelvin-Voigt dissipation model. In article [5] it was considered a simple example of a Maxwell material. In work [6] it was used several rheological models, i.e. Andrade, extended Burgers,
Sundberg-Cooper, and power-law to compute the dissipative properties within Mars and calculate the global quality factor. It would be interesting to further apply these rheological models in our approach.

Tidal orbital evolution of the Mars-Phobos system and the Keplerian elements in two-body problem was studied in [6] and [7] respectively without taking into account their rotational motion. But we considered not only the orbital evolution of the Earth-Moon system but also their rotation evolution. We caught the moment when the Moon entered the spin-orbit resonance and continues to remain in it in the future. This is the advantage of the method used.

Papers [1, 2] and our manuscript are a single whole. So, using their results, we are going to get the global characteristics of the bodies, deformed by tidal forces: their shape, moments of inertia, tidal torques, the tidal lag angle, the relationship between the viscosity coefficient and lag angle, Love number $k_2$. The vast majority of works are limited, as a rule, Love number $k_2$, for example, during experiment simulation the Ganymede’s tides Love number $k_2$ is determined with an accuracy $10^{-4}$ [8]. We intend to apply our theory to the Earth and find global internal material properties, depending on elastic constants and coefficient of internal viscous friction. All the goals set are new, because the celestial-mechanical model and methods of its research are new.

2. Main notations and assumptions in statement of the problem

We consider the motion of two homogeneous and isotropic viscoelastic bodies (the Earth and the Moon) with masses $m_1$ and $m_2$, respectively, in the central field of massive material point $O$ (the Sun) with mass $M$ and $m_2 = m_1 < M$ relation. We connect with point $O$ inertial frame $OXYZ$, with points $O_1$—Koenig’s system of coordinates $OXYZ$ and $O_iX_iY_iZ_i$, system fixed in the rotating body, where $O_i$ is the centre of mass of the $i$-th body (everywhere in the text $i=1$ is reffered to the first body, $i=2$—the second body). The $OX$, $OY$, $O_iX_i$ and $O_iY_i$ axes lie in the orbital plane of motion of the bodies. $O_iZ$ and $OZ$, axes coincide, and are perpendicular to the orbital plane. $i$-th body rotates about the $OZ$ axis with angular speed. We denote $\varphi_i$ the angle between $O_iX_i$ and $OX$ axes. It is measured from $OX$ axis of Koenig’s system of coordinates $OXYZ$. Let $R_i(R_i \cos \lambda_i, R_i \sin \lambda_i, 0)$ denote the position vector of the centre of mass $C$ of the bodies in the inertial frame. Here $\lambda_i$ is the angle measure from the $OX$ axis, $R_i$ is the separation of $O_i$ and $C$ points. In the $OXYZ$ Koenig’s system vector $R_2(R_2 \cos \lambda_2, R_2 \sin \lambda_2, 0)$ denote the position vector of $O_2$ point, where $\lambda_2$ angle is measured from the $OX$ axis, $R_2$ is the separation of $O_2$ and $O_1$ points. The centre of mass $C$ moves with respect to origin $O$ of inertial frame with orbital angular velocity $\omega_1$. The second body moves about the first body with $\omega_2$ angular velocity.

In the unperturbed state the bodies are solid, homogeneous and spherical of density $\rho_i$ and $r_{i0}$ radius, and $r_{i0} \ll R_2 \ll R_i$ relation. The centre of mass $C$ moves in circular keplerian orbit with respect to origin $O$ of inertial frame with constant angular velocity $\omega_1$, the second body moves in circular keplerian orbit about the first body with $\omega_2$ constant angular velocity, $i$-th body rotates about the $OZ_i$ axis at $\omega_{2,i}$ constant rate. All four movements are independent of each other.

If the bodies are viscoelastic then the shape of each body is distorted by gravitational forces of massive point and another body treated as material point. Due to distribution of matter associated with displacements of the points at each body, the tidal potential perturbation (non-central part of the external potential) arises. We call it just perturbing potential. In the perturbed motion the centre of mass $C$ moves in quasicircular orbit with respect to massive point, the second body moves about the first body in quasicircular orbit too. All $\omega_1$, $\omega_2$, $\omega_3$, $\omega_4$ angular velocities are very slow function of time, since perturbations are small.
3. Viscoelastic displacements and perturbing potential

The full statement of the problem and solution of equation of motion is in our work [1]. We got there the displacements \( u_i \) of \( i \)-th deformed body, potential \( \Pi \) of the system and perturbing potential \( \Pi_p \).

\[
u_i(r,t) = u_{i1} + u_{i2} + u_{i3}
\]

The displacements of points \( u_{i1} \) are caused by centrifugal forces, \( u_{i2} \) – elastic displacements are caused by massive point and another body in \( i \)-th body, \( u_{i3} \) – dissipative displacements are caused by massive point and another body in \( i \)-th body. We will consider only non-stationary \( u_{i2} \) and \( u_{i3} \) displacements and \( u_{ins}(r,t) = u_{i1} + u_{i2} + u_{i3} \) their sum.

\[
u_{i2} = \sum_{k=1}^{3} f_{ki} G \frac{m_k}{r_i^3} \left[ a_{i1} (B_{ki} r_i r_j) + (a_{i2} r_i^2 + a_{i3} r_{i0}) B_{ki} r_j \right]
\]

\[
u_{i3} = \sum_{k=1}^{3} f_{kii} G \frac{m_i}{r_i^3} \left[ a_{i1} \left( \frac{d B_{ki}}{d \psi_{ki}} r_i r_j \right) r_i + (a_{i2} r_i^2 + a_{i3} r_{i0}^2) \frac{d B_{ki}}{d \psi_{ki}} r_j \right],
\]

\[
a_{i1} = (1 + v_i)/ (5v_i + 7), \ a_{i2} = -a_{i1}(2 + v_i), a_{i3} = a_{i1}(2v_i + 3),
\]

\[
B_{ki} = \frac{1}{6} \begin{pmatrix}
3\cos 2\psi_{ki} + 1 & 3\sin 2\psi_{ki} & 0 \\
3\sin 2\psi_{ki} & -3\cos 2\psi_{ki} + 1 & 0 \\
0 & 0 & -2
\end{pmatrix}
\]

Here \( f_{ki} = f = GM, f_{kii} = Gm_{k-1}, \) \( G \) – universal gravitational constant. In the \( OXYZ \) system vector \( r_i(x_i, y_i, z_i) \) denote the position vector of any point of the non-deformed \( i \)-th body. \( E_i \) – Young’s modulus, \( v_i \) – Poisson’s ratio, \( \chi_i \) – the coefficient of internal viscous friction of \( i \)-th body, \( \psi_{ki} = \lambda_k - \phi_i \). In formulae (2), (3) \( u_{a12}, u_{a13} \) – elastic and dissipative displacements are raised in \( i \)-th body by \( k \)-th material point, index of summation \( k = 1 \) means – the massive point, \( k = 2 \) – another body treated as material point different \( i \)-th body.

\[
\Pi = -m \int R_i - Gm_i m_j / R_{ij} + \Pi_p.
\]

We will consider \( \Pi_p \) as a function of \( u_{ins} \) only.

\[
\Pi_p(u_{ins}) = \sum_{k=1}^{3} \sum_{i=1}^{m} \Pi_{pki}(u_{ins}), \quad \Pi_{pki}(u_{ins}) = f_{ki} G \frac{m_k}{r_i^3} \left[ r_i u_{ins} - 3(\xi_{ki} r_i) (\xi_{ki} u_{ins}) \right] \rho i \, d r_i
\]

Here \( m = m_i + m_j, \) \( \Pi_{pki} \) – is the perturbing potential of \( k \)-th material point and \( i \)-th body, deformed by massive point and \( k \)-th material point; \( V_i \) – sphere of radius \( r_{i0} \) of the non-deformed \( i \)-th body, \( d r_i \) – the element of volume.

\[
\xi_{ki} = r_i (\cos \psi_{ki}, \sin \psi_{ki}, 0), \quad \tau_{21} = -1, \quad \tau_{21} = 1 (k \neq 2, i \neq 1).
\]

We will transform the sum \( u_{a12} + u_{a13} = u_{ins} \) – viscoelastic displacements are raised in \( i \)-th body by \( k \)-th material point. For this purpose we will introduce a new variable \( \alpha_{ki} \) and new matrix \( \tilde{B}_{ki} \) as follows

\[
tan 2\alpha_{ki} = 2\chi_i (\omega_k - \omega_{2i,i})
\]
\[ \mathbf{B}_{ki} = B_{ki} \left[ 1 + \frac{1}{2} \tan 2 \alpha_i \right] \left( \begin{array}{ccc} 3 \cos 2 \tilde{\psi}_{ki} + \cos 2 \alpha_i & 3 \sin 2 \tilde{\psi}_{ki} & 0 \\ 3 \sin 2 \tilde{\psi}_{ki} & -3 \cos 2 \tilde{\psi}_{ki} + \cos 2 \alpha_i & 0 \\ 0 & 0 & -2 \cos 2 \alpha_i \end{array} \right) , \]

where \( \tilde{\psi}_{ki} = \psi_{ki} - \alpha_i \).

Then \( \mathbf{u}_{\text{ins}} \) and \( \mathbf{u}_{\text{ins}}^{\text{ins}} \) take the form

\[ \mathbf{u}_{\text{ins}} = \sum_{k=1}^{2} \mathbf{u}_{\text{ins}}^{k} , \quad \mathbf{u}_{\text{ins}}^{k} = \rho_1 E_i^{-1} f_{ki} R_i^3 \left[ a_{11} (\mathbf{B}_{ki} r_i, r_i) r_i + (a_{12} r_i^2 + a_{13} r_i^3) \mathbf{B}_{ki} r_i \right]. \]

We substitute expression \( \mathbf{u}_{\text{ins}}^{(10)} \) in \( \Pi_{\text{pki}} (\mathbf{u}_{\text{ins}}^{(6)}) \) and after cumbersome and tedious calculations obtain tidal potential perturbation

\[ \Pi_{\text{pki}} (\mathbf{u}_{\text{ins}}^{(10)}) = -\frac{3}{2} \beta_{ki} \rho D_{12} \sum_{q=1}^{2} f_{qi} R_i^3 \left[ 3 \cos (\lambda_q - \alpha_q - \lambda_q) + 1 \right] , \]

where

\[ \beta_{ki} = \rho_1 E_i^{-1} f_{ki} R_i^3 , \quad D_{12} = 4 \pi r_i^3 a_{11} (9 v_r + 13) / 105 . \]

4. The shape of the body, deformed by the viscoelastic displacements

\[ \mathbf{u}_{\text{ins}} = \sum_{k=1}^{2} \mathbf{u}_{\text{ins}}^{k} , \quad \mathbf{u}_{\text{ins}}^{k} = \rho_1 E_i^{-1} f_{ki} R_i^3 \left[ a_{11} (\mathbf{B}_{ki} r_i, r_i) r_i + (a_{12} r_i^2 + a_{13} r_i^3) \mathbf{B}_{ki} r_i \right]. \]

Figure 1. Schematic presentations by dashed circular line the shape of \( i \)-th non-deformed body with center \( O_i \) and by solid line the shape of \( i \)-th body (oblate spheroid) deformed \( k \)-th material point (a) and (c). On part (b) the shape of \( i \)-th body is deformed by another body as material point and massive point. \( O_i X_i Y_i Z_i \) – coordinate system, fixed in \( i \)-th rotating body, \( O_i \tilde{X}_i \tilde{Y}_i \tilde{Z}_i \) – system, connected with deformed body (a). The \( O_i \tilde{X}_i \) axis is the axis of symmetry of deformed body (a) and (c). \( |\alpha_i| \) – tidal lag angle (a). The points \( P_i \) transit to points \( \tilde{P}_i \) in the process of deformation, \( \theta_i \) – polar angle of points \( P_i \) on the surface of \( i \)-th non-deformed body (c).

We will consider the shape of \( i \)-th body distorted by the only one \( k \)-th material point (figure 1a, 1c). The arbitrary point of the non-deformed body with position vector \( r_i (x_i, y_i, z_i) \) transits to the point with position vector \( r_k = r_i + \mathbf{u}_{\text{ins}} \) in \( O_i X_i Y_i Z_i \) coordinate system. We want to find the shape of the body
distorted by \( \mathbf{u}_{\text{m}} \) displacements. Let’s go to new \( O_i \bar{X}_i \bar{Y}_i \bar{Z}_i \) coordinate system. The axes \( O_i \bar{X}_i \) and \( O_i \bar{Y}_i \) lie in the plane \( O_i X_i Y_i \), the axes \( O_i \bar{Z}_i \) and \( O_i Z_i \) coincide. The angle between axes \( O_i \bar{X}_i \) and \( O_i X_i \) is equal to \( \psi_{ik} \). It is measured from \( O_i X_i \) axis. The transformation from \( O_i \bar{X}_i \bar{Y}_i \bar{Z}_i \) system to \( O_i X_i Y_i Z_i \) system is given by the matrix

\[
\mathbf{\Gamma}_i (\bar{\psi}_{ik}) = \begin{pmatrix}
\cos \bar{\psi}_{ik} & -\sin \bar{\psi}_{ik} & 0 \\
\sin \bar{\psi}_{ik} & \cos \bar{\psi}_{ik} & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

where \( \bar{\psi}_{ik} \) is the vector \( \bar{r}_i \) in \( O_i \bar{X}_i \bar{Y}_i \bar{Z}_i \) system.

Let \( r_{0i}(\bar{x}_{0i}, \bar{y}_{0i}, \bar{z}_{0i}) \) are the coordinates of an arbitrary point \( P_i \) on the surface of the non-deformed body (the sphere with radius \( r_{0i} \)) in \( O_i \bar{X}_i \bar{Y}_i \bar{Z}_i \) system. We introduce the angle \( \theta_i (0 \leq \theta_i \leq \pi) \) a such that \( \bar{x}_{0i} = r_{0i} \cos \theta_i, \sqrt{\bar{y}_{0i}^2 + \bar{z}_{0i}^2} = r_{0i} \sin \theta_i \) (figure 1c). In the process of deformation the points \( P_i(\bar{x}_{0i}, \bar{y}_{0i}, \bar{z}_{0i}) \) transit to \( \bar{P}_i(\bar{x}_{1i}, \bar{y}_{1i}, \bar{z}_{1i}) \) points (figure 1c). They form the surface of \( i \)-th body in \( O_i \bar{X}_i \bar{Y}_i \bar{Z}_i \) system distorted by \( k \)-th material point.

\[
\bar{x}_i(\theta) = r_{0i} \cos \theta \left[ 1 + 4 \gamma_{ik} r_{0i}^2 \left[ a_1 P_i(\cos \theta) + (a_2 + a_3) \right] \right],
\]

\[
\bar{R}_i(\theta) = \sqrt{\bar{x}_i^2 + \bar{z}_i^2} = r_{0i} \sin \theta \left[ 1 + 2 \gamma_{ik} r_{0i}^2 \left[ 2 a_1 P_i(\cos \theta) - (a_2 + a_3) \right] \right],
\]

\[
O_i \bar{P}_i(\theta) = \sqrt{\bar{x}_i^2 + \bar{z}_i^2} = r_{0i} \left[ 1 + 4 \gamma_{ik} r_{0i}^2 a_1 (2 + \nu_i) P_i(\cos \theta) \right].
\]

where

\[
\gamma_{ik} = \beta_{ik} / (2 \cos 2\alpha_i) = \rho E_i^{-1} f_i R_i^{-3} / (2 \cos 2\alpha_u).
\]

So, the deformed body is axisymmetric body of rotation at first approximation, an oblate spheroid, with \( O_i \bar{X}_i \) axis of symmetry. Its long semi-axis \( \bar{x}_i(0) \) is lying along the \( O_i \bar{X}_i \) axis and \( \bar{R}_i(\pi/2) \) short semi-axis gives a circular cross section.

\[
\bar{x}_i(0) = O_i \bar{P}_i(0) = r_{0i} \left[ 1 + 4 \gamma_{ik} r_{0i}^2 a_1 (2 + \nu_i) \right],
\]

\[
\bar{R}_i(\pi/2) = r_{0i} \left[ 1 - 2 \gamma_{ik} r_{0i}^2 a_1 (2 + \nu_i) \right].
\]

The largest amplitude \( \Delta \bar{x}_i(0) \) of the tidal bulge at \( \theta_i = 0 \) or \( \pi \) is equal:

\[
\Delta \bar{x}_i(0) = \bar{x}_i(0) - r_{0i} = 4 \gamma_{ik} r_{0i}^3 a_1 (2 + \nu_i).
\]

The moments of inertia of the deformed body are equal:

\[
\bar{A}_{ik} = \int_{V_i} \left( \bar{y}_i^2 + \bar{z}_i^2 \right) \rho \, dV_i = 0.4 m_i r_{0i}^2 - 8 \gamma_{ik} \rho_i D_{12},
\]

\[
\bar{B}_{ik} = \int_{V_i} \left( \bar{y}_i^2 + \bar{z}_i^2 \right) \rho \, dV_i = 0.4 m_i r_{0i}^2 + 4 \gamma_{ik} \rho_i D_{12},
\]

\[
\bar{C}_{ik} = \int_{V_i} \left( \bar{y}_i^2 + \bar{z}_i^2 \right) \rho \, dV_i = 0.4 m_i r_{0i}^2 + 4 \gamma_{ik} \rho_i D_{12}.
\]

We have calculated the volume \( V_i \) of the distorted \( i \)-th body.

\[
V_i = \frac{4}{3} \pi r_{0i}^3 \left[ 1 + 12 \gamma_{ik} r_{0i}^3 a_1 (1 + \nu_i) \sin^2 \alpha_u \right]
\]

The expressions (14)-(21) are obtained, neglecting the higher order terms \( \gamma_{ik}^2 \) and \( \gamma_{ik} \sin^2 \alpha_u \). But in the expression (22) we left the term \( \gamma_{ik} \sin^2 \alpha_u \) to show that the body volume increased. If we take them into account then the body shape will be triaxial. It is caused by viscous forces.
5. Tidal torques

So we have got the perturbing potential \( \Pi_{pki} (u_{ni}) \) (11) of \( k \)-th material point and \( i \)-th body, deformed by massive point and \( k \)-th material point, in other words, the potential of interaction of \( k \)-th material point with the tidal bulge arised by it \((q=k)\) and the bulge arised by massive point \((q \neq k)\) in \( i \)-th body (figure 1b). We can rewrite (11):

\[
\Pi_{pki} (u_{ni}) = -6 \beta_k \rho, D_{ij} \sum_{q=1}^{2} f_{q_i} R_q^3 P_z \cos (\lambda_q - \alpha_q - \lambda_i) . \tag{23}
\]

where \( P_z (\cos \theta) = (3 \cos 2\theta + 1)/4 \) – Legendre polynomial of degree two in \( \cos \theta \).

In (23) the angle \((\lambda_q - \alpha_q)\) is the angle between the axis of symmetry of \( i \)-th body deformed by \( q \)-th point and the axis \( OX \) of the Koenig’s coordinate system. It is measured from \( OX \) axis. Here the potential \( \Pi_{pki} \) can give the only one non-zero component \( T_{kiz} \) of the \( \{ T_{qkl}, T_{qkl} \} \) torque, which is directed along the \( O_i Z_i \) axis and is orthogonal to the plane of orbit:

\[
T_{qkl} = T_{klq} = 0 ,
\]

\[
T_{qkl} = \delta (-\Pi_{pki}) / \delta (\lambda_q - \alpha_q) = -9 \beta_k \rho, D_{ij} \sum_{q=1}^{2} f_{q_i} R_q^3 \sin 2 \left( (\lambda_q - \alpha_q) - \lambda_i \right) . \tag{24}
\]

We consider two cases for fixed \( q \).

In first case \( q \neq k \). Then the torque that acts from the \( k \)-th point on \( i \)-th body deformed by the \( q \)-th point is shortperiodic value and its component is equal

\[
T_{qklz} = -9 \beta_k \rho, D_{ij} f_{q_i} R_q^3 \sin 2 \left( (\lambda_q - \alpha_q) - \lambda_i \right) . \tag{25}
\]

This torque creates the tidal libration of \( i \)-th body and does not affect its evolution.

In second case \( q=k \). Then the torque, that acts from the \( k \)-th point on \( i \)-th body deformed by it, has longperiodic component and is equal

\[
T_{klz} = 9 \beta_k \rho, D_{ij} f_{k_i} R_q^3 \sin 2 \alpha_i . \tag{26}
\]

Since \( \sin 2\alpha_i \approx 2(\omega_i - \omega_{z+i}) \), it follows when angular rate \( \omega_{z+i} \) of \( i \)-th body is greater the orbital velocity \( \omega_k \) of \( k \)-th point, we have negative torque acting clockwise and slowing down the rotation of \( i \)-th body (a planet). The same, but positive torque (counterclockwise) acts on \( k \)-th point (the body) and accelerates its movement in orbit. This leads to the removal of \( k \)-th point (satellite) from \( i \)-th body (a planet). When angular rate \( \omega_{z+i} \) of \( i \)-th body is lower the orbital velocity \( \omega_k \) of \( k \)-th point, we have positive torque acting counterclockwise and accelerating the rotation of \( i \)-th body (a planet). The same, but negative torque (clockwise) acts on \( k \)-th point (satellite) and slowing down its movements in orbit. This causes \( k \)-th point (satellite) to approach to \( i \)-th body (a planet).

6. Comparison of results

1. Angle \( \alpha_i \) is the angle between the axis of symmetry of \( i \)-th body deformed by \( k \)-th point (tidal bulge) and the direction to the \( k \)-th point (figure 1a). In other words, the module of \( \alpha_i \) is the tidal lag angle

\[
|\alpha_i| \approx \chi_i |\omega_k - \omega_2z+i| . \tag{27}
\]

In work [9] we can see the tidal lag angle in the form

\[
\delta = \Delta t |\omega_p - n| . \tag{28}
\]

In this formula \( \Delta t \) is time lag, \( \omega_p \) is the planetary spin rate, \( n \) is the satellite’s mean motion. The formulas (27) and (28) are identical. So the coefficient internal viscosity is equal time lag:

\[
\chi_i = \Delta t . \tag{29}
\]
This is the mechanical meaning of coefficient internal viscosity $\chi_i$.

2. In work [10] (formulae 4.44) it is supposed the surface of a body deformed by tidal forces is described by equation

$$R(\theta') = C \left[ 1 + \varepsilon_i P_i \left( \cos \theta' \right) \right],$$

where $\varepsilon_i (<1)$ – constant, $C$ – indicates the average radius of the planet. But in our model we not supposed we proved this equality

$$O_{\theta_i} = r_i \left[ 1 + 4 \chi_i \varepsilon_i P_i \left( 2 + \nu_i \right) \right].$$

The expressions (30) and (31) are almost identical. The surface of deformed body is described by Legendre polynomial $P_i$ in both cases at first approximation.

3. Let us compare the torque $T_{kiiz}$ (26) and the same torque given by (4.159) in [10].

$$\Gamma = \frac{3}{2} k_i \frac{Gm^2}{a} C \sin 2\varepsilon_i,$$

where $k_i$ is Love number. In our notations, $m_i = m_k$, $a = R_k$, $C = r_0$, $\alpha_{ki} = \varepsilon_i$. We rewrite torque (32)

$$\Gamma = \frac{3}{2} k_i \frac{Gm^2}{R_k^2} r_0^3 \sin 2\alpha_{ki}.$$

We present the torque $T_{kiiz}$ (26) in the same structure as expression (33)

$$T_{kiiz} = \frac{3}{2} \left[ 8 \pi G \rho_i E_i^{-1} r_0^2 \left( 1 + \nu_i \right) \left( 9 \nu_i + 13 \right) / \left( 35 \left( 5 \nu_i + 7 \right) \right) \right] \frac{Gm^2}{R_k^2} r_0^3 \sin 2\alpha_{ki}. \tag{34}$$

Equating torques $\Gamma$ (33) and $T_{kiiz}$ (34), we can see that in our model $k_i$ has the following expression

$$k_i = 8 \pi G \rho_i E_i^{-1} r_0^2 \left( 1 + \nu_i \right) \left( 9 \nu_i + 13 \right) / \left( 35 \left( 5 \nu_i + 7 \right) \right). \tag{35}$$

7. Global values of Poisson’s ratio and viscosity coefficients for the Earth

It is known from observations that the mean value Love number $k_2 = 0.299$ for the Earth [10], gravitational constant $G = 6.67 \times 10^{-11} \text{m}^3 \text{kg}^{-1} \text{sek}^{-2}$, density $\rho_i = 5.52 \times 10^3 \text{kg} \text{m}^{-3}$, mean radius $r_0 = 6.37 \times 10^6 \text{m}$. But we don’t know Young’s modulus $E_i$ and $\nu_i$ – Poisson’s ratio. Let $E_i = 10^1 \text{N m}^{-2}$ Young’s modulus. This is a reasonable assumption. This modulus is a quarter more than marble and two times less than iron. Then we can find $E_i$ from equation (35). Its value $E_i = -0.73$ is negative. Is the Earth auxetic? What it means? How it may be explained? It may be explained from physicochemical point of view. Inside the Earth is a steam boiler. When we begin to stretch the globe from the poles, its outer walls will begin to thin out and internal pressure will begin lead to the expansion of the body. Formula (22) confirms this: body volume increased.

We got that coefficient of internal friction $\chi_i$ is equal to time lag and we would like to estimate its value for the Earth. It follows from (27)

$$\chi_i \approx \left| \alpha_{ki} / \left( r_0 - r_2 \right) \right|. \tag{36}$$

The tidal lag angle $\alpha_{ki} = 2.52^\circ = 0.044 \text{rad}$ for Moon’s tides on the Earth, orbital angular velocity the Moon $\omega_k = 266 \times 10^{-8} \text{rad/s}$, angular polar rate of rotation the Earth $\omega_{2vi} = 727 \times 10^{-7} \text{rad/s}$. Hence we obtain from (36) viscosity coefficient $\chi_i = 629 \text{s} = 0.007 \text{d}.$

8. Conclusion

The very problem of creating an analytical model of the planet (the Earth), deformed by the tide, is determined by the complexity of its internal structure. In our model we find the global characteristics of
the bodies. In scientific literature the most popular version of internal structure of the Earth is given by Love [11]. He considered a spherical, homogeneous, incompressible body and showed that its surface must be described the second zonal harmonic $P_2 \left( \cos \theta \right)$. We studied more realistic model of viscoelastic body and found that its surface also is described by the same harmonic at first approximation. But when we take into account viscous forces then its shape is triaxial.

We have discovered the effect of the tidal libration, when on the tidal bulge caused, for example, on the Earth by the Moon is acting the torque from the Sun, formula (25). It means the libration of the tidal lag angle with $T = \pi / (\omega_2 - \omega_1)$ period, where $\omega_2$ is the orbital angular velocity the Moon about the Earth and $\omega_1$ is the orbital angular velocity the Earth about the Sun.

We have obtained the new formula for the tidal lag angle (27), which is equal to the product of the viscosity coefficient of a given body and the absolute value of the difference in the angular velocities of the interacting bodies. The mechanical meaning of viscosity coefficient is that it is equal tidal time lag (29).

We have set very interesting fact that the Earth’s substance (in global) behaves like an auxetic. It means when the body is stretched, it becomes thicker perpendicular to the applied force. And from the formula (22) follows that the volume of the deformed body increased due to action of viscous forces.

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