Effective electromagnetic actions for Lorentz violating theories exhibiting the axial anomaly

Andrés Gómez
Facultad de Ciencias, Universidad Nacional Autónoma de México,
04510 México, Distrito Federal, México

A. Martín-Ruiz† and Luis F. Urrutia‡
Instituto de Ciencias Nucleares,
Universidad Nacional Autónoma de México,
04510 México, Distrito Federal, México

The CPT odd contribution to the effective electromagnetic action deriving from the vacuum polarization tensor in a large class of fermionic systems exhibiting Lorentz invariance violation (LIV) is calculated using thermal field theory methods, focusing upon corrections depending on the chemical potential. The systems considered exhibit the axial anomaly and their effective actions are described by axion electrodynamics whereby all the LIV parameters enter in the coupling $\Theta(x)$ to the unmodified Pontryagin density. A preliminary application to type-I tilted Weyl semimetals is briefly presented.

Keywords: Lorentz violation, Vacuum polarization, Thermal field theory, Axial anomaly

I. INTRODUCTION

The study of quantum corrections in the QED fermionic sector of the Standard Model Extension (SME) [1, 2] due to the electromagnetic interaction was sparkled in Ref. [2] which posed the problem of obtaining corrections to the Chern-Simons interaction $\frac{1}{2} \epsilon^{\mu\nu\alpha\beta} \Delta k_{\mu} A_{\nu} F_{\alpha\beta}$ as arising from the one loop vacuum polarization tensor of fermions with the additional coupling $\bar{\Psi} \gamma^5 \gamma^\mu b_\mu \Psi$. Since then, a large number of authors have carried the calculation of the CPT odd contribution to the vacuum polarization obtaining a result that can be presented as $\Delta k_{\mu} = \zeta (e^2/\pi^2) b_\mu$, where $\zeta$ is always a finite coefficient adopting different values, including $\zeta = 0$, according to the regularization prescription chosen to deal with the superficially linear divergence of the vacuum polarization tensor $\Pi_{\mu\nu}$ [3–16]. A survey of the principal approaches regarding this issue can be found in Refs. [11, 16–18]. The search of a physical criterion to select one specific value of $\zeta$ has been unsuccessful and the current understanding is that $\zeta$ would be fixed either by an experimental condition or by a more fundamental theory. In order to briefly summarize some of the previous results we assume that $b_\mu$ is timelike. The case $\zeta = 0$ is obtained by demanding gauge invariance of the effective Lagrangian density [3], and has the drawback of eliminating the Chern-Simons contribution from the outset since this term changes by a total derivative under gauge transformations. Using the full non-perturbative expression in $b_\mu$ for the fermion propagator and different regulators consistent with symmetric integration in four dimensions, the authors of Refs. [4–7] find $|\zeta| = 3/16$. In an attempt to elucidate a further significance of the non-perturbative approach, which demands the use of a unique regulator for all the contributions of $\Pi_{\mu\nu}$, the CPT even contribution to $\Pi_{\mu\nu}$ (second order in $b_\mu$) was calculated in the $m = 0$ case, finding that gauge invariance is violated [14]. The use of Pauli-Villars regularization was subsequently introduced in the non-perturbative calculation of the massive case, yielding gauge invariance to second order but two possibilities for the CPT odd contribution: either the expected value $\zeta = 0$ or $|\zeta| = 3/8$ [15]. Again, there is no unique way of fixing the $\zeta$ contribution by demanding gauge invariance of $\Pi_{\mu\nu}$. An alternative regularization prescription based upon the maximal residual symmetry group of the vector $b_\mu$, proposed in Ref. [13], provides a definition on how to perform the loop integration $d^4k$ in $\Pi_{\mu\nu}$. For example, in the timelike case, the maximal residual symmetry group is $SO(3)$ which calls for a spherically symmetric integration in the tri-momentum $k$ instead of on the four-momentum $k_\mu$. This prescription yields $|\zeta| = 1/4$.

In this work we extend previous calculations of effective electromagnetic action induced by radiative correction in the SME by including additional terms of the fermionic sector in the minimal QED extension of the SME [19].

*Electronic address: andresgz@ciencias.unam.mx
†Electronic address: alberto.martin@nucleares.unam.mx
‡Electronic address: urrutia@nucleares.unam.mx
starting point is the action

\[ S = \int d^4x \bar{\Psi} (\Gamma^\mu i \partial_\mu - M - e\Gamma^\mu A_\mu) \Psi, \]

which is coupled to the electromagnetic field \( A^\mu = (A^0, A^i) = (A^0, \mathbf{A}) \), with the notation \( \mathbf{A} = (A_x, A_y, A_z) \) in terms of the Cartesian components. The metric is \( \eta_{\mu\nu} = \text{diag}(+,-,-,-) \) and we use \( \hbar = c = 1 \) units henceforth. We set \( m = m_5 = H_{\mu\nu} = \epsilon_\lambda = f_\lambda = g_{\lambda\alpha\nu} = 0 \), in the notation of Table XVI of Ref. [19]. Thus we restrict ourselves to

\[ \Gamma^\mu = c^{\mu\nu}\gamma^\nu + d^{\mu\nu}\gamma^5\gamma^\nu, \quad M = a_\mu\gamma^\mu + b_\mu\gamma^5\gamma^\mu, \]

with \( \gamma^\mu \) being the standard gamma matrices. The term \( c^{\mu\nu} \) already includes the \( \delta_\mu^\nu \) contribution corresponding to the free Dirac action. The terms \( \bar{\Psi} \Gamma^\mu i \partial_\mu \Psi \) are CPT even, while those in \( \bar{\Psi} M \Psi \) are CPT odd. Still, since \( \gamma^5 \) is PT odd, each of them separately contains a mixture of PT even and odd terms.

Let us emphasize that the methods and techniques employed in the calculation of these extended effective actions in high energy physics, besides being valuable in their own, can be of relevance in some areas of condensed matter physics. Indeed, the identification of fermionic quasiparticles of Dirac and/or Weyl type in the linearized approximation of Hamiltonians in topological phases of matter provides the opportunity of studying them under the perspective of the SME. Such approach has been particularly fruitful in the case of Weyl semimetals (WSMs) whose electronic Hamiltonians naturally include some of the LIV terms considered in the fermion sector of the SME. Nevertheless, in this case, the LIV parameters need not be highly suppressed, since they are determined by the electronic structure of the material and are subjected to experimental determination [20–25]. Our choice of the LIV parameters in Eq. (2) are those relevant for the description of a general WSM. The simplest example arises in the Hamiltonian of a Weyl semimetal with no tilting and with an isotropic Fermi velocity, which can be embedded in the fermionic action

\[ S = \int d^4x \bar{\Psi}(x) \left( i\gamma^\mu \partial_\mu - \tilde{b}_\mu\gamma^5\gamma^\mu \right) \Psi(x), \]

where \( \tilde{b}_\mu = (\tilde{b}_0, \mathbf{\tilde{b}}) \). Starting from the action (3) coupled to an electromagnetic field, the chiral rotation method of Ref. [21], which provides an alternative quite different from the standard vacuum polarization calculation, yields

\[ S_{\text{eff}} = \frac{e^2}{32\pi^2} \int d^4x \Theta(x) \epsilon^{\alpha\beta\mu\nu} F_{\alpha\beta} F_{\mu\nu}, \quad \Theta(x) = 2\tilde{b}_\mu x^\mu, \]

which determines \( |\zeta| = 1/4 \). Another outstanding example relevant in condensed matter is the case of the superfluid \(^3\text{He}-\Lambda\), where the value \( \zeta = 1/2 \) is reported [26].

The action (4) captures the electromagnetic response of WSMs. While the axion coupling \( \Theta(x) \) arises from the nontrivial topology of the band structure of the material, the appearance of the abelian Pontryagin density (APD) \( \epsilon^{\alpha\beta\mu\nu} F_{\alpha\beta} F_{\mu\nu} \) is related to the axial anomaly

\[ \partial_\mu J^5_\mu = -\frac{e^2}{16\pi^2} \epsilon^{\alpha\beta\mu\nu} F_{\alpha\beta} F_{\mu\nu}. \]

From Eq. (4) we obtain the effective sources [27]

\[ \rho = \frac{\delta S_{\text{eff}}}{\delta A_0} = \frac{e^2}{2\pi^2} \tilde{b} \cdot \mathbf{B}, \quad J = \frac{\delta S_{\text{eff}}}{\delta \mathbf{A}} = \frac{e^2}{2\pi^2} (\mathbf{\tilde{b}} \times \mathbf{E} - \tilde{b}_0 \mathbf{B}), \]

where \( \mathbf{E} \) and \( \mathbf{B} \) are the electromagnetic fields. In our index notation the above current is \( J_x = \delta S_{\text{eff}}/\delta A^0 \). In particular, for \( \tilde{b}_0 = 0 \), this yields \( J_x = \sigma_{xy} E_y \) with the transverse anomalous Hall conductivity \( \sigma_{xy} = -\frac{e^2}{2\pi^2} \tilde{b}_z \), a distinguishing feature of Weyl semimetals [28]. As it will become clear in Section III A, the knowledge of this quantity allows to unambiguously fix the otherwise arbitrary coefficient \( \zeta \) in this case. This additional information is a general consequence of the underlying microscopic theory describing the material, as opposed to the lack of a more fundamental theory containing the SME.

Recently, the appearance of the APD in effective actions arising from (1) has promoted the use of anomaly calculations to obtain them. For example, in the path integral approach, \( S_{\text{eff}} \) was obtained by introducing the electromagnetic coupling in Eq. (3) and subsequently eliminating the fermionic term proportional to \( \tilde{b}_\mu \) through a chiral rotation. Nevertheless, this produces an electromagnetic contribution to the action arising from the nonzero Jacobian of the chiral rotation which is proportional to the Pontryagin density [21]. Following this idea, the Fujikawa prescription to obtain the chiral anomalies [29–30] has also been used to calculate the effective electromagnetic action of different materials.
with the eigenvalues to second order in the electromagnetic potential. This introduces the vacuum polarization tensor $\Pi_{\mu\nu}$ which we write as $\gamma$ which project onto right- and left-handed spinors, respectively. Note that the decomposition of the Dirac spinor into the right and left Weyl spinors is manifest. These latter are eigenspinors of $\gamma$.

The reasons indicated above suggest the convenience of applying and extending the quantum field theory methods developed in high energy physics to obtain the required effective electromagnetic actions corresponding to fermionic systems described by the action (1) which are of interest in condensed matter physics. In particular this procedure should clarify how the LIV corrections enter in the effective action, while the chiral anomaly remains insensitive to them [32–35]. Motivated by the inclusion of temperature effects in the odd contribution of the vacuum polarization tensor arising from the $b_\mu$ coupling in the action (4), reported in Refs. [36–41], we provide the first steps to incorporate thermal field theory in the case of the more general LIV couplings defined in Eqs. (2). In this way, we incorporate non-zero chemical potential effects, but still remain in the zero temperature limit. We follow the conventions of Ref. [42].

II. THE EFFECTIVE ACTION

Integrating the fermions in Eq. (1) defines the effective action

$$\exp(iS_{\text{eff}}) = \det \left[ \Psi(x) \left( \Gamma^\mu i\partial_\mu - M - e\Gamma^\mu A_\mu \right) \bar{\Psi}(x) \right],$$

which we write as

$$S_{\text{eff}}^{(2)}(A) = \frac{1}{2} \int \frac{d^4 p}{(2\pi)^4} A_\mu(-p) \Pi^{\mu\nu}(p) A_\nu(p),$$

(8)

to second order in the electromagnetic potential. This introduces the vacuum polarization tensor $\Pi^{\mu\nu}(p)$

$$i\Pi^{\mu\nu}(p) = e^2 \int \frac{d^4 k}{(2\pi)^4} \text{tr} \left[ S(k - p)\Gamma^\mu S(k)\Gamma^\nu \right],$$

(9)

where $S(k) = i/(\Gamma^\mu k_\mu - M)$ is the exact fermion propagator in momentum space including LIV modifications. Since $S_{\text{eff}}^{(2)}$ is real we must have $\Pi^{\mu\nu}(p) = \Pi_{\nu\mu}(p)$. As we will show, the new effective action (8) will keep the form of Eq. (4) with all the modifications entering through a new vector $B_\lambda$ to be determined. The link between both expressions is accomplished with the identifications

$$\Pi^{\mu\nu}(p) = -i\frac{e^2}{2\pi^2} B_\lambda p_\lambda e^{\mu\nu\lambda\kappa}, \quad \Theta(x) = 2B_\lambda x^\lambda.$$  

(10)

In the following we calculate the CPT odd contribution of the effective action (8). In the massless case ($m = 0$), $\Gamma^\mu k_\mu - M$ is linear in $\gamma^\mu$ and $\gamma^5\gamma^\mu$. The appearance of the matrix $\gamma^5$ suggests the convenience of using left and right chiral projectors in order to evaluate $\gamma^5 = \pm 1$ [14 34 37]. Therefore, we can work in the chiral basis, where the decomposition of the Dirac spinor into the right and left Weyl spinors is manifest. These latter are eigenspinors of $\gamma^5$ with the eigenvalues $\pm 1$. We now define the projection operators

$$P_R = \frac{1 + \gamma^5}{2}, \quad P_L = \frac{1 - \gamma^5}{2}, \quad \gamma^5_R = 1,$$

(11)

which project onto right- and left-handed spinors, respectively. Note that $\gamma^\mu P_L = P_R\gamma^\mu$. The projectors (11) allow us to define the matrices $\Gamma_R^\mu$ such that

$$\Gamma^\mu P_R = (\epsilon^\mu_{\nu} - d^\mu_{\nu}) \gamma^\nu P_R \equiv \Gamma_R^\mu P_R,$$

(12)

which identifies $\Gamma^\mu_R = r^\mu_{\nu} \gamma^\nu$ with $r^\mu_{\nu} = \epsilon^\mu_{\nu} - d^\mu_{\nu}$. In an analogous way we define the left-handed part of the matrices $\Gamma^\mu$ as

$$\Gamma^\mu P_L = \Gamma_L^\mu P_L, \quad \Gamma_L^\mu = l^\mu_{\nu} \gamma^\nu, \quad l^\mu_{\nu} = \epsilon^\mu_{\nu} + d^\mu_{\nu}.$$  

(13)
Following the same idea, we can also split the fermion propagator $S(k)$ into its right- and left-handed parts, i.e.

\[
\frac{i}{\Gamma^\mu k^\mu - M} \gamma^\alpha P_R = P_R S_R(k) \gamma^\alpha, \quad S_R(k) = \frac{i}{(k^\mu r^\mu - a_\nu + b_\nu) \gamma^\nu},
\]

\[
\frac{i}{\Gamma^\mu k^\mu - M} \gamma^\alpha P_L = P_L S_L(k) \gamma^\alpha, \quad S_L(k) = \frac{i}{(k^\mu l^\mu - a_\nu - b_\nu) \gamma^\nu}.
\]

(14)

Note that the propagators $S_L$ and $S_R$, having the generic form $i/(Z_\nu \gamma^\nu)$, can be readily rationalized as $i(Z_\nu \gamma^\nu)/Z^2$.

To proceed forward with the calculation we now split the combination $T^\mu\nu(k, p) = S(k - p) \Gamma^\rho S(k) \Gamma^\nu$ under the trace in $\Pi^{\mu\nu}(p)$ into its left- and right-handed parts, i.e.

\[
T^\mu\nu_{L(R)}(k, p) = (S(k - p) \Gamma^\rho S(k) \Gamma^\nu)_{L(R)},
\]

which implies that the vacuum polarization can be written as the sum $\Pi^{\mu\nu}(p) = \Pi^{\mu\nu}_L(p) + \Pi^{\mu\nu}_R(p)$, where

\[
i \Pi^{\mu\nu}_L(p) = e^2 \int \frac{d^4k}{(2\pi)^4} \text{tr} \left[ T^\mu\nu_{L(R)}(k, p) \right]
\]

(16)

is the vacuum polarization for a left-(right-)handed massless fermion. We now concentrate in the calculation of $\Pi^{\mu\nu}_{L(R)}(p)$. Using Eqs. (12)-(14) and the cyclic property of the trace, the left-handed part $\Pi^{\mu\nu}_L$ can be written as

\[
i \Pi^{\mu\nu}_L(p) = e^2 m^\mu m^\nu \int \frac{d^4k}{(2\pi)^4} \text{tr} \left[ S_L(k - p) \gamma^\beta S_L(k) \gamma^\alpha P_L \right].
\]

(17)

A similar procedure yields the right-handed part $\Pi^{\mu\nu}_R(p)$ by means of the replacements $L \rightarrow R$, $l^\mu \rightarrow r^\mu$, $b_\mu \rightarrow -b_\mu$, $P_L \rightarrow P_R$ in Eq. (17). In the following we restrict ourselves to the axial contributions $\Pi^{\mu\nu}_{A,L}$ and $\Pi^{\mu\nu}_{A,R}$ of the left- and right-handed terms, respectively, which are obtained by isolating the terms $P_R \rightarrow +\gamma^5/2$ and $P_L \rightarrow -\gamma^5/2$ in the corresponding expressions for the vacuum polarization. Both expressions can be summarized in the general form

\[
i \Pi^{\mu\nu}_{A,\chi}(p) = \frac{\chi}{2} e^2 m^\mu m^\nu \int \frac{d^4k}{(2\pi)^4} \text{tr} \left[ S_{\chi}(k - p) \gamma^\beta S_{\chi}(k) \gamma^\alpha \gamma^5 \right],
\]

(18)

with the following assignments

\[
R : \chi = +1 \quad m^\mu = r^\mu, \quad C_\rho = a_\rho - b_\rho,
\]

\[
L : \chi = -1 \quad m^\mu = l^\mu, \quad C_\rho = a_\rho + b_\rho.
\]

(19)

Clearly, the full axial contribution to the vacuum polarization is the sum of the $L$ and $R$ parts, i.e. $\Pi^{\mu\nu} = \Pi^{\mu\nu}_{A,L} + \Pi^{\mu\nu}_{A,R}$.

For the sake of simplicity, we do not introduce an additional subindex $\chi$ either in the matrix $m^\alpha \beta$ or in the vector $C_\rho$, which are to be restored at the end of the calculations according to Eq. (19). The calculation in Eq. (18) proceeds as follows: simplify the notation we introduce the primed vectors $q'_\mu = q_\mu m^\mu$, maintaining the original measure $d^4k$. Also we rationalize the denominators in the propagators and take the trace $\text{tr}(\gamma^\rho \gamma^\beta \gamma^\sigma \gamma^5) = -4 \epsilon^{\rho\beta\sigma5}$, with $\epsilon^{0123} = +1$. We use the antisymmetry of the Levi-Civita symbol to eliminate the term $\epsilon^{\alpha\beta\sigma}(k' - C)_\rho(k' - C)_\sigma$ in the numerator. Since we are interested in the contribution to the effective action including the product of two electromagnetic tensors without additional derivatives, we calculate the integral Eq. (18) only to first order in $p_\alpha$. Finally, and assuming that $m^\mu$ is invertible, the identity $m^\mu \beta m^\nu \alpha m^\nu \sigma \epsilon^{\beta\alpha\sigma} = (\det m)(m^{-1})^\rho \chi^{\mu\nu\lambda\kappa}$ yields the further simplification

\[
\Pi^{\mu\nu}_{A,\chi}(p) = -2 \chi e^2 (\det m)(m^{-1})^\rho \chi^{\mu\nu\lambda\kappa} I_\rho(C),
\]

(20)

where we have defined the integral

\[
I_\rho(C) = \int \frac{d^4k}{(2\pi)^4} g_\rho(k_0, k), \quad g_\rho(k_0, k) = \frac{(k' - C)_\rho}{[(k' - C)^2]^2},
\]

(21)

and we recall that $k'_\mu = k_\alpha m^\alpha \mu$. Previous to regularization, the above expression is our final result for the vacuum polarization tensor in Minkowski spacetime, which can be evaluated for the left- and right-handed fermions according to Eq. (20) with the assignments in Eq. (19). The result (20) holds for arbitrary LIV terms $e^\mu_\nu, d^\mu_\nu, a_\mu$ and $b_\mu$ as long as these produce invertible matrices $P_\nu^\rho$ and $r^\mu_\nu$. 
III. THE REGULARIZATION

As a first step in the inclusion of thermal field theory methods in the description of the radiative corrections to the fermionic action (1) under the new conditions (3), here we shall consider the case of a non-zero chemical potential \( \mu \) but still remain in the zero-temperature limit. The inclusion of both will be published elsewhere. To this end we adopt the finite temperature approach in the imaginary time formulation [43], which we recover after the substitution

\[
\int \frac{d^4k}{(2\pi)^4} \rightarrow +iT \sum_{n=-\infty}^{\infty} \int \frac{d^3k}{(2\pi)^3},
\]

with \( k_0 \rightarrow k_0 = i\omega_n + \Lambda \).

In order to have a specific system which determines the choice of the finite but undetermined parameters, together with a better physical understanding of the subsequent steps, we introduce the Hamiltonian

\[
H_\chi(p) = v_\chi \cdot (p + \chi \hat{b}) - \chi \tilde{b}_0 + \chi v_F \sigma \cdot (p + \chi \hat{b}),
\]

borrowed from condensed matter, which describes a Weyl semimetal with two linear 3D band crossings of chirality \( \chi = \pm 1 \) close to \( \pm \hat{b} \) in momentum space and \( \pm b_0 \) in energy. Here \( v_F \) is the isotropic Fermi velocity at each band crossing, \( \sigma = (\sigma_x, \sigma_y, \sigma_z) \) is the triplet of spin-1/2 Pauli matrices, \( v_\chi \) is the tilting parameter and \( p \) is the momentum. In the Weyl basis for the gamma matrices

\[
\gamma^0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -\sigma_0 & 0 \\ 0 & \sigma_0 \end{pmatrix},
\]

both 2 \( \times \) 2 Hamiltonians in Eq. (23) can be embedded in the action (1) with the choices \( c^{\bar{i}0} = 1, c^{\bar{i}i} = d^{\bar{i}i} = d^{\bar{i}0} = 0 \), i.e. \( \Gamma^0 = \gamma^0 \) together with \( \Gamma^i \) and \( M \) being determined by the parameters in (23). For simplicity, the anisotropy in the Fermi velocities is not included in (23) but enters naturally in our final result through the coefficients \( c^{\bar{i}j} \) and \( d^{\bar{i}j} \). The Hamiltonian (23) is realized in terms of chiral fermions whose dispersion relation behaves linearly, with band crossing points localized both in momentum and energy, as schematically indicated in the Fig. (a). As usual, the position of the chemical potential determines most of the transport properties of a given material. In a metal for instance, it has to be measured from the gap closing, since it represents the filling of either the conduction or the valence bands. This suggests that in WSMs, the transport properties will depend on the band filling, i.e. on the chemical potential as measured from the node of the cones. Therefore, in order to apply the finite temperature approach (22) to WSMs, the parameter \( \Lambda \) in Eq. (22) has to be understood as the chemical potential measured from the band-crossing points, as shown in Fig. (a). To be precise, \( \Lambda = \mu - E_\chi(p_\chi) \), where \( p_\chi \) is the location in momentum of the node with chirality \( \chi \), and \( E_\chi(p_\chi) \) the corresponding energy. Both \( p_\chi \) and \( E_\chi(p_\chi) \) will be determined later for the problem at hand. In the simplified model (3), it is clear that \( p_\chi = \chi \hat{b} \) and \( E_\chi(p_\chi) = -\chi b_0 \).

The sum in Eq. (22) is over the Matsubara frequencies \( \omega_n = (2n + 1)\pi T \) required to produce anti-periodic boundary conditions for the fermions [48]. Next we focus in Eq. (21) and we make use of the additional relation [48]

\[
\lim_{T \to 0} T \sum_{n=-\infty}^{\infty} g_\mu(k_0 = i\omega_n + \Lambda, k) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dk_0 g_\mu(k_0, k) + \frac{1}{2\pi i} \int_{\Omega} dk_0 g_\mu(k_0, k),
\]

where the contour \( \Omega \) is shown in the Fig. (b).
A. The $\mu$-independent contribution

The first term in the right hand side of Eq. (25) reduces to the standard zero-temperature, zero-chemical potential contribution, which has been previously discussed in the literature as extensively reported in section I. Going back to Eq. (20), this corresponds to the direct evaluation of the integral $I_\rho(C)$ after the change of integration variables $k_0' = \nu m_\nu \mu$. Since the only vector at our disposal is $C_\mu$ we have

$$I_\rho^{(1)}(C) = \frac{i}{\det m} \bar{N} C_\mu, \quad \bar{N} = \frac{1}{C^2} \left[ \int \frac{d^4 k'}{(2\pi)^4} \frac{(k' - C) \cdot C}{(k' - C)^2} \right] E,$$

where the integral inside the square brackets is in Euclidean space and the factor $+i$ comes from the Wick rotation. The factor $\bar{N}$ is regularization dependent and could only be a function of the magnitude of the four-vector $C_\mu$. However, a change of scale $C_\sigma \rightarrow \lambda C_\sigma$ followed by an additional change of variables $k'_\mu = \lambda k_\mu$ shows that $\bar{N}$ is just a numerical factor, independent of $C_\mu$. Therefore, $\bar{N}$ is the same for both left- and right-handed fermions, i.e. $\bar{N}_L = \bar{N}_R = \bar{N}$. In this way, the total contribution to the vacuum polarization in this case is summarized in the four-vector

$$B_\lambda^{(1)} = -4\pi^2 \bar{N} \left[ C_{L\rho} (l^{-1})^\rho_\lambda - C_{R\rho} (r^{-1})^\rho_\lambda \right],$$

according to Eq. (10). As shown previously in the literature the factor $\bar{N}$ is finite but undetermined. In the case of WSMs, its dependence upon the regularization procedure has been studied in Ref. [22] and the final choice is made by selecting the anomalous Hall conductivity $\sigma_{xy} = -e^2 b_z/(2\pi^2)$ as the physical quantity to be reproduced in the zero-tilting limit of the Hamiltonian (23) [21, 22, 28]. To this end it is necessary to take a cut-off in the direction of the spatial component of $C$, with the result $\bar{N} = -1/(8\pi^2)$. The suitability of this choice will be demonstrated later by comparing our field-theoretic results with those obtained from a semiclassical Boltzmann approach, which has proven to be successful in the study the transport properties of WSMs.

B. The $\mu$-dependent contribution

Next we consider the second term in the right-hand side of Eq. (25) and calculate

$$I_\rho^{(2)}(C) = \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\pi i} \oint_{\Omega} d k_0 g_\mu(k_0, k).$$

Let us start by finding the double poles of the integrand in the $k_0$-plane. They are located at the two points $k_{0+} = C_0 \pm |k' - C|$, where we recall the relations $k_{0+} = k_\mu m^\mu_\mu$ and $k'_{\mu} = k_0 m^0_\mu + k_j m^j_\mu$. In the general case the solution for $k_0$ involves a rather cumbersome second order equation which will not be very illuminating for our purposes. In order to illustrate the full procedure we restrict ourselves to the simpler situation where $m^0_\mu = \delta^0_\mu$. Under this assumption the poles are located at

$$k_{0\xi} = -k_j m^j_\mu + C_0 + \xi |k' - C|, \quad \xi = \pm 1, \quad k'_{\mu} = k_\mu m^\mu_\mu,$$

$$+i\infty$$

Figure 1: (a) The effective chemical potential. (b) The contour of integration in the $k_0$-plane.
which corresponds to the dispersion relation of the model, wherefrom we can determine the position in momentum
and energy of the Dirac/Weyl node. Clearly, the band touching point (i.e. the node) occurs at \( k' = C \), and the
corresponding energy is \( E_0 \equiv k_{0\xi}(k' = C) \), which can be explicitly expressed as

\[
E_0 = -\mathbf{V} \cdot C + C_0 = \mathbf{\nu} \cdot C + C_0, \quad \mathbf{\nu} = (m^{-1})^i m^i_0.
\]  

(30)

Applying the residue theorem we obtain

\[
\frac{1}{2\pi i} \oint_{\Omega} dk_0 g_\rho = \sum_{\xi = \pm 1} \text{Res}(g_\rho, k_0 = k_{0\xi}) H(k_{0\xi}) H(\Lambda - k_{0\xi}),
\]

(31)

where the Heaviside functions \( H \) guarantee that the poles \( k_{0\xi} \) fall inside the contour \( \Omega \) of Fig. 1
i.e. \( 0 < k_{0\xi} < \Lambda \), where \( \Lambda = \mu - E_0 \), with \( E_0 \) given by Eq. (30). The residues are

\[
\text{Res}(g_\rho, k_0 = k_{0\xi}) = -\frac{1}{4} \delta_\rho^\xi |k'_i - C_i|.
\]

(32)

Substitution of the residues (32) in Eq. (28) determines \( I_{\rho}(C) \), and combining the result with Eq. (20) it follows
that the vacuum polarization becomes

\[
\Pi_{\lambda\chi}^{\mu\nu}(p) = i\chi \frac{e^2}{2} (\det m)(m^{-1})^i \chi p_\kappa \epsilon^{\mu\nu\lambda\kappa} \times \sum_{\xi = \pm 1} \xi \int \frac{d^3k}{(2\pi)^3} \frac{k'_i - C_i}{|k' - C|^3} H(k_{0\xi}) H(\Lambda - k_{0\xi}),
\]

(33)

which includes the factor +i arising from the Eq. (22). It is convenient to make the change of variables

\[
k''_i = k'_i - C_i, \quad d^3k'' = d^3k' = \det(m) d^3k
\]

(34)

where we used that \( \det(m^i_j) = \det(m^{\mu\nu}) = \det(m) \) since \( m^{\mu\nu} = \delta^0_0 \). This yields

\[
\Pi_{\lambda\chi}^{\mu\nu}(p) = i\chi \frac{e^2}{2} (m^{-1})^i \chi p_\kappa \epsilon^{\mu\nu\lambda\kappa} (I^+_i - I^-_i)
\]

(35)

with

\[
I^+_i = \int \frac{d^3k''}{(2\pi)^3} \frac{k''_i}{|k''|^3} H(k_{0\xi}) H(\Lambda - k_{0\xi}).
\]

(36)

In the new double-primed variables \( k'' \) the constraints \( 0 < k_{0\xi} < \Lambda \) read

\[
0 < -(k''_j + C_j)(m^{-1})^j_i m^i_0 + C_0 + \xi |k''| < \Lambda.
\]

(37)

From the identifications in Eq. (30) these constrains become

\[
0 < \mathbf{V} \cdot k'' + \xi |k''| + E_0 < \Lambda.
\]

(38)

It is convenient to perform the 3-momentum integration in spherical coordinates. Hence we present the condition (38) as

\[
0 < |k''| \{ |\mathbf{\nu}| \cos \theta + \xi \} + E_0 < \Lambda,
\]

(39)

Writing \( I^+_i \) as a linear combination of the vectors \( V_i \) and \( C_i \), one can show that the projection upon \( C_i \) is zero, so that

\[
I^+_i = N^\xi V_i,
\]

(40)

where \( N^\xi \) depends only on \( (|\mathbf{\nu}|, E_0, \Lambda) \) as

\[
N^\xi = \frac{1}{(2\pi)^2 |\mathbf{\nu}|} \int^\pi_0 d\theta \sin \theta \cos \theta \int^\infty_0 d|k''| H(k_{0\xi}) H(\Lambda - k_{0\xi}).
\]

(41)
Note that for $|\mathbf{V}| \geq 1$ there exists an angular direction for which $|\mathbf{V}| \cos \theta + \xi$ is zero whereby the integral over $|k''|$ diverges and which would imply the need of a nontrivial renormalization procedure. Thus, from now on we take the simpler convergent case $|\mathbf{V}| < 1$ (relevant for type-I WSMs) which yields $|\mathbf{V}| \cos \theta + \xi$ with definite sign: positive (negative) for $\xi = +1(\xi = -1)$. This in turn fixes the inequalities indicated in Eq. (39) as

$$\xi = +1 : \frac{-E_0}{|\mathbf{V}| \cos \theta + 1} < |k''| < \frac{\Lambda - E_0}{|\mathbf{V}| \cos \theta + 1} \tag{42}$$

$$\xi = -1 : \frac{\Lambda - E_0}{|\mathbf{V}| \cos \theta - 1} < |k''| < \frac{-E_0}{|\mathbf{V}| \cos \theta - 1} \tag{43}$$

The above equations demand us to distinguish the signs of $E_0$ and of $\Lambda - E_0$, since we must ensure that $|k''| \geq 0$. This leaves us with four cases according to the combinations $\Lambda - E_0 \geq 0 \text{ and } E_0 \geq 0$. Here we make use of the assumption that $\Lambda > 0$, as it constrains the possibilities of these four cases. Under this condition a detailed analysis yields the global result

$$I^+_i - I^+_i = \frac{\Lambda}{\pi^2} N_{\chi} V_{\chi}, \quad N_{\chi} = \frac{1}{2|\mathbf{V}|^3} \left(|\mathbf{V}| - \arctanh(|\mathbf{V}|)\right). \tag{44}$$

The resulting contribution to the vacuum polarization is

$$\Pi^\mu_\nu = i e^2 \epsilon^{\mu
u\lambda\kappa} \left\{ \Lambda R_N R_{\nu} (r^{-1})_\lambda - \Lambda L_N L_{\nu} (l^{-1})_\lambda \right\} \tag{45}$$

where $\Lambda$ has acquired a chirality dependence through the splitting of the zero excitation energy modes given by

$$E_{0L(R)} = \mathbf{V}_{L(R)} \cdot \mathbf{C}_{L(R)} + C_{0L(R)}, \quad \Lambda_{L/R} = \mu - E_{0L(R)}. \tag{46}$$

We thus obtain the additional contribution

$$\mathbf{B}^{(2)}_{\lambda} = -\left[ \Lambda R_N R_{\nu} (r^{-1})_\lambda - \Lambda L_N L_{\nu} (l^{-1})_\lambda \right] \tag{47}$$

to the coupling $\Theta(x)$ in Eq. (10).

Summarizing, the full effective electromagnetic action of the system described by the fermionic action (1), with the only restrictions $c^0_\nu = \delta^0_\nu, \quad d^0_\nu = 0, \quad l^\mu_\nu \text{ and } r^\mu_\nu \text{ invertible and } |\mathbf{V}| < 1$, is given by the action (1) with $\Theta(x) = x^\lambda (\mathbf{B}^{(1)}_{\lambda} + \mathbf{B}^{(2)}_{\lambda})$. Notice that the contribution proportional to $\mu$ is not regularization dependent. The coupling $\Theta$ is CPT and PT odd as reflected in Eqs. (6) with $\tilde{\mathbf{E}}_\nu \rightarrow B_\nu$ since $\mathbf{B}_i$ breaks T but not P, while $\mathbf{E}_i$ does the opposite. Then, even if we start with only the CPT even part of the Lagrangian in Eq. (1) (the $\Gamma^\mu$ term ) by setting $a_\mu = b_\mu = 0$ it is not surprising that we obtain a non-zero PT odd $\Theta$ because $\Gamma^\mu$ already includes both PT even and odd contributions. In this particular case $\mathbf{B}^{(1)}_{\lambda} = 0, \Lambda_{\chi} = \mu$, but $\mathbf{B}^{(2)}_{\lambda} \neq 0$ because $\mathbf{V}_R$ and $\mathbf{V}_L$ remain arbitrary. In other words, the source of the PT (CPT) odd effective electromagnetic action here is the $\gamma^5$ PT odd contribution in $\Gamma^\mu$.

One particularly interesting and simple system takes place when we take $c^i_i = \delta^i_i$ and $d^i_i = 0$, which corresponds to the case of arbitrary tilting $\mathbf{V}_R$ and $\mathbf{V}_L$, but with equal isotropic Fermi velocity $v_F = 1$ at each node. That is, we take

$$C_{R\nu} = a_\nu - b_\nu, \quad C_{L\nu} = a_\nu + b_\nu, \quad r^\mu_\nu = \delta^\mu_\nu + \mathbf{V}_R^i \delta^\mu_\nu \delta^0_\nu, \quad l^\mu_\nu = \delta^\mu_\nu + \mathbf{V}_L^i \delta^\mu_\nu \delta^0_\nu, \quad (r^{-1})^\mu_\nu = \delta^\mu_\nu - \mathbf{V}_R^i \delta^\mu_\nu \delta^0_\nu, \quad (l^{-1})^\mu_\nu = \delta^\mu_\nu - \mathbf{V}_L^i \delta^\mu_\nu \delta^0_\nu. \tag{48}$$

Under these conditions $\mathbf{V}_i = \mathbf{V}^i_r$. Putting together the contributions for $\mathbf{B}_{\lambda} = \mathbf{B}_{\lambda}^{(1)} + \mathbf{B}_{\lambda}^{(2)}$ we obtain

$$\mathbf{B}_0 = b_0 + \mathbf{b} \cdot \mathbf{U}_+ + \mathbf{a} \cdot \mathbf{U}_- - \sum_{\chi = \pm 1} \frac{\chi \Lambda_{\chi}}{2|\mathbf{V}_{\chi}|} \left(|\mathbf{V}_{\chi}| - \arctanh(|\mathbf{V}_{\chi}|)\right), \tag{49}$$

$$\mathbf{B}^i = \mathbf{b}^i - \sum_{\chi = \pm 1} (\mathbf{V}_{\chi})^i \frac{\chi \Lambda_{\chi}}{2|\mathbf{V}_{\chi}|^3} \left(|\mathbf{V}_{\chi}| - \arctanh(|\mathbf{V}_{\chi}|)\right), \tag{50}$$

with $\mathbf{U}_\pm = \frac{1}{2}(\mathbf{V}_- \pm \mathbf{V}_+)$ and $\Lambda_{\chi} = \mu - E_{0\chi}$ according to Eq. (46). We recall that $\chi = 1(\chi = -1)$ denote the $R(L)$ contributions. Consistently with the property that the axial anomaly is insensitive to LIV modifications $[32][33]$, our
results (51) show that the Pontryagin density remains unchanged, and that the additional LIV terms in Eq. (2) which defines the fermionic action [1], as well as the chemical potential, modify only the \( \Theta \) coupling.

In order to check the consistency of our results with a condensed matter approach, we first establish the conditions under which our general model reduces to that of a WSM as described by the Hamiltonian (23). Indeed, the equivalence is achieved by setting \( V_\chi = v_\chi \), \( U_\chi = u_\chi \), and

\[
a = 0, \quad b = \tilde{b}, \quad a_0 = -\tilde{b} \cdot u_-, \quad b_0 = \tilde{b}_0 - \tilde{b} \cdot u_+ \tag{52}
\]

such that \( \Lambda_\chi = \mu + \chi \tilde{b}_0 \). In particular, for a type-I Weyl semimetal (i.e. with \( v_\chi < v_F = 1 \)) with the tilting vector \( \tilde{b} \) parallel to \( \tilde{b} \), the semiclassical Boltzmann approach (47) leads to the same effective action (4) with the Weyl node separation shifted by (48)

\[
\tilde{b} \rightarrow \tilde{b} - \sum_{\chi = \pm 1} \frac{\chi \Lambda_\chi}{2|v_\chi|^2} \left[ v_\chi - \text{arctanh}(v_\chi) \right], \tag{53}
\]

where \( \tilde{b} = |\tilde{b}| \) and \( \Lambda_\chi = \mu + \chi \tilde{b}_0 > 0 \) is the chemical potential measured from the nodal point. Clearly, the spatial components in Eq. (51) successfully simplify to the result of Eq. (53).

IV. SUMMARY AND CONCLUSIONS

We extend the vacuum polarization method of high energy physics, used in the calculation of the electromagnetic response of fermionic systems, to a large class of fermionic couplings included in the Standard Model Extension (SME) describing Lorentz symmetry violations [1]. Emphasis is made in the CPT odd contribution to the effective electromagnetic action arising from the fermionic sector of the SME considered in Eq. (2).

Finally, the recent calculation of the one-loop Heisenberg-Euler effective action for zero chemical potential in two of the most studied minimal Lorentz-violating extensions of QED (19), motivates the challenge of extending the thermal field theory approach to the calculation of the CPT even contributions to the effective electromagnetic action arising from the fermionic sector of the SME considered in Eq. (2).

Acknowledgements

L.F.U. and A.M.-R. acknowledge support from the project CONACYT (México) # CF-428214. A.M.-R. has been partially supported by DGAPA-UNAM Projects # IA101320 and # IA102722. L.F.U. and A.G.A. were supported in part by Project DGAPA-UNAM # 103319.
[1] D. Colladay and V.A. Kostelecký, CPT violation and the standard model, Phys. Rev. D 55 (1997) 6760.
[2] D. Colladay and V.A. Kostelecký, Lorentz-violating extension of the standard model, Phys. Rev. D 58 (1998) 116002.
[3] S. Coleman and S. L. Glashow, High-energy tests of Lorentz invariance, Phys. Rev. D 59 (1999) 116008.
[4] R. Jackiw and V. A. Kostelecký, Radiatively Induced Lorentz and CPT Violation in Electrodynamics, Phys. Rev. Lett. 82 (1999) 3572.
[5] M. Pérez-Victoria, Exact Calculation of the Radiatively Induced Lorentz and CPT Violation in QED, Phys. Rev. Lett. 83 (1999) 2518.
[6] J. M. Chung, Radiatively-Induced Lorentz and CPT violating Chern-Simons Terms in QED, Phys. Lett. B 461 (1999) 138.
[7] J. M. Chung and P. Oh, Lorentz and CPT Violating Chern-Simons Term in the Derivative Expansion of QED, Phys. Rev. D 60 (1999) 067702.
[8] W. F. Chen, Understanding radiatively induced Lorentz-CPT violation in differential regularization, Phys. Rev. D 60 (1999) 085007.
[9] G. Bonneau, Regularisation : many recipes, but a unique principle : Ward identities and Normalisation conditions. The case of CPT violation in QED, Nucl. Phys. B 593 (2001) 398.
[10] M. Chaichian, W. F. Chen, and R. G. Felipe, Radiatively induced Lorentz and CPT violation in Schwinger constant field approximation, Phys. Lett. B 503 (2001) 215.
[11] M. Pérez-Victoria, Physical (ir)relevance of ambiguities to Lorentz and CPT violation in QED, J. High Energy Phys. 04 (2001) 032.
[12] O. A. Battistel and G. Dallabona, Role of Ambiguities and Gauge Invariance in the Calculation of the Radiatively Induced Chern-Simons Shift in Extended Q.E.D., Nucl. Phys. B 610 (2001) 316.
[13] A. A. Andrianov, P. Giacconi, and R. Soldati, Lorentz and CPT violations form Chern-Simons modifications of QED, J. High Energy Phys. 02 (2002) 030.
[14] B. Altschul, Failure of gauge invariance in the nonperturbative formulation of massless Lorentz-violating QED, Phys. Rev. D 69 (2004) 125009.
[15] B. Altschul, Gauge invariance and the Pauli-Villars regulator in Lorentz- and CPT-violating electrodynamics, Phys. Rev. D 70 (2004) 101701(R).
[16] J. Alfaro, A. A. Andrianov, M. Cambiaso, P. Giacconi, R. Soldati, Bare and Induced Lorentz & CPT Invariance Violations in QED, Int. J. Mod. Phys. A 25 (2010) 3271.
[17] R. Jackiw, When radiative corrections are finite but undetermined, Int. J. Mod. Phys. B 14 (2000) 2011.
[18] B. Altschul, There is no ambiguity in the radiatively induced gravitational Chern-Simons term, Phys. Rev. D 99 (2019) 125009.
[19] V. A. Kostelecký and N. Russell, Data tables for Lorentz and CPT violation, Rev. Mod. Phys. 83 (2011) 11.
[20] Grushin, A.G. Consequences of a condensed matter realization of Lorentz violating QED in Weyl semi-metals. Phys. Rev. D 86 (2012) 045001.
[21] A.A. Zyzuin, A.A Burkov, Topological response in Weyl semimetals and the chiral anomaly, Phys. Rev. B. 86 (2012) 115133.
[22] P. Goswami, S. Terawi, Axionic field theory of (3 + 1)-dimensional Weyl semimetals, Phys. Rev. B 88 (2013) 245107.
[23] A. Sekine and K. Nomura, Axion electrodynamics in topological materials, J. Appl. Phys. 129 (2021) 141101.
[24] M. N. Chernodub, Y. Ferreira, A. G. Grushin, K. Landsteiner, María A. H. Vozmediano. Thermal transport, geometry, and anomalies, arXiv:2110.05471.
[25] V. A. Kostelecký, R. Lehniert, N. McGinnis, M. Schreck and B. Seradjeh, Lorentz violation in Dirac and Weyl semimetals, arXiv:2112.14293.
[26] G. E. Volovik, On induced CPT-odd Chern-Simons terms in 3 + 1 effective action, Sov. Phys. JETP Lett. 70 (1999) 1.
[27] M. M. Vazifeh and M. Franz, Electromagnetic response of Weyl Semimetals, Phys. Rev. Lett. 111 (2013) 027201.
[28] A. A. Burkov and L. Balents, Weyl Semimetal in a Topological Insulator Multilayer, Phys. Rev. Lett. 111 (2013) 237205.
[29] K. Fujikawa, H. Suzuki, Path Integrals and Quantum Anomalies, Clarendon Press, Oxford, 2004, pp. 65, 92, 149.
[30] R.A. Bertlmann, Anomalies in Quantum Field Theory, Clarendon Press, Oxford, 1996.
[31] J. M. Chung, Lorentz- and CPT-violating Chern-Simons term in the functional integral formalism, Phys. Rev. D 60 (1999) 127901.
[32] A. Gómez, L. Urrutia, The Axial Anomaly in Lorentz Violating Theories: Towards the Electromagnetic Response of Weakly Tilted Weyl Semimetals, Symmetry 13 (2021) 1181.
[33] P. Arias, H. Falomir, J. Gamboa, F. Mendez, F. Schaposnik, Chiral anomaly beyond Lorentz invariance, Phys. Rev. D. 76 (2007) 025019.
[34] A. Salvio, Relaxing Lorentz invariance in general perturbative anomalies, Phys. Rev. D 78 (2008) 085023.
[35] A.P.B. Scarpelli, T. Mariz, J.R. Nascimento, A.Y. Petrov, On the anomalies in Lorentz-breaking theories, Int. J. Mod. Phys. A, 31 (2016) 1650063.
[36] D. Ebert, V. Ch. Zhukovsky and A. S. Razumovsky, Chern-Simons like term generation in an extended model of QED under external conditions, Phys. Rev. D 70 (2004) 025003.
[37] F. A. Brito, L. S. Grigorio, M. S. Guimaraes, E. Passos, and C. Wotzasek, Induced Chern-Simons-like action in Lorentz-violating massless QED, Phys. Rev. D 78 (2008) 125023.
[38] T. Mariz, J. R. Nascimento, E. Passos, R. F. Ribeiro and F. A. Brito, A remark on Lorentz violation at finite temperature,
[39] J. R. Nascimento, E. Passos, A. Yu. Petrov and F. A. Brito, Lorentz-CPT violation, radiative corrections and finite temperature, JHEP 06 (2007) 016
[40] J. Leite and T. Mariz, Induced Lorentz-violating terms at finite temperature, EPL 99 (2012) 21003
[41] V. Ch. Zhukovsky, A. E. Lobanov, and E. M. Murchikova, Radiative effects in the standard model extension, Phys. Rev. D 73 (2006) 065016.
[42] M. E. Peskin, D. V. Schroeder, An introduction to Quantum Field Theory, Perseus Books, Cambridge, 1995.
[43] J. I. Kaputsa, C. Gale, Finite-temperature Field Theory: Principles and Applications, second ed., Cambridge University Press, Cambridge, 2006.
[44] C. W. Bernard, Feynman rules for gauge theories at finite temperature, Phys. Rev. D 9 (1974) 3312.
[45] W. Ditrich, Effective Lagrangians at finite temperature, Phys. Rev. D 19 (1979) 2385.
[46] A. Das, Finite Temperature Field Theory, World Scientific, Singapore, 1997.
[47] R. Kubo, Statistical Mechanics, seventh ed., North-Holland Physics Publishing, Amsterdam, 1988, pp. 363-366.
[48] A. Gómez et al, The electromagnetic effective action for weakly tilted Weyl semimetals, (Unpublished results).
[49] A. F. Ferrari, J. Furtado, J. F. Assunção, T. Mariz and A. Yu. Petrov, One-loop calculations in Lorentz-breaking theories and proper-time method, EPL 136 (2021) 21002.