Vortex Glass is a Metal: Unified Theory of the Magnetic Field and Disorder-Tuned Bose Metals

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We consider the disordered quantum rotor model in the presence of a magnetic field. We analyze the transport properties in the vicinity of the multicritical point between the superconductor, phase glass and paramagnetic phases. We find that the magnetic field leaves metallic transport of bosons in the glassy phase in tact. In the vicinity of the multicritical point of the superconductivity-to-Bose metal transition, the resistivity turns on as \((H - H_c)^\mu\) with \(H_c\). This functional form is in excellent agreement with the experimenterally observed turn-on of the resistivity in the metallic state in MoGe, namely \(R \approx R_c(H - H_c)^\mu\), \(1 < \mu < 3\). The metallic state is also shown to persist in three spatial dimensions. In addition, we also show that the metallic state remains intact in the presence of Ohmic dissipation in spite of recent claims to the contrary. As the phase glass in \(d = 3\) is identical to the vortex glass, we conclude that the vortex glass is, in actuality, a metal rather than a superconductor at \(T = 0\). Our analysis unifies the recent experiments on vortex glass systems in which the linear resistivity remained non-zero below the putative vortex glass transition and the experiments on thin films in which a metallic phase has been observed to disrupt the direct transition from a superconductor to an insulator.

I. INTRODUCTION

When superconductivity is quenched in thin metal alloy films, a metallic\(^{1,2,3,4}\) rather than an insulating phase ensues. The subsequent metallic phase is robust as it exists over a wide range of magnetic field and film thickness (disorder). Regardless of the statistics of the charge carriers, a \(T = 0\) metallic phase in two dimensions does not mesh easily with conventional wisdom. In the case of bosons, the uncertainty principle dictates that in the absence of some new state, bosons either condense or localize as in conventional treatments\(^{5}\) of the insulator-superconductor problem. Further, non-interacting electrons\(^{6}\) carry no current in two dimensions in the presence of disorder. Precisely what the new exotic metallic state is is hotly debated\(^{6,7}\). While many proposals have been made\(^{7}\), only the phase glass model\(^{7,8}\) has been demonstrated through direct calculation to have a finite resistivity at \(T = 0\) in the presence of disorder. In this model, the charge carriers are strictly bosonic. Metallic transport arises from the coupling of the bosonic degrees of freedom to the low-lying excitations of the glassy phase. The density\(^{9,11,12,13}\) of such low-lying excitations scales as \(|\omega|\) and hence gives rise to a dynamical exponent of \(z = 2\). Indeed, other models have been proposed which could give rise to \(z = 2\) dynamics. For example, a popular view is that the metallic state for the bosons is driven by dissipation arising from normal electronic excitations\(^{11,12}\). Such fermionic excitations are typically modeled by a dissipative Ohmic heat bath. However, bosons described by an Ohmic propagator of the form,

\[
G_0 = (k^2 + \eta |\omega_n| + m^2)^{-1},
\]

are the replica indices,

\[
G_{ab}^{(0)}(k, \omega_n) = G_0(k, \omega_n)\delta_{ab} + \beta G_0^2(k, \omega_n)q\delta_{\omega_n,0}
\]

where \(a\) and \(b\) are the replica indices, \(q\) is the Edwards-Anderson order parameter for the glassy phase and \(\omega_n\) are the Matsubara frequencies. The first term is the standard bosonic propagator. However, the second term arises entirely from the glassy landscape. Although Eq. \(2\) can be derived explicitly, it also follows directly from dimensional reduction. That the second term contains the square of the bare bosonic propagator, Zinn-Justin\(^{16}\) surmised holds profound consequences for disordered systems. Indeed this is true. It is from this term that the metallic state arises.

To constitute a complete theory of the metallic state that disrupts the insulator-superconductor transition, the phase glass must be applicable to systems in the presence of a magnetic field as well. It is precisely this question that we address here. We show that the metallic state persists in the presence of a magnetic field. In the weak-field limit, we find that the dominant role of the field is to renormalize the inverse correlation length. The low-field limit of the
resistivity is described by a power law of the form $R_s(H - H_s)^{2\nu}$, where $z = 2$ and $\nu = 1/2$ at the mean-field level. This functional form is in excellent agreement with the experimentally observed turn-on of the resistivity in MoGe\textsuperscript{25,26}. We also find that the metallic state survives in 3D. In 3D the relevant comparison is with the vortex glass\textsuperscript{25} which has been argued extensively to be a superconducting phase. We find instead that this state is metallic\textsuperscript{25,27,28}.

1. Phase Glass Overview

We begin by summarizing the key elements of the zero-field theory we will need when we consider the effects of a non-zero magnetic field. The phase glass arises naturally in the context of the disordered quantum rotor model

\[ H = -E_C \sum_i \left( \frac{\partial}{\partial \theta_i} \right)^2 - \sum_{i,j} J_{ij} \cos(\theta_i - \theta_j), \]

with random Josephson couplings $J_{ij}$ but fixed on-site energies, $E_C$. The phase of each superconducting island is $\theta_i$. Note that additional on-site disorder of the form $iv_j \partial/\partial \theta_j$ results in the equivalent particle-hole symmetric field theory provided that the distribution of on-site energies has zero mean. For the on-site energies, the non-zero mean case is irrelevant here as this corresponds to a density-driven insulator-superconductor transition (IST)\textsuperscript{28}.

If the Josephson couplings are chosen from a distribution with zero mean, only two phases are possible: 1) a glass arising from the distribution of positive and negative $J_{ij}$'s and 2) a disordered paramagnetic state. A superconducting phase obtains if the distribution

\[ P(J_{ij}) = \frac{1}{\sqrt{2\pi J^2}} \exp \left[ -\frac{(J_{ij} - J_0)^2}{2J^2} \right] \]

of $J_{ij}$'s has non-zero mean, $J_0$, and $J$ the variance. To distinguish between the phases, it is expedient to introduce the set of variables $S_i = (\cos \theta_i, \sin \theta_i)$ which allows us to recast the interaction term in the random Josephson Hamiltonian as a spin problem with random magnetic interactions, $\sum_{i,j} J_{ij} S_i \cdot S_j$. Let $\langle \ldots \rangle$ and $\langle \ldots \rangle$ represent averages over the thermal degrees of freedom and over the disorder, respectively. Integrating over the random interactions will introduce two auxiliary fields

\[ Q_{\mu \nu}^{ab}(k, k', \tau, \tau') = (S_{\mu}^a(k, \tau) S_{\nu}^b(k', \tau')) \]

and $\Psi_{\mu}^a(k, \tau) = (S_{\mu}^a(k, \tau))$, respectively. The superscripts represent the replica indices. A non-zero value of $\Psi_{\mu}^a(k, \tau)$ implies phase ordering of the charge $2e$ degrees of freedom. For quantum spin glasses, it is the diagonal elements of the $Q$-matrix $D(\tau - \tau') = \lim_{n \to 0} \frac{1}{2n} \langle Q_{\mu \nu}^{ab}(k, k', \tau, \tau') \rangle$ in the limit that $|\tau - \tau'| \to \infty$ that serve as the effective Edwards-Anderson spin-glass order parameter\textsuperscript{11,12,13,14} within Landau theory. The essential aspect of the quantum rotor spin glass is that the saddle point solution for the corresponding action is minimized by a solution of the form

\[ Q_{\mu \nu}^{ab}(k, \omega_1, \omega_2) = \beta(2\pi)^d \delta^d(k) \delta_{\mu \nu} \left[ D(\omega_1) \delta_{\omega_1 + \omega_2, 0} \delta_{ab} + \beta\delta_{\omega_1, 0} \delta_{\omega_2, 0} q_{ab}^{\omega_1} \right] \]

where the diagonal elements are given by

\[ D(\omega) = -\sqrt{\omega^2 + \Delta^2}/\kappa, \]

with $\kappa$ a coupling constant in the Landau free energy for the spin glass. The diagonal elements of the $Q$-matrices describe the excitation spectrum. Throughout the glassy phase, $\Delta = 0$ and hence the spectrum is ungapped and given by $D(\omega) = -|\omega|/\kappa$. The linear dependence on $|\omega|$ arises because the correlation function $Q_{\mu \nu}^{ab}(\tau)$ decays $\propto \tau^{-2}$ as $\tau^{-2}$. This dependence results in a fundamental change in the dynamical critical exponent from $z = 1$ to $z = 2$. In a recent treatment of the disordered quantum rotor problem, Ikeda\textsuperscript{28} included an Ohmic dissipative $|\omega|$ term at the level of the Hamiltonian. He has found that the excitation spectrum in the spin-glass order parameter scales as $|\omega|^{1/2}$. With the $|\omega|^{1/2}$ ansatz, Ikeda\textsuperscript{28} has found that the resultant phase is a superconductor. We shown in Appendix B, however, that this claim is without merit. Namely, even if $|\omega|$ dissipation is put in by hand at the outset, the universality class of the problem does not change. Such a contribution simply renormalizes the coefficient of $|\omega|$.

To analyze the bosonic conductivity, we focus on the part of the free energy

\[ \mathcal{F} = \sum_{\mu, k, \omega_n} (k^2 + \omega_2^2 + m^2) |\Psi_{\mu}^a(k, \omega_n)|^2 \]
\[-\frac{1}{rt} \int d^d x \int d\tau_1 d\tau_2 \sum_{a,b,\mu,\nu} \Psi^a_\mu(x,\tau_1) \Psi^b_\nu(x,\tau_2) Q^{ab}_{\mu\nu}(x,\tau_1,\tau_2)\]
\[+ U \int d\tau \sum_{a,\mu} [\Psi^a_\mu(x,\tau) \Psi^*_a(x,\tau)]^2\] (8)

governed by the fluctuations of the superconducting order parameter. The coupling between the spin-glass and bosonic degrees of freedom fundamentally changes the dynamics as can be seen by substituting the saddle-point solution for the Q-matrices into Eq. \(5\). The resulting expression for the free energy

\[F_{\text{Gauss}} = \sum_{a,k,\omega_n} (k^2 + \omega_n^2 + \eta|\omega_n| + m^2)|\psi^a(k,\omega_n)|^2\]
\[- \beta q \sum_{a,b,k,\omega_n} \delta_{\omega_n,0} \psi^a(k,\omega_n)|\psi^b(k,\omega_n)|^*,\] (9)

reflects the damped dynamics introduced by the glass degrees of freedom. As remarked earlier, the \(|\omega|\) term dominates giving rise to \(\varepsilon = 2\) dynamics. Certainly, dissipative models with \(|\omega|\) dynamics put in by hand have been studied explicitly in the context of the insulator-superconductor transition partly in the hope of finding new phases. However, direct calculations of the conductivity reveal that \(|\omega|\) dynamics alone cannot give rise to a metallic phase. What is new here is that the \(|\omega|\) term is not put in by hand but arises naturally from the coupling between the glassy dynamics of the phase and the charge degrees of freedom. Further, as is evident from Eq. (2), the Gaussian propagator changes the \(Q\)-matrices into Eq. (8). The resulting expression for the free energy demonstrates that both the \(|\omega|\) dynamics and the glassy landscape conspire to yield a non-zero bosonic conductivity at \(T = 0\). This expression crosses over smoothly to the \(\sigma = \infty\) conductivity in the superconducting phase \((m = 0)\) to the vanishing conductivity of the insulating state \((q = 0)\). Hence, the quantum rotor model in which the disordered Josephson couplings possess a non-zero mean describes a metal, insulator and a superconductor.

2. Magnetic Field: Summary of Findings

In this paper, we extend the formalism summarised above to include a magnetic field. Since disorder is also included, our results in 3D are relevant to the vortex glass state. In the weak field limit, we find that the transport properties are well approximated by replacing \(m^2\) in the conductivity by

\[\tilde{m}^2 = m^2 + \frac{1}{2} m_H^2\] (11)

where \(m_H^2 = \frac{2e^2 H}{\hbar} = 2/\ell^2\) is the inverse square of the magnetic length \(\ell\) in units of \(c = 1\). Physically, this result is transparent as the predominant role of the magnetic field is to provide a confining potential for the electrons on a scale set by \(1/\sqrt{m_H^2}\). The new condition for the onset of superconductivity is

\[m^2 + \frac{1}{2} m_H^2 = 0\] (12)

which has a non-trivial solution because \(m^2 < 0\) demarcates the superconducting phase. Hence, we find quite generally that metallic transport survives in the presence of a magnetic field. Further, we find that metallic transport persists even in \(d=3\) where the vortex glass phase has been proposed to exist in the presence of disorder. Because of the equivalence of the phase glass to a vortex glass in the presence of magnetic field, our results imply that the linear conductivity should remain finite in the vortex glass. These results are discussed extensively in the context of the insulator-superconductor transition and the recent experiments that indicate that the vortex glass state is not accompanied by a vanishing of the linear I-V characteristics.

II. PHASE GLASS CONDUCTIVITY IN A MAGNETIC FIELD

In the presence of a magnetic field, the \(O(2)\) quantum rotor model becomes

\[H = -EC \sum_i \left(\frac{\partial}{\partial \theta_i}\right)^2 - \sum_{(i,j)} J_{ij} \cos(\theta_i - \theta_j - A_{ij}),\] (13)
where \( A_{ij} = (e^* / h) \int d^3 A \cdot dl \) \((e^* = 2e)\). The Josephson couplings are assumed to be random and governed by the distribution Eq. 1. In the current work, we treat \( A \) to be a static time-independent magnetic field as this is the most experimentally relevant situation. For a random system, the technique for treating disorder is now standard. We provide additional steps that are necessary to determine how the electromagnetic gauge couples to the spin glass order parameter. We write the replicated partition function as

\[
\overline{Z^a} = \int \mathcal{D} \theta_i \mathcal{D} J_{ij} e^{-S}
\]

(14)

where the Euclidean action is given by

\[
S = \int_0^\beta d\tau \left\{ \sum_{i,a} \frac{1}{4E_C} \left( \frac{\partial \theta_i^a(\tau)}{\partial \tau} \right)^2 - \sum_a \sum_{(ij)} J_{ij} \cos \left[ \theta_i^a(\tau) - \theta_j^a(\tau) - A_{ij}(\tau) \right] \right\}
\]

(15)

where the superscript \( a \) represents the replica index. The integration over \( J_{ij} \) in Eq. 14 results in the effective action,

\[
S_{\text{eff}} = \int_0^\beta d\tau \sum_{i,a} \frac{1}{4E_C} \left( \frac{\partial \theta_i^a(\tau)}{\partial \tau} \right)^2
\]

\[
+ J_0^2 \sum_a \sum_{(ij)} \int_0^\beta d\tau d\tau' \frac{1}{4} \sum_{\alpha=+1,-1} \exp \left\{ i \left[ \theta_i^\alpha(\tau) - \alpha \theta_i^\alpha(\tau') - (\theta_j^\alpha(\tau) - \alpha \theta_j^\alpha(\tau')) - (A_{ij}(\tau) - A_{ij}(\tau')) \right] \right\}
\]

(16)

where \( \alpha = +1, -1 \). As a result of the sum over \( \alpha \), we see that the vector potential enters both symmetrically and anti-symmetrically. As in the zero-field case, we introduce the two-component spinor, \( S^a(\tau) \) and the auxiliary field defined in Eq. 3. The remaining steps involve performing the cumulant expansion and taking the continuum limit. The final action can be separated into the local and gradient parts:

\[
S_{\text{eff}} = S_{\text{loc}} + S_{\text{gr}}
\]

(17)

where the local part

\[
S_{\text{loc}} = \int d^d x \left\{ \frac{1}{\kappa} \int d\tau \sum_a \left( r + \frac{\partial}{\partial \tau_1} \frac{\partial}{\partial \tau_2} \right) Q_{\mu\nu}^{aa}(x, \tau_1, \tau_2) \right|_{\tau_1=\tau_2=\tau}
\]

\[
- \frac{\kappa}{3} \int d\tau_1 d\tau_2 d\tau_3 \sum_{a,b,c} \left[ Q_{\mu\nu}^{ab}(x, \tau_1, \tau_2) Q_{\nu\rho}^{bc}(x, \tau_2, \tau_3) Q_{\mu\rho}^{ca}(x, \tau_3, \tau_1) + Q_{\mu\nu}^{ab}(x, \tau_1, \tau_2) Q_{\nu\rho}^{ca}(x, \tau_2, \tau_3) Q_{\mu\rho}^{bc}(x, \tau_3, \tau_1) \right]
\]

\[
+ \int d^d x \left\{ \int d\tau \sum_a \left[ u Q_{\mu\nu}^{aa}(x, \tau_1, \tau_2) Q_{\mu\nu}^{aa}(x, \tau, \tau) + v Q_{\mu\nu}^{aa}(x, \tau_1, \tau_2) Q_{\nu\rho}^{aa}(x, \tau, \tau) \right] \right\}
\]

\[
- \frac{y_1}{6\ell} \int d^d x \int d\tau_1 d\tau_2 \sum_{a,b} \left[ Q_{\mu\nu}^{ab}(x, \tau_1, \tau_2) \right]^4
\]

\[
+ \int d^d x \int d\tau \sum_a \left[ \frac{\partial}{\partial \tau_1} \Psi_\mu^a(x, \tau_1) \frac{\partial}{\partial \tau_2} \Psi_\mu^a(x, \tau_2) \right|_{\tau_1=\tau_2=\tau} + \frac{\zeta}{2} \int d\tau \sum_a \left[ \Psi_\mu^a(x, \tau) \Psi_\mu^a(x, \tau) \right]^2
\]

\[
- \frac{1}{\kappa t g} \int d^d x \int d\tau_1 d\tau_2 \sum_{a,b} \Psi_\mu^a(x, \tau_1) \Psi_\mu^b(x, \tau_2) Q_{\mu\nu}^{ab}(x, \tau_1, \tau_2) + \cdots
\]

(18)
contains a spin-glass part that is identical to that of Read, Sachdev and Ye, a superconducting part derived previously and a gradient part

\[ S_{GR} = \int d^4 x \int_0^\beta d\tau_1 d\tau_2 \sum_{a,b} \left| \left( \nabla - \frac{ie^*}{\hbar} A(x, \tau_1) + \frac{ie^*}{\hbar} A(x, \tau_2) \right) Q^{ab}_{-}(x, \tau_1, \tau_2) \right|^2 \\
+ \int d^4 x \int_0^\beta d\tau_1 d\tau_2 \sum_{a,b} \left| \left( \nabla - \frac{ie^*}{\hbar} A(x, \tau_1) - \frac{ie^*}{\hbar} A(x, \tau_2) \right) Q^{ab}_{+}(x, \tau_1, \tau_2) \right|^2 \\
+ \frac{1}{2g} \int d^4 x \left\{ d\tau \sum_a \Psi^a_\mu(x, \tau) \left[ -\gamma - \left( \nabla - \frac{ie^*}{c\hbar} A(x, \tau) \right) \right] \Psi^a_\mu(x, \tau) \right\} \tag{19} \]

in which the vector potential couples both symmetrically and asymmetrically to combinations of the \(Q\)-matrices of the same parity as well as to the bosonic part of the order parameter. Using the fact that \(Q^{ab}_{\pm}(\tau_1, \tau_2) \sim \langle \exp \left[i(\theta^a_r(\tau) \pm \theta^b_r(\tau'))\right]\rangle\), we write the parity combinations of the \(Q\)-matrices as follows

\[ Q^{ab}_{\pm}(x, \tau_1, \tau_2) = \frac{1}{2} \left[ Q^{ab}_{11}(x, \tau_1, \tau_2) \mp Q^{ab}_{22}(x, \tau_1, \tau_2) \right] + \frac{i}{2} \left[ Q^{ab}_{12}(x, \tau_1, \tau_2) \pm Q^{ab}_{21}(x, \tau_1, \tau_2) \right]. \tag{20} \]

It is evident that the vector potential enters in a non-time translationally invariant manner. This is a direct consequence of the fact that the \(Q\)-matrices themselves are a function of two independent times, not simply the difference of \(t_1 - t_2\).

1. Transport in 2D

Our immediate focus is on the bosonic fluctuation conductivity in the glass phase. To this end, it suffices to focus on the \(\psi\)-dependent part of the action:

\[ S_\psi = \int d^d x \left\{ d\tau \sum_a \Psi^a_\mu(x, \tau) \left[ -\gamma - \left( \nabla - \frac{ie^*}{c\hbar} A(x, \tau) \right) \right] \Psi^a_\mu(x, \tau) \right\} \\
+ \int d^d x \int d\tau \sum_a \frac{\partial}{\partial \tau_1} \Psi^a_\mu(x, \tau_1) \frac{\partial}{\partial \tau_2} \Psi^a_\mu(x, \tau_2)|_{\tau_1=\tau_2=\tau} + \frac{\zeta}{2} \int d\tau \sum_a \left[ \Psi^a_\mu(x, \tau) \Psi^a_\mu(x, \tau) \right]^2 \\
- \frac{1}{2g} \int d^d x \int d\tau_1 d\tau_2 \sum_{a,b} \Psi^a_\mu(x, \tau_1) \Psi^b_\mu(x, \tau_2) Q^{ab}_{\mu\nu}(x, \tau_1, \tau_2). \tag{21} \]

A precursor to the final result can be obtained by assuming that \(\psi\) is independent of both space and time. In this case, the only vector-potential dependent terms that survive in the total action are those arising from the pure diamagnetic term, \(A^2\), in the bosonic part of the action. For the field along the \(z\)-axis, we write the vector potential as

\[ A = \frac{1}{2}(B \times r). \tag{22} \]

Consequently, the integral of the diamagnetic term over a rectangle with dimensions \(L_1 \times L_2\) yields \((L_1^2 + L_2^2)/3\ell^2\). This result can be combined into the gradient part of the action for the bosonic degrees of freedom simply by redifining the mass term as

\[ \gamma \rightarrow \gamma + \frac{2}{3}(L_1^2 + L_2^2)/\ell^2 \equiv \tilde{\gamma}. \tag{23} \]

At mean field

\[ \psi^2 = \frac{1}{\xi}(\gamma - \Delta) \tag{24} \]

provides a non-trivial phase boundary for the superconducting state. In the presence of a magnetic field, our preliminary considerations yield to a modified boundary

\[ \tilde{\gamma} \geq \Delta. \tag{25} \]
Because $\tilde{\gamma} > \gamma$, we find that the presence of magnetic field suppresses the superconducting region. Because $\gamma$ acts as the mass term for the superconducting order, an intimate connection

$$\gamma = \frac{2J_0d - E_C}{2J_0da^2}$$

exists with the basic coupling constants, $E_C$ and $J_0$. Here we have adopted the mean-field expression$^{34}$ with $a$ is the distance between the superconducting islands and $d$ is the spatial dimension. Use of $\tilde{\gamma}$ in this expression results in a mean-field expression for the critical field. Hence, the renormalization of the mass by the inverse magnetic length has immediate physical consequences. For example, by simply replacing $m$ by $\tilde{\gamma}$, we obtain the leading magnetic-field dependence of the conductivity. A more rigorous treatment will show that this result is almost correct.

A correct treatment of the conductivity requires that we relax the constraint that $\psi$ be spatially independent. In this case, we need to expand $\psi$ into basis of Landau levels. To this end, we define

$$\psi^a(l, x, y, p_y) = e^{ip_yy} \phi_l(x) C^a(l, p_y)$$

where the $\phi_l(x)$ are the Hermite polynomials. Within the Landau gauge, the gradient part of the action requires the solution to a Landau level problem of the form,

$$\left[ -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + \frac{1}{2\beta^2} p_y \frac{\partial}{\partial x} + \frac{m^2_H x^2}{4} \right] = \left( l + \frac{1}{2} \right) m^2_H \phi_l(x).$$

Expanding the free energy in the Landau level basis results in the replacement of momentum summations with sums over Landau levels. The resultant free energy

$$F_\psi = \sum_{n,l,p_\nu,\omega_m} \left[ (l + \frac{1}{2}) m^2_H + m^2 + \omega^2_m + \eta |\omega_m| \right] |C^a(l, p_y, \omega_m)|^2$$

$$+ \frac{U m^4_H}{4\pi^2} \int d\tau \sum_{a,p,\nu,\omega_m} |C(l, p_y, \omega_m)|^4 + \beta q \delta_{\omega_m,0} C^{a*}(n, p_y, \omega_m) C^b(l, p_y, \omega_m)$$

can be written entirely in terms of the expansion coefficients $C^a(l, p_y, \omega_m)$ once the Landau level problem is solved. Our expression for the free energy lays plain that the naive association of the mass with $m^2 + m^2_H/2$ is not entirely correct as an $l$-dependent term exists as well. The corresponding Gaussian propagator is of the form

$$G^{ab}_0(l, \omega_n) = G_{0}(l, \omega_n) \delta_{ab} + \beta G_{0}(l, \omega_n) q \delta_{\omega_n,0}$$

in the $n \to 0$ (zero replica) limit with $G_0(l, \omega_n) = (m^2_H(l + 1/2) + \omega^2_n + \eta |\omega| + \tilde{m}^2)^{-1}$. To calculate the conductivity, we add to the vector potential a small perturbation $\delta A(x,t)$ which we use as a source to calculate the current and the subsequent conductivity which according to the Kubo formula

$$\sigma(i\omega_n) = -\frac{\hbar}{\omega_n} \lim_{n \to 0} \frac{1}{n} \int d^d(x - x') \int_0^\beta d(\tau - \tau') \delta_{A_x(x,\tau)} \delta_{A_x(x',\tau')} e^{i\omega_n(\tau - \tau')},$$

involves two derivatives of the free energy with respect to the vector potential. The resultant Kubo formula for the conductivity

$$\sigma(i\omega_n) = \frac{\hbar e^2 m^2_H}{2}\sum_{a,b,\omega_m,l=0} \sum_{l=0}^{\infty} \left[ 2G^{ab}_0(l, \omega_m) G^{a*}_0(l + 1, \omega_m) + G^{ab}_0(l, \omega_m + \omega_n) G^{a*}_0(l + 1, \omega_m) \right]$$

involves an explicit sum, $l$, over the Landau levels. In the absence of the glassy landscape (that is, $q^{ab} = 0$), this expression has been evaluated previously.$^{35,36}$ The result in the quantum disordered phase$^{35,36}$ is that an insulating phase obtains regardless of the magnitude of the dissipative dynamics $|\omega|$. Consequently, the only terms that might give rise to a metallic phase are those proportional to the Edwards-Anderson order parameter, $q$. The leading such term gives rise to a conductivity of the form,

$$\sigma^{2D}(\omega = 0, T \to 0) = \frac{(e^2 m^2_H)^2}{\hbar} \sum_{l=0}^{\infty} \left[ 2q \eta \left( m^2 + (l + \frac{1}{2}) m^2_H \right)^2 \left( m^2 + (l + \frac{1}{2}) m^2_H \right)^2 \frac{2\eta x}{x^3 - \Psi(1, \frac{\eta x^2}{x^3})} \right].$$
Where $x$ is defined as $x = m^2/M$. The summation over $l$ give rise a digamma function defined as $\Psi(1, x) = \frac{d \ln \Gamma(x)}{dx}$ where $\Gamma(x)$ is the Gamma function. The large and small field limits relative to $m^2$ can be found exactly. In the weak field regime, we find that the dc conductivity

$$
\sigma_{2D}(\omega = 0, T \to 0) = 4 e^2 \frac{\eta}{3h} \frac{m^2}{m^4} \left( 1 + \frac{m^2_H}{m^4} \right) \simeq 4 e^2 \frac{\eta}{3h} \frac{m^2}{m^4} \left[ 1 - \frac{7}{20} x^2 \right] \quad x \equiv m^2_H/m^2 \ll 1
$$

(34)
can be written compactly in terms of the renormalized mass scale, $\tilde{m}$. In obtaining this expression, we worked in the correct order of limits for the dc conductivity; that is, $\omega = 0, T \to 0$. When the field vanishes, this conductivity goes over smoothly to Eq. (10). In the opposite limit, $m^2_H \gg m^2$, the conductivity reduces to a simple sum

$$
\sigma_{2D}(\omega = 0, T \to 0) = \frac{(e^*)^2}{h} \frac{\eta}{m^2_H} \left[ 2 + \frac{\pi^2}{x} \right], \quad x \gg 1
$$

(35)

Consequently, we find that the conductivity of the Bose metal remains finite in both the weak and strong field regimes.

2. Transport in 3D

That the fluctuation conductivity reaches a plateau in the phase glass in the presence of a magnetic field does not appear to have been expected previously. Before we consider the effect of the quartic term, we show that the metallic state persists in three spatial dimensions as well. The key in generalizing the formalism presented thus far to 3D is to include in the expansion for the order parameter, $\psi^a(l, x, y, z, p_y, p_z) = e^{ip_y y + ip_z z} \psi_l(x) C^a(l, p_y, p_z)$ a plane wave for the third direction. For a vector potential of the form, $A(\mathbf{r}) = (0, Hx, 0)$, the spatially varying part of the action involves the eigenvalue problem,

$$
- \frac{\partial^2}{\partial x^2} \psi_l(x) + \left( \frac{e^* H}{h} - x_0 \right)^2 \psi_l(x) = \tilde{E}_l(p_z) \psi_l(x)
$$

(37)

where

$$
\tilde{E} = m^2_H (l + \frac{1}{2}) + p_z^2.
$$

(38)

Except for the $p_z$-dependence, this oscillator problem is identical to the 2D analogue. In fact, expanding the free energy in terms of the states in Eq. (36) and computing the derivatives with respect to the vector potential in the Kubo formula reveals that the primary difference in the longitudinal conductivity

$$
\sigma_{3D}(i \omega_n) = \frac{(e^* m^2_H)^2}{2h} \sum_{a,b,p_z,\omega_m} \sum_{l=0}^{\infty} (l + 1) \left[ 2G^{ab}_{0}(l, \omega_m, p_z)G^{ab}_{0}(l + 1, \omega_m, p_z) - G^{ab}_{0}(l, \omega_m + \omega_n, p_z)G^{ab}_{0}(l + 1, \omega_m + \omega_n, p_z) \right]
$$

(39)
is the presence of a sum over $p_z$. Because the $p_z$ dependence in the bare Green function

$$
G_0(l, p_z, \omega_n) = (l m^2_H + \omega_n^2 + \eta |\omega| + \tilde{m}^2 + p_z^2)^{-1}
$$

(40)
amounts to a shift in $m^2$, the final expression for the conductivity in $D = 3$

$$
\sigma_{3D}(\omega = 0, T \to 0) = \frac{(e^* m^2_H)^2}{h} \sum_{p_z} \sum_{l=0}^{\infty} (l + 1) \left( \frac{2\eta}{(m^2 + p_z^2 + (l + 3/2)m^2_H)^2} \right) (m^2 + p_z^2 + (l + 1/2)m^2_H)^2.
$$

(41)
is easily obtained from Eq. (39). Let us define

$$
A_n = m^2 + \left( n + \frac{1}{2} \right) m^2_H.
$$

(42)
FIG. 1: Conductivity of in the presence of magnetic field. In 2D, the longitudinal conductivity is given by
\[ \sigma_{0D} = \sigma_0 f_2(x) + \sigma_1 g_2(x) \]
with \( \sigma_0 = 8e^2 q\eta/hm^4 \) and \( \sigma_1 = 8e^2 q^2 U\eta/h\pi m^6 \). The corresponding expression in 3D is
\[ \sigma_{3D} = m \sigma_0 f_3(x) + m \sigma_1 g_3(x) \]
Performing the integral over \( p_z \) using residues reduces the conductivity
\[ \sigma(\omega = 0, T \to 0) = \frac{\pi(e^*)^2}{2h} m_H^4 \sum_{n=0}^{\infty} (n + 1) \left[ \frac{1}{A_n^{3/2} A_{n+1}^{3/2} (A_n^{1/2} + A_n^{1/2})} + \frac{1}{A_n A_{n+1} (A_n^{1/2} + A_{n+1}^{1/2})^3} \right] \]
to a single sum over the Landau index.
In 3D, the leading asymptotic inverse correlation length dependence scales as \( m^{-3} \) as opposed to the \( m^{-4} \) dependence obtained in 2D. In the opposite limit,
\[ \sigma_{3D} = \frac{(e^*)^2 q\eta}{h} \frac{\pi}{m^3} \left[ \frac{1}{6} + \frac{1}{8} x \right] \]
x \( \ll 1 \).

For the sake of comparison, we have evaluated the conductivities in 2D and 3D numerically and plotted them in Fig. 1 as a function of \( x \). As is observed experimentally, both are decreasing functions of the magnetic field.

3. Role of interactions:

Thus far, we have worked at the Gaussian level in which interactions are strictly ignored. At the tree level, a dynamical exponent of \( z = 2 \) renders the quartic interaction \( U \) marginally irrelevant. However, considering the last
term in Eq. 4 on equal footing with $U$ in the one-loop renormalization group scheme, we reach the conclusion that the RG equations flow to strong coupling. The relevance of $q$ at all dimensions manifests itself also by the increasing singularity of relevant contributions from higher order diagrams in the perturbation series in $U$. Consequently, we consider the role of interactions. Consider first the linear $U$ correction which is given by the diagrams in Fig. 2. We use the definition of $A_n$ to simplify the notation in the evaluation of the diagrams. In two dimensions, the sum of these diagrams is given by

$$\sigma_1^{(1)} = \frac{(e^*)^2 q^2 U m_H^2}{2h} \left\{ 2 \sum_n (n+1)(A_n)^{-3}A_{n+1}^{-1} + A_{n+1}^{-3}A_{n+1}^{-1} + 2A_n^{-2}A_{n+1}^{-2} \right\}$$

$$- \sum_n (n+1)m_H^2 \left( 2A_n^{-2}G_0^0(n, \omega_n) + 4A_n^{-3}G_0(n+1, \omega_n) \right)$$

$$- \sum_n (n+1)m_H^2 \left( 2A_n^{-2}G_0^0(n+1, \omega_n) + 4A_{n+1}^{-3}G_0(n, \omega_n) \right) \sum_l A_l^{-2},$$

(47)

which simplifies in the zero-frequency limit to

$$\sigma_1^{(1)} = \frac{32e^2 q^2 U \eta}{h} \sum_{n,l} \frac{(n+1)x^3}{A_n^3 A_{n+1}^3 A_l^3}.$$ 

(48)

Two relevant features are that the leading $U$ correction has a non-divergent contribution to the conductivity in the $T \to 0$ limit and the dependence on $m$ is more singular than in the $U = 0$ case. This points to a scaling function of the form $\sigma \approx (e^2/h)(\eta q/m^2)F(U, \omega_H)$ where $F(y) \sim y^n$ for large $y$, which yields the critical behavior $\sigma \sim m^{-x}$ with $x = 4 + 2p$ and $F(U, \omega_H)$ is a general some function of the interactions and the magnetic field. The value of the exponent $p$ cannot be inferred at any finite order in perturbation theory.

The analogous expression can be derived for the three-dimensional case. The only complication here is the integration over $p_z$. The result,

$$\sigma_{3D}(\omega_n = 0) = \frac{4e^2 q^2 \eta U}{h m^5} \sum_{n,l} \left[ \frac{3\pi n + 1}{8 A_n^{5/2}} + \frac{\pi (n+1)(A_n A_{n+1}^{1/2} - 2A_n^{1/2} A_{n+1})}{2 A_n^{3/2} A_{n+1}^{3/2} (A_n^{1/2} + A_{n+1}^{1/2})^2} \right.$$

$$- \frac{\pi}{2} \frac{(A_n + \omega_H)(n+1)}{A_n^{3/2} A_{n+1}^{3/2} (A_n^{1/2} + A_{n+1}^{1/2})} \left. \right] \frac{1}{A_l^{3/2}}.$$ 

(49)
expressed in terms of the function $A_n$ is once again finite as $T \to 0$. Hence, the bosonic conductivity is robust to interactions (at least in leading order) in both $D = 2$ and $D = 3$ in the presence of a magnetic field.

Of course, the mass, $m$, is also renormalized by the interactions. To understand its dependence on the interactions at large $N$. The self-consistent $1/N$ expansion for the correction to the mass gap,

$$\frac{\delta m^2}{U} = \frac{T m_H^2}{4\pi} \sum_{\omega_m, n} G_0(\omega_m, n) + \beta q \delta_{\omega_0, 0} G_0(\omega_m, n)^2$$

contains two terms. The term independent of the glass environment has been calculated previously. The second term, written succinctly as,

$$\frac{\delta m_{2D}^2}{U} = \frac{\omega_H T \beta q}{4\pi m^2} \sum_n \frac{1}{(1 + (n + 1/2)x)^2}$$

is explicitly independent of temperature and provides hence only an added magnetic field dependence to the gap equation. Here $x = m_H^2/m^2$. In the 3D case, the integral over $p_z$ leads to slightly different power in the denominator,

$$\frac{\delta m_{3D}^2}{U} = \frac{q}{8m} \sum_{n=0}^{\infty} \frac{x}{(1 + (n + 1/2)x)^{3/2}}$$

with, once again, no added temperature dependence. Hence, the interaction terms that couple to the glassy degrees of freedom lead to an innocuous renormalization of the mass as depicted in Fig. (3).

4. Absence of phase stiffness

In the strict sense, a physical system possesses a non-zero phase rigidity if upon a non-uniform rotation of the phase, the free energy increase is of the form,

$$\Delta F = \frac{\rho_s}{2} \int d^2 r |\nabla \theta|^2,$$
where $\rho_s$ is the spin or superfluid stiffness and $\theta$ is the collective phase variable. For the problem at hand, it suffices to expand the gradient part of the free energy for the spin-glass part in terms of a local part given by Eq. (6) and a spatially varying part we refer to as $Q^{ab}(k, \omega_1, \omega_2)$.

Firstly, let us consider the contribution of the spatial independent part of $Q$. It has been shown recently\textsuperscript{31} that a $Q^{4}$th term which is responsible for replica symmetry breaking in the spin-glass part changes the mean-field solution for the excitation spectrum, $D(\omega)$. In this case, the excitation spectrum,

$$D(\omega) = c_{EA}^{ab} - |\omega|/\kappa$$

(54)

remains gapped in the spin glass phase. Here $c$ is a constant and $q_{EA}$ is the diagonal part of $q^{ab}$. In the absence of the $Q^{4}$th term, the stiffness has been shown to vanish\textsuperscript{10}. However, recently, Ikeda\textsuperscript{32} has disputed this claim and has proposed that the constant term in Eq. (54) gives leads to a non-zero stiffness. We show that this claim is false. Following our earlier calculation\textsuperscript{10} (which Ikeda\textsuperscript{32} also follows), we find that the relevant contribution to the spin-glass stiffness containing the $D(\omega)$ term reduces to

$$\sigma_{PS}(i\omega_n) = \frac{16e^2}{\hbar\omega_n} \left[ T \sum_{\omega_m} D(\omega_m)(D(\omega_m) - D(\omega_m + \omega_n)) - q_{EA}(D(\omega_n) - D(0)) \right]$$

(55)

in the presence of the constant term. As we have shown previously\textsuperscript{10}, $\sigma_{PS}$ can be evaluated by analytically continuing the first term using the definition of $D(\omega)$. The result\textsuperscript{10} is non-critical and metallic even in the presence of the constant contribution to $D(\omega)$. This state of affairs arises because as long as difference of $D$’s always appear, any constant that is added to $D(0)$ cannot yield a divergent contribution to the spin-glass conductivity, contrary to the claims of Ikeda\textsuperscript{32}.

The contribution of spatially dependent part of $Q$ can be extracted from the propagator for the spatially varying part of the action. Note that from Eq. (54), $Q^{ab}_{-}$ is determined by the longitudinal components of the spin-glass order parameter only. Further, $Q^{ab}_{-}$ does not couple to the magnetic field. The transverse components enter $Q^{ab}_{+}$ only and couple directly to the field. Hence, it is only the spatial fluctuations of $Q^{ab}_{+}$ that are of interest here. The associated propagator for $Q^{ab}_{+}(k, \omega_1, \omega_2)$ at the Gaussian level is simply

$$g^{ab}_{+}(n, \omega_1, \omega_2) = \frac{1}{(n + \frac{1}{2})m_H + |\omega_1| + |\omega_2|}$$

(56)

Because $\omega$ is a Matsubara frequency, the propagator describes the motion of mode whose energy dispersion scales as $i(n + 1/2)m_H^2$. Such a mode is purely imaginary and hence does not possess a stiffness. For $Q^{ab}_{-}, (n + 1/2)m_H^2$ is replaced by $k^2$ and diffusive transport ensues. Hence, neither the transverse nor the longitudinal components possess a stiffness. Consequently, the phase glass is not a superconductor and only the fluctuation conductivity computed previously contributes to the dc transport. A corollary of this work is that the vortex glass, in which the disorder in the $J_{ij}$’s (caused by a random vector potential) is bounded between $[-1, 1]$ does not have a stiffness and hence is not a superconductor.

5. Comparison with Experiment

Our theory makes precise claims as to how the resistivity turns on in both the disordered and magnetic-field tuned transitions. Hence, given the plethora of experimental data, it behooves us to put our predictions to the test. Consider first the disorder-tuned transition. In this case, the disorder resistivity turns on as the fourth power of the mass according to Eq. (11). As $m^2 \propto \xi^{-2}$ and $\xi \propto (d - d_c)^{-\nu}$, we find that the conductivity turns on as $(d - d_c)^{2\nu}$. For the problem at hand, the mean-field exponents are given by $z = 2$ and $\nu = 1/2$. Any attempt to go beyond mean-field will result in an uncontrolled expansion as a result of the coupling to the $Q^{ab}$ degrees of freedom\textsuperscript{11}. If a controlled expansion were possible, $\nu$ would be renormalized not $z$. Hence, to understand how close the transition is to mean-field, we treat $\nu$ as a fitting parameter. Consequently, we used the expression $\rho(T = 0) \propto (d - d_c)^{\mu}$ to fit the data on three different samples. The results shown in Fig. 11 point to a value of $\mu$ ranging between $[1/2, 2/3]$. To corroborate this prediction, we turn to the magnetic field tuned transition. Mason and Kapitulnik\textsuperscript{23} find that the resistance per square turns on as

$$R \propto R_c(H - H_c)^{\mu} \quad 1 < \mu < 3.$$  

(57)

Since we are interested in the region close to the lower critical field, it suffices to investigate the critical properties of the conductivity using the weak-field expression, Eq. (38). A rough scaling of the conductivity can be obtained by noting that in the weak-field expression, $m^2$ is replaced by $m^2 + m_H^2 = \tilde{m}^2$. If we replace $\tilde{m}^2$ by $H^{-2\nu}$, with
FIG. 4: Critical scaling of the turn-on of the resistivity near the superconductor-metal transition on the metallic side as a function of the sample thickness. $d_c$ represents the critical thickness needed to destroy superconductivity. Based on Eq. (10), we are able to extract a value of the correlation length exponent, $\nu$. The fits establish a value of $\nu$ roughly between $[1/2, 2/3]$. The data were extracted from Ref. (1).

$h = (H - H_c)/H_c$, we find that $\rho \approx h^{4\nu}$. To fit Eq. (57), we require that $4\nu \approx 2.6$ or alternatively, $\nu = 0.63$, a value not inconsistent with the disorder-tuned transition. The transition to the metal state occurs when $m^2 + m_H^2 = 0$. For $m^2 < 0$, we can define $x = m_H^2/|m^2|$. Then the Eq. (34) can be written as

$$\sigma_{xx}(\omega = 0, T \to 0) = \left(\frac{e^*m_H^2}{m^4} \right)^2 \sum_{l=0}^{\infty} \frac{2\eta(l + 1)}{(-1 + (l + \frac{3}{2})^2 x^2)(-1 + (l + \frac{1}{2})^2 x^2)} = \frac{(e^*m_H^2)^2}{h} \frac{1}{m^4} \left(\frac{2}{x} + 2 \frac{\Psi(1, \frac{x^2-2}{2x})}{x^3}\right)$$

The first singularity appears at $x = 2$ which is the superconductor-metal transition point and it yields a critical magnetic field $H_c = \frac{m^2 h}{e^*}$. Expanding the conductivity around $x = 2$, we obtain for $h \to 0$

$$\rho_{xx} = \frac{1}{\sigma_{xx}} = 2\pi \frac{h}{(e^*)^2} \frac{m^4}{\eta} \left(\frac{h^2 + h^4}{2}\right)$$

Hence, the leading term is $h^2$ which is consistent with the turn-on of the resistivity in the disorder-tuned transition. A comparison of this data and a best fit plot to the magnetic-field tuned data is shown in Fig. 5. The consilience of these results unifies the disorder and magnetic-field tuned transitions. Both have the same cause and their turn-on exponents resistivity as a function of the disorder and magnetic field are equivalent. Our prediction that $\nu \approx 1/2$ can certainly be tested further experimentally. In comparing with experiment, it is certainly best to know the width of the critical region. However, since heat capacity measurements are not available, we can only obtain a lower bound using the standard Ginzburg criterion. In our work, we calculated the conductivity in the zero temperature limit in terms of the inverse correlation length, $m^2$ as, which to leading order in the field is given by

$$\tilde{m}^2(T = 0) = m^2 + \frac{e^*H}{\hbar c} = \alpha(-T_c) + \frac{e^*H}{\hbar c} = \frac{e^*H_c (H - H_c)}{\hbar c} = e^*H_c h$$

Substituting this expression into the Ginzburg criterion and assuming that the specific heat term is of order unity, we find that $h \equiv (H - H_c)/H_c \approx 10^{-6}$ sets the lower bound for the width of the critical region.
FIG. 5: The turn-on of the resistivity as a function of magnetic field for MoGe near the superconductor-metal transition on the metallic side. The theory (solid line) curve is determined by Eq. 60. The best-fit curve, $700(H - H_c)^2\mu$, $1 < \mu < 3$. The data were taken from N. Mason’s thesis.

III. GLOBAL SUMMARY

All the bosonic models considered here are obtained from an action of the form,

$$S_\psi = \int d^d x \left\{ d\tau \sum_a \Psi^a_\mu(x,\tau) \left[ m^2 - \left( \nabla - \frac{i e^*}{\hbar} A(x,\tau) \right)^2 \right] \Psi^a_\mu(x,\tau) \right\}$$

$$+ \int d^d x \int d\tau \sum_a \frac{\partial^2}{\partial \tau_1^2} \Psi^a_\mu(x,\tau_1) \frac{\partial}{\partial \tau_2} \Psi^a_\mu(x,\tau_2)_{\tau_1=\tau_2=\tau} + U \int d\tau \sum_a [\Psi^a_\mu(x,\tau) \Psi^a_\mu(x,\tau)]^2$$

$$+ \eta \int d^d x \int d\tau_1 d\tau_2 \sum_a \frac{\partial \Psi^a_\mu(x,\tau_1)}{\partial \tau_2} [\Psi^a_\mu(x,\tau_2)]_{\tau_1=\tau_2=\tau} + \beta \int d^d x \int d\tau \sum_{a,b} q^{ab} \Psi^a_\mu(x,\tau) [\Psi^b_\mu(x,\tau)]^*. \quad (61)$$

The parameters which determine the conductivity are the dissipation $\eta$, the Edwards-Anderson order parameter $q^{ab}$, the quartic interaction $U$, temperature $\beta = 1/K_B T$ and magnetic field $B$. In the following, we provide a brief table which catalogues all the distinct cases that arise and the dependence of the conductivity on the above factors. The table exhausts all known Bose models in which the conductivity has or can be calculated.

IV. FINAL REMARKS

We have shown here that the bosonic conductivity in the phase glass is robust to both interactions and a non-zero magnetic field. Further, we find complete agreement with between theory and experiment for the disorder-tuned and the magnetic-field tuned turn-on of the resistivity in 2D films. In general, the resistivity turns on as

$$\rho(T = 0) \propto \rho_c (g - g_c)^{2\nu}$$

where $g$ is either disorder or magnetic field. The significance that a single expression with universal exponents describes both the disorder-tuned data as well as the field-tuned data is two-fold: 1) both the disorder and magnetic-field transitions share a common origin and 2) both are driven by superconducting fluctuations, that is, both phases are bosonic rather than fermionic in character. In fact, the power-law turn-on of the resistivity places severe restrictions
is particularly revealing in light of the recent experiments which find that in the putative vortex glass phase, the films that nominally display a superconductor-insulator transition. The fact that the metallic phase persists in $D=3$ phase glass is a candidate to explain the experimental findings of the leveling of the resistivity as any permissible theory of the Bose metal. Ultimately, if our proposal is correct, the metallic phase in the thin films has been proven conclusively that the stiffness vanishes. In light of this result and the results derived here, we conclude that

\begin{table}[h]
\centering
\caption{Conductivity for bosonic models as a function of system parameters.}
\begin{tabular}{|c|c|c|c|c|c|}
\hline
Distinct Cases & $\eta = 0$ & $q = 0$ & $\eta \neq 0$ & $q = 0$ & $\eta \neq 0$ & $q \neq 0$ \\
\hline
$B = 0$ & $T = 0$ & $U = 0$ & $0$ & $I$ & $0$ & $I$ & \frac{4\eta}{\pi m^4} M \\
$B = 0$ & $T = 0$ & $U \neq 0$ & $0$ & $I$ & $0$ & $I(QD) \propto SC(RC)\uparrow$ & \frac{4\eta}{\pi m^4} (1 + \frac{qU}{m^2}) M \\
$B = 0$ & $T \neq 0$ & $U = 0$ & $\infty$ & PC & $\frac{2\eta m^2}{\pi m^4} e^{-m/T} + \frac{4\eta}{\pi m^4} [T \ll m]$ & Box left+Box above \\
$B \neq 0$ & $T = 0$ & $U = 0$ & $0$ & $I$ & $0$ & $I$ & \frac{4\eta}{\pi m^4} (1 + \frac{2n^2}{m^2}) M \\
$B \neq 0$ & $T = 0$ & $U \neq 0$ & $0$ & $I$ & $0$ & $I$ & \frac{4\eta}{\pi m^4} (1 + \frac{2n^2}{m^2} + \frac{qU}{m^2} f(x)) M \\
$B \neq 0$ & $T \neq 0$ & $U = 0$ & $\infty$ & PC & $\frac{2\eta m^2}{\pi m^4} e^{-m/T} + \frac{4\eta}{\pi m^4} [T \ll m]$ & Box left+Box above \\
$B \neq 0$ & $T \neq 0$ & $U \neq 0$ & $\infty$ & SC & $\frac{2\eta m^2}{\pi m^4} e^{-m/T} + \frac{4\eta}{\pi m^4} [T \ll m]$ & U and q renormalize $m^a$ \\
$B \neq 0$ & $T \neq 0$ & $U \neq 0$ & $\infty$ & SC & $\frac{2\eta m^2}{\pi m^4} e^{-m/T} + \frac{4\eta}{\pi m^4} [T \ll m]$ & U and B renormalize $m^a$ \\
\hline
\end{tabular}
\end{table}

\textsuperscript{a}U,B and q will renormalize $m$ and change the dependence of temperature $T$ thus change the behavior of the conductivity indirectly. See section:Role of interaction.

\textsuperscript{b}The conductivity in this case is equal to $\sigma (\eta \neq 0, q = 0; T \neq 0, U = 0)$ plus $\sigma \eta \neq 0, q \neq 0; T = 0, U = 0$.

\textsuperscript{c}PC, SC, I, M,QD and RC represent perfect conductor, superconductor, insulator, metal, quantum disorder regime and renormalized classical regime respectively.

any permissible theory of the Bose metal. Further, identical results hold in $D=2$ as well as in $D=3$. Hence, the phase glass is a candidate to explain the experimental findings of the leveling of the resistivity as $T \to 0$ in thin films that nominally display a superconductor-insulator transition. The fact that the metallic phase persists in $D=3$ is particularly revealing in light of the recent experiments which find that in the putative vortex glass phase, the $I-V$ characteristics remain linear as opposed to the non-linear behaviour anticipated for a true superconducting phase. As it turns out, no explicit calculation of the conductivity has ever been performed for the vortex glass. The only difference between the vortex glass and the phase glass considered here is that in the former, the disorder in the $J_{ij}$’s is bounded in the interval $[-1,1]$. This is not a substantive difference. Hence, the model considered here contains the complexity of the vortex glass and hence should suffice to describe this model. Further, as we have shown above and elsewhere\textsuperscript{10}, coupling the electromagnetic gauge directly to the spin glass order parameter does not proves conclusively that the stiffness vanishes. In light of this result and the results derived here, we conclude that the vortex glass is in fact a metal. Ultimately, if our proposal is correct, the metallic phase in the thin films has been seen before. Namely, it is the vortex glass.

V. APPENDIX A:

The propagator, Eq. (2), describes bosonic transport for the free energy given by Eq. (9):

$$F_{\text{gauss}} = \sum_{a,k,\omega_n} (k^2 + \omega_n^2 + \eta |\omega_n| + m^2) |\psi^a(k,\omega_n)|^2$$

$$-\beta \eta \sum_{a,b,k,\omega_n} \delta_{\omega_n,0} \psi^a(k,\omega_n) \psi^b(k,\omega_n)^*.$$  \hspace{1cm} (63)

Consequently, in replica space, the inverse propagator is

$$G_{ab}^{(0)}(k,\omega_n) = (m^2 + k^2 + \omega^2 + \eta |\omega| - \beta q^{ab} \delta_{\omega,0})^{-1}. \hspace{1cm} (64)$$

That is,

$$\sum_b G_{ab}^{(0)}(k,\omega_n) (m^2 + k^2 + \omega^2 + \eta |\omega| - \beta q^{bc} \delta_{\omega,0}) = 1_{ac} \hspace{1cm} (65)$$
To invert the matrix, we set
\[ G_{ab}^{0}(k, \omega_n) = A(k, \omega_n)\delta_{ab} + B(k, \omega_n)\beta q^{ab} \]
(66)
and
\[ C(k, \omega_n) = (m^2 + k^2 + \omega^2 + \eta|\omega|) \]
(67)
and then substitute into Eq. (64) to obtain
\[
1_{ac} = \sum_b \left( A(k, \omega_n)\delta_{ab} + B(k, \omega_n)\beta q^{ab} \right) \left( C(k, \omega_n)\delta_{bc} - \beta q^{bc}\delta_{\omega,0} \right)
\]
\[ = A(k, \omega_n)C(k, \omega_n)\delta_{ac} + \left( B(k, \omega_n)C(k, \omega_n) - A(k, \omega_n)\delta_{\omega,0} \right)\beta q^{ac} + B(k, \omega_n)\delta_{\omega,0} \sum_b q^{ab}q^{bc}. \]
(68)
The last term is a product of Parisi matrices which is parameterized by the function \( \tilde{q} \) and \( q(x) \) \( (0 < x < 1) \), where \( \tilde{q} \) and \( q(x) \) are the diagonal and off-diagonal elements of \( q \) in the continuous representation. The sum over the replica index \( b \) in the last term in Eq. (68) will be proportional to \( n \) and hence vanish in the \( n \to 0 \) limit. Consequently, we simplify Eq. (68) to obtain
\[
A(k, \omega_n)C(k, \omega_n) = 1
\]
\[ B(k, \omega_n)C(k, \omega_n) - A(k, \omega_n)\delta_{\omega,0} = 0. \]
(69)
The solution to these equations is
\[
A(k, \omega_n) = \frac{1}{C(k, \omega_n)} = \frac{1}{m^2 + k^2 + \omega^2 + \eta|\omega|} = G_{0}(k, \omega_n)
\]
\[ B(k, \omega_n) = \frac{A(k, \omega_n)\delta_{\omega,0}}{C(k, \omega_n)} = \frac{\delta_{\omega,0}}{(m^2 + k^2)^2} = G_{0}^2(k, \omega_n)\delta_{\omega,0}. \]
(70)
Eq.(2) is then derived.

VI. APPENDIX B:

The point of this Appendix is to show that even if an Ohmic dissipation term is put in by hand as done recently,\(^{29}\) nothing of a substantive nature changes. Hence, the claim that \( D \propto |\omega|^{1/2} \) leads to a superconducting state instead of a metallic one is false. To avoid any confusion, we redo the cumulant expansion.

The effective action for our problem is given by
\[
S_{\text{eff}} = \int_{0}^{\beta} d\tau \sum_{a,b} \frac{1}{4E_{C}} \left( \frac{\partial \theta_{a}^{\mu}}{\partial \tau} \right)^2
\]
\[ + \frac{J_0^2}{2} \sum_{a,b} \sum_{ij} \int_{0}^{\beta} d\tau d\tau' \frac{1}{4} \sum_{\alpha=+1,-1} \exp \left\{ i \left[ \theta_{i}^{\alpha}(\tau) - \theta_{i}^{\alpha}(\tau') \right] - \left[ \theta_{j}^{\alpha}(\tau) - \theta_{j}^{\alpha}(\tau') \right] - (A_{ij}(\tau) - \alpha A_{ij}(\tau')) \right\} \]
\[ + \text{c.c.}
\]
\[ - \frac{J_0}{2} \sum_{a} \sum_{ij} \int_{0}^{\beta} d\tau \exp \left( \theta_{i}^{a} - \theta_{j}^{a}(\tau) - A_{ij} \right) + \text{c.c.}. \]
(71)
Using the Hubbard-Stratanovich transformation and introducing the order parameters \( \Psi_{\mu}^{a}(k, \tau) \) and \( Q_{\mu\nu}^{ab}(k, \tau_1, \tau_2) \), we have the effective free energy\(^{33}\),
\[
F_{\text{eff}}[\Psi, Q] = \sum_{a,k} \int_{0}^{\beta} d\tau \Psi_{\mu}^{a}(k, \tau) [\Psi_{\mu}^{a}(k, \tau)]^\ast + \sum_{a,b,k,k'} \int_{0}^{\beta} d\tau \int_{0}^{\beta} d\tau' Q_{\mu\nu}^{ab}(k, \tau, \tau') [Q_{\mu\nu}^{ab}(k, \tau, \tau')]^\ast
\]
\[ - \ln \left[ \hat{T} \exp \left( 2 \int_{0}^{\beta} d\tau \sum_{a,k} \sqrt{J_0(k)} \Psi_{\mu}^{a}(k, \tau) S_{\mu}^{a}(-k, \tau) \right) \right]. \]
Here, the new free energy, \( F_{\text{dis}} = \int_0^\beta \int_0^\beta d\tau \int_0^\beta d\tau' \frac{\eta}{\pi(\tau - \tau')^2} \left( 1 - \sum_{a,b,k} S^a_{\mu}(k,\tau) S^a_{\mu}(-k,\tau') \right) \) (72)

where \( J(k) = J \sum_k \cos(kR), J_0(k) = J_0 \sum_k \cos(kR) \) (\( R \) is the displacement vector of nearest neighbour) and

\[
\langle A \rangle_0 = \frac{1}{Z_0} Tr(e^{-\beta H_0^{eff}} A)
\] (73)

Here, \( Z_0 = Tr[\exp(-\beta H_0)] \). The Hamiltonian \( H_0 \) and \( H_0^{eff} \) are defined as,

\[
H_0 = -\frac{1}{4E_c} \sum_i \left( \frac{\partial}{\partial \theta_i} \right)^2, \quad H_0^{eff} = -\frac{1}{4E_c} \sum_i \left( \frac{\partial}{\partial \theta_i} \right)^2
\] (74)

where \( a \) is the index of replica and \( i \) is the site label.

The only difference from Ref\(^{23} \) with respect to the cumulant expansion is the dissipation term. However, inspection of the free energy indicates that this term can be combined with the second-to-last term by defining a new \( Q \) matrix, \( \hat{Q}_{\mu\nu} \) defined as,

\[
\sqrt{2J(k)J(k')} Q_{\mu\nu}(k,\tau,\tau') - \delta_{\mu\nu} \delta_{k+k',0} \frac{\eta}{\pi(\tau - \tau')^2} \rightarrow \sqrt{2J(k)J(k')} \hat{Q}_{\mu\nu}(k,\tau,\tau')
\] (75)

The new free energy,

\[
\left\langle \hat{T} \exp \left( 2 \int_0^\beta d\tau \sum_{a,k} \sqrt{J_0(k)} \Psi^a_{\mu}(k,\tau) S^a_{\mu}(-k,\tau) \right. \right.
\]

\[
\left. + \sum_{a,b,k,k'} \int_0^\beta d\tau \int_0^\beta d\tau' \left[ \sqrt{2J(k)J(k')} \hat{Q}_{\mu\nu}(k,\tau,\tau') \right] S^a_{\mu}(k,\tau) S^b_{\nu}(k',\tau') \right) \rightangle_0
\] (76)

is identical with \( Q \) replaced with \( \hat{Q} \). Hence, the solution to \( \hat{Q} \) is the same as that for \( Q \). Once the transformation, Eq. (75) is undone, we recover that the Ohmic dissipation term simply renormalizes the total coefficient of \( |\omega| \), leaving the glass and metal intact, contrary to the claims of Ikeda\(^{29} \).

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1. H. M. Jaeger, D. B. Haviland, B. G. Orr, and A. M. Goldman, Phys. Rev. B 40, 182 (1989).
2. L. J. Gerlits et al., Phys. Rev. Lett. 63, 326 (1989).
3. D. Ephron, A. Yazdani, A. Kapitulnik, and M. R. Beasley, Phys. Rev. Lett. 76, 1529 (1996).
4. N. Mason and A. Kapitulnik, Phys. Rev. Lett. 82, 41 (1999); ibid, cond-mat/0006138.
5. M. P. A. Fisher, Phys. Rev. Lett. 65, 923 (1990).
6. E. Abrahams, P. W. Anderson, D. C. Licciardello, T. V. Ramakrishnan, Phys. Rev. Lett. 42, 673 (1979).
7. For a review of numerous proposed models, see, P. Phillips and D. Dalidovich, Science 302 243 (2003); V. M. Galitski, G. Refael, M. P. A. Fisher, and T. Senthil, Phys. Rev. Lett. 95, 077002 (2005).
8. D. Dalidovich and P. Phillips, Phys. Rev. Lett. 89, 27001/1-4 (2002).
9. P. Phillips and D. Dalidovich, Phys. Rev. B 68, 104427 (2003).
10. N. Read, S. Sachdev, and J. Ye, Phys. Rev. B 52, 384 (1995); J. Ye, S. Sachdev, and N. Read, Phys. Rev. Lett. 70, 4011 (1993).
11. J. Miller and D. A. Huse, Phys. Rev. Lett. 70, 3147 (1993).
12. A. J. Bray and M. A. Moore, Phys. Rev. B 31 631-633 (1985).
14 S. Chakravarty, G.-L. Ingold, S. Kivelson, and G. Zimanyi, Phys. Rev. B 37, 3283 (1988).
15 A. Kapitulnik, N. Mason, S. A. Kivelson, S. Chakravarty, Phys. Rev. B 63, 125322-1 (2001).
16 K. Wagenblast, A. van Otterlo, G. Schön, and G. Zimanyi, Phys. Rev. Lett. 78, 1779 (1997).
17 D. Dalidovich and P. Phillips, Phys. Rev. Lett. 84, 737, (2000).
18 M. Steiner and A. Kapitulnik, Physica C 422, 16 (2005).
19 J. Zinn-Justin, Quantum Field Theory and Critical Phenomena, Ch. 23, (Clarendon Press, Oxford, 1996); The factor of $\omega^2$ in the numerator of the replica propagator in Appendix 23 should be $\omega^2$, the strength of the disorder.
20 J. Jain, Phys. Rev. B 40, 8079 (1989).
21 B. I. Halperin, P. A. Lee, and N. Read, Phys. Rev. B 47, 7312 (1993).
22 M. V. Feigelman, V. B. Geshkenbein, L. B. Ioffe, A. I. Larkin, Phys. Rev. B 48, 16641 (1993).
23 N. Mason, PhD thesis, Stanford, 2003.
24 D. Fisher, M. P. A. Fisher, D. Huse, Phys. Rev. B 43, 130 (1991).
25 C. Reichhardt, A. van Otterlo, G. T. Zimányi, Phys. Rev. Lett. 84, 1994 (2000).
26 W. Jian, et. al., J. Phys. Cond. Matt. 9, 8085 (1997).
27 D. R. Strachan, et al. Phys. Rev. Lett. 87, 67007 (2001).
28 See p. 201 in S. Sachdev, Quantum Phase Transitions (Cambridge University Press, New York, 1999).
29 R. Ikeda, cond-mat/0510380 (see version 3 of this paper). In this version and the previous version v2, the author uses the same equation to “easily find the only physically meaningful solution”, one is $D \propto |\omega|$ , the other is $D \propto |\omega|^{1/2}$. Appendix B shows that only the $|\omega|$ solution is correct.
30 It has been claimed by R. Ikeda, cond-mat/0307563 that a magnetic field destroys metallic transport in the phase glass. This calculation is based on a time-dependent electromagnetic gauge and hence is irrelevant to the current work.
31 D. Dalidovich and V. Dobrosavljevic Phys. Rev. B 66, 081107 (2002).
32 R. Ikeda, cond-mat/0510380 v2. In this reference, the $D(0)$ term was dropped inadvertently, thereby leading to the erroneous claim of a stiffness.
33 D. Dalidovich and P. Phillips, Phys. Rev. B 59, 11925 (1999).
34 D. Dalidovich and P. Phillips, Phys. Rev. B 59, 11925 (1999).
35 S. Doniach, Phys. Rev. B 24, 5063 (1981).
36 A. van Otterlo, K.-H. Wagenblast, R. Fazio, and G. Schön, Phys. Rev. B 48, 3316 (1993).
37 D. Dalidovich and P. Phillips, Phys. Rev. B 64, 184511-1 (2001).
38 K. Damle and S. Sachdev, Phys. Rev. B 56, 8714 (1997).
39 D. Dalidovich and P. Phillips, Phys. Rev. B 64, 184511-1 (2001).
40 A. Van Otterlo, et.al,Phys.Rev.B.48,3316 (1993).
41 J. Scalapino, et.al,Phys.Rev.Lett.68,2830 (1992).
42 D. Dalidovich, unpublished results.