Covariant theory of Bose-Einstein condensates in curved spacetimes with electromagnetic interactions: the hydrodynamic approach

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We develop a hydrodynamic representation of the Klein-Gordon-Maxwell-Einstein equations. These equations combine quantum mechanics, electromagnetism, and general relativity. We consider the case of an arbitrary curved spacetime, the case of weak gravitational fields in a static or expanding background, and the nonrelativistic (Newtonian) limit. The Klein-Gordon-Maxwell-Einstein equations govern the evolution of a complex scalar field, possibly describing self-gravitating Bose-Einstein condensates, coupled to an electromagnetic field. They may find applications in the context of dark matter, boson stars, and neutron stars with a superfluid core.

I. INTRODUCTION

The fundamental equations of quantum mechanics (Schrödinger, Klein-Gordon, Dirac) have an interesting history. A short historic was written by Dirac\textsuperscript{[1]} who retraced the early development of quantum mechanics. According to Dirac, what we now call the Klein-Gordon (KG) equation [see Eq. (F1)] was actually introduced by de Broglie. He proposed this wave equation because he noticed that there was an interesting connection between its solutions and the relativistic motion of a particle. De Broglie postulated that these waves were associated with the motion of the particle. In his thesis\textsuperscript{[2]}, he introduced the correspondences $E = \hbar \omega$ and $\mathbf{p} = \hbar \mathbf{k}$ between the energy and impulse of a particle and the pulsation and wave vector of a wave. He aimed at performing a real physical synthesis, valid for all particles, of the notion of wave-corpuscle duality that Einstein\textsuperscript{[3]} had introduced for photons in his theory of light quanta in 1905.\textsuperscript{1}

Schrödinger who was studying the motion of an electron in an atom used the wave equation of de Broglie appropriately modified to take into account the electromagnetic field in which the electron was moving. He guessed what we now call the electromagnetic KG equation [see Eq. (F2)]. It is interesting to note that Schrödinger first attempted to develop a relativistic wave theory of the hydrogen atom and derived Eq. (F2). However, he abandoned it when he realized that it was not giving the correct energy levels of the hydrogen atom. He was very depressed about it. He came back to the problem a few months later and considered the nonrelativistic limit of Eq. (F2). He obtained, in the absence of magnetic field, what is now called the Schrödinger equation [see Eq. (F3)]. With this non-relativistic approximation, he obtained results in agreement with the observations apart from the fine structure of the hydrogen spectrum which depends on the relativistic corrections.

If things really happened as Dirac describes in his historic\textsuperscript{[1]}, it is curious to note that the manner Schrödinger introduced his equation in his published papers\textsuperscript{[6,7]} is totally different from his original approach retraced by Dirac. He used only nonrelativistic arguments and did not rely on Eq. (F2). In his first paper (27 January 1926)\textsuperscript{2} he obtained the eigenvalue equation (F8) from a variational principle. In his second paper (23 February 1926)\textsuperscript{7} he recovered this eigenvalue equation from an ingenious procedure obtained by combining the de Broglie relations and the standard wave equation (second order in time) with a space dependent phase velocity (see Appendix F for a brief presentation of his historical derivation). In his papers\textsuperscript{[6,7]}, Schrödinger solved the eigenvalue equation (F8) for several potentials. In addition to recovering the energy spectrum of the hydrogen atom heuristically obtained by Bohr\textsuperscript{[10,11]}, he also derived the energy spectrum of the harmonic oscillator and rotator, and found agreement with the result obtained by Heisenberg\textsuperscript{[12]} from his more abstract matrix mechanics.\textsuperscript{3} In his third paper (10 May 1926)

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\textsuperscript{1} As reported by Bhaumik\textsuperscript{[4]}, “Langevin, who was de Broglie’s thesis advisor, was very skeptical about de Broglie’s theory and contacted Einstein to have his opinion. Einstein strongly supported de Broglie’s work and suggested to physicists to look for an evidence of the matter wave. A proof was furnished soon by the accidental discovery of electron waves by Davission and Germer\textsuperscript{[5]} in observing a diffraction pattern in a nickel crystal. Later, Einstein mentioned to Rabi that he had thought about the equation for matter waves before de Broglie but did not publish it because there was no evidence for it at that time.”

\textsuperscript{2} Here and in the following, the dates correspond to the dates of submission.

\textsuperscript{3} In this brief review, we stick to what was originally called the “wave (or undulatory) mechanics” initiated by de Broglie\textsuperscript{[2]} and developed by Schrödinger\textsuperscript{[6,7]}. We shall not discuss in detail what was originally called the “quantum mechanics” (a term introduced by Born) based on the matrix theory of Heisenberg\textsuperscript{[12]} further developed by Born and Jordan\textsuperscript{[13,14]}. In Ref.\textsuperscript{[15]} (18 March 1926), Schrödinger showed the equivalence between his wave mechanics and the Heisenberg-Born-Jordan quantum mechanics.
Schrödinger applied his theory to the perturbation of the hydrogen atom caused by an external homogeneous electric field (Stark effect). In his fourth paper (21 June 1926)\(^4\), he obtained for the first time the time-dependent equation that now bears his name. From this equation, he introduced a (charge) density and a current (of charge) \[\text{see Eqs. (14) and (17)}\] and derived a local conservation equation for \(\rho = |\psi|^2\) \[\text{see Eq. (173)}\]. This gives an interpretation to the wave function in the sense that \(|\psi|^2(r, t)\) characterizes the “presence” of the particle at some point. Schrödinger thought that the wave function represents a particle that is spread out, most of the particle being where the modulus of the wave function \(|\psi|^2\) is large. For example, according to Schrödinger’s view, the charge of the electron is not concentrated in a point, but is spread out through the whole space, proportional to the quantity \(|\psi|^2\).

On the other hand, Born (21 July 1926, 16 October 1926)\(^4\) and de Broglie (19 July 1926, 5 August 1926)\(^5\) developed a probabilistic interpretation of the wave function.\(^4\) Born proposed that the magnitude of the wavefunction \(\psi(r, t)\) does not tell us how much of the particle is at position \(r\) at time \(t\), but rather the probability that the particle is at \(r\) at time \(t\). This gives an interpretation to the wave function in the sense that \(|\psi|^2\) represents the probability of presence of the electron at some point. The correspondence principle \[\text{see Eq. (20)}\] was introduced by Schrödinger (18 March 1926)\(^6\) and de Broglie (19 July 1926, 5 August 1926)\(^5\). On the other hand, by making the transformation of Eq. (80) introduced by Wentzel (18 June 1926)\(^7\), Brillouin (5 July 1926)\(^8\) and Kramers (9 September 1926)\(^9\), it was found that the Schrödinger equation reduces to the Hamilton-Jacobi equation in the semi-classical limit \(\hbar \to 0\).\(^5\) Although the fundamental papers of Schrödinger\(^6\)\(^9\) are written in German, he also wrote an English summary of his theory in Ref. 22 (3 September 1926).

The electromagnetic Klein-Gordon (KG) equation \[\text{see Eq. (F2)}\] appeared in many publications in a very short lapse of time: Klein\(^2\) (28 April 1926), Fock\(^3\) (11 June 1926), Schrödinger\(^4\) (21 June 1926), De Donder and Dungen\(^5\) (5 July 1926), de Broglie\(^5\) (19 July 1926), Fock\(^3\) (30 July 1926), de Broglie\(^5\) (5 August 1926), Kudar\(^6\) (30 August 1926), Ehrenfest and Uhlenbeck\(^7\) (16 September 1926), Gamow and Iwanenko\(^8\) (19 September 1926), Gordon\(^9\) (29 September 1926), De Donder\(^5\) (11 October 1926), Schrödinger\(^9\) (30 November 1926), Klein\(^5\) (6 December 1926), and Schrödinger\(^9\) (10 December 1926).\(^5\) These authors obtained the electromagnetic KG equation in a flat spacetime (except\(^5\) who developed their theory in a five dimensional curved spacetime). The KG equation taking into account electromagnetism and general relativity in a four dimensional curved spacetime \[\text{see Eq. (11)}\] was first written by De Donder\(^5\) and Fock\(^3\). His work was further developed by Rosenfeld\(^10\) (in a four or five dimensional spacetime) and by de Broglie\(^9\). The KG equation was introduced by these authors in different manners, either from the correspondence principle\(^5\),\(^6\),\(^8\),\(^9\) or from a variational principle\(^5\),\(^6\),\(^8\),\(^9\). The authors of Refs.\(^3\),\(^5\),\(^7\),\(^8\),\(^9\),\(^10\) introduced a charge density \[\text{see Eq. (24)}\] and a current of charge \[\text{see Eq. (25)}\] and showed that the KG equation conserves the charge \[\text{see Eq. (26)}\].

This brief review shows that there was an impressive activity during this period. Clearly, 1926 is the year of quantum mechanics! We also note that researchers were interested since the start by finding a relativistic wave equation. This is why the classical wave equation of Schrödinger (first order in time and involving the complex number \(i\) and the complex wavefunction \(\psi\)) was rather original, and unexpected, at that time. As we have seen, Schrödinger himself was originally interested by finding a relativistic wave equation. Klein\(^2\),\(^5\) made the transformation of Eq. (73) and showed the connection between the KG equation and the Schrödinger equation in the nonrelativistic limit \(c \to +\infty\) \[\text{see also Eq. (31)}\]. On the other hand, by making the WKB transformation \[\text{see Eq. (59)}\], it was found that the KG equation reduces to the relativistic Hamilton-Jacobi equation in the semi-classical limit \(\hbar \to 0\).\(^7\) Therefore, the KG equation appeared to be a physically relevant relativistic wave equation since it returned the correct results in the nonrelativistic limit \(c \to +\infty\) and in the semi-classical limit \(\hbar \to 0\).

\(^4\) As reported by Bhaumik\(^8\), Born was strongly influenced by Einstein in his interpretation as he stated in his Nobel lecture: “Again an idea of Einstein’s gave me the lead. He had tried to make the duality of particles - light quanta or photons - and waves comprehensible by interpreting the square of the optical wave amplitudes as probability density for the occurrence of photons. This concept could at once be carried over to the \(\psi\)-function: \(|\psi|^2\) ought to represent the probability density for electrons (or other particles).”

\(^5\) This is the so-called WKB approximation which allows one to study the semi-classical regime of a quantum system. One recovers the classical mechanics from the wave mechanics in the limit \(\hbar \to 0\) in the same manner that one recovers geometric optics from the theory of undulatory optics when the wavelength \(\lambda \to 0\).

\(^6\) We note that Klein\(^5\) and Fock\(^3\) derived their wave equation in a five dimensional spacetime in an attempt to unify electromagnetism and gravitation. This is the so-called Kaluza-Klein theory. The works of Klein\(^5\) and Fock\(^3\) were done independently to each other, and independently from the earlier work of Kaluza\(^9\). Their approach was followed by Ehrenfest and Uhlenbeck\(^7\), Gamow and Iwanenko\(^8\). We note that the standard form of the KG equation formulated in a four dimensional spacetime \[\text{see Eq. (F2)}\] first appeared in Schrödinger’s paper\(^6\).

\(^7\) The quantum Hamilton-Jacobi equation\(^9\) was written by Klein\(^5\) and De Donder\(^5\) but they did not write its complex hydrodynamic representation \[\text{see Eq. (15)}\]. It is interesting to note that De Donder\(^5\) postulated the relation\(^10\) to derive the wave equation\(^10\). In his approach, Eq. (22) is a fundamental equation connecting the classical mechanics to the wave mechanics.
At that time, most physicists were satisfied with the KG equation. However, Dirac was not happy with it because the KG equation is second order in time and so one cannot apply to it the transformation theory that Dirac cherished so much (it was his “darling”[1]). In the Introduction of his paper on the quantum theory of the electron, Dirac pointed out several problems with the KG equation. From the KG equation, one can introduce a density that satisfies a local conservation equation. However, this density is not definite positive so it cannot be interpreted as a density probability. As a result, the interpretation of the scalar field (SF) \( \varphi \) governed by the KG equation is unclear. Another difficulty with the KG equation is that it allows negative kinetic energies as solution. These negative kinetic energies cannot have a physical reality. Finally, the wave theory based on the KG equation, when applied to the hydrogen atom, does not give results in agreement with experiments because the observed number of stationary states for an electron in an atom is twice the number given by the theory. To account for this observation, Uhlenbeck and Goudsmit introduced the idea that the electron has a spin. This idea has been incorporated in the nonrelativistic wave theory by Pauli and Darwin, leading to the Pauli equation. This equation gives results in agreement with the experiment for hydrogen-like spectra to the first order of accuracy. However, this approach remains heuristic. Furthermore it is not fully relativistic.

In order to solve the first difficulty (the indeterminate sign of the density), Dirac proposed another relativistic extension of the Schrödinger equation. He obtained an equation, the Dirac equation, that satisfies the requirements of relativity and that is first order in time (his equation involves matrices that generalize the matrices that Pauli introduced in his equation). Therefore, one can apply the transformation theory to it. In Dirac’s theory, one can introduce a density that is positive definite and that satisfies a local conservation equation. This density can be interpreted as the probability density of presence of the electron. Therefore, the Dirac wave function \( \psi \) has a clear physical interpretation. Furthermore, when including the electromagnetic field, the Dirac equation gave the electron a spin \( s = 1/2 \) and the correct magnetic moment without ad hoc assumption. That was for Dirac a surprise and an unexpected bonus. Finally, the Dirac equation gave results in total agreement with observation of the spectrum of hydrogen.

There remained a difficulty with the Dirac equation. It was possible for the electron of charge \(-e\) to have states of positive and negative kinetic energy, this last case having no physical meaning. One cannot arbitrarily exclude solutions with negative energy (as we do in classical relativity theory) since, in general, a perturbation will cause transitions from states with positive energy to states with negative energy. Dirac solved the problem of negative energies via the “hole” theory. He assumed that all the negative energy states are occupied by electrons (Dirac sea) on account of Pauli’s exclusion principle, except for a few states with a low velocity. These vacant states are “holes”. These holes behave as particles with positive energy and charge \(+e\). Dirac did not dare to postulate a new particle. He published his work as a theory of electrons and protons, hoping that in some unexplained way the Coulomb interaction between the particles would account for the big difference of mass between electrons and protons. This consideration was criticized by Weyl who published a categorical statement that the new particle should have the same mass as the electron. This led to the concept of antimatter. The first antiparticle, the positron, was experimentally discovered by Anderson in 1932. Antimatter was unsuspected before Dirac’s work.

Although the KG equation is not a successful relativistic generalization of the Schrödinger equation to describe the electron and other spin-1/2 particles (fermions), this equation was resurrected in the context of quantum field theory where it was shown to describe spin-0 particles (bosons) such as \( \pi \)-mesons, pions, or the Higgs boson. There are therefore two possible theories for particles, both relativistic, one for the particles of zero spin satisfying the Bose-Einstein statistics, the other for particles of spin \( s = 1/2 \) satisfying the Fermi-Dirac statistics. The Dirac equation applies to electrons and to other particles of spin 1/2 like protons and quarks. The KG equation applies to certain kinds of mesons with zero spin. These particles are neutral for a real SF and charged for a complex SF.

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8 In his historic, Dirac mentions that, although Schrödinger discovered the KG equation, he was not “bold enough to publish it” because it did not give results in agreement with observations for the hydrogen spectrum. Actually, as we have previously indicated, Schrödinger did publish the KG equation in Ref. 23. This is even the first paper where the KG equation appears under its standard form. However, at the end of Ref. 23, Schrödinger points out difficulties with the relativistic theory: (i) the difficulty to extend it to more than one electron; (ii) its inability to account for the fine structure of the hydrogen atom. Concerning point (ii), Schrödinger foresaw that the deficiency may be solved by taking the spin of the electron into account.

9 As indicated previously, the conserved density was interpreted as a charge density (de Broglie interpreted it as a density of corpuscles, apparently not realizing the problem of sign).

10 The idea of the spin of the electron goes back to Compton. It was also imagined by Kronig but Pauli discouraged him to publish his results (see the very interesting anecdotes of Dirac on this subject).

11 The application of the KG equation to spin-0 particles was first proposed by Pauli and Weisskopf. They worked in terms of the energy density (different from the charge density that is definite positive and satisfies a local conservation equation).
In the 1960s, the equations of quantum mechanics found new applications in the context of Bose-Einstein condensates (BECs). The subject started with the work of Bogoliubov \[60\] who attempted to explain the phenomenon of superfluidity with a model of imperfect Bose gas with weak repulsion between atoms. At \( T = 0 \), all the bosons are in the same quantum state described by a single wavefunction \( \psi(r,t) \).\(^{12}\) When the particles interact through a pair contact potential \[61\,62\], the evolution of the wavefunction \( \psi(r,t) \) is described by a nonlinear version of the Schrödinger equation called the Gross-Pitaevskii (GP) equation \[63\,67\] which involves a cubic nonlinearity \[see Eq. (C30)\]. A relativistic extension of the GP equation is provided by the KG equation incorporating a potential \( V(|\psi|^2) \) that takes self-interaction into account \[see Eq. (C2)\]. A pair contact interaction between the bosons corresponds to a quartic potential in the KG equation \[see Eq. (3)\].

The coupling between the KG equation and gravity through the Einstein equations, leading to the Klein-Gordon-Einstein (KGE) equations, was considered by Kaup \(1968\) \[68\] and Ruffini and Bonazzola \(1969\) \[69\] in the context of boson stars. In a sense, boson stars are the descendant of the so-called geons of Wheeler \(1955\) \[70\] except that they are built from scalar particles (spin-0) instead of electromagnetic fields, i.e., spin-1 bosons. These authors determined the maximum mass of boson stars in general relativity. Ruffini and Bonazzola \[69\] also considered the nonrelativistic limit of their theory, leading to the Schrödinger-Poisson equation, describing Newtonian self-gravitating BEC stars. In the works of Kaup \[68\] and Ruffini and Bonazzola \[69\], it was assumed that the bosons have no self-interaction. Self-interacting boson stars were studied later by Colpi et al. \(1986\) \[71\]. They considered the case of a self-interacting SF described by the KGE equations with a quartic potential and showed that the self-interaction of the bosons can considerably increase the maximum mass of boson stars.

More recently, it was proposed that dark matter (DM) halos could be made of a SF described by the KGE equations. Actually, at the galactic scale, the Newtonian limit is valid so that DM halos can be described by the Schrödinger-Poisson (SP) equations or by the Gross-Pitaevskii-Poisson (GPP) equations. Therefore, DM halos could be gigantic quantum objects made of BECs. The wave properties of bosonic DM may stabilize the system against gravitational collapse, providing halo cores and sharply suppressing small-scale linear power. This may solve the problems of the cold dark matter (CDM) model such as the cusp problem and the missing satellite problem. The scalar field dark matter (SFDM) model and the BEC dark matter (BECDM) model, also called ΨDM models, have received much attention in the last years.\(^{13}\)

The Schrödinger equation has always been regarded as an abstract equation because it describes the evolution of a complex wavefunction \( \psi(r,t) \) whose interpretation is unclear and still makes debate. For that reason, other representations of the Schrödinger equation have been proposed. In a very intriguing early work, Madelung \(25\,October\,1926\) \[76\] made the transformation of Eqs. \[80\] and \[81\] and showed that the Schrödinger equation is mathematically equivalent to hydrodynamic equations, namely the equation of continuity \[see Eq. (190)\] and the Euler equation \[see Eq. (192)\] for a pressureless irrotational perfect fluid with an additional quantum potential \[see Eq. (194)\] arising from the finite value of \( h \). The quantum potential, or quantum force, accounts for the Heisenberg uncertainty principle. Because of this quantum term, the particle’s motion does not follow the laws of classical mechanics. The paper of Madelung was welcome with skepticism.\(^{14}\) For example, in his review of quantum mechanics, Pauli expresses the opinion that the hydrodynamic approach of Madelung is not very interesting \(see\ the\ comment\ in\ Ref. \[77\]\).

At about the same period, de Broglie \[43\,44\,72\] developed a relativistic hydrodynamic representation of the KG equation \( see\ the\ work\ of\ Madelung\) \[76\]. He made the transformation of Eqs. \[80\] and \[81\] and derived the relativistic quantum Euler equations \[84\] and \[85\] that contain the Lorentz invariant quantum potential \[87\].\(^{15}\) This is the relativistic version of the classical Madelung quantum potential \[88\]. De Broglie \[72\] interpreted the quantum force as a force of internal tensions existing around the corpuscles \( see\ also\ Rosenfeld \[43\,44\]\). He showed that everything happens as if the particles had an effective mass depending on the quantum potential \[see Eq. (193)\]. He also interpreted the continuity equation \[195\] as a conservation equation for a density transported by a velocity \[see Eq. (196)\]. The aim of de Broglie was to provide a causal and objective interpretation of wave

\(^{12}\) For a system of \( N \) bosons in interaction one has to consider in principle the \( N \)-body Schrödinger equation. It reduces to a self-consistent one-body Schrödinger equation when we neglect fluctuations and implement a mean field approximation. In that case, the \( N \)-body wave function is equal to a product of \( N \) one-body wave functions \( see\ the\ Hartree\ approximation\). This approximation is exact in the case of BECs when \( N \to +\infty \). This is equivalent to starting from the Heisenberg equation of motion for the wave function operator in the formalism of second quantification, as in the theory of Bogoliubov \[60\], and neglecting excitations.

\(^{13}\) The bibliography on boson stars and on SF/BEC dark matter is extensive. We refer the interested reader to the Introduction of \[72\] for a short historic and to the reviews \[76\,79\] for an exhaustive list of references.

\(^{14}\) His hydrodynamic approach was further developed by Kennard \[1927\] \[77\] in 1927. However, apart from that work, the paper of Madelung was very little quoted.

\(^{15}\) Actually, these relativistic hydrodynamic equations were derived by London \(25\,February\,1927\) \[80\] a little before de Broglie’s first paper on the subject \(1st\ April\,1927\) \[43\].
mechanics, in accordance with the wish expressed many times by Einstein, and in contrast to the purely probabilistic interpretation of quantum mechanics put forward by Born, Bohr, and Heisenberg. This is what he called the pilot wave theory because, in virtue of Eqs. (55) and (58), the particle is guided by the wave $\psi$. The pilot wave theory of de Broglie was criticized by Pauli during the October 1927 Solvay Physics Conference (see the comment in [81]), and de Broglie abandoned it.

The results of Madelung [76] and de Broglie [45, 46, 79] were rediscovered by Bohm [82, 83] in 1952 in relation to his interpretation of the quantum theory in terms of “hidden” variables. For that reason, the quantum potential is sometimes called the Bohm potential. Developing the works of Madelung [76], de Broglie [45, 46, 79] and Bohm [82, 83], Takabayasi [84, 85] proposed a formulation of classical and relativistic quantum mechanics in terms of hydrodynamic equations based on the Schrödinger and KG equations. He emphasized the role of the quantum force that is responsible for the “blurring” of the classical trajectory. He interpreted the diffusion of wave packets, interference effects and tunnel effects in terms of this quantum force. He also criticized certain aspects of Bohm’s interpretation. This renewal of interest for a causal interpretation of quantum mechanics stimulated de Broglie to return to the problem again and undertake a fresh examination of his old ideas [86–89].

The hydrodynamic representation of Madelung [76] and de Broglie [45, 46, 79] may lack a clear physical interpretation in the case where the Schrödinger equation or the KG equation describes just one particle. However, it takes more sense when the Schrödinger equation, the GP equation, or the KG equation describes a BEC made of many particles in the same quantum state. In that case, the BEC can be interpreted as a real fluid described by the quantum Euler equation. For self-interacting BECs, there is an additional pressure force coming from the potential of self-interaction. This hydrodynamic representation was developed by Gross [64, 65]. He writes [65]: “We have a model of a fluid which is more satisfactory than that of classical fluid dynamics”.

More recently, the Madelung hydrodynamic representation of the (nonrelativistic) Schrödinger, or GP, equation has been used by Böhmer and Harko [93], Chavanis [72, 96, 98] and Rindler-Daller and Shapiro [74, 92] among others to describe BEC dark matter (in a static or expanding background), and by Chavanis and Harko [100] to describe BEC stars such as boson stars or neutron stars with a superfluid core. On the other hand, the de Broglie hydrodynamic representation of the (relativistic) KG equation has been used by Suárez and Matos [101, 102] to study the formation of structures in the universe, assuming that dark matter is in the form of a fundamental SF with a quartic potential. In the works of Suárez and Matos [101, 102], the SF is taken to be real and the gravitational potential is introduced by hand in the KG equation, and assumed to be determined by the classical Poisson equation where the source is the rest-mass density $\rho$. This leads to the Klein-Gordon-Poisson (KGP) equations. However, this treatment is not self-consistent since it combines relativistic and nonrelativistic equations. A self-consistent relativistic treatment was developed by Suárez and Chavanis [103, 104], who derived the hydrodynamic representation of a complex SF evolving through the KG equation and coupled to gravity through the Einstein equations in the weak field approximation. This corresponds to the Klein-Gordon-Einstein (KGE) equations. They used the conformal Newtonian gauge which takes into account metric perturbations up to first order and considered only scalar perturbations. This is sufficient to calculate observational cosmological consequences of the SF dynamics in the linear regime.

On the other hand, the case where the SF interacts with an electromagnetic field in a flat spacetime has been considered recently by Matos and collaborators [105, 106]. They derived the hydrodynamic representation of the Klein-Gordon-Maxwell (KGM) equations. We note that the electromagnetic field was already taken into account in the pioneering works of de Broglie [45, 46, 79] and Takabayasi [84, 85].

In this paper, we combine these different approaches (electromagnetism and gravity) and consider the general case where the SF interacts with an electromagnetic field in a curved spacetime. Therefore, we derive the hydrodynamic representation of the Klein-Gordon-Maxwell-Einstein (KGME) equations. To our knowledge, this general case has not been treated previously. We consider the fully nonlinear Einstein equations and the weak field approximation in a static and expanding background. We recover the nonrelativistic (Newtonian) results when $c \to +\infty$. This is probably the most complete model that we can consider. The KGME equations describe self-gravitating Bose-Einstein condensates coupled to an electromagnetic field. They may find applications in the context of dark matter, boson stars, and neutron stars with a superfluid core.

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16 One interesting aspect of BECs is related to their superfluid properties. Their velocity field $v = \nabla S/m$ (where $S$ is the action) is irrotational but there may exist vortical motion due to singular point vortices with circulation quantized in units of $h/m$ as argued by Onsager in a footnote [90] and Feynman [91] (this fact was actually discovered by Dirac [92] in a more general context in which an electromagnetic field is present). In contrast to classical hydrodynamics, the cores of vortices are completely determined by the de Broglie length and all energies are finite.

17 A quantum hydrodynamics for a many body system was previously developed by Landau [93] and London [94] in connection to superfluidity and superconductivity. It seems that neither Landau, London and Gross were aware of the works of Madelung and de Broglie.
The paper is organized as follows. In Sec. II we consider the KGME equations in an arbitrary curved spacetime. In Sec. III we make the Klein transformation and derive the Gross-Pitaevskii-Maxwell-Einstein (GPME) equations. In Sec. IV we consider the nonrelativistic limit $c \rightarrow +\infty$ of the GPME equations leading to the Gross-Pitaevskii-Maxwell-Poisson (GPMP) equations. In each case, we derive a complex and real hydrodynamic representation of the field equations. The Appendices reorganize complementary results. In Appendix A we list various identities that are useful in our calculations. In Appendices B and C we consider the generalized Klein-Gordon-Maxwell-Poisson (KGMP) equations and the generalized GPMP equations in an expanding and static background. In Appendix D we derive the relativistic and nonrelativistic eigenvalues equations associated with the KG and Schrödinger equations. In Appendix E we discuss the relation between our approach and the pilot wave theory of de Broglie. In Appendix F we recall the historical derivation of the Schrödinger equation in complement to the discussion given in our Introduction.

II. THE KLEIN-GORDON-MAXWELL-EINSTEIN EQUATIONS

A. The Lagrangian of the scalar field coupled to the electromagnetic field

We consider a complex SF which is a continuous function of space and time defined at each point by $\varphi(x^\mu) = \varphi(x, y, z, t)$. The action of the relativistic SF is

$$S_\varphi = \int \mathcal{L}_\varphi \sqrt{-g} \, dx,$$

where $\mathcal{L}_\varphi = \mathcal{L}_\varphi(\varphi, \varphi^*, \partial_\mu \varphi, \partial_\mu \varphi^*)$ is the Lagrangian density and $g = \det(g_{\mu\nu})$ is the determinant of the metric tensor. In the absence of electromagnetic field, the Lagrangian of the SF is

$$\mathcal{L}_\varphi^{(0)} = \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi^* \partial_\nu \varphi - \frac{m^2 c^2}{2h^2} |\varphi|^2 - V(|\varphi|^2).$$

(2)

The first term is the kinetic energy, the second term (quadratic) is the rest-mass energy, and the third term is the self-interaction energy. In some applications, we shall consider a quartic SF potential of the form $V(|\varphi|^2) = \frac{\lambda_s}{m^4} |\varphi|^4$. If the SF describes a BEC at $T = 0$, the quartic potential can be rewritten as

$$V(|\varphi|^2) = \frac{2\pi a_s m}{\hbar^2} |\varphi|^4,$$

(3)

where $a_s$ denotes the s-scattering length of the bosons (in that case $\lambda_s = 4\pi a_s \hbar^2 / m$).

We assume that the SF interacts with an electromagnetic field described by the quadripotential $A^\mu$. We introduce the Faraday tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$  

(4)

The Lagrangian of the SF coupled to an electromagnetic field is obtained from the Lagrangian of the SF in the absence of electromagnetic field by making the substitution $\partial_\mu \rightarrow \partial_\mu + \partial_\mu F_{\mu\nu}$, where $e$ is the elementary charge, and adding the Lagrangian of the electromagnetic field $-\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu}$, where $\mu_0$ is the permeability of free space (for future reference we recall that the vacuum permeability $\mu_0$ and the vacuum permittivity $\epsilon_0$ satisfy the relation $\epsilon_0 \mu_0 = 1/c^2$). This yields

$$\mathcal{L}_\varphi = \frac{1}{2} g^{\mu\nu} \left( \partial_\mu \varphi^* - i \frac{e}{\hbar} A_\mu \varphi^* \right) \left( \partial_\nu \varphi + i \frac{e}{\hbar} A_\nu \varphi \right) - \frac{m^2 c^2}{2h^2} |\varphi|^2 - V(|\varphi|^2) - \frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu}.$$  

(5)

The Lagrangian of the SF can be expanded as

$$\mathcal{L}_\varphi = \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi^* \partial_\nu \varphi + \frac{1}{2} g^{\mu\nu} i \frac{e}{\hbar} A_\mu \varphi \partial_\nu \varphi^* - \frac{1}{2} g^{\mu\nu} i \frac{e}{\hbar} A_\mu \varphi^* \partial_\nu \varphi + \frac{1}{2} g^{\mu\nu} \frac{e^2}{h^2} A_\mu A_\nu |\varphi|^2 - \frac{m^2 c^2}{2h^2} |\varphi|^2 - V(|\varphi|^2) - \frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu}.$$  

(6)

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18 In general relativity, the Faraday tensor should be written as $F_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu$ where $D$ is the covariant derivative defined by Eq. (A1). Using Eqs. (A2) and (A3), we have $D_\mu A_\nu - D_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$ leading to Eq. (4).
B. The energy-momentum tensor

Taking the variation of the SF action \( \delta S \) with respect to the metric \( g_{\mu\nu} \), we get

\[
\delta S = \frac{1}{2} \int T_{\mu\nu} \delta g^{\mu\nu} \sqrt{-g} \, d^4x,
\]

where

\[
T_{\mu\nu} = \frac{1}{2} \partial_{\nu} \partial^{\alpha} - \frac{m^2 c^2}{\hbar^2} |\phi|^2 - V(|\phi|^2) \]

is the energy-momentum tensor of the SF. For the Lagrangian (6), it takes the form

\[
T_{\mu\nu} = \frac{1}{2} \left( \partial_{\mu} \phi \partial_{\nu} \phi - \frac{m^2 c^2}{\hbar^2} |\phi|^2 - \nabla^2 \phi \right) + \frac{1}{2} \left( \partial_{\mu} A_{\nu} \phi \partial_{\nu} \phi - \frac{m^2 c^2}{\hbar^2} |\phi|^2 - \nabla^2 \phi \right) + \left( F_{\mu\alpha} F_{\nu}^{\alpha} + \frac{1}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \right). \]

We can also obtain the energy-momentum tensor of a SF coupled to an electromagnetic field from the energy-momentum tensor \( T_{\mu\nu}^{(0)} \) of a SF in the absence of electromagnetic field by making the substitution \( \partial_{\mu} \to \partial_{\mu} + ie A_{\mu} \) and adding the energy-momentum tensor of the electromagnetic field \( \frac{1}{\mu_0} \left( F_{\mu\alpha} F_{\nu}^{\alpha} + \frac{1}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \right) \).

The conservation of the energy-momentum tensor, which results via the Noether theorem from the invariance of the Lagrangian density under continuous translations in space and time, writes

\[
D_\nu T^{\mu\nu} = 0.
\]

By analogy with the energy-momentum tensor of a perfect fluid, the energy density and the pressure tensor of the SF are defined by

\[
\epsilon = T_0^0, \quad P_i^j = -T_i^j.
\]

On the other hand, using Eq. (9), one can show that the quantity \( (m^2/\hbar^2)|\phi|^2 \) tends to the rest-mass density \( \rho \) in the non-relativistic limit \( c \to +\infty \) (see Appendix A of [103]). Therefore, we shall interpret

\[
\rho = \frac{m^2}{\hbar^2} |\phi|^2
\]

as a pseudo rest-mass density. In the relativistic regime, the density \( \rho \) defined by Eq. (12) has not a clear physical interpretation (it is not the rest-mass density and \( \rho c^2 \) is not the energy density). However, it can always be introduced as a convenient notation [103]. It is only in the non-relativistic limit \( c \to +\infty \) that \( \rho \) can be identified with the rest-mass density.

C. The Klein-Gordon equation

The equation of motion for the SF can be derived from the principle of least action. Imposing \( \delta S = 0 \) for arbitrary variations \( \delta \phi \) and \( \delta \phi^* \) of the field, we obtain the Euler-Lagrange equation

\[
D_\mu \left( \frac{\partial L_\phi}{\partial (\partial_\mu \phi)} \right) - \frac{\partial L_\phi}{\partial \phi^*} = 0.
\]

For the Lagrangian (6), this leads to the electromagnetic KG equation

\[
\Box \phi + \frac{m^2 c^2}{\hbar^2} \phi + 2 \frac{dV}{d|\phi|^2} \phi = 0,
\]
\[ \square e \phi = \left( D_\mu + i \frac{e}{\hbar} A_\mu \right) \left( \partial^\mu + i \frac{e}{\hbar} A^\mu \right) \phi. \] (15)

The KG equation of a SF coupled to an electromagnetic field can also be obtained from the KG equation of a SF in the absence of electromagnetic field by making the substitution \( D_\mu \rightarrow D_\mu + i \frac{e}{\hbar} A_\mu \). For the quartic potential \((3)\), the KG equation \((13)\) becomes

\[ \square e \phi + \frac{m^2 c^2}{\hbar^2} \phi + \frac{8 \pi a_s m}{\hbar^2} |\phi|^2 \phi = 0. \] (16)

Using the identities of Appendix A, the electromagnetic d’Alembertian operator \((15)\) can be written in the equivalent forms

\[ \square e \phi = D_\mu \partial^\mu \phi + i \frac{e}{\hbar} (D_\mu A^\mu) \phi + 2i \frac{e}{\hbar} A_\mu \partial^\mu \phi - \frac{e^2}{\hbar^2} A^\mu A^\mu \phi, \] (17)

\[ \square e \phi = g^{\mu \nu} \partial_\mu \partial_\nu \phi - g^{\mu \nu} \Gamma^\sigma_{\mu \nu} \partial_\sigma \phi + i \frac{e}{\hbar} g^{\mu \nu} \left( \partial_\mu A_\nu - \Gamma^\sigma_{\mu \nu} A_\sigma \right) \phi + 2i \frac{e}{\hbar} g^{\mu \nu} A_\mu \partial_\nu \phi - \frac{e^2}{\hbar^2} g^{\mu \nu} A_\mu A_\nu \phi, \] (18)

\[ \square e \phi = g^{\mu \nu} \left( \partial_\mu + i \frac{e}{\hbar} A_\mu \right) \left( \partial_\nu + i \frac{e}{\hbar} A_\nu \right) \phi - g^{\mu \nu} \Gamma^\sigma_{\mu \nu} \left( \partial_\sigma + i \frac{e}{\hbar} A_\sigma \right) \phi. \] (19)

We note that the first term in Eq. \((17)\) corresponds to the d’Alembertian operator in a curved spacetime in the absence of electromagnetic field

\[ \square \phi = D_\mu \partial^\mu \phi = g^{\mu \nu} D_\mu \partial_\nu \phi = g^{\mu \nu} \left( \partial_\mu \partial_\nu \phi - \Gamma^\sigma_{\mu \nu} \partial_\sigma \phi \right). \] (20)

We also note that the second term in Eq. \((17)\) disappears if we make the choice of the Lorentz gauge

\[ D_\mu A^\mu = 0. \] (21)

\[ J_\mu = -\frac{m}{2\hbar} \left( \phi^* \partial^\mu \phi - \phi \partial^\mu \phi^* \right) - \frac{em}{\hbar^2} |\phi|^2 A_\mu. \] (22)

As recalled in the Introduction, there is a difficulty with the interpretation of the KG equation because the density \( J_0 \) is not definite positive. As a result, it cannot be interpreted as a density probability, or as a mass density. However, it can be interpreted as a charge density that can take positive or negative values. Therefore, we introduce the quadricurrent of charge

\[ (J_\phi)_\mu \equiv \frac{e}{m} J_\mu = -\frac{e}{2\hbar} \left( \phi^* \partial^\mu \phi - \phi \partial^\mu \phi^* \right) - \frac{e^2}{\hbar^2} |\phi|^2 A_\mu. \] (23)

We note that the current vanishes for a real scalar field. Therefore, only complex scalar fields are charged. We define the charge density by \( \rho_c = (J_\phi)_0 / c \) and the current of charge by \( J_c = ((J_\phi)_x, (J_\phi)_y, (J_\phi)_z) = -(J_c)_1, -(J_c)_2, -(J_c)_3 \) with the lower indices.\(^{19}\) Similarly, we define the electric potential by \( U / c = A_0 \) and the potential vector by \( A = (A_x, A_y, A_z) = (-A_1, -A_2, -A_3) \). The charge density and the current of charge are then given by

\[ \rho = \frac{e}{2\hbar c^2} \left( \phi \frac{\partial \phi^*}{\partial t} - \phi^* \frac{\partial \phi}{\partial t} \right) - \frac{e^2}{\hbar^2 c^2} |\phi|^2 U, \] (24)

\(^{19}\) This is not the usual convention but we find that the equations are simpler when we use this convention. Note that our convention (with the minus sign) reduces to the usual one in the case of a flat spacetime.
\[ J_e = \frac{e}{2i\hbar} (\phi^* \nabla \varphi - \varphi \nabla \phi^*) - \frac{e^2}{\hbar^2} |\varphi|^2 A. \]  

(25)

Taking the divergence of Eq. (23) and using the KG equation (14), one can show that

\[ D_\mu J^\mu_e = 0. \]  

(26)

This equation expresses the local conservation of the charge of a complex SF. The global conservation of charge can be obtained as follows. Integrating Eq. (26) over the whole physical space and using Eq. (A1), we get

\[ 0 = \int D_\mu J^\mu_e \sqrt{-g} \, d^3 x = \int \partial_\mu \left( \sqrt{-g} J^\mu_e \right) \, d^3 x = \int \partial_0 \left( \sqrt{-g} J^0_e \right) \, d^3 x + \int \partial_i \left( \sqrt{-g} J^i_e \right) \, d^3 x = \frac{1}{e} \int \partial_i \left( \sqrt{-g} J^0_e \right) \, d^3 x, \]  

(27)

where the last equality is obtained because the second integral in the second line can be converted into a surface term that vanishes at infinity. If we define the total charge by

\[ Q = \frac{1}{e} \int J^0_e \sqrt{-g} \, d^3 x, \]  

(28)

we find that Eq. (27) takes the form

\[ \frac{dQ}{dt} = 0. \]  

(29)

This equation expresses the global conservation of the charge of a complex SF. The conserved quadricurrent of charge and the conserved charge result via the Noether theorem from the invariance of the Lagrangian density under a global phase transformation \( \varphi \to \varphi e^{-i\theta} \) of the complex SF.

E. The Maxwell equations

The Maxwell equations can be obtained from the principle of least action. Imposing \( \delta S_\varphi = 0 \) for arbitrary variations \( \delta A^\nu \) of the quadripotential, we obtain the Euler-Lagrange equations

\[ D_\mu \left[ \frac{\partial L_\varphi}{\partial (\partial_\mu A^\nu)} \right] - \frac{\partial L_\varphi}{\partial A^\nu} = 0. \]  

(30)

For the Lagrangian (5), we have

\[ \frac{\partial L_\varphi}{\partial A^\nu} = -J^\nu_e, \quad \frac{\partial L_\varphi}{\partial (\partial_\mu A^\nu)} = -\frac{1}{\mu_0} F^{\mu\nu}, \]  

(31)

leading to the Maxwell equations

\[ D_\mu F^{\mu\nu} = \mu_0 J^\nu_e. \]  

(32)

For a complex SF, the quadricurrent of charge \( J^\mu_e \) is given by Eq. (23). The conservation of charge expressed by Eq. (26) is included in the Maxwell equations (it results from the anti-symmetry of the Faraday tensor). Substituting Eq. (4) into the Maxwell equations (32), we obtain the field equations satisfied by the quadripotential

\[ D_\mu D^{\mu\nu} A_\nu - D_\mu D_\nu A_\mu = \mu_0 (J^\nu_e). \]  

(33)

Using the identities (A11) and (A12), we can rewrite Eq. (33) as

\[ \Box A_\nu - D_\nu D^\mu A_\mu - R_{\mu\nu} A^\mu = \mu_0 (J^\nu_e), \]  

(34)

where \( R_{\mu\nu} \) is the Ricci tensor. With the choice of the Lorentz gauge (21), Eq. (34) reduces to

\[ \Box A_\nu - R_{\mu\nu} A^\mu = \mu_0 (J^\nu_e). \]  

(35)
F. The Einstein equations

The Einstein-Hilbert action in general relativity is defined by
\[ S_g = \frac{c^4}{16\pi G} \int R \sqrt{-g} \, d^4x, \]  
where \( R \) is the Ricci scalar and \( G \) is Newton’s gravitational constant. Its variation with respect to the metric \( g_{\mu\nu} \) is given by [107]:
\[ \delta S_g = -\frac{c^4}{16\pi G} \int \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \delta g^{\mu\nu} \sqrt{-g} \, d^4x. \]  
\[ (37) \]

The total action \((\text{SF} + \text{gravity})\) is \( S = S_\phi + S_g \). The field equations can be obtained from the principle of least action. Imposing \( \delta S = 0 \) for arbitrary variations \( \delta g_{\mu\nu} \) of the metric, and using Eqs. (7) and (37), we obtain the Einstein equations
\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G \frac{T_{\mu\nu}}{c^4}. \]  
\[ (38) \]

These are a set of 10 equations that describe the fundamental interaction between gravity and matter as a result of the curvature of spacetime. The energy-momentum tensor \( T_{\mu\nu} \) is the source of the gravitational field in the Einstein field equations of general relativity in the sense that it determines the metric \( g_{\mu\nu} \). The conservation of the energy-momentum tensor expressed by Eq. (10) is included in the Einstein equations. Eqs. (14), (32) and (38) form the KGME equations.

G. The complex hydrodynamic equations

In this section, we restrict ourselves to the electromagnetic KG equation [14] without self-interaction \((V = 0)\). We write the SF in the WKB form
\[ \varphi \propto e^{iS_{\text{tot}}/\hbar}, \]  
where \( S_{\text{tot}} \) is a complex action. Substituting Eq. (39) into Eq. (14), we obtain the complex quantum relativistic Hamilton-Jacobi equation
\[ (\partial_\mu S_{\text{tot}} + eA_\mu)(\partial^\mu S_{\text{tot}} + eA^\mu) - m^2 c^2 = i\hbar D_\mu(\partial^\mu S_{\text{tot}} + eA^\mu) = i\hbar \Box S_{\text{tot}} + i\hbar D_\mu A^\mu. \]  
\[ (40) \]

For \( \hbar = 0 \), we recover the relativistic Hamilton-Jacobi equation (in that case \( S_{\text{tot}} \) is real):
\[ [S_{\text{tot}}] = (\partial_\mu S_{\text{tot}} + eA_\mu)(\partial^\mu S_{\text{tot}} + eA^\mu) - m^2 c^2 = 0. \]  
\[ (41) \]

With the choice of the Lorentz gauge [21], the complex quantum relativistic Hamilton-Jacobi equation (40) can be written as
\[ [S_{\text{tot}}] = i\hbar \Box S_{\text{tot}}. \]  
\[ (42) \]

We introduce a complex quadrivelocity
\[ U_\mu = -\frac{\partial_\mu S_{\text{tot}} + eA_\mu}{m} \]  
\[ (43) \]
whose components are
\[ U_0 = -\frac{1}{mc} \left( \frac{\partial S_{\text{tot}}}{\partial t} + eU \right), \quad U = \frac{\nabla S_{\text{tot}} - eA}{m}. \]  
\[ (44) \]

In consistency with our previous conventions, we have defined \( U = (U_x, U_y, U_z) = (-U_1, -U_2, -U_3) \). We also introduce a complex energy \( E_{\text{tot}} \) such that \( U_0 = E_{\text{tot}}/mc \). According to Eq. (43), we have
\[ E_{\text{tot}} = -\frac{\partial S_{\text{tot}}}{\partial t} - eU. \]  
\[ (45) \]
Substituting Eq. (43) into Eq. (40), we find that the complex quantum relativistic Hamilton-Jacobi equation takes the form

\[ U_\mu U^\mu - c^2 = -i \frac{\hbar}{m} D_\mu U^\mu. \] (46)

For \( \hbar = 0 \), it reduces to (in that case \( u_\mu \) is real):

\[ u_\mu u^\mu = c^2. \] (47)

This corresponds to the relativistic equation of mechanics \( p_\mu p^\mu = m^2 c^2 \) where \( m u_\mu \) represents the impulse \( p^\mu \). Taking the gradient of Eq. (46) and using the identities (A10), (A11) and (A12), we obtain the equation

\[ U^\mu D_\nu U_\mu = -i \frac{\hbar}{2m} (D_\mu D_\nu U^\mu - R_{\mu
u} U^\mu). \] (48)

Using the relation

\[ D_\mu U_\nu - D_\nu U_\mu = -\frac{e}{m} F_{\mu\nu} \] (49)

obtained from Eqs. (4), (43) and (A13), we can rewrite Eq. (48) as

\[ \frac{dU_\nu}{d\tau} \equiv U^\mu D_\mu U_\nu = -\frac{e}{m} F_{\mu\nu} - i \frac{\hbar}{2m} \Box U_\nu - i \frac{\hbar e}{2m^2} D_\mu F_{\mu\nu} + \frac{\hbar}{2m} R_{\mu\nu} U^\mu. \] (50)

Equation (50) can be interpreted as a complex quantum relativistic Euler-Lorentz equation. The first term on the r.h.s. is a relativistic viscous term with a complex viscosity

\[ \nu = \frac{i\hbar}{2m}, \] (51)

the second term is a complex Lorentz force, the third term is a peculiar complex electromagnetic quantum force and the fourth term arises from the curvature of spacetime. The KG equation (14) is equivalent to the complex hydrodynamic equation (50). For \( \hbar = 0 \), we recover the relativistic Euler-Lorentz equation (in that case \( u_\mu \) is real):

\[ \frac{du_\nu}{d\tau} \equiv u_\mu D_\mu u_\nu = -\frac{e}{m} F_{\mu\nu}. \] (52)

**H. The real hydrodynamic equations**

We write the SF in the de Broglie form

\[ \varphi = \frac{\hbar}{m} \sqrt{\rho e} e^{iS_{\text{tot}}/\hbar}, \] (53)

where \( \rho \) is the pseudo rest-mass density (12) and

\[ S_{\text{tot}} = \frac{\hbar}{2i} \ln \left( \frac{\varphi}{\varphi^*} \right) \] (54)

is a real action. Making the de Broglie transformation (53) in the electromagnetic KG equation (14), and separating real and imaginary parts, we obtain the pair of equations

\[ D_\mu \left[ \rho (\partial^\mu S_{\text{tot}} + e A^\mu) \right] = 0, \] (55)

\[ (\partial_\mu S_{\text{tot}} + e A_\mu) (\partial^\mu S_{\text{tot}} + e A^\mu) - \hbar^2 \frac{\Box}{\sqrt{\rho}} - m^2 c^2 - 2m^2 \beta^2 (\rho) = 0. \] (56)

Equation (55) can be interpreted as a continuity equation and Eq. (56) can be interpreted as a quantum relativistic Hamilton-Jacobi equation with a relativistic covariant quantum potential

\[ Q = -\frac{\hbar^2}{2m} \frac{\Box}{\sqrt{\rho}}. \] (57)
For $\hbar = 0$, we recover the relativistic Hamilton-Jacobi equation (11) with the additional term $-2m^2V'(\rho)$. Following de Broglie, we introduce the quadrivelocity

$$u_\mu = -\frac{\partial_\mu S_{tot} + eA_\mu}{m}$$

(58)

whose components are

$$u_0 = -\frac{1}{mc} \left( \partial S_{tot} \partial t + eU \right), \quad u = \nabla S_{tot} - eA,$$

(59)

where we have defined $u = (u_x, u_y, u_z) = (-u_1, -u_2, -u_3)$. We also introduce an energy $E_{tot}$ such that $u_0 = E_{tot}/mc$. According to Eq. (59), we have

$$E_{tot} = -\frac{\partial S_{tot}}{\partial t} - eU.$$  

(60)

Using Eq. (58), we can rewrite Eqs. (55) and (56) as

$$D_\mu (\rho u^\mu) = 0,$$

(61)

$$u_\mu u^\mu = \frac{\hbar^2}{m^2} \nabla^2 \sqrt{\rho} + c^2 + 2V'(\rho).$$

(62)

Under this form, it is clear that Eq. (61) can be interpreted as a continuity equation (we show in Sec. III that it is equivalent to the local charge conservation). On the other hand, Eq. (62) can be interpreted as a quantum relativistic Hamilton-Jacobi or Bernoulli equation. For $\hbar = 0$, we recover the relativistic Hamilton-Jacobi equation (47) with the additional term $2V'(\rho)$. Taking the gradient of Eq. (62) and using the identity (A10) and the relation

$$D_\mu u_\nu - D_\nu u_\mu = -\frac{e}{m} F_\mu\nu$$

(63)

obtained from Eqs. (4), (58) and (A13), we get

$$\frac{du_\nu}{d\tau} \equiv u_\mu D_\mu u_\nu = \frac{\hbar^2}{2m^2} D_\nu \left( \frac{\nabla \rho}{\sqrt{\rho}} \right) - \frac{e}{m} u_\mu F_\mu\nu + D_\nu V'(\rho).$$

(64)

Equation (64) can be interpreted as a quantum relativistic Euler-Lorentz equation. The first term on the r.h.s. is the relativistic quantum force, the second term is the Lorentz force and the third term is a pressure force arising from the self-interaction of the SF. The KG equation (14) is equivalent to the hydrodynamic equations (61), (62) and (64). For $\hbar = 0$, we recover the relativistic Euler-Lorentz equation (52) with the additional term $D_\nu V'(\rho)$.

I. The hydrodynamic representation of the current

According to Eq. (54), we have

$$\partial_\mu S_{tot} = \frac{\hbar}{2i|\varphi|^2} \left( \varphi^* \partial_\mu \varphi - \varphi \partial_\mu \varphi^* \right),$$

(65)

$$\frac{\partial S_{tot}}{\partial t} = \frac{\hbar}{2i|\varphi|^2} \left( \varphi^* \frac{\partial \varphi}{\partial t} - \varphi \frac{\partial \varphi^*}{\partial t} \right), \quad \nabla S_{tot} = \frac{\hbar}{2i|\varphi|^2} \left( \varphi^* \nabla \varphi - \varphi \nabla \varphi^* \right).$$

(66)

As a result, the quadricurrent of charge, the charge density and the current of charge defined by Eqs. (23)-(25) can be rewritten as

$$(J_e)_\mu = -\frac{e}{m} \rho \frac{\partial S_{tot}}{\partial t} + eA_\mu,$$

$$\rho = -\frac{e}{mc} \rho \frac{\partial S_{tot}}{\partial t} + eU,$$

$$J_e = \frac{e}{m} \nabla S_{tot} - eA.$$  

(67)

Comparing these expressions with Eqs. (55)-(60), we obtain

$$(J_e)_\mu = \frac{e}{m} \rho u_\mu, \quad \rho = \frac{e}{mc} \rho u_0 = \frac{e}{mc^2} \rho E_{tot}, \quad J_e = \frac{e}{m} \rho u.$$  

(68)

Using the relation (68) between the quadricurrent of charge and the quadrivelocity, we find that the continuity equation (61) is equivalent to the local charge conservation equation (26). On the other hand, the Maxwell equations (32) can be rewritten as

$$D_\mu F^{\mu\nu} = \mu_0 \frac{e}{m} \rho u^\nu.$$  

(69)
J. The general relativistic London equation

In the case where the term $\partial_\mu S_{\text{tot}}$ can be neglected as compared to $eA_\mu$ in Eq. (67), the quadricurrent of charge, the charge density and the current of charge reduce to

$$(J_e)_\mu = -\frac{e^2}{m^2}eA_\mu, \quad \rho_e = -\frac{e^2}{m^2}eU, \quad J_e = -\frac{e^2}{m^2}eA.$$  \hspace{1cm} (70)

These equations are very similar to the London equations that were introduced phenomenologically by the brothers London \cite{108} in 1934 in their theory of superconductivity.\footnote{In their Concluding Remarks, they discuss the link between their phenomenological equations and the formulae for the charge density and the current of charge associated with the KG equation. This exactly corresponds to the presentation that we have given here (we arrived at these results independently). It is often said that the London equations are purely phenomenological. Actually, they can be derived from the KG equation. This gives a precise mathematical meaning to the density $\rho$ that appears in the London equations. In their phenomenological approach, $\rho$ is identified with the particle density. In our approach (and in their Concluding Remarks), $\rho$ corresponds to the pseudo rest-mass density $\rho_e$ that has not a straightforward interpretation. However, in many cases, it is expected to be of the same order of magnitude as the particle density.}

Substituting Eq. (70) into Eq. (34), we obtain the general relativistic London equation

$$\Box A_\nu - D_\nu D_\mu A^\mu - R_{\mu\nu} A^\mu = -\mu_0 \frac{e^2}{m^2}eA_\nu.$$ \hspace{1cm} (71)

With the choice of the Lorentz gauge (21), Eq. (71) reduces to

$$\Box A_\nu - R_{\mu\nu} A^\mu = -\mu_0 \frac{e^2}{m^2}eA_\nu.$$ \hspace{1cm} (72)

III. THE GROSS-PITAEVSKII-MAXWELL-EINSTEIN EQUATIONS

A. The general relativistic Gross-Pitaevskii equation

The KG equation (14) expressed in terms of the SF $\varphi$ does not have a well-defined limit when $c \to +\infty$. In order to recover the GP equation in the nonrelativistic limit $c \to +\infty$ we have to make the Klein transformation

$$\varphi(r,t) = \frac{\hbar}{m} e^{-imc^2 t/\hbar} \psi(r,t).$$ \hspace{1cm} (73)

The new field $\psi$ will be called the pseudo wavefunction. It is related to the pseudo rest-mass density (12) by

$$\rho = |\psi|^2.$$ \hspace{1cm} (74)

It is only in the nonrelativistic limit $c \to +\infty$ that $\psi$ can be identified with the wavefunction and that $\rho$ can be identified with the rest-mass density. However, we can always make the Klein transformation (73) in the KG equation (14) even if we are not in the nonrelativistic limit \cite{108}. In that case, we obtain the general relativistic GP equation (see Appendix \textit{G}):

$$i\hbar \frac{\partial \psi}{\partial t} - \frac{\hbar^2}{2m} \Box \psi + \frac{1}{2} mc^2 \left( g^{00} - 1 \right) \psi - ecA^0 \psi - m \frac{dV}{d|\psi|^2} \psi - \frac{1}{2} i\hbar cg^{\mu\nu} \Gamma^0_{\mu\nu} \psi = 0.$$ \hspace{1cm} (75)

It can also be written as

$$i\hbar g^{00} \frac{\partial \psi}{\partial t} - \frac{\hbar^2}{2m} \left( D_\mu + ie \frac{e}{\hbar} A_\mu \right) \left( \partial^\mu + ie \frac{e}{\hbar} A^\mu \right) \psi + \frac{1}{2} mc^2 \left( g^{00} - 1 \right) \psi - ecg^{00} A_0 \psi - m \frac{dV}{d|\psi|^2} \psi
+ i\hbar cg^{0j} \left( \partial_j + ie \frac{e}{\hbar} A_j \right) \psi - \frac{1}{2} i\hbar cg^{\mu\nu} \Gamma^0_{\mu\nu} \psi = 0.$$ \hspace{1cm} (76)

The GP equation (75) coupled to the Maxwell equations (32) and to the Einstein equations (38) written in terms of $\psi$ form the GPME equations.
B. The charge density and the current of charge

Using the Klein transformation (73), the quadricurrent of charge, the charge density and the current of charge defined by Eqs. (23)-(25) can be expressed in terms of the pseudo wavefunction $\psi$ as

\[
(J_e)_\mu = -\frac{e\hbar}{2im^2} \left( \psi^* \partial_\mu \psi - \psi \partial_\mu \psi^* - \frac{2imc}{\hbar} |\psi|^2 \delta_\mu^0 \right) - \frac{e^2}{m^2} |\psi|^2 A_\mu,
\]

(77)

\[
\rho_e = -\frac{e\hbar}{2im^2c^2} \left( \psi^* \frac{\partial}{\partial t} \psi - \psi \frac{\partial}{\partial t} \psi^* - \frac{2imc^2}{\hbar} |\psi|^2 \right) - \frac{e^2}{m^2c^2} |\psi|^2 U,
\]

(78)

\[
J_e = -\frac{e\hbar}{2im^2} (\psi^* \nabla \psi - \psi \nabla \psi^*) - \frac{e^2}{m^2} |\psi|^2 A.
\]

(79)

C. The complex hydrodynamic equations

In this section, we restrict ourselves to the electromagnetic KG equation (14) without self-interaction ($V = 0$). We write the pseudo wave function under the WKB form

\[
\psi \propto e^{iS/\hbar},
\]

(80)

where $S$ is a complex action. We introduce a complex quadrivelocility

\[
V_\mu = -\frac{\partial_\mu S + eA_\mu}{m}, \quad V_0 = -\frac{1}{mc} \left( \frac{\partial S}{\partial t} + eU \right), \quad V = \nabla S - \frac{eA}{m}.
\]

(81)

We also introduce a complex energy $\mathcal{E}$ such that $V_0 = \mathcal{E}/mc$. According to Eq. (81), we have

\[
\mathcal{E} = -\frac{\partial S}{\partial t} - eU.
\]

(82)

Combining Eqs. (73) and (80), we obtain

\[
\varphi \propto e^{i(S-mc^2t)/\hbar}.
\]

(83)

Comparing this equation with Eq. (39), we get

\[
S_{\text{tot}} = S - mc^2 t.
\]

(84)

Therefore, we find the relations

\[
U_\mu = V_\mu + c_\mu^0, \quad U^\mu = V^\mu + cg^\mu_0,
\]

\[
U_0 = V_0 + c, \quad U = V, \quad \mathcal{E}_{\text{tot}} = \mathcal{E} + mc^2.
\]

(85)

(86)

In order to obtain the complex hydrodynamic representation of the general relativistic GP equation (75), we can perform the WKB transformation (80), introduce the quadrivelocility (81) and proceed as in Sec. II G. Alternatively, using the relations (84)-(86), we can directly rewrite the complex hydrodynamic equations of Sec. II G in terms of $S$ and $V_\mu$ instead of $S_{\text{tot}}$ and $U_\mu$. The complex quantum relativistic Hamilton-Jacobi equation (46) and the complex quantum relativistic Euler-Lorentz equation (50) become

\[
V_\mu V^\nu = c^2(1 - g^{\mu\nu}) - 2cV^0 - \frac{\hbar}{m} D_\mu V^\nu + i \frac{\hbar c}{m} g^{\mu\nu} \Gamma_0^0,
\]

\[
\frac{dV_\nu}{d\tau} \equiv V^\mu D_\mu V_\nu = -cD^0 V_\nu - \frac{\hbar}{2m} \square V_\nu - \frac{c}{m} V^\mu F_{\mu\nu} - \frac{c}{m} cF_0^\nu - i \frac{\hbar c}{2m^2} D^\mu F_{\mu\nu} + i \frac{\hbar}{2m} R_{\nu\mu} V^\mu + i \frac{\hbar c}{2m} R_0^\nu V^\mu + c^2 g^{\mu0} \Gamma_0^\nu + c^2 g^{00} \Gamma_0^\mu + i \frac{\hbar c}{2m} D^\mu \Gamma_0^0.
\]

(87)

(88)

These complex hydrodynamic equations are equivalent to the general relativistic GP equation (75).
D. The real hydrodynamic equations

We write the pseudo wave function under the Madelung form

$$\psi(r, t) = \sqrt{\rho(r, t)} e^{iS(r, t)/\hbar}, \quad (89)$$

where $\rho$ is the pseudo rest-mass density and $S = \frac{1}{2} i \hbar \ln \left( \frac{\psi^*}{\psi} \right)$ is a real action. We introduce a quadrivelocity

$$v^\mu = -\frac{\partial S + eA^\mu}{m}, \quad v_0 = -\frac{1}{mc} \left( \frac{\partial S}{\partial t} + eU \right), \quad v = \nabla S - eA^\mu. \quad (91)$$

We also introduce an energy $E$ such that $v_0 = E/mc$. According to Eq. (91), we have

$$E = -\frac{\partial S}{\partial t} - eU. \quad (92)$$

Combining Eqs. (73) and (89), we obtain

$$\phi = m \hbar \sqrt{\rho} e^{i(S - mc^2 t)/\hbar}. \quad (93)$$

Comparing this equation with Eq. (53), we get

$$S_{\text{tot}} = S - mc^2 t. \quad (94)$$

Therefore, we find the relations

$$u^\mu = v^\mu + c \delta^\mu_0, \quad u^\mu = v^\mu + cg^\mu_0, \quad \rho v^\mu = \rho v^\mu + c \delta^\mu_0, \quad \frac{1}{2} \rho v^\mu v_\mu = -2c^2 - 2 \frac{\hbar^2}{m^2} \nabla^2 \rho + 2 \frac{dV}{d\rho}, \quad (95)$$

$$u_0 = v_0 + c, \quad u = v, \quad E_{\text{tot}} = E + mc^2. \quad (96)$$

In order to obtain the real hydrodynamic representation of the general relativistic GP equation (75), we can perform the Madelung transformation (89), introduce the quadrivelocity (91) and proceed as in Sec. II H. Alternatively, using the relations (94)-(96), we can directly rewrite the complex hydrodynamic equations of Sec. II H in terms of $S$ and $v^\mu$ instead of $S_{\text{tot}}$ and $u^\mu$. The continuity equation (61), the quantum relativistic Hamilton-Jacobi equation (62), and the quantum relativistic Euler-Lorentz equation (64) become

$$D_\mu (\rho v^\mu) + c \delta^\mu_0 \rho = 0, \quad (97)$$

$$v_\mu v^\mu = -2c^2 + 2 \left( 1 - g^0_0 \right) + 2 \frac{\hbar^2}{m^2} \nabla^2 \rho + 2 \frac{dV}{d\rho}, \quad (98)$$

$$\frac{dv^\mu}{d\tau} = v^\mu D_\mu v^\nu = -cD^0_\nu v^\nu + \frac{\hbar^2}{2m^2} D_\nu \left( \frac{\nabla \sqrt{\rho}}{\sqrt{\rho}} \right) - \frac{e}{m} v^\nu F_{\nu\mu} + \frac{e}{m} F^0_\nu + D_\nu V'(\rho) + c v^\nu \Gamma^0_\mu + c^2 g^\nu_0 \Gamma^0_\mu. \quad (99)$$

These hydrodynamic equations are equivalent to the general relativistic GP equation (75).

E. The hydrodynamic representation of the current

According to Eq. (90), we have

$$\partial^\mu S = \frac{1}{2} \frac{i}{|\psi|^2} \left( \psi \partial^\mu \psi^* - \psi^* \partial^\mu \psi \right), \quad (100)$$
\[
\frac{\partial S}{\partial t} = \frac{1}{2i} \frac{1}{|\psi|^2} \left( \psi \frac{\partial \psi^*}{\partial t} - \psi^* \frac{\partial \psi}{\partial t} \right), \quad \nabla S = \frac{1}{2i} \frac{1}{|\psi|^2} (\psi \nabla \psi^* - \psi^* \nabla \psi).
\]

(101)

As a result, the quadricurrent of charge, the charge density and the current of charge defined by Eqs. (77)-(79) can be rewritten as

\[
(J_e)_\mu = -\frac{e}{m^2} \rho(\partial_\mu S - mc\delta^0_\mu + eA_\mu), \quad \rho_e = -\frac{e}{mc^2} \rho\left(\frac{\partial S}{\partial t} - mc^2 + eU\right), \quad J_e = \frac{e}{m^2} \rho(\nabla S - eA). \quad (102)
\]

Comparing these expressions with Eq. (91) we obtain

\[
(J_e)_\mu = \frac{e}{m} \rho(v_\mu + \delta_\mu^0 c), \quad \rho_e = \frac{e}{mc} \rho(v_0 + c) = \frac{e}{m} \rho \left(1 + \frac{E}{mc^2}\right), \quad J_e = \frac{e}{m} \rho \nu. \quad (103)
\]

These equations can be directly obtained from the results of Sec. III by using Eqs. (94)-(96).

IV. THE WEAK FIELD APPROXIMATION

In this section, we consider the KGME equations in the weak field approximation \(\Phi/c^2 \ll 1\). The equations that we derive are valid at the order \(O(\Phi/c^2)\).

A. The conformal Newtonian Gauge

We shall work with the conformal Newtonian gauge which is a perturbed form of the Friedmann-Lemaître-Robertson-Walker (FLRW) line element \([102]\). We consider the simplest form of Newtonian gauge, only taking into account scalar perturbations which are the ones that contribute to the formation of structures in cosmology. We neglect anisotropic stresses. We assume that the Universe is flat in agreement with the observations of the cosmic microwave background (CMB). Under these conditions, the line element is given by

\[
d s^2 = c^2 \left(1 + 2 \frac{\Phi}{c^2}\right) dt^2 - a(t)^2 \left(1 - 2 \frac{\Phi}{c^2}\right) \delta_{ij} dx^i dx^j, \quad (104)
\]

where \(\Phi/c^2 \ll 1\). In this metric, \(\Phi(r,t)\) represents the gravitational potential of classical Newtonian gravity and \(a(t)\) is the scale factor. The expression of the Christoffel symbols necessary to derive the equations of the following subsections can be found, e.g., in Appendix A of \([110]\).

B. The electromagnetic field and the first couple of Maxwell equations

We define the electromagnetic field as a function of the potentials by

\[
E = -\frac{\partial A}{\partial t} - \nabla U, \quad B = \frac{1}{a} \nabla \times A. \quad (105)
\]

Taking the curl of the electric field and the divergence of the magnetic field, we obtain the first couple of Maxwell equations

\[
\frac{1}{a} \nabla \times E = -\frac{\partial B}{\partial t} - HB, \quad \nabla \cdot B = 0, \quad (106)
\]

where \(H = \dot{a}/a\) is the Hubble parameter. The Faraday tensor \([4]\) can be expressed in terms of the electromagnetic field as

\[
F_{\mu \nu} = \begin{pmatrix}
0 & E_x/c & E_y/c & E_z/c \\
-E_x/c & 0 & -aB_z & aB_y \\
-E_y/c & aB_z & 0 & -aB_x \\
-E_z/c & -aB_y & aB_x & 0
\end{pmatrix}. \quad (107)
\]
C. The Klein-Gordon equation

In the weak field approximation, using the Newtonian gauge, the electromagnetic d’Alembertian operator can be written as

\[
\left(1 + \frac{2\Phi}{c^2}\right) \Box \varphi = \frac{1}{c^2} \left(\frac{\partial}{\partial t} + ieU\right)^2 \varphi - \frac{1}{a^2} \left(1 + \frac{4\Phi}{c^2}\right) \left(\nabla - \frac{i}{\hbar} A\right)^2 \varphi + \left(\frac{3H}{c^2} - \frac{4\partial\Phi}{c^2\partial t}\right) \left(\frac{\partial}{\partial t} + ieU\right) \varphi \tag{108}
\]

or, equivalently, as

\[
\left(1 + \frac{2\Phi}{c^2}\right) \Box \varphi = \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} + \frac{3H \partial \varphi}{c^2 \partial t} - \frac{1}{a^2} \left(1 + \frac{4\Phi}{c^2}\right) \Delta \varphi - \frac{4\partial\Phi}{c^2\partial t} \frac{\partial \varphi}{\partial t} + \frac{i\epsilon}{\hbar} \left[1 + \frac{4\Phi}{c^2}\right] \nabla \cdot A + \frac{1}{c^2} \frac{\partial U}{\partial t} \varphi
\]

\[-i\frac{\hbar}{c^2} \left(\frac{\partial\Phi}{\partial t} - \frac{3}{c^2} HU\right) \varphi + \frac{\epsilon^2}{\hbar^2} \left[1 + \frac{4\Phi}{c^2}\right] A^2 - \frac{1}{c^2} U^2 \right] \varphi + 2i\hbar \left[1 + \frac{4\Phi}{c^2}\right] A \cdot \nabla \varphi + \frac{U \partial \varphi}{c^2 \partial t}. \tag{109}
\]

We note that the four first terms in Eq. (109) correspond to the d’Alembertian operator in the absence of electromagnetic field

\[
\left(1 + \frac{2\Phi}{c^2}\right) \Box = \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} + \frac{3H \partial \varphi}{c^2 \partial t} - \frac{1}{a^2} \left(1 + \frac{4\Phi}{c^2}\right) \Delta - \frac{4\partial\Phi}{c^2\partial t} \frac{\partial \varphi}{\partial t} \tag{110}
\]

The KG equation (11) takes the form

\[
\frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} + \frac{3H \partial \varphi}{c^2 \partial t} - \frac{1}{a^2} \left(1 + \frac{4\Phi}{c^2}\right) \Delta \varphi - \frac{4\partial\Phi}{c^2\partial t} \frac{\partial \varphi}{\partial t} + \frac{i\epsilon}{\hbar} \left[1 + \frac{4\Phi}{c^2}\right] \nabla \cdot A + \frac{1}{c^2} \frac{\partial U}{\partial t} \varphi
\]

\[-i\frac{\hbar}{c^2} \left(\frac{\partial\Phi}{\partial t} - \frac{3}{c^2} HU\right) \varphi + \frac{\epsilon^2}{\hbar^2} \left[1 + \frac{4\Phi}{c^2}\right] A^2 - \frac{1}{c^2} U^2 \right] \varphi + 2i\hbar \left[1 + \frac{4\Phi}{c^2}\right] A \cdot \nabla \varphi + \frac{U \partial \varphi}{c^2 \partial t}
\]

\[+ \left(1 + 2\frac{\Phi}{c^2}\right) \frac{m^2c^2}{\hbar^2} \varphi + 2 \left(1 + 2\frac{\Phi}{c^2}\right) \frac{dV}{d|\varphi|^2} \varphi = 0. \tag{111}
\]

Using the expression (11) of the energy-momentum tensor, we find that the energy density and the pressure are given by

\[
\epsilon = T_0^0 = \frac{1}{2c^2} \left(1 - \frac{2\Phi}{c^2}\right) \left|\frac{\partial \varphi}{\partial t}\right|^2 + \frac{1}{2a^2} \left(1 + \frac{2\Phi}{c^2}\right) |\nabla \varphi|^2 + \frac{i\epsilon}{2\hbar c^2} \left(1 - \frac{2\Phi}{c^2}\right) \left(\frac{\partial \varphi^*}{\partial t} - \varphi^* \frac{\partial \varphi}{\partial t}\right)
\]

\[+ \frac{e^2}{2\hbar^2} \frac{U^2}{c^2} \left(1 - \frac{2\Phi}{c^2}\right) |\varphi|^2 - \frac{1}{2a^2} \left(1 + \frac{2\Phi}{c^2}\right) \frac{i\epsilon}{\hbar} A \cdot (\nabla \varphi^* - \varphi^* \nabla \varphi) + \frac{1}{2a^2} \left(1 + \frac{2\Phi}{c^2}\right) \frac{e^2}{\hbar^2} A^2 |\varphi|^2
\]

\[+ \frac{m^2c^2}{2\hbar^2} |\varphi|^2 + V(|\varphi|^2) + \frac{\epsilon_0}{2a^2} \mathbf{E}^2 + \frac{1}{2\mu_0 a^2} \left(1 + \frac{4\Phi}{c^2}\right) \mathbf{B}^2. \tag{112}
\]

\[
P = -\frac{1}{3} (T_1^1 + T_2^2 + T_3^3) = \frac{1}{2c^2} \left(1 - \frac{2\Phi}{c^2}\right) \left|\frac{\partial \varphi}{\partial t}\right|^2 - \frac{1}{6a^2} \left(1 + \frac{2\Phi}{c^2}\right) |\nabla \varphi|^2 + \frac{e^2}{2\hbar c^2} \left(1 - \frac{2\Phi}{c^2}\right) \left(\frac{\partial \varphi^*}{\partial t} - \varphi^* \frac{\partial \varphi}{\partial t}\right)
\]

\[-\frac{e^2}{2\hbar^2} \left(1 - \frac{2\Phi}{c^2}\right) |\varphi|^2 + \frac{1}{6a^2} \left(1 + \frac{2\Phi}{c^2}\right) \frac{i\epsilon}{\hbar} A \cdot (\nabla \varphi^* - \varphi^* \nabla \varphi) - \frac{1}{6a^2} \left(1 + \frac{2\Phi}{c^2}\right) \frac{e^2}{\hbar^2} A^2 |\varphi|^2
\]

\[-\frac{m^2c^2}{2\hbar^2} |\varphi|^2 + V(|\varphi|^2) + \frac{\epsilon_0}{6a^2} \mathbf{E}^2 + \frac{1}{6\mu_0 a^2} \left(1 + \frac{4\Phi}{c^2}\right) \mathbf{B}^2. \tag{113}
\]

They can be written in the simpler form

\[
\epsilon = \frac{1}{2c^2} \left(1 - \frac{2\Phi}{c^2}\right) \left|\frac{\partial \varphi}{\partial t}\right|^2 + \frac{i\epsilon}{\hbar} U \varphi \left|\nabla \varphi - \frac{i\epsilon}{\hbar} A \varphi\right|^2
\]

\[+ \frac{m^2c^2}{2\hbar^2} |\varphi|^2 + V(|\varphi|^2) + \frac{\epsilon_0}{2a^2} \mathbf{E}^2 + \frac{1}{2\mu_0 a^2} \left(1 + \frac{4\Phi}{c^2}\right) \mathbf{B}^2. \tag{114}
\]
The local charge conservation equation \( \text{(26)} \) can be written as

\[
\rho_e \frac{\partial \rho_e}{\partial t} + \frac{1}{a^2} \left( 1 + \frac{2\Phi}{c^2} \right) \mathbf{J}_e - \frac{\partial}{\partial t} \left( \frac{4\Phi}{c^2} \right) \rho_e + 3H\rho_e = 0. \tag{119}
\]

The total charge is given by

\[
Q = a^3 \int \rho_e \left( 1 - \frac{4\Phi}{c^2} \right) dr. \tag{120}
\]

E. The second couple of Maxwell equations

The Maxwell equations \( \text{(32)} \) can be written as

\[
D_\mu F_{\nu} = D_\mu (g^{\mu\sigma} F_{\sigma\nu}) = \mu_0 (J_e)_\nu. \tag{121}
\]

Using Eqs. \( \text{(A8)} \) and \( \text{(A9)} \), we obtain

\[
g^{\mu\sigma} \left( \partial_\mu F_{\sigma\nu} - \Gamma^\rho_{\sigma\mu} F_{\rho\nu} - \Gamma^\rho_{\mu\rho} F_{\sigma\nu} \right) = \mu_0 (J_e)_\nu. \tag{122}
\]

In the weak field approximation, using the conformal Newtonian gauge, the second couple of Maxwell equations writes

\[
\frac{1}{a^2} \left( 1 + \frac{2\Phi}{c^2} \right) \nabla \cdot \mathbf{E} = \frac{\rho_e}{\epsilon_0},
\]

\[
- \left( 1 - \frac{2\Phi}{c^2} \right) \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} + \frac{2}{c^2} \frac{\partial \Phi}{\partial t} \mathbf{E} - \left( 1 - \frac{2\Phi}{c^2} \right) \frac{H}{c^2} \mathbf{E} + \frac{2}{c^2 a} \nabla \Phi \times \mathbf{B} + \frac{1}{a} \left( 1 + \frac{2\Phi}{c^2} \right) \mathbf{B} \times \mathbf{E} = \mu_0 \mathbf{J}_e. \tag{124}
\]
On the other hand, the Lorentz gauge [21] takes the form
\[
\frac{1}{c^2} \frac{\partial U}{\partial t} + \frac{1}{a^2} \left( 1 + \frac{4\Phi}{c^2} \right) \nabla \cdot \mathbf{A} - \frac{4U}{c^2} \frac{\partial \Phi}{\partial t} + \frac{3H}{c^2} U = 0.
\] (125)

Combining Eqs. (105), (110), (123), and (124), and making the choice of the Lorentz gauge [125], we find that the field equations satisfied by the potentials \(U\) and \(A\) are
\[
\Box U - \frac{2}{a^2} H \left( 1 + \frac{2\Phi}{c^2} \right) \nabla \cdot \mathbf{A} + \frac{4}{a^2 c^2} \frac{\partial \Phi}{\partial t} \nabla \cdot \mathbf{A} - \frac{4U}{c^2} \frac{\partial \Phi}{\partial t} + \frac{3H}{c^2} U - \frac{2}{a^2 c^2} \mathbf{E} \cdot \nabla \Phi = \frac{\rho_e}{\epsilon_0}.
\] (126)
\[
\Box \mathbf{A} + \left( 1 - \frac{2\Phi}{c^2} \right) \frac{2H}{c^2} \mathbf{E} - \frac{2}{a^2 c^2} \frac{\partial \Phi}{\partial t} \mathbf{E} + \frac{2}{c^2 a^2} \nabla \Phi \times (\nabla \times \mathbf{A}) - \frac{4}{c^4} \nabla \Phi \cdot \nabla \mathbf{A} + \frac{4U}{c^4} \frac{\partial \nabla \Phi}{\partial t} = \mu_0 \mathbf{J}_e.
\] (127)

F. The Einstein equations

The time-time component of the Einstein equations (38) is
\[
R^0_0 - \frac{1}{2} R = \frac{8\pi G}{c^4} T^0_0,
\] (128)
where the time-time component of the energy-momentum tensor \(T^0_0\) is equal to the energy density \(\epsilon\). In the weak field approximation, using the Newtonian conformal gauge, we find
\[
R^0_0 - \frac{1}{2} R = \frac{3H^2}{c^4} \Phi - \frac{6}{c^4} H \left( \frac{\partial \Phi}{\partial t} + H \Phi \right).
\] (129)

Therefore, Eq. (128) can be written as
\[
\frac{\Delta \Phi}{4\pi Ga^2} = \frac{\epsilon}{c^2} - \frac{3H^2}{8\pi G} + \frac{3H}{4\pi G c^2} \left( \frac{\partial \Phi}{\partial t} + H \Phi \right).
\] (130)

Substituting Eq. (124) into Eq. (130), we obtain
\[
\frac{\Delta \Phi}{4\pi Ga^2} = \frac{1}{2c^4} \left( 1 - \frac{2\Phi}{c^2} \right) \left| \frac{\partial \phi}{\partial t} \right|^2 + \frac{1}{2a^2 c^2} \left( 1 + \frac{2\Phi}{c^2} \right) \nabla \varphi^2 + \frac{m^2}{2a^2} \varphi^2 + \frac{1}{c^2} V(\varphi^2) - \frac{3H^2}{8\pi G} - \frac{3H}{4\pi G c^2} \left( \frac{\partial \Phi}{\partial t} + H \Phi \right)
\]
\[
+ \frac{\epsilon}{2 \hbar c^4} \left( 1 - \frac{2\Phi}{c^2} \right) \left( \varphi \frac{\partial \phi}{\partial t} - \varphi^* \frac{\partial \phi^*}{\partial t} \right) + \frac{e^2}{2 \hbar c^2} U^2 \left( 1 - \frac{2\Phi}{c^2} \right) \left| \varphi \right|^2 - \frac{1}{2a^2} \left( 1 + \frac{2\Phi}{c^2} \right) \frac{\epsilon_0}{2a^2 c^2} \frac{\mathbf{E}^2}{2} + \frac{1}{2a^2 c^2} \frac{\hbar^2 c^2}{2} \mathbf{A}^2 |\varphi|^2 + \frac{\epsilon_0}{2a^2 c^2} \frac{\mathbf{E}^2}{2} + \frac{1}{2a^2 c^2} \left( 1 + \frac{4\Phi}{c^2} \right) \frac{\mathbf{B}^2}{c^2}.
\] (131)

Equations (111), (123), (124) and (131) form the KGME equations in the weak field approximation. The charge density and the current of charge appearing in the Maxwell equations (123) and (124) are given in terms of \(\varphi\) by Eqs. (24) and (25).

G. The Gross-Pitaevskii-Maxwell-Einstein equations

Making the Klein transformation (23) in the KGME equations (111) and (131), we obtain
\[
\frac{ih}{\hbar} \frac{\partial \psi}{\partial t} - \frac{\hbar^2}{2m c^2} \frac{\partial^2 \psi}{\partial t^2} - \frac{3}{2} H \frac{\hbar^2}{mc^2} \frac{\partial \psi}{\partial t} + \frac{\hbar^2}{2ma^2} \left( 1 + \frac{4\Phi}{c^2} \right) \Delta \psi - m \frac{dV}{d|\psi|^2} \psi - \frac{3}{2} \frac{\partial \Phi}{c^2} \frac{\partial \psi}{\partial t} - \frac{m}{2} \frac{dV}{d|\psi|^2} \psi + \frac{3}{2} \frac{\partial H}{c^2} \psi
\]
\[
+ \frac{2m^2 \epsilon_0}{mc^4} \frac{\partial}{\partial t} \left( \frac{\partial \psi}{\partial t} - \frac{imc^2}{\hbar} \psi \right) - \frac{e \hbar}{2m} \left( 1 + \frac{4\Phi}{c^2} \right) \nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial U}{\partial t} \psi + \frac{i}{m} \frac{e \hbar}{2m} \left( \frac{U}{c^4} \frac{\partial \Phi}{\partial t} - \frac{3}{2} H U \right) \psi
\]
\[
- \frac{c^2}{2m} \left[ \frac{1}{a^2} \left( 1 + \frac{4\Phi}{c^2} \right) \mathbf{A}^2 - \frac{1}{c^2} U^2 \right] \psi - i \frac{e \hbar}{m} \left[ \frac{1}{a^2} \left( 1 + \frac{4\Phi}{c^2} \right) \mathbf{A} \cdot \nabla \psi + \frac{U}{c} \left( \frac{\partial \psi}{\partial t} - \frac{imc^2}{\hbar} \psi \right) \right] = 0,
\] (132)
We must be careful that this definition differs from that of Eq. (91) because of the presence of the scale factor $a$. We introduce the velocity field

$$\psi$$

Equations (123), (124), (132) and (133) form the GPME equations in the weak field approximation. The charge and the current of charge appearing in the Maxwell equations (123) and (124) are given in terms of the pseudo wavefunction $\psi$ by Eqs. (78) and (79).

Remark: By introducing the electromagnetic d’Alembertian $\Box$, the GP equation (132) can be written in the more compact form

$$ih\frac{\partial \psi}{\partial t} + \frac{3}{2}i\hbar \left( H - \frac{4}{3c^2} \frac{\partial \psi}{\partial t} \right) \psi - \frac{\hbar^2}{2m} \left( 1 + \frac{2\Phi}{c^2} \right) \Box \psi - eU\psi - m\Phi \psi - \left( 1 + \frac{2\Phi}{c^2} \right) m \frac{dV}{d(|\psi|^2)} \psi = 0.$$  

(137)

This equation can also be directly derived from Eq. (75) in the weak field approximation.

H. Hydrodynamic representation of the Gross-Pitaevskii equation

We can obtain a hydrodynamic representation of the GPME equations (132) and (133) by making the Madelung transformation (89). We introduce the velocity field

$$\psi = \nabla S - eA.$$  

(138)

21 We must be careful that this definition differs from that of Eq. (111) because of the presence of the scale factor $a$. However, in order to avoid the proliferation of notations, we have named the velocity in Sec. III D and in the present section by the same symbol $v$. 

We note that
\[
\nabla \times v = -\frac{e}{ma} \nabla \times A = -\frac{e}{m} B. \tag{139}
\]

This relation shows that the magnetic field creates a vorticity field. This vorticity is equal to twice the Larmor pulsation \(\omega_L = -(e/2m)B\). In the absence of magnetic field, the velocity field is irrotational. We also introduce the energy
\[
E = -\frac{\partial S}{\partial t} - eU. \tag{140}
\]

We note that
\[
\nabla E + ma \frac{\partial v}{\partial t} = -maHv + eE. \tag{141}
\]

Substituting Eq. \(\text{(9)}\) into the GPME equations \(\text{(132)}\) and \(\text{(133)}\), and separating real and imaginary parts, we obtain the hydrodynamic equations
\[
\frac{\partial \rho}{\partial t} + 3H \left( 1 + \frac{E}{mc^2} \right) \rho + \frac{1}{a} \left( 1 + \frac{4\Phi}{c^2} \right) \nabla \cdot (\rho v) = -\frac{1}{mc^2} \frac{\partial}{\partial t} (\rho E) + \frac{4}{c^2} \left( 1 + \frac{E}{mc^2} \right) \rho \frac{\partial \Phi}{\partial t}, \tag{142}
\]
\[
\frac{\partial S}{\partial t} + \frac{1}{2ma^2} \left( 1 + \frac{4\Phi}{c^2} \right) (\nabla S - eA)^2 = -\frac{h^2}{2mc^2} \frac{\partial^2 \Phi}{\partial \rho^2} + \left( 1 + \frac{4\Phi}{c^2} \right) \frac{h^2}{2ma^2} \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} - m\Phi - eU - \left( 1 + \frac{2\Phi}{c^2} \right) m\rho \frac{E^2}{2mc^2} - \left( 3H - \frac{4}{c^2} \frac{\partial \Phi}{\partial t} \right) \frac{h^2}{4mc^2 \rho} \frac{\partial \rho}{\partial t}, \tag{143}
\]
\[
\frac{\partial v}{\partial t} + Hv + \frac{1}{a} (v \cdot \nabla) v = -\frac{h^2}{2ma^2c^2} \nabla \left( \frac{\rho^2 \sqrt{\rho}}{\sqrt{\rho}} \right) + \frac{h^2}{2ma^2c^2} \nabla \left[ \left( 1 + \frac{4\Phi}{c^2} \right) \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right] - \frac{1}{a} \nabla \Phi - \frac{1}{a} \nabla p - \frac{2}{ac^2} \nabla (h\Phi) - \frac{2}{ac^2} \nabla (\Phi v^2) + \frac{e}{ma} (E + v \times B) + \frac{1}{2am^2c^2} \nabla (E^2) - \frac{3}{4} \frac{h^2}{am^2c^2} H \nabla \left( \frac{1}{\rho} \frac{\partial \rho}{\partial t} \right) + \frac{h^2}{am^2c^2} \nabla \left( \frac{\partial \Phi}{\partial t} \frac{\partial \rho}{\partial t} \right), \tag{144}
\]
\[
\frac{\Delta \Phi}{4\pi Ga^2} = \left( 1 - \frac{\Phi}{c^2} \right) \rho + \frac{h^2}{2mc^2c^2} \left( 1 - \frac{2\Phi}{c^2} \right) \left[ \frac{1}{4\rho} \left( \frac{\partial \rho}{\partial t} \right)^2 + \frac{\rho}{h^2} \left( \frac{\partial S}{\partial t} \right)^2 \right] + \frac{h^2}{2ma^2c^2m^2} \left( 1 + \frac{2\Phi}{c^2} \right) \left[ \frac{1}{4\rho} (\rho \partial \rho)^2 + \frac{\rho}{h^2} (\nabla S)^2 \right] + \frac{1}{c^2} V(\rho) - \frac{1}{m^2c^2} \left( 1 - \frac{2\Phi}{c^2} \right) \rho \frac{\partial S}{\partial t} + \frac{3}{8\pi} \frac{H\Phi}{c^2} \left( \frac{\partial \Phi}{\partial t} + H\Phi \right) \rho + \frac{e}{mc^2} \frac{U^2}{c^2} \left( 1 - \frac{2\Phi}{c^2} \right) \rho - \frac{1}{c^2} \left( 1 + \frac{2\Phi}{c^2} \right) \frac{e}{m^2c^2} \rho A \cdot \nabla S + \frac{1}{2a^2} \left( 1 + \frac{2\Phi}{c^2} \right) \frac{e^2}{m^2c^2} A^2 \rho + \frac{e_0}{2a^2} \frac{E^2}{c^2} + \frac{1}{2\mu_0a^2} \left( 1 + \frac{4\Phi}{c^2} \right) B^2, \tag{145}
\]

where \(h(\rho) = V'(\rho)\) is a pseudo enthalpy and \(p(\rho)\) is a pseudo pressure defined by the relation \(h'(\rho) = p'(\rho)/\rho\). The pseudo pressure is explicitly given by \(p(\rho) = \rho h(\rho) - \int h(\rho) \, d\rho\), i.e.,
\[
p(\rho) = \rho V'(\rho) - V(\rho). \tag{146}
\]

The pseudo velocity of sound is \(c_s^2 = p'(\rho) = pV''(\rho)\). In order to obtain Eq. \(\text{(144)}\), we have taken the gradient of Eq. \(\text{(143)}\) and we have used the identity \((v \cdot \nabla)v = \nabla(v^2/2) - v \times (\nabla \times v)\) together with Eq. \(\text{(139)}\). For the quartic potential \(\text{(144)},\) we get
\[
V(\rho) = \frac{2\pi a_s h^2}{m^3} \rho^2, \quad h(\rho) = \frac{4\pi a_s h^2}{m^3} \rho, \quad p(\rho) = \frac{2\pi a_s h^2}{m^3} \rho^2, \quad c_s^2 = \frac{4\pi a_s h^2}{m^3} \rho. \tag{147}
\]
We note that, in general, the pressure
Introducing the energy (140), we can write the quantum relativistic Hamilton-Jacobi equation (151) as

not obvious since Eqs. (142)-(145) are valid in the relativistic regime. The interpretation of this equation of state is, however, not direct because \( \rho \) and \( p \) are a pseudo rest-mass density and a pseudo pressure that coincide with the real rest-mass density and the real pressure of a BEC only in the nonrelativistic limit \( c \to +\infty \).

The energy density and the pressure can be written in terms of hydrodynamic variables as

\[
e = \left(1 - \frac{\Phi}{c^2}\right) \rho c^2 + \frac{\hbar^2}{2m^2c^2} \left(1 - \frac{2\Phi}{c^2}\right) \left[\frac{1}{4\rho} \left(\frac{\partial \rho}{\partial t}\right)^2 + \frac{\rho}{\hbar^2} \left(\frac{\partial \mathbf{S}}{\partial t}\right)^2\right] + \frac{\hbar^2}{2a^2m^2c^2} \left(1 + \frac{2\Phi}{c^2}\right) \left[\frac{1}{4\rho} \left(\nabla \rho\right)^2 + \frac{\rho}{\hbar^2} \left(\nabla S\right)^2\right] + V(\rho) - \frac{1}{m} \left(1 - \frac{2\Phi}{c^2}\right) \rho \frac{\partial S}{\partial t} + \frac{e}{m^2} \left(1 + \frac{2\Phi}{c^2}\right) \rho \frac{\partial \mathbf{A}}{\partial t} - \frac{e}{m} U \left(1 - \frac{2\Phi}{c^2}\right) S
\]

\[
P = -\rho \Phi + \frac{\hbar^2}{2m^2c^2} \left(1 - \frac{2\Phi}{c^2}\right) \left[\frac{1}{4\rho} \left(\frac{\partial \rho}{\partial t}\right)^2 + \frac{\rho}{\hbar^2} \left(\frac{\partial \mathbf{S}}{\partial t}\right)^2\right] - \frac{\hbar^2}{6a^2m^2c^2} \left(1 + \frac{2\Phi}{c^2}\right) \left[\frac{1}{4\rho} \left(\nabla \rho\right)^2 + \frac{\rho}{\hbar^2} \left(\nabla S\right)^2\right] - V(\rho) - \frac{1}{m} \left(1 - \frac{2\Phi}{c^2}\right) \rho \frac{\partial S}{\partial t} + \frac{e}{m^2} \left(1 + \frac{2\Phi}{c^2}\right) \rho \frac{\partial \mathbf{A}}{\partial t} - \frac{e}{m} U \left(1 - \frac{2\Phi}{c^2}\right) S
\]

\[
\frac{\partial S}{\partial t} + \frac{1}{2ma^2} \left(1 + \frac{4\Phi}{c^2}\right) \left(\nabla S - e \mathbf{A}\right)^2 = -\frac{\hbar^2}{2m} \left(1 + \frac{2\Phi}{c^2}\right) \frac{\nabla \rho}{\sqrt{\rho}} - m \Phi - eU - \left(1 + \frac{2\Phi}{c^2}\right) m \rho \frac{\nabla \rho}{\rho} + \frac{E^2}{2mc^2},
\]

\[
\frac{\partial \mathbf{v}}{\partial t} + H \mathbf{v} + \frac{1}{a} (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{\hbar^2}{2ma^2} \nabla \left[ \left(1 + \frac{2\Phi}{c^2}\right) \frac{\nabla \rho}{\sqrt{\rho}} \right] - \frac{1}{a} \nabla \Phi - \frac{1}{\rho a} \nabla \rho - \frac{2}{ac^2} \nabla (\Phi \mathbf{v}) - \frac{e}{ma} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) + \frac{1}{2am^2c^2} \nabla (E^2).
\]

Introducing the energy (149), we can write the quantum relativistic Hamilton-Jacobi equation (150) as

\[
E + \frac{E^2}{2mc^2} = \frac{1}{2m} \left(1 + \frac{4\Phi}{c^2}\right) \mathbf{v}^2 + \frac{\hbar^2}{2m} \left(1 + \frac{2\Phi}{c^2}\right) \frac{\nabla \rho}{\sqrt{\rho}} + m \Phi + \left(1 + \frac{2\Phi}{c^2}\right) m \rho \frac{\nabla \rho}{\rho}.
\]
This is a second degree equation for $E$ whose solutions are

$$E = me^2 \left[ -1 \pm \sqrt{1 + \left(1 + \frac{4\Phi}{c^2}\right) \frac{v^2}{c^2} + \frac{\hbar^2}{m^2 c^4} \left(1 + \frac{2\Phi}{c^2}\right) \frac{\sqrt{\beta}}{\sqrt{\rho}} + 2 \frac{\Phi}{c^2} + 2 \left(1 + \frac{2\Phi}{c^2}\right) \frac{1}{c^2} \hbar(\rho) \right].$$

(154)

This expression can be substituted into Eqs. (150) and (152) to obtain a closed system of hydrodynamic equations.

I. Vortices and magnetic monopoles

The velocity field corresponding to the Madelung transformation is defined by Eq. (138). The vorticity is then given by Eq. (139). When $A = 0$, the flow is irrotational ($\nabla \times \mathbf{v} = 0$). More generally, the equation

$$\mathbf{v} + \left(\frac{e}{ma}\right) \mathbf{A} = \nabla S/m$$

implies the relation

$$\nabla \times \left(\mathbf{v} + \left(\frac{e}{ma}\right) \mathbf{A}\right) = 0 \quad \forall \mathbf{r} \text{ where } \rho(\mathbf{r}) \neq 0.$$

(155)

This relation is valid only at the points where $\nabla S/m$ is well defined, i.e., at the points where the wavefunction (or the density) does not vanish. When the wavefunction vanishes, its phase does not have any meaning and neither $S$ nor $\nabla S$ is well defined (the velocity is singular). At such points, known as nodal points, $\nabla \times \left[\mathbf{v} + \left(\frac{e}{ma}\right) \mathbf{A}\right]$ does not vanish in general, leading to the appearance of singularities. If we consider the circulation of $\mathbf{v} + \left(\frac{e}{ma}\right) \mathbf{A}$ around a nodal point, we have

$$\Gamma = \oint (\mathbf{v} + \left(\frac{e}{ma}\right) \mathbf{A}) \cdot d\mathbf{l} = \frac{1}{ma} \oint \nabla S \cdot d\mathbf{l} = \frac{1}{ma} \oint dS = 2\pi n \frac{\hbar}{ma} \quad n = 0, \pm 1, \pm 2, ...$$

(156)

since the phase $S/\hbar$, when it exists, is defined up to a multiple of $2\pi$. This relation shows that the circulation around a nodal point is quantized in units of $\hbar/ma$. The integer $n$ is the circulation number. Using the Stokes theorem, we get

$$\Gamma = \int \left(\nabla \times \mathbf{v} + \left(\frac{e}{m}\right) \mathbf{B}\right) \cdot dS = 2\pi n \frac{\hbar}{ma}.$$  

(157)

Therefore, $\nabla \times \mathbf{v} + \left(\frac{e}{m}\right) \mathbf{B}$ vanishes everywhere except on certain singular points (nodal points) where it has singularities of the $\delta$-type. We note that the derivation of Eqs. (155)-(157) does not depend upon the dynamical equations so the results of this section are valid for many types of waves and flows.

The previous arguments were developed by Onsager [90] and Feynman [91] in the case of a quantum fluid without electromagnetic field. In that case, the singularities correspond to vortices. When $\mathbf{B} = 0$, Eqs. (156) and (157) reduce to (from now on, we take $a = 1$):

$$\oint \mathbf{v} \cdot d\mathbf{l} = \int (\nabla \times \mathbf{v}) \cdot dS = 2\pi n \frac{\hbar}{m},$$

(158)

so that the circulation of the velocity is quantized in units of $\hbar/m$. Dirac [92] previously developed these arguments in a more general context in which an electromagnetic field can be present. He focused on the situation where

$$\oint \mathbf{v} \cdot d\mathbf{l} = \int (\nabla \times \mathbf{v}) \cdot dS = 0.$$  

In that case, Eqs. (156) and (157) reduce to

$$\oint \mathbf{A} \cdot d\mathbf{l} = \int \mathbf{B} \cdot dS = 2\pi n \frac{\hbar}{e}.$$  

(159)

Following Dirac [92], if the magnetic field is created by a magnetic monopole, Eq. (159) implies that its magnetic charge $\mu$ satisfies the relation

$$\frac{e\mu}{4\pi \epsilon_0 hc} = \frac{1}{2} n.$$  

(160)

This is known as the Dirac quantization condition. The hypothetical existence of a magnetic monopole would imply that the electric charges must be quantized in certain units. Inversely, the existence of the electric charges implies that the magnetic charges of the hypothetical magnetic monopoles, if they exist, must be quantized in units inversely
proportional to the elementary electric charge. Equation (160) can be compared with the famous relation of quantum electrodynamics

\[
\frac{e^2}{4\pi \varepsilon_0 \hbar c} = \alpha \approx 137,
\]

where \(\alpha\) is Sommerfeld’s fine structure constant. The initial goal of Dirac [92] was to unravel the meaning of Eq. (161). At the end of his seminal paper on magnetic monopoles, he expressed his disappointment that his theory leads to the reciprocity relation (160) between electricity and magnetism, instead of the purely electronic quantum condition (161).

J. The charge and the current in the hydrodynamic representation

The charge density and the current of charge are given by Eq. (102). Comparing Eq. (102) with Eqs. (138) and (140), they can be expressed in terms of hydrodynamic variables as

\[
\rho_e = \frac{e}{m} \rho \left(1 + \frac{E}{mc^2}\right), \quad J_e = \frac{e}{m} \rho v.
\]

Using Eq. (162), we can check that the equation of continuity (150) is equivalent to the local charge conservation equation (119).

K. The London equations

In the case where the charge density and the current of charge can be approximated by Eq. (70), we obtain from Eqs. (126) and (127) the London equations in the weak field approximation

\[
\Box \Phi - \frac{2}{a^2} H \left(1 + \frac{4\Phi c^2}{a^2}ight) \nabla \cdot A + \frac{4}{a^2 c^2} \frac{\partial \Phi}{\partial t} \nabla \cdot A - \frac{4}{a^2 c^2} \frac{\partial^2 \Phi}{\partial t^2} + \left(1 - \frac{2\Phi c^2}{a^2}ight) \frac{3H}{c^2} U - \frac{2}{a^2 c^2} E \cdot \nabla \Phi = -\frac{e^2}{m^2 c^2 \epsilon_0} \rho U,
\]

\[
\Box A + \left(1 - \frac{2\Phi c^2}{a^2}ight) \frac{2H}{c^2} E - \frac{2}{a^2 c^2} \frac{\partial \Phi}{\partial t} E + \frac{2}{a^2 c^2} \nabla \Phi \times (\nabla \times A) - \frac{4}{a^2 c^2} \frac{\partial \Phi}{\partial t} E = -\frac{e^2 \mu_0}{m^2} \rho A.
\]

L. Hydrodynamic representation of the Klein-Gordon equation

We can obtain a hydrodynamic representation of the KGME equations (111) and (131) by making the Broglie transformation (53) and introducing the velocity field

\[
u = \frac{\nabla S_{\text{tot}} - eA}{ma} \quad (165)
\]

and the energy

\[
E_{\text{tot}} = -\frac{\partial S_{\text{tot}}}{\partial t} - eU. \quad (166)
\]

A first possibility to obtain the hydrodynamic equations is to substitute Eq. (53) into Eqs. (111) and (131), and proceed as in Sec. IV H. Alternatively, using the relations (144)-(146), we can directly rewrite the hydrodynamic equations of Sec. IV H in terms of \(S_{\text{tot}}, E_{\text{tot}}\) and \(u\) instead of \(S, E\) and \(v\). The continuity equation (150), the quantum Hamilton-Jacobi equation (151), and the Euler-Lorentz equation (152) become

\[
\frac{\partial}{\partial t} \left(\frac{E_{\text{tot}}}{mc^2 \rho}\right) + 3 \left(H - \frac{4}{3c^2} \frac{\partial \Phi}{\partial t}\right) \frac{E_{\text{tot}}}{mc^2 \rho} + \frac{1}{a} \left(1 + \frac{4\Phi}{c^2}\right) \nabla \cdot (\rho \nu) = 0,
\]

\[
\frac{\partial S_{\text{tot}}}{\partial t} + \frac{1}{2ma^2} \left(1 + \frac{4\Phi}{c^2}\right) (\nabla S_{\text{tot}} - eA)^2 = -\frac{\hbar^2}{2m} \left(1 + \frac{2\Phi}{c^2}\right) \Box \sqrt{\rho} - m\Phi - \epsilon U
\]

\[-\left(1 + \frac{2\Phi}{c^2}\right) \frac{m\hbar}{2c^2} - \frac{E_{\text{tot}}^2}{2mc^2} - E_{\text{tot}} - \frac{1}{2} mc^2,
\]

\[
\text{(167)}
\]

\[
\text{(168)}
\]
\[
\frac{\partial u}{\partial t} + Hu + \frac{1}{a}(u \cdot \nabla)u = -\frac{\hbar^2}{2m^2a} \nabla \left[ \left( 1 + \frac{2\Phi}{c^2} \right) \sqrt{\rho} \right] - \frac{1}{a} \nabla \Phi - \frac{1}{pa} \nabla p - \frac{2}{ac^2} \nabla (h\Phi) - \frac{2}{ac^2} \nabla (\Phi u^2) + \frac{e}{ma} (E + u \times B) + \frac{1}{2am^2c^2} \nabla (E_{\text{tot}}^2) - \frac{1}{am} \nabla E_{\text{tot}}.
\]  

Introducing the energy (166), we can write the quantum relativistic Hamilton-Jacobi equation (168) as

\[
E_{\text{tot}}^2 = m^2c^4 + m^2c^2 \left( 1 + \frac{4\Phi}{c^2} \right) u^2 + \hbar^2c^2 \left( 1 + \frac{2\Phi}{c^2} \right) \frac{\nabla^2}{\sqrt{\rho}} + \frac{1}{2m^2c^2} \Phi + \frac{2}{m^2c^2} \hbar (\rho).
\]  

This equation can be interpreted as the quantum generalization (also taking into account general relativity) of the relativistic equation of mechanics

\[
E_{\text{tot}}^2 = p^2c^2 + m^2c^4
\]

where \(m\) represents the impulse \(p\). Finally, the charge density and the current of charge (162) can be expressed in terms of hydrodynamic variables as

\[
\rho_e = \frac{e\rho_{\text{tot}}}{m^2c^2}, \quad J_e = a \frac{e}{m} \rho u.
\]

V. THE NONRELATIVISTIC LIMIT

In this section, we consider the nonrelativistic limit \(c \to +\infty\) of the KGME equations. As explained at the beginning of Sec. III in order to derive the nonrelativistic equations from the KGME equations, we first have to make the Klein transformation (73) and take the limit \(c \to +\infty\) in the GPME equations.

A. The Gross-Pitaevskii-Maxwell-Poisson equations

For \(c \to +\infty\), the GPME equations (132) and (133) reduce to

\[
\frac{i\hbar}{2m} \frac{\partial \psi}{\partial t} + \frac{3}{2} \frac{i\hbar}{2m} H \psi = -\frac{\hbar^2}{2ma^2} \left( \nabla - i\frac{e}{\hbar} A \right)^2 \psi + m\Phi \psi + eU \psi + m \frac{dV}{d|\psi|^2} \psi,
\]

\[
\frac{\Delta \Phi}{4\pi G a^2} = |\psi|^2 - \frac{3H^2}{8\pi G}.
\]

B. The mass density and the current of mass

One can associate to the GP equation (172) a density and a current defined by

\[
\rho = |\psi|^2, \quad J = \frac{\hbar}{2im} (\psi^* \nabla \psi - \psi \nabla \psi^*) - \frac{e}{m} |\psi|^2 A.
\]

They satisfy the local conservation equation

\[
\frac{\partial \rho}{\partial t} + \frac{1}{a^2} \nabla \cdot J + 3H \rho = 0.
\]

Since the density \(\rho = |\psi|^2\) is definite positive, it can be interpreted as a probability density, or as a mass density. Accordingly, the current \(J\) can be interpreted as a current of probability, or as a current of mass. In that case, Eq. (175) expresses the local conservation of the normalization condition for the probability density, or the local conservation of mass. The total mass is \(M = a^3 \int \rho d\mathbf{r}\).

C. The charge density and the current of charge

For \(c \to +\infty\), the charge density and the current of charge defined by Eqs. (78) and (79) reduce to

\[
\rho_e = \frac{e}{m} |\psi|^2, \quad J_e = \frac{e\hbar}{2im^2} (\psi^* \nabla \psi - \psi \nabla \psi^*) - \frac{e^2}{m^2} |\psi|^2 A.
\]

(176)
Comparing Eqs. (174) and (176), we obtain the relations
\[ \rho_e = \frac{e}{m} \rho, \quad J_e = \frac{e}{m} J \]
(177)
between the density and current of charge and the density and current of mass. The local charge conservation equation (119) reduces to
\[ \frac{\partial \rho_e}{\partial t} + \frac{1}{a^2} \nabla \cdot J_e + 3H \rho_e = 0. \]
(178)
It is equivalent to the local mass conservation equation (175). The total charge is
\[ Q = a^3 \int \rho_e \, d\mathbf{r}. \]
The Maxwell equations are given by Eqs. (B12) and (B13), the Lorentz gauge by Eq. (B14), and the field equations for \( U \) and \( A \) by Eqs. (B15) and (B16). Of course, we cannot take the limit \( c \to +\infty \) in the Maxwell equations since they have a relativistic origin. Equations (172), (173), (B12) and (B13) form the GPMP equations.

D. The complex hydrodynamic equations

In this section, we restrict ourselves to the electromagnetic Schrödinger equation corresponding to the GP equation (172) without self-interaction (\( V = 0 \)). It can be written as
\[ i\hbar \frac{\partial \psi}{\partial t} + \frac{3}{2} i\hbar \mathbf{H} \psi = -\frac{\hbar^2}{2ma^2} \Delta \psi + \frac{ie}{ma^2} \mathbf{A} \cdot \nabla \psi + \frac{ihe}{2ma^2} (\nabla \cdot \mathbf{A}) \psi + \frac{e^2}{2ma^2} \mathbf{A}^2 \psi + m\Phi \psi + eU \psi. \]
(179)
We note that the third term on the r.h.s. disappears if we make the choice of the Coulomb gauge
\[ \nabla \cdot \mathbf{A} = 0. \]
(180)
Making the WKB transformation (80) in the Schrödinger equation (179), we obtain the complex quantum Hamilton-Jacobi equation
\[ \frac{\partial S}{\partial t} + \frac{1}{a} (\mathbf{V} \cdot \nabla) \mathbf{V} + \mathbf{H} \mathbf{V} = \frac{i\hbar}{2ma^2} \nabla \cdot (\nabla S - e\mathbf{A}) + 3i\hbar \mathbf{H}. \]
(181)
We introduce the complex velocity
\[ \mathbf{V} = \frac{\nabla S - e\mathbf{A}}{ma}. \]
(182)
We note that
\[ \nabla \times \mathbf{V} = -\frac{e}{ma} \nabla \times \mathbf{A} = \frac{e}{m} \mathbf{B}. \]
(183)
Using Eq. (183), the general identities \((\mathbf{V} \cdot \nabla) \mathbf{V} = \nabla(\mathbf{V}^2/2) - \mathbf{V} \times (\nabla \times \mathbf{V})\) and \(\Delta \mathbf{V} = \nabla(\nabla \cdot \mathbf{V}) - \nabla \times (\nabla \times \mathbf{V})\) reduce to
\[ (\mathbf{V} \cdot \nabla) \mathbf{V} = \nabla \left( \frac{\mathbf{V}^2}{2} \right) + \frac{e}{m} \mathbf{V} \times \mathbf{B}, \quad \Delta \mathbf{V} = \nabla(\nabla \cdot \mathbf{V}) + \frac{e}{m} \nabla \times \mathbf{B}. \]
(184)
Taking the gradient of Eq. (181), and using Eqs. (105), (182) and (184), we obtain the complex hydrodynamic equation
\[ \frac{\partial \mathbf{V}}{\partial t} + \frac{1}{a} (\mathbf{V} \cdot \nabla) \mathbf{V} + \mathbf{H} \mathbf{V} = \frac{i\hbar}{2ma^2} \nabla \cdot (\nabla \cdot \mathbf{V}) + \frac{e}{ma} (\mathbf{E} + \mathbf{V} \times \mathbf{B}) - \frac{1}{a} \nabla \Phi. \]
(185)
It can also be written as
\[ \frac{\partial \mathbf{V}}{\partial t} + \frac{1}{a} (\mathbf{V} \cdot \nabla) \mathbf{V} + \mathbf{H} \mathbf{V} = \frac{i\hbar}{2ma^2} \Delta \mathbf{V} + \frac{e}{ma} (\mathbf{E} + \mathbf{V} \times \mathbf{B}) - \frac{i\hbar}{2ma^2} \nabla \times \mathbf{B} - \frac{1}{a} \nabla \Phi. \]
(186)
Equation (186) can be interpreted as a complex quantum Euler-Lorentz equation. The first term on the r.h.s. is similar to a viscous term with a complex viscosity
\[ \nu = \frac{i\hbar}{2ma^2}. \]
(187)
the second term is a complex Lorentz force \((e/ma)(E + V \times B)\), the third term in a peculiar complex electromagnetic quantum force \(- (i\hbar/2ma^2) \nabla \times B\), and the last term is the gravitational force.

**Remark:** Introducing the complex energy

\[
E = - \frac{\partial S}{\partial t} - eU,
\]

the quantum Hamilton-Jacobi equation \(^{(181)}\) takes the form

\[
E = \frac{1}{2} m V^2 + m\Phi - \frac{i\hbar}{2a} \nabla \cdot V - \frac{3}{2} i\hbar H.
\]

**E. The real hydrodynamic equations**

Making the Madelung transformation defined by Eqs. \(^{(89)}\) and \(^{(138)}\) in the GPMP equations \(^{(172)}\) and \(^{(173)}\), or taking the limit \(c \to +\infty\) in the quantum barotropic ELME equations \(^{(142)}-^{(145)}\), we obtain the hydrodynamic equations

\[
\frac{\partial \rho}{\partial t} + 3H\rho + \frac{1}{a} \nabla \cdot (\rho V) = 0,
\]

\[
\frac{\partial S}{\partial t} + \frac{1}{2ma^2} (\nabla S - eA)^2 = \frac{\hbar^2}{2ma^2} \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} - m\Phi - eU - mh(\rho),
\]

\[
\frac{\partial v}{\partial t} + Hv + \frac{1}{a} (v \cdot \nabla) v = \frac{\hbar^2}{2ma^3} \nabla \left( \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) - \frac{1}{a} \nabla \Phi - \frac{1}{\rho a} \nabla p + \frac{e}{ma} (E + v \times B),
\]

\[
\frac{\Delta \Phi}{4\pi Ga^2} = \rho - \frac{3H^2}{8\pi G}.
\]

These equations have a clear physical interpretation. Eq. \(^{(190)}\), corresponding to the imaginary part of the GP equation, is the continuity equation. It accounts for the conservation of mass. Eq. \(^{(191)}\), corresponding to the real part of the GP equation, is the quantum Hamilton-Jacobi or Bernoulli equation. It involves the Madelung quantum potential

\[
Q = - \frac{\hbar^2}{2ma^2} \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}}.
\]

Equation \(^{(192)}\), obtained by taking the gradient of Eq. \(^{(191)}\), is the quantum Euler-Lorentz equation. It includes the quantum force \(-(1/a)\nabla Q\), the pressure force \(-(m/\rho a)\nabla p\) arising from the self-interaction of the bosons, the Lorentz force \((e/m)(E + v \times B)\) and the gravitational force \(-\nabla \Phi\). Equation \(^{(192)}\) is the Poisson equation. These equations are coupled to the Maxwell equations \(^{(B12)}\) and \(^{(B13)}\) in which the charge density and the current of charge are expressed in terms of hydrodynamic variables according to Eq. \(^{(197)}\). We stress that the hydrodynamic equations \(^{(190)}-^{(193)}\) are equivalent to the GPMP equations \(^{(172)}\) and \(^{(173)}\). Equations \(^{(190)}-^{(193)}\), \(^{(B12)}\) and \(^{(B13)}\) form the quantum barotropic Euler-Lorentz-Maxwell-Poisson (ELMP) equations. For \(\hbar = 0\), we recover the barotropic ELMP equations.

**Remark:** Introducing the energy

\[
E = - \frac{\partial S}{\partial t} - eU,
\]

the quantum Hamilton-Jacobi equation \(^{(191)}\) takes the form

\[
E = \frac{1}{2} m v^2 - \frac{\hbar^2}{2ma^2} \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} + m\Phi + mh(\rho).
\]

This is the sum of the kinetic energy, the quantum potential energy, the gravitational energy, and the enthalpy. Equation \(^{(196)}\) can be interpreted as the quantum generalization of the classical equation of mechanics \(E = p^2/2m + m\Phi\) where \(mv\) represents the impulse \(p\).
F. Hydrodynamic representation of the current

Combining Eqs. (74), (101) and (138), or taking the nonrelativistic limit $c \to +\infty$ in Eq. (162), we find that the density and current of mass and charge defined by Eqs. (174) and (176) can be expressed in terms of hydrodynamic variables as

$$\begin{align*}
J &= a \rho v, \\
\rho_e &= e m \rho, \\
J_e &= a e m \rho v = a \rho_e v.
\end{align*}$$

Using these relations, we can check that the equation of continuity (190) is equivalent to the local charge conservation equation (178).

VI. CONCLUSION

In this paper, we have developed a hydrodynamic representation of the KGME equations describing the evolution of a complex (charged) SF. The KG equation can be viewed as a relativistic generalization of the Schrödinger and GP equations for spin-0 bosons forming a BEC at $T = 0$. These equations may find applications in the context of dark matter, boson stars, and neutron stars with a superfluid core. Most authors work in terms of the SF $\varphi$ and solve the KG wave equation. However, the hydrodynamic equations derived in the present paper and in previous works may be easier to implement numerically and interpret physically. When applied to cosmology, they can be viewed as a quantum generalization of the standard hydrodynamic equations of the CDM model. The fundamental difference with the CDM model is the existence of a quantum force arising from the Heisenberg uncertainty principle that acts at small scales in order to prevent singularities. Even if this term is difficult to calculate numerically with a good accuracy as it involves third order derivatives, it provides a small-scale regularization that stabilizes the system at the scale of DM halos. Our hydrodynamic equations also involve a pressure force arising from the self-interaction of the bosons. When the self-interaction is repulsive (scattering), this force has a stabilizing role. On a physical point of view, these quantum forces (uncertainty principle and scattering) may solve the small-scale problems of the CDM model such as the cusp problem and the missing satellite problem. On a practical point of view, they can stabilize the numerical algorithm used to solve the hydrodynamic equations. In the present paper, we have developed a general formalism associated with the KGME equations. Astrophysical applications of this formalism will be given in future contributions. For example, in cosmology, a charged SF could explain the existence of the magnetic fields in galaxies. On the other hand, if the centers of neutron stars condensate and form BECs, they could become superfluid and/or superconducting. This could explain the high values of the magnetic fields observed in neutron stars because, under these conditions, the remanent electrons are very energetic and could generate such high fields. In a different context, boson stars have been proposed as black hole mimickers. Finally, some people have discussed the possibility that the Higgs field has led some imprints in the physics of the early universe. These are some of the applications of charged SF in cosmology and astrophysics that could be developed in future works.

Appendix A: Useful identities

In this Appendix, we list various identities that are useful in our calculations:

$$D_\mu V^\mu = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} V^\mu),$$  \hspace{1cm} (A1)

$$D_\mu \varphi = \partial_\mu \varphi,$$  \hspace{1cm} (A2)

$$D_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\sigma_{\mu \nu} V_\sigma,$$  \hspace{1cm} (A3)

$$\Gamma^\sigma_{\mu \nu} = \Gamma^\sigma_{\nu \mu},$$  \hspace{1cm} (A4)

$$D_\mu (\varphi V_\nu) = \varphi D_\mu V_\nu + V_\nu \partial_\mu \varphi,$$  \hspace{1cm} (A5)

$$D_\mu V^\mu = D_\mu (g^{\mu \nu} V_\nu) = g^{\mu \nu} D_\mu V_\nu = g^{\mu \nu} \partial_\mu V_\nu - g^{\mu \nu} \Gamma^\sigma_{\mu \nu} V_\sigma,$$  \hspace{1cm} (A6)
\[ D_\mu(\varphi V^\mu) = \varphi D_\mu V^\mu + V^\mu \partial_\mu \varphi, \]  
\[ D_\mu(g^{\mu\sigma} F_{\sigma\nu}) = g^{\mu\sigma} D_\mu F_{\sigma\nu}, \]  
\[ D_\mu F_{\sigma\nu} = \partial_\mu F_{\sigma\nu} - \Gamma^\rho_{\sigma\mu} F_{\rho\nu} - \Gamma^\rho_{\nu\mu} F_{\sigma\rho}, \]  
\[ D_\nu(V_\mu V^\mu) = 2 V_\mu D_\nu V^\mu, \]  
\[ D_\mu D_\nu V_\alpha - D_\nu D_\mu V_\alpha = -R^\beta_{\alpha\mu\nu} V_\beta, \]  
\[ g^{\mu\alpha} R^\beta_{\alpha\mu\nu} V_\beta = -R^\beta_{\beta\nu} V_\beta, \]  
\[ D_\mu D_\nu \varphi = D_\nu D_\mu \varphi. \]  

**Appendix B: The generalized Klein-Gordon-Maxwell-Poisson equations in an expanding background**

In this Appendix, we consider a simplified model in which we introduce the gravitational potential \( \Phi(r, t) \) in the ordinary KG equation by hand, as an external potential, and assume that this potential is produced by the SF itself via a generalized Poisson equation in which the source is the energy density \( \epsilon \). This leads to the generalized KGMP equations. We then show that these equations can be rigorously justified from the KGME equations in the weak field limit \( \Phi/c^2 \to 0 \).

1. **The Klein-Gordon equation**

We consider the FLRW metric that describes an isotropic and homogeneous expanding background. We assume that the Universe is flat in agreement with the observations of the CMB. The line element in the comoving frame is

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = c^2 dt^2 - a(t)^2 \delta_{ij} dx^i dx^j. \]  

In order to take the self-gravity of the SF into account, we introduce a Lagrangian of interaction that couples the gravitational potential \( \Phi(r, t) \) to the SF \( \varphi(r, t) \) according to

\[ \mathcal{L}_{\text{int}} = -\frac{m^2 \hbar^2}{2} \Phi |\varphi|^2. \]  

The total Lagrangian of the system (SF + gravity) is given by \( \mathcal{L} = \mathcal{L}_\varphi + \mathcal{L}_{\text{int}} \). The equation of motion resulting from the stationarity of the total action \( S = S_\varphi + S_{\text{int}} \), obtained by writing \( \delta S = 0 \), is the KG equation

\[ \Box_e \varphi + \frac{m^2 c^2}{\hbar^2} \varphi + \frac{2m^2}{\hbar^2} \Phi \varphi + 2 \frac{dV}{d|\varphi|^2} \varphi = 0. \]  

In Eq. (B3) the gravitational potential \( \Phi(r, t) \) acts as an external potential. This amounts to introducing a potential of interaction of the form

\[ V_{\text{int}}(|\varphi|^2) = \frac{m^2 \hbar^2}{2} \Phi |\varphi|^2 \]  

in the KG equation [14]. The electromagnetic d’Alembertian operator in a FLRW spacetime can be written as

\[ \Box_e \varphi = \frac{1}{c^2} \left( \frac{\partial}{\partial t} + \frac{e}{\hbar} U \right)^2 \varphi - \frac{1}{a^2} \left( \nabla - \frac{e}{\hbar} A \right)^2 \varphi + \frac{3H}{c^2} \left( \frac{\partial}{\partial t} + \frac{e}{\hbar} U \right) \varphi. \]  

(B5)
or, equivalently, as
\[
\Box \varphi = \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} + \frac{3H}{c^2} \frac{\partial \varphi}{\partial t} - \frac{1}{a^2} \Delta \varphi + \frac{e}{\hbar} \left( \frac{1}{a^2} \nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial U}{\partial t} + 3H \frac{U}{c^2} \right) \varphi + \frac{e^2}{\hbar^2} \left( \frac{1}{a^2} \mathbf{A}^2 - \frac{1}{c^2} U^2 \right) \varphi + 2i \frac{e}{\hbar} \left( \frac{1}{a^2} \mathbf{A} \cdot \nabla \varphi + \frac{U}{c^2} \frac{\partial \varphi}{\partial t} \right). \tag{B6}
\]

The first three terms in Eq. (B6) correspond to the d’Alembertian in a FLRW spacetime
\[
\Box = D_\mu \partial^\mu = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \frac{3H}{c^2} \frac{\partial}{\partial t} - \frac{1}{a^2} \Delta. \tag{B7}
\]

The KG equation (B3) takes the form
\[
\frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} + \frac{3H}{c^2} \frac{\partial \varphi}{\partial t} - \frac{1}{a^2} \Delta \varphi + \frac{i e}{\hbar} \left( \frac{1}{a^2} \nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial U}{\partial t} \right) \varphi + \frac{3e}{\hbar c^2} HU \varphi + \frac{e^2}{\hbar^2} \left( \frac{1}{a^2} \mathbf{A}^2 - \frac{1}{c^2} U^2 \right) \varphi
\]
\[
+ 2i \frac{e}{\hbar} \left( \frac{1}{a^2} \mathbf{A} \cdot \nabla \varphi + \frac{U}{c^2} \frac{\partial \varphi}{\partial t} \right) + \left( 1 + \frac{2\Phi}{c^2} \right) \frac{m^2 c^2}{\hbar^2} \varphi + 2 \frac{dV}{d|\varphi|} \varphi = 0. \tag{B8}
\]

The energy density and the pressure defined from the diagonal part of the energy-momentum tensor \[(B11)\] are given by
\[
\epsilon = \frac{1}{2c^2} \left| \frac{\partial \varphi}{\partial t} \right|^2 + \frac{1}{2a^2} \left| \nabla \varphi \right|^2 + i e \frac{U}{2c^2} \left( \varphi \frac{\partial \varphi^*}{\partial t} - \varphi^* \frac{\partial \varphi}{\partial t} \right) + \frac{e^2}{2a^2} \frac{U^2}{c^2} \left| \varphi \right|^2 + \frac{m^2 c^2}{2a^2} \left| \varphi \right|^2 + V(|\varphi|^2) + \frac{\epsilon_0}{2a^2} \mathbf{E}^2 + \frac{1}{2\mu_0 a^2} \mathbf{B}^2, \tag{B9}
\]

\[
P = \frac{1}{2c^2} \left| \frac{\partial \varphi}{\partial t} \right|^2 - \frac{1}{6a^2} \left| \nabla \varphi \right|^2 + \frac{i e}{2c^2} \frac{U}{2c^2} \left( \varphi \frac{\partial \varphi^*}{\partial t} - \varphi^* \frac{\partial \varphi}{\partial t} \right) + \frac{e^2}{6a^2} \frac{U^2}{c^2} \left| \varphi \right|^2 - \frac{m^2 c^2}{2a^2} \left| \varphi \right|^2 - V(|\varphi|^2) + \frac{\epsilon_0}{6a^2} \mathbf{E}^2 + \frac{1}{6\mu_0 a^2} \mathbf{B}^2. \tag{B10}
\]

2. The local conservation of charge

The charge conservation equation can be written as
\[
\frac{\partial \rho_e}{\partial t} + \frac{1}{a^2} \nabla \cdot \mathbf{J}_e + 3H \rho_e = 0. \tag{B11}
\]

The total charge is \( Q = a^3 \int \rho_e \, dr \).

3. The Maxwell equations

The Maxwell equations are given by
\[
\frac{1}{a} \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} - H \mathbf{B}, \quad \nabla \cdot \mathbf{B} = 0. \tag{B12}
\]

\[
\frac{1}{a^2} \nabla \cdot \mathbf{E} = \frac{\rho_e}{\epsilon_0}, \quad \frac{1}{a} \nabla \times \mathbf{B} = \mu_0 J_e + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} + \frac{H}{c^2} \mathbf{E}. \tag{B13}
\]
The Lorentz gauge takes the form
\[
\frac{1}{c^2} \frac{\partial U}{\partial t} + \frac{3H}{c^2} U + \frac{1}{a^2} \nabla \cdot \mathbf{A} = 0.
\] (B14)

Within the Lorentz gauge, the field equations satisfied by the potentials \( U \) and \( \mathbf{A} \) are given by
\[
\Box U - \frac{2}{a^2} H \nabla \cdot \mathbf{A} + \frac{3\dot{H}}{c^2} U = \frac{\rho_e}{\epsilon_0},
\] (B15)
\[
\Box \mathbf{A} + \frac{2H}{c^2} \mathbf{E} = \mu_0 \mathbf{J}_e.
\] (B16)

We also note that the fourth term in Eq. (B6) disappears if we make the choice of the Lorentz gauge.

4. The generalized Poisson equation

Equation (B8) is the electromagnetic KG equation for a SF in an external potential \( \Phi(\mathbf{r}, t) \) in an expanding background. We now state that \( \Phi(\mathbf{r}, t) \) is actually the gravitational potential produced by the SF itself. We phenomenologically assume that the gravitational potential is determined by a generalized Poisson equation of the form
\[
\frac{\Delta \Phi}{4\pi Ga^2} = \frac{1}{c^2}(\epsilon - \epsilon_b)
\] (B17)
in which the source of the gravitational potential is the energy density \( \epsilon \) of the SF (more precisely, its deviation from the homogeneous background density \( \epsilon_b(t) \)). Using Eq. (B9) for the energy density of a SF, and recalling the Friedmann equation (107):
\[
H^2 = \frac{8\pi G}{3c^2} \epsilon_b,
\] (B18)
the generalized Poisson equation can be written as
\[
\frac{\Delta \Phi}{4\pi Ga^2} = \frac{1}{2c^4} \left| \frac{\partial \varphi}{\partial t} \right|^2 + \frac{1}{2a^2 c^2} \left| \nabla \varphi \right|^2 + \frac{m^2}{2h^2} \left| \varphi \right|^2 + \frac{1}{c^2} V(\left| \varphi \right|^2) - \frac{3H^2}{8\pi G} + \frac{i e}{2\hbar c^3} \left( \varphi \frac{\partial \varphi^*}{\partial t} - \varphi^* \frac{\partial \varphi}{\partial t} \right) + \frac{e^2}{2\hbar^2 c^4} \mathbf{A} \cdot \left( \varphi \nabla \varphi^* - \varphi^* \nabla \varphi \right) + \frac{e^2}{2a^2 h^2 c^2} \mathbf{A}^2 \left| \varphi \right|^2 + \frac{\epsilon_0}{2a^2 c^2} \mathbf{E}^2 + \frac{1}{2\mu_0 a^2 c^2} \mathbf{B}^2.
\] (B19)

Equations (B8), (B12), (B13) and (B19) form the generalized KGMP equations. They have been introduced in an ad hoc manner by introducing the Newtonian potential \( \Phi \) by hand, as an external potential, but they can be rigorously justified from the KGME equations (103), (111), (123), (124) and (131) in the weak field limit \( \Phi/c^2 \to 0 \) (which, of course, is different from the nonrelativistic limit \( c \to +\infty \)). We note that we cannot neglect the term \( 2\Phi/c^2 \) in the last but one term of Eq. (111) because it is multiplied by \( c^2 \). This is how the Newtonian potential \( \Phi \) arises in the KG equation (B8). Therefore, we do not have to introduce \( \Phi \) by hand: the generalized KGMP equations can be obtained from the KGME equations by simply neglecting terms of order \( \Phi/c^2 \) (when they are not multiplied by \( c^2 \)) in these equations. The equations derived from the generalised KGMP equations can be obtained from the ones derived from the KGME equations in the same manner.

Remark: We could consider a simplified model in which the source of the Newtonian potential is the pseudo rest mass density \( \rho \) instead of the energy density \( \epsilon/c^2 \). In that case, the generalized Poisson equation (B17) would be replaced by the ordinary Poisson equation
\[
\frac{\Delta \Phi}{4\pi Ga^2} = \rho - \frac{\epsilon_b}{c^2}.
\] (B20)

However, there is no rigorous justification of this equation in the relativistic regime.
5. The generalized Gross-Pitaevskii-Maxwell-Poisson equations

Making the Klein transformation \( T \) in the generalized KGMP equations \( (B2) \) and \( (B19) \), or taking the weak field limit \( \Phi/c^2 \rightarrow 0 \) in the GPME equations \( (132) \) and \( (133) \), we obtain the generalized GPMP equations

\[
\begin{align*}
\hbar \frac{\partial \psi}{\partial t} - \frac{\hbar^2}{2mc^2} \frac{\partial^2 \psi}{\partial t^2} - \frac{3}{2} \frac{\hbar^2}{mc^2} \frac{\partial \psi}{\partial t} + \frac{\hbar}{2ma^2} \Delta \psi - m \Phi \psi - m \frac{dV}{d|\psi|^2} \psi + \frac{3}{2} i \hbar H \psi - i \frac{\hbar}{2m} \left( 1 \frac{\alpha^2}{a^2} \nabla \cdot A - \frac{1}{c^2} \frac{\partial \psi}{\partial t} \right) \psi \\
- i \frac{3e \hbar}{2mc^2} H U \psi - \frac{e^2}{2m} \left( \frac{1}{a^2} A^2 - \frac{1}{c^2} U^2 \right) \psi - i \frac{e \hbar}{m} \left[ \frac{1}{a^2} A \cdot \nabla \psi + U \left( \frac{\partial \psi}{\partial t} - \frac{imc^2}{h} \psi \right) \right] = 0, \quad (B21)
\end{align*}
\]

\[
\frac{\Delta \Phi}{4\pi Ga^2} = |\psi|^2 + \frac{\hbar^2}{2mc^2} \frac{\partial |\psi|^2}{\partial t} + \frac{\hbar^2}{2a^2 c^2} |\nabla \psi|^2 + \frac{1}{c^2} V(|\psi|^2) - \frac{h}{mc} \text{Im} \left( \frac{\partial \psi}{\partial t} \psi^* \right) + \frac{3H^2}{8\pi G} + i \frac{\hbar}{2mc^2} \left( \psi \frac{\partial \psi^*}{\partial t} - \psi^* \frac{\partial \psi}{\partial t} \right)
\]

\[
+i \frac{e \hbar}{2mc^2} \left( \psi \frac{\partial \psi^*}{\partial t} - \psi^* \frac{\partial \psi}{\partial t} \right) - \frac{e}{mc} U |\psi|^2 + \frac{e^2}{2m^2 c^2} |\psi|^2 - \frac{1}{2a^2 c^2} A^2 (\psi \nabla \psi^* - \psi^* \nabla \psi) + \frac{1}{2a^2 c^2} \frac{e^2}{2a^2 c^2} A^2 |\psi|^2 + \frac{e|E|^2}{2a^2 c^2} + \frac{1}{2\mu_0 a^2 c^2}. \quad (B22)
\]

The energy density and the pressure are given by

\[
\epsilon = e^2 |\psi|^2 + \frac{\hbar^2}{2mc^2} \frac{\partial |\psi|^2}{\partial t} + \frac{\hbar^2}{2a^2 c^2} |\nabla \psi|^2 + \frac{1}{c^2} V(|\psi|^2) - \frac{h}{mc} \text{Im} \left( \frac{\partial \psi}{\partial t} \psi^* \right) + i \frac{\hbar}{2mc^2} \left( \psi \frac{\partial \psi^*}{\partial t} - \psi^* \frac{\partial \psi}{\partial t} \right)
\]

\[
+ \frac{e}{mc} U |\psi|^2 + \frac{e^2}{2m^2 c^2} |\psi|^2 - \frac{1}{2a^2} \frac{ie \hbar}{2mc^2} (\psi \nabla \psi^* - \psi^* \nabla \psi) + \frac{1}{2a^2 c^2} \left( A^2 |\psi|^2 \right) + \frac{e_0 |E|^2}{2a^2 c^2} + \frac{1}{2\mu_0 a^2 c^2}, \quad (B23)
\]

\[
P = \frac{\hbar^2}{2mc^2} \frac{\partial |\psi|^2}{\partial t} - \frac{\hbar^2}{6a^2 c^2} |\nabla \psi|^2 - \frac{h}{mc} \text{Im} \left( \frac{\partial \psi}{\partial t} \psi^* \right) + i \frac{\hbar}{2mc^2} \left( \psi \frac{\partial \psi^*}{\partial t} - \psi^* \frac{\partial \psi}{\partial t} \right)
\]

\[
- \frac{e}{mc} U |\psi|^2 + \frac{e^2}{2m^2 c^2} |\psi|^2 + \frac{1}{6a^2} \frac{ie \hbar}{2mc^2} (\psi \nabla \psi^* - \psi^* \nabla \psi) - \frac{e^2}{6a^2 c^2} \left( A^2 |\psi|^2 \right) + \frac{e_0 |E|^2}{6a^2 c^2} + \frac{1}{6\mu_0 a^2 c^2}. \quad (B24)
\]

Remark: by introducing the electromagnetic d’Alembertian operator \( \Box \), the generalized GP equation \( (B21) \) can be written in the more compact form

\[
i \hbar \frac{\partial \psi}{\partial t} + \frac{3}{2} i \hbar H \psi - \frac{\hbar}{2mc} \Box_e \psi - eU \psi - m \Phi \psi - m \frac{dV}{d|\psi|^2} \psi = 0. \quad (B25)
\]

In the nonrelativistic limit \( c \rightarrow +\infty \), Eqs. \( (B22) \) and \( (B24) \) return the GPP equations \( (172) \) and \( (173) \).

6. The hydrodynamic equations

Making the Madelung transformation defined by Eqs. \( (89) \) and \( (158) \) in the generalized GPMP equations \( (B21) \) and \( (B22) \), or taking the weak field limit \( \Phi/c^2 \rightarrow 0 \) in the quantum barotropic ELME equations \( (142)-(145) \), we obtain the hydrodynamic equations

\[
\frac{\partial \rho}{\partial t} + 3H \left( 1 + \frac{E}{mc^2} \right) \rho + \frac{1}{a} \nabla \cdot (\rho v) = - \frac{1}{mc^2} \frac{\partial}{\partial t} (\rho E), \quad (B26)
\]

\[
\frac{\partial S}{\partial t} + \frac{1}{2ma^2} (\nabla S - eA)^2 = - \frac{\hbar^2}{2mc^2} \frac{\partial^2 \sqrt{\rho}}{\partial \sqrt{\rho}} + \frac{\hbar^2}{2ma^2} \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}}
\]

\[
- m \Phi - eU - mh(\rho) + \frac{E^2}{2mc^2} - \frac{3H^2}{4mc^2} \frac{\partial \rho}{\partial t}, \quad (B27)
\]
\[ \frac{\partial \mathbf{v}}{\partial t} + H \mathbf{v} + \frac{1}{a} (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{\hbar^2}{2m^2ac^2} \nabla \left( \frac{\partial^2 \mathbf{r}}{\sqrt{\rho}} \right) + \frac{\hbar^2}{2m^2a^3} \nabla \left( \frac{\nabla \sqrt{\rho}}{\sqrt{\rho}} \right) - \frac{1}{a} \nabla \Phi - \frac{1}{\rho a} \nabla p + \frac{e}{ma} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \]
\[ + \frac{1}{2am^2c^2} \nabla (E^2) - \frac{3}{4am^2c^2} H \nabla \left( \frac{1}{\rho \partial t} \right), \quad (B28) \]

\[ \frac{\Delta \Phi}{4\pi Ga^2} = \rho + \frac{\hbar^2}{2m^2c^4} \left[ \frac{1}{4\rho} \left( \frac{\partial \rho}{\partial t} \right)^2 + \frac{\rho}{\hbar^2} \left( \frac{\partial S}{\partial t} \right)^2 \right] + \frac{\hbar^2}{2a^2m^2} \left[ \frac{1}{4\rho} (\nabla \rho)^2 + \frac{\rho}{\hbar^2} (\nabla S)^2 \right] \]
\[ + \frac{e^2}{c^2} U^2 - \frac{1}{a^2 m^2 c^2} \rho \mathbf{A} \cdot \nabla S + \frac{1}{2a^2 m^2 c^2} \mathbf{A}^2 \rho + \frac{e_0}{2a^2 c^2} + \frac{1}{2\mu_0a^2} \mathbf{B}^2. \quad (B29) \]

The energy density and the pressure can be written in terms of hydrodynamic variables as

\[ \epsilon = \rho c^2 + \frac{\hbar^2}{2m^2c^2} \left[ \frac{1}{4\rho} \left( \frac{\partial \rho}{\partial t} \right)^2 + \frac{\rho}{\hbar^2} \left( \frac{\partial S}{\partial t} \right)^2 \right] + \frac{\hbar^2}{2a^2m^2} \left[ \frac{1}{4\rho} (\nabla \rho)^2 + \frac{\rho}{\hbar^2} (\nabla S)^2 \right] + V(\rho) - \frac{1}{m} \frac{\partial S}{\partial t} + \frac{e U}{m^2 c^4} \frac{\partial S}{\partial t} - \frac{e}{m} U \rho \]
\[ + \frac{e^2}{c^2} U^2 - \frac{1}{a^2 m^2 c^2} \rho \mathbf{A} \cdot \nabla S + \frac{1}{2a^2 m^2 c^2} \mathbf{A}^2 \rho + \frac{e_0}{2a^2 c^2} + \frac{1}{2\mu_0a^2} \mathbf{B}^2, \quad (B30) \]

\[ P = \frac{\hbar^2}{2m^2c^2} \left[ \frac{1}{4\rho} \left( \frac{\partial \rho}{\partial t} \right)^2 + \frac{\rho}{\hbar^2} \left( \frac{\partial S}{\partial t} \right)^2 \right] - \frac{\hbar^2}{6a^2m^2} \left[ \frac{1}{4\rho} (\nabla \rho)^2 + \frac{\rho}{\hbar^2} (\nabla S)^2 \right] - V(\rho) - \frac{1}{m} \frac{\partial S}{\partial t} + \frac{e U}{m^2 c^4} \frac{\partial S}{\partial t} - \frac{e}{m} U \rho \]
\[ + \frac{e^2}{c^2} U^2 + \frac{1}{a^2 m^2 c^2} \rho \mathbf{A} \cdot \nabla S - \frac{1}{6a^2 m^2} \mathbf{A}^2 \rho + \frac{e_0}{6a^2 c^2} + \frac{1}{6\mu_0a^2} \mathbf{B}^2. \quad (B31) \]

Equations (B12), (B13), and (B26)-(B29) form the generalized quantum barotropic ELMP equations. In the nonrelativistic limit \( c \to +\infty \), we recover the quantum barotropic ELMP equations (190)-(193), (B12), and (B13).

Remark: By introducing the d’Alembertian operator (B7), the hydrodynamic equations (B26)-(B28) can be written in the more compact form

\[ \frac{\partial}{\partial t} \left[ \left( 1 + \frac{E}{mc^2} \right) \rho \right] + 3H \left( 1 + \frac{E}{mc^2} \right) \rho + \frac{1}{a} \nabla \cdot (\rho \mathbf{v}) = 0, \quad (B32) \]

\[ \frac{\partial S}{\partial t} + \frac{1}{2ma^2} (\nabla S - e \mathbf{A})^2 = -\frac{\hbar^2}{2m} \sqrt{\rho} - m \Phi - eU - mh(\rho) + \frac{E^2}{2mc^2}, \quad (B33) \]

\[ \frac{\partial \mathbf{v}}{\partial t} + H \mathbf{v} + \frac{1}{a} (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{\hbar^2}{2m^2a} \nabla \left( \frac{\sqrt{\rho}}{\sqrt{\rho}} \right) - \frac{1}{a} \nabla \Phi - \frac{1}{\rho a} \nabla p + \frac{e}{ma} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) + \frac{1}{2am^2c^2} \nabla (E^2). \quad (B34) \]

Using Eq. (B12), we can check that the equation of continuity (B22) is equivalent to the local charge conservation equation (B11). Introducing the energy (B10), we can write Eq. (B33) as

\[ E + \frac{E^2}{2mc^2} = \frac{1}{2} mv^2 + \frac{\hbar^2}{2m} \sqrt{\rho} + m \Phi + mh(\rho). \quad (B35) \]

This is a second degree equation for \( E \) whose solutions are

\[ E = mc^2 \left[ -1 \pm \sqrt{1 + \frac{c^2}{e^2} + \frac{\hbar^2}{m^2c^2} \sqrt{\rho} + \frac{2e}{c^2} + 2\frac{1}{c^2} h(\rho)} \right]. \quad (B36) \]

This expression can be substituted into Eqs. (B32) and (B34) to obtain a closed system of hydrodynamic equations.
7. The London equations

The London equations are given by

\[ \Box U - \frac{2}{a^2} H \nabla \cdot A + \frac{3H}{c^2} U = -\frac{e^2}{m^2 c^2 \epsilon_0} \rho U, \]  

(B37)

\[ \Box A + \frac{2H}{c^2} E = -\frac{e^2}{m^2 \rho} A. \]  

(B38)

We note that these equations are coupled through the terms arising from the expansion of the universe.

8. The magnetohydrodynamic equations

The magnetohydrodynamic (MHD) equations can be formally obtained by taking \( E = 0 \) in the hydrodynamic equations of the previous subsections. Using Eq. (103), the Lorentz force in Eq. (B28) can be written as

\[ \frac{1}{\rho \alpha} \mathbf{J}_e \times \mathbf{B}. \]  

On the other hand, the Maxwell equation (B13) reduces to

\[ \frac{1}{c^2} \nabla \times \mathbf{B} = \mu_0 \mathbf{J}_e. \]  

Combining these two relations and using the identity \( (\mathbf{B} \cdot \nabla) \mathbf{B} = \nabla (\mathbf{B}^2/2) - \mathbf{B} \times (\nabla \times \mathbf{B}) \), we obtain

\[ \frac{1}{\rho \alpha} \mathbf{J}_e \times \mathbf{B} = -\frac{1}{\rho \alpha^2} \nabla \left( \frac{\mathbf{B}^2}{2\mu_0} \right) + \frac{1}{\rho \mu_0 \alpha^2} (\mathbf{B} \cdot \nabla) \mathbf{B}. \]  

(B39)

The first term on the r.h.s. is the magnetic pressure force and the second term is the magnetic tension force. We can define a magnetic pressure as

\[ p_m = \frac{\mathbf{B}^2}{2\mu_0 \alpha^2}. \]  

We note that the magnetic pressure in Eq. (B31) is

\[ P_m = \frac{\mathbf{B}^2}{6\mu_0 \alpha^2}. \]  

Therefore, \( P_m = p_m/3 \). The difference between \( P_m \) and \( p_m \) basically comes from the fact that the energy-momentum tensor \( T_{ij} \) is not diagonal (see Sec. 12.10 of [114]).

Appendix C: The generalized Klein-Gordon-Maxwell-Poisson equations in a static background

In this Appendix, we consider a flat and static spacetime with the Minkowskian metric

\[ ds^2 = g_{\mu \nu} dx^\mu dx^\nu = c^2 dt^2 - \delta_{ij} dx^i dx^j, \]  

(C1)

and we introduce the Newtonian potential \( \Phi \) by hand as an external potential (see Appendix B). This is equivalent to taking the weak field limit \( \Phi/c^2 \rightarrow 0 \) in the equations of Sec. IV and setting \( a = 1 \) and \( H = 0 \). This particular case will allow us to simplify certain equations of Secs. II and III.

1. The Klein-Gordon equation

The KG equation is given by

\[ \Box_e \varphi + \frac{m^2 c^2}{\hbar^2} \varphi + \frac{2m^2}{\hbar^2} \Phi \varphi + \frac{2}{d|\varphi|^2} \varphi = 0. \]  

(C2)

The electromagnetic d’Alembertian operator can be written under the equivalent forms

\[ \Box_e \varphi = \left( \partial_\mu + i \frac{e}{\hbar} A_\mu \right) \left( \partial^\mu + i \frac{e}{\hbar} A^\mu \right) \varphi, \]  

(C3)

\[ \Box_e \varphi = \partial_\mu \partial^\mu \varphi + i \frac{e}{\hbar} (\partial_\mu A^\mu) \varphi + 2i \frac{e}{\hbar} A_\mu \partial^\mu \varphi - \frac{e^2}{\hbar^2} A_\mu A^\mu \varphi, \]  

(C4)

\[ \Box_e \varphi = \frac{1}{c^2} \left( \frac{\partial}{\partial t} + i \frac{e}{\hbar} U \right)^2 \varphi - \left( \nabla - i \frac{e}{\hbar} A \right)^2 \varphi. \]  

(C5)
\[ \Box \varphi = \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} - \Delta \varphi + \frac{e}{\hbar} \left( \frac{1}{c^2} \frac{\partial U}{\partial t} + \nabla \cdot A \right) \varphi + 2 \frac{e}{\hbar} \left( \frac{U \partial \varphi}{c^2 \partial t} + A \cdot \nabla \varphi \right) - \frac{e^2}{\hbar^2} \left( \frac{U^2}{c^2} - A^2 \right) \varphi. \]  
\( (C6) \)

The first term in Eq. \((C4)\) and the first two terms in Eq. \((C6)\) correspond to the d’Alembertian in a Minkowskian spacetime

\[ \Box = \partial_{\mu} \partial^{\mu} = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta. \]  
\( (C7) \)

2. The local conservation of charge

The local charge conservation equation writes

\[ \partial_{\mu} J_{\mu}^{e} = 0 \quad \Leftrightarrow \quad \frac{\partial \rho_{e}}{\partial t} + \nabla \cdot J_{e} = 0 \]  
\( (C8) \)

and the total charge is given by

\[ Q = \frac{1}{e} \int J_{e}^{0} d^{3}x = \int \rho_{e} d^{3}x. \]  
\( (C9) \)

3. The Maxwell equations

The Maxwell equations are given by

\[ E = -\frac{\partial A}{\partial t} - \nabla U, \quad B = \nabla \times A, \]  
\( (C10) \)

\[ \nabla \times E = -\frac{\partial B}{\partial t}, \quad \nabla \cdot B = 0, \]  
\( (C11) \)

\[ \nabla \cdot E = \frac{\rho_{e}}{\epsilon_{0}}, \quad \nabla \times B = \mu_{0} J + \frac{1}{c^2} \frac{\partial E}{\partial t}. \]  
\( (C12) \)

The Lorentz gauge takes the form

\[ \partial_{\mu} A_{\mu} = 0 \quad \Leftrightarrow \quad \frac{1}{c^2} \frac{\partial U}{\partial t} + \nabla \cdot A = 0. \]  
\( (C13) \)

Within the Lorentz gauge, the field equations satisfied by the potentials \(U\) and \(A\) are given by

\[ \Box A_{\nu} = \mu_{0} (J_{e})_{\nu}, \quad \Box U = \frac{\rho_{e}}{\epsilon_{0}}, \quad \Box A = \mu_{0} J_{e}. \]  
\( (C14) \)

We also note that the second term in Eq. \((C4)\) and the third term in Eq. \((C6)\) disappear if we make the choice of the Lorentz gauge.

4. The generalized Poisson equation

The generalized Poisson equation writes (see Appendix \([B4]\)):

\[ \frac{\Delta \Phi}{4\pi G} = \frac{\epsilon}{c^2} = \frac{1}{2c^4} \left| \frac{\partial \varphi}{\partial t} \right|^2 + \frac{1}{2c^2} |\nabla \varphi|^2 + \frac{m^2}{2\hbar^2} |\varphi|^2 + \frac{1}{c^2} V(|\varphi|^2) - \frac{3H^2}{8\pi G} + i \frac{e}{2\hbar c^2} \left( \frac{\varphi}{\partial t} - \varphi \frac{\partial \varphi}{\partial t} \right) \]

\[ + \frac{e^2 U^2}{2\hbar^2 c^2} |\varphi|^2 - \frac{1}{2} \frac{i e}{\hbar c^2} \nabla \cdot (\varphi \nabla \varphi^* - \varphi^* \nabla \varphi) + \frac{e^2}{2\hbar^2 c^2} A^2 |\varphi|^2 + \frac{e_0 E^2}{2c^2} + \frac{1}{2\mu_0 c^2} A^2. \]  
\( (C15) \)

Equations \((C2), (C11), (C12)\) and \((C15)\) form the generalized KGMP equations.
5. The complex hydrodynamic equations

Making the WKB transformation [39] in the KG equation (C2), we obtain the complex quantum relativistic Hamilton-Jacobi equation (we assume $V = 0$):

$$\frac{1}{c^2} \left( \frac{\partial S_{\text{tot}}}{\partial t} + eU \right)^2 - (\nabla S_{\text{tot}} - eA)^2 - m^2 c^2 - 2m^2 \Phi = i\hbar \Box S_{\text{tot}} + i\hbar \left( \frac{1}{c^2} \frac{\partial U}{\partial t} + \nabla \cdot A \right).$$  \hspace{1cm} (C16)

Introducing the complex quadrivelocity [40], it can be rewritten as

$$U_\mu U^\mu - c^2 - 2\Phi = -i \frac{\hbar}{m} \partial_\mu U^\mu.$$  \hspace{1cm} (C17)

Taking the gradient of this equation and proceeding as in Sec. II G, we obtain the complex quantum relativistic Euler-Lorentz equation

$$dU_\nu/d\tau \equiv U_\mu \partial_\mu u_\nu = -i \frac{\hbar}{m} \Box \sqrt{\rho} \sqrt{\rho} + \frac{c}{m} U_\mu F_{\mu \nu} - i \frac{\hbar}{m} \partial_\mu \Phi + \partial_\nu \Phi.$$  \hspace{1cm} (C18)

The first term on the r.h.s. is a relativistic viscous term with a complex viscosity given by Eq. (51), the second term is a complex electromagnetic quantum force and the fourth term is the gravitational force. The KG equation (C2) is equivalent to the complex hydrodynamic equation (C18). For $\hbar = 0$, we recover the relativistic Euler-Lorentz equation (in that case $u_\mu$ is real):

$$dU_\nu/d\tau = u_\mu \partial_\mu u_\nu = -e \frac{c}{m} U_\mu F_{\mu \nu} + \partial_\nu \Phi.$$  \hspace{1cm} (C19)

Introducing the complex energy [45], the complex quantum relativistic Hamilton-Jacobi equation (C16) can be written as

$$E_{\text{tot}}^2 = m^2 c^4 + m^2 c^2 U^2 + 2m^2 c^2 \Phi - i\hbar \frac{\partial E_{\text{tot}}}{\partial t} - i\hbar mc^2 \nabla \cdot U.$$  \hspace{1cm} (C20)

6. The real hydrodynamic equations

Making the de Broglie transformation [53] in the KG equation (C2), separating real and imaginary parts, and introducing the quadrivelocity [58], we obtain the continuity equation

$$\partial_\mu (\rho u^\mu) = 0$$  \hspace{1cm} (C21)

and the quantum relativistic Hamilton-Jacobi equation

$$u_\mu u^\mu = \frac{\hbar^2}{m^2} \frac{\Box \sqrt{\rho}}{\sqrt{\rho}} + c^2 + 2\Phi + 2V'(\rho).$$  \hspace{1cm} (C22)

Taking the gradient of Eq. (C22) and proceeding as in Sec. II H, we obtain the quantum relativistic Euler-Lorentz equation

$$dU_\nu/d\tau = u_\mu \partial_\mu u_\nu = \frac{\hbar^2}{2m^2} \partial_\nu \left( \frac{\Box \sqrt{\rho}}{\sqrt{\rho}} \right) - \frac{e}{m} u^\mu F_{\mu \nu} + \partial_\nu \Phi + \partial_\nu V'(\rho).$$  \hspace{1cm} (C23)

The first term on the r.h.s. is the relativistic quantum force, the second term is the Lorentz force, the third term is the gravitational force and the fourth term is a pressure force arising from the self-interaction of the SF. Therefore, the KG equation (C2) is equivalent to the hydrodynamic equations (C21)-(C23). For $\hbar = 0$, we recover the relativistic Euler-Lorentz equation (C19) with the additional term $+\partial_\nu V'(\rho)$. Introducing the energy [60], the real quantum relativistic Hamilton-Jacobi equation (C22) can be written as

$$E_{\text{tot}}^2 = m^2 c^4 + m^2 c^2 U^2 + \frac{\hbar^2 c^2}{\sqrt{\rho}} + 2m^2 c^2 \Phi + 2m^2 c^2 V'(\rho).$$  \hspace{1cm} (C24)

This equation can be interpreted as the quantum version of the relativistic equation of mechanics $E_{\text{tot}}^2 = p^2 c^2 + m^2 c^4$ where $mu_\mu$ represents the impulse $p_\mu$. 

7. The generalized Gross-Pitaevskii-Maxwell-Poisson equations

Making the Klein transformation \(^{(23)}\) in the generalized KGMP equations \((C2)\) and \((C15)\), we obtain

\[
\frac{i\hbar}{\partial t} \frac{\partial \psi}{\partial t} - \frac{\hbar^2}{2mc^2} \frac{\partial^2 \psi}{\partial t^2} + \frac{\hbar^2}{2m} \Delta \psi - m\Phi \psi - m \frac{dV}{d|\psi|^2} \psi - i\hbar \left( \nabla \cdot A + \frac{1}{c^2} \frac{\partial U}{\partial t} \right) \psi
- \frac{\epsilon^2}{2m} \left( A^2 - \frac{1}{c^2} U^2 \right) \psi - i\frac{\epsilon h}{m} \left[ A \cdot \nabla \psi + U \frac{\partial \psi}{\partial t} - \frac{\epsilon m c^2}{\hbar} \psi \right] = 0.
\]

\[
(C25)
\]

\[
\frac{\Delta \Phi}{4\pi G} = |\psi|^2 + \frac{\hbar^2}{2mc^2} \left| \frac{\partial \psi}{\partial t} \right|^2 + \frac{\hbar^2}{2c^2 m^2} |\nabla \psi|^2 + \frac{1}{c^2} V(|\psi|^2) - \frac{\hbar}{mc} \text{Im} \left( \frac{\partial \psi}{\partial t} \psi^* \right)
+ \frac{i\hbar}{2mc^2} \left( \psi \frac{\partial \psi^*}{\partial t} - \psi^* \frac{\partial \psi}{\partial t} \right) - \frac{\epsilon}{mc^2} U |\psi|^2 + \frac{e^2}{2mc^2} |\nabla \psi|^2 - \frac{1}{2m^2 c^2} A : (\psi \nabla \psi^* - \psi^* \nabla \psi)
+ \frac{1}{2m^2 c^2} A^2 |\psi|^2 + \frac{\epsilon_0}{c^2} \frac{E^2}{2} + \frac{1}{2\mu_0} \frac{B^2}{c^2}.
\]

\[
(C26)
\]

Equations \((C11), (C12), (C25)\) and \((C26)\) form the generalized GPMP equations. The relativistic GP equation \((C25)\) can be written more compactly as

\[
\frac{i\hbar}{\partial t} \frac{\partial \psi}{\partial t} - \frac{\hbar^2}{2m} \left( \nabla - i\frac{e}{\hbar} A \right)^2 \psi + m\Phi \psi - m \frac{dV}{d|\psi|^2} \psi = 0.
\]

\[
(C27)
\]

This equation can be directly obtained from Eq. \((75)\) by considering that the metric is approximately flat. Since the term \(g^{00} - 1\) in Eq. \((84)\) is multiplied by \(c^2\), we have to use the expression of the metric \(g^{00} \approx 1 - 2\Phi/c^2\) valid at first order in \(\Phi/c^2\) to evaluate this term. This is how the Newtonian potential \(\Phi\) enters into Eq. \((C27)\).

In the nonrelativistic limit \(c \to +\infty\), Eqs. \((C25)\) and \((C26)\) reduce to

\[
\frac{i\hbar}{\partial t} \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \left( \nabla - i\frac{e}{\hbar} A \right)^2 \psi + m\Phi \psi + eU \psi + m \frac{dV}{d|\psi|^2} \psi.
\]

\[
(C28)
\]

\[
\Delta \Phi = 4\pi G |\psi|^2.
\]

\[
(C29)
\]

Equations \((C11), (C12), (C28)\) and \((C29)\) form the GPMP equations. The GPMP equations are usually derived in a different manner, from the mean field Schrödinger equation (see, e.g., Sec. II.A of \([72]\)). Here, we have rigorously derived these equations from the KGME equations in the nonrelativistic limit \(c \to +\infty\). This derivation shows that the nonlinearity in the GP equation \((C28)\) is related to the potential of self-interaction \(V\) in the KG equation \((C2)\).

For the quartic potential \((84)-(86)\) of Sec. III.C, the complex quantum relativistic Hamilton-Jacobi equation and the complex Euler-Lorentz equation write (we assume \(V = 0\)):

\[
\frac{1}{c^2} \left( \frac{\partial S}{\partial t} + eU \right)^2 - 2m \left( \frac{\partial S}{\partial t} + eU \right) - (\nabla S - eA)^2 - 2m^2 \Phi = i\hbar \Box S + ie \hbar \left( \frac{1}{c^2} \frac{\partial U}{\partial t} + \nabla \cdot A \right).
\]

\[
(C31)
\]
\[ V_\mu V^\mu = 2\Phi - 2cV^0 - i\frac{\hbar}{m}\partial_\mu V^\mu, \] (C32)

\[ \frac{dV_\nu}{d\tau} \equiv V^\mu \partial_\mu V_\nu = -c\partial^0 V_\nu - i\frac{\hbar}{2m} \Box V_\nu - \frac{e}{m} V^\mu F_{\mu\nu} - \frac{e}{m} cF^0_\nu - i\frac{\hbar e}{2m^2} \partial_\mu F_{\mu\nu} + \partial_\nu \Phi, \] (C33)

\[ \frac{\mathcal{E}^2}{2mc^2} + \mathcal{E} = \frac{1}{2} m\mathbf{v}^2 + m\Phi - \frac{i\hbar}{2mc^2} \partial_t \mathcal{E} - i\frac{\hbar}{2} \nabla \cdot \mathbf{v}. \] (C34)

9. The real hydrodynamic equations

The real hydrodynamic equations corresponding to the relativistic GP equation (C25) can be obtained by making the Madelung transformation \[ \text{(89)} \] and proceeding as in Appendix \[ \text{C6} \]. Alternatively, they can be directly obtained from the real hydrodynamic equations \[ \text{(C21)-(C24)} \] corresponding to the KG equation \[ \text{(C2)} \] by using the relations \[ \text{(92)-(96)} \] of Sec. \[ \text{III D} \]. The continuity equation, the quantum relativistic Hamilton-Jacobi equation, and the quantum relativistic Euler-Lorentz equation write

\[ \frac{\partial \rho}{\partial t} + \partial_\mu (\rho v^\mu) = 0, \] (C35)

\[ v_\mu v^\mu = -2cV^0 + 2\Phi + \frac{\hbar^2}{m^2} \Box \rho + 2V' (\rho), \] (C36)

\[ \frac{dv_\nu}{d\tau} \equiv v^\mu \partial_\mu v_\nu = -c\partial^0 v_\nu + \frac{\hbar^2}{2m^2} \partial_\nu \left( \frac{\Box \rho}{\sqrt{\rho}} \right) - \frac{e}{m} v^\mu F_{\mu\nu} - \frac{e}{m} cF^0_\nu + \partial_\nu \Phi + \partial_\nu V'(\rho), \] (C37)

\[ \frac{E^2}{2mc^2} + E = \frac{1}{2} m\mathbf{v}^2 + \frac{\hbar^2}{2m} \Box \rho + m\Phi + mV'(\rho). \] (C38)

10. The hydrodynamic equations

Making the Madelung transformation defined by Eqs. \[ \text{(89)} \] and \[ \text{(138)} \] in the generalized GPMP equations \[ \text{(C25)} \] and \[ \text{(C26)} \], or taking the weak field limit \[ \Phi/c^2 \to 0 \] in the quantum barotropic ELME equations \[ \text{(145)} \] and \[ \text{(150)-(152)} \] and setting \( a = 1 \) and \( H = 0 \), we obtain the hydrodynamic equations

\[ \frac{\partial}{\partial t} \left[ \left( 1 + \frac{E}{mc^2} \right) \rho \right] + \nabla \cdot (\rho \mathbf{v}) = 0, \] (C39)

\[ \frac{\partial S}{\partial t} + \frac{1}{2m} (\nabla S - eA)^2 = -\frac{\hbar^2}{2m} \Box \rho - m\Phi - eU - mh(\rho) + \frac{E^2}{2mc^2}, \] (C40)

\[ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{\hbar^2}{2m^2} \nabla \left( \frac{\Box \rho}{\sqrt{\rho}} \right) - \nabla \Phi - \frac{1}{\rho} \nabla \rho + \frac{e}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) + \frac{1}{2mc^2} \nabla (E^2), \] (C41)

\[ \frac{\Delta \Phi}{4\pi G} = \rho + \frac{\hbar^2}{2m^2c^4} \left[ \frac{1}{4\rho} \left( \frac{\partial \mathbf{v}}{\partial t} \right)^2 + \rho \left( \frac{\partial S}{\partial t} \right)^2 \right] + \frac{\hbar^2}{2c^2m^2} \left[ \frac{1}{4\rho} (\nabla \rho)^2 + \frac{\rho}{h^2} (\nabla S)^2 \right] \]
\[ + \frac{e^2}{m^2c^2} V(\rho) - \frac{1}{mc^2} \frac{\partial S}{\partial t} + \frac{e}{mc^2} U \frac{\partial S}{\partial t} - \frac{e}{mc^2} U \rho \]
\[ + \frac{e^2}{2m^2} \frac{U^2}{c^4} - \frac{e}{m^2c^2} \rho A \cdot \nabla S + \frac{1}{2m^2c^2} \frac{e^2}{2} A^2 \rho + \frac{e_0}{2} E^2 + \frac{1}{2\mu_0} B^2. \] (C42)
Equations (C11), (C12), and (C39)-(C42) form the generalized quantum barotropic ELMP equations. Introducing the energy (140), the quantum relativistic Hamilton-Jacobi equation (C40) can be written as

$$E + \frac{E^2}{2mc^2} = \frac{1}{2}mv^2 + \frac{\hbar^2}{2m} \frac{\Box \sqrt{\rho}}{\sqrt{\rho}} + m\Phi + mV'(\rho).$$

(C43)

Solving for $E$ in Eq. (C43), we obtain

$$E = mc^2 \left[ -1 \pm \sqrt{1 + \frac{v^2}{c^2} + \frac{\hbar^2}{m^2 c^2} \frac{\Box \sqrt{\rho}}{\sqrt{\rho}} + 2 \frac{\Phi}{c^2} + 2 \frac{1}{c^2} h(\rho) \right].$$

(C44)

This expression can be substituted into Eqs. (C39) and (C41) to obtain a closed system of hydrodynamic equations. In the nonrelativistic limit $c \to +\infty$, the hydrodynamic equations (C39)-(C42) reduce to

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0,$$

(C45)

$$\frac{\partial S}{\partial t} + \frac{1}{2m} (\nabla S - eA)^2 = \frac{\hbar^2}{2m} \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} - m\Phi - eU - mh(\rho),$$

(C46)

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v = \frac{\hbar^2}{2m^2} \nabla \left( \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) - \nabla \Phi - \frac{1}{\rho} \nabla p + \frac{e}{m} (E + v \times B),$$

(C47)

$$\Delta \Phi = 4\pi G \rho.$$

(C48)

Equations (C11), (C12), and (C45)-(C48) form the quantum barotropic ELMP equations. Introducing the energy (140), the quantum Hamilton-Jacobi equation (C46) can be written as

$$E = \frac{1}{2}mv^2 - \frac{\hbar^2}{2m} \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} + m\Phi + mh(\rho).$$

(C49)

This equation can be interpreted as the quantum generalization of the classical equation of mechanics $E = p^2 / 2m + m\Phi$ where $mv$ represents the impulse $p$.

11. The London equations

The London equations are given by

$$\Box A_v = -\mu_0 \frac{e^2}{m^2 \rho} j_A, \quad \Box U = -\frac{e^2}{m^2 c^2 \epsilon_0} \rho U, \quad \Box A = -\frac{e^2 \mu_0}{m^2} \rho A.$$

(C50)

In a static spacetime, the equations for $U$ and $A$ are decoupled. In the absence of electric field, $A$ is stationary and Eq. (C50) reduces to

$$\Delta A = \frac{e^2 \mu_0}{m^2} \rho A.$$

(C51)

This is the standard London equation which displays a characteristic length scale $\lambda = (m^2 / \mu_0 e^2)^{1/2}$ over which the potential vector, hence the magnetic field, is exponentially suppressed. This accounts for the Meissner effect [113] in the theory of superconductivity wherein a material exponentially expels all internal magnetic fields as it crosses the superconducting threshold. The length scale $\lambda$ is the London penetration depth.
Appendix D: The relativistic and nonrelativistic eigenvalue equations

In the absence of magnetic field \((A = 0)\) and self-interaction \((V = 0)\), the KG equation \((C2)\) can be written as
\[
\frac{1}{c^2} \left( \frac{\partial}{\partial t} + i\frac{e}{\hbar} U \right)^2 \varphi - \Delta \varphi + \frac{m^2 c^2}{\hbar^2} \varphi + \frac{2m^2}{\hbar^2} \Phi \varphi = 0, \tag{D1}
\]
or, equivalently, as
\[
\frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} - \Delta \varphi + i\frac{e}{\hbar c^2} \frac{\partial U}{\partial t} \varphi + 2i\frac{e}{\hbar c^2} U \frac{\partial \varphi}{\partial t} - \frac{e^2}{\hbar^2 c^2} U^2 \varphi + \frac{m^2 c^2}{\hbar^2} \varphi + \frac{2m^2}{\hbar^2} \Phi \varphi = 0. \tag{D2}
\]

Looking for a stationary solution of the form
\[
\varphi = \frac{\hbar}{m} \sqrt{\rho} e^{-iE_{\text{tot}}t}/\hbar, \tag{D3}
\]
where \(E_{\text{tot}}\) is a constant, we obtain the eigenvalue equation
\[
(E_{\text{tot}} - eU)^2 \varphi + c^2 \hbar^2 \Delta \varphi - 2m^2 c^2 \Phi \varphi - m^2 c^4 \varphi = 0. \tag{D4}
\]
This equation can also be obtained from the Hamilton-Jacobi equation \((C22)\) with \(A = 0, V = 0\) and \(S_{\text{tot}} = -E_{\text{tot}}t\).22

In the absence of magnetic field \((A = 0)\) and self-interaction \((V = 0)\), the GP equation \((C28)\) reduces to the Schrödinger equation
\[
i\frac{\hbar}{2m} \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m\Phi \psi + eU \psi. \tag{D5}
\]
Looking for a stationary solution of the form
\[
\psi = \sqrt{\rho} e^{-iEt}/\hbar, \tag{D6}
\]
where \(E\) is a constant, we obtain the eigenvalue equation
\[
-\frac{\hbar^2}{2m} \Delta \psi + (eU + m\Phi) \psi = E \psi. \tag{D7}
\]
This equation can also be obtained from the Hamilton-Jacobi equation \((C46)\) with \(A = 0, V = 0\) and \(S = -Et\). Similarly to the remark made in Sec. \((C7)\) the gravitational (external) potential \(\Phi\) and the electric potential \(U\) appear on the same footing in the nonrelativistic eigenvalue equation \((D7)\) while they appear at different places in the relativistic eigenvalue equation \((D4)\). On the other hand, if we set \(E_{\text{tot}} = E + mc^2\) in Eq. \((D4)\) and take the nonrelativistic limit \(c \to +\infty\), we recover Eq. \((D7)\).

If we consider a free particle \((U = \Phi = 0)\), expand the wave functions \(\varphi\) and \(\psi\) as plane waves of the form \(e^{i(k \cdot r - \omega t)}\), and use the de Broglie relations \(E_{\text{tot}} = h\omega\) and \(p = \hbar k\), we obtain from the KG equation \((D1)\) the relativistic dispersion relation
\[
\omega^2 = c^2 k^2 + \frac{m^2 c^4}{\hbar^2} \quad \Leftrightarrow \quad E^2_{\text{tot}} = p^2 c^2 + m^2 c^4, \tag{D8}
\]
and from the Schrödinger equation \((D5)\) the classical dispersion relation
\[
\omega = \frac{\hbar k^2}{2m} \quad \Leftrightarrow \quad E = \frac{p^2}{2m}. \tag{D9}
\]
In the relativistic case, we find that \(E_{\text{tot}} = \pm \sqrt{p^2 c^2 + m^2 c^4}\). There exist solutions both for positive and negative energies. The solutions yielding negative energies led Dirac 52 to postulate the existence of antiparticles.

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22 We note that the eigenenergy \(E_{\text{tot}}\) defined by Eq. \((D3)\) is different from the energy defined by Eq. \((60)\) because of the presence of the term \(eU\). However, in order to avoid the proliferation of notations, we use the same symbol. The same remark applies to the eigenenergy \(E\) defined by Eq. \((D6)\) below.
Appendix E: The pilot wave theory of de Broglie

De Broglie \cite{45,46} performed the transformation \cite{63} on the KG equation \cite{14} with $V = 0$ and derived the quantum relativistic Hamilton-Jacobi equation \cite{63} which contains the relativistic covariant quantum potential \cite{57}. He also proposed to formally rewrite Eq. \cite{55} in the form of a relativistic Hamilton-Jacobi equation

\begin{equation}
(\partial_{\mu}S_{\text{tot}} + eA_{\mu}) (\partial^{\mu}S_{\text{tot}} + eA^{\mu}) - M^2 c^2 = 0 \tag{E1}
\end{equation}

by introducing an effective mass

\begin{equation}
M = \sqrt{m^2 + \frac{\hbar^2 \Box}{c^2}} \tag{E2}
\end{equation}

which includes the relativistic quantum potential. Equation \cite{61} can be rewritten as $u_{\mu}u^{\mu} = c^2$ where $u_{\mu}$ is defined by Eq. \cite{55} with $M$ instead of $m$.

De Broglie also developed a pilot wave theory that is closely related to the results obtained in Sec. III. We mention here some connections with his work. As shown in Sec. III the quadricurrent of charge is $(J_{\mu})_0 = (e/m)\rho_{0\mu}$, the density of charge is $\rho = \rho_{0\mu}/c$, and the current of charge is $J_{\mu} = (e/m)\rho_{\mu}u$. We note that, in the relativistic regime, $(J_{\mu})_0 \neq \rho_{\mu}u$, $(J_{\mu})_0 \neq \rho_{\mu}u$ as we could naively expect (these relations are only valid in the nonrelativistic limit $c \rightarrow +\infty$). We have instead $(J_{\mu})_0 = \rho_{c}(u_{\mu}/u_0)c$, $(J_{\mu})_0 = \rho_{c}c$, and $J_{\mu} = \rho_{c}(u/u_0)c$.

We can write the quadricurrent of charge as

\begin{equation}
(J_{\mu})_0 = \rho_{c}(u_{\mu})_0 \tag{E3}
\end{equation}

by introducing a charge quadrivelocity

\begin{equation}
(u_{\mu})_0 = \frac{u_{\mu}}{u_0}c = \frac{\partial_{\mu}S_{\text{tot}} + eA_{\mu}}{\partial_{0}S_{\text{tot}} + eA_{0}c}. \tag{E4}
\end{equation}

We note that $(u_{\mu})_0 = c$. The components of the quadricurrent of charge satisfy the relations

\begin{equation}
(J_{\mu})_0 = \rho_{c}c, \quad J_{\mu} = \rho_{c}u_{\mu}. \tag{E5}
\end{equation}

The charge density and the charge velocity are given by

\begin{equation}
\rho_{c} = -\frac{e}{m} \frac{\partial S_{\text{tot}}}{\partial t} + eU, \quad \frac{u_{\mu}}{u_0}c = -\frac{\nabla S_{\text{tot}} - eA}{\partial_{t}S_{\text{tot}} + eU}c^2. \tag{E6}
\end{equation}

Using these relations, we find that the local charge conservation equation \cite{20}, which is equivalent to the equation of continuity \cite{41}, can be written as

\begin{equation}
D_{\mu}(\rho_{c}u^{\mu}) = 0. \tag{E7}
\end{equation}

We found these equations by ourselves. However, by studying the early history of quantum mechanics to prepare the Introduction, we discovered with surprise that these equations were first obtained by de Broglie \cite{45} a very long time ago in relation to his pilot wave theory [see his Eqs. (I) and (II)] and presented as formules fondamentales of his theory.

Appendix F: The historical derivation of the Schrödinger equation

In most textbooks of quantum mechanics, the Schrödinger equation is derived from the correspondence principle. This is not, however, how Schrödinger initially derived it. In this Appendix, we give a short account of the manner how Schrödinger obtained his famous equation in his historical papers since his original approach is not well-known.

1. The pre-historic derivation of the Schrödinger equation

We first begin by the manner how Schrödinger obtained his equation according to Dirac \cite{1}. However, we could not find any trace of this derivation (or even a mention of it) in any published paper. Therefore, the claim of Dirac must be considered with circumspection. According to Dirac \cite{1}, de Broglie introduced the relativistic wave equation

\begin{equation}
\frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} - \Delta \varphi + \frac{m^2 c^2}{\hbar^2} \varphi = 0. \tag{F1}
\end{equation}
This is what we now call the KG equation. Then, still according to Dirac, Schrödinger extended this equation in order to include the electromagnetic field and “guessed” the equation
\[
\frac{1}{c^2} \left( \frac{\partial}{\partial t} + \frac{ie}{\hbar} U \right)^2 \varphi - \left( \nabla - \frac{ie}{\hbar} A \right)^2 \varphi + \frac{m^2c^2}{\hbar^2} \varphi = 0. \tag{F2}
\]
He then considered the nonrelativistic limit of this equation (Dirac does not explain how he did) and obtained, in the absence of magnetic field, the wave equation
\[
\frac{i\hbar}{\partial t} \psi = -\frac{\hbar^2}{2m} \Delta \psi + eU\psi. \tag{F3}
\]
This is the Schrödinger equation. However, in his published papers, Schrödinger introduced this equation in a completely different manner, adopting from the start a nonrelativistic approach. He introduced the KG equation (F2) in his fourth paper, but did not explicitly take the nonrelativistic limit of this equation to make the connection with Eq. (F3). To our knowledge, this connection was first made by Klein.

2. The first derivation of the Schrödinger equation

In his first paper, Schrödinger derives the eigenvalue equation from a variational principle. He starts from the classical Hamilton-Jacobi equation
\[
E = \frac{1}{2m} (\nabla S)^2 + m\Phi(r), \tag{F4}
\]
where $E$ is the classical energy (which is a constant) and $S = S(r)$ is the classical action which is related to the classical impulse by $p = \nabla S$. He introduces a (real) wave function $\psi(r)$ through the substitution
\[
S = \hbar \ln \psi. \tag{F5}
\]
Equation (F4) is then rewritten in terms of $\psi$ as
\[
(\nabla \psi)^2 - \frac{2m}{\hbar^2} (E - m\Phi)\psi^2 = 0. \tag{F6}
\]
At that point, Schrödinger introduces the functional
\[
J = \int \left[ (\nabla \psi)^2 - \frac{2m}{\hbar^2} (E - m\Phi)\psi^2 \right] d\mathbf{r} \tag{F7}
\]
and considers its minimization with respect to variations on $\psi$. The condition $\delta J = 0$ gives
\[
\Delta \psi + \frac{2m}{\hbar^2} (E - m\Phi)\psi = 0, \tag{F8}
\]
which is the time-independent Schrödinger equation. This is an eigenvalue equation for the energy $E$. This is how the energy becomes quantized. Indeed, for a given potential $\Phi$, this equation has physical solutions only for some particular values of the energy. Therefore, this equation determines the quantification of the energy through the boundary conditions. In the Zusatz bei der Korrektur, Schrödinger remarks that his variational principle can be written as the minimization of an energy functional
\[
E_{\text{tot}} = \int \left[ \frac{\hbar^2}{2m} (\nabla \psi)^2 + m\Phi\psi^2 \right] d\mathbf{r} \tag{F9}
\]

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23 We could not find any published paper of de Broglie containing the KG equation prior to 1926. There is an equation similar to the KG equation in but it has the opposite sign: $\Box \varphi - \frac{(m^2c^2/\hbar^2)}{\varphi = 0}$ instead of $\Box \varphi + \frac{(m^2c^2/\hbar^2)}{\varphi = 0}$. On this point again, the claim of Dirac (the attribution of the KG equation to de Broglie) must be considered with circumspection.

24 This can be written as $\psi = e^{S/\hbar}$ which corresponds to the WKB transformation with a purely imaginary action $S = -iS$.

25 Schrödinger puts much emphasis on this equation in his papers, so this equation really is the Schrödinger equation (more than the time-dependent equation introduced later).
under the constraint
\[ \int \psi^2 d\mathbf{r} = 1. \] (F10)

Indeed, if we write the variational principle as
\[ \delta E_{\text{tot}} - E \delta \int \psi^2 d\mathbf{r} = 0, \] (F11)
we obtain Eq. (F8) where the eigenenergy \( E \) appears as the Lagrange multiplier associated with the constraint (F10).

Remark: Nowadays, we derive the time-independent Schrödinger equation as follows. We write the Hamiltonian (total energy) as
\[ E_{\text{tot}} = \int \frac{\hbar^2}{2m} (\nabla \psi)^2 d\mathbf{r} + \int m \Phi |\psi|^2 d\mathbf{r}, \] (F12)
where the first term is the kinetic energy and the second term is the potential energy. One can show that the Hamiltonian (F12) and the integral \( \int |\psi|^2 d\mathbf{r} \) (normalization) are conserved by the Schrödinger equation (F23). As a result, a minimum of energy \( E_{\text{tot}} \) under the normalization condition \( \int |\psi|^2 d\mathbf{r} = 1 \) is a stationary solution of the Schrödinger equation that is formally nonlinearly dynamically stable.\textsuperscript{26} Writing the variational principle as
\[ \delta E_{\text{tot}} - \mu \delta \int |\psi|^2 d\mathbf{r} = 0, \] (F13)
where \( \mu \) is a Lagrange multiplier (chemical potential), we obtain the time-independent Schrödinger equation (F8) provided that \( \mu = E \). This shows that the chemical potential \( \mu \) can be identified with the eigenenergy \( E \) and \textit{vice versa}.

3. The second derivation of the Schrödinger equation

In his second paper \[ \text{[7]} \], Schrödinger proposes another, more physical, derivation of his equation by developing an analogy between geometric and undulatory mechanics. His derivation combines arguments coming from mechanics, optics and dispersive waves using the principles of Hamilton, Huygens, and Fermat. We very succinctly recall the main steps of his derivation.

Schrödinger was strongly inspired by the fundamental researches of de Broglie on matter waves (see his Introduction in \[ \text{[23]} \]). Following de Broglie, he assumes that each particle (like an electron or a proton) is described by a wave of the form
\[ \psi(\mathbf{r}, t) = Ae^{i(k \cdot \mathbf{r} - \omega t)}. \] (F14)

Using the de Broglie relations
\[ E = \hbar \omega, \quad \mathbf{p} = \hbar \mathbf{k}, \] (F15)
and the classical expression of the energy
\[ E = \frac{p^2}{2m} + m \Phi(\mathbf{r}) \] (F16)
of a particle of mass \( m \) moving in a potential \( \Phi(\mathbf{r}) \), Schrödinger finds that the phase velocity of the matter wave is given by\textsuperscript{27}
\[ v_\omega = \frac{\omega}{k} = \frac{E}{p} = \frac{E}{\sqrt{2m(E - m \Phi(\mathbf{r}))}}. \] (F17)

\textsuperscript{26} This is how we now justify the minimization problem \( \min \{ E_{\text{tot}} | \int |\psi|^2 d\mathbf{r} \text{ fixed} \} \). Schrödinger introduced this minimization problem as a postulate, without any justification. This remarkably led him to the Schrödinger equation. However, this cannot be considered as a derivation of this equation.

\textsuperscript{27} Note that the phase velocity \[ \text{[F17]} \] differs from the group velocity \( v_g = d\omega/dk = dE/dp = p/m \) which coincides with the velocity \( v = p/m \) of the particle.
Therefore, the phase velocity of the wave depends on the position \( r \). This is at variance with the electromagnetic wave for which \( v_\phi = c \) is a constant. Schrödinger then assumes that the evolution of the wave function \( \psi(r, t) \) is given by the ordinary (second order in time) wave equation in a dispersive medium

\[
\frac{1}{v_\phi^2(r)} \frac{\partial^2 \psi}{\partial t^2} = \Delta \psi.
\]  

(F18)

He also argues that the wave function must be of the form

\[
\psi(r, t) \propto e^{-iEt/\hbar},
\]  

(F19)

which corresponds to the temporal term in Eq. (F14) when we use the de Broglie relation (F15). From Eq. (F19), we get

\[
\frac{\partial^2 \psi}{\partial t^2} = -\frac{E^2}{\hbar^2} \psi.
\]  

(F20)

Combining Eqs. (F17), (F18) and (F20), Schrödinger recovers the fundamental eigenvalue equation

\[
-\frac{\hbar^2}{2m} \Delta \psi + m\Phi \psi = E \psi
\]  

(F21)

that he previously obtained from a variational principle [6]. This is the stationary Schrödinger equation. In his fourth paper [9], using the identity

\[
\frac{\partial \psi}{\partial t} = -i\frac{E}{\hbar} \psi,
\]  

(F22)

he eliminates the energy from Eq. (F21) and rewrites this equation under the form

\[
i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m\Phi \psi.
\]  

(F23)

This is the time-dependent Schrödinger equation.

Remark: Before arriving at Eq. (F23), Schrödinger derives from Eq. (F21) the equation

\[
\left( -\frac{\hbar^2}{2m} \Delta + m\Phi \right)^2 \psi = E^2 \psi.
\]  

(F24)

Using Eq. (F20) he eliminates the energy from Eq. (F24) and obtains an equation the form

\[
\hbar^2 \frac{\partial^2 \psi}{\partial t^2} + \left( -\frac{\hbar^2}{2m} \Delta + m\Phi \right)^2 \psi = 0
\]  

(F25)

in which there is no complex \( i \). Of course, Eq. (F25) can also be obtained from Eq. (F23) by time derivation.

4. The correspondence principle

The nowadays standard procedure to derive the Schrödinger equation is based on the correspondence principle (this procedure can also be extended to the KG equation). The correspondence principle first appeared in the works of Schrödinger [15] and de Broglie [18, 19], and was rapidly adopted by many other authors. The quantum equations (KG and Schrödinger) can be obtained from the non-quantum ones by using the correspondence principle

\[
p_\mu \leftrightarrow i\hbar \partial_\mu, \quad E_{(tot)} \leftrightarrow i\hbar \frac{\partial}{\partial t}, \quad \mathbf{p} \leftrightarrow -i\hbar \nabla,
\]  

(F26)

where \( p_0 = E_{(tot)}/c \) and \( \mathbf{p} = (-p_1, -p_2, -p_3) \) are the components of the quadri-impulse \( p_\mu \). In this way, the classical equation of mechanics \( E = p^2/2m + m\Phi \) yields the Schrödinger equation (F23) and the relativistic equation of mechanics \( E_{(tot)}^2 = p^2c^2 + m^2c^4 \), equivalent to \( p_\mu p^\mu = m^2c^2 \), yields the KG equation (F1). The electromagnetic field can be taken into account through the substitutions \( \partial_\mu \rightarrow \partial_\mu + \frac{i}{e} A_\mu \), \( \partial_t \rightarrow \partial_t + \frac{i}{e} U \) and \( \nabla \rightarrow \nabla - \frac{i}{e} \mathbf{A} \), leading to the electromagnetic Schrödinger equation (C22) and to the electromagnetic KG equation (F2). An external potential \( \Phi \) can be introduced in the KG equation by writing the relativistic equation of mechanics as \( E_{(tot)}^2 = p^2c^2 + m^2c^4 + 2m^2c^2\Phi \). This leads to Eq. (C2). However, the rigorous way to take gravity into account in the KG equation is to consider a curved spacetime according to Einstein’s theory of general relativity as we have done in the present paper. In the weak field limit \( \Phi/c^2 \rightarrow 0 \), it is found that the KG equation reduces to Eq. (C2).
5. The relativistic generalization Schrödinger’s variational principle

Schrödinger’s variational principle described in Sec. [2] was generalized to relativistic particles by De Donder [26, 32, 37, 40] and Gordon [31]. They started from the relativistic Hamilton-Jacobi equation

$$\partial_{\mu}S_{\text{tot}}\partial^{\mu}S_{\text{tot}} - m^2c^2 = 0 \quad (F27)$$

and performed the transformation

$$S_{\text{tot}} = \hbar \ln \varphi, \quad (F28)$$

thereby obtaining

$$\partial_{\mu}\varphi \partial^{\mu}\varphi - \frac{m^2c^2}{\hbar^2} \varphi^2 = 0. \quad (F29)$$

Integrating this equation over the four dimensional spacetime, they obtained the functional

$$S_{\varphi} = \int \left( \frac{1}{2} \partial_{\mu}\varphi \partial^{\mu}\varphi - \frac{m^2c^2}{2\hbar^2} \varphi^2 \right) d^4x \quad (F30)$$

which can be interpreted as the action of a real SF (Gordon [31] used an extension of this method to deal with a complex SF). The principle of least action$$\delta S_{\varphi} = 0$$ then leads to the KG equation (F1).

Appendix G: Detail of the calculations leading to Eq. (75)

From Eq. (73) we obtain

$$D_{\mu} \partial_{\nu} \varphi \equiv D_{\mu} (\partial_{\nu} \psi - i\frac{mc}{\hbar} \delta^0_{\nu} \psi) e^{-imc^2t/\hbar}. \quad (G1)$$

Using

$$D_{\mu} \partial^{\nu} \varphi = D_{\mu} (g^{\mu\nu} \partial_{\nu} \varphi) = g^{\mu\nu} D_{\mu} \partial_{\nu} \varphi \quad (G2)$$

and Eq. (G1), we get (see Appendix A):

$$D_{\mu} \partial^{\nu} \varphi = g^{\mu\nu} \frac{\hbar}{m} e^{-imc^2t/\hbar} \left[ D_{\mu} \left( \partial_{\nu} \psi - i \frac{mc}{\hbar} \delta^0_{\nu} \psi \right) - \frac{imc}{\hbar} \delta_0^{\nu} \left( \partial_{\mu} \psi - i \frac{mc}{\hbar} \delta^0_{\mu} \psi \right) \right]. \quad (G3)$$

Performing the tensorial product of the term in bracket with $g^{\mu\nu}$, and using the identity

$$D_{\mu} \left( \delta^0_{\nu} \psi \right) = \partial_{\nu} \left( \delta^0_{\nu} \psi \right) - \Gamma_{\mu\nu}^\sigma \delta^0_{\sigma} \psi = \delta^0_{\nu} \partial_{\mu} \psi - \Gamma^0_{\mu\nu} \psi, \quad (G4)$$

we obtain

$$D_{\mu} \partial^{\nu} \varphi = \frac{\hbar}{m} e^{-imc^2t/\hbar} \left[ D_{\mu} \partial^{\nu} \psi - 2i \frac{mc}{\hbar} \partial^0 \psi - \frac{m^2c^2}{\hbar^2} g^{00} \psi + \frac{imc}{\hbar} \Gamma_{\mu\nu}^0 g^{\mu\nu} \psi \right]. \quad (G5)$$

Substituting Eqs. (G1) and (G5) into Eq. (17) and rearranging terms, we get

$$\Box \varphi = \frac{\hbar}{m} e^{-imc^2t/\hbar} \left[ \Box \varphi - 2i \frac{mc}{\hbar} \partial^0 \varphi - \frac{m^2c^2}{\hbar^2} g^{00} \varphi + 2 \frac{mc}{\hbar} \Gamma^0 \psi + \frac{imc}{\hbar} \Gamma_{\mu\nu}^0 g^{\mu\nu} \psi \right]. \quad (G6)$$

Finally, substituting Eq. (G6) into the KG equation (14), we obtain the general relativistic GP equation (75). We can also proceed slightly differently. Substituting Eq. (G1) into the identity

$$D_{\mu} \partial_{\nu} \varphi = \partial_{\mu} \partial_{\nu} \varphi - \Gamma_{\mu\nu}^\sigma \partial_{\sigma} \varphi \quad (G7)$$

and expanding the terms, we get

$$D_{\mu} \partial_{\nu} \varphi = \frac{\hbar}{m} e^{-imc^2t/\hbar} \left[ D_{\mu} \partial_{\nu} \psi - i \frac{mc}{\hbar} (\delta^0_{\mu} \partial_{\nu} \psi + \delta^0_{\nu} \partial_{\mu} \psi) - \frac{m^2c^2}{\hbar^2} \delta^0_{\nu} \delta^0_{\mu} \psi + i \frac{mc}{\hbar} \Gamma^0_{\mu\nu} \psi \right]. \quad (G8)$$

Performing the tensorial product of Eq. (G8) with $g^{\mu\nu}$ and using Eq. (G2), we recover Eq. (G3), then Eq. (17).
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