An experimental test of Corrsin’s conjecture and some related ideas

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Abstract. We present an experimental test of Corrsin’s conjecture against experimental data obtained by a particle tracking technique in approximately homogeneous and isotropic turbulent flow at Reynolds numbers $R_\lambda \approx 100$. The conjecture states that

$$R_L(t) \approx \int R_E(x, t) G(x, t) \, d^3x,$$

where $R_L(t-t') = \langle v(t) \cdot v(t') \rangle$ is the Lagrangian velocity covariance function, $G$ is the single particle mean Green’s function, and $R_E(x-x', t-t') \equiv \langle u(x, t) \cdot u(x', t') \rangle$ is the Eulerian two-point, two-time velocity covariance function. All terms in the relation have been measured in the experiment. The equation is exact if a conditional Lagrangian velocity function $R_L(t|x)$ is inserted in place of $R_E(x, t)$ on the right-hand side. $R_L(t|x)$ is obtained by restricting sampling of the two velocities to situations where both belong to the same fluid particle trajectory. The experimental data show that the $R_E(x, t)$ and $R_L(t|x)$ behave fundamentally differently, thereby seriously questioning the rationale of the conjecture. The estimate of $R_L(t)$, based on Corrsin’s conjecture and the experimentally determined $R_E$ and $G$, is found to decrease too fast compared to the directly measured $R_L$, thus underestimating the Lagrangian timescale by about 40%. Even asymptotically ($t \to \infty$) the estimate is considerably lower than the measured Lagrangian correlation function. The simpler relation $R_L(t) = R_E(0, t)$, which has also been attributed to Corrsin, appears to agree much better with data. Various simple physical models of the spatiotemporal Eulerian correlation

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function have been compared with data, and models inspired by eddy sweeping seem to perform well.

Contents

1. Introduction 2
2. The experiment 7
   2.1. Data processing .................................. 8
3. Experimental results 12
   3.1. Eulerian statistics ................................. 12
   3.2. Lagrangian statistics ............................... 15
4. Conclusions 20
Acknowledgments 22
Appendix A. $R_L(t|t')$ for Gaussian displacements 22
Appendix B. A counterexample 23
References 24

1. Introduction

Corrsin’s conjecture [1, 2] states that

$$R_L(t) = \int R_E(x, t)G(x, t) \, d^3x.$$  \hspace{1cm} (1)

Here $R_L(t) \equiv \langle v_i(t') v_i(t' + t) \rangle = \langle v(t') \cdot v(t' + t) \rangle$ is the Lagrangian auto-covariance function for a particle being advected by the statistically stationary and homogeneous flow field $u(x, t)$ along a trajectory $x(t)$ given by

$$\frac{dx(t)}{dt} = v(t) = u(x(t), t), \quad x(0) = 0.$$  \hspace{1cm} (2)

In other words, we assume without loss of generality that the particle goes through the origin for $t = 0$, and we furthermore suppose that the coordinate system follows the mean flow. $R_E(x - x', t - t') \equiv \langle u(x, t) \cdot u(x', t') \rangle$ is the Eulerian velocity correlation function and $G(x, t)$ is the mean Green’s function for the dispersion of the tracer particle, i.e. the pdf of the fluid particle displacement at time $t$. More generally, both sides could be made second-order tensors formed by the velocity components, but the traces of the tensors suffice for the present discussion.

Inserting spatial Fourier transforms, e.g. $G(x, t) = \int G(k, t) \exp(ik \cdot x) \, d^3k$, we may restate (1) as

$$R_L(t) = (2\pi)^3 \int R_E(-k, t)G(k, t) \, d^3k.$$  \hspace{1cm} (3)

An alternative formulation can be made in terms of the second-order Lagrangian structure function

$$S_L(t) \equiv \langle |v(t + t') - v(t')|^2 \rangle = 2\langle |u|^2 \rangle - R_L(t).$$  \hspace{1cm} (4)
and the second-order Eulerian structure function

$$S_E(x, t) \equiv \langle |u(x + x', t + t') - u(x', t')|^2 \rangle = 2(\langle |u|^2 \rangle - R_E(x, t)).$$  \hspace{1cm} (5)

Using these quantities Corrsin’s conjecture becomes

$$S_L(t) = \int S_E(x, t)G(x, t)\, d^3x.$$  \hspace{1cm} (6)

This equation can be argued for in the following way. Let

$$C(x, t) \equiv \delta(x - x(t))$$  \hspace{1cm} (7)

denote the instantaneous particle concentration and let \(\Delta u(x, t) \equiv u(x, t) - u(0, 0)\), then by the properties of the \(\delta\)-function

$$S_L(t) = \int \langle \Delta u(x, t)^2 C(x, t) \rangle \, d^3x.$$  \hspace{1cm} (8)

If the particle displacement can be regarded as statistically independent of \(\Delta u\) then \(C\) can be averaged separately and, since \(\langle C \rangle = G\), we get

$$S_L(t) = \int \langle \Delta u(x, t)^2 \rangle \langle C(x, t) \rangle \, d^3x = \int S_E(x, t)G(x, t)\, d^3x.$$  \hspace{1cm} (9)

If we assume a statistical stationary Lagrangian velocity increment \(\Delta v \equiv v(t) - v(0)\) then it is uncorrelated with the displacement \(x\) (see appendix A). If we assume further that \(x\) and \(\Delta v\) are joint Gaussian process variables then it follows that they are in fact independent. It is well known that the diffusion becomes Gaussian for large \(t\), as demonstrated by Yeung [3] for DNS data, so we should indeed expect \(x\) and \(\Delta v\) to become independent for large \(t\). Random flight models based on a Langevin type of equation, which represent \(x(t)\) as a Gaussian process throughout, also perform quite well. It is therefore tempting to accept that Corrsin’s conjecture must tend to be exact for large \(t\). Shlien and Corrsin [2] expressed this belief by the statement that the conjecture ‘must be correct at large times and sometimes crudely useful at smaller \(t\)’. However, more is required to rigorously prove (1) for large \(t\), namely that \(x\) becomes independent not only of \(\Delta v\), but also of \(\Delta u\), which is defined even outside the trajectory. In appendix B, we give an example of a random flow field for which \(x\) and \(\Delta v\) are independent, but Corrsin’s conjecture is not valid.

We note that (1) turns into an exact relation if \(R_E(x, t)\) is replaced by the conditional Lagrangian velocity covariance \(R_L(t|x)\), viz.

$$R_L(t) = \int R_L(t|x)G(x, t)\, d^3x.$$  \hspace{1cm} (10)

Both \(R_L(t|x)\) and \(R_E(x, t)\) are covariances of velocities at two spacetime points, but \(R_L(t|x)\) is restricted to situations where a trajectory connects the origin to \(x\). Equation (1) would be exact if the velocity covariance were unbiased with respect to such a restriction.

Corrsin’s conjecture was used by Saffman [4] in a closure scheme, which has since been adopted by many authors, e.g. [5]–[11]. It should also be noted that (1) was derived
theoretically by Roberts [5] and Kraichnan [12] by applying the principles of the direct interaction approximation (DIA) to the problem of passive scalar diffusion.

Along with (1) Saffman assumed a model for $R_E(k, t)$. $R_E(k, 0)$ is given by the spectrum and, therefore, strictly positive and we may therefore define $F(k, t)$ so that

$$R_E(k, t) = R_E(k, 0)F(k, t). \quad (11)$$

Saffman used

$$F(k, t) = \exp\left(-\frac{1}{2}|k|^2\sigma_u^2 t^2\right), \quad (12)$$

where $\sigma_u = \sqrt{\langle|u|^2\rangle/3}$ and

$$R_E(k, 0) \propto k^2 e^{-k^2/k_0^2}. \quad (13)$$

Other choices can of course be made. In the DIA framework, only the spectrum is needed since $R_E(k, t)$ and $G(k, t)$ are determined as part of the scheme.

Furthermore, Saffman assumed a Gaussian form of $G$. This assumption has strong experimental support from wind tunnel studies (somewhat less from atmospheric dispersion experiments which are hampered by the presence of large-scale eddy structures). An isotropic Gaussian $G$ is uniquely determined by the dispersion parameter $\sigma(t)$ given by

$$3\sigma^2(t) = \int G(x, t)|x|^2 d^3x. \quad (14)$$

Finally, we have Taylor’s [13] relation

$$\sigma^2(t) = 2/3 \int (t - t') R_L(t') dt'. \quad (15)$$

Equations (3), (11), (14) and (15), with known forms of the spectrum $R_E(k, 0)$ and $F$, constitute a closed set of equations from which we can determine $R_L(t)$ and $\sigma^2(t)$. In the following, we will refer to this as Saffman’s closure.

Saffman’s closure, and variants of it, contain Corrsin’s conjecture as a crucial element. The assumptions made about $R_E(k, t)$, are also important, and both should be checked experimentally. Assuming a Gaussian shape of $G$ seems fairly innocent in the light of various experimental evidence.

Arguments can be given in favour and disfavour of Corrsin’s conjecture. Kraichnan [12] tested the DIA closure using artificial, Gaussian flow fields and found good agreement between $R_L$ computed by DIA and computed directly (except for some 2D frozen turbulence cases, where particles were trapped in closed streamlines). $G$ also turned out to be very nearly Gaussian. Lundgren and Pointin [14] tested Saffman’s closure, as described above, on the same flow fields and found likewise good agreement. These results of course speak in favour of Corrsin’s conjecture. More recently, Dentz et al [15] simulated frozen 2D flow in a porous medium and tested Corrsin’s conjecture. It was found to perform poorly in the large $t$ limit for strong turbulence, thus questioning the assumption that it should yield the correct asymptotic behaviour for large $t$. Kaneda and Ishida [8] used Corrsin’s hypothesis to compute the vertical dispersion of particles in flow with strong density stratification and found good agreement with DNS results.
This flow is dominated by randomly superimposed gravity waves rather than turbulence, and any vertical sweeping of eddies is blocked by the density gradient.

Contrary to the theoretical arguments leading to (9) and speaking in favour of Corrsin’s hypothesis, we shall now give a series of reasons for it not to be valid.

In appendix B we prove that
\[ R_L(t|r) = A(t) + B(t) r^2 \]
when \( x(t) \) is a Gaussian process. The parabolic growth with \( r \) indicates a strong bias with respect to the separation. Particles that have travelled far in a given time therefore tend to have large velocities at the endpoints and \( R_L(t|r) \) becomes an **increasing** function of \( r \), irrespective of whether \( t \) is large or small. Contrary to this, we can expect \( R_E(x,t) \) to decrease to zero for large separations. In view of this, it appears less obvious that \( R_E(x,t) \) can replace \( R_L(t|r) \) in the exact relation (10). It requires at least that the two functions cross close to where \( G \) attains its maximum, which may seem almost too much to hope for.

If \( t \) is small enough then \( x \approx vt \) so that \( G(x,t) \approx t^{-3} P_v(x/t) \), where \( P_v \) is the velocity pdf. Inserting this into the right-hand side of (6) leads to
\[
\int S_E(x,t) G(x,t) \, d^3 x \approx \int S_E(vt,t) P_v(v) \, d^3 v. \tag{16}
\]
Dividing both sides of (6) by \( t^2 \) and letting \( t \to 0 \) therefore yields
\[
\left< \left| \frac{du}{dt} \right|^2 \right> = \int \left< \left( \frac{\partial u}{\partial t} + v \cdot \nabla u \right)^2 \right> P_v(v) d^3 v = \left< \left| \frac{\partial u}{\partial t} \right|^2 \right> + \frac{\varepsilon \sigma_v^2}{v}. \tag{17}
\]
According to Tennekes’ [16] analysis, the two terms on the right-hand side of (17) are approximately of equal magnitude so that their sum is \( \approx 2 \sigma_v^2 \varepsilon/v \). The acceleration variance, appearing on the left-hand side, is a Galilean invariant and should obey Kolmogorov scaling [17] so that for large \( R_\lambda \)
\[
\left< \left| \frac{du}{dt} \right|^2 \right> = 3a_0 \frac{\varepsilon^{3/2}}{v^{1/2}}, \tag{18}
\]
where \( a_0 \) is a numerical constant. The relation has been verified experimentally by Voth et al [18] and by Gylfason et al [19] who found \( a_0 \approx 5 \) for \( R_\lambda \) approaching \( \sim 1000 \). Using \( R_\lambda = \sqrt{15 \sigma_u^4/\varepsilon v} \), valid for isotropic turbulence, we find that the right-hand side of (17) is about \( \frac{2}{3\sqrt{15a_0}} R_\lambda \sim 0.03 R_\lambda \) times larger than the left-hand side. This is of course a serious discrepancy.

When \( t \) is larger than a few times the Kolmogorov time scale \( \tau_\eta \equiv \sqrt{\nu/\varepsilon} \), \( S_L(t) \) enters the inertial range, where it should attain a constant slope. This is a consequence of Kolmogorov scaling since it amounts to saying that
\[
S_L(t) \propto \varepsilon t. \tag{19}
\]
Measurements at \( R_\lambda = 740 \) made by Mordant et al [20] show a nearly exponential decay of \( R_L(t) \). A plot of \( S_L(t)/(\varepsilon t) \) against \( t/\tau_\eta \) shows a maximum, but not a genuine plateau. This indicates that very high Reynolds numbers are required in order to observe (19) over an extended range. Even so, we must expect (19) to be valid for \( t \) in the inertial range when \( R_\lambda \) is large enough.

*New Journal of Physics* 7 (2005) 142 ([http://www.njp.org/](http://www.njp.org/))
Using (11), (6) can be written as

\[ S_L(t) = 2 \int R_E(k, 0)(1 - (2\pi)^3 G(k, t) F(k, t)) \, d^3 x. \] (20)

The integrand contains a high pass filter since \((2\pi)^3 G(k, 0) = F(k, 0) = 1\). For \(t\) small, but still large compared to the Kolmogorov time scale, \(1 - G F\) therefore cuts off the energy containing part of the spectrum. This is a healthy feature because we do not expect large, spatially coherent eddies to affect a Galilean invariant such as \(S_L(t)\). We may therefore insert a Kolmogorov spectrum, i.e. \(R_E(k, 0) = (2\pi)^{-1/2} k^{-1/3}\), where \(\alpha_K \approx 1.7\) is the spectral Kolmogorov constant. As long as \(R_L(t) \approx \langle v^2 \rangle\), we have \(\sigma(t) \approx \sigma_n t\) and \((2\pi)^3 G(k, t) \approx e^{-\langle \sigma_n k^2 \rangle}/2\). Inserting this into (20) along with Saffman’s suggestion for \(F\) yields

\[ S_L(t) \approx 2 \int_0^\infty \alpha_K e^{2/3} k^{-5/3} (1 - e^{-\langle \sigma_n k^2 \rangle}) \, dk = 6\Gamma(2/3)\alpha_K (\varepsilon \sigma_n t)^{2/3}. \] (21)

This differs from (19), which we believe to be the correct result. According to (21), \(S_L(t)/(\varepsilon t)\) should decrease as \(t^{-1/3}\) for small (but not too small) \(t\). This contradicts the experiments by Mordant et al [20] who found \(S_L(t)/(\varepsilon t)\) to be increasing from a finite, positive value at \(t = 0\).

The following summarizes the arguments disfavouring the conjecture:

(i) \(R_L(t|r)\) and \(R_E(r, t)\) appearing in the integrand of the exact relation (10) and Corrsin’s hypothesis, respectively, are very different.

(ii) The hypothesis leads to a wrong scaling of the acceleration variance.

(iii) For very large Reynolds numbers it predicts \(S_L(t) \propto t^{2/3}\), while \(t^{1/3}\) is observed.

Although variants of Saffman’s closure have been often used, and the scheme probably has been rediscovered many times over the years, there have been few attempts to test the steps of the procedure experimentally. This involves testing both ((1) or (3)) and (11) separately. The purpose of the present investigation was to do this and to confront the conflicting lines of reasoning presented above with experiments. The particle tracking experiment described below is particularly well suited for measurements of Lagrangian statistics. Eulerian statistics can also be obtained, but with less accuracy. For conventional atmospheric measurements (instruments on masts) the situation is reversed since only certain Eulerian statistics can be obtained. These include the spectrum, but not the time dependence of \(R_E(x, t)\), and Lagrangian velocity statistics are altogether unobtainable. Saffman’s closure is an attempt to devise a machinery that can translate Eulerian data, which exist in vast amounts, into Lagrangian statistics, and as such it is very valuable.

There is yet another similar conjecture which has been attributed to Corrsin [21]. He argued for the relation

\[ R_L(t) \approx R_E(0, t) \] (22)

and noted that he had ‘heard a number of people offer this equality as a guess’. Cambon et al [9] note that (22) follows from (1) if \(G(r, t)\) is so narrow compared to \(R_E(r, t)\) that it can be approximated by a \(\delta\)-function. It should be noted that the objections to (1) regarding the microscale and the inertial range behaviour also apply to (22) [16]. Shlien and Corrsin [2] measured...
In decaying turbulence behind a grid \((62 \leq R_{\lambda} \leq 72)\) and compared with measurement of \(R_{E}(0, t)\) made by Comte-Bellot and Corrsin [22] in the same wind tunnel. The study gave little support to (22) and basically confirmed the criticism raised by Tennekes.

Finally, we emphasize that the assumption of stationarity of the Eulerian velocity field does not automatically imply stationarity of the Lagrangian velocity of a diffusing particle. In a water tank experiment with a steady turbulence generator (e.g. propellers or oscillating grids) a particle visits all parts of the tank and its Lagrangian history will of course be statistically stationary. However, when we observe only those parts of the trajectories that pass through a central volume, away from boundaries and the turbulence generator, these parts will no longer necessarily be stationary. In fact, the kinetic energy decays with a rate that matches the energy dissipation, as we shall see in section 3. This complicates data analysis because care must be taken not to blend in an assumption of stationarity. For example, we cannot take (4) for granted, because in reality \(\langle |u(x, t + t')|^2 \rangle < \langle |u(x, t)|^2 \rangle\) when \(t' > 0\). Likewise, we cannot take (15) as a ‘kinematic fact’, even if it might be a fairly good approximation.

In section 2, we describe the experiment. Section 3 contains experimental results. First we check the Gaussianity of \(G\) and equation (A.6) (which is exact when \(x(t)\) is a Gaussian process). Then we test Corrsin’s conjecture by comparing the two sides of (1) using experimental data. Finally, we compare measurements of \(R_{E}(r, t)\) with a simple model that emphasizes the sweeping effect. Conclusions are drawn in section 4.

2. The experiment

Data were taken from the water tank experiments described in detail in Mann et al [23] and Ott and Mann [24]. In these experiments, turbulent flow was generated by two oscillating grids, see figure 1. This produces turbulence with a very small mean flow (<10% of \(\sigma_{u}\)), and turbulence
in the central parts of the tank is approximately homogeneous and isotropic. The flow was seeded with small, neutrally buoyant polystyrene particles which were located stereoscopically by four CCD cameras, and the individual particle trajectories were reconstructed subsequently from the particle positions. The measurements involved between 500 and 1000 simultaneous trajectories.

2.1. Data processing

The main problems to tackle in the data analysis are: the relatively small number of measurements, the limited measuring volume and lost tracks.

In the data analysis, we need statistics that depend on time delay \( t \) and vectorial separation \( x \), i.e. \( R_E(x, t) \), \( G(x, t) \) and \( R_L(t|x) \). In principle, this is a matter of dividing the \((x, t)\) space into bins and average for each bin, but the number of bins becomes too large for the limited amount of data we have. The angular dependence is, however, generally weak for the present data and it is profitable to change to spherical coordinates and expand in terms of spherical harmonics, e.g.

\[
R_E(x, t) = R_E(r, \theta, \phi, t) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} R_{lm}^E(r, t)Y_{lm}(\phi, \theta)
\]  

It is most practical to normalize the spherical harmonics so that \( \int_0^\pi d\theta \int_0^{2\pi} d\phi |Y_{lm}(\phi, \theta)|^2 = 4\pi \) in which case \( R_{lm}^E(r, t) \) is obtained by averaging over a shell of radius \( r \), namely

\[
R_{lm}^E(r, t) = \langle u(x_0, t_0) \cdot u(x_0 + x, t_0 + t) Y_{lm}|_{|x| = r} \rangle
\]  

where \( Y_{lm} = Y_{l,-m} \) is the complex conjugate of \( Y_{lm} \). The problem with limited amount of data now manifests itself as an increasing uncertainty of \( R_{lm}^E(r, t) \) as \( l \) increases. A previous investigation, showed that the flow is nearly isotropic so the first term with \( l = m = 0 \) dominates [23]. The analysis also revealed an almost perfect rotation symmetry around the polar axis (perpendicular to the grid planes) which implies that terms with \( m \neq 0 \) can be neglected. Of the remaining terms, those with uneven \( l \) are ‘forbidden’ by the symmetry of the tank. In the analysis, we have included terms with \( m = 0 \) and \( l = 0, 1, 2, 3 \) and 4. Apart from the symmetric term, the only asymmetric term that contributes appreciably (to \( G \), say) is the one with \( l = 2 \) and \( m = 0 \).

In spherical coordinates, (1) becomes

\[
R_L(t) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \int_0^{\infty} \overline{R_{lm}^E(r, t)} G_{lm}^E(r, t) 4\pi r^2 \, dr.
\]  

Averaging over directions could have been obtained by formally referring averages to an ensemble consisting of all rotated versions of the experiment. Since this enlarged ensemble is obviously invariant under rotations, it seems legitimate to compare data averaged over directions in this way with theoretical results for the isotropic turbulence. Using the enlarged ensemble does not affect the isotropic term, but the anisotropic terms, those with \( l \neq 0 \), all become identically equal to zero. \( R_L(t) \), on the other hand, remains the same for the two ensembles, hence we get a different version of Corrsin’s conjecture for the enlarged ensemble, namely

\[
R_L(t) = \int_0^{\infty} R_{E00}^0(r, t) G_{00}^0(r, t) 4\pi r^2 \, dr.
\]
This non-invariance with respect to random rotations is a spurious feature, analogous to the well-known non-invariance to random Galilean transformations. However, the two versions of Corrsin’s conjecture yield almost identical results for the present data. We find that the anisotropic terms never contribute more than 1% to the right-hand side of (25), which renders them insignificant. Taken individually, \( G^{20}(r, t) \) and \( R_E^{20}(r, t) \) do attain values comparable to \( G^{00}(r, t) \) and \( R_E^{00}(r, t) \), but for different values of \( r \), so not much is left after the radial integration. Having noted this, we skip any discussion of anisotropic terms in the following.

The measurements reported are restricted to lie within a sampling volume \( B \) chosen as a sphere with radius \( R_B = 60 \) mm centred, where the velocity variance is minimal (a point located fairly close to the geometric centre of the tank). \( B \) is within view of all four cameras, and inside \( B \) the density of detected particle positions is in close agreement with a random, uniform distribution. Particles can be tracked even outside \( B \), but the tracking gradually becomes more inefficient away from the sampling volume. Particle tracks that leave \( B \) and re-enter later on are likely to be detected, although not with 100% certainty.

The tracking occasionally fails even for particles inside \( B \). Inside \( B \), the tracking efficiency (the probability of finding the next position on a track) is better than 99.5%, which still leaves an appreciable probability of losing long tracks. The endpoints of lost tracks have been found to be evenly distributed inside \( B \). The reason for the lost tracks is that the tracking procedure occasionally fails. The next position on a track is found by forecasting it from previous points on the track and picking the particle position closest to the forecast position as the next point. This can fail if one of the positions is inaccurate or if a particle position is missing, which may happen when particles overlap on one or more of the camera images. When two particle images completely overlap, only one position is generated and consequently there will be a missing position in one of the two tracks. The point is that this happens irrespective of particle position and velocity and hence lost tracks should not bias Lagrangian statistics. The main effects of lost tracks are therefore eliminated by a normalization of \( G \) so that \( \int_{\text{tank}} G_{\exp}(x, t) \, d^3x = 1 \). When \( \sigma \ll R_B \) this can be done approximately by letting \( \int_{|x| < 2R_B} G_{\exp}(x, t) \, d^3x = 1 \). It is possible that the missing tail of \( G \) could be successfully estimated from, e.g. by assuming self-similarity, but we prefer this approach because it is simple and because it avoids extrapolations beyond what was actually measured.

Trajectories may go in and out of \( B \), but points on a trajectory that are outside \( B \) are neglected. In calculating \( G(r, t) \), we take any two points on a trajectory which are both inside \( B \), calculate their separation \( r \) and let the observation count with an appropriate weight \( 1/W(r) \) in order to compensate for the missing observations where the second point is outside \( B \). Thus, \( W(r) \) is the probability that the second point falls within \( B \) given the first point is inside \( B \). In this way, we can determine \( G(x, t) \) for \( |x| < 2R_B \). Assuming homogeneity, \( W(r) \) is given by

\[
W(r) = \begin{cases} 
\left(1 - \frac{r}{2R_B}\right)^2 \left(1 + \frac{r}{4R_B}\right) & \text{if } r < 2R_B, \\
0 & \text{otherwise}.
\end{cases}
\]  

(27)

Note that due to the spherical shape of \( B \) it is not necessary to assume isotropy in order to get (27). Figure 2 shows (27) as well as \( W(r) \) determined from data. The agreement is good except for very small separations where data are sparse. For separations much larger than 60 mm, the degree of compensation is large and results should be taken with a grain of salt. It should be noted that the compensation for the finite size of \( B \) does not apply to quantities such as \( R_E(r, t) \).
or averages conditioned on $x$, which are obtained simply by averaging the content of each bin separately.

Statistics for separations larger than $2R_B$ are not obtained. With the compensation described above (and neglecting possible inhomogeneity), we effectively emulate a situation where a particle is located at time $t_0$ at a fixed position and followed for as long as it is within a distance of $2R_B$ from the starting point. The experimentally determined value of $\langle x^2 \rangle$ is therefore equal to

$$
\langle x^2 \rangle_{\exp} = \frac{\int_0^\infty G_{\exp}(r, t)4\pi r^4 \, dr}{\int_0^{2R_B} G(r, t)4\pi r^2 \, dr},
$$

(28)

where,

$$
G_{\exp}(r, t) = \begin{cases} G(r, t) & \text{if } r < 2R_B, \\ \int_0^{2R_B} G(r, t)4\pi r^2 \, dr & \text{if } r > 2R_B. \end{cases}
$$

(29)

For small $t$, where no particles are observed to travel nearly as far as $2R_B$, there is no problem because $R_B$ is effectively infinite. As $t$ increases, separations larger than $2R_B$ may occur, but are not counted, hence $\langle x^2 \rangle_{\exp}$ is an underestimate. The error can be estimated by using a Gaussian $G$ in which case

$$
\langle x^2 \rangle_{\exp} \approx \langle x^2 \rangle - \frac{32\pi}{3} \langle x^2 \rangle R_B^3 G(2R_B, t).
$$

(30)

This is illustrated in figure 3. The black curve is a ‘good guess’ on $\sigma(t)$ in run 20. The blue curve is $\sigma_{\exp}(t)$ obtained from (30) using the ‘best guess’ $\sigma(t)$ and a Gaussian $G$. Thus the blue curve represents simply the averaging of the observed separations, i.e. as in (28). For $t < 3$ s there is practically no finite volume effect, but for larger $t$, $\sigma_{\exp}$ underpredicts substantially. The red curve in figure 3 shows the result of an alternative method, which uses Taylor’s relation (15) with the experimentally obtained $R_L$. An estimate of the error of this method can be made by assuming
that $x(t)$ and $v(t)$ are joint Gaussian stochastic variables, so that $R_L(t|r)$ is given by (A.9). After some manipulations, this yields

$$R_{L,\text{exp}}(t) = \int_0^\infty R_L(t|r)G_{\text{exp}}(r, t)4\pi r^2 \, dr$$

$$= R_L(t) - \frac{4\pi}{3} (2R_B)^3 G_{\text{exp}}(2R_B, t) \frac{\langle x \cdot v \rangle^2}{\langle x^2 \rangle}.$$  \hspace{1cm} (31)

Judged from figure 3, this method only has a deviation of $-6\%$ at $t = 6$ s. Taylor’s relation is of course exact for homogeneous, stationary turbulence, but it does not hold exactly for the experimental estimates. This is basically due to the fact that the time derivative does not commute with the experimental averaging. But, ironically, the estimate of $\sigma$ based on it is actually better than the more direct estimate. Actually, the black, ‘good guess’ curve was obtained using (31) to make a tentative correction of $R_{L,\text{exp}}$.

The fact that observations are restricted to the subvolume $B$ has an impact on statistical stationarity. Even if Eulerian statistics are stationary it is possible, and even likely, that Lagrangian statistics based on observations inside $B$ are non-stationary. With no sources of kinetic energy inside $B$, there must be a net energy flux into $B$ which can take three forms: advection of kinetic energy, ‘massage’ by pressure fluctuations on the surface of $B$ and viscous transport. Viscous transport is small at high Reynolds numbers, and we suspect pressure transport to be less efficient than advection. Particles should therefore, tend to carry more kinetic energy when they enter $B$ than when they leave, and the Lagrangian kinetic energy should therefore be non-stationary. This kind of non-stationarity is, in other words, not merely an artifact caused by experimental limitations, but a reflection of physical processes going on inside $B$. We will call this local Lagrangian non-stationary.

The Lagrangian velocity increment and the displacement are uncorrelated in statistically stationary turbulence (see appendix A). Local Lagrangian non-stationarity may upset this result.
3. Experimental results

3.1. Eulerian statistics

Figure 4 shows the velocity distribution in $B$. The graph roughly follows a Gaussian although more sharply peaked at the maximum and with a longer tail. The tail rolls off as an exponential rather than a Gaussian.

Figure 5 shows the experimental results for $R_E(r, 0)$ (actually the symmetric $R_{00}^E$). The full line is a fit to $R_E$ obtained from a von Kármán spectrum (for details see [24])

$$E(k) = \alpha_k \varepsilon^{2/3} \frac{k^4 L^{17/3}}{(1 + k^2 L^2)^{17/6}}$$

(32)

From the fit, we estimate the dissipation rate to be $\varepsilon = 260 \text{ mm}^2 \text{s}^{-3}$ and the length scale to be $L = 27.3 \text{ mm}$. These estimates are consistent with those given in [23] and [24], where

\[\textit{New Journal of Physics} \; 7 \; (2005) \; 142 \; (\text{http://www.njp.org/})\]
box-shaped measuring volumes were used. The bins used for averaging are 2 mm thick shells, except for the first, which is a sphere of radius 1 mm. Due to the varying bin size, statistical errors increase as \( r \to 0 \). \( R_E(r, 0) \) is an exception to this since it is a one-particle statistics and actually the most certain measurement of \( R_E \). The measurements also get uncertain beyond \( r \sim R_B \) because of the finite size of the measuring volume. Given these uncertainties, the fit reproduces data surprisingly well. Alternatively, \( \varepsilon \) can be estimated by the relation \( \varepsilon = -\langle a \cdot \nu \rangle \), valid for decaying, homogeneous turbulence. This yields \( \varepsilon \sim 259 \text{ mm s}^{-2} \) in agreement with the first estimate. The perfect match is probably fortuitous.

Figure 6(a) shows \( R_E \) versus \( r \) for several \( t \) and figure 6(b) shows \( R_E \) versus \( t \) for several \( r \). The lines are obtained using the model

\[
R_E(k, t) = R_E(k, 0) \exp \left[ -\frac{1}{2} (k \sigma_u t)^2 \right]. \tag{33}
\]

where \( R_E(k, 0) \) is obtained from the fitted spectrum as described above and \( k = |k| \). This is essentially the model proposed by Saffman \cite{4} and others, see references in \cite{25}. It appears to capture the behaviour reasonably well for \( t \lesssim 0.2 \text{ s} \), while it underestimates for larger \( t \).

Hunt \textit{et al} \cite{26} tried to improve the model by introducing an additional parameter \( a \) in (33):

\[
R_E(k, t) = R_E(k, 0) \exp \left[ -\frac{1}{2} (ak \sigma_u t)^2 \right]. \tag{34}
\]

Based on DNS, they estimated \( a \approx 0.5 \). As seen from figure 7, the behaviour of \( R_E \) for large times is improved but at the cost of deteriorating the good short time match seen in figure 7(b).

In figure 8, yet another model is compared with data:

\[
R_E(k, t) = R_E(k, 0) \exp \left[ -\frac{1}{2} (k \sigma(t))^2 \right]. \tag{35}
\]
where $\sigma(t)$ is the experimental value. One can argue for this model in the following way. Consider $\langle u(0, 0)u(x, t) \rangle_{r}$, which should be understood as the conditional average given that a particle travels from $(0, 0)$ to $(r, t)$. We may estimate the velocity field at $(x, t)$ very roughly by assuming that it simply suffers the same translation $r$ as the particle, in other words

$$u(x, t) \sim u(x - r, 0). \quad (36)$$
Ignoring any bias from $r$ on $u$ we obtain

$$R_E(x, t) = \int R_E(x - r, 0) G(r, t) \, d^3 r.$$  \hfill (37)

Taking the Fourier transform of both sides and using a Gaussian $G$ then leads to (35). The model works just as well as (33) for small $t$, which is not surprising since (33) is obtained from (35) if we let $\sigma(t) \approx \sigma_u t$. For larger $t$, (35) is a slight improvement compared to (33), but for larger $t$ the values are still too low. The model essentially represents the second-order structure as frozen eddies being moved around as if they were particles. This appears to be realistic for short times, but for large $t$ the crude assumptions made to derive (35) evidently do not hold. Note that the model is not improved if we allow eddies to decay, since this would only reduce $R_E$. It therefore seems to be the assumption that $u$ is unbiased by $x$ and/or the assumption that eddies move like particles which are at stake.

We finally mention that a model based on Kolmogorov scaling of the time dependence, i.e.

$$R_E(k, t) = R_E(k, 0) \exp(-\mu \varepsilon^{1/3} k^{2/3} t)$$

where $\mu$ is an adjustable parameter, has also been tested. It performs poorer than both (34) and (35).

3.2. Lagrangian statistics

Figure 9 shows $4\pi r^2 \sigma(t) G^{(0)}(r, t)$ against $r/\sigma(t)$ for several $t$. The points collapse nicely on a single curve which is close to a Gaussian. Comparing with figure 4, the displacement seems to be more Gaussian than the velocity. A closer inspection of figure 9 shows, however, that the Gaussian fit is best for large $t$, while for small $t$ the shape tends to resemble that of the velocity distribution. The tails are also thicker than that for a Gaussian even for large $t$. There is a small anisotropic contribution mainly from $G^{(2)}(r, t)$ stretching $G$ in the directions towards the grids, which is consistent with a slightly higher variance of the corresponding velocity component.

The measurements were taken after a few minutes operation of the oscillating grids to avoid any start-up effects, and there is no sign that the Eulerian velocity field $u(x, t)$ is not statistically stationary. For example, the total kinetic energy of particles inside $B$ plotted against $t$ shows...
no trends apart from the expected fluctuations. The Lagrangian kinetic energy \( \frac{1}{2} v^2(t) \) is, on the other hand, non-stationary. Figure 10 shows \( \langle v^2(t + t_0) \rangle \) versus \( t \), where the average is taken over particles which were observed inside \( B \) both at \( t_0 \) and \( t_0 + t \). As mentioned above, the initial decrease reflects the advection of kinetic energy into \( B \). Since \( \langle v^2(0) \rangle = \langle u^2 \rangle \), the non-stationarity of the Lagrangian kinetic energy means that

\[
S_L(t) = \langle v^2(t) \rangle + \langle u^2 \rangle - 2R_L(t) < 2(\langle u^2 \rangle - R_L(t)),
\]

while for the Eulerian structure function, we have the usual

\[
S_E(x, t) = 2(\langle u^2 \rangle - R_E(x, t)).
\]

This means that (1) and (6) are in fact not equivalent and should yield different results when applied to data. In the following, we have chosen to use (1) based on \( R_L \) and \( R_E \). Note that if (6) is used to obtain \( S_L \) from \( S_E \), and \( R_L \) is subsequently obtained from \( S_L \) using the observed \( \langle v^2(t) \rangle \) then we end up with a smaller \( R_L \).

The red curve in figure 11 shows the Lagrangian velocity auto-covariance obtained from direct measurements. The results become progressively more uncertain as \( t \) increases because of lost tracks and because of the finite volume of \( B \). The (negative) error due to clipping, as estimated in section 2.1, is essentially zero for \( t < 3 \) s and increases to \(-35 \text{ mm}^2 \text{s}^{-2}\) at \( t = 6 \) s. Judged by eye, this seems insignificant compared to the statistical scatter at large \( t \). At the given experimental time resolution, the curve appears rounded near \( t = 0 \), but in fact it has a cusp. This is because \( R_L(t) \approx R_L(0) + |t|\langle a \cdot v \rangle \), for small \( t \) where, as mentioned above, we find \( \langle a \cdot v \rangle \sim -\varepsilon \) from data. Since the slope of \( R_L \) is discontinuous at \( t = 0 \), the curvature is infinite and does not define a timescale. The experimental \( S_L(t) \), on the other hand, should have zero slope at \( t = 0 \) and a finite curvature at \( t = 0 \) related to the acceleration variance. It can be determined, i.e. by plotting \( t^2/S_L(t) \) versus \( t^2 \) and extrapolate to zero. However, it is better to determine the acceleration variance directly as \( \langle (x(t - \Delta t) - 2x(t) + x(t + \Delta t))^2 \rangle /\Delta t^4 \) since this involves a narrower time filter. In this way, we obtain

\[
\left( \frac{\text{d}v}{\text{d}t} \right)^2 \approx 4.6 \times 10^4 \text{mm}^2 \text{s}^{-4}.
\]
Comparing with (18) this corresponds to $a_0 \approx 3.4$, in agreement with the low Reynolds number runs by Voth \textit{et al} [18]. The present experiment does not allow us to measure $\frac{\partial u}{\partial t}$, but the second term on the right-hand side of (17) is accessible. We find

$$\frac{\varepsilon \sigma^2}{\nu} \approx 1.1 \times 10^5\text{mm}^2\text{s}^{-4}$$

which is more than twice as large as the measured acceleration variance and in clear support of Tennekes’ ideas.

The blue curve in figure 11 shows $R_L$ obtained from (1). The predicted $R_L$ is consistently smaller than the experimental $R_L$, and there is no sign of asymptotic convergence for large $t$.

Plots of $\sigma$ versus $t$ obtained by the two methods discussed in section 2.1 are shown in figure 12. The same trends are found as in figure 3, but with a larger difference between the
results of the two methods for the measurements. The difference also sets in earlier, and can be seen already at \( t \sim 1 \) s as opposed to \( t \sim 3 \) s for the Gaussian model. This must be caused by the tail of \( G \) being broader than for a Gaussian. The estimated error of \(-6\%\) at \( t = 6 \) s for the method based on \( R_L \) is therefore too optimistic. The real error is probably at least twice as large at \( t = 6 \) s, but it seems safe to state that \( \sigma \), using the method based on \( R_L \), is accurate to within a few per cent for \( t < 3 \) s. \( T_L \) can be estimated as \( \frac{1}{R_L(0)} \int_0^{6s} R_L(t) \, dt \approx 0.97 \) s. This is most likely an underestimate both because \( R_L \) is underestimated and because the integration has been cut at \( t = 6 \) s. The figure also shows \( \sigma \) estimated from Corrsin’s conjecture using the experimental \( G \) and \( R_E \). The data are somewhat uncertain for \( t > 3 \) s, but already at \( t = 3 \) s Corrsin’s conjecture is 15% below the experimental value, which itself is most probably an underestimate. From Corrsin’s conjecture, we find \( \frac{1}{R_L(0)} \int_0^{6s} R_L(t) \, dt \approx 0.60 \) s, almost 40% below the result for the direct measurement. The situation is even worse when (6) is used (yellow curve).

Although the particle separation and the velocity increment may separately be close to Gaussian, they need not form a Gaussian pair of variables. The measured correlation coefficient \( C_{\Delta v, \mathbf{r}}(t) \) (defined in appendix A) is non-zero, but very small; smaller than \( \pm 0.06 \) for \( t < 6 \) s. The correlation between the velocity increment \( \Delta \mathbf{v} \) and the separation \( \mathbf{r} \) is therefore close to zero as expected. Independence of the two variables would imply \( S_L(t) = S_L(t|\mathbf{r}) \). The small value of the factor \( C_{\Delta v, \mathbf{r}}^2(0) \) in (A.6) means that the Gaussian theory predicts that the derivations between \( S_L(t) \) and \( S_L(t|\mathbf{r}) \) are smaller than 2% for \( 0.5 < r/\sigma < 2.5 \). Figure 13, where \( S_L(t|\mathbf{r})/S_L(t) \) is plotted against \( r/\sigma \), shows much larger deviations. The curves joining points with constant \( r \) (the curves generally get steeper as \( r \) decreases). Due to the approximate self-similarity of \( G \), the 10% fractile of \( G \) is located around \( r/\sigma \sim 0.5 \) for all \( t \) while the 90% fractile is located around \( r/\sigma \sim 2.5 \). Observations in this interval are therefore of primary interest in relation to Corrsin’s conjecture. The figure contains data points for \( t < 3 \) s and \( r < 60 \) mm, where the compensations for the finite size of \( B \) and for lost tracks are not too large. Blue colour indicates observations where \( t < 0.6 \) s, while red colour indicates \( t > 0.6 \) s. The red points more or less collapse on a single parabolic curve as suggested by (A.6), but it deviates much more from 1 than suggested by Gaussian theory. The blue points, representing observations for \( t \) somewhat smaller than the Lagrangian time scale, deviate even more from 1 and do not collapse on a single curve. The tendency is that for large \( r/\sigma \), i.e. situations where the particle travels unusually far, the velocity

![Figure 13](http://www.njp.org/)

**Figure 13.** \( S_L(t|\mathbf{r})/S_L(t) \) versus \( r/\sigma \) shown as curves each for constant \( t \). The colouring is explained in the text.
Figure 14. $R_L(t|r)/R_L(t)$ versus $r/\sigma$ shown as curves each for constant $t$. The colouring is as in figure 13.

Figure 15. $R_E(r, t)/R_L(t)$ versus $r/\sigma$ shown as curves for constant $t$ ($< 3$ s). The curves generally get steeper as $t$ increases.

increment is also large, and this tendency is strongest in earlier stages of the diffusion process. Conversely, particles that travel a distance somewhat shorter than average tend to experience a smaller velocity increment, even if all curves seem to return to $S_L(t|r)/S_L(t) = 1$ as $r/\sigma \to 0$.

Figure 14 shows $R_L(t|r)/R_L(t)$ plotted against $r/\sigma$ for several $t$ using a similar colouring as in figure 13. Again we see roughly parabolic shapes, $y = a(t) + b(t)x^2$, where $a \sim 0$ and $b(t) \sim 0.37$ fit all curves fairly well. Gaussian theory predicts parabolas with $b(t) = B(t)/R_L(t)$ as described in appendix A. However, from data we find that $B(t)/R_L(t) \approx 4$ at $t = 0$ s increasing to about 10 at $t = 3$ s, and these values are more than an order of magnitude too large to explain figure 14. There is no sign of a convergence towards a joint Gaussian behaviour for large $t$.

In figure 15, we have plotted $R_E(r, t)/R_L(t)$ against $r/\sigma$. $R_E(r, t)/R_L(t)$ is clearly below the ‘nominal’ value 1 in most of the interesting range of $r/\sigma$. Moreover, $R_E(r, t)/R_L(t)$ decreases as $t$ increases. This tendency should reverse itself in order to get Corrsin’s conjecture asymptotically, but again we see no sign of this.
Finally, figure 16 shows $R_E(0, t)$ and $R_L(t)$ for comparison. Here the values of $R_E(0, t)$ were estimated by sampling situations where two different particles pass the same position to within 3 mm. $R_E(0, t)$ is clearly smaller than $R_L(t)$ for small $t$ as expected from Tennekes’ arguments. However, $R_E(0, t)$ fits $R_L(t)$ quite well for large $t$.

4. Conclusions

Corrsin’s conjecture has been tested against data from a double oscillating grid experiment, which allows simultaneous measurements of $R_E$, $G$ and $R_L$. Two oscillating grids generate approximately homogeneous, isotropic and turbulence with an approximately Gaussian pdf.

Although the Eulerian velocity field is stationary, we observe that the Lagrangian velocity of a particle is locally non-stationary. Thus the kinetic energy of a particle is observed to decay while it is inside the measuring volume $B$, and the energy input, due to the difference of kinetic energy between particles entering and leaving $B$, matches the energy dissipation. This indicates that local Lagrangian non-stationarity can be expected in any geophysical flow, where kinetic energy is produced and dissipated at different places. Lagrangian stationarity could have been restored, if measurements were taken in the entire tank, but then (approximate) homogeneity would have to be sacrificed. The assumption of Lagrangian stationarity is inherent in the theory outlined in section 1, and the implications of allowing for local Lagrangian non-stationarity are not clear to us, but it should probably be treated in conjunction with a (small) violation of homogeneity. In this paper we present local Lagrangian non-stationarity merely as an experimental fact.

The angular dependence of $R_E$ and $G$ was taken into account by expanding in terms of spherical harmonics, and we found that contribution to (1) from non-isotropic components is negligible.

The principal use that has been made of Corrsin’s conjecture for more than 40 years has been in context with a variant of Saffman’s closure scheme, which should be regarded as the prototype example of an application involving the conjecture. Saffman’s closure involves assumptions in addition to Corrsin’s conjecture, which have been tested experimentally. These are (1) Gaussianity of the velocity and separation, which is by and large substantiated by data, and (2) a model relating $R_E(r, t)$ to the spectrum.

New Journal of Physics 7 (2005) 142 (http://www.njp.org/)
Three simple models were used to parametrize the experimentally obtained $R_E$. A model based on Kolmogorov scaling of the $t$ dependence was tested and found to be clearly inferior to those presented here, which all highlight the sweeping effect. The model based on the simple idea that eddies are dispersed as fixed objects much like particles works best. The model also assumes that the flow structure is unbiased by an observation of the distance travelled by a nearby particle in the flow. In this model, there is no eddy decay which therefore seems to have little implications for the time dependence of $R_E(r, t)$. The sweeping effect alone seems adequate to explain the temporal behaviour, at least for $t$ somewhat smaller than $T_L$. For larger $t$ the model underestimates $R_E(r, t)$ which is puzzling; one might expect that the neglect of eddy decay should lead to an overestimation of $R_E(r, t)$. The explanation for the observed underestimation seems to be that the flow field cannot be assumed to be independent of the distance travelled by one of the particles in the flow.

Shlien and Corrsin [2] regarded Corrsin’s conjecture as exact in the limit of large $t$ and characterized it as ‘perhaps sometimes crudely useful’ for all $t$. Judging possible usefulness is always subjective, but relative to the results reported for synthetic 3D Gaussian flow fields generated by kinematic simulation, our results show a much less successful performance of the conjecture. We generally observe $R_L(t)$ to be larger than predicted by the conjecture. For small and moderate $t$, our results are consistent with Tennekes’s [16] analysis and confirm his criticism. For large $t$ (up to $3T_L$ and tentatively up to $6T_L$) we find no sign of recovery. The predicted $R_L$ is consistently below the measured $R_L(t)$ yielding $T_L \approx 0.6$ s as opposed to the observed $T_L \approx 1$ s. We imagine that this discrepancy in many circumstances would render Corrsin’s conjecture not very useful. It should be noted that errors introduced by corrections made to data due to the finite size of $B$ should tend to diminish the observed $R_L$ because the turbulence is generally slightly stronger outside $B$ (in directions towards the grids) than inside $B$. The observed $T_L$ is therefore probably underestimated rather than overestimated. We speculate that the relatively more successful outcome of the tests with artificial Gaussian fields is connected with the fact that there is no sweeping of smaller eddies by larger eddies in these flows, where eddy decay, represented as decay of Fourier modes, determines the temporal behaviour of $R_E(r, t)$. It is well known that DIA has problems representing the sweeping effect correctly, and it is therefore likely to make better performance for flows where sweeping is absent.

The possible exactness of the conjecture in the limit $t \to \infty$ is a genuine hypothesis and should be treated as such. In section 1, we gave arguments in favour of it and pointed out that, although persuasive, they are not compelling. In appendix B, we outline the construction of an ensemble for which Corrsin’s conjecture does not hold. The example also suggests that $G$ and $R_L$ are not in general related to $R_E$, since it is possible for two ensembles to have the same $R_L$ and $G$ but different $R_E$. This rules out the possibility that a general relation exists between them valid for any ensemble of flows. However, the counterexample is not a Navier–Stokes flow, and the hypothesis could still be valid for a more restricted class of flows, possibly even for certain Navier–Stokes flows. The measurements show, as expected, that the velocity increment $\Delta u$ and the displacement $x$ are each approximately Gaussian and practically uncorrelated. Furthermore, $\sigma^2(t)$ is practically linear with $t$ for $t > 3T_L$. These conditions are characteristics of Brownian motion and it seems natural to expect the diffusion process to resemble a Gaussian process. Relations implied by this are presented in appendix A. Comparing these to data, we find that the diffusion is in fact not a Gaussian process even at the relatively late stages covered in the experiment. This does not preclude Brownian motion at a later stage, but we find it surprising that no signs of an approach to a Gaussian process can be seen. The diffusion process may eventually approach Brownian motion, but not until $R_L$ is essentially equal to zero.

*New Journal of Physics* 7 (2005) 142 (http://www.njp.org/)
Finally, we tested the relation $R_L(t) \approx R_E(0, t)$. It suffers from the same defects as Corrsin’s hypothesis with respect to the behaviour for small $t$, but for large $t$ it is actually quite good. This is remarkable since the relation has been derived from Corrsin’s hypothesis. We speculate that the agreement is coincidental and that the relation might fail at higher Reynolds numbers.

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**Appendix A. $R_L(t \mid x)$ for Gaussian displacements**

In this appendix we prove the parabolic form

$$R_L(t \mid r) = A(t) + B(t)r^2$$

(A.1)

if the turbulence is isotropic and the displacement is a Gaussian process.

For notational convenience, we let $v_0 \equiv v(t_0)$, $v \equiv v(t_0)$, $\Delta v \equiv v - v_0$ and $r \equiv x(t + t_0) - x(t_0)$ in the following. Then the assumption is that $r$, $v_0$ and $v$ are stochastic variables with a joint Gaussian distribution.

Firstly, we note that the three components are uncorrelated due to the assumed isotropy. Uncorrelated, Gaussian variables are automatically independent, hence $s_i$ and $v_j$ are independent for $i \neq j$, but $s_i$ and $v_i$ are correlated since $\langle s_i v_i \rangle = \frac{1}{2}d(\langle s_i^2 \rangle)/dt$. However, if we define two new velocities

$$w \equiv v - \frac{\langle r \cdot v \rangle}{\langle r^2 \rangle} r$$

and

$$w_0 \equiv v_0 - \frac{\langle r \cdot v_0 \rangle}{\langle r^2 \rangle} r,$$

(A.2)

then any component of $r$ is uncorrelated with any component of $w$ or $w_0$, and it follows that $r$ is independent of both $w$ and $w_0$. The statistics of $w$ and $w_0$ are therefore, unaffected by conditioning on $r$. For the first-order statistics, this implies that

$$0 = \langle w \rangle = \langle w \rangle_r = \langle v \rangle_r - \frac{\langle r \cdot v \rangle}{\langle r^2 \rangle} r,$$

$$0 = \langle w_0 \rangle = \langle w_0 \rangle_r = \langle v_0 \rangle_r - \frac{\langle r \cdot v_0 \rangle}{\langle r^2 \rangle} r,$$

(A.3)

where $\langle \ast \rangle_r$ denotes an average conditioned on the displacement $r$. Because $w$ and $w_0$ both are independent of $r$ we also have

$$\langle v \cdot v_0 \rangle - \frac{\langle v \cdot r \rangle \langle v_0 \cdot r \rangle}{\langle r^2 \rangle} = \langle w \cdot w_0 \rangle = \langle w \cdot w_0 \rangle_r = \langle v \cdot v_0 \rangle_r - \frac{\langle v \cdot r \rangle \langle v_0 \cdot r \rangle}{\langle r^2 \rangle^2} r^2,$$

(A.4)

where (A.3) was used. From (A.4) it follows that

$$R_L(t \mid r) \equiv \langle v \cdot v_0 \rangle_r = \langle v \cdot v_0 \rangle + \frac{\langle v \cdot r \rangle \langle v_0 \cdot r \rangle}{\langle r^2 \rangle^2} (r^2 - \langle r^2 \rangle).$$

(A.5)
Using the same principles, a similar equation can be derived for $S_L(t|r)$. We find

$$S_L(t|r) \equiv \langle (\Delta v)^2 \rangle_r = \langle (\Delta v)^2 \rangle + \frac{\langle \Delta v \cdot r \rangle^2}{\langle r^2 \rangle} - \langle r^2 \rangle$$

$$= S_L(t) \left[ 1 + C_{\Delta v,r}^2(t) \frac{r^2}{\langle r^2 \rangle} \right], \quad (A.6)$$

where

$$C_{\Delta v,r}(t) \equiv \frac{\langle \Delta v \cdot r \rangle}{\sqrt{\langle (\Delta v)^2 \rangle \langle r^2 \rangle}} \quad (A.7)$$

is the correlation coefficient for $v - v_0$ and $r$.

If we assume stationarity in the sense that the separation is independent of the absolute start time $t_0$ then we have

$$\langle (v - v_0) \cdot r \rangle = \langle (v(t_0 + t) - v(t_0)) \cdot (x(t_0 + t) - x(t_0)) \rangle$$

$$= \frac{\partial}{\partial t_0} \frac{1}{2} \langle (x(t_0 + t) - x(t_0))^2 \rangle = 0 \quad (A.8)$$

and therefore $\langle v_0 \cdot r \rangle = \langle v \cdot r \rangle = \frac{1}{2} d\langle r^2 \rangle/dt = 3\sigma d\sigma/dt$. Differentiating (15) twice we get

$$d^2\sigma^2/dt^2 = \frac{3}{2} R_L(t), \quad \text{so} \quad A(t) = \frac{3}{2} d^2\sigma^2/dt^2 - 3\sigma^2 = 3\sigma \dot{\sigma}. \quad \text{Inserting into} \quad (A.5) \text{then yields}$$

$$R_L(t|r) = 3\sigma \dot{\sigma} + \left( \frac{\dot{\sigma}}{\sigma} \right)^2 r^2. \quad (A.9)$$

Equation (A.8) implies that $C_{\Delta v,r}(t) = 0$ so that (A.6) reduces to the simple result

$$S_L(t|r) = S_L(t). \quad (A.10)$$

**Appendix B. A counterexample**

In this appendix we construct an example of flows with identical $R_L(t)$ and $G(x, t)$ but with differing $R_E(x, t)$.

To make it as simple as possible we consider a time independent, chaotic, random, incompressible flow field $u(x)$. The field is characterized by $R_E(x)$ which does not depend on $t$. The Lagrangian covariance function $R_L(t)$, however, depends on $t$ and tends to 0 for large $t$ due to the irregular turning of streamlines. How fast $R_E$ and $R_L$ tend to zero as $x$ and $t$ tend to infinity depends on the way $u$ is generated. We now consider a new flow field $\tilde{u}(x) \equiv \phi(x) u(x)$. Here $\phi$ is a stochastic field which attains the values $-1$ or $1$ almost everywhere. We require that $\nabla \cdot \tilde{u} = 0$, which implies that $u \cdot \nabla \phi = 0$. In other words, $\phi$ must be constant along the streamlines, but we are otherwise free to choose independent, random values of $\phi$ for different streamlines. Because the geometry of the streamlines is unchanged, the flow along a line may or may not change direction, $R_L(t)$ and $G(x, t)$ are the same for $u$ and $\tilde{u}$. However, if we choose $\phi$ to vary in an erratic way from streamline to streamline $R_E(x)$ will almost certainly become zero, except at $x = 0$.

We have thus constructed two ‘flow fields’ with identical Lagrangian statistics but with differing Eulerian, implying that Corrsin’s hypothesis (1) cannot in general be true for random, incompressible fields.
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