A Primer on Zadoff Chu Sequences

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I. INTRODUCTION AND OVERVIEW

Zadoff Chu (ZC) sequences are a principal manifestation of spread spectrum in modern cellular systems including LTE and 5G NR, largely displacing PN and Walsh sequences which were the mainstays of 3G cellular (WCDMA and cdma2000) and the 2G-era IS-95. ZC sequences are complex sequences with unit amplitude and particular phase shifts, as opposed to Walsh and PN codes which are real and binary valued, most commonly ±1 when used in communication systems.

ZC sequences have a number of remarkable and desirable properties that we define in the next section. Because of these properties, they are used for many key functions in current cellular systems, and are likely to be prevalent in future cellular systems as well. In LTE and 5G NR, they are widely used for a number of important initial access and overhead channel functions that are often overlooked by engineers who focus on data transmission. For example, ZC sequences are used for initial access in both the downlink (synchronization) and uplink (random access), uplink control information, uplink channel sounding, and for the reference symbols (pilots) used for fine-grained channel estimation. It is not an exaggeration to say that most types of signals other than the data transmissions in modern cellular standards utilize ZC sequences.

II. DEFINITION AND KEY PROPERTIES

A. Definitions

Zadoff Chu sequence. A Zadoff Chu sequence has two key parameters, (i) the root index $q = 1, 2, \ldots, \mathcal{N}_{zc} - 1$ and (ii) the length of the sequence, $\mathcal{N}_{zc}$, which must be an odd number (and is often a prime number). Given these two parameters, the $q^{th}$ ZC sequence $s_q[n]$ is defined as

$$s_q[n] = \exp\left[-j\pi q \frac{n(n + 1)}{\mathcal{N}_{zc}}\right]$$

where $n = 0, 1, 2, \ldots, \mathcal{N}_{zc} - 1$. Note that each sequence has length $\mathcal{N}_{zc}$, while the number of such sequences is $\mathcal{N}_{zc} - 1$. For example, if the ZC is used as a time domain spreading code as in DS-CDMA, the spreading factor would be $\mathcal{N}_{sf} = \mathcal{N}_{zc}$ and $n$ denotes time, and each user utilizes a different value of $q$.
Cyclic shifts and correlations. A cyclic shift, also known as a circular shift, is simply the rotation of a finite length sequence. Specifically, given a sequence $x[n]$ of length $N$, we shall write the $m$th cyclic shift of $x[n]$ as

$$x^{(m)}[n] = x[(n + m) \mod N].$$

Thus, while defined for all integer $m$, there are $N$ unique cyclic shifts.

The cyclic (periodic) autocorrelation of a length $N$ sequence $x[n]$ is defined as

$$R_{xx}[\tau] = \sum_{n=0}^{N-1} x^*[n] x^{(\tau)}[n],$$

for $\tau = 0, 1, \ldots N - 1$. Note that $x^*[n]$ means the complex conjugate of each entry of $x[n]$.

The normalized cyclic autocorrelation is

$$\bar{R}_{xx}[\tau] = \frac{1}{N} R_{xx}[\tau],$$

which results in $\bar{R}_{xx}[0] = 1$ and $\bar{R}_{xx}[\tau \neq 0] \leq 1$.

The cyclic (periodic) cross correlation of two sequences $x[n]$ and $y[n]$, which are each of length $N$, is

$$R_{xy}[\tau] = \sum_{n=0}^{N-1} x^*[n] y^{(\tau)}[n],$$

for $\tau = 0, 1, \ldots N - 1$. For $y[n] = x[n]$ this reduces to the above autocorrelation function.

Similarly the normalized cyclic cross correlation is

$$\bar{R}_{xy}[\tau] = \frac{1}{N} R_{xy}[\tau].$$

Note that it is common in practice to call the single value $\bar{R}_{xy}[0]$ the “cross correlation” since this gives the relative correlation between the two sequences without any shifting.

B. Key Properties

ZC sequences have several highly desirable properties.

Property 1: Constant Amplitude. Obviously, from (1) since they are unit amplitude complex numbers, all values of $s_q[n]$ have a constant amplitude of 1 and only the phase changes from sample to sample. This is desirable for several implementation reasons, for example the peak to average power ratio of the sequence is also 1, and also limits the PAPR of the eventual continuous time signal as well. Since the amplitude need not be computed, only the phase need s to be stored.

Property 2: Zero Cyclic Autocorrelation. The cyclic autocorrelation of a ZC sequence is optimal, in that it is zero for all nonzero shifts of the sequence.. Namely, the unnormalized cyclic autocorrelation
function of $s_q[n]$ is $N_{zc}\delta[\tau]$, where $\tau \in \mathbb{Z}$ is the cyclic shift. The normalized cyclic autocorrelation function of $s_q[n]$ is $\delta[\tau]$.

These first two properties are sometimes combined to be called Constant Amplitude Zero Autocorrelation (CAZAC). ZC sequences thus are \textit{CAZAC sequences}.

\textbf{Property 3: Fixed Cyclic Cross-Correlation.} For any two distinct ZC sequences of the same length – i.e. the same value of $N_{zc}$ but one having root index $q = q_1$ while the other with root index $q_2 \neq q_1$ – the normalized cyclic cross correlation is exactly $1/\sqrt{N_{zc}}$. This assumes that $N_{zc}$ is prime, or more generally that $|q_1 - q_2|$ is relatively prime to $N_{zc}$, i.e. the only positive integer evenly dividing $N_{zc}$ and $|q_1 - q_2|$ is 1. The unnormalized cyclic cross correlation is $\sqrt{N_{zc}}$. This is in fact the optimal cross correlation for any two sequences with the optimal autocorrelation defined above. So it is somewhat remarkable that ZC sequences, provided $N_{zc}$ is a prime number, can furnish $N_{zc} - 1$ of such sequences. Note that if $N_{zc}$ is not a prime number, the $|q_1 - q_2|$ constraint reduces the number of “good” sequences, which is why prime numbers are generally preferred.

\textbf{Property 4: The DFT or IDFT of a ZC Sequence is a ZC Sequence.} Since the DFT of a sequence $x[n]$ is a sum of complex exponentials with rotating phase shifts weighted by the sequence $x[n]$, if the sequence $x[n]$ is a ZC sequence, which itself is a rotating sequence of phase shifts, the result is also a ZC sequence. Similarly, the IDFT of a ZC sequence is also a ZC sequence. The exact mapping for the IDFT or DFT of a ZC sequence depends on the length $N_{zc}$ of the sequence and can be found in \cite{1}.

The implementation benefit of this property is that a ZC sequence can be generated directly in the frequency domain without actually taking the DFT of the sequence. This is particularly useful for OFDMA or SC-FDMA waveforms that utilize the frequency domain for signaling. Note that computing the FFT for sequences of prime length is quite inefficient, which makes this property especially appealing.

\textit{C. Sequences not of prime length}

We end this section by emphasizing that these properties listed above only apply to ZC sequences meeting the above stated conditions. Most commonly, this means the sequences are of length $N_{zc}$ wherein $N_{zc}$ is a prime number, although there are additional possibilities as discussed under Property 3. This limitation is not necessarily convenient. For example in many practical cases the desired spreading factor will not be a prime number. An arbitrary length ZC sequence can be created from a prime length one using either cyclic extension or truncation, however the above important Properties 2 and 3 no longer precisely hold, although the loss may be tolerable.

Confusingly, such truncated or extended ZC sequences are often still called ZC sequences by practicing engineers (and their wireless standards), even though they do not satisfy the above theoretical
properties and are not technically ZC or CAZAC sequences. We explore this more below in Example 2. We will distinguish these imperfect ZC sequences by denoting them as $z_q[n]$.

To create a cyclically extended ZC sequence of arbitrary length $N_{sf}$, using (1), compute

$$z_q[n] = s_q[n \mod N_{zc}]$$

for $n = 0, 1, \ldots, N_{sf} - 1$ and where $N_{zc}$ is the largest prime number less than or equal to $N_{sf}$. That is, we simply wrap the necessary number of values of the sequence $s_q[n]$ around to the back of the sequences to “top it off” to the desired length.

To create a truncated ZC sequence of arbitrary length is also simple. Just create a ZC sequence of length $N_{zc} \geq N_{sf}$ and cut off the last $N_{zc} - N_{sf}$ values of the sequence. In practice, cyclically extended sequences tend to be preferred to truncated sequences for reasons that are not entirely clear to me.

## III. Examples

**Example 1: Length 5 ZC sequences.** To begin with a simple example, we consider $N_{zc} = 5$ and $q = 1$. This gives a sequence

$$s_1[0] = \exp(0) = 1$$
$$s_1[1] = \exp(-j2\pi/5)$$
$$s_1[2] = \exp(-j6\pi/5)$$
$$s_1[3] = \exp(-j12\pi/5) = \exp(-j2\pi/5)$$
$$s_1[4] = \exp(-j4\pi) = 1$$

The sequence may seem strange, as the values 1 and $\exp(-j2\pi/5)$ each appear twice. But indeed, one can readily verify that this sequence, when multiplied by any shifted version of itself and summed, gives 0. For example, define vectors for the $q = 1$ case with shifts 0 and 2 as

$$s_1^{(0)} = [1 \ \exp(-j2\pi/5) \ \exp(-j6\pi/5) \ \exp(-j2\pi/5) \ 1]$$
$$s_1^{(2)} = [\exp(-j6\pi/5) \ \exp(-j2\pi/5) \ 1 \ 1 \ \exp(-j2\pi/5)]$$

One can readily compute, with the assistance of Euler’s formula, that $(s_1^{(0)})^*s_1^{(2)} = 0 + 0j$, where $s^*$ means the complex conjugate transpose of $s$. It is important to not forget that unlike for real sequences, the auto and cross correlation for complex sequences involves taking the complex conjugate of one of the sequences.
Meanwhile, the sequence for \( q = 4 \) can be found to be

\[
\begin{align*}
s_4[0] &= \exp(0) = 1 \\
s_4[1] &= \exp(j2\pi/5) \\
s_4[2] &= \exp(-j4\pi/5) \\
s_4[3] &= \exp(j2\pi/5) \\
s_4[4] &= \exp(-j4\pi) = 1
\end{align*}
\]

Needless to say, this sequence has the same cyclic autocorrelation property as for \( q = 1 \). Although not very obviously so, the normalized shifted cross correlation of \( s_1[n] \) and \( s_4[n] \) can readily be computed to indeed be \( 1/\sqrt{5} = 0.4472 \) for all possible such shifts.

**Example 2: Length 12 ZC Sequences.** Because LTE and 5G NR resource blocks are based upon 12 subcarriers, length 12 ZC sequences are commonly used in these standards, particularly for uplink control channels and uplink demodulation reference symbols as described in more detail in Sect. V. These \( N_{sf} = 12 \) sequences are created via cyclic extension of a \( N_{zc} = 11 \) ZC sequence.

The unshifted \( q = 1 \) and \( q = 4 \) such length 12 sequences are given below, focusing on the relative phase shifts

\[
\begin{align*}
z_1^{(0)} &= \exp \left( \frac{i\pi}{11} \cdot [0 \ -2 \ -6 \ 10 \ 2 \ -8 \ 2 \ 10 \ -6 \ -2 \ 0 \ 0] \right) \\
z_4^{(0)} &= \exp \left( \frac{i\pi}{11} \cdot [0 \ -8 \ -2 \ -4 \ 8 \ -10 \ 8 \ -4 \ -2 \ -8 \ 0 \ 0] \right)
\end{align*}
\]

Both of these sequences, along with the other (note there are 11 total) different length 12 cyclically extended ZC sequences, have a normalized autocorrelation that is no longer a delta function, and the autocorrelation of each sequence is different. Their normalized autocorrelations are plotted in Fig. II where it can be seen that the autocorrelation for \( z_4[n] \) is much worse than for \( z_1[n] \). A common non-zero value for each is \( 1/N_{sf} = 1/12 \) (observed at shifts 1, 2, 10, and 11 in the plot), as opposed to the zero autocorrelation for a prime length ZC sequence, although it can be quite a bit higher as seen in the plot. Note that in terms of the energy remaining after shifted autocorrelation, these values are squared, which is helpful in terms of maximizing the peak.

**IV. Comparison to Other Common Spread Spectrum Codes**

PN sequences are probably the most commonly used sequences for spread spectrum, and are also commonly used for “scrambling”, which means \( N_{sf} = 1 \). Effectively, they are deterministic \( \pm 1 \) sequences that appear statistically to be nearly identical\(^1\) to iid Bernoulli(0.5) sequences, i.e. each value of the sequence

\(^1\)The longer the period of the sequence, the more closely any segment of the sequence resembles a purely random signal
sequence is 1 or −1 with equal probability and independent from all the others. For two arbitrary shifts or segments of the sequence, correlated over $N_{sf}$ values, both the normalized autocorrelation and cross correlations are random variables with variance $1/\sqrt{N_{sf}}$. We recall that the cross correlation energy is the square of the normalized cyclic cross correlation, so $1/N_{zc}$, which is equivalent to the average cross correlation of ZC sequences of the same spreading factor. Thus, on average, PN sequences have the same cross correlation as ZC sequences, but much higher autocorrelation, in addition to fluctuations around that average. In this sense, PN sequences are a major degradation from ZC sequences, their loss stemming in large part from the restriction that the sequence be real and binary. When applied in complex format, for example a different PN codes is used on each of the I and Q channels as in 2G and 3G CDMA systems, each functions as an independent PN code and there is no loss nor synergy between them. In contrast ZC sequences take advantage of the complex modulation to directly encode the phase between the I and Q branches.

Walsh codes are another commonly used spreading code. They are orthogonal codes, meaning they have ideal (zero) cross-correlation, providing $N_{sf}$ such orthogonal codes for spreading factor $N_{sf}$, where $N_{sf}$ is necessarily a power of two. Meanwhile, they have an atrocious (normalized) autocorrelation, not infrequently equal to the maximum value of 1. The exact autocorrelation value can be anywhere

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**Fig. 1.** Normalized cyclic autocorrelation of the $q = 1$ and $q = 4$ length 12 ZC sequences.
between 0 and 1 depending on the code and the shift amount. A similarly poor performance occurs for shifted cross correlations, e.g. when the codes between two users are not perfectly synchronized. This is why Walsh codes are always combined with PN codes when transmitted over a dispersive channel.

In summary, ZC sequences are complex sequences with unit amplitude and arbitrary phase, as opposed to Walsh and PN codes which are real and binary valued (for our purposes, at ±1). ZC sequences consist of particular phase shifts of a unit amplitude complex exponential. In this sense, ZC sequences are more closely related to the otherwise obscure Complementary Code Keying (CCK) modulation used in the first commercially successful WiFi standard, IEEE 802.11b.

V. ZC IN LTE AND 5G

ZC sequences are utilized in LTE and 5G NR for many key functions.

Initial downlink synchronization. This is the first step for a UE to begin a connection with a BS. Synchronization is not only how the UE acquires the BS timing, but also how it receives the key system information necessary for subsequent communication. Specifically, the Primary Synchronization Sequence (PSS) in LTE is based on a \( N_{zc} = 63 \) sequence, and values \( q = 29, 34, 25 \) are used (a different value of \( q \) for each sector, usually). The reasons for choosing an odd but not prime number for \( N_{zc} \) are unclear to me. The rationale for choosing those three specific root indices \( q \) is based on their superior auto and crosscorrelations under fractional timing and frequency offsets, as demonstrated in [2]. That is, in the event of a frequency offset of say, 7.5 KHz (half a subcarrier) between either different sectors or due to carrier misalignment for a specific sector, the desirable theoretical properties of the true ZC sequences are compromised anyway. Also, the sequences for \( q = 29 \) and \( q = 34 \) are complex conjugates of each other and so a single correlator can be used to detect both, which allows some desirable complexity reduction at the UE.

In 5G, the PSS is however no longer a ZC sequence but rather is based on m-sequences (PN codes) of length 127.

Random access. This is how the UE gains access to the network. Because the UE has not yet been admitted to the network, it does not have a scheduled time/frequency slot for uplink transmissions. Thus the uplink random access channel should be robust to many UEs simultaneously transmitting at slightly different timing offsets: a perfect setting for ZC sequences which have both strong shifted cross correlation properties.

Specifically, the Physical Random Access Channel (PRACH) uses \( N_{zc} = 839 \) for the long preamble and \( N_{zc} = 139 \) for the short preamble, in both LTE and 5G NR.

Uplink control information. The physical uplink control channel (PUCCH) is used by the UEs to convey channel state information and ACK/NAKs, as well as requests to transmit. All the UEs share the
PUCCH, so they periodically use it and also benefit from the use of a spreading code to allow multiple UEs to send control information at the same time and frequency.

In LTE, there are 4 PUCCH formats: 0, 1, 2, and 3. ZC sequences (length 12, extended from $N_{zc} = 11$ sequences as in the above example) are used for all but format 3, which is a supplemental format for use in carrier aggregation. In 5G NR, there are 5 PUCCH formats. Extended length 12 ZC sequences are used for formats 0 and 1.

**Uplink reference signals (pilot signals).** Uplink reference symbols are utilized by the BS for channel estimation, synchronization, and are necessary for demodulating the UE’s data transmission. In particular they are used for the Sounding Reference Symbols (SRS), which are sent periodically when the UE is not transmitting data, as well as the Demodulation Reference Symbols (DM-RS) which are embedded with data transmissions to aid with precise channel estimation.

In LTE and 5G NR, length $N_{zc} = 31$ ZC sequences are cyclically extended in the frequency domain to achieve length 36 sequences for SRS transmissions.

As far as the DM-RS, whenever SC-FDMA is used, then the DM-RS are length 12 extended ZC sequences. This is always the case in the LTE uplink. In 5G NR, the uplink can be either OFDMA or SC-FDMA. When the 5G NR uplink uses OFDMA, then a Gold code (based on 2 maximal length PN codes) is used instead.

VI. **Bibliographic Notes**

This primer draws from [2], which has a summary of Zadoff Chu sequences in Sect. 7.2.1, and also discusses all their myriad uses in the LTE standard in detail in the relevant sections. We also gratefully acknowledge the 5G discussions in [3]. The two seminal papers by Zadoff (actually B. Frank is first author) in 1962 [4] and then Chu in 1972 [5]. We should pause to reflect on the fact that an important piece of work in 1962 and 1972 (the pre-Latex age) could be authored in under 2 pages. Perhaps a return to brevity would be desirable for our field.

**References**

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