\( \mathcal{H}_2 \) Performance of Series-Parallel Networks: A Compositional Perspective

Mathias Hudoba de Badyn, and Mehran Mesbahi

Abstract—We examine the \( \mathcal{H}_2 \) norm of matrix-weighted leader-follower consensus on series-parallel networks. By using an extension of electrical network theory on matrix-valued resistances, voltages and currents, we show that the computation of the \( \mathcal{H}_2 \) norm can be performed efficiently by decomposing the network into atomic elements and composition rules. Lastly, we examine the problem of efficiently adapting the matrix-valued edge weights to optimize the \( \mathcal{H}_2 \) norm of the network.

I. INTRODUCTION

Networked systems are ubiquitous in many disciplines, including robotic swarms \(^2\), animal flocking \(^3\), oscillator networks \(^4\). A widely analyzed algorithm in networked systems is consensus, a distributed information-sharing protocol over a network. Consensus has diverse applications in control and estimation, such as multi-agent systems \(^5\), \(^6\), distributed Kalman filtering \(^7\), and swarm deployment \(^8\). System-theoretic properties of this algorithm have been extensively examined in the literature, typically using the graph Laplacian \(^9\). This setup allows one to draw connections between systems aspects of distributed algorithms and the graph structure, e.g., highlighting the role of symmetries in inducing uncontrollable modes \(^10\). Various performance measures for networks have been examined in the literature, such as entropy and other so-called spectral measures \(^9\), \(^11\), \(^12\), as well as disturbance rejection \(^13\)–\(^15\). One can use such measures to modify the topology of the network to adapt to antagonistic influences \(^14\).

Most of the existing literature on consensus considers the case where each node in the network has a scalar state. Recently, several extensions to the case of vector-valued node states have been proposed; such networks have matrix-weighted edges. Properties of such networks, and their use in formation control were considered in \(^16\). Further applications of matrix-valued weighted graphs are found in the control of coupled oscillators \(^17\), opinion dynamics \(^18\), spacecraft formation control \(^19\), and distributed estimation \(^20\), \(^21\).

The notions of performance and control of consensus rely on the underlying topology of the network. As such, it is desirable to devise algorithms for system theoretic analysis and synthesis that are scalable and modular. One approach is to take smaller, atomic elements and build large-scale graphs from them; one can then use graph-growing operations that preserve properties such as controllability \(^22\), \(^23\).

In this paper, we approach the \( \mathcal{H}_2 \) performance problem on leader-follower consensus by utilizing the paradigms of matrix-weighted resistor networks and in particular series-parallel networks. Such networks can be decomposed into smaller networks via simple operations in sublinear time \(^24\). Many NP-hard problems on general classes of graphs become linear on series-parallel graphs, such as finding maximum matchings induced subgraphs and independent sets, and the maximum disjoint triangle problem \(^25\). This has been exploited in a number of other disciplines. Efficient algorithms have been derived for the Quadratic Assignment Problem \(^26\), the discrete time-cost tradeoff problem \(^27\), and resource allocation by dynamic programming \(^28\).

The contributions of the paper are as follows. Using the matrix-valued resistance extension of electrical networks, we show that the analogous notion of effective resistance is related to the \( \mathcal{H}_2 \) norm on a leader-follower consensus network with vector states. By exploiting the decomposability of series-parallel networks, we present a way of computing the \( \mathcal{H}_2 \) norm of a leader-follower consensus network in best-case \( \mathcal{O}(k^\omega |\mathcal{R}| \log |\mathcal{N}|) \) (worst case \( \mathcal{O}(k^\omega |\mathcal{R}| |\mathcal{N}|) \)) complexity, where \(|\mathcal{R}|, |\mathcal{N}|\) are the number of leaders and followers, respectively, and \( \mathcal{O}(k^\omega) \) is the complexity of inverting a \( k \times k \) symmetric positive-definite matrix; the current best lower bound for \( \omega \) is 2.3728639 \(^29\). We also provide a gradient descent method for adaptively re-weighting the network to optimize \( \mathcal{H}_2 \) performance that utilizes computations of similar complexity by again using the decomposition of series-parallel networks.

The outline of the paper is as follows: in §\( \text{II} \) the notation, preliminaries and problem setup are presented. Our main results on \( \mathcal{H}_2 \) computation/adaptive re-weighting are in §\( \text{III} \) with conclusions in §\( \text{IV} \).

II. PRELIMINARIES & SETUP

A. Mathematical Preliminaries

For a matrix \( A \), we respectively denote its inverse, pseudoinverse, transpose and conjugate transpose as \( A^{-1}, A^\dagger, A^T, A^* \). A graph \( \mathcal{G} \) is a triple of sets \( (\mathcal{N}, \mathcal{E}, \mathcal{W}) \), where \( \mathcal{N} \) is a set of nodes, \( \mathcal{E} \subseteq \mathcal{N}^2 \) is a set of edges denoting pairwise connections between nodes, and \( \mathcal{W} = \{W_{vu} \in S_{++}^k : \{u,v\} \in \mathcal{E}\} \) is a set of matrix-valued weights on the edges, where \( S_{++}^k \) denotes the set of \( k \times k \) symmetric positive-definite matrices with real entries. A graph is called directed if the edge \( \{i,j\} \neq \{j,i\} \), and for edge \( \{i,j\} \) the node \( i \) (\( j \)) is called the head (tail) of...
The incidence matrix $E$ is a $|\mathcal{N}| \times |\mathcal{E}|$ matrix, where each column of $E$ corresponding to an edge $\{i, j\}$ is denoted by $a_{ij}$. For each edge $l := \{i, j\}$, where $i$ is the tail and $j$ is the head, $E_{il} = 1$ and $E_{jl} = -1$. If $G$ is undirected, by convention we write that $E_{il} = 1$ and $E_{jl} = -1$ for $i > j$.

Since we are dealing with matrix-valued weights, for defining the graph Laplacian below, we need the matrices $E \equiv E \otimes I_k$ and $A_{ij} \equiv a_{ij} \otimes I_k$, where $I_k$ is the $k \times k$ identity matrix, and $\otimes$ denotes the Kronecker product. The weight matrix $W$ is a $k|\mathcal{E}| \times k|\mathcal{E}|$ blockwise diagonal matrix containing the weights $W_{ij}$ of each edge $e$. The graph Laplacian $L$ of an undirected graph $G$ can be defined by the incidence and weight matrix as $L \equiv EWE^\top = \sum_{e \in \mathcal{E}} A_{ij} W_{ij} A_{ji}^\top$.

A tree is a connected graph with no cycles, and a leaf is designated as a node of degree 1. A binary tree is a tree where one node is designated as the root, and all nodes of $\mathcal{T}$ are either leaves or parents. Each parent in a binary tree has one parent and at most two children, except the root which has no parent. The height $h$ of a binary tree is the length of the longest path from the root to a leaf. A complete (sometimes called full) binary tree is one where each node has either zero or two children. Finally, the parallel addition of two symmetric matrices $A, B$ is defined as $A \oplus B = A + (A^\top B)^\top$.

### B. Problem Setup

In this paper, we examine the $H_2$ performance of leader-follower consensus problem on matrix-valued weighted series-parallel networks. Consider a connected weighted graph $G = (\mathcal{N}, \mathcal{E}, W)$ with the Laplacian $L$. Each node $i$ has a state $x_i \in \mathbb{R}^k$. Denote a set of leaders $\mathcal{R} \subset \mathcal{N}$ and a set of followers $\mathcal{N} \setminus \mathcal{R}$. Suppose that one is able to take over the state $x_i \in \mathbb{R}^k$ of a leader, and thereby exert control over the followers. Further, suppose that each leader is connected by an edge with identity weight to a unique node in $\mathcal{N} \setminus \mathcal{R}$, that collectively will be called the source nodes and designated as the set $\mathcal{R}$ (see Figure 1 for a schematic of the setup). Then, using $B := B \otimes I_k$, the graph Laplacian of $G$ can be partitioned as:

$$L = \begin{pmatrix} L_{\mathcal{R}} & -B \\ -B^\top & L_{\mathcal{N} \setminus \mathcal{R}} + \sum_{e \in R} e_i W_{ij} e_j^\top \end{pmatrix},$$

where $e_i \equiv e_i \otimes I_k$. The control matrix is given by $B^\top = [e_{i_1}, \ldots, e_{i_m}]$ where $R = \{i_1, \ldots, i_m\}$ are the nodes attached to leaders. The graph Laplacian is written as:

$$L_{\mathcal{N} \setminus \mathcal{R}} = E_{\mathcal{N} \setminus \mathcal{R}} WE_{\mathcal{N} \setminus \mathcal{R}}^\top = \sum_{\{i,j\} \in \mathcal{E}(\mathcal{N} \setminus \mathcal{R})} A_{ij} W_{ij} A_{ji}^\top,$$

where $E_{\mathcal{N} \setminus \mathcal{R}} = E_{\mathcal{N} \setminus \mathcal{R}} \otimes I_k$ and $E_{\mathcal{N} \setminus \mathcal{R}}$ is the result of removing the rows from $E$ corresponding to the nodes in the leader set $\mathcal{R}$. The matrix $W = \text{Blkdiag}[W_e]$ denotes the matrix consisting of the positive-definite weights $W_e$ on the block-diagonal. The corresponding leader-follower consensus dynamics are now given by:

$$x = - \left( L_{\mathcal{N} \setminus \mathcal{R}} + \sum_{e \in R} e_i W_{ij} e_j^\top \right) x + B^\top u, \quad (1)$$

where $x$ is the vector containing the stacked states of the nodes, and $u$ is the stacked vector of the leader node states—the control inputs to the followers. We note that the Dirichlet Laplacian $A(W) \equiv (L_{\mathcal{N} \setminus \mathcal{R}} + \sum_{e \in R} e_i W_{ij} e_j^\top)$ is positive definite if $G$ is connected.

For a linear system with measurement $y = Cx$ and transfer function $G(s) = C(sI - A)^{-1} B$, the $H_2$ norm is defined as:

$$\|G(s)\|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Tr}(G(j\omega)^* G(j\omega)) d\omega.$$

The $H_2$ norm is a measure of the input-output energy excitation of the system. If we want to examine the holistic response of the system, we can measure all of its internal states; as such, we subsequently assume that $C = I$. In the scalar case of $I$ (i.e., $k = 1$), the $H_2$ norm of this system is given by [30],

$$(H_2 G, B)^2 = \frac{1}{2} \text{Tr}(B^\top A(W)^{-1} B).$$

In the case of matrix-valued edge weights, we have the following proposition.

**Proposition 1.** Consider the leader-follower consensus dynamics in Figure 1 for positive integer $k$. Suppose that all nodes are observed, i.e. $y = x$. Then the $H_2$ norm is given by

$$(H_2 G, B)^2 = \frac{1}{2} \text{Tr}(B^\top A(W)^{-1} B).$$

**Proof.** The $H_2$ norm is given by $\text{Tr}(B^\top PB)$, where $P$ satisfies the Lyapunov equation

$$-A(W)P - PA(W)^T + I = 0.$$

The ansatz $P = \frac{1}{2} A(W)^{-1}$ yields the solution. \qed
The contributions of the paper are the following. First, we use an electrical perspective on the leader-follower consensus dynamics, as well as a decomposition of series-parallel networks, to compute the $H_2$ performance on a class of two-terminal series-parallel graphs. In particular, we show that when adding networks in series and parallel, the $H_2$ norm follows a similar composition procedure as adding resistors in series and parallel. Secondly, we provide an adaptive procedure to re-weight the network to minimize the $H_2$ norm—this procedure computes the ‘power’ dissipated across the network as part of a gradient update, and so can also be computed efficiently using a decomposition of series-parallel networks.

This extends the work in [1] by considering vector-valued node states, and therefore matrix-valued edge weights. We show that by using a generalization of electrical network theory with matrix-valued resistors, such as in [31], the electrical interpretation of the $H_2$ performance of leader-follower consensus holds, and that the computations in the scalar-state case can naturally be extended to analogous, but non-trivial, electrical computations in the vector-state case.

C. Electrical Network Models

One can view consensus through the lens of electrical network theory [14, 31]. In the leader-follower setup of [13], consider the graph $G = (N, E, W)$ with $W_e \in S_{++}^n$. The weight $W_e$ represents a matrix-valued resistor on each edge $e := ij$ with matrix-valued conductance $W_{ij}$. A generalized current (see [31]) from node $u$ to $v$ with intensity $i \in \mathbb{R}^{k \times k}$ is an edge function $I : E \to \mathbb{R}^{k \times k}$ such that

$$
\sum_{(k,l) \in E} I_{(k,l)} - \sum_{(l,k) \in E} I_{(l,k)} = \begin{cases} 
1 & p = u \\
-1 & p = v \\
0 & \forall p \in N, \quad k \in \mathbb{R}^{k \times k}
\end{cases}
$$

and there exists a node function $V : N \to \mathbb{R}^{k \times k}$ satisfying $R_{eff}^u I_{(u,v)} = V_u - V_v, \quad \forall \{u, v\} \in E$.

In this setting, the power dissipated across an edge with a matrix weight $R_{eff}$ and current $I_e$ is given by the inner product, $P_e(i_e) = \text{Tr}(I_e^\top R_{eff} I_e)$, which reduces to the familiar formula $P = i^2 R$ when $k = 1$.

**Proposition 2.** Let $R_1$ and $R_2$ denote two-matrix valued resistances in $S_{++}^n$, and consider the current $I \in \mathbb{R}^{k \times k}$ across $R_1$ and $R_2$ in parallel. Then, the effective resistance across the join is given by $R_{eff} = R_1 \cup R_2$.

**Proof.** Recall that power with current $I$ dissipates across a parallel join according to the infimal convolution,

$$
P_p(I) = \min \{ \text{Tr}(I_1^\top R_1 I_1) + \text{Tr}(I_2^\top R_2 I_2) \} \quad \text{s.t.} \quad I_1 + I_2 = I \quad (2)
$$

Applying the identity $(f \circ g)^* = f^* + g^*$, where $f^*$ denotes the Fenchel conjugate, to Problem 2 yields the minimum $P_p(I) = \text{Tr}(I^\top (R_1 \cup R_2) I)$. 

Furthermore, the $ith$ $k \times k$ block on the diagonal of the matrix,

$$
A(W)^{-1} = \left( \sum_{i,j \in E} A_{ij} W_{ij} A_{ij}^\top + \sum_{i \in R} e_i W_i e_i^\top \right)^{-1}
$$

form the matrix-valued effective resistances from node $i \in N \setminus R$ to $R$, denoted $R^i_{eff}(R)$. Note that when $x \in \mathbb{R}^{n \times k}$ is a matrix of stacked $k \times k$ current matrices injected into $n$ nodes of $G$, then the matrix $A^{-1} x$ is the stacked matrix of the voltage drops from each node to the grounded leader node set. In particular, the $sth$ $k \times k$ block of $A^{-1} x$ is denoted $B_{ik}^{k \times k}[A^{-1} x]$, and if $x = (e_s \otimes I_k) = e_s$, then this corresponds to a current of identity intensity injected into node $s$; again, this setup is shown in Figure 1. Finally, we denote the quantity $Y_s^i = B_{ik}^{k \times k}[A^{-1} e_s]$, as the voltage drop from node $i$ to $R$ under identity current injected into node $s$; this reduces to the corresponding scalar definition seen in [30] when $k = 1$.

D. Series-Parallel Graph Models

In this paper, we consider the class of graphs known as series-parallel graphs. Given a series-parallel graph, there exist efficient (i.e., $O(\log |N|)$) algorithms that decompose the graph into atomic structures and simple composition operations on them [24, 32, 33].

**Definition 1** (Two-Terminal Series-Parallel Graphs). An acyclic graph is called two-terminal series-parallel (TTSP) if it can be defined recursively as follows:

1. The graph defined by two vertices connected by an edge (a 1-path) is a TTSP graph, where one node is labeled the source, and the other the sink.
2. If $G_1 = (N_1, E_1)$ and $G_2 = (N_2, E_2)$ are TTSP where $S_1 = \{s_1\}, T_1 = \{t_1\}$ are the unique source and sink of $G_1$, then the following operations produce TTSP graphs:
   a) **Parallel Addition:** $G_2 \leftarrow s_1 \sim s_2, \quad t_1 \sim t_2$.
   b) **Series Addition:** $G_2 \leftarrow t_1 \sim s_2$.

Denote the parallel join of $G_1$ and $G_2$ as $G_1 \oplus G_2$, and the corresponding series join as $G_1 \odot G_2$, see Fig 2.

The two recursive operations defining the TTSP graph model allow for a simple constructive approach for defining
graphs from atomic elements. Indeed, efficient algorithms exist that decompose TTSP graphs into a decomposition tree with the following structure [24, 32, 33].

**Definition 2 (TTSP Decomposition Tree).** A TTSP decomposition tree of a TTSP graph $G$ is a binary tree $T(G)$ with the following properties:

1) $T$ is a complete (sometimes called full) binary tree, in that every node has either 2 or 0 children.
2) Every leaf of $T$ corresponds to a 1-path.
3) Every parent of $T$ corresponds to either a series or parallel addition operation from Def. 7 on its children.

In the following proposition, we quantify the height of the tree in terms of the size of the resulting graph.

**Proposition 3 (Properties of $T(G)$ [11]).** Let $G$ be a TTSP graph with $N$ nodes constructed from $l$-paths with $p$ parallel joins and $s$ series joins. Then, the height of $T(G)$ is bounded by $\log_2 (N + 2p - s) \leq h \leq \frac{1}{2} (N + 2p + s) - 2$.

### III. System-Theoretic Analysis on Series-Parallel Graphs

#### A. Synthesis of $H_2$-Optimal Networks

In this section, we discuss an efficient algorithm on series-parallel networks for an a priori synthesis computation of the $H_2$ norm. First, we define an all-input series-parallel graph (AITTSP), an example of which is shown in Figure 5.

**Definition 3 (All-Input TTSP Graphs).** Consider a graph $G$ in the setup of the leader-follower consensus dynamics. Identify each leader node $i_1, i_2, \ldots, i_r \in R$ as one node $l$ connected to all source nodes $s \in R$, as depicted in Figure 7. The graph $G$ is called an all-input two-terminal series-parallel graph if, for all source nodes $s \in R$, $G$ is TTSP with $s$ as the source and $l$ as the sink.

Next, let us examine the structure of the $H_2$ norm in the context of the leader-follower consensus setup in $\S$II-B.

**Lemma 1.** Consider the leader-follower consensus setup with $C = I \otimes I_k$. Then,

$$
(H_{2}^{G,B})^2 = \frac{1}{2} \sum_{s \in R} Tr [Y_s^*] = \frac{1}{2} \sum_{s \in R} Tr [\rho_{s,R}^{eff}].
$$

**Proof.** We note that,

$$(H_{2}^{G,B})^2 = \frac{1}{2} Tr (B^T A^{-1} B) = \frac{1}{2} Tr \left( \sum_{s \in R} (e_s \otimes I_k) A_{s,1}^{-1} (e_s \otimes I_k)^T \right)$$

$$= \frac{1}{2} Tr \left( \sum_{s \in R} (e_s \otimes I_k) A_{s,1}^{-1} (e_s \otimes I_k)^T \right)$$

$$= \frac{1}{2} Tr \left[ Blk_{s \times k} [A^{-1} e_s] \right]$$

$$= \frac{1}{2} Tr \left[ Y_s^* \right] = \frac{1}{2} \sum_{s \in R} Tr [\rho_{s,R}^{eff}].$$

Each quantity $Y_s^*$ on the left side of the sum in 3 is the voltage drop from the source node $s \in R$ to the grounded leader node set $R$ (depicted in Figure 1 Right, if $v_s = v_t$). This is precisely the voltage dropped over the last parallel join of the series-parallel decomposition of an all-input TTSP graph; the join in question is exactly the one depicted in Figure 5. We can utilize this observation to efficiently compute the $H_2$ norm a priori knowing only the weights of the edges and the decomposition tree of $G$.

We use the following setup. Consider a TTSP graph $G$ with source node $s$ and sink node $t$. Ground the source node $s$, and consider the grounded Laplacian $A$ with respect to the grounded source $s$. This is a leader-follower system with a single leader.

This parallel join, depicted in Figure 5, effectively makes one of the terminals of the resulting graph an element of $R$, and the other terminal (the sink) an element of $R$. The control matrix of the leader-follower consensus problem corresponds to exactly those elements in $R$, which are the ‘sinks’ of the TTSP graph used in that computation. Therefore, our choice of $B$ must always select the state of the sink vector $t$; hence $B = e_t = e_1 \otimes I_k$.

We now proceed in two steps. First, we need a lemma that essentially states that for an arbitrary TTSP graph, there exists an equivalent 1-path TTSP graph with the same effective resistance. Then, any composition rule on arbitrary TTSP graphs can be reduced to a composition on the equivalent 1-paths, simplifying the analysis. Afterward, we show that the series-parallel composition graphs produces a similar series-parallel computation of the $H_2$ performance.

**Lemma 2.** Consider two graphs: an arbitrary TTSP graph $G_1$ with source $s_1$ and sink $t_1$ with effective resistance $\rho_{s_1,t_1}^{eff}$, and a 1-path TTSP graph $G_2$ with source $s_2$ and sink $t_2$ with effective resistance $\rho_{s_2,t_2}^{eff}$. Let their respective control matrices be $B_1 = e_{t_1}$ and $B_2 = e_{s_2}$. Further suppose that $\rho_{s_1,t_1}^{eff} = \rho_{s_2,t_2}^{eff}$. Then, $(H_{2}^{1})^2 = (H_{2}^{2})^2$.

**Proof.** The setup of the graphs in the statement of the lemma is a leader-follower consensus with grounded (leader) nodes $s_1, s_2$. Therefore, we can invoke Lemma 1 denoting the graphs’ respective Dirichlet Laplacians as $A_1, A_2$ we can compute,

$$(H_{2}^{1})^2 = \frac{1}{2} Tr \left( e_{t_1}^T \otimes I_k \right) [A_1]^{-1} \left( e_{t_1} \otimes I_k \right)$$

$$= \frac{1}{2} \rho_{s_1,t_1}^{eff} = \frac{1}{2} \rho_{s_2,t_2}^{eff} = \frac{1}{2} Tr \left[ B_2^T [A_2^{-1}] B_2 \right] = (H_{2}^{2})^2.$$

**Lemma 2** will allow us to reduce the computation of the $H_2$ norm of a composite TTSP graph to the computation of $H_2$ norms of an equivalent 1-path. This allows us to prove the following theorem.

**Theorem 1.** Consider the leader-follower consensus setup in $\S$II-B on two graphs: an arbitrary TTSP graph $G_1$ with source $s_1$ and sink $t_1$, and a second arbitrary TTSP graph $G_2$ with source $s_2$ and sink $t_2$. Let the Dirichlet Laplacian of $G_i$ grounded with respect to its source $s_i$ be $A_i$, and
Inverting this expression and substituting into (6) yields,
\[(H_2 G_1 G_2)^2 = (H_2 G_1)^2 + (H_2 G_2)^2\]
\[(H_2 G_1 G_2)^2 \leq (H_2 G_1)^2 : (H_2 G_2)^2,\]
with equality in the last display if and only if \(\rho_{s_1,t_1} = c \rho_{s_2,t_2}\) for \(c > 0\).

Proof. By Lemma [1], the \(H_2\) norm of an arbitrary graph can be computed from an equivalent 1-path. Hence, we need to show (4) and (5) for series and parallel joins of 1-paths.

Consider the 1-paths in Figure 3 with weights \(W_1, W_2 \in S_{k+1}\), between the sources (square nodes) and sinks (circular nodes). The Dirichlet Laplacians of both with respect to the grounded source nodes are simply \(L_{G_1} = [W_1]\), \(L_{G_2} = [W_2]\), and their respective control matrices are \(B_{G_1} = B_{G_2} = I_k\) and \(H_2\) norms are \((H_2 G_1)^2 = \frac{1}{2} \text{Tr}[W_1^{-1}]\), \((H_2 G_2)^2 = \frac{1}{2} \text{Tr}[W_2^{-1}]\).

Similarly, the Laplacians of the series and parallel joins in Figure 3 are,
\[L_{G_1 G_2} = [W_1 + W_2], \quad L_{G_1 \otimes G_2} = [W_1 + W_2 - W_2, -W_2].\]
The control matrices are \(B_{G_1 G_2} = I_k, B_{G_1 \otimes G_2} = [0_{k \times k} I_k]\). Therefore, the \(H_2\) norm in the series join case is
\[(H_2 G_1 G_2)^2 = \frac{1}{2} \text{Tr} \left[ \begin{bmatrix} 0_{k \times k} & I_k \end{bmatrix} \begin{bmatrix} W_1 + W_2 & -W_2 \\ -W_2 & W_2 \end{bmatrix}^{-1} \begin{bmatrix} 0_{k \times k} & I_k \end{bmatrix} \right],\]
\[= \frac{1}{2} \left( W_2 - W_2 (W_1 + W_2)^{-1} W_2 \right)^{-1},\]
where (6) follows from standard block matrix inversion formulae. Note that we can write
\[W_1 (W_1 + W_2) W_2 = W_2 W_1 (W_1 + W_2)^{-1} W_2.\]
Inverting this expression and substituting into (6) yields,
\[(H_2 G_1 G_2)^2 = \frac{1}{2} W_1 (W_1 + W_2)^{-1} W_2) \]
\[= \frac{1}{2} W_1^{-1} + \frac{1}{2} W_2^{-1} = (H_2 G_1)^2 + (H_2 G_2)^2,\]
as desired. We can compute the \(H_2\) norm in the parallel join case by invoking Theorem 13 from [34], which states that for \(A, B \in S_{k+1}\), \(\text{Tr}(A : B) \leq \text{Tr}(A) \cdot \text{Tr}(B)\), with equality if and only if \(A = c B\) for some \(c > 0\). By this result, and by Lemma [1] we can compute:
\[(H_2 G_1 G_2)^2 = \frac{1}{2} \text{Tr} \left[ \rho_1^\text{eff} : \rho_2^\text{eff} \right] \leq \frac{1}{2} \left( \text{Tr} \left[ \rho_1^\text{eff} \right] : \text{Tr} \left[ \rho_2^\text{eff} \right] \right),\]
\[= (H_2 G_1)^2 : (H_2 G_2)^2.\]
This concludes the proof. \(\square\)

We now propose Algorithm [1] for computing the \(H_2\) norm of an AITTSP graph with control input \(e_s\). If the matrix weights across the graph are scalar multiples of each other (which is the case when \(k = 1\)), Algorithm [1] computes precisely the \(H_2\) norm; otherwise it computes an upper bound.

Algorithm 1: \(H_2\) norm of AITTSP Graph with \(B = e_i\)

Input: Decomposition tree \(T(G)\), weights \(W(G), e_i\)
Result: Upper bound on \((H_2 G)^2\)
for each leaf \(L\) of \(T(G)\) do
if \(j\) is a series join then
Output \((H_2 G)^2 \leftarrow (H_2 G_1)^2 + (H_2 G_2)^2\) to parent;
else
Output \((H_2 G)^2_{\text{est}} \leftarrow (H_2 G_1)^2 : (H_2 G_2)^2\) to parent;
else
wait;
return \((H_2 G)^2\) at root node of \(T(G)\).

Theorem 2. Consider the leader-follower consensus dynamics as in §II-B on an all-input TTSP graph \(G\). Then, the \(H_2\) norm of this system is given by
\[(H_2 G^B)^2 = \frac{1}{2} \text{Tr} \left[ B^T A^{-1} B \right] = \sum_{s \in \mathcal{R}} (H_2 G_{s,e_s})^2,\]
and the best-case complexity of computing this \(H_2\) norm is \(O(k^{\omega} |R| \log |\mathcal{N}|)\), and the worst-case complexity is \(O(k^{\omega} |R| |\mathcal{N}|)\).

Proof. We can compute:
\[(H_2 G^B)^2 = \frac{1}{2} \sum_{s \in \mathcal{R}} \text{Tr} \left[ e_s^T A^{-1} e_s \right] = \sum_{s \in \mathcal{R}} (H_2 G_{s,e_s})^2.\]
This is a sum of \(|R|\) \(H_2\) norms of leader-follower consensus networks with control input \(e_s\). Since at each layer of the decomposition tree \(T(G)\) each computation happens independently, the complexity depends on the height \(h\) of the decomposition tree. Hence, the complexity is determined by \(h\), which by Proposition [3] is \(O(\log |\mathcal{N}|)\), and \(O(|\mathcal{N}|)\) for best and worst-case, respectively. Each call of the three algorithms must be done \(|R|\) times. Algorithm 1 may perform an inversion of a \(k \times k\) matrix, which has complexity \(O(k^{\omega})\). Algorithm 2 requires computing \((I_1, I_2) = \arg P_1 \circ P_2\), given by
\[(R^{-1}_{\text{eff}} : R^{-1}_{\text{eff}} : R^{-1}_{\text{eff}} : R^{-1}_{\text{eff}} I_m),
R^{-1}_{\text{eff}} : R^{-1}_{\text{eff}} : R^{-1}_{\text{eff}} : R^{-1}_{\text{eff}} I_m),\]
over every parallel join, which also requires inverting two \(k \times k\) matrices (the parallel addition of resistances is computed in Algorithm 1).
\(\square\)

B. Noise Rejection and Adaptive Weight Design

It is often the case that one wishes to adapt the network in order to reject noise. For example, in a swarm of UAVs
performing consensus on heading, it is undesirable for the swarm to be influenced by wind gusts [30]. Similarly, one may wish to design matrix-weighted networks in order to reject noise during distributed attitude control and estimation [16].

In this section, we discuss a protocol that allows the network to quickly adapt its interaction edge weights to minimize the collective network response to such disturbances.

Consider the leader-follower consensus dynamics setup from [16] and the task of re-assigning weights to the edges of the network to minimize the $\mathcal{H}_2$ norm. This can be done via the optimization problem,

$$
\min \left( \frac{1}{2} \text{Tr} \left( \mathbf{B}^\top \mathbf{A}(W)^{-1} \mathbf{B} \right) \right)
$$

s.t. \quad \mathbf{W} := \text{Blkdiag}(\mathbf{W}_c), \quad \forall e \in \mathcal{E}

$$
\mathbf{W}_c \in \mathbb{S}_+^{k \times k}, \quad \forall e \in \mathcal{E}

U_e \geq \mathbf{W}_c \geq L_c, \quad \forall e \in \mathcal{E}, \quad L_c, \mathbf{P}_c \in \mathbb{S}_+^{k \times k}.
$$

Remark 1. One must include the bounds in Problem (7) in order to prevent edges from becoming disconnected, or from becoming arbitrarily ‘large’ (in the sense of the Loewner order on $\mathbf{W}_c$). This also motivates adding a regularization term $\frac{1}{2} \sum_{e \in \mathcal{E}} \|\mathbf{W}_e\|^2$ for some matrix norm $\|\cdot\|$ to the cost function.

The key features of optimization problem (7) are highlighted in the following proposition and proven in the appendix.

**Proposition 4.** Consider the setting of Problem (7). Then,

1. The objective $f_W = \frac{1}{2} \text{Tr} \left( \mathbf{B}^\top \mathbf{A}(W)^{-1} \mathbf{B} \right)$ is strongly convex on the cone of positive-definite matrices $\mathbf{W}_c \in \mathbb{S}_+^{k \times k}$.

2. The gradient of the objective function with respect to a single edge weight $\mathbf{W}_c$ at a point $H \in \mathbb{S}_+^{k}$ is given by $\nabla f_{\mathbf{W}_c}(H) = -\frac{1}{2} \mathbf{Q}^\top \mathbf{Q}$, where $\mathbf{Q} = \mathbf{A}_c^{-1} \mathbf{A}_c^{-1} \mathbf{e}_s$, and $\mathbf{Y}^s_i = \text{Blkdiag}(A^{-1}(\mathbf{e}_s \otimes \mathbf{I}_k))$, where $A^{-1}$ is evaluated with $H$ in place of $\mathbf{W}_c$.

We can solve Problem (7) using a projected gradient descent algorithm:

$$
x^{t+1} = \text{Proj}_{\mathcal{C}} \left( x^t + \frac{1}{ht^2} \nabla_x f(x^t) \right),
$$

where $\mathcal{C}$ is the cone generated by the constraints of Problem (7).

Following Remark 1, let us include the Frobenius norm penalty on the edge weights in the cost:

$$
f(W) = \frac{1}{2} \text{Tr} \left( \mathbf{B}^\top \mathbf{A}^{-1} \mathbf{B} \right) + \frac{1}{2} \sum_{e \in \mathcal{E}} \text{Tr} \left( \mathbf{W}_e^\top \mathbf{W}_e \right).
$$

The gradient of the penalty term with respect to $\mathbf{W}_e$ is simply $htW_e$, so each edge in the network updates its weight according to the dynamics given by the gradient update

$$
W_e^{t+1} = \text{Proj}_{\mathcal{C}} \left( \left( 1 - \frac{1}{\sqrt{t}} \right) W_e^t + \frac{1}{ht^2} \nabla f_{\mathbf{W}_c}(W_e^t) \right).
$$

Problem (7) in the scalar edge weight case was examined in [16]. In this case, an efficient projection onto the constraint set is via the $\|\cdot\|_\infty$ norm. In the case of matrix edge weights, there is no such natural projection. Instead, at each step one solves the problem $\text{Proj}_{\mathcal{C}}(X) = \min_{Y \in \mathcal{C} \cap \mathbb{S}_+^{k \times k}} \|Y - X\|$ for some matrix norm $\|\cdot\|$. However, if $k$ is relatively small compared to the size of the overall network, then this operation is not the most computationally expensive part of the setup. The complexity in solving Problem (7) lies in computing the gradient $\nabla f_{\mathbf{W}_c}(H)$, as it requires the voltage drops $\mathbf{Y}_i^s, \mathbf{Y}_j^s$ for each edge $(i, j) \in \mathcal{E}$.

We now present an algorithm for computing this quantity on all-input two-terminal series-parallel graphs, and a characterization of its complexity. Informally, the algorithm is as follows. For each source node $s$, the algorithm utilizes the decomposition tree of $\mathcal{G}$ with $s$ as the source and the grounded leader set as the sink; this is why $\mathcal{G}$ needs to be TTSP with respect to all source nodes. The voltage drops can be computed from resistances and currents across each join, and the currents can be extracted from the power dissipated across each join. Hence, the effective resistances are computed first, as in Algorithm 2. Starting from the root of the decomposition tree, the currents at each join can be computed by Eq. (2), as in Algorithm 3 and depicted in Figure 4. Finally, the voltage drops over each branch are computed starting from the leaves of the decomposition tree, as in Algorithm 4.

**Algorithm 2: Effective Resistance over AITTS Graph**

**Input:** Decomposition tree $\mathcal{T}(\mathcal{G})$

Edge weights $\mathcal{W}(\mathcal{G})$

**Result:** Effective resistances $\rho(\mathcal{G})$ over $\mathcal{T}(\mathcal{G})$

for each leaf of $\mathcal{T}(\mathcal{G})$ do

Output $R_{\mathcal{T}} = W_{\mathcal{T}}^{-1}$ to parent;

for each parent $j$ of $\mathcal{T}(\mathcal{G})$ do

if received $R_{\mathcal{T}}$, from both children $i = 1, 2$ then

if $j$ is a series join then

Output $R_{\mathcal{T}} = R_{\mathcal{U}_1} + R_{\mathcal{U}_2}$ to parent;

else

Output $R_{\mathcal{T}} = R_{\mathcal{U}_1} : R_{\mathcal{U}_2}$ to parent;

else

wait;
Fig. 5: Left: An all-input TTSP graph. Right: Parallel joins across \( R \) to \( \mathcal{R} \) used to compute \( \mathbf{Y}_s^* \).

Algorithm 3: Branch Currents over \( \text{AITTSP Graph} \)

**Input:** Decomposition tree \( \mathcal{T}(\mathcal{G}) \)

Effective resistances \( \rho(\mathcal{G}) \) over \( \mathcal{T}(\mathcal{G}) \)

**Result:** Currents \( I(\mathcal{G}) \) over \( \mathcal{T}(\mathcal{G}) \)

for each parent \( j \) of \( \mathcal{T}(\mathcal{G}) \) do

if \( j \) is root then

if \( j \) is a series join then

Output \( I_{\text{out}} = I_k \) to children;

else

Output \( (I_1, I_2) = \arg P_1 \circ P_2 \) to children;

else if received \( I_{\text{in}} \) from parent then

if \( j \) is a series join then

Output \( I_{\text{out}} = I_{\text{in}} \) to children;

else

Output \( (I_1, I_2) = \arg P_1 \circ P_2 \) to children;

else

wait;

Algorithm 4: Voltage Drops over \( \text{AITTSP Graph} \)

**Input:** Decomposition tree \( \mathcal{T}(\mathcal{G}) \)

Effective resistances \( \rho(\mathcal{G}) \) over \( \mathcal{T}(\mathcal{G}) \)

**Result:** Voltage drops \( \mathbf{Y}_s^* \) over \( \mathcal{T}(\mathcal{G}) \)

for each leaf of \( \mathcal{T}(\mathcal{G}) \) do

Output \( \mathbf{V}_e = W^{-1}_e \mathbf{I}_e \) to parent;

for each parent \( j \) of \( \mathcal{T}(\mathcal{G}) \) do

if received \( \mathbf{V}_{ei} \) from both children \( i = 1, 2 \) then

if \( j \) is a series join then

Output \( \mathbf{V}_{\text{out}} = \mathbf{V}_{\text{in}_1} + \mathbf{V}_{\text{in}_2} \) to parent;

else

Output \( \mathbf{V}_{\text{out}} = \mathbf{V}_{\text{in}_2} \) to parent;

else

wait;

Remark 2. Following Theorem 2, the best/worst-case complexity of computing the voltage drops \( y_s^* \) in the update scheme of \( 8 \) is \( \mathcal{O}(|\mathcal{R}| \log |\mathcal{N}|) \), and \( \mathcal{O}(|\mathcal{R}| |\mathcal{N}|) \), respectively.

IV. Example

We perform the gradient descent on the TTSP graph in Figure 6 with 3 leader nodes, identified to a single node depicted as an orange square. The weights are assumed to be elements of \( S^2_{+,*} \), which have 3 independent elements. These independent elements of each weight \( W_{\{i,j\}} \) are represented as a multi-edge, with 3 edges between each node \( i \) and its neighbour \( j \). The edges connecting the leader nodes to the source nodes are identity weights, and the remaining weights and bounds were randomly initialized, with weight penalty of \( \lambda = 0.05 \). The algorithm convergence is shown in Figure 7.

In this setup, each node in \( \mathcal{G} \) computes one of the independent calculations at each layer of the decomposition tree \( \mathcal{T} \). This is possible as each layer has at most \( \log \mathcal{L} \) computations, and \( \mathcal{G} \) has at least \( \log \mathcal{L} + 2 \) nodes, where \( \mathcal{L} \) is the number of leaves in the decomposition tree. Algorithms 2-4 are then executed in this manner for each \( s \in \mathcal{R} \), ultimately outputting the voltage drops \( \mathbf{Y}_s^* \) which are used to compute \( \nabla_W f_{W_{\{i,j\}}} \). Each edge \( \{i,j\} \) is then updated according to \( 8 \).

V. Conclusion

In this paper we discussed how to utilize the compositional aspects of two-terminal series-parallel graphs to efficiently compute performance measures on leader-follower consensus networks. In particular, we have shown that given a decomposition tree of a TTSP graph, one can compute the \( \mathcal{H}_2 \) norm, and a gradient update for network reweighting, in best-case and worst-case complexity of \( \mathcal{O}(k^\omega |\mathcal{R}| \log |\mathcal{N}|) \) and \( \mathcal{O}(k^\omega |\mathcal{R}| |\mathcal{N}|) \), respectively.
The authors thank Jingjing Bu and Airlie Chapman for useful conversations.

REFERENCES

[1] M. Hudoba de Badyn and M. Mesbahi, “Efficient Computation of Performance on Series-Parallel Networks,” in Proc. American Control Conference, Philadelphia, USA, 2019, pp. 3364–3369.

[2] H. Li, C. Feng, H. Ehrbard, Y. Shen, B. Cobos, F. Zhang, K. Elamvazhuthi, S. Berman, M. Haberland, and A. L. Bertozzi, “Decentralized stochastic control of robotic swarm density: Theory, simulation, and experiment,” in Proc. IEEE International Conference on Intelligent Robots and Systems, Vancouver, Canada, 2017, pp. 4341–4347.

[3] T. Vicsak, A. Czirk, E. Ben-Jacob, I. Cohen, and O. Shochet, “Novel type of phase transition in a system of self-driven particles,” New York, vol. 75, no. 6, pp. 1226–1229, 1995.

[4] M. H. Matheny, J. Emerheiser, W. Fon, A. Chapman, M. Rohden, A. Salova, J. Li, M. Hudoba de Badyn, L. Dueñas-Osorio, M. Mesbahi, J. P. Crutchfield, M. C. Cross, R. M. D’Souza, and M. L. Roukes, “Exotic states in a simple network of nanoelectromechanical oscillators,” Science, vol. 363, no. 6431, p. aav7932, 2019.

[5] Y. Chen, J. Lu, X. Yu, and D. J. Hill, “Multi-agent systems with dynamical topologies: Consensus and applications,” IEEE Circuits and Systems Magazine, vol. 13, no. 3, pp. 21–34, 2013.

[6] R. Oliffati-Saber and V. M. Murray, “Consensus protocols for networks of dynamic agents,” Proc. of the American Control Conference, vol. 2, pp. 951–956, 2003.

[7] R. Oliffati-Saber, “Distributed Kalman filter with embedded consensus filters,” in Proceedings of the 44th IEEE Conference on Decision and Control and the European Control Conference, Seville, Spain, 2005, pp. 8179–8184.

[8] M. Hudoba de Badyn, U. Eren, B. Açıkmese, and M. Mesbahi, “Optimal mass transport and kernel density estimation for state-dependent networked dynamic systems,” in Proc. 57th IEEE Conference on Decision and Control, Miami Beach, USA, 2018.

[9] M. Mesbahi and M. Egerstedt, “Graph-Theoretic Methods in Multiagent Networks,” Princeton University Press, 2010.

[10] A. Rahmani, M. Ji, M. Mesbahi, and M. Egerstedt, “Controllability of multi-agent systems from a graph-theoretic perspective,” SIAM Journal on Control and Optimization, vol. 48, no. 1, pp. 162–186, 2009.

[11] M. Hudoba de Badyn, A. Chapman, and M. Mesbahi, “Network entropy: A system-theoretic perspective,” in Proc. 54th IEEE Conference on Decision and Control, Osaka, Japan, 2015, pp. 5512–5517.

[12] M. Siami and N. Motie, “Growing linear consensus networks endowed by spectral systemic performance measures,” IEEE Transaction on Automatic Control, vol. 63, no. 7, pp. 2091 – 2106, 2018.

[13] ———, “Fundamental limits and tradeoffs on disturbance propagation in linear dynamical networks,” IEEE Transaction on Automatic Control, vol. 64, no. 12, pp. 4055–4062, 2019.

[14] A. Chapman and M. Mesbahi, “Semi-autonomous consensus: Network measures and adaptive trees,” IEEE Transactions on Automatic Control, vol. 58, no. 1, pp. 19–31, 2013.

[15] B. Bamiieh, M. R. Jovanović, P. Mitra, and S. Patterson, “Coherence in Large-Scale Networks: Dimension-Dependent Limitations of Local Feedback,” IEEE Transactions on Automatic Control, vol. 57, no. 9, pp. 2315–2329, 2012.

[16] M. H. Trinh, C. Van Nguyen, Y. H. Lim, and H. S. Ahn, “Matrix-weighted consensus and its applications,” Automatica, vol. 89, pp. 415–419, 2018.

[17] S. E. Tuna, “Synchronization under matrix-weighted Laplacian,” Automatica, vol. 73, pp. 76–81, 2016.

[18] E. H. Serres, A. V. Proskurnikov, R. Tempo, and N. E. Friedkin, “Novel multidimensional models of opinion dynamics in social networks,” IEEE Transactions on Automatic Control, vol. 62, no. 5, pp. 2270–2285, 2017.

[19] J. L. Ramirez, M. Pavone, E. Frazzoli, and D. W. Miller, “Controlled operation of spacecraft formation via cyclic pursuit: Theory and experiment,” Proc. American Control Conference, pp. 4811–4817, 2011.

[20] B. H. Lee and H. S. Ahn, “Distributed formation control via whiskering and submodular optimization,” in Proc. 55th IEEE Conference on Decision and Control, Las Vegas, USA, 2016, pp. 867–872.

[21] A. Chapman, M. Nabi-Abdolyousefi, and M. Mesbahi, “Controllability and observability of network-of-networks via Cartesian products,” IEEE Transaction on Automatic Control, vol. 59, no. 10, pp. 2668–2679, 2014.

[22] D. Eppstein, “Parallel Recognition of Series-Parallel Graphs,” Information and Computation, vol. 98, pp. 41–55, 1992.

[23] K. Takamizawa, T. Nishizeki, and N. Saito, “Linear-time computability of combinatorial problems on series-parallel graphs,” Journal of the Association for Computing Machinery, vol. 29, no. 3, pp. 623–641, 1986.

[24] E. Cela, The Quadratic Assignment Problem: Theory and Algorithms. Springer Science & Business Media, 2013, vol. 1.

[25] P. De, E. J. Dunne, J. B. Ghosh, and C. E. Wells, “Complexity of the discrete time-cost tradeoff problem for project networks,” Operations Research, vol. 45, no. 2, pp. 302–306, 1997.

[26] S. E. Elmaghraby, “Resource allocation via dynamic programming in activity networks,” European Journal of Operational Research, vol. 64, no. 2, pp. 199–215, 1993.

[27] F. Le Gall, “Powers of Tensors and Fast Matrix Multiplication,” in Proc. of the 39th International Symposium on Symbolic and Algebraic Computation, Kobe, Japan, 2014, pp. 296–303.

[28] M. Hudoba de Badyn and A. Chapman, “Distributed formation control via cyclic pursuit: Theory and experiment,” in Proc. 55th IEEE Conference on Decision and Control, Las Vegas, USA, 2016, pp. 867–872.

APPENDIX

Proof of Proposition 4
In this setup, $H_2^2$ is given by

$$\frac{1}{2} \sum_{s \in R} \text{Tr} \left[ e_s^T A_s^T e_s \right], \quad A = \sum_{s \in \mathcal{E}} A_s W_s A_s^T.$$ 

We want to find the gradient of the $H_2$ norm with respect to $W_s$ for each $c \in \mathcal{E}$, and we do this by computing the gradients for each $s \in R$ of $\text{Tr} \left[ e_s^T A_s^T e_s \right] = (\alpha \circ \beta \circ \gamma) [W_s]$, where,

$$\alpha(X) = \text{Tr}(e_s^T X e_s), \quad d_{\alpha}(X)[H] = \text{Tr}(e_s^T H e_s),$$

$$\beta(X) = X^{-1}, \quad d_{\beta}(X)[H] = -X^{-1} H X^{-1},$$

$$\gamma(X) = A_s X A_s^T + \sum_{c \in \mathcal{E}} A_s W_s A_s^T, \quad d_{\gamma}(X)[H] = A_s H A_s^T.$$ 

Let $Q = A_s^T A_s^T A_s^T e_s$. Then, the gradient at a point $H$ is

$$d(\alpha \circ \beta \circ \gamma)[H] = -\text{Tr} \left[ Q^T H Q \right] = \langle -Q^T Q, H \rangle,$$

so the gradient at $H$ is $-Q^T Q$. Let $l = \{i, j\}$. Then,

$$Q = A_s^T A_s A_s^T e_s = (e_s^T \otimes I_k - e_s^T \otimes I_k) A_s^T (e_s \otimes I_k).$$

This is precisely $Q = \text{Blk}^{k \times k} \left[ A_s^T A_s^T e_s - \text{Blk}^{k \times k} \left[ A_s^T A_s^T e_s \right] \right] = Y_s^T - Y_s$, or the difference in matrix-valued voltage drops to the grounded leader node from node $i$ to node $j$. Hence, the gradient with respect to weight $W_s$ is given by $\nabla_{W_s} H_2^2[H] = -\sum_{c \in \mathcal{E}} (Y_s - Y_s^T)^T (Y_s - Y_s^T)$, where $A_s$ is computed with $H$ in place of $W_s$.

