TIGHT NEIGHBORHOODS OF CONTACT SUBMANIFOLDS

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Abstract. We prove that any small enough neighborhood of a closed contact submanifold is always tight under a mild assumption on its normal bundle. The non-existence of $C^0$–small positive loops of contactomorphisms in general overtwisted manifolds is shown as a corollary.

1. Introduction

A contact manifold $(M, \xi)$ is an $(2n + 1)$–dimensional manifold equipped with a maximally non–integrable codimension 1 distribution $\xi \subset TM$. If we assume that $\xi$ is coorientable, as will be the case in the article, the hyperplane distribution can be written as the kernel of a global 1–form $\alpha$, $\xi = \ker(\alpha)$, and the maximal non–integrable condition reads as $\alpha \wedge (d\alpha)^n \neq 0$. These conditions imply that $(\xi, d\alpha)$ is a symplectic vector bundle over $M$. However, a contact structure on $M$ cannot be directly recovered from a hyperplane distribution $\xi$ and a symplectic structure $\omega$ on the fibers. The formal data $(\xi, \omega)$ is called formal contact structure.

Let $\text{Cont}(M)$ and $\text{FCont}(M)$ denote the set of contact and formal contact structures, respectively. Gromov proved that if $M$ is open the natural inclusion is a homotopy equivalence. The statement does not readily extend to closed manifolds. In dimension 3, Eliashberg introduced a subclass $\text{Cont}_{OT}(M)$ of $\text{Cont}(M)$, the so–called overtwisted contact structures, and proved that any formal contact homotopy class contains a unique, up to isotopy, overtwisted contact structure. Recently, this result has been extended to arbitrary dimension in [2] so the notion of overtwisted contact structure has been settled in general.

Prior to [2], different proposals for the definition of the overtwisting phenomenon appeared in the literature. The plastikstufe, introduced in [12], resembled the overtwisted disk in the sense that it provides an obstruction to symplectic fillability. The presence of a plastikstufe has been shown to be equivalent to the contact structure being overtwisted (check [3, Theorem 1.1] and [11] for a list of disguises of an overtwisted structure). One of the corollaries obtained in [3] is a stability property for overtwisted structures: if $(M, \ker \alpha)$ is overtwisted then $(M \times \mathbb{D}^2(R), \ker(\alpha + r^2d\theta))$ is also overtwisted provided $R > 0$ is large enough, where $\mathbb{D}^2(R)$ denotes the open 2–disk of radius $R$ and $r^2d\theta$ denotes the standard radial Liouville form in $\mathbb{R}^2$.

1.1. Statements of the results. This paper explores the other end of the previous discussion, can small neighborhoods of contact submanifolds be overtwisted? We provide a negative answer to the question in several instances. The main result presented in the article is the following:

Theorem 1. Let $(M, \ker \alpha)$ be a contact manifold. Then there exists $\varepsilon > 0$ such that $(M \times \mathbb{D}^2(\varepsilon), \ker(\alpha + r^2d\theta))$ is tight.

This theorem was previously obtained by Gironella [9, Corollary H] in the case of 3–manifolds with a completely different approach. An interesting consequence is stated in the next corollary:

Corollary 2. Given any overtwisted contact manifold $(M, \alpha)$, there exists a radius $R_0 \in \mathbb{R}^+\setminus\{0\}$ such that $(M \times \mathbb{D}^2(R), \alpha + r^2d\theta)$ is tight if $R \in (0, R_0)$ and is overtwisted if $R > R_0$.

Note that a similar statement was already proven in [13] but in the case of GPS–overtwisted. Theorem 1 can be extended to arbitrary neighborhoods of codimension 2 contact submanifolds $M$ whose normal bundle has a nowhere vanishing section:

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Theorem 3. Suppose $M$ is a contact submanifold of the contact manifold $(N, \xi)$. Assume that the normal bundle of $M$ has a nowhere vanishing section. Then, there is a neighborhood of $M$ in $N$ that is tight.

The proof of Theorems 1 and 3 is based on Theorem 10. That theorem states that for $m$ large enough $(M \times P^{2m}(\varepsilon), \ker(\alpha + \sum_{i=1}^{m} r_i^2 \partial_{h_i}))$ admits a contact embedding in a closed contact manifold of the same dimension that is Stein fillable, therefore $M \times P^{2m}(\varepsilon)$ is obviously tight. However, Theorems 1 and 3 do not prove such a strong result. Their proof uses [3, Theorem 1.1.(ii)] and some packing lemmas to obtain a contradiction by stabilizing and reducing to Theorem 10.

1.2. Applications.

1.2.1. Remarks about contact submanifolds. We are assuming a choice of contact forms whenever the measure of a radius of the tubular neighborhoods of a contact submanifold is required.

1. Assume that $(M, \xi)$ contact embeds into an overtwisted contact manifold $(N, \xi_{OT})$ as a codimension 2 submanifold with trivial normal bundle. By Theorem 1, it is clear that the overtwisted disk cannot be localized on arbitrary small neighborhoods of $M$, even assuming that $M$ itself is overtwisted. This stands in sharp contrast with [11] and [3] in which it is shown that the overtwisted disk can be localized around a very special kind of codimension $n$ submanifold: a plastikstufe [12].

2. Assume now that $(M, \xi_{OT})$ is overtwisted and contact embeds into a tight contact manifold $(N, \xi)$ as codimension 2 submanifold with trivial normal bundle. Then we can perform a fibered connected sum of $(N, \xi)$ with itself along $(M, \xi_{OT})$. The gluing region is $M \times (-\varepsilon, \varepsilon) \times S^1$, for some $\varepsilon > 0$, and coordinates $(p, t, \theta)$ can be chosen such that the glued contact structure admits an associated contact form $\alpha = \alpha_{OT} + t \, d\theta$.

It is clear that the contact connection associated to the contact fibration $M \times (-\varepsilon, \varepsilon) \times S^1 \to (-\varepsilon, \varepsilon) \times S^1$ induces the identity when we lift by parallel transport the loop $\{0\} \times S^1$. The parallel transport of an overtwisted disk of the fiber induces a plastikstufe, see [14] for more details. By [11], the manifold is overtwisted.

Call $R_M > 0$ the biggest radius for which $M \times \mathbb{D}^2(R_M)$ contact embeds in $N$. The connected sum $N \#_M N$ readily increases the biggest radius to be $R_{N \#_M N} \geq \sqrt{2} R_M$: the annulus has twice the area of the original disk and therefore you can embed a disk of radius $\sqrt{2} R_M$. However we get much more, since we actually obtain $R_{N \#_M N} = \infty$. This is because we can always formally contact embed $M \times \mathbb{R}^2$ into $N \#_M N$. Moreover, we can assume that the embedding restricted to a very small neighborhood $U$ of the fiber $M \times \{0\}$ provides a honest fibered contact embedding into $M \times (-\varepsilon, \varepsilon) \times S^1$. Indeed, applying [2, Corollary 1.4] relative to the domain $U$ we obtain a contact embedding of $M \times \mathbb{R}^2$ thanks to the fact that $N \#_M N$ is overtwisted. This just means that the contact embedding of the tubular neighborhood can be really sophisticated and its explicit construction is far from obvious.

1.2.2. Small loops of contactomorphisms. Theorem 1 allows to extend the result of non-existence of small positive loops of contactomorphisms in overtwisted 3–manifolds contained in [4] to arbitrary dimension. A loop of contactomorphisms or, more generally, a contact isotopy is said to be positive if it moves every point in a direction positively transverse to the contact distribution. The notion of positivity induces for certain manifolds, called orderable, a partial order on the universal cover of the contactomorphism group and it is related with non-squeezing and rigidity in contact geometry, see [6, 8]. As explained in [6], orderability is equivalent to the non-existence of a positive contractible loop of contactomorphisms.

Any contact isotopy is generated by a contact Hamiltonian $H_t : \mathbb{R} \to \mathbb{R}$ that takes only positive values in case the isotopy is positive. The main result of [4] states that if $(M, \ker \alpha)$ is an overtwisted 3–manifold there exists a constant $C(\alpha)$ such that any positive loop of contactomorphisms generated by a Hamiltonian $H : M \times S^1 \to \mathbb{R}^+$ satisfies $\|H\|_{C^0} \geq C(\alpha)$. The result has been recently extended to arbitrary hypertight or Liouville (exact symplectically) fillable
contact manifolds in [1]. As a consequence of Theorem 1, we can eliminate the restriction on the dimension in the overtwisted case:

**Theorem 4.** Let \((M, \ker \alpha)\) be an overtwisted contact manifold. There exists a constant \(C(\alpha)\) such that the norm of a Hamiltonian \(H : M \times S^1 \to \mathbb{R}^+\) that generates a positive loop \(\{ \phi_g \}\) of contactomorphisms on \(M\) satisfies

\[
\|H\|_{C^0} \geq C(\alpha)
\]

The strategy of the proof copies that of [4]. The first step is to prove that \(M \times \mathbb{D}^2(\varepsilon)\) is tight, this is provided by Theorem 1. The second step shows that a small positive loop provides a way to lift a plastikstufe in \(M\) (whose existence is equivalent to overtwistedness as discussed above [11]) to a plastikstufe in \(M \times \mathbb{D}^2(\varepsilon)\). This is exactly Proposition 9 in [4]. This provides a contradiction that forbids the existence of the small positive loop.

It is worth mentioning that the argument forbids the existence of (possibly non–contractible) small positive loops. This is in contrast with [1] and the work in progress by S. Sandon [15] in which they need to add the contractibility hypothesis in order to conclude.

**Remark 5.** The hypothesis in Theorem 4 can be changed by the probably weaker notion of GPS–overtwisted, see [13]. Indeed, assume that the manifold \((M, \xi)\) is GPS–overtwisted. This means that there is an immersed GPS in the manifold. The positive loop produce a GPS in \(M \times \mathbb{D}^2(\varepsilon)\) by parallel transport of the GPS around a closed loop in the base \(D^2(\varepsilon)\). In this case, we need to iterate the process \(k\) times to produce a GPS in \(M \times P^{2k}(\varepsilon)\). Now, Theorem 10 concludes that this manifold embeds into a Stein fillable one providing a contradiction with the main result in [13].

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2. \(M \times P^{2m}(\varepsilon)\) admits a Stein fillable smooth compactification

2.1. Construction of a formal contact embedding \(M \to \partial W\) with trivial normal bundle. Recall that \((\xi, \omega_0)\) defines a symplectic vector bundle over \(M\), thus it is equipped with a complex bundle structure unique up to homotopy. Denote \(\xi^*\) the dual complex vector bundle of \(\xi\). A standard result on the theory of vector bundles guarantees the existence of a complex vector bundle \(\tau \to M\) such that \(\xi^* \oplus \tau \to M\) is trivial, that is, there is an isomorphism of complex vector bundles over \(M\) between \(\xi^* \oplus \tau\) and \(M \times \mathbb{C}^k = \mathbb{C}^k\), where \(k\) is a positive integer large enough.

Denote \(\pi : T^*M \to M\) the cotangent bundle projection and denote \(pr : \pi^*\tau \to T^*M\) the bundle projection. Define \(\tilde{\pi} = \pi \circ pr\). Let us understand \(\widetilde{W} = \pi^*\tau\) as a smooth almost complex manifold. Choosing a \(\xi\)–compatible contact form \(\alpha\), i.e \(\xi = \ker \alpha\), it is clear that

\[
T\widetilde{W} \cong \tilde{\pi}^*\tau \oplus pr^*T(TM) \cong \tilde{\pi}^*\tau \oplus \tilde{\pi}^*TM \cong \tilde{\pi}^*(\xi^* \oplus \langle \alpha \rangle) \oplus \tilde{\pi}^*TM
\]

In particular, the vector bundle \(\pi^*\tau \xrightarrow{\tilde{\pi}} M\) is isomorphic to \(\mathbb{C}^k \oplus \langle \alpha \rangle\). Fix a direct sum bundle metric \(h\) in \(\pi^*\tau\) such that \(h(\alpha, \alpha) = 1\). Now define

\[
W = \{(v, p) \in \widetilde{W} : h(v, v) \leq 1\}.
\]

Given a complex structure \(j\) in \(\xi\) compatible with \(\omega_0\), we can extend it to a complex structure on \(T^*M\) and by a direct sum with a complex structure in \(\tau\) we obtain a complex structure \(J\)
in $T\partial W$. Then, $(W, J)$ is an almost complex manifold with boundary $\partial W$ that has a natural formal contact structure $\xi_0 = T\partial W \cap J(T\partial W)$. Consider the embedding $e_0: M \to \partial W = S(\mathbb{C}^k_1 \oplus \langle \alpha \rangle) : p \mapsto (0, 1)$.

We claim that its normal bundle is trivial because it is equal to $\tilde{\pi}^*\mathbb{C}^k_1$. The reason is that the normal bundle to a section of a vector bundle is the restriction of the vertical bundle to the section. In our case the restriction of the vertical bundle $T(S(\mathbb{C}^k_1) \oplus \langle \alpha \rangle)$ to the image of $e_0$ is clearly $(\tilde{\pi}^*\mathbb{C}^k_1)_{|\text{im}(e_0)}$.

2.2. $W$ is Stein fillable and $\partial W$ is contact. The distribution $T\partial W \cap J(T\partial W)$ is not necessarily a contact structure in $\partial W$. However, we will deform this distribution to a genuine contact structure following the result.

**Theorem 6** (Eliashberg [7]). Let $(V^{2n}, J)$ be an almost complex manifold with boundary of dimension $2n > 4$ and suppose that $f: V \to [0, 1]$ is a Morse function constant on $\partial V$ such that $\text{ind}_p(f) \leq n$ for every $p \in \text{Crit}(f)$. Then, there exists a homotopy of almost complex structures $\{J_t\}_{t=0}^1$ such that $J_0 = J$, $J_1$ is integrable and $f$ is $J_1$-convex.

We are clearly in the hypothesis since our manifold $W$ is almost complex, has dimension $2k + 1 + \dim M > 4$ (because $2k \geq \dim \xi = \dim M - 1$) and deformation retracts to $M$.

From Theorem 6 we obtain a homotopy of almost complex structures $\{J_t\}$ in $W$ such that $J_0 = J$, $J_1$ is integrable. Moreover $(W, J_1)$ is a Stein domain and $\partial W$ inherits a contact structure given by $\xi_1 = J_1(T\partial W) \cap T\partial W$. In fact, there is a homotopy of formal contact structures between $\xi_0$ and $\xi_1$ provided by $\xi_t = J_t(T\partial W) \cap T\partial W$.

2.3. Properties of the embedding $e_0: (M, \xi) \to (\partial W, \xi_1)$.

**Definition 7.** An embedding $e: (M_0, \xi_0, J_0) \to (M_1, \xi_1, J_1)$ is called formal contact if there exists an homotopy of monomorphisms $\{\Psi_t: TM_0 \to TM_1\}_{t=0}^1$ such that $\Psi_0 = de$, $\xi_0 = \Psi_1^{-1}(\xi_1)$ and $\Psi_1: (\xi_0, J_0) \to (\xi_1, J_1)$ is complex.

So far we have produced an embedding $e_0: (M, \xi, J) \to (\partial W, \xi_0, J_0)$ that is formal contact with the constant homotopy equal to $de_0$. Indeed, $de_0^{-1}(\xi_0) = \xi$ and $de_0(\xi)$ is a complex subbundle of $\xi_0$. There is a family of complex isomorphisms $\Phi_t: \xi_0 \to \xi_1$ such that $\Phi_0 = \text{id}$. Fix a Reeb vector field $R$ associated to $\xi$ and define $R_0 = de_0(R)$. Build a family $\{R_t\}$ of vector fields in $T\partial W$ satisfying $R_0|_{\text{im}(e_0)} = R_0$ and $\langle R_t \rangle \perp \xi_1 = T\partial W$. We take a family of metrics $g_t$ in $\partial W$ defined in the following way: its restriction to $\xi_1$ is hermitic for the complex bundle $(\xi_1, J_1)$ and $R_t$ is unitary and orthogonal to $\xi_t$.

Extend $\Phi_t$ to an isomorphism of $T\partial W|_{\text{im}(e_0)}$ in such a way that $\Phi_t(R_0) = R_t$. Define $E_t = \Phi_t \circ de_0: TM \to T\partial W$.

The family $\{E_t\}_{t=0}^1$ is composed of bundle monomorphisms and clearly satisfies that $E_t^{-1}(\xi_t) = \xi$ and $E_1(\xi)$ is a complex subbundle of $\xi_1$. Therefore, $(e_0, E_t)$ is a formal contact embedding.

Define $N_t = E_t(TM)^{g_t}$ that is a bundle over $\text{im}(e_0)$ which is complex by construction. $N_0$ is isomorphic to $\mathbb{C}^k_1$ and therefore all the bundles $N_t$ are trivial complex bundles.

2.4. Obtaining a contact embedding via $h$-principle. The only missing piece to complete the puzzle is to prove that the embedding $e_0$ can be made contact.

Using $h$-principle it is possible to deform $(e_0, E_t)$ to a contact embedding thanks to the following theorem (cf. [5, Theorem 12.3.1]):

**Theorem 8.** Let $(e, E_0)$, $e: (M_0, \xi_0 = \text{ker} \alpha_0) \to (M_1, \xi_1 = \text{ker} \alpha_1)$, be a formal contact embedding between closed contact manifolds such that $\dim M_0 + 2 < \dim M_1$. Then, there exists a family of embeddings $\tilde{e}_t: M_0 \to M_1$ such that:

- $\tilde{e}_0 = e$ and $\tilde{e}_1$ is contact,
- $de_1$ is homotopic to $E_1$ through monomorphisms $G_t: TM_0 \to TM_1$, lifting the embeddings $\tilde{e}_t$, such that $G_t(\xi_0) \subset \xi_1$ and the restrictions $G_t|_{\xi_0}: (\xi_0, d\alpha_0) \to (\xi_1, d\alpha_1)$ are symplectic.
Theorem 8 applied to $(e_0, E_t)$ provides a family of embeddings $\{e_t\}$ in which $e_t: (M, \xi) \rightarrow (\partial W, \xi_1)$ is a contact embedding and a family of monomorphisms $G_t: TM \rightarrow T\partial W$ that lift $e_t$ such that $G_0 = E_1$, $G_1 = de_1$ and $G_t(\xi) \subset \xi_1$ is a complex subbundle.

**Lemma 9.** The normal bundle of $\text{im}(e_1)$ in $(\partial W, \xi_1)$ is trivial.

**Proof.** Recall that $N_1 = E_1(TM)^{\perp g_1} = G_0(TM)^{\perp g_1}$ is a trivial complex vector bundle. Define, for $t \in [1, 2]$, $N_t = G_{t-1}(TM)^{\perp g_1}$. Clearly, $N_2$ is the normal bundle of the contact embedding $e_1$. Since $N_1$ is a trivial vector bundle so is $N_2$.

Denote the $2m$–dimensional polydisk by $P^{2m}(r_1, \ldots, r_m) = \mathbb{D}^2(r_1) \times \cdots \times \mathbb{D}^2(r_m)$ and abbreviate it as $P^{2m}(r)$ when $r_1 = \cdots = r_m = r$. The following result summarizes the work completed in this section and an important consequence (namely, the title of the section): $M \times P^{2m}(\varepsilon)$ admits a smooth compactification into a Stein fillable contact manifold.

**Theorem 10.** Any closed contact manifold $(M, \ker \alpha)$ contact embeds in the boundary of a Stein fillable manifold with trivial normal bundle. Furthermore, there exists $k \geq 1$ such that for any $m \geq k$

\[
\left( M \times P^{2m}(\varepsilon), \ker \left( \alpha + \sum_{i=1}^{m} r_i^2 d\theta_i \right) \right)
\]

is tight with $\varepsilon > 0$ small enough depending only on $\alpha$ and $k$.

**Proof.** The map $e_1$ proves the first part because by Lemma 9 the normal bundle of the contact embedding $e_1: (M, \xi) \rightarrow (\partial W, \xi_1)$ is trivial. Notice that the codimension of the embedding is equal to $2k = \dim \tau$ and by replacing $\tau$ with $\tau' = \tau \oplus \mathbb{C}^{m-k}$ we obtain embeddings of arbitrary codimension $2m \geq 2k$.

Suppose henceforth that $m \geq k$. By an standard neighborhood theorem in contact geometry it follows that there is a contactomorphism between a neighborhood of $\text{im}(e_1)$ in $(\partial W, \xi_1)$ and a neighborhood of $M \times \{0\}$ in $(M \times \mathbb{R}^{2m}, \ker(\alpha + \sum_{i=1}^{k} r_i^2 d\theta_i))$. Therefore, for some $\varepsilon_0 > 0$, the previous contactomorphism provides an embedding from $M \times P^{2m}(\varepsilon_0)$ into $\partial W$.

Finally, since $(\partial W, \xi_1)$ is Stein fillable, it is tight. Thus, any of its open subsets is also tight and the conclusion follows.

3. $M \times \mathbb{D}^2(\varepsilon)$ is tight if $\varepsilon$ is small

The argument leading to Theorem 10 provided no bound on the first positive integer $k$ such that $M \times P^{2k}(\varepsilon, \ldots, \varepsilon)$ is tight. Indeed, $k$ was fixed at the beginning of Section 2, depending on the rank of $\tau \rightarrow M$, the bundle constructed to make the sum $\xi^* \oplus \tau$ trivial.

The insight needed to prove Theorem 1 is supplied by the understanding of overtwisted contact manifolds briefly discussed in the introduction. To be more concrete, the precise statement we will use in this section, extracted from [3], is the following:

**Theorem 11.** Suppose that $(M, \ker \alpha)$ is an overtwisted contact manifold. Then, if $R$ is large enough, $(M \times \mathbb{D}^2(R), \ker(\alpha + r^2 d\theta))$ is also overtwisted.

The idea is to embed $\partial W \times \mathbb{D}^2(R)$ in the boundary $\partial V$ of a Weinstein manifold. Using the embedding constructed in the previous section we obtain then an embedding $M \times \mathbb{D}^2(R) \rightarrow \partial V$ that has trivial normal bundle. This leads to the proof of a statement similar to Theorem 10 in which we replace $(M, \ker \alpha)$ by $(M \times \mathbb{D}^2(R), \ker(\alpha + r^2 d\theta))$. Note that it is key to make sure that $R$ is arbitrarily large.

A Weinstein manifold $(W, \omega, f, Y)$ is a manifold with boundary $W$ equipped with a symplectic structure $\omega$, a Morse function $f: W \rightarrow \mathbb{R}$ and a Liouville vector field $Y$ that is a pseudo-gradient for $f$. Notice that the symplectic form is automatically exact, $\omega = L_Y \omega = d i_Y \omega$, so the boundary of a Weinstein manifold is exact symplectically fillable.

The product of Weinstein manifolds $(W_1, \omega_1, g_1, Y_1)$ and $(W_2, \omega_2, g_2, Y_2)$ can be equipped with a Weinstein structure. Indeed, define $\omega' = \omega_1 + \omega_2$ and $Y' = Y_1 + Y_2$. Clearly, $Y'$ is Liouville
for $\omega'$. Suppose for simplicity that $g_1$ and $g_2$ are strictly positive (a rescaling would make the argument work in general) and define a function on $W = W_1 \times W_2$ by

$$f_q = (g_1^q + g_2^q)^{1/q}$$

for an arbitrary $q > 1$. It is easy to check that $\text{Crit}(f_q) = \text{Crit}(g_1) \times \text{Crit}(g_2)$, the function $f_q$ is Morse and $Y'$ is pseudogradient for $f_q$.

The Stein fillable manifold $W$ supplied by Theorem 10 is naturally equipped with a Weinstein structure $(W = f^{-1}(0, 1), \omega, f, Y)$ that satisfies $\xi_1 = \ker(i_Y \omega|_{\partial W})$. By the preceding discussion, a Weinstein structure in $W \times \mathbb{R}^2$ is given by $\omega + dx \wedge dy$, $X = Y + r \frac{\partial}{\partial r}$ and

$$f_q = \left( f^q + \left( \frac{x^2 + y^2}{(2R)^2} \right)^q \right)^{1/q}$$

The critical points of $f_q$ have the form $(p, 0, 0)$, where $p \in \text{Crit}(f)$.

The Liouville vector field $X$ is transverse to $\partial W \times \mathbb{R}^2$. Our aim is now to embed $\partial W \times \mathbb{D}^2(R)$ into a level set of $f_q$ by following $\phi_q$, the flow of $X$. We can easily show:

**Proposition 12.** For any $\delta > 0$, there exists $q > 1$ large enough and a function $\mu: \partial W \times \mathbb{D}^2(R) \to \mathbb{R}^+$ such that $||\mu||_{C^0} \leq \delta$ and $\phi_\mu: \partial W \times \mathbb{D}^2(R) \to W \times \mathbb{D}^2(R)$ satisfies $\phi_\mu(\partial W \times \mathbb{D}^2(R)) \subset f_q^{-1}(1)$.

**Proof.** For $q \to \infty$, the level set $f_q^{-1}(1)$ gets $C^\infty$-close to the submanifold $\partial W \times \mathbb{D}^2(R)$. Since $X$ is transverse to both of them, the result follows. \qed

![Figure 1. Contact embedding of $\partial W \times \mathbb{D}^2(R)$ into $f_q^{-1}(1)$.](image)

**Lemma 13.** Let $e: H \hookrightarrow M$ be a hypersurface transverse to a nowhere vanishing Liouville vector field $X$ in $(M, \omega)$, the 1-form $e^* i_X \omega$ defines a contact structure on $H$. Moreover, if $\phi_t$ denotes the Liouville flow starting at $H$ and $s: H \to \mathbb{R}$ is a fixed function, then $\phi_s \circ e : H \hookrightarrow M$ is contactomorphic to $e$ provided the flow $\phi_s$ is well-defined.

Notice that the level set $f_q^{-1}(1)$ is the boundary of the Weinstein manifold $V = f_q^{-1}(0, 1)$. Denote $\alpha' = i_X (\alpha + dx \wedge dy)$. A straightforward application of Lemma 13 concludes the following:

**Proposition 14.** For any $R > 0$, the contact manifold $(\partial W \times \mathbb{D}^2(R), \ker(\alpha'|_{\partial W \times \mathbb{D}^2(R)})$ admits a contact embedding into the boundary of a Weinstein manifold.

Combining the last proposition and the results from the previous section we obtain:

**Corollary 15.** Given a contact manifold $(M, \alpha)$ there exists $k \in \mathbb{N}$ and $\varepsilon_0 > 0$ such that for every $R > 0$ the contact manifold $(M \times \mathbb{P}^{2k+2}(\varepsilon_0, \ldots, \varepsilon_0, R), \ker(\alpha + \sum_{i=1}^{k+1} r_i^2 d\theta_i))$ is tight.
Let us emphasize that $\varepsilon_0$ does not depend on $R$: for any $R > 0$, $M \times P^{2k+2}(\varepsilon_0, \ldots, \varepsilon_0, R)$ is tight.

**Proof.** The integer $k$ and the number $\varepsilon_0$ both come from Theorem 10. Denote by $e'$ the contact embedding from $(M \times P^{2k}(\varepsilon_0), \ker(\alpha + \sum_{j=1}^{k} r_j^2 d\theta_j))$ into $(\partial W, \xi_1 = \ker(i_Y \omega))$ and let $\eta$ be the conformal factor of $e'$, $(e')^*i_Y\omega = \exp(\eta)\alpha'$. If necessary, decrease the value of $\varepsilon_0$ to guarantee that sup $\eta$ is finite.

Proposition 14 supplies a Weinstein manifold $(V = f_q^{-1}(0,1], \omega + dx \wedge dy, f_q, X)$ and contact embedding

$$\varphi: (\partial W \times \mathbb{D}^2(\exp(\text{sup } \eta/2)R), \alpha') \hookrightarrow (\partial V, \alpha').$$

Therefore, the map $\tilde{\varphi}: M \times P^{2k+2}(\varepsilon_0, \ldots, \varepsilon_0, R) \rightarrow \partial V$ given by

$$\tilde{\varphi}(p, x, y, x_{k+1}, y_{k+1}) = \varphi(e'(p, x, y), \exp(\eta/2)x_{k+1}, \exp(\eta/2)y_{k+1})$$

is a contact embedding. Since $\partial V$ is exact symplectically fillable the conclusion follows. \qed

We are ready to prove Theorem 1. To ease the notation, we shall understand the contact form is equal to $\alpha + \sum r_i^2 d\theta_i$ in case it is omitted.

Let us proceed by contradiction. Suppose that $M \times \mathbb{D}^2(\varepsilon)$ is overtwisted for $\varepsilon$ smaller than $\varepsilon_0$. Applying Theorem 11 $k$ times consecutively we obtain a radius $R_k > 0$ such that $M \times P^{2k+2}((\varepsilon, R_1, \ldots, R_k)$ is overtwisted. As we will show below, this manifold contact embeds into $M \times P^{2k+2}(\varepsilon_0, \ldots, \varepsilon_0, R)$ provided $R$ is large enough. From Corollary 15 we know that the latter manifold is tight so we reach a contradiction. Therefore, $M \times \mathbb{D}^2(\varepsilon)$ is tight.

The only missing ingredient is the announced contact embedding:

$$(1) \quad M \times P^{2k+2}(\varepsilon, R_1, \ldots, R_k) \rightarrow M \times P^{2k+2}(\varepsilon_0, \ldots, \varepsilon_0, R)$$

Its existence, subject to the conditions $\varepsilon < \varepsilon_0$ and $R$ large enough, is a consequence of the following packing theorem in symplectic geometry proved by Guth [10, Theorem 1].

**Theorem 16.** For every $m \in \mathbb{N}$ there is a constant $C(m) \geq 1$ such that for any pair of ordered $m$–tuples of positive numbers $R_1 \leq \ldots \leq R_m$ and $R'_1 \leq \ldots \leq R'_m$ that satisfy

- $C(m)R_1 \leq R'_1$ and
- $C(m)R_1 \cdots R_k \leq R'_1 \cdots R'_m$,

there is a symplectic embedding

$$P^{2m}(R_1, \ldots, R_m) \hookrightarrow P^{2m}(R'_1, \ldots, R'_m)$$

The symplectic embedding supplied by Theorem 16 is automatically extended to our desired contact embedding (1) thanks to the following lemma:

**Lemma 17.** Let $\Psi: (D_1, d\lambda_1) \rightarrow (D_2, d\lambda_2)$ be an exact symplectic embedding. For any contact manifold $(M, \ker \alpha)$ with a choice of contact form $\alpha$ that makes the associated Reeb flow complete, $\Psi$ induces a (strict) contact embedding

$$(M \times D_1, \alpha + \lambda_1) \rightarrow (M \times D_2, \alpha + \lambda_2).$$

**Proof.** Since $\Psi$ is exact, there exists a smooth function $H : D_1 \rightarrow \mathbb{R}$ such that $dH = \Psi^* \lambda_2 - \lambda_1$. If we denote the Reeb flow in $M$ by $\Phi$,

$$\varphi: (M \times D_1, \alpha + \lambda_1) \rightarrow (M \times D_2, \alpha + \lambda_2), \quad \varphi(p, x) = (\Phi_{-H(x)}(p), \Psi(x))$$

is a contact embedding. \qed
4. Extension to contact submanifolds

The results from the previous sections can be extended to a more general setting: contact submanifolds with arbitrary normal bundle. In the presence of a nowhere vanishing section of the normal bundle we will prove that the contact submanifold has a tight neighborhood. This is the content of Theorem 3.

Let \( \pi: E \rightarrow M \) be a complex vector bundle over a contact manifold equipped with an hermitian metric and a unitary connection \( \nabla \). The associated vertical bundle is denoted by \( V = \ker(d\pi) \). The standard Liouville form in \( \mathbb{R}^{2n} \) is \( U(n) \)-invariant and induces a global 1–form in \( V \) that will be denoted \( \tilde{\lambda} \). This real 1–form can be extended to \( TE \) by the expression \( \lambda = \tilde{\lambda} \circ \pi_V \) after we choose a projection onto the vertical direction \( \pi_V: TE \rightarrow V \). The map \( \pi_V \) is determined by the choice of unitary connection so it is not canonical. The 1–form in \( TE \) associated to the connection \( \nabla \) is \( \tilde{\lambda} = \pi^*\alpha + \lambda \).

Even though \( \tilde{\alpha} \) can be seen as the lift of the contact form \( \alpha \) to \( E \), it is not a globally defined contact form in general. However, it defines a contact form around the zero section in \( E_0 \) of the vector bundle.

**Lemma 18.** \( \tilde{\alpha} \) is a contact form in a neighborhood of \( E_0 \). The restriction \( (E_0, \ker(\tilde{\alpha}|_{E_0})) \) is contactomorphic to \( (M, \xi = \ker(\alpha)) \). Moreover, given any other contact structure \( \ker(\beta) \) that coincides with \( \ker(\tilde{\alpha}) \) in \( E_0 \) and with the same complex structure in the normal bundle, there exist neighborhoods \( U, V \) of \( E_0 \) such that \((U, \ker(\beta|_U))\) and \((V, \ker(\tilde{\alpha}|_V))\) are contactomorphic.

Suppose henceforth that \( \pi \) has a global nowhere vanishing section \( s: M \rightarrow E \). The section \( s \) creates a complex line subbundle \( \pi|_L: L \rightarrow M \). Then, the bundle \( E \) splits as \( E = F \oplus L \) and \( L \) is trivial, i.e. there is an isomorphism \( \phi: L \rightarrow \mathbb{C} \) that sends \( s(p) \) to \( 1_p \in \mathbb{C} \) in the fiber above every point \( p \in M \).

A suitable choice of unitary connection on \( \pi: E \rightarrow M \) ensures that the associated contact form can be written as \( \tilde{\alpha} = \alpha' + \lambda \), where \( \alpha' \) is a contact form in \( F \) and \( \lambda \) is the radial Liouville form in \( \mathbb{R}^2 \).

**Proposition 19.** There exists \( U \), a neighborhood of the zero section \( F_0 \) of \( F \), and \( \varepsilon > 0 \) such that \((U \times \mathbb{D}^2(\varepsilon), \ker(\alpha' + \lambda))\) is tight.

Note that this statement is exactly Theorem 1 except from the fact that \( F \) is not closed. The proof of Proposition 19 follows by embedding \((U, \ker(\alpha'))\) in a closed contact manifold \((\tilde{F}, \ker(\tilde{\alpha}'))\) and then applying Theorem 1 to this manifold to deduce that \((\tilde{F} \times \mathbb{D}^2(\varepsilon), \ker(\tilde{\alpha}' + \lambda))\) is tight if \( \varepsilon > 0 \) is small. This result evidently implies that \((U \times \mathbb{D}^2(\varepsilon), \ker(\alpha' + \lambda))\) is also tight.

The aforementioned embedding is defined by the natural inclusion of \( F \) in the projectivization of \( F \oplus \mathbb{C} \):

\[
F \hookrightarrow Q = \mathbb{P}(F \oplus \mathbb{C})
\]

The complex bundle \( \pi_Q: Q \rightarrow M \) carries a natural formal contact structure \( \xi' = (d\pi_Q)^{-1}(\xi) \) where \( \xi \) is an almost complex structure in \( \mathbb{C}^2 \) obtained as the sum of the pullback of a complex structure in \( \xi \) compatible with \( \alpha' \) and a complex structure on the fibers of \( \pi_Q \). This formal contact structure is genuine (i.e., it is a true contact structure) in a neighborhood \( U \) of \( F_0 \) by Lemma 18. The \( h \)-principle for closed manifolds proved in [2, Theorem 1.1] provides a homotopy from any formal contact structure to a contact structure. Furthermore, the homotopy can be made relative to a closed set in which the formal contact structure is already genuine. Applying this theorem we obtain a contact structure \( \xi' \) on \( Q \) that agrees with \( \ker(\alpha') \) in \( U \).

We can reformulate Proposition 19 in the following way:

**Theorem 20.** Let \( \pi: E \rightarrow M \) be a complex vector bundle over a closed contact manifold \((M, \xi)\). Suppose that \( \pi \) has a global nowhere vanishing section. Then, there exists a neighborhood \( U \) of the zero section of the bundle such that \((U, \xi)\) is tight for any contact structure \( \xi \) extending \( \xi \) and preserving the complex structure of \( E \).

An immediate application of Theorem 20 to the case in which \( M \) is a contact submanifold and \( \pi \) is its normal bundle yields Theorem 3.
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