Monodromy deformation approach to the scaling limit of the Painlevé first equation

Andrei A. Kapaev
St. Petersburg Department of Steklov Mathematical Institute,
Fontanka 27, St. Petersburg, 191011, Russia
E-mail: kapaev@pdmi.ras.ru

Abstract

The isomonodromy deformation equation for a $2 \times 2$ matrix linear ODE with a large parameter can be locally reduced to a (hyper)elliptic equation. To globalize this result, we apply the isomonodromy deformation method and obtain the modulation equations for the asymptotic algebraic curve. The method is applied to the degenerate solution of the Painlevé first equation.

1 Introduction

The classical Painlevé equations are found by P. Painlevé and B. Gambier as the only up to Möbius transformations irreducible second-order ODEs $y_{xx} = R(x,y,y_x)$ with the rational in $y_x$, algebraic in $y$ and analytic in $x$ r.h.s. whose general solutions are free from movable critical points. The last condition is usually referred to as the Painlevé property while the equations which enjoy this property are called the Painlevé-type equations. The term irreducible means that the generic solutions of the Painlevé equations can not be expressed in terms of elementary or classical special functions and hence deserve the name of Painlevé transcendents.

The main tool in the classical works is the $\alpha$-test of Painlevé based on the assertion that the limiting form of the scaled Painlevé-type equation, or the reduced form, also possesses the Painlevé property. E.g., the Painlevé second equation PII,

\[ y_{xx} = 2y^3 + xy - \alpha, \]

scaled in accord with the relations

\[ x = \delta^{-2/3}x_0 + \delta^{1/3}t, \quad y = \delta^{-1/3}v, \quad y_x = \delta^{-2/3}v_t, \quad \alpha = \delta^{-1}\beta_0 + \beta_1, \]

where $t$ is the new independent variable, $v$ is the new dependable, and $x_0$, $\beta_0$, $\beta_1$ are some constant parameters, becomes

\[ v_{tt} = 2v^3 + x_0v - \beta_0 + \delta(tv - \beta_1). \]

At $\delta = 0$, the latter is known to have the Painlevé property since it can be integrated by the use of elliptic functions. If $x_0 = -6$ and $\beta_0 = -4$, the additional scaling changes $t = \delta^{-1/5}\tau$, $v = 1 + \delta^{2/5}z$, $\beta = -\delta^{-1/5}\gamma$, turn equation (3) into $z_{\tau\tau} = 6z^2 + \tau + \gamma + \delta^{2/5}(2z^3 + \tau z)$, which, at $\delta = 0$, coincides with the Painlevé first equation PI.

The scaling reduction of the Painlevé equation to the elliptic equation means that the Painlevé function is described locally by the elliptic function. This fact was justified for equations PI and PII by P. Boutroux and later by P. Doran-Wu and N. Joshi. In our terms, P. Doran-Wu and N. Joshi studied the local
behavior of the function \( v(t) \) \((x_0 = 0)\) with the initial condition \((t, v, v_t) = (t_0, \eta, \eta')\)
where \(t_0, \eta, \eta'\) are such constants that
\[
D := \eta'^2 - \eta^4 + 2\beta_0\eta \neq 3(\beta_0/2)^{1/3}e^{\frac{\pi}{\sqrt{3}}k},
\]
k \(\in\mathbb{Z}\). They proved that, in the region \(|t - t_0| < |\ln \delta|\), in the leading order w.r.t. \(\delta \to 0\), the function \(v(t)\) is uniformly approximated by the elliptic function \(V(t)\),
\[
V'^2 = V^4 - 2\beta_0 V + D.
\]

Any extension of the above result beyond the indicated boundary involves modulation of the elliptic curve, i.e. certain change of \(D\) when \(x_0\) in (8) varies. For the first time, the modulation equations for the elliptic asymptotics as \(|x| \to \infty\)
of the Painlevé first and second functions were found by P. Boutroux \([5]\). Namely, letting in (1) \(y = \sqrt{x}v, t = \frac{2}{\pi}x^{3/2}\) and integrating, he has found the equation
\[
(v')^2 = v^4 + v^2 + D(t),
\]
where \(D(t)\) is a transcendent function. Using integral estimates, P. Boutroux obtained the variation of \(D(t)\) on a period \(\omega_k = \oint_L (V^4 + V^2 + D_k)^{-1/2}dV\) of the local elliptic ansatz, \((v')^2 = V'^4 + V^2 + D_k\), \(D_k = \text{const}\), and has shown that, along the unbounded “line of periods”, there exists \(\lim_{k \to \infty} D_k = D_\infty\).
The latter satisfies the transcendent Boutroux equation
\[
\oint_L \sqrt{V^4 + V^2 + D_\infty}dV = 0. \tag{4}
\]

Nowadays it is clear that (4) is strongly related to the Hamiltonian structure associated to the Painlevé equation and appears as the asymptotic form of the relative Poincaré invariant \([2]\) for PII.

Another approach to the modulation equations is provided by the method of G. Kuzmak \([56]\) who proposed to expand \(v\) in an ascending series in \(\delta\), \(v = V + \delta v_1 + \ldots\) with a periodic leading term \(V\), and to introduce, aside from the “quick” variable \(t\), another, formally independent, “slow” variable \(T = \delta t\) for description of the variation of the period and phase shift in \(V(t, T)\). Eliminating “secular” terms from the linear differential equation for \(v_1\), one finds an integral condition for \(V(t, T)\) which is usually interpreted as the constancy of the Lagrange’s action. This condition is the desired modulation equation for \(D(T)\). Using this way, N. Joshi and M. Kruskal \([23]\) described the modulation of the elliptic asymptotic solutions of P1 and PII w.r.t. \(\varphi = \arg x\) as \(|x| \to \infty\).

Applying the Kuzmak’s method and using the connection formulae of M. Ablowitz and H. Segur \([1, 73]\) for the class of the physically interesting decreasing as \(x \to +\infty\) solutions of PII \((\alpha = 0)\), \(y(x, a) \sim a A(x) \sim \frac{a}{2\sqrt{\pi}}x^{-1/4}e^{-\frac{1}{4}x^{3/2}}\), J. Miles \([10]\) described the asymptotic dependence of the solution on the amplitude \(a\) as \(a \to i\infty\) and \(a \to 1 - 0\).

Using the same ideas, O. Kiselev \([51]\) constructed the imaginary on the real axis solution of PII \((\alpha \to i\infty)\) which is monotonic as \(x \to +\infty\) and oscillates as \(x \to -\infty\). The oscillating tail is described by the modulated elliptic function,
\[
V'^2 = V^4 + x_0V^2 + 2IV + D,
\]
where \(D\) depends on \(x_0\) in such a way that \(\oint \sqrt{V^4 + x_0V^2 + 2IV + D}dV \equiv 2\pi\).

An equivalent way leading to the modulation equations in a differential form is described by J. Whitham \([22]\). This approach utilizes averaging w.r.t. the “fast” variable rather than considers a single period of the elliptic function. This idea was applied by V. Novokshenov \([14, 55]\) and V. Vereshchagin \([71, 80]\) to the asymptotic as \(|x| \to \infty\) description of equations PII, PIII and PIV, and to description of the scaling limits in the classical Painlevé equations. The differential modulation equations for the scaling limits are found in the explicit form \(\partial_T D = f(D, K, E)\), where \(T\) is the “slow” variable, \(D\) is the module, and \(K, E\) are the complete elliptic integrals.
In the most systematic way, the averaging method for the integrable “soliton” equations was developed by H. Flaschka, M. Forest and D. McLaughlin [18] and I. Krichever [54]. Applications of their ideas to equation PI $y_{xx} = 6y^2 + x$ are presented in the papers of I. Krichever [55], S. Novikov [63] and F. Fucito, A. Gamba, M. Martellini and O. Ragnisco [24]. In particular, the differential Whitham equations for PI imply the integral conditions for the asymptotic elliptic curve [55],

$$\text{Im} \int_{a,b} w(\lambda) d\lambda = h_{a,b} = \text{const}, \quad w^2 = 4\lambda^3 + 2x\lambda - g_3,$$

which yield the invariant $g_3$ in the ansatz $y(x) = \wp(x - x_0; -2x, g_3(x))$ as a transcendent function of $x$.

Possibly, the most effective approach to the Painlevé equations is based on their isomonodromy interpretation. R. Fuchs [23] constructed a linear scalar second-order ODE with four Fuchsian singularities and one apparent singular point whose isomonodromy deformation is governed by the Painlevé sixth equation PVI. R. Garnier [26] has found the linear equations associated to the lower Painlevé equations imposing on the equation of R. Fuchs the scaling changes of variables of P. Painlevé [69] for the cascade of the successive reductions of the Painlevé equations to each other. These linear equations also form a cascade of reductions arising in some neighborhoods of the merging singularities and turning points. Comments and some technical details missed in the original paper of R. Garnier can be found in the article of P. Boutroux [5]. L. Schlesinger [72] developed the isomonodromy deformation theory for generic linear matrix equations with Fuchsian singular points, while R. Garnier [28] transformed his scalar equations to the matrix form and discovered the link between the isomonodromy and “isospectral” deformations [27]. In the papers of M. Jimbo, T. Miwa and K. Ueno [35] and H. Flaschka and A. Newell [19] the isomonodromy deformation theory was developed for the equations with non-Fuchsian singularities.

The asymptotics of the Painlevé functions as $x$ tends to one of its fixed singularities were studied by M. Jimbo, A. Its, V. Novokshenov, B. McCoy, Sh. Tang, B. Suleymanov, A. Kitaev, A. Kapaev and others, see [37, 34, 58, 74, 40, 52]. The modulation equations for the elliptic asymptotic solutions as $|x| \to \infty$ were discussed from the isomonodromy deformation viewpoint in [64, 42, 43, 49, 65]. The majority of the mentioned authors used the WKB method as the basic ingredient of the asymptotic investigation of the linear equation.

H. Flaschka and A. Newell [20] observed that the WKB approximation to a solution of the $\lambda$-equation in the Lax pair, $\Psi_\lambda = \delta^{-1}A\Psi$, $d\Psi = U\Psi$, satisfies as $\delta \to 0$ the “isospectral” problem, $A\psi = \mu\psi$, $d\psi = U\psi$. Making use of this fact, K. Takasaki [75] avoided the averaging procedure and proposed the differential modulation equations simply comparing the WKB approximation for the first equation in the Lax pair and the Baker-Akhiezer function for the second equation.

The scaling limits in Painlevé equations find important applications in topological field theories and quantum gravity [11, 61, 6, 29, 70], in the theory of semiclassical orthogonal polynomials and random matrices [22, 62, 59, 71, 57]. In the series of papers [21], A. Fokas, A. Its and A. Kitaev described the continuous limit in the matrix model of quantum gravity as the isomonodromy scaling limit induced by the sequence of the Bäcklund transformations in PIV. Similarly, A. Kapaev and A. Kitaev [48] described the isomonodromy scaling limit from PII to PI, P. Bleher and A. Its [1] computed the asymptotics of the semi-classical orthogonal polynomials with the quartic weight related to the isomonodromy scaling limit in PIV, while
J. Baik, P. Deift and K. Johansson [3] found the distribution law of the increasing subsequence of random permutations related to the isomonodromy scaling limit from PIII to PII. The elliptic asymptotic solutions for the isomonodromy scaling limits in PII and the corresponding modulation equations are found in [4]. All the modulation equations which previously appeared in the isomonodromy deformation context, see [4, 12, 16, 18, 35, 36, 44], have the integral form (1) with the proper integrands \( w(\lambda) \) and the trivial r.h.s., \( h_a = h_b = 0 \).

In Section 2, we recall the basic notions of the isomonodromy deformation theory of the linear systems with rational coefficients, introduce the scaling changes of variables which lead to the “isospectral” problem, and define the WKB approximation. In Section 3, we introduce the spectral curve and find the modulation equations for the (hyper)elliptic asymptotic solutions of the isomonodromy deformation equation, describe the domain of their validity and discuss their solvability. In Section 4, we apply the method to equation PI.

## 2 Isomonodromy deformations and scaling limits

Here, we recall the basic facts of the theory of the matrix equations with rational coefficients following the text books of W. Wasow [81], F. Olver [67] and M. Fedorjuk [16], and of the isomonodromy deformation theory following M. Jimbo, T. Miwa and K. Ueno [35, 36] and H. Flaschka and A. Newell [19].

Consider a first order \( 2 \times 2 \) matrix equation with rational coefficients,

\[
\frac{d\Psi}{d\lambda} = A(\lambda)\Psi, \quad A(\lambda) = \sum_{k=0}^{\infty} \sum_{\nu=1}^{n} A_{\nu,k} (\lambda - a^{(\nu)})^{k+1} - \sum_{k=1}^{\infty} A_{\infty,k} \lambda^{k-1}, \quad \text{Tr} A(\lambda) = 0. \tag{6}
\]

Without loss of generality, \( \lambda = \infty \) is the singular point of the highest Poincaré rank, i.e. \( r_{\infty} \geq r_{\nu}, \nu = 1, \ldots, n \). We call equation (6) generic if the eigenvalues of \( A_{\nu,\infty} \) are distinct for \( r_{\nu} \neq 0 \) and if they are distinct modulo integers for \( r_{\nu} = 0 \). Let the generic equation (6) be diagonal at infinity, i.e. \( A_{\infty,\nu} = a^{(\nu)}_\infty \sigma_3 \) for \( r_{\infty} \geq 1 \), or \( A_{\infty,0} = -\sum_{\nu=1}^{n} A_{\nu,0} = x_0^{(\infty)} \sigma_3 \) for \( r_{\infty} = 0 \), where \( \sigma_3 = \text{diag}(1, -1) \).

The generic equation (6) has the formal solution near the singularity \( a^{(\nu)} \),

\[
\Psi_{\nu}(\lambda) = W^{(\nu)}(\lambda) e^{\theta^{(\nu)}(\lambda)\sigma_3}, \quad \nu = 1, \ldots, n, \infty,
\]

\[
\hat{\Psi}^{(\nu)}(\lambda) = I + \sum_{j=1}^{\infty} x^{(\nu)}_j \xi^j, \quad \theta^{(\nu)}(\lambda) = \sum_{j=1}^{r_{\nu}} x^{(\nu)}_j \xi^{-j} + x^{(\nu)}_0 \ln \xi, \tag{7}
\]

where \( \xi = \lambda - a^{(\nu)} \) for a finite singular point \( a^{(\nu)} \) and \( \xi = 1/\lambda \) otherwise. The coefficients \( x^{(\nu)}_j, x^{(\nu)}_j \) of the formal expansion (7) are determined uniquely by the eigenvector matrix \( W^{(\nu)} \) of \( A_{\nu,\nu} \). The formal solution \( \Psi_{\infty}(\lambda) \) is fixed by the normalization condition \( W^{(\infty)} = I \). The eigenvector matrix \( W^{(\nu)} \) for the finite singularity \( a^{(\nu)} \) is unique modulo permutation and scaling of the eigenvectors. Thus, \( \Psi_{\nu}(\lambda) \) (7) is unique modulo the right constant diagonal and permutation matrix multipliers. In practice, this ambiguity does not make any difficulty, and we assume below that the eigenvector matrices \( W^{(\nu)} \) are chosen in some convenient way.

If \( r_{\nu} = 0 \), the singularity is called Fuchsian. For Fuchsian singular point, the series \( \hat{\Psi}^{(\nu)}(\lambda) \) converges in a disk centered at \( a^{(\nu)} \) [11], and the rational terms in the sum for the diagonal matrix \( \theta^{(\nu)}(\lambda) \) are absent. If \( r_{\nu} \neq 0 \), expansion (7) gives
an asymptotic representation for \( \Psi(\lambda) \) in one of the Stokes sectors near \( \xi = 0 \), i.e. for \( \arg \xi \in (\frac{1}{r}\nu(k - 1) + \varphi + \varepsilon, \frac{1}{r}\nu(k + 1) + \varphi - \varepsilon) \), where \( \varphi = \frac{1}{r\nu}(\text{arg} x^{(\nu)}_{-r\nu} - \frac{\pi}{2}) \), \( k = 1, \ldots, 2r\nu, \varepsilon > 0 \).

A comprehensive description of the fundamental solutions of the non-generic equation can be found in [81].

The ratio \( \tilde{\Psi}^{-1}(\lambda)\Psi(\lambda) \) of any two solutions \( \Psi \) and \( \tilde{\Psi} \) of (6) does not depend on \( \lambda \). Ratios of the fundamental solutions normalized by (7) are usually called the monodromy data. The latter include the monodromy matrices \( M_\nu \) which describe the branching of \( \Psi(\lambda) \) near the singularities, the Stokes matrices \( S^{(k)}_\nu \), \( k = 1, \ldots, 2r\nu \), which describe the Stokes phenomenon near the non-Fuchsian singularities, and the connection matrices \( E^{\nu}_{\rho} \) which are the ratios of the fundamental solutions normalized at distinct singular points. At this stage, we do not need more detailed description of the monodromy data, see [35, 19].

The set of deformation parameters is specified for generic equation (6) in [35]. These include the positions \( a^{(\nu)} \) of the singular points and the coefficients \( x^{(\nu)}_{-j}, j \neq 0 \), for \( \theta^{(\nu)}(\lambda) \) in (7), and form together the vector \( x \) (the extension of this definition to some non-generic equations is given in [17]). Remaining parameters \( x^{(\nu)}_{0} \) are usually called the formal monodromy exponents.

Using the linear transformation of the complex \( \lambda \)-plane, it is always possible to fix two of the deformation parameters, e.g. to move two of the finite singular points \( a^{(\nu)} \) to 0 and 1. Alternatively, if \( r_\infty \geq 2 \), one may put \( x^{(\infty)}_{-r_\infty} = 1, x^{(\infty)}_{1-r_\infty} = 0 \). Thus the number of the actual deformation parameters is equal to

\[
m_c = n + r - 2, \quad r = \sum_{\nu=1,\ldots,n,\infty} r_\nu. \tag{8}\]

Let \( d \) denote the exterior differentiation w.r.t. entries of the vector of deformation parameters \( x \). The monodromy data of (8) do not depend on \( x \) if and only if there exist 1-forms \( \Omega \) and \( \Theta^{(\nu)} \) such that the fundamental solutions above additionally satisfy equations [35]

\[
d\Psi = \Omega \Psi, \quad dW^{(\nu)} = \Theta^{(\nu)}W^{(\nu)}, \quad \nu = 1, \ldots, n, \infty. \tag{9}\]

The compatibility condition of (8), (9),

\[
dA = \frac{\partial \Omega}{\partial \lambda} + [\Omega, A], \quad d\Omega = \Omega \land \Omega, \tag{10}\]

is the completely integrable differential system [35] whose fixed singularities are the planes \( a^{(\nu)} = a^{(\nu)}, \nu \neq \rho, \) and \( x^{(\nu)}_{-r_\nu} = 0, r_\nu \neq 0 \) [35]. The generic system (10) corresponding to \( m_c = 1 \) is equivalent to one of the classical Painlevé equations [36, 19].

An extension of the isomonodromy deformation theory to the non-generic equations is given in [17]. We also mention an exhaustive study [13] of a quite important particular non-generic equation (8) with the matrix \( A(\lambda) \) given by

\[
A(\lambda) = \sum_{\nu=1}^{3} \frac{A_\nu}{\lambda - a^{(\nu)}}, \tag{11}\]

where the residue matrices \( A_\nu, \nu = 1, 2, 3 \), are all nilpotent and satisfy the normalization condition \( A_1 + A_2 + A_3 = -\alpha \sigma_3 \). If \( a^{(1)} = 0, a^{(2)} = 1, a^{(3)} = x, \)

\[
A_1 + A_2 + A_3 = -\alpha \sigma_3 \tag{12}\]





then $\Omega = -\frac{1}{\lambda} dx$, and the Schlesinger system which governs the isomonodromy deformations of (\ref{eq:0}), (\ref{eq:1}) is reduced to the particular case of the PVI equation,

$$
y_{xx} = \frac{1}{2} \left( \frac{1}{x} + \frac{1}{y - 1} + \frac{1}{y - x} \right) y_x^2 - \left( \frac{1}{x} + \frac{1}{y - 1} + \frac{1}{y - x} \right) y_x + \frac{y(y - 1)(y - x)}{2x^2(x - 1)^2} \left( (2\alpha - 1)^2 + x(x - 1) \right),
$$

(\ref{eq:12})

where $y$ is the only zero of the entry $(A(\lambda))_{12}$. Physical applications of (\ref{eq:12}) include the self-dual Bianchi IX model \cite{78, 1, 20} and 2D-topological field theories \cite{12}.

Let us make in (\ref{eq:13}) changes of variables,

$$
\lambda = \delta^\nu \zeta, \quad a^{(\nu)} = \delta^\nu (b^{(\nu)} + \delta c^{(\nu)}), \quad \nu = {\text{const}}, \quad \nu = 1, \ldots, n,
$$

(\ref{eq:13})

$$
A_{\nu, -k} = \delta^{k\nu - 1} B_{\nu, -k}, \quad A_{\nu, -k} = \delta^{-k\nu - 1} B_{\nu, -k}, \quad k = 0, \ldots, r_{\nu},
$$

so that, for generic equation (\ref{eq:11}),

$$
x^{(\nu)}_{-k} = \delta^{k\nu - 1} (t^{(\nu)}_{-k} + \delta r^{(\nu)}_{-k}), \quad x^{(\nu)}_{-k} = \delta^{-k\nu - 1} (t^{(\nu)}_{-k} + \delta r^{(\nu)}_{-k}).
$$

(\ref{eq:14})

The constants $b^{(\nu)}$, $t^{(\nu)}_{-k}$, $k \neq 0$, form the vector $t$ which mark the center of the asymptotic domain as $\delta \to +0$. Inequalities $|c^{(\nu)}|, |r^{(\nu)}_{-k}| < \text{const}$, or, in the vector form, $\|t\| < \text{const}$, give the range of the “local” deformation. The entries of $t$ are called the “slow” variables, while the entries of $\tau$ are usually called the “quick” or “fast” variables. The parameters $t^{(\nu)}_0$ and $\tau^{(\nu)}_0$ form the vectors $\alpha$ and $\beta$.

Remark 1. If $r_{\infty} \neq 0$ and $x^{(\nu)}_{-r_{\infty}}$ is fixed by the use of the proper change of the independent variable $\lambda$ then equation (\ref{eq:13}) yields $\nu = -1/r_{\infty}$. If all the singularities are Fuchsian, i.e., $r_{\nu} = 0$, $\nu = 1, \ldots, n, \infty$, one may put for simplicity $\nu = 0$ which yields the famous autonomous Garnier system \cite{27}. E.g., using in (\ref{eq:12}) the changes $x = t + \delta \tau$, $\alpha = \delta^{-1} \alpha_0 + \alpha_1$, $\delta \to 0$, we obtain the reduced autonomous equation

$$
y_{\tau \tau} = \frac{1}{2} \left( \frac{1}{y} + \frac{1}{y - 1} + \frac{1}{y - \tau} \right) y_{\tau}^2 + 2\alpha_0^2 \frac{y(y - 1)(y - \tau)}{\tau(t - 1)^2},
$$

(\ref{eq:15})

and therefore is solvable in terms of elliptic functions.

Remark 2. Interpreting the change of the formal monodromy exponents $x^{(\nu)}_0$ as the result of the Schlesinger transformation action which preserve all the monodromy data \cite{56, 21}, we arrive at the quantization condition, $x^{(\nu)}_0 = \delta^{-1} t^{(\nu)}_0 + \tau^{(\nu)}_0 = \rho^{(\nu)} + l^{(\nu)}$ with the complex constants $\rho^{(\nu)}$ and integers $l^{(\nu)}$, $\nu = 1, \ldots, n, \infty$. Hence $l^{(\nu)}$ and $\delta^{-1}$ are large and comparable, $\lim_{\delta \to +0} l^{(\nu)} = \text{const}$, while $\tau^{(\nu)}_0$ remain bounded as $\delta \to +0$. Furthermore, since $\delta > 0$ and $l^{(\nu)} \in \mathbb{Z}$,

$$
l^{(\nu)}_0 \in \mathbb{R}.
$$

(\ref{eq:16})

Thus the quantized real part of the vector $\alpha$ of the formal monodromy exponents becomes the vector of discrete deformation parameters with the step $O(\delta)$. Redefining $\delta$, we normalize the vector $\text{Re} \alpha$ which thus contains $n$ free parameters.

Using (\ref{eq:13}) in (\ref{eq:0}), (\ref{eq:1}), we obtain

$$
\frac{d\Psi}{d\zeta} = \delta^{-1} B(\zeta) \Psi, \quad d\Psi = \omega \Psi,
$$

(\ref{eq:17})

$$
B(\zeta) = \sum_{\nu=1}^{n} \sum_{k=0}^{r_{\nu}} \frac{B_{\nu, -k}}{\zeta - b^{(\nu)} - \delta c^{(\nu)} k + 1} - \sum_{k=1}^{r_{\infty}} B_{\infty, -k} \zeta^{k - 1}.
$$
Compatibility of (17) reads

$$dB = [\omega, B] + \delta \frac{\partial \omega}{\partial \zeta}, \quad d\omega = \omega \wedge \omega.$$

The method of constructing the approximate solution of the first of equations (17) is comprehensively described in [81, 16]. Let $R \subset \mathbb{C}$ be a closed simply connected domain satisfying the following conditions: $B(\zeta)$ is holomorphic in $R$, and eigenvalues $\mu_j(\zeta)$ of $B(\zeta)$, $j = 1, 2$, have no zero in $R$. This domain can be obtained by removal of independent of $\delta$ neighborhoods of all the singularities and turning points, i.e. zeros of $\mu_1(\zeta) - \mu_2(\zeta) = 2\mu(\zeta)$, supplemented by certain cuts of the complex $\zeta$-plane. Given $N \in \mathbb{N}$, equation (17) can be diagonalized in $R$ up to the order $O(\delta^N)$. Integrating the leading diagonal part, one finds the WKB approximation, which is holomorphic and invertible in $R$ [16],

$$\Psi_{WKB}(\zeta) = W(\zeta) \left[ I + \sum_{k=1}^{N-1} \delta^k W^{(k)}(\zeta) \right] \exp \left\{ \delta^{-1} \int_{a}^{\zeta} \Lambda(\zeta) d\zeta \right\},$$

$$\Lambda(\zeta) = \mu \sigma_3 - \delta \text{diag}(W^{-1}W_\zeta) + \sum_{k=2}^{N-1} \delta^k \Lambda^{(k)}(\zeta),$$

$$W = (W_1, W_2), \quad BW_j = \mu_j W_j, \quad j = 1, 2, \quad \mu = \mu_1 = -\mu_2. \quad (20)$$

Here the off-diagonal $W^{(k)}(\zeta)$ and diagonal $\Lambda^{(k)}(\zeta)$ matrices are recursively determined by the eigenvector matrix $W(\zeta)$.

The contour $\gamma_j(a, b) \subset R$, $j = 1, 2$, connecting $a$ and $b$ is called the $j$-canonical path if $\text{Re} \int_{a}^{b} \mu_j(\zeta) d\zeta$, $\zeta \in \gamma_j(a, b)$, does not decrease as $\zeta$ runs from $a$ to $b$ along $\gamma_j(a, b)$. A closed simply connected domain $R \subset R$ is called the $j$-canonical domain if there exists such a point $a_1$ that $\forall \delta \in R$ the contour $\gamma(a_1, b) \subset R$ connecting $a_1$ and $b$ is homotopy equivalent to a $j$-canonical path. If $R$ is $j$-canonical for both $j = 1$ and $j = 2$, it is called the canonical domain. In the canonical domain $R$, the WKB approximation (19) approaches the true solution of (17) with the uniform in $\zeta \in R$ accuracy $O(\delta^N)$ [16].

Any small enough neighborhood of a given point $\zeta_0 \in R$ is a canonical domain. This domain can be extended as follows. Let $\gamma_0(a_1, a_2) = \{ \zeta \in R : \text{Im} \int_{a_1}^{\zeta} \mu(\zeta) d\zeta = 0 \}$ be a segment of the anti-Stokes level line which passes through $\zeta_0$ and has endpoints $a_1$ and $a_2$. The numbering of the endpoints is chosen in such a way that the oriented contour $\gamma_0(a_1, a_2)$ is the 1-canonical path (i.e. for $\mu_1 = \mu$) while $\gamma_0(a_2, a_1)$ is the 2-canonical path (i.e. for $\mu_2 = -\mu$). Consider an arbitrary point of the segment, $\zeta_1 \in \gamma_0(a_1, a_2)$, and introduce the contour $\gamma_1 = \{ \zeta \in R : \text{Re} \int_{\zeta_1}^{\zeta} \mu(\zeta) d\zeta = 0 \}$, which is the segment of the Stokes level line. By construction, the union $R = \cup_{\zeta_1 \in \gamma_0(a_1, a_2)} \gamma_1$ of all the contours $\gamma_1$ is the canonical domain.

Remark 3. Near the non-Fuchsian singularity, the canonical domain $R$ can be extended beyond the boundary of $R$ to fill out one of the Stokes sectors [16].

The approximate solution of the system (17) in the canonical domain can be constructed multiplying the WKB approximation (19) in a right diagonal matrix $Q(\tau)(I + \sum_{n=1}^{N} Q_n(\tau)\delta^n + O(\delta^N))$ independent of $\zeta$. Below, we assume that the diagonal matrix is absorbed into the eigenvector matrix $W(\zeta)$, so that the approximate solution of the system (17) is given by (19).
3 Modulation of the spectral curve

In the canonical domain, integration of the system \( (17) \) is reduced in the leading order w.r.t. \( \delta \) to the eigenvalue problem. Equations \( (18) \) at \( \delta = 0 \) with the eigenvalue problem \( (20) \) are basic ingredients of the algebro-geometric integration of the “soliton” equations, see [33, 14, 53, 13]. The “spectral curve” \( \Gamma \) is determined by the characteristic equation

\[
\det(B(\lambda) - \mu I) = 0. \tag{21}
\]

In our \( 2 \times 2 \) traceless case, this is the hyperelliptic curve \( \mu^2 = -\det B(\lambda) \) of genus \( g = n + r - 2 \). The spectral curve of the original system \( (8) \) is clearly not stationary since \( d(\det A) = -d(A^2) = -A \frac{\partial A}{\partial \alpha} + \frac{\partial A}{\partial H} B \), but if scaled via \( (13) \), it varies “slowly”,

\[
d(\det B) = -\delta \left( B \frac{\partial \omega}{\partial \zeta} + \frac{\partial \omega}{\partial \zeta} B \right). \tag{22}
\]

We adopt the term singular for the curve \( \Gamma \) whose topological properties as \( \delta \neq 0 \) differ from that at \( \delta = 0 \), e.g. when the singularities of \( (13) \) coalesce, or turning points merge with each other or with the singular points. Given a parameterization of the curve \( \Gamma \), we define the discriminant set in the total space of the parameters as the set determining the singular curve.

Let \( \Gamma_{as} \) be the limiting \( \delta = 0 \) form of the spectral curve \( \Gamma \) \( (21) \) for generic equation \( (17) \),

\[
\mu_{as}^2 = -\lim_{\delta \to 0} \det B = \frac{P_{2g+2}(\zeta)}{\prod_{\nu=1}^{n}(\zeta - b(\nu))^{2(\nu_r - 1)}}, \tag{23}
\]

where \( P_{2g+2}(\zeta) \) is a polynomial of degree \( 2g + 2 \). Totally, the curve \( (23) \) depends on \( 2g + n + 1 \) complex parameters which form the space \( \mathcal{T} \otimes \mathcal{D} \). Here, \( \mathcal{T} \) is the space of \( g \) complex deformation parameters \( t \) and of \( n \) real normalized deformation parameters \( \text{Re} \alpha \), see Remark 2. Below, we use the notation \( t \) for the combined vector \( (t, \text{Re} \alpha) \). \( \mathcal{D} \) is the space of \( g \) complex invariants and of \( n + 1 \) imaginary parts \( \text{Im} \alpha \). We call the asymptotic spectral curve \( \Gamma_{as} \) \( (23) \) generic if \( 1) b(\nu) \neq b(\rho) \) for \( \nu \neq \rho, 2) \text{det} B_{\nu, r_r} \neq 0 \), and, \( 3) \) all the asymptotic turning points are simple.

Two first conditions hold true for the generic equation \( (17) \) with the parameters lying apart from the fixed singularities of \( (10) \) while the third condition can be violated. However, because the non-generic curves \( \Gamma_{as} \) are localized on a union of algebraic surfaces of co-dimension one, a small variation of the parameters turns the non-generic spectral curve into generic one. In contrast, the generic curve \( \Gamma_{as} \) is stable w.r.t. the small variation of the parameters, and the corresponding curve \( \Gamma \) is not singular at \( \delta = 0 \).

To illustrate the introduced notions, let us consider the spectral curve \( (g = 1) \) for the non-generic equation \( (8), (11) \),

\[
\mu^2 = \frac{\lambda \alpha^2 + H}{\lambda(\lambda - 1)(\lambda - x)}, \quad H = \frac{(x(1) \frac{dy}{dx} - x(y - 1))}{4y(y - 1)(y - x)} - \alpha^2 y. \tag{24}
\]

Letting \( x = t + \delta \tau, \alpha = \delta^{-1} \alpha_0 + \alpha_1, \alpha_0 \neq 0 \), we find that \( D = \delta^2 H \to D_0 \) \( (13) \) as \( \delta \to 0 \). The asymptotic spectral curve for \( (24) \), \( \mu_{as}^2(\lambda) = \frac{\lambda^2(\lambda - \lambda_0)}{\lambda(\lambda - 1)(\lambda - x)}, \lambda_0 = -\frac{2D}{\alpha_0} \), depends on the only deformation parameter \( t \). The set of the formal monodromy exponents includes the parameter \( \alpha_0 \), where \( \text{Re} \alpha_0 = 1 \). The complex module
Corollary 3.2. Let \( I \) be generic at the point \((t_0, D_0) \in \mathcal{T} \otimes \mathcal{D} \). Then there exists an open neighborhood \( U \subset \mathcal{T} \) of the point \( t_0 \) such that, for any closed path \( \ell \in \Gamma_{as} \),

\[
J_{\ell}(t, D) := \text{Re} \int_\ell m_{as}(\zeta) \, d\zeta = h_\ell, \quad \forall t \in U, \quad h_\ell = J_{\ell}(t_0, D_0).
\]

Proof. Let \( \ell \in \Gamma \) be an arbitrary closed contour with the base point \( \zeta_0 \). Consider the solution \( \Psi(\zeta) \) of (17) as a multivalued mapping of \( \mathbb{C} \setminus \{b^{(1)}+\delta^{(1)}, \ldots, b^{(n)}+\delta^{(n)}\} \) into \( SL(2, \mathbb{C}) \) and the function \( \Psi_{WKB}(\zeta) \) as a multivalued mapping of the Riemann surface of the curve \( \Gamma \) into \( SL(2, \mathbb{C}) \).

The projection \( \pi(\ell) \) of the contour \( \ell \) on the complex \( \zeta \)-plane passes through a finite number of the overlapping canonical regions \( \mathcal{R}_k, k = 1, \ldots, s, \mathcal{R}_{s+1} = \mathcal{R}_1 \). In each region \( \mathcal{R}_k \), the function (18) gives a uniform in \( \zeta \) approximation for the exact solutions \( \Psi_k(\zeta) \) of the system (17). The exact solutions for the adjacent canonical regions \( \mathcal{R}_k, \mathcal{R}_{k+1} \) differ from each other in a right multiplier \( G_k \) which is independent of \( \zeta \) and \( t \in \mathcal{T} \), i.e. \( \Psi_{k+1}(\zeta) = \Psi_k(\zeta)G_k \). Thus

\[
m_{\ell}(\Psi_{s+1}(\zeta)) = \Psi_1(\zeta)M_{\ell}G_1 \cdots G_s,
\]

where \( m_{\ell} \) is the operator of the analytic continuation along \( \ell \), and \( M_{\ell} \) is the monodromy matrix for \( \Psi_1(\zeta) \) along \( \ell \). Using for \( \Psi_k(\zeta) \) the WKB approximations, we find

\[
\exp \left\{ \delta^{-1} \int_\ell \Lambda(\zeta) \, d\zeta \right\} = (I + \mathcal{O}(\delta))G(\delta).
\]

Due to continuous dependence of \( \Psi_k(\zeta) \) and of the generic curve (23) on all the parameters, the topology of the curve and of the contour \( \ell \in \Gamma \) preserve as soon as the point \((t, D)\) is close enough to the initial point \((t_0, D_0)\). Therefore the r.h.s. of (24) preserves while the deformation parameters \( t \) vary in a neighborhood of \( t_0 \). Equating the leading orders of the l.h.s. for (26) at the initial point \( t_0 \), and nearby points \( t \), we arrive at (25). \( \square \)

Theorem 3.1 immediately provides us with the following assertions:

Corollary 3.2. Let \( U \subset \mathcal{T} \) be an open domain and let (24) holds true. Let the curve \( \Gamma_{as} \) be generic at the point \( t_1 \in \partial U \). Then there exists an open domain \( W \subset \mathcal{T} \) such that \( U \subset W, t_1 \in W \), and (24) is valid \( \forall t \in W \).

Corollary 3.3. Let \( U, U' \subset \mathcal{T} \) be contiguous open domains and \( \Sigma \subset \{ \partial U \cap \partial U' \} \neq \emptyset \). Let \( J_{\ell}(t, D) = h_\ell \) \( \forall t \in U \), \( J_{\ell}(t, D) = h'_\ell \) \( \forall t \in U' \), and the integral \( J_{\ell}(t, D) \) is continuous across \( \Sigma \). Then \( h_\ell = h'_\ell \).

In accord with the Corollaries 3.2 and 3.3, the equation (25) is valid in a subdomain \( U \subset \mathcal{T} \) bounded by the points where the curve \( \Gamma_{as} \) degenerates. In other words, if \( \mathcal{I} \) is the integral manifold for (24) parameterized by \( t \in \mathcal{T} \), then the boundary \( \partial \mathcal{I} \) is the projection on \( \mathcal{T} \) of the intersection of \( \mathcal{I} \) with the discriminant set of \( \Gamma_{as} \). Beyond this boundary, the equation (25) may be invalid.

Varying the contour \( \ell \) in (25), we obtain the system of equations \( J_{\ell_j} = h_j, j = 1, \ldots, 2g+n+1 \), where the contours \( \ell_j \) form a homology basis of the Riemann surface.
of $\Gamma_{as}$. For instance, taking a small circle $c_\nu$ around $\zeta = b^{(\nu)}$ as the integration path $\ell$, we obtain the equation for entries of Im $\alpha$,

$$\text{Im} \, t_0^{(\nu)} = \frac{h_{c_\nu}}{2\pi} = \text{const}, \quad \nu = 1, \ldots, n, \infty,$$

(27)

which contains the reality condition (10) as a particular case.

To discuss (25) further, it is convenient to impose the conditions (27) from the very beginning and to remove $n+1$ small circles from the “sufficient” set of contours. The remaining contours $\ell_j$, $j = 1, \ldots, 2g$, form a homology basis of the Riemann surface of the algebraic curve $w^2 = P_{2g+2}(\zeta)$.

Remark 4. We refer to the systems of $2g$ real transcendent equations (24) over the homology basis of the curve $w^2 = P_{2g+2}(\zeta)$ as the Krichever’s system. Its particular case with the trivial r.h.s. $h_j = 0$, $j = 1, \ldots, 2g$, is called the Boutroux’ system.

The real dimension $2g$ of $D$ coincides with the number of the real equations in the Krichever’s system (24). The functions $J_{\ell_j}(t,D)$, $j = 1, \ldots, 2g$, are the independent first integrals of the completely integrable Pfaffian system $dJ = 0$,

$$\omega \left( \frac{dD}{dD} \right) = -\Omega \left( \frac{dt}{d\ell} \right),$$

(28)

where $\omega = \{\omega_{ij}, \tilde{\omega}_{ij}\}$ and $\Omega = \{\Omega_{ik}, \tilde{\Omega}_{ik}\}$ are the matrices of the partial derivatives, $\omega_{ij} = \partial J_{\ell_i}/\partial D_j$, $\Omega_{ik} = \partial J_{\ell_i}/\partial t_k$, $i,j = 1, \ldots, g$, $k = 1, \ldots, 2g+n$. Here, the bar means the complex conjugation. The Pfaffian system (28) provides us with the modulation equation in the differential form. If contracted to a real plane Im $t = 0$, (28) yields the system which is equivalent to the modulation equations of Whitham type.

The scaling limits in the classical Painlevé equations are described generically by elliptic equations, $g = 1$. If $g = 1$ and $\theta := \frac{\partial}{\partial \zeta} h_{as}(\zeta,t,D) \, d\zeta$ is a holomorphic 1-form on the Riemann surface of $\Gamma_{as}$, then the classical inequality $\text{Im} \left( \int_\alpha \theta \cdot \bar{\theta} \right) > 0$ provides us with the condition det $\omega \neq 0$. Since the curve $\Gamma_{as}$ is generic, the integral manifold for the Pfaffian system (28) passing through the initial point $(t_0,D_0)$ is well parameterized by $2g+n$ real deformation parameters $t$, $\bar{t}$, and the Krichever’s system (28) determines a differentiable in the real sense complex function $D(t,\bar{t})$. Below, we omit the dependence of $D$ on $\bar{t}$.

In some scaling limit problems, the Painlevé function is fixed by the choice of the monodromy data rather then by initial conditions. The dependence of the unknowns on the additional parameter $\delta$ means that we consider a fiber space $\mathcal{E} = (\mathcal{P}, \pi, B)$ where $\pi$ is the projection of the total space $\mathcal{P}$ on the base space $B$ of the scaling parameter $\delta$. Each fiber $\mathcal{P}(\delta)$ is the functional space of the Painlevé functions which is isomorphic to the monodromy surface $\mathcal{M}$ for the associated linear system. The function $y(.,\delta)$ appears as a section of the fiber space $\mathcal{E}$. For instance, the sections of the fiber space $\mathcal{E}$ for the isomonodromy scaling limits yield a trivial, “horizontal” foliation. In this respect, it is quite important to relate the quantities $h_j$ in (28) with the monodromy data of the associated linear system.

Let us introduce the following notions [10]. The Stokes line is a Stokes level line emanating from a turning point. The Stokes graph is the union of all the Stokes lines. The Stokes complex is a connected component of the Stokes graph. The Stokes chain is a broken line whose links are the Stokes lines with the common turning points. We associate the singular point to a Stokes complex if there is a
Stokes chain which approaches or encircles this point. The singular point can be associated to more than one Stokes complex. If the Stokes complex approaches or encircles the only singular point then this singularity is non-Fuchsian and there exist at least two Stokes lines of the Stokes complex approaching this point.

In our next assertion, we assume the generosity of the monodromy data. The assumption is specified in [42, 44, 49, 64, 43] for particular asymptotic problems in equations PII and PIV as the nontriviality of some of the Stokes multipliers and their combinations. Below, the assumption means that the ratios of the fundamental solutions normalized at the singular points associated to more than one Stokes complex are neither diagonal nor off-diagonal. It is clear that a small admissible variation of the monodromy data, i.e. the variation preserving the monodromy surface $\mathcal{M}$, turn the non-generic monodromy data into generic ones while generic monodromy data are stable w.r.t. the small admissible variation.

**Theorem 3.4** Let the parameters in generic equation (12) lie apart from the fixed singularities of (10). Let the corresponding monodromy data be generic and do not depend on $\delta$ as $\delta \to +0$. Then, for any closed path $\ell \in \Gamma_{as}$, the asymptotic curve satisfies the condition

$$J_\ell(t, D) := \text{Re} \int_\ell \mu_{as}(\zeta) d\zeta = 0. \quad (29)$$

**Proof.** We give here a sketch proof. Assume for a moment that for a small circle $\gamma_\ell$ around $b^{(v)}$ taken as the integration path $\ell$ in (28) the equation (10) is violated. Then the formula $M_{\nu}S^{(1)}_b \ldots S^{(2\nu)}_b = e^{2\pi i \sigma_3/\nu}$, which relates the exponential dependence of the product on $\delta$, shows the exponential dependence of the product on $\delta^{-1}$ which is a contradiction.

Let $J_\ell = h_\ell \neq 0$ for a contour $\ell$ which encircles two turning points. Thus the Stokes graph has at least two connected components $S_j$, $j = 1, 2$. Without loss of generality, the domain $\mathcal{R}$ separating $S_j$, $j = 1, 2$, does not contain neither Stokes lines nor singular points, and its boundary consists of two Stokes chains $\partial \mathcal{R}_j \subset S_j$, $j = 1, 2$. Choose the turning points $a_1 \in \partial \mathcal{R}_1$ and $a_2 \in \partial \mathcal{R}_2$. By construction, $2\text{Re} \int_{a_1}^{a_2} \mu_{as}(\zeta) d\zeta = h_\ell \neq 0$.

First compute the ratio of two fundamental solutions normalized by (2) at the singular points $b^{(j)}$ associated to the Stokes complexes $S_1$ and $S_2$. With this aim, choose a Stokes chain from a neighborhood of $b^{(j)}$ to $a_j$, $j = 1, 2$. Since the subdomain of $\mathcal{R}$ bounded by the Stokes level lines is a canonical domain for the WKB approximation (19) (4), we find the ratio as the product

$$G_{12} = e^{F_1 \sigma_3} E_1 \exp \left\{ \delta^{-1} \int_{a_1}^{a_2} \mu_{as}(\zeta) d\zeta \sigma_3 \right\} E_2^{-1} e^{-F_2 \sigma_3},$$

where $F_j = \left\{ \delta^{-1} \int_{c_j}^{\zeta} \mu d\zeta - \theta^{(j)}(\lambda) \right\}_{\lambda \to b^{(j)}}$. Modulo the terms which may behave like a power of $\delta^{-1}$, the matrices $E_j$, $j = 1, 2$, can be represented by the products of the monodromy data of the reduced linear equations approximating (17) near the turning points and of the exponents of the phase integrals $\exp \left\{ \delta^{-1} \int_{c_j}^{\zeta} \mu d\zeta \sigma_3 \right\}$ between the successive turning points of the Stokes chain, see [34, 43]. Therefore, if neither of the matrices $E_j$ is diagonal nor anti-diagonal, then, in the leading order, $G_{12}$ exponentially depends on $\delta^{-1}$ which is a contradiction. If both or one of $E_j$
is diagonal or anti-diagonal then \( \text{Re} F_j \neq 0 \) for one of \( j = 1, 2 \). In the latter case, compute the ratio of two fundamental solutions normalized at the singular points of the same Stokes complex \( S_j \),

\[
G_j = \frac{F_j \sigma_3 N e^{-F_k \sigma_3}}{e^{F_j \sigma_3 N e^{-F_k \sigma_3}}},
\]

where \( N \) may behave like a power of \( \delta^{-1} \). Due to assumption on \( G_j \), the matrix \( N \) is neither diagonal nor anti-diagonal, thus, in the leading order, \( G_j \) exponentially depends on \( \delta^{-1} \) that is a contradiction. \( \Box \)

**Remark 5.** Equation (28) implies that the asymptotic as \( \delta \to 0 \) Stokes graph for (17) is connected.

**Remark 6.** The assumption of Theorem 3.4 can be relaxed to the condition of boundedness or slow dependence of the monodromy data on \( \delta^{-1} \).

Absence of the initial condition for the Pfaffian system in the assumptions of Theorem 3.4 does not prevent the solvability of (29) for \( D \). Indeed, the degeneration of the spectral curve due to merging of the turning points encircled by the contour \( \ell \) allows us to find a point \((t_0, D_0)\) on the discriminant set which could serve as the initial point for (28). For equation PII, this idea was used in [44] to prove the existence of the differentiable in the real sense function \( D(t) \) satisfying (29). There is no principal obstacle to extend the proof for other classical Painlevé equations. The preliminary considerations for PIII and PIV which include the description of the corresponding discriminant sets can be found in [47, 46].

To illustrate the said above, consider again equation (17), (11) and its spectral curve (24). Assuming (29), we describe its Stokes graph as the union of three finite Stokes lines emanating from \( \lambda_0 \) and terminating at the simple poles \( 0, 1, t_0 \). The discriminant set, which is the union of the planes \( t = 0, t = 1, t = \lambda_0, \lambda_0 = 0, \lambda_0 = 1 \), intersects with the integral manifold for (29) along the real axis \( \text{Im} t = 0 \), moreover the values \( t < 0, 0 < t < 1 \) and \( t > 1 \) correspond to the planes \( \lambda_0 = 0, t = \lambda_0 \) and \( \lambda_0 = 1 \), respectively. Thus if the initial data for (12) at \( t = t_0, \text{Im} t_0 < 0 \) satisfy the conditions \( \lambda_0 \neq 0, 1, t_0 \), then the asymptotic as \( \delta \to 0 \) Painlevé function is elliptic in the lower half of the complex \( t \)-plane and degenerates on the real line \( \text{Im} t = 0 \). Whether the asymptotics is elliptic or remains degenerate above the real line depends on the chosen initial data, or, equivalently, on the corresponding monodromy data.

### 4 Scaling limits in PI

In this section, we present the monodromy deformation approach to the scaling limits in the Painlevé first equation PI,

\[
y_{xx} = 6y^2 + x. \tag{30}
\]

Equation PI governs the isomonodromy deformations of the linear system (3) with

\[
A(\lambda) = (4\lambda^4 + x + 2y^2) \sigma_3 - i(4y\lambda^2 + x + 2y^2) \sigma_2 - (2y_x \lambda + \frac{1}{2\lambda}) \sigma_1. \tag{31}
\]

Here \( \sigma_3 = (1 0 \quad 0 1), \sigma_2 = (0 -i \quad i 0), \sigma_1 = (0 1 \quad 1 0) \). The set of the monodromy data for equation (31) consists of the Stokes matrices \( S_k = \Psi_{k+1}^{-1}(\lambda)\Psi_{k+1}(\lambda) \) where \( \Psi_k(\lambda) \sim \exp[\left(\frac{4}{5} \lambda^5 + x\lambda\right)\sigma_3] \) as \( |\lambda| \to \infty \), \( \text{arg} \lambda \in (\frac{\pi}{5}(k - \frac{3}{2}), \frac{\pi}{5}(k + \frac{1}{2})) \). The Stokes matrices
satisfy the cyclic relation \( S_1S_2S_3S_4S_5 = i\sigma_1 \), thus the Painlevé function set is parameterized by the points of the 2-dimensional complex monodromy surface:

\[
\begin{align*}
\text{if } 1 + s_2s_3 \neq 0 & \text{ then } s_1 = \frac{i - s_3}{1 + s_2s_3}, \quad s_4 = \frac{i - s_2}{1 + s_2s_3}, \quad s_5 = i(1 + s_2s_3), \\
\text{if } 1 + s_2s_3 = 0 & \text{ then } s_2 = s_3 = i, \quad s_5 = 0, \quad s_1 + s_4 = i.
\end{align*}
\] (32)

Physically interesting solutions of equation (30) have the non-oscillating asymptotic behavior \( y \sim \pm \sqrt{-x/6} + O(x^{-2}) \) as \( x \to -\infty \). Their existence has been proved by Ph. Holmes and D. Spence [30]. The solution with the asymptotics \( y \sim \sqrt{-x/6} + O(x^{-2}) \) is unique and corresponds to the values \( s_2 = s_3 = 0, \ s_1 = s_4 = s_5 = i \), while the solutions with the asymptotics \( y \sim \sqrt{-x/6} + O(x^{-2}) \) form a 1-parametric family distinguished by the condition \( 1 + s_2s_3 = 0 \) [41]. In more details,

\[
y(x) \sim y_-(x) + a(-x)^{-1/8} e^{-\frac{i}{2}(3/2)^{1/4}(-x)^{1/4}}, \quad x \to -\infty,
\] (33)

where \( y_-(x) = \sqrt{-x/6} + O(x^{-2}) \) does not contain any free parameter, and

\[
a = -\frac{(2/3)^{1/8}}{2\sqrt{2\pi}} \frac{s_1 - s_4}{2}, \quad s_1 + s_4 = i.
\] (34)

On the complex \( x \)-plane, the Painlevé function for \( s_2 = s_3 = i, \ s_5 = 0, \ s_1 + s_4 = i \) has the asymptotics (33) as \( |x| \to \infty, \arg x \in \left( \frac{\pi}{4}, \frac{3\pi}{4} \right) \) with a piece-wise constant amplitude \( a \) which shows a jump across the negative real axis [11] [76]; on the rays \( \arg x = \pm \frac{\pi}{4}, \frac{3\pi}{4} \), the asymptotics is trigonometric; in the interior of the sectors between the indicated rays, the asymptotics is elliptic, see [11] [10].

The scaling change of variables

\[
x = \delta^{-4/5}(t_0 + \delta t), \quad y = \delta^{-2/5}v, \quad y_x = \delta^{-3/5}v_x,
\] (35)

transforms (30) into

\[
v_{tt} = 6v^2 + t_0 + \delta t.
\] (36)

The reduced form of (30), i.e. equation (36) at \( \delta = 0 \), is easy to integrate,

\[
v_t^2 = 4v^3 - g_2v - g_3, \quad g_2 = -2t_0, \quad g_3 = \text{const}.
\] (37)

Thus, at \( \delta = 0 \), \( v(t) = \varphi(t - \varphi; g_2, g_3) \) is the elliptic function of Weierstraß. Let \( v_0 = \sqrt{-t_0/6} \). For the particular value \( g_3 = -8v_0^3 \), the elliptic function degenerates,

\[
v(t) = v_0 + \frac{3v_0}{\sinh^2\left(\sqrt{3v_0}(t - \varphi)\right)}.
\] (38)

Below, we find the asymptotics of the degenerate Painlevé function, \( s_2 = s_3 = i, \ s_5 = 0, \ s_1 + s_4 = i \), when \( |s_4| \) is large. Also, we find the asymptotics of the “almost degenerate” Painlevé functions when \( s_5 \neq 0 \) is small. In order to solve these problems, let us introduce the elliptic curve

\[
w^2 = (z - z_1)(z - z_3)(z - z_5) = z^3 + \frac{t_0}{2}z + \frac{D_0}{4}.
\] (39)

The Riemann surface \( \Gamma_{as} \) for (39) consists of two complex \( z \)-planes pasted along the cuts \([-\infty, z_5]\) and \([z_1, z_3]\). Let \( \arg w(z) \to 0 \) as \( z \to +\infty \) on the upper sheet.
Let the $a$-cycle encircles counter-clockwise the cut $[z_1, z_3]$, and the upper half of the $b$-cycle connects $z_5$ and $z_3$. Introduce the integrals over $a$- and $b$-cycles

$$\omega_a = \frac{1}{2} \oint_a dz, \quad \omega_b = \frac{1}{2} \oint_b dz, \quad \tau = \frac{\omega_b}{\omega_a}$$

$$I_\ell = \oint_\ell w(z) dz, \quad \Omega_\ell = \oint w(z) dz, \quad \Omega_\ell^{(j)} = \Omega_\ell - 2z_j \omega_\ell, \quad \ell \in \{a, b\}. \quad (40)$$

Introduce the system of the Krichever’s equations for the function $D_0(t_0)$,

$$2 \Re I_a = 0, \quad 2 \Re I_b = h_b, \quad h_b = -\frac{8}{5}(3/2)^{1/4}. \quad (41)$$

Due to triviality of $h_a$, the system (11) admits the degeneration of the curve (22), $z_1 = z_3 = (-t_0/6)^{1/2}, D_0 = 8(-t_0/6)^{5/2}$. Thus the intersection of the discriminant set with the integral manifold of (11) satisfies equation $\Re (-t_0)^{5/4} = 1$. Among five lines satisfying the latter equation, we denote by symbol $\gamma$ the line passing through the point $t_0 = -1$ and asymptotic to the rays $\arg t_0 = \pm \frac{3\pi}{5}$.

Finally, let us introduce the perforated region $R(c_0, \epsilon_1)$,

$$R(c_0, \epsilon_1) = \{ t \in \mathbb{C} : |t| < c_0, |t - n\omega_a - m\omega_b| > \epsilon_1 > 0, n, m \in \mathbb{Z} \}. \quad (42)$$

**Theorem 4.1** Let $|s_4| > 1$ and $0 < c' \leq |\frac{s_4}{s_1^2}| \leq c''$ for some constants $c', c''$. Then there exist the positive constant $c_0$ and the positive small constants $\epsilon, \epsilon_1$ such that, to the right of $\gamma$, the Painlevé function is given by

$$y(x) = \delta^{-2/5} \varphi(t; \varphi; -2t_0, -D_0), \quad t = \delta^{-1/5} x - \delta^{-1} t_0, \quad (43)$$

where $D_0(t_0)$ is determined by (44) and equations

$$\delta^{-1} = -\frac{\ln |s_4|}{h_b}, \quad (44)$$

$$t - \varphi = \delta^{-1/5} t_0 + t + \frac{\omega_a}{2\pi i} \ln(is_4) - \frac{\omega_b}{2\pi i} \ln \frac{s_1}{s_4} + O(\delta^{-1-\epsilon}) \in R(c_0, \epsilon_1). \quad (45)$$

**Proof.** It follows from the integral estimates similar to presented in [3, 11, 8] that there exists the solution $y(x) = \delta^{-2/5} \varphi(t; \varphi; -2t_0, -D_0), |t| < \text{const.}$, of the Cauchy problem for equation PI with the initial data $x_0 = \delta^{-4/5} t_0, y_0 = \delta^{-2/5} v_0, y'_0 = \delta^{-3/5} v'_0$, where $|\delta| \ll 1$ and $D_0 = \epsilon_0^2 - 4v_0^3 - 2t_0 v_0$. Using (45) and (48) with $\kappa = -1/5$ (see Remark 1) in [3, 11], we obtain equation (17), $\Psi_\zeta \Psi^{-1} = \delta^{-1} B$, where

$$B(\zeta) = (4\zeta^4 + t_0 + 4v^2 + \delta t)s_3 - i(4v \zeta^2 + t_0 + 2v^2 + \delta t)s_2 - (2v \zeta + \frac{\delta}{2\zeta})s_1. \quad (46)$$

The corresponding spectral curve is given by

$$\mu^2 = \mu_{as}^2 + \delta r(\zeta), \quad \mu_{as}^2 = 4\zeta^2(4\zeta^6 + 2t_0 \zeta^2 + D_0) = 16\zeta^2 w^2(\zeta^2), \quad (47)$$

$$D := v_t^2 - 4v^3 - 2t_0 v + D_0 = \delta D_1, \quad (48)$$

$$r(\zeta) = 4\zeta^2 D_1 + 8t \zeta^2 (\zeta^2 - v) + 2v + \frac{\delta}{4\zeta^2}. \quad (49)$$
Theorem [3.1] implies that the system (24), 2Re $I_a = h_a$, 2Re $I_b = h_b$, holds true, and the leading order $D_0(t_0)$ of the parameter $D$ does not depend on $t$. Thus the definition of $D$ (48) is consistent with the elliptic asymptotic ansatz (37):

$$e^2 = 4v^3 + 2t_0 v + D_0 = 4w^2(v), \quad v(t) = \varphi(t - \varphi; -2t_0, -D_0).$$  \hspace{1cm} (49)

To find the phase shift $\varphi$, we apply the isomonodromy deformation method [43]. Let us introduce notations $a_3 = (B)_{11}$, $a_+ = (B)_{12}$, $a_- = (B)_{21}$ for the entries of the matrix $B$ (10) and assume the following: 1) $|v|, |v_\ell| \leq c$ for some constant $c > 0$, and 2) zeros of $a_+(\zeta)$, or, alternatively, of $a_-(\zeta)$, lie apart from the paths of integration below. Let us assume that the parameters $t_0$, $D_0$ determining the elliptic curve (34) and (44) are such that the asymptotic at $\delta = 0$ Stokes lines for the system (37), (44), emanating from $\zeta_1 = z_1^{1/2}$ and $\zeta_3 = z_3^{1/2}$ form the continuous chains $e^{-i\pi}\infty \rightarrow \zeta_1 \rightarrow e^{-i\pi}\infty$, $e^{-i\pi}\infty \rightarrow \zeta_3 \rightarrow e^{i\pi}\infty$ and $e^{i\pi}\infty \rightarrow \zeta_3 \rightarrow e^{i\pi}\infty$. Then the result of [45] implies the expressions for the Stokes multipliers as $\text{Re}(\delta^{-1}) \rightarrow +\infty, |\text{Im}(\delta^{-1})| < \text{const},$

$$\ln s_4 = 2\delta^{-1} \left[\int_{\zeta^{(c)}}^{\zeta^{(n)}} \nu_\sigma d\zeta - \theta + \frac{\delta}{2} \ln \frac{a_\sigma}{\mu} \right]_{\zeta \rightarrow e^{-i\pi}\infty} - \sigma \ln(-2i) + O(\delta),$$
$$\ln s_1 = 2\delta^{-1} \left[\int_{\zeta^{(c)}}^{\zeta^{(n)}} \nu_\sigma d\zeta - \theta + \frac{\delta}{2} \ln \frac{a_\sigma}{\mu} \right]_{\zeta \rightarrow e^{i\pi}\infty} - \sigma \ln(-2i) + O(\delta),$$  \hspace{1cm} (50)

where $\zeta^{(c)}$ are zeros of $\nu_\sigma(\zeta)$, $\left|\frac{\nu_\sigma}{\nu_{\sigma_0}}\right|_{\delta=0} = z_k^{1/2}$, $k = 1, 3,$

$$\nu^2 = \mu^{2}_{\sigma_0} + \delta \tau + \sigma \delta \left( a_{3} - a_{3}^\prime \right) a_{3}^\prime, \quad \theta = \frac{4}{5} \epsilon^5 + (t_0 + \delta t) \zeta, \quad \sigma \in \{+, -\}. \hspace{1cm} (51)$$

Since the Stokes multipliers do not depend on $t$, and, at the critical point $t = t_5$, $v(t) = 0$, $v = z_5$, the assumptions above are not violated for the non-degenerate curve (59), we may compute the r.h.s. of (39) at this point. Using the change $\zeta^2 = z$, we transform the elliptic integrals in (39) as follows (look for similar details in [42, 44]):

$$\ln(-is_4) = -2\delta^{-1}I_b - \frac{t_5}{2} \Omega(b)^{(5)} - \frac{D_1(t_5)}{2} \omega_b + O(\delta^{\frac{5}{2} - \epsilon}),$$
$$\ln \frac{s_1}{s_4} = -2\delta^{-1}I_a - \frac{t_5}{2} \Omega(a)^{(5)} - \frac{D_1(t_5)}{2} \omega_a + O(\delta^{\frac{5}{2} - \epsilon}), \hspace{1cm} (52)$$

where $\epsilon > 0$ is small. The conditions $\ln|s_4| \gg 1$ and $\epsilon' \leq \left|\frac{\delta}{s_4}\right| \leq \epsilon''$ are consistent with (52) if $h_a = 0$ and $h_b < 0$. Substitutions $\delta \mapsto \alpha^{5/4} \delta$, $t_0 \mapsto \alpha t_0$, $t \mapsto \alpha^{-1/4} t$, $v \mapsto \alpha^{5/4} v$, $v \mapsto \alpha^{3/4} v_\ell$, which leave $x$, $y$ and $y_\ell$ invariant, allow us to demand for $h_a$ the value in (44). The conditions on the Stokes graph imposed above are consistent with the reality condition, $D_0(t_0) \in \mathbb{R}$ for $t_0 > -1$. The latter implies that the chosen branch of $D_0(t_0)$ yields degeneration of the curve (34) on $\gamma$, and the validity domain of the elliptic asymptotic solution is located to the right of $\gamma$. Using in (52) the identities

$$\omega_a I_b - \omega_b I_a = \frac{4\pi i}{5} t_0, \quad \omega_a \Omega_b - \omega_b \Omega_a = 4\pi i, \hspace{1cm} (53)$$

which follow from the Legendre identity, we find the critical point $t_5$,

$$t_5 = -\delta^{-1} \frac{4}{5} t_0 - \frac{\omega_a}{2\pi \iota} \ln(-is_4) + \frac{\omega_b}{2\pi \iota} \ln \frac{s_1}{s_4} + O(\delta^{\frac{5}{2} - \epsilon}),$$
and the expression for the argument of the \( \varphi \)-function of Weierstrass \([43]\). \( \square \)

**Remark 7.** Because \( \omega_{\alpha} \) and \( \omega_{\beta} \) are periods of the elliptic function, the asymptotics \([43]\) is invariant w.r.t. the choice of the argument of the Stokes multipliers.

**Theorem 4.2** Let \( 0 < |s_5| \ll 1, 0 < c' \leq |s_2| \leq c'' \) for some constants \( c', c'' \). Then there exist positive constant \( c_0 \) and positive small constants \( \epsilon, \epsilon_1 \) such that, to the left of \( \gamma \), the Painlevé function is given by \([44]\) where \( D_0(t_0) \) is determined by \([44]\) and

\[
\delta^{-1} = \frac{\ln |s_5|}{h_b},
\]

\[
t - \varphi = \delta^{-1} \frac{4}{5} t_0 + t - \frac{\omega_a}{2\pi i} \ln(is_5) + \frac{\omega_b}{2\pi i} \ln(is_2) + O(\delta^{2-\epsilon}) \in \mathcal{R}(c_0, \epsilon_1). (55)
\]

**Proof.** In essential, we repeat the proof of Theorem 4.1. Existence of the elliptic asymptotic solution of the Cauchy problem follows from the integral estimates similar to presented in \([3, 4, 8, 81]\). Using \([27, 28] \) in \([8, 81]\), we obtain equation \([17]\), \( \Psi \zeta \Psi^{-1} = \delta^{-1} B \), with the matrix \( B \) \([46]\) whose spectral curve is given by \([17]\). Applying Theorem 3.1, we arrive at the system \([25]\) for the function \( D_0(t_0) \), \( 2 \Re I_a = h_a, 2 \Re I_b = h_b \). The definition of \( D_0 \) \([48]\) is consistent with the elliptic asymptotic ansatz \([37], [49]\). Assume that \( |v_1|, |v_3| \leq c \) and zeros of \( a_+(\zeta) \) or \( a_-(\zeta) \) lie apart from the paths of integration below. Let us assume also that the parameters \( t_0, D_0 \) determining the elliptic curve \([39]\) and \([13]\) are such that the asymptotic at \( \delta = 0 \) Stokes lines for the system \([17]\), \([14]\), emanating from \( \zeta_k = \zeta_k^{\pm 1/2}, k = 1, 3, 5 \), form the continuous chains \( e^{-i\pi} \infty \rightarrow \zeta_1 \rightarrow e^{i\pi} \infty, e^{i\pi} \infty \rightarrow \zeta_1 \rightarrow \zeta_3 \rightarrow e^{i\pi} \infty \) and \( e^{i\pi} \infty \rightarrow \zeta_5 \rightarrow e^{i\pi} \infty \). Then, using the result of \([45]\), we find the Stokes multipliers as \( \Re(\delta^{-1}) \rightarrow +\infty, |\Im(\delta^{-1})| < \text{const}, \)

\[
\begin{align*}
\ln s_2 &= -2\delta^{-1} \left[ \int_{\zeta_2}^{\zeta_3} \nu_\sigma d\zeta - \theta + \frac{\sigma}{2} \ln \frac{a_\sigma}{\mu} \bigg|_{\zeta=-\infty}^{\zeta=+\infty} + \sigma \ln(2i) + O(\delta) \right], \\
\ln s_5 &= -2\delta^{-1} \left[ \int_{\zeta_5}^{\zeta_3} \nu_\sigma d\zeta - \theta + \frac{\sigma}{2} \ln \frac{a_\sigma}{\mu} \bigg|_{\zeta=-\infty}^{\zeta=+\infty} + \sigma \ln(2i) + O(\delta), \quad \sigma \in \{+,-\}. \right.
\end{align*}
\]

We compute the r.h.s. of \([56]\) at the critical point \( t = t_3, v = 0, v = z_3 \) since the Stokes multipliers do not depend on \( t \) and the assumptions above are not violated for the non-degenerate curve \([53]\). Using the method of \([22, 14]\), we transform the elliptic integrals above as follows,

\[
\begin{align*}
\ln(-is_2) &= 2\delta^{-1} I_a + \frac{t_3}{2} \Omega_a^{(3)} + \frac{D_1(t_3)}{2} \omega_a + O(\delta^{2-\epsilon}), \\
\ln(-is_5) &= 2\delta^{-1} I_b + \frac{t_3}{2} \Omega_b^{(3)} + \frac{D_1(t_3)}{2} \omega_b + O(\delta^{2-\epsilon}),
\end{align*}
\]

where \( \epsilon > 0 \) is small. The condition \( |s_5| \ll 1 \) and the boundedness of \( s_2 \) are consistent with \([57]\) if \( h_a = 0 \) and \( h_b < 0 \). Substitutions \( \delta \rightarrow \alpha^{3/4} \delta, \ t_0 \rightarrow \alpha t_0, \ t \rightarrow \alpha^{-1/4} t, \ v_0 \rightarrow \alpha^{1/4} v, \ y_0 \rightarrow \alpha^{3/4} y, \) which leave \( x, y \) and \( y_x \) invariant, allow us to scale \( h_b \) to the value in \([14]\). The conditions on the Stokes graph imposed above are consistent with \( D_0(t_0) \in \mathbb{R} \) for \( t_0 < -1 \). Thus, it is chosen the branch of \( D_0(t_0) \) which yields the degeneration of the curve \([39]\) on \( \gamma \), and the validity domain of the elliptic asymptotic solution is located to the left of \( \gamma \).
Using in (57) the identities (53), we find the critical point $t_3$,

$$t_3 = -\delta^{-1} \frac{4}{3} t_0 + \frac{\omega_a}{2\pi i} \ln(-i\delta) - \frac{\omega_b}{2\pi i} \ln(-i\delta) + O(\delta^{-\varepsilon}),$$

and the argument of the $\wp$-function of Weierstrass (53). □

The description of the Painlevé function on the line $\gamma$ is given by the following assertion.

**Theorem 4.3** Let $t_0 \in \gamma$ where the asymptotic spectral curve (49) degenerates via $z_1 = z_3$. Let the functions $v, v_1$ introduced in (49) satisfy the estimates $v - z_1, v_1 = O(\delta^{-\varepsilon}), \varepsilon \in [0, \frac{1}{2}]$. Then the Stokes multipliers of the associated linear system (49), (54) are as follows:

i) if $0 \leq \varepsilon < \frac{1}{2}$ and $v_1 - \delta^2(v - z_1)\sqrt{v + 2z_1} = O(\delta^{1-\varepsilon}), \delta^2 = 1$,

$$s_4 = -i\sqrt{v + 2z_1 - \delta^2(3z_1)}\sqrt{v + 2z_1} + \frac{e^{i\pi/3}}{\Gamma(\rho_b)} e^{F_b}(1 + O(\delta^{-3\varepsilon})),
$$

$$s_5 = -i\sqrt{v + 2z_1 + \delta^2(3z_1)}\sqrt{v + 2z_1} + \frac{e^{i\pi/3}}{\Gamma(-\rho_b)} e^{-F_b}(1 + O(\delta^{3\varepsilon})),
$$

$$s_1 = -e^{-2\pi i \rho_c} s_4 (1 + O(\delta^{-\frac{1}{2}})),
$$

$$1 + s_4 s_5 = -e^{2\pi i \rho_c} (1 + O(\delta^{1-\frac{1}{2}})), \quad \arg(-1 - s_4 s_5) \in \left(-\frac{2\pi}{3}, \frac{2\pi}{3}\right),$$

where

$$F_b = \delta^{-1} \frac{8}{3} (3/2)^\frac{1}{3}(t_0)^\frac{2}{3} + \rho_b \ln[i\delta^{-1/2}3^{2/3}(t_0)^{1/2}] - 2(3/2)^{1/3}(t_0)^{1/3} t_1, \tag{59}$$

$$\rho_b = \frac{1}{4\sqrt{3z_1}}(\delta^{-1} v^2 - 4(v - z_1)(v + 2z_1) - 2t(v - z_1) + \frac{v_1}{v - z_1}), \quad \Re \rho_b \in \left(-\frac{1}{3}, \frac{1}{3}\right);$$

ii) if $\frac{1}{3} < \varepsilon \leq \frac{1}{2}$ then

$$s_4 = \frac{3^{1/8} 2^{7/8} (-t_0)^{1/8} \delta^{1/2}}{\sqrt{v_1 - 2\sqrt{3z_1} (v - z_1)}} \sqrt{2\pi e^{i\pi/3} e^{-\pi i/4}} \Gamma(\rho_c) e^{F_c}(1 + O(\delta^{3\varepsilon} + O(\delta^{3\varepsilon} - 1))),
$$

$$s_5 = \frac{3^{1/8} 2^{7/8} (-t_0)^{1/8} \delta^{1/2}}{\sqrt{v_1 + 2\sqrt{3z_1} (v - z_1)}} \sqrt{2\pi e^{i\pi/3} e^{-\pi i/4}} \Gamma(-\rho_c) e^{-F_c}(1 + O(\delta^{3\varepsilon} + O(\delta^{3\varepsilon} - 1))),
$$

$$s_1 = -e^{-2\pi i \rho_c s_4} (1 + O(\delta^{1-\varepsilon})),
$$

$$1 + s_4 s_5 = e^{2\pi i \rho_c} (1 + O(\delta^{1-\frac{1}{2}})), \quad \arg(1 + s_4 s_5) \in \left(-\frac{\pi}{3}, \frac{\pi}{3}\right),$$

where

$$F_c = \delta^{-1} \frac{8}{3} (3/2)^{1/4}(t_0)^{1/4} + \rho_c \ln(i\delta^{-1/4}3^{5/4}(t_0)^{5/4}) - 2\sqrt{3z_1} t_1 + i\frac{\pi}{4}, \tag{61}$$

$$\rho_c = \frac{\delta^{-1}}{4\sqrt{3z_1}}(v_1^2 - 12z_1(v - z_1)^2) + O(\delta^3), \quad \Re \rho_c \in \left(-\frac{1}{6}, \frac{1}{6}\right).$$

In other notations, this assertion was obtained in (61) following the conventional arguments of the isomonodromy deformation method (61). Applying the method of (62), it is possible to justify the expressions above. It is also possible to improve
the estimates for the error terms. To avoid the repetition and to save space, we skip this known derivation.

The inversion of (58) yields the asymptotics of the Painlevé function (38) with $v_0 = z_1 = \sqrt{-t_0/6}$ and the phase given in terms of the Stokes multipliers by

$$\sqrt{3v_0(t - \varphi)} = -\frac{F_b}{2} - \frac{1}{2} \ln A, \quad A = -i\sqrt{2\pi} \frac{e^{i\frac{\pi}{2} \rho_b}}{s_4 \Gamma\left(\frac{1}{2} + \rho_b\right)},$$  \hspace{1cm} (62)$$

where $F_b$ is defined in (59), and

$$\rho_b = \frac{1}{2\pi i} \ln(-1 - s_4 s_5), \quad \arg(-1 - s_4 s_5) \in \left(-\frac{2\pi}{3}, \frac{2\pi}{3}\right).$$

For our purposes, the inversion of (60) is more interesting. In the leading order, this yields

$$v = \sqrt{-t_0/6 + \delta^{1/2}(Ae^{-F_x} - Be^{F_x})},$$ \hspace{1cm} (63)

$$AB = 3^{-1/4} 2^{-7/4} (-t_0)^{-1/4} \rho_c (1 + O(\delta^{1/2 + \pi/4} + O(\delta^{3\pi/4 - 1}))),$$

$$\rho_c = \frac{1}{2\pi i} \ln(1 + s_4 s_5) + O(\delta^{1 - \pi/4}), \quad \arg(1 + s_4 s_5) \in \left(-\frac{\pi}{3}, \frac{\pi}{3}\right),$$

$$A = 3^{-1/8} 2^{-7/8} (-t_0)^{-1/8} \sqrt{2\pi} \frac{e^{i\frac{\pi}{2} \rho_c e^{-i\pi/4}}}{s_5 \Gamma(-\rho_c)} (1 + O(\delta^{1/4 + \pi/4} + O(\delta^{3\pi/4 - 1}))),$$

$$B = 3^{-1/8} 2^{-7/8} (-t_0)^{-1/8} \sqrt{2\pi} \frac{e^{i\frac{\pi}{2} \rho_c e^{-i\pi/4}}}{s_4 \Gamma(\rho_c)} (1 + O(\delta^{1/4 + \pi/4} + O(\delta^{3\pi/4 - 1}))),$$

and $F_x$ is defined in (51).

Now, we are prepared to discuss the problems announced earlier.

i) The large amplitude separatrix solution.

Let us consider the Painlevé function $y(x)$ corresponding to the Stokes multipliers (32) satisfying $1 + s_2 s_3 = 0$, i.e. $s_2 = s_3 = i$, $s_5 = 0$, $s_1 + s_4 = i$, when $|s_4| \gg 1$. The asymptotics (33), (44) explain the used term the large amplitude separatrix solution. Introduce $t_0$, $t$ and $v$ as in (33) and consider the boundary $\gamma$ for the domain of validity of (41). Then applying Theorem 4.1, we see that to the right of $\gamma$, the asymptotics of the Painlevé function is elliptic and is described by the equations (43), (44), (45). The corresponding elliptic curve is determined by the system of the Krichever’s equations (41).

On $\gamma$, the elliptic curve degenerates, and, since $1 + s_4 s_5 = 1$, the asymptotics of the Painlevé function is described by the limiting as $\rho_c \to 0$ form of (43),

$$v = \sqrt{-t_0/6 + \delta^{1/2} A e^F}, \quad \delta^{-1} = -\frac{\ln |s_4|}{h_b}, \quad h_b = -\frac{8}{5} (3/2)^{1/4},$$

$$A = 3^{-1/8} 2^{-7/8} (-t_0)^{-1/8} \frac{s_4}{\sqrt{2\pi}} \Gamma(1 + O(\delta^{1/16})), \quad (64)$$

$$F = -\delta^{-1} \frac{8}{5} (3/2)^{1/4} (-t_0)^{5/4} + 2^{3/4} \delta^{1/4} (-t_0)^{1/4} t.$$ 

To the left of $\gamma$, the curve (33) remains degenerate, and the Krichever’s system is not applicable. Observing however, that the complete series for the solution with the leading terms (44) contains the positive degrees of $e^F$ only, we may conjecture that the asymptotics of the Painlevé function to the left of $\gamma$, i.e. for $\text{Re} (-t_0)^{5/4} > 1$, is given by the analytic continuation of (43). This conjecture can be confirmed
by the use of the Riemann-Hilbert problem approach, see \[3, 22\]. An alternative investigation of such series based on the Borel summation method is given in \[76, 8\].

ii) The small perturbation of the degenerate solution.

Let \(0 < |s_3| \ll 1, c' \leq |s_3| < 1\), \(c' \leq |s_2| \leq |s_4| \leq c''\), for some positive finite constants \(c', c''\), e.g., when \(s_2 = i(1 + \varepsilon), s_3 = i(1 + \varepsilon x), 0 < |\varepsilon| \ll 1, c' \leq |x| \leq c''\). Using Theorem 4.2, we see that, to the left of the line \(\gamma_l = \gamma\) defined above, the asymptotics of \(y(x)\) is described by (43), (54), (55). The elliptic curve (39) is determined by the Krichever’s system (41) with \(h_a = 0, h_b = -8(3/2)^{1/4}\).

Due to Theorem 4.1, the right boundary \(\gamma_r\) of the discriminant set consists of the points satisfying the equation \(\text{Re} (\gamma_l) \gamma_r = \frac{5}{4} < 1\) and is asymptotic to the rays \(\text{arg} t_0 = \pm \frac{3}{5}\pi\). If \(s_4\) is bounded as \(\varepsilon \to 0\), the right boundary \(\gamma_r\) coincides with these rays. To the right of \(\gamma_r\), the asymptotic solution \(y(x)\) is given by (14), (15) where the elliptic curve (39) is determined by the Krichever’s system with the parameters \(h'_a = 0, h'_b = -h_b \frac{\ln |s_4|}{\ln |s_5|}\). If \(s_4\) is bounded, then \(h'_a = h'_b = 0\), and the description of the Painlevé function can be obtained by the use of changes (53) in the formulae of (11), (14) for \(y(x)\) as \(|x| \to \infty\). On the boundaries \(\gamma_l\) and \(\gamma_r\), we may use (53) with \(\delta^{-1} = \frac{\ln |s_4|}{h_b}, h_b = -\frac{1}{2}(3/2)^{1/4}, \rho_c \to 0\). As easy to see, both the non-constant terms in (53) decrease as \(\varepsilon \to 0\) for \(-\frac{\ln |s_4|}{\ln |s_5|} < \text{Re} (\gamma_l) \gamma_r < 1\). Because the complete expansion of the Painlevé function involves the positive degrees of these terms only, we may conjecture the validity of (53) between \(\gamma_l\) and \(\gamma_r\) as well.

In particular, the discussion above provides us with the possibility to give the analytic interpretation for the formal 2-parametric “instanton type” series with the initial terms given by (53) and extensively studied in \[5, 76, 77\].

Acknowledgments. This work was partially supported by RFBR under grant number 99–01–00687. The author is grateful to a referee for his valuable comments and to A. Kitaev for remarks.

References

[1] Ablowitz M J, Segur H 1977 Exact linearization of a Painlevé transcendent Phys. Rev. Lett. 38 1103-1106

[2] Arnold V I 1989 Mathematical methods of classical mechanics Nauka, Moscow

[3] Baik J, Deift P, Johansson K 1998 On the distribution of the length of the longest increasing subsequence of random permutations math.CO/9810103

[4] Bleher P, Its A 1997 Semiclassical asymptotics of orthogonal polynomials, Riemann-Hilbert problem, and universality of the matrix model Preprint # 97-2, Dept. of Mathematical Sciences IUPUI

[5] Boutroux P 1913 Recherches sur les transcendantes de M. Painlevé et l’étude asymptotique des équations différentielles du second ordre Ann. Sci. Écol. Norm. Supér. 30 255-376; 1914 31 99-159

[6] Brézin E, Kazakov V A 1990 Exactly solvable field theories of closed strings Phys. Lett. 236 B 144-50
[7] Chakravarty S 1994 A class of integrable conformally self-dual metrics
Classical Quantum Gravity 11 no 1 L1-L6

[8] Costin O 1997 Correlation between pole location and asymptotic behavior for Painlevé I solutions MSRI preprint 1997-094

[9] Deift P A, Zhou X 1995 A steepest descent method for oscillatory Riemann-Hilbert problems. Asymptotics for the MKdV equation Ann. of Math. 137 295-368

[10] Doran-Wu P, Joshi N 1997 Direct asymptotic analysis of the second Painlevé equation: three limits J. Phys. A 30 4701-4708
Joshi N 1999 The second Painlevé equation in the large-parameter limit I: local asymptotic analysis Stud. Appl. Math. 102 345-73

[11] Douglas M, Shenker S 1989 Strings in less than one dimension, Rutgers preprint RU-89-34

[12] Dubrovin B 1995 Geometry of 2D topological field theories Springer Lect. Notes Math. 1620 120-348

[13] Dubrovin B A, Krichever I M, Novikov S P 1982 Topological and algebraic geometry methods in contemporary mathematical physics. II Soviet Scient. Reviews, Math. Phys. Reviews 3 1-150

[14] Dubrovin B A, Matveev V B, Novikov S P 1976 Nonlinear equations of Korteweg-de Vries type, finite band operators and Abelian varieties Russian Math. Surveys 31 59-146

[15] Dubrovin B and Mazzocco M 2000 Monodromy of certain Painlevé-VI transcendents and reflection groups Invent. Math. 141 55-147

[16] Fedorjuk M V 1983 Asymptotic methods for linear ordinary differential equations Moscow, Nauka

[17] Fedorjuk M V 1986 Isomonodromy deformations of equations with an irregular singularity Diff. Uravnen. 22 961-7

[18] Flaschka H, Forest M G, McLaughlin D W 1980 Multiphase averaging and the inverse spectral solution of the Korteweg-de Vries equation Comm. Pure Appl. Math. 33 739-84

[19] Flaschka H, Newell A C 1980 Monodromy- and spectrum-preserving deformations. I Commun. Math. Phys. 76 65-116

[20] Flaschka H, Newell A 1981 Multiphase similarity solutions of integrable evolution equations Physica D 3 203-221

[21] Fokas A S, Its A R, Kitaev A V 1990 Isomonodromic approach in the theory of two-dimensional quantum gravity Usp. Matem. Nauk 45 135-6

——— 1991 Discrete Painlevé equations and their appearance in quantum gravity Comm. Math. Phys. 142 313-44
——— 1992 The isomonodromy approach to matrix models in 2D quantum gravity Comm. Math. Phys. 147 395-430
[22] Freud G 1976 On the coefficients in the recursion formulae of orthogonal polynomials *Proc. Royal Irish Acad. Sect. A* 76 1-6

[23] Fuchs R 1906 Über lineare homogene Differentialgleichungen zweiter Ordnung mit drei im Endlichen gelegenen wesentlich singulären Stellen *Math. Annalen* 63 301-21

[24] Fucito F, Gamba A, Martellini M, Ragnisco O 1992 Nonlinear WKB analysis of the string equations *Int. J. Mod. Phys. B* 6 2123-48

[25] Gambier B 1910 Sur les équations différentielles du second ordre et du premier degré dont l’intégrale générale est a points critiques fixes *Acta Math.* 33 1-55

[26] Garnier R 1912 Sur les équationes différentielles du troisième ordre dont l’intégrale générale est uniforme et sur une classe d’équationnes nouvelles d’ordre supérieur dont l’intégrale générale a ses points critique fixes *Ann. Sci. Ec. Norm. Super.* 29 1-126

[27] Garnier R 1917 Etude de l’intégrale générale de l’équation VI de M. Painlevé *Ann. Sci. École Norm. Sup.* 34 239-53

[28] Garnier R 1926 Solution du problème de Riemann pour les systèmes différentielles linéaires du second ordre *Ann. Sci. École Norm. Sup.* 43 177-307

[29] Gross D J and Migdal A A 1990 Nonperturbative two-dimensional quantum gravity *Phys. Rev. Lett.* 64 127-30

[30] Holmes Ph, Spence D 1984 On a Painlevé-type boundary-value problem *Quart. J. Mech. Appl. Math.* 37 525-38

[31] Ince E L 1956 *Ordinary Differential Equations*, Dover, New York, 1956

[32] Its A R, Fokas A S, Kapaev A A 1994 On the asymptotic analysis of the Painlevé equations via the isomonodromy method, *Nonlinearity* 7 1291-1325

[33] Its A R, Matveev V B 1976 On a class of solutions of the KdV equation, in: *Problemy Matem. Fiziki* 8, Leningrad State Univ., Leningrad

[34] Its A R, Novokshenov V Yu 1986 The Isomonodromic Deformation Method in the Theory of Painlevé Equations in “Lecture Notes in Mathematics” 1191 Springer-Verlag, Berlin-Heidelberg-New York-Tokyo

[35] Jimbo M, Miwa T, Ueno K 1981 Monodromy preserving deformation of linear ordinary differential equations with rational coefficients *Physica D* 2 306-52

[36] Jimbo M, Miwa T 1981 Monodromy preserving deformation of linear ordinary differential equations with rational coefficients II *Physica D* 2 407-48

[37] Jimbo M 1982 Monodromy problem and the boundary condition for some Painlevé equations *Publ. RIMS, Kyoto Univ.* 18 1137-61
[38] Joshi N 1999 The second Painlevé equation in the large-parameter limit I: Local asymptotic analysis Stud. Appl. Math. 102 345-73

[39] Joshi N, Kruskal M D 1988 An asymptotic approach to the connection problem for the first and the second Painlevé equations Phys. Lett. A 130 129-37

[40] Kapaev A A 1986 Singular Solutions of the Painleve II Equations, Lect. Notes in Math, Springer-Verlag, 1191 261-83

[41] Kapaev A A 1988 Asymptotics of solutions of the Painlevé equation of the first kind Diff. Uravnenija 24 1684-95

________ 1993 Global asymptotics of the first Painlevé transcendent Inst. Nonlinear Studies Preprint INS # 225

[42] Kapaev A A 1991 The essential singularity of the Painlevé function of the second kind and nonlinear Stokes phenomenon Zap. Nauch. Semin. LOMI 187 139-70

[43] Kapaev A A 1992 Global asymptotics of the second Painlevé transcendent Phys. Lett. A 167 356-62

________ 1996 Global asymptotics of the fourth Painlevé transcendent POMI Preprint 6/1996; Steklov Math. Inst. and IUPUI Preprint # 96-5

[44] Kapaev A A 1994 Scaling limits in the second Painlevé transcendent Zap. Nauch. Semin. LOMI 209 60-101

[45] Kapaev A A 1996 WKB method for $2 \times 2$ systems of linear ordinary differential equations with rational coefficients Steklov Math. Inst. and IUPUI Preprint # 96-6

[46] Kapaev A A 1996 Scaling limits in the fourth Painlevé transcendent Steklov Math. Inst. Preprint POMI 15/1996

[47] Kapaev A A 1997 Discriminant set for the scaling limits in the third Painlevé transcendent Steklov Math. Inst. Preprint PDMI 21/1997

[48] Kapaev A A, Kitaev A V 1991 The limit transition $P_2 \rightarrow P_1$ Zap. Nauch. Semin. LOMI 187 75-87

[49] Kapaev A A, Kitaev A V 1993 Connection formulae for the first Painlevé transcendent in the complex domain Lett. Math. Phys. 27 243-52

[50] Kawai T, Takei Y 1996 WKB analysis of Painlevé transcendents with a large parameter. I Adv. in Math. 118 1-33

________ 1998 WKB analysis of Painlevé transcendents with a large parameter. III. — Local equivalence of 2-parameter Painlevé transcendents Adv. in Math. 134 178-218

[51] Kiselev O M 2001 Hard loss of stability in Painlevé-2 equation J. Nonlinear Math. Phys. 8 65-95
[52] Kitaev A V 1988 Asymptotic description of the fourth Painlevé equation. Solutions on the Stokes rays analogs Zap. Nauch. Semin. LOMI 169 84-9

[53] Krichever I M 1977 Integration of nonlinear equations by the method of algebraic geometry Funct. Anal. Appl. 11 12-26

[54] Krichever I M 1988 Method of averaging for two-dimensional “integrable” equations Funct. Anal. Appl. 22 200-13

[55] Krichever I M 1990 On Heisenberg relations for the ordinary linear differential operators ETH preprint, Zürieh, IHES preprint, Bur-Sur-Yvette

[56] Kuzmak G E 1959 Asymptotic solutions of nonlinear second-order differential equations with variable coefficients J. Appl. Math. Mech. 23 730-744

[57] Magnus A P 1995 Painlevé-type differential equations for the recurrence coefficients of semi-classical orthogonal polynomials J. Comp. Appl. Math. 57 215-37

[58] McCoy B M, Tang Sh 1986 Connection formulae for Painlevé V functions. I. Physica 19 D 42-72

…….., Connection formulae for Painlevé V functions. II. The δ function Bose gas problem. Physica 20 D 187-216

[59] Mehta M L 1994 Painlevé transcendent in the theory of random matrices An introduction to methods of complex analysis and geometry for classical mechanics and non-linear waves (Chamonix, 1993) 197-208 Frontières, Gif-sur-Yvette

[60] Miles J W 1978 On the second Painlevé transcendent Proc. R. Soc. Lond. A 361 277-91

[61] Moor G 1990 Matrix models of 2D gravity and isomonodromic deformations Prog. Theor. Phys. Suppl. 102 255-85

[62] Nevai P 1986 Géza Freud, orthogonal polynomials and Christoffel functions. A case study J. Approx. Theory 48 3-167

[63] Novikov S P 1990 Quantization of finite-gap potentials and nonlinear quasiclassical approximation in nonperturbative string theory Funct. Anal. Appl. 24 296-306

[64] Novokshenov V Yu 1990 Ansatz Boutroux for the second Painlevé equation and elliptic solutions Izv. Akad. Nauk SSSR ser. math. 54 1229-51

[65] Novokshenov V Yu 1998 Radial-symmetric solution of the cosh-Laplace equation and the distribution of its singularities Russian J. Math. Phys. 5 211-26

[66] Okumura S 1998 The self-dual Einstein-Weyl metric and classical solution of Painlevé VI Lett. Math. Phys. 46 no 3 219-32
[67] Olver F W J 1974 *Asymptotics and special functions* N. Y., L., Academic Press

[68] Painlevé P 1900 *Mémoire sur les équations différentielles dont l’intégrale générale est uniforme* Bull. Soc. Math. France 28 201-61

[69] Painlevé P 1906 *Sur les équations différentielles du second ordre à points critiques fixes* Comptes Rendus 143 1111-17

[70] Paniak L D, Szabo R J, 2000 *Fermionic Quantum Gravity* arXiv:hep-th/0005128

[71] Pastur L 1992 *On the universality of the level spacing distribution for some ensembles of random matrices* Lett. Math. Phys. 25 259-265

[72] Schlesinger L 1912 *Über eine Klasse von Differentialensystemen beliebiger Ordnung mit festen kritischen Punkten* J. für Math. 141 96-145

[73] Segur H, Ablowitz M J 1981 *Asymptotic solutions of nonlinear evolution equations and a Painlevé transcendent* Physica D 3 165-184

[74] Suleimanov B I 1986 *On Asymptotics of Regular Solutions for a Special Kind of Painlevé V Equation.* Lect. Notes in Math., Springer Verlag, 1191, 230-61.

[75] Takasaki K 1998 *Spectral curves and Whitham equations in isomonodromic problems of Schlesinger type* Asian J. Math. 4 1049-78

[76] Takei Y 1995 *On the connection formula for the first Painlevé equation – from the viewpoint of the exact WKB analysis* Sūrikaisekikenkyūshō Kōkyūroku 931 70-99

[77] Takei Y 2000 *An explicit description of the connection formula for the first Painlevé equation Toward the exact WKB analysis of differential equations, linear or non-linear (Kyoto, 1998)*, Kyoto Univ. Press, Kyoto, 271-96

[78] Tod K P 1994 *Self-dual Einstein metrics from the Painlevé VI equation* Phys. Lett. A 190 221-4

[79] Vereschagin V L 1997 *Global asymptotics for the fourth Painlevé transcendent* Mat. Sbornik 188 11-32

[80] Vereschagin V L 2000 *Asymptotic behavior of solutions, singular at zero. of the sine-Gordon equation* Mat. Zametki 67 329-342

[81] Wasow W 1965 *Asymptotic Expansions for Ordinary Differential Equations* Interscience-Wiley, New York

[82] Whitham J B 1974 *Linear and nonlinear waves* Wiley-Interscience, New York