HORIZONTAL DISPLACEMENT OF CURVES IN BUNDLE
\[ \text{SO}(n) \to \text{SO}_0(1, n) \to \mathbb{H}^n \]

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Abstract. The Riemannian submersion \( \pi : \text{SO}_0(1, n) \to \mathbb{H}^n \) is a principal bundle and its fiber at \( \pi(e) \) is the imbedding of \( \text{SO}(n) \) into \( \text{SO}_0(1, n) \), where \( e \) is the identity of both \( \text{SO}_0(1, n) \) and \( \text{SO}(n) \). In this study, we associate a curve, starting from the identity, in \( \text{SO}(n) \) to a given surface with boundary, diffeomorphic to the closed disk \( D^2 \), in \( \mathbb{H}^n \) such that the starting point and the ending point of the curve agree with those of the horizontal lifting of the boundary curve of the given surface with boundary, respectively, and that the length of the curve is as same as the area of the given surface with boundary.

Introduction

Let \( O(1, n) = \{ A \in \text{GL}(n + 1; \mathbb{R}) \mid A^t S A = S \} \), where \( S = \begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix} \).

Let \( \text{SO}_0(1, n) \) be the identity component of \( O(1, n) \), which is also the identity component of \( \text{SO}(1, n) \), and consider a subgroup of \( \text{SO}_0(1, n) \) consisting of all matrices of the form \( \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} \), where \( B \in \text{SO}(n) \). Call the embedded subgroup \( \text{SO}(n) \) again.

Note the Lie algebra \( \mathfrak{o}(1, n) \) is given by
\[
\mathfrak{o}(1, n) = \{ X \in \mathfrak{gl}(n + 1; \mathbb{R}) | X^t S + SX = 0 \}.
\]

Now, think of a left-invariant metric on \( \text{SO}_0(1, n) \), induced from an inner product \( \langle \cdot , \cdot \rangle \) on the Lie algebra, \( \mathfrak{o}(1, n) \), defined as follows:
\[
\langle A, B \rangle = \frac{1}{2} \text{trace}(A^t B) \quad \text{for } A, B \in \mathfrak{o}(1, n).
\]

If \( \phi \) is a Killing-Cartan form, then
\[
\langle X, Y \rangle = \frac{n - 1}{2} \phi(X, Y) \quad \text{for } X, Y \in \mathfrak{o}(n)^\perp \subset \mathfrak{o}(1, n).
\]

And the right actions of \( \text{SO}(n) \) become isometries and \( \text{SO}_0(1, n)/\text{SO}(n) \) becomes isometric to \( \mathbb{H}^n \).

Under this metric, we have a principal bundle structure
\[
\text{SO}(n) \to \text{SO}_0(1, n) \to \mathbb{H}^n,
\]
where \( \pi : \text{SO}_0(1, n) \to \mathbb{H}^n \) is a Riemannian submersion.
If n=2, then it can be easily shown that for a given geodesic triangle in \( \mathbb{H}^2 \), the distance by the horizontal displacement of the boundary curve of the given geodesic triangle in the fiber is as same as the area of the triangle. Furthermore, the direction of the boundary curve of the given geodesic triangle in \( \mathbb{H}^2 \) will determine the direction of its holonomy displacement. All of these are dealt with in Section 3.

Can a similar result be obtained in a topological disk in \( \mathbb{H}^n \)?

If it is a geodesic triangle, something similar can be easily said from the result for the case n=2 and ‘Fact 2’, mentioned in section 4. But, what can be done for a general disk in \( \mathbb{H}^n \)?

To answer this question, we intend to approximate the given disk with geodesic triangles, since there exists a unique totally geodesic triangle for any 3 different points in \( \mathbb{H}^n \). And then we intend to construct a curve in the fiber by using the property for the case n=2. But how can we approximate it? Though each geodesic triangle and its boundary curve determine the direction of each horizontal displacement, some linear ordering of geodesic triangles and the induced ordering of their boundary curves may not represent the boundary curve of their union. If the given disk is contained in an isometrically embedded plane \( \mathbb{H}^2 \) in \( \mathbb{H}^n \), something similar can be said from a curve in the fiber SO(n), made from the result for the case n=2 and ‘Fact 2’, mentioned in section 4, since horizontal displacements are happening in the one-dimensional vertical subgroup. Though the different orderings of triangles give different curves in the vertical space, they will meet at the same ending point. So, with respect to any ordering, the horizontal displacement of the boundary curve of the given disk can be approximated. But in other cases, what can be obtained? Something similar could be done if the fiber SO(n) were abelian, which would make the ending points of any other different two curves in the fiber, induced from different linear orderings, be the same. But the fiber SO(n) is not abelian for \( n \geq 3 \). The difficult part is that not only the approximation of the area but also the linear ordering of the triangles on each step for the approximation of the boundary curve of the disk should be considered at the same time. This is one of the hardest parts in this paper, which is dealt with in Section 4 and Appendices A and B. Furthermore, can holonomy displacements by the lifts of piecewise geodesics approaching to the boundary of the given topological disk in the base space converge to the holonomy displacement by the lift of the boundary? It will be discussed in Subsection 4.3.

After the case n=2 is explained in section 3 our following main result for the general case will be explained in section 4.

**Theorem 0.1.** Let \( \pi : SO_0(1,n) \rightarrow \mathbb{H}^n \) be the Riemannian submersion given as before. Then, given a smooth disk \( S \), with \( \bar{e} = \pi(e) \) on its boundary, in \( \mathbb{H}^n \), there is a \( C^1 \)- curve \( f : [0,1] \rightarrow SO(n) \subset SO_0(1,n) \) with \( f(0) = e \) such that
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- \( f(1) = f(0)^{-1} f(1) \) is the difference by the holonomy induced from the boundary of \( S \) in view of right multiplication
- the length of the curve \( f \) is the area of \( S \).

1. **Strategy for approximation**

‘Factorization Lemma’, given by Lichnerowicz, *Theorie Globale des Connexions et des Groupes d’Holonomie*, [3, vol 1, p.284], will be helpful to understand this section. For the difference, focus on properties of triangles mentioned in number 6. The reason for introducing another approximation will be given in subsection 4.4.7.

1. For any 3 points in \( \mathbb{H}^n \), there exists a unique totally geodesic triangle with these vertices.

2. Let \( \triangle ABC \) be a totally geodesic triangle in \( \mathbb{H}^n \) and consider a piecewise geodesic from \( \bar{e} = \pi(e) \) to \( A \), where \( e \) is the identity of \( \text{SO}(1,n) \).

Then, horizontal displacement of \( \gamma = \bar{e}A \cdot AB \cdot BC \cdot CA \cdot A\bar{e} \) \( \in \text{SO}(n) \), where

\[
\text{the length of } \bar{e}g = \text{ the area of } \triangle ABC.
\]

3. Let \( \triangle ABC \) and \( \triangle ACD \) be two given geodesic triangles in \( \mathbb{H}^n \) and consider a piecewise geodesic curve from \( \bar{e} \) to \( A \).

Consider two curves \( \gamma_1 = \bar{e}A \cdot AB \cdot BC \cdot CA \cdot A\bar{e} \) and \( \gamma_2 = \bar{e}A \cdot AC \cdot CD \cdot DA \cdot A\bar{e} \). Then the horizontal displacement of \( \gamma_1 \ast \gamma_2 \) equals to that of \( \gamma_3 = \bar{e}A \cdot AB \cdot BC \cdot CD \cdot DA \cdot A\bar{e} \).
In general, if $\gamma_1 = \bar{e}A \cdot AB \cdot BC \cdot CD \cdot DA \cdot \bar{AE}$ and $\gamma_2 = \bar{e}A \cdot AD \cdot DC \cdot CE \cdot ED \cdot DA \cdot \bar{AE}$ are two curves in $\mathbb{H}^n$, then the horizontal displacement of $\gamma_1 \ast \gamma_2$ equals to that of $\gamma_3 = \bar{e}A \cdot AB \cdot BC \cdot CE \cdot ED \cdot DA \cdot \bar{AE}$.

4. What’s the difficulty of the approximation?
Consider three given geodesic triangles $\triangle ABC$, $\triangle ACE$, $\triangle CDE$ in $\mathbb{H}^n$.

Then, for three curves $\gamma_1 = \bar{e}A \cdot AB \cdot BC \cdot CA \cdot \bar{AE}$, $\gamma_2 = \bar{e}A \cdot AC \cdot CE \cdot EA \cdot \bar{AE}$ and $\gamma_3 = \bar{e}A \cdot AE \cdot EC \cdot CD \cdot DE \cdot EA \cdot \bar{AE}$, the horizontal lift of $\gamma_1 \ast \gamma_2 \ast \gamma_3$ equals to that of $\gamma_4 = \bar{e}A \cdot AB \cdot BC \cdot CD \cdot DE \cdot EA \cdot \bar{AE}$, which relates to the boundary of the polygon $ABCDE$. But the horizontal lift of $\gamma_1 \ast \gamma_3 \ast \gamma_2$ equals to $\gamma_5 = \bar{e}A \cdot AB \cdot BC \cdot CA \cdot AE \cdot EC \cdot CD \cdot DE \cdot EA \cdot AC \cdot CE \cdot EA \cdot \bar{AE}$, which does not relate to the boundary of the polygon $ABCDE$. Thus, for our object, the order of curves is important, which relates to the order of triangles.

5. Refer to the number 4.
Consider a curve $\tilde{\gamma}_3 = \bar{e}A \cdot AC \cdot CD \cdot DE \cdot EC \cdot CA \cdot \bar{AE}$. Though the order ($\gamma_1$, $\gamma_2$, $\tilde{\gamma}_3$) of curves relates to the order of the triangles, induced by the order ($\gamma_1$, $\gamma_2$, $\gamma_3$) of curves, the horizontal lift of $\gamma_1 \ast \gamma_2 \ast \tilde{\gamma}_3$ equals to $\tilde{\gamma}_4 = \bar{e}A \cdot AB \cdot BC \cdot CE \cdot EA \cdot AC \cdot CD \cdot DE \cdot EC \cdot CA \cdot \bar{AE}$, which does not relate to the boundary of the polygon $ABCDE$. Thus, it is also important how to make a curve that represents a given triangle. This problem in the construction of a curve for each triangle will be solved by introducing the starting point and the ending point of each triangle.
6. Instead of approximating a given topological disk in $\mathbb{H}^n$ directly, we will approximate $D^2$ by triangles, and approximate the given disk in $\mathbb{H}^n$ by the diffeomorphism from $D^2$ to it. In fact, in Appendix, for each $n = 0, 1, 2, \cdots$, we will construct a subdivision $D_n$ of the interval $[0, 1]$ and an ordered set $A_n$ consisting of triangles having the following properties:

Property 1.) Given a non-first element $L$ in $A_n$, the boundary of $\bigcup\{M \in A_n | M < L\}$ contains a side of $L$, which will be divided into two line segments in its barycentric subdivision, where one of two line segments will become a side of the first triangle and the other one will become a side of the second triangle in the barycentric subdivision of $L$.

Property 2.) Given $L \in A_n$, $\bigcup\{M \in A_n | M \leq L\}$ is diffeomorphic to the disk $D^2$.

Property 3.) Assume $L \in A_n$ and six triangles $M_1, M_2, \cdots, M_6 \in A_{n+1}$, obtained from the barycentric subdivision of $L$, follows the order of $i = 1, 2, \cdots, 6$ in $A_{n+1}$. Then the starting points of $M_1$ and $L$ are same. Also are the ending points of $M_6$ and $L$.

Property 4.) Assume $L, M \in A_n$ and that $M$ is the next element of $L$ in $A_n$ for $n \geq 1$.

Then, The ending point of $L$ and the starting point of $M$ are same.

2. Definitions, Triangles and Curves

All materials in this section will be dealt with in Appendix concretely.

2.1. Notations. $f \ast g : [0, 1] \rightarrow \mathbb{H}^n$ is an ordinary juxtaposition of curves $f, g : [0, 1] \rightarrow \mathbb{H}^n$. And, for a given curve $c : [0, 1] \rightarrow \mathbb{H}^n$, $\overline{c}$ represents a curve whose direction is opposite to that of $c$, that is, $\overline{c} : [0, 1] \rightarrow \mathbb{H}^n$ is given by $\overline{c}(t) = c(1 - t)$.

2.2. Simplification $\gamma$ of a curve $g : [a, b] \rightarrow \mathbb{H}^n$. Given a curve $g : [a, b] \rightarrow S$, we can think of a curve $\gamma : [a, b] \rightarrow S$ whose direction is one-sided as follows:

If we can find $c, d, e \in (a, b)$ such that $a < c < d < e < b$ and $\text{Im}(g|_{[c,d]}) = \text{Im}(g|_{[d,e]})$ and that the directions of $g|_{[c,d]}$ and $g|_{[d,e]}$ are one-sided but opposite from each other, then we can think of the new curve $\bar{g} : [a, b] \rightarrow D^2$ from the remaining part $g|_{[a,c]}$ and $g|_{[c,b]}$ by translating in the domain and rescaling as follows:

Note $g(c) = g(e)$.

Consider two curves $g_1 : [a, d] \rightarrow \mathbb{H}^n$ and $g_2 : [d, b] \rightarrow \mathbb{H}^n$ given by

$$g\left(\frac{c-a}{d-a}(t-a)+a\right) = g_1(t) \text{ for } t \in [a, d]$$
and
\[ g\left(\frac{b-e}{b-d}(t-b)+b\right) = g_2(t) \text{ for } t \in [d,b], \]
and then let \( \tilde{g} = g_1 * g_2 \).

From a curve obtained by doing this work again and again and by reparametrizing it, we can think of a constant speed curve \( \gamma : [a,b] \to S \) which we want.

2.3. The definition of \( D_n, j_n, t^n_1, t^n_2 \).

\[
D_n = \left\{ \frac{1}{2} \cdot \frac{j}{6^n} \mid j = 0,1,2,\cdots,6^n \right\} \bigcup \left( \bigcup_{k=1}^{n} \left\{ \sum_{i=1}^{k} \frac{1}{2^i} + \frac{1}{2^{k+1}} \cdot \frac{j}{2^{k-1} \cdot 6^{n-k+1}} \mid j = 0,1,2,\cdots,2^{k-1} \cdot 6^{n-k+1} \right\} \right) \]

Think of the usual order \( D_n \) and regard
\[
0, \frac{1}{2} \cdot \frac{1}{6^n}, \frac{1}{2} \cdot \frac{2}{6^n}, \cdots, \frac{1}{2} = \frac{1}{2} \cdot \frac{6^n}{6^n}, \frac{1}{2} + \frac{1}{2^2} \cdot \frac{1}{2^0} \cdot \frac{1}{6^n}, \cdots \in D_n
\]
as 0th, 1st, 2nd, \cdots, \( 6^n \)th, \( 6^n+1 \)th, \cdots element, respectively.

Now, define functions

\[
j_n : D_n \to \{0,1,2,3,\cdots\} \\
t^n_1 : (D_n - \{0\}) \cup \{1\} \to D_n \\\nt^n_2 : D_n - \{ \text{the last element of } D_n \} \to D_n
\]
as follows:

\[
j_n(s) = j \text{ for the } j\text{-th element } s \in D_n.
\]

\[
t^n_1(s) \text{ is the } (j-1)\text{-th element in } D_n \text{ for a given } j\text{-th element } s \in D_n - \{0\}\text{ and } t^n_2(1) \text{ is the last element in } D_n.
\]

\[
t^n_2(s) \text{ is the } (j+1)\text{-th element in } D_n \text{ for a given } j\text{-th element } s \in D_n - \{ \text{the last element of } D_n \}.
\]

2.4. Definition of \( \gamma^n_{t_0}, c^n_{t_0}, \tilde{c}^n_{t_0}, \varepsilon^n_{t_0}, \varphi^n_{t_0} \) and \( \psi^n_{t_0} \) on the disk \( D^2 \).

Recall from Proposition ?? that the union \( U_i \) of triangles from 1st one to \( i \)-th one is diffeomorphic to a disk.

Let \( n \in \{1,2,3,\cdots\} \) and \( t_0 \in D_n \) be given.

With respect to the ordering of \( D_n \), we will define \( \gamma^n_{t_0}, c^n_{t_0}, \tilde{c}^n_{t_0}, \varphi^n_{t_0} \) inductively for each fixed \( n \):

Case 1) \( t_0 \) is the first element in \( D_n \), in fact, \( t_0 = \frac{1}{2} \cdot \frac{1}{6^n} \).

The orientation at the barycenter of \( T_0 \in A_0 \) will give the direction of the boundary curve of the first triangle in \( A_n \).
Then
\[ c^n_{t_0} : [0, 1] \to \{ \text{basepoint} \} \subset D^2 \]
\[ \tilde{c}^n_{t_0} : [0, 1] \to \{ \text{basepoint} \} \subset D^2 \]
and
\[ \varphi^n_{t_0} : [0, 1] \to D^2 \]
\[ \gamma^n_{t_0} : [0, 1] \to D^2 \]
can be thought, where \( \varphi^n_{t_0} \) and \( \gamma^n_{t_0} \) are the piecewise smooth boundary curve of the first triangle in \( A_n \) with constant speed and the direction of the boundary curve is induced from the given orientation.

Note \( \gamma^n_{t_0} \) can be regarded as the simplification of \( c^n_{t_0} * \varphi^n_{t_0} * \tilde{c}^n_{t_0} \).

We will call \( \gamma^n_{t_0} \) the holonomy curve at time \( t = t_0 \).

Now, consider the path from the basepoint to the ending point of the first triangle in \( n\)-step along the opposite direction of the holonomy curve \( \gamma^n_{t_0} \) at \( t = t_0 \), which is a piecewise smooth curve with constant speed. Then from the path, we can define a piecewise smooth curve
\[ 1c^n_{t_0} : [0, 1] \to D^2 \]
with constant speed. And its opposite direction can make us define
\[ 1\tilde{c}^n_{t_0} : [0, 1] \to D^2 \]

Define a piecewise smooth curve
\[ \psi^n_{t_0} : [0, 1] \to D^2 \]
with constant speed as the boundary curve of the 1st triangle in the \( n\)-th step, where the curve is a loop at the ending point of the first triangle and the direction of the boundary curve is induced from the given orientation.

Case 2) \( t_0 \) is the \( j \)-th element in \( D_n \), \( i.e., \, j_n(t_0) = j \), where \( j \geq 2 \)

Let \( t_1 \) be the \( (j - 1)\)-th element in \( D_n \), \( i.e., \, t_1^n(t_0) = t_1 \) and \( j_n(t_1) = j - 1 \), where \( j - 1 \geq 1 \).

Consider the path from the basepoint to the starting point of the \( j \)-th triangle in the \( n\)-th step along the opposite direction of the holonomy curve \( \gamma^n_{t_1} \) at \( t = t_1 \), which is a piecewise smooth one with constant speed. Then from the path, we can define a piecewise smooth curve
\[ c^n_{t_0} : [0, 1] \to \partial U_{j-1} \subset D^2 \]
with constant speed, where \( U_{j-1} \) is the union of triangle in \( A_n \) from the 1st one to the \( (j - 1)\)-th one.

And its opposite direction can make us define
Define a piecewise smooth curve

\[ \varphi^n_{t_0} : [0, 1] \to D^2 \]

with constant speed as the boundary curve of the \( j \)-th triangle in the \( n \)-th step, where the curve is a loop at the starting point of the triangle and the direction of the boundary curve is induced from the given orientation.

Now define a piecewise smooth curve

\[ \gamma^n_{t_0} : [0, 1] \to \partial U_j \subset D^2 \]

with constant speed from the simplification of \( \gamma^n_{t_1} * \varphi^n_{t_0} * \varphi^n_{t_0} * \varphi^n_{t_0} \), where \( U_j \) is the union of triangle in \( A_n \) from the 1st one to the \( j \)-th one. The new curve will be also called the holonomy curve at time \( t = t_0 \).

Now, consider the path from the basepoint to the ending point of the \( j \)-th triangle in the \( n \)-th step along the opposite direction of the holonomy curve \( \gamma^n_{t_0} \) at \( t = t_0 \), which is a piecewise smooth one with constant speed. Then from the path, we can define a piecewise smooth curve

\[ 1c^n_{t_0} : [0, 1] \to \partial U_j \subset D^2 \]

with constant speed. And its opposite direction can make us define

\[ 1\tilde{c}^n_{t_0} : [0, 1] \to \partial U_j \subset D^2. \]

Define a piecewise smooth curve

\[ \psi^n_{t_0} : [0, 1] \to D^2 \]

with constant speed as the boundary curve of the \( j \)-th triangle in the \( n \)-th step, where the curve is a loop at the ending point of the \( j \)-th triangle and the direction of the boundary curve is induced from the given orientation.

2.5. the simplification of \( \tilde{c}^n_{t_0} * 1c^n_{t_0} \). For each \( n \geq 1 \) and \( 0 \neq t_0 \in D_n \), where \( t_0 \) is the \( j_n(t_0) \)-th element in \( D_n \), the simplification of \( \tilde{c}^n_{t_0} * 1c^n_{t_0} \) is a curve along the boundary of \( j_n(t_0) \)-th triangle in \( A_n \) with opposite direction to the given orientation such that it starts from the starting point of the triangle and that its image consists of the following sets:

- one point, one side, two sides or the boundary of the triangle.
2.6. The induced curves on the surface $S \subset \mathbb{H}^n$ and totally geodesic planes in $\mathbb{H}^n$. Let $\Phi : D^2 \to S$ be a given diffeomorphism. Then we can think of triangles in $S$ induced from the barycentric subdivision on $D^2$ on each $n$-th step. We will use $' \sim ' \text{ notation for the induced triangles and curves in } S$, that is,

$$\tilde{T} = \Phi(T) \quad \text{for} \quad T \in A_n$$

and

$$\tilde{A}_n = \{ \Phi(T) \mid T \in A_n \}$$

which are piecewise smooth curves with constant speed such that

$$\text{Im}(\tilde{\gamma}^n_{t_0}) = \text{Im}(\Phi \circ \gamma^n_{t_0})$$

$$\text{Im}(\tilde{\varphi}^n_{t_0}) = \text{Im}(\Phi \circ \varphi^n_{t_0})$$

$$\text{Im}(\tilde{\psi}^n_{t_0}) = \text{Im}(\Phi \circ \psi^n_{t_0})$$

and whose direction relates to that of $\gamma^n_{t_0}, \varphi^n_{t_0}, \psi^n_{t_0}, 1c^n_{t_0}, 1c^n_{t_0}$, respectively.

Now with respect to each triangle in $S$, we can think of a totally geodesic triangle with same vertices in $\mathbb{H}^n$. So, each step will induce the similar concept, i.e. triangles and curves, on the induced pleated surface consisting of totally geodesic triangles and we’ll use $' \land ' \text{ notation for them. In other words, we can think of}$

$$\tilde{T} \in \tilde{A}_n, \tilde{\gamma}^n_{t_0}, \tilde{\varphi}^n_{t_0}, \tilde{\psi}^n_{t_0}, 1c^n_{t_0}, 1c^n_{t_0},$$

where the curves $\tilde{\gamma}^n_{t_0}, \tilde{\varphi}^n_{t_0}, \tilde{\psi}^n_{t_0}, 1c^n_{t_0}, 1c^n_{t_0}$ are piecewise geodesics in $\mathbb{H}^n$, induced from the boundaries of totally geodesic triangles $\tilde{T}$, and are relating to the previous curves $\gamma^n_{t_0}, \varphi^n_{t_0}, \psi^n_{t_0}, 1c^n_{t_0}, 1c^n_{t_0}$ in $S$ and $\gamma^n_{t_0}, \varphi^n_{t_0}, \psi^n_{t_0}, 1c^n_{t_0}, 1c^n_{t_0}$ in $D^2$. 

3. In \( \text{SO}(2) \to \text{SO}_0(1, 2) \to \mathbb{H}^2 \)

For

\[
E_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad E_3 = [E_1, E_2] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},
\]

let \( \{E_1, E_2, E_3\} \) be an ordered orthonormal basis of \( \mathfrak{so}(1, n) \), which induces the canonical orientation on \( \text{SO}_0(1, 2) \), and so induces the canonical orientation on \( \mathbb{H}^2 \), that is, the counterclockwise one.

For \( t \in \mathbb{R} \), put

\[
\Psi(\alpha) = \exp(tE_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix}.
\]

Let \( c : [t_0, t_3] \to \mathbb{H}^2 \) be a simple-closed arc-length parameterized piecewise-smooth curve representing a geodesic triangle in \( \mathbb{H}^2 \):

More precisely,

\( c \) is continuous on \([t_0, t_3]\) and smooth on \((t_0, t_1) \cup (t_1, t_2) \cup (t_2, t_3)\), where \( c(t_0) = c(t_3) \), \( c(t_1) \) and \( c(t_2) \) are vertices of the given geodesic triangle.

Let

\[
\begin{align*}
\alpha & \text{ be the angle from } \dot{c}(t_0^+) \text{ to } -\dot{c}(t_3^-), \\
\beta & \text{ be the angle from } \dot{c}(t_1^+) \text{ to } -\dot{c}(t_1^-), \text{ and} \\
\gamma & \text{ be the angle from } \dot{c}(t_2^+) \text{ to } -\dot{c}(t_2^-).
\end{align*}
\]

Then, either \( \alpha, \beta, \gamma > 0 \) or \( \alpha, \beta, \gamma < 0 \) holds.

**Lemma 3.1.** Under the above condition, let \( \tilde{c} : [t_0, t_3] \to \text{SO}_0(1, 2) \) be a horizontal lift of \( c \). Then, the relation between the holonomy and the area of a geodesic triangle is given by

\[
\tilde{c}(t_0)^{-1} \cdot \tilde{c}(t_3) = \left( \Psi\left( \pi - |\alpha + \beta + \gamma| \right) \right)^\delta,
\]

where

\[
\delta = \begin{cases} 
1 & \text{if } \alpha, \beta, \gamma > 0 \\
-1 & \text{if } \alpha, \beta, \gamma < 0
\end{cases}
\]
Furthermore, $\pi - |\alpha + \beta + \gamma|$ is the area of the geodesic triangle.

Proof. Let $\pi : \text{SO}_0(1, 2) \to \mathbb{H}^2$ be the given Riemannian submersion.

Recall that for any $k \in \text{SO}(2)$, the restriction $Ad_k|_{\mathfrak{so}(2)}^\perp$ of $Ad_k(\cdot) : \mathfrak{so}(1, 2) \to \mathfrak{so}(2)^\perp$ is an automorphism of $\mathfrak{so}(2)^\perp$. For $-\dot{c}(t_3^-)$ and its horizontal lift $x$ at $\tilde{c}(t_0)$, find $A \in \mathfrak{so}(2)^\perp$ satisfying

$$L_{\tilde{c}(t_0)^{-1}} \dot{x} = A.$$

Then $\dot{\tilde{c}}(t_0^+), \dot{\tilde{c}}(t_1^+)$ and $\dot{\tilde{c}}(t_2^+)$ will be related to $Ad_{\Psi(\alpha)}A, Ad_{\Psi(\pi)\cdot\Psi(\alpha)\cdot\Psi(\beta)}A$ and $Ad_{\Psi(\alpha)\cdot\Psi(\beta)\cdot\Psi(\gamma)}A$, respectively, by

$$L_{\tilde{c}(t_0)^{-1}} \dot{\tilde{c}}(t_0^+) = (Ad_{\Psi(\alpha)}A)_e$$

$$L_{\tilde{c}(t_1)^{-1}} \dot{\tilde{c}}(t_1^+) = (Ad_{\Psi(\beta)}(-Ad_{\Psi(\alpha)}A))_e = (Ad_{\Psi(\pi)\cdot\Psi(\alpha)\cdot\Psi(\beta)}A)_e$$

$$L_{\tilde{c}(t_2)^{-1}} \dot{\tilde{c}}(t_2^+) = (Ad_{\Psi(\gamma)}(-Ad_{\Psi(\pi)\cdot\Psi(\alpha)\cdot\Psi(\beta)}A))_e = (Ad_{\Psi(\alpha)\cdot\Psi(\beta)\cdot\Psi(\gamma)}A)_e,$$

in other words,

$$\dot{\tilde{c}}(t_0^+) = L_{\tilde{c}(t_0)}(Ad_{\Psi(\alpha)}A)_e$$

$$\dot{\tilde{c}}(t_1^+) = L_{\tilde{c}(t_1)}(Ad_{\Psi(\pi)\cdot\Psi(\alpha)\cdot\Psi(\beta)}A)_e$$

$$\dot{\tilde{c}}(t_2^+) = L_{\tilde{c}(t_2)}(Ad_{\Psi(\alpha)\cdot\Psi(\beta)\cdot\Psi(\gamma)}A)_e$$

And so
\[-\dot{c}(t_3^-)\] will be related to

\[-Ad \Psi(\alpha) \cdot \Psi(\beta) \cdot \Psi(\gamma) A = Ad \Psi(\pi) \cdot \Psi(\alpha) \cdot \Psi(\beta) \cdot \Psi(\gamma) A\, ,\]

that is,

\[-\dot{c}(t_3^-) = L\dot{c}(t_3) \cdot (Ad \Psi(\pi) \cdot \Psi(\alpha) \cdot \Psi(\beta) \cdot \Psi(\gamma) A) \, e^A .\]

Therefore,

\[
\pi(\dot{c}(t_0) \cdot e^{tA}) = \pi(\dot{c}(t_3) \cdot e^{t(Ad \Psi(\pi) \cdot \Psi(\alpha) \cdot \Psi(\beta) \cdot \Psi(\gamma) A)})
\]

\[
= \pi(\dot{c}(t_0) \cdot e^{tAd \dot{c}(t_0)^{-1} \cdot \dot{c}(t_3)} \cdot e^{tAd \Psi(\pi) \cdot \Psi(\alpha) \cdot \Psi(\beta) \cdot \Psi(\gamma) A} \cdot e^{tA \cdot \dot{c}(t_0)^{-1} \cdot \dot{c}(t_3)})
\]

because

\[
\dot{c}(t_0)^{-1} \cdot \dot{c}(t_3) \in SO(2)
\]

and

\[
B \in so(2)^\perp \text{ and } k \in SO(2) \Rightarrow k \cdot e^{tB} \cdot k^{-1} = e^{tAd_k B} .
\]

Thus, we get

\[
A = Ad \dot{c}(t_0)^{-1} \cdot \dot{c}(t_3) \cdot \Psi(\pi) \cdot \Psi(\alpha) \cdot \Psi(\beta) \cdot \Psi(\gamma) A
\]

and so

\[
\dot{c}(t_0)^{-1} \cdot \dot{c}(t_3) \cdot \Psi(\pi + (\alpha + \beta + \gamma)) = \Psi(2n\pi) \quad \text{for some } n \in \mathbb{Z} .
\]

Therefore,

\[
\dot{c}(t_0)^{-1} \cdot \dot{c}(t_3) = \Psi(2n\pi) \cdot \left( \Psi(\pi + (\alpha + \beta + \gamma)) \right)^{-1}
\]

\[
= \left( \Psi(\pi + (\alpha + \beta + \gamma)) \right)^{-1}
\]

\[
= \bigg\{ \begin{array}{ll}
\Psi(-\pi - (\alpha + \beta + \gamma)) & \text{if } \alpha, \beta, \gamma > 0 \\
\Psi(\pi + (\alpha + \beta + \gamma))^{-1} & \text{if } \alpha, \beta, \gamma < 0
\end{array}
\]

\[
= \bigg\{ \begin{array}{ll}
\Psi(\pi - (\alpha + \beta + \gamma)) & \text{if } \alpha, \beta, \gamma > 0 \\
\left( \Psi(\pi - ((-\alpha) + (-\beta) + (-\gamma))) \right)^{-1} & \text{if } \alpha, \beta, \gamma < 0
\end{array}
\]

\[
= \left( \Psi(\pi - |\alpha + \beta + \gamma|) \right)^\delta , \text{ where } \delta = \bigg\{ \begin{array}{ll}
1 & \text{if } \alpha, \beta, \gamma > 0 \\
-1 & \text{if } \alpha, \beta, \gamma < 0
\end{array}
\]

\[
= \Psi(\pi) \cdot \Psi(\alpha) \cdot \Psi(\beta) \cdot \Psi(\gamma) A
\]

\[
\square
\]

4. **Liftings in SO(n) \rightarrow SO_0(1, n) \rightarrow \mathbb{H}^n**

This section is the proof of Theorem 0.1.
4.1. Preliminaries on the Riemannian submersion $\pi : \text{SO}_0(1,n) \to \mathbb{H}^n$. Let $\pi : \text{SO}_0(1,n) \to \mathbb{H}^n$ be the given Riemannian submersion. In fact, this is the quotient of the isometric right translation by $\text{SO}(n)$ and $\mathbb{H}^n$ is isometric to $\text{SO}_0(1,n)/\text{SO}(n)$.

Let $G = \text{SO}_0(1,n)$, $K = \text{SO}(n)$, and $\mathfrak{g}$ and $\mathfrak{k}$ be their Lie algebras, respectively.

Fact 1) $X \in \mathfrak{k} \perp \Rightarrow t \mapsto g \cdot \exp(tX) : (-\infty, \infty) \to G$ is a horizontal geodesic for any $g \in G$.

Fact 2) $X,Y \in \mathfrak{k} \perp \Rightarrow [X,Y] \in \mathfrak{k}$ and $\operatorname{Span}\{X,Y,[X,Y]\} \subset \mathfrak{g}$ is a Lie subalgebra of $\mathfrak{g}$ and its related subgroup $H$ of $\text{SO}_0(1,n)$ is isometric to $\text{SO}_0(1,2)$. Furthermore, the riemannian submersion $\text{SO}_0(1,n) \to \text{SO}(n)$ can be restricted to $(H = \text{SO}_0(1,2)) \to \text{SO}(2)$.

Fact 3) $U \in \mathfrak{k} \Rightarrow t \mapsto k \cdot \exp(tU) : (-\infty, \infty) \to K \subset G$ is a vertical geodesic for any $k \in K$.

Fact 4) For any $k \in K$, the right translation $R_k : G \to G$ by $k, R_k(g) = gk$, is an isometry. Or, equivalently, $\operatorname{Ad}_k : \mathfrak{g} \to \mathfrak{g}$ is a linear isometry for any $k \in K$.

4.2. Definition of $\bar{f} : \bigcup_{m=1}^{\infty} D_m \to K = \text{SO}(n), \bar{f}_m : D_m \to \text{SO}(n)$ and $\hat{f}_m : [0,1] \to \text{SO}(n)$ and the property of $\hat{f}_m$.

4.2.1. Definition of $\bar{f}, \bar{f}_m$. Let $\bar{f}(0) = e.$ Fix $t_0 \in \bigcup_{m=1}^{\infty} D_m - \{0\}$. Then we can find a positive integer $m_0 = \min \{m_1 \mid m + 1 \geq m_1 \Rightarrow t_0 \in D_m\}$.

Note that on the given surface $S$,

$$\bar{\gamma}_{t_0}^n = \bar{\gamma}_{t_0}^{n_0} \text{ and } \bar{c}_{t_2}^n(t_0) = 1^{n_0} \cdot \bar{c}_{t_2}^{n_0} = 1^{n_0} \cdot \bar{c}_{t_2}^{n_0}(t_0)$$

for all $n \geq n_0$. So let

$$\bar{\gamma}_{t_0} := \bar{\gamma}_{t_0}^{n_0} \text{ and } 1^{n_0} \cdot \bar{c}_{t_0} := 1^{n_0} \cdot \bar{c}_{t_0}$$

Define

$$\hat{f}(t_0) := \text{ the value, at } t = 1, \text{ of the horizontal lifting of } \bar{\gamma}_{t_0} \text{ at } e.$$

Put $\hat{f}_n$ as the restriction $\hat{f}_n |_{\bigcup_{n=1}^{\infty} D_n}$ of $\hat{f}_n$, defined below, to $D_n$.  


4.2.2. Definition of \( \hat{f}_n \) and its property. Let’s define a curve \( \hat{f}_n : [0, 1] \to K = \text{SO}(n) \) with \( \hat{f}_n(0) = e \) inductively as follows:

Step 1) Assume \( t_0 \in D_n \) is the 1st element in \( D_n \), in fact, \( t_0 = \frac{1}{2} \cdot \frac{1}{e^n} \). Then, \( t_1^n(t_0) = 0 \).

Consider the first triangle in \( \hat{A}_n \), its starting point and the horizontal lifting of

\[
x := \lim_{t \to 0^+} \frac{1}{|\dot{\hat{\varphi}}_{t_0}^n(t)|} \cdot \dot{\hat{\varphi}}_{t_0}^n(t)
\]

and

\[
y := - \lim_{t \to 1^-} \frac{1}{|\dot{\hat{\varphi}}_{t_0}^n(t)|} \cdot \dot{\hat{\varphi}}_{t_0}^n(t)
\]

at \( e \), respectively and find

\[
X_0^n = X_{t_1^n(t_0)}^n, \quad Y_0^n = Y_{t_1^n(t_0)}^n \in \mathfrak{t}^\perp
\]

with

\[
\pi_* X_0^n|_e = \pi_* X_{t_1^n(t_0)}^n|_e = x
\]

and

\[
\pi_* Y_0^n|_e = \pi_* Y_{t_1^n(t_0)}^n|_e = y.
\]

Then, define

\[
\hat{f}_n(t) := \exp \left( t \cdot \frac{(\text{Area of the 1st triangle in } \hat{A}_n)}{t_0 \cdot |X_0^n, Y_0^n|} \cdot [X_0^n, Y_0^n] \right)
\]

for \( t \in [0, t_0] \).
which is a geodesic in $K = \text{SO}(n)$ from Fact 3.

Step 2) Assume $t_0 \in D_n$ is the $j$-th element in $D_n$, where $j \geq 2$.
Note $t^n_1(t_0)$ is the $(j-1)$-th element in $D_n$, where $j-1 \geq 1$.

Let $f_n(t^n_1(t_0))\hat{c}_t^n : [0,1] \to \text{SO}_0(1,n)$ be the horizontal lifting of $\hat{c}_t^n$ at $\hat{f}_n(t^n_1(t_0))$ and then consider the $j$-th triangle in $\hat{A}_n$, its starting point and the horizontal lifting of

$$x := \lim_{t \to 0^+} \frac{1}{|\dot{\phi}_t^n(t)|} \cdot \dot{\phi}_t^n(t)$$

and

$$y := -\lim_{t \to 1^-} \frac{1}{|\dot{\phi}_t^n(t)|} \cdot \dot{\phi}_t^n(t)$$

at $g := \hat{f}_n(t^n_1(t_0))\hat{c}_t^n(1)$, respectively and find

$$X_{t^n_1(t_0)}^n, Y_{t^n_1(t_0)}^n \in \mathfrak{k}^\perp$$

with

$$\pi_* X_{t^n_1(t_0)}^n |_g = x$$

and

$$\pi_* Y_{t^n_1(t_0)}^n |_g = y.$$
\[ \hat{f}_n(t) := \hat{f}_n(t_1^n(t_0)) \cdot \exp \left( (t - t_1^n(t_0)) \cdot \frac{\text{(Area of } j\text{-th triangle in } \hat{A}_n)}{\left| (t_0 - t_1^n(t_0)) \cdot |X_1^n(t_0), Y_1^n(t_0)| \right|} \cdot [X_1^n(t_0), Y_1^n(t_0)] \right) \]

for \( t \in [t_1^n(t_0), t_0] \).

Step 3) \( t_0 = 1 \)

Note \( t_1^n(1) \) be the last element in \( D_n \), in other words, \( t_1^n(1) = \sum_{i=1}^{n+1} \frac{1}{2^i} \).

Then define \( \hat{f}_n(t) := \hat{f}_n(t_1^n(1)) \)

for \( t \in [t_1^n(1), 1] \).

Now check the property of \( \hat{f}_n \).

Assume \( 0 \neq t_0 \in D_n \) is a \( j \)-th element in \( D_n \), where \( j \geq 1 \). Then \( t_1^n(t_0) \) is the \((j-1)\)-th elements in \( D_n \), where \( j-1 \geq 0 \), and from Facts, mentioned early in this section, and from the property in Section 3 we get

\[ \hat{f}_n(t_0) = \text{the value, at } t = 1, \text{ of the horizontal lifting of } \hat{c}_0^n \ast \hat{\varphi}_0^n \ast \hat{\gamma}_0^n \text{ at } \hat{f}_n(t_1^n(t_0)) \]

\[ = \text{the value, at } t = 1, \text{ of the horizontal lifting of } \hat{\gamma}_0^n \text{ at } e. \]

Define, for any \( g \in G \), \( l_g : K \to G \) by \( l_g(k) = gk \), which is an isometric imbedding of \( K \) onto the fiber \( gK \).
And let $\omega$ and $\Omega$ be the connection form and the curvature form of the connection of the principal bundle $\pi : SO_0(1, n) \to \mathbb{H}^n$, respectively.

Then, under the identification of $T_c G$ and $g$, for $t \in (t^n_1(t_0), t_0)$ and $g = f_n(t^n_1(t_0)) \tilde{c}^n_{t_0}(1)$, which is the value, at $t = 1$, of the horizontal lifting of $\tilde{c}^n_{t_0}$ at $\hat{f}_n(t^n_1(t_0))$,

$$\omega \left( \frac{1}{|f_n(t)|} \cdot \dot{f}_n(t) \right) = (l_{f_n(t)})*^{-1} \left( \frac{1}{|f_n(t)|} \cdot \dot{f}_n(t) \right)$$

$$= (l_{f_n(t)})*^{-1} \left( \frac{1}{|f_n(t)|} \cdot \dot{f}_n(t) \right)$$

$$= (l_{f_n(t)})*^{-1} \left( \frac{1}{|f_n(t)|} \cdot \dot{f}_n(t) \right)$$

$$= \frac{1}{|f_n(t)|} \cdot \dot{f}_n(t)$$

and

$$\omega \left( \frac{1}{|f_n(t)|} \cdot \dot{f}_n(t) \right) \mid e = L_{(f_n(t))^{-1}} \left( \frac{1}{|f_n(t)|} \cdot \dot{f}_n(t) \right)$$

Roughly speaking, the unit tangent vector $\frac{1}{|f_n(t)|} \cdot \dot{f}_n(t)$, $t \in (t^n_1(t_0), t_0)$, is the negative of the unit curvature of the 2-dimensional horizontal plane

$$\dot{H}_g = \text{Span}\{X^n_{t^n_1(t_0)} \mid g, Y^n_{t^n_1(t_0)} \mid g\},$$

where $g = f_n(t^n_1(t_0)) \tilde{c}^n_{t_0}(1)$, which projects to the tangent plane of the $j_n(t_0)$-th triangle in $\hat{A}_n$ at $\pi(g) = \tilde{c}^n_{t_0}(1)$ = the starting point of the $j_n(t_0)$-th triangle in $\hat{A}_n$. And, the length of $\dot{f}_n \mid [t^n_1(t_0), t_0]$ is the area of the $f_n(t_0)$-th triangle in $\hat{A}_n$.

4.3. The convergence of $\hat{f}_n(t_0)$ to $\tilde{f}(t_0)$. Recall

$$\hat{f}(t_0) = \text{the value , at } t = 1, \text{ of the horizontal lifting of } \hat{\gamma}_{t_0} \text{ at } e$$

and

$$\check{f}_n(t_0) = \hat{f}_n(t_0) = \text{the value , at } t = 1, \text{ of the horizontal lifting of } \hat{\gamma}^n_{t_0} \text{ at } e .$$

Consider our Riemannian submersion

$$SO(n) \longrightarrow SO_0(1, n) \longrightarrow SO_0(1, n)/SO(n).$$

This bundle has a global cross section $s : \mathbb{H} \to NA \subset G$, which comes from the Iwasawa decomposition $NAK$, where $K = SO(n)$. That is, every element of $G$ is uniquely written as $nak$, and the projection maps this to $n a K \in \mathbb{H}$.
The equation (4–1) comes about as follows. The curve ˜
implies the following equalities should hold.

\[ \text{V for every } V \text{ satisfying (4–1)} \]

By abusing of notation, express \( s \circ h \) by \( h \). For a curve \( h : [0, 1] \to \mathbb{H}^n \), the unique horizontal lift \( h : [0, 1] \to SO_0(n, 1) \) is given by

\[ h(t) \cdot a(t) = \tilde{h}(t) \]

for a unique curve \( a(t) \) in \( SO(n) \). Such an \( a(t) \) is obtained by solving the differential equation

\[ \langle h^{-1} h' + a' a^{-1}, V \rangle = 0 \]

for every \( V \in \mathfrak{k} \), where \( ' \) means the derivative with respect to \( t \). Note that the first entry \( h^{-1} h' + a' a^{-1} \) is an element of the Lie algebra \( \mathfrak{so}(n, 1) \).

The equation (4–1) comes about as follows. The curve \( \tilde{h}(t) \) being horizontal implies the following equalities should hold.

\[ 0 = \langle (h(t)a(t))', (h(t)a(t))V \rangle = \langle h'(t)a(t) + h(t)a'(t), (h(t)a(t))V \rangle = \langle (h(t)a(t)) (a(t)^{-1} h(t)^{-1} h'(t)a(t) + a^{-1}(t)a'(t)) , (h(t)a(t))V \rangle, \]

for every \( V \in \mathfrak{k} \), on the tangent space at \( h(t)a(t) \). Since the metric on \( G \) is left-invariant, this implies

\[ 0 = \langle a(t)^{-1} h(t)^{-1} h'(t)a(t) + a^{-1}(t)a'(t), V \rangle, \]

for every \( V \in \mathfrak{k} \), on the tangent space at \( e \), \( G_e = \mathfrak{g} \). Since this holds for all \( V \in \mathfrak{k} \) and \( a(t) \in K \), by taking conjugation by \( a(t) \), the above is equivalent to the equality (4-1) above.

We examine the equalities (4-1) more closely. The equality holds for every \( V \in \mathfrak{k} \) implies that \( h(t)^{-1} h'(t) + a'(t)a^{-1}(t) \) does not have any vertical component. That is, \( -a'(t)a^{-1}(t) \) is the vertical component of \( h(t)^{-1} h'(t) \) so that

\[ h(t)^{-1} h'(t) = -a'(t)a^{-1}(t) + X_1 \in \mathfrak{k} \oplus \mathfrak{t}^\perp. \]

is a vertical and horizontal splitting.

Let \( g(t) \) be another path with a unique horizontal lift \( \tilde{g}(t) = g(t)b(t) \), satisfying

\[ 0 = \langle g^{-1} g', b' b^{-1}, V \rangle, \]

for every \( V \in \mathfrak{k} \). Again, we have a splitting

\[ g(t)^{-1} g'(t) = -b'(t)b^{-1}(t) + X_2 \in \mathfrak{k} \oplus \mathfrak{t}^\perp. \]

From

\[ ||h(t)^{-1} h'(t) - g(t)^{-1} g'(t)|| = ||a'(t)a^{-1}(t) - b'(t)b^{-1}(t)|| + ||X_1 - X_2||, \]
we get
\[
(4-3) \quad ||a'(t)a^{-1}(t) - b'(t)b^{-1}(t)|| \leq ||h(t)^{-1}h'(t) - g(t)^{-1}g'(t)||.
\]
These are norms on the Lie algebra \(\mathfrak{so}(1, n)\).

On the space of piecewise \(C^k\)-curves \((k \geq 1)\) in \(\text{SO}_0(1, n)\) with initial point \(e\), we define a distance function by
\[
\rho(h, g) = \int_0^1 ||h'(t)\cdot h(t)^{-1} - g'(t)\cdot g(t)^{-1}|| \, dt.
\]
Note that \(h'(t)\cdot h(t)^{-1} \in \mathfrak{so}(n, 1)\) and \(||.||\) is the norm there. We argue that this is a metric. Suppose \(\rho(h, g) = 0\). Then, by continuity, \(h'(t)\cdot h(t)^{-1} = g'(t)\cdot g(t)^{-1}\) for every \(t\). Now we apply the following Lemma to the \(C^1\)-curves piece by piece to conclude \(h(t) = g(t)\) for all \(t \in [0, 1]\), see [3], vol 1, p69.

**Lemma 4.1.** Let \(G\) be a Lie group and \(\mathfrak{g}\) its Lie algebra identified with \(T_e(G)\). Let \(Y_t, 0 \leq t \leq 1, \) be a continuous curve in \(T_e(G)\). Then there exists in \(G\) a unique curve \(a_t\) of class \(C^1\) such that \(a_0 = e\) and \(a_t a_t^{-1} = Y_t\) for \(0 \leq t \leq 1\).

Let \(h\) be a curve in \(\mathbb{H}^n\) (or in \(NA\), by abuse of notation). The unique curve \(a : [0, 1] \to \text{SO}(n)\) such that \(h(t)\cdot a(t)\) is the horizontal lift of \(h(t)\) will be called \(w_h\).

For two curves \(h\) and \(g\), the inequality \((4-3)\) shows that \(\rho(w_h, w_g) \leq \rho(h, g)\). Let \(\mathfrak{P}\) be the space of all piecewise \(C^k\)-curves on \(NA\) with the initial point \(e\).

**Proposition 4.2.** The map \(\mathfrak{P} \to G\) sending \(f\) to \(w_h(1)\) is continuous. More precisely, let \(h : [0, 1] \to NA\) be a piecewise \(C^k\)-curve. For every \(\epsilon > 0\), there exists \(\delta > 0\) such that, if \(g \in \mathfrak{P}\) and \(\rho(h, g) < \delta\), then \(d(e, w_h(1)^{-1}\cdot w_g(1)) = d(w_h(1), w_g(1)) < \epsilon\).

**Proof.** For simplicity, we write \(w_h(t), w_g(t)\) by \(a(t), b(t)\), respectively.
\[
(4-4) \quad 0 = (aa^T)' = a'a^T + a(a^T)'\]
Clearly \(a(a^T)b^T\) is a well-defined element of the Lie algebra \(\mathfrak{k}\) and
\[
a(a^T)b^T = a((a^T)'b + a^Tb')b^T
\]
\[
= a(a^T)' + b'b^T
\]
\[
= -a'a^T + b'b^T \quad \text{(from the equality (4-4))}
\]
\[
= -a'a^{-1} + b'b^{-1}
\]
Thus,
\[
||a'a^{-1} - b'b^{-1}|| = ||a(a^T)b'\cdot b^T||.
\]
Observe that \((aTb') \in T_e r_T(K)\). The left translation \(\ell_a\) and the right translation \(r_T\) maps this vector to a tangent vector at \(0 \in T_e(K)\). However, both these translations are isometries so that they preserve the norms. We have,

\[ ||a'a^{-1} - b'b^{-1}|| = ||a(aTb)b'T|| = ||(aTb')|| = ||(a^{-1}b')||.\]

Consequently, if

\[ \int ||(a^{-1}b')|| dt = \int ||a'a^{-1} - b'b^{-1}|| dt \]

is small, the arc-length of the path \(a(t)^{-1}b(t)\) is small. Therefore, if \(a(0)\) and \(b(0)\) were close (or if \(a(0) = b(0)\), then \(a(1)\) and \(b(1)\) are close. This finishes the proof. \(\square\)

Since

\[ \bar{f}_n(t_0) = \text{the value, at } t = 1, \text{ of the horizontal lifting of } \zeta^n_{t_0} \text{ at } e \]

and

\[ \zeta^n_{t_0} \text{ converges to } \zeta^{n_0}_{t_0} = \zeta_{t_0} \text{ as } n \text{ goes to } \infty, \]

we get, by reparameterizing \(\zeta^{n_0}_{t_0}\) and \(\zeta_{t_0}\) respectively if needed,

\[ \bar{f}(t_0) = \text{the value, at } t = 1, \text{ of the horizontal lifting of } \zeta_{t_0} \text{ at } e \]

\[ = \lim_{n \to \infty} \bar{f}_n(t_0). \]

4.4. Preliminaries for the main proof. Fix \(t_0 \in \bigcup_{n=1}^{\infty} D_n - \{0\}\) and find a positive integer \(n_0 = \min \{ n_1 \mid n + 1 \geq n_1 \Rightarrow t_0 \in D_n \}\).

Assume \(n \geq n_0\).

Note \(t_0\) is not the last element in \(D_n\) for \(n \geq n_0\). Notice that with respect to totally geodesic triangles, \(c_{t_2}^{n_0}(t_0)(1) = 1 \zeta_{t_0}^{n_0}(1) = 1 \zeta_{t_0}(1) = c_{t_2}^{n_0}(1)\) for all \(n \geq n_0\), which is the ending point of the \(j_n(t_0)-\)th triangle in \(\hat{A}_n\) and also the starting point of the \((j_n(t_2)(t_0)) = j_n(t_0) + 1\)-th triangle in \(\hat{A}_n\) for all \(n \geq n_0\).

4.4.1. A new curve \(c^{\text{short}}_{t_0}\) for the comparison of triangles. Define \(c^{\text{short}}_{t_0} : [0, 1] \to \mathbb{H}^n\) as the shortest geodesic from \(\pi(e) \in \mathbb{H}^n\) to \(c_{t_2}^{n_0}(t_0)(1) = \zeta_{t_0}^{n_0}(1) = \zeta_{t_0}(1) = c_{t_2}^{n_0}(1)\), in other words, to the starting point of \((j_n(t_2)(t_0)) = j_n(t_0) + 1\)-th triangle in \(\hat{A}_n\), which is also the ending point of \(j_n(t_0)-\)th triangle in \(\hat{A}_n\). Consider its horizontal lift

\[ e^{c^{\text{short}}_{t_0}} : [0, 1] \to \text{SO}_0(1, n) \]

at \(e\).
4.4.2. **the comparison of** \((j_n(t_{2n}(t_0))) = j_n(t_0) + 1\)-**th totally geodesic triangles** . For each \(n \geq n_0\), consider

\[ e \hat{c}^n_{t_2}(t_0) := R_{f_n(t_0)^{-1}} \circ f_n(t_0) \hat{c}^n_{t_2}(t_0) : [0, 1] \rightarrow SO(1, n) , \]

which is the horizontal lifting of \( \hat{c}^n_{t_2}(t_0) \) at \( e \), that is,

\[ \pi \circ e \hat{c}^n_{t_2}(t_0) = \hat{c}^n_{t_2}(t_0) = \pi \circ f_n(t_0) \hat{c}^n_{t_2}(t_0) . \]

Note \( e \hat{c}^n_{t_2}(t_0) \) and \( f_n(t_0) \hat{c}^n_{t_2}(t_0) \) are piecewise geodesics, since the right translation \( R_k : G \rightarrow G \) by \( k \) is an isometry for any \( k \in K = SO(n) \) and \( \hat{c}^n_{t_0} \) are piecewise geodesics.

Consider the \((j_n(t_{2n}(t_0))) = j_n(t_0) + 1\)-th triangle in \( \hat{A}_n \), its starting point and the horizontal lifting of

\[ x := \lim_{t \rightarrow 0^+} \frac{1}{\| \hat{\varphi}^n_{t_2}(t_0) \|} \cdot \hat{\varphi}^n_{t_2}(t_0)(t) \]

and

\[ y := -\lim_{t \rightarrow 1} \frac{1}{\| \hat{\varphi}^n_{t_2}(t_0) \|} \cdot \hat{\varphi}^n_{t_2}(t_0)(t) \]

at \( e \hat{c}^n_{t_2}(t_0)(1) =: g \), respectively and find

\[ \hat{X}^n_{t_0}, \hat{Y}^n_{t_0} \in T^\perp \]

with

\[ \pi_* \hat{X}^n_{t_0} \big|_g = x \]

and

\[ \pi_* \hat{Y}^n_{t_0} \big|_g = y . \]
Also consider the horizontal lifting of $x$ and $y$ at $\exp_{\tilde{c}_{t_0}^{\text{short}}(1)} =: g_{t_0}$ and find

$$\tilde{X}_{t_0}^n, \tilde{Y}_{t_0}^n \in \mathfrak{t}^\perp$$

with

$$\pi_* \tilde{X}_{t_0}^n | g_{t_0} = x$$

and

$$\pi_* \tilde{Y}_{t_0}^n | g_{t_0} = y.$$  

Note

$$f_n(t_0) e_{t_2}^n (t_0) = f_n(t_1^n(t_2^n(t_0))) e_{t_2}^n (t_0),$$

so

$$\pi_* X_{t_0}^n | f_n(t_0) e_{t_2}^n (t_0) (1) = \pi_* X_{t_1^n(t_2^n(t_0))}^n | f_n(t_1^n(t_2^n(t_0))) e_{t_2}^n (t_0) (1)$$

$$= \lim_{t \to 0^+} \frac{1}{\tilde{\varphi}_{t_2}^n (t_0) (t)} \cdot \hat{\varphi}_{t_2}^n (t_0) (t)$$

$$= x$$

$$= \pi_* \tilde{X}_{t_0}^n | g$$

$$= \pi_* \tilde{X}_{t_0}^n | e_{t_2}^n (t_0) (1),$$

which implies

$$X_{t_0}^n = Ad(f_n(t_0))^{-1} \tilde{X}_{t_0}^n$$

from $f_n(t_0) e_{t_2}^n (t_0) = R f_n(t_0) \circ e_{t_2}^n (t_0).$

Similarly,

$$Y_{t_0}^n = Ad(f_n(t_0))^{-1} \tilde{Y}_{t_0}^n.$$

And by considering a loop

$$\tilde{c}_{t_0}^{\text{short}} \ast \tilde{c}_{t_0}^n (t_2(t_0)) : [0, 1] \to \mathbb{H}^n,$$

where $\tilde{c}_{t_0}^{\text{short}} : [0, 1] \to \mathbb{H}^3$ is given by

$$\tilde{c}_{t_0}^{\text{short}} (t) = \tilde{c}_{t_0}^{\text{short}} (1 - t),$$

and its horizontal lifting at $\exp_{\tilde{c}_{t_0}^{\text{short}}(1)}$, we obtain

$$\tilde{X}_{t_0}^n = Ad((\exp_{\tilde{c}_{t_0}^{\text{short}}(1)})^{-1} \ast \exp_{t_2^n(t_0)} (1))^{-1} \tilde{X}_{t_0}^n$$

$$\tilde{Y}_{t_0}^n = Ad((\exp_{\tilde{c}_{t_0}^{\text{short}}(1)})^{-1} \ast \exp_{t_2^n(t_0)} (1))^{-1} \tilde{Y}_{t_0}^n.$$

Then we get

$$X_{t_0}^n = Ad((\exp_{\tilde{c}_{t_0}^{\text{short}}(1)})^{-1} \ast \exp_{t_2^n(t_0)} (1) \circ f_n(t_0))^{-1} \tilde{X}_{t_0}^n$$

$$Y_{t_0}^n = Ad((\exp_{\tilde{c}_{t_0}^{\text{short}}(1)})^{-1} \ast \exp_{t_2^n(t_0)} (1) \circ f_n(t_0))^{-1} \tilde{Y}_{t_0}^n.$$
Since both \((c_2^{\text{short}(1)})^{-1} \cdot c_n^{\text{short}(1)}(1)\) and \(\hat{f}_n(t_0)\) are elements in \(K = \text{SO}(n)\), we get
\[
[X^m_{t_0}, Y^m_{t_0}] = Ad((c_2^{\text{short}(1)})^{-1} \cdot c_n^{\text{short}(1)}(1) \cdot \hat{f}_n(t_0))^{-1} [X^m_{t_0}, Y^m_{t_0}].
\]

Note \(c_n^{\text{short}}(t_0) = 1\). So we can rewrite \(X^m_{t_0}\) and \(Y^m_{t_0}\) as
\[
X^m_{t_0} = Ad((c_2^{\text{short}(1)})^{-1} \cdot c_n^{\text{short}(1)}(1) \cdot \hat{f}_n(t_0))^{-1} \hat{X}^n_{t_0}
\]
\[
Y^m_{t_0} = Ad((c_2^{\text{short}(1)})^{-1} \cdot c_n^{\text{short}(1)}(1) \cdot \hat{f}_n(t_0))^{-1} \hat{Y}^n_{t_0}.
\]

Since \(c_n^{\text{short}}\) converges to 1 in \(\mathbb{H}^n\) as \(n\) goes to \(\infty\), \(\hat{f}_n(t_0)\) converges to \(\hat{f}(t_0)\) in \(\text{SO}(0,1, n)\).

Since \(\hat{f}_n(t_0) = \) the value , at \(t = 1\), of the horizontal lifting of \(\gamma^m_{t_0}\) at \(e\)
and \(\hat{f}_n(t_0)\) converges to \(\hat{f}_{n_0} = \hat{f}_{t_0}\) as \(n\) goes to \(\infty\), we get, from Proposition 4.2
\[
\hat{f}(t_0) = \) the value , at \(t = 1\), of the horizontal lifting of \(\gamma_t\) at \(e\)
\[
= \lim_{n \to \infty} \hat{f}_n(t_0).
\]

Then we also get
\[
f_n(t_0)(1) = R_{f_n(t_0)} \circ e(1)_{t_0} = e(1)_{t_0} \cdot \hat{f}_n(t_0),
\]
which will converge to
\[
e(1)_{t_0} \cdot \hat{f}(t_0) = R_{f(t_0)} \circ e(1)_{t_0} = f(t_0)(1)_{t_0}.
\]

4.4.3. the comparison of \((j_n(t_0^2(t_0))) = j_n(t_0) + 1\)-th triangles on the given surface \(S\). Now, consider the \((j_{n_0}(t_{n_0}^2(t_0))) = j_{n_0}(t_0)+1\)-th triangle, lying on \(S\), in \(\tilde{A}_{n_0}\), its starting point and the horizontal lifting of
\[
x := \lim_{t \to 0^+} \frac{1}{\varphi_{t_2}^{n_0}(t)} \cdot \varphi_{t_2}^{n_0}(t)
\]
and
\[
y := - \lim_{t \to 1^-} \frac{1}{\varphi_{t_2}^{n_0}(t)} \cdot \varphi_{t_2}^{n_0}(t).
\]

at \(g_t = e^{c_2^{\text{short}}(1)}\) and at \(g := f(t_0)(\hat{c})_{t_0}(1)\), respectively, and find
\[
0_y X_{t_0}^{n_0}, 0_y Y_{t_0}^{n_0}, 0_x X_{t_0}^{n_0}, 0_x Y_{t_0}^{n_0} \in \mathbb{F}^1
\]
with
\[
\pi_s \ |_{\tilde{X}_{t_0}^{n_0}} g_t = x = \pi_s \ |_{\tilde{X}_{t_0}^{n_0}} g
\]
and
\[
\pi_s \ |_{\tilde{Y}_{t_0}^{n_0}} g_t = y = \pi_s \ |_{\tilde{Y}_{t_0}^{n_0}} g.
\]
Then,
\[ g = f(t_0)(1\hat{c})t_0(1) = e(1\hat{c})t_0(1) \cdot \tilde{f}(t_0) = g_{t_0} \cdot (e^{c_{\text{short}}}(1))^{-1} \cdot e(1\hat{c})t_0(1) \cdot \tilde{f}(t_0) \]
implies that
\[ 0X_{t_0}^{n_0} = Ad((e^{c_{\text{short}}}(1))^{-1} \cdot e(1\hat{c})t_0(1) \cdot \tilde{f}(t_0))^{-1} 0\dot{X}_{t_0}^{n_0} \]
\[ 0Y_{t_0}^{n_0} = Ad((e^{c_{\text{short}}}(1))^{-1} \cdot e(1\hat{c})t_0(1) \cdot \tilde{f}(t_0))^{-1} 0\dot{Y}_{t_0}^{n_0} . \]

4.4.4. \textit{The convergence of tangent planes induced by} \((j_n(t_0)) = j_n(t_0) + 1\)-\textit{th triangles at} \(t = t_0\) \textit{and the convergence of} \(\hat{f}_n\) \textit{under} \(\lim_{n \to \infty \ t \to t_0} \).
Now, for \(g \in \pi^{-1}(\pi(f_n(t_0)(1\hat{c})t_0(1)))\), let
\[ \dot{H}_g^n := \text{Span}\{\hat{x}, \hat{y}\} , \]
where \(\hat{x}, \hat{y}\) are horizontal vectors at \(g\) satisfying
\[ \pi_* \hat{x} = \lim_{t \to 0^+} \frac{1}{|\varphi_{t_0}^{n}(t)|} \cdot \dot{\varphi}_{t_0}^{n}(t) \]
\[ \pi_* \hat{y} = -\lim_{t \to 1^-} \frac{1}{|\varphi_{t_0}^{n}(t)|} \cdot \dot{\varphi}_{t_0}^{n}(t) . \]
Also, for \(g \in \pi^{-1}(\pi(f_n(t_0)(1\hat{c})t_0(1)))\), let
\[ \dot{H}_g^n := \text{Span}\{\hat{x}, \hat{y}\} , \]
where \(\hat{x}, \hat{y}\) are horizontal vectors at \(g\) satisfying
\[ \pi_* \hat{x} = \lim_{t \to 0^+} \frac{1}{|\varphi_{t_0}^{n}(t)|} \cdot \dot{\varphi}_{t_0}^{n}(t) \]
HORIZONTAL DISPLACEMENT OF CURVES IN BUNDLE SO(n) → SO₀(1, n) → ℝⁿ

\[
\pi_* \tilde{y} = \lim_{t \to 1^-} \frac{1}{|\varphi_t^n(t_0'(t))|} \cdot \tilde{y}^{n}_{\varphi_t^n(t_0)}(t).
\]

Note, for \( t \in (t₀, t_{2(n)}(t₀)) = (t_1^n(t_2^n(t₀)), t_2^n(t₀)), \)

\[
\omega \left( \frac{1}{|f_n(t)|} \cdot \dot{f}_n(t) \right)
\]

\[= (-1) \cdot \left( \text{the unit curvature of the 2-dimensional horizontal oriented tangent plane}, \right.
\]

\[
\hat{H}^{n_{f_n(t₀)}(1)}_{f_n(t₀)}(1) = \hat{H}^{n_{f_n(t₀)}(1)}_{f_n(t₀)}(1)
\]

\[
= \text{Span}\{X^n_{t_0}'(t_0) | f_n(t_0') Y^n_{t_0}'(t_0) | f_n(t_0') \}
\]

\[
= \text{Span}\{X^n_{t_0} | f_n(t_0) Y^n_{t_0} | f_n(t_0) \}
\]

\[
\text{at } \hat{f}_n(t_2^n(t₀)) \hat{c}_n^{n}_{t_0}(1) = f_{n(t₀)}(1)_{t₀}(1),
\]

which projects to the tangent plane

\[
\text{of the } (j_{n}(t_2^n(t₀))) = j_{n(t₀)} + 1)-th \text{ triangle in } \hat{A}_n
\]

\[
\text{at } \pi(f_{n(t₀)}) 1 = 1_{t₀}(1) = \hat{c}_n^{n}_{t_0}(1)
\]

\[
\text{the starting point of the } (j_{n}(t_2^n(t₀))) = j_{n(t₀)} + 1)-th \text{ triangle in } \hat{A}_n
\]

with respect to the connection of the principal bundle \( \pi : SO₀(1, n) \to ℝⁿ \).
\[ \begin{align*}
&= \frac{1}{\left| X_{t_0}^{n}(t_2(t_0)) \right|} \cdot \left[ X_{t_1}^{n}(t_2(t_0)), Y_{t_1}^{n}(t_2(t_0)) \right] \\
&= \frac{1}{\left| X_{t_0}^{n}, Y_{t_0}^{n} \right|} \cdot \left[ X_{t_0}^{n}, Y_{t_0}^{n} \right] \\
&= Ad((e^{-\epsilon \hat{c}_{t_0}(1)})^{-1}.e(t_0)^{n+1}(1).f_n(t_0))^{-1} \cdot \frac{1}{\left| X_{t_0}^{n}, Y_{t_0}^{n} \right|} \cdot \left[ X_{t_0}^{n}, Y_{t_0}^{n} \right]
\end{align*} \]

Note the tangent plane of the \( (j_n(t_2(t_0)) = j_n(t_0) + 1) \)-th triangle in \( \tilde{A}_n \) at \( i\hat{c}_{t_0}(1) = \tilde{c}_{t_2(t_0)}^{n}(1) = \tilde{c}_{t_2(t_0)}^{n}(1) = i\hat{c}_{t_0}(1) = \tilde{c}_{t_2(t_0)}^{n}(1) = i\hat{c}_{t_0}(1) \) for all \( n \geq n_0 \), the starting point of the \( (j_n(t_2(t_0)) = j_n(t_0) + 1) \)-th triangle in \( \tilde{A}_n \) and the ending point of the \( j_n(t_0) \)-th triangle in \( \tilde{A}_n \) for all \( n \geq n_0 \) at the same time, which is also the starting point of the \( (j_{n_0}(t_2(t_0)) = j_{n_0}(t_0) + 1) \)-th triangle, lying on \( S \) in \( \tilde{A}_{n_0} \) and the ending point of the \( j_{n_0}(t_0) \)-th triangle in \( \tilde{A}_{n_0} \) at the same time, will converge to the tangent plane of \( S \) at \( \tilde{c}_{t_2(t_0)}^{n}(1) = i\hat{c}_{t_0}(1) \).

The tangent plane of \( S \) at \( \tilde{c}_{t_2(t_0)}^{n}(1) = \tilde{c}_{t_2(t_0)}^{n}(1) \) for \( n \geq n_0 \)

\[ \tilde{H}^{n}_{e^{\hat{c}_{t_0}(1)}} = \text{Span}\{ \tilde{X}_{t_0}^{n} \mid e^{\hat{c}_{t_0}(1)}, \tilde{Y}_{t_0}^{n} \mid e^{\hat{c}_{t_0}(1)} \} \]

And, note, in general, if \( \lim_{n \to \infty} g_n = g_0 \) in \( G \) and \( \lim_{n \to \infty} X_n = X_0 \) in \( g \), then
HORIZONTAL DISPLACEMENT OF CURVES IN BUNDLE $\text{SO}(n) \to \text{SO}_0(1, n) \to \mathbb{H}^n$

$$\lim_{n \to \infty} t \cdot \text{Ad}_{g_n} X_n = \lim_{n \to \infty} \exp^{-1}(\exp(t \cdot \text{Ad}_{g_n} x_n))$$
$$= \lim_{n \to \infty} \exp^{-1}(g_n \cdot \exp(t \cdot X_n) \cdot g_n^{-1})$$
$$= \exp^{-1}(g_0 \cdot \exp(t \cdot X_0) \cdot g_0^{-1})$$
$$= \exp^{-1}(\exp(t \cdot \text{Ad}_{g_0} X_0))$$
$$= t \cdot \text{Ad}_{g_0} X_0.$$

Now, refer to previous three pictures. Then we get, for $t \in (t_0, t^n_2(t_0))$,

$$\omega \left( \frac{1}{|f_n(t)|} \cdot \dot{f}_n(t) \right)$$
$$= \frac{1}{|X^n_0 - Y^n_0|} \cdot [X^n_0, Y^n_0]$$
$$= \text{Ad}_{((e^{c_{\hat{t}_0}(1)} - 1, e^{c_{\hat{t}_0}(1)} - 1) \cdot f_n(t_0))^{-1}} \left( \frac{1}{|X^n_0 - Y^n_0|} \cdot [X^n_0, Y^n_0] \right)$$
$$= (-1) \cdot \text{Ad}_{((e^{c_{\hat{t}_0}(1)} - 1, e^{c_{\hat{t}_0}(1)} - 1) \cdot f_n(t_0))^{-1}} \left( \text{the unit curvature of the 2-dimensional horizontal oriented tangent plane}, \right.$$
$$\hat{H}^{n_0}_{c^{\hat{t}_0}(1)} = \text{Span}\{X^n_{t_0} \mid e^{c_{\hat{t}_0}(1)} - 1, Y^n_{t_0} \mid e^{c_{\hat{t}_0}(1)} - 1\}\right),$$

which will converge to

$$(-1) \cdot \text{Ad}_{((e^{c_{\hat{t}_0}(1)} - 1, e^{c_{\hat{t}_0}(1)} - 1) \cdot f(t_0))^{-1}} \left( \text{the unit curvature of the 2-dimensional horizontal oriented tangent plane}, \right.$$
under \( \lim_{n \to \infty} \lim_{t \to t_0^+} \).

So, under the identification of \( T_e K \) with \( \mathfrak{k} \),

\[
\lim_{n \to \infty} \lim_{t \to t_0^+} L(f_n(t))^{-1} \cdot \frac{1}{|f_n(t)|} \cdot \dot{f}_n(t)
\]

\[
= \lim_{n \to \infty} \left| \frac{X_{t_0}^n}{Y_{t_0}^n} \right| \cdot [X_{t_0}^n, Y_{t_0}^n]
\]

\[
= \left| \frac{0X_{t_0}^{n_0}, 0Y_{t_0}^{n_0}}{} \right| \cdot [0X_{t_0}^{n_0}, 0Y_{t_0}^{n_0}]
\]

\[
= (-1) \cdot \left( \text{the unit curvature of the 2-dimensional horizontal oriented tangent plane, } \hat{H}_{f(t_0)}(1) \right.
\]

which projects to the tangent plane of \( S \) at \( \pi(f(t_0)(1\bar{e}_{t_0}(1))) = 1\bar{e}_{t_0}(1) \)

with respect to the connection of the principal bundle \( \pi : \text{SO}(1, n) \to \mathbb{H}^n \)

4.4.5.

\[
\lim_{n \to \infty} \lim_{t \to t_0^+} L((f_n(t))^{-1}) \cdot \frac{1}{|f_n(t)|} \cdot \dot{f}_n(t) = \lim_{n \to \infty} \lim_{t \to t_0^+} L((f_n(t))^{-1}) \cdot \frac{1}{|f_n(t)|} \cdot \dot{f}_n(t)
\]

To show

\[
\lim_{n \to \infty} \lim_{t \to t_0^+} L(f_n(t))^{-1} \cdot \frac{1}{|f_n(t)|} \cdot \dot{f}_n(t) = \lim_{n \to \infty} \lim_{t \to t_0^+} L(f_n(t))^{-1} \cdot \frac{1}{|f_n(t)|} \cdot \dot{f}_n(t)
\]

for \( g \in \pi^{-1}(\pi(f_n(t_0))((1\bar{e}_{t_0}(1))) = \pi^{-1}(\pi(f_n(t_0))((1\bar{e}_{t_0}(1)))) \), let

\[
\hat{H}_g^n := \text{Span}\{\hat{x}, \hat{y}\},
\]

where \( \hat{x}, \hat{y} \) are horizontal vectors at \( g \) satisfying

\[
\pi_* \hat{x} = \lim_{t \to t_0^+} \frac{1}{|\psi_{t_0}^n(t)|} \cdot \hat{\psi}_{t_0}^n(t)
\]

\[
\pi_* \hat{y} = -\lim_{t \to t_0^+} \frac{1}{|\psi_{t_0}^n(t)|} \cdot \hat{\psi}_{t_0}^n(t).
\]

Also, for \( g \in \pi^{-1}(\pi(f_n(t_0))((1\bar{e}_{t_0}(1))) \), let

\[
\hat{H}_g^n := \text{Span}\{\hat{x}, \hat{y}\},
\]

where \( \hat{x}, \hat{y} \) are horizontal vectors at \( g \) satisfying

\[
\pi_* \hat{x} = \lim_{t \to t_0^+} \frac{1}{|\hat{\psi}_{t_0}^n(t)|} \cdot \hat{\psi}_{t_0}^n(t)
\]

\[
\pi_* \hat{y} = -\lim_{t \to t_0^+} \frac{1}{|\hat{\psi}_{t_0}^n(t)|} \cdot \hat{\psi}_{t_0}^n(t).
\]

Now, consider the horizontal lifting of

\[
z := \lim_{t \to t_0^+} \frac{1}{|\hat{\psi}_{t_0}^n(t)|} \cdot \hat{\psi}_{t_0}^n(t)
\]
and
\[ w := - \lim_{t \to 1^+} \frac{1}{\| \psi_{t_0}(t) \|} \cdot \dot{\psi}_{t_0}(t) \]
at \( g := f_n(t_1^n(t_0)) \hat{c}_{t_0}^n(1) = f_n(t_1^n(t_0))c_{t_2^n(t_0)}^{\hat{c}_n}(1) \), respectively and find
\[ Z^n_{t_0}, W^n_{t_0} \in t^\perp \]
with
\[ \pi_* Z^n_{t_0} \big| g = z \]
and
\[ \pi_* W^n_{t_0} \big| g = w . \]
Also consider the horizontal lifting of \( z \) and \( w \) at \( e(1 \hat{c})_{t_0}^n(1) = e\hat{c}^n_{t_2(t_0)}(1) =: g \), respectively and find
\[ \hat{Z}^n_{t_0}, \hat{W}^n_{t_0} \in t^\perp \]
with
\[ \pi_* \hat{Z}^n_{t_0} \big| g = z \]
and
\[ \pi_* \hat{W}^n_{t_0} \big| g = w . \]
And consider the horizontal lifting of \( z \) and \( w \) at \( e\hat{c}^{\text{short}}_{t_0}(1) = g_{t_0} \) and find
\[ \hat{Z}^n_{t_0}, \hat{W}^n_{t_0} \in t^\perp \]
with
\[ \pi_* \hat{Z}^n_{t_0} \big| g_{t_0} = z \]
and
\[ \pi_* \hat{W}^n_{t_0} \big| g_{t_0} = w . \]
Note \( \text{Im} \varphi^n_{t_0} = \text{Im} \psi^n_{t_0} \) is the boundary of a geodesic triangle in \( \mathbb{H}^n \).
Then, from Facts, mentioned earlier in this section, and from the property in Section 3, we get
\[
\hat{f}_n(t) = \hat{f}_n(t_1^n(t_0)) \cdot \exp \left( (t - t_1^n(t_0)) \cdot \frac{\text{Area of } \hat{j}^n(t_0)-\text{th triangle in } \hat{A}_n}{\langle (t_0 - t_1^n(t_0)) \cdot [X^n_{t_2^n(t_0)}, Y^n_{t_2^n(t_0)}] \rangle} \cdot [X^n_{t_1^n(t_0)}, Y^n_{t_1^n(t_0)}] \right) 
\]
\[
= \hat{f}_n(t_1^n(t_0)) \cdot \exp \left( (t - t_1^n(t_0)) \cdot \frac{\text{Area of } \hat{j}^n(t_0)-\text{th triangle in } \hat{A}_n}{\langle (t_0 - t_1^n(t_0)) \cdot [Z^n_{t_0}, W^n_{t_0}] \rangle} \cdot [Z^n_{t_0}, W^n_{t_0}] \right) 
\]
for \( t \in [t_1^n(t_0), t_0] \).

Note
\[
\pi_* Z^n_{t_0} |_{\tilde{f}_n(t_1^n(t_0)) (1)} = \lim_{t \to 0^+} \frac{1}{| \psi^n_{t_0}(t) |} \cdot \psi^n_{t_0}(t)
\]

which implies

\[
Z^n_{t_0} = Ad_{\tilde{f}_n(t_1^n(t_0))}^{-1} \tilde{Z}^n_{t_0}
\]

from \( \tilde{f}_n(t_1^n(t_0)) (1) = R_{\tilde{f}_n(t_1^n(t_0))} \circ e(1 \tilde{c})_{t_0} \cdot \).

Similarly,

\[
W^n_{t_0} = Ad_{\tilde{f}_n(t_1^n(t_0))}^{-1} \tilde{W}^n_{t_0}
\]

And by considering a loop

\[
\tilde{c}_{t_0}^{\text{short}} : [0, 1] \to \mathbb{H}^n,
\]

where \( \tilde{c}_{t_0}^{\text{short}} : [0, 1] \to \mathbb{H}^n \) is given by

\[
\tilde{c}_{t_0}^{\text{short}}(t) = c_{t_0}^{\text{short}}(1 - t),
\]

and its horizontal lifting at \( e \tilde{c}_{t_0}^{\text{short}}(1) \), we obtain

\[
\tilde{Z}^n_{t_0} = Ad_{(e \tilde{c}_{t_0}^{\text{short}}(1)-1, e(1 \tilde{c})_{t_0} (1))}^{-1} \tilde{Z}^n_{t_0}
\]

\[
\tilde{W}^n_{t_0} = Ad_{(e \tilde{c}_{t_0}^{\text{short}}(1)-1, e(1 \tilde{c})_{t_0} (1))}^{-1} \tilde{W}^n_{t_0}.
\]

Then we get

\[
Z^n_{t_0} = Ad_{((e \tilde{c}_{t_0}^{\text{short}}(1))^{-1}, e(1 \tilde{c})_{t_0} (1))}^{-1} \tilde{Z}^n_{t_0}
\]

\[
W^n_{t_0} = Ad_{((e \tilde{c}_{t_0}^{\text{short}}(1))^{-1}, e(1 \tilde{c})_{t_0} (1))}^{-1} \tilde{W}^n_{t_0}.
\]

Since both \( (e \tilde{c}_{t_0}^{\text{short}}(1))^{-1}, e(1 \tilde{c})_{t_0} (1) \) and \( \tilde{f}_n(t_1^n(t_0)) \) are elements in \( K = SO(n) \), we get

\[
[Z^n_{t_0}, W^n_{t_0}] = Ad_{((e \tilde{c}_{t_0}^{\text{short}}(1))^{-1}, e(1 \tilde{c})_{t_0} (1))}^{-1} [\tilde{Z}^n_{t_0}, \tilde{W}^n_{t_0}].
\]

Note the tangent plane of the \( \tilde{f}_n(t_0) \)-th triangle in \( \hat{A}_n \) at \( 1 \tilde{c}_{t_0}^{n}(1) = \tilde{c}_{t_2^n(t_0)}(1) = \tilde{c}_{t_2^n(t_0)}^{n_0}(1) = 1 \tilde{c}_{t_2^n(t_0)}^{n_0}(1) = 1 \tilde{c}_{t_2^n(t_0)}^{n_0}(1) \) for all \( n \geq n_0 \), the starting point of the \( (\tilde{f}_n(t_1^n(t_0)) = \tilde{f}_n(t_0) + 1) \)-th triangle in \( \hat{A}_n \) and also the ending point of the \( \tilde{f}_n(t_0) \)-th triangle in \( \hat{A}_n \) for all \( n \geq n_0 \), which is also the starting point of the \( (\tilde{f}_{n_0}(t_2^{n_0}(t_0)) = \tilde{f}_{n_0}(t_0) + 1) \)-th triangle, lying on \( S \), in \( \hat{A}_{n_0} \) and also the ending point of the \( \tilde{f}_{n_0}(t_0) \)-th triangle in \( \hat{A}_{n_0} \), will converge to the tangent plane of \( S \) at \( \tilde{c}_{t_2^{n_0}(t_0)}(1) = 1 \tilde{c}_{t_2^{n_0}(t_0)}(1) \), which implies that for \( t \in (t_1^n(t_0), t_0) \)
\[ \omega \left( \frac{1}{|f_n(t)|} \cdot \dot{f}_n(t) \right) = \frac{1}{\|Z_{t_0}^n, W_{t_0}^n\|} \cdot \left[ Z_{t_0}^n, W_{t_0}^n \right] \]

\[ = Ad_{((e^\varepsilon_\theta(1))_{t_0})^{-1} \cdot \varepsilon(1)_{t_0}(1) \cdot f_n(t_0))^{-1}} \left( \frac{1}{\|Z_{t_0}^n, W_{t_0}^n\|} \cdot \left[ \dot{Z}_{t_0}^n, \dot{W}_{t_0}^n \right] \right) \]

\[ = (-1) \cdot Ad_{((e^\varepsilon_\theta(1))_{t_0})^{-1} \cdot \varepsilon(1)_{t_0}(1) \cdot f_n(t_0))^{-1}} \left( \text{the unit curvature of the 2-dimensional horizontal oriented tangent plane,} \right. \]

\[ 1 \mathcal{H}^n_{\varepsilon_\theta(1)} = \text{Span}\{ \dot{Z}_{t_0}^n \mid _{e^\varepsilon_\theta(1)}, \dot{W}_{t_0}^n \mid _{e^\varepsilon_\theta(1)} \} \]

will converge to

\[ (-1) \cdot Ad_{((e^\varepsilon_\theta(1))_{t_0})^{-1} \cdot \varepsilon(1)_{t_0}(1) \cdot f(t_0))^{-1}} \left( \text{the unit curvature of the 2-dimensional horizontal oriented tangent plane,} \right. \]

\[ \mathcal{H}^n_{\varepsilon_\theta(1)} = \text{Span}\{ \dot{X}_{t_0}^n \mid _{e^\varepsilon_\theta(1)}, \dot{Y}_{t_0}^n \mid _{e^\varepsilon_\theta(1)} \} \]

\[ = Ad_{((e^\varepsilon_\theta(1))_{t_0})^{-1} \cdot \varepsilon(1)_{t_0}(1) \cdot f(t_0))^{-1}} \left( \frac{1}{\|X_{t_0}^n, Y_{t_0}^n\|} \cdot \left[ \dot{X}_{t_0}^n, \dot{Y}_{t_0}^n \right] \right) \]

\[ = (-1) \cdot \left( \text{the unit curvature of the 2-dimensional horizontal oriented tangent plane,} \right. \]

\[ \mathcal{H}^n_{f(t_0)(1)_{t_0}} = \text{Span}\{ \dot{X}_{t_0}^n \mid _{f(t_0)(1)_{t_0}(1)}, \dot{Y}_{t_0}^n \mid _{f(t_0)(1)_{t_0}(1)} \} \]

at \( f(t_0)(1)_{t_0}(1) \),

which projects to the tangent plane

\[ \text{of the } (j_{n_0}(t_{n_0}^2(t_0))) = j_{n_0}(t_0) + 1 \text{-th triangle in } \mathcal{A}_{n_0}, \]

where \( n_0 = \text{min } \{ n_1 \mid n + 1 \geq n_1 \Rightarrow t_0 \in D_n \} \),

- so tangent to the given disk \( S \)

at \( \pi(f(t_0)(1)_{t_0}(1)) = \hat{c}_{t_0}(1) = c_{t_0}^{n_0}(1) \)

= the starting point of the \( (j_{n_0}(t_{2}^{n_0}(t_0))) = j_{n_0}(t_0) + 1 \)-th triangle in \( \mathcal{A}_{n_0} \)

with respect to the connection of the principal bundle \( \pi : SO(1, n) \rightarrow \mathbb{H}^n \)

Thus we get

\[ \lim_{n \to \infty} \lim_{t \to t_0^-} L_{(f_n(t))^{-1}} \left( \frac{1}{|f_n(t)|} \cdot \dot{f}_n(t) \right) = \lim_{n \to \infty} \frac{1}{\|Z_{t_0}^n, W_{t_0}^n\|} \cdot \left[ Z_{t_0}^n, W_{t_0}^n \right] \]

\[ = \frac{1}{\|X_{t_0}^n, Y_{t_0}^n\|} \cdot \left[ X_{t_0}^n, Y_{t_0}^n \right] \]

\[ = \lim_{n \to \infty} \lim_{t \to t_0^+} L_{(f_n(t))^{-1}} \left( \frac{1}{|f_n(t)|} \cdot \dot{f}_n(t) \right). \]
4.4.6. **Main Part.** Define a function \( s_n : D_n - \{0\} \to (0, \infty) \) as follows: Given \( t \in D_n - \{0\} \), assume \( t \) is the \( j \)-th element in \( D_n \), i.e., \( j = j_n(t) \). Then,

\[
s_n(t) = \sum_{i=1}^{j = j_n(t)} \text{ (the area of \( i \)-th triangle in } \hat{A}_n) = \text{ the area of the region surrounded by } \hat{\gamma}_t^n \text{ in the } n \text{-th step pleated surface}.
\]

Note in \( S \), for \( n \geq n_0 \), the region surrounded by \( \hat{\gamma}_t^n \) in \( S = \) the region surrounded by \( \hat{\gamma}^{n_0}_{t_0} \) in \( S \), so we get

\[
\lim_{n \to \infty} s_n(t_0) = \text{ the area of the region surrounded by } \hat{\gamma}^{n_0}_{t_0} \text{ in } S =: s(t_0).
\]

Thus, we obtain a function

\[
s : \cup_{n=1}^{\infty} D_n - \{0\} \to (0, \infty).
\]

Now, induce a function

\[
f_n : [0, \text{ the area of the } n \text{-step pleated surface}] \to K,
\]

which is the reparametrization of \( \hat{f}_n \) with \( |\hat{f}_n(t)| = 1 \) on \( [0, \text{ the area of the } n \text{-step pleated surface}] \) —

\[
\left\{ \sum_{i=1}^{j} \text{ (the area of the } i \text{-th triangle in } \hat{A}_n) \mid j = 1, 2, \ldots, |\hat{A}_n| \right\}.
\]

Then we get

\[
f_n(s_n(t)) = \hat{f}_n(t) = \bar{f}_n(t) \quad \text{for } t \in D_n - \{0\}.
\]

Define a function

\[
f : \{ s(t) \mid t \in \cup_{n=1}^{\infty} D_n - \{0\} \} \to K
\]

by

\[
f(s(t)) = \bar{f}(t).
\]

Then we get

\[
f(s(t_0)) = \bar{f}(t_0) = \lim_{n \to \infty} \hat{f}_n(t_0) = \lim_{n \to \infty} f_n(s_n(t_0)).
\]

Note, for \( t_1 \in \cup_{n=1}^{\infty} D_n - \{0\} \),

\[
\bar{f}(t_1) = \lim_{n \to \infty} \hat{f}_n(t_1)
\]

\[
= \lim_{n \to \infty} (\text{the value, at } t = 1, \text{ of the horizontal lifting of } \hat{\gamma}_{t_1}^{n} \text{ at } e)
\]

\[
= \text{ the value, at } t = 1, \text{ of the horizontal lifting of } \hat{\gamma}_{t_1}^{n} \text{ at } e.
\]
HORIZONTAL DISPLACEMENT OF CURVES IN BUNDLE $\text{SO}(n) \to \text{SO}_0(1, n) \to \mathbb{R}^n$

Since $\tilde{\gamma}_{t_1}$ converges to $\tilde{\gamma}_{t_0}$ as $t_1$ approaches $t_0$ in $\bigcup_{n=1}^{\infty} D_n$, Proposition 4.2 implies that $\tilde{f}$ will be continuous on $\bigcup_{n=1}^{\infty} D_n - \{0\}$ and we can extend $\tilde{f}$ on $[0,1]$. And from $f(s(t_0)) = \tilde{f}(t_0)$, $f$ will be continuous on $\{s(t) \mid t \in \bigcup_{n=1}^{\infty} D_n - \{0\}\}$.

Note $s$ is continuous on $\bigcup_{n=1}^{\infty} D_n - \{0\}$ and so it can be extended on $[0,1]$. Since $\{s(t) \mid t \in \bigcup_{n=1}^{\infty} D_n - \{0\}\}$ is a dense subset of $[0, \text{the area of } S]$, we can extend $f$ on $[0, \text{the area of } S]$ continuously. Call it $f$ as well. Then we get $f \circ s = \tilde{f}$ is continuous on $[0,1]$ and

$$f(\text{the area of } S) = \tilde{f}(1) = \lim_{t \to 1, t \in \bigcup_{n=1}^{\infty} D_n} e \tilde{\gamma}_{t}(1) = e\tilde{\gamma}(1),$$

where $\tilde{\gamma} : [0,1] \to S$ is the boundary curve of $S$ and $e\tilde{\gamma}$ is its horizontal lifting at $e$.

Let's show $f$ is a $C^1$ curve. Define a function $F_n$ from $[0, \text{the area of the } n\text{-step pleated surface}]$ - 

$$\left\{ \sum_{i=1}^{j} (\text{the area of the } i\text{-th triangle in } \hat{A}_n) \mid j = 1, 2, \ldots, |\hat{A}_n| \right\}$$

to the unit sphere in $\mathbb{k}$ by

$$F_n(t) = L(f_n(t)^{-1}) \hat{f}_n(t).$$

And define a function

$$F : \{s(t) \mid t \in \bigcup_{n=1}^{\infty} D_n - \{0\}\} \to \text{the unit sphere in } \mathbb{k}$$

by

$$F(s(t_0)) = \lim_{n \to \infty, t \to t_0^-} L(f_n(t_0)^{-1}) \left( \frac{1}{|f_n(t_0)|} \hat{f}_n(t) \right) = \lim_{n \to \infty, t \to t_0^+} L(f_n(t_0)^{-1}) \left( \frac{1}{|f_n(t_0)|} \hat{f}_n(t) \right).$$

Then, $F_n$ is constant on the interval

$$\left(0, \text{the area of the first triangle in } \hat{A}_n\right)$$

and on the interval

$$\left(\sum_{i=1}^{j} (\text{the area of the } i\text{-th triangle in } \hat{A}_n) , \sum_{i=1}^{j+1} (\text{the area of the } i\text{-th triangle in } \hat{A}_n)\right)$$
for each $j = 1, 2, \cdots, |A_n|$, and

\[
F(s(t_0)) = \lim_{n \to \infty} \lim_{t \to t_0^-} L(f_n(t)) \left( \frac{1}{|f_n(t)|} \cdot \dot{f}_n(t) \right) = \lim_{n \to \infty} \lim_{t \to t_0^+} L(f_n(t)) \left( \frac{1}{|f_n(t)|} \cdot \dot{f}_n(t) \right) \\
= \lim_{n \to \infty} \lim_{t \to s_n(t_0)^-} L((f_n(t))^{-1} \cdot \dot{f}_n(t)) = \lim_{n \to \infty} \lim_{t \to s_n(t_0)^+} L((f_n(t))^{-1} \cdot \dot{f}_n(t)) \\
= \lim_{n \to \infty} \lim_{t \to s_n(t_0)} L(f_n(t)) \\
= \lim_{n \to \infty} \lim_{t \to s_n(t_0)} F_n(t).
\]

Also

\[
F(s(t_0)) = \lim_{n \to \infty} \lim_{t \to t_0} \omega \left( \frac{1}{|f_n(t)|} \cdot \dot{f}_n(t) \right) = (-1) \cdot \left( \text{the unit curvature of the 2-dimensional horizontal oriented tangent plane,} \right)
\]

\[
\tilde{H}^n_{f(t_0)}(1) \cdot \dot{f}(t_0) = \tilde{H}^n_{f(t_0)}(1) \cdot \dot{f}(t_0)
\]

which projects to the tangent plane of $S$ at $(1\tilde{c})_{t_0}(1)$.

Note paths $1\tilde{c}_t$ on $S$ gives us

\[
\lim_{t \to t_0, t \in \cup_{n=1}^{\infty} D_n} 1\tilde{c}_t(1) = 1\tilde{c}_{t_0}(1)
\]

and

\[
\lim_{t \to t_0, t \in \cup_{n=1}^{\infty} D_n} f(t)(1\tilde{c})_{t_0}(1) = \lim_{t \to t_0, t \in \cup_{n=1}^{\infty} D_n} e(1\tilde{c})_{t_0}(1) \cdot \dot{f}(t) = e(1\tilde{c})_{t_0}(1) \cdot \dot{f}(t_0) = f(t_0)(1\tilde{c})_{t_0}(1).
\]

Then we get

\[
\lim_{t \to t_0, t \in \cup_{n=1}^{\infty} D_n} F(s(t)) = \lim_{t \to t_0, t \in \cup_{n=1}^{\infty} D_n} (-1) \cdot \left( \text{the unit curvature of the 2-dimensional horizontal oriented tangent plane,} \right) \tilde{H}^n_{f(t_0)}(1) \cdot \dot{f}(t_0)
\]

for some $n(t) \in \mathbb{N}$ depending on $t$.

whose projection is the tangent plane of the surface $S$ at $1\tilde{c}_{t_0}(1)$

\[
= (-1) \cdot \left( \text{the unit curvature of the 2-dimensional horizontal oriented tangent plane,} \right) \tilde{H}^n_{f(t_0)}(1) \cdot \dot{f}(t_0)
\]

whose projection is the tangent plane of the surface $S$ at $1\tilde{c}_{t_0}(1)$.\]
= F(s(t_0)).

So, we get

\[ F : \{ s(t) \mid t \in \bigcup_{n=1}^{\infty} D_n - \{0\} \} \to \text{the unit sphere in } \mathbb{R} \]

is a continuous function. Since \{ s(t) \mid t \in \bigcup_{n=1}^{\infty} D_n \} is a dense subset of \([0, \text{the area of } S]\), we can extend \( F \) on \([0, \text{the area of } S]\) continuously. Call it also \( F \). Consider the \( C^1 \) curve

\[ \alpha : [0, \text{the area of } S] \to K \]

satisfying

\[ \alpha(0) = e \quad \text{and} \quad L_{(\alpha(t)^{-1})} \dot{\alpha}(t) = F(t). \]

Note the function

\[ f_n : [0, \text{the area of the } n \text{- step pleated surface}] \to K \]

can be regarded as the piecewise integral curve of

\[ \dot{f}_n(t) = L_{(f_n(t))} (L_{(f_n(t))}^{-1} \dot{f}_n(t)) = L_{f_n(t)} F_n(t), \]

or equivalently the piecewise solution of the ODE

\[ L_{(\alpha_n(t))}^{-1} \dot{\alpha}_n(t) = F_n(t). \]

Then

\[ F(s(t_0)) = \lim_{n \to \infty} \lim_{t \to s_n(t_0)} L_{(f_n(t))}^{-1} \dot{f}_n(t) = \lim_{n \to \infty} \lim_{t \to s_n(t_0)} F_n(t) \]

implies that

\[ \alpha(s(t_0)) = \lim_{n \to \infty} \lim_{t \to s_n(t_0)} \alpha_n(t) = \lim_{n \to \infty} \lim_{t \to s_n(t_0)} f_n(t) = \lim_{n \to \infty} f_n(s_n(t_0)) = f(s(t_0)). \]

Since \{ s(t) \mid t \in \bigcup_{n=1}^{\infty} D_n \} is a dense subset of \([0, \text{the area of } S]\) and \( F \) is continuous on \([0, \text{the area of } S]\), we get

\[ f = \alpha \text{ is a } C^1 \text{ curve on } [0, \text{the area of } S]. \]

Also, we obtain

the length of the curve \( f = \text{the length of the curve } \alpha \)

\[ = \int_{0}^{\text{the area of } S} | \dot{\alpha}(t) | \, dt \]

\[ = \int_{0}^{\text{the area of } S} | F(t) | \, dt \]

\[ = \text{the area of } S, \]

which proves Theorem 0.1.
4.4.7. Remarks on Factorization Lemma. ‘Factorization Lemma’, introduced by Lichnerowicz, *Theorie Globale des Connexions et des Groupes d’Holonomie*, [3, vol 1, p.284], can give us another sequence of piece-wise smooth loops $\mu_m : [0,1] \to \mathbb{H}^n$, $m = 1, 2, \cdots$, with $\mu_m(0) = \pi(e)$ such that it converges to $\partial S$. And a similar way to make the sequence of curves $f_n : [0,1] \to K$, $n = 1, 2, \cdots$, can give us a sequence of curves $g_n : [0,1] \to K$, $n = 1, 2, \cdots$, with $g_n(0) = e$ such that $g_n(1)$ is the ending point of the horizontal lifting of $\mu_n$ at $e$ and that the length of $g_n$ is the area of the pleated surface, the union of totally geodesic triangles obtained in the construction of $g_n$. Since the sequence of the areas converges to the area of $S$, Prop 4.2 will say that $g_n(1)$ will converge to $e\tilde{\gamma}(1)$ and that the distance from $e$ to $e\tilde{\gamma}(1)$ is less that equal to the area of $S$. But the sequence $\{g_n\}$ may not converge to some curve from $e$ to $e\tilde{\gamma}(1)$.

References

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Appendices

A. About Triangles

For each $n = 0, 1, 2, \cdots$, all triangles inside $2^n \cdot 3$-gon will consist of two kinds of triangles, interior ones and exterior ones.

A.1. The definition of interior triangles and the definition of their starting points and ending points. Consider a regular triangle whose vertices lie on the boundary of the given disk $D^2$ and one of whose vertices is the base point of the disk. Call the triangle $T_0$. And the base point will be called its starting and ending point.

Now let’s define triangles $T_{a_0a_1\cdots a_n}$ inductively as follows:

Case 1) $n = 1$:

The given orientation at the center of $D^2$ and the base point, or equivalently the starting and ending point of $T_0$, will give the order $b_0$ of sides of $T_0$, where $b_0 = 1.2.3$, in the counter-clockwise or clockwise order. For the barycentric subdivision of $T_0$, thinking of the triangle with the base point as its vertex and with one side lying on the first side of $T_0$ as the first triangle will give the order of triangles in the counter-clockwise or clockwise order. The $i$-th triangle will be called $T_{a_0a_1}$, where $a_0 = 0$ and $a_1 = i$, for $i = 1, 2, \cdots, 6$. 

[Diagrams showing the orientations of triangles $T_0$ and $T_1$ with their corresponding labels $b_0$, $b_1$, etc.]
For $T_{01}$, the base point, or equivalently the starting point of $T_0$, will be called the starting point of $T_{01}$ and the barycenter of $T_0$ will be called the ending point of $T_{01}$.

For $T_{0i}$, where $i = 2, 3, 4, 5$, the barycenter of $T_0$ will be called the starting and ending point of $T_{0i}$ for $i = 2, 3, 4, 5$.

For $T_{06}$, the barycenter of $T_0$ will be called the starting point of $T_{06}$ and the base point , or equivalently the ending point of $T_0$, will be called the ending point of $T_{06}$.

Case 2) $n \geq 2$ :
Let $L_{n-1} := T_{a_0a_1\cdots a_{n-2}a_{n-1}}$ be given, where $a_0 = 0$ and $a_1, \cdots, a_{n-1} \in \{1, 2, 3, 4, 5, 6\}$. Let $L_{n-2} := T_{a_0\cdots a_{n-2}}$ and assume the following properties:
- $M_j := T_{a_0a_1\cdots a_{n-2}j}$, $j \in \{1, 2, 3, 4, 5, 6\}$, consists one of six triangles obtained by the barycentric subdivision of $L_{n-2}$,
- $L_{n-1}$ is also one of those, in other words,
$$L_{n-1} = T_{a_0a_1\cdots a_{n-2}a_{n-1}} = T_{a_0a_1\cdots a_{n-2}j_0} = M_{j_0},$$
for some $j_0 \in \{1, 2, 3, 4, 5, 6\}$.

- common vertex of $L_{n-2}$ and $M_1$ is the starting point of each of them,
- the barycenter of $L_{n-2}$ is the starting point of $M_i$ for $i = 2, 3, 4, 5, 6$, and the ending point of $M_i$ for $i = 1, 2, 3, 4, 5$,
- the common vertex of $L_{n-2}$ and $M_6$ is the ending point of each of them,
- if the starting and the ending point of $L_{n-2}$ are same, then they are the common vertex of $L_{n-2}$ and $M_6$,
- if the starting and the ending point of $L_{n-2}$ are different, then $M_1$ and $M_6$ are mutually opposite ones inside $L_{n-2}$,
- one side of $L_{n-2}$, which contains a side of $M_1$, is divided into two line segments, each of which is one side of $M_i$ for $i = 1, 2$, respectively.
Notice all the above assumptions hold for \( n=2 \).

Note that the line segment connecting the barycenter and the starting point of \( L_{n-2} \) is one side of \( M_1 \) from the assumption that the common vertex of \( L_{n-2} \) and \( M_1 \) is the starting point of each of them.

Under the above assumptions, we have two choices such that the order of \( M_1 \) and \( M_2 \) is either the counter-clockwise order or the clockwise order with respect to the barycenter of \( L_{n-2} \) and the line segment connecting the barycenter and the starting point of \( L_{n-2} \).

Subcase 2-1) \( a_{n-1} = 1 \), that is, \( L_{n-1} = M_1 = T_{a_0 \cdots a_{n-2} 1} \):

Assume the order of \( L_{n-1} = M_1 \) and \( M_2 = T_{a_0 \cdots a_{n-2} 2} \) is the counter-clockwise order with respect to the barycenter of \( L_{n-2} = T_{a_0 \cdots a_{n-2}} \) and the line segment connecting the barycenter and the starting point of \( L_{n-2} \). Out of six triangles obtained from the barycentric subdivision of \( L_{n-1} = M_1 \), choose the triangle with a part of one side of \( L_{n-2} \) as its side and with the starting point and the barycenter of \( L_{n-1} = M_1 \) as its vertices, and call it \( T_{a_0 \cdots a_{n-2} 1} \). At the barycenter of \( L_{n-1} = M_1 \), consider the counter-clockwise order of the 6 triangles from the \( N_1^1 \). The 5 triangles from the next one of \( N_1^1 \) will be called

\[
T_{a_0 \cdots a_{n-2} 12}, T_{a_0 \cdots a_{n-2} 13}, T_{a_0 \cdots a_{n-2} 16}, T_{a_0 \cdots a_{n-2} 15}, T_{a_0 \cdots a_{n-2} 14}
\]

in order. Let \( N_i^1 := T_{a_0 \cdots a_{n-2} 1 i} \) for \( i = 2, 3, 4, 5, 6 \).

If the order of \( L_{n-1} = M_1 \) and \( M_2 \) is the clockwise order, then the order will be given from the symmetry by the line connecting the barycenter and the starting point of \( L_{n-1} = M_1 \):

Recall the assumptions for \( L_{n-1} = M_1 \), lying between the phrase ‘Case 2) \( n \geq 2 \)’ and the one ‘Subcase 2-1) \( a_{n-1} = 1, \cdots \)’, and let \( L_n := N_j^1 \), \( j = 1, \cdots, 6 \).

Note the common vertex of \( L_{n-1} = M_1 \) and \( N_1^1 \) is the starting point of \( L_{n-1} = M_1 \) from the definition of \( N_1^1 \). Now, call the vertex the starting point of \( N_1^1 \). And call the barycenter of \( L_{n-1} = M_1 \) the starting point of \( N_i^1 \) for \( i = 2, 3, 4, 5, 6 \). Also, call the barycenter the ending point of \( N_i^1 \) for \( i = 1, 2, 3, 4, 5 \).
Note the common vertex of $L_{n-1} = M_1$ and $N^1_6$ is the barycenter of $L_{n-2}$, so the ending point of $L_{n-1} = M_1$ from the assumption for $M_1$. Call the vertex the ending point of $N^1_6$.

Note that the starting and the ending point of $L_{n-1} = M_1$ are different and the positions of $N^1_1$ and $N^1_6$ are mutually opposite inside $L_{n-1} = M_1$.

And the side of $L_{n-1} = M_1$, which contains a side of $N^1_1$, is divided into two line segments, each of which is one side of $N^1_i$ for $i = 1, 2$, respectively.

Subcase 2-2 ) $a_{n-1} = 6$, that is, $L_{n-1} = M_6 = T_{a_0 \cdots a_{n-2}}$:

Assume the order of $M_1$ and $M_2$ is the counter-clockwise order with respect to the barycenter of $L_{n-2}$ and the line segment connecting the barycenter and the starting point of $L_{n-2}$. From the assumptions, lying between the phrase ‘Case 2) $n \geq 2$’ and the one ‘Subcase 2-1 ) $a_{n-1} = 1, \cdots$’ the vertex of $L_{n-1} = M_6$, which is also the barycenter of $L_{n-2}$, is the starting point of $L_{n-1} = M_6$. The counter-clockwise angle of $L_{n-1} = M_6$ at its starting point determines its initial side and the terminal side. Out of six triangles obtained from the barycentric subdivision of $L_{n-1} = M_6$, choose the triangle with a part of the initial side of $L_{n-1} = M_6$ as its side and with the starting point and the barycenter of $L_{n-1} = M_6$ as its vertices, and call it $T_{a_0 \cdots a_{n-2}61}$ and let $N^6_1 := T_{a_0 \cdots a_{n-2}61}$. At the barycenter of $L_{n-1} = M_6$, consider the counter-clockwise order of the 6 triangles from the $N^6_1$. The 5 triangles from the next one of $N^6_1 = T_{a_0 \cdots a_{n-2}61}$ will be called

$$T_{a_0 \cdots a_{n-2}62}, T_{a_0 \cdots a_{n-2}63}, T_{a_0 \cdots a_{n-2}66}, T_{a_0 \cdots a_{n-2}65}, T_{a_0 \cdots a_{n-2}64}$$
in order. Let $N_6^i := T_{a_0 \cdots a_{n-2}i}$ for $i = 2, 3, 4, 5, 6$.

If the order of $M_1$ and $M_2$ is the clockwise order, then the order will be given from the symmetry by the line connecting the barycenter and the starting point of $L_{n-1} = M_6$:

Recall the assumptions for $L_{n-1} = M_6$, lying between the phrase ‘Case 2) $n \geq 2$’ and the one ‘Subcase 2-1 ) $a_{n-1} = 1, \cdots,$’ and let $L_n := N_j^6$, $j = 1, \cdots, 6$.

Note the common vertex of $L_{n-1} = M_6$ and $N_6^1$ is the starting point of $L_{n-1} = M_6$ from the definition of $N_6^1$. Now, call the vertex the starting point of $N_6^1$. And call the barycenter of $L_{n-1} = M_6$ the starting point of $N_6^i$ for $i = 2, 3, 4, 5, 6$. Also, call the barycenter the ending point of $N_6^i$ for $i = 1, 2, 3, 4, 5$.

To consider the common vertex of $L_{n-1} = M_6$ and $N_6^i$, we have the following two possibilities:

The starting point and the ending point of $L_{n-2}$ are same or different.

But in any possibilities, the common vertex of $L_{n-1} = M_6$ and $N_6^i$ is also the common vertex of $L_{n-2}$ and $L_{n-1} = M_6$, so the ending point of $L_{n-1} = M_6$ from the assumption for $M_6$. Call the vertex the ending point of $N_6^i$.

Note that the starting and the ending point of $L_{n-1} = M_6$ are different and the positions of $N_6^1$ and $N_6^i$ are mutually opposite inside $L_{n-1} = M_6$.

Notice the side of $L_{n-1} = M_6$, which contains a side of $N_6^1$, is divided into two line segments, each of which is one side of $N_6^i$ for $i = 1, 2$, respectively.
Subcase 2-3) $a_{n-1} \in \{2, 3\}$ or $(a_{n-1} \in \{4, 5\}$ and $a_{n-2} \in \{0, 2, 3, 4, 5\}$),
that is,
\[
L_{n-1} = M_{i} = T_{a_{0} \cdots a_{n-2} i} \text{ for } i = 2, 3
\]
or
\[
L_{n-1} = M_{i} = T_{a_{0} \cdots a_{n-2} i} \text{ for } i = 4, 5 \text{ and } a_{n-2} \in \{0, 2, 3, 4, 5\}:
\]

Let $i = a_{n-1}$.

Assume the order of $M_{1}$ and $M_{2}$ is the counter-clockwise order with respect to the barycenter of $L_{n-2}$ and the line segment connecting the barycenter and the starting point of $L_{n-2}$. From the assumptions, lying between the phrase `Case 2) $n \geq 2$' and the one `Subcase 2-1) $a_{n-1} = 1, \cdots$', the vertex of $L_{n-1} = M_{i}$, which is also the barycenter of $L_{n-2}$, is the starting point of $L_{n-1} = M_{i}$. The counter-clockwise angle of $L_{n-1} = M_{i}$ at its starting point determines the initial side and the terminal side. Out of six triangles obtained from the barycentric subdivision of $L_{n-1} = M_{i}$, choose the triangle with a part of the initial side of $L_{n-1} = M_{i}$ as its side and with the starting point and the barycenter of $L_{n-1} = M_{i}$ as its vertices, and call it $T_{a_{0} \cdots a_{n-2} a_{n-1} 1}$, in other words, $T_{a_{0} \cdots a_{n-2} 1}$, and let $N_{i}^{1} := T_{a_{0} \cdots a_{n-2} 1}$. At the barycenter of $L_{n-1} = M_{i}$, consider the counter-clockwise order of the 6 triangles from the $N_{i}^{1}$. The 5 triangles from the next one of $N_{i}^{1} = T_{a_{0} \cdots a_{n-2} a_{n-1} 1}$ will be called
\[
T_{a_{0} \cdots a_{n-2} a_{n-1} 2}, T_{a_{0} \cdots a_{n-2} a_{n-1} 3}, T_{a_{0} \cdots a_{n-2} a_{n-1} 4}, T_{a_{0} \cdots a_{n-2} a_{n-1} 5}, T_{a_{0} \cdots a_{n-2} a_{n-1} 6}
\]
in order. Let $N_{j}^{i} := T_{a_{0} \cdots a_{n-2} i j} = T_{a_{0} \cdots a_{n-2} a_{n-1} j}$ for $j = 2, 3, 4, 5, 6$.

If the order of $M_{1}$ and $M_{2}$ is the clockwise order, then the order will be given from the symmetry by the line connecting the barycenter and the starting point of $L_{n-1} = M_{i} = T_{a_{0} \cdots a_{n-2} a_{n-1} 1}$.
Recall the assumptions for \( L_{n-1} = M_i \), lying between the phrase ‘Case 2) \( n \geq 2 \)’ and the one ‘Subcase 2-1) \( a_{n-1} = 1, \ldots \)’, and let \( L_n := N^i_j \), \( j = 1, \cdots, 6 \).

Note the common vertex of \( L_{n-1} = M_i \) and \( N^i_1 \) is the starting point of \( L_{n-1} = M_i \) from the definition of \( N^i_1 \). Now, call the vertex the starting point of \( N^i_1 \). And call the barycenter of \( L_{n-1} = M_i \) the starting point of \( N^i_j \) for \( j = 2, 3, 4, 5, 6 \). Also, call the barycenter the ending point of \( N^i_j \) for \( j = 1, 2, 3, 4, 5 \).

Note the common vertex of \( L_{n-1} = M_i \) and \( N^i_6 \) is the starting point of \( L_{n-1} = M_i \) from the assumption for \( M_i \). Call the vertex the ending point of \( N^i_6 \).

Note that the starting and the ending point of \( L_{n-1} = M_i \) are same and they are the common vertex of \( L_{n-1} = M_i \) and \( N^i_6 \).

And the side of \( L_{n-1} = M_i \), which contains a side of \( N^i_1 \), is divided into two line segments, each of which is one side of \( N^i_j \) for \( j = 1, 2, 3, 4, 5 \), respectively.

Subcase 2-4) \( a_{n-1} \in \{4, 5\} \) and \( a_{n-2} \in \{1, 6\} \), that is, \( L_{n-1} = M_i = T_{a_0 \cdots a_{n-2} i} \) for \( i = 4, 5 \) and \( a_{n-2} \in \{1, 6\} \):

Let \( i = a_{n-1} \).

Assume the order of \( M_1 \) and \( M_2 \) is the counter-clockwise order with respect to the barycenter of \( L_{n-2} \) and the line segment connecting the barycenter and the starting point of \( L_{n-2} \). From the assumptions, lying between the
phrase ‘Case 2) $n \geq 2$’ and the one ‘Subcase 2-1) $a_{n-1} = 1, \cdots$’, the vertex of $L_{n-1} = M_i$, which is also the barycenter of $L_{n-2}$, is the starting point of $L_{n-1} = M_i$. The clockwise angle of $L_{n-1} = M_i$ at its starting point determines the initial side and the terminal side. Out of six triangles obtained from the barycentric subdivision of $L_{n-1} = M_i$, choose the triangle with a part of the initial side of $L_{n-1} = M_i$ as its side and with the starting point and the barycenter of $L_{n-1} = M_i$ as its vertices, and call it $T_{a_0 \cdots a_{n-2} a_{n-1} 1}$, in other words, $T_{a_0 \cdots a_{n-2} 211}$, and let $N_i^j := T_{a_0 \cdots a_{n-2} 211}$. At the barycenter of $L_{n-1} = M_i$, consider the clockwise order of the 6 triangles from the $N_i^j$. The 5 triangles from the next one of $N_i^1 = T_{a_0 \cdots a_{n-2} 211} = T_{a_0 \cdots a_{n-2} a_{n-1} 1}$ will be called $T_{a_0 \cdots a_{n-2} a_{n-1} 2}, T_{a_0 \cdots a_{n-2} a_{n-1} 3}, T_{a_0 \cdots a_{n-2} a_{n-1} 4}, T_{a_0 \cdots a_{n-2} a_{n-1} 5}, T_{a_0 \cdots a_{n-2} a_{n-1} 6}$ in order. Let $N_{j}^i := T_{a_0 \cdots a_{n-2} ij} = T_{a_0 \cdots a_{n-2} a_{n-1} j}$ for $j = 2, 3, 4, 5, 6$.

If the order of $M_1$ and $M_2$ is the clockwise order, then the order will be given from the symmetry by the line connecting the barycenter and the starting point of $L_{n-1} = M_i$:

Recall the assumptions for $L_{n-1} = M_i$, lying between the phrase ‘Case 2) $n \geq 2$’ and the one ‘Subcase 2-1) $a_{n-1} = 1, \cdots$’, and let $L_{n} := N_j^i$, $j = 1, \cdots, 6$.

Note the common vertex of $L_{n-1} = M_i$ and $N_i^1$ is the starting point of $L_{n-1} = M_i$ from the definition of $N_i^1$. Now, call the vertex the starting point of $N_i^1$. And call the barycenter of $L_{n-1} = M_i$ the starting point of $N_j^i$ for $j = 2, 3, 4, 5, 6$. Also, call the barycenter the ending point of $N_j^i$ for $j = 1, 2, 3, 4, 5, 6$.

Note that the starting and the ending point of $L_{n-1} = M_i$ are same and they are the common vertex of $L_{n-1} = M_i$ and $N_i^1$.

And the side of $L_{n-1} = M_i$, which contains a side of $N_i^1$, is divided into two line segments, each of which is one side of $N_j^i$ for $j = 1, 2, 3, 4, 5, 6$.

Under the counterclockwise orientation, interior triangles for $n = 2, 3$ will be given as follows:
Now let’s review the definition of the starting points and ending points of triangles made right before as follows:
To begin with, note that the common vertex of $T_{a_0a_1\ldots a_{n-1}}$ and $T_{a_0a_1\ldots a_{n-1}}$ is the starting point of $T_{a_0a_1\ldots a_{n-1}}$ and that the common vertex of $T_{a_0a_1\ldots a_{n-1}}$ and $T_{a_0a_1\ldots a_{n-16}}$ is the ending point of $T_{a_0a_1\ldots a_{n-1}}$.

For $T_{a_0a_1\ldots a_{n-11}}$, the starting point of $T_{a_0a_1\ldots a_{n-1}}$ is the common vertex with $T_{a_0a_1\ldots a_{n-11}}$ and will be called the starting point of $T_{a_0a_1\ldots a_{n-11}}$. And the barycenter of $T_{a_0a_1\ldots a_{n-11}}$ will be called the ending point of $T_{a_0a_1\ldots a_{n-11}}$.

For $T_{a_0a_1\ldots a_{n-11}}$, where $i = 2, 3, 4, 5$, the barycenter of $T_{a_0a_1\ldots a_{n-1}}$ will be called the starting and ending point of $T_{a_0a_1\ldots a_{n-11}}$.

For $T_{a_0a_1\ldots a_{n-16}}$, the barycenter of $T_{a_0a_1\ldots a_{n-1}}$ will be called the starting point of $T_{a_0a_1\ldots a_{n-16}}$. And the ending point of $T_{a_0a_1\ldots a_{n-1}}$ is the common vertex with $T_{a_0a_1\ldots a_{n-16}}$ and will be called the ending point of $T_{a_0a_1\ldots a_{n-16}}$.

To check whether we can define triangles inductively:

Recall the assumptions for $T_{a_0a_1\ldots a_{n-1}}$, lying between the phrase ‘Case 2) $n \geq 2$’ and the one ‘Subcase 2-1) $a_{n-1} = 1, \ldots$’.

Note that the common vertex of $T_{a_0a_1\ldots a_{n-1}}$ and $T_{a_0a_1\ldots a_{n-11}}$ is the starting point of each of them. The barycenter of $T_{a_0a_1\ldots a_{n-1}}$ is the starting point of $T_{a_0a_1\ldots a_{n-1}}$ for $i = 2, 3, 4, 5, 6$, and the ending point of $T_{a_0a_1\ldots a_{n-1}}$ for $i = 1, 2, 3, 4, 5$. The common vertex of $T_{a_0a_1\ldots a_{n-1}}$ and $T_{a_0a_1\ldots a_{n-16}}$ is the ending point of each of them.

Notice that if the starting and the ending point of $T_{a_0\ldots a_{n-2a_{n-1}}}$ are same then $a_{n-1} \neq 1, 6$ and they are the common vertex of $T_{a_0\ldots a_{n-2a_{n-1}}}$ and $T_{a_0\ldots a_{n-2a_{n-1}}}$.

Also note that if the starting and the ending point of $T_{a_0\ldots a_{n-2a_{n-1}}}$ are different then $a_{n-1} \in \{1, 6\}$ and $T_{a_0\ldots a_{n-2a_{n-1}}}$ and $T_{a_0\ldots a_{n-2a_{n-1}}}$ are mutually opposite ones inside $T_{a_0\ldots a_{n-2a_{n-1}}}$.

And one side of $T_{a_0a_1\ldots a_{n-1}}$, which contains a side of $T_{a_0a_1a_2\ldots a_{n-1}}$ is divided into two line segments, each of which is one side of $T_{a_0a_1\ldots a_{n-1}}$, for $i = 1, 2$, respectively. Thus we can define triangles inductively.

A.2. The definition of exterior triangles and the definition of their starting points and ending points.

A.2.1. The definition of $S_{0}^{b_0b_1\cdots b_n}$. The given orientation at the center of $D^2$ and the base point, or equivalently the starting and ending point of $T_0$, will give the order $b_0$ of sides of $T_0$, where $b_0 = 1, 2, 3$, as explained early in ‘Section 1.’

Case 1) $n = 1$:

From the given orientation at the center of $D^2$, consider the direction of each side of $T_0$, which will give the starting point and the ending point of each side.

For the side $b_0$ of $T_0$ and the (line) segment on the boundary of $D^2$, which faces the side $b_0$ and has common terminal points with the side $b_0$, consider the midpoint of the side $b_0$ and of the boundary segment, respectively. Then a given half of the side $b_0$, the straight line segment between the midpoint of the boundary segment and the common terminal point of the side $b_0$ and
of the given half of the side $b_0$, and the straight line segment between the midpoint of the side $b_0$ and that of the boundary segment will determine a triangle, so we can obtain two triangles from each half of the side $b_0$. Let’s call them $S_{b_0}^{a_1}$ and $S_{b_0}^{a_2}$, where for $S_{b_0}^{a_i}$, $i$ is determined by the order with respect to the orientation at the center of $D^2$ and the line segment connecting the center of $D^2$ and the starting point of $T_0$.

Under the counterclockwise orientation, exterior triangles for $n = 1$ will be given as follows:

Case 2) $n \geq 2$:

Let $S_{b_0}^{b_1 \cdots b_{n-1}}$ be given.

The side of $S_{b_0}^{b_1 \cdots b_{n-1}}$, which faces the boundary of $D^2$, will give two triangles as follows:

Consider the direction of the side of $S_{b_0}^{b_1 \cdots b_{n-1}}$, which faces the boundary of $D^2$, and that of the line segment on the boundary of $D^2$, which is being faced by the side, respectively, from the orientation at the center of $D^2$ and the line segment connecting the center of $D^2$ and the starting point of $T_0$. Then we can think of the starting point, midpoint and ending point of the side of $S_{b_0}^{b_1 \cdots b_{n-1}}$, which faces the boundary of $D^2$, and those of the boundary segment, respectively. Now, refer to the construction of two triangles in ‘case 1.’ Then the triangle with the midpoints and common starting point of the side and the boundary segment as vertices will be called $S_{b_0}^{b_1 \cdots b_{n-1}1}$ and the triangle with the midpoints and common ending point of the side and the boundary segment as vertices will be called $S_{b_0}^{b_1 \cdots b_{n-1}2}$.

Under the counterclockwise orientation, exterior triangles for $n = 2, 3$ will be given as follows:
Now, let’s define the starting point and the ending point of the triangles made right before as follows:

Let \( n \geq 1 \).

For \( S_0^{b_0b_1\ldots b_{n-1}} \), \( i = 1, 2 \), consider the direction of its side facing the boundary with respect to the orientation at the center of \( D^2 \).

If \( n \) is odd, the ending point of the side, facing the boundary of \( D^2 \), will be called the starting point of \( S_0^{b_0b_1\ldots b_{n-1}} \) and the starting point of the side, facing the boundary of \( D^2 \), will be called the ending point of \( S_0^{b_0b_1\ldots b_{n-1}} \).

If \( n \) is even, the starting point of the side, facing the boundary of \( D^2 \), will be called the starting point of \( S_0^{b_0b_1\ldots b_{n-1}} \) and the ending point of the side, facing the boundary of \( D^2 \), will be called the ending point of \( S_0^{b_0b_1\ldots b_{n-1}} \).

A.2.2. The definition of \( S_{a_0a_1\ldots a_m}^{b_0b_1\ldots b_k} \). Let \( 1 \leq k < n \) be given. Let \( m = n - k \). To define \( S_{a_0a_1\ldots a_m}^{b_0b_1\ldots b_k} \), consider a triangle \( \tilde{T}_0 \) whose orientation is the opposite one of \( T_0 \) (considering \( T_{014} \) might be helpful). Then the \( m \)-step barycentric subdivision makes us think of \( \tilde{T}_{0a_1\ldots a_m} \), which is the mirror-symmetry of \( T_{0a_1\ldots a_m} \) (for example \( T_{014a_1\ldots a_m} \)). Note the orientation of the triangle \( \tilde{T}_0 \) is the opposite one of \( T_0 \) (considering \( T_{014} \) might be helpful), and its \( m \)-step barycentric subdivision \( \tilde{T}_{01a_2\ldots a_m} \) is also the mirror-symmetry of \( T_{01a_2\ldots a_m} \) (for example \( T_{014a_2\ldots a_m} \)).

We want to define \( S_{a_0a_1\ldots a_m}^{b_0b_1\ldots b_k} \) as follows:

Case 1-1) \( k \) is odd and \( b_k = 1 \):

Consider the barycentric subdivision of \( S_0^{b_0b_1\ldots b_k} \). By comparing it with that of \( T_{01} \), define

- \( S_{0j}^{b_0b_1\ldots b_k} \), which matches \( T_{01j} \) for \( j \in \{1, 2, 3, 4\} \),
- \( S_{05}^{b_0b_1\ldots b_k} \), which matches \( T_{016} \),
- \( S_{06}^{b_0b_1\ldots b_k} \), which matches \( T_{015} \),
and their starting and ending points.

For \( m \geq 2 \), the respective identification of

\[ S_{0j}^{b_0b_1\ldots b_k}, S_{0j}^{b_0b_1\ldots b_k}, S_{04}^{b_0b_1\ldots b_k}, S_{06}^{b_0b_1\ldots b_k} \text{ with } T_{01}, T_0, \tilde{T}_0, \tilde{T}_{01}, \]

where \( j \in \{2, 3, 5\} \), and their \( m \)-step barycentric subdivision can make us define \( S_{a_0a_1\ldots a_m}^{b_0b_1\ldots b_k} \), where \( a_0 = 0 \).

Case 1-2) \( k \) is odd and \( b_k = 2 \):

Identify \( S_0^{b_0b_1\ldots b_k} \) with \( T_{01} \), where the starting point and ending point of \( S_0^{b_0b_1\ldots b_k} \) is also identified to those of \( T_{01} \).

Consider the \( m \)-step barycentric subdivision of \( S_0^{b_0b_1\ldots b_k} \) and \( T_{01} \) respectively. The identification, then, can make us define \( S_{a_0a_1\ldots a_m}^{b_0b_1\ldots b_k} \) from \( T_{01a_1\ldots a_m} \), where \( a_0 = 0 \).
Under the counterclockwise orientation, the triangles for $k = 1$ and $m = 1$ will be given as follows:

\[
\begin{array}{c}
S_0^u S_1^u S_2^u \\
S_3^u S_4^u S_5^u \\
S_6^u S_7^u S_8^u \\
S_9^u S_{10}^u S_{11}^u \\
S_{12}^u S_{13}^u S_{14}^u \\
S_{15}^u S_{16}^u S_{17}^u
\end{array}
\]

Case 2-1) $k$ is even and $b_k = 1$:
By identifying $S_{01}^{b_0b_1\ldots b_k}$ with $\tilde{T}_{01}$, we can define $S_{0a_0a_1\ldots a_m}^{b_0b_1\ldots b_k}$ from $\tilde{T}_{01a_1\ldots a_m}$, where $a_0 = 0$ (for example $T_{0141a_1\ldots a_m}$).

Case 2-2) $k$ is even and $b_k = 2$:
Consider the barycentric subdivision of $S_{01}^{b_0b_1\ldots b_k}$. By comparing it with that of $\tilde{T}_{01}$, define
- $S_{0j}^{b_0b_1\ldots b_k}$, which matches $\tilde{T}_{01j}$ for $j \in \{1, 2, 3, 4\}$,
- $S_{05}^{b_0b_1\ldots b_k}$, which matches $\tilde{T}_{015}$,
- $S_{06}^{b_0b_1\ldots b_k}$, which matches $\tilde{T}_{015}$,
and their starting and ending points.

For $m \geq 2$, the respective identification of
\[
S_{01}^{b_0b_1\ldots b_k}, S_{0j}^{b_0b_1\ldots b_k}, S_{04}^{b_0b_1\ldots b_k}, S_{06}^{b_0b_1\ldots b_k}
\]
with $\tilde{T}_{01}, \tilde{T}_0, T_0, T_{01}$,

where $j \in \{2, 3, 5\}$, and their $m$-step barycentric subdivision can make us define $S_{a_0a_1a_2\ldots a_m}^{b_0b_1\ldots b_k}$, where $a_0 = 0$.

Under the counterclockwise orientation, the triangles for $k = 2$ and $m = 1$ will be given as follows:
A.3. The ordering of triangles in the \( n \)-th step. For \( n = 1, 2, \ldots \), let

\[
A_n = \{ T_{a_0a_1\ldots a_n} \mid a_0 = 0, a_i \in \{1, 2, 3, 4, 5, 6\} \text{ for } i = 1, \ldots, n \} \cup \\
(\cup_{k+m=n, 1\leq k\leq n, 0\leq m\leq n-1} \{ S_{b_0b_1\ldots b_k} \mid b_0 \in \{1, 2, 3\}, b_i \in \{1, 2\} \text{ for } i = 1, \ldots, k, \\
c_0 = 0, c_j \in \{1, \ldots, 6\} \text{ for } 1 \leq j \leq m \text{ if } m \geq 1 \})
\]

, which is regarded as the set of all triangles in the \( n \)-th step.

Now refer to the following pictures for 0th, 1st, 2nd and 3rd step under the counterclockwise orientation:

Case 1) \( T_{a_0\ldots a_n} < S_{c_0c_1\ldots c_m} \), where \( k + m = n \)

Case 2) \( T_{a_0\ldots a_n} < T_{b_0\ldots b_n} \) if \((a_0, \ldots, a_n) < (b_0, \ldots, b_0)\) with respect to the dictionary order

Case 3) The order of \( S_{b_0b_1\ldots b_k} \) and \( S_{c_0c_1\ldots c_t} \), where \( k + m = n = s + t \)

Case 3-1) \( k < s \):

\[
S_{b_0b_1\ldots b_k} < S_{c_0c_1\ldots c_s}
\]

Case 3-2) \( k = s \) (so, \( m = t \)) and \((b_0, b_1, \ldots, b_k) < (c_0, c_1, \ldots, c_k)\) with respect to the dictionary order:
If $k$ is odd, then $S_{a_0 a_1 \cdots a_m}^{b_0 b_1 \cdots b_k} > S_{d_0 d_1 \cdots d_m}^{c_0 c_1 \cdots c_k}$

If $k$ is even, then $S_{a_0 a_1 \cdots a_m}^{b_0 b_1 \cdots b_k} < S_{d_0 d_1 \cdots d_m}^{c_0 c_1 \cdots c_k}$
Case 3-3) \( k = s \), \((b_0, b_1, \cdots, b_k) = (c_0, c_1, \cdots, c_k)\) and \((a_0, \cdots, a_m) < (d_0, \cdots, d_m)\) with respect to the dictionary order:

\[
S_{a_0 \cdots a_m}^{b_0 b_1 \cdots b_k} < S_{d_0 \cdots d_m}^{c_0 c_1 \cdots c_k}
\]

A.4. **The properties of triangles in** \( A_n \). We can easily check the following three properties from the definition of triangles.

Property 1.) Given a non-first element \( L \) in \( A_n \), the boundary of \( \bigcup \{ M \in A_n | M < L \} \) contains a side of \( L \), which will be divided into two line segments in its barycentric subdivision, where one of two line segments will become a side of the first triangle and the other one will become a side of the second triangle in the barycentric subdivision of \( L \).
Property 2.) Given \( L \in A_n \), \( \bigcup \{ M \in A_n \mid M \leq L \} \) is diffeomorphic to the disk \( D^2 \).

Property 3.) Assume \( L \in A_n \) and six triangles \( M_1, M_2, \cdots, M_6 \in A_{n+1} \), obtained from the barycentric subdivision of \( L \), follows the order of \( i = 1, 2, \cdots, 6 \) in \( A_{n+1} \). Then the starting points of \( M_1 \) and \( L \) are same. Also are the ending points of \( M_6 \) and \( L \).

And we also have the next property :

Property 4.) Assume \( L, M \in A_n \) and that \( M \) is the next element of \( L \) in \( A_n \) for \( n \geq 1 \).
Then, The ending point of \( L \) and the starting point of \( M \) are same.

Proof )
Case 1) \( L = T_{a_0 \cdots a_{n-1} a_n} \) for some \( (a_0, \cdots, a_{n-1}, a_n) \)
Subcase 1-1 ) \( a_n \neq 6 \)
Note \( M = T_{b_0 \cdots b_{n-1} b_n} \), where
\[
\begin{align*}
b_i &= a_i \text{ for } 0 \leq i < n \text{ and } b_n = a_n + 1. \\
\end{align*}
\]
Then inside the triangle \( Ta_0 \cdots a_{n-1} a_n \), the barycenter of \( T_{a_0 \cdots a_{n-1}} \) is the ending point of \( L = T_{a_0 \cdots a_{n-1} a_n} \) and the starting point of \( M = T_{b_0 \cdots b_{n-1} b_n} \) at the same time.

Subcase 1-2 ) \( a_n = 6 \)
If \( a_0 = 0 \) and \( a_1 = \cdots = a_n = 6 \), then the ending point of \( L = T_{a_0 \cdots a_{n-1} a_n} \) is the ending point of \( T_{06} \) by induction and also the ending point of \( T_0 \), that is, the basepoint, which is the starting point of \( S_0^{32} \) and so the starting point of \( M = S_0^{32} \) if \( n = 1 \) and the starting point of \( M = S_{b_0 \cdots b_{n-1}}^{32} \) with \( b_0 = 0 \) and \( b_1 = \cdots = b_{n-1} = 1 \) if \( n \geq 2 \).
Now assume \( n \geq 2 \) and \( a_i \neq 6 \) for some \( i \) with \( 1 \leq i < n \).
We can find \( i_0 \) satisfying \( 1 \leq i_0 < n \), \( a_{i_o} \neq 6 \) and \( a_i = 6 \) for all \( i_0 < i \leq n \). Then \( M = T_{b_0 \cdots b_{n-1} b_n} \) satisfies
\[
\begin{align*}
b_i &= a_i \text{ for all } 0 \leq i < i_0 \\
b_{i_0} &= a_{i_0} + 1 \\
b_i &= 1 \text{ for all } i_0 < i \leq n \\
\end{align*}
\]
Note the ending point of \( L = T_{a_0 \cdots a_{n-1} a_n} \) is the ending point of \( T_{a_0 \cdots a_{i_0}} \) by induction.
Notice the starting point of \( M = T_{b_0 \cdots b_{n-1} b_n} \) is the starting point of \( T_{b_0 \cdots b_{i_0}} \) by induction.
Since \( a_i = b_i \) for \( 0 \leq i < i_0 \) and \( b_{i_0} = a_{i_0} + 1 \), the ending point of \( L = T_{a_0 \cdots a_{i_0-1} a_{i_0}} \) is the barycenter of \( T_{a_0 \cdots a_{i_0-1}} \), which is the starting point of \( T_{a_0 \cdots a_{i_0-1} b_{i_0}} = T_{b_0 \cdots b_{i_0-1} b_{i_0}} = M \). Thus, we get
the ending point of $L$ is the starting point of $M$.

Case 2) $L = S_{b_0b_1\cdots b_k}$ where $k + m = n$.

Subcase 2-1) $m = 0$ and $k = n$ is odd.

Note $(b_0, b_1, \cdots, b_n) \neq (1, 1, \cdots, 1)$, because if $(b_0, b_1, \cdots, b_n) = (1, 1, \cdots, 1)$ then $L = S_{0b_0b_1\cdots b_n} = S_{01\cdots 1}$ is the last element in $A_n$.

If $n = 1$, then we can trivially obtain that the ending point of $L$ is the starting point of $M$ from the definition of triangles.

Assume $n \geq 2$.

If $M = S_{d_0d_1\cdots d_n}$, then we get $(b_0, b_1, \cdots, b_n) > (d_0, d_1, \cdots, d_n)$ and so

either $(d_0 = b_0 - 1, d_1 = \cdots = d_n = 2$ and $b_1 = \cdots = b_n = 1)$$

or

$\exists i_0$ with $1 \leq i_0 \leq n$ such that

- $d_i = b_i$ for all $0 \leq i < i_0$
- $d_{i_0} = 1, b_{i_0} = 2$
- $d_i = 2, b_i = 1$ for all $i_0 < i \leq n$ if $1 \leq i_0 < n$.

In the first possibility, the side of $L$ and the side of $M$, both of which faces the boundary, meet at one point where $S_{b_0b_1\cdots b_n} = S_{b_0}b_0$ and $S_{d_0d_1\cdots d_n} = S_{d_0}d_0$ meet.

In the second possibility, the side of $L$ and the side of $M$, both of which faces the boundary, meet at one point which is contained in such line segment as the intersection of $S_{d_0d_1\cdots d_{i_0-1}b_{i_0}} = S_{b_0b_1\cdots b_{i_0-1}2}$ and $S_{b_0b_1\cdots b_{i_0-1}} = S_{d_0d_1\cdots d_{i_0-1}d_{i_0}}$

In any possibilities, the side of $L$ and the side of $M$, both of which faces the boundary, meet at one point. Since $n$ is odd, the point is the starting point of the side of $L$, facing the boundary, and the ending point of the side of $M$, facing the boundary, so

the ending point of $L$ is the starting point of $M$.

Subcase 2-2) $m = 0$ and $k = n$ is even.

Note $(b_0, b_1, \cdots, b_n) \neq (3, 2, \cdots, 2)$, because if $(b_0, b_1, \cdots, b_n) = (3, 2, \cdots, 2)$ then $L = S_{b_0b_1\cdots b_n} = S_{32\cdots 2}$ is the last element in $A_n$.

If $M = S_{d_0d_1\cdots d_n}$, then we get $(b_0, b_1, \cdots, b_n) < (d_0, d_1, \cdots, d_n)$ and so

either $(d_0 = b_0 + 1, b_1 = \cdots = b_n = 2$ and $d_1 = \cdots = d_n = 1)$$

or
\[ i_0 \text{ with } 1 \leq i_0 \leq n \text{ such that} \]

\[ d_i = b_i \text{ for all } 0 \leq i < i_0 \]
\[ d_{i_0} = 2, b_{i_0} = 1 \]
\[ d_i = 1, b_i = 2 \text{ for all } i_0 < i \leq n \text{ if } 1 \leq i_0 < n \]

In the first possibility, the side of \( L \) and the side of \( M \), both of which faces the boundary, meet at one point where \( S_0^{b_0b_1} = S_0^{b_02} \) and \( S_0^{d_0d_1} = S_0^{d_01} \) meet.

In the second possibility, the side of \( L \) and the side of \( M \), both of which faces the boundary, meet at one point which is contained in such line segment as the intersection of \( S_0^{b_0b_1\cdots-b_{i_0}-1b_{i_0}} = S_0^{b_0b_1\cdots-b_{i_0}-1} \) and \( S_0^{b_0b_1\cdots-b_{i_0}-12} = S_0^{d_0d_1\cdots d_{i_0-1}d_{i_0}} \).

In any possibilities, the side of \( L \) and the side of \( M \), both of which faces the boundary, meet at one point. Since \( n \) is even, the above condition implies that the point is the ending point of the side of \( L \), facing the boundary and the starting point of the side of \( M \), facing the boundary, so

the ending point of \( L \) is the starting point of \( M \).

Subcase 2-3 ) \( m \geq 1 \) and \( c_m \neq 6 \)

If \( m = 1 \), then the barycenter is both the ending point of \( L \) and the starting point of \( M \) from the definition.

Assume \( m \geq 2 \). Note \( L = S_{c_0c_1c_2\cdots c_{m-1}c_m}^{b_0b_1\cdots b_k} \) and its next element \( M \) are inside the triangle \( S_{c_0c_1c_2\cdots c_{m-1}c_m}^{b_0b_1\cdots b_k} \), which is one of the triangles obtained by the barycentric subdivision \( S_0^{b_0b_1\cdots b_k} = S_0^{b_0b_1\cdots b_k} \).

Compare it with the proper one of \( T_0, T_{01}, \tilde{T}_0 \) and \( \tilde{T}_{01} \). By referring to subcase 1-1, - by restricting it to the first triangle if needed-, we get

the ending point of \( L \) is the starting point of \( M \).

Subcase 2-4 ) \( m \geq 1 \) and \( c_m = 6 \)

Note \( L = S_{c_0c_1\cdots c_m}^{b_0b_1\cdots b_k} \) is inside the triangle \( S_0^{b_0b_1\cdots b_k} \), where \( c_0 = 0 \).

Assume \( c_1 = c_2 = \cdots = c_{m-1} = c_m = 6 \).

If \( m = 1 \), then the ending point of \( L = S_{c_0c_m}^{b_0b_1\cdots b_k} = S_{06}^{b_0b_1\cdots b_k} \) will be the ending point of \( S_{06}^{b_0\cdots b_k} \) tautologically. If \( m \geq 2 \), then compare \( S_{c_0c_1}^{b_0b_1\cdots b_k} \) with the proper one of \( T_0, T_{01}, \tilde{T}_0 \) and \( \tilde{T}_{01} \). Then from the comparison, the ending point of \( L \) will be the ending point of \( S_{06}^{b_0\cdots b_k} \), which is also the ending point of \( S_0^{b_0\cdots b_k} \).

If \( M = S_{a_0\cdots a_t}^{d_0d_1\cdots d_s} \), with \( a_0 = 0 \), then we get
either

\[(k = s, a_1 = \cdots = a_t = 1 \text{ and } S_0^{d_0 d_1 \cdots d_s} \text{ is the next element of } S_0^{b_0 \cdots b_k} \text{ in } A_{k+1})\]
or

\[(s = k + 1, \ t = m - 1, \ a_i = 1 \text{ for } 1 \leq i \leq t \text{ in case of } m \geq 2 \text{ and}\]

\[S_0^{d_0 d_1 \cdots d_s} \text{ is the next element of } S_0^{b_0 \cdots b_k} \text{ in } A_{k+1}).\]

In the first possibility, \(S_0^{d_0 d_1 \cdots d_s}\) will be also the next element of \(S_0^{b_0 b_1 \cdots b_k}\) in \(A_k\) and so the ending point of \(S_0^{b_0 b_1 \cdots b_k}\) will be the starting point of \(S_0^{d_0 d_1 \cdots d_s}\) from subcase 2-1 and 2-2, which implies

the ending point of \(S_0^{b_0 b_1 \cdots b_k}\) will be the starting point of \(S_0^{d_0 d_1 \cdots d_s}\).

In the second possibility, note one of \(k\) and \(s\) is odd and the other one is even, which implies that \(S_0^{b_0 b_1 \cdots b_k}\) is the last element in \(A_k\) and that \(S_0^{d_0 d_1 \cdots d_s}\) is the first element in the subset

\[\{S_0^{x_0 x_1 \cdots x_s} \mid x_0 \in \{1, 2, 3\}, \ x_i \in \{1, 2\} \text{ for } i = 1, \cdots s\}\]
of \(A_s = A_{k+1}\). Also, notice that the ending point of \(S_0^{b_0 b_1 \cdots b_k}\) is also the ending point of \(S_0^{b_0 b_1 \cdots b_k}\) from the definition of triangles. By thinking of the side of \(S_0^{b_0 b_1 \cdots b_k}\), which faces the boundary, and the side of \(S_0^{d_0 d_1 \cdots d_s}\), which faces the boundary, we get

the ending point of \(S_0^{b_0 b_1 \cdots b_k}\) will be the starting point of \(S_0^{d_0 d_1 \cdots d_s}\),

so

the ending point of \(S_0^{b_0 b_1 \cdots b_k}\) will be the starting point of \(S_0^{d_0 d_1 \cdots d_s}\).

In any possibilities, the ending point of \(S_0^{b_0 \cdots b_k}\), which is also the ending point of \(L\), is the starting point of its next element in \(A_{k+1}\), which will be the starting point of \(M\) from \(a_1 = \cdots = a_t = 1\) if \(t \geq 1\). Thus, we get

the ending point of \(L\) is the starting point of \(M\).

Now, assume \(m \geq 2\) and \(c_i \neq 6\) for some \(1 \leq i < m\). From the comparison of \(S_0^{b_0 b_1 \cdots b_k}\) with the proper one of \(T_0, T_0^1, \tilde{T}_0\) and \(\tilde{T}_0^1\), we get \(L\) and \(M\) are inside the triangle \(S_0^{b_0 \cdots b_k}\) and from subcase 1-2, we get

the ending point of \(L\) is the starting point of \(M\).
B. ABOUT CURVES

B.1. Notations. \( f \ast g : [0, 1] \rightarrow \mathbb{H}^n \) is an ordinary juxtaposition of curves \( f, g : [0, 1] \rightarrow \mathbb{H}^n \). And, for a given curve \( c : [0, 1] \rightarrow \mathbb{H}^n \), \( \bar{c} \) represents a curve whose direction is opposite to that of \( c \), that is, \( \bar{c} : [0, 1] \rightarrow \mathbb{H}^n \) is given by \( \bar{c}(t) = c(1 - t) \).

B.2. Simplification \( \gamma \) of a curve \( g : [a, b] \rightarrow \mathbb{H}^n \). Given a curve \( g : [a, b] \rightarrow S \), we can think of a curve \( \gamma : [a, b] \rightarrow S \) whose direction is one-sided as follows:

If we can find \( c, d, e \in (a, b) \) such that \( a < c < d < e < b \) and \( \text{Im}(g|_{[c, d]}) = \text{Im}(g|_{[d, e]}) \) and that the directions of \( g|_{[c, d]} \) and \( g|_{[d, e]} \) are one-sided but opposite from each other, then we can think of the new curve \( \tilde{g} : [a, b] \rightarrow D^2 \) from the remaining part \( g|_{[a, c]} \) and \( g|_{[e, b]} \) by translating in the domain and rescaling as follows:

\[ \text{Note } g(c) = g(e). \]

Consider two curves \( g_1 : [a, d] \rightarrow \mathbb{H}^n \) and \( g_2 : [d, b] \rightarrow \mathbb{H}^n \) given by

\[ g \left( \frac{c - a}{d - a} (t - a) + a \right) = g_1(t) \text{ for } t \in [a, d] \]

and

\[ g \left( \frac{b - e}{b - d} (t - b) + b \right) = g_2(t) \text{ for } t \in [d, b], \]

and then let \( \tilde{g} = g_1 \ast g_2 \).

From a curve obtained by doing this work again and again, we can think of a constant speed curve \( \gamma : [a, b] \rightarrow S \) which we want.

B.3. The definition of \( D_n, j_n, t_1^n, t_2^n \).

\[ D_n = \left\{ \frac{1}{2} \cdot \frac{j}{6^n} \mid j = 0, 1, 2, \ldots, 6^n \right\} \bigcup \left( \bigcup_{k=1}^{n} \left\{ \sum_{i=1}^{k} \frac{1}{2i} + \frac{1}{2^{k+1}} \cdot \frac{j}{2^{k-1} \cdot 6^n-k+1} \mid j = 0, 1, 2, \ldots, 2^{k-1} \cdot 6^n-k+1 \right\} \right) \]

Think of the usual order \( D_n \) and regard

\[ 0, \frac{1}{2}, \frac{1}{6^n}, \frac{2}{2}, \frac{2}{6^n}, \ldots, \frac{1}{2} = \frac{1}{2} \cdot \frac{6^n}{6^n}, \frac{1}{2} + \frac{1}{2^2} \cdot \frac{1}{2^0} \cdot \frac{6^n}{6^n}, \ldots \in D_n \]

as 0th, 1st, 2nd, \ldots, \( 6^n \)th, \( 6^{n+1} \)th, \ldots element, respectively.

Now, define functions

\[ j_n : D_n \rightarrow \{0, 1, 2, 3, \ldots \} \]

\[ t_1^n : D_n - \{0\} \rightarrow D_n \]

\[ t_2^n : D_n - \{ \text{the last element of } D_n \} \rightarrow D_n \]

as follows:
\( j_n(s) = j \) for the \( j \)-th element \( s \in D_n \).

\( t^n_1(s) \) is the \((j - 1)\)-th element in \( D_n \) for a given \( j \)-th element \( s \in D_n - \{0\} \).

\( t^n_2(s) \) is the \((j + 1)\)-th element in \( D_n \) for a given \( j \)-th element \( s \in D_n - \{ \) the last element of \( D_n \} \).

B.4. **Definition of \( \gamma^n_{t_0}, c^n_{t_0}, \bar{c}^n_{t_0}, \varphi^n_{t_0}, \bar{\varphi}^n_{t_0}, \phi^n_{t_0} \) and \( \psi^n_{t_0} \) on the disk \( D^2 \).**

Let \( n \in \{1, 2, 3, \cdots \} \) and \( t_0 \in D_n \) be given. With respect to the ordering of \( D_n \), we’ll define \( \gamma^n_{t_0}, c^n_{t_0}, \bar{c}^n_{t_0}, \varphi^n_{t_0}, \bar{\varphi}^n_{t_0} \) and \( \phi^n_{t_0} \) inductively for each fixed \( n \):

- **Case 1)** \( t_0 \) is the first element in \( D_n \), in fact, \( t_0 = \frac{1}{2} \cdot \frac{1}{6^n} \)

The orientation at the barycenter of \( T_0 \in A_0 \) will give the direction of the boundary curve of the first triangle in \( A_n \). Then

\[
\begin{align*}
c^n_{t_0} : [0, 1] & \to \{ \text{basepoint} \} \subset D^2 \\
\bar{c}^n_{t_0} : [0, 1] & \to \{ \text{basepoint} \} \subset D^2 \\
\varphi^n_{t_0} : [0, 1] & \to D^2 
\end{align*}
\]

and

\[
\gamma^n_{t_0} : [0, 1] \to D^2
\]

can be thought, where \( \varphi^n_{t_0} \) and \( \gamma^n_{t_0} \) are the piecewise smooth boundary curve of the first triangle in \( A_n \) with constant speed and the direction of the boundary curve is induced from the given orientation.

Note \( \gamma^n_{t_0} \) can be regarded as the simplification of \( c^n_{t_0} \ast \varphi^n_{t_0} \ast \bar{c}^n_{t_0} \).

We will call \( \gamma^n_{t_0} \) the **holonomy curve at time** \( t = t_0 \).

Now, consider the path from the basepoint to the ending point of the first triangle in \( n \)-step along the opposite direction of the holonomy curve \( \gamma^n_{t_0} \) at \( t = t_0 \), which is a piecewise smooth curve with constant speed. Then from the path, we can define a piecewise smooth curve

\[
\psi^n_{t_0} : [0, 1] \to D^2
\]

with constant speed. And its opposite direction can make us define

\[
\psi^n_{t_0} : [0, 1] \to D^2.
\]

Define a piecewise smooth curve

\[
\psi^n_{t_0} : [0, 1] \to D^2
\]
with constant speed as the boundary curve of the 1st triangle in the n-th step, where the curve is a loop at the ending point of the first triangle and the direction of the boundary curve is induced from the given orientation.

Case 2) $t_0$ is the $j$-th element in $D_n$, where $j \geq 2$

Let $t_1$ be the $(j - 1)$-th element in $D_n$, where $j - 1 \geq 1$.

Consider the path from the basepoint to the starting point of the $j$-th triangle in the n-th step along the opposite direction of the holonomy curve $\gamma_{t_1}^n$ at $t = t_1$, which is a piecewise smooth one with constant speed. Then from the path, we can define a piecewise smooth curve

$$c_{t_0}^n : [0, 1] \rightarrow \partial U_{j-1} \subset D^2$$

with constant speed, where $U_{j-1}$ is the union of triangle in $A_n$ from 1st one to $(j - 1)$-th one.

And its opposite direction can make us define

$$\overline{c}_{t_0}^n : [0, 1] \rightarrow \partial U_{j-1} \subset D^2.$$

Define a piecewise smooth curve

$$\varphi_{t_0}^n : [0, 1] \rightarrow D^2$$

with constant speed as the boundary curve of the $j$-th triangle in the n-th step, where the curve is a loop at the starting point of the triangle and the direction of the boundary curve is induced from the given orientation.

Now define a piecewise smooth curve

$$\gamma_{t_0}^n : [0, 1] \rightarrow \partial U_j \subset D^2$$

with constant speed from the simplification of $\gamma_{t_1}^n \ast c_{t_0}^n \ast \varphi_{t_0}^n \ast \overline{c}_{t_0}^n$, where $U_j$ is the union of triangle in $A_n$ from 1st one to $j$-th one. The new curve will be also called the holonomy curve at time $t = t_0$.

Now, consider the path from the basepoint to the ending point of the $j$-th triangle in the n-th step along the opposite direction of the holonomy curve $\gamma_{t_0}^n$ at $t = t_0$, which is a piecewise smooth one with constant speed. Then from the path, we can define a piecewise smooth curve

$$1c_{t_0}^n : [0, 1] \rightarrow \partial U_j \subset D^2$$

with constant speed. And its opposite direction can make us define

$$1\overline{c}_{t_0}^n : [0, 1] \rightarrow \partial U_j \subset D^2.$$

Define a piecewise smooth curve

$$\psi_{t_0}^n : [0, 1] \rightarrow D^2.$$
with constant speed as the boundary curve of the \( j \)-th triangle in the \( n \)-th step, where the curve is a loop at the ending point of the \( j \)-th triangle and the direction of the boundary curve is induced from the given orientation.

B.5. the simplification of \( \bar{c}_{t_0}^n \ast 1c_{t_0}^n \). For each \( n \geq 1 \) and \( t_0 \neq 0 \), where \( t_0 \) is the \( j_n(t_0) \)-th element in \( D_n \), the simplification of \( \bar{c}_{t_0}^n \ast 1c_{t_0}^n \) is a curve along the boundary curve of \( j_n(t_0) \)-th triangle in \( A_n \) with opposite direction to the given orientation such that it starts from the starting point of the triangle and that its image consists of the following sets:

- one point, one side, two sides or the boundary of the triangle.

Proof )

If \( n=1 \), then it can be easily checked.

Assume \( n \geq 2 \). If \( t_0 \) is greater than the maximum of \( D_{n-1} - \{1\} \), then the above property can be easily checked.

Now assume \( n \geq 2 \) and \( t_0 \) is less than or equal to the maximum of \( D_{n-1} - \{1\} \). Now find \( \delta_n(t_0) \in D_{n-1} \) such that \( t_{n-1}^{n-1}(\delta_n(t_0)) < t_0 \leq \delta_n(t_0) \), where \( t_{n-1}^{n-1}(\delta_n(t_0)) \) is the previous element of \( \delta_n(t_0) \) in \( D_{n-1} \). Then, the \( j_n(t_0) \)-th triangle in \( A_n \) is one of the barycentric subdivision of the \( j_{n-1}(\delta_n(t_0)) \)-th triangle in \( A_{n-1} \).

And find a value \( \epsilon(j_n(t_0)) \) such that, for the given \( L = j_n(t_0) \)-th triangle in \( A_n \),

\[
\text{if } L = T_{a_0a_1\cdots a_n}, \text{ then } \epsilon(j_n(t_0)) = a_n
\]

and

\[
\text{if } L = S_{a_0a_1\cdots a_s}^{b_1\cdots b_k}, \text{ where } n = k + s, \text{ then } \epsilon(j_n(t_0)) = a_s.
\]

Assume \( n \geq 2 \) and that the property, mentioned early in this subsection, holds for \( n - 1 \).

Then we obtained the following result.

Case 1) Assume the image of the simplification of \( \bar{c}_{\delta_n(t_0)}^{n-1} \ast 1c_{\delta_n(t_0)}^{n-1} \) consists of one point.

Now refer to the following picture under the counterclockwise orientation.

The thick line is a part of the image of \( \gamma_{t_0}^{n-1}(\delta_n(t_0)) \) and the outer triangle is the \( j_{n-1}(\delta_n(t_0)) \)-th triangle in \( A_{n-1} \).
Note the direction of the line segment of the \( j_{n-1}(\delta_n(t_0)) \)-th triangle along \( \gamma_{t_1}^{n-1}(\delta_n(t_0)) \), mentioned in the Property 1 in the subsection A.4 of the section A, lying on the boundary curve \( \gamma_{t_1}^{n-1}(\delta_n(t_0)) \), is from the common vertex of the \( j_{n-1}(\delta_n(t_0)) \)-th triangle in \( A_{n-1} \) with the second triangle of its barycentric subdivision to its common vertex with the first triangle of its barycentric subdivision, and

\[
\epsilon(j_n(t_0)) = 1, 6 \Rightarrow \overline{c_{t_0}^n} \ast 1c_{t_0}^n \text{ consists of one side}
\]
\[
\epsilon(j_n(t_0)) = 2, 3 \Rightarrow \overline{c_{t_0}^n} \ast 1c_{t_0}^n \text{ consists of one point}
\]
\[
\epsilon(j_n(t_0)) = 4, 5 \Rightarrow \overline{c_{t_0}^n} \ast 1c_{t_0}^n \text{ consists of one point}
\]

Case 2) Assume the image of the simplification of \( \overline{c_{\delta_n(t_0)}^{n-1}} \ast 1c_{\delta_n(t_0)}^{n-1} \) consists of one side.

Now refer to the following picture under the counterclockwise orientation. The thick line is a part of the image of \( \gamma_{t_1}^{n-1}(\delta_n(t_0)) \) and the outer triangle is the \( j_{n-1}(\delta_n(t_0)) \)-th triangle in \( A_{n-1} \). Don’t forget that the ending point of \( j_{n-1}(\delta_n(t_0)) \) in \( A_{n-1} \) will lie on the image of \( \gamma_{\delta_n(t_0)}^{n-1} \), even though it might not lie on the image of \( \gamma_{t_1}^{n-1}(\delta_n(t_0)) \).
Note the direction of the line segment of the $j_{n-1}(\delta_n(t_0))$-th triangle along $\gamma_{t_1}^{n-1}(\delta_n(t_0))$, mentioned in the Property 1 in the subsection A.4 of the section A, lying on the boundary curve $\gamma_{t_1}^{n-1}(\delta_n(t_0))$, is from the common vertex of the $j_{n-1}(\delta_n(t_0))$-th triangle in $A_{n-1}$ with the second triangle of its barycentric subdivision to its common vertex with the first triangle of its barycentric subdivision, and

$$
\epsilon(j_n(t_0)) = 1 \Rightarrow \bar{c}_{t_0}^n * 1c_{t_0}^n \text{ consists of one side}
$$
$$
\epsilon(j_n(t_0)) = 2, 3 \Rightarrow \bar{c}_{t_0}^n * 1c_{t_0}^n \text{ consists of one point}
$$
$$
\epsilon(j_n(t_0)) = 4 \Rightarrow \bar{c}_{t_0}^n * 1c_{t_0}^n \text{ consists of the boundary}
$$
$$
\epsilon(j_n(t_0)) = 5 \Rightarrow \bar{c}_{t_0}^n * 1c_{t_0}^n \text{ consists of either the boundary or one side}
$$
$$
\epsilon(j_n(t_0)) = 6 \Rightarrow \bar{c}_{t_0}^n * 1c_{t_0}^n \text{ consists of either one side or two sides}
$$

Remark B.1. The last 2 pictures in the bottom seem to be possible under the induction hypothesis. But it might not happen in fact.

Remark B.2. The following picture in the bottom can’t happen from Property 2 in the subsection A.4 of the section A.

Case 3) Assume the image of the simplification of $\bar{c}_{t}^{n-1} * 1c_{t_0}^{n-1}$ consists of two sides.

Now refer to the following picture under the counterclockwise orientation. The thick line is a part of the image of $\gamma_{\delta_n(t_0)}^{n-1}$ and the outer triangle is the $j_{n-1}(\delta_n(t_0))$-th triangle in $A_{n-1}$. Don’t forget that the ending point of $j_{n-1}(\delta_n(t_0))$ in $A_{n-1}$ will lie on the image of $\gamma_{\delta_n(t_0)}^{n-1}$, even though it might not lie on the image of $\gamma_{t_1}^{n-1}(\delta_n(t_0))$. 
Note the direction of the line segment of the $j_{n-1}(\delta_n(t_0))$-th triangle along $\gamma_{t_{n-1}}^{n-1}(\delta_n(t_0))$, mentioned in the Property 1 in the subsection A.4 of the section A, lying on the boundary curve $\gamma_{t_{n-1}}^{n-1}(\delta_n(t_0))$, is from the common vertex of the $j_{n-1}(\delta_n(t_0))$-th triangle in $A_{n-1}$ with the first triangle of its barycentric subdivision to its common vertex with the second triangle of its barycentric subdivision, and

\[ \epsilon(j_n(t_0)) = 1 \Rightarrow \bar{c}^n_{t_0} \ast 1 \bar{c}^n_{t_0} \text{ consists of two sides} \]
\[ \epsilon(j_n(t_0)) = 2, 3 \Rightarrow \bar{c}^n_{t_0} \ast 1 \bar{c}^n_{t_0} \text{ consists of the boundary} \]
\[ \epsilon(j_n(t_0)) = 4 \Rightarrow \bar{c}^n_{t_0} \ast 1 \bar{c}^n_{t_0} \text{ consists of one point} \]
\[ \epsilon(j_n(t_0)) = 5 \Rightarrow \bar{c}^n_{t_0} \ast 1 \bar{c}^n_{t_0} \text{ consists of either one point or the boundary} \]
\[ \epsilon(j_n(t_0)) = 6 \Rightarrow \bar{c}^n_{t_0} \ast 1 \bar{c}^n_{t_0} \text{ consists of either two sides or one side} \]

**Remark B.3.** The last 3 pictures in the bottom seem to be possible under the induction hypothesis. But it might not happen in fact.

**Remark B.4.** The following picture in the bottom can’t happen from Property 2 in the subsection A.4 of the section A.
Case 4) Assume the image of the simplification of $\bar{c}_{\delta_n(t_0)}^{n-1} \ast 1c_{\delta_n(t_0)}^{n-1}$ consists of the boundary.

Now refer to the following picture under the counterclockwise orientation. The thick line is a part of the image of $\gamma_{t_1}^{n-1}(\delta_n(t_0))$ and the outer triangle is the $j_{n-1}(\delta_n(t_0))$-th triangle in $A_{n-1}$.

Note the direction of the line segment of the $j_{n-1}(\delta_n(t_0))$-th triangle along $\gamma_{t_1}^{n-1}(\delta_n(t_0))$, mentioned in the Property 1 in the subsection A.3 of the section A, lying on the boundary curve $\gamma_{t_1}^{n-1}(\delta_n(t_0))$, is from the common vertex of the $j_{n-1}(\delta_n(t_0))$-th triangle in $A_{n-1}$ with the first triangle of its barycentric subdivision to its common vertex with the second triangle of its barycentric subdivision, and

$$\epsilon(j_{\alpha}(t_0)) = 1, 6 \Rightarrow \bar{c}_{t_0}^{n} \ast 1c_{t_0}^{n} \text{ consists of two sides}$$
$$\epsilon(j_{\alpha}(t_0)) = 2, 3 \Rightarrow \bar{c}_{t_0}^{n} \ast 1c_{t_0}^{n} \text{ consists of the boundary}$$
$$\epsilon(j_{\alpha}(t_0)) = 4, 5 \Rightarrow \bar{c}_{t_0}^{n} \ast 1c_{t_0}^{n} \text{ consists of the boundary}$$

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