1. The statement. – Let $\mathbf{k}$ be an algebraically closed field of characteristic 0. If $f_1$ and $f_2$ are two endomorphisms of a projective surface $X$ over $\mathbf{k}$ and $f_1$ is conjugate to $f_2$ by a birational transformation of $X$, then $f_1$ and $f_2$ have the same topological degree. When $X$ is the projective plane $\mathbb{P}^2_\mathbf{k}$, $f_1$ (resp. $f_2$) is given by homogeneous formulas of the same degree $d$ without common factor, and $d$ is called the degree, or algebraic degree of $f_1$; in that case the topological degree is $d^2$, so, $f_1$ and $f_2$ have the same degree $d$ if they are conjugate.

Theorem A. Let $\mathbf{k}$ be an algebraically closed field of characteristic 0. Let $f_1$ and $f_2$ be dominant endomorphisms of $\mathbb{P}^2_\mathbf{k}$ over $\mathbf{k}$. Let $h : \mathbb{P}^2_\mathbf{k} \dashrightarrow \mathbb{P}^2_\mathbf{k}$ be a birational map such that $h \circ f_1 = f_2 \circ h$. If the degree $d$ of $f_1$ is $\geq 2$, there exists an isomorphism $h' : \mathbb{P}^2_\mathbf{k} \rightarrow \mathbb{P}^2_\mathbf{k}$ such that $h' \circ f_1 = f_2 \circ h'$.

Moreover, $h$ itself is in $\text{Aut}(\mathbb{P}^2_\mathbf{k})$, except may be if $f_1$ is conjugate by an element of $\text{Aut}(\mathbb{P}^2_\mathbf{k})$ to

1. the composition of $g_d : [x : y : z] \mapsto [x^d : y^d : z^d]$ and a permutation of the coordinates,

2. or the endomorphism $(x, y) \mapsto (x^d, y^d + \sum_{j=2}^{d} a_j y^{d-j})$ of the open subset $\mathbb{A}^1_\mathbf{k} \setminus \{0\} \times \mathbb{A}^1_\mathbf{k} \subset \mathbb{P}^2_\mathbf{k}$, for some coefficients $a_j \in \mathbf{k}$.

Theorem A is proved in Sections 2 to 6. A counter-example is given in Section 7 when $\text{char}(\mathbf{k}) \neq 0$. The case $d = 1$ is covered by [11]; in particular, there are automorphisms $f_1, f_2 \in \text{Aut}(\mathbb{P}^2_\mathbf{k})$ which are conjugate by some birational transformation but not by an automorphism.

Example 1. When $f_1 = f_2$ is the composition of $g_d$ and a permutation of the coordinates and $h$ is the Cremona involution $[x : y : z] \mapsto [x^{-1} : y^{-1} : z^{-1}]$, we have $h \circ f_1 = f_2 \circ h$.

Example 2. When

$$f_1(x, y) = (x^d, y^d + \sum_{j=2}^{d} a_j y^{d-j}) \quad \text{and} \quad f_2(x, y) = (x^d, y^d + \sum_{j=2}^{d} a_j (B/A)^j x^j y^{d-j})$$

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with \( a_j \in \mathbb{k} \) then \( h(x, y) = (Ax, Bxy) \) conjugates \( f_1 \) to \( f_2 \) if \( A \) and \( B \) are roots of unity of order dividing \( d - 1 \), and \( \deg(h) = 2 \). On the other hand, \( h'[x : y : z] = [Az/B : y : x] \) is an automorphism of \( \mathbb{P}^2 \) that conjugates \( f_1 \) to \( f_2 \).

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### 2. The exceptional locus

If \( h : \mathbb{P}^2 \to \mathbb{P}^2 \) is a birational map, we denote by \( \text{Ind}(h) \) its **indeterminacy locus** (a finite subset of \( \mathbb{P}^2(\mathbb{k}) \)), and by \( \text{Exc}(h) \) its **exceptional set**, i.e. the union of the curves contracted by \( h \) (a finite union of irreducible curves). Let \( U_h = \mathbb{P}^2_\mathbb{k} \setminus \text{Exc}(h) \) be the complement of \( \text{Exc}(h) \); it is a Zariski dense open subset of \( \mathbb{P}^2_\mathbb{k} \). If \( C \subset \mathbb{P}^2_\mathbb{k} \) is a curve, we denote by \( h_* (C) \) the **strict transform** of \( C \), i.e. the Zariski closure of \( h(C) \setminus \text{Ind}(f) \).

**Proposition 3.** If \( h \) is a birational transformation of the projective plane, then

1. \( \text{Ind}(h) \subseteq \text{Exc}(h) \),
2. \( h|_{U_h}(U_h) = U_{h^{-1}} \), and
3. \( h|_{U_h} : U_h \to U_{h^{-1}} \) is an isomorphism.

**Proof.** There is a smooth projective surface \( X \) and two birational morphisms \( \pi_1, \pi_2 : X \to \mathbb{P}^2 \) such that \( h = \pi_2 \circ \pi_1^{-1} \); we choose \( X \) minimal, in the sense that there is no \((-1)\)-curve \( C \) of \( X \) which is contracted by both \( \pi_1 \) and \( \pi_2 \) ([15]).

Pick a point \( p \in \text{Ind}(h) \). The divisor \( \pi_1^{-1}(p) \) is a tree of rational curves of negative self-intersections, with at least one \((-1)\)-curve. If \( p \notin \text{Exc}(h) \), any curve contracted by \( \pi_2 \) that intersects \( \pi_1^{-1}(p) \) is in fact contained in \( \pi_1^{-1}(p) \). But \( \pi_2 \) may be decomposed as a succession of contractions of \((-1)\)-curves: since it does not contract any \((-1)\)-curve in \( \pi_1^{-1}(p) \), we deduce that \( \pi_2 \) is a local isomorphism along \( \pi_1^{-1}(p) \). This contradicts the minimality of \( \mathbb{P}^2_\mathbb{k} \), hence \( \text{Ind}(h) \subset \text{Exc}(h) \). Thus \( h|_{U_h} : U_h \to \mathbb{P}^2 \) is regular. Since \( U_h \cap \text{Exc}(h) = \emptyset \), \( h|_{U_h} \) is an open immersion, \( h^{-1} \) is well defined on \( h|_{U_h}(U_h) \), and \( h^{-1} \) is an open immersion on \( h|_{U_h}(U_h) \). It follows that \( h|_{U_h}(U_h) \subseteq U_{h^{-1}} \). The same argument shows that \( h^{-1}|_{U_{h^{-1}}^{-1}} : U_{h^{-1}} \to \mathbb{P}^2 \) is well defined and its image is in \( U_h \). Since \( h^{-1}|_{U_{h^{-1}}^{-1}} \circ h|_{U_h} = \text{id} \) and \( h|_{U_h} \circ h^{-1}|_{U_{h^{-1}}} = \text{id} \); this concludes the proof. \( \square \)

Let \( f_1 \) and \( f_2 \) be dominant endomorphisms of \( \mathbb{P}^2_\mathbb{k} \). Let \( h : \mathbb{P}^2 \to \mathbb{P}^2 \) be a birational map such that \( f_1 = h^{-1} \circ f_2 \circ h \). Let \( d \) be the common (algebraic) degree of \( f_1 \) and \( f_2 \). Recall that an algebraic subset \( D \) of \( \mathbb{P}^2_\mathbb{k} \) is **totally invariant** under the action of the endomorphism \( g \) if \( g^{-1}(C) = C \) (then \( g(C) = C \), and if \( \deg(g) \geq 2 \), \( g \) ramifies along \( C \)).

**Lemma 4.** The exceptional set of \( h \) is totally invariant under the action of \( f_1 : f_1^{-1}(\text{Exc}(h)) = \text{Exc}(h) \).
Proof. Since \( h \circ f_1 = f_2 \circ h \), the strict transform of \( f_1^{-1}(\text{Exc}(h)) \) by \( f_2 \circ h \) is a finite set, but every dominant endomorphism of \( \mathbb{P}^2_k \) is a finite map, so the strict transform of \( f_1^{-1}(\text{Exc}(h)) \) by \( h \) is already a finite set. This means that \( f_1^{-1}(\text{Exc}(h)) \) is contained in \( \text{Exc}(h) \); this implies \( f_1(\text{Exc}(E)) \subset E \) and then \( f_1^{-1}(\text{Exc}(h)) = \text{Exc}(h) = f_1(\text{Exc}(h)) \) because \( f_1 \) is onto.  

\[ \square \]

**Lemma 5.** If \( d \geq 2 \) then \( \text{Exc}(h) \) and \( \text{Exc}(h^{-1}) \) are two isomorphic configurations of lines, and this configuration falls in the following list:

(P0) the empty set;
(P1) one line in \( \mathbb{P}^2 \);
(P2) two lines in \( \mathbb{P}^2 \);
(P3) three lines in \( \mathbb{P}^2 \) in general position.

Proof. Assume \( \text{Exc}(h) \) is not empty; then, by Lemma 4, the curve \( \text{Exc}(h) \) is totally invariant under \( f_1 \). According to [6, §4] and [4, Proposition 2], \( \text{Exc}(h) \) is one of the three curves listed in (P1) to (P3).

Changing \( h \) into \( h^{-1} \) and permuting the role of \( f_1 \) and \( f_2 \), we see that \( \text{Exc}(h^{-1}) \) is also a configuration of type (Pi) for some \( i \). Proposition 5 shows that \( U_h \simeq U_{h^{-1}} \). Since the four possibilities (Pi) correspond to pairwise non-isomorphic complements, we deduce that \( \text{Exc}(h) \) and \( \text{Exc}(h^{-1}) \) have the same type.  

\[ \square \]

**Remark 6.** One can also refer to [7] to prove this lemma. Indeed, \( f_1 \) induces a map from the set of irreducible components of \( \text{Exc}(h) \) into itself, and since \( f_1 \) is onto, this map is a permutation; the same applies to \( f_2 \). Thus, replacing \( f_1 \) and \( f_2 \) by \( f_1^m \) and \( f_2^m \) for some suitable \( m \geq 1 \), we may assume that \( f_1(C) = C \) for every irreducible component \( C \) of \( \text{Exc}(h) \). Since \( f_1 \) is finite, \( \text{Exc}(h) \) has only finitely many irreducible components, and \( f_1(\text{Exc}(h)) = \text{Exc}(h) \), we obtain \( f_1^{-1}(C) = C \) for every component. Since \( f_1 \) acts by multiplication by \( d \) on \( \text{Pic}(\mathbb{P}^2_k) \), the ramification index of \( f_1 \) along \( C \) is \( d > 1 \), and the main theorem of [7] implies that \( C \) is a line.

**Remark 7.** Totally invariant hypersurfaces of endomorphisms of \( \mathbb{P}^3 \) are unions of hyperplanes, at most four of them (we refer to [9] for a proof and important additional references, notably the work of J.-M. Hwang, N. Nakayama and D.-Q. Zhang). So, an analog of Lemma 5 holds in dimension 3 too; but our proof in case (P1), see § 4 below, does not apply in dimension 3, at least not directly. (Note that [2] contains an important gap, since its main result is based on a wrong lemma from [3]).

3. Normal forms. – Two configurations of the same type (Pi) are equivalent under the action of \( \text{Aut}(\mathbb{P}^2_k) = \text{PGL}_3(k) \). If we change \( h \) into \( A \circ h \circ B \) for some well chosen pair of automorphisms \( (A, B) \), or equivalently if we change \( f_1 \) into
Theorem A is proved.

\( \text{(P0)} \) – \( \text{Exc}(h) = \text{Exc}(h^{-1}) = \emptyset \). – Then \( h \) is an automorphism of \( \mathbb{P}^2_k \) and Theorem A is proved.

\( \text{(P1)} \) – \( \text{Exc}(h) = \text{Exc}(h^{-1}) = \{ z = 0 \} \). – Then \( h \) induces an automorphism of \( \mathbb{A}^2_k \) and \( f_1 \) and \( f_2 \) restrict to endomorphisms of \( \mathbb{A}^2_k = \mathbb{P}^2_k \setminus \{ z = 0 \} \) (that extend to endomorphisms of \( \mathbb{P}^2_k \)).

\( \text{(P2)} \) – \( \text{Exc}(h) = \text{Exc}(h^{-1}) = \{ x = 0 \} \cup \{ z = 0 \} \). – Then, \( U_h \) and \( U_{h^{-1}} \) are both equal to the open set \( U := \{ (x,y) \in \mathbb{A}^2 \mid x \neq 0 \} \). Moreover,

\[
    h|_U(x,y) = (Ax, Bx^my + C(x))
\]

for some regular function \( C(x) \) on \( \mathbb{A}^1_k \setminus \{ 0 \} \) and \( m \in \mathbb{Z} \), and

\[
    f_i|_U(x,y) = (x^{\pm d}, F_i(x,y))
\]

for some rational functions \( F_i \in k(x)[y] \) which are regular on \( (\mathbb{A}^1_k \setminus \{ 0 \}) \times \mathbb{A}^1 \) and have degree \( d \) (more precisely, \( f_i \) must define an endomorphism of \( \mathbb{P}^2 \) of degree \( d \)). Moreover, the signs of the exponent \( \pm d \) in Equation (2) are the same for \( f_1 \) and \( f_2 \).

\( \text{(P3)} \) – \( \text{Exc}(h) = \text{Exc}(h^{-1}) = \{ x = 0 \} \cup \{ y = 0 \} \cup \{ z = 0 \} \). – In this case, each \( f_i \) is equal to \( a_i \circ g_d \) where \( g_d([x : y : z]) = [x^d : y^d : z^d] \) and each \( a_i \) is an automorphism of \( \mathbb{P}^2_k \) acting by permutation of the coordinates, while \( h \) is an automorphism of \( (\mathbb{A}^1 \setminus \{ 0 \}) \times (\mathbb{A}^1 \setminus \{ 0 \}) \).

4. Endomorphisms of \( \mathbb{A}^2_k \). – This section proves Theorem A in case (P1):

**Proposition 8.** Let \( f_1 \) and \( f_2 \) be endomorphisms of \( \mathbb{A}^2 \) that extend to endomorphisms of \( \mathbb{P}^2 \) of degree \( d \geq 2 \). If \( h \) is an automorphism of \( \mathbb{A}^2 \) that conjugates \( f_1 \) to \( f_2 \) then \( h \) is an affine automorphism i.e. \( \deg h = 1 \).

We follow the notation from [5] and denote by \( V_\infty \) the valuative tree of \( \mathbb{A}^2 = \text{Spec}(k[x,y]) \) at infinity. If \( g \) is an endomorphism of \( \mathbb{A}^2 \), we denote by \( g_\bullet \) its action on \( V_\infty \).

Set \( V_1 = \{ v \in V_\infty \mid \alpha(v) \geq 0, A(v) \leq 0 \} \), where \( \alpha \) and \( A \) are respectively the skewness and thinness function, as defined in page 216 of [5]; the set \( V_1 \) is a closed subtree of \( V_\infty \). For \( v \in V_1 \), \( v(F) \leq 0 \) for every \( F \in k[x,y] \setminus \{ 0 \} \). Then \( V_1 \) is invariant under each \( (f_i)_\bullet \), and if we set

\[
    \mathcal{T}_i = \{ v \in V_1 : (f_i)_\bullet v = v \}
\]

then \( \mathcal{T}_2 = h_\bullet \mathcal{T}_1 \). Since each \( f_i \) extends to an endomorphism of \( \mathbb{P}^2_k \), the valuation \( \deg \) is an element of \( \mathcal{T}_1 \cap \mathcal{T}_2 \). Also, in the terminology of [5], \( \lambda_2(f_i) = \)
\[\lambda_1(f_i)^2 = d^2 \text{ and } \deg(f_i^n) = \lambda_i^n = d^n \text{ for all } n \geq 1 \text{ and for } i = 1 \text{ and } 2, \]

because \(f_1\) and \(f_2\) extend to regular endomorphisms of \(\mathbb{P}_k^2\) of degree \(d\). So by [5, Proposition 5.3 (a)], \(T_i\) is a single point or a closed segment.

A valuation \(v \in V_\infty\) is monomial of weight \((s,t)\) for the pair of polynomial functions \((P,Q) \in k[x,y]^2\) if

1. \(P\) and \(Q\) generate \(k[x,y]\) as a \(k\)-algebra,
2. if \(F\) is any non-zero element of \(k[x,y]\) and
   \[F = \sum_{i,j \geq 0} a_{ij}P^iQ^j\]
   is its decomposition as a polynomial function of \(P\) and \(Q\) then
   \[v(F) = -\max\{si + tj : a_{i,j} \neq 0\}.\] 

We say that \(v\) is monomial for the basis \((P,Q)\) of \(k[x,y]\), if \(v\) is monomial for \((P,Q)\) and some weight \((s,t)\). In particular, \(-\deg\) is monomial for \((x,y)\), of weight \((1,1)\).

**Lemma 9.** If \(v \in V_1\) is monomial for \((P,Q)\) of weight \((s,t)\), then \(s,t \geq 0\), and \(\min\{s,t\} = \min\{-v(F) : F \in k[x,y] \setminus k\}\).

**Proof.** First, assume that \((P,Q) = (x,y)\). For an element \(v\) of \(V_1\), \(v(F) \leq 0\) for every \(F\) in \(k[x,y]\), hence \(s = -v(x)\) and \(t = -v(y)\) are non-negative; and the formula for \(\min\{s,t\}\) follows from the inequality \(-v(F) \geq \min\{s,t\}\). To get the statement for any pair \((P,Q)\), change \(v\) into \(g^{-1}v\) where \(g\) is the automorphism defined by \(g(x,y) = (P(x,y),Q(x,y))\).

**Lemma 10.** If \(-\deg\) is monomial for \((P,Q)\), of weight \((s,t)\), then \(s = t = 1\) and \(P\) and \(Q\) are of degree one in \(k[x,y]\).

**Proof.** By Lemma 9 we may assume that \(1 \leq s \leq t\); thus, after an affine change of variables, we may assume that \(P = x\). Since \(k[x,y]\) is generated by \(x\) and \(Q\), \(Q\) takes form \(Q = ay + C(x)\) where \(a \in k^*\) and \(C \in k[x]\). If \(C\) is a constant, we conclude the proof. Now we assume \(\deg(C) \geq 1\). Then \(t = \deg(Q) = \deg(C)\). Since \(y = a^{-1}(Q - C(x))\) and \(-\deg\) is monomial for \((x,Q)\) of weight \((1,t)\), we get \(1 = \deg(y) = \max\{t,\deg C\} = t\). It follows that \(t = \deg Q = 1\), which concludes the proof.

**Proof of Proposition 8** By [5, Proposition 5.3 (b), (d)], there exists \(P\) and \(Q \in k[x,y]\) such that for every \(v \in T_1\), \(v\) is monomial for \((P,Q)\). Moreover, \(-\deg\) is in \(T_1 \cap T_2\). By Lemma 10 \(P = x\) and \(Q = y\) after an affine change of coordinates. Since \(T_2 = h^*T_1\), for every \(v \in T_2\), \(v\) is monomial for \((h^*x,h^*y)\). Since \(-\deg\in T_2\), Lemma 10 implies \(\deg h^*x = \deg h^*y = 1\) and this concludes the proof.

5. **Endomorphisms of** \((\mathbb{A}_k^1 \setminus \{0\}) \times \mathbb{A}_k^1\). – We now arrive at case (P2), namely \(\text{Exc}(h) = \text{Exc}(h^{-1}) = \{x = 0\} \cup \{z = 0\}\), and keep the notations from Section 4. Our first goal is to prove that,
Lemma 11. If \( h \) is not an affine automorphism of the affine plane, then after a conjugacy by an affine transformation of the plane,

- Either \( f_1 \) and \( f_2 \) are equal to \((x^d, y^d)\) and \( h(x, y) = (Ax, Bx^m y) \) with \( A \) and \( B \) two roots of unity of order dividing \( d - 1 \) and \( m \in \mathbb{Z} \setminus \{0\} \).
- Or, up to a permutation of \( f_1 \) and \( f_2 \),

\[
f_1(x, y) = (x^d, y^d + \sum_{j=2}^{d} a_j y^{d-j}) \quad \text{and} \quad f_2(x, y) = (x^d, y^d + \sum_{j=2}^{d} a_j (B/A)^j x^j y^{d-j})
\]

with \( a_j \in \mathbb{k} \), and \( h(x, y) = (Ax, Bxy) \) with \( A \) and \( B \) two roots of unity of order dividing \( d - 1 \); then \( h'(x : y : z) = [Az/B : y : x] \) is an automorphism of \( \mathbb{P}^2 \) that conjugates \( f_1 \) to \( f_2 \).

Proof. We split the proof in two steps.

**Step 1.** We assume that \( f_1' \mid U(x, y) = (x^d, F_1(x, y)) \), with \( d > 0 \).

Since \( f_1 \) extends to a degree \( d \) endomorphism of \( \mathbb{P}^2 \), we can write \( F_1(x, y) = a_0 y^d + \sum_{j=1}^{d} a_j(x) y^{d-j} \) where \( a_0 \in \mathbb{k}^* \) and the \( a_j \in \mathbb{k}[x] \) satisfy \( \text{deg}(a_j) \leq j \) for all \( j \). Changing the coordinates to \((x, by)\) with \( b^d = a_0 \), we assume \( a_0 = 1 \). We can also conjugate \( f_1 \) by the automorphism

\[
(x, y) \mapsto \left( x, y + \frac{1}{d} a_1(x) \right)
\]

and assume \( a_1 = 0 \). Altogether, the change of coordinates \((x, y) \mapsto (x, by + \frac{1}{d} a_1(x))\) is affine because \( \text{deg}(a_1) \leq 1 \), and conjugates \( f_1 \) to an endomorphism \((x^d, F_1(x, y))\) normalized by \( F_1(x, y) = y^d + \sum_{j=2}^{d} a_j(x) y^{d-j} \) with \( \text{deg}(a_j) \leq j \).

Similarly, we may assume that \( F_2(x, y) = y^d + \sum_{j=2}^{d} b_j(x) y^{d-j} \) for some polynomial functions \( b_j \) with \( \text{deg}(b_j) \leq j \) for all \( j \).

Now, with the notation used in Equation (1), the two terms of the conjugacy relation \( h \circ f_1 = f_2 \circ h \) are

\[
h \circ f_1 = (Ax^d, Bx^m y^d + \sum_{j=2}^{d} a_j(x) y^{d-j} + C(x^d))
\]

\[
f_2 \circ h = (A^d x^d, (Bx^m y + C(x))^d + \sum_{j=2}^{d} b_j(Ax)(Bx^m y + C(x))^{d-j}).
\]

This gives \( A^{d-1} = 1 \), and comparing the terms of degree \( d \) in \( y \) we get \( B^{d-1} = 1 \). Then, looking at the term of degree \( d - 1 \) in \( y \), we obtain \( C(x) = 0 \). Thus \( h(x, y) = (Ax, Bx^m y) \) for some roots of unity \( A \) and \( B \), the orders of which divide \( d - 1 \). Since \( h \) is not an automorphism, we have

\[
m \neq 0.
\]
Permuting the role of \( f_1 \) and \( f_2 \) (or changing \( h \) in its inverse), we suppose \( m \geq 1 \). Coming back to (6) and (7), we obtain the sequence of equalities

\[
b_j(Ax) = a_j(x)(Bx^m)^j
\]

for all indices \( j \) between 2 and \( d \). On the other hand, \( a_j \) and \( b_j \) are elements of \( k[x] \) of degree at most \( j \). Since \( m \geq 1 \), there are only two possibilities.

(a) All \( a_j \) and \( b_j \) are equal to 0; then \( f_1(x, y) = f_2(x, y) = (x^d, y^d) \), which concludes the proof.

(b) Some \( a_j \) is different from 0 and \( m = 1 \). Then all coefficients \( a_j \) are constant, and \( b_j(x) = a_j(Bx/A)^j \) for all indices \( j = 2, \ldots, d \).

In case (b), we set \( \alpha = B/A \) (a root of unity of order dividing \( d - 1 \)), and use homogeneous coordinates to write

\[
f_1[x : y : z] = [x^d : y^d + \sum_{j=2}^{d} a_j z^j y^{d-j} : z^d]
\]

(10)

\[
f_2[x : y : z] = [x^d : y^d + \sum_{j=2}^{d} a_j \alpha^j z^j y^{d-j} : z^d].
\]

(11)

The conjugacy \( h[x : y : z] = [Axz : Bxy : z^2] \) is not a linear projective automorphism of \( \mathbb{P}^2 \), but the automorphism defined by \( [x : y : z] \mapsto [z/\alpha : y : x] \) conjugates \( f_1 \) to \( f_2 \).

**Step 2.**– The only remaining case is when \( f_i = (x^{-d}, F_i(x,y)) \), for \( i = 1, 2 \), with

\[
F_1(x,y) = \sum_{j=0}^{d} a_j(x)x^{-d}y^{d-j} \text{ and } F_2(x, y) = \sum_{j=0}^{d} b_j(x)x^{-d}y^{d-j}
\]

(12)

for some polynomial functions \( a_j, b_j \in k[x] \) that satisfy \( \deg(a_j), \deg(b_j) \leq j \) and \( a_0b_0 \neq 0 \). Writing the conjugacy equation \( h \circ f_1 = f_2 \circ h \) and looking at the term of degree \( d \) in \( y \), we get the relation

\[
Bx^{-md}a_0x^{-d}y^d = b_0(Ax)^{-d}(Bx^m y)^d.
\]

(13)

Comparing the degree in \( x \) we get \( -md - d = md - d \), hence \( m = 0 \). Moreover, \( h \) conjugates \( f_1^2 \) to \( f_2^2 \); thus, by the first step, \( h \) should be an affine automorphism since \( m = 0 \) (see Equation (8)).

\[
\square
\]

6. **Endomorphisms of \((\mathbb{A}^1_k \setminus \{0\})^2 \).** – Denote by \([x : y : z]\) the homogeneous coordinates of \( \mathbb{P}^2_k \) and by \((x, y)\) the coordinates of the open subset \( V := (\mathbb{A}^1_k \setminus \{0\})^2 \) defined by \( xy \neq 0, z = 1 \). We write \( f_i = a_i \circ g_d \), as in case (P3) of Section 3. Since \( h \) is an automorphism of \((\mathbb{A}^1_k \setminus \{0\})^2 \), it is the composition \( t_h \circ m_h \) of a
diagonal map \( t_h(x, y) = (ux, vy) \), for some pair \((u, v) \in (k^*)^2\), and a monomial map \( m_h(x, y) = (x^ay^b, x^cy^d) \), for some matrix

\[
M_h := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(z).
\] (14)

Also, note that the group \( S_3 \subset \text{Bir}(P^2_k) \) of permutations of the coordinates \([x : y : z]\) corresponds to a finite subgroup \( S_3 \) of \( \text{GL}_2(z) \).

Since \( m_h \) commutes to \( g_d \) and \( g_d \circ t_h = t_h^d \circ g_d \), the conjugacy equation is equivalent to

\[
t_h \circ (m_h \circ a_1 \circ m_h^{-1}) \circ (g_d \circ m_h) = a_2 \circ t_h^d \circ (g_d \circ m_h).
\] (15)

The automorphisms \( a_1 \) and \( a_2 \) are monomial maps, induced by elements \( A_1 \) and \( A_2 \) of \( S_3 \), and Equation (15) implies that \( M_h \) conjugates \( A_1 \) to \( A_2 \) in \( \text{GL}_2(z) \); indeed, the matrices can be recovered by looking at the action on the set of units \( wx^my^n \) in \( k(V) \) (or on the fundamental group \( \pi_1(V(C)) \) if \( k = C \)). There are two possibilities:

(a) either \( A_1 = A_2 = \text{Id} \), there is no constraint on \( m_h \);

(b) or \( A_1 \) and \( A_2 \) are non-trivial permutations, they are conjugate by an element \( P \in S_3 \), and \( M_h = \pm A_j^t \circ P \), for some \( j \in \mathbb{Z} \).

In both cases, \( u \) and \( v \) are roots of unity (there order is determined by \( d \) and the \( A_i \)). Let \( p \) be the monomial transformation associated to \( P \); it is a permutation of the coordinates, hence an element of \( \text{Aut}(\mathbb{P}^2_k) \). Then, \( h'(x, y) = t_h \circ p \) is an element of \( \text{Aut}(\mathbb{P}^2_k) \) that conjugates \( f_1 \) to \( f_2 \).

7. An example in positive characteristic. – Assume that \( q = p^s \) with \( s \geq 2 \). Set \( G := xy^p + (x - 1)y \). Then,

\[
f_1(x, y) = (x^q, y^q + G(x, y))
\]
defines an endomorphism of \( A^2 \) that extends to an endomorphism of \( \mathbb{P}^2 \).

Consider a polynomial \( P(x) \in F_q[x] \) such that \( 2 \leq \deg(P) \leq \frac{q}{p} - 1 \). Observe that \( \deg(G) < \deg(G(x, y + P(x))) < q \). Then \( g(x, y) = (x, y - P(x)) \) is an automorphism of \( A^2_k \) that conjugates \( f_1 \) to

\[
f_2(x, y) := g \circ f_1 \circ g^{-1}(x, y)
\]

\[
= (x^q, y^q + P(x)^q + G(x, y + P(x)) - P(x^q))
\]

\[
= (x^q, y^q + G(x, y + P(x))).
\] (16)

As \( f_1, f_2 \) is an endomorphism of \( A^2 \) that extends to a regular endomorphism of \( \mathbb{P}^2 \) (here we use the inequality \( \deg(G(x, y + P(x))) < q \)).

Let us prove that \( f_1 \) and \( f_2 \) are not conjugate by any automorphism of \( \mathbb{P}^2 \). We assume that there exists \( h \in \text{PGL}_3(F_q) \) such that \( h \circ f_1 = f_2 \circ h \) and seek a
contradiction. Consider the pencils of lines through the point $[0 : 1 : 0]$ in $\mathbb{P}^2$; for $a \in \mathbb{F}_q$ we denote by $L_a$ the line \{ $x = az$ \}, and by $L_\infty$ the line \{ $z = 0$ \}. Then
\begin{align}
\{ L_a : a \in \mathbb{F}_q \cup \{ \infty \} \} &= \{ \text{lines } L \text{ such that } f_1^{-1}L = L \} \\
&= \{ \text{lines } L \text{ such that } f_2^{-1}L = L \};
\end{align}
in other words, the lines $L_a$ for $a \in \mathbb{F}_q \cup \{ \infty \}$ are exactly the lines which are totally invariant under the action of $f_1$ (resp. of $f_2$). Since $h$ conjugates $f_1$ to $f_2$, it permutes these lines. In particular, $h$ fixes the point $[0 : 1 : 0]$, and if we identify $L_a \cap \mathbb{A}^2$ to $\mathbb{A}^1$ with its coordinate $y$ by the parametrization $y \mapsto (a, y)$ then $h$ maps $L_a$ to another line $L_{a'}$ in an affine way: $h(a, y) = (a', \alpha y + \beta)$.

Since $g$ conjugates $f_1$ to $f_2$ and $g$ fixes each of the lines $L_{a'}$, we know that $f_1|_{L_a}$ is conjugated to $f_2|_{L_a}$ for every $a \in \mathbb{F}_q$; for $a = \infty$, both $f_1|_{L_\infty}$ and $f_2|_{L_\infty}$ are conjugate to $y \mapsto y^q$. Moreover
\begin{itemize}
  \item $a = \infty$ is the unique parameter such that $f_1|_{L_a}$ is conjugate to $y \mapsto y^q$ by an affine map $y \mapsto \alpha y + \beta$;
  \item $a = 0$ is the unique parameter such that $f_1|_{L_a}$ is conjugate to $y \mapsto y^q - y$ by an affine map;
  \item $a = 1$ is the unique parameter such that $f_1|_{L_a}$ is conjugate to $y \mapsto y^q + y^p$ by an affine map.
\end{itemize}
And the same properties hold for $f_2$. As a consequence, we obtain $h(L_\infty) = L_\infty$, $h(L_0) = L_0$ and $h(L_1) = L_1$; this means that there are coefficients $\alpha \in \mathbb{F}_q^*$ and $\beta, \gamma \in \mathbb{F}_q$ such that $h(x, y) = (x, \alpha y + \beta x + \gamma)$. Writing down the relation $h \circ f_1 = f_2 \circ h$ we obtain the relation
\begin{align}
\alpha y^q + \alpha G(x, y) + \beta x^q + \gamma &= \alpha^q y^q + \beta^q x^q + \gamma^q \\
+ G(x, \alpha y + \beta x + \gamma + P(x)).
\end{align}
We note that $1 < \deg G(x, y) < \deg G(x, \alpha y + \beta x + \gamma + P(x)) < q$. Compare the terms of degree $q$, we get $\alpha y^q + \beta x^q = \alpha^q y^q + \beta^q x^q$. It follows that
\begin{align}
\alpha G(x, y) + \gamma &= \gamma^q + G(x, \alpha y + \beta x + \gamma + P(x)).
\end{align}
Then $\deg G(x, y) = \deg G(x, \alpha y + \beta x + \gamma + P(x))$, which is a contradiction.

References

[1] Jérémie Blanc. Conjugacy classes of affine automorphisms of $\mathbb{K}^n$ and linear automorphisms of $\mathbb{P}^n$ in the Cremona groups. Manuscripta Math., 119(2):225–241, 2006.
[2] Jean-Yves Briend, Serge Cantat, and Mitsuhiro Shishikura. Linearity of the exceptional set for maps of $\mathbb{P}_n(\mathbb{C})$. Math. Ann., 330(1):39–43, 2004.
[3] Jean-Yves Briend and Julien Duval. Deux caractérisations de la mesure d’équilibre d’un endomorphisme de $\mathbb{P}^n(\mathbb{C})$. Publ. Math. Inst. Hautes Études Sci., (93):145–159, 2001.
[4] Dominique Cerveau and Alcides Lins Neto. Hypersurfaces exceptionnelles des endomorphismes de $\mathbb{C}P(n)$. Bol. Soc. Brasil. Mat. (N.S.), 31(2):155–161, 2000.
[5] Charles Favre and Mattias Jonsson. Dynamical compactifications of \( C^2 \). *Ann. of Math. (2)*, 173(1):211–248, 2011.

[6] John Erik Fornæss and Nessim Sibony. Complex dynamics in higher dimension. I. Number 222, pages 5, 201–231. 1994. Complex analytic methods in dynamical systems (Rio de Janeiro, 1992).

[7] Rajendra Vasent Gurjar. On ramification of self-maps of \( P^2 \). *J. Algebra*, 259(1):191–200, 2003.

[8] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[9] Andreas Höring. Totally invariant divisors of endomorphisms of projective spaces. *Manuscripta Math.*, 153(1-2):173–182, 2017.

Serge Cantat, IRMAR, Campus de Beaulieu, Bâtiments 22-23 263 Avenue du Général Leclerc, CS 74205 35042 Rennes Cédex

E-mail address: serge.cantat@univ-rennes1.fr

Junyi Xie, IRMAR, Campus de Beaulieu, Bâtiments 22-23 263 Avenue du Général Leclerc, CS 74205 35042 Rennes Cédex

E-mail address: junyi.xie@univ-rennes1.fr