LOW-LYING ZEROS IN FAMILIES OF HOLOMORPHIC CUSP FORMS:
THE WEIGHT ASPECT

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Abstract. We study low-lying zeros of $L$-functions attached to holomorphic cusp forms of level 1 and large weight. In this family, the Katz–Sarnak heuristic with orthogonal symmetry type was established in the work of Iwaniec, Luo and Sarnak for test functions $\phi$ satisfying the condition $\text{supp}(\hat{\phi}) \subset (-2, 2)$. We refine their density result by uncovering lower-order terms that exhibit a sharp transition when the support of $\hat{\phi}$ reaches the point 1. In particular the first of these terms involves the quantity $\hat{\phi}(1)$ which appeared in previous work of Fouvry–Iwaniec and Rudnick in symplectic families. Our approach involves a careful analysis of the Petersson formula and circumvents the assumption of GRH for $GL(2)$ automorphic $L$-functions. Finally, when $\text{supp}(\hat{\phi}) \subset (-1, 1)$ we obtain an unconditional estimate which is significantly more precise than the prediction of the $L$-functions Ratios Conjecture.

1. Introduction

Katz and Sarnak [KS] conjectured that the distribution of low-lying zeros in a family $\mathcal{F}$ of $L$-functions is governed by a certain random matrix model $G(\mathcal{F})$ called the symmetry type of $\mathcal{F}$. This symmetry type has been determined in many families; see for example [FI, HR, ILS, M1, OS, Y], as well as the references in [SST]. Sarnak, Shin and Templier [SST] recently refined the Katz–Sarnak heuristics and introduced invariants which allow for a conjectural determination of the symmetry type.

In the current paper we focus on the family of classical holomorphic cusp forms of level 1 and large even weight $k$. As in [ILS, Chapter 10], this will ease the exposition and allow for a more transparent analysis. For this family, the predictions of Katz and Sarnak were confirmed in the influential work of Iwaniec, Luo and Sarnak [ILS] for a certain class of test functions, under the assumption of the Riemann Hypothesis for Dirichlet $L$-functions and holomorphic cusp form $L$-functions. Our first goal is to relax these conditions by only assuming the Riemann Hypothesis for Dirichlet $L$-functions. Our second and main goal is to refine the Iwaniec–Luo–Sarnak density results by determining lower-order terms up to an arbitrary negative power of $\log k$.

More precisely, we fix a basis $B_k$ of Hecke eigenforms in the space $H_k$ of holomorphic modular forms of level 1 and even weight $k$. We normalize so that for every

$$f(z) = \sum_{n=1}^{\infty} a_f(n)n^{\frac{k-1}{2}} e^{2\pi i nz} \in B_k,$$

the first coefficient satisfies $a_f(1) = 1$. Hence, the Hecke eigenvalues of $f$ are given by $\lambda_f(n) = a_f(n)$ and for $\Re(s) > 1$ the $L$-function of $f$ takes the form

$$L(s, f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s}.$$

This classically extends to an entire function and satisfies a functional equation relating the values at $s$ to those at $1 - s$. In the sums over $f \in B_k$ to be considered in this paper, we will scale each
term with the harmonic weight
\[
\omega_f := \frac{\Gamma(k-1)}{(4\pi)^{k-1}(f,f)},
\]
where
\[
(f, f) := \int_{SL_2(\mathbb{Z})/\mathbb{Z}} |y^{k-2} f(z)|^2 dx dy.
\]
Note that \(k^{-1-\epsilon} \leq \epsilon \omega_f \leq k^{-1+\epsilon}\) (see [ILS, Lemma 2.5], [HL, p.164] and [Iw1, Theorem 2]), and moreover
\[
\Omega_k := \sum_{f \in B_k} \omega_f = 1 + O(2^{-k})
\]  \(1.1\)
(take \(m = n = 1\) in Proposition 3.3). The use of these essentially constant weights is standard (see for instance [ILS, Chapter 10], [M2])

For an even Schwartz test function \(\phi\), we define the 1-level density
\[
\mathcal{D}_k(\phi; X) := \frac{1}{\Omega_k} \sum_{f \in B_k} \omega_f \sum_{\gamma_f} \phi\left(\frac{\log X}{2\pi}\right).
\]
Moreover, for \(h\) a non-negative, not identically zero smooth weight function with compact support in \(\mathbb{R}_{>0}\) we define the following averages of the 1-level densities over families of constant sign\(^2\) of the functional equation:
\[
\mathcal{D}_{K,h}^\pm(\phi) := \frac{1}{H^\pm(K)} \sum_{k \equiv 3\pm1 \mod 4} h\left(\frac{k-1}{K}\right) \mathcal{D}_k(\phi; K^2),
\]
where \(H^\pm(K) = \sum_{k \equiv 3\pm1 \mod 4} h\left(\frac{k-1}{K}\right)\). The Katz–Sarnak prediction for this family (see [ILS], [SST, Conjecture 2, Section 2.7]) states that
\[
\lim_{K \to \infty} \mathcal{D}_{K,h}^\pm(\phi) = \int_{\mathbb{R}} \tilde{\phi} \cdot \tilde{W}^\pm,
\] \(1.2\)
with
\[
\tilde{W}^+(t) = \tilde{W}(SO(\text{even}))(t) = \delta_0(t) + \frac{\eta(t)}{2}; \quad \tilde{W}^-(t) = \tilde{W}(SO(\text{odd}))(t) = \delta_0(t) - \frac{\eta(t)}{2} + 1,
\]
where \(\eta(t) = 1\) for \(|t| < 1\), \(\eta(\pm1) = \frac{1}{2}\) and \(\eta(t) = 0\) for \(|t| > 1\), and \(\tilde{\phi}(\xi) := \int_{\mathbb{R}} \phi(x)e^{-2\pi i \xi x} dx\). Under the Riemann Hypothesis for Dirichlet and holomorphic cusp form \(L\)-functions, the estimate (1.2) was confirmed\(^3\) in [ILS, Theorem 1.3] under the condition \(\text{supp}(\tilde{\phi}) \subset (-2, 2)\).

We now state our main theorem which, in the case when the level \(N = 1\), refines the estimate in [ILS, Theorem 1.3] by weakening its assumptions and obtaining lower-order terms which contain a phase transition as the support of \(\tilde{\phi}\) reaches 1.

**Theorem 1.1.** Let \(\phi\) be an even Schwartz test function for which \(\text{supp}(\tilde{\phi}) \subset (-2, 2)\). Assuming the Riemann Hypothesis for Dirichlet \(L\)-functions, we have the estimate
\[
\mathcal{D}_{K,h}^\pm(\phi) = \int_{\mathbb{R}} \tilde{\phi} \cdot \tilde{W}^\pm + \sum_{1 \leq j \leq J} R_{j,h} \tilde{\phi}^{(j-1)}(0) \pm S_{j,h} \tilde{\phi}^{(j-1)}(1) \left(\frac{1}{\log K}\right)^{j+1} + O_{\phi,h,J}\left(\frac{1}{\log K}\right)^{J+1},
\] \(1.3\)
where the constants \(R_{j,h}\) and \(S_{j,h}\) appearing in the lower-order terms only depend on the weight function \(h\) (see (6.8), (6.9), and (6.10)).

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\(^1\)Note that the situation can be drastically different with arithmetic weights as in [KR].

\(^2\)Note that for any \(f \in H_k\), the sign of the functional equation of \(L(s,f)\) is given by \((-1)^{\frac{k+1}{2}}\).

\(^3\)This work has been extended to families of more general automorphic \(L\)-functions in [ST].
We deduce Theorem 1.1 from a power-saving formula for the 1-level density (see (6.1)), which we combine with an asymptotic evaluation of the resulting terms (see Theorem 6.6). We were able to circumvent the use of the Riemann Hypothesis for holomorphic cusp form $L$-functions by refining the estimate [ILS, Corollary 2.2] on the remainder in the Petersson trace formula (see Proposition 3.3). The specific estimate we obtain is the following:

$$
\sum_{f \in B_k} \omega_f \lambda_f(m) \lambda_f(n) = \delta(m,n) + O_\varepsilon \left( \frac{(m,n)^{\frac{3}{2}}(mn)^{\frac{1}{2}+\varepsilon}}{k} + \frac{k^\frac{1}{2}(m,n)^\frac{1}{2}}{(mn)^{\frac{1}{2}-\varepsilon}} \right),
$$

In particular, when $(m,n) = 1$ this estimate is nontrivial in the range $mn \leq k^{4-\varepsilon}$, whereas [ILS, Corollary 2.2] is nontrivial up to $mn \leq k^{\frac{1}{2}-\varepsilon}$.

The terms involving $\widehat{\phi}(j)(1)$ in (1.3) are responsible for a sharp transition at 1 in these orthogonal families and are analogous to those obtained in symplectic families in [FI, FPS2, FPS3, R, Wax]. Indeed, in the family of real Dirichlet characters considered in [FPS2], after applying the explicit formula and treating the resulting sums over primes by repeatedly using the Poisson summation formula, one obtains lower-order terms involving $\widehat{\phi}(j)(1)$. This work was inspired by the function field case considered in [R], in which, using Poisson summation, the 1-level density is turned into an average of the trace of the Frobenius class in the hyperelliptic ensemble, from which a transition term is isolated using the explicit formula. Transition terms also surface in predictions coming from the $L$-function Ratios Conjecture [FPS3, Wax]; in this case one needs to compute averages of ratios of local factors at infinity. In the current situation, these terms come from a significantly different source, namely from a careful analysis of averages of Bessel functions and Kloosterman sums coming from the Petersson trace formula. Independently of the use of different methods, this seems to indicate that a transition in lower-order terms should exist whenever the symmetry type of a family is even or odd orthogonal, or symplectic.

Interestingly, averaging over all even values of the weight $k$, we find that

$$
\frac{1}{H(K)} \sum_{k \equiv 0 \mod 2} h \left( \frac{k-1}{K} \right) D_k(\phi; K^2) = \int_{\mathbb{R}} \hat{\phi} \cdot \hat{W} + \sum_{1 \leq j \leq J} R_{j,h} \hat{\phi}^{(j-1)(0)}(0) + O_{\phi,h,J} \left( \frac{1}{(\log K)^{t+1}} \right),
$$

where $\hat{W} = \hat{W}(O) := \frac{1}{2} + \delta_0$ and $H(K) := H^+(K) + H^-(K)$. Hence, as expected we see that there is no transition at 1 in this mixed signs family (see also [M2, Theorem 1.6]). We should point out that for similar reasons, there is no transition in mixed sign families of holomorphic cusp forms of fixed weight and of large level [M2, RR].

**Remark 1.2.** One can compute explicitly the constants $R_{j,h}$ and $S_{j,h}$ in Theorem 1.1. In particular, the first of these are given by

$$
R_{1,h} := 1 + \int_0^\infty h \cdot \log \left( \frac{\theta(t) - t}{t^2} \right) dt,
$$

$$
S_{1,h} := -\gamma + \int_0^\infty h \cdot \log \left( \frac{\theta(t) - t}{t^2} \right) dt \log(4\pi) - \sum_{p \leq t} \log p,
$$

where $\theta(t) := \sum_{p \leq t} \log p$.

We now state our results for test functions whose Fourier transform is supported in the interval $(-1, 1)$. Under this restriction our estimates are substantially more precise. Indeed, we do not need the GRH assumption, the error term is exponentially small in the weight $k$ and we do not need the average over $k$ (we set $X = k^2$ in $D_k(\phi; X)$).
Lemma 2.1. Let \( \phi \) be an even Schwartz test function for which \( \text{supp}(\hat{\phi}) \subset (-1,1) \). Then the (unaveraged) 1-level density satisfies the estimate

\[
D_k(\phi; k^2) = \frac{1}{\log(k^2)} \int_{\mathbb{R}} \left( \frac{1}{\Gamma} \left( \frac{1}{4} + \frac{k + 1}{4} + \frac{2\pi it}{\log(k^2)} \right) + \frac{1}{\Gamma} \left( \frac{1}{4} + \frac{k - 1}{4} + \frac{2\pi it}{\log(k^2)} \right) \right) \phi(t) dt \\
+ 2 \sum_{p} \frac{1}{p} \log p \left( \frac{2 \log(p)}{\log(k^2)} \right) \log p \frac{\log \pi}{\log k} \phi(0) + O(k^{-2} 2^{-k}).
\] (1.5)

Remark 1.4.

1. We emphasize that Theorem 1.3 is unconditional. Moreover, the error term in (1.5) is exponentially small, in particular this is significantly more precise than predictions from the Ratios Conjecture [CFZ, CS, M2]. This comes from exponential bounds on the Bessel functions occurring in the Petersson trace formula (see (3.2)).

2. The Katz–Sarnak main term in this case is given by \( \hat{\phi}(0) + \frac{\phi(0)}{2} \). One can extract this term from (1.5) by applying Lemmas 2.2 and 4.1.

3. An estimate for the 1-level density \( D_k(\phi; k^2) \) was previously obtained by Miller [M2, Lemmas 4.2 and 4.4] with the error term \( O(k^{\frac{9}{2} - \frac{7}{6} + \epsilon}) \), under the same conditions.

Lower-order terms in the level aspect were previously studied in [M2, MM, RR]. In an upcoming paper we will refine the Iwaniec–Luo–Sarnak and Miller–Montague estimates for fixed weight and large level and isolate a transition term of the same type as in Theorem 1.1.

The paper is divided as follows. In Sections 2 and 3 we discuss prerequisites, establish (1.4) and discard higher prime powers in the explicit formula. Section 4 is dedicated to the proof of Theorem 1.3. Finally, in Section 5 we apply estimates on averages of Bessel functions to isolate a transition term, which we carefully evaluate in Section 6.

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2. Explicit formula

We begin by recalling the explicit formula for holomorphic cusp form \( L \)-functions in the case where the level equals 1.

Lemma 2.1. Let \( \phi \) be an even Schwartz test function. We have the formula

\[
D_k(\phi; X) = -2\hat{\phi}(0) \frac{\log \pi}{\log X} + \frac{1}{\log X} \int_{\mathbb{R}} \left( \frac{1}{\Gamma} \left( \frac{1}{4} + \frac{k + 1}{4} + \frac{2\pi it}{\log X} \right) + \frac{1}{\Gamma} \left( \frac{1}{4} + \frac{k - 1}{4} + \frac{2\pi it}{\log X} \right) \right) \phi(t) dt \\
- \frac{2}{\Omega_k} \sum_{f \in B_k} \omega_f \sum_{p, \nu} \frac{\alpha_f(p) + \beta_f(p)}{p^{\frac{\nu}{2}}} \left( \frac{\nu \log p}{\log X} \right) \log p \frac{\log \pi}{\log X}.
\] (2.1)

Here, \( \alpha_f(p), \beta_f(p) \) are the local coefficients of the \( L \)-function

\[
L(s, f) = \prod_p \left( 1 - \frac{\alpha_f(p)}{p^{s}} \right)^{-1} \left( 1 - \frac{\beta_f(p)}{p^{s}} \right)^{-1} \quad (\Re(s) > 1);
\]

in particular we have that \( |\alpha_f(p)| = |\beta_f(p)| = 1 \).
Proof. For \( f \in B_k \), the formula [ILS, (4.11)] reads
\[
\sum_{\gamma_f} \phi\left( \frac{\gamma_f \log X}{2\pi} \right) = \frac{1}{\log X} \int_{\mathbb{R}} \left( \frac{\Gamma'(1/4 + k + 1/4 + 2\pi it)}{\Gamma(1/4 + k + 1/4 + 2\pi it)} + \frac{\Gamma'(1/4 + k - 1/4 + 2\pi it)}{\log X} \right) \phi(t) dt 
- 2 \hat{\phi}(0) \frac{\log X}{\log X} - 2 \sum_{p, \nu} \frac{\alpha_f^\nu(p) + \beta_f^\nu(p)}{p^\nu} \phi\left( \frac{\log p}{\log X} \right) \frac{\log p}{\log X}.
\]
Summing over \( f \in B_k \) against the weight \( \omega_f \) we obtain the desired formula. \( \square \)

We now estimate the integral involving the logarithmic derivative of the gamma function in (2.1).

Lemma 2.2. Let \( \varepsilon > 0 \) and let \( \phi \) be an even Schwartz test function. In the range \( k \leq X^5 \), we have the estimate\(^4\)
\[
\frac{1}{\log X} \int_{\mathbb{R}} \left( \frac{\Gamma'(1/4 + k + 1/4 + 2\pi it)}{\log X} + \frac{\Gamma'(1/4 + k - 1/4 + 2\pi it)}{\log X} \right) \phi(t) dt = \hat{\phi}(0) \left( \frac{\log(k^2) - 16}{\log X} \right) + O_{\varepsilon}(k^{-1+\varepsilon}).
\]
Proof. Applying Stirling’s formula
\[
\Gamma'(z) = \log z + O(|z|^{-1})
\]
in the region \( \Re(z) > 0 \), we see that
\[
\frac{1}{\log X} \int_{\mathbb{R}} \left( \frac{\Gamma'(1/4 + k + 1/4 + 2\pi it)}{\log X} + \frac{\Gamma'(1/4 + k - 1/4 + 2\pi it)}{\log X} \right) \phi(t) dt
= \frac{1}{\log X} \int_{\mathbb{R}} \left( \log \left( \frac{1}{4} + \frac{k + 1}{4} + \frac{2\pi it}{\log X} \right) + \log \left( \frac{1}{4} + \frac{k - 1}{4} + \frac{2\pi it}{\log X} \right) \right) \phi(t) dt + O(k^{-1})
= \frac{1}{\log X} \int_{-k^\varepsilon}^{k^\varepsilon} \left( \log \left( \frac{1}{4} + \frac{k + 1}{4} + \frac{2\pi it}{\log X} \right) + \log \left( \frac{1}{4} + \frac{k - 1}{4} + \frac{2\pi it}{\log X} \right) \right) \phi(t) dt + O_{\varepsilon}(k^{-1})
= \frac{1}{\log X} \int_{-k^\varepsilon}^{k^\varepsilon} \log \left( \frac{k^2}{16} \right) \phi(t) dt + O_{\varepsilon}(k^{-1+\varepsilon}).
\]
The result follows from extending the integral to \( \mathbb{R} \). \( \square \)

3. The Petersson trace formula and related estimates

In order to handle the term involving sums over prime powers in (2.1) we will apply the Petersson trace formula. For \( m, n \in \mathbb{Z} \) and \( c \in \mathbb{N} \) we define the Kloosterman sum
\[
S(m, n; c) := \sum_{x \mod c \atop (x, c) = 1} e\left( \frac{mx + nx}{c} \right).
\]
We will repeatedly use the classical Weil bound (see for instance [IK, Corollary 11.12])
\[
S(m, n; c) \leq \tau(c)(m, n, c)^{1/2} c^{1/2}.
\]

Lemma 3.1. Let \( m, n, k \in \mathbb{N} \), with \( 2 \mid k \). We have the exact formula
\[
\sum_{f \in B_k} \omega_f \lambda_f(m) \lambda_f(n) = \delta(m, n) + 2\pi i k \sum_{c \geq 1} c^{-1} S(m, n; c) J_{k-1}\left( \frac{4\pi \sqrt{mn}}{c} \right),
\]
where \( J_{k-1} \) is the Bessel function.
Proof. See [P], [IK, Proposition 14.5]. \( \square \)

\(^4\)Using more terms in the Stirling approximation, this estimate can be refined to an asymptotic series in descending powers of \( k \).
We recall the following bound on the Bessel function.

**Lemma 3.2.** Let \( k \in \mathbb{N} \). We have the bound

\[
J_{k-1}(x) \ll \min \left( \frac{1}{(k-1)!} \left( \frac{x}{2} \right)^{k-1}, x^{-\frac{1}{2}} (|x - k + 1| + k^{\frac{1}{2}})^{-\frac{1}{2}} \right).
\]

**Proof.** See [ILS, (2.11\textsuperscript{'}); (2.11\textsuperscript{")}], which for the range \( x \geq k^2 \) follows from [Wat], specifically equations (1) p.49, (2) p.77, (6) p.78, (1), (3) p.199, (1) p.202, (4) p.250, (5) p.252, and for the remaining range follows from [K, Theorem 2]. \( \square \)

In [ILS, Chapter 2], this bound is shown to imply the estimate

\[
\sum_{f \in B_k} \omega_f \lambda_f(m) \lambda_f(n) = \delta(m,n) + O(k^{-\frac{7}{2}} (mn)^{\frac{1}{2}} \tau_3((m,n)) \log(2mn)),
\]

which is non-trivial in the range \( mn \leq k^{\frac{10}{3}} - \varepsilon \). By a more careful decomposition of the sum over \( c \) in (3.1), we establish a more precise estimate which is non-trivial in the wider range \( mn \leq k^{4-\varepsilon} \).

**Proposition 3.3.** Let \( \varepsilon > 0 \), and let \( m, n, k \in \mathbb{N} \), with \( 2 \mid k \). We have the estimate

\[
\sum_{f \in B_k} \omega_f \lambda_f(m) \lambda_f(n) = \delta(m,n) + O\left( \left( \frac{m,n}{k} \right)^{\frac{1}{k}} \left( \frac{1}{k} \right)^{\frac{1}{k}} \log(2mn) \prod_{p \mid (m,n)} \left( 1 + \frac{3}{\sqrt{p}} \right) \right).
\]

In the range \( mn \leq k^2/(4\pi)^2 \), we have the exponentially precise estimate

\[
\sum_{f \in B_k} \omega_f \lambda_f(m) \lambda_f(n) = \delta(m,n) + O\left( 2^{-k} (mn)^{\frac{1}{2}} \log(2mn) \prod_{p \mid (m,n)} \left( 1 + \frac{3}{\sqrt{p}} \right) \right).
\]

**Proof.** We bound the rightmost term in the statement of Lemma 3.1 by combining the Weil bound with Lemma 3.2, as follows:

\[
\sum_{c \geq 1} c^{-1} S(m,n,c) J_{k-1} \left( \frac{4\pi \sqrt{mn}}{c} \right) \ll \sum_{c \leq \frac{4\pi \sqrt{mn}}{k^{1/3}}} c^{-\frac{1}{4}} \tau(c) \left( \frac{m,n,c}{c} \right)^{\frac{1}{2}} \left| \frac{4\pi \sqrt{mn}}{c} - k + 1 \right|^{-\frac{1}{4}} + \sum_{\frac{4\pi \sqrt{mn}}{k^{1/3}} < c < \frac{4\pi \sqrt{mn}}{k^{1/3}} + \frac{4\pi \sqrt{mn}}{k^{1/3}}} c^{-\frac{1}{4}} \tau(c) \left( \frac{m,n,c}{c} \right)^{\frac{1}{2}} k^{-\frac{1}{4}} + \sum_{\frac{4\pi \sqrt{mn}}{k^{1/3}} + \frac{4\pi \sqrt{mn}}{k^{1/3}} < c < \frac{4\pi \sqrt{mn}}{k^{1/3}}} c^{-\frac{1}{4}} \tau(c) \left( \frac{m,n,c}{c} \right)^{\frac{1}{2}} \left| \frac{4\pi \sqrt{mn}}{c} - k + 1 \right|^{-\frac{1}{4}} + \sum_{c \geq \frac{4\pi \sqrt{mn}}{k^{1/3}}} \frac{1}{(k-1)!} \left( \frac{2\pi \sqrt{mn}}{c} \right)^{k-1} = S_1 + S_2 + S_3 + S_4.
\]
We first bound $S_4$. To do so, note that

$$S_4 \ll \frac{1}{(k-1)!} (2\pi \sqrt{mn})^{k-1} \sum_{d|(m,n)} d^{\frac{1}{2}} \sum_{c \geq \frac{4\pi \sqrt{mn}}{c(m,n)=d}} \tau(c) c^{-k+\frac{1}{2}} \leq \frac{1}{(k-1)!} (2\pi \sqrt{mn})^{k-1} \sum_{d|(m,n)} \tau(d) d^{-k+1} \sum_{f \geq \frac{4\pi \sqrt{mn}}{d(f)}} \tau(f) f^{-k+\frac{1}{2}} \ll 2^{-k}(mn)^{\frac{1}{4}} \log(2mn) \prod_{p|(m,n)} \left(1 + \frac{3}{\sqrt{p}}\right),$$

by Stirling’s approximation. Note that $S_1, S_2$ and $S_3$ are all empty whenever $mn \leq k^2/(4\pi \varepsilon)^2$ and hence (3.2) follows.

We now assume that $mn > k^2/(4\pi \varepsilon)^2$. A computation similar to the one above shows that

$$S_2 \ll \varepsilon k^{-\frac{1}{2}}(mn)^{\frac{1}{4} + \varepsilon} + (mn)^{-\frac{1}{4} + \varepsilon} k^{\frac{1}{2}}(m,n)^{\frac{1}{2}}$$

(second term accounts for the possibility that the sum contains only one term).

As for $S_1$, we compute that

$$S_1 \ll \sum_{c \leq 2\pi \sqrt{mn}} \tau(c) \frac{(m,n,c)}{(mn)^{\frac{1}{4}}} + \sum_{2\pi \sqrt{mn} \leq c \leq 4\pi \sqrt{mn} - \frac{4\pi \sqrt{mn}}{k^{1/3}}} \tau(c) \frac{(m,n,c)}{(mn)^{\frac{1}{4}}} \left|4\pi \sqrt{mn} - c(k-1)\right|^{-\frac{1}{4}} \ll_{\varepsilon} \frac{(mn)^{\frac{1}{4}} \log(2mn)}{k} \prod_{p|(m,n)} \left(1 + \frac{3}{\sqrt{p}}\right) + (mn)^{-\frac{1}{4} + \varepsilon} k^{\frac{1}{2}}(m,n)^{\frac{1}{2}} + \sum_{2\pi \sqrt{mn} \leq c \leq 4\pi \sqrt{mn} - \frac{4\pi \sqrt{mn}}{k^{1/3}}} \tau(c) \frac{(m,n,c)}{(mn)^{\frac{1}{4}}} \left|4\pi \sqrt{mn} - c(k-1)\right|^{-\frac{1}{4}}.$$

Making the change of variables $b = \left[\frac{4\pi \sqrt{mn}}{k-1}\right] - c$, we see that the sum over $c$ is

$$\ll_{\varepsilon} (m,n)^{\frac{1}{2}}(mn)^{-\frac{1}{4} + \varepsilon} \sum_{\frac{4\pi \sqrt{mn}}{k^{1/3}} \leq b < \frac{2\pi \sqrt{mn}}{k-1}} |b(k-1)|^{-\frac{1}{4}} \ll k^{-1}(mn)^{\frac{1}{4} + \varepsilon}(m,n)^{\frac{1}{2}}.$$

In a similar way we see that $S_3 \ll_{\varepsilon} k^{-1}(mn)^{\frac{1}{4} + \varepsilon}(m,n)^{\frac{1}{2}} + (mn)^{-\frac{1}{4} + \varepsilon} k^{\frac{1}{2}}(m,n)^{\frac{1}{2}}$, and the proof is finished.

In the next lemma we apply Proposition 3.3 in order to discard higher prime powers in the explicit formula (2.1).

**Lemma 3.4.** Assume that $k \in 2\mathbb{N}$, $X \in \mathbb{R}_{\geq 2}$ and the even Schwartz test function $\phi$ are such that $X^\sigma < k^4$, where $\sigma := \sup(\text{supp}(\tilde{\phi}))$. Then we have the following estimate on the 1-level density:

$$D_k(\phi; X) = \tilde{\phi}(0) \left(\frac{\log(k^2) - \log(16\pi^2)}{\log X}\right) + 2 \sum_{p} \frac{1}{p} \phi\left(\frac{2\log p}{\log X}\right) \log p \log X$$

$$- 2 \sum_{f \in B_k} \omega_f \sum_{p} \frac{\lambda_f(p)}{p^{\sigma}} \tilde{\phi}\left(\frac{\log p}{\log X}\right) \log p \log X + O\left(\frac{X^{\frac{1}{4} + \varepsilon}}{k} + \frac{1}{k^{\frac{1}{4} - \varepsilon}}\right). \tag{3.3}$$
Assuming the stronger condition \( X^\sigma < (k/4\pi e)^2 \), we have the more precise estimate

\[
D_k(\phi; X) = \frac{1}{\log X} \int_R \left( \frac{p'}{\Gamma} \left( \frac{1}{4} + \frac{k + 1}{4} + \frac{2\pi it}{\log X} \right) + \frac{p'}{\Gamma} \left( \frac{1}{4} + \frac{k - 1}{4} + \frac{2\pi it}{\log X} \right) \right) \phi(t) dt
\]

\[ -2\tilde{\phi}(0) \frac{\log \pi}{\log X} + 2 \sum_p \frac{1}{p} \tilde{\phi} \left( \frac{2 \log p}{\log X} \right) \frac{\log p}{\log X} - 2 \sum_{f \in B_k} \omega_f \sum_p \frac{\lambda_f(p)}{p^2} \tilde{\phi} \left( \frac{\log p}{\log X} \right) \frac{\log p}{\log X} + O\left( \frac{k^{4+\varepsilon}}{2^k} \right). \tag{3.4}
\]

**Proof.** The goal of this proof is to estimate the terms \( p, \nu \geq 2 \) in (2.1). By the Hecke relations, the sum of those terms is equal to

\[
-\frac{2}{\Omega_k} \sum_{f \in B_k} \omega_f \sum_{p, \nu \geq 2} \frac{\lambda_f(p^\nu)}{p^2} \tilde{\phi} \left( \nu \frac{\log p}{\log X} \right) \frac{\log p}{\log X}.
\]

From Proposition 3.3 and (1.1), we see that

\[
-\frac{2}{\Omega_k} \sum_{f \in B_k} \omega_f \sum_{p, \nu \geq 2} \frac{\lambda_f(p^\nu)}{p^2} \tilde{\phi} \left( \nu \frac{\log p}{\log X} \right) \frac{\log p}{\log X} \ll \varepsilon 2^{-k}
\]

\[
\ll 2^{-k} + \left( X^{\sigma} < (k/4\pi e)^2 \right) (k - 1)^{-\frac{1}{2}} + k^{\frac{1}{2} + \varepsilon}.
\]

Similarly,

\[
\frac{2}{\Omega_k} \sum_{f \in B_k} \omega_f \sum_{p, \nu \geq 3} \frac{\lambda_f(p^{\nu-2})}{p^2} \tilde{\phi} \left( \nu \frac{\log p}{\log X} \right) \frac{\log p}{\log X} \ll 2^{-k} + I_{[X^\sigma > (k/4\pi e)^2]} k^{-\frac{1}{2} + \varepsilon}.
\]

The only terms left are

\[
\frac{2}{\Omega_k} \sum_{f \in B_k} \omega_f \sum_p \frac{\lambda_f(1)}{p} \tilde{\phi} \left( \frac{2 \log p}{\log X} \right) \frac{\log p}{\log X} = 2 \sum_p \frac{1}{p} \tilde{\phi} \left( \frac{2 \log p}{\log X} \right) \frac{\log p}{\log X}.
\]

We conclude the proof by applying Lemmas 2.1 and 2.2, and (1.1). \( \square \)

4. 1-LEVEL DENSITY: UNCONDITIONAL RESULTS

In this section we evaluate the 1-level density \( D_k(\phi; X) \) for test functions satisfying \( \sup(\text{supp}(\phi)) < 1 \), unconditionally. We begin by asymptotically evaluating the second term on the right-hand side of (3.3).

**Lemma 4.1.** Let \( \phi \) be an even Schwartz test function. For any fixed \( J \geq 1 \), we have the estimate

\[
2 \sum_p \frac{1}{p} \tilde{\phi} \left( \frac{2 \log p}{\log X} \right) \frac{\log p}{\log X} = \phi(0) + \sum_{1 \leq j \leq J} c_j \phi^{(j-1)}(0) \frac{1}{(\log X)^j} + O\left( \frac{1}{(\log X)^{J+1}} \right),
\]

where

\[
c_1 := 2 \int_1^\infty \frac{\theta(t) - t}{t^2} dt + 2
\]

and for \( j \geq 2 \),

\[
c_j := \frac{2^j}{(j - 2)!} \int_1^\infty (\log t)^{j-2} \left( \frac{\log t}{j - 1} - 1 \right) \frac{\theta(t) - t}{t^2} dt.
\]
Proof. Performing summation by parts, we reach the following identity:

$$\frac{2}{\log X} \sum_p \log \frac{p}{K} \phi \left( \frac{\log p}{\log X} \right) = \frac{\phi(0)}{2} + \frac{2\phi(0)}{\log X} - \frac{2}{\log X} \int_1^\infty \left( 2 \frac{\partial \phi \left( \frac{\log t}{\log X} \right)}{\log X} - \phi \left( \frac{\log t}{\log X} \right) \right) \frac{\theta(t) - t}{t^2} dt. $$

By the prime number theorem in the form $\theta(t) - t \ll t \exp(-2\sqrt{\log t})$, we see that for any $0 < \xi < 1$,

$$\int_{X^{\xi/2}}^\infty \left( 2 \frac{\partial \phi \left( \frac{\log t}{\log X} \right)}{\log X} - \phi \left( \frac{\log t}{\log X} \right) \right) \frac{\theta(t) - t}{t^2} dt \ll \exp(-c \sqrt{\log t}).$$

Moreover, taking Taylor series and applying the prime number theorem, we see that

$$- \frac{2}{\log X} \int_1^{X^{1/2}} \left( 2 \frac{\partial \phi \left( \frac{\log t}{\log X} \right)}{\log X} - \phi \left( \frac{\log t}{\log X} \right) \right) \frac{\theta(t) - t}{t^2} dt$$

$$= 2 \sum_{0 \leq j \leq J} \frac{1}{j!} \left( \phi^{(j)}(0) - \phi^{(j+1)}(0) \right) \int_1^{X^{1/2}} \frac{(2 \log t)^{j+1}}{(\log X)^{j+2}} \frac{\theta(t) - t}{t^2} dt + O_J \left( \int_1^{X^{1/2}} \frac{(2 \log t)^{j+1}}{(\log X)^{j+2}} \frac{\theta(t) - t}{t^2} dt \right)$$

$$= \sum_{1 \leq j \leq J+1} \frac{c_j \phi^{(j-1)}(0)}{(\log X)^j} + O_J \left( \frac{1}{(\log X)^{j+2}} + \exp(-c \sqrt{\log X}) \right).$$

The result follows from selecting $\xi = (\log X)^{-1+\delta}$ for some $\delta > 0$. \hfill $\square$

We now set $X = k^2$ and prove Theorem 1.3.

Proof of Theorem 1.3. We apply Proposition 3.3 and obtain that the second prime sum in (3.4) satisfies the bound

$$2 \sum_{f \in \mathcal{B}_k} \omega_f \sum_p \lambda_f(p) \phi \left( \frac{\log p}{\log(k^2)} \right) \frac{\log p}{\log(k^2)} \ll \frac{\log k}{2} 2^{-k}.$$

The proof follows. \hfill $\square$

5. 1-level density averaged over the weight: Extended support

In this section we study the quantities $\mathcal{D}_{K,h}^+(\phi)$ and $\mathcal{D}_{K,h}^-(\phi)$, that is we average the 1-level density $\mathcal{D}_k(\phi; K^2)$ over $k \asymp K$ against the weight $h(k - 1)K$. 

Lemma 5.1 ([Iw2, Lemma 5.8], [ILS, Corollary 8.2]). For $h$ a non-negative, smooth function with compact support in $R_{>0}$ and for any $K \geq 2$, we have the estimates

$$2 \sum_{k \equiv 0 \mod 2} h \left( \frac{k - 1}{K} \right) J_{k-1}(x) = h \left( \frac{x}{K} \right) + O \left( \frac{x}{K^3} \right);$$

$$2 \sum_{k \equiv 0 \mod 2} i^{k} h \left( \frac{k - 1}{K} \right) J_{k-1}(x) = \frac{K}{\sqrt{x}} \delta_8 \left( \zeta_8 e^{ix} h \left( \frac{K^2}{2x} \right) \right) + O \left( \frac{x}{K^4} \right),$$

where $\zeta_8 = e^{2\pi i/8}$ and $h(x) = \int_0^\infty h \left( \frac{\sqrt{u}}{K} \right) e^{ixu} du$.

In the next lemma we estimate the total weight $H^\pm(K)$ and a related sum.

Lemma 5.2. For $h$ a non-negative, smooth function with compact support in $R_{>0}$ and for any $K, N \geq 2$, we have the estimates

$$H^\pm(K) = \frac{K}{4} \int_{R^+} h \left( \frac{u}{K^2} \right) + O(N(K-N));$$
Proof. More generally, we will show that for any \(a\) mod 4,
\[
\sum_{k \equiv a \text{ mod } 4} h\left(\frac{k}{K}\right) = \frac{K \int_{\mathbb{R}^+} h}{4} + O_N(K^{-N});
\]  
(5.1)

and for \(b\) mod 4,
\[
\sum_{k \equiv a \text{ mod } 4} h\left(\frac{k}{K}\right) \log(k+1) = K \log K \int_{\mathbb{R}^+} h + K \int_{\mathbb{R}^+} h \cdot \log + \sum_{\ell=1}^{N} \frac{(-1)^{\ell+1}}{\ell K^{\ell-1}} \int_{\mathbb{R}^+} t^{-\ell} h(t) dt + O_N(K^{-N}).
\]  
(5.2)

Now, for any \(K\) and \((5.2)\) follows.

Finally, the integral on the right-hand side of (5.3) equals
\[
\int_{\mathbb{R}^+} h\left(\frac{t}{K}\right) \log(t+1)e(-\xi t) dt \ll N \frac{\log K}{(|\xi|K)^N}.
\]

Finally, the integral on the right-hand side of (5.3) equals
\[
K \int_{\mathbb{R}^+} h(t) \log(Kt+1) dt = K \log K \int_{\mathbb{R}^+} h + K \int_{\mathbb{R}^+} h \cdot \log + \sum_{\ell=1}^{N} \frac{(-1)^{\ell+1}}{\ell K^{\ell-1}} \int_{\mathbb{R}^+} t^{-\ell} h(t) dt + O_N(K^{-N}),
\]
and (5.2) follows.

In the next lemma we estimate the average of (3.3) over \(k\). In order to do so, we will apply Lemmas 5.1 and 5.2.

**Lemma 5.3.** Let \(\phi\) be an even Schwartz test function and let \(h\) be a non-negative, smooth function with compact support in \(\mathbb{R}_{>0}\). Under the condition \(\sigma = \text{sup}(\text{supp}(\hat{\phi}))) < 2\) and for \(K \geq 2\), we have the estimate
\[
\mathcal{D}_{K,h}^\pm(\phi) = \hat{\phi}(0) \left(1 + \frac{f_{K,h} \cdot \log}{f_{K,h} \cdot \log K} - \log(4\pi)\right) + 2 \sum_p \frac{1}{p} \hat{\phi}\left(\frac{2 \log p}{\log(K^2)}\right) \frac{\log p}{\log(K^2)}
\]
\[
\pm \frac{\pi}{\log(K^2)H^+(K)} \sum_p \frac{\log p}{p^2 \pi} \frac{\hat{\phi}(\log(p^2))}{\log(K^2)} \sum_{c=1}^\infty S(p,1;c) \frac{4\pi \sqrt{p}}{cK} \hat{\phi}\left(\frac{4\pi \sqrt{p}}{cK}\right) + O_K\left(K^\frac{2}{4} - 1 + + K^{-\frac{1}{4}}\right).
\]

**Proof.** From combining Lemmas 3.4 and 5.2, we have that
\[
\mathcal{D}_{K,h}^\pm(\phi) = \hat{\phi}(0) \left(1 + \frac{f_{K,h} \cdot \log}{f_{K,h} \cdot \log K} - \log(4\pi)\right) + 2 \sum_p \frac{1}{p} \hat{\phi}\left(\frac{2 \log p}{\log(K^2)}\right) \frac{\log p}{\log(K^2)}
\]
\[
- \frac{2}{H^+(K)} \sum_{k \equiv 3 \pm 1 \text{ mod } 4} h\left(\frac{k-1}{K}\right) \sum_{f \in B_k} \omega(f) \sum_p \frac{\lambda(f(p))}{p^2 \pi} \hat{\phi}\left(\frac{\log p}{\log(K^2)}\right) \frac{\log p}{\log(K^2)} + O_K\left(K^\frac{2}{4} - 1 + + K^{-\frac{1}{4}}\right).
\]
By the Petersson trace formula (Lemma 3.1), the third term is equal to
\[- \frac{2\pi}{H^\pm(K)} \sum_{k \equiv 0 \mod 2} (i^k \pm 1) h\left(\frac{k-1}{K}\right) \sum_p \frac{1}{p^2} \phi\left(\frac{\log p}{\log(K^2)}\right) \log p \sum_{c \geq 1} c^{-1} S(p, 1, c) J_{k-1}\left(\frac{4\pi \sqrt{p}}{c}\right) \cdot\]

(5.4)

Applying Lemma 5.1, we see that
\[- \frac{2}{H^\pm(K)} \sum_{k \equiv 0 \mod 2} i^k h\left(\frac{k-1}{K}\right) J_{k-1}\left(\frac{4\pi \sqrt{p}}{c}\right) = - \frac{-K c^4}{H^\pm(K)2\pi \frac{i}{p^2}} \sum_{0 \leq i \leq 2} (n e^{i \pi \sqrt{p}} \frac{\left(\frac{K^2 c}{8\pi \sqrt{p}}\right)}{O\left(\frac{\sqrt{p}}{c K^8}\right)}).
\]

(5.5)

Since \( p \leq K^{4-\epsilon} \), we see by the rapid decay of \( h \) that for any \( A \geq 1 \), the first term in this expression is
\[\ll_A \frac{e^\frac{1}{p^2}}{\sqrt{p}} \frac{(K^2 c)}{A^A},\]
and hence the contribution of this term to (5.4) is
\[\ll_A K^{A(\sigma-2)+\frac{2}{5}}.\]

As for the sum of the error terms in (5.5), the contribution is \( \ll K^{2\sigma-5} \), which is an admissible error term. Moreover, applying Lemma 5.1 once more,
\[- \frac{2}{H^\pm(K)} \sum_{k \equiv 0 \mod 2} h\left(\frac{k-1}{K}\right) J_{k-1}\left(\frac{4\pi \sqrt{p}}{c}\right) = - \frac{1}{H^\pm(K)} \frac{4\pi \sqrt{p}}{cK} + O\left(\frac{\sqrt{p}}{c K^4}\right),\]
resulting in a main term as well as the admissible error term \( O(K^{2\sigma-4}) \).

We now end this section by evaluating the second sum over primes in Lemma 5.3, under GRH for Dirichlet \( L \)-functions. This term will be responsible for the phase transition at 1, and will be investigated more closely in Section 6.

**Lemma 5.4.** Let \( \phi \) be an even Schwartz test function, let \( h \) be a non-negative, smooth function with compact support in \( \mathbb{R}_{>0} \), and assume the Riemann Hypothesis for Dirichlet \( L \)-functions. Then for any \( K \geq 2 \) we have the estimate
\[\sum_{c \geq 1} \frac{1}{c} \sum_p \log p \frac{\phi}{\log(K^2)} S(p, 1; c) h\left(\frac{4\pi \sqrt{p}}{cK}\right) = \log(K^2) \int_0^\sigma K^{u-1} \phi(u) \sum_{c \geq 1} \mu^2(c) c \left(\frac{4\pi K^{u-1}}{c}\right) du + O(K^{\sigma-1}(\log K)^3),\]

(5.6)

where \( \varphi \) is Euler’s totient function.

**Proof.** If \( \sigma < 1 \), then for large enough \( K \) the left-hand side of (5.6) is identically zero. We may thus assume that \( \sigma \geq 1 \). The sum over \( p \) equals
\[\int_0^\infty \frac{1}{t^2} \frac{\phi}{\log(K^2)} h\left(\frac{4\pi \sqrt{t}}{cK}\right) dT(t),\]
where, by [ILS, Lemma 6.1],
\[T(t) := \sum_{p \leq t} S(p, 1, c) \log p = \int_0^t \frac{\mu^2(c)}{\varphi(c)} + O(\varphi(c)t^\frac{1}{2}(\log(ct))^2).\]

Note that our restriction on the support of \( h \) implies that \( c \asymp \sqrt{t}/K \), and hence the restriction on the support of \( \tilde{\phi} \) implies that for squarefree values of \( c \) and for \( t \leq K^{4-\epsilon} \), the main term in this
We have the unconditional estimate
\[
\sum_{c \geq 1} \frac{\mu^2(c)}{c \varphi(c)} \int_1^\infty \frac{1}{t^2} \phi\left(\frac{\log t}{\log c}\right) h\left(\frac{4\pi \sqrt{t}}{cK}\right) dt + O(K^{-1}(\log K)^3).
\]

The proof follows. \qed

6. Evaluation of the Transition Term

The goal of this section is to evaluate the integral in Lemma 5.4. This will be done using different techniques depending on the range of the variable \(u\). To this end, for \(a, b \in \mathbb{R}_{\geq 0}\) we define
\[
I_{a,b} := \frac{\pi}{H^\pm(K)} \int_a^b K^u \widehat{\phi}(u) \sum_{c \geq 1} \frac{\mu^2(c)}{c \varphi(c)} h\left(\frac{4\pi K^{u-1}}{c}\right) du.
\]
Notice that the inner sum is long only when \(u\) is larger and far away from 1. By Lemmas 5.3 and 5.4, we see that when \(\sigma = \sup(\text{supp}(\hat{\phi})) < 2\) and under the assumption of GRH for Dirichlet \(L\)-functions,
\[
\Phi_{K,h}^\pm(\phi) = \hat{\phi}(0) \left(1 + \frac{f_x}{f_x + h} - \frac{\log(4\pi)}{\log K}\right) + 2 \sum_p \frac{1}{p} \hat{\phi} \left(\frac{2 \log p}{\log(K^2)}\right) \log p \log(K^2) + I_{0,\infty} + O_{\epsilon}(K^{\frac{2}{3} + \epsilon} + K^{-1\frac{1}{2} + \epsilon}).
\]

We now move on to evaluating the integral \(I_{0,\infty}\). We let \(\delta_K\) be a positive parameter which satisfies \(\delta_K \gg h/\log K\). Recall that \(h\) is supported in \(\mathbb{R}_{+}\), and hence for \(K\) large enough the integrand in \(I_{0,\infty}\) is zero in the interval \([0,1 - \delta_K]\). Hence,
\[
I_{0,\infty} = I_{1-\delta_K,\sigma},
\]
where, as before, \(\sigma = \sup(\text{supp}(\hat{\phi}))\).

**Lemma 6.1.** We have the unconditional estimate\(^5\)
\[
\sum_{c \leq x} \frac{c \mu^2(c)}{c \varphi(c)} = x + O(x^{\frac{1}{2}}).
\]

**Proof.** We first establish the following estimate, for squarefree values of \(d\):
\[
S_d(x) := \sum_{\substack{m \leq x \atop (m,d) = 1}} \frac{\mu^2(m)}{m} = C_1(d) \log x + C_2(d) + O\left(x^{-\frac{1}{2}}d^\frac{1}{2} \prod_{p|d} \left(1 - p^{-\frac{1}{2}}\right)^{-1}\right),
\]
where
\[
C_1(d) := \frac{1}{\zeta(2)} \prod_{p|d} \left(1 - \frac{1}{p + 1}\right); \quad C_2(d) := C_1(d) \left(\gamma - 2 \frac{\zeta'}{\zeta}(2) - \sum_{p|d} \frac{p \log p}{p + 1}\right).
\]
To do so, note that
\[
S_d(x) = \sum_{\ell_1|d} \frac{\lambda(\ell_1)}{\ell_1} S_{\ell_1}\left(\frac{x}{d}\right).
\]

---

\(^5\)The error term here can be improved to \(O(x^{\frac{1}{2}} \exp(-c(\log x)^{\frac{1}{2}}(\log \log x)^{-\frac{1}{2}}))\) by replacing (6.4) with the stronger estimate obtained from combining [MV, Exercise 19, §6.2.1] with the Korobov-Vinogradov zero-free region.
Lemma 6.2 (Range \( u > 1 + \delta_K \)). Let \( \phi \) be an even Schwartz test function and let \( K \geq 2 \). We have that
\[
I_{1+\delta_K,\infty} = \int_{1+\delta_K}^{\infty} \tilde{\phi} + O(K^{-\frac{5}{2}}).
\]

Proof. By Lemma 6.1, we have that
\[
S(y) := \sum_{c \leq y} \frac{c \mu^2(c)}{\varphi(c)} = y + O(y^{\frac{3}{4}}),
\]
hence, for \( u > 1 + \delta_K \),
\[
\sum_{c \geq 1} \frac{\mu^2(c)}{c \varphi(c)} h\left(\frac{4\pi K^{u-1}}{c}\right) = -\int_0^{\infty} S(y) \left(\frac{1}{y^2} h\left(\frac{4\pi K^{u-1}}{y}\right)\right)' \, dy
\]
\[
= \int_0^{\infty} \frac{1}{y^2} h\left(\frac{4\pi K^{u-1}}{y}\right) \, dy
\]
\[
+ O\left(\int_0^{\infty} \left(\frac{1}{y^3} h\left(\frac{4\pi K^{u-1}}{y}\right) + \frac{K^{u-1}}{y^4} h'\left(\frac{4\pi K^{u-1}}{y}\right)\right)y^{\frac{3}{2}} \, dy\right)
\]
\[
= \frac{1}{4\pi K^{u-1}} \int_0^{\infty} h + O(K^{-\frac{5}{2}(u-1)}).
\]
The desired estimate follows by integrating over \( u \) against \( K^u \tilde{\phi}(u) \) and applying Lemma 5.2. \( \square \)

---

6The precise value of the constant is deduced from writing \( S_1(x) = \frac{1}{2\pi i} \int_{(1)} \zeta(s+1) \frac{x^s}{s} \, ds \) and shifting the contour of integration to the left.
We now evaluate the part of the integral \( I_{0,\infty} \) in which \( u \) is close to 1. In this range we can expand \( \hat{\phi}(u) \) into Taylor series around \( u = 1 \) and recover the transition terms \( \hat{\phi}^{(j)}(1) \) (see Lemma 6.5). The resulting integrals are evaluated in Lemma 6.4 by applying the inverse Mellin transform, truncating the resulting integrals and shifting the contours of integration.

**Lemma 6.3.** Whenever \( x \in \mathbb{R}_{\geq 0}, j \in \mathbb{N}, \Re(s) > 0 \) and \( |x| \geq 2 \),

\[
\int_{x}^{\infty} u^{j} e^{-us} du \ll \frac{j! e^{-R(s)x} x^{j}}{|s|},
\]

where the implied constant is absolute.

**Proof.** Applying integration by parts, we reach the exact formula

\[
\int_{x}^{\infty} u^{j} e^{-us} du = \frac{e^{-x s} x^{j}}{s} \sum_{0 \leq \ell \leq j} \frac{j! x^{-\ell}}{(j - \ell)! s^{\ell}}.
\]

The proof follows. \( \square \)

**Lemma 6.4.** For \( j \geq 0, K \geq 2 \) and \( 20(\log K)^{-1} \leq \delta_K \leq \frac{1}{2} \) we have the estimate

\[
\sum_{c \geq 1} \frac{\mu^{2}(c)}{c} \int_{K^{-\delta_K}}^{K^{\delta_K}} \frac{\log v}{c} h\left(\frac{4\pi v}{c}\right) dv = \mathcal{M}h(1) \left(\frac{\delta_K \log K}{j+1}\right) + c_{j,h} + O_{\varepsilon}\left(j^{1}\delta_K^{j}K^{\delta_K}\right),
\]

where

\[
C_{j,h} := \left. \frac{(-1)^{j}}{j+1} \left( ds \right)^{j+1} \left( s \zeta(s) (4\pi)^{s-1} \mathcal{M}h(1-s) \right) \right|_{s=0},
\]

with

\[
Z(s) = \zeta(s+1) \prod_{p} \left( 1 + \frac{1}{p-1} \left( \frac{1}{p^{s+1}} - \frac{1}{p^{2s+1}} \right) \right).
\]

**Proof.** Define

\[
f_{K,j}(c) := \int_{K^{-\delta_K}}^{K^{\delta_K}} \frac{\log v}{c} h\left(\frac{4\pi v}{c}\right) dv.
\]

By the restriction on the support of \( h \), the function \( f_{K,j} \) also has compact support on \( \mathbb{R}_{>0} \). We conclude that its Mellin transform \( \varphi_{K,j}(s) \) is entire. Moreover,

\[
\varphi_{K,j}(s) := \int_{0}^{\infty} x^{s-1} f_{K,j}(x) dx = (4\pi)^{s-1} \mathcal{M}h(1-s) \int_{K^{-\delta_K}}^{K^{\delta_K}} \left( \log v \right)^{j} v^{s-1} dv = \varphi_{K,j}^{+}(s) + \varphi_{K,j}^{-}(s),
\]

where

\[
\varphi_{K,j}^{\pm}(s) = \pm (4\pi)^{s-1} \mathcal{M}h(1-s) \int_{1}^{K^{\pm\delta_K}} \left( \log v \right)^{j} v^{s-1} dv
\]

are also entire. For any \( N \geq 1 \), applying [FPS1, Lemma 2.1] yields the crude bound

\[
\varphi_{K,j}^{\pm}(s) \ll_{N,j} K|s|^{-N} \quad (|3(s)| \geq 1, \Re(s) \leq 1). \tag{6.6}
\]

Next, Mellin Inversion gives the formula

\[
f_{K,j}(c) = \frac{1}{2\pi i} \int_{\frac{1}{2}} c^{-s} \left( \varphi_{K,j}^{+}(s) + \varphi_{K,j}^{-}(s) \right) ds.
\]

Hence, by absolute convergence,

\[
\sum_{c \geq 1} \frac{\mu^{2}(c)}{c} f_{K,j}(c) = \frac{1}{2\pi i} \int_{\frac{1}{2}} Z(s) \left( \varphi_{K,j}^{+}(s) + \varphi_{K,j}^{-}(s) \right) ds,
\]
where
\[ Z(s) := \sum_{c=1}^{\infty} \frac{\mu^2(c)}{c^s \varphi(c)} = \zeta(s + 1) \prod_p \left( 1 + \frac{1}{p-1} \left( \frac{1}{p^{s+1}} - \frac{1}{p^{2s+1}} \right) \right). \]

By applying Lemma 6.3, we obtain the estimate
\[ \varphi_{K,j}(s) = (4\pi)^{s-1} \mathcal{M}h(1-s) \left( \int_0^1 (\log v)^j v^{s-1} dv + O\left( j!(\delta_K \log K)^j K^{-\delta_K \Re(s)} \right) \right) \quad (\Re(s) > 0, |s| \geq \frac{1}{10}). \]

Moreover,
\[ \int_0^1 (\log v)^j v^{s-1} dv = \frac{(-1)^j j!}{s^{j+1}} \quad (\Re(s) > 0). \]

From the rapid decay of \( \mathcal{M}h(1-s) \) on vertical lines (see [FPS1, Lemma 2.1]), we see that
\[ \frac{1}{2\pi i} \int_{(\frac{1}{2})} Z(s) \varphi_{K,j}(s) ds = \frac{(-1)^j j!}{2\pi i} \int_{(\frac{1}{2})} Z(s) (4\pi)^{s-1} \mathcal{M}h(1-s) \frac{ds}{s^{j+1}} + O\left( j!(\delta_K \log K)^j K^{-\delta_K / 2} \right). \]

As for the other part of the integral, by applying (6.6) we can shift the contour to the left until the line \( \Re(s) = -\frac{1}{2} + \frac{\epsilon}{2} \), and reach the identity
\[ \frac{1}{2\pi i} \int_{(\frac{1}{2})} Z(s) \varphi_{K,j}^+(s) ds = (4\pi)^{s-1} \mathcal{M}h(1) \int_1^{K^j} (\log v)^j v^{s-1} dv + \frac{1}{2\pi i} \int_{(-\frac{1}{2} + \frac{\epsilon}{2})} Z(s) \varphi_{K,j}^+(s) ds. \]

In a similar fashion as before, we see that
\[ \varphi_{K,j}^+(s) = (4\pi)^{s-1} \mathcal{M}h(1-s) \left( \frac{(-1)^j j!}{s^{j+1}} + O\left( j!(\delta_K \log K)^j K^{\delta_K \Re(s)} \right) \right) \quad (\Re(s) < 0, |s| \geq \frac{1}{10}) \]
and deduce that
\[ \frac{1}{2\pi i} \int_{(\frac{1}{2})} Z(s) \varphi_{K,j}^+(s) ds = \mathcal{M}h(1) \left( \frac{(\delta_K \log K)^{j+1}}{4\pi(j+1)} + \frac{(-1)^j j!}{2\pi i} \int_{(-\frac{1}{2} + \frac{\epsilon}{2})} \frac{Z(s)(4\pi)^{s-1} \mathcal{M}h(1-s) \frac{ds}{s^{j+1}}}{s^{j+1}} \right) + O_{\epsilon} \left( j!(\delta_K \log K)^j K^{(-\frac{1}{2} + \epsilon)\delta_K} \right). \]

Putting these estimates together, we conclude that
\[ \sum_{c \geq 1} \frac{\mu^2(c)}{\varphi(c)} f_{K,j}(c) = \mathcal{M}h(1) \left( \frac{(\delta_K \log K)^{j+1}}{4\pi(j+1)} + \frac{(-1)^j j!}{2\pi i} \left( \int_{(\frac{1}{2})} - \int_{(-\frac{1}{2} + \frac{\epsilon}{2})} \right) \frac{Z(s)(4\pi)^{s-1} \mathcal{M}h(1-s) \frac{ds}{s^{j+1}}}{s^{j+1}} \right) + O_{\epsilon} \left( j!(\delta_K \log K)^j K^{(-\frac{1}{2} + \epsilon)\delta_K} \right). \]

The result follows. \( \square \)

**Lemma 6.5** (Range \( 1 - \delta_K < u < 1 + \delta_K \)). Let \( \phi \) be an even Schwartz test function. We have for \( K \geq 2 \) and odd \( J \geq 1 \) that
\[ I_{1-\delta_K,1+\delta_K} = \int_1^{1+\delta_K} \hat{\phi} + \frac{\pi K}{H^2(K)} \sum_{0 \leq j \leq J} \frac{(\hat{\phi}(j)(1) C_{j,h} + O_{\epsilon,j} \left( \delta_{K}^{j+2} + \frac{1}{(\log K)^{J+2}} + \frac{K^{\delta_K(-\frac{1}{2} + \epsilon)}}{\log K} \right))}{j!(\log K)^{j+2}}, \]

where the constants \( C_{j,h} \) are defined in (6.5).

**Proof.** By definition of \( I_{1-\delta_K,1+\delta_K} \), we need to evaluate the sum
\[ \sum_{c \geq 1} \frac{\mu^2(c)}{\varphi(c)} \int_{1-\delta_K}^{1+\delta_K} \frac{K u}{c} \hat{\phi}(u) h \left( \frac{4\pi K u - 1}{c} \right) du = \frac{K}{\log K} \sum_{c \geq 1} \frac{\mu^2(c)}{\varphi(c)} \int_{K^{-\delta_K}}^{K^\delta_K} \frac{1}{c} \hat{\phi} \left( \frac{\log v}{\log K} + 1 \right) \left( \frac{4\pi v}{c} \right) dv. \]

Taking Taylor series and applying Lemma 6.4, this is
\[ = K \sum_{0 \leq j \leq J} \frac{\hat{g}(j)(1)}{j!(\log K)^{j+1}} \sum_{c \geq 1} \mu^2(c) \int_{K^{-\delta_K}}^{K^\delta_K} \frac{(\log v)^j}{c} h \left( \frac{4\pi v}{c} \right) dv \]
\[ + O_{\varepsilon,J} \left( K^{\delta_K} + \frac{K}{(\log K)^{J+2}} + \frac{K^{1+\delta_K(-\frac{1}{2}+\varepsilon)}}{\log K} \right). \]

Applying Lemma 6.4 once more, we reach the expression
\[ K M h(1) \sum_{0 \leq j \leq J} \frac{\hat{g}(j)(1)\delta_{j+1}}{(j+1)!} + K \sum_{0 \leq j \leq J} \frac{\hat{g}(j)(1)C_{j,h}}{(\log K)^{J+1}} + O_{\varepsilon,J} \left( K^{\delta_K} + \frac{K}{(\log K)^{J+2}} + \frac{K^{1+\delta_K(-\frac{1}{2}+\varepsilon)}}{\log K} \right). \]

Applying Lemma 5.2, the proof follows.

Collecting the estimates in this section, we reach the following theorem.

**Theorem 6.6.** Let \( \phi \) be an even Schwartz test function for which \( \text{supp}(\hat{\phi}) \subset (-2,2) \). Assuming the Riemann Hypothesis for Dirichlet L-functions, for \( K \geq 2 \) we have the estimate
\[ \varphi_{K,h}^{\pm}(\phi) = \hat{\phi}(0) \left( 1 + \frac{\int_{-\infty}^{\infty} h \log \frac{4\pi h}{n} \log \frac{4\pi}{K}}{\log K} \right) + \frac{\hat{\phi}(0)}{2} + \sum_{1 \leq j \leq J} c_j \varphi_{j-1}(0) \frac{2^j}{j!(\log K)^{J+1}} \]
\[ + \int_{1}^{\infty} \hat{\phi} \pm \sum_{1 \leq j \leq J} D_{j,h}(\hat{\phi}) \frac{(1)}{(\log K)^{J+1}} + O_{\varepsilon,J} \left( \frac{1}{(\log K)^{J+1}} \right), \]
where the \( c_j \) are defined in Lemma 4.1, and
\[ D_{j,h} = -\frac{4\pi}{\int_{-\infty}^{\infty} h \cdot (j-1)!} C_{j-1,h} = \frac{4\pi(1)}{j! \int_{-\infty}^{\infty} h \cdot (ds)^{j}} \left( s Z(s)(4\pi)^{s-1} M h(1-s) \right) \bigg|_{s=0}, \]
with
\[ Z(s) = \zeta(s+1) \prod_{p} \left( 1 + \frac{1}{p-1} \left( \frac{1}{p^{s+1}} - \frac{1}{p^{2s+1}} \right) \right). \]  

**Proof.** Recall (6.1), which is valid for \( \sigma = \text{supp}(\text{supp}(\hat{\phi})) < 2 \):
\[ \varphi_{K,h}^{\pm}(\phi) = \hat{\phi}(0) \left( \frac{1}{\log K} + \frac{4\pi}{\int_{-\infty}^{\infty} h \log \frac{4\pi h}{n} \log \frac{4\pi}{K}} \right) + 2 \sum_{p} \frac{1}{p} \varphi \left( \frac{2 \log p}{\log(K^2)} \right) \frac{\log p}{\log(K^2)} + I_{0,\infty} + O_{\varepsilon}(K^{\frac{2}{2}+\varepsilon} + K^{\frac{1}{2}+\varepsilon}). \]

We can clearly assume without loss of generality that \( J \) is odd. The sum over primes is estimated in Lemma 4.1. Moreover, we recall that for \( K \) large enough \( I_{0,1-\delta_K} = 0 \) and therefore we have that
\[ I_{0,\infty} = I_{1-\delta_K,1+\delta_K} + I_{1+\delta_K,\infty}, \]
which together with Lemmas 6.2 and 6.5 and the choice \( \delta_K = 3(J+3) \log \log K / \log K \) implies the desired result.

**Proof of Theorem 1.1.** It follows immediately from Theorem 6.6 with
\[ S_{j,h} = D_{j,h} = \frac{4\pi(1)}{j! \int_{-\infty}^{\infty} h \cdot (ds)^{j}} \left( s Z(s)(4\pi)^{s-1} M h(1-s) \right) \bigg|_{s=0} \]  

\[ \sum_{c \geq 1} \mu^2(c) \int_{K^{-\delta_K}}^{K^\delta_K} |(\log v)^{J+1} h(\frac{4\pi v}{c})| dv, \]
which can be evaluated using Lemma 6.4 whenever \( J + 1 \) is even.
and for \( j \geq 2 \),

\[
R_{j,h} = \frac{1}{(j-2)!} \int_1^\infty (\log t)^{j-2} \left( \frac{\theta(t) - t}{t^2} \right) \theta(t) - t \, dt
\]

(note that these do not depend on \( h \)).

\[
(6.10)
\]

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