Abstract.

In this paper, we study a series of $L^2$-torsion invariants from the viewpoint of the mapping class group of a surface. We establish some vanishing theorems for them. Moreover we explicitly calculate the first two invariants and compare them with hyperbolic volumes.

§1. Magnus representation

Let $\Sigma_{g,1}$ be a compact oriented smooth surface of genus $g$ with a boundary $\partial \Sigma_{g,1} \cong S^1$. In this paper, we always assume that $g \geq 1$. We take and fix a base point $* \in \partial \Sigma_{g,1}$ of $\Sigma_{g,1}$. Let $\mathcal{M}_{g,1}$ be the mapping class group of $\Sigma_{g,1}$, namely, the group of all isotopy classes of orientation preserving diffeomorphisms of $\Sigma_{g,1}$ relative to the boundary. We denote $\pi_1(\Sigma_{g,1},*)$ by $\Gamma$, which is a free group of rank $2g$, and fix a generating system $\Gamma = \langle x_1, \ldots, x_{2g} \rangle$. Let $\mathbb{Z}\Gamma$ be the group ring of $\Gamma$ over $\mathbb{Z}$. We write $\varphi_* \in \text{Aut}(\Gamma)$ to the automorphism induced from $\varphi \in \mathcal{M}_{g,1}$. The following result, usually called the Dehn-Nielsen-Baer theorem, is classical and fundamental to study the mapping class group $\mathcal{M}_{g,1}$ by using combinatorial group theories (see [9] Section 2.9).

**Proposition 1.1** (Zieschang [27]). *The above induced homomorphism $\mathcal{M}_{g,1} \ni \varphi \mapsto \varphi_* \in \text{Aut}(\Gamma)$ is injective.*

As a corollary, we see that $\varphi$ can be determined by the words $\varphi_*(x_1), \ldots, \varphi_*(x_{2g}) \in \Gamma$. Since the fundamental formula $\gamma = 1 + \ldots$
\[ \sum_{i=1}^{2g} (\partial x/\partial x_i)(x_i - 1) \text{ holds in } \mathbb{Z} \Gamma \text{ for any } \gamma \in \Gamma, \text{ the word } \varphi_*(x_j) \text{ is determined by } \{ \partial \varphi_*(x_j)/\partial x_i \}. \] Here \( \partial/\partial x_i : \mathbb{Z} \Gamma \to \mathbb{Z} \Gamma \) denotes Fox's free differential. See [1] Section 3.1 for a systematic treatment of the subject. The Magnus representation of the mapping class group is defined as follows.

**Definition 1.2.** The Magnus representation of \( M_{g,1} \) is defined by the assignment

\[
\varphi \mapsto \left( \frac{\partial \varphi_*(x_j)}{\partial x_i} \right)_{ij} \in GL(2g, \mathbb{Z} \Gamma),
\]

where \( \sum_g \lambda gg = \sum_g \lambda gg^{-1} \) for any element \( \sum_g \lambda gg \in \mathbb{Z} \Gamma \).

**Remark 1.3.** By the expression \( \gamma = 1 + \sum_i (\partial \gamma/\partial x_i)(x_i - 1) \), it follows that \( r \) is injective. However, it is not a group homomorphism, just a crossed homomorphism. According to the practice, we call it simply the Magnus representation of \( M_{g,1} \).

Now for a matrix \( B \in M(n, \mathbb{C}) \), let us recall that its characteristic polynomial

\[ \det(tI - B) \]

is one of the fundamental tools in the linear algebra. Here \( I \) denotes the identity matrix of degree \( n \). If we can define a characteristic polynomial of \( r(\varphi) \), it may be useful tool to study the mapping class group. In order to define it for a Magnus matrix \( r(\varphi) \), we need to clarify the following two points.

1. What is the determinant over a non-commutative group ring?
2. What is the meaning of a variable "t" in the group?

As an answer to these problems, we can formulate that

- the variable \( t \) lives in the fundamental group of the mapping torus of \( \varphi \),
- a characteristic polynomial "det" \( (tI - r(\varphi)) \) with respect to the Fuglede-Kadison determinant.

In the later sections, we explain that the characteristic polynomial of \( r(\varphi) \) is defined as a real number and it essentially gives the \( L^2 \)-torsion and the hyperbolic volume of the mapping torus of \( \varphi \). Moreover taking the lower central series of the surface group \( \Gamma \), we obtain a family of Magnus representations, so that we can introduce a sequence of \( L^2 \)-torsion invariants as an approximate sequence of the hyperbolic volume.

This paper is organized as follows. In the next section, we briefly recall the definition of the Fuglede-Kadison determinant. In Section 3,
we summarize some properties of the $L^2$-torsion of 3-manifolds and explain a relation to the Magnus representation. We introduce a sequence of $L^2$-torsion invariants for a surface bundle over the circle in Section 4 and give some formulas for them in Section 5. In the last section, we discuss some vanishing theorems for $L^2$-torsion invariants.

§2. Fuglede-Kadison determinant

In this section, we review the combinatorial definition of the Fuglede-Kadison determinant over a non-commutative group ring and its basic properties (see [19] for details).

The idea to define a determinant over a group ring comes from the following observation. That is, for a matrix $B \in GL(n, \mathbb{C})$ with the (non-zero) eigenvalues $\lambda_1, \ldots, \lambda_n$, we can formally calculate

$$\log | \det B |^2 = \log \prod_{i=1}^{n} \lambda_i \bar{\lambda}_i = \sum_{i=1}^{n} \log \lambda_i \bar{\lambda}_i$$

$$= \sum_{i=1}^{n} \left( \sum_{p=1}^{\infty} \frac{(-1)^{p+1}}{p} (\lambda_i \bar{\lambda}_i - 1)^p \right)$$

$$= - \sum_{p=1}^{\infty} \left( \sum_{i=1}^{n} \frac{1}{p} (1 - \lambda_i \bar{\lambda}_i)^p \right)$$

$$= - \sum_{p=1}^{\infty} \frac{1}{p} \text{tr} ((I - BB^*)^p)$$

by the power series expansion of log, where $B^*$ is the adjoint matrix of $B$. More precisely, if we take a sufficiently large constant $K > 0$, we obtain

$$| \det B | = K^n \exp \left( - \frac{1}{2} \sum_{p=1}^{\infty} \frac{1}{p} \text{tr} \left( (I - K^{-2}BB^*)^p \right) \right) \in \mathbb{R}_{>0}.$$

Thus if we can define a certain "trace" over a group ring, we get a determinant by using this formula.

Let $\pi$ be a discrete group and $\mathbb{C} \pi$ denote its group ring over $\mathbb{C}$. For an element $\sum g \in \pi \lambda_g g \in \mathbb{C} \pi$, we define the $\mathbb{C} \pi$-trace $\text{tr}_{\mathbb{C} \pi} : \mathbb{C} \pi \rightarrow \mathbb{C}$ by

$$\text{tr}_{\mathbb{C} \pi} \left( \sum_{g \in \pi} \lambda_g g \right) = \lambda_e \in \mathbb{C},$$
where \( e \) is the unit element in \( \pi \). For an \( n \times n \)-matrix \( B = (b_{ij}) \in M(n, \mathbb{C}_\pi) \), we extend the definition of \( \mathbb{C}_\pi \)-trace by means of

\[
\text{tr}_{\mathbb{C}_\pi}(B) = \sum_{i=1}^{n} \text{tr}_{\mathbb{C}_\pi}(b_{ii}).
\]

Next let us recall the definition of the \( L^2 \)-Betti number of an \( n \times m \)-matrix \( B \in M(n, m, \mathbb{C}_\pi) \). We consider the bounded \( \pi \)-equivariant operator

\[
R_B : \bigoplus_{i=1}^{n} l^2(\pi) \rightarrow \bigoplus_{i=1}^{m} l^2(\pi)
\]
defined by the natural right action of \( B \). Here \( l^2(\pi) \) is the complex Hilbert space of the formal sums \( \sum_{g \in \pi} \lambda_g g \) which are square summable. We fix a positive real number \( K \) so that \( K \geq \|R_B\|_\infty \) holds, where \( \|R_B\|_\infty \) is the operator norm of \( R_B \).

**Definition 2.1.** The \( L^2 \)-Betti number of a matrix \( B \in M(n, m, \mathbb{C}_\pi) \) is defined by

\[
b(B) = \lim_{p \to \infty} \text{tr}_{\mathbb{C}_\pi} \left( (I - K^{-2}BB^*)^p \right) \in \mathbb{R}_{\geq 0},
\]

where \( B^* = (\bar{b}_{ji}) \) and \( \sum_{g \in \pi} \lambda_g g = \sum \bar{\lambda}_g g^{-1} \) for each entry.

Roughly speaking, the \( L^2 \)-Betti number \( b(B) \) measures the size of the kernel of a matrix \( B \). Hereafter we assume \( b(B) = 0 \). Then, for a matrix with coefficients in a non-commutative group ring, we can introduce the desired determinant as follows.

**Definition 2.2.** The Fuglede-Kadison determinant of a matrix \( B \in M(n, m, \mathbb{C}_\pi) \) is defined by

\[
det_{\mathbb{C}_\pi}(B) = K^n \exp \left( -\frac{1}{2} \sum_{p=1}^{\infty} \frac{1}{p} \text{tr}_{\mathbb{C}_\pi} \left( (I - K^{-2}BB^*)^p \right) \right) \in \mathbb{R}_{>0},
\]

if the infinite sum of non-negative real numbers in the above exponential converges to a real number.

**Remark 2.3.** It is shown that the \( L^2 \)-Betti number \( b(B) \) and the Fuglede-Kadison determinant \( \det_{\mathbb{C}_\pi}(B) \) are independent of the choice of the constant \( K \) (see [16] for example).

Here we consider the condition of the convergence. For any matrix \( B \in M(n, \mathbb{C}) \), the condition

\[
\lim_{p \to \infty} \text{tr} \left( (I - K^{-2}BB^*)^p \right) = 0
\]
implies that $B$ has no zero eigenvalues, and then $|\det B|$ converges. In the case of group rings, if $\det_{C\pi}(B)$ converges, then $b(B) = 0$. But it is not a sufficient condition, so that we need additional one. It is a problem to decide when $\det_{C\pi}(B)$ converges. Under the assumption that $b(B) = 0$, such a sufficient condition is given by the positivity of the Novikov-Shubin invariant $\alpha(B)$. Then the convergence of the infinite sum in the Fuglede-Kadison determinant is guaranteed. The Novikov-Shubin invariant of an operator $R_B$ measures how concentrated the spectrum of $R_B^*R_B$ is. However, in general, it is hard to check the positivity of the Novikov-Shubin invariant.

To avoid the difficulty, we need to consider the determinant class condition for groups (see [19], [24] for details). A group $\pi$ is of $\det \geq 1$-class if for any $B \in M(n, m, \mathbb{Z}_\pi)$ the Fuglede-Kadison determinant of $B$ satisfies $\det_{C\pi}(B) \geq 1$. There are no known examples of groups which are not of $\det \geq 1$-class. Further recently it was proved that there is a certain large class $\mathcal{G}$ of groups for which they are of $\det \geq 1$-class. It includes amenable groups and countable residually finite groups. If we can see that $\pi$ belongs to $\mathcal{G}$, namely it is of $\det \geq 1$-class, the convergence of the Fuglede-Kadison determinant is guaranteed when the $L^2$-Betti number is vanishing. See [18], [19], [24] for definitions and properties of these subjects.

§3. $L^2$-torsion of 3-manifolds

In this section, we quickly recall the definition of the $L^2$-torsion of 3-manifolds. It is an $L^2$-analogue of the Reidemeister and the Ray-Singer torsion and essentially gives Gromov's simplicial volume under certain general conditions [2], [3], [4], [8], [14], [15], [20], [21], [22]. See [19] and its references for historical background, related works and so on.

Let $M$ be a compact connected orientable 3-manifold. We fix a $CW$-complex structure on $M$. We may assume that the action of $\pi_1 M$ on the universal covering $\widetilde{M}$ is cellular (if necessary, we have only to take a subdivision of the original structure). We consider the $C\pi_1 M$-chain complex

$$
0 \longrightarrow C_3(\widetilde{M}, \mathbb{C}) \xrightarrow{\partial_3} C_2(\widetilde{M}, \mathbb{C}) \xrightarrow{\partial_2} C_1(\widetilde{M}, \mathbb{C}) \xrightarrow{\partial_1} C_0(\widetilde{M}, \mathbb{C}) \longrightarrow 0
$$

of $\widetilde{M}$. Since the boundary operator $\partial_i$ is a matrix with coefficients in $C\pi_1 M$, if we take the adjoint operator $\partial_i^* : C_{i-1}(\widetilde{M}, \mathbb{C}) \to C_i(\widetilde{M}, \mathbb{C})$ as in the previous section, we can define the $i$th (combinatorial) Laplace operator $\Delta_i : C_i(\widetilde{M}, \mathbb{C}) \to C_i(\widetilde{M}, \mathbb{C})$ by

$$
\Delta_i = \partial_{i+1} \circ \partial_i^* + \partial_i^* \circ \partial_i.
$$
Let us suppose that all the $L^2$-Betti numbers $b(\Delta_i)$ vanish and the fundamental group $\pi_1 M$ is of $\det \geq 1$-class. Thereby as a generalization of the classical Reidemeister torsion, the $L^2$-torsion $\tau(M)$ is defined by

**Definition 3.1.**

$$\tau(M) = \prod_{i=0}^{3} \det_{\pi_1 M} (\Delta_i)^{(-1)^{i+1}} \in \mathbb{R}_{>0}.$$ 

As for the positivity of Novikov-Shubin invariants $\alpha(\Delta_i)$ for the Laplace operator $\Delta_i$, it is known that $\alpha(\Delta_i) > 0$ holds under some general assumptions (see [15]). For example, if a compact connected orientable 3-manifold $M$ satisfies

1. $\pi_1 M$ is infinite,
2. $M$ is an irreducible 3-manifold or $S^1 \times S^2$ or $\mathbb{RP}^3 \# \mathbb{RP}^3$,
3. if $\partial M \neq \phi$, it consists of tori,
4. if $\partial M = \phi$, $M$ is finitely covered by a 3-manifold which is a hyperbolic, Seifert or Haken 3-manifold,

then $b(\Delta_i) = 0$ and $\alpha(\Delta_i) > 0$ for each $i$. Therefore, we see that the $L^2$-torsion $\tau(M)$ is also well-defined in view of these conditions.

**Remark 3.2.** The above condition (4) is automatically satisfied by Perelman’s proof of Thurston’s Geometrization Conjecture.

As a notable property of the $L^2$-torsion, it is known that $\log \tau(M)$ can be interpreted as Gromov’s simplicial volume $\|M\|$ and hyperbolic volume $\text{vol}(M)$ (see [7]) of $M$. See [21] for the heart of the proof.

**Theorem 3.3.** Let $M$ be a compact connected orientable irreducible 3-manifold with an infinite fundamental group such that $\partial M$ is empty or a disjoint union of incompressible tori. Then it holds that

$$\log \tau(M) = C\|M\|,$$

where $C$ is the universal constant not depending on $M$. In particular, if $M$ is a hyperbolic 3-manifold, we obtain

$$\log \tau(M) = -\frac{1}{3\pi} \text{vol}(M).$$

Next we review Lück’s formula for the $L^2$-torsion of 3-manifolds ([16] Theorem 2.4). From this formula, we see that $\log \tau$ is a characteristic polynomial of the Magnus representation of the mapping class group.
**Theorem 3.4.** Let $M$ be as in the above theorem. We suppose that $\partial M$ is non-empty and $\pi_1M$ has a deficiency one presentation

$$\langle s_1, \ldots, s_{n+1} \mid r_1, \ldots, r_n \rangle.$$ 

Put $A$ to be the $n \times n$-matrix with entries in $\mathbb{Z}\pi_1M$ obtained from the matrix $(\partial r_i/\partial s_j)$ by deleting one of the columns. Then the logarithm of the $L^2$-torsion of $M$ is given by

$$\log \tau(M) = -2 \log \det_{\mathbb{C}\pi_1M}(A)$$

$$= -2n \log K + \sum_{p=1}^{\infty} \frac{1}{p} \text{tr}_{\mathbb{C}\pi_1M} \left( (I - K^{-2}AA^*)^p \right),$$

where $K$ is a constant satisfying $K \geq ||R_A||_\infty$.

To see a relation between the Magnus representation and the $L^2$-torsion, we describe the above Lück’s formula for a surface bundle over the circle.

For an orientation preserving diffeomorphism $\varphi$ of $\Sigma_{g,1}$, we form the mapping torus $M_\varphi$ by taking the product $\Sigma_{g,1} \times [0,1]$ and gluing $\Sigma_{g,1} \times \{0\}$ and $\Sigma_{g,1} \times \{1\}$ via $\varphi$. This gives a surface bundle over $S^1$. Its diffeomorphism type is determined by the monodromy map $\varphi$, and conversely the monodromy map $\varphi$ is determined by a given surface bundle up to conjugacy and isotopy. Here an isotopy fixes setwisely the points on the boundary $\partial \Sigma_{g,1}$. We take a deficiency one presentation of the fundamental group $\pi = \pi_1(M_\varphi, \ast)$,

$$\pi = \langle x_1, \ldots, x_{2g}, t \mid r_i : tx_it^{-1} = \varphi_*(x_i), 1 \leq i \leq 2g \rangle,$$

where the base point $\ast$ of $\pi$ and $\Gamma = \pi_1(\Sigma_{g,1}, \ast)$ is the same one on the fiber $\Sigma_{g,1} \times \{0\} \subset M_\varphi$ and $\varphi_* : \Gamma \to \Gamma$ is the automorphism induced by $\varphi : \Sigma_{g,1} \to \Sigma_{g,1}$. It should be noted that $\pi$ is isomorphic to the semi-direct product of $\Gamma$ and $\pi_1S^1 \cong \mathbb{Z} = \langle t \rangle$.

Applying the free differential calculus to the relations $r_i$ ($1 \leq i \leq 2g$), we obtain the Alexander matrix

$$A = \left( \frac{\partial r_i}{\partial x_j} \right) \in M(2g, \mathbb{Z}\pi).$$

Then Lück’s formula for a surface bundle over the circle is given by

$$\log \tau(M_\varphi) = -2 \log \det_{\mathbb{C}\pi}(A)$$

$$= -4g \log K + \sum_{p=1}^{\infty} \frac{1}{p} \text{tr}_{\mathbb{C}\pi} \left( (I - K^{-2}AA^*)^p \right),$$

where $K$ is a constant satisfying $K \geq ||R_A||_\infty$. 
where $K$ is a constant satisfying $K \geq ||R_A||_\infty$.

This formula enables us to interpret the $L^2$-torsion $\log \tau$ of a surface bundle over the circle as the characteristic polynomial of the Magnus representation $r(\varphi)$. In fact, an easy calculation shows that

$$A = \left( \frac{\partial r_i}{\partial x_j} \right) = tI - t r(\varphi).$$

Then if we take the Fuglede-Kadison determinant in $M(2g, \mathbb{C})$, we have

$$\det_{\mathbb{C}^\pi} \left( tI - tr(\varphi) \right) = \det_{\mathbb{C}^\pi} \left( tI - tr(\varphi) \right)^* = \det_{\mathbb{C}^\pi} \left( t^{-1}I - r(\varphi) \right)$$

because $\tr_{\mathbb{C}^\pi}(BB^*) = \tr_{\mathbb{C}^\pi}(B^*B)$ holds. Therefore the $L^2$-torsion is interpreted as the characteristic polynomial of $r(\varphi)$.

§4. Definition of $L^2$-torsion invariants

As was seen in Section 3, Lück's formula gives a way to calculate the simplicial volume from a presentation of the fundamental group. However, in general, it seems to be difficult to evaluate the exact values from the formula. In this section, we introduce a sequence of $L^2$-torsion invariants which approximates the original one for a surface bundle over the circle. See [12] for details.

In order to construct such a sequence of $L^2$-torsion invariants, we consider the lower central series of $\Gamma$. Namely, we take the descending infinite sequence

$$\Gamma_1 = \Gamma \supset \Gamma_2 \supset \cdots \supset \Gamma_k \supset \cdots,$$

where $\Gamma_k = [\Gamma_{k-1}, \Gamma_1]$ for $k \geq 2$. Let $N_k$ be the $k$th nilpotent quotient $N_k = \Gamma/\Gamma_k$ and $p_k : \Gamma \to N_k$ be the natural projection.

In the previous section, we considered a chain complex $C_*(\tilde{M}_\varphi, \mathbb{C})$ of $\mathbb{C}\pi$-modules. Instead of this complex, we can use the chain complex

$$C_*(M_\varphi, l^2(\pi)) = l^2(\pi) \otimes_{\mathbb{C}\pi} C_*(\tilde{M}_\varphi, \mathbb{C})$$

to define the same $L^2$-torsion $\tau(M_\varphi)$. This point of view allows us to introduce a sequence of the $L^2$-torsion invariants.

The group $\Gamma_k$ is a normal subgroup of $\pi$, so that we can take the quotient group $\pi(k) = \pi/\Gamma_k$. It should be noted that $\pi(k)$ is isomorphic to the semi-direct product $N_k \rtimes \mathbb{Z}$. We denote the induced projection
\( \pi \to \pi(k) \) by the same letter \( p_k \). Thereby we can consider the chain complex
\[
C_* (M_\varphi, l^2 (\pi(k))) = l^2 (\pi(k)) \otimes_{C_\pi} C_* (\overline{M_\varphi}, C)
\]
through the projection \( p_k \). By using the Laplace operator
\[
\Delta_i^{(k)} : C_i (M_\varphi, l^2 (\pi(k))) \to C_i (M_\varphi, l^2 (\pi(k)))
\]
on this complex, we can formally define the \( k \)th \( L^2 \)-torsion invariant \( \tau_k(M_\varphi) \) as follows.

**Definition 4.1.**
\[
\tau_k(M_\varphi) = \prod_{i=0}^{3} \det_{C_\pi(k)}(\Delta_i^{(k)})^{(-1)^{i+1}i}.
\]

Of course, this definition is well-defined if every \( L^2 \)-Betti number \( b(\Delta_i^{(k)}) \) vanishes and every \( \pi(k) \) is of \( \det \geq 1 \)-class. The next lemma is easily proved (see [12], [17]).

**Lemma 4.2.** The \( L^2 \)-Betti numbers of \( \Delta_i^{(k)} \) are all zero.

Recall the class \( G \) of groups. It is the smallest class of groups which contains the trivial group and is closed under the following processes: (i) amenable quotients, (ii) colimits, (iii) inverse limits, (iv) subgroups and (v) quotients with finite kernel (see [19], [24]). It is known that \( G \) contains all amenable groups. By definition, \( N_k = \Gamma / \Gamma_k \) is a nilpotent group and in particular an amenable group. Hence every \( N_k \) belongs to \( G \). Further for any automorphism \( \varphi_* : N_k \to N_k \), its mapping torus extension (HNN-extension) \( N_k \rtimes \mathbb{Z} \) also belongs to \( G \). Therefore we have

**Lemma 4.3.** The group \( \pi(k) \) belongs to \( G \).

As a result, we can conclude that our \( L^2 \)-torsion invariants \( \tau_k \) can be defined for any \( k \geq 1 \) and they are all homotopy invariants (see [19], [24]).

Now let us describe a formula of the \( k \)th \( L^2 \)-torsion invariant \( \tau_k(M_\varphi) \) and establish a relation to the Magnus representation of the mapping class group. Let \( p_{k_*} : C_\pi \to C_\pi(k) \) be an induced homomorphism over the group rings. For \( k \geq 1 \), we put
\[
A_k = \left( p_{k_*} \left( \frac{\partial r_i}{\partial x_j} \right) \right) \in M(2g, C_\pi(k)).
\]
Moreover we fix a constant $K_k$ satisfying $K_k \geq \|R_{A_k}\|_\infty$. Then we have

$$
\log \tau_k(M_\psi) = -2 \log \det_{\mathbb{C}_\pi(k)}(R_{A_k})
$$

$$
= -4g \log K_k + \sum_{p=1}^{\infty} \frac{1}{p} \text{tr}_{\mathbb{C}_\pi(k)}((I - K_k^{-2}A_k A_k^*)^p),
$$

by virtue of the same argument as Theorem 3.4.

For the $k$th invariant $\tau_k$, we have taken the lower central series $\{\Gamma_k\}$ of $\Gamma$ and the nilpotent quotients $\{N_k\}$. These quotients induce a sequence of representations (more precisely, crossed homomorphisms)

$$
r_k : \mathcal{M}_{g,1} \to GL(2g, \mathbb{Z}N_k)
$$

for $k \geq 1$ (see [23]). They naively approximate the original Magnus representation $r : \mathcal{M}_{g,1} \to GL(2g, \mathbb{Z} \Gamma)$. By the similar observation as before, the $k$th invariant $\log \tau_k(M_\psi)$ can be regarded as the characteristic polynomial of $r_k(\varphi)$ with respect to the Fuglede-Kadison determinant in $M(2g, \mathbb{C}_\pi(k))$.

From the viewpoint of the Magnus representation of the mapping class group, it seems natural to raise the following problem.

**Problem 4.4.** Show that the sequence $\{\tau_k(M_\psi)\}$ converges to $\tau(M_\psi)$ when we take the limit on $k$.

In general, such an approximation problem for the $L^2$-torsion seems to be difficult. However, similar convergence results are known for the $L^2$-Betti numbers. In fact, Lück shows in [18] a theorem relating $L^2$-Betti numbers to ordinary Betti numbers of finite coverings. This result is generalized to more general settings by Schick in [24].

As for the Fuglede-Kadison determinant, Lück proves in [19] the following. Let $f : \mathbb{Q}[Z] \to \mathbb{Q}[Z]$ be the $\mathbb{Q}[Z]$-map given by multiplication with $p(t) \in \mathbb{Q}[Z]$ and $f(2) : l^2(Z) \to l^2(Z)$ be the linear operator obtained from $f$ by tensoring with $l^2(Z)$ over $\mathbb{Q}[Z]$. Further let $f_{[n]} : \mathbb{C}[Z/n] \to \mathbb{C}[Z/n]$ be the linear operator obtained from $f$ by taking the tensor product with $\mathbb{C}[Z/n]$ over $\mathbb{Q}[Z]$. We then get an approximation result:

$$
\log \det_{\mathbb{C}[Z]}(f(2)) = \lim_{n \to \infty} \frac{\log \det_{\mathbb{C}[Z/n]}(f_{[n]})}{n}
$$

(see [11] for a similar statement). In [19] Lück also points out that there exists a purely algebraic example where Fuglede-Kadison determinants do not converge.

On the other hand, in general, we have at least an inequality for the Fuglede-Kadison determinant in the limit statement (see [24]). That is,
for the operator $R_{A_k}$ we see that
\[
\log \det_{C_\pi}(R_A) \geq \limsup_k \log \det_{C_\pi(k)}(R_{A_k})
\]
holds. In the last section, we shall discuss Problem 4.4 again and give an affirmative answer under certain conditions.

§5. Formulas of $\tau_1$ and $\tau_2$

In this section, we give explicit formulas of the first two invariants of a sequence of our $L^2$-torsion invariants. They are really computable formulas, so that we can make a systematic calculation for low genus cases. In particular, we compare them with hyperbolic volumes. The results discussed here are a summary of our previous paper [12] (see also [10], [11]).

First we consider the Magnus representation
\[
r_1 : \mathcal{M}_{g,1} \to GL(2g, \mathbb{Z}N_1).
\]
Here $N_1 = \Gamma/\Gamma_1$ is the trivial group and then the above representation is the same as the usual homological action of $\mathcal{M}_{g,1}$ on $H_1(\Sigma_{g,1}, \mathbb{Z})$. Namely we have the representation
\[
r_1 : \mathcal{M}_{g,1} \to \text{Aut}(H_1(\Sigma_{g,1}, \mathbb{Z}), \langle , \rangle) \cong \text{Sp}(2g, \mathbb{Z}),
\]
where $\langle , \rangle$ denotes the intersection form on the first homology group. Further $\pi(1) = \pi/\Gamma_1 \cong \mathbb{Z} = \langle t \rangle$ and its group ring $\mathbb{C}(t)$ is a commutative Laurent polynomial ring $\mathbb{C}[t, t^{-1}]$. Then the matrix $A_1$ is nothing but the usual characteristic matrix of $r_1(\varphi)$. In this case, it is described by the usual determinant for a matrix with commutative entries.

In order to state the theorem, we recall a definition from number theory (see [6] and its references). For a Laurent polynomial $F(t) \in \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$, the Mahler measure of $F$ is defined by
\[
m(F) = \int_0^1 \cdots \int_0^1 \log |F(e^{2\pi \sqrt{-1}\theta_1}, \ldots, e^{2\pi \sqrt{-1}\theta_n})| \, d\theta_1 \cdots d\theta_n,
\]
where we assume that undefined terms are omitted. Namely we define the integrand to be zero whenever we hit a zero of $F$.

**Theorem 5.1** ([12]). The logarithm of the first invariant $\tau_1$ is given by
\[
\log \tau_1(M_\varphi) = -2m(\Delta_{r_1(\varphi)}),
\]
where $\Delta_{r_1(\varphi)}(t) = \det A_1 = \det(tI - r_1(\varphi))$. Moreover if $\Delta_{r_1(\varphi)}(t)$ has a factorization $\Delta_{r_1(\varphi)}(t) = \prod_{i=1}^{2g}(t - \alpha_i)$ ($\alpha_i \in \mathbb{C}$), then we have

$$\log \tau_1(M_\varphi) = -2 \sum_{i=1}^{2g} \log \max\{1, |\alpha_i|\}.$$

**Remark 5.2.** In other words, $\log \tau_1(M_\varphi)$ is given by the integral of the Alexander polynomial of $M_\varphi$ over the circle $S^1$ (see [16], for the exterior of a knot $K$ in the 3-sphere $S^3$). Further, $\log \tau_1(M_\varphi)$ can be described by the asymptotic behavior of the order of the first homology group of a cyclic covering (see [11]).

The point of the proof is to identify the Hilbert space $l^2(\mathbb{Z})$ with $L^2(\mathbb{R}/\mathbb{Z})$ in terms of the Fourier transforms. Then the $\mathbb{C}\langle t \rangle$-trace $\tr_{\mathbb{C}\langle t \rangle} : l^2(\mathbb{Z}) \to \mathbb{C}$ can be realized as the integration

$$L^2(\mathbb{R}/\mathbb{Z}) \ni f(\theta) \mapsto \int_0^1 f(\theta)d\theta \in \mathbb{C}$$

(see [12] for details). From this description and Kronecker’s theorem ([6] Theorem 2), we obtain a certain vanishing theorem of the first invariant.

**Corollary 5.3.** The logarithm of $\tau_1(M_\varphi)$ vanishes if and only if every eigenvalue of $r_1(\varphi) \in \text{Sp}(2g, \mathbb{Z})$ is a root of unity.

This corollary seems to be interesting from the viewpoint of Problem 4.4. Because in some case, we can say that the first invariant $\tau_1$ already approximates the simplicial volume. In particular, Corollary 5.3 implies that a torus bundle $M_\varphi$ ($g = 1$) with a hyperbolic structure (namely, $|\tr(r_1(\varphi))| \geq 3$) has always non-trivial $L^2$-torsion invariant $\tau_1(M_\varphi)$. Summing up, we have

**Corollary 5.4.** For any $\varphi \in \mathcal{M}_{1,1}$, its mapping torus $M_\varphi$ admits a hyperbolic structure if and only if $M_\varphi$ has a non-trivial $L^2$-torsion invariant $\tau_1(M_\varphi)$.

Therefore, the first invariant $\tau_1$ already approximates the simplicial volume in genus one case.

**Remark 5.5.** It is known that if the characteristic polynomial of $r_1(\varphi) \in \text{Sp}(2g, \mathbb{Z})$ is irreducible over $\mathbb{Z}$, has no roots of unity as eigenvalues and is not a polynomial in $t^n$ for any $n > 1$, then $\varphi$ is pseudo-Anosov (see Casson-Bleiler [5]). In this case, $\text{vol}(M_\varphi) \neq 0$ and further $\log \tau_1(M_\varphi) \neq 0$ by Corollary 5.3.
Example 5.6. It is well-known that the mapping class group of the two dimensional torus $T^2 = \mathbb{R}^2 / \mathbb{Z}^2$ is isomorphic to $SL(2, \mathbb{Z})$. Taking a matrix \( \begin{pmatrix} q & 1 \\ -1 & 0 \end{pmatrix} \in SL(2, \mathbb{Z}) \), it gives a diffeomorphism $\varphi$ on $T^2$. We may assume that it is the identity on some embedded disk by an isotopic deformation and it gives an element of $\mathcal{M}_{1,1}$. We use the same symbol $\varphi$ for this mapping class. An easy calculation shows that

$$r_1(\varphi) = \begin{pmatrix} q & 1 \\ -1 & 0 \end{pmatrix}$$

and

$$\Delta_{r_1(\varphi)}(t) = \det(tI - r_1(\varphi)) = t^2 - qt + 1.$$ 

We put $\xi_{\pm} = \left( q \pm \sqrt{q^2 - 4} \right) / 2$ (the eigenvalues of the matrix $r_1(\varphi)$). If $|q| \leq 2$, then $|\xi_{\pm}| = 1$. Hence $\log \tau_1(M_\varphi) = 0$ in these cases. On the other hand, either $|\xi_+|$ or $|\xi_-|$ is greater than one when $|q| \geq 3$, so that $M_\varphi$ has a non-trivial $L^2$-torsion invariant $\tau_1$ in these cases. In fact, the logarithm of the first invariant is given by

$$\log \tau_1(M_\varphi) = -2 \log \max \{|\xi_+|, |\xi_-|\}.$$ 

The values of $\log \tau_1$ for the traces $q$ and $-q$ are the same, so that it is a function of $|\text{tr}(r_1(\varphi))|$. We put a graph of the $L^2$-torsion invariant
−3π log τ₁(ℳₓ) and the hyperbolic volume vol(ℳₓ) as a function of |q| in Fig 1.

**Example 5.7.** Next we consider the genus two case. Let \( t₁, \ldots, t₅ \) be the Lickorish-Humphries generators of \( ℳ₂₁ \). We take the element \( \varphi = t₁t₃t₅²t₂⁻¹t₄⁻¹ \in ℳ₂₁ \). As was shown in [5], the characteristic polynomial of \( r(\varphi) \) is

\[
Δ_{r₁(\varphi)}(t) = \det(tI - r₁(\varphi)) = t⁴ - 9t³ + 21t² - 9t + 1
\]

and irreducible over \( ℤ \). Moreover it has no roots of unity as zeros. Hence, \( \varphi \) is pseudo-Anosov and \( ℳₓ \) has a non-trivial \( L² \)-torsion invariant \( τ₁(ℳₓ) \). In fact, we have

\[
-3π log τ₁(ℳₓ) = 52.954,... \quad \text{and} \quad vol(ℳₓ) = 11.466, ...
\]

**Remark 5.8.** In the above two examples, we used SnapPea [26] to compute the hyperbolic volumes.

Now in the following, we consider the second invariant \( τ₂ \). In the case of genus one, we can prove the vanishing of \( log τ₂(Mₓ) \).

**Theorem 5.9 ([11]).** \( log τ₂(Mₓ) = 0 \) for any \( \varphi \in ℳ₁₁ \).

This follows from the fact that the group \( π(2) \) is isomorphic to the fundamental group of a closed torus bundle over the circle. Such a 3-manifold admits no hyperbolic structures, so that the original \( L² \)-torsion is trivial and we obtain the assertion.

On the other hand, in the case of \( g ≥ 2 \), it is difficult to describe \( log τ₂ \) explicitly on the full mapping class group \( ℳ_{g,1} \). However, we can do it on the Torelli group. Let \( \varphi \) be an element of the Torelli group \( ℐ_{g,1} \), namely \( \varphi \) acts trivially on the first homology group \( H₁(Σ_{g,1}, ℤ) \). Then we notice that \( log τ₁(ℳₓ) = 0 \) holds for any \( \varphi \in ℐ_{g,1} \) (see Corollary 5.3). To give an explicit formula of \( log τ₂ \), we consider the Magnus representation

\[
r₂ : ℳ_{g,1} \rightarrow GL(2g, ℤN₂),
\]

where \( N₂ = Γ/⟨Γ, Γ⟩ \cong H₁(Σ_{g,1}, ℤ) \). If we restrict \( r₂ \) to the Torelli group \( ℐ_{g,1} \), this is really a homomorphism (see [23] Corollary 5.4). Then our formula for the second \( L² \)-torsion invariant is the following. The proof is similar to one for Theorem 5.1.

**Theorem 5.10 ([12]).** For any mapping class \( \varphi \in ℐ_{g,1} \), the logarithm of the second \( L² \)-torsion invariant \( τ₂(Mₓ) \) is given by

\[
log τ₂(Mₓ) = -2m(Δr₂(\varphi)),
\]
where $\Delta_{r_2(\varphi)}(y_1, \ldots, y_{2g}, t) = \det A_2 = \det(tI - r_2(\varphi))$ and $y_i$ denotes the homology class corresponding to $x_i$.

Now we suppose $F(t) \in \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ is primitive. We define $F$ to be a generalized cyclotomic polynomial if it is a monomial times a product of one-variable cyclotomic polynomials evaluated at monomials.

The next corollary immediately follows from the theorem of Boyd, Lawton and Smyth (see [6] Theorem 4).

**Corollary 5.11.** For any mapping class $\varphi \in \mathcal{I}_{g,1}$, $\log \tau_2(M_\varphi) = 0$ if and only if $\Delta_{r_2(\varphi)}$ is a generalized cyclotomic polynomial.

As a typical element of the Torelli group $\mathcal{I}_{g,1}$, we first consider a BSCC-map $\varphi_h$ ($1 \leq h \leq g$) of genus $h$. That is, a Dehn twist along a bounding simple closed curve on $\Sigma_{g,1}$ which separates $\Sigma_{g,1}$ into $\Sigma_{h,1}$ and genus $g - h$ surface with two boundaries. We then see from [25] Corollary 4.3 that $\Delta_{r_2(\varphi_h)} = (t - 1)^{2g}$. This is clearly a generalized cyclotomic polynomial, so that $\log \tau_2(M_{\varphi_h}) = 0$.

Second we consider a BP-map $\psi_h = D_c D_{c'}^{-1}$ of genus $h$ ($1 \leq h \leq g - 1$), where $c$ and $c'$ are disjoint homologous simple closed curves on $\Sigma_{g,1}$ and $D_c$ denotes the Dehn twist along $c$. Since

$$\Delta_{r_2(\psi_h)} = (t - 1)^{2g - 2h}(t - y_{g+h+1})^{2h}$$

holds (see [25]), where $y_{g+h+1}$ denotes the homology class corresponding to the $(h + 1)$th meridian of $\Sigma_{g,1}$, we also have $\log \tau_2(M_{\psi_h}) = 0$.

The next example shows the non-triviality of the second $L^2$-torsion invariant $\log \tau_2$.

**Example 5.12.** Let $\varphi = t_3^3 \varphi_1 t_3^{-1} \varphi_1 \in \mathcal{I}_{2,1}$. Then we see from a computation in [25] that

$$\Delta_{r_2(\varphi)} = (t - 1)^4 + t(t - 1)^2(y_1 - 2 + y_1^{-1})(y_2 - 2 + y_2^{-1}).$$

This is not a generalized cyclotomic polynomial, so that the mapping torus $M_\varphi$ has a non-trivial $L^2$-torsion invariant $\tau_2(M_\varphi)$. In fact we can numerically compute it by means of Lawton’s result (see [13]). More precisely we have

$$-3\pi \log \tau_2(M_\varphi) = 6\pi m(\Delta_{r_2(\varphi)}) = 19.28...$$
§6. Vanishing of $\log \tau_k$ for reducible mapping classes

From the Nielsen-Thurston theory (see [5]), the mapping classes of a surface are classified into the following three types: (i) periodic, (ii) reducible and (iii) pseudo-Anosov. In our point of view, the most interesting object is a pseudo-Anosov map $\varphi$. Because the corresponding mapping torus $M_\varphi$ has non-trivial hyperbolic volume.

In this final section, we show two vanishing theorems for $\log \tau_k$. We introduced an infinite sequence $\{\tau_k\}$ as an approximation of the hyperbolic volume. Thus if it behaves well with the index $k$, we ought to prove

$$\lim_{k \to \infty} \log \tau_k = 0$$

for non-hyperbolic 3-manifolds (see Problem 4.4). As a first step of this observation, we obtain the following.

**Theorem 6.1.** If $\varphi \in M_{g,1}$ is the product of Dehn twists along any disjoint non-separating simple closed curves on $\Sigma_{g,1}$ which are mutually non-homologous, then $\log \tau_k(M_\varphi) = 0$ for any $k \geq 1$.

**Remark 6.2.** The mapping torus $M_\varphi$ for $\varphi \in M_{g,1}$ as above admits no hyperbolic structures, so that $\text{vol}(M_\varphi) = 0$ holds.

**Proof.** At first, we prove the theorem for the genus one case. After that we give the outline of the proof in the higher genus case.

Let $D_c$ be a Dehn twist along a non-separating simple closed curve $c$ on $\Sigma_{1,1}$. Taking a conjugation, we can assume that the curve $c$ is one of the standard generators of $\pi_1(\Sigma_{1,1})$. We then see that $\varphi = D_c^q$ is represented by a matrix $\begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} \in SL(2,\mathbb{Z})$. Thus we can choose a deficiency one presentation

$$\langle x, y, t \mid txt^{-1} = x, tyt^{-1} = x^qy \rangle$$

of $\pi_1(M_\varphi)$. Applying the free differential calculus to the relators $txt^{-1}x^{-1}$ and $tyt^{-1}(x^qy)^{-1}$, we obtain the Alexander matrix

$$A = \begin{pmatrix} t - 1 & 0 \\ -\partial(x^q)/\partial x & t - x^q \end{pmatrix}.$$ 

Here we remark that the generators $t$ and $x$ can be commuted by the relation $txt^{-1} = x$. Hence in this case, the $k$th Alexander matrix $A_k$ coincides with the original matrix $A$. In particular, $t$ and $x$ always commute. As we saw in Section 5, the $L^2$-torsion invariant $\tau_k(M_\varphi)$ ($k \geq \ldots$
1) can be computed by using the usual determinant and the Mahler measure in such a situation. Since

$$\det A = (t - 1)(t - x^q)$$

is a generalized cyclotomic polynomial, we obtain \(\log \tau_k(M_\varphi) = 0\) as desired (see Corollary 5.11).

In the higher genus case, we can assume that the mapping class \(\varphi\) is given by

\[
\begin{align*}
\varphi_*(x_1) &= x_1, \\
\varphi_*(x_2) &= x_1^q x_2, \\
\varphi_*(x_{2l-1}) &= x_{2l-1}, \\
\varphi_*(x_{2l}) &= x_{2l-1}^q x_{2l}, \\
\varphi_*(x_{2l+1}) &= x_{2l+1}, \ldots, \\
\varphi_*(x_{2g}) &= x_{2g}
\end{align*}
\]

by taking a conjugation, where \(q_1, \ldots, q_l \in \mathbb{Z}\) and \(1 \leq l \leq g - 1\). We then obtain the following presentation of \(\pi_1(M_\varphi)\):

\[
\langle x_1, \ldots, x_{2g}, t \mid tx_it^{-1} = \varphi_*(x_i), \ 1 \leq i \leq 2g \rangle.
\]

Since the Alexander matrix \(A\) is the direct sum of the \(2 \times 2\)-matrix in the genus one case, we obtain \(\log \tau_k(M_\varphi) = 0\) by the similar arguments.

Q.E.D.

As another affirmative answer to Problem 4.4, we can show the vanishing of \(\log \tau_k\) for the following mapping classes (see [12]). That is, we consider the case where there exists an integer \(n\) such that \(M_\varphi^n\) is topologically the product of \(\Sigma_{g,1}\) and \(S^1\). Here its bundle structure is non-trivial in general. Namely the \(n\)th power \(\varphi^n\) of a given monodromy \(\varphi\) is not trivial. A typical example is the Dehn twist along the simple closed curve on \(\Sigma_{g,1}\) parallel to the boundary. The difference between an isotopy fixing the boundary pointwisely and such one setwisely, it gives birth to the difference between a bundle structure and a topological type. We then obtain

**Theorem 6.3 ([12]).** \(\log \tau_k(M_\varphi) = 0\) for any \(k \geq 1\).

It is easy to see that such a 3-manifold does not admit a hyperbolic structure. Hence it has trivial simplicial volume.

The above two examples are both non-hyperbolic cases, so that we conclude the present paper with the following problem.

**Problem 6.4.** Show

\[
\lim_{k \to \infty} \log \tau_k(M_\varphi) = \log \tau(M_\varphi)
\]

for a pseudo-Anosov diffeomorphism \(\varphi\).
Acknowledgements. The authors are grateful to the referee for his/her numerous and helpful comments which greatly improved this paper.

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