Analytical and Experimental Investigations of Crack width for RC Beams in Bending

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Abstract. Although cracks do not considerably impair the resistance of reinforced concrete elements, they definitely affect the durability and appearance of structures. The control of crack width and spacing is a convenient approach in engineering practice. The width of a crack is characterized by high variability and the formulas suggested by design codes vary significantly. In the presented paper, the crack widths are derived using analytical methods recommended by several design codes and compared with the results of experimental investigations.

1. Introduction

Cracking of reinforced concrete elements used to be described with deterministic methods, while its variability is taken into consideration by application of safety factors. The deterministic approach to the definition of a crack leads to the conclusion that at reaching the cracking moment all the cracks occur at the same time, of the same width, height, and spacing. Such an approach is a gross simplification of a complex process. The experimental investigations show that cracks keep propagating during the entire time of loading and the values describing the process (such as cracking moment, crack width, and spacing) are random variables.

The deterministic approach to calculation of crack width is convenient in engineering practice, because it allows for rational control of limiting values. However, the ability to understand the process of crack propagation is important. Despite the fact that cracks do not significantly influence the resistance of reinforced concrete elements, they affect durability and appearance of the structure, as well as the comfort of people. In the limit state theory, crack width control is considered a serviceability limit state.

In the presented paper, the calculation of crack widths according to different design codes’ requirements (PN-B-03264, EC2, ACI 318) is presented. The results are discussed and compared with the results of experimental investigations.

2. Width of cracks perpendicular to axis of structural element

Due to the influence of cracks on the durability of reinforced concrete structures, cracking phenomena remain in the area of interest of numerous researchers. The investigations are carried out on the cracking of concrete elements, reinforced concrete elements, fibre-reinforced concrete, and elements reinforced with FRP (Fibre Reinforcement Polymer) in the states of normal and complex
stress. The same phenomena are being described from different perspectives, using inconsistent definitions and denomination [1-11]. However, constantly the same three parameters are taken into consideration: the cracking moment, the crack width and the spacing.

The cracking moment of a flexured RC beam is derived based on the stage Ib model, with the assumption of strain compatibility at the contact area between concrete and steel. Using the compatibility equations in the cross-section, the following formulas are obtained:

\[
\int_{-v'}^{v'} \sigma_{c}(\varepsilon_{c})dA_{c} + \sum_{i=1}^{n} A_{si} \sigma_{si}(\varepsilon_{si}) = 0, \tag{1}
\]

\[
M = \int_{-v'}^{v'} z\sigma_{c}(\varepsilon_{c})dA_{c} + \sum_{i=1}^{n} A_{si} \sigma_{si}(\varepsilon_{si})z_{si}, \tag{2}
\]

where:
- \( v \) is the distance between the neutral axis and the edge of the beam in compression,
- \( v' \) is the distance between the neutral axis and the edge of the beam in tension,
- \( z_{si} \) is the distance between the fiber and the neutral axis,
- \( A_{c} \) is the area of the concrete cross-section,
- \( A_{si} \) is the area of steel reinforcement,
- \( \sigma_{c}(\varepsilon_{c}) \) is the stress in concrete,
- \( \sigma_{si}(\varepsilon_{si}) \) is the stress in steel,
- \( \varepsilon_{c} \) is the strain in concrete,
- \( \varepsilon_{si} \) is the strain in steel, and
- \( M \) is the bending moment.

Assuming the Euler–Bernoulli theory that plane sections remain plane, it is assumed that:

\[
\varepsilon = z\theta, \tag{3}
\]

For the derivation of the cracking moment the strain damage condition used is:

\[
\varepsilon_{i} = v'\varepsilon_{i} = \varepsilon_{ci}, \tag{4}
\]

where:

\[
\sigma(\varepsilon_{ci}) = f_{ct}, \tag{5}
\]

where: \( \theta \) is the curvature ratio, \( \varepsilon_{i} \) is the maximum tensile strain, \( \varepsilon_{ci} \) is the non-destructive strain of the concrete in tension, and \( f_{ct} \) is the concrete tensile strength.

In a reinforced concrete element subjected to flexure, the first cracks appear when concrete in the tension zone reaches its tensile strength. In figure 1, the schematic distributions of the stresses in concrete and steel at the initial and final stage of the crack propagation are presented.

**Figure 1.** Distribution of strains in concrete (\( \varepsilon_{c} \)) and steel reinforcement in tension (\( \varepsilon_{s} \)) in the vicinity of cracks: a) initial stage of crack propagation, b) final stage of crack propagation.
The crack width \( w \) can be determined from the expression:

\[
w = \frac{x}{2} \int_{-\frac{x}{2}}^{\frac{x}{2}} [\varepsilon_s(l) - \varepsilon_c(l)] dl,
\]

where: \( x \) is the spacing between cracks, \( \varepsilon_s(l) \) is the strain in steel in the vicinity of the crack, \( \varepsilon_c(l) \) is the strain in concrete in the sections between cracks, and \( \varepsilon_s \) is the concrete shrinkage strain.

Taking into consideration the fact that the coefficients of thermal expansion for concrete and steel are similar, the influence of temperature is omitted in the calculation of the crack width at the level of steel reinforcement (5).

The solution for problem (6) may be found by introducing the relation between the strains and stresses in the cracked section:

\[
\varepsilon_s(x\eta) = \varepsilon_s - \Delta \varepsilon_s(x\eta),
\]

where: \( \varepsilon_s \) is the steel strain in a cracked section, and \( \eta \) is the transformed coordinator.

Assuming \( l = x\eta \) it results that \( dl = xd\eta \), then substituting relation (7) for \( \varepsilon_s(l) \) to expression (6), the following relation may be obtained:

\[
w = x\eta \int_0^{1} \left[ 1 - \frac{\Delta \varepsilon_s(x\eta)}{x} \right] d\eta + x\varepsilon_{cs} = x\varepsilon_{cs}(1-c) + x\varepsilon_{cs} = \frac{\sigma_s}{E_s} x(1-c) + x\varepsilon_{cs},
\]

where \( \varepsilon_s \) is a function or constant taking into consideration influence of concrete on the crack width.

The value \( c \), which defines the impact of concrete on deceleration of steel strain, can be determined in different manners [12-14]. It is caused by the difference in definition of spacing between cracks. Theoretical spacing of cracks is derived as a distance between the potential location of a new crack and an already cracked cross-section, and it depends mostly on bonding between concrete and steel.

Three design codes (PN-B-03264, EC2, ACI 318-95) were chosen for the calculation of crack width and comparison with experimental investigations.

2.1. Calculation of crack width according to PN-B-03264

According to PN-B-03264:1999 [15] the width of cracks perpendicular to the axis of the structural element is given by the expression:

\[
w_k = \beta s_{rm} \varepsilon_{sm},
\]

where: \( \beta \) is a coefficient dependent on the relation of the crack width to the mean value [15], \( s_{rm} \) is the mean final crack spacing, and \( \varepsilon_{sm} \) is the mean strain in the tensile reinforcement.

Values of \( s_{rm} \) and \( \varepsilon_{sm} \) are calculated from the following expressions:

\[
s_{rm} = 50 + 0.25k_s \frac{\phi}{\rho},
\]

\[
\varepsilon_{sm} = \frac{\phi}{\rho}.
\]
\[ \varepsilon_{sm} = \frac{\sigma_s}{E_s} \left[ 1 - \beta_1 \beta_2 \left( \frac{\sigma_{sr}}{\sigma_s} \right)^2 \right], \]  
\[ (11) \]

where: \( k_1 \) is a coefficient dependent on the bond properties of the reinforcement, \( k_2 \) is a coefficient dependent on distribution of tensile stresses, \( \rho \) is the effective reinforcement ratio, \( \sigma_s \) and \( \sigma_{sr} \) correspond to the stresses in the tension reinforcement calculated on the basis of a cracked section under the loading conditions causing first cracking and cracked, \( \beta_1 \) is a coefficient depending on bond properties of the reinforcement, and \( \beta_2 \) is a coefficient taking into account the influence of the duration of the loading or of repeated loading.

2.2. Calculation of crack width according to EC2

According to the EC2 [16], at the serviceability limit state check the characteristic crack width \( w_k \) may be obtained from the relation:

\[ w_k = s_{r,max} (\varepsilon_{sm} - \varepsilon_{cm}), \]  
\[ (12) \]

where: \( s_{r,max} \) is the maximum crack spacing, \( \varepsilon_{sm} \) is the mean strain in the reinforcement including the effect of imposed deformations and taking into account the effects of tension stiffening, and \( \varepsilon_{cm} \) is the mean strain in the concrete between cracks.

The difference between strains \( \varepsilon_{sm} \) and \( \varepsilon_{cm} \) is calculated from the following expression:

\[ (\varepsilon_{sm} - \varepsilon_{cm}) = \max \left\{ \frac{1}{E_s} \left[ \sigma_s - k_i f_{ct,eff} \rho_{p,eff} (1 + \alpha \rho_{p,eff}) \right], \right. \]  
\[ 0,6 \frac{\sigma_s}{E_s} \right\}, \]  
\[ (13) \]

where: \( \sigma_s \) is the stress in the tension reinforcement assuming a cracked section, \( k_i \) is a factor dependent on the duration of the load, \( \alpha \) is the ratio \( E_s/E_{cm} \), \( \rho_{p,eff} \) is the effective reinforcement ratio calculated according to [16], \( f_{ct,eff} \) is the mean value of the tensile strength of the concrete effective at the time when the cracks may first be expected to occur \( (f_{ct,eff} = f_{ct,m} \) may be assumed).

The maximum final crack spacing may be calculated from expression:

\[ s_{r,max} = 3.4c + 0.425k_1k_2 \frac{\phi}{\rho_{p,eff}}, \]  
\[ (14) \]

where: \( k_1 \) is a coefficient which takes into account the bond properties of the reinforcement, \( k_2 \) is a coefficient which takes into account the distribution of strain, \( \phi \) is the bar diameter, and \( c \) is the cover to the reinforcement.

2.3. Calculation of crack width according to ACI 318

The ACI 318 Code gives importance to reinforcement details rather than crack width \( w \), per se. Before 1999, the Code’s provisions were formulated for distribution of reinforcement, which were based on empirical equations using a calculated maximum crack width of 0.016 in. The ACI 318-95 [17] provides a distribution that reasonably controls flexural cracking. The Code recommends distributing flexural tension reinforcement within maximum flexural tension zones of a member cross section. The quantity limiting distribution of flexural reinforcement \( z \) should not exceed 175 kips/in. for interior exposure and 145 kips/in. for exterior exposure:
\[ z = f_s \sqrt[d_c]{A}, \]  
which is based on the Gergely-Lutz expression for crack width in units of in.:

\[ w = 0.076 \beta f_s \sqrt[d_c]{A} \times 10^{-3}, \]

where: \( f_s \) is the stress in reinforcement at service load, \( d_c \) is the thickness of concrete cover measured from extreme tension fibre to centre of bar or wire located closest thereto, and \( A \) is effective tension area of concrete surrounding the flexural tension reinforcement and having the same centroid as that reinforcement divided by the number of bars or wires.

Although the newest ACI 318-19 Code [18] recognizes crack width to be related to reinforcement stress, thickness of concrete cover, and the spacing of reinforcement, its approach to crack control is significantly different. The Code does not define a limit of the crack width. It is stated that the crack widths in structures are inherently subject to wide scatter and they are considered highly variable. The provisions for spacing are intended to limit surface cracks to a width that is acceptable in practice but may vary widely in a given structure. Another significant difference is that the ACI 318-19 does not differentiate between interior and exterior exposures. It is stated that the role of cracks in the corrosion of reinforcement is considered controversial and the corrosion is not clearly correlated with surface crack widths in the range normally found with reinforcement stresses at service load levels.

3. Experimental tests

The experimental investigations were carried out and described by Włodarczyk M. [19]. The object of laboratory tests were reinforced concrete rectangular beams of cross-section \( 120 \times 300 \text{ mm} \) and length \( 3000 \text{ mm} \). The assumed statical schema was a simply supported beam loaded with two concentrated forces at \( 1/3 \) of the span length.

The tests included a series of four reinforced concrete beams reinforced with \( 2 \phi 20 \) of steel 34GS in the tension zone (reinforcement ratio \( \rho_1 = 1.94\% \); characteristic yield strength \( f_{yk} = 435.7 \text{ MPa} \), elastic modulus \( E_s = 210 \text{ GPa} \)) and \( 2 \phi 10 \) of steel St3S in the compression zone. The stirrups \( \phi 6 \) of steel St3S were applied. Between the two concentrated forces, the stirrups were spaced every 225 mm, and between the forces and the supports, the stirrups were spaced every 100 mm. For all the beams the same concrete mix recipe was used with concrete compressive strength \( f_{ck} = 34.41 \text{ MPa} \div 36.22 \text{ MPa} \), mean tensile strength \( f_{ctm} = 2.58 \text{ MPa} \div 2.69 \text{ MPa} \), and modulus of elasticity \( E_{cm} = 31478 \text{ MPa} \div 31955 \text{ MPa} \).

The loading procedure assumed the increase of loading in multiple steps every 2.5 kN (up to 15 kN) and then every 5 kN: \( F: 0.0 \rightarrow 2.5 \rightarrow 5.0 \rightarrow 7.5 \rightarrow 10.0 \rightarrow 12.5 \rightarrow 15.0 \rightarrow 20.0 \text{ up to failure.} \) The failure of beams occurred in the range between 63.60 kN and 66.40 kN, and the cracking in the range between 5.0 kN and 12.5 kN.

The results of average crack width in relation to the load level are summarized in table 1, the maximum values are summarized in table 2.
4. Results and discussion

The theoretical crack widths are calculated for beams B1, B2, B3, and B4 in relation to the applied loading according to (10), (12), (15) and (16) and presented in table 3. In table 4, the theoretical values of cracking moments and resistance moments are compared with the results of laboratory tests.
Table 3. Theoretical values of crack widths

| Force [kN] | Crack width [mm] | Beam B1 | Beam B2 | Beam B3 | Beam B4 |
|------------|------------------|---------|---------|---------|---------|
|            | ACI318-95        | EC2-2008| PN-B-03264-99 |
| 0          | -                | -       | -       | -       |
| 2.5        | -                | -       | -       | -       |
| 5.0        | 0.0603           | 0.163   | 0.0089  |
| 7.5        | 0.0804           | 0.0270  | 0.0124  |
| 10.0       | 0.1005           | 0.0377  | 0.0217  |
| 12.5       | 0.1207           | 0.0483  | 0.0302  |
| 15.0       | 0.1609           | 0.0697  | 0.0382  |
| 20.0       | 0.2011           | 0.0910  | 0.0534  |
| 25.0       | 0.2413           | 0.1123  | 0.0683  |
| 30.0       | 0.2815           | 0.1337  | 0.0830  |
| 35.0       | 0.3218           | 0.1550  | 0.1119  |
| 40.0       | 0.3394           | 0.1763  | 0.1263  |
| 45.0       | 0.3394           | 0.1977  | 0.1406  |
| 50.0       | 0.3394           | 0.1993  | 0.1550  |
| 55.0       | 0.3394           | 0.1993  | 0.1693  |
| 60.0       | 0.3394           | 0.1993  | 0.2121  |
| 65.0       | 0.3394           | 0.1993  |         |

Table 4. Theoretical values of cracking moments and resistance moments

| Value of the moment [kNm] | Beam B1 | Beam B2 | Beam B3 | Beam B4 |
|---------------------------|---------|---------|---------|---------|
| Cracking moment           |         |         |         |         |
| \( M_{cr,ACI} \)          | 5.82    | 5.82    | 5.82    | 5.82    |
| \( M_{cr,EC2} \)          | 5.90    | 5.90    | 5.90    | 5.90    |
| \( M_{cr,PN-B} \)         | 4.84    | 4.84    | 4.84    | 4.84    |
| \( 10.0 < M_{cr} \leq 12 \) | 5      | 5       | 5       | 5       |
| \( M_{cr,EXP} \)          | 5.0 < \( M_{cr} \) \leq 7.5 | 5 | 5 | 5 |
| Resistance moment         |         |         |         |         |
| \( M_n \)                 | 68.00   | 67.93   | 67.54   | 67.72   |
| \( M_n,EXP \)             | 66.40   | 64.80   | 64.60   | 63.60   |

In figures 2 to 5, the theoretical values of crack widths as well as the average and the maximum values from the laboratory tests were presented. Comparing the diagrams it can be noticed that for all the tested beams, the laboratory results are generally smaller than the theoretical values calculated according to [15, 16, 17]. Analysing the results for beams B1, B2, B3, and B4, the fastest crack propagation can be observed at the level of 30% to 70% of the resistance moment. For the theoretical values derived for EC 2 and ACI the crack widths propagate to about 70% for the resistance moment, and then they stabilize. The top branches of both diagrams are parallel. EC2 and ACI define the maximum crack widths. Also the maximum crack widths from the laboratory test at a certain level stabilize. At the level of 85% of resistance moment, there is a sudden increase and drop of the laboratory measured maximum crack widths. It could be originated by slip of the reinforcement in zones, where the bond between concrete and steel was lost. Maximum crack widths from the laboratory results are lower than maximum values calculated based for the design codes [16, 17]. According to PN-B-03264, the crack widths are calculated as average values. Those values are close
to the averages of test results (see figures 2 to 5). Regardless of the approach chosen for the calculation of crack width, the results can be considered safe.

**Figure 2.** Crack widths in beam B1  
**Figure 3.** Crack widths in beam B2  
**Figure 4.** Crack widths in beam B3  
**Figure 5.** Crack widths in beam B4

5. Conclusions

Cracks do not influence resistance of structures but they are an important factor affecting durability and appearance of structures. Crack propagation is a complex process developing during the entire time of loading of an element. The deterministic approaches to calculation of crack width, recommended in the design codes, are a simplification, which is conservative and convenient in engineering practice.

In this paper, the crack widths were calculated for three design codes (PN-B-03264, EC2, ACI 318) in relation to the applied load level and compared with laboratory test results for four RC beams. The tested simply supported beams were loaded in multiple steps with two concentrated forces at 1/3 of the span length. The results were presented in the form of tables and diagrams, and discussed. The fastest crack propagation was observed at 30% to 70% of the resistance moment. The diagrams for EC2, ACI, and the results of the laboratory tests have a similar pattern with the horizontal top branch when they stabilize. A sudden increase and drop of the laboratory measured maximum crack widths at the level of 85% of resistance moment was justified by the bond-slip. Generally, the analytical values were smaller than the experimental ones, which were considered conservative. For more detailed comparison of the level of safety provided by the design codes for the selected limit state, the safety factors on both sides of the performance function, the resistance and the effects of action, should be taken into consideration.
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