Transient Fluid Dynamics of the Quark-Gluon Plasma According to AdS/CFT

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We argue, using the AdS/CFT correspondence, that the transient dynamics of the shear stress tensor in a strongly coupled $\mathcal{N} = 4$ SYM plasma is not described by relaxation-type, fluid dynamical equations: at long times the equations of motion should contain a second-order comoving derivative of the shear stress tensor. This occurs because in this strongly-coupled system the lowest “non-hydrodynamical” quasinormal modes associated with shear stress possess a nonzero real part at zero wavenumber. We use Weyl invariance to obtain the most general equations of motion containing 2 comoving derivatives of the shear stress tensor in the transient regime that are compatible with the symmetries. We show that the asymptotic solution of this theory valid at times much larger than the timescale associated with the “non-hydrodynamical” modes reproduces the well-known results previously obtained directly from the AdS/CFT correspondence. If the QGP formed in heavy ion collisions can be at least qualitatively understood in terms of strongly-coupled $\mathcal{N} = 4$ SYM theory, the second time derivative present in the equations of motion of the fluid may lead to an unexpected dependence on the initial conditions for the shear stress tensor needed in numerical hydrodynamic simulations.
I. INTRODUCTION

Relativistic fluid dynamical models have played a key role in our current understanding of the nearly perfect fluid behavior displayed by the Quark-Gluon Plasma (QGP) formed in heavy ion collisions \[1\]. In these models, exact energy-momentum conservation, \( \nabla_\mu T^{\mu \nu} = 0 \), is supplied by another phenomenological dynamical equation for the macroscopic shear stress tensor, \( \pi^{\mu \nu} \), which is defined as

\[
\pi^{\mu \nu} \equiv T^{\mu \nu} - \varepsilon u^{\mu} u^{\nu} + P \Delta^{\mu \nu},
\]

where \( T^{\mu \nu} \equiv \langle \hat{T}^{\mu \nu} \rangle \), \( \varepsilon \) is the local energy density, \( P \) is the local pressure, \( u^{\mu} \) is the local fluid 4-velocity, and \( \Delta^{\mu \nu} = g^{\mu \nu} - u^{\nu} u^{\mu} \) is a spatial projector (our metric signature is (+, −, −, −)), i.e., \( u^{\mu} \Delta_{\mu \nu} = 0 \) (we shall not consider here the contribution from the bulk viscous pressure or the effects from nonzero baryonic density). Since we are going to consider only conformal fluids in this paper, the trace of \( \pi^{\mu \nu} \) is equal to zero. In fact, in terms of the doubly symmetric and traceless projection operator, \( \Delta^{\mu \alpha \beta} = (\Delta^{\mu \alpha} \Delta^{\nu \beta} + \Delta^{\mu \beta} \Delta^{\alpha \nu}) / 2 - \Delta^{\mu \nu} \Delta^{\alpha \beta} / 3 \), one can see that \( \Delta^{\mu \nu} \pi^{\alpha \beta} = \pi^{\mu \nu} \) and \( u_\mu \pi^{\mu \nu} = 0 \).

In relativistic fluids causality is intimately connected to stability \[2, 3\] and Israel and Stewart \[4\] were among the first to understand that the characteristic times within which fluid dynamical dissipative currents, such as \( \pi^{\mu \nu} \), relax towards their asymptotic Navier-Stokes values cannot be arbitrarily small. The so-called Israel-Stewart (IS) equations for the shear stress tensor are relaxation-type equations (in terms of the co-moving derivative \( d / d \tau \equiv u^\mu \nabla_\mu \)) of the following general form

\[
\tau_1 \dot{\pi}^{(\mu \nu)} + \pi^{\mu \nu} = 2 \eta \sigma^{\mu \nu} + \ldots,
\]

where \( \tau_1 \) is the relaxation time coefficient, \( \eta \) is the shear viscosity, \( \sigma^{\mu \nu} \equiv \nabla_\mu u^\nu \) is the shear tensor, \( \nabla_\mu \equiv \Delta^\mu_\nu \nabla^\nu \), \( \hat{A}^{(\mu \nu)} = \Delta^{\alpha \beta} d A_{\alpha \beta} / d \tau \), and the dots denote nonlinear terms involving \( \pi^{\mu \nu} \) and gradients of \( T \) and \( u^\mu \) \[4, 5\]. The major step taken by Israel and Stewart in \[4\] was to realize that causality demands that the dissipative currents obey dynamical equations of motion (which introduce the linear transport coefficient \( \tau_1 \)) that describe their transient dynamics towards their respective asymptotic relativistic Navier-Stokes solution. A few years earlier, Kadanoff and Martin \[6\] argued that a similar relaxation transport coefficient should appear in the description of spin diffusion in a way that is consistent with well-known sum rules.

Using the 14-moments method, it is possible to show \[4, 5\] that in relativistic gases \( \tau_1 \) is of the order of the microscopic collision time. Therefore, one expects that in physical situations where the time variation of the fluid flow is comparable to this microscopic scale, the fluid is in the transient regime where the relaxation dynamics described by \( \tau_1 \) becomes important. For dilute gases, such as air, under normal circumstances the collision time is orders of magnitude smaller than the typical time variation of the flow. However, it is possible to create physical systems under which the flow of a fluid varies in a timescale of the order of the mean free time (such as in microflows \[7\]). Given the rapid expansion experienced by the QGP formed in heavy ion collisions, it is reasonable to investigate the dependence of hydrodynamic predictions on the actual value of \( \tau_1 \) (see, for example, \[8\]).

If the dynamical properties of the strongly-coupled QGP can be (at least qualitatively) understood using \( N = 4 \) Supersymmetric Yang Mills (SYM) theory, we shall see in this paper that this would imply that the relaxation equations for \( \pi^{\mu \nu} \) commonly used in numerical simulations must be replaced by new equations for transient dynamics involving second-order comoving derivatives of \( \pi^{\mu \nu} \).

II. NON-HYDRODYNAMIC POLES AND TRANSIENT FLUID DYNAMICS

It is well-known that retarded correlators can have singularities such as simple poles and also branch cuts. Of particular relevance for fluid dynamics are the so-called “hydrodynamic” poles, \( \omega_0 (k) \), which appear in retarded correlators of conserved currents \[9\]. The \( k \) dependence of these modes can be used to obtain the corresponding diffusion transport coefficient \( D \) associated with a given conserved quantity through the relation \( \lim_{k \to 0} \omega_0 (k) \sim -i D k^2 + \ldots \[9\] \). The hydrodynamic modes are characterized by the momentum dependence \( \sim -i k^2 \) and, consequently, vanish in the limit \( k \to 0 \). Because of their appearance in Navier-Stokes theory, the existence of hydrodynamic modes is quite often taken as evidence for fluid dynamical behavior. Furthermore, modes that do not share this behavior, i.e., modes in which \( \lim_{k \to 0} \omega_0 (k) \neq 0 \), are known as “non-hydrodynamic modes”.

Surprisingly enough, a stable theory of relativistic fluid dynamics cannot be formulated using only hydrodynamic modes. In order to obtain a causal and stable theory of fluid dynamics, the shear stress tensor has to be promoted to an independent dynamical variable, as in Israel and Stewart theory, e.g., Eq. \( 2 \). Since the shear stress tensor is not a conserved quantity, when it becomes an independent dynamical variable non-hydrodynamic modes must appear in the
theory. In the case of Israel-Stewart theory, the non-hydrodynamic modes describe the relaxation of the dissipative currents towards their respective Navier-Stokes asymptotic solutions and they can be directly related to the relaxation times \(\tau_1\).

Clearly, in order to fully describe the complicated many-body dynamics of an interacting system down to arbitrarily small time scales, all the infinite number of non-hydrodynamic modes must be taken into account. For sufficiently long times, however, only the slowest modes should contribute significantly and one should be able to systematically neglect the effect of faster modes to the system’s dynamics (at infinite times, no non-hydrodynamic mode should be required at all). This type of truncation should be possible at sufficiently long times and does not depend on whether or not the non-hydrodynamic modes are parametrically separated (as long as the distribution of poles is discrete).

As a matter of fact, it was shown in Ref. [10] that the long-distance, long-time linearized dynamics of the shear stress tensor in any system that can be described via the Boltzmann equation must follow the general ansatz from Israel and Stewart, Eq. 2. In other words, at long times \(\pi^{\mu\nu}\) must obey a first-order differential equation in the comoving derivative that describes how it relaxes towards its steady state Navier-Stokes value. This means that in relativistic dilute gases the relaxation time can be extracted directly from the microscopic theory and does not necessarily have to be considered as a regulator for the gradient expansion, as advocated in Ref. [11] (for a recent discussion see Ref. [12]). We remark that this relaxation behavior at long times should not be taken for granted: it is a direct consequence of the fact that the retarded Green’s function associated with shear stress tensor, obtained via the linearized Boltzmann equation, is a meromorphic function where all the poles lie on the (negative) imaginary axis and, in this case, for times \(t_\omega(0) \gg 1\) one obtains that \(\tau_1 = -i/\omega_1(0)\) with \(\omega_1\) being the non-hydrodynamic mode with smallest frequency.

The near-equilibrium dynamics of weakly-coupled QCD at very large \(T\) is expected to be described by a Boltzmann equation involving quark and gluon scattering. Therefore, according to the general discussion in Ref. [10], one should expect that deep in the deconfined phase long distance, long time disturbances of the shear stress tensor follow IS-type equations although the correct value of \(\tau_1\) in this case, as determined by the first non-hydrodynamic pole of the retarded Green’s function, has not yet been computed.

Sufficiently below the phase transition, say \(T \sim 130\) MeV, lattice data for QCD thermodynamics seems to be consistent with the predictions from non-interacting hadron resonance models. Thus, it is natural to assume that at low \(T\) QCD behaves as a weakly coupled gas of hadrons and resonances, which then implies that a description via the Boltzmann equation (including different particle species) may be appropriate. Using the formalism derived in Ref. [10], it is possible to show that the total shear stress tensor in this case, which is a sum over all the hadronic species, follows an IS-like equation of motion with a single relaxation time coefficient given by the slowest non-hydrodynamic mode. Therefore, we expect that in QCD the non-hydrodynamic pole of the retarded Green’s function closest to the origin should be a purely imaginary number at very high or low \(T\) and in the transient regime one will find relaxation equations for \(\pi^{\mu\nu}\).

However, it is clear that the surprising “perfect fluid” character of the QGP appears not at low or high temperatures (where \(\eta/s \sim 1\)) but rather near the phase transition, say \(T \sim 160 - 200\) MeV, where the number of degrees of freedom increases very rapidly and \(\eta/s\), the figure of merit for perfect fluid behavior, is expected to be rather small [13]. Given that the correct physical mechanism that leads to the perfect fluid behavior of the QGP is not yet known and that the relevant \(t'\)Hooft coupling near the phase transition, \(\lambda_{QCD} \equiv g_{QCD}^2 N_c \sim 10\) for \(N_c = 3\) and \(\alpha_{QCD} \sim 0.3\), we find it useful to investigate what the AdS/CFT correspondence [16] has to say about the fluid dynamical equations for the shear stress tensor in strongly-coupled gauge theories in the transient regime. In other words, how do strongly coupled plasmas (which possess gravity duals) relax towards their asymptotic, universal Navier-Stokes solution? This question will be answered in the following sections.

III. WEYL INVARIANCE AND THE EQUATIONS OF MOTION FOR THE \(\mathcal{N} = 4\) SYM FLUID IN THE TRANSIENT REGIME

Recently, a new way to derive the equations of motion of relativistic fluid dynamics based on Weyl invariance was put forward by Baier et al in Ref. [11]. The main idea is to use the fact that the dynamics of conformal plasmas (with equations of motion involving less than 4 derivatives) should be invariant under Weyl transformations in which the metric changes as \(g_{\mu\nu} \rightarrow g_{\mu\nu}(x) e^{-2\Omega(x)}\), and \(\Omega(x)\) is an arbitrary scalar function. Since the energy momentum tensor scales classically, it is easy to prove that under a Weyl transformation it scales homogeneously with conformal weight equals 6, i.e, \(T^{\mu\nu} \rightarrow e^{6\Omega} T^{\mu\nu}\). The basic hydrodynamic variables change under Weyl transformations as follows: from \(u_\mu u^\mu = 1\) one obtains that \(u^\mu \rightarrow e^{\Omega} u^\mu\) and, since the ideal energy-momentum tensor has conformal weight equals 6, the temperature scales as \(T \rightarrow e^{\Omega} T\) and, thus, the dissipative part of the energy-momentum tensor also transforms homogeneously, i.e, \(\pi^{\mu\nu} \rightarrow e^{6\Omega} \pi^{\mu\nu}\).

It is important to notice, however, that while \(T\) and \(u^\mu\) scale homogeneously, their conventional spacetime covariant derivative does not. Things get significantly easier with the aid of the Weyl covariant derivative defined in Ref. [17], which
acts on the basic hydrodynamic variables as follows

\[ D_\alpha T = \nabla_\alpha T - T \dot{u}_\alpha + \frac{u_\alpha \theta T}{3}, \]
\[ D_\alpha u^\nu = \nabla_\alpha u^\nu - u_\alpha \dot{u}^\nu - \frac{\Delta^\nu_\alpha \theta}{3}, \]  
(3)

where \( \theta = \nabla_\nu u^\nu \). In this case, these derivatives are homogeneous under Weyl transformations, i.e., \( D_\alpha T = e^{\Omega} D_\alpha T \) and \( D_\alpha u^\nu = e^{\Omega} D_\alpha u^\nu \). Defining the coming Weyl invariant derivative as \( D = u_\mu D^\mu \), one can see that \( D u^\nu = 0 \), \( D_\mu u^\mu = 0 \), and \( DT = \dot{T} + \theta T/3 \). Moreover, since \( \pi^{\mu\nu} \) is traceless and of conformal weight equals 6, it is possible to show that \( D_\alpha \pi^{\alpha\beta} = D_\alpha \pi^{\alpha\beta} - u^\alpha \delta^{\alpha\beta} + u^\mu \pi_{\alpha\beta}^{\mu} \) and we can now write the general conservation laws for a conformal fluid (where \( \varepsilon = 3\rho \)) as

\[ DP = \frac{\pi_{\alpha\beta} \sigma^{\alpha\beta}}{3}, \]
\[ D^\mu_\alpha P = D_{\perp \alpha} \pi^{\alpha\mu} + u^\mu \pi_{\alpha\beta}^{\mu} \sigma^{\alpha\beta}, \]  
(4)

A. The Gradient Expansion Approach in Conformal Fluid Dynamics

In order to solve the conservation laws \( \square \) we must provide the equation satisfied by the shear stress tensor. One possibility to derive such equation is via the gradient expansion in which \( \pi^{\mu\nu} \) is assumed to be solely expressed in terms of \( P \) (or temperature), \( u^\mu \) and their gradients. In this framework, it is possible to express \( \pi^{\mu\nu} \) in terms of a controlled expansion in powers of derivatives or order of derivatives of \( P \) and \( u^\mu \),

\[ \pi^{\mu\nu} = \eta_1 \Pi_1^{\mu\nu} + \eta_2 \Pi_2^{\mu\nu} + \cdots , \]
(5)

where the quantities \( \Pi_1^{\mu\nu} \) and \( \Pi_2^{\mu\nu} \) correspond to terms of first and second order in gradients of \( P \) and \( u^\mu \), respectively, and the dots denote possible terms with higher order derivatives.

This derivative expansion is controlled by a small parameter called the Knudsen number, \( \text{Kn} = \ell_{\text{micro}}/L_{\text{macro}} \), which is basically the ratio between a microscopic length scale (e.g., the inverse temperature for conformal fluids or the mean free-path for gases) \( \sim \ell_{\text{micro}} \) and the overall macroscopic length scale of the fluid \( \sim L_{\text{macro}} \) (the inverse of the gradient of velocity or temperature). The term \( \Pi_1^{\mu\nu} \) is assumed proportional to the gradient of a macroscopic variable and should be of order \( \text{Kn}^{\frac{1}{2}} \). Every additional derivative brings in another inverse power of \( L_{\text{macro}} \) and, thus, \( \Pi_2^{\mu\nu} \sim L_{\text{macro}}^{-1} \). The microscopic scale \( \ell_{\text{micro}} \) is contained in the coefficients \( \eta_i \). Up to some overall power of \( \ell_{\text{micro}} \) (which restores the correct scaling dimension), \( \eta_i \sim \ell_{\text{micro}}^n \). Therefore, the terms \( \Pi_1^{\mu\nu} \) and \( \Pi_2^{\mu\nu} \) multiplied by their corresponding coefficients in Eq. \( \square \) are of order \( \text{Kn} \) and \( \text{Kn}^2 \), respectively. Subsequent terms would be of a higher order in \( \text{Kn} \) and, therefore, when the system exhibits a clear separation between \( \ell_{\text{micro}} \) and \( L_{\text{macro}} \), i.e., when \( \text{Kn} \ll 1 \), it is possible to truncate this expansion. Ideal fluid dynamics corresponds to the zeroth order truncation of this series, i.e., when no terms are included at all.

The first term in the gradient expansion can be obtained by constructing all possible tensors that can be made using first order derivatives of \( P \) and \( u^\mu \). These can be easily obtained and are:

\[ D_\mu P \quad \text{and} \quad D_\mu u^\nu . \]
(6)

Next, one has to build, using these gradients, tensors that have the same properties satisfied by \( \pi^{\mu\nu} \) and, at the same time, transform homogeneously under Weyl transformations. Here, the only possibility is the shear tensor, which can be written in terms of the Weyl derivative of \( u^\mu \) as

\[ \sigma^{\mu\nu} = D^{\mu}_{\alpha\beta} \nabla^\alpha u^\beta = \frac{D^\mu_\alpha u^\nu + D^\nu_\alpha u^\mu}{2}. \]

(7)

Therefore, the most general equation allowed by symmetry that can be satisfied by \( \sigma^{\mu\nu} \), up to first order in \( \text{Kn} \), is

\[ \pi^{\mu\nu} = \eta_1 \sigma^{\mu\nu}, \]

which corresponds to the relativistic Navier-Stokes theory, with \( \eta_1/2 \) being identified as the shear viscosity coefficient \( \eta \). It is easy to see that the shear tensor scales as \( \sigma^{\mu\nu} \sim e^{2\text{Kn} \sigma^{\mu\nu}} \) and, therefore, \( \eta \sim T^3 \).
In the framework of the gradient expansion, relativistic Navier-Stokes theory can be extended by including terms of second order in gradients of \( P \) and \( u^\mu \). In order to do so, one has to obtain all the possible terms involving Weyl derivatives of \( P \) and \( u^\mu \) that contribute to \( \Pi^{\mu\nu}_{\text{eff}} \). All the independent terms of second order in gradients of \( P \) and \( u^\mu \) that are symmetric, transverse, traceless, and that transform homogeneously under Weyl transformations are,

\[
\begin{align*}
O_1^{\mu\nu} &= D\sigma^{(\mu\nu)} = \delta^{(\mu\nu)} + \sigma^{\mu\nu} \theta/3, \\
O_2^{\mu\nu} &= R^{(\mu\nu)} + 2u_\alpha R^{(\alpha\nu\mu)} u_\beta, \\
O_3^{\mu\nu} &= \sigma^{(\mu\nu)} \lambda, \\
O_4^{\mu\nu} &= \sigma^{(\mu\nu)} \Omega^\nu, \\
O_5^{\mu\nu} &= \Omega^{(\mu\nu)},
\end{align*}
\]

where \( \Omega^{\mu\nu} = (\nabla_+^{\mu} u^\nu - \nabla_+^{\nu} u^\mu)/2 \) is the vorticity operator, \( R^{\mu\nu} \) is the Ricci tensor, and \( R^{\mu\alpha\nu\beta} \) is the Riemann tensor. All the terms above have conformal weight 4 and were first found and listed in Ref. [11]. Note that terms such as \( \Delta^{\mu\nu\rho} \Delta^{\rho\nu\alpha} P \), \( \Delta^{\mu\nu\rho} (D^\beta P)(D^\alpha P) \), and \( \Delta^{\mu\nu\rho\sigma} \sigma^{\alpha\beta} \) contribute only to \( \mathcal{O}(K^3) \), as can be seen by substituting the leading order relation \( \pi^{\mu\nu} \sim \sigma^{\mu\nu} \) together with the general conservation laws [4].

Therefore, the most general equation allowed by symmetry that can be satisfied by \( \pi^{\mu\nu} \), up to second order in \( K \), is

\[
\pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} - \sum_{i=1}^{5} 2\eta b_i O_i^{\mu\nu} + \mathcal{O}(K^3).
\]

The 6 coefficients \( \eta \) and \( b_i \) can be calculated via Kubo formulas for the correlators of the energy-momentum tensor derived using metric perturbations. In strongly-coupled \( \mathcal{N} = 4 \) SYM theory, all the coefficients above (including those associated with nonlinear terms) were determined using the AdS/CFT correspondence [11, 18, 19]. For instance, for strongly-coupled SYM one finds \( \eta(2\pi T)/P_0 = 2 \) \( (P_0 \sim T^4 \) is the pressure at equilibrium) and \( b_1(2\pi T) = 2 - \ln 2 \) [11].

### B. Going Beyond the Gradient Expansion via the Inclusion of Transient Effects in Relativistic Fluid Dynamics

Relativistic Navier-Stokes theory and its extensions via the gradient expansion are hindered by acausal behavior which complicates their usage in relativistic problems [2]. In Ref. [11], a stable and causal fluid-dynamical theory was obtained from the gradient expansion by substituting in all second-order terms the “inverted” first-order solution, \( \sigma^{\mu\nu} = \pi^{\mu\nu}/(2\eta) \). Then, the following equation of motion for \( \pi^{\mu\nu} \) appears,

\[
b_1 D\pi^{(\mu\nu)} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} - 2\eta b_2 \sigma^{\mu\nu} - 2\eta b_3 \tilde{\sigma}^{\mu\nu} - 2\eta b_4 \check{\sigma}^{\mu\nu} - 2\eta b_5 \tilde{\sigma}^{\mu\nu},
\]

where \( \check{\sigma}^{\mu\nu} \) corresponds to \( \sigma^{\mu\nu} \) with the substitution \( \sigma^{\mu\nu} \rightarrow \pi^{\mu\nu}/(2\eta) \) and we used that \( DT \sim \mathcal{O}(K^2) \). Note, however, that in order to render the gradient expansion stable, the shear stress tensor had to be promoted to an independent dynamical variable. On the other hand, Eq. (10) was proved to be the most general equation allowed by symmetry only when \( \pi^{\mu\nu} \) was not an independent dynamical variable. Therefore, for causal theories of fluid dynamics the analysis first proposed in Ref. [11] and reproduced in the previous section has to be revisited.

In this section, we use Weyl invariance to obtain the full set of nonlinear differential equations that describe a conformal fluid when the shear stress tensor is a dynamical variable. Now, the idea is to extend Navier-Stokes theory by including all possible terms that can be constructed from gradients of \( P \), \( u^\mu \), and \( \pi^{\mu\nu} \) that are symmetric, transverse, traceless, and that transform homogeneously under Weyl transformations. Then, in addition to the terms constructed in the previous section, we can also build new terms, e.g.,

\[
D\pi^{(\mu\nu)}, \ D^2\pi^{(\mu\nu)}, \ldots, \pi^{(\mu\nu)} \alpha, \ \pi^{(\mu\nu)\alpha}, \ \pi^{(\mu\nu)\alpha}, \ldots.
\]

(11)

Note, however, that the shear stress tensor can no longer be expressed in terms of a series in powers of Knudsen number. By including terms of the form, e.g. \( D\pi^{(\mu\nu)} \) and \( D^2\pi^{(\mu\nu)} \), the shear stress tensor satisfies a partial differential equation and its relation with the Knudsen number is dynamical, as happens with Israel-Stewart theory, and not algebraic, as occurred in the gradient expansion. We organize the most general equation of motion for \( \pi^{\mu\nu} \) in the following form,

\[
\begin{align*}
\ldots + \chi_2 D^2\pi^{(\mu\nu)} + \chi_1 D\pi^{(\mu\nu)} + \pi^{\mu\nu} \\
= 2\eta\varphi^{\mu\nu} + e_1 \pi^{(\mu\nu)} \alpha + e_2 \pi^{(\mu\nu)} \alpha + e_3 \pi^{(\mu\nu)} \alpha - \sum_{i=1}^{5} 2\eta c_i O_i^{\mu\nu} + \ldots,
\end{align*}
\]

(13)
where the dots denote additional possible terms. Note that,

\[ D\pi^{(\mu\nu)} = \frac{4}{3}\pi^{\mu\nu}\theta , \]

\[ D^2\pi^{(\mu\nu)} = \frac{4}{3}\pi^{\mu\nu} + \frac{20}{9}\pi^{\mu\nu}\theta + \frac{4}{3}\pi^{\mu\nu}\theta . \]  

(14)

The truncation of Eq. (13) is not trivial, as was discussed in Ref. [10]. The terms on the right hand side serve as source terms for the shear stress tensor while the terms on the left hand side describe the relaxation/oscillation of the shear stress tensor when perturbed by gradients. The right hand side of Eq. (13), i.e., the source terms, can be organized as a series in Knudsen number and the so-called inverse “Reynolds number” \( \text{Re}^{-1} = |\pi^{\mu\nu}\pi_{\mu\nu}|^{1/2}/P_0 \). Since \( \pi^{\mu\nu} \) is an independent dynamical variable, the inverse Reynolds number can be considered as an independent small parameter that gives additional information on how equilibrium is approached. Therefore, it is possible to systematically organize the source terms of the equation of motion as an expansion in powers of both \( \text{Kn} \) and \( \text{Re}^{-1} \).

In this case, the terms \( e_1\pi^{(\mu\nu)}_\alpha \) and \( e_3\pi^{(\mu\nu)}_\alpha \), the term \( e_2\pi^{(\mu\nu)}_\alpha \), and the terms \( \eta\gamma O_\mu^{\mu\nu} \), are all the possible terms of order \( O(\text{Re}^{-1}\text{Kn}) \), \( O(\text{Re}^{-2}) \), and \( O(\text{Kn}^2) \), respectively. If we wish to describe the source terms only up to order \( O(\text{Re}^{-2}, \text{Re}^{-1}\text{Kn}, \text{Kn}^2) \), they are enough.

The truncation of the left hand side is more complicated since it cannot be organized as an algebraic series in powers of small quantities, such as Knudsen number or inverse Reynolds number. However, the order of the differential equation in the comoving derivative on the left hand side is equal to the number of non-hydrodynamic modes included in the dynamical description of the system. For example, if we include only the first comoving derivative of \( \pi^{\mu\nu} \), e.g. \( D\pi^{(\mu\nu)} \), we have only one non-hydrodynamic mode, while if we also include the second order comoving derivative we would have two non-hydrodynamic modes.

The main purpose of the gradient expansion is to correct Navier-Stokes theory in cases where the Knudsen number is not very small, i.e., the microscopic scale is no longer very separated from the macroscopic scales of interest. By including second order gradients of \( P \) and \( u^\mu \), the Navier-Stokes theory is extended to describe the dynamics at larger wavenumbers or smaller wavelengths. However, if the separation between the microscopic and macroscopic scales is no longer optimal, it is not enough to extend the applicability of the theory to describe higher wavenumbers, but one should also extend it to describe higher frequencies. This is the role played by the left hand side of the Eq. (13). When more comoving derivatives of \( \pi^{\mu\nu} \) are included, more non-hydrodynamic modes are introduced in the theory, and a description of higher frequencies is obtained. Therefore, the structure of the left hand of Eq. (13) has to be determined by carefully matching the modes introduced in the macroscopic theory with the modes of the underlying microscopic theory, taking into account what are the relevant frequencies in the macroscopic domain. Also, one has to make sure that such matching can be done, i.e., the modes included in the macroscopic theory exist in the microscopic one.

For dilute gases described by the Boltzmann equation, all the non-hydrodynamic modes lie on the imaginary axis in the complex \( \omega \)-plane (in the limit of vanishing wavenumber) [10]. In the long-time limit it is only necessary to include the non-hydrodynamic mode with the smallest frequency (at zero wavenumber) and the equation of motion for \( \pi^{\mu\nu} \) with source terms up to order \( O(\text{Re}^{-2}, \text{Re}^{-1}\text{Kn}, \text{Kn}^2) \) becomes

\[ \tau_1 D\pi^{(\mu\nu)} + \pi^{\mu\nu} = 2\eta u^{\mu\nu} + e_1\pi^{(\mu\nu)}_\alpha \alpha + e_2\pi^{(\mu\nu)}_\alpha \alpha + e_3\pi^{(\mu\nu)}_\alpha \alpha - \sum_{i=1}^{5} 2\eta c_i O_i^{\mu\nu} . \]  

(15)

One should remark that, in general, the coefficients \( c_i \)’s in the equation above are different than the \( b_i \)’s in Eq. (9). One can see that the transient theory defined in Eq. (15) reduces to the well known result (9) in the limit of vanishing relaxation time. In this limit, we can obtain an asymptotic solution for \( \pi^{\mu\nu} \) by substituting the first order solution \( \pi^{\mu\nu} \approx 2\eta u^{\mu\nu} \) into all terms in Eq. (15), which then implies that, asymptotically, \( \tau_\pi D\pi^{(\mu\nu)} \sim 2\eta \pi^{(\mu\nu)} + O(\text{Kn}^3) \). In fact, one can relate the new coefficients with those in Eq. (9) as follows: \( b_1 = \tau_1 + c_1, b_2 = c_2, b_3 = c_3 - c_1 - c_2, b_4 = c_4 - c_3, \) and \( b_5 = c_5 \). Therefore, Eq. (15) leads to the appropriate asymptotic limit up to \( O(\text{Kn}^2) \). Also, one can show that the general theory (in flat spacetime) obtained from the Boltzmann equation using the moments method, as recently derived in [20], has the exact same form as (15) in the conformal limit (massless limit and cross section \( \sigma \sim 1/T^2 \)). Note also that Eq. (16) can be seen as a particular case of our transient theory in which \( c_1 = c_3 = c_4 = 0 \).

Equation (15) can be extended by including one more comoving derivative of \( \pi^{\mu\nu} \),

\[ 2\eta c_i O_i^{\mu\nu} + \sum_{i=1}^{5} f_i \pi^{(\mu\nu)}_i \rho - \xi D^{(\mu} D\pi^{\nu)} \chi = 0 . \]  

(16)
where \(D^{(\mu} D^{\nu)} \pi^\lambda \) is the only source term found of conformal weight 8,

\[
D^{(\mu} D^{\nu)} \pi^\lambda = \Delta^\mu_\alpha^\nu_\beta (\nabla^\alpha - 6\eta^\alpha) \nabla^\lambda \pi^{\lambda\beta}.
\] (17)

Above, we included source terms of order \(O(Re^{-1}Kn), O(Re^{-2}), O(Kn^2)\), and \(O(Re^{-1}Kn^2)\). In principle, we could have also included source terms of order \(O(Re^{-3})\) and \(O(Kn^3)\), but this is time consuming and outside the purposes of this paper. Again, it is easy to see that the theory displayed above reproduces the \(O(Kn^2)\) gradient expansion obtained in [11], since all the new terms included are of order \(O(Kn^3)\) when the asymptotic solution \(\pi^{\mu\nu} \sim 2\eta^{\mu\nu}\) is substituted.

It is interesting to observe that, since the transient theories in (15) and (16) reduce to (9), several results previously derived using (9) are automatically valid also for the transient theories derived here. For instance, the expansion around \(k \to 0\) for the sound mode present in the theories defined via Eqs. (15), (16), and (9) is

\[
\omega_{\text{sound}}(k) = \pm \frac{k}{\sqrt{3}} - i \frac{k^2 \eta}{6P_0} \pm \frac{k^3}{6\sqrt{3}} \left( \frac{b_1 \eta}{P_0} - \frac{\eta^2}{4P_0^2} \right) + O(k^4).
\] (18)

As mentioned above, the transient theory obtained from the Boltzmann equation taking into account only the slowest non-hydrodynamical mode assumes the form of Eq. (15). We shall see in the following that for the strongly coupled SYM fluid the transient theory assumes the form displayed in Eq. (16), i.e., it includes at least the 2 slowest non-hydrodynamic modes.

IV. NON-HYDRODYNAMIC POLES IN AN \(N = 4\) SYM PLASMA

The analytical properties of thermal retarded correlators at strong coupling and their calculation via the AdS/CFT correspondence [16] have been discussed in the literature in great detail [21, 22]. Through the duality [23], the poles of the retarded thermal 2-point function of \(T^{xy}\) correspond to the quasinormal frequencies in the so-called scalar channel, which amounts to solving the equation of motion for a massless scalar field minimally coupled to gravity in the bulk [24]. It was found that for strongly coupled \(N = 4\) SYM theory the poles always come in pairs with the same imaginary part and opposite real parts [25], which is very different than the weak coupling behavior inferred from the Boltzmann equation [10] (see Fig. 1). It would be interesting to check if this retarded correlator in weakly coupled \(N = 4\) SYM indeed possesses poles only on the imaginary axis. As shown in [23], due to rotational invariance, the three independent Green’s functions that describe the fluctuations become identical at zero wavenumber and, thus, in this limit the non-hydrodynamical poles in all of these channels become the same.

In the supergravity approximation, the retarded correlator of the glueball operator \(TrF^2\) in strongly coupled \(N = 4\) SYM is also found by solving the same equation of motion for a minimally coupled scalar in the bulk. This occurs because both operators have their UV scaling dimension equals to 4, which corresponds to a massless scalar field in the bulk [16]. The poles of this glueball correlator determine the masses and the decay properties of the glueballs at finite \(T\). It is clear in this context, however, that the fundamental mode of this glueball correlator must indeed be doubly degenerate. This occurs because the mass of the state, which is nonzero, only appears as \(m^2\) (i.e., the poles have opposite real parts) and in the deconfined phase glueballs must decay (hence the poles must have a nonzero imaginary part). Therefore, while the numerical value of the poles of the \(TrF^2\) (or \(\tilde{T}^{xy}\)) correlator will change according to the conformal theory used in the calculation, the general structure discussed above concerning the distribution of the poles in the complex plane should remain valid for conformal plasmas. Note, however, that while in the supergravity approximation the singularities of the Green’s functions do not depend on the ’t Hooft coupling or \(N_c\), the analytical properties of the Green’s functions are expected to change outside the supergravity limit (see, for instance, the discussion in [26]). It would be interesting to investigate the analytical structure of this correlator in non-conformal plasmas described by bottom-up gravity theories constructed to mimic some properties displayed by QCD at nonzero temperature [27].

A crucial insight derived in [10], which will be used extensively here, was that the zero wavenumber limit of the poles of the Green’s function determines the coefficients of the linear terms in the equations of motion of relativistic dissipative fluid dynamics.

V. LINEARIZED FLUID DYNAMIC EQUATIONS FOR THE \(N = 4\) SYM PLASMA

We choose to derive the macroscopic equations of motion for \(\Pi^{\mu\nu}\) via linear response to small metric perturbations, as discussed in [21]. The linear transport coefficients are determined from perturbations \(h^{\mu\nu}\) of the metric tensor
\[ \eta^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu} \] in the gauge theory. While this method can be equally used in weak and strongly coupled gauge theories, we will focus on the results for strongly-coupled $N = 4$ SYM. Within linear response, the variation of the energy-momentum tensor $T^{\mu\nu}$ due to the metric perturbations is

\[ \delta T^{\mu\nu}(X) = \frac{1}{2} \int_{-\infty}^{\infty} d^4X' \, G^{\mu\nu\alpha\beta}_{R}(X - X') \, h_{\alpha\beta}(X') , \]  

(19)

where $G^{\mu\nu\alpha\beta}_{R}(X - X')$ is the retarded Green’s function, whose properties will be obtained from the AdS/CFT correspondence. We consider only the following metric perturbation of $h_{xy}$ and all other components of the metric tensor left unperturbed (11). In this case, all the other components of $\delta T^{\mu\nu}$ decouple from the $xy$ component and we obtain the following expression for $\delta T^{xy}$

\[ \delta T^{xy}(t, z) = \int_{-\infty}^{\infty} dt' \, dz' \, G^{xy}_{R}(t' - t, z' - z) \, h_{xy}(t', z') . \]  

(20)

The energy-momentum tensor $T^{\mu\nu}$ is assumed to have the traditional fluid-dynamical structure shown in Eq. (11), which then implies that

\[ \delta T^{xy} = T^{xy}(\eta^{\mu\nu} + h^{\mu\nu}) - T^{xy}(\eta^{\mu\nu}) = -P_{0} h^{xy} + \delta \pi^{xy} . \]  

(21)

where $\delta \pi^{xy}$ is the $xy$ component of the shear stress tensor created by the metric perturbations and $P_{0}$ is the pressure of the unperturbed state. We set the shear stress tensor of the unperturbed state to zero. Using the energy-momentum equations of motion we arrive at the following equation (10)

\[ \delta \pi^{xy} = P_{0} h^{xy} + \int_{-\infty}^{\infty} dt' \, dz' \, G^{xy}_{R}(t' - t, z' - z) h_{xy}(t', z') . \]

or, equivalently, in Fourier space, $\delta \pi^{xy}(t, z) = 1/(2\pi)^{2} \int_{-\infty}^{\infty} d\omega \, dk \, e^{-i\omega t + ikz} \, \delta \tilde{\pi}^{xy}(\omega, k)$ where

\[ \delta \tilde{\pi}^{xy}(\omega, k) = \tilde{G}_{R}(\omega, k) \tilde{h}_{xy}(\omega, k) , \]  

(22)

with $\tilde{G}_{R}(\omega, k) = -P_{0} + \tilde{G}^{xy}_{R}(\omega, k)$. Note that $\tilde{G}_{R}(\omega, k)$ has the same analytic structure as $\tilde{G}^{xy}_{R}(\omega, k)$ because $P_{0}$ does not depend on $\omega$ and $k$.

Given that $\tilde{G}_{R}(\omega, k)$ is a meromorphic function with only non-hydrodynamic poles, in the case of homogeneous relaxation (where $k = 0$) one can formally write (10)

\[ \tilde{G}_{R}(\omega) = F(\omega) + \frac{1}{4\pi i} \sum_{n=1}^{\infty} \left( \frac{f_{n}(\omega)}{\omega - \omega_{n}(0)} + \frac{f_{n}^{*}(-\omega^{*})}{\omega + \omega_{n}^{*}(0)} \right) , \]  

(23)

where $F(\omega)$ and $f_{n}(\omega)$ are analytical functions (and we used that $\tilde{G}_{R}(\omega) = \tilde{G}_{R}(-\omega^{*})$). Performing the Fourier transform and picking up the residues one finds (using that $\tilde{h}^{*}(\omega) = \tilde{h}(-\omega^{*})$)

\[ \delta \pi^{xy}(t) = P_{0} h^{xy}(t) + \theta(t) \sum_{n=1}^{\infty} |f_{n}(\omega_{n}(0)) \tilde{h}_{xy}(\omega_{n}(0))| e^{-\Gamma_{n} t} \cos(\Omega_{n} t + \delta_{n}) , \]  

(24)
where $\delta_n$ is a constant phase shift. Clearly, one must be careful when dealing with the representation above because, in general, the sum in the Eq. (23) may not converge. Since there is only a few explicit examples where all the poles and residues are known analytically, in general, the convergence properties of the sum employed in $\tilde{G}_R$ are not known. However, note that in the equation for $\delta \pi^{xy}(t)$ derived above, $\hbar$ enters in the coefficients of the sum. Therefore, the sum in Eq. (24) may converge as long as the metric disturbance varies sufficiently slowly in time, i.e., $\hbar$ goes to zero fast enough for $\omega \neq 0$. Note, however, that this is indeed the case we are interested in since we want to study the response of the system to an external agent (the metric variations) that varies sufficiently slow in time (near the fluid regime). Thus, it is simple to show that under these conditions, only the first two poles ($n = 1$ in the sum, i.e, two distinct time scales) will contribute at sufficiently long (though finite) times $t \Gamma \gg 1$.

Including only the 2 slowest non-hydrodynamic modes close to the origin in the $\omega$–complex plane, we obtain the following linearized equation of motion for $\delta \pi^{xy}$ in an AdS/CFT configuration (see Fig. 1-b) directly from Eq. (22),

$$[\Phi_2(0)\partial_t^2 + \Phi_1(0)\partial_t + 1] \delta \pi^{xy} = C_0(0)\tilde{h}_{xy} - \left( \frac{\partial^2 C_0(k)}{2} \right)_{k=0} \partial^2 h(t, z)$$

$$+ [C_2(0) + C_1(0)\Phi_1(0)] \tilde{h}(t, z) + O(\tilde{h}(t, z), \partial^2 h(t, z)),$$

where we defined

$$C_p(k) = \frac{ip}{p!} \partial_p \tilde{G}_R(\omega, k) \bigg|_{\omega = 0}, \quad \Phi_1(k) = \frac{-i}{\omega_1(k)} - \frac{i}{\omega_2(k)}, \quad \Phi_2(k) = \frac{-1}{\omega_1(k)\omega_2(k)}.$$  \hspace{1cm} (25)

Note that, due to the symmetries of the retarded Green’s function, it was not possible to include only one non-hydrodynamic mode, as was possible in the Boltzmann equation and happened in Israel-Stewart theory, since the first two poles are symmetric relative to the $\omega$ imaginary axis and are equally distant from the origin in the complex $\omega$–plane. Eq. (25) came directly from the underlying microscopic theory since our starting point was Eq. (22). We assumed in the derivation of Eq. (24) that the Green’s function contains only non-hydrodynamic poles (that are functions of $k^2$ due to rotational invariance), $C_0(0) = \tilde{G}_R(0, 0) = 0$, and we also limited ourselves to only display the terms containing at most 2 derivatives.

It is easy to show that the macroscopic theory in Eq. (16) when linearized via metric perturbations becomes (note that due to these metric perturbations the shear tensor becomes $\sigma^{xy} \sim \partial_t h^{xy}$)

$$[\chi_1 \partial_t^2 + \chi_1 \partial_t + 1] \delta \pi^{xy}(t, z) = \eta \partial_t h(t, z) + \eta \sigma_2 \partial_t^2 h(t, z) + \eta(c_1 + c_2) \partial_t^2 h(t, z).$$

(27)

Note that the coefficient $\chi$ in Eq. (16) does not appear in this case because of the specific way we chose to disturb the metric. However, it can be shown by calculating the general dispersion relations for the sound and shear channels for a conformal fluid that a term like $D^{(\mu} D_{\lambda) \pi^{xy \lambda}}$ cannot appear and therefore we must take $\chi = 0$ \hspace{1cm} (28).

We can now match the long distance, long time limit of the microscopic theory in Eqs. (24) to the macroscopic theory in (27) to derive the well-known Kubo formula $\eta = C_1(0) = \delta \partial_t h(t, z)$ and, thus, we recover that $\eta/s = 1/(4\pi)$ \hspace{1cm} (24) in the supergravity limit. Moreover, $\chi_{1, 2} = C_{1, 2}(0), 2\eta c_2 = -\left( \partial^2 C_0(k) \right)_{k=0} = -\left( \partial^2 G_R(0, k) \right)_{k=0}$ and $2\eta(c_1 - c_2) = (\partial^2 - \partial_0^2)G_R(\omega, k)_{\omega k=0}$. Using the results for the Taylor expansion of $G_R^{xyxy}$ derived by \hspace{1cm} (11) and the calculation of the poles at zero wavevectors from \hspace{1cm} (22), one obtains the following values for the transport coefficients in $N = 4$ SYM, $\chi_1 \sim 0.63/(2\pi T)$ and $\chi_2 \sim 0.23/(2\pi T)^2$, $c_2 = 1/(2\pi T)$ and $c_1 = \chi_1 - (2 - \ln 2)/(2\pi T)$. Thus, all the coefficients associated with the linear terms in the transient theory in Eq. (16) have been determined. There are, however, still 10 coefficients in the transient theory that remain to be computed: $e_i$’s, $f_i$’s, $c_3$ and $c_4$. Since they correspond to nonlinear terms, they cannot be determined from linear response theory.

As was mentioned before, since the transient theory derived in this paper automatically reduces to the asymptotic theory derived in \hspace{1cm} (11) several results derived within that theory are contained within the general transient theory displayed in \hspace{1cm} (10). For instance, results derived within the fluid-gravity correspondence \hspace{1cm} (22) or the Bjorken expanding systems studied in \hspace{1cm} (31) can be readily recovered. In fact, in the case of a Bjorken expanding system, the transient theory defined by Eq. (16) should give the same expressions obtained from the Burnett-like theory in Eq. (19) at sufficiently large times.

VI. FINAL COMMENTS

In summary, in this paper we derived the most general equation of motion (see Eq. (16)) compatible with conformal invariance satisfied by the shear stress tensor of strongly coupled $\mathcal{N} = 4$ SYM theory in the transient regime at $\mathcal{O}(\text{Re}^{-2}, \text{Kn}^2, \text{Re}^{-1}\text{Kn}^2)$. This equation contains 17 transport coefficients (of which 7 were determined in this paper).
and it describes the transient regime experienced by the fluid as it evolves towards its universal asymptotic solution given by the gradient expansion computed to $\mathcal{O}(\text{Kn}^2)$ in \cite{Denicol:2009am}. Equation \cite{Denicol:2009am} is a second-order differential equation with respect to propertime for $\pi^\alpha$ and, as such, it is structurally different than the relaxation-type equations expected to describe the transient fluid dynamics of weakly-coupled systems \cite{Denicol:2010xn}.

An important point concerns the stability of Eq. \cite{Denicol:2010xn} with respect to hydrostatic equilibrium. As mentioned above, the stability of transient, relativistic fluid dynamics is a nontrivial problem and, in fact, so far only the stability conditions for relaxation-type equations have been checked \cite{Denicol:2010xn}. The inclusion of additional time derivatives affects the previous studies and, thus, one must generalize these calculations to verify under which conditions Eq. \cite{Denicol:2010xn} describe a stable and causal fluid.

The novel transient physics contained in Eq. \cite{Denicol:2010xn} may bring some light into the description of the early time dynamics of the strongly coupled QGP formed in ultrarelativistic heavy ion collisions. The authors thank H. Niemi, H. Waringa, G. Torrieri, and D. Rischke for discussions. We thank the Helmholtz International Center for FAIR within the framework of the LOEWE program for support. J. N. thanks Conselho Nacional de Desenvolvimento Cientifico e Tecnologico (CNPq) and Fundacao de Amparo a Pesquisa do Estado de Sao Paulo (FAPESP) for support.

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