Abstract.
In this paper, we will use the Kohn’s $\bar{\partial}_b$-theory on CR-hypersurfaces to derive some new results in CR-geometry.

Main Theorem. Let $M^{2n-1}$ be the smooth boundary of a bounded strongly pseudo-convex domain $\Omega$ in a complete Stein manifold $V^{2n}$. Then (1) For $n \geq 3$, $M^{2n-1}$ admits a pseudo-Einstein metric; (2) For $n \geq 2$, $M^{2n-1}$ admits a Fefferman metric of zero CR $Q$-curvature; and (3) for a compact strictly pseudoconvex CR emendable 3-manifold $M^3$, its CR Paneitz operator $P$ is a closed operator.

There are examples of non-emendable strongly pseudoconvex CR manifold $M^3$, for which the corresponding $\bar{\partial}_b$-operator and Paneitz operators are not closed operators.

0. Introduction

In this paper, we study several questions, including the existence of $Q$-flat metrics, pseudo-Einstein metrics and the closedness of the CR Paneitz operators.

First, we will use an approach proposed by Fefferman and his school to prove that “the complete Kähler-Einstein $g_\infty$ on an open domain $\Omega$ induces a metric on $M = \partial \Omega$ with zero CR $Q$-curvature, where $\Omega$ is a smooth, bounded strictly pseudo-convex domain in a Stein manifold $V^{2n}$.” To achieve this goal, we solve a $\partial \bar{\partial}$-Poincaré-LeLong equation via the $\bar{\partial}$-theory. Although this part does not produce new hard a-priori estimates, it is still valuable for other potential applications.

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The second purpose is to prove the existence of pseudo-Einstein metrics on strictly pseudo-convex CR-hypersurface of real dimension $\geq 5$ through solving the $\bar{\partial}_b$ Poincaré-LeLong equations.

The last part of our paper is to study the closedness of CR Paneitz operator, which is a fourth-order differential operator. It is known that the positivity of CR Paneitz operator is related to the deformation of $Q$-curvatures under the conformal change of metrics on Riemannian manifold $M^m$. In particular, the positivity of CR Paneitz operator is also related to the lower bound of the first eigenvalue of sub-Laplace on a CR manifold $M^3$, see [CC], [CCC] and [LL]. It will be shown that, if $M^3 = b\Omega^4$ is the smooth boundary of bounded strictly pseudo-convex domain $\Omega$ in a Stein manifold $V^4$, then its CR-Paneitz operator on $M^3$ is closed, for any metric on $M^3$.

**Main Theorem.** Let $M^{2n{-}1}$ be the smooth boundary of a bounded strongly pseudo-convex domain $\Omega$ in a complete Stein manifold $V^{2n}$. Then

(1) For $n \geq 2$, $M^{2n{-}1}$ admits a metric of zero CR $Q$-curvature;

(2) For $n \geq 3$, $M^{2n{-}1}$ admits a pseudo-Einstein metric;

(3) In addition, for a compact strictly pseudoconvex CR embeddable 3-manifold $M^3$, its CR Paneitz operator $P$ is a closed operator.

Earlier work in this direction for the case of $V^{2n} = \mathbb{C}^n$ can be found in [L2], [FH] and [GG]. In a very recent paper [LL], Li and Luk obtained an explicit formula for Webster’s pseudo-Ricci curvature on real hypersurfaces in $\mathbb{C}^n$. Thus, their result could lead another proof of Cheng-Yau’s result ([CY]) and Mok-Yau’s theorem [MY], which will be used in Section 2 below.

Among other things, we introduce some new methods to handle pseudo-Einstein metric and Paneitz operators in this paper. For example, we use the closeness of $\bar{\partial}_b$ and $\bar{\partial}_b^*$ operators provided by Kohn’s theory, in order to complete the proof. When $\dim_{\mathbb{R}}[M] = 3$, we decompose the Paneitz operator $P$ as a product of closed
operators. Thus, the closed property of $P$ will follow immediately, see Lemma 1.4 and Section 4 below.

1. Preliminary results

It is well-known that the real Laplace $\triangle$ on a Kähler manifold $V^{2n}$ satisfies

$$\triangle = 2\Box = 2\bar{\Box},$$

where $\Box = \bar{\partial} \bar{\partial}^* + \bar{\partial}^* \partial$ is the complex Laplace operator. However, it may happen that $\triangle_b \neq 2\Box_b$ in some cases. Let us recall the notions of $\triangle_b$ and $\Box_b$.

Since $M^{2n-1} = bV^{2n}$ has odd real dimension, it is a Cauchy-Riemann manifold. The $\bar{\partial}_b$ operator induces a sub-elliptic operator

$$\Box_b = \bar{\partial}^*_b \bar{\partial}_b + \bar{\partial}_b \partial^*_b$$

acting on $L^2_{(p,q)}(M)$. Similarly, there is a real sub-Laplace operator, which can be viewed as partial trace of the hessian operator (or can be viewed a sum of the squares of $(2n-2)$ vectors):

$$\triangle_b u|_z = \sum_{k=1}^{2n-2} \langle \nabla e_k (\nabla^b u), e_k \rangle|_z$$

where $e_{2n}$ is the outward real unit normal vector of $\Omega$ along boundary $M = b\Omega$, $e_{2j} = J e_{2j-1}$ for $j = 1, \ldots, n$, $z \in M$, $J$ is the complex structure of $V^{2n}$, \{e_1, e_2, \cdots, e_{2n-3}, e_{2n-2}, e_{2n-1}, e_{2n}\} is an orthonormal basis of $[T_z (V)]_\mathbb{R}$ and

$$\nabla^b u = \sum_{k=1}^{2n-2} du(e_k)e_k.$$

When $\Omega$ has the strongly pseudo-convex boundary in a Stein manifold $V^{2n}$ with $n = 2$, it has been observed that

$$\Box_b u = \frac{1}{2} [\triangle_b u + \sqrt{-1} Tu]$$

(1.1)
for all $u \in L^2(M^3)$, where $T = \lambda e_3$ is the Reeb vector of the CR 3-manifold $M^3$ for some real valued function $\lambda$, see [L1, p414].

The operator $\Box_b$ is a Lewy type operator, which may not be locally solvable.

If the Reeb vector $T$ induces an infinitesimal pseudo-conformal with respect to the Tanaka-Webster metric, then the torsion of $M^3$ is zero, see [Web, p33]. In this case, the operator $\Box_b$ is related to the so-called CR Paneitz operator $P$, where $P$ is given by

$$Pu = \Delta_b^2 u + T^2 u = 4\Box_b \bar{\Box}_b u,$$

for $u \in L^2(M^3)$. More generally, if $M^3$ has torsion free in the sense of Tanaka (cf. [Ta1-2] [Web]), then (1.2) holds.

The eigenvalues of the Paneitz operator and CR Paneitz operators have been considered various authors ([Ch], [CC]). The eigenvalue estimate plays an important role to the study of the so-called Q-curvature flow, see [Br] [CCC].

**Definition 1.1.** (1) The CR-Paneitz operator $P : L^2(M^3) \to L^2(M^3)$ is called essentially positive, if there is a positive constant $\lambda_1 > 0$ such that

$$\langle Pu, u \rangle \geq \lambda_1 \|u\|^2,$$

for all $u \perp \ker(P)$.

(2) The operator $\mathcal{F} : L^2_{(p,q)}(M) \to L^2_{(p,q)}(M)$ is said to have positive spectrum gap at 0 (or is said to be a closed operator) if there is a positive constant $\lambda_{p,q} > 0$ such that

$$\|\mathcal{F}u\| \geq \lambda_{p,q} \|u\|,$$

for all $u \perp [L^2_{(p,q)}(M) \cap \ker(\mathcal{F})]$.

(3) A smooth function $f : U_\varepsilon(M) \to \mathbb{R}$ is called a defining function of $M$ if $f^{-1}(0) = M$ and if 0 is not a critical value of $f$, where $U_\varepsilon(M) \subseteq V^{2n}$ is a neighborhood of $M$ in a Stein manifold $V^{2n}$.

(4) Let $\theta$ be a contact 1-form of $M^{2n-1}$ and $J : \ker \theta \to \ker \theta$ be the almost complex structure on the CR-distribution $\ker \theta$ such that $J^2 \bar{v} = -\bar{v}$ for all $\bar{v} \in \ker \theta$. 
In what follows, we always let

\[ [T^{(1,0)}(M) \oplus T^{(0,1)}(M)] = [\ker \theta] \otimes \mathbb{C}. \]

(5) A CR manifold \( M^{2n-1} \) is said to have transverse symmetry or torsion-free if it admits a CR Reeb vector field \( \xi \) such that \( \xi \notin \ker \theta \) with

\[ \mathcal{L}_\xi J = 0 \]

where \( \mathcal{L} \) is the Lie derivative and \( J \) is the complex structure of \( [T^{(1,0)}(M) \oplus T^{(0,1)}(M)] \).

If \( \xi \) is the real part of a holomorphic vector field \( \tilde{X} \) on a neighborhood \( U_\varepsilon(M) \) of \( M \), then \( \xi \) induces an automorphism on \( U_\varepsilon(M) \). Any real part \( \xi \) of a holomorphic vector field restricted to \( M \) induces a CR-automorphism of \( M \).

In the Hörmand-Kohn \( L^2 \)-theory and the Kohn-Rossi theory, the essential spectrum of \( \Box \) and \( \Box_b \) have been extensively investigated.

A smooth \((p, q)\)-form \( u \) on \( \Omega \) with \( q \geq 1 \) is said to satisfy the \( \bar{\partial} \)-Neumann boundary condition if

\[ u((\bar{\partial}\rho)\#,...)|_z = 0 \]

for all \( z \in M = \partial \Omega \), where \( (\bar{\partial}\rho)\# \) is the complex normal vector field of type \((0,1)\) along the boundary \( M^{2n-1} \).

**Theorem 1.2.** ([CS], [CaWS]) Let \( \Omega \) be a bounded domain with smooth pseudo-convex boundary \( M \) in a complete Hermitian manifold \( V^{2n} \). Suppose that \( V^{2n} \) is either a Stein manifold or \( \mathbb{C}P^n \). Then the complex Laplace operator \( \Box \) is

(1) positive for on \( L^2_{(p,q)}(\Omega) \) with \( (n-1) \geq q \geq 1 \); and

(2) essentially positive on \( L^2_{(p,0)}(\Omega) \) and \( L^2_{(p,n)}(\Omega) \)

with respect to \( \bar{\partial} \)-Neumann boundary condition on \( M = \partial \Omega \).

Moreover, for any Hermitian metric on \( \Omega \), the operator \( \Box \) is essentially positive on \( \Omega \) with respect to \( \bar{\partial} \)-Neumann boundary condition on \( M \).
For the $L^2$ estimates of $\Box$, the domains $\Omega$ in Theorem 1.2 are not necessarily strictly pseudo-convex. However, for estimates of $\Box_b$ on the boundary $M^{2n-1}$ of $\Omega$, we need extra assumptions on $M^{2n-1}$.

The dual of $\bar{\partial}$-Neumann problem is the so-called $\bar{\partial}$-Cauchy problem. A $(p, q)$-form $u$ is said to satisfy the Cauchy boundary condition on $M = b\Omega$ if

$$u(\xi, ...)|_z = 0$$

for all $\xi \in T^{(0,1)}_z(M)$ and $z \in M$. If a $\bar{\partial}$-closed form $f \in C^\infty_{(p, q+1)}(\Omega)$ with a compact support in $\Omega$, then one consider to solve $\bar{\partial}u = f$ such that $u$ has a compact support in $\Omega$ as well. Solving $\bar{\partial}u = f$ with compact support is related to the $\bar{\partial}$-extension problem, via the Kohn-Rossi theory. Using the solution to the $\bar{\partial}$-extension problem and Theorem 1.2, we are able to solve $\bar{\partial}_b u = f$ on a special class of CR-manifolds:

**Theorem 1.3.** ([CS], [CaSW]) Let $\Omega$ be a bounded Hermitian manifold with a smooth pseudo-convex boundary $M$. Suppose that one of the following conditions holds:

(1) $\Omega$ is a domain of a complete Stein manifold $V^{2n}$;

(2) $\Omega \subset \mathbb{C}P^n$, and $M = b\Omega$ admits a pluri-subharmonic defining function.

Then the $\bar{\partial}$-Cauchy boundary problem is solvable on $\Omega$. Furthermore, (1) $\bar{\partial}_b$-operator is closed; and (2) the operator $\Box_b : L^2_{(p, q)}(M) \to L^2_{(p, q)}(M)$ is positive for $1 \leq q \leq n - 2$ and essentially positive for $q = 0$ or $q = n - 1$.

When $M = b\Omega$ is strongly pseudo-convex, it is well-known that $M$ admits a pluri-subharmonic defining function, see [DF].

If $\mathcal{L} : H_1 \to H_2$ is a linear operator, we let $\text{Dom}(\mathcal{L})$ be its domain and $\mathcal{R}(\mathcal{L})$ be its range. If $A \subset H$ is a subset of a Hilbert space $H$, the closure of $A$ in $H$ is denoted by $\bar{A}$.

We begin with an elementary but useful criterion for closed operators.

**Lemma 1.4.** ([CS, p60] or [Hö1-2]) Let $\mathcal{L} : H_1 \to H_2$ be a linear, closed, densely
defined operator from the Hilbert space $H_1$ to another Hilbert space $H_2$. The following conditions on $\mathcal{L}$ are equivalent:

(1) The range $\mathcal{R}(\mathcal{L})$ of $\mathcal{L}$ is closed;

(2) There is a constant $C$ such that

$$\| f \|_1 \leq C \| \mathcal{L} f \|_2$$

for all $f \in \text{Dom}(\mathcal{L}) \cap \mathcal{R}(\mathcal{L}^*)$;

(3) The range $\mathcal{R}(\mathcal{L}^*)$ of $\mathcal{L}^*$ is closed;

(4) There is a constant $C$ such that

$$\| f \|_2 \leq C \| \mathcal{L}^* f \|_1$$

for all $f \in \text{Dom}(\mathcal{L}^*) \cap \mathcal{R}(\mathcal{L})$.

2. The existence of CR $Q$-flat metrics on strictly pseudo-convex CR-hypersurfaces in a Stein manifold

In this section, we first recall an existence result of CR $Q$-flat metrics on CR-hypersurfaces in Euclidean space $\mathbb{C}^n$ due to Fefferman and others. Afterwards, we will extend such a result to CR-hypersurfaces in an arbitrary Stein Manifold $V^{2n}$. One of our key steps is to use the $\bar{\partial}$-theory to introduce the generalized Fefferman’s functional $u \to \hat{J}(u)$, which is independent of the choice of local holomorphic coordinates, see (2.5) below.

2.a. A sufficient condition for existence of $Q$-flat metrics on real hypersurfaces.

Let us recall a sufficient condition for existence of $Q$-flat metrics on real hypersurfaces, which were derived by Fefferman and others.

**Proposition 2.0.** ([FG1-2], [GG]) Let $\Omega \subset \mathbb{C}^n$ be a compact domain with smooth boundary $M^{2n-1} = b\Omega$ in the complex Euclidean space $\mathbb{C}^n$. Suppose $\Sigma^{2n}$ is an unit
circle bundle defined on a CR-hypersurface $M^{2n-1}$ and suppose that $\Sigma^{2n}$ admits an $S^1$-invariant Einstein-Lorentz metric $g_u^+ = i\partial \bar{\partial} H_u|_{\Sigma^{2n}}$ defined as below. Then $M^{2n-1}$ admits a metric of zero CR $Q$-metric.

We now provide a description of the metric $g_u^+$ stated in Proposition 2.0, which will be used for any real hypersurface $M^{2n-1}$ in a Stein manifold $V^{2n}$ as well.

Let $K^*$ be the canonical bundle of $V^{2n}$ restrict to $M$ and let $\Sigma^{2n} = K^*/\mathbb{R}^+$ be the unit circle bundle of $K^*$. Thus there is a fibration

$$S^1 \to \Sigma^{2n} \to M^{2n-1}$$

and $\dim_{\mathbb{R}}(\Sigma^{2n}) = 2n$.

We may assume that $\Omega \subset V^{2n}$ is an open strictly pseudo-convex domain with compact smooth boundary $M^{2n-1} = b\Omega$. Suppose that $\hat{u}$ is a defining function of $M^{2n-1}$. For example, we can choose $\hat{u}$ as a signed distance function form $M$:

$$\hat{u}(z) = \begin{cases} 
- d(z, M), & \text{if } z \in \Omega \\
  d(z, M), & \text{if } z \notin \Omega
\end{cases}$$

Any other defining function $u$ can be expressed as

$$u(z) = e^{\eta} \hat{u}$$

for some real valued function $\eta$.

The contact structure on $M$ is an 1-form given by

$$\theta_u(\xi) = du(J\xi)$$

for all $\xi \in [T(M)]_{\mathbb{R}}$, where $J$ is the complex structure of $V^{2n}$.

There are two types of metrics which we will use. The first one is the Cheng-Yau metric on $\Omega$; and the second one is introduced by Fefferman on a line bundle over $b\Omega$.
Let us first consider complete Kähler metrics on an open domain Ω. Suppose that
\[ \omega_u = i \partial \bar{\partial} [\log(-\frac{1}{u})] \]
is a Kähler form on Ω. Such a Kähler form \( \omega_u \) corresponds to a Kähler metric
\[ g_u(X, Y) = \omega_u(X, JY) = i \partial \bar{\partial} [\log(-\frac{1}{u})](X, JY), \]
where \( J \) is the complex structure of Ω.

Secondly, Fefferman and his school considered a class of Lorentz metrics on canonical bundle on \( K^* \) mentioned above.

We will use an extrinsic way to define such metrics, along the line described in a new book [DT, p150]. Suppose that \( \Lambda_{(n,0)}(V^{2n}) \) be the canonical line bundle of open domain \( V^{2n} \). Clearly, \( \mathcal{L}_{V^{2n}} = \Lambda_{(n,0)}(V^{2n}) \) is a complex manifold of complex dimension \((n + 1)\).

When \( \xi \) is a cross-section of \( \mathcal{L}_{V^{2n}} \) over \( V^{2n} \), the norm \( |\xi|_{g_u} \) induced by \( g_u \) is well-defined. We further define
\[ H_u(z, \xi) = |\xi|^\frac{n}{2} g_u u(z) \]

There is an \((1,1)\)-form defined on \( \mathcal{L}_{V^{2n}} \) given by \( i \partial \bar{\partial} H_u \).

Similarly, there is a Hermitian form
\[ G_u(\tilde{X}, \tilde{Y}) = i \partial \bar{\partial} H_u(\tilde{X}, \tilde{J}Y), \]
where \( \tilde{J} \) is the complex structure of line bundle \( \mathcal{L}_{V^{2n}} \). The Hermitian form \( G_u \) is not necessarily positive definite on the complex manifold \( \mathcal{L}_{V^{2n}} \).

We now consider a subset
\[ \Sigma^{2n} = \{(z, \xi) \in \mathcal{L}_{V^{2n}} \mid z \in b\Omega, |\xi| = 1\} \]
where \( \Omega \) is an open, bounded and strictly pseudo-convex domain in \( V^{2n} \).
Finally, when $i\partial\bar{\partial}u > 0$ on $M = b\Omega$, we consider

$$g_+^u = G_u|\Sigma^{2n}.$$  \hspace{1cm} (2.4)

It was shown that $g_+^u$ is a Lorentz metric on $\Sigma^{2n}$. Clearly, $\Sigma^{2n}$ is diffeomorphic to the unit circle bundle $K^*$ mentioned above.

We remark that the function $u = 0$ vanishes on $M^{2n-1}$. The leading term of the metric $g_+^u$ is

$$i\partial\bar{\partial}u.$$

In [FH], Fefferman and Hirachi studied the so-called $Q$-curvature of $CR$-manifold $M^3$:

$$Q_{\theta_u}^{CR} = \frac{4}{3}(\Delta_b R - 2Im\nabla^\alpha \nabla^\beta A_{\alpha\beta}),$$

where $R$ is the Tanaka-Webster scalar curvature, $A$ is the torsion, $\Delta_b$ is the sub-Laplacian computed in terms of the contact 1-form $\theta_u$ and $\theta_u(\xi) = du(J\xi)$ for all $\xi \in T(M)$.

For higher dimensional manifolds, the $Q$-curvatures of higher order have been studied in [FH] and [GG].

The notations above will be used in the next two sub-sections.

2.b. Relations between the Fefferman’s Lorentz metric and the Cheng-Yau’s Kähler-Einstein metric.

In this sub-section, we illustrate a strategy to obtain the existence of $Q$-flat metrics on real hypersurfaces in $\mathbb{C}^n$.

Let us now recall a result obtained by Fefferman and his school.

**Proposition 2.1.** ([FG1, Chapter III]) Let $\Omega \subset V^{2n}$, $M = b\Omega \subset \mathbb{C}^n$, $u = \hat{u}e^\eta$ and $\{g_u, g_+^u\}$ be as above. If the Cheng-Yau metric $g_u$ is a complete Kähler-Einstein on $\Omega$, then the Lorentz metric $g_+^u$ is Einstein on $\Sigma^{2n}$.

Here is a direct application of Propositions 2.0-2.1.
Corollary 2.2. ([FH], [GG]) Let $\Omega \subset \mathbb{C}^{2n}$ be an open strictly pseudo-convex domain with compact closure and let $M^{2n-1} = b\Omega$ be its boundary. Then $M$ admits a metric of zero CR $Q$-curvature.

Proposition 2.1 and Corollary 2.2 were stated for strictly pseudo-convex and bounded domain $\Omega$ in $\mathbb{C}^n$. We would like to extend these results to any strictly pseudo-convex and bounded domain $\Omega$ in a Stein manifold $V^{2n}$.

2.c. Compact smooth real hypersurfaces in a Stein manifold.

Our goal of this section is to verify the following theorem.

Proposition 2.3. Let $\Omega$ be a bounded, open and strictly pseudo-convex domain with a smooth boundary in a Stein manifold $V^{2n}$. If the metric $g_u$ above is a complete Kähler-Einstein metric on $\Omega$, then $g_u$ induces a metric $\tilde{g}_u^\infty$ on $M = b\Omega$ with zero CR $Q$-curvature.

Proof. Since $V^{2n}$ is Stein, we may assume that $V^{2n} \subset \mathbb{C}^m$ is a complete submanifold of $\mathbb{C}^m$, for sufficiently large $m$. Let $\hat{g}$ be induced metric on $\Omega \subset V^{2n} \subset \mathbb{C}^m$.

For each local holomorphic coordinate system $\{(z_1, ..., z_n)\}$ of $\Omega$, the Ricci tensor $\hat{\text{Ric}}$ of $\hat{g}$ is given by

$$\hat{\text{Ric}} = -i\partial\bar{\partial} \log[\det \hat{g}_{i\bar{j}}].$$

It is clear that $\hat{\text{Ric}}$ is well-defined and independent of the choice of local holomorphic coordinate system $\{(z_1, ..., z_n)\}$. Moreover, $\hat{Ric}$ is a closed $(1, 1)$-form on $\Omega$. In what follows, we first would like to solve Poincare-Lelong equation $i\partial\bar{\partial}f = \hat{\text{Ric}}$.

For this purpose, we recall a theorem of Dolbeault:

$$H^{(1,1)}(\Omega) = H^{(0,1)}(\Omega, O|_{\Omega})$$

where $O|_{\Omega}$ is the bundle of holomorphic $(1, 0)$-forms.

Since $\Omega$ is strictly pseudo-convex and bounded domain in a Stein manifold $V^{2n}$, by a theorem of Andreotti and Vesentini [AV], we have

$$H^{(0,1)}(\Omega, O|_{\Omega}) = 0.$$
In fact, Proposition A.4 of [CaWS, p218] is also applicable for (0, q)-forms with values in $O|\Omega$. Thus, $H^{(1,1)}(\Omega) = H^{(0,1)}(\Omega, O|\Omega) = 0$. Professor Siu also handled similar formula with values in a vector bundle $E$, although the weighted functions were not discussed there (cf. [Siu, Chapters 2-3]). Hence, the first Chern class $c_1(O|\Omega) = 0$. Recall that, by Chern-Weil theory, the co-homology class $c_1(O|\Omega)$ is independent of the choices of affine connections, (cf. [Mi]). Therefore, $c_1(O|\Omega) = 0$ implies that the Chern-Weil form $\hat{Ric}$ is $d$-exact on $\Omega$.

Therefore, we have $\hat{Ric} = d\beta$ for some 1-form $\beta$. Let us consider the decomposition of $\beta = \beta^{(1,0)} + \beta^{(0,1)}$, where $\beta^{(0,1)}$ is the $(0,1)$-component of $\beta$. If $\hat{Ric} = d\beta$ and if $\beta = \beta^{(0,1)} + \beta^{(1,0)}$, then $\bar{\partial}\beta^{(0,1)} = 0$, where we used the fact that $\hat{Ric}$ is an (1,1)-form. Choosing $f$ with $\bar{\partial}f = i\beta^{(0,1)}$, we get a solution $i\bar{\partial}\bar{\partial}f = \hat{Ric}$.

Recall that $\hat{Ric}$ is real valued. Replacing $f$ by $Re\{f\}$ if needed, we conclude that the Poincare-Lelong equation

$$i\bar{\partial}\bar{\partial}f = \hat{Ric} = -i\bar{\partial}\bar{\partial}\log[\det \hat{g}_{ij}].$$

has a smooth real-valued solution $f$ on $\Omega \cup b\Omega$. Such a solution $f$ is unique up to adding a pluri-subharmonic function. If we require that $f$ has the smallest $L^2(\Omega)$-norm, then such a solution is unique, see Chapters 4-5 of [CS]. Such a solution $f$ is called a Ricci potential of $\hat{g}$.

Following Fefferman [F2], we consider

$$\hat{J}(u) = (-1)^ne^{-f} \frac{1}{\det \hat{g}_{ij}} \det \begin{pmatrix} u & u_j \\ u_i & u_{ij} \end{pmatrix}$$  \( (2.5) \)

where $f$ is the Ricci potential of $\hat{g}$ as above, $u_i = \frac{\partial u}{\partial z_j}$, $u_{ij} = \frac{\partial^2 u}{\partial z_i \partial z_j}$ and \{z_1, ..., z_n\} is a local holomorphic frame.

When $\Omega \subset \mathbb{C}^n$, we choose the standard coordinate system. Thus, in this case, $\det \hat{g}_{ij} = 1$ and we can choose $f = 0$. Therefore, our definition coincides with Fefferman’s definition for the case of $\Omega \subset \mathbb{C}^n$, see [F2] and [CY].
A calculation similar to [CY, p508] further shows that the metric $g_u$ is Kähler-Einstein of negative curvature $-(n+1)$ if
\[
\frac{\det \varphi_{ij}}{\det \hat{g}_{ij}} = e^f e^{(n+1)\varphi}
\]
holds, where $\varphi = \log(-\frac{1}{u})$.

A further calculation shows that the above equation holds if and only if
\[
\hat{J}(u)|_z \equiv 1
\]
holds for all $z \in \Omega$.

It is known that if $\hat{J}(u)|_z \equiv 1$ in $\Omega$, then $M = b\Omega$ has zero CR Q-curvature, see [FH, Chapter 3]. This completes the proof of Proposition 2.3. □

**Corollary 2.4.** Suppose that $\Omega \subset V^{2n}$ be a bounded, open and strictly pseudo-convex domain with smooth boundary in a Stein manifold $V^{2n}$ with $n \geq 2$. Then its boundary $M^{2n-1} = b\Omega$ admits a metric of zero Q-curvature.

**Proof.** By Proposition 2.3, it remains to verify that there is a complete Kähler-Einstein metric $g_u$ on $\Omega$. The existence of such a complete Kähler-Einstein metric $g_u$ is provided by Mok-Yau in [MY, p52]. In fact, Mok and Yau found desired solutions $u = e^\eta \hat{u}$ and $\varphi = \log(-\frac{1}{u})$ satisfying $\frac{\det \varphi_{ij}}{\det \hat{g}_{ij}} = e^f e^{(n+1)\varphi}$. □

### 3. Existence of Pseudo-Einstein metrics on CR-hypersurfaces of real dimension $\geq 5$

In this section, we discuss the existence of pseudo-Einstein metrics on CR-hypersurfaces of real dimension $\geq 5$. A metric $g$ defined on a CR-manifold $M^{2n-1}$ is said to be *pseudo-Einstein (or partially Einstein)* if its Ricci tensor satisfies
\[
Ric_g(X, Y)|_z = \lambda g(X, Y)|_z
\]
for some constant $\lambda = \lambda(z)$ and for all real vectors $\{X, Y\}$ in the CR-distribution $ker(\theta)|_z$, where $\theta$ is the contact form of $M^{2n-1}$. 15
One of our new contributions in this section is to use the $\bar{\partial}_b$-theory to solve boundary version of Poincaré-Lelong equation related to the partially Einstein equation, see Proposition 3.4 and Corollary 3.5 below.

When $\dim_{\mathbb{R}}[M^{2n-1}] = 3$, any metric $g$ on $M^3$ is pseudo-Einstein (i.e., partially Einstein). Therefore, we only consider the case of $\dim_{\mathbb{R}}[M^{2n-1}] \geq 5$.

We emphasize that a pseudo-Einstein metric $g$ on $M^{2n-1}$ is not necessarily Einstein. The pseudo-Einstein condition puts no restriction on its Ricci curvature in the directions which are transversal to $CR$-distribution. It might happen that

$$Ric_g(Z, Y) \neq \lambda g(Z, Y)$$

for some transversal vector $Z \perp \ker(\theta)$.

In [L2], Lee already showed that, if a compact strongly pseudo-convex CR-manifold $M^{2n-1}$ admits a closed, nowhere vanishing $(n, 0)$-form, then $M^{2n-1}$ admits a pseudo-Einstein metric. In particular, if $M = b\Omega$ and $\Omega \subset \mathbb{C}^n$, then $M$ admits a pseudo-Einstein structure.

We make extra observations to extend Lee’s result to the case of $\Omega \subset V^{2n}$ for any Stein manifold $V^{2n}$. The new ingredient of our approach will use the fact that the Chern curvature forms $\Theta$ are type of $(1, 1)$ for Lorentz-Kähler metrics.

In addition, we will use Kohn’s $\bar{\partial}_b$-theory to solve the boundary version of Poincare-Lelong equation

$$i\partial_b \bar{\partial}_b f = \Theta$$  \hspace{1cm} (3.1)

for any $\bar{\partial}_b$-closed $(1, 1)$-form $\Theta$.

The equation (3.1) above is related to the existence of pseudo-Einstein metrics, as described in [L2, p173]. Such an equation was previously studied in [CaWS] for other purposes.

It is well-known that, for any function $u$, one has

$$(dc^u)(\xi) = (du)(J\xi)$$

and

$$dd^c u = i\partial \bar{\partial} u.$$

We begin with an elementary observation.
Lemma 3.1. Let \( \hat{u} \) be a defining function of \( M = b\Omega \). Suppose that \( \Omega \subset V^{2n} \) is a strictly pseudoconvex bounded domain in a Stein manifold. Then

1. There is another defining function \( u = e^\varphi \hat{u} \) such that \( u \) is a strictly plurisubharmonic in a neighborhood of \( M = b\Omega \), i.e., \( i\partial \bar{\partial} u > 0 \).

2. When \( i\partial \bar{\partial} u > 0 \) and \( \theta_u = dc^u \), then \( i\partial \bar{\partial} u \) gives rise to a Kähler metric \( g_u \) in a neighborhood of \( M \).

3. If \( u = e^\varphi \hat{u} \), \( \theta = dc^u \) and \( \hat{\theta} = dc^{\hat{u}} \), then one has

\[ \theta = e^\varphi \hat{\theta} \text{ on } M. \]

Proof. Assertion (1) was stated in Theorem 3.4.4 of [CS, p45-46].

The verification of Assertions (2)-(3) is straightforward. □

Proposition 3.2. Let \( \Omega \) be a bounded, strictly pseudo-convex domain with a smooth boundary \( M = b\Omega \) in a Stein manifold \( V^{2n} \), let \( \mathcal{O} \) be the holomorphic \((1,0)\)-form bundle of \( V^{2n} \), and let \( K^* \) be the canonical line bundle of \( V^{2n} \). Suppose that \( \dim_{\mathbb{R}}[V^{2n}] = 2n \geq 6 \). Then the following is true.

1. The first Chern class of \( \mathcal{O}|\Omega \) is equal to zero, i.e., \( c_1(\mathcal{O}|\Omega) = 0 \); Moreover, the first Chern class of canonical line \( c_1(K^*|_M) = 0 \);

2. The Ricci curvature form \( \text{Ric}_g \) of any metric \( g = d\theta \) on \( \mathcal{O}|\Omega \) is a \( d \)-exact \((1,1)\)-form on \( M \). Furthermore, \( \text{Ric}(\xi, \bar{\xi}) \) is a real number for all \( \xi \in T^{(1,0)}(M) \).

Proof. (1) We will use curved version of Kohn-Morrey formula to verify that

\[ c_1(\mathcal{O}|\Omega) = 0. \]  \( (3.2) \)

Recall that the closure \( \bar{\Omega} \) of \( \Omega \) is compact. Since \( V^{2n} \) is a Stein manifold, there is a strictly pluri-subharmonic function \( \phi_0 \). Let \( \phi = \lambda \phi_0 \) for sufficiently large \( \lambda > 0 \). Using Bochner-Hörmander-Kohn-Morrey formula, we obtain

\[ H^{(p,q)}(\Omega) = 0, \]  \( (3.3) \)
for all $0 < q < n$, (cf. Proposition A.4 of [CaWS, p218]).

It is well-known that, for $\dim_{\mathbb{C}}(\Omega) = n > 2$

$$H^1(\Omega, \mathcal{O}|_{\Omega}) = H^{(1,1)}(\Omega) = 0. \quad (3.4)$$

It follows that the first Chern class of $\mathcal{O}|_{\Omega}$ is zero.

Choose a Kähler metric $\hat{g}$ on $\Omega$. Then the Ricci curvature form $\hat{\Theta}$ is a $d$-exact $(1,1)$-form.

The classical Kohn-Rossi theory states that any $\bar{\partial}_b$-closed $(1,0)$-form on $M = b\Omega$ can be extend to a unique holomorphic $(1,0)$-form on the whole $\Omega$. Thus,

$$H^{(1,1)}(M) = 0, \quad (3.5)$$

see [KoR].

It is also known that $c_1(\mathcal{O}|_M) = c_1(K^*|_M) = 0$.

(2) Let $g_u$ be the Kähler metric associated with the Kähler form $i\partial\bar{\partial}u$. The corresponding first Chern curvature form $\Theta_u$ of the Kähler metric $g_u$ is a closed $(1,1)$-form in a neighborhood of $M$ in $V^{2n}$.

The classical Chern-Weil theory implies that the cohomology class of the first Chern curvature form $\Theta|_M$ is independent of the choice of the choice of affine connections on $M$.

In fact, if $\theta_u = d^c u$ and $i\partial\bar{\partial}u > 0$, then $d\theta_u = dd^c u = i\partial\bar{\partial}u > 0$ gives rise a Kähler metric in a neighborhood of $M$. For any other $\tilde{\theta} = e^{2\varphi}\theta$, the Ricci curvature form corresponding to $\tilde{\theta}$ remains to be of type $(1,1)$, see Lemma 2.4 of [L2]. □

We now recall that a result of Lee [L2].

**Proposition 3.3.** ([L2, Lemma 6.1, p173-174]) Let $M = b\Omega$ and $\Omega \subset V^{2n}$ be as in Main Theorem. Suppose that $\hat{\theta} = e^{2u}\hat{\tilde{\theta}}$ and $\hat{\Ric}$ is the Ricci curvature form corresponding to $\hat{\tilde{\theta}}$. Then $\hat{\theta}$ is pseudo-Einstein if and only if there is a real solution $u$ satisfying

$$i\partial_b\bar{\partial}_b u = \hat{\Ric}$$
Proof. By (6.3) of [L2], the trace-less part of \( \tilde{Ric} \) is zero if there is \( \varphi \) satisfying
\[
(n + 1)i\bar{\partial}_b \partial_b \varphi = \tilde{Ric}
\]
Since \( \tilde{Ric} \) is a real valued \( d \)-exact real-valued \((1,1)\)-form by Proposition 3.2 above, we can choose \( \varphi \) to be real-values as well. (Otherwise, let \( v = \frac{1}{2}(\varphi + \bar{\varphi}) \) instead). \( \square \)

**Proposition 3.4.** Let \( M = b\Omega \) and \( \Omega \subset V^{2n} \) be as in Main Theorem. Suppose that the Ricci curvature form \( \hat{Ric} \) form is a \( d \)-exact \((1,1)\)-form for the contact \( 1 \)-form \( \hat{\theta} \). Then there always a real-valued function \( u \) satisfying
\[
i\bar{\partial}_b \partial_b u = \hat{Ric}
\] (3.6)

**Proof.** Choose \( \sigma \) such that
\[
d\sigma = \hat{Ric}.
\] (3.7)
Let \( \sigma = \sigma^{(0,1)} + \sigma^{(1,0)} + \lambda \theta \), where \( \sigma^{(0,1)} \) is \((0,1)\)-component of \( \sigma \). Since \( \hat{Ric} \) is of type \((1,1)\), by (3.7) we have
\[
\bar{\partial}_b \sigma^{(0,1)} = 0.
\] (3.8)

Because \( \text{dim}_\mathbb{C}(\Omega) > 2 \), by a Theorem of Kohn that there is complex-valued function \( f \) with
\[
i\bar{\partial}_b f = \sigma^{(0,1)},
\] (3.9)
see [CS, Ch9].

It follows that
\[
i\partial_b \bar{\partial}_b f = \partial \sigma^{(0,1)} = (d\sigma)_b = (\hat{Ric})_b.
\] (3.10)
Since \( (\hat{Ric})_b \) is real-valued \((1,1)\)-form, choosing \( u = Re\{f\} \), we are done. \( \square \)

We now summarize our result of this section.

**Corollary 3.5.** Suppose that \( \Omega \subset V^{2n} \) be a compact strictly pseudo-convex domain with smooth boundary in a Stein manifold \( V^{2n} \). Then its boundary \( M^{2n-1} = b\Omega \) admits an intrinsic pseudo-Einstein (i.e., partially Einstein) metric.

**Proof.** This is a direct consequence of Lemma 3.1 and Propositions 3.2-3.4. \( \square \)
4. Estimates for CR Paneitz operators on $M^3$

In the remaining of this paper, we study the so-called CR Paneitz operator

$$P_u f = \triangle_b^2 f + T^2 f + 4Im\nabla_\beta (A^{\alpha\beta}\nabla_\alpha f), \tag{4.1}$$

where $T = J\nabla u$ is the Reeb vector and $A$ is the torsion tensor of the contact form $\theta_u$.

When the torsion $A$ vanishes, the formula (4.1) reduces to (1.2).

It remains to verify that CR Paneitz operator $P_u$ is a closed operator.

If $\hat{\theta} = e^\varphi \theta_u$ on $M^3$ and $\hat{Q}$ is the corresponding CR $Q$-curvature of the metric associated with the contact form $\hat{\theta}$, then

$$e^{2\varphi} \hat{Q} = Q + P_u \varphi,$$

see (5.14) of [GG].

Our goal is to show the following result.

**Proposition 4.1.** Let $\Omega \subset V^4$ be an open strictly pseudo-convex domain with compact closure in a Stein manifold $V^4$ and let $M^3 = b\Omega$ be its boundary. Suppose that $g_u$ is the Cheng-Yau Einstein metric on $\Omega$ and $\theta_u(\cdot) = du(J\cdot)$ is the corresponding contact 1-form on $M^3$. Then the Paneitz operator $P_u$ is closed:

$$\int_{M^3} |P_u f|^2 \geq c \int_{M^3} |f|^2; \tag{4.2}$$

for any real valued function $f \perp \ker P_u$, where $c > 0$ is a constant independent of $f$.

**Remark 4.2:** The constant $c$ in Proposition 4.1 depends mostly on the Tanaka-Webster curvature $R$ and pseudo-hermitian torsion $A_{11}$ of $(M^3, J, \theta_u)$ respectively. In fact, the following holds:

$$\int_M 2(Pf) f \theta_u \wedge d\theta_u = \int_M [3(\triangle_b f)^2 - |Hess_b f|^2 - R|\nabla_b f|^2 - 6Im\{A_{11} f_1\}] \theta_u \wedge d\theta_u,$$

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where $\nabla_b$ and $\text{Hess}_b^2$ denotes the sub-gradient and sub-Hessian with respect to $(J, \theta_u)$ respectively, see [CC].

For the proof of Proposition 4.1, we need some notations.

In what follows, we let $\theta = \theta_u$ be the given contact form. The vector $T$ is the characteristic vector in $T(M)$ such that $\theta(T) = 1$, $(d\theta)(T, .) = 0$.

An $(1, 0)$-form $\theta^1 \in \Lambda_{(1, 0)}(M^3)$ is called admissible if

$$\theta^1(T) = 0, d\theta = ih_{1,\bar{1}}\theta^1 \wedge \theta^{\bar{1}}$$

for some hermitian metric function $h_{1,\bar{1}}$.

It is known that

$$\Delta_b f = -f_\alpha^\alpha - f_{\bar{\alpha}}^{\bar{\alpha}}$$

and

$$\Box_b f = 2(\bar{\partial}^*_b \partial_b + \partial_b \bar{\partial}^*_b)f = (\Delta_b + iT)f = -2f_{\bar{\alpha}}^{\bar{\alpha}}.$$ 

Inspired by proof of Proposition 3.4 of [L2], we will express the CR Paneitz operator $P$ as a product of several closed operators.

We first consider

$$\mathcal{L} f = d_b f + (\Delta_b f)\theta,$$ 

(4.3)

where $\theta$ is the contact 1-form described above.

**Lemma 4.2.** Let $M^3 = b\Omega$, $\theta$, $A$ and $\mathcal{L}$ be as above. Suppose that $\Omega \subset V^4$ is a strictly pseudo-convex domain in a Stein manifold $V^4$ and that $\Omega$ has compact closure. Then $\mathcal{L}$ is a closed operator.

Moreover, one has

$$d[\mathcal{L} f] = 2(f^\dagger_1 + iA_{11}f^\dagger_1)\theta \wedge \theta^{\dagger} + 2(f^{\dagger}_1 - iA_{\bar{1}1}f^{\dagger}_{\bar{1}})\theta \wedge \theta^{\bar{1}}.$$ 

**Proof.** By Theorem 9.4.2 of [CS], both $d_b^c$ and $\Delta_b$ are closed operators for strictly pseudo-convex compact CR-hypersurfaces. Notice that $d_b^c f \in [\Lambda_{(1, 0)}(M^3) \oplus \Lambda_{(0, 1)}(M^3)]$ is always orthogonal to the 1-form $(\Delta_b f)\theta$. Hence, $\mathcal{L}$ is a closed operator.
We will use the proof of Proposition 3.4 of [L2].

The \( \theta^1 \wedge \bar{\theta}^1 \)-component of \( d[\mathcal{L}f] \) is

\[
i[f_{1\bar{1}} + f_{1\bar{1}} - (f_{1\bar{1}} + f_{1\bar{1}})h_{1\bar{1}}] \theta^1 \wedge \bar{\theta}^1 = 0.
\]

On the other hand, the \( \theta \wedge \bar{\theta} \)-component of \( d[\mathcal{L}f] \) is

\[
[f_{1\bar{1}} + f_{1\bar{1}} - if_{1,0} + iA_{11}f^1] \theta \wedge \bar{\theta}.
\] (4.4)

It is known (cf. [L2, Section 2]) that

\[
-f_{1\bar{1}} + f_{1\bar{1}} + if_{1,0} + iA_{11}f^1 = 0.
\] (4.5)

It follows from (4.4) and (4.5) that the \( \theta \wedge \theta^1 \)-component of \( d[\mathcal{L}f] \) is equal to

\[
2(f_{1\bar{1}} + iA_{11}f^1) \theta \wedge \theta.
\] (4.6)

For the same reason, the \( \theta \wedge \bar{\theta}^1 \)-component of \( d[\mathcal{L}f] \) is equal to

\[
2(f_{1\bar{1}} - iA_{1\bar{1}}f^\bar{1}) \theta \wedge \bar{\theta}.
\] (4.7)

This completes the proof. □

Proof of Proposition 4.1. We now consider the composition of operators:

\[
\tilde{P} f = \partial_b^* [ (d(\mathcal{L}f))[T].
\] (4.8)

It follows from that

\[
(d(Lf))[T] = 2(f_{1\bar{1}} + iA_{11}f^1) \theta^1 + 2(f_{1\bar{1}} - iA_{1\bar{1}}f^\bar{1}) \bar{\theta}.
\] (4.9)

We observe that \( \partial_b^* \) acts on \( \Lambda_{(1,0)}(M^3) \) trivially. For real valued function \( f \), we further consider

\[
Re[\tilde{P} \circ f] = Re[\Box_b \Box_b f] + 4Im(A_{1\bar{1}}f_1)_1,
\] (4.10)

where \( Re\{z\} \) is the real part of complex number of \( z \).
Therefore, it follows from (4.8)-(4.10) that, for real valued function $f$, we have

$$\text{Re}[\tilde{P}f] = \Delta^2_b f + T^2 f + 4\text{Im}(A_{1\bar{1}}f_1) = Pf.$$ \hfill (4.11)

Thus, the CR Paneitz operator $P$ satisfies

$$Pf = \text{Re}[\tilde{P}f],$$ \hfill (4.12)

where

$$\tilde{P} = \partial^*_b [(d(Lf)) |_T].$$

A composition of closed operators remains to be a closed operator.

If $M^3 = b\Omega$ is a compact strictly pseudo-convex hypersurface in a Stein manifold $V^4$, then $\{\bar{\partial}_b, d, \partial^*_b, L\}$ are closed operators, by Kohn’s $\bar{\partial}_b$-theory (cf. [CS, Theorem 9.4.2, p231]). Theorem 9.4.2 of [CS] was stated for $\Omega \subset \C^2$, but its proof is applicable to $\Omega$ in all Stein manifolds $V^4$ including $\C^2$. It is clear that the operator $\text{Re}$ is a closed operator. Therefore, $P = \text{Re}\tilde{P}$ is a closed operator as well. \quad \square

**Proof of Main Theorem.** Main Theorem now follows from Corollary 2.4, Corollary 3.5 and Proposition 4.1. \quad \square

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