Quantum Information in Space and Time *

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Abstract

Many important results in modern quantum information theory have been obtained for an idealized situation when the spacetime dependence of quantum phenomena is neglected. However the transmission and processing of (quantum) information is a physical process in spacetime. Therefore such basic notions in quantum information theory as the notions of composite systems, entangled states and the channel should be formulated in space and time. We emphasize the importance of the investigation of quantum information in space and time. Entangled states in space and time are considered. A modification of Bell’s equation which includes the spacetime variables is suggested. A general relation between quantum theory and theory of classical stochastic processes is proposed. It expresses the condition of local realism in the form of a noncommutative spectral theorem. Applications of this relation to the security of quantum key distribution in quantum cryptography are considered.

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1 Introduction

Recent remarkable experimental and theoretical results have shown that quantum effects can provide qualitatively new forms of communication and computation, sometimes more powerful then the classical ones. Interesting and important results obtained in quantum computing, teleportation and cryptography are based on the investigation of basic properties of quantum mechanics. Especially important are properties of nonfactorized entangled states which were named by Schrodinger as the most characteristic feature of quantum mechanics.

Modern quantum information theory is built on ideas of classical information theory of C. Shannon and on the notions of von Neumann quantum mechanical entropy and of entangled states as formulated by J. Bell, see [1, 2, 3] for recent discussions. The spacetime dependence is not explicitly indicated in this approach. As a result, many important achievements in modern quantum information theory have been obtained for an idealized situation when the spacetime dependence of quantum phenomena is neglected.

We emphasize the importance of the investigation of quantum information effects in space and time. Transmission and processing of (quantum) information is a physical process in spacetime. Therefore a formulation of such basic notions in quantum information theory as the notions of composite systems, entangled states and the channel should include the spacetime variables.

In this paper entangled states in space and time are considered. A modification of Bell's equation which includes the spacetime variables is suggested. A general relation between

\[1\] The importance of the investigation of quantum information effects in space and time and especially the role of relativistic invariance in classical and quantum information theory was stressed in the talk by the author at the First International Conference on Quantum Information which was held at Meijo University, Japan, November 4-8, 1997.
quantum theory and classical theory of stochastic processes is proposed which expresses the condition of local realism in the form of a noncommutative spectral theorem. Applications of this relation to the security of quantum key distribution in quantum cryptography are considered.

Entangled states, i.e. the states of two particles with the wave function which is not a product of the wave functions of single particles, have been studied in many theoretical and experimental works starting from works of Einstein, Podolsky and Rosen, Bohm and Bell, see e.g. [4].

Bell’s theorem [5] states that there are quantum spin correlation functions that can not be represented as classical correlation functions of separated random variables. It has been interpreted as incompatibility of the requirement of locality with the statistical predictions of quantum mechanics [5]. For a recent discussion of Bell’s theorem see, for example [4] and references therein. It is now widely accepted, as a result of Bell’s theorem and related experiments, that ”Einstein’s local realism” must be rejected.

Let us note however that, evidently, the very formulation of the problem of locality in quantum mechanics is based on ascribing a special role to the position in ordinary three-dimensional space. It is rather surprising therefore that the space dependence of the wave function is neglected in discussions of the problem of locality in relation to Bell’s inequalities. Actually it is the space part of the wave function which is relevant to the consideration of the problem of locality.

We know that the wave function of particle includes not only the spin part but also the part depending on spacetime variables. Recently it was pointed out [10] that in fact the spacetime part of the wave function was neglected in the proof of Bell’s theorem. However just the spacetime part is crucial for considerations of property of locality of quantum system. Actually the spacetime part leads to an extra factor in quantum correlations and as a result the ordinary proof of Bell’s theorem fails in this case. We present a modification of Bell’s equation which includes space and time variables.

We present a criterion of locality (or nonlocality) of quantum theory in a realist model of hidden variables. We argue that predictions of quantum mechanics can be consistent with Bell’s inequalities for some Gaussian wave functions and hence Einstein’s local realism is restored in this case. Moreover we show that due to the expansion of the wave packet the locality criterion is always satisfied for nonrelativistic particles if regions of detectors are far enough from each other. This result has applications to the security of certain quantum cryptographic protocols.

We will consider an important connection between quantum mechanics and theory of classical stochastic processes. Consider for example an equation

\[ \cos(\alpha - \beta) = E \xi_\alpha \eta_\beta \]

where \( \xi_\alpha \) and \( \eta_\beta \) are two random processes [11] and \( E \) is the expectation. Bell’s theorem states that there exists no solution of the equation for bounded stochastic processes such that \( |\xi_\alpha| \leq 1, \ |\eta_\beta| \leq 1. \)

The function \( \cos(\alpha - \beta) \) describes the quantum mechanical correlation of spins of two entangled particles. It was shown in [11] that if one takes into account the space part of the wave function then the quantum correlation in the simplest case will take the form
$g \cos(\alpha - \beta)$ instead of just $\cos(\alpha - \beta)$ where the parameter $g$ describes the location of the system in space and time. In this case one gets a modified equation

$$g \cos(\alpha - \beta) = E \xi_\alpha \eta_\beta$$

One can prove (see below) that if $g$ is small enough then there exists a solution of the modified equation.

It is important to study also a more general question: which class of functions $f(s,t)$ admits a representation of the form

$$f(s,t) = E x_s y_t$$

where $x_s$ and $y_t$ are bounded stochastic processes and also analogous question for the functions of several variables $f(t_1, ..., t_n)$.

Such considerations could provide a noncommutative generalization of von Neumann’s spectral theorem.

Bell’s theorem constitutes an important part in quantum cryptography [12]. It is now generally accepted that techniques of quantum cryptography can allow secure communications between distant parties [13] - [20]. The promise of secure cryptographic quantum key distribution schemes is based on the use of quantum entanglement in the spin space and on quantum no-cloning theorem. An important contribution of quantum cryptography is a mechanism for detecting eavesdropping.

However in certain current quantum cryptography protocols the space part of the wave function is neglected. But just the space part of the wave function describes the behaviour of particles in ordinary real three-dimensional space. As a result such schemes can be secure against eavesdropping attacks in the abstract spin space but could be insecure in the real three-dimensional space. We will discuss how one can try to improve the security of quantum cryptography schemes in space by using a special preparation of the space part of the wave function, see [19].

Spacetime description is important for quantum computation [21]. Some problems of quantum teleportation in space have been discussed in [22].

2 Bell’s Theorem

2.1 Bell’s Theorem and Stochastic Processes

In the presentation of Bell’s theorem we will follow [10] where one can find also more references. Bell’s theorem reads:

$$\cos(\alpha - \beta) \neq E \xi_\alpha \eta_\beta$$  \hspace{1cm} (1)

where $\xi_\alpha$ and $\eta_\beta$ are two random processes such that $|\xi_\alpha| \leq 1$, $|\eta_\beta| \leq 1$ and $E$ is the expectation. In more details:

**Theorem 1.** There exists no probability space $(\Lambda, \mathcal{F}, d\rho(\lambda))$ and a pair of stochastic processes $\xi_\alpha = \xi_\alpha(\lambda)$, $\eta_\beta = \eta_\beta(\lambda)$, $0 \leq \alpha, \beta \leq 2\pi$ which obey $|\xi_\alpha(\lambda)| \leq 1$, $|\eta_\beta(\lambda)| \leq 1$ such that the following equation is valid

$$\cos(\alpha - \beta) = E \xi_\alpha \eta_\beta$$  \hspace{1cm} (2)
for all $\alpha$ and $\beta$.

Here $\Lambda$ is a set, $\mathcal{F}$ is a sigma-algebra of subsets and $d\rho(\lambda)$ is a probability measure, i.e. $d\rho(\lambda) \geq 0$, $\int d\rho(\lambda) = 1$. The expectation is

$$E_{\xi, \eta} = \int_{\Lambda} \xi(\lambda) \eta(\lambda) d\rho(\lambda)$$

One can write Eq. (2) as an integral equation

$$\cos(\alpha - \beta) = \int_{\Lambda} \xi(\lambda) \eta(\lambda) d\rho(\lambda) \quad (3)$$

We say that the integral equation (3) has no solutions $(\Lambda, \mathcal{F}, d\rho(\lambda), \xi, \eta)$ with the bound $|\xi| \leq 1$, $|\eta| \leq 1$.

We will prove the theorem below. Let us discuss now the physical interpretation of this result.

Consider a pair of spin one-half particles formed in the singlet spin state and moving freely towards two detectors. If one neglects the space part of the wave function then one has the Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^2$ and the quantum mechanical correlation of two spins in the singlet state $\psi_{\text{spin}} \in \mathbb{C}^2 \otimes \mathbb{C}^2$ is

$$D_{\text{spin}}(a, b) = \langle \psi_{\text{spin}} | \sigma \cdot a \otimes \sigma \cdot b | \psi_{\text{spin}} \rangle = -a \cdot b \quad (4)$$

Here $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ are two unit vectors in three-dimensional space $\mathbb{R}^3$, $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\sigma \cdot a = \sum_{i=1}^{3} \sigma_i a_i$$

and

$$\psi_{\text{spin}} = \frac{1}{\sqrt{2}} \left( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

If the vectors $a$ and $b$ belong to the same plane then one can write $-a \cdot b = \cos(\alpha - \beta)$ and hence Bell’s theorem states that the function $D_{\text{spin}}(a, b)$ Eq. (4) can not be represented in the form

$$P(a, b) = \int_{\Lambda} \xi(a, \lambda) \eta(b, \lambda) d\rho(\lambda) \quad (5)$$

i.e.

$$D_{\text{spin}}(a, b) \neq P(a, b) \quad (6)$$

Here $\xi(a, \lambda)$ and $\eta(b, \lambda)$ are random fields on the sphere, $|\xi(a, \lambda)| \leq 1$, $|\eta(b, \lambda)| \leq 1$ and $d\rho(\lambda)$ is a positive probability measure, $\int d\rho(\lambda) = 1$. The parameters $\lambda$ are interpreted as hidden variables in a realist theory. It is clear that Eq. (3) can be reduced to Eq. (1).
2.2 CHSH Inequality

To prove Theorem 1 we will use the following

**Theorem 2.** Let $f_1, f_2, g_1$ and $g_2$ be random variables (i.e. measured functions) on the probability space $(\Lambda, \mathcal{F}, d\rho(\lambda))$ such that

$$|f_i(\lambda)g_j(\lambda)| \leq 1, \quad i, j = 1, 2.$$ 

Denote

$$P_{ij} = Ef_{i}g_{j}, \quad i, j = 1, 2.$$ 

Then

$$|P_{11} - P_{12}| + |P_{21} + P_{22}| \leq 2.$$ 

**Proof of Theorem 2.** One has

$$P_{11} - P_{12} = Ef_{1}g_{1} - Ef_{1}g_{2} = E(f_{1}g_{1}(1 \pm f_{2}g_{2})) - E(f_{1}g_{2}(1 \pm f_{2}g_{1}))$$

Hence

$$|P_{11} - P_{12}| \leq E(1 \pm f_{2}g_{2}) + E(1 \pm f_{2}g_{1}) = 2 \pm (P_{22} + P_{21})$$

Now let us note that if $x$ and $y$ are two real numbers then

$$|x| \leq 2 \pm y \rightarrow |x| + |y| \leq 2.$$ 

Therefore taking $x = P_{11} - P_{12}$ and $y = P_{22} + P_{21}$ one gets the bound

$$|P_{11} - P_{12}| + |P_{21} + P_{22}| \leq 2.$$ 

The theorem is proved.

The last inequality is called the Clauser-Horn-Shimony-Holt (CHSH) inequality. By using notations of Eq. \(\mathbb{5}\) one has

$$|P(a, b) - P(a', b')| + |P(a', b) + P(a', b')| \leq 2$$  \(\mathbb{7}\)

for any four unit vectors $a, b, a', b'$.

**Proof of Theorem 1.** Let us denote

$$f_{i}(\lambda) = \xi_{\alpha_{i}}(\lambda), \quad g_{j}(\lambda) = \eta_{\beta_{j}}(\lambda), \quad i, j = 1, 2$$

for some $\alpha_{i}, \beta_{j}$. If one would have

$$\cos(\alpha_{i} - \beta_{j}) = Ef_{i}g_{j}$$

then due to Theorem 2 one should have

$$|\cos(\alpha_{1} - \beta_{1}) - \cos(\alpha_{1} - \beta_{2})| + |\cos(\alpha_{2} - \beta_{1}) + \cos(\alpha_{2} - \beta_{2})| \leq 2.$$ 

However for $\alpha_{1} = \pi/2$, $\alpha_{2} = 0$, $\beta_{1} = \pi/4$, $\beta_{2} = -\pi/4$ we obtain

$$|\cos(\alpha_{1} - \beta_{1}) - \cos(\alpha_{1} - \beta_{2})| + |\cos(\alpha_{2} - \beta_{1}) + \cos(\alpha_{2} - \beta_{2})| = 2\sqrt{2}$$
which is greater than 2. This contradiction proves Theorem 1.

It will be shown below that if one takes into account the space part of the wave function then the quantum correlation in the simplest case will take the form $g \cos(\alpha - \beta)$ instead of just $\cos(\alpha - \beta)$ where the parameter $g$ describes the location of the system in space and time. In this case one can get a representation

$$g \cos(\alpha - \beta) = E \xi_\alpha \eta_\beta$$  \hspace{1cm} (8)$$

if $g$ is small enough. The factor $g$ gives a contribution to visibility or efficiency of detectors that are used in the phenomenological description of detectors.

3 Localized Detectors

3.1 Modified Bell’s equation

In the previous section the space part of the wave function of the particles was neglected. However exactly the space part is relevant to the discussion of locality. The Hilbert space assigned to one particle with spin 1/2 is $\mathbb{C}^2 \otimes L^2(\mathbb{R}^3)$ and the Hilbert space of two particles is $\mathbb{C}^2 \otimes L^2(\mathbb{R}^3) \otimes \mathbb{C}^2 \otimes L^2(\mathbb{R}^3)$. The complete wave function is $\psi = (\psi_{\alpha\beta}(r_1, r_2, t))$ where $\alpha$ and $\beta$ are spinor indices, $t$ is time and $r_1$ and $r_2$ are vectors in three-dimensional space.

We suppose that there are two detectors (“Alice” and ”Bob”) which are located in space $\mathbb{R}^3$ within the two localized regions $O_A$ and $O_B$ respectively, well separated from one another. One could apply here an approach of local algebras [23].

Quantum correlation describing the measurements of spins by Alice and Bob at their localized detectors is

$$G(a, O_A, b, O_B) = \langle \psi | \sigma \cdot a P_{O_A} \otimes \sigma \cdot b P_{O_B} | \psi \rangle$$  \hspace{1cm} (9)$$

Here $P_O$ is the projection operator onto the region $O$.

Let us consider the case when the wave function has the form of the product of the spin function and the space function $\psi = \psi_{spin} \phi(r_1, r_2)$. Then one has

$$G(a, O_A, b, O_B) = g(O_A, O_B) D_{spin}(a, b)$$  \hspace{1cm} (10)$$

where the function

$$g(O_A, O_B) = \int_{O_A \times O_B} |\phi(r_1, r_2)|^2 dr_1 dr_2$$  \hspace{1cm} (11)$$

describes correlation of particles in space. It is the probability to find one particle in the region $O_A$ and another particle in the region $O_B$.

One has

$$0 \leq g(O_A, O_B) \leq 1$$  \hspace{1cm} (12)$$

Remark 1. In relativistic quantum field theory there is no nonzero strictly localized projection operator that annihilates the vacuum. It is a consequence of the Reeh-Schlieder
theorem. Therefore, apparently, the function $g(O_A, O_B)$ should be always strictly smaller than 1.

Now one inquires whether one can write the representation

$$g(O_A, O_B)D_{\text{spin}}(a, b) = \int \xi(a, O_A, \lambda)\eta(b, O_B, \lambda)d\rho(\lambda)$$  \hspace{1cm} (13)

Note that if we are interested in the conditional probability of finding the projection of spin along vector $a$ for the particle 1 in the region $O_A$ and the projection of spin along the vector $b$ for the particle 2 in the region $O_B$ then we have to divide both sides of Eq. (13) by $g(O_A, O_B)$.

Instead of Eq (2) in Theorem 1 now we have the modified equation

$$g \cos(\alpha - \beta) = E\xi_\alpha \eta_\beta$$  \hspace{1cm} (14)

The factor $g$ is important. In particular one can write the following representation [9] for $0 \leq g \leq 1/2$:

$$g \cos(\alpha - \beta) = \int_0^{2\pi} \sqrt{2g} \cos(\alpha - \lambda)\sqrt{2g} \cos(\beta - \lambda)\frac{d\lambda}{2\pi}$$  \hspace{1cm} (15)

Therefore if $0 \leq g \leq 1/2$ then there exists a solution of Eq. (14) where

$$\xi_\alpha(\lambda) = \sqrt{2g} \cos(\alpha - \lambda), \quad \eta_\beta(\lambda) = \sqrt{2g} \cos(\beta - \lambda)$$

and $|\xi_\alpha| \leq 1, |\eta_\beta| \leq 1$. If $g > 1/\sqrt{2}$ then it follows from Theorem 2 that there is no solution to Eq. (14). We have obtained

**Theorem 3.** If $g > 1/\sqrt{2}$ then there is no solution $(\Lambda, \mathcal{F}, d\rho(\lambda), \xi_\alpha, \eta_\beta)$ to Eq. (14) with the bounds $|\xi_\alpha| \leq 1, |\eta_\beta| \leq 1$. If $0 \leq g \leq 1/2$ then there exists a solution to Eq. (14) with the bounds $|\xi_\alpha| \leq 1, |\eta_\beta| \leq 1$.

**Remark 2.** Further results on solutions of the modified equation have been obtained by A.K. Guschchin, S. V. Bochkarev and D. Prokhorenko. Local variable models for inefficient detectors are presented in [24, 25].

**Remark 3.** A local modified equation reads

$$|\phi(r_1, r_2, t)|^2 \cos(\alpha - \beta) = E\xi(\alpha, r_1, t)\eta(\beta, r_2, t).$$

### 3.2 Relativistic Particles

We can not immediately apply the previous considerations to the case of relativistic particles such as photons and the Dirac particles because in these cases the wave function cannot be represented as a product of the spin part and the spacetime part. Let us show that the wave function of photon cannot be represented in the product form. Let $A_i(k)$ be the wave function of photon, where $i = 1, 2, 3$ and $k \in \mathbb{R}^3$. One has the gauge condition $k^iA_i(k) = 0$ [26]. If one supposes that the wave function has a product form $A_i(k) = \phi_i f(k)$ then from the gauge condition one gets $A_i(k) = 0$. Therefore the case of relativistic particles requires a separate investigation.
3.3 Noncommutative Spectral Theory and Local Realism

As a generalisation of the previous discussion we would like to suggest here a general relation between quantum theory and theory of classical stochastic processes [11] which expresses the condition of local realism. Let \( \mathcal{H} \) be a Hilbert space, \( \rho \) is the density operator, \( \{ A_\alpha \} \) is a family of self-adjoint operators in \( \mathcal{H} \). One says that the family of observables \( \{ A_\alpha \} \) and the state \( \rho \) satisfy to the condition of local realism if there exists a probability space \( (\Lambda, \mathcal{F}, d\rho(\lambda)) \) and a family of random variables \( \{ \xi_\alpha \} \) such that the range of \( \xi_\alpha \) belongs to the spectrum of \( A_\alpha \) and for any subset \( \{ A_i \} \) of mutually commutative operators one has a representation

\[
\text{Tr}(\rho A_{i_1}...A_{i_n}) = E\xi_{i_1}...\xi_{i_n}
\]

The physical meaning of the representation is that it describes the quantum-classical correspondence. If the family \( \{ A_\alpha \} \) would be a maximal commutative family of self-adjoint operators then for pure states the previous representation can be reduced to the von Neumann spectral theorem [27]. In our case the family \( \{ A_\alpha \} \) consists from not necessary commuting operators. Hence we will call such a representation a noncommutative spectral representation.

Of course one has a question for which families of operators and states a noncommutative spectral theorem is valid, i.e. when we can write the noncommutative spectral representation. We need a noncommutative generalization of von Neumann’s spectral theorem.

It would be helpful to study the following problem: describe the class of functions \( f(t_1, ..., t_n) \) which admits the representation of the form

\[
f(t_1, ..., t_n) = Ex_{t_1}...z_{t_n}
\]

where \( x_t, ..., z_t \) are random processes which obey the bounds \( |x_t| \leq 1, ..., |z_t| \leq 1 \).

From the previous discussion we know that there are such families of operators and such states which do not admit the noncommutative spectral representation and therefore they do not satisfy the condition of local realism. Indeed let us take the Hilbert space \( \mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \) and four operators \( A_1, A_2, A_3, A_4 \) of the form (we denote \( A_3 = B_1, A_4 = B_2 \))

\[
A_1 = \begin{pmatrix} \sin \alpha_1 & \cos \alpha_1 \\ \cos \alpha_1 & -\sin \alpha_1 \end{pmatrix} \otimes I, \quad A_2 = \begin{pmatrix} \sin \alpha_2 & \cos \alpha_2 \\ \cos \alpha_2 & -\sin \alpha_2 \end{pmatrix} \otimes I
\]

and

\[
B_1 = I \otimes \begin{pmatrix} -\sin \beta_1 & -\cos \beta_1 \\ -\cos \beta_1 & \sin \beta_1 \end{pmatrix}, \quad B_2 = I \otimes \begin{pmatrix} -\sin \beta_2 & -\cos \beta_2 \\ -\cos \beta_2 & \sin \beta_2 \end{pmatrix}
\]

Here operators \( A_i \) correspond to operators \( \sigma \cdot a \) and operators \( B_i \) corresponds to operators \( \sigma \cdot b \) where \( a = (\cos \alpha, 0, \sin \alpha) \), \( b = (-\cos \beta, 0, -\sin \beta) \). Operators \( A_i \) commute with operators \( B_j \), \( [A_i, B_j] = 0, \ i, j = 1, 2 \) and one has

\[
\langle \psi_{\text{spin}} | A_i B_j | \psi_{\text{spin}} \rangle = \cos(\alpha_i - \beta_j), \ i, j = 1, 2
\]

We know from Theorem 2 that this function can not be represented as the expected value \( E\xi_i \eta_j \) of random variables with the bounds \( |\xi_i| \leq 1, |\eta_j| \leq 1 \).

However, as it was discussed above, the space part of the wave function was neglected in the previous consideration. We suggest that in physics one could prepare only such states...
and observables which satisfy the condition of local realism. Perhaps we should restrict
ourself in this proposal to the consideration of only such families of observables which satisfy
the condition of relativistic local causality. If there are physical phenomena which do not
satisfy this proposal then it would be important to describe quantum processes which satisfy
the above formulated condition of local realism and also processes which do not satisfy this
condition.

Let us now apply these considerations to quantum cryptography.

4 The Quantum Key Distribution

4.1 Protocol

There are quantum cryptographic protocols with one and with two particles, for a review see
for example [2]. Here we shall consider the quantum key distribution with two particles.
Ekert [12] showed that one can use the Einstein-Podolsky-Rosen correlations to establish a
secret random key between two parties ("Alice" and "Bob"). Bell’s inequalities are used to
check the presence of an intermediate eavesdropper ("Eve"). There are two stages to the
quantum cryptographic protocol, the first stage over a quantum channel, the second over a
public channel.

The quantum channel consists of a source that emits pairs of spin one-half particles,
in a singlet state. The particles fly apart towards Alice and Bob, who, after the particles
have separated, perform measurements on spin components along one of three directions,
given by unit vectors \( a \) and \( b \). In the second stage Alice and Bob communicate over a
public channel. They announce in public the orientation of the detectors they have chosen
for particular measurements. Then they divide the measurement results into two separate
groups: a first group for which they used different orientation of the detectors, and a second
group for which they used the same orientation of the detectors. Now Alice and Bob can
reveal publicly the results they obtained but within the first group of measurements only.
This allows them, by using Bell’s inequality, to establish the presence of an eavesdropper
(Eve). The results of the second group of measurements can be converted into a secret key.
One supposes that Eve has a detector which is located within the region \( \mathcal{O}_E \) and she is
described by hidden variables \( \lambda \).

We will interpret Eve as a hidden variable in a realist theory and will study whether the
quantum correlation can be represented in the form Eq. (13). From Theorem 3 one can see
that if the following inequality

\[
g(\mathcal{O}_A, \mathcal{O}_B) \leq 1/2
\]

is valid for regions \( \mathcal{O}_A \) and \( \mathcal{O}_B \) which are well separated from one another then there is
no violation of the CHSH inequalities (7) and therefore Alice and Bob can not detect the
presence of an eavesdropper. On the other side, if for a pair of well separated regions \( \mathcal{O}_A \)
and \( \mathcal{O}_B \) one has

\[
g(\mathcal{O}_A, \mathcal{O}_B) > 1/\sqrt{2}
\]

then it could be a violation of the realist locality in these regions for a given state. Then, in
principle, one can hope to detect an eavesdropper in these circumstances.
Note that if we set \( g(O_A, O_B) = 1 \) in (13) as it was done in the original proof of Bell’s theorem, then it means we did a special preparation of the states of particles to be completely localized inside of detectors. There exist such well localized states (see however the previous Remark) but there exist also another states, with the wave functions which are not very well localized inside the detectors, and still particles in such states are also observed in detectors. The fact that a particle is observed inside the detector does not mean, of course, that its wave function is strictly localized inside the detector before the measurement. Actually one has to perform a thorough investigation of the preparation and the evolution of our entangled states in space and time if one needs to estimate the function \( g(O_A, O_B) \).

4.2 Gaussian Wave Functions

Now let us consider the criterion of locality for Gaussian wave functions. We will show that with a reasonable accuracy there is no violation of locality in this case. Let us take the wave function \( \phi \) of the form \( \phi = \psi_1(r_1)\psi_2(r_2) \) where the individual wave functions have the moduli
\[
|\psi_1(r)|^2 = \left( \frac{m^2}{2\pi} \right)^{3/2} e^{-m^2 r^2 / 2}, \quad |\psi_2(r)|^2 = \left( \frac{m^2}{2\pi} \right)^{3/2} e^{-m^2 (r-\ell)^2 / 2}
\]
We suppose that the length of the vector \( \ell \) is much larger than \( 1/m \). We can make measurements of \( P_{O_A} \) and \( P_{O_B} \) for any well separated regions \( O_A \) and \( O_B \). Let us suppose a rather nonfavorite case for the criterion of locality when the wave functions of the particles are almost localized inside the regions \( O_A \) and \( O_B \) respectively. In such a case the function \( g(O_A, O_B) \) can take values near its maximum. We suppose that the region \( O_A \) is given by \( |r_1| < 1/m, r = (r_1, r_2, r_3) \) and the region \( O_B \) is obtained from \( O_A \) by translation on \( \ell \). Hence \( \psi_1(r_1) \) is a Gaussian function with modules appreciably different from zero only in \( O_A \) and similarly \( \psi_2(r_2) \) is localized in the region \( O_B \). Then we have
\[
g(O_A, O_B) = \left( \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-x^2/2} dx \right)^6
\]
One can estimate (13) as
\[
g(O_A, O_B) < \left( \frac{2}{\pi} \right)^3
\]
which is smaller than 1/2. Therefore the locality criterion (13) is satisfied in this case.

4.3 Expansion of Wave Packet

Let us remind that there is a well known effect of expansion of wave packets due to the free time evolution. If \( \epsilon \) is the characteristic length of the Gaussian wave packet describing a particle of mass \( M \) at time \( t = 0 \) then at time \( t \) the characteristic length \( \epsilon_t \) will be
\[
\epsilon_t = \epsilon \sqrt{1 + \frac{\hbar^2 t^2}{M^2 \epsilon^4}}.
\]
It tends to \( (\hbar/M\epsilon)t \) as \( t \to \infty \). Therefore the locality criterion is always satisfied for nonrelativistic particles if regions \( O_A \) and \( O_B \) are far enough from each other.
5 Conclusions

We have studied some problems in quantum information theory which requires the inclusion of spacetime variables. In particular entangled states in space and time were considered. A modification of Bell's equation which includes the spacetime variables is suggested and investigated. A general relation between quantum theory and theory of classical stochastic processes was proposed which expresses the condition of local realism in the form of a noncommutative spectral theorem. Applications of this relation to the security of quantum key distribution in quantum cryptography were considered.

There are many interesting open problems in the approach to quantum information in space and time discussed in this paper. Some of them related with the noncommutative spectral theory and theory of classical stochastic processes have been discussed above.

In quantum cryptography there are important open problems which require further investigations. In quantum cryptographic protocols with two entangled photons to detect the eavesdropper's presence by using Bell's inequality we have to estimate the function \( g(\mathcal{O}_A, \mathcal{O}_B) \). To increase the detectability of the eavesdropper one has to do a thorough investigation of the process of preparation of the entangled state and then its evolution in space and time towards Alice and Bob. One has to develop a proof of the security of such a protocol.

In the previous section Eve was interpreted as an abstract hidden variable. However one can assume that more information about Eve is available. In particular one can assume that she is located somewhere in space in a region \( \mathcal{O}_E \). It seems one has to study a generalization of the function \( g(\mathcal{O}_A, \mathcal{O}_B) \), which depends not only on the Alice and Bob locations \( \mathcal{O}_A \) and \( \mathcal{O}_B \) but also depends on the Eve location \( \mathcal{O}_E \), and try to find a strategy which leads to an optimal value of this function.

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