RANDOM ITERATIONS OF MAPS ON $\mathbb{R}^k$: ASYMPTOTIC STABILITY, SYNCHRONIZATION AND FUNCTIONAL CENTRAL LIMIT THEOREM

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Abstract. We study independent and identically distributed random iterations of continuous maps defined on a connected closed subset $S$ of the Euclidean space $\mathbb{R}^k$. We assume the maps are monotone (with respect to a suitable partial order) and a “topological” condition on the maps. Then, we prove the existence of a pullback random attractor whose distribution is the unique stationary measure of the random iteration, and we obtain the synchronization of random orbits. As a consequence of the synchronization phenomenon, a functional central limit theorem is established.

1. Introduction

Let $X = \{X_n\}$ be an independent and identically distributed (i.i.d) sequence of random variables taking values on a measurable space $E$ and consider a family $\{f_\alpha\}_{\alpha \in E}$ of maps $f_\alpha : S \to S$. Under appropriate measurability assumptions, these two ingredients specify a homogeneous Markov chain with state space $S$ given by

\[ Z_n = f_{X_{n-1}} \circ \cdots \circ f_{X_0}(Z_0) \]

called an i.i.d. random iteration of maps, where $Z_0$ is a random variable independent of $X = \{X_n\}$ taking values on $S$. In the case $Z_0 = x$, we denote the random iteration (1.1) by $Z_n^x$.

We study i.i.d. random iterations of maps defined on a connected closed subset $S$ of the Euclidean space $\mathbb{R}^k$. For every subset $J \subset \{1, \ldots, k\}$ we define a partial order on $\mathbb{R}^k$ and we introduce the class of strictly $J$-monotone maps (see Section 2.1 for the precise definition). For $J = I_k$, the partial order is just the strict componentwise order. Several notions of convergence for the Markov chain induced by a family of strictly $J$-monotone maps are investigated, provided the maps also satisfy a “topological condition” called the ping-pong property, see Definition 2.2. This condition is closely related to the splitting property considered in [7,17]. In [7], the authors consider i.i.d. random iterations of non-decreasing continuous maps (w.r.t. the weak componentwise order) defined on a closed subset $S \subset \mathbb{R}^k$ satisfying the splitting property. Then they prove asymptotic stability of the Markov chain, i.e., that there exists a unique initial distribution (the distribution of $Z_0$) for which the Markov chain $Z_n^x$ is stationary, and under every initial distribution the sequence $Z_n$ converges in distribution to the stationary measure. This result was

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generalized for monotone maps, see [5]. Furthermore, in [5, 7] the authors show that for i.i.d. random iterations of non-decreasing maps and observables $\phi$ that may be expressed as a difference of two non-decreasing bounded functions, the sequence $\phi(Z_n)$ satisfies a functional central limit theorem (FCLT), in the sense that the process $Y_n$ given by

$$Y_n(t) = \frac{1}{\sqrt{n}} \sum_{j=0}^{\lfloor nt \rfloor} \phi(Z_j), \quad 0 \leq t < \infty$$

converges to a Brownian motion under every initial distribution. These results have found applications in mathematical economics and nonlinear autoregressive models, see [5, 7].

In this paper, by using a different approach, we prove similar results for i.i.d. random iterations of strictly $J$-monotone maps satisfying the ping-pong property. Our theorems extend the results in [5, 7]. First, we prove the asymptotic stability for a wider class of monotone maps (the $J$-monotone maps) and we give a characterization of the unique stationary measure through Letac principle [27], see Theorem 1. Then, we prove that a synchronization phenomenon holds (see Theorem 2) and we explore this property to obtain a functional central limit theorem, see Theorem 3. Unlike the FCLT obtained in [5, 7], in our Theorem 3 the family of maps may include both types: increasing and decreasing maps.

The functional central limit theorem is obtained for i.i.d. random iterations of $J$-monotone maps and Lipschitz observables. We only assume the ping-pong property and a certain boundedness condition on the Markov chain $Z_n$, see Theorem 3. This result is a consequence of Theorem 2, which states the synchronization of i.i.d. random iterations of strictly $J$-monotone maps satisfying the ping-pong property. Namely, we prove that, with probability 1, for every pair $x, y \in S$, the i.i.d. random iterations $Z^x_n$ and $Z^y_n$ satisfy

$$d(Z^x_n, Z^y_n) \xrightarrow{n \to \infty} 0$$

exponentially fast, where $d$ is the Euclidean distance. The synchronization effect (or some contraction property) usually leads to a central limit theorem (CLT). In [4, 33], central limit theorems are obtained for a certain class of i.i.d. random iterations of Lipschitz maps having negative Lyapunov exponent. For an i.i.d. random iteration of homeomorphisms on the circle, Malicet in [28] shows that a local synchronization holds under the assumption that the maps do not have an invariant measure in common. This result was later used to prove a CLT in [32]. In [26], a central limit theorem is obtained for contractive iterated function systems with place-dependent probabilities.

The phenomenon of synchronization was first observed by Huygens [21] in the movement of two pendulum clocks hanging from a wall and since then has been investigated in several areas, see [30]. For random iterations, results on synchronization were obtained in several settings where no contraction-like property is given a priori, see [1, 19, 20, 24, 28].

Usually, the synchronization effect is derived from negative Lyapunov exponents, with the suitable definition in each setting, see discussion in Section 2.3. However, in some situations Lyapunov exponents play no role. This is the case of the i.i.d.
random iteration of double rotations studied in [19], where the authors obtain a synchronization on average using properties of simple random walks on \( Z \). The synchronization of random orbits obtained in our Theorem 2 for i.i.d. random iterations of \( J \)-monotone maps, is another example where an analysis of Lyapunov exponents does not take place. We emphasize that the maps are not assumed to be Lipschitz throughout this paper. We refer to [16, 23] for related results for Markovian random iterations of finitely many maps defined on a connected compact subset of a Euclidean space.

We observe that, even at the intersection of the class of random iterations considered in this paper with the one in [7], we provide new results. Indeed, concerning the asymptotic stability, we show that the unique stationary measure is the distribution of a minimal random pullback attractor, in a sense to be clarified in Corollary 1. This characterization of stationary measures is already known for random iterations having negative Lyapunov exponents (in the case where the maps are at least Lipschitz), see [3, 27, 31]. Finally, concerning the FCLT, the family of maps in our Theorem 3 may include both types: increasing and decreasing ([5, 7] considers only non-decreasing maps).

**Organization of the paper.** In Section 2 we state precisely the main definitions and results of this work. In Section 3 we prove Theorem 3 and Corollary 1. In Section 4 we prove Theorem 2 and Corollaries 2 and 3. Theorem 3 is proved in Section 5.

### 2. Statements of results

#### 2.1. General setting.

Let \((E, \mathcal{F}, \nu)\) be a probability space and consider an i.i.d. sequence of random variables \(X = \{X_n\}\) with state space \(E\). Throughout, \(\nu\) denote the common distribution of \(X\) and \((\Omega, \mathcal{F}, \mathbb{P})\) is the probability space where \(X\) is defined. Let \(S \subset \mathbb{R}^k\) be a connected subspace and consider a measurable map \(f: E \times S \to S\). We denote by \(f_\alpha\) the map \(f_\alpha(x) = f(\alpha, x)\). For every random variable \(Z_0: \Omega \to S\) independent of \(X\), the pair \((f, X)\) induces a homogeneous Markov chain \(Z_n\) as defined in (1.1), whose transition probability is given by

\[
p(x, A) = \int \mathbb{1}_A(f_\alpha(x)) \, d\nu(\alpha).
\]

The pair \((f, X)\), as well as any of its induced Markov chain, will be called an *i.i.d. random iteration of maps*.

Now, let us introduce the class of maps that we study in this paper. Consider a subset \(J \subset \{1, \ldots, k\}\) and define a partial order as follows: given \(x, y \in \mathbb{R}^k\), we write \(x <_J y\) if and only if

\[
x_i < y_i \text{ for } i \in J \text{ and } x_i > y_i \text{ for } i \notin J.
\]

Consider a map \(f: S \to S\). We say that \(f\) is *strictly \(J\)-increasing* if

\[
x <_J y \Rightarrow f(x) <_J f(y).
\]

Similarly, we say that \(f\) is *strictly \(J\)-decreasing* if

\[
x <_J y \Rightarrow f(y) <_J f(x).
\]

The map \(f\) is called *strictly \(J\)-monotone* if \(f\) is either strictly \(J\)-increasing or strictly \(J\)-decreasing. Given subsets \(S_1, S_2 \subset \mathbb{R}^k\), we write \(S_1 <_J S_2\) if

\[
x <_J y \text{ for every } x \in S_1 \text{ and } y \in S_2.
\]
Let $\pi_s: \mathbb{R}^k \to \mathbb{R}$ be the natural projection $\pi_s(x) = x_s$, $s = 1, \ldots, k$.

**Remark 2.1.** A fundamental property of the partial order $<_J$ is the following: if $S_1 <_J S_2$, then

$$\pi_s(S_1) \cap \pi_s(S_2) = \emptyset$$

for every $s = 1, \ldots, k$. That is, the respective projections of $S_1$ and $S_2$ are disjoints. In particular, for every strictly $J$-monotone map $f$ we have that $f(S_1)$ and $f(S_2)$ have disjoints projections because either $f(S_1) <_J f(S_2)$ or $f(S_2) <_J f(S_1)$.

In this paper, we study random iterations of strictly $J$-monotone maps satisfying the following “topological” property:

**Definition 2.2.** We say that an i.i.d. random iteration $(f, X)$ of strictly $J$-monotone maps satisfies the ping-pong property if there are $m \in \mathbb{N}$ and measurable subsets $A$ and $B$ of $E^m$ with $\nu^m(A) > 0$ and $\nu^m(B) > 0$ such that for every $(\alpha_0, \ldots, \alpha_{m-1}) \in A$ and $(\beta_0, \ldots, \beta_{m-1}) \in B$ we have

$$f_{\alpha_0} \circ \cdots \circ f_{\alpha_{m-1}}(S) <_J f_{\beta_0} \circ \cdots \circ f_{\beta_{m-1}}(S).$$

This condition is related to the splitting condition considered in [5, 7], see the discussion after Theorem 4 in Section 2.2.

2.2. **Stationary measures.** Let $p$ be the transition probability as defined in (2.1). Associate with $p$ there is an operator acting on the space of probability measures on $S$ given by $\mu \mapsto T\mu$, where $T\mu$ is the probability measure defined by

$$T\mu(A) = \int p(x, A) \, d\mu(x)$$

for every Borel set $A \subset S$. A fixed point for $T$ is called a stationary measure. That is, a probability measure $\mu$ on $S$ is a stationary measure if

$$\mu(A) = \int p(x, A) \, d\mu(x)$$

for every Borel set $A \subset S$.

We say that $T$ is asymptotically stable if there is a stationary measure $\mu$ such that for every probability measure $\zeta$ we have that $T^n\zeta$ converges to $\mu$ in the weak-star topology. Results on the stability of Markov operators for i.i.d. random iterations
of monotone continuous maps go back to Dubins and Freedman \[17\]. Therein, asymptotic stability is proved for Markov operators associated with i.i.d. random iterations of monotone maps on \([0, 1]\) satisfying the following condition called \textit{splitting property}: there are \(x_0 \in \mathbb{R}\) and \(m \geq 1\) such that

\[
P(Z_m^x \leq x_0 \ \forall x) > 0 \quad \text{and} \quad P(Z_m^x \geq x_0 \ \forall x) > 0.
\]

This result was generalized in higher dimensions in \[5, 7\] for i.i.d. random iterations of monotone maps defined on a closed subset \(S \subset \mathbb{R}^k\) satisfying the multidimensional analog of the splitting property, where the total order \(\leq\) of \(\mathbb{R}\) is replaced by the weak componentwise order on \(\mathbb{R}^k\). Recall that the weak componentwise order is a partial order \(\leq\) defined by: \(x \leq y\) if and only if \(x_i \leq y_i\) for every \(i = 1, \ldots, k\).

Our first result states the asymptotic stability of Markov operators associated with i.i.d. random iterations of \(J\)-strictly monotone continuous maps satisfying the ping-pong property. This extends the results in \[5, 17\] for a wider class of monotone maps. Throughout this paper, if \(\pi: \Omega \to S\) is a measurable map, then we denote by \(\pi \mathbb{P}\) the image of \(\mathbb{P}\) by \(\pi\), that is, the probability measure on \(S\) given by \(\pi \mathbb{P}(A) = \mathbb{P}(\pi^{-1}(A))\) for every Borel set \(A \subset S\). The probability \(\pi \mathbb{P}\) is also called the distribution of \(\pi\).

**Theorem 1.** Let \(S\) be a connected closed subset of \(\mathbb{R}^k\) and let \((f, X)\) be an i.i.d. random iteration of strictly \(J\)-monotone continuous maps on \(S\) satisfying the ping-pong property. Then

(i) There is a measurable map \(\pi: \Omega \to S\) such that for \(\mathbb{P}\)-almost every \(\omega\) we have

\[
\pi(\omega) = \lim_{n \to \infty} f_{X_0(\omega)} \circ \cdots \circ f_{X_n(\omega)}(x)
\]

for every \(x \in S\).

(ii) The probability measure \(\pi \mathbb{P}\) is the unique stationary measure and for every probability measure \(\varsigma\) on \(S\) we have

\[
T^n \varsigma \to \pi \mathbb{P}
\]

in the weak-star topology, where \(T\) is the Markov operator.

The second item of Theorem 1 says that the Markov operator is asymptotically stable. We observe that the proof of item (ii) is a straightforward consequence of item (i) and the Letac principle, see \[27\].

Note that if \((f, X)\) is an i.i.d. random iteration of strictly monotone maps (w.r.t. the weak componentwise order \(\leq\)) satisfying

\[
P(Z_m^x < x_0 \ \forall x) > 0 \quad \text{and} \quad P(Z_m^x > x_0 \ \forall x) > 0
\]

for some \(m\), then \((f, X)\) is of both types: an i.i.d. random iteration of monotone maps (w.r.t. the weak componentwise order) satisfying the splitting property and an i.i.d. random iteration of strictly \(J\)-monotone maps satisfying the ping-pong property with \(J = \{1, \ldots, k\}\). Thus, the intersection between these two classes is indeed non-empty. Note that at this intersection the asymptotic stability stated in Theorem 1 is not a new result. Nevertheless, Theorem 1 states a little more than the asymptotic stability. In fact, it shows that the unique stationary measure arises as the distribution of a pullback random attractor, see Section 2.2.1.
2.2.1. Pullback attractor. In this section, we take an alternative point of view of random iterations in order to show that the map $\pi$ of Theorem 1 is a pullback random attractor in the sense of [15]. To this end, let $\Omega = E^Z$ endowed with the product $\sigma$-algebra and the product measure $\mathbb{P} = \nu^Z$. Consider a measurable map $f: E \times S \to S$ and denote by $f_\omega$ the map $f_\omega(x) = f(\omega, x)$. Following Arnold [2], the map $\phi: \mathbb{N} \times \Omega \times S \to S$ given by

$$ (2.3) \quad \phi(n, \omega, x) = f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_0}(x) \overset{def}{=} f_\omega^n(x) $$

is called an i.i.d. random iteration of maps.

Note that the sequence of natural projections $X_n(\omega) = \omega_n$, $n \geq 0$, is an i.i.d. sequence of random variables with distribution $\nu$. Hence the sequence $\omega \mapsto f_\omega^n(x)$ is also an i.i.d. random iteration as in (1.1). We say that the map $\phi$ satisfies the ping-pong property if $(f, X)$ does satisfy.

We recall that a pullback random attractor of $\phi$ is a $\phi$-invariant random compact set $\omega \mapsto K(\omega)$ such that for $P$-almost every $\omega$

$$ \lim_{n \to \infty} d(\phi(n, \sigma^{-n}(\omega), x), K(\omega)) = 0, $$

where $\sigma$ is the shift map on $\Omega$, see for instance [15]. For general random dynamical systems, pullback random attractors and several notions of random attractors have been extensively studied by Crauel et al. in [9, 10, 11, 12, 13, 14]. For discrete random dynamical systems, the study of pullback random attractors goes back to [3, 22, 27, 31].

A straightforward consequence of Theorem 1 is the following:

**Corollary 1.** Let $S$ be a connected closed subset of $\mathbb{R}^k$ and let $\phi$ be an i.i.d. random iteration of strictly $J$-monotone continuous maps on $S$ satisfying the ping-pong property. Then, there is a measurable map $\pi: \Omega \to S$ such that for every $x \in S$

$$ \lim_{n \to \infty} d(\phi(n, \sigma^{-n}(\omega), x), \pi(\omega)) = 0 $$

for $\mathbb{P}$-almost every $\omega$. Moreover, the distribution of $\pi$ is the unique stationary measure.

The assertion of Corollary 1 means that the random compact set $\omega \mapsto \{\pi(\omega)\}$ is a pullback random attractor. As an example, consider the maps $f_1(x) = e^x$ and $f_2(x) = -e^x$ on $\mathbb{R}$. We take $E = \{1, 2\}^Z$ and $\mathbb{P} = \nu^Z$, where $\nu$ is a probability measure on $\{1, 2\}$ such that $\nu(\{i\}) > 0$ for $i = 1, 2$. Then, the i.i.d. random iteration $\phi$ given by

$$ \phi(n, \omega, x) = f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_0}(x) $$

satisfies the ping-pong property. Hence, it follows from Theorem 1 that $\phi$ has a pullback random attractor given by a measurable map $\pi: \Omega \to S$. 
2.3. **Synchronization.** Consider an i.i.d. random iteration \((f, X)\) of maps on \(S\). We use \(f^n_\omega(x)\) to denote \(Z^n_\omega(x)\), where \(Z^n_\omega\) is the Markov chain induced by \((f, X)\) as defined in (1.1) with \(Z_0 = x\). We say that \((f, X)\) is **synchronizing** if for every \(x\) and \(y\), there is a measurable set \(\Omega^{x,y} \subset \Omega\) with \(\mathbb{P}\)-full measure, such that

\[
\lim_{n \to \infty} d(f^n_\omega(x), f^n_\omega(y)) = 0
\]

for every \(\omega \in \Omega^{x,y}\). In other words, given any \(x\) and \(y\), with probability 1, the random orbits \(f^n_\omega(x)\) and \(f^n_\omega(y)\) converge to each other.

A standard class of random iterations in which synchronization takes place is the class of random iterations of Lipschitz maps with negative maximal Lyapunov exponent. Namely, let \((f, X)\) be an i.i.d. random iteration of Lipschitz maps and assume that there is an integrable map \(c: \Omega \to \mathbb{R}\) such that for \(\mathbb{P}\)-almost every \(\omega\) we have

\[
d(f_\omega(x), f_\omega(y)) \leq c(\omega)d(x, y)
\]

for every \(x, y \in S\). The **maximal Lyapunov exponent** of the i.i.d. random iteration \((f, X)\) is defined by

\[
\lambda(\omega) = \lim_{n \to \infty} \frac{1}{n} \log \sup_{x, y} \frac{d(f^n_\omega(x), f^n_\omega(y))}{d(x, y)}.
\]

The above limit exits for \(\mathbb{P}\)-almost every \(\omega\) by Kingman’s theorem, see [31]. Note that if the Lyapunov exponent is bounded (a.e) by a negative constant, then it follows from the definition that there are a measurable map \(C: \Omega \to \mathbb{R}\) and \(\lambda < 1\) such that

\[
d(f^n_\omega(x), f^n_\omega(y)) \leq C(\omega)\lambda^n d(x, y).
\]

In other words, negative Lyapunov exponent means exponentially fast synchronization. A practical way to verify the negativity of the Lyapunov exponent is the following estimate

\[(2.4)\]

\[
\lambda(\omega) \leq \int \log c(\omega)\,d\mathbb{P}(\omega) \overset{\text{def}}{=} \lambda_0
\]

for \(\mathbb{P}\)-almost every \(\omega\). This seems to be a folklore result and we do not found a reference stating (2.4) explicitly. We observe that this estimate holds for any random iteration of Lipschitz maps (not necessarily i.i.d. random iterations). A proof can be performed using ideas from [31 Section 5].

In our next theorem, we present a result stating exponentially fast synchronization. Since we do not assume the maps to be Lipschitz, the maximal Lyapunov exponent plays no role. We consider an i.i.d. random iteration of strictly \(J\)-monotone continuous maps. Under an additional boundedness condition on the maps we show that the ping-pong property implies (uniform) synchronization.

**Assumption 1.** There are a bounded set \(B\) and \(m_0\) such that

\[f^{m_0}_\omega(S) \subset B\]

for \(\mathbb{P}\)-almost every \(\omega\).

Clearly, Assumption 1 holds when either \(S\) is bounded or the images of the maps \(f_\omega\) are bounded. For an example where neither \(S\) is bounded nor the images of the maps are bounded, consider the maps \(f_1, f_2: \mathbb{R} \to \mathbb{R}\) where \(f_1(x) = e^{-x}\) and \(f_2\) is a strictly monotone map whose image is a bounded subset \(B_0\) of \((-\infty, 0)\). Then, for every \(i, j \in \{1, 2\}\) the composition \(f_i \circ f_j\) has image contained in \(B \overset{\text{def}}{=} B_0 \cup [0, 1]\).
Theorem 2. Let $S$ be a connected subset of $\mathbb{R}^k$ and let $(f, X)$ be an i.i.d. random iteration of strictly $J$-monotone continuous maps on $S$ satisfying the ping-pong property. Then, under Assumption 1, there exist constants $r < 1$ and $m_0 \geq 1$, and an integrable map $c \colon \Omega \to [0, \infty)$ such that for $\mathbb{P}$-almost every $\omega$

$$\operatorname{diam} f^n_\omega(S) \leq c(\omega) r^n$$

for every $n \geq m_0$.

This theorem was proved in [28] for the case $S = [0, 1]$ using different ideas. Theorem 2 can be useful even in the case the maps are Lipschitz maps. To illustrate this assertion, let us consider a simple example. Take an i.i.d. random iteration of two Lipschitz maps $f_1$ and $f_2$ with Lipschitz constants equal to 2 and $\frac{1}{2}$, respectively. If we denote $p_i \overset{\text{def}}{=} \mathbb{P}(X_0 = i) > 0$, $i = 1, 2$, we have

$$\lambda_0 = p_1 \cdot \log 2 + p_2 \cdot \log \frac{1}{2} = (p_1 - p_2) \cdot \log 2.$$  

Then, $\lambda_0$ is negative, if and only if, $p_1 < p_2$. On the other hand, if the maps are $J$-monotone and the induced i.i.d. random iteration satisfies the ping-pong property (if the images of $f_1$ and $f_2$ are disjoint for instance), then Theorem 2 applies independently of the signal of $p_1 - p_2$.

2.3.1. Exponentially fast convergence in the Wasserstein distance. Under assumptions of Theorem 2 we are able to improve item (ii) of Theorem 1 by showing exponentially fast convergence in the Wasserstein distance. Recall that for any pair of probability measures $\mu_1$ and $\mu_2$ on $S$ of bounded supports, the 1-Wasserstein distance $W_1$ is given by

$$W_1(\mu_1, \mu_2) = \sup \left| \int f \, d\mu_1 - \int f \, d\mu_2 \right|,$$

where the sup is taken over all Lipschitz maps with Lipschitz constant 1.

Corollary 2. Under assumptions of Theorem 2 we have the following:

(i) $\pi_{\mathbb{P}}$ has bounded support and there is $m_0$ such that for every probability measure $\varsigma$ the probability measure $T^n \varsigma$ has bounded support for every $n \geq m_0$.

(ii) There are $C$ and $r < 1$ such that for every probability measure $\varsigma$ on $S$

$$W_1(T^n \varsigma, \pi_{\mathbb{P}}) \leq C r^n$$

for every $n \geq m_0$.

An exponentially fast convergence is also obtained in [5, 7] for the $n$-step transition probability of an i.i.d. random iteration of monotone maps (w.r.t. the weak componentwise order) satisfying the splitting property. The authors consider a Kolmogorov type distance, and Assumption 1 is not required.

2.3.2. Forward random attractor. We now return to the setting of Section 2.2.1. We show that the map $\pi$ of Corollary 3.3 is also a forward random attractor. Given an i.i.d. random iteration $\varphi$, we recall that a forward random attractor of $\varphi$ is a $\varphi$-invariant random compact set $\omega \mapsto K(\omega)$ such that

$$\lim_{n \to \infty} d(\varphi(n, \omega, x), K(\sigma^n(\omega))) = 0,$$

where $\sigma$ is the shift map, see for instance Crauel and Scheutzow [15]. As a consequence of Theorem 2 we have the following:
Corollary 3. Let \( \varphi \) be an i.i.d. random iteration of strictly \( J \)-monotone maps satisfying the ping-pong property. Let \( \pi: \Omega \to S \) be the map as in Corollary 1. Then, under Assumption 1, we have that for every \( x \in S \)
\[
\lim_{n \to \infty} d(\varphi(n, \omega, x), \pi(\sigma^n(\omega))) = 0
\]
for \( \mathbb{P} \)-almost every \( \omega \).

Hence, by Corollaries 1 and 3 we obtain that \( \omega \mapsto \{ \pi(\omega) \} \) is a pullback and a forward random attractor.

2.4. Functional central limit theorem. We now present a functional central limit theorem for the i.i.d. random iteration considered in Theorem 2.

In the study of central limit theorems for a general homogeneous Markov chain, there are several results that reduce the problem to the verification of some analytical condition on the associated transfer operator, see for instance [6, 18, 29]. A classical one is to find a solution of the Poisson equation, which is the approach we follow in the FCLT presented in this paper. To this end, let us recall the definitions of the transfer operator and the Poisson equation. Consider a homogeneous Markov chain \( Z_n \) with state space \( M \) and transition probability \( p \). The transfer operator \( P \) induced by the transition probability \( p \) is defined as follows: given a non-negative measurable map \( f \), the action of \( P \) in \( f \) is a non-negative measurable map \( Pf \) given by
\[
Pf(x) = \int f(y) p(x, dy).
\]
For a measurable map \( f \) not necessarily non-negative, we write \( f = f^+ - f^- \) as a difference of non-negative measurable maps and we define
\[
Pf(x) = Pf^+(x) - Pf^-(x),
\]
if \( Pf^+(x) \) and \( Pf^-(x) \) are both finite.

In [7, Page 1340] is formulated a simple condition on the transfer operator \( P \) guaranteeing that a FCLT holds. Namely, assume that \( Y_n \) is an ergodic stationary Markov chain whose stationary distribution is \( \mu \). Note that \( P \) takes \( L^2(\mu) \) into \( L^2(\mu) \). Given a non-constant (a.e.) map \( \phi: S \to \mathbb{R} \) with \( \phi \in L^2(\mu) \) and \( \int \phi d\mu = 0 \), consider the Poisson equation
\[
(I - P)\psi = \phi.
\]
Assume that there is a solution \( \psi \in L^2(\mu) \) and let \( \sigma \) be a solution of \( \int (\psi - P\psi)^2 d\mu > 0 \) (the ergodicity implies \( \sigma > 0 \)). Consider the process \( Y_n \) given by
\[
Y_n(t) = \frac{1}{\sigma \sqrt{n}} \sum_{j=0}^{[nt]} \phi(Z_j), \quad 0 \leq t < \infty
\]
taking values on the space \( D[0, \infty) \) of real-valued right continuous function on \( [0, \infty) \) having left limits endowed with the Skorohod topology. Then, the process \( Y_n \) converges in distribution (weak-star convergence) to the Wiener measure in \( D[0, \infty) \). The idea is to use the solution of the Poisson equation to reduce the problem to the Martingale case, see also [6, 18] and Billingsley [8, Theorem 18.3].
Theorem 3. Let $S$ be a connected closed subset of $\mathbb{R}^k$ and let $(f, X)$ be an i.i.d. random iteration of strictly $J$-monotone continuous maps on $S$ satisfying the ping-pong property. Let $\mu$ be the unique stationary measure and consider a non-constant (a.e.) Lipschitz map $\phi: S \to \mathbb{R}$ in $L^2(\mu)$ with $\int \phi \, d\mu = 0$. Then, under Assumption 1:

(i) there is $\psi \in L^2(\mu)$ such that $(I - P)\psi = \phi$.
(ii) If $Z_n$ is a stationary Markov chain associated with $(f, X)$, then the process $Y_n$ given by

$$Y_n(t) = \frac{1}{\sigma \sqrt{n}} \sum_{j=0}^{[nt]} \phi(Z_j), \quad 0 \leq t < \infty,$$

converges in distribution to the Wiener measure in $D[0, \infty)$, where $\sigma = \int (\psi - P\psi)^2 \, d\mu$.

Remark 2.3. It follows from Theorem 2 that the FCLT in Theorem 3 holds under every initial distribution.

Theorem 3 extends [7, Theorem 3.1] and [5, Theorem 4.2]. Therein, for i.i.d. random iterations of non-decreasing maps on a closed subset $S \subset \mathbb{R}^k$ satisfying the splitting property, the authors show that the Poisson equation $(I - P)\psi = \phi$ has a solution provided that $\phi$ may be expressed as a difference of two non-decreasing bounded functions. Moreover, item (ii) is also obtained for non-stationary Markov chains $Z_n$ in their setting. See also [5, Remark 4.2].

3. Proof of Theorem 3

We start with a preliminary result. Let $\pi_s: \mathbb{R}^k \to \mathbb{R}$ be the natural projection $\pi_s(x) = x_s$, $s = 1, \ldots, k$.

Theorem 3.1. Let $S$ be a connected closed subset of $\mathbb{R}^k$ and let $(f, X)$ be an i.i.d. random iteration of $J$-monotone continuous maps on $S$ satisfying the ping-pong property. Then, there exists $0 < r < 1$ such that for every finite Borel measure $\eta$ on $\mathbb{R}$ there is $C \geq 0$ such that

$$\int \eta(\pi_s(f_{X_n}^n(S))) \, dP(\omega) = \int \eta(\pi_s(f_{X_0} \circ \cdots \circ f_{X_{n-1}}(S))) \, dP \leq r^n C$$

for every $n \geq 0$ and every $s = 1, \ldots, k$.

This theorem is an important step of the proof of Theorem 1 and its proof is inspired by the ideas in [25].

Proof of Theorem 3.1. For every $s \in \{1, \ldots, k\}$, $x \in S$ and $n \geq 1$, define

$$\Sigma_n^x(s) = \{\omega \in \Omega: x \in \pi_s(f_{X_0(\omega)} \circ \cdots \circ f_{X_{n-1}(\omega)}(S))\}.$$ 

Note that for every $n \geq 1$, we have the following useful property

$$\Sigma_{n+1}^x(s) \subset \Sigma_n^x(s)$$

for every $s = 1, \ldots, k$.

Lemma 3.2. There are $\lambda < 1$ and an integer $m \geq 1$ such that

$$\mathbb{P}(\Sigma_j^x(s)) \leq \lambda^j$$

for every $j \geq 1$ and $s = 1, \ldots, k$. 

Proof: Fix $s \in \{1, \ldots, k\}$. The proof is by induction on $j$. Let $m$, $A$ and $B$ as in the definition of the ping-pong property, recall Definition 2.2. For $j = 1$, it follows from the ping-pong property and Remark 2.1 that

$$\Sigma_m^x(s) \subset \Omega - \Gamma^1,$$

where either

$$\Gamma^1 = \{\omega \in \Omega: (X_0(\omega), \ldots, X_{m-1}(\omega)) \in A\} \overset{\text{def}}{=} A_1,$$

or

$$\Gamma^1 = \{\omega \in \Omega: (X_0(\omega), \ldots, X_{m-1}(\omega)) \in B\} \overset{\text{def}}{=} B_1.$$

In particular,

$$P(\Sigma_m^x(s)) \leq 1 - \rho \overset{\text{def}}{=} \lambda,$$

where $\rho = \min\{P(A_1), P(B_1)\} > 0$.

Assume that the lemma holds for $j \geq 1$. We now prove the lemma for $j + 1$. Consider the random variable $Z = (X_0, \ldots, X_{jm-1})$ taking values in $E^{jm}$. For every $z \in E^{jm}$, we claim that

$$(3.1) \quad \Sigma_{(j+1)m}^x(s) \cap [Z = z] \subset \Sigma_{jm}^x(s) - [Z = z] \cap \Gamma_j \cap \Sigma_{jm}^x(s)$$

where either

$$\Gamma_j = \{\omega \in \Omega: (X_{jm}(\omega), \ldots, X_{(j+1)m-1}(\omega)) \in A\} \overset{\text{def}}{=} A_j,$$

or

$$\Gamma_j = \{\omega \in \Omega: (X_{jm}(\omega), \ldots, X_{(j+1)m-1}(\omega)) \in B\} \overset{\text{def}}{=} B_j.$$

Indeed, if (3.1) does not hold, then there are $\omega, \hat{\omega} \in \Omega$ such that

$$\omega \in \Sigma_{(j+1)m}^x(s) \cap [Z = z] \cap \Gamma_j \quad \text{and} \quad \hat{\omega} \in \Sigma_{(j+1)m}^x(s) \cap [Z = z] \cap B_j.$$

If we write $z = (\alpha_0, \ldots, \alpha_{jm-1})$, we have that

$$x \in \pi_s(f_{\alpha_0} \circ \cdots \circ f_{\alpha_{jm-1}} \circ f_{X_{jm}(\omega)} \circ \cdots \circ f_{X_{(j+1)m}(\omega)}(S))$$

and

$$x \in \pi_s(f_{\alpha_0} \circ \cdots \circ f_{\alpha_{jm-1}} \circ f_{X_{jm}(\hat{\omega})} \circ \cdots \circ f_{X_{(j+1)m}(\hat{\omega})}(S)).$$

The inclusions above can not hold simultaneously. Again, this follows from the ping-pong property and Remark 2.1. This proves that (3.1) holds.

Now, let $P(\cdot|Z = \cdot)$ denote the regular conditional probability given by $Z$. Since $P([Z = z]|Z = z) = 1$, it follows from (3.1) that

$$P(\Sigma_{(j+1)m}^x(s)|Z = z) \leq P(\Sigma_{jm}^x(s)|Z = z) - P(\Gamma_j \cap \Sigma_{jm}^x(s)|Z = z).$$

In particular, since $z$ is arbitrary, we have

$$(3.2) \quad P(\Sigma_{(j+1)m}^x(s)|Z = z) \circ Z \leq P(\Sigma_{jm}^x(s)|Z = z) \circ Z - P(\Gamma_j \cap \Sigma_{jm}^x(s)|Z = z) \circ Z.$$

It follows from the definition of the regular conditional probability that

$$P(\Gamma_j \cap \Sigma_{jm}^x(s)|Z = z) \circ Z = E(1_{\Gamma_j \cap \Sigma_{jm}^x(s)}|Z).$$

Since $\Gamma_j$ and $Z$ are independent random variables, we conclude

$$E(1_{\Gamma_j \cap \Sigma_{jm}^x(s)}|Z) = 1_{\Gamma_j} E(1_{\Sigma_{jm}^x(s)}|Z) = 1_{\Gamma_j} P(\Sigma_{jm}^x(s)|Z = z) \circ Z.$$ 

Note that $1_{\Gamma_j}$ and $P(\Sigma_{jm}^x(s)|Z = z) \circ Z$ are also independent random variables. Therefore, integrating (3.2) we get

$$(3.3) \quad P(\Sigma_{(j+1)m}^x(s)) \leq P(\Sigma_{jm}^x(s)) - P(\Gamma_j)P(\Sigma_{jm}^x(s)) = P(\Sigma_{jm}^x(s))(1 - P(\Gamma_j)).$$
Since $X = \{X_n\}$ is an i.i.d. sequence of random variables, we have $P(A_j) = P(A_1)$ and $P(B_j) = P(B_1)$. Hence, we conclude from (3.3) that

$$P(\Sigma_{r+1}^x(s)) \leq \lambda^j \cdot (1 - \rho) = \lambda^{j+1}.$$  

Note that $\lambda$ does not depend on $s$. The proof of the lemma is now complete. □

We now conclude the proof of Theorem 3.1. Let $\lambda$ as in Lemma 3.2 and define $\hat{\lambda} = \lambda^{\frac{1}{m}}$. Thus, for every $s$ we have

$$P(\Sigma_{jm}(s)) \leq \hat{\lambda}^{mj}.$$  

Now, choose any $r > 0$ such that $r < 1$ and $\hat{\lambda} \leq \min\{r, r^2, \ldots, r^m\}$. Let $n \geq m$. Then, there is $e \in \{0, \ldots, m - 1\}$ such that $n = jm + e$. Note that $\Sigma_n^x(s) \subset \Sigma_{jm}(s)$. Therefore

$$P(\Sigma_n^x(s)) \leq P(\Sigma_{jm}(s)) \leq \hat{\lambda}^{mj} = \hat{\lambda}^{mj-1} \hat{\lambda} \leq r^{mj-1} r^e = r^n.$$  

Now, it follows from Fubini theorem that

$$\int \eta(\pi_s(f^n_\omega(S))) dP(\omega) = \int \sum_{x} \sum_{s} \eta(\pi_s(f^n_\omega(S))) d\eta(x) = \int \sum_{x} \sum_{s} \sum_{x} \sum_{s} \eta(\pi_s(f^n_\omega(S))) d\eta(x) \leq r^n \eta(\mathbb{R})$$  

for every $n \geq m$. This implies that there is $C \geq 0$ such that

$$\int \eta(\pi_s(f^n_\omega(S))) dP(\omega) \leq r^n C$$  

for every $n \geq 0$ for every $s$. □

Before proving Theorem 1 we need two technical lemmas. The first one says that with probability 1 the set $\pi_s(f_{X_0} \circ \cdots \circ f_{X_{n-1}}(S))$ is bounded for $n$ sufficiently large. The second one is a general result from measure theory that will be used to state that $\pi_s(f_{X_0} \circ \cdots \circ f_{X_{n-1}}(S))$ is “contracting” exponentially fast with respect to any finite Borel measure.

**Lemma 3.3.** For $P$-almost every $\omega$, there is $n_0$ (depending on $\omega$) such that

$$\pi_s(f_{X_0}(\omega) \circ \cdots \circ f_{X_{n-1}(\omega)}(S))$$  

is bounded for every $n \geq n_0$.

**Proof.** Let $A, B \subset E$ and $m$ as in the definition of the ping-pong property. Consider the set $A \times B$. For every $(\alpha, \beta)$ we define the sets

$$G_{\alpha \beta} = \pi_s(f_{\alpha_0} \circ \cdots \circ f_{\alpha_{m-1}} \circ f_{\beta_0} \circ \cdots \circ f_{\beta_{m-1}}(S))$$  

and

$$G_{\alpha \alpha} = \pi_s(f_{\alpha_0} \circ \cdots \circ f_{\alpha_{m-1}} \circ f_{\alpha_0} \circ \cdots \circ f_{\alpha_{m-1}}(S))$$  

and

$$G_{\beta \alpha} = \pi_s(f_{\beta_0} \circ \cdots \circ f_{\beta_{m-1}} \circ f_{\alpha_0} \circ \cdots \circ f_{\alpha_{m-1}}(S))$$  

Associate to these set there are the subsets of $A \times B$:

$$E_{\alpha \beta} = \{(\alpha, \beta) \in A \times B : G_{\alpha \beta} \text{ is bounded}\}$$  

and

$$E_{\alpha \alpha} = \{(\alpha, \beta) \in A \times B : G_{\alpha \alpha} \text{ is bounded}\}$$

For every $e_0 \in A$, there is $\rho > 0$ such that $P(e_0 \in A) \geq 1 - \rho$. Let $s$ be as in (3.2) such that $s \geq n_0$. Then, for any $\eta > 0$ there is $r > 0$ such that $r^s \eta < \rho$. Hence, for any $(\alpha, \beta)$ with $s \geq n_0$ we have

$$P(G_{\alpha \beta}) \geq 1 - \rho$$  

and

$$P(G_{\alpha \alpha}) \geq 1 - \rho.$$  

Thus, for any $(\alpha, \beta)$ we conclude that $G_{\alpha \beta}$ and $G_{\alpha \alpha}$ are bounded for $n \geq n_0$.

**Lemma 3.4.** For any $\eta > 0$ there is $r > 0$ such that $r^s \eta < \rho$.
and
\[ E_{\beta\alpha} = \{ (\alpha, \beta) \in A \times B : G_{\beta\alpha} \text{ is bounded} \}. \]

**Claim 3.4.** We have
\[ A \times B = E_{\alpha\beta} \cup E_{\alpha\alpha} \cup E_{\beta\alpha}. \]

**Proof.** Given \((\alpha, \beta) = (\alpha_0, \ldots, \alpha_{m-1}, \beta_0, \ldots, \beta_{m-1}) \in A \times B\), it follows from the ping-pong property that
\[ f_{\alpha_0} \circ \cdots \circ f_{\alpha_{m-1}}(S) < J f_{\beta_0} \circ \cdots \circ f_{\beta_{m-1}}(S). \]

In particular, the sets \(G_{\alpha\beta}, G_{\alpha\alpha}, G_{\beta\alpha}\) are disjoints. Since \(S\) is connected, they also are intervals, which implies that some of them must be bounded. \(\square\)

Now, since \(\nu^{2m}(A \times B) > 0\), we have that at least one of the sets \(G_{\alpha\beta}, G_{\alpha\alpha}, G_{\beta\alpha}\) has positive \(\nu^{2m}\)-measure. Without loss of generality, we can assume that \(\nu^{2m}(G_{\alpha\beta}) > 0\). In particular, it follows from Birkhoff’s Ergodic Theorem that for \(P\)-almost every \(\omega\), there is \(n_0 \geq 0\) such that
\[ (X_{n_0}(\omega), \ldots, X_{n_0+2m-1}(\omega)) \in G_{\alpha\beta}. \]

By definition of \(G_{\alpha\beta}\), we have that \(\pi_s(f_{X_{n_0}(\omega)} \circ \cdots \circ f_{X_{n_0+2m-1}(\omega)}(S))\) is bounded.

Since \(S\) is closed, the image of a bounded set by a continuous maps is also bounded and then we conclude that for every \(n \geq n_0 \overset{\text{def}}{=} n_0 + 2m\) we have that the set
\[ \pi_s(f_{X_n(\omega)} \circ \cdots \circ f_{X_n(\omega)}(S)) \]

is bounded. \(\square\)

**Lemma 3.5.** Let \(Y_n : \Omega \to [0, \infty)\) be a sequence of measurable maps and assume that there exists \(0 < \lambda < 1\) such that \(\mathbb{E}Y_n \leq \lambda^n\). Then, there exist an integrable function \(c : \Omega \to [0, \infty)\) and \(q < 1\) such that for \(P\)-almost every \(\omega\) it holds \(Y_n(\omega) \leq c(\omega) \cdot q^n\) for every \(n \geq 0\).

**Proof.** We take any \(q < 1\) with \(\lambda < q\) and apply the Monotone Convergence Theorem to obtain that
\[ \int \sum_{n=1}^{\infty} \frac{Y_n(\omega)}{q^n} dP(\omega) = \sum_{n=1}^{\infty} \frac{\mathbb{E}Y_n}{q^n} < \infty. \]

Therefore
\[ c(\omega) \overset{\text{def}}{=} \sum_{n=0}^{\infty} \frac{Y_n(\omega)}{q^n} < \infty, \]

for \(P\)-almost every \(\omega\), which implies that
\[ Y_n(\omega) \leq c(\omega)q^n \]

\(P\)-almost every \(\omega\). \(\square\)
Denote by $\text{Leb}$ the Lebesgue measure on $\mathbb{R}$ and consider the finite Borel measure $m_\ell$ given by $m_\ell(A) = \text{Leb}([-\ell,\ell] \cap A)$. Thus, it follows from Theorem 3.1 and Lemma 3.3 that for every $s$, there is an integrable map $c: \Omega \to [0, \infty)$ (which we can assume that does not depend on $s$, because $s$ varies on a finite set) such that for $\mathbb{P}$-almost every $\omega$ it holds

\begin{equation}
(3.4) \quad m_\ell(\pi_s(f_{X_0(\omega)} \circ \cdots \circ f_{X_{n-1}(\omega)}(S))) \leq c(\omega)q^n
\end{equation}

for every $n \geq 1$.

Now, let $\omega$ as in Lemma 3.3 satisfying eq. (3.3). Then, there is $n_0$ such that

$$
\pi_s(f_{X_0(\omega)} \circ \cdots \circ f_{X_{n-1}(\omega)}(S))
$$

is bounded for every $n \geq n_0$ and $s$. Since the sequence $\left(\pi_s(f_{X_0(\omega)} \circ \cdots \circ f_{X_{n-1}(\omega)}(S))\right)_n$ is nested, we conclude that there is $\ell \in \mathbb{N}$ such that

$$
\pi_s(f_{X_0(\omega)} \circ \cdots \circ f_{X_{n-1}(\omega)}(S)) \subset [-\ell, \ell]
$$

for every $n \geq n_0$ and $s$.

Note that for every connected subset $I$ of $[-\ell, \ell]$ we have $m_\ell(I) = \text{diam}(I)$. Since $\pi_s(f_{X_0(\omega)} \circ \cdots \circ f_{X_{n-1}(\omega)}(S))$ is a connected subset of $\mathbb{R}$, we get that

$$
m_\ell(\pi_s(f_{X_0(\omega)} \circ \cdots \circ f_{X_{n-1}(\omega)}(S))) = \text{diam}(\pi_s(f_{X_0(\omega)} \circ \cdots \circ f_{X_{n-1}(\omega)}(S)))
$$

for every $n \geq n_0$ and $s$. Therefore, it follows from eq. (3.4) that

$$
\text{diam}(\pi_s(f_{X_0(\omega)} \circ \cdots \circ f_{X_{n-1}(\omega)}(S))) \leq c(\omega)q^n
$$

for every $n \geq n_0$ and $s$.

Now, if $\mathbb{R}^k$ is endowed with the sup distance, we have

$$
\text{diam } f_{X_0(\omega)} \circ \cdots \circ f_{X_{n-1}(\omega)}(S) = \max_s \text{diam}(\pi_s(f_{X_0(\omega)} \circ \cdots \circ f_{X_{n-1}(\omega)}(S))) \leq c(\omega)q^n
$$

for every $n \geq n_0$. This implies that

\begin{equation}
(3.5) \quad \lim_{n \to \infty} \text{diam } f_{X_0(\omega)} \circ \cdots \circ f_{X_{n-1}(\omega)}(S) = 0.
\end{equation}

To conclude the proof of item (i), note that it follows from (3.5) that for every $x$ the sequence $\hat{X}_n(\omega) = X_0(\omega) \circ \cdots \circ X_{n-1}(\omega)(x)$ is a Cauchy sequence.

Item (ii) follows from Letac principle [27] and item (i). \hfill \Box

3.2. **Proof of Corollary** Since the sequence $\hat{X}_n(\omega) = X_0(\omega) \circ \cdots \circ X_{n-1}(\omega)$, $n \geq 1$, is an i.i.d. sequence of random variables with distribution $\nu$, it follows from Theorem 1 that for $\mathbb{P}$-almost every $\omega$, the limit

$$
\lim_{n \to \infty} f_{\omega^{-1}} \circ \cdots \circ f_{\omega^{-n}}(x) \overset{\text{def}}{=} \pi(\omega)
$$

exists and is independent of the point $x$. \hfill \Box

4. **Proof of Theorem 2**

In this section, we consider $\mathbb{R}^k$ endowed with the taxcab metric. Then, we have for every subset $B \subset \mathbb{R}^k$ that

\begin{equation}
(4.1) \quad \text{diam } B \leq \sum_{s=1}^{k} \text{diam } \pi_s(B)
\end{equation}

By Assumption 1 there are a bounded set $B$ and $m_0$ such that for $\mathbb{P}$-almost every $\omega$ we have

$$
f_{\omega}^{m_0}(S) \subset B.
$$
In particular, since the random variables $(X_0, \ldots, X_{m_0-1})$ and $(X_{m_0-1}, \ldots, X_0)$ have the same distribution, we also have

$$f_{X_0}(\omega) \circ \cdots \circ f_{X_{m_0-1}}(\omega)(S) \subset B$$

for $\mathbb{P}$-almost every $\omega$. This implies that for every $n \geq m_0$, it holds

$$f_{X_0}(\omega) \circ \cdots \circ f_{X_{n-1}}(\omega)(S) \subset B$$

for $\mathbb{P}$-almost every $\omega$. Indeed, recall that the sequence $(f_{X_0}(\omega) \circ \cdots \circ f_{X_{n-1}}(\omega)(S))_n$ is nested.

Now, let $\ell$ be such that $\pi_s(B) \subset [-\ell, \ell]$ for every $s$. In particular,

$$\pi_s(f_{X_0}(\omega) \circ \cdots \circ f_{X_{n-1}}(\omega)(S)) \subset [-\ell, \ell]$$

for every $n \geq m_0$ and $s$. Let $m_\ell$ be the finite Borel measure given by $m_\ell(A) = \text{Leb}([-\ell, \ell] \cap A)$. Note that if $I$ is a connected set with $I \subset [-\ell, \ell]$, then $m_\ell(I) = \text{diam} I$. Then, applying Theorem 3.1 to the measure $m_\ell$, we get that there is $C$ such that

$$\int \text{diam}(\pi_s(f^n_\omega(S))) \, d\mathbb{P}(\omega) = \int \text{diam}(\pi_s(f_{X_0} \circ \cdots \circ f_{X_{n-1}}(S))) \, d\mathbb{P}$$

$$= \int m_\ell(\pi_s(f_{X_0} \circ \cdots \circ f_{X_{n-1}}(S))) \, d\mathbb{P} \leq r^n C$$

(4.2)

for every $n \geq m_0$ and every $s$, where the first equality follows from the fact that $(X_0, \ldots, X_{n-1})$ and $(X_{n-1}, \ldots, X_0)$ have the same distribution.

Thus, it follows from eq. (4.1) and (4.2), that

$$\int \text{diam}(f^n_\omega(S)) \, d\mathbb{P}(\omega) = \int \text{diam}(f_{X_0} \circ \cdots \circ f_{X_{n-1}}(S)) \, d\mathbb{P} \leq kCr^n$$

for every $n \geq m_0$. Now, it follows from Lemma 3.5 that there is an integrable map $c: \Omega \to \mathbb{R}$ such that for $\mathbb{P}$-almost every $\omega$ we have

$$\text{diam}(f^n_\omega(S)) \leq c(\omega) r^n$$

for every $n \geq m_0$.

\[ \square \]

4.1. \textbf{Proof of Corollary 2} Arguing as in the proof of Theorem 2 we have that there are $m_0 \geq 1$ and a bounded set such that for every $n \geq m_0$, it holds

$$f_{X_0}(\omega) \circ \cdots \circ f_{X_{n-1}}(\omega)(S) \subset B$$

(4.3)

for $\mathbb{P}$-almost every $\omega$. This implies that $\text{supp} \pi \mathbb{P} \subset \hat{B}$. Note that $\mathbb{P}(\{\omega: f^n_\omega(x) \in B\}) = \mathbb{P}(\{\omega: f_{X_0}(\omega) \circ \cdots \circ f_{X_{n-1}}(\omega)(x) \in B\}) = 1$, for every $n \geq 1$. Also, by the definition of $T$ we have that

$$T^n \varsigma(B) = \int \mathbb{P}(\{\omega: f^n_\omega(x) \in B\}) \, d\varsigma(x)$$

for every probability measure $\varsigma$ on $S$. Therefore, it follows from eq. (4.3) that

$$T^n \varsigma(B) = 1$$

for every $n \geq m_0$. This implies that $\text{supp} T^n \varsigma \subset \hat{B}$ for every $n \geq m_0$.

We now prove item (ii). Let $\{X_n\}_{n \in \mathbb{Z}}$ be a bilateral sequence of i.i.d. random variables with distribution $\nu$. For every $n \geq 0$, the sequence $X_{n-1}, X_0, X_{-1}, \ldots$...
is an i.i.d. sequence of random variables with distribution \( \nu \). Then, it follows from Theorem 1 that for every \( n \geq 0 \) there is a measurable map \( \pi_n \) such that
\[
\pi_n(\omega) = \lim_{k \to \infty} f_{X_{n-1}(\omega)} \circ \cdots \circ f_{X_0(\omega)} \circ f_{X_{-1}(\omega)} \circ \cdots \circ f_{X_{-k}(\omega)}(p)
\]
for \( \mathbb{P} \)-almost every \( \omega \). Note that \( \pi_n(\omega) = f_{\omega}^n(\pi_0(\omega)) \) and for every \( n \), and the maps \( \pi \) and \( \pi_n \) have the same distribution. In particular, for every \( \pi \mathbb{P} \)-integrable map \( \phi : S \to \mathbb{R} \) we have
\[
\int \phi(\pi(\omega)) \, d\mathbb{P}(\omega) = \int \phi(f_{\omega}^n(\pi_0(\omega))) \, d\mathbb{P}(\omega).
\]
Assume now that \( \phi \) is a Lipschitz map with Lipschitz constant \( 1 \). It follows from the definition of the transfer operator \( P \) and the Markov operator \( T \) that,
\[
P^n \phi(x) = \int \phi(f_0^n(x)) \, d\mathbb{P}(\omega) \quad \text{and} \quad \int P^n \phi(x) \, d\mu(x) = \int \phi(x) \, d\mathbb{P}(\omega)
\]
for every \( n \geq m_0 \) and \( x \in S \). Hence, we have
\[
(4.4) \quad \left| \int \phi \, dT^n \varsigma - \int \phi \, d\pi \mathbb{P} \right| = \left| \int P^n \phi(x) \, d\mu(x) - \int \phi(f_0^n(\pi_0(\omega))) \, d\mathbb{P}(\omega) \right|
\]
\[
= \left| \int \int \phi(f_0^n(x)) \, d\mathbb{P}(\omega) \, d\varsigma(x) - \int \phi(f_0^n(\pi_0(\omega))) \, d\mathbb{P}(\omega) \right|
\]
\[
= \left| \int (\phi(f_0^n(x)) - \phi(f_0^n(\pi_0(\omega)))) \, d\mathbb{P}(\omega) \, d\varsigma(x) \right|.
\]
It follows from Theorem 2 that there are an integrable map \( c : \Omega \to \mathbb{R} \) and \( r < 1 \) such that
\[
\text{diam } f_\omega^n(S) \leq c(\omega) r^n
\]
for \( \mathbb{P} \)-almost every \( \omega \) and \( n \geq m_0 \). Then,
\[
|\phi(f_\omega^n(x)) - \phi(f_\omega^n(\pi_0(\omega)))| \leq \text{diam } f_\omega^n(S) \leq c(\omega) r^n
\]
for \( \mathbb{P} \)-almost every \( \omega \) and \( n \geq m_0 \), and thus it follows from eq. (4.4) that
\[
\left| \int \phi \, dT^n \mu - \int \phi \, d\pi \mathbb{P} \right| \leq \int \text{diam } f_\omega^n(S) \, d\mathbb{P}(\omega) \leq Cr^n
\]
for every \( n \geq m_0 \). This implies that
\[
W_1(T^n \mu, \pi \mathbb{P}) \leq Cr^n
\]
for every \( n \geq m_0 \). \( \square \)

4.2. **Proof of Corollary 3.** Using the notation of Section 2.3.2 Theorem 2 can be rewrite as: there are \( c : \Omega \to \mathbb{R} \), \( r < 1 \) and a constant \( C \geq 0 \) such that for \( \mathbb{P} \)-almost every \( \omega \),
\[
d(\phi(n, \omega, x), \phi(n, \omega, y)) \leq c(\omega) r^n
\]
for every \( x, y \in S \) and \( n \geq m_0 \).

Since the sequence \( X_n(\omega) = \omega_{-n}, n \geq 1 \), is an i.i.d. sequence with distribution \( \nu \), we get from Theorem 1 that for \( \mathbb{P} \)-almost every \( \omega \), the limit
\[
\lim_{n \to \infty} f_{\omega_{-1}} \circ \cdots \circ f_{\omega_{-n}}(x) \overset{\text{def}}{=} \pi(\omega)
\]
exists and is independent of the point $x$. Note that the map $\pi$ satisfies the following invariance equation

$$f_\omega(\pi(\omega)) = \pi(\sigma(\omega))$$

for $\mathbb{P}$-almost every $\omega$, where $\sigma$ is the shift map on $E^2$. By induction, we get

$$\varphi(n, \omega, \pi(\omega)) = \pi(\sigma^n(\omega))$$

for $\mathbb{P}$-almost every $\omega$ and every $n$. Therefore,

$$\lim_{n \to \infty} d(\varphi(n, \omega, x), \pi(\sigma^n(\omega))) = 0$$

for $\mathbb{P}$-almost every $\omega$. \hfill \square

5. Solving the Poisson equation. Proof of Theorem 3

As observed in Section 2.4 we only need to prove item (i). Let $(f, X)$ be an i.i.d. random iteration as in Theorem 3 and $\mu$ be the transition probability given by (2.1). Let $P$ be the associated transfer operator as defined in 2.5. Let $\phi$ be the unique Lipschitz map $\phi \in L^2(\mu)$ with $\int \phi \, d\mu = 0$. It follows from the definition of the transfer operator $P$ that for every $x$ and every $n \geq 1$ we have

$$P^n \phi(x) = \int \phi(f^n_\omega(x)) \, d\Phi(\omega) \quad \text{and} \quad P^n \phi(x) \, d\mu(x) = \int \phi(x) \, dT^n \mu(x),$$

where $T$ is the Markov operator. Then, for every $x$ we have

$$|P^n \phi(x)| = \left| P^n \phi(x) - \int \phi(y) \, d\mu(y) \right|$$

$$= \left| P^n \phi(x) - \int P^n \phi(y) \, d\mu(y) \right|$$

$$= \left| \int P^n \phi(x) \, d\mu(y) - \int P^n \phi(y) \, d\mu(y) \right|$$

$$= \left| \int \phi(f^n_\omega(x)) \, d\Phi(\omega) \, d\mu(y) - \int \phi(f^n_\omega(x)) \, d\Phi(\omega) \, d\mu(y) \right|$$

$$\leq \int \int |\phi(f^n_\omega(x)) - \phi(f^n_\omega(y))| \, d\Phi(\omega) \, d\mu(y).$$

Let $L$ be the Lipschitz constant of $\phi$. Thus, we have

$$|\phi(f^n_\omega(x)) - \phi(f^n_\omega(y))| \leq L \cdot d(f^n_\omega(x), f^n_\omega(y)) \leq L \cdot \text{diam} f^n_\omega(S)$$

for every $\omega \in \Omega$. On the other hand, it follows from Theorem 2 that there are constants $C \geq 0$, $0 < r < 1$ and an integer $m_0 \geq 1$ such that

$$\int \text{diam} f^n_\omega(S) \, d\Phi(\omega) \leq Cr^n$$

for every $n \geq m_0$.

Therefore, for every $x$ and $n \geq m_0$ we have

$$(P^n \phi(x))^2 \leq C^2 \cdot L^2 \cdot r^{2n}.$$
This implies that \( \|P^n \phi\|_2 \leq C_0 \lambda^n \) for every \( n \geq m_0 \), where \( C_0 = C^2 \cdot L^2 \) and \( \lambda = r^2 < 1 \). In particular,
\[
\sum_{n=0}^{\infty} \|P^n \phi\|_2 < \infty.
\]
Then, the map \( \psi = -\sum_{n=0}^{\infty} P^n \phi \) is a well defined element of \( L^2(\mu) \) and solves the Poisson equation \( (I - P)\psi = \phi \).

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