A Local Energy Identity for Parabolic Equations with Divergence-Free Drift

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Abstract

We prove a local energy identity for a class of distributional solutions, in \( L_{2,\infty} \cap W^{1,0}_2 \), of parabolic equations with divergence-free drift.

1 Introduction

We are considering the parabolic equations of the type

\[
\partial_t u - \text{div}(a \nabla u) + b \cdot \nabla u = 0,
\]

where \( a \) is a bounded, symmetric and uniformly elliptic matrix and \( b \) a divergence-free vector field belonging to \( L_\infty(BMO^{-1}) \). We say that a divergence-free vector field \( b \) belongs to the space \( BMO^{-1} \) if there exists a skew symmetric matrix \( d \) belonging to \( BMO \) such that \( b = \text{div}(d) \).

Therefore, the above equation can be rewritten as follows:

\[
\partial_t u - \text{div}(A \nabla u) = 0, \tag{1}
\]

where \( A = a + d \), with \( a \) as before and \( d \in L_\infty(BMO) \) a skew symmetric matrix.

G. Seregin and co-authors introduced, in their paper [1], the notion of suitable weak solutions to equation (1), which are distributional solutions that belong to the energy class \( L_{2,\infty} \cap W^{1,0}_2 \) and that satisfy a particular local energy inequality. In this paper, we establish a local energy identity for distributional solutions of (1) which belong to the energy class \( L_{2,\infty} \cap W^{1,0}_2 \), and therefore, we prove at the same time that the local energy inequality required in the definition of suitable weak solutions, introduced in [1], is a direct consequence of being a distributional solution in the above energy class.

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2 Preliminaries

In what follows, we will use the following abbreviated notations: \( B := B(0, 1) \) (the unit ball of \( \mathbb{R}^n \)), \( Q := B \times (-1, 0) \), as well as \( z := (x, t) \).

We recall that a function \( d \) is in the space \( BMO(\Omega; \mathbb{R}^{n \times n}) \) if the following quantity
\[
\sup \left\{ \frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} |d - [d]_{x_0, r}| \, dx : B(x_0, r) \subset \subset \Omega \right\},
\]
with \([d]_{x_0, r}\) the average of \( d \) over \( B(x_0, r) \), is bounded; and a function \( u \) belongs to the Hardy space \( H^1(\mathbb{R}^n) \) if there exists a function \( \phi \in C^\infty_0(B) \) such that \( u \phi \in L^1(\mathbb{R}^n) \), where \( u \phi(x) := \sup_{t > 0} |(\phi_t \ast u)(x)| \), and \( \phi_t(x) := t^{-n} \phi(x/t) \).

For simplicity we adopt the following notation convention \( \partial_i f = f, i \). We have the following classical div-curl type lemma for Hardy spaces, which is a direct consequence of Theorem II.1 in [5].

**Lemma 1.** Let \( u \in W^{1, p}_p(\mathbb{R}^n) \) and \( v \in W^{1, q}_q(\mathbb{R}^n) \), with \( 1 < p < \infty \) and \( 1/p + 1/q = 1 \). Then \( u, j v, i - v, j u, i \in H^1(\mathbb{R}^n) \) for all \( i, j = 1, \ldots, n \) and we have
\[
\|u, j v, i - v, j u, i\|_{H^1} \leq C \|\nabla u\|_{L^p} \|\nabla v\|_{L^q}, \quad \forall i, j = 1, \ldots, n.
\]

We recall also some basic facts related to the spectral decomposition of the Laplace operator on a bounded domain \( \Omega \) of \( \mathbb{R}^n \), with smooth boundary. The Laplacian viewed as an unbounded operator from \( L^2(\Omega) \) into itself has a discrete spectrum; we denote by \( 0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots \) (with \( \lambda_n \to \infty \)), its eigenvalues and \( \{\phi_k\}_{k=1}^\infty \) the corresponding eigenvectors which form a Hilbert basis of \( L^2(\Omega) \). Setting \( \dot{L}^1_2(\Omega) \) to be the completion of \( C^\infty_0(\Omega) \) with respect to the Dirichlet semi-norm \( \|u\|_{\dot{L}^1_2}^2 := \int |\nabla u|^2 \), and \( H^{-1}(\Omega) \) to be the dual of \( \dot{L}^1_2(\Omega) \), we have the following classical lemma, which gives us a Hilbert basis of \( \dot{L}^1_2(\Omega) \) and a representation of the norm of \( H^{-1} \) by means of the eigenvectors and eigenvalues of the Laplace operator.

**Lemma 2.** \( \{\phi_k/\sqrt{\lambda_k}\}_{k=1}^\infty \) is a Hilbert basis of \( \dot{L}^1_2(\Omega) \) and as a direct consequence, we have that, if \( f \in H^{-1}(\Omega) \), then
\[
\|f\|^2_{H^{-1}(\Omega)} = \sum_{k=1}^\infty f_k^2/\lambda_k,
\]
where \( f_k = \langle f, \phi_k \rangle \).

**Proof.** The proof of this result is quiet classical, therefore we skip it. \( \square \)
3 Main Theorem

We now state the main result of this paper.

**Theorem 3.1.** Let $u$ belonging to the energy class

$$L_{2,\infty}(Q) \cap W^{1,0}_2(Q),$$

such that

$$\int_Q u \partial_t \phi dz = \int_Q (A \nabla u).\nabla \phi dz \quad \forall \phi \in C_0^\infty(Q),$$

(2)

where $A = a + d$, with $a \in L_{\infty}(Q; \mathbb{R}^{n \times n})$ a symmetric matrix satisfying

$$\nu I_n \leq a \leq \nu^{-1} I_n$$

and $d \in L_{\infty}(-1,0; \text{BMO}(B; \mathbb{R}^{n \times n}))$ a skew symmetric matrix. Then the following energy identity holds for all $t_0 \in (-1,0)$ and for all test functions $\phi \in C_0^\infty(B \times (-1,1))$:

$$\frac{1}{2} \int_B \phi(x,t_0)|u(x,t_0)|^2 dx + \int_{-1}^{t_0} \int_B \phi \nabla u. a \nabla u dz = \frac{1}{2} \int_{-1}^{t_0} \int_B |u|^2 \partial_t \phi dz$$

$$- \int_{-1}^{t_0} \int_B (A \nabla u).\nabla \phi u dz.$$ 

4 Proof of Theorem 3.1

The method we use for this proof are due to Seregin, in his lecture notes: "Parabolic Equations". We start by proving a simple regularity result for the time derivative of $u$ defined as in Theorem 3.1.

**Proposition 4.1.** Let $u$ defined as in Theorem 3.1. Then

$$\partial_t u \in L_2(-1,0; H^{-1}(B)).$$

**Proof.** **Step 1.** Let us set

$$g(x,t) = A(x,t)\nabla u(x,t),$$

and consider the problem

$$\begin{cases}
-\Delta U(\cdot,t) = \text{div} \, g(\cdot,t) \quad \text{for a.e } t \in (-1,0) \\
U(\cdot,t)|_{\partial B} = 0.
\end{cases}$$

(3)

Let $v \in C_0^\infty(B)$, we have:

$$\int_B \text{div} \, g(\cdot,t) v dx = - \int_B g(\cdot,t) \cdot \nabla v dx$$

$$= - \int_B (a \nabla u) \cdot \nabla v dx - \int_B (d \nabla u) \cdot \nabla v dx$$

$$=: A_1 + A_2.$$
We have by a straightforward computation that 
\[ |A_1| \leq \|a\|_{L^\infty(Q)} \|\nabla u(\cdot, t)\|_{L^2(B)} \|\nabla v\|_{L^2(B)} \quad \text{for a.e } t \in (-1, 0). \]

On the other hand, we have thanks to the skew symmetry of \( d \), that \( A_2 \) can be rewritten as follows 
\[ -A_2 = \frac{1}{2} \sum_{i,j=1}^{n} \int_{B} d_{ij}(u_{;j} v_{,i} - v_{;j} u_{,i}) \, dx. \]

Denote by \( \bar{u} \) the extension of \( u \) from \( B \) to \( \mathbb{R}^n \) such that 
\[ \|\bar{u}(\cdot, t)\|_{W^1_2(\mathbb{R}^n)} \leq c \|u(\cdot, t)\|_{W^1_2(B)} \quad \text{for a.e } t \in (-1, 0), \]
where \( c \) depends only on \( n \). Similarly, let us denote by \( \bar{d} \) the extension of \( d \) from \( B \) to \( \mathbb{R}^n \) such that 
\[ \|\bar{d}(\cdot, t)\|_{BMO(\mathbb{R}^n;\mathbb{R}^{n \times n})} \leq c \|d(\cdot, t)\|_{BMO(\mathbb{B};\mathbb{R}^{n \times n})} \quad \text{for a.e } t \in (-1, 0), \]
where, again, \( c \) depends only on \( n \). In the later case, to construct such an extension, one can use a reflection on the boundary (See, e.g., Theorem 2 in [2], where this is done for very general domains \( \Omega \subset \mathbb{R}^n \)). Therefore, because \( v \) is compactly supported in \( B \), we have that 
\[ -A_2 = \frac{1}{2} \sum_{i,j=1}^{n} \int_{\mathbb{R}^n} \bar{d}_{ij}(\bar{u}_{;j} v_{,i} - \bar{v}_{;j} \bar{u}_{,i}) \, dx. \]

We have from Lemma 1 that \( \bar{u}_{;j} v_{,i} - \bar{v}_{;j} \bar{u}_{,i} \in H^1(\mathbb{R}^n) \) with 
\[ \|\bar{u}_{;j} v_{,i} - \bar{v}_{;j} \bar{u}_{,i}\|_{H^1(\mathbb{R}^n)} \leq C \|\nabla \bar{u}\|_{L^2(\mathbb{R}^n)} \|\nabla v\|_{L^2(\mathbb{R}^n)}, \]
and since \( BMO(\mathbb{R}^n) \) is the dual of the Hardy space \( H^1(\mathbb{R}^n) \), we derive that 
\[ |A_2| \leq C \|\bar{d}\|_{L^\infty(-1,0;BMO(\mathbb{R}^n;\mathbb{R}^{n \times n}))} \|\nabla \bar{u}\|_{L^2(\mathbb{R}^n)} \|\nabla v\|_{L^2(\mathbb{R}^n)}, \]
and a fortiori 
\[ |A_2| \leq C \|d\|_{L^\infty(-1,0;BMO(\mathbb{B};\mathbb{R}^{n \times n}))} \|\nabla u\|_{L^2(B)} \|\nabla v\|_{L^2(B)}, \]
(with \( C \) depending only on \( n \)). Hence, we have that \( \text{div } g(\cdot, t) \in H^{-1}(B) \), with 
\[ \|\text{div } g(\cdot, t)\|_{H^{-1}(B)} \leq C(n, a, d) \|\nabla u(\cdot, t)\|_{L^2(B)} \quad \text{for a.e } t \in (-1, 0). \]

Therefore, there exists a unique \( U(\cdot, t) \in \tilde{L}^1_2(B) \) which solves (3) and such that 
\[ \|\nabla U(\cdot, t)\|_{L^2(B)} \leq C(n, a, d) \|\nabla u(\cdot, t)\|_{L^2(B)} \quad \text{for a.e } t \in (-1, 0). \]

We also deduce that \( \nabla U \in L^2(Q) \) and 
\[ \|\nabla U\|_{L^2(Q)} \leq C(n, a, d) \|\nabla u\|_{L^2(Q)}. \]

**Step 2.** Now, we can rewrite (2) as follows 
\[ \int_{Q} u \partial_t \phi dz = \int_{Q} \nabla U. \nabla \phi dz \quad \forall \phi \in C_0^\infty(Q). \quad (4) \]
By a density arguments, we can test (4) with functions $\phi(x,t) = \chi(t)\phi_k(x)$, where $\chi \in C_0^\infty(-1,0)$ and $\phi_k$ is an eigenfunction (introduced in the second part of the preliminaries
section; here we choose $\Omega = B$). Since $(\phi_k)_{k=1}^\infty$ is a Hilbert basis of $L_2(B)$, we can write $u$ as follows

$$u(x,t) = \sum_{k=1}^\infty d_k(t)\phi_k(x),$$

where $d_k(t) = \int_B u(\cdot,t)\phi_k dx$; we also have

$$U(x,t) = \sum_{k=1}^\infty b_k(t)\phi_k(x),$$

where $b_k(t) = \int_B U(\cdot,t)\phi_k dx$. So we have, thanks to Lemma 2, that

$$\|\nabla U\|_{L_2(Q)}^2 = \int_{-1}^0 \sum_{k=1}^\infty b_k^2(t)\lambda_k dt \leq C(n,a,d)\|\nabla u\|_{L_2(Q)}^2 < \infty.$$

We have now

$$\int_{-1}^0 d_k(t)\chi'(t)dt = \int_{-1}^0 \chi(t)\int_B \nabla U\cdot\nabla \phi_k dx dt$$

$$= \int_{-1}^0 \chi(t)b_k(t)\lambda_k dt.$$

So, $d_k'(t) = -b_k(t)\lambda_k$ and we derive that

$$\partial_t u(x,t) = \sum_{k=1}^\infty d_k'(t)\phi_k(x),$$

where the convergence of this sum occurs in the space of distributions; thus we have, for every $w \in C_0^\infty(B)$ and $\chi \in C_0^\infty(-1,0)$, that

$$\int_{-1}^0 \langle \partial u(\cdot,t), w \rangle \chi(t) dt = -\lim_{N \to \infty} \int_{-1}^0 \sum_{k=1}^N b_k(t)\lambda_k \int_B \phi_k(x)w(x)dx \chi(t) dt$$

$$= \lim_{N \to \infty} \int_{-1}^0 \sum_{k=1}^N b_k(t) \int_B \nabla \phi_k(x)\cdot\nabla w(x) dx \chi(t) dt$$

$$\leq \|\nabla w\|_{L_2(B)} \int_{-1}^0 \left( \sum_{k=1}^\infty b_k^2(t)\lambda_k \right)^{1/2} |\chi(t)| dt \quad \text{by Lemma 2}$$

$$\leq c(n, a, d)\|\nabla u\|_{L_2(Q)}\|\chi\|_{L_2(-1,0)}\|\nabla w\|_{L_2(B)},$$

and the statement follows.

Remark 1. Let $\varphi \in C_0^\infty(B)$. Then, we readily deduce from the above proposition that

$$\partial_t (u\varphi) \in L_2(-1,0; H^{-1}(B)).$$

Since obviously $u\varphi \in L_2(-1,0; \tilde{L}^1(B))$, we conclude that

$$u\varphi \in C([-1,0]; L_2(B)).$$
Proof of Theorem 3.1. Step 1. Consider the following auxiliary equation

\[
\begin{align*}
\partial_t w - \text{div}(A\nabla w) &= F - \text{div} G \quad \text{in } B \times (-1,0), \\
|w|_{\partial Q} &= 0,
\end{align*}
\]

where \( F = u \partial_t \varphi - (A \nabla u) \nabla \varphi \) and \( G = u (A \nabla \varphi) \). We have that the distribution \( F - \text{div} G \) belongs to \( L_2(-1,0; H^{-1}(B)) \). This is a direct consequence of the fact that \( u \varphi \) is distributional solution of (5) together with Remark 1 and Step 1 of the proof of Proposition 4.1. But, for our purpose we are interested, more precisely, in the bounds of the terms which appear in the definition of \( F - \text{div} G \) and which belong to \( L_2(-1,0; H^{-1}(B)) \). Therefore let us consider a function \( w \in C_0^\infty(B) \); then for a.e \( t \in (-1,0) \)

\[
\int_B F w dx + \int_B G \nabla w dx = \int_B u \partial_t \varphi w dx - \int_B (a \nabla u) \nabla \varphi w dx - \int_B (d \nabla u) \nabla \varphi w dx \\
+ \int_B (a \nabla \varphi) \nabla w dx + \int_B (d \nabla \varphi) \nabla w dx = J_1 + J_2 + J_3,
\]

where

\[
J_1 := \int_B u \partial_t \varphi w, \\
J_2 := -\int_B (a \nabla u) \nabla \varphi w dx + \int_B (a \nabla \varphi) \nabla w dx, \\
J_3 := -\int_B (d \nabla u) \nabla \varphi w dx + \int_B (d \nabla \varphi) \nabla w dx.
\]

We rewrite the term \( J_3 \) as follows

\[
J_3 = -\int_B d \nabla (uw) \nabla \varphi dx + \int_B (d \nabla w) \nabla \varphi w dx + \int_B (d \nabla \varphi) \nabla w dx = -\int_B d \nabla (uw) \nabla \varphi dx \quad \text{(by the skew symmetry of } d). \]

But again, thanks to the skew symmetry of \( d \), we have that

\[
J_3 = -\frac{1}{2} \sum_{i,j=1}^n \int_B b_{i,j} [ (uw)_{,ij} \phi_{,i} - \phi_{,ij} (uw)_{,i} ] dx
\]

and making the same computations as for \( A_2 \) in Step 1 of the proof of Proposition 4.1, with the only difference being that we keep \( p \) arbitrary (instead of choosing \( p = 2 \) as in the proof of Proposition 4.1), we obtain

\[
|J_3| \leq C(n,a,d) \| \nabla (uw) \|_{L_p(B)} \| \varphi \|_{L_p/(p-1)(B)},
\]

(for \( 1 < p < \infty \) to be suitably chosen in function of \( n \)) which implies that

\[
|J_3| \leq C(n,a,d,\varphi) \left( \| w \nabla u \|_{L_p(B)} + \| u \nabla w \|_{L_p(B)} \right).
\]
If \( n \geq 3 \), we steadily have for
\[
1 < p < \min(2, \frac{n}{n-1}),
\]
that (here \( 2^* = \frac{2n}{n-2} \))
\[
\| w \nabla u \|_{L^p(B)} + \| u \nabla w \|_{L^p(B)} \leq c(n) \left[ \| w \|_{L^{2^*}(B)} \| \nabla u \|_{L^2(B)} + \| u \|_{L^{2^*}(B)} \| \nabla w \|_{L^2(B)} \right]
\leq c(n) \left( \| \nabla u \|_{L^2(B)} + \| u \|_{L^2(B)} \right) \| \nabla w \|_{L^2(B)},
\]
where Sobolev embedding and Poincaré’s inequality are used in the last estimate.

The case \( n = 2 \) is a straightforward adaptation of the previous (since \( H^1(B) \) embeds continuously in every \( L^s(B), \ 1 \leq s < \infty \)), whereas for the case \( n = 1 \), we take \( p = 2 \), use the fact that \( H^1(B) \) is continuously embedded in \( L^\infty(B) \) and Poincaré’s inequality for the term in \( w \).

Next, we have the following easy bound for the terms \( J_1 \) and \( J_2 \):
\[
|J_1 + J_2| \leq C(n, \varphi) \left( \| \nabla u \|_{L^2(B)} + \| u \|_{L^2(B)} \right) \| \nabla w \|_{L^2(B)}.
\]
So in conclusion, we have, for a.e \( t \in (-1, 0) \) that
\[
\| F(\cdot, t) - \text{div} G(\cdot, t) \|_{H^{-1}(B)} \leq C(n, a, d, \varphi) \left( \| \nabla u(\cdot, t) \|_{L^2(B)} + \| u(\cdot, t) \|_{L^2(B)} \right),
\]
and we get a fortiori
\[
\| F - \text{div} G \|_{L^2(-1,0; H^{-1}(B))} \leq C(n, a, d, \varphi) \left( \| \nabla u \|_{L^2(Q)} + \| u \|_{L^2,\infty(Q)} \right)
\tag{6}
\]

**Step 2.** Let us now tackle the question of well-posedness of (5). Consider the time-indexed family of bilinear forms
\[
\delta_t(w, v) := \int_B (A \nabla w). \nabla v dx.
\]
Let us first notice that the map \( t \in (-1, 0) \mapsto \delta_t(w, v) \) is measurable for every \( w, v \in \dot{L}^1(B) \). Furthermore, we have by similar computations as those made in Step 1 in the proof of Proposition 4.1, that there exists a constant \( C = C(n, a, d) > 0 \) independent of \( t \) such that
\[
|\delta_t(w, v)| \leq C \| \nabla w \|_{L^2(B)} \| \nabla v \|_{L^2(B)},
\]
for all \( w, v \in \dot{L}^1(B) \) i.e \( \delta_t \) is a bounded bilinear operator on \( \dot{L}^1(B) \). We have, additionally, the following coercivity estimate
\[
\delta_t(w, w) = \int_B (A \nabla w). \nabla w dx
= \int_B (a \nabla w). \nabla w dx \quad (\text{by the skew symmetry of } d),
\geq \nu \int_B |\nabla w|^2 dx.
\]
In view of these previous estimates and the regularity proved for the right-hand side of (5) and considering the evolution triple $L^1_2(B) \subset L_2(B) \subset H^{-1}$, we have by applying J-L. Lions abstract theorem for well-posedness of evolution equations (see e.g.,[4], Theorem 4.1, Chapter 3, section 4) that there exists a unique solution

$$ w \in C([-1,0]; L_2(B)) \cap L_2(-1,0; L^1_2(B)), $$

with

$$ \partial_t w \in L_2(-1,0; H^{-1}) $$

such that

$$ \int_Q \partial_t w v dz + \int_Q (A \nabla w). \nabla v dz = \int_Q (F - \text{div} \, G) v dz \quad (7) $$

for any $v \in C_0^\infty(Q)$. Let us notice that, from Remark 1, the fact that $u\phi$ is a distributional solution of (5) and by the above uniqueness result:

$$ w = u\phi. $$

On another hand, by the regularity obtained for $w$, we can extend identity (7) to functions $v$ in $L_2(-1,0; L^1_2(B))$ and therefore, test (7) with $w$ itself. Thus, we get

$$ \int_Q (A \nabla w). \nabla w dz = \int_Q (F - \text{div} \, G) w dz. \quad (8) $$

Denote by $L$, the left-hand side of the above identity. By the skew symmetry of $d$, we obtain that

$$ L = \int_Q (a \nabla w). \nabla w dz; $$

therefore coming back to $u$, we get

$$ L = \int_Q a(\varphi \nabla u + u \nabla \varphi). (\varphi \nabla u + u \nabla \varphi) dz $$

$$ = \int_Q \varphi^2 (a \nabla u). \nabla u dz + 2 \int_Q u \varphi (a \nabla u). \nabla \varphi dz + \int_Q u^2 (a \nabla \varphi). \nabla \varphi. $$

Now, denote by $R$ the right hand side of (8); we easily obtain that

$$ R = \int_Q u \partial_t \varphi w dz - \int_Q (a \nabla u). \nabla \varphi w dz + \int_Q (a \nabla \varphi). \nabla w u - \int_Q d \nabla (u w). \nabla \varphi dz $$

$$ = \int_Q u^2 \varphi \partial_t \varphi dz - \int_Q w \varphi (a \nabla u). \nabla \varphi dz + \int_Q \varphi u (a \nabla u). \nabla \varphi dz + \int_Q u^2 (a \nabla \varphi). \nabla \varphi dz $$

$$ - 2 \int_Q u \varphi (d \nabla u). \nabla \varphi dz. $$

Therefore, (8) implies that

$$ \int_Q \varphi^2 (a \nabla u). \nabla u dz + 2 \int_Q w \varphi (a \nabla u). \nabla \varphi dz = \int_Q u^2 \varphi \partial_t \varphi dz - 2 \int_Q u \varphi (d \nabla u). \nabla \varphi dz, \quad (9) $$

8
for all $\varphi \in C_0^\infty(Q)$. Let us notice that all the integrals in the above identity are finite, especially the last one of the right hand side of (9). To see this we rewrite

$$\int_Q w\varphi(d\nabla u)\cdot \nabla \varphi dz = \int_Q d\nabla(u^2/2)\cdot \nabla(\varphi^2/2)dz$$

and use the same method as in the estimation of $J_3$ in the previous step.

**Step 3.** Now, we choose $\varphi(x,t) = \chi_\epsilon(t)\phi(x,t)$, where $\phi \in C_0^\infty(B \times (-1,1))$ and $\chi_\epsilon(t) = 1$ if $t \leq t_0 - \epsilon$, $\chi_\epsilon(t) = (t_0 + \epsilon - t)/(2\epsilon)$, if $t_0 - \epsilon < t < t_0 + \epsilon$, and $\chi_\epsilon(t) = 0$ when $t \geq t_0 + \epsilon$, with $t_0 \in (-1,0)$. Therefore passing to the limit $\epsilon \to 0$ in (9), we have that Theorem 3.1 is proved.

\[\square\]

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