Discriminating between effective theories of $U_A(1)$ symmetry breaking

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We address the question if one can empirically distinguish between the two proposed solutions to the “$U_A(1)$ problem”: the ’t Hooft, and the Veneziano-Witten $U_A(1)$ symmetry breaking effective interactions. Two hadronic observables are offered as discriminants: (1) The scalar $(0^+)$ meson spectrum; (2) Weinberg’s second spectral sum rule. Their present experimental status is discussed.

I. WHAT IS THE “$U_A(1)$ PROBLEM”?

The $U_A(1)$ problem consists of two parts: (i) the discrepancy between the left- and right-hand side in the “Gell-Mann–Okubo” type mass relation:

$$m_{\eta'}^2 + m_\pi^2 = (1111 \text{ MeV})^2 \neq 2m_K^2 = (700 \text{ MeV})^2,$$

and (ii) the $\eta - \eta'$ mixing angle being negative and far from the ideal one:

$$\theta_{ps} \simeq -20^\circ \neq 35.3^\circ.$$

The presently accepted solution postulates an explicit breaking of the $U_A(1)$ symmetry, i.e., a new interaction $H_{U(1)}$ that satisfies

$$\left( f m_{U(1)}^2 \right)_{ab} = -\langle 0 | [Q_5^a, [Q_5^b, H_{U(1)}(0)]] | 0 \rangle = 0, \quad a, b = 1, \ldots 8;$$

$$\neq 0, \quad a = b = 0. \quad (1)$$

Here $a, b$ are the flavour indices of the axial charges corresponding to the appropriate pseudoscalar (ps) meson(s). This interaction raises the mass of the SU(3) flavour-singlet and thus provides for the mass difference

$$m_{U(1)}^2 = m_{\eta'}^2 + m_\pi^2 - 2m_K^2 \simeq (855 \text{ MeV})^2.$$

The $U_A(1)$ symmetry-breaking mass $m_{U(1)}$ is 855 MeV, provided all pseudoscalar decay constants are equal. Inclusion of the variation in ps decay constants leads to $830 \pm 60$ MeV. The same interaction $H_{U(1)}$ solves the ps mixing angle problem:

$$\tan 2\theta_{ps} = \frac{(2\sqrt{2}/3) \Delta_{ps}^2}{(1/3) \Delta_{ps}^2 - f_0^2 m_{U(1)}^2}, \quad (2)$$

where

$$\Delta_{ps}^2 = f_K^2 (m_{K^0}^2 + m_{K^+}^2) - f_\pi^2 (m_{\pi^0}^2 + m_{\pi^+}^2), \quad (3)$$

$$f_0^2 m_{U(1)}^2 = f_\eta^2 m_{\eta'}^2 + f_\pi^2 m_{\pi^0}^2 - f_K^2 (m_{K^0}^2 + m_{K^+}^2) + f_\pi^2 (m_{\pi^+}^2 - m_{\pi^0}^2), \quad (4)$$

which leads to a negative ps. mixing angle $\theta_{ps} = -(25 \pm 10)^\circ$.

II. TWO SOLUTIONS

Two $U_A(1)$ symmetry breaking effective operators are discussed in the literature: (1) the ’t Hooft-Kobayashi-Kondo-Maskawa (“’t Hooft”, for short) effective interaction, and (2) the “Veneziano-Witten” effective interaction.
The 't Hooft interaction

This interaction is believed to be induced by instantons in QCD. It reads

\[ \mathcal{L}_{tH}^{(N_f=3)} = -K_{tH} \left[ \det_f \left( \bar{\psi}(1 + \gamma_5)\psi \right) + \det_f \left( \bar{\psi}(1 - \gamma_5)\psi \right) \right] \]  

(5)

where, \( \det_f \left( \bar{\psi}(1 \pm \gamma_5)\psi \right) \) is a determinant of the flavour-space matrix. Eq. (5) then leads to

\[ m_{00}^2(tH) f_0^2 = 6 \langle 0 | \mathcal{L}(3)_{tH}(0) | 0 \rangle + O(1/N_C) \]
\[ = -12K_{tH} \langle \bar{q}q \rangle_0 + O(1/N_C) , \]

(6)

The symbol \( O(1/N_C) \) remind us that we have neglected \( 1/N_C \) suppressed terms. So long as the respective coupling constant is sufficiently large, the U(1) problem is solved.

The Veneziano-Witten interaction

This interaction

\[ \mathcal{L}_{VW}^{(N_f=3)} = K_{VW} \left[ \det_f \left( \bar{\psi}(1 + \gamma_5)\psi \right) - \det_f \left( \bar{\psi}(1 - \gamma_5)\psi \right) \right]^2 \]  

(7)

is not induced by instantons, but rather by \( 1/N_C \) effects in QCD. Again from Eq. (5) we find

\[ m_{00}^2(VW) f_0^2 = 48K_{VW} \langle \bar{q}q \rangle_0 + O(1/N_C) . \]

(8)

The same comments about the coupling constant hold as for the 't Hooft interaction. Once again, the flavour-singlet pseudoscalar mass moves up above the octet one.

III. DISCRIMINATING BETWEEN THE SOLUTIONS

Manifestly, no study of the pseudoscalar \( \eta, \eta' \) mesons’ properties alone can resolve this issue. We offer two new tests discriminating between the two effective interactions in a chiral quark model. Differences between models with the 't Hooft- and the Veneziano-Witten (VW) \( U_A(1) \) symmetry-breaking interactions arise in: (i) the scalar mesons spectra, [7, 8, 9]. The 't Hooft interaction leads to a mass shift within the scalar nonet that is identical in size, but opposite in sign to that found in pseudoscalars, whereas the VW one does not shift the scalar meson masses at all. (ii) the second spectral (Weinberg) sum rule [10]. The 't Hooft interaction strongly modifies this sum rule, whereas the VW one does not change it at all.

A. Scalar meson spectrum

In the following we use an effective chiral field theory of quarks with a non-trivial ground state characterized by a finite quark condensate and an effective \( U_A(1) \) symmetry-breaking interaction, following Nambu and Jona-Lasinio (NJL) [11]. The \( U_L(3) \times U_R(3) \) symmetric gluon-exchange interaction (the first line) is modelled by a quartic quark selfinteraction

\[ \mathcal{L}_{NJL}^{(3)} = \bar{\psi}[i\partial - m^0]\psi + G \sum_{i=0}^{8} \left[ \left( \bar{\psi} \lambda_i \psi \right)^2 + \left( \bar{\psi} i \gamma_5 \lambda_i \psi \right)^2 \right] + \mathcal{L}_{U(1)} , \]

(9)

amended by the \( U(1)_A \) symmetry breaking effective interaction. There are at present no easily applicable nonperturbative methods for a direct approach to the 6- or 12-point operators in Eq. (9). Therefore one has to construct an “effective mean-field quartic self-interaction Lagrangian” \( \mathcal{L}_{eff}^{(4)} \) from Eq. (5,7) following the procedure employed in Ref. [7]. This leads to consistent chiral dynamics in the sense that the Goldstone theorem and other chiral Ward–Takahashi identities pertaining to the ps octet remain intact in the chiral limit. Mathematically that procedure is equivalent to taking a quark and an antiquark external line and closing them into a loop using Feynman rules for the Lagrangian (9) in all possible ways while taking into account the proper symmetry number of the diagram. The meson masses are read off from the poles of their propagators, which in turn are constrained by the gap equation. This model has turned out to be a reliable laboratory for calculating light spinless meson mass relations induced by \( U_A(1) \) symmetry-breaking, as can be seen from the comparison between the NJL model results [7] and a confining potential model’s predictions [8]. The close agreement of the spectra is the best justification of the NJL model.
The ps meson flavour singlet - octet mass shift due to the ’t Hooft interaction in the NJL model has been established to be in exact agreement with the general result (6), see Ref. [7]. One finds, however, that the singlet - octet mass splitting in the scalar (0\(^+\)) channel is just as large as, though of opposite sign to the ps one. This statement is embodied in the \(N_f = 3\) scalar – pseudoscalar meson mass sum rule

\[
m^2_\eta + m^2_\eta' - m^2_\eta + m^2_\eta' = m^2_{K^*_\eta} + m^2_{K^*_\eta'} - m^2_f - m^2_{f'}.
\] (10)

Equivalent results were found in a confining chiral quark model, Ref. [8], also as an effect of the ’t Hooft \(U_A(1)\) symmetry breaking interaction. For two flavours the ’t Hooft interaction predicts a mass splitting between the isoscalar (\(f_0\)) and isovector (\(a_0\)) scalar (0\(^+\)) states that has been observed on the lattice, see Fig. 6 in Ref. [12]. The corresponding pseudoscalar mass splitting with two flavours has not been calculated, so direct comparison with the \(N_f = 2\) sum rule

\[
m^2_a - m^2_{a'} = m^2_{f_0' - a_0}
\] (11)

is not possible at the moment. The sum rule (11) shifts the masses of the physical iso-singlet scalar states \(f_0, f_0'\) from their simple quark model values, see Fig. 1.

Assuming that the \(f_0(1500)\) is one of the two isoscalar scalar states, the sum rule (11) predicts the mass of the other scalar state as 1000±50 MeV. As there are two iso-singlet scalar states \(f_0\) in the Particle Data tables with mass(es) very close to 1 GeV, the \(f_0(980)\) and the (”σ”) \(f_0(1600)\), one is left with an ambiguity. The \(f_0(1500)\) is in better shape: Ritter et al. [3] have explained the puzzling absence of \(K\bar K\) pairs from the \(f_0(1500)\) two-body decay products as a consequence of the ’t Hooft interaction. This explanation depends critically on the scalar mixing angle \(\theta_s\) being small and \textit{positive}, which follows from

\[
\tan 2\theta_s = \frac{(4\sqrt{2}/3)\left(m^2_{K^*_f} - m^2_{a_0}\right)}{m^2_{U(1)} + (2/3)\left(m^2_{K^*_f} - m^2_{a_0}\right)}.
\] (12)
One must say that higher-order corrections are known to modify our sum rule \( \text{(10)} \). For example, upon taking into account of vector- and axial-vector mesons, the r.h.s. (scalar masses) of Eqs. \( \text{(10),(11)} \) are reduced by a multiplicative factor equal to the flavour-singlet axial coupling constant of the constituent quarks \( \text{[14]} \).

**Veneziano-Witten interaction** To leading order in \( N_C \), we find again that the general result Eq. \( \text{(8)} \) holds for the meson masses, see Ref. \[ 8 \]. This time scalar mesons are unaffected by the VW interaction. In the absence of flavour singlet-octet mass splitting in the scalar sector, the flavour-singlet scalar mesons mix ideally (see Eq. \( \text{(12)} \), but with \( m_{(1)}^0 \) omitted from the denominator) and one finds one \( \bar{u}u, \bar{d}d \) and one \( s\bar{s} \) state, with a mass splitting of about 300 MeV. The lower-lying state is degenerate with the isovector scalar mesons, i.e., around 1320 MeV in this model. Curiously, there is an \( f_0 \) state at 1370 MeV. Then the heavy scalar meson ought to be near 1600 MeV. There are two candidates in the vicinity: (a) the familiar \( f_0(1500) \), and (b) the new \( f_0(1720) \). The former has a puzzling absence, for an \( s\bar{s} \) state, of the \( K\bar{K} \) decay mode. This has prompted suggestions that it is not an ordinary \( q\bar{q} \) octet member, as the Veneziano-Witten model predicts. This evidence and the apparent success of the ’t Hooft model at explaining the \( f_0(1500) \) decay pattern \[ 13 \] seem to rule against the Veneziano-Witten model.

**B. The second spectral sum rule**

Weinberg’s second sum rule (Wsr II) \[ 15 \]

\[
\int_0^\infty ds \left( \rho^V(s) - \rho^A(s) \right) = 0, \tag{13}
\]

for the (difference of) vector and axial vector spectral functions is a statement about the chiral symmetry of the underlying theory at asymptotically large momenta. Here \( \rho_{V,A}^b \) are the spectral functions as defined by

\[
\langle 0 | \left[ A^a_i(x), A^b_j(y) \right] | 0 \rangle &= -i \int d\mu^2 \rho_A^b(\mu^2) \left( g_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\mu^2} \right) \Delta(x-y;\mu^2) \\
&+ i \int d\mu^2 \rho_V^b(\mu^2) \partial_\mu \partial_\nu \Delta(x-y;\mu^2), \tag{14}
\]

\[
\langle 0 | \left[ V^a_i(x), V^b_j(y) \right] | 0 \rangle &= -i \int d\mu^2 \rho_V^b(\mu^2) \left( g_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\mu^2} \right) \Delta(x-y;\mu^2), \tag{15}
\]

\( V^a_i, A^b_j \) are the vector- and axial currents and \( \Delta \) is the commutator of the free scalar fields at two space-time points.

The two Weinberg spectral sum rules have been examined in an effective field theoretical model of QCD, Ref. \[ 10 \], with the result that the first sum rule is exactly satisfied, and the second one is broken even in the exact \( SU_L(2) \times SU_R(2) \) chiral limit, i.e., with current quark masses \( m_u^0 = m_d^0 = 0 \).

**Violations of the second Weinberg sum rule** Nieh \[ 17 \] gave a critical assessment of this sum rule very early, though it passed largely unnoticed. One can express the violation of the Wsr II due to the Hamiltonian \( H \) as

\[
\delta_{ij} \int_0^\infty ds \left( \rho^b_{V}(s) - \rho^A_{A}(s) \right) = \int d^3 x (0) \left[ [H, A^a_i(0, x)], A^b_j(0, y) \right] | 0 \rangle \\
- \int d^3 x (0) \left[ [H, V^a_i(0, x)], V^b_j(0, y) \right] | 0 \rangle, \tag{16}
\]

where \( i,j = 1,2,3 \) are the spatial dimension indices. Thus, Eq. \( \text{(14)} \) shows that the second sum rule actually tests the commutators of the *spatial current components* and the Hamiltonian, i.e., the invariance of the theory under \( U_L(6) \times U_R(6) \) current algebra symmetry transformations \[ 13 \], rather than merely the usual \( U_L(3) \times U_R(3) \) chiral charge algebra, which is its subalgebra. Manifestly, any term in the Hamiltonian that breaks the \( U_L(3) \times U_R(3) \) symmetry will also break the \( U_L(6) \times U_R(6) \) symmetry. There are three sources of \( U_L(3) \times U_R(3) \) symmetry breaking in QCD: (i) the current quark masses; (ii) \( U_A(1) \) symmetry-breaking effective interaction; and (iii) the electroweak (EW) interactions. In the following we shall examine only the first two.

The second spectral sum rule Eq. \( \text{(16)} \), upon Nieh’s correction \[ 17 \], is also satisfied in the effective model of Ref. \[ 10 \]. That model already contains the \( U_A(1) \) symmetry breaking ’t Hooft interaction in its two-flavor version. Thus there is a large violation of the Wsr II even in the chiral limit, due the \( U_A(1) \) symmetry breaking ’t Hooft interaction in this model \[ 10 \].
Current quark mass terms  Inserting the current quark mass Hamiltonian $\mathcal{H}_{\chi SB}(0) = \bar{\Psi}(0)M_q^0\Psi(0)$ into Eq. (16) we find

$$\int_0^\infty ds \left( \rho^{ab}_{\tau}(s) - \rho^{ab}_A(s) \right) = \int d^3x(0) \left[ \left[ H_{\chi SB}, A^a_i(0, x) \right], A^b_j(0, y) \right] |0\rangle - \int d^3x(0) \left[ \left[ H_{\chi SB}, V^a_i(0, x) \right], V^b_j(0, y) \right] |0\rangle$$

$$= -\langle 0 | \bar{\Psi} \left\{ \left\{ M_q^0, \frac{\lambda^a}{2} \right\}, \frac{\lambda^b}{2} \right\} \Psi |0\rangle$$

$$= -\langle 0 | [Q^a_{\tau}, Q^b_{\tau}, \bar{\Psi}M_q^0\Psi] |0\rangle$$

$$= (f_{ps}m_{ps}^2(mech) f_{ps})_{ab}. \quad (17)$$

The expression on the right-hand side of (17) is the same as the one entering the Gell-Mann-Oakes-Renner (GMOR) formula relating the pseudoscalar (ps) and scalar channels. Here one

Note that this result holds for all others as we have come to expect its influence only in the flavour-singlet ps and scalar channels. Thus the current quark mass induced corrections to the right-hand side of Eq. (19) we find

$$\int d^3x(0) \left[ \left[ H^{(3)}_{\chi_{1H}}, A^a_i(0, x) \right], A^b_j(0, y) \right] |0\rangle - \int d^3x(0) \left[ \left[ H^{(3)}_{\chi_{1H}}, V^a_i(0, x) \right], V^b_j(0, y) \right] |0\rangle$$

$$= -\langle 0 | H^{(3)}_{\chi_{1H}} |0\rangle \delta_{ij} \delta^{ab}. \quad (18)$$

The interacting ground state ("vacuum") expectation value of the 't Hooft interaction is related to the 't Hooft mass with three light flavours via Eqs. (14), see Ref. 12. The empirical value of $f_{ps}^2m_{ps}^2$ was determined above as (300MeV)$^4$ from the definition Eq. (6). This leads to

$$\int d^3x(0) \left[ \left[ H^{(3)}_{\chi_{1H}}, A^a_i(0, x) \right], A^b_j(0, y) \right] |0\rangle - \int d^3x(0) \left[ \left[ H^{(3)}_{\chi_{1H}}, V^a_i(0, x) \right], V^b_j(0, y) \right] |0\rangle$$

$$= \delta_{ij} \delta^{ab} f_{ps}^2 m_{U(1)}^2. \quad (19)$$

Note that this result holds for all $a, b = 0, \ldots, 8$, i.e., not only in the flavour-singlet channel ($a, b = 0$). This is something of a surprise, as we have come to expect its influence only in the flavour-singlet ps and scalar channels. Here one is sensitive to the violation of the $U_L(6) \times U_R(6)$ current algebra, rather than that of the (usual) $SU_L(3) \times SU_R(3)$ algebra of chiral charges, and that the 't Hooft interaction violates the $U_L(6) \times U_R(6)$ symmetry. Adding now the current quark mass contribution to the right-hand side of Eq. (14) we find

$$\int_0^\infty ds \left( \rho^{ab}_{\tau}(s) - \rho^{ab}_A(s) \right) = \delta_{ab} f_{ps}^2 m_{U(1)}^2 + f_a m_{ab}(mech). f_b. \quad (20)$$

There is, however, another way to effectively break the $U_A(1)$ symmetry with quark degrees of freedom: the Veneziano-Witten effective quark interaction Eq. (7) Insert this into the double commutators in Eq. (10); direct calculation yields zero, to leading order in $1/N_C$,

$$\int d^3x(0) \left[ \left[ H^{(3)}_{VW}, A^a_i(0, x) \right], A^b_j(0, y) \right] |0\rangle - \int d^3x(0) \left[ \left[ H^{(3)}_{VW}, V^a_i(0, x) \right], V^b_j(0, y) \right] |0\rangle$$

$$= 0 + O(1/N_C), \quad (21)$$

thus leaving only the current quark mass induced corrections to the Wsr II

$$\int_0^\infty ds \left( \rho^{ab}_{\tau}(s) - \rho^{ab}_A(s) \right) = f_a m_{ab}(mech). f_b, \quad (22)$$

in the Veneziano-Witten model. Hence there is an order of magnitude difference between these two models of $U_A(1)$ symmetry breaking in all the flavour channels ($a, b = 0, 1, \ldots, 8$).

Just one precise measurement, say in the isovector channel, should discriminate between the two models. Some data at low energies already exist, see Fig. 1. in Ref. 13, but the range of energy integration is limited by the lepton mass, and saturation of the sum rule is not achieved. Thus far, the results are inconclusive. New kinds of experiments seem necessary. There is hope, however, that the methods described by Hatsuda in these proceedings 14 will allow an "exact" calculation of the second spectral sum rule in lattice QCD.
IV. CONCLUSIONS

1. There are two effective $U_A(1)$ symmetry breaking interactions: 't Hooft and Veneziano-Witten.
2. Scalar meson spectrum and the second spectral sum rule discriminate between them.
3. Flavour-singlet scalar mesons can be identified in accord with either the 't Hooft or Veneziano-Witten interactions. Decay properties seem to slightly prefer 't Hooft, but more work is necessary.
4. Present ($\tau$ lepton decay) data on the second Weinberg sum rule does not extend high enough in energy to differentiate between these two interactions.

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