EQUIVARIANT EULER CHARACTERISTICS OF PARTITION POSETS

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ABSTRACT. We compute all the equivariant Euler characteristics of the \( \Sigma_n \)-poset of partitions of the \( n \) element set.

1. INTRODUCTION

Let \( G \) be a finite group and \( \Pi \) a finite \( G \)-poset. For \( r \geq 1 \), let \( C_r(G) \) denote the set of tuples \( X = (x_1, \ldots, x_r) \) of \( r \) commuting elements of \( G \). Write \( \Pi^X \) for the subposet consisting of all elements of \( \Pi \) fixed by all elements of \( X \in C_r(G) \). The \( r \)th reduced equivariant Euler characteristic of the \( G \) poset \( \Pi \), as defined by Atiyah and Segal [2], is the normalized sum

\[
\tilde{\chi}_r(\Pi, G) = \frac{1}{|G|} \sum_{X \in C_r(G)} \tilde{\chi}(\Pi^X)
\]

of the reduced Euler characteristics of the subposets \( \Pi^X \) as runs through the set \( C_r(G) \) of commuting \( r \)-tuples.

In this note we focus on equivariant Euler characteristics of partition posets. Let \( \Sigma_n \) denote the symmetric group of degree \( n \geq 2 \). The set \( \Pi(\Sigma_{n-1}\setminus \Sigma_n) \) of partitions of the standard right \( \Sigma_n \)-set \( \Sigma_{n-1}\setminus \Sigma_n \) is a (contractible) right \( \Sigma_n \)-lattice with smallest element \( \hat{0} \), the discrete partition, and largest element \( \hat{1} \), the indiscrete partition. We let \( \Pi^*(\Sigma_{n-1}\setminus \Sigma_n) = \Pi(\Sigma_{n-1}\setminus \Sigma_n) - \{\hat{0}, \hat{1}\} \) be the (non-contractible) \( \Sigma_n \)-poset obtained by removing \( \hat{0} \) and \( \hat{1} \).

We now state the result and defer the the explanation of the undefined expressions till after theorem.

**Theorem 1.1.** The \( r \)th reduced equivariant Euler characteristic of the \( \Sigma_n \)-poset \( \Pi^*(\Sigma_{n-1}\setminus \Sigma_n) \) of partitions is

\[
\tilde{\chi}_r(\Pi^*(\Sigma_{n-1}\setminus \Sigma_n), \Sigma_n) = \frac{1}{n}(a \ast b_{r})(n)
\]

when \( n \geq 2 \) and \( r \geq 1 \).

The multiplicative arithmetic sequence \( a \ast b_{r} \) is the Dirichlet convolution

\[
(a \ast b_{r})(n) = \sum_{d_1d_2=n} a(d_1)b_{r}(d_2)
\]

of the the multiplicative arithmetic sequences \( a \) and \( b_{r} \) given by

\[
a(n) = (-1)^{n+1}, \quad b_{r}(n) = \prod_p (-1)^{n_p} \binom{r}{n_p}_p, \quad r \geq 1, n \geq 1
\]

where \( n = \prod_p p^{n_p} \) is the prime factorization of \( n \) and the \( p \)-binomial coefficient

\[
\binom{r}{d}_p = \frac{(p^r - 1)(p^r - p^{r-1}) \cdots (p^d - p^{d-1})}{(p^d - 1)(p^d - p^{d-1}) \cdots (p^1 - 1)}
\]

is the number of \( d \)-dimensional subspaces of the \( r \)-dimensional \( \mathbb{F}_p \)-vector space [9, Proposition 1.3.18].

2. PARTITIONS OF FINITE G-SETS

Let \( G \) be a group and \( S \) a finite right \( G \)-set.

**Definition 2.1.** (1) A partition \( \pi \) of \( S \) is an equivalence relation on \( S \). The blocks of \( \pi \) are the equivalence classes of \( \pi \). For any \( x \in S \), \([x]_\pi\), or simply \([x]\), is the \( \pi \)-block of \( x \). The set of \( \pi \)-blocks is denoted \( \pi \setminus S \).

(2) \( \Pi(S) \) is the \( G \)-lattice of all partitions of \( S \) and \( \Pi^*(S) = \Pi(S) - \{\hat{0}, \hat{1}\} \) the \( G \)-poset of all partitions of \( S \) but the discrete and the indiscrete partitions, \( \hat{0} \) and \( \hat{1} \).

(3) A partition of \( S \) is a \( G \)-partition if \( x \sim y \iff xg \sim yg \) for all \( x, y \in S \) and \( g \in G \).
(4) $\Pi(S)^G$ is the lattice of all $G$-partitions of $S$ and $\Pi^+(S)^G = \Pi(G)^G - \{0, 1\}$ the poset of all $G$-partitions of $S$ but the discrete and indiscrete partitions.

(5) The isotropy subgroup at $x \in S$ is the subgroup $xG = \{g \in G \mid xg = x\}$ of $G$.

(6) If $\pi$ is a $G$-partition, the block isotropy subgroup at $x \in S$ is the isotropy subgroup $[x]G = \{g \in G \mid (xg)\pi x\}$ at the $\pi$-block $x$ of $x$ in the $G$-set $\pi\backslash S$ of $\pi$-blocks.

(7) The $G$-set $S$ is isotypical if all isotropy subgroups are conjugate.

(8) The $G$-partition $\pi \in \Pi(S)^G$ is isotypical if the $G$-set $\pi\backslash S$ of $\pi$-blocks is isotypical. $\Pi^+(S)^G$ is the poset of all isotypical $G$-partitions and $\Pi^+(\Pi^+)^G = \Pi^+(\Pi)^G - \{0, 1\}$.

The set $\Pi(S)$ of partitions of $S$ is partially ordered by refinement:

$$\pi_1 \leq \pi_2 \iff \forall x \in S: [x]_{\pi_1} \subseteq [x]_{\pi_2}$$

The meet of $\pi_1$ and $\pi_2$ is the partition $\pi_1 \cap \pi_2$ with blocks $[x]_{\pi_1 \cap \pi_2} = [x]_{\pi_1} \cap [x]_{\pi_2}$, $x \in S$. The discrete partition is $\emptyset$ with blocks $[x]_\emptyset = \{x\}$, $x \in S$, and the indiscrete partition is 1 with block $[x]_1 = S$, $x \in S$.

**Example 2.2.** Let $K$ be a subgroup of $G$. The partition $\omega_K$, whose blocks $[x]_{\omega_K} = xK$ are the $K$-orbits in $S$, is an $N_G(K)$-partition of $S$. In particular, the partition $\omega_G$ whose blocks are the $G$-orbits is a $G$-partition.

The set $\Pi(S)$ of partitions of $S$ is a right $G$-lattice: For any partition $\pi$ of $S$ and any $g \in G$, $\pi g$ is the partition given by $x(\pi g)y \iff (xg)\pi(xg)$. Then $[x]_{\pi g} = \{yg \mid x(\pi g)y\} = \{yg \mid (y(g)(\pi(xg))\} = \{y \mid y\pi(xg)\} = [x]_\pi$.

Obviously,

$$\pi \text{ is a } G\text{-partition} \iff \forall g \in G: \pi g = \pi \iff \forall g \in G \forall x \in X: [x]_{\pi g} = [x]_{\pi} \iff \forall g \in G \forall b \in \pi: bg \in \pi$$

Thus the fixed poset for this $G$-action on $\Pi(S)$, $\Pi(S)^G$, is the set of all $G$-partitions. The discrete and the indiscrete partitions are $G$-partitions.

**Proposition 2.3.** Let $\pi$ be a $G$-partition of $S$.

1. There is a right $G$-action on the set $\pi\backslash S$ of $\pi$-blocks such that $S \to \pi\backslash S$ is a $G$-map.
2. $xG \leq [x]G$ for any $x \in S$.
3. $xgG = xG^g$ and $[xg]G = [x]G^g$
4. $xG \leq [x]G$ for any $x \in S$ and any $g \in G$.

**Proof.** The $G$-action on $\pi\backslash S$ is given by $[x]g = [xg]$ for all $x \in S$ and $g \in G$. $\square$

**Definition 2.4.** Let $P$ be a subposet of a lattice. An element $c$ of $P$ is a contractor if $x \lor c \in P$ or $x \land c \in P$ for all $x \in P$.

If $c$ is a contractor for $P$ then $x \leq x \lor c \leq c$ or $x \leq x \land c \leq c$ are homotopies between the identity map of $P$ and the constant map $c$. We view $P$ as a finite topological space with the order right ideals as open sets.

**Lemma 2.5.** [1, Lemma 7.1] $\Pi^+(S)^G$ is contractible unless $S$ is isotypical.

**Proof.** Let $\omega_G$ be the $G$-partition represented by the $G$-map $S \to S/G$ to the $G$-set of $G$-orbits and $\theta_G$ the $G$-partition represented by the $G$-map $S \to S/G \to \Pi(S)/G$ to the set of isomorphism classes of $G$-orbits. Explicitly, $x\omega_G y$ if and only if $x$ and $y$ are in the same $G$-orbit, and $x\theta_G y$ if and only if $x$ and $y$ have conjugate isotropy subgroups. We shall prove that $\theta_G$ is a contractor (Definition 2.4) for $\Pi^+(S)^G$ when $S$ is not isotypical.

We first make some small observations. Obviously, $\omega_G \leq \theta_G$. The $G$-action is trivial if and only if $\omega_G = \emptyset$. The $G$-action is isotypical if and only if $\theta_G = 1$. If the $G$-action is trivial, all isotropy subgroups are equal to $G$, and therefore $\theta_G = 1$. We may summarize these observation in a string

$$\theta_G = \emptyset \Rightarrow \omega_G = \emptyset \iff \forall x \in S: xG = G \Rightarrow \theta_G = 1 \iff S \text{ is isotypical}$$

of implications.

Let $\pi$ be any $G$-partition of $S$. We claim that

$$\pi \land \theta_G = \emptyset \Rightarrow \pi = \emptyset$$

To see this first note that

$$\forall x, y \in S: x\pi y \Rightarrow y \cdot xG \subseteq [y]_{\pi \land \theta_G}$$

Indeed, let $x\pi y$ and $g \in xG$. Then $y\pi(xg)$ for $y\pi x$, $x = xg$, and $(xg)\pi(yg)$. Thus $y$ and $yg$ are both in $[y]_{\pi}$ and in $[y]_{\theta_G}$. Now assume that $\pi \land \theta_G = \emptyset$. Then

$$\forall x, y \in S: x\pi y \Rightarrow xG \leq yG$$
for the block $[y]_{|\theta_G} = [y]_0 = \{y\}$ consists of $y$ alone which forces $yg = y$ for all $g \in xG$. This can be sharpened to
\[ \forall x, y \in S : xGy \iff xG = yG \]
as the equivalence relation $\pi$ is symmetric, of course. Now, when $x$ and $y$ have the same isotropy subgroups, $x$ and $y$ belong to the same block under $\theta_G$. Thus we have shown $\pi \leq \theta_G$. Then $\pi = \pi \wedge \theta_G = 0$. This proves claim (2.6).

Suppose that $S$ is not isotypical. Then $\theta_G \neq 0, 1$ and $\theta_G$ belongs to the poset $\Pi^*(S)^G$. From claim (2.6) we know that $\pi \wedge \theta_G \neq 0$ for all $\pi \in \Pi^*(S)^G$. Thus $\theta_G$ is a contractor for $\Pi^*(S)^G$.

There are, of course, isotypical $G$-sets $S$ for which $\Pi^*(S)^G$ is contractible.

**Example 2.7** (An isotypical $G$-set $S$ such that $\Pi^*(S)^G$ is contractible). Suppose that the Frattini subgroup $\Phi(G)$ of $G$ is nontrivial and proper. The $G$-set $S = G$ is transitive and hence isotypical. But still the poset $\Pi^*(S)^G$ is contractible: By Proposition 2.8, $\Pi^*(S)^G$ is the poset $(1, G)$ of non-identity proper subgroups of $G$, and $\Phi(G)$ is a contractor of $(1, G)$. (I thank Matthew Gelvin for pointing out this example.)

A $G$-partition of a transitive $G$-set $S$ is uniquely determined by its block isotropy subgroup at a single point.

**Proposition 2.8.** [10, Lemma 3] Let $S$ be a transitive $G$-set and $x$ a point of $S$. The block isotropy map
\[ \Pi(S)^G \to [xG, G] = xG/S_G : \pi \mapsto [x]_\pi G \]
is an isomorphism of posets.

**Proof.** Let $H = xG$ be the isotropy subgroup of $x$. For every subgroup $K$ of $G$ containing $H$, let $\pi_K$ be the $G$-partition of $S$ with blocks $xKg, g \in G$ (the fibres of $S = H \cap K\setminus \{G\}$). The $\pi_K$-block of $x$, $[x]_{\pi_K} = xK$, has isotropy subgroup $\{g \in G \mid xg \in xK\} = K$. Conversely, let $\pi$ be any $G$-partition of $S$. The orbit through $x$ of the block isotropy subgroup $[x]_\pi G$ is $x : [x]_\pi G = [x]_\pi$ as $S$ is transitive. These observations show that $K \mapsto \pi_K$ is an inverse to the block isotropy subgroup map $\pi \mapsto [x]_\pi G$. It is clear that these bijections respect the partial orderings. □

**Definition 2.9.** $\mathcal{O}_G$ is the category of finite $G$-sets with surjective $G$-maps as morphisms.

We may consider $G$-partitions as morphisms in the category $\mathcal{O}_G$. To any $G$-partition $\pi$ of the $G$-set $S$ we associate the surjective $G$-map $S \to \pi S$. Conversely, the blocks of the partition represented by the surjective $G$-map $\pi : S \to T$ are the fibres of $\pi$. The block of $x \in S$ is $\pi^{-1}(\pi(x))$. The overlap of the block and the $G$-orbit of $x$ is the orbit through $x$ of the block isotropy subgroup, $\pi^{-1}(\pi(x)) \cap xG = x\pi(x)G$.

3. Euler characteristics of posets of $G$-partitions

Let $\Pi$ be a finite poset. For $a, b \in \Pi$ let
\[ a/\Pi = \{p \in \Pi \mid a \leq p\} \quad a/\Pi = \{p \in \Pi \mid a < p\} \quad k_a = -\chi(a/\Pi) \]
\[ \Pi/b = \{p \in \Pi \mid p \leq b\} \quad \Pi/b = \{p \in \Pi \mid p < b\} \quad k_b = -\chi(\Pi/b) \]
denote the coslice of $\Pi$ under $a$, the proper coslice of $\Pi$ under $a$, and the weighting at $a$, and, dually, the slice of $\Pi$ over $b$, the proper slice of $\Pi$ over $b$, and the coweighting at $b$ [4, Corollary 3.8]. The Euler characteristic of $\Pi$
\[ \sum_{a \in \Pi} k_a = \chi(\Pi) = \sum_{b \in \Pi} k_b \]
is the sum of the values of the weighting or coweighting. In particular, for a finite $G$-set $S$, we can compute the Euler characteristic of $\Pi^*(S)^G$,
\[ \sum_{\pi \in \Pi^*(S)^G} -\chi(\pi/\Pi^*(S)^G) = \chi(\Pi^*(S)^G) = \sum_{\pi \in \Pi^*(S)^G} -\chi(\Pi^*(S)^G/\pi) \]
from its weighting or coweighting [4, Corollary 3.8]. We shall now determine these functions.

**Proposition 3.2** (Slices in $\Pi^*(S)^G$). For any $G$-partition $\pi$ of the right $G$-set $S$ \[ \pi/\Pi(S)^G = \Pi(\pi\setminus S)^G, \quad \pi/\Pi^*(S)^G = \Pi^*(\pi\setminus S)^G \]
The weighting for $\Pi^*(S)^G$
\[ k^\pi = -\chi(\Pi^*(\pi\setminus S)^G), \quad \pi \in \Pi^*(S)^G, \]
vanishes at $\pi$ unless $\pi$ is isotypical (Definition 2.1.8).

**Proof.** Let $\rho$ be a partition of the right $G$-set $\pi\setminus G$ of blocks of $\pi$. There is then a partition of $S$ with blocks $[x] = ([x]_{\rho}), x \in S$. This new partition is a $G$-partition if and only if $\rho$ is a $G$-partition of $\pi \setminus S$. Any $G$-partition $\geq \pi$ of $S$ arises in this way. □
Proposition 3.3 (Coslices in $\Pi^*(S)^G$). For any $G$-partition $\pi$ of the right $G$-set $S$

$$\Pi(S)^G/\pi = \prod_{BG \in \pi \backslash S/G} \Pi(B)^G,$$  

$$\Pi^*(S)^G/\pi = \left( \prod_{BG \in \pi \backslash S/G} \Pi(B)^G \right)^*$$

The coweighting for $\Pi^*(S)^G$

$$k_{\pi} = - \prod_{BG \in \pi \backslash S/G} \chi(\Pi^*(B)^G), \quad \pi \in \Pi^*(S)^G,$$

vanishes at $\pi$ unless all blocks $B$ of $\pi$ are isotypical $G$-sets.

Proof. Let $\pi$ be a $G$-partition and $B$ one its blocks. Observe first that the blocks contained in $B$ of a $G$-partition $\lambda \leq \pi$ determine all blocks of $\lambda$ contained in any of the blocks of the orbit $BG$ through $B$ for the $G$-action on $\pi \backslash S$.

Let $B$ be a block, with isotropy subgroup $BG$, of the $G$-partition $\pi$. Let $\lambda$ be a $BG$ partition of $\pi$. Extend $\lambda$ to a $G$-partition of the orbit $BG$ of $\pi$ by $[xg]_\lambda = [x]_\lambda g$. We must argue that this extension is well-defined. Suppose that $x_1g_1 = x_2g_2$ for some $x_1, x_2 \in B$ and $g_1, g_2 \in G$. We must show that $[x_1]_\lambda g_1 = [x_2]_\lambda g_2$. We have $x_2 = x_1g_1g_1^{-1} = x_1g_2g_2^{-1}$. From $B = [x_2]_\lambda = [x_1g_1g_1^{-1}]_\lambda = [x_1]_\lambda g_1g_1^{-1} = Bg_1g_1^{-1}$ we get that $g_1g_1^{-1}$ stabilizes the block $B$. As $\lambda$ is a $BG$-partition, $[x_1]_\lambda g_1 = [x_1]_\lambda g_2g_2^{-1}$ implies $[x_1]_\lambda = [x_2]_\lambda g_2$ as we wanted.

Conversely, if $\lambda$ is a $G$-partition and $\lambda \leq \pi$ then the blocks of $\lambda$ inside a fixed block $B$ of $\pi$ form a $BG$-partition of $B$, of course.

According to Quillen the reduced Euler characteristic is multiplicative: $\chi((\prod L_i)^{\ast}) = \prod \chi(L_i^*)$ for lattices $L_i$ of more than one element [1, Proposition 2.8].

If the block $B$ of partition $\pi$ consists of a single element of $S$, then also the partition poset $\Pi(B)$ consists of a single element so it can be omitted from the poset product $\prod_{B \in \pi \backslash S} \Pi(B)$.

In all cases,

$$\sum_{\pi \in \Pi^*(S)^G} \chi(\Pi^*(\pi \backslash S)^G) = -\chi(\Pi^*(S)^G) = \sum_{\pi \in \Pi^*(S)^G} \prod_{BG \in \pi \backslash S/G} \chi(\Pi^*(B)^G)$$

where the sum on the left can be restricted to the $G$-partitions $\pi$ with $G$-isotypical block set $\pi \backslash S$, and the sum on the right can be restricted to the $G$-partitions $\pi$ for which $BG$ acts isotypically on every block $B$ of $\pi$. If $G$ acts non-isotypically on $S$ then these sums equal 0.

Example 3.5 (Two examples of G-partition posets). The poset $\Pi^*(S)^G$ of nontrivial $G$-partitions for $S = \{1, 2, \ldots, 4\}$ and $G = \langle (1, 2), (4, 5) \rangle \leq \Sigma_4$ (isotypical):

$$13 - 24 \quad \quad 12 - 34 \quad \quad 14 - 23$$

$$\begin{array}{c}
(k^*, k_*) = (1, 1) \\
(k^*, k_*) = (1, -1) \\
(k^*, k_*) = (1, 1) \\
\end{array}$$

$$\sum k^* = 3 = \sum k_*$$

$$1 - 2 - 34 \quad \quad 12 - 3 - 4$$

$$\begin{array}{c}
(k^*, k_*) = (0, 1) \\
(k^*, k_*) = (0, 1) \\
\end{array}$$

The poset $\Pi^*(S)^G$ of nontrivial $G$-partitions for $S = \{1, 2, \ldots, 6\}$ and $G = \langle (1, 2, 3), (4, 5) \rangle \leq \Sigma_6$ (non-isotypical):

$$1236 - 45 \quad \quad 12345 - 6 \quad \quad 1234 - 56$$

$$\begin{array}{c}
(k^*, k_*) = (1, 0) \\
(k^*, k_*) = (1, 0) \\
(k^*, k_*) = (1, 0) \\
\end{array}$$

$$1236 - 4 - 5 \quad \quad 12345 - 6 = \theta_G \quad \quad 123 - 4 - 56$$

$$\begin{array}{c}
(k^*, k_*) = (0, 0) \\
(k^*, k_*) = (-2, -1) \\
(k^*, k_*) = (0, 0) \\
\end{array}$$

$$\sum k^* = 1 = \sum k_*$$

Corollary 3.6. The inclusion $\Pi^{\ast \text{iso}}(S)^G \hookrightarrow \Pi^*(S)^G$ is a homotopy equivalence.
Proof. This follows immediately from Bouc’s theorem [3] since \( \pi/\Pi^*(S)^G \) is contractible unless \( \pi \) is isotypical by Proposition 3.2 and Lemma 2.5.

Because of Corollary 3.6 we now restrict attention to isotypical \( G \)-partitions of isotypical \( G \)-sets.

For any \( G \)-orbit \( S \) and any natural number \( n \geq 1 \), let \( nS = \coprod_n S \) be the isotypical \( G \)-set with \( n \) \( G \)-orbits isomorphic to \( S \).

**Definition 3.7.** Let \( S \) and \( T \) be \( G \)-orbits.

- An \( nS/kT \)-partition is an isotypical \( G \)-partition of \( nS \) with block \( G \)-set isomorphic to \( kT \).
- The \( G \)-Stirling number of the second kind

\[
S_G(nS,kT) = |\{\pi \in \Pi(nS)^G \mid \pi(nS) \cong kT\}|
\]

is the number \( nS/kT \)-partitions.

In the following, \( S_G \) is the poset of subgroups, and \([S_G]\) the poset of subgroup conjugacy classes of \( G \). We write \( \zeta_G \), or just \( \zeta \), for the poset incidence matrix (with \( \zeta_G(H,K) = 1 \) if \( H \leq K \) and \( \zeta_G(H,K) = 0 \) otherwise) and \( \mu = \mu_G = \zeta^{-1} \) for the Möbius matrix of \( S_G \).

**Definition 3.8.** The \( G \)-Stirling matrix of degree \( n \) is the square \( (n||S_G|| \times n||S_G||) \)-matrix

\[
[\zeta]_G \otimes S_G = ( (S_G(sH\backslash G,tK\backslash G))_{1 \leq s,t \leq n} )_{H,K \in S_G}
\]

obtained as the \((||S_G|| \times ||S_G||)\)-matrix of \((n \times n)\)-block matrices \((S_G(sH\backslash G,tK\backslash G))_{1 \leq s,t \leq n}\) of Stirling numbers with fixed \( G \)-orbits \( G,H \) and \( K,G \).

If we order the subgroups of \( G \) in decreasing order starting with \( G \) itself, the \( G \)-Stirling matrix is lower triangular. If we in Equation 3.1 insert the values from Proposition 3.2 we obtain formulas for the reduced Euler characteristic of the poset \( \Pi^*(nS)^G \),

\[
\chi(\Pi^*(nS)^G) = -1 - \sum_{T,k} \chi(\Pi^*(kT)^G)S_G(nS,kT), \quad 1 = \sum_{k|T|>1} -\chi(\Pi^*(kT)^G)S_G(nS,kT)
\]

with \( T \) ranging over the set of isomorphism classes of \( G \)-orbits and \( k \geq 1 \) over natural numbers with \( k|T| > 1 \). (Observe that \( S_G(nS,nS) = 1 \).) In matrix notation

\[
(S_G(sH\backslash G,tK\backslash G))_{H,K \in S_G} = \left( \begin{array}{c} \vdots \\ -\chi(\Pi^*(sH\backslash G)^G) \\ \vdots \end{array} \right)_{1 \leq s,t \leq n} = \left( \begin{array}{c} 0 \\ 1 \\ 1 \end{array} \right)
\]

we see that minus the reduced Euler characteristics of the \( G \)-partitions of the isotypical \( G \)-sets are a weighting for the Stirling matrix of \( G \). Equation (3.10) comes with the caveat that the top entry of the left column vector is 0 and not \(-\chi(\Pi^*(1G\backslash G)^G) = 1 \).

**Example 3.11** \((G\text{-Stirling matrices of degree 1})\). The Stirling number for single orbits \( S = H\backslash G \) and \( T = K\backslash G \),

\[
S_G(H\backslash G,K\backslash G) = S_G(H,[K]) = \left| \frac{N_G(H,K)}{|N_G(K,K)|} \right| = \left| \frac{G(H\backslash G,K\backslash G)}{G(K\backslash G)} \right| = \frac{|(K\backslash G)^H|}{|K\backslash G|} = \frac{\text{TOM}(H,K)}{\text{TOM}(K,K)}
\]

is the number, \( S_G(H,[K]) = |\{L \in [K] \mid H \leq L\}|, \) of conjugates of \( K \) containing \( H \) [7, Definition 3.5, Lemma 3.6]. This number is determined by the table of marks \( \text{TOM}(H,K) = \|(K\backslash G)^H\| \) for \( G \). Proposition 2.8 or [7] show that the entries of the column vector in Equation (3.10) are

\[
-\chi(\Pi^*(H\backslash G)^G) = -\chi(H,G) = -\mu(H,G)
\]

for all proper subgroups \( H \) of \( G \). (In any finite poset, \( \mu(x,y) = \chi(x,y) \) whenever \( x < y \) [9, Proposition 3.8.5]).

For instance, \( G = \Sigma_3 \) has \|\( \Sigma_3 \)\| = 4 orbits \( S_1,S_2,S_3,S_6 \) of sizes 1, 2, 3, 6. The \( \Sigma_3 \)-Stirling matrix of degree 1 is

| \( S_{\Sigma_3}(S,T) \) | \( \Sigma_3 \backslash \Sigma_3 \) | \( A_3 \backslash \Sigma_3 \) | \( C_2 \backslash \Sigma_3 \) | \( C_1 \backslash \Sigma_3 \) | \(-\chi(\Pi^*(H\backslash \Sigma_3)^\Sigma_3)\) |
|---|---|---|---|---|---|
| \( \Sigma_3 \backslash \Sigma_3 \) | 1 | | | | 0 |
| \( A_3 \backslash \Sigma_3 \) | 1 | 1 | | | 1 |
| \( C_2 \backslash \Sigma_3 \) | | 1 | 1 | | |
| \( C_1 \backslash \Sigma_3 \) | | 1 | 3 | | |

and (remembering the caveat that the top entry of the column to the far right is 0 when solving Equation (3.10)) we read off that \( \mu(A_3,\Sigma_3) = -1 \), \( \mu(C_2,\Sigma_3) = -1 \), \( \mu(1,\Sigma_3) = 3 \).
Since $\bar{\chi}(\Pi^*(1H \setminus G)^G) = \bar{\chi}(H, G) = \mu(H, G)$ for proper subgroups $H$ of $G$ by Proposition 2.8, it seems natural to define the higher Möbius numbers to be the solutions to the linear equation (3.10).

**Definition 3.12** (Higher Möbius numbers). For every subgroup $H$ of $G$ and every natural number $n \geq 1$ let

$$\mu_n(H, G) = \bar{\chi}(\Pi^*(nH \setminus G)^G)$$

with the convention that $\mu_1(G, G) = 1$.

For any group $G$, $\mu_n(G, G) = (-1)^n(n-1)! = \mu_n(1, 1)$ for $n \geq 2$, and $\mu_n(1, G) = \bar{\chi}(\Pi^*(\prod_n G)^G)$ for $n \geq 1$. With $n = 1$, $\mu_1(H, G) = \mu(H, G)$ is the usual Möbius function of $S_G$ as considered in Example 3.11.

The higher Möbius numbers $\mu_h(H, G)$ for $1 \leq h \leq n$ are determined by the $G$-Stirling matrix of degree $n$. We shall now consider the problem of determining the entries of this matrix.

Let $S(n, k)$ stand both for the poset of partitions of the $n$ element set with $k$ blocks and for the Stirling number (Example 3.17) of such partitions. Then

$$S_G(nH \setminus G, kK \setminus G) = \sum_{\pi \in S(n, k)} \prod_{b \in \pi} \frac{|O_G(H \setminus G, K \setminus G)|^{[b]}}{|O_G(K \setminus G)|^{[b]}} S(n, k) = \frac{TOM(H, K)^n}{TOM(K, K)^k} S(n, k)$$

In particular

$$S_G(nS, kT) = \begin{cases} |T|^{n-k} S(n, k) & O_G(S, T) \neq \emptyset \\ 0 & O_G(S, T) = \emptyset \end{cases}$$

when $G$ is abelian.

**Lemma 3.14.** If $H \leq G$ is normal in $G$, then $\mu_n(H, G) = \mu_n(1, H \setminus G)$ for all $n \geq 1$.

**Proof.** $H$ acts trivially on $H \setminus G$ as $Hgh = Hghg^{-1}g = Hg$ for all $h \in H$, $g \in G$. Thus a partition of $nH \setminus G$ is a $G$-partition if and only if it is a $H \setminus G$-partition. \qed

The higher Möbius numbers $\mu_1(H, G), \ldots, \mu_n(H, G)$ for $H \leq G$ (except for $\mu_1(G, G)$ which by decree equals 1) solve the system of linear equations (3.10) which we now rewrite as

$$[\zeta]_G \otimes S_G$$

with the $G$-Stirling matrix as coefficient matrix. We shall adapt the convention that in the Stirling matrix the groups will be listed with decreasing order. The group $G$ itself occurs as the first group in the Stirling matrix which is lower triangular. The first $n$ columns are made up of the block matrices $(S(i, j))_{1 \leq i, j \leq n}$ of classical Stirling numbers. All entries of the first column, in particular, equal $S(n, 1) = 1$. Thus

$$[\zeta]_G \otimes S_G$$

$$= \begin{bmatrix} \mu_1(G, G) \\ \mu_n(G, G) \\ \vdots \\ \mu_1(1, G) \\ \mu_n(1, G) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 0 \end{bmatrix}$$
or

\[
\begin{bmatrix}
\mu_1(G,G) \\
\vdots \\
\mu_n(G,G) \\
\vdots \\
\mu_1(1,G) \\
\vdots \\
\mu_n(1,G)
\end{bmatrix}
= ([\zeta]_G \otimes S_G)^{-1}
\begin{bmatrix}
1 \\
0 \\
\vdots \\
0
\end{bmatrix}
\]  

(3.16)

The entries of the inverted matrix \(([\zeta]_G \otimes S_G)^{-1}\) are the \textit{G-Stirling numbers of the first kind} [9, p 36].

**Example 3.17** (Higher Möbius numbers of the trivial group). The \(C_1\)-Stirling matrix of the second kind (Definition 3.7) is the matrix

\[
S =
\begin{bmatrix}
1 \\
1 & 1 \\
1 & 3 & 1 \\
1 & 7 & 6 & 1 \\
1 & 15 & 25 & 10 & 1 \\
1 & 31 & 90 & 65 & 15 & 1 \\
\end{bmatrix}
\]

of classical Stirling numbers \(S(n,k) = |\{ \pi \in \Pi_n \mid |\pi| = k \}\) of the second kind. The higher Möbius numbers of the trivial group are by Equation (3.16) equal to the Stirling numbers of the first kind [9, p 36]

\[
\mu_n(1,1) = (S^{-1})(n,1) = s(n,1) = (-1)^{n-1}(n-1)!, \quad n \geq 1
\]

We have re-derived the classical formula [9, Example 3.10.4] for the reduced Euler characteristic of the partition poset.

**Lemma 3.18.** If the group \(G\) is abelian then

\[
\mu_n(H,G) = \mu(H,G)|G : H|^{n-1} \mu_n(1,1)
\]

for all \(n \geq 1\) and all subgroups \(H \leq G\).

**Proof.** Since \(G\) is abelian, \(S_G(iH \backslash G, jK \backslash G) = |G : K|^{i-j} S(i,j)\) by Equation (3.13), and the \(G\)-Stirling matrix of degree \(n\) is the block matrix

\[
((\zeta(H,K)|G : K|^{i-j} S(i,j))_{1 \leq i,j \leq n})_{H,K \in [S_G]}
\]

The vector \(((\mu(G,H))_{1 \leq i \leq n})_{H \in [S_G]}\) is (Equation (3.16)) the first column

\[
((\mu(G,H)|G : H|^{i-1} S^{-1}(i,1))_{1 \leq i \leq n})_{H \in [S_G]} = ((\mu(H,K)|G : H|^{i-1} \mu_i(1,1))_{1 \leq i \leq n})_{H \in [S_G]}
\]

in the inverse matrix

\[
((\mu(G,H)|G : H|^{i-j} S^{-1}(i,j))_{1 \leq i,j \leq n})_{H,K \in [S_G]}
\]

of the \(G\)-Stirling matrix. \(\square\)

In the example below we consider an example of a Stirling matrix for a non-abelian group.

**Example 3.19.** The \(\Sigma_3\)-Stirling matrix of degree 3 (reusing the notation of Example 3.11) is

| \(S_{\Sigma_3}(S,T)\) | 1S₁ | 2S₁ | 3S₁ | 1S₂ | 2S₂ | 3S₂ | 1S₃ | 2S₃ | 3S₃ | 1S₆ | 2S₆ | 3S₆ | \(-\mu_i(H, \Sigma_3)\) |
|-----------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|------------------|
| 1S₁             | 1   | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0                |
| 2S₁             | 1   | 1   | 0   | 1   | 1   | 0   | 1   | 1   | 1   | 1   | 1   | 1   | 1                |
| 3S₁             | 1   | 3   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0                |
| 1S₂             | 1   | 0   | 0   | 1   | 0   | 0   | 1   | 0   | 0   | 1   | 0   | 0   | 1                |
| 2S₂             | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0                |
| 3S₂             | 1   | 3   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1                |
| 1S₃             | 1   | 0   | 0   | 1   | 0   | 0   | 3   | 0   | 0   | 1   | 0   | 0   | 0                |
| 2S₃             | 1   | 1   | 0   | 2   | 1   | 0   | 9   | 9   | 0   | 6   | 1   | 0   | 18               |
| 3S₃             | 1   | 3   | 1   | 4   | 6   | 1   | 27  | 81  | 27  | 36  | 18  | 1   | -216             |
We read off that \( \mu_n(A_3, \Sigma_3) = \mu_n(1, C_2) = -2^{n-1} \mu_n(1, 1) \) (Lemma \ref{lem:4.14}) and that \( \mu_n(1, \Sigma_3) = -3^n \mu_n(1, 1) \). This last result shows that Lemma \ref{lem:4.18} does not in general extend to non-abelian groups.

4. Equivariant Euler characteristics of G-posets

Let \( \Pi \) be a finite G-poset. The \( r \)th, \( r \geq 1 \), equivariant Euler characteristic of \( \Pi \) is \([2] [7, \text{Proposition 2.9}]
\[
\chi_r(\Pi, G) = \frac{1}{|G|} \sum_{X \in C_r(G)} \chi(\Pi^X) = \frac{1}{|G|} \sum_{A \leq G} \chi(\Pi^A) \varphi_r(A)
\]

The first sum runs over the set \( C_r(G) \) of all commuting \( r \)-tuples \( X = (x_1, \ldots, x_r) \) of elements of \( G \). The second sum runs over all abelian subgroups \( A \) of \( G \) and \( \varphi_r(A) \) is the number of generating \( r \)-tuples \( (a_1, \ldots, a_r) \) of elements of \( A \) [5] [7, Remark 2.20].

We now specialize from general poset to posets of partitions. Let \( S \) be a finite \( G \)-set, \( \Pi(S) \) the \( G \)-poset of partitions of \( G \), and \( \Pi^*(S) = \Pi(S) - \{ \emptyset, \hat{1} \} \) the \( G \)-poset of non-extreme partitions of \( S \).

**Definition 4.1.** The group \( G \) acts effectively on \( S \) if only the trivial element of \( G \) fixes all elements of \( S \).

**Lemma 4.2.** Suppose that the abelian group \( A \) acts effectively on \( S \). The following conditions are equivalent:

1. \( A \) acts isotypically on \( S \)
2. \( A \) acts freely on \( S \)
3. The degree of any non-identity element of \( A \) is \( |S| \)
4. The cycle structure of any element of \( A \) is \( d^m \) for some natural numbers \( d \) and \( m \) with \( dm = |S| \)

If \( A \) acts isotypically on \( S \) then the order of \( A \) divides \( |S| \).

**Proof.** If \( A \) acts isotypically and \( A \) is abelian, the isotropy subgroup at any point of \( S \) is the same subgroup, \( B \), of \( A \). The group \( B \) acts trivially on \( S \), so \( B \) is the trivial subgroup since the action is effective. Thus \( A \) acts freely on \( S \).

If \( A \) acts isotypically on \( S \) then \( S = m1 \backslash A \) as right \( A \)-sets and \( |S| = m|A| \).

**Lemma 4.3.** Let \( A \) be any abelian subgroup of \( \Sigma_n \) acting freely on \( \Sigma_{n-1} \backslash \Sigma_n \). Put \( m = \frac{n}{|\Sigma_n|} \).

1. The number of conjugates of \( A \) in \( \Sigma_n \) is
\[
|\Sigma_n : N_{\Sigma_n}(A)| = \frac{1}{|\text{Aut}(A)||A|^m m!}
\]
2. \( \chi(\Pi^*(\Sigma_{n-1} \backslash \Sigma_n)^A) = (\Pi^*(\Sigma_{n-1} \backslash \Sigma_n)^A) = (1)^{m-1} \mu(1, A)|A|^{m-1}(m - 1)! \) when \( n \geq 2 \).
3. \( \chi(\Pi^*(\Sigma_{n-1} \backslash \Sigma_n)^A)\Sigma_n : N_{\Sigma_n}(A)| \Sigma_n : N_{\Sigma_n}(A)| = \frac{(-1)^{n/|\Pi|}}{|\text{Aut}(A)|} (n - 1)! \)

**Proof.** (1) It is a standard result that the normalizer of \( A \) in the right regular permutation representation of \( A \) is the holomorphic \( A \times \text{Aut}(A) \) of \( A \) [8, pp 36–37]. Similarly, the normalizer of \( A \) in \( m \) times the right regular representation is \( (A \rtimes \Sigma_n) \rtimes \text{Aut}(A) \) of order \( |A||\Sigma_n| \).

(2) As an \( A \)-set \( \Sigma_{n-1} \backslash \Sigma_n = m1 \backslash A \) consists of \( m \) free \( A \)-orbits. According to Lemma \ref{lem:4.3}
\[
\chi(\Pi^*(\Sigma_{n-1} \backslash \Sigma_n)^A) = \chi(\Pi^*(m1 \backslash A)^A) = \mu(1, A)|A|^{m-1} \mu(m, A)|A|^{m-1}(m - 1)!
\]

This formula also holds when \( A \) is trivial group. In this case, the left hand side is \( \chi(\Pi^*(\Sigma_{n-1} \backslash \Sigma_n)) = (1)^{n-1}(n - 1)! \), and the right hand side is \( (1)^{n-1}(n - 1)! \) as \( \mu(1) = 1 \).

(3) This is an immediate consequence of (1) and (2).

**Proof of Theorem 1.1.** on Combine the expression
\[
\chi_r(\Pi^*(\Sigma_{n-1} \backslash \Sigma_n), \Sigma_n) = \frac{1}{n!} \sum_{\text{free and abelian}} \chi(\Pi^*(\Sigma_{n-1} \backslash \Sigma_n)^A) \varphi_r(A)|\Sigma_n : N_{\Sigma_n}(A)|
\]

for the \( r \)th equivariant Euler characteristic with Lemma 4.3(3). Note also that any abelian group of order dividing \( n \) is realizable as a unique subgroup conjugacy class in the symmetric group \( \Sigma_n \) acting freely on \( \Sigma_{n-1} \backslash \Sigma_n \). This gives
\[
\chi_r(\Pi^*(\Sigma_{n-1} \backslash \Sigma_n), \Sigma_n) = -\frac{1}{n} \sum_{|A|=n} (-1)^{n/|A|} \mu(1, A) \frac{\varphi_r(A)}{|\text{Aut}(A)|}
\]

where the sum ranges over the set of isomorphism classes of abelian groups \( A \) of order dividing \( n \). The Möbius function \( \mu(1, A) \) is completely known [5, 2.8]. Indeed, write \( A = \prod A_p \) as the product of its Sylow \( p \)-subgroups \( A_p \).
Then $\mu(1, A) = \prod \mu(1, A_p)$ and $\mu(1, A_p) = 0$ unless $A_p$ is an elementary abelian $p$-group. For an elementary abelian $p$-group of rank $d$,

$$\mu(1, C_p^d) = (-1)^d p^{(\frac{d}{2})}$$

Suppose now that $A = \prod A_p$ where each Sylow $p$-subgroup $A_p = C_p^{d_p}$ is elementary abelian of rank $d_p$. By [6, Lemma 2.1], $\text{Aut}(A) = \prod_p \text{Aut}(A_p) = \prod_p \text{GL}_{d_p}(p)$ and clearly $\varphi_r(\prod A_p) = \prod \varphi_r(A_p)$. The number of surjections of $C_p^r$ onto $C_p^d$ is

$$\varphi_r(C_p^d) = \binom{r}{d} |\text{GL}_d(p)|$$

and consequently

$$\frac{\varphi_r(C_p^d)}{|\text{Aut}(C_p^d)|} = \binom{r}{d}.$$ 

This finishes the proof. □

Let $c_r(n) = (a * b_r)(n)$ denote Dirichlet convolution of the multiplicative arithmetic function $a(n)$ and $b_r(n)$. The function $a$ is $-1 (+1)$ on any even (odd) prime power and the multiplicative function $b_r$ has value

$$(4.4) \quad b_r(p^e) = (-1)^e p^{\frac{e}{2}} \binom{r}{e}$$

on any prime power $p^e$.

**Proposition 4.5.** The multiplicative arithmetic sequences $b_r$ are given by $b_1 = \mu$ and the recurrence relations

$$b_{r+1}(p^d) = p^d b_r(p^d) - p^{d-1} b_r(p^{d-1})$$

valid for all $r \geq 1$ and all prime powers $p^d$, $d \geq 0$.

**Proof.** Use Pascal’s identities for ordinary and Gaussian binomial coefficients [9, Equation 17b]

$$\binom{d}{2} = \binom{d-1}{2} + (d-1), \quad \binom{r+1}{d} = p^d \binom{r}{d} + \binom{r}{d-1}$$

and the definition (4.4) of $b_r$. □

In the following proposition, 1 is the constant sequence with value 1 on all $n \geq 1$.

**Corollary 4.6.** $(1 * b_{r+1})(n) = nb_r(n)$ for all $r, n \geq 1$.

**Proof.** The telescopic sum

$$(1 * b_{r+1})(p^d) = \sum_{e=0}^{d} b_r(p^e) = \sum_{e=0}^{d} (p^e b_r(p^e) - p^{e-1} b_r(p^{e-1})$$

evaluates to $p^d b_r(p^d)$ at any prime power $p^d$. □

**Proposition 4.7.** The multiplicative arithmetic sequences $c_r$ are given by $c_1 = 1, -2, 0, 0, \ldots$ and

$$c_{r+1}(n) = n b_r(n) - b_r(n/2) \quad (\text{where } b_r(n/2) = 0 \text{ for odd } n)$$

for all $r, n \geq 1$.

**Proof.** The two multiplicative sequences $c_1 = a * \mu$ and $1, -2, 0, 0, \ldots$ are identical since they agree on all prime powers. For odd $n$, $c_{r+1}(n) = (a * b_{r+1})(n) = (1 * b_{r+1})(n) = nb_r(n)$ by Corollary 4.6. For powers of 2,

$$c_{r+1}(2^d) = (a * b_{r+1})(2^d) = b_{r+1}(2^d) - \sum_{e=0}^{d-1} b_r(2^e) = 2^d b_r(2^d) - 2^{d-1} b_r(2^{d-1}) - 2^{d-1} b_r(2^{d-1}) = 2^d (b_r(2^d) - b_r(2^{d-1}))$$

by the recurrence relation of Proposition 4.5. Thus $c_{r+1}(n) = n b_r(n) - b_r(n/2)$ for even $n$ by multiplicativity. □

The multiplicative sequences $c_r$ can be defined recursively. The initial sequence is $c_1 = 1, -2, 0, 0, \ldots$. For $r \geq 1$,

$$c_{r+1}(2^d) = \begin{cases} 2 c_r(2) & d = 1 \\ 2^d c_r(2^d) + \sum_{j=2}^{d} 2^{d-j} 2 c_r(2^{d-j}) & d \geq 2 \end{cases}$$

for powers of 2. At powers of an odd prime $p$, $c_{r+1}(p^d) = p^d c_r(p^d) - p^{d-1} c_r(p^{d-1})$ as the sequences $b_r$ and $c_r$ coincide and we can refer to Proposition 4.5.
Corollary 4.8. The Dirichlet series of the multiplicative arithmetic functions $b_r$ and $c_r$ are
\[
\sum_{n=1}^{\infty} \frac{b_r(n)}{n^s} = \frac{1}{\zeta(s)(s-1)\cdots\zeta(s-r+1)}, \quad \sum_{n=1}^{\infty} \frac{c_r(n)}{n^s} = \frac{2^s - 2}{2^s\zeta(s-1)\cdots\zeta(s-r+1)}
\]
where $\zeta(s)$ is the Riemann zeta function and $r \geq 1$.

Proof. Write $\beta_r(s)$ for the Dirichlet series of $b_r(n)$. Corollary 4.6 implies the recurrence
\[
\zeta(s)\beta_{r+1}(s) = \beta_r(s-1)
\]
as $\beta_r(n)$, with series $\beta_r(s-1)$, is the Dirichlet convolution of 1, with series $\zeta(s)$, and $b_{r+1}(n)$. (The Dirichlet series of a Dirichlet convolution is the product of the Dirichlet series of the factors.) The expression for the Dirichlet series of $b_r(n)$ follows by induction starting with the series, $\zeta(s)^{-1}$, for $b_1 = \mu$. The Dirichlet series of the Dirichlet convolution $c_r = a \ast b_r$ is the product of this series and the series, $\zeta(s)(1 - 2^{1-s})$, of $a = 1 \ast c_1$. \(\square\)

It is easy to make explicit computations on a computer. The values of the multiplicative arithmetic function $\frac{1}{n} c_r(n) = \chi_r (\Pi^*(\Sigma_{n-1} \setminus \Sigma_n), \Sigma_n)$, $2 \leq n \leq 15$ and $1 \leq r \leq 5$, are

| $\frac{1}{n} c_r(n)$ | $n = 2$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---------------------|--------|---|---|---|---|---|---|---|----|----|----|----|----|----|
| $r = 1$             | 1      | -1| 0 | 0 | 0 | 0 | 0 | 0 | 0  | 0  | 0  | 0  | 0  | 0  |
| $r = 2$             | -2     | -1| 1 | -1| 2 | -1| 0 | 2  | -1 | -1 | -1 | 2  | 1  |    |
| $r = 3$             | -4     | -4| 5 | -6| 16| -8| -2| 3  | 24 | -12| -20| -14| 32 | 24 |
| $r = 4$             | -8     | -13| 21| -31| 104| -57| -22| 39 | 248| -133| -273| -183| 456| 403|
| $r = 5$             | -16    | -40| 85| -156| 640| -400| -190| 390| 2496| -1464| -3400| -2380| 6400| 6240|

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