Mean field teams and games with correlated types

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Abstract

Mean field games have traditionally been defined [1], [2] as a model of large scale interaction of players where each player has a private type that is independent across the players. In this paper, we introduce a new model of mean field teams and games with correlated types where there are a large population of homogeneous players sequentially making strategic decisions and each player is affected by other players through an aggregate population state. Each player has a private type that only she observes and types of any \( N \) players are correlated through a kernel \( Q \). All players commonly observe a correlated mean-field population state which represents the empirical distribution of any \( N \) players’ correlated joint types. We define the Mean-Field Team optimal Strategies (MFTO) as strategies of the players that maximize total expected joint reward of the players. We also define Mean-Field Equilibrium (MFE) in such games as solution of coupled Bellman dynamic programming backward equation and Fokker Planck forward equation of the correlated mean field state, where a player’s strategy in an MFE depends on both, her private type and current correlated mean field population state. We present sufficient conditions for the existence of such an equilibria. We also present a backward recursive methodology equivalent of master’s equation to compute all MFTO and MFES of the team and game respectively. Each step in this methodology consists of solving an optimization problem for the team problem and a fixed-point equation for the game. We provide sufficient conditions that guarantee existence of this fixed-point equation for the game for each time \( t \).

I. INTRODUCTION

To model the behavior of large population strategic interactions, Mean Field Games (MFGs) were introduced independently by [1] and [2]. In such games, there are a large number of homogeneous strategic players, each with an independent type that evolves as a controlled Markov process, where each player has infinitesimal effect on system dynamics and is affected
by other players through a mean-field population state. Since its introduction, there have been a large number of applications such as economic growth, security in networks, oil production, volatility formation, population dynamics (see [3], [4], [5], [6], [7], [8], [9] and references therein).

However, many times in the real world the players have correlated preferences. For instance, when people are voting for a candidate, or buying a product, or are getting infected by a malware, their state can be correlated across players, for example, your voting preferences may be correlated with that of your neighbors, whose preferences maybe correlated with their neighbors and so on. Such a scenario with correlated types can not be captured by the traditional MFG models presented in [2], [1], which significantly reduces the applicability of such models.

In this paper, we introduce a new model that corrects this shortcoming. We consider discrete-time mean field teams and games where each player sequentially makes strategic decisions and is affected by other players through a correlated mean-field population state. Each player has a private type that evolves through a controlled Markov process which only she observes and all players observe the current population state which is the distribution of other players’ types. We assume that types of the players are correlated in the society such that for any $N$ player, their types evolve through a known symmetric kernel.

In the team version of the problem, players have an objective to maximize the total expected common reward of the players. In the corresponding games, when each agent has a homogeneous reward function that only depends on her own type, action and the common correlated mean field state, a Mean Field Equilibrium (MFE) is defined through a coupled backward-forward equation as follows: the correlated mean-field state evolves through Fokker Planck forward equation given an MFE policy profile of the players. And MFE policy satisfies the Bellman backward equation, given the correlated mean-field states. As a result, in order to compute an MFE, one needs to solve a coupled backward and forward fixed-point equation in the space of correlated mean-field states and equilibrium policies.

In this paper, we consider a non-stationary model where players are cognizant i.e. they actively observe the current population state (which need not have converged) and act based on that population state and their own private state. For the team problem, we provide a backward recursive dynamic program to compute optimum homogeneous Markovian strategies of the players within the class of those strategies. For the game problem, we provide a backward recursive methodology to compute all (non-stationary) MFE of that game which involves solving
a smaller fixed point equation for each time $t$. Since this methodology computes all MFE of the game, there exists a solution to this smaller fixed point equation for each time $t$, whenever there exists an MFE. Our methodology is motivated by the developments in the theory of dynamic games with asymmetric information in [10], [11], [12], [13], [14], [15], where authors in these works have considered different models of such games and provided a sequential decomposition framework to compute Markovian perfect Bayesian equilibria or Mean field equilibria of such games.

The paper is structured as follows. In Section II, we present model, notation and background. In section III, we present a dynamic program to compute optimum Markovian homogeneous strategies for the finite horizon and infinite horizon team problem. In section IV, we present a methodology to compute MFE for the finite horizon game. In Section V, we extend the sequential decomposition idea to infinite horizon games. In Section VII, we discuss the existence of per time fixed-point equation. We conclude in Section VII.

A. Notation

We use uppercase letters for random variables and lowercase for their realizations. For any variable, subscripts represent time indices and superscripts represent player identities. We use notation $-i$ to represent all players other than player $i$ i.e. $-i = \{1, 2, \ldots, i-1, i+1, \ldots, N\}$. We use notation $a_{t:t'}$ to represent the vector $(a_t, a_{t+1}, \ldots, a_{t'})$ when $t' \geq t$ or an empty vector if $t' < t$. We use $a_{t-i}$ to mean $(a^1_t, a^2_t, \ldots, a^{-1}_t, a^{+1}_t, \ldots, a^N_t)$. We remove superscripts or subscripts if we want to represent the whole vector, for example $a_t$ represents $(a^1_t, \ldots, a^N_t)$. We denote the indicator function of any set $A$ by $\mathbb{1}\{A\}$. For any finite set $S$, $P(S)$ represents the space of probability measures on $S$ and $|S|$ represents its cardinality. We denote by $P^\sigma$ (or $E^\sigma$) the probability measure generated by (or expectation with respect to) strategy profile $\sigma$. We denote the set of real numbers by $\mathbb{R}$. All equalities and inequalities involving random variables are to be interpreted in a.s. sense.

II. MODEL AND BACKGROUND

We consider a discrete time large population sequential team and game as follows. There are $M$ homogeneous players, where $M$ tends to $\infty$. We denote the set of homogeneous players by $[M]$ and with some abuse of notation, set of time by $[T]$ for both finite and infinite time horizon. In each period $t \in [T]$, player $i \in [M]$ observes a private type $x_t \in \mathcal{X} = \{1, 2, \ldots, N_x\}$ and
a common observation $z_t \in \mathcal{Z}$, takes action $a_t \in \mathcal{A} = \{1, 2, \ldots, N_a\}$, and receives a reward $R(x_t, a_t, z_t)$ which is a function of its current type $x_t$, action $a_t$ and the common observation $z_t$. Any $N$ players’ types evolve as a correlated controlled Markov process,

$$
(x_{t+1}^1, x_{t+1}^2, \ldots, x_{t+1}^N) = f_x(x_t^1, x_t^2, \ldots, x_t^N, a_t^{1:N}, z_t, w_t).
$$

The random variables $(W_t)_t$ are assumed to be mutually independent across players and across time, and independent of initial random variables $x_1, z_1$. We also write the above update of $x_t$ through a kernel,

$$
(x_{t+1}^1, x_{t+1}^2, \ldots, x_{t+1}^N) \sim Q_x(\cdot|z_t, x_t^1, x_t^2, \ldots, x_t^N, a_t^1, a_t^2, \ldots, a_t^N).
$$

where $Q_x$ is a kernel symmetric across all agents i.e. one can use any permutation of the order of the agents without any change in the output of the kernel. She takes action $a_i^t$ according to a behavioral strategy $\sigma = (\sigma_t)_t$, where $\sigma_t : (\mathcal{Z})^t \times \mathcal{A}^t \rightarrow \mathcal{P}(\mathcal{A})$. We denote the space of such measurable strategies as $\mathcal{S}_\sigma$. This implies $A_t \sim \sigma_t(\cdot|z_{1:t}, x_{1:t})$. We denote $\mathcal{H}_t = \mathcal{Z}^t \times \mathcal{A}^t$ to be the set of observed histories $(z_{1:t}, x_{1:t})$ of a player.

For finite time-horizon team, $\mathbb{T}_T$, all players together want to maximize their total expected discounted reward over a time horizon $T$, discounted by discount factor $0 < \delta \leq 1$,

$$
J_T^{Team} := \mathbb{E}^\sigma \left[ \sum_{t=1}^{T} \delta^{t-1} R(X_t, A_t, Z_t) \right].
$$

For finite time-horizon game, $\mathbb{G}_T$, each player wants to maximize its total expected discounted reward over a time horizon $T$, discounted by discount factor $0 < \delta \leq 1$,

$$
J_T^{Game} := \mathbb{E}^\sigma \left[ \sum_{t=1}^{T} \delta^{t-1} R(X_t, A_t, Z_t) \right].
$$

Similarly we define an infinite time-horizon team and game, $\mathbb{T}_\infty$ and $\mathbb{G}_\infty$, respectively, by replacing $T$ above by $\infty$.

In the following, we define the appropriate solution concepts to analyze this system.

A. Solution concept: Team optimal solution

A team optimal solution is defined as set of symmetric Markovian strategies $\tilde{\sigma} = \{\tilde{\sigma}_t\}_{t \in [T]}$, where $\tilde{\sigma}_t : \mathcal{Z} \times \mathcal{X} \rightarrow \Delta(\mathcal{A})$ i.e. $A_t \sim \sigma_t(\cdot|z_t, x_t)$, and mean field states $z = \{z_t\}_{t \in [T]}$ that satisfy the following optimization problem.
• A policy $\tilde{\sigma}$ is Mean Field Team Optimal (MFTO) if for all $t \in [T], \sigma_{t:T}, z_{1:t}$,

\[
\mathbb{E}^{\tilde{\sigma}_{t:T}} \left[ \sum_{n=t}^{T} \delta^{n-t} \sum_{x_n} Z_n(x_n) R(x_n, A_n, Z_n) | z_{1:t} \right] \geq \\
\mathbb{E}^{\sigma_{t:T}} \left[ \sum_{n=t}^{T} \delta^{n-t} \sum_{x_n} Z_n(x_n) R(x_n, A_n, Z_n) | z_{1:t} \right],
\]

(5)

We note that in the above equation, team optimality is defined only within the class of Markovian policies that depend on the current state $x_t$ and the mean field population state $z_t$.

MFTO for $T_\infty$ are defined in a similar way where summation in the above equations is taken such that $T$ is replaced by $\infty$.

B. Solution concept: Mean Field Equilibrium

Mean-Field Equilibrium (MFE) is defined as set of symmetric Markovian strategies $\tilde{\sigma} = \{\tilde{\sigma}_t\}_{t \in [T]}$, where $\tilde{\sigma}_t : \mathcal{Z} \times \mathcal{X} \rightarrow \Delta(\mathcal{A})$ i.e. $A_t \sim \sigma_t(\cdot | z_t, x_t)$, and mean field states $z = \{z_t\}_{t \in [T]}$ that satisfy the forward-backward equations defined through following equations

• A policy $\tilde{\sigma}$ is optimal for $z := z_{1:t}$ if for all $t \in [T], \sigma_{t:T}, x_{1:t}$,

\[
\mathbb{E}^{\tilde{\sigma}_{t:T}} \left[ \sum_{n=t}^{T} \delta^{n-t} R(X_n, A_n, Z_n) | z_{1:t}, x_{1:t} \right] \geq \\
\mathbb{E}^{\sigma_{t:T}} \left[ \sum_{n=t}^{T} \delta^{n-t} R(X_n, A_n, Z_n) | z_{1:t}, x_{1:t} \right],
\]

(6)

• Define the above backward optimization equation $\psi : S_z \rightarrow 2^{S_\sigma}$ as

\[
\psi(z) := \{\sigma \in S_\sigma : \sigma \text{ is optimal given } z\}.
\]

(7)

• Conversely, define a forward mapping $\Lambda : S_\sigma \rightarrow S_z$ as follows: given $\sigma \in S_\sigma, z = \Lambda(\sigma)$, is constructed recursively as

\[
\begin{align*}
&z_{t+1}(x^1_{t+1}, x^2_{t+1}, \ldots, x^N_{t+1}) = \sum_{x^1_{t+1}, x^2_{t+1}, \ldots, x^N_{t+1}} \sigma_t(x^1_{t+1}, x^2_{t+1}, \ldots, x^N_{t+1}) \times \\
&Q_{x}(x^1_{t+1}, x^2_{t+1}, \ldots, x^N_{t+1} | x_t^{[1:N]}, a^1_t^{[1:N]}, x_t^{[1:N]}) \prod_{i=1}^{N} \sigma_t(a^i_t | z_t, x^i_t)
\end{align*}
\]

(8)

Definition 1: A pair $(\sigma, z)$ is an MFE if $\sigma \in \psi(z)$ and $z = \Lambda(\sigma)$.

MFE for $\mathbb{G}_\infty$ are defined in a similar way where summation in the above equations is taken such that $T$ is replaced by $\infty$. We note that standard MFE with independent types as defined in [1], [2] is a special case where $N = 1$. 

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C. Existence of MFE

Assumption 1 (A1): Suppose the reward function \( R(z_t, x_t, a_t) \) is continuous in \( z_t \).

We note that the above assumption implies that the reward function is bounded.

It was shown in [16, Theorem 1] that an MFE exists for a game with independent types under Assumption A1. In the following, we show that similar arguments under kernel \( Q_x \) go through to show existence of MFE with the correlated types.

**Proposition 1:** Under Assumption A1, there exists an MFE of the game.

**Proof:** Please see Appendix A.

D. Common agent approach

Similar to the common agent approach in [17], an alternate and equivalent way of defining the strategies of the players is as follows. We first generate partial function \( \gamma_t : \mathcal{X} \rightarrow \mathcal{P}(A) \) as a function of \( z_t \) through an equilibrium generating function \( \theta_t : \mathcal{Z} \rightarrow (\mathcal{X} \rightarrow \mathcal{P}(A)) \) such that \( \gamma_t = \theta(z_t) \). Then action \( A_t \) is generated by applying this prescription function \( \gamma_t \) on player \( i \)'s current private information \( x_t \), i.e. \( A_t \sim \gamma_t(\cdot|x_t) \). Thus \( A_t \sim \sigma_t(\cdot|z_t, x_t) = \theta(z_t)(\cdot|x_t) \).

For a given symmetric prescription function \( \gamma_t = \theta(z_t) \), the statistical correlated mean-field \( z_t \) evolves according to the discrete-time Fokker Planck equation [18], \( \forall y \in \mathcal{X} \):

\[
\begin{align*}
z_{t+1}(x_{t+1}^1, x_{t+1}^2, \ldots, x_{t+1}^N) &= \sum_{x_t^{[1:N]}, \alpha_t^{[1:N]}} z_t(x_t^1, x_t^2, \ldots, x_t^N) \times \\
& Q_x(x_{t+1}^1, x_{t+1}^2, \ldots, x_{t+1}^N \mid z_t, x_t^{[1:N]}, \alpha_t^{[1:N]}) \prod_{i=1}^N \gamma_t(a_t^i \mid x_t^i)
\end{align*}
\]

which implies

\[
\begin{align*}
z_{t+1} &= \phi(z_t, \gamma_t).
\end{align*}
\]

III. A METHODOLOGY TO COMPUTE MFTO POLICIES

In this section, we will provide a dynamic program to compute MFTO for both \( T_T \) and \( T_\infty \). This allows one to solve smaller optimization problem for each time \( t \) that equivalently solves the dynamic optimization problem across time.

As mentioned before, in MFTO, strategies of player \( i \) which depend on the mean field population state at time \( t \), \( z_t \), and on its current type \( x_t \).
A. Dynamic program for $T_T$

In this subsection, we will provide a dynamic programming methodology to generate symmetric Markovian MFTO strategies of $G_T$ of the form described above. We generate a reward-to-go function $(V_t)_{t \in [T]}$, where $V_t : \mathcal{Z} \times \mathcal{X} \to \mathbb{R}$. These quantities are generated through the optimization problem as follows.

1. Initialize $\forall z_{T+1}$,

$$V_{T+1}(z_{T+1}) \overset{\Delta}{=} 0. \quad (11)$$

2. For $t = T, T-1, \ldots, 1$, $\forall z_t$, let $\theta_t[z_t]$ be generated as follows. Set $\gamma^*_t = \theta_t[z_t]$, where $\gamma^*_t$ is the solution of the following optimization problem,

$$\gamma^*_t \in \arg \max_{\gamma_t} \mathbb{E}^{\gamma_t} [R(x_t, A_t, z_t) + \delta V_{t+1}(\phi(z_t, \gamma_t))|z_t], \quad (12)$$

where expectation in (26) is with respect to random variable $(X_t, A_t)$ through the measure $z_t(x_t)\gamma_t(a_t|x_t)$.

Furthermore, using the quantity $\gamma^*_t$ found above, define

$$V_t(z_t) \overset{\Delta}{=} \mathbb{E}^{\gamma^*_t} [R(x_t, A_t, z_t) + \delta V_{t+1}(\phi(z_t, \gamma^*_t))|z_t]. \quad (13)$$

Then, an optimum strategy is defined as

$$\sigma^*_t(a_t|z_{1:t}, x_{1:t}) = \gamma^*_t(a_t|x_t), \quad (14)$$

where $\gamma^*_t = \theta[z_t]$.

In the following theorem, we show that the strategy thus constructed is an MFTO strategy

**Theorem 1:** A strategy $(\tilde{\sigma})$ constructed from the above methodology is an MFTO i.e. $\forall t \in [T], h_t \in H_t, \sigma$,

$$\mathbb{E}^\tilde{\sigma} \left[ \sum_{n=t}^{T} \delta^{n-t} R(X_n, A_n, Z_n)|z_{1:t}\right] \geq \mathbb{E}^\sigma \left[ \sum_{n=t}^{T} \delta^{n-t} R(X_n, A_n, Z_n)|z_{1:t}\right], \quad (15)$$

and $z_{n+1} = \phi(z_n, \tilde{\sigma}(\cdot|z_n, \cdot)) \forall n \in [T]$ such that $n \geq t$.

**Proof:** We first note that $\{z_t, \gamma_t\}_t$ is a controlled Markov process of this system since

$$z_{t+1} = \phi(z_t, \gamma_t) \quad (16)$$
and
\[
\mathbb{E}[R(X_t, A_t, Z_t)|z_t] = \sum_{x_t} z_t(x_t) \gamma_t(a_t|x_t) R(x_t, a_t, z_t)
\]
\[
= \hat{R}(z_t, \gamma_t)
\] (17)

Thus one can find the optimal policies of the players using the dynamic program in (18)–(20) using standard Markov decision theory \[19\].

B. Dynamic program for $T_{\infty}$

In this subsection, we will provide a dynamic programming methodology to generate symmetric Markovian MFTO strategies of $G$ of the form described above. We generate a reward-to-go function $(V_t)_{t \in [T]}$, where $V_t : Z \times X \to \mathbb{R}$. These quantities are generated through the optimization problem as follows.

1. Initialize $\forall z_{T+1}$,
\[
V_{T+1}(z_{T+1}) \triangleq 0.
\] (18)

2. For $t = T, T - 1, \ldots, 1$, $\forall z_t$, let $\gamma_t[z_t]$ be generated as follows. Set $\gamma_t^* = \theta_t[z_t]$, where $\gamma_t^*$ is the solution of the following optimization problem,
\[
\gamma_t^* \in \arg \max_{\gamma_t} \mathbb{E}^{\gamma_t} [R(X_t, A_t, z_t) + \delta V_{t+1}(\phi(z_t, \gamma_t))|z_t],
\] (19)

where expectation in (26) is with respect to random variable $(X_t, A_t)$ through the measure $z_t(x_t) \gamma_t(a_t|x_t)$.

Furthermore, using the quantity $\gamma_t^*$ found above, define
\[
V_t(z_t) \triangleq \mathbb{E}^{\gamma_t^*} [R(x_t, A_t, z_t) + \delta V_{t+1}(\phi(z_t, \gamma_t^*))|z_t].
\] (20)

Then, an optimum strategy is defined as
\[
\sigma_t^*(a_t|z_{1:t}, x_{1:t}) = \gamma_t^*(a_t|x_t),
\] (21)

where $\gamma_t^* = \theta[z_t]$.

**Theorem 2:** A strategy $(\bar{\sigma})$ constructed from the above methodology is an MFTO i.e. $\forall t \in [T], h_t \in \mathcal{H}_t, \sigma$,
\[
\mathbb{E}^\bar{\sigma} \left[ \sum_{n=t}^{T} \delta^{n-t} R(X_n, A_n, Z_n)|z_{1:t} \right] \geq \mathbb{E}^\sigma \left[ \sum_{n=t}^{T} \delta^{n-t} R(X_n, A_n, Z_n)|z_{1:t} \right],
\] (22)
and \( z_{n+1} = \phi(z_n, \sigma(\cdot|z_n, \cdot)) \forall n \in [T] \) such that \( n \geq t \).

**Proof:** We first note that \( \{\hat{z}_t, \gamma_t\}_t \) is a controlled Markov process of this system since

\[
\hat{z}_{t+1} = \phi(\hat{z}_t, \gamma_t)
\]

and

\[
\mathbb{E}[R(X_t, A_t, Z_t)|z_t] = \sum_{x_t} z_t(x_t) \gamma_t(a_t|x_t) R(x_t, a_t, z_t) = \hat{R}(z_t, \gamma_t)
\]

Thus one can find the optimal policies of the players using the dynamic program in (18)–(20) using standard Markov decision theory [19].

**IV. A METHODOLOGY TO COMPUTE MFE**

We first note that in the definition of MFE in Definition 1, \( \sigma \) and \( z \) are coupled through a fixed point equation defined through a backward equation \( \psi \) and a forward equation \( \Lambda \). This is a fixed point equation *across time* whose complexity increases exponentially with time, and thus suffers from the same curse of dimensionality as any dynamic optimization problem.

In this section, we will provide a backward recursive sequential decomposition methodology to compute MFE for both \( G_T \) and \( G_\infty \). This allows one to solve smaller fixed-point equations for each time \( t \) that equivalently solves this bigger fixed point equation across time (and is thus equivalent to dynamic program for a dynamic optimization problem where one can solve for the bigger optimization across time by solving for smaller optimization problem for each time \( t \)).

As mentioned before, in MFE, strategies of player \( i \) which depend on the mean field population state at time \( t \), \( z_t \), and on its current type \( x_t \). Equivalently, player \( i \) takes action of the form

\[ A_t \sim \sigma_t(\cdot|z_t, x_t) \]

We are only interested in symmetric equilibria of such games such that

\[ A_t \sim \gamma_t(\cdot|x_t) = \theta_t[z_t](\cdot|x_t) \]

i.e. there is no dependence of the identity of the players on their strategies.

**A. Backward recursive methodology for \( G_T \)**

In this subsection, we will provide a methodology to generate symmetric MFE of \( G_T \) of the form described above. We define an equilibrium generating function \( (\theta_t)_{t \in [T]} \), where \( \theta_t : \)

\( ^1 \)Note however, that the unilateral deviations of the player are considered in the space of all strategies.
\[ \mathcal{Z} \rightarrow \{ \mathcal{X} \rightarrow \mathcal{P}(A) \} \], where for each \( z_t \), we generate \( \tilde{\gamma}_t = \theta_t[z_t] \). In addition, we generate a reward-to-go function \( (V_t)_{t \in [T]} \), where \( V_t : \mathcal{Z} \times \mathcal{X} \rightarrow \mathbb{R} \). These quantities are generated through a fixed-point equation as follows.

1. Initialize \( \forall z_{T+1}, x_{T+1} \in \mathcal{X} \),
\[
V_{T+1}(z_{T+1}, x_{T+1}) \triangleq 0. \tag{25}
\]

2. For \( t = T, T - 1, \ldots, 1 \), \( \forall z_t \), let \( \theta_t[z_t] \) be generated as follows. Set \( \tilde{\gamma}_t = \theta_t[z_t] \), where \( \tilde{\gamma}_t \) is the solution of the following fixed-point equation,
\[
\forall i \in [N], x_t \in \mathcal{X},
\]
\[
\tilde{\gamma}_t(\cdot | x_t) \in \arg \max_{\gamma_t(\cdot | x_t)} \mathbb{E}^{\gamma_t(\cdot | x_t)} \left[ R(X_t, A_t, z_t) + \delta V_{t+1}(\phi(z_t, \tilde{\gamma}_t), X_{t+1}) | z_t, x_t \right], \tag{26}
\]
where expectation in \( (26) \) is with respect to random variable \( (X_t^{1:N-1}, A_t, X_{t+1}) \) through the measure \( \tilde{z}_t(x_t^{1:N-1}) \prod_{i=1}^{N-1} \gamma_t(\cdot | x_t^{1:N-1}) Q_\phi(x_{t+1}|z_t, x_t^{1:N-1}, x_t, a_t^{1:N-1}, a_t) \). We note that the solution of \( (26) \), \( \tilde{\gamma}_t \), appears both on the left of \( (26) \) and on the right side in the update of \( z_t \), and is thus unlike the fixed-point equation found in Bayesian Nash equilibrium.

Furthermore, using the quantity \( \tilde{\gamma}_t \) found above, define
\[
V_t(z_t, x_t) \triangleq \mathbb{E}^{\tilde{\gamma}_t(\cdot | x_t)} \left[ R(x_t, A_t, z_t) + \delta V_{t+1}(\phi(z_t, \tilde{\gamma}_t), X_{t+1}) | z_t, x_t \right]. \tag{27}
\]

Then, an equilibrium strategy is defined as
\[
\tilde{\sigma}_t(a_t|z_{1:t}, x_{1:t}) = \tilde{\gamma}_t(a_t|x_t), \tag{28}
\]
where \( \tilde{\gamma}_t = \theta[z_t] \).

In the following theorem, we show that the strategy thus constructed is an MFE of the game.

**Theorem 3:** A strategy \( (\tilde{\sigma}) \) constructed from the above methodology is an MFE of the game i.e. \( \forall t \in [T], h_t \in \mathcal{H}_t, \sigma, \)
\[
\mathbb{E}^{\tilde{\sigma}} \left[ \sum_{n=t}^{T} \delta^{n-t} R(X_n, A_n, Z_n) | z_{1:t}, x_{1:t} \right] 
\geq \mathbb{E}^{\sigma} \left[ \sum_{n=t}^{T} \delta^{n-t} R(X_n, A_n, Z_n) | z_{1:t}, x_{1:t} \right], \tag{29}
\]
and \( z_{n+1} = \phi(z_n, \tilde{\sigma}(\cdot | z_n, \cdot)) \forall n \in [T] \) such that \( n \geq t \).

**Proof:** Please see Appendix B.

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1 We discuss the existence of solution of this fixed-point equation in Section VI.
The intuition for (26) is as follows. It notes that at equilibrium, the update of the mean field $z_t$ is defined by the equilibrium strategies of the players, and no user has an incentive to unilaterally deviate in its action. Thus the equilibrium strategy maximizes a user’s utility-to-go when the mean field is updated by the same equilibrium policy, which explains the occurrence of $\tilde{\gamma}$ at both left and right side of (26).

In the following, we show that every MFE can be found using the above backward recursion.

**B. Converse**

*Theorem 4 (Converse):* Let $\tilde{\sigma}$ be an MFE of the mean field game. Then there exists an equilibrium generating function $\theta$ that satisfies (26) in backward recursion such that $\tilde{\sigma}$ is defined using $\theta$.

*Proof:* Please see Appendix D. ■

In the following we consider the infinite horizon game $G_\infty$ and provide a similar methodology as before to compute its MFE.

**V. METHODOLOGY FOR THE INFINITE HORIZON PROBLEM $G_\infty$**

In this section, we consider the infinite-horizon problem $G_\infty$, for which we assume the reward function $R$ to be absolutely bounded.

We define an equilibrium generating function $\theta : \mathcal{Z} \rightarrow \{\mathcal{X} \rightarrow \mathcal{P}(A)\}$, where for each $z_t$, we generate $\tilde{\gamma}_t = \theta[z_t]$. In addition, we generate a reward-to-go function $V : \mathcal{Z} \times \mathcal{X} \rightarrow \mathbb{R}$. These quantities are generated through a fixed-point equation as follows.

For all $z$, set $\tilde{\gamma} = \theta[z]$. Then $(\tilde{\gamma}, V)$ are solution of the following fixed-point equation.

\begin{align}
\tilde{\gamma}(.|x) &\in \arg \max_{\gamma(.|x)} \mathbb{E}^{\tilde{\gamma}(.|x)} \left[ R(x, A, z) + \delta V(\phi(z, \tilde{\gamma}), X')|z, x \right], \\
V(z, x) &= \mathbb{E}^{\tilde{\gamma}(.|x)} \left[ R(x, A, z) + \delta V(\phi(z, \tilde{\gamma}), X')|z, x \right].
\end{align}

(30) (31)

where expectation in (30) is with respect to random variable $(X^{[1:N-1]}, A, X')$ through the measure $z(x^{[1:N-1]}) \prod_{i=1}^{N-1} \gamma(a^i|x^i) Q_x(x'|z, x^{[1:N-1]}, x, a^{[1:N-1]}, a)$.

$^3$We discuss the existence of solution of this fixed-point equation in Section VI.

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Then an equilibrium strategy is defined as
\[ \tilde{\sigma}(a_t|z_{1:t}, x_{1:t}) = \tilde{\gamma}(a_t|x_t), \] (32)
where \( \tilde{\gamma} = \theta[z_t] \).

The following theorem shows that the strategy thus constructed is an MFE of the game.

**Theorem 5:** A strategy \((\tilde{\sigma})\) constructed from the above methodology is an MFE of the game i.e. \( \forall t, h_t \in H_t, \sigma, \)
\[
\mathbb{E}^{\tilde{\sigma}} \left[ \sum_{n=t}^{\infty} \delta^{n-t} R(X_n, A_n, Z_n)|z_{1:t}, x_{1:t}] \right] \\
\geq \mathbb{E}^{\sigma} \left[ \sum_{n=t}^{\infty} \delta^{n-t} R(X_n, A_n, Z_n)|z_{1:t}, x_{1:t}] \right], \quad (33)
\]
and \( z_{n+1} = \phi(z_n, \tilde{\sigma}(\cdot|z_n, \cdot)) \forall n \in [T] \) such that \( n \geq t \).

**Proof:** Please see Appendix E.

In the following, we show that every mean field equilibria can be found using the above backward recursion.

**A. Converse**

**Theorem 6 (Converse):** Let \( \tilde{\sigma} \) be an MFE of the mean field game. Then there exists an equilibrium generating function \( \theta \) that satisfies (30)-(31) in backward recursion such that \( \tilde{\sigma} \) is defined using \( \theta \).

**Proof:** Please see Appendix G.

### VI. Existence of Per Stage Fixed-point Equation

In this section, we discuss sufficient conditions for the existence of a solution of the fixed-point equations (26) and (30)-(31).

**Theorem 7:** Under assumption (A1), for every \( t \) there exists solution of the fixed-point equations (26), and for (30)-(31).

**Proof:** Under the assumption (A1), it was shown in Theorem 1 that there exists an MFE of both the finite and infinite horizon games. Furthermore, Theorem 4 and Theorem 6 show that all MFE can be found using backward recursion for the finite and infinite horizon problems respectively. This proves that under (A1), for every \( t \), there exists a solution of (26), and for (30)-(31).

---

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VII. CONCLUSION

In this paper, we define both finite and infinite horizon, large population dynamic game where each player is affected by others through a correlated mean-field population state. We prove the existence of MFG under appropriate sufficient conditions. We present a novel backward recursive methodology to compute Mean-field equilibria (MFE) for such games, where each player’s strategy depends on its current private type and current correlated mean-field population state. We also prove the existence of each fixed-point equation \( t \). This new framework opens door to studying many new applications where players have correlated types.

APPENDIX A

We first note that as mentioned before, the proof in this Appendix is adapted from the proof of MFE in [20]. As shown in [20], we first note that a (mixed) strategy is a measurable function \( \sigma: \mathcal{Z} \times \mathcal{X} \to \mathcal{P}(\mathcal{A}) \), that associates to each state \( x \in \mathcal{X} \) and each time \( t \) a probability measure on the set of possible actions. We also denote by \( \pi^\sigma(x_t, a_t) \) the probability that, at time \( t \), a player in state \( x_t \) takes the action \( a_t \), under strategy \( \sigma \). For all \( t \) and all \( x \in \mathcal{X} \), we have \( \sum_{a \in \mathcal{A}} \pi^\sigma(x, a) = 1 \). The set of all possible strategies is denoted by \( \mathcal{S} \).

The set \( \mathcal{S} \) is a bounded subset of the Hilbert space of the functions \( \mathcal{Z} \times \mathcal{X} \to \mathcal{P}(\mathcal{A}) \) equipped with the inner product the exponentially weighted inner product: \( \langle f, g \rangle = \sum_{t=0}^{\infty} \delta^t f_t g_t \). This shows that \( \mathcal{S} \) is weakly compact, where the weak topology is defined as follows: a sequence of policy \( \sigma^n \) converges to a policy \( \sigma \) if for any bounded function \( g \):

\[
\lim_{n \to \infty} \sum_{t=0}^{\infty} \delta^t \sigma^n_t g_t = \sum_{t=0}^{\infty} \delta^t \sigma_t g_t dt.
\]

1) Proof of Theorem 1

Proof:

- Let

\[
W(z_{1:T}, \sigma_{1:T}) = \mathbb{E}^{\sigma_{1:T}} \left[ \sum_{n=1}^{T} \delta^{n-t} R(X_n, A_n, z_n) \right] \tag{34}
\]

- A policy \( \tilde{\sigma} \) is optimal for \( z := z_{1:t} \) if for all \( t \in [T], \sigma_{t:T}, x_{1:t}, \)

\[
\mathbb{E}^{\tilde{\sigma}_{t:T}} \left[ \sum_{n=t}^{T} \delta^{n-t} R(X_n, A_n, Z_n)|z_{1:t}, x_{1:t}] \right] \geq
\]

\[
\mathbb{E}^{\sigma_{t:T}} \left[ \sum_{n=t}^{T} \delta^{n-t} R(X_n, A_n, Z_n)|z_{1:t}, x_{1:t}] \right]. \tag{35}
\]
• Define the above backward optimization equation $\psi : S_z \rightarrow 2^{S_\sigma}$ as

$$\psi(z) := \{\sigma \in S_\sigma : \sigma \text{ is optimal given } z\}.$$  \hfill (36)

• Conversely, define a forward mapping $\Lambda : S_\sigma \rightarrow S_z$ as follows: given $\sigma \in S_\sigma, z = \Lambda(\sigma)$, is constructed recursively as

$$z_{t+1}(x_{t+1}^{1}, x_{t+1}^{2}, \ldots, x_{t+1}^{N}) = \sum_{x_{t}^{1} \in \mathcal{A}_{t}^{1}, a_{t}^{1} \notin \mathcal{A}_{t}^{1}} z_t(x_t^{1}, x_t^{2}, \ldots, x_t^{N}) \times$$

$$Q_x(x_{t+1}^{1}, x_{t+1}^{2}, \ldots, x_{t+1}^{N}|z_t, x_t^{[1:N]}, a_t^{[1:N]}) \prod_{i=1}^{N} \sigma_t(a_t^i|z_t, x_t^i)$$  \hfill (37)

Define $\Omega : S_z \rightarrow 2^{S_z}$ as the best response to a population distribution $z$ i.e.

$$\Omega(z) = \Lambda(\psi(z))$$  \hfill (38)

**Definition of $\Omega(z)$** – Since $W$ is continuous in $\sigma$ (which is implied by the continuity of $\Lambda(\sigma)$). This shows that there exists $\sigma$ that attains the maximum in Equation (36), which shows that $\Omega(z)$ is well defined and non-empty.

**Compactness of $\Omega(z)$** – Let us consider the following optimization problem:

$$\min_{\pi, x} \sum_{t=0}^{T} \left( \sum_{x_t, a_t} \delta^t \pi_t(x_t, a_t) R(z_t, x_t, a_t) \right)$$  \hfill (39)

such that $\pi_t$ satisfies

$$\begin{align*}
\sum_{a_t} \pi_t(x_t, a_t) &= x_t, \\
\pi_t(x_t, a_t) &\geq 0, \\
x_{t+1} &= \sum_{x_t, a_t} \pi_t(x_t, a_t) Q_x(x_{t+1}|z_t, x_t, a_t) \quad \forall x_{t+1}
\end{align*}$$  \hfill (40)

The above problem is a linear problem, which implies that the set of optimal solutions is convex and compact. Let us show that the set of optimal solution of the optimization problem (39) is $\Omega(z_t)$. To show this, let us remark that the constraints (34) are equivalent to the constraints (40) by replacing the variables $x_t \sigma_t(a_t|z_t, a_t)$ by $\pi(x_t, a_t)$. Then, the constraint $\sigma_t \in S$ of (34), that corresponds to $\sigma_t \in \mathcal{P}(\mathcal{A})$, is replaced with $\pi_t \geq 0$ and $\sum_{a_t} \pi_t(x_t, a_t) = x_t$. 

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Upper-semi continuity of $\Omega$. To prove that $\Omega$ is upper-semi continuous, let us show that the graph of $z \mapsto \Omega(z)$ is closed. Let $z_n \in S_z$ and $x_n \in \Omega(z_n)$ be two sequences such that $\lim_{n \to \infty} z_n = z_\infty$ and $\lim_{n \to \infty} x_n = x_\infty$. We want to show that $x_\infty \in \Omega(z_\infty)$.

As $W$ is continuous, for all $x_n \in \Omega(z_n)$, there exists a strategy $\sigma_n$ that minimizes $W(z_n, \sigma_n)$ and such that $x_n = \Lambda(\sigma_n, z_n)$. As the set $S$ is weakly compact, this sequence of strategies has a sub sequence that converges weakly to a strategy $\sigma_*$. Moreover, we have:

- As $W$ is continuous, $\sigma_*$ minimizes $W(\pi, z_\infty)$. This shows that $x^{\sigma_*} \in \Omega(z_\infty)$.
- The solution of (34) is continuous in $\sigma$ and $z$, which shows that $x_\infty = x^{\sigma_*, z_\infty}$.

Combining these two facts shows that $x_\infty \in \Omega(z_\infty)$ which implies that the graph of $\Omega$ is closed.

Since for all $z \in S_z$, $\Omega(z)$ is well defined and non empty (since the minimum is attained in $\mathcal{E}$), is convex and compact. Moreover, the function $\Omega(\cdot)$ is upper-semi-continuous. As $S_z$ is compact [21, Prop. 11.11], this shows that $\Omega(\cdot)$ satisfies the conditions of the fixed point theorem given in [22, Theorem 8.6] and therefore has a fixed point $z^*$. By the definition of $\Omega$, this implies that there exists a strategy $\sigma$ that is a best-response to $z$, which implies that $\sigma$ is a mean field equilibrium.

Similar arguments are used for the infinite horizon game.

\section*{Appendix B}

Proof: We prove Theorem 3 using induction and the results in Lemma 1 and 2 proved in C. Let $\tilde{\sigma}$ be the strategies computed by the methodology in Section III and let $z_{n+1} = \phi(z_n, \tilde{\sigma}(\cdot|z_n, \cdot)) \forall n \in [T]$.

For base case at $t = T$, $\forall i \in [N], x_{1:T}, \sigma$

\begin{align}
\mathbb{E}^{\tilde{\sigma}_T}\{ R(X_T, A_T, Z_T) | z_{1:T}, x_{1:T} \} &= V_T(z_T, x_T) \\
&\geq \mathbb{E}^{\sigma_*} \{ R(X_T, A_T, Z_T) | z_{1:T}, x_{1:T} \},
\end{align}

where (41a) follows from Lemma 2 and (41b) follows from Lemma 1 in C.
Let the induction hypothesis be that for $t + 1, \forall i \in [N], z_{1:t+1}, x_{1:t+1} \in (\mathcal{A})^{t+1}, \sigma$,

$$
\mathbb{E}^{\sigma_{t+1}} \left\{ \sum_{n=t+1}^{T} \delta^{n-t-1} R(X_n, A_n, Z_n) \big| z_{1:t+1}, x_{1:t+1} \right\} \tag{42a}
$$

$$
\geq \mathbb{E}^{\sigma_{t+1}} \left\{ \sum_{n=t+1}^{T} \delta^{n-t-1} R(X_n, A_n, Z_n) \big| z_{1:t+1}, x_{1:t+1} \right\}, \tag{42b}
$$

Then $\forall i \in [N], x_{1:t}, \sigma$, we have

$$
\mathbb{E}^{\sigma_{t}} \left\{ \sum_{n=t}^{T} \delta^{n-t-1} R(X_n, A_n, Z_n) \big| z_{1:t+1}, x_{1:t} \right\}
$$

$$
= V_t(z_t, x_t) \tag{43a}
$$

$$
\geq \mathbb{E}^{\sigma_{t}} \left\{ R(X_t, A_t, Z_t) + \delta V_{t+1}(z_{t+1}, X_{t+1}) \big| z_{1:t+1}, x_{1:t} \right\} \tag{43b}
$$

$$
= \mathbb{E}^{\sigma_{t}} \left\{ R(X_t, A_t, Z_t) + \delta \mathbb{E}^{\sigma_{t+1}} \left\{ \sum_{n=t+1}^{T} \delta^{n-t-1} R(X_n, A_n, Z_n) \big| z_{1:t+1}, x_{1:t}, X_{t+1} \right\} \big| z_{1:t+1}, x_{1:t} \right\} \tag{43c}
$$

$$
\geq \mathbb{E}^{\sigma_{t}} \left\{ R(X_t, A_t, Z_t) + \delta \mathbb{E}^{\sigma_{t+1}} \left\{ \sum_{n=t+1}^{T} \delta^{n-t-1} R(X_n, A_n, Z_n) \big| z_{1:t+1}, x_{1:t}, X_{t+1} \right\} \big| z_{1:t+1}, x_{1:t} \right\} \tag{43d}
$$

$$
= \mathbb{E}^{\sigma_{t}} \left\{ R(X_t, A_t, Z_t) + \delta \mathbb{E}^{\sigma_{t+1}} \left\{ \sum_{n=t+1}^{T} \delta^{n-t-1} R(X_n, A_n, Z_n) \big| z_{1:t+1}, x_{1:t}, X_{t+1} \right\} \big| z_{1:t}, x_{1:t} \right\} \tag{43e}
$$

$$
= \mathbb{E}^{\sigma_{t}} \left\{ \sum_{n=t}^{T} \delta^{n-t} R(X_n, A_n, Z_n) \big| z_{1:t}, x_{1:t} \right\}, \tag{43f}
$$

where (43a) follows from Lemma 2, (43b) follows from Lemma 1, (43c) follows from Lemma 2, (43d) follows from induction hypothesis in (42b) and (43e) follows since the random variables involved in the right conditional expectation do not depend on strategies $\sigma_t$.

**APPENDIX C**

*Lemma 1*: Let $\bar{\sigma}$ be the strategies computed by the methodology in Section III and let $z_{n+1} = \phi(z_n, \bar{\sigma}(z_n)) \forall n = 1 \ldots t$. Then $\forall t \in [T], i \in [N], x_{1:t}, \sigma_t$

$$
V_t(z_t, x_t) \geq \mathbb{E}^{\sigma_t} \left\{ R(X_t, A_t, Z_t) + \delta V_{t+1}(z_{t+1}, X_{t+1}) \big| z_{1:t}, x_{1:t} \right\}. \tag{44}
$$

**Proof**: We prove this lemma by contradiction.
Suppose the claim is not true for $t$. This implies $\exists i, \tilde{\sigma}_t, \tilde{z}_{1:t}, \tilde{x}_{1:t}$ such that
\[
\mathbb{E}^{\tilde{\sigma}_t} \left\{ R(X_t, A_t, Z_t) + \delta V_{t+1}(Z_{t+1}, X_{t+1}) \mid \tilde{z}_{1:T}, \tilde{x}_{1:t} \right\} > V_t(\tilde{z}_t, \tilde{x}_t). \tag{45}
\]
We will show that this leads to a contradiction. Construct
\[
\tilde{\gamma}_t(a_t \mid x_t) = \begin{cases} \tilde{\sigma}_t(a_t \mid \tilde{z}_{1:t}, \tilde{x}_{1:t}) & x_t = \tilde{x}_t \\ \text{arbitrary otherwise} & \end{cases}
\] (46)

Then for $\tilde{x}_{1:t}$, we have
\[
V_t(\tilde{z}_t, \tilde{x}_t) = \max_{\tilde{\gamma}_t} \mathbb{E}^{\tilde{\gamma}_t(\cdot \mid \tilde{x}_t)} \left\{ R(\tilde{x}_t, A_t, \tilde{z}_t) + \delta V_{t+1}(\phi(\tilde{z}_t, \tilde{\gamma}_t), X_{t+1}) \mid \tilde{z}_t, \tilde{x}_t \right\}, \tag{47a}
\]
\[
\geq \mathbb{E}^{\tilde{\gamma}_t(\cdot \mid \tilde{x}_t)} \left\{ R(x_t, A_t, \tilde{z}_t) + \delta V_{t+1}(\phi(\tilde{z}_t, \tilde{\gamma}_t), X_{t+1}) \mid \tilde{z}_t, \tilde{x}_t \right\} \tag{47b}
\]
\[
= \sum_{x_t^{[1:N-1]}, a_t} \left\{ R(\tilde{x}_t, a_t, \tilde{z}_t) + \delta V_{t+1}(\phi(\tilde{z}_t, \tilde{\gamma}_t), x_{t+1}) \right\} \tilde{\gamma}_t(a_t \mid \tilde{z}_{1:t}, \tilde{x}_{1:t}) z_t(x_t^{[1:N-1]}) \tag{47c}
\]
\[
\prod_{i=1}^{N-1} \tilde{\gamma}_t(a_t^i \mid x_t^i) Q_x(x_{t+1} \mid z_t, x_t^{[1:N-1]}, \tilde{x}_t, a_t^{[1:N-1]}, a_t) \tag{47d}
\]
\[
= \mathbb{E}^{\tilde{\gamma}_t} \left\{ R(\tilde{x}_t, a_t, \tilde{z}_t) + \delta V_{t+1}(\phi(\tilde{z}_t, \tilde{\gamma}_t), X_{t+1}) \mid \tilde{z}_{1:t}, \tilde{x}_{1:t} \right\} \tag{47e}
\]
\[
> V_t(\tilde{z}_t, \tilde{x}_t), \tag{47f}
\]
where (47a) follows from definition of $V_t$ in (27), (47d) follows from definition of $\tilde{\gamma}_t$ and (47f) follows from (45). However this leads to a contradiction. \hfill \qed

\textbf{Lemma 2:} Let $\tilde{\sigma}$ be the strategies computed by the methodology in Section III and let $z_{n+1} = \phi(z_n, \tilde{\sigma}_n(\cdot \mid z_n, )) \forall n \in [T]$. Then $\forall i \in [N], t \in [T], x_{1:t},$
\[
V_i(z_t, x_t) = \mathbb{E}^{\tilde{\sigma}_{t:T}} \left\{ \sum_{n=t}^{T} \delta^{n-t} R(X_n, A_n, Z_n) \mid z_{1:t}, x_{1:t} \right\}. \tag{48}
\]

\textbf{Proof:} We prove the lemma by induction. For $t = T,$
\[
\mathbb{E}^{\tilde{\sigma}_T} \left\{ R(X_T, A_T, Z_T) \mid z_{1:T}, x_{1:T} \right\} = \sum_{a_T} R(x_T, a_T, z_T) \tilde{\sigma}_T(a_T \mid z_T, x_T) \tag{49a}
\]
\[
= V_T(z_T, x_T), \tag{49b}
\]
where (49b) follows from the definition of $V_t$ in (27). Suppose the claim is true for $t + 1$, i.e.,

$$V_{t+1}(z_{t+1}, x_{t+1}) = E^\hat{\theta}_{t+1:T}\{ \sum_{n=t+1}^{T} \delta^{n-t-1} R(X_n, A_n, Z_n) \mid z_{1:t+1}, x_{1:t+1} \}.$$  

(50)

Then $\forall i \in [N], t \in [T], x_{1:t}$, we have

$$E^\hat{\theta}_{t:T}\{ \sum_{n=t}^{T} \delta^{n-t} R(X_n, A_n, Z_n) \mid z_{1:t}, x_{1:t} \} = E^\hat{\theta}_{t:T}\{ R(X_t, A_t, Z_t) + \delta E^\hat{\theta}_{t:T}\{ \sum_{n=t}^{T} \delta^{n-t-1} R(X_n, A_n, Z_n) \mid z_{1:t+1}, x_{1:t}, X_{t+1} \} \mid z_{1:t}, x_{1:t} \} + \delta E^\hat{\theta}_{t:T}\{ \sum_{n=t+1}^{T} \delta^{n-t-1} R(X_n, A_n, Z_n) \mid z_{1:t+1}, x_{1:t}, X_{t+1} \} \mid z_{1:t}, x_{1:t} \}$$

(51a)

$$= E^\hat{\theta}_{t:T}\{ R(X_t, A_t, Z_t) + \delta V_{t+1}(z_{t+1}, X_{t+1}) \mid z_{1:t}, x_{1:t} \}$$

(51b)

$$= E^\hat{\theta}_{t:T}\{ R(X_t, A_t, z_t) + \delta V_{t+1}(z_{t+1}, X_{t+1}) \mid z_{1:t}, x_{1:t} \}$$

(51c)

$$= E^\hat{\theta}_{t:T}\{ R(x_t, A_t, z_t) + \delta V_{t+1}(z_{t+1}, X_{t+1}) \mid z_{1:t}, x_{1:t} \}$$

(51d)

$$= V_t(z_t, x_t),$$

(51e)

(51c) follows from the induction hypothesis in (50), (51d) follows because the random variables involved in expectation, $X_t, A_t, X_{t+1}$ do not depend on $\hat{\theta}_{t+1:T}$ and (51e) follows from the definition of $V_t$ in (27).

**APPENDIX D**

**Proof:** We prove this by contradiction. Suppose for any equilibrium generating function $\theta$ that generates an MFE $\tilde{\sigma}$ and for $z_{n+1} = \phi(z_n, \tilde{\sigma}(\cdot | z_n, \cdot))\forall n \in [T]$, there exists $t \in [T], i \in [N]$ such that (26) is not satisfied for $\theta$ i.e. for $\hat{\gamma}_t = \theta_t[z_t] = \tilde{\sigma}_t(\cdot | z_t, \cdot),$

$$\hat{\gamma}_t(\cdot | x_t) \notin \arg \max_{\gamma_t(\cdot | x_t)} E^{\gamma_t(\cdot | x_t)}\{ R_t(X_t, A_t, z_t) + V_{t+1}(\phi(z_t, \hat{\gamma}_t), X_{t+1}) \mid x_t, z_t \}.$$  

(52)

Let $t$ be the first instance in the backward recursion when this happens. This implies $\exists \hat{\gamma}_t(\cdot | x_t)$ such that

$$E^{\hat{\gamma}_t(\cdot | x_t)}\{ R_t(x_t, A_t, z_t) + V_{t+1}(\phi(z_t, \hat{\gamma}_t), X_{t+1}) \mid z_{1:t}, x_{1:t} \} > E^{\hat{\gamma}_t(\cdot | x_t)}\{ R_t(x_t, A_t) + V_{t+1}(\phi(z_t, \hat{\gamma}_t), X_{t+1}) \mid z_{1:t}, x_{1:t} \}$$

(53)
This implies for $\tilde{\sigma}_t(\cdot | z_t, \cdot) = \tilde{\gamma}_t$,

\[
\mathbb{E}^{\tilde{\sigma}_t, T} \left\{ \sum_{n=t}^{T} R_n(X_n, A_n, Z_n) \mid z_{1:t}, x_{1:t} \right\} \\
= \mathbb{E}^{\tilde{\sigma}_t} \left\{ R_t(x_t, A_t, z_t) + \\
\mathbb{E}^{\tilde{\sigma}_t, T} \left\{ \sum_{n=t+1}^{T} R_n(X_n, A_n, Z_n) \mid z_{1:t+1}, x_{1:t}, X_{t+1} \right\} \mid z_{1:t}, x_{1:t} \right\}
\]

(54)

\[
= \mathbb{E}^{\tilde{\gamma}_t(x_t)} \left\{ R_t(x_t, A_t, z_t) + V_{t+1}(\phi(z_t, \tilde{\gamma}_t), X_{t+1}) \mid z_t, x_t \right\}
\]

(55)

\[
\leq \mathbb{E}^{\tilde{\sigma}_t(x_t, x_t)} \left\{ R_t(x_t, A_t, z_t) + V_{t+1}(\phi(z_t, \tilde{\gamma}_t), X_{t+1}) \mid z_t, x_t \right\}
\]

(56)

\[
= \mathbb{E}^{\tilde{\sigma}_t} \left\{ R_t(x_t, A_t, z_t) + \\
\mathbb{E}^{\tilde{\sigma}_t, T} \left\{ \sum_{n=t+1}^{T} R_n(X_n, A_n, Z_n) \mid z_{1:t+1}, x_{1:t}, X_{t+1} \right\} \mid z_{1:t}, x_{1:t} \right\}
\]

(57)

\[
= \mathbb{E}^{\tilde{\sigma}_t, T} \left\{ \sum_{n=t}^{T} R_n(X_n, A_n, Z_n) \mid z_{1:t}, x_{1:t} \right\}
\]

(58)

where (78) follows from the definitions of $\tilde{\gamma}_t$ and Lemma 2, (79) follows from (75) and the definition of $\tilde{\sigma}_t$, (80) follows from Lemma 1. However, this leads to a contradiction since $\tilde{\sigma}$ is an MFE of the game.

\section*{APPENDIX E}

Let $\tilde{\sigma}$ be the strategies computed by the methodology in Section IV and let $z_{n+1} = \phi(z_n, \tilde{\sigma}_n(\cdot | z_n, \cdot)) \forall n \in [T]$. We divide the proof into two parts: first we show that the value function $V$ is at least as big as any reward-to-go function; secondly we show that under the strategy $\tilde{\sigma}$, reward-to-go is $V$. Note that $h_t := (z_{1:t}, x_{1:t})$.

\textbf{Part 1:} For any $i \in [N], \sigma$ define the following reward-to-go functions

\[
W_t^\sigma(h_t) = \mathbb{E}^{\sigma} \left\{ \sum_{n=t}^{\infty} \delta^{n-t} R(X_n, A_n, Z_n) \mid h_t \right\}
\]

(59a)

\[
W_t^{\sigma, T}(h_t) = \mathbb{E}^{\sigma} \left\{ \sum_{n=t}^{T} \delta^{n-t} R(X_n, A_n, Z_n) + \delta^{T+1-t} V(Z_{T+1}, X_{T+1}) \mid h_t \right\}
\]

(59b)

Since $\mathcal{X}, \mathcal{A}$ are finite sets the reward $R$ is absolutely bounded, the reward-to-go $W_t^\sigma(h_t)$ is finite $\forall i, t, \sigma, h_t$.

For any $i \in [N], x_{1:t},$

\[
V(z_t, x_t) - W_t^\sigma(h_t) = \left[ V(z_t, x_t) - W_t^{\sigma, T}(h_t) \right] + \left[ W_t^{\sigma, T}(h_t) - W_t^\sigma(h_t) \right]
\]

(60)
Combining results from Lemmas 4 and 5 in \[E\] the term in the first bracket in RHS of (60) is non-negative. Using (59), the term in the second bracket is

\[
\left( \delta^{T+1-t} \right) \mathbb{E}^\sigma \left\{ - \sum_{n=T+1}^{\infty} \delta^{n-(T+1)} R(X_n, A_n, Z_n) + V(Z_{T+1}, X_{T+1}) \mid h_t \right\}.
\] (61)

The summation in the expression above is bounded by a convergent geometric series. Also, \(V\) is bounded. Hence the above quantity can be made arbitrarily small by choosing \(T\) appropriately large. Since the LHS of (60) does not depend on \(T\), which implies,

\[
V(z_t, x_t) \geq W_t^\sigma (h_t).
\] (62)

**Part 2:** Since the strategy the equilibrium strategy \(\tilde{\sigma}\) generated in (32) is such that \(\tilde{\sigma}_t\) depends on \(h_t\) only through \(z_t\) and \(x_t\), the reward-to-go \(W_t^{\tilde{\sigma}}\), at strategy \(\tilde{\sigma}\), can be written (with abuse of notation) as

\[
W_t^{\tilde{\sigma}}(h_t) = W_t^{\tilde{\sigma}}(z_t, x_t) = \mathbb{E}^\tilde{\sigma} \left\{ \sum_{n=t}^{\infty} \delta^{n-t} R(X_n, A_n, Z_n) \mid z_t, x_t \right\}.
\] (63)

For any \(x_{1:t}\),

\[
W_t^{\tilde{\sigma}}(z_t, x_t) = \mathbb{E}^\tilde{\sigma} \left\{ R(X_t, A_t, Z_t) + \delta W_{t+1}^\tilde{\sigma} (\phi(z_t, \theta[z_t]), X_{t+1}) \mid z_t, x_t \right\}
\] (64a)

\[
V(z_t, x_t) = \mathbb{E}^\tilde{\sigma} \left\{ R(X_t, A_t, Z_t) + \delta V(\phi(z_t, \theta[z_t]), X_{t+1}) \mid z_t, x_t \right\}.
\] (64b)

Repeated application of the above for the first \(n\) time periods gives

\[
W_t^{\tilde{\sigma}}(z_t, x_t) = \mathbb{E}^\tilde{\sigma} \left\{ \sum_{m=t}^{t+n-1} \delta^{m-t} R(X_m, A_m, Z_m) + \delta^n W_{t+n}^\tilde{\sigma} (Z_{t+n}, X_{t+n}) \mid z_t, x_t \right\}
\] (65a)

\[
V(z_t, x_t) = \mathbb{E}^\tilde{\sigma} \left\{ \sum_{m=t}^{t+n-1} \delta^{m-t} R(X_m, A_m, Z_m) + \delta^n V(Z_{t+n}, X_{t+n}) \mid z_t, x_t \right\}.
\] (65b)

Taking differences results in

\[
W_t^{\tilde{\sigma}}(z_t, x_t) - V(z_t, x_t) = \delta^n \mathbb{E}^\tilde{\sigma} \left\{ W_{t+n}^\tilde{\sigma} (Z_{t+n}, X_{t+n}) - V(Z_{t+n}, X_{t+n}) \mid z_t, x_t \right\}.
\] (66)

Taking absolute value of both sides then using Jensen’s inequality for \(f(x) = |x|\) and finally taking supremum over \(h_t\) reduces to

\[
\sup_{h_t} |W_t^{\tilde{\sigma}}(z_t, x_t) - V(z_t, x_t)| \leq \delta^n \sup_{h_t} \mathbb{E}^\tilde{\sigma} \left\{ |W_{t+n}^\tilde{\sigma} (Z_{t+n}, X_{t+n}) - V(Z_{t+n}, X_{t+n})| \mid z_t, x_t \right\}.
\] (67)

Now using the fact that \(W_{t+n}, V\) are bounded and that we can choose \(n\) arbitrarily large, we get \(\sup_{h_t} |W_t^{\tilde{\sigma}}(z_t, x_t) - V(z_t, x_t)| = 0\).
APPENDIX F

In this section, we present three lemmas. Lemma 3 is intermediate technical results needed in the proof of Lemma 4. Then the results in Lemma 4 and 5 are used in (D) for the proof of Theorem 5. The proof for Lemma 3 below isn’t stated as it analogous to the proof of Lemma 1 from (C) used in the proof of Theorem 3 (the only difference being a non-zero terminal reward in the finite-horizon model).

Let $\tilde{\sigma}$ be the strategies computed by the methodology in Section IV and let $z_{n+1} = \phi(z_n, \tilde{\sigma}_n(z_n)) \forall n \in [T]$. Define the reward-to-go $W_{\sigma,T}^\alpha$ for any agent $i$ and strategy $\sigma$ as

$$W_{\sigma,T}^\alpha(z_{1:t}, x_{1:t}) =$$

$$\mathbb{E}^{\sigma} \left[ \sum_{n=t}^{T} \delta^{n-t} R(X_n, A_n, Z_n) + \delta^{T+1-t} G(Z_{T+1}, X_{T+1}) \mid z_{1:t}, x_{1:t} \right].$$

(68)

Since $X, A$ are assumed to be finite and $G$ absolutely bounded, the reward-to-go is finite $\forall i, t, \sigma, x_{1:t}$. In the following, any quantity with a $T$ in the superscript refers the finite horizon model with terminal reward $G$.

Let $V_{T}^\alpha(z_t, x_t)$ be the value function for the finite time horizon problem with horizon $T$ defined in (27).

**Lemma 3:** For any $t \in [T]$, $i \in [N]$, $x_{1:t}$ and $\sigma$,

$$V_{T}^\alpha(z_t, x_t) \geq \mathbb{E}^{\sigma} \left[ R(x_t, A_t, z_t) + \delta V_{t+1}^\alpha \left( \phi(z_t, \theta[z_t]), X_{t+1} \right) \mid z_{1:t}, x_{1:t} \right].$$

(69)

The result below shows that the value function from the backwards recursive methodology is higher than any reward-to-go.

**Lemma 4:** For any $t \in [T]$, $i \in [N]$, $x_{1:t}$ and $\sigma$,

$$V_{t}^\alpha(z_t, x_t) \geq W_{t}^{\alpha,T}(z_{1:t}, x_{1:t}).$$

(70)

**Proof:** We use backward induction for this. At time $T$, using the maximization property from (26) (modified with terminal reward $G$),

$$V_{T}^\alpha(z_T, x_T) \equiv \mathbb{E}^{\tilde{\gamma}_T} \left[ R(X_T, A_T, Z_T) + \delta G \left( \phi(z_T, \tilde{\gamma}_T), X_{T+1} \right) \mid z_T, x_T \right]$$

(71a)

$$\geq \mathbb{E}^{\tilde{\gamma}_T} \left[ R(X_T, A_T, Z_T) + \delta G \left( \phi(z_T, \tilde{\gamma}_T), X_{T+1} \right) \mid z_{1:T}, x_{1:T} \right]$$

(71b)

$$= W_{T}^{\alpha,T}(h_T)$$

(71d)
Here the second inequality follows from (26) and (27) and the final equality is by definition in (68).

Assume that the result holds for all $n \in \{t + 1, \ldots, T\}$, then at time $t$ we have

$$V_t^T(z_t, x_t) \geq \mathbb{E}^\sigma_t \left[ R(X_t, A_t, Z_t) + \delta V_{t+1}^T \left( \phi(z_t, \theta[z_t]), X_{t+1} \right) \mid z_{1:t}, x_{1:t} \right]$$  \quad (72a)

$$\geq \mathbb{E}^\sigma_t \left[ R(X_t, A_t, Z_t) + \delta \mathbb{E}^{\sigma_{t+1:T}} \left[ \sum_{n=t+1}^T \delta^{n-(t+1)} R(X_n, A_n, Z_n) \right. \right.$$

$$\left. \left. + \delta^{T-t} G(Z_{T+1}, X_{T+1}) \mid z_{1:t}, x_{1:t}, X_{t+1} \right) \mid z_{1:t}, x_{1:t} \right]$$  \quad (72b)

$$= \mathbb{E}^{\sigma_{t:T}} \left[ \sum_{n=t}^T \delta^{n-t} R(X_n, A_n, Z_n) + \delta^{T+1-t} G(Z_{T+1}, X_{T+1}) \mid z_{1:t}, x_{1:t} \right]$$  \quad (72c)

$$= W_{t}^{\sigma,T}(z_{1:t}, x_{1:t})$$  \quad (72d)

Here the first inequality follows from Lemma 3, the second inequality from the induction hypothesis, the third equality follows since the random variables on the right-hand side do not depend on $\sigma_t$, and the final equality by definition (68).

The following result highlights the similarities between the fixed-point equation in infinite-horizon and the backwards recursion in the finite-horizon.

**Lemma 5:** Consider the finite horizon game with $G \equiv V$. Then $V_t^T = V$, $\forall \; i \in [N], \; t \in \{1, \ldots, T\}$ satisfies the backwards recursive construction stated above (adapted from (26) and (27)).

**Proof:** Use backward induction for this. Consider the finite horizon methodology at time $t = T$, noting that $V_{T+1}^T = G \equiv V$,

$$\tilde{\gamma}_T^T(\cdot \mid x_T) \in \arg \max_{\gamma_T^T(\cdot \mid x_T)} \mathbb{E}^{\gamma_T^T(\cdot \mid x_T)} \left[ R(x_T, A_T, z_T) + \delta V \left( \phi(z_T, \tilde{\gamma}_T^T), X_{T+1} \right) \mid z_T, x_T \right]$$  \quad (73a)

$$V_T^T(z_T, x_T) = \mathbb{E}^{\tilde{\gamma}_T^T(\cdot \mid x_T)} \left[ R(x_T, A_T, z_T) + \delta V \left( \phi(z_T, \tilde{\gamma}_T^T), X_{T+1} \right) \mid z_T, x_T \right].$$  \quad (73b)

Comparing the above set of equations with (30), we can see that the pair $(V, \tilde{\gamma})$ arising out of (30) satisfies the above. Now assume that $V_n^T \equiv V$ for all $n \in \{t + 1, \ldots, T\}$. At time $t$, in the finite horizon construction from (26), (27), substituting $V$ in place of $V_{t+1}^T$ from the induction hypothesis, we get the same set of equations as (73). Thus $V_t^T \equiv V$ satisfies it.
APPENDIX G

Proof: We prove this by contradiction. Suppose for the equilibrium generating function \( \theta \) that generates MFE \( \hat{\sigma} \) and for \( z_{n+1} = \phi(z_n, \sigma_n(\cdot|z_n, \cdot)) \forall n \in [T] \), there exists \( t \in [T], i \in [N] \), such that (30)–(31) is not satisfied for \( \theta \) i.e. for \( \tilde{\gamma}_t = \theta[z_t] = \hat{\sigma}(\cdot|z_t, \cdot) \),

\[
\tilde{\gamma}_t \not\in \arg \max _{\gamma_t(\cdot|z_t)} \mathbb{E}^{\gamma_t(\cdot|z_t)} \left\{ R(X_t, A_t, Z_t) + \delta V(\phi(Z_t, \tilde{\gamma}_t), X_{t+1}) | x_t, z_t \right\} .
\]  
(74)

Let \( t \) be the first instance in the backward recursion when this happens. This implies \( \exists \tilde{\gamma}_t \) such that

\[
\mathbb{E}^{\tilde{\gamma}_t(\cdot|z_t)} \left\{ R(X_t, A_t, Z_t) + \delta V(\phi(Z_t, \tilde{\gamma}_t), X_{t+1}) | z_t, x_t \right\} > \mathbb{E}^{\hat{\gamma}_t(\cdot|z_t)} \left\{ R(X_t, A_t, Z_t) + \delta V(\phi(Z_t, \hat{\gamma}_t), X_{t+1}) | z_t, x_t \right\} 
\]  
(75)

This implies for \( \tilde{\sigma}(\cdot|z_t, \cdot) = \tilde{\gamma}_t \),

\[
\mathbb{E}^{\tilde{\gamma}} \left\{ \sum _{n=t}^{\infty} \delta^{n-t} R(X_n, A_n, Z_n) | z_{1:t}, x_{1:t} \right\} = \mathbb{E}^{\tilde{\gamma}_t} \left\{ R(X_t, A_t, Z_t) + \mathbb{E}^{\tilde{\gamma}_t+1:T} \left\{ \sum _{n=t+1}^{\infty} \delta^{n-t} R(X_n, A_n, Z_n) | z_{1:t}, x_{1:t}, X_{t+1} \right\} | z_{1:t}, x_{1:t} \right\} 
\]  
(76)

\[
= \mathbb{E}^{\tilde{\gamma}_t} \left\{ R(X_t, A_t, Z_t) + \mathbb{E}^{\tilde{\gamma}_t+1:T} \left\{ \sum _{n=t+1}^{\infty} \delta^{n-t} R(X_n, A_n, Z_n) | z_{1:t}, x_{1:t}, X_{t+1} \right\} | z_{1:t}, x_{1:t} \right\} 
\]  
(77)

\[
= \mathbb{E}^{\tilde{\gamma}_t} \left\{ R(X_t, A_t, Z_t) + \delta V(\phi(Z_t, \tilde{\gamma}_t), X_{t+1}) | z_t, x_t \right\} > \mathbb{E}^{\hat{\gamma}_t} \left\{ R(X_t, A_t, Z_t) + \delta V(\phi(Z_t, \hat{\gamma}_t), X_{t+1}) | z_t, x_t \right\} 
\]  
(78)

\[
< \mathbb{E}^{\hat{\gamma}_t} \left\{ R(X_t, A_t, Z_t) + \mathbb{E}^{\hat{\gamma}_t+1:T} \left\{ \sum _{n=t+1}^{\infty} \delta^{n-t} R(X_n, A_n, Z_n) | z_{1:t}, x_{1:t}, X_{t+1} \right\} | z_{1:t}, x_{1:t} \right\} 
\]  
(79)

\[
= \mathbb{E}^{\hat{\gamma}_t} \left\{ R(X_t, A_t, Z_t) + \mathbb{E}^{\tilde{\gamma}_t+1:T} \left\{ \sum _{n=t+1}^{\infty} \delta^{n-t} R(X_n, A_n, Z_n) | z_{1:t}, x_{1:t}, X_{t+1} \right\} | z_{1:t}, x_{1:t} \right\} 
\]  
(80)

\[
= \mathbb{E}^{\tilde{\gamma}_t \hat{\gamma}_t+1:T} \left\{ \sum _{n=t}^{\infty} \delta^{n-t} R(X_n, A_n, Z_n) | z_{1:t}, x_{1:t} \right\} .
\]  
(81)

where (78) follows from the definitions of \( \tilde{\gamma}_t \) and Appendix E, (79) follows from (75) and the definition of \( \hat{\gamma}_t \), (80) follows from Appendix E. However, this leads to a contradiction since \( \tilde{\sigma} \) is an MFE of the game. 

\[ \square \]
REFERENCES

[1] J.-M. Lasry and P.-L. Lions, “Mean field games,” *Japanese Journal of Mathematics*, vol. 2, no. 1, pp. 229–260, 2007.

[2] M. Huang, R. P. Malhamé, and P. E. Caines, “Large population stochastic dynamic games: closed-loop mckean-vlasov systems and the nash certainty equivalence principle,” *Communications in Information & Systems*, vol. 6, no. 3, pp. 221–252, 2006.

[3] J.-M. Lasry, P.-L. Lions, and O. Guéant, “Application of mean field games to growth theory,” 2008.

[4] O. Guéant, J.-M. Lasry, and P.-L. Lions, “Mean field games and applications,” in *Paris-Princeton lectures on mathematical finance 2010*. Springer, 2011, pp. 205–266.

[5] J. Subramanian and A. Mahajan, “Reinforcement learning in stationary mean-field games,” in *International Conference on Autonomous Agents and Multiagent Systems (AAMAS)*, 2019.

[6] M. Huang and Y. Ma, “Mean field stochastic games: Monotone costs and threshold policies,” in *2016 IEEE 55th Conference on Decision and Control (CDC)*. IEEE, 2016, pp. 7105–7110.

[7] ——, “Mean field stochastic games with binary action spaces and monotone costs,” *arXiv preprint arXiv:1701.06661*, 2017.

[8] ——, “Mean field stochastic games with binary actions: Stationary threshold policies,” in *2017 IEEE 56th Annual Conference on Decision and Control (CDC)*. IEEE, 2017, pp. 27–32.

[9] S. Adlakha, R. Johari, and G. Y. Weintraub, “Equilibria of dynamic games with many players: Existence, approximation, and market structure,” *Journal of Economic Theory*, vol. 156, pp. 269–316, 2015.

[10] D. Vasal, A. Sinha, and A. Anastasopoulos, “A systematic process for evaluating structured perfect bayesian equilibria in dynamic games with asymmetric information,” *IEEE Transactions on Automatic Control*, 2018.

[11] D. Vasal and A. Anastasopoulos, “Decentralized Bayesian learning in dynamic games,” in *Allerton Conference on Communication, Control, and Computing*, 2016. [Online]. Available: [https://arxiv.org/abs/1607.06847](https://arxiv.org/abs/1607.06847)

[12] ——, “Signaling equilibria of dynamic LQG games with asymmetric information,” in *Conference on Decision and Control*, 2016.

[13] Y. Ouyang, H. Tavafoghi, and D. Teneketzis, “Dynamic games with asymmetric information: Common information based perfect bayesian equilibria and sequential decomposition,” *IEEE Transactions on Automatic Control*, vol. 62, no. 1, pp. 222–237, 2017.

[14] H. T. Jahormi, “On design and analysis of cyber-physical systems with strategic agents,” Ph.D. dissertation, University of Michigan, Ann Arbor, 2017.

[15] D. Vasal and R. Berry, “alpha– robust equilibrium in anonymous games,” *arXiv preprint arXiv:2005.06812*, 2020.

[16] J. Doncel, N. Gast, and B. Gaujal, “Discrete mean field games: Existence of equilibria and convergence,” *arXiv preprint arXiv:1909.01209*, 2019.

[17] A. Nayyar, A. Mahajan, and D. Teneketzis, “Decentralized stochastic control with partial history sharing: A common information approach,” *Automatic Control, IEEE Transactions on*, vol. 58, no. 7, pp. 1644–1658, 2013.

[18] J. Arabneydi and A. Mahajan, “Team optimal control of coupled subsystems with mean-field sharing,” in *53rd IEEE Conference on Decision and Control*. IEEE, 2014, pp. 1669–1674.

[19] P. Kumar and P. Varaiya, “Stochastic systems,” 1986.

[20] F. Delarue, D. Lacker, and K. Ramanan, “From the master equation to mean field game limit theory: a central limit theorem,” *Electron. J. Probab.*, vol. 24, p. 54 pp., 2019. [Online]. Available: [https://doi.org/10.1214/19-EJP298](https://doi.org/10.1214/19-EJP298)

[21] K. C. Border, *Fixed point theorems with applications to economics and game theory*. Cambridge university press, 1989.

[22] A. Granas and J. Dugundji, *Fixed point theory*. Springer Science & Business Media, 2013.