The n:m resonance dual pair

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In honor of Tudor Ratiu’s sixtieth birthday.

AMS Classification: 53D17, 53D20
Keywords: dual pairs, Poisson brackets

Abstract

1 Introduction

The dual pairs for 1:1 and 1:-1 resonance are presented in [Marsden(1987)], reformulating results from [Cushman and Rod(1982)] and [Iwai(1985)]. The dual pair for 1:1 resonance is the pair of momentum maps associated to the commuting Hamiltonian actions of the Lie groups $S^1$ and $SU(2)$ on $\mathbb{C}^2$ endowed with the opposite $\omega$ of the canonical symplectic form:

$$\mathbb{R} \xrightarrow{R} (\mathbb{C}^2, \omega) \xrightarrow{J} su(2)^*.$$

The momentum map $J$ maps the fibers of $R$, which are 3-spheres, into 2-spheres, coadjoint orbits of $SU(2)$. The restriction of $J$ to these 3-spheres is a Hopf fibration.

A similar construction works for the $S^1$ and $SU(1,1)$ actions on $\mathbb{C}^2$ endowed with the symplectic form $\omega_- = -dx_1 \wedge dy_1 + dx_2 \wedge dy_2$, thus obtaining the 1:-1 resonance dual pair:

$$\mathbb{R} \xleftarrow{R_-} (\mathbb{C}^2, \omega_-) \xrightarrow{J_-} su(1,1)^*.$$

The momentum map $J_-$ maps the fibers of $R_-$, which are 3-hyperboloids, into 2-hyperboloids, coadjoint orbits of $SU(1,1)$. The restriction of $J_-$ to these 3-hyperboloids is a hyperbolic Hopf fibration.

In this paper we build dual pairs of Poisson maps

$$\mathbb{R} \xleftarrow{R_{\pm}} (D, \omega_{\pm}) \xrightarrow{\Pi_{\pm}} B$$

associated to $n : m$ resonance, as well as to $n : -m$ resonance. Except for the above mentioned cases $1 : \pm 1$, these are not pairs of momentum maps. Here $D$ is an open subset of $\mathbb{C}^2$ with the above mentioned symplectic forms $\omega_{\pm}$, and $B$ an open subset of $\mathbb{R}^3$. The Poisson structure on $B$, which depends on the natural numbers $n$ and $m$, is not Lie-Poisson. Instead, its symplectic leaves are the Kummer shapes: bounded surfaces for $n : m$ resonance, and unbounded surfaces for $n : -m$ resonance [Kummer(1986)].

Under some extra hypothesis, to each integrable system in the non-commutative sense (also called superintegrable system) one can associate a dual pair whose right leg is the map defined by the independent first integrals [Fassò(2005)] [Ortega and Ratiu(2004)]. Beside the rigid body and the Kepler system, the two uncoupled oscillators in $m : n$ resonance comprise a well known example of superintegrable system. The dual pairs we present in this article are of this type.

Acknowledgements. We are grateful to Andreas Kriegl for very helpful suggestions regarding Lemma 5.1 and Lemma 9.1, and to Francesco Fassò for turning our attention to dual pairs in the context of superintegrable systems. We also acknowledge partial support by the Royal Society of London’s Wolfson Scheme and hospitality at the Institute for Mathematical Sciences, Imperial College London. Finally, we are grateful to our late friend Jerry Marsden and we fondly remember the many wonderful discussions of geometric mechanics we had together with both Jerry and Tudor over the years.

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2 Dual pairs

Let \((M, \omega)\) be a symplectic manifold and \(P_1, P_2\) be two Poisson manifolds. A pair of Poisson mappings

\[
P_1 \xrightarrow{J_1} (M, \omega) \xrightarrow{J_2} P_2
\]

is called a dual pair [Weinstein(1983)] if \(\ker TJ_1\) and \(\ker TJ_2\) are symplectic orthogonal complements of one another. That is

\[
(\ker TJ_1)^\omega = \ker TJ_2. \tag{1}
\]

A systematic treatment of dual pairs can be found in Chapter 11 of [Ortega and Ratiu(2004)]. The infinite dimensional case is treated in [Gay-Balmaz and Vizman(2009)].

**Proposition 2.1.** Let \(J_1\) and \(J_2\) be momentum maps arising from the canonical actions of two connected Lie groups \(G_1\) and \(G_2\) on a symplectic manifold \((M, \omega)\). We assume that both momentum maps are equivariant, so they are Poisson maps with respect to the (+) Lie-Poisson structure on the dual Lie algebras. Moreover we assume that \(J_1\) is \(G_2\)-invariant, and the \(G_2\) action is transitive on level sets of \(J_1\). Then the pair of momentum maps

\[
\mathfrak{g}_1^* \xleftarrow{J_1^*} (M, \omega) \xrightarrow{J_2^*} \mathfrak{g}_2^*
\]

is a dual pair.

**Proof.** The transitivity of the \(G_2\) action on level sets of \(J_1\) is written infinitesimally as \((\mathfrak{g}_2)_M = \ker TJ_1\). The dual pair property is seen upon writing \((\ker TJ_1)^\omega = ((\mathfrak{g}_2)_M)^\omega = \ker TJ_2\). □

As a consequence we obtain that the actions of the Lie groups \(G_1\) and \(G_2\) on \(M\) commute.

The dual pair is called full if \(J_1 : M \to P_1\) and \(J_2 : M \to P_2\) are surjective submersions. A key result in the context of dual pairs is the symplectic leaf correspondence for full dual pairs with connected fibers. Namely, there is a bijective correspondence between the symplectic leaves of \(P_1\) and those of \(P_2\) [Weinstein(1983)]:

\[
\mathcal{L}_1 \mapsto J_2(J_1^{-1}(\mathcal{L}_1)) \text{ with inverse } \mathcal{L}_2 \mapsto J_1(J_2^{-1}(\mathcal{L}_2)).
\]

3 The 1 : 1 resonance dual pair

Let \(\langle , \rangle\) be the canonical Hermitian inner product on \(\mathbb{C}^2\). This means that \(\langle a, b \rangle = g(a, b) + i\omega(a, b)\), with \(g\) the euclidean metric on \(\mathbb{C}^2\) and \(\omega\) the opposite of the canonical symplectic form on \(\mathbb{C}^2\). The Lie group \(U(2)\) of unitary \(2 \times 2\) matrices, i.e. complex matrices \(g\) with the property \(\langle ga, gb \rangle = \langle a, b \rangle\) for all \(a, b \in \mathbb{C}^2\), acts in a Hamiltonian way on \((\mathbb{C}^2, \omega)\) with momentum map

\[
\tilde{J} : \mathbb{C}^2 \to u(2)^*, \quad \langle \tilde{J}(a), \xi \rangle_{u(2)} = \frac{i}{2} \langle a, \xi(a) \rangle. \tag{3}
\]

This follows from the computation

\[
d_a \langle \tilde{J}, \xi \rangle_{u(2)} = \frac{i}{2} \langle a, \xi(a) \rangle + \frac{i}{2} \langle \xi(a), a \rangle = \Re \langle \xi(a), a \rangle = \omega(\xi^2(a), a),
\]

where \(\xi \in u(2)\), the Lie algebra of skew Hermitian \(2 \times 2\) matrix, i.e. \(\langle \xi(a), b \rangle + \langle a, \xi(b) \rangle = 0\) for all \(a, b \in \mathbb{C}^2\). Another way to deduce this is from the general form of the momentum map for linear Hamiltonian actions on linear symplectic spaces \(\langle J(a), \xi \rangle_{u(2)} = \frac{i}{2} \omega(\xi(a), a)\), since \(g(\xi(a), a) = 0\) for all \(\xi \in u(2)\).

The Lie group \(U(2)\) is the direct product of its center, which is isomorphic to the circle \(S^1\), and the special unitary group

\[
\text{SU}(2) = \left\{ \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \mid |\alpha|^2 + |\beta|^2 = 1 \right\}.
\]

The momentum map for the circle action is

\[
R : \mathbb{C}^2 \to \mathbb{R}, \quad R(a) = \frac{1}{2}(|a_1|^2 + |a_2|^2) = \frac{1}{2} \langle a, a \rangle.
\]
To compute the momentum map for the SU(2) action we consider the linear isomorphism

\[ v \in \mathbb{R}^3 \mapsto \xi_v = \begin{bmatrix} iv_3 & iv_1 + v_2 \\ iv_1 - v_2 & -iv_3 \end{bmatrix} \in \mathfrak{su}(2) \subset \mathfrak{u}(2). \]

With this identification, using again the expression (3) of \( \bar{J} \), the momentum map becomes

\[ J : \mathbb{C}^2 \to \mathfrak{su}(2)^* = \mathbb{R}^3, \quad J(a) = \left( \Re(a_1 \bar{a}_2), -\Im(a_1 \bar{a}_2), \frac{1}{2}|a_1|^2 - \frac{1}{2}|a_2|^2 \right). \tag{4} \]

This follows from the computation:

\[ \langle J(a), v \rangle_{\mathbb{R}^3} = \langle \bar{J}(a), \xi_v \rangle_{\mathfrak{su}(2)} = \frac{i}{2} \langle a, \xi_v(a) \rangle = \frac{i}{2} v_1 (a_1 \bar{a}_2 + \bar{a}_1 a_2) + \frac{i}{2} v_2 (a_1 \bar{a}_2 - \bar{a}_1 a_2) + \frac{1}{2} v_3 (a_1 \bar{a}_1 - a_2 \bar{a}_2). \]

It is easy to see that the momentum map \( \bar{J} \) is \( U(2) \)-equivariant:

\[ \langle \bar{J}(g \cdot a), \xi_u \rangle_{\mathfrak{su}(2)} = \frac{1}{2} i \langle g \cdot a, \xi(g \cdot a) \rangle = \frac{1}{2} i \langle a, \xi_g a \rangle = \langle \bar{J}(a), \Ad_{g^{-1}} \xi_u \rangle_{\mathfrak{su}(2)} = \langle \Ad_{g^{-1}} \bar{J}(a), \xi_u \rangle_{\mathfrak{su}(2)}. \]

Therefore, \( \bar{J}(g \cdot a) = \Ad_{g^{-1}} \bar{J}(a) \) for all \( g \in U(2) \). From the equivariance of \( \bar{J} \) follows the \( S^1 \) equivariance of \( R \) and the SU(2) equivariance of \( J \).

We denote the components of the momentum map \( J \) by \( X, Y, Z \), so

\[ X(a) - iY(a) = a_1 \bar{a}_2 \quad \text{and} \quad Z(a) = \frac{1}{2} |a_1|^2 - \frac{1}{2} |a_2|^2. \]

They satisfy \( X^2 + Y^2 + Z^2 = R^2 \). In real coordinates we recognize the three first integrals of the integrable system of two uncoupled oscillators:

\[
\begin{align*}
X(x_1, y_1, x_2, y_2) &= x_1 x_2 + y_1 y_2 \\
Y(x_1, y_1, x_2, y_2) &= x_1 y_2 - x_2 y_1 \\
Z(x_1, y_1, x_2, y_2) &= \frac{1}{2} \left( x_1^2 + y_1^2 - x_2^2 - y_2^2 \right).
\end{align*}
\]

**Proposition 3.1.** The pair of momentum maps

\[ \mathbb{R} \ni R \mapsto (\mathbb{C}^2, \omega) \xrightarrow{J} \mathfrak{su}(2)^* = \mathbb{R}^3 \tag{5} \]

is a dual pair.

**Proof.** The momentum maps for the commuting Hamiltonian actions of SU(2) and \( S^1 \) on \( (\mathbb{C}^2, \omega) \) are equivariant, hence they form a pair of Poisson maps. \( R \) is obviously SU(2) invariant, so the dual pair property \( (\ker TR)^\omega = \ker T J \) follows from Proposition 2.1 if we show that SU(2) acts transitively on fibers of \( R \).

To each element \( a \in \mathbb{C}^2 \) we associate a complex matrix \( h_a = \begin{bmatrix} a_1 & -\bar{a}_2 \\ a_2 & \bar{a}_1 \end{bmatrix} \), so the action of an element \( g \in \text{SU}(2) \) on \( \mathbb{C}^2, g \cdot a = b \), can be rewritten as matrix multiplication \( g \cdot h_a = h_b \). For any \( r > 0 \), the fiber \( R^{-1}(\frac{1}{r^2}) \) is the 3-sphere \( S^3_r \) of radius \( r \). We notice that \( \frac{1}{r} h_a \in \text{SU}(2) \) for all \( a \in S^3_r \). Given two elements in the same fiber, \( a, b \in S^3_r \), the matrix

\[ g = h_b h_a^{-1} = \left( \frac{1}{r} h_b \right) \left( \frac{1}{r} h_a \right)^{-1} \in \text{SU}(2) \]

satisfies \( g \cdot a = b \), hence SU(2) acts transitively on fibers of \( R \). \( \square \)

Because \( X^2 + Y^2 + Z^2 = R^2 \), the momentum map \( J = (X, Y, Z) \) maps the fibers of \( R \), which are 3-spheres, into 2-spheres, coadjoint orbits of SU(2). The restriction of \( J \) to these 3-spheres is the Hopf fibration. For this dual pair the symplectic leaf correspondence becomes \( \{ c^2 \} \mapsto J(R^{-1}(c^2)) = S^2_c \).
4 Poisson brackets on $\mathbb{R}^3$

Vector fields $\mathbf{v} = (v_1, v_2, v_3)$ on $\mathbb{R}^3$ with $v_1, v_2, v_3 \in F(\mathbb{R}^3)$ are in 1-1 correspondence with bivector fields on $\mathbb{R}^3$

$$\pi_{\mathbf{v}} = v_1 \partial_y \wedge \partial_z + v_2 \partial_z \wedge \partial_x + v_3 \partial_x \wedge \partial_y.$$ The following are necessary and sufficient conditions for the bivector field $\pi_{\mathbf{v}}$ to be Poisson:

1. $v_1 (\partial_y v_3 - \partial_z v_2) + v_2 (\partial_z v_1 - \partial_x v_3) + v_3 (\partial_x v_2 - \partial_y v_1) = 0.$
2. $\mathbf{v}^\top \wedge d(\mathbf{v}) = 0$, where $\mathbf{v}^\top = v_1 dx + v_2 dy + v_3 dz$.
3. The distribution $\mathbf{v}^\perp$ on $\mathbb{R}^3$ is integrable.

Under these circumstances the Hamiltonian vector field with Hamiltonian function $H$ on the Poisson manifold $(\mathbb{R}^3, \pi_{\mathbf{v}})$ is $X_H = \mathbf{v} \times \nabla H$, with $\times$ denoting the usual vector product on $\mathbb{R}^3$, so the Poisson bracket on $\mathbb{R}^3$ associated to the bivector field $\pi_{\mathbf{v}}$ can be written as

$$\{F, G\}_{\mathbf{v}} = \mathbf{v} \cdot (\nabla F \times \nabla G).$$

All Hamiltonian vector fields are orthogonal to $\mathbf{v}$, hence the symplectic leaves of the Poisson structure $\pi_{\mathbf{v}}$ are leaves of the integrable distribution $\mathbf{v}^\perp$.

The equivalent conditions 1., 2., and 3. are satisfied for gradient vector fields $\mathbf{v} = \nabla C$ with $C \in F(\mathbb{R}^3)$. The associated Poisson bracket is the Nambu bracket

$$\{F, G\}_C = \nabla C \cdot (\nabla F \times \nabla G) = \text{Jac}(C, F, G),$$

where $\text{Jac}$ denotes the Jacobian determinant. The function $C$ is a Casimir and the symplectic leaves are the level surfaces $C = \text{constant}$. A similar result holds in a more general setting:

**Proposition 4.1.** The vector field $\mathbf{v} = f \nabla C$, where $f$ is a nonvanishing function on $\mathbb{R}^3$, determines a Poisson structure $\pi_{\mathbf{v}}$ on $\mathbb{R}^3$ with symplectic leaves the level surfaces of the function $C$.

**Proof.** From the three equivalent conditions, the third one is the easiest to check: the distribution $\mathbf{v}^\perp$ coincides with the orthogonal distribution to the gradient vector field of $C$, hence it is integrable.

5 Kummer shapes as symplectic leaves

The **Kummer shapes** in $n : m$ resonance, $n, m > 0$, are the bounded surfaces defined by the equation [Kummer(1986)]

$$x^2 + y^2 - \left(\frac{c + z}{n}\right)^m \left(\frac{c - z}{m}\right)^n = 0, \quad |z| < c,$$

(6)

where $c$ is a positive constant (see Figure 2). They are obtained by rotating around the $z$ axis the algebraic curve (see Figure 1)

$$y^2 = \left(\frac{c + z}{n}\right)^m \left(\frac{c - z}{m}\right)^n, \quad |z| < c.$$

Let $\Phi \in F(\mathbb{R}^4)$ be given by

$$\Phi(x, y, z, r) = x^2 + y^2 - \left(\frac{r + z}{n}\right)^m \left(\frac{r - z}{m}\right)^n.$$

(7)

One can obtain the Kummer shapes also by slicing with hyperplanes $r = c$ of $\mathbb{R}^4$ that part of the hypersurface $\Phi = 0$ included in the intersection of the halfspaces $z < r$ and $z > -r$.

**Lemma 5.1.** The Kummer shapes can be expressed as level sets of a smooth function $C$ defined on $\mathbb{R}^3$ with the $z$ axis removed.
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The bivector field \( \pi \) is of the form \( \pi = -\partial_r \Phi \) and \( \Phi = 0 \) is a smooth function defined on \( \mathbb{R}^3 \setminus Oz \) such that

\[
\Phi(x, y, z, C) = \frac{c(y, z) + z}{n} - \frac{c(y, z) - z}{m}, \quad |z| < c(y, z).
\]

Then \( C(x, y, z) = c(\sqrt{x^2 + y^2}, z) \) is a smooth function defined on \( \mathbb{R}^3 \setminus Oz \) whose level sets are the Kummer shapes.

The polynomial function \( p(r) = \left( \frac{c(y, z)}{n} \right)^m \left( \frac{c(y, z)}{m} \right)^n - y^2 \), with coefficients smoothly depending on \((y, z) \in (0, \infty) \times \mathbb{R}_+\), has at least one zero in the interval \((|z|, \infty)\) because \( p(|z|) = -y^2 < 0 \) and \( \lim_{r \to \infty} p(r) = +\infty \). But \( p \) is a monotone increasing function on \((|z|, \infty)\), so there is a unique zero of \( p \) in the interval \((|z|, \infty)\), denoted by \( c(y, z) \).

This gives the smooth function \( c \) on \((0, \infty) \times \mathbb{R}\) we were seeking.

From the implicit identity (8) we deduce that the gradient vector field \( \nabla C \) can be written as

\[
\nabla C = \left( -\frac{1}{\partial_r \Phi} \nabla (x, y, z) \Phi \right) \bigg|_{r = C}.
\]

It follows that the vector field \( v \), defined on \( \mathbb{R}^3 \setminus Oz \) by

\[
v := \nabla (x, y, z) \Phi \bigg|_{r = C} = \left( 2x, 2y, -(x^2 + y^2) \left( \frac{m}{C(x, y, z) + z} - \frac{n}{C(x, y, z) - z} \right) \right),
\]

is of the form \( v = f \nabla C \), where \( f \) is the nonvanishing function

\[
f = -\partial_r \Phi \bigg|_{r = C}.
\]

**Proposition 5.2.** The Kummer shapes (6), with the singular points \((0, 0, \pm c)\) removed, are symplectic leaves of the Poisson manifold \((\mathbb{R}^3 \setminus Oz, \pi_v)\) associated to the vector field \( v \) given by (9).

**Proof.** We know from (9) that \( v = f \nabla C \), so by proposition 4.1 the bivector field \( \pi_v \) is a Poisson bivector field. Its symplectic leaves are the surfaces \( C = \text{constant} \), i.e. the Kummer shapes. \( \square \)
6 Poisson maps for \( n : m \) resonance

Let \( n \) and \( m \) be non-zero natural numbers. The action
\[
z \cdot (a_1, a_2) = (z^n a_1, z^m a_2), \quad z \in S^1 \subset \mathbb{C}
\]
of the circle \( S^1 \) on \( \mathbb{C}^2 \), with the opposite \( \omega \) of the canonical symplectic form,
\[
\omega = -\frac{i}{2}(da_1 \wedge d\bar{a}_1 + da_2 \wedge d\bar{a}_2) = -dx_1 \wedge dy_1 - dx_2 \wedge dy_2
\]
is Hamiltonian with infinitesimal action \((a_1, a_2) \mapsto (ia_1, ima_2)\). The associated momentum map
\[
R : \mathbb{C}^2 \to \mathbb{R}, \quad R(a) = \frac{n}{2}|a_1|^2 + \frac{m}{2}|a_2|^2
\]
is equivariant, which implies that \( R \) is a Poisson map.

Let \( X, Y, Z \) be the functions on \( \mathbb{C}^2 \) uniquely defined by the identities
\[
X(a) = a_1^m \bar{a}_2^n \quad \text{and} \quad Z = \frac{n}{2}|a_1|^2 - \frac{m}{2}|a_2|^2.
\]
An easy computation reveals that
\[
X^2 + Y^2 = \left( \frac{R + Z}{n} \right)^m \left( \frac{R - Z}{m} \right)^n.
\]
This can be written as \( \Phi \circ (X, Y, Z) = 0 \) on \( \mathbb{C}^2 \), with \( \Phi \) the function (7), which means that \( C \circ (X, Y, Z) = R \) on \( (\mathbb{C} \setminus \{0\})^2 \). Here we have to restrict the functions \( X, Y, Z \) to \( (\mathbb{C} \setminus \{0\})^2 \) because \( C \) is not defined on the \( z \) axis.

**Proposition 6.1.** The map \( \Pi = (X, Y, Z) : (\mathbb{C} \setminus \{0\})^2 \to \mathbb{R}^3 \setminus Oz \) is a Poisson map with respect to \( \omega \), the opposite of the canonical symplectic form on \( (\mathbb{C} \setminus \{0\})^2 \) and the Poisson bivector field \( \pi_{mnv} \) on \( \mathbb{R}^3 \setminus Oz \), with vector field \( v \) defined by (9).

**Proof.** The following Poisson brackets on the symplectic manifold \((\mathbb{C}^2, \omega)\) are computed in [Holm(2008)]:
\[
\{Y, Z\} = 2mnX \\
\{Z, X\} = 2mnY \\
\{X, Y\} = -mn(X^2 + Y^2) \left( \frac{m}{R + Z} - \frac{n}{R - Z} \right).
\]
Knowing that
\[
\pi_{mnv} = 2mn x \partial_y \wedge \partial_z + 2mny \partial_z \wedge \partial_x - mn(x^2 + y^2) \left( \frac{m}{C(x, y, z) + z} - \frac{n}{C(x, y, z) - z} \right) \partial_x \wedge \partial_y,
\]
the result follows from the functional identity \( C \circ (X, Y, Z) = R \).

One may also verify that \( \Pi \) is a surjective submersion.

**Remark 6.2.** For \( n = m = 1 \) there are no singularities along the \( z \) axis, so one obtains the Poisson structure
\[
2x \partial_y \wedge \partial_z + 2y \partial_z \wedge \partial_x + 2z \partial_x \wedge \partial_y
\]
on all of \( \mathbb{R}^3 \). This is isomorphic to the Lie-Poisson structure on \( su(2)^* \), the dual of the Lie algebra of \( SU(2) \). The Kummer shapes are spheres: the coadjoint orbits of \( SU(2) \). Moreover, in this case the map \( \Pi \) becomes the equivariant momentum map \( J : \mathbb{C}^2 \to su(2)^* \) from (4), for the canonical Hamiltonian \( SU(2) \)-action on \( \mathbb{C}^2 \).
7 The $n : m$ resonance dual pair

We saw in Proposition 3.1 that the dual pair for $1:1$ resonance is the pair $(R, J)$ of momentum maps associated to the natural commuting Hamiltonian actions of $S^1$ and $SU(2)$ on $C^2$ with the opposite $\omega$ of the canonical symplectic form:

$$\mathbb{R} \xleftarrow{R} (C^2, \omega) \rightarrow (\text{SU}(2)^*) = \mathbb{R}^3.$$ 

There is a dual pair also for general $n : m$ resonance, but it is a dual pair of Poisson maps, rather than momentum maps.

On $\mathbb{R}^3 \setminus Oz$ we consider the Poisson bivector field $\pi_{mnv}$, with $v$ the vector field (9).

**Theorem 7.1.** The pair of Poisson maps

$$\mathbb{R} \xleftarrow{R} ((\mathbb{C} \setminus \{0\})^2, \omega) \rightarrow (\mathbb{R}^3 \setminus Oz, \pi_{mnv})$$

is a dual pair for all pairs $(m, n)$ of nonzero natural numbers.

**Proof.** We know already from the previous section that both $R$ and $II$ are Poisson maps, we only have to check the dual pair property

$$\ker T_a R = (\ker T_a II)^\omega, \quad \forall a \in (\mathbb{C} \setminus \{0\})^2. \quad (14)$$

The symplectic form $\omega$ and the canonical Riemannian metric $g$ on $C^2$ introduced in Section 3 are related by $\omega(a, b) = g(a, ib)$, so the symplectic and Riemannian orthogonals to a real vector subspace $V \subset C^2$ are also related: $V^\perp = (iV)^\perp$. If the vector subspace $V$ is generated by the vector $a = (a_1, a_2) \in C^2$, then $V^\perp = a^\perp$ and $V^\perp = a\perp$. Thus we get

$$\ker T_a R = (\text{nia}_1, m\text{ia}_2)^\perp = (\text{nia}_1, m\text{ia}_2)^\omega. \quad (15)$$

Using the expression (13) of the functions $X, Y, Z$, it is not hard to verify that

$$(\text{nia}_1, m\text{ia}_2) \in \ker T_a II = \ker T_a X \cap \ker T_a Y \cap \ker T_a Z$$

We check it here for the function $X(a) = \frac{1}{2}(a_m a_2^* + a_1^m a_2^*)$, the computations being similar for $Y$ and $Z$ from (13):

$$T_a X (\text{nia}_1, m\text{ia}_2) = \frac{1}{2} (m a_1^m a_2^* (\text{nia}_1) + n a_1^m a_2^n (\text{nia}_2) + m a_1 a_2^{m-1} (-m\text{ia}_2) + n a_1 a_2^{n-1} (m\text{ia}_2) = 0$$

implies $(\text{nia}_1, m\text{ia}_2) \in \ker T_a X$. The kernel of $T_a II$ is 1-dimensional ($II$ is a submersion), so it must be generated by the nonzero vector $(\text{nia}_1, m\text{ia}_2)$. We get

$$(\ker T_a II)^\omega = (\text{nia}_1, m\text{ia}_2)^\omega,$$

which, together with (15), ensures the dual pair property (14). \qed

The symplectic leaf correspondence theorem for dual pairs, applied to the $n : m$ resonance, says that, for each $c > 0$, the symplectic leaf $\{c\}$ of $\mathbb{R}$ corresponds to the symplectic leaf $\Pi(R^{-1}(c))$ of $\mathbb{R}^3$, i.e. to the Kummer surface $C(x, y, z) = c$, because $C \circ II = R$.

8 The $1 : -1$ resonance dual pair

In this section we give an alternative approach to [Iwai(1985)] for the $1 : -1$ resonance dual pair. The Lie group $U(1, 1)$ of complex $2 \times 2$ matrices preserving the Hermitian inner product

$$\langle a, b \rangle_+ = a_1 \bar{b}_1 - a_2 \bar{b}_2 \text{ on } C^2.$$ \quad (16)

has a 1-dimensional center, isomorphic to $S^1$, and a normal subgroup $SU(1, 1)$ consisting of complex matrices with determinant 1:

$$SU(1, 1) = \left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \right\} \text{ with } |\alpha|^2 - |\beta|^2 = 1.$$ 

We endow $C^2$ with the symplectic form

$$\omega_- = -\frac{i}{2}(da_1 \wedge d\bar{a}_1 - da_2 \wedge d\bar{a}_2) = -dx_1 \wedge dy_1 + dx_2 \wedge dy_2.$$ \quad (17)
the imaginary part of the inner product (16). The natural action of the group \text{U}(1,1) by multiplication on \( \mathbb{C}^2 \) is Hamiltonian, with \( \text{U}(1,1) \)-equivariant momentum map

\[
\bar{J}_- : \mathbb{C}^2 \to u(1,1)^*, \quad \langle \bar{J}_-(a), \xi \rangle_{u(1,1)} = \frac{1}{2} (-\omega_\cdot)(\xi(a), a) = \frac{i}{2} (a, \xi(a))_-. \tag{18}
\]

The map

\[
R_- : \mathbb{C}^2 \to \mathbb{R}, \quad R_- = \frac{1}{2} |a_1|^2 - \frac{1}{2} |a_2|^2.
\]

is the momentum map for the natural \( S^1 \)-action \( z \cdot (a_1, a_2) = (za_1, za_2) \) on \((\mathbb{C}^2, \omega_- )\). This is the action of the center of \( \text{U}(1,1) \). There is a linear isomorphism between \( \mathbb{R}^3 \) and the Lie algebra \( \text{su}(1,1) \) given by

\[
u \in \mathbb{R}^3 \mapsto \begin{bmatrix} iu_3 & iu_1 + u_2 & -iu_3 \\ -iu_1 + u_2 & iu_3 & \end{bmatrix} \in \text{su}(1,1).
\]

With this identification, from the expression (18) of \( \bar{J}_- \) we deduce the following expression of the momentum map for the \( \text{SU}(1,1) \) action:

\[
J_- : \mathbb{C}^2 \to \text{su}(1,1)^* = \mathbb{R}^3, \quad J_-(a) = \left( \text{Re}(a_1 \bar{a}_2), -\text{Im}(a_1 \bar{a}_2), -\frac{1}{2} |a_1|^2 + \frac{1}{2} |a_2|^2 \right).
\]

(20)

Denoting by \( (X, Y, Z_-) \) the three components of the momentum map \( J_- \), we get that \( X^2 + Y^2 - Z_-^2 = R_-^2 \).

**Proposition 8.1.** The pair of momentum maps (19) and (20) for the commuting actions of \( S^1 \) and \( \text{SU}(1,1) \) on \((\mathbb{C}^2, \omega_-)\)

\[
\mathbb{R} \xleftarrow{R_-} (\mathbb{C}^2, \omega_-) \xrightarrow{J_-} \text{su}(1,1)^* = \mathbb{R}^3
\]

is a dual pair.

The proof is similar to that of Proposition 3.1. It uses the fact that the action of an element \( g \in \text{SU}(1,1) \) on \( \mathbb{C}^2 \), \( g \cdot a = b \), can be rewritten as matrix multiplication \( g \cdot k = k_b \), where \( k_a = \begin{bmatrix} a_1 \\ a_2 \\ a_1 \end{bmatrix} \). Given two elements \( a, b \) in the same fiber of \( R_- \), the matrix \( g = k_b k_a^{-1} \in \text{SU}(1,1) \) satisfies \( g \cdot a = b \), hence \( \text{SU}(1,1) \) acts transitively on fibers of \( R_- \).

The momentum map \( J_- \) maps the fibers of \( R_- \), which are 3-hyperboloids, into 2-hyperboloids, coadjoint orbits of \( \text{SU}(1,1) \). The restriction of \( J_- \) to these 3-hyperboloids is the hyperbolic Hopf fibration.

## 9 The \( n : -m \) resonance dual pair

In this section we build a dual pair for the more general \( n : -m \) resonance, a dual pair in which the second map is not a momentum map, but only a Poisson map. The first map is

\[
R_- : \mathbb{C}^2 \to \mathbb{R}, \quad R_-(a) = \frac{n}{2} |a_1|^2 - \frac{m}{2} |a_2|^2,
\]
Lemma 9.1. The unbounded Kummer shapes can be expressed as level sets of a smooth function
and
and
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and
provided
and
The Kummer shapes in 
n resonance are the unbounded surfaces defined by the equation (see Figures 4 and 5):

\[ x^2 + y^2 - \left( \frac{z + c}{n} \right)^m \left( \frac{z - c}{m} \right)^n = 0, \quad |z| > c, \]

where \( c \) is a positive constant. Those with \( n \) and \( m \) of the same parity have two connected components, the others are connected (see Figure 3). They are obtained by rotating around the \( z \) axis the algebraic curve

\[ y^2 = \left( \frac{z + c}{n} \right)^m \left( \frac{z - c}{m} \right)^n, \quad |z| > c. \]

Let \( \Psi \in \mathcal{F}(\mathbb{R}^4) \) be given by

\[ \Psi(x, y, z, r) = x^2 + y^2 - \left( \frac{z + r}{n} \right)^m \left( \frac{z - r}{m} \right)^n. \]  

By slicing with hyperplanes \( r = c \) of \( \mathbb{R}^4 \) that part of the hypersurface \( \Psi = 0 \) included in the union of the halfspaces \( z > r \) and \( z < -r \), one obtains these Kummer shapes.

Lemma 9.1. The unbounded Kummer shapes can be expressed as level sets of a smooth function \( C_- \) defined on an open subset of \( \mathbb{R}^3 \setminus Oz \):

\[ B = \{(x, y, z) \in \mathbb{R}^3 \setminus Oz | n^m m^n (x^2 + y^2) < z^{n+m}\}. \]  

Proof. We have to show that there exists a smooth function \( C_- : B \to \mathbb{R} \) which satisfies

\[ \Psi(x, y, z, C_-(x, y, z)) = 0, \quad C_-(x, y, z) < |z|. \]

In other words the hypersurface \( \Psi = 0 \) coincides with the graph of the function \( C_- \), on the union of halfspaces \( z > r \) and \( z < -r \).

To prove this, we observe that \( \Psi \) and \( B \) are rotationally symmetric in \( (x, y) \), so the problem reduces to proving the existence and uniqueness of a smooth function \( c_- \) on \( (0, \infty) \times \mathbb{R} \) such that

\[ y^2 = \left( \frac{z + c_-(y, z)}{n} \right)^m \left( \frac{z - c_-(y, z)}{m} \right)^n, \quad |z| > c_-(y, z), \]

provided \( n^m m^n y^2 < z^{n+m} \). Then \( C_-(x, y, z) = c_-(\sqrt{x^2 + y^2}, z) \) is a smooth function with level sets the unbounded Kummer shapes.

Figure 4: Curves generating upper Kummer shapes for 1:-1, 2:-1, 3:-1, 3:-2 and 4:-2 resonance

Figure 5: Curves generating lower Kummer shapes for 1:-1, 3:-1 and 4:-2 resonance
The polynomial function
\[ p(r) = \left( \frac{z + r}{n} \right)^m \left( \frac{z - r}{n} \right)^n - y^2, \]
with coefficients depending smoothly on \((y, z) \in (0, \infty) \times \mathbb{R}\), has at least one zero in the interval \((0, |z|)\) because \(p(0) = \frac{z^m}{n^m} - y^2 > 0\) and \(p(|z|) = -y^2 < 0\). The existence is clear, but for the uniqueness one has to consider separately the two cases \(n \geq m\) and \(n < m\). In the first case \(p\) is a monotone decreasing function on \((0, z)\), in the second case there is a critical point \(r_0 \in (0, |z|)\) of \(p\), with \(p(r_0) > 0\), and \(p\) is monotone increasing on \((0, r_0)\) and monotone decreasing on \((r_0, |z|)\). In conclusion there is a unique zero of \(p\) in the interval \((0, |z|)\), denoted by \(c_-(y, z)\). This gives the smooth function \(c_-\) we were seeking.

\[ \square \]

The unbounded Kummer shapes in \(n = m\) resonance are symplectic leaves of the Poisson manifold \((B, \pi_w)\), where \(B\) is defined in (23) and the bivector field \(\pi_w\) is associated to the vector field
\[ w := \nabla_{(x, y, z)} \Psi|_{r=C_-} = \left( 2x, 2y, -(x^2 + y^2) \left( \frac{m}{C_-(x, y, z)} + z - \frac{n}{C_-(x, y, z) - z} \right) \right). \]

Indeed, \(w = g \nabla C_-\) for \(g\) the nowhere zero function \(g = -\partial_\Psi|_{r=C_-}\) on \(B\).

As for the \(n > m\) resonance, but now with the roles of \(Z\) and \(R\) switched (\(Z_- = R\) and \(R_- = Z\)), we find that
\[ X^2 + Y^2 = \left( \frac{Z_- + R_-}{n} \right)^m \left( \frac{Z_- - R_-}{m} \right)^n \]
This means that \(\Psi \circ (X, Y, Z, R) = 0\), so that \(C_- \circ (X, Y, Z_-) = R_-\) on the open set
\[ D = \left\{ (a_1, a_2) \in (\mathbb{C}^2 \setminus \{0\})^2 : (m|a_1|^2)^n(m|a_2|^2)^m < \left( \frac{n}{2} |a_1|^2 + \frac{m}{2} |a_2|^2 \right)^{m+n} \right\}. \]
The inequality defining \(D\) comes from \(m^n m^n (X(a_1)^2 + Y(a_1)^2) < Z_- (a_1)^{n+m}\), a necessary condition for the existence of \(C_- (X(a_1), Y(a_1), Z_- (a_1))\).

**Lemma 9.2.** For \(\{\ ,\ \}_{-}\) the Poisson bracket on \(\mathbb{C}^2\) induced by the symplectic form \(\omega_-\), the following identities hold:
\[ \{Z_-, X - iY\} = 2imn(X - iY) \]
\[ \{X, Y\} = -mn(X^2 + Y^2) \left( \frac{m}{R_- + Z_-} - \frac{n}{R_- - Z_-} \right) \]

Proof. Using the fact that on \(C\) we have \(\{z^n, z^m\} = 2imn z^n z^m\), and \(\{|z|^2, z^n\} = 2inz^n\) as well as \(\{|z|^2, z^n\} = -2inz^n\), we compute
\[ \{Z_-, X - iY\} = \frac{n}{2} \tilde{a}_2 \{a_1^m, a_1^m\} - \frac{m}{2} a_1 \{a_2^m, a_1^m\} - \frac{n}{2} (2im) a_1^m \tilde{a}_2 \tilde{a}_2 = 2imn(X - iY) \]
and
\[ \{X, Y\} = \frac{i}{2} \{a_1^m a_2^m, a_1^m \tilde{a}_2 \} = \frac{i}{2} \{a_1^m, a_1^m\} - |a_2|^{2m} - \frac{n}{2} \{a_2^m, a_2^m\} - |a_1|^{2m} \]
\[ = \frac{i}{2} (2im |a_1|^{2m-2}) |a_2|^{2m} - \frac{i}{2} (-2im |a_2|^{2m-2}) |a_1|^{2m} \]
\[ = - |a_1|^{2m} |a_2|^{2m} \left( \frac{m^2}{|a_1|^2} + \frac{n^2}{|a_2|^2} \right) = -mn(X^2 + Y^2) \left( \frac{m}{R_- + Z_-} - \frac{n}{R_- - Z_-} \right). \]

\[ \square \]

**Proposition 9.3.** The map \(\Pi_- = (X, Y, Z_-) : D \subset \mathbb{C}^2 \to B \subset \mathbb{R}^3\) is a Poisson map with respect to the symplectic form \(\omega_-\) on \(\mathbb{C}^2\) and the Poisson bivector field \(\pi_{mnw}\) on \(B\).
Proof. From lemma 9.2 we have that:
\[
\begin{align*}
\{Y, Z_\cdot\}_\cdot &= 2mnX \\
\{Z_\cdot, X\}_\cdot &= 2mnY \\
\{X, Y\}_\cdot &= -mn(X^2 + Y^2) \left( \frac{m}{R^- + Z^-} - \frac{n}{R^- - Z^-} \right)
\end{align*}
\]
Knowing that
\[
\pi_{mnw} = 2mny\partial_y \wedge \partial_z + 2mny\partial_z \wedge \partial_x - mn(x^2 + y^2) \left( \frac{m}{C_-(x,y,z) + z} - \frac{n}{C_-(x,y,z) - z} \right) \partial_x \wedge \partial_y,
\]
the result follows from the functional identity $C_\cdot \circ (X, Y, Z_\cdot) = R_-$.  

Theorem 9.4. The pair of momentum maps
\[
\mathbb{R} \overset{R_-}{\leftarrow} (D, \omega_-) \overset{\Pi_-}{\twoheadrightarrow} (B, \pi_{mnw})
\]
is a dual pair for all pairs $(m, n)$ of nonzero natural numbers, with $B$ and $D$ given in (23) and (25).

Proof. We know already that both $R_-$ and $\Pi_-$ are Poisson maps. We have to show the dual pair property
\[
\ker TR_- = (\ker T\Pi_-)^{\omega_-}.
\]  
(26)

The proof is similar to that of Theorem 7.1. The symplectic orthogonal for $\omega_-$ and the Riemannian orthogonal for the canonical Riemannian metric $g$ on $\mathbb{C}^2$ are related by: $(a_1, a_2) \perp = (ia_1, -ia_2)^{\omega_-}$ because of the identity $\omega_-(a, b) = g((a_1, a_2), (ib_1, -ib_2))$ for all $a = (a_1, a_2)$ and $b = (b_1, b_2)$ on $\mathbb{C}^2$. Thus we get
\[
\ker T_a R_- = (na_1, -ma_2) \perp = (nia_1, mia_2)^{\omega_-}.
\]  
(27)

As in the proof of Theorem 7.1 one sees that
\[
(nia_1, mia_2) \in \ker T_a \Pi_- = \ker T_a X \cap \ker T_a Y \cap \ker T_a Z_-
\]
The kernel of $T_a \Pi_-$ being 1-dimensional, it must be generated by $(nia_1, mia_2)$. We get
\[
(\ker T_a \Pi_-)^{\omega_-} = (nia_1, mia_2)^{\omega_-},
\]
which, together with (27), ensures the dual pair property (26). 

The symplectic leaf correspondence theorem for dual pairs, applied to the $n : -m$ resonance, says that for each $c \in \mathbb{R}$, the symplectic leaf $\{c\}$ of $\mathbb{R}$ corresponds to the symplectic leaf $\Pi(R^1(c))$ of $B$, i.e. to the unbounded Kummer shape $C_-(x,y,z) = c$, because $C_\cdot \circ \Pi_- = R_-$. 

10 Conclusions

A Hamiltonian system having a number of independent integrals of motion bigger than the dimension of its phase space is called superintegrable. More precisely we are given a symplectic 2$d$-dimensional symplectic manifold $(M, \omega)$, and a submersion $f = (f_1, \ldots, f_{2d-n}) : M \to \mathbb{R}^{2d-n}$ with compact connected fibers, with two properties:

1. $\{f_i, f_j\} = \pi_{ij} \circ f$ for $\pi_{ij} : B \subset \mathbb{R}^{2d-n} \to \mathbb{R}$, $i, j = 1, \ldots, 2d - n$,
2. $\text{rank}(\pi_{ij}) = 2d - 2n$.

The Michenko-Fomenko theorem [Mishenkov and Fomenko(1978)] states that under these circumstances the fibers of $f$ are diffeomorphic to the $n$-dimensional torus, and locally there exist generalized action-angle coordinates $(p, q, a, \alpha)$ on the symplectic manifold $M$ ($p$ and $q$ have $d - n$ components, while the actions $a$ and the angles $\alpha$ have $n$ components), that is $\omega = dp \wedge dq + da \wedge d\alpha$. Integrable systems with $d$ integrals of motion in involution are obtained for $n = d$. 

As explained in [Fassò(2005)][Ortega and Ratiu(2004)], the two conditions have a geometric interpretation. The functions \(\pi_{ij}\) are the components of a Poisson bivector field \(\pi_B\) on the open subset \(B \subset \mathbb{R}^{2d-n}\), such that the map \(f : (M, \omega) \rightarrow (B, \pi_B)\) is Poisson. By dimension counting follows that \(\text{rank}\ \pi_B = \dim M - \dim \ker Tf - \dim (\ker Tf \cap (\ker Tf)^\omega)\), so condition 2. implies that \(\ker Tf \subset (\ker Tf)^\omega\), which means that the submersion \(f\) has isotropic fibers (the isotropic tori).

Condition 1. also ensures that the orthogonal distribution \((\ker Tf)^\omega\) is integrable. This follows from its involutivity: for the local basis \(X_{f_1}, \ldots, X_{f_{2d-n}}\) of \((\ker Tf)^\omega\) consisting of the Hamiltonian vector fields with Hamiltonian functions given by the \(2d-n\) integrals of motion, all commutators \([X_{f_i}, X_{f_j}] = X_{f_i(f_j)} = \pi_{ij} \circ f\) are again sections of \((\ker Tf)^\omega\). The other way around, the obvious integrability of \(\ker Tf\) ensures that there is a Poisson bivector field \(\pi_A\) on the space \(A\) of leaves of the integrable distribution \((\ker Tf)^\omega\) (\(A\) is assumed to be a manifold) such that the projection on the space of leaves \(p : (M, \omega) \rightarrow (A, \pi_A)\) is Poisson.

The dual pair of Poisson maps associated to the superintegrable system is

\[
(A, \pi_A) \leftrightarrow (M, \omega) \xrightarrow{f} (B \subset \mathbb{R}^{2d-n}, \pi_B).
\]

The angles \(\alpha\) are coordinates on the fibers of \(f\), while the actions \(\alpha\) are local coordinates on \(A\) and \((p, q)\) are local coordinatess on the symplectic leaves of \(B \subset \mathbb{R}^{2d-n}\). These two Poisson maps coincide in the integrable case \(n = d\).

The dual pair for the superintegrable system of two uncoupled oscillators in \(m : n\) resonance is the one presented in Theorem 7.1 for positive \(n\), resp. Theorem 9.4 for negative \(n\). The functions \(X, Y, Z\) (13), resp. \(X, Y, Z\) (21), are the three independent integrals of motion on the 4-dimensional symplectic manifold \((M = (\mathbb{C} \setminus \{0\})^2, \omega)\), resp. \((M = \{(a_1, a_2) \in (\mathbb{C} \setminus \{0\})^2 : (n|a_1|^2 + m|a_2|^2)^n < \left(\frac{n|a_1|^2 + m|a_2|^2}{2}\right)^{n+m}\}, \omega),\) where \(\omega\) is the opposite of the canonical symplectic form on \(\mathbb{C}^2\), and \(\omega_\perp = \frac{i}{2}(da_1 \wedge da_1 - da_2 \wedge da_2)\).

The Poisson manifold \(B\) is an open subset of \(\mathbb{R}^5; B = \mathbb{R}^3 \setminus O\) with Poisson bivector field \(\pi_B = 2mnx_\partial y \wedge \partial z + 2mny_\partial z \wedge \partial x - mn(x_3^2 + y_3^2)
\left(\frac{m}{(x, y, z)^{2n}} + \frac{n}{(x, y, z)^{2m}}\right) \partial x \wedge \partial y\), where \(C\) is a smooth function on \(B\) implicitly defined by

\[
x^2 + y^2 - \left(\frac{C(x, y, z)}{m}\right)^m - \left(\frac{C(x, y, z)}{n}\right)^n = 0 \text{ and } |z| \leq C(x, y, z),\text{ resp. } B = \{(x, y, z) \in \mathbb{R}^3 \setminus O \mid n^m m^n (x^2 + y^2) < C_\perp (x, y, z)^n = 0 \text{ and } C_\perp (x, y, z) < |z|\}.
\]

The symplectic leaves of the Poisson manifold \((B, \pi_B)\) are the Kummer surfaces, bounded for positive \(n\), resp. unbounded for negative \(n\).

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