Bipartite graphs with a perfect matching and digraphs *

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Abstract

In this paper, we introduce a corresponding between bipartite graphs with a perfect matching and digraphs, which implicates an equivalent relation between the extendibility of bipartite graphs and the strongly connectivity of digraphs. Such an equivalent relation explains the similar results on \(k\)-extendable bipartite graphs and \(k\)-strong digraphs. We also study the relation among \(k\)-extendable bipartite graphs, \(k\)-strong digraphs and combinatorial matrices. For bipartite graphs that are not \(1\)-extendable and digraphs that are not strong, we prove that the elementary components and strong components are counterparts.

\textbf{Key words:} \(k\)-extendable, strongly \(k\)-connected, indecomposable, irreducible, strong component, elementary component

1 Introduction and terminologies

In this paper, all graphs (digraphs) considered have no loop and multiple edge (arc) unless explicitly stated. For all terminologies not defined, we refer the reader to [2], [3] and [4]. All matrices considered are zero-one matrices.

We use \(V(G)\) and \(E(G)\) to denote the vertex set and edge set of a graph \(G\). Let \(G\) be a bipartite graph with bipartition \((U, W)\) where \(U = \{u_1, \ldots, u_n\}\) and \(W = \{w_1, \ldots, w_n\}\). The matrix \(A = (a_{ij})_{n \times n}\), where \(a_{ij} = 1\) if and only if \(u_i w_j \in E(G)\), is called the \textit{reduced adjacency matrix} of \(G\). We denote \(A\) by \(R(G)\). We call \(G\) the \textit{reduced associated bipartite graph} of \(A\) and denote \(G\) by \(B(A)\).

A connected graph is \textit{elementary} if the union of its perfect matchings forms a connected subgraph. A connected graph \(G\) is called \textit{k-extendable}, for \(k \leq ([|V(G)|-1]/2\), if \(G\) has a matching of size \(k\), and every matching of size \(k\) of \(G\) is contained in a perfect matching of \(G\). \(G\) is said to be \textit{minimal k-extendable} if \(G\) is \(k\)-extendable but \(G - e\) is not \(k\)-extendable for any \(e \in E(G)\). An edge of \(G\) is called a \textit{fixed single (fixed double) edge} if it belongs to no (all) perfect matchings of \(G\). An edge of \(G\) is called \textit{fixed} if it is either a fixed single or a fixed double edge of \(G\). All non-fixed edges of \(G\) form a subgraph \(H\), each component of which is elementary and is therefore called an \textit{elementary component}.

Let \(D\) be a digraph. We denote by \(V(D)\), \(A(D)\) and \(M(D)\) the vertex set, arc set and the adjacent matrix of \(D\). Let \(M\) be an adjacent matrix of \(D\), we call \(D\) the \textit{associated digraph} of \(M\) and denote \(D\) by \(D(M)\). \(D\) is \textit{strongly connected}, or \textit{strong}, if there exists a path from \(x\) to \(y\) and a path from \(y\) to \(x\) in \(D\) for any \(x, y \in V(D), x \neq y\). A set \(S \subset V(D)\) is a separator if \(D - S\) is not strong. \(D\) is \textit{k-strongly connected}, or \textit{k-strong}, if \(|V(D)| \geq k + 1\) and \(D\) has no separator of order less than \(k\). \(D\) is \textit{minimal k-strong} if \(D\) is k-strong, but \(D - a\) is not k-strong for any arc \(a \in A(D)\). A strong component is a maximal subdigraph of \(D\) which is strong.

We call a path, directed or undirected, from \(u\) to \(v\) a \((u, v)\)-path. The set of the end-vertices of the edges in a matching \(M\) is denoted by \(V(M)\), or \(V(e)\) if \(M = \{e\}\). The symmetric difference of two sets \(S_1\) and \(S_2\), is denoted by \(S_1 \triangle S_2\).

Let \(B_n\) denote the set of all matrices of order \(n\) over the Boolean algebra \(\{0, 1\}\). We call a matrix \(A \in B_n\) \textit{reducible} if there exists a permutation matrix \(P\), such that

\[
P^TAP = \begin{bmatrix} B & 0 \\ C & D \end{bmatrix},
\]
where \( B \) is an \( l \times l \) matrix and \( D \) is an \( (n-l) \times (n-l) \) matrix, for some \( 1 \leq l \leq n-1 \). \( A \) is irreducible if it is not reducible. Let \( k \) be an integer with \( 1 \leq k \leq n \). \( A \) is called \( k \)-reducible if there exists a permutation matrix \( P \), such that

\[
P^T A P = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & A_{23} \end{bmatrix},
\]

where \( A_{11} \) and \( [A_{22} \ A_{23}] \) are square matrices of order at least one and the size of the zero submatrix at the upper right corner is \( l \times (n-k+1-l) \), \( 1 \leq l \leq n-1 \). If \( A \) is not \( k \)-reducible, then \( A \) is called \( k \)-irreducible.

A matrix \( A \in B_n \) is call partly decomposable if there exist permutation matrices \( P \) and \( Q \), such that

\[
PAQ = \begin{bmatrix} B & 0 \\ C & D \end{bmatrix},
\]

where \( B \) is an \( l \times l \) matrix and \( D \) is an \( (n-l) \times (n-l) \) matrix, for some \( 1 \leq l \leq n-1 \). \( A \) is fully indecomposable if it is not partly decomposable. Let \( k \) be an integer with \( 0 \leq k \leq n \). \( A \) is called \( k \)-partly decomposable if it contains an \( l \times (n-k+1-l) \) zero submatrix, for some \( 1 \leq l \leq n-1 \). A matrix which is not \( k \)-partly decomposable is called \( k \)-indecomposable.

A diagonal of a matrix \( A = (a_{ij}) \in B_n \) is a collection \( T \) of \( n \) entries \( a_{1i_1}, a_{2i_2}, \ldots, a_{ni_n} \) of \( A \) such that \( \{i_1, i_2, \ldots, i_n\} = \{1, 2, \ldots, n\} \). If \( i_j = j \) for \( j = 1, 2, \ldots, n \), we call the diagonal main diagonal of the matrix.

Let \( G \) be a bipartite graph with bipartition \((U, W)\), where \( U = \{u_1, \ldots, u_n\} \) and \( W = \{w_1, \ldots, w_n\} \), and \( M = \{u_iw_j, 1 \leq i \leq n\} \) a perfect matching of \( G \). We form \( R(G) = (a_{ij})_{n \times n} \), where \( a_{ij} = 1 \) if and only if \( u_iw_j \in E(G) \). Then \( R(G) \) has a positive main diagonal, which corresponds to \( M \). We obtain a digraph \( D = D(R(G) - I) \), where \( I \) denote the identity matrix. On the contrary, given a digraph \( D \), we can get a bipartite graph \( G = B(M(D) + I) \), which has a perfect matching. Hence we have a corresponding between bipartite graphs with a perfect matching and digraphs. We may get different \( D \) from \( G \), depending on how we choose the perfect matching \( M \), therefore we denote \( D \) by \( D(D, M) \). While \( G \) is uniquely determined by \( D \), we denote it by \( G = B(D) \). Clearly, such a corresponding includes a bijection between \( M \) and \( V(D) \), and a bijection between \( E(G) \setminus M \) and \( A(D) \). \( D \) can also be understood as obtained from \( G \) by orienting all edges of \( G \) towards the same partition and then contracting all edges of \( M \).

There is a well-known equivalent property between the 1-extendibility of \( G \) and the strong connectivity of \( D \).

**Theorem 1.1.** (\cite{5}, Exercise 4.1.5) Let \( G \) be a bipartite graph and \( M \) a perfect matching of \( G \). Then \( D = D(G, M) \) is strong if and only if \( G \) is 1-extendable.

The following is another interesting relation between \( G \) and \( D \).

**Theorem 1.2.** (\cite{5}, Exercise 4.3.3) Let \( G \) be a bipartite graph with a unique perfect matching \( M \). Then \( D = D(G, M) \) is acyclic.

In this paper we further discuss the relation between \( G \) and \( D \), as well as their relations with combinatorial matrices.

## 2 Extendibility versus Connectivity

Below is a generalization of Theorem 1.1 which has been stated in \cite{12} without a proof.

**Theorem 2.1.** Let \( G \) be a bipartite graph and \( M \) a perfect matching of \( G \). Then \( D = D(G, M) \) is \( k \)-strong if and only if \( G \) is \( k \)-extendable.

We prove Theorem 2.1 in this section and show some interesting applications of it. We need Menger’s Theorem in our proof.

**Theorem 2.2.** (Menger \cite{10}) Let \( D \) be a digraph. Then \( D \) is \( k \)-strong if and only if \( |V(D)| \geq k + 1 \) and \( D \) contains \( k \) internally vertex disjoint \((s, t)\)-paths for every choice of distinct vertices \( s, t \in V \).

Actually we use an equivalent form of Menger’s Theorem. Further more, we only need the following weaken form, which appears as an exercise in \cite{2}.
Lemma 2.3. ([3], Exercise 7.17) Let \( D \) be a \( k \)-strong digraph. Let \( x_1, x_2, \ldots, x_{k-1}, y_1, y_2, \ldots, y_{k-1} \) be distinct vertices of \( D \), then there are \( k \) independent paths in \( D \), starting at \( x_i \), \( 0 \leq i \leq k-1 \) and ending at \( y_j \), \( 0 \leq j \leq k-1 \).

Now comes the proof of Theorem 2.1.

Proof. Let \( D \) be \( k \)-strong. We use induction on \( k \) to prove that \( G \) is \( k \)-extendable. When \( k = 1 \), the conclusion follows from Theorem 2.1. Suppose that the conclusion holds for all integers \( 1 \leq m < k \). Now we prove that an arbitrary matching \( M_0 \) of size \( k \) in \( G \) is contained in a perfect matching of \( G \).

Firstly we assume that \( |M_0 \cap M| \geq 1 \). Let \( e \in M_0 \cap M \) and the vertex in \( D \) corresponding to \( e \) be \( v_e \). Let \( G' = G - V(e), D' = D - v_e \), and \( M' = M \setminus e \). Then \( D' \) is \((k-1)\)-strong and \( D' = D(G', M') \). By the induction hypothesis, \( G' \) is \((k-1)\)-extendable. Hence \( M_0 \setminus \{e\} \), which is a matching of size \( k - 1 \) in \( G' \), is contained in a perfect matching \( M' \) of \( G' \). Then \( M' \cup \{e\} \) is a perfect matching of \( G \) containing \( M_0 \).

Now we handle the case that \( M_0 \cap M = \emptyset \). In this case, \( M_0 \) corresponds to an arc set \( A_0 \) of order \( k \) of \( D \). The arcs in \( A_0 \) form some independent cycles and paths in \( D \). Let the set of cycles formed be \( C_0 = \{C_0, C_1, \ldots, C_{x-1}\} \) and the set of paths formed be \( P_0 = \{P_0, P_1, \ldots, P_{t-1}\} \). Let the starting and ending vertices of \( P_i \) be \( u_i \) and \( v_i \), \( 0 \leq i \leq t-1 \). Let \( V_0 \) be the union of the set of vertices of cycles in \( C_0 \) and the set of internal vertices of paths in \( P_0 \). Then \( |V_0| = k - t \). By definition, \( D - V_0 \) is \( t \)-strong.

By Lemma 2.3 there are \( t \) independent paths in \( D \) starting at \( v_i \), \( 0 \leq i \leq t - 1 \), and ending at \( u_j \), \( 0 \leq j \leq t - 1 \). Such paths, together with the paths in \( P_0 \), form some independent cycles in \( D \). Denote the set of such cycles by \( C_1 \). Then \( C_0 \cup C_1 \) is a set of independent cycles in \( D \) which covers all arcs in \( A_0 \). Let the set of edges of cycles in \( C \) be \( E(C) \), then \( E(C) \cup M \) is a perfect matching of \( G \) containing \( M_0 \). Hence \( G \) is \( k \)-extendable.

Conversely, suppose that \( G \) is \( k \)-extendable. To see that \( D \) is \( k \)-strong, let \( \{v_1, v_2, \ldots, v_{k-1}\} \) be a set of \( k-1 \) vertices in \( D \). Denote by \( e_i \), the edge in \( G \) corresponding to \( v_i \), \( 1 \leq i \leq k-1 \). Let \( G' = G - \cup_{i=1}^{k-1} V(e_i), D' = D - \{v_i : 1 \leq i \leq k-1\} \) and \( M' = M \setminus \{e_i : 1 \leq i \leq k-1\} \). Then \( D' = G(G', M') \). Since \( G \) is \( k \)-extendable, \( G' \) is \( 1 \)-extendable. Hence \( D' \) is strong by Theorem 1.1 and \( D \) is \( k \)-strong.

Theorem 2.4. Let \( G \) be a bipartite graph and \( M \) a perfect matching of \( G \). If \( G \) is minimal \( k \)-extendable then \( D = D(G, M) \) is minimal \( k \)-strong.

Proof. Suppose that \( G \) is minimal \( k \)-extendable. By Theorem 2.1 \( D \) is \( k \)-strong. Let \( a \) be an arc of \( D \) and \( e \) be the edge corresponding to \( a \) in \( G \). Then \( D - a = D(G - e, M) \). By the minimality of \( G \), \( G - e \) is not \( k \)-extendable, hence \( D - a \) is not \( k \)-strong by Theorem 2.1. By the arbitrary of \( a \), \( D \) is minimal \( k \)-strong.

The converse of Theorem 2.4 does not generally hold, that is, \( G \) does not need to be minimal \( k \)-extendable if \( D = D(G, M) \) is minimal \( k \)-strong. For example, we show a minimal strong digraph \( D_0 \) in Figure 1 and \( G_0 = B(D_0) \), which is not minimal 1-extendable, in Figure 2.

![Figure 1](image1.png)  
![Figure 2](image2.png)

There are many parallel results on \( k \)-extendable bipartite graphs and \( k \)-strong digraphs. Theorem 2.4 and Theorem 2.1 help to explain such a similarity between these two classes of graphs. In the rest of this section, we will illustrate some such results.

Our first demonstrations are the well-known ear decompositions of strong digraphs and 1-extendable bipartite graphs.

An ear decomposition of a digraph \( D \) is a sequence \( \mathcal{E} = \{P_0, P_1, \ldots, P_t\} \), where \( P_0 \) is a cycle and each \( P_i \) is a path, or a cycle with the following properties:

- (a) \( P_i \) and \( P_j \) are arc disjoint when \( i \neq j \).
- (b) For each \( i = 1, \ldots, t \), if \( P_i \) is a cycle, then it has precisely one vertex in common with \( V(D_{i-1}) \). Otherwise the end-vertices of \( P_i \) are distinct vertices of \( V(D_{i-1}) \) and no other vertex of \( P_i \) belongs to
$V(D_{i-1})$. Here $D_i$ denotes the digraph with vertices $\bigcup_{j=0}^{i} V(P_j)$ and arcs $\bigcup_{j=0}^{i} A(P_j)$.

(c) $\bigcup_{j=0}^{i} V(P_j) = V(D)$ and $\bigcup_{j=0}^{i} A(P_j) = A(D)$.

**Theorem 2.5.** ([2], Theorem 7.2.2) A digraph is strong if and only if it has an ear decomposition. Furthermore, if $D$ is strong, then for every vertex $v$, every cycle $C$ containing $v$ can be used as starting cycle $P_i$ for an ear decomposition of $D$.

Let $e$ be an edge and $G_0$ be the graph containing $e$ only. Join the end-vertices of $e$ by an odd path $P_1$ we obtain a graph $G_1$. Now if $G_{i-1} = e + P_1 + \ldots + P_{i-1}$ has already been constructed, join any two vertices in different color classes of $G_{i-1}$ by an odd path $P_i$ having no other vertices in common with $G_{i-1}$ we obtain $G_i$. The decomposition $G_r = e + P_1 + \ldots + P_r$ is called a bipartite ear decomposition of $G_r$.

**Theorem 2.6.** ([3], Theorem 4.1.6) A bipartite graph is 1-extendable if and only if it has a bipartite ear decomposition. Such an ear decomposition may be started with any edge $e$ of $G$.

It is remarked in [3] that, given a bipartite ear decomposition $G = e + P_1 + \ldots + P_r$ of a bipartite graph $G$, there is exactly one perfect matching $M$ in $G$ such that $M \cap E(G_i)$ is a perfect matching of $G_i$ for every $i$, $0 \leq i \leq r$. It is not hard to check that the given bipartite ear decomposition corresponds to an ear decomposition of the digraph $D = D(G, M)$.

Next, we show two corresponding characterizations.

**Theorem 2.7.** (Plummer [11]) Let $G$ be a connected bipartite graph with bipartition $(U, W)$, $k$ a positive integer such that $k \leq (|V(G)| - 2)/2$. Then $G$ is $k$-extendable if and only if $|U| = |W|$ and for all non-empty subset $X$ of $U$ with $|X| \leq |U| - k$, $|N(X)| \geq |X| + k$.

Let $D$ be a digraph. Let $X, Y$ be disjoint non-empty proper subsets of $V(D)$, the ordered pair $(X, Y)$ is called a one-way pair in $D$ if $D$ has no arc with tail in $X$ and head in $Y$. Let $h(X, Y) = |V - X - Y|$.

**Theorem 2.8.** (Frank and Jordán [3]) A digraph $D$ is $k$-strong if and only if $h(X, Y) \geq k$ for every one-way pair $(X, Y)$ in $D$.

The condition in Theorem 2.8 is equivalent to that $N^+(X) \geq k$ for any set $X \subseteq V(D)$ with $|X| \leq |V(D)| - k$, which is similar to the condition in Theorem 2.7.

The counterpart of Menger’s Theorem for bipartite $k$-extendable graphs was proved by Aldred et al. in [1]. The original proof is a little involved. Now, with Theorem 2.7 we can deduce it from Menger’s Theorem straightforwardly.

**Theorem 2.9.** Let $G$ be a bipartite graph with bipartition $(U, W)$ and a perfect matching. Then $G$ is $k$-extendable if and only if for any perfect matching $M$ and for each pair of vertices $u \in U$ and $w \in W$, there are $k$ internally disjoint $M$-alternating paths connecting $u$ and $w$, furthermore, these $k$ paths start and end with edges in $E(G) \setminus M$.

**Proof.** Let $M$ be any perfect matching of $G$, and $D = D(G, M)$ be obtained by orienting all edges of $G$ towards $W$ then contracting all edges in $M$. Suppose that $G$ is $k$-extendable. Firstly we prove the below claim.

**Claim 1.** Let $D$ be a $k$-strong digraph and $x$ a vertex of $D$, then $D$ contains $k$ cycles, any two of which intersect at $x$ only.

**Proof.** Let $x'$ be a vertex not in $V(D)$. Construct $D'$ such that $V(D') = V(D) \cup \{x'\}$, $A(D') = A(D) \cup \{ux' : ux \in A(D)\} \cup \{x'u : xu \in A(D)\}$. We prove that $D'$ is $k$-strong. If $D'$ is not $k$-strong, then there exists a separator $S$ of size less than $k$. If $S$ contains $x'$, then $S - x'$ is a separator of $D$ of size less than $k - 1$, contradicting the strong connectivity of $D$. Assume that $S$ does not contain $x'$, then any vertex $y$ which is separated from $x'$ by $S$ is separated from $x$ by $S$ as well, hence $S$ is a separator of $D$, again contradicting the strong connectivity of $D$. Therefore $D'$ is $k$-strong. By Menger’s Theorem there are $k$ internally disjoint $(x, x')$-paths in $D$. Replacing every arc $ux'$ in these paths with the arc $ux$, we obtain the cycles as claimed.

By Theorem 2.7 $D$ is $k$-strong. If $uw \notin M$, let $uw', w'w \in M$, and $u_0, w_0 \in V(D)$ be the vertices of $D$ corresponding to edges $uu'$ and $ww'$. By Menger’s Theorem there are $k$ internally disjoint paths in $D$ from $u_0$ to $w_0$, which correspond to $k$ $M$-alternating paths in $G$ from $u'$ to $w'$, starting and ending
with the edges $u'u$ and $ww'$, respectively. Furthermore, any two of these $M$-alternating paths intersect at the edges $u'u$ and $ww'$ only. Removing $u'u$ and $ww'$ from these paths we obtain $k$ internally disjoint $M$-alternating paths from $u$ to $w$ in $G$, starting and ending with edges in $E(G) \setminus M$. If $ww \in M$, let $v \in V(D)$ be the vertices of $D$ corresponding to $ww$. By Claim 1 there are $k$ cycles in $D$, any two of which intersect at $v$ only. The cycles correspond to $k$ $M$-alternating cycles in $G$, any two of which intersect at the edge $ww$ only. Removing $ww$ from the cycles we obtain the paths we want.

Conversely, suppose that for $M$, any vertices $u$ and $w$ in $G$, we can always find the $M$-alternating paths as stated. Let $v_1, v_2$ be any two vertices in $D$ and $u_1w_1, u_2w_2$ be the edges in $M$ corresponding to $v_1$ and $v_2$, where $u_i \in U$ and $w_i \in W$, $i = 1, 2$. Then there are $k$ internally disjoint $M$-alternating paths from $u_1$ to $w_2$, starting and ending with edges in $E(G) \setminus M$. Adding edges $u_1w_1$ and $u_2w_2$ to each of the paths, we get $k$ $M$-alternating paths, corresponding to $k$ internally disjoint paths in $D$ from $v_1$ to $v_2$. Since $v_1, v_2$ is arbitrarily chosen, by Menger’s Theorem, $D$ is $k$-strong. By Theorem 2.13, $G$ is $k$-extendable. □

When considering minimal $k$-extendable bipartite graph and minimal $k$-strong digraphs, we find the following similar results.

**Theorem 2.10.** (Mader [5]) Every minimal $k$-strong digraph contains at least $k$ vertices of out-degree $k$ and at least $k$ vertices of in-degree $k$.

**Theorem 2.11.** (Lou [7]) Every minimal $k$-extendable bipartite graph $G$ with bipartition $(U, W)$ has at least $2k + 2$ vertices of degree $k + 1$. Furthermore, both $U$ and $W$ contain at least $k + 1$ vertices of degree $k + 1$.

Neither of them implies the other but striking analogical techniques were used in [5] and [7]. We cite two corresponding structural lemmas here.

Let $h(a)$ and $t(a)$ denote the head and tail of an arc $a$, respectively. An arc set $a_1, a_2, \ldots, a_m$, where $m$ is even, is call an anti-directed trail if for all $i$, $h(a_{2i+1}) = h(a_{2i+2})$ and $t(a_{2i+2}) = t(a_{2i+3})$, or for all $i$, $t(a_{2i+1}) = t(a_{2i+2})$ and $h(a_{2i+2}) = h(a_{2i+3})$ (indexes modulo $m$).

**Theorem 2.12.** (Mader [5]) Let $D$ be a minimal $k$-strong digraph. Then the subgraph of $D$ induced by all arcs whose tail is of outdegree at least $k + 1$ and whose head is of indegree at least $k + 1$ does not contain an anti-directed trail.

**Theorem 2.13.** (Lou [7]) In a minimal $k$-extendable bipartite graph, the subgraph induced by the edges both ends of which have degree at least $k + 2$ is a forest.

It can be verified that an anti-directed trail in $D$ corresponds to a closed trail in $G = B(D)$, while a closed trail in $G$ does not always corresponds to an anti-directed trail in $D$.

### 3 Combinatorial Matrices

In this section, we show the equivalence among $k$-connected digraphs, $k$-extendable bipartite graphs and combinatorial matrices.

**Theorem 3.1.** (Petersen, Theorem 2.1.1) Let $A \in B_n$, then $A$ is irreducible if and only if the associated digraph $D(A)$ is strong.

**Theorem 3.2.** (Brualdi et al. [4]) Let $A \in B_n$. Then $A$ is fully indecomposable if and only if every one entry of $A$ lies in a nonzero diagonal, and every zero entry of $A$ lies in a diagonal with exactly one zero member.

A nonzero diagonal of $A$ corresponds to a perfect matching of the reduced associated bipartite graph $B(A)$. The condition in Theorem 3.2 is equivalent to that $B(A)$ is 1-extendable.

**Theorem 3.3.** (Petersen, Theorem 2.1.3) Let $A \in B_n$, then

1. If $A$ is fully indecomposable, then $A$ is irreducible.
2. $A$ is irreducible if and only if $A + I$ is fully indecomposable.

The followings are generalized results for $k$-indecomposable matrices and $k$-irreducible matrices.

**Theorem 3.4.** (You et al. [13]) Suppose $k \geq 1$. Then a matrix $A \in B_n$ is $k$-irreducible if and only if $D(A)$ is $k$-strong.
Theorem 3.5. Suppose $0 \leq k \leq n - 1$ and $A \in B_n$. Then $A$ is $k$-indecomposable if and only if $G = B(A)$ is $k$-extendable.

Proof. Suppose that $A$ is $k$-indecomposable. Let the bipartition of $G$ be $(U,W)$. Let $U_1$ be a subset of $U$ such that $|U_1| \leq n - k$. If $|N(U_1)| \leq |U_1| + k - 1$, then $|W \setminus N(U_1)| \geq n - |U_1| - k + 1$, and the submatrix of $A$ indexed by $U_1$ and $W \setminus N(U_1)$ is a zero matrix of size at least $|U_1| \times (n - k + 1 - |U_1|)$. By definition, $A$ is $k$-partly decomposable, a contradiction. Hence $|N(U_1)| \geq |U_1| + k$. By Theorem 2.7 $G$ is $k$-extendable.

Conversely, suppose that $G$ is $k$-extendable. If $A$ is $k$-partly decomposable then $A$ has an $l \times (n-k+1-l)$ zero submatrix, for some $1 \leq l \leq n-k$. Let the subset of $V(G)$ indexing the row of the submatrix be $U_1$, then $|U_1| = l \leq n - k$ and $|N(U_1)| \leq n - (n-k+1-l) = l + k - 1 = |U_1| + k - 1$, contradicting Theorem 2.7.

Lemma 3.6. (You et al. [15]) Suppose $k \geq 1$ and $A \in B_n$ has a positive main diagonal. Then $A$ is $k$-indecomposable if and only if $D(A)$ is $k$-strong.

Theorem 3.7. Let $A \in B_n$. Then

1. If $A$ is $k$-indecomposable, then $A$ is $k$-irreducible.
2. A is $k$-irreducible if and only if $A + I$ is $k$-indecomposable.

Proof. By definition, if $A$ is $k$-reducible then $A$ is $k$-decomposable. Hence if $A$ is $k$-indecomposable, $A$ is $k$-irreducible and (1) holds.

By Theorem 3.1 $A$ is irreducible if and only if $D(A)$ is $k$-strong. Since adding a loop to a vertex or removing a loop from a vertex does not affect the strongly connectivity of a digraph, $D(A)$ is $k$-strong if and only if $D(A + I)$ is $k$-strong. By Lemma 3.6 $D(A + I)$ is $k$-strong if and only if $A + I$ is $k$-indecomposable.

4 Elementary components versus strong components

Let $G$ be a bipartite graph with a perfect matching $M$, but not 1-extendable. By Theorem 2.1 $D = D(G,M)$ is not strong. In this section, we consider the elementary components of $G$ and the strong components of $D$.

Lemma 4.1. Let $G$ be a bipartite graph with a perfect matching $M$, and $G_1$ an elementary component of $G$, then $E(G_1) \cap M$ is a perfect matching of $G_1$.

Proof. An edge $e \in E(G) \setminus E(G_1)$ incident to a vertex in $G_1$ is fixed. However it can not be a fixed double edge, since every edge adjacent to a fixed double edge must be a fixed single edge. Hence, all edges in $M$ saturating vertices in $V(G_1)$ must be in $E(G_1)$ and $E(G_1) \cap M$ is a perfect matching of $G_1$.

Let $M_1 = E(G_1) \cap M$, then $D_1 = D(G_1, M_1)$ is a subdigraph of $D$. Moreover, let $G_1$ be a subgraph of $G$ consisting of only a fixed double edge of $G$, then $e \in M$ and $D_1 = D(G_1, \{e\})$ contains only one vertex of $D$.

Theorem 4.2. Let $G$ be a bipartite graph with a perfect matching $M$, $G_1$ a subgraph of $G$ such that $M_1 = E(G_1) \cap M$ is a perfect matching of $G_1$. Let $D = D(G,M)$ and $D_1 = D(G_1,M_1)$. Then the followings are equivalent.

1. $G_1$ is an elementary component of $G$, or consists of a fixed double edge only.
2. $D_1$ is a strong component of $D$.

Proof. Suppose that $G_1$ is an elementary component of $G$. Then $G_1$ is 1-extendable and hence $D_1$ is strong. Assume that $D_1$ is properly contained in a strong subdigraph $D'_1$ of $D$. Then $G'_1 = B(D'_1)$ is a 1-extendable subgraph of $G$ containing $G_1$. Furthermore, any perfect matching of $G'_1$ is contained in a perfect matching of $G$. Therefore any edge of $G'_1$ is contained in a perfect matching of $G$. However any edge in $E(G'_1) \setminus E(G_1)$ incident to a vertex of $G_1$ must be a fixed single edge and can not be contained in any perfect matching of $G$, which leads to a contradiction. Hence $D_1$ is a maximal strong subdigraph, that is, a strong component, of $D$.

Suppose that $G_1$ consists of a fixed double edge $e$ only. Then $e \in M$ and $D(G_1, \{e\})$ contains exactly a vertex $v$ in $D$. If $v$ is properly contained in a strong component $D'_1$ of $D$, then $G'_1 = B(D'_1)$ is a 1-extendable subgraph of $G$ containing $e$. Furthermore, every perfect matching of $G'_1$ is contained in a perfect matching of $G$. Hence every edge of $G'_1$ is contained in a perfect matching of $G$. However $e$ is contained in every perfect matching of $G$, so all edges adjacent to $e$ are fixed single edges and cannot be
contained in a perfect matching of $G$, a contradiction. Hence $v$ composes a strong component of $D$ with only one vertex.

Conversely, let $D_1$ be a strong component of $D$. Then $G_1 = B(D_1)$ is 1-extendable. To prove that $G_1$ is an elementary component or consist of a fixed double edge, we need only to prove that an edge $e = u_1u_2 \in E(G) \setminus E(G_1)$ associated with a vertex $u_1 \in V(G_1)$ is a fixed single edge. Suppose that $e$ is not a fixed single edge and contained in a perfect matching $M'$ of $G$. Let $u_1w_1, u_2w_2 \in M$, which correspond to vertices $v_1$ and $v_2$ in $D$ respectively, then $v_1 \in V(D_1)$ and $v_2 \notin V(D_1)$. $M \triangle M'$ consists of nonadjacent edges and alternating cycles. The edges $e, u_1w_1$ and $u_2w_2$ must be contained in an alternating cycle $C$. However $C$ corresponds to a directed cycle in $D$, which contains $v_1$ and $v_2$. This contradicts the fact that $D_1$ is a strong component of $D$. □

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