MINIMAL RESOLUTIONS OF LATTICE IDEALS

YUPENG LI, EZRA MILLER, AND ERIKA ORDOG

Abstract. A canonical minimal free resolution of an arbitrary co-artinian lattice ideal over the polynomial ring is constructed over any field whose characteristic is 0 or any but finitely many positive primes. The differential has a closed-form combinatorial description as a sum over lattice paths in $\mathbb{Z}^n$ of weights that come from sequences of faces in simplicial complexes indexed by lattice points. Over a field of any characteristic, a non-canonical but simpler resolution is constructed by selecting choices of higher-dimensional analogues of spanning trees along lattice paths. These constructions generalize sylvan resolutions for monomial ideals by lifting them equivariantly to lattice modules.

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1. Introduction

One of the goals of commutative algebra is to construct free resolutions of ideals over the polynomial ring $R = \mathbb{k}[x_1, \ldots, x_n] = \mathbb{k}[\mathbf{x}]$ in $n$ variables over a field $\mathbb{k}$. Combinatorial settings, such as that of determinantal or toric ideals, provide models for the general theory. For the class of lattice ideals, of which the toric ideals are the prime examples, the input data is a lattice $L$ (that is, a subgroup) in $\mathbb{Z}^n$ whose intersection with the nonnegative orthant is trivial: $L \cap \mathbb{N}^n = \{0\}$; see [MS05, Chapter 7]. The lattice ideal $I_L = \langle x^u - x^v \mid u, v \in \mathbb{N}^n \text{ and } u - v \in L \rangle$ is homogeneous with respect to a grading in which the degree of each variable is a positive integer because $L \cap \mathbb{N}^n = \{0\}$.

Many beautiful constructions of free resolutions of lattice ideals are known in various settings. For example, Peeva and Sturmfels [PS98a] present a minimal free resolution built from combinatorial quadrangle resolutions given any codimension 2 lattice ideal.

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The same authors [PS98b] construct the Scarf complex, which yields a canonical combinatorial resolution of any lattice ideal over a field of any characteristic; that resolution is minimal when the ideal is generic. One can also associate a lattice ideal $I_L$ with a graph $G$ via the image $L$ of the graph Laplacian matrix. For complete undirected graphs, Manjunath and Sturmfels [MS13] study the minimal free resolution of $I_L$ by constructing the complex $\mathcal{CYC}_G$, which coincides with the Scarf complex. Subsequently, for connected undirected graphs, Manjunath–Schreyer–Wilmes [MSW15] and Mohammadi–Shokrieh [MS14], specify a minimal free resolution of $I_L$ using combinatorial graph theory and Gröbner bases. More recently, for directed graphs, O’Carroll and Planas-Vilanova [OP18] construct a free resolution of $I_L$ as the chain complex $\mathcal{CYC}_G$ associated to a finite, strongly connected, weighted digraph. The resolution is minimal if and only if the digraph is strongly complete. Other properties and combinatorial descriptions of various resolutions of lattice ideals can be found in [BPV01, CT10, Pis03] and references therein.

Along the lines of canonical but perhaps not minimal resolutions, Bayer and Sturmfels [BS98] (see also [MS05, Chapter 9]) generalize the Scarf construction to the hull resolution: a canonical cellular resolution for any monomial module. The monomials with exponent vectors in a lattice $L$ generate the lattice module $M_L = R\{x^a \mid a \in L\}$. When the lattice ideal $I_L$ is generic, the resolution of $M_L$ is minimal. Any free resolution of the lattice module $M_L$ over $R[L]$ descends functorially to a resolution of the lattice ideal $I_L$ over $R$ in a way that preserves minimality [BS98, Corollary 3.3].

Although the module structures of minimal resolutions of lattice ideals were explicitly described by Briales-Morales, Písón-Casares, and Vigneron-Tenorio [BPV01], with differentials filled in algorithmically, none of the known closed-form combinatorial constructions for arbitrary lattice ideals are minimal. The best result along these lines is by Tchernev [Tch19]: an explicit recursive algorithm for canonical minimal resolutions of toric rings, where the lattice ideal is prime, using dynamical systems on chain complexes. His method works over a field of any characteristic by using a transcendental extension of the base field when necessary.

Until now there has been no closed-form description of the differentials in any family of minimal resolutions that encompasses all toric ideals, let alone all lattice ideals. The main result of this paper is the construction of a canonical minimal free resolution of an arbitrary positively graded lattice ideal with a closed-form combinatorial description of the differential in characteristic 0 and all but finitely many positive characteristics (Theorem 4.5.1 and Remark 4.6.1). It generalizes the sylvan resolution for monomial ideals [EMO20, Theorem 3.7], which is the first canonical closed-form combinatorial minimal free resolution for arbitrary monomial ideals; it works in characteristic 0 and all but finitely many positive characteristics. Our resolutions of lattice ideals first use the sylvan construction to minimally resolve the lattice module $M_L$ over the polynomial ring $R$ in a canonical way, so that the resolution is equipped with a natural free action of the lattice $L$. Thus the $R$-resolution of $M_L$ is a canonical minimal resolution of $M_L$. 

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as an \( R[L] \) module, so it descends via the Bayer–Sturmfels functor to a resolution of \( I_L \) over \( R \) by \([\text{BS}98 \text{ Corollary 3.3}]\).

We provide a similar construction over a field of any characteristic (Theorem \([\text{BS}98 \text{ Theorem 4.5.2}]\) and Remark \([\text{BS}98 \text{ Remark 4.6.2}]\), but the resolution in this case is non-canonical, since it involves choices of higher-dimensional analogues of spanning trees (Definition \([\text{BS}98 \text{ Definition 3.1.4}]\)).

Acknowledgments.

2. Overview of the construction

A lattice module is a type of monomial module, an \( R \)-submodule of the Laurent polynomial ring \( k[x_1^\pm 1, \ldots, x_n^\pm 1] \) generated by monomials \( x^a \) for vectors \( a \in \mathbb{Z}^n \). In addition to ensuring a positive grading, the condition \( L \cap \mathbb{N}^n = \{0\} \) means that the lattice module \( M_L \) is co-artinian, which for a general monomial module \( M \) means that it is generated by its set of minimal monomials:

\[
\text{min}(M) = \{x^a \in M \mid x^a / x_i \notin M \text{ for all } i\};
\]
equivalently, the set of monomials in \( M \) with degree \( \preceq b \) is finite for all \( b \in \mathbb{Z}^n \).

The \( \mathbb{Z}^n \)-graded Betti numbers of any co-artinian monomial module can be computed by taking the homology of its Koszul simplicial complexes for degrees \( b \in \mathbb{Z}^n \).

**Proposition 2.1** ([BS98 Corollary 1.13]). The \( i \)-th Betti number of any co-artinian monomial module \( M \) in degree \( b \in \mathbb{Z}^n \) is

\[
\beta_{i,b}(M) = \dim_k \text{Tor}_i^R (k, M)_b = \dim_k \widetilde{H}_{i-1}(K^b M; k),
\]
where \( K^b M = \{\tau \in \{0,1\}^n \mid x^{b-\tau} \in M\} \) is the Koszul simplicial complex of \( M \) in degree \( b \) and \( \widetilde{H} \) denotes reduced homology.

Thus in a minimal \( \mathbb{Z}^n \)-graded free resolution of \( M \) over \( R \), the \( i \)-th free module with basis in degree \( b \) can be expressed as

\[
F_{i,b} = \widetilde{H}_{i-1}(K^b M; k) \otimes_k R(-b),
\]
where \( N(-b) \) is the \( \mathbb{Z}^n \)-graded shift of any \( R \)-module \( N \) up by \( b \), so \( N(-b)_a = N_{a-b} \). What remains is the central problem: specify differentials in the free resolution over \( R \).

In fact, for the current purpose, where \( M = M_L \) is a lattice module, arbitrary differentials \( \partial_i \) over \( R \) do not suffice: they must in addition be \( L \)-equivariant, in the sense that \( \partial_i \) should commute with translation by \( \ell \) for \( \ell \in L \). Equivalently, the resolution should carry an action of the group algebra \( R[L] \), making it an \( R[L] \)-free resolution of \( M_L \). The reason is to be able to quotient modulo the action of \( L \) to get a \( \mathbb{Z}^n/L \)-graded free resolution of \( R/I_L \) by free \( R \)-modules \([\text{BS}98 \text{ Corollary 3.3}]\).

Once the module structure of an \( R \)-free resolution of a co-artinian module \( M \) is specified by Eq. \([2.1]\), the sylvan method \([\text{EMO}20]\) yields differentials: for each pair
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of comparable lattice points in \( \mathbb{Z}^n \), construct a sylvan homology morphism \( \tilde{\mathcal{H}}_{i-1}K^aM \leftarrow \tilde{\mathcal{H}}_iK^bM \) (Definition [3.2]) in such a way that the induced homomorphisms

\[
\tilde{\mathcal{H}}_{i-1}K^aM \otimes k[x](-a) \leftarrow \tilde{\mathcal{H}}_iK^bM \otimes k[x](-b)
\]

defines a minimal free resolution [EMO20, Theorem 3.7 and Corollary 9.5]. Alas, the sylvan method was developed for monomial ideals \( I \) rather than for monomial modules. Our first observation is therefore that the sylvan method extends with no difficulty to the case where the monomial ideal \( I \) is replaced by a co-artinian monomial module \( M \) (Proposition 3.8).

When \( M = M_L \) is a lattice module, the set of sylvan morphisms is automatically \( L \)-invariant (Proposition 4.3). For our main result, Theorem 4.5, it remains only to take the quotient modulo \( L \) of the sylvan \( R \)-free resolution of \( M_L \) thus constructed.

3. The sylvan method

The sylvan method [EMO20] constructs explicit, closed-form minimal free resolutions of a monomial ideal \( I \) from combinatorial information [EMO20, Definitions 3.5 and 9.4] intrinsic to the Koszul simplicial complexes \( K^bI \) for all lattice points \( b \in \mathbb{Z}^n \). This procedure works when the field \( k \) has arbitrary characteristic if certain combinatorial choices are allowed in each Koszul simplicial complex [EMO20, Corollary 9.5]. In addition, the procedure works canonically, without making any choices at all, as long as the characteristic of \( k \) avoids finitely many primes [EMO20, Theorem 3.7]. Beyond some easily stated basic properties of the differentials in these sylvan resolutions—canonical or non-canonical—the details are not relevant to the constructions of sylvan resolutions of lattice ideals here. We therefore isolate the properties of sylvan resolutions required for the extension to monomial modules and then to lattice ideals.

**Definition 3.1** ([EMO20 Definition 2.1, Example 2.2, and Definition 9.2]). Fix a simplicial complex \( K \) with reduced differential \( \partial_i : \tilde{C}_iK \rightarrow \tilde{C}_{i-1}K \) over a given ring \( A \) which is assumed to be a field \( k \) or the integers \( \mathbb{Z} \).

1. A shrubbery for \( \partial_i \) is a maximal set \( T \) of \( i \)-dimensional faces of \( K \) whose image \( \partial_i(T) \) is independent in the boundaries \( \tilde{B}_iK = \partial_i(\tilde{C}_iK) \).
2. A stake set for \( \partial_i \) is a minimal set \( S \) of \( (i-1) \)-dimensional faces of \( K \) such that the composite \( \tilde{B}_iK \leftarrow \tilde{C}_iK \rightarrow A\{S\} \) is injective.
3. A hedge for \( \partial_i \) is a pair consisting of a stake set and a shrubbery for \( \partial_i \). A hedge for \( \partial_i \) may be expressed as \( ST_i = (S_{i-1}, T_i) \) to indicate that the faces in \( S \) have dimension \( i-1 \) while the faces in \( T \) have dimension \( i \).
4. A community in \( K \) is a sequence \( ST_\bullet = (ST_0, ST_1, ST_2, \ldots) \) of hedges for \( \partial_0, \partial_1, \partial_2 \ldots \) with \( T_i \cap S_i = \emptyset \) for all \( i \).
Definition 3.2. Fix a co-artinian monomial $R$-module $M$. A family of homomorphisms
\[ \tilde{C}_{i-1}K^aM \xleftarrow{D_{ab}} \tilde{C}_iK^bM \]
between chain groups of the Koszul simplicial complexes of $M$ over $k$ for all comparable pairs $a \preceq b$ of lattice points and all homological degrees $i \in \mathbb{Z}$ is canonical sylvan if
1. $D_{ab}$ induces morphisms $\tilde{H}_{i-1}K^aM \xleftarrow{} \tilde{H}_iK^bM$ whose induced homomorphisms
\[ \tilde{H}_{i-1}K^aM \otimes_k R(-a) \xleftarrow{} \tilde{H}_iK^bM \otimes_k R(-b) \]
of $\mathbb{N}^n$-graded free $R$-modules constitute a minimal free resolution of $M$, and
2. $D_{ab}$ depends only on the Koszul simplicial complexes $K^cM$ indexed by $c$ in the interval $[a, b]$.

The family $D_{ab}$ is noncanonical sylvan if, instead of condition 2,
3. $D_{ab}$ depends only on the Koszul simplicial complexes $K^cM$, along with a community therein, indexed by $c$ in the interval $[a, b]$.

The family $\{D_{ab}\}_{a \preceq b}$ is sylvan if it is canonical or noncanonical sylvan.

In other words, in a canonical sylvan resolution of a monomial module $M$, the homomorphism $\tilde{H}_{i-1}K^aM \otimes R(-a) \xleftarrow{} \tilde{H}_iK^bM \otimes R(-b)$ is constructed entirely from information intrinsic to the restriction of $M$ to the interval $[a - 1, b]$, where $1 = (1, \ldots, 1)$, with no external choices; the $1$ is needed because $K^cM$ reflects the interval $[c - 1, c]$. In contrast, noncanonical sylvan resolutions involve choices of communities.

Example 3.3. Any monomial ideal $I \subseteq R$ has a noncanonical sylvan family [EMO20, Corollary 9.5] in which $D_{ab}$ is specified by an explicit, closed-form sum over all saturated decreasing lattice paths from $b$ to $a$ [EMO20, Definition 9.4], once communities have been specified in the Koszul simplicial complexes of $I$; any communities suffice.

To specify restrictions on the characteristic of the field $k$ in canonical sylvan constructions, we summarize the relevant points from [EMO20] Definitions 2.1, 2.11, and 2.13.

Definition 3.4. Fix a simplicial complex $K$.

1. Write $\tau(T)$ for the index of the subgroup of $\tilde{B}_i$ generated over $Z$ by the images of the faces in a shrubbery $T$ for $\partial_i$ over $Z$.
2. Dually, write $\sigma(S)$ for the index of the image of $\tilde{B}_iK$ in the span $Z\{S\}$ of a stake set $S$ for $\partial_i$ over $Z$.

A field $k$ is torsionless for $K$ if for all $i$ the characteristic of $k$ does not divide $\tau_i = \sum_T \tau(T)^2$ or $\sigma_i = \sum_S \sigma(S)^2$ or the order of the torsion subgroup of $\tilde{C}_iK/\tilde{B}_iK$, where the sums are respectively over all shrubberies $T$ and stake sets $S$ for $\partial_i$.

Example 3.5. Any monomial ideal $I \subseteq R$ has a canonical sylvan family [EMO20, Theorem 3.7] in which $D_{ab}$ is specified by an explicit, closed-form sum over all saturated decreasing lattice paths from $b$ to $a$ [EMO20, Definition 3.6], once $k$ is torsionless for the Koszul simplicial complexes of $I$. 
Remark 3.6. There are only finitely many simplicial complexes on $n$ vertices, so a field $k$ is torsionless for an arbitrary co-artinian monomial module $M$ if its characteristic avoids finitely many positive integer primes.

The main purpose of this section is to extend the sylvan and canonical sylvan families in Examples 3.3 and 3.5 to co-artinian monomial modules. The key but nonetheless elementary observation is that sylvan morphisms are preserved by translation.

Lemma 3.7. Fix a monomial module $M$ with a sylvan family (canonical or noncanonical). Translation by a vector $\ell \in \mathbb{Z}^n$ naturally induces a sylvan family on the $\mathbb{Z}^n$-graded shift $M(-\ell)$ in which the sylvan homology morphism

$$\tilde{C}_i K^{a+\ell}M(-\ell) \xleftarrow{D^{a+\ell}b+\ell} \tilde{C}_i K^{b+\ell}M(-\ell)$$

is the sylvan homology morphism $D^{ab}$ on $M$ itself.

Proof. Translation up by $\ell$ takes that interval to $[a+\ell, b+\ell]$. The lemma is therefore immediate from Definition 3.2 and the fact that $K^{c+\ell}M(-\ell) = K^{c}M$. □

This observation has two important manifestations, detailed in Propositions 3.8 and 4.3 which respectively use translation of

- any co-artinian module up so that an arbitrarily large subset of it sits in the nonnegative orthant, and
- a lattice module along the lattice.

Proposition 3.8. Fix a co-artinian monomial module $M$. The noncanonical and canonical sylvan families in Examples 3.3 and 3.5 work verbatim when the monomial ideal $I$ is replaced by an arbitrary co-artinian monomial module $M$.

Proof. The co-artinian hypothesis implies that only finitely many free modules

$$\tilde{H}_{i-1} K^{a}M \otimes R(-a)$$

contribute nonzero $\mathbb{Z}^n$-graded degree $b$ components. That the displayed homomorphisms in Definition 3.2 constitute a complex and that this complex is exact can hence be verified at any given $\mathbb{Z}^n$-graded degree $b$ by translating the monomial module $M$ up so that the nonnegative orthant contains all of the degrees $a$ beneath $b$ such that $K^{a}M$ has at least one face. □

4. Equivariant sylvan resolutions

Lemma 4.1. When $M = M_L$ is a co-artinian lattice module, the Koszul simplicial complexes $K^{a}M_L$ and $K^{a+\ell}M_L$ are equal as simplicial subcomplexes of the simplex on $\{1, \ldots, n\}$ whenever $\ell \in L$.

Proof. This is immediate from the $L$-invariance of $M_L$ itself. □
Remark 4.2. When \( M = M_L \) is a lattice module in Proposition 3.8, any canonical sylvan family is automatically \( L \)-equivariant, in a sense to be made precise in Proposition 4.3. However, noncanonical sylvan families need not be \( L \)-equivariant: different communities can be selected in Koszul simplicial complexes indexed by lattice points in the same coset of \( L \) even though the Koszul simplicial complexes themselves are the same. Equivariant noncanonical sylvan families still exist, which is crucial for constructing closed-form combinatorial minimal resolutions over fields of arbitrary characteristic, but additional care is required to construct them.

Proposition 4.3. Fix a co-artinian lattice module \( M_L \).

1. In any canonical sylvan family the maps satisfy \( D_{ab} = D_a^{n+\ell} b^{\ell} \) for all \( \ell \in L \) as homomorphisms from chains of \( K^b M_L = K^{b+\ell} M_L \) to those of \( K^a M_L = K^{a+\ell} M_L \).

2. A noncanonical sylvan family can be constructed so that \( D_{ab} = D_a^{n+\ell} b^{\ell} \) for all \( \ell \in L \) by selecting one community in each of the Koszul simplicial complexes \( K^c M_L \) for \( c \) in a set of representatives for the cosets of \( L \).

These sylvan families are called equivariant.

Proof. The canonical case follows from Lemma 3.7 by Definition 3.2.2. For the noncanonical case, copy the selected community for each coset representative into the Koszul simplicial complexes for all other lattice points in the coset. This yields a sylvan family as in Example 3.3 by way of Proposition 3.8. \( \square \)

Corollary 4.4. The minimal free resolution over \( R \) of a co-artinian lattice module \( M_L \) arising from an equivariant sylvan family (Proposition 4.3) is \( L \)-equivariant; equivalently, it is a minimal \( \mathbb{Z}^n \)-graded \( R[L] \)-free resolution of \( M_L \). \( \square \)

Now comes the main result: the construction of combinatorial, closed-form minimal free resolutions of co-artinian lattice ideals in arbitrary characteristic, and canonical such resolutions when the characteristic avoids finitely many positive primes.

Theorem 4.5. Fix a field \( k \) and a lattice ideal \( I_L \subseteq R = \mathbb{k}[x_1, \ldots, x_n] \).

1. If \( k \) is torsionless (Definition 3.4) for the Koszul simplicial complexes of the lattice module \( M_L \) and \( F_* \) is the free resolution of the lattice module \( M_L \) afforded by the canonical sylvan family in Proposition 3.8, then \( F_* \otimes_{R[L]} R \) is a minimal \( \mathbb{Z}^n / L \)-graded \( R[L] \)-free resolution of \( M_L \).

2. If \( F_* \) is the free resolution of the lattice module \( M_L \) afforded by any noncanonical sylvan family as in Proposition 4.3, then \( F_* \otimes_{R[L]} R \) is a minimal \( \mathbb{Z}^n / L \)-graded \( R[L] \)-free resolution of \( M_L \).

Proof. By Corollary 4.4 the Bayer–Sturmfels functor [BS98 Corollary 3.3] applies. \( \square \)
Remark 4.6. Unwinding the definitions leading to Theorem 4.5 helps exhibit the explicit nature of the minimal resolutions it constructs. For $\alpha \preceq \beta \in \mathbb{Z}^n/L$, choose coset representatives $a \preceq b \in \mathbb{Z}^n$. To $\alpha$ and $\beta$ are associated Koszul simplicial complexes $K^\alpha L = K^a M_L$ and $K^\beta L = K^b M_L$ on $\{1, \ldots, n\}$. Any sylvan resolution of $I_L$, be it canonical or noncanonical, is expressed by specifying $\mathbb{Z}^n/L$-graded homomorphisms

$$\tilde{H}_{i-1} K^\alpha L \otimes_k R(-\alpha) \leftarrow \tilde{H}_i K^\beta L \otimes_k R(-\beta).$$

These are induced by sylvan homology morphisms $\tilde{\mathcal{H}}_{i-1} K^\alpha L \leftarrow \tilde{\mathcal{H}}_i K^\beta L$ which are explicitly enacted on (Koszul simplicial) cycles by sylvan morphisms $\tilde{\mathcal{C}}_{i-1} K^\alpha L \leftarrow \tilde{\mathcal{C}}_i K^\beta L$.

Each $D^{\alpha\beta}$ is given by its sylvan matrix, whose rows and columns are indexed by faces (i.e., by subsets of $\{1, \ldots, n\}$). To be completely precise requires notions from [EMO20], which we use henceforth without further comment.

1. In the canonical sylvan free resolution of $I_L$ the entry of $D_{\sigma\tau}$ indexed by faces $\sigma \in K^\alpha_{i-1} L$ and $\tau \in K^\beta_i L$ is a normalized sum of the weights $w_{\varphi}$ of all chain-link fences $\varphi$ from $\tau$ to $\sigma$ along all saturated decreasing lattice paths $\lambda$ from $b$ to $a$:

$$D_{\sigma\tau} = \sum_{\lambda \in \Lambda(a, b)} \frac{1}{\Delta_{i, \lambda} M_L} \sum_{\varphi \in \Phi_{\sigma\tau}(\lambda)} w_{\varphi}.$$

2. A noncanonical sylvan free resolution of $I_L$ requires a choice of community (Definition 3.1.1) in the Koszul simplicial complex $K^\alpha L$ for each coset $\alpha \in \mathbb{Z}^n/L$. The entry of $D_{\sigma\tau}$ indexed by faces $\sigma \in K^\alpha_{i-1} L$ and $\tau \in K^\beta_i L$ is the sum, over all saturated decreasing lattice paths $\lambda$ from $b$ to $a$, of the weights $w_{\varphi}^k$ over $k$ of all chain-link fences $\varphi$ from $\tau$ to $\sigma$ that are subordinate to the hedgerow $ST^\lambda_i$ derived from the communities along $\lambda$:

$$D_{\sigma\tau} = \sum_{\lambda \in \Lambda(a, b)} \sum_{\varphi \in \Phi_{\sigma\tau}(\lambda)} \sum_{\varphi \in ST^\lambda_i} w_{\varphi}^k.$$

5. Extended example

The lattice $L$ that is the kernel of the matrix $[435]$ has lattice ideal

$$I_L = \langle xy^2 - z^2, xz - y^3, yz - x^2 \rangle \subseteq R = k[x, y, z].$$

The corresponding lattice module is

$$M_L = \langle x^ay^b z^c | 4a + 3b + 5c = 0 \rangle \subseteq k[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}].$$

Note that $M_L$ is a $\mathbb{Z}^3$-graded $R[L]$-submodule of $k[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$. The hull resolution of $M_L$ is constructed in [MS05, Example 9.21], which also contains a picture of $M_L$. 
For the moment, treat $M_L$ as an $R$-module with generators $\{x^{l_1}y^{l_2}z^{l_3} \mid (l_1, l_2, l_3) \in L\}$. The first goal is to list all possible sylvan morphisms between Koszul simplicial complexes up to translation by vectors in $L$. Our computation of sylvan morphisms is based on the explicit formulas for three variables in [Ord20, Theorem 3.1 and Theorem 4.1].

There are six different nonzero sylvan morphisms from $\widetilde{C}_0$ to $\widetilde{C}_{-1}$:

(a) $\widetilde{C}_{-1}K^{000} \otimes \langle 1 \rangle \xleftarrow{\mathcal{O} \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}} \widetilde{C}_0K^{120} \otimes \langle xy^2 \rangle$

(b) $\widetilde{C}_{-1}K^{12\{-2\}} \otimes \langle xyz^{-2} \rangle \xleftarrow{\mathcal{O} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}} \widetilde{C}_0K^{120} \otimes \langle xy^2 \rangle$

(c) $\widetilde{C}_{-1}K^{000} \otimes \langle 1 \rangle \xleftarrow{\mathcal{O} \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}} \widetilde{C}_0K^{101} \otimes \langle xz \rangle$

(d) $\widetilde{C}_{-1}K^{1\{\{-3\}\}1} \otimes \langle xy^{-3}z \rangle \xleftarrow{\mathcal{O} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}} \widetilde{C}_0K^{101} \otimes \langle xz \rangle$

(e) $\widetilde{C}_{-1}K^{000} \otimes \langle 1 \rangle \xleftarrow{\mathcal{O} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}} \widetilde{C}_0K^{011} \otimes \langle yz \rangle$

(f) $\widetilde{C}_{-1}K^{\{-2\}11} \otimes \langle x^{-2}yz \rangle \xleftarrow{\mathcal{O} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}} \widetilde{C}_0K^{011} \otimes \langle yz \rangle$.

There are also six different nonzero sylvan morphisms from $\widetilde{C}_1$ to $\widetilde{C}_0$:

(a) $\widetilde{C}_0K^{002} \otimes \langle z^2 \rangle \xleftarrow{\begin{bmatrix} xy & yx & xz \\ x \begin{bmatrix} 1/3 & 2/3 & 0 \\ y \begin{bmatrix} 1/3 & -1/3 & 0 \\ z \begin{bmatrix} -2/3 & -1/3 & 0 \end{bmatrix} \end{bmatrix} \end{bmatrix}} \widetilde{C}_1K^{012} \otimes \langle yz^2 \rangle$

(b) $\widetilde{C}_0K^{(-1)\{12\}} \otimes \langle x^{-1}yz^2 \rangle \xleftarrow{\begin{bmatrix} xy & yx & xz \\ x \begin{bmatrix} 0 & 1/3 & -1/3 \\ y \begin{bmatrix} 0 & -2/3 & -1/3 \\ z \begin{bmatrix} 0 & 1/3 & 2/3 \end{bmatrix} \end{bmatrix} \end{bmatrix}} \widetilde{C}_1K^{012} \otimes \langle yz^2 \rangle$

(c) $\widetilde{C}_0K^{011} \otimes \langle yz \rangle \xleftarrow{\begin{bmatrix} xy & yx & xz \\ x \begin{bmatrix} -1/3 & 0 & -2/3 \\ y \begin{bmatrix} 2/3 & 0 & 1/3 \\ z \begin{bmatrix} -1/3 & 0 & 1/3 \end{bmatrix} \end{bmatrix} \end{bmatrix}} \widetilde{C}_1K^{012} \otimes \langle yz^2 \rangle$
In the staircase diagrams, the relevant Koszul simplicial complexes are drawn (in color, where available), with the source degree at the center of the simplicial complex and the target degrees labeled with letters corresponding to the order in which the sylvan morphisms are listed. In the first list of six, each Koszul simplicial complex accounts for two sylvan morphisms that have the same source degree and different target degrees. In the second list of six, each Koszul simplicial complex accounts for three sylvan morphisms that have the same source degree and different target degrees.

All other nonzero sylvan morphisms are translations by lattice points in $L$ of these twelve sylvan morphisms. By Proposition 4.3, all nonzero sylvan matrices are identical to one of these twelve matrices, the difference residing only in their $\mathbb{Z}^3$-graded degrees.

For instance, the sylvan morphism

$$\tilde{C}_0 K^{011} \otimes \langle yz \rangle \rightarrow \tilde{C}_1 K^{031} \otimes \langle y^3 z \rangle$$

agrees with the sylvan morphism (c) in the second group of six, namely

$$\tilde{C}_0 K^{(-1)31} \otimes \langle x^{-1} y^3 z \rangle \rightarrow \tilde{C}_1 K^{031} \otimes \langle y^3 z \rangle.$$
The $L$-invariance of these sylvan morphisms implies that the sylvan resolution of $M_L$ as an $R$-module is endowed with the additional structure of an $R[L]$-free resolution of $M_L$. Picking a representative for each relevant coset of $L$ in $\mathbb{Z}^3/L$ yields the resolution

$$
\begin{align*}
\tilde{C}_0K^{002} & \oplus \tilde{C}_-1K^{000} \xleftarrow{\varphi_1^1} \tilde{C}_0K^{\{−1\}12} \oplus \tilde{C}_1K^{012} \\
& \oplus \tilde{C}_0K^{011} \xleftarrow{\varphi_1^2} \tilde{C}_1K^{031}
\end{align*}
$$

of $M_L$ over $R[L]$, where $\varphi_1$ and $\varphi_2$ conglomerate the previous sylvan data.

Why conglomeration? Suppose that $\alpha$ and $\beta$ are the source and target lattice points of a sylvan morphism for $M_L$. If $\alpha' \in \alpha + L$ and $\beta$ are the source and target of another sylvan morphism for $M_L$, with target degree $\alpha' \neq \alpha$ but the same source degree $\beta$, then the relevant $R$-free modules generated in degrees $\alpha$ and $\alpha'$ are independent of one another, but modulo the action of $L$ they become identified. For instance, the two sylvan morphisms (a) and (b) from $\tilde{C}_0$ to $\tilde{C}_-1$ in the first group of six have the same source degree $\beta = 120$ but different target degrees $\alpha = 000$ and $\alpha' = 12\{-2\}$ that become identified modulo $L$ because $\alpha' - \alpha = \alpha' \in L$. (It is worth checking this in the staircase diagram.) Translating this setup by $L$, the same situation can be viewed as letting $\alpha$ remain the same while taking $\beta \neq \beta' = \beta + L$.

Using variables $u, v, w$ for the Laurent $L$-monomials in $R[L]$ corresponding to $x, y, z$ in $R$, the lattice module $M_L$ is isomorphic, as an $R[L]$-module, to the quotient

$$
R[L]\big/\langle x^{a_1}y^{a_2}z^{a_3} - x^{b_1}y^{b_2}z^{b_3}, a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle = \langle (a_1, a_2, a_3), (b_1, b_2, b_3) \in \mathbb{N}^3 \text{ and } (a_1, a_2, a_3) - (b_1, b_2, b_3) \in L \rangle.
$$

The homological degree 0 syzygy module

$$
im(\varphi_1) = \langle \frac{w^2}{uw^2}xy^2 - z^2, \frac{vw}{u^2}xz - \frac{w^2}{uv^2}y^3, yz - \frac{vw}{u^2}x^2 \rangle
$$

can be read from the first group of six sylvan morphisms of the $R$-module $M_L$. For instance, we consider the sylvan morphisms from $\beta = 002$ to $\alpha = 000$ and $\alpha' = \{-1\}\{-1\}2$ which are the translations of the first two morphisms $\tilde{C}_-1 \leftarrow \tilde{C}_0$ in the list. We pick a generator for $\tilde{H}_0K^{002}$ and it can be either $x - z$ or $y - z$. Here $x, y, z$ are the 0-faces of the Koszul complex not the variables in the polynomial ring. Two sylvan morphisms are listed here and we look at the images of the $x - z = [1 \ 0 \ 1]^T$ in each morphism

$$
\begin{align*}
\tilde{C}_-1K^{000} \otimes (1) & \xleftarrow{x \frac{w^2}{uw^2}} \tilde{C}_-1K^{\{-1\}2} \otimes \frac{w^2}{uw^2} \xleftarrow{\varphi_2} \tilde{C}_0K^{002} \otimes \frac{w^2}{uw^2}xy^2 \\
\varnothing \otimes \frac{w^2}{uv^2}xy^2 & \xleftarrow{1 \ 1 \ 0} \varnothing \otimes x^2y^2 \xleftarrow{w^2} (x - z) \otimes \frac{w^2}{uw^2}xy^2
\end{align*}
$$
\[ \tilde{C}_{-1}K^{000} \otimes (1) \leftarrow \mathbb{C} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tilde{C}_0K^{002} \otimes (z^2) \]
\[ -\mathbb{C} \otimes z^2 \leftarrow (x - z) \otimes z^2 \]

which correspond to the first generator of syzygy module. Applying the functor \( \pi_L \) from \( \mathbb{Z}^3 \)-graded \( R[L] \)-modules to \( \mathbb{Z}^3/L \)-graded \( R \)-modules [BS98, Corollary 3.3] (see also [MS05, Theorem 9.17]) yields the resolution of the lattice ideal \( I_L \) constructed in [MS05, Example 9.21]. We write down the resolutions of \( M_L \) as \( R[L] \)-module

\[
0 \leftarrow R[L] \leftarrow \left[ \begin{array}{ccc}
\frac{w^2}{u^2}xy^2 - z^2 & \frac{uv}{u^2}xz - \frac{w^2}{u^2}y^3 & yz - \frac{uv}{u^2}x^2
\end{array} \right] R[L]^3 \leftarrow \left[ \begin{array}{c}
y \\
x \\
0
\end{array} \right] R[L] \leftarrow 0
\]

and resolution of \( R/I_L \) as \( R \)-module

\[
0 \leftarrow R \leftarrow \left[ \begin{array}{ccc}
xy^2 - z^2 & xz - y^3 & yz - x^2
\end{array} \right] R^3 \leftarrow \left[ \begin{array}{c}
y \\
x \\
0
\end{array} \right] R^2 \leftarrow 0.
\]

**Remark 5.1.** The fact that the \( L \)-equivariant free resolution of \( M_L \) in this section—and indeed, in Theorem 4.5—is sylvan is largely irrelevant to the main point of the paper: any canonical minimal \( R \)-free \( \mathbb{Z}^n \)-graded resolution of the lattice module \( M_L \) is automatically \( L \)-equivariant, so the quotient functor of Bayer and Sturmfels [BS98] yields a similarly canonical \( \mathbb{Z}^n/L \)-graded minimal \( R \)-free resolution of the lattice ideal \( I_L \).

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