Coulomb corrections to the bremsstrahlung and electron pair production cross section of high-energy muons on extended nuclei

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Abstract

The energy reconstruction of high-energy muons depends on the energy loss characteristics. Accurate knowledge of the cross sections of the energy loss processes is necessary for precise measurements of the energy spectrum of muons and muon-induced neutrinos.

The cross sections of the two most dominant processes, electron pair production and bremsstrahlung are calculated exactly in the coupling parameter $Z\alpha$ to the electromagnetic field of a nucleus for realistic extended screened nuclei. An analytical parametrization of the mass and nuclear-charge dependence of the cross section is given.

Keywords: muon cross sections, pair production, bremsstrahlung, Coulomb corrections, QED

1. Introduction

The energy reconstruction of high-energy muons is a central task in cosmic-ray and neutrino astronomy experiments. The energy is reconstructed based on the energy loss. The dominant energy loss processes of high-energy muons are electron pair production \cite{1,2,3}, bremsstrahlung \cite{4,5}, and inelastic nuclear interaction \cite{6,7,8}. The energy is lost stochastically, however the average energy lost per distance is well described by a quasi-linear function of the energy

$$\frac{dE}{dx} = a(E) + b(E)E,$$

(1)
where $a, b$ only weakly depend on the energy. The uncertainties of the cross sections influence the systematic uncertainties of the energy reconstruction [9].

Currently used parametrizations of the pair production and bremsstrahlung cross section are calculations in the Born approximation which take into account the screening of the nucleus by atomic electrons [1][10], the effect of the extended nucleus [2][10][4] and the contribution of atomic electrons as target particles [3][5].

The effect of higher-order corrections in the nuclear coupling constant $Z \alpha$, where $Z$ is the nuclear charge and $\alpha$ the fine structure constant, so-called Coulomb corrections, has been considered for pair production in [11][12] for a point-like nucleus. However, in [11], it was pointed out that the effect of a form factor can be sizeable. In this article, the corrections are calculated for a realistic nuclear charge distribution. This process has been considered recently in [13] in the quasiclassical approximation, neglecting the nuclear form factor, but taking into account higher-order corrections to the interaction of the initial charged particle with the nucleus, which will be neglected in the following, because only muons are considered as initial charged particle, while in [13] emphasis was put also on ions as initial particles. In this article, the correction to the spectrum of secondary particles and also the average energy loss of the muon is calculated.

The effect of Coulomb corrections on the bremsstrahlung cross section has been calculated in [14] for electrons and was shown to be large (from $\sim 1\%$ for medium nuclei such as iron to $\sim 10\%$ for heavy nuclei such as uranium). In [15] Coulomb corrections for muon bremsstrahlung on extended nuclei were calculated in the approximation of a homogeneously charged sphere for the nuclear charge density and were found to be small ($\lesssim 0.5\%$). This process was also considered more recently in [16], in whose approximation the correction for muons vanishes identically, independently of the form of the nuclear potential.
2. Higher order corrections in $Z\alpha$ for a point-like nucleus

The coupling to the field of the nucleus is governed by the coupling constant $\nu = Z\alpha$, which for high $Z$ can achieve values which are not very small compared to 1, and therefore should be treated non-perturbatively. The first work on this subject was [14], where the corrections to the cross sections of bremsstrahlung and pair production by real photons were calculated using wave functions which are approximate solutions of the Dirac equation for a Coulomb field. The final result is in the case of pair production by a real photon the expression

$$\frac{d\sigma}{dx} = \frac{4}{3} Z^2 \alpha^2 \varepsilon \left( 1 + 2x^2 + (1 - x)^2 \right) \left( \ln \frac{2x(1-x)}{\varepsilon/m} - \frac{1}{2} - f(Z\alpha) \right),$$

where $x = \epsilon_+ / \omega$ is the ratio of the initial photon energy $\omega$ and the positron energy $\epsilon_+$. It is very difficult to extend this treatment to a more realistic description of the nucleus as the wave functions would have to be determined for the given potential. The term $\ln[2x(1-x)/\varepsilon/m] - 1/2$ arising from the leading order calculation is called the main logarithm in the following[1] the term $f(Z\alpha)$ arising from higher-order corrections in the coupling to the nuclear field is called Coulomb correction.

In [11], this result was obtained again in a much simpler way by resummation of the perturbation series. Moreover, the approach in [11] allows for the inclusion of realistic atomic and nuclear form factors. In [17] this approach was applied to the problem of electron-positron photoproduction in the field of a screened nucleus. In [12], the results of [11] were used to calculate the Coulomb correction to the pair production cross section by high-energy muons on a pointlike nucleus. First, the calculations of [11] are briefly reviewed and then this treatment is extended to calculate the corrections for a screened extended nucleus which allows also to determine the Coulomb correction to muon bremsstrahlung.

\footnote{For an atomic field different from the Coulomb case, the main logarithm changes also.}
The main contribution to the cross section arises from small scattering angles, therefore the momenta (cf. Fig. 1) are expressed in Sudakov variables \[ k_i = \alpha_i \tilde{p}_1 + \beta_i \tilde{p}_2 + k_{i\perp}, \]
\[ q_i = x_i \tilde{p}_1 + y_i \tilde{p}_2 + q_{i\perp}, \]
where \( \tilde{p}_1 = p_1 + \frac{Q^2}{s} p_2, \tilde{p}_2 = p_2 - \frac{m^2}{s} p_1 \) are almost light-like vectors, \( Q^2 = -p_2^2 \) is the virtuality of the photon and \( s = 2p_1 p_2 \gg Q^2, m^2 \) is the center of momentum energy. As a simplification, the mass of the nucleus and of the produced lepton are set equal \( p_2^2 = q_1^2 = q_2^2 = m^2 \). The mass of the nucleus does not enter the final result, where its mass is considered infinite. Denoting again by \( x \) the fraction of the energy of the initial photon \( \omega \) which is transferred to the antilepton,
\[ x_1 = x, \quad x_2 = 1 - x, \]
\[ y_1 = \frac{m^2 + q_1^2}{xs}, \quad y_2 = \frac{m^2 + q_2^2}{(1-x)s} \]
with \( q_i^2 = -q_{i\perp}^2 \). The amplitude for the diagram with \( N \) exchanged photons is given in the impact representation [19] by
\[ \mathcal{M}_N = \frac{8\pi^2 s (-i)^{N-1}}{N!} \int \prod_{i=1}^{N} \frac{d^2 k_i}{(2\pi)^2 k_i^2} \delta \left( \sum_{j=1}^{N} k_j - q_1 - q_2 \right) J_{\gamma \gamma \rightarrow \ell\bar{\ell}}^N J_{\ell A}^N. \]
The impact factors are given by
\[ J^N_{\gamma \to \ell \bar{\ell}} = \int \prod_{i=1}^{N-1} \left( \frac{d(\beta_i s)}{2\pi i} \right) (iA)_{\mu_1 \ldots \mu_N} \frac{\hat{p}_2^{\mu_1} \ldots \hat{p}_2^{\mu_N}}{s^N}, \] (5)
\[ J^N_A = \int \prod_{i=1}^{N-1} \left( \frac{d(\alpha_i s)}{2\pi i} \right) (iB)_{\mu_1 \ldots \mu_N} \frac{\hat{p}_1^{\mu_1} \ldots \hat{p}_1^{\mu_N}}{s^N}. \] (6)

where \((iA)_{\mu_1 \ldots \mu_N}\) is the amplitude corresponding to the upper part of the diagram in Fig. 1 and \((iB)_{\mu_1 \ldots \mu_N}\) to the lower part.

For an infinitely heavy point nucleus, the impact factor is given by
\[ J^N_A = i(-1)^N (eZ)^N. \] (7)

Accounting for an extended nucleus can be either carried out by modifying the impact factor or equivalently by modifying the Coulomb propagator \(1/k^2_N\) in (4).

The impact factors for the lepton part of the diagram can be determined by a recurrence relation. The impact factor for one exchanged photon is given by
\[ J^1_{\gamma \to \ell \bar{\ell}}(q_1, q_2) = ie^2 u_1 [m \hat{e} S^1 - 2x(T^1 e) - T^1 \hat{e} \hat{p}_2] u_2 \] (8)
for a transversely polarized photon with polarization vector \(e\) and for a longitudinally polarized photon by
\[ J^1_{\gamma \to \ell \bar{\ell}}(q_1, q_2) = -ie^2 \sqrt{Q^2 x(1-x)} S^1(q_1, q_2) u_1 \hat{p}_2 u_2, \] (9)
where
\[ S^1 = S^1(q_1, q_2) = \frac{1}{\mu^2 + q_1^2} - \frac{1}{\mu^2 + q_2^2}; \] (10)
\[ T^1 = T^1(q_1, q_2) = \frac{q_1}{\mu^2 + q_1^2} + \frac{q_2}{\mu^2 + q_2^2}; \] (11)
\[ \mu^2 = m^2 + Q^2 x(1-x). \] (12)

The scalar \(S^N\) and vector \(T^N\) structures are related by the recurrence relations
\[ S^N(q_1, q_2, k_N) = S^{N-1}(q_1, q_2 - k_N) - S^{N-1}(q_1 - k_N, q_2), \] (13)
\[ T^N(q_1, q_2, k_N) = T^{N-1}(q_1, q_2 - k_N) - T^{N-1}(q_1 - k_N, q_2). \] (14)
because due to Bose symmetry the $N$-th $t$-channel photon can be considered as the last one attached to the lepton line, from which the relations follow immediately [20]. The dependence on the other $t$-channel photon momenta $k_1, \ldots, k_{N-1}$ is omitted for clarity. The integral over the $t$-channel momenta

$$J^N_S(q_1, q_2) = \int \prod_{i=1}^{N} \frac{d^2k_i}{k_i^2} F(k_i) \delta \left( \sum_{j=1}^{N} k_j - q \right) S^N$$

with the formfactor $F(k)$ can be recast using the recurrence relations as

$$J^N_S(q_1, q_2) = \int \frac{d^2k}{k^2} F(k) [J^N_{S-1}(q_1, q_2 - k) - J^N_{S-1}(q_1 - k, q_2)]$$

such that for the Fourier transform of $J^N_S(q_1, q_2)$

$$J^N_S(r_1, r_2) = \frac{1}{(2\pi)^2} \int d^2q_1 d^2q_2 J^N_S(q_1, q_2) e^{-i\vec{q}_1 \cdot \vec{r}_1 + i\vec{q}_2 \cdot \vec{r}_2}$$

the recurrence relation assumes the form

$$J^N_S(r_1, r_2) = J^{N-1}_S(r_1, r_2) \pi \phi(r_1, r_2)$$

$$\phi(r_1, r_2) = \frac{1}{\pi} \int \frac{d^2k}{k^2} (e^{ikr_2} - e^{ikr_1})$$

Using the Fourier transform of $J^1_S$

$$J^1_S(r_1, r_2) = \frac{1}{2} K_0(\mu|\vec{r}_1 - \vec{r}_2|) \phi(r_1, r_2),$$

the total impact factor to all orders, inverting the Fourier transform, is given by

$$J_S(q_1, q_2) = \frac{i}{(2\pi)^2 2\nu} \int d^2r_1 d^2r_2 e^{-i\vec{q}_1 \cdot \vec{r}_1 - i\vec{q}_2 \cdot \vec{r}_2}$$

$$\times K_0(\mu|\vec{r}_1 - \vec{r}_2|)[e^{-i\nu \phi(\vec{r}_1, \vec{r}_2)} - 1],$$

and analogously for the vector structure by

$$J_T(q_1, q_2) = -\frac{1}{(2\pi)^2 2\nu} \int d^2r_1 d^2r_2 e^{-i\vec{q}_1 \cdot \vec{r}_1 - i\vec{q}_2 \cdot \vec{r}_2}$$

$$\times \frac{\mu(|\vec{r}_1 - \vec{r}_2|)}{|\vec{r}_1 - \vec{r}_2|} K_1(\mu|\vec{r}_1 - \vec{r}_2|)[e^{-i\nu \phi(\vec{r}_1, \vec{r}_2)} - 1].$$

Here $K_\nu(z)$ is the modified Bessel function. To obtain the amplitude out of the impact factor, according to [21] the impact factor is multiplied by a universal
phase factor and the amplitude is given by
\[ M = 8\pi e\nu s \left( \frac{x}{1-x} \right)^{-iv} \tilde{u}_1 \{ m\hat{e}J_S(q_1, q_2) \]
\[ - 2x\{ J_T(q_1, q_2)|e| - \tilde{J}_T(q_1, q_2)|e| \} \frac{\hat{p}_2}{s} u_2 \]
\[ (22) \]
for a transversely polarized incident photon and by
\[ M = -16\pi e\nu s \left( \frac{x}{1-x} \right)^{-iv} \sqrt{Q^2}x(1-x)\tilde{u}_1 J_S(q_1, q_2) \frac{\hat{p}_2}{s} u_2 \]
\[ (23) \]
for a longitudinally polarized incident photon.

The total cross section is obtained by integration over the transversal momenta \( q_1, q_2 \) and the energy fraction \( x \) as
\[ d\sigma = \frac{2\nu^2\alpha}{\pi^2} \left\{ m^2|J_S|^2 + |J_T|^2[x^2 + (1-x)^2] \right\} dx \ d^2q_1 \ d^2q_2 \]
\[ (24) \]
for the transversely polarized photon, summed over all polarization states. To obtain the Coulomb correction to the Born cross section, the Born approximation cross section has to be subtracted. Therefore the correction \( d\sigma_2 \), for \( d\sigma = d\sigma_1 + d\sigma_2 \) with \( d\sigma_1 \) the Born approximation cross section, is given for the transversely and longitudinally polarized photon by
\[ \frac{d\sigma_T}{dx} = \frac{2\nu^2\alpha}{\pi^2} \left\{ m^2A_1 + [x^2 + (1-x)^2]A_2 \right\}, \]
\[ (25) \]
\[ \frac{d\sigma_S}{dx} = \frac{2\nu^2\alpha}{\pi^2} 4Q^2x^2(1-x)^2A_1, \]
\[ (26) \]
respectively, where
\[ A_1 = \int d^2q_1 \ d^2q_2 (|J_S|^2 - |J_S^1|^2), \]
\[ (27) \]
\[ A_2 = \int d^2q_1 \ d^2q_2 (|J_T|^2 - |J_T^1|^2). \]
\[ (28) \]

3. Higher-order corrections for an extended screened nucleus

In the calculation of corrections in a Coulomb field, the expressions for \( A_1, A_2 \) contain terms which diverge and have to be regularized, which leads to not well-defined expressions when attempting a numerical integration. As pointed out by [17], the divergences are removed when screening is taken into account.
Using the form factor \[ F(k) = F_n(k) - F_e(k), \]
\[ F_n(k) = \left(1 + \frac{a^2 k^2}{12}\right)^{-2}, \quad a = (0.58 + 0.82 A^{1/3}) \cdot 5.07 \text{GeV}^{-1} \]
\[ F_e(k) = \frac{1}{1 + b^2 k^2}, \quad b = \frac{184.15 Z^{-1/3}}{m_e \sqrt{\alpha}}, \]
\[ \phi is given by \]
\[ \phi(r_1, r_2) = \frac{1}{\pi} \int \frac{d^2 k}{k^2} F(|k|) (e^{ikr_2} - e^{ikr_1}) \]
\[ = 2[K_0(\Lambda_e r_2) - K_0(\Lambda_e r_1)] + 2[K_0(\Lambda_n r_1) - K_0(\Lambda_n r_2)] + \Lambda_n r_1 K_1(\Lambda_n r_1) - \Lambda_n r_2 K_1(\Lambda_n r_2), \]
\[ \Lambda_e = \frac{1}{b}, \quad \Lambda_n = \frac{\sqrt{12}}{a}. \]

The quantities \(A_1, A_2\) are given by the expressions
\[ A_1 = \frac{\pi}{2 \nu^2 \mu^4} \int_0^{\infty} dx \int_0^{\infty} dR \int_0^{2\pi} d\theta x^3 K_0^2(x) \{2 - 2 \cos(\nu \phi_{12}) - \nu^2 \phi_{12}^2\}, \]
\[ A_2 = \frac{\pi}{2 \nu^2 \mu^2} \int_0^{\infty} dx \int_0^{\infty} dR \int_0^{2\pi} d\theta x^3 K_1^2(x) \{2 - 2 \cos(\nu \phi_{12}) - \nu^2 \phi_{12}^2\}, \]
\[ \phi_{12} = \phi \left( \frac{x R}{\mu}, \frac{x R^2 + 1 - 2 R \cos \theta}{\mu} \right). \]

In the case of a Coulomb field, \( \phi = \ln(r_1^2/r_2^2) \) and \( A_1, A_2 \) assume the values
\[ A_1^C = -\frac{2\pi^2}{3\mu^4} f(\nu), \]
\[ A_2^C = -\frac{4\pi^2}{3\mu^2} f(\nu), \]
\[ f(\nu) = \frac{1}{2} \{\Psi(1 - i\nu) + \Psi(1 + i\nu) - 2\Psi(1)\} \]
\[ = \nu^2 \sum_{n=1}^{\infty} \frac{1}{n(n^2 + \nu^2)}. \]

When realistic form factors are employed, it is no longer possible to evaluate the Coulomb corrections in closed form. The numerical results can be approximated.
by

\[ A_1 = A_1^C g_1(\mu/\text{MeV}, \nu), \quad A_2 = A_2^C g_2(\mu/\text{MeV}, \nu), \]

\[ g_i(x, \nu) = \frac{a_i(\nu) + b_i(\nu)x}{1 + c_i(\nu)x + d_i(\nu)x^2}, \]

where \(a_i, b_i, c_i, d_i\) are approximately cubic polynomials for \(Z > 5\)

\[ a_1(\nu) = 1.0026 - 2.2789 \times 10^{-2}\nu + 2.9437 \times 10^{-2}\nu^2 - 4.1536 \times 10^{-2}\nu^3, \]
\[ b_1(\nu) = 1.9465 \times 10^{-2} - 7.7063 \times 10^{-2}\nu + 1.9979 \times 10^{-1}\nu^2 - 1.4107 \times 10^{-1}\nu^3, \]
\[ c_1(\nu) = 3.6785 \times 10^{-2} + 5.4466 \times 10^{-2}\nu - 9.2971 \times 10^{-2}\nu^2 + 2.7357 \times 10^{-2}\nu^3, \]
\[ d_1(\nu) = 9.9382 \times 10^{-4} + 2.4601 \times 10^{-3}\nu + 2.6733 \times 10^{-3}\nu^2 - 2.8198 \times 10^{-3}\nu^3; \]
\[ a_2(\nu) = 1.0046 - 1.9267 \times 10^{-2}\nu + 4.5255 \times 10^{-2}\nu^2 - 5.1603 \times 10^{-2}\nu^3, \]
\[ b_2(\nu) = 8.8223 \times 10^{-3} - 5.2931 \times 10^{-2}\nu + 1.4854 \times 10^{-1}\nu^2 - 1.0764 \times 10^{-1}\nu^3, \]
\[ c_2(\nu) = 3.7141 \times 10^{-2} + 1.2897 \times 10^{-1}\nu - 2.2677 \times 10^{-1}\nu^2 + 1.0776 \times 10^{-1}\nu^3, \]
\[ d_2(\nu) = 7.1444 \times 10^{-4} + 1.7710 \times 10^{-3}\nu + 5.0240 \times 10^{-3}\nu^2 - 4.5527 \times 10^{-3}\nu^3. \]

Since the correction is small for low \(Z\), it is possible to use this parametrization for all \(Z\).

This Coulomb correction for the virtual photon pair production can be used to calculate several cross sections. Setting \(Q^2 = 0, \mu^2 = m^2\), one obtains the corrections for real photoproduction of particles with mass \(m\) on a screened extended nucleus. The numerical examples show that for electrons, the result of \[14\] is reproduced with a small correction for heavy nuclei (see Fig. 2), while for muons the correction due to multiphoton exchange is very small (see Fig. 3).

Since the main logarithm assumes the value \(\ln[BZ^{-1/3}(m_\mu/m_e)] - \ln(1.54\times 10^{0.27})\) with \(B \approx 183\) \[4\] in the full-screening limit, the correction to the energy loss spectrum due to Coulomb corrections is negligible with very high accuracy

\[ \max_Z \frac{f(\nu)g_{1,2}(m_\mu, \nu)}{\ln \left( B\frac{m_\mu}{m_e} Z^{-1/3} \right) - \ln(1.54\times 10^{0.27})} < 0.004. \]

The small influence of the nuclear form factor on electrons and the smallness of the corrections for heavy particles was already observed by \[11\] in the limiting
Figure 2: Correction to the main logarithm of bremsstrahlung and photoproduction for electrons due to Coulomb corrections for a Coulomb field and a screened nucleus. Shown are the correction $f(\nu)$ for a Coulomb field (solid line) and the corrections which account for the screened nucleus $g_1(m_e, \nu)f(\nu)$ (dashed line), $g_2(m_e, \nu)f(\nu)$ (dotted line).
Figure 3: Correction to the main logarithm of bremsstrahlung and photoproduction for muons due to Coulomb corrections for a Coulomb field and a screened nucleus. Shown are the correction $f(\nu)$ for a Coulomb field (solid line) and the corrections which account for the screened nucleus $g_1(m_\mu, \nu) f(\nu)$ (dashed line), $g_2(m_\mu, \nu) f(\nu)$ (dotted line).
cases $\Lambda \gg m$ for electrons and $\Lambda \ll m$ for muons, using a nuclear form factor

$$F(k) = \frac{\Lambda^2}{\Lambda^2 + k^2}.$$ (35)

From the corrections to the real photoproduction cross section, the corrections to the bremsstrahlung cross section are obtained via the substitution rules $\epsilon_+ \to -\epsilon_1, \epsilon_- \to \epsilon_2, \omega \to -\omega$, $d\sigma \to (\omega^2 d\omega / \epsilon_+^2 d\epsilon_+) d\sigma$, where $x = \epsilon_+ / \omega$ (e.g., [23]). Again, the classical result for electrons is obtained, that the function $f(\nu)$ is subtracted from the main logarithm, and it is observed that the correction for muon bremsstrahlung is small, as was found in [15] for a simplified nuclear form factor.

Using the result for the process of pair production by a virtual photon, one can calculate the Coulomb corrections to the cross section of pair production by a charged particle, thus generalizing the corrections calculated by [12] for pair production in a Coulomb field. The correction to the cross section for pair production by a muon is given by

$$d\sigma_2 = d\sigma_T(\omega, Q^2)\sigma_T^2(\omega, Q^2) + d\sigma_S(\omega, Q^2)\sigma_S^2(\omega, Q^2),$$ (36)

where the virtual photon fluxes are given by [23]

$$d\sigma_T(\omega, Q^2) = \frac{\alpha}{\pi} (1-v) \left( 1 - \frac{Q_{\min}^2}{Q^2} + \frac{v^2}{2(1-v)} \right) \frac{d\omega}{\omega} \frac{dQ^2}{Q^2}.$$ (37)

$$d\sigma_S(\omega, Q^2) = \frac{\alpha}{\pi} (1-v) \frac{d\omega}{\omega} \frac{dQ^2}{Q^2},$$ (38)

$$Q_{\min}^2 = \frac{m^2_{\mu} v^2}{1-v} \leq Q^2 < \infty.$$ (39)

where $v = \omega / E_{\mu}$ is the fractional energy loss of the muon. Since $\sigma_T^2, \sigma_S^2$ are independent of $\omega$ and $d\sigma_T, d\sigma_S, Q_{\min}^2$ only depend on the fractional energy loss, the correction itself is independent of the incident muon energy, because the singularity for $x \to 0, x \to 1$ is only logarithmic and therefore integrable. Since the contribution in Born approximation is dependent on energy, however, the relative importance of the correction is a function of the energy. Also, the integration over $x$ should only be carried out in the range where the Born contribution is non-negative. The influence of Coulomb corrections on the differential
Figure 4: Differential cross section \( \frac{d\sigma}{dv} \) for a muon of 100 TeV primary energy in standard rock. Shown are the cross section in Born approximation [2] (solid line), our Coulomb corrections (dashed line), and the corrections of [12] (dotted line).

The influence of Coulomb corrections on the average energy loss

\[
- \frac{dE}{dX} = \frac{N_A}{A} \int \omega \frac{d\sigma}{d\omega} d\omega, \tag{40}
\]

where \( N_A \) is Avogadro’s constant, \( A \) is the mass number of the material, and \( X = x/\rho \) is the depth, is shown in Fig. 6 for standard rock \(^2\) and in Fig. 7 for lead, integrated in the appropriate energy-dependent limits of the Born approximation cross section of [2].

\(^2\)Standard rock is assumed as a mixture of MgCO\(_3\) and CaCO\(_3\) consisting of 52% oxygen, 27% calcium and 9% magnesium.
Figure 5: Differential cross section $d\sigma/dv$ for a muon of 100 TeV primary energy in lead. Shown are the cross section in Born approximation [2] (solid line), our Coulomb corrections (dashed line), and the corrections of [12] (dotted line).
Figure 6: Average energy loss through pair production in standard rock, calculated using the Born cross section of \[2\] (solid line) and the negative Coulomb corrections calculated in this work (dashed line) and in \[12\] (dotted line).
Figure 7: Average energy loss through pair production in lead, calculated using the Born cross section of [2] (solid line) and the negative Coulomb corrections calculated in this work (dashed line) and in [12] (dotted line).
4. Discussion

We have calculated Coulomb corrections to the cross sections of pair production and bremsstrahlung on extended screened nuclei. These calculations generalize the work of [11, 12, 17] with regard to pair production and the work of [15] with regard to muon bremsstrahlung.

Our results confirm that the Coulomb corrections to the muon bremsstrahlung cross section are negligible with very high accuracy. This coincides qualitatively with the results of [15], who applied a very simple model for the charge distribution of the nucleus and used a different method based on wave functions. However, here a more realistic charge distribution was used; therefore a direct comparison of the numerical results is difficult. In contrast to the results of [16], the corrections do not vanish identically in our calculation.

Our results on electron pair production by high-energy muons confirm the importance of Coulomb corrections established by [12] for this process in precise calculations of muon transport. Our calculations differ in two aspects from [12]:

- the correction in the cross section is integrated only over values of $x, v$, for which the Born cross section is positive;
- the atomic and nuclear form factor is taken into account.

The effect of the first aspect decreases with energy; however, as shown in Fig. 4, the effect of correct limits is still noticeable at a muon energy of 100 TeV in standard rock. The second point leads to an additional decrease of the Coulomb correction which does not decrease with energy. As shown in Fig. 7, for lead the correction to the energy loss is smaller by more than 10%, amounting to about a percent of the Born loss. For the differential cross section, the effect is even greater, as shown in Fig. 5.

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