Abstract. Estimating the drift parameter of the Ornstein–Uhlenbeck process is a classical problem in statistics. It can be approached in a number of ways, but only likelihood based estimators are known to attain the optimal accuracy in the large sample asymptotic regime. Construction and analysis of such estimators can be quite challenging, depending on the properties of the driving noise. Essentially the only class for which a fairly complete picture is available consists of the fractional Brownian motions (fBm). In this paper we construct the Maximum Likelihood Estimator (MLE), when the driving noise contains both standard and fractional components. Such mixtures arise in mathematical finance in the context of arbitrage modelling. We show how asymptotic analysis can be carried out in this case, using some recent results on the canonical representation and spectral structure of the mixed fBm.

1. Introduction

Estimating drift parameter $\theta \in \mathbb{R}$ from a sample path of the Ornstein–Uhlenbeck type process $X^T = (X_t, t \in [0, T])$, generated by the equation

$$X_t = X_0 + \theta \int_0^t X_s ds + V_t, \quad t \geq 0$$

(1.1)

is a prototypical problem in statistical inference of random processes. In its classical form, with the driving process $V$ being the standard Brownian motion $B = (B_t, t \in [0, T])$, this problem has been extensively studied since the late 60’s. In this case the probability measures $\mu^T_\theta$, $\theta \in \mathbb{R}$ induced by $X^T$ are equivalent and the corresponding likelihood function is given by the Girsanov theorem:

$$\frac{d\mu^T_\theta}{d\mu^T_0}(X^T) = \exp \left( \theta \int_0^T X_t dX_t - \frac{1}{2} \theta^2 \int_0^T X_t^2 dt \right).$$

(1.2)

Consequently, the MLE, being the unique maximizer of the likelihood, is given by the simple formula

$$\hat{\theta}_T = \frac{\int_0^T X_t dX_t}{\int_0^T X_t^2 dt}, \quad T > 0.$$  

(1.3)

It is known to attain the optimal asymptotic accuracy in the local minimax sense as $T \to \infty$ and its limit behaviour is determined by the sign of the drift parameter $\theta$. In the stable
case, corresponding to \( \theta < 0 \), the estimation error \( \hat{\theta}_T - \theta \) is asymptotically normal at the usual parametric rate \( \sqrt{T} \):

\[
\sqrt{T}(\hat{\theta}_T - \theta) \xrightarrow{d} N(0, 2\theta), \quad \forall \theta < 0 \quad (1.4)
\]

where the convergence is in distribution. Entirely different asymptotics emerges in the neutrally stable and unstable cases, \( \theta = 0 \) and \( \theta > 0 \). A comprehensive account of the early and more recent developments concerning the Brownian case can be found in [21], [19].

The drift estimation problem can be approached in a number of alternative ways, which typically produce reasonable estimators (see e.g. [14], [25]). However, without being based on the likelihood function, these estimators are asymptotically subefficient as \( T \to \infty \), at best up to a finite gap with respect to the information bound. Construction and analysis of the likelihood based estimators, on the other hand, requires a convenient formula for the likelihood function, which can be hard to find for a given driving process \( V \).

The MLE for \( \theta \) can be constructed, taking advantage of the Girsanov theorem, if the process \( V \) is Gaussian and it admits a canonical innovation representation, [12]. More precisely, suppose that there exists a pair of deterministic kernels \( g(s, t) \) and \( \tilde{g}(s, t) \), so that

\[
V_t = \int_0^t \tilde{g}(s, t) dM_s \quad \text{and} \quad M_t = \int_0^t g(s, t) dV_s, \quad t \geq 0 \quad (1.5)
\]

and \( M = (M_t, t \geq 0) \) is a continuous martingale with a strictly increasing quadratic variation \( \langle M \rangle = (\langle M \rangle_t, t \geq 0) \). Here the stochastic integrals are defined in some reasonable sense, e.g., through approximation by simple functions.

Then integrating the kernel \( g(s, t) \) with respect to \( X \), we obtain a semimartingale:

\[
Z_t := \int_0^t g(s, t) dX_s = \theta \int_0^t Q_s(X) d\langle M \rangle_s + M_t
\]

where

\[
Q_t(X) := \frac{d}{d\langle M \rangle_t} \int_0^t g(s, t) X_s ds.
\]

The filtrations generated by \( X \) and \( Z \) coincide, \( \mathcal{F}_t^X = \mathcal{F}_t^Z, t \geq 0 \), and by the Girsanov theorem the measures \( \mu_\theta^T, \theta \in \mathbb{R} \) are equivalent with the likelihood function of the form (cf. (1.2))

\[
\frac{d\mu_\theta^T}{d\mu_0^T}(X_T) = \exp \left( \theta \int_0^T Q_t(X) dZ_t - \frac{1}{2} \theta^2 \int_0^T Q_t(X)^2 d\langle M \rangle_t \right).
\]

The MLE is therefore given by (cf. (1.3)):

\[
\hat{\theta}_T := \frac{\int_0^T Q_t(X) dZ_t}{\int_0^T Q_t(X)^2 d\langle M \rangle_t}. \quad (1.6)
\]

The principle difficulty with this method is that the canonical representation (1.5) may not exist or can be hard to find in a suitable form for a given process \( V \). Moreover, even
if such a representation is available, it may not be immediately clear how to approach the asymptotic analysis of the estimation error

$$\hat{\theta}_T - \theta = \frac{\int_0^T Q_t(X)\,dM_t}{\int_0^T Q_t(X)^2\,d\langle M \rangle_t}. \quad (1.7)$$

Consequently likelihood based estimators have been studied only for a few processes beyond the standard Brownian motion.

This program has been realized in [17] for the Ornstein–Uhlenbeck process (1.1), driven by the fractional Brownian motion $B^H = (B^H_t, t \geq 0)$, that is, a centered Gaussian process with covariance function

$$\mathbb{E}B^H_t B^H_s = \frac{1}{2} \left( t^{2H} + s^{2H} - |t-s|^{2H} \right), \quad s, t \geq 0,$$

where $H \in (0, 1]$ is the Hurst parameter. For $H = \frac{1}{2}$ the fBm coincides with the standard Brownian motion, but otherwise has quite different properties, see e.g. [11], [22], [4]. In particular, for $H > \frac{1}{2}$ its increments exhibit long range dependence, which makes fBm an important modelling tool (see, e.g., [3]). The MLE in [17] is shown to satisfy the asymptotics (1.4), using the canonical representation for the fBm, also known in the literature as Molchan-Golosov transformation (see also [23], [15]).

A large class for which the canonical representation is available consists of mixtures

$$V_t = B_t + G_t, \quad t \geq 0 \quad (1.8)$$

where $B$ is the standard Brownian motion and $G$ is an independent Gaussian process. Such processes emerge in engineering applications, including signal detection problems (see, e.g., [16]). The kernels $g(s, t)$ and $\tilde{g}(s, t)$ in this case were constructed in [13], using solutions of certain integral equations, under the assumption that the covariance function of $G$ satisfies

$$\mathbb{E}G_t G_s = \int_0^t \int_0^s K(u, v)\,dudv \quad (1.9)$$

with a square integrable kernel

$$\int_0^T \int_0^T K(u, v)^2\,dudv < \infty. \quad (1.10)$$

A natural question in this regard is whether the asymptotics (1.4) remains intact for such processes?

In this paper we will answer this question affirmatively for the mixture with $G_t := B^H_t$, $H \in (0, 1)$, often referred in the literature as the mixed fBm. The interest in this process has been triggered by the paper [6], which revealed a number of its curious properties, useful in mathematical finance (see [7], [2]). Further related results can be found in [8], [1], [27], [5], [10].

We will show that the large sample behaviour of the MLE in (1.6) is governed in this case by a certain singularly perturbed integro–differential equation. While some elements of our approach remain applicable to more general mixtures of the form (1.8) (see Remark 2.2 below), the results also indicate that the ultimate answer to the posed question requires quite delicate estimates on its solution, which might be hard to obtain in general.
2. The main result

Consider the mixed fractional Ornstein–Uhlenbeck process (1.1) with \( V = B + B^H \), where \( B \) and \( B^H \), \( H \in (0, 1) \setminus \{ \frac{1}{2} \} \) are independent standard and fractional Brownian motions. The conditions (1.9)-(1.10) are satisfied in this case only for \( H > \frac{3}{4} \) and therefore the innovations construction from [13] cannot be used for the whole range of \( H \). Instead we will use an alternative canonical representation suggested in [5] based on the martingale \( M_t := \mathbb{E}(B_t | \mathcal{F}_t^V) \). (2.1)

To this end consider the integro–differential Wiener-Hopf type equation:

\[
g(s, t) + \frac{d}{ds} \int_0^t g(r, t) H |s - r|^{2H - 1} \text{sign}(s - r) dr = 1, \quad 0 < s \neq t \leq T. \tag{2.2}
\]

By Theorem 5.1 in [5] this equation admits a unique solution for any \( H \in (0, \frac{1}{2}) \). It is continuous on \([0, T]\), and the martingale, defined in (2.1), satisfies

\[
M_t = \int_0^t g(s, t) dV_t \quad \text{and} \quad \langle M \rangle_t = \int_0^t g(s, t) ds, \quad t \in [0, T],
\]

where the stochastic integral is defined for \( L^2(0, T) \) deterministic integrands in the usual way. By Corollary 2.9 in [5], the representation (1.5) of \( V \) in terms of \( M \) holds with

\[
\tilde{g}(s, t) := 1 - \frac{d}{d \langle M \rangle_s} \int_0^t g(r, s) dr,
\]

and consequently the MLE of \( \theta \) is given by (1.6).

**Theorem 2.1.** The MLE for the drift parameter of the stable mixed fractional Ornstein–Uhlenbeck process satisfies (1.4).

**Remark 2.2.** In the course of the proof we will show that the assertion of this theorem holds if the following two conditions are met:

\[
\int_1^\infty \left( \frac{d}{dt} \log \frac{d}{dt} \langle M \rangle_t \right)^2 dt < \infty, \tag{2.4}
\]

and

\[
\frac{1}{t} \max \left( \frac{dt}{d \langle M \rangle_t}, \frac{d \langle M \rangle_t}{dt} \right) \xrightarrow{t \to \infty} 0. \tag{2.5}
\]

This part of the proof can be carried out in a greater generality, at least for sufficiently regular \( G \) with stationary increments. The equation (2.2) in this case takes the form

\[
g(s, t) + \frac{d}{ds} \int_0^t g(r, t) \frac{d}{dr} v(s - r) dr = 1, \quad 0 < s \neq t \leq T, \tag{2.6}
\]

where \( v(t) := \mathbb{E}G_t^2 \). Assuming that it has the unique solution, such that the aforementioned construction remains valid, the conditions (2.4) and (2.5) would still guarantee asymptotic normality (1.4).
singly perturbed equation, obtained from (2.6) by an appropriate scaling. In the fBm case with \( v(t) = t^{2H} \), asymptotic analysis of this equation is carried out using spectral approximations, recently obtained in [9]. It is not clear at this point, how similar results can be derived in the more general case.

3. PROOF OF THEOREM 2.1

The proof is inspired by the approach in [18]. In view of (1.7), in order to prove (1.4) it is enough to show that (Theorem 1.19 in [19])

\[
\frac{1}{T} \int_0^T Q_t^2 \, d\langle M \rangle_t \xrightarrow{p} \frac{1}{2|\theta|}.
\]

We will check the stronger condition in terms of the Laplace transform

\[
\mathcal{L}_T(\mu) := \mathbb{E} \exp \left( -\mu \frac{1}{T} \int_0^T Q_t^2 \, d\langle M \rangle_t \right) \xrightarrow{T \to \infty} \exp \left( -\frac{\mu}{2|\theta|} \right), \quad \mu \in \mathbb{R}_+,
\]

which in addition to the weak convergence (1.4), also implies the convergence of moments.

The main difficulty in implementation of the method from [18], is that in the setup under consideration the kernels \( g(s,t) \) and \( \tilde{g}(s,t) \) do not admit explicit formulas. We will split the proof into several lemmas. Our first lemma shows that the process \( Q \) can be expressed as a stochastic integral with respect to \( Z \), whose integrand involves the derivative \( d\langle M \rangle_t / dt \). This derivative exists and is continuous by Theorem 2.4 in [5].

**Lemma 3.1.**

\[
Q_t = \int_0^t \psi(s,t) \, dZ_s,
\]

where

\[
\psi(s,t) = \frac{1}{2} \left( \frac{dt}{d\langle M \rangle_t} + \frac{ds}{d\langle M \rangle_s} \right).
\]

**Proof.** By Corollary 2.9 in [5], \( \mathcal{F}^X_t = \mathcal{F}^Z_t \) and \( X_t = \int_0^t g(s,t) \, dZ_s \) with \( \tilde{g}(s,t) \) given in (2.3). Consequently,

\[
Q_t = \frac{d}{d\langle M \rangle_t} \int_0^t g(s,t) X_s \, ds = \frac{d}{d\langle M \rangle_t} \int_0^t g(s,t) \int_r^s \tilde{g}(r,s) \, dZ_r \, ds
\]

\[
= \frac{d}{d\langle M \rangle_t} \int_0^t \left( \int_r^t g(s,t) \tilde{g}(r,s) \, ds \right) dZ_r \xrightarrow{\dagger} \int_0^t \psi(r,t) \, dZ_r,
\]

with

\[
\psi(r,t) := \frac{d}{d\langle M \rangle_t} \left( \int_r^t g(s,t) \tilde{g}(r,s) \, ds \right),
\]

where the equality \( \dagger \) holds, since the integrand vanishes at \( r = t \). Note that \( \psi(s,t) \) does not depend on \( \theta \) and thus we can set \( \theta = 0 \) in all further calculations. For this choice \( Q_t = \int_0^t \psi(r,t) \, dM_r \) and hence

\[
\mathbb{E} Q_t \mathbb{M}_s = \int_0^s \psi(r,t) \, d\langle M \rangle_r, \quad s \leq t.
\]
Gathering all parts together, we get
\[
\psi(s, t) = \frac{\partial}{\partial (M)_s} \mathbb{E} Q_t M_s = \frac{\partial}{\partial (M)_s} \frac{\partial}{\partial (M)_t} \mathbb{E} \int_0^t g(r, t) V_r dr \int_0^s g(r, s) dV_r,
\]
and
\[
\mathbb{E} \int_0^s g(\tau, t) V_\tau d\tau \int_0^s g(r, s) dV_r = \mathbb{E} \int_0^t \int_0^s g(\tau, t) g(r, s) \frac{\partial}{\partial r} \mathbb{E} V_r V_r d\tau dr = \int_0^s \int_0^t g(r, s) g(\tau, t) \frac{\partial}{\partial r} (\tau \wedge r + \frac{1}{2} \tau^{2H} + \frac{1}{2} r^{2H} - \frac{1}{2} |r - \tau|^{2H}) d\tau dr = \int_0^s \int_0^t g(r, s) g(\tau, t) (1_{\{r \leq \tau\}} + H r^{2H-1} - H |r - \tau|^{2H-1} \text{sign}(r - \tau)) d\tau dr = \int_0^s g(r, s) \int_0^t g(\tau, t) d\tau dr + H \int_0^s g(r, s) r^{2H-1} dr \int_0^t g(\tau, t) d\tau - H \int_0^s g(r, s) \int_0^t g(\tau, t) |r - \tau|^{2H-1} \text{sign}(r - \tau) d\tau dr.
\]
Define
\[
\phi(r, t) := \int_r^t g(\tau, t) d\tau - H \int_0^t g(\tau, t) |r - \tau|^{2H-1} \text{sign}(r - \tau) d\tau,
\]
then
\[
\frac{\partial}{\partial r} \phi(r, t) = -g(r, t) - H \frac{\partial}{\partial r} \int_0^t g(\tau, t) |r - \tau|^{2H-1} \text{sign}(r - \tau) d\tau = -1
\]
and
\[
\phi(r, t) = \phi(0, t) - r = \int_0^t g(\tau, t) d\tau + H \int_0^t g(\tau, t) r^{2H-1} d\tau - r \equiv (M)_t + H (N)_t - r.
\]
Gathering all parts together, we get
\[
\psi(s, t) = \frac{\partial}{\partial (M)_s} \frac{\partial}{\partial (M)_t} \left( H (M)_t (N)_s + \int_0^s g(r, s) \left( (M)_t + H (N)_t - r \right) d\tau \right) = \frac{d(N)_s}{d(M)_s} + \frac{d}{d(M)_t} \left( (M)_t + H (N)_t \right) \frac{d}{d(M)_s} \int_0^s g(r, s) dr - \frac{\partial}{\partial (M)_s} \frac{\partial}{\partial (M)_t} \int_0^s g(r, s) r dr = \frac{d(N)_s}{d(M)_s} + H \frac{d(N)_t}{d(M)_t}.
\]
Integrating the equation (2.2) gives
\[
\int_0^t g(s, t) ds + \int_0^t \frac{d}{ds} \int_0^t g(r, t) H |s - r|^{2H-1} \text{sign}(s - r) dr ds = t,
\]
or, equivalently, 
\[
(M)_t + 2H (N)_t = t,
\]
where we used the symmetry \(g(s, t) = g(t - s, t)\). Consequently,
\[
1 + 2H \frac{d(N)_t}{d(M)_t} = \frac{dt}{d(M)_t}.
\]
and
\[
\psi(s, t) = H \frac{d(N)_s}{d(M)_s} + 1 + H \frac{d(N)_t}{d(M)_t} = \frac{1}{2} \left( \frac{dt}{d(M)_t} + \frac{ds}{d(M)_s} \right).
\]
Lemma 3.2. Assume that $\langle M \rangle_t$ is twice continuously differentiable. Then the convergence (3.1) holds under the conditions (2.4) and (2.5).

Proof. By Lemma 3.1

$$Q_t = \int_0^t \psi(r, t) dZ_r = \frac{1}{2} \psi(t, t) Z_t + \frac{1}{2} \int_0^t \psi(r, r) dZ_r.$$ 

Denote $Y_t = \int_0^t \psi(r, r) dZ_r$, then

$$dZ_t = -|\theta| Q_t d\langle M \rangle_t + dM_t = -\frac{1}{2} |\theta| \psi(t, t) Z_t d\langle M \rangle_t + \frac{1}{2} |\theta| Y_t d\langle M \rangle_t + dM_t =$$

$$-\frac{1}{2} |\theta| Z_t dt - \frac{1}{2} |\theta| Y_t \psi(t, t) dt + \frac{1}{\sqrt{\psi(t, t)}} dW_t,$$

where $W = (W_t, t \in [0, T])$ is a standard Brownian motion. Similarly,

$$dY_t = -\frac{1}{2} |\theta| \psi(t, t) Z_t d\langle M \rangle_t - |\theta| \psi(t, t) \frac{1}{2} Y_t d\langle M \rangle_t + \psi(t, t) dM_t =$$

$$-\frac{1}{2} |\theta| \psi(t, t) Z_t dt - |\theta| \frac{1}{2} Y_t dt + \sqrt{\psi(t, t)} dW_t.$$

The vector $\zeta_t = (Z_t, Y_t)^T$ solves the linear system of Itô stochastic differential equations

$$d\zeta_t = -\frac{1}{2} |\theta| A(t) \zeta_t dt + b(t) dW_t$$

with

$$A(t) = \begin{pmatrix} 1 & \frac{1}{\psi(t, t)} \\ \psi(t, t) & 1 \end{pmatrix} \quad \text{and} \quad b(t) = \begin{pmatrix} 1 \\ \frac{1}{\sqrt{\psi(t, t)}} \end{pmatrix}.$$ 

The Laplace transform in (3.1) reads

$$\mathcal{L}_T(\mu) = \mathbb{E} \exp \left( -\mu \frac{1}{T} \int_0^T Q_t^2 d\langle M \rangle_t \right) = \mathbb{E} \exp \left( -\mu \frac{1}{T^2} \int_0^T (\psi(t, t) Z_t + Y_t)^2 d\langle M \rangle_t \right) =$$

$$\mathbb{E} \exp \left( -\mu \frac{1}{T^2} \int_0^T \left( \sqrt{\psi(t, t)} Z_t + \frac{1}{\sqrt{\psi(t, t)}} Y_t \right)^2 dt \right) = \mathbb{E} \exp \left( -\mu \frac{1}{T^2} \int_0^T \zeta_t^T R(t) \zeta_t dt \right),$$

where

$$R(t) = \begin{pmatrix} \psi(t, t) & \frac{1}{\psi(t, t)} \\ \frac{1}{\psi(t, t)} & 1 \end{pmatrix}.$$ 

By the Cameron-Martin type formula (see Section 4.1 of [18]),

$$\mathcal{L}_T(\mu) = \exp \left( -\mu \frac{1}{T} \int_0^T \text{tr} \left( \Gamma(s) R(s) \right) ds \right), \quad (3.2)$$

where $\Gamma(t)$ is the solution of the Riccati equation

$$\dot{\Gamma}(t) = -\frac{|\theta|}{2} A(t) \Gamma(t) - \frac{|\theta|}{2} \Gamma(t) A(t)^T + B(t) - \frac{\mu}{2T} \Gamma(t) R(s) \Gamma(t), \quad t \in [0, T], \quad (3.3)$$

□
with \( B(t) = b(t)b(t)^T \), subject to the initial condition \( \Gamma(0) = 0 \).

The equation (3.3) has a unique positive definite solution for any \( \mu \in \mathbb{R} \) and all \( T \) large enough, which can be expressed as the ratio \( \Gamma(t) = \Phi_1^{-1}(t)\Phi_2(t) \), where \( \Phi_1(t) \) and \( \Phi_2(t) \) solve
\[
\dot{\Phi}_1(t) = \frac{\theta}{2}\Phi_1(t)A(t) + \frac{\mu}{2T}\Phi_2(t)R(t)
\]
\[
\dot{\Phi}_2(t) = \Phi_1(t)B(t) - \frac{\theta}{2}\Phi_2(t)A(t)^T,
\]
subject to \( \Phi_1(0) = I \) and \( \Phi_2(0) = 0 \). Since \( \text{tr}\ A(t) = 2 \) and
\[
\Phi_1^{-1}(t)\dot{\Phi}_1(t) = \frac{\theta}{2}A(t) + \frac{\mu}{2T}\Gamma(t)R(t)
\]
it follows that
\[
\frac{\mu}{2T}\text{tr}\ (\Gamma(t)R(t)) = \text{tr}\ (\Phi_1^{-1}(t)\dot{\Phi}_1(t)) - |\theta|.
\]
By Liouville’s formula \( \text{tr}\ (\Phi_1^{-1}(t)\dot{\Phi}_1(t)) = \frac{d}{dt}\log\det(\Phi_1(t)) \) and
\[
\frac{\mu}{2T}\int_0^T \text{tr}\ (\Gamma(t)R(t))\,dt = \log\det(\Phi_1(T)) - |\theta|T. \tag{3.5}
\]

Let us now calculate \( \log\det(\Phi_1(T)) \). To this end, define
\[
J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]
and note that \( R(t) = JA(t), B(t) = A(t)J \) and \( JA(t)J = A(t)^T \). If we set \( \Phi_2(t) := \Phi_2(t)J \) and multiply the second equation in (3.4) by \( J \) from the left, we obtain the system
\[
\dot{\Phi}_1(t) = \frac{\theta}{2}\Phi_1(t)A(t) + \frac{\mu}{2T}\Phi_2(t)A(t)
\]
\[
\dot{\Phi}_2(t) = \Phi_1(t)A(t) - \frac{\theta}{2}\Phi_2(t)A(t),
\]
subject to \( \Phi_1(0) = I \) and \( \Phi_2(0) = 0 \). The matrix
\[
\begin{pmatrix} \frac{\theta}{2} & \frac{\mu}{2T} \\ 1 & -\frac{\theta}{2} \end{pmatrix}
\]
has two real eigenvalues \( \pm \lambda_T \) with \( \lambda_T = \sqrt{\left(\frac{\theta}{2}\right)^2 + \frac{\mu}{2T}} \) and the corresponding eigenvectors
\[
v_+ = \begin{pmatrix} a_+^+ \\ 1 \end{pmatrix}, \quad v_- = \begin{pmatrix} a_-^- \\ 1 \end{pmatrix},
\]
where
\[
a_\pm^\pm = \frac{|\theta|}{2} \pm \sqrt{\left(\frac{\theta}{2}\right)^2 + \frac{\mu}{2T}}.
\]
Diagonalizing (3.6) we get
\[
\Phi_1(t) = a_+^+\Upsilon_1(t) + a_-^-\Upsilon_2(t),
\]
where
\[
\dot{\Upsilon}_1(t) = \lambda_T\Upsilon_1(t)A(t)
\]
\[
\dot{\Upsilon}_2(t) = -\lambda_T\Upsilon_2(t)A(t) \tag{3.7}
\]
subject to \( \Upsilon_1(0) = -\Upsilon_2(0) = \frac{1}{2}/(\theta/2)^2 + \frac{\mu}{2T}I \). Therefore

\[
\log \det (\Phi_1(T)) = \log \det (a_T^{-1} \Upsilon_1(t) + a_T^{-1} \Upsilon_2(t)) = \log \det (a_T^{-1} \Upsilon_1(t)) + \log \det \left( I + \frac{a_T^{-1} \Upsilon_1(t)}{a_T} \Upsilon_2(t) \right). \tag{3.8}
\]

Note that \( a_T^{-1} = |\theta| + O(T^{-1}) \) and det \( \Upsilon_1(0) = \theta^{-2} + O(T^{-1}) \) and thus

\[
\log \det (a_T^{-1} \Upsilon_1(t)) - |\theta|T := \log(a_T^{-1})^2 + \log \det \Upsilon_1(0) + \int_0^T \lambda_T \text{tr} A(t) dt - |\theta|T = 2T\lambda_T - |\theta|T + O(T^{-1}) = T \left( \sqrt{\theta^2 + \frac{2\mu}{T}} - |\theta| \right) + O(T^{-1}) = \frac{\mu}{|\theta|} + O(T^{-1}).
\]

The desired limit (3.1) now holds by (3.2) and (3.5), if we check that the last term in (3.8) vanishes:

\[
\frac{a_T}{a_T} \| \Upsilon_1^{-1}(T) \Upsilon_2(T) \|_{T \rightarrow \infty} \rightarrow 0,
\]

or, equivalently,

\[
\frac{1}{T} \| \Upsilon_1^{-1}(T) \Upsilon_2(T) \|_{T \rightarrow \infty} \rightarrow 0, \tag{3.9}
\]

since \( |a_T/a_T^+| \sim T^{-1} \) as \( T \rightarrow \infty \). Here and below we use the \( \ell^2 \) operator norm for matrices.

We will need an estimate for the growth rate of the solution of the second equation from (3.7). To this end, denote \( g_t := 1/\sqrt{\psi(t,t)} \) and fix an arbitrary vector \( v \in \mathbb{R}^2 \), then

\[
v_t := v^\top \Upsilon_2(t) \begin{pmatrix} g_t & 0 \\ 0 & 1/g_t \end{pmatrix}
\]

satisfies the ODE \( \dot{v}_t = H(t)v_t \) with the symmetric matrix

\[
H(t) = \begin{pmatrix} -\lambda_T + \dot{g}_t/g_t & -\lambda_T \\ -\lambda_T & -\lambda_T - g_t/g_t \end{pmatrix}.
\]

The maximal eigenvalue of \( H(t) \) is

\[
-\lambda_T + \sqrt{\lambda_T^2 + (\dot{g}_t/g_t)^2} \leq \frac{1}{2\lambda_T} (\dot{g}_t/g_t)^2
\]

and thus, under the assumption (2.4),

\[
\|v_T\|_2 \leq \|v_1\|_2 \exp \left( \frac{1}{2\lambda_T} \int_1^T (\dot{g}_t/g_t)^2 dt \right) \leq \|v_1\|_2 C_1, \quad T \geq 1
\]

with a constant \( C_1 \) independent of \( T \). Therefore

\[
\|\Upsilon_2(T)\| \leq C_2 \max(g_T, 1/g_T),
\]

with a constant \( C_2 \), which depends only on the initial condition \( \Upsilon_2(0) \).

Further, note that

\[
\frac{d}{dt} \Upsilon_1^{-1}(t) = -\lambda_T A(t) \Upsilon_1^{-1}(t)
\]
which under transposition and multiplication by $J$ from the right becomes

$$
\frac{d}{dt}(\Upsilon^{-\top}_1(t)J) = -\lambda_T \Upsilon^{-\top}_1(t)A^\top(t)J = -\lambda_T (\Upsilon^{-\top}_1(t)J)A(t),
$$

i.e. $\Upsilon^{-\top}_1(t)J$ and $\Upsilon_2(t)$ solve the same equation with possibly different initial conditions. Therefore

$$
\|\Upsilon^{-\top}_1(T)\Upsilon_2(T)\| \leq \|\Upsilon^{-\top}_1(T)\|\|\Upsilon_2(T)\| \leq C_3 \max(g_T^2, 1/g_T^2),
$$

with a constant $C_3$. The claim of the lemma now follows from $(3.9)$. \hfill \Box

It is left to check conditions of Lemma 3.2, which we do separately for $H > 1/2$ and $H < 1/2$.

**Lemma 3.3.** For $H > 1/2$

$$
\frac{d}{dT}(\langle M \rangle_T) \sim T^{1-2H} \quad \text{and} \quad \left( \frac{d}{dT} \log \frac{d}{dT}(\langle M \rangle_T) \right)^2 \sim T^{-2}, \quad \text{as} \ T \to \infty
$$

and thus the conditions of Lemma 3.2 hold.

**Proof.** For $H > 1/2$ the derivative and integration in (2.2) can be interchanged and it takes the form of integral equation

$$
g(s,t) + \int_0^t g(r,t)c_H|s-r|^{2H-2}dr = 1, \quad 0 < s < t \leq T,
$$

where $c_H = H(2H - 1)$. By Theorem 2.4, [5] in this case

$$
\langle M \rangle_T = \int_0^T g^2(t,t)dt,
$$

with $g(t,t) > 0$ for all $t \geq 0$.

Define small parameter $\varepsilon := T^{1-2H}$. A simple calculation shows that the function $u_\varepsilon(x) := T^{2H-1}g(xT,T)$ solves the integral equation

$$
\varepsilon u_\varepsilon(x) + \int_0^1 c_H|y-x|^{2H-2}u_\varepsilon(y)dy = 1, \quad x \in [0,1] \quad (3.10)
$$

and, moreover,

$$
\frac{d\langle M \rangle_T}{dT} = g^2(T,T) = \varepsilon^2 u_\varepsilon^2(1). \quad (3.11)
$$

For any $\varepsilon > 0$ the equation of the second kind (3.10) has a unique solution, continuous on the closed interval $[0,1]$ (see e.g. [26]). For $\varepsilon = 0$ it degenerates to the equation of the first kind, whose unique solution is known in a closed form [20]:

$$
u_0(x) = a_H x^{\frac{1}{2}-H}(1-x)^{\frac{1}{2}-H}, \quad x \in (0,1)
$$

where $a_H$ is an explicit constant. Note that $u_0$ explodes at the endpoints of the interval and therefore it is reasonable to suggest that $u_\varepsilon(1) \to \infty$ as $\varepsilon \to 0$. 

To estimate the growth of $u_\varepsilon(1)$ we will use the following asymptotic approximations for the ordered sequence of eigenvalues and the scalar products with the corresponding eigenfunctions for the integral operator in (3.11) (see Theorem 2.3, [24]): for $H > \frac{1}{2}$

$$
\begin{align*}
\lambda_n & \sim C_1 n^{1-2H} \\
\langle (\cdot)^{-\beta} , \varphi_n \rangle & \sim C_3 n^{\beta - 1} \quad n \to \infty \\
\langle 1, \varphi_{2n+1} \rangle & \sim C_2 n^{-\frac{1}{2}-H}
\end{align*}
$$

(3.12)

where $C_i$’s are nonzero constants and $\beta \in (0, 1)$. The eigenfunctions with even indices are antisymmetric around the midpoint of the interval and hence $\langle 1, \varphi_{2n} \rangle = 0$.

Taking scalar product of both sides of (3.10) and using these estimates we obtain:

$$
u_\varepsilon(1) = \varepsilon^{-1} \int_0^1 (u_0(x) - u_\varepsilon(x)) c_H (1-x)^{2H-2} dx =
$$

$$
\sum_{n \text{ odd}} \frac{1}{\lambda_n(\varepsilon + \lambda_n)} \int_0^1 \varphi_n(x) c_H (1-x)^{2H-2} dx \sim
$$

$$
\sum_{n \text{ odd}} \frac{n^{-\frac{1}{2}-H}}{\varepsilon + n^{1-2H}} \sim \int_1^{\infty} \frac{x^{-\frac{1}{2}-H}}{\varepsilon + x^{1-2H}} dx \sim \varepsilon^{-\frac{1}{2}} \int_0^{\infty} \frac{y^{H-\frac{3}{2}}}{y^{2H-1} + 1} dy,
$$

which yields the claimed asymptotics

$$
\frac{d\langle M \rangle_T}{dT} = \varepsilon^2 u_\varepsilon^2(1) \sim \varepsilon = T^{1-2H}.
$$

(3.13)

The second condition is verified similarly:

$$
\frac{d}{d\varepsilon} u_\varepsilon(1) = \frac{d}{d\varepsilon} \sum_{n \text{ odd}} \frac{\langle \varphi_n, 1 \rangle}{\lambda_n(\varepsilon + \lambda_n)} \int_0^1 \varphi_n(x) c_H |x - 1|^{2H-2} dx \sim
$$

$$
- \sum_{n \text{ odd}} \frac{n^{-\frac{1}{2}-H}}{(\varepsilon + n^{1-2H})^2} \sim - \int_1^{\infty} \frac{x^{-\frac{1}{2}-H}}{\varepsilon + x^{1-2H}} dx \sim -\varepsilon^{-\frac{1}{2}} \int_0^{\infty} \frac{y^{3H-\frac{5}{2}}}{(y^{2H-1} + 1)^2} dy,
$$

and hence

$$
\left( \frac{d}{dT} \log \frac{d}{dT} \langle M \rangle_T \right) \leq \left( \frac{d}{dT} \log g(T,T) \right) \leq \left( \frac{d}{dT} \log T^{1-2H} u_\varepsilon(1) \right)^2 \leq \frac{2}{T^2} + \left( \frac{d}{dT} \log u_\varepsilon(1) \right)^2 = \frac{2}{T^2} + \left( \frac{dT^{1-2H} u_\varepsilon(1)}{dT} \right) \sim \frac{1}{T^2}.
$$

Lemma 3.4. For $H < \frac{1}{2}$,

$$
\frac{d}{dT} \langle M \rangle_T \sim \text{const.} \quad \text{and} \quad \left( \frac{d}{dT} \log \frac{d}{dT} \langle M \rangle_T \right)^2 \sim T^{-2} \quad \text{as} \ T \to \infty
$$

and thus the conditions of Lemma 3.2 hold.
Proof. For $H < \frac{1}{2}$ the equation (2.2) takes the form (see Theorem 5.1 in [5]):

$$c_H(\Phi g)(s) + \frac{2 - 2H}{\lambda_H}(\Psi g)(s,t)s^{1-2H} = c_H(\Phi 1)(s), \quad s \in (0,t],$$

where

$$(\Psi g)(s,t) = -2H \frac{d}{ds} \int_s^t g(r,t)r^{H-\frac{1}{2}}(r-s)^{H-\frac{1}{2}} dr$$

and

$$(\Phi f)(s) = \frac{d}{ds} \int_0^s f(r)r^{\frac{1}{2}-H}(s-r)^{\frac{1}{2}-H} dr.$$ 

Moreover, by Theorem 2.4 in [5],

$$\sqrt{\frac{d}{dt} \langle M \rangle_t} = \sqrt{\frac{2 - 2H}{\lambda_H} t^{\frac{1}{2}-H}} \langle \Psi g \rangle(t,t) =: p(t,t),$$

and it follows from (3.14) that

$$p(t,t) = \sqrt{\frac{\lambda_H}{2 - 2H}} t^{H-\frac{1}{2}} c_H ((\Phi 1)(t) - (\Phi g)(t)) =$$

$$c_H \sqrt{\frac{\lambda_H}{2 - 2H} (\frac{1}{2} - H) t^{\frac{1}{2}}} \int_0^t (1 - g(r,t))r^{\frac{1}{2}-H}(t-r)^{-\frac{1}{2}-H} dr.$$ 

Let $\varepsilon := T^{2H-1}$ and define $u_\varepsilon(u) := g(uT,T)$, $u \in [0,1]$, then

$$p(T,T) = C \varepsilon^{\frac{1}{2}} \int_0^1 (1 - u_\varepsilon(x))(x^{\frac{1}{2}-H}(1-x)^{-\frac{1}{2}-H}dx,$$

with a constant $C > 0$. The function $u_\varepsilon$ solves the equation

$$\varepsilon u_\varepsilon + K_H^{-1} u_\varepsilon = K_H^{-1} 1,$$

where $K_H$ stands for the operator in (2.2). For $H < \frac{1}{2}$ the inverse $K_H^{-1}$ turns to be an integral operator with a certain weakly singular kernel (see (iv) of Theorem 5.1 in [5]).

The limit equation is obviously solved by $u_0 \equiv 1$ and hence

$$p(T,T) = C \varepsilon^{\frac{1}{2}} \int_0^1 (u_0(x) - u_\varepsilon(x))(x^{\frac{1}{2}-H}(1-x)^{-\frac{1}{2}-H}dx.$$ 

Since $u_0, u_\varepsilon \in L^2(0,1),$

$$u_0 - u_\varepsilon = \sum_n \langle 1, \varphi_n \rangle \varphi_n - \sum_n \frac{\langle K_H^{-1}, \varphi_n \rangle}{\varepsilon + \lambda_n^{-1}} \varphi_n$$

$$= \sum_n \langle 1, \varphi_n \rangle \varphi_n - \sum_n \frac{\lambda_n^{-1}(1, \varphi_n)}{\varepsilon + \lambda_n^{-1}} \varphi_n = \sum_n \frac{\varepsilon}{\varepsilon + \lambda_n^{-1}} (1, \varphi_n) \varphi_n.$$ 

Define $h(u) := Cu^{\frac{1}{2}-H}(1-u)^{-\frac{1}{2}-H}$, then

$$p(T,T) = \varepsilon^{\frac{1}{2}} \sum_n \frac{1}{\varepsilon + \lambda_n^{-1}} (1, \varphi_n) \langle h, \varphi_n \rangle.$$ 

By Theorem 2.3, [24], for $H < \frac{1}{2}$ the eigenvalues satisfy the same asymptotics as in (3.12) and therefore form an increasing sequence, in agreement with the fact that in this case
the operator $K_{\alpha}$ is not compact. Also we have $\langle (\cdot)^{-\beta}h, \varphi_n \rangle \sim C_h n^{\beta-1}$ for any $\beta \in (0, 1)$ and any continuous function with $h(0) \neq 0$. However the asymptotics of the averages for symmetric eigenfunctions is completely different in this case: $\langle 1, \varphi_n \rangle \sim C_4 n^{-1}$.

Now we can estimate the growth rate of the series from (3.16):

$$r(\epsilon) = \sum_{n \text{ odd}} \frac{1}{\epsilon + \lambda_n} \langle 1, \varphi_n \rangle \langle h, \varphi_n \rangle \sim \sum_{n \text{ odd}} \frac{n^{-\frac{3}{2} + H}}{\epsilon + n^{2H-1}} \sim \epsilon^{-\frac{3}{2}} \int_0^\infty \frac{y^{-\frac{1}{2} - H}}{y^{1 - 2H} + 1} dy$$

and hence

$$p(T, T) \sim \text{const.}, \quad T \to \infty,$$

which verifies the condition (2.5).

Further, differentiating the series in (3.16), we get

$$\frac{d}{d\epsilon} r(\epsilon) = d \sum_n \frac{1}{\epsilon + \lambda_n} \langle 1, \varphi_n \rangle \langle h, \varphi_n \rangle = \sum_n \frac{n^{-\frac{3}{2} + H}}{(\epsilon + n^{2H-1})^2} \sim -\epsilon^{-\frac{3}{2}} \int_0^\infty \frac{y^{1 - 3H}}{(y^{1 - 2H} + 1)^2} dy,$$

and

$$\left( \frac{d}{dT} \log \frac{d}{dT} \langle M \rangle_T \right)^2 = \left( \frac{d}{dT} \log p(T, T) \right)^2 \sim \left( \frac{d\epsilon}{dT} \frac{d}{d\epsilon} \log \epsilon^{-\frac{1}{2} r(\epsilon)} \right)^2 \sim T^{-2},$$

which verifies (2.4).

**Remark 3.5.** In the simpler regression problem

$$X_t = \theta t + V_t, \quad t \in [0, T]$$

the MLE of $\theta \in \mathbb{R}$ is given by

$$\hat{\theta}_T(X) = \int_0^T g(t, T) dX_t.$$

Consequently the estimation error is normal with zero mean and its variance is controlled by the growth rate of $\langle M \rangle_T$, rather than the derivative $d\langle M \rangle_T/dT$ as in the Ornstein-Uhlenbeck problem. Finding asymptotics of the bracket $\langle M \rangle_T$ amounts to singular perturbation analysis of the equations (3.10) and (3.15) with respect to weak convergence (cf. (3.13)), which can be carried out either directly (see the discussion concluding Section 7.1 in [24]) or using the spectral asymptotics as above.

The corresponding limit variance is

$$\mathbb{E}(\hat{\theta}_T - \theta)^2 \simeq \begin{cases} v_H T^{2H-2} / T^{-1} & H > \frac{1}{2} \\ v_H & H < \frac{1}{2} \end{cases} \quad \text{with} \quad v_H = \frac{2 H \Gamma(H + \frac{1}{2}) \Gamma(3 - 2H)}{\Gamma(\frac{3}{2} - H)}$$

and it follows that the asymptotic is dominated by the fractional component for $H > \frac{1}{2}$ and by the standard Brownian component for $H < \frac{1}{2}$.
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