LINE GRAPHS OF COMPLEX UNIT GAIN GRAPHS WITH LEAST EIGENVALUE $-2^*$

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Abstract. Let $\Gamma$ be a simple graph with vertex set $V(\Gamma) = \{v_1, v_2, \ldots, v_n\}$, and let $E(\Gamma)$ be the set of oriented edges. Each edge of $\Gamma$ determines two different elements in $E(\Gamma)$. Namely, if $v_i$ and $v_j$ are adjacent in $\Gamma$, we find in $E(\Gamma)$ the oriented edge $e_{ij}$ which goes from $v_i$ to $v_j$, and $e_{ji}$ going in the opposite direction. Given any group $\mathfrak{G}$, a ($\mathfrak{G}$-)gain graph is a triple $\Phi = (\Gamma, \mathfrak{G}, \gamma)$ consisting of an underlying graph $\Gamma$, the gain group $\mathfrak{G}$, and a map $\gamma : E(\Gamma) \to \mathfrak{G}$ such that $\gamma(e_{ij}) = \gamma(e_{ji})^{-1}$ called the gain function. The gain graph $\Phi$ is said to be balanced if for every direct cycle $C = e_{i_1i_2} \cdots e_{i_ki_1}$ in $\Gamma$ (if any), we have $\gamma(e_{i_1i_2}) \cdots \gamma(e_{i_ki_1}) = 1$. A gain graph is said to be unbalanced if it is not balanced. Most of the concepts defined for simple graphs directly extend to gain graphs. For instance, we say that a gain graph $\Phi = (\Gamma, \mathfrak{G}, \gamma)$ is of order $n$ and size $m$ if its underlying graph $\Gamma$ has $n$ vertices and $m$ edges; moreover, we say that a gain graph $(\Gamma, \mathfrak{G}, \gamma)$ is $k$-cyclic if the underlying graph $\Gamma$ is connected and $|E(\Gamma)| = |V(\Gamma)| + k - 1$. As usual, the words unicyclic and bicyclic stand as synonyms for 1-cyclic and 2-cyclic, respectively.

Gain graphs (also known in the literature as voltage graphs) are studied in many research areas (see [21] and the annotated bibliography [22]).

In particular, a complex unit gain graph is a $\mathfrak{G}$-gain graph with $\mathfrak{G}$ being equal to the multiplicative group $\mathbf{T}$ of all complex numbers with norm 1. The theory of complex unit gain graphs embodies those of signed graphs and mixed graphs (as defined in [14]). In fact, a signed graph (resp. mixed graph) can be seen as a particular $\mathbf{T}$-gain graph with gains in the subset $\{\pm 1\}$ (resp. $\{1, \pm i\}$) of $\mathbf{T}$.

Over the last decade, there has been a growing interest for the study of matrices and eigenvalues associated with $\mathbf{T}$-gain graphs. For instance, in [17], Reff studied many properties of the adjacency and the Laplacian matrix of $\mathbf{T}$-gain graphs. Further spectral results concerning $\mathbf{T}$-gain graphs have been obtained in [2, 16] (where $\mathbf{T}$-gain graphs are called weighted directed graphs). More recently, in [4] the authors figured out how the least Laplacian eigenvalue of a $\mathbf{T}_4$-gain graph (i.e. a $\mathbf{T}$-gain graph with gains in $\{\pm 1, \pm i\}$) is

*Received by the editors on March 11, 2020. Accepted for publication on October 17, 2020. Handling Editor: Froilán Dopico. Corresponding Author: Maurizio Brunetti.

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related to its frustration index and number. Moreover, Godsil-McKay-like switchings have been described in [3] for the purpose of identifying pairs of non-isomorphic cospectral $T$-gain graphs.

In [18], Reff introduced a notion of orientation for gain graphs in order to provide a suitable setting to build up line graphs of gain graphs. In the wake of his seminal ideas, the authors of this paper specialized in [1] Reff’s results to $T_4$-gain graphs.

The starting point of this paper is Theorem 2.14, which extends Theorem 4 in [1] to complex unit gain graphs. It turns out that, for every complex unit gain graph $\Phi$, the minimum possible eigenvalue for the adjacency matrix of an associated line graph $L(\Phi)$ is $-2$. We prove that such minimum is attained whenever $\Phi$ has a connected component which is neither a tree nor a balanced unicyclic gain graph. In these cases, we study the $-2$-eigenspace, detecting a basis by using the star complement technique and generalizing the routine successfully applied in the past to simple graphs (see [10, 11, 12]) and to signed graphs (see [5, 6]).

The remainder of the paper is organized as follows. In Section 2, we recall some background theory on $T$-gain graphs, the star complement technique, and the basic properties of line graphs associated with $T$-gain graphs. In Section 3, we explicitly compute the components of $-2$-eigenvectors in all cases when $-2$ belongs to the adjacency spectrum of the line graph $L(\Phi)$. The final section contains two examples.

2. Preliminaries.

2.1. Gain graphs. From now on, a $T$-gain graph will be simply denoted by $\Phi = (\Gamma, \gamma)$. Given a $T$-gain graph $\Phi = (\Gamma, \gamma)$ of order $n$ and size $m$, we adopt the notation

$$V(\Gamma) = \{ v_1, \ldots, v_n \} \quad \text{and} \quad E(\Gamma) = \{ e_1, \ldots, e_m \},$$

for the set of vertices and the set of (unoriented) edges of $\Gamma$, respectively.

Let $M_{m,n}(\mathbb{C})$ be the set of $m \times n$ complex matrices. For a matrix $A = (a_{ij}) \in M_{m,n}(\mathbb{C})$, we denote by $A^\ast = (a^\ast_{ij}) \in M_{n,m}(\mathbb{C})$ its conjugate (or Hermitian) transpose, that is, $a^\ast_{ij} = \overline{a_{ji}}$.

The adjacency matrix $A(\Phi) = (a_{ij}) \in M_{n,n}(\mathbb{C})$ of a $T$-gain graph $\Phi = (\Gamma, \gamma)$ is defined by

$$a_{ij} = \begin{cases} \gamma(e_{ij}) & \text{if } v_i \text{ is adjacent to } v_j, \\ 0 & \text{otherwise.} \end{cases}$$

If $v_i$ is adjacent to $v_j$, then $a_{ij} = \gamma(e_{ij}) = \gamma(e_{ji})^{-1} = \overline{\gamma(e_{ji})} = \overline{\pi_{ji}}$. Consequently, $A(\Phi)$ is Hermitian and its eigenvalues $\lambda_1(\Phi) \geq \cdots \geq \lambda_n(\Phi)$ are real. The Laplacian matrix $L(\Phi)$ is defined as $D(\Gamma) - A(\Phi)$, where $D(\Gamma)$ stands for the diagonal matrix of vertex degrees of $\Gamma$. Therefore, $L(\Phi)$ is also Hermitian. According to [17], the matrix $L(\Phi)$ is positive semidefinite, and all its eigenvalues $\mu_1(\Phi) \geq \cdots \geq \mu_n(\Phi)$ are nonnegative. We write $\phi(\Phi, x)$ and $\psi(\Phi, x)$ to denote the characteristic polynomial of $A(\Phi)$ and $L(\Phi)$, respectively. By definition, the spectrum $\text{Spec}(A(\Phi))$ (resp. $\text{Spec}(L(\Phi))$) is the multiset of eigenvalues of $A(\Phi)$ (resp. of $L(\Phi)$). For every eigenvalue $\lambda$ of $A(\Phi)$, the corresponding eigenspace is denoted by $E(\Phi, \lambda)$.

A switching function of a given $T$-gain graph $\Phi$ is any map $\zeta : V(\Gamma) \to T$. Switching the $T$-gain graph $\Phi = (\Gamma, \gamma)$ means replacing $\gamma$ by $\gamma^\zeta$, where $\gamma^\zeta(e_{ij}) = \zeta(v_i)^{-1}\gamma(e_{ij})\zeta(v_j)$ and obtaining the new $T$-gain graph $\Phi^\zeta = (\Gamma, \gamma^\zeta)$. We say that $\Phi_1 = (\Gamma, \gamma_1)$ and $\Phi_2 = (\Gamma, \gamma_2)$ (and their corresponding gain functions) are switching equivalent if there exists a switching function $\zeta$ such that $\Phi_2 = \Phi_1^\zeta$. By writing $\Phi_1 \sim \Phi_2$ or $\gamma_1 \sim \gamma_2$, we mean that $\Phi_1$ and $\Phi_2$ are switching equivalent.
To each switching function $\zeta$, we associate a diagonal matrix $D(\zeta) = \text{diag}(\zeta(v_1), \ldots, \zeta(v_n))$. Note that

$$A(\Phi_2) = D(\zeta)^* A(\Phi_1) D(\zeta) \quad \text{and} \quad L(\Phi_2) = D(\zeta)^* L(\Phi_1) D(\zeta).$$

Therefore, given any pair $(\Phi_1, \Phi_2)$ of switching equivalent $T$-gain graphs, we get the following equality between their spectra:

$$\text{Spec}(A(\Phi_1)) = \text{Spec}(A(\Phi_2)) \quad \text{and} \quad \text{Spec}(L(\Phi_1)) = \text{Spec}(L(\Phi_2)).$$

One of the key notions in the theory of gain graphs (and of the more general theory of biased graphs) is the property of balance (see [9, 21, 23]). An oriented edge $e_{i_{k-1}i_k} \in \overrightarrow{E}(\Gamma)$ is said to be neutral for $\Phi = (\Gamma, \gamma)$ if $\gamma(e_{i_{k-1}i_k}) = 1$. Similarly, the walk $W = e_{i_1i_2} e_{i_2i_3} \cdots e_{i_{l-1}i_l}$ is said to be neutral if its gain

$$\gamma(W) := \gamma(e_{i_1i_2}) \gamma(e_{i_2i_3}) \cdots \gamma(e_{i_{l-1}i_l}),$$

is equal to 1. We write $(\Gamma, 1)$ for the $T$-gain graph with all neutral edges.

An edge set $S \subseteq E$ is said to be balanced if every directed cycle $\overrightarrow{C}$ with edges in $S$ is neutral. A subgraph is balanced if its edge set is balanced (see [1, 4, 17] for further details).

The following proposition gives necessary and sufficient conditions for a $T$-gain graph to be balanced.

**Proposition 2.1** ([17, Lemma 2.1]). Let $\Phi = (\Gamma, \gamma)$ be a $T$-gain graph. Then the following are equivalent:

1. $\Phi$ is balanced.
2. $\Phi \sim (\Gamma, 1)$.
3. There exists a function $\theta : V(\Gamma) \to \mathbb{T}$ such that

$$\theta(v_i)^{-1} \theta(v_j) = \gamma(e_{ij}) \quad \forall e_{ij} \in \overrightarrow{E}(\Gamma).$$

By Proposition 2.1 (2), or [20, Theorem 2.8] we deduce the following corollary.

**Corollary 2.2.** A connected $T$-gain graph $\Phi$ of order $n$ is balanced if and only if its least Laplacian eigenvalue $\mu_n(\Phi)$ is 0.

The next proposition specializes [18, Lemma 2.2] to the case of $T$-gain graphs.

**Proposition 2.3.** Let $\Phi_1 = (\Gamma, \gamma_1)$ and $\Phi_2 = (\Gamma, \gamma_2)$ be $T$-gain graphs with the same underlying graph $\Gamma$. If for every cycle $C$ in $\Gamma$ there exists a directed cycle with base vertex $v$ such that $\gamma_1(\overrightarrow{C_v}) = \gamma_2(\overrightarrow{C_v})$, then there exists a switching function $\zeta$ such that $\Phi_2 = \Phi_1^\zeta$.

By Proposition 2.3, it follows that a gain graph $\Phi$ is balanced if and only if all its directed cycles are neutral.

Let $\Phi = (\Gamma, \gamma)$ be a complex unit gain graph, and let $X$ be a subset of $V(\Gamma)$. We write $\Phi[X]$ to denote the induced subgraph of $\Phi$ with vertex set $X$, and write $\Phi - X$ to denote $\Phi[V(\Gamma) \setminus X]$. As a consequence of the Cauchy’s Interlacing Theorem for Hermitian matrices (see, for instance, [15, Theorem 4.3.17]), we arrive at the following result.
Proposition 2.4. Let $\Phi = (\Gamma, \gamma)$ be a $T$-gain graph of order $n$. For every $v \in V(\Gamma)$, the elements of $\text{Spec}(A(\Phi))$ and $\text{Spec}(A(\Phi - \{v\}))$ interlace as follows.

\begin{equation}
\lambda_1(\Phi) \geq \lambda_1(\Phi - \{v\}) \geq \lambda_2(\Phi) \geq \lambda_2(\Phi - \{v\}) \geq \cdots \geq \lambda_{n-1}(\Phi - \{v\}) \geq \lambda_n(\Phi).
\end{equation}

From (2.1), it follows that the multiplicity of every eigenvalue $\lambda \in \text{Spec}(A(\Phi))$ can change at most by 1 if some vertex is deleted. In view of this, a vertex $v$ is called downer, neutral, or Parter for $\lambda$ if the multiplicity of $\lambda$ decreases, remains the same, or increases, respectively. For some general results on the latter topic, we refer the reader to [19].

2.2. Star sets and star complements. Let $\Phi = (\Gamma, \gamma)$ be a complex unit gain graph, and let $m(\lambda)$ denote the multiplicity of the eigenvalue $\lambda \in \text{Spec}(A(\Phi))$. A star set for $\lambda$ in $\Phi$ is a subset $X$ of $V(\Gamma)$ such that $\lambda \notin \text{Spec}(A(\Phi - X))$ and $|X| = m(\lambda)$. The graph $\Phi - X$ is called a star complement of $\Phi$ with respect to $\lambda$.

In order to apply the star complement technique to complex unit gain graphs, we need to extend to Hermitian matrices some arguments given in [10, 12], where the authors only deal with real symmetric matrices.

Proposition 2.5. Let $\Phi = (\Gamma, \gamma)$ be a complex unit gain graph with $n$ vertices. For every eigenvalue $\lambda \in \text{Spec}(A(\Phi))$, there exists a star set $X$ for $\lambda$.

Proof. Let $m(\lambda)$ be the multiplicity of a fixed $\lambda \in \text{Spec}(A(\Phi))$. Since $\lambda I - A(\Phi)$ is a Hermitian matrix of rank $n - m(\lambda)$, one of its principal submatrices of order $n - m(\lambda)$, say $P$, is non-singular. Note that $P$ has the form $\lambda I - C$, where $C$ is a principal submatrix of $A(\Phi)$. This means that the vertices not corresponding to rows and columns in $C$ determine a star set for $\lambda$, and the remaining ones, that is, those corresponding to $C$, a star complement. 

Here and throughout the rest of the paper, $N_{v}(v)$ (or simply $N(v)$ when it is clear which graph we are referring to) denotes the set of neighbors in a graph $\Gamma$ of a vertex $v \in V(\Gamma)$. The proof of the following theorem is constructive and resembles the one of Theorem 5.1.6 in [12].

Proposition 2.6. A connected complex unit gain graph $\Phi = (\Gamma, \gamma)$ has a connected star complement for each $\lambda \in \text{Spec}(A(\Phi))$.

Proof. Since $\Gamma$ is connected, we can fix a labeling $\{v_1, \ldots, v_n\}$ for its vertices such that, for each $i \geq 2$, there exists a $v_j \in N(v_i)$ with $j < i$. Let $m(\lambda)$ be the multiplicity of a fixed $\lambda \in \text{Spec}(A(\Phi))$, and let $c_i$ (resp. $c^i$) denote the $i$-th row (resp. the $i$-th column) of the matrix $\lambda I - A(\Phi)$. We now choose a subset of vertices $Y = \{v_{j_1}, \ldots, v_{j_{n-m(\lambda)}}\}$ according to the following procedure. We set $j_1 = 1$ and

\[ j_h = \min\{k > j_{h-1} \mid c^k \notin \text{Span}(c^{j_1}, \ldots, c^{j_{h-1}})\} \quad \text{for} \quad 1 < h \leq n - m(\lambda). \]

The columns $c^{j_1}, \ldots, c^{j_{n-m(\lambda)}}$ are linearly independent and generate the column space of $\lambda I - A(\Phi)$. Since such matrix is Hermitian, the rows $c_{j_1}, \ldots, c_{j_{n-m(\lambda)}}$ are linearly independent as well and generate the row space of $\lambda I - A(\Phi)$. Thus, the principal submatrix determined by the sequence $j_1 < \cdots < j_{n-m(\lambda)}$ is non-singular. This is equivalent to say that $\Phi[Y]$ is a star complement. We now show that $\Phi[Y]$ is connected by proving that each of its vertices (apart from the first one) is adjacent to a preceding one. For each $h > 1$ let $k = \min\{i \mid v_i \in N(v_{j_h})\}$. In our assumptions $k < j_h$. By definition of $k$, $-\gamma(e_{j_h,k})$ is the first non-zero element on the $j_h$-th row of $\lambda I - A$. This implies that $c^k \notin \text{Span}(c_1, \ldots, c^{k-1})$. Hence, $v_k$ belongs to $Y$. 

Proposition 2.7. Let $\Phi = (\Gamma, \gamma)$ be a complex unit gain graph of order $n$, let $X = \{v_1, \ldots, v_{m(\lambda)}\}$ be a star set for $\lambda \in \text{Spec}(A(\Phi))$, and let $X_h$ denote the set $\{v_1, \ldots, v_{i_h}\}$, for $1 \leq h \leq m(\lambda)$. The multiplicity of $\lambda$ for $A(\Phi - X_h)$ is $m(\lambda) - h$. 
Proof. By equation (2.1), the deletion of a vertex changes the multiplicity of every eigenvalue at most by 1. The statement now comes from the fact that the multiplicity of $\lambda$ for the first and the last graph of the nested sequence

$$
\Phi - X = \Phi - X_{m(\lambda)} \subset \Phi - X_{m(\lambda) - 1} \subset \cdots \subset \Phi - X_2 \subset \Phi - X_1 \subset \Phi,
$$
is 0 and $m$, respectively.

Corollary 2.8. Let $\Phi = (\Gamma, \gamma)$ be a complex unit gain graph, and let $X$ be a star set for $\lambda \in \text{Spec}(A(\Phi))$. Denoted by $Y$ the set $V(\Gamma) \setminus X$, the multiplicity of $\lambda$ for the graph $\Phi[Y \cup \{v\}]$ is 1 for every $v \in X$.

Thanks to Corollary 2.8, we can extend to $T$-gain graphs Theorem 7.3.1 in [10] without making use of projection maps and their properties.

Corollary 2.9. Let $\Phi = (\Gamma, \gamma)$ be a complex unit gain graph, $X$ be star set for $\lambda \neq 0$, and $\Phi[Y]$ be the corresponding star complement. Then, each vertex of $X$ has a neighbor in $Y$.

Proof. Assuming the contrary, a suitable vertex $v \in X$ would be isolated in $\Phi[Y \cup \{v\}]$; therefore, the multiplicity of $\lambda$ for both $\Phi[Y]$ and $\Phi[Y \cup \{v\}]$ would be 0 contradicting Corollary 2.8.

A basis for the eigenspace of $\lambda \in \text{Spec}(A(\Phi))$ can be constructed as follows from the star complement $\Phi[Y]$; for each $v \in X$ we consider a generator $y_v$ of the $\lambda$-eigenspace of $\Phi[Y \cup \{v\}]$ (its dimension is 1 by Corollary 2.8). A $\lambda$-eigenvector $x_v$ for $\Phi$ is obtained from $y_v$ by adding zero entries in correspondence of vertices in $X \setminus \{v\}$. By Proposition 2.7, the vertex $v \in X$ is a downer for $\lambda$; therefore, the $v$-component of $x_v$ is non-zero. It follows that the several $x_v$’s for $v \in X$ are linearly independent and form a basis for $\mathcal{E}_\Phi(\lambda)$.

2.3. Line graphs associated with $T$-gain graphs. Let $\Phi = (\Gamma, \gamma)$ be a $T$-gain graph of order $n$ and size $m$. As in [17], the $n \times m$ complex matrix $H(\Phi) = (\eta_{vc})$ with entries in $T \cup \{0\}$ is said to be an incidence matrix of $\Phi$ if

$$
\eta_{vc} = \begin{cases} 
-\eta_{ij} \gamma(e_{ij}) & \text{if the endpoints of } e_h \text{ are precisely } v_i \text{ and } v_j, \\
0 & \text{otherwise.}
\end{cases}
$$

In the case when $e_h$ joins $v_i$ and $v_j$, we also require that $\eta_{vc_{eh}}$ is non-zero. We say ‘an’ incidence matrix, because by this definition $H(\Phi)$ is unique only if $\Gamma$ is empty, that is, if it is of size 0. If each column is multiplied by any element in $T$, the resulting matrix is still an incidence matrix. Indeed, Proposition 2.10, whose proof is straightforward, says that all the other possible incidence matrices can be obtained from a fixed $H(\Phi)$ in such a way.

Proposition 2.10. Let $H(\Phi) = (\eta_{vc})$ and $H(\Phi)' = (\eta_{vc}')$ be two incidence matrices both related to the $T$-gain graph $\Phi = (\Gamma, \gamma)$. There exists an $m \times m$ diagonal matrix $S$ with entries in $T \cup \{0\}$ such that $H(\Phi)' = H(\Phi)S$ and $S^*S = I$.

By Proposition 2.10, for a fixed edge $e_h \in E(\Gamma)$ with endpoints $v_i$ and $v_j$, the possibilities for the non-zero elements on the corresponding column of $H(\Phi)$ are

$$
(\eta_{v_i e_h}, \eta_{v_j e_h}) = (e^{i\theta}, e^{i(\theta + \pi)}) \gamma(e_{ij}) \quad \text{for } 0 \leq \theta < 2\pi.
$$

In what follows, we denote by $H$ a specific incidence matrix related to the $T$-gain graph $\Phi = (\Gamma, \gamma)$. We next explain how $H$ determines a $T$-gain structure on the line graph $L(\Gamma)$. It is well known that
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$V(\mathcal{L}(\Gamma)) = E(\Gamma)$, and $ef \in E(\mathcal{L}(\Gamma))$ whenever $e$ and $f$ share an endpoint. We denote by $\mathcal{L}_H(\Phi)$ the $T$-gain graph $(\mathcal{L}(\Gamma), \gamma^c_H)$, where

$$\gamma^c_H : \{ef \in \vec{E}(\mathcal{L}(\Gamma)) \rightarrow \eta_{we} \eta_{wf} \in T,$$

where $w$ is the endpoint shared by the edges $e$ and $f$. It is easy to verify that $\gamma^c_H$ is a gain function. In fact, $\gamma^c_H(ef) = \overline{\gamma^c_H(ef)}$.

Given any Abelian group $\mathcal{G}$, the gains for the line graph associated with a $\mathcal{G}$-gain graph in [18] do not only depend on the chosen incidence matrix but also on the pick of a weak involution in $\mathcal{G}$, that is, on an element $s \in \mathcal{G}$ such that $s^2 = 1_{\mathcal{G}}$. Our definition of $\mathcal{L}_H(\Phi)$ is consistent with N. Reff’s for $s = 1_{\mathcal{G}}$ and $\mathcal{G} = T$.

**THEOREM 2.11 ([18, Theorem 5.1]).** Let $H$ be one of the incidence matrices related to the $T$-gain graph $\Phi = (\Gamma, \gamma)$. Then,

$$H(\Phi)^*H(\Phi) = 2I_m + A(\mathcal{L}_H(\Phi)).$$

In a private communication to the authors, Tom Zaslavsky gave several arguments in favor of defining $\mathcal{L}_H(\Phi)$ by picking a non-trivial weak involution in $\mathcal{G}$ whenever it exists. Chosen $s = -1 \in T$, Equation 2.3 should be replaced by [23, Theorem 5.1], and everything we say in Sections 3 and 4 on $E_{\mathcal{L}_H(\Phi)}(-2)$ would hold for $E_{\mathcal{L}_H(\Phi)}(2)$. Yet, we prefer to pick $s = 1_T$. In this way, our conclusions are more directly related to the classical results of Spectral Graph Theory collected in [12, 13]. Moreover, when $\gamma(\vec{E}(\Gamma)) \subseteq \{-1, 1\}$, that is, when the $T$-gain graph $\Phi$ is actually a signed graph, and $\gamma^c_H$ is the gain function defined as in (2.2), we retrieve the same signature on $\mathcal{L}(\Phi)$ as assigned in [5, Section 1] and [6, Section 2].

We omit the proofs of Propositions 2.12, 2.13 and Theorem 2.14, since they are conceptually identical to those written down in [1] in the more restrictive context of $T_4$-gain graphs.

**PROPOSITION 2.12 ([1, Proposition 5]).** Let $H$ and $H'$ be two of incidence matrices both associated with the same $T$-gain graph $\Phi = (\Gamma, \gamma)$. Then $\mathcal{L}_H(\Phi)$ and $\mathcal{L}_{H'}(\Phi)$ share the same adjacency spectrum. Moreover, if $S$ is the diagonal matrix such that $H(\Phi)' = H(\Phi)S$, then

$$A(\mathcal{L}_{H'}(\Phi)) = S^*A(\mathcal{L}_H(\Phi))S.$$

**PROPOSITION 2.13 ([1, Proposition 6 and its proof]).** Line graphs of switching equivalent $T$-gain graphs $\Phi_1 = (\Gamma, \gamma_1)$ and $\Phi_2 = (\Gamma, \gamma_2)$ are switching equivalent. Moreover, if $\zeta : V(\Gamma) \rightarrow T$ is the switching function such that $\Phi_2 = \Phi_1^\zeta$, and $H_1$ is an incidence matrix for $\Phi_1$, then $D(\zeta)^{-1}H_1$ is an incidence matrix for $\Phi_2$,

$$\mathcal{L}_H(\Phi_1) = \mathcal{L}_{D(\zeta)^{-1}H_1}(\Phi_2).$$

The final result of this section concerns the mutual interrelationships between the Laplacian polynomial of a $T$-gain graph $\Phi$ and the adjacency polynomial of its line graphs. Proposition 2.12 allows us to drop the incidence matrix out of notations in the statements.

**THEOREM 2.14 ([1, Theorem 4]).** Let $\Gamma$ be a graph of order $n$ and size $m$, and $\Phi$ a $T$-gain graph having $\Gamma$ as underlying graph. Then

$$\phi(\mathcal{L}(\Phi), x) = (x + 2)^{m-n}\psi(\Phi, x + 2).$$

Since the Laplacian eigenvalues of a complex unit graph are all nonnegative, from (2.4) it immediately follows that no eigenvalue in $\text{Spec}(A(\mathcal{L}(\Phi)))$ is less than $-2$. 
3. An eigenbasis for \(-2\) in complex unit line graphs. Let \(\Phi = (\Gamma, \gamma)\) be a complex unit gain graph, and let \(\mathcal{L}(\Phi) = (\mathcal{L}(\Gamma), \gamma^\mathcal{L})\) be the associated line graph arising from a fixed incidence matrix \(H\) of \(\Phi\). The first theorem of this section identifies the structural conditions on \(\Phi\) ensuring the presence of \(-2\) in \(\text{Spec}(A(\mathcal{L}(\Phi)))\).

**Theorem 3.1.** Let \(\Phi = (\Gamma, \gamma)\) be a connected complex unit gain graph of order \(n\) and size \(m\), and \(\overline{C}(\Gamma)\) be the set of directed cycles in \(\Gamma\) in the last paragraph of Section 2.2 to find an eigenbasis for \(\mathcal{L}(\Phi)\).

\[
(-1)^m \phi(\mathcal{L}(\Phi), -2) = \begin{cases} 
  m + 1 & \text{if } \Gamma \text{ is a tree}, \\
  2 - 2 \cos \theta & \text{if } (\Gamma, \gamma) \text{ is unbalanced unicyclic and } \gamma(\overline{C}) = e^{i\theta} \text{ for a } \overline{C} \in \overline{C}(\Gamma), \\
  0 & \text{otherwise}.
\end{cases}
\]

**Proof.** If \(\Gamma\) is a tree, then \(\Phi\) is balanced. Therefore, \(\Phi \sim (\Gamma, 1)\), and by Proposition 2.13, we get \(\phi(\mathcal{L}(\Phi)) = \phi(\mathcal{L}(\Gamma, 1))\). The equality \((-1)^m \phi(\mathcal{L}(\Phi), -2) = m + 1\) now comes from [12, Lemma 7.5.2(i)] or [7, Lemma 3.8]. In fact, \((\Gamma, 1)\) can be regarded as an unsigned graph.

Let now \(\Gamma\) be unicyclic. Equation (2.4) specializes to

\[
\phi(\mathcal{L}(\Phi), -2) = \psi(\Phi, 0) = \det(L(\Phi)).
\]

Since \(\Gamma\) is unicyclic, the directed cycles in \(\overline{C}(\Gamma)\) have just two possible gains. Such gains are complex conjugate, say \(e^{i\theta}\) and \(e^{-i\theta}\). Fixed any \(\overline{C} \in \overline{C}(\Gamma)\), from [20, Lemmas 2.2 and 2.4] we deduce

\[
\det(L(\Phi)) = \det(\overline{C}) = |1 - \gamma(\overline{C})|^2 = (1 - \gamma(\overline{C}))(1 - \gamma^{-1}(\overline{C})) = 2 - 2 \cos \theta.
\]

Finally, if \(\Gamma\) is neither a tree nor a unicyclic graph, then \(m > n\), and \((-1)^m \phi(\mathcal{L}(\Phi), -2) = 0\) by Theorem 2.14.

**Corollary 3.2.** Let \(\Phi\) be a connected complex unit gain graph. The least eigenvalue of \(\mathcal{L}(\Phi)\) is \(-2\) if and only if \(\Phi\) contains as a complex unit subgraph at least one balanced cycle or two unbalanced cycles.

In what follows, we attempt to stick as close as possible to the way of arguing of [5, Section 3], where an eigenbasis for \(-2\) in signed lined graphs is detected.

Unless told otherwise, we assume the underlying graph \(\Gamma\) of the complex unit gain graph \(\Phi\) (and therefore \(\mathcal{L}(\Gamma)\)) is connected, and \(-2\) belongs to \(\text{Spec}(A(\mathcal{L}(\Phi)))\) with multiplicity \(k > 0\). We now use the ideas explained in the last paragraph of Section 2.2 to find an eigenbasis for \(\lambda = -2\). Such basis will arise from a connected star complement in \(\mathcal{L}(\Phi)\) related to \(-2\).

By Proposition 2.6, \(\mathcal{L}(\Phi)\) has a connected induced subgraph which is a star complement with respect to \(\lambda = -2\). The corresponding edges in \(\Phi\) induce the ‘line star complement’, which is also connected apart from isolated vertices, if any. In the spirit of [5, 11], every line star complement in \(\Phi\) with respect to \(-2\) is also called a foundation. Henceforth, we assume \(\Psi = (\Lambda, \gamma|_{E(\Lambda)})\) is a fixed foundation. Since the isolated vertices do not affect \(\mathcal{L}(\Psi) \subset \mathcal{L}(\Phi)\), it is not restrictive to assume \(\Psi\) is connected. If this is the case, by Theorem 3.1, \(\Psi\) is either a tree or an unbalanced unicyclic graph.

As discussed in Section 2.2, the procedure to obtain a \((-2)\)-eigenbasis of \(\mathcal{L}(\Phi)\) consists of enriching the induced subgraph \(\mathcal{L}(\Psi)\) by a vertex in \(V(\mathcal{L}(\Gamma)) \setminus V(\mathcal{L}(\Lambda))\), or equivalently in adding an edge \(e \in E(\Gamma) \setminus E(\Lambda)\) to \(\Lambda\). We set \(\Psi_e := (\Lambda_e, \gamma|_{E(\Lambda_e)})\), where \(V(\Lambda_e) = V(\Lambda)\) and \(E(\Lambda_e) = E(\Lambda) \cup \{e\}\). Let now \(x_e\) be a \((-2)\)-eigenvector of \(\mathcal{L}(\Psi_e)\). Each of its coordinates is labeled by a suitable edge in \(\Psi_e\). By Corollary 2.8 applied
to $L(\Psi_e)$, the $(-2)$-eigenspace of its adjacency matrix is one-dimensional. Thus, every $(-2)$-eigenvector $v$ of $L(\Psi_e)$ is proportional to $x_e$, in particular $v$ shares with $x_e$ the same non-zero versus zero pattern.

In view of the latter observation, we can distinguish two types of edges in $\Psi_e$. We say that an edge is heavy (resp. light) when the corresponding entry in $x_e$ is non-zero (resp. zero). The unique subgraph $\Theta_e$ of $\Psi_e$ induced by its heavy edges will be called the core of $\Psi_e$. Throughout the rest of this section, the words ‘downer’, ‘neutral’, and ‘Parter’ will always be used to qualify the vertices of a certain line graph with respect to the eigenvalue $-2$.

**Proposition 3.3.** The vertices in $L(\Psi_e)$ corresponding to edges of $\Theta_e$ are downers, the remaining ones are neutrals.

*Proof.* Since $\lambda = -2$ is the least eigenvalue for $A(L(\Psi_e))$, from (2.1) it follows that the graph $L(\Psi_e)$ has no Parter vertices. It is routine to check that vertices corresponding to light edges are neutral. Now, assume by contradiction that an edge $f$ of $\Theta_e$ corresponds to a neutral vertex for $L(\Psi_e)$. There would exist a $-2$-eigenvector $y_e$ for $A(L(\Psi_e) - \{f\})$, and a $-2$-eigenvector $y'_e$ for $A(L(\Psi_e))$ obtained from $y_e$ by inserting a 0-entry in correspondence of $f \in V(L(\Psi_e))$. Clearly $y'_e$ would not be proportional to $x_e$, against the one-dimensionality of the $-2$-eigenspace for $A(L(\Psi_e))$. Hence, such a ‘downer’ $f$ in $\Theta_e$ does not exist. $\square$

The next proposition collects some properties of the core $\Theta_e$.

**Proposition 3.4.** Let $\Theta_e$ be the core of the graph $\Psi_e$ built from a connected foundation $\Psi = (\Lambda, \gamma)$ of a connected complex unit graph $\Phi = (\Gamma, \gamma)$ and an edge $e \in E(\Gamma) \setminus E(\Lambda)$. The following properties hold.

(i) The edge $e$ belongs to $\Theta_e$.

(ii) $\Theta_e$ is connected.

(iii) No edge in $\Theta_e$ is pendant.

(iv) The edge $e$ belongs to some cycle of $\Theta_e$.

*Proof.* Since the graph $\Psi$ is a foundation, the vertex in $L(\Psi_e)$ corresponding to $e$ is a downer. Therefore, Part (i) comes from Proposition 3.3.

Let $X_e$ be the connected component of $\Theta_e$ containing $e$. Note that $-2 \notin \text{Spec}(A(L(X_e)))$, otherwise $e$ would not be a downer for $L(\Psi_e)$. The multiplicity of $-2$ in $\text{Spec}(A(L(X_e)))$ is necessarily one, being one in $\text{Spec}(A(L(\Psi_e)))$. This implies that every edge in $\Theta_e \setminus X_e$, if existing, would be neutral against Proposition 3.3. It follows that $X_e = \Theta_e$, proving Part (ii).

Since $\Theta_e$ is connected and $-2$ is an eigenvalue for $A(L(\Theta_e))$ (of multiplicity one), Corollary 3.2 implies that $\Theta_e$ contains as complex unit subgraph at least a balanced cycle or at least two unbalanced cycles. For each pendant edge $f$, the same thing would be true for the connected complex unit graph $\Theta_e - \{f\}$. Again by Corollary 3.2, we would infer that $-2$ is an eigenvalue for $A(L(\Theta_e - \{f\}))$, and the edge $f$ would be neutral against Proposition 3.3. Hence, no pendant edges exist as stated in Part (iii).

Part (iv) is proved by contradiction. Since $\Theta_e$ does not contain pendant edges, if $e$ does not belong to a cycle, then it should be a bridge. By Part (i), we know that $-2 \notin \text{Spec}(A(L(\Theta_e - \{e\})))$. By Corollary 3.2, no component of $\Theta_e - \{e\}$ contains a balanced cycle or two unbalanced cycles. This implies that $e$ would belong to a path connecting two unbalanced cycles. Recall now that, in our hypotheses, the graph foundation $\Psi$ is connected, and $\Theta_e - \{e\} \subset \Psi$. Hence, we can find in $\Psi$ two unbalanced cycles joined by a path against Corollary 3.2. $\square$
From Proposition 3.4, and by Corollary 3.2 applied to $\Psi$, we conclude that the core $\Theta_e$ is either a balanced cycle, or a dumbbell whose two cycles are both unbalanced, or an $\infty$-graph with two unbalanced cycles. Recall that a dumbbell is a graph consisting of two disjoint cycles joined by a non-trivial path, whereas an $\infty$-graph consists of two cycles with just one vertex in common.

So the problem of constructing $(-2)$-eigenvectors in complex unit line graphs is reduced to finding those eigenvectors arising from the cores described above.

**Theorem 3.5.** Let the core $\Theta_e = (C, \gamma|E(C))$ be a balanced cycle. After labeling the $q \geq 3$ vertices of $C$ and its edges as in Fig. 1, a generator $a = (a_0, a_1, \ldots, a_{q-1})^\top$ of the $-2$-eigenspace of $A(L(\Theta_e))$ is given by the formulæ:

$$a_i = (-1)^i \prod_{s=1}^{i} \nu(s) a_0 \quad \text{for } 1 \leq i \leq q - 1 \quad \text{and} \quad a_0 \neq 0,$$

where the component $a_i$ corresponds to the edge $e_i$, and

$$\nu(i) = \gamma(e_{i-1}e_i) = \eta_{e_{i-1}}\eta_{e_i} \in T \quad \text{for } 1 \leq i \leq q - 1.$$

Moreover, the vector $a$ can be extended to a $(-2)$-eigenvector of $A(L(\Phi))$ by inserting zeros at the remaining entries.

**Proof.** Vertices and edges of the cycle $C$ are labeled as follows:

$$V(C) = \{v_0, \ldots, v_{q-1}\}, \quad \text{and} \quad E(C) = \{e_i = v_iv_{i+1} | 0 \leq i \leq q - 2\} \cup \{e_{q-1} = v_{q-1}v_0\}.$$  

Let $x = (x_0, x_1, \ldots, x_{q-1})^\top$ be a $(-2)$-eigenvector of $L(\Theta_e)$. By using (3.5), the equation $A(L(\Theta_e))x = -2x$ yields

$$-2x_0 = \nu(0)x_{q-1} + \nu(1)x_1,$$

$$-2x_i = \nu(i)x_{i-1} + \nu(i + 1)x_{i+1} \quad \text{for } 0 < i < q - 1,$$

$$-2x_{q-1} = \nu(q - 1)x_{q-2} + \nu(0)x_0,$$

where we set $\nu(0) = \gamma(e_{q-1}e_0) = \eta_{0e_{q-1}}\eta_{0e_0}$.

Now we fix a non-zero complex number $a_0$ and choose as a ‘guessing solution’ the vector

$$a = (a_0, a_1, \ldots, a_{q-1})^\top.$$  

![Figure 1. Vertex and edge labeling for the core $\Theta_e$ being a cycle.](image-url)
If \( P \) and in our hypotheses \( w \) and its end-vertices \( P \) in the statement of Theorem 3.5 have an intriguing geometric meaning: they compute the gains of the several which actually holds, since in general are both equivalent to

\[
-2a_0 = \nu(0)a_{q-1} + \nu(1)a_1 \quad \text{and} \quad -2a_{q-1} = \nu(q-1)a_{q-2} + \nu(0)a_0,
\]

are both equivalent to

\[
\prod_{s=0}^{q-1} \nu(s) = (-1)^q,
\]

which actually holds, since in general

\[
\prod_{s=0}^{q-1} \nu(s) = (-1)^q \gamma(\bar{C}_0),
\]

where \( \bar{C}_0 = e_{01}e_{12} \cdots e_{(q-1)0} \), and in our hypotheses \( \gamma(\bar{C}_0) = 1 \).

As Tom Zaslavsky privately pointed out to the authors, for \( 1 \leq i \leq q-1 \), the numbers \( \prod_{s=1}^{i} \nu(s) \) appearing in the statement of Theorem 3.5 have an intriguing geometric meaning: they compute the gains of the several paths \( P_{0i} \)'s in \( L(\Phi) \) where \( P_{0i} := e_ie_{i-1} \cdots e_0 \).

We now fix some notation to investigate the cases when the underlying graph of \( \Theta_{e} \) consists of two cycles \( C' \) and \( C'' \) (of length \( q' \) and \( q'' \), respectively) joined by a path \( P \) of length \( p \geq 0 \). In literature, this bicyclic graph is often denoted by \( B(q',p,q'') \) (see, for instance, [8, 10]). We label vertices and edges of \( \Theta_{e} \) as follows:

\[
V(C') = \{v_0', \ldots, v_{q'-1}'\}, \quad V(C'') = \{v_0'', \ldots, v_{q''-1}'\},
\]

\[
E(C') = \{e'_i = v'_i v'_{i+1} \mid 0 \leq i \leq q' - 2\} \cup \{e'_{q'-1} = v'_{q'-1} v'_0\},
\]

\[
E(C'') = \{e''_i = v''_i v''_{i+1} \mid 0 \leq i \leq q'' - 2\} \cup \{e''_{q''-1} = v''_{q''-1} v''_0\}.
\]

If \( P \) is non-trivial, that is, if its length is \( p > 0 \), we assume that

\[
V(P) = \{w_0, \ldots, w_p\}, \quad E(P) = \{f_i = w_i w_{i+1} \mid 0 \leq i \leq p - 1\},
\]

and its end-vertices \( w_0 \) and \( w_p \) are, respectively, identified with vertices \( v'_0 \in V(C') \) and \( v''_0 \in V(C'') \) (see Figs. 2 and 3).

Let \( x \) be a \(-2\)-eigenvector for \( A(L(\Theta_{e})) \). For convenience, we split its ordered set of components into three (resp. two) parts if \( p > 0 \) (resp. \( p = 0 \)), each corresponding to its constituents \( C', P \) (if non-trivial), and \( C'' \). Namely, we write \( x = a' + b + a'' \) where \( a' = (a'_0, a'_1, \ldots, a'_{q'-1})^\top \), \( b = (b_0, b_1, \ldots, b_{p-1})^\top \), and \( a'' = (a''_0, a''_1, \ldots, a''_{q''-1})^\top \), and the components \( a'_i, b_i, \) and \( a''_i \) respectively correspond to the edges \( e'_i, f_i, \) and \( e''_i \). In the statements of Theorems 3.6 and 3.7, the following two directed cycles

\[
\bar{C}_0'' = e_{01}e_{12} \cdots e'_{(q'-1)0} \quad \text{and} \quad \bar{C}_0'' = e_{01}e_{12} \cdots e''_{(q''-1)0},
\]

where \( e'_{ij} = v'_i v'_j \) and \( e''_{ij} = v''_i v''_j \), play an important role.
Theorem 3.6. Let the core \( \Theta_e = (B(q', p, q''), \gamma|\overline{E}(B(q', p, q''))|) \) be a complex unit dumbbell with two unbalanced cycles. Under the above notation (see also Fig. 2), for each non-zero complex number \( b_0 \), a generator \( a' + b + a'' \) of the \(-2\)-eigenspace of \( A(\mathcal{L}(\Theta_e)) \) is given by the formulæ:

\[
\begin{align*}
a'_0 &= -\left(1 - \gamma(C_0')\right)^{-1} \gamma \mathcal{C}(e'_0 f_0) b_0, \\
a''_0 &= -\left(1 - \gamma(C_0'')\right)^{-1} \gamma \mathcal{C}(e''_0 f_{p-1}) b_{p-1},
\end{align*}
\]

and

\[
\begin{align*}
a'_i &= (-1)^i \prod_{s=1}^{i} \nu'(s) a'_0 & \text{for } 1 \leq i \leq q' - 1, \\
b_i &= (-1)^i \prod_{s=1}^{i} \nu(s) b_0 & \text{for } 1 \leq i \leq p - 1 \text{ and } b_0 \neq 0, \\
a''_i &= (-1)^i \prod_{s=1}^{i} \nu''(s) a''_0 & \text{for } 1 \leq i \leq q'' - 1,
\end{align*}
\]

where

\[
\begin{align*}
\nu'(i) &= \gamma \mathcal{C}(e'_{i-1} e'_i) = \pi_{\omega e'_i} \eta_{e'_i} \in \mathbb{T} & \text{for } 1 \leq i \leq q' - 1, \\
\nu(i) &= \gamma \mathcal{C}(f_{i-1} f_i) = \pi_{\omega f_i} \eta_{f_i} \in \mathbb{T} & \text{for } 1 \leq i \leq p - 1, \\
\nu''(i) &= \gamma \mathcal{C}(e''_{i-1} e''_i) = \pi_{\omega e''_i} \eta_{e''_i} \in \mathbb{T} & \text{for } 1 \leq i \leq q'' - 1.
\end{align*}
\]

Moreover, \( a' + b + a'' \) can be extended to a \((-2)\)-eigenvector of \( A(\mathcal{L}(\Phi)) \) by putting zeros at all other entries.
Proof. We start by setting

\[(3.14) \quad \nu'(0) = \gamma^2(e_{q'1} - e_0') = \eta_0 \eta_0' e_0' \quad \text{and} \quad \nu''(0) = \gamma^2(e_{q''1} - e_0'') = \eta_0 \eta_0' e_0''.\]

By definition, we get

\[(3.15) \quad \prod_{s=0}^{q'1-1} \nu'(s)^q = (-1)^q \gamma(C^q_0) \quad \text{and} \quad \prod_{s=0}^{q''1-1} \nu''(s)^q = (-1)^q \gamma(C^q_0).\]

We have to check that \(A(L(\Theta_q))(a' + b + a'') = -2(a' + b + a'').\) The eigenvalue equations at vertices of degree 2 in \(L(\Theta_q)\) resemble the middle equation in (3.6), and it is not hard to show that they actually hold by looking at (3.8)–(3.10). The non-trivial checks involve the vertices in correspondence of the edges \(e_0', e_{q'1-1}, e_0'', e_{q''1-1}, f_0\) (all incident to \(v_0')\), and \(e_0', e_{q'1-1}, f_{p-1}\) (all incident to \(v_0'\)). By virtue of symmetry, we provide the verification just for \(e_0'\) and \(f_0\).

Consider first the edge \(e_0'\). We have to check the equality

\[(3.16) \quad (2) a_0' = \nu'(1) a_1' + \nu''(0) a_{q'1-1} + \gamma^2(e_{0'} f_0) b_0.\]

When you make the substitutions

\[a_1' = -\nu''(1) a_0', \quad a_{q'1-1} = (-1)^q \nu'(1) \left[ \prod_{s=1}^{q'1-1} \nu'(s) \right] a_0', \quad \text{and} \quad b_0 = -(1 - \gamma(C_0)) \gamma^2(f_0 e_0'),\]

coming from (3.7) and (3.8), Equality 3.16 becomes in fact equivalent to the first equation of (3.15).

Consider secondly the edge \(f_0\). We have to check the equality

\[(3.17) \quad -2b_0 = \gamma^2(f_0 e_0') a_0' + \gamma^2(f_0 e_{q'1-1}) a_{q'1-1} + \begin{cases} \gamma^2(f_0 e_{q'1-1}) a_{q'1-1} \\ \nu'(1) b_1 \end{cases} \quad \text{if} \quad p = 1, \quad \text{and} \quad \text{if} \quad p > 1.\]

To this aim, we observe that

\[\gamma^2(f_0 e_{q'1-1}) a_{q'1-1} = -\left(1 - \gamma(C_0')\right)^{-1} b_0 \quad \text{by (3.7)},\]

and

\[\gamma^2(f_0 e_{q'1-1}) a_{q'1-1} = \eta_{0'0} f_0 \eta_{0'0} e_{q'1-1} \cdot (-1)^q \left[ \prod_{s=1}^{q'1-1} \nu'(s) \right] a_0' \quad \text{by (2.2) and (3.8)},\]

\[= \eta_{0'0} f_0 \eta_{0'0} e_{q'1-1} \nu'(0) \cdot (-1)^q \left[ \prod_{s=0}^{q'1-1} \nu'(s) \right] a_0' \]

\[= \eta_{0'0} f_0 \eta_{0'0} a_{q'1-1} (1 - \gamma(C_0')) a_0' \quad \text{by (3.14) and (3.15)},\]

\[= \gamma^2(f_0 e_{q'1-1}) \gamma^2(f_0 e_{q'1-1}) \left(1 - \gamma(C_0')\right)^{-1} b_0 \quad \text{by (2.2) and (3.7)},\]

\[= \gamma(C_0') \left(1 - \gamma(C_0')\right)^{-1} b_0.\]
Hence, (3.17) is equivalent to

\begin{equation}
- b_0 = \begin{cases} 
\gamma^L (f_0 e_0') a_0' + \gamma^L (f_0 e_{q''-1}') a_0'' & \text{if } p = 1, \\
\nu(1) b_1 & \text{if } p > 1.
\end{cases}
\end{equation}

For \( p > 1 \), (3.18) follows from \( b_1 = -\nu(1) b_0 \), which is (3.9) specialized to the case \( i = 1 \). For \( p = 1 \), note that

\[
\gamma^L (f_0 e_0') a_0'' = -\left(1 - \gamma(C_0^0)\right)^{-1} b_0 \quad \text{by (3.7)},
\]

and, arguing as above,

\[
\gamma^L (f_0 e_{q''-1}') a_0'' = \gamma^L (f_0 e_{q''-1}') (-1)^{q''-1} \prod_{s=1}^{q''-1} \nu''(s) a_0''
\]

\[
= (-1)^{q''-1} \gamma^L (f_0 e_0') \prod_{s=0}^{q''-1} \nu''(s) a_0''
\]

\[
= -\gamma(C_0^0) \left(1 - \gamma(C_0^0)\right)^{-1} b_0.
\]

Hence, (3.18) holds for \( p = 1 \) as well.

**Theorem 3.7.** Let the core \( \Theta_c = (B(q', 0, q''), \gamma|_{\hat{E}(B(q', 0, q''))}) \) be a complex unit \( \infty \)-graph with two unbalanced cycles. Under the above notation (see also Fig. 3), for each non-zero complex number \( a_0' \), a generator \( a' + a'' \) of the \(-2\)-eigenspace of \( A(\mathcal{L}(\Theta_c)) \) is given by the formula:

\begin{equation}
a_0'' = -\left(1 - \gamma(C_0^0)\right)^{-1} \left(1 - \gamma(C_0^0)\right) \gamma^L (e_0'' e_0') a_0',
\end{equation}

and

\begin{equation}
a_i' = (-1)^i \prod_{s=1}^{i} \nu''(s) a_0' \quad \text{for } 1 \leq i \leq q' - 1,
\end{equation}

\begin{equation}
a_i'' = (-1)^i \prod_{s=1}^{i} \nu''(s) a_0'' \quad \text{for } 1 \leq i \leq q'' - 1,
\end{equation}

where the \( \nu''(i) \)'s and the \( \nu''(i) \)'s satisfy (3.11) and (3.13).

Moreover, \( a' + a'' \) can be extended to a \((-2)\)-eigenvector of \( A(\mathcal{L}(\Phi)) \) by putting zeros at all other entries.

**Proof.** Let \( \nu'(0) \) and \( \nu''(0) \) be as in (3.14). In order to check that \( A(\mathcal{L}(\Theta_c))(a' + a'') = -2(a' + a'') \), it suffices to verify the eigenvalue equations at the vertices corresponding to the four edges incident to \( v_0' = v_0'' \).

Once again, by virtue of symmetry, we only consider the edge \( e_0' = v_0' v_1' \). We have to verify the equality

\begin{equation}
-2 a_0' = \nu'(1) a_0' + \nu''(0) a_{q'-1}'' + \gamma^L (e_0'' e_0') a_0'' + \gamma^L (e_0'' e_{q''-1}') a_{q''-1}''.
\end{equation}

This can be done once you observe that

\[
\nu'(1) a_0' = -a_0' \quad \text{by (3.20) when } i = 1,
\]

\[
\nu''(0) a_{q'-1}'' = -\gamma(C_0^0) a_0'' \quad \text{by (3.15) and (3.20)},
\]

\[
\gamma^L (e_0'' e_0') a_0'' = -\left(1 - \gamma(C_0^0)\right)^{-1} \left(1 - \gamma(C_0^0)\right) a_0' \quad \text{by (3.19)},
\]
Line Graphs of Complex Unit Gain Graphs with Least Eigenvalue $-2$

and

$$
\gamma^C(e'_0e''_{i'-1})a''_{i'-1} = \left(1 - \gamma(C''_{0'})\right) \left(1 - \gamma(C'_{0'})\right) \gamma(C''_{0'})a'_0,
$$

which comes by (3.21), the equality $\gamma^C(e'_0e''_{i'-1}) = \gamma^C(e'_0e''_0)(0)$, and (3.15).

Remark 3.8. If the two unbalanced cycles $C'$ and $C''$ of a bicyclic core $\Theta_e$ have both gain $-1$, Theorems 3.5, 3.6 and 3.7 return the same formulæ stated in Theorems 3.1, 3.3, and 3.5 in [5], where the $-2$-eigenvectors for signed line graphs are described.

For clarity, we recap the explained procedure for constructing an eigenbasis for $-2$ of $L(\Phi)$ from the structure of the root graph $\Phi = (\Gamma, \gamma)$.

Step 1: Choose in $\Phi$ any connected line star complement, say $\Psi = (\Lambda, \gamma_{E(\Lambda)})$.

Step 2: For each edge $e$ of $\Gamma$ not belonging to $\Phi$, form the one-edge extension $\Psi_e := (\Lambda_e, \gamma_{E(\Lambda_e)})$ of $\Psi$, where $V(\Lambda_e) = V(\Lambda)$ and $E(\Lambda_e) = E(\Lambda) \cup \{e\}$, and identify its core $\Theta_e$: the eigenvector $x_e$ corresponding to $e$ is constructed by using an appropriate formula from one of Theorems 3.5, 3.6, and 3.7. These eigenvectors, if the $e$'s are added in turn (one edge per each eigenvector), comprise an eigenbasis for $-2$ in $L(\Phi)$.

We end this section by explaining how the $-2$-eigenspace of a complex unit line graph $L(\Phi)$ changes when $\Phi$ is replaced by a switching equivalent graph. Let $\Phi_1 = (\Gamma, \gamma_1)$ and $\Phi_2 = (\Gamma, \gamma_2)$ be two complex unit gain graphs such that $\Phi_2 = \Phi_1\gamma$ for a suitable switching function $\gamma : V(\Gamma) \rightarrow T$, and let $H_1$ (resp. $H_2$) be an incidence matrix of the complex unit gain graph $\Phi_1$ (resp. $\Phi_2$). By Proposition 2.13, $D(\gamma)^{-1}H_1$ is an incidence matrix for $\Phi_2$ such that $L(H_1(\Phi_1)) = L(D(\gamma)^{-1}H_1(\Phi_2))$. Hence, it follows from Proposition 2.10 that there exists a diagonal matrix $S$ such that $H_2 = D(\gamma)^{-1}H_1S$. Finally, by Proposition 2.12 applied to $\Phi_2$, if $x$ is an eigenvector of $A(L(H_1(\Phi_1))) = A(L(D(\gamma)^{-1}H_1(\Phi_2)))$, then $S^*x$ is an eigenvector of $A(L(H_2(\Phi_2)))$.

4. Examples. In order to depict $T$-gain graphs in Figs. 4 and 5, each continuous (resp., dashed) thick undirected line represents two opposite oriented edges with gain 1 (resp., $-1$), whereas the arrows detect the oriented edges $uv$’s with an imaginary gain. The value $\gamma(uv)$ is specified near the correspondent arrow.

![Diagram 4](image1.png)

**Figure 4. A complex unit dumbbell $\Phi$ and one of its associated line graphs $L(\Phi)$.**

![Diagram 5](image2.png)

**Figure 5. A complex unit $\infty$-graph $\Phi$ and one of its associated line graphs $L(\Phi)$.**
EXAMPLE 4.1. Let \( \Phi = (\Gamma, \gamma) \) be the complex unit gain graph depicted in Fig. 4. The vertex and the edge labeling are consistent with the one used in Fig. 2. Namely, \( e_i' = v_i'v_{i+1}' \) and \( e_i'' = v_i''v_{i+1}'' \) for \( i \in \{0, 1\} \); \( e_2' = v_2'v_0', e_2'' = v_2''v_0'' \) and \( f_0 = v_0'v_0'' \). In order to write down an incidence matrix \( H \) for \( \Phi \) and the adjacency matrix of the corresponding line graph \( L(\Phi) \), we choose the ordering \( v_0', v_1', v_2', v_0'', v_1'', v_2'' \) for the elements in \( V(\Gamma) \), and the ordering \( e_0', e_1', e_2', f_0, e_0'', e_1'', e_2'' \) for those in \( E(\Gamma) \). The gains of the directed cycles \( C_0' := e_0'e_1'e_2'20 \) and \( C_0'' := e_0''e_1''e_2''20 \) are
\[
\gamma(C_0') = e^{i\frac{\pi}{2}} \quad \text{and} \quad \gamma(C_0'') = -1.
\]

An incidence matrix \( H \) for \( \Phi \) is given by
\[
H = \begin{pmatrix}
1 & 0 & 1 & 1 & 0 & 0 & 0 \\
-1 & e^{i\frac{\pi}{2}} & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix},
\]

According to the rules explained in Section 2.3, the graph \( L_H(\Phi) \) is depicted in Fig. 4 and its adjacency matrix is
\[
A(L_H(\Phi)) = \begin{pmatrix}
0 & e^{\frac{i\pi}{2}} & 1 & 1 & 0 & 0 & 0 \\
e^{\frac{i\pi}{2}} & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & -1 & 0 & -1 \\
0 & 0 & 0 & -1 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & -1 & 0 & -1 \\
0 & 0 & 0 & -1 & 1 & 1 & 0
\end{pmatrix}.
\]

For instance, \( \gamma_L(\gamma_0'\gamma_1') = \prod_{v_i'v_i''} \gamma_i' \gamma_i'' = -e^{i\frac{\pi}{2}} = e^{i\frac{3\pi}{2}} \). By Theorem 3.1 and Corollary 3.2, the graph \( \Psi = \Phi - \{e_0'\} \) is a connected foundation, and \( \Phi \) has the form \( \Psi_{e_0}' \). Hence, we expect to find \( -2 \) as eigenvalue of \( A(L_H(\Phi)) \) of multiplicity 1. A MATLAB computation confirms that the characteristic polynomial
\[
\phi(L(\Phi), x) = x^7 - 10x^5 - 5x^4 + 24x^3 + 17x^2 - 9x - 6,
\]
has seven distinct roots of multiplicity one, namely,
\[
\text{Spec}(A(L_H(\Phi))) = \left\{ -2, -\sqrt{3}, -1, 1 - 2 \cos \left( \frac{2\pi}{9} \right), 1 - 2 \sin \left( \frac{\pi}{18} \right), \sqrt{3}, 1 - 2 \cos \left( \frac{2\pi}{9} \right) \right\}.
\]

The row-column product confirms that the vector
\[
(a_0', a_1', a_2', b_0, a_0'', a_1'', a_2'')^\top = (2e^{i\frac{2\pi}{9}}, 2e^{i\frac{\pi}{3}}, 2e^{i\frac{2\pi}{9}}, 2, 1, 1, 1)^\top,
\]
is an \(-2\)-eigenvector for \( A(L_H(\Phi)) \). We leave to reader to check that its components satisfy the formulæ given in the statement of Theorem 3.6., after noting that
\[
\left( 1 - \gamma(C_0') \right)^{-1} = e^{i\frac{\pi}{2}} = e^{i\frac{3\pi}{2}}.
\]

EXAMPLE 4.2. Let \( \tilde{\Phi} = (\tilde{\Gamma}, \tilde{\gamma}) \) be the complex unit gain graph depicted in Fig. 5. The vertex and the edge labeling are consistent with the ones used in Fig. 3. Namely, \( v_0' = v_0'', e_i' = v_i'v_{i+1}' \), and \( e_i'' = v_i''v_{i+1}'' \) for
Line Graphs of Complex Unit Gain Graphs with Least Eigenvalue $-2$

$i \in \{0, 1\}; e_2' = v_2'v_0'$ and $e_2'' = v_2''v_0''$. Once we choose the ordering $v_0', v_1', v_2', v_1'', v_2''$ for the elements in $V(\Gamma)$, and the ordering $e_0', e_1', e_2', e_0'', e_1'', e_2''$ for those in $E(\Gamma)$, an incidence matrix $\tilde{H}$ for $\tilde{\Phi}$ is given by

$$
\tilde{H} = \begin{pmatrix}
1 & 0 & 1 & 1 & 0 & 1 \\
-1 & e^{i\frac{2\pi}{3}} & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & i & 0 \\
0 & 0 & 0 & 0 & -1 & -1
\end{pmatrix}.
$$

The graph $\mathcal{L}_{\tilde{H}}(\tilde{\Phi})$ is depicted in Fig. 5 and its adjacency matrix is

$$
A(\mathcal{L}_{\tilde{H}}(\tilde{\Phi})) = \begin{pmatrix}
0 & e^{i\frac{4\pi}{3}} & 1 & 1 & 0 & 1 \\
e^{i\frac{2\pi}{3}} & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & -i & 1 \\
0 & 0 & 0 & i & 0 & 1 \\
1 & 0 & 1 & 1 & 1 & 0
\end{pmatrix}.
$$

Note that the gains of the directed cycles $C_0' := e_0'v_1'e_2'v_0'$ and $C_0'' := e_0''v_1''e_2''v_0''$ are

$$
\gamma(C_0') = e^{i\frac{2\pi}{3}} \quad \text{and} \quad \gamma(C_0'') = i.
$$

In this example too, by Theorem 3.1 and Corollary 3.2, the graph $\hat{\Phi} = \tilde{\Phi} - \{e_0'\}$ is a connected foundation and $\hat{\Phi}$ has the form $\hat{\Psi}_{e_0'}$. Hence, we expect to find $-2$ as eigenvalue of $A(\mathcal{L}_{\tilde{H}}(\hat{\Phi}))$ of multiplicity 1. Our expectation is confirmed by a MATLAB computation, which gives

$$
\phi(\mathcal{L}_{\tilde{H}}(\hat{\Phi}), x) = (x + 2)(x + 1)(x - 1)(x^3 - 2x^2 - 5x + 3).
$$

With hand calculations, it is not hard to verify that

$$
x = (a_0', a_1', a_2', a_0'', a_1'', a_2'')^\top = (2, 2e^{-i\frac{2\pi}{3}}, 2e^{i\frac{2\pi}{3}}, (1 - i)e^{i\frac{4\pi}{3}}, (1 + i)e^{i\frac{4\pi}{3}}, (1 + i)e^{i\frac{4\pi}{3}})^\top,
$$

is an $-2$-eigenvector for $A(\mathcal{L}_{\tilde{H}}(\hat{\Phi}))$. Its components satisfy the formulae given in the statement of Theorem 3.7.

In fact, since $\hat{\gamma}(e_0''e_0') = 1$, for $a_0' = 2$, Equation 3.19 reads

$$
a_0'' = -2 \left( 1 - \gamma(C_0'') \right)^{-1} \left( 1 - \gamma(C_0') \right) = -2(1 + i)^{-1}(1 - e^{-i\frac{2\pi}{3}}) = (1 - i)e^{i\frac{4\pi}{3}}.
$$

A simple check shows that the other components of $x$ verify (3.20) and (3.21).

**Acknowledgments.** The authors thank the anonymous referee for his/her careful reading. His/her suggestions greatly improved the presentation of this paper. The research that led to the present paper was partially supported by a grant of the group GNSAGA of INdAM.

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