ON THE NUMBER OF ZEROS TO THE EQUATION
\[ f(x_1) + \ldots + f(x_n) = a \] OVER FINITE FIELDS

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Abstract. Let \( p \) be a prime, \( k \) a positive integer and let \( \mathbb{F}_q \) be the finite field of \( q = p^k \) elements. Let \( f(x) \) be a polynomial over \( \mathbb{F}_q \) and \( a \in \mathbb{F}_q \). We denote by \( N_s(f, a) \) the number of zeros of \( f(x_1) + \cdots + f(x_s) = a \). In this paper, we show that

\[
\sum_{s=1}^\infty N_s(f, 0)x^s = \frac{x}{1 - qx} - \frac{xM_f(x)}{qM_f(x)},
\]

where

\[
M_f(x) := \prod_{m \in \mathbb{F}_q^* \text{ s.t. } S_{f, m} \neq 0} (x - \frac{1}{S_{f, m}})
\]

with \( S_{f, m} := \sum_{x \in \mathbb{F}_q} \zeta^p \text{Tr}(mf(x)) \), \( \zeta_p \) being the \( p \)-th primitive unit root and \( \text{Tr} \) being the trace map from \( \mathbb{F}_q \) to \( \mathbb{F}_p \). This extends Richman’s theorem which treats the case of \( f(x) \) being a monomial. Moreover, we show that the generating series

\[
\sum_{s=1}^\infty N_s(f, a)x^s
\]

is a rational function in \( x \) and also present its explicit expression in terms of the first \( 2d+1 \) initial values \( N_1(f, a), \ldots, N_{2d+1}(f, a) \), where \( d \) is a positive integer no more than \( q - 1 \). From this result, the theorems of Chowla-Cowles-Cowles and of Myerson can be derived.

1. Introduction

Let \( p \) be a prime number and let \( \mathbb{F}_q \) be the finite field of \( q = p^k \) elements with \( k \) being a positive integer. Let \( F(x_1, \ldots, x_n) \) be a polynomial with \( n \) variables in \( \mathbb{F}_q \). We set \( N(F = 0) \) to be the number of \( \mathbb{F}_q \)-rational points of the affine hypersurface \( F(x_1, \ldots, x_n) = 0 \) over \( \mathbb{F}_q \). Calculating the exact value of \( N(F = 0) \) is an important topic in number theory and finite fields. In general, it is difficult to give an explicit formula for \( N(F = 0) \). The \( p \)-adic behavior of \( N(F = 0) \) has been deeply investigated by lots of authors (see, for example, [1], [2], [4], [6], [13], [15], [20] and [22]). Finding the explicit formula for \( N(F = 0) \) under certain conditions received attention from many authors for many years. See, for examples, [5], [7] to [12], [16] to [19] and [23] to [25].

Let \( f(x) \) be a polynomial over \( \mathbb{F}_q \) and \( a \in \mathbb{F}_q \). Denote by \( N_s(f, a) \) the number of \( s \)-tuples \( (x_1, \ldots, x_s) \in \mathbb{F}_q^s \) such that

\[
f(x_1) + \ldots + f(x_s) = a.
\]

If \( q = p \) and \( a = 0 \), then Chowla, Cowles and Cowles [7] proved that \( \sum_{s=1}^\infty N_s(f, 0)x^s \) is a rational function in \( x \) but its explicit expression is unknown. In the case \( f(x) = x^3 \),
q = p and \( p \equiv 1 \pmod{3} \), Chowla, Cowles and Cowles [27] showed that

\[
\sum_{s=1}^{\infty} N_s(x^3, 0)x^s = \frac{x}{1 - px} + \frac{(p - 1)(2 + bx)x^2}{1 - 3px^2 - pbx^3},
\]

where \( b \) is uniquely determined by \( 4p = b^2 + 27c^2 \) and \( b \equiv 1 \pmod{3} \). From this, one can read an expression of \( N_s(x^3, 0) \) for each integer \( s \geq 1 \). Myerson [10] extended the Chowla-Cowles-Cowles theorem from \( \mathbb{F}_p \) to \( \mathbb{F}_q \). Let \( a \in \mathbb{F}_q^* := \mathbb{F}_q \setminus \{0\} \). If \( q = 2 \pmod{3} \), it is known that every element in \( \mathbb{F}_q \) is a cube, and so \( N_s(x^3, a) = q^{s-1} \). If \( q = 1 \pmod{3} \) with \( p = 2 \pmod{3} \), then Wolfmann [25] gave a formula for \( N_s(x^3, a) \) but did not present the explicit expression for \( \sum_{s=1}^{\infty} N_s(x^3, a)x^s \). By using Gauss sum, Jacobi sum and the Hasse-Davenport relation, Hong and Zhu [10] showed that if \( q = 1 \pmod{3} \), then the generating function \( \sum_{s=1}^{\infty} N_s(x^3, a)x^s \) is a rational function in \( x \) and also presented its explicit expression. In [26], Zhao, Feng, Hong and Zhu used the cyclotomic theory and exponential sums to show that the generating function \( \sum_{s=1}^{\infty} N_s(x^4, a)x^s \) is a rational function in \( x \) and also provided its explicit expression.

If \( f(x) = x^e \) is a monomial and \( q = p \) with \( p = 1 \pmod{e} \) and \( e \geq 2 \) being an integer, then Richman [18] extended the Chowla-Cowles-Cowles theorem by showing that

\[
\sum_{s=1}^{\infty} N_s(x^e, 0)x^s = \frac{x}{1 - px} - \frac{(p - 1)y(x)x}{p^2 y(x)},
\]

where \( y(x) \) is the reciprocal polynomial of the minimal polynomial \( g(x) \) of the exponential sum \( \sum_{k=0}^{p-1} \exp(2\pi ik^e/p) \), i.e., \( y(x) = x^{\deg(g(x))}g(1/x) \), and \( y'(x) \) stands for the derivative of \( y(x) \). Richman pointed out also that this result can be easily extended to \( \mathbb{F}_q \) by replacing \( \sum_{k=0}^{p-1} \exp(2\pi ik^e/p) \) with \( \sum_{k \in \mathbb{F}_q} \exp(2\pi i \text{Tr}(k^e)/p) \), where \( \text{Tr} \) denotes the trace map from \( \mathbb{F}_q \) to its prime subfield \( \mathbb{F}_p \). This reveals the relationship between the numerator and denominator of the rational expression of \( \sum_{s=1}^{\infty} N_s(x^e, 0)x^s \).

In this paper, we are mainly concerned with the number \( N_s(f, a) \) of zeros of equation \( f(x) = 0 \) and the rationality of the generating series \( \sum_{s=1}^{\infty} N_s(f, a)x^s \). As usual, let \( \mathbb{Z}, \mathbb{Q} \) and \( \mathbb{C} \) denote the ring of integers, the field of rational numbers and the field of complex numbers. Let \( \mathbb{N} \) and \( \mathbb{N}^* \) stand for the set of all nonnegative integers and the set of all positive integers. Let \( \{a_s\}_{s=1}^{\infty} \) be a sequence with \( a_s \in \mathbb{Z} \). If there exists a polynomial \( g(x) = \sum_{i=0}^{d} k_ix^i \in \mathbb{Z}[x] \) with \( k_d \neq 0 \) such that

\[
k_0a_{j+1} + k_1a_{j+2} + \ldots + k_{d-1}a_{j+d} + k_da_{j+d+1} = 0
\]

holds for all integers \( j \geq 0 \), then \( \{a_s\}_{s=1}^{\infty} \) is called a linear recursion sequence and \( g(x) \) is called a generating polynomial of \( \{a_s\}_{s=1}^{\infty} \). We also say that the sequence \( \{a_s\}_{s=1}^{\infty} \) is generated by \( g(x) \). It is easy to see that if \( \{a_s\}_{s=1}^{\infty} \) is a linear recursion sequence, and both of \( g_1(x) \) and \( g_2(x) \) are generating polynomials of \( \{a_s\}_{s=1}^{\infty} \), then \( g_1(x) + g_2(x) \) is a generating polynomial of \( \{a_s\}_{s=1}^{\infty} \) and \( kx^e \cdot g_1(x) \) is a generating polynomial of \( \{a_s\}_{s=1}^{\infty} \) for any \( k \in \mathbb{Z} \) and \( e \in \mathbb{N} \). This infers that for any \( f(x) \in \mathbb{Z}[x] \), \( f(x)g_1(x) \) is a generating polynomial of \( \{a_s\}_{s=1}^{\infty} \). It then follows that the set \( \varphi \) consisting of all the generating polynomials of the sequence \( \{a_s\}_{s=1}^{\infty} \) forms an ideal of \( \mathbb{Z}[x] \). Furthermore, by Euclidean algorithm in \( \mathbb{Z}[x] \), one can easily deduce that if \( h(x) \in \varphi \) satisfies that the degree of \( h(x) \) is minimal and the greatest common divisor of all the coefficients of \( h(x) \) is equal to 1, then \( h(x)g(x) \) for any \( g(x) \in \varphi \). Therefore \( \varphi \) is a principle ideal of \( \mathbb{Z}[x] \) generated by \( h(x) \). Such \( h(x) \) is called the minimal polynomial of the sequence \( \{a_s\}_{s=1}^{\infty} \). We define the degree of the sequence \( \{a_s\}_{s=1}^{\infty} \), denoted by \( \deg\{a_s\}_{s=1}^{\infty} \), to be the degree of the minimal polynomial of the sequence \( \{a_s\}_{s=1}^{\infty} \).
We denote by $\text{Tr}(b) := \sum_{i=0}^{k-1} b^i$ the trace map from $\mathbb{F}_{p^k}$ to $\mathbb{F}_p$, where $b \in \mathbb{F}_{p^k}$. Take $\zeta_p := \exp(\frac{2\pi i}{p})$ to be the $p$-th primitive root of unity for convenience. For any $m \in \mathbb{F}_q$, one defines the exponential sum $S_{f,m}$ over $\mathbb{F}_q$ as follows:

$$S_{f,m} := \sum_{x \in \mathbb{F}_q} \zeta_p^{\text{Tr}(mf(x))}.$$

Let

$$\Omega_f := \{S_{f,m} : m \in \mathbb{F}_q^* \text{ and } S_{f,m} \neq 0\}$$

be the set of all distinct nonzero exponential sums $S_{f,m}$. Associated to the polynomial $f(x)$, we introduce an auxiliary polynomial $m_f(x)$ as follows:

$$m_f(x) := \prod_{\lambda \in \Omega_f} (x - \lambda).$$

One can show that $m_f(x)$ is of integer coefficients. For any given $m \in \mathbb{F}_q^*$ and $f(x)$, the minimal polynomial of $S_{f,m}$ divides $m_f(x)$. Myerson [17] and Wan [21] investigated the degree of the minimal polynomial of $S_{f,1}$. In what follows, we let

$$u_s(f,a) := N_s(f,a) - q^{s-1}$$

for all positive integers $s$. The first main result of this paper can be stated as follows.

**Theorem 1.1.** Let $a \in \mathbb{F}_q$. Then each of the following is true:

(i). The sequence $\{u_s(f,a)\}_{s=1}^{\infty}$ is a linear recursion sequence with $m_f(x)$ being its generating polynomial. Furthermore, $m_f(x)$ is the minimal polynomial of the sequence $\{u_s(f,0)\}_{s=1}^{\infty}$.

(ii). We have

$$\sum_{s=1}^{\infty} N_s(f,a)x^s = \frac{x}{1-qx} - \frac{x\tilde{M}_f,a(x)}{qM_f(x)},$$

where

$$M_f(x) := \prod_{m \in \mathbb{F}_q^* \setminus S_{f,m} \neq 0} \left(x - \frac{1}{S_{f,m}}\right)$$

and

$$\tilde{M}_{f,a}(x) := \sum_{n \in \mathbb{F}_q^* \setminus S_{f,m} \neq 0} \prod_{m \in \mathbb{F}_q^* \setminus S_{f,m} \neq 0} \zeta_p^{\text{Tr}(-na)}\left(x - \frac{1}{S_{f,m}}\right).$$

In particular, if $a = 0$, then $\tilde{M}_{f,0}(x) = \tilde{M}_{f,0}(x)$ is equal to the derivative of $M_f(x)$.

By using Theorem 1.1, we can deduce an explicit expression of $\sum_{s=1}^{\infty} N_s(f,a)x^s$ in terms of the initial values $N_1(f,a), N_2(f,a), \ldots, N_{\deg(u_s(f,a))_{s=1}^{\infty}+1}(f,a)$. That is, we have the following second main result of this paper.

**Theorem 1.2.** Let $a \in \mathbb{F}_q$. Then the generating series $\sum_{s=1}^{\infty} N_s(f,a)x^s$ is a rational function in $x$. Furthermore, we have

$$\sum_{s=1}^{\infty} N_s(f,a)x^s = \frac{x}{1-qx} + \frac{\sum_{i=1}^{d} \left(\sum_{k \geq 0, j \geq 1} c_k u_j(f,a)\right)x^i}{\sum_{i=0}^{d} c_i x^i},$$

where $d := \deg(u_s(f,a))_{s=1}^{\infty}$ and $X := (c_d, \ldots, c_1, c_0)^T$ is any given nonzero integer solution of $AX = 0$ with $A := (u_{i+j-1}(f,a))_{1 \leq i, j \leq d+1}$ being the Hankel matrix of order $d+1$ associated with the sequence $\{u_s(f,a)\}_{s=1}^{\infty}$. 

Remark 1.3. The positive integer $d$ in Theorem 1.2 can be taken as any integer greater than $\deg\{u_s(f, a)\}_{s=1}^\infty$, and the rational expression of $\sum_{s=1}^\infty N_s(f, a)x^s$ is unchanged.

This paper is organized as follows. First of all, in Section 2, we show several preliminary lemmas that are needed in the proofs of Theorems 1.1 and 1.2. In Section 3, we present the proofs of Theorems 1.1 and 1.2. Two examples are given in the last section to demonstrate the validity of Theorems 1.1 and 1.2.

2. Preliminary lemmas

In this section, we present several preliminary lemmas that are needed in proving Theorems 1.1 and 1.2.

Lemma 2.1. [10] Let $p$ be a prime number and $k$ be a positive integer. Let $\mathbb{F}_q$ be the finite field of $q = p^k$ elements and $\mathbb{F}_q^*$ its multiplicative group. Then for any $x_0 \in \mathbb{F}_q$, we have

$$\sum_{x \in \mathbb{F}_q} e_{p}(x x_0) = \begin{cases} q & \text{if } x_0 = 0, \\ 0 & \text{if } x_0 \neq 0. \end{cases}$$

Lemma 2.2. [6,12,14] The trace function $Tr$ satisfies the following properties:

(i). $Tr(\alpha + \beta) = Tr(\alpha) + Tr(\beta)$ for all $\alpha, \beta \in \mathbb{F}_q$.
(ii). $Tr(c\alpha) = cTr(\alpha)$ for all $c \in \mathbb{F}_p$ and $\alpha \in \mathbb{F}_q$.

Lemma 2.3. Let $R$ be a singular integer square matrix. Then the matrix equation $RX = 0$ has a nonzero integer solution.

Proof. It is a standard result from linear algebra over $\mathbb{Z}$. For the completeness, here we still provide a detailed proof.

Let the order of $R$ be $n$. Since $RX = 0$ is solvable over $\mathbb{Q}$, one may let $X_0 \in \mathbb{Q}^n$ be a nonzero rational solution of $RX = 0$. Then multiplying by the least common denominator $m$ of all the components of $X_0$ gives us that $mX_0 \in \mathbb{Z}^n$ is a nonzero integer solution of $RX = 0$. Thus Lemma 2.3 is proved.

Lemma 2.4. Let $h(x) = \prod_{i=1}^n (x - \lambda_i)^{k_i}$, with $\lambda_i \in \mathbb{C}$ and $k_i \in \mathbb{N}^*$ for $1 \leq i \leq n$, and $\lambda_i \neq \lambda_j$ for $1 \leq i \neq j \leq n$. Let $H(x) = \prod_{i=1}^n (x - \lambda_i)$ be the radical of $h(x)$. If $h(x) \in \mathbb{Z}[x]$, then $H(x) \in \mathbb{Z}[x]$.

Proof. Since $h(x) \in \mathbb{Z}[x]$ and $\mathbb{Z}[x]$ is a unique factorization domain (U.F.D.), by the arithmetic fundamental theorem over the ring $\mathbb{Z}[x]$, one may let

$$h(x) = h_{r_1}^{e_1}(x) \cdots h_{r_r}^{e_r}(x)$$

with $r, e_1, \ldots, e_r \in \mathbb{N}^*$ and $h_1(x), \ldots, h_r(x) \in \mathbb{Z}[x]$ being $r$ distinct irreducible polynomials. Since each of $h_1(x), \ldots, h_r(x)$ has no repeated complex roots and any two of $h_1(x), \ldots, h_r(x)$ have no common complex root, the product $h_1(x) \cdots h_r(x)$ has no repeated complex roots. Hence the set of complex roots of $h_1(x) \cdots h_r(x)$ is equal to the set of complex roots of $h(x)$. But the set of complex roots of $h_1(x)$ equals the set of complex roots of $H(x)$. Thus the set of complex roots of $h_1(x) \cdots h_r(x)$ is equal to the set of complex roots of $H(x)$.

Notice that $H(x)$ has also no repeated complex roots. It then follows from the assumption that $H(x)$ and $h(x)$ are monic that

$$H(x) = h_1(x) \cdots h_r(x).$$

Thus $H(x) \in \mathbb{Z}[x]$ as required.

The proof of Lemma 2.4 is complete.
For a polynomial \( f(x) \in \mathbb{Z}[x] \) of degree \( d \), we denote by \( \overline{f}(x) \) the reciprocal polynomial of \( f(x) \), i.e., \( \overline{f}(x) := x^d f(x^{-1}) \).

**Lemma 2.5.** Let \( r(x) \in \mathbb{Z}[x] \) be a polynomial and let \( \{a_n\}_{n=1}^\infty \) be a linear recursion sequence of integers. Then \( \{a_n\}_{n=1}^\infty \) is generated by \( r(x) \) if and only if \( \overline{r}(x) \sum_{s=1}^\infty a_s x^{s-1} \) is a polynomial of degree \( < \deg r(x) \).

**Proof.** Let \( r(x) = \sum_{i=0}^m b_{m-i} x^i \in \mathbb{Z}[x] \), where \( m \geq 1 \) is an integer. Then \( \overline{r}(x) = \sum_{i=0}^m b_i x^i \).

It follows that
\[
\overline{r}(x) \sum_{s=1}^\infty a_s x^{s-1} = \left( \sum_{i=0}^m b_i x^i \right) \left( \sum_{s=1}^\infty a_s x^{s-1} \right) = \sum_{j=0}^{m-1} \left( \sum_{i=0}^j b_i a_{j-i+1} \right) x^j + \sum_{j=m}^\infty \left( \sum_{i=0}^j b_i a_{j-i+1} \right) x^j.
\]

(2.1)

Notice that \( \{a_n\}_{n=1}^\infty \) is generated by \( r(x) \) if and only if \( \sum_{i=0}^m b_i a_{m-i+1} = 0 \) for all integers \( j \geq m \). But by (2.1), the latter one is true if and only if
\[
\overline{r}(x) \sum_{s=1}^\infty a_s x^{s-1} = \sum_{j=0}^{m-1} \left( \sum_{i=0}^j b_i a_{j-i+1} \right) x^j.
\]

Thus \( \{a_n\}_{n=1}^\infty \) is generated by \( r(x) \) if and only if \( \overline{r}(x) \sum_{s=1}^\infty a_s x^{s-1} \) is a polynomial of degree \( < \deg r(x) \). So Lemma 2.5 is proved. \( \square \)

**Lemma 2.6.** Let \( r(x) \in \mathbb{Z}[x] \) be a monic polynomial of degree \( d \) with \( r(0) \neq 0 \) and having \( d \) different complex roots \( \alpha_1, ..., \alpha_d \). Let \( \{a_n\}_{n=1}^\infty \) be the linear recursion sequence of integers generated by \( r(x) \). Then there are \( d \) complex numbers \( \lambda_1, ..., \lambda_d \) which are uniquely determined by the sequence \( \{a_n\}_{n=1}^\infty \) such that
\[
\sum_{s=1}^\infty a_s x^{s-1} = \frac{\lambda_1}{1 - \alpha_1 x} + ... + \frac{\lambda_d}{1 - \alpha_d x}.
\]

(2.2)

Furthermore, we have \( a_s = \sum_{i=1}^d \lambda_i \alpha_i^{s-1} \) for each integer \( s \geq 1 \), and \( r(x) \) is the minimal polynomial of \( \{a_n\}_{n=1}^\infty \) if and only if all of \( \lambda_1, ..., \lambda_d \) are nonzero.

**Proof.** Let \( t(x) := \overline{r}(x) \sum_{s=1}^\infty a_s x^{s-1} \). Since \( \{a_n\}_{n=1}^\infty \) is generated by \( r(x) \), by Lemma 2.5 one knows that \( t(x) \) is a polynomial of integer coefficients and \( \deg(t(x)) < \deg(r(x)) \).

Noticing that
\[
\overline{r}(x) = x^d (x^{-1} - \alpha_1)...(x^{-1} - \alpha_d) = (1 - \alpha_1 x) \cdots (1 - \alpha_d x),
\]

one derives that
\[
\sum_{s=1}^\infty a_s x^{s-1} = \frac{t(x)}{\overline{r}(x)} = \frac{t(x)}{(1 - \alpha_1 x) \cdots (1 - \alpha_d x)}.
\]

(2.3)

Since \( \alpha_1, ..., \alpha_d \) are pairwise distinct and \( r(0) \neq 0 \) implying that none of \( \alpha_1, ..., \alpha_d \) is zero, we have \( \prod_{j=1}^d (1 - \alpha_j \alpha_i^{-1}) \neq 0 \). So for any integer \( k \) with \( 1 \leq k \leq d \), one may let
\[
\lambda_k := \frac{t(\alpha_k^{-1})}{\prod_{j \neq k} (1 - \alpha_j \alpha_k^{-1})}.
\]

(2.4)
Then
\[ t(\alpha_k^{-1}) = \lambda_k \prod_{j=1, j \neq k}^d (1 - \alpha_j \alpha_k^{-1}) = \sum_{i=1}^d \lambda_i \prod_{j=i, j \neq i}^d (1 - \alpha_j \alpha_k^{-1}). \]
Hence \( \alpha_1^{-1}, \ldots, \alpha_d^{-1} \) are \( d \) distinct zeros of the polynomial
\[ t(x) - \sum_{i=1}^d \lambda_i \prod_{j=i, j \neq i}^d (1 - \alpha_j x). \] (2.5)

But the degree of the polynomial in (2.5) is clearly no more than \( d - 1 \). Hence the polynomial in (2.5) is equal to zero. One then derives that
\[ t(x) = \sum_{i=1}^d \lambda_i \prod_{j=1, j \neq i}^d (1 - \alpha_j x) = \left( \frac{\lambda_1}{1 - \alpha_1 x} + \ldots + \frac{\lambda_d}{1 - \alpha_d x} \right) \prod_{i=1}^d (1 - \alpha_i x). \] (2.6)

Thus (2.2) follows immediately from (2.3) and (2.6). So (2.2) is proved.

Now by (2.2), we can deduce that
\[ \sum_{s=1}^\infty a_s x^{s-1} = \sum_{s=1}^\infty \lambda_1 (\alpha_1 x)^{s-1} + \ldots + \sum_{s=1}^\infty \lambda_d (\alpha_d x)^{s-1} = \sum_{s=0}^\infty \left( \sum_{i=1}^d \lambda_i \alpha_i^s \right) x^s. \]
Comparing the coefficients of \( x^{s-1} \) on both sides gives us \( a_s = \sum_{i=1}^d \lambda_i \alpha_i^{s-1} \) as desired.

In what follows, we show that \( r(x) \) is the minimal polynomial of \( \{a_n\}_{n=1}^\infty \) if and only if all the \( \lambda_i \) \( (1 \leq i \leq d) \) are nonzero. To do so, we first show that \( r(x) \) is the minimal polynomial of \( \{a_n\}_{n=1}^\infty \) if and only if \( \text{gcd}(\tau(x), t(x)) = 1 \).

Suppose that \( r(x) \) is the minimal polynomial of \( \{a_n\}_{n=1}^\infty \). Let \( \text{gcd}(\tau(x), t(x)) = \overline{d}(x) \). If \( \overline{d}(x) \neq 1 \), then \( \deg(\overline{d}(x)) \geq 1 \) since the greatest common divisor of all the coefficients of \( r(x) \) is equal to 1. Moreover, we write \( t(x) = t_0(x) \overline{d}(x) \) and \( \tau(x) = \tau_0(x) \overline{d}(x) \) with \( t_0(x), \tau_0(x) \in \mathbb{Z}[x] \). Since \( t(x) = \tau(x) \sum_{s=1}^\infty a_s x^{s-1} \in \mathbb{Z}[x] \) and \( \deg(t(x)) < \deg(r(x)) \), one has \( t_0(x) = \overline{\tau_0}(x) \sum_{s=1}^\infty a_s x^{s-1} \). But \( r(0) \neq 0 \) tells us that \( \deg(\overline{\tau_0}(x)) = \deg(r) \) and \( \overline{\tau_0}(0) \neq 0 \). It then follows that
\[ \deg(t_0(x)) = \deg(t(x)) - \deg(\overline{d}(x)) < \deg(r(x)) - \deg(\overline{d}(x)) = \deg(\tau(x)) - \deg(\overline{d}(x)) = \deg(\overline{\tau_0}(x)) = \deg(r_0(x)), \]
where \( r_0(x) \) is the reciprocal polynomial of \( \tau_0(x) \). So by Lemma 2.5, one knows that \( r_0(x) \) is a generating polynomial of \( \{a_n\}_{n=1}^\infty \). This contradicts with the assumption that \( r(x) \) is the minimal polynomial of \( \{a_n\}_{n=1}^\infty \) since \( \deg(r_0(x)) < \deg(r(x)) \). Hence we must have \( \text{gcd}(\tau(x), t(x)) = 1 \).

Conversely, let \( \text{gcd}(\tau(x), t(x)) = 1 \). Assume that \( r(x) \) is not the minimal polynomial of \( \{a_n\}_{n=1}^\infty \). Since \( r(x) \) is monic, there exists a polynomial \( r_1(x) \) of degree \( < \deg(r(x)) \) which is the minimal polynomial of \( \{a_n\}_{n=1}^\infty \). By the fact that all the generating polynomials of \( \{a_n\}_{n=1}^\infty \) forms an ideal generated by \( r_1(x) \), one may let \( r(x) = r_1(x) g(x) \) with \( g(x) \in \mathbb{Z}[x] \) and \( \deg(g(x)) \geq 1 \). Evidently, we have \( \tau(x) = \tau_1(x) g(x) \). By Lemma 2.5 one may let
\[ \sum_{s=1}^\infty a_s x^{s-1} = \frac{t_1(x)}{\tau_1(x)} \] for some \( t_1(x) \in \mathbb{Z}[x] \).
Then

\[ 0 = \frac{t(x) - t_1(x)}{\overline{f}(x)} = \frac{t(x) - t_1(x)/\overline{f}(x)}{\overline{f}(x)} = \frac{t(x) - t_1(x)/\overline{f}(x)}{\overline{f}(x)}. \]

This implies that \( t(x) = t_1(x)/\overline{f}(x) \). So \( \overline{f}(x)|t(x) \) which contradicts with the fact that \( \gcd(\overline{f}(x), t(x)) = \gcd(\overline{f}(x), t(x)) = 1 \).

Thus \( r(x) \) is the minimal polynomial of \( \{a_n\}_{n=1}^\infty \). This ends the proof of the statement that \( r(x) \) is the minimal polynomial of \( \{a_n\}_{n=1}^\infty \) if and only if \( \gcd(\overline{f}(x), t(x)) = 1 \).

Finally, by (2.4) we derive that \( \lambda_i \neq 0 \) for all \( 1 \leq i \leq d \) if and only if none of the roots \( \alpha^{-1}_i, 1 \leq i \leq d, \) of \( \overline{f}(x) \) is a zero of \( t(x) \) which is equivalent to \( \gcd(\overline{f}(x), t(x)) = 1 \). Therefore \( r(x) \) is the minimal polynomial of \( \{a_n\}_{n=1}^\infty \) if and only if all the \( \lambda_i (1 \leq i \leq d) \) are nonzero.

This completes the proof of Lemma 2.6.

\[ \square \]

3. Proofs of Theorems 1.1 and 1.2

In this section, we present the proofs of Theorems 1.1 and 1.2. We begin with the proof of Theorem 1.1.

Proof of Theorem 1.1 (i). Let \( m_f(x) := \prod_{\lambda \in \Omega_f} (x - \lambda) \). First of all, we prove that \( m_f(x) \) is of integer coefficients. By Lemma 2.1 we have

\[ N_s(f, a) = \frac{1}{q} \sum_{m \in \mathbb{F}_q^*} \sum_{(x_1, \ldots, x_s) \in \mathbb{F}_q^s} \zeta_p^{\text{Tr}(m(f(x_1) + \cdots + f(x_s) - a))} \]

\[ = \frac{1}{q} \sum_{m \in \mathbb{F}_q^*} \left( \sum_{x \in \mathbb{F}_q} \zeta_p^{\text{Tr}(m(f(x))} \right)^s \zeta_p^{\text{Tr}(-ma)} \]

\[ = q^{s-1} + \frac{1}{q} \sum_{m \in \mathbb{F}_q^*} \left( \sum_{x \in \mathbb{F}_q} \zeta_p^{\text{Tr}(m(f(x))} \right)^s \zeta_p^{\text{Tr}(-ma)} \]

\[ = q^{s-1} + \frac{1}{q} \sum_{m \in \mathbb{F}_q^*} S_{f,m}^s \zeta_p^{\text{Tr}(-ma)} \]

(3.1)

Then it follows from (3.1) that

\[ q u_s(f, a) = q N_s(f, a) - q^s = \sum_{m \in \mathbb{F}_q^*} S_{f,m}^s \zeta_p^{\text{Tr}(-ma)} = \sum_{m \in \mathbb{F}_q^*} S_{f,m}^s \zeta_p^{\text{Tr}(-ma)} \]

(3.2)

Let

\[ g(x) := \prod_{m \in \mathbb{F}_q^*} (x - S_{f,m}) := \sum_{i=0}^{q-1} b_i x^i. \]

(3.3)

Then \( b_i \in \mathbb{Q}(\zeta_p) \) for all integers \( i \) with \( 0 \leq i \leq q - 1 \).

Now pick a \( \sigma \in \text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \), where \( \text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \) is the Galois group of the cyclotomic field \( \mathbb{Q}(\zeta_p) \) over \( \mathbb{Q} \). One may let \( \sigma(\zeta_p) = \zeta_p^h \) for some integer \( h \) with \( 1 \leq h \leq p - 1 \).

Then one can deduce that

\[ \sigma(S_{f,m}) = \sigma \left( \sum_{x \in \mathbb{F}_q} \zeta_p^{\text{Tr}(m(f(x))} \right) = \sum_{x \in \mathbb{F}_q} \zeta_p^{\text{Tr}(m(f(x))} = \sum_{x \in \mathbb{F}_q} \zeta_p^{\text{Tr}(hm(f(x))} = S_{f,m}^h. \]

\[ \square \]
Since $1 \leq h \leq p - 1$, one has $\{hm | m \in \mathbb{F}_q^*\} = \mathbb{F}_q^*$. It then follows that for any $\sigma \in \text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$, we have
\[
\sigma(g(x)) = \prod_{m \in \mathbb{F}_q^*} (x - \sigma(S_{f,m})) = \prod_{m \in \mathbb{F}_q^*} (x - S_{f,hm}) = \prod_{m \in \mathbb{F}_q^*} (x - S_{f,m}) = g(x).
\]

Hence we must have $b_i \in \mathbb{Q}$ for all integers $i$ with $0 \leq i \leq q - 1$.

On the other hand, it is well known that all the algebraic integers in $\mathbb{Q}(\zeta_p)$ form a ring which is $\mathbb{Z}[\zeta_p]$. By (3.3), we know that each coefficient $b_i$ ($0 \leq i \leq q - 1$) is a linear $\mathbb{Z}$-combination of powers of $\zeta_p$. In other words, $b_i \in \mathbb{Z}[\zeta_p]$ for each integer $i$ with $0 \leq i \leq q - 1$. Hence
\[
b_i \in \mathbb{Q} \cap \mathbb{Z}[\zeta_p] = \mathbb{Z}
\]
for all $0 \leq i \leq q - 1$ and so $g(x) \in \mathbb{Z}[x]$. Write $g(x) = x^e h(x)$ with $e$ being a nonnegative integer, $h(x) \in \mathbb{Z}[x]$ and $h(0) \neq 0$. Evidently, by (3.3) we have
\[
h(x) = \prod_{\lambda \in \Omega_f} (x - \lambda)^{k_\lambda} \tag{3.4}
\]
with all $k_\lambda$ being positive integers. Then Lemma 2.4 applied to $h(x)$ gives us that $m_f(x) \in \mathbb{Z}[x]$.

Consequently, we show that the integral coefficients polynomial $m_f(x)$ is a generating polynomial of $\{u_s(f,a)\}_{s=1}^\infty$. Let $d = \text{deg}(m_f(x))$. Since $m_f(x)$ is monic, one may let
\[
m_f(x) = x^d + \sum_{i=0}^{d-1} a_i x^i, \quad a_i \in \mathbb{Z}.
\]

If $S_{f,m} \neq 0$ for $m \in \mathbb{F}_q^*$, then $m_f(S_{f,m}) = 0$ that infers that
\[
S_{f,m}^d + a_{d-1} S_{f,m}^{d-1} + \cdots + a_1 S_{f,m} + a_0 = 0.
\]

Multiplying by $S_{f,m}^{-s} \zeta_p^{-s} (\text{Tr}(-ma))$ on both sides gives us that
\[
S_{f,m}^s \zeta_p^{-s} (\text{Tr}(-ma)) + a_{d-1} S_{f,m}^{s-1} \zeta_p^{-s} (\text{Tr}(-ma)) + \cdots + a_0 S_{f,m}^{-d} \zeta_p^{-s} (\text{Tr}(-ma)) = 0
\]
for any integer $s \geq d$. Then taking sum tells that
\[
\sum_{m \in \mathbb{F}_q^*} S_{f,m}^{s} \zeta_p^{-s} (\text{Tr}(-ma)) + a_{d-1} \sum_{m \in \mathbb{F}_q^*} S_{f,m}^{s-1} \zeta_p^{-s} (\text{Tr}(-ma)) + \cdots + a_0 \sum_{m \in \mathbb{F}_q^*} S_{f,m}^{-d} \zeta_p^{-s} (\text{Tr}(-ma)) = 0 \tag{3.5}
\]
for all integers $s \geq d$. Putting (3.2) into (3.5) gives that for any integer $s \geq d + 1$, one derives that
\[
u_s(f,a) + a_{d-1} \nu_{s-1}(f,a) + \cdots + a_0 \nu_{s-d}(f,a) = 0. \tag{3.6}
\]

Thus $\{u_s(f,a)\}_{s=1}^\infty$ is a linear recursion sequence, and $m_f(x)$ is a generating polynomial of $\{u_s(f,a)\}_{s=1}^\infty$. This implies that the minimal polynomial of $\{u_s(f,a)\}_{s=1}^\infty$ divides $m_f(x)$ as desired.
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Let us now show that \( m_f(x) \) is the minimal polynomial of \( u_s(f, 0) \). By (3.2), we have

\[
\sum_{s=1}^{\infty} u_s(f, a)x^{s-1} = \sum_{s=1}^{\infty} \sum_{m \in \mathbb{Q}^*_{F_q}} \frac{S_{f,m}^{Tr(-ma)}}{q} x^{s-1}
\]

\[
= \sum_{m \in \mathbb{Q}^*_{F_q}} \frac{\zeta_p^{Tr(-ma)}}{q} \sum_{s=1}^{\infty} S_{f,m} x^{s-1}
\]

\[
= \sum_{m \in \mathbb{Q}^*_{F_q}} \frac{\zeta_p^{Tr(-ma)}}{q} \frac{S_{f,m}}{1 - S_{f,m} x}
\]

\[
= \sum_{\lambda \in \Omega_f} \frac{\lambda}{q} \sum_{m \in \{ x \in \mathbb{Q}^*_{F_q} : S_{f,m,x} = \lambda \}} \frac{\zeta_p^{Tr(-ma)}}{1 - \lambda x}.
\]

Let \( a = 0 \). Then \( \zeta_p^{Tr(-ma)} = 1 \) for all \( m \in \mathbb{Q} \). Notice that \( \lambda \neq 0 \) for all \( \lambda \in \Omega_f \). Hence

\[
= \sum_{\lambda \in \Omega_f} \frac{\lambda}{q} \sum_{m \in \{ x \in \mathbb{Q}^*_{F_q} : S_{f,m,x} = \lambda \}} \frac{\zeta_p^{Tr(-ma)}}{1 - \lambda x}.
\]

Thus by Lemma 2.6, one knows that \( m_f(x) \) equals the minimal polynomial of \( \{ u_s(f, 0) \}_{s=1}^{\infty} \).

Part (i) is proved.

(ii). By (3.7), we deduce that

\[
\sum_{s=1}^{\infty} u_s(f, a) x^s = \frac{1}{q} \sum_{m \in \mathbb{Q}^*_{F_q}} \frac{\zeta_p^{Tr(-ma)}}{S_{f,m}} \frac{S_{f,m,x}}{1 - S_{f,m} x}
\]

\[
= \frac{x}{q} \sum_{m \in \mathbb{Q}^*_{F_q}} \frac{\zeta_p^{Tr(-ma)}}{S_{f,m}} \frac{S_{f,m,x}}{1 - S_{f,m} x}
\]

\[
= \frac{x}{q} \sum_{m \in \mathbb{Q}^*_{F_q \setminus \{m\}}} \frac{\zeta_p^{Tr(-ma)}}{S_{f,m \neq 0}} \frac{S_{f,m,x}}{1 - S_{f,m} x} \prod_{m \in \mathbb{Q}^*_{F_q \setminus \{m\}}} \left( x - \frac{1}{S_{f,m}} \right)
\]

\[
= -\frac{x\dot{M}_f,a(x)}{qM_f(x)}.
\]

It then follows that

\[
\sum_{s=1}^{\infty} N_{f,s}(a) x^s = \sum_{s=1}^{\infty} (u_{f,s}(a) + q^{s-1}) x^s = \frac{x}{1 - qx} - \frac{x\dot{M}_f,a(x)}{qM_f(x)}.
\]

as required.

In particular, if \( a = 0 \), then

\[
\dot{M}_f,a(x) = \dot{M}_f,0(x) = \sum_{m \in \mathbb{Q}^*_{F_q \setminus \{m\}}} \frac{\zeta_p^{Tr(-ma)}}{S_{f,m \neq 0}} \prod_{m \in \mathbb{Q}^*_{F_q \setminus \{m\}}} \left( x - \frac{1}{S_{f,m}} \right) = M_f(x).
\]

In other words, \( \dot{M}_f,0(x) \) equals the derivative of \( M_f(x) \). So part (ii) is proved.
This finishes the proof of Theorem 1.1.

Finally, we show Theorem 1.2 as the conclusion of this section.

**Proof of Theorem 1.2.** For brevity, we write \( u_s(f, a) \) as \( u_s \) for all positive integer \( s \) in what follows. Since \( d = \deg \{ u_s \}_{s=1}^{\infty} \), by Theorem 1.1 one knows that the sequence \( \{ u_s(f, a) \}_{s=1}^{\infty} \) is a linear recursion sequence. So one may let

\[
g(x) = \sum_{i=0}^{d-1} a_i x^i + x^d
\]

be any given generating polynomial of \( \{ u_s \}_{s=1}^{\infty} \). Then

\[
a_0 u_s + \cdots + a_{d-1} u_{s+d-1} + u_{s+d} = 0 \tag{3.8}
\]

holds for all positive integer \( s \). It follows that

\[
a_0 A_1 + \cdots + a_{d-1} A_d + A_{d+1} = 0,
\]

where for any integer \( i \) with \( 1 \leq i \leq d + 1 \), \( A_i := (u_i, u_{i+1}, \ldots, u_{i+d})^T \) stands for the \( i \)-th column of the \((d+1) \times (d+1)\) matrix \( A = (u_{ij})_{1 \leq i,j \leq d+1} \). Since \( X = (a_0, \ldots, a_{d-1}, 1)^T \in \mathbb{Z}^{d+1} \) is a nonzero solution of the matrix equation \( AX = 0 \), \( A \) is singular. Then Lemma 2.3 tells us that \( AX = 0 \) has nonzero integer solutions. Now we pick \( X_0 = (c_d, \ldots, c_1, c_0) \) to be such an arbitrary solution. Then

\[
c_d u_i + \cdots + c_1 u_{i+d-1} + c_0 u_{i+d} = 0 \tag{3.9}
\]

for all integers \( i \) with \( 1 \leq i \leq d + 1 \).

In what follows, we use induction on \( i \) to show that (3.9) is true for all positive integers \( i \). First of all, since (3.9) holds for all positive integers \( i \leq d + 1 \), one may let \( r \) be an integer with \( r \geq d + 1 \), and we assume that (3.9) is true for all positive integers \( i \leq r \). In the following, we prove that (3.9) remains true for the \( r + 1 \) case.

Letting \( s = r + 1 - d, r + 2 - d, \ldots, r, r + 1 \) in (3.8) and then applying the inductive hypothesis, one arrives at

\[
c_d u_{r+1} + c_{d-1} u_{r+2} + \cdots + c_1 u_{r+d} + c_0 u_{r+d+1}
\]

\[
= -c_d \sum_{i=0}^{d-1} a_i u_{r+1-d+i} - c_{d-1} \sum_{i=0}^{d-1} a_i u_{r+2-d+i} - \cdots - c_0 \sum_{i=0}^{d-1} a_i u_{r+1+i}
\]

\[
= -a_0 \sum_{t=0}^{d} c_t u_{r+1-t} - a_1 \sum_{t=0}^{d} c_t u_{r+2-t} - \cdots - a_{d-2} \sum_{t=0}^{d} c_t u_{r+d-2-t} - a_{d-1} \sum_{t=0}^{d} c_t u_{r+d-t}
\]

\[= 0.
\]

Hence (3.9) is valid for the \( r + 1 \) case. Hence (3.9) holds for all positive integers \( i \).
Finally, applying (3.9) we can deduce that
\[(c_0 + c_1 x + \cdots + c_d x^d) \sum_{s=1}^{\infty} u_s x^s\]
\[= \sum_{i=1}^{d} \left( \sum_{j=k=0, j \geq 1} c_k u_j \right) x^i + \sum_{i=d+1}^{\infty} \left( \sum_{j=k=1, 0 \leq k \leq j \geq 1} c_k u_j \right) x^i\]
\[= \sum_{i=1}^{d} \left( \sum_{j=k=0, j \geq 1} c_k u_j \right) x^i.

However, \(u_s = N_s(f, a) - q^{s-1}\) for any integer \(s \geq 1\). It then follows that
\[\sum_{s=1}^{\infty} N_s(f, a) x^s = \frac{x}{1 - qx} + \sum_{s=1}^{\infty} u_s x^s\]
\[= \frac{x}{1 - qx} + \frac{\sum_{i=1}^{d} \left( \sum_{j=k=0, j \geq 1} c_k u_j \right) x^i}{c_0 + c_1 x + \cdots + c_d x^d}\]
as expected.

This concludes the proof of Theorem 1.2. \(\Box\)

4. Examples

In this section, we supply two examples to illustrate the validity of Theorems 1.1 and 1.2. We write \(N_s(a)\) as \(N_s(f, a)\) and \(u_s\) as \(u_s(f, a)\) for convenience.

Example 4.1. Let \(q = p = 1 \pmod{3}\) and \(f(x) = x^4\). By [7], we have \(N_1(0) = 1, N_2(0) = 3p - 2, N_3(0) = p^2 + b(p - 1), N_4(0) = p^3 + 6pb(p - 1), N_5(0) = p^4 + 5pb(p - 1), N_6(0) = p^5 + (18p^2 + pb^2)(p - 1)\) and \(N_7(0) = p^6 + 21p^2b(p - 1)\), where \(4p = b^2 + 27\) and \(b \equiv 1 \pmod{3}\). Then \(u_1 = 0, u_2 = 2(p - 1), u_3 = b(p - 1), u_4 = 6p(p - 1), u_5 = 5pb(p - 1), u_6 = (18p^2 + pb^2)(p - 1)\) and \(u_7 = 21p^2b(p - 1)\). Then the Hankel matrix \(A\) is given by
\[A = (p - 1) \begin{pmatrix} 0 & 2 & b & 6p \\ 2 & b & 6p & 5pb \\ b & 6p & 5pb & 18p^2 + pb^2 \\ 6p & 5pb & 18p^2 + pb^2 & 21p^2b \end{pmatrix}.
\]

It follows that \((-pb, -3p, 0, 1)^T := (c_0, c_2, c_1, c_0)^T\) is a solution of the matrix equation \(AX = 0\). By Theorem 1.2 we obtain that
\[\sum_{i=1}^{4} \left( \sum_{j=k=0, j \geq 1} c_k u_j \right) x^i = c_0 u_1 x + (c_0 u_2 + c_1 u_1) x^2 + (c_0 u_3 + c_1 u_2 + c_2 u_1) x^3 = 2(p - 1)x^2 + b(p - 1)x^3.
\]

Hence
\[\sum_{s=1}^{\infty} N_s(0)x^s = \frac{x}{1 - 7x} + \frac{2(p - 1)x^2 + b(p - 1)x^3}{1 - 3px^2 - pbx^3}.
\]
This is exactly Chowla, Cowles and Cowles’ formula presented in [7].

**Example 4.2.** Let \( q = 5 \) and \( f(x) = x^2 + x^3 \). By calculations, one finds that \( N_1(1) = 1, N_2(1) = 4, N_3(1) = 20, N_4(1) = 120, N_5(1) = 650, N_6(1) = 3225, N_7(1) = 15750, N_8(1) = 78000 \) and \( N_9(1) = 390000 \). Then we have \( u_1 = 0, u_2 = -1, u_3 = -5, u_4 = -5, u_5 = 25, u_6 = 100, u_7 = 125, u_8 = -125 \) and \( u_9 = -625 \). So the Hankel matrix \( A \) is given by

\[
A = \begin{pmatrix}
0 & -1 & -5 & 25 \\
-1 & -5 & -5 & 25 \\
-5 & 25 & 100 & 125 \\
25 & 100 & 125 & -125 & -625
\end{pmatrix}.
\]

It then follows that \((25, -25, 15, -5, 1)^T = (c_4, c_3, c_2, c_1, c_0)^T\) is a solution of \( AX = 0 \). Applying Theorem [12], one gets that

\[
\sum_{i=1}^{4} \left( \sum_{j+k=i} c_k u_j \right) x^i = c_0 u_1 x + (c_0 u_2 + c_1 u_1) x^2 + (c_0 u_3 + c_1 u_2 + c_2 u_1) x^3 + (c_0 u_4 + c_1 u_3 + c_2 u_2 + c_3 u_1) x^4 = -x^2 + 5x^4.
\]

Therefore

\[
\sum_{s=1}^{\infty} N_s(1) x^s = \frac{x}{1-5x} + \frac{-x^2 + 5x^4}{1-5x + 15x^2 - 25x^3 + 25x^4}.
\]

On the other hand, let \( g(x) := 1 - 5x + 15x^2 - 25x^3 + 25x^4 \). Then \( \gcd(-x^2 + 5x^4, g(x)) = 1 \).

It follows from the proof of Lemma 2.5 that \( \mathcal{g}(x) = x^4 g\left(\frac{1}{x}\right) \) equals the minimal polynomial of \( \{N_s(1)\}_{s=1}^{\infty} \). By Theorem [11], one then deduces that \( g(x) \) divides \( M_f(x) \) and \( \deg(M_f(x)) \leq \left| P_{\mathbb{F}_5} \right| = 4 \). Since \( \deg g(x) = 4 \), one has \( M_f(x) = \frac{1}{25} g(x) \). Therefore

\[
\sum_{s=1}^{\infty} N_s(0) x^s = \frac{x}{1-5x} - \frac{x (1 - 5x + 15x^2 - 25x^3 + 25x^4)'}{5 (1-5x + 15x^2 - 25x^3 + 25x^4)} = \frac{x}{1-5x} + \frac{x - 6x^2 + 15x^3 - 20x^4}{1-5x + 15x^2 - 25x^3 + 25x^4}.
\]

**Acknowledgements**

The authors would like to thank the anonymous referee for careful reading of the manuscript and helpful suggestions and comments that improve the presentation of the paper.

**References**

1. A. Adolphson and S. Sperber, \( p \)-Adic estimates for exponential sums and the theorem of Chevalley-Warning, *Ann. Sci. Ecole Norm. Sup.* 20 (1987), 545-556.
2. J. Ax, Zeros of polynomials over finite fields, *Amer. J. Math.* 86 (1964), 255-261.
3. B. Berndt, R. Evans and K. Williams, *Gauss and Jacobi sums*, Wiley-Interscience, New York, 1998.
4. W. Cao, A partial improvement of the Ax-Katz theorem, *J. Number Theory* 132 (2012), 485-494.
ON THE NUMBER OF ZEROS TO THE EQUATION $f(x_1) + \ldots + f(x_n) = a$

5. L. Carlitz, The numbers of solutions of a particular equation in a finite field, *Publ. Math. Debrecen* 4 (1956), 379-384.
6. C. Chevalley, Démonstration dûe hypothèse de M. Artin (French), *Abh. Math. Sem. Univ. Hamburg* 11 (1935), 73-75.
7. S. Chowla, J. Cowles and M. Cowles, On the number of zeros of diagonal cubic forms, *J. Number Theory* 9 (1977), 502-506.
8. L. Carlitz, The numbers of solutions of a particular equation in a finite field, *Publ. Math. Debrecen* 4 (1956), 379-384.
9. C. Chevalley, Démonstration dûe hypothèse de M. Artin (French), *Abh. Math. Sem. Univ. Hamburg* 11 (1935), 73-75.
10. S.F. Hong and C.X. Zhu, On the number of zeros of diagonal cubic forms over finite fields, *Forum Math.* 33 (2021), 697-708.
11. S.N. Hu, S.F. Hong and W. Zhao, The number of rational points of a family of hypersurfaces over finite fields, *J. Number Theory* 156 (2015), 135-153.
12. K. Ireland and M. Rosen, *A classical introduction to modern number theory*, Second Edition, Springer-Verlag New York, Inc. 1990.
13. N.M. Katz, On a theorem of Ax, *Amer. J. Math.* 93 (1971), 485-499.
14. R. Lidl and H. Niederreiter, *Finite fields*, Second Ed., Encyclopedia Math. Appl., vol. 20, Cambridge University Press, Cambridge, 1997.
15. O. Moreno and C.J. Moreno, Improvements of Chevalley-Warning and the Ax-Katz theorem, *Amer. J. Math.* 117 (1995), 241-244.
16. G. Myerson, On the number of zeros of diagonal cubic forms, *J. Number Theory* 11 (1979), 95-99.
17. G. Myerson, Period polynomials and Gauss sums for finite fields, *Acta Arith.* 39 (1981), 251-264.
18. D.R. Richman, Some remarks on the number of solutions to the equation $f(x_1) + \ldots + f(x_n) = 0$, *Stud. Appl. Math.* 71 (1984), 263-266.
19. D.Q. Wan, Zeros of diagonal equations over finite fields, *Proc. Amer. Math. Soc.* 103 (1988), 1049-1052.
20. D.Q. Wan, An elementary proof of a theorem of Katz, *Amer. J. Math.* 111 (1989), 1-8.
21. D.Q. Wan, Algebraic theory of exponential sums over finite fields, Lecture Notes at 2019 HIT Undergraduate Number Theory Summer School, available at https://www.math.uci.edu/~dwan/Wan_HIT_2019.pdf.
22. E. Warning, Bermerkung zur Vorstehenden Arbeit von Herr Chevalley, *Abh. Math. Sem. Univ. Hamburg* 11 (1936), 76-83.
23. A. Weil, On some exponential sums, *Proc. Natu. Acad. Sci. U.S.A.* 34 (1948), 204-207.
24. J. Wolfmann, The number of solutions of certain diagonal equations over finite fields, *J. Number Theory* 42 (1992), 247-257.
25. J. Wolfmann, New results on diagonal equations over finite fields from cyclic codes, *Contemp. Math.* 168 (1994), 387-395.
26. J.Y. Zhao, Y.L. Feng, S.F. Hong and C.X. Zhu, On the number of zeros of diagonal quartic forms over finite fields, [arXiv:2108.00396].