Research Article

Classical Inference of the Cubic Transmuted Lindley Distribution under Type-II Censored Sample

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1. Introduction

Probability distributions are used to explain real-world phenomena. There are several practical life problems that do not fit any of the basic probability models. In order to generate more flexible statistical models, the generalization of well-known distributions has been widely used.

These real-world data sets are properly fitted using generalized distributions. Amoroso [1] developed the generalized gamma distribution to explain the income rate distribution. Since then, a number of researchers have discussed the generalizations of distributions. There are many researches that studied generalized distributions such as the inverse Gaussian distribution proposed by Good [2]; Lindley distribution developed by Lindley [3]; generalization of Pareto distribution by Ljubo [4]; Pickands III [5]; Hosking and Wallis [6] and McDonald and Xu [7] created the first and second types of generalized beta to examine income distribution. A generalized form of the exponential distribution was proposed by Gupta et al. [8]; Gupta and Kundu [9]; and Gupta and Kundu [10]. The Quadratic Transmutation Map (QTM) was introduced by Shaw and Buckley [11] and used to generate non-Gaussian distributions. Rahman et al. [12, 13] have introduced another two cubic transmuted families of distributions as the extension of the transmuted family of distributions. The Lindley distribution (LD) can be used in biology, engineering, and medicine sectors [14].

There are several generalizations that have been made over Lindley distribution, such as beta Lindley distribution proposed by Merovci and Sharma [15]; cubic ranked transmuted Lindley proposed by [23]. Our model is a a special case of [23], by making some reparameterization; However, we did not propose a new distribution; we only
made inferences on the cubic transmuted Lindley, which was generated from the cubic transmuted family proposed by Rahman et al. [16]; although our model is a special case from the cubic ranked transmuted Lindley proposed by [23], our distribution performs better in fitting real data; however, the proposed model has less number of parameters.

The Weibull Lindley distribution by Asgharzadeh et al. [17], wrapped modified Lindley distribution, is the latest modification of Lindley distribution by Chesneau et al. [18]. The transmuted generalization of Lomax distribution has been introduced by Abu El Azm et al. [19]. They introduced a new one-parameter circular distribution based on the wrapping method.

Their main objective was to extend the scope in astronomy, geology, and meteorology, that is, over circling objects. Hence, the main goal of this paper is to expand the Lindley distribution’s goodness of fit test on further aspects over lifetime, time series, environmental, production sectors, and so on, so we applied the Type-II censored sample to the CT-Lindley.

There are various instances in life testing and dependability studies when observations are overlooked or trials are abandoned prior to failure. The tester may be unable to get comprehensive data on failure times for all experimental observations. The data from all of these trials are referred to as censored data. There are several sorts of censored tests; the most common and widely utilised are Type-I censored and Type-II censored; see, for example, [20]. While on product spacing (PS), many authors used PS based on censored samples; for example, El-Sherpieny et al. [21] introduced progressive Type-II hybrid censored schemes based on PS. Alshenawy et al. [22] introduced progressive Type-II censoring schemes for the PS method by binomial random removal. Almetwally et al. [23] introduced PS estimation of Weibull distribution under adaptive Type-II progressive censoring schemes. Alshenawy et al. [24] obtained the PS estimation for a stress-strength model under progressive hybrid censored. Almongy et al. [25] used the PS method under progressive Type-II censoring schemes. Almetwally et al. [26] estimated parameters by using the PS method of flexible extension Weibull distribution based on Type-I and Type-II censoring. In the presence of fuzzy system, the PS has been used by Sabry et al. [27].

Based on the complete sample, many authors used the PS method, for example, Almetwally et al. [28]; Ahmad and Almetwally [29]; Almetwally [30]; and Almongy et al. [31]. This paper firstly aims to insert and treatise a new form distribution defined by Lindley distribution and cubic transmuted family. Considerable statistical properties of CT-Lindley distribution are shown. Secondly, the estimate of parameters for the CT-Lindley distribution is discussed using the MLE and MPS approaches. Thirdly, we examined the estimation of parameters for the CT-Lindley distribution using Type-II censored data. Monte Carlo simulation is used to determine the estimators’ efficiency. Fourthly, real-world data studies are conducted to verify the model and scheme’s validity.

A second-order Lindley distribution is developed to accomplish so, which can capture the complex behavior in real-life experiences covering the lifetime, environmental, biological, engineering, and production data sets.

The layout plan of the paper is as follows. The equations of the CT-Lindley distribution using the aid of the cubic transmuted family that was presented by Rahman et al. [16] are introduced in Section 2. Some of the distributional properties are illustrated in Section 3. The statistical inferences using both classical methods are described in Section 5. The simulation study associated with its results is recorded in Section 6. Four real-life applications of the distribution are shown in Section 7. In the end, some concluding remarks are listed in Section 8.

2. Cubic Transmuted Lindley Distribution

In this section, the Lindley distribution is introduced, and some properties are derived.

The Lindley distribution was first introduced in the literature by Lindley [3]. The pdf and corresponding cdf are given as follows:

\[
g(x; \theta) = \frac{\theta^2}{\theta + 1} (x + 1)e^{-\theta x}, \quad x \in \mathbb{R}^+, \tag{1}
\]

and

\[
G(x; \theta) = 1 - \frac{\theta + \theta x + 1}{\theta + 1} e^{-\theta x}, \quad x \in \mathbb{R}^+, \tag{2}
\]

where \( \theta \in \mathbb{R}^+ \) [32] and it presents the shape of the distribution. The quadratic transmuted family of distribution has the following cdf form:

\[
F(x) = (\lambda + 1)G(x) - \lambda G^2(x), \quad x \in \mathbb{R}, \tag{3}
\]

where \( \lambda \in [-1, 1] \) and \( G(x) \) is the base distribution function of any standard probability distribution. For, \( \lambda = 0 \), (3) becomes the baseline probability distribution. Many new distributions have already been developed by using (3) in the literature of probability models, including Aryall G. R. [33]; Merovci and Elbatal [34]; Mahmoud and Mandouh [35]; Alizadeh et al. [36]; Ahmed and Qasim [37]. Merovci and Elbatal [34] developed transmuted Lindley distribution that has the following cdf:

\[
Q(x; \theta, \lambda) = \left[ 1 - \frac{\theta + \theta x + 1}{\theta + 1} e^{-\theta x} \right] \left[ 1 + \lambda \frac{\theta + \theta x + 1}{\theta + 1} e^{-\theta x} \right], \quad x \in \mathbb{R}^+, \tag{4}
\]
where \( \theta \in \mathbb{R}^+ \) and \( \lambda \in [0, 1] \). Rahman et al. [16] proposed a cubic transmuted family, of which the distribution function is written as

\[
F(x) = (1 - \lambda)G(x) + 3\lambda G^2(x) - 2\lambda G^3(x), \quad x \in \mathbb{R},
\]

where \( G(x) \) is the distribution function of any baseline probability distribution. The cubic transmuted Lindley is a special case from the ranked transmuted Lindley introduced by [23], by making some reparameterization; however, we used the cubic transmuted family proposed by Rahman et al. [38] to get the equations of the cubic transmuted Livley, which is a special case of the distribution proposed in [23]. Distribution is a special case from equation (4) that caused other new distributions to be introduced by Ansari et al. [39]; Ogunde et al. [40]; Wang and Bao [41]; Rahman et al. [38]; Akter et al. [42]; and so on, in order to introduce the proposed distribution, by substituting equation (2) into (4), which has the following distribution function:

\[
F(x; \theta, \lambda) = \frac{1}{(\theta + 1)^3} e^{-3\theta x} \left[ (\theta + 1) e^{\theta x} - 1 - (x + 1)\theta \right] \times \left[ e^{2\theta x} (\theta + 1)^2 + e^{\theta x} (\theta + 1) \lambda \theta (\theta + \theta x + 1) - 2\lambda (\theta + \theta x + 1)^2 \right], \quad x \in \mathbb{R}^+,
\]

where \( \theta \in \mathbb{R}^+ \) and \( \lambda \in [-1, 1] \). The corresponding pdf of the cubic transmuted Lindley distribution is obtained by differentiating (6) with respect to \( x \), which is defined as follows.

\[
f(x; \theta, \lambda) = \frac{\theta^2}{\theta + 1} (x + 1) e^{-x\theta} \left[ 1 + \lambda \right] e^{-2\theta x} \left[ (\theta + 1)^2 \right] \times \left[ 6(\theta + 1)e^{\theta x} (\theta + \theta x + 1) - 6(\theta + \theta x + 1)^2 \right] - 1], \quad x \in \mathbb{R}^+,
\]

where \( \theta \in \mathbb{R}^+ \) and \( \lambda \in [-1, 1] \). Figure 1 shows some possible shapes of the CT-Lindley distribution that indicates the capability of capturing the complex behavior of the real-life data sets. It can be seen that the shapes of the distribution approach zero as \( \theta \rightarrow \infty \) and also become narrow.

### 3. Distributional Properties

Some important distributional properties of the CT-Lindley distribution that was not introduced by Rahman et al. [38], given in (7), are discussed in the following subsections.

\[
\mu_r' = \frac{\Gamma(r + 1)}{3\theta^r (\theta + 1)^2} e^{-(r+2)} \left[ -2^{r+3} \left\{ 27(\theta + 1)^3 + r^3 (3\theta + 4)r^2 + (9\theta(3\theta + 7) + 38)r \right\} \lambda - 3^{r+3} \right] \\
\times (\theta + 1) \left\{ (\theta + 1)2^{r+2} (\theta + r + 1) (\lambda - 1) - 3\lambda \left\{ 4(\theta + 1)^2 + r^2 + (4\theta + 5)r \right\} \right\}.
\]

Let \( r = 1 \) in (8); we have

\[
\mu = \frac{1}{108\theta(\theta + 1)^2} \left[ 108(\theta + 1)^2 (\theta + 2) - [3\theta(6\theta(\theta + 4) + 19) + 19]\lambda \right].
\]

The variance of CT-Lindley distribution is obtained as

\[
\sigma^2 = \mu^2 - \mu.
\]
It is possible to obtain all other higher-order moments using $r > 2$ in equation (7).

3.2. Moment Generating Function. A moment is calculated by either a sum or an integral function of a distribution. The computation of moments tends to be time-consuming. On the other hand, it is simply generated by the derivatives of a single expected value function. This function is called a moment generating function, which is stated by the following theorem.

**Theorem 3.1.** The moment generating function $M_x(t)$ is represented by the symbol $M_x(t)$ and has the following definition:
\[ M_X(t) = \sum_{r=0}^{\infty} t^r \frac{6^{-(r+2)}\Gamma(r+1)}{3\theta^r(\theta+1)^3} \{-2^{r+3}\{27(\theta + 1)^3 + r^3(3\theta + 4)r^2 + (9\theta(3\theta + 7) + 38)r\}\lambda \\
-3^{r+3}(\theta + 1)[(\theta + 1)2^{r+2}(\theta + r + 1)(\lambda - 1) - 3\lambda(4(\theta + 1)^2 + r^2 + (4\theta + 5)r)]} \]  

(11)

**Proof 1.** The moment generating function is defined as

\[ M_X(t) = \mathbb{E}[e^{tX}], \]

\[ = \int_{0}^{\infty} e^{tx} f(x) \, dx, \quad (12) \]

where \( f(x) \) is given in (7). By using the series expansion of \( e^{tx} \) given by Zwillinger and Jeffrey [43], we have

\[ M_X(t) = \sum_{r=0}^{\infty} t^r \int_{0}^{\infty} e^{tx} f(x) \, dx, \quad \text{(13)} \]

By substituting (7) into (13), we have the moment generating function \( M_X(t) \).

### 3.3. Characteristic Function

The Fourier transform of the distribution’s density function is the characteristic function, which is analogous to the “logarithm table trick” for convolution. Characteristic functions are useful in probability theory because they may be used to derive the properties of distributions.

The characteristic function of the CT-Lindley distribution is stated by the following theorem.

**Theorem 3.2.** Let a continuous random variable \( X \) follow the cubic transmuted Lindley distribution, and then the characteristic function \( \phi_X(t) \) is

\[ \phi_X(t) = \sum_{r=0}^{\infty} \frac{(it)^r 6^{-(r+2)}\Gamma(r+1)}{3\theta^r(\theta+1)^3} \{-2^{r+3}\{27(\theta + 1)^3 + r^3(3\theta + 4)r^2 + (9\theta(3\theta + 7) + 38)r\}\lambda \\
-3^{r+3}(\theta + 1)[(\theta + 1)2^{r+2}(\theta + r + 1)(\lambda - 1) - 3\lambda(4(\theta + 1)^2 + r^2 + (4\theta + 5)r)]} \]  

(14)

where \( i = \sqrt{-1} \) is the imaginary unit and \( t \in \mathbb{R} \).

**Proof 2.** The proof is simple.

Instead of calculating the density functions directly, characteristic functions are an alternate way for simplifying the probability distributions function. The manipulation of probability distributions, the formation of moments, or the shape of a group of points for any random variable, as well as the fitting of probability distributions to large samples of data, are all examples of practical applications of characteristic functions.

### 3.4. Quantile Function and Median

The quantile function for the CT-Lindley distribution is obtained by solving \( F(x) = q \); see, for example, Rahman et al. [16] and further proceed as follows:

\[ \frac{1}{(\theta + 1)}e^{-3\theta x} [(\theta + 1)e^{\theta x} - 1 - (x + 1)\theta] \times [e^{2\theta x}(\theta + 1)^2 + e^{\theta x}(\theta + 1)\lambda(\theta + \theta x + 1) - 2\lambda(\theta + \theta x + 1)^2] = q, \]

(15)

which can be written as

\[ x_q = -\frac{1}{\theta} \left\{ \theta + 1 + W[e^{-\theta q} - 1] \right\}, \]

(16)

where

\[ y = \frac{1}{2} \left( (\lambda + 2) \sqrt{\Delta_1 + 9\Delta_2} - \frac{1}{2} \right) \left( 3\sqrt{\Delta_1 + 27\Delta_2} \right)^{-1/3}, \]

\[ \Delta_1 = \lambda^3 \{ \lambda(\lambda + 6) - 108(q - 1)q - 15 \} + 8 \],

\[ \Delta_2 = \lambda^2 (2q - 1), \]

\[ W = \text{Lambert W function.} \]
Using (16), it is easy to obtain the first quartile \(Q_1\), second quartile \(Q_2\), or median and third quartile \(Q_3\) by setting \(q = 0.25, 0.50, \text{ and } 0.75\), respectively.

3.5. Reliability Analysis. The complement of a distribution function is known as the reliability function, and for CT-Lindley distribution, it is defined as follows:

\[
R(t; \theta, \lambda) = 1 - \frac{1}{(\theta + 1)^2} e^{-3\theta t} \left[ (\theta + 1)e^{\theta t} - 1 - (t + 1)\theta \right] \\
\times \left[ e^{2\theta t} (\theta + 1)^2 + e^{\theta t} (\theta + 1)\lambda (\theta + \theta t + 1) \\
- 2\lambda (\theta + \theta t + 1)^2 \right].
\]

(18)

The hazard function is defined as the ratio of the probability distribution function as in equation (6) to the reliability function as in (18) and expressed as

\[
h(t) = \frac{e^{2\theta t} \theta^2 (t + 1) \left[ 1 + \left. -1 + e^{-2\theta t} (6\theta e^{\theta t} + 1 - t - 1 - 6(\theta + \theta t + 1)^2) \right/ (\theta + 1)^2 \right] \lambda}{(\theta + \theta t + 1) \left[ -e^{2\theta t} (\lambda - 1) + 3\lambda e^{\theta t} (\theta + \theta t + 1)/\theta + 1 - 2\lambda (\theta + \theta t + 1)^2/((\theta + 1)^2) \right]}. \tag{19}
\]

Using a sequence of values for the random variable, several plots of the reliability and hazard rate functions for the proposed distribution are shown in Figure 2. This distribution has the capacity to accommodate various increasing and decreasing hazard rate functions, as shown in the previous figure. It can be seen that this data range is suitable for the combination of parameter values used to draw these graphs.

\[
\frac{1}{(\theta + 1)^2} e^{-3\theta x} \left[ (\theta + 1)e^{\theta x} - 1 - (x + 1)\theta \right] \left[ e^{2\theta x} (\theta + 1)^2 + e^{\theta x} (\theta + 1)\lambda (\theta + \theta x + 1) - 2\lambda (\theta + \theta x + 1)^2 \right] = u.
\]

(20)

where \(u \sim U(0, 1)\) and the above can be further obtained as

\[
x = \frac{1}{\theta} \left\{ \theta + 1 + W\left[ e^{-\theta - 1} (\theta + 1) (j - 1) \right] \right\}, \tag{21}
\]

with

\[
j = \frac{1}{2} - \left( 2^{3/\lambda} \lambda \right)^{-1} \sqrt{3\sqrt{3} + \sqrt{\Delta_1}} + 9\Delta_2 - \frac{1}{2} \left( \frac{\lambda + 2}{\lambda} \right) (3\sqrt{3} \sqrt{\Delta_1} + 27\Delta_2)^{-1/3},
\]

\[
\Delta_1 = -\lambda^3 [\lambda (\lambda + 6) - 108 (u - 1) u - 15] + 8],
\]

\[
\Delta_2 = \lambda^2 (2u - 1),
\]

\[
W = \text{Lambert W function.}
\]

4. Order Statistics

The probability density function of the \(r\)th order statistic for the proposed cubic transmuted Lindley distribution is given as follows:
where \( r = 1, 2, \ldots, n \) and \( r = 1 \) gives the lowest order statistic \( X_{1:n} \) density function, which is

\[
f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!} \frac{e^{\theta(-x)}}{\theta+1} \left[ 1 + \left( \frac{6(\theta+1)e^{\theta x}(\theta + \theta x + 1) - 6(\theta + \theta x + 1)^2}{(\theta + 1)^2\lambda} \right) \right] \\
\times e^{-\lambda x} \left( [(\theta + 1)\lambda e^{\theta x} (\theta + \theta x + 1) - 2(\theta + \theta x + 1)^2 \right) \\
\times \left( [(\theta + 1)^3 e^{\theta x} - 1] - e^{-\theta x} \left( \theta(-x + 1) \right) + (\theta + 1)^3 e^{\theta x} - 1 \right) \lambda \\
\times e^{\theta x} (\theta + \theta x + 1) - 2(\theta + \theta x + 1)^2 + (\theta + 1)^2 e^{2\theta x} \right) \left( \theta + 1 \right)^{-3} t^{n-r},
\]

(23)
\[ f_{1:n}(x) = \frac{n!}{(n-1)!} \frac{e^{\theta(x)}}{10^2 \theta + 1} \left[ 1 - e^{-2\theta x} \left( 6(\theta + 1) e^{\theta x} (\theta + \theta x + 1) - 6(\theta + \theta x + 1)^2 \right) / (\theta + 1)^2 \right] \]

\[ \times \left[ 1 - e^{-3\theta x} \left( \theta(-x + 1) + (\theta + 1)e^{\theta x} - 1 \right) \right] (\theta + 1) e^{\theta x} (\theta + \theta x + 1) \]

\[ -2\lambda \left( \theta + \theta x + 1 \right)^2 + (\theta + 1)^2 e^{2\theta x} \right] (\theta + 1)^{-3} \]

Also, for using \( r = n \) in (23), the density function of the highest order statistic \( X_{1:n} \) is obtained by

\[ f_{1:n}(x) = \frac{n!}{(n-1)!} \frac{\theta^2 (x + 1)}{10^2 \theta + 1} \left[ e^{\theta(x)} (\theta(-x + 1) + (\theta + 1)e^{\theta x} - 1) \right] (\theta + 1) e^{\theta x} (\theta + \theta x + 1) \]

\[ -2\lambda \left( \theta + \theta x + 1 \right)^2 e^{2\theta x} \right] (\theta + 1)^{-3} \]

The \( k^{th} \) order moment of \( X_{r:n} \) for the proposed cubic transmuted Lindley distribution is obtained by using the following equation:

\[ E(X_{r:n}^k) = \int_0^\infty x^k f_{x:n}(x) dx. \]

For \( \lambda = 0 \), it has the density function of the \( r^{th} \) order statistic for CT-Lindley distribution as follows; for more details, see Ghitany et al. [32].

5. Estimation Based on Complete Sample and Censored Sample

The estimation of model parameters for the CT-Lindley distribution has been done by the MLE method. For doing this, consider a random sample \( x_1, x_2, \ldots, x_n \) of size \( n \) from the proposed cubic transmuted Lindley distribution, which has the likelihood function as

\[ L = \prod_{i=1}^n \theta^2 (x_i + 1) e^{-x_i \theta} \left[ 1 + \lambda \left( e^{-2\theta x_i} \left( 6(\theta + 1) e^{\theta x_i} (\theta + \theta x_i + 1) - 6(\theta + \theta x_i + 1)^2 \right) / (\theta + 1)^2 \right) \right] \]

\[ \times \left[ 1 - e^{\theta(x_i)} \right] (\theta + 1) e^{\theta x_i} (\theta + \theta x_i + 1) \]

\[ -2\lambda \left( \theta + \theta x_i + 1 \right)^2 + (\theta + 1)^2 e^{2\theta x_i} \right] (\theta + 1)^{-3} \]

and the log-likelihood function is

\[ l = 2n \log(\theta) - n \log(\theta + 1) + \sum_{i=1}^n \log(x_i + 1) - \theta \sum_{i=1}^n x_i + n \sum_{i=1}^n \log \left[ -6\lambda e^{-\theta x_i} (\theta + \theta x_i + 1) \right] + 5\lambda + 1 - 6\lambda \left( 1 - e^{-\theta x_i} \right) \]

\[ \left( \theta + 1 \right)^2 \right] + n \sum_{i=1}^n \log \left[ -6\lambda e^{-\theta x_i} (\theta + \theta x_i + 1) \right] + 5\lambda + 1 - 6\lambda \left( 1 - e^{-\theta x_i} \right) \]

\[ \left( \theta + 1 \right)^2 \right] + n \sum_{i=1}^n \log \left[ -6\lambda e^{-\theta x_i} (\theta + \theta x_i + 1) \right] + 5\lambda + 1 - 6\lambda \left( 1 - e^{-\theta x_i} \right) \]

\[ \left( \theta + 1 \right)^2 \right] + n \sum_{i=1}^n \log \left[ -6\lambda e^{-\theta x_i} (\theta + \theta x_i + 1) \right] + 5\lambda + 1 - 6\lambda \left( 1 - e^{-\theta x_i} \right) \]
So, by differentiating and maximizing the log-likelihood function in (29), with respect to unknown parameters, \( \theta \) and \( \lambda \) are obtained as follows:

\[
\frac{\partial l}{\partial \theta} = \frac{(\theta + 2)n}{\theta(\theta + 1)} - \sum_{i=1}^{n} x_i + \left[-6\lambda \psi_1 - \psi_2 + 5\lambda + 1\right]^{-1} \sum_{i=1}^{n} \left[\psi_3 + \psi_4 + \psi_5 - \psi_6\right],
\]

and

\[
\frac{\partial l}{\partial \lambda} = \sum_{i=1}^{n} \frac{-6[1 - \Omega]^{-2} - 6\Omega + 5}{-6[1 - \Omega]^{-2} - 6\Omega + 5 + 1}
\]

where

\[
\psi_1 = \left[1 - \frac{e^{-\theta x_i}(\theta + \theta x_i + 1)}{\theta + 1}\right]^2, \quad \psi_2
\]

\[
\psi_3 = \frac{6\lambda e^{-\theta x_i}}{\theta + 1} - x_i(\theta + \theta x_i + 1), \quad \psi_4
\]

\[
\psi_5 = \frac{1}{\theta + 1} \left[2\theta - \theta^2 x_i^2 - 2\theta^2 x_i^2 - \theta x_i - \theta x_i^2 + 2\theta x_i + 2\right], \quad \psi_6
\]

\[
\psi_6 = \frac{1}{(\theta + 1)^2} \left[\theta^2 + 2\theta + 2\theta x_i + \theta^2 x_i^2 + 2\theta^2 x_i + 1\right], \Omega
\]

The maximum value is obtained by setting \( \partial l/\partial \theta = 0 \) and \( \partial l/\partial \lambda = 0 \), and solving the associated nonlinear system of equations gives the maximum likelihood estimate \( \hat{\Theta} = (\hat{\theta}, \hat{\lambda})^T \) of \( \Theta = (\theta, \lambda)^T \). The theoretical solutions are often extremely complicated, and in order to get the numerical solution, we applied R-package “bbmle” [44]. Also as \( n \to \infty \), the asymptotic distribution of the MLE \((\hat{\theta}, \hat{\lambda})\) is given by [12, 38]

\[
(\hat{\theta}, \hat{\lambda}) \sim \mathcal{N} \left(\left[\begin{array}{c} \bar{\theta} \\ \bar{\lambda} \end{array}\right], \left[\begin{array}{cc} V_{11} & V_{12} \\ V_{21} & V_{22} \end{array}\right] \right)
\]

(33)

The asymptotic variance-covariance matrix \( V \) of the estimates \( \hat{\theta}, \hat{\lambda} \) is obtained by inverting the Hessian matrix. Approximate 100(1 - \( \alpha \)) two-sided confidence intervals for \( \theta \) and \( \lambda \) are given by

\[
\hat{\theta} \pm Z_{\alpha} \sqrt{V_{11}}, \quad \hat{\lambda} \pm Z_{\alpha} \sqrt{V_{22}}
\]

(34)

where \( Z_{\alpha} \) is the \( \alpha \) percentile of the standard normal distribution.

5.1. Estimation Methods Based on Type-II Censored Sample.

In this section, the parameter estimation for the CT-Lindley distribution based on Type-II censored using MLE and MPS estimation methods will be discussed in detail.

Let \( x_1, x_2, \ldots, x_n \) be a random sample of size \( n \) from the PDF of the CT-Lindley model based on Type-II censored sample; then, the likelihood function takes the form

\[
L(\theta, \lambda) = \frac{n!}{(n-r)!} [1 - F(x_{r,n}; \theta, \lambda)]^{n-r} \prod_{i=1}^{r} f(x_{i,n}; \theta, \lambda).
\]

(35)

The data is made up on the observations: \( x_{1,n}, x_{2,n}, \ldots, x_{r,n} \), and the information that \( (n-r) \) objects persist further than the expiry of the period \( x_{r,n} \), where \( r \) represents the total number of uncensored items. Assume we have \( n \) CT-Lindley distribution observation, which is located in life testing. The number of errors \( r \) is random. In Type-II censorship, a life test is terminated after a predetermined number of errors here, \( n \) and \( r \) are fixed and predetermined, but \( x_{r,n} \) is random. The following formula (19) represents the likelihood function of the CT-Lindley distribution, based on Type-II censoring.
\[ L(\theta, \lambda) = \prod_{i=1}^{r} (x_{i:n} + 1) \left[ 1 + \lambda \left\{ e^{-2\theta x_{i:n}} \left[ \frac{6(\theta + 1)e^{\theta x_{i:n}} - 6(\theta + \theta x_{i:n} + 1)^2}{(\theta + 1)^3} \right] - 1 \right\} \right] \]

(36)

where

\[ R(x_{r:n}; \theta, \lambda) = \frac{\theta^{2r}}{(\theta + 1)^r} e^{-\theta \sum_{i=1}^{r} x_{i:n}}. \]

The log-likelihood function for may be maximised directly (36). Using the software to solve the nonlinear likelihood equations produced, which is called R-package to solve the nonlinear likelihood equations obtained by differentiating (36) with respect to \( \theta, \lambda \) and being equal to zero.
5.2. MPS under Censoring Sample. The general form for the MPS function under Type-II censored samples is given as

$$PS(\theta, \lambda) = \frac{n!}{(n - r)!} \left[ 1 - F(x_{r; n}; \theta, \lambda) \right]^{r-1} \prod_{i=1}^{r+1} F(x_{i; n}; \theta, \lambda) - F(x_{i-1; n}; \theta, \lambda) \quad i = 2, \ldots, r. \quad (37)$$

| $\theta$ = 2 | MLE | MPS |
|-------------|-----|-----|
| $\lambda$ | Mean | Bias | MSE | Mean | Bias | MSE |
| 50 | $\theta$ | 2.0392 | 0.0392 | 0.1217 | 2.0205 | 0.0205 | 0.1186 |
| 70 | $\lambda$ | -0.4487 | 0.0513 | 0.1684 | -0.5323 | -0.0323 | 0.1478 |
| 100 | $\theta$ | 2.0429 | 0.0429 | 0.0617 | 2.0334 | 0.0334 | 0.0599 |
| 200 | $\lambda$ | -0.4756 | 0.0244 | 0.0695 | -0.5160 | -0.0160 | 0.0649 |
| 50 | $\theta$ | 2.0314 | 0.0314 | 0.0290 | 2.0265 | 0.0265 | 0.0286 |
| 70 | $\lambda$ | -0.4832 | 0.0168 | 0.0377 | -0.5030 | -0.0030 | 0.0362 |
| 100 | $\theta$ | 1.9975 | -0.0025 | 0.0730 | 1.9708 | -0.0292 | 0.0699 |
| 200 | $\lambda$ | -0.4738 | 0.0262 | 0.1447 | -0.5365 | -0.0365 | 0.1295 |
| 50 | $\theta$ | 2.0208 | 0.0208 | 0.0384 | 2.0114 | 0.0114 | 0.0286 |
| 70 | $\lambda$ | -0.4906 | 0.0094 | 0.0656 | -0.5209 | -0.0209 | 0.0619 |
| 100 | $\theta$ | 2.0190 | 0.0190 | 0.0196 | 2.0114 | 0.0114 | 0.0191 |
| 200 | $\lambda$ | -0.4939 | 0.0061 | 0.0347 | -0.5086 | -0.0086 | 0.0337 |
| 50 | $\lambda$ | 1.9928 | -0.0072 | 0.0659 | 1.9399 | -0.0601 | 0.0640 |
| 70 | $\theta$ | -0.4766 | 0.0234 | 0.1450 | -0.5269 | -0.0269 | 0.1292 |
| 100 | $\lambda$ | 2.0166 | 0.0166 | 0.0333 | 1.9830 | -0.0170 | 0.0317 |
| 200 | $\theta$ | -0.4919 | 0.0081 | 0.0638 | -0.5125 | -0.0125 | 0.0603 |
| 50 | $\lambda$ | 2.0181 | 0.0181 | 0.0174 | 1.9983 | -0.0017 | 0.0167 |
| 70 | $\theta$ | 2.0068 | 0.0068 | 0.0323 | 2.0073 | 0.0073 | 0.0194 |
| 100 | $\lambda$ | 1.0013 | 0.1013 | 0.1123 | 0.8380 | -0.0620 | 0.0388 |
| 200 | $\theta$ | 2.0130 | 0.0130 | 0.0090 | 2.0105 | 0.0105 | 0.0089 |
| 50 | $\lambda$ | 0.9324 | 0.0324 | 0.0286 | 0.8552 | -0.0448 | 0.0183 |
| 70 | $\theta$ | 1.9732 | -0.0268 | 0.0386 | 1.9947 | -0.0053 | 0.0314 |
| 100 | $\lambda$ | 1.0406 | 0.1406 | 0.3183 | 0.8016 | -0.0984 | 0.0743 |
| 200 | $\theta$ | 2.0017 | 0.0017 | 0.0171 | 2.0106 | 0.0106 | 0.0169 |
| 50 | $\lambda$ | 0.9719 | 0.0719 | 0.0621 | 0.8415 | -0.0585 | 0.0322 |
| 70 | $\theta$ | 2.0091 | 0.0091 | 0.0078 | 2.0131 | 0.0131 | 0.0078 |
| 100 | $\lambda$ | 0.9232 | 0.0232 | 0.0193 | 0.8589 | -0.0411 | 0.0143 |
| 200 | $\theta$ | 1.9762 | -0.0238 | 0.0347 | 1.9943 | -0.0057 | 0.0311 |
| 50 | $\lambda$ | 1.0600 | 0.1600 | 0.2419 | 0.7729 | -0.1271 | 0.0615 |
| 70 | $\theta$ | 1.9997 | -0.0003 | 0.0171 | 2.0112 | 0.0112 | 0.0165 |
| 100 | $\lambda$ | 0.9544 | 0.0544 | 0.0472 | 0.8199 | -0.0801 | 0.0267 |
| 200 | $\theta$ | 2.0080 | 0.0080 | 0.0079 | 2.0140 | 0.0140 | 0.0077 |
| 50 | $\lambda$ | 0.9199 | 0.0199 | 0.0159 | 0.8500 | -0.0500 | 0.0120 |
We employ a numerical methodology like the Newton-Raphson method to evaluate the MPS of the natural logarithm of the product spacing function under Type-II censored sample \( \theta, \lambda \).

6. Simulation

In this section, Monte Carlo simulation is done to estimate the parameters of CT-Lindley distribution based on Type-II censoring by using MLE and MPS methods. In the simulation algorithm, Monte Carlo experiments were carried out under the following data generated from CT-Lindley distribution by using the quantile (13), where \( x \) has CT-Lindley distributed for different parameters \( \theta, \lambda \) as follows.

In Table 1, \( \theta = 0.5 \) and \( \lambda = -0.5 \) and 0.9.

In Table 2, \( \theta = 2 \) and \( \lambda = -0.5 \) and 0.9.

![Figure 3: Estimated distribution functions for the selected models along with the proposed cubic transmuted Lindley distribution over the empirical distribution functions for the (a) Wheaton River and (b) Extreme Wind Speed data sets.](image)

We employ a numerical methodology like the Newton-Raphson method to evaluate the MPS of the natural logarithm of the product spacing function under Type-II censored sample \( \theta, \lambda \).

### Table 4: MLE of the parameters and respective SE with log-likelihood for selected distributions.

| Distribution                          | Parameters | Estimate | SE   | Log-likelihood |
|---------------------------------------|------------|----------|------|----------------|
| Cubic transmuted Lindley              | \( \theta \) | 0.1731   | 0.0159 | -252.207      |
|                                       | \( \lambda \) | -0.9999  | 0.2756 |                |
| Cubic ranked transmuted Lindley       | \( \theta \) | 0.1885001| 0.0034 | -253.723      |
|                                       | \( \lambda_1 \) | 0.9999   | 0.4943 | -            |
|                                       | \( \lambda_2 \) | -0.9999  | 0.4943 | -            |
| Transmuted Lindley                    | \( \theta \) | 0.1386   | 0.0165 | -263.115      |
|                                       | \( \lambda \) | 0.2218   | 0.2218 | -            |
| Lindley                               | \( \theta \) | 0.1530   | 0.0128 | -264.202      |

Different samples sizes have been obtained; \( n = 50, 100, \) and 200; also we used different censored samples schemes, where under Type-II, when \( p = 0.7 \) and 0.9 are the ratios of sample size. We can find the parameter estimation by using log-equations (35) and (36), \( R \)-package using 10000 iterations of the Newton-Raphson algorithm. The optimum technique is the one that minimizes bias and mean squared error.

From Tables 1-2, we can conclude the following:

(i) As \( n \) increases, so does bias and MSE.

(ii) In Type-II censored samples, as the percentage of failure (p) grows, increases, then the values of the Bias and MSE for the CT-Lindley parameters go down.

### Table 3: The summary statistics along with Skewness and Kurtosis of the data sets.

| Data set        | Min   | Q1    | Median | Mean  | Q3    | Max   | Skewness | Kurtosis |
|-----------------|-------|-------|--------|-------|-------|-------|----------|----------|
| Wheaton River   | 0.100 | 2.125 | 9.50   | 12.20 | 20.12 | 64.00 | 1.442    | 2.725    |
| Extreme Wind Speed | 15.92 | 25.80 | 33.90  | 41.45 | 52.34 | 94.34 | 0.885    | -0.009   |
6.1. Simulation Algorithm. In this part of the paper, we will discuss the simulation steps:

(1) The first step is to set the parameters’ starting values
(2) Second, generate a random sample of size \( n \) for complete sample or size \( m \) for censored sample
(3) Specify the number of failures in the censored sample
(4) Specify the censoring scheme that will be used
(5) Solve the likelihood equation and the MPS equations to get the estimates

(6) Do these steps \( N = 1000 \) times, find the average values of the parameters every time, and compute any other measures related to the simulation

7. Applications

In this section, we applied different kinds of data such as engineering data, scientific data, wind speed data, and agriculture data; all these kinds of data represent different shapes of skewness. Three real-life applications of the CT-Lindley distribution are conducted to check its applicability by the following four subsections. To evaluate the applicability of the CT-Lindley distribution, the two most related distributions, transmuted Lindley and Lindley and cubic transmuted Lindley [23], are selected. To know the

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{Distribution} & \text{−2 log-likelihood} & \text{AIC} & \text{AICc} & \text{BIC} \\
\hline
\text{Cubic transmuted Lindley} & 504.415 & 508.414 & 508.588 & 512.968 \\
\text{Cubic ranked transmuted Lindley} & 519.415 & 525.9232 & 526.588 & 532.7532 \\
\text{Transmuted Lindley} & 526.230 & 530.230 & 530.404 & 534.783 \\
\text{Lindley} & 528.405 & 530.405 & 530.462 & 532.682 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{Distribution} & \text{Parameters} & \text{Estimate} & \text{SE} & \text{Log-likelihood} \\
\hline
\text{Cubic transmuted Lindley} & \theta & 0.0430 & 0.0034 & -223.723 \\
& \lambda & 0.9999 & 0.4943 & \\
\text{ Ranked transmuted Lindley} & \theta & 0.0997419 & 0.0034 & -243.123 \\
& \lambda_1 & 0.0000001 & 0.4943 & \\
& \lambda_2 & 0.9999999 & 0.4943 & \\
\text{Transmuted Lindley} & \theta & 0.0644 & 0.0116 & -224.415 \\
& \lambda & -0.9999 & 0.979 & \\
\text{Lindley} & \theta & 0.0471 & 2.1 e - 05 & -232.165 \\
\hline
\end{array}
\]
To assess the practicality of the proposed model, certain model selection criteria such as $-2\log$-likelihood, Akaike’s information criterion (AIC), corrected Akaike information criterion (AICc), and Bayesian information criterion (BIC) are used.

7.1. The Wheaton River Data. The data from Bourguignon et al. [45] refer to the Wheaton River at Carcross, Yukon Territory, Canada, exceeding flood peaks (in $m^3/s$). The data are comprised of 72 exceedances from 1958 to 1984, rounded to the nearest decimal place. This data set is highly skewed and heavy-tailed platykurtic time series data. For comparison with other models, estimates of the model parameters with corresponding standard error and the log-likelihood values of the proposed model are shown in Table 4.

The estimated plots for the selected models, as well as the proposed CT-Lindley distribution, are placed over the empirical cdf plot in Figure 3(b), as well as the estimated density function in Figure 4(b). When compared to different models, the data set fits well with the proposed distribution, as seen in the figures.

The obtained values of the model selection criterion, shown in Table 5, imply that the proposed CT-Lindley distribution is more suitable to fit this highly skewed and heavy-tailed platykurtic time series data set than any other selected models used in this study.

7.2. Extreme Wind Speed Data. One of EWS (Extreme Wind Speed) Chiodo and Noia [46] data of 52 weekly maximum wind speeds in (m/s) has been used in this study. This moderately skewed and light-tailed platykurtic data set is an example of environmental data. Estimated model parameters with the appropriate SE and log-likelihood values are displayed in Table 6. Figure 4 describes the estimated distribution functions for the selected models along with the proposed cubic transmuted Lindley distribution over the empirical distribution functions for the (a) Wheaton River and (b) Extreme Wind Speed data sets.

The values of model selection criterion values are shown in Table 7, which prove that the proposed CT-Lindley model provides better performance than other models in case of environmental data.

7.3. Fatigue Life. The data are given by Birnbaum and Saunders (19) on the fatigue life of 6061-T6 aluminum coupons cut parallel to the direction of rolling and oscillated at 18 cycles per second. The data set consists of 101 observations with maximum stress per cycle of 31,000 psi. The data are 17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.80, 51.84, 51.96, 54.12, 55.56, 67.80, 68.44, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, and 173.40. The AIC for the proposed distribution is 230.0895, and the BIC is 232.3605, while for the cubic ranked transmuted Lindley, AIC is 232.4806, and BIC is 235.8871. As a result, we may conclude that the suggested distribution beats its rivals.

8. Conclusions and Recommendations

The statistical inference has been made using the cubic transmuted Lindley distribution. We used two famous classical methods such as the MLE and MPS. We made a simulation study using both methods. To know how this distribution is performing, four different real-life data sets from the lifetime, time series, environmental, and production sectors have been considered. Reviewing the characteristics of the data set, it can be seen that they are not normal but highly skewed, negatively skewed, moderately skewed, heavy-tailed, or light-tailed platykurtic. Using estimated graphs as well as well-known model selection criterion values, it was shown both graphically and numerically that the proposed model gives better performance and captures more sectors of real-life problems than other models selected for comparison in this study. So, in the end, based on the analysis shown throughout this paper, this distribution is flexible enough to capture more complex real-life data sets that arise in various areas of life [47, 48].

9. Future Work

In the next paper, we will apply an accelerated life test on the CT-Lindley; under Type-II censored sample, we will apply different kinds of classical and Bayesian estimators. We will use real data from Nelson’s book for accelerated life tests. We will apply different kinds of acceleration models such as constant and partially accelerated experiments, and we will extend our work to get the optimal censoring scheme for the experiment and the optimal sample size. We can look for applying competing risk data to the proposed model and see its flexibility in fitting the data.

One last work will be done. We will work on the related parts between our paper and neutrosophic statistics.

Data Availability

The document contains all of the relevant data as well as references to that data.
Conflicts of Interest

The authors declare that they have no conflicts of interest.

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