Upper bounds for the moduli of polynomial-like maps

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Abstract
We establish a version of the Pommerenke–Levin–Yoccoz inequality for the modulus of a polynomial-like (PL) restriction of a polynomial and give two applications. First we show that if the modulus of a PL restriction of a polynomial is bounded from below then this restricts the combinatorics of the polynomial. The second application concerns parameter slices of cubic polynomials given by the non-repelling multiplier of a fixed point. Namely, the intersection of the so-called Main Cubioid and the multiplier slice lies in the closure of the principal hyperbolic domain, with possible exception of queer components.

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1. Introduction

The Pommerenke–Levin–Yoccoz inequality [23, 30, 39, 41], often abbreviated as PLY, is a powerful and widely used tool in polynomial dynamics. It implies the local connectivity of the Mandelbrot set at all points of the Main Cardioid [23] and, in fact, at all boundary points of any hyperbolic component. Recently, it has been realized that the importance of the PLY inequality extends well beyond rational dynamics at least to Kleinian groups [40]

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and holomorphic correspondences [15]; it also transcended the boundary between 1D and multi-dimensional worlds [2, 3, 26]. We start with a gentle historical introduction to the PLY inequality (section 1.1) followed by a brief discussion of possible generalizations. Our contribution combines the PLY inequality with Douady and Hubbard’s theory of polynomial-like (PL) maps [17] (see sections 1.2 and 1.3).

1.1. The PLY inequality

Our results concern complex polynomials of any degree. However, for the purpose of this introduction and in order to emphasize the main ideas in the simplest possible context, we restrict our attention to quadratic polynomials (Douady and Hubbard’s ‘Orsay Notes’ [18] present a foundational treatise on the subject). The quadratic Mandelbrot set $\mathcal{M}$ is a famous fractal in complex dynamics. It is the subset of the plane $\mathbb{C}$ consisting of all $c$ such that the critical point $0$ of the polynomial $Q_c(z) = z^2 + c$ has bounded orbit. A key observation is that the critical orbits of a complex polynomial are the most important orbits to look at: they determine, to a large extent, the dynamics of the entire polynomial. For example, a classical result of Fatou and Julia implies that $c \in \mathcal{M}$ if and only if the filled Julia set

$$K_c = K_{Q_c} := \{ z \in \mathbb{C} \mid Q^n_c(z) \not\to \infty \text{ as } n \to \infty \}$$

is connected. If $c \notin \mathcal{M}$, then $K_c$ is a Cantor set, and $Q_c : K_c \to K_c$ is topologically conjugate to a one-sided dyadic shift. The (filled) Main Cardioid consists of all $c$ such that $K_c$ has no repelling fixed cutpoints (equivalently, $Q_c$ has a non-repelling fixed point $\alpha_c$). Recall that $\alpha_c$ is non-repelling if $|Q'_c(\alpha_c)| \leq 1$; otherwise this fixed point would be repelling. A cutpoint of $K_c$ is a point $z \in K_c$ such that $K_c \setminus \{z\}$ is disconnected. For a quadratic polynomial $Q_c$, having no repelling fixed cutpoints turns out to be equivalent to having no repelling periodic cutpoints (that is, fixed cutpoints of some iterate $Q^n_c$); this is no longer true in higher degree cases (in [22], general topological criteria for the existence of fixed cutpoints are discussed).

Among the fixed points of $Q_c$, only one is a non-repelling point or a cutpoint; this distinguished fixed point can be consistently denoted by $\alpha_c$. Suppose now that $\alpha_c$ is a repelling cutpoint of $Q_c$; let $\lambda_c := Q'_c(\alpha_c)$ be its multiplier. Then the set $K_c \setminus \{\alpha_c\}$ has finitely many components. The simplest version of the PLY inequality relates the multiplier $\lambda_c$ (a continuous invariant of analytic conjugacy) with the number $q$ of connected components of the set $K_c \setminus \{\alpha_c\}$ (a discrete invariant of topological conjugacy):

$$\log |\lambda_c| \leq 2 \log 2 \frac{q}{d},$$

This result goes back to Pommerenke [41]. In fact, Pommerenke proved a more general result, applicable to a Fatou component of a rational function, from which several accesses to the same periodic boundary point are possible (see theorem 2.8 for a fuller version of Pommerenke’s statement). For now, it suffices to remark that the requirement $\deg(Q_c) = 2$ is not essential: for the degree $d$ case $\log d$ in the numerator is simply replaced with $\log d$.

Pommerenke’s version of the PLY inequality was generalized and improved by Levin [30, 31] (see section 2.3 for more detailed overview of these improvements). The contribution of Yoccoz is remarkable but less relevant to this paper and, for this reason, we skip the precise formulas in this case. If $\alpha_c$ is a repelling cutpoint of $K_c$, then the components of $K_c \setminus \{\alpha_c\}$ undergo a combinatorial rotation with rotation number $p/q$. The inequalities of Pommerenke and Levin contain only the denominator $q$ of $p/q$, whereas the version of Yoccoz relates the full combinatorial rotation number $p/q$ with the multiplier $\lambda_c$. 

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1.2. PL renormalization

The fractal nature of the Mandelbrot set is manifested through self-similarity as there are infinitely many ‘copies’ of $M$ in $M$; these copies are called baby Mandelbrot sets. For each baby Mandelbrot set $M'$, there is a natural homeomorphism from $M'$ to $M$ called renormalization (the inverse homeomorphism is called tuning). The renormalization homeomorphism has strong analytic properties but may fail to be differentiable at some boundary points of $M'$.

The description of the renormalization homeomorphisms given in [17] goes as follows. Suppose that there is a positive integer $n_c$ and a pair of Jordan disks $U_c \subset V_c$ such that $Q_c^n : U_c \to V_c$ is a branched covering of degree 2. If $Q_c^{n_c}(0) \in U_c$ for all $k = 0, 1, 2, \ldots$, then $c$ belongs to some baby Mandelbrot set $M'$ and $c$ is called renormalizable. Let $K_c$ be the set of all points of $U_c$ that do not escape from $U_c$ under iterates of $Q_c^n$. This set is called a PL filled Julia set of $Q_c$, and will be abbreviated as a PL set. A special case of a theorem from [17] is that there exists a $c_+ \in M$ such that the restriction of $Q_c^{n_c}$ to $K_c$ is topologically conjugate to the restriction of $Q_{c_+}$ to $K_{c_+}$. Moreover, $c_+$ is unique if one imposes some additional ‘smoothness’ requirements on the conjugacy (for example, the conjugacy must be holomorphic at all interior points of the PL set). The map $c \mapsto c_+$ from $M'$ to $M$ is the desired renormalization homeomorphism.

Polynomial renormalization plays a key role in the modern study of parameter spaces in particular, in most results related to MLC, the famous conjecture stating that the Mandelbrot set is locally connected. Despite spectacular developments on the way of resolving this conjecture, it remains open up to date, see e.g. [16, 19, 27, 28].

A key notion in the just described renormalization procedure is that of a PL map [17]: this is a proper holomorphic map from Jordan disk $U$ to a Jordan disk $V$ so that $U$ is compactly contained in $V$. Note that a repelling fixed point $\alpha$ of any complex polynomial $P$ provides an example of a PL map: if $U$ is a small disk around $\alpha$, then $P : U \to P(U)$ is a homeomorphism, and $U$ is compactly contained in $P(U)$. In other words, repelling fixed points and degree 1 PL restrictions of a complex polynomial are two different interpretations of the same phenomenon. Most papers dealing with PL maps discard the degree one case as trivial, because the associated PL set is a fixed point. However, from our point of view, this is a non-trivial case, and in fact the main motivating example.

1.3. Merging the two theories

In this paper, we aim at merging the PLY inequality (at least the version of Pommerenke and Levin) with Douady and Hubbard’s theory of PL maps. Let $P$ be a complex degree $D$ polynomial (the reader is referred to the appendix, section 5, for basic notions related to the dynamics of $P$). The role of a repelling cutpoint is played by a connected PL set $K^*$ of some PL restriction $P : U_1 \to U_0$ of $P$. Note that this is a more general setting than that of the original PLY inequality: the original setting is recovered if the degree of the PL restriction is one. If the degree $P : U_1 \to U_0$ is one, then the multiplier $\lambda$ of $P$ at the only fixed point $\alpha \in U_1$ is related to the modulus of the annulus $U_0 \setminus U_1$. Indeed, a straightforward computation using the linearizing coordinate of $P$ near $\alpha$ (in which $P$ acts as the multiplication by $\lambda$, and the disks $U_0$, $U_1$ correspond to round disks about zero of radii 1 and $|\lambda|$, respectively) yields

$$\frac{\log |\lambda|}{2\pi} = \text{mod} \left( U_0 \setminus U_1 \right).$$

The modulus in the right-hand side is called the modulus of the PL map $P : U_1 \to U_0$, and it plays an important role in polynomial renormalization.
If the degree of $P : U_1 \to U_0$ is bigger than one, then there is no combinatorial rotation number $p/q$. However, there is a natural analog of $q$, its denominator. Namely, consider components of $K_P \setminus K^*$, where $K_P$ is the filled Julia set of $P$, and $K^*$ is the PL set of the PL map $P : U_1 \to U_0$. These components are called decorations of $K^*$. Suppose that there are $q^*$ decorations attached to $K^*$ at points of a repelling periodic $q^*$-cycle. Then $q^*$ can be viewed as analogous (in the context of PL maps) to the denominator $q$ of the combinatorial rotation number $p/q$ from above. In this paper we prove the following inequality assuming that $K_P$ is connected:

$$\frac{1}{\text{mod}(U_0 \setminus U_1)} \geq \frac{\pi q}{\log D},$$

where $D$ is the degree of $P$. This can be viewed as similar to the PLY inequality in the PL context, and will be called PL-PLY inequality see theorem 2.4 for a more precise statement. An improved detailed version is theorem 2.7, which is also parallel to a version of the PLY inequality.

1.4. Applications and further directions

The PL-PLY inequality gives an upper bound of the modulus $\text{mod}(U_0 \setminus U_1)$. Lower bounds of the same modulus are called a priori bounds; they play an important role in studies of local connectivity of Julia sets and rigidity. An upper bound of the modulus often implies negative results (the absence of a priori bounds); a new development of this approach is presented in [6].

On the other hand, we want to emphasize that the PL-PLY inequality also implies positive results. We obtain two such results in this paper. Here we give a conceptual overview of the results while precise formulations can be found in section 2.

1.4.1. Bounded geometry implies bounded combinatorics; rational PL restrictions. Periodic decorations of $K^*$ attached to $K^*$ at periodic points can be described combinatorially through their periods and associated external angles (more precisely, such a decoration is separated from $K^*$ by a cut consisting of two periodic rays and their common landing point; the external arguments of the rays encode the position of the cut and of the decoration). By the PL-PLY inequality, if the modulus of $U_0 \setminus U_1$ is bounded from below, then there are only finitely many possibilities for the combinatorial description of the decorations. This property can be loosely referred to as ‘bounded geometry implies bounded combinatorics of $P : U_1 \to U_0$’. Roughly speaking, if $K_P$ is connected, then, in the presence of a priori bounds, the number of periodic decorations of $K^*$ cannot be arbitrarily large, and their possible positions range through a finite set of possibilities.

To give an application of the above result, consider an invariant PL set $K^*$ defined by a PL restriction $P|_{U_1} : U_1 \to U_0$. A small analytic perturbation $\tilde{P}$ of $P$ yields a PL map $\tilde{P} : \tilde{U}_1 \to \tilde{U}_0$ with a PL set $\tilde{K}^*$ close to $P : U_1 \to U_0$, cf [17] (example (4) on p. 295) and of modulus not much smaller than that between $U_0$ and $U_1$. Hence, if $K_P$ remains connected, then the number of periodic decorations attached to $\tilde{K}^*$ (if any) is bounded. If $\tilde{P}$ has locally connected Julia set (e.g. if $\tilde{P}$ is a hyperbolic polynomial), then it is easy to see that the desired periodic decorations exist and, by the above, the number of them is bounded. Pushing $\tilde{P}$ back to $P$ we would prove
that there are periodic decorations defined for $K^*$ itself, showing that $K^*$ is separated from the rest of $K_F$ by finitely many periodic cuts and their preimages.

We hope to remove/relax some assumptions essential to the above argument aiming at proving that in general $K^*$ has this property. This conclusion can be restated in terms of the general notion of a rational PL restriction. Namely, the following description often applies to PL sets. Suppose that $K^*$ is an invariant PL set of a polynomial $P$. Then there exist finitely many periodic cuts whose vertices are points of repelling cycles from $K^*$ such that (1) the set $K^*$ is contained in the closure of a unique complementary component $T$ of the union of the cuts, and (2) the set of points from $K_F$ whose forward orbits are contained in $T$ coincides with $K^*$. In this case we will call $K^*$ a rational PL restriction of $P$. Is it true that every invariant PL set of $P$ is a rational PL restriction of $P$? For some special classes of cubic polynomials, it is answered affirmatively in this paper.

Inou and Kiwi [25] initiated the study of combinatorially defined renormalization scheme for higher degree polynomials. Further developments of Inou and Kiwi’s renormalization scheme, describing properties of the associated higher-dimensional renormalization maps, can be found in [24, 45, 46]. The notion of a renormalizable polynomial in [25] is closely related to rational PL restrictions as above and a priori is more restrictive than a renormalizable polynomial in the sense of Douady and Hubbard [17]. It would be important to know whether the two notions are the same in general. In this paper, coincidence is established for many cubic polynomials, see the next subsection.

1.4.2. Cubic polynomials. After the parameter $c$-plane of quadratic polynomials $Q_c$, the next simplest case is the moduli space of all complex cubic polynomials, i.e. the set of all affine conjugacy classes of cubic polynomials. Since every cubic polynomial is affinely conjugate to a polynomial of the form

$$f_{\lambda,b}(z) = \lambda z + bz^2 + z^3.$$

It suffices to study the complex 2D family $F$ of maps $f_{\lambda,b}$. It is convenient to visualize the complex 2D (= real 4D) space through complex 1D (= real 2D) slices. The family $F$ splits into naturally defined complex 1D slices $F_\lambda$, where $F_\lambda$ is the set of all $f_{\lambda,b}$ with fixed $\lambda$ and variable $b$. Moreover, the parameter $\lambda$ has a dynamical meaning: it is the multiplier of the fixed point 0 of $f_{\lambda,b}$. There has been extensive research of slices $F_\lambda$, see [4, 7–9, 11, 14, 21, 43, 44, 47–49].

The principal hyperbolic component $P^\circ$ of $F$ is by definition the set of all $f \in F$, for which both critical points are in the immediate attracting basin of 0. Thus, $P^\circ$ is analogous to the interior of the Main Cardioid. It follows from the definition of $P^\circ$ that $P^\circ \cap F_\lambda = \emptyset$ unless $|\lambda| < 1$. In [21, 43], the zero slice $F_0$ is considered (with a slightly different parameterization), and it is shown that $P^\circ \cap F_0$ is a Jordan disk $\Omega_0$. Moreover, the connectedness locus (see section 5.6 of the appendix) of the slice $F_0$ is the union of $\Omega_0$ and decorations, each of which contain copies of the quadratic Mandelbrot set. The attracting slices $F_\lambda$ with $|\lambda| < 1$ look essentially the same as $F_0$, as can be shown by a Branner–Hubbard motion [42]. On the other hand, the structure of the neutral slices $F_\lambda$ with $|\lambda| = 1$ is much more delicate. There are still many unanswered questions about them (cf [11]); see [10, 48, 49] for very recent results concerning the simplest neutral slices (where $\lambda = e^{2\pi i \theta}$, with $\theta$ being rational or irrational of bounded type). An earlier paper [44] studies the dynamics of polynomials from $F_1$. 

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The closure of the Main Cardioid in the c-plane, the parameter plane of quadratic polynomials $Q_c$, can be described dynamically. Namely, it consists of all values of $c$ satisfying the following equivalent conditions:

- there is a non-repelling fixed point of $Q_c$;
- there are no repelling periodic cutpoints of $K_c$.

Similarly, let us define a set $CU \subset F$ as the set of all $f = f_{\lambda,b}$ with $|\lambda| \leq 1$ such that

- the filled Julia set $K_f$ of $f$ has no repelling periodic cutpoints,
- nonrepelling periodic points of $f$ different from 0 have multiplier 1.

The set $CU$ is called the Main Cubioid of $F$. The term is inspired by the analogy with the (filled) Main Cardioid of the Mandelbrot set. It is known [5] that $P$, the closure of the principal hyperbolic component $P^\circ$, is a subset of $CU$; it is still unknown whether $P = CU$.

In order to better describe a possible difference $CU \backslash P$, consider slices $P_{\lambda} = P \cap F_{\lambda}$ and $CU_{\lambda} = CU \cap F_{\lambda}$. By the above, it is enough to look at neutral slices $|\lambda| = 1$. It is proved in [5, Theorem B] that

$$P_{\lambda} \subset \text{Th}(P_{\lambda}) \subset CU_{\lambda}.$$ 

Here $\text{Th}(P_{\lambda})$ stands for the topological hull of $P_{\lambda}$, i.e. the union of the compact set $P_{\lambda}$ and all its bounded complementary components in $F_{\lambda}$. Both inclusions shown above are conjectured to be equalities. In this paper, we deduce from the PL-PLY inequality that the second inclusion is indeed equality (theorem 2.10).

The difference $\text{Th}(P_{\lambda}) \backslash P_{\lambda}$, if nonempty, consists of so called queer components. It is conjectured in a more general context that queer components do not exist. For each queer component $U$, all polynomials from $U$ have the same topological dynamics on the Julia sets, their Julia sets have positive area and support invariant line fields. All the (conjecturally non-existing) queer components of $\text{Th}(P_{\lambda}) \backslash P_{\lambda}$ can be classified into two types: renormalizable and non-renormalizable. A queer component $U$ of $\text{Th}(P_{\lambda}) \backslash P_{\lambda}$ being renormalizable means that, for every $P \in U$, there is a PL restriction $P : U_1 \to U_0$. Being non-renormalizable means, of course, not being renormalizable.

For a neutral slice $F_{\lambda}$ with $|\lambda| = 1$, all $P \in F_{\lambda}$ admitting PL restrictions also admit rational PL restrictions, except when $P$ belongs to a renormalizable queer component of $\text{Th}(P_{\lambda}) \backslash P_{\lambda}$, see theorem 2.12.

2. Statement of the main results

The appendix (section 5) contains the necessary background. We use standard notation ($\mathbb{R}$, $\mathbb{C}$, etc). The boundary (in $\mathbb{C}$) of a set $X \subset \mathbb{C}$ is denoted by $\partial X$. For a set $Z$, let $|Z|$ be its cardinality. For a polynomial $f$, let $J_f$ be its Julia set, and $K_f$ be its filled Julia set. Throughout the paper, $P$ denotes a polynomial of degree $D > 1$ with connected Julia set $J_P$.

2.1. Cuts and wedges

If external rays $R$ and $L$ of $P$ land at the same point $a$, then the union $\Gamma = R \cup L \cup \{a\}$ is called a cut. The point $a$ is called the vertex of $\Gamma$. The cut $\Gamma$ is degenerate if $R = L$ and nondegenerate otherwise. Nondegenerate cuts separate $K_P$. The period of a periodic cut is the period of an
Given a PL restriction $P$, let $\Gamma = \{ f^m \} \subset K^*$ be a cut with a repelling/parabolic $P$-periodic vertex $a \in U$ where $U$ is an open Jordan disk. Choose a small open Jordan disk $\Delta' \subset W \cap U$ whose boundary consists of two small initial arcs of $L$ and $R$ with common endpoint $a$ and other endpoints $x_L \in L$, $x_R \in R$, and a curve $I$ connecting $x_L$ and $x_R$ inside $W \cap U$. Denote by $\Delta_{U,W}$ the component of $W \cap U$ containing $\Delta'$ and call $\Delta_{U,W}$ the core component of $W \cap U$. The core component $\Delta_{U,W}$ is independent of the choice of $\Delta'$.

To study the PL (filled) Julia sets, we need a few other concepts.

**Definition 2.3.** For a PL restriction $P : U_1 \to U_0$ of $P$, let $Z$ be a finite $P$-invariant set of periodic nondegenerate cuts attached to $K^*$ (whose vertices, called $Z$-vertices, are repelling or parabolic $P$-periodic points), and let $W_Z$ be the set of the associated wedges ($Z$-wedges). If the wedges $W_T$ from $W_Z$ are pairwise disjoint, and for each $\Gamma \in Z$ the restriction $P : \Delta_{\Gamma,U_1} \to C$ is univalent, $Z$ is called paralegal, see figure 1.

Below $Z$ always denotes a paralegal set of cuts of a PL restriction $P : U_1 \to U_0$ of $P$. A $Z$-wedge may contain external rays landing at its vertex as the external rays forming cuts of $Z$ with vertex $z$ do not have to be all external rays landing at $z$.

**Theorem 2.4.** Let $P : U_1 \to U_0$ be a PL restriction of $P$. Then

$$\frac{1}{\text{mod}(U_0 \setminus U_1)} \geq \frac{\pi|Z|}{\log D}.$$
impossible since the dynamics on geodesic rays is that of an irrational rotation of the circle. The set $\Omega \cap K^*$ is empty. Indeed, otherwise it is easy to see that all domains from the orbit of $\Omega$ are contained in $K^*$. Thus, all domains from the orbit of $\Omega$ are contained in the core components of the corresponding wedges, a contradiction with the fact that $P$ on these core components is univalent. If $a$ is parabolic (with corresponding parabolic Fatou domains $\Omega^* \subset K^*$ and $a \in \partial \Omega^*$), the domain $\Omega \subset W_\Gamma$ is not its immediate basin of attraction.

Definition 2.6. Define three sets of periodic accesses to the vertices of $Z$:

- the set $\mathcal{B}_Z$ of all periodic accesses to $Z$-vertices from $Z$-domains,
- the set $\mathcal{A}_Z$ of accesses from infinity to $Z$-vertices represented by $R$ for every $\Gamma = R \cup L \cup \{a\} \in Z$,
- the set $\mathcal{C}_Z$ of accesses from infinity to $Z$-vertices corresponding to external rays contained in $Z$-wedges.

Note that there is a collection of pairwise disjoint arcs representing all three kinds of accesses. The paralegal set $Z$ of cuts endowed with sets $\mathcal{A}_Z$, $\mathcal{B}_Z$, $\mathcal{C}_Z$ is said to be legal. Note, that, by definition, $|\mathcal{A}_Z| = |Z|$.

Let $a$ be a vertex of $Z$ and $\alpha$ be a $Z$-access to $a$ from a $Z$-domain $\Omega$. A Riemann map $\phi : \mathbb{D} \to \Omega$ depends on $\Omega$, not on $\alpha$. By [41], there is a unique point $b \in \mathbb{S}$ such that $\phi^{-1}(\alpha)$ is an access to $b$ from $\mathbb{D}$. The Blaschke product $\phi^{-1} \circ P^{\alpha} \circ \phi$ has a multiplier $\lambda_\alpha^* \in \mathbb{R}_{>1}$ at $b$ called the conjugate multiplier (of $\alpha$). Necessarily, $\lambda_\alpha^* > 1$ as $\lambda_\alpha^* = 1$ means that the $Z$-domain $\Omega_1$ with vertex $a$ is an immediate basin of the parabolic point $a$, which is impossible.
Theorem 2.7. Let $P: U_1 \to U_0$ be a PL restriction of $P$; let $Z$ be a legal set of cuts. Then

$$
\frac{1}{\text{mod} \left( U_0 \setminus U_1 \right)} \geq \sum_{\alpha \in \mathcal{A}_x \cup \mathcal{B}_x \cup \mathcal{C}_x} \frac{m_\alpha \pi}{\log \lambda_\alpha} = \frac{\pi (|Z| + |\mathcal{C}_x|)}{\log D} + \sum_{\alpha \in \mathcal{B}_x} \frac{m_\alpha \pi}{\log \lambda_\alpha} \geq \frac{\pi (|Z| + |\mathcal{C}_x|)}{\log D}.
$$

The assumption that the wedges $W_\Gamma$ contain no critical points $\Delta_{U_1, \Gamma}$ is satisfied if, e.g. the filled Julia set of the PL map $P: U_1 \to U_0$ is connected and disjoint from all $W_\Gamma$. The right hand side of the inequality is independent of the PL degree $d$ of $P: U_1 \to U_0$. The case $d = 1$ is not excluded, rather it is closely related with a special case of theorem 3 from [41] (see below).

To illustrate how these theorems work, we give two applications: the first is dynamical and valid for any degree $D$, the second deals with 1-dimensional parameter slices of the space of cubic polynomials.

2.3. Further discussion

To relate theorem 2.7 to known results we use our machinery and specialize theorem 3 of [41] in the polynomial case. Consider a polynomial $g$ such that $g: U_1 \to U_0$ is a PL map of degree one with repelling fixed point $a \in U_1$ of combinatorial rotation number 0. In this case, $Z_a$ will be formed by all pairs of neighboring external rays of $g$ landing on $a$. Collections of accesses $\mathcal{B}_a$ and $\mathcal{A}_a$ are as in definition 2.6; this defines a legal set of cuts.

Theorem 2.8 ([41, theorem 3]). Consider a degree $D > 1$ complex polynomial $g$ with connected Julia set. Let $a$ be a repelling $g$-fixed point of combinatorial rotation number 0. Then

$$
\frac{2}{\log |g'(a)|} \geq \frac{2 \log |g'(a)|}{|\log g'(a)|^2} \geq \sum_{\alpha \in \mathcal{A}_a \cup \mathcal{B}_a} \frac{1}{\log \lambda_\alpha} = \frac{|\mathcal{A}_a|}{\log D} + \sum_{\alpha \in \mathcal{B}_a} \frac{1}{\log \lambda_\alpha}.
$$

Let $g: U_1 \to U_0$ be a degree one PL restriction of a polynomial $g$ with $a \in U_1$. A straightforward computation shows that $2 \pi \text{mod}(U_0 \setminus U_1) \leq \log |g'(a)|$; equality is attained if $U_1$ is represented by a round disk in the linearizing coordinate for $f$ near $a$. Substituting this expression into theorem 2.8, removing intermediate terms, and using the fact that $|\mathcal{A}_a| = |Z_a|$ and $m_\alpha = 1$ for any access $\alpha \in \mathcal{B}_a$, we obtain the inequality

$$
\frac{1}{\pi \text{mod} \left( U_0 \setminus U_1 \right)} \geq \frac{|Z_a|}{\log D} + \sum_{\alpha \in \mathcal{B}_a} \frac{1}{\log \lambda_\alpha},
$$

which is precisely the case $d = 1$ of theorem 2.7 for rotation number 0.

Theorem 3 of [41] was later generalized by Levin [30, 31] and is now a part of the more general Pommerenke–Levin–Yoccoz (PLY) inequality [23, 39]. All versions of the PLY inequality deal with a single fixed (or periodic) point $a$ of $P$. The generalization of theorem 2.8 by Levin [31] has two improvements. Firstly, the Riemann maps $\phi$ are replaced with $\infty$-quasiconformal maps. Then, in the right hand side of the inequality, the term $(\log \lambda_\alpha)^{-1}$ is replaced with $(\log \lambda_\alpha)^{-1}$. Secondly, the left hand side can be replaced with $2 \beta/(\log |g'(a)|)$.

Here $\beta \in [0, 1]$ is the asymptotic density of $E_\alpha \cup \Omega_\alpha$ near $a$ with respect to the metric $|dz|/|z-a|$, and $\Omega_\alpha$ is the Fatou component containing $\alpha$. Similar improvements can also be made to theorem 2.7 with essentially the same methods.
The contribution of Yoccoz [23, 39] deals with nonzero rotation numbers. Though the \( d = 1 \) case of theorem 2.7 includes the possibility of a nonzero rotation number, it is essentially reduced (via the summation trick, see section 3.1) to the non-rotational case and, as a consequence, is weaker than the full PLY inequality. Note that, for higher degrees, theorem 2.7 describes the influence of several different cycles, which is not the case for the PLY inequality. Another interesting analog of the PLY inequality is obtained in [13], however, it is not compatible with PL behavior. There are also versions of the PLY inequality in the case when the Julia set of \( g \) is allowed to be disconnected, see [20, 32].

2.4. Bounded geometry implies bounded combinatorics

Theorem 2.9 is a dynamical application of theorem 2.7.

**Theorem 2.9.** Let \( P \) be a degree \( D \) polynomial with connected \( K_p \). If for a PL restriction \( P : U_1 \to U_0 \) of modulus \( \text{mod}(U_0 \setminus U_1) \geq \mu \) and filled PL Julia set \( K^* \) there exists a cycle of cuts \( Z \) of minimal period \( s \) attached to \( K^* \), then \( s \leq \frac{\log b}{\mu \pi} \). In particular, there are only finitely many possible pairs of arguments of external rays that form \( Z \).

**Proof.** By lemma 4.2, the collection \( Z \) is legal. By theorem 2.4, we have \( \mu \leq \text{mod}(U_0 \setminus U_1) \leq (s\pi)^{-1} \log D \). It follows that \( s \leq \frac{\log b}{\mu \pi} \).

2.5. Slices of cubic polynomials

Consider the space of complex cubic polynomials with fixed point 0. By a linear conjugacy (that is, a map \( z \mapsto \alpha z \) with \( \alpha \in \mathbb{C} \setminus \{0\} \), any such polynomial can be reduced to the form

\[
f_{\lambda,b}(z) = \lambda z + bz^2 + z^3.
\]

Recall that \( F \) is the space of all such polynomials, and \( F_{\lambda} \) is the space of \( f_{\lambda,b} \) with fixed \( \lambda \). Then \( F_{\lambda} \) is isomorphic to \( \mathbb{C} \), and \( b \in \mathbb{C} \) is a natural complex coordinate on \( F_{\lambda} \). The maps \( f_{\lambda,\pm b} \) are linearly conjugate by the map \( z \mapsto -z \) while no other polynomials from \( F_{\lambda} \) are linearly conjugate. Thus, if maps from \( F_{\lambda} \) are regarded up to linear conjugacies (preserving 0), then the corresponding parameter space is the quotient of \( \mathbb{C} \) with coordinate \( b \) under the involution \( b \mapsto -b \). The principal hyperbolic component \( \mathcal{P}^o \) of \( F \) is the set of \( f_{\lambda,b} \) with \( |\lambda| < 1 \) such that both critical points of \( f_{\lambda,b} \) are in the Fatou component of 0. It is similar to the interior of the (filled) Main Cardioid in the (quadratic) Mandelbrot set. On the other hand, the closure \( \overline{\mathcal{P}} \) of \( \mathcal{P}^o \) has much more interesting and delicate topology than its quadratic analog.

We study \( \mathcal{P} \) through its slices \( \mathcal{P}_{\lambda} = \mathcal{P} \cap F_{\lambda} \) that are nonempty if and only if \( |\lambda| \leq 1 \) — an assumption always made in this paper. By the main theorem of [9], a bounded complementary component \( \mathcal{U} \) of \( \mathcal{P}_{\lambda} \) is stable (see section 5.4) and for any \( f \in \mathcal{U} \), the Julia set \( J_f \) of \( f \) is connected, has positive measure and carries a measurable \( f \)-invariant line field. One critical point of \( f \) is in the immediate (attracting or parabolic) basin of 0 or in the Julia set while the other one is always in the Julia set. Such stable components are called queer (see section 5.5). For any compact set \( E \subset \mathbb{C} \) define its *topological hull* \( \text{Th}(E) \) as the complement of the unique unbounded component of \( \mathbb{C} \setminus E \); say that \( E \) is full if \( E = \text{Th}(E) \). Conjecturally, there are no queer components, and so \( \mathcal{P}_{\lambda} \) is full. Recall that \( \mathcal{CU} \subset F \) is the set of all \( f = f_{\lambda,b} \) with \( |\lambda| \leq 1 \) such that

- the filled Julia set \( J_f \) of \( f \) has no repelling periodic cutpoints,
- nonrepelling periodic points of \( f \) different from 0 have multiplier 1.
Setting \( \mathcal{CU}_\lambda = \mathcal{F}_\lambda \cap \mathcal{U}_\lambda \), we have \( \text{Th}(\mathcal{P}_\lambda) \subset \mathcal{CU}_\lambda \) by Theorem B of [5]. Theorem 2.10 verifies a conjecture from [5]; it is the main result of this paper concerning polynomial parameter spaces. By the Main Theorem of [11] (see theorem 5.14), the set \( \mathcal{CU}_\lambda \) is a full continuum.

**Theorem 2.10.** We have \( \text{Th}(\mathcal{P}_\lambda) = \mathcal{CU}_\lambda \).

Theorem 2.12 is a dynamical application of theorem 2.7; it describes the dynamics of some cubic polynomials. From now on we abbreviate ‘quadratic-like’ to ‘QL’. If \( P \in \mathcal{F}_\lambda \) has a QL restriction whose filled Julia set contains 0, then \( P \) is said to be immediately renormalizable (at 0). Lemma 2.11 relies upon theorem 5.11 from [36].

**Lemma 2.11 (Lemma 7.2 [11]).** If \( f \) is a complex cubic polynomial with a non-repelling fixed point \( a \), and there exists a quadratic-like filled Julia set \( K' \) with \( a \in K' \), then \( K' \) is unique.

The critical points of \( P \) are denoted by \( \omega_1 = \omega_1(P) \) and \( \omega_2 = \omega_2(P) \); they are numbered so that \( \omega_1 \in K' \) and \( \omega_2 \notin K' \) (one omits \( P \) from the notation whenever the choice of \( P \) is clear). By lemma 2.11, this numbering of the critical points is unambiguous. Suppose that \( P \in \mathcal{F}_\lambda \), where \( |\lambda| \leq 1 \), and that \( K_\lambda \) is connected. If \( P \notin \text{Th}(\mathcal{P}_\lambda) \), then \( P \) is immediately renormalizable at 0 by [7, theorem C]. Recall that some terminology and notation (e.g. the concept of a (parameter) wake) is introduced in the appendix.

**Theorem 2.12.** Consider \( P \in \mathcal{F}_\lambda \setminus \text{Th}(\mathcal{P}_\lambda) \) with \( |\lambda| \leq 1 \) and connected \( K_\lambda \). Then there is a nondegenerate paralegal cycle of cuts \( \mathcal{Z} \) separating \( K' \) from \( \omega_2(P) \). If the vertices of \( \mathcal{Z} \) are parabolic then they all equal 0.

**Proof.** By theorem 2.10, we have \( P \notin \mathcal{CU}_\lambda \). By the main theorem of [11] (see theorem 5.14), the polynomial \( P \) belongs to a wake \( \mathcal{W}_\lambda(\theta_1, \theta_2) \), and the rays \( R_\lambda(\theta_1 + 1/3), R_\lambda(\theta_2 + 2/3) \) land at a repelling or parabolic point \( a \); in the latter case \( a = 0 \). These rays and \( a \) form a cut whose cycle is the desired \( \mathcal{Z} \) because the wedge formed by the rays \( R_\lambda(\theta_1 + 1/3), R_\lambda(\theta_2 + 2/3) \) has the angle measure greater than 1/3 and, hence, must contain \( \omega_2(P) \).

Theorem 2.12 provides a combinatorial framework for renormalization; see [25] for a consistent combinatorial approach.

### 2.6. Plan of the paper

Section 3 contains the proof of theorem 2.7. Next, in section 4, we prove theorem 2.10. Finally, section 5 is the appendix.

### 3. Proof of theorem 2.7

Let all assumptions of theorem 2.7 be satisfied.

**Definition 3.1 (Curve families \( K \) and \( K_\Gamma \)).** Let \( K \) be the family of all rectifiable curves in \( U_0 \setminus \overline{U}_1 \) that wind once. By theorem 5.5, we have \( (\text{mod}(U_0 \setminus \overline{U}_1))^{-1} = \text{EL}(K) \). Figure 1 shows \( U_0 \setminus \overline{U}_1 \) as a dark shaded annulus, and a curve \( \gamma \) from \( K \). For each \( \Gamma = R \cup L \cup \{a\} \in \mathcal{Z} \), let \( K_\Gamma \) denote the set of rectifiable curves in \( (U_0 \setminus \overline{U}_1) \cap \Delta_{\Gamma, U_0} \) that connect \( R \) with \( L \). One representative \( \gamma_\Gamma \in K_\Gamma \) is shown in figure 1.

**Lemma 3.2.** The family \( K \) overflows each of the families \( K_\Gamma \). Therefore, \( \text{EL}(K) \geq \sum_{\Gamma \in \mathcal{Z}} \text{EL}(K_\Gamma) \).

**Proof.** Take \( \gamma \in K \). Connect \( \partial U_0 \) with \( \partial U_1 \) in \( \Delta_{\Gamma, U_0} \) by an arc \( \beta \) in \( U_0 \setminus \overline{U}_1 \), except the endpoints. Clearly, \( \beta \) must cross \( \gamma \) — otherwise \( \gamma \) is contractible, since \( U_0 \setminus (\overline{U}_1 \cup \beta) \) is simply
connected. By small perturbations, arrange that \( \gamma \) and \( \beta \) are smooth and transverse at all intersection points (see, e.g. theorems II.11.7 and II.15.4 of [12]). Let \( \gamma_T \) be a component of \( \gamma \cap \Delta T_{u_0} \) containing a point \( b \in \beta \). The endpoints of \( \gamma_T \) are in \( R \cup L \). If both endpoints are in \( R \) or both in \( L \), then the intersection index of \( \gamma_T \) and \( \beta \) in \( W_T \) is even. On the other hand, the intersection index of \( \gamma \) and \( \beta \) is 1. Therefore, there exists a \( \gamma_T \) as above with one endpoint in \( R \) and the other endpoint in \( L \). Finally, the Series Law implies the desired inequality. \( \square \)

Thus, we need to estimate the extremal lengths of \( K_T \) for all \( \Gamma \in \mathcal{Z} \).

3.1. Summation trick

Suppose that \( V_0 \subset U_0 \) and \( V_1 = V_0 \cap U_1 \) are open sets such that \( P : V_1 \to V_0 \) is a conformal isomorphism, and \( V_0 \) has components \( V^0, \ldots, V^{m-1} \), \( m < \infty \), each containing a unique component of \( V_1 \). Suppose that \( P(V \cap U_1) = V^{r+1} \pmod{m} \). Let \( K_i \) be the set of rectifiable curves \( \gamma \) in \( U_0 \setminus U_1 \) with the following properties. Firstly, \( \gamma \) connects two boundary points of \( V \) and is otherwise contained in \( V \). Secondly, \( \gamma \) separates \( V \cap U_1 \) from \( \partial U_0 \) in \( V^i \). The summation trick shown below allows to estimate the sum \( \sum_{i=0}^{m-1} \text{EL}(K_i) \).

From now on, for any positive integer \( n \), define \( U_n \) inductively as the full preimage of \( U_{n-1} \) under the PL map \( P : U_1 \to U_0 \). Define \( A_i \) as \( U_i \setminus U_{i+1} \). The set \( A_0 \) is an annulus by definition of a PL map. However, sets \( A_i \) may have more complicated topology if the Julia set of this PL map is disconnected. Let \( K_i^* \) be the pullback of \( K_i \) under the homeomorphism \( P^i : A_1 \cap V^0 \to A_0 \cap V^i \). Set \( K^* = \bigcup K_i^* \). We need lemma 3.3 in which we use the conventions \( 1/0 = \infty \), \( 1/\infty = 0 \), and \( \infty + t = \infty \) for any \( t \in \mathbb{R}_{>0} \).

**Lemma 3.3.** Let \( \ell \) be a given nonnegative real number. Suppose that \( \ell_i \), where \( i = 0, \ldots, m - 1 \), are nonnegative numbers. Then \( \sum \ell_i \geq m^2 \ell \) provided \( \ell = 0 \) or \( \sum \ell_i^{-1} = \ell^{-1} \).

**Proof.** Assume that \( \ell > 0 \) (the case \( \ell = 0 \) is obvious). Setting \( x_i^2 = \ell_i^{-1} \) for \( 0 \leq i \leq m - 1 \), the desired inequality can be restated as \( (x_1^2 + \cdots + x_m^2)(x_1^{-2} + \cdots + x_m^{-2}) \geq m^2 \), which is the Cauchy–Schwarz inequality. Alternatively, the lemma reduces to a classical inequality between the arithmetic mean and the harmonic mean. \( \square \)

**Proposition 3.4.** We have \( \sum_{i=0}^{m-1} \text{EL}(K_i) \geq m^2 \text{EL}(K^*) \)

**Proof.** Since \( P^i \) is a conformal univalent map on \( A_1 \cap V^0 \), then \( \text{EL}(K_i^*) = \text{EL}(K_i) \). The families \( K_i^* \) are disjoint. Hence, the Parallel Law (theorem 5.6) applies to \( K^* = \bigcup K_i^* \). Therefore, \( \ell_i = \text{EL}(K_i^*) \). By the Parallel Law, \( \sum \ell_i^{-1} = \ell^{-1} \), and the claim follows from lemma 3.3. \( \square \)

3.2. Fatou accesses

Consider a periodic access \( \alpha \) of period \( m_\alpha \) from a Fatou domain \( \Omega \) to a periodic point \( a \in J_P \) (see definition 2.5 and the paragraph just below it). Let \( O(a) \) be a small disk neighborhood of \( a \). Then \( f = P^{m_\alpha}|_{O(a)} \) is univalent. Identify two points of \( \Omega \cap (O(a) \setminus \{a\}) \) if they belong to the same \( f \)-orbit. Let \( p_\alpha \) be the quotient map. The component of the quotient space containing \( p_\alpha(a) \) is an annulus denoted by \( \mathcal{A}_\alpha \). Write \( \lambda^*_\alpha \) for the conjugate multiplier of \( \alpha \).

**Lemma 3.5 (Proposition 4.3 of [39]).** \( \text{The modulus of the annulus } \mathcal{A}_\alpha \text{ is } \pi / \log \lambda^*_\alpha. \) In particular, \( \mod(\mathcal{A}_\alpha) = \pi / (m_\alpha \log D) \) if \( \alpha \) is in \( \mathbb{C} \setminus K_P \).

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If \( a \) is the vertex of \( \Gamma \in \mathcal{Z} \), and a periodic access \( \alpha \) to \( a \) is in a \( \mathcal{Z} \)-wedge \( W_T \), then \( \alpha \subset \Omega \) where \( \Omega \subset W_T \) is a bounded \( \mathcal{Z} \)-domain (and \( \alpha \) is determined by \( \Omega \)) or \( \Omega = \mathbb{C} \setminus K_P \) and there may be many different accesses to \( a \) from \( \Omega \cap W_T \). Let \( K_\alpha \) be the collection of rectifiable curves \( \gamma \) in \( U_0 \setminus \overline{U}_1 \) that (1) connect two boundary points of \( \Omega \) and otherwise lie in \( \Omega \), and (2) separate some representatives of \( \alpha \) from \( \partial U_0 \cap \Omega \) in \( \Omega \).

**Lemma 3.6.** The collection \( K_\alpha \) is nonempty, that is, \( \Omega \) cannot lie entirely in \( U_0 \). Moreover, \( K_\alpha \) overflows \( K_\alpha \).

**Proof.** If \( \Omega \subset U_0 \), then \( \partial U_m \) is disjoint from \( \Omega \) as otherwise \( \Omega \cap \partial U_0 \neq \emptyset \); hence \( \Omega \subset U_{m_\alpha} \).

Repeating this, we see that \( \Omega \subset K^* \), a contradiction. The last claim of the lemma is immediate (cf lemma 3.2).

Finally, we use the summation trick of section 3.1 to estimate the total contribution of all \( K_{P(\alpha)} \).

**Lemma 3.7.** For \( \alpha \) as above \( \sum_{i=0}^{m_\alpha-1} EL(K_{P(\alpha)}) \geq \frac{m_\alpha \pi}{\log \lambda_\alpha} = \sum_{i=0}^{m_\alpha-1} \frac{m_\alpha \pi}{\log \lambda_{P(\alpha)}} \).

Note: it is not claimed that \( EL(K_{P(\alpha)}) \geq \frac{m_\alpha \pi}{\log \lambda_{P(\alpha)}} \). However, this estimate holds true after averaging over the cycle of \( \alpha \).

**Proof.** Apply the summation trick to \( V_1 \) defined as the union of components of \( U_1 \cap \bigcup_{i=0}^{m_\alpha-1} P(\Omega) \) attached to \( \mathcal{Z} \)-vertices and \( V_0 = P(V_1) \). Note that \( V_0 \) has \( m_\alpha \) components even though \( P(\Omega) \) are the same if \( \Omega \) is the basin of infinity. By proposition 3.4

\[
\sum_{i=0}^{m_\alpha-1} EL(K_{P(\alpha)}) \geq m_\alpha^2 EL(K^*) .
\]

The curve family \( K^* \) lies in \( V_0 \setminus \overline{U}_{m_\alpha} \). Replacing the latter set by its suitable univalent \( P^{m_\alpha} \)-pullback, if necessary, and taking the corresponding pullback of \( K^* \), assume that \( K^* \) lies in \( O(a) \cap \Omega \), where \( O(a) \) is a Jordan disk around \( a \) on which \( P^{m_\alpha} \) is univalent. Under this assumption, \( P_\alpha \) is defined and injective on an open set containing \( K^* \). Note also that \( P_\alpha(K^*) \subset K(K_\alpha) \), where \( K(K_\alpha) \) is as in definition 5.4, therefore, \( EL(K^*) = EL(P_\alpha(K^*)) \geq \mod(K_\alpha) = \pi / \log \lambda_\alpha \). The inequality of the lemma follows. The equality holds because \( \lambda_{P(\alpha)} = \lambda_\alpha \). \( \Box \)

### 3.3. Side annuli

A periodic access \( \alpha \) from \( \mathbb{C} \setminus K_P \) to a periodic point \( a \in K_P \) corresponds to a unique periodic external ray \( X_\alpha \) of \( P \) with \( P^{m_\alpha} X_\alpha = X_\alpha \) (conversely, given an external ray \( Y \) landing at \( \gamma \in J_P \) denote by \( \alpha_Y \) the corresponding access). The set \( \mathbb{A}_\alpha \setminus p_\alpha(X_\alpha) \) consists of two \textit{side annuli} \( \mathbb{A}_\alpha^+ \) and \( \mathbb{A}_\alpha^- \) (i.e. \( X_\alpha \) divides \( \mathbb{C} \setminus K_P \) locally near \( a \) into two sectors projecting to \( \mathbb{A}_\alpha^\pm \) by \( p_\alpha \)). Choose the labeling so that \( \mathbb{A}_\alpha^+ \) corresponds to the positive (counterclockwise) side of \( X_\alpha \). The image of \( X_\alpha \) in \( \mathbb{A}_\alpha \) is the unique simple closed Poincaré geodesic (cf the proof of theorem I.A in [23]). It divides \( \mathbb{A}_\alpha \) into two annuli of modulus \( \mod(\mathbb{A}_\alpha) / 2 \), by lemma 3.8.

**Lemma 3.8.** Let \( A \) be a topological annulus, and let \( \gamma \subset A \) be the unique simple closed geodesic in \( A \). Then \( A \setminus \gamma \) consists of two annuli, each of modulus \( \mod(A) / 2 \).

**Proof.** Let \( A \) be a topological annulus, and let \( \gamma \subset A \) be the unique simple closed geodesic in \( A \). Then \( A \setminus \gamma \) consists of two annuli, each of modulus \( \mod(A) / 2 \).

**Lemma 3.8** is well-known; since \( A \) is isomorphic to the flat cylinder \( \{ \text{Im}(z) < h / 2 \} / \mathbb{Z} \), where \( b = \mod(A) \), the statement follows immediately from the reflection symmetry of the cylinder \( \gamma \) is then represented by \( \mathbb{R} / \mathbb{Z} \), cf remark 2.41 and section 2.6.1 of [35].
For a $Z$-access $\alpha$ let $K_\alpha^+$ (resp., $K_\alpha^-$) be the family of rectifiable curves $\gamma$ in $U_0 \setminus \overline{U}_1$ with $p_\alpha(\gamma) \in K(A^\alpha_+)$ (resp., $p_\alpha(\gamma) \in K(A^\alpha_-)$).

**Lemma 3.9.** We have $\sum_{i=0}^{m-1} \text{EL}(K_{p(\alpha_i)}) \geq \frac{m \pi}{2 \log D}$, and similarly for $K_{\alpha_i}^-$. 

The proof is similar to that of lemma 3.7. We use the fact that the families $K_{\alpha_i}^\pm$ identify with certain subfamilies of $K(A^\alpha_\pm)$, as well as the summation trick of section 3.1 for properly chosen $V_0, V_1 \subset \mathbb{C} \setminus K_P$. Observe that since $\lambda_{\alpha_i}^\pm = m_\alpha \log D$, then there is $m_\alpha$ rather than $m_\alpha^2$ in the numerator.

**3.4. Proof of theorem 2.7**

For a cut $\Gamma \in R \cup L \cup \{a\} \subset Z$ of period $m$ and its wedge $W = W_\Gamma$, let $\Omega_1, \ldots, \Omega_t$ be all bounded $W$-domains. Write $\lambda_j^\pm$ for the corresponding conjugate multipliers ($j = 1, \ldots, t$). Let $k$ be the number of external rays in $W$ landing at $a$; clearly, $k + 1 \geq t$.

**Proposition 3.10.** In the above situation we have

$$\sum_{i=0}^{m-1} \text{EL}(K_{p(\Gamma)}) \geq \frac{m (k+1) \pi}{\log D} + m \sum_{j=1}^{t} \frac{m \pi}{\log \lambda_j^+},$$

$$= \frac{m (k+1) \pi}{\log D} + \sum_{i=0}^{m-1} \sum_{j=1}^{t} \frac{m \pi}{\log \lambda_j^+}.$$

It follows that the average (over time) contribution of each $\Omega_i$ is at least $m \pi / \log \lambda_j^+$. If $a$ is not accessible from any bounded Fatou component in $W$, then the second term on the right hand side is zero.

**Proof.** For every $j = 1, \ldots, t$, there is a single access $\alpha_j$ from $\Omega_j$ to $a$. Also, let $\gamma_1, \ldots, \gamma_k$ be the accesses to $a$ from $W \setminus K(P)$ corresponding to the external rays in $W$ landing at $a$. Recall that $\alpha_R$ and $\alpha_L$ are accesses represented by $R$ and $L$, respectively. For every $i = 0, \ldots, m-1$, the extremal length $\text{EL}(K_{p(\Gamma)})$ is bounded below by

$$\sum_{j=1}^{t} \text{EL}(K_{p(\alpha_j)}) + \sum_{j=1}^{k} \text{EL}(K_{p(\gamma_j)}) + \text{EL}(K_{p(\alpha_R)}) + \text{EL}(K_{p(\alpha_L)}).$$

Since $K_{p(\Gamma)}$ overflows each of the disjoint families $K_{p(\alpha_j)}, K_{p(\gamma_j)}, K_{p(\alpha_R)}$, and $K_{p(\alpha_L)}$ (According to our orientation conventions, $K_{\alpha_R}$ and $K_{\alpha_L}$ both lie in $W_1$) Taking the sum of both parts as $i$ runs from 0 to $m-1$ and applying lemmas 3.7 and 3.9 we obtain the desired inequality.

We can now conclude the proof of theorem 2.7.

**Proof of theorem 2.7.** By lemma 3.2, we have $\text{EL}(K) \geq \sum_{\Gamma \in Z} \text{EL}(K_{\Gamma})$. The sum in the right-hand side splits into blocks corresponding to different cycles of cuts. Applying proposition 3.10 to every cycle of cuts thus obtained yields the desired.

4. **Applications**

We prove lemma 4.2 that is used in theorem 2.9, and theorem 2.10.
4.1. A sufficient condition of being legal

Let $f: \mathbb{C} \to \mathbb{C}$ be a branched covering, and $U \subset \mathbb{C}$ be an open Jordan disk. A closed ray $R \subset \mathbb{C}$ is the image of $\mathbb{R}_{\geq 0} = \{ x \mid x \geq 0 \}$ under an embedding $g$ such that $g(t) \to \infty$ as $t \to \infty$, and an open ray is a closed ray with the $g$-image of $0$ removed; in either case the $g$-image of $0$ is called the endpoint of $R$. Finally, the union of finitely many rays that share an endpoint and are otherwise disjoint is called a non-compact star, and the common endpoint of the rays forming the star is called the vertex thereof. Recall that we write $|Z|$ for the cardinality of a set $Z$.

**Lemma 4.1.** Let $f: \mathbb{C} \to \mathbb{C}$ be a branched covering, and $U \subset \mathbb{C}$ be an open Jordan disk. Suppose that $f(U)$ is a Jordan disk, there are no critical values of $f$ in $f(\partial U)$, and every component $V$ of $f(U) \setminus f(\partial U)$ contains at most $|\partial V \cap \partial f(U)|$ critical values of $f$. If $f: U \to f(U)$ is not a homeomorphism, then $f$ has a critical point in $U$.

Note that, as there are no critical values in $f(\partial U)$, there are no critical values in $\partial f(U) \subset f(\partial U)$. The assumption on the number of critical values of $f$ in $V$ is vacuous unless $\partial V \cap \partial f(U)$ is a finite set. Even though this assumption can be somewhat relaxed, it cannot be dropped altogether, see figure 3.

**Proof.** See figure 2. Connect the critical values of $f$ to infinity with pairwise disjoint closed rays as described below. Let $v_1, \ldots, v_k \in f(U)$ be all critical values in $f(U)$. Then, by the assumption on the number of critical values in every component of $f(U) \setminus f(\partial U)$, each $v_j$ for $j = 1, \ldots, k$ can be connected to infinity with a closed ray $R_{v_j} = R_j$ so that

1. sets $R_j \cap \overline{f(U)}$ are simple arcs avoiding $f(\partial U) \cap f(U)$,
2. sets $R_j$ are pairwise disjoint, and
3. rays $R_j$ contain no critical values other than their endpoints $v_j$.

Then connect each other critical value $v \not\in f(U)$ to infinity with a ray $R_v$ so that $R_v$ is disjoint from $f(U)$ and all the rays are pairwise disjoint. This is possible because $f(U)$ is a disk.

For every critical point $c$ of $f$, let $S_c$ be the pullback of $R_{f(c)}$ containing $c$. Clearly, $S_c$ is a non-compact star, and $S_c \cap S_{c'}$ for $c \neq c'$. The stars $S_c$ partition $\mathbb{C}$ into open pieces each of which maps onto its image homeomorphically. Since $f: U \to f(U)$ is not a homeomorphism,
if may be bigger than An example (due to Michał Misiurewicz) of a branched covering of $\Gamma$ and such that $P_a$ in and $U$ is different from $\emptyset = \emptyset$: for some corresponding to $2 = \Delta = \rightarrow$ except for the endpoint except for endpoint but it is a union of pseudo-chords and pseudo-are homeomorphic to the interval $\Delta = \emptyset$ is only one point, we must have $D_{\beta}$ By a small perturbation, we may assume that and, by construction, $I$ such that $A$ Blokh with $C_2 f$ is partitioned into $C_2 f$ is univalent. Thus, an invariant set of cuts is paralegal $c_n f$ is injective near as the complementary components of $P \partial$ is cannot connect $c$ must cross some $2$ is also a Jordan disk. containing the critical value $C$ gaps $S = \Delta$ is $\Gamma$ can be straightened in the following sense: replace the pullback to $\partial$ of $C$ is isomorphic to the open unit disk $[f \subset \{f \in K\} \ni \partial$ is smooth and transversal to the cut $\partial f$ is violated: namely, the component $U$ lie outside of $\emptyset$ with endpoints $\partial \setminus \{x, y\} = \emptyset \subset U$. This implies that $f(\emptyset) = f(U)$ and, by construction, $f(\emptyset) \subset R_j$ for some $j = 1, \ldots, k$. Since $R_j \cap \partial f(U)$ is only one point, we must have $f(x) = f(y)$. Therefore, $c \in \emptyset$ as claimed.

Say that $U_0$ is $\Gamma$-adapted if $\partial U_0$ is smooth and transversal to the cut $\Gamma$ so that $\Gamma \cap \partial U_0$ is finite. One can arrange that $U_0$ is $\Gamma$-adapted by a small perturbation.

**Lemma 4.2.** For every cut $\Gamma = R \cup L \cup \{a\}$ attached to $K^*$ and such that $P^*(a) \neq 0$, the restriction of $P$ to the core component $\Delta_{\Gamma, U_0}$ is univalent. Thus, an invariant set of cuts is paralegal if all its cuts are attached to $K^*$.

**Proof.** By a small perturbation, we may assume that $U_0$ is $P(\Gamma)$-adapted. Set $\Delta_1 = \Delta_{\Gamma, U_1}$ and $\Delta_0 = \Delta_{P(\Gamma), U_0}$. Components of $P(\Gamma) \cap U_0$ are homeomorphic to the interval $(0, 1)$; call them pseudo-chords of $U_0$. Define pseudo-gaps of $U_0$ as the complementary components of $U_0$ to the union of pseudo-chords.

The image $P(\Delta_1)$ may be bigger than $\Delta_0$ but it is a union of pseudo-chords and pseudo-gaps of $U_0$. By the Riemann mapping theorem, $U_0$ is isomorphic to the open unit disk $\mathbb{D}$. The corresponding partition of $\mathbb{D}$ can be straightened in the following sense: replace the pullback of every pseudo-chord with a straight chord connecting the same boundary points of $\mathbb{D}$. The thus obtained chords are clearly disjoint. Now, $\mathbb{D}$ is partitioned into chords and gaps corresponding to the pseudo-chords and pseudo-gaps of $U_0$. Consider the union of chords and gaps of $\mathbb{D}$ corresponding to $P(\Delta_1)$. This set is necessarily convex, hence homeomorphic to $\mathbb{D}$. We conclude that $P(\Delta_1)$ is also a Jordan disk.

Suppose that there is a critical point $c$ of $P$ in $\Delta_1$. Connect $P(c)$ with $P(a)$ by an arc $\beta$ disjoint from $P(\Gamma)$ except for the endpoint $P(a)$. There are several ($> 1$) pullbacks of $\beta \setminus \{P(c)\}$ with endpoint $c$. On the other hand, two different pullbacks of $\beta \setminus \{P(c)\}$ cannot connect $c$ with $a$ — since $P$ is injective near $a$. Choose a pullback $\alpha$ whose other endpoint $b$ is different from $a$. By definition, $b \in K^*$. Since $\beta$ is disjoint from $P(\Gamma)$ except for endpoint $P(a)$, the arc $\alpha$ lies

Figure 3. An example (due to Michal Misiurewicz) of a branched covering $f: \mathbb{C} \to \mathbb{C}$ and a topological disk $U \subset \mathbb{C}$ such that $f(U)$ is a disk, but there are no critical points of $f$ in $U$. Note that an assumption of lemma 4.1 is violated: namely, the component $V$ of $f(U) \setminus f(\partial U)$ containing the critical value $v = f(c)$ does not touch $\partial f(U)$, and similarly with the component $V' \ni v' = f(c')$. The critical points $c, c'$ lie outside of $U$.
entirely in $W_\Gamma$. Therefore, $b \in W_\Gamma$, a contradiction with $\Gamma$ being attached to $K^\ast$. The desired statement now follows from lemma 4.1 applied to $P : \Delta_1 \to P(\Delta_1)$. Since every component of $P(\Delta_1) \setminus P(\partial \Delta_1)$ contains non-degenerate arcs of $\partial P(\Delta_1)$ on the boundary, the assumptions of lemma 4.1 are fulfilled (in a trivial way: the cardinality of $\partial V \cap \partial f(U)$ is infinite).

4.2. Proof of theorem 2.10

Lemma 4.3 below is based on theorem 2.9.

**Lemma 4.3.** Let $P : U_1 \to U_0$ be a PL map with no critical points in $\partial U_1$. Set $\text{mod}(U_0 \setminus \overline{U}_1) = \mu$. If $P_n \to P$ is a sequence of polynomials and $U_n^\ast \ni 0$ is a $P_n$-pullback of $U_0$ for any $n$, then $P_n : U_n^\ast \to U_0$ is PL for large $n$, and any cycle of periodic cuts attached to $K^\ast_n$ has period at most $\frac{\log D}{\mu n}$.

**Proof.** Since $P_n \to P$, then for any $\varepsilon > 0$ there is $N = N_\varepsilon$ such that if $n > N$ then $P_n : U_n^\ast \to U_0$ is PL and an annulus $A$ of modulus $\text{mod}(A) > \mu - \varepsilon$ is essentially embedded into $U_0 \setminus \overline{U}_1$ so that $\text{mod}(U_0 \setminus \overline{U}_1) \geq \mu - \varepsilon > 0$. By theorem 2.9, we have $m \leq \frac{\log D}{(\mu - \varepsilon)n}$ for a period $m$ cycle of cuts attached to $K^\ast(P_n)$, $n > N$. Choosing $\varepsilon$, we guarantee that $\frac{\log D}{(\mu - \varepsilon)n}$ is less than the integer part of $\frac{\log D}{\mu n} + \frac{1}{2}$ which implies the desired.

To prove theorem 2.10, it suffices to show that any $P \in C_\lambda \setminus \text{Th}(\mathbb{P}_\lambda)$ is outside of $\text{ClU}_\lambda$ (see section 5.6 for the notation; in particular, $C_\lambda$ stands for the connectedness locus in the slice $\mathcal{F}_\lambda$). Such $P$ is immediately renormalizable by theorem 5.13 from the appendix. The corresponding filled Julia set $K^\ast$ is connected. The two critical points of $P$ are $\omega_1(P) \in K^\ast$ and $\omega_2(P) \notin K^\ast$. There are two cases to consider: either the critical point $\omega_2 = \omega_2(P)$ is active or it is passive (see section 5.5 for definitions of active and passive). The former case is considered in the following proposition.

**Proposition 4.4.** Take $P \in C_\lambda \setminus \text{Th}(\mathbb{P}_\lambda)$. If $\omega_2(P)$ is active and $P$ is not the root point of a wake of $C_\lambda$, then $P \notin \text{ClU}_\lambda$.

Wakes are defined in section 5.6 of the appendix. The proof of proposition 4.4 is based on lemma 4.5, which implements a standard normal family argument. A similar claim for Misurowsicz rather than critically periodic parameters is given in proposition 2.1 of [37].

**Lemma 4.5.** Under assumptions of proposition 4.4, there is a sequence $P_n \in C_\lambda \setminus \text{Th}(\mathbb{P}_\lambda)$ converging to $P$ and such that $\omega_2(P_n)$ is $P_n$-periodic.

**Proof.** If $U$ is a Jordan disk neighborhood of $P$ in $\mathcal{F}_\lambda$, disjoint from $\text{Th}(\mathbb{P}_\lambda)$, then $\omega_2(P)$ is well defined for all $P \in U$ and depends holomorphically on $P$. Note that $\omega_2(P)$ is never mapped to $\omega_2(P)$ since $\omega_2(P)$ lies in the $P$-invariant continuum $K^\ast$, which does not contain $\omega_2(P)$. By way of contradiction, assume that $\omega_2(P)$ is not periodic for all $P \in U$. Then the backward orbit of $\omega_2(P)$ moves holomorphically with $P \in U$. Choose three distinct elements $a(P)$, $b(P)$, $c(P)$ from this backward orbit. Since the functions $P \mapsto P^a(\omega_2(P))$ do not form a normal family on $U$, they cannot avoid the points $a(P)$, $b(P)$, $c(P)$. Thus there is a $P \in U$ and $n$ such that $P^n(\omega_2(P))$ coincides, say, with $a(P)$, which implies that $\omega_2(P)$ is periodic.

**Proof of proposition 4.4.** Choose $P_n \to P$ as in lemma 4.5. Since $\omega_2(P_n)$ is periodic and by definition of $\text{ClU}$, it follows that $P_n \notin \text{ClU}_\lambda$. By theorem 5.14, the polynomial $P_n$ lies in a wake $\mathcal{W}_\lambda([\theta_1^n, \theta_2^n])$, where $\theta_1^n + 1/3$ and $\theta_2^n + 2/3$ are periodic. Let $a_n$ be the common landing point of the rays $R_P(\theta_1^n + 1/3)$ and $R_P(\theta_2^n + 2/3)$. Write $A_n$ for the cut formed by these two rays and $a_n$. If infinitely many of $P_n$ are in the same wake, then $P$ is in this wake too, and hence
$P \notin CU_\lambda$ (the only point of $CU_\lambda$ in the closure of $W_\lambda(\theta_1, \theta_2)$ is the root point of $W_\lambda(\theta_1, \theta_2)$, and $P$ is not that root point by the assumption). Passing to a subsequence, we may assume that all pairs $\{\theta^n_1, \theta^n_2\}$ are different. Also, we may assume that all $a_n$ are repelling, since there are only finitely many wakes associated with the parabolic vertex 0 (see theorem 5.14).

Let $P : U_1 \to U_0$ be a QL restriction of $P$ with connected QL filled Julia set $K'$. Replacing $U_1$ and $U_0$ with smaller disks if necessary, we may assume that there are no critical points on the boundary of $U_1$. Set $U^n_1$ to be the component of $P^{-1}(U_0)$ containing 0. Since $P_n \to P$ and by lemmas 5.9 and 5.11, the map $P_n : U^n_1 \to U_0$ is a QL map if $n$ is large. Moreover, all $P_n$’s have connected filled Julia sets $K^n_\ast$, near which they are hybrid equivalent to $Q_\lambda(z) = \lambda z + z^2$. It now follows from lemma 4.3 that the cuts $\Lambda_n$ have bounded periods. This is a contradiction, since there are only finitely many wakes of any given period. (Recall that the period of the wake $W_\lambda(\theta^n_1, \theta^n_2)$ is defined as the period of the cut $\Lambda_n$)

We can now complete the proof of theorem 2.10.

**Proof of theorem 2.10.** By way of contradiction, let $CU_\lambda \setminus \text{Th}(P_\lambda) \neq \emptyset$. Since $CU_\lambda$ is a full continuum, and $\text{Th}(P_\lambda)$ is compact, there are uncountably many boundary points of $CU_\lambda$ outside of $\text{Th}(P_\lambda)$. Choose a boundary point $P$ of $CU_\lambda$ so that $P$ is not in $\text{Th}(P_\lambda)$ and not a root point of a wake. Such $P$ exists since there are only countably many wakes, hence they have only countably many root points altogether. The critical point $\omega_2(P)$ is active since $P \in \partial CU_\lambda$.

Theorem 2.10 now follows from proposition 4.4.

5. **Appendix: background material**

This section gives an overview of known results used in the paper including classical foundations as well as more recent specific developments.

5.1. **Moduli and extremal length**

Most of results in section 5.1 can be found in classical textbooks, e.g. [1].

Let $A$ be a Riemann surface homeomorphic to an annulus. Then, by the Uniformization Theorem, there is a conformal isomorphism between $A$ and a Euclidean cylinder of height $\mu$ and circumference 1. In this case, $\mu$ is called the *modulus* of $A$ and is denoted by $\text{mod}(A)$. This is a conformal invariant. It is a straightforward computation using the complex logarithm function that the modulus of the round annulus $A = \{z \in \mathbb{C} \mid r_1 < |z| < r_2\}$ is given by $\text{mod}(A) = \log(r_2/r_1)/(2\pi)$.

**Definition 5.1 (Extremal length).** Let $K$ be a family of locally rectifiable curves in $\mathbb{C}$ or in a Riemann surface. The *extremal length* of $K$ is defined as

$$\text{EL}(K) = \sup_\rho \frac{L_\rho(K)^2}{\text{area}(\rho)}.$$ 

Here $\rho$ ranges through all measurable conformal metrics on $\mathbb{C}$ (or on the chosen Riemann surface) of finite positive area $\text{area}(\rho)$, and $L_\rho(K)$ is the infimum length of a curve from $K$ with respect to $\rho$.

**Definition 5.2 (Overflow).** For two families of curves $K_1$ and $K_2$ we say that $K_2$ *overflows* $K_1$, and write $K_1 < K_2$ if every curve from $K_2$ is an extension of a curve from $K_1$.

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Informally, $K_1 < K_2$ means that $K_2$ has fewer curves that are longer. Note that $K_2 \subset K_1$ implies $K_1 < K_2$ (‘reversion of the inequality’). The next proposition follows immediately from definition.

**Proposition 5.3.** If $K_1 < K_2$, then $\text{EL}(K_1) \leq \text{EL}(K_2)$.

**Definition 5.4.** For an open annulus $A \subset \mathbb{C}$ let $K(A)$ be the family of all rectifiable curves in $A$ connecting the boundary components of $A$, and let $K_o(A)$ be the family of all closed rectifiable curves that wind once in $A$.

For the following classical result see, e.g. [1].

**Theorem 5.5.** We have $\text{mod}(A) = \text{EL}(K(A)) = \frac{1}{\text{EL}(K_o(A))}$.

Now recall the parallel and series laws for extremal lengths, cf the appendix in [29]. Two families of curves $K_1, K_2$ are disjoint if any curve from $K_1$ is disjoint from any curve from $K_2$.

**Theorem 5.6 (Parallel Law).** Suppose that $K_1, \ldots, K_m$ are pairwise disjoint families of rectifiable curves in $\mathbb{C}$. Then

$$\frac{1}{\text{EL}(K_1 \cup \cdots \cup K_m)} = \sum_{i=1}^{m} \frac{1}{\text{EL}(K_i)}.$$ 

**Theorem 5.7 (Series Law).** Suppose that $K_1, \ldots, K_m$ are pairwise disjoint families of rectifiable curves in $\mathbb{C}$. If a family $K$ of rectifiable curves overflows each of the families $K_1, \ldots, K_m$, then

$$\text{EL}(K) \geq \sum_{i=1}^{m} \text{EL}(K_i).$$

The Series Law is essentially equivalent to the Grötzsch inequality on the moduli of annuli. An annulus $A_2$ is essentially embedded into an annulus $A_1$ if $A_2 \subset A_1$, and the identical embedding of $A_2$ into $A_1$ induces an isomorphism of fundamental groups.

**Lemma 5.8 (Grötzsch inequality).** If $A_1, \ldots, A_n$ are pairwise disjoint annuli essentially embedded into an annulus $A$, then

$$\text{mod}(A) \geq \text{mod}(A_1) + \cdots + \text{mod}(A_n).$$

### 5.2. External rays

Let $P$ be a degree $D > 1$ complex polynomial. The filled Julia set $K_P$ is the set $\{z \in \mathbb{C} \mid P^n(z) \not\to \infty\}$. This is a nonempty compact set; the Julia set $J_P$ is its boundary $\partial K_P$. A classical theorem of Böttcher states that $P$ is conjugate to $z \mapsto z^D$ near infinity. If $K_P$ is connected, then the conjugacy extends to a conformal isomorphism between $\overline{\mathbb{C}} \setminus K_P$ and the open unit disk $D = \{z \in \mathbb{C} \mid |z| < 1\}$. Without loss of generality we may assume that $P$ is monic, i.e. the highest term of $P$ is $z^D$. Then there is a conformal isomorphism $\psi_P : D \to \overline{\mathbb{C}} \setminus K_P$ conjugating $z \mapsto z^D$ with $P$ and normalized so that $\psi_P(0) = \infty$ and $\psi_P'(0) > 0$. The inverse map $\phi_P = \psi_P^{-1}$ is called the Böttcher coordinate. An external ray $R_P(\theta)$ of argument $\theta \in \mathbb{R}/\mathbb{Z}$ is the $\psi_P$-image of $\{e^{2\pi i \rho} \mid \rho \in (0, 1)\}$; clearly, $P(R_P(\theta)) = R_P(D\theta)$. A ray $R_P(\theta)$ lands at $a \in K_P$ if $a = \lim_{n \to 1^-} \psi_P(e^{2\pi i \rho})$ is the only accumulation point of $R_P(\theta)$ in $\mathbb{C}$. By the Douady–Hubbard–Sullivan landing theorem, if $\theta$ is rational, then $R_P(\theta)$ lands at a (pre)periodic
point that is eventually mapped to a repelling or parabolic periodic point. A periodic point \( a \) with \( P^n(a) = a \) is called repelling if \( |(P^n)'(a)| > 1 \) and parabolic if \( (P^n)'(a) \) is a root of unity. Conversely, any point that eventually maps to a repelling or parabolic periodic point is the landing point of at least one and at most finitely many external rays with rational arguments.

5.3. PL maps

Let \( U \) and \( V \) be Jordan disks such that \( U \Subset V \) (i.e. \( \overline{U} \subset V \)). The following classical definition is due to Douady and Hubbard [17]. A proper holomorphic map \( f : U \to V \) is said to be polynomial-like (PL); if the degree of \( f \) is two it is called quadratic-like (QL). The filled Julia set \( F(f) \) of \( f \) is defined as the set of points in \( U \), whose forward \( f \)-orbits stay in \( U \). Similarly to polynomials, the set \( F(f) \) is connected if and only if all critical points of \( f \) are in \( F(f) \). Let \( f_1 : U_1 \to V_1 \) and \( f_2 : U_2 \to V_2 \) be two PL maps. Consider Jordan neighborhoods \( W_1 \) of \( F(f_1) \) and \( W_2 \) of \( F(f_2) \). A quasiconformal homeomorphism \( \phi : W_1 \to W_2 \) is called a hybrid equivalence between \( f_1 \) and \( f_2 \) if \( f_2 \circ \phi = \phi \circ f_1 \) whenever both parts are defined, and \( \partial \phi = 0 \) on \( F(f_1) \). By the Straightening theorem of [17], a PL map \( f : U \to V \) is hybrid equivalent to a polynomial of the same degree restricted on a Jordan neighborhood of its filled Julia set. (Abusing the language, we will simply say ‘hybrid equivalent to a polynomial’.)

Lemma 5.9. If \( P \in \mathcal{F}_A, |\lambda| \leq 1 \) has a QL restriction \( Q : U^* \to V^* \) with \( 0 \in U^* \), then the corresponding filled QL Julia set \( K^* \) is connected and unique; also, \( P \) and the map \( Q_A(z) = \lambda z + z^2 \) are hybrid equivalent near their (QL) filled Julia sets.

5.4. Stability

Suppose that \( P_n : U^n_1 \to U^n_0 \) is a sequence of continuous maps between open domains \( U^n_1 \) and \( U^n_0 \) of \( \mathbb{C} \). Convergence of \( P_n \) to \( P : U_1 \to U_0 \) is understood in the following sense: every compact subset \( C \subset U_1 \) is contained in \( U^n_1 \) for all sufficiently large \( n \), and \( P_n : C \to \mathbb{C} \) converge uniformly to \( P : C \to \mathbb{C} \).

Lemma 5.10. Let \( P : U_1 \to U_0 \) be a PL map of degree \( d \). Suppose that a sequence of PL maps \( P_n : U^n_1 \to U^n_0 \) converges to \( P \) as \( n \to \infty \). Then the degree of \( P_n \) is \( d \) for all sufficiently large \( n \).

Proof. Choose a compact Jordan disk \( C \) contained in \( U_1 \) and such that \( C = P^{-1}(P(C)) \), and interpret \( d \) as a winding number of \( P|_{\partial C} \). ■

The next lemma is more specific for our setup.

Lemma 5.11. Let \( P \) be a polynomial, \( P : U_1 \to U_0 \) be a PL map of degree \( d \) with connected filled PL Julia set \( K^* \) and no critical points in \( \partial U_1 \). Suppose that \( P_n \to P \) is a sequence of polynomials such that for some Jordan disks \( U^n_i \) \( \Subset U_0 \) disjoint from \( U_1 \) the maps \( P_n : U^n_i \to U_0 \) are PL with connected PL Julia sets. Then the PL maps \( P_n : U^n_i \to U_0 \) are of degree \( d \) for large \( n \), and their filled PL Julia sets \( K^n_0 \) converge to \( K^* \).

Proof. Follows from lemma 5.10, continuity and the definitions. ■

Recall now a stability result about (pre)periodic points.

Lemma 5.12 ([18], cf lemma B.1 [22]). Let \( g \) be a polynomial of degree \( > 1 \), and \( z \) be a repelling periodic point of \( g \). If an external ray \( R_g(\theta) \) with rational argument \( \theta \) lands at \( z \),

\[ i.e. \text{such that } \exists C > 0 \text{ with the property } C^{-1} \text{mod}|A| \leq \text{mod}(\phi(A)) \leq C \text{mod}|A| \text{ for any annulus } A \subset W_1. \]
then for every polynomial $\tilde{g}$ sufficiently close to $g$, the ray $R_{\tilde{g}}(0)$ lands at a repelling periodic point $\tilde{z}$ of $\tilde{g}$ close to $z$. Also, $\tilde{z}$ depends holomorphically on $\tilde{g}$ and has the same period as $z$.

Let $A \subset \mathbb{C}$ be any subset and $\Upsilon$ be a metric space with a marked base point $\tau_0$. A map $(\tau, z) \mapsto \iota_\tau(z)$ from $\Upsilon \times A$ to $\mathbb{C}$ is an equicontinuous motion (of $A$ over $\Upsilon$) if $\iota_{\tau_0} = id_A$, the family of maps $\tau \mapsto \iota_\tau(z)$ parameterized by $z \in A$ is equicontinuous, and $\iota_\tau$ is injective for every $\tau \in \Upsilon$. An equicontinuous motion is holomorphic if $\Upsilon$ is a Riemann surface, and each function $\tau \mapsto \iota_\tau(z)$, where $z \in A$, is holomorphic. By the $\lambda$-lemma of [38] (see also [33]), to define a holomorphic motion, it is enough to require that every map $\iota_\tau$ is injective, and $\iota_\tau(z)$ depends holomorphically on $\tau$, for every fixed $z$. Then the family of maps $\iota_\tau$ is automatically equicontinuous. Suppose now that $F_\tau : \mathbb{C} \to \mathbb{C}$ is a family of rational maps such that $F_{\tau_0}(A) \subset A$. An equicontinuous motion $(\tau, z) \mapsto \iota_\tau(z)$ is equivariant with respect to the family $F_\tau$ if $\iota_\tau(F_{\tau_0}(z)) = F_{\tau_0}(\iota_\tau(z))$ for all $z \in A$. An $F_{\tau_0}$-invariant set $A$ is called stable if it admits an equivariant (with respect to the family $F_\tau$) holomorphic motion over some neighborhood of $\tau_0$ in $\Upsilon$.

5.5. Cubic case

First we state results from [7] related to subsection 5.4; set $\Upsilon = \mathcal{F}_{\lambda}$ (see subsection 2.5) and $F_P = P$ for $P \in \Upsilon$.

**Theorem 5.13 (Summary of some results of [7]).** Any $P \in \mathcal{F}_{\lambda} \setminus \text{Th}(\mathcal{P}_{\lambda})$ is immediately renormalizable. Its QL Julia set $J^* = \partial \mathcal{K}^*$ admits an equivariant holomorphic motion over $\mathcal{F}_{\lambda} \setminus \text{Th}(\mathcal{P}_{\lambda})$.

The first statement of this theorem follows from Theorem C, and the second statement follows from theorem A and lemma 3.12 of [7].

A map $f \in \mathcal{F}_{\lambda}$ is stable if $J_f = \partial \mathcal{K}_f$ is stable. This notion is a special case of $J$-stability [33, 38]. Following [37], we say that a simple critical point $c_f$ of $f \in \mathcal{F}_{\lambda}$ is active if, for every small neighborhood $\mathcal{U}$ of $f$ in $\mathcal{F}_{\lambda}$, the sequence of the mappings $g \mapsto g^{\omega}(c_g)$ fails to be normal in $\mathcal{U}$. Here $c_g$ is the critical point of $g$ close to $c_f$. If the critical point $c_f$ is not active, then it is passive. The map $f$ with simple critical points is stable if and only if both critical points of $f$ are passive [37].

The set of all stable maps in $\mathcal{F}_{\lambda}$ is open; its components are called stable components. A classification of stable components of $\mathcal{F}_{\lambda}$ is given in section 3 of [47]: a stable component $\mathcal{U}$ can be hyperbolic-like, capture, or queer. If $\mathcal{U}$ is hyperbolic-like, then any $f \in \mathcal{U}$ has an attracting or super-attracting cycle whose immediate basin contains $\omega_2$. If $\mathcal{U}$ is capture, then $\omega_2(f)$ is eventually mapped to a Fatou component $V$ containing 0 or being an immediate parabolic basin of 0. Finally, if $\mathcal{U}$ is queer, then, for every $f \in \mathcal{U}$, the Julia set $J_f$ has positive measure and carries an $f$-invariant measurable line field. The critical points of $f \in \mathcal{U}$ can be consistently denoted by $\omega_1(f), \omega_2(f)$ so that $\omega_1(f) \in J_f$, and $\omega_1(f)$ is either in a (super)attracting/parabolic basin associated with 0, or in $J_f$. Moreover, the orbit of $\omega_1(f)$ accumulates either on 0 or on the boundary of the Siegel disk around 0. Conjecturally, there are no queer components.

5.6. The structure of slices

Let us overview some results of [11]. Write $\mathcal{C}_{\lambda}$ for the connectedness locus in $\mathcal{F}_{\lambda}$, i.e. the set of all $f \in \mathcal{F}_{\lambda}$ with $K_f$ connected. If $f \in \mathcal{F}_{\lambda}$ has disconnected $K_f$, then the Böttcher coordinate $\phi_f$ extends to a disk containing $\omega_2(f)$. The latter is the so-called co-critical point of $f$, the unique point different from $\omega_2$ and mapping to $f(\omega_2)$. The map $\Phi_{\lambda}(f) = \phi_f(\omega_2^1(f))$ is a conformal
isomorphism between $\mathcal{F}_\lambda \setminus \mathcal{C}_\lambda$ and the complement of the closed unit disk, see [14]. Define parameter rays $\mathcal{R}_\lambda(\theta)$ as the $\Phi_\lambda$-preimages of $\{re^{\pi i} | r > 1\}$. There is an explicit [11] set $\Omega$ of angle pairs $\theta_1, \theta_2 \in \mathbb{R}/\mathbb{Z}$ such that the parameter rays $\mathcal{R}_\lambda(\theta_1), \mathcal{R}_\lambda(\theta_2)$ land at the same point of $\mathcal{P}_\lambda$. It is essential that the set $\Omega$ does not depend on $\lambda$ provided that $|\lambda| \leq 1$. For $\{\theta_1, \theta_2\} \in \Omega$, the domain $\mathcal{W}_\lambda(\theta_1, \theta_2)$ bounded by rays $\mathcal{R}_\lambda(\theta_1), \mathcal{R}_\lambda(\theta_2)$ and their common landing point so that $\mathcal{W}_\lambda(\theta_1, \theta_2) \cap \mathcal{C}_\lambda = \emptyset$ is called a (parameter) wake. The common landing point of $\mathcal{R}_\lambda(\theta_1)$ and $\mathcal{R}_\lambda(\theta_2)$ is called the root point of $\mathcal{W}_\lambda(\theta_1, \theta_2)$. Observe that the terminology used for cuts in the dynamic plane is different: a component of the complement to a cut is called a wedge and the common landing point of the two external rays that form the cut is said to be the vertex of that cut. Limbs are defined as intersections of $\mathcal{C}_\lambda$ with parameter wakes. The following theorem summarizes the main results of [11].

**Theorem 5.14.** The connectedness locus $\mathcal{C}_\lambda$ is the disjoint union of $\mathcal{C}_\lambda \cup \mathcal{L} \cup \mathcal{S}$ and all limbs. The set $\mathcal{C}_\lambda \cup \mathcal{L}$ is a full continuum. For every wake $\mathcal{W}_\lambda(\theta_1, \theta_2)$ and every $f \in \mathcal{W}_\lambda(\theta_1, \theta_2)$, the dynamic rays $\mathcal{R}_f(\theta_1 + 1/3)$ and $\mathcal{R}_f(\theta_2 + 2/3)$ lie in the same periodic cut attached to $K^*(f)$. The vertex of this cut is either repelling or parabolic; in the latter case it coincides with 0.

**Data availability statement**

All data that support the findings of this study are included within the article (and any supplementary files).

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