Robust Output Feedback Stabilization of Multivariable Invertible Nonlinear Systems: A Feedback Linearization-Based Method

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Abstract—This note studies the robust output feedback stabilization problem of a class of multi-input multi-output invertible nonlinear systems, for which an “ideal” state feedback based on feedback linearization can be designed under certain mild assumptions. By systematically designing a set of extended low-power high-gain observers, we show that this “ideal” linearizing feedback law can be approximately estimated, which provides a robust output feedback stabilizer such that the origin of the resulting closed-loop system is semiglobally asymptotically stable.

Index Terms—Multi-input multi-output; Extended high-gain observer; Output feedback; Feedback linearization

I. INTRODUCTION

In recent decades, several methodologies for stabilization to the zero equilibrium for nonlinear systems have been developed such as feedback linearization, backstepping, and passivity-based control [1], that differ in the kind of system structure (normal form and lower triangular form) and in assumptions on the internal stability (input-output stability and output-to-state stability). Among these methods, feedback linearization, which utilizes a (dynamic) feedback to impose a linear input-output behavior, has received much attention due to its simplicity, particularly for single-input single-output (SISO) systems.

For SISO nonlinear systems having the normal form, feedback linearization can be achieved by cancelling the undesired nonlinear terms. Moreover, the input-output transient performance can be shaped using linear control theory. Thus, one is able to obtain an “ideal” state feedback, which, together with a suitable minimum-phase assumption, solves the stabilization problem and forces an “ideal” system performance. If only output information is available for the feedback and there exist uncertainties, [2] proposes to utilize an extended high-gain observer (EHGO) in order to estimate not only the unavailable states but also the perturbed nonlinear terms to solve the robust output feedback stabilization problem. This yields an estimate of the “ideal” state feedback and thus recovers the “ideal” system performance in a semiglobal sense.

The technique of [2] is extended to multi-input multi-output (MIMO) nonlinear systems with a well-defined vector relative degree in [3]. A similar extension is also done in [4] with a different assumption on the high-frequency gain matrix. However, the class of MIMO nonlinear systems considered in [3], [4] is a very particular one, while for more general classes of MIMO nonlinear systems there are few relevant results available in the literature. Recently, several authors have studied a general class of MIMO nonlinear systems [5], [6], [7], [8], [9], referred to as invertible MIMO nonlinear systems [10], [11], for which a well-defined vector relative degree is not necessary. In [5], with a static input-output linearizable assumption, invertible MIMO nonlinear systems can be transformed to an “intermediate” form with a vector relative degree \( \{1, \ldots, 1\} \), for which the corresponding stabilization problem via state/output feedback can be solved under a strong minimum-phase assumption. In [7], a more general invertibility property is studied for which input-output linearization is achieved by dynamic state feedback, but at the price of requiring a trivial zero dynamics. In spite of these impressive results, it is nontrivial to apply the EHGO technique to robustify the stabilizer while recovering the “ideal” feedback linearizing performance for invertible MIMO nonlinear systems. For the same class of invertible MIMO nonlinear systems as in [7], a recursive design method of EHGOs is proposed by [9], but additionally requiring the high-frequency gain matrix to be lower triangular.

On the other hand, in [2], [3], [9] the maximum power of the high-gain parameter increases as the number of states increases, which in practice may create numerical implementation problems when the dimension of the system to be estimated is very large. To solve this problem, the low-power technique in [12], [13] can be employed. However, the combination with the low-power technique is nontrivial, particularly for invertible MIMO nonlinear systems.

Motivated by the previous analysis, this technical note studies the problem of robust output feedback stabilization for the class of MIMO invertible nonlinear systems as in [5]. Though compared to [9] this note requires a stronger invertibility property, a lower triangular high-frequency gain matrix is not necessary and a nontrivial zero dynamics is permitted. Taking advantage of both EHGO and low-power [12] techniques, we design a set of EHGOs for all input-output channels such that the linearizing feedback can be estimated, which thus enables the desired recovery of the performance.
by the feedback linearizing design. Meanwhile, each EHGO only has the power of its high-gain parameter up to 2, which to some extent solves the numerical implementation problem.

This note is organized as follows. Section II presents the problem statement and the standing assumptions, under which the feedback linearization method can be used. In Section III a robust output feedback stabilizer is proposed by employing a set of extended low-power high-gain observers, and then in Section IV the design parameters in the controller and the corresponding stability analysis are given. A brief conclusion is made in Section V.

Notations: \( | \cdot | \) denotes the standard Euclidean norm and \( | \cdot |_{[a, b]} \) denotes the essential supremum norm of a signal restricted to an interval \([a, b]\). A continuous function \( \alpha : \mathbb{R}_+ := [0, \infty) \to \mathbb{R}_+ \) is said to be of class \( K \) if \( \alpha \) is strictly increasing and \( \alpha(0) = 0 \), and of class \( K_\infty \) if it is also unbounded. A continuous function \( \beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \) is of class \( KL \) if, for each fixed \( t \geq 0 \), the function \( \beta(\cdot, t) \) is of class \( K \) and, for each fixed \( r > 0 \), \( \beta(r, \cdot) \) is strictly decreasing and \( \lim_{t \to \infty} \beta(r, t) = 0 \). For any positive integer \( d \), \( 0 \leq d \leq n \), \( d \times 1 \) vector, whose entries are all zero, and \( (A_d, B_d, C_d) \) is used to denote the matrix triplet in the prime form. Namely, \( A_d \) denotes a shift matrix of dimension \( d \times d \), \( B_d = (0 \ldots 0 1)^T \in \mathbb{R}^d \), and \( C_d = (1 0 \ldots 0) \in \mathbb{R}^{1 \times d} \).

II. PRELIMINARIES

A. Problem Formulation

Consider multivariable nonlinear systems of the form

\[
\dot{x} = f(x) + g(x)u
\]

\[
y = h(x)
\]

where state \( x \in \mathbb{R}^n \), control input \( u \in \mathbb{R}^m \), output \( y \in \mathbb{R}^m \), and all mappings \( f(x), g(x), \text{and } h(x) \) are smooth. As in [3], this paper considers the class of systems satisfying the following conditions.

Assumption 1: For system (1), there exists a smooth state feedback law \( u = \alpha(x) + \beta(x) \) with \( \alpha : \mathbb{R}^n \to \mathbb{R}^m \), invertible \( \beta : \mathbb{R}^n \to \mathbb{R}^{m \times m} \), and \( \dot{u} \in \mathbb{R}^m \) such that the resulting system has a linear input-output behavior between \( \dot{u} \) and \( y \).

Assumption 2: The system (1) is strongly invertible, in the sense of [10], [11].

With Assumptions 1 and 2 by using the Structure Algorithm [5], [8], we can define a set of new variables, obeying equations of the form

\[
\dot{\xi}_1 = A_r \xi_1 + B_r [a_1(x) + b_1(x)u]
\]

\[
\dot{\xi}_k = A_r \xi_k + \sum_{i=1}^k M^k_{i,j} [a_i(x) + b_i(x)u] + B_r [a_k(x) + b_k(x)u], \quad 2 \leq k \leq m
\]

with \( r := r_1 + r_2 + \ldots + r_m \leq n \), in which the partial state \( \xi := \text{vec}(\xi_1, \ldots, \xi_m) \in \mathbb{R}^r \) with \( \xi_1 \in \mathbb{R}^r \), output \( y = \text{col}(y_1, \ldots, y_m) \) with \( y_i = C_r \xi_i \in \mathbb{R}^r \) for \( 1 \leq i \leq m \), and \( M^k_{i,j} \in \mathbb{R}^{r \times r} \) denotes the vector of “multipliers” in the Structure Algorithm [5], [8], defined by

\[
M^k_{i,j} = (0_{i,j-1} \delta^j_{i,r_j+1} \ldots \delta^j_{i,r_k} 0), \quad 1 \leq j < i \leq m,
\]

and \( a_i(0) = 0 \). As in [2], [9], the explicit expressions of functions \( a(\cdot) \) and \( b(\cdot) \) are not required in the following output feedback stabilizer design, which in turn allows the existence of uncertainties and thus indicates that the following design is robust. Occasionally, we will denote by \( \xi_i \) the vector of \( \xi_1, \ldots, \xi_i \), by \( M_{i,j}^k \) the \( j \)-th element of vector \( M^k_1 \), by \( a_i(x) \) the \( i \)-th entry of the vector \( a(x) \in \mathbb{R}^m \) and by \( b_i(x) \) the \( i \)-th row of the invertible matrix \( b(x) \in \mathbb{R}^{m \times m} \). As shown in [8], Proposition 9.1, one can obtain a smooth map \( \Phi : \mathbb{R}^r \to \mathbb{R}^r \) with \( \xi = \Phi(x) \) and \( \Phi(0) = 0 \).

Moreover, as in [9] we also assume that system (1) is strongly—and also locally exponentially—minimum-phase, as formulated below.

Assumption 3: There exist \( \beta_1 \in KL \) and \( \alpha_1 \in K_\infty \) such that for every \( x(0) \in \mathbb{R}^n \),

\[
|x(t)| \leq \beta_1(|x(0)|, t) + \alpha_1(|\xi|, t)
\]

holds, uniformly in \( u \), and for some constants \( d, k, M, \alpha > 0 \)

\[
\alpha_1(s) \leq k r, \quad \beta(r, t) \leq M e^{-\alpha t} r \quad \text{for } |r| \leq d.
\]

Assumption 3 in fact characterizes the output-to-state stability of system (1), by recalling that \( \xi \) can be expressed as a function of outputs \( y \) and their derivatives. From (3), it is seen that \( |x(t)| \) eventually becomes small when \( \xi \) is small. With this in mind, had \( x \) been available for feedback and the functions \( a(\cdot) \) and \( b(\cdot) \) been known, by feedback linearization we can design an “ideal” control law

\[
u^* = b^{-1}(x)[-a(x) + v]
\]

with the residual control \( v \).

This ideal control reduces the input-output model (2) to

\[
\dot{\xi} = A\xi + Bv
\]

where \( A = \text{blkdiag}(A_{r_1}, \ldots, A_{r_m}) \), and

\[
B = \begin{pmatrix} B_{r_1} & 0 & 0 \\ M_{r_2}^1 & B_{r_2} & 0 \\ \vdots & \vdots & \vdots \\ M_{r_m}^1 & M_{r_m}^m & B_{r_m} \\ \end{pmatrix}.
\]

It can be easily verified that the pair \((A, B)\) is controllable. Thus a natural design of \( v \) is the linear feedback control

\[
v = -K\xi
\]

where the choice of \( K \in \mathbb{R}^{m \times r} \) can be selected via a linear control design method to make the matrix \( A - BK \) Hurwitz. With this choice of \( K \), it immediately follows that \( \xi(t) \) is globally exponentially stable at the origin, which, according to [14] and with Assumption 3 implies that the zero equilibrium point of system (1) with the ideal control law (4) is globally asymptotically stable. According to a converse Lyapunov theorem [20], the resulting ideal closed-loop system permits a Lyapunov function \( V_\epsilon \) and an \( \alpha_\epsilon \in K_\infty \) such that

\[
\dot{V}_\epsilon \leq -\alpha_\epsilon(|x|).
\]

The above ideal feedback law (4)-5 is not implementable due to the inaccessibility of full knowledge of \( x \) and the fact that functions \( a(\cdot), b(\cdot) \) might be affected by uncertainties. In
this respect, to recover the transient performance by the ideal feedback law, an estimate of this ideal feedback law \[4, 5\] is required. Motivated by this, this note develops a new set of high-gain observers, which enables us to obtain such an estimate of the ideal controller \[4, 5\].

Remark 1: In \[2\], an interesting framework was established to reconstruct the ideal control by using an EHGO to estimate the unavailable state and also the matched perturbations that appear only in the last equation. This idea was later extended to multivariable nonlinear systems in normal form in \[3\]. However, it is noted that the methods in \[2, 3\] cannot be directly used for systems having partial normal form \(2\), mainly because the resulting "perturbations" to be estimated are unmatched in the presence of the vectors \(M^j_i\).

Remark 2: Though all multipliers \(\delta^j_{1,k}\) are limited to be constants, we stress that the extension to the case that \(\delta^j_{1,k}\) are bounded functions of the output \(y\) (i.e., \(\delta^j_{1,k}(y)\)) is straightforward by replacing the constant multipliers by the corresponding output-dependent multipliers. In this respect, it is worth noting that \(9\) studies a class of multivariable invertible nonlinear systems with special state-dependent multipliers. Using dynamic extension, and under the trivial zero dynamics assumption, the system can be fully decoupled into multiple standard normal forms via state feedback. This is different from the case we consider in this note where the zero dynamics is non-trivial and the feedback linearization is performed by static state feedback, resulting in an ideal system that is not in the normal form. Moreover, in \(9\) to achieve performance recovery the high-frequency gain matrix \(b(x)\) is required to be lower triangular, while this is not necessary in this note.

### III. Observer and Control Design

Let \(C_x \subset \mathbb{R}^n\) be any compact set, and \(c > 0\) be such that

\[ C_x \subset \Omega_c := \{ x \in \mathbb{R}^n : V_x(x) \leq c \} \]

where \(V_x(x)\) is defined in \(6\). As in \(6\), we assume that the high-frequency gain matrix \(b(x)\) satisfies the property below.

Assumption 4: There exist a constant nonsingular matrix \(\tilde{B} \in \mathbb{R}^{m \times m}\) and a number \(0 < \mu_0 < 1\) such that

\[ \|((b(x) - \tilde{B})\tilde{B}^{-1})\| \leq \mu_0 , \quad \text{for all} \quad x \in \Omega_{c+1}. \]  

Define the perturbation term \(6\)

\[ \sigma(x, u) := a(x) + [b(x) - \tilde{B}]u \]

which indicates that \(a(x) + b(x)u = \tilde{B}u + \sigma\), and the ideal feedback control \(u^*\) in \(4\) can be rewritten as

\[ u^* = -\tilde{B}^{-1}(\sigma(x, u) + K(x)). \]

In view of this, if there is a desired observer that can provide estimates for both the partial states \(\xi\) and the perturbations \(\sigma\), an estimate of the ideal feedback control \(u^*\) can be obtained. However, it is noted that the perturbations \(\sigma\) defined in \(4\) are in fact a function of the control input \(u\), and appears not only in the bottom equation of each set of \(2\), but also at the middle equations of the \(k\)th set, \(k = 2, \ldots, m\). This makes the observer design and the stability analysis challenging.

Bearing in mind the previous analysis, we design a set of observers having the form

\[
\begin{align*}
\hat{y}_{1,1}^1 &= \eta_{1,1}^2 + \xi_1\gamma_{1,1}(y_1 - \eta_{1,1}^1) \\
\hat{y}_{1,2}^1 &= \eta_{1,2}^2 + (\xi_1)^2\gamma_{1,1}(y_1 - \eta_{1,1}^1) \\
\hat{y}_{1,i}^1 &= \eta_{i,1}^2 + \xi_1\gamma_{i,1}(\hat{y}_{i-1,1}^1 - \eta_{1,1}^1), \quad 1 \leq i \leq r_1 - 2 \\
\hat{y}_{1,i}^1 &= \eta_{i,1}^2 + (\xi_1)^2\gamma_{i,1}(\hat{y}_{i-1,1}^1 - \eta_{1,1}^1), \quad 1 \leq i \leq r_1 - 2 \\
\hat{y}_{1,r_1-1}^1 &= \eta_{r_1-1,1}^2 + \xi_1\gamma_{r_1-1,1}(\hat{y}_{r_1-2,1}^1 - \eta_{1,1}^1) \\
\hat{y}_{1,r_1-1}^1 &= \eta_{r_1-1,1}^2 + \tilde{B}_1u + (\xi_1)^2\gamma_{r_1-1,1}(\hat{y}_{r_1-2,1}^1 - \eta_{1,1}^1 - \eta_{1,r_1}^1) \\
\hat{y}_{1,r_1-1}^1 &= \eta_{r_1-1,1}^2 + \tilde{B}_1u + \xi_1\gamma_{r_1-1,1}(\hat{y}_{r_1-2,1}^1 - \eta_{1,r_1}^1) \\
\hat{y}_{1,r_1-1}^1 &= (\xi_1)^2\gamma_{r_1-1,1}(\hat{y}_{r_1-2,1}^1 - \eta_{1,r_1}) \\
\hat{y}_{1,r_1-1}^1 &= (\xi_1)^2\gamma_{r_1-1,1}(\hat{y}_{r_1-2,1}^1 - \eta_{1,r_1}) \\
\end{align*}
\]

and for \(k = 2, \ldots, m\),

\[
\begin{align*}
\hat{y}_{k,1}^1 &= \hat{y}_{k,1}^1 + \sum_{j=1}^{k-1} M_{k,1}^j(\eta_{j,r_j}^2 + \tilde{B}_j u) + \xi_k\gamma_{k,1}(y_k - \eta_{k,1}^1) \\
\hat{y}_{k,2}^1 &= \hat{y}_{k,2}^1 + \sum_{j=1}^{k-1} M_{k,2}^j(\eta_{j,r_j}^2 + \tilde{B}_j u) + (\xi_k)^2\gamma_{k,1}(y_k - \eta_{k,1}^1) \\
\hat{y}_{k,i}^1 &= \hat{y}_{k,i}^1 + \sum_{j=1}^{k-1} M_{k,i}^j(\eta_{j,r_j}^2 + \tilde{B}_j u) + \xi_k\gamma_{k,i}(\eta_{k,i-1}^2 - \eta_{k,i}^1), \\
&\quad 1 \leq i \leq r_k - 2 \\
\hat{y}_{k,r_k-1}^1 &= \hat{y}_{k,r_k-1}^1 + \sum_{j=1}^{k-1} M_{k,r_k-1}^j(\eta_{j,r_j}^2 + \tilde{B}_j u) \\
&\quad + (\xi_k)^2\gamma_{k,r_k-1}(\eta_{k,r_k-2}^2 - \eta_{k,r_k-1}^1) \\
\hat{y}_{k,r_k-1}^1 &= \hat{y}_{k,r_k-1}^1 + \tilde{B}_k u + (\xi_k)^2\gamma_{k,r_k-1}(\eta_{k,r_k-2}^2 - \eta_{k,r_k-1}^1) \\
\hat{y}_{k,r_k-1}^1 &= \hat{y}_{k,r_k-1}^1 + \tilde{B}_k u + \xi_k\gamma_{k,r_k-1}(\eta_{k,r_k-2}^2 - \eta_{k,r_k-1}^1) \\
\end{align*}
\]

where the observer state \(\eta_k = \text{vec}(\eta_{k,1}, \ldots, \eta_{k,r_k})\) with \(\eta_{k,j} := \text{col}(\eta_{1,k,j}^1, \eta_{2,k,j}^1) \in \mathbb{R}^2\), and high-gain parameters \(\xi_k\), parameters \(\gamma_{k,1}^1, \gamma_{k,i}^2, k = 1, \ldots, m\) will be determined later.

The above set of observers \(10-11\) is comprised of \(m\) high-gain observers, the \(k\)th of which is used to estimate not only the partial state \(\xi_k\), but also the perturbation term \(\sigma_k\) (i.e., the \(k\)-th entry of \(\sigma\)). In this respect, the observer \(10-11\) is a kind of extended high-gain observer \(2, 3\). On the other hand, the design of \(10-11\) also utilizes the low-power technique developed in \(12\) for the purpose of solving the implementation problem when \(r_k\) is very large. As one can see, the high-gain parameter \(\xi_k\) of each observer is powered up to only 2, rather than \(r_k + 1\), as in \(2, 3\), although the dimension of the observer increases to \(2r_k\).

Let \(\xi_k\) and \(\sigma_k\) denote the estimates of \(\xi_k\) and \(\sigma_k\), respectively, the expressions of which are given by

\[ \xi_k = (I_{r_k} \otimes C_2)\eta_k, \quad \sigma_k = \eta_{k,r_k}^2, \quad k = 1, \ldots, m \]

where \(\otimes\) denotes the Kronecker product.

Setting \(\eta := \text{vec}(\eta_1, \ldots, \eta_m) \in \mathbb{R}^{2r}, \hat{\xi} := \text{vec}(\xi_1, \ldots, \xi_m) \in \mathbb{R}^r\) and \(\sigma := \text{col}(\sigma_1, \ldots, \sigma_m) \in \mathbb{R}^m\),
rather than the ideal feedback 9, we propose an implementable feedback law as

\[ u = -\tilde{B}^{-1}\text{satv}_l \left( \bar{\sigma} + K\dot{\bar{\xi}} \right) \quad (12) \]

where satv_l(\cdot) is a vector-valued saturation function, each element of which is an odd and monotonic saturation function satv_l(\cdot), characterized as follows: satv_l(s) = s if |s| \leq t; 0 < \frac{d\text{satv}_l(s)}{ds} < 1 for all |s| > t; and \lim_{s \to \infty} \text{satv}_l(s) = l + \varepsilon_0 with 0 < \varepsilon_0 \ll 1. It is noted that the saturation level l is a design parameter, whose value will be determined in the next section. For convenience, we use \nabla\text{satv}_l to denote the Jacobian matrix of function satv_l(\cdot). Clearly, by definition we have \|\nabla\text{satv}_l\| \leq 1.

IV. STABILITY ANALYSIS

A. Change of Coordinates

The aim of this subsection is to derive the estimation error dynamics, whose stability will be analyzed in the next subsection.

Define the scaled estimation errors as

\[
\begin{align*}
\eta_{k,1}^1 &= (\ell_k)^{r_k} (y_k - \eta_{k,1}^1) \\
\eta_{k,1}^2 &= (\ell_k)^{r_k - i} (\xi_k,2 - \eta_{k,1}^2) \\
\eta_{k,i}^1 &= (\ell_k)^{r_k - i+1} (\xi_{k,i} - \eta_{k,i}^1), \quad 1 \leq i \leq r_k - 2 \\
\eta_{k,1}^{r_k-1} &= (\ell_k)^2 (\xi_{k,1} - \eta_{k,1}^{r_k-1}) \\
\eta_{k,1}^{r_k} &= \ell_k (\xi_{k,1} - \eta_{k,1}^{r_k}) \\
\eta_{k,r_k}^1 &= \sigma_k - \eta_{k,r_k}^1
\end{align*}
\]

(13)

with \sigma_k being the k-th element of vector \sigma defined in 8, for 1 \leq k \leq m.

Observe that setting \bar{\sigma}_k := \sigma_k - \bar{\sigma}_k, we have \bar{\sigma}_k = \eta_{k,r_k}^1. To be consistent with the previous notations, we set 
\bar{\sigma} = \text{col} (\bar{\sigma}_1, \ldots, \bar{\sigma}_m), \bar{\eta}_{k,j} = \text{col} (\bar{\eta}_{k,j,1}, \bar{\eta}_{k,j,2}), \bar{\eta}_k = \text{vec}(\bar{\eta}_{k,1}, \bar{\eta}_{k,r_k}), for 1 \leq k \leq m and 1 \leq j \leq r_k and \bar{\eta} = \text{vec}(\bar{\eta}_1, \ldots, \bar{\eta}_m).

Remark 3: From the bottom equation of (13), it can be seen that \bar{\sigma} is used to estimate the entire perturbation \sigma, which is motivated by (16), (17) and is significantly different from [2], [3], where the extra observer state (i.e., \eta_{n+1} in [2] and \sigma in [3]) is used to partially estimate the perturbations, that is to estimate a term independent of u. An obvious benefit of using this complete estimation is that the perturbations that appear in the middle equations of the kth set can be controlled towards zero with the asymptotic gain adjusted by the high-gain parameters \ell_j, j = 1, \ldots, k - 1. This in turn enables us to analyze the closed-loop stability by appropriately designing gain parameters \ell_j, j = 1, \ldots, m. However, since the perturbations \sigma to be estimated depend on the control input u, the corresponding stability analysis will be more complicated than that of [2], [3], due to the need to compute the derivative of the estimation errors.

Bearing in mind the change of variables (13), we observe that (12) implicitly defines u as a solution of the equation

\[ u = -\tilde{B}^{-1}\text{satv}_l(\sigma(x,u) + K\xi - \bar{\sigma} - K(\Lambda_{\ell}^{-1} \otimes C_2)\eta) \quad (14) \]

where \Lambda_{\ell} = \text{blkdiag}(\Lambda_{\ell,1}, \ldots, \Lambda_{\ell,n}) with \Lambda_{\ell,k} = \text{diag}(\ell_{k}^{r_k}, \ldots, \ell_{k}), and \sigma, as defined in 8, depends on x, u. The following lemma shows that the equation (14) has the unique solution u.

Lemma 1: Set \psi(u) = u + \tilde{B}^{-1}\text{satv}_l(\sigma + K\xi - \bar{\sigma} - K(\Lambda_{\ell}^{-1} \otimes C_2)\eta) and suppose Assumption 4 holds. Then there exists a unique solution of the equation \psi(u) = 0 for all x \in \Omega_{\ell+1}.

With Assumption 4 some simple calculations show that the Jacobian \partial \psi(\cdot)/\partial u is uniformly nonsingular, which in turn proves Lemma 1. We omit the corresponding details. Then, let u = \pi(x, \eta) denote the unique solution of (14), which allows us to rewrite (1)-(2) as

\[
\begin{align*}
\dot{x} &= f(x) + g(x)\pi(x, \eta) \\
\dot{\bar{\eta}} &= (A - BK)\bar{\eta} + B(\phi(x, \eta) - \text{satv}_l(\phi(x, \eta))) \\
&= (A - BK)\bar{\eta} - \bar{\sigma} - K(\Lambda_{\ell}^{-1} \otimes C_2)\eta
\end{align*}
\]

in which

\[ \phi(x, \eta) = K\Phi(x) + a(x) + (b(x) - \tilde{B})\pi(x, \eta) \]

Then, let \ell_i \geq 1, i = 1, \ldots, m and the saturation level l be

\[ l = \sup_{x \in \Omega_{\ell+1}, |\eta| \leq 1} |\phi(x, \eta) - \bar{\sigma} - K(\Lambda_{\ell}^{-1} \otimes C_2)\eta| \quad (16) \]

Taking the time derivative of the estimation errors in (13) yields

\[
\begin{align*}
\dot{\eta}_{k,1}^1 &= \ell_k (\ell_k - \eta_{k,1}^1) + (\ell_k)^{r_k} \sum_{j=1}^{k-1} M_{k,j}\eta_{k,r_j}^1 \\
\dot{\eta}_{k,1}^2 &= \ell_k (\ell_k - \eta_{k,1}^2), \quad 1 \leq i \leq r_k - 1 \\
\dot{\eta}_{k,i}^1 &= \ell_k (\ell_k - \eta_{k,i}^1), \quad 1 \leq i \leq r_k - 1 \\
\dot{\eta}_{k,1}^{r_k-1} &= \ell_k (\ell_k - \eta_{k,1}^{r_k-1}) \\
\dot{\eta}_{k,1}^{r_k} &= \ell_k (\ell_k - \eta_{k,1}^{r_k}) \\
\dot{\eta}_{k,r_k}^1 &= \sigma_k - \eta_{k,r_k}^1
\end{align*}
\]

for \eta_{n+1} in [2] and \sigma in [3]) is used to partially estimate the perturbations, that is to estimate a term independent of u. An obvious benefit of using this complete estimation is that the perturbations that appear in the middle equations of the kth set can be controlled towards zero with the asymptotic gain adjusted by the high-gain parameters \ell_j, j = 1, \ldots, k - 1. This in turn enables us to analyze the closed-loop stability by appropriately designing gain parameters \ell_j, j = 1, \ldots, m. However, since the perturbations \sigma to be estimated depend on the control input u, the corresponding stability analysis will be more complicated than that of [2], [3], due to the need to compute the derivative of the estimation errors.

Putting all bottom equations of (17) and (18) together, and recalling the fact that \bar{\sigma}_k = \eta_{k,r_k}^1, we have

\[ \dot{\bar{\sigma}} = H\bar{\sigma} + \bar{\sigma} \quad (19) \]

where \Lambda_\ell = \text{diag}(\ell_1 I_{r_1}, \ldots, \ell_m I_{r_m}), and

\[ H = \text{blkdiag}(H_1, \ldots, H_m), \quad H_k = (0 \quad \cdots \quad 0 \quad -\gamma_{k,r_k}^2 \quad -\gamma_{k,r_k}^2) \in \mathbb{R}^{2r_k}. \quad (20) \]

Recalling (8) and (12), we observe that

\[ \sigma = a(x) - [b(x) - \tilde{B}]\tilde{B}^{-1}\text{satv}_l(\sigma + K\xi - \bar{\sigma} - K(\Lambda_{\ell}^{-1} \otimes C_2)\eta). \]


whose time derivative is given by
\[
\dot{\sigma} = \dot{a}(x) - \dot{b}(x)\dot{B}^{-1}\nabla \sigma(v) - \Delta_0(\dot{\sigma} + K(\dot{\sigma} - \dot{B}^{-1}\nabla \sigma(v)))
\]
where
\[
\Delta_0 = [b(x) - \dot{B}]\dot{B}^{-1}\nabla \sigma(v).
\]
(22)

By adding the term \(\Delta_0\dot{\sigma}\) on both sides of equation (21), and setting
\[
\Delta_1 = \dot{a}(x) - \dot{b}(x)\dot{B}^{-1}\nabla \sigma(v) - K(\dot{\sigma} - \dot{B}^{-1}\nabla \sigma(v)) - \Delta_0(\dot{\sigma} - \dot{B}^{-1}\nabla \sigma(v)),
\]
the equation (21) can be rewritten as
\[
(I_m + \Delta_0)\dot{\sigma} = \Delta_1 + \Delta_0(\dot{\sigma} - K(\dot{\sigma} - \dot{B}^{-1}\nabla \sigma(v))).
\]
(23)

We then observe that \(\Delta_0\) and \(\Delta_1\) have the following properties. **Lemma 2:** Suppose Assumption 4 holds, then for all \(x \in \Omega_{c+1}\),
(i) \(\|\Delta_0\| \leq \mu_0 < 1\), and \(I_m + \Delta_0\) is invertible,
(ii) there exists a constant \(\delta_1 > 0\), independent of \(\ell\), such that \(|\Delta_1| \leq \delta_1\) holds for all \(\dot{\eta} \in \mathbb{R}^{2r}\).

The proof of Lemma 2(i) is straightforward using Assumption 4 and the fact that \(\nabla \sigma(v)\) is a diagonal matrix whose entries are less than one, while the proof of (ii) can be easily concluded by deriving the explicit expression of \(\Delta_1\) and is also omitted. Using the first part of Lemma 2 (24) implies
\[
\dot{\sigma} = (I_m + \Delta_0)^{-1}[\Delta_1 + \Delta_0(\dot{\sigma} - K(\dot{\sigma} - \dot{B}^{-1}\nabla \sigma(v)))].
\]
(25)

On the other hand, it can be verified that \((\dot{\sigma} - \dot{B}^{-1}\nabla \sigma(v))\) is independent of \(\sigma\), and \((\dot{\sigma} - \dot{B}^{-1}\nabla \sigma(v))\) can be expressed as a linear function of \(\dot{\eta}\). In other words, there exists \(J(\ell)\), dependent on \(\ell\) such that
\[
(\dot{\sigma} - \dot{B}^{-1}\nabla \sigma(v)) = J(\ell)\dot{\eta},
\]
(26)

where, bearing in mind the definition of \(\Lambda_\ell\) given after (14), \(J(\ell)\) has the property that for any \(\ell_i \geq 1, i = 1, \ldots, m\), there exists \(\delta_2 > 0\), independent of \(\ell_i\)'s, such that
\[
|J(\ell)| \leq \delta_2.
\]
(27)

Substituting (25) and (26) into (19), we obtain
\[
[I_m - (I_m + \Delta_0)^{-1}\Delta_0] \dot{\sigma} = \dot{H}L\dot{\eta} + (I_m + \Delta_0)^{-1}[\Delta_1 + \Delta_0KJ(\ell)\dot{\eta}].
\]
(28)

By observing that \([I_m - (I_m + \Delta_0)^{-1}\Delta_0] = (I_m + \Delta_0)^{-1}\), we further obtain
\[
\dot{\sigma} = (I_m + \Delta_0)HL\dot{\eta} + \Delta_1 + \Delta_0KJ(\ell)\dot{\eta}.
\]
(29)

where for \(i = 1, \ldots, m, F_{ij} = A_2 - \Gamma_{ij}C_2, j = 1, \ldots, r_i,\)
\[
D_2 = \begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix}, \quad \Gamma_{ij} = \begin{pmatrix}
\gamma_{ij}^1 & 0 \\
0 & \gamma_{ij}^2
\end{pmatrix}.
\]

Thus, the equations of the re-scaled estimation errors (17) and (18) can be compactly described by
\[
\dot{\eta} = [F(\ell) + G\Delta_0H + G\Delta_0KJ(\ell)L_\ell\eta + G\Delta_1]L_\ell\dot{\eta} + G\Delta_1
\]
(30)

where
\[
F(\ell) = \begin{pmatrix}
F_1 \\
\frac{1}{\ell_1}L_{21}(\ell_2)B_{21}^T & F_2 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & \frac{1}{\ell_m}L_{m1}(\ell_m)B_{m1}^T & 0 \\
0 & \cdots & \frac{1}{\ell_m}L_{m2}(\ell_m)B_{m2}^T & \cdots & F_m
\end{pmatrix},
\]
\[
G = \text{blkdiag}(B_{21}, \ldots, B_{2m}),
\]
\[
L_{ij}(\ell_i) = \begin{pmatrix}
\ell_i^{-r_i+1} & 1 \\
\ell_i^{-2} & \ell_i^{-r_i+1} & 1 \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & \ell_i^{-r_i+1} & 1 \\
0 & \cdots & \ell_i^{-r_i+1} & 0
\end{pmatrix}, \quad 1 \leq j \leq i \leq m.
\]

It is noted that there exists \(i_\ell > 0\), independent of \(\ell\), such that \(|L_{ij}(\ell_i)| \leq \epsilon_i'i_\ell^{-r_i+1}\) holds, given \(\ell_i \geq 1\).

**B. Stability Analysis of the Estimation Error Dynamics**

Before presenting the main result of this subsection, a fundamental lemma, proven in Appendix A, is given below.

**Lemma 3:** Suppose Assumption 4 holds. There exist \(\gamma_{i,j}^1, \gamma_{i,j}^2 > 0, j = 1, \ldots, r_i, i = 1, \ldots, m\), and symmetric positive definite matrices \(P_i\) and positive constants \(\lambda_i > 0, i = 1, \ldots, m\) such that
\[
\sum_{i=1}^{m} \eta_i^T(P_iF_i + I_{r_i}^T)\eta_i + 2\eta^TPG\Delta_0H\eta \leq -\sum_{i=1}^{m} \lambda_i|\eta_i|^2
\]

with \(P = \text{blkdiag}(P_1, \ldots, P_m)\), holds for all \(x \in \Omega_{c+1}\). With the choice of \(\Gamma_{ij} = \begin{pmatrix} \gamma_{ij}^1 & 0 \\ 0 & \gamma_{ij}^2 \end{pmatrix}\) in Lemma 3, the stability property of (20) is formulated as below.

**Proposition 1:** Given any \(\tau_{\text{max}} > 0\) and \(R > 0\), suppose \(x \in \Omega_{c+1}\) for all \(t \in [0, \tau_{\text{max}}]\), and the initial conditions \(|\eta(0)| \leq R\). Let \(\Gamma_{ij}\) be chosen as in Lemma 3 so that (30) is satisfied, and choose the design parameters as
\[
\ell_i = g_i\cdot((\ell_i+1)^{r_i} - r_i)\quad \text{for } 1 \leq i \leq m - 1.
\]

Then for every \(\tau_2 < \tau_{\text{max}}\) and every \(\epsilon > 0\), there exist \(q_i > 0, i = 1, \ldots, m\), independent of \(\kappa\), and a \(\kappa^* \geq 1\) such that for all \(\kappa \geq \kappa^*\),
\[
|\eta(t)| \leq 2\epsilon, \quad \text{for all } t \in [\tau_2, \tau_{\text{max}}].
\]

**Proof.** Let \(V_c(\eta) = \eta^T L_cP\eta\), and \(\alpha_1 = \min\{\text{eig}(P)\}\) and \(\alpha_2 = \max\{\text{eig}(P)\}\) with \(\text{eig}(P)\) denoting the set of all eigenvalues of matrix \(P\). It is clear that
\[
V_c(\dot{\eta}) \geq \alpha_1\sum_{i=1}^{m} \ell_i|\eta_i|^2 \geq \alpha_1\ell_{\text{min}}|\eta|^2
\]
\[
V_c(\eta) \leq \alpha_2\sum_{i=1}^{m} \ell_i|\eta_i|^2 \leq \alpha_2\ell_{\text{max}}|\eta|^2
\]

where \(\ell_{\text{max}} = \max\{\ell_1, \ldots, \ell_m\}\) and \(\ell_{\text{min}} = \min\{\ell_1, \ldots, \ell_m\}\).
We compute the derivative of $V_c$ along the system (29) as

$$
\dot{V}_c = 2\eta^T L_1 P \left[ F(\ell) + G \Delta_0 H + G \Delta_0 K J(\ell) L_1^{-1} \right] L_1 \eta
+ 2\eta^T L_1 PG \Delta_i
$$

$$
= \sum_{i=1}^{m-1} \ell_i \eta_i^T (P_i F_i + F_i^T P_i) \ell_i \eta_i
+ 2\eta^T L_1 PG \Delta_0 H L_1 \eta
+ 2\eta^T L_1 PG \Delta_0 K J(\ell) \eta + \Delta_i
$$

$$
\leq \sum_{i=1}^{m-1} \lambda_i \ell_i^2 |\eta_i|^2 + 2 \sum_{i=1}^{m-1} \sum_{j=i+1}^m \ell_i \eta_i^T P_i \ell_j \eta_j
+ 2\eta^T L_1 PG \Delta_0 K J(\ell) \eta + \Delta_i
$$

where the inequality is obtained by using Lemma 3 and the fact that $|L_1(\ell_i)| \leq \ell_i \lambda_i^{\ell_i^2}$.

Then letting $\lambda_{\min} = \min\{\lambda_1, \ldots, \lambda_m\}$, and using Young’s Inequality, (27) and Lemma 2 we have

$$
2\ell_i \eta_i^T P_i \ell_j \eta_j \leq \frac{2(\ell_i - 1) \lambda_j \eta_j^T P_i \ell_j \eta_j}{2(\ell_i - 1) \lambda_j^2 \eta_j^T P_i \ell_j \eta_j}
$$

$$
2\eta^T L_1 PG \Delta_0 K J(\ell) \eta \leq \frac{2 \lambda_{\min} \eta_j^T P_i \ell_j \eta_j}{2 \lambda_{\min} \eta_j^T P_i \ell_j \eta_j}
$$

$$
2\eta^T L_1 \eta_j^T PG \Delta_0 \leq \frac{2 \lambda_{\min} |\eta_j|^2}{2 \lambda_{\min} |\eta_j|^2}
$$

The first of the above inequalities further indicates that

$$
2 \sum_{i=1}^{m-1} \sum_{j=i+1}^m \ell_i \eta_i^T P_i \ell_j \eta_j \leq \lambda_{\min} \sum_{i=1}^{m-1} \sum_{j=i+1}^m \ell_i \eta_i^T P_i \ell_j \eta_j
$$

Therefore, we have

$$
\dot{V}_c \leq -\frac{\lambda_{\min}}{2} \eta_m^2 - \theta_0 |\eta_m|^2 + \theta_1
$$

where $\theta_0 = \frac{2 \lambda_{\min}}{2} \eta_j^T P_i \ell_j \eta_j$ and $\theta_1 = \frac{8 \lambda_{\min}}{2} \eta_j^T P_i \ell_j \eta_j$.

To further elaborate the right side of (34), we need the following lemma with the proof given in Appendix B

**Lemma 4:** There exist constants $g_i > 0$, $i = 1, \ldots, m$, independent of $\kappa$, and $\theta^* > 0$ such that for all $\kappa \geq \theta^*$, all $\ell_i$, $i = 1, \ldots, m$ given in (31) satisfy the inequalities

$$
\frac{\lambda_{\min}}{2} \ell_i^2 - \theta_0 \geq \kappa \ell_i
$$

$$
\frac{\lambda_{\min}}{4} \ell_i^2 - \theta_0 \geq \kappa \ell_i
$$

with $i = 1, \ldots, m - 1$.

With $g_i$ and $\theta^*$ as designed as in the above lemma, it is seen that for $\kappa \geq \theta^*$, the derivative of $V_c$ in (34) can be further bounded by

$$
\dot{V}_c \leq -\frac{\kappa}{\alpha_2} \sum_{i=1}^{m} \ell_i |\eta_i|^2 + \theta_1 \leq -\frac{\kappa}{\alpha_2} V_c(\eta) + \theta_1.
$$

Bearing in mind inequalities (33), standard arguments then show that

$$
V_c(\eta(t)) \leq e^{-\frac{\alpha_1}{\alpha_2}t} V_c(\eta(0)) + \frac{\theta_1}{\alpha_2}
$$

$$
\implies |\eta(t)| \leq \frac{\alpha_2 \max_{\ell_1}}{\alpha_{\ell_{\min}}} e^{-\frac{\alpha_1}{\alpha_2}t} |\eta(0)| + \frac{\theta_1}{\alpha_2}.
$$

With (31), and recalling the fact that all coefficients $g_i$’s in (31) are independent of parameter $\kappa$, it is immediate that there exists a $\kappa^* > 0$ such that for all $\kappa \geq \kappa^*$, $\ell_{\min} = \ell_m$ and $\ell_{\max} = \ell_1$, and there exists a constant $\kappa > 0$, independent of $\kappa$ such that

$$
\frac{\alpha_2 \max_{\ell_1}}{\alpha_{\ell_{\min}}} \leq \kappa \kappa^* = \kappa^* \kappa^* + \kappa^* + \alpha_2
$$

$$
\kappa \kappa^* \kappa^* + \kappa^* + \alpha_2 \leq \kappa \kappa^* \kappa^* + \kappa^* + \alpha_2.
$$

Thus choosing $\kappa^* = \max\{1, \theta^*, \kappa^*, \sqrt{\frac{\kappa^*}{\alpha_2}}\}$ yields (32), which completes the proof. $\Box$

**C. Stability Analysis of the Closed-Loop System**

In this subsection, we analyze the asymptotic stability of the resulting closed-loop system using the nonlinear separation principles [13, 19].

**Theorem 1:** Consider the closed-loop system consisting of the plant (1), the observers (10)-(11), and the controller (12). Suppose Assumptions (11) are satisfied. Given any compact set $C \subset \mathbb{R}^{n+2r}$, there exist $\ell_1 > 1, i = 1, \ldots, m$, such that all trajectories of the closed-loop system with initial conditions $(x(0), \eta(0)) \in C$ remain bounded and satisfy $\lim_{t \to \infty} |x(t)| = 0$.

**Proof:** Substituting the actual control (12) into (1), we have

$$
\dot{x} = f(x) + g(x)u + g(x)\tilde{u}
$$

where $u^*$ is defined in (9). $\tilde{u} = b^{-1}(x)(a(x) + K\xi) - B^{-1}s(t)\Phi(x)$ and note that $\xi = \Phi(x)$ by definition. With (6), computing the time derivative of $V_c(x)$ along (37) yields

$$
\dot{V}_c(x) = -\alpha_1(x) + \frac{\partial V_c}{\partial x}(x) \dot{x}
$$

$$
\leq \frac{\partial V_c}{\partial x}(x) \left( b^{-1}(x) \|a(x) + K\xi\| + (l + e_0)\|B^{-1}\| \right).
$$

It is clear that there exists a number $\delta_0 > 0$, independent of the high-gain parameters $\ell_k$, such that the inequality $V_c(x) \leq \delta_0$ holds for all $x \in \Omega_{\ell_{\max}}$. 

Therefore, it can be concluded that given any initial condition $x(0) \in C_x \subset \Omega_x$, there exists $\tau_1 \geq \frac{4}{\delta_0}$ such that $x(t) \in \Omega_{c+1}$ for all $t \in [0, \tau_1]$.

Now, let us consider the resulting closed-loop system as
\[
\dot{x} = f(x) + g(x)u^* + g(x)[\phi(x, \tilde{y}) - \text{sat}_v(\phi(x, \tilde{y}))]
- \sigma - K(\Lambda_{x}^{-1} \otimes C_2)\tilde{y}
\]
\[
\dot{\tilde{y}} = [F(\ell) + G\Delta_0H + G\Delta_0KL\ell^{-1}]\ell_\tau \tilde{y} + G\Delta_1.
\]
(38)

Given any $\tau_2 < \tau_1$, according to Proposition 1 for any sufficiently small $\varepsilon > 0$ there exists a sufficiently large $\kappa$ such that $|\tilde{y}(t)| \leq 2\varepsilon$ for all $t \in (\tau_2, \tau_1]$. This implies that
\[
\text{sat}_v(\phi(x, \tilde{y})) - \sigma - K(\Lambda_{x}^{-1} \otimes C_2)\tilde{y} = (\phi(x, \tilde{y}) - \sigma - K(\Lambda_{x}^{-1} \otimes C_2)\tilde{y})
\]
for $t \in (\tau_2, \tau_1]$. Thus, the upper system in (38) can be simplified as
\[
\dot{x} = f(x) + g(x)u^* + g(x)[\phi(x, \tilde{y}) - \text{sat}_v(\phi(x, \tilde{y}))]
- \sigma - K(\Lambda_{x}^{-1} \otimes C_2)\tilde{y}
\]
\[
\dot{\tilde{y}} = [F(\ell) + G\Delta_0H + G\Delta_0KL\ell^{-1}]\ell_\tau \tilde{y} + G\Delta_1.
\]
(39)

Given any $\tau_2 < \tau_1$, according to Proposition 1 for any sufficiently small $\varepsilon > 0$ there exists a sufficiently large $\kappa$ such that $|\tilde{y}(t)| \leq 2\varepsilon$ for all $t \in (\tau_2, \tau_1]$. This implies that
\[
\text{sat}_v(\phi(x, \tilde{y})) - \sigma - K(\Lambda_{x}^{-1} \otimes C_2)\tilde{y} = (\phi(x, \tilde{y}) - \sigma - K(\Lambda_{x}^{-1} \otimes C_2)\tilde{y})
\]
for $t \in (\tau_2, \tau_1]$. Thus, the upper system in (38) can be simplified as
\[
\dot{x} = f(x) + g(x)u^* + g(x)[\phi(x, \tilde{y}) - \text{sat}_v(\phi(x, \tilde{y}))]
- \sigma - K(\Lambda_{x}^{-1} \otimes C_2)\tilde{y}
\]
\[
\dot{\tilde{y}} = [F(\ell) + G\Delta_0H + G\Delta_0KL\ell^{-1}]\ell_\tau \tilde{y} + G\Delta!.
\]
(38)

V. CONCLUSIONS

This note studies the robust stabilization problem of multivariable invertible nonlinear systems 1 via output feedback. By utilizing feedback linearization, a state feedback law can render a linear input-output behavior and force an “ideal” system performance. By systematically designing a set of extended low-power high-gain observers, we show that this “ideal” linearizing feedback law can be approximately estimated, providing a robust output feedback stabilizer such that the origin of the resulting closed-loop system is semiglobally asymptotically stable. Moreover, each EHG has the own its high-gain parameter up to 2, which to some extent solves the numerical implementation problem.

APPENDIX

A. Proof of Lemma 3

According to [15], it can be inferred that there exist $\gamma_{i,j}^1 > 0$ and $\gamma_{i,j}^2 > 0$ such that matrix $F_i$ is Hurwitz. With these choices of $(\gamma_{i,j}^1, \gamma_{i,j}^2)$, we then consider the system
\[
\dot{\tilde{y}}_i = F_i\tilde{y}_i + G_iu_i, \quad i = 1, \ldots, m
\]
\[
y_i = H_i\tilde{y}_i, \quad i = 1, \ldots, m
\]
\[
u = \Delta_0y
\]
\[
(39)
\]
where state $\tilde{y} = (\tilde{y}_1, \ldots, \tilde{y}_m)$ with $\tilde{y}_i = \text{vec}(\tilde{y}_{i,1}, \ldots, \tilde{y}_{i,r_i})$ and $\tilde{y}_{i,j} = \text{col}(\tilde{y}_{i,j,1}^1, \tilde{y}_{i,j,2}^2)$, output $y := \text{col}(y_1, \ldots, y_m)$ and input $u := \text{col}(u_1, \ldots, u_m)$. By taking the change of variables
\[
\chi_i, k = (\Pi_j \gamma_{i,r_i+1-j}^1 \tilde{y}_{i,r_i-k}^1 - \tilde{y}_{i,r_i-k}^1), \quad k = 1, \ldots, r_i - 1
\]
\[
\chi_i, r_i = -\Pi_j \gamma_{i,r_i+1-j}^2 \tilde{y}_{i,r_i}^1
\]
\[
\chi_i, r_i + k = -\Pi_j \gamma_{i,r_i+j}^2 \tilde{y}_{i,r_i}^2, \quad k = 1, \ldots, r_i
\]
in which we have defined $y_i = \chi_i, 1$, system (39) is transformed into the lower-triangular form
\[
\dot{\chi}_i, k = -\gamma_{i,1} \chi_i, k + \chi_i, k+1, \quad 1 \leq k < r_i
\]
\[
\dot{\chi}_i, r_i = -\Pi_j \gamma_{i,r_i-j}^2 \chi_i, j - \Pi_j \gamma_{i,r_i-j}^2 u_i^j
\]
\[
y_i = \chi_i, 1
\]
\[
u = \Delta_0y.
\]
(40)

Since $\gamma_{i,1}^1$ and $\gamma_{i,1}^2$ are nonzero constants, the above change of variables defines a nonsingular matrix $T_i \in \mathbb{R}^{2r_i, 2r_i}$ such that
$\chi_i = T_i\tilde{y}^i$ with $\chi_i = \text{col}(\chi_{i,1}, \ldots, \chi_{i,2r_i})$.

For compactness, system (40) can be rewritten as
\[
\dot{\chi}_i = F_i\chi_i + G_iu_i
\]
\[
y_i = H_i\chi_i
\]
\[
u = \Delta_0y
\]
\[
(41)
\]
in which $\tilde{F}_i = \tilde{T}_i\tilde{F}_i\tilde{T}_i^{-1}$, $\tilde{G}_i = \tilde{T}_i\tilde{G}_i$ and $\tilde{H}_i = H_i\tilde{T}_i^{-1}$.

From (40), it can be easily seen that the triplet $(\tilde{F}_i, \tilde{G}_i, \tilde{H}_i)$ is controllable and observable. Denote the minimal polynomial of Hurwitz $\tilde{F}_i$ as $P_i(s) = p_{i,0} + s p_{i,1} + \ldots + p_{i,2r_i} s^{2r_i-1} + s^{2r_i}$. By some straightforward but lengthy calculations, we can deduce that $p_{i,0} = \Pi_j \gamma_{i,1}^2 \tilde{y}_{i,j}^2$. With this being the case, let $\tilde{G}(s)$ denote the state transfer function of system (41), given by $\tilde{G}(s) = \text{diag}(\tilde{G}_1(s), \ldots, \tilde{G}_m(s))$ where
\[
\tilde{G}_i(s) = -\Pi_j \gamma_{i,1}^2 \tilde{y}_{i,j}^2 / P_i(s), \quad i = 1, \ldots, m.
\]
It is clear that $|\tilde{G}_i(\infty)| = 1$ if $\rho_0$.}

By the Bounded Real Lemma 3 Theorem 3.1), there is a positive definite and symmetric matrix $\tilde{P}_i$ and a number $\lambda_i$ such that
\[
2\chi_i^\top \tilde{P}_i(F_i\chi_i + \tilde{G}_i u_i) \leq -\lambda_i |\chi_i|^2 + \frac{1}{\mu_0} |u_i|^2 - |y_i|^2
\]
for $i = 1, \ldots, m$. This then suggests
\[
\sum_{i=1}^m 2\chi_i^\top \tilde{P}_i(F_i\chi_i + \tilde{G}_i u_i) \leq -\sum_{i=1}^m \lambda_i |\chi_i|^2 + \frac{1}{\mu_0} |u|^2 - |y|^2.
\]
Since $|u| = |\Delta_0y| \leq ||\Delta_0|| |y| \leq \mu_0 |y|$ by Lemma 2 we have
\[
\sum_{i=1}^m 2\chi_i^\top \tilde{P}_i(F_i\chi_i + \tilde{G}_i u_i) \leq -\sum_{i=1}^m \lambda_i |\chi_i|^2.
\]

Thus, letting $P_i = \tilde{T}_i\tilde{P}_i\tilde{T}_i^{-1}$ and $\lambda_i = \tilde{\lambda}_i|T_i|^2$ for $i = 1, \ldots, m$, the inequality (30) can be obtained, which completes the proof.
B. Proof of Lemma 2

Let
\[
\Psi_m = \frac{\lambda_m}{2} r_m^2 - q_0 - \kappa \ell_m \\
\Psi_i = \frac{\lambda_i}{4} r_i^2 - \sum_{j=i+1}^{m} \frac{2(j-1)\|P_j\|^2_i}{\lambda_j} r_j^2(r_j-r_{i+1}) - q_0 - \kappa \ell_i
\]
with \(1 \leq i \leq m\), which indicates that the proof is completed if it is shown that \(\Psi_i \geq 0\) for all \(1 \leq i \leq m\). We proceed to show this by a recursive method.

Step 1: Let us consider the case that \(i = m\). With the choice of \(\ell_m\) given in (31), choosing \(g_m > \frac{q_0}{m}\) and letting \(\mu_m = \frac{\lambda_m}{2} g_m^2 - g_m\), we observe that \(\mu_m > 0\). Thus, it can be seen that \(\Psi_m \geq 0\) for all \(\kappa \geq \theta_m\) with \(\theta_m = \max\{1, \frac{q_0}{\mu_m}\}\).

Step 2: With the choice of \(\ell_{m-1}\) in (31), \(\Psi_{m-1}\) reads as
\[
\Psi_{m-1} = \left[\frac{\lambda_{m-1}}{4} g_{m-1}^2 - \frac{2(m-1)\|P_m\|^2_{m-1}}{\lambda_m} r_m^2(r_m-r_{m-1}+1)\right] - q_0 - \kappa g_{m-1}^2(r_{m-1}-r_m+1) \\
\geq \mu_{m-1} r_{m-1}^2(r_{m-1}-r_m+1) - q_0
\]
where the inequality is obtained using \(\kappa \geq 1\) and defining
\[
\mu_{m-1} := \left[\frac{\lambda_{m-1}}{4} g_{m-1}^2 - \frac{2(m-1)\|P_m\|^2_{m-1}}{\lambda_m} r_m^2(r_m-r_{m-1}+1)\right] g_{m-1} r_{m-1}^2(r_{m-1}-r_m+1) - q_0
\]
Given any fixed \(g_m\), it is clear that there exists a positive constant \(g_{m-1} > 0\), independent on \(\kappa\) such that \(\mu_{m-1} > 0\) for all \(g_{m-1} > g_{m-1}\). This further indicates \(\Psi_{m-1} \geq 0\) for all \(\kappa \geq \theta_{m-1}\) with \(\theta_{m-1} = \max\{1, \frac{q_0}{\mu_{m-1}}\}\).

Step \(m-i+1\): Following the previous design, we now proceed to the \(m-i+1\)-th step, \(i = 1, \ldots, m\), and have fixed \(g_j\) and \(\theta_j\) for \(j = i+1, \ldots, m\). With (31), we observe that
\[
\Psi_i(\kappa) = \omega_i \kappa^2 \Pi_{k=1}^{m-i} (r_{i+k} - r_{i+k-1} + 1) + \sum_{j=i+1}^{m} \omega_{i,j} \kappa^2 (r_j - r_{i+1}) \Pi_{k=1}^{m-j} (r_{j+k} - r_{j+k-1} + 1) - \omega_{i,0} \kappa \Pi_{j=i+1}^{m} (r_{j+k} - r_{j+k-1} + 1) - q_0
\]
where
\[
\omega_{i,i} = \frac{\lambda_i}{4} g_i^2 \Pi_{k=1}^{m-i} (g_k)^2 \\
\omega_{i,j} = \frac{2(j-1)\|P_j\|^2_i}{\lambda_j} \Pi_{k=j+1}^{m} (g_k)^2(r_j-r_{i+1}), \quad j = i+1, \ldots, m \\
\omega_{i,0} = g_i \Pi_{j=i+1}^{m} (g_k)^2(r_j-r_{i+1})
\]
In this way, the function \(\Psi_i\) in (42) is expressed as a polynomial of \(\kappa\). Moreover, it is noted that \(r_i \leq r_{i+1} \leq \cdots \leq r_m\), and the inequality
\[
\Pi_{k=1}^{m-i} (r_{i+k} - r_{i+k-1} + 1) \geq (r_j - r_{i+1}) \Pi_{k=1}^{m-j} (r_{j+k} - r_{j+k-1} + 1) \geq 1
\]
holds for all \(j = i+1, \ldots, m\). Thus, given \(\kappa \geq 1\) we have
\[
\Psi_i \geq \mu_i \kappa^2 \Pi_{k=1}^{m-i} (r_{i+k} - r_{i+k-1} + 1) - q_0
\]
with \(\mu_i = \omega_i \kappa - \sum_{j=i+1}^{m} \omega_{i,j} \kappa - \omega_{i,0}\). Recalling (43), it can be seen that given any fixed \(g_j, j = i+1, \ldots, m\), there always exists \(q^*_i > 0\), independent on \(\kappa\) such that \(\mu_i > 0\) for all \(g_j > g^*_j\). With the above choice of \(g_i\) being the case, it then can be easily shown that there exists a \(\theta_i > 0\) such that for all \(\kappa \geq \theta_i\), the polynomial function \(\Psi_i\) is positive.

Finally, following the previous recursive design, at the step \(m\) we can fix \(g_1\) and \(\theta_1\). Therefore, choosing \(\theta^* = \max\{\theta_1, \ldots, \theta_m\}\), we can conclude that for any \(\kappa \geq \theta^*\), \(\Psi_i \geq 0\) hold for all \(i = 1, \ldots, m\), which completes the proof.

References

[1] H. Khalil, Nonlinear Systems, 3rd ed. Upper Saddle River, NJ: Prentice-Hall, 2002.
[2] L. Freidovich, and H.K. Khalil, “Performance recovery of feedback-linearization-based designs,” IEEE Trans. Automat. Contr., vol.53 no.10, pp. 2324-2334, 2008.
[3] L. Wang, A. Isidori and H. Su, “Output feedback stabilization of nonlinear MIMO systems having uncertain high-frequency gain matrix,” Syst. Control Lett., vol.83, pp.1-8, 2015.
[4] B. Guo, Z. Zhao, “On convergence of the nonlinear active disturbance rejection control for MIMO systems,” SIAM Journal on Control and Optimization, vol. 51, no.2, pp.1727-1757, 2013.
[5] L. Wang, A. Isidori and H. Su, “Global Stabilization of a Class of Invertible MIMO Nonlinear Systems,” IEEE Trans. Automat. Contr., vol.60 no.3, pp. 616-631, 2015.
[6] L.Wang, A. Isidori, Z. Liu and H. Su, “Robust output regulation for invertible nonlinear MIMO systems,” Automatica, vol.82, pp. 278-286, 2017.
[7] L.Wang, A. Isidori, L. Marconi and H. Su, “Stabilization by output feedback of multivariable invertible nonlinear systems,” IEEE Trans. Automat. Contr., vol.62, pp. 2419-2433, 2017.
[8] A. Isidori, Lectures in Feedback Design for Multivariable Systems, Springer, 2017.
[9] Y. Wu, A. Isidori, R. Lu and H. Khalil, “Performance recovery of dynamic feedback-linearization methods for multivariable nonlinear systems,” IEEE Trans. Automat. Cont., DOI: 10.1109/TAC.2019.2924176.
[10] R. M. Hirschorn, “Invertibility of multivariable nonlinear control systems,” IEEE Trans. Automat. Cont., vol.24, , pp. 855-865, 1979.
[11] S. N. Singh, “A modified algorithm for invertibility in nonlinear systems,” IEEE Trans. Automat. Contr., vol.26, no.2, pp. 595-598, 1981.
[12] D. Astolfi, and L. Marconi, “A High-Gain Nonlinear Observer with Limited Gain Power,” IEEE Trans. Automat. Contr., vol.53, no.10, pp. 2324-2334, 2016.
[13] H.K. Khalil, High-Gain Observers in Nonlinear Feedback Control, Society for Industrial and Applied Mathematics, 2017.
[14] D. Liberzon, “Output-input stability implies feedback stabilization,” Syst. Control Lett., vol. 53, no. 3, pp. 237C248, 2004.
[15] L.Wang, D. Astolfi, L. Marconi and H. Su, “High-gain observers with limited gain power for systems with observability canonical form,” Automatica, vol.75, pp. 16-23, 2017.
[16] L. Praly, Z.P. Jiang, “Further results on robust semiglobal stabilization with dynamic input uncertainties,” in Proceedings of the 37th IEEE Conference on Decision and Control, pp.891 - 896, 1998.
[17] L. Praly and Z.P. Jiang, “Semiglobal stabilization in the presence of minimum-phase dynamic input uncertainties,” in Proceedings of the 4th IFAC Symposium on Nonlinear Control Systems, vol. 2 pp. 325-330, July 1998.
[18] A. Teel, L. Praly, “Tools for semiglobal stabilization by partial state and output feedback,” SIAM Journal on Control and Optimization, vol.33, no.5, pp. 1443-1488, 1994.
[19] A. Isidori, Nonlinear Control Systems II, Springer, 1999.
[20] C. M. Kellett, “Classical converse theorems in Lyapunov’s second method,” Discrete and Continuous Dynamical Systems Series B, vol.20, no.8, pp.2333-2360, 2015.
Robust Output Feedback Stabilization of Multivariable Invertible Nonlinear Systems: Feedback Linearization-Based Method

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Abstract—abc

Index Terms—Multi-input multi-output; Extended high-gain observer; Output feedback; Feedback linearization

I. INTRODUCTION

Stabilization of nonlinear systems to the zero equilibrium point has been a rather fundamental, sophisticated and open problem in the field of systems and control theory. In the last decades, several methodologies have been developed such as feedback linearization, backstepping and passivity-based control [16], that differ in the kind of system structure (normal form and lower triangular form), and in assumptions of the internal stability (input-output stability and output-to-state stability). Among them, the feedback linearization method, which utilizes a (dynamic) state feedback to derive a linear input-output behavior, has received much attention due to its simplicity, particularly for single-input single-output (SISO) systems.

For SISO nonlinear systems having the normal form, it is universally known that the feedback linearization is achieved in such a way that the undesired nonlinear terms is cancelled, resulting in an “ideal” state feedback law, which, together with a mild minimum-phase assumption, solves the stabilization problem at hand and can shape the transient performance using the linear control theory to appropriately design the linear stabilizing term. If only the output information is available for the feedback and there exist uncertainties, to solve the robust output feedback stabilization problem, [8] proposes to utilize an extended high-gain observer (EHGO) to the purpose of approximately estimating not only the unavailable states but also the perturbed nonlinear terms. This enables to obtain an estimate of the “ideal” state feedback law and recover the system performance by the “ideal” state feedback. It is noted that the power of high-gain parameter in [8] equals to \( r + 1 \) with \( r \) denoting the relative degree, which in practice may bring numerical implementation problem when \( r \) is very large. To solve this problem, the low-power technique in [9], [19] can be directly employed.

As far as multi-input multi-output (MIMO) nonlinear systems are concerned, the technique of [8] is extended in [7] to a multivariable setting with a well-defined vector relative degree. A similar result is also done in [18] with a different assumption on the high-frequency gain matrix. However, it is noted that the class of MIMO nonlinear systems considered in [7], [18] is a very particular one, where for more general class of MIMO nonlinear systems, there are very few results available in the literature. In this respect, much attention and some efforts have recently been given to a rather general class of MIMO nonlinear systems [15], [6], [12], [17] referred to as invertible MIMO nonlinear systems [1], [2], for which the vector relative degree is not necessary. In [6], with an static input-output linearizable assumption, the invertible MIMO nonlinear systems are shown to be transformed to an “intermediate” one with a vector relative degree \( \{1, \ldots, 1\} \), for which the corresponding stabilization problem via state/output feedback can be solved under a strongly minimum-phase assumption. In [12], the input-output linearizable assumption is weakened from static to dynamic, but at the price of assuming a trivial zero dynamics. Furthermore, in [17] if in addition to a “positivity condition”, the high-frequency gain matrix fulfills a lower triangular structure, a series of EHGOs can be recursively designed in such a way that the dynamic state feedback can be approximately estimated so as to guarantee the robustness and recover the performance that is obtained by the dynamic state feedback. It is noted that in all these works on invertible MIMO nonlinear systems, the aforementioned numerical implementation problem due to large powers of high-gain parameters still exists.

This paper is mainly interested in the robust stabilization for the class of MIMO invertible nonlinear systems as in [6], where the linear input-output behavior can be achieved by a static state feedback and the high-frequency gain matrix fulfills a suitable “positivity condition”. Different from [12], the “intermediate” systems with a vector relative degree \( \{1, \ldots, 1\} \) is not employed, which simplifies the corresponding output feedback stabilizer. On the other hand, due to the use of the technique of the EHGO, the “ideal” static state feedback law, that contains not only (perturbed) nonlinear terms but also unmeasurable states, can be estimated, which in turn guarantees the robustness and recovers the performance by the feedback-linearizing design. Moreover, in order to reduce the power of high-gain parameters for each observer, the low-power technique, proposed in [9], is also employed.

Notations: \( | \cdot | \) denotes the standard Euclidean norm and \( | \cdot |_{[a,b]} \) denotes the essential supremum norm of a signal restricted to an interval \( [a, b] \). For any positive integer \( d \), \( 0_d \) denotes a \( d \times 1 \) vector, whose entries are all zero, and \( (A_d, B_d, C_d) \) is used to denote the matrix triplet in the prime

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form. Namely, \( A_d \) denotes a shift matrix of dimension \( d \times d \), \( B_d = (0 \cdots 0 1)^T \in \mathbb{R}^d \), and \( C_d = (1 0 \cdots 0) \in \mathbb{R}^{1 \times d} \).

II. PRELIMINARIES

A. Problem Formulation

Consider multivariable nonlinear systems of the form

\[
\begin{align*}
\dot{x} &= f(x) + g(x)u \\
y &= h(x)
\end{align*}
\]

where \( x \in \mathbb{R}^n \), control input \( u \in \mathbb{R}^m \), output \( y \in \mathbb{R}^m \), and all mappings \( f(x), g(x) \), and \( h(x) \) are smooth. As in [4], this paper considers the class of systems satisfying the following conditions.

**Assumption 1:** For system (1), there exists a state feedback law \( u = \hat{a}(x) + \hat{b}(x)\hat{u} \) with an invertible \( \hat{b}(x) \) such that the resulting system has a linear input-output behavior between \( \hat{u} \) and \( y \).

**Assumption 2:** The system (1) is strongly invertible, in the sense of [1], [2].

With Assumptions 1 and 2 by implementing the Structure Algorithm [6, 15], we can define a set of new variables, obeying equations of the form

\[
\begin{align*}
\dot{\xi}_1 &= A_r \xi_1 + B_r (a_1(x) + b_1(x)u) \\
\dot{\xi}_k &= A_r \xi_k + \sum_{i=1}^{k-1} M_i^j (a_i(x) + b_i(x)u) + B_r (a_k(x) + b_k(x)u), \quad 2 \leq k \leq m
\end{align*}
\]

with \( r := r_1 + r_2 + \cdots + r_m \leq n \), in which the partial state \( \xi = \text{vec}(\xi_1, \cdots, \xi_m) \) with \( \xi_i \in \mathbb{R}^{r_i} \), output \( y \in \text{col}(y_1, \cdots, y_m) \) with \( y_i = C_r \xi_i \) for \( 1 \leq i \leq m \), and \( M_i^j \in \mathbb{R}^{r_i \times r_j} \) denotes the vector of “multipliers” when implementing the Structure Algorithm [6, 15], defined by

\[
M_i^j = (0_{r_j-1}^{r_j} \delta_{i,r_j+1}^{r_i} \cdots \delta_{i,r_j}^{r_j} 0), \quad 1 \leq j < i \leq m ,
\]

and \( a_i(0) = 0 \). Occasionally, we will denote by \( \xi_i \) the vector of \( \text{col}(\xi_1, \cdots, \xi_i) \), by \( M_{i,j}^k \) the \( j \)-th element of vector \( M_i^j \), by \( a_i(x) \) the \( i \)-th entry of the vector \( a(x) \in \mathbb{R}^m \) and by \( b_i(x) \) the \( i \)-th row of the invertible matrix \( b(x) \in \mathbb{R}^{m \times m} \). It is noted that as shown in [13] Proposition 9.1, one can obtain a smooth map \( \Phi : \mathbb{R}^n \to \mathbb{R}^r \) with \( \xi = \Phi(x) \) and \( \Phi(0) = 0 \).

Moreover, as in [15] we also assume that system (1) is strongly—and also locally exponentially—minimum-phase, that is formulated as below.

**Assumption 3:** There exist \( \beta_1 \in K\mathcal{L} \) and \( \alpha_1 \in K_{\infty} \) such that for every \( x(0) \in \mathbb{R}^n \),

\[
|x(t)| \leq \beta_1(|x(0)|, t) + \alpha_1(|\xi|_{0,t})
\]

holds, uniformly in \( u \), and for some constants \( d, k, M, \alpha > 0 \),

\[
\alpha_1(s) \leq k r, \quad \beta(r, t) \leq Me^{-\alpha t} r \quad \text{for} \quad |r| \leq d.
\]

Assumption 3 in fact characterizes the output-to-state stability of system (1), by recalling that \( \xi \) can be expressed as a function of outputs \( y \) and their derivatives. From (3), it is seen that \( |x(t)| \) eventually becomes small when \( \xi \) is small. With this in mind, \( x \) has been available for feedback and the functions \( a(\cdot) \) and \( b(\cdot) \) been known, by feedback linearization we could have an “ideal” control law

\[
u^* = b^{-1}(x)[-a(x) + v]
\]

with the residual control \( v \).

This “ideal” control reduces the input-output model (2) to

\[
\dot{\xi} = A\xi + B\nu
\]

where \( A = \text{blkdiag}(A_r_1, \ldots, A_r_m) \), and

\[
B = \begin{pmatrix} B_{r_1} & 0 & \cdots & 0 \\
M_1^1 & B_{r_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
M_m^1 & M_1^2 & \cdots & B_{r_m}
\end{pmatrix}.
\]

It can be easily verified that the pair \((A, B)\) is controllable. Thus a natural design of \( v \) is the linear feedback control

\[
\dot{v} = -K\xi
\]

where the choice of \( K \in \mathbb{R}^{m \times r} \) can be decided via a linear control design method to make the matrix \( A - BK \) Hurwitz. With this choice of \( K \), it immediately follows that \( \xi(t) \) is globally exponentially stable at the origin. According to [3], this, together with Assumption 3, implies that the zero equilibrium point of system (1) with the “ideal” control law (4) is globally asymptotically stable. According to the converse Lyapunov theorem [15], the resulting “ideal” closed-loop system permits a Lyapunov function \( V_2(x) \) such that

\[
\dot{V}_2 \leq -\alpha_2(|x|)
\]

for some \( \alpha_2 \leq K_{\infty} \).

The above “ideal” feedback law (4)-(6) is not implementable due to the unaccessibility of full knowledge of \( x \). In this respect, to recover the transient performance by the “ideal” feedback law, an estimate of this “ideal” feedback law (4)-(6) is required. Motivated by this, this paper develops a new set of high-gain observers, which enables us to obtain such an estimate of “ideal” controller (4)-(6).

**Remark 1:** In [3], an interesting framework was established to “reconstruct” the “ideal” control by using an extended high-gain observer to estimate the unavailable state and also the matched perturbations that appear only in the last equation. This idea was later extended to multivariable nonlinear systems in normal form in [7]. However, it is noted that the methods in [8, 7] cannot be directly used for systems having partial normal form (3), mainly because the resulting “perturbations” to be estimated are unmatched in presence of vectors \( M_i^j \).

**Remark 2:** Though all multipliers \( \delta_{i,k}^j \)'s are limited to be constants, we stress that its extension to the case that \( \delta_{i,k}^j \)'s are bounded functions of output \( y \) (i.e., \( \delta_{i,k}^j(y) \)) is straightforward by replacing the constant multipliers by the corresponding output-dependent multipliers. In this respect, it is worth noting that [17] studies a class of multivariable invertible nonlinear systems with a special state-dependent multipliers. Using dynamic extension, and under the trivial zero dynamics assumption, the system can be fully decoupled into multiple standard normal forms via state feedback. This difference is from the case in the current paper, where the zero dynamics is non-trivial, and the feedback linearization is
performed by static state feedback, resulting an “ideal” system that is not in the normal form. Moreover, in [17] to achieve the performance recovery the high-frequency gain matrix \( b(x) \) is required to be lower triangular, while it is not necessary in this paper.

III. OBSERVER AND CONTROL DESIGN

Let \( \hat{b} \) be a nonsingular matrix to be fixed later and define the “perturbation” term

\[
\sigma := a(x) + [b(x) - \hat{b}]u
\]

which indicates that \( a(x) + b(x)u = \hat{b}u + \sigma \), and the “ideal” feedback control \( u^* \) can be rewritten as

\[
u^* = -\hat{b}^{-1}(\sigma + K\xi).
\]

In view of this, if there is a desired observer that can provide estimates for both the partial states \( \xi \) and the “perturbations” \( \sigma \), an estimate of the “ideal” feedback control \( u^* \) can be obtained. However, it is noted that the “perturbations” \( \sigma \) defined in [8] are in fact a function of control input \( u \), and appears not only at the bottom equation of each set of [2], but also at the middle equations of the \( k \)-th set, \( k = 2, \ldots, m \), which suggests that the “perturbations” \( \sigma \) are not matched. As a consequence, both issues make the observer design and the stability analysis challenging.

Bearing in mind the previous analysis, we design a set of observers having the form

\[
\begin{align*}
\eta_{1,1}^k &= \eta_{1,1}^k + \ell_1 \gamma_{1,1}^k (y_1 - \eta_{1,1}^k) \\
\eta_{1,2}^k &= \eta_{1,2}^k + (\ell_1^2/2)\gamma_{1,1}^k (y_1 - \eta_{1,1}^k) + (\ell_1 \gamma_{1,1}^k/2) (\eta_{1,2}^k - \eta_{1,1}^k) \\
\eta_{2,1}^k &= \eta_{2,1}^k + \ell_2 \gamma_{2,1}^k (y_2 - \eta_{2,1}^k) \\
\eta_{2,2}^k &= \eta_{2,2}^k + (\ell_2^2/2)\gamma_{2,1}^k (y_2 - \eta_{2,1}^k) \\
\eta_{1,k}^{2,i} &= \eta_{1,k}^{2,i} + \ell_1 \gamma_{1,1}^k (y_1 - \eta_{1,1}^k) \\
\eta_{1,k}^{2,i+1} &= \eta_{1,k}^{2,i+1} + (\ell_1^2/2)\gamma_{1,1}^k (y_1 - \eta_{1,1}^k)
\end{align*}
\]

and for \( k = 2, \ldots, m \),

\[
\begin{align*}
\eta_{k,i}^{1,k} &= \eta_{k,i}^{1,k} + \sum_{j=1}^{k-1} M_{k,j}^1 (\eta_{j,r,j} + \hat{b}_j u) + \ell_1 \gamma_{1,k}^1 (y_k - \eta_{k,1}^k) \\
\eta_{k,i}^{2,k} &= \eta_{k,i}^{2,k} + \sum_{j=1}^{k-1} M_{k,j}^2 (\eta_{j,r,j} + \hat{b}_j u) + (\ell_2 \gamma_{2,k}^2) (y_k - \eta_{k,1}^k) \\
\eta_{k,i}^{2,i} &= \eta_{k,i}^{2,i} + \sum_{j=1}^{k-1} M_{k,j}^2 (\eta_{j,r,j} + \hat{b}_j u) + \ell_2 \gamma_{2,k}^1 (y_k - \eta_{k,1}^k), \\
\eta_{k,i}^{2,i+1} &= \eta_{k,i}^{2,i+1} + \sum_{j=1}^{k-1} M_{k,j}^2 (\eta_{j,r,j} + \hat{b}_j u) + (\ell_2^2/2)\gamma_{2,k}^1 (y_k - \eta_{k,1}^k)
\end{align*}
\]

where the observer state \( \eta_k = \text{vec}(\eta_{k,1}, \ldots, \eta_{k,r,k}) \) with \( \eta_{k,j} := \text{col}(\eta_{j,k}^1, \eta_{j,k}^2) \in \mathbb{R}^2 \) for \( k = 1, \ldots, m \).

The above set of observers (10)-(11) is comprised of \( m \) high-gain observers, the \( k \)-th of which is used to estimate not only partial state \( \xi_k \), but also the “perturbation” term \( \sigma_k \) (i.e., the \( k \)-th entry of \( \sigma \)). In this respect, the observer (10)-(11) is a kind of extended high-gain observer [8], [7]. On the other hand, the design of (10)-(11) also utilizes the lower-power technique developed in [9] for the purpose of solving the implementation problem when \( r_k \) is very large. As one can see, the high-gain parameter \( \ell_k \) of each observer is powered up to only 2, rather than \( r_k + 1 \), as in [8], [7], although the dimension increases to \( 2r_k \).

With this being the case, we let \( \hat{\xi}_k \) and \( \hat{\sigma}_k \) denote the estimates of \( \xi_k \) and \( \sigma_k \), respectively, the expressions of which are given by

\[
\hat{\xi}_k = (I_{r_k} \otimes C_2)\eta_k, \quad \hat{\sigma}_k = \eta_{k,r_k}^2, \quad k = 1, \ldots, m.
\]

Towards this end, by setting \( \eta := \text{vec}(\eta_1, \ldots, \eta_m) \), \( \hat{\xi} := \text{vec}(\hat{\xi}_1, \ldots, \hat{\xi}_m) \) and \( \hat{\sigma} := \text{col}(\hat{\sigma}_1, \ldots, \hat{\sigma}_m) \), instead of (9) we propose an implementable feedback law as

\[
u = -\hat{b}^{-1}(\hat{\sigma} + K\hat{\xi})
\]

where \( \text{sat}_{\nu}(\cdot) \) is a vector-valued saturation function, each element of which is an odd and monotonic saturation function \( \text{sat}_{\nu}(\cdot) \), characterised as follows: \( \text{sat}_{\nu}(s) = s \) if \( |s| \leq l; \ 0 < \frac{d}{ds}\text{sat}_{\nu}(s) < 1 \) for all \( |s| > l; \) and \( \lim_{s \to \infty} \text{sat}_{\nu}(s) = l + \epsilon_0 < l \). It is noted that the saturation level \( l \) is a design parameter, whose value will be determined in the next section. For convenience, we use \( \nabla \text{sat}_{\nu} \) to denote the Jacobian matrix of function \( \text{sat}_{\nu} \). Clearly, by definition we have \( \| \nabla \text{sat}_{\nu} \| \leq 1 \).

IV. STABILITY ANALYSIS

A. Stability Analysis of Observer (10)-(11)

Let \( C_x \subset \mathbb{R}^n \) be any compact set, and \( c > 0 \) be such that

\[
C_x \subset \Omega_x := \{ x \in \mathbb{R}^n : V(x) \leq c \}.
\]

As in [7], we assume that the high-frequency gain matrix \( b(x) \) fulfills the following property.

Assumption 4: There exist a constant nonsingular matrix \( \hat{b} \in \mathbb{R}^{m \times m} \) and a number \( 0 < \mu_0 < 1 \) such that

\[
\| (b(x) - \hat{b})\hat{b}^{-1} \| \leq \mu_0, \quad \text{ for all } x \in \Omega_{c+1}.
\]

With this being the case, we now proceed to study the stability property of the proposed observer (10)-(11).

Define the scaled estimation errors as

\[
\begin{align*}
\hat{\eta}_{k,1}^{1,k} &= (\ell_1 \gamma_{1,k}^1) (y_k - \eta_{k,1}^k) \\
\hat{\eta}_{k,1}^{2,k} &= (\ell_1 \gamma_{1,k}^1) (y_k - \eta_{k,1}^k) \\
\hat{\eta}_{k,1}^{1,i} &= (\ell_1 \gamma_{1,k}^1) (y_k - \eta_{k,1}^k) \\
\hat{\eta}_{k,1}^{2,i} &= (\ell_1 \gamma_{1,k}^1) (y_k - \eta_{k,1}^k) \\
\hat{\eta}_{k,1}^{1,i+1} &= (\ell_1 \gamma_{1,k}^1) (y_k - \eta_{k,1}^k), \quad 1 \leq i \leq r_k - 2 \\
\hat{\eta}_{k,1}^{2,i+1} &= (\ell_1 \gamma_{1,k}^1) (y_k - \eta_{k,1}^k), \quad 1 \leq i \leq r_k - 2 \\
\hat{\eta}_{k,1}^{1,r_k-1} &= (\ell_1 \gamma_{1,k}^1) (y_k - \eta_{k,1}^k) \\
\hat{\eta}_{k,1}^{2,r_k-1} &= (\ell_1 \gamma_{1,k}^1) (y_k - \eta_{k,1}^k) \\
\hat{\eta}_{k,1}^{1,r_k} &= (\ell_1 \gamma_{1,k}^1) (y_k - \eta_{k,1}^k) \\
\hat{\eta}_{k,1}^{2,r_k} &= (\ell_1 \gamma_{1,k}^1) (y_k - \eta_{k,1}^k)
\end{align*}
\]

(14)
with \( \sigma_k \) being the \( k \)-th element of vector \( \sigma \) defined in (8), and for \( 1 \leq k \leq m \).

It is observed that setting \( \tilde{\sigma}_k := \sigma_k - \hat{\sigma}_k \), we have \( \tilde{\sigma}_k = \tilde{\eta}_{m,r_k}^2 \). To be consistent with the previous notations, we set \( \hat{\sigma} = \operatorname{col}(\tilde{\eta}_{1,m}, \tilde{\eta}_{m,m}^2) \), \( \hat{\eta}_{k,i} = \operatorname{col}(\hat{\eta}_{k,j}, \hat{\eta}_{k,j}^2) \), \( \eta_{k,i} = \operatorname{vec}(\tilde{\eta}_{k,1}, \tilde{\eta}_{k,r_k}) \), for \( k = 1, \ldots, m \) and \( j = 1, \ldots, r_k \) and \( \tilde{\eta} = \operatorname{vec}(\eta_{1}, \ldots, \eta_{m}) \).

**Remark 3:** From the bottom equation of (14), it can be seen that \( \hat{\sigma} \) is used to estimate the whole “perturbation” \( \hat{\sigma} \), which is motivated by [13] and is significantly different from [8], [7], where the extra observable state (i.e., \( \eta_{n+1} \) in [8] and \( \sigma \) in [7]) is used to partially estimate the “perturbations”, that is to estimate a term independent of \( u \). An obvious benefit of using this complete estimation is that the “perturbations” that appear at the middle equations of the \( k \)-th set can be controlled towards zero with the asymptotic gain adjusted by the parameters \( \ell_j \), \( j = 1, \ldots, k-1 \). This in turn enables us to analyse the closed-loop stability by appropriately designing gain parameters \( \ell_j \), \( j = 1, \ldots, m \). However, regarding the “perturbations” \( \sigma \) to be estimated depend on the control input \( u \), the corresponding stability analysis will be more complicated than that of [8], [7], since computing the derivative of their estimation errors is needed.

Bearing in mind the change of variables (14), we observe that (12) implicitly defines \( u \) as a solution of the equation

\[
u = -\hat{b}^{-1}\text{sat}_1(\sigma + K\xi - \hat{\sigma} - K(\Lambda_\ell^{-1} \otimes C_2)\hat{\eta})
\] (15)

where \( \Lambda_\ell = \operatorname{blkdiag}(\Lambda_\ell, \ldots, \Lambda_\ell_m) \) with \( \Lambda_\ell_k = \operatorname{diag}(\ell_k^2, \ldots, \ell_k) \), and \( \sigma \), as defined in (8), depends on \( u \). The following lemma shows that the equation (15) has the unique solution \( u \).

**Lemma 1:** Set \( \psi(u) = u + \hat{b}^{-1}\text{sat}_1(\sigma + K\xi - \hat{\sigma} - K(\Lambda_\ell^{-1} \otimes C_2)\hat{\eta}) \) and suppose Assumption [8] holds. Then there exists a unique solution of the equation \( \psi(u) = 0 \) for all \( x \in \Omega_{+1} \).

With Assumption [4] some simple calculations can show that the Jacobian \( \partial\psi(u) \) is uniformly nonsingular, which in turn proves Lemma 1. For simplicity we omit the corresponding details. Then, let \( u = \pi(x, \hat{\eta}) \) denote the unique solution of (15). Then, we can rewrite (12) and (3) as

\[
\begin{align*}
\dot{x} &= f(x) + g(x)\pi(x, \hat{\eta}) \\
\dot{\xi} &= (A - BK)\xi + B[\phi(x, \hat{\eta}) - \text{sat}_1(\phi(x, \hat{\eta}) - \hat{\sigma} - K(\Lambda_\ell^{-1} \otimes C_2)\hat{\eta})]
\end{align*}
\] (16)
in which

\[
\phi(x, \hat{\eta}) = K\Phi(x) + a(x) + (b(x) - \hat{b})\pi(x, \hat{\eta}).
\]

Then, let \( \ell_i \geq 1, \ i = 1, \ldots, m \) and the saturation level \( l \) be

\[
l = \sup_{x \in \Omega_{+1}, \hat{\eta} \leq 1} |\phi(x, \hat{\eta}) - \hat{\sigma} - K(\Lambda_\ell^{-1} \otimes C_2)\hat{\eta}|.
\] (17)

To this end, taking the time derivative of the estimate errors in (14) yields

\[
\begin{align*}
\hat{\eta}_{k,1} &= \ell_1(\gamma_{k,1}\eta_{k,1}^2 + \eta_{k,1}^2) \\
\hat{\eta}_{k,1} &= \ell_1(\gamma_{k,1}\eta_{k,1}^2 + \eta_{k,1}^2) \\
\hat{\eta}_{k,i} &= \ell_1(\gamma_{k,i}\eta_{k,i}^2 + \eta_{k,i}^2) \\
\hat{\eta}_{k,i} &= \ell_1(\gamma_{k,i}\eta_{k,i}^2 + \eta_{k,i}^2) \\
\hat{\eta}_{k,r_k} &= \ell_1(\gamma_{k,r_k}\eta_{k,r_k}^2 + \eta_{k,r_k}^2) \\
\hat{\eta}_{k,r_k} &= \ell_1(\gamma_{k,r_k}\eta_{k,r_k}^2 + \eta_{k,r_k}^2) + \hat{\sigma}_k
\end{align*}
\] (18)
and for \( k = 2, \ldots, m, \)

\[
\begin{align*}
\hat{\eta}_{k,1} &= \ell_k(\gamma_{k,1}\eta_{k,1}^2 + \eta_{k,1}^2) + (\ell_k)\gamma_{k,1}\eta_{k,1}^2 + \eta_{k,1}^2 \\
\hat{\eta}_{k,1} &= \ell_k(\gamma_{k,1}\eta_{k,1}^2 + \eta_{k,1}^2) + (\ell_k)\gamma_{k,1}\eta_{k,1}^2 + \eta_{k,1}^2 \\
\hat{\eta}_{k,i} &= \ell_k(\gamma_{k,i}\eta_{k,i}^2 + \eta_{k,i}^2) \\
\hat{\eta}_{k,i} &= \ell_k(\gamma_{k,i}\eta_{k,i}^2 + \eta_{k,i}^2) \\
\hat{\eta}_{k,r_k} &= \ell_k(\gamma_{k,r_k}\eta_{k,r_k}^2 + \eta_{k,r_k}^2) \\
\hat{\eta}_{k,r_k} &= \ell_k(\gamma_{k,r_k}\eta_{k,r_k}^2 + \eta_{k,r_k}^2) + \hat{\sigma}_k
\end{align*}
\] (19)

Putting all bottom equations of (18) and (19) together, and recalling the fact that \( \hat{\sigma}_k = \tilde{\eta}_{k,r_k}^2 \), we have

\[
\hat{\sigma} = H\ell_1\hat{\eta} + \hat{\sigma}
\] (20)
where \( H = \operatorname{blkdiag}(H_1, \ldots, H_m) \),

\[
H_k = (0 \ldots 0 \gamma_{k,r_k}^2 - 2\gamma_{k,r_k}^2 0) \in \mathbb{R}^{2r_k}.
\] (21)

Recalling (8) and (12), we observe that

\[
\sigma = a(x) - |b(x)| - \hat{b}\ell^{-1}\text{sat}_1(\hat{\sigma} + K\xi) \\
\sigma = a(x) - |b(x)| - \hat{b}\ell^{-1}\text{sat}_1(\sigma + K\xi - \hat{\sigma} - K(\Lambda_\ell^{-1} \otimes C_2)\hat{\eta})
\]
whose time derivative can be described by

\[
\dot{\sigma} = \dot{a}(x) - \hat{b}\ell^{-1}\text{sat}_1(\phi(x, \hat{\eta}) - \hat{\sigma} - K(\Lambda_\ell^{-1} \otimes C_2)\hat{\eta}) \\
- \Delta_0(\sigma + K\dot{\xi} - \hat{\sigma} - K(\Lambda_\ell^{-1} \otimes C_2)\hat{\eta})
\] (22)
where

\[
\Delta_0 = \|b(x) - \hat{b}\ell^{-1}\nabla\text{sat}_1\|.
\] (23)

By adding the term \( \Delta_0\dot{\sigma} \) on the both sides of equation (22), and setting

\[
\Delta_1 = \dot{a}(x) - \hat{b}\ell^{-1}\text{sat}_1(\phi(x, \hat{\eta}) - \hat{\sigma} - K(\Lambda_\ell^{-1} \otimes C_2)\hat{\eta}) \\
- \Delta_0(\sigma + K\dot{\xi} - \hat{\sigma} - K(\Lambda_\ell^{-1} \otimes C_2)\hat{\eta})
\] (24)
the equation (22) can be rewritten as

\[
(I_m + \Delta_0)\dot{\sigma} = \Delta_1 + \Delta_0(\hat{\sigma} + K(\Lambda_\ell^{-1} \otimes C_2)\hat{\eta}).
\] (25)

We then observe that \( \Delta_0 \) and \( \Delta_1 \) have the following properties.

**Lemma 2:** Suppose Assumption [H] holds, then for all \( x \in \Omega_{+1} \),

(i) \( \|\Delta_0\| \leq \mu_0 < 1 \), and \( I_m + \Delta_0 \) is invertible,

(ii) there exists a constant \( \delta_1 > 0 \), independent of \( \ell = (\ell_1, \ldots, \ell_k) \) such that \( |\Delta_1| \leq \delta_1 \) holds for all \( \hat{\eta} \in \mathbb{R}^{2r} \).
The proof of Lemma 2(ii) is straightforward using Assumption 4 and the fact that \( \nabla \text{satv}_i \) is a diagonal matrix whose entries are less than one, while as for the proof of (ii), it can be easily concluded by deriving the explicit expression of \( \Delta_1 \) and is also omitted for simplicity. Using the first part of Lemma 2 implies

\[
\hat{\sigma} = (I_m + \Delta_0)^{-1}[\Delta_1 + \Delta_0(\hat{\sigma} + K(\Lambda_\ell^{-1} \otimes C_2)\hat{\eta})]
\]

(26)

On the other hand, it can be verified that \((\Lambda_\ell^{-1} \otimes C_2)\hat{\eta}\) is independent of \(\hat{\sigma}\), and \((\Lambda_\ell^{-1} \otimes C_2)\hat{\eta}\) can be expressed as a linear function of \(\hat{\eta}\), that is, there exists \(J(\ell)\), dependent of \(\ell\) such that

\[
(\Lambda_\ell^{-1} \otimes C_2)\hat{\eta} = J(\ell)\hat{\eta}
\]

(27)

where bearing in mind the definition of \(\Lambda_\ell\) given after (15), \(J(\ell)\) has the property that for any \(\ell_i \geq 1, i = 1, \ldots, m\), there exists \(\delta_2 > 0\), independent of \(\ell_i\)'s, such that

\[
|J(\ell)| \leq \delta_2.
\]

(28)

Substituting (26) and (27) into (20), we can obtain

\[
[I_m - (I_m + \Delta_0)^{-1}\Delta_0]\hat{\sigma} = HLL\hat{\eta} + (I_m + \Delta_0)^{-1}[\Delta_1 + \Delta_0 KJ(\ell)\hat{\eta}] .
\]

By observing that

\[
[I_m - (I_m + \Delta_0)^{-1}\Delta_0] = (I_m + \Delta_0)^{-1},
\]

we can further obtain

\[
\hat{\sigma} = (I_m + \Delta_0)HLL\hat{\eta} + \Delta_1 + \Delta_0 KJ(\ell)\hat{\eta}.
\]

(29)

Towards this end, let

\[
F_i = \begin{pmatrix}
F_{i,1} & D_2 & \cdots & 0 & 0 & 0 \\
\Gamma_{i,2}B_2^T & F_{i,2} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & F_{i,r_i - 2} & D_2 & 0 \\
0 & 0 & \cdots & \Gamma_{i,r_i - 1}B_2^T & F_{i,r_i - 1} & D_2 \\
0 & 0 & \cdots & \Gamma_{i,r_i}B_2^T & F_{i,r_i} & \cdots & D_2 \\
\end{pmatrix}
\]

(30)

where for \(i = 1, \ldots, m\), \(F_{ij} = A_2 - \Gamma_{i,j}C_2, j = 1, \ldots, r_i\),

\[
D_2 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}, \quad \Gamma_{i,j} = \begin{pmatrix}
\gamma_{i,j}^1 & \gamma_{i,j}^2 \\
\end{pmatrix}.
\]

Thus, the equations of the scaled estimate errors (18) and (19) can be compactly described by

\[
\hat{\eta} = [F(\ell) + G\Delta_0H + G\Delta_0KJ(\ell)L_\ell^{-1}]L_\ell\hat{\eta} + G\Delta_1
\]

(31)

where

\[
F(\ell) = \begin{pmatrix}
\frac{1}{\ell_1}L_{21}(\ell_2)B_2^T & 0 & \cdots & 0 \\
\Gamma_{1,2}B_2^T & F_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{\ell_m}L_{m1}(\ell_m)B_2^T & \cdots & \cdots & F_m \\
\end{pmatrix}
\]

\[
G = \text{blkdiag}(B_{21}, \ldots, B_{2r_m})
\]

\[
L_{ij}(\ell_i) = \begin{pmatrix}
(\ell_i)^{\ell_i - r_i + 1}\delta^{i_{r_i + 1}1} & \cdots & 0 \\
(\ell_i)^{\ell_i - r_i + 1}\delta^{i_{r_i + 1}1} & \cdots & 0 \\
\end{pmatrix}, \quad 1 \leq j < i \leq m
\]

It is clear that there exists \(\epsilon_{ij} > 0\), independent of \(\ell_i\) such that

\[
|L_{ij}(\ell_i)| \leq \epsilon_{ij}e^{\ell_i - r_i + 1}, \text{ for } \ell_i \geq 1.
\]

Before presenting the main result of this subsection, a fundamental lemma is given as below.

**Lemma 3:** Suppose Assumption 4 holds. There exist \(\gamma_{1,j}^1, \gamma_{1,j}^2 > 0, j = 1, \ldots, r_i, i = 1, \ldots, m\), and symmetric positive definite matrices \(P_i\) and positive constants \(\lambda_i > 0, i = 1, \ldots, m\) such that

\[
\sum_{i=1}^{m} \eta_i^T(P_iF_i + F_i^T P_i)e_i + 2\eta_i^T P_i^T G\Delta_0H\eta_i \leq -\sum_{i=1}^{m} \lambda_i|\eta_i|^2
\]

(32)

with \(P = \text{blkdiag}(P_1, \ldots, P_m)\), holds for all \(x \in \Omega_{c+1}\).

**Proof.** The proof is given in Appendix A.

With the choice of \(\Gamma_{i,j} = (\gamma_{1,j}^1, \gamma_{1,j}^2)\) in Lemma 3, we then choose a positive definite function \(V_c(\hat{\eta}) = \eta^T L\eta\hat{\eta}\). Setting \(\alpha_1 = \min\{\sigma(P)\}\) and \(\alpha_2 = \max\{\sigma(P)\}\) with \(P = \text{blkdiag}(P_1, \ldots, P_m)\), \(\sigma(P)\) denoting the set of all eigenvalues of matrix \(P\), we can obtain

\[
V_c(\hat{\eta}) \geq \alpha_1 \sum_{i=1}^{m} \ell_i|\hat{\eta}_i|^2 \geq \alpha_1 \ell_{\max}|\hat{\eta}|^2
\]

(33)

where \(\ell_{\max} = \max\{\ell_1, \ldots, \ell_m\}\) and \(\ell_{\min} = \min\{\ell_1, \ldots, \ell_m\}\).

With this in mind, we compute the derivative of \(V_c\) along system (31) as

\[
\dot{V}_c = 2\eta^T L\eta P[F(\ell) + G\Delta_0H + G\Delta_0KJ(\ell)L_\ell^{-1}]L_\ell\eta
\]

\[
+ 2\eta^T L\eta P G\Delta_1
\]

\[
+ \frac{2}{\ell_r} \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \ell_j\eta_j^2 P_j L_j(\ell_j)B_{2r_i}^T \eta_i
\]

\[
+ 2\eta^T L\eta P G[\Delta_0KJ(\ell)\eta + \Delta_1]
\]

\[
\leq -\sum_{i=1}^{m} \lambda_i \ell_i^2|\eta_i|^2 + 2\sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \ell_{ij}\|P_j\|e^{2(\ell_i - \ell_j + 1)}|\eta_j| \cdot |\eta_i|
\]

\[
+ 2\eta^T L\eta P G[\Delta_0KJ(\ell)\eta + \Delta_1]
\]

(34)

where the inequality is obtained by using Lemma 3 and the fact that \(|L_{ij}(\ell_i)| \leq \epsilon_{ij}e^{\ell_i - r_i + 1}\).

Then letting \(\lambda_{\min} = \min\{\lambda_1, \ldots, \lambda_m\}\), and using Young’s Inequality, (28) and Lemma 2, we have

\[
2\epsilon_{ij}\|P_j\|e^{\ell_i - r_i + 1}|\eta_j| \cdot |\eta_i| \leq \frac{\lambda_{\min} \sum_{k=1}^{m} \ell_k^2|\eta_k|^2 + \frac{8}{\lambda_{\min}} \|P\|^2 \|K\|^2 \sum_{k=1}^{m} |\eta_k|^2}{\ell_r^2}
\]

\[
2\eta^T L\eta P G\Delta_1 \leq \frac{\lambda_{\min} \sum_{k=1}^{m} \ell_k^2|\eta_k|^2 + \frac{8}{\lambda_{\min}} \|P\|^2 |\delta_1|^2}{\ell_r^2}
\]
The first of the above inequalities further indicates that
\[
2 \sum_{i=1}^{m-1} j \sum_{j=i+1}^{m} i_{ij} \| \tilde{P}_j \| \| \tilde{r}_i - r_j \| + \| \tilde{h}_i \| \cdot \| \tilde{h}_j \| \leq
\sum_{i=1}^{m-1} \left( \frac{\lambda_i}{2} \tilde{\epsilon}_i^2 + \frac{m}{\lambda_j} \sum_{j=i+1}^m (j-1)^2 \| \tilde{P}_j \| ^2 \tilde{\epsilon}_j^2 (r_j-r_i+1) \right) \| \tilde{h}_i \|^2 + \frac{\lambda_m \tilde{\epsilon}_m^2}{2} \| \tilde{h}_m \|^2.
\]

Therefore, we have
\[
\dot{V}_c \leq - \frac{\lambda_m}{2} \tilde{\epsilon}_m^2 (\tilde{h}_m - \theta_0) \| \tilde{h}_m \|^2 + \tilde{g}_1
- \sum_{i=1}^{m-1} \left( \frac{\lambda_i}{4} \tilde{\epsilon}_i^2 - \frac{m}{\lambda_j} \sum_{j=i+1}^m (j-1)^2 \| \tilde{P}_j \| ^2 \tilde{\epsilon}_j^2 (r_j-r_i+1) - \theta_0 \right) \| \tilde{h}_i \|^2
\]
where \( \theta_0 = \frac{\lambda \min}{2} \| P \| ^2 \| h \| ^2 \| \xi \| ^2 \) and \( \tilde{g}_1 = \frac{8}{\lambda \min} (\delta_1) \| P \| ^2 \).

Towards this end, we have the stability of system (31), that is summarized as below.

Lemma 4: Given any \( \tau_{\text{max}} > 0 \) and \( R > 0 \), suppose \( x \in \Omega_{\epsilon+1} \) for all \( t \in [0, \tau_{\text{max}}] \), and the initial conditions \( \| \tilde{\eta}(0) \| \leq R \). Let \( G_{ij} \) be chosen as in Lemma 3 so that (32) is satisfied, and choose the design parameters as
\[
\ell_m = \max \{ g \eta, 4 \}
\]
\[
\ell_i = b_i \cdot (\ell_{i+1})^{\ell_i-1} - r_i \quad \text{for } 1 \leq i \leq m - 1.
\]

Then for every \( \tau_2 < \tau_{\text{max}} \) and every \( \epsilon > 0 \), there exist \( g_i > 0 \), \( i = 1, \ldots, m \), independent of \( \kappa \), and a \( \kappa^* \geq 1 \) such that for all \( \kappa \geq \kappa^* \),
\[
| \tilde{\eta}(t) | \leq 2 \epsilon, \quad \text{for all } t \in [\tau, \tau_{\text{max}}].
\]

Proof. According to Lemma 5 (given in Appendix B), we can deduce that there exist \( g_i > 0 \), \( i = 1, \ldots, m \), independent of \( \kappa \), and \( \theta^* > 0 \) such that for all \( \kappa \geq \kappa^* \),
\[
\lambda_i \tilde{\epsilon}_i^2 - \sum_{j=i+1}^{m} (j-1)^2 \| \tilde{P}_j \| ^2 \tilde{\epsilon}_j^2 (r_j-r_i+1) - \theta_0 \geq \kappa \ell_i,
\]
\[
\lambda_m \tilde{\epsilon}_m^2 - \theta_0 \geq \kappa \ell_m.
\]

This then implies that the derivative of \( V_c \) in (35) can be further bounded by
\[
\dot{V}_c \leq - \kappa \sum_{i=1}^{m} \tilde{\epsilon}_i | \tilde{\eta}_i | ^2 + \tilde{g}_1
\leq - \frac{\alpha_2}{\alpha_1 \ell_{\text{min}}} V_c(\tilde{\eta}) + \tilde{g}_1.
\]

Bearing in mind the inequalities (33), standard arguments then show that
\[
V_c(\tilde{\eta}(t)) \leq e^{- \frac{\alpha_2}{\alpha_1 \ell_{\text{min}}} t} V_c(\tilde{\eta}(0)) + \frac{\alpha_1 \ell_{\text{max}}}{\alpha_2} \tilde{g}_1 \epsilon^2 \Rightarrow | \tilde{\eta}(t) | \leq \frac{\alpha_2 \ell_{\text{max}}}{\alpha_1 \ell_{\text{min}}} e^{- \frac{\alpha_2}{\alpha_1 \ell_{\text{min}}} t} | \tilde{\eta}(0) | + \frac{\alpha_1 \ell_{\text{max}}}{\alpha_2} \tilde{g}_1 \epsilon.
\]

With (36), and recalling the fact that all coefficients \( g_i \)'s in (36) are independent of high-gain parameter \( \kappa \), it is immediate to follow that there exists a \( \kappa^* > 0 \) such that for all \( \kappa \geq \kappa^* \), \( \ell_{\min} = \ell_m \) and \( \ell_{\max} = \ell_1 \), and there exists a constant \( \varsigma > 0 \), independent of \( \kappa \) such that
\[
\frac{\alpha_2 \ell_{\text{max}}}{\alpha_1 \ell_{\text{min}}} \leq \varsigma \kappa \varpi_1,
\]
where \( \varpi_1 = \prod_{k=1}^{m-1}(r_{k+1} - r_k + 1) - 1 \). On the other hand, since \( | \tilde{\eta}(0) | \leq R \) and \( x \in \Omega_{\epsilon+1} \), it can be seen from (31) that
\[
| \tilde{\eta}(0) | \leq \varsigma_2 \kappa \varpi_2
\]
for some \( \varsigma_2 > 0 \) and \( \varpi_2 > 0 \). Thus, we have
\[
| \tilde{\eta}(t) | \leq \varsigma_2 \kappa \varpi_1 + \varpi_2 \epsilon^2 + \frac{\theta_0 \| \tilde{\xi} \|}{g_i \alpha \kappa^2} (40)
\]

Fix any \( \epsilon > 0 \), and we know that for any \( \tau_2 \in (0, \tau_{\text{max}}) \) there always exists a \( \kappa^* > 0 \) such that for all \( \kappa \geq \kappa^* \),
\[
| \tilde{\eta}(t) | \leq 2 \epsilon, \quad \text{for all } t \in [\tau_2, \tau_{\text{max}}]. \quad (41)
\]

B. Stability Analysis of Closed-Loop System

In this subsection, we will analyze the asymptotic stability of the resulting closed-loop system using the nonlinear separation principles (14), (5).

Substituting the actual control (12) to (1), we have
\[
\dot{x} = f(x) + g(x) u^* + g(x) \tilde{u}
\]
where \( \tilde{u} = b^{-1}(x)(a(x) + K \xi) - b^{-1} \text{sat}_v(\bar{\sigma} + K \hat{\xi}) \) and note that \( \xi = \Phi(x) \), by definition. With (7), computing the time derivative of \( V_x(x) \) along (42) yields
\[
\dot{V}_x(x) = - \alpha_x(\| x \|) + \frac{\partial V_x}{\partial x}(g(x) \tilde{u})
\leq \frac{\partial V_x}{\partial x}(g(x)) \left( \| b^{-1}(x) \| (\| a(x) \| + | K \xi |) + (l + e_0) \| \tilde{u}^{-1} \| \right).
\]

It is clear that there exists a number \( \delta_0 > 0 \), independent of high-gain parameters \( \ell_i \)'s, such that the inequality
\[
V_x(x) \leq \delta_0
\]
holds for all \( x \in \Omega_{\epsilon+1} \).

Therefore, it can be concluded that given any initial condition \( x(0) \in C_x \subset \Omega_x \), there exists \( \tau_1 \geq \frac{1}{\delta_0} \) such that \( x(t) \in \Omega_{\epsilon+1} \) for all \( t \in [0, \tau_1] \).

Then let us consider the resulting closed-loop system, described as
\[
\dot{x} = f(x) + g(x) u^* + g(x) [\phi(x, \tilde{\eta}) - \text{sat}_v(\bar{\sigma}(x, \tilde{\eta})) - \bar{\sigma} - K(\Lambda^1 \otimes C_2) \tilde{\eta}]
\]
\[
\tilde{\eta} = [\mathbf{f}(\ell) + G \Delta_0 \mathbf{H} + G \Delta_0 K J(\ell) L_2^{-1}] L_2 \tilde{\eta} + G \Delta_1.
\]

Given any \( \tau_2 < \tau_1 \), according to Lemma 4 for any sufficiently small \( \epsilon > 0 \) there exists a sufficiently large \( \kappa \) such that \( | \tilde{\eta}(0) | \leq 2 \epsilon < 1 \) for all \( t \in (\tau_2, \tau_1] \). This implies that
\[
\text{sat}_v(\phi(x, \tilde{\eta}) - \bar{\sigma} - K(\Lambda^1 \otimes C_2) \tilde{\eta}) = \phi(x, \tilde{\eta}) - \bar{\sigma} - K(\Lambda^1 \otimes C_2) \tilde{\eta}
\]
for some \( \varsigma_2 > 0 \) and \( \varpi_2 > 0 \). Thus, we have
\[
| \tilde{\eta}(t) | \leq \varsigma_2 \kappa \varpi_2 + \varpi_2 \epsilon^2 + \frac{\theta_0 \| \tilde{\xi} \|}{g_i \alpha \kappa^2} (40)
\]
for $t \in (\tau_2, \tau_3]$. Thus, the upper system in (43) can be simplified as

$$\dot{x} = f(x) + g(x)u^* + g(x)[\sigma + K(A^{-1}_{\ell} \otimes C_2)\hat{\eta}]$$.

If $|\hat{\eta}| \leq 2c$, then there exists a $\rho > 0$, independent of $\ell_i \geq 1$, $i = 1, \ldots, m$ such that $|\sigma + K(A^{-1}_{\ell} \otimes C_2)\hat{\eta}| \leq 2\rho e$.

Pick any number $0 < c' \ll c$ and consider the “annular” compact set

$$S_{c'}^{c+1} = \{x : c' \leq V_x(x) \leq c + 1\}.$$

Let $\nu_{\min}$ be

$$\nu_{\min} = \min_{x \in S_{c'}^{c+1}} \alpha_x(|x|).$$

If $c$ is such that $2\rho e \leq \frac{1}{2}\nu_{\min}$, it then follows that

$$V_x(x) \leq -\frac{1}{2}\nu_{\min}(t - \tau_1)$$

so long as $x \in S_{c'}^{c+1}$. This, in turn, implies

$$V_x(x(t)) \leq V_x(x(\tau_1)) - \frac{1}{2}\nu_{\min}(t - \tau_1) \leq c + 1 - \frac{1}{2}\nu_{\min}(t - \tau_1)$$

so long as $x \in S_{c'}^{c+1}$. Clearly, there exists a time $\delta_3 > \delta_1$ such that $x(t) \in \Omega_{c+1}$ for all $t \in [\tau_2, \tau_3]$ and $V_x(x(\tau_3)) = c'$. Since $V_x$ is negative on the boundary of $\Omega_{c'}$, it is concluded that $x(t) \in \Omega_{c'}$ for all $t \geq \tau_3$ and $x(t) \in \Omega_{c+1}$ for all $t \geq 0$.

In summary, according to the arguments in [5, 14], the following result can be easily concluded.

**Proposition 1:** Consider the closed-loop system consisting of the plant ([1]), the observers ([10, 11]) and the controller ([12]). Suppose Assumptions [11, 12] are satisfied. Given any compact set $\mathcal{C} \subset \mathbb{R}^{n+2r}$, there exist $\ell_i > 1$, $i = 1, \ldots, m$ such that all trajectories of the closed-loop system with initial conditions $(x(0), \eta(0)) \in \mathcal{C}$ remain bounded and satisfy $\lim_{t \to \infty} |x(t)| = 0$.

**V. Conclusions**

**APPENDIX**

**A. Proof of Lemma 5**

First of all, according to [10], it can be implied that there exist $\gamma_{i,j}^1 > 0$ and $\gamma_{i,j}^2 > 0$ such that matrix $F_i$ is Hurwitz. With these choices of $(\gamma_{i,j}^1, \gamma_{i,j}^2)$ being the case, we then consider the system

$$\begin{align*}
\dot{\tilde{y}}_i &= F_i \tilde{y}_i + G_i u_i, \quad i = 1, \ldots, m \\
y_i &= H_i \tilde{y}_i, \quad i = 1, \ldots, m \\
u &= \Delta_0 y 
\end{align*}$$

with state $\tilde{y} = (\tilde{y}_1, \ldots, \tilde{y}_m)$ with $\tilde{y}_i = \text{vec}(\tilde{y}_{i,1}, \ldots, \tilde{y}_{i,r_i})$ and $\tilde{y}_{i,r_i} = \text{col}(\tilde{y}_{i,1}, \tilde{y}_{i,r_i})$, input $u = \text{col}(u_1, \ldots, u_m)$ and output $y = \text{col}(y_1, \ldots, y_m)$. By taking the change of variables

$$\begin{align*}
\chi_{i,k} &= -\frac{1}{2}(r_i - 1)^2 \gamma_{i,r_i-1,k} - \frac{1}{2} \gamma_{i,r_i-1,k+1} \\
\chi_{i,r_i} &= -\frac{1}{2}(r_i - 1)^2 \gamma_{i,r_i-1} \gamma_{i,r_i} \\
\chi_{i,r_i+k} &= -\frac{1}{2}(r_i - 1)^2 \gamma_{i,r_i}^2 \end{align*}$$

with $k = 1, \ldots, r_i - 1$, in which we have defined $y_i = \chi_{i,1}$, system (45) is transformed into the form

$$\begin{align*}
\dot{\chi}_{i,k} &= -\frac{1}{2}(r_i - 1)^2 \gamma_{i,r_i-1,k} + \chi_{i,k+1}, \quad 1 \leq k \leq r_i \\
\dot{\chi}_{i,r_i+k} &= -\frac{1}{2}(r_i - 1)^2 \gamma_{i,r_i-1} \chi_{i,r_i-1,k} + \chi_{i,r_i,k+1}, \\
\dot{\chi}_{i,2r_i} &= -\frac{1}{2}(r_i - 1)^2 \gamma_{i,r_i} \chi_{i,r_i} - \frac{1}{2}(r_i - 1)^2 \gamma_{i,r_i} \gamma_{i,r_i} u_i \\
y_i &= \chi_{i,1} \\
u &= \Delta_0 y
\end{align*}$$

It is worth noting that since $\gamma_{i,j}^1$ and $\gamma_{i,j}^2$ are nonzero constants, the above change of variables actually defines a nonsingular matrix $T_i \in \mathbb{R}^{2r_i \times 2r_i}$ such that $\chi_i = T_i \tilde{x}_i$ with $\chi_i = \text{col}(\chi_{i,1}, \ldots, \chi_{i,2r_i})$.

For compactness, system (46) can be rewritten as

$$\begin{align*}
\dot{\tilde{y}}_i &= F_i \tilde{y}_i + G_i u_i \\
y_i &= H_i \tilde{y}_i \\
u &= \Delta_0 y
\end{align*}$$

in which $\tilde{F}_i = T_i F_i T_i^{-1}$, $\tilde{G}_i = T_i G_i$ and $\tilde{H}_i = H_i T_i^{-1}$. Then it can be easily verified that the triplet $(\tilde{F}_i, \tilde{G}_i, \tilde{H}_i)$ is controllable and observable. Denote the minimal polynomial of Hurwitz $F_i$ as

$$P_i(s) = p_{i,0} + p_{i,1}s + \ldots + p_{i,2r_is^{2r_i-1}} + s^{2r_i}$$

By some calculations, we can deduce that $p_{i,0} = \frac{1}{\mu_0} \gamma_{i,j}^2$. With this being the case, let $G(s)$ denote the state transfer function of system (47), given by

$$G(s) = \text{diag}(G_1(s), \ldots, G_m(s))$$

in which for $i = 1, \ldots, m$

$$G_i(s) = -\frac{(\Pi_{j=1}^{r_i} \gamma_{i,j}^2)}{P_i(s)}$$

It is clear that $|G_i(\infty)| = 1 < \frac{1}{\mu_0}$.

By the Bounded Real Lemma, there is a positive definite and symmetric matrix $\tilde{P}_i$ and a number $\lambda_i$ such that

$$2\chi_i^T \tilde{P}_i (\tilde{F}_i \tilde{y}_i + \tilde{G}_i u_i) \leq -\lambda_i |\chi_i|^2 + \frac{1}{\mu_0^2} |u|^2 - |y|^2$$

for $i = 1, \ldots, m$. This then suggests

$$\sum_{i=1}^{m} 2\chi_i^T \tilde{P}_i (\tilde{F}_i \tilde{y}_i + \tilde{G}_i u_i) \leq -\sum_{i=1}^{m} \lambda_i |\chi_i|^2 + \frac{1}{\mu_0^2} |u|^2 - |y|^2$$

Since $|u| = |\Delta_0 y| \leq ||\Delta_0||y| \leq \mu_0 |y|$ by Lemma 2 we have

$$\sum_{i=1}^{m} 2\chi_i^T \tilde{P}_i (\tilde{F}_i \tilde{y}_i + \tilde{G}_i u_i) \leq -\sum_{i=1}^{m} \lambda_i |\chi_i|^2$$

Thus, letting $P_i = T_i \tilde{P}_i T_i$ and $\lambda_i \leq |\lambda_i||T_i||^2$ for $i = 1, \ldots, m$, the inequality (32) can be obtained, which completes the proof.
B. A Technical Lemma

Lemma 5: There exist constants $g_i > 0$, $i = 1, \ldots, m$, independent of $\kappa$ and $\theta^* > 0$ such that all $\ell_i$, $i = 1, \ldots, m$ given in (56) satisfy the inequalities

$$\frac{\lambda_m}{2} \ell_m^2 - \varrho_0 \geq \kappa \ell_m$$

$$\frac{\lambda_i}{4} \ell_i^2 - \sum_{j=1}^{m} \frac{2(j-1)\|P_j\|^2 \ell_j^2}{\lambda_j} (r(j-r_j+r_i+1) - \varrho_0) \geq \kappa \ell_i$$

with $i = 1, \ldots, m-1$, hold for all $\kappa \geq \theta^*$.

Proof. Let

$$\Psi_m = \frac{\lambda_m}{2} \ell_m^2 - \varrho_0 - \kappa \ell_m$$

$$\Psi_i = \frac{\lambda_i}{4} \ell_i^2 - \sum_{j=1}^{m} \frac{2(j-1)\|P_j\|^2 \ell_j^2}{\lambda_j} (r(j-r_j+r_i+1) - \varrho_0 - \kappa \ell_i$$

with $1 \leq i \leq m-1$, which indicates that the proof is completed if it is shown that $\Psi_i \geq 0$ for all $1 \leq i \leq m$.

Now we proceed to show this by recursive method. First of all, let us consider the case that $i = m$. With the choice of $\ell_m$ given in (56) and choosing $g_m = \frac{\lambda_m}{4}$, it is observed that

$$\Psi_m = \frac{3}{2} \lambda_m \kappa^2 - \varrho_0$$

Clearly, letting $\theta_m = \max\{1, \sqrt{\frac{2 \lambda_m \varrho_0}{3}}\}$, we can deduce that

$$\frac{\lambda_m}{2} \ell_m^2 - \varrho_0 \geq \kappa \ell_m$$

holds for all $\kappa \geq \theta_m$.

Then consider the case that $1 \leq i \leq m-1$, and assume $g_j$, $j = i + 1, \ldots, m$ are fixed. It is observed that

$$\Psi_i(\kappa) = \psi_i(\kappa) \Pi_{k=i+1}^{m} (r_{k+i} - r_{i+k-1} + 1)$$

$$- \sum_{j=i+1}^{m} \omega_{ij} \kappa 2(r_j - r_i + 1) \Pi_{k=j}^{m-1} (r_{j+k} - r_{j+k-1} + 1)$$

$$- \omega_{i0} \kappa \Pi_{k=i+1}^{m} (r_{i+k} - r_{i+k-1} + 1) + 1 - \varrho_0$$

where

$$\omega_{i,i} = \frac{\lambda_i}{4} \Pi_{k=i}^{m} (g_k)^2$$

$$\omega_{i,j} = \frac{2(j-1)\|P_j\|^2 \ell_j^2}{\lambda_j} \Pi_{k=j}^{m-1} (g_k)^2 (r_j-r_i+1)$$

$$\omega_{i,0} = \Pi_{k=i+1}^{m} (g_k)^2$$

In this way, the function $\Psi_i$ in (49) is expressed as a polynomial of $\kappa$. Moreover, it is noted that $r_i \leq r_{i+1} \leq \cdots \leq r_m$, and the inequality

$$\Pi_{k=i+1}^{m} (r_{i+k} - r_{i+k-1} + 1)$$

$$\geq (r_j - r_i + 1) \Pi_{k=j}^{m-1} (r_{j+k} - r_{j+k-1} + 1) \geq 1$$

holds for all $j = i+1, \ldots, m$. This further implies that the highest degree of $\kappa$ in (49) is $\Pi_{k=i+1}^{m} (r_{i+k} - r_{i+k-1} + 1)$, and its coefficient can be described by

$$\omega_i - \omega_{i+1} - \cdots - \omega_{i+i^*}$$

where $i^*$ denotes the largest number such that $r_i = r_{i+1} = \cdots = r_{i+i^*}$. If this coefficient (51) is strictly larger than zero, then there always exists large enough $\kappa$ such that $\Psi_i \geq 0$.

With this in mind, by recalling (50), it can be easily verified that the coefficient (51) in question is in fact a polynomial function of $g_i$ with degree 2, and the coefficient of $(g_i)^2 \geq \Delta - \Pi_{k=i+1}^{m} (g_k)^2 > 0$. Thus, there always exists a $g_i^* > 0$ such that for all $g_i \geq g_i^*$, the coefficient (51) is strictly positive. It is noted that the value of $g_i^*$ is independent of $\kappa$.

With the above choice of $g_i$, being the case, it then can be easily shown that there exists a $\theta_i > 0$ such that for all $\kappa \geq \theta_i$, the polynomial function $\Psi_i$ is positive.

Therefore, choosing $\theta^* = \max\{\theta_1, \ldots, \theta_m\}$, we can conclude that for all $\kappa \geq \theta^*$ and all $i = 1, \ldots, m$, the inequalities $\Psi_i \geq 0$, which completes the proof.

References

[1] R. M. Hirschorn, “Invertibility of multivariable nonlinear control systems,” IEEE Trans. Automat. Contr., vol.24, pp. 855-865, 1979.

[2] S. N. Singh, “A modified algorithm for invertibility in nonlinear systems,” IEEE Trans. Automat. Contr., vol.26, no.2, pp. 595-598, 1981.

[3] D. Liberzon, “Output-input stability implies feedback stabilization,” Syst. Control Lett., vol. 53, no. 3, pp. 237-248, 2004.

[4] L. Praly, Z.P. Jiang, “Further results on robust semiglobal stabilization with dynamic input uncertainties,” Proceedings of the 37th IEEE Conference on Decision and Control, pp.891 - 896, 1998.

[5] A. Teel, L. Praly, “Tools for semiglobal stabilization by partial state and output feedback,” SIAM Journal on Control and Optimization, vol.33, no.5, pp. 1443-1488, 1994.

[6] L. Wang, A. Isidori and H. Su, “Global Stabilization of a Class of Invertible MIMO Nonlinear Systems,” IEEE Trans. on Automatic Control, vol.60 no.3, pp. 616-631, 2015.

[7] L. Wang, A. Isidori and H. Su, “Output feedback stabilization of nonlinear MIMO systems having uncertain high-frequency gain matrix,” Systems & Control Letters, vol.83, pp.1-8, 2015.

[8] L. Fridovich, and H.K. Khalil, “Performance recovery of feedback-linearization-based designs,” IEEE Trans. on Automatic Control, vol.53 no.10, pp. 2324-2334, 2008.

[9] D. Astolfi, and L. Marconi, “A High-Gain Nonlinear Observer with Limited Gain Power,” IEEE Trans. on Automatic Control, vol.53 no.10, pp. 2324-2334, 2016.

[10] L. Wang, D. Astolfi, L. Marconi and H. Su, “High-gain observers with limited gain power for systems with observability canonical form,” Automatica, vol.75, pp. 16-23, 2017.

[11] L. Wang, A. Isidori, Z. Liu and H. Su, “Robust output regulation for invertible nonlinear MIMO systems,” Automatica, vol.82, pp. 278-286, 2017.

[12] L. Wang, A. Isidori, L. Marconi and H. Su, “Stabilization by output feedback of multivariable invertible nonlinear systems,” IEEE Trans. on Automatic Control, vol.62, pp. 2419-2433, 2017.

[13] L. Praly and Z.P. Jiang, “Semiglobal stabilization in the presence of minimum-phase dynamic input uncertainties,” NOLCOS98. vol. 2 pp. 325- 330, July 1998.

[14] A. Isidori, Nonlinear control systems II, Springer, 1999.

[15] A. Isidori, Lectures in Feedback Design for Multivariable Systems, Springer, 2017.

[16] H. Khalil, Nonlinear Systems, 3rd ed. Upper Saddle River, NJ: Prentice-Hall, 2002.

[17] Y. Wu, A. Isidori, R. Lu and H. Khalil, “Performance recovery of dynamic feedback-linearization methods for multivariable nonlinear systems,” IEEE Trans. on Automatic Control, DOI: 10.1109/TAC.2019.2924176.

[18] B. Guo, Z. Zhao, “On convergence of the nonlinear active disturbance rejection control for MIMO systems,” SIAM Journal on Control and Optimization, vol. 51, no.2, pp.1727-1757, 2013.

[19] H.K. Khalil, High-Gain Observers in Nonlinear Feedback Control, Society for Industrial and Applied Mathematics, 2017.