Some Structural Properties of the Standard Quantized Matrix Algebra $M_q(n)$

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Abstract. Let $M_q(n)$ be the standard quantized matrix algebra $M_q(n)$ introduced by Faddeev, Reshetikhin, and Takhtajan. It is shown explicitly that the defining relations of $M_q(n)$ form a Gröbner-Shirshov basis. Consequently, several structural properties of $M_q(n)$ are derived.

Key words: quantized matrix algebra; Gröbner-Shirshov basis; PBW basis

1. Introduction

Let $K$ be a field of characteristic 0. The standard quantized matrix algebra $M_q(n)$, introduced in [2], has been widely studied and generalized in different contexts, for instance, [3], [4], [5], and [6]. In this note, we show explicitly that the defining relations of $M_q(n)$ form a Gröbner-Shirshov basis. Consequently, this result enables us to derive several structural properties of $M_q(n)$, such as having a PBW $K$-basis, being of Hilbert series $\frac{1}{(1-t)^n}$, of Gelfand-Kirillov dimension $n^2$, of global homological dimension $n^2$, being a classical Koszul algebra, and having elimination property for (one-sided) ideals in the sense of [8] (see also [9, A3]).

For classical Gröbner-Shirshov basis theory of noncommutative associative algebras, one is referred to, for instance [1].

Throughout this note, $K$ denotes a field of characteristic 0, $K^* = K - \{0\}$, and all $K$-algebras considered are associative with multiplicative identity 1. If $S$ is a nonempty subset of an algebra $A$, then we write $\langle S \rangle$ for the two-sided ideal of $A$ generated by $S$.

2. The defining relations of $M_q(n)$ form a Gröbner-Shirshov basis

In this section, all terminologies concerning Gröbner-Shirshov bases, such as composition, ambiguity, and normal word, etc., are referred to [1].
Let $K$ be a field of characteristic 0, $I(n) = \{(i, j) | i, j = 1, 2, \ldots, n\}$ with $n \geq 2$, and let $M_q(n)$ be the standard quantized matrix algebra with the set of $n^2$ generators $z = \{z_{ij} | (i, j) \in I(n)\}$, in the sense of [2], namely, $M_q(n)$ is the associative $K$-algebra generated by the $n^2$ given generators subject to the relations:

\[
\begin{align*}
    &z_{ij}z_{ik} = qz_{ik}z_{ij}, & \text{if } j < k, \\
    &z_{ij}z_{kj} = qz_{kj}z_{ij}, & \text{if } i < k, \\
    &z_{ij}z_{st} = z_{st}z_{ij}, & \text{if } i < s, \ t < j, \\
    &z_{ij}z_{st} = z_{st}z_{ij} + (q - q^{-1})z_{ts}z_{sj}, & \text{if } i < s, \ j < t,
\end{align*}
\]

where $i, j, k, s, t = 1, 2, \ldots, n$ and $q \in K^*$ is the quantum parameter.

Now, let $Z = \{Z_{ij} | (i, j) \in I(n)\}$, $K\langle Z \rangle$ the free associative $K$-algebra generated by $Z$, and let $S$ denote the set of defining relations of $M_q(n)$ in $K\langle Z \rangle$, that is, $S$ consists of elements

\[
\begin{align*}
    (a) \ f_{ij} &= Z_{ij}Z_{ik} - qZ_{ik}Z_{ij}, & \text{if } j < k, \\
    (b) \ g_{ij} &= Z_{ij}Z_{kj} - qZ_{kj}Z_{ij}, & \text{if } i < k, \\
    (c) \ h_{ij} &= Z_{ij}Z_{st} - Z_{st}Z_{ij}, & \text{if } i < s, \ t < j, \\
    (d) \ h'_{ij} &= Z_{ij}Z_{st} - Z_{st}Z_{ij} - (q - q^{-1})Z_{ts}Z_{sj}, & \text{if } i < s, \ j < t.
\end{align*}
\]

Then, $M_q(n) \cong K\langle Z \rangle/\langle S \rangle$ as $K$-algebras, where $\langle S \rangle$ denotes the (two-sided) ideal of $K\langle Z \rangle$ generated by $S$, i.e., $M_q(n)$ is presented as a quotient of $K\langle Z \rangle$. Our aim below is to show that $S$ forms a Gröbner-Shirshov basis with respect to a certain monomial ordering on $K\langle Z \rangle$. To this end, let us take the deg-lex ordering $\prec_{d,\text{lex}}$ (i.e., the degree-preserving right lexicographic ordering) on the set $Z^*$ of all mono words in $Z$, i.e., $Z^*$ consists of all words of finite length like $u = Z_{ij}Z_{kl} \cdots Z_{st}$. More precisely, we first take the right lexicographic ordering $\prec_{\text{lex}}$ on $Z^*$ which is the natural extension of the ordering on the set $Z$ of generators of $K\langle Z \rangle$: for $Z_{ij}$, $Z_{kl} \in Z$,

\[
Z_{kl} < Z_{ij} \iff \begin{cases} k < i, \\
 \text{or } k = i \text{ and } l < j, \end{cases}
\]

and for two words $u = Z_{k_1l_1} \cdots Z_{k_ml_m}Z_{k_1l_1}, \ v = Z_{l_1j_1} \cdots Z_{l_2j_2}Z_{i_1j_1} \in Z^*$,

\[u \prec_{\text{lex}} v \iff \text{there exists an } m \geq 1, \text{ such that}
\]

\[
Z_{k_1l_1} = Z_{l_1j_1}, \ Z_{k_2l_2} = Z_{l_2j_2}, \ldots, Z_{k_{m-lm-lm-1}} = Z_{l_{m-lm-1}}.
\]

(but $Z_{km.lm} < Z_{lm.jm}$)

(note that conventionally the empty word $1 < Z_{ij}$ for all $Z_{ij} \in Z$). For instance

\[
Z_{43}Z_{21}Z_{31} \prec_{\text{lex}} Z_{41}Z_{23}Z_{11} \prec_{\text{lex}} Z_{42}Z_{13}Z_{34}Z_{41}.
\]

And then, by assigning each $Z_{ij}$ the degree 1, $1 \leq i, j \leq n$, and writing $|u|$ for the degree of a word $u \in Z^*$, we take the degree-preserving right lexicographic ordering $\prec_{d,\text{lex}}$ on $Z^*$: for $u, v \in Z^*$,

\[u \prec_{d,\text{lex}} v \iff \begin{cases} |u| < |v|, \\
 \text{or } |u| = |v| \text{ and } u \prec_{\text{lex}} v. \end{cases}\]
It is straightforward to check that \(<_{d, rlex}\) is a monomial ordering on \(Z^*\), namely, \(<_{d, rlex}\) is a well-ordering and

\[ u <_{d, rlex} v \text{ implies } wuv <_{d, rlex} wvr \text{ for all } u, v, w, r \in Z^*. \]

With this monomial ordering \(<_{d, rlex}\) in hand, we are ready to prove the following result.

**Theorem 2.1** With notation as fixed above, let \(J = \langle S \rangle\) be the ideal of \(M_q(n)\) generated by \(S\). Then, with respect to the monomial ordering \(<_{d, rlex}\) on \(K\langle Z \rangle\), the set \(S\) is a Gröbner-Shirshov basis of the ideal \(J\), i.e., the defining relations of \(M_q(n)\) form a Gröbner-Shirshov basis.

**Proof** By [1], it is sufficient to check that all compositions determined by elements in \(S\) are trivial modulo \(S\). In doing so, let us first fix two more notations. For an element \(f \in K\langle Z \rangle\), we write \(\overline{f}\) for the leading mono word of \(f\) with respect to \(<_{d, rlex}\); i.e., if \(f = \sum_{i=1}^n \lambda_i u_i\) with \(\lambda_i \in K, u_i \in Z^*\), such that \(u_1 <_{d, rlex} u_2 <_{d, rlex} \cdots <_{d, rlex} u_n\), then \(\overline{f} = u_n\). Thus, the set \(S\) of defining relations of \(M_q(n)\) has the set of leading mono words

\[
\overline{S} = \left\{ \frac{T_{ijk}}{k_{ijst}} = Z_{ij}Z_{ik}Z_{is}, \quad \frac{g_{ijk}}{h_{ijst}} = Z_{ij}Z_{kj}, \quad i < k, \quad j < k, \quad i < j, \quad i < s, \quad j < k, \quad j < s \right\}
\]

Also let us write \((a \land b)\) for the composition determined by defining relations \((a)\) and \((b)\) in \(S\). Similar notations are made for compositions of other pairs of defining relations in \(S\).

By means of \(\overline{S}\) above, we start by listing all possible ambiguities \(w\) of compositions of intersections determined by elements in \(S\), as follows:

\[
\begin{align*}
(a \land a) & \quad w = Z_{ij}Z_{ik}Z_{is}, \quad \text{if } j < k < s, \\
(a \land b) & \quad w_1 = Z_{ij}Z_{ik}Z_{sk}, \quad \text{if } j < k, \quad i < s, \\
(a \land b) & \quad w_2 = Z_{ij}Z_{kj}Z_{ks}, \quad \text{if } i < k, \quad j < s, \\
(b \land b) & \quad w = Z_{ij}Z_{kj}Z_{sj}, \quad \text{if } i < k < s, \\
(a \land c) & \quad w_1 = Z_{ij}Z_{st}Z_{sk}, \quad \text{if } i < s, \quad t < j, \quad t < k, \\
(a \land c) & \quad w_2 = Z_{ij}Z_{ik}Z_{st}, \quad \text{if } j < k, \quad i < s, \quad t < k, \\
(c \land c) & \quad w = Z_{ij}Z_{st}Z_{kl}, \quad \text{if } i < s < k, \quad l < t < j, \\
(b \land c) & \quad w_1 = Z_{ij}Z_{kj}Z_{st}, \quad \text{if } i < k < s, \quad t < j, \\
(b \land c) & \quad w_2 = Z_{ij}Z_{st}Z_{kt}, \quad \text{if } i < s < k, \quad t < j, \\
(a \land d) & \quad w_1 = Z_{ij}Z_{st}Z_{sk}, \quad \text{if } i < s, \quad j < t < k, \\
(a \land d) & \quad w_2 = Z_{ij}Z_{sk}Z_{st}, \quad \text{if } i < s, \quad j < k < t, \\
(b \land d) & \quad w_1 = Z_{ij}Z_{kj}Z_{st}, \quad \text{if } i < k < s, \quad j < t, \\
(b \land d) & \quad w_2 = Z_{st}Z_{ij}Z_{kj}, \quad \text{if } s < i < k, \quad t < j, \\
(c \land d) & \quad w_1 = Z_{ij}Z_{st}Z_{kl}, \quad \text{if } i < s < k, \quad t < j, \quad t < l, \\
(c \land d) & \quad w_2 = Z_{kl}Z_{ij}Z_{st}, \quad \text{if } k < i < s, \quad t < j, \quad l < j, \\
(d \land d) & \quad w = Z_{ij}Z_{st}Z_{kl}, \quad \text{if } i < s < k, \quad j < t < l.
\end{align*}
\]
Instead of writing down all tedious verification processes, below we shall record only the verification processes of five typical cases:

(a ∧ b) with \( w_1 = Z_{ij}Z_{ik}Z_{sk} \),
(a ∧ c) with \( w_1 = Z_{ij}Z_{st}Z_{sk} \),
(a ∧ d) with \( w_1 = Z_{ij}Z_{st}Z_{kl} \),
(c ∧ d) with \( w = Z_{ij}Z_{st}Z_{kl} \),

because other cases can be checked in a similar way (the interested reader may contact the author directly in order to see other verification processes).

- The case \((a ∧ b)\) with \( w_1 = Z_{ij}Z_{ik}Z_{sk} \), where \( j < k, i < s \).
  Since \( w_1 = \overline{f}_{ijik}Z_{sk} = Z_{ij}g_{ijks} \), we have
  \[
  (f_{ijik}, g_{ijsk})_{w_1} = f_{ijik}Z_{sk} - Z_{ij}g_{ijsk}
  = -qZ_{ik}Z_{ij}Z_{sk} + qZ_{ij}Z_{sk}Z_{ik}
  = \equiv -qZ_{ik}[Z_{sk}Z_{ij} + (q - q^{-1})Z_{ik}Z_{sj}] + q[Z_{sk}Z_{ij} + (q - q^{-1})Z_{ik}Z_{sj}]Z_{ik}
  = -q^2Z_{sk}Z_{ik}Z_{ij} - q^2Z_{sj}Z_{ik}^2 + Z_{sj}Z_{ik}^2 + q^2Z_{sk}Z_{ik}Z_{ij} + q^2Z_{sk}Z_{ik}^2 - Z_{sj}Z_{ik}^2
  \equiv 0 \mod(S, w_1).
  
- The case \((a ∧ c)\) with \( w_1 = Z_{ij}Z_{st}Z_{sk} \), where \( i < s, t < j, t < k \).
  Since \( w_1 = Z_{ij}f_{stsk} = \overline{h}_{ijst}Z_{sk} \), there are three cases to deal with.
  Case 1. If \( j = k \), then
  \[
  (h_{ijst}, f_{st})_{w_1} = h_{ijst}Z_{sk} - Z_{ij}f_{st}
  = -Z_{st}Z_{ij}Z_{sk} + qZ_{ij}Z_{sk}Z_{st}
  = -q^2Z_{sk}Z_{st}Z_{ik} + q^2Z_{sk}Z_{st}Z_{ik}
  = 0 \mod(S, w_1).
  
  Case 2. If \( j < k \), then
  \[
  (h_{ijst}, f_{st})_{w_1} = -Z_{st}Z_{ij}Z_{sk} + qZ_{ij}Z_{sk}Z_{st}
  = -Z_{st}[Z_{sk}Z_{ij} + (q - q^{-1})Z_{ik}Z_{sj}] + q[Z_{sk}Z_{ij} + (q - q^{-1})Z_{ik}Z_{sj}]Z_{st}
  = -qZ_{sk}Z_{st}Z_{ij} - q^2Z_{st}Z_{sj}Z_{ik} + q^{-1}Z_{st}Z_{sj}Z_{ik} + qZ_{sk}Z_{st}Z_{ij}
  + q^2Z_{sj}Z_{st}Z_{ik} - Z_{sj}Z_{st}Z_{ik}
  = 0 \mod(S, w_1).
  
  Case 3. If \( j > k \), then
  \[
  (h_{ijst}, f_{st})_{w_1} = -Z_{st}Z_{ij}Z_{sk} + qZ_{ij}Z_{sk}Z_{st}
  = -qZ_{sk}Z_{st}Z_{ij} + qZ_{sk}Z_{st}Z_{ij}
  = 0 \mod(S, w_1).
  
- The case \((a ∧ d)\) with \( w_1 = Z_{ij}Z_{st}Z_{sk} \), where \( i < s, j < t < k \).
Since $w_1 = \overline{h'}_{ijst}Z_{sk} = Z_{ij}\overline{f}_{stkl}$, we have

$$(h'_{ijst}, f_{stkl})_{w_1} = h'_{ijst}Z_{sk} - Z_{ij}f_{stkl}$$

$$= -Z_{st}Z_{ij}Z_{sk} - (q - q^{-1})Z_{st}Z_{sk}Z_{st} + qZ_{ij}Z_{sk}Z_{st}$$

$$= -Z_{st}Z_{sk}Z_{ij} - (q - q^{-1})Z_{st}Z_{sk}Z_{sk} - (q - q^{-1})Z_{st}Z_{st}Z_{st} + qZ_{sk}Z_{ij}Z_{st} + q(q - q^{-1})Z_{sk}Z_{ij}Z_{st}$$

$$\equiv -qZ_{sk}Z_{sk}Z_{ij} - (q - q^{-1})Z_{st}Z_{sk}Z_{ik} - (q - q^{-1})Z_{sk}Z_{st}Z_{sk}$$

$$= -(q - q^{-1})^2Z_{sk}Z_{sk}Z_{st}Z_{sk} + qZ_{sk}Z_{st}Z_{st} + q(q - q^{-1})Z_{sk}Z_{st}Z_{sk}$$

$$\equiv -(q - q^{-1})^2Z_{sk}Z_{sk}Z_{st}Z_{sk} - q(q - q^{-1})Z_{sk}Z_{skZ_{st}Z_{st}} + q(q - q^{-1})Z_{sk}Z_{st}Z_{sk}$$

$$\equiv 0 \mod (S, w_1).$$

- The case $(c \land d)$ with $w_1 = Z_{ij}Z_{st}Z_{kl}$, where $i < s < k$, $t < j$, $t < l$.

Since $w_1 = \overline{h'}_{ijst}Z_{kl} = Z_{ij}\overline{h'}_{stkl}$, we have three cases to consider.

Case 1. If $l = j$, then

$$(h_{ijst}, h'_{stkl})_{w_1} = h_{ijst}Z_{kl} - Z_{ij}h'_{stkl}$$

$$= -Z_{st}Z_{ij}Z_{kl} + Z_{ij}Z_{kl}Z_{st} + (q - q^{-1})Z_{ij}Z_{kl}Z_{st}$$

$$= -Z_{st}Z_{ij}Z_{kl} + Z_{ij}Z_{kl}Z_{st} + (q - q^{-1})Z_{st}Z_{kl}Z_{st}$$

$$= -qZ_{ij}Z_{kl}Z_{st}Z_{kl} - (q - q^{-1})Z_{st}Z_{kl}Z_{kl} + Z_{kl}Z_{kl}Z_{st}Z_{st} + (q - q^{-1})Z_{kl}Z_{kl}Z_{st}$$

$$\equiv -qZ_{kl}Z_{kl}Z_{kl}Z_{kl} - (q - q^{-1})Z_{st}Z_{kl}Z_{kl} + Z_{kl}Z_{kl}Z_{st}Z_{st} + (q - q^{-1})Z_{kl}Z_{kl}Z_{st}$$

$$\equiv -(q - q^{-1})^2Z_{kl}Z_{kl}Z_{kl}Z_{kl} - (q - q^{-1})Z_{st}Z_{kl}Z_{kl} + Z_{kl}Z_{kl}Z_{st}Z_{st} + (q - q^{-1})Z_{kl}Z_{kl}Z_{st}$$

$$\equiv 0 \mod (S, w_1).$$

Case 2. If $l > j$, then $i < s < k$, $t < j < l$, and

$$(h_{ijst}, h'_{stkl})_{w_1} = h_{ijst}Z_{kl} - Z_{ij}h'_{stkl}$$

$$= -Z_{st}Z_{ij}Z_{kl} + Z_{ij}Z_{kl}Z_{st} + (q - q^{-1})Z_{ij}Z_{kl}Z_{st}$$

$$= -Z_{st}Z_{kl}Z_{ij} - (q - q^{-1})Z_{st}Z_{kl}Z_{kl} + Z_{kl}Z_{ij}Z_{st} + (q - q^{-1})Z_{kl}Z_{kl}Z_{st}$$

$$\equiv -Z_{kl}Z_{kl}Z_{kl}Z_{kl} - (q - q^{-1})Z_{st}Z_{kl}Z_{kl} + Z_{kl}Z_{kl}Z_{st}Z_{st} + (q - q^{-1})Z_{kl}Z_{kl}Z_{st}$$

$$\equiv -(q - q^{-1})^2Z_{kl}Z_{kl}Z_{kl}Z_{kl} - (q - q^{-1})Z_{st}Z_{kl}Z_{kl} + Z_{kl}Z_{kl}Z_{st}Z_{st} + (q - q^{-1})Z_{kl}Z_{kl}Z_{st}$$

$$\equiv 0 \mod (S, w_1).$$

Case 3. If $l < j$, then $i < s < l$, $t < l < j$, and

$$(h_{ijst}, h'_{stkl})_{w_1} = h_{ijst}Z_{kl} - Z_{ij}h'_{stkl}$$

$$= -Z_{st}Z_{ij}Z_{kl} + Z_{ij}Z_{kl}Z_{st} + (q - q^{-1})Z_{ij}Z_{kl}Z_{st}$$

$$= -Z_{st}Z_{kl}Z_{ij} - (q - q^{-1})Z_{st}Z_{kl}Z_{kl} + Z_{kl}Z_{ij}Z_{st} + (q - q^{-1})Z_{kl}Z_{kl}Z_{st}$$

$$\equiv -Z_{kl}Z_{kl}Z_{kl}Z_{kl} - (q - q^{-1})Z_{st}Z_{kl}Z_{kl} + Z_{kl}Z_{kl}Z_{st}Z_{st} + (q - q^{-1})Z_{kl}Z_{kl}Z_{st}$$

$$\equiv -(q - q^{-1})^2Z_{kl}Z_{kl}Z_{kl}Z_{kl} - (q - q^{-1})Z_{st}Z_{kl}Z_{kl} + Z_{kl}Z_{kl}Z_{st}Z_{st} + (q - q^{-1})Z_{kl}Z_{kl}Z_{st}$$

$$\equiv 0 \mod (S, w_1).$$
By means of Theorem 2.1, we derive several structural properties of $3$. Some applications of Theorem 2.1

This finishes the proof of the theorem.

3. Some applications of Theorem 2.1

By means of Theorem 2.1, we derive several structural properties of $M_q(n)$ in this section. All notations used in Section 2 are maintained.

**Corollary 3.1** The standard quantized matrix algebra $M_q(n) \cong K\langle Z \rangle / J$ has the linear basis, or more precisely, the PBW basis

$$B = \left\{ z_{nn}^{k_{nn}} z_{nn-1}^{k_{nn-1}} \cdots z_{n_1}^{k_{n_1}} z_{n_1-1}^{k_{n_1-1}} \cdots z_{1n}^{k_{1n}} \cdots z_{11}^{k_{11}} \mid k_{ij} \in \mathbb{N}, (i, j) \in I(n) \right\}.$$ 

**Proof** With respect to the monomial ordering $\prec_{d, \text{rllex}}$ on the set $Z^*$ of mono words of $K\langle Z \rangle$, we note that

$$Z_{11} \prec_{d, \text{rllex}} Z_{12} \prec_{d, \text{rllex}} \cdots \prec_{d, \text{rllex}} Z_{1n} \prec_{d, \text{rllex}} Z_{21} \prec_{d, \text{rllex}} Z_{22} \prec_{d, \text{rllex}} \cdots \prec_{d, \text{rllex}} Z_{2n} \prec_{d, \text{rllex}} \cdots \prec_{d, \text{rllex}} Z_{n1} \prec_{d, \text{rllex}} Z_{n2} \prec_{d, \text{rllex}} \cdots \prec_{d, \text{rllex}} Z_{nn},$$

and the Gröbner-Shirshov basis $S$ of the ideal $J = \langle S \rangle$ has the set of leading mono words consisting of

- $Z_{ij}Z_{ik}$ with $Z_{ij} \prec_{d, \text{rllex}} Z_{ik}$ where $j < k$,
- $Z_{ij}Z_{kj}$ with $Z_{ij} \prec_{d, \text{rllex}} Z_{kj}$ where $i < k$,
- $Z_{ij}Z_{st}$ with $Z_{ij} \prec_{d, \text{rllex}} Z_{st}$ where $i < s$, $t < j$,
- $Z_{ij}Z_{st}$ with $Z_{ij} \prec_{d, \text{rllex}} Z_{st}$ where $i < s$, $j < t$.

It follows from classical Gröbner-Shirshov basis theory that the set of normal forms of $Z^*$ (mod $S$) is given as follows:

$$\left\{ z_{nn}^{k_{nn}} z_{nn-1}^{k_{nn-1}} \cdots z_{n_1}^{k_{n_1}} z_{n_1-1}^{k_{n_1-1}} \cdots z_{1n}^{k_{1n}} \cdots z_{11}^{k_{11}} \mid k_{ij} \in \mathbb{N}, (i, j) \in I(n) \right\}.$$
Therefore, $M_q(n)$ has the desired PBW basis. □

Before giving next result, we recall three results of [7] in one proposition below, for the reader’s convenience.

**Proposition 3.2** Adopting notations used in [7], let $K\langle X \rangle = K\langle X_1, X_2, \ldots, X_n \rangle$ be the free $K$-algebra with the set of generators $X = \{X_1, X_2, \ldots, X_n\}$, and let $<$ be a monomial ordering on $K\langle X \rangle$. Suppose that $\mathcal{G}$ is a Gröbner-Shirshov basis of the ideal $I = \langle \mathcal{G} \rangle$ with respect to $<$, such that the set of leading monomials

$$\text{LM}(\mathcal{G}) = \{X_j X_i \mid 1 \leq i < j \leq n\},$$

or

$$\text{LM}(\mathcal{G}) = \{X_i X_j \mid 1 \leq i < j \leq n\}.$$

Considering the algebra $A = K\langle X \rangle / I$, the following statements hold.

(i) [7, P.167, Example 3] The Gelfand-Kirillov dimension $\text{GK.dim}A = n$.

(ii) [7, P.185, Corollary 7.6] The global homological dimension $\text{gl.dim}A = n$, provided $\mathcal{G}$ consists of homogeneous elements with respect to a certain $\mathbb{N}$-gradation of $K\langle X \rangle$. (Note that in this case $G^{\mathbb{N}}(A) = A$, with the notation used in loc. cit.)

(iii) [7, P.201, Corollary 3.2] $A$ is a classical Koszul algebra, provided $\mathcal{G}$ consists of quadratic homogeneous elements with respect to the $\mathbb{N}$-gradation of $K\langle X \rangle$ such that each $X_i$ is assigned the degree 1, $1 \leq i \leq n$. (Note that in this case $G^{\mathbb{N}}(A) = A$, with the notation used in loc. cit.) □

**Remark** Let $j_1 j_2 \cdots j_n$ be a permutation of $1, 2, \ldots, n$. One may notice from the respectively quoted references in Proposition 3.2 that if, in the case of Proposition 3.2, the monomial ordering $<$ employed there is such that

$$X_{j_1} < X_{j_2} < \cdots < X_{j_n},$$

and

$$\text{LM}(\mathcal{G}) = \{X_{j_k} X_{j_l} \mid X_{j_l} < X_{j_k}, \ 1 \leq j_k, j_l \leq n\},$$

or

$$\text{LM}(\mathcal{G}) = \{X_{j_k} X_{j_l} \mid X_{j_k} < X_{j_l}, \ 1 \leq j_k, j_l \leq n\},$$

then all results still hold true.

Applying Proposition 3.2 and the above remark to $M_q(n) \cong K\langle Z \rangle / J$, we are able to derive the result below.

**Theorem 3.3** The standard quantized matrix algebra $M_q(n)$ has the following structural properties.

(i) The Hilbert series of $M_q(n)$ is \[ \frac{1}{(1-t)^{n^2}}. \]

(ii) The Gelfand-Kirillov dimension $\text{GK.dim}M_q(n) = n^2$. 

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(iii) The global homological dimension $\text{gl.dim} M_q(n) = n^2$.

(iv) $M_q(n)$ is a classical quadratic Koszul algebra.

**Proof** Recalling from Section 2 that with respect to the monomial ordering $\prec_{d\text{-}\text{rlex}}$ on the set $Z^*$ of mono words of $K\langle Z \rangle$, we have

$$Z_{ij} \prec_{d\text{-}\text{rlex}} Z_{sk} \iff \begin{cases} i = s, & \text{if } j < k, \\ i < s, & \text{if } j = k, \\ i < s, & \text{if } k < j, \\ i < s, & \text{if } j < k. \end{cases}$$

$Z_{11} \prec_{d\text{-}\text{rlex}} Z_{12} \prec_{d\text{-}\text{rlex}} \cdots \prec_{d\text{-}\text{rlex}} Z_{1n} \prec_{d\text{-}\text{rlex}} Z_{21} \prec_{d\text{-}\text{rlex}} Z_{22} \prec_{d\text{-}\text{rlex}} \cdots \prec_{d\text{-}\text{rlex}} Z_{2n} \prec_{d\text{-}\text{rlex}} \cdots \prec_{d\text{-}\text{rlex}} Z_{n1} \prec_{d\text{-}\text{rlex}} Z_{n2} \prec_{d\text{-}\text{rlex}} \cdots \prec_{d\text{-}\text{rlex}} Z_{nn},$

and thus, all leading mono words of the Gröbner-Shirshov basis $S$ of the ideal $J = \langle S \rangle$ are established as follows:

$Z_{ij}Z_{ik}$ with $Z_{ij} \prec_{d\text{-}\text{rlex}} Z_{ik}$ where $j < k$, 
$Z_{ij}Z_{kj}$ with $Z_{ij} \prec_{d\text{-}\text{rlex}} Z_{kj}$ where $i < k$,
$Z_{ij}Z_{st}$ with $Z_{ij} \prec_{d\text{-}\text{rlex}} Z_{st}$ where $i < s$, $t < j$,  
$Z_{ij}Z_{st}$ with $Z_{ij} \prec_{d\text{-}\text{rlex}} Z_{st}$ where $i < s$, $j < t$.

This means that $M_q(n)$ satisfies the conditions of Proposition 3.2. Therefore, the assertions (i) – (iv) are established as follows.

(i) Since $M_q(n)$ has the PBW $K$-basis as described in Corollary 3.1, it follows that the Hilbert series of $M_q(n)$ is $\frac{1}{(1-t)^n^2}$.

(ii) This follows from Theorem 2.1 and Proposition 3.2(i).

Note that $M_q(n)$ is an $\mathbb{N}$-graded algebra defined by a quadratic homogeneous Gröbner basis (Theorem 2.1), where each generator $Z_{ij}$ is assigned the degree $1$, $(i,j) \in I(n)$. The assertions (iii) and (iv) follow from Proposition 3.2(ii) and Proposition 3.2(iii), respectively. 

We end this section by concluding that the algebra $M_q(n)$ also has the elimination property for (one-sided) ideals in the sense of [8] (see also [9, A3]). To see this, let us first recall the Elimination Lemma given in [8]. Let $A = K[a_1, \ldots, a_n]$ be a finitely generated $K$-algebra with the PBW basis $B = \{a^\alpha = a_1^{\alpha_1} \cdots a_n^{\alpha_n} \mid \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n\}$ and, for a subset $U = \{a_{i_1}, \ldots, a_{i_r}\} \subseteq \{a_1, \ldots, a_n\}$ with $i_1 < i_2 < \cdots < i_r$, let

$$T = \left\{ a_{i_1}^{\alpha_{i_1}} \cdots a_{i_r}^{\alpha_{i_r}} \mid (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r \right\}, \quad V(T) = K\text{-span}T.$$

**Lemma 3.4** [8, Lemma 3.1] Let the algebra $A$ and the notations be as fixed above, and let $L$ be a nonzero left ideal of $A$ and $A/L$ the left $A$-module defined by $L$. If there is a subset $U = \{a_{i_1}, \ldots, a_{i_r}\} \subseteq \{a_1, \ldots, a_n\}$ with $i_1 < i_2 < \cdots < i_r$, such that $V(T) \cap L = \{0\}$, then

$$\text{GK.dim}(A/L) \geq r.$$

Consequently, if $A/L$ has finite GK dimension $\text{GK.dim}(A/L) = d < n$ (= the number of generators of $A$), then

$$V(T) \cap L \neq \{0\}.$$
Also by Theorem 3.3(ii), $\text{GK.dim } M$ for every $U$ holds true for every subset $U$ mentioned above.

By Corollary 3.1, the proof is as follows. With notation as fixed above, let $T$ be a left ideal of $M_q(n)$. Then $\text{GK.dim } M_q(n)/L \leq n^2$. If furthermore $\text{GK.dim } M_q(n)/L = m < n^2$, then

$$V(T) \cap L \neq \{0\}$$

holds true for every subset $U = \{z_1, z_2, \ldots, z_m\} \subset Z$ with $i_1 < i_2 < \cdots < i_m + 1$, in particular, for every $U = \{z_1, z_2, \ldots, z_m\}$ with $m + 1 \leq s \leq n - 1$, we have $V(T) \cap L \neq \{0\}$.

**Proof** By Corollary 3.1, $M_q(n)$ has the PBW basis

$$\mathcal{B} = \{z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_{n^2}^{\alpha_{n^2}} | \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n^2}) \in \mathbb{N}^{n^2}\}.$$

Also by Theorem 3.3(ii), $\text{GK.dim } M_q(n) = n^2$, thereby $\text{GK.dim } M_q(n)/L \leq n^2$. If furthermore $\text{GK.dim } M_q(n)/L = d < n^2$, then the desired elimination property follows from Lemma 3.4 mentioned above.

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