On Fermat’s principle for causal curves in time oriented Finsler spacetimes

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Abstract

Fermat’s principle for causal curves with the same energy in time orientable Finsler spacetimes is obtained. We calculate the second variation of the time arrival time functional along a geodesic in terms of the index form associated with the Lagrangian $L$. Then, we study of the character of the geodesics and a Morse index theorem is presented.

1 Introduction

Although currently general relativity explains most of the present known gravitational phenomena, it is generally accepted its incompatibility with quantum mechanics. Attempts to solve such incompatibility are usually called quantum gravity theories. Examples of these theories are string theory, canonical quantum gravity or loop quantum gravity, among others. Some of the quantum gravity theories, such as string field theory, non-commutative geometry particle models, spin foam models indicate a universal feature: modified dispersion relations at the Planck scale and violations of Lorentz invariance symmetry. The phenomenological investigation of such relations has shown that such dispersion relations can be associated with a Finsler-Lorentz geometry type model \textsuperscript{13}.

Finsler geometry is now well established, and indeed it admits a complete set of tools to be investigated in analogy with Riemannian geometry \textsuperscript{5}. In a similar situation, although at a less developed stage, appears to be Finsler-Lorentz geometry, where the above examples can be included.

Some interesting applications of Lorentzian geometry in gravitational physics come via Fermat’s principle in general relativity (see \textsuperscript{21} and references therein). Indeed, there is a version of the principle for Finsler spacetimes in the sense of J. Beem’s in \textsuperscript{22} for lightlike curves. Therefore, Fermat’s principle becomes a powerful tool in the search for observations in quantum phenomenology, since

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different geometric models could imply significant different observational predictions.

The aim of this work is twofold. Firstly, we aim at obtaining a general version of the Fermat principle for causal curves of fixed energy, and its first consequences for time orientable Finsler spacetimes, and then we calculate the second variation of the time arrival functional. We will also discuss the character of the critical point of the time arrival functional. Finally, the Morse index theorem for Finsler spacetimes is introduced and related with the second variation of the time arrival functional. Apart from a mathematical interest, this is justified by the discussion above on the possible applicability of the results in testing quantum gravity phenomenology models, for instance by their predictions on gravitational lensing. Secondly, we want to show that some techniques can be transplanted from the Lorentzian setting to the Finsler spacetime framework.

As we mentioned, it is difficult to find a framework for Finsler spacetimes that meets all the properties required by the current physical models, containing lightlike, timelike and spacelike trajectories that are not singular, and that are able to capture the gauge symmetry principle. In this context, we take as definition of Finsler spacetime the classical definition of J. Beem [6]. We observe that Beem’s framework does not contemplate all the models which are interesting in quantum gravity. For instance, some of the Finsler spacetime geometries that appear in physical models contain singularity sectors on the tangent bundle manifold. In order to deal with such models, one needs to restrict Beem’s definition to operate only on the region where there is enough regularity for the geometric objects. Apart from this problem, Beem’s framework seems a good model for applications.

2 Geometric framework

2.1 Finsler spacetimes

We introduce the basic notation and fundamental notions of Finsler spacetimes, following J. Beem [6]. Let M be a differentiable manifold and TM the tangent bundle of M. Local coordinates (U, x) on M induce local natural coordinates (TU, x, y) on TM. The slit tangent bundle is N = TM \ {0}.

Definition 2.1 A Finsler spacetime is a pair (M, L) such that
1. M is an n-dimensional real, second countable, Hausdorff C∞-manifold.
2. L : TM \ {0} → R is a real smooth function such that
   (a) L(x, ·) is positive homogeneous of degree two in the variable y,
   \[ L(x, ky) = k^2 L(x, y), \quad \forall k \in ]0, \infty[, \quad (2.1) \]
   (b) The vertical Hessian
   \[ g_{ij}(x, y) = \frac{\partial^2 L(x, y)}{\partial y^i \partial y^j}, \quad (2.2) \]
   is non-degenerate and with signature \((-+, +, ..., +)\) for all \((x, y) \in N\).

Direct consequences of this definition and Euler’s theorem for positive homogeneous functions are the following relations,
\[ \frac{\partial L(x, y)}{\partial y^k} y^k = 2L(x, y), \quad \frac{\partial L(x, y)}{\partial y^i} = g_{ij}(x, y) y^j, \quad L(x, y) = \frac{1}{2} g_{ij}(x, y) y^i y^j. \quad (2.3) \]
A Finsler space-time \((M, L)\) is reversible if \(L(x, y) = L(x, -y)\) for any \((x, y) \in N\). Otherwise \(L\) is called non-reversible. Note that because homogeneity on \(y\) it is equivalent give the function \(L\) or the corresponding fundamental tensor \(g\).

**Remark 2.2** There are several alternative definitions of Finsler spacetime in the literature. One of them goes back to Asanov [1]. His notion of Finsler spacetime is convenient when dealing with timelike trajectories, and it can be used also with Randers type spacetimes [24], but it does not allow a covariant notion of lightlike curve. Another definition of Finsler spacetime was introduced recently by Pfeifer and Wohlfarth [23]. These authors restricted to reversible metrics, based on physical grounds, although most of their methods apply to the general case of non-reversible metrics. However, as those authors note, their definition cannot deal with Randers type spacetimes (indeed, it is not known if Randers spacetimes admit a description in Beem’s framework).

### 2.2 Basic causality notions in Finsler spacetimes

The causal framework that we use is based on the Finsler spacetime structure \((M, L)\) and is a natural generalization of the Lorentzian causal framework. A vector field \(X \in \Gamma TM\) is timelike if \(L(x, X(x)) < 0\) at all point \(x \in M\) and a curve \(\lambda : I \to M\) is timelike if the tangent vector field is timelike \(L(\lambda, \dot{\lambda}) < 0\). A vector field \(X \in \Gamma\) is lightlike if \(L(x, X(x)) = 0\), \(\forall x \in M\) and a curve is lightlike if its tangent vector field is lightlike. Similar notions are for spacelike. A curve is causal if either is timelike and has constant speed \(g^{ij}(\dot{\lambda}, \dot{\lambda})\) or if it is lightlike.

A time like orientation will be a smooth timelike vector field \(T \in \Gamma TM\). A timelike curve \(\lambda : I \to M\) is future pointed (resp. past-pointed) if its tangent vector field \(g^{ij}(\lambda(t), \dot{\lambda}(t)) T^i \dot{\lambda}^j < 0\) (resp. \(g^{ij}(\lambda(t), \dot{\lambda}(t)) T^i \dot{\lambda}^j > 0\)). Similar notions hold for lightlike vectors and curves. An observer is described by a timelike curve \(\gamma : [a, b] \to M\) which is future pointed.

Note the following facts

1. The function \(L(x, y)\) defines a positive definite, homogeneous of degree one in \(y\) function \(F(x, y)\) in the sub-bundle of time-like vectors \(T^+M := \{(x, y) \in TM, y \in T_xM \text{ s.t. } L(x, y) < 0\}\).

2. Each connected component of \(T^+M\) is an open convex cone [22].

The first fact suggests to define a Finsler function \(F(x, y)\) on \(T^+M\) as \(F(x, y) = \sqrt{L(x, y)}\), meanwhile for spacelike vector it is defined by \(F(x, y) = \sqrt{L(x, y)}\) and it is zero zero on the null cone (this is a generalization of one definition of Lorentzian Randers space given in [9]).

\(NM := \{(x, y) \in TM, y \in T_xM \text{ s.t. } L(x, y) = 0\}\). 

### 2.3 Examples of Finsler spacetimes

We collect several examples of Finsler spacetimes investigated in the literature. The examples below do not exhaust the intense use of Finsler geometries in physical applications. On the other hand, such a bunch of examples partially motivates the mathematical investigation of Finsler spacetimes.

**Example 2.3** The first example to consider are Lorentzian spacetimes \((M, h)\), where \(h\) is a Lorentzian metric. In this case, the Lagrangian is given by \(L(x, y) = h_x(y, y), \ y \in T_xM\).
Example 2.4 Let $M$ be an $n$-dimensional manifold and let us consider the following Lagrangian function,

$$L(x, y) = \frac{1}{2} \left( \ell(x, y)^2 - U_i(x)U_j(x)y^i y^j \right)$$

(2.6)

where $U(s)$ defines a 1-form on $M$ and

1. $\ell((x, ky)) = k \ell(x, y)$ for positive $k$,

2. $\frac{\partial^2 \ell^2(x, y)}{\partial y^i \partial y^j} w^i w^j > 0$ if $U(x)(w) > 0$ and

3. There is a unique vector field $V(x)$ defined by $U(x)(V(x)) = -1$ and $\frac{\partial^2 \ell^2(x, y)}{\partial y^i \partial y^j} V^i(x) = 0$. These conditions guarantee that the matrix of fundamental tensor components

$$g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 \ell^2(x, y)}{\partial y^i \partial y^j} - U^i U^j$$

(2.7)

is non-degenerate and with signature $(−1, 1, ..., 1)$.

The relevance of this example to Physics resides in that it describes light propagation in a linear, dielectric and permeable medium [20].

Example 2.5 A family of Finsler spacetimes that have been recently considered in the physics literature are based on Berwald-Moore Finsler metrics [16]. Let $(M, η)$ be the Minkowski spacetime and $W$ a timelike vector field on $M$. An Euclidean metric induced by $W$ is

$$\hat{η}_z(x, y) = η_z(x, y) - 2 \frac{η_z^2(y, W)}{η_z(W, W)}$$

(2.8)

Let $\hat{y}$ be the orthogonal component of $y$ to $W$ using $\hat{η}$ and $φ$ a 2p-tensor. Then the fundamental tensor $g$ is of the form

$$g_z(x, y) = η_z(x, y) + \hat{η}_z(x, y) \left( 1 + \frac{1}{p} \frac{φ(\hat{y}, ..., \hat{y})}{η_z(y, y)^p} \right).$$

(2.9)

This tensor determines a Finsler spacetime iff $φ$ is small enough compared with $\hat{η}$. Experimentally, Finsler spacetimes of Berwald-Moore type are constrain to be Lorentzian with a very high accuracy [16].

Example 2.6 Not directly related with Physics is the following example [6]. The spacetime manifold is $M = R^3$ and the Lagrangian is the highly non-reversible function $L$

$$L(x) = \frac{(x^1)^3 - x^1(x^2)^2}{((x^1)^2 + (x^2)^2)^{\frac{3}{2}}}.$$ 

(2.10)

Then $L(−x) = −L(x)$ and the indicatrix has six connected components.

The following examples share the common fact that they do not define Finslerian in the whole tangent space of a spacetime manifold. In order to consider such examples one needs to relax the conditions of the Finsler spacetime (some notions of weak Finsler structures can be found in [24, 23, 4, 24]).

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4Compare with the formulae in [16].
Example 2.7 S. F. Rutz found a non-Riemannian solutions of a Einstein-Finsler theory in vacuum [25]. It is a spherical symmetric, Finsler space-time, with spacetime manifold \( M = \mathbb{R} \times \mathbb{R}^+ \times S^2 \) and fundamental tensor

\[
ds_R^2 = -\left(1 - \frac{2m}{r}\right)\left(1 - \delta \frac{d\Omega}{dt}\right) dt^2 + \frac{1}{\left(1 - \frac{2m}{r}\right)}dr^2 + r^2 d\Omega^2,
\]

(2.11)

where \((t, r, \theta, \varphi)\) are the local spherical coordinate system, used to describe the Schwarzschild’s solution [7],

\[
ds_S^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{2m}{r}\right)} dr^2 + r^2 d\Omega^2,
\]

(2.12)

If in spherical coordinates a tangent vector \( y \in TM \) is expressed as

\[
y = y_t \frac{\partial}{\partial t} + y_r \frac{\partial}{\partial r} + y_\theta \frac{\partial}{\partial \theta} + y_\varphi \frac{\partial}{\partial \varphi},
\]

then one has that

\[
\frac{d\Omega}{dt} := \sqrt{y_\theta^2 + \sin^2 \theta y_\varphi^2}.
\]

The parameter \( \delta \) is small compared with 1. It is clear from the last term that \((M, g_R)\) is a reversible Finsler spacetime. It is smooth on \( M \setminus A \), where \( A \) contains the set where \( \frac{d\Omega}{dt} = 0 \) and at the Schwarzschild radius \( r_S = 2m \). Therefore, the domain of regularity of (2.11) is the same than the domain of regularity of (2.12).

Example 2.8 The rainbow metric is a phenomenological description of the modification of the dispersion relations produced by possible quantum gravity corrections [13]. Let \((M, \eta)\) be a stationary Lorentzian spacetime such that \( \mathcal{L}_W \eta = 0 \). There is a foliation on \( M \) given by integral curves of \( W \). The orthogonal spacelike hypersurfaces \( \Sigma_t \) furnish an induced Riemannian metric \( \bar{\eta} \) by isometric embedding. The rainbow metric is determined by the following lagrangian function \( L \) (compare with [13]),

\[
L(x, y) = \left(\eta(x, y) - C_1(m) \frac{\bar{\eta}(\bar{y}, \bar{y})^{\frac{2}{3}}}{\eta(x, y)}\right)^2.
\]

(2.13)

This metric is not regular in the light cone \( \eta(x, y) = 0 \). This singularity is related with the mass of the particle \( m \). Therefore, each specie of elementary particle has its particular metric (this is why the name rainbow metric). The rainbow metric is non-reversible.

2.4 Variational setting

Definition 2.9 An affine parameterized geodesic of a Lagrangian \( L \) is a solution of the Euler-Lagrange equation

\[
\frac{d}{ds} \frac{\partial L(\lambda(s), \dot{\lambda}(s))}{\partial \dot{\lambda}^i} - \frac{\partial L(\lambda(s), \dot{\lambda}(s))}{\partial \lambda^i} = 0, \quad i = 1, ..., n
\]

(2.14)

with \( \dot{\lambda}^i(s) = \frac{d\lambda^i(s)}{ds} \). In this case \( s \) is an affine parameter.
An arbitrarily parameterized geodesic is a solution of the differential equation

\[
\frac{d}{ds} \left( \frac{\partial L(\lambda(s), \dot{\lambda}(s))}{\partial \lambda^i} \dot{\lambda}^i + \frac{\partial L(\lambda(s), \dot{\lambda}(s))}{\partial \dot{\lambda}^i} \ddot{\lambda}^i \right) = f(s) \dot{\lambda}^i, \quad i = 1, \ldots, n \tag{2.15}
\]

for a given function \( f : I \rightarrow M \) and with \( \dot{\lambda}(s) = \frac{d\lambda^i(s)}{ds} \).

Given an arbitrarily parameterized geodesic of an affine connection on \( M \), it is possible to find a positive re-parameterization such that with the new parameter the curve is an affine geodesic.

Using the equation (2.14) and the homogeneity condition (2.1), one can show that \( L(\lambda, \dot{\lambda}) \) is preserved along affinely parameterized geodesics,

\[
\frac{d}{ds} L(\lambda(s), \dot{\lambda}(s)) = \frac{\partial L(\lambda(s), \dot{\lambda}(s))}{\partial \lambda^i} \dot{\lambda}^i + \frac{\partial L(\lambda(s), \dot{\lambda}(s))}{\partial \dot{\lambda}^i} \ddot{\lambda}^i = \frac{d}{ds} \left( \frac{\partial L(\lambda(s), \dot{\lambda}(s))}{\partial \dot{\lambda}^i} \dot{\lambda}^i \right) = 2 \frac{d}{ds} \left( L(\lambda(s), \dot{\lambda}(s)) \right).
\]

from which follows that \( L(\lambda, \dot{\lambda}) \) is constant along \( \lambda \) (and therefore, also along any equivalent arbitrarily reparameterized geodesic). Therefore, a causal geodesic is a geodesic with \( g_{\lambda(s)}(\dot{\lambda}, \dot{\lambda}) \leq 0 \); for a timelike geodesic \( g_{\lambda(s)}(\dot{\lambda}, \dot{\lambda}) \leq 0 \) and for a lightlike geodesic \( g_{\lambda(s)}(\dot{\lambda}, \dot{\lambda}) = 0 \). Note that the causal character of a geodesic is preserved by reparameterization and that time orientation is preserved by monotone increasing reparameterizations.

Let us consider a point \( q \in M \), a constant \( c \leq 0 \) and a timelike curve \( \gamma : I \rightarrow M \) with tangent vector \( \gamma(t)' = \frac{d\gamma(t)}{dt} \). Then the space of admissible curves is

\[
\mathcal{C}_{q, \gamma, c} := \left\{ \lambda : [0, 1] \rightarrow M, \text{ smooth such that} \right. \\
1. \lambda(0) = q, \\
2. \exists \tau(\lambda) \in [0, 1] \text{ s.t. } \lambda(1) = \gamma(\tau(\lambda)), \\
3. L(\lambda(s), \dot{\lambda}(s)) = -c^2, \forall s \in [0, 1], \\
4. g_{\lambda(s)}(\gamma'(\tau(s))) \dot{\lambda}'(s) (\gamma'(\tau(s))) < 0 \left. \right\}.
\]

Note that if \( \lambda \in \mathcal{C}_{q, \gamma, c}, \lambda \) will not be parameterized necessarily by the proper time,

\[
t_\lambda(\tilde{\lambda}(r)) = \int_{d_1}^{d_2} \sqrt{-g_{\lambda(s)}(\dot{\lambda}(s), \dot{\lambda}(s)) \dot{\lambda}'(s) \dot{\lambda}'(s)} dr. \tag{2.16}
\]

**Definition 2.10** An allowed variation of \( \lambda \in \mathcal{C}_{q, \gamma, c} \) is a smooth map \( \Lambda : (-\epsilon_0, \epsilon_0) \times [0, 1] \rightarrow M \) such that
1. Each of the curves $\Lambda(\epsilon, \cdot)$ is allowed,

$$(\epsilon, s) \mapsto \Lambda(\epsilon, \cdot) \in C_{q, \gamma, c} \quad \forall \epsilon \in [-\epsilon_0, \epsilon_0], \epsilon_0 > 0.$$ 

2. The central curve is $\lambda(s), \Lambda(0, s) = \lambda(s)$.

We introduce two functionals relevant to ours purposes,

**Definition 2.11** Let $C^\infty([0, 1], M)$ be the space of smooth parameterized curves of $M$ parameterized in the interval $[0, 1]$. The energy functional is

$$E : C^\infty([0, 1], M) \longrightarrow R, \quad \lambda \mapsto E(\lambda) := \int_0^1 L(\lambda(s), \dot{\lambda}(s)) \, ds,$$

(2.17)

Note that for any allowed variation, the energy $E$ of each curve is $-c^2$. Therefore, we are considering causal curves with prescribed energy. As a consequence of the prescription of the energy one has that

$$\frac{d}{d\epsilon} \bigg|_{\epsilon=0} \left( \int_0^1 L(\Lambda(\epsilon, s), \dot{\Lambda}(\epsilon, s)) \, ds \right) = 0.$$ 

(2.18)

All the curves in the class $C_{q, \gamma, c}$ have constant energy and equal to $E = -c^2$.

**Definition 2.12** Let $C_{q, \gamma, c}$ be the space of admissible curves. The time arrival functional is

$$\tau : C_{q, \gamma, c} \longrightarrow R, \quad \lambda \mapsto \tau(\lambda).$$

(2.19)

3 Fermat’s principle for causal curves in time oriented Finsler spacetimes

3.1 Regularity of the time arrival functional

In some standard treatments of Fermat’s principle for lightlike geodesics it is assumed that the time arrival functional acting on any allowed variation $\Lambda(\epsilon, s)$ is of class $C^1$ in the variable $\epsilon$ [18, 22]. Such regularity holds when $L$ is a Lorenztian metric and one consider $\lambda$ timelike curves [12]. Indeed one has the following result,

**Proposition 3.1** Let $(M, L)$ be a Finsler spacetime and $\Lambda : (-\epsilon, \epsilon) \times [0, 1] \longrightarrow M$ be a variation of a causal geodesic $\lambda$ and $\gamma : I \longrightarrow M$ a timelike, positive temporary oriented curve. Then the function $\tau(\Lambda(\epsilon, \cdot))$ is smooth on $\epsilon$.

**Proof.** Let us consider the time arrival functional acting on the variation $\Lambda(\epsilon, s)$, i.e., the function

$$t : (-\epsilon_0, \epsilon_0) \longrightarrow R$$

$$\epsilon \mapsto \gamma^{-1}(\Lambda(\epsilon, 1)) = \gamma^{-1} \circ \Lambda(\epsilon, 1),$$

The function

$$\Lambda(\cdot, 1) : (-\epsilon_0, \epsilon_0) \longrightarrow M$$

$$\epsilon \mapsto \Lambda(\epsilon, 1)$$

is smooth. Since $\gamma(\sigma)$ is smooth, $\gamma' \neq 0$ for any $\sigma \in I$ and $\gamma(\sigma)$ does not have self-intersections, $\gamma^{-1} : \gamma(I) \longrightarrow M$ is smooth. Therefore, since $t(\epsilon) = \tau(\Lambda(\epsilon, \cdot))$ is smooth on $\epsilon$ the result follows. \[\square\]
\subsection{3.2 Fermat’s principle in Finsler spacetimes for causal curves}

Let us fix the time positive oriented timelike curve $\gamma: I \rightarrow M$. Fermat’s principle for causal curves can be stated as follows

\begin{proposition}
Let $(M, L)$ be a time orientable Finsler spacetime. Then the causal curve $\lambda: [0, 1] \rightarrow M$ is a geodesic (pre-geodesic in the lightlike case) of $L$ iff it is a critical point of the time arrival functional (2.19),
\begin{equation}
\left. \frac{d}{de} \right|_{e=0} \tau(\Lambda(e, s)) = 0,
\end{equation}
for any allowed variation $\Lambda(e, s)$ of $\lambda(s)$.
\end{proposition}

\begin{remark}
This is the Finslerian version of the Fermat principle for lightlike curves obtained in [18] and for timelike causal curves in [12].
\end{remark}

\begin{remark}
In the case of a timelike geodesic, the parameter of $\lambda(s)$ is an affine parameter. This is not the case if $\lambda$ is a lightlike curve.
\end{remark}

Before we go to the proof of proposition 3.2, let us write some intermediate formulas. First note that a smooth curve $\lambda$ is a critical point of the functional energy $E$ iff
\begin{equation}
0 = \int_0^1 \left. \frac{d}{de} \right|_{e=0} L(\Lambda(e, s), \dot{\Lambda}(e, s)) \, ds = \int_0^1 \left. \frac{d}{de} \right|_{e=0} (L(\Lambda(e, s), \dot{\Lambda}(e, s))) \, ds
\end{equation}
for any allowed variation $\Lambda(e, s)$. Also note that since $\Lambda(e, 1) = \gamma(\tau(\Lambda(e, 1)))$ one has that in local coordinates
\begin{equation}
\left. \frac{d}{de} \right|_{e=0} \Lambda^i(e, 1) = (\gamma^i)'(\tau(\lambda)) \left. \frac{d}{de} \right|_{e=0} (\tau(\Lambda(e, s))), \quad i = 1, \ldots, n,
\end{equation}
for any allowed variation $\Lambda(e, s)$.

\textbf{Proof of proposition 3.2} The “only if” is proven following a similar argument as in [22]. The condition that all the curves in the allowed variation $\Lambda$ are of fixed energy and that the allowed variation $\Lambda$ is indeed a smooth function on $e$ and $s$ implies the following relation,
\begin{align*}
0 &= \int_0^1 \left. \frac{d}{de} \right|_{e=0} L(\Lambda(e, s), \dot{\Lambda}(e, s)) \, ds \\
&= \int_0^1 \left. \frac{\partial L}{\partial \lambda^i} \right|_{e=0} (\Lambda^i(e, s)) \, ds + \left. \frac{\partial L}{\partial \dot{\lambda}^i} \right|_{e=0} (\dot{\Lambda}(e, s)) \, ds \\
&= \int_0^1 \left( \left. \frac{\partial L}{\partial \lambda^i} \right|_{e=0} \left( \frac{d}{ds} \right) \left. \frac{\partial L}{\partial \dot{\lambda}^i} \right|_{e=0} (\dot{\Lambda}(e, s)) \right) \, ds + \left( \frac{\partial L}{\partial \lambda^i} \right|_{e=0} \left. \frac{d}{ds} \right|_{e=0} \Lambda^i(e, s)) \, ds.
\end{align*}
Then using the relation (3.3), one obtains
\begin{align*}
\int_0^1 &\left( \frac{\partial L}{\partial \lambda^i} \right|_{e=0} \left( \frac{d}{ds} \right) \left. \frac{\partial L}{\partial \dot{\lambda}^i} \right|_{e=0} (\dot{\Lambda}(e, s)) \right) \, ds + \left( \frac{\partial L}{\partial \lambda^i} \right|_{e=0} \left. \frac{d}{ds} \right|_{e=0} \Lambda^i(e, s)) \right. \right|_{e=0} \right|_{e=0} (\tau(\epsilon, s)) = 0
\end{align*}
The fact that
\[ \sum_0^1 \left( \frac{\partial L}{\partial \lambda^i} - \frac{d}{ds} \frac{\partial L}{\partial \dot{\lambda}^i} \right) \frac{d}{de} \bigg|_{e=0} (\Lambda^i) \, ds + \left( g_{ij}(\lambda(1), \lambda'(1)) \dot{\lambda}^i(\gamma^j)'(\tau(\lambda)) \right) \frac{d}{de} \bigg|_{e=0} \left( \tau(\epsilon, s) \right) = 0. \] (3.4)

Given a curve \( \lambda \in C_{\gamma, \tau, \epsilon} \), one has the condition
\[ g_{\lambda(\epsilon)}(\dot{\lambda}(1), \gamma(\tau(\lambda))) = g_{\beta}(\lambda(1), \dot{\lambda}(1))(\gamma^j)'(\tau(\lambda)) \neq 0 \] (3.5)

Let us parameterize the geodesic by an affine parameter \( s \in [0, 1] \), which means that equation (2.11) holds. Then it is clear from (3.1) that for curves of fixed energy \( E = -c^2 \leq 0 \), the solutions of the Euler-Lagrange equations are critical points of the time arrival functional,
\[ \frac{\partial L(\lambda, \dot{\lambda})}{\partial \dot{\lambda}^i} - \frac{d}{ds} \frac{\partial L(\lambda, \dot{\lambda})}{\partial \dot{\lambda}^i} = 0, \quad E = -c^2 \Rightarrow \frac{d}{de} \bigg|_{e=0} \tau(\Lambda(\epsilon, s)) = 0. \]

This implication is independent of the signature of the metric. It strongly depends on the requirement that the energy \( E \) has a fixed valued for all the allowed curves.

The “if” condition in proposition (3.2) can be proved as follows. Let us consider a variation of \( C_{\gamma, \tau, \epsilon} \ni \lambda : [0, 1] \mapsto M \) defined by
\[ \Lambda^\alpha(\epsilon, s) = \lambda^\alpha(s) + \epsilon A^\alpha(s), \quad \alpha = 1, ..., n - 1, \] (3.6)

with \( A^\alpha \) arbitrary smooth functions. In order to be an allowed variation along the curve \( \lambda(s) \in C_{\gamma, \tau, \epsilon}, \Lambda(\epsilon, s) \) must satisfy the condition,
\[ L(\Lambda, \dot{\Lambda}) = -c^2 \] (3.7)

with initial condition \( \Lambda(\epsilon, 0) = -c^2 \). This equation can be expressed explicitly in an convenient local coordinate system such that the last coordinate \( n \) corresponds to the integral curves of the time orientation vector field \( T(x) \). In such coordinate system \( \sum_{i=1}^n g_{ni}(\Lambda, \dot{\Lambda}) \dot{\Lambda}^i \neq 0 \) holds and one has the constrain
\[ c^2 + \sum_{\alpha, \beta=1}^{n-1} g_{\alpha \beta}(\Lambda, \dot{\Lambda}) \dot{\Lambda}^\alpha \dot{\Lambda}^\beta + \sum_{i=1}^n g_{ni}(\Lambda, \dot{\Lambda}) \dot{\Lambda}^i \dot{\Lambda}^n = 0. \] (3.8)

The fact that \( \sum_{i=1}^n g_{ni}(\Lambda, \dot{\Lambda}) \dot{\Lambda}^i \neq 0 \), \( \dot{\Lambda}^n \) can be solved. Let us fix the value of \( \epsilon \). Since \( c^2 \) does not contain any derivative \( \frac{d}{ds} \), the variable \( \epsilon \) can be considered as a continuous parameter of the differential equation (3.8) that \( s \) considered as an ODE. Then one uses standard ODE theory to establish local smoothness of \( \Lambda(\epsilon, s) \) on the parameter \( \epsilon \) for \( s \in [0, s_0] \) for some \( s_0 \) (see for instance [8, Chapter 1]). Such solution can be extended further. Indeed, the solution can be extended to an interval \( [s_0 - \delta, s_0 + \delta] \), with the convenient initial data at the point \( s_0 - \delta \in [0, s_0] \) and with a smooth dependence on \( 0 < \epsilon < \epsilon_1 \leq \epsilon_0 \). Repeating this procedure one can extend the solution to a finite collection of open sets of \( R \) which is maximal and contained in \( \lambda([0, 1]) \). Let \( s_{\max} \leq 1 \) be the maximal value of \( s \) such that the dependence on \( \epsilon \in \Lambda(\epsilon, s) \) is smooth. There are two possibilities,
1. \( \lambda \in C_{q, \gamma, c} \). In this case, \( s_{\text{max}} = 1 \), since otherwise one can extend \( s_{\text{max}} \) leading to a contradiction with ODE theory.

2. \( \lambda \) does not intersect \( \gamma \). This is in contradiction with the hypothesis \( \lambda(1) = \gamma(\tau(\lambda)) \).

Let us consider the minimum \( \epsilon_{\text{min}} \) of the above \( \epsilon_k \), that by compactness of \( [0, 1] \) must be finite. Using again a compactness argument, one can show that there is a finite open cover of \( \lambda : [0, 1] \rightarrow M \) such that the differential equation (3.8) contains an unique solution on each local chart. Because paracompact property of \( M \), there is an adapted partition of the unity. Using bump functions [20], one can patch in a smooth way a solution in an open subset of \( [0, 1] \), obtaining a global solution \( \lambda \) for (3.8) in \([0, 1] \). This proves the existence of allowed variations of \( \lambda \) satisfying the ODE (3.8) (but does not shows uniqueness of the solutions).

Formula (3.8) can be read as a second order polynomial equation in the variables \( \epsilon A_i(s) \). Therefore, using the local existence and uniqueness of ODE’s theory locally one can write the expression

\[
\dot{\Lambda}^n(\epsilon, s) = \int_0^1 \left( \sum_{\alpha=1}^{n-1} A^n_{\alpha}(s) \tilde{h}_\alpha(s) \right) + \epsilon^2 f_2(s, A(s), \epsilon)
\]

for some unique, smooth functions \( \{\tilde{h}_\alpha(s)\} \). Integrating respect to \( s \) both sides (3.9) one obtains

\[
\Lambda^n(\epsilon, s) = \int_0^1 \left( \sum_{\alpha=1}^{n-1} A^n_{\alpha}(s) h_\alpha(s) \right) + \int_0^1 \epsilon^2 f_2(s, A(s), \epsilon)
\]

Then the variation of the relation (3.8) is equivalent to

\[
\int_0^1 \left( \sum_{\alpha=1}^{n-1} \frac{\partial L}{\partial \lambda_i} \frac{d}{ds} \frac{\partial L}{\partial \lambda_i} A^n_{\alpha}(s) + \left( \frac{\partial L}{\partial \lambda_i} - \frac{d}{ds} \frac{\partial L}{\partial \lambda_i} \right) \left( \sum_{\alpha=1}^{n-1} A^n_{\alpha}(s) h_\alpha(s) \right) \right) ds
\]

\[
= \int_0^1 \left( \epsilon \gamma'(\tau(\lambda)) \frac{\partial L}{\partial \lambda_i} \frac{d}{d\epsilon} \lambda_i(\epsilon) \right) \frac{d}{d\epsilon} \bigg|_{\epsilon=0} \left( \tau(\Lambda(\epsilon, s)) \right)
\]

For critical points of the arrival time functional it holds that

\[
\frac{d}{d\epsilon} \bigg|_{\epsilon=0} \left( \tau(\Lambda(\epsilon, s)) \right) = 0.
\]

This implies:

\[
\int_0^1 \left( \sum_{\alpha=1}^{n-1} \frac{\partial L}{\partial \lambda_i} \frac{d}{ds} \frac{\partial L}{\partial \lambda_i} A^n_{\alpha}(s) + \left( \frac{\partial L}{\partial \lambda_i} - \frac{d}{ds} \frac{\partial L}{\partial \lambda_i} \right) \left( \sum_{\alpha=1}^{n-1} A^n_{\alpha}(s) h_\alpha(s) \right) \right) ds = 0,
\]

therefore, one can write the relation

\[
0 = \int_0^1 \left( \frac{\partial L}{\partial \lambda_i} - \frac{d}{ds} \frac{\partial L}{\partial \lambda_i} \right) \sum_{\alpha=1}^{n-1} B^i(S) ds.
\]
for any arbitrary, small enough functions \(B_1(s), ..., B^{n-1}(s)\). The space of functions \(\{B_1(s), ..., B^{n-1}\}\) determine a \((n - 1)\)-dimensional vector space at each point \(\lambda(s)\). It is also true that such \((n - 1)\)-vector space is orthogonal to the vector field \(\dot{\lambda}(s)\). Let us consider a basis of such sub-space \(\{B_1(s), ..., B_{n-1}\}\).

Using the fact that \(g\) is non-degenerate, the expression (3.10) can be rewritten as

\[
0 = \int_0^1 g_\lambda(\omega(s), B_\alpha(s)) \, ds
\]

for some vector field \(\omega: [0, 1] \rightarrow M\) defined by the relation

\[
g_\lambda(\omega(s), B(s)) = \left(\frac{\partial L}{\partial \lambda^\alpha} - \frac{d}{ds} \frac{\partial L}{\partial \dot{\lambda}^i}\right) B^i(s).
\]

Since the functions \(B^i(s)\) are smooth, can extract from the integral the local condition

\[
g_\lambda(\omega(s), B_\alpha(s)) = 0.
\]

This implies that for a fixed \(s\) the vector \(\omega(s)\) is parallel to \(\dot{\lambda}(s)\). This can be seen as follows. If \(\omega(s)\) is not parallel to \(\dot{\lambda}(s)\), there is vector \(C(s)\) in \(V^\perp(s)\) such that \(g_\lambda(C(s), B(s)) = 0\) for all \(B(s)\). As we say, the dimension of the variational fields \(B(s)\) orthogonal to \(\dot{\lambda}(s)\) is \(n - 1\) for all \(s\). The dimension of \(\text{span}\{\dot{\lambda}(s), B_i(s), C(s)\}\) must be maximum \(n\). Therefore, \(C(s) = 0\).

3.3 Formula for the first variation of the time arrival functional

The above calculation provides the formula of the first variation of the time arrival functional. Writing

\[
g_\lambda(\dot{\lambda}(1, \epsilon), \dot{\lambda}(1, \epsilon)\dot{\lambda}'(1, \epsilon)(\gamma')'(\tau(\Lambda(\epsilon, s))) = g_\lambda(\dot{\lambda}', \gamma'(\tau(\Lambda)))
\]

the first variation of the functional \(\tau(\Lambda)\) is given by the expression:

\[
g_\lambda(\dot{\lambda}, \gamma'(\tau(\Lambda))) \frac{d}{d\epsilon} \left(\tau(\Lambda(\epsilon, s))\right)
\]

\[
= - \int_0^1 \left(\frac{\partial L(\Lambda(\epsilon, s), \dot{\Lambda}(\epsilon, s))}{\partial \Lambda^i} - \frac{d}{ds} \frac{\partial L(\Lambda(\epsilon, s), \dot{\Lambda}(\epsilon, s))}{\partial \dot{\Lambda}^i}\right) \frac{d}{d\epsilon} \Lambda^i(\epsilon, s) \, ds.
\]

Note that in this formula \(\epsilon\) has not been fixed to have the value \(\epsilon = 0\).

4 Second variation formula

4.1 The Chern connection on the pull-back bundle \(\pi^*TM\)

Our way to introduce Chern’s connection is motivated by a suggestion from S. Vacaru to one of the authors (R.G.T.) on the definition of the non-linear connection. In particular Cartan tensor components are defined as

\[
C_{ijk} := \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k}, \quad i, j, k = 1, ..., n,
\]
differently to the way it is introduced in [5]. Therefore, because of the homogeneity of the tensor \( g \), Euler’s theorem implies

\[
C_{(x,y)}(y, \cdot, \cdot) = \frac{1}{2} y^k \frac{\partial g_{ij}}{\partial y^k} = 0. \tag{4.2}
\]

The formal second kind Christoffel symbols \( \gamma^i_{jk}(x, y) \) are defined by the expression

\[
\gamma^i_{jk} = \frac{1}{2} g^{is}(\frac{\partial g_{sj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^s} + \frac{\partial g_{sk}}{\partial x^j}), \quad i, j, k, s = 1, \ldots, n.
\]

The non-linear connection coefficients are defined on \( N := TM \setminus \{0\} \) to be

\[
N^i_{jk} = \gamma^i_{jk} y^k - C^i_{jk} \gamma^k_{rs} \frac{y^r}{F} \frac{y^s}{F}, \quad F \neq 0, \quad i, j, k, r, s = 1, \ldots, n,
\]

where \( C^i_{jk} = g^{il} C_{ljk} \). An adapted frame is determined by the smooth tangent basis for \( T_u N, u \in N \) (see [5]):

\[
\left\{ \frac{\delta}{\delta x^i}|u, \ldots, \frac{\delta}{\delta x^j}|u, F \frac{\partial}{\partial y^i}|u, \ldots, F \frac{\partial}{\partial y^j}|u \right\},
\]

\[
\frac{\delta}{\partial x^j}|u = \frac{\partial}{\partial x^j}|u - N^i_{jk} \frac{\partial}{\partial y^i}|u, \quad i, j = 1, \ldots, n. \tag{4.3}
\]

Given a tangent vector \( X \in T_x M \) and \( u \in \pi^{-1}(x) \), there is a unique horizontal tangent vector \( h(X) \in T_u N \) with \( d\pi(h(X)) = X \) (horizontal lift of \( X \)).

The pull-back bundle \( \pi^*TM \) is the maximal subsect of the cartesian product \( N \times TM \) such that the diagram

\[
\begin{array}{ccc}
\pi^*TM & \xrightarrow{\pi_2} & TM \\
\pi_1 \downarrow & & \downarrow \pi \\
N & \xrightarrow{\pi} & M
\end{array}
\]

is commutative. The projections on the first and second factors are

\( \pi_1 : \pi^*TM \rightarrow N, (u, \xi) \mapsto u, \quad \pi_2 : \pi^*TM \rightarrow TM, (u, \xi) \mapsto \xi \).

The Chern connection is defined through the following

**Theorem 4.1** Let \((M, F)\) be a Finsler structure. The pull-back vector bundle \( \pi_1 : \pi^*TM \rightarrow N \) admits a unique linear connection determined by the connection 1-forms \( \{\omega^i_j, i, j = 1, \ldots, n\} \) such that the following structure equations hold:

1. **Torsion free condition**

\[
d(dx^i) - dx^j \wedge w^i_j = 0, \quad i, j = 1, \ldots, n. \tag{4.4}
\]

2. **Almost \( g \)-compatibility condition**

\[
dg_{ij} - g_{kj} w^k_i - g_{ik} w^k_j = 2A_{ijk} \frac{\delta y^k}{F}, \quad i, j, k = 1, \ldots, n. \tag{4.5}
\]
This result is proved along the same lines as in [5] for the existence of the Chern connection. This is because the hypothesis that one uses and the Christoffel trick is the same as in the positive case.

**Corollary 4.2** Let \( h(X) \) and \( v(X) \) be the horizontal and vertical lifts of \( X \in \Gamma TM \) to \( T N \), and \( \pi^*g \) the pull back-metric. For the Chern connection the following properties hold:

1. The almost \( g \)-compatibility metric condition is equivalent to
   \[
   \nabla_{v(X)}\pi^*g = 2C(X, \cdot, \cdot), \quad \nabla_{h(X)}\pi^*g = 0,
   \]
   (4.6)

2. The torsion-free condition of the Chern connection is equivalent to the following:
   (a) Null vertical covariant derivative of sections of \( \pi^*TM \):
   \[
   \nabla_{V(X)}\pi^*Y = 0,
   \]
   (4.7)
   for any vertical component \( V(X) \) of \( X \).
   (b) Let us consider \( X, Y \in TM \) and their horizontal lifts \( h(X) \) and \( h(Y) \). Then
   \[
   \nabla_{h(X)}\pi^*Y - \nabla_{h(Y)}\pi^*X - \pi^*([X, Y]) = 0.
   \]
   (4.8)
   The proof of this corollary is similar to the positive case that one can be find in [10].

Given a Finsler structure, the curvature endomorphism of the Chern connection have two pieces different from zero [5]. One of the pieces correspond to the \( hh \)-curvature and its components are given by

\[
R^i_{jkl} = \frac{\delta \Gamma^i_{jk}}{\delta x^l} - \frac{\delta \Gamma^i_{jl}}{\delta x^k} - \Gamma^h_{jl} \Gamma^i_{hk} - \Gamma^i_{hk} \Gamma^h_{jl}, \quad h, i, j, k, l = 1, ..., n. \quad (4.9)
\]

**Lemma 4.3** Let \( \lambda : [0, 1] \to M \) be a critical point of the energy \( -L \) with fixed energy. Then along a timelike curve \( \lambda : [0, 1] \to M \) the following relations are true for the Chern connection,

1. The connection preserves \( g \) along \( \lambda \) in the sense that
   \[
   \frac{d}{dr}(g_{\lambda}(Y, W)) = g_{\lambda}(\nabla_{h(X)}(\dot{\lambda})\pi^*Y, \pi^*W) + g_{\lambda}(\pi^*Y, \nabla_{h(X)}(\dot{\lambda})\pi^*W),
   \]
   (4.10)
   holds for all \( X, Y, W \in \Gamma TM \).

2. The connection \( \nabla \) is torsion free: if \( [X, Y] = 0 \), then
   \[
   \nabla_{h(X)}(\lambda, \dot{\lambda})\pi^*Y = \nabla_{h(Y)}(\lambda, \dot{\lambda})\pi^*X, \quad X \in \Gamma TM.
   \]
   (4.11)

**Proof.** The first property is the evaluation of the almost metric compatibility of the Chern connection along \( u(s) = (\lambda(s), \dot{\lambda}) \), and with \( \frac{d}{ds} \) the derivative along the vector field \( X \in \Gamma \pi^*TM \). When the condition (115) is evaluated at \( u \), the left hand side is

\[
(dg_{ij} - g_{ik}w_i^k - g_{ik}w_j^k)(X) = dg_{ij}|_u(X) - g_{kj}|_u w_i^k(X) - g_{ik}|_u w_j^k(X)
   = \frac{d}{dr}(g_{ij}|_u(X)) - g_{kj}|_u w_i^k(X) - g_{ik}|_u w_j^k(X).
\]
This expression is proportional to the Cartan tensor along the causal geodesic $\lambda$ evaluated in the first entry at $\dot{\lambda}(s)$. However, the Cartan tensor along a critical point of $L = -F^2$ is zero,

$$2C_{ij}k \frac{\delta y^k}{F} \bigg|_u (X) = 0.$$  

This last statement was proved for positive definite Finsler metrics in [5]. It remains valid for the tensor $g$ along a causal geodesic in the Lorentzian case. The second statement is a consequence of the torsion free condition of the Chern connection. 

**Lemma 4.4** Let $\nabla$ be the Chern connection on $\pi^*TM \to N$. The first variation of the functional $L$ is given by the expression

$$0 = g_{\lambda}(\dot{\Lambda}(1, \epsilon), \gamma'(\tau(\lambda))) \frac{d}{d\epsilon} \left( \tau(\lambda, s) \right) - \int_0^1 g_{\lambda}(\nabla_{h(\lambda)}\Lambda, \frac{d}{d\epsilon} \Lambda(\epsilon, s)) \, ds,$$

where $h(\dot{\lambda}) \in TN$ is the horizontal lift of $\dot{\lambda} \in T\lambda M$.

**Proof.** It is a standard result in the calculus of variations that if a connection is torsion-free and preserves the metric along a geodesic (as it is the case because of (4.6) and **lemma 4.3**), one obtains the relation

$$\int_0^1 ds \left( \frac{\partial L(\Lambda(s), \dot{\Lambda}(\epsilon, s))}{\partial \Lambda^i} \right) \Lambda^i(s) = \int_0^1 ds \frac{1}{2} g_{\lambda}(\nabla_{h(\lambda)}\Lambda, \frac{d}{d\epsilon} \Lambda(\epsilon, s)),$$

from which follows the result. 

**Definition 4.5** Let $\lambda \in C_{\epsilon, \gamma, \epsilon}$ and $\Lambda_1, \Lambda_2$ be two allowed variations of $\lambda$ such the corresponding vector fields along $\lambda$ are that $A(s) = \frac{d\Lambda_1(s, \epsilon)}{d\epsilon} |_{\epsilon = 0}$ and $B(s) = \frac{d\Lambda_2(s, \epsilon)}{d\epsilon} |_{\epsilon = 0}$. We define the index form acting on $A(s)$ and $B(s)$ to be

$$J_\lambda(A, B) := \int_0^1 \left( g_{\lambda}(B(s), R_{\lambda}(A(s), \dot{\lambda}(s))(\dot{\lambda}(s))) - g_{\lambda}(\nabla_{\dot{\lambda}} A(s), \nabla_{\dot{\lambda}} B(s)) \right) ds,$$

(4.13)

where $R$ is the $hh$-curvature of the covariant derivative $+\nabla$ along $\lambda$ induced from the Chern connection.

Since the Cartan tensor along a geodesic is zero, the $hh$-curvature tensor along $\lambda$ is equal to the Riemann curvature tensor along $\lambda$,

$$R^i jkl = \frac{\partial \gamma^i jk}{\partial x^l} - \frac{\partial \gamma^i jk}{\partial x^l} - \gamma^h jk \gamma^i h \gamma^j l + \gamma^h jk \gamma^i h \gamma^j l, \quad h, i, j, k, l = 1, \ldots, n.$$  

(4.14)
One can relate the space $T_\lambda \mathcal{C}_{q,\gamma,c}$ with the space of tangent vector fields along the geodesic $\lambda$,

$$T_\lambda := \left\{ A : [0, 1] \rightarrow TM, \quad C^\infty \text{ s.t.} \right\}
1. \pi(A) = \lambda,
2. A(0) = 0,
3. A(1) = 0,
4. g(\dot{\lambda}(s), A(s)) = 0 \right\}.$$

**Lemma 4.6** Given a causal geodesic $\lambda : [0, 1] \rightarrow M$, the vector spaces $T_\lambda$ and $T_\lambda \mathcal{C}_{q,\gamma,c}$ coincide.

**Proof.** We prove that the vector spaces coincide. For both spaces $A(0) = 0$.

From property 2 in the definition of $\mathcal{C}_{q,\gamma,c}$ one has that for any $A \in T_\lambda \mathcal{C}_{q,\gamma,c}$ and homogeneity of $L$

$$\frac{d}{ds}L\left(\Lambda(s, \epsilon), \dot{\Lambda}(s, \epsilon)\right)_{|_{\epsilon = 0}} = \frac{\partial L}{\partial \Lambda^i(s)} A^i(s) + \frac{\partial L}{\partial \dot{\Lambda}^i(s)} \dot{A}^i(s) + \frac{\partial L}{\partial \dot{\Lambda}^i(s)} A^i(s) + g_{ij}(\lambda, \dot{\lambda}(s)) \dot{\lambda}^j(s) \dot{A}^i(s) = 0.$$

Using the condition that $\lambda$ is a geodesic this reduces to

$$\frac{d}{ds}L\left(\Lambda(s, \epsilon), \dot{\Lambda}(s, \epsilon)\right)_{|_{\epsilon = 0}} = \frac{d}{ds}\left(\frac{\partial L}{\partial \dot{\Lambda}^i(s)} \dot{A}^i(s) + g_{ij}(\lambda, \dot{\lambda}(s)) \dot{\lambda}^j(s) \dot{A}^i(s)\right) = 0.$$

This conservation law proves that $A(s)$ satisfies point 3. To check point 3, let us note that $A(1) = a\gamma'(\tau(\lambda))$. By point 3,

$$0 = g(\dot{\lambda}(s), A(s)) = \frac{d}{ds}\left(a\gamma'(\tau(\lambda)), \dot{\lambda}(s)\right) = a g_X(\gamma'(\tau(\lambda)), \dot{\lambda}(s)).$$

In the last product the factor $g_X(\gamma'(\tau(\lambda)), \dot{\lambda}(s))$ cannot be zero, since $T$ and $\dot{\gamma}$ are both not mutually non-orthogonal timelike vectors and the hypothesis that $g_X(\dot{\lambda}(T(\lambda(1))), \dot{\lambda}(1)) < 0$. Therefore, $a = 0$.

**Lemma 4.6** is used in the proof of the second variation formula and in the index formula in the next section.

### 4.2 Second variation of the time arrival functional

Note that for each allowed variation $A(s)\epsilon$ the time arrival functional is a smooth function on the variable $\epsilon$. Therefore, the hessian is defined along a critical point.

**Proposition 4.7** Let $\lambda : [0, 1] \rightarrow M$ be in $\mathcal{C}_{q,\gamma,c}$ a causal geodesic. Let $A(s,\epsilon)$ an allowed variation with $A = \frac{d}{d\epsilon} |_{\epsilon = 0} A(s,\epsilon)$. Then

$$\frac{d^2}{d\epsilon^2} \tau(\Lambda(s,\epsilon)) |_{\epsilon = 0} = \frac{J_X(A, A)}{g(\gamma'(\tau(\lambda)), \dot{\lambda}(1))}, \quad (4.15)$$
In the last line, the first term can be computed more explicitly:

\[ 0 = \frac{d}{d\epsilon} \left( g_{\hat{\Lambda}}(\hat{\Lambda}(\epsilon, 1), \gamma'(\tau(\Lambda))) \right) \frac{d}{d\epsilon} (\tau\Lambda(\epsilon, s)) \]

\[- \int_0^1 2 g_{\hat{\Lambda}}(\nabla_{h(\hat{\Lambda})}(\hat{\Lambda}) \Lambda, \frac{d}{d\epsilon} \Lambda(s, \epsilon)) \left|_{\epsilon=0} \right. \]

\[ = \frac{d}{d\epsilon} \left( g_{\hat{\Lambda}}(\hat{\Lambda}(\epsilon, 1), \gamma'(\tau(\Lambda))) \right) \left|_{\epsilon=0} \right. + \frac{d}{d\epsilon} (\tau\Lambda(\epsilon, s)) \]

\[- \frac{d}{d\epsilon} \left( \int_0^1 g_{\hat{\Lambda}}(\nabla_{h(\hat{\Lambda})}(\hat{\Lambda}) \Lambda, \frac{d}{d\epsilon} \Lambda(s, \epsilon)) \left|_{\epsilon=0} \right. \right. \]

\[ \text{Evaluated on a geodesic, the first term is zero, since by Proposition 4.2} \]

\[ \frac{d}{d\epsilon} \left|_{\epsilon=0} \right. (\tau(\Lambda(\epsilon, s))) = 0. \]

Using the metric compatibility and the torsion free conditions along \( \lambda \), the condition \( [\hat{\Lambda}, \frac{d}{d\epsilon}\Lambda] = 0 \) and the geodesic equation in terms of the connection \( \nabla_{\hat{\lambda}}(\hat{\lambda}) \hat{\lambda} = 0 \), one obtains the following expression

\[ \frac{d}{d\epsilon} \left( \int_0^1 g_{\hat{\Lambda}}(\nabla_{h(\hat{\Lambda})}(\hat{\Lambda}) \hat{\lambda}, \frac{d}{d\epsilon} \Lambda(s, \epsilon)) \right. \left|_{\epsilon=0} \right. \]

\[ = \int_0^1 \frac{d}{d\epsilon} \left( g_{\hat{\Lambda}}(\nabla_{h(\hat{\Lambda})}(\hat{\Lambda}) \hat{\lambda}, \frac{d}{d\epsilon} \Lambda(s)) \right. \left|_{\epsilon=0} \right. \]

\[ = \int_0^1 g_{\hat{\Lambda}}(\nabla_{h(\hat{\Lambda})}(\hat{\Lambda}) \hat{\lambda}, \frac{d}{d\epsilon} \Lambda(s)) \left|_{\epsilon=0} \right. \]

\[ = \int_0^1 g_{\hat{\Lambda}}(\nabla_{h(\hat{\Lambda})}(\hat{\Lambda}) \hat{\lambda}, \frac{d}{d\epsilon} \Lambda(s)) \left|_{\epsilon=0} \right. \]

\[ = \int_0^1 g_{\hat{\Lambda}}(\nabla_{h(\hat{\Lambda})}(\hat{\Lambda}) \hat{\lambda}, \frac{d}{d\epsilon} \Lambda(s)) \left|_{\epsilon=0} \right. \]

\[ = \int_0^1 g_{\hat{\Lambda}}(\nabla_{h(\hat{\Lambda})}(\hat{\Lambda}) \hat{\lambda}, \frac{d}{d\epsilon} \Lambda(s)) \left|_{\epsilon=0} \right. \]

\[ = \int_0^1 g_{\hat{\Lambda}}(\nabla_{h(\hat{\Lambda})}(\hat{\Lambda}) \hat{\lambda}, \frac{d}{d\epsilon} \Lambda(s)) \left|_{\epsilon=0} \right. \]

\[ = \int_0^1 g_{\hat{\Lambda}}(\nabla_{h(\hat{\Lambda})}(\hat{\Lambda}) \hat{\lambda}, \frac{d}{d\epsilon} \Lambda(s)) \left|_{\epsilon=0} \right. \]

\[ = \int_0^1 g_{\hat{\Lambda}}(\nabla_{h(\hat{\Lambda})}(\hat{\Lambda}) \hat{\lambda}, \frac{d}{d\epsilon} \Lambda(s)) \left|_{\epsilon=0} \right. \]

In the last line, the first term can be expanded more explicitly:

\[ \int_0^1 g_{\hat{\Lambda}}(\nabla_{h(\hat{\Lambda})}(\hat{\Lambda}) \hat{\lambda}, \frac{d}{d\epsilon} \Lambda(s)) \left|_{\epsilon=0} \right. \]

\[ = - \int_0^1 g_{\hat{\Lambda}}(\nabla_{h(\hat{\Lambda})}(\hat{\Lambda}) \hat{\lambda}, \nabla_{h(\hat{\Lambda})}(\hat{\Lambda}) A) \left|_{\epsilon=0} \right. \]

\[ + \int_0^1 \frac{d}{ds} \left( g_{\hat{\Lambda}}(\nabla_{h(\hat{\Lambda})}(\hat{\Lambda}) A) \right) \left|_{\epsilon=0} \right. \]

\[ = - \int_0^1 g_{\hat{\Lambda}}(\nabla_{h(\hat{\Lambda})}(\hat{\Lambda}) A, \nabla_{h(\hat{\Lambda})}(\hat{\Lambda}) A) \left|_{\epsilon=0} \right. \]

where in the last equality we have used Lemma 4.7. Combining this last relation with the definition of the index \( J_\lambda \) one obtains the result. \( \square \)

The proof of \( \ref{4.7} \) suggests to compile the following properties in a lemma.

**Lemma 4.8** Let \( \lambda \in C_q \gamma, \epsilon \) be a causal geodesic. Then along any geodesic \( \lambda \) the following properties hold:
1. The connection is torsion-free along \( \lambda \).
2. The Cartan tensor is zero along a geodesic. Therefore, it is metric compatible along \( \lambda \).
3. The \( hh \)-curvature reduces to the Riemann curvature (4.14) on a geodesic.

**Proof.** For Finslerian quantities, the base point vector is fixed along a geodesic. Then 1. For the Chern’s connection, the connection is torsion-free on whole \( N \). 2. The Cartan tensor along a geodesic is zero \[5\]. 3. As consequence that the Cartan tensor is zero, the \( hh \)-curvature reduces to a formal Riemann curvature \[4.14\]. 
\[\square\]

### 5 Applications

In order to investigate the character of the critical points of the time arrival functional, we borrow some techniques from \[7, 18\]. By lemma \[4.8\] the proofs from \[7, 18\] can be adapted to Finsler spacetimes with minimal changes. Using this fact, we provide two applications. First, since the relation between the second variation of time functional and the index form along a causal geodesic is the same than in \[18\], one can study the character of the critical points of the time arrival functional as in the Lorentzian case. These results serve to illustrate that indeed one can transplant the methods of the Lorentzian case to the Finsler spacetime category. Also note that the result is valid for causal geodesics, not only timelike geodesics.

Second, one can translate the techniques from \[7\] to obtain a Morse index theorem for Finsler spacetime causal geodesics. This is related with the index of the Hessian of the time arrival functional.

#### 5.1 The character of the critical points of the time arrival functional

Let us consider the vector spaces

\[ V^\perp(\lambda) := \{ \text{all piecewise smooth vector fields } A \text{ along } \lambda \text{ s.t. } g_{\lambda}(\dot{\lambda}, A(s)) = 0 \} \]

\[ V^\perp_0(\lambda) := \{ A \in V^\perp, \text{ s.t. } A(0) = A(1) = 0 \} \]

A direct application of \[4.8\] is that for arbitrary smooth vector fields along \( \lambda \), the index form \[4.13\] is

\[
J_{\lambda}(A, B) = -g_{\lambda}(\nabla_{h(\lambda)}A, B)|^1_0 - \int_0^1 \left( g_{\lambda}(B(s), \nabla_{\lambda}A(s) + R_{\lambda}(A(s), \dot{\lambda}(s))(\dot{\lambda}(s))) \right) ds.
\]

(5.1)

If \( B \in V^\perp_0(\lambda) \) and \( A \) is smooth, the index form is

\[
J_{\lambda}(A, B) = \int_0^1 \left( g_{h(\lambda)}(B(s), \nabla h(\lambda)A(s) + R_{\lambda}(A(s), \dot{\lambda}(s))(\dot{\lambda}(s))) \right) ds.
\]

(5.2)

**Definition 5.1** Let \((M, L)\) be a Finsler spacetime and \( \mathcal{C}_{q, \gamma, c} \ni \lambda: [a, b] \rightarrow M \) a causal geodesic.
1. A Jacobi field is a vector field $Y$ along $\lambda$ such that it is solution of the Jacobi equation
\[
\nabla_{h(\dot{\lambda})} \nabla_{h(\dot{\lambda})} Y + R_{\lambda}(\dot{Y}, \dot{\lambda}) \dot{\lambda} = 0,
\]
(5.3)

2. Let $C_{q,\gamma,c} \ni \lambda : [a, b] \rightarrow M$ be a causal geodesic. Then $\lambda(t_1)$ and $\lambda(t_2)$ are conjugate points along $\lambda$ iff there is a non-zero Jacobi field such that $Y(t_1) = Y(t_2) = 0$.

**Lemma 5.2** Let $\lambda \in C_{q,\gamma,c}$ be a causal geodesic without conjugate points. Then $J_{\lambda}(A, A) < 0$ for any $A \in V_0^\perp(\lambda)$.

**Proof.** We learnt from the proof of proposition 10.9 and lemma 10.10 that Lorentzian properties along geodesics carry over from the Lorentzian case to Finsler-Lorentz case. This makes the proof of the lemma completely similar to the Lorentzian case (see [7] for the timelike case and [7, 18] for the light-like case). \(\square\)

The following proposition is a restatement of known results in Lorentzian geometry.

**Proposition 5.3** Let $(M, L)$ be a Finsler spacetime and $\lambda : [0, 1] \rightarrow M$ a causal geodesic and $Y$ a Jacobi field along $\lambda$. Then

1. The function along $\lambda$ given by $g_\lambda(Y, \dot{\lambda})$ is an affine function.

2. Let $Y$ such that $Y(t_0) = Y(t_1) = 0$ for different $t_1, t_2 \in [0, 1]$. Then
   (a) If $\lambda$ is a timelike geodesic, then $Y \in V_0^\perp(\lambda)$.
   (b) If $\lambda$ is a lightlike geodesic, then $Y$ is either orthogonal or parallel to $\dot{\lambda}$ (therefore, it is lightlike).

3. Let $Y$ such that $Y(0) = Y(1) = 0$ for different $t_1, t_2 \in [0, 1]$. Then
   (a) If $\lambda$ is a timelike geodesic, then $\nabla_{h(\dot{\lambda})} Y \in V_0^\perp(\lambda)$.
   (b) If $\lambda$ is a lightlike geodesic, then $\nabla_{h(\dot{\lambda})} Y$ is orthogonal to $\dot{\lambda}$ (therefore, it is lightlike).

**Proof.** The proof follows closely the proof for lemma 10.9, corollary 10.10 and 10.11 in [7], through the use of lemma 4.8. \(\square\)

The following theorem is proved in a similar way as in [18].

**Theorem 5.4** Let $\lambda \in C_{q,\gamma,c} \ni \lambda : [a, b] \rightarrow M$ be a causal geodesic. Then

1. If $\lambda$ does not have conjugate points, then it is a local minimum of the time arrival functional $\tau$.

2. If $\lambda$ has intermediate conjugate points, then it is a local saddle point of the time arrival functional $\tau$.

**Proof.** The proof of the first statement follows the same steps than the Lorentzian case (see [18]): from the formula for the second variation of the arrival time functional (4.15), it follows that if $J_{\lambda} < 0(A; A)$ (by lemma 5.2), then the time arrival is a local minimal.

The prove of the second statement is identical to the proof in [7] for the case of timelike geodesics and to [18] for lightlike geodesics and will not be rewrite here. Note that it is essential in the proof both lemma 4.8 and proposition 5.3 which provide exactly the same tools as in the Lorentzian case. \(\square\)
Section 5.2 Morse index theorem for the arrival time functional in Finsler-Lorentz spacetimes

The common feature of the results and techniques in the Finsler and Lorentzian case suggests that there is also a Finsler spacetime version of the Morse index theorem. Indeed, there is such result as we indicate below. The index of $\lambda$ is equal to the number on conjugate points along $c$ counted with multiplicity, that is, counting the dimension of the vector space of Jacobi fields $J_i \in V_0^\perp(\lambda)$ vanishing at each conjugate point $\lambda(s_i)$,

$$I(\lambda) = \sum_{s \in (0,1)} \dim(J_i). \quad (5.4)$$

The index of the bilinear form $\tau$ along $\lambda$ denoted by $I(\tau, \lambda)$ is the supremum of the dimensions of all subspaces of $V_0^\perp(\lambda)$ on which the Hessian $\text{Hess}(\tau)$ is negative. Then one has the following result,

**Theorem 5.5** Given a causal geodesic $\lambda: [0,1] \rightarrow M$, the number of conjugate points $I(\lambda)$ is given by

$$I(\lambda) = I(\tau, \lambda). \quad (5.5)$$

**Proof.** The methods and results described before suggest the existence of a version of the Morse index theorem for the functional energy for general Finsler-Lorentz spacetimes. That this is the case will be seen in [11]. By lemma 4.6, one relates such index with $I(\tau, \lambda)$ by formula (4.15). $\square$

**Remark 5.6** It is worth to mention that one can establish theorem 5.4 as a direct consequence of the index theorem 5.5.

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