Space-variant Shack-Hartmann wavefront sensing based on affine transformation estimation

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The space-variant wavefront reconstruction problem inherently exists in deep tissue imaging. In this paper, we propose a framework of Shack-Hartmann wavefront space-variant sensing with extended source illumination. The space-variant wavefront is modeled as a four-dimensional function where two dimensions are in the spatial domain and two in the Fourier domain with priors that both gently vary. Here, the affine transformation is used to characterize the wavefront space-variant function. Correspondingly, the zonal and modal methods are both escalated to adapt to four-dimensional representation and reconstruction. Experiments and simulations show double to quadruple improvements in space-variant wavefront reconstruction accuracy compared to the conventional space-invariant correlation method. © 2022 Optica Publishing Group

1. INTRODUCTION

The space-variant problem in deep tissue imaging has bothered biologists and microscopists for decades. Various methods have been developed to resolve the problem and are roughly categorized into two types: isotropic approximation and the conjugate method. The isotropic approximation method, which assumes an isotropic zone within a certain range to be space-invariant, has been used in laser point scanning microscopy [1], selected plane illumination microscopy [2], structured illumination microscopy (SIM) [3], Fourier ptychography microscopy (FPM) [4], etc. However, the isotropic zone is usually small; thus, this method is not efficient for large field of view (FOV) applications that take numerous snapshots for a single image. The counterpart conjugate method assumes that the wavefront fluctuations originate from one plane near the sample, instead of the pupil plane, as used in widefield microscopy [5, 6]. Alternatively, the pupil wavefronts from different FOVs are separated in space and then sensed and corrected, as used in laser point scanning microscopy [7]. However, the sample-induced wavefront is hardly simply modeled by a phase screen. Otherwise, the images may not be fully restored due to this model error. In addition, sample-induced aberration could be modeled by multiple planes, as used in astronomy and called multiconjugate adaptive optics [8], which could definitely improve the system performance but also dramatically increase the complexity and cost of the imaging system; thus, this technology has not yet been used in microscopy.

The space-variant imaging problem can be divided into two steps: wavefront sensing and image restoration. This paper focuses on how to model and reconstruct space-variant wavefront. For direct wavefront measurement, the Shack-Hartmann wavefront sensor (SHWFS) is the most popular way. The SHWFS consists of a microlens array and a camera, and the camera is usually located on the back focal plane of the microlens. The subspot
images on the camera reveal the gradient of the incident wavefront by its centroid drifts within the corresponding lenslet area under the point source illumination as a guide star. However, this conventional Shack-Hartmann (SH) scheme only represents the wavefront at one designated field of angle, that is, the angle of the guide star [9]. Besides, the averaged wavefront across the isotropic zone is represented by the local shifts of the optical spot images based on the two-photon laser scanning technique [1]. In addition, the averaged wavefront across the entire FOV can also be calculated by the correlation between the subimages with and without wavefront aberrations under extended source illumination [10, 11]. Similar to obtaining the averaged wavefront, the difference is that the descanning technique adds all intensities of the subimages to improve the signal-to-noise ratio since the fluorescence is very weak, while the correlation method uses the subimages themselves. However, all these Shack-Hartmann schemes cannot present a space-variant wavefront.

In this paper, we illustrate an SHWFS scheme to reconstruct a space-variant wavefront with extended source illumination. The affine transformation is used to evaluate subimage polynomials pupil plane, or Fourier domain,

\[ \phi(x, y, \xi, \eta) = \sum_{p=1}^{P} c_p R_p(\xi, \eta), \]  

(1)

where \((\xi, \eta)\) are the global normalized lateral coordinates of the pupil plane, or Fourier domain, \(R_p(\xi, \eta)\) are the \(p\)th orthogonal polynomials \(c_p\) and are the corresponding coefficients, e.g., the Zernike polynomials \(Z_p(\xi, \eta)\) for the circular domain or the Legendre polynomials \(L_p(\xi, \eta)\) for the rectangular domain, \(P\) is the maximum representation term for the pupil plane.

Therefore, we define the space-variant wavefront,

\[ \phi(x, y, \xi, \eta) = \sum_{p=1}^{P} c_p (x, y) R_p(\xi, \eta). \]  

(2)

Similarly, we assume that the space-variant function \(c_p(x, y)\) varies slowly; thus, the dual-orthogonal modal of the space-variant wavefront function is given as

\[ \phi(x, y, \xi, \eta) = \sum_{q=1}^{Q} c_{p,q} T_q(x, y) R_p(\xi, \eta), \]  

(3)

where

\[ c_p(x, y) = \sum_{q=1}^{Q} c_{p,q} T_q(x, y), \]  

(4)

and \(T_q(x, y)\) are the \(q\)th orthogonal polynomials, \(c_{p,q}\) are the corresponding coefficients, and \(Q\) is the maximum representation term for the pupil plane. The purpose of our work is to reconstruct the space-variant wavefront \(\phi(x, y, \xi, \eta)\) by retrieving the dual-orthogonal coefficients \(c_{p,q}\), as the modal method, or discrete \(\phi(x_m, y_n, \xi_p, \eta_q)\) on position \((x_m, y_n, \xi_p, \eta_q)\), as the zonal method.

B. Affine transformation estimation for evaluating the space-variant function in Shack-Hartmann subapertures

In SH, the light field projects onto a micro lens array and forges an image on the back focal plane with a subimage array, as shown in Fig. 2(a1-c1). For the scenario of the conventional space-invariant wavefront, only one gradient vector is obtained from subimages in one lenslet, as shown in Fig. 2(a2-b2). The centroid shift or correlation algorithm, depending on which light source is used, the point or extended, can be used for representing the local slope. On the other hand, for sensing space-variant wavefront, the subimages deform as shown in Fig. 2(c2), rather than just translation, as shown in Fig. 2(b2). We used the affine transformation to characterize the deformation of the subimages in one lenslet

\[ \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ 1 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}, \]  

(5)

where \((x_1, y_1)\) is an arbitrary point on the \((x, y)\) plane of the corresponding lenslet area since the back focal plane of the lenslet is also the image plane, \((x_2, y_2)\) is the coordinate after deformation, as shown in Fig. 2(c2) and Fig. 3(a,b), \([x_2 - x_1, y_2 - y_1]^T\) presents the centroid shift of point \((x_1, y_1)\), and \([a_1, a_2, a_3, b_1, b_2, b_3]^T\) are the coefficients of an affine transformation. The affine coefficients from all lenslets can reconstruct the dual-orthogonal coefficients \(c_{p,q}\) or discrete \(\phi(x_m, y_n, \xi_p, \eta_q)\) (details in Appendix).

Various approaches, such as intensity-based optimization, control points, and feature detection and matching, can be used to register images on many platforms. In this paper, the MATLAB® internal function ‘imregtform’ is used to estimate affine transformation coefficients, in which the image similarity metric is optimized during registration.

3. NUMERICAL SIMULATIONS

In this section, the numerical simulations in space-variant Shack-Hartmann wavefront sensing are given as the simulation configuration, the forward process of image formation, and wavefront reconstruction based on zonal and modal methods.
by matrix multiplication. This process is almost identical to that used in light-field microscopy [13]. Before the vectorization, the object matrix is a $37 \times 37$ matrix distributed by sampling the light source. The simulations are programmed in MATLAB® R2021a and run on a PC on the primary configurations of Intel i7-8700 CPU, 16G RAM with the Windows 10 professional operating system. The function ‘imregtform’, we used to obtain affine transform coefficients, is tested with many light sources, such as squares, circles, ellipses, octagons, and peaks (a function for producing continuous and smooth distribution by the MATLAB® internal function ‘peaks’) by simulations, and the results present consistent performance. This algorithm shows robustness for our applications. (details in Supplemental document).

In space-variant wavefront reconstruction, the dual Legendre polynomials are naturally adopted in the modal method for the dual-orthogonal wavefront representation and reconstruction since the wavefront area and the object field are both settled squares in the simulations. Here, the two-dimensional Legendre polynomial formula is presented as

$$L_{m,n}(x, y) = \frac{1}{2^{m+n+1}m!n!} \frac{d^m}{dx^m} \left(x^2 - 1\right)^m \frac{d^n}{dy^n} \left(y^2 - 1\right)^n,$$

where $(x, y)$ are the lateral coordinates defined in the unit square $[-1,1] ^2$ and $(m, n)$ the corresponding order. For mathematical convenience, the single index $k$ is used to replace the double order $(m, n)$ by the following rule:

$$k = \frac{(m + n)(m + n + 1)}{2} + n,$$

that is,

$$L_k(x, y) = L_{m,n}(x, y).$$

The total reconstruction term on the pupil plane is set at 45, namely, $P = 45$ in Eq. (3), to represent the wavefront function since the spatial sampling is sufficient where the lenslet numbers are up to $15 \times 15$. On the other hand, only 3 terms, also $Q = 3$ in Eq. (3), are used and reconstructed for representing the space-variant function on the image plane because the affine transformation only characterizes very slow changes.

B. Simulated results of the Shack-Hartmann sensor, space-variant function estimation, and wavefront reconstruction

By using the forward process algorithm described in Section A and Fig. 3, the simulation results of SH images are shown in Fig. 4 under the square and point array illumination sources with and without phase perturbation.

Two subimages with and without the phase screen are marked blue and green, respectively, as shown in Fig. 4(c, f). For the point array source, the centroids of all subspots are extracted conventionally to calculate centroid shifts, For the square source, the centroid shifts are calculated at the blue points in Fig. 4(c) based on the space-variant function obtained by affine transform estimation, as shown in Fig. 4(f).

For the zonal method, the corresponding wavefront can be estimated based on the Southwell approach [14] once we have the centroid shifts or the local slopes, also known as the zonal method. The wavefront reconstruction results under $3 \times 3$ point array illumination are provided in Fig. 5(b1) as well as the residual wavefronts in Fig. 5(b2). The RMS of the residual wavefronts varies from 0.108 to 0.197 rad across different views, whereas that of the input wavefront varies from 3.681 to 4.765 rad. The reconstructed wavefronts by the zonal method under

A. Configuration and parameters for simulating space-variant Shack-Hartmann sensing

In this subsection, we set up the environments for simulating space-variant Shack-Hartmann sensing and wavefront reconstruction, and this configuration and parameters settled here are mainly consistent with the experiments in Section 4.

In our simulations, the SH has a microlens array with a $15 \times 15$ lenslet, while each lenslet has a $500 \mu$m pitch and $46.7$ mm focal length. A $465 \times 465$ pixel camera is located at the focal plane of the microlens array. For simplicity, every lenslet covers $31 \times 31$ pixels, while the pixel size is set at $500/31 \mu$m. The wavelength of the incident light is $462$ nm. Additionally, a $1.2 \times 1.2$ mm$^2$ square light source lies on the front focal plane of a lens with a focal length of $200$ mm, while the microlens array is located on its back focal plane, as shown in Fig. 3(a). In addition, a $3 \times 3$ star array source with a interval of $0.5$ mm is used as a reference for comparison with our method, while the central star coincides with the center of the square source.

In order to generate the SH images of the square and the star array source, the measurement matrix of the SH system is calculated in advance. First, the space-variant wavefront is designed by using the geometrical projection method with different incident angles, where the variation of a wavefront is calculated in advance. First, the space-variant wavefront gradients represented by (a2) subspot shift, (b2) subimage shift, and (c2) subimage transformation. (a2-c2) Subimages zoom of sections in the box of (a1-c1).

![Fig. 2. Schematic of wavefront sensing in Shack-Hartmann sensor. Space-invariant wavefront sensing with (a1) point and (b1) extended source illumination. (c1) Space-variant wavefront sensing with extended source illumination. Local wavefront gradients represented by (a2) subspot shift, (b2) subimage shift, and (c2) subimage transformation. (a2-c2) Subimages zoom of sections in the box of (a1-c1).](image-url)
Fig. 3. Simulations on Shack-Hartmann space-variant wavefront sensing system and geometrical projection schematic for generating wavefronts with different views. (a) SH system with phase screen perturbation, (b) subimages formation in a single lenslet with and without phase perturbation, (c) phase screen distribution used in simulations, scaled in radians, (d) 3D rendering for geometrical projection schematic that the phase screen projects onto the microlens array. (e) Forward process of SH imaging by using the measurement matrix system representation.

Fig. 4. Shack-Hartmann simulation and space-variant function estimation. The SH images are illuminated by the 3 × 3 star array (a) without and (b) with the phase screen. (c) Subimages zoom of sections in the box of (a) and (b) labeled blue and green, centroid shift marked for each star. SH images illuminated by the square extended source (d) without and (e) with phase screen. (f) Subimages zoom of sections in the box of (d) and (e) labeled blue and green, position shift marked by the affine transform estimation.

In addition, the correlation method for conventional space-invariant Shack-Hartmann wavefront sensing is carried out with an extended light source for comparison. In this method, the correlation algorithm is performed between the subimages with and without aberration to evaluate the local shift. The wavefronts reconstructed by the correlation method are presented in Fig. 5(e1) as well as the residual wavefronts in Fig. 5(e2). The RMS of the residual wavefronts varies from 0.383 to 0.465 rad.

Table 1. Comparison of the RMS of residual wavefronts referring to different cases in numerical simulations, all scaled in radians.

|                | Star | Zonal | Modal | Corr |
|----------------|------|-------|-------|------|
| Minimal RMS    | 0.108| 0.275 | 0.383 | 0.511|
| Maximal RMS    | 0.197| 0.589 | 0.465 | 1.260|
| Mean RMS       | 0.147| 0.421 | 0.417 | 0.830|
Fig. 5. Comparisons of simulation results of wavefront reconstruction, all scaled in radians. (a) Projected wavefronts with different angles of view, the ground truth. Reconstruction wavefronts by using (b1) the zonal method with $3 \times 3$ stars illuminated, proposed (c1) zonal and (d1) modal methods, and (e1) the correlation method with square source illuminated. (b2-e2) Residual wavefronts between (a) the ground truth and (b1-e1) the corresponding reconstruction wavefronts. (f) RMS of the corresponding residual wavefronts (b2-e2).

4. EXPERIMENTS

In this section, an experimental demonstration is implemented to validate our proposal for space-variant Shack-Hartmann wavefront sensing. The configuration of the experiments and wavefront reconstruction results are presented.

A. Experimental setup

As shown in Fig. 6, we build the demonstration system for space-variant Shack-Hartmann wavefront sensing. To perform the designated light source, an LED diffuser and a mask are combined, in which the LED with a center wavelength of 462 nm provides a large area of illumination with a broad illumination field of view and the mask shapes the images on the camera. Hence, two customized masks are used, including a $1.2 \times 1.2 \text{ mm}^2$ square for performing the extended light source and a $3 \times 3$ pinhole array for reference, where each pinhole has a 50 µm diameter and intervals of 0.5 mm. The image of the two masks overlapped is also shown in Fig. 6. The mask is located on the front focal plane of lens L1 with a focal length of 200 mm, and the LED diffuser and phase screen lie above and below the mask by approximately 10 mm and 20 mm, respectively. The light beam is collimated by the lens L1 and projected onto a homemade Shack-Hartmann wavefront sensor. The SH is made by a microlens array (Edmund, #64-483, pitch 0.5 mm, focal length 46.7 mm, size $10 \times 10 \text{ mm}^2$) and an electron-multiplying CCD (Hamamatsu, C9100-23B, $512 \times 512$ pixels, cell size $16 \times 16 \text{ µm}^2$), and the camera is located on the back focal plane of the microlens array. The exposure times are set at 3.8 s and 150 ms to acquire images to the pinhole array and square mask, respectively. The SH images acquired by our experimental setup are shown in Fig. 7. The periphery pixels of raw $512 \times 512$ images were chopped to $469 \times 469$ images and then resized to $465 \times 465$ based on the bicubic interpolation method for nearly $15 \times 15$ pixels in each lenslet. This processing is implemented to conveniently extract the subimages from each lenslet. Thus, the affine transform estimation is used to provide the subtle shifts compared to the reference.

The reconstruction results of wavefronts are shown in Fig.
The reconstructed wavefronts under star array illumination exhibit obvious and continuously slow space-variant features. The reconstructed wavefronts under square hole illumination are presented in Fig. 8(b1-d1) via our zonal and modal method as well as the correlation method. By using the wavefronts of star array illumination as a reference, the corresponding residual wavefronts are obtained and shown in Fig. 8(b2-d2). The RMS of the residual wavefronts via our zonal method varies from 0.558 to 2.450 rad. The modal method provides close results that vary from 0.617 to 2.449 rad; thus, the zonal and modal methods used in our approach present similar performance in reconstructing wavefronts. The correlation method shows that the RMS varies from 1.850 to 10.228 rad. Further comparisons could be exploited with different RMSs or PVs of wavefront and space-variant functions in the future.

**Table 2.** Comparison of the RMS of residual wavefronts referring to different cases in imaging experiments, all scaled in radians.

|                | Zonal | Modal | Corr |
|----------------|-------|-------|------|
| Minimal RMS    | 0.558 | 0.617 | 1.850|
| Maximal RMS    | 2.450 | 2.449 | 10.228|
| Mean RMS       | 1.378 | 1.461 | 5.984|

**5. DISCUSSION**

In the experiments, the absolute values of RMS of residual wavefronts are still high via our methods, as shown in Tab. 2. This phenomenon is mainly because the amplitude of the input phase is significant; correspondingly, its high-frequency components are also high. In addition, point array illumination reveals the spatially local wavefront, while the square source with affine transformation only carried out the spatially averaged global wavefront. Thus, the spatially high-frequency components may not be well reconstructed within the square source, resulting in a high absolute value of residual wavefronts RMS.

Another interesting result is that the reconstructed wavefront at any FOV did not represent much of the high-frequency component in the back pupil plane, as shown in 8(a). There are three main contributions to these results. The first one is the epoxy glue-made phase screen, while the strong tension of the glue made its surface very smooth. In addition, every point of the extended light source only bites a small piece on the screen with a very narrow angle since the source has a close distance to the phase screen. Moreover, the transformation from the screen plane to the back focal plane, including two free-space propagations and a lens, further smooths this partial phase.

It is worth mentioning that the optical flow method in [15] and our affine transformation estimation method could be classified into the same category. The traditional light field camera is essentially equivalent to the SH with the difference that the microlens is located on the imaging plane or the pupil plane [16, 17]. After transferring the light-field camera data from the spatial domain to the phase-space domain by corresponding pixel rearrangement, each light-field subimage is conjugated to the object with local wavefront perturbations at every object point due to the space-variant wavefront, as in SH. In phase space, light-field images and SH images all present so-called instantaneous phase information. With point light source illumination, the instantaneous phase reveals the gradient of the wavefront phase and thus the phase itself. With the extended light source illumination, the optical flow method is implemented by searching the maximum correlation positions for 7 × 7 segmented areas and then using cubic interpolation to obtain the shifts of every pixel. The same a priori assumption is used here that aberrations change slowly across the whole FOV. In our proposal, we designed a uniform extended guide source that filled on the entire FOV, but the optical flow method directly used the distribution of the objective block by a block that could be unevenly distributed. The uniformity of subimages may affect the accuracy of wavefront reconstruction. Additionally, the optimization-based algorithm in affine transformation estimation generally outperforms the blocked correlation methods and cubic interpolation.

Furthermore, the affine transformation can only characterize the low-frequency deformation of the subimages. If the sample changes rapidly, our method may lose fidelity. In these cases, other nonrigid intensity-based techniques, such as diffeomorphic demons [18] and free form deformation [19], could provide the localized displacement field with more high-frequency components. However, with no modal characterization, the modal method and representation may not feasible to the displacement field directly.

**6. CONCLUSION**

In this paper, we propose the space-variant Shack-Hartmann wavefront sensing approach based on affine transformation estimation. Two main contributions are made. First, the affine
transformation is adopted to estimate local shifts at every FOV under extended source illumination since the conventional correlation estimation only expresses the overall displacement with no local details. Second, we provide the dual-orthogonal model for space-variant wavefront representation and build a framework of wavefront reconstruction with this model. For comparison, the space-variant wavefront was previously represented by many wavefronts distributed at different FOVs with no efficiency and no elegance.

We carried out simulations and experiments on the space-variant Shack-Hartmann wavefront sensing approach. In the simulations, the average RMS of 0.83 rad of the resident wavefront reconstructed by using the conventional space-invariant Shack-Hartmann correlation method is reduced to 0.42 rad by using our methods, showing double performance improvements.

In the experiments, the average RMS 6 rad of the correlation presented is required to reconstruct the wavefront by the modal method under the continuous condition of the global wavefront. Thus, the local slope drives the arbitrary point \((x^m_1, y^m_1)\) to \((x^m_2, y^m_2)\) with the relation

\[
\begin{bmatrix}
    x^m_2 - x^m_1 \\
    y^m_2 - y^m_1
\end{bmatrix} =
\begin{bmatrix}
    a_1^m & a_2^m & a_3^m \\
    b_1^m & b_2^m & b_3^m
\end{bmatrix}
\begin{bmatrix}
    x^m_1 \\
    y^m_1
\end{bmatrix},
\]

or

\[
\begin{bmatrix}
    x^m_2 \\
    y^m_2
\end{bmatrix} =
\begin{bmatrix}
    a_1^m + 1 & a_2^m & a_3^m \\
    b_1^m & b_2^m + 1 & b_3^m
\end{bmatrix}
\begin{bmatrix}
    x^m_1 \\
    y^m_1
\end{bmatrix},
\]

where the coefficients \(A\) and \(B\) in Eq. (10) are concealed for mathematical simplicity. As shown in Fig. 2(c), the coefficients in Eq. (14) are characterized and estimated by an affine transformation estimation between the subimages with and without the aberration.

The wavefront is about to be reconstructed from the map of the estimated local information obtained via zonal and modal representation based on the smooth prior of the global wavefront. The zonal method assumes that the phases on the edges of the neighboring lenslets are equal, and the modal method decomposes the wavefront into a sum of a series of continuous orthogonal polynomials.

For the zonal method, the local slope \(a^m_n, b^m_n\) of any point \((x_0, y_0)\) on the object plane can be obtained by the local slope function in Eq. (11) based on the result of the affine transformation estimation. Then, the space-variant wavefront \(\phi^{m,n}(x_0, y_0)\) can be reconstructed from the map of the local slope \(a^m_n, b^m_n\) by using the Southwell approach [14].

For the modal method, a proper dimensionality-reduced presentation is required to reconstruct the wavefront by the modal method effectively. Through several form transformations with Eq. (3) and Eq. (13), the relation between the dual-orthogonal coefficients \(e_{p,q}\) and the local slope function coefficients \(a_i^m, b_i^m, i = 1, 2, 3\) is given as

\[
\begin{align*}
    a_1^m &= \sum_{q=1}^{Q} \sum_{p=1}^{P} e_{p,q} \frac{\partial T_q(\xi, \eta)}{\partial x} \frac{\partial R_p(\xi, \eta)}{\partial \xi}, \\
    a_2^m &= \sum_{q=1}^{Q} \sum_{p=1}^{P} e_{p,q} \frac{\partial T_q(\xi, \eta)}{\partial y} \frac{\partial R_p(\xi, \eta)}{\partial \xi}, \\
    a_3^m &= \sum_{p=1}^{P} e_{p,1} \frac{\partial R_p(\xi, \eta)}{\partial \xi}, \\
    b_1^m &= \sum_{q=1}^{Q} \sum_{p=1}^{P} e_{p,q} \frac{\partial T_q(\xi, \eta)}{\partial x} \frac{\partial R_p(\xi, \eta)}{\partial \eta}, \\
    b_2^m &= \sum_{q=1}^{Q} \sum_{p=1}^{P} e_{p,q} \frac{\partial T_q(\xi, \eta)}{\partial y} \frac{\partial R_p(\xi, \eta)}{\partial \eta}, \\
    b_3^m &= \sum_{p=1}^{P} e_{p,1} \frac{\partial R_p(\xi, \eta)}{\partial \eta}.
\end{align*}
\]

Thus, Eq. (15) can be written as follows:

\[
\mathbf{d} = \mathbf{M} e,
\]

where

\[
\mathbf{d} = \begin{bmatrix}
    a_1^m & a_2^m & a_3^m & b_1^m & b_2^m & b_3^m
\end{bmatrix}^T,
\]

\[
\mathbf{M} = \begin{bmatrix}
    a_1^m & a_2^m & a_3^m & b_1^m & b_2^m & b_3^m
\end{bmatrix},
\]

\[
e = \begin{bmatrix}
    e_{p,q}
\end{bmatrix}
\]
\[ e = \begin{bmatrix} e_{p,1} & e_{p,2} & \cdots & e_{p,q} \end{bmatrix}^T \]

\[ M = \begin{bmatrix}
\frac{\partial R_i(y, \eta)}{\partial \eta} & \frac{\partial R_i(y, \eta)}{\partial y} & \frac{\partial R_i(y, \eta)}{\partial x_1} & \cdots & \frac{\partial R_i(y, \eta)}{\partial x_q} \\
\frac{\partial R_i(x, \eta)}{\partial \eta} & \frac{\partial R_i(x, \eta)}{\partial y} & \frac{\partial R_i(x, \eta)}{\partial y} & \cdots & \frac{\partial R_i(x, \eta)}{\partial x_q} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{\partial R_i(y, \eta)}{\partial \eta} & \frac{\partial R_i(y, \eta)}{\partial y} & \frac{\partial R_i(y, \eta)}{\partial x_1} & \cdots & \frac{\partial R_i(y, \eta)}{\partial x_q}
\end{bmatrix} \]

\[ \frac{\partial R_p(\xi, \eta)}{\partial k} = \begin{bmatrix}
\frac{\partial R_i(\xi, \eta)}{\partial k} & \frac{\partial R_i(\xi, \eta)}{\partial k} & \cdots & \frac{\partial R_i(\xi, \eta)}{\partial k}
\end{bmatrix}, \quad k = \{ \xi, \eta \}. \]

Finally, the solution of Eq. (16) is given as

\[ e = \text{pinv} \left( M \right) d \]  

(17)

where pinv means pseudoinverse operation.

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