Spatial structure and angular momentum of electro-magnetic wave radiated from a relativistic electron moving on a spiral orbit

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Abstract. We derive analytic expressions for the vector potential of electromagnetic wave radiated from a relativistic electron moving on a spiral orbit under an approximation that the velocity along the spiral orbit is much larger than that perpendicular to the axis. Based on this expression, we show that the electromagnetic wave has a spiral wave front and carries angular momentum.

Keywords: Electromagnetic wave, Relativistic electron, Angular momentum, Optical vortex

1. Introduction

Twisted photon beams (optical vortices) have attracted enormous interests because of their applications in many fields, such as optical manipulations, optical communication/information technologies, imaging technologies [1, 2, 3]. Moreover, their possible roles in natural sciences have been discussed [4, 5]. A twisted beam carries orbital angular momentum (OAM), which is attributed to its helical phase structure accompanied by a phase singularity at the center, in adding to the spin angular momentum (SAM) which is attributed to its polarization [6]. The study was started with Laguerre-Gaussian mode of the electromagnetic field, which has a spiral phase term, \( \exp(-i\phi) \), and carries OAM of \( \hbar l \). Here, \( l \) is called topological charge. Later the study was extended to other modes, such as Mathieu, Ince-Gaussian and Bessel beams [7, 8, 9].

Generations of the twisted beam have been demonstrated in various methods. One direction is to use optical elements, such as holographic filters or spiral zone plates, and convert Hermit-Gauss beams to Laguerre-Gauss beams [10, 11, 12]. Another direction is to generate
twisted beams directly from free electrons. There are two basic concepts on this direction. One is to generate a twisted beam by controlling the spatial distribution of relativistic electrons in a beam. In this case, each electron radiates normal light, but, as a consequence of the interference between the radiation field emitted from the spatially ordered electrons, a twisted beam is formed [13]. Another is to utilize the basic characteristics of the radiation field emitted from a single electron, which is the subject of this work.

In the modern synchrotron light sources, a device called undulator is widely used. It produces a periodically alternating magnetic field on the electron orbit. The electrons execute undulating motion and radiate quasi-monochromatic light. It was shown that this undulator radiation has characteristic phase structures, depending on the configuration of the magnetic field [14]. In particular, the radiation from a device called helical undulator has spiral phase structure. Later, it was discussed theoretically in the context of optical vortex [15], and it was verified experimentally [16, 17, 18, 19].

In the helical undulator, electrons run on a spiral orbit. The spiral motion of the electrons can be separated into a drift motion along the undulator axis and a circular motion around it. In the moving reference frame with the drift velocity, the radiation field should be that from the circular motion. Its Lorentz transformation to the laboratory frame is the helical undulator radiation. Therefore, the origin of the peculiar characteristics of the radiation field from helical undulators should be found in the radiation from the circular motion. Indeed, it was shown that such a radiation field has spiral phase structure and carries angular momentum [20, 21]. Since the undulator radiation and inverse Thomson scattering are physically equivalent, it was proposed to produce vortex gamma-rays by nonlinear inverse Thomson scattering of a circular polarized intense laser beam [22, 23]. It was also proposed to produce vortex microwaves based on cyclotron radiation [24].

In the previous theoretical work [20], an analytic expression was derived for the vector potential of the radiation emitted from an electron in circular motion, and the angular momentum carried by the radiation field was discussed. Then, the radiation field and its angular momentum for the case that the electron is running on a spiral orbit was discussed based on Lorentz transformation. In this paper, we derive the analytic expression directly from the spiral electron motion and discuss the spatial structure and angular momentum of the radiation emitted from this electron. We also show some numerical examples of the spatial structure of the radiation field, based on the analytic expression.

2. Vector potential and electromagnetic field

The electron motion on a helical orbit may be expressed as follows:

$$\vec{r}_e = \rho \left\{ \cos \omega t \hat{e}_z + \sin \omega t \hat{e}_y \right\} + z_e(t) \hat{e}_z$$  (1)
\[
\vec{\beta} = \frac{\alpha, \rho}{c} \left( -\sin \omega t \vec{e}_x + \cos \omega t \vec{e}_y \right) + \beta_z(t) \vec{e}_z \\
= -\beta_\perp \sin \omega t \vec{e}_x + \beta_\perp \cos \omega t \vec{e}_y + \beta_z(t) \vec{e}_z \quad \left( \beta_\perp \equiv \frac{\alpha, \rho}{c} \right)
\]

(2)

in a coordinate system shown in Fig. 1. Here, the velocity perpendicular to the z-axis, \( \beta_\perp \), is constant. Therefore, the velocity along the z-axis (the center of the helical motion) is also constant and can be written as follows;

\[
\beta_z(t) = \sqrt{\beta^2 - \beta^2_\perp} = \text{const.} \\
\approx 1 - \frac{1}{2\gamma^2} - \frac{1}{2} \beta^2_\perp \quad (\beta_\perp \ll 1)
\]

(3)

Figure 1: Coordinate System. The electron executes the helical motion along z-axis.

The vector potential of the radiation from an electron whose motion is expressed by Eqs. (1) and (2) may be expressed as follows; [25]
\[
\vec{A}(t) = \int_{-\infty}^{+\infty} A_n e^{-i\omega t} \frac{dt}{2\pi}
\]

\[
\vec{A}_0 = \int_{-\infty}^{+\infty} A(t) e^{-i\omega t} dt = e^{i\frac{e}{r}} \int_{-\infty}^{+\infty} e^{i \left\{ r\omega_r - \frac{e}{r} \right\}} d\vec{r} + o\left(\frac{1}{r^2}\right) \equiv \vec{A}_0^{(1)} + o\left(\frac{1}{r^2}\right)
\]

Here, \( \vec{A}_0^{(1)} \) shows the vector potential of the first order of \( \frac{1}{r} \). As expressing the electron motion in the spherical coordinate system, the following expression can be obtained;

\[
\vec{A}_0^{(1)} = e^{i\frac{e}{r}} \int_{-\infty}^{+\infty} dte^{-\omega t} \exp\left[-i\left(\xi \sin(\omega t - \varphi) + \omega \beta_z \cos(\theta \cdot t)\right)\right]
\times \left\{ \begin{array}{l}
\beta_\perp \sin \theta \cos(\omega t - \varphi) + \{\beta_\perp \cos \theta\}

\beta_\perp \cos \theta \cos(\omega t - \varphi) - \{\beta_\perp \sin \theta\}

\beta_\perp \sin(\omega t - \varphi)
\end{array} \right\}_\theta_\varphi
\]

\[
= e^{i\frac{e}{r}} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{+\infty} dte^{i(\omega(1-\beta_\perp \cos \theta)-m\omega_\theta)t} e^{i\omega t} \left\{ \begin{array}{l}
a^m_\varphi(\theta)
a^m_\theta(\theta)
\end{array} \right\}_\theta_\varphi
\]

where;

\[
a^m_\varphi(\theta) = \beta_\perp \sin \theta \frac{1}{2} \left\{ J_{m+1}(\xi) + J_{m-1}(\xi) \right\} + \beta_\perp \cos \theta J_m(\xi)
\]
\[
a^m_\theta(\theta) = \beta_\perp \cos \theta \frac{1}{2} \left\{ J_{m+1}(\xi) + J_{m-1}(\xi) \right\} - \beta_\perp \sin \theta J_m(\xi)
\]
\[
a^m_\varphi(\theta) = \beta_\perp \cos \theta \frac{1}{2i} \left\{ J_{m+1}(\xi) - J_{m-1}(\xi) \right\}
\]

and \( \xi = \beta_\perp \sin \theta \). Here, we have used a relation;

\[
e^{-i \omega \sin \theta} = \sum_{m=-\infty}^{\infty} J_m(z)e^{-im\theta}
\]

where \( J_m(z) \) is Bessel function.

In case that the helical motion continues infinitely, the integral in Eq. (5) is reduced to Delta function;
\[ \int_{-\infty}^{+\infty} dt e^{i\omega(1-\beta_z \cos \theta) - m \omega_z t} = 2\pi \delta \left( \omega \left(1-\beta_z \cos \theta\right) - (m \omega_2) \right) \]  

which means that the vector potential has sharp peaks at the following frequency;

\[ \omega_m(\theta) = \frac{m \omega_2}{1 - \beta_z \cos \theta} \]  

This relationship represents the relativistic Doppler effect on the frequency. When the motion is finite, the spectral peaks have finite width expressed by;

\[ \int_{-T}^{+T} dt e^{i\omega(1-\beta_z \cos \theta) - m \omega_z t} = 2 \sin \left[ \frac{\omega \left(1-\beta_z \cos \theta\right) - (m \omega_2) \right]_T \]  

At the \( m \)-th peak, the vector potential can be expressed as follows;

\[ \bar{A}_\omega^{(m)} = e^{\frac{i \omega_m(\theta)r_{\text{mp}}}{c}} \frac{2\pi}{r} \begin{pmatrix} a_e^m(\theta) \\ a_e^m(\theta)_{\text{rhp}} \end{pmatrix} \]  

which may be called \( m \)-th harmonic.

Fourier components of the electric and magnetic fields may be expressed by that of the vector potential as follows:

\[ \vec{H}_\omega = \nabla \times \bar{A}_\omega \]

\[ \vec{E}_\omega = \frac{c}{-i \omega} \nabla \times H_\omega \]  

By inserting the vector potential to Eq. (11), the electromagnetic fields can be expressed as follows;
\[ H_r^m = \frac{1}{r \sin \theta} \left\{ \frac{\partial}{\partial \theta} \left( \sin \theta A_{\phi}^{m(1)} \right) - \frac{\partial A_{\theta}^{m(1)}}{\partial \phi} \right\} + o \left( \frac{1}{r^3} \right) \]

\[ H_{\theta}^m = -i \frac{\omega_m(\theta)}{c} A_{\phi}^{m(1)} + o \left( \frac{1}{r^2} \right) \] (12)

\[ H_{\phi}^m = i \frac{\omega_m(\theta)}{c} A_{\theta}^{m(1)} + o \left( \frac{1}{r^2} \right) \]

\[ E_r^m = \frac{1}{r \sin \theta} \left\{ -\frac{1}{\omega_m(\theta)} \frac{\partial}{\partial \theta} \left( \sin \theta \omega_m(\theta) A_{\phi}^{m(1)} \right) - \frac{\partial A_{\theta}^{m(1)}}{\partial \phi} \right\} + o \left( \frac{1}{r^3} \right) \]

\[ E_{\theta}^m = i \frac{\omega_m(\theta)}{c} A_{\phi}^{m(1)} + o \left( \frac{1}{r^2} \right) \] (13)

\[ E_{\phi}^m = i \frac{\omega_m(\theta)}{c} A_{\theta}^{m(1)} + o \left( \frac{1}{r^2} \right) \]

It should be noted here, that the transverse components (the \( \theta \) and \( \phi \) components) of the electromagnetic fields have the first-order terms of \( 1/r \) but the longitudinal (\( r \)) component does not. The non-zero terms of longitudinal starts from the second order of \( 1/r \). This 2\textsuperscript{nd} order term, which is usually neglected in the radiation field calculation, contributes to angular momentum as shown in the next section.

3. **Angular momentum of radiation field**

The energy and the linear momentum carried by the \( m \)-th harmonic of the radiation field can be expressed with Poynting vector;

\[ \vec{S}^m = \frac{c}{4\pi} (\vec{E}^m \times \vec{H}^{m*}) = \frac{c}{4\pi} \begin{pmatrix} E_{\theta}^{m(1)} H_{\phi}^{m(1)*} - E_{\phi}^{m(1)} H_{\theta}^{m(1)*} \\ 0 \\ 0 \end{pmatrix}_{r \in \phi} + o \left( \frac{1}{r^3} \right) \] (14)

and its \( r \) component is given by;

\[ S_r^m = \frac{c}{4\pi} \left( \frac{\omega_m(\theta)}{c} \right)^2 \begin{pmatrix} A_{\theta}^{m(1)} A_{\phi}^{m(1)*} + A_{\phi}^{m(1)} A_{\theta}^{m(1)*} \end{pmatrix} + o \left( \frac{1}{r^2} \right) \] (15)

The energy carried by the radiation field can be written as follows;

\[ \frac{dU^m}{dt} = \int S_r^m r^2 d\Omega = \frac{c}{4\pi} \int r^2 d\Omega \left( \frac{\omega_m(\theta)}{c} \right)^2 \begin{pmatrix} A_{\theta}^{m(1)} A_{\phi}^{m(1)*} + A_{\phi}^{m(1)} A_{\theta}^{m(1)*} \end{pmatrix} + o \left( \frac{1}{r} \right) \] (16)
As dividing the integrant of Eq. (15) by the energy per photon, \( \hbar \omega_m(\theta) \), the number of photons may be expressed as follows;

\[
\frac{dN^m}{dt} = \int \frac{1}{4\pi \hbar c} r^2 d\Omega \omega_m(\theta) \left( A_{\phi}^{m(1)} A_{\phi}^{m(1)*} + A_{\phi}^{m(1)} A_{\phi}^{m(1)*} \right) + o\left(\frac{1}{r}\right)
\]

(17)

The angular momentum component parallel to the axis of the helical motion carried by the radiation field may be expressed as follows:

\[
\frac{dJ_z}{dt} = \int cr^2 j_z d\Omega = \int cr^2 \left( \frac{1}{4\pi c} \left( \tilde{r} \times (\tilde{E} \times \tilde{H}^*) \right) \right) d\Omega
\]

\[
= \int cr^2 \left( \frac{1}{4\pi c} r \sin \theta \left( E_r^{(2)} H_\phi^{(1)*} - E_\phi^{(1)} H_r^{(2)*} \right) + o\left(\frac{1}{r^3}\right) \right) d\Omega
\]

(18)

As described above, the second order terms regarding to \( 1/r \) appear in the integrant. The angular momentum carried by \( m \)-th harmonic can be expressed as below:

\[
\frac{dJ^m}{dt} = \int c \left[ \frac{1}{4\pi c} + \frac{\omega_m(\theta)}{c} \left( m \left( A_{\phi}^{m(1)} A_{\phi}^{m(1)*} + A_{\phi}^{m(1)} A_{\phi}^{m(1)*} \right) \right) \right] r^2 \sin \theta d\theta d\phi
\]

\[
= \frac{1}{4\pi} \left( \frac{i}{c} \right) \frac{\partial}{\partial \theta} \left( \sin \theta \omega_m(\theta) A_{\phi}^{m(1)} \cdot \sin \theta A_{\phi}^{m(1)*} \right) r^2 d\theta d\phi
\]

\[
+ (m) \frac{1}{4\pi} \int \frac{\omega_m(\theta)}{c} \left( A_{\phi}^{m(1)} A_{\phi}^{m(1)*} + A_{\phi}^{m(1)} A_{\phi}^{m(1)*} \right) r^2 d\Omega + o\left(\frac{1}{r}\right)
\]

\[
= m \frac{1}{4\pi} \int \frac{\omega_m(\theta)}{c} \left( A_{\phi}^{m(1)} A_{\phi}^{m(1)*} + A_{\phi}^{m(1)} A_{\phi}^{m(1)*} \right) r^2 d\Omega + o\left(\frac{1}{r}\right)
\]

(19)

Eqs. (16) and (18) may be summarized as follows;

\[
\frac{dJ^m}{dt} = m\hbar \frac{dN^m}{dt}
\]

(20)

which indicates that \( m \)-th harmonic component carries angular momentum, \( m\hbar \) per photon.

Here, we have obtained the same result as obtained in the previous work [20]. In the previous work, the angular momentum was calculated for a circular motion. Then, it was simply discussed that, because of the invariance of the angular momentum in Lorentz transformation along the motion of the reference frame. This work verifies the treatment in the previous work.

4. Spatial structure of intensity, polarization and phase

In Cartesian coordinate, the electric field can be expressed as follows;

\[
\bar{E} = E_{\phi} \hat{e}_{\phi} + E_{\phi} \hat{e}_{\phi} \equiv E_x \hat{e}_x + E_y \hat{e}_y + E_z \hat{e}_z
\]

(21)

where,
\[
E_\pm \equiv \frac{1}{2} \left( E_o \mp i E_o e^{\theta \pm \phi} \right) \cos \theta e^{i \phi}
\]

and \( \vec{e}_\pm \equiv \vec{e}_x \pm i \vec{e}_y \). When the electron motion is ultra-relativistic along the \( z \)-axis (\( \beta_z \approx 1 \)), the radiation is concentrated around the \( z \)-axis (\( \theta \ll 1 / \gamma \)), and only the transverse electric field is dominant, which can be decomposed into circularly polarized components with positive and negative helicities, \( E_\pm \). By inserting Eq. (10) to Eq. (22) via Eq. (13), it is straightforward to show that those components have phase terms given below:

\[
E_\pm \vec{e}_\pm \propto e^{i(m \pm 1)\theta} \vec{e}_\pm
\]

In the photon picture, this may be intuitively understood as that the positive/negative helicity component has \( \pm \hbar \) SAM and \((m \mp 1)\hbar \) OAM.

In the following, we present the results from numerical calculations to show typical spatial structures of the phase and intensity of the radiation emitted from an electron in a spiral motion. In the calculations, typical parameters for an undulator at synchrotron light sources were chosen such that the period length and the strength of the magnetic field \( \lambda_0 = 2\pi c / \omega_e = 5 \text{ cm} \), \( B_0 = 0.5 T \), the number of periods, 20 and the electron beam with Lorentz factor, \( \gamma = 1000 \), which corresponds to about 500 MeV. Since the electron motion is finite, the harmonic components of the radiation have finite spectral width as given by Eq. (9). The following numerical examples are calculated for the peak frequency.

Figure 2: Spatial structure of 1\textsuperscript{st} harmonic component \((m=1)\) (a)-(d) and of 2\textsuperscript{nd} harmonic component \((m=2)\) (e)-(h).
Figure 2 shows the spatial structure of the radiation from an electron in spiral motion for the harmonic number \( m = 1 \) (Figs. (a)-(c)) and \( m = 2 \) (Figs. (e)-(f)). The distributions of the radiation intensity on the transverse plane for the positive and negative helicities of the first harmonic are shown in Figs. 2(a) and 2(b) and those of the second harmonic are shown in Figs. 2(e) and 2(f). The intensity distributions are normalized to the maximum values. For non-zero OAM, the spatial distribution of the radiation is “donut-shaped” with zero intensity at the center, which is one of the major characteristics of optical vortex [1, 2, 3]. Indeed, the intensity of the positive helicity component of the first harmonic distributes like Gaussian (Fig. 2(a)). But the intensity profile of the negative helicity of the first harmonic and both the positive and negative helicity components of the second harmonic distribute like “donut” as can be seen in Figs. 2(b), 2(e), and 2(f). The diameter of donut shape becomes larger as the topological charge increases.

The phase structures of the first harmonic and second harmonic for positive and negative helicities are shown in Figs. 2(c), 2(d), 2(g), and 2(h). The phase structure of the positive helicity of the first harmonic \( (m = 1) \) exhibits concentric circular distribution and it is uniform around the center (Fig. 2(c)), which indicates that the field does not carry OAM, which means that the topological charge \( l = 0 \). On the other hand, its negative helicity component shows a double spiral phase structure (Fig. 2(d)), which means \( l = 2 \). For the second harmonics \( (m = 2) \), the positive helicity component shows a single spiral structure \( (l = 1, \text{Fig. 2(g)}) \) and the negative a triple spiral one \( (l = 3, \text{Fig. 2(h)}) \). These results are consistent with Eq. (23). It should be noted that the spiral phase pattern on the transverse plane (perpendicular to \( z \)-axis) arises from the phase term in Eq. (9). As \( r \) in the phase term is replaced with \( z/\cos\theta \), the phase depends not only on the azimuthal angle but also on the distance from the center on the transverse plane through \( \theta \).

5. Conclusion

We have derived an analytic expression of the electromagnetic field radiated from a free electron running on a spiral orbit. We have shown that the radiation field carries angular momentum and possesses a helical phase structure, depending on the harmonic number and the helicity. Such a physical situation, in which relativistic free electron are running on spiral orbits, can be found in laboratories and nature, such as in particle accelerators, nuclear fusion plasma, astrophysical objects and so on. Electromagnetic waves which have helical wave fronts and carry orbital angular momenta are ubiquitous.
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