Solvability of the Fractional Hyperbolic Keller-Segel System

Gerardo Huaroto and Wladimir Neves

Abstract

We study a new nonlocal approach to the mathematical modelling of the Chemotaxis problem, which describes the random motion of a certain population due to a substance concentration. Considering the initial-boundary value problem for the fractional hyperbolic Keller-Segel model, we prove the solvability of the problem. The solvability result relies mostly on fractional calculus and kinetic formulation of scalar conservation laws.

1 Introduction

We introduce and study in this paper the Fractional Hyperbolic Keller-Segel (FHKS for short) model for chemotaxis described by the following system

\[
\begin{align*}
\partial_t u + \text{div}(g(u) \nabla K_s c) &= 0, \quad \text{in } (0, \infty) \times \Omega, \\
(-\Delta_N)^{1-s} c + c &= u, \quad \text{in } \Omega, \\
u(t=0) &= u_0, \quad \text{in } \Omega, \\
\nabla K_s c \cdot \nu &= 0, \quad \text{on } \Gamma,
\end{align*}
\]

where \( u(t,x) \) is the density of cells and \( c(t,x) \) is the chemoattractant concentration, which is responsible for the cell aggregation. The problem is posed in a bounded open subset \( \Omega \subset \mathbb{R}^n \), \( n = 1, 2, \text{ or } 3 \), with \( C^2 \) boundary denoted by \( \Gamma \), and as usual we denote by \( \nu(r) \) the outward normal to \( \Omega \) at \( r \in \Gamma \). The given measurable bounded function \( u_0 \) is the initial condition of the cells, and we assume

\[
0 \leq u_0(x) \leq 1, \quad \text{for a.e. } x \in \Omega.
\]

Moreover, since the normal fractional flux in the equation on \( u \) vanishes on \( \Gamma \), that is, we assume that the boundary is characteristic, it is not necessary to prescribe boundary conditions for \( u \). This assumption besides the natural one, it prevents some specific difficulties related to the trace problem, see [18] for instance.

Here for \( 0 < s < 1, (-\Delta_N)^s \) denotes the Neumann spectral fractional Laplacian (NSFL for short) operator, which characterizes long-range diffusion effects.
We also consider the non-local operator $K_s$ given by definition in (2.5), which cannot be interpreted as the inverse of $(-\Delta_N)^s$. Indeed, we are dealing with the spectral fractional Laplacian $(-\Delta_N)^s$, which is defined from the eigenvalues of the Laplacian with Neumann boundary conditions, where the first eigenvalue is zero and the correspondent eigenfunction is a constant. Hence the NSFL is not an injective operator in its domain, actually it becomes injective when restricted to the set
\[
\left\{ f \in D\left((-\Delta_N)^s\right); \int_\Omega f(x) \, dx = 0 \right\}. \tag{1.3}
\]
Therefore, the operator $((-\Delta_N)^s)^{-1} = (-\Delta_N)^{-s}$ does not exist in general, and thus we are not allowed to write, $(-\Delta_N)^{1-s}c = (-\Delta_N)(-\Delta_N)^{-s}c$, unless $c(t, x)$ satisfies (1.3). The reader is addressed to Section 2 for a comprehensive description of the NSFL operator.

Now, due to the first eigenfunction of $(-\Delta_N)^{1-s}$ be $\varphi_0 = 1/\sqrt{|\Omega|}$, it is not difficult to show that
\[
\int_\Omega (-\Delta_N)^{1-s}c(t, x) \, dx = 0. \tag{1.4}
\]
Indeed, we have for almost all $t > 0$
\[
\int_\Omega (-\Delta_N)^{1-s}c(t, x) \, dx = \sqrt{|\Omega|} < (-\Delta_N)^{1-s}c(t, \cdot), \varphi_0 \geq 0,
\]
where we have used Proposition (2.1) item (2). Consequently, from the second equation in (1.1) it follows that, for a.a. $t > 0$
\[
\int_\Omega c(t, x) \, dx = \int_\Omega u(t, x) \, dx, \tag{1.5}
\]
which implies that $c$ satisfies condition (1.3) if, and only if, $u$ satisfies it.

Last but not least, we consider (conveniently) the function $g(u) = u(1-u)$, which prevents blow up. Indeed, from a maximum principle established result, and the assumption (1.2) we are going to show that, $0 \leq u(t, x) \leq 1$ almost everywhere. Consequently, we could not impose that the chemoattractant concentration $c(t, x)$ satisfies condition (1.3), otherwise $u = 0$ a.e.. This implies an inherent fractional characterization of the problem studied here.

The theory of chemotaxis modeling goes back to E. F. Keller and L. A. Segel [11, 12, 13], where a detailed description of the movement of cells oriented by chemical cues can be found. In fact, a nonlocal version of the Keller-Segel model has been proposed by Caffarelli, Vazquez in [3] (Section 8. Comments and Extensions). Clearly, the study of a nonlocal version of the Keller-Segel model becomes interesting as a proposed open problem in that seminal cited paper [3]. But not just because of that, it seems natural to assume that the random motion of a certain population due a substance concentration can be described by Lévy process instead of the Brownian motion, which is in fact a special case of the continuous Lévy process. The fractional model proposed here in (1.1) is
a fractional generalization of the model considered in Perthame-Dalibard [19]. Indeed, in that paper they studied the following system
\[
\begin{aligned}
\partial_t u + \text{div}(g(u) \nabla S) &= 0, \quad \text{in } (0, \infty) \times \Omega, \\
(-\Delta)S + S &= u, \quad \text{in } \Omega, \\
u \cdot \nabla c(t, x) &= 0, \quad \text{on } \Gamma, \\
\end{aligned}
\]
which follows from the system (1.1), at least formally passing to the limit as \(s \to 0^+\). In particular, it is allowed to have jumps in the chemoattractant concentration \(c(t, x)\) in the proposed model (1.1), which is not the case for the original model (1.6). This is very important for practical applications.

The mathematical analysis developed in this paper combine two different aspects (fractional calculus and kinetic formulation of scalar conservation laws). The first one is quite new, where well-posedness and several useful estimates are obtained for a fractional parabolic-elliptic system, see Section 3. Then, the rigidity result of Perthame-Dalibard [19] is well adapted in Section 4 to tackle the limiting problem to obtain the proof of the Theorem 1.1 (Main Theorem). Indeed, the non-local description of the chemotaxis model studied here brings new difficulties to the entropy structure of the first equation in (1.1), see (1.7). In particular, because of the non-local space dependence of the entropy flux, which leads to specific difficulties. More precisely, due to the lack of strong compactness or contraction properties, we apply the kinetic formulation associated to the propagation of oscillations of the transport equations (Theorem 4.1). To this end, it is essential to use the renormalization procedure in the kinetic equation, and we are allowed to do that for any \(s \in (0, 1/2]\).

We would like to remark that the uniqueness property is not addressed in this paper, neither the asymptotic behavior. First, let us stress that no uniqueness result is expected even for the standard system (1.6). On the other hand, long-time behavior was establish to (1.6) in [19], and similarly the main ingredient require here is
\[
\int_\Omega g(u(t)) \nabla K_s c(t) \cdot \nabla c(t, x) \, dx \geq 0
\]
for each \(t > 0\), which does not seen that such inequality holds. Indeed, we may write
\[
\int_\Omega g(u(t)) \nabla K_s c(t) \cdot \nabla c(t) \, dx = \int_\Omega g(c(t)) \nabla K_s c(t) \cdot \nabla c(t) \, dx \\
+ \int_\Omega (1 - 2\xi)(u(t) - c(t)) \nabla K_s c(t) \cdot \nabla c(t) \, dx,
\]
where \(\xi \in (\min\{u, c\}, \max\{u, c\})\). Formally, the first integral in the right hand side is positive (see Caffarelli, Soria, Vázquez [4] p.1706), although we do not have sign control in the second one. We leave this question for future studies.
Finally, we discuss in Section 5 some related problems that can be handled similarly. In particular, one of the systems considered does not satisfy exactly the condition (1.5), compare it with (5.3).

1.1 Statement of the FHKS system

The aim of this section is to formulate the mathematical problem for the FHKS system. We begin observing that, the first equation in (1.1) is a hyperbolic scalar conservation law, thus the density of cells function $u$ may admit shocks. Therefore, in order to select the more correct physical solution, we need an admissible criteria, which is given by the entropy condition.

**Definition 1.1.** A pair $F(u) = (\eta(u), q(u))$ is called an entropy pair for the first equation in (1.1), if there exists $\eta : \mathbb{R} \to \mathbb{R}$ a Lipschitz continuous and also convex function and the function $q : \mathbb{R} \to \mathbb{R}$, which satisfies

$$q'(u) = \eta'(u)g'(u),$$

for almost all $u \in \mathbb{R}$. Also, we call $\eta(u)$ an entropy and $q(u)\nabla K_s c$ the associated entropy flux for the first equation in (1.1).

Analogously to the most important example of the Kružkov’s entropies, we consider $F(u, v) = (|u - v|, \text{sgn}(u - v)(g(u) - g(v)))$ for each $v \in \mathbb{R}$. Another two examples of parameterized family of entropy pairs for (1.1), which will be conveniently used for the kinetic formulation, are given by

$$F^\pm(u, v) = (|u - v| \pm, \text{sgn}^\pm(u - v)(g(u) - g(v)))$$

for each $v \in \mathbb{R}$, where $|v| \pm := \max\{\pm v, 0\}$, and

$$\text{sgn}^+(v) := \begin{cases} 1, & \text{if } v > 0 \\ 0, & \text{if } v \leq 0 \end{cases}, \quad \text{sgn}^-(v) := \begin{cases} -1, & \text{if } v < 0 \\ 0, & \text{if } v \geq 0 \end{cases}.$$

In order to formulate the mathematical problem for the system (1.1), we formally assume enough regularity to the pair $(u(t, x), c(t, x))$. Then for any entropy $\eta \in C^2$, we multiply by $\eta'(u)$ the first equation in (1.1) to obtain

$$\partial_t \eta(u) + \text{div}(g(u)\nabla K_s c) + (u - c)[q - g\eta'](u) \leq 0,$$

(1.7)

where we have used that, $-\text{div}\nabla K_s c = u - c$. One recalls that, any smooth entropy pair $F(u) = (\eta(u), q(u))$ for (1.1) can be recovered by the family of Kružkov’s entropies. Therefore, the following definition tells us in which sense a pair of functions $(u, c)$ is a weak solution of FHKS system: (1.1)-(1.2).

**Definition 1.2.** Given an initial data $u_0 \in L^\infty(\Omega)$ satisfying (1.2) and any $s \in (0, 1)$, a pair of functions

$$(u, c) \in L^\infty((0, \infty) \times \Omega) \times L^\infty((0, \infty); D(((-\Delta)^{(1-s)/2})))$$

satisfying (1.7) is a weak solution of FHKS system: (1.1)-(1.2).
is called a weak solution to the FHKS system, if for almost all \( t > 0 \), the pair \((u, c)\) satisfies the condition (1.5), and the integral inequality

\[
\int_0^\infty \int_\Omega |u(t, x) - v| \partial_t \phi \, dx \, dt + \int_\Omega |u_0(x) - v| \phi(0) \, dx \\
+ \int_0^\infty \int_\Omega \text{sgn}(u(t, x) - v) \left( (g(u(t, x)) - g(v)) \nabla K_s c(t, x) \cdot \nabla \phi + g(v)\phi \right) \geq 0
\]

(1.8)

for any fixed \( v \in \mathbb{R} \), and each nonnegative function \( \phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d) \) and also the following integral identity

\[
\int_\Omega (\nabla K_s c \cdot \nabla \psi + c \psi) \, dx = \int_\Omega u \psi \, dx, \quad \text{for a.a. } t > 0
\]

(1.9)

holds for any \( \psi \in H^1(\Omega) \).

Now, we are able to state plainly the main result of this paper. Then, we have the following

**Theorem 1.1 (Main Theorem).** Let \( u_0 \in L^\infty(\Omega) \) be an initial data satisfying (1.2) and \( 0 < s \leq 1/2 \). Then, there exists a pair of functions

\[
(u, c) \in L^\infty((0, \infty) \times \Omega) \times L^\infty((0, \infty); D((-\Delta_N)^{1-s})),
\]

which is a weak solution to the FHKS system, and it satisfies

\[ 0 \leq u(t, x) \leq 1, \quad 0 \leq c(t, x) \leq 1, \]

for almost all \( t > 0 \) and \( x \in \Omega \).

**Remark 1.1.** In fact, the condition (1.5) follows from (1.9) with an integration by parts, due to the regularity of the function \( c(t, \cdot) \in D((-\Delta_N)^{1-s}) \) for almost all \( t > 0 \). Indeed, the second equation in (1.1) is satisfied almost everywhere, and (1.5) follows integrating it on \( \Omega \).

### 1.2 Notation and Functional Spaces

Let \( \Omega \) be a bounded open set in \( \mathbb{R}^n \). We denote by \( dx \) (or \( d\xi \), etc.) the Lebesgue measure, and by \( \mathcal{H}^\theta \) the \( \theta \)-dimensional Hausdorff measure. By \( L^p(\Omega) \) we denote the set of real \( p \)-summable functions with respect to the Lebesgue measure, and the vector counterparts of these spaces are denoted by \( L^p(\Omega)^n \).

- **The space** \( W^{s,p}(\Omega) \)

  The fractional Sobolev space is denoted by \( W^{s,p}(\Omega) \), where a real \( s \geq 0 \) is the smoothness index, and a real \( p \geq 1 \) is the integrability index. More precisely, for
s ∈ (0, 1), p ∈ [1, +∞), the fractional Sobolev space of order s with Lebesgue exponent p is defined by
\[ W^{s,p}(\Omega) := \left\{ u ∈ L^p(\Omega) : \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n + sp}} \, dx \, dy < +\infty \right\}, \]
endowed with norm
\[ \|u\|_{W^{s,p}(\Omega)} = \left( \int_{\Omega} |u|^p \, dx + \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n + sp}} \, dx \, dy \right)^{\frac{1}{p}}. \]
Moreover, for s > 1 we write \( s = m + \sigma \), where \( m \) is an integer and \( \sigma \in (0, 1) \). In this case, the space \( W^{s,p}(\Omega) \) consists of those equivalence classes of functions \( u \in W^{m,p}(\Omega) \) whose distributional derivatives \( D^\alpha u \), with \( |\alpha| = m \), belong to \( W^{\sigma,p}(\Omega) \), that is
\[ W^{s,p}(\Omega) = \left\{ u \in W^{m,p}(\Omega) : \sum_{|\alpha| = m} \|D^\alpha u\|_{W^{\sigma,p}(\Omega)} < \infty \right\}. \]
which is a Banach space with respect to the norm
\[ \|u\|_{W^{s,p}(\Omega)} = \left( \|u\|^p_{W^{m,p}(\Omega)} + \sum_{|u| = m} \|D^\alpha u\|^p_{W^{\sigma,p}(\Omega)} \right)^{\frac{1}{p}}. \]
If \( s = m \) is an integer, then the space \( W^{s,p}(\Omega) \) coincides with the Sobolev space \( W^{m,p}(\Omega) \). Also, it is very interesting the case when \( p = 2 \), i.e. \( W^{s,2}(\Omega) \). In this case, the (fractional) Sobolev space is also a Hilbert space, and we can consider the inner product
\[ \langle u, v \rangle_{W^{s,2}(\Omega)} = \langle u, v \rangle + \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y)) \cdot (v(x) - v(y))}{|x - y|^{2 + s}} \, dx \, dy, \]
where \( \langle \cdot, \cdot \rangle \) is the inner product in \( L^2(\Omega) \).

- **The space** \( H^s(\Omega) \)

  Following Lions, Magenes [15], we can define for \( s \in (0, 1) \), the spaces \( H^s(\Omega) \) by interpolation between \( H^1(\Omega) \) and \( L^2(\Omega) \), i.e.
\[ H^s(\Omega) = [H^1(\Omega), L^2(\Omega)]_{1-s}. \]
According to this definition, this space is a Hilbert space with the natural norm given by the interpolation. We recall that, when \( \Omega \) has Lipschitz boundary regularity, then the spaces \( W^{s,2}(\Omega) \) and \( H^s(\Omega) \) are equivalent.

## 2 The NSFL Operator

Here and subsequently \( \Omega \subset \mathbb{R}^n \) is a bounded open set with \( C^2 \)-boundary \( \Gamma \). Following [9][10], we are interested in fractional powers of a strictly positive self-adjoint operators defined in a domain, which is dense in a (separable) Hilbert
space. More precisely, let us denote by \((-\Delta_N)\) the operator \((-\Delta)\) subject to Neumann boundary conditions. One observes that \((-\Delta_N)\) is a nonnegative and self-adjoint operator defined in

\[
D(-\Delta_N) = \{ u \in H^1(\Omega) : (-\Delta)u \in L^2(\Omega), \text{ with } \nabla u \cdot \nu = 0 \text{ on } \Gamma \} = \{ u \in H^2(\Omega) : \nabla u \cdot \nu = 0 \text{ on } \Gamma \}.
\]

By the spectral theory, there exists a complete orthonormal basis \(\{\varphi_k\}_{k=0}^{\infty}\) of \(L^2(\Omega)\), where \(\varphi_k\) satisfies the following eigenvalue problem

\[
\begin{aligned}
-\Delta \varphi_k &= \lambda_k \varphi_k, & \text{in } \Omega, \\
\nabla \varphi_k \cdot \nu &= 0, & \text{on } \Gamma.
\end{aligned}
\]

(2.1)

Therefore, we have that \(\varphi_k\) is the eigenfunction corresponding to eigenvalue \(\lambda_k\) for each \(k \geq 0\), where one repeats each eigenvalue \(\lambda_k\) according to its (finite) multiplicity:

\[
0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots, \quad \lambda_k \to \infty \text{ as } k \to \infty.
\]

Moreover, it is not difficult to show that, \(\varphi_0 = 1/\sqrt{|\Omega|}\), that is a constant value, and

\[
\int_{\Omega} \varphi_k(x) \, dx = 0, \quad \text{for all } k \geq 1.
\]

(2.2)

Then, we may write

\[
D(-\Delta_N) = \{ u \in L^2(\Omega) : \sum_{k=1}^{\infty} \lambda_k^2 |\langle u, \varphi_k \rangle|^2 < \infty \},
\]

\[
(-\Delta_N)u = \sum_{k=1}^{\infty} \lambda_k \langle u, \varphi_k \rangle \varphi_k, \quad \text{for each } u \in D(-\Delta_N).
\]

Now, applying the functional calculus, we define for each \(s \in (0,1)\), the Neumann spectral fractional Laplacian operator, that is

\[
(-\Delta_N)^s : D((-\Delta_N)^s) \subset L^2(\Omega) \to L^2(\Omega),
\]

given by

\[
(-\Delta_N)^s u = \sum_{k=1}^{\infty} \lambda_k^s \langle u, \varphi_k \rangle \varphi_k,
\]

(2.3)

\[
D((-\Delta_N)^s) = \{ u \in L^2(\Omega) : \sum_{k=1}^{\infty} \lambda_k^{2s} |\langle u, \varphi_k \rangle|^2 < +\infty \}.
\]

Moreover, \(D((-\Delta_N)^s)\) is a Hilbert space, with the inner product

\[
\langle u, v \rangle_s := \langle u, v \rangle + \int_{\Omega} (-\Delta_N)^s u(x) (-\Delta_N)^s v(x) \, dx.
\]
In particular, the norm $| \cdot |_s$ is defined by
\[ |u|_s^2 := \|u\|_{L^2(\Omega)}^2 + \|(-\Delta)^s u\|_{L^2(\Omega)}^2. \] (2.4)

Analogously, we define $K_s : D(K_s) = L^2(\Omega) \to L^2(\Omega)$ by
\[ K_s u := \sum_{k=1}^{\infty} \lambda_k^{-s} \langle u, \varphi_k \rangle \varphi_k, \]
\[ D(K_s) = \left\{ u \in L^2(\Omega) : \sum_{k=1}^{\infty} \lambda_k^{-2s} |\langle u, \varphi_k \rangle|^2 < +\infty \right\}. \] (2.5)

The next proposition gives us the main properties of the $(-\Delta)^s$, and $K_s$ operators defined above.

**Proposition 2.1.** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $s \in (0, 1)$, and consider $(-\Delta)^s$, and $K_s$ the operators defined respectively by (2.3) and (2.5). Then, we have:

1. $D(-\Delta) \subset D((-\Delta)^s)$, thus $D((-\Delta)^s)$ is dense in $L^2(\Omega)$.
2. The operator $(-\Delta)^s$ and $K_s$ are self-adjoint.
3. If $0 < s_1 \leq s_2 \leq 1$, then $D((-\Delta)^{s_2}) \hookrightarrow D((-\Delta)^{s_1})$, and $D((-\Delta)^{s_2})$ is dense in $D((-\Delta)^{s_1})$.
4. For any $\lambda > 0$, and $s \in [0, 1]$, the operator $I_d + \lambda(-\Delta)^s$ is bijective from $D((-\Delta)^s)$ to $L^2(\Omega)$.

**Proof.** The proofs of items (1) – (3) follow analogously to Proposition 2.1 in [9], hence we omit them. Let us show item (4), first we note that for any $u \in D((-\Delta)^s)$, we have
\[ \langle (-\Delta)^s u, u \rangle = \sum_{k=1}^{\infty} \lambda_k^s |\langle u, \varphi_k \rangle|^2 \geq \lambda_1^s \sum_{k=1}^{\infty} |\langle u, \varphi_k \rangle|^2 \geq 0, \] (2.6)
which implies for any $\lambda > 0$
\[ \|u + \lambda(-\Delta)^s u\|_{L^2(\Omega)} \geq \|u\|_{L^2(\Omega)}. \] (2.7)

Therefore, the linear operator $I_d + \lambda(-\Delta)^s$ is injective. Moreover, for each $f \in L^2(\Omega)$ there exists $v \in D((-\Delta)^s)$, such that
\[ v + \lambda(-\Delta)^s v = f. \]

Indeed, it is enough to take
\[ v(x) = \sum_{k=0}^{\infty} \frac{\langle f, \varphi_k \rangle}{1 + \lambda \lambda_k^s} \varphi_k(x) \] (2.8)
and check that, $v \in D((-\Delta)^s)$ and satisfies the above equation. Therefore, $I_d + \lambda(-\Delta)^s$ is a bijective operator. \qed
Remark 2.1. 1. One remarks that, for each \( \lambda > 0 \) the operator
\[
\lambda I_d + (-\Delta_N)^s : D((-\Delta_N)^s) \to L^2(\Omega)
\]
is invertible. For the extremal case \( (\lambda = 0) \), this assertion is false which is due to the fact that, \((-\Delta_N)^s\) is not injective in \( D((-\Delta_N)^s) \).

2. Thanks to the above observation, we have that \((-\Delta_N)^s\) is not invertible in its domain. Then, the operator \( K_s \) could not be seen as the inverse of \((-\Delta_N)^s\). However, if we restrict the domain of the fractional Laplacian to a specific subset of \( D((-\Delta_N)^s) \), we obtain the existence of the inverse \((-\Delta_N)^{-s}\).

Let us mention an important result, which help us to show the existence of solutions for the parabolic regularization of the system \((1.1)\).

Proposition 2.2. Given \( v \in L^2(\Omega) \), then for all \( s \in (0,1) \)
\[
K_s(I_d + (-\Delta_N)^{1-s})^{-1} v \in D(-\Delta_N).
\]

Proof. It follows from Proposition 2.1 together with the definition of the \( K_s \) operator. \( \square \)

2.1 The inverse of the restricted NSFL operator

Here, we consider a subset of the domain \( D((-\Delta_N)^s) \), such that there exists the inverse of the NSFL operator when restricted to this set. To this end, let us define for each \( s \in [0,1) \) the following set
\[
H^2_N := \left\{ u \in D((-\Delta_N)^s); \int_\Omega u(x) \, dx = 0 \right\}.
\]

Hence we have the following

Proposition 2.3. Under the conditions of Proposition 2.1, it follows that:

1. For all \( u \in H^2_N \), there exists \( \alpha > 0 \) such that
\[
((-\Delta_N)^s u, u) \geq \alpha^s \|u\|_{L^2(\Omega)}^2,
\]
where \( \alpha \) is the coercivity constant of \((-\Delta_N)\).

2. The operator \((-\Delta_N)^s\) is bijective from \( H^2_N \) to \( H^0_N \). In particular, the inverse of the fractional Neumann spectral Laplacian, i.e. \((-\Delta_N)^{s-1}\), exists.

Proof. 1. First, for \( u \in H^2_N \) we have
\[
((-\Delta_N)^s u, u) = \sum_{k=1}^{\infty} \lambda_k \|u, \varphi_k\|^2 \geq \lambda_1 \sum_{k=1}^{\infty} \|u, \varphi_k\|^2 = \lambda_1 \|u\|_{L^2(\Omega)}^2,
\]
(2.11)
where we have used in the last step, $\varphi_0 = 1/\sqrt{|\Omega|}$ and $\int_\Omega u(x)dx = 0$.

2. From item (1), it follows that $(-\Delta_N)^s$ is injective in $\mathcal{H}^0_N$. Now, we show that $(-\Delta_N)^s$ is also surjective. Indeed, for any $u \in \mathcal{H}^0_N$ let $v$ be defined by

$$v := \sum_{k=1}^{\infty} \lambda_k^{-s}(u, \varphi_k)\varphi_k.$$  

Then, $v \in D((-\Delta_N)^s)$ and

$$\int_\Omega v(x)dx = \left(\sum_{k=1}^{\infty} \lambda_k^{-s}(u, \varphi_k)\varphi_k, 1\right) = 0,$$

where we have used that $\langle \varphi_k, 1 \rangle = 0$, for each $k \geq 1$. Consequently, $v \in \mathcal{H}^s_N$ and also $(-\Delta_N)^s v = u$. Then the operator $(-\Delta_N)^s$ is surjective, and thus $((-\Delta_N)^s)^{-1}$ exists.

**Remark 2.2.** Applying Proposition 2.3, it follows that for each $u \in \mathcal{H}^0_N$ the inverse of the NSFL is given by

$$((-\Delta_N)^s)^{-1}u = (-\Delta_N)^{-s}u = \sum_{k=1}^{\infty} \lambda_k^{-s}(u, \varphi_k)\varphi_k. \quad (2.12)$$

From now own, we write $(-\Delta_N)^{-s}$ to denote the inverse of the NSFL operator, whenever this makes sense.

**Proposition 2.4.** Let $(-\Delta_N)^{-s}$ be the operator defined by (2.12) for any fixed $s \in (0, 1)$. Then, we have:

1. The operator $(-\Delta_N)^{-s}$ is self-adjoint in $\mathcal{H}^0_N$.
2. The operator $(-\Delta_N)^{-s}$ is continuous from $\mathcal{H}^0_N$ to itself.
3. If $\sigma > 0$ and $u \in \mathcal{H}^{2s}_N$, then $\mathcal{H}^{2(s+\sigma)}_N \ni v = (-\Delta_N)^{-\sigma} u$.

**Proof.** The proof proceeds analogously to the Proposition 2.1 in [9], and hence we omit it.

### 2.2 Some auxiliary results

First, we recall that using the language of semigroups, as introduced in [20] (see also [21]), one can check that $(-\Delta_N)^s$ is indeed a nonlocal operator. In fact, the NSFL is also given by

$$(-\Delta_N)^s u(x) = \frac{1}{\Gamma(-s)} \int_0^{\infty} (e^{t\Delta_N} u(x) - u(x)) \frac{dt}{t^{1+s}}, \quad x \in \Omega,$$

where $e^{t\Delta_N} u(x)$ is the heat diffusion semigroup generated by the Neumann Laplacian acting on $u$. 


Now, the aim is to characterize the space $D((-\Delta_N)^{s})$. To begin, we study $D((-\Delta_N)^{1/2})$, indeed, by using the $L^2$ normalization and the weak formulation of the equation (2.1), we see that $\|\varphi_k\|_{H^1(\Omega)}^2 = 1 + \lambda_k$. It is easy to check that $\{\varphi_k\}_{k\in\mathbb{N}_0}$ is also an orthogonal basis of $H^1(\Omega)$. Hence, we find 

$$H^1(\Omega) = \left\{ u \in L^2(\Omega) : \|u\|_{H^1(\Omega)}^2 = \sum_{k=0}^{\infty} (1 + \lambda_k) |\langle u, \varphi_k \rangle|^2 < \infty \right\}.$$ 

Therefore $H^1(\Omega) = D((-\Delta_N)^{1/2})$. In particular, from (2.4) we have 

$$\int_{\Omega} (-\Delta_N)^{1/2} u(x) (-\Delta_N)^{1/2} u(x) \, dx = \int_{\Omega} \nabla u(x) \cdot \nabla u(x) \, dx$$ 

(2.13) 

for all $u \in H^1(\Omega)$. Similarly, we have the following

**Proposition 2.5.** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Then for any $s \in (0,1)$,

$$H^s_N = \left\{ u \in H^s(\Omega) : \int_{\Omega} u(x) \, dx = 0 \right\}.$$ 

(2.14) 

In particular, for each $u \in H^s_N$, there exist $C_1, C_2 > 0$ such that

$$C_1 \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy \leq \|(-\Delta_N)^{s/2} u\|_{L^2(\Omega)}^2$$

$$\leq C_2 \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy.$$ 

**Proof.** See Theorem 2.5 in [22], and Lemma 7.1 in [2].

Here and subsequently, we denote for each $s \in (0,1)$ the operator $H_s = K_s^{1/2}$. Then, we consider the following

**Lemma 2.1.** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $s \in (0,1)$ and $u \in D((-\Delta_N)^{1-s})$, then $K_s u \in D((-\Delta_N))$. In particular, we have in trace sense

$$\nabla K_s u \cdot \nu = 0 \text{ on } \Gamma.$$ 

**Proof.** The proof follows directly from Proposition 2.1, item (3).

**Proposition 2.6.** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $0 < s < 1$. 

1. There exists a constant $C_1 > 0$, such that, for each $u \in H^1(\Omega)$

$$\int_{\Omega} |\nabla K_s u(x)|^2 \, dx \leq C_1 \int_{\Omega} |\nabla u(x)|^2 \, dx.$$ 

(2.15)

Similarly, for each $u \in H^1(\Omega)$, then $\nabla H_s u \in L^2(\Omega)$ and

$$\int_{\Omega} |\nabla H_s u(x)|^2 \, dx \leq C_1^{1/2} \int_{\Omega} |\nabla u(x)|^2 \, dx.$$ 

(2.16)
(2) If $u \in H^1(\Omega)$, then
\[
\int_{\Omega} \nabla K_s u(x) \cdot \nabla u(x) dx = \int_{\Omega} |\nabla H_s u(x)|^2 dx.
\] (2.17)

Proof. 1. To show item 1, we use the equivalence between $D((-\Delta_N)^{1/2})$ and $H^1(\Omega)$ (see (2.13)). Then, we have
\[
\int_{\Omega} |\nabla K_s u(x)|^2 dx = \sum_{k=1}^{\infty} \lambda_k |\langle K_s u, \varphi_k \rangle|^2 = \sum_{k=1}^{\infty} \lambda_k |\lambda_k^{-s} \langle u, \varphi_k \rangle|^2
\]
\[
\leq \lambda_1^{-2s} \sum_{k=1}^{\infty} \lambda_k |\langle u, \varphi_k \rangle|^2 = \lambda_1^{-2s} \int_{\Omega} |\nabla u(x)|^2 dx < \infty,
\]
and analogously for $\nabla H_s u$.

2. Now, we show item 2. Since $u \in H^1(\Omega)$, it is enough to consider that $u \in D((-\Delta_N^s))$ and thus apply a standard density argument. First, we integrate by parts to obtain
\[
\int_{\Omega} \nabla K_s u(x) \cdot \nabla u(x) dx = \int_{\Omega} (-\Delta_N) K_s u(x) u(x) dx = \int_{\Omega} (-\Delta_N)^{1-s} u(x) u(x) dx,
\]
where we have used the definition of $K_s u$ and $\nabla K_s u \cdot \nu = 0$ on $\Gamma$. Due to the fractional Laplacian being self-adjoint, it follows that
\[
\int_{\Omega} \nabla K_s u(x) \cdot \nabla u(x) dx = \int_{\Omega} |(-\Delta_N)^{(1-s)/2} u(x)|^2 dx.
\]
Therefore, using the equivalence norm (2.13) together with the definition of $H_s u$, we have
\[
\int_{\Omega} \nabla K_s u(x) \cdot \nabla u(x) dx = \int_{\Omega} |\nabla H_s u(x)|^2 dx.
\]

3 On a Perturbed Problem

The aim of this section is to introduce and study the properties of a perturbed system associated to (1.1). More precisely, given $\varepsilon > 0$ we consider the following fractional parabolic-elliptic system
\[
\begin{aligned}
\partial_t u_\varepsilon + \text{div} (g(u_\varepsilon) \nabla K_s c_\varepsilon) &= \varepsilon \Delta u_\varepsilon, &\text{in } (0, \infty) \times \Omega, \\
(-\Delta_N)^{1-s} c_\varepsilon + c_\varepsilon &= u_\varepsilon, &\text{in } \Omega, \\
u_\varepsilon(0) &= u_{0,\varepsilon}, &\text{in } \Omega, \\
\nabla K_s c_\varepsilon \cdot \nu &= 0 \text{ and } \nabla u_\varepsilon \cdot \nu = 0, &\text{on } \Gamma,
\end{aligned}
\] (3.1)
where \( u_{0, \varepsilon} \) is a regularized initial data, such that
\[
  u_{0, \varepsilon} \to u_0 \text{ strongly in } L^1(\Omega) \text{ as } \varepsilon \to 0, \text{ and } \|u_{0, \varepsilon}\|_{L^\infty} \leq \|u_0\|_{L^\infty}. \tag{3.2}
\]

Then, we show existence and uniqueness of \((u_\varepsilon, c_\varepsilon)\). To this end, we are going to apply the Banach Fixed Point Theorem to prove the local in time existence of solution to (3.1), and thus applying a contradiction argument we extend it to be global in time. Since (3.1) is a fractional non-standard parabolic-elliptic system we present the proof in details. To begin, we consider the following

**Lemma 3.1.** Let \( s \in (0,1) \) be fixed and \( \tilde{u} \in L^\infty((0, \infty) \times \Omega) \). Then for each \( u_{0, \varepsilon} \in L^2(\Omega) \) the fractional parabolic-elliptic system
\[
  \begin{cases}
    \partial_t u_\varepsilon - \varepsilon \Delta u_\varepsilon = - \text{div} \left(g(\tilde{u}) \nabla K_s c_\varepsilon\right), & \text{in } (0, \infty) \times \Omega, \\
    (-\Delta_N)^{1-s} c_\varepsilon + c_\varepsilon = \tilde{u}, & \text{in } \Omega, \\
    u_\varepsilon(0) = u_{0, \varepsilon}, & \text{in } \Omega, \\
    \nabla K_s c_\varepsilon \cdot \nu = 0, & \text{and } \nabla u_\varepsilon \cdot \nu = 0 \quad \text{on } \Gamma,
  \end{cases} \tag{3.3}
\]

admits a unique weak solution
\((u_\varepsilon, c_\varepsilon) \in L^2((0, \infty); H^1(\Omega)) \cap C([0, \infty); L^2(\Omega)) \times L^\infty((0, \infty); D((-\Delta_N)^{1-s}))). \)

**Proof.**
1. Given \( \tilde{u} \in L^\infty((0, \infty) \times \Omega) \) from (2.8) and Remark 2.1, it is possible to write, for almost all \( t > 0 \), the chemoattractant density as follows
\[
  c_\varepsilon(t) = \left(I_d + (-\Delta_N)^{1-s}\right)^{-1} \tilde{u}(t) = \sum_{k=0}^{\infty} \frac{1}{1 + \lambda_k^{-s}} \langle \tilde{u}(t), \varphi_k \rangle \varphi_k \tag{3.4}
\]

In particular \( c_\varepsilon \in L^\infty((0, \infty); D((-\Delta_N)^{1-s})) \) and, for almost all \( t > 0 \) and almost everywhere \( x \in \Omega \), satisfies the equation
\[
  (-\Delta_N)^{1-s} c_\varepsilon(t, x) + c_\varepsilon(t, x) = \tilde{u}(t, x).
\]

Moreover, due to Proposition 2.2 we have that \( \mathcal{K}_s c_\varepsilon(t) \in D(-\Delta_N) \) for a.a. \( t > 0 \). In particular, \( \nabla \mathcal{K}_s c_\varepsilon \cdot \nu = 0 \) on \( \Gamma \), which is due to the definition of \( D(-\Delta_N) \). Consequently, the second equation in (3.3) is solved. The uniqueness follows easily from the linearity of the equation.

2. Now, we show the existence of solution for the first equation in (3.3). First, since we have the explicit form of \( c_\varepsilon \), we may write
\[
  \mathcal{K}_s c_\varepsilon = \mathcal{K}_s \left(I_d + (-\Delta_N)^{1-s}\right)^{-1} \tilde{u} = \sum_{k=1}^{\infty} \frac{\lambda_k^{-s}}{1 + \lambda_k^{-s}} \langle \tilde{u}, \varphi_k \rangle \varphi_k = \sum_{k=1}^{\infty} \frac{1}{\lambda_k + \lambda_k^s} \langle \tilde{u}, \varphi_k \rangle \varphi_k, \tag{3.6}
\]
where (3.6) follows by the definition of $K_s$ together with (3.4). Observe that, $K_s c \varepsilon$ is an operator which depends on $\tilde{u}$, so we define

$$L \tilde{u} := K_s c \varepsilon = \sum_{k=1}^{\infty} \frac{1}{\lambda_k + \lambda_k^s} \langle \tilde{u}, \varphi_k \rangle \varphi_k,$$

where the last assertion is obtained by (3.5). Then the first equation in (3.3) together with the initial-boundary condition can be written as follows

$$\begin{cases}
\partial_t u \varepsilon - \varepsilon \Delta u \varepsilon = -\text{div} \left( g(\tilde{u}) \nabla L \tilde{u} \right), & \text{in } (0, \infty) \times \Omega, \\
u \varepsilon(0) = u_0 \varepsilon, & \text{in } \Omega, \\
 \nabla u \varepsilon \cdot \nu = 0, & \text{on } \Gamma.
\end{cases}$$

(3.7)

Therefore, it is enough to show the existence of a unique solution for (3.7).

We claim that, for any $\tilde{u} \in L^\infty((0, \infty) \times \Omega)$,

$$\text{div} \left( g(\tilde{u}) \nabla L \tilde{u} \right) \in L^\infty((0, \infty); H^{-1}(\Omega)).$$

Indeed, from (3.6), the definition of $L \tilde{u}$ and the equivalence norm (see (2.18)), we have for almost all $t > 0$

$$\int \Omega |\nabla L(\tilde{u}(t,x))|^2 dx = \int \Omega |(-\Delta N)^{1/2} L(\tilde{u}(t,x))|^2 dx$$

$$= \sum_{k=1}^{\infty} \lambda_k \left( \frac{1}{\lambda_k + \lambda_k^s} \right)^2 |\langle \tilde{u}(t), \varphi_k \rangle|^2$$

$$= \sum_{k=1}^{\infty} \lambda_k^{-1} \left( \frac{\lambda_k}{\lambda_k + \lambda_k^s} \right)^2 |\langle \tilde{u}(t), \varphi_k \rangle|^2$$

$$\leq \lambda_1^{-1} \sum_{k=1}^{\infty} |\langle \tilde{u}(t), \varphi_k \rangle|^2 \leq \lambda_1^{-1} \int |\tilde{u}(t,x)|^2 dx$$

$$\leq \lambda_1^{-1} |\Omega| \|\tilde{u}\|_{L^\infty}^2,$$

where $|\Omega|$ is the Lebesgue measure of $\Omega$ and $\lambda_1$ is the first eigenvalue of (2.1). Therefore, we obtain $g(\tilde{u}) \nabla L \tilde{u} \in L^\infty((0, \infty); L^2(\Omega))$, from which follows the claim.

Finally, applying a standard parabolic theory (see Lions, Magenes [16]), there exists a unique weak solution $u \varepsilon \in L^2((0, \infty); H^1(\Omega)) \cap C([0, \infty); L^2(\Omega))$ of (3.1), and thus the proof is complete. \[\square\]

From the proof of the previous lemma, one observes that the fractional parabolic-elliptic system (3.1) can be decoupled. Therefore, equivalently to
show existence and uniqueness of solution \((u_\varepsilon, c_\varepsilon)\) for (3.1), we study the following initial-boundary Neumann value problem

\[
\begin{aligned}
\frac{\partial u_\varepsilon}{\partial t} - \varepsilon \Delta u_\varepsilon &= -\text{div} \left( g(u_\varepsilon) \nabla L(u_\varepsilon) \right), \quad \text{in } \Omega_T, \\
\nabla u_\varepsilon \cdot \nu &= 0, \quad \text{on } \Gamma, \\
u_\varepsilon(0) &= u_{0,\varepsilon}, \quad \text{in } \Omega.
\end{aligned}
\] (3.8)

### 3.1 Local existence of solution

For convenience, let us denote for any \(T > 0\),

\[
W(T) = L^2((0, T); H^1(\Omega)) \cap C([0, T); L^2(\Omega)),
\]

and also \(\Omega_T = (0, T) \times \Omega\). Then, we consider the following

**Theorem 3.1 (Local existence).** Given \(u_{0,\varepsilon}\) satisfying (3.2), then there exists a positive time \(T = T(u_0)\), such that the problem (3.8) admits a unique weak solution

\[
u_\varepsilon \in L^\infty(\Omega_T) \cap W(T).
\]

**Proof.** 1. Hereupon, we denote by \(B^T_R\) the following set

\[
B^T_R := \{ \tilde{u} \in L^\infty(\Omega_T) : ||\tilde{u}||_{L^\infty(\Omega_T)} \leq R \},
\]

where \(T > 0\) and \(R > 0\) are chosen a posteriori. Also we define the mapping \(T : B^T_R \to W(T), \tilde{u} \mapsto u_\varepsilon = T(\tilde{u})\) where \(u_\varepsilon\) is the unique solution of (3.7) (for each \(\varepsilon > 0\) fixed), that is to say, for each \(\tilde{u} \in L^\infty(\Omega_T)\) and \(t \in [0, T)\), we may write

\[
u_\varepsilon(t, x) \equiv T(\tilde{u})(t, x) = \int_\Omega K(t, x, y)u_{0,\varepsilon}(y) \, dy \\
+ \int_0^t \int_\Omega g(\tilde{u}(\tau, y)) \nabla_y K(t - \tau, x, y) \cdot \nabla L\tilde{u}(\tau, y) \, dy \, d\tau,
\] (3.9)

where \(K(t, x, y)\) is the heat kernel of \((-\Delta)\) with Neumann boundary condition, namely (see [7], Theorem 2.1.4)

\[
K(t, x, y) = \sum_{k=0}^\infty e^{-t\varepsilon\lambda_k} \varphi_k(x) \varphi_k(y).
\]

2. Now, we show that \(T(\tilde{u}) \in B^T_R\) for each \(\tilde{u} \in B^T_R\). Indeed, from (3.9) and
applying Hölder inequality, we obtain
\[ |u_\varepsilon(t, x)| \leq C \| u_0 \|_{L^\infty(\Omega)} + C_1 \int_0^t (t - \tau)^{-1/2} \| g(\tilde{u}(\tau)) \|_{L^\infty(\Omega)} \| \nabla L \tilde{u}(\tau) \|_{L^\infty(\Omega)} \, d\tau \]
\[ \leq C \| u_0 \|_{L^\infty(\Omega)} + C_1 \int_0^t (t - \tau)^{-1/2} \| g(\tilde{u}(\tau)) \|_{L^\infty(\Omega)} \| \nabla L \tilde{u}(\tau) \|_{L^\infty(\Omega)} \, d\tau, \]
where \( C, C_1 \) are positive constants and we have used estimates of heat kernels with Neumann boundary conditions, (see [25], Lemma 3.3).

To follow, from (3.6) let \( G_s(x, y) \) be the kernel of \( L \), that is to say, for each \( s \in (0, 1) \), we define
\[ G_s(x, y) := \sum_{k=1}^{\infty} \frac{1}{\lambda_k + \lambda_k^s} \varphi_k(x) \varphi_k(y), \quad (x, y \in \Omega). \]
Consequently, \( \nabla L \tilde{u} \) could be written as follows
\[ \nabla L \tilde{u}(t, x) = \int_\Omega \nabla G_s(x, y) \tilde{u}(t, y) \, dy, \]
for almost everywhere \( t \in (0, T), x \in \Omega \). Therefore, applying Theorem 6.4 in [1] (see also [2], Proposition 5.2), we have
\[ \| \nabla L \tilde{u}(t) \|_{L^\infty(\Omega)} \leq C_2 \| \tilde{u}(t) \|_{L^\infty(\Omega)}, \]
where \( C_2 \) is a positive constant. From the above estimate and (3.10), we obtain
\[ |u_\varepsilon(t, x)| \leq C \| u_0 \|_{L^\infty(\Omega)} + C_1 C_2 \int_0^t (t - \tau)^{-1/2} \| g(\tilde{u}(\tau)) \|_{L^\infty(\Omega)} \| \tilde{u}(\tau) \|_{L^\infty(\Omega)} \, d\tau \]
\[ \leq C \| u_0 \|_{L^\infty(\Omega)} + C_1 C_2 R (R + R^2) \frac{1}{2} T^{1/2}. \]

Therefore, taking \( R = \frac{6}{5} C \| u_0 \|_{L^\infty(\Omega)} \) and \( T = \frac{1}{(12 C_1 C_2 (R + R^2)^2)} \), it follows that \( \mathcal{T}(\tilde{u}) \in B^T_R \).

3. Finally, we prove that \( \mathcal{T} \) is a contraction on \( B^T_R \). Indeed, we consider \( \tilde{u}_i \in B^T_R \), \( (i = 1, 2) \), and similarly to item (2) we have
\[ |\mathcal{T}(\tilde{u}_1)(t, x) - \mathcal{T}(\tilde{u}_2)(t, x)| \]
\[ \leq \int_0^t \| \nabla K(t - \tau, x, \cdot) \|_{L^1(\Omega)} \| g(\tilde{u}_1) \nabla L(\tilde{u}_1)(\tau) - g(\tilde{u}_2) \nabla L(\tilde{u}_2)(\tau) \|_{L^\infty(\Omega)} \, d\tau \]
\[ \leq 3C_1 C_2 (1 + R) R T^{1/2} \| \tilde{u}_1 - \tilde{u}_2 \|_{L^\infty(\Omega)} \leq \frac{1}{4} \| \tilde{u}_1 - \tilde{u}_2 \|_{L^\infty(\Omega)}. \]
Therefore, the mapping $T$ is a contraction, and hence we can apply the Banach Fixed Point Theorem. Thus $T$ has a fixed point, which is by construction the unique solution of (3.8).

**Remark 3.1.** Let $T_M$ be the maximal time of existence of the solution $u_\varepsilon$ for the problem (3.8). If $T_M < \infty$, then there exists an increase sequence $\{t_j\}_{j=1}^\infty$, such that, $t_j \to T_M^-$ as $j \to \infty$ and

$$\lim_{j \to \infty} \|u_\varepsilon(t_j, \cdot)\|_{L^\infty(\Omega)} = +\infty.$$ 

### 3.2 Global existence of solution

The main issue of this section is to show that, under the condition (1.2) for the initial data $u_0$, we obtain (by contradiction) global in time existence of solution of the problem (3.1). To this end, we show first the uniform boundedness of $(u_\varepsilon, c_\varepsilon)$.

**Proposition 3.1.** Let $u_0 \in L^\infty(\Omega)$ be satisfying (1.2), and consider

$$(u_\varepsilon, c_\varepsilon) \in W(T_M) \cap L^\infty(\Omega_T \cap L^\infty((0,T_M); D((-\Delta_N)^{1-s}))$$

the unique weak solution of (3.1). Then, it satisfies

$$0 \leq u_\varepsilon(t,x) \leq 1, \quad a.e. \text{ in } [0,T_M) \times \Omega,$$  

(3.11)

$$0 \leq c_\varepsilon(t,x) \leq 1, \quad a.e. \text{ in } [0,T_M) \times \Omega.$$  

(3.12)

**Proof.**

1. First, we observe that, div $(g(u_\varepsilon)\nabla u_\varepsilon) \in L^2((0,T_M); L^2(\Omega))$. Therefore, from equation (3.8) and standard parabolic regularity theory, we obtain

$$u_\varepsilon \in C \left([0,T_M); H^1(\Omega) \right) \cap L^2 \left((0,T_M); H^2(\Omega) \right) \quad \text{and} \quad \partial_t u_\varepsilon \in L^2(\Omega_{T_M}).$$

Consequently, the pair $(u_\varepsilon, c_\varepsilon)$ satisfies the partial differential equations in (3.1) in the strong sense, that is, for almost all $(t,x) \in \Omega_{T_M}$.

2. To show (3.15), we multiply the first equation in (3.1) by $\varphi_\delta'(u_\varepsilon)$, where for $\delta > 0$

$$\varphi_\delta(z) = \begin{cases} 
((z-1)^2 + \delta^2)^{1/2} - \delta, & \text{for } z \geq 1, \\
0, & \text{for } z \leq 1.
\end{cases}$$

Then, we obtain for each $t \in (0,T_M)$

$$\int_\Omega \varphi_\delta(u_\varepsilon(t,x))dx - \int_{\Omega_t} \varphi_\delta'(u_\varepsilon(t,x)) g(u_\varepsilon(t,x)) \nabla K_s c_\varepsilon(\tau,x) \cdot \nabla u_\varepsilon(\tau,x) dxd\tau$$

$$+ \varepsilon \int_{\Omega_t} |\nabla u_\varepsilon(\tau,x)|^2 \varphi_\delta''(u_\varepsilon(\tau,x))dxd\tau = 0,$$
where we have used that $0 \leq u_0 \leq 1$, and the boundary condition in (3.1). On the other hand, one observes that

$$\int_\Omega \varphi_\delta(u_\varepsilon(t)) dx \leq \delta \int_\Omega u_\varepsilon^2 |\nabla c_\varepsilon|^2 dx dt,$$

and passing to the limit as $\delta \to 0^+$, we have

$$\int_\Omega |u_\varepsilon(t, x) - 1|^+ dx \leq 0.$$

Thus for a.e. $(t, x) \in \Omega_{T_M}$, $|u_\varepsilon(t, x) - 1|^+ = 0$ and similarly we show that $|u_\varepsilon(t, x)|^- = 0$, from which follows (3.15).

3. Now, we recall that $\Omega \subset \mathbb{R}^n$ (with $n = 1, 2, 3$), then by the regularity of $c_\varepsilon$ together with Morrey’s Inequality, we have that $c_\varepsilon \in L^\infty(0, T_M; C(\bar{\Omega}))$, thus the infimum is finite, where the infimum is taking over $(0, T_M) \times \bar{\Omega}$.

On the other hand, without loss of generality, let $(t_0, x_0) \in (0, T_M) \times \Omega$ be a point where $c_\varepsilon(t, \cdot)$ is a minimum, which is to say

$$c_\varepsilon(t, x) \geq c_\varepsilon(t_0, x_0) \quad \text{for each} \quad (t, x) \in (0, T_M) \times \bar{\Omega}.$$

We claim that $c_\varepsilon(t_0, x_0) \geq 0$, which implies that $c_\varepsilon$ is non-negative. Indeed, suppose that $c_\varepsilon(t_0, x_0) < 0$, and evaluating $(t_0, x_0)$ in the second equation [3.1], we obtain

$$(-\Delta_N)^{1-s} c_\varepsilon(t_0, x_0) + c_\varepsilon(t_0, x_0) = u_\varepsilon(t_0, x_0). \quad (3.13)$$

Now, we recall that

$$(-\Delta_N)^{1-s} c_\varepsilon(t_0, x_0) = \frac{1}{\Gamma(s - 1)} \int_0^\infty \left( e^{t\Delta_N} c_\varepsilon(t_0, x_0) - c_\varepsilon(t_0, x_0) \right) \frac{dt}{t^{2-s}}, \quad (3.14)$$

where $\Gamma(s - 1) < 0$ ($s < 1$) and $v(t, x) = e^{t\Delta_N} c_\varepsilon(t_0, x)$ is the weak solution of the IBVP

$$\begin{cases}
\partial_t v - \Delta v = 0, & \text{in} \ (0, \infty) \times \Omega, \\
\nabla v \cdot \nu = 0, & \text{on} \ [0, \infty) \times \Gamma, \\
v(0, x) = c_\varepsilon(t_0, x), & \text{in} \ \Omega.
\end{cases}$$

Hence applying the (weak) maximum principle (see [17], Theorem 7), we get that the minimum occur on the parabolic boundary of \((0, \infty) \times \Omega\), which comprises \(\{0\} \times \Omega\) and \([0, \infty) \times \Gamma\). Moreover, the minimum of \(v\) could not occur on \([0, \infty) \times \Gamma\). Consequently, we obtain

\[
e^{t \Delta} c_e(t_0, x) \geq c_e(t_0, x_0),
\]

for all \((t, x) \in (0, \infty) \times \Omega\). Therefore from (3.14) we deduce that,

\[
(-\Delta_N)^{1-s} c_e(t_0, x_0) \leq 0,
\]

this together with (3.13) and \(c_e(t_0, x_0) < 0\), implies \(u_\varepsilon(t_0, x_0) < 0\), which is a contradiction, hence \(0 \leq c_e\). By an analogous argument, we obtain that \(c_e(t, x) \leq 1\), which finish the proof.

**Theorem 3.2** (Global Existence). Given \(u_0 \in L^\infty(\Omega)\) satisfying (1.2), then there exists a unique solution \((u_\varepsilon, c_\varepsilon)\) of the problem (3.1), and it satisfies the following uniform bounds

\[
0 \leq u_\varepsilon(t, x) \leq 1, \quad a.e. \text{ in } [0, \infty) \times \Omega,
\]

\[
0 \leq c_\varepsilon(t, x) \leq 1, \quad a.e. \text{ in } [0, \infty) \times \Omega.
\]

**Claim:** The maximal existence time \(T_M = +\infty\).

**Proof of Claim:** Indeed, let us suppose that \(T_M < +\infty\). Therefore, applying Remark 3.1 there exists \(t_j \to T_M^+\) as \(j \to \infty\) such that

\[
\lim_{j \to \infty} \|u_\varepsilon(t_j, \cdot)\|_{L^\infty(\Omega)} = +\infty,
\]

which is a contradiction, thus \(T_M = +\infty\).

### 3.3 Perturbed problem estimates

The aim of this section is to investigate some important properties of the global solution \((u_\varepsilon, c_\varepsilon)\) of the problem (3.1) as given by Theorem 3.2. Henceforth, we consider that \((u_\varepsilon, c_\varepsilon)\) satisfies the partial differential equations of the problem (3.1) in the strong sense, (see item 1 in the proof of Proposition 3.1).

**Lemma 3.2.** Let \((u_\varepsilon, c_\varepsilon)\) be the unique solution of the problem (3.1). Then for any entropy \(\eta \in C^2\),

\[
\frac{\partial}{\partial t} \eta(u_\varepsilon) + \text{div} (q(u_\varepsilon) \nabla K_s c_\varepsilon) + (u_\varepsilon - c_\varepsilon) [g - g\eta'](u_\varepsilon) + \varepsilon(-\Delta)\eta(u_\varepsilon) = -\varepsilon\eta''|\nabla u_\varepsilon|^2 \leq 0.
\]
Proof. First, multiplying the first equation in (3.1) by \( \eta'(u_\varepsilon) \) we obtain
\[
\partial_t \eta(u_\varepsilon) + \varepsilon(-\Delta) \eta(u_\varepsilon) + \eta'(u_\varepsilon) \text{div}(g(u_\varepsilon) \nabla K_{c_\varepsilon}) = -\varepsilon \eta''|\nabla u_\varepsilon|^2. \tag{3.18}
\]

On the other hand, one observes that
\[
\text{div}(q(u_\varepsilon) \nabla K_{s_\varepsilon} u_\varepsilon) = q'(u_\varepsilon) \nabla u_\varepsilon \cdot \nabla K_{s_\varepsilon} - q(u_\varepsilon)(-\Delta)^{1-s}c_\varepsilon
\]
where we have used \( q'(u) = \eta'(u)g'(u) \). Then from the second equation in (3.1),
\[
\text{div}(q(u_\varepsilon) \nabla K_{s_\varepsilon} u_\varepsilon) = \eta'(u_\varepsilon) \text{div}(g(u_\varepsilon) \nabla K_{s_\varepsilon}) + (\eta'(u_\varepsilon)g(u_\varepsilon) - q(u_\varepsilon))(-\Delta)^{1-s}c_\varepsilon
\]
which substituting in (3.18) concludes the proof.

**Lemma 3.3** (Mass conservation). For almost all \( t > 0 \),
\[
\int_\Omega c_\varepsilon(t, x) dx = \int_\Omega u_\varepsilon(t, x) dx = \int_\Omega u_{0, \varepsilon}(x) dx.
\]

**Proof.** It is enough to integrate (3.1) (first and second equations) over \( \Omega \).

**Proposition 3.2.** For any \( T > 0 \),
\[
\iint_{\Omega_T} |\sqrt{\varepsilon} \nabla u_\varepsilon(t, x)|^2 dx dt \leq C, \tag{3.19}
\]
where \( C = C(T, |\Omega|, \|u_0\|_{L^\infty}) \) is a positive constant.

**Proof.** Let us multiply (3.1) (first equation) by \( u_\varepsilon \) and integrating by parts over \( \Omega_T \), we obtain
\[
\frac{1}{2} \int_\Omega u_\varepsilon(T)^2 dx + \varepsilon \int_{\Omega_T} |\nabla u_\varepsilon(t, x)|^2 dx dt
= \frac{1}{2} \int_\Omega u_{0, \varepsilon}(x)^2 dx + \int_{\Omega_T} g(u_\varepsilon) \nabla K_{s_\varepsilon} \cdot \nabla u_\varepsilon(t, x) dx dt. \tag{3.20}
\]

Now, multiplying (3.1) (second equation) by \( \frac{u_\varepsilon^2}{2} - \frac{u_\varepsilon^3}{3} \), and integrating by parts we have
\[
\int_\Omega g(u_\varepsilon) \nabla K_{s_\varepsilon} \cdot \nabla u_\varepsilon(t, x) dx = \int_\Omega ((u_\varepsilon - u_\varepsilon^2) \nabla K_{s_\varepsilon} \cdot \nabla u_\varepsilon)(t, x) dx
= \int_\Omega \left( \frac{u_\varepsilon^3}{2} - \frac{u_\varepsilon^4}{3} - \frac{c_\varepsilon u_\varepsilon^2}{2} + \frac{c_\varepsilon u_\varepsilon^3}{3} \right)(t, x) dx.
\]
This, together with (3.20) implies that
\[
\varepsilon \int \int_{\Omega_T} |\nabla u_{\varepsilon}(t,x)|^2 \, dx \, dt \leq \frac{1}{2} \int_{\Omega} u_{0,\varepsilon}(x)^2 \, dx + \int \int_{\Omega_T} \left( \frac{u_{\varepsilon}^3}{2} + \frac{u_{\varepsilon}^4}{3} - \frac{c_{\varepsilon} u_{\varepsilon}^2}{2} + \frac{c_{\varepsilon} u_{\varepsilon}^3}{3} \right)(t,x) \, dx \, dt,
\]
which proves the assertion. \( \square \)

**Proposition 3.3.** For each \( T > 0 \) and \( s \in (0,1) \), there exist positive constants \( C_1 = C_1(|\Omega|) \), \( C_2 = C_2(T,|\Omega|) \), such that

\[
\|c_{\varepsilon}\|_{L^\infty((0,\infty); D((-\Delta_N)^{1-s}))} \leq C_1, \tag{3.21}
\]

\[
\|\partial_t c_{\varepsilon}\|_{L^2((0,T); ((-\Delta_N)^{1-s}))^\ast (H^1(\Omega))}} \leq C_2. \tag{3.22}
\]

**Proof.** 1. In order to show the first estimate, we recall from (3.4) that
\[
c_{\varepsilon} = (I_d + (-\Delta_N)^{1-s})^{-1} u_{\varepsilon} = \sum_{k=0}^{\infty} \frac{1}{1 + \lambda_{k}^{-s}} \langle u_{\varepsilon}, \varphi_k \rangle \varphi_k,
\]
hence we obtain \( c_{\varepsilon} \in L^\infty((0,\infty); D((-\Delta_N)^{1-s})) \). Moreover, for almost all \( t > 0 \)
\[
\|(-\Delta_N)^{1-s}c_{\varepsilon}(t)\|_{L^2(\Omega)}^2 = \sum_{k=0}^{\infty} \frac{\lambda_{k}^{2(1-s)}}{(1 + \lambda_{k}^{-s})^2} |\langle u_{\varepsilon}(t), \varphi_k \rangle|^2 \leq \sum_{k=0}^{\infty} |\langle u_{\varepsilon}(t), \varphi_k \rangle|^2 < \infty.
\]

2. To show (3.22), we first observe that from (3.4), we have for almost all \( t > 0 \)
\[
\partial_t c_{\varepsilon}(t) = (I + (-\Delta_N)^{1-s})^{-1} \partial_t u_{\varepsilon}(t)
\]
\[
= (I + (-\Delta_N)^{1-s})^{-1} \left( -\text{div}(g(u_{\varepsilon}(t))\nabla K \xi c_{\varepsilon}(t)) + \varepsilon \Delta u_{\varepsilon}(t) \right)
\]
in distributional sense.

On the other hand, from Proposition 3.2 together with (3.21), we have that
\[
-\text{div}(g(u_{\varepsilon}(t))\nabla K \xi c_{\varepsilon}) + \varepsilon \Delta u_{\varepsilon}
\]
is uniformly bounded (with respect to \( \varepsilon > 0 \)) in \( L^2((0,T); (H^1(\Omega))^\ast) \). One recalls that \( (H^1(\Omega))^\ast = (-\Delta_N)(H^1(\Omega)) \). Finally, applying Proposition 2.1 we get the result. \( \square \)
4 Proof of Main Theorem

In this section we prove Theorem 1.1, and to this end we are mostly concerned to pass to the limit in (3.17) as $\varepsilon \to 0$. More precisely, we write (3.17) in weak sense (using the entropy pair $F(u,v)$), and jointly with (3.1) (second equation) we obtain (1.8), (1.9) respectively, after pass to the limit as $\varepsilon \to 0$. Since (3.17) has non-linear terms, the uniform estimates on the sequence ${u_\varepsilon}$ are not sufficient to take the limit transition on $\varepsilon$ as it goes to 0. In fact, we need strong convergence, and as usual for scalar conservation laws we apply (following closer Chemetov, Neves [6] and also Perthame, Dalibard [19]) the Kinetic Theory.

First, we take in Lemma 3.2, $\eta(u) = |u-k|^+$, $q(u) = \text{sgn}^+(u-k)(g(u) - g(k))$ with $k \in \mathbb{R}$. Then, we obtain from equation (3.17) and system (3.1), for each test function $\phi \in C_0^\infty((-\infty,T) \times \mathbb{R}^n)$, (for simplicity of exposition and abuse of notation we may have $T = \infty$),

$$
\int\int_{\Omega_T} \left( \eta(u_\varepsilon) \phi_t + q(u_\varepsilon) \nabla K_s c_\varepsilon \cdot \nabla \phi - \varepsilon \nabla \eta(u_\varepsilon) \cdot \nabla \phi \right) dxdt \\
\quad + \int_{\Omega} \eta(u_{0,\varepsilon}) \phi(0) dx + \int\int_{\Omega_T} (u_\varepsilon - c_\varepsilon) \eta'(u_\varepsilon) g(k) \phi dxdt = m^+_\varepsilon(\phi),
$$

where we have used $[q - g\eta'](u_\varepsilon) = -\eta'(u_\varepsilon) g(k)$, and $m^+_\varepsilon$ is a real non-negative Radon measure, defined by

$$
m^+_\varepsilon(\phi) := \int\int_{\Omega_T} \varepsilon \eta''(u_\varepsilon) |\nabla u_\varepsilon|^2 \phi dxdt.
$$

Now, we differentiate in the distributional sense equation (3.17) with respect to $k$, hence we obtain (as now a standard procedure in the kinetic theory) the following transport like equation

$$
\frac{\partial f_\varepsilon}{\partial t} + (k - c_\varepsilon) g(k) \frac{\partial f_\varepsilon}{\partial k} + g'(k) \nabla K_s c_\varepsilon \cdot \nabla f_\varepsilon + \varepsilon(-\Delta)f_\varepsilon = \partial_k m^+_\varepsilon,
$$

where $f_\varepsilon(t,x,k) := \text{sgn}^+(u_\varepsilon(t,x) - k)$. Rigorously, from (4.1) we get that the function $f_\varepsilon(t,x,k)$ satisfies the following equation

$$
\int\int_{\Omega_T} \left\{ \int_k^1 f_\varepsilon(t,x,v)[\phi_t + g'(v)\nabla K_s c_\varepsilon \cdot \nabla \phi] dv - \varepsilon \nabla \eta(u_\varepsilon) \cdot \nabla \phi \right\} dxdt \\
\quad + \int_{\Omega} |u_{0,\varepsilon} - k|^+ \phi(0) dx \\
\quad + \int\int_{\Omega_T} (u_\varepsilon - c_\varepsilon) g(k) f_\varepsilon \phi dxdt = m^+_\varepsilon(\phi) \geq 0,
$$
for all nonnegative function $\phi \in C^\infty_0 ((-\infty, T) \times \mathbb{R}^n)$. Furthermore, it follows that for any function $G \in C^1([0, 1])$, with $G(0) = 0$,

$$G(u_\varepsilon) = \int_0^1 G'(v)f_\varepsilon(\cdot, \cdot, v) \, dv \quad \text{a.e. in } \Omega_T,$$

$$0 \leq f_\varepsilon \leq 1 \quad \text{on } \Omega_T \times \mathbb{R}, \quad f_\varepsilon(t, x, k) = \begin{cases} 1, & \text{for } k \leq 0, \\ 0, & \text{for } k \geq 1, \end{cases}$$

$$\partial_k f_\varepsilon \leq 0 \quad \text{in } D'((\Omega_T \times \mathbb{R})).$$

(4.5)

From (3.15), (3.19), (4.2) and the Riesz Representation Theorem, we get that $m^+\varepsilon$ is a real positive Radon measure, defined on $\Omega_T \times \mathbb{R}$, and

$$\int_{\Omega_T \times \mathbb{R}} \phi \, dm^+\varepsilon \leq C \quad \text{continuously on } \Omega_T \times \mathbb{R},$$

(4.6)

where $C$ is a positive constant independent of $\varepsilon$.

Similarly to derive the inequality (4.4), we now consider $\eta(u) = |u - k|^{-}$, and $q(u) = \text{sgn}^-(u - k)(g(u) - g(k))$. Then, we obtain

$$\frac{\partial}{\partial t} (1 - f_\varepsilon) + (k - c_\varepsilon) g(k) \frac{\partial}{\partial k} (1 - f_\varepsilon) + g'(k)\nabla K_s c_\varepsilon \cdot \nabla (1 - f_\varepsilon)$$

$$+ \varepsilon(\Delta)(1 - f_\varepsilon) = -\partial_k m^-\varepsilon$$

(4.7)

in the distribution sense, and for all nonnegative function $\phi \in C^\infty_0 ((-\infty, T) \times \mathbb{R}^n)$, we have the following identity

$$\int_{\Omega_T} \int_0^k \left\{ (1 - f_\varepsilon(t, x, v)) [\phi_t + g'(v)\nabla K_s c_\varepsilon \cdot \nabla \phi] dv - \varepsilon \nabla \eta(u_\varepsilon) \cdot \nabla \phi \right\} dx dt$$

$$+ \int_{\Omega} |u_0,\varepsilon - k|^ {-} \phi(0) \, dx$$

$$+ \int_{\Omega_T} (u_\varepsilon - c_\varepsilon) g(k)(1 - f_\varepsilon) \phi \, dx dt = m^-\varepsilon(\phi) \geq 0,$$

(4.8)

where $m^-\varepsilon$ is defined in the same way by (4.2). Moreover, the real positive Radon measure $m^-\varepsilon$, defined on $\Omega_T \times \mathbb{R}$, satisfies the following properties

$$m^-\varepsilon(\cdot, \cdot, k) = 0 \quad \text{for any } k < 0 \text{ on } \Omega_T, \text{ and for any } |\phi| \leq 1$$

$$\int_{\Omega_T \times \mathbb{R}} \phi \, dm^-\varepsilon \leq C \quad \text{continuously on } \Omega_T \times \mathbb{R}.$$
that (passing to a subsequence)

\[ c_\varepsilon \rightharpoonup c \quad \text{weakly} - \star \text{ in } L^\infty_{\text{loc}}((0, \infty); L^\infty(\Omega)). \] (4.10)

Although, we need more than convergence in averages for the sequence \( \{c_\varepsilon\} \).

Then, we have the following

**Proposition 4.1.** Given \( s \in (0, 1) \), there exists \( c \in L^\infty((0, \infty); D((-\Delta_N)^{1-s})) \), such that

\[ c_\varepsilon \rightarrow c \quad \text{strongly in } L^2_{\text{loc}}((0, \infty); D((-\Delta_N)^{1-s}/2)). \] (4.11)

**Proof.** First, from the uniform estimate (3.21), and passing to a convenient subsequence, there exists a function \( c \in L^\infty((0, \infty); D((-\Delta_N)^{1-s})) \), such that

\[ c_\varepsilon \rightharpoonup c \quad \text{weakly in } L^2_{\text{loc}}((0, \infty); D((-\Delta_N)^{1-s})). \]

Now, due to (3.22) \( \partial_t c_\varepsilon \) is uniform bounded in \( L^2_{\text{loc}}((0, \infty); (-\Delta_N)^s(H^1(\Omega))) \).

Finally, from definitions (2.9), (2.14) and applying the Aubin-Lions’ Theorem, we get (4.11).

Therefore, in view of the above results (passing to subsequences still denoted by \( \varepsilon \)), there exist functions \( u \in L^\infty((0, \infty) \times \Omega), f \in L^\infty((0, \infty) \times \Omega \times \mathbb{R}), c \in L^\infty((0, \infty); D((-\Delta_N)^{1-s})), \) and non-negative measures \( m^\pm = m^\pm(t, x, k) \), such that (locally in time),

\[
\begin{align*}
    u_\varepsilon & \rightharpoonup u, \quad \text{weakly} - \star \text{ in } L^\infty, \\
    f_\varepsilon & \rightharpoonup f, \quad \text{weakly} - \star \text{ in } L^\infty, \\
    m^\pm_\varepsilon & \rightarrow m^\pm, \quad \text{weakly in } \mathcal{M}^+, \\
    c_\varepsilon & \rightarrow c, \quad \text{strongly in } L^2_{\text{loc}}((0, \infty); L^2(\Omega)), \\
    \nabla K_sc_\varepsilon & \rightharpoonup \nabla K_sc, \quad \text{strongly in } L^2_{\text{loc}}((0, \infty); L^2(\Omega)).
\end{align*}
\]

The above convergences are enough to pass to the limit (as \( \varepsilon > 0 \) goes to zero) in the second equation of the system (3.1), that is to say

\[ (-\Delta_N)^{1-s}c + c = u \] (4.12)

also in equations (4.3) and (4.7), the only exception is the term \( g'(k)\nabla K_sc_\varepsilon \cdot \nabla f_\varepsilon \), which yields an extra effort. First, we can write

\[
g'(k)\nabla K_sc_\varepsilon \cdot \nabla f_\varepsilon = \text{div} \left( g'(k)f_\varepsilon \nabla K_sc_\varepsilon \right) + g'(k)(-\Delta_N)^{1-s}c_\varepsilon f_\varepsilon = \text{div} \left( g'(k)f_\varepsilon \nabla K_sc_\varepsilon \right) + (u_\varepsilon - c_\varepsilon) g'(k)f_\varepsilon.
\]

Moreover, we have in the sense of distributions as \( \varepsilon \to 0 \),

\[
\begin{align*}
    \text{div} \left( g'(k)f_\varepsilon \nabla K_sc_\varepsilon \right) & \rightarrow \text{div} \left( g'(k)f \nabla K_sc \right), \\
    c_\varepsilon g'(k)f_\varepsilon & \rightarrow c g'(k)f.
\end{align*}
\]
Although, from the moment, we cannot assert that the weak limit of \( u_\varepsilon f_\varepsilon \) is \( uf \).
However, we know that \( \{ u_\varepsilon f_\varepsilon \}_{\varepsilon > 0} \) is uniformly bounded in \( L^\infty((0, \infty) \times \Omega \times \mathbb{R}) \). Then, extracting a further subsequence (if necessary), there exists a function \( \rho = \rho(t, x, k) \in L^\infty((0, \infty) \times \Omega \times \mathbb{R}) \), such that (locally in time)
\[
\quad u_\varepsilon f_\varepsilon \rightharpoonup \rho, \quad \text{weakly}^{−\ast} \text{ in } L^\infty.
\]

**Remark 4.1.** Thanks to the definition of \( f_\varepsilon \), together with (4.13), one observes that
\[
\rho(t, x, k) = \begin{cases} 
0, & \text{when } k \geq 1, \\
\ u(t, x), & \text{when } k \leq 0,
\end{cases}
\]
almost everywhere in \( (0, \infty) \times \Omega \times \mathbb{R} \).

Consequently, we obtain in distribution sense
\[
\quad g'(k)\nabla K_s c \cdot \nabla f_\varepsilon \rightarrow g'(k)\nabla K_s c \cdot \nabla f + g'(k)(\rho - uf),
\]
where we have used (4.12). Therefore, from (4.4) and (4.8), it follows respectively that, for any nonnegative function \( \phi \in C^\infty_0((−\infty, T) \times \mathbb{R}^{n+1}) \),
\[
\int_{\Omega_T} \int_k^1 f(t, x, v)[\phi_t + g'(v)\nabla K_s c \cdot \nabla \phi] dv \, dx \, dt
\]
\[+ \int_{\Omega_T} (\rho - uf)g(k)\phi \, dx \, dt + \int_{\Omega_T} (u - c)g(k)f \phi \, dx \, dt \tag{4.14}\]
\[+ \int_{\Omega} |u_0 - k|^+ \phi(0) \, dx = m^+(\phi) =: \int_{\Omega_T} m^+(t, x, k)\phi \, dx \, dt \geq 0, \]

and
\[
\int_{\Omega_T} \int_0^k (1 - f(t, x, v))[\phi_t + g'(v)\nabla K_s c \cdot \nabla \phi] dv \, dx \, dt
\]
\[- \int_{\Omega_T} (\rho - uf)g(k)\phi \, dx \, dt + \int_{\Omega_T} (u - c)g(k)(1 - f) \phi \, dx \, dt \tag{4.15}\]
\[+ \int_{\Omega} |u_0 - k|^− \phi(0) \, dx = m^−(\phi) =: \int_{\Omega_T} m^−(t, x, k)\phi \, dx \, dt \geq 0.
\]
Moreover, for any function \( G \in C^1([0, 1]) \), with \( G(0) = 0 \), it follows that
\[
G(u) = \int_0^1 G'(v)f(\cdot, \cdot, v) \, dv \quad \text{a.e. in } \Omega_T,
\]
\[
0 \leq f \leq 1 \quad \text{on } \Omega_T \times \mathbb{R}, \quad f(t, x, k) = \begin{cases} 
1, & \text{for } k \leq 0, \\
0, & \text{for } k \geq 1,
\end{cases}
\]
\[
\partial_k f \leq 0 \quad \text{in distribution sense}, \tag{4.16}
\]
also we have
\[
\int_{\Omega_T \times \mathbb{R}} \phi \, dm^\pm \leq C \quad \text{for any } |\phi| \leq 1,
\]
where the continuity of \( m^\pm \) on \( k \) follows from (4.14), (4.15).

Finally, we take \( \phi = \partial_k \psi \) in (4.14) and (4.15), with \( \psi \) being a nonnegative function in \( C_c^\infty(\Omega_T \times \mathbb{R}) \). Then integrating by parts on \( k \), we obtain that \( f \) satisfies, respectively, the following transport equations (in distribution sense)
\[
\frac{\partial}{\partial t} f + b \cdot \nabla_{(k,x)} f + g'(k)(\rho - uf) = \partial_k m^+,
\]
and
\[
\frac{\partial}{\partial t} (1 - f) + b \cdot \nabla_{(k,x)} (1 - f) - g'(k)(\rho - uf) = -\partial_k m^-.
\]
Here \( \nabla_{(k,x)} = \left( \frac{\partial}{\partial k}, \nabla_x \right) \) and the vector field \( b : (0,T) \times \Omega \times [0,1] \to \mathbb{R} \times \mathbb{R}^n \), called drift, is given by
\[
b(t,x,k) = \left( (k - c(t,x))g(k), g'(k)\nabla K_s c(t,x) \right).
\]
Moreover, we have for \( 0 < s \leq 1/2 \)
\[
b \in L^\infty((0,\infty);H^1(\Omega \times [0,1])), \quad \text{and}
\]
\[
b \cdot \nabla_{(k,x)} f = \text{div}_{(k,x)}(bf) \quad \text{in distribution sense.}
\]

**Lemma 4.1.** Let \( b \) be the drift vector field defined by (4.20), and \( 0 < s \leq 1/2 \). Then, the function \( F = f(1 - f) \) satisfies in the sense of distributions
\[
\frac{\partial}{\partial t} F + \text{div}_{(k,x)}(bf) + R (1 - 2f) \leq 0,
\]
where \( R := g'(k)(\rho - uf) \).

**Proof.** Under the conditions of the vector field \( b \) given by (4.21), we can apply the renormalization procedure which means that, the equations (4.18) and (4.19) are regularized on a parameter \( \theta \), and respectively multiplied by \( (1 - f^\theta) \) and \( f^\theta \), \( (f^\theta \) being the regularization of \( f \). Then, the obtained equations are added and, taking the limit as \( \theta \to 0 \) we obtain (4.22), where the inequality follows from the following relation
\[
\int_{\mathbb{R}} ((1 - f^\theta)\partial_k (m^+)^\theta - f^\theta \partial_k (m^-)^\theta) \, dk = \int_{\mathbb{R}} (m^+ + m^-)^\theta \partial_k f^\theta \, dk \leq 0,
\]
in view of (4.16), (4.17). Actually, we omit the details as now it is a standard procedure in the renormalization theory for transport equations. \( \square \)
Now, let us study the trace concept of $f$ at time $t = 0$.

**Proposition 4.2.** The function $f(t,x,k)$ has the trace $f_0(x,k)$ at time $t = 0$, such that

$$f_0(x,k) := \text{ess lim}_{t \to 0} f(t,x,k) \text{ almost everywhere in } \Omega \times \mathbb{R}.$$  

Moreover, $f_0 = (f_0)^2$.

**Proof.** 1. First, let $\mathcal{E}$ be a countable dense subset of $C^1_0(\Omega)$. Then, for each $\zeta \in \mathcal{E}$ and $k \in \mathbb{Q} \cap [0,1]$, we define the following set of full measure in $(0,T)$,

$$E_{\zeta,k} := \{ t \in (0,T) / t \text{ is a Lebesgue point of } I(t) = \int_{\Omega} \int_k^1 f(t,x,v) \zeta(x) dv dx \},$$

and consider

$$E := \bigcap_{(\zeta,k)} E_{\zeta,k},$$

where the intersection is taken over $\mathcal{E} \times (\mathbb{Q} \cap [0,1])$. Also $E$ is a set of full measure in $(0,T)$.

2. To show the existence of the essential limit of $f(t,x,k)$, as $t$ goes to zero, we use the inequalities (4.14), (4.15). Indeed, we consider the test function $\phi(t,x,k) = \zeta_j(t) \psi(x)$, $\zeta_j(t) = H_j(t + t_0) - H_j(t - t_0)$ for any $t_0 \in \mathcal{E}$ (fixed), and $j \geq 1$, where $H_j$ is a standard regularization of the Heaviside function, and $\psi$ is a non-negative function which belongs to $\mathcal{E}$. Then, we have from (4.14)

$$\int_0^T \int_{\Omega} \int_k^1 f(t,x,v) \zeta_j'(t) \psi(x) dv dx dt + \int_0^T \Phi_k(t) \zeta_j(t) dt$$

$$+ \int_{\Omega} |u_0 - k|^+ \psi(x) dx \geq 0,$$

where

$$\Phi_k(t) = \int_{\Omega} \left( \int_k^1 f(t,x,v) g'(v) \nabla K_s c \cdot \nabla \psi dv ight. \left. + ((\rho - uf)g(k) + (u - c)) g(k) f(t,x,k) \psi(x) \right) dx.$$ 

Passing to the limit in the above equation as $j \to \infty$, and taking into account that $t_0$ is Lebesgue point of $I(t)$, we obtain

$$\int_{\Omega} \psi(x) \left\{ - \int_k^1 f(t_0,x,v) dv + |u_0(x) - k|^+ \right\} dx + \int_0^{t_0} \Phi_k(t) dt \geq 0, \quad (4.23)$$

where we have used the Dominated Convergence Theorem. Since $t_0 \in \mathcal{E}$ is arbitrary, and in view of the density of $\mathcal{E}$ in $L^1(\Omega)$, it follows from (4.23) that

$$\text{ess lim}_{t \to 0} \int_{\Omega} \left\{ - \int_k^1 f(t,x,v) dv + |u_0(x) - k|^+ \right\} \psi(x) dx \geq 0.$$
for all non-negative $\psi \in L^1(\Omega)$, which implies for almost everywhere $x$ in $\Omega$,
$$\text{ess lim}_{t \to 0} f(t, x, k) = 0, \quad \text{if } k > u_0(x).$$
Similarly, we obtain from (4.15)
$$\text{ess lim}_{t \to 0} \int_0^k \left\{ - \int_0^k (1 - f(t, x, v)) dv + |u_0(x) - k| \right\} \psi(x) \, dx \geq 0$$
for all non-negative $\psi \in L^1(\Omega)$, which implies for almost everywhere $x$ in $\Omega$,
$$\text{ess lim}_{t \to 0} f(t, x, k) = 1, \quad \text{if } k < u_0(x).$$
Therefore, the $\text{ess lim}_{t \to 0} f(t, x, k)$ exists, and in particular we have
$$f_0(x, k) = \text{sgn}^+(u_0(x) - k)$$
almost everywhere in $\Omega \times \mathbb{R}$, which concludes the proof. \(\square\)

One remarks that, since $f \in L^\infty((0, \infty) \times \Omega \times \mathbb{R})$, it follows that
$$\text{ess lim}_{t \to 0^+} f(t, x, k) = \lim_{\delta \to 0^+} \frac{1}{\delta} \int_0^\delta f(\tau, x, k) \, d\tau$$
(4.24)
almost everywhere in $\Omega \times \mathbb{R}$.

Before we gain the strong convergence of $u_\varepsilon$, which is obtained showing that,
$f = f^2$, which is to say, $F(t, x, k) = 0$ almost everywhere in $(0, \infty) \times \Omega \times \mathbb{R}$, it remains to study the remainder term $R$, that is
$$R(t, x, k) = g'(k)(\rho - uf)(t, x, k).$$
In fact, this study has been done in [19], and we recall here the main details with minor modifications. First, from Remark [18] we only need to obtain the formula for $\rho$, once $k \in (0, 1)$, since $R \equiv 0$ for $k \leq 0$ and $k \geq 1$. Then, considering the test functions $\varphi_1 \in \mathcal{C}_0^\infty((-\infty, T) \times \Omega)$, $\varphi_2 \in \mathcal{C}_0^\infty(0, 1)$, for any $T > 0$, we have
$$\int_{\Omega_T} \int_0^1 \rho(t, x, k) \varphi_1(t, x) \varphi_2'(k) \, dk \, dx \, dt$$
$$= \lim_{\varepsilon \to 0} \int_{\Omega_T} \int_0^1 u_\varepsilon(t, x) f_\varepsilon(t, x, k) \varphi_1(t, x) \varphi_2'(k) \, dk \, dx \, dt$$
$$= \lim_{\varepsilon \to 0} \int_{\Omega_T} u_\varepsilon(t, x) \varphi_1(t, x) \left\{ \int_0^1 \varphi_2'(k) f_\varepsilon(t, x, k) \, dk \right\} \, dx \, dt$$
$$= \lim_{\varepsilon \to 0} \int_{\Omega_T} u_\varepsilon(t, x) \varphi_2(\varphi_2(t, x)) \varphi_1(t, x) \, dx \, dt$$
$$= \lim_{\varepsilon \to 0} \int_{\Omega_T} \left\{ \int_0^1 \frac{d}{dk} (k \varphi_2(k)) f_\varepsilon(t, x, k) \, dk \right\} \varphi_1(t, x) \, dx \, dt$$
$$= \int_{\Omega_T} \left\{ \int_0^1 \frac{d}{dk} (k \varphi_2(k)) f(t, x, k) \, dk \right\} \varphi_1(t, x) \, dx \, dt,$
where we have used (4.5). Consequently,
\[-\frac{\partial}{\partial k}[\rho - kf] = f\]
and integrating this equation on \((0, 1)\), we obtain
\[
\rho(t, x, k) = kf(t, x, k) + \int_{k}^{1} f(t, x, v) dv.
\]

(4.25)

**Lemma 4.2.** Let \(\rho\) be given by (4.25). Then, for any \(T > 0\)
\[|R(t, x, k)| \leq F(t, x, k)\]
for almost everywhere \(t \in (0, T), x \in \Omega,\) and \(k \in (0, 1)\).

**Proof.** Follows from Lemma 3.1 in [19].

Now, we are ready to show the strong convergence of the family \(\{u_\varepsilon\}\), which
is the main issue to prove Theorem 1.1 (Main Theorem).

**Theorem 4.1.** We have \(F = 0\) almost everywhere in \((0, \infty) \times \Omega \times \mathbb{R}\).

**Proof.** First, let us recall that \(R\) and \(F\) are identically zero for \(k \leq 0\) and \(k \geq 1\).

Now let us define for \(\delta > 0\) sufficiently small, the function \(\zeta_\delta\) as follows
\[
\zeta_\delta(z) := \begin{cases} 
0, & \text{if } z < 0, \\
\frac{z}{\delta}, & \text{if } 0 \leq z \leq \delta, \\
1, & \text{if } z > \delta,
\end{cases}
\]
and consider \(\Psi_\delta(t, x, k) = \phi_\delta(t) \psi_\delta(x) \xi_\delta(k)\), where
\[
\phi_\delta(t) := \zeta_\delta(t) - \zeta_\delta(t - t_0 + \delta), \quad \text{for } t_0 \in (2\delta, T),
\]
\[
\psi_\delta(x) := \zeta_\delta(d(x)), \quad \text{for } x \in \Omega,
\]
\[
\xi_\delta(k) := \zeta_\delta(k + \delta^{-1}) - \zeta_\delta(k - \delta^{-1}), \quad \text{for } k \in \mathbb{R},
\]
where \(d(x) = \min_{y \in \Gamma} |x - y|\) is the distance function from \(x \in \overline{\Omega}\) to \(\Gamma\), and also \(t_0\) is a Lebesgue point of the function
\[
t \mapsto \int_{\Omega \times \mathbb{R}} F(t, x, k) \, dx \, dk.
\]

Then, choosing \(\Psi_\delta(t, x, k)\) as a test function in the respective integral form of
(4.22), we get the inequality
\[
-\int_{\Omega_T} \int_{[0,1]} F \left( \partial_t \Psi_\delta(t, x, k) + b \cdot \nabla_{(x,k)} \Psi_\delta(t, x, k) \right) \, dx \, dk \, dt
\]
\[
+ \int_{\Omega_T} \int_{[0,1]} R(1 - 2f) \Psi_\delta(t, x, k) \, dx \, dk \, dt \leq 0.
\]
From the definition of the drift vector field $b$, i.e. equation 4.20, it follows that

$$\frac{1}{\delta} \int_{t_0-\delta}^{t_0} \int_{\Omega \times [0,1]} F \psi_d(x) \, dk \, dx \, dt \leq \frac{1}{\delta} \int_{0}^{\delta} \int_{\Omega \times [0,1]} F \psi_d(x) \, dk \, dx \, dt$$

$$+ \frac{1}{\delta} \int_{\{0 \leq d(x) \leq \delta\}} \int_{[0,T] \times [0,1]} F \phi_\delta(t) \, g'(k) \nabla K_{sc} \cdot \nabla d(x) \, dt \, dk \, dx$$

$$+ \int_{\Omega_T} \int_{[0,1]} F \phi_\delta(t) \, \psi_d(x) \, dk \, dx \, dt$$

$$= I^\delta_1 + I^\delta_2 + \int_{\Omega_T} \int_{[0,1]} F \phi_\delta(t) \, \psi_d(x) \, dk \, dx \, dt,$$

with the obvious notation and we have used Lemma 4.2.

Now, due to Proposition 4.2 and applying the Dominated Convergence Theorem, we have

$$\limsup_{\delta \to 0} I^\delta_1 \leq \int_{\Omega \times [0,1]} (f_0 - (f_0)^2) \, dk \, dx = 0. \quad (4.28)$$

For the term $I^\delta_2$ one observes that, since $\Gamma$ is a $C^2$–boundary, there exists a sufficiently small $\delta > 0$ such that, each point $x \in \Omega_\delta := \{x \in \Omega : d(x) < \delta\}$ has a unique projection $x_\delta = x_\delta(x)$ on the boundary $\Gamma$. For every $x \in \Omega_\delta$, we have

$$\nabla d(x) = -\nu(x_\delta) + O(\delta)$$

and the Jacobian of the change of variables

$$\Omega_\delta \ni x \leftrightarrow (x_\delta, \tau) \in \Gamma \times (0, \delta)$$

is equal to $\frac{D(x)}{D(x_\delta, \tau)} = 1 + O(\delta)$, where $\tau = d(x)$. Therefore, we obtain

$$I^\delta_2 \leq \int_{0}^{T} \int_{\Gamma} \int_{0}^{1} F^\delta \phi_\delta(t) \, |\nabla K_{sc} \cdot \nu| \, dk \, dx_\delta \, dt + O(\delta)$$

$$= O(\delta), \quad (4.29)$$

where we have used that $\nabla K_{sc} \in L^2((0,T); H^1(\Omega))$, $\nabla K_{sc} \cdot \nu = 0$ on $\Gamma$,

$$F^\delta := \frac{1}{\delta} \int_{0}^{\delta} F(\cdot, (x_\delta, \tau), \cdot) \, d\tau,$$

and $(x_\delta, \tau)$ forms an orthogonal coordinate system in a neighborhood of $\tau = 0$.

Finally, from (4.28) and (4.29) we get passing to the limit in (4.27) as $\delta \to 0$,

$$\int_{\Omega \times [0,1]} F(t, x, k) \, dk \, dx \leq \int_{0}^{T} \int_{\Omega \times [0,1]} F(t', x, k) \, dk \, dx \, dt' \quad (4.30)$$
for almost everywhere \( t \in [0, T] \). Therefore, applying the Gronwall’s Lemma, we obtain from (4.30)

\[
\int_{[0,T]} \int_{\Omega} F(t, x, k) \, dk \, dx \, dt \leq 0,
\]

which implies the result.

The above theorem implies that, the kinetic function \( f \) takes only the values 0 and 1 almost everywhere in \((0, \infty) \times \Omega \times \mathbb{R} \), and since \( f \) is monotone decreasing on \( k \), there exists a function \( w = w(t, x) \), such that

\[
f(t, x, k) = \text{sgn}^+(w(t, x) - k).
\]

Therefore for any \( G \in C^1([0, 1]) \), with \( G(0) = 0 \), it follows that

\[
G(u_\varepsilon) = \int_0^1 G'(v)f_\varepsilon(\cdot, \cdot, v) \, dv \rightarrow \int_0^1 G'(v)f(\cdot, \cdot, v) \, dv = G(w)
\]

weakly star in \( L^\infty(\Omega_T) \), which implies \( w = u \) almost everywhere, and the strong convergence of the family \( \{u_\varepsilon\} \) to \( u \) in \( L^p(\Omega_T) \) for any \( p < \infty \). Then, we write (3.17) in weak sense using the entropy pair \( F(u, v) \), and jointly with the second equation in (3.1), also written in the weak sense, we derive passing to the limit as \( \varepsilon \to 0 \) that, the pair \( (u, c) \) satisfies (1.8) and (1.9), which ends the proof of Theorem 1.1.

5 Comments and Extensions

One remarks that, the results established in this paper apply to some interesting correlated versions of the system (1.1).

1. Let us consider for \( s \in (0, 1) \) the following system

\[
\begin{aligned}
\partial_t u + \text{div} \left( g(u) \nabla K_s c \right) &= 0, \quad \text{in} \ (0, \infty) \times \Omega, \\
(-\Delta_N + I_d)^{1-s} c &= u, \quad \text{in} \ \Omega, \\
u|_{t=0} = u_0, & \quad \text{in} \ \Omega, \\
\nabla K_s c \cdot \nu &= 0, \quad \text{on} \ \Gamma,
\end{aligned}
\]

(5.1)

where the operator \((-\Delta_N + I_d)^{1-s}\) is analogously defined by the spectral theory. Indeed, there exists a complete orthonormal basis \( \{\varphi_k\}_{k=0}^\infty \) of \( L^2(\Omega) \), where \( \varphi_k \) satisfies the following eigenvalue problem

\[
\begin{aligned}
(-\Delta + I_d)\varphi_k &= \mu_k \varphi_k, \quad \text{in} \ \Omega, \\
\nabla \varphi_k \cdot \nu &= 0, \quad \text{on} \ \Gamma.
\end{aligned}
\]

\]
Therefore, we have that $\varphi_k$ is the eigenfunction corresponding to eigenvalue $\mu_k$, which is given by $\mu_k = \lambda_k + 1$ for each $k \geq 0$, where the pair $(\varphi_k, \lambda_k)$ is the solution of (2.1). Thus, applying the functional calculus we can define

$$(-\Delta_N + I_d)^s u := \sum_{k=0}^{\infty} (\lambda_k + 1)^s \langle u, \varphi_k \rangle \varphi_k.$$ 

Now, we are allowed to take $K_s = ((-\Delta_N + I_d)^s)^{-1} = (-\Delta_N + I_d)^{-s}$. Similar to the system (1.1), it is not difficult to show that the condition (1.5) is satisfied, that is to say

$$\int_{\Omega} c(t, x) \, dx = \int_{\Omega} u(t, x) \, dx.$$ 

2. Finally, let $0 \leq \sigma \leq 1$ be fixed and consider for $s \in (0, 1)$ the following system

$$\begin{aligned}
\partial_t u + \text{div}(g(u) \nabla K_s c) &= 0, \quad \text{in } (0, \infty) \times \Omega, \\
(-\Delta_N + \sigma I_d)^{1-s} c + (1 - \sigma)c &= u, \quad \text{in } \Omega, \\
u|_{\{t=0\}} &= u_0, \quad \text{in } \Omega, \\
\nabla K_s c \cdot \nu &= 0, \quad \text{on } \Gamma.
\end{aligned}$$

(5.2)

Clearly, for $\sigma = 0$ we get the system (1.1) and for $\sigma = 1$ we have (5.1). For $0 < \sigma < 1$ we may similarly define the operators

$$(-\Delta_N + \sigma I_d)^s \quad \text{and} \quad K_s = (-\Delta_N + \sigma I_d)^{-s}.$$ 

This system does not satisfy exactly the condition (1.5). Indeed, we have

$$(\sigma^{1-s} + 1 - \sigma) \int_{\Omega} c(t, x) \, dx = \int_{\Omega} u(t, x) \, dx,$$ 

(5.3)

and for each $\sigma \in (0, 1)$, it follows that $(\sigma^{1-s} + 1 - \sigma) \in (0, 1)$. Moreover, for any $\sigma \in [0, 1]$ the system (5.2) turns into (1.6) (at least formally) passing to the limit as $s \to 0^+$.

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Wladimir Neves  
Instituto de Matemática  
Universidade Federal do Rio de Janeiro  
Av. Athos da Silveira Ramos, 149  
Rio de Janeiro, RJ, Brazil  
CEP 21941-909  
wladimir@im.ufrj.br

Gerardo Huaroto  
Departamento de Matemática  
Universidade Federal de Alagoas  
Av. Lourival Melo Mota, S/N  
Maceio, AL, Brazil  
CEP 57072-970  
gerardo.cardenas@im.ufal.br