Extra current and integer quantum Hall conductance in the spin-orbit coupling system

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Abstract - We study the extra term of particle current in a 2D k-cubic Rashba spin-orbit coupling system and the integer quantization of the Hall conductance in this system. We provide a correct formula of charge current in this system and the careful consideration of extra currents provides a stronger theoretical basis for the theory of the quantum Hall effect which has not been considered before. The non-trivial extra contribution to the particle current density and local conductivity, which originates from the cubic dependence on the momentum operator in the Hamiltonian, will have no effect on the integer quantization of the Hall conductance. The extension of Noether’s theorem for the 2D k-cubic Rashba system is also addressed. The two methods reach to exactly the same results.

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Introduction. – Recent experimental demonstrations of the spin Hall effect in some semiconductors [1–5] may create a way to manipulate the spin of carriers in terms of electric field that presents potential in future applications. It has stimulated many scientists’ interest. The experiments clearly show that the spin-orbit coupling (SOC) of carriers in some semiconductors plays a key role in disclosing the spin Hall effect. Several theoretical models of SOC have been suggested to study the charge and spin transport for different kinds of semiconductor systems, such as 2-dimensional (2D) linear k-dependent Rashba [6] and Dresselhaus [7,8] models, quadratic k-dependent Luttinger model [9], 3D k-cubic Dresselhaus model [10] and 2D k-cubic Rashba SOC model which was found in a GaAs-AlGaAs interface of a typical semiconductor heterojunction, where the k-cubic Rashba SOC effect for heavy holes cannot be neglected in a high-density regime [11]. For some Hamiltonians including terms with high-order power (>2) of momentum operators (MO) \( \hat{p} \), like the 3D k-cubic Dresselhaus model, we have proved that the conventional expression of particle current density (CD) \( j_{\text{conv}}(r,t) = Re\{\psi^\dagger(r,t)(i\hbar\partial\hat{H}/\partial \psi^\dagger(r,t))\} \) is no longer valid [12]. In that system, for the sake of current conservation, a non-trivial extra term of CD \( j_{\text{extra}}(r,t) \) \((\nabla \cdot j_{\text{extra}}(r,t) \neq 0)\) should be added to the conventional one. Then the continuity equation of conserved particle CD \( j(r,t) = j_{\text{conv}}(r,t) + j_{\text{extra}}(r,t) \) can be satisfied. Thus, the extra term \( j_{\text{extra}}(r,t) \) is a physical quantity and has effect on the conductivity of the system. It is natural to extend it to the k-cubic Rashba system where the extra terms of CD may also appear due to cubic \( k \), the term with high-order power of MO \( \hat{p} \) in its Hamiltonian. However, 2D k-cubic Rashba is a real system that can demonstrate the integer quantum Hall conductance. The high precision of the integer quantum Hall effect [13] was explained in some famous papers [14–16] where the expression of charge CD is implicitly based upon the conventional form. One naturally questions whether some correction to the quantum Hall conductance could come from the additional term of current \( j_{\text{extra}}(r,t) \) in 2D k-cubic Rashba SOC semiconductors. In this paper, we would rigorously deduce the exact expression of charge CD that shows the existence of a non-trivial extra term \( j_{\text{extra}}(r,t) \) \((\nabla \cdot j_{\text{extra}}(r,t) \neq 0)\). It is not a local circular current and does have the contribution to electric conductivity. Further, we prove that it has no contribution to the quantum Hall conductance. So the explanation of the integer quantum Hall conductance is extended to a more general case that includes MO of triple power in the Hamiltonian, though whose formula of charge CD must be corrected by a nontrivial extra term due to the requirement of its continuity. Our paper shows a more clear understanding of the property of...
integer quantization of Hall conductance no matter the Hamiltonian including additional cubic \( k \)-dependent SOC which is a realizable 2D quantum Hall system.

This paper is organized as follows. Firstly, we simply introduce the formulae of the calculation of particle CD in the first section. The deduction of the expression of extra term \( j_{\text{extra}}(\mathbf{r}, t) \) in addition to \( j_{\text{conv}}(\mathbf{r}, t) \) for a 2D cubic Rashba Hamiltonian is presented. The second section gives a proof that there is no contribution to the integer quantum Hall conductance from the extra term \( j_{\text{extra}}(\mathbf{r}, t) \). Our expression of particle CD confirmed by extended Noether’s theorem is attached in the appendix.

**Density of particle current.** – We study the 2D cubic Rashba system that is a promising model system for an ultra-thin film of \( p \)-doped semiconductor [11]. In a perpendicular magnetic field, the single particle Hamiltonian is

\[
\hat{H} = \hat{H}_N (\mathbf{\hat{p}}, \mathbf{r}) + \hat{H}_R ,
\]

\[
\hat{H}_N (\mathbf{\hat{p}}, \mathbf{r}) = \frac{\hbar^2}{2m^*} V(\mathbf{r}) - e\mathbf{A}\cdot\mathbf{\hat{p}} + \frac{e}{c} E_y \mathbf{\hat{p}}_y ,
\]

\[
\hat{H}_R = i\lambda (\hat{p}_z \sigma^+ - \hat{p}_z \sigma^-) ,
\]

where \( V(\mathbf{r}) \) is a local spin-independent potential and could contain an impurity potential, \( \lambda = \alpha/2\hbar^3 \) is the spin-orbit coupling constant, \( E_y \) is the transverse Hall electric field, \( \hat{p}_x = \mathbf{\hat{p}}_x \pm i\mathbf{\hat{p}}_y \), \( \sigma^\pm = \sigma_x \pm i\sigma_y \) where \( \sigma_x \) and \( \sigma_y \) are Pauli matrices, and \( \mathbf{\hat{p}} = (\hat{p}_x, \mathbf{\hat{p}}_y - eA_y) \). The corresponding Schrödinger (or, say, Pauli) equation is

\[
\frac{\partial}{\partial t} \psi (\mathbf{r}, t) = \frac{\hbar}{i} \hat{H} \psi (\mathbf{r}, t) ,
\]

where the Hamiltonian \( \hat{H} \) is a \( 2 \times 2 \) matrix. The particle density for a pure quantum state is \( n(\mathbf{r}, t) = \psi^\dagger (\mathbf{r}, t) \psi (\mathbf{r}, t) \) in which we have performed the inner product for spin space, but not for position. This rule of inner product is also used in the following deductions implicitly. Since the number of particles is conserved, the total number of particles \( N = \int n(\mathbf{r}, t) d\mathbf{r} \) should be a constant. The conserved particle CD \( j(\mathbf{r}, t) \) is defined by the following continuity equation:

\[
\frac{\partial n(\mathbf{r}, t)}{\partial t} = -\nabla \cdot j(\mathbf{r}, t) .
\]

For simplifying the notations, in the paper we will not discriminate the notions of particle CD and charge CD which only differ by a factor of charge \( e \) and can be self-explanatory according to the context.

For a mixed state, the density matrix \( \hat{\rho} = \sum_n |\psi_n\rangle \rho_n \langle \psi_n| \), where \( \rho_n \) is the probability of the state \( |\psi_n\rangle \). \( \rho_{\text{max}} > 0 \), \( \sum_n \rho_n = 1 \). The density of particle is defined by

\[
n(\mathbf{r}, t) = \langle \mathbf{r}, t | \hat{\rho} | \mathbf{r}, t \rangle = \sum_n \langle \mathbf{r}, t | \psi_n \rangle \rho_n \langle \psi_n | \mathbf{r}, t \rangle
\]

\[
= \sum_n \rho_n \psi_n (\mathbf{r}, t) \psi^\dagger (\mathbf{r}, t) ,
\]

We discuss the case of \( \rho_n \), being time independent. Then, based on the Schrödinger equation, the left-hand side of eq. (5) can be expressed as

\[
\frac{\partial n(\mathbf{r}, t)}{\partial t} = \sum_n \rho_n \left\{ -\nabla \cdot \hat{J}_N^\dagger (\mathbf{r}, t) + \frac{1}{\hbar} \hat{H}_R \psi_n (\mathbf{r}, t) \right\} \psi^\dagger (\mathbf{r}, t)
\]

\[
+ \psi^\dagger (\mathbf{r}, t) \left\{ \frac{1}{\hbar} \hat{H}_R \psi_n (\mathbf{r}, t) \right\} \}
\]

(7)
Hall conductance. – Now we study the charge CD along the x-direction which is \( j^{(x)}(r) = j^{(x)}_{\text{conv}}(r) + j^{(x)}_{\text{extra}}(r) \). We take the integral with respect to \( y \) for \( j^{(x)}(r) \) to get the charge current

\[
I^{(x)} = \int_{L_y} j^{(x)}_{\text{conv}}(r) dy + \int_{L_y} j^{(x)}_{\text{extra}}(r) dy,
\]

where \( L_y \) is the width of the system, and denote the length of the system as \( L_x \). Finally, \( L_x \) and \( L_y \) can approach infinity if the system becomes macroscopic. \( I^{(x)} \) should not be position-\( x \) dependent, because of the particle conservation. Then we take the integral of \( x \) for \( I^{(x)} \):

\[
I^{(x)} = \frac{1}{\Omega} \int_{\Omega} j^{(x)}_{\text{conv}}(r) L_y dr + \frac{1}{\Omega} \int_{\Omega} j^{(x)}_{\text{extra}}(r) L_y dr
\]

\[
= I^{(x)}_{\text{conv}} + I^{(x)}_{\text{extra}},
\]

where \( \Omega = L_x L_y \), \( J^{(x)}_{\text{conv}} = \frac{1}{\Omega} \int_{\Omega} j^{(x)}_{\text{conv}}(r) L_y dr \) and \( J^{(x)}_{\text{extra}} = \frac{1}{\Omega} \int_{\Omega} j^{(x)}_{\text{extra}}(r) L_y dr \). From eq. (10), the extra part of the current is

\[
I^{(x)}_{\text{extra}} = e \lambda^2 y^2 \sum_{n} \rho_n \frac{1}{\Omega} \int_{\Omega} \left[ \partial_x \partial_y (\Psi_n^+(r) \sigma^x \Psi_n(r)) + \partial_y^2 (\Psi_n^+(r) \sigma^y \Psi_n(r)) \right] dr
\]

\[
= -e \lambda L_y \sum_{n} \rho_n \frac{1}{\Omega} \int_{\Omega} \left[ \partial_x \partial_y (\Psi_n^+(r) \sigma^x \Psi_n(r)) + \partial_y^2 (\Psi_n^+(r) \sigma^y \Psi_n(r)) \right] dr
\]

\[
+ 2 \Psi_n^+(r) \sigma_x \partial_y \Psi_n(r) - 2 \partial_y \Psi_n(r) \sigma_x \partial_x \Psi_n(r) - 2 \partial_y \Psi_n(r) \sigma_x \partial_x \Psi_n(r)
\]

\[
- 2 \Psi_n^+(r) \sigma_y \partial_x \Psi_n(r) + 2 \partial_x \Psi_n(r) \sigma_y \partial_y \Psi_n(r)
\]

\[
+ (\partial_y \Psi_n(r)) \sigma_y \Psi_n(r) + \Psi_n(r) \sigma_y \partial_y^2 \Psi_n(r) - 2 \partial_y \Psi_n(r) \sigma_y \partial_y \Psi_n(r).
\]

The terms on the right-hand side of the above equation become the spatial inner product after the integration of \( r \) over the whole space of the system. Since the operators \( \{\partial_x, \partial_y\} \) are Hermitian, as an example, we have

\[
\frac{1}{\Omega} \int_{\Omega} \left[ \partial_x \partial_y (\Psi_n^+(r) \sigma^x \Psi_n(r)) + \partial_y^2 (\Psi_n^+(r) \sigma^y \Psi_n(r)) \right] dr
\]

\[
= \frac{1}{\Omega} \int_{\Omega} \left[ \partial_x \partial_y (\Psi_n^+(r) \sigma^x \Psi_n(r)) + \partial_y^2 (\Psi_n^+(r) \sigma^y \Psi_n(r)) \right] dr
\]

\[
= \frac{1}{\Omega} \int_{\Omega} \left[ \partial_x \partial_y (\Psi_n^+(r) \sigma^x \Psi_n(r)) + \partial_y^2 (\Psi_n^+(r) \sigma^y \Psi_n(r)) \right] dr
\]

Considering the above property for the inner product of position space in eq. (15), we can easily obtain \( I^{(x)}_{\text{extra}} = 0 \).
No contribution to Hall conductance from the extra term of charge CD is proved. Finally, we have

\[ I^{(x)} = \frac{1}{\Omega} \int \int_{\Omega} j_{\text{con}}^{(x)}(r) dr \]

\[ = eL_R \text{Re} \sum_{n} p_n \frac{1}{\Omega} \int \int \{ \Psi^\dagger_n(r) \frac{1}{i\hbar} [\hat{x}, \hat{H}] \Psi_n(r) \} \ dr \]

\[ = eL_R \text{Re} \sum_{n} p_n \langle \Psi_n | \frac{1}{i\hbar} [\hat{x}, \hat{H}] | \Psi_n \rangle, \]

\[ H_0 = \tilde{p}_x^2/(2m^*) + i\lambda(\tilde{p}_+ \sigma^+ - \tilde{p}_- \sigma^-). \]

Thus, the quantum Hall conductance is only from the conventional term \( j_{\text{con}}(r, t) \). For \( H_0 \), the cubic 2D Rashba model without transverse electric field, its Schrödinger equation is

\[ H_0 |\Psi_n^0(0)\rangle = E_n^0 |\Psi_n^0\rangle. \]

It has been solved exactly [17],

\[ E_n^0 = (n + 1/2) \hbar \omega, \quad n \leq 2, \]

\[ E_{n,s}^0 = \left[ (n - 1) + s \sqrt{\gamma^2 n(n - 1)(n - 2) + \frac{9}{4}} \right] \hbar \omega, \quad n \geq 3, \]  

where \( \omega = eB/(mc) \), \( s = \pm 1 \) and \( \gamma = 4m^*/\sqrt{2\hbar eB/c} \). The eigenenergies in the traditional quantum Hall effect are Landau levels separated by gaps. Now the “Landau levels” of a 2D cubic Rashba model have some modification for \( n \geq 3 \), but they keep essential the feature of the gap separation. The corresponding eigenfunctions are

\[ |\Psi_n^0\rangle = \left( \begin{array}{c} 0 \\ \phi_n \end{array} \right), \quad n \leq 2, \]

\[ |\Psi_n^0\rangle = |\Psi_n^{0,s}\rangle = \left( \begin{array}{c} C_{n+1}\phi_n \\ C_{n+2}\phi_n \end{array} \right), \quad n \geq 3, \]

where \( \{C_{n+1}, C_{n+2}\} \) are normalized constants,

\[ C_{n+1} = \sqrt{\frac{1}{c_n^2 + 1}}, \quad C_{n+2} = \frac{1}{\sqrt{c_n^2 + 1}}, \]

\[ c_n = \frac{1}{\gamma \sqrt{n(n - 1)(n - 2)}} \times \left( \frac{3}{2} + s \sqrt{\gamma^2 n(n - 1)(n - 2) + \frac{9}{4}} \right), \]

and \( \phi_n \) is the wave function of the harmonic oscillation type. Impurities may result in widening out the “Landau levels”. The conventional velocity operator in position space is \( \tilde{v} = 1/(i\hbar) [\hat{x}, \hat{H}] = 1/(i\hbar) [\hat{x}, \hat{H}_0] \). It is easy to have \( \tilde{v}(k) = 1/\hbar V_{k} E^{(0)}(k) \) in \( k \) space. Following Laughlin [14] or Kohmoto’s [16] deduction, the integer quantization of quantum Hall conductance can be obtained. Here we will present a different approach to reach to the conclusion of integer quantum Hall conductance for such a specific 2D cubic Rashba system.

Since the eigenenergies and wave functions of the Schrödinger equation in the second quantization representation can be found exactly, we also calculate the Hall conductance in linear response approximation and it shows excellent consistency with the integer quantization of the Hall conductance. More specifically, the Hall conductance \( \sigma_{xy} \) of this system can be written as

\[ \sigma_{xy} = \sum_{n,s} N_{n,s} (\sigma_{xy})_{n,s}, \]

where \( N_{n,s} \) is the number of particles occupying the \((n, s)-th \) “Landau level” (here the “Landau level” is marked by two indices, \( n \) indicating the energy level of the system without SOC, \( s \) indicating the energy level splitting due to SOC). And \((\sigma_{xy})_{n,s}\) is the one particle’s contribution from the \((n, s)-th \) “Landau level”, by linear response theory,

\[ (\sigma_{xy})_{n,s} = \sum_{\langle n', s' \rangle \neq \langle n, s \rangle} \frac{\langle \Psi_n^0 | j_x | \Psi_{n', s'}^0 \rangle \langle \Psi_{n', s'}^0 | H' | \Psi_n^0 \rangle + \text{h.c.}}{E_{n,s} - E_{n', s'}^0}. \]

where \( H' = -e\tilde{g}E_y \) and the electric field is uniform. Now we adopt the Landau gauge \( \tilde{p}_x = h k_x - eB y/c, \tilde{p}_y = \tilde{p}_y \), and introduce the operator of bosonic quasi-particles \( a = \sqrt{2mc^2} (\tilde{p}_x - i\tilde{p}_y), a^\dagger = \sqrt{2mc^2} (\tilde{p}_x + i\tilde{p}_y), [a, a^\dagger] = 1 \). Then we get

\[ \hat{H}_0 = \hbar \omega \left( \begin{array}{c} a^\dagger a + \frac{1}{2} i\gamma a^3 \\ -i\gamma a^3 \end{array} \right). \]

Then

\[ \tilde{j}_x = \frac{e}{i\hbar} \left[ x, \hat{H}_0 \right] \]

\[ = e\omega \sqrt{\frac{\hbar c}{2eB}} (a + a^\dagger) + \frac{3ie\gamma}{m} \sqrt{\frac{\hbar eB}{2c}} \left( \begin{array}{c} 0 \\ -a^2 \end{array} \right). \]

and

\[ \hat{H}' = -e\tilde{g}E_y = -eE_y \left( h k_x - \sqrt{\frac{\hbar eB}{2c}} (a + a^\dagger) \right) I, \]

where \( I \) is a unit matrix. Then, using eqs. (17), (19) and (20) for matrix elements \( \langle \Psi_{n, \pm}^0 | \tilde{j}_x | \Psi_{n+1, \pm}^0 \rangle \) and \( \langle \Psi_{n, \pm}^0 | H' | \Psi_{n+1, \pm}^0 \rangle \), the selection rules will be found. And the summation over states in the Hall conductance in
eq. (18) can be simplified as
\[
\langle \sigma_{xy} \rangle_{n, \pm} = \frac{1}{E L_x L_y} \left[ \langle \Psi_0^{(0)} | \hat{J}_x | \Psi_0^{(0)} \rangle_{n-1, +} \langle H' | \Psi_0^{(0)} \rangle_{n+, \pm} \right.
\]
\[+ \langle \Psi_0^{(0)} | \hat{J}_x | \Psi_0^{(0)} \rangle_{n+, \pm} \langle H' | \Psi_0^{(0)} \rangle_{n-1, +} \]
\[+ \langle \Psi_0^{(0)} | \hat{J}_x | \Psi_0^{(0)} \rangle_{n-, \pm} \langle H' | \Psi_0^{(0)} \rangle_{n+, \pm} \]
\[+ \langle \Psi_0^{(0)} | \hat{J}_x | \Psi_0^{(0)} \rangle_{n+, \pm} \langle H' | \Psi_0^{(0)} \rangle_{n-, \pm} \]
\[+ \langle \Psi_0^{(0)} | \hat{J}_x | \Psi_0^{(0)} \rangle_{n-, \pm} \langle H' | \Psi_0^{(0)} \rangle_{n-, \pm} \] + h.c.
\] Then using eqs. (16) and (17), all the elements \( \langle \Psi_0^{(0)} | \hat{J}_x | \Psi_0^{(0)} \rangle_{n, \pm} \) and \( \langle H' | \Psi_0^{(0)} \rangle_{n, \pm} \) can be calculated without difficulty. After a long but straightforward algebraic deduction, we can finally obtain
\[
\langle \sigma_{xy} \rangle_{n, \pm} = \frac{e^2 \Phi_0}{h} \phi, \tag{21}
\]
where \( \Phi_0 = B L_x L_y, \phi = \frac{\hbar c}{e} \). By summing up all the contributions from different energy levels, the total Hall conductance will be
\[
\sigma_{xy} = \sum N_{n, s} e^2 \frac{c^2}{h} \frac{1}{\Phi_0} = \sum N_{n, s} e^2 \frac{c^2}{h} \phi. \tag{22}
\]
Here \( N_0 \) is the total number of carriers, and the filling factor \( \nu = N_0 / (\Phi_0) \).

Due to the existence of impurities in practical samples, localized states appear in the region between the “Landau levels”. It leads to the appearance of the plateau when the Fermi level lies in that region. The gap between two conductance plateaus is obviously \( e^2/h \). It is concluded that the cubic SOC do induce the extra term of CCD that yields the contribution to the electric conductivity, but no contribution is given to the quantum Hall conductance.

**Conclusions.** We have derived an exact formula of particle current density for a 2D cubic Rashba model that appears in some \( p \)-doped semiconductors. In addition to the conventional current expression, there must be an extra term that ensures the current continuity equation. The extra term must have the contribution to the electric conductivity, but no contribution to the charge quantum Hall conductance that is proved rigorously. So, it can be clearly shown that no effect is made on the topological property of the integer quantization of the Hall conductance due to the existence of extra terms in the 2D cubic Rashba coupling system. Further experimentally detectable effects of the new term are still on research.

**Appendix: Deduction of extra terms from Noether’s theorem.** In this appendix, we point out that, for a 2D cubic Rashba Hamiltonian where the highest order of derivatives is higher than 2, it is necessary to generalize the expression of conserved current in Noether’s theorem. Applying the generalized Noether’s theorem [12], we can get the expressions of conserved particle CD of a \( k \)-cubic Rashba SOC system from \( U (1) \) gauge invariance.

Noether’s theorem not only indicates the relation between conserved currents and symmetries of the Lagrangian, but also implies that the expression of conserved current depends on the form of the Lagrangian from the beginning of its deduc-

\[ L = \frac{1}{2} \sum \frac{\partial \mathcal{L}}{\partial \phi_{\mu \nu}} \partial_{\mu \nu} \phi - \frac{1}{2} \sum \frac{\partial \mathcal{L}}{\partial \phi_{\mu \nu}} \delta^{\mu \nu} \phi \]

the Lagrangian from the beginning of its deduction. In usual cases, Lagrangians are expressed as \( \mathcal{L}(\phi(x), \partial_{\mu} \phi(x), \phi^\dagger(x), \partial_{\mu} \phi^\dagger(x)) \), \( x^\mu = (t, \mathbf{r}) \), \( \mu = 0, 1, 2, 3 \) —such as the Lagrangian of complex scalar field—which only include fields \( \phi(x), \phi^\dagger(x) \) and their first-order derivatives \( \partial_{\mu} \phi(x), \partial_{\mu} \phi^\dagger(x) \) as independent variables. But in our case, the Hamiltonian \( \hat{H} \) includes higher-order derivatives. So its Lagrangian should be written in the form \( \mathcal{L}(\phi(x), \partial_{\mu} \phi(x), \partial_{\mu} \partial_{\nu} \phi(x), \ldots, \phi^\dagger(x), \partial_{\mu} \phi^\dagger(x), \partial_{\mu} \partial_{\nu} \phi^\dagger(x), \ldots) \), where higher-order derivatives are also included as independent variables. For simplicity, we denote \( \phi(x) \) and \( \phi^\dagger(x) \) as \( \phi \) and \( \phi^\dagger \). The Hamiltonian of a \( k \)-cubic Rashba system studied here is \( \hat{H}_{\text{R}} = \hat{p}^2/2m + i \lambda [\hat{p}^2, \sigma^+ - \hat{p}^2_\perp \sigma^-] \). The corresponding Lagrangian can be

\[ \mathcal{L}[\phi, \partial_{\mu} \phi, \partial_{\mu} \partial_{\nu} \phi, \partial_{\mu} \partial_{\nu} \phi^\dagger, \partial_{\mu} \partial_{\nu} \phi^\dagger, \partial_{\mu} \partial_{\nu} \phi^\dagger, \partial_{\mu} \partial_{\nu} \phi^\dagger] = \]

\[ \phi^\dagger (i \partial_{\mu} \phi) + \frac{1}{2m} \phi \partial^2_{\mu \nu} \phi \]

\[ + 2i \lambda \phi^\dagger \left( \sigma^2 \partial^2_\perp + \sigma^y \partial^2_\perp \right) \]

\[ - 2i \lambda \phi^\dagger \left( 3 \sigma^x \partial^2_\perp \phi + 3 \sigma^y \partial^2_\perp \phi \right). \]

According to the least action principle, one can easily obtain an Euler-Lagrange equation

\[ 0 = \sum \left( \frac{\partial \mathcal{L}}{\partial \phi_{\mu \nu}} \partial_{\mu \nu} \phi - \left( \frac{\partial \mathcal{L}}{\partial \partial_{\mu} \phi} \right) \partial_{\mu} \phi \right) - \left( \frac{\partial \mathcal{L}}{\partial \partial_{\mu} \phi} \right) \partial_{\mu} \phi \]

which yields the Schrödinger equation. Actually, the first two terms on the right-hand side of the above equation give the conventional formula of particle CD. The remaining parts lead to the extra terms. The corresponding conserved current \( F_{\mu} \) is

\[ F_{\mu} = \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \phi)} \delta \phi + \left( \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \phi)} \delta \phi \right) \partial_{\mu} \phi \]

\[ + \left( \frac{\partial \mathcal{L}}{\partial (\partial^2_{\mu \nu} \phi)} \right) \delta \phi + (\phi \rightarrow \phi^*), \tag{A.1} \]

which satisfies the continuity equation \( \partial_{\mu} F_{\mu} = 0 \). We concentrate on the deduction of the conserved particle current corresponding to \( U (1) \) gauge symmetry. From an infinitesimal variation of fields \( \delta \phi = i \alpha \phi, \delta \phi^\dagger = -i \alpha \phi^\dagger \), the
expression of conserved particle CD for a \( k \)-cubic Rashba system is

\[
j^x = -F^x = \phi^\dagger \left( \frac{i \partial_y}{2m} + \frac{i \partial_x}{2m} \right) \phi
- 2\lambda \phi^\dagger \phi \sigma^y \left( \partial_y \phi \right)
- \left( \partial_x \phi^\dagger \right) \sigma^y \left( \partial_x \phi \right)
+ 6\lambda \phi^\dagger \phi \sigma^y \partial_y \phi
- \left( \partial_y \phi^\dagger \right) \sigma^y \partial_y \phi
+ \left( \partial_y^2 \phi^\dagger \right) \sigma^y \phi.
\]

Comparing the above formulae with the conventional one \( J_{\text{conv}} = \text{Re} \left\{ \phi^\dagger \left( \frac{i}{2} \left[ r, H_{\text{R}} \right] \phi \right) \right\} \), we get the extra term of particle CD \( J_{\text{extra}} = j - J_{\text{conv}} \):

\[
j_{\text{extra}}^x = -\lambda \partial_y^2 \left( \phi^\dagger \sigma^y \phi \right)
+ 6\lambda \left( \partial_x \phi^\dagger \right) \sigma^y \left( \partial_y \phi \right)
+ 6\lambda \left( \partial_x \phi^\dagger \right) \sigma^y \left( \partial_y \phi \right)
+ 3\lambda \phi^\dagger \left( \sigma^y \partial_y \phi \right)
+ 3\lambda \phi^\dagger \left( \sigma^y \partial_y \phi \right)
+ 3\lambda \phi^\dagger \left( \sigma^y \partial_y \phi \right). \tag{A.2}
\]

\[
j_{\text{extra}}^y = -\lambda \partial_y^2 \left( \phi^\dagger \sigma^y \phi \right)
+ 6\lambda \left( \partial_x \phi^\dagger \right) \sigma^y \left( \partial_y \phi \right)
+ 6\lambda \left( \partial_x \phi^\dagger \right) \sigma^y \left( \partial_y \phi \right)
+ 3\lambda \phi^\dagger \left( \sigma^y \partial_y \phi \right)
+ 3\lambda \phi^\dagger \left( \sigma^y \partial_y \phi \right). \tag{A.3}
\]

Further, it is not difficult to check that the extra term \( J_{\text{extra}} \) deduced here by extended Noether’s theorem and \( J_{\text{extra}} \) in the second section do satisfy the equation \( \nabla \cdot (J_{\text{extra}} - J_{\text{extra}}) = 0 \). Thus we conclude that our result of the extra term is rigorous.

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