ON INTERVALS \((kn, (k + 1)n)\) CONTAINING A PRIME FOR ALL \(n > 1\)

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Abstract. We study values of \(k\) for which the interval \((kn, (k + 1)n)\) contains a prime for every \(n > 1\). We prove that the list of such integers \(k\) includes \(k = 1, 2, 3, 5, 9, 14\), and no others, at least for \(k \leq 50,000,000\). For every known \(k\) of this list, we give a good upper estimate of the smallest \(N_k(m)\), such that, if \(n \geq N_k(m)\), then the interval \((kn, (k+1)n)\) contains at least \(m\) primes.

1. Introduction and main results

In 1850, P. L. Chebyshev proved the famous Bertrand postulate (1845) that every interval \([n, 2n]\) contains a prime (for a very elegant version of his proof, see Theorem 9.2 in [10]). Other nice proofs were given by S. Ramanujan in 1919 [8] and P. Erdős in 1932 (reproduced in [4], pp.171-173). In 2006, M. El. Bachraoui [1] proved that every interval \([2n, 3n]\) contains a prime, while A. Loo [6] proved the same statement for every interval \([3n, 4n]\). Moreover, A. Loo found a lower estimate for the number of primes in the interval \([3n, 4n]\). Note also that already in 1952 J. Nagura [7] proved that, for \(n \geq 25\), there is always a prime between \(n\) and \(\frac{6}{5}n\). From his result it follows that the interval \([5n, 6n]\) always contains a prime. In this paper we prove the following.

Theorem 1. The list of integers \(k\) for which every interval \((kn, (k+1)n)\), \(n > 1\), contains a prime includes \(k = 1, 2, 3, 5, 9, 14\) and no others, at least for \(k \leq 50,000,000\).

Besides, in this paper, for every \(k = 1, 2, 3, 5, 9, 14\), we give an algorithm for finding the smallest \(N_k(m)\), such that, for \(n \geq N_k(m)\), the interval \((kn, (k+1)n)\) contains at least \(m\) primes.

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2. Case \( k = 1 \)

Ramanujan [8] not only proved Bertrand’s postulate but also indicated the smallest integers \( \{R(m)\} \), such that, if \( x \geq R(m) \), then the interval \((\frac{x}{2}, x]\) contains at least \( m \) primes, or, the same, \( \pi(x) - \pi(x/2) \geq m \). It is easy to see that here it is sufficient to consider integer \( x \) and it is evident that every term of \( \{R(m)\} \) is prime. The numbers \( \{R(m)\} \) are called Ramanujan primes [14]. It is the sequence (A104272 in [13]):

\[
\begin{align*}
(1) & \quad 2, 11, 17, 29, 41, 47, 59, 67, 71, 97, \\
\end{align*}
\]

Since \( \pi(x) - \pi(x/2) \) is not a monotonic function, to calculate the Ramanujan numbers one should have an effective upper estimate of \( R(m) \). In [8] Ramanujan showed that

\[
\pi(x) - \pi(x/2) > \frac{1}{\ln x} \left( \frac{x}{6} - 3\sqrt{x} \right), \quad x > 300.
\]

In particular, for \( x \geq 324 \), the left hand side is positive and thus \( \geq 1 \). Using direct descent, he found that \( \pi(x) - \pi(x/2) \geq 1 \) already from \( x \geq 2 \). Thus \( R(1) = 2 \) which proves the Bertrand postulate. Further, e.g., for \( x \geq 400 \), the left hand side of (2) is more than 1 and thus \( \geq 2 \). Again, using direct descent, he found that \( \pi(x) - \pi(x/2) \geq 2 \) already from \( x \geq 11 \). Thus \( R(2) = 11 \), etc. Sondow [14] found that \( R(m) < 4m \ln(4m) \) and conjectured that

\[
R(m) < p_{3m}
\]

which was proved by Laishram [5]. Since, for \( n \geq 2 \), \( p_n \leq en \ln n \) (cf. [3], Section 4), then (3) yields \( R(m) \leq 3em \ln(3m), \ m \geq 1 \). Set \( x = 2n \). Then, if \( 2n \geq R(m) \), then \( \pi(2n) - \pi(n) \geq m \). Thus the interval \((n, 2n]\) contains at least \( m \) primes, if

\[
n \geq \left\lceil \frac{R(m) + 1}{2} \right\rceil = \begin{cases} 2, & \text{if } m = 1, \\
\frac{R(m)+1}{2}, & \text{if } m \geq 2.
\end{cases}
\]

Denote by \( N_1(m) \) the smallest number such that, if \( n \geq N_1(m) \), then the interval \((n, 2n]\) contains at least \( m \) primes. It is clear, that \( N_1(1) = R(1) = 2 \). If \( m \geq 2 \), formally the condition \( x = 2n \geq 2N_1(m) \) is not stronger than the condition \( x \geq R(m) \), since the latter holds for \( x \) even and odd. Therefore, for \( m \geq 2 \), we have \( N_1(m) \leq \frac{R(m)+1}{2} \). Let us show that in fact we have here the equality.
Proposition 2. For \( m \geq 2 \),

\[
N_1(m) = \frac{R(m) + 1}{2}.
\]

Proof. Note that the interval \( \left( \frac{R(m)-1}{2}, R(m) - 1 \right) \) cannot contain more than \( m - 1 \) primes. Indeed, it is an interval of type \( (\frac{x}{2}, x) \) for integer \( x \) and the following such interval is \( \left( \frac{R(m)}{2}, R(m) \right) \). By the definition, \( R(m) \) is the smallest number such that if \( x \geq R(m) \), then \( \{ (\frac{x}{2}, x) \} \) contains \( \geq m \) primes. Therefore, the supposition that already interval \( \left( \frac{R(m)-1}{2}, R(m) - 1 \right) \) contains \( \geq m \) primes contradicts the minimality of \( R(m) \). Since the following interval of type \( (y, 2y) \) with integer \( y \geq R(m) - 1 \) is \( \left( \frac{R(m)+1}{2}, R(m) + 1 \right) \), then (4) follows. \( \square \)

So the sequence \( \{ N_1(m) \} \), by (4), is (A084140 in [13])

\[
2, 6, 9, 15, 21, 24, 30, 34, 36, 49, ... \]

3. Generalized Ramanujan numbers

Further our research is based on a generalization of Ramanujan’s method. With this aim, we define generalized Ramanujan numbers (cf. [12], Section 10, and earlier (2009) comment in A164952 [13]).

Definition 3. Let \( v > 1 \) be a real number. A \( v \)-Ramanujan number \( (R_v(m)) \), is the smallest integer such that if \( x \geq R_v(m) \), then \( \pi(x) - \pi(x/v) \geq m \).

It is known [10] that all \( v \)-Ramanujan numbers are primes. In particular, \( R_2(m) = R(m) \), \( m = 1, 2, ... \), are the proper Ramanujan primes.

Definition 4. For a real number \( v > 1 \) the \( v \)-Chebyshev number \( C_v(m) \) is the smallest integer, such that if \( x \geq C_v(m) \), then \( \vartheta(x) - \vartheta(x/v) \geq m \ln x \), where \( \vartheta(x) = \sum_{p \leq x} \ln p \) is the Chebyshev function.

Since \( \frac{\vartheta(x) - \vartheta(x/v)}{\ln x} \) can enlarge on 1 only when \( x \) is prime, then all \( v \)-Chebyshev numbers \( C_v(m) \) are primes.

Proposition 5. We have

\[
R_v(m) \leq C_v(m).
\]
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Proof. Let \(x \geq C_v(m)\). Then we have

\[
m \leq \frac{\vartheta(x) - \vartheta(x/v)}{\ln x} = \sum_{\frac{x}{p} \leq x} \frac{\ln p}{\ln x} \leq \sum_{\frac{x}{p} \leq x} 1 = \pi(x) - \pi(x/v).
\]

Thus, if \(x \geq C_v(m)\), then always \(\pi(x) - \pi(x/v) \geq m\). By the Definition 3, this means that \(R_v(m) \leq C_v(m)\). \(\square\)

Now we give an upper estimates for \(C_v(m)\) and \(R_v(m)\).

Proposition 6. Let \(x = x_v(m) \geq 2\) be any number for which

\[
x \ln x \left(1 - \frac{1300}{\ln^4 x}\right) \geq \frac{vm}{v - 1}.
\]

Then

\[
R_v(m) \leq C_v(m) \leq x_v(m).
\]

Proof. We use the following inequality of Dusart \([3]\) (see his Theorem 5.2):

\[
|\vartheta(x) - x| \leq \frac{1300x}{\ln^4 x}, \ x \geq 2.
\]

Thus we have

\[
\vartheta(x) - \vartheta(x/v) \geq x \left(1 - \frac{1}{v} - 1300 \left(\frac{1}{\ln^4 x} - \frac{1}{v \ln^4 \frac{x}{v}}\right)\right)
\geq x \left(1 - \frac{1}{v}\right) \left(1 - \frac{1300}{\ln^4 x}\right).
\]

If now

\[
x \left(1 - \frac{1}{v}\right) \left(1 - \frac{1300}{\ln^4 x}\right) \geq m \ln x, \ x \geq x_v(m),
\]

then

\[
\vartheta(x) - \vartheta(x/v) \geq m \ln x, \ x \geq x_v(m)
\]

and, by the Definition 4, \(C_v(m) \leq x_v(m)\). So, according to (6), we conclude that \(R_v(m) \leq x_v(m)\). \(\square\)

Remark 7. In fact, in Theorem 5.2 \([3]\) Dusart gives several inequalities of the form

\[
|\vartheta(x) - x| \leq \frac{ax}{\ln^b x}, \ x \geq x_0(a,b).
\]

In the proof we used the maximal value \(b = 4\). However, with the computer point of view, the values \(a = 1300, b = 4\) from Dusart’s theorem not always are the best. The analysis for \(x \geq 25\) shows that the condition

\[
x \left(1 - \frac{1}{v}\right) \left(1 - \frac{ax}{\ln^b x}\right) \geq m \ln x
\]
is the weakest and thus satisfies for the smallest \( x_v(x) = x_v(a, b) \), if to use the following values of \( a \) and \( b \) from Dusart’s theorem:

\[
\begin{align*}
& a = 3.965, \ b = 2 \text{ for } x \text{ in range } (25, \ 7 \cdot 10^7]; \\
& a = 1300, \ b = 4 \text{ for } x \text{ in range } (7 \cdot 10^7, \ 10^9]; \\
& a = 0.001, \ b = 1 \text{ for } x \text{ in range } (10^9, \ 8 \cdot 10^9]; \\
& a = 0.78, \ b = 3 \text{ for } x \text{ in range } (8 \cdot 10^9, \ 7 \cdot 10^{33}] ; \\
& a = 1300, \ b = 4 \text{ for } x > 7 \cdot 10^{33}. 
\end{align*}
\]

Proposition \[6\] gives the terms of sequences \( \{C_v(m)\}, \ \{R_v(m)\} \) for every \( v > 1, \ m \geq 1 \). In particular, if \( k = 1 \) we find \( \{C_2(m)\} : \)

\[
\begin{align*}
11, \ 17, \ 29, \ 41, \ 47, \ 59, \ 67, \ 71, \ 97, \ 101, \ 107, \ 127, \ 149, \ 151, \ 167, \ 179, \ 223, \\
229, \ 233, \ 239, \ 241, \ 263, \ 269, \ 281, \ 307, \ 311, \ 347, \ 349, \ 367, \ 373, \ 401, \ 409, \\
419, \ 431, \ 433, \ 443, \ldots. \end{align*}
\]

This sequence requires a separate comment. We observe that up to \( C_2(100) = 1489 \) only two terms of this sequence (\( C_2(17) = 223 \) and \( C_2(36) = 443 \)) are not Ramanujan numbers, and the sequence is missing only the following Ramanujan numbers: 181, 227, 439, 491, 1283, 1301 and no others up to 1489. The latter observation shows how much the ratio \( \frac{\vartheta(x)}{\ln x} \) exactly approximates \( \pi(x) \).

Further, for \( v = \frac{k+1}{k} \), we find the following sequences:

for \( k = 2, \ \{C_v(m)\} , \)

\[
\begin{align*}
(11) \quad & 13, \ 37, \ 41, \ 67, \ 73, \ 97, \ 127, \ 137, \ 173, \ 179, \ 181, \ 211, \ 229, \ 239, \ldots; \\
(12) \quad & 2, \ 13, \ 37, \ 41, \ 67, \ 73, \ 97, \ 127, \ 137, \ 173, \ 179, \ 181, \ 211, \ 229, \ 239, \ldots; \\
(13) \quad & 29, \ 59, \ 67, \ 101, \ 149, \ 157, \ 163, \ 191, \ 227, \ 269, \ 271, \ 307, \ 379, \ldots; \\
(14) \quad & 11, \ 29, \ 59, \ 67, \ 101, \ 149, \ 157, \ 163, \ 191, \ 227, \ 269, \ 271, \ 307, \ 379, \ldots; \\
(15) \quad & 59, \ 137, \ 139, \ 149, \ 223, \ 241, \ 347, \ 353, \ 383, \ 389, \ 563, \ 569, \ 593, \ldots; \\
(16) \quad & 29, \ 59, \ 137, \ 139, \ 149, \ 223, \ 241, \ 347, \ 353, \ 383, \ 389, \ 563, \ 569, \ 593, \ldots; 
\end{align*}
\]
for \( k = 9 \), \( \{ C_v(m) \} \),

\[ \{223, 227, 269, 349, 359, 569, 587, 739, 809, 857, 991, 1009, \ldots\} ; \]

for \( k = 9 \), \( \{ R_v(m) \} \),

\[ \{127, 223, 227, 269, 349, 359, 569, 587, 739, 809, 857, 991, 1009, \ldots\} ; \]

for \( k = 14 \), \( \{ C_v(m) \} \),

\[ \{307, 347, 563, 569, 733, 821, 1427, 1429, 1433, 1447, 1481, \ldots\} ; \]

for \( k = 14 \), \( \{ R_v(m) \} \),

\[ \{127, 307, 347, 563, 569, 733, 1423, 1427, 1429, 1433, 1439, 1447, \ldots\} . \]

4. Estimates of type \( \text{(3)} \)

**Proposition 8.** We have

\[ \text{(21)} \quad C_2(m - 1) \leq p_{3m}, \quad m \geq 2; \]

\[ \text{(22)} \quad R_{\frac{m}{2}}(m) \leq p_{4m}, \quad m \geq 1; \quad C_{\frac{m}{2}}(m - 1) \leq p_{4m}, \quad m \geq 2; \]

\[ \text{(23)} \quad R_{\frac{m}{3}}(m) \leq p_{6m}, \quad m \geq 1; \quad C_{\frac{m}{3}}(m - 1) \leq p_{6m}, \quad m \geq 2; \]

\[ \text{(24)} \quad R_{\frac{m}{5}}(m) \leq p_{11m}, \quad m \geq 1; \quad C_{\frac{m}{5}}(m - 1) \leq p_{11m}, \quad m \geq 2; \]

\[ \text{(25)} \quad R_{\frac{m}{10}}(m) \leq p_{31m}, \quad m \geq 1; \quad C_{\frac{m}{10}}(m - 1) \leq p_{31m}, \quad m \geq 2; \]

\[ \text{(26)} \quad R_{\frac{15}{11}}(m) \leq p_{32m}, \quad m \geq 1; \quad C_{\frac{15}{11}}(m - 1) \leq p_{32m}, \quad m \geq 2. \]

**Proof.** Firstly let us find some values of \( m_0 = m_0(k) \), such that, at least, for \( m \geq m_0 \) all formulas \( \text{(21)-(26)} \) hold. According to \( \text{(8)-(9)} \), it is sufficient to show that, for \( m \geq m_0 \), we can take \( p_{tm} \), where \( t = 3, 4, 6, 11, 31, 32 \) for formulas \( \text{(21)-(26)} \) respectively, in the capacity of \( x_v(m) \). As we noted in Remark \( \text{[7]} \), in order to get possibly smaller values of \( m_0 \), we use, instead of \( \text{(8)} \), the estimate

\[ \frac{x}{\ln x} \left(1 - \frac{3.965}{\ln^2 x}\right) \geq \frac{vm}{v - 1}. \]

In order to get \( x = p_{mt} \) satisfying this inequality, note that \( \text{[11]} \)

\[ p_n \geq n \ln n. \]

Therefore, it is sufficient to consider \( p_{mt} \) satisfying the inequality

\[ \ln p_{tm} \leq \left(1 - \frac{1}{v}\right) t \ln(tm) \left(1 - \frac{3.965}{\ln^2(tm \ln(tm))}\right). \]
On the other hand, for \( n \geq 2 \), (see (4.2) in [3])

\[
\ln p_n \leq \ln n + \ln \ln n + 1.
\]

Thus it is sufficient to choose \( m \) so large that the following inequality holds

\[
\ln(tm) + \ln \ln(tm) + 1 \leq \left(1 - \frac{1}{v}\right)t \ln(tm) \left(1 - \frac{3.965}{\ln^2(tm \ln(tm))}\right),
\]

or, since \( 1 - \frac{1}{v} = \frac{1}{k+1} \), that

\[ (28) \quad \frac{\ln(tm) + \ln \ln(tm) + 1}{\ln(tm)(1 - \frac{3.965}{\ln^2(tm \ln(tm))})} \leq \frac{t}{k + 1}. \]

Let, e.g., \( k = 1 \), \( t = 3 \). We can choose \( m_0 = 350 \). Then the left hand side of \((28)\) equals 1.4976... < 1.5. This means that that at least, for \( m \geq 350 \), the estimate \((3)\) and, for \( m \geq 351 \), the estimate \((21)\) are valid. Using a computer verification for \( m \leq 350 \), we obtain both of these estimates. Note that another short proof of \((3)\) was obtained in [12] (see there Remark 32). Other estimates of the proposition are proved in the same way. □

5. Estimates and formulas for \( N_k(m) \)

**Proposition 9.**

\[ (29) \quad N_k(1) = 2, \quad k = 2, 3, 5, 9, 14. \]

For \( m \geq 2 \),

\[ (30) \quad N_k(m) \leq \left\lceil \frac{R_k + 1(m)}{k + 1} \right\rceil; \]

besides, if \( R_k + 1(m) \equiv 1 \pmod{k + 1} \), then

\[ (31) \quad N_k(m) = \left\lceil \frac{R_k + 1(m)}{k + 1} \right\rceil = \frac{R_k + 1(m) + k}{k + 1}, \]

and, if \( R_k + 1(m) \equiv 2 \pmod{k + 1} \), then

\[ (32) \quad N_k(m) = \left\lceil \frac{R_k + 1(m)}{k + 1} \right\rceil = \frac{R_k + 1(m) + k - 1}{k + 1}. \]

**Proof.** If \( m \geq 2 \), formally the condition \( x = (k + 1)n \geq (k + 1)N_k(m) \) is not stronger than the condition \( x \geq R_k + 1(m) \), since the first one is valid only for \( x \) multiple of \( k + 1 \). Therefore, for \( m \geq 2 \), \((30)\) holds. It allows to calculate the terms of sequence \( \{N_k(m)\} \) for every \( k > 1, \ m \geq 2 \). Since \( N_k(1) \leq N_k(2) \), then, having \( N_k(2) \), we also can prove \((29)\), using direct calculations. Now
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let \(R_{k+1}^k(m) \equiv 1 \pmod{k+1}\). Note that, for \(y = (R_{k+1}^k(m) - 1)/(k+1)\) the interval

\[
\left( ky, (k+1)y \right) = \left( \frac{k}{k+1} \left( R_{k+1}^k(m) - 1 \right), R_{k+1}^k(m) - 1 \right)
\]

cannot contain more than \(m - 1\) primes. Indeed, it is an interval of type \(\left( \frac{k}{k+1}x, x \right)\) for integer \(x\) and the following such interval is

\[
\left( \frac{k}{k+1} \left( R_{k+1}^k(m) \right), R_{k+1}^k(m) \right).
\]

By the definition, \(R_{k+1}^k(m)\) is the \textit{smallest} number such that if \(x \geq R_{k+1}^k(m)\), then \(\{ (\frac{k}{k+1}x, x) \} \) contains \(\geq m\) primes. Therefore, the supposition that already interval (33) contains \(\geq m\) primes contradicts the minimality of \(R_{k+1}^k(m)\). Since the following interval of type \(\left( ky, (k+1)y \right)\) with integer \(y \geq \frac{k}{k+1}(R_{k+1}^k(m) - 1)\) is

\[
\left( \frac{k}{k+1} \left( R_{k+1}^k(m) + k \right), R_{k+1}^k(m) + k \right)
\]

then (31) follows.

Finally, let \(R_{k+1}^k(m) \equiv 2 \pmod{k+1}\). Again show that, for \(y = (R_{k+1}^k(m) - 2)/(k+1)\) the interval

\[
\left( ky, (k+1)y \right) = \left( \frac{k}{k+1} \left( R_{k+1}^k(m) - 2 \right), R_{k+1}^k(m) - 2 \right)
\]

cannot contain more than \(m - 1\) primes. Indeed, comparing interval (34) with interval (33), we see that they contain the same integers except for \(R_{k+1}^k(m) - 2\) which is multiple of \(k + 1\). Therefore, they contain the same number of primes and this number does not exceed \(m - 1\). Again, since the following interval of type \(\left( ky, (k+1)y \right)\) with integer \(y \geq \frac{k}{k+1}(R_{k+1}^k(m) - 2)\) is

\[
\left( \frac{k}{k+1} \left( R_{k+1}^k(m) + k - 1 \right), R_{k+1}^k(m) + k - 1 \right)
\]

then (32) follows.  

Remark 10. Obviously formulas (30)-(32) are valid for not only for the considered values of \(k\), but for arbitrary \(k \geq 1\).

As a corollary from (29), (31)-(32), we obtain the following formula in case \(k = 2\).
Proposition 11.

\[
N_2(m) = \begin{cases} 
2, & \text{if } m = 1, \\
\left\lceil \frac{R_3(m)}{3} \right\rceil, & \text{if } m \geq 2. 
\end{cases}
\]

Formula (35) shows that the case \( k = 2 \) over its regularity not concedes to a classic case \( k = 1 \). Note that, if \( k \geq 3 \) and \( R_{k+1}(m) \equiv j \pmod{k+1} \), \( 3 \leq j \leq k \), then, generally speaking, (30) is not an equality. Evidently, \( N_k(m) \geq N_k(m-1) \) and it is interesting that the equality is attainable (see below sequences (37)-(40)).

Example 12. Let \( k = 3 \), \( m = 2 \). Then \( v = \frac{4}{3} \) and, by (14), \( R_4(2) = 29 \equiv 1 \pmod{4} \). Therefore, by (31), \( N_3(2) = 29 + \frac{3}{4} = 8 \). Indeed, interval \((3\cdot7, 4\cdot7)\) already contains only prime 23.

Example 13. Let \( k = 3 \), \( m = 3 \). Then, by (14), \( R_3(3) = 59 \equiv 3 \pmod{4} \). Here \( N_3(3) = 11 \) which is essentially less than \( \left\lceil \frac{R_3(3)}{4} \right\rceil = 15 \). Indeed, each interval \((3\cdot15, 4\cdot15), (3\cdot14, 4\cdot14), (3\cdot13, 4\cdot13), (3\cdot12, 4\cdot12), (3\cdot11, 4\cdot11)\) contains more than 2 primes and only interval \((3\cdot10, 4\cdot10)\) contains only 2 primes.

In any case, Proposition 9 allows to calculate terms of sequence \( \{N_k(m)\} \) for every considered values of \( k \). So, we obtain the following few terms of \( \{N_k(m)\} \):

for \( k = 2 \),
\[
2, 5, 13, 14, 23, 25, 33, 43, 46, 58, 60, 61, 71, 77, 80, 88, 103, 104, \ldots;
\]
for \( k = 3 \),
\[
2, 8, 11, 17, 26, 38, 40, 41, 48, 57, 68, 68, 70, 87, 96, 100, 108, 109, \ldots;
\]
for \( k = 5 \),
\[
2, 7, 17, 24, 25, 38, 41, 58, 59, 64, 65, 73, 95, 97, 103, 106, 107, 108, \ldots;
\]
for \( k = 9 \),
\[
2, 14, 23, 23, 34, 36, 57, 58, 60, 60, 77, 86, 100, 100, 102, 123, 149, \ldots;
\]
for \( k = 14 \),
\[
2, 11, 24, 37, 38, 39, 50, 96, 96, 96, 96, 97, 97, 125, 125, 132, 178, 178, \ldots.
\]
Remark 14. If, as in [1], [6], instead of intervals \((kn, (k + 1)n)\), to consider intervals \([kn, (k + 1)n]\), then sequences \((5), (36)-(38)\) would begin with 1.

6. Method of small intervals

If we know a theorem of the type: for \(x \geq x_0(\Delta)\), the interval \((x, (1 + \frac{1}{\Delta})x]\) contains a prime, then we can calculate a bounded number of the first terms of sequences \((5)\) and \((36)-(40)\). Indeed, put \(x_1 = kn\), such that \(n \geq x_0k\). Then \((k + 1)n = \frac{k+1}{k}x_1\) and, if \(1 + \frac{1}{\Delta} < \frac{k+1}{k}\), i.e., \(\Delta > k\), then

\[
\left(x_1, (1 + \frac{1}{\Delta})x_1\right) \subset (kn, (k + 1)n).
\]

Thus, if \(n \geq \frac{x_0}{k}\), then the interval \((kn, (k + 1)n)\) contains a prime, and, using method of finite descent, we can find \(N_k(1)\). Further, put \(x_2 = (1 + \frac{1}{\Delta})x_1\). Then interval \((x_2, (1 + \frac{1}{\Delta})x_2]\) also contains a prime. Thus the union

\[
\left(x_1, (1 + \frac{1}{\Delta})x_1\right] \cup \left(x_2, (1 + \frac{1}{\Delta})x_2\right] = \left(x_1, (1 + \frac{1}{\Delta})^2x_1\right]
\]

contains at least two primes. This means that if \((1 + \frac{1}{\Delta})^2x_1 < (k + 1)n\) or \((1 + \frac{1}{\Delta})^2 < 1 + \frac{1}{k}\), then

\[
\left(x_1, (1 + \frac{1}{\Delta})^2x_1\right] \subset (kn, (k + 1)n)
\]

and the interval \((kn, (k + 1)n)\) contains at least two primes; again, using method of finite descent, we can find \(N_k(2)\), etc., if \((1 + \frac{1}{\Delta})^m < 1 + \frac{1}{k}\), then

\[
\left(x_1, (1 + \frac{1}{\Delta})^m x_1\right] \subset (kn, (k + 1)n)
\]

and the interval \((kn, (k + 1)n)\) contains at least \(m\) primes and we can find \(N_k(m)\). In this way, we can find \(N_k(m)\) for \(m < \frac{\ln(1 + \frac{1}{\Delta})}{\ln(1 + \frac{1}{k})}\). In 2002, Ramaré and Saouter [9] proved that interval \((x(1 - 28314000^{-1}), x)\) always contains a prime if \(x > 10726905041\), or, equivalently, interval \((x, 1 + 28313999^{-1})\) contains a prime if \(x > 10726905419\). This means that, e.g., we can find \(N_{14}(m)\) for \(m \leq 1954471\). Unfortunately, this method cannot give the exact estimates and formulas for \(N_k(m)\) as \((30)-(32)\).

We can also to consider a more general application of this method. Consider a fixed infinite set \(P\) of primes which we call \(P\)-primes. Furthermore, consider the following generalization of \(v\)-Ramanujan numbers.

Definition 15. For \(v > 1\), a \((v, P)\)-Ramanujan number \((R_v^P(m))\), is the smallest integer such that if \(x \geq R_v^P(m)\), then \(\pi_P(x) - \pi_P(x/v) \geq m\), where \(\pi_P(x)\) is the number of \(P\)-primes not exceeding \(x\).
Note that every \((v, P)\)-Ramanujan number is \(P\)-prime. If we know a theorem of the type: for \(x \geq x_0(\Delta)\), the interval \((x, (1 + \frac{1}{P})x]\) contains a \(P\)-prime, then, using the above described algorithm we can calculate a bounded number of the first \((v, P)\)-Ramanujan numbers. For example, let \(P\) be the set of primes \(p \equiv 1\) (mod 3). From the result of Cullinan and Hajir \([2]\) it follows, in particular, that for \(x \geq 106706\), the interval \((x, 1.048x]\) contains a \(P\)-prime. Using the considered algorithm, we can calculate the first 14 \((2, P)\)-Ramanujan numbers. They are

\[
(41) \quad 7, 31, 43, 67, 97, 103, 151, 163, 181, 223, 229, 271, 331, 337.
\]

Analogously, if \(P\) is the set of primes \(p \equiv 2\) (mod 3), then the sequence of \((2, P)\)-Ramanujan numbers begins

\[
(42) \quad 11, 23, 47, 59, 83, 107, 131, 167, 227, 233, 239, 251, 263, 281, \ldots;
\]

if \(P\) is the set of primes \(p \equiv 1\) (mod 4), then the sequence of \((2, P)\)-Ramanujan numbers begins

\[
(43) \quad 13, 37, 41, 89, 97, 109, 149, 229, 233, 241, 257, 277, 281, 317, \ldots;
\]

and, if \(P\) is the set of primes \(p \equiv 3\) (mod 4), then the sequence of \((2, P)\)-Ramanujan numbers begins

\[
(44) \quad 7, 23, 47, 67, 71, 103, 127, 167, 179, 191, 223, 227, 263, 307, \ldots.
\]

Denote by \(N_{k}^{(P)}(m)\) the smallest number such that, for \(n \geq N_{k}^{(P)}(m)\), the interval \((kn, (k+1)n]\) contains at least \(m\) \(P\)-primes. It is easy to see that formulas \((30)-(32)\) hold for \(N_{k}^{(P)}(m)\) and \(R_{n+1}^{(P)}(m)\). In particular, in cases \(k = 1, 2\) we have the formulas

\[
(45) \quad N_{1}^{(P)}(m) = \frac{R_{2}^{(P)}(m) + 1}{2}, \quad N_{2}^{(P)}(m) = \left\lfloor \frac{R_{3}^{(P)}(m)}{3} \right\rfloor.
\]

Therefore, the following sequences for \(N_{1}^{(P)}(m)\) for the considered cases of set \(P\) correspond to sequences \((41)-(44)\) respectively:

\[
(46) \quad 4, 16, 22, 34, 49, 52, 76, 82, 91, 112, 115, 136, 166, 169, \ldots;
\]

\[
(47) \quad 6, 12, 24, 30, 42, 54, 66, 84, 114, 117, 120, 126, 132, 141, \ldots;
\]

\[
(48) \quad 7, 19, 21, 45, 49, 55, 75, 115, 117, 121, 129, 139, 141, 159, \ldots;
\]

\[
(49) \quad 4, 12, 24, 34, 36, 52, 64, 84, 90, 96, 112, 114, 132, 154, \ldots.
\]
7. Proof of Theorem 1

For \( k \geq 1 \), denote by \( a(k) \) the least integer \( n > 1 \) for which the interval \( (kn, (k+1)n) \) contains no prime; in the case, when such \( n \) does not exist, we put \( a(k) = 0 \). Taking into account (21), note that \( a(k) = 0 \) for \( k = 1, 2, 3, 5, 9, 14, \ldots \). Consider sequence \( \{a(k)\} \). Its first few terms are (A218831 in [13])

\[
(50) \ 0, 0, 0, 2, 0, 4, 2, 3, 0, 2, 3, 2, 0, 6, 2, 2, 3, 2, 6, 3, 2, 4, 2, 7, 2, 2, 4, 3, \ldots .
\]

Calculations of \( a(k) \) in the range \( \{15, \ldots, 5 \times 10^7\} \) lead to values of \( a(k) \) in the interval \( [2, 16] \) which completes the proof. \( \square \)

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