LEVEL 1 PERFECT CRYSTALS
AND PATH REALIZATIONS OF
BASIC REPRESENTATIONS AT \( q = 0 \).

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Abstract. We present a uniform construction of level 1 perfect crystals \( \mathcal{B} \) for all affine Lie algebras. We also introduce the notion of a crystal algebra and give an explicit description of its multiplication. This allows us to determine the energy function on \( \mathcal{B} \otimes \mathcal{B} \) completely and thereby give a path realization of the basic representations at \( q = 0 \) in the homogeneous picture.

1. Introduction

In the last two decades, intensive study of solvable lattice models in statistical mechanics has led to new constructions of representations of affine Lie algebras and their quantum counterparts. One noteworthy example is a conjectural realization of the basic representations for the quantum affine algebra \( U_q(\widehat{\mathfrak{g}}) \) for \( \mathfrak{g} = \mathfrak{sl}_2 \) via the states of the XXZ model,

\[
\cdots \otimes V \otimes V \otimes V,
\]

where \( V \) is the natural two-dimensional representation of the quantum group \( U_q(\mathfrak{sl}_2) \) extended to a representation of the quantum affine algebra \( U'_q(\widehat{\mathfrak{sl}}_2) \) (the subalgebra of \( U_q(\widehat{\mathfrak{sl}}_2) \) without the “degree operator”). While its rigorous mathematical meaning is not fully understood for arbitrary values of the parameter \( q \), there exists a well-established theory of perfect crystals developed to handle the limiting case \( q = 0 \) of this construction. In particular, the two-dimensional representation \( V \) of \( U'_q(\widehat{\mathfrak{sl}}_2) \) in (1.1) admits the structure of the simplest nontrivial perfect crystal of level 1. This perfect crystal gives rise to a path realization of the basic representations of \( U'_q(\widehat{\mathfrak{sl}}_2) \) and hence to a construction of their crystal graphs at \( q = 0 \). Moreover, this construction has undergone various generalizations: first to the case where \( V \) is the natural \( n \)-dimensional representation of \( U'_q(\widehat{\mathfrak{sl}}_n) \), then to other classical types and to higher level perfect crystals for them.

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(see \((KMN)^2\), \([HK]\) and the references therein), and lastly to perfect crystals and path realizations for some of the exceptional types (\(G_2^{(1)}\) in \([\Xi]\), \(E_6^{(1)}\) and \(E_7^{(1)}\) in \([\Xi]\), and \(D_4^{(3)}\) as announced in \([MOY]\)).

In this paper, we give a uniform construction of level 1 perfect crystals that yields a path realization of the crystal graphs for the basic representations of all affine Lie algebras. Our perfect crystals arise in the following way. Associated to the quantum affine algebra \(U_q(\hat{g})\) is a certain finite-dimensional simple Lie algebra \(g\), which is of type \(X_n\) in the untwisted type \(X_n^{(1)}\) case and is given by (2.4) in the twisted cases. In our construction, \(V\) as a module for \(U_q(g)\) is a direct sum

\[
V = L_g(0) \oplus L_g(\theta)
\]

of the trivial one-dimensional \(U_q(g)\)-representation \(L_g(0)\) with the representation \(L_g(\theta)\) of \(U_q(g)\) having highest weight \(\theta\), which is the highest root of \(g\) in the untwisted case and the highest short root of \(g\) in all other cases but \(A_{2n}^{(2)}\). Our first result, Theorem 3.5, asserts that the action of \(U_q(g)\) on the module \(V\) in (1.2) can be extended to an action of the quantum affine algebra \(U'_q(\hat{g})\), and the union

\[
\mathcal{B} = \mathcal{B}(0) \sqcup \mathcal{B}(\theta)
\]

of the corresponding crystals has the structure of a perfect crystal of level 1 for \(V\).

In order to introduce the homogeneous grading in the path realization of the crystal graph of the basic representations resulting from our perfect crystal of level 1, we study in detail the energy function,

\[
H : \mathcal{B} \otimes \mathcal{B} \to \mathbb{Z}.
\]

The definition of \(H\) implies that its values are constant on each connected component of the crystal graph \(\mathcal{B} \otimes \mathcal{B}\) with the 0-arrows omitted. Thus, the computation of the energy function (1.4) easily reduces to specifying its values on the connected components of \(\mathcal{B}(\theta) \otimes \mathcal{B}(\theta)\). We show that \(H = 1\) on all the components except those isomorphic to \(\mathcal{B}(0)\), \(\mathcal{B}(\theta)\), and \(\mathcal{B}(2\theta)\), where \(H = 0, 0,\) and \(2\), respectively. We describe explicitly the component of \(\mathcal{B}(\theta) \otimes \mathcal{B}(\theta)\) isomorphic to \(\mathcal{B}(\theta)\) in Proposition 5.3 and the component isomorphic to \(\mathcal{B}(2\theta)\) in Proposition 6.4, and thereby conclude the determination of the energy function in Theorem 6.5.

The description of the component isomorphic to \(\mathcal{B}(\theta)\) (there is only one except for type \(A_n, n \geq 2\)) gives rise to the projection

\[
m : \mathcal{B}(\theta) \otimes \mathcal{B}(\theta) \to \mathcal{B}(\theta),
\]
which can be viewed as a multiplication endowing \( B(\theta) \) with a crystal algebra structure. When \( \theta \) is the highest root of \( g \), then \( B(\theta) \) is the crystal graph of the adjoint representation of \( g \), and \((1.5)\) yields the structure of a crystal Lie algebra. When \( \theta \) is the highest short root for \( g \), then the multiplication in \((1.5)\) gives a crystal version of the exceptional Jordan algebra for type \( F_4 \) and of the octonions for type \( G_2 \), without their unit element.

Our construction suggests various generalizations. First, one can replace the module \( V \) in \((1.2)\) by more general finite-dimensional indecomposable modules for \( U_q(\hat{g}) \) and study when they admit the structure of a perfect crystal. In particular, a family of perfect crystals based on Kirillov-Reshetikhin modules was conjectured in [HKOTY]. When a perfect crystal structure exists, the path realization should provide a relation between the finite-dimensional and infinite-dimensional representations of quantum affine algebras. Second, one can try to “melt the crystals” and find the quantum versions of the classical identities that characterize crystal Lie algebras, the crystal exceptional Jordan algebra, and the crystal octonion algebra. Finally, one can attempt to make sense of the tensor product construction of the basic representations of quantum affine algebras.

Our paper is organized as follows: Section 2 is devoted to a review of the notion of a perfect crystal and its corresponding energy function. In Section 3, we present our construction of a perfect crystal of level 1 and verify that all the axioms of a perfect crystal are satisfied except for the connectedness of \( B \otimes B \). The next section establishes that \( B \otimes B \) is indeed connected. In Section 5, we discuss the crystal algebras that arise from forgetting the 0-arrows and looking at the connected component of \( B \otimes B \) which is isomorphic to \( B(\theta) \). In the final section, we evaluate the energy function on the various components of the crystal \( B \otimes B \) minus its 0-arrows.

2. Basics on Perfect Crystals

In this section, we describe the theory of perfect crystals for affine Lie algebras. Our discussion here sets the stage for the next part where we give a uniform construction of level 1 perfect crystals for all affine Lie algebras.

Let \( I = \{0, 1, \ldots, n\} \) be an index set, and let \( A = (a_{i,j})_{i,j \in I} \) be a Cartan matrix of affine type. Thus, \( A \) can be characterized by the following properties: \( a_{i,i} = 2 \) for all \( i \in I \), \( a_{i,j} \in \mathbb{Z}_{\leq 0} \), and \( a_{i,j} = 0 \) if and only if \( a_{j,i} = 0 \) for all \( i \neq j \) in \( I \). The rank of \( A \) is \( n \), and if \( v \in \mathbb{R}^{n+1} \) and \( Av \geq 0 \) (componentwise), then \( v > 0 \) or \( v = 0 \).

We assume \( A \) is indecomposable so that if \( I = I' \cup I'' \) where \( I' \) and \( I'' \) are nonempty, then for some \( i \in I' \) and \( j \in I'' \), the entry \( a_{i,j} \neq 0 \). An affine Cartan matrix is always symmetrizable – there exists a diagonal matrix \( D = \text{diag}(s_i \mid i \in I) \) of positive integers such that \( DA \) is symmetric.

\[\text{We thank V. Chari for this remark.}\]
The free abelian group
\begin{equation}
Q^\vee = \mathbb{Z}h_0 \oplus \mathbb{Z}h_1 \oplus \cdots \oplus \mathbb{Z}h_n \oplus \mathbb{Z}d
\end{equation}
is the extended coroot lattice. The linear functionals \(\alpha_i\) and \(\Lambda_i\) \((i \in I)\) on the complexification \(\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} Q^\vee\) of \(Q^\vee\) given by
\begin{align}
\langle h_j, \alpha_i \rangle &:= \alpha_i(h_j) = a_{j,i} \\
\langle h_j, \Lambda_i \rangle &:= \Lambda_i(h_j) = \delta_{i,j}
\end{align}
are the simple roots and fundamental weights, respectively. Let \(\Pi = \{\alpha_i \mid i \in I\}\) denote the set of simple roots and \(\Pi^\vee = \{h_i \mid i \in I\}\) the set of simple coroots. The weight lattice
\begin{equation}
P = \{\lambda \in \mathfrak{h}^* \mid \lambda(Q^\vee) \subset \mathbb{Z}\}
\end{equation}
contains the set \(P^+ = \{\lambda \in P \mid \lambda(h_i) \in \mathbb{Z}_{\geq 0} \text{ for all } i \in I\}\) of dominant integral weights.

The affine Lie algebra \(\widehat{\mathfrak{g}}\) attached to the data \((A, \Pi, \Pi^\vee, P, Q^\vee)\) has generators \(e_i, f_i\) \((i \in I)\), \(h \in \mathfrak{h}\), which satisfy certain relations (see for example [K] or [HK, Prop. 2.1.6]). The algebra \(\widehat{\mathfrak{g}}\) can be of three types:
\begin{align*}
\widehat{\mathfrak{g}} = \left\{ \begin{array}{l}
A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}, F_4^{(1)}, G_2^{(1)}, \\
A_{2n}^{(2)}, A_{2n-1}^{(2)}, D_{n+1}^{(2)}, E_6^{(2)}, \\
D_4^{(3)}
\end{array} \right\}.
\end{align*}

Associated to \(\widehat{\mathfrak{g}}\) is a finite-dimensional simple Lie algebra \(\mathfrak{g}\) over \(\mathbb{C}\) given by
\begin{equation}
\begin{array}{cccccccc}
\widehat{\mathfrak{g}} & X_n^{(1)} & A_n^{(2)} & A_{2n-1}^{(2)} & D_{n+1}^{(2)} & E_6^{(2)} & D_4^{(3)} \\
\mathfrak{g} & X_n & C_n & C_n & B_n & F_4 & G_2
\end{array}
\end{equation}
The superscript \(t\) is to indicate that the Cartan matrix is the transpose of the one used for the \(F_4\) associated to \(F_4^{(1)}\).

The canonical central element \(c\) and the null root \(\delta\) are given by the expressions
\begin{align}
c &= c_0h_0 + c_1h_1 + \cdots + c_nh_n, \\
\delta &= d_0\alpha_0 + d_1\alpha_1 + \cdots + d_n\alpha_n,
\end{align}
where \(c_0 = 1\), and \(d_0 = 1\) except for type \(A_{2n}^{(2)}\) where \(d_0 = 2\). The first term comes from the fact that the center of the corresponding affine Lie algebra \(\widehat{\mathfrak{g}}\) is generated by \(c\), while the second comes from the fact that the vector \([d_0, d_1, \ldots, d_n]^t \in \mathbb{C}^{n+1}\) spans the null space of the Cartan matrix \(A\). We say that a dominant weight \(\lambda \in P^+\) has level \(\ell\) if \(\langle c, \lambda \rangle := \lambda(c) = \ell\).
Given any \( n \in \mathbb{Z} \) and an indeterminate \( x \), let
\[
[n]_x = \frac{x^n - x^{-n}}{x - x^{-1}}.
\]
Set \([0]_x = 1\) and \([n]_x = [n]_x [n-1]_x \cdots [1]_x\) for \( n \geq 1 \), and for \( m \geq n \geq 0 \), let
\[
\begin{bmatrix} m \\ n \end{bmatrix}_x = \frac{[m]_x!}{[m-n]_x!}.
\]

**Definition 2.6.** The quantum affine algebra \( U_q(\hat{\mathfrak{g}}) \) associated with \((A, \Pi, \Pi', P, Q, Q')\) is the associative algebra with unit element over \( \mathbb{C}(q) \) (where \( q \) is an indeterminate) with generators \( e_i, f_i \) \((i \in \mathcal{I})\), \( q^h \) \((h \in Q')\) satisfying the defining relations

1. \( q^0 = 1 \), \( q^h q^{h'} = q^{h+h'} \) for \( h, h' \in Q' \),
2. \( q^h e_i q^{-h} = q^{\alpha_i(h)} e_i \) for \( h \in Q' \), \( i \in \mathcal{I} \),
3. \( q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i \) for \( h \in Q' \), \( i \in \mathcal{I} \),
4. \( e_i f_j - f_j e_i = \frac{\delta_{i,j}}{q_i - q_i^{-1}} K_i - K_i^{-1} \) for \( i, j \in \mathcal{I} \),
5. \( \sum_{k=0}^{1-a_{i,j}} \left[ \begin{array}{c} 1 - a_{i,j} \\ k \end{array} \right]_{q_i} e_i^{1-a_{i,j}-k} e_j e_i^k = 0 \) for \( i \neq j \),
6. \( \sum_{k=0}^{1-a_{i,j}} \left[ \begin{array}{c} 1 - a_{i,j} \\ k \end{array} \right]_{q_i} f_i^{1-a_{i,j}-k} f_j f_i^k = 0 \) for \( i \neq j \),

where \( q_i = q^{a_i} \) and \( K_i = q^{s_i h_i} \).

Crystal base theory has been developed for \( U_q(\hat{\mathfrak{g}}) \)-modules in the category \( \Theta_{\text{int}} \) of integrable modules. This is the category of \( U_q(\hat{\mathfrak{g}}) \)-modules \( M \) such that

(a) \( M \) has a weight space decomposition: \( M = \bigoplus_{\lambda \in \mathcal{P}} M_\lambda \), where \( M_\lambda = \{ v \in M \mid q^h v = q^{\lambda(h)} v \} \) for all \( h \in Q' \).

(b) there are finitely many \( \lambda_1, \ldots, \lambda_k \in P \) such that \( \text{wt}(M) \subseteq \Omega(\lambda_1) \cup \cdots \cup \Omega(\lambda_k) \), where \( \text{wt}(M) = \{ \lambda \in P \mid M_\lambda \neq 0 \} \) and \( \Omega(\lambda_j) = \{ \mu \in P \mid \mu \in \lambda_j + \sum_{i \in \mathcal{I}} \mathbb{Z}_{\leq 0} \alpha_i \} \);

(c) the elements \( e_i \) and \( f_i \) act locally nilpotently on \( M \) for all \( i \in \mathcal{I} \).

If \( M \) is a module in category \( \Theta_{\text{int}} \), then for each \( i \in \mathcal{I} \), a weight vector \( u \in M_\lambda \) has a unique expression \( u = \sum_{k=0}^{N} f_i^{(k)} u_k \), where \( u_k \in M_{\lambda + k \alpha_i} \cap \ker e_i \) for \( k = 0, 1, \ldots, N \), and \( f_i^{(k)} = f_i^k/[k]_q! \). The Kashiwara operators are defined on \( u \) using these expressions according to the rules

\[
\tilde{e}_i u = \sum_{k=1}^{N} f_i^{(k-1)} u_k, \quad \tilde{f}_i u = \sum_{k=0}^{N} f_i^{(k+1)} u_k.
\]
Let $A_0 = \{ f = g/h \mid g, h \in \mathbb{C}[q], h(0) \neq 0 \}$ be the localization of $\mathbb{C}[q]$ at the ideal $(q)$. Every module $M \in \mathcal{O}_{\text{int}}$ has a special type of $A_0$-lattice called a crystal lattice.

**Definition 2.8.** Assume $M$ is a $U_q(\mathfrak{g})$-module in category $\mathcal{O}_{\text{int}}$. A free $A_0$-submodule $\mathcal{L}$ of $M$ is a crystal lattice if

(i) $\mathcal{L}$ generates $M$ as a vector space over $\mathbb{C}(q)$;
(ii) $\mathcal{L} = \bigoplus_{\lambda \in \mathcal{P}} \mathcal{L}_\lambda$ where $\mathcal{L}_\lambda = M_\lambda \cap \mathcal{L}$;
(iii) $\hat{e}_i \mathcal{L} \subseteq \mathcal{L}$ and $\hat{f}_i \mathcal{L} \subseteq \mathcal{L}$.

Since the operators $\hat{e}_i$ and $\hat{f}_i$ preserve the lattice $\mathcal{L}$, they give well-defined operators on the quotient $\mathcal{L}/q\mathcal{L}$, which we denote by the same symbols.

**Definition 2.9.** A crystal base for a $U_q(\mathfrak{g})$-module $M \in \mathcal{O}_{\text{int}}$ is a pair $(\mathcal{L}, \mathcal{B})$ such that

(1) $\mathcal{L}$ is a crystal lattice of $M$;
(2) $\mathcal{B}$ is a $\mathbb{C}$-basis of $\mathcal{L}/q\mathcal{L} \cong \mathbb{C} \otimes_{A_0} \mathcal{L}$;
(3) $\mathcal{B} = \bigsqcup_{\lambda \in \mathcal{P}} \mathcal{B}_\lambda$, where $\mathcal{B}_\lambda = \mathcal{B} \cap (\mathcal{L}_\lambda/q\mathcal{L}_\lambda)$;
(4) $\hat{e}_i \mathcal{B} \subseteq \mathcal{B} \cup \{0\}$ and $\hat{f}_i \mathcal{B} \subseteq \mathcal{B} \cup \{0\}$ for all $i \in \mathcal{I}$;
(5) $\hat{f}_ib = b'$ if and only if $b = \hat{e}_ib'$ for $b, b' \in \mathcal{B}$ and $i \in \mathcal{I}$.

As each $M \in \mathcal{O}_{\text{int}}$ has such a crystal base, we can associate to each $M$ a crystal graph having $\mathcal{B}$ as the set of vertices. Vertices $b, b' \in \mathcal{B}$ are connected by an arrow labelled by $i$ pointing from $b$ to $b'$ if and only if $\hat{f}_ib = b'$. The crystal graph encodes much of the combinatorial information about $M$.

For $i \in \mathcal{I}$, let $\varepsilon_i, \varphi_i : \mathcal{B} \to \mathbb{Z}$ be defined by

\begin{align}
\varepsilon_i(b) &= \max\{k \geq 0 \mid \hat{e}_kb \in \mathcal{B}\}, \\
\varphi_i(b) &= \max\{k \geq 0 \mid \hat{f}_kb \in \mathcal{B}\}.
\end{align}

From property (5) we see that $\varepsilon_i(b)$ is just the number of $i$-arrows coming into $b$ in the crystal graph and $\varphi_i(b)$ is just the number of $i$-arrows emanating from $b$. Moreover, $\varphi_i(b) - \varepsilon_i(b) = \lambda(h_i)$ for all $b \in \mathcal{B}_\lambda$. Thus, if

$$\varepsilon(b) = \sum_{i \in \mathcal{I}} \varepsilon_i(b)\Lambda_i, \quad \varphi(b) = \sum_{i \in \mathcal{I}} \varphi_i(b)\Lambda_i,$$

then $\text{wt } b = \varphi(b) - \varepsilon(b) = \lambda$ for all $b \in \mathcal{B}_\lambda$.

Morphisms between two crystals $\mathcal{B}_1$ and $\mathcal{B}_2$ associated to $U_q(\mathfrak{g})$ are maps $\Psi : \mathcal{B}_1 \cup \{0\} \to \mathcal{B}_2 \cup \{0\}$ such that $\Psi(0) = 0$; $\text{wt } \Psi(b) = \text{wt } b$, $\varepsilon_i(\Psi(b)) = \varepsilon_i(b)$ and $\varphi_i(\Psi(b)) = \varphi_i(b)$ for all $i \in \mathcal{I}$; and if $b, b' \in \mathcal{B}_1$, $\hat{f}_i b = b'$, and $\Psi(b), \Psi(b') \in \mathcal{B}_2$, then $\hat{f}_i \Psi(b) = \Psi(b')$, $\hat{e}_i \Psi(b') = \Psi(b)$. A morphism $\Psi$ is called strict if it commutes with the Kashiwara operators $\hat{e}_i, \hat{f}_i$ for all $i \in \mathcal{I}$.
One of the most striking features of crystal bases is their behavior under tensor products. If \( M_j \in \mathcal{O}_{\text{int}} \) for \( j = 1, 2 \) and \((\mathcal{L}_j, \mathcal{B}_j)\) are the corresponding crystal bases, set \( \mathcal{L} = \mathcal{L}_1 \otimes_{\mathcal{A}_0} \mathcal{L}_2 \) and \( \mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \). Then \((\mathcal{L}, \mathcal{B})\) is a crystal base of \( M_1 \otimes_{\mathcal{C}(q)} M_2 \), where the action of the Kashiwara operators on \( \mathcal{B} \) is given by

\[
\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} 
\tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\
 b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), 
\end{cases}
\]

(2.11)

\[
\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} 
\tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\
 b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). 
\end{cases}
\]

Corresponding to any \( \lambda \in \mathcal{P} \) is a one-dimensional module \( \mathbb{C}v_\lambda \) for the subalgebra \( U_{\mathbb{Z}}^0 \) of \( U_q(\hat{\mathfrak{g}}) \) generated by \( e_i (i \in J), q^h (h \in Q^\vee) \), where the \( U_{\mathbb{Z}}^0 \)-action is given by \( e_i.v_\lambda = 0 \) and \( q^h.v_\lambda = q^{\lambda(h)}v_\lambda \). The induced module \( V(\lambda) := U_q(\hat{\mathfrak{g}}) \otimes_{U_{\mathbb{Z}}^0} \mathbb{C}v_\lambda \) (the so-called Verma module) has a unique maximal submodule and a unique irreducible quotient \( L(\lambda) \). The modules \( L(\lambda) \) for \( \lambda \in \mathcal{P}^+ \) account for all the irreducible modules in category \( \mathcal{O}_{\text{int}} \). Let \((\mathcal{L}(\lambda), \mathcal{B}(\lambda))\) denote the crystal base corresponding to \( L(\lambda) \). Since the weight space of \( L(\lambda) \) corresponding to the weight \( \lambda \) is one-dimensional, the crystal \( \mathcal{B}(\lambda) \) has a unique element of weight \( \lambda \), which we denote by \( u_\lambda \) in the sequel. It has the property that \( \tilde{e}_i u_\lambda = 0 \) for all \( i \in J \).

The subalgebra \( U'_q(\hat{\mathfrak{g}}) \) of \( U_q(\hat{\mathfrak{g}}) \) generated by \( e_i, f_i, K_i^{\geq 1} (i \in J) \) is often also referred to as the quantum affine algebra. The main difference between \( U_q(\hat{\mathfrak{g}}) \) and \( U'_q(\hat{\mathfrak{g}}) \) is that \( U'_q(\hat{\mathfrak{g}}) \) admits nontrivial finite-dimensional irreducible modules, while \( U_q(\hat{\mathfrak{g}}) \) does not. The theory of perfect crystals, which we introduce next, requires us to work with \( U'_q(\hat{\mathfrak{g}}) \) for that reason. Here we will need the coroot lattice

\[
\tilde{Q}^\vee = \mathbb{Z}h_0 \oplus \mathbb{Z}h_1 \oplus \cdots \oplus \mathbb{Z}h_n,
\]

and its complexification \( \tilde{h} = \mathbb{C} \otimes_{\mathbb{Z}} \tilde{Q}^\vee \). When elements of the \( \mathbb{Z} \)-submodule

\[
\tilde{P} = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \cdots \oplus \mathbb{Z}\Lambda_n
\]

of \( P \) are restricted to \( \tilde{Q}^\vee \), they give the dual lattice of classical weights. Let \( \tilde{P}^+ := \sum_{i=0}^n \mathbb{Z}_{\geq 0}\Lambda_i \) denote the corresponding set of dominant weights.

Every symmetrizable Kac-Moody Lie algebra has a crystal base theory. In particular, the finite-dimensional simple Lie algebras \( \mathfrak{g} \) over \( \mathbb{C} \) have a crystal base theory, and every finite-dimensional \( \mathfrak{g} \)-module has a crystal base. We refer to such crystal bases as finite classical crystals.

We recall the definition of a perfect crystal (see for example [HK, Defn. 10.5.1]).

**Definition 2.14.** For a positive integer \( \ell \), we say a finite classical crystal \( \mathcal{B} \) is a perfect crystal of level \( \ell \) for the quantum affine algebra \( U_q(\hat{\mathfrak{g}}) \) if
(1) there is a finite-dimensional $U_q(\widehat{g})$-module with a crystal base whose crystal graph is isomorphic to $\mathcal{B}$ (when the 0-arrows are removed);
(2) $\mathcal{B} \otimes \mathcal{B}$ is connected;
(3) there exists a classical weight $\lambda_0$ such that
\[
\text{wt}(\mathcal{B}) \subset \lambda_0 + \frac{1}{d_0} \sum_{i \neq 0} Z_{\leq 0} \alpha_i \quad \text{and} \quad |\mathcal{B}_{\lambda_0}| = 1;
\]
(4) for any $b \in \mathcal{B}$, we have
\[
\langle c, \varepsilon(b) \rangle = \sum_{i} \varepsilon_i(b) \Lambda_i(c) \geq \ell;
\]
(5) for each $\lambda \in \bar{P}_+^{\ell} := \{ \mu \in P_+ \mid \langle c, \mu \rangle = \ell \}$, there exist unique vectors $b^\lambda$ and $b_\lambda$ in $\mathcal{B}$ such that $\varepsilon(b^\lambda) = \lambda$ and $\varphi(b_\lambda) = \lambda$.

The significance of perfect crystals is that they provide a means of constructing the crystal base $\mathcal{B}(\lambda)$ of any irreducible $U_q(\widehat{g})$-module $L(\lambda)$ corresponding to a classical weight $\lambda \in \bar{P}_+^{\ell}$.

Theorem 2.15. \cite{KMN} Assume $\mathcal{B}$ is a perfect crystal of level $\ell > 0$. Then for any classical weight $\lambda \in \bar{P}_+^{\ell}$, there is a crystal isomorphism
\[
\Psi : \mathcal{B}(\lambda) \xrightarrow{\sim} \mathcal{B}(\varepsilon(b_\lambda)) \otimes \mathcal{B}
\]
\[
u_\lambda \mapsto \varepsilon(b_\lambda) \otimes b_\lambda.
\]
As a consequence of this theorem, any $\lambda \in \bar{P}_+^{\ell}$ gives rise to a sequence of weights and corresponding elements in the perfect crystal $\mathcal{B}$,

\[
\lambda_0 = \lambda \quad b_0 = b_\lambda
\]
\[
\lambda_{k+1} = \varepsilon(b_{\lambda_k}) \quad b_{k+1} = b_{\lambda_{k+1}} \quad \text{for all} \quad k \geq 1,
\]
such that

\[
\mathcal{B}(\lambda_k) \xrightarrow{\sim} \mathcal{B}(\lambda_{k+1}) \otimes \mathcal{B}
\]
\[
u_{\lambda_k} \mapsto \nu_{\lambda_{k+1}} \otimes b_k.
\]
Iterating this isomorphism, we have

\[
\mathcal{B}(\lambda) \xrightarrow{\sim} \mathcal{B}(\lambda_1) \otimes \mathcal{B} \xrightarrow{\sim} \mathcal{B}(\lambda_2) \otimes \mathcal{B} \otimes \mathcal{B} \xrightarrow{\sim} \cdots
\]
\[
u_\lambda \mapsto \nu_{\lambda_1} \otimes b_0 \mapsto \nu_{\lambda_2} \otimes b_1 \otimes b_0 \mapsto \cdots.
\]

Definition 2.19. For $\lambda \in \bar{P}_+^{\ell}$, the ground state path of weight $\lambda$ is the tensor product
\[
p_\lambda = (b_k)_{k=0}^\infty = \cdots \otimes b_{k+1} \otimes b_k \otimes \cdots \otimes b_1 \otimes b_0,
\]
where the elements \( b_k \in B \) are as in (2.16). A tensor product \( p = (p_k)_{k=0}^{\infty} = \cdots \otimes p_{k+1} \otimes p_k \otimes \cdots \otimes p_1 \otimes p_0 \) of elements \( p_k \in B \) is said to be a \( \lambda \)-path if \( p_k = b_k \) for all \( k \gg 0 \).

**Theorem 2.20.** (KMN)c Assume \( \lambda \in \mathbb{P}_+^\ell \). Then there is a crystal isomorphism \( B(\lambda) \overset{\sim}{\to} \mathcal{P}(\lambda) \), with \( u_\lambda \mapsto p_\lambda \), between the crystal base \( B(\lambda) \) of \( L(\lambda) \) and the set \( \mathcal{P}(\lambda) \) of \( \lambda \)-paths.

The crystal structure of \( \mathcal{P}(\lambda) \) referred to in Theorem 2.20 may be described as follows. Given any \( p = (p_k)_{k=0}^{\infty} \in \mathcal{P}(\lambda) \), let \( N > 0 \) be such that \( p_k = b_k \) for all \( k \geq N \). As in [HK, (10.48)], set

\[
\overline{\text{wt}} p = \lambda_N + \sum_{k=0}^{N-1} \overline{\text{wt}} p_k,
\]

\[
\bar{e}_i p = \cdots \otimes p_{N+1} \otimes \bar{e}_i (p_N \otimes \cdots \otimes p_0),
\]

\[
\bar{f}_i p = \cdots \otimes p_{N+1} \otimes \bar{f}_i (p_N \otimes \cdots \otimes p_0),
\]

\[
\varepsilon_i(p) = \max(\varepsilon_i(p') - \varphi_i(b_N), 0),
\]

\[
\varphi_i(p) = \varphi_i(p') + \max(\varphi_i(b_N) - \varepsilon_i(p'), 0),
\]

where \( p' := p_{N-1} \otimes \cdots \otimes p_1 \otimes p_0 \) and \( \overline{\text{wt}} \) signifies the classical weight of an element of \( B \) or \( \mathcal{P}(\lambda) \).

Equation (2.21) describes the classical weight \( \overline{\text{wt}} p \) (i.e. the element of \( \tilde{P} \) attached to each \( p \in \mathcal{P}(\lambda) \)). We would like to calculate the actual affine weight \( \text{wt} p \) in \( P \). For this, we need the notion of an energy function.

**Definition 2.22.** Let \( V \) be a finite-dimensional \( U'_q(\hat{g}) \)-module with crystal base \((\mathcal{L}, B)\). An energy function on \( B \) is a map \( H : B \otimes B \to \mathbb{Z} \) satisfying

\[
H(\bar{e}_i(b_1 \otimes b_2)) = \begin{cases} 
H(b_1 \otimes b_2) & \text{if } i \neq 0, \\
H(b_1 \otimes b_2) + 1 & \text{if } i = 0 \text{ and } \varphi_0(b_1) \geq \varepsilon_0(b_2) \\
H(b_1 \otimes b_2) - 1 & \text{if } i = 0 \text{ and } \varphi_0(b_1) < \varepsilon_0(b_2).
\end{cases}
\]

for all \( b_1, b_2 \in B \) with \( \bar{e}(b_1 \otimes b_2) \in B \otimes B \).

**Example 2.24.** Let \( \hat{g} \) be the affine Lie algebra \( A_2^{(1)} \), and let \( B \) be the crystal with 3 elements:
Then

\[ H( a \otimes b ) = \begin{cases} 1 & \text{if } a \geq b \\ 0 & \text{if } a < b. \end{cases} \]

**Theorem 2.25.** ([KMN]²a) Assume \( \lambda \in P^+ \) and \( p = (p_k)_{k=0}^{\infty} \in \mathcal{P}(\lambda) \). Then the weight of \( p \) and the character of the irreducible \( U_q(\hat{\mathfrak{g}}) \)-module \( L(\lambda) \) are given by the following expressions:

\[ \text{wt} p = \lambda + \sum_{k=0}^{\infty} (\text{wt} p_k - \text{wt} b_k) \]

\[ - \left( \sum_{k=0}^{\infty} (k+1) \left( H(p_{k+1} \otimes p_k) - H(b_{k+1} \otimes b_k) \right) \right) \delta, \]

\[ \text{ch} L(\lambda) = \sum_{p \in \mathcal{P}(\lambda)} e^{\text{wt} p}. \]

(Note that in Equation (2.26), we are viewing \( \text{wt} p_k \) and \( \text{wt} b_k \) as classical weights, i.e. elements of the \( \mathbb{Z} \)-submodule \( \mathbb{Z} \Lambda_0 \oplus \mathbb{Z} \Lambda_1 \oplus \cdots \oplus \mathbb{Z} \Lambda_n \) of \( P \) rather than considering their restriction to \( \overline{Q}' \).)

Since perfect crystals reveal much about the structure of crystal bases for irreducible modules, which in turn can be used to compute their weights and characters, our goal in the subsequent sections will be to construct perfect crystals for all affine Lie algebras and to calculate the corresponding energy functions.
3. A Uniform Construction of Level 1 Perfect Crystals

Let \( \hat{g} \) be an affine Lie algebra and let

\[
\theta = \begin{cases} 
  d_1\alpha_1 + \cdots + d_n\alpha_n & \text{if } \hat{g} \neq A_{2n}^{(2)}, \\
  \frac{1}{2}(d_1\alpha_1 + \cdots + d_n\alpha_n) & \text{if } \hat{g} = A_{2n}^{(2)},
\end{cases}
\]

where the \( d_i \) are as in (2.5). Thus, when \( \hat{g} = X_n^{(1)} \) (the so-called untwisted case), \( \theta \) is the highest root of \( g \). In all other cases except for \( A_{2n}^{(2)} \), \( \theta \) is the highest short root of \( g \). The specific values of the \( d_i \) can be read from [HK, Ex. 10.1.1] or they can be seen from the marks above the roots \( \alpha_1, \ldots, \alpha_n \) in [K, Tables Aff 1-3].

Let \( B(\theta) \) denote the crystal graph of the irreducible \( U_q(\hat{g}) \)-module \( L_{\hat{g}}(\theta) \). Thus, the crystal graph \( B(\theta) \) corresponds to the adjoint representation of \( g \) in the untwisted case and to the “little” adjoint representation of \( g \) (with highest weight the highest short root) in all other cases but \( A_{2n}^{(2)} \). The set \( \Lambda := \text{wt} B(\theta) \) of weights of \( B(\theta) \) is a subset of \( \Phi \cup \{0\} \), where \( \Phi \) is the root system of \( g \) (except when \( \hat{g} \) is of type \( A_{2n}^{(2)} \)). In the untwisted case equality holds, \( \Lambda = \Phi \cup \{0\} \).

Let \( \Phi^+ \) and \( \Phi^- = -\Phi^+ \) denote the positive and negative roots respectively of \( g \). Set \( \Lambda^+ = \Lambda \cap \Phi^+ \), \( \Lambda^- = -\Lambda^+ \), so that \( \Lambda = \Lambda^+ \cup \{0\} \cup \Lambda^- \). Note if \( \hat{g} = A_{2n}^{(2)} \), then

\[
\Lambda = \Lambda^+ \cup \Lambda^- = \{ \pm (\alpha_i + \cdots + \alpha_{n-1} + \frac{1}{2}\alpha_n) \mid i = 1, \ldots, n-1 \} \cup \{ \pm \frac{1}{2}\alpha_n \}.
\]

Correspondingly, we write

\[
B(\theta) = \begin{cases} 
  \{x_\alpha \mid \alpha \in \Lambda^+\} \cup \{y_i \mid \alpha_i \in \Lambda^+\} \cup \{x_{-\alpha} \mid \alpha \in \Lambda^+\} & \text{if } \hat{g} \neq A_{2n}^{(2)}, \\
  \{x_\alpha \mid \alpha \in \Lambda^+\} \cup \{x_{-\alpha} \mid \alpha \in \Lambda^+\} & \text{if } \hat{g} = A_{2n}^{(2)}.
\end{cases}
\]

Hence in the untwisted case, \( B(\theta) = \{x_{\pm\alpha} \mid \alpha \in \Phi^+\} \cup \{y_i \mid i = 1, \ldots, n\} \).

Set \( B(0) = \{\emptyset\} \), which we identify with the crystal graph of the one-dimensional \( U_q(\hat{g}) \)-module \( L_{\hat{g}}(0) \). As we argue below, the set

\[
B = B(\theta) \cup B(0)
\]

can be endowed with a crystal structure as follows:

\[
(i \neq 0) \quad x_\alpha \xrightarrow{i} x_\beta \iff \alpha - \alpha_i = \beta \quad (\alpha, \beta \in \Lambda),
\]

\[
(i = 0) \quad x_\alpha \xrightarrow{0} x_\theta \iff \alpha + \theta = \beta \quad (\alpha, \beta \neq \pm \theta),
\]

\[
x_{-\theta} \xrightarrow{0} \emptyset \xrightarrow{0} x_\theta.
\]

\[
\text{Thus, when } \hat{g} = X_n^{(1)} \text{ (the so-called untwisted case), } \theta \text{ is the highest root of } g. \text{ In all other cases except for } A_{2n}^{(2)} \text{, } \theta \text{ is the highest short root of } g. \text{ The specific values of the } d_i \text{ can be read from [HK, Ex. 10.1.1] or they can be seen from the marks above the roots } \alpha_1, \ldots, \alpha_n \text{ in [K, Tables Aff 1-3].}
\]

Let \( B(\theta) \) denote the crystal graph of the irreducible \( U_q(\hat{g}) \)-module \( L_{\hat{g}}(\theta) \). Thus, the crystal graph \( B(\theta) \) corresponds to the adjoint representation of \( g \) in the untwisted case and to the “little” adjoint representation of \( g \) (with highest weight the highest short root) in all other cases but \( A_{2n}^{(2)} \). The set \( \Lambda := \text{wt} B(\theta) \) of weights of \( B(\theta) \) is a subset of \( \Phi \cup \{0\} \), where \( \Phi \) is the root system of \( g \) (except when \( \hat{g} \) is of type \( A_{2n}^{(2)} \)). In the untwisted case equality holds, \( \Lambda = \Phi \cup \{0\} \).

Let \( \Phi^+ \) and \( \Phi^- = -\Phi^+ \) denote the positive and negative roots respectively of \( g \). Set \( \Lambda^+ = \Lambda \cap \Phi^+ \), \( \Lambda^- = -\Lambda^+ \), so that \( \Lambda = \Lambda^+ \cup \{0\} \cup \Lambda^- \). Note if \( \hat{g} = A_{2n}^{(2)} \), then

\[
\Lambda = \Lambda^+ \cup \Lambda^- = \{ \pm (\alpha_i + \cdots + \alpha_{n-1} + \frac{1}{2}\alpha_n) \mid i = 1, \ldots, n-1 \} \cup \{ \pm \frac{1}{2}\alpha_n \}.
\]

Correspondingly, we write

\[
B(\theta) = \begin{cases} 
  \{x_\alpha \mid \alpha \in \Lambda^+\} \cup \{y_i \mid \alpha_i \in \Lambda^+\} \cup \{x_{-\alpha} \mid \alpha \in \Lambda^+\} & \text{if } \hat{g} \neq A_{2n}^{(2)}, \\
  \{x_\alpha \mid \alpha \in \Lambda^+\} \cup \{x_{-\alpha} \mid \alpha \in \Lambda^+\} & \text{if } \hat{g} = A_{2n}^{(2)}.
\end{cases}
\]

Hence in the untwisted case, \( B(\theta) = \{x_{\pm\alpha} \mid \alpha \in \Phi^+\} \cup \{y_i \mid i = 1, \ldots, n\} \).

Set \( B(0) = \{\emptyset\} \), which we identify with the crystal graph of the one-dimensional \( U_q(\hat{g}) \)-module \( L_{\hat{g}}(0) \). As we argue below, the set

\[
B = B(\theta) \cup B(0)
\]

can be endowed with a crystal structure as follows:

\[
(i \neq 0) \quad x_\alpha \xrightarrow{i} x_\beta \iff \alpha - \alpha_i = \beta \quad (\alpha, \beta \in \Lambda),
\]

\[
(i = 0) \quad x_\alpha \xrightarrow{0} x_\theta \iff \alpha + \theta = \beta \quad (\alpha, \beta \neq \pm \theta),
\]

\[
x_{-\theta} \xrightarrow{0} \emptyset \xrightarrow{0} x_\theta.
\]
We remark that in the $\hat{g} = A_{2n}^{(2)}$ case, no $\alpha_i$ belongs to $\Lambda^+$. Now we are ready to state our main theorem.

**Theorem 3.5.** $\mathcal{B} = \mathcal{B}(\theta) \sqcup \mathcal{B}(0)$ with the structure given in (3.4) is a perfect crystal of level 1 for every quantum affine algebra $U'_q(\hat{g})$.

Before embarking on the proof of Theorem 3.5, we present several examples. In doing so, we use $\varpi_1, \varpi_2, \ldots$ to denote the fundamental weights of the finite-dimensional algebra $g$.

**Examples 3.6.** (1) $\hat{g} = A_2^{(1)}$, $g = A_2$, $\theta = \alpha_1 + \alpha_2 = \varpi_1 + \varpi_2$, and $c = h_0 + h_1 + h_2$.

(2) $\hat{g} = D_4^{(3)}$, $g = G_2$, $\theta = 2\alpha_1 + \alpha_2 = \varpi_1$, and $c = h_0 + 2h_1 + 3h_2$.

(3) $\hat{g} = C_2^{(1)}$, $g = C_2$, $\theta = 2\alpha_1 + \alpha_2 = 2\varpi_1$, and $c = h_0 + h_1 + h_2$.

(4) $\hat{g} = A_1^{(2)}$, $g = C_2$, $\theta = \alpha_1 + \frac{1}{2}\alpha_2 = \varpi_1$, and $c = h_0 + 2h_1 + 2h_2$. 
More generally, for arbitrary $n$ we have

$$(5) \quad \hat{\mathfrak{g}} = \Lambda_{2n}^{(2)}, \quad \mathfrak{g} = C_n, \quad \theta = \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} + \frac{1}{2}\alpha_n = \varpi_1, \quad \text{and} \quad c = h_0 + 2h_1 + 2h_2 + \cdots + 2h_n.$$
for all $\beta \in \Lambda^\pm$. By [KMPY], it suffices to check that relations (1)-(4) in Definition 2.6.1 hold. This can be done on a case-by-case basis. Since the action of $U_q(g)$ on $V$ is the same as on the $U_q(g)$-module $L_\theta(0) \oplus L_\theta(0)$, the relations that need to be verified are the ones involving $e_0$, $f_0$, and $K_0^{\pm 1}$. Here are a few sample calculations.

In the second one, we will use the fact that $\langle \beta \rangle = 1$, then $\beta - \theta \in \Lambda$ but $\beta + \theta \notin \Lambda$, and analogously, if $\beta(h_0) = -1$, then $\beta + \theta \in \Lambda$ but $\beta - \theta \notin \Lambda$.

\[
\begin{align*}
(e_0 f_0 - f_0 e_0).b & = [2]_{q_0} e_0 x_\theta - [2]_{q_0} f_0 x_{-\theta} = [2]_{q_0} (\emptyset - \emptyset) = 0, \\
\left( K_0 - K_0^{-1} \right). \emptyset & = 0, \\
(e_0 f_0 - f_0 e_0).x_\beta & = \begin{cases} -f_0 e_0 x_\beta = -x_\beta & \text{if } \beta(h_0) = -1, \\
e_0 f_0 x_\beta = x_\beta & \text{if } \beta(h_0) = 1, \\
0 & \text{otherwise}, \\
\end{cases} \\
\left( K_0 - K_0^{-1} \right).x_\beta & = \frac{q_{\theta} - q_{-\theta}}{q_0 - q_0^{-1}} x_\beta = \begin{cases} -x_\beta & \text{if } \beta(h_0) = -1, \\
x_\beta & \text{if } \beta(h_0) = 1, \\
0 & \text{otherwise}.
\end{cases}
\end{align*}
\]

**Step 2.** There exists a classical weight $\lambda_0 \in \bar{P}$ such that $|B_{\lambda_0}| = 1$ and $\text{wt} \lambda \subset \lambda_0 + \frac{1}{\alpha} \sum_{i \neq 0} Z \leq \alpha_i$.

This is easily seen by taking $\lambda_0 = \theta$.

**Step 3.** For all $b \in B$, we have $\langle c, \varepsilon(b) \rangle \geq 1$.

First suppose that $b \in B(\theta)$ and $b \neq x_\theta$. Then there exists an $i \neq 0$ such that $\varepsilon_i b \neq 0$; i.e., $\varepsilon_i(b) \geq 1$, so that $\langle c, \varepsilon(b) \rangle \geq 1$ must hold. When $b = x_\theta$, we have $\varepsilon_0(b) = 2$ (see (3.2))) so that $\langle c, \varepsilon(b) \rangle \geq 1$. Finally, when $b = \emptyset$, then $\varepsilon_0(\emptyset) = 1$, so that $\langle c, \varepsilon(b) \rangle \geq 1$ in this case also.

**Step 4.** For all $\lambda \in P^+$ with $\langle \lambda \rangle = 1$, there exist unique elements $b^\lambda$ and $b_\lambda \in B$ such that $\varepsilon(b^\lambda) = \lambda$ and $\Phi(b_\lambda) = \lambda$.

When $\lambda = \Lambda_0$, we can take $b^\lambda_0 = \emptyset = b_\Lambda_0$. Now when $\lambda = \Lambda_i$ for $i \neq 0$, then setting $b^\lambda_i = y_i = b_{\Lambda_i}$ will give the desired result.

All that remains in the proof of Theorem 3.3 is to show that the crystal graph $B \otimes B$ is connected. We devote the next section to this task.
4. \( \mathcal{B} \otimes \mathcal{B} \) is connected

Our approach to proving this can be summarized as follows. We forget the 0-arrows in \( \mathcal{B} \otimes \mathcal{B} \) and view it as a crystal graph for the quantum algebra \( U_q(\mathfrak{g}) \) associated to the simple Lie algebra \( \mathfrak{g} \):

\[
(4.1) \quad \mathcal{B} \otimes \mathcal{B} = (\mathcal{B}(\theta) \otimes \mathcal{B}(\theta)) \sqcup (\mathcal{B}(\theta) \otimes \mathcal{B}(0)) \sqcup (\mathcal{B}(0) \otimes \mathcal{B}(\theta)) \sqcup (\mathcal{B}(0) \otimes \mathcal{B}(0)).
\]

Since crystals corresponding to simple modules are connected, it suffices to locate the maximal vectors (\( \tilde{e}_i b = 0 \) for all \( i \in \mathcal{I} \setminus \{0\} \)) inside the components on the right and show that they are all connected to one another by various \( i \)-arrows for \( i \in \mathcal{I} \).

There are obvious maximal vectors inside \( \mathcal{B} \otimes \mathcal{B} \),

1. \( x_\theta \otimes x_\theta \)
2. \( x_\theta \otimes \emptyset \)
3. \( \emptyset \otimes x_\theta \)
4. \( \emptyset \otimes \emptyset \)
5. \( x_\theta \otimes x_{-\theta} \),

and they can be connected as displayed below:

\[
\emptyset \otimes \emptyset \rightarrow 0 \otimes x_\theta \rightarrow \emptyset \otimes x_{-\theta} \rightarrow 0 \rightarrow x_\theta \otimes x_{-\theta} \rightarrow 0 \rightarrow \emptyset \otimes \emptyset \rightarrow x_\theta \otimes x_\theta,
\]

where \( \rightarrow \rightarrow \rightarrow \) indicates that an appropriate sequence of Kashiwara operators \( \tilde{f}_i \) with \( i \in \mathcal{I} \setminus \{0\} \) has been applied. All other maximal vectors have the form

6. \( x_\theta \otimes x_{-\theta} - \alpha \) for some \( \alpha \in \Lambda^+ \)
7. \( x_\theta \otimes y_i \) for some \( i \) such that \( \alpha_i \in \Lambda^+ \). (When \( \widehat{\mathfrak{g}} \) is of type \( \text{A}_{2n}^{(2)} \) or \( \text{D}_{n+1}^{(2)} \), this case does not occur, because no \( \alpha_i \) belongs to \( \Lambda^+ \).)

Since they can be connected to \( \emptyset \otimes x_\theta \) via

\[
x_\theta \otimes x_{-\theta} - \alpha \rightarrow \emptyset \otimes x_{\alpha} \rightarrow \emptyset \otimes x_{-\alpha} \rightarrow x_\theta \otimes x_{-\theta} \rightarrow 0 \rightarrow \emptyset \otimes y_i \rightarrow x_\theta \otimes y_i,
\]

the entire crystal graph \( \mathcal{B} \otimes \mathcal{B} \) must be connected. \( \square \)

5. Crystal Algebras

The irreducible \( U_q(\mathfrak{g}) \)-module \( L_q(\theta) \) is special, since except for type \( \text{A}_n, n \geq 2 \), there is a unique (up to scalar factor) \( U_q(\mathfrak{g}) \)-invariant projection,

\[
(5.1) \quad \mathbf{m} : L_q(\theta) \otimes L_q(\theta) \rightarrow L_q(\theta),
\]

giving a canonical algebra structure on \( L_q(\theta) \). (Imposing skew-symmetry will also ensure that such a map is unique for type \( \text{A}_n, n \geq 2 \).) When \( \theta \) is the highest root of \( \mathfrak{g} \) and \( q = 1 \) (i.e. in the classical \( \mathfrak{g} \)-module case), the module \( L_q(\mathfrak{g}) \) is isomorphic to the adjoint module \( \mathfrak{g} \) and (5.1) is just the Lie algebra structure.
When $\theta$ is the highest short root of $\mathfrak{g}$ (and again $q = 1$), we obtain the 27-dimensional exceptional simple Jordan algebra minus its unit element if $\mathfrak{g}$ is of type $F_4$, and the 8-dimensional octonion algebra minus its unit element if $\mathfrak{g}$ is of type $G_2$. Thus, $(\ref{5.4})$ defines quantum analogues of the above algebras for generic values of $q$. Taking the limit $q = 0$, we obtain a strict morphism of crystals:

\begin{equation}
(\ref{5.2})
m : B(\theta) \otimes B(\theta) \to B(\theta).
\end{equation}

We will describe this morphism (and its inverse) explicitly in terms of the root vectors, since it is of independent interest. It will also be crucial in determining the energy function $H$.

Our computations of the energy function $H$ in the next section will rely heavily on knowing the various connected components that result from omitting the 0-arrows of $B \otimes B$, since $H$ must be constant on those components. The connected components $\mathcal{C}(x_0 \otimes y_i)$, for values of $i$ such that vertex $i$ is connected to vertex 0 in the Dynkin diagram of $\mathcal{g}$, are of particular interest, for they are isomorphic to the crystal $B(\theta)$. In the next proposition, we construct the isomorphism $\Psi : B(\theta) \to \mathcal{C}(x_0 \otimes y_i)$ explicitly. The strict morphism $m : B(\theta) \otimes B(\theta) \to B(\theta)$ is the inverse of $\Psi$ on $\mathcal{C}(x_0 \otimes y_i)$ and is zero on all other components of $B(\theta) \otimes B(\theta)$. We regard the morphism $m$ as a multiplication on $B(\theta)$ resulting in a “crystal algebra” $(B(\theta), m)$. We will discuss this algebra and provide an interesting example.

**Proposition 5.3.** There is a crystal isomorphism $B(\theta) \sim \mathcal{C}(x_0 \otimes y_i)$ whenever vertex $i$ is connected to vertex 0 in the Dynkin diagram of $\mathcal{g}$ and $\alpha_i$ belongs to $\Lambda^+$. 

**Remark 5.4.** When $\mathcal{g}$ is of type $\Lambda_n^{(2)}$ or $D_n^{(2)}$, there is no value of $i$ satisfying the hypotheses of the proposition, and there are no components of $B(\theta) \otimes B(\theta)$ isomorphic to $B(\theta)$.

**Proof.** Our proof of Proposition $5.3$ will be broken into a number of cases. We first establish that a crystal isomorphism $B(\theta) \sim \mathcal{C}(x_0 \otimes y_i)$ exists for $\mathcal{g} \neq \Lambda_n^{(1)}$, $C_n^{(1)}$. For such $\mathcal{g}$, the root $\theta$ is equal to the fundamental weight $\varpi_i$. We begin the argument by introducing a “grading” on $\Lambda^+$ as follows: set

$$\Lambda^+_j = \{ \gamma \in \Lambda^+ | \text{ coefficient of } \alpha_i \text{ in } \gamma \text{ is } j \}. $$

Then $\Lambda^+ = \Lambda^+_0 \sqcup \Lambda^+_1 \sqcup \Lambda^+_2$ where $\Lambda^+_2 = \{ \theta \}$. We construct a map $\Psi : B(\theta) \to \mathcal{C}(x_0 \otimes y_i)$ using this decomposition. First, for $\gamma = \theta \in \Lambda^+_2$, set

\begin{equation}
(\ref{5.5})
\Psi(x_\gamma) = x_0 \otimes y_i.
\end{equation}

Next, if $\gamma \in \Lambda^+_1$, then there is a sequence $(i_1 = i, i_2, \ldots, i_\ell)$ with $1 \leq i_\ell \neq i \leq n$ for $\ell > 1$ so that $f_{i_1} \cdots f_{i_\ell} x_\gamma = x_\gamma$. Set $\beta := \alpha_{i_1} + \cdots + \alpha_{i_\ell}$. Then by the tensor
product rules, \( \tilde{f}_i \cdots \tilde{f}_1 \tilde{f}_j (x_\theta \otimes \emptyset) = x_\theta \otimes x_{-\beta} \in \mathcal{C}(x_\theta \otimes y_i) \subset \mathcal{B}(\theta) \otimes \mathcal{B}(\theta) \), and we define

\[
(5.6) \quad \Psi(x_\gamma) = x_\theta \otimes x_{-\beta}.
\]

Finally, suppose \( \gamma \in \Lambda_0^+ \), and let \( \text{supp}(\gamma) \) denote the set of indices \( k \) such that \( \alpha_k \) occurs in \( \gamma \) with nonzero coefficient. We take \( j = (j_1, \ldots, j_t = i) \) to be the sequence of nodes in the Dynkin diagram connecting the nodes corresponding to the indices in \( \text{supp}(\gamma) \) with \( i \) as in the examples below.

**Examples 5.7.**

(1) The root \( \gamma = \alpha_1 + \alpha_2 \) for \( E_6^{(1)} \) has \( \text{supp}(\gamma) = \{1, 2\} \). Since \( \theta = \varpi_6 \), we have \( j = (3, 6) \).

(2) The root \( \gamma = \alpha_2 + 2\alpha_3 \) for \( F_4^{(1)} \) has \( \text{supp}(\gamma) = \{2, 3\} \). Since \( \theta = \varpi_1 \), the sequence \( j \) is a singleton \( j = (1) \).

Once we have the sequence \( j \) for \( \gamma \in \Lambda_0^+ \), we set \( \alpha = \alpha_{j_1} + \cdots + \alpha_{j_t} = \alpha_{j_1} + \cdots + \alpha_{j_{t-1}} + \alpha_i \) so that \( \gamma + \alpha \in \Lambda_1^+ \). Then we may write \( \gamma + \alpha = \theta - \beta \). Note \( \beta \neq 0 \), since the coefficient of \( \alpha_i \) in \( \gamma + \alpha \) is 1. We set

\[
(5.8) \quad \Psi(x_\gamma) = x_{\theta-\alpha} \otimes x_{-\beta}.
\]

This is well-defined since \( \alpha \) is uniquely determined.

On the remaining elements in the crystal, we specify the values of \( \Psi \) as follows:

\[
(5.9) \quad \begin{align*}
\Psi(x_{-\theta}) &= y_i \otimes x_{-\theta} \\
\Psi(x_{-\gamma}) &= x_\beta \otimes x_{-\theta+\alpha} \quad \text{if} \quad \Psi(x_\gamma) = x_{\theta-\alpha} \otimes x_{-\beta} \quad \text{and} \quad \alpha \geq 0, \\
\Psi(y_i) &= x_{\theta-\alpha_i} \otimes x_{-\theta+\alpha_i} \quad \text{if vertex} \ i \ \text{is connected to vertex} \ 0, \\
\Psi(y_j) &= x_{\theta-\alpha_i \cdots - \alpha_i} \otimes x_{-\theta+\alpha_i \cdots + \alpha_i} \quad \text{if} \quad j \neq i,
\end{align*}
\]

where \( \{i_1 = i, i_2, \ldots, i_t = j\} \) is the unique (ordered) sequence of vertices connecting vertex \( i \) and vertex \( j \) in the Dynkin diagram.

By its construction, the map \( \Psi \) preserves the weight. We claim that \( \Psi \) commutes with the Kashiwara operators and so is a crystal morphism. We will check our assertion for \( \tilde{f}_k \) \( (k = 1, \ldots, n) \) by considering the various possibilities for \( \gamma \in \Lambda_j^+ \) \( j = 0, 1, 2 \):
(Case $j = 2$) If $\gamma = \theta$, then $f_i x_\theta = x_{\theta - \alpha_i}$ while $f_k x_\theta = 0$ for all $k \neq i$. Then $f_i \Psi(x_\theta) = f_i (x_\theta \otimes y_i) = x_\theta \otimes x_{-\alpha_i}$. But $\Psi(f_i x_\theta) = \Psi(x_{\theta - \alpha_i}) = x_\theta \otimes x_{-\alpha_i}$ as $\theta - \alpha_i \in \Lambda^+_\theta$. For $k \neq i$, $f_k \Psi(x_\theta) = 0 = \Psi(f_k x_\theta)$, so $\Psi$ commutes with $f_k$ for all $k$ when applied to $x_\theta$.

(Case $j = 1$) We may assume $\gamma = \theta - \beta \in \Lambda^+_\theta$ and $\Psi(x_\gamma) = x_\theta \otimes x_{-\beta}$. If $f_k x_\gamma \neq 0$, then $\gamma - \alpha_k \in \Lambda$, and if $k \neq i$, then $\gamma - \alpha_k \in \Lambda^+$. In that case, $\Psi(f_k x_\gamma) = \Psi(x_{\gamma - \alpha_k}) = x_\theta \otimes x_{-\beta - \alpha_k}$. Since $f_k \Psi(x_\gamma) = f_k (x_\theta \otimes x_{-\beta}) = x_{\theta - \alpha_k} \otimes x_{-\alpha_i}$, the result holds in this case. Now when $k = i$ and $f_i x_\gamma \neq 0$, then $\gamma - \alpha_i \in \Lambda^+_\theta$. The “$\alpha$” corresponding to $\gamma - \alpha_i$ is $\alpha_i$ in this situation, and so $\Psi(f_i x_\gamma) = \Psi(x_{\gamma - \alpha_i}) = x_{\theta - \alpha_i} \otimes x_{-\beta}$. Note that $f_i \Psi(x_\gamma) = f_i (x_\theta \otimes x_{-\beta}) = x_{\theta - \alpha_i} \otimes x_{-\beta}$ because $-\beta + \alpha_i \not\in \Lambda$. Indeed, if $-\beta + \alpha_i \in \Lambda$, then $\theta = \gamma - \beta = (\gamma + \alpha_i) - (\beta - \alpha_i)$ implies that $\gamma + \alpha_i \in \Lambda^+$. Moreover, since $\gamma + \alpha_i \in \Lambda^+_\theta$, it must be that $\gamma + \alpha_i = \theta$. But then $\varphi_i(x_\theta) \geq 2$, a contradiction. Consequently, $\Psi$ commutes with $f_i$ also, and the $j = 1$ case is handled.

(Case $j = 0$) We assume now that $\gamma \in \Lambda^+_\theta$ and let $j = (j_1, \ldots, j_l = i)$ be the corresponding sequence. Then $\Psi(x_\gamma) = x_{\theta - \alpha} \otimes x_{-\beta}$ (where $\alpha = \alpha_j + \cdots + \alpha_{j_l}$), $\gamma + \alpha \in \Lambda^+_\theta$, and $\theta = \gamma + \alpha + \beta$. If $f_k x_\gamma \neq 0$, then $k \neq i$, $k \in \text{supp}(\gamma)$, and $\gamma - \alpha_k \in \Lambda^+_\theta$. We consider two possible scenarios:

1. Node $k$ is connected to node $j_1$ in the Dynkin diagram of $\tilde{G}$: In this case, $(\gamma - \alpha_k) + (\alpha + \alpha_k) \in \Lambda^+_\theta$, and therefore $\Psi(f_k x_\gamma) = \Psi(x_{\gamma - \alpha_k}) = x_{\theta - \alpha_k} \otimes x_{-\beta}$. On the other hand, we claim that $f_k \Psi(x_\gamma) = f_k (x_{\theta - \alpha} \otimes x_{-\beta}) = x_{\theta - \alpha_k} \otimes x_{-\beta}$. Indeed, $\theta = \gamma + \alpha + \beta = (\gamma - \alpha_k) + (\alpha + (l + 1) \alpha_k) + (\beta - \lambda_k)$ whenever $\beta - \lambda_k \in \Lambda^+$. That is to say $\theta - \alpha = (l + 1) \alpha_j \in \Lambda$, which implies $\varphi_k(\theta - \alpha) \geq \varepsilon_k(-\beta) + 1$. Thus $f_k$ should act on the first component of $x_{\theta - \alpha} \otimes x_{-\beta}$ according to the tensor product rules; hence $f_k (x_{\theta - \alpha} \otimes x_{-\beta}) = x_{\theta - \alpha_k} \otimes x_{-\beta}$ as asserted, and $f_k$ commutes with $\Psi$ when applied to $x_\gamma$.

2. Node $k$ is not connected to node $j_1$: Then $\gamma - \alpha_k + \alpha \in \Lambda^+_\theta$. It follows from the $\Lambda^+_\theta$ case above that $\Psi(f_k x_\gamma) = x_{\theta - \alpha} \otimes x_{-\beta - \alpha_k}$. Now $\varphi_k(\theta - \alpha) = 0$ since $k$ is not connected to $j_1$. So $f_k$ should act on the second component of $x_{\theta - \alpha} \otimes x_{-\beta}$, which implies $f_k \Psi(x_\gamma) = f_k (x_{\theta - \alpha} \otimes x_{-\beta}) = x_{\theta - \alpha_k} \otimes x_{-\beta - \alpha_k}$. Thus, $f_k$ commutes with $\Psi$ for such $x_\gamma$. As this is the last case that needed to be considered, we have shown that $f_k$ commutes with $\Psi$ on all of $\mathcal{B}(\theta)$.

Now $\varphi_k$ and $\varepsilon_k$ have the same values on $x_\theta$ and $x_\theta \otimes y_i$ for each $k$. Since $\Psi$ commutes with the Kashiwara operators, these functions will have the same values on elements in the crystals $\mathcal{B}(\theta)$ and $\mathcal{C}(x_\theta \otimes y_i)$ which correspond under $\Psi$, because they are connected to $x_\theta$ and $x_\theta \otimes y_i$ respectively by the same sequences of Kashiwara operators. Hence $\Psi : \mathcal{B}(\theta) \rightarrow \mathcal{C}(x_\theta \otimes y_i)$ is a crystal isomorphism for all types of affine algebras except for $\Lambda_n^{(1)}$ and $C_n^{(1)}$, which we now address.
Case $A_n^{(1)}$: When $n = 1$, we have $\theta = \alpha_1$, and

$$
\begin{align*}
x_{\alpha_1} &\mapsto x_{\alpha_1} \otimes y_1 \\
y_1 &\mapsto y_1 \otimes y_1 \\
x_{-\alpha_1} &\mapsto y_1 \otimes x_{-\alpha_1}
\end{align*}
$$

defines a crystal isomorphism in this case.

For $n \geq 2$, there are two vertices connected to the 0 vertex in the Dynkin diagram, namely the first and last. When $i = 1$, define a weight-preserving map $\Psi : \mathcal{B}(\theta) \rightarrow \mathcal{C}(x_{\theta} \otimes y_1)$ by assigning

$$
(5.10) \quad \begin{align*}
x_{\alpha_j+\cdots+\alpha_k} &\mapsto x_{\alpha_1+\cdots+\alpha_k} \otimes x_{-\alpha_1-\cdots-\alpha_{j-1}} \\
x_{\alpha_j} &\mapsto x_{\alpha_1+\cdots+\alpha_j} \otimes x_{-\alpha_1-\cdots-\alpha_{j-1}} \\
x_{\alpha_1} &\mapsto x_{\alpha_1} \otimes y_1 \\
y_1 &\mapsto y_1 \otimes y_1 \\
y_j &\mapsto x_{\alpha_1+\cdots+\alpha_{j-1}} \otimes x_{-\alpha_1-\cdots-\alpha_{j-1}}
\end{align*}
$$

and by using (5.9) for the negative roots. One can check that the Kashiwara operators $\tilde{f}_j$ commute with $\Psi$ and proceed as before to show that $\Psi$ is a crystal isomorphism. For example, in the case of $x_{\alpha_j+\cdots+\alpha_k}$, there are exactly two Kashiwara operators $\tilde{f}_j$ and $\tilde{f}_k$ whose action on each side of (5.10) is nonzero. For the first,

$$
\Psi(\tilde{f}_j x_{\alpha_j+\cdots+\alpha_k}) = x_{\alpha_j+\cdots+\alpha_k+\alpha} \mapsto x_{\alpha_1+\cdots+\alpha_k} \otimes x_{-\alpha_1-\cdots-\alpha_j}, \quad \text{while}
$$

$$
\tilde{f}_j \Psi(x_{\alpha_j+\cdots+\alpha_k}) = \tilde{f}_j \left(x_{\alpha_1+\cdots+\alpha_k} \otimes x_{-\alpha_1-\cdots-\alpha_{j-1}}\right) = x_{\alpha_1+\cdots+\alpha_k} \otimes x_{-\alpha_1-\cdots-\alpha_j}.
$$

For the second,

$$
\Psi(\tilde{f}_k x_{\alpha_j+\cdots+\alpha_k}) = x_{\alpha_j+\cdots+\alpha_k} \mapsto x_{\alpha_1+\cdots+\alpha_k} \otimes x_{-\alpha_1-\cdots-\alpha_{j-1}}, \quad \text{while}
$$

$$
\tilde{f}_k \Psi(x_{\alpha_j+\cdots+\alpha_k}) = \tilde{f}_k \left(x_{\alpha_1+\cdots+\alpha_k} \otimes x_{-\alpha_1-\cdots-\alpha_{j-1}}\right) = x_{\alpha_1+\cdots+\alpha_k} \otimes x_{-\alpha_1-\cdots-\alpha_{j-1}}.
$$

The remaining calculations to verify that $\Psi$ commutes with the Kashiwara operators and is a crystal isomorphism are similar and are left to the reader.

Assume now that $i = n$ and define a weight-preserving map $\Psi : \mathcal{B}(\theta) \rightarrow \mathcal{C}(x_{\theta} \otimes y_n)$ by assigning

$$
(5.11) \quad \begin{align*}
x_{\alpha_j+\cdots+\alpha_k} &\mapsto x_{\alpha_j+\cdots+\alpha_n} \otimes x_{-\alpha_{k+1}-\cdots-\alpha_n} \\
x_{\alpha_j} &\mapsto x_{\alpha_j+\cdots+\alpha_n} \otimes x_{-\alpha_{j+1}-\cdots-\alpha_n} \\
x_{\alpha_n} &\mapsto x_{\alpha_n} \otimes y_n \\
y_j &\mapsto x_{\alpha_{j+1}+\cdots+\alpha_n} \otimes x_{-\alpha_{j+1}-\cdots-\alpha_n} \\
y_n &\mapsto y_n \otimes y_n.
\end{align*}
$$
The argument is exactly as before. For example, the computations involving the first expression in (5.11) are these:

\[
\begin{align*}
\Psi(\tilde{f}_j x_{\alpha_j + \cdots + \alpha_k}) &= \Psi(x_{\alpha_j+1 + \cdots + \alpha_k}) = x_{\alpha_j+1 + \cdots + \alpha_n} \otimes x_{-\alpha_{k+1} - \cdots - \alpha_n} \\
\tilde{f}_j \Psi(x_{\alpha_j + \cdots + \alpha_k}) &= \tilde{f}_j (x_{\alpha_j+1 + \cdots + \alpha_n} \otimes x_{-\alpha_{k+1} - \cdots - \alpha_n}) = x_{\alpha_j+1 + \cdots + \alpha_n} \otimes x_{-\alpha_{k+1} - \cdots - \alpha_n} \\
\Psi(\tilde{f}_k x_{\alpha_j + \cdots + \alpha_k}) &= \Psi(x_{\alpha_j + \cdots + \alpha_k}) = x_{\alpha_j+1 + \cdots + \alpha_n} \otimes x_{-\alpha_{k} - \cdots - \alpha_n} \\
\tilde{f}_k \Psi(x_{\alpha_j + \cdots + \alpha_k}) &= \tilde{f}_k (x_{\alpha_j+1 + \cdots + \alpha_n} \otimes x_{-\alpha_{k+1} - \cdots - \alpha_n}) = x_{\alpha_j+1 + \cdots + \alpha_n} \otimes x_{-\alpha_{k+1} - \cdots - \alpha_n}.
\end{align*}
\]

**Case** $C_n^{(1)}$: In this case, $\theta = 2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n$ and vertex 0 is connect to vertex 1. We define a weight-preserving map $\Psi : B(\theta) \rightarrow C(x_\theta \otimes y_1)$ by assigning

\[
(5.12)
\]

\[
x_{\alpha_j + \cdots + \alpha_{k-1} + 2\alpha_k + \cdots + 2\alpha_{n-1} + \alpha_n} \mapsto x_{\alpha_j + \cdots + \alpha_{k-1} + 2\alpha_k + \cdots + 2\alpha_{n-1} + \alpha_n} \otimes x_{-\alpha_1 - \cdots - \alpha_{j-1}} \quad (1 \leq j \leq k \leq n - 1)
\]

\[
x_{\alpha_j + \cdots + \alpha_k} \mapsto x_{\alpha_j + \cdots + \alpha_k} \otimes x_{-\alpha_1 - \cdots - \alpha_{j-1}} \quad (1 \leq j \leq k \leq n)
\]

\[
x_{\alpha_1} \mapsto x_{\alpha_1} \otimes y_1
\]

\[
x_{\alpha_j} \mapsto x_{\alpha_j} \otimes y_1 \otimes x_{-\alpha_1 - \cdots - \alpha_{j-1}} \quad (2 \leq j \leq n)
\]

\[
y_1 \mapsto y_1 \otimes y_1.
\]

\[
y_{\gamma} \mapsto x_{\alpha_j + \cdots + \alpha_k} \otimes x_{-\alpha_1 - \cdots - \alpha_j} \quad (2 \leq j \leq n)
\]

and using (5.9). The arguments are straightforward as in the previous cases. Here is a sample computation for $x_{\gamma}$, where $\gamma = \alpha_j + \cdots + \alpha_{k-1} + 2\alpha_k + \cdots + 2\alpha_{n-1} + \alpha_n$:

\[
\begin{align*}
\Psi(\tilde{f}_j x_{\gamma}) &= \Psi(x_{\gamma - \alpha_j}) \\
&= x_{\alpha_j + \cdots + \alpha_{k-1} + 2\alpha_k + \cdots + 2\alpha_{n-1} + \alpha_n} \otimes x_{-\alpha_1 - \cdots - \alpha_{j-1}} \\
\tilde{f}_j \Psi(x_{\gamma}) &= \tilde{f}_j (x_{\alpha_j + \cdots + \alpha_{k-1} + 2\alpha_k + \cdots + 2\alpha_{n-1} + \alpha_n} \otimes x_{-\alpha_1 - \cdots - \alpha_{j-1}}) \\
&= x_{\alpha_j + \cdots + \alpha_{k-1} + 2\alpha_k + \cdots + 2\alpha_{n-1} + \alpha_n} \otimes x_{-\alpha_1 - \cdots - \alpha_{j-1}} \\
\Psi(\tilde{f}_k x_{\gamma}) &= \Psi(x_{\gamma - \alpha_k}) \\
&= x_{\alpha_1 + \cdots + \alpha_{k-1} + \alpha_k + 2\alpha_{k+1} + \cdots + 2\alpha_{n-1} + \alpha_n} \otimes x_{-\alpha_1 - \cdots - \alpha_{j-1}} \\
\tilde{f}_k \Psi(x_{\gamma}) &= \tilde{f}_k (x_{\alpha_j + \cdots + \alpha_{k-1} + 2\alpha_k + \cdots + 2\alpha_{n-1} + \alpha_n} \otimes x_{-\alpha_1 - \cdots - \alpha_{j-1}}) \\
&= x_{\alpha_1 + \cdots + \alpha_{k-1} + \alpha_k + 2\alpha_{k+1} + \cdots + 2\alpha_{n-1} + \alpha_n} \otimes x_{-\alpha_1 - \cdots - \alpha_{j-1}}.
\end{align*}
\]

**Example 5.13.** When $\hat{g} = D_4^{(3)}$ and $g = G_2$, then $B(\theta)$ is the crystal of the 7-dimensional irreducible $U_q(g)$-module, and the crystal isomorphism $C(x_\theta \otimes y_1) \cong B(\theta)$ gives the “crystal octonion algebra” (without the unit element). Displayed
below is a portion of the multiplication table in this algebra. All other products between basis elements are zero.

|     | $x_{-\theta}$ | $x_{-\alpha_1 - \alpha_2}$ | $x_{-\alpha_1}$ | $y_1$ |
|-----|---------------|-----------------------------|-----------------|------|
| $x_{\theta}$ | 0             | $x_{\alpha_1}$             | $x_{\alpha_1 + \alpha_2}$ | $x_{\theta}$ |
| $x_{\alpha_1 + \alpha_2}$ | $-x_{-\alpha_1}$ | $y_1$ | 0 | 0 |
| $x_{\alpha_1}$ | $x_{-\alpha_1 - \alpha_2}$ | 0 | 0 | 0 |
| $y_1$ | $-x_{-\theta}$ | 0 | 0 | 0 |

6. The Energy Function

The energy function on the perfect crystal $B$ enables us to determine the weight of a path (see Theorem 2.25). Here we compute the energy function, which we assume has been normalized so that $H(\emptyset \otimes \emptyset) = 0$. First, observe how the maximal vectors of the right side of (4.1) are connected:

\[\begin{align*}
x_{\theta} \otimes x_{\theta} &\leftarrow x_{\theta} \otimes \emptyset \leftarrow x_{\theta} \otimes x_{-\theta} \\
x_{\theta} \otimes x_{\theta-\alpha} &\leftarrow x_{\theta} \otimes x_{-\alpha} \leftarrow \emptyset \otimes x_{-\theta} \leftarrow \emptyset \otimes x_{\theta} \leftarrow \emptyset \otimes \emptyset.
\end{align*}\]

Thus, by (2.23) we deduce

\[\begin{array}{|c|c|c|c|c|}
\hline
\text{max'l vec.} & \emptyset \otimes \emptyset & \emptyset \otimes x_{\theta} & x_{\theta} \otimes x_{-\theta} & x_{\theta} \otimes \emptyset \\
\text{H-value} & 0 & 1 & 0 & 1 & 2 & 1 & 0 \\
\hline
\end{array}\]

Since the values of $H$ are constant on each connected component of the crystal graph $B \otimes B$ without the $0$-arrows, we need to describe the connected components containing the maximal vectors above.

\[\begin{align*}
(6.3) &\quad (1) \text{ The connected component } C(x_{\theta} \otimes x_{-\theta}) \text{ containing the maximal vector } x_{\theta} \otimes x_{-\theta} \text{ is a singleton } \{x_{\theta} \otimes x_{-\theta}\} \text{ and by the above table, } H = 0. \\
&\quad (2) \text{ Similarly, } C(\emptyset \otimes \emptyset) = \{\emptyset \otimes \emptyset\} \text{ and } H = 0.
\end{align*}\]
(3) The connected components containing $x_\theta \otimes \emptyset$ and $\emptyset \otimes x_\theta$ have a simple description,

$$C(x_\theta \otimes \emptyset) = \{ b \otimes \emptyset | b \in B(\theta) \}$$

$$C(\emptyset \otimes x_\theta) = \{ \emptyset \otimes b | b \in B(\theta) \}$$

and on both, $H = 1$.

Next, we will describe the component $C(x_\theta \otimes x_\theta)$. The components of the form $C(x_\theta \otimes x_\theta - \alpha)$ will then consist of all the tensor products of elements of $B$ which have not been already specified, and on them, $H = 1$.

**Proposition 6.4.** If $\hat{g} \neq A^{(2)}_{2n}$, the connected component of the maximal vector $x_\theta \otimes x_\theta$ is given by

$$C(x_\theta \otimes x_\theta) = \{ x_\alpha \otimes x_\beta | \alpha, \beta \in \Lambda^+ \sqcup \Lambda^-, \alpha \leq \beta \}$$

$$\bigcup \{ y_i \otimes x_\beta | \theta(h_i) > 0, \beta = \alpha_i + \gamma \in \Lambda^+ \text{ for some } \gamma \in \Lambda \}$$

$$\bigcup \{ x_\alpha \otimes y_i | \theta(h_i) > 0, \alpha = \alpha_i + \gamma \in \Lambda^+ \text{ for some } \gamma \in \Lambda \},$$

where $\alpha \leq \beta$ if and only if $\beta - \alpha \in \sum_{i \in J \setminus \{0\}} \mathbb{Z}_{\geq 0} \alpha_i$.

If $\hat{g} = A^{(2)}_{2n}$, we have

$$C(x_\theta \otimes x_\theta) = \{ x_\alpha \otimes x_\beta | \alpha, \beta \in \Lambda^+ \sqcup \Lambda^-, \alpha \leq \beta \}.$$

**Proof.**

**Step 1.** For any $\alpha \in \Lambda^+ \sqcup \Lambda^-$, there exist sequences $i = (i_1, \ldots, i_r)$ and $j = (j_1, \ldots, j_t)$ of indices in $J \setminus \{0\}$ such that $x_\theta \otimes x_\theta \overset{i_1}{\to} x_\alpha \otimes x_\theta$ and $x_\theta \otimes x_\theta \overset{j}{\to} x_\alpha \otimes x_\beta$.

(Here the shorthand $\overset{i}{\to}$ signifies that the sequence of arrows $i_1 \to i_2 \to \cdots$ has been applied.)

By the tensor product rule, the Kashiwara operator $\tilde{e}_k$ acts on the second component first, so using that and the connectedness of $B(\theta)$, we see that both $x_\alpha \otimes x_\theta$ and $x_\alpha \otimes x_\alpha$ can be connected to $x_\theta \otimes x_\theta$ by applying a suitable sequence of operators $\tilde{e}_k$.

**Step 2.** Suppose that $\alpha, \beta \in \Lambda^+ \sqcup \Lambda^-$ and $\alpha \leq \beta$. Then there exist $\gamma \in \Lambda$ and a sequence $j$ such that $x_\gamma \otimes x_\theta \overset{j}{\to} x_\alpha \otimes x_\beta$. Hence $x_\alpha \otimes x_\beta \in C(x_\theta \otimes x_\theta)$. 

Working in the crystal base $\mathcal{B}(\theta)$, we may choose sequences $i_{\alpha}$ and $i_{\beta}$ such that $x_\theta \mapsto x_\alpha$ and $x_\theta \mapsto x_\beta$. We induct on $r = |i_{\alpha}|$ and $s = |i_{\beta}|$. To begin, if $r = 0$, then $\theta = \alpha \leq \beta \leq \theta$, which forces all of them to be equal. Thus, $x_\alpha \otimes x_\beta = x_\theta \otimes x_\theta$ in this case, and we may take $\gamma = \theta$ and $j$ to be the empty sequence. Similarly, if $s = 0$, then $\beta = \theta$ and $x_\alpha \otimes x_\beta = x_\alpha \otimes x_\theta$, so the result follows from Step 1.

Suppose then that $r > 0$ and $s > 0$. Now if there exists an $\ell \in i_{\beta}$ such that $\beta + \alpha_\ell \in \Lambda$, but $\alpha - \alpha_\ell \not\in \Lambda$, then $\tilde{e}_\ell(x_\alpha \otimes x_\beta) = x_\alpha \otimes x_{\beta + \alpha_\ell}$. By induction, there exist $\gamma \in \Lambda$ and a sequence $j$ such that $x_\gamma \otimes x_\theta \overset{i}{\mapsto} x_\alpha \otimes x_\beta$. Adjoining $\ell$ to the end of the sequence gives the desired sequence $\gamma$. Since by Step 1, there exists a sequence $i$ such that $x_\theta \otimes x_\theta \overset{i}{\mapsto} x_\alpha \otimes x_\beta$, it follows that $x_\alpha \otimes x_\beta \in \mathcal{C}(x_\theta \otimes x_\theta)$.

Suppose there exists a value of $k$ such that $\alpha + \alpha_k \in \Lambda$ and $\alpha + \alpha_k \leq \beta$. Then $\tilde{e}_k(x_\alpha \otimes x_\beta) = x_\alpha + \alpha_k \otimes x_\beta$. By induction, there exist $x_\gamma$ with $\gamma \in \Lambda$ and a sequence $y$ such that $x_\gamma \otimes x_\theta \overset{j}{\mapsto} x_{\alpha + \alpha_k} \otimes x_\beta$. Adjoining $k$ to the end of the sequence and using Step 1 then gives the result.

Hence we may suppose that no such $\ell$ and $k$ exist. Therefore, (1) for all $k \in i_{\alpha}$, we have $\alpha + \alpha_k \not\in \Lambda$, i.e. $\tilde{e}_k x_\alpha = 0$; and (2) for all $\ell \in i_{\beta}$ with $\beta + \alpha_\ell \in \Lambda$, we have $\alpha - \alpha_\ell \not\in \Lambda$. By the tensor product rule, this implies that $\tilde{e}_m(x_\alpha \otimes x_\beta) = 0$ for all $m \in J \setminus \{0\}$. Hence $x_\alpha \otimes x_\beta$ is a maximal vector in $\mathcal{B}(\theta) \otimes \mathcal{B}(\theta)$. But by [HK Cor. 4.4.4(2)], it must be that $x_\alpha$ is a maximal vector of $\mathcal{B}(\theta)$. Thus, $\theta = \alpha \leq \beta \leq \theta$, which forces $r = 0 = s$. Since we are assuming both $r$ and $s$ are positive, this case cannot happen. The proof of Step 2 is now complete.

**Step 3.** The elements $y_i \otimes x_\beta$ with $\theta(h_i) > 0$ and $\beta = \alpha_i + \gamma \in \Lambda^+$ for some $\gamma \in \Lambda$ belong to $\mathcal{C}(x_\theta \otimes x_\theta)$.

The assumptions imply that $\tilde{e}_i(y_i \otimes x_\beta) = x_{\alpha_i} \otimes x_\beta$. Then by Steps 1 and 2, we can find $\gamma \in \Lambda$ and sequences $i$ and $j$ so that $x_\theta \otimes x_\theta \overset{i}{\mapsto} x_\gamma \otimes x_\theta \overset{j}{\mapsto} x_{\alpha_i} \otimes x_\beta \overset{i}{\mapsto} y_i \otimes x_\beta$. Hence, $y_i \otimes x_\beta \in \mathcal{C}(x_\theta \otimes x_\theta)$, as claimed.

**Step 4.** The elements $x_{-\alpha} \otimes y_i$ such that $\theta(h_i) > 0$ and $\alpha = \alpha_i + \gamma \in \Lambda^+$ for some $\gamma \in \Lambda$ belong to $\mathcal{C}(x_\theta \otimes x_\theta)$.

Observe here that because $-\alpha - \alpha_i \in \Lambda^-$, $\tilde{f}_i(x_{-\alpha} \otimes y_i) = x_{-\alpha} \otimes x_{-\alpha_i}$, where $-\alpha \leq -\alpha_i$. Thus, by Step 2, we can find a sequence $j$ such that $x_\gamma \otimes x_\theta \overset{j}{\mapsto} x_{-\alpha} \otimes x_{-\alpha_i} \overset{i}{\mapsto} x_{-\alpha} \otimes y_i$, which implies that $x_{-\alpha} \otimes y_i \in \mathcal{C}(x_\theta \otimes x_\theta)$ by Step 1.

We have argued in Steps 1-4 that the sets listed in the statement of the lemma belong to the connected component $\mathcal{C}(x_\theta \otimes x_\theta)$. It remains to show that the set in the right-hand side of our statement is closed under the action of Kashiwara operators, which can be done in a rather straightforward way using the tensor product rule.
product rule. For instance, suppose $\alpha, \beta \in \Lambda$ and $\alpha \leq \beta$. Then by the tensor product rule, we have $\tilde{f}_i(x_\alpha \otimes x_\beta) = x_{\alpha - \alpha_i} \otimes x_\beta$ or $\tilde{f}_i(x_\alpha \otimes x_\beta) = x_\alpha \otimes x_{\beta - \alpha_i}$. In the first case, we are done. In the latter case, since $\alpha \leq \beta$, we have $\underline{\alpha} \subset \underline{\beta}$, and $i \in \underline{\alpha} \setminus \underline{\beta}$. Hence $\alpha \leq \beta - \alpha_i$. The other cases can be verified similarly.

Combining the results of the last two sections, we arrive at the following:

**Theorem 6.5.** The energy function $H$ on the level 1 perfect crystal $\mathcal{B}$ is given explicitly by Table (6.2), Proposition 5.3, and Proposition 6.4.

The energy function formula for the weight of a path in Theorem 2.20 (in particular, the coefficient of $\delta$) gives the homogeneous grading in the path realization of the crystal base of any level 1 (basic) representation of $U_q(\hat{\mathfrak{g}})$. Theorem 6.5 and the general character formula in Theorem 2.25 then yield new character formulas for the basic representations. Their explicit combinatorial expressions in terms of the roots deserve further study.

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