A NEW NUMERICAL SCHEME FOR RESISTIVE RELATIVISTIC MAGNETOHYDRODYNAMICS USING METHOD OF CHARACTERISTICS

Makoto Takamoto$^1$ and Tsuyoshi Inoue$^2$

1 Theoretical Astrophysics Group, Department of Physics, Kyoto University, Kyoto 606-8502, Japan
2 Division of Theoretical Astronomy, National Astronomical Observatory of Japan, Mitaka, Tokyo 181-8588, Japan

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ABSTRACT

We present a new numerical method of special relativistic resistive magnetohydrodynamics with scalar resistivity that can treat a range of phenomena, from non-relativistic to relativistic (shock, contact discontinuity, and Alfvén wave). The present scheme calculates the numerical flux of fluid by using an approximate Riemann solver and electromagnetic field by using the method of characteristics. Since this scheme uses appropriate characteristic velocities, it is capable of accurately solving problems that cannot be approximated as ideal magnetohydrodynamics and whose characteristic velocity is much lower than the velocity of light. The numerical results show that our scheme can solve the above problems as well as nearly ideal MHD problems. Our new scheme is particularly well suited to systems with initially weak magnetic field and mixed phenomena of relativistic and non-relativistic velocity, for example magnetorotational instability in an accretion disk and super Alfvénic turbulence.

Key words: magnetohydrodynamics (MHD) – methods: numerical – plasmas – relativistic processes

1. INTRODUCTION

The magnetohydrodynamics (MHD) approximation has some interesting properties, for example the flux freezing and magnetic pressure; the former can be used for the collimation of the jet, and the latter for the acceleration of the plasma. Thus, the magnetic field is considered an essential ingredient for many astrophysical phenomena. In particular, many observations indicate that most of the high-energy phenomena in astrophysics are related to the strongly magnetized relativistic plasma around some compact objects, for example active galactic nuclei (Antonucci 1993; Urry & Padovani 1995), relativistic jets (Blandford & Konigl 1979; Mirabel & Rodríguez 1999), pulsar winds (Rees & Gunn 1974; Camus et al. 2009), gamma-ray bursts (GRBs; Woosley 1993; Piran 2004), and so on. Since it is extremely difficult to solve the relativistic MHD (RMHD) equations analytically, the theoretical investigations in fully nonlinear regimes are mainly based on numerical simulations (McKinney & Gammie 2004; Inoue et al. 2011). Most of these studies approximate the plasma as an ideal RMHD fluid. One reason for this is that the ideal RMHD is an excellent approximation of high-energy phenomena for ordinary parameters. However, when one considers extreme phenomena, such as neutron star mergers, or the central engines of GRB, the electrical conductivity can be small, and highly resistive regions may appear. In addition, when one considers magnetic reconnection, the resistivity plays an essential role in this phenomenon. Magnetic reconnection is one of the most important phenomena, since it is highly dynamic, and it changes magnetic field energy into fluid energy (Zweibel & Yamada 2009; Zenitani et al. 2009; Li et al. 2007). Though numerical results of ideal RMHD exhibit magnetic reconnection, this originates in the purely numerical resistivity, and this is unphysical. For this reason, using resistive RMHD is important for the understanding of reconnection and related phenomena.

In order to consider ohmic dissipation, one only has to take into account an additional term $-\nabla \times (\nabla \times B)/\sigma$ in the induction equation of non-relativistic MHD. However, similar to other non-relativistic dissipation, this induction equation is parabolic and it is well known that it is acausal. As a result, if one takes into account ohmic dissipation in an RMHD in a similar way, the equation inevitably includes unphysical exponential growing modes, and is unstable for small perturbations similar to other dissipation (Hiscock & Lindblom 1983, 1985). This unphysical divergence results from the fact that one neglects the time derivative of the electric field in the induction equation with ohmic dissipation. For this reason, when one takes into account the ohmic dissipation, one has to consider the time evolution of the electric field, that is, one has to deal with the relativistic electromagnetic hydrodynamic equation. This equation is a telegrapher equation and satisfies causality.

In this paper, we present a new numerical scheme for resistive RMHD. There are several examples of pioneering work for resistive RMHD, for example Komissarov (2007, hereafter K07) proposed a numerical method that solves hyperbolic fluxes by using the Harten–Lax–van Leer (HLL) prescription, and damping of the electric field by ohmic dissipation that is very stiff by using Strang-splitting techniques; Palenzuela et al. (2009, hereafter P09) proposed a numerical method that solves hyperbolic fluxes by the local Lax–Friedrichs approximate Riemann solver, and the stiff part by using implicit–explicit Runge–Kutta methods. However, these methods use the velocity of light as the characteristic velocity, and their numerical solutions are diffusive when one considers problems whose characteristic velocity is much lower than that of light. This indicates that their numerical solutions are diffusive in many important high plasma β dynamics, and also their solutions become highly diffusive when the characteristic velocity of phenomena is much lower than the velocity of light. In particular, when one solves the dynamics of the accretion disk around a black hole with a relativistic jet, one has to use relativistic resistive MHD code that can solve both highly relativistic and non-relativistic dynamics with resistivity for the following three reasons: (1) the saturation of the magnetorotational instability (MRI) depends on the resistivity; (2) the dynamics of an accretion disk are not ordinarily relativistic, especially, the dynamics of the MRI are sub-Alfvénic; and (3) the dynamics of the jet are highly relativistic. For these reasons, previous schemes are diffusive in such phenomena, and we need more accurate numerical schemes. We are developing a new numerical scheme capable of accurately solving problems.
whose characteristic velocity is quite different from the velocity of light. In this scheme, we obtain the numerical flux of a fluid by using the velocity of sound as the characteristic velocity, and the numerical flux of electromagnetic field by using appropriate characteristic velocities of RMHD. This enables us to obtain accurate numerical results when we consider problems whose characteristic velocity is much lower than the velocity of light. In addition, P09 pointed out that the Strang-splitting method used in the Komissarov method is unstable when applied to discontinuous flows with large conductivities. However, we find that this problem is not related to the Strang-splitting method, but to the evolution of electric field \( \mathbf{E} \) during the primitive recovery that is introduced in the method by P09. By considering this procedure, we can apply the Strang-splitting method to discontinuous flows with large conductivities.

This paper is organized as follows. In Section 2, the basic equations of resistive RMHD are presented. In Section 3, we present the numerical method. Results of numerical test problems previously presented are shown in Section 4. In Section 5, we present results of numerical test problems that cannot be solved accurately by previous codes.

2. BASIC EQUATIONS

Throughout this paper, we use the units
\[
c = 1. 
\]
In Cartesian coordinates, the Minkowski metric tensor \( \eta_{\mu\nu} \) is given by
\[
\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1). \tag{2}
\]
Variables indicated by Greek letters take values from 0 to 3, and those indicated by Roman letters take values from 1 to 3.

2.1. The Maxwell Equations

The covariant Maxwell equations can be written as
\[
\partial_\nu F^{\mu\nu} = I^\mu, \tag{3}
\]
\[
\partial_\nu ^* F^{\mu\nu} = 0, \tag{4}
\]
where \( F^{\mu\nu} \) is the Maxwell tensor, \(^* F^{\mu\nu} \) is the Faraday tensor, and \( I^\mu \) is the four-vector of electric current.

If we consider highly ionized plasma, the electric and magnetic susceptibilities can be neglected. Then, one has
\[
^* F^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}, \tag{5}
\]
where
\[
\epsilon^{\mu\nu\rho\sigma} = \sqrt{-g} \epsilon_{\mu\nu\rho\sigma}, \tag{6}
\]
is the Levi–Civita alternating tensor of spacetime, and \( \epsilon_{\mu\nu\rho\sigma} \) is the four-dimensional Levi–Civita symbol.

We introduce a future-directed unit time-like vector \( n^\mu \) normal to a space-like hypersurface \( \Sigma \). Using \( n^\mu \), we can decompose the Maxwell tensor into following forms:
\[
F^{\mu\nu} = n^\mu E^\nu - n^\nu E^\mu + n_\rho \epsilon^{\mu\nu\rho\sigma} B_\sigma \tag{7}
\]
Similarly, the current four-vector \( I^\mu \) can be decomposed into
\[
I^\mu = q n^\mu + J^\mu, \tag{8}
\]
where \( q \) is the charge density observed in the rest frame of \( n^\mu \) and \( J^\mu \) is the conduction current satisfying \( J^\mu n_\mu = 0 \).

In the following, we consider only Minkowski spacetime, so \( n^\mu = (1, 0, 0, 0) \).

By using the decomposition of the Maxwell tensor equation (7) and the current four-vector (8), the Maxwell equations can be split into the familiar set:
\[
\nabla \cdot \mathbf{E} = q, \tag{9}
\]
\[
\nabla \cdot \mathbf{B} = 0, \tag{10}
\]
\[
\partial_t \mathbf{E} - \nabla \times \mathbf{B} = -\mathbf{J}, \tag{11}
\]
\[
\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0. \tag{12}
\]
From the Maxwell equations, we can derive the electric charge conservation law:
\[
\partial_t q + \nabla \cdot \mathbf{J} = 0. \tag{13}
\]

2.2. The Hydrodynamic Equations

The relativistic hydrodynamic equations can be obtained from the conservation of mass, momentum, and energy:
\[
\partial_\mu N^\mu = 0, \tag{14}
\]
\[
\partial_\mu T_{\mu\nu} = 0, \tag{15}
\]
where \( N^\mu \) is the mass density current and \( T_{\mu\nu} \) the energy–momentum tensor defined respectively as
\[
N^\mu = \rho u^\mu, \tag{16}
\]
\[
T_{\mu\nu} = T_{\mu\nu}^{\text{fluid}} + T_{\mu\nu}^{\text{EM}}, \tag{17}
\]
where
\[
T_{\mu\nu}^{\text{fluid}} \equiv \rho hu^\mu u^\nu + \rho \eta_{\mu\nu}, \tag{18}
\]
\[
T_{\mu\nu}^{\text{EM}} \equiv F^{\mu\rho} F_{\rho\nu} - \frac{1}{4} (F_{\rho\sigma} F_{\rho\sigma}) n_{\mu\nu}. \tag{19}
\]
Here, \( h = 1 + \epsilon + p/\rho \) is the specific enthalpy, \( \rho \) is the proper rest mass density, \( p \) is the thermodynamic pressure, and \( \epsilon \) is the specific internal energy.

The evolution equation of a relativistic resistive MHD is
\[
\partial_t \left( \begin{array}{c} D \\ m^i \\ e \end{array} \right) + \partial_j \left( \begin{array}{c} F^i_j \\ F^{0i}_m \\ F^i_c \end{array} \right) = 0, \tag{20}
\]
where \( D, m^i, \) and \( e \) are the density, momentum density, and total energy density, respectively. In the laboratory frame, \( D, m, \) and \( e \) are given by
\[
D = \gamma \rho, \tag{21}
\]
\[
m = \rho \gamma^2 v + \mathbf{E} \times \mathbf{B}, \tag{22}
\]
\[
e = \rho \gamma^2 v - p + \frac{1}{2}(E^2 + B^2), \tag{23}
\]
where \( \mathbf{v} \) is the fluid three-velocity, \( \gamma = (1 - \mathbf{v}^2)^{-1/2} \) is the Lorentz factor, and the numerical fluxes are

\[
F^{\text{i}}_D = D\mathbf{v}^i, \quad (24)
\]

\[
F^{\text{ij}}_m = m^i \mathbf{v}^j + p\eta^i_j - E^i E^j - B^i B^j + \frac{1}{2} (E^2 + B^2) \eta^i_j, \quad (25)
\]

\[
F^{\text{i}}_e = m^i. \quad (26)
\]

This is the most common form of perfect fluid equations for numerical hydrodynamics.

### 2.3. Ohm’s Law

The system of Equations (9)–(12), and (20) is closed by means of Ohm’s law. Although there are various forms of Ohm’s law, we consider only the simplest kind of relativistic Ohm’s law that accounts only for the plasma resistivity, and that assumes that it is isotropic similar to previous studies K07 and P09. In the covariant form, it is given by

\[
I^\mu = \sigma F^{\mu\nu} u_\nu + q_0 u^\mu, \quad (27)
\]

where \( \sigma = 1/\eta \) is the conductivity, \( \eta \) is the resistivity, and \( q_0 = -I_\mu u^\mu \) is the electric charge density as measured in the fluid frame.

As for the Maxwell equations and fluid equations, we can decompose Equation (27) into 3 + 1 form, and then the space component of Equation (27) is given by

\[
\mathbf{J} = \sigma \gamma [\mathbf{E} + \mathbf{v} \times \mathbf{B} - (\mathbf{E} \cdot \mathbf{v}) \mathbf{v}] + q\mathbf{v}. \quad (28)
\]

In the fluid rest frame, Equation (28) becomes

\[
\mathbf{J} = \sigma \mathbf{E}. \quad (29)
\]

The ideal MHD limit of Ohm’s law can be obtained in the limit of infinite conductivity (\( \sigma \to \infty \)). In this limit, Equation (28) reduces to

\[
\mathbf{E} + \mathbf{v} \times \mathbf{B} - (\mathbf{E} \cdot \mathbf{v}) \mathbf{v} = 0. \quad (30)
\]

Splitting this equation into the components that are normal and parallel to the velocity vector, it becomes

\[
\mathbf{E}_\parallel + \mathbf{v} \times \mathbf{B} = 0, \quad (31)
\]

\[
\mathbf{E}_\perp - (\mathbf{E} \cdot \mathbf{v}) \mathbf{v} = 0. \quad (32)
\]

From these equations, we can obtain the usual result

\[
\mathbf{E} = -\mathbf{v} \times \mathbf{B}. \quad (33)
\]

### 3. NUMERICAL METHOD

In this section, we present our new numerical scheme for the resistive RMHD. Since the pioneering studies of resistive RMHD K07 and P09 use the velocity of light as the characteristic velocity, their solution becomes highly diffusive when the characteristic velocity is much lower than the velocity of light. In our new scheme, we obtain the numerical flux of fluid by using the velocity of sound as the characteristic velocity, and the numerical flux of the electromagnetic field by using the Alfvén velocity as the characteristic velocity. This enables us to obtain accurate numerical results even when the characteristic velocity is much lower than the velocity of light. In the following sections, we consider the one-dimensional case. The extension to the multi-dimensional scheme using the constrained transport method (Evans & Hawley 1988; Stone & Norman 1992) will be shown in our next paper.
\[
\partial_t E - \nabla \times B = -qv,
\]

(46)\[
\partial_t E = -J_c.
\] (47)

In component form, Equations (45) and (46) reduce to

\[
\partial_t B^x = 0,
\]

(48)\[
\partial_t B^y - \partial_x E^z = 0,
\] (49)\[
\partial_t B^z + \partial_x E^y = 0,
\]

(50)\[
\partial_t E^x = -qv^x,
\]

(51)\[
\partial_t E^y + \partial_x B^z = -qv^y,
\]

(52)\[
\partial_t E^z - \partial_x B^y = -qv^z.
\]

(53)

We solve Equations (49), (50), (52), and (53) using the MOC, which will be shown in Section 3.2. Equation (51) is solved using the Runge–Kutta method. The numerical scheme for the stiff equation, Equation (47), will be shown in Section 3.3.

3.2. Method of Characteristics

The MOC can be used to solve the initial value problems of advective and hyperbolic equations. As is well known, the Maxwell equations are hyperbolic, so we can solve the Maxwell equations accurately by using this method.

The Maxwell equations for the transverse fields are Equations (49), (50), (52), and (53). By adding and subtracting these equations, for \(E^y\), \(B_z\), and \(J^y\), we obtain

\[
[\partial_t \pm c_{ch} \partial_x]F = -\frac{1}{2} J^y,
\]

(54)\[
\pm F \equiv \frac{1}{2} (E^y \pm B^z),
\]

(55)

where \(c_{ch}\) is the characteristic velocity, and this is equal to the speed of light in ordinal Maxwell equations.

The transverse fields are recovered from \(\pm F\) by

\[
E^y = + F - F,
\]

(56)\[
B^z = + F - F.
\]

(57)

The left-hand side of Equation (54) is the total derivative \(dF/dt\) for an observer moving at velocity \(\pm c_{ch}\).

Let us consider conservative discretizations of Equations (50) and (52):

\[
\bar{E}_{y,i}^{n+1} = \bar{E}_{y,i}^n + \frac{\Delta n}{\Delta x_i} \left[ \left( E_{y,i+1/2}^{n+1/2} - E_{y,i-1/2}^{n+1/2} \right) \right],
\]

(58)\[
\bar{E}_{y,i}^{n+1} = \bar{E}_{y,i}^n - \frac{\Delta n}{\Delta x_i} \left[ \left( B_{z,i+1/2}^{n+1/2} - B_{z,i-1/2}^{n+1/2} \right) \right] - q_{n+1/2} \frac{\Delta r}{\Delta v_{y,i}^{n+1/2}},
\]

(59)

where the superscript \(n\) means the time step and subscript \(i\) means the coordinate of the cell center. Using Equations (56) and (57), we can obtain the numerical flux of Equations (58) and (59). (See Figure 1.) The same procedure can be done for time advance of \(E_z\) and \(B^y\).

The characteristic velocity of the Maxwell equations in vacuum is the velocity of light. However, since we consider the electromagnetic hydrodynamics equations, an appropriate characteristic velocity has to be used for them. Also, because we consider resistive systems, the characteristic velocity varies with the conductivity \(\sigma\) and the scale of wave modes. For example, as shown in Appendix A, the transverse electromagnetic hydrodynamic waves propagate with the velocity of light when \(k/\sigma\) is large, where \(k\) is the wave number, and they propagate with the Alfvén velocity when \(k/\sigma\) is smaller than a critical value \(\chi_c\). I nt h i s c a s e, and on the right is the supersonic case. These figures show that the half time-step transverse electromagnetic field \(E^y\) and \(B_z\) are determined by the fields at the base of two characteristics.

\[
\begin{align*}
\text{Figure 1. Schematic drawing of Eulerian-like characteristics when one uses piecewise linear interpolation.} \\
\text{ch is the characteristic velocity, On the left is the subsonic case, and on the right is the supersonic case. These figures show that the half time-step transverse electromagnetic field } E^y \text{ and } B_z \text{ are determined by the fields at the base of two characteristics.}
\end{align*}
\]
A detailed procedure to judge whether we use the velocity of light or magnetohydrodynamic characteristic velocities is given in Appendix A.

In addition to the numerical flux of the Maxwell equations, the characteristic velocity is also required to construct the Maxwell stress terms and the Poynting flux term. When the characteristic velocity obtained from the analysis of the transverse waves is the velocity of light, we use it as the characteristic velocity for them; when the transverse wave characteristic velocity is the Alfvén velocity, we use the characteristic velocities for them as follows. Note that if the following characteristic velocities are not used, numerical integration becomes unstable, and unstable numerical oscillation occurs. Then, the necessary procedures are as follows.

1. For the numerical flux of the Maxwell equations, Equations (58) and (59), we use the Alfvén wave velocity in the laboratory frame because the information of transverse electromagnetic fields is transmitted by the Alfvén wave. In RMHD, the Alfvén velocity in the laboratory frame can be obtained by solving (Anile 1990)

\[ H a^2 - B^2 = 0, \]  
\[ \text{where } H = \rho h + b^2, a = \gamma (v_{\text{AL}} - v^t), B = b^x - v_{\text{AL}}b^0, \] 
and \( b^x \) is the covariant magnetic field defined as

\[ b^\mu = \left[ \gamma v \cdot B, \frac{B}{\gamma} + \gamma (v \cdot B)v \right]. \]  
\[ \text{(60)} \]

2. For the Maxwell tension terms \( -EE - BB \) in Equation (38), we use the Alfvén wave velocity in the fluid comoving frame because the magnetic tension force is originated by the Alfvén wave. In RMHD, the Alfvén velocity in the fluid comoving frame is given by

\[ v_{\text{AC}} = \frac{B^x}{\sqrt{H}}. \]  
\[ \text{(62)} \]

3. For the Poynting flux \( \mathbf{E} \times \mathbf{B} \) of the energy equation, Equation (39), and the Maxwell pressure terms \( E^2/2 + B^2/2 \) in Equation (38), we use the fast magnetosonic wave velocity in the laboratory frame because the magnetic pressure originates in the magnetosonic wave. In RMHD, the fast magnetosonic wave velocity in the laboratory frame can be obtained by solving

\[ \rho h (1 - c_s^2)a^4 = (1 - v_{\text{FM}}^2)((|b|^2 + \rho h c_s^2 a^2 - c_s^2 B^2). \]  
\[ \text{(63)} \] 

Equation (63) is a quartic equation, and ordinarily one has to use the Newton–Raphson method or the quartic formula for obtaining solutions. However, since our scheme splits the fluid part and the electromagnetic part, the sound velocity \( c_s \) can be set equal to zero. Then, the characteristic equation, Equation (63), reduces to

\[ \rho h \gamma^2 (v^t - v_{\text{FM}})^2 = (1 - v_{\text{FM}}^2)|b|^2. \]  
\[ \text{(64)} \]

By using the quadratic formula, one can obtain solutions of the above equation:

\[ v_{\text{FM}} = \frac{\rho h \gamma^2 v^t \pm |b| \sqrt{|b|^2 + (1 - (v^t)^2)\rho h \gamma^2}}{\rho h \gamma^2 + |b|^2}. \]  
\[ \text{(65)} \]

To sum up, we only have to substitute the appropriate characteristic velocities \( v_{\text{AL}}, v_{\text{AC}}, \) and \( v_{\text{FM}} \) into \( c_s^2 \) in Equation (54), and calculate the electromagnetic field \( E, B \) at half time step. Then, the numerical fluxes of electromagnetic hydrodynamics equations are given by

\[ F_{\text{in}, \text{EM}}^{ix} = -E_{\text{AC}}^x E_{\text{AC}}^x - B_{\text{AC}}^x B_{\text{AC}}^x + \frac{1}{2}[(E_{\text{FM}}^2 + B_{\text{FM}}^2) \cdot b^x, \]  
\[ \text{(66)} \]

\[ F_{\text{em}, \text{EM}}^{ix} = (E_{\text{FM}} \times B_{\text{FM}})^x, \]  
\[ \text{(67)} \]

where \( E_{\text{AC}}, B_{\text{AC}} \) means that they are calculated by using the Alfvén velocity in the comoving frame, and \( E_{\text{FM}}, B_{\text{FM}} \) by using the fast magnetosonic wave velocity in the laboratory frame. For the numerical flux of the Maxwell equation, one has to use the Alfvén velocity in the laboratory frame \( v_{\text{AL}} \) for the calculation.

3.3. Stiff Part

As explained Section 3.1, Equation (44) contains stiff terms. Following the previous work K07, we split the equation into components normal and parallel to the velocity vector:

\[ \partial_t E_\perp + \sigma \gamma [E_\perp - (E \cdot v)v] = 0, \]  
\[ \text{(68)} \]

\[ \partial_t E_\parallel + \sigma \gamma [E_\parallel + v \times B]] = 0. \]  
\[ \text{(69)} \]

Since we use the Strang-splitting method, the right-hand side of the above equations can be considered constant apart from the electric field \( E \). As a result, these equations can be solved analytically:

\[ E_\parallel = E_\parallel^0 \exp \left[ -\frac{\sigma t}{\gamma} \right], \]  
\[ \text{(70)} \]

\[ E_\perp = E_\perp^0 + (E_\perp^0 - E_\perp^0) \exp \left[ -\sigma \gamma t \right], \]  
\[ \text{(71)} \]

where \( E_\perp^* = -v \times B \) and the suffix 0 indicates the initial component. If we use the explicit integrator, the stiff equation has to be solved in very small time steps \( \Delta t \). However, since Equations (70) and (71) are formal solutions, we can avoid the stability constraints of the time step. In the context of ambipolar diffusion in partially ionized plasma, a similar numerical technique using the piecewise formal solution of the stiff part is known to be a useful scheme (Inoue et al. 2007; Inoue & Inutsuka 2008, 2009).

3.4. Constraint Equations

It is well known that Equations (9) and (10) are constraints on the Cauchy surface. Though Maxwell equations ensure that these constraints are preserved at all times, straightforward numerical integration of Maxwell equations does not preserve these properties because of the accumulated numerical error. This causes corruption of numerical results and results in a crash in the end. For this reason, there are a number of numerical
techniques for avoiding this problem. We have implemented hyperbolic divergence cleaning for the electric field. The main idea of hyperbolic divergence cleaning is that one defines new variable \( \Psi \) as the deviation from constraint equations and arranges a system of equations to decay or carry the deviation \( \Psi \) out of the computational domain by high-speed waves. For the magnetic field, if one sets \( B^i \) constant, the constraint equation can be satisfied in the one-dimensional case. In the multidimensional case, we can implement the constrained transport method (Evans & Hawley 1988; Stone & Norman 1992). The detailed implementation will be presented in our next paper.

For hyperbolic divergence cleaning, we modify Equations (9) and (51):

\[
\frac{\partial}{\partial t} \Psi + \nabla \cdot \mathbf{E} = q - \kappa \Psi,
\]

\[
\frac{\partial}{\partial t} \mathbf{E}^\perp + \kappa \frac{\partial}{\partial x} \mathbf{E}^\perp = -qv^s,
\]

where \( \Psi \) is a new dynamic variable and \( \kappa \) a positive constant. Clearly, when we set \( \Psi = 0 \), we can recover the standard Maxwell equation, Equation (9). From these equations, we can obtain the telegrapher equation for \( \Psi \):

\[
\frac{\partial^2}{\partial t^2} \Psi + \kappa \frac{\partial}{\partial t} \Psi - \nabla^2 \Psi = 0.
\]

Thus, \( \Psi \) propagates at the speed of light and decays exponentially over a timescale \( 1/\kappa \).

Similar to Equation (44), Equation (72) contains stiff source terms. Thus, we split the equation into a stiff part and non-stiff part:

\[
\frac{\partial}{\partial t} \Psi + \nabla \cdot \mathbf{E} = q,
\]

\[
\frac{\partial}{\partial t} \Psi = -\kappa \Psi.
\]

The analytical solution of Equation (76) is

\[
\Psi = \Psi_0 \exp(-\kappa t),
\]

where \( \Psi_0 \) is the initial value of \( \Psi \).

3.5. Primitive Recovery

In order to compute numerical flux (35)–(39), the primitive variables \( \{ \rho, \mathbf{v}, \rho, \mathbf{B}, \mathbf{E} \} \) have to be recovered from the conserved variables \( \{ D, \mathbf{m}, \mathbf{e}, \mathbf{B}, \mathbf{E} \} \). In conserved variables, \( \mathbf{E} \) and \( \mathbf{B} \) can be obtained by evolving the Maxwell equations. However, as pointed out by P09, it is more stable to perform evolution of the stiff part, Equations (70) and (71), during this primitive recovery process when \( \sigma \) is large, i.e., the ideal MHD approximation is valid. This is because when we consider the MHD approximation, the electric field \( \mathbf{E} \) is equal to \( -\mathbf{v} \times \mathbf{B} \); however, in general, primitive recovered \( \mathbf{E} \) does not satisfy this relation. In what follows we explain the primitive recovery procedure following P09.

1. Set an initial guess for the velocity by using the previous time-step value, then evolve electric field \( \mathbf{E} \) using Equations (70) and (71).

2. Subtract the Poynting flux and electromagnetic energy density from conserved variables, and new variables can be defined as follows:

\[
\mathbf{m}' = \rho \gamma^2 \mathbf{v},
\]

\[
e' = \rho \gamma^2 - p.
\]

Then, variables \( \{ D, \mathbf{m}', e' \} \) are the ideal relativistic fluid conserved variables and can be recovered by using the ordinary procedures.

3. Replace the initial guess for the velocity with the obtained velocity \( \mathbf{v} \) and repeat the steps 1–3 until the primitive variables converge.

3.6. Algorithm

In this section, we provide the detailed numerical algorithm. In Cartesian coordinates, the relativistic resistive MHD equations written in conservative fashion are

\[
\frac{\partial}{\partial t} \left( \begin{array}{c} D \\ m^i \\ E^j \\ F_{\text{EM}}^i \\ \Psi \\ \mathbf{q} \end{array} \right) + \frac{\partial}{\partial x} \left( \begin{array}{c} \frac{F_{\text{EM}}^i}{m^j,\text{fluid}} \\ F_{\text{EM}}^i \\ 0 \\ 0 \\ \mathbf{B}^i \times \mathbf{E}^j \\ 0 \end{array} \right) = 0,
\]

where

\[
D = \gamma \rho,
\]

\[
\mathbf{m} = \rho \gamma^2 \mathbf{v} + \mathbf{E} \times \mathbf{B},
\]

\[
e = \rho \gamma^2 - p + \frac{1}{2}(E^2 + B^2),
\]

\[
F_{\text{D}}^i = D v^i,
\]

\[
F_{\text{m,fluid}}^i = m^i v^i + \rho \eta_i^x,
\]

\[
F_{\mathbf{E},\text{fluid}}^i = \rho \gamma^2 v^i,
\]

\[
F_{\text{m,EM}}^i = -E^i E^j - B^i B^j + \left[ \frac{1}{2} (E^2 + B^2) \right] \eta_i^x,
\]

\[
F_{\mathbf{E},\text{EM}}^i = (\mathbf{E} \times \mathbf{B})^i.
\]

The electric field \( \mathbf{E} \) and magnetic field \( \mathbf{B} \) are evolved by the Maxwell equations. If the ohmic dissipation is considered, the Maxwell equations have stiff and non-stiff parts. The non-stiff part is

\[
\partial_t \mathbf{U}_{\text{Maxwell}} + \frac{\partial}{\partial x} \mathbf{F}_{\text{Maxwell}} = S_{\text{non-stiff}},
\]

\[
\begin{pmatrix}
B^x \\
B^y \\
B^z \\
E^x \\
E^y \\
E^z \\
\Psi \\
q
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 \\
-E^z \\
E^y \\
\Psi \\
B^z \\
-B^y \\
E^x \\
J^x
\end{pmatrix},
\]

\[
S_{\text{non-stiff}} = \begin{pmatrix}
0 \\
0 \\
0 \\
-q v^x \\
-q v^y \\
-q v^z \\
q \\
0
\end{pmatrix}.
\]
where \( J^x = \sigma [E^x + (v \times B)^x - (E \cdot v) v_x^x] + q v^x \). The Maxwell equations are consistent with the equation of charge conservation. However, numerical errors in general destroy the conservation law in a way similar to the constraint equations. Thus, the above equation contains the equation of charge conservation.

As explained in Sections 3.3 and 3.4, the stiff part is evolved by using the formal solution

\[
E_\parallel = E_\parallel^0 \exp \left[ -\frac{\sigma}{\gamma} t \right],
\]

\[
E_\perp = E_\perp^0 + (E_\perp^0 - E_\perp^*) \exp \left[ -\sigma \gamma t \right],
\]

\[
\Psi = \Psi_0 \exp[-\kappa t].
\]

Using the above system of equations, the second-order numerical algorithm is given as follows.

1. Advance the stiff-part equations over \( \Delta t/4 \) by using the formal solutions, Equations (91)–(93).
2. Advance the non-stiff part of the Maxwell equations, Equations (89) and (90), over \( \Delta t/2 \) by using the MOC as explained in Section 3.2 and calculate numerical flux \( F_{EM} \) (38) and (39). On the other hand, numerical flux \( F_{fluid} \) (35)–(37) can be calculated by using the approximate Riemann solver (Martí & Müller 1994, 2003; Martí 1996; Banyuls et al. 1997; Aloy et al. 1999; Pons et al. 2000; Font et al. 2000; Del Zanna & Bucciantini 2002; Mignone & Bodo 2005; Mignone et al. 2005). In this paper, we use the HLLC solver.
3. Advance conserved variables \( D, m, \) and \( e \) over the half time step \( \Delta t/2 \) by using Equations (34)–(39). Then, calculate primitive variables of half time step \( U^{n+1/2} \) by primitive recovery explained in Section 3.5. In our scheme, electric field \( \mathbf{E} \) has to be evolved \( \Delta t/4 \) by using formal solution (91) and (92) during primitive recovery. Primitive variables obtained through this procedure are used for the calculation of the numerical flux at \( t = t + \Delta t/2 \).
4. Again, advance initial stiff variables over \( \Delta t/2 \) by using the formal solutions (91)–(93).
5. Calculate temporal second-order numerical flux (35)–(39) by using primitive variables obtained through procedure 3. Then, advance conserved variables \( D, m, e \) over \( \Delta t \) by Equation (34), and electric field \( \mathbf{E} \) and magnetic field \( \mathbf{B} \) by the Maxwell equations of the stiff part (Equation (90)).
6. Calculate primitive variables by a primitive recovery process. During this process, the electric field \( \mathbf{E} \) is advanced over \( \Delta t/2 \) by using formal solutions (91) and (92).

For the spatial second order, we use the MUSCL scheme by van Leer explained in Appendix B.

Note that if we evolve electric field \( \mathbf{E} \) in the integration of stiff equations or the primitive recovery procedure, we have to evolve other primitive variables. This is because conserved variables are not changed during those procedures, and this means that the change of electric field \( \mathbf{E} \) affects all other primitive variables.

4. TEST SIMULATIONS

In this section, several one-dimensional test simulations given in previous studies K07 and P09 are presented. For the numerical flux of fluid, we use the HLLC solver (Mignone & Bodo 2005). We use an ideal equation of state \( \rho e = p/(\Gamma - 1) \) with \( \Gamma = 2 \), and Courant number, CFL = 0.25.

**Figure 2.** Results of large amplitude circularly polarized Alfvén wave test with large conductivity \( \sigma = 10^6 \). This test is carried out for three different grid points \( N = 50, 100, 200 \).

### 4.1. Large Amplitude CP Alfvén Waves

This test consists of the propagation of a large amplitude circularly polarized (CP) Alfvén wave along a uniform background field \( B_0 \). The analytical exact solution of this problem is given by Del Zanna et al. (2007); and this problem is used as the ideal-MHD limit test problem by P09. We use the same condition as P09:

\[
(B^x, B^y) = \eta A B_0 (\cos[k(x-v_At)], \sin[k(x-v_At)]),
\]

\[
(v^x, v^y) = -\frac{v_A}{B_0} (B^x, B^y),
\]

where \( B^x = B_0, v^y = 0, k \) is the wave number, and \( \eta_A \) is the amplitude of the wave. The special relativistic Alfvén speed \( v_A \) is given by

\[
v_A^2 = \frac{2B_0^2}{h + B_0^2(1 + \eta_A^2)} \left[ 1 + \sqrt{1 - \left( \frac{2\eta_A B_0^2}{h + B_0^2(1+\eta_A^2)} \right)^2} \right]^{-1}.
\]

For the initial data parameters, we have used \( \rho = p = \eta_A = 1 \), and \( B_0 = 1.1547 \). Using these parameters, the Alfvén velocity is \( v_A = 1/2 \). For the boundary condition, the periodic one is used. In addition, we use a high uniform conductivity \( \sigma = 10^6 \) following P09, since this is the exact solution of ideal RMHD.

Figure 2 shows results of our new code at \( t = 2.0 \) (one Alfvén crossing time) for three different resolution cases with \( N = [50, 100, 200] \). The computational domain is \( x \in [-0.5, 0.5] \). This result indicates that our new code reproduces ideal RMHD solutions when the conductivity \( \sigma \) is high.

In these test problems, we cannot achieve full second-order accuracy. The left-hand side of Figure 3 is the \( L_1 \) norm errors of the tangential magnetic field \( B_y \) of this test problem. This figure shows that our numerical result is nearly 1.5-order convergence. We estimate that this is because our scheme uses many operator splittings, and the time accuracy of our scheme worsens. Note that this problem is one of the most difficult to solve in relativistic resistive MHD, since this is the limit of large conductivity \( \sigma \). The right-hand side of Figure 3 is the \( L_1 \) norm errors of the \( B_y \) of the next test problem. Since the conductivity \( \sigma \) has a moderate value in that test problem, our new scheme achieves second-order convergence.
Figure 3. $L_1$ norm errors of the tangential magnetic field $B_y$ under different grid resolution for the second-order schemes using the new scheme. The left-hand side is the result of large amplitude CP Alfvén waves, and the right-hand side is the result of the self-similar current sheet.

4.2. Self-similar Current Sheet

This problem is used as the test problem of highly resistive cases in K07 and P09. In this test, it is assumed that the magnetic pressure is much smaller than the gas pressure, so that the background fluid is not influenced by the evolution of the magnetic field. We assume that the magnetic field has only a tangential component $B = (0, B(x, t), 0)$, and $B(x, t)$ changes its sign within this current sheet. Since we are interested only in the evolution of the magnetic field, the background fluid is set initially in equilibrium, $p = \text{const}$. In addition, we assume that the conductivity $\sigma$ is high, and the diffusion timescale is much longer than the light propagating timescale. Although the resistive RMHD equation is hyperbolic, this assumption allows us to neglect the displacement currents at least in the rest frame. As the result, the evolution equation is reduced to

$$\partial_t B - \frac{1}{\sigma} \partial_x^2 B = 0.$$  \hspace{1cm} (97)

This equation has exact solution

$$B(x, t) = B_0 \text{erf} \left( \frac{1}{\sqrt{2 \xi}} \right),$$  \hspace{1cm} (98)

$$\xi = \frac{t}{x^2},$$  \hspace{1cm} (99)

where erf is the error function. Following K07 and P09, we set the initial condition at $t = 1$ with $p = 50$, $\rho = 1$, $E = v = 0$, and $\sigma = 100$. The computational domain is $[-1.5, 1.5]$, and the number of grid points is $N = 200$. Figure 4 is the numerical result at $t = 9$. This figure shows that our scheme can solve a highly resistive problem accurately. The convergence rate is consistent with the second-order spatial and temporal discretization.

4.3. The Propagation of Alfvénic Transverse Waves with Ohmic Dissipation

In order to confirm the capability of our method for the relativistic resistive MHD, we perform the test calculation of the propagation of Alfvénic transverse waves with ohmic dissipation and compare the results with the exact dispersion relation (A10) obtained in Appendix A.

As explained in Appendix A, the resistive relativistic magnetohydrodynamic equation contains transverse wave modes that become the light wave in the large $k/\sigma$ region and become the Alfvén wave in the small $k/\sigma$ region. To demonstrate the propagation of transverse waves, we set the initial condition by eigenfunctions of the mode obtained from Equations (A6) to (A9):

$$B_z = 0.05 \cos(kx),$$  \hspace{1cm} (100)

$$v^z = \frac{B_z}{\rho h} \left( \left( \frac{\omega^/}{k^/} \right)^2 - 1 \right) B^z,$$  \hspace{1cm} (101)

$$E^x = \frac{B^x}{1 - i \omega^/} v^z,$$  \hspace{1cm} (102)

$$E^y = \frac{\omega^/}{k^/} B^z,$$  \hspace{1cm} (103)

where $\omega^/ \equiv \omega/\sigma$ and $k^/ \equiv k/\sigma = 2\pi/\sigma$, and $\omega$ is the solution of the dispersion relation (A10). We set the same parameters in Appendix A,

$$(\rho h, B^z) = (1.5, 0.55).$$  \hspace{1cm} (104)

Since the enthalpy includes the information of the equation of state, one can take any value of $\Gamma$. In this calculation, we set
$\Gamma = 2$ and $p = 1$. The computational domain covers the region $[-0.5, 0.5]$ where the periodic boundary condition is imposed, and the number of grid points is $N = 200$.

The propagation speed of the numerical waves can be determined by tracing the position where $B^c$ is maximum. We measure the propagation speed and evaluate $\text{Re}[\omega]$ based on the time when the maximum of $B^c$ reaches $x = 0$ again, i.e., one-wave crossing period. The damping rate $\text{Im}[\omega]$ is measured by using $B^c_M = B^c_0 \exp[\text{Im}[\omega]t]$ where $B^c_M$ is the maximum of $B^c$ after the one-wave crossing time.

In Figure 5, we plot the real and imaginary parts of $\omega/\sigma$ against $k/\sigma$. The solid line is the exact dispersion relation obtained in Appendix A. We have performed the calculation in the cases of $k/\sigma = 0.01, 0.1, 0.5, 1, 4, 10, 100$. These figures show that our new numerical code can reproduce the propagation of Alfvénic transverse waves accurately for any value of the conductivity $\sigma$.

4.4. Shock-tube Problem

For the shock-tube test problem, we consider the simple MHD version of the Brio and Wu test as P09. The initial left and right states are given by

\[(\rho^L, p^L, (B^L)^j) = (1.0, 1.0, 0.5) \quad \text{for} \quad x < 0.5 \quad (105)\]

\[(\rho^R, p^R, (B^R)^j) = (0.125, 0.1, -0.5) \quad \text{for} \quad x \geq 0.5. \quad (106)\]

All the other fields are set to 0.

Figure 6 shows the numerical results at $t = 0.4$ for different grid points $N = 100, 200, 400$. The computational domain covers the region $[0, 1]$. We also plot an ideal RMHD solution by the solid line computed by a publicly available code developed by Giacomazzo & Rezzolla (2006). The conductivity is uniform with $\sigma = 10^6$. The solution of this Riemann problem contains a rarefaction moving to the left, a shock moving to the right, and a tangential discontinuity between them. Figure 6 shows that our numerical solution of the resistive MHD can reproduce the profile of an ideal MHD shock-tube problem using high conductivity $\sigma$. In addition, our numerical solution captures contact discontinuity as sharp as P09.

Figure 7 shows the numerical results of the same problem for different conductivity $\sigma = 0, 10, 10^2, 10^3, 10^6$. The number of grid points is $N = 400$. The solid line is the ideal solution. The ideal RMHD solution by the solid line. The number of grid points is $N = 400$. This result shows that our numerical solution reproduces nearly the same results as P09.
P09 report that Strang’s splitting method becomes unstable for moderately high values of the conductivity for this shock-tube problem, and one has to use the implicit method. However, this is not related to whether one uses Strang’s splitting or the implicit method, but to the revision of the electric field during the iteration of the primitive recovery (H. R. Takahashi 2010, private communication). Our scheme uses Strang’s splitting, but can solve this shock-tube problem stably even when \( \sigma \gtrsim 10^8 \), if we revise the electric field during the primitive recovery as explained in Section 3.5.

5. TEST SIMULATIONS FOR FLUID DOMINATED CASE

The previous studies K07 and P09 use the velocity of light for the characteristic velocity. Thus, their numerical solutions become highly diffusive when one considers problems whose sound velocity or Alfvén velocity is much lower than the velocity of light. In this section, we perform test problems in such cases and compare the results of the HLL code with that of our code.

5.1. Shock-tube Test Problem

In this section, we compute a high plasma \( \beta \) shock-tube problem and compare the results of our code with those of the HLL code. The initial left and right states are given by

\[
\begin{align*}
(\rho^L, p^L, (B^\gamma)^L) &= (10^3, 1.0, 0.05) \quad \text{for } x < 0.5, \\
(\rho^R, p^R, (B^\gamma)^R) &= (10^4, 0.1, -0.05) \quad \text{for } x \geq 0.5.
\end{align*}
\]

All the other fields are set to 0.

Figure 8 shows the numerical results of our code and the HLL one, being compared with ideal solutions at \( t = 30.0 \). The number of grid points is \( N = 400 \). These figures show that the HLL solver becomes more diffusive than our code. In addition, Figure 8 shows that the density profile of the shock heated region somewhat overshoots that of the ideal solution, and the tangential magnetic field \( B^\gamma \) slightly undershoots that of the ideal solution. These results show that when the plasma \( \beta \) is high, the HLL solver becomes highly diffusive and does not reproduce the correct value of the shock heated region. In contrast, our numerical results reproduce ideal solutions very well even for high \( \beta \) problems.

5.2. The Propagation of Contact Discontinuity

In this section, we calculate the propagation of a contact discontinuity and study the accuracy of capturing the contact discontinuity for various advection velocities. When one uses the HLL code by Komissarov, the numerical results can be expected to be diffusive for the case of very slow advection velocity, since the HLL code uses the velocity of light for the characteristic velocity. In contrast, our new code uses the velocity of sound for the fluid characteristic velocity, and the numerical results will be more accurate for any advection velocity.

We consider the propagation of contact discontinuity of magnetohydrodynamics. The initial condition is

\[
\begin{align*}
(\rho^L, p^L, (B^\gamma)^L) &= (1.0, 1.0, 0.1) \quad \text{for } x < 0, \\
(\rho^R, p^R, (B^\gamma)^R) &= (1.5, 1.0, 0.05) \quad \text{for } x \geq 0.
\end{align*}
\]

All the other fields are set to 0.

Since we want to consider the ideal fluid case, we consider high conductivity \( \sigma = 10^8 \). We use an equation of state with \( \Gamma = 5/3 \), and the computational domain covers the region \([-0.5, 0.5]\) with 100 grid points. The CFL number is 0.25, and the integration is carried out until two fluid crossing time. For the boundary condition, the periodic one is used. For the advection velocity, we use the following velocities:

\[
\nu = 0.9, 0.5, 0.1, 0.05, 0.01.
\]

On the left of Figure 9 are the numerical results of the density profile calculated by using our new code, and on the left of Figure 10 are the numerical results of the density profile calculated by using the HLL code. The solid lines are the ideal solution. This figure shows that the numerical results of density profiles by the HLL code of \( \nu = 0.9 \) are nearly equal to that of our new code. However, the numerical results by the HLL code become more diffusive than those by our code as the advection velocity becomes small; in contrast, the accuracy of the numerical results by our code is nearly independent of the advection velocity. The right-hand side of Figure 9 are the numerical results of tangential magnetic field \( B^\gamma \) calculated by using our new code, and the right-hand side of Figure 10 are the numerical results of tangential magnetic field \( B^\gamma \) calculated by
using the HLL code. Similar to the density profile, the numerical results by using the HLL code become more diffusive than by using our code as the advection velocity becomes small; in contrast, the accuracy of the numerical results by our code is nearly independent of the advection velocity.

In conclusion, the HLL code is not capable of accurately solving problems whose advection velocity is smaller than the velocity of light, since the HLL code uses the latter for the characteristic velocity. The diffusive result of the HLL code is always problematic for any discontinuity when the propagation velocity is much smaller than the velocity of light. In contrast, since our code uses appropriate characteristic velocities, the numerical dissipation does not depend on the characteristic velocity. For this reason, our new code can solve any advection velocity problems accurately, especially problems including discontinuities.

5.3. The Propagation of Small Amplitude Alfvén Waves

In this section, we consider the propagation of small amplitude Alfvén waves in high $\beta$ plasma. The integration is performed for different resolutions, and we compare the numerical results of Komissarov's HLL code and our code. For the application to the numerical simulation of MRI, the integration is performed for a small number of grid points: $N = 16, 32, 64$ for one wavelength of the Alfvén wave; this corresponds to the number of grid points for resolving the wavelength of maximum growth rate of MRI.

For the initial condition, we consider

$$ (\rho, p, B^x, B^y) = (10, 0.05, 0.1, 0.1), $$

$$ B^z = 0.01 \sin(2\pi x / L), $$

$$ v^z = -\frac{B^z}{\sqrt{\rho h + |B|^2}}. $$

In this case, the Alfvén velocity $v_A$ and plasma beta $\beta$ are given by

$$ v_A = 3.14 \times 10^{-2}, \quad \beta = 5.02 \times 10^2, $$

where the Alfvén velocity and the plasma beta are defined as

$$ v_A = \frac{B^z}{\sqrt{\rho h + |B|^2}}, $$

$$ \beta = \frac{\rho h}{|B|}. $$

---

Figure 9. Numerical results of the propagation of contact discontinuity of RMHD by using our new scheme for different advection velocities: $v^x = 0.9, 0.5, 0.1, 0.05, 0.01$. The number of grid points is $N = 100$.

Figure 10. Numerical results of the propagation of contact discontinuity of RMHD by using the HLL code for different advection velocities: $v^x = 0.9, 0.5, 0.1, 0.05, 0.01$. The number of grid points is $N = 100$. 
Since the initial magnetic field is very weak for most of the MRI phenomenon, a weak magnetic field is considered. In order to consider the ideal fluid case, we set a high conductivity \( \sigma = 10^6 \). We use an equation of state with \( \Gamma = 2 \). The computational domain covers the region \([-0.5, 0.5]\). The CFL number is 0.1, and the integration is carried out until 1 Alfvén wave crossing time. For the boundary condition, the periodic one is used.

The numerical results of our code and HLL are presented in Figure 11. Although the amplitude of both results falls because of the numerical diffusion, it can be seen that HLL results are more diffusive than our numerical results when the number of grid points is \( N = 16, 32, 64 \). When the number of grid points is \( N = 64 \), the numerical result of HLL code is a little more accurate than that of our code. This is because our new scheme uses an operator split for the accuracy, and the convergence rate is a little less than second order in time. However, from a practical point of view, it is impossible to cost 64 grid points for the wavelength of maximum growth rate of MRI in many cases, and still our new code can integrate the growth of magnetic field by MRI more accurately.

In conclusion, when one considers the high \( \beta \) plasma, our code is more accurate than the HLL code because our code uses the velocity of sound and the Alfvén velocity as the characteristic velocity. In particular, the above results show that our new method is useful for application to phenomena including MRI. This instability occurs in the system whose angular momentum changes as \( r^{-n} \) \((0 < n < 2)\), and the amplitude of the perturbative Alfvén waves grows exponentially over the duration of nearly one Kepler rotation. Since the above condition is satisfied in most of the differential rotating systems in gravity, MRI is one of the most important astrophysical phenomena. In order to reproduce this instability numerically, one has to resolve the wavelength of maximum growth rate. However, this is difficult for most problems, since this wavelength is proportional to the initial weak magnetic field. For this reason, in order to reproduce MRI numerically, one has to use numerical schemes that can integrate small amplitude Alfvén waves accurately by the smaller number of grid points. Then, the results of test problems in this section show that our new numerical scheme can deal such problems more accurately than previous codes.

In these three test problems, we consider extremely high density cases in order to distinguish differences easily. However, this can always happen when the magnetic field is weak. As a result, if one considers problems including an initially weak magnetic field like MRI in the accretion disk, our code can produce more accurate results.

6. CONCLUSION

In this paper, we have presented a new numerical scheme of resistive RMHD for the one-dimensional case which can solve matter dominated problems more accurately than the existing numerical method. Since this new scheme uses a different characteristic velocity for obtaining the numerical flux of fluid and electromagnetic field, one can solve accurately and stably problems whose characteristic velocity is much lower than that of light.

When one considers relativistic problems, one has to solve stiff equations for electric fields. In general, it is difficult to deal with stiff equations, and special methods have been presented; for example, K07 uses Strang’s splitting method, and P09 use the implicit method. P09 report that Strang’s splitting method is incapable of solving problems that include discontinuity, such as a shock. However, we find that this is not related to the method for the stiff equations, and one can solve problems including a shock if one evolves the electric field during the primitive recovery; we use Strang’s splitting method, and the solver is well behaved for shock-tube problems. The results of other test problems show that our new scheme is capable of accurately solving both highly resistive problems and nearly ideal MHD ones. In addition, it has been shown that our code can solve low characteristic velocity problems more accurately than the HLL code.

The problems of high density and high plasma \( \beta \) appear when one considers MRI in the accretion disk with a relativistic jet, for example. In this case, one has to use relativistic resistive MHD code that can solve both highly relativistic and non-relativistic dynamics with resistivity for the following three reasons: (1) the saturation of the MRI depends on the resistivity; (2) the dynamics of an accretion disk are not ordinarily relativistic, especially, the dynamics of the MRI is sub-Alfvénic; and (3) the dynamics of the jet are highly relativistic. Our new scheme can solve such problems accurately even when the initial magnetic field is very weak.

We present a multi-dimensional extension of our scheme in our next paper.
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APPENDIX A
THE DISPERSION RELATION OF THE RELATIVISTIC ELECTROMAGNETIC FLUID

As explained in Section 3.2, we solve the evolution of electric and magnetic field by the MOC. For the characteristic velocity, we use the appropriate MHD characteristic velocity when \( \sigma \) is large, that is, the ideal MHD approximation is valid. However, when the conductivity \( \sigma \) is not so large, we have to replace the characteristic velocity with the speed of light. In this section, we discuss when to switch the characteristic velocity from appropriate MHD characteristic velocity to the velocity of light. In the following, we calculate the linear perturbation of the relativistic electromagnetic equation in order to obtain the characteristic velocity.

The relativistic electromagnetic fluid equations are given by

\[ p \mu u^\mu \partial_\mu u^i = -\nabla p + (qE + J \times B), \quad (A1) \]

\[ \partial t B = -\nabla \times E, \quad (A2) \]

\[ \partial t E = \nabla \times B - J, \quad (A3) \]

\[ J = \sigma \gamma [E + v \times B] - (E \cdot v)v + qv, \quad (A4) \]

\[ q = \nabla \cdot E, \quad \nabla \cdot B = 0. \quad (A5) \]

To obtain the dispersion relation, we start by expanding physical variables around an unperturbed state in the following frame.

1. The fluid is at rest: \( v_0 = 0 \).
2. The \( x \)-coordinate is parallel to \( k: k = ke_x \).
3. The magnetic field is in the \( x \)-direction: \( B_0 = B^x e_x \).
4. Charge neutrality: \( q_0 = 0, E_0 = 0 \).

Since we only want to judge when to switch characteristic velocity, we consider the propagation of transverse waves along the magnetic field. When one uses this procedure during the numerical simulation, one only has to calculate \( B^2 - E^2 \) of the simulation data and substitute its square root into the above \( B^x \).

This is because \( B^2 - E^2 \) is a scalar and becomes the square of the magnetic field in the fluid comoving frame because of the assumption of the charge neutrality. Since the magnetic field appears only in the form of \( B^x \) in the following procedure, one can neglect the sign of magnetic field. In the following, we consider only the characteristic velocity of transverse waves.

In the above condition, the Alfvén mode is included in the \( z \)-component of the velocity \( \delta v^z \) and magnetic field \( \delta B^z \) and decouples from other variables. For this reason, we consider only variables related to \( \delta v^z \) and \( \delta B^z \).

We replace the current vector in Equation (A1) with Equation (A3). Then, the perturbed equations are

\[ i \omega \phi \delta v^z + ikB^x \delta B^z - i\omega B^x \delta E^z = 0, \]

\[ i\omega \delta B^z - ik \delta E^z = 0, \]

\[ \sigma B^x \delta v^z + (i\omega - \sigma) \delta E^z = 0, \]

\[ \sigma B^z \delta v^z + ik \delta B^z + (\sigma - i\omega) \delta E^z = 0. \]

From these equations, the following dispersion relation is obtained:

\[ \omega^4 + i\sigma (B^2 + 2\rho) \omega^3 - [k^2 \rho + \sigma^2 (B^2 + \rho \omega)]\omega^2 - i\sigma (B^2 + \rho \omega) k^2 \omega + \sigma^2 (B^x)^2 k^2 = 0. \]

Equation (A10) is the biquadratic equation with respect to \( \omega \) and has the formula of radicals. However, the analytical formula is very complex and hard to analyze and is not suitable for obtaining the characteristic velocity. As shown below, the transverse wave becomes an Alfvén wave in the long wavelength regime and a light wave in the short wavelength regime shown in Figure 12. Figure 12 shows that the damping rate is a.
monotonically increasing function of \( k \). For this reason, we establish the following criterion for the characteristic velocity: when all light modes damp during one time step \( \Delta t^n \), we use appropriate MHD characteristic velocity for the MOC; when some light modes do not damp during one time step \( \Delta t^n \), we use the velocity of light for the MOC. We will discuss this method in detail in the following.

First, we substitute \( \omega = \omega_R + i \omega_I \) into Equation (A10) and divide the dispersion relation into a real part and imaginary part. Then, the real part is

\[
\rho h \omega_R^4 - \rho k h^2 + \sigma^2 (B^2 + \rho h) + 3 \sigma (B^2 + 2 \rho h) \omega_I \\
+ 6 \rho h \omega_R^2 \omega_I^2 + \rho h \omega_I^4 + \sigma (B^2 + 2 \rho h) \omega_I^3 \\
+ \rho k h^2 + \sigma^2 (B^2 + \rho h) \omega_I^2 + \sigma (B^2 + \rho h) k^2 \omega_I \\
+ (B^2 \sigma + \rho \sigma + 2 \omega_I) \omega_I^2 = 0.
\]

and the imaginary part is

\[
[B^2 \sigma + 2 \rho (\sigma + 2 \omega_I)] \omega_R^2 - [\rho (\sigma + 2 \omega_I)k^2 + 2 \omega_I (\sigma + \omega_I)] \\
+ B^2 \sigma [k^2 + \omega_I (2\sigma + 3 \omega_I)] \omega_I = 0.
\]

Equation (A12) implies that the solution for \( \omega_R \) is 0 and a conjugate complex number. The solutions of \( \omega_R \) are pure decaying modes, so the other modes are the desired propagating ones that become the velocity of light in the limit of small \( \sigma \) and the Alfvén velocity in the limit of large \( \sigma \). Figure 12 shows the dispersion relation for the propagation modes of the following parameters:

\[
(\rho h, B) = (1.5, 0.55).
\]

These figures show that this mode becomes light in the limit of small \( \sigma \) and an Alfvén wave in the limit of large \( \sigma \). Although the form of \( \omega_R \) does not become as Figure 12 for some parameters, this mode always becomes light waves in the limit of small \( \sigma \).

From Equation (A12), this desired mode can be obtained as follows:

\[
\omega_R^2 = \frac{\rho h (\sigma + 2 \omega_I) [k^2 + 2 \omega_I (\sigma + \omega_I)] + B^2 \sigma [k^2 + \omega_I (2\sigma + 3 \omega_I)]}{B^2 \sigma + 2 \rho h (\sigma + 2 \omega_I)}.
\]

We substitute this \( \omega_R^2 \) into Equation (A11) and obtain

\[
\alpha_4 k^4 + \alpha_2 k^2 + \alpha_0 = 0,
\]

where

\[
\alpha_4 = -B^2 \rho^2 h^2 \sigma (\sigma + 2 \omega_I) - \rho^3 h^3 (\sigma + 2 \omega_I)^2,
\]

\[
\alpha_2 = B^4 (B^2)^2 \sigma^4 - B^6 \sigma^3 (\sigma + 2 \omega_I) \\
- 2 B^4 \rho h \sigma^2 (2 \sigma^2 + 7 \sigma \omega_I + 6 \omega_I^2) \\
- \rho^2 h^2 (\sigma + 2 \omega_I)^2 \{-4 (B^4)^2 \sigma^2 + 2 \rho h (\sigma + 2 \omega_I)^2 \}
\]

\[
- B^2 \rho h \sigma (\sigma + 2 \omega_I) [-4 (B^4)^2 \sigma^2 \\
+ \rho h (5 \sigma^2 + 18 \sigma \omega_I + 16 \omega_I^2)],
\]

\[
\alpha_0 = -2 B^6 \sigma^3 \omega_I (\sigma + 2 \omega_I)^2 - 4 \rho^3 h^2 \omega_I (\sigma + \omega_I) (\sigma + 2 \omega_I)^3 \\
- 2 B^2 \rho^2 h^2 \omega_I (\sigma + 2 \omega_I) (5 \sigma + 6 \omega_I) \\
- 4 B^4 \rho h \sigma^2 \omega_I (\sigma + 2 \omega_I) (2 \sigma^2 + 7 \sigma \omega_I + 6 \omega_I^2).
\]

This equation should include the propagation modes.

Equation (A15) is the biquadratic equation with respect to \( k \), but includes unknown quantity \( \omega_I \). Figure 12 shows that the propagation mode becomes light waves in the short wavelength region, and the damping rate \( -\omega_I \) is a monotonically increasing function of \( k \). Note that what we want to know is whether the undamped shortest wavelength mode is light waves or Alfvén waves, and we do not necessarily have to solve the biquadratic equation directly.

For this reason, we substitute \(-2 \pi/\Delta t \) into \( \omega_I \) of Equation (A15) and solve it with respect to \( k^2 \):

\[
k^2 = (\beta_1 + \sqrt{\beta_2})/\beta_3
\]

\[
\beta_1 = B^4 (B^2)^2 \sigma^4 - B^6 \sigma^3 (\sigma + 2 \omega_I) \\
- 2 B^4 \rho h \sigma^2 (2 \sigma^2 + 7 \sigma \omega_I + 6 \omega_I^2) + \rho^2 h^2 (\sigma + 2 \omega_I)^2 \\
\times [-4 (B^4)^2 \sigma^2 + 2 \rho h (\sigma + 2 \omega_I)^2] \\
- B^2 \rho h \sigma (\sigma + 2 \omega_I) [-4 (B^4)^2 \sigma^2 \\
+ \rho h (5 \sigma^2 + 18 \sigma \omega_I + 16 \omega_I^2)]
\]

\[
\beta_2 = -8 \rho^2 h^2 \omega_I (\sigma + 2 \omega_I) [B^2 \sigma + 2 \rho h (\sigma + 2 \omega_I)] \\
\times [B^2 \sigma + \rho h (\sigma + 2 \omega_I)^2] \\
+ [-B^6 \sigma^3 (\sigma + 2 \omega_I) + 2 \rho^2 h^2 (\sigma + 2 \omega_I)^2 (2 B^4)^2 \sigma^2 \\
- \rho h (\sigma + 2 \omega_I)^2] \\
+ B^2 \rho h \sigma (\sigma + 2 \omega_I) [4 (B^4)^2 \sigma^2 \\
- \rho h (5 \sigma^2 + 18 \sigma \omega_I + 16 \omega_I^2)]
\]

\[
\beta_3 = \rho^2 h^2 (\sigma + 2 \omega_I) [B^2 \sigma + \rho h (\sigma + 2 \omega_I)]
\]

\[
\omega_I = -\frac{2 \pi}{\Delta t}
\]

where Equation (A15) has two solutions of \( k^2 \), and we adopt the larger one since \( k \) is a real number.

Substituting the above \( k^2 \) into Equation (A14), one can obtain the desired characteristic velocity. Since what we need is appropriate MHD characteristic velocity, the obtained velocity cannot be used as the characteristic velocity. However, if the obtained velocity is not the Alfvén velocity, it shows that we should use the velocity of light for the characteristic velocity.

This method requires some further explanation.

First, note that the above method needs \( B^2, B^4, \) and \( \Delta t^n \) in the comoving frame, and one should transform numerical data from the laboratory frame to the comoving frame.

Second, numerical experiments indicate that \( \omega_R \) becomes 0 for some range of \( k \) for some parameter, and \( k^2 \) of Equation (A19) becomes negative. In this case, we use the speed of light as the characteristic velocity.

Finally, Figure 12 implies that \(-\omega_I/\sigma \) has some maximum value. This can be proved as follows. First, dividing Equation (A10) by \( \sigma \), one obtains

\[
\rho h \omega + i (B^2 + 2 \rho h) \omega_3 - [k^2 \rho h + (B^2 + \rho h)] \omega^2 \\
- i (B^2 + \rho h) k^2 \omega + (B^4)^2 k^2 = 0
\]

where \( \omega = \omega/\sigma \) and \( k = k/\sigma \).
Figure 12 implies that the maximum value of \(-\omega_I/\sigma\) is obtained in the limit of large \(k\), and the propagation mode is light wave in this limit. For this reason, we substitute \(\omega = \bar{k} - i\bar{\omega}_I\) into Equation (A24). Then it reduces to
\[
-i\rho(-1 + 2\bar{\omega}_I)^\dagger + \left[(B^\dagger)^2 + B^2(-1 + 2\bar{\omega}_I')\right]
+ \rho(-1 + 5\bar{\omega}_I' - 5\bar{\omega}_I'^2)\bar{k}^2
+ i\left[-B^2\bar{\omega}_I'(-2 + 3\bar{\omega}_I') + 2\rho\bar{\omega}_I'(1 - 3\bar{\omega}_I' + 2\bar{\omega}_I'^2)\right]\bar{k}^2
- B^2(-1 + \bar{\omega}_I')\bar{\omega}_I'^2 + \rho(-1 + \bar{\omega}_I')\bar{\omega}_I'^2 = 0.
\]
(A25)

Since this is in the limit of large \(\bar{k}\), what we have to consider is only the highest degree of \(\bar{k}\). Then, we set its coefficient equal to 0, and it reduces to
\[
\bar{\omega}_I' = \frac{1}{2}.
\]
(A26)

This shows that \(-\omega_I/\sigma\) becomes 1/2 in the limit of large \(k/\sigma\), and we use the speed of light as the characteristic velocity when \(\omega_I = -2\pi/\Delta t\) is less than \(-\sigma/2\).

**APPENDIX B**

**MUSCL**

For the second-order scheme, one has to compute the cell boundary numerical flux using the Riemann solver or MOC with left and right states obtained by using MUSCL. In this section, we explain MUSCL of van Leer (1979). Since we need a second-order scheme, the left and right states of primitive variables \(Q\) are
\[
Q_{i+1/2} = Q_{i} + \frac{\delta Q_i}{2},
\]
(B1)
\[
Q_{i+1/2} = Q_{i+1} - \frac{\delta Q_{i+1}}{2},
\]
(B2)

where \(\delta Q_i\) follows from a predictor step:
\[
U_i^{n+1/2} = U_i^n - \frac{\Delta t}{2\Delta x_i} \left[ F(Q_{i+1/2}^n) - F(Q_{i-1/2}^n) \right].
\]
(B3)

where \(U\) is the conserved variables. In the above equation, \(Q_{i+1/2}^n\) can be computed from Equations (B1) and (B2) by replacing \(Q_{i+1/2}^n\) with \(Q^n\).

When one uses MUSCL, the \(\delta Q_i\) in Equations (B1) and (B2) are computed as follows:
\[
\delta Q_i = \begin{cases} 
\min \left(2\left|\Delta Q_{i+1/2}^n\right|, \left|\Delta Q_i^n\right|, 2\left|\Delta Q_{i-1/2}^n\right|\right) \text{sgn}\Delta Q_i^n & \text{if } \text{sgn}\Delta Q_{i+1/2}^n = \text{sgn}\Delta Q_i^n = \text{sgn}\Delta Q_{i-1/2}^n, \\
0 & \text{otherwise,}
\end{cases}
\]
(B4)

where
\[
\Delta Q_{i+1/2} = Q_{i+1} - Q_i,
\]
(B5)
\[
\Delta Q_i = \frac{Q_{i+1} - Q_{i-1}}{2}.
\]
(B6)

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