On the reconstruction of inclusions in a heat conductive body from dynamical boundary data over a finite time interval

Masaru Ikehata\textsuperscript{1} and Mishio Kawashita\textsuperscript{2}

\textsuperscript{1} Department of Mathematics, Graduate School of Engineering, Gunma University, Kiryu 376-8515, Japan
\textsuperscript{2} Department of Mathematics, Graduate School of Sciences, Hiroshima University, Higashi-Hiroshima 739-8526, Japan

E-mail: ikehata@math.sci.gunma-u.ac.jp and kawasita@math.sci.hiroshima-u.ac.jp

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Abstract
The enclosure method was originally introduced for inverse problems concerning non-destructive evaluation governed by elliptic equations. It was developed as one of the useful approaches in inverse problems and applied for various equations. In this paper, an application of the enclosure method to an inverse initial boundary value problem for a parabolic equation with a discontinuous coefficient is given. A simple method to extract the depth of unknown inclusions in a heat conductive body from a single set of the temperature and heat flux on the boundary observed over a finite time interval is introduced. Other related results with infinitely many data are also reported. One of them gives the minimum radius of the open ball centred at a given point that contains the inclusions. The formula for the minimum radius is newly discovered.

1. Introduction

A part in the body that has a different conductivity from the known reference one is called an inclusion. In this paper, we consider a mathematical formulation of an inverse problem for the heat equation with inclusion. Assume that we have a set of the pair of temperature fields on the boundary of a heat conductive body and the corresponding heat flux across the boundary of the body over a finite time interval. This is the set of observed data in our inverse problem. The problem in this paper is to find out what information about inclusions in the body one can extract from the set. The solution to this problem may have a possible application to non-destructive evaluation by thermal imaging. We study this problem from a mathematical point of view and aim to find an analytical approach for extracting information about the location and shape of the inclusions.
Let $\Omega$ be a bounded domain of $\mathbb{R}^3$ with a smooth boundary. We denote the unit outward normal vectors to $\partial \Omega$ by the symbol $\nu$. Let $T$ be an arbitrary fixed positive number.

Given $f = f(x, t), (x, t) \in \partial \Omega \times [0, T]$ let $u = u(x, t)$ be the solution of the initial boundary value problem for the parabolic equation

$$
\begin{align*}
  u_t - \nabla \cdot \gamma \nabla u &= 0 & & \text{in} & & \Omega \times [0, T], \\
  \gamma \nabla u \cdot \nu &= f & & \text{on} & & \partial \Omega \times [0, T], \\
  u(x, 0) &= 0 & & \text{in} & & \Omega,
\end{align*}
$$

where $\gamma = \gamma(x) = (\gamma_{ij}(x))$ satisfies

(G1) for each $i, j = 1, 2, 3$ $\gamma_{ij}(x) = \gamma_{ji}(x) \in L^\infty(\Omega)$;

(G2) there exists a positive constant $C$ such that $\gamma(x)\xi \cdot \xi \geq C||\xi||^2$ for all $\xi \in \mathbb{R}^3$ and a.e. $x \in \Omega$.

This paper is concerned with the extraction of information about ‘discontinuity’ of $\gamma$ from $u$ and $\gamma \nabla u \cdot \nu$ on $\partial \Omega \times [0, T]$ for some $f$ and an arbitrary fixed $T < \infty$. However, we do not consider completely general $\gamma$. Instead we assume that there exists an open set $D$ with a smooth boundary such that $\overline{D} \subset \Omega$ and $\gamma(x)$ a.e. $x \in \Omega \setminus D$ coincides with the $3 \times 3$ identity matrix $I_3$ and satisfies one of the following two conditions:

(A1) there exists a positive constant $C'$ such that $-(\gamma(x) - I_3)\xi \cdot \xi \geq C'||\xi||^2$ for all $\xi \in \mathbb{R}^3$ and a.e. $x \in D$;

(A2) there exists a positive constant $C'$ such that $(\gamma(x) - I_3)\xi \cdot \xi \geq C'||\xi||^2$ for all $\xi \in \mathbb{R}^3$ and a.e. $x \in D$.

Write $h(x) = \gamma(x) - I_3$ a.e. $x \in D$. In this paper we consider the following problem.

**Inverse problem.** Assume that both $D$ and $h$ are unknown. Extract information about the location and shape of $D$ from a set of the pair of temperature $u(x, t)$ and heat flux $f(x, t)$ for $(x, t) \in \partial \Omega \times [0, T]$.

$D$ is a model of the union of unknown inclusions where the heat conductivity is anisotropic, different from that of the surrounding homogeneous isotropic conductive medium. The problem is a mathematical formulation of a typical inverse problem in thermal imaging.

Elayyan–Isakov [5] investigated the uniqueness issue of this type of problem. As a corollary of their uniqueness theorem we know that the lateral Neumann-to-Dirichlet map: $f \mapsto u|_{\partial \Omega \times [0, T]}$ uniquely determines $D$ together with $h$ inside $D$ if $\Omega \setminus \overline{D}$ is connected and $h$ is given by $hI_3$ with a smooth function $b$ on $\overline{D}$. However, their purpose is to recover the full information about the location and shape of $D$ and for the purpose their proof requires infinitely many pairs of the temperature and heat flux on $\partial \Omega \times [0, T]$ even just for determining a single point on $\partial D$. This shows the difficulty of obtaining the detailed image of inclusions from boundary measurements.

Also note that in [2] an approach to the inverse problem in a one-space dimensional case, which is an analogue of the probe method introduced by Ikehata [7], has been proposed. Therein $\Omega$ and $D$ are given by open intervals. Their approach introduces the pre-indicator function denoted by $I(y', s'; y, s)$ with their notation which depends on two points $(y, s), (y', s') \in (\Omega \setminus \overline{D}) \times [0, T]$. The pre-indicator function can be computed from an integral over the lateral boundary $\partial \Omega \times [0, T]$ which involves the Neumann-to-Dirichlet map acting on infinitely many special heat fluxes prescribed on the lateral boundary. The procedure in [2] for the determination of the right-end point of $D$ consists of two steps: (i) computation of $I(y + \epsilon, s + \epsilon^2; y, s)$ for each $\epsilon > 0$ at the given point $(y, s)$ in the spacetime and $\lim_{\epsilon \to 0} I(y + \epsilon, s + \epsilon^2; y, s)$; (ii) moving $(y, s)$ in (i) to the left along a path on the spacetime
and do the same computation until the computation result becomes large. This procedure has been tested numerically in [3].

In this paper we mainly seek a simpler method that yields partial or rough information about the location and shape of $D$ from $u(x, t)$ on $\partial \Omega \times [0, T]$ for a single fixed heat flux or explicit heat fluxes prescribed on $\partial \Omega \times [0, T]$. We think that this type of study provides us with knowledge about good heat fluxes on the boundary of the body to get such information. In [12] we have already developed an argument based on the enclosure method, which was originally introduced for elliptic equations in [8, 9], to derive two types of formulae in the case when the inclusion has the zero conductivity, that is a cavity. The argument yields the values of the support function of the cavity at a given direction and the distance of a given point outside the body to the cavity from the temperature fields and special explicit heat fluxes. In this paper, we see that the argument also works for the inclusion case and also yields new information: the minimum radius of the open ball centred at a given point that contains the inclusions.

The main new point of this paper is as follows: an introduction of another argument which is also based on the enclosure method and gives a formula which has not been considered in [12]. It makes use of a single set of a general heat flux and the corresponding temperature field on the surface of the body over a finite time interval. It yields a depth of unknown inclusions in a heat conductive body from the surface of the body. We give a heat flux $f(x, t)$ on the surface of the body as it moves from the beginning of the experiments (cf (1.2) below). But, there is no other condition on $f$. Hence note that we do not need to prescribe any explicit heat flux.

Note that in theorem 2.1 of [10] the enclosure method has been applied to a one-space dimensional version of inverse problem. Therein complex exponential solutions of the backward heat equation with a large parameter are used. In this paper we use only real exponential solutions.

1.1. A formula with a general heat flux

The new point of this paper is a derivation of the following formula which can be considered as the main result of this paper. It makes use of a single set of a heat flux and the corresponding temperature on $\partial \Omega \times [0, T]$ and gives partial or rough information about the location of inclusions.

**Theorem 1.1.** Let $f = f(x, t)$, being square integrable in $(x, t)$, satisfy: there exists $\mu \in \mathbb{R}$ such that

$$0 < \inf_{x \in \partial \Omega} \lim_{\tau \to \infty} \inf \tau^\mu \int_0^T e^{-\tau t} f(x, t) \, dt \leq \sup_{x \in \partial \Omega} \lim_{\tau \to \infty} \sup \tau^\mu \int_0^T e^{-\tau t} f(x, t) \, dt < \infty$$

(1.2)

and that the function

$$\partial \Omega \ni x \mapsto \int_0^T e^{-\tau t} f(x, t) \, dt$$

is continuous.

Let $u = u_f(x, t)$ be the weak solution of (1.1) for this $f$ and let $v = v_g(x; \tau)$ be the solution of

$$\frac{\partial^2 v}{\partial t^2} - \Delta v = 0 \quad \text{in } \Omega, \quad \frac{\partial v}{\partial \nu} = g(x; \tau) \quad \text{on } \partial \Omega.$$  

(1.3)

Then, there exists $\tau_0 > 0$ such that
• if (A1) is satisfied, then for all $\tau \geq \tau_0$

$$\int_{\partial\Omega} \int_0^T e^{-\tau t} (v(x; \tau) f(x, t) - u_f(x, t) g(x; \tau)) \, dt \, dS < 0;$$

• if (A2) is satisfied, then for all $\tau \geq \tau_0$

$$\int_{\partial\Omega} \int_0^T e^{-\tau t} (v(x; \tau) f(x, t) - u_f(x, t) g(x; \tau)) \, dt \, dS > 0.$$

In both cases the formula

$$\lim_{\tau \to \infty} \frac{1}{2\sqrt{\tau}} \log \left| \int_{\partial\Omega} \int_0^T e^{-\tau t} (v(x; \tau) f(x, t) - u_f(x, t) g(x; \tau)) \, dt \, dS \right| = -\text{dist}(D, \partial\Omega)$$

is valid, where

$$\text{dist} (D, \partial\Omega) = \inf \{|y - x| | y \in \partial\Omega, x \in D\}.$$

Simple examples of $f(x, t)$ are $f(x, t) = \varphi(t) h(x)$ where $h \in C(\partial\Omega)$ with a positive lower bound and $\varphi(t) = t^n$, $n = 0, 1, 2, 3, \ldots$, for $0 < t < T$. This is because of

$$\lim_{\tau \to \infty} \tau^{n+1} \int_0^T e^{-\tau t} t^n \, dt = \int_0^\infty e^{-\xi} \xi^n \, d\xi > 0$$

and thus (1.2) is satisfied for $\mu = n + 1$. It is also possible to give a more general example.

Note that Varadhan [15] considered the asymptotic behaviour as $\tau \to \infty$ of the solution of the problem

$$(\Delta - \tau)v = 0 \quad \text{in} \quad \Omega,$$

$v = 1 \quad \text{on} \quad \partial\Omega.$

He used the behaviour to establish the short-time asymptotics of the heat kernel. See also [14] and references therein for the subject itself. Theorem 1.1 shows that this type of solution can be applied to inverse initial boundary value problems for parabolic equations over a finite time interval.

In [11] Ikehata considered an inverse obstacle scattering problem whose governing equation is given by the wave equation in three dimensions. The observation data are given by a wave field measured on a known surface surrounding unknown obstacles over a finite time interval. The wave is generated by an initial data with compact support outside the surface. Applying the idea of the enclosure method, he established an extraction formula of the distance from a given point outside the surface to obstacles from the data. To establish the formula he made use of the solution $v \in H^1(\mathbb{R}^3)$ of the inhomogeneous modified Helmholtz equation

$$(\Delta - \tau^2)v + f(x) = 0 \quad \text{in} \quad \mathbb{R}^3,$$

where $f(x)$ is an initial data of the wave field. Thus the equations in (1.3) correspond to this equation. However, in contrast to the solution of this equation, that of (1.3) has not an explicit form in general. In this paper, we solve (1.3) by using the potential theory and study its behaviour as $\tau \to \infty$ to get a necessary estimate.
1.2. Other three formulae with special heat fluxes

If one uses special heat fluxes, then one can explicitly obtain more information about the location and shape of $D$. The idea for the derivation of the following formulae come from [12].

The second result is the following.

**Theorem 1.2.** Given $ω \in S^2$ let $f$ be the function of $(x, t) \in \partial \Omega \times ]0, T]$ having a parameter $\tau > 0$ defined by the equation

$$f(x, t; \tau) = \frac{\partial v(x; \tau)}{\partial ν}(x; \tau) ϕ(t), \quad (1.5)$$

where $v(x; \tau) = e^{\sqrt{\tau} \cdot ω}$ and $ϕ \in L^2(0, T)$ satisfying the following condition: there exists $μ \in \mathbb{R}$ such that

$$\liminf_{\tau \to \infty} \tau^{\mu} \left| \int_0^T e^{-\tau t} ϕ(t) \, dt \right| > 0. \quad (1.6)$$

Let $u_f = u_f(x, t)$ be the weak solution of (1.1) for $f = f(x, t; \tau)$ and $h_D(ω) = \sup_{x \in D} x \cdot ω$. Then the formula

$$\lim_{\tau \to \infty} \frac{1}{2\sqrt{\tau}} \log \left| \int_{\partial \Omega} \int_0^T e^{-\tau t} \left( v(x; τ) f(x, t; τ) - u_f(x, t) \frac{\partial v}{\partial ν}(x; τ) \right) \, dt \, dS \right| = h_D(ω) \quad (1.7)$$

is valid.

Note that if $ϕ(t)$ is smooth on $[0, T']$ with $0 < T' \leq T$ and $t = 0$ is not a zero point with infinite order of $ϕ(t)$, then (1.6) is satisfied for an appropriate $μ > 0$.

Next we choose a third solution of the equation $(\triangle - \tau)v = 0$ in $\Omega_1$: given $p \in \mathbb{R}^3 \setminus \Omega_1$

$$v(x; τ) = \begin{cases} e^{\sqrt{\tau} |x - p|} - e^{-\sqrt{\tau} |x - p|} & \text{if } x \in \mathbb{R}^3 \setminus \{y\}, \\ 2\sqrt{\tau} & \text{if } x = y. \end{cases} \quad (1.8)$$

Using this $v$, we obtain the third formula.

**Theorem 1.3.** Let $p \in \mathbb{R}^3 \setminus \Omega_1$ and replace $v$ of (1.5) with (1.8). Let $u_f = u_f(x, t)$ be the weak solution of (1.1) for this $f = f(x, t; τ, p)$. Then assuming (1.6), one has the formula

$$\lim_{\tau \to \infty} \frac{1}{2\sqrt{\tau}} \log \left| \int_{\partial \Omega} \int_0^T e^{-\tau t} \left( v(x; τ) f(x, t; τ, p) - u_f(x, t) \frac{\partial v}{\partial ν}(x; τ) \right) \, dt \, dS \right| = -d_D(p), \quad (1.9)$$

where $d_D(p)$ denotes the distance from $p$ to $D$:

$$d_D(p) = \inf_{y \in D} ||y - p||. \quad (1.10)$$

Finally we introduce another formula which is also new and not given in [12]. Let $y \in \mathbb{R}^3$ be an arbitrary fixed point. We choose the function $v$ given by

$$v(x; τ) = \begin{cases} e^{\sqrt{\tau} |x - y|} - e^{-\sqrt{\tau} |x - y|} & \text{if } x \in \mathbb{R}^3 \setminus \{y\}, \\ 2\sqrt{\tau} & \text{if } x = y. \end{cases} \quad (1.10)$$

Note that $v(x; τ)$ is smooth as the function of $x$ and satisfies the modified Helmholtz equation in the whole space. Hence we can choose the reference point $y \in \mathbb{R}^3$ without any restriction. Note that theorem 1.3 gives $d_D(p)$; however, we have to take $p \in \mathbb{R}^3 \setminus \Omega$. 

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Theorem 1.4. Let \( y \in \mathbb{R}^3 \) and replace \( v \) of \((1.5)\) with \((1.10)\). Let \( u_f = u_f(x, t) \) be the weak solution of \((1.1)\) for \( f = f(x, t; \tau, y) \). Then assuming \((1.6)\), one has the formula

\[
\lim_{\tau \to \infty} \frac{1}{2\sqrt{\tau}} \log \left| \int_{\partial \Omega} \int_0^T e^{-\tau t} \left( v(x; \tau) f(x, t; \tau, y) - u_f(x, t) \frac{\partial v}{\partial \nu}(x; \tau) \right) dt dS \right| = R_D(y),
\]

where \( R_D(y) = \sup_{x \in D} |x - y| \).

The above theorem makes use of a smooth solution of the modified Helmholtz equation that grows every point as \( \tau \to \infty \). The function \( R_D(y) \), \( y \in \Omega \), is a newcomer and gives the minimum radius of the ball centred at \( y \) that contains \( D \). Moreover we have the estimate of \( D \) from above as

\[
D \subset \cap_{y \in \Omega} \{ x \in \mathbb{R}^3 | |x - y| < R_D(y) \}.
\]

1.3. Construction of the paper

A brief outline of this paper is as follows. Theorem 1.1 is proved in subsection 2.3 after formulating the notion of the weak solution of \((1.1)\) together with a related estimate in subsection 2.1. The proof is based on an integral identity which is described in subsection 2.2. Using the identity, we give an asymptotic representation formula of the integral

\[
\int_{\partial \Omega} \int_0^T e^{-\tau t} \left( v(x; \tau) f(x, t) - u_f(x, t) \frac{\partial v}{\partial \nu}(x; \tau) \right) dt dS
\]

whose leading term is given by using two Neumann-to-Dirichlet maps for the operators \( \Delta - \tau \) and \( \nabla \cdot \nabla - \tau \) in \( \Omega \). Then with the help of a system of integral inequalities \([11]\) which is widely used in previous applications of the enclosure method to elliptic equations \([9]\) we see that the problem is reduced to giving some asymptotic estimates for the integral of the gradient of \( v_g \) over \( D \). In some sense, this is an indirect verification of the hypothesis: \( v_g(x; \tau) \sim e^{-\sqrt{\tau}d_{\partial \Omega}(x)} \) as \( \tau \to \infty \). The estimates are stated in subsection 2.3 and their proof is given in subsection 2.4. It is based on the integral representation of \( v_g \) with a single layer potential over \( \partial \Omega \). The proof of theorems 1.2, 1.3 and 1.4 can be done along with the same line as \([12]\) in which the case when \( \partial D \) is perfectly insulated is considered. For reader’s convenience we describe an outline of the proof in section 3. In the appendix we give detailed proofs of four claims used in subsection 2.4.

2. Extracting depth

2.1. Preliminaries about the direct problem

In this subsection, following \([4]\) we describe what we mean by solution \((1.1)\). The presentation here is almost parallel to subsection 2.1 in \([12]\).

We put \( W(0, T; H^1(\Omega), (H^1(\Omega))') = \{ u \in L^2(0, T; H^1(\Omega)) | u' \in L^2(0, T; (H^1(\Omega))') \} \).

Given \( f \in L^2(0, T; H^{-1/2}(\partial \Omega)) \) we say that \( u \in W(0, T; H^1(\Omega), (H^1(\Omega))') \) satisfy

\[
\frac{\partial u}{\partial t} - \nabla \cdot \gamma \nabla u = 0 \quad \text{in} \quad \Omega \times (0, T],
\]

\[
\gamma \nabla u \cdot \nu = f \quad \text{on} \quad \partial \Omega \times (0, T] \tag{2.1}
\]

in the weak sense if \( u \) satisfies

\[
(\langle u'(t), \varphi \rangle + \int_{\Omega} \gamma(x) \nabla u(x, t) \cdot \nabla \varphi(x) \, dx = \langle f(t), \varphi|_{\partial \Omega} \rangle \quad \text{in} \quad (0, T), \tag{2.2}
\]

\[6\]
in the sense of distribution on \((0, T)\) for all \(\varphi \in H^1(\Omega)\) and a.e. \(t \in [0, T]\). We see that every \(u \in W(0, T; H^1(\Omega), (H^1(\Omega))^\prime)\) is almost everywhere equal to a continuous function of \([0, T]\) in \(L^2(\Omega)\) (theorem 1.1 on p 473 in [4]). Further, we have
\[
W(0, T; H^1(\Omega), (H^1(\Omega))^\prime) \hookrightarrow C^0([0, T]; L^2(\Omega)),
\]
the space \(C^0([0, T]; L^2(\Omega))\) being equipped with the norm of uniform convergence. Thus one can consider \(u(0)\) and \(u(T)\) as elements of \(L^2(\Omega)\). Then we see that given \(u_0 \in L^2(\Omega)\) there exists unique \(u\) such that \(u\) satisfies (2.1) in the weak sense and satisfies the initial condition \(u(0) = u_0\) (theorems 1 and 2 on p 512 in [4]).

Let \(u_0 = 0\). Remark 2 on p 512 and theorem 3 on p 520 in [4] yield the continuity of \(u\) on \(f\): there exists \(C_T > 0\) independent of \(f\) such that
\[
\|u\|_{L^2(0, T; H^1(\Omega))} \leq C_T \|f\|_{L^2([0, T]; H^{-1/2}(\partial\Omega))}.
\]
Moreover, from (2.2) and (2.4) we have
\[
\|u\|_{L^2(0, T; H^1(\Omega))} \leq C_T \|f\|_{L^2([0, T]; H^{-1/2}(\partial\Omega))}.
\]
This together with (2.3) and (2.4) yields one of the important estimates in the enclosure method:
\[
\|u(T)\|_{L^1(\Omega)} \leq C_T \|f\|_{L^2([0, T]; H^{-1/2}(\partial\Omega))}.
\]
In the following subsection, we denote by \(u_T\) the weak solution of (2.1) with \(u(0) = 0\) and this is the meaning of the weak solution of (1.1).

2.2. A basic identity

Define
\[
w_f(x; \tau) = \int_0^T e^{-\tau t} u_f(x, t) \, dt, \quad x \in \Omega
\]
and
\[
g_f(x; \tau) = \int_0^T e^{-\tau t} f(x, t) \, dt, \quad x \in \partial\Omega,
\]
where \(\tau > 0\) is a parameter. This type of transform has been used in the study [10] for the corresponding problem in a one-space dimensional case.

In this subsection, we derive an identity that connects the data for the parabolic equation with the Cauchy data of the solutions of the modified Helmholtz-type equations.

Let \(v = v(x)\) satisfy \((\Delta - \tau) v = 0\) in \(\Omega\). Integration by parts yields
\[
\int_{\partial\Omega} \left( g_f v - w_f \frac{\partial v}{\partial n} \right) \, dS = \int_{\Omega} (\gamma - I_3) \nabla v \cdot \nabla u_f \, dx + e^{-\tau T} \int_{\Omega} u_f (x, T) v(x) \, dx.
\]
Let \(p_f = p\) be the unique solution of the boundary value problem:
\[
(\nabla \cdot (\gamma - I_3) \nabla p) = 0 \quad \text{in} \quad \Omega,
\]
\[
\gamma \nabla p \cdot v = g_f \quad \text{on} \quad \partial\Omega.
\]
Set \(\epsilon_f = w_f - p_f\). Since we have
\[
\int_{\partial\Omega} \left( g_f v - p_f \frac{\partial v}{\partial n} \right) \, dS = \int_{\Omega} (\gamma - I_3) \nabla v \cdot \nabla p_f \, dx,
\]
from (2.6) it follows that
\[
\int_{\partial \Omega} \left(gfv - wf \frac{\partial v}{\partial \nu}\right) dS = \int_{\partial \Omega} \left(gfv - pf \frac{\partial v}{\partial \nu}\right) dS + \int_{\Omega} (\gamma - I_3) \nabla v \cdot \nabla \epsilon_f \, dx + e^{-\tau T} \int_{\Omega} u_f(x, T) v(x) \, dx.
\] (2.7)

Note that \(\epsilon_f = \epsilon\) satisfies
\[
(\nabla \cdot (\gamma \nabla - \tau)\epsilon = e^{-\tau T} u_f(x, T) \text{ in } \Omega,
\]
(2.8)
\[
\gamma \nabla \epsilon \cdot \nu = 0 \text{ on } \partial \Omega.
\]

Let \(R_{I_3}(\tau)\) and \(R_\gamma(\tau)\) denote the Neumann-to-Dirichlet maps on \(\partial \Omega\) for the operators \(\triangle - \tau\) and \(\nabla \cdot (\gamma \nabla - \tau)\) in \(\Omega\), respectively. We have
\[
\int_{\partial \Omega} \left(gv - wf \frac{\partial v}{\partial \nu}\right) dS = \int_{\partial \Omega} g(R_{I_3}(\tau) - R_\gamma(\tau)) \frac{\partial v}{\partial \nu} dS + \int_{\Omega} (\gamma - I_3) \nabla v \cdot \nabla \epsilon_f \, dx + e^{-\tau T} \int_{\Omega} u_f(x, T) v(x) \, dx.
\] (2.9)

This is our basic identity. In the proof of theorems 1.1–1.4 we show that, in some sense, one can ignore the second and third terms of the right-hand side. Thus the basic identity provides us a relationship between the boundary data for the parabolic equation over a finite time interval and the Cauchy data for the modified Helmholtz-type equations.

2.3. Proof of theorem 1.1

Since \(\epsilon\) satisfies (2.8), one gets
\[
\|\nabla \epsilon\|_{L^2(\Omega)} \leq C e^{-\tau T} \|u_f(\cdot, T)\|_{L^2(\Omega)},
\] (2.10)
where \(C\) is a positive constant. Since \(f\) is independent of \(\tau\), it follows from (2.5) and (2.10) that \(\|\nabla \epsilon\|_{L^2(\Omega)} = O(e^{-\tau T})\) as \(\tau \to \infty\).

Now substitute \(v = v_g\) into (2.9). From (1.2) and (1.3) one gets \(\|v_g(\cdot, \tau)\|_{H^1(\Omega)} = O(\tau^{-\mu})\) as \(\tau \to \infty\). From these one gets the estimate on the second and third terms in (2.9) as \(\tau \to \infty\):
\[
\int_{\Omega} (\gamma - I_3) \nabla v_g \cdot \nabla \epsilon_f \, dx + e^{-\tau T} \int_{\Omega} u_f(x, T) v_g \, dx = O(\tau^{-\mu} e^{-\tau T}).
\]

Note that this is a very rough estimate; however, for our purpose it is enough; at this step we never make use of the assumption that \(\gamma(x) = I_3\) outside \(D\).

Summing up, we have obtained the asymptotic formula:
\[
\int_{\partial \Omega} (gv_g - wf \frac{\partial v}{\partial \nu}) dS = \int_{\partial \Omega} g(R_{I_3}(\tau) - R_\gamma(\tau))g \, dS + O(\tau^{-\mu} e^{-\tau T}).
\] (2.11)

The following system of inequalities is quite useful to give an estimation of the first term of the right-hand side.

**Proposition 2.1.** Let \(\gamma_0\) and \(\gamma\) satisfy (G1) and (G2). Let \(u\) solve
\[
\nabla \cdot (\gamma \nabla u - \tau u) = 0 \text{ in } \Omega,
\]
\[
\gamma \nabla u \cdot v = g \text{ on } \partial \Omega
\]
and $v$

$$\nabla \cdot \gamma_0 \nabla v - \tau v = 0 \quad \text{in} \quad \Omega ,$$

$$\gamma_0 \nabla v \cdot v = g \quad \text{on} \quad \partial \Omega .$$

Then it holds that

$$\int_\Omega (\gamma_0^{-1} - \gamma^{-1}) \gamma_0 \nabla v \cdot \nabla v \, dx \leq \int_{\partial \Omega} g(v - u) \, dS \leq \int_\Omega (\gamma - \gamma_0) \nabla v \cdot \nabla v \, dx. \quad (2.12)$$

For the proof see [6]. In the present situation $\gamma_0(x) \equiv I_3$ and $\gamma(x) = I_3$ a.e. $x \in \Omega \setminus D$ and thus from (2.12) we obtain

$$\int_D (I_3 - \gamma^{-1}) \nabla v_g \cdot \nabla v_g \, dx \leq \int_{\partial \Omega} g(R_{I_3}(\tau) - R_{\gamma}(\tau)) g \, dS \leq \int_D (\gamma - I_3) \nabla v_g \cdot \nabla v_g \, dx. \quad (2.13)$$

Here we describe a key lemma whose proof is given in the next subsection.

**Lemma 2.1.** There exist real numbers $\lambda_1$ and $\lambda_2$ independent of $\tau$ such that

$$\limsup_{\tau \to \infty} \tau^{\lambda_1} e^{2\sqrt{T_{\text{dist}}(D, \partial \Omega)}} \int_D |\nabla v_g|^2 \, dx < \infty \quad (2.14)$$

and

$$\liminf_{\tau \to \infty} \tau^{\lambda_2} e^{2\sqrt{T_{\text{dist}}(D, \partial \Omega)}} \int_D |\nabla v_g|^2 \, dx > 0. \quad (2.15)$$

From the proof one can choose $\lambda_1 = 2\mu - 1$ and $\lambda_2 = 2\mu + 5/2$; however, the exact values of $\lambda_1$, $\lambda_2$ are not important for the derivation of formula (1.4) itself.

From (2.11), (2.13) and (2.14) one gets

$$\limsup_{\tau \to \infty} \tau^{\lambda_1} e^{2\sqrt{T_{\text{dist}}(D, \partial \Omega)}} \left| \int_{\partial \Omega} (g v_g - w_f g) \, dS \right| < \infty. \quad (2.16)$$

Now consider the case when (A1) is satisfied. It follows from the right half of (2.13) and (2.15) that

$$\liminf_{\tau \to \infty} \tau^{\lambda_2} e^{2\sqrt{T_{\text{dist}}(D, \partial \Omega)}} \left( - \int_{\partial \Omega} g(R_{I_3}(\tau) - R_{\gamma}(\tau)) g \, dS \right) > 0. \quad (2.17)$$

This together with (2.11) gives

$$\liminf_{\tau \to \infty} \tau^{\lambda_2} e^{2\sqrt{T_{\text{dist}}(D, \partial \Omega)}} \left( - \int_{\partial \Omega} (g v_g - w_f g) \, dS \right) > 0. \quad (2.17)$$

This also implies that there exists $\tau_0 > 0$ such that for all $\tau \geq \tau_0$

$$- \int_{\partial \Omega} (g v_g - w_f g) \, dS > 0.$$

Next consider the case when (A2) is satisfied. Since $I_3 - \gamma(x)^{-1} = \gamma(x)^{-1/2}(\gamma(x) - I_3)\gamma(x)^{-1/2}$, one can find a positive constant $C$ such that for all $\xi \in \mathbb{R}^n$, $(I_3 - \gamma(x)^{-1})\xi \cdot \xi \geq C|\xi|^2$. Hence a similar argument yields that

$$\liminf_{\tau \to \infty} \tau^{\lambda_2} e^{2\sqrt{T_{\text{dist}}(D, \partial \Omega)}} \left( \int_{\partial \Omega} (g v_g - w_f g) \, dS \right) > 0 \quad (2.18)$$

and this implies that there exists $\tau_0 > 0$ such that for all $\tau \geq \tau_0$

$$\int_{\partial \Omega} (g v_g - w_f g) \, dS > 0.$$

Now formula (1.4) is a direct consequence of (2.16), (2.17), (2.18) and the identity

$$\int_{\partial \Omega} \int_0^T e^{-\tau t} \left( v(x, t; \tau) - u_f(x, t) \frac{\partial v}{\partial \nu}(x) \right) \, dt \, dS = \int_{\partial \Omega} \left( g v - w_f \frac{\partial v}{\partial \nu} \right) \, dS.$$
2.4. Proof of lemma 2.1

Let μ be the constant in (1.2). Set \( \tilde{v}(x; \tau) = \tau^\mu v_g(x; \tau) \) and
\[
\tilde{g}(x; \tau) = \tau^\mu \int_0^\tau e^{-\tau t} f(x, t) \, dt, \quad x \in \partial \Omega.
\]
It suffices to prove (2.14) and (2.15) for \( \tilde{v} \) instead of \( v_g \).

\( \tilde{v} \) satisfies \((\Delta - \tau) \tilde{v} = 0 \) in \( \Omega \) and \( \partial \tilde{v} / \partial s = \tilde{g} \) on \( \partial \Omega \). In what follows we simply write \( \tilde{v} \) and \( \tilde{g} \) as \( v \) and \( g \), respectively. We think that this creates no confusion. Thus from (1.2) one has the following: there exist positive constants \( C \) and \( \tau_0 \) independent of \( x \in \partial \Omega \) such that, for all \( x \in \partial \Omega \) and all \( \tau \geq \tau_0 \),
\[
C^{-1} \leq g(x; \tau) \leq C. \tag{2.19}
\]

Using the potential theory (cf [13]), one has the expression
\[
v(x; \tau) = \frac{1}{2\pi} \int_{\partial \Omega} \frac{e^{-\sqrt{\tau}|x-y|}}{|x-y|} \psi(y; \tau) \, dS_y, \quad x \in \Omega,
\]
where \( \psi(\cdot; \tau) \in C(\partial \Omega) \) is the unique solution of the integral equation of the second kind on \( \partial \Omega \):
\[
\psi(y; \tau) + \frac{1}{2\pi} \int_{\partial \Omega} \frac{\partial}{\partial v_y} \left( \frac{e^{-\sqrt{\tau}|y-y'|}}{|y-y'|} \right) \psi(y'; \tau) \, dS_{y'} = g(y; \tau), \quad y \in \partial \Omega. \tag{2.20}
\]

It is well known that the operator
\[
C(\partial \Omega) \ni \varphi \mapsto S_{\partial \Omega}(\tau) \varphi \in C(\partial \Omega),
\]
where
\[
S_{\partial \Omega}(\tau) \varphi(y) = \frac{1}{2\pi} \int_{\partial \Omega} \frac{\partial}{\partial v_y} \left( \frac{e^{-\sqrt{\tau}|y-y'|}}{|y-y'|} \right) \varphi(y') \, dS_{y'}, \quad y \in \partial \Omega
\]
is bounded and its operator norm has a bound \( O(\tau^{-1/2}) \) as \( \tau \to \infty \). Thus it follows from (2.19) and (2.20) that \( \psi(\cdot; \tau) \) also has the following: there exist positive constants \( C' \) and \( \tau_0 \) independent of \( x \in \partial \Omega \) such that, for all \( x \in \partial \Omega \) and all \( \tau \geq \tau_0 \),
\[
C'^{-1} \leq \psi(x; \tau) \leq C'. \tag{2.21}
\]

Since
\[
\nabla v(x; \tau) = -\frac{1}{2\pi} \int_{\partial \Omega} \psi(y; \tau) \frac{e^{-\sqrt{\tau}|x-y|}}{|x-y|} \left( \sqrt{\tau} + \frac{1}{|x-y|} \right) \frac{x-y}{|x-y|} \, dS_y,
\]
one has
\[
\int_D |\nabla v(x; \tau)|^2 \, dx = \int_{\partial \Omega} dS_y \int_{\partial \Omega} dS_{y'} \int_D dx \, e^{-\sqrt{\tau}|x-y|} e^{-\sqrt{\tau}|x-y'|} \Phi(x, y, y'; \tau), \tag{2.22}
\]
where
\[
\Phi(x, y, y'; \tau) = \frac{1}{(2\pi)^2} \psi(y; \tau) \psi(y'; \tau) \left( \sqrt{\tau} + \frac{1}{|x-y|} \right) \left( \sqrt{\tau} + \frac{1}{|x-y'|} \right) \times \left( \frac{x-y}{|x-y|} \right) \left( \frac{x-y'}{|x-y'|} \right). \tag{2.23}
\]

From (2.21), (2.22) and (2.23) we can easily obtain (2.14) with \( \lambda_1 = -1 \).

The problem is the proof of (2.15). We divide the integrand of (2.22) into two parts. Set \( d_0 = \text{dist}(D, \partial \Omega) \) and \( \mathcal{M} = \{(x, y) \in \overline{D} \times \partial \Omega : |x-y| = d_0\} \). It is easy to see that \( \mathcal{M} \) coincides with the set of all \( (x, y) \in \partial D \times \partial \Omega \) such that \( |x-y| = d_0 \).
In what follows we denote by \( B_R(z) \) the open ball centred at a point \( z \) with radius \( R \). Given \( \delta > 0 \) define

\[
\mathcal{W}_\delta = \bigcup_{(x_0,y_0) \in \mathcal{M}} (\overline{D} \cap B_\delta(x_0)) \times (\partial \Omega \cap B_\delta(y_0)) \times (\partial \Omega \cap B_\delta(y_0)).
\]

The set \( \mathcal{W}_\delta \) is open in \( \overline{D} \times \partial \Omega \times \partial \Omega \) and contains the set of all \((x,y,y)\) with \((x,y) \in \mathcal{M} \).

Here we state two claims concerning \( \mathcal{W}_\delta \) whose proof is given in the appendix.

**Claim 1.** Given \( \epsilon > 0 \) there exists \( \delta_1 > 0 \) such that for all \((x,y,y) \in \mathcal{W}_{\delta_1}\), it holds that

\[
\frac{x-y}{|x-y|} - \frac{x-y'}{|x-y'|} \geq 1 - \epsilon, \quad |x-y| \leq d_0 + \epsilon, \quad |x-y'| \leq d_0 + \epsilon.
\]

**Claim 2.** Given \( \delta_1 > 0 \) there exists \( \delta_2 > 0 \) such that if \((x,y,y) \in \mathcal{W}_{\delta_1}\) and \( \tau \geq \tau_0 \),

\[
\Phi(x,y,y';\tau) \geq C_1 \tau.
\]

Thus giving \( \epsilon = 1/2 \) in claim 1 and choosing the corresponding \( \delta_1 \), we have the following:

if \((x,y,y') \in \mathcal{W}_{\delta_1}\), then

\[
\frac{x-y}{|x-y|} - \frac{x-y'}{|x-y'|} \geq \frac{1}{2}.
\]

It follows from this together with the left half of (2.21) and (2.23) that there exist constants \( C_1 \) and \( \tau_0 > 0 \) such that, for all \((x,y,y') \in \mathcal{W}_{\delta_1}\) and \( \tau \geq \tau_0 \),

\[
\Phi(x,y,y';\tau) \geq C_1 \tau.
\]

On the other hand, using the right half of (2.21), it is easy to see that there exist positive constants \( C_2 \) and \( \tau_1 > \tau_0 \) such that, for all \((x,y,y') \in \mathcal{W}_{\delta_1}\) and \( \tau \geq \tau_1 \),

\[
\Phi(x,y,y';\tau) \leq C_2 \tau.
\]

Now choose \( \delta_2 \) in claim 2 corresponding to \( \delta_1 \) already chosen.

Then we have

\[
e^{-\sqrt{\tau}(|x-y|+|x-y'|)} \leq e^{-2\sqrt{\tau_0} - \sqrt{\tau\tau_1}} \quad \text{for any} \quad (x,y,y') \in \overline{D} \times \partial \Omega \times \partial \Omega \setminus \mathcal{W}_{\delta_1}.
\]

Hence dividing the integral (2.22) into \( \mathcal{W}_{\delta_1} \) and its compliment, one gets as \( \tau \to \infty \)

\[
t^{-1} \int_D |\nabla v(x;\tau)|^2 \, dx \geq C_1 \int_{\mathcal{W}_{\delta_1}} dS_x \, dS_y \, dx \, e^{-\sqrt{\tau}(|x-y|+|x-y'|)} + O(e^{-2\sqrt{\tau_0}} - \sqrt{\tau_1} \tau). \tag{2.24}
\]

Choose \((x_0,y_0) \in \mathcal{M}\). It follows from the definition of \( \mathcal{W}_{\delta_1} \) and the inequality \(|x-y|+|x-y'| \leq 2|x-x_0| + 2d_0 + |y_0-y| + |y_0-y'|\) that

\[
\int_{\mathcal{W}_{\delta_1}} dS_x \, dS_y \, dx \, e^{-\sqrt{\tau}(|x-y|+|x-y'|)}
\]

\[
\geq \int_{\overline{D} \cap B_\delta(x_0)} dx \int_{\partial \Omega \cap B_\delta(y_0)} dS_y \, e^{-\sqrt{\tau}(|x-y|+|x-y'|)}
\]

\[
\geq e^{-2\sqrt{\tau_0}} \int_{\overline{D} \cap B_\delta(x_0)} e^{-2\sqrt{\tau}(|x-x_0|)} \, dx \left( \int_{\partial \Omega \cap B_\delta(y_0)} e^{-\sqrt{\tau}(|y_0-y|)} dS_y \right)^2.
\]

Now (2.15) with \( \lambda_2 = 5/2 \) is a direct consequence of this together with (2.24) and the following two claims.

**Claim 3.** For all \( \delta > 0 \) we have

\[
\liminf_{\tau \to \infty} \tau \int_{D \cap B_\delta(x_0)} e^{-2\sqrt{\tau}(|x-x_0|)} \, dx > 0.
\]

**Claim 4.** For all \( \delta > 0 \) we have

\[
\liminf_{\tau \to \infty} \tau \int_{\partial \Omega \cap B_\delta(y_0)} e^{-\sqrt{\tau}(|y_0-y|)} dS_y > 0.
\]

For the proof of these claims see the appendix.
3. Outline of the proof of theorems 1.2, 1.3 and 1.4

Since from (1.5) we have
\[ g_f(x; \tau) = \frac{\partial v}{\partial \nu}(x; \tau) \int_0^T e^{-\tau t} \varphi(t) \, dt, \]
it follows from (2.9) that
\[ \int_{\partial \Omega} \left( g_f v - u_f \frac{\partial v}{\partial \nu} \right) \, dS = \int_0^T \frac{\partial v}{\partial \nu}(R_f(\tau) - R_f(\tau)) \left( \frac{\partial v}{\partial \nu} \right) \big|_{\partial \Omega} \, dS \]
\[ + \int_{\partial \Omega} (\gamma - I_3) \nabla v \cdot \nabla \epsilon_f \, dx + e^{-\tau T} \int_{\Omega} u_f(x, T) v(x; \tau) \, dx. \quad (3.1) \]
By virtue of (2.5) one knows that for both \( v \) in theorems 1.2–1.4 there exists a constant \( \kappa \) such that
\[ \| u_f(\cdot, T) \|_{L^2(\Omega)} = O(e^{\kappa \sqrt{\tau}}) \text{ as } \tau \to \infty. \]
This together with (2.10) yields that as \( \tau \to \infty \)
\[ \int_{\Omega} (\gamma - I_3) \nabla v \cdot \nabla \epsilon_f \, dx + e^{-\tau T} \int_{\Omega} u_f(x, T) v(x; \tau) \, dx = O(e^{-\tau T/2}). \quad (3.2) \]
Here we recall the following lemma.

**Lemma 3.1.** There exist real numbers \( \mu_1, \mu_2, \mu_3 \in \mathbb{R} \) such that, for all \( \omega \in S^2, p \in \mathbb{R}^3 \setminus \Omega \) and \( y \in \mathbb{R}^3 \),
\[ \liminf_{\tau \to \infty} \tau^{\mu_1} e^{-2\sqrt{\tau}d_0(\omega)} \int_D e^{2\sqrt{\tau}x \cdot \omega} \, dx > 0, \quad (3.3) \]
\[ \liminf_{\tau \to \infty} \tau^{\mu_2} e^{2\sqrt{\tau}d(p)} \int_D e^{-2\sqrt{\tau}|x-p|} \, dx > 0 \quad (3.4) \]
and
\[ \liminf_{\tau \to \infty} \tau^{\mu_3} e^{-2\sqrt{\tau}d(y)} \int_D e^{2\sqrt{\tau}|x-y|} \, dx > 0. \quad (3.5) \]

It is not necessary for us to know the values of \( \mu_1, \mu_2, \mu_3 \) precisely. Every case can be reduced to the case when \( D \) is given by an open ball since we are assuming that \( \partial D \) is smooth. See [9] for the proof of (3.3) and [12] (or [11]) for the proof of (3.4) and (3.5). Now it is a due course to see that a combination of (1.6), (2.12), (3.1), (3.2) and (3.3)/(3.4)/(3.5) yields (1.7)/(1.9)/(1.11).

4. Conclusion and open problems

We showed how the enclosure method can be applied to inverse initial boundary value problems over a finite time interval for parabolic equations with discontinuous coefficients. We established four types of formulae. It should be emphasized that in all formulae the initial temperature field \( \text{inside the body} \) is assumed to be a known constant. We think that this is a natural condition and can be realized without special care in practice. In fact, just make it cold by using a refrigerator if the size of the body is not too large!

Two of them are new in idea and yield the following: (I) a depth of unknown inclusions in a heat conductive body from the surface of the body with a single set of a heat flux and the corresponding temperature on the surface over a finite time interval (see theorem 1.1); (II) the minimum radius of the open ball centred at a given point that contains unknown inclusions with a special explicit heat flux with a large parameter (see theorem 1.4).
The point is the choice of the heat flux $f$ and a solution $v$ of the modified Helmholtz equation $(\triangle - \tau)v = 0$ in $\Omega$ in the integral

$$\int_{\partial\Omega} \int_0^T e^{-\tau t} \left( vf - uf \frac{\partial v}{\partial v} \right) dt \, dS.$$  \hspace{1cm} (4.1)

In (I) first $f$ is given and we choose $v$ by solving the Neumann problem for the modified Helmholtz equation in $\Omega$ whose Neumann data can be calculated from $f$; in (II) using a special $v$ which is growing everywhere as $\tau \to \infty$, we specify the form of $f$.

The procedure suggested from (I) of extracting $\text{dist}(D, \partial\Omega)$ is extremely simple and summarized as follows.

(i) Give the heat flux $f$ satisfying (1.2) for $\mu \in \mathbb{R}$ across $\partial\Omega$ over the time interval $[0, T]$.
(ii) Measure the temperature $u_f(x, t)$ on $\partial\Omega$ over the time interval $[0, T]$.
(iii) Fix large $\tau > 0$ and compute the solution $v_g$ of (1.3).
(iv) Compute the quantity

$$-\frac{1}{2\sqrt{\tau}} \log \left| \int_{\partial\Omega} \int_0^T e^{-\tau t} \left( v(x; \tau)f(x, t) - uf(x, t)\frac{\partial v}{\partial v}(x; \tau) \right) dt \, dS \right|$$

as an approximation of $\text{dist}(D, \partial\Omega)$.

If $D$ is near surface $\partial\Omega$ and isolated in a small part, the information $\text{dist}(D, \partial\Omega)$ may not be so useful; however, if $D$ is deep inside or occupies a large part, then the set of all $x \in \Omega$ such that $\text{dist}(D, \partial\Omega) < d_{\partial\Omega}(x)$ may give a good estimation of $D$ from above.

The method can also be applied to more complicated situations, for example, inclusions in a body with a known inhomogeneous isotropic or anisotropic conductivity apart from some technical difficulties or similar problems with acoustic/elastic/electromagnetic waves, etc. Such applications belong to our future study.

The next challenging problem is to clarify what information about $D$ can be extracted from the asymptotic behaviour of integral (4.1) as $\tau \to \infty$ if $f$ is fixed; $v$ is one of the three special solutions of the modified Helmholtz equation in theorems 1.2, 1.3 and 1.4, that is one of $v = e^{\sqrt{\tau}x \cdot \omega}$, (1.8) and (1.10). This remains open at the present time.

Finally it should be pointed out that in Beilina and Klibanov [1] an inverse problem for the Cauchy problem for a hyperbolic equation with a variable smooth coefficient in the whole space has been considered. The initial data are given by the delta function concentrated at a single point located outside of the domain of interest. They gave a numerical procedure for the reconstruction of the coefficient in the domain of interest by observing the solution on the surface surrounding the domain. This procedure is also tested numerically. It would be interesting to construct and test a numerical method based on the formulae obtained in our paper. The study is not in the scope of our paper, but rather is left for future work.

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Appendix. Proof of claims

Claim 1

Define

\[ F(x, y, y') = \left( 1 - \frac{x - y}{|x - y|} \cdot \frac{x - y'}{|x - y'} \right) + (|x - y| - \delta) \]

\[ + (|x - y'| - \delta_0), \ (x, y, y') \in \overline{D} \times \partial\Omega \times \partial\Omega. \]

Since \(|x - y| \geq \delta_0\) and \((x - y)/|x - y|, (x - y')/|x - y'| \leq 1\) for \((x, y, y') \in \overline{D} \times \partial\Omega \times \partial\Omega\), it suffices to prove that given \(\epsilon > 0\) there exists \(\delta_1 > 0\) such that \(F(x, y, y') \leq \epsilon\) for all \((x, y, y') \in \mathcal{W}_h\). Assume that this is not true. There exist \(\epsilon_0 > 0\) and a sequence \((x_l, y_l, y_l') \in \mathcal{W}_h/l, l = 1, 2, \ldots\), such that \(F(x_l, y_l, y_l') \geq \epsilon_0\). The definition of \(\mathcal{W}_h/l\) we know that for each \(l\) there exists \((p_l, q_l) \in \mathcal{M}\) such that \(|x_l - p_l| < 1/l, |y_l - q_l| < 1/l\) and \(|y_l' - q_l| < 1/l\). Since \(\overline{D}\) and \(\partial\Omega\) are compact, one can choose a subsequence \(l_1, l_2, \ldots\) of \(l = 1, 2, \ldots\) in such a way that the limits \(x_{l_j} = x_j \in \overline{D}, y_{l_j} = y_j \in \partial\Omega, y_{l_j}' = y_j' \in \partial\Omega\) exist. Clearly it holds that \(x = p\) and \(y = y' = q\). Since \(\mathcal{M}\) is closed, one gets \((x, y) \in \mathcal{M}\), and thus \(|x - y| = \delta_0\). This together with \(y = y'\) gives \(F(x, y, y') = 0\). On the other hand, since \(F(x_l, y_l, y_l') \geq \epsilon_0\), one gets \(F(x, y, y') \geq \epsilon_0\). This is a contradiction.

Claim 2

Assume that the statement is not true. There exist \(\delta_0 > 0\) and a sequence \((x_l, y_l, y_l') \in \overline{D} \times \partial\Omega \times \partial\Omega|\mathcal{W}_h, l = 1, 2, \ldots\), such that \(|x_l - y_l| + |x_l - y_l'| < 2\delta_0 + 1/l\). Since \(\overline{D}\) and \(\partial\Omega\) are compact, if necessary replacing the sequence with a suitable subsequence, one may assume that the limits \( \lim_{j \to \infty} x_j = x \in \overline{D}, \lim_{j \to \infty} y_j = y \in \partial\Omega \), and \( \lim_{j \to \infty} y_j' = y' \in \partial\Omega \). Since \(\mathcal{W}_h/l\) is open, one has \((x, y, y') \in \overline{D} \times \partial\Omega \times \partial\Omega|\mathcal{W}_h, l = 1, 2, \ldots\). On the other hand, since \(|(x_l - y_l) - \delta_0| + |(x_l - y_l') - \delta_0| < 1/l, |x_l - y_l'| \geq \delta_0\) and \(|x_l - y_l'| \geq \delta_0, \ (x_l, y_l, y_l') \in \mathcal{W}_h\), we obtain \(|x - y| = |x - y'| = \delta_0\). This means that \((x, y) \in \mathcal{M}\), and \((x, y) \in \mathcal{M}\). Since \(x \in \partial D\), using local coordinates at \(x\) and \(y\), one can easily show that \((y-x)/\delta_0 = v_s\) and similarly \((y'-x)/\delta_0 = v_s\). This yields \(y = y'\), and thus \((x, y, y') \in \mathcal{W}_h\). This is a contradiction.

Claim 3

Given \(\delta > 0\) one can choose \(\delta_0 > 0\) with \(\delta_0 < \delta\) such that there exists a smooth function \(g\) on \(\mathbb{R}^2\) with compact support such that \(g(0,0) = 0\) and \(D \cap B_{\delta_0}(0) = \{x_0 + \sigma_1 e_1 + \sigma_2 e_2 - s v_{\sigma_0} \sigma_1^2 + \sigma_2^2 + s^2 < \delta_0^2, s > g(\sigma_1, \sigma_2)\}\), where \(e_1\) and \(e_2\) are the unit tangent vectors at \(x_0\) and orthogonal to each other.

One can choose a positive constant \(C\) in such a way that for all \(\sigma = (\sigma_1, \sigma_2) \in \mathbb{R}^2\) it holds that \(|g(\sigma)| \leq C|\sigma|\). Let \(\tau \geq 1\). We can easily see that if \(s > C|\sigma|\), then \(s > \sqrt{T \sigma}/\sqrt{\tau}\).

This together with the change of variables yields

\[ \int_{D \cap B_{\delta_0}(0)} e^{-\sqrt{T|x-x_0|}} \, dx \geq \int_{s > \sqrt{T \sigma}/\sqrt{\tau}} e^{-\sqrt{T \sigma^2 + s^2}} \, d\sigma \, ds \]

\[ \geq \int_{s > C|\sigma|, \sigma^2 + s^2 < \sqrt{T \sigma}^2} e^{-\sqrt{T \sigma^2 + s^2}} \, d\sigma \, ds. \]

Since

\[ \lim_{\tau \to \infty} \int_{s > C|\sigma|, \sigma^2 + s^2 < \sqrt{T \sigma}^2} e^{-\sqrt{T \sigma^2 + s^2}} \, d\sigma \, ds = \int_{s > C|\sigma|} e^{-\sqrt{T \sigma^2 + s^2}} \, d\sigma \, ds < \infty, \]

we have the desired conclusion.
Claim 4

Given \( \delta > 0 \) one can choose \( \delta_0 > 0 \) with \( \delta_0 < \delta \) such that there exists a smooth function \( h \) on \( \mathbb{R}^2 \) with compact support such that \( h(0, 0) = 0 \) and \( \partial \Omega \cap B_{\delta_0}(y_0) = \{ y_0 + \sigma_1 e_1 + \sigma_2 e_2 - h(\sigma_1, \sigma_2) \nu_0 | \sigma_1^2 + \sigma_2^2 + h(\sigma_1, \sigma_2)^2 < \delta_0^2 \} \), where \( e_1 \) and \( e_2 \) are unit tangent vectors at \( y_0 \) and orthogonal to each other. Note that \( h \) also satisfies that for all \( \sigma = (\sigma_1, \sigma_2) \in \mathbb{R}^2 \), 

\[ h(\sigma) \leq C|\sigma|, \]

where \( C \) is a positive constant. We see that if 

\[ |\sigma| < \delta_0 / (\sqrt{1 + C^2}), \]

then \( \sigma_1^2 + \sigma_2^2 + h(\sigma_1, \sigma_2)^2 < \delta_0^2 \). This together with a change of variables yields 

\[
\int_{\partial \Omega \cap B_{\delta_0}(y_0)} e^{-\sqrt{\tau}|y-y_0|} dSy \geq \int_{|\sigma| < \delta_0 / (\sqrt{1 + C^2})} e^{-\sqrt{\tau} \sqrt{|\sigma|^2 + h(\sigma)^2} \sqrt{1 + |\nabla h(\sigma)|^2}} d\sigma
\]

\[
\geq \int_{|\sigma| < \delta_0 / (\sqrt{1 + C^2})} e^{-\sqrt{\tau} \sqrt{|\sigma|^2 + C^2} |\sigma|} d\sigma
\]

\[ = (\sqrt{\tau} \sqrt{1 + C^2})^{-2} \int_{|\sigma| < \sqrt{\delta_0}} e^{-|\sigma|} d\sigma. \]

Now one gets the desired conclusion.

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