On the Semisimplicity of the Action of the Frobenius on Etale Cohomology

Marcelo Gomez Morteo

August 2, 2011

Abstract

We give a proof of the semisimplicity of the action of the geometric frobenius on etale cohomology. The proof is based on [MGM11] and on the Weil Conjectures, i.e., on the Riemann Hypothesis for nonsingular projective varieties over finite fields.

Keywords: Local Spectra. Algebraic and Topological K Theory l-adic Completion of Spectra.

Introduction:

We are going to work with the abelian category $B(l)_*$. See the definition below and also [B83]. In [B83] Bousfield defined an universal functor $U: Z(l) \rightarrow B(l)_*$ which will be crucial here. We start with the isomorphism ([T89])

\[ \pi_0(\mathcal{L}(E(1)) K(X_\infty)) \bigotimes Q \simeq \oplus_{i=0}^{d} H^2_{et}(X_\infty, Q(i))(i) \]

(1)

Here $L_{(E(1))}$ is is the Bousfield localization of $E(1)$ where $E(1)$ is such that $K(l) = \bigvee_{i=1}^{\infty} \Sigma^{2i} E(1)$. (See below the definition of $E(1)$ and also [B83]) $K(l)$ is the $l$ localized topological K spectrum $K$. In (1), $K(X_\infty)$ is the Quillen’s algebraic K theory spectrum on $X_\infty = X \otimes F$, where $F$ is the algebraic closure of the finite field $F_q$ with $q = p^s$ and $p \neq l$. $X$ is a nonsingular projective variety over $F_q$ with finite dimension and $d$ is the dimension of $X_\infty$. $L_{E(1)}$ denotes $E(1)$ localization, and $[L_{E(1)} K(X_\infty)]^l$ denotes the $l$-adic completion of the spectra $L_{E(1)} K(X_\infty)$. Here $H^2_{et}(X_\infty, Q(i))$ are the etale cohomology groups with coefficients in $Q_l$, which are $Q_l$ vector spaces.

We show that $\mathcal{U}(\pi_0([L_{E(1)} K(X_\infty)]^l) \bigotimes Q) \simeq [E_0(1)(K(X_\infty))]^l \bigotimes Q$ (2)
where on the right hand side of the equality, $[E_0(1)(K(X_{\infty}))]^l$ is the $l$-adic completion of the $Z_{(l)}$-module $E_0(1)(K(X_{\infty}))$ which is an object in $B$ (See the definition of $B$ below). This isomorphism is dependent on highly non trivial facts, such as the fact that $\pi_0 L_{E(1)} K(X_{\infty})$ is an $l$-reduced group, which follows from [MGM11]. Being $l$-reduced group means:

$$\text{Hom}(\mathbb{Q}/\mathbb{Z}(l), \pi_0 L_{E(1)} K(X_{\infty})) = 0$$

We then obtain the isomorphism: $[(E_0(1)(K(X_{\infty})))^l \otimes \mathbb{Q}] \simeq \mathcal{U}(\bigoplus_{i=0}^{2d} H_{et}(X_{\infty}, \mathbb{Q}_{(l)}(i)))$. Therefore the functor $\mathcal{U}$ allows to study the action of the geometric Frobenius $\Phi_X : X \mapsto X$ on $\bigoplus_{i=0}^{2d} H_{et}(X_{\infty}, \mathbb{Q}_{(l)}(i))$ through the action of $\mathcal{U}(\Phi_X)$ on $[E_0(1)(K(X_{\infty})))^l \otimes \mathbb{Q}$. We prove that $E_0(1)(K(X_{\infty})) \otimes \mathbb{Q}$ is dense in $[E_0(1)(K(X_{\infty})))^l \otimes \mathbb{Q}$ with the $l$-adic topology and finally show that $U(\Phi_X)$ is an Adams operation on $[E_0(1)(K(X_{\infty}))))^l \otimes \mathbb{Q}$ because of the Weil Conjectures and we conclude that the action of $\Phi_X$ on the étale cohomological spaces $H_{et}^2(X_{\infty}, \mathbb{Q}_{(l)}(i))$ is semisimple.

The Category $B(l)_*$

We begin by describing an abelian category, denoted $B(l)_*$, equivalent to the category of $E(1)_*, E(1)$-comodules (see [B83], 10.3) Bousfield describes $B(l)_*$ as follows: Let $l$ be an odd prime and let $B$ denote the category of $Z_{(l)}[Z^*_{(l)}]$-modules for the group ring $Z_{(l)}[Z^*_{(l)}]$, where $Z^*_{(l)}$ are the units in $Z_{(l)}$, with the action by the group ring defined by Adams operations $\Psi^k : M \mapsto M$ which are automorphisms and satisfy the following:

i) There is an eigenspace decomposition

$$M \otimes \mathbb{Q} \cong \bigoplus_{j \in \mathbb{Z}} W_{j(l-1)}$$

such that for all $w \in W_{j(l-1)}$ and $k \in Z_{(l)}$,

$$(\Psi^k \otimes id)w = k^{j(l-1)}w$$

ii) For all $x \in M$ there is a finitely generated submodule $C(x)$ containing $x$, satisfying: for all $m \geq 1$ there is an $n$ such that the action of $Z^*_{(l)}$ on
$C(x)/l^nC(x)$ factors through the quotient of $(Z/l^{n+1})^*$ by a subgroup of order $l-1$.

To build the category $\mathcal{B}(l)_*$ out of the above category $\mathcal{B}$, we additionally need the following:

Let $T^j(l^{-1}) : \mathcal{B} \mapsto \mathcal{B}$ with $j \in Z$ denote the following equivalence:

For all $M$ in $\mathcal{B}$, $T^j(l^{-1})(M) = M$ as $Z(l)$-module, but not as $Z(l)[Z(l)^*]$-module since the Adams operations in $T^j(l^{-1})(M)$ are now $k^j(l^{-1})\Psi^k : M \mapsto M$ where $\Psi^k$ is the Adams operation of multiplication by $k$ in $\mathcal{B}$. Now an object in $\mathcal{B}(l)_*$ is defined as a collection of modules $M = (M_n)_{n \in Z}$, with $M_n$ in $\mathcal{B}$ together with a collection of isomorphisms for all $n \in Z$,

$$T^{l-1}(M_n) \mapsto M_{n+2(l-1)}$$

Note that the category $\mathcal{B}$ can be viewed as the subcategory of $\mathcal{B}(l)_*$ consisting of those objects $(M_n)_{n \in Z}$ such that $M_n = M$ if $n$ is congruent to 0 mod $2(l-1)$ and 0 otherwise.

In [B83] Bousfield constructs a functor $\mathcal{U} : \pi_*(E(1) - \text{Mod}) \mapsto \mathcal{B}(l)_*$. For $H \in \pi_*(E(1) - \text{Mod})$, let $\mathcal{U}$ in $\mathcal{B}$ consist of the objects $\mathcal{U}(H_n)$ in $\mathcal{B}$ for all $n \in Z$.

The Spectrum $E(1)$ and its homology theory $E(1)_*$:

Given $E(1)$, which by construction depends on the prime $l$, there is a map $E(1) \mapsto K_i$ which is a ring morphism (see [R] Chapter VI Theorem 3.28) and verifies the equivalence $K_i = \vee_{l=1}^{l-2} \Sigma^2 E(1)$. There are Adams operations $\Psi^k : E(1) \mapsto E(1)$ with $k \in Z(l)^*$ which are the units in $Z(l)$. These Adams operations are ring spectra equivalences and $\Psi^k$ carries $\nu^j$ to $k^{j(l-1)}\nu^j$ in $\pi_{2j(l-1)}E(1)$ for each integer $j$ where $\nu$ is such that $\pi_1 E(1) = Z[l][\nu, \nu^{-1}]$ and $\nu$ has degree $2(l-1)$. Another property of $E(1)$ is that $E(1)$ localization is the same as $calK(l)$ localization.

The homology $E(1)_*(X)$ with $X$ a spectrum also has Adams operations $\Psi^k : E(1)_*(X) \mapsto E(1)_*(X)$. One checks that $\Psi^k(\nu^j x) = k^{j(l-1)}\nu^j \Psi^k(x)$ for each integer $j$ and $k \in Z(l)^*$ and $x \in E(1)_*(X)$. The multiplication by $\nu^j$ induces an isomorphism $\nu^j : T^{j(l-1)}E(1)_n(X) \mapsto E(1)_{n+2j(l-1)}(X)$ in $\mathcal{B}(l)_*$ for each $j, n \in Z$. It follows that $E(1)_*(X)$ is in $\mathcal{B}(l)_*$ for each spectrum $X$ in $\mathcal{S}$ by taking $E(1)_*(X) = M$ defined in 1.1 and by taking as Adams operations, the Adams operations just mentioned.
Remarks 1.

a) We know from ([B83] page 929) that \( \mathcal{U}: \pi_*(E(1) - \text{Mod}) \to \mathcal{B}(l)_* \) verifies:

\[
\mathcal{U}(G) = E(1)_*E(1) \otimes_{\pi_*E(1)} G
\]

for all \( \pi_*(E(1)) \)-module \( G \). In particular taking 0 component, \( \mathcal{U}_0(G) = E_0(1)E(1) \otimes_{\mathbb{Z}/(l)} G \).

Therefore if \( \phi: G \mapsto G \) is a map of \( \mathbb{Z}(l) \)-modules with an eigenvalue \( \lambda \), then \( \mathcal{U}(\phi) \) has also eigenvalue \( \lambda \) in \( \mathcal{U}(G) \).

b) The next proposition is also proven in [T89]

Proposition 1: Under the hypothesis that \( \pi_0(L_{E(1)}(K(X_\infty))) \) is \( l \)-reduced we get:

\[
\pi_0([L_{E(1)}K(X_\infty)]^l) \otimes \mathbb{Q} \simeq [\pi_0(L_{E(1)}K(X_\infty))]^l \otimes \mathbb{Q}
\]

Proof: Let \( G_\nu = \pi_0(Y \wedge M(Z/l') \simeq \pi_0(T_\nu) = \pi_0(Y/l') \), where \( Y = L_{E(1)}K(X_\infty) \).

There is an exact sequence ([BK72] Chap 9)

(3)

\[
0 \mapsto \lim^{1}(G_\nu) \mapsto \pi_0(\text{homlim}T_\nu) \mapsto \lim(G_\nu) \mapsto 0
\]

and (4)

\[
G_\nu = \pi_0(Y \wedge M(Z/l')) \simeq (\pi_0(Y) \otimes Z/l') \oplus \text{Tor}^1(Z/l', Y)
\]

Now \( \lim \text{Tor}^1(Z/l', Y) = 0 \) since the limit is equal to

\[
\Pi_{i=1}^\infty \{g_i = l^i - \text{torsion} - \text{element} \in \pi_0(L_{E(1)}K(X_\infty))/lg_i+1 = g_i\}
\]

and each coordinate in this limit is 0, for it belongs to the intersection of all \( l^i(\pi_0L_{E(1)}K(X_\infty)) \) which is 0 because \( \pi_0L_{E(1)}K(X_\infty) \) is reduced. Then by (4), I get:

(5)

\[
\lim(G_\nu) = \lim(\pi_0(Y) \otimes Z/l') = [\pi_0(Y)]^l
\]

Obviously (6): \( \lim^{1}(\pi_0(Y) \otimes Z/l') = 0 \). On the other hand, \( \lim^1 \text{Tor}^1(Z/l', Y) \) has bounded \( l \)-torsion. Let me show why this is so:
Let \( M_\nu = Tor^1(Z/l^\nu, Y) \). The map \( M_{\nu+1} \rightarrow M_\nu \) is the map which goes from the \( l^{\nu+1} \)-torsion elements of \( Y \) to the \( l^{\nu} \)-torsion elements of \( Y \) given by \( x \mapsto lx \). It is in general not surjective, so that it is difficult to prove that \( \text{lim}^1 M_\nu = 0 \). Anyway, (3) has simplified because of (5) and (6) to

\[
0 \rightarrow \text{lim}^1(Tor^1(Z/l^\nu, \pi_0(Y))) \rightarrow \pi_0(\text{hom}lim Y \land M(Z/l^\nu)) = \pi_0(Y^1) \rightarrow \text{lim}(G_\nu) = [\pi_0(Y)]^l \rightarrow 0
\]

where \( Y = L_{E(1)}K(X_\infty) \) and \( \text{lim}(G_\nu) = [\pi_0(Y)]^l \) is reduced since it is the projective limit of the reduced groups \( G_\nu \). See [MGM10]. \( [\pi_0(Y)]^l \) is also a cotorsion group (see [F1]), since it is the epimorphic image of a cotorsion group in the exact sequence (7): \( \pi_0(Y^1) \) is equal to the cotorsion reduced group \( Ext^1(Q/Z_0, Y) \) since (see [B79])

\[
Ext^1(Q/Z_0, Y) \rightarrow \pi_0(Y^1) \rightarrow Hom(Q/Z_0, \pi_{-1}(Y))
\]

and \( Hom(Q/Z_0, \pi_{-1}(Y)) = 0 \) since \( \pi_{-1}(Y) \) is \( l \)-reduced, see ([MGM10]) and therefore, \( Ext^1(Q/Z_0, Y) \simeq \pi_0(Y^1) \) and \( Ext^1(Q/Z_0, Y) \) is a cotorsion group. The torsion group of the cotorsion reduced group \( [\pi_0(Y)]^l \) noted \( T([\pi_0(Y)]^l) \) is in the terminology of [F2] an \( \ell \)-complete torsion group. Now \( [\pi_0(Y)]^l \) is a reduced algebraically compact group since it is complete in the terminology of [R08] page 440. It is complete because it is the closure in the \( \ell \)-adic topology of the topological Hausdorff group \( \pi_0(Y) \). Being reduced and algebraically compact implies it is a direct summand of a direct product of cyclic \( \ell \)-groups by [F2] Corollary 38.2 page 161. Henceforth, \( T([\pi_0(Y)]^l) \) is contained in a direct product of cyclic \( \ell \)-groups, and so the torsion part of \( \pi_0(Y) \), noted \( T(\pi_0(Y)) \) is contained in a direct sum of cyclic \( \ell \)-groups. Since \( T(\pi_0(Y)) \) is reduced, then it has bounded \( \ell \)-torsion, i.e., there exists \( \nu_0 \) such that \( l^{\nu_0}T(\pi_0(Y)) = 0 \). Then by definition of \( \text{lim}^1 \), \( \text{lim}^1(Tor^1(Z/l^\nu, \pi_0(Y))) \) has bounded \( \ell \)-torsion, as wanted. Then, tensoring with \( Q \) in the exact sequence (7) becomes,

\[
0 \rightarrow \pi_0(Y^1) \otimes Q \rightarrow [\pi_0(Y)]^l \otimes Q \rightarrow 0
\]

and therefore Proposition 1 has been proved.

Remark 2: We conjecture that \( \pi_0(Y^1) \) and therefore also \( [\pi_0(Y)]^l \) are without torsion, in which case by [R08] page 445 \( [\pi_0(Y)]^l \), is a direct summand of copies of \( Z_l \). Since tensored by \( Q \), i.e., \( [\pi_0(Y)]^l \otimes Q \), using (1) and the above Proposition 1, is a finite direct sum of copies of \( Q_l \), \( [\pi_0(Y)]^l \) has to be a finite direct sum of copies of \( Z_l \).

Theorem 1: \( \mathcal{U}([\pi_0(L_{E(1)}K(X_\infty))]^l \otimes Q) \simeq [E(1)_0(K(X_\infty))]^l \otimes Q \)
Corollary 2: \( \mathcal{U}(\oplus_{i=0}^d H^{2i}_{et}(\mathcal{X}, \mathbb{Q}_l(i))) \cong [E(1)_0(K(X_\infty))]^t \otimes \mathbb{Q} \).

Proof: \( \mathcal{U}(\oplus_{i=0}^d H^{2i}_{et}(\mathcal{X}, \mathbb{Q}_l(i))) \cong \mathcal{U}([\pi_0(L_{E(1)}K(X_\infty))]^t \otimes \mathbb{Q}) \). This fact follows from Proposition 1 and from (1). Hence, corollary 2 follows immediately from theorem 1.

Proof of theorem 1:

Since \( \pi_0(L_{E(1)}K(X_\infty)) \) is \( l \)-reduced the kernel of the \( l \)-completion \( \pi_0(L_{E(1)}K(X_\infty)) \mapsto [\pi_0(L_{E(1)}K(X_\infty))]^t \) is equal to 0 and the completion map is injective. Also the \( l \)-adic topology in \( \pi_0(L_{E(1)}K(X_\infty)) \) is Hausdorff and this space is dense in its \( l \)-completed space. The functor \( \mathcal{U} \) is exact and 0 \( \mapsto \mathcal{U}(\pi_0(L_{E(1)}K(X_\infty)) \otimes \mathbb{Q}) \cong E(1)_0K(X_\infty) \otimes \mathbb{Q} \mapsto \mathcal{U}([\pi_0(L_{E(1)}K(X_\infty))]^t \otimes \mathbb{Q} \) has dense image. On the other hand since \( \pi_0(L_{E(1)}K(X_\infty)) \) is \( l \)-reduced, \( \mathcal{U}(\pi_0(L_{E(1)}K(X_\infty)) \otimes \mathbb{Q} \cong E(1)_0K(X_\infty) \otimes \mathbb{Q} \) is \( l \)-reduced (See [B83]) and the isomorphism holds as an isomorphism of \( \mathbb{Q} \)-vector spaces. Then,

\[
0 \mapsto E(1)_0K(X_\infty) \otimes \mathbb{Q} \mapsto [E(1)_0K(X_\infty)]^t \otimes \mathbb{Q}
\]

with dense image. By uniqueness of the \( l \)-completed space, (9) \( \mathcal{U}([\pi_0(L_{E(1)}K(X_\infty))]^t \otimes \mathbb{Q} \cong [E(1)_0K(X_\infty)]^t \otimes \mathbb{Q} \), and the theorem follows.

Remark 3: If the conjecture stated in remark 2 holds, then by (9), remark 2, and remark 1 b), \( [E(1)_0K(X_\infty)]^t \) is a finite direct sum of copies of \( \mathbb{Z}_l[[t]] \), a fact which was proved for nonsingular complete curves in [DM95].

Theorem 2: The geometric Frobenius \( \Phi_X \) acts semisimply in \( \oplus_{i=0}^d H^{2i}_{et}(\mathcal{X}, \mathbb{Q}_l(i)) \).

Before we give a proof we need to state one definition and two remarks:

Definition 1: Let \( E \) and \( F \) be cohomology theories. A natural transformation from \( E^n(-) \) to \( F^m(-) \) is called a cohomology operation from \( E^n(-) \) to \( F^m(-) \). If it is compatible with the suspension isomorphisms then it is called a stable operation. (See [Bo95])

Remark 4: Let \( E \) and \( F \) be spectra. Then the set of stable cohomology operations from \( E \) to \( F \) can be identified with \( F^*(E) \) [Bo95]. Therefore the ring \( E(1)^*E(1) \) may be identified with the stable operations of degree 0, \( \phi : E(1)^*_*(-) \mapsto E(1)^*_*(-) \) which in turn are induced by map of spectra \( \phi : E(1) \mapsto E(1) \) (See [KJ84] page 57) Therefore given a base for \( E(1)^*E(1) \), we obtain a base for the stable operations of degree 0 on \( E(1)^*_*(-) \).

Remark 5: From ([CCW05] page 13), we know that \( \widehat{E(1)}E(1) \) is isomorphic
to $\mathbb{Z}[[Y]]$, where $Y = \Psi^r - 1$ for the Adams operation $\Psi^r$ with $r$ a primitive modulo $l^2$ and where $\tilde{E}(1)$ is the $l$-adic completion of $E(1)$. It is in particular from this fact that Bousfield obtains in ([B83] page 908) an equivalence between the category $\mathcal{B}(l)$ and the category $\mathcal{B}(l)^r$. This isomorphism gives us a base for the ring $\tilde{E}(1)E(1)$, and by remark 4, it gives us a base for the stable degree 0 operations $\phi : E(1)_*(\cdot) \to E(1)_*(\cdot)$, and in particular for the 0 component degree 0 operations $\phi : E(1)_0(\cdot) \to E(1)_0(\cdot)$.

Proof of theorem 2:

By the Weil Conjectures the eigenvalues of $\Phi_X$ acting on $H^2_{et}(X_\infty, Q_l(i))$ are algebraic numbers all of whose complex conjugates have absolute value $q^i$. Then $U(\Phi_X)$ acting on $U(\bigoplus_{i=0}^d H^2_{et}(X_\infty, Q_l(i)))$ has eigenvalues whose complex conjugates have absolute value $q^i$, $i \in 1, 2, \ldots d$.

We will prove in a moment that this map can be identified with the Adams operation $\Psi^q$ on $[E(1)_0 K(X_\infty)]^l$ which is an object in $\mathcal{B}$. Now, this Adams operation, $\Psi^q$, is diagonalizable on $[E(1)_0 K(X_\infty)]^l \otimes Q$ Then, by corollary 2, it is diagonalizable in $U(\bigoplus_{i=0}^d H^2_{et}(X_\infty, Q_l(i)))$. This fact in turn implies that $U(\Phi_X)$ is also diagonalizable in $U(\bigoplus_{i=0}^d H^2_{et}(X_\infty, Q_l(i)))$, which then implies that $\Phi_X$ is diagonalizable in $\bigoplus_{i=0}^d H^2_{et}(X_\infty, Q_l(i))$ as wanted.

$U(\Phi_X)$ can be identified with the Adams operation $\Psi^q$ because by the isomorphism of remark 5, $U(\Phi_X)$ is an infinite combination of the elements, $(\Psi^r - 1)^s$ $s = 0, 1, 2, 3, \ldots$. This last fact says that the eigenvalues of $\Phi_X$ on $[E(1)_0 K(X_\infty)]^l \otimes Q$ are a combination of the of the eigenvalues of $(\Psi^r - 1)^s$, $s = 0, 1, 2, 3, \ldots$ on the same space, which are real numbers, implying that the eigenvalues of $U(\Phi_X)$ are real numbers. Since on the other hand they are algebraic numbers whose complex conjugate have absolute value $q^i$, they must be equal to $q^i$ and hence it is the Adams operation $\Psi^q$, as stated above.

References

[1] B83 A.K.Bousfield. On The Homotopy Theory of K-Local Spectra at an Odd Prime. J.Math 107 pp 895-932.
[B79] A.K.Bousfield. The Localization of Spectra with Respect to Homotopy Theory. Topology 18 pp 257-281.
[Bo95] M J Boardman. Stable Operations in Generalized Cohomology. Handbook of Algebraic Topology. North Holland Amsterdam 1995
[CCW05] F Clarke, M Crossley, and S Whitehouse. Algebras of Operations in K-Theory. Topology 44 issue 1, january 2005, pp 151-174
[DM95] W.G.Dwyer and S.A.Mitchell. On the K-Theory Spectrum of a Smooth Curve Over a Finite Field Topology 36 pp 899-929.
[F1] L Fuchs. *Infinite Abelian Groups* Academic Press Vol 1, 1970.

[F2] L Fuchs. *Infinite Abelian Groups* Academic Press Vol 2, 1973.

[KJ84] K Johnson *The Action of the Stable Operations of Complex K-Theory on Coefficient Groups.* Illinois Journal of Mathematics 21, issue 1, 1984, pp 57-63.

[MGM11] M. Gomez Morteo. *The Tate Thomason Conjecture* ArXiv: 1007.0427 v3.

[R] Y Rudyak *On Thom Spectra, Orientability and Cobordism.* Springer-Verlag

[R08] J.J Rotman. *An Introduction to Homological Algebra* Springer-Verlag, 2008.

[T89] R.W. Thomason. *A Finiteness Condition Equivalent to the Tate Conjecture over Fq* Contemporary Mathematics 83 pp 385-392.

*E-mail address:* valmont8ar@hotmail.com