A Note on Lebesgue Solvability of Elliptic Homogeneous Linear Equations with Measure Data

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Abstract
In this work, we present new results on solvability of the equation $A^*(D)f = \mu$ for $f \in L^p$ and positive measure data $\mu$ associated to an elliptic homogeneous linear differential operator $A(D)$ of order $m$. Our method is based on $(m, p)$-energy control of $\mu$ giving a natural characterization for solutions when $1 \leq p < \infty$. We also obtain sufficient conditions in the limiting case $p = \infty$ using new $L^1$ estimates on measures for elliptic and canceling operators.

Keywords Divergence-measure vector fields · Lebesgue solvability · $L^1$ estimates · Elliptic equations · Canceling operators

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1 Introduction
Phuc and Torres [8] characterized the existence of solutions in Lebesgue spaces for the divergence equation

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\[ \text{div } f = \nu, \quad (1.1) \]

where \( \nu \in \mathcal{M}_+(\mathbb{R}^N) \), the set of scalar positive Borel measures on \( \mathbb{R}^N \), and \( f \in L^p(\mathbb{R}^N, \mathbb{R}^N) \). The method is based on controlling the \((1, p)\)-energy of \( \nu \) defined by \( \| I_1 \nu \|_{L^p} \), where \( I_1 \) is the Riesz potential operator. In fact, \( \| I_1 \nu \|_{L^p} \) finite is a necessary condition for solvability in \( L^p \), since from (1.1), we have

\[ I_1 \nu = c_N \sum_{j=1}^{N} R_j f_j \quad (1.2) \]

and the control in norm follows as a direct consequence of the continuity of Riesz transform operators \( R_j \) in \( L^p(\mathbb{R}^N) \) for \( 1 < p < \infty \). The following result was proved in [8, Theorems 3.1 and 3.2]:

**Theorem** If \( f \in L^p(\mathbb{R}^N, \mathbb{R}^N) \) satisfies (1.1) for some \( \nu \in \mathcal{M}_+(\mathbb{R}^N) \), then

(i) \( \nu = 0 \), assuming \( 1 \leq p \leq N/(N-1) \);

(ii) \( \nu \) has finite \((1, p)\)-energy, assuming \( N/(N-1) < p < \infty \). Conversely, if \( \nu \in \mathcal{M}_+(\mathbb{R}^N) \) has finite \((1, p)\)-energy, then there is a vector field \( f \in L^p(\mathbb{R}^N, \mathbb{R}^N) \) satisfying (1.1).

The previous result does not cover the case \( p = \infty \), since the proof breaks down once the Riesz transform is not bounded in \( L^\infty(\mathbb{R}^N) \). However, from Gauss–Green theorem, if \( f \in L^\infty(\mathbb{R}^N, \mathbb{R}^N) \) is a solution of (1.1), then for any ball \( B(x, r) \), there exists \( C = C(N) > 0 \) such that

\[ \nu(B(x, r)) = \int_{\partial B(x, r)} f \cdot n \, d\mathcal{H}^{N-1} \leq C \| f \|_{L^\infty} r^{N-1}. \]

It is easy to check that \( \| I_1 \nu \|_{L^\infty} < \infty \) implies the following control of the measure \( \nu \) on balls

\[ \nu(B(x, r)) \leq C r^{N-1}, \quad (1.3) \]

where the constant is independent of \( x \in \mathbb{R}^N \) and \( r > 0 \). Indeed,

\[ I_1 \nu(x) \geq C \int_{B(x, r)} \frac{1}{|x - y|^{N-1}} \, d\nu(y) \geq C \int_{B(x, r)} \frac{1}{r^{N-1}} \, d\nu(y) = \frac{C \nu(B(x, r))}{r^{N-1}}, \]

then we have (1.3). A nontrivial argument (see [8]) is sufficient to show that (1.3) implies that

\[ \left| \int_{\mathbb{R}^n} u(x) \, dv \right| \leq C \| \nabla u \|_{L^1}, \quad \forall u \in C^\infty_c(\mathbb{R}^N) \quad (1.4) \]
and from a standard duality argument, a solution $f \in L^\infty(\mathbb{R}^N, \mathbb{R}^N)$ for (1.1) is obtained (see [9] for further results on signed measures). Measures satisfying the Morrey control for $1 \leq \lambda < \infty$ given by

$$
\|\mu\|_\lambda := \sup_B \frac{|\mu|(B(x, r))}{r^\lambda} < \infty,
$$

where the supremum is taken for all open balls $B = B(x, r)$ with $x \in \mathbb{R}^N$ and $r > 0$, and $|\mu|$ is the total variation on $\mu$ are referred to as $\lambda$-Ahlfors regular.

Let $A(D)$ be a homogeneous linear differential operator of order $m$ on $\mathbb{R}^N$, $N \geq 2$, i.e., a linear differential operator in which all the partial derivatives are of the same order $m$, from a finite-dimensional complex vector space $E$ to a finite-dimensional complex vector space $F$, given by

$$
A(D) = \sum_{|\alpha|=m} a_\alpha \partial^\alpha : C^\infty_c(\mathbb{R}^N, E) \to C^\infty_c(\mathbb{R}^N, F),
$$

where the coefficients $a_\alpha \in \mathcal{L}(E, F)$, the set of linear transformations from $E$ to $F$, are constant (in the sense that they do not depend on $x \in \mathbb{R}^N$).

Inspired by the previous theorem, in this paper, we carry further the study of Lebesgue solvability for the equation:

$$
A^*(D) f = \mu,
$$

where $A^*(D)$ is the (formal) adjoint operator associated to the homogeneous linear differential operator $A(D)$. Naturally, the concept of energy of the measure $\mu$ associated to (1.5) can be extended in accordance to the order of $A(D)$ defining the $(m, p)$-energy of $\mu$ by the functional $\|I_m \mu\|_{L^p}$ (see the Definition 2.1 for complete details).

Our first result concerns the Lebesgue solvability for the Eq. (1.5) when $1 \leq p < \infty$.

**Theorem A** Let $A(D)$ be a homogeneous linear differential operator of order $1 \leq m < N$ on $\mathbb{R}^N$, $N \geq 2$, from $E$ to $F$ and $\mu \in \mathcal{M}(\mathbb{R}^N, E^*)$.

(i) If $1 \leq p \leq N/(N - m)$, $f \in L^p(\mathbb{R}^N, F^*)$ is a solution for (1.5) and $\mu \in \mathcal{M}(\mathbb{R}^N, E^*)$, then $\mu \equiv 0$.

(ii) If $N/(N - m) < p < \infty$ and $f \in L^p(\mathbb{R}^N, F^*)$ is a solution for (1.5), then $\mu$ has finite $(m, p)$ energy. Conversely, if $|\mu|$ has finite $(m, p)$-energy and $A(D)$ is elliptic, then there exists a function $f \in L^p(\mathbb{R}^N, F^*)$ solving (1.5).

We recall that ellipticity means the symbol $A(\xi) : E \to F$ given by

$$
A(\xi) := \sum_{|\alpha|=m} a_\alpha \xi^\alpha
$$

is injective for $\xi \in \mathbb{R}^N \setminus \{0\}$. In particular, the Theorem A recovers [8, Theorems 3.1 and 3.2] taking $A(D) = \nabla$, where $E = \mathbb{R}$ and $F = \mathbb{R}^N$, which is elliptic and $A^*(D) = \text{div}$.
Our second and main result deals with the case $p = \infty$.

**Theorem B** Let $A(D)$ be a homogeneous linear differential operator of order $1 \leq m < N$ from $E$ to $F$ and $\mu \in \mathcal{M}(\mathbb{R}^N, E^*)$. If $A(D)$ is elliptic and canceling, and $\mu$ satisfies

$$
\| \mu \|_{0, N-m} := \sup_{r>0} \frac{|\mu|(B_r)}{r^{N-m}} < \infty, \quad (1.7)
$$

and the potential control

$$
\int_0^{\lfloor y \rfloor/2} \frac{|\mu|(B(y, r))}{r^{N-m+1}} \, dr \lesssim 1, \quad \text{uniformly on } y, \quad (1.8)
$$

then there exists $f \in L^\infty(\mathbb{R}^N, F^*)$ solving (1.5).

The canceling property means

$$
\bigcap_{\xi \in \mathbb{R}^N \setminus \{0\}} A(\xi)[E] = \{0\}. \quad (1.9)
$$

The theory of canceling operators is due to J. Van Schaftingen (see [13]), motivated by studies of some $L^1$ a priori estimates for vector fields with divergence free and chain of complexes.

We point out that the assumption (1.7) is weaker in comparison to $\| \mu \|_{N-m} < \infty$, since it is only necessary to take the supremum over balls centered at the origin. The condition (1.8) can be understood as an uniform control of the truncated Wolff’s potential associated to positive Borel measures on $\mathbb{R}^N$ originally defined

$$
W_{\alpha, p}^t \nu(x) = \int_0^t \left[ \frac{\nu(B(x, r))}{r^{N-\alpha p}} \right] \frac{1}{r^{\frac{1}{p-1}}} \, dr
$$

for $1 < p < \infty$ and $\alpha > 0$ (see [6] for the original introduction of Wolff’s potential and [1, 10] for applications).

The main ingredient in the proof of Theorem B is to investigate sufficient conditions on $\mu$ in order to obtain

$$
\left| \int_{\mathbb{R}^N} u(x) \, d\mu(x) \right| \lesssim \| A(D)u \|_{L^1}, \quad \forall u \in C_c^\infty(\mathbb{R}^N, E). \quad (1.10)
$$

Inequalities of this type were studied by P. de Nápoli and T. Picon in [3] in the setting of vector fields associated to cocanceling (see the Definition (4.1)) operators where $d\mu = |x|^{-\beta} \, dx$, i.e., the (scalar) positive measure is given by special weighted power for some $\beta > 0$. More recently, F. Gmeineder, B. Raitañ and J. Van Schaftingen (see...
that the set of complex-valued Borel measures on $\Omega_1$ measures with finite control.

For any $0 < s < 1$ Measures with Finite Energy measures, precisely: if $q = \frac{N-s}{N-1}$ and $0 \leq s < 1$ then the estimate

$$\left( \int_{\mathbb{R}^N} \left| D^{m-1} u(x) \right|^q d\nu(x) \right)^{1/q} \lesssim \|\nu\|_{q(N-1)}^{1/q} \|A(D)u\|_{L^1}, \quad (1.11)$$

for all $u \in C_c^\infty(\mathbb{R}^N, E)$ and all $q(N-1)$-Ahfors regular measure $\nu$, holds if and only if $A(D)$ is elliptic and canceling. Besides the authors claim that it seems to be no simple a generalization for $s = 1$, i.e., $q = 1$, in particular, the inequality holds for the total derivative operator $A(D) = D^m$ that is elliptic and canceling (see Remark 5.1). We also point out that in a different fashion from the previous result, we obtain sufficient conditions on $\mu$ to fulfill (1.10) that come naturally from $(m, p)$-energy control.

The paper is organized as follows. In Sect. 2 we briefly study properties of measures with finite $(m, p)$-energy. The proof of Theorem A is presented in Sect. 3. The Sect. 4 is devoted to the proof of Theorem B, where a Fundamental Lemma 4.3, with own interest, is presented. Finally in Sect. 5, we present some general comments, in particular, we discuss an extension for inequality (1.11) when $q = 1$ for elliptic and canceling operators and a reciprocal to Theorem B for first-order operators.

**Notation** Throughout this work, $\Omega$ always denotes an open subset of $\mathbb{R}^N$. The symbol $f \lesssim g$ means that there exists a constant $C > 0$, neither depending on $f$ nor $g$, such that $f \leq C g$. Given a set $A \subset \mathbb{R}^N$, we denote by $|A|$ its Lebesgue measure. We write $B = B(x, R)$ for the open ball with center $x$ and radius $R > 0$. By $B_R$, we mean the ball centered at the origin with radius $R$. We fix $\int_Q f(x) dx := \frac{1}{|Q|} \int_Q f(x) dx$. We denote by $\mathcal{M}(\Omega)$ the set of signed Borel measures on $\Omega$. We add the subscript $\mathcal{M}_+(\Omega)$ to denote the set of positive Borel measures on $\Omega$. We write $\mathcal{M}(\Omega, \mathbb{C})$ for the set of complex-valued Borel measures on $\Omega$ given by $\mu = \mu^\text{Re} + i \mu^\text{Im}$, where $\mu^\text{Re}, \mu^\text{Im} \in \mathcal{M}(\Omega)$. By $\mathcal{M}_+(\Omega, \mathbb{C})$ we mean the set of measures $\mu \in \mathcal{M}(\Omega, \mathbb{C})$ such that $\mu^\text{Re}, \mu^\text{Im} \in \mathcal{M}_+(\Omega)$.

### 2 Measures with Finite Energy

For any $0 < m < N$ and $f$ a function in the Schwartz space $S(\mathbb{R}^N)$, consider the fractional integrals also called Riesz potential operators defined by

$$I_m f(x) = \frac{1}{\gamma(m)} \int_{\mathbb{R}^N} \frac{f(y)}{|x - y|^{N-m}} dy,$$

with $\gamma(m) := \pi^{N/2} 2^m \Gamma(m/2) / \Gamma((N-m)/2)$.

Let $X$ be a complex vector space with dim$_\mathbb{C} X = d < \infty$. We denote by $\mathcal{M}(\Omega, X)$ the set of all $X$-valued complex measures on $\Omega$, $\mu = (\mu_1, \ldots, \mu_d)$ where $\mu_\ell = \mu_\ell^\text{Re} + i \mu_\ell^\text{Im} \in \mathcal{M}(\Omega, \mathbb{C})$ for all $\ell = 1, \ldots, d$. Similarly, $\mathcal{M}_+(\Omega, X)$ is the set of measures $\mu \in \mathcal{M}(\Omega, X)$ such that $\mu_\ell \in \mathcal{M}_+(\Omega, \mathbb{C})$ for all $\ell = 1, \ldots, d$. If
\( \eta \in \mathcal{M}(\Omega, \mathbb{C}) \), we define
\[
I_m \eta(x) := \frac{1}{\gamma(m)} \int_{\Omega} \frac{1}{|x - y|^{N-m}} d\eta(y)
\]
and, for \( \mu \in \mathcal{M}(\Omega, X) \), \( I_m \mu := (I_m \mu_1, \ldots, I_m \mu_d) \). We denote the total variation of \( \mu \) by \( |\mu| \). The total variation of a vector-valued measure is defined similarly to that of a complex measure. Details can be seen in [2, 11].

**Definition 2.1** Let \( 1 \leq p < \infty \) and \( 0 < m < N \). We say that \( \mu \in \mathcal{M}(\Omega, X) \) has finite \((m, p)\)-energy if
\[
\|I_m \mu\|_{L^p} := \left( \int_{\mathbb{R}^N} |I_m \mu(x)|^p \, dx \right)^{1/p} < \infty,
\]
and \( \mu \) has finite \((m, 1)\)-weak energy if
\[
\|I_m \mu\|_{L^1, \infty} = \sup_{\lambda > 0} \lambda \left\{ x \in \mathbb{R}^N : |I_m \mu(x)| > \lambda \right\} < \infty.
\]

From the previous definition follows \( \|I_m \mu\|_{L^p} \leq \|I_m \mu\|_{L^{p, \infty}} \). The same estimate holds replacing \( L^p \) by \( L^{p, \infty} \).

**Proposition 2.1** If \( \mu \in \mathcal{M}_+(\Omega, X) \) has finite \((m, p)\)-energy for some \( 1 < p \leq N/(N-m) \) or \((m, 1)\)-weak energy, then \( \mu \equiv 0 \) on \( \Omega \).

**Proof** Let \( R > 0 \) and by simplicity, we assume \( \mu_\ell \in \mathcal{M}_+(\Omega) \) for each \( \ell \in \{1, \ldots, d\} \). We have
\[
I_m \mu_\ell(x) \gtrsim \int_{B_R \cap \Omega} \frac{1}{|x - y|^{N-m}} \, d\mu_\ell(y) \geq \frac{\mu_\ell(B_R \cap \Omega)}{(|x| + R)^{N-m}}.
\]
Thus,
\[
\int_{\mathbb{R}^N} |I_m \mu_\ell(x)|^p \, dx \gtrsim \int_{\mathbb{R}^N} [I_m \mu_\ell(x)]^p \, dx \geq \int_{\mathbb{R}^N} \left[ \frac{\mu_\ell(B_R \cap \Omega)}{(|x| + R)^{N-m}} \right]^p \, dx
\]
\[
= [\mu_\ell(B_R \cap \Omega)]^p \int_{\mathbb{R}^N} \frac{1}{(|x| + R)^{N-m}} \, dx.
\]
Observe that for \( 1 < p \leq N/(N-m) \) the last integral blows up to infinity, thus, we must have \( \mu_\ell(B_R \cap \Omega) = 0 \), since \( \|I_m \mu\|_{L^p} < \infty \). For the case \( p = 1 \), we have
\[
\sup_{\lambda > 0} \lambda \left\{ x \in \mathbb{R}^N : \left( \frac{\mu_\ell(B_R \cap \Omega)}{(|x| + R)^{N-m}} > \lambda \right) \right\} \lesssim \|I_m \mu\|_{L^1, \infty} < \infty.
\]
Thus,
\[
\lambda \left| \left\{ x : \frac{\mu_\ell (B_R \cap \Omega)}{|x| + R}^{N-m} > \frac{1}{\lambda} \right\} \right| = \lambda \left| B \left( 0, \left( \frac{\mu_\ell (B_R \cap \Omega)}{\lambda} \right)^{\frac{1}{N-m}} - R \right) \right|
\]
\[
= \lambda^{-\frac{m}{N-m}} \left| B \left( 0, \mu_\ell (B_R \cap \Omega)^{\frac{1}{N-m}} - \lambda^{\frac{1}{N-m}} R \right) \right|
\]
which blows up to infinity when \( \lambda > 0 \) is small and \( \mu_\ell (B_R \cap \Omega) \neq 0 \). Given that \( R > 0 \) was arbitrarily chosen, and that \( \Omega = \bigcup_{k \in \mathbb{N}} [B_k \cap \Omega] \), we conclude that \( \mu_\ell \equiv 0 \) on \( \Omega \) for every \( \ell \in \{1, \ldots, d\} \). Therefore, \( \mu \equiv 0 \).

\[\square\]

3 Proof of Theorem A

Throughout this section, \( A(D) \) denotes an elliptic homogeneous linear differential operator of order \( m \) on \( \mathbb{R}^N \), \( N \geq 2 \) and \( 1 \leq m < N \), with constant coefficients from a finite-dimensional complex vector space \( E \) to a finite-dimensional complex vector space \( F \). Since the vector spaces have finite dimension, we will use the identification \( X \) instead of \( X^* \), for simplicity.

**Proposition 3.1** Let \( 1 \leq p \leq \frac{N}{N-m} \). If \( \mu \in \mathcal{M}_+(\mathbb{R}^N, E) \) and \( f \in L^p (\mathbb{R}^N, F) \) is a solution for \( A^*(D) f = \mu \), then \( \mu \equiv 0 \).

**Proof** From the identity \( (N-m) \int_{|x-y|}^{\infty} \frac{1}{r^{N-m+1}} dr = \frac{1}{|x-y|^{N-m}} \) and the Fubini’s theorem, we may write

\[
I_m \mu (x) = c_{N,m} \int_{\mathbb{R}^N} \left( \int_{|x-y|}^{\infty} \frac{1}{r^{N-m+1}} dr \right) d\mu (y)
\]
\[
= c_{N,m} \int_{\mathbb{R}^N} \left( \int_{0}^{\infty} \frac{\chi_{\{r > |x-y|\}} (r)}{r^{N-m+1}} d\mu (y) \right) dr
\]
\[
= c_{N,m} \lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{\infty} \frac{\mu (B (x, r))}{r^{N-m+1}} dr.
\]

Now, using the Gauss–Green theorem with weak derivatives, we have

\[
\mu (B (x, r)) = \int_{B(x,r)} A^*(D) f (y) dy = \sum_{|\alpha| = m} a^*_\alpha \int_{\partial B(x,r)} \partial^\alpha f (y) dy
\]
\[
= \sum_{|\alpha| = m} a^*_\alpha \int_{\partial B(x,r)} \partial^\alpha - e_j f (y) \frac{y_j - x_j}{|y-x|} d\omega (y),
\]
where we choose, for each multi-index \( \alpha = (\alpha_1, \ldots, \alpha_N) \), a number \( j_\alpha \in \{1, \ldots, N\} \) such that \( \alpha_{j_\alpha} \neq 0 \) in a way that \( \partial^\alpha f = \partial_{x_{j_\alpha}} (\partial^{\alpha - e_{j_\alpha}} f) \). Summarizing

\[
I_m \mu(x) = c_{N,m} \sum_{|\alpha|=m} a^* \alpha \left. \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} \partial^{\alpha - e_{j_\alpha}} f(y) \frac{x_j - y_j}{|x-y|^{N+1}} \, dy \right|_{y \to x}
\]

\[
= c_{N,m} \sum_{|\alpha|=m} a^* \alpha (K_{j_\alpha} * \partial^{\alpha - e_{j_\alpha}} f)(x),
\]

where \( K_{j_\alpha}(x) := x_{j_\alpha}/|x|^{N+2}. \) Thus, from [12, p. 73], we have \( \hat{K}_{j_\alpha}(\xi) = c_{N,m} \xi_{j_\alpha}/|\xi|^m \) and hence, recalling the constant \( c_{N,m} \), we have

\[
(K_{j_\alpha} * \partial^{\alpha - e_{j_\alpha}} f)\hat{f}(\xi) = c_{N,m} \xi_{j_\alpha} \xi^m \hat{f}(\xi) = c_{N,m} \frac{\xi^\alpha}{|\xi|^m} \hat{f}(\xi) = (R^\alpha \hat{f})(\xi)
\]

where \( R^\alpha := R_1^{\alpha_1} \circ R_2^{\alpha_2} \circ \cdots \circ R_N^{\alpha_N} \) is the \( \alpha \)-order Riesz transform operator. In this way,

\[
I_m \mu = c_m,N \sum_{|\alpha|=m} a^* \alpha R^\alpha f.
\]

In particular for \( m = 1 \),

\[
I_1 \mu(x) = c_N \sum_{j=1}^{N} a^* j \left. \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} f(y) \frac{x_j - y_j}{|x-y|^{N+1}} \, dy \right|_{y \to x}
\]

\[
= c_N \sum_{j=1}^{N} a^* j R_j f(x)
\]

for almost every \( x \in \mathbb{R}^N \), where \( R_j \) is the \( j^{th} \) Riesz transform operator.

Since each \( R_j \) is bounded from \( L^p \) to itself for \( 1 < p < \infty \) and of type weak(1, 1), we conclude that \( \|I_m \mu\|_{L^p} \lesssim \|f\|_{L^p} < \infty \) that is \( \mu \) has finite \((m, p)\)-energy for \( 1 < p \leq N/(N-m) \) and \( \|I_m \mu\|_{L^{1,\infty}} \lesssim \|f\|_{L^1} < \infty \) for \( p = 1 \). Notice that up until this point, we only needed \( \mu \in \mathcal{M}(\mathbb{R}^N, E) \). If \( \mu \in \mathcal{M}_+(\mathbb{R}^N, E) \), it follows from Proposition 2.1 that \( \mu \equiv 0 \) in \( \mathbb{R}^N \).

Next we prove the second part of the Theorem A.

**Proposition 3.2** Let \( N/(N-m) < p < \infty \) and \( \mu \in \mathcal{M}(\mathbb{R}^N, E) \). If \( f \in L^p(\mathbb{R}^N, F) \) is a solution for \( A^* (D) f = \mu \), then \( \mu \) has finite \((m, p)\)-energy. Conversely, if \( |\mu| \) has finite \((m, p)\)-energy, then there exists a function \( f \in L^p(\mathbb{R}^N, F) \) solving the equation \( A^* (D) f = \mu \).

**Proof** The first part follows from identity (3.1) and the boundedness of \( \alpha \)-order Riesz transform operators. For the converse, consider the function \( \xi \mapsto H(\xi) \in \mathcal{L}(F, E) \)
defined by

\[ H(\xi) = (A^* \circ A)^{-1}(\xi)A^*(\xi) \]

that is smooth in \( \mathbb{R}^N \setminus \{0\} \) and homogeneous of degree \(-m\). Here, \( A^*(\xi) \) is the symbol of the adjoint operator \( A^*(D) \). Since we are assuming that \( 1 \leq m < N \), then \( H \) is a locally integrable tempered distribution and its inverse Fourier transform \( K(x) \) is a locally integrable tempered distribution homogeneous of degree \(-N + m\) (see [4, p. 71]) that satisfies

\[
\int K(x-y)[A(D)u(y)] dy, \quad u \in C_\infty^c(\mathbb{R}^N, E) \tag{3.2}
\]

and clearly \(|u(x)| \leq I_m|A(D)u|(x)\).

Let \( w_A^{m,p'}(\mathbb{R}^N, E) \) be the closure of \( C_\infty^c(\mathbb{R}^N, E) \) with respect to the norm \(|u|_{m,p'} = \|A(D)u\|_{L^{p'}}\). Thus,

\[
\left| \int_{\mathbb{R}^N} u(x) d\mu(x) \right| \leq \int_{\mathbb{R}^N} \left[ \int_{\mathbb{R}^N} \frac{|A(D)u(y)|}{|x-y|^{N-m}} dy \right] d|\mu|(x) \\
\leq \int_{\mathbb{R}^N} |A(D)u(y)| I_m|\mu|(y) dy \\
\leq \|u\|_{m,p'} \|I_m|\mu\|_{L^p} \lesssim \|u\|_{m,p'},
\]

since \(|\mu|\) has finite \((m, p)\)-energy following that \( \mu \in [w_A^{m,p'}(\mathbb{R}^N, E)]^* \). Since \( A(D) : w_A^{m,p'}(\mathbb{R}^N, E) \to L^{p'}(\mathbb{R}^N, F) \) is a linear isometry, hence, its adjoint \( A^*(D) : L^p(\mathbb{R}^N, F) \to [w_A^{m,p'}(\mathbb{R}^N, E)]^* \) is surjective. Therefore, there exists \( f \in L^p(\mathbb{R}^N, F) \) such that \( A^*(D)f = \mu \). \( \square \)

The Theorem A becomes a characterization when \(|\mu| = \mu\). This is the case for positive scalar measures:

**Corollary 3.1** Let \( A(D) \) be an elliptic homogeneous linear differential operator of order \( 1 \leq m < N \) on \( \mathbb{R}^N \), \( N \geq 2 \), from \( E \) to \( F \), where \( E \) and \( F \) are finite-dimensional real vector spaces, with \( \dim_{\mathbb{R}} E = 1 \). Let \( \mu \in \mathcal{M}_+(\mathbb{R}^N, E^*) \) and \( N/(N - m) < p < \infty \). Then \( \mu \) has finite \((m, p)\)-energy if and only if there exists a function \( f \in L^p(\mathbb{R}^N, F^*) \) solving \( A^*(D)f = \mu \).

### 4 Proof of Theorem B

In order to prove Theorem B, it is enough to show that (1.10) holds. In fact, assuming the validity of this inequality, we conclude that \( \mu \in [w_A^{m,1}(\mathbb{R}^N, E)]^* \) and \( \text{bis in idem} \) the argument used in the proof of Proposition 3.2, there exists \( f \in L^\infty(\mathbb{R}^N, F) \) such that \( A^*(D)f = \mu \). From the identity (3.2), since \( A(D) \) is elliptic, the inequality (1.10)
is equivalent to
\[
\left| \int_{\mathbb{R}^N} \left[ \int_{\mathbb{R}^N} K(x - y)g(y) \, dy \right] d\mu(x) \right| \lesssim \|g\|_{L^1} \tag{4.1}
\]
where \(g := A(D)u\) for all \(u \in C^\infty_c(\mathbb{R}^N, E)\) and moreover
\[
|K(x - y)| \leq C |x - y|^{m-N}, \quad x \neq y \tag{4.2}
\]
and
\[
|\partial_y K(x - y)| \leq C |x - y|^{m-N-1}, \quad 2|y| \leq |x|. \tag{4.3}
\]
The proof reduces to obtaining inequality (4.1) invoking a special class of vector fields in \(L^1\) norm associated to an elliptic and canceling operator \(A(D)\) and \(\mu\) satisfying (1.7) and (1.8).

The first step is an extension of Hardy-type inequality [3, Lemma 2.1] on two measures, which we present for the sake of completeness.

**Lemma 4.1** Let \(1 \leq q < \infty\) and \(\nu\) be a \(\sigma\)-finite real positive measure. Suppose \(\tilde{u}\) and \(\tilde{v}\) are measurable and nonnegative almost everywhere. Then
\[
\left[ \int_{\mathbb{R}^N} \left( \int_{B_{|x|/2}} \tilde{g}(y) \, dy \right)^q \tilde{u}(x) \, d\nu(x) \right]^{1/q} \lesssim \int_{\mathbb{R}^N} \tilde{g}(x)\tilde{v}(x) \, dx \tag{4.4}
\]
holds for all \(\tilde{g} \geq 0\) if and only if
\[
C := \sup_{R > 0} \left( \int_{(B_R)^c} \tilde{u}(x) \, d\nu(x) \right)^{1/q} \left( \sup_{x \in B_R} [\tilde{v}(x)]^{-1} \right) < \infty. \tag{4.5}
\]

**Proof** By Minkowski inequality, we have
\[
\left[ \int_{\mathbb{R}^N} \left( \int_{B_{|x|/2}} \tilde{g}(y) \, dy \right)^q \tilde{u}(x) \, d\nu(x) \right]^{1/q} \leq \int_{\mathbb{R}^N} \tilde{g}(y) \left( \int_{(B_{2|y|})^c} \tilde{u}(x) \, d\nu(x) \right)^{1/q} \, dy
\]
\[
\leq C \int_{\mathbb{R}^N} \tilde{g}(y) \tilde{v}(y) \, dy,
\]
since
\[
\left( \int_{(B_{2|y|})^c} \tilde{u}(x) \, d\nu(x) \right)^{1/q} [\tilde{v}(y)]^{-1}
\]
\[
\leq \left( \int_{(B_{2|y|})^c} \tilde{u}(x) \, d\nu(x) \right)^{1/q} \left( \sup_{x \in B_{2|y|}} [\tilde{v}(x)]^{-1} \right) \leq C.
\]
Conversely, for $R > 0$ consider $S(R) := \operatorname{ess} \sup_{z \in B_R} [\widetilde{v}(z)]^{-1}$. For each $n \in \mathbb{N}$, we define the set $\widetilde{M}_n := \left\{ z \in B_R : [\widetilde{v}(z)]^{-1} > S(R) - \frac{1}{n} \right\}$. From the definition follows $|\widetilde{M}_n| > 0$, hence there exist $M_n \subseteq \widetilde{M}_n$ with $0 < |M_n| < \infty$. Choosing $\widetilde{g}(y) = \chi_{M_n}(y)$ and using (4.4), we have

$$\left( \int_{(B_{2R})^c} \tilde{u}(x) \, d\nu(x) \right)^{1/q} \leq \frac{1}{|M_n|} \left[ \int_{\mathbb{R}^N} \left( \int_{B_{1/2}} \chi_{M_n}(y) \, dy \right)^q \tilde{u}(x) \, d\nu(x) \right]^{1/q} \lesssim \int_{M_n} \tilde{v}(x) \, dx \lesssim \left( S(R) - \frac{1}{n} \right)^{-1}.$$  

Taking $n \to \infty$ we get $\left( \int_{(B_{2R})^c} \tilde{u}(x) \, d\nu(x) \right)^{1/q} S(R) \lesssim 1$ and the result follows since the control is uniform on $R > 0$. \hfill \qed

One fundamental property of elliptic and canceling operators $A(D)$ is the existence of a homogeneous linear differential operator $L(D) : C^\infty(\mathbb{R}^N, F) \to C^\infty(\mathbb{R}^N, V)$ of order $\kappa$ for some finite-dimensional complex vector space $V$ such that

$$\bigcap_{\xi \in \mathbb{R}^N \setminus \{0\}} \ker L(\xi) = \bigcap_{\xi \in \mathbb{R}^N \setminus \{0\}} A(\xi)[E] = \{0\}. \quad (4.6)$$

The next definition was also introduced by Van Schaftingen in [13]:

**Definition 4.1** Let $L(D)$ be a homogeneous linear differential operator of order $\kappa$ on $\mathbb{R}^N$ from $F$ to $V$. The operator $L(D)$ is cocanceling if

$$\bigcap_{\xi \in \mathbb{R}^N \setminus \{0\}} \ker L(\xi) = \{0\}.$$ 

An example of cocanceling operator on $\mathbb{R}^N$ from $F = \mathbb{R}^N$ to $V = \mathbb{R}$ is the divergence operator $L(D) = \text{div}$. Indeed, for every $e \in \mathbb{R}^N$ we have $L(\xi)[e] = \xi \cdot e$ and then clearly

$$\bigcap_{\xi \in \mathbb{R}^N \setminus \{0\}} \ker L(\xi) = \bigcap_{\xi \in \mathbb{R}^N \setminus \{0\}} \xi \perp = \{0\}.$$ 

The following peculiar estimate for vector fields belonging to the kernel of some cocanceling operator was presented at [3, Lemma 3.1].
Lemma 4.2. Let $L(D)$ be a cocanceling homogeneous linear differential operator of order $m$ on $\mathbb{R}^N$ from $F$ to $V$. Then there exists $C > 0$ such that for every $\varphi \in C^m_c(\mathbb{R}^N, F)$, we have

$$\left| \int_{\mathbb{R}^N} \varphi(y) \cdot f(y) \, dy \right| \leq C \sum_{j=1}^{m} \int_{\mathbb{R}^N} |f(y)| |y|^j \left| D^j \varphi(y) \right| \, dy$$

(4.7)

for all functions $f \in L^1(\mathbb{R}^N, F)$ satisfying $L(D)f = 0$ in the sense of distributions.

The second step to obtain (4.1) is an improvement of [3, Lemma 3.2] and [7, Lemma 2.1] in the setting of positive Borel measures.

Lemma 4.3. Assume $N \geq 2$, $0 < \ell < N$ and $K(x, y) \in L^1_{loc}(\mathbb{R}^N \times \mathbb{R}^N, \mathcal{L}(F, V))$ satisfying

$$|K(x, y)| \leq C |x - y|^\ell - N, \quad x \neq y$$

(4.8)

and

$$|K(x, y) - K(x, 0)| \leq C \frac{|y|}{|x|^{N-\ell+1}}, \quad 2|y| \leq |x|.$$  

(4.9)

Suppose $1 \leq q < \infty$ and let $\nu \in \mathcal{M}_+(\mathbb{R}^N)$ satisfying

$$\|\nu\|_{0, (N-\ell)q} < \infty,$$

(4.10)

and the following uniform potential condition

$$[[\nu]]_{(N-\ell)q} := \sup_{y \in \mathbb{R}^N} \int_0^{\frac{|y|}{2}} \frac{\nu(B(y, r))}{r^{(N-\ell)q+1}} \, dr < \infty.$$  

(4.11)

If $L(D)$ is cocanceling, then there exists $\tilde{C} > 0$ such that

$$\left( \int_{\mathbb{R}^N} \left| \int_{\mathbb{R}^N} K(x, y)g(y) \, dy \right|^q \, dv(x) \right)^{1/q} \leq \tilde{C} \int_{\mathbb{R}^N} |g(x)| \, dx,$$

(4.12)

for all $g \in L^1(\mathbb{R}^N, F)$ satisfying $L(D)g = 0$ in the sense of distributions.

Remark 4.1. A stronger condition satisfying (4.11) is given by

$$\nu(B(y, R)) \leq C_2 |y|^{(N-\ell)q-N} R^N$$

(4.13)

when $R < |y|/2$. The integration boundary $|y|/2$ in (4.11) can be swapped to $a|y|$, where $a$ is a fixed constant $0 < a < 1$. In this case, (4.13) must hold for $R < a|x|$ to imply (4.11).
Let us present an example of positive measure satisfying (4.10) and (4.11). Suppose $N \geq 2$, $0 < \ell < N$, $1 \leq q \leq N/(N - \ell)$, and define $d\nu = |x|^{q(N-\ell)-N}dx$. The control (4.10) is obvious for the case when $q = N/(N - \ell)$, since $\nu$ is simply the Lebesgue measure and $(N - \ell)q = N$. Otherwise,

$$v(B_R) = \int_{B_R} |x|^{q(N-\ell)-N} dx \lesssim \int_0^R r^{q(N-\ell)-1} dr \lesssim R^{(N-\ell)q}.$$

For (4.11), we note that if $y \in B_R$ and $R < |x|/2$, then $|x|/2 < |x + y| < 3|x|/2$, thus,

$$v(B(x, R)) = \int_{B_R} |x + y|^{q(N-\ell)-N} dy \lesssim |x|^{(N-\ell)q-N} R^N.$$

In order to prove the inequality (4.1), and consequently the Theorem B, we estimate

$$\left| \int_{\mathbb{R}^N} \left[ \int_{\mathbb{R}^N} K(x - y)g(y) dy \right] d\mu(x) \right| \leq \int_{\mathbb{R}^N} \left| \int_{\mathbb{R}^N} K(x - y)g(y) dy \right| d|\mu|(x)$$

and we apply the Lemma 4.3 for $q = 1$ and $\nu = |\mu|$, taking $K(x, y) = K(x - y)$ given by identity (3.2) that satisfies (4.2) and (4.3). Note that (4.10) and (4.11) come naturally from (1.7) and (1.8). The conclusion follows taking $g := A(D)u$ that belongs to the kernel of some cocanceling operator $L(D)$ from (4.6), since $A(D)$ is canceling.

Now we present the proof of Lemma 4.3.

**Proof** Let $\psi \in C^\infty_c(B_{1/2}, \mathbb{R})$ be a cut-off function such that $0 \leq \psi \leq 1$, $\psi \equiv 1$ on $B_{1/4}$, and write $K(x, y) = K_1(x, y) + K_2(x, y)$ with $K_1(x, y) = \psi(y/|x|)K(x, 0)$. We claim that

$$J_j \doteq \left( \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} K_j(x, y)g(y) dy \right)^q d\nu(x) \right)^{1/q} \lesssim \int_{\mathbb{R}^N} |g(x)| dx \quad (4.14)$$

for $j = 1, 2$ and $g \in L^1(\mathbb{R}^N, F)$ satisfying $L(D)g = 0$ in the sense of distributions. Using the control (4.8), we may estimate

$$J_1 = \left( \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \psi \left( \frac{y}{|x|} \right) g(y) dy \right)^q |K(x, 0)|^q d\nu(x) \right)^{1/q}$$

$$\lesssim \left( \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \psi \left( \frac{y}{|x|} \right) g(y) dy \right)^q |x|^{(\ell-N)q} d\nu(x) \right)^{1/q}$$

$$\lesssim \left( \int_{\mathbb{R}^N} \left[ \int_{B_1} |g(y)| dy \right]^q |x|^{(\ell-N)q} d\nu(x) \right)^{1/q}$$

$$= \left( \int_{\mathbb{R}^N} \left[ \int_{B_1} |y| |g(y)| dy \right]^q |x|^{(\ell-N-1)q} d\nu(x) \right)^{1/q},$$

which completes the proof.
where the second inequality follows from (4.7). In order to control the previous term, we use Lemma 4.1, taking \( \tilde{u}(x) = |x|^{(\ell - N - 1)q} \), \( \tilde{g}(x) = |g(x)| \) and \( \tilde{v}(x) = |x|^{-1} \). So checking (4.5) we have

\[
\left( \int_{(B_R)^c} \tilde{u}(x) \, d\nu(x) \right)^{1/q} = \left( \sum_{k=1}^{\infty} \int_{2^{k-1}R \leq |x| < 2^k R} |x|^{(\ell - N - 1)q} \, d\nu(x) \right)^{1/q} \\
\leq \left( \sum_{k=1}^{\infty} (2^{k-1}R)^{(\ell - N - 1)q} v(B_{2^k R}) \right)^{1/q} \\
\leq \|v\|_{0,(N - \ell)q} \left( \sum_{k=1}^{\infty} (2^{k-1}R)^{(\ell - N - 1)q} (2^k R)^{(N - \ell)q} \right)^{1/q} \\
\lesssim \|v\|_{0,(N - \ell)q} \left\{ \sup_{x \in B_R} [\tilde{v}(x)]^{-1} \right\}^{-1},
\]

where the last step follows from \( \sup_{x \in B_R} [\tilde{v}(x)]^{-1} = R \). Hence,

\[
J_1 \lesssim \left( \int_{\mathbb{R}^N} \left[ \int_{|y|/2} |y| \, |g(y)| \, dy \right]^q |x|^{(\ell - N - 1)q} \, d\nu(x) \right)^{1/q} \\
\lesssim \|v\|_{0,(N - \ell)q} \int_{\mathbb{R}^N} |g(x)| \, dx.
\]

Now for \( J_2 \), using Minkowski’s Inequality, we get

\[
J_2 \leq \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} |K_2(x, y)|^q \, d\nu(x) \right)^{1/q} |g(y)| \, dy.
\]

It remains to be shown that

\[
\int_{\mathbb{R}^N} |K_2(x, y)|^q \, d\nu(x) \leq C \quad (4.15)
\]

for some constant \( C > 0 \) uniformly on \( y \). For each \( y \in \mathbb{R}^N \) we get the following upper estimate for the previous integration

\[
\int_{|x| < 4|y|} |K(x, y)|^q \, d\nu(x) + \int_{|x| \geq 2|y|} |K(x, y) - K(x, 0)|^q \, d\nu(x) := (I) + (II).
\]

From conditions (4.9) and (4.10), we have

\[
(II) \lesssim |y|^q \int_{(B_{2|y|})^c} |x|^{(\ell - N - 1)q} \, d\nu(x) \lesssim \|v\|_{0,(N - \ell)q}.
\]
while from condition (4.8)

\[
(I) \lesssim \int_{B_{4|y|}} |x - y|^{(\ell - N)q} \, dv(x) \\
= \int_{B(y, |y|/2)} |x - y|^{(\ell - N)q} \, dv(x) + \int_{B_{4|y|} \setminus B(y, |y|/2)} |x - y|^{(\ell - N)q} \, dv(x).
\]

The second part is straightforward:

\[
(I_b) \leq \frac{1}{(|y|/2)^{(N-\ell)q}} \int_{B_{4|y|}} dv(x) = \frac{v(B_{4|y|})}{(|y|/2)^{(N-\ell)q}} \lesssim \|v\|_{0,(N-\ell)q}.
\]

Finally, writing \( A_x := \{ r \in \mathbb{R} : r > |x-y| \} \) and pointing out that \( B(y, |y|/2) \subset B_{2|y|} \), we obtain from (4.10) and (4.11)

\[
(I_a) = (N-\ell)q \int_{\mathbb{R}^N} \chi_{B(y, |y|/2)}(x) \left( \int_0^\infty \frac{\chi_{A_x}(r)}{r^{(N-\ell)q+1}} \, dr \right) \, dv(x) \\
= (N-\ell)q \int_0^\infty \left( \int_{B(y, |y|/2) \cap B(y, r)} \frac{1}{r^{(N-\ell)q+1}} \, dv(x) \right) \, dr \\
= (N-\ell)q \left( \int_0^{3|y|/2} \frac{v(B(y, r))}{r^{(N-\ell)q}} \, dr + v(B(y, |y|/2)) \right) \int_0^\infty \frac{1}{r^{(N-\ell)q+1}} \, dr \\
\lesssim [v](N-\ell)q + \|v\|_{0,(N-\ell)q},
\]

concluding (4.15) and thus \( J_2 \leq ([v](N-\ell)q + \|v\|_{0,(N-\ell)q})^{1/q} \int_{\mathbb{R}^N} |g(y)| \, dy. \)

A natural question arises about the necessity of the potential condition (4.11). Next we present a self improvement of the previous lemma where we assume an extra decay of Ahlfors regularity hypothesis (4.10) to every ball in order to remove the potential condition.

**Lemma 4.4** Assume \( N \geq 2, 0 < \ell < N \) and \( K(x, y) \in L^1_{loc}(\mathbb{R}^N \times \mathbb{R}^N, \mathcal{L}(F, V)) \) satisfying (4.8) and (4.9). Suppose \( 0 < \alpha < 1, 1 \leq q < \infty \), and let \( v \in \mathcal{M}_+(\mathbb{R}^N) \) satisfying

\[
\|v\|_{(N-\ell+\alpha)q} < \infty. \quad (4.16)
\]

If \( L(D) \) is cocanceling, then there exists \( \tilde{C} > 0 \) such that

\[
\left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K(x, y) g(y) \, dy \right)^{1/q} \, dv(x) \leq \tilde{C} \int_{\mathbb{R}^N} |g(x)| \, |x|^\alpha \, dx, \quad (4.17)
\]

for all \( g |x|^\alpha \in L^1(\mathbb{R}^N, F) \) satisfying \( L(D)g = 0 \) in the sense of distributions.
Proof With the necessary adaptations, the proof follows the same steps of Lemma 4.3. The noteworthy change to estimate $J_1$ is choosing $\tilde{v}(x) = |x|^{\alpha-1}$ for $0 \leq \alpha < 1$ when we apply the Lemma 4.1. For $J_2$, the same calculations from the previous proof give $(II) \lesssim \|v\|_{(N-\ell+\alpha)q} |y|^{|a|q}$ and $(I_b) \lesssim \|v\|_{(N-\ell+\alpha)q} |y|^{|a|q}$. The main change happens when estimating

$$(I_a) = \sum_{k=1}^{\infty} \int_{B(y,2^{-k}|y|)\setminus B(y,2^{-(k+1)}|y|)} |x - y|^{(\ell-N)q} \, dv(x)$$

$$(\leq) = \sum_{k=1}^{\infty} \int_{B(y,2^{-k}|y|)} (2^{-(k+1)}|y|)\!(\ell-N)q \, dv(x)$$

$$= |y|^{(\ell-N)q} \sum_{k=1}^{\infty} (2^{-(k+1)}\!(\ell-N)q) \, v(B(y, 2^{-k}|y|))$$

$$\leq 2^{(N-\ell)q} \|v\|_{(N-\ell+\alpha)q} |y|^{\alpha q} \sum_{k=1}^{\infty} (2^{-\alpha q})^k$$

$$= 2^{(N-\ell)q} (2^{\alpha q} - 1)^{-1} \|v\|_{(N-\ell+\alpha)q} |y|^{\alpha q},$$

where in the second inequality we use the strong assumption (4.16), in particular $v(B(y, 2^{-k}|y|)) \leq \|v\|_{(N-\ell+\alpha)q} (2^{-k}|y|)^{(N-\ell+\alpha)q}$. 

5 Applications and General Comments

5.1 Limiting Case for Trace Inequalities for Vector Fields

Next we present the validity of the inequality (1.11) for $q = 1$ (see [5, Theorem 1.1]) under $(N - 1)$- Ahlfors regularity and an additional uniform potential condition on $v$.

Theorem 5.1 Let $A(D)$ be a homogeneous linear differential operator of order $m$ on $\mathbb{R}^N$, $N \geq 2$, from $E$ to $F$. If $A(D)$ is elliptic and canceling, then for all $v \in \mathcal{M}_+(\mathbb{R}^N)$ satisfying (4.10) and (4.11), there exists $C > 0$ such that

$$\int_{\mathbb{R}^N} |D^{m-1}u(x)| \, dv \leq C \|A(D)u\|_{L^1}, \quad \forall u \in C_c^\infty(\mathbb{R}^N, E). \tag{5.1}$$

Proof The inequality follows by the combination of the identity $D^{m-1}u(x) = \int_{\mathbb{R}^N} K(x - y)|A(D)u(y)| \, dy$ where $\hat{K}(\xi) := \sum_{|\alpha|=m-1} \xi^\alpha (A^* \circ A)^{-1}(\xi)A^*(\xi)$ that satisfies (4.8) and (4.9) for $\ell = 1$ and then the estimate (5.1) follows by Lemma 4.3 for $q = 1$, as showed in the proof of inequality (4.1).

As a consequence of the previous proof, we can estimate the constant at inequality (5.1) by

$$C \lesssim \|v\|_{0,N-1} + [[v]]_{N-1}.$$
Remark 5.1 Let $D^m := (D^{|\alpha|})_{|\alpha|=m}$ the total derivative operator that is an elliptic and canceling homogeneous linear differential operator. Using (1.4) follows directly that

$$\int_{\mathbb{R}^N} \left| D^{m-1} u(x) \right| \, dv \lesssim \|v\|_{N-1} \|D^m u\|_{L^1}, \quad (5.2)$$

for all $u \in C_0^\infty(\mathbb{R}^N)$ and $v \in \mathcal{M}_+(\mathbb{R}^N)$. Although the assumption that $v$ is $(N-1)$-Ahlfors regular contrasts with $\|v\|_{0,N-1} < \infty$ at Theorem 5.1, the uniform potential condition (4.11) is not necessary to the validity of (5.2).

In the same spirit of [7, Theorem A] the inequality (5.1) can be extended for the following:

Theorem 5.2 Let $A(D)$ be a homogeneous linear differential operator of order $m$ on $\mathbb{R}^N$, $N \geq 2$, from $E$ to $F$, and assume that $1 \leq q < \infty$, $0 < \ell < N$ and $\ell \leq m$. If $A(D)$ is elliptic and canceling, then for all $v \in \mathcal{M}_+^+(\mathbb{R}^N)$ satisfying (4.10) and (4.11) there exists $C > 0$ such that

$$\left( \int_{\mathbb{R}^N} \left| (-\Delta)^{(m-\ell)/2} u(x) \right|^q \, dv \right)^{1/q} \leq C \|A(D) u\|_{L^1}, \quad \forall \ u \in C_c^\infty(\mathbb{R}^N, E). \quad (5.3)$$

The proof follows the same steps when proving Theorem 5.1 and will be omitted. In particular, the inequality (5.3) recovers the inequality (1.5) in [7, Theorem A] taking $d\mu = |x|^{-N+(N-\ell)q} \, dx$ for $1 \leq q < N/(N-\ell)$ (see Remark 4.1).

5.2 First-Order Operators

It remains as an open question whether (1.7) or (1.8) are necessary conditions to obtain a $L^\infty$ solution to (1.5) for homogeneous differential operator $A(D)$ with order $m > 1$. For $m = 1$, however, we show that certain (expected) decay regularity on $\mu$ is necessary:

Theorem 5.3 Let $A(D)$ be a first-order homogeneous linear differential operator on $\mathbb{R}^N$ from $E$ to $F$ and $\mu \in \mathcal{M}(\mathbb{R}^N, E^*)$. If there exists $f \in L^\infty(\mathbb{R}^N, F^*)$ solving (1.5), then there exists a constant $C > 0$ such that

$$|\mu(B(x,r))| \leq Cr^{N-1} \quad (5.4)$$

for every $x \in \mathbb{R}^N$ and $r > 0$.

Proof Denoting $A(D) = \sum_{j=1}^N a_j \partial_j$, we have, for every $x \in \mathbb{R}^N$ and almost every $r > 0$,

$$\mu(B(x,r)) = \int_{B(x,r)} A^*(D) f(y) \, dy = -\sum_{j=1}^N \int_{\partial B(x,r)} a_j^* f(y) \frac{y_j - x_j}{|y-x|} \, dS(y),$$

where $A^*(D)$ is the adjoint of $A(D)$.
hence, $|\mu(B(x, r))| \leq C_N \| f \|_{L^\infty} r^{N-1}$.

To extend this estimate for every $r > 0$, let $M \subset \mathbb{R}_+$ be the zero-measure set of values $r > 0$ for which the previous estimate does not hold. Given $x \in \mathbb{R}^N$ and $r > 0$, we can write $B[x, r] = \cap_j B(x, r_j)$, where $(r_j)_j \subset \mathbb{R}_+ \setminus M$ is a decreasing sequence converging to $r$ (note that $\mathbb{R}_+ \setminus M$ is dense in $\mathbb{R}_+$). Thus, simplifying the notation assuming $\mu_\ell \in \mathcal{M}(\mathbb{R}^N)$ for each $j = 1, \ldots, d$ we have

$$\mu_\ell(B(x, r)) \leq \lim_{j \to \infty} |\mu(B(x, r_j))| \leq C_N \| f \|_{L^\infty} \lim_{j \to \infty} r_j^{N-1} = C_N \| f \|_{L^\infty} r^{N-1}.$$  

Summarizing

$$|\mu(B(x, r))| \leq (2d)^{1/2} C_N \| f \|_{L^\infty} r^{N-1}.$$  

\[\square\]

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