On the fractional stochastic integration for random non-smooth integrands

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ABSTRACT
The paper suggests a way of stochastic integration of random integrands with respect to fractional Brownian motion with the Hurst parameter $H > 1/2$. The integral is defined initially on the processes that are "piecewise" predictable on a short horizon. Then the integral is extended on a wide class of square integrable adapted random processes. This class is described via a mild restriction on the growth rate of the conditional mean square error for the forecast on an arbitrarily short horizon given current observations. On the other hand, a pathwise regularity, such as Hölder condition, etc., is not required for the integrand. The suggested integration can be interpreted as foresighted integration for integrands featuring certain restrictions on the forecasting error. This integration is based on Itô’s integration and does not involve Malliavin calculus or Wick products. In addition, it is shown that these stochastic integrals depend right continuously on $H$ at $H = 1/2$.

KEYWORDS
Stochastic integration; fractional Brownian motion; random integrands; Hurst parameter; forecast error

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1. Introduction
The paper considers stochastic integration of random integrands with respect to fractional Brownian motion. These integrals can be defined using different approaches; see review and discussion in [1–14]. This integration has many applications in statistical modeling, especially for quantitative finance; see e.g. [10, 15–27]. Special statistical inference methods developed for these models; see e.g. [28–31].

Naturally, the integral can be defined as a Riemann sum for piecewise constant in time integrands; the problem is an extension on more general classes of integrands. There is a special approach based on the so-called the Wick product rather than Riemann sums; see, e.g. [8, 15–17, 19]. This approach allows integrands of quite general type but the features the Wick product makes the corresponding integrals quite distinctive from the integrals based on the Riemann sums.

Currently, stochastic integrals with respect to the fractional Brownian motion $B_H$ with a Hurst parameter $H \in (1/2, 1)$ are defined for random integrands in the following cases.
The integral is defined for the integrands that are pathwise Hölder with index $p > 1 - H$; see, e.g., Theorem 21 in [9] and [4, 14].

(ii) The integral is defined pathwise for integrands that has $q$-bounded variation with $q < 1/(1 - H)$; see, e.g., [32, 33].

(iii) The integral is defined as a Skorohod integral for integrands $\gamma$ such that $\nabla \gamma$ is $L_p$-integrable for $p > (1/2 - H)^{-1}$, where $\nabla$ is the Gross-Sobolev derivative (Theorem 3.6 [6] (2003) or Theorem 6.2 [7]). This approach is based on anticipating integrals (see, e.g., [5, 8, 15, 19], and review in [7]). It can be noted that this requires certain differentiability of the integrand in the sense of existence of $\nabla g$ or the fractional derivative [1].

(iv) The integral is a special case of Russo–Vallois integral for integrands with paths from the Besov spaces, which still means some relaxed generalized Hölder type property [34].

We exclude from this list the integrals based on the Wick product and integrals for piecewise constant integrands.

In this paper, we readdress stochastic integration of random integrands with respect to fractional Brownian motion. We suggest an integration scheme allowing to extend the class of admissible random integrands known in the literature. In particular, we show that stochastic integral with respect to the fractional Brownian motion $B_H$ with $H \in (1/2, 1)$ is well defined on a wide class of $L_2$-integrable processes with a mild restriction on the growth rate for conditional variance for a short term forecast. It is not required that the integrands $\gamma$ satisfy Hölder condition, or have finite $p$-variation, or $\nabla \gamma$ is $L_p$-integrable, or a fractional derivative exists. The description of this class does not require to use Malliavin calculus as in [6, 7] and does not use any kind of derivatives.

We use a modification of the classical Riemann sums. Instead of the standard extension of the Riemann sums from the set of piecewise constant integrands, we used an extension of different sums from processes being “piecewise predictable” on a short horizon that are not necessarily piecewise constant. More precisely, these integrands are adapted to the filtration generated by the observations being frozen at grid time points. In other words, this “piecewise predictable” class includes all integrands that are predictable without error on a fixed time horizon that can be arbitrarily short. The corresponding stochastic integral is represented via sums of integrals of two different types: one type is a standard Itô’s integral, and another type is a Lebesgue integral for random integrands.

In the second step, we extended this integral on a wide class of $L_2$-integrable processes (Theorem 3.1 below); the resulting integrals are denoted as $\int \cdot d\mathcal{F} B_H$. The corresponding condition allows a simple formulation that does not require Malliavin calculus used in [6, 7]. This theorem implies prior estimates of the stochastic integral via a norm of a random integrand (Corollary 3.1).

Furthermore, it is shown that the stochastic integrals depend right continuously on $H$ at $H = 1/2$ under some additional mild restrictions on the growth rate for the conditional variance of the future values given current observations (Theorem 4.1 below).
The paper is organized as follows. Section 2 presents some definitions. In Section 3, we present the definition of the new type of integral and some convergence results and prior estimates. In Section 4, we show some continuity of the new integral with respect to a variable Hurst parameter. The proofs are given in Section 5.

2. Some definitions

We are given a probability space \((\Omega, \mathcal{F}, P)\), where \(\Omega\) is a set of elementary events, \(\mathcal{F}\) is a complete \(\sigma\)-algebra of events, and \(P\) is a probability measure.

We assume that \(\{B_H(t)\}_{t \in \mathbb{R}}\) is a fractional Brownian motion with the Hurst parameter \(H \in (1/2, 1)\) defined as described in [10, 30] such that \(B_H(0) = 0\) and

\[
B_H(t) = \int_{-\infty}^{t} f(t, r) dB(r),
\]

where \(t \geq 0\) and

\[
f(t, r) = c_H(t - r)^{H-1/2} \mathbb{I}_{r \geq 0} + c_H((t - r)^{H-1/2} - (-r)^{H-1/2}) \mathbb{I}_{r < 0}.
\]

Here \(c_H = \sqrt{2H \Gamma(3/2 - H) / \Gamma(1/2 + H) \Gamma(2 - 2H)}\), \(\Gamma\) is the Gamma function, \(\mathbb{I}\) is the indicator function, and \(\{B(t)\}_{t \in \mathbb{R}}\) is a standard Brownian motion such that \(B(0) = 0\); we denote by \(\int \cdot dB\) the standard Itô’s integration.

Let \(d_H \triangleq c_H(H - 1/2)\). For \(T > 0\), \(\tau \in [0, T]\) and \(g \in L_2(0, T)\), set

\[
G_H(\tau, T, g) \triangleq d_H \int_{\tau}^{T} (t - \tau)^{H-3/2} g(t) dt.
\]

Let \(\{\mathcal{G}_t\}\) be the filtration generated by the process \(B(t)\). Let \(T > 0\) be given.

Let \(\mathcal{L}_{22}\) be the linear normed space formed as the completion in \(L_2\)-norm of the set of all \(\mathcal{G}_t\)-adapted bounded measurable processes \(\gamma(t), \ t \in [0, T]\), with the norm \(\|\gamma\|_{\mathcal{L}_{22}} = (E \int_{0}^{T} \gamma(t)^2 dt)^{1/2}\).

For \(\varepsilon > 0\), let \(\mathcal{X}_\varepsilon\) be the set of all \(\gamma \in \mathcal{L}_{22}\) such that there exists an integer \(n > 0\) and a set of nonrandom times \(T = \{T_k\}_{k=1}^{n} \subset [0, T]\), where \(T_0 = 0\), \(T_n = T\), and \(0 < T_{k+1} - T_k \leq \varepsilon\), such that \(\gamma(t)\) is \(\mathcal{G}_{T_k}\)-measurable for \(t \in [T_k, T_{k+1})\). Clearly, we need \(n \geq T/\varepsilon\).

In particular, the set \(\mathcal{X}_\varepsilon\) includes all \(\gamma \in \mathcal{L}_{22}\) such that \(\gamma(t)\) is \(\mathcal{G}_{T_{k-\varepsilon}}\)-measurable for all \(t \in [0, T]\).

Let \(\mathcal{X} = \bigcup_{\varepsilon > 0} \mathcal{X}_\varepsilon\). Let \(\mathcal{X}_{k, PC}\) be the set of all \(\gamma \in \mathcal{L}_{22}\) such that there exists an integer \(n > 0\) and a set of nonrandom times \(T = \{T_k\}_{k=1}^{n} \subset [0, T]\), where \(T_0 = 0\), \(T_n = T\), and \(0 < T_{k+1} - T_k \leq \varepsilon\), such that \(\gamma(t) = \gamma(T_k)\) for \(t \in [T_k, T_{k+1})\).

For the brevity, we sometimes denote \(L_p(\Omega, \mathcal{G}_T, P)\) by \(L_p(\Omega)\), \(p \geq 1\).

3. The main result: integration for random integrands

For any \(\gamma \in \mathcal{X}_{k, PC}\), it is natural to define the stochastic integral with respect to \(B_H\) in \(L_1(\Omega, \mathcal{G}_T, P)\) as the Riemann sum.
\[
\sum_{k=0}^{n} \gamma(T_k)(B_H(T_{k+1}) - B_H(T_k)).
\]

If \( \gamma \in \mathcal{L}_{22} \) is such that this sum has a limit in probability as \( n \to +\infty \), and this limit is independent on the choice of \( \{T_k\}_{k=1}^{n} \), then we call this limit the integral \( \int_{0}^{T} \gamma(t)d_{KS}B_H(t) \).

The classes of admissible deterministic integrands \( \gamma \) are known; see, e.g. [11, 12]. However, there are some difficulties with identifying classes of admissible random \( \gamma \). The present paper suggests a modification of the stochastic integral based on the extension from \( \mathcal{X} \), i.e. from the set of random functions that are not necessarily piecewise constant but rather “piecewise predictable”. This modification will allow to establish a new extended class of random integrands that are not necessarily “piecewise predictable”.

### 3.1. The case of nonrandom integrands

As the first step, let us construct a stochastic integral over the time interval \( [s, T] \) for \( \mathcal{G}_s \)-measurable integrands \( \gamma \in L_2(\Omega, \mathcal{G}_s, \mathbf{P}, L_2(s, T)) \), where \( s \in [0, T) \). These integrands can be regarded as nonrandom on the conditional probability space given \( \mathcal{G}_s \).

By (2.1), we have that

\[
B_H(t) = W_H(t) + R_H(t),
\]

where \( t > s \),

\[
W_H(t) = \int_{s}^{t} f(t, r)dB(r), \quad R_H(t) = \int_{-\infty}^{s} f(t, r)dB(r).
\]

The processes \( W_H(t) \) and \( R_H(t) \) are independent Gaussian processes with zero mean. In addition, the process \( W_H \) is \( \mathcal{G}_t \)-adapted, \( R_H(t) \) is \( \mathcal{G}_t \)-measurable for all \( t > s \), and \( W_H(t) \) is independent on \( \mathcal{G}_s \) for all \( t > s \).

To define integration with respect to \( dB_H \) for \( \mathcal{G}_s \)-measurable integrands \( \gamma \in L_2(\Omega, \mathcal{G}_s, \mathbf{P}, L_2(s, T)) \) we define integration with respect to \( W_H \) and \( R_H \) separately.

First, it can be noted that if we had \( f(t, \cdot) \in L_2(s, t) \) then integration with respect to \( W_H \) would be straightforward, since we would be able to find the Itô’s differential \( dW_H(t) \) as

\[
f(t, t)dB(t) + \int_{0}^{t} f(t, r)dB(r) \cdot dt = 0 \cdot dB(t) + \int_{0}^{t} f(t, r)dB(r) \cdot dt,
\]

which would allow us to accept \( \int_{s}^{T} \gamma(t)\left[\int_{0}^{t} f(t, r)dB(r)\right]dt \) as \( \int_{s}^{T} \gamma(t)dW_H(t) \). However, the expression (3.1) cannot be regarded as an Itô’s differential, since \( f(t, \cdot) \not\in L_2(s, t) \). Nevertheless, we will be using a modification of this version of the integral with respect to \( W_H \) amended with some approximations to overcome insufficient integrability of \( f(t, \cdot) \).
For $\varepsilon > 0$, let

$$W_{H,\varepsilon}(t) = \int_s^t f(t, r - \varepsilon) dB(r).$$

In this case, there exists a usual Itô’s differential

$$dW_{H,\varepsilon}(t) = f(t, t - \varepsilon) dB(t) + \int_0^t f_i(t, r - \varepsilon) dB(r) \cdot dt,$$

representing a “regularized” approximation of the right hand part of (3.1).

**Proposition 3.1.** For any $\gamma \in L^2(\Omega, G_s, P, L^2(s, T))$,

$$\lim_{\varepsilon \to 0} \int_s^T \gamma(t) dW_{H,\varepsilon}(t) = \int_s^T G_H(t, \tau, \gamma) dB(\tau);$$

the limit holds in $L^2(\Omega, G_T, P)$.

This result justifies the following definition.

**Definition 3.1.** We regard the limit in Definition 3.1 as the stochastic integral with respect to $W_H$, and we denote it as $\int_s^T \gamma(t) dF W_H(t)$, i.e.

$$\int_s^T \gamma(t) dF W_H(t) \overset{\Delta}{=} \int_s^T G_H(t, \tau, \gamma) dB(\tau).$$

It appears that this choice for the case of nonrandom integrands leads to a new version of a stochastic integral for random integrands constructed below.

**Proposition 3.2.**

(i) $R_H(t)$ is $G_s$-measurable for all $t > s$ and differentiable in $t > s$ in the sense that

$$\lim_{\delta \to 0} E \left| \frac{R_H(t + \delta) - R_H(t)}{\delta} - DR_H(t) \right| = 0,$$

where

$$DR_H(t) \overset{\Delta}{=} \int_{-\infty}^t f_i(t, q) dB(q).$$

The process $DR_H$ is such that

(a) $DR_H(t)$ is $G_s$-measurable for all $t > s$;

(b) for any $t > s$,

$$E [DR_H(t)]^2 = \frac{d^2 H}{2 - 2H} (t - s)^{2H - 2},$$

$$E \int_s^t [DR_H(r)]^2 dr = \frac{c_H d_H}{2(2 - 2H)} (t - s)^{2H - 1}.$$
**Definition 3.2.** For $s \leq T$ and $\gamma \in L_2(\Omega, \mathcal{G}_s, \mathbb{P}, L_2(s, T))$, we define the integral
\[
\int_s^T \gamma(t) dF_B(t) = \int_s^T \gamma(t) dF_W(t) + \int_s^T \gamma(t) dR_H(t) dt
\]
\[
\leq \int_s^T G_H(t) dB(t) + \int_s^T \gamma(t) dR_H(t) dt.
\]

The first integral in the sum above is described in Definition 3.1, and the second one is a pathwise Lebesgue integral on $[s, T]$. The sum belongs to $L_1(\Omega, \mathcal{G}_T, \mathbb{P})$ thanks to Propositions 3.1 and 3.2.

**Proposition 3.3.** Under the assumptions and notation of Definition 3.2,
\[
E \left| \int_s^T \gamma(t) dF_W(t) \right|^2 \leq c E \int_s^T \gamma(t)^2 dt,
\]
\[
E \left| \int_s^T \gamma(t) dR_H(t) dt \right| \leq c \left( E \int_s^T \gamma(t)^2 dt \right)^{1/2},
\]
\[
E \left| \int_s^T \gamma(t) dF_B(t) \right| \leq c \left( E \int_s^T \gamma(t)^2 dt \right)^{1/2},
\]
for some $c = c(H, T) > 0$.

**Remark 3.1.** For the purposes of the proofs below, we need stronger estimates for $\int \gamma(t) dF_W dt$ and $\int \gamma(t) dF_B(t)$ than for $\int \gamma(t)^2 dt$, such as is given in Proposition 3.3. It can be noted that combined estimates from Proposition 3.3 would lead to estimate $E|I_H(\gamma)| \leq \text{const} \left( E \int_s^T \gamma(t)^2 dt \right)^{1/2}$ which is weaker than known estimates [5, 11].

**Proposition 3.4.** We have that
\[
\int_s^T 1 \cdot dF_B(t) = B_H(T) - B_H(s).
\]

### 3.2. Extension on piecewise-predictable integrands from $\mathcal{X}_e$

**Definition 3.3.** Let $\gamma \in \mathcal{X}_e$, where $e > 0$. By the definitions, there exists a finite set $\Theta$ of nonrandom times $\Theta = \{T_k\}_{k=0}^n \subset [s, T]$, where $n > 0$ is an integer, $T_0 = 0$, $T_n = T$, and $T_{k+1} \in (T_k, T_k + e]$ such that $\gamma(t)$ is $\mathcal{G}_{T_k}$-measurable for $t \in [T_k, T_{k+1})$. Let $\int_{T_{k-1}}^{T_k} \gamma(t) dF_B(t)$ be defined according to Definition 3.2 with the interval $[s, T]$ replaced by $[T_{k-1}, T_k]$. We call the sum
\[
I_H(\gamma) = \sum_{k=1}^n \int_{T_{k-1}}^{T_k} \gamma(t) dF_B(t).
\]
the foresighted integral of $\gamma$ and denote it as $\int_0^T \gamma(t) dF_B(t)$.

The integral in the above definition belongs to $L_1(\Omega, \mathcal{G}_T, \mathbb{P})$ thanks to Propositions 3.1 and 3.2.
Remark 3.2. It follows from Proposition 3.4 that

\[
\int_0^T \gamma(t) dF(t) = \int_0^T \gamma(t) dR(t)
\]

for piecewise constant \(\gamma \in \cup_{c>0} \mathcal{Y}_{c,PC}\). However, it appears that converges of Riemann sums requires more restriction for non-piecewise constant \(\gamma\) than the convergence for the suggested new integral. This is because this approximation is finer that approximation by the piecewise constant functions.

3.3 Extension on random integrands of a general type with a mild restriction on prediction error

Let \(E_t\) and \(\text{Var}_t\) denote the conditional expectation and the conditional variance given \(\mathcal{G}_t\), respectively.

For \(\nu > 0\) and \(\varepsilon > 0\), let \(\mathcal{Y}_{\nu,\varepsilon}\) be the set of all processes \(\gamma \in \mathcal{L}_{22}\) such that

\[
\sup_{t \in [0,T]} \sup_{t \in [\tau, T \wedge (\tau + \varepsilon)]} [\text{EVar}_t \gamma(t)]^{1/2} \leq C(t - \tau)^{1-H+\nu} \quad \text{a.s.}
\]

for some \(C = C(\gamma) > 0\).

It can be noted that \(E_t \gamma(t)\) can be interpreted as the forecast at time \(\tau\) of \(\gamma(t)\) for \(t > \tau\); the forecast is based on observations of the events from \(\mathcal{G}_t\). Respectively, \(\text{Var}_t \gamma(t)\) can be interpreted as the conditional means-square error of this forecast given \(\mathcal{G}_t\).

This means that, on the short horizon \(\varepsilon\), processes from \(\mathcal{Y}_{\nu,\varepsilon}\) feature stronger predictability for \(\nu > 0\) than for \(\nu = 0\).

Proposition 3.5. For any \(\nu > 0\) and \(\varepsilon > 0\), the space \(\mathcal{Y}_{\nu,\varepsilon}\) with the norm

\[
\|\gamma\|_{\mathcal{Y}_{\nu,\varepsilon}} \overset{\Delta}{=} \|\gamma\|_{\mathcal{L}_{22}} + \sup_{t \in [0,T]} \sup_{t \in [\tau, T \wedge (\tau + \varepsilon)]} \frac{[\text{EVar}_t \gamma(t)]^{1/2}}{(t - \tau)^{1-H+\nu}}
\]

is a Banach space.

It follows from the definitions that if \(\varepsilon_0 \in (0,\varepsilon)\) and \(\gamma \in \mathcal{Y}_{\nu,\varepsilon}\) then \(\gamma \in \mathcal{Y}_{\nu,\varepsilon_0}\) and \(\|\gamma\|_{\mathcal{Y}_{\nu,\varepsilon_0}} \leq \|\gamma\|_{\mathcal{Y}_{\nu,\varepsilon}}\). Also, it can be seen that \(\mathcal{X}_\varepsilon \subset \mathcal{Y}_{\nu,\varepsilon}\) for any \(\nu > 0\).

Let \(\mathcal{Y} = \Delta \cup_{\nu > 0, \varepsilon > 0} \mathcal{Y}_{\nu,\varepsilon}\). Clearly, the set \(\mathcal{Y}\) is everywhere dense in \(\mathcal{L}_{22}\).

Example 3.1. We have that \(B|_{[0,T]} \in \mathcal{Y}_{0,\varepsilon}\) but \(B|_{[0,T]} \notin \mathcal{Y}\). On the other hand, \(B_H|_{[0,T]} \in \mathcal{Y}_{2H-1,\varepsilon}\) for any \(\varepsilon > 0\).

For \(\gamma \in \mathcal{L}_{22}\), let \(Z(\gamma)\) be the set of processes \(\{\gamma_n \in \mathcal{X}, \ n = 0, 1, 2, \ldots\}\), such that \(\gamma_n(t) = E_{T_k} \gamma(t)\) for \(t \in [T_k, T_{k+1})\), where \(k = 0, 1, \ldots, 2^n\) and where \(T_k = kT/2^n\).

Theorem 3.1.

(i) Let \(\gamma \in \mathcal{Y}\), and let \(\{\gamma_n\}_{n=1}^\infty = Z(\gamma)\). Then the sequence \(\{I_H(\gamma_n)\}_{n=1}^\infty\) converges to a limit in \(L_1(\Omega, \mathcal{G}_T, \mathcal{P})\) uniformly over \(H \in (1/2, c)\) for any \(c \in (1/2, 1)\). Let \(I_H(\gamma)\) denote this limit.

(ii) For any \(\varepsilon > 0\), \(H \in (1/2, 1)\), and \(\nu > 0\), the operator \(I_H(\cdot): \mathcal{Y}_{\nu,\varepsilon} \rightarrow L_1(\Omega, \mathcal{G}_T, \mathcal{P})\) defined in statement (i) is a linear continuous operator. For any
\( \varepsilon > 0 \), the norms of these operators are bounded in \( H \in (1/2, \varepsilon) \), for any \( \varepsilon \in (1/2, 1) \).

We will regard \( I_H(\gamma) \) defined in Theorem 3.1 as the stochastic integral

\[
I_H(\gamma) = \int_0^T \gamma(t) dF_B(t), \quad \gamma \in \mathcal{Y}.
\]

**Example 3.2.** The class of processes \( \gamma \) for which the stochastic integral is defined includes processes with arbitrarily irregular paths. For example, let

\[
\gamma(t) = \psi(t) \int_0^t (t - \tau)^{\kappa/2} dB(t),
\]

where \( \kappa \geq 1 - H + \nu \) for some \( \nu > 0 \), and where \( \psi \) is a nonrandom function. Then \( \gamma \in \mathcal{Y}_{\nu, T} \) for any measurable bounded process \( \psi(\cdot) : [0, T] \to \mathbb{R} \), and the integral \( I_H(\gamma) \) is defined for all these choices of \( \psi \). Similarly, if \( \psi(t) \equiv \psi(t - \varepsilon) \) for some \( \varepsilon > 0 \) and some \( G_t \)-adapted measurable bounded process \( \psi(\cdot) : [-\varepsilon, T - \varepsilon] \times \Omega \to \mathbb{R} \), then \( \gamma \in \mathcal{Y}_{\kappa, T} \), and the integral \( I_H(\gamma) \) is defined again. For these integrands \( \gamma \), there are no restrictions on paths regularity. On the other hand, these \( \gamma \) satisfy the restrictions on the predicability imposed in Theorem 3.1, i.e. restrictions on the conditional variance of the short-term forecast.

**Corollary 3.1.** For any \( \varepsilon > 0 \) and \( \nu > 0 \), there exists a constant \( c > 0 \) depending on \( T, \varepsilon, \nu \) only such that

\[
E \left| \int_0^T \gamma(t) dF_B(t) \right| \leq c \| \gamma \|_{\mathcal{Y}_{\nu, \varepsilon}} \quad \forall \gamma \in \mathcal{Y}_{\nu, \varepsilon}.
\]

**Corollary 3.1** follows immediately from Theorem 3.1.

For \( \nu > 0 \) and \( r > 1 \), let \( \mathcal{H}_{\nu, r} \) be the set of all \( \gamma \in \mathcal{L}_{22} \) such that \( \sup_{s, t \in [0, T]} \| \gamma(s) - \gamma(t) \|_{L_r(\Omega)} \leq C |t - s|^{1-H+\nu} \) for some \( C = C(\gamma) > 0 \).

It can be seen that \( \mathcal{H}_{\nu, r} \subset \mathcal{Y}_{\nu, \varepsilon} \) for \( r \geq 2 \) for all \( \varepsilon > 0 \).

For \( \gamma \in \mathcal{H}_{\nu, r} \), let \( \mathcal{Z}(\gamma) \) be the set of processes \( \{ \gamma_n \in \mathcal{X} \colon n = 0, 1, 2, \ldots \} \), such that, for \( t \in [T_k, T_{k+1}) \), either \( \gamma_n(t) = \gamma(T_k) \), or \( \gamma_n(t) = E_{T_k} \gamma(t) \), where \( k = 0, 1, \ldots, 2^n \) and where \( T_k = kT/2^n \).

**Proposition 3.6.** For any \( r \in (1, 2] \) and \( \nu > 0 \), the conclusions of Theorem 3.1 hold for \( \gamma \in \mathcal{H}_{\nu, r} \) if \( \mathcal{Y}, \mathcal{Y}_{\nu, \nu}, \) and \( \mathcal{Z}(\gamma) \), are replaced by \( \cup_{\nu > 0} \mathcal{H}_{\nu, r}, \mathcal{H}_{\nu, r}, \) and \( \mathcal{Z}(\gamma) \), respectively.

### 4. Continuity of the foresighted integral in \( H \to 1/2 + 0 \)

The following theorem describes some classes of random integrands where the stochastic integrals are continuous with respect to the Hurst parameter \( H \to 1/2 + 0 \).

**Theorem 4.1.** For any \( \gamma \in \mathcal{Y} \),

\[
E \left| \int_0^T \gamma(t) dF_B(t) - \int_0^T \gamma(t) dB(t) \right| \to 0 \quad \text{as} \quad H \to 1/2 + 0. \quad (4.1)
\]
In fact, the question about continuity at \( H = 1/2 \) of stochastic integrals with respect to \( dB_H \) is quite interesting. In particular, it is known that

\[
E \int_0^T B_H(t) dR_S B_H(t) \sim E \int_0^T B(t) dB(t) \quad \text{as} \quad H \to 1/2 + 0.
\] (4.2)

This follows from the equality

\[
2 \int_0^T B(t) dB(t) = B(T)^2 - T
\]
combined with the equalities [26]

\[
2 \int_0^T B_H(t) dR_S B_H(t) = B_H(T)^2, \quad H \in (1/2, 1).
\]

**Remark 4.1.** Theorem 4.1 does not contradict to the divergence stated in (4.2) since \( B_{[0,T]} \notin \mathcal{Y} \). On the other hand, this theorem ensures that, for any \( H_1 > 1/2 \),

\[
E \int_0^T B_{H_1}(t) dB_{H_1}(t) \to E \int_0^T B_H(t) dB(t) \quad \text{as} \quad H \to 1/2 + 0,
\]

since \( B_H|_{[0,T]} \notin \mathcal{Y} \).

### 5. Proofs

It can be noted, the operator (2.3) is such that there exists \( c > 0 \) such that

\[
\|G_{H,\epsilon}(\cdot, T, g)\|_{L_2(s, T)} \leq c \|g\|_{L_2(s, T)};
\] (5.1)

see [35, 36] and some developments in [35]). Moreover, this \( c \) is independent on \( H \in (1/2, 1) \) and \( s \in [0, T] \); see Theorem 2.6 in [37], p.48.

Consider the derivative

\[
f_t(t, r) = d_H(t - r)^{H - 3/2}, \quad t > r.
\]

Since \( H - 3/2 \in (-1, -1/2) \), it follows that \( 2(H - 3/2) \in (-2, -1) \) and \( \|f_t(\cdot, \cdot)\|_{L_2(-\infty, s)} < +\infty \) for all \( s < t \).

**Proof of Proposition 3.1.** For \( \tau \in [s, T] \), \( \epsilon \geq 0 \), and \( g \in L_2(s, T) \), set

\[
G_{H,\epsilon}(\tau, T, g) \overset{\Delta}{=} c_H f(t, t - \epsilon) g(\tau) + d_H \int_{\tau}^T (t - \tau + \epsilon)^{H - 3/2} g(\tau) d\tau.
\]

By the restrictions on \( \gamma \) and by (5.1), we have that \( G_{H,\epsilon}(\cdot, T, \gamma) \) is \( \mathcal{G}_s \)-measurable for any \( \tau \), that \( \int_s^T dB(\tau) G_{H}(\tau, T, \gamma) \) is well defined as an Itô’s integral, and that \( \int_s^T \gamma(t) dW_{H,\epsilon}(\tau) \) is also well defined as the Itô’s integral

\[
\int_s^T \gamma(t) dW_{H,\epsilon}(\tau) = c_H \int_s^T \gamma(t) f(t, t - \epsilon) dB(t) + d_H \int_s^T \gamma(t) dt \int_s^t (t - \tau + \epsilon)^{H - 3/2} dB(\tau),
\]
i.e.
\[
\int_s^T \gamma(t) dW_{H,c}(t) = \int_s^T dB(\tau) G_{H,c}(\tau, T, \gamma).
\]

Furthermore, let
\[
D_{\epsilon} \Delta \int_s^T dB(\tau) G_{H}(\tau, T, \gamma) - \int_s^T \gamma(t) dW_{H,c}(t).
\]

We have that \( D_{\epsilon} = \bar{D}_{\epsilon} + \tilde{D}_{\epsilon} \), where \( \bar{D}_{\epsilon} \Delta - \int_s^T \gamma(t)f(t, t - \epsilon) dB(t) \) and where
\[
\tilde{D}_{\epsilon} \Delta \int_s^T DB(\tau)|G_{H}(\tau, T, \gamma) - G_{H,c}(\tau, T, \gamma)|.
\]

Clearly, \( \mathbb{E}\tilde{D}_{\epsilon}^2 \rightarrow 0 \) as \( \epsilon \rightarrow 0 \). Let us show that \( \mathbb{E}\bar{D}_{\epsilon}^2 \rightarrow 0 \) as \( \epsilon \rightarrow 0 \). For this, it suffices to consider \( \epsilon = \epsilon_j \) for a monotonically decreasing sequence \( \{\epsilon_j\}_{j=1}^\infty \).

Assume first that \( \gamma(t) \geq 0 \) a.e. In this case, \((t - \tau + \epsilon_j)^{H-3/2} \gamma(t) \geq 0 \) a.e. if \( i > j \), i.e., \( \epsilon_i < \epsilon_j \).

It follows that \( G_{H}(\tau, T, \gamma) - G_{H,c}(\tau, T, \gamma) \geq 0 \) a.s. for almost all \( \tau \). It also follows that \( \|G_{H,c}(\cdot, T, \gamma)\|_{L_2(s, T)} \leq \|\gamma\|_{L_2(s, T)} \) with the same \( c \) as in (5.1).

We have that \( G_{H}(\tau, T, \gamma) - G_{H,c}(\tau, T, \gamma) \rightarrow 0 \) a.s. for almost all \( \tau \) as \( \epsilon = \epsilon_j \rightarrow 0 \) and that \( 0 \leq G_{H,c}(\tau, T, \gamma) \leq G_{H}(\tau, T, \gamma) \) for a.e. \( \tau \). By the Lebesgue Dominated Convergence Theorem, it follows that \( \mathbb{E}\bar{D}_{\epsilon}^2 \rightarrow 0 \) as \( \epsilon \rightarrow 0 \).

The case where \( \gamma \leq 0 \) can be considered similarly. In the case of a sign variable \( \gamma \), apply the proof above for \( \gamma_+ = \gamma \mathbb{1}_{\gamma \geq 0} \) and for \( \gamma_- = -\gamma \mathbb{1}_{\gamma \leq 0} \) separately. Then the proof for \( \gamma = \gamma_+ - \gamma_- \) follows. This completes the proof of Proposition 3.1.

**Proof of Proposition 3.2.** Let us prove statement (i). We need to verify the properties related to the differentiability of \( R_{H}(t) \).

Let \( t > s \) and \( r < s \). Let
\[
f^{(1)}(t, r, \delta) = \frac{f(t + \delta, r) - f(t, r)}{\delta},
\]
where \( \delta \in (-t - s)/2, (t - s)/2 \).

Clearly, \( f_i(t, r) - f^{(1)}(t, r, \delta) \rightarrow 0 \) as \( \delta \rightarrow 0 \) for all \( t > s \) and \( r < s \). Let us show that
\[
\|f_i(t, \cdot) - f^{(1)}(t, \cdot, \delta)\|_{L_2(-\infty, s)} \rightarrow 0 \text{ as } \delta \rightarrow 0.
\]
We have that
\[
f^{(1)}(t, r, \delta) = \delta^{-1} \int_t^{t+\delta} f_i(s, r) ds = f_i(\theta(t, \delta), r)
\]
for some \( \theta(t, \delta) \in (t, t + \delta) \). Hence
\[
|f_i(t, r) - f^{(1)}(t, r, \delta)| \leq \sup_{h \in (t, t + \delta)} |f_i(t, r) - f_i(h, r)| \leq \delta \sup_{h \in (t, t + \delta)} |f_i(h, r)|,
\]
where \( f_i(h, r) = d_H(H - 3/2)(h - r)^{H-5/2} \). For \( \delta > 0 \), we have that
\[
\sup_{h \in (t, t + \delta)} |f_i(h, r)| \leq d_H|H - 3/2| (t - r)^{H-5/2}.
\]

For \( \delta \in (-t - s)/2, 0], \) we have that
\[
\sup_{h \in (t, t + \delta)} |f_i(h, r)| \leq d_H|H - 3/2| (t + \delta - r)^{H-5/2}.
\]

It follows that \( \|f_i(t, \cdot)\|_{L_2(-\infty, s)} < +\infty \).
By (5.2), it follows for all $t > s$

$$\mathcal{D}R_H(t) = \lim_{\alpha \to 0} \frac{R_H(t + \alpha) - R_H(t)}{\alpha} = \int_{-\infty}^{t} f_{t}(t, r) dB(r),$$

for the mean square limit described in statement (ii).

Further, we have that

$$E\mathcal{D}R_H(t)^2 = \int_{-\infty}^{t} |f_{t}(t, r)|^2 dr = d_H^2 \int_{-\infty}^{t} (t - r)^{2H - 3} dr = \frac{d_H^2}{2 - 2H} (t - s)^{2H - 2}.$$ 

Hence, for $t > s$,

$$E \int_{s}^{t} \mathcal{D}R_H(r)^2 dr = \frac{d_H^2}{2 - 2H} \int_{s}^{t} (r - s)^{2H - 2} dr = \frac{d_H^2}{(2 - 2H)(2H - 1)} (t - s)^{2H - 1} = \frac{c_H d_H}{2(2 - 2H)} (t - s)^{2H - 1}.$$ 

This completes the proof of Proposition 3.2.

Proof of Proposition 3.3 follows from (5.1) and Proposition 3.2.

Proof of Proposition 3.4. Let $h \triangleq H - 1/2$. By the definition of the integral $\int d_F B_H$, 

$$\int_{s}^{T} 1 \cdot d_F B_H(t) = J_1 + J_2,$$

where

$$J_1 \triangleq \int_{s}^{T} 1 \cdot d_F W_H(t) = d_H \int_{s}^{T} dB(\tau) \int_{\tau}^{T} (t - \tau)^{h-1} dt = \int_{s}^{T} dB(\tau) G_H(\tau, T, 1)$$

and

$$J_2 \triangleq \int_{s}^{T} 1 \cdot \mathcal{D}R_H(t) dt = \int_{s}^{T} dt \int_{-\infty}^{s} f_{t}(t, r) dB(r) = d_H \int_{s}^{T} dt \int_{-\infty}^{s} (t - \tau)^{h-1} dB(\tau).$$

We have that

$$J_1 = c_H \int_{s}^{T} dB(\tau)(T - \tau)^{h}$$

and

$$J_2 = d_H \int_{-\infty}^{s} dB(\tau) \int_{s}^{T} (t - \tau)^{h-1} dt = c_H \int_{-\infty}^{s} dB(\tau)[(T - \tau)^{h} - (s - \tau)^{h}].$$

Hence

$$\int_{s}^{T} 1 \cdot d_F B_H(t) = J_1 + J_2 = c_H \int_{s}^{T} dB(\tau)(T - \tau)^{h} + c_H \int_{-\infty}^{s} dB(\tau)[(T - \tau)^{h} - (s - \tau)^{h}].$$
It follows from the well known properties of fractional Brownian motions that this value is \( B_H(T) - B_H(s) \). Let us show this for the sake of completeness. We have that
\[
B_H(T) - B_H(s) = c_H \int_0^T dB(\tau) (T - \tau)^h + c_H \int_0^s dB(\tau) [ (T - \tau)^h - (-\tau)^h ]
- c_H \int_0^s dB(\tau) (s - \tau)^h - c_H \int_0^0 dB(\tau) [ (s - \tau)^h - (-\tau)^h ] \\
= c_H \int_0^T dB(\tau) (T - \tau)^h + c_H \int_s^T dB(\tau) [ (T - \tau)^h - (s - \tau)^h ].
\]

This completes the proof of Proposition 3.4. \( \square \)

**Proof of Proposition 3.5.** We denote by \( \bar{\ell}_1 \) the \( \sigma \)-algebra of Lebesgue sets in \( \mathbb{R} \), and we denote by \( \bar{B}_1 \) the \( \sigma \)-algebra of Lebesgue sets in \( \mathbb{R} \). Let \( D = \{ (t, r) : 0 \leq r \leq t \leq T \} \).

Let \( \mathcal{V}_1 = L_2([0, T], \bar{B}_1, \bar{\ell}_1, L_2(\Omega, \mathcal{G}_0, \mathbb{P})) \), and let \( \mathcal{V}_2 \) be the linear normed space of all measurable function (classes of equivalence) \( g : D \times \Omega \to \mathbb{R} \) such that \( g(t, r) \in L_2(\Omega, \mathcal{G}_r, \mathbb{P}) \) for a.e. \( t, r \), with the norm
\[
\| g \|_{\mathcal{V}_2} = \left( \mathbb{E} \int_0^T dt \int_0^t g(t, r)^2 dr \right)^{1/2} \sup_{t \in [0, T]} \sup_{r \in [t, (t+\varepsilon) \wedge T]} \left( \mathbb{E} \int_t^{t+\varepsilon} g(t, r)^2 d\theta \right)^{1/2} (t - 1)^{-H+\nu}.
\]

By the Ito’s representation theorem, it follows that \( \gamma \in \mathcal{Y}_{\nu, \varepsilon} \) can be represented as
\[
\gamma(t) = \mathbb{E}_0 \gamma(t) + \int_0^t g(t, r) dB(r)
\]
for some \( g(\cdot, \cdot) \in \mathcal{V}_2 \); here \( \mathbb{E}_0 \gamma(\cdot) \in \mathcal{V}_1 \); see, e.g., Theorem 4.3.3 in [38]. In this case, \( \text{Var} \cdot \gamma(t) = \mathbb{E}_0 \int_0^t g(t, r)^2 dr \). To prove the proposition, it suffices to observe that the space \( \mathcal{V}_1 \times \mathcal{V}_2 \) is complete and is in a continuous and continuously invertible bijection with the space \( \mathcal{Y}_{\nu, \varepsilon} \). This completes the proof of Proposition 3.5. \( \square \)

To prove Theorems 3.1, Proposition 3.6, and Theorem 4.1, we will need some notation.

We will be using functions
\[
\hat{\rho}(t) \triangleq \int_{-\infty}^0 f_1(t, r) dB(r), \quad \rho(t, \tau) \triangleq \int_0^\tau f_1(t, r) dB(r), \quad \tau > t > 0.
\]

In the proofs below, we consider an integer \( n > 0 \) and \( \gamma_n \in \mathcal{X} \) such that there exist some \( \varepsilon > 0 \) and a set \( \Theta_n = \{ T_k \}_{k=1}^n \subset [0, T] \), where \( T_0 = 0, T_n = T, \) and \( T_{k+1} \in (T_k, T_k + \varepsilon) \) such that \( \gamma_n(t) \in L_2(\Omega, \mathcal{G}_{T_k}, \mathbb{P}) \) for \( t \in [T_k, T_{k+1}) \).

Let
\[
I_{W, H, k} = \int_{T_{k-1}}^{T_k} \gamma_n(t) dW_{H, k}(t), \quad I_{R, H, k} = \int_{T_{k-1}}^{T_k} \gamma_n(t) dR_{H, k}(t) dt,
\]
where \( W_{H, k}, R_{H, k}, \) and \( D R_{H, k} \) are defined similarly to \( W_H, R_H, \) and \( D R_H \), with \([s, T]\) replaced by \([T_{k-1}, T_k]\).
Let
\[ I_{W,H}(\gamma_n) \triangleq \sum_{k=1}^{n} I_{W,H,k}, \quad I_{R,H}(\gamma_n) \triangleq \sum_{k=1}^{n} I_{R,H,k}. \]

Clearly,
\[ I_{H}(\gamma_n) = \int_{T_{k-1}}^{T_k} \gamma_n(t) dF_B(t) = I_{W,H,k} + I_{R,H,k}, \]
and
\[ I_{H}(\gamma_n) = I_{W,H}(\gamma_n) + I_{R,H}(\gamma_n). \]

By the definitions of \( \tilde{\rho}(\cdot) \) and \( \rho(\cdot) \), we have
\[
I_{R,H,k} = \int_{T_{k-1}}^{T_k} \gamma_n(t) dR_k(t) dt = \int_{T_{k-1}}^{T_k} \gamma_n(t) \left( \int_{-\infty}^{T_k} f(t,s) dB(s) \right) dt.
\]

Hence
\[
I_{H}(\gamma_n) = I_{W,H}(\gamma_n) + \hat{I}_{R,H}(\gamma_n) + \tilde{I}_{R,H}(\gamma_n),
\]
where
\[
\hat{I}_{R,H}(\gamma_n) = \int_{0}^{T} \gamma_n(t) \tilde{\rho}(t) dt, \quad \tilde{I}_{R,H}(\gamma_n) = \sum_{k=1}^{n} I_{R,H,k},
\]
and where
\[
J_{R,H,k} = \int_{T_k}^{T_{k+1}} \gamma_n(t) \rho(t,T_k) dt.
\]

For \( k = 0,...,n-1 \), consider operators \( \Gamma_k(\cdot) : L_2(0,T_{k+1}) \to L_2(0,T_{k+1}) \) such that
\[
\Gamma_k(\cdot,g) = G_H(\cdot,T_{k+1},g),
\]
i.e.
\[
\Gamma_k(\tau,g) = d_H \int_{\tau}^{T_{k+1}} (t-\tau)^{-3/2} g(t) dt.
\]

Similarly to (5.1), we have that \( \| \Gamma_k(\cdot,g) \|_{L_2(T_k,T_{k+1})} \leq \hat{c} \| g \|_{L_2(T_k,T_{k+1})} \) for some \( \hat{c} > 0 \) that is independent on \( g \in L_2(T_k,T_{k+1}) \) and \( H \in (1/2,1) \); see Theorem 2.6 in [37], p.48.

**Lemma 5.1.** For any \( c \in (1/2,1) \), there exists some \( C = C(c) > 0 \) such that, for any \( \gamma_n \in \mathcal{X} \) and \( H \in (1/2,1) \),
\[
E|I_{W,H}(\gamma_n)| + E|\hat{I}_{R,H}(\gamma_n)| \leq C\| \gamma_n \|_{L_2}.
\]
Proof of Lemma 5.1. For \( k = 1, \ldots, n \), we have that

\[
I_{W,H,k} = d_H \int_{T_{k-1}}^{T_k} \gamma'_n(t) dt \int_{T_{k-1}}^{t} (t - \tau)^{-3/2} dB(\tau)
\]

\[
= d_H \int_{T_{k-1}}^{T_k} dB(\tau) \int_{\tau}^{T_k} (t - \tau)^{-3/2} \gamma'_n(t) dt = \int_{T_{k-1}}^{T_k} dB(\tau) \Gamma_{k-1}(\tau, \gamma_n).
\]

The last integral here converges in \( L_2(\Omega, \mathcal{G}_T, P) \). Hence

\[
E \| I_{W,H}(\gamma_n) \|_{L_2(\Omega)}^2 = E \left( \sum_{k=1}^{n} I_{W,H,k} \right)^2 = \sum_{k=1}^{n} E I_{W,H,k}^2 = E \sum_{k=1}^{n} \Gamma_{k-1}(\tau, \gamma_n)^2 d\tau
\]

\[
\leq \delta \sum_{k=1}^{n} \int_{T_{k-1}}^{T_k} \gamma_n(\tau)^2 d\tau = \delta \| \gamma_n \|_{L_2}^2.
\]

Further, we have that

\[
E|\tilde{I}_{R,H}(\gamma_n)| \leq \left( E \int_{0}^{T} \gamma'_n(t)^2 dt \right)^{1/2} \left( E \int_{0}^{T} \tilde{\rho}(t)^2 dt \right)^{1/2}.
\]

By (3.4), \( E \int_{0}^{T} \tilde{\rho}(t)^2 dt \leq \frac{d_0}{2(2-2H)} T^{2H-1} \). This completes the proof of Lemma 5.1.

The following proofs will be given for Theorem 3.1 and 4.1 simultaneously with the proof of Proposition 3.6.

For the sake of the proofs of Theorem 3.1 and 4.1, we assume below that that \( r = 2, \ p = 2, \ \gamma \in \mathcal{V}_{\nu,e} \) and \( \{\gamma_n\}_{n=1}^{\infty} = \mathcal{Z}(\gamma) \). For the sake of the proof of Proposition 3.6, we assume below that \( r \in (1, 2), \ p = (1 - 1/r)^{-1}, \ \gamma \in \mathcal{H}_{\nu,r} \) and \( \{\gamma_n\}_{n=1}^{\infty} = \mathcal{Z}(\gamma) \).

We consider below positive integers \( n, m \to +\infty \) such that \( n \geq m \). We assume below that \( T_k = kT/2^n, \ k = 0, 1, \ldots, 2^n \). This means that the grid \( \{T_k\}_{k=0}^{2^n} \) is formed as defined for \( n \) rather than for \( m \); since \( n \geq m \), Definition 3.3 is applicable to the integral \( \int_{0}^{T} \gamma_m(t) dt F_H(t) \) with this grid as well.

We denote

\[ \epsilon_m \Delta T/2^n = T_{k+1} - T_k, \ \epsilon_n \Delta T/2^n = T_{k+1} - T_k. \]

We assume that \( m \) is such that \( \epsilon_m \leq \epsilon \). It implies that \( \epsilon_n \leq \epsilon \) as well.

We denote by \( I_{R,k,n} \) and \( J_{R,k,n} \) the corresponding values \( J_{R,H,k} \) defined for \( \gamma = \gamma_n \) and \( \gamma = \gamma_m \) respectively obtained using the same grid \( \{T_k\}_{k=0}^{2^n} \).

Lemma 5.2. The sequence \( \{I_{R,H}(\gamma_n)\}_{n=1}^{\infty} \) has a limit in \( L_1(\Omega, \mathcal{G}_T, P) \); it converges to this limit uniformly in \( H \in (1/2, c) \), for any \( c \in (1/2, 1) \).

Proof of Lemma 5.2. Clearly,

\[
\| \gamma_n - \gamma_m \|_{L_2}^2 \to 0 \quad \text{as} \quad n, m \to +\infty
\]

and

\[
\| \gamma_n - \gamma_m \|_{L_2}^2 \to 0 \quad \text{as} \quad m \to +\infty \quad \text{uniformly in} \ n > m.
\]
By Lemma 5.1, we have that
\[
E\left[|W,H(\gamma_n) - W,H(\gamma_m)|^2\right]_{L_2(\Omega)} + E\left[\hat{J}_{R,H}(\gamma_n) - \hat{J}_{R,H}(\gamma_m)\right] \to 0 \text{ as } b, m \to +\infty.
\]
This implies that the sequences \(\{W,H(\gamma_n)\}_{n=1}^{\infty}\) and \(\{\hat{J}_{R,H}(\gamma_n)\}_{n=1}^{\infty}\) have limits in \(L_1(\Omega, G_T, P)\), and that they converge to these limits uniformly in \(H \in (1/2, c)\), for any \(c \in (1/2, 1)\).

Therefore, to prove Lemma 5.2, it suffices to prove that the sequence \(\{\hat{J}_{R,H}(\gamma_n)\}_{n=1}^{\infty}\) have a limit in \(L_1(\Omega, G_T, P)\) as well, and that it converges to this limit uniformly in \(H \in (1/2, c)\), for any \(c \in (1/2, 1)\).

Let
\[
\xi_k(t) \overset{\Delta}{=} \rho(t, T_k) = d_H \int_0^{T_k} (t - s)^{H-3/2} dB(s).
\]
We have that
\[
\psi_{n,m,k} \overset{\Delta}{=} J_{R,k,n} - J_{R,k,m} = \int_{T_k}^{T_{k+1}} [\gamma_n(t) - \gamma_m(t)]\xi_k(t) dt,
\]
Remind that \(p > 0\) is such that \(1/p + 1/r = 1\). We have that
\[
\|\psi_{n,m,k}\|_{L_1(\Omega)} \leq \int_{T_k}^{T_{k+1}} \|\gamma_n(t) - \gamma_m(t)\|_{L_1(\Omega)} \|\xi_k(t)\|_{L_p(\Omega)} dt.
\]
Further, we have that
\[
\|\xi_k(t)\|^2_{L_2(\Omega)} = d_H^2 \int_0^{T_0} (t - s)^{2H-3} ds = \frac{d_H^2}{2H - 2} \left[ (t - T_k)^{2H - 2} - t^{2H - 2} \right]
= \frac{d_H^2}{2 - 2H} \left[ t^{2H - 2} - (t - T_k)^{2H - 2} \right], \quad t \in (T_k, T_{k+1}].
\]
Hence
\[
\int_{T_k}^{T_{k+1}} \|\xi_k(t)\|^2_{L_2(\Omega)} dt = \frac{d_H^2}{2 - 2H}(2H - 1) \left[ (T_{k+1} - T_k)^{2H - 1} - T_k^{2H - 1} + T_{k+1}^{2H - 1} \right]
= \frac{c_H d_H}{4 - 4H} \left[ (T_{k+1} - T_k)^{2H - 1} - T_k^{2H - 1} + T_{k+1}^{2H - 1} \right].
\]
Hence
\[
\left( \int_{T_k}^{T_{k+1}} \|\xi_k(t)\|^2_{L_2(\Omega)} dt \right)^{1/2} \leq \bar{C}_0 C_{H,n}^{H - 1/2}, \quad (5.3)
\]
where
\[
C_H \overset{\Delta}{=} \frac{\sqrt{c_H d_H}}{2 - 2H},
\]
and where \(\bar{C}_0 > 0\) is independent on \(\gamma\), \(k\) and \(H\); it depends on \(T\) only.

By the properties of Gaussian distributions, we have that
\[
\|\xi_k(t)\|_{L_p(\Omega)} \leq C(p) \|\xi_k(t)\|_{L_2(\Omega)}
\]
for some $C(p) > 0$. Hence
\[
\int_{T_k}^{T_{k+1}} \| \xi_k(t) \|_{L_p(\Omega)} dt \leq C(p) \int_{T_k}^{T_{k+1}} \| \chi_k(t) \|_{L_2(\Omega)} dt \leq C(p) \left( \int_{T_k}^{T_{k+1}} \| \xi_k(t) \|_{L_2(\Omega)}^2 dt \right)^{1/2} \varepsilon_n^{1/2} \\
\leq C(p) \tilde{C}_0 \varepsilon_n^{H-1/2} \varepsilon_n^{1/2} = C(p) \tilde{C}_0 \varepsilon_n^H.
\]

Let
\[
T_d^{(m)} \overset{\Delta}{=} \varepsilon_m d, \quad d = 0, 1, ..., 2^m,
\]

Here $\varepsilon_m = T/2^m$. Let
\[
\tau_m(t) = \inf \{ T_d^{(m)} : t \in [T_d^{(m)}, T_{d+1}^{(m)}], d = 0, 1, ..., 2^m - 1 \}.
\]

Clearly, the function $\tau_m(t)$ is non-decreasing, and $\tau_m(t) \leq \tau_n(t)$.

By the definitions of $\gamma_m$ and $\tau_m$, we have that $\gamma_m(t) = E_{\tau_m(t)} \gamma(t) = E_{\tau_m(t)} \gamma_n(t)$ and $\gamma_n(t) = E_{\tau_n(t)} \gamma(t)$. Hence
\[
\| \gamma_n(t) - \gamma_m(t) \|_{L_2(\Omega)} = \| \gamma_n(t) - E_{\tau_m(t)} \gamma_n(t) \|_{L_2(\Omega)} \leq \| \gamma(t) - E_{\tau_m(t)} \gamma(t) \|_{L_2(\Omega)}.
\]

For the sake of the proof of Theorem 3.1, we have assumed that $\gamma \in \mathcal{Y}_{\nu, \varepsilon}$. It follows that
\[
\sup_{t \in [0, T]} \| \gamma_m(t) - \gamma_n(t) \|_{L_2(\Omega)} \leq \sup_{t \in [0, T]} (E \text{Var } \tau_m(t) \gamma(t))^{1/2} \leq c \varepsilon_m^{1-H+\nu} \| \gamma_m \|_{\mathcal{Y}_{\nu, \varepsilon}}, \tag{5.4}
\]

where $c > 0$ are independent on $\gamma$ and $H \in (1/2, 1)$.

Let $n = m + 1$. In this case, we have that $\varepsilon_m = 2 \varepsilon_n$. By (5.3) and (5.4), we have that
\[
\| \psi_{k,m+1} \|_{L_1(\Omega)} \leq \int_{T_k}^{T_{k+1}} \| \gamma_{m+1}(t) - \gamma_m(t) \|_{L_2(\Omega)} \| \xi_k(t) \|_{L_p(\Omega)} dt \\
\leq \sup_{t \in [0, T]} \| \gamma_{m+1}(t) - \gamma_m(t) \|_{L_2(\Omega)} \int_{T_k}^{T_{k+1}} \| \xi_k(t) \|_{L_p(\Omega)} dt \\
\leq \int_{T_k}^{T_{k+1}} \| \gamma_{m+1}(t) - \gamma_m(t) \|_{L_2(\Omega)} \| \xi_k(t) \|_{L_p(\Omega)} dt \\
\leq c_\phi C_H \varepsilon_m^{1+\nu} \| \gamma_m \|_{\mathcal{Y}_{\nu, \varepsilon}},
\]

where $c_\phi > 0$ is independent on $\gamma$, $k$, and $H \in (1/2, 1)$. We have that $2^n = 2^{m+1} = 2T/\varepsilon_m$. Hence
\[
\| J_{R,H}(\gamma_{m+1}) - J_{R,H}(\gamma_m) \|_{L_1(\Omega)} \leq \sum_{k=0}^{2^{n-1}} \| \psi_{k,n,m} \|_{L_1(\Omega)} \leq 2^n c_\phi C_H \varepsilon_m^{1+\nu} \| \gamma \|_{\mathcal{Y}_{\nu, \varepsilon}} \tag{5.5}
\]
\[
= 2T \varepsilon_m^{-1} c_\phi C_H \varepsilon_m^{1+\nu} \| \gamma \|_{\mathcal{Y}_{\nu, \varepsilon}} = c_f C_H (2^{-m})^\nu \| \gamma \|_{\mathcal{Y}_{\nu, \varepsilon}},
\]

where $c_f > 0$ is independent on $m$, $\gamma$, $H$, and $\nu \geq 0$. 

Further, let \( m \in \{1, 2, \ldots, n\} \). Clearly,

\[
\overline{J}_{R, H}(\gamma_n) - \overline{J}_{R, H}(\gamma_m) = \sum_{k=m+1}^{n} (\overline{J}_{R, H}(\gamma_k) - \overline{J}_{R, H}(\gamma_{k-1})).
\]

(5.6)

It follows that

\[
\|\overline{J}_{R, H}(\gamma_n) - \overline{J}_{R, H}(\gamma_m)\|_{L^1(\Omega)} \leq c_J C_H \sum_{k=m+1}^{n} (2^{-k})^\nu \|\gamma\|_{\mathcal{Y}_{\nu, \varepsilon}} \to 0
\]

as \( m \to +\infty \)

(5.7)

uniformly in \( n > m \) and in the case where \( \nu > 0 \), uniformly in \( H \in (1/2, c) \), for any \( c \in (1/2, 1) \). Hence \( \{\overline{J}_{R, H}(\gamma_n)\} \) is a Cauchy sequence in \( L_q(\Omega, \mathcal{F}, \mathbb{P}) \), and has a limit in this space, uniformly in \( H \in (1/2, c) \), for any \( c \in (1/2, 1) \).

For the sake of the proof of Proposition 3.6, we use, instead of (5.4), the estimates

\[
\sup_t \|\gamma_n(t) - \gamma_m(t)\|_{L^1(\Omega)} = \sup_{k \in \{0, \ldots, 2^{k-1}\}} \sup_{t \in [T, T_1]} \|\gamma_m(t) - \gamma_n(t)\|_{L^1(\Omega)} \leq c_m^{1-H+\nu} \|\gamma\|_{\mathcal{H}_{\nu, \varepsilon}}.
\]

Then the proof above can be repeated with minor changes. In particular, the corresponding constant \( c_J \) depends on \( r \).

This completes the proof of Lemma 5.2.

\[ \square \]

**Proof of Theorem 3.1.** It follows immediately from Lemma 5.2 that the sequence \( \{I_H(\gamma_n)\}_{n=1}^{\infty} \) converges to a limit in \( L_1(\Omega, G_T, \mathbb{P}) \), uniformly in \( H \in (1/2, c) \), for any \( c \in (1/2, 1) \). This proves statement (i) of Theorem 3.1.

Let us prove statement (ii) of Theorem 3.1. It follows from Lemma 5.1 that the operators \( I_{W, H}(\cdot) : \mathcal{X} \to L_1(\Omega, G_T, \mathbb{P}) \) and \( I_{R, H}(\cdot) : \mathcal{X} \to L_1(\Omega, G_T, \mathbb{P}) \) allow continuous extension into continuous operators \( I_{W, H}(\cdot) : \mathcal{L}_{22} \to L_1(\Omega, G_T, \mathbb{P}) \) and \( I_{R, H}(\cdot) : \mathcal{L}_{22} \to L_1(\Omega, G_T, \mathbb{P}) \), that are bonded uniformly in \( H \in (1/2, c) \), for any \( c \in (1/2, 1) \).

It suffices to show that, for any \( \nu > 0 \) and \( \varepsilon > 0 \),

\[
\sup_{n \geq 0} \mathbb{E}|\overline{J}_{R, H}(\gamma_n)| \leq \overline{C} \|\gamma\|_{\mathcal{Y}_{\nu, \varepsilon}}
\]

for some \( \overline{C} = \overline{C}(\varepsilon, \nu) > 0 \).

Assume that \( \gamma \in \mathcal{Y}_{\nu, \varepsilon} \) for some \( \varepsilon > 0 \). Let

\[
m_\varepsilon \overset{\Delta}{=} \min \{m : 2^{-m} T \leq \varepsilon\}.
\]

(5.8)

It follows from (5.6) that, for all \( n > m_\varepsilon \),

\[
\mathbb{E}|\overline{J}_{R, H}(\gamma_n)| \leq \|\overline{J}_{R, H}(\gamma_m)\|_{L^1(\Omega)} + c_J C_H \sum_{k=m+1}^{n} (2^{-k})^\nu \|\gamma\|_{\mathcal{Y}_{\nu, \varepsilon}}
\]

\[
\leq \|\overline{J}_{R, H}(\gamma_m)\|_{L^1(\Omega)} + \overline{C}_{H, \nu, m_\varepsilon} \|\gamma\|_{\mathcal{Y}_{\nu, \varepsilon}},
\]

(5.9)

where \( c_J \) is the same as in (5.5), and where

\[
\overline{C}_{H, \nu, m_\varepsilon} \overset{\Delta}{=} c_J C_H \sum_{k=2^{m_\varepsilon+1}}^{\infty} (2^{-k})^\nu.
\]
Clearly, $C_{H,v,m}$ is independent on $\gamma \in \mathcal{Y}_{v,c}$ and, for any $c \in (1/2,1)$, $C_{H,v,m}$ is bounded by a constant for all $H \in (1/2,c), \varepsilon > 0$.

Further, let

$$\xi_k^{(m_\varepsilon)}(t) = \rho(t, T_k^{(m_\varepsilon)}) = d_H \int_0^{T_k^{(m_\varepsilon)}} (t - s)^{H-3/2} dB(s).$$

Let $M_\varepsilon = C_{0}^2 C_{H}^2 t^{2H-1}$ and

$$a_k = \int_{T_k^{(m_\varepsilon)}}^{T} \| \gamma_{m_\varepsilon}(t) \|^2_{L^2(\Omega)} dt, \quad b_k = \int_{T_k^{(m_\varepsilon)}}^{T} \| \xi_k^{(m_\varepsilon)}(t) \|^2_{L^2(\Omega)} dt.$$

Clearly,

$$\sum_{k=1}^{n} a_k = \| \gamma_{m_\varepsilon}(t) \|^2_{L^2(\Omega)} dt \leq \int_0^{T} \| \gamma(t) \|^2_{L^2(\Omega)} dt \leq \| \gamma \|^2_{Y_{v,c}}.$$

As was shown for $\xi_k(t)$ in (5.3), we have that $b_k \leq M_\varepsilon$ for all $k$.

We have that, for any $c \in (1/2,1),

$$E|J_{R,H}(\gamma_{m_\varepsilon})| \leq \sum_{k=1}^{2m_\varepsilon} \int_{T_k^{(m_\varepsilon)}}^{T} \| \gamma_{m_\varepsilon}(t) \|_{L^2(\Omega)} \| \xi_k^{(m_\varepsilon)}(t) \|_{L^2(\Omega)} dt \leq \sum_{k=1}^{2m_\varepsilon} a_k^{1/2} b_k^{1/2}$$

$$\leq \left( \sum_{k=1}^{2m_\varepsilon} a_k \right)^{1/2} \left( \sum_{k=1}^{2m_\varepsilon} b_k \right)^{1/2} \leq M_\varepsilon^{1/2} \cdot 2^{m_\varepsilon/2} \| \gamma \|_{Y_{v,c}} \leq \hat{C} \| \gamma \|_{Y_{v,c}}$$

(5.10)

for some $\hat{C} = \hat{C}(c,m_\varepsilon) > 0$. We have used here the H"older’s inequality.

It can be noted that the value $m_\varepsilon$ in (5.10) is not increasing, since $\varepsilon > 0$ is fixed.

By the definitions, $\gamma_0(t)$ is $G_0$-measurable. By the second estimate in Proposition 3.3,

$$E|J_{R,H}(\gamma_0)| \leq \hat{C}_0 \| \gamma \|_{L^2} \leq \hat{C}_0 \| \gamma \|_{Y_{v,c}}$$

for some $\hat{C}_0 = \hat{C}_0(c)$.

The proof of Theorem 3.1(ii) follows from (5.9) and (5.10). This completes the proof of Theorem 3.1.

Proof of Proposition 3.6 repeats the proof of Theorem 3.1, given the adjustments mentioned in the proof of Lemma 5.2.

The remaining part of the paper is devoted to the proof of Theorem 4.1. We will use the notation from the proof of Theorem 3.1 with the following amendment: since we consider variable $H \in [1/2,1)$, we include corresponding $H$ as an index for a variable.

In particular, it follows from the notation that

$$I_{W,H}(\gamma_n) = \sum_{k=1}^{n} P_{W,H,k} + I_{1/2}(\gamma_n),$$
We have that
\[ d_H = (H - 1/2)c_H \to 0, \quad C_H = \frac{\sqrt{c_H d_H}}{2 - 2H} \to 0 \quad \text{as} \quad H \to 1/2 + 0. \]

**Lemma 5.3.** For any \( \gamma_n \in \mathcal{X}_c \),
\[ ||I_{W,H}(\gamma_n) - I_{1/2}(\gamma_n)||_{L^2(\Omega)} + ||\hat{I}_{R,H}(\gamma_n)||_{L^1(\Omega)} \to 0 \quad \text{as} \quad H \to 1/2 + 0 \]
uniformly over any bounded in \( L^2 \) set of \( \gamma_n \in \mathcal{X}_c \).

**Proof of Lemma 5.3.** For the operators \( \Gamma_k(\cdot, \cdot) = G_{H}(\cdot, T_{k+1}, \cdot) \) introduced before Lemma 5.1, we have that \( ||\Gamma_k(\cdot, g)||_{L^2(T_k, T_{k+1})} \leq \hat{c}||g||_{L^2(T_k, T_{k+1})} \) for some \( \hat{c} > 0 \) that is independent on \( H \in (1/2, 1) \). Similarly to the proof of Lemma 5.1, we have that
\[ P_{W,H,k} = \int_{T_{k-1}}^{T_k} dB(\tau) [\Gamma_{k-1}(\tau, \gamma_n) - \gamma_n(\tau)], \quad k = 1, \ldots, n. \]

These integrals converge in \( L^2(\Omega, G_T, P) \).

Let
\[ \alpha_{H,k} = \frac{\Delta}{T_{k-1}} \int_{T_{k-1}}^{T_k} |\Gamma_{k-1}(\tau, \gamma_n) - \gamma_n(\tau)|^2 d\tau. \]

We have that \( E\alpha_{H,k} = E\alpha_{W,H,k}^2 \) and
\[ E||I_{W,H}(\gamma_n) - I_{W,1/2}(\gamma_n)||_{L^2(\Omega)}^2 = E\left( \sum_{k=1}^{n} P_{W,H,k} \right)^2 = E\sum_{k=1}^{n} \alpha_{H,k}. \]

By the properties of the Riemann–Liouville integral, we have that
\[ ||\gamma_n - \Gamma_k(\cdot, \gamma_n)||_{L^2(T_k, T_{k+1})} \to 0 \quad \text{a.s. as} \quad H \to 1/2 + 0 \]
and
\[ \|\Gamma_k(\cdot, \gamma_n)\|_{L^2(T_{k-1}, T_k)} \leq ||\gamma_n||_{L^2(T_k, T_{k+1})} \quad \text{a.s.} \]
Hence \( 0 \leq \alpha_{H,k} \leq \sqrt{2}||\gamma_n||_{L^2(T_{k-1}, T_k)} \) a.s. By Lebesgue’s Dominated convergence Theorem, it follows that
\[ E\sum_{k=1}^{n} \alpha_{H,k} \to 0 \quad \text{a.s. as} \quad H \to 1/2 + 0. \]

Hence
\[ E||I_{W,H}(\gamma_n) - I_{W,1/2}(\gamma_n)||_{L^2(\Omega)}^2 \to 0 \quad \text{as} \quad H \to 1/2 + 0. \]

Further, we have that
\[ E|\hat{I}_{R,H}(\gamma_n)| \leq \left( E \int_{0}^{T} \gamma_n(t)^2 dt \right)^{1/2} \left( E \int_{0}^{T} \hat{\rho}(t)^2 dt \right)^{1/2}. \]

Similarly to the proof of Proposition 3.2, we obtain that
\[ E\hat{\rho}(t)^2 = \int_{-\infty}^{t} |f_i(t, r)|^2 dr = \frac{d_H^2}{2 - 2H} \frac{1}{2H - 2}. \]

and
\[
E \int_0^T \rho(t)^2 dt = \frac{d_H^2}{2(2 - 2H)} T^{2H - 1} = \frac{c_H d_H}{4} T^{2H - 1} \to 0 \quad \text{as} \quad H \to 1/2 + 0.
\]

This completes the proof of Lemma 5.3. \hfill \Box

**Lemma 5.4.** Let \( \nu > 0, \gamma \in \mathcal{Y}_\nu, \varepsilon \), and \( \{\gamma_n\}_{n=1}^\infty = \mathcal{Z}(\gamma) \). In the notation introduced above, we have that

\[
\|J_{R,H}(\gamma_n)\|_{L^1(\Omega)} \to 0 \quad \text{as} \quad H \to 1/2 + 0
\]

uniformly in \( n > 0 \).

**Proof of Lemma 5.4.** Assume that \( \gamma \in \mathcal{Y}_{\nu,\varepsilon} \) for some \( \varepsilon > 0 \), and that \( m_\varepsilon \) is defined by (5.8). It follows from equation (5.6) applied to \( J_R = J_{R,H} \) that, for any and any \( n > m_\varepsilon \),

\[
E|J_{R,H}(\gamma_n)| \leq \|J_{R,H}(\gamma_{m_\varepsilon})\|_{L^1(\Omega)} + c_j c_H \sum_{k=m_\varepsilon+1}^n (2^{-k})^{\nu/2+H-1/2} \|\gamma\|_{Y_{\nu,\varepsilon}}
\]

\[
\leq \|J_{R,H}(\gamma_{m_\varepsilon})\|_{L^1(\Omega)} + C_{H,\nu,m_\varepsilon} \|\gamma\|_{Y_{\nu,\varepsilon}},
\]

where \( C_{H,\nu,m_\varepsilon} \) is the same as in (5.9); if \( \nu > 0 \), then \( C_{H,\nu,m_\varepsilon} \) is bounded by a constant for all \( H \in (1/2,1), \varepsilon > 0 \). In addition, we have that \( C_{H,\nu,m_\varepsilon} \to 0 \) as \( H \to 1/2+ \) uniformly in \( n \). By (5), \( \|J_{R,H}(\gamma_{m_\varepsilon})\|_{L^1(\Omega)} \to 0 \) as \( H \to 1/2 + . \) This completes the proof of Lemma 5.4. \hfill \Box

**Proof of Theorem 4.1.** Let \( \gamma \in \mathcal{Y}_{\nu,\varepsilon} \) for any \( \nu > 0 \) and \( \varepsilon > 0 \). Let \( \gamma_n = \mathcal{Z}(\gamma) \). We have to show that \( E|I_H(\gamma) - I_{1/2}(\gamma)| \to 0 \) as \( H \to 1/2 + . \) We have that

\[
E|I_H(\gamma) - I_{1/2}(\gamma)| \leq A_{1,H,n} + A_{2,H,n} + A_{3,n}
\]

where

\[
A_{1,H,n} \triangleq E|I_H(\gamma) - I_H(\gamma_n)|, \quad A_{2,H,n} \triangleq E|I_H(\gamma_n) - I_{1/2}(\gamma_n)|, \quad A_{3,n} \triangleq E|I_{1/2}(\gamma_n) - I_{1/2}(\gamma)|.
\]

Clearly, \( \|\gamma - \gamma_n\|_{Y_{\nu,\varepsilon}} \to 0 \) as \( n \to +\infty \) for any \( \varepsilon > 0 \).

Let \( c \in (1/2,1) \) be given. By Theorem 3.1, \( A_{1,H,n} \to 0 \) as \( n \to +\infty \) uniformly in \( H \in (1/2,c) \). By Lemmata 5.3-5.4, \( A_{2,H,n} \to 0 \) as \( H \to 1/2+ \) uniformly in \( n \). Finally, by the properties of the Itô integral, it follows that \( A_{3,n} \to 0 \) as \( n \to +\infty \). This completes the proof of Theorem 4.1. \hfill \Box

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