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A dynamical decomposition of the torus into pseudo-circles

François Béguin, Sylvain Crovisier, Tobias Jäger

September 10, 2018

To the memory of Dmitri V. Anosov.

Abstract

We build an irrational pseudo-rotation of the 2-torus which is semiconjugate to an irrational rotation of the circle in such a way that all the fibres of the semi-conjugacy are pseudo-circles. The proof uses the well-known ‘fast-approximation method’ introduced by Anosov and Katok.

1 Introduction

It is well known that continua (connected compact metric spaces) with complicated structure naturally appear in smooth surface dynamics. A striking example is provided by the pseudo-circle, introduced by Bing [Bi1] and characterized by Fearnley [Fe]. It is a continuum which:

- can be embedded in $S^2$ and separates,
- is circularly chainable: it admits coverings into compact subsets $(A_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$ whose diameter are arbitrarily small, such that $A_i \cap A_j \neq \emptyset$ if and only if $i = j \pm 1$ or $i = j$,
- is indecomposable: it cannot be written as the union of two proper continua,
- and whose non-trivial proper subcontinua are indecomposable, homogeneous (any point can be sent on any other point by some homeomorphism) and all homeomorphic to the same topological space (called the pseudo-arc).

Handel [Ha] has built a smooth diffeomorphism of $S^2$ preserving a minimal invariant set homeomorphic to the pseudo-circle. Later, Prajs [Pr] has constructed a partition of the annulus into pseudo-arcs, and likewise his method could be used to produce partitions of the torus into pseudo-circles. It was not known, however, if such a pathological foliation could be ‘dynamical’, that is, invariant under the dynamics of a torus homeomorphism or diffeomorphism that permutes the leaves of the foliation. Conversely, if a homeomorphism of the two-torus is semiconjugate to an irrational rotation of the circle, one may wonder whether most, or at least some, of the fibres of the semi-conjugacy must have a simple structure or even be topological circles. We give a positive answer to the first and a negative to the second of these questions. Denote by $T^d = \mathbb{R}^d/\mathbb{Z}^d$ the $d$-dimensional torus and by $\text{Diff}^\infty_{\text{vol},0}(T^2)$ the space of real-analytic diffeomorphisms of $T^2$ that are isotopic to the identity and preserve the canonical volume.
Theorem. There exists a minimal diffeomorphism \( f \in \text{Diff}^\omega_{\text{vol},0}(\mathbb{T}^2) \) whose rotation set is reduced to a unique totally irrational vector and which preserves a partition \( C \) of \( \mathbb{T}^2 \) into pseudo-circles.

Moreover there exists a continuous map \( p: \mathbb{T}^2 \to \mathbb{T}^1 \) which semi-conjugates \( f \) to an irrational rotation. The elements of \( C \) are the pre-images \( p^{-1}(x) \).

This result has implications for a number questions that naturally come up in the rotation theory on the torus and, more specifically, the dynamics of irrational pseudo-rotations. We discuss these issues in more detail in Section \[4\] alongside with the uniqueness of the semi-conjugacy.

Idea of the construction. The diffeomorphism \( f \) is obtained as limit of a sequence of diffeomorphisms \( f_n \) that are conjugated to rational rotations \( R_{\alpha_n} \) by diffeomorphisms \( H_n \) isotopic to the identity, following the celebrated Anosov-Katok method, see [FK1]. As a side effect, this means that its dynamics can be made uniquely ergodic, although we will not expand on this. Note that most of the constructions using this method deal with the \( C^\infty \) category; some cases, as [FK2] allow to work in the real-analytic category. The sequence \( f_n = H_n^{-1} \circ R_{\alpha_n} \circ H_n \) is obtained inductively. At stage \( n \), the diffeomorphism \( f_n \) preserves the foliation \( V_n \) by vertical circles \( H_n^{-1}(\{x\} \times \mathbb{T}^1) \). The main requirement is to have the circles of the foliation \( V_{n+1} \) arbitrarily close to the circles of \( V_n \) in the Hausdorff topology, but more crooked. Then the partitions \( V_n \) will converge to the partition into pseudo-circles. The foliation \( V_{n+1} \) is built inductively in the chart defined by the conjugacy \( H_n \), as the preimage of the foliation into vertical lines under a new homeomorphism \( h_{n+1} \). A sketch of the construction is given in Figure [1]. Since we then let \( H_{n+1} = h_{n+1} \circ H_n \), this ensures that the foliation \( H_{n+1}(V_{n+1}) \) is again the one given by vertical circles. In order to obtain \( h_{n+1} \), one first builds a leaf of \( V_{n+1} \) in the chart given by \( V_n \) which is crooked with respect to the vertical circles in such a way that the leaf is transverse to a linear flow \( \varphi_{n+1} \). This first leaf is the image of some vertical circle. One then obtains the complete foliation (and thus the definition of \( h_{n+1} \)) by pushing this initial leaf by the flow.

Figure 1: In the coordinates defined by \( H_n \) (given by the dashed lines), \( h_{n+1}^{-1} \) maps straight lines (solid lines on the left) to crooked leaves of \( V_{n+1} \) (on the right). The latter are translates of each other along the flow lines of \( \varphi_{n+1} \).

We note that A. Avila has announced recently the construction of an element of \( \text{Diff}^\omega_{\text{vol},0}(\mathbb{T}^2) \) whose rotation set is a non-trivial compact interval contained in line with irrational slope and
which does not contain any rational point (a counter-example to one case of a conjecture by Franks and Misuincrewicz [FM]). His construction may somewhat be compared to ours: the diffeomorphism is obtained as the limit of diffeomorphisms acting periodically on the leaves of a foliation by circles. In his case however the homotopy class of the leaves is modified at each stage of the construction.

2 A criterion for the existence of a partition into pseudo-circles

2.a – Crooking. The construction of the pseudo-circle uses the following notions.

Definitions. A **circular chain** is finite family of sets $\mathcal{D} = \{D_\ell, \ell \in \mathbb{Z}/N\mathbb{Z}\}$ such that $D_k$ intersects $D_\ell$ if and only if $k - \ell \in \{-1, 0, +1\}$. A circular chain $\mathcal{D}' = \{D'_\ell, i \in \mathbb{Z}/N'\mathbb{Z}\}$ said to be **crooked inside** another circular chain $\mathcal{D} = \{D_\ell, \ell \in \mathbb{Z}/N\mathbb{Z}\}$ if there exists a map $\ell: \mathbb{Z}/N'\mathbb{Z} \to \mathbb{Z}/N\mathbb{Z}$ with the following properties.

- $D'_i \subset D_{\ell(i)}$ for each $i \in \mathbb{Z}$;
- if $i < j$ are such that for all $i < k < j$ the element $\ell(k)$ belongs to the same interval bounded by $\ell(i)$ and $\ell(j)$ (either positively or negatively oriented in $\mathbb{Z}/N\mathbb{Z}$) and the length of this interval is greater than 4, then there exists $u, v$ with $i < u < v < j$ such that $d(\ell(u), \ell(j)) \leq 1$ and $d(\ell(v), \ell(i)) \leq 1$. (Here $d$ denotes the canonical distance on $\mathbb{Z}/N\mathbb{Z}$.)

The pseudo-circle can then be obtained as follows. For the sake of consistency with the later sections, we work in the torus instead of $\mathbb{R}^2$ and require that the circular chains – and thus the resulting pseudo-circle – are homologically non-trivial.

**Theorem ([Bi1, Fe]).** Consider a sequence $(\mathcal{D}_n)_{n \geq 0}$ of circular chains of open topological disks in $\mathbb{T}^2$. Assume that

- $\mathcal{D}_{n+1}$ is crooked inside $\mathcal{D}_n$ for each $n$,
- the closure of $\bigcup_{D \in \mathcal{D}_{n+1}} D$ is contained in $\bigcup_{D \in \mathcal{D}_n} D$ for every $n$,
- the maximal diameter of the elements of $\mathcal{D}_n$ goes to zero as $n \to +\infty$.
- the union $\bigcup_{D \in \mathcal{D}_n} D$ contains homotopically non-trivial loops of a unique homotopy type $v \in \mathbb{Z}^2 \setminus \{0\}$.

Then the compact set $X := \bigcap_{n \geq 0} \bigcup_{D \in \mathcal{D}_n} D$ is homeomorphic to the pseudo-circle. Moreover, $X$ is an annular continuum of homotopy type $v$ (see [JP, JT]).

At some point later on, we will have to speak about lifts of circular chains in the torus to the universal covering $\mathbb{R}^2$, and similarly about lifts of circular chains of intervals in the circle to $\mathbb{R}$. Suppose that $\mathcal{D}$ is a circular chain of topological disks in $\mathbb{T}^2$ as above and denote by $\pi: \mathbb{R}^2 \to \mathbb{T}^2$ the canonical projection. Note that for each $D_\ell \in \mathcal{D}$, the preimage $\pi^{-1}(D)$ consists of a countable number of connected components, each of which is a topological disk homeomorphic to $D_\ell$ and disjoint from all its integer translates. Suppose in addition that $\text{Diam}(D_\ell) < 1/4$ for all $\ell \in \mathbb{Z}/N\mathbb{Z}$, so that none of the unions $D_\ell \cup D_{\ell+1}$ is essential in the torus (contains a homotopically non-trivial loop).
Definitions. A lift $\tilde{D}$ of $D$ is a sequence of topological disks $(\tilde{D}_\ell)_{\ell \in \mathbb{Z}}$ of $\mathbb{R}^2$ such that

- for all $\ell \in \mathbb{Z}$ the disk $\tilde{D}_\ell$ is a connected component of $\pi^{-1}(D_\ell)$;
- $\tilde{D}_k$ intersects $\tilde{D}_\ell$ if and only if $k - \ell \in \{-1, 0, 1\}$.

The disc $\tilde{D}_{k+N}$ is the image of $\tilde{D}_k$ by translation by a vector $v \in \mathbb{Z}^2$ which does not depend on the lift, nor on $k$, and is called the homotopy type of $D$.

Note that if $D$ and $D'$ are circular chains of topological disks with homotopy type $v \in \mathbb{Z}^2 \setminus \{0\}$ as above, $D'$ is crooked inside $D$ and $\tilde{D}, \tilde{D}'$ are lifts in the above sense, then there exists a function $\ell : \mathbb{Z} \to \mathbb{Z}$ (to which we refer as a lift of $\ell : \mathbb{Z}/N'\mathbb{Z} \to \mathbb{Z}/\mathbb{Z}$) such that

- $\tilde{D}_{i+N'} = \tilde{D}_i + v$ and $\tilde{D}_{i+N} = \tilde{D}_i + v$;
- $\hat{\ell}(i + N') = \hat{\ell}(i) + N$;
- $\tilde{D}_i \subseteq \tilde{D}_{\hat{\ell}(i)}$ for all $i \in \mathbb{Z}$;
- if $i < j < i + N$ are such that for all $i < k < j$ the integer $\hat{\ell}(k)$ belongs to the interval bounded by $\hat{\ell}(i)$ and $\hat{\ell}(j)$ and $|\hat{\ell}(j) - \hat{\ell}(i)| > 4$, then there exists $u, v$ with $i < u < v < j$ such that $|\hat{\ell}(u) - \hat{\ell}(j)| < 1$ and $|\hat{\ell}(v) - \hat{\ell}(i)| < 1$.

All these remarks apply in an analogous way to circular chains of intervals in the circle and their lifts to $\mathbb{R}$.

During the construction, we will also use another notion of the crooking.

Definition. For $\varepsilon > 0$, a continuous map $g : I \to \mathbb{R}$ on the interval $I$ is $\varepsilon$-crooked if for any $a < b$ in $I$, there are $a < c < d < b$ such that $|g(d) - g(a)| < \varepsilon$ and $|g(c) - g(b)| < \varepsilon$.

Note that $\varepsilon$-crooked maps exist for any $\varepsilon$ (see [Bi2]).

2.b – Elements of the construction. Let $\pi : \mathbb{T}^2 \to \mathbb{T}^1$ be the projection on the first coordinate. Let $\mathcal{B}(N)$ be the covering of the circle by $N$ open intervals defined as follows

$$\mathcal{B}(N) = \{B_i, i \in \mathbb{Z}/N\mathbb{Z}\} \quad \text{where} \quad B_i = \left(\frac{i - 5/4}{N}, \frac{i + 1/4}{N}\right). \quad (2.1)$$

We will build inductively:

- a sequence of integers $(N_n)_{n \geq 0}$,
- a sequence of positive real numbers $(\varepsilon_n)_{n \geq 0}$,
- a sequence of conjugating diffeomorphisms $(H_n)_{n \geq 0}$ in $\text{Diff}^c_{\text{vol},0}(\mathbb{T}^2)$,
- a sequence of rational rotations $(R_{\alpha_n})_{n \geq 0}$ of $\mathbb{T}^2$.

To $N_n, \varepsilon_n, H_n, R_{\alpha_n}$, we will associate:

- for each $x \in \mathbb{T}^1$, the annulus $A_{n,x}$ which is the image under $H_n^{-1}$ of the vertical annulus $(x - \varepsilon_n, x + \varepsilon_n) \times \mathbb{T}^1$, 


– for each \( x \in T^1 \), the covering \( D_{n,x} \) of the annulus \( A_{n,x} \) defined by

\[
D_{n,x} = \left\{ H_n^{-1}(x - \varepsilon_n, x + \varepsilon_n) \times B_i \mid B_i \in \mathcal{B}(N_n) \right\}
\]

(note that \( D_{n,x} \) is a circular chain with \( N_n \) elements),

– the projection \( p_n = \pi_1 \circ H_n : T^2 \to T^1 \) (where \( \pi_1 : T^2 \to T^1 \) is the projection to the first coordinate),

– the diffeomorphism \( f_n = H_n^{-1} \circ R_{\alpha_n} \circ H_n \in \text{Diff}_{vol, 0}^\omega(T^2) \).

We will denote by \( \left( \frac{c_n}{q_n}, \frac{s_n}{q_n} \right) \) the coordinates of \( \alpha_n \), with \( r_n, s_n \in \mathbb{Z} \) and \( q_n \in \mathbb{N} \setminus \{0\} \).

2.c – Inductive properties. The torus \( T^2 \) is embedded as the subset \( \{(z_1, z_2) \in \mathbb{C}^2, |z_1| = |z_2| = 1\} \) of \( \mathbb{C}^2 \). One will consider homeomorphisms \( f \) of \( T^2 \) such that both \( f \) and \( f^{-1} \) extend as holomorphic functions defined on a neighborhood of \( \Delta = \{(z_1, z_2) \in \mathbb{C}^2, |z_1|, |z_2| \in \left[\frac{1}{2}, 2\right]\} \).

One then introduces the supremum norm \( \| \cdot \|_\Delta \) on \( \Delta \) and the metric

\[
d_0(f, f') = \max(\|f, f'\|_\Delta, \|f^{-1}, f'^{-1}\|_\Delta).
\]

The sequences \( (N_n), (\varepsilon_n), (H_n), (R_{\alpha_n}) \) will be constructed inductively so that the following properties hold.

1. For each \( x \in T^1 \), the circular chain \( D_{n+1,x} \) is crooked inside the circular chain \( D_{n,x} \) (in particular, the annulus \( A_{n+1,x} \) is contained in the interior of the annulus \( A_{n,x} \)), and the supremum of the diameter of the elements of the coverings \( D_{n+1,x} \) is less than \( \frac{1}{n+1} \).

2. The angle \( \alpha_{n+1} \) is close, but not equal, to \( \alpha_n \). More precisely: \( \left| \frac{r_{n+1}}{q_{n+1}} - \frac{r_n}{q_n} \right| \) and \( \left| \frac{s_{n+1}}{q_{n+1}} - \frac{s_n}{q_n} \right| \) are smaller than \( 1/(2^{n+1}q_n) \) and the orbits of \( R_{\alpha_{n+1}} \) are \( \frac{1}{2n+1} \)-dense in \( T^2 \).

3. Every orbit of the diffeomorphism \( f_{n+1} \) is \( \frac{1}{n+1} \)-dense in \( T^2 \).

4. The diffeomorphism \( f_{n+1} \) is (very) close to \( f_n \). More precisely, both \( f_{n+1} \), \( H_n \) and their inverses extend holomorphically on \( (\mathbb{C} \setminus \{0\})^2 \) and satisfy:

   (a) \( d_0(f_{n+1}^i, f_n^i) < \min\left(\frac{1}{2}d_0(f_n, f_n^{-1}), \frac{1}{n}\right) \) for \( i = 1, \ldots, q_n \);  

   (b) \( d_0(f_{n+1}, f_n) < \frac{\eta_n}{2} \) where \( \eta_n \) is chosen such that, for every homeomorphism \( g \) in the ball (for \( d_0 \) centered at \( f_n \) of radius \( \eta_n \)), the rotation set of \( g \) is contained in the ball centered at \( \alpha_n \) of radius \( \frac{1}{n} \).

5. The projection \( p_{n+1} \) is close to \( p_n \) for the \( C^0 \)-topology. More precisely: \( d_0(p_{n+1}, p_n) < \frac{1}{2n} \).

Remarks. The existence of the real number \( \eta_n \) used in property 2(b) is a consequence of the upper semi-continuity of the rotation set \( \rho(F) \) with respect to \( F \) \cite[Corollary 3.7]{MZ}.

Property 4(b) (more precisely, the fact that \( D_{n+1,x} \) is crooked inside \( D_{n,x} \)) implies that the sequence of conjugating diffeomorphisms \( (H_n) \) will necessarily diverge. Nevertheless, the sequence of diffeomorphisms \( (f_n) = (H_n^{-1} \circ R_{\alpha_n} \circ H_n) \) will converge (Property 4). This convergence is obtained by using the well-known ingredients of the Anosov-Katok method:
– one first chooses a conjugating diffeomorphism $H_{n+1}$ of the form $H_{n+1} = h_{n+1} \circ H_n$, where $h_{n+1}$ might be very wild, but commutes with the rotation $R_{\alpha_n}$; this implies that $f_n = H_{n+1}^{-1} \circ R_{\alpha_n} \circ H_{n+1}$;

– then, choosing $\alpha_{n+1}$ close enough to $\alpha_n$ is enough to ensure that $f_{n+1} = H_{n+1}^{-1} \circ R_{\alpha_{n+1}} \circ H_{n+1}$ is close to $f_n = H_{n+1}^{-1} \circ R_{\alpha_n} \circ H_{n+1}$.

A specific point in our construction is that, although the sequence of diffeomorphisms $(H_n)$ will diverge, we require that the sequence of maps $(\pi \circ H_n)$ converges (Property 5). Indeed, we want that the fibers of $\pi_1 \circ H_n$ converge to pseudo-circles “foliating” $T^2$.

2.d – Proof of the theorem. One can easily check that the theorem follows from the inductive properties stated above. Properties [(i)] imply that, for every $x \in T^1$, the sequence of annuli $(A_{n,x})$ decreases and converges in Hausdorff topology to a pseudo-circle $C_x$. Moreover, the collection of pseudo-circles $C = \{C_x\}_{x \in T^1}$ is a partition of $T^2$; this follows from the following fact:

– for every $n$, the collection of annuli $\{A_{n,x}, x \in T^1\}$ covers $T^2$,

– for any $x \neq x'$, the annuli $A_{n,x}$ and $A_{n,x'}$ are disjoint if $n$ is large enough.

Property [(i)] implies that the sequence $(f_n)$ converges to an holomorphic function $f$ on the interior of $\Delta$. The same holds for $(f_n^{-1})$. Consequently, the restriction of $f$ to $T^2$ is a real-analytic diffeomorphism. Since each $f_n$ is volume-preserving, $f$ belongs to \text{Diff}^{\omega}_{\text{vol},0}(T^2).

Given $x \in T^1$, let $x' := x + \pi(\alpha_n)$. Then $f_n(A_{n,x}) = A_{n,x'}$. From property [(i)] one deduces that $A_{n+1,x}$ is mapped by $f_n$ inside $A_{n,x'}$ and $A_{n+1,x'}$ is mapped by $f_n^{-1}$ inside $A_{n,x}$. Hence, the annulus $f(A_{n+1,x})$ is contained in the $d_0(f,f_n)$-neighbourhood of the annulus $A_{n,x'}$ and the annulus $f^{-1}(A_{n+1,x'})$ is contained in the $d_0(f,f_n)$-neighbourhood of the annulus $A_{n,x}$. Since $d_0(f,f_n)$ tends to 0 as $n$ goes to infinity, this implies that $f$ preserves the partition in pseudo-circles $C = \{C_x\}_{x \in T^1}$.

Property [(ii)] implies that the sequence $(\alpha_n)$ converges towards some $\alpha \in \mathbb{R}^2$. It also implies that the $q_n$ first iterates of $R_\alpha$ are $1/2^{n+1}$ close to those of $R_{\alpha_{n+1}}$, hence are $\frac{1}{2^n}$-dense in $T^2$. Consequently $\alpha$ is totally irrational.

Properties [(iii)] imply that the rotation set of $f$ is reduced to $\{\alpha\}$ (indeed, they imply that $d_0(f,f_n) < \eta_n$ and therefore the rotation set of $f$ is contained in the ball of radius $\frac{1}{n}$ centered at $\alpha_n$ for every $n$). Hence $f$ is an irrational pseudo-rotation.

Consider a point $z \in T^2$. For every $n$, according to property [(iii)] the orbit of $z$ under $f_n$ is $\frac{1}{n}$-dense in $T^2$. But the orbit of $z$ under $f_n$ is periodic of period less than $q_n$ (since $f_n$ is conjugate to the rotation $R_{\alpha_n}$). Using property [(i)]a, we obtain that the orbit of $z$ under $f$ remains at distance less than $\frac{\alpha}{n}$ of the orbit of $z$ under $f_n$ for a time $q_n$. Combined with property 3, this means that the orbit of $z$ under $f$ is $\frac{\alpha}{n}$-dense in $T^2$. Since $n$ is arbitrary, $f$ is minimal.

Property [(iv)] implies that the sequence of maps $(p_n)$ converges in topology $C^0$ towards a continuous map $p$. For each $n$, the map $p_n$ semi-conjugates $f_n$ to the rotation of $\mathbb{T}^1$ with angle $\pi(\alpha_n)$. Passing to the limit, it follows that the map $p$ semi-conjugates $f$ to the rotation of angle $\pi(\alpha)$. So we get all the conclusions of the theorem. \qed
3 Inductive construction

Now we explain how to construct a sequence of integers \((N_n)_{n \geq 0}\), a sequence of real numbers \((\varepsilon_n)_{n \geq 0}\), a sequence of conjugating diffeomorphisms \((H_n)_{n \geq 0}\) and a sequence of vectors \((\alpha_n)_{n \geq 0}\), so that properties 1 \ldots 5 are satisfied. We assume that the sequences have already been constructed up to rank \(n\). We will construct \(N_{n+1}, \varepsilon_{n+1}, H_{n+1}, \alpha_{n+1}\).

3.a – Preliminary constructions. Recall that we denote by \(\left(\frac{p_n}{q_n}, \frac{r_n}{q_n}\right)\) the coordinates of \(\alpha_n\). We introduce a periodic linear flow \(\varphi_{n+1} : (t, (x, y)) \mapsto (x, y) + t \cdot \alpha_n + t \cdot (0, p_nb_{n+1})\) on \(\mathbb{T}^2\) where \(b_{n+1}\) is an integer which will be specified below. Observe that the time 1 map of this flow is the rotation \(R_{\alpha_n}\). The first return map of \(\varphi_{n+1}\) on the vertical circle \(\{x\} \times \mathbb{T}^1\) is the time \(\frac{q_n}{p_n}\) map of \(\varphi_{n+1}\). Denote by \(m_n\) the period of this first return map, and observe that \(m_n\) depends on \(\alpha_n\), but does not depend on the choice of the integer \(b_{n+1}\).

We also introduce a \(C^\omega\) map \(\theta_n : \mathbb{R} \to \mathbb{R}\) satisfying the following properties.

- \(x \mapsto \theta_n(m_nx) - m_nx\) is a trigonometric polynomial (hence \(\theta_n - \text{Id}\) is \(1/m_n\)-periodic),
- \(\theta_n(0) = 0, \theta_n\left(\frac{1}{2m_n}\right) = 2, \theta_n\left(\left[0, \frac{1}{2m_n}\right]\right) = [0, 2]\) and \(\theta_n\) is \(\frac{1}{4m_n}\)-crooked on \([0, \frac{1}{2m_n}]\),
- \(\theta_n\left(\frac{1}{m_n}\right) = \frac{1}{m_n}, \theta_n\left(\left[\frac{1}{2m_n}, \frac{1}{m_n}\right]\right) = \left[\frac{1}{m_n}, 2\right]\) and \(\theta_n\) is \(\frac{1}{4m_n}\)-crooked on \([\frac{1}{2m_n}, \frac{1}{m_n}]\).

Note that \(\theta_n - \text{Id}\) induces a function on \(\mathbb{T}^1\) which extends holomorphically to \(\mathbb{C} \setminus \{0\}\). Recall that \(\mathcal{B}(N)\) denotes the covering of the circle by \(N\) compact intervals defined by \([2, 1]\). Moreover, \(\theta\) itself induces a degree one map on the circle, which we denote by \(\theta\) again for simplicity.

Claim. If \(N_{n+1}\) is large enough, then, for any \(\omega \in \mathbb{T}^1\), the circular chain of intervals \(\{\theta_n(B - \omega) + \omega, B \in \mathcal{B}(N_{n+1})\}\) is crooked inside the circular chain \(\mathcal{B}(N_n)\).

Proof. We work with a lift of the family \(\mathcal{B}(N)\), in the sense discussed in Section 2, obtained as a covering \(\hat{\mathcal{B}}(N)\) of the real line by intervals of the form:

\[
\hat{B}_i = \left(\frac{i - 5/4}{N}, \frac{i + 1/4}{N}\right), \quad i \in \mathbb{Z}.
\]

If \(N_{n+1}\) is large enough, each interval \(\theta_n(\hat{B}_i - \omega) + \omega\) with \(\hat{B}_i \in \hat{\mathcal{B}}(N_{n+1})\) has length strictly less than \(\frac{1}{2N_n}\) and therefore is contained in an interval \(\hat{B}'_{\ell(i)}\) of the family \(\hat{\mathcal{B}}(N_n)\). Since \(\theta_n\) has degree 1, one can choose the function \(\hat{\ell}\) such that \(\hat{\ell}(i + N_{n+1}) = \hat{\ell}(i) + N_n\).

Let \(i < j\) be two integers such that \(\hat{\ell}(k)\) belongs to the interval bounded by \(\hat{\ell}(i)\) and \(\hat{\ell}(j)\) for each \(i < k < j\) and such that \(4 < |\hat{\ell}(j) - \hat{\ell}(i)| < N_n\). Let us choose \(i \leq i' < j' \leq j\) such that \(\hat{\ell}(i') = \hat{\ell}(i), \hat{\ell}(j') = \hat{\ell}(j)\) and some points \(a \in \hat{B}_{i'}\) and \(b \in \hat{B}_{j'}\). One considers \(a \leq a' < b' \leq b\) such that \(a, a'\) (resp. \(b, b'\)) have the same image by \(x \mapsto \theta_n(x - \omega)\). One can assume that \(a' - \omega, b' - \omega\) belong to the same interval \(I_k := \left[\frac{k}{2m_n}, \frac{k+1}{2m_n}\right]\); indeed, for each \(k\), the image under \(\theta_n\) of \(I_k\) has length \(> 1\) and the point \(\theta_n\left(\frac{k}{2m_n}\right)\) is an end point of \(\theta_n(I_{k-1} \cup I_k)\).

Now the points \(a' - \omega, b' - \omega\) belong to the same interval \(I_k = \left[\frac{k}{2m_n}, \frac{k+1}{2m_n}\right]\). Since \(\theta_n\) is \((4N_n)^{-1}\)-crooked on this interval, this implies that there exists \(a' < c < d < b'\) such that
- \( \theta_n(a - \omega) = \theta_n(a' - \omega) \) and \( \theta_n(d - \omega) \) are \((4N_n)^{-1}\)-close,
- \( \theta_n(b - \omega) = \theta_n(b' - \omega) \) and \( \theta_n(c - \omega) \) are \((4N_n)^{-1}\)-close.

Hence \( d \) is contained in an interval \( \hat{B}_u \) such that \( |\hat{\ell}(v) - \hat{\ell}(i)| \leq 1 \). Similarly, \( c \) is contained in an interval \( \hat{B}_u \) such that \( |\hat{\ell}(u) - \hat{\ell}(i)| \leq 1 \). Since \( a < c < d < b \) and \( 2 < |\hat{\ell}(j) - \hat{\ell}(i)| \) one gets \( i < u < v < j \).

In the projection to \( \mathbb{T}^1 \), the above shows that the circular chain \( \{\theta_n(B - \omega) + B, B \in \mathcal{B}(N_{n+1})\} \) is crooked inside the circular chain \( \mathcal{B}(N_n) \) as announced. \( \square \)

Since \( m_n \) only depends on \( \alpha_n \), one can fix the map \( \theta_n \) but choose \( b_{n+1} \) and \( N_{n+1} \) arbitrarily large later in the construction.

3.b – Construction of \( H_{n+1} \). Consider the map \( \Theta_{n+1} : \{0\} \times \mathbb{T}^1 \rightarrow \mathbb{T}^1 \) defined by \( \Theta_{n+1} : (0, y) \mapsto \theta_n(y) - y \). Since \( \theta_n - \text{Id} \) is \(1/m_n\)-periodic, and since the period of the return map associated to the linear flow \( \varphi_{n+1} \) on \( \{0\} \times \mathbb{T}^1 \) is equal to \( m_n \), this maps extends to a map \( \Theta_{n+1} : \mathbb{T}^2 \rightarrow \mathbb{T}^1 \) which is constant along the orbits of \( \varphi_{n+1} \). We define \( H_{n+1} \) by setting \( H_{n+1} := h_{n+1} \circ H_n \), where

\[
h_{n+1}(x, y) = \varphi_{n+1} \left( \frac{\Theta_{n+1}(x, y)}{p_n b_{n+1}}, (x, y) \right) = \left( x, y - \Theta_{n+1}(x, y) \right) - \frac{\Theta_{n+1}(x, y)}{q_n b_{n+1}} \left( 1, \frac{r_n}{p_n} \right).
\]

Since \( \Theta_{n+1} \) is constant along the orbits of \( \varphi_{n+1} \), one has

\[
h_{n+1}^{-1}(x, y) = \left( x, y + \Theta_{n+1}(x, y) \right) + \frac{\Theta_{n+1}(x, y)}{q_n b_{n+1}} \left( 1, \frac{r_n}{p_n} \right). \tag{3.1}
\]

Clearly, \( h_{n+1} \) (and hence \( H_{n+1} \)) belongs to \( \text{Diff}_{\text{vol},0}^\omega(\mathbb{T}^2) \). Note also that both \( h_{n+1} \) and \( h_{n+1}^{-1} \) extend holomorphically to the domain \( (\mathbb{C} \setminus \{0\})^2 \).

3.c – Choice of \( b_{n+1}, \varepsilon_{n+1} \) and \( N_{n+1} \). Now, we explain how to fix the values of \( b_{n+1}, \varepsilon_{n+1}, \) and \( N_{n+1} \) so that properties 1 and 5 hold. From (3.1), if \( b_{n+1} \) is large enough, the image under \( h_{n+1}^{-1} \) of every point \( (x, y) \in \mathbb{T}^2 \) is arbitrarily close to \( (x, y + \Theta_{n+1}(x, y)) \). Now observe that, for each \( x \in \mathbb{T}^1 \), there exists \( \omega_x \in \mathbb{T}^1 \) such that

\[
\Theta_{n+1}(x, y) = \theta_n(y - \omega_x) - y + \omega_x.
\]

As a consequence, if \( b_{n+1} \) is large enough, the image under \( h_{n+1}^{-1} \) of every point \( z = (x, y) \in \mathbb{T}^2 \) is arbitrarily close to \( (x, \theta_n(y - \omega_x) + \omega_x) \). Hence, if \( b_{n+1} \) is large enough and \( \varepsilon_{n+1} \) is small enough, the image under \( h_{n+1}^{-1} \) of a rectangle \( (x - \varepsilon_{n+1}, x + \varepsilon_{n+1}) \times B \) is contained in an arbitrary small neighbourhood of the square \( \{x\} \times (\theta_n(B - \omega_x) + \omega_x) \). Using the claim above, this implies that the family \( \mathcal{D}_{n+1,x} \) is crooked inside \( \mathcal{D}_{n,x} \), provided \( N_{n+1} \) is large enough. By continuity of \( H_{n+1}^{-1} \), the diameter of the elements of the covering \( \mathcal{D}_{n+1,x} \) is less than \( \frac{1}{n+1} \) if \( b_{n+1}, N_{n+1} \) are large enough and \( \varepsilon_{n+1} \) is small enough. We have thus checked that property 5 holds, provided that \( b_{n+1}, N_{n+1} \) are chosen large enough and \( \varepsilon_{n+1} \) is chosen small enough.

By definition of \( p_n, p_{n+1} \), and \( H_{n+1} \), in order to check that property 4 is satisfied, it is enough to check that \( \pi \circ h_{n+1} \) is close to \( \pi \) for the \( C^0 \)-topology. This is a direct consequence of the definition of \( h_{n+1} \), provided that \( b_{n+1} \) is chosen large enough.
Figure 2: Choice of the function $\theta$ (on the right) and the image of a vertical line $\{x\} \times \mathbb{R}$ under the lift $\hat{h}_{n+1}$ of $h_{n+1}$ (schematic picture with $m_n = 2$). Note that the preimages of vertical lines under $h_{n+1}$ have a similar ‘crookedness’, which is the fact that is needed for our construction (see next section). The horizontal size of images (and preimages) of vertical lines under $h_{n+1}$ is small compared to $\varepsilon_{n+1}$. This size is controlled by the flow lines of $\varphi_{n+1}$ (dashed lines), whose direction is given by the almost vertical vector $\left(\frac{1}{\varepsilon_{n+1}}q_{n+1} + (r_n/p_n), 1\right)$.

3.d – Choice of $\alpha_{n+1}$. Clearly, one can choose $\alpha_{n+1}$ arbitrarily close to $\alpha_n$ so that property 2 holds. By uniform continuity of $H_{n+1}$ and $H_{n+1}^{-1}$, there exists $\eta$ so that the orbits of $f_{n+1} = H_{n+1} \circ R_{\alpha_{n+1}} \circ H_{n+1}^{-1}$ are $\frac{1}{n+1}$-dense in $\mathbb{T}^2$ provided that the orbits of the rotation $R_{\alpha_{n+1}}$ are $\eta$-dense in $\mathbb{T}^2$. One can thus choose $\alpha_{n+1}$ arbitrarily close to $\alpha_n$ so that property 3 is satisfied. By construction both $f_{n+1}$ and $f_{n+1}^{-1}$ extend holomorphically to $\left(\mathbb{C} \setminus \{0\}\right)^2$. Moreover the diffeomorphism $h_{n+1}$ commutes with the flow $\varphi_{n+1}$, hence with the rotation $R_{\alpha_n}$. Consequently $f_{n} = H_{n+1}^{-1} \circ R_{\alpha_n} \circ H_{n+1}$. This shows that $f_{n+1}$ is arbitrarily close to $f_n$ when $\alpha_{n+1}$ is chosen arbitrarily close to $\alpha_n$. In particular, property 4 holds provided that $\alpha_{n+1}$ is chosen close enough to $\alpha_n$. 

9
4 Uniqueness of the semi-conjugacy, non-existence of wandering curves and further remarks

The aim of this last section is to discuss, somewhat informally, the implications of our construction for some questions arising in the context of dynamics and rotation theory on the two-torus. Throughout this section, we assume \( f \) is an irrational pseudo-rotation with an invariant foliation of pseudo-circles, consisting of the fibres of a semi-conjugacy \( p : \mathbb{T}^2 \to \mathbb{T}^1 \) to an irrational rotation \( R_\alpha \). Moreover, we will freely add further assumptions on \( f \) if these can easily be ensured in the preceding Anosov-Katok-construction. We first note that the semi-conjugacy in Theorem 1 is unique, modulo post-composition by rotations.

Proposition 4.1. The semi-conjugacy \( p \) in Theorem 1 is uniquely determined: any continuous map \( p' \) which is homotopic to \( p \) and semi-conjugates \( f \) to the same circle rotation as \( p \) can be written as \( p' = R \circ p \) where \( R \) is a rotation of the circle.

This follows directly by combining [JP, Corollary 4.3] (uniqueness of the semi-conjugacy provided the non-wandering set is externally transitive) with [Po, Theorem A] (external transitivity of the non-wandering set of irrational pseudo-rotations).

The uniqueness of the semi-conjugacy further allows to see that \( f \) does not admit any loop which is wandering (i.e. disjoint from all its iterates) and has the same homotopy type as the pseudo-circles (that is, homotopy vector \( v_2 = (0, 1) \) in our construction). The reason for this is the fact that the existence of such a loop \( \Gamma \) would allow to construct a semi-conjugacy \( \tilde{p} \) to the rotation \( R_\alpha \) such that \( \Gamma \) is contained in a single fibre of the semi-conjugacy. Details of this construction can be found in [JP, Lemma 3.2] (the fact that \( \Gamma \) is contained in a single fibre is not mentioned explicitly, but is obvious from the proof). Due to the uniqueness of the semi-conjugacy (modulo rotations) and the fact that none of the pseudo-circles of the foliation contains any non-degenerate curves, this yields a contradiction.

More generally, it is even possible to show that \( f \) does not admit any loops disjoint of all its iterates, regardless of the homotopy type. This is slightly more subtle, and we only sketch the argument. The crucial observation is the fact that we may construct \( f \) such that

\[
\sup \left| \langle F^n(z) - z - n\rho, v \rangle \right| < \infty \quad \text{iff} \quad v = (1, 0), \tag{4.1}
\]

where \( F : \mathbb{R}^2 \to \mathbb{R}^2 \) is a lift of \( f \) and \( \rho \in \mathbb{R}^2 \) is the corresponding rotation vector. Now, if there exists a wandering loop of homotopy type \( w \in \mathbb{Z}^2 \setminus \{0\} \), then it is not hard to see that

\[
\sup \left| \left\langle F^n(z) - z - n\rho, w^\perp \right\rangle \right| < \infty.
\]

However, according to (4.1) this is only possible if \( w = (0, 1) \), and this is exactly the homotopy type of the pseudo-circles which was excluded before. Homotopically trivial wandering loops cannot exist by minimality, and therefore no loop of any homotopy type can be wandering.

Roughly speaking, in order to prove (4.1) one has to use the fact that since the leaves of the foliations \( \mathcal{V}_k \) are increasingly crooked, connected fundamental domains of these circles in the lift become arbitrarily large in diameter. Since an iterate of \( f_k \) acts as a rotation on these leaves, this allows to see that for suitable integers \( n_k \) the vertical deviations \( \left| \langle F^k(z) - z - n_k\rho, w^\perp \rangle \right| \) become arbitrarily large. If the \( f_k \) converge to \( f \) sufficiently fast, then this carries over to the limit and yields unbounded vertical deviations for \( f \). At the same time horizontal deviations
(that is, \( v = v_1 = (1,0) \) in (4.1)) are bounded due to the existence of the semi-conjugacy (e.g. [JT, Lemma 3.1]). Together, these two facts yield unbounded deviations for all \( v \neq v_1 \).

The fact that \( f \) does not admit any wanding loops is of some interest in the context of the Arc Translation Theorem due to Kwapisz [Kw] [BCL], which asserts that given an irrational pseudo-rotation and any integer \( n \), there exist essential loops which are disjoint from their first \( n \) iterates. It is natural to ask under what additional assumptions this statement can be strengthened by passing from a finite number to all iterates. A natural obstruction is certainly to have unbounded deviations in all directions, as discussed above. However, our example shows that even if the deviations are bounded in some direction, the existence of a wandering curve is not guaranteed. Thus, in this sense the statement of the Arc Translation Theorem is optimal, and essential loops have to be replaced by more general classes of essential continua in order to obtain results in infinite time.

Finally, we want to mention a loose connection of our construction to the Franks-Misiurewicz Conjecture [FM]. The latter asserts that if the rotation set of a torus homeomorphism is a line segment of positive length, then either it contains infinitely many rational points, or it has a rational endpoint. As mentioned in the introduction, Avila has recently announced a counterexample to this conjecture for the case where the rotation segment has irrational slope, but the line it defines does not pass through a rational point. Conversely, Le Calvez and Tal have announced the first positive partial result on the conjecture: if the rotation set is a segment with irrational slope, it cannot contain a rational point in its relative interior. One case which is still completely open, however, is whether the rotation set can be a line segment with rational slope, but without rational points – for example, of the form \( \{\alpha\} \times [a,b] \) with \( \alpha \in \mathbb{R} \) irrational and \( a < b \). Now, in the situation where there exists a semi-conjugacy \( p \), homotopic to \( \pi_1 \), to the rotation \( R_\alpha \) on the circle, the rotation set has to be contained in the line \( \{\alpha\} \times \mathbb{R} \) [JT, Lemma 3.1]. Hence, this is a natural class of maps to look for counterexamples to this subcase of the Franks-Misiurewicz Conjecture. However, it is known that in order to have a non-degenerate rotation interval, the fibres of the semi-conjugacy need to have a complicated structure – more precisely, they need to be indecomposable [JP]. Our example shows that such a rich fibre structure is possible in principle. Yet, whether a non-degenerate rotation interval can be achieved remains open. Here, the fact that the Anosov-Katok method typically leads to uniquely ergodic examples suggests that a different approach would be needed to produce such examples, if these exists at all.

References

[AK] D. Anosov, A. Katok. New examples in smooth ergodic theory. Ergodic diffeomorphisms. *Trans. Moscow Math. Soc.* **23** (1970), 1–35.

[BCL] F. Béguin, S. Crovisier, F. Le Roux. Pseudo-rotations of the closed annulus: variation on a theorem of J. Kwapisz. *Nonlinearity* **17** (2004), 1427–1453.

[Bi1] R. H. Bing. Concerning hereditarily indecomposable continua. *Pacific J. Math.* **1** (1951), 43–51.

[Bi2] R. H. Bing. Higher-dimensional hereditarily indecomposable continua. *Trans. AMS.* **71** (1951), 267–273.
[FK1] B. Fayad, A. Katok. Constructions in elliptic dynamics. *Ergodic Theory Dynam. Systems* 24 (2004), 1477–1520.

[FK2] B. Fayad, A. Katok. Analytic uniquely ergodic volume preserving maps on odd spheres. *Comment. Math. Helv.* 89 (2014), 963–977.

[FM] J. Franks, M. Misiurewicz. Rotation sets of toral flows. *Proc. AMS* 109 (1990), 243–249.

[Fe] L. Fearnley. Classification of all hereditarily indecomposable circularly chainable continua. *Trans. AMS* 168 (1972), 387–401.

[Ha] M. Handel. A pathological area preserving $C^\infty$ diffeomorphism of the plane. *Proc. AMS* 86 (1982), 163–168.

[JP] T. Jäger, A. Passeggi. On torus homeomorphisms semiconjugate to irrational circle rotations. [arXiv:1305.0723](https://arxiv.org/abs/1305.0723) To appear in *Ergodic Theory Dynam. Systems*.

[JT] T. Jäger, F. Tal. Irrational rotation factors for conservative torus homeomorphisms. Preprint 2014. [arXiv:1410.3662](https://arxiv.org/abs/1410.3662).

[Kw] J. Kwapisz. A priori degeneracy of one-dimensional rotation sets for periodic point free torus maps. *Trans. AMS* 354 (2002), 2865–2895.

[MZ] M. Misiurewicz and K. Ziemian. Rotation sets for maps of tori. *J. Lond. Math. Soc.*, 40:490–506, 1989.

[Po] R. Potrie. Recurrence of non-resonant torus homeomorphisms. *Proc. AMS* 140 (2012), 3973–3981.

[Pr] J. Prajs. A continuous circle of pseudo-arcs filling up the annulus. *Trans. AMS* 352 (1999), 1743–1757.

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