On \((p_1(x), p_2(x))\)-Laplace Equations *

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Abstract. In this paper, we investigate the \((p_1(x), p_2(x))\)-Laplace operator, the properties of the corresponding integral functional and weak solutions to the related differential equations. We show that the integral functional admits a derivative of type \((S_+)\) which induces a homeomorphism between duality space pairs. As applications of the above results, we gave some existence results of the \((p_1(x), p_2(x))\)-Laplace equation

\[-\text{div}(|\nabla u|^{p_1(x)-2}\nabla u) - \text{div}(|\nabla u|^{p_2(x)-2}\nabla u) = f(x, u)\]

in a bounded smooth domain \(\Omega \subset \mathbb{R}^N\) with Dirichlet boundary condition.

Keywords Critical points; Variable exponent Lebesgue-Sobolev Space; Mountain-Pass Lemma; Fountain Theorem.

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0. Introduction

In the mathematical modeling of electrorheological fluids (see [1, 2, 3] and the references therein), the \(p(x)\)-Laplace operator, defined as \(\Delta_{p(x)}u(x) := \text{div}(|\nabla u(x)|^{p(x)-2}\nabla u)\), plays an important role. Based on these application backgrounds, the study of differential equations involving \(p(x)\)-Laplace operators have been a very interesting and exciting topic in recent years (see

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in particular the nice survey [4, 5] and the references therein for further details). During the development of last several decades, it turns out that the use of variational methods in dealing with problems involving nonstandard growth conditions is a far-reaching field. The $p(x)$—growth conditions can be regarded as an important particular case of the nonstandard $(p, q)$—growth conditions. Many results have been obtained on this kind of problems, for example [6, 7, 8, 11, 12, 13] and so on.

In this paper, continuing our former investigations on these topics (see [6, 7, 8, 11, 12, 13, 14, 15]), we shall study the joint effects of different $(p_1(x), p_2(x))$-Laplace operators and investigate the properties of the corresponding integral functional in proper framework of variable exponent Lebesgue-Sobolev spaces (see Section 1). We choose to present here these properties as it is clear that not only these results have their own interests from the point view of pure functional analysis but also they have direct applications in the study of differential equation with $(p_1(x), p_2(x))$—growth conditions.

As an illustration of the aforementioned properties, we choose to present some results on the existence of weak solutions to the following $(p_1(x), p_2(x))$-Laplace equation

$$\begin{cases}
  -\text{div}(|\nabla u|^{p_1(x)-2}\nabla u) - \text{div}(|\nabla u|^{p_2(x)-2}\nabla u) = f(x, u), & \text{in } \Omega; \\
  u = 0, & \text{on } \partial \Omega,
\end{cases}
$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain and $p_i(x) \in C(\overline{\Omega})$ with $p_i(x) > 1$ for any $x \in \overline{\Omega}$ and for $i = 1, 2$. These results themselves are interesting and new. Under proper growth conditions on nonlinear terms, we showed the corresponding functional enjoys coercive property (Theorem 3.3), mountain pass geometry (Theorem 3.6) and fountain geometry (Theorem 3.7) respectively, which yields rich existence results to our problem. The role of the results on $(p_1(x), p_2(x))$-Laplace operator lies in that they supply us a neat means to prove the required compactness result (Lemma 3.5) when we use variational devices. It is important to notice that the growth conditions on the nonlinear term are given according to the function $p_M(x)$ and $p_m(x)$ other than $p_1^- \wedge p_2^- \lor p_1^+ \lor p_2^+$ (see Section 3).

For the ease of exposition and to keep the paper in a reasonable length, we only give existence results on equations with $(p_1(x), p_2(x))$-growth conditions on bounded domain and with Dirichlet boundary condition. With some symmetry conditions (see [16]) and weighted exponent Sobolev spaces method, we could also give many results on unbounded domain with other types of boundary conditions. Besides, under proper conditions, the functionals will satisfy some other geometry structures. We will address these issues in a following paper.
This paper is organized as follows. In Section 1, for the convenience of the readers, we recall the definitions of the variable exponent Lebesgue-Sobolev spaces which can be regarded as a special class of generalized Orlicz-Sobolev spaces and introduce some basic properties of these spaces; In Section 2, we investigate properties of the \((p_1(x), p_2(x))\)-Laplace operator and the corresponding integral functional; In Section 3, we show the existence of weak solutions to problem \((P)\) by variational arguments.

1. The spaces \(W^{1,p(x)}_0(\Omega)\)

In this section, we will give out some theories on spaces \(W^{1,p(x)}_0(\Omega)\) which we call generalized Lebesgue-Sobolev spaces. Firstly we state some basic properties of spaces \(W^{1,p(x)}_0(\Omega)\) which will be used later (for details see [17, 14, 15, 4] and the references therein).

We write
\[ C_+ = \{ h | h \in C(\Omega), h(x) > 1 \text{ for any } x \in \Omega \}, \]
\[ h^+ = \max_{\Omega} h(x), h^- = \min_{\Omega} h(x) \text{ for any } h \in C(\Omega), \]
\[ L^{p(x)}(\Omega) = \{ u | u \text{ is a measurable real-valued function}, \int_{\Omega} |u(x)|^{p(x)} dx < \infty \}. \]

The linear vector space \(L^{p(x)}(\Omega)\) can be equipped by the following norm
\[ |u|_{p(x)} = \inf \left\{ \lambda > 0 \ : \ \int_{\Omega} \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}, \]
then \((L^{p(x)}, | \cdot |_{p(x)})\) becomes a Banach space and we call it variable exponent Lebesgue space.

In the following we shall collect some basic propositions concerning the variable exponent Lebesgue spaces. These propositions will be used throughout our analysis.

**Proposition 1.1** (see Fan and Zhao [17] and Zhao et al. [14]). (i) The space \((L^{p(x)}, | \cdot |_{p(x)})\) is a separable, uniform convex Banach space, and its conjugate space is \(L^{q(x)}(\Omega)\), where \(\frac{1}{p(x)} + \frac{1}{q(x)} = 1\). For any \(u \in L^{p(x)}\) and \(v \in L^{q(x)}\), we have
\[ \left| \int_{\Omega} u v dx \right| \leq \left( \frac{1}{p^+} + \frac{1}{q^-} \right) |u|_{p(x)} |v|_{q(x)}. \]
(ii) If \( p_1, p_2 \in C_+(\Omega), p_1(x) \leq p_2(x) \) for any \( x \in \Omega \), then \( L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega) \), and the imbedding is continuous.

**Proposition 1.2** (see Fan and Zhao [17] and Zhao et al. [15]). If \( f : \Omega \times \mathbb{R} \rightarrow \mathbb{R} \) is a Caratheodory function and satisfies

\[
|f(x, s)| \leq a(x) + b|s|^{\frac{p_1(x)}{p_2(x)}} \quad \text{for any } x \in \Omega, s \in \mathbb{R},
\]

where \( p_1, p_2 \in C_+(\Omega), a(x) \in L^{p_2(x)}(\Omega), a(x) \geq 0 \) and \( b \geq 0 \) is a constant, then the Nemytsky operator from \( L^{p_2(x)}(\Omega) \) to \( L^{p_2(x)}(\Omega) \) defined by

\[
(N_f(u))(x) = f(x, u(x))
\]

is a continuous and bounded operator.

**Proposition 1.3** (see Fan and Zhao [17] and Zhao et al. [14]). If we denote

\[
\rho(u) = \int_{\Omega} |u|^{p(x)} \, dx, \forall u \in L^{p(x)}(\Omega),
\]

then

(i) \( |u|_{p(x)} < 1 (= 1; > 1) \Leftrightarrow \rho(u) < 1 (= 1; > 1) \);

(ii) \( |u|_{p(x)} > 1 \Rightarrow |u|_{p^-(x)}^{p^-(x)} \leq \rho(u) \leq |u|_{p^+(x)}^{p^+(x)} \); \( |u|_{p(x)} < 1 \Rightarrow |u|_{p^-(x)}^{p^-(x)} \geq \rho(u) \geq |u|_{p^+(x)}^{p^+(x)} \);

(iii) \( |u|_{p(x)} \to 0 \Leftrightarrow \rho(u) \to 0; \ |u|_{p(x)} \to \infty \Leftrightarrow \rho(u) \to \infty \).

**Proposition 1.4** (see Fan and Zhao [17] and Zhao et al. [14]). If \( u, u_n \in L^{p(x)}(\Omega), n = 1, 2, ..., \) then the following statements are equivalent to each other:

1. \( \lim_{k \to \infty} |u_k - u|_{p(x)} = 0 \);
2. \( \lim_{k \to \infty} \rho(u_k - u) = 0 \);
3. \( u_k \rightharpoonup u \) in measure in \( \Omega \) and \( \lim_{k \to \infty} \rho(u_k) = \rho(u) \).

The variable exponent Sobolev space \( W^{1,p(x)}(\Omega) \) is defined by

\[
W^{1,p(x)}(\Omega) = \{ u \in L^{p(x)}(\Omega) | \| \nabla u \| \in L^{p(x)}(\Omega) \}
\]

and it is equipped with the norm

\[
\| u \|_{p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}, \forall u \in W^{1,p(x)}(\Omega).
\]

If we denote by \( W^{1,p(x)}_0(\Omega) \) the closure of \( C_c^\infty(\Omega) \) in \( W^{1,p(x)}(\Omega) \) and

\[
p^*(x) = \begin{cases} \frac{N_{p(x)}}{N - p(x)}, & p(x) < N, \\ \infty, & p(x) \geq N, \end{cases}
\]
we have the following

**Proposition 1.5** (see Fan and Zhao [17]). (i) $W^{1,p(x)}(\Omega)$ and $W^{1,p(x)}_0(\Omega)$ are separable reflexive Banach spaces; (ii) If $q \in C_+^{+}(\Omega)$ and $q(x) < p^*(x)$ for any $x \in \overline{\Omega}$, then the imbedding from $W^{1,p(x)}(\Omega)$ to $L^{q(x)}(\Omega)$ is compact and continuous; (iii) There is a constant $C > 0$ such that

$$|u|_{p(x)} \leq C|\nabla u|_{p(x)}, \forall u \in W^{1,p(x)}_0(\Omega).$$

**Remark 1.6.** By (iii) of Proposition 1.5, we know that $|\nabla u|_{p(x)}$ and $\|u\|_{p(x)}$ are equivalent norms on $W^{1,p(x)}_0(\Omega)$. We will use $|\nabla u|_{p(x)}$ instead of $\|u\|_{p(x)}$ in the following discussions.

### 2. Properties of $(p_1(x), p_2(x))$-Laplace operator

In this section we give the properties of the $(p_1(x), p_2(x))$-Laplace operator $(-\Delta_{p_1(x)} - \Delta_{p_2(x)})u := -\text{div}(|\nabla u|^{p_1(x)-2}\nabla u) - \text{div}(|\nabla u|^{p_2(x)-2}\nabla u)$. Consider the following functional,

$$J(u) = \int_{\Omega} \frac{1}{p_1(x)}|\nabla u|^{p_1(x)}dx + \int_{\Omega} \frac{1}{p_2(x)}|\nabla u|^{p_2(x)}dx, \forall u \in X,$$

where $X := W^{1,p_1(x)}_0(\Omega) \cap W^{1,p_2(x)}_0(\Omega)$ with its norm given by $\|u\| := \|u\|_{p_1(x)} + \|u\|_{p_2(x)}, \forall u \in X$.

It is obvious that $J \in C^1(X, \mathbb{R})$ (see [18]), and the $(p_1(x), p_2(x))$-Laplace operator is the derivative operator of $J$ in the weak sense. Denote $L = J' : X \to X^*$, then

$$\langle L(u), v \rangle = \int_{\Omega} |\nabla u|^{p_1(x)-2}\nabla u \nabla vdx + \int_{\Omega} |\nabla u|^{p_2(x)-2}\nabla u \nabla vdx, \forall u, v \in X,$$

in which $\langle \cdot, \cdot \rangle$ is the dual pair between $X$ and its dual $X^*$.

**Remark 2.1.** $(X, \| \cdot \|)$ is a separable reflexive Banach space.

**Theorem 2.2.** (i) $L : X \to X^*$ is a continuous, bounded and strictly monotone operator; (ii) $L$ is a mapping of type $(S_+)$, namely: if $u_n \to u$ in $X$ and $\lim_{n \to \infty} \langle L(u_n) - L(u), v \rangle = 0$, then $u_n \to u$ in the norm of $X$. In particular, $L : X \to X^*$ is a $\sigma$-compact and continuous operator.
\[ L(u), u_n - u \leq 0, \text{ then } u_n \to u \text{ in } X; \]

(iii) \( L : X \to X^* \) is a homeomorphism.

**Proof.** (i) It is obvious that \( L \) is continuous and bounded. Considering the duality pair \( \langle Lu - Lv, u - v \rangle \), we have

\[
\langle Lu - Lv, u - v \rangle \\
= \int_\Omega (|\nabla u|^{p_1(x)-2}\nabla u - |\nabla v|^{p_1(x)-2}\nabla v)(\nabla u - \nabla v)dx \\
+ \int_\Omega (|\nabla u|^{p_2(x)-2}\nabla u - |\nabla v|^{p_2(x)-2}\nabla v)(\nabla u - \nabla v)dx \\
\geq \int_\Omega h_1(x)dx + \int_\Omega h_2(x)dx \geq 0,
\]

where

\[
0 \leq h_i(x) = \begin{cases} \\
\frac{(p_i(x)-1)|\nabla u - \nabla v|^2}{(|\nabla u|^{p_i(x)} + |\nabla v|^{p_i(x)})^{\frac{2}{p_i(x)}}}, & 1 < p_i(x) < 2, \\
\frac{1}{2^{p_i(x)}}|\nabla u - \nabla v|^{p_i(x)}, & p_i(x) \geq 2,
\end{cases}
\]

\( i = 1, 2 \). Here we have applied the following inequalities (see [19, 20]): for any \( \xi, \eta \in \mathbb{R}^N \),

\[
[(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta) \cdot (\xi - \eta)](|\xi|^p + |\eta|^p)^{\frac{2-p}{p}} \geq (p-1)|\xi - \eta|^2, & 1 < p < 2; \\
(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta) \cdot (\xi - \eta) \geq \frac{1}{2p}|\xi - \eta|^p, & p \geq 2,
\]

in which the equal-sign holds if and only if \( \xi = \eta \).

(ii) From (i), if \( u_n \rightharpoonup u \) in \( X \) and \( \lim_{n \to \infty} \langle L(u_n) - L(u), u_n - u \rangle \leq 0 \), then \( \lim_{n \to \infty} \langle L(u_n) - L(u), u_n - u \rangle = 0 \). This implies by the proof of (i)

\[
o_n(1) = \int_{\{1 < p_i(x) < 2\}} \frac{(p_i(x)-1)|\nabla u_n - \nabla v|^2}{(|\nabla u_n|^{p_i(x)} + |\nabla v|^{p_i(x)})^{\frac{2}{p_i(x)}}}dx + \int_{\{p_i(x) \geq 2\}} \frac{1}{2^{p_i(x)}}|\nabla u_n - \nabla u|^{p_i(x)}dx.
\]

The above equality implies \( \nabla u_n \) converges in measure to \( \nabla u \) in \( \Omega \). So we may assume up to a subsequence (still denoted by \( \{\nabla u_n\} \)) that \( \nabla u_n(x) \to \nabla u(x) \), a. e. \( x \in \Omega \). By Fatou’s Lemma, we get

\[
\lim_{n \to \infty} \int_\Omega \frac{1}{p_i(x)}|\nabla u_n|^{p_i(x)}dx \geq \int_\Omega \frac{1}{p_i(x)}|\nabla u|^{p_i(x)}dx. \tag{1}
\]

In view of the assumption \( u_n \rightharpoonup u \), we have

\[
\lim_{n \to \infty} \langle L(u_n), u_n - u \rangle = \lim_{n \to \infty} \langle L(u_n) - L(u), u_n - u \rangle. \tag{2}
\]
We also have
\[ \langle L(u_n), u_n - u \rangle \]
\[ = \int_{\Omega} |\nabla u_n|^{p_1(x)} dx - \int_{\Omega} |\nabla u_n|^{p_1(x)-2} \nabla u_n \nabla u dx + \int_{\Omega} |\nabla u_n|^{p_2(x)} dx - \int_{\Omega} |\nabla u_n|^{p_2(x)-2} \nabla u_n \nabla u dx \]
\[ \geq \int_{\Omega} |\nabla u_n|^{p_1(x)} dx - \int_{\Omega} |\nabla u_n|^{p_1(x)-1} |\nabla u| dx + \int_{\Omega} |\nabla u_n|^{p_2(x)} dx - \int_{\Omega} |\nabla u_n|^{p_2(x)-1} |\nabla u| dx \]
\[ \geq \int_{\Omega} |\nabla u_n|^{p_1(x)} dx - \int_{\Omega} \left( \frac{p_1(x)}{p_1(x)} |\nabla u_n|^{p_1(x)} + \frac{1}{p_1(x)} |\nabla u|^{p_1(x)} \right) dx \]
\[ + \int_{\Omega} |\nabla u_n|^{p_2(x)} dx - \int_{\Omega} \left( \frac{p_2(x)}{p_2(x)} |\nabla u_n|^{p_2(x)} + \frac{1}{p_2(x)} |\nabla u|^{p_2(x)} \right) dx \]
\[ \geq \int_{\Omega} 1 \left| |\nabla u_n|^{p_1(x)} dx - \int_{\Omega} 1 \left| |\nabla u|^{p_1(x)} dx + \int_{\Omega} 1 \left| |\nabla u_n|^{p_2(x)} dx - \int_{\Omega} 1 \left| |\nabla u|^{p_2(x)} dx. \right. \right. \]

By (1) and (2), the above inequality implies
\[ \lim_{n \to \infty} \int_{\Omega} 1 \left| |\nabla u_n|^{p_1(x)} dx = \int_{\Omega} 1 \left| |\nabla u|^{p_1(x)} dx. \right. \right. \]

(3)

From (3) it follows that the integrals of the function family \( \frac{1}{p_1(x)} |\nabla u_n|^{p_1(x)} \)
possesses absolutely equicontinuity on \( \Omega \) (see [21]). Since
\[ \frac{1}{p_1(x)} |\nabla u_n - \nabla u|^{p_1(x)} \leq C \left( \frac{1}{p_1(x)} |\nabla u_n|^{p_1(x)} + \frac{1}{p_1(x)} |\nabla u|^{p_1(x)} \right), \]
the integrals of the family \( \frac{1}{p_1(x)} |\nabla u_n - \nabla u|^{p_1(x)} \) are also absolutely equicontinuous on \( \Omega \) and therefore
\[ \lim_{n \to \infty} \int_{\Omega} \frac{1}{p_1(x)} |\nabla u_n - \nabla u|^{p_1(x)} dx = 0. \]

(4)

By (4),
\[ \lim_{n \to \infty} \int_{\Omega} |\nabla u_n - \nabla u|^{p_1(x)} dx = 0. \]

(5)

From (5), we have \( u_n \to u \) in \( X \). This implies that \( L \) is of type \( (S_+) \).

(iii) By (i), \( L \) is an injection. Since
\[ \lim_{\|u\| \to \infty} \frac{\langle Lu, u \rangle}{\|u\|} = \lim_{\|u\| \to \infty} \frac{\int_{\Omega} |\nabla u|^{p_1(x)} dx + \int_{\Omega} |\nabla u|^{p_2(x)} dx}{\|u\|} = \infty, \]
\( L \) is coercive, thus \( L \) is a surjection by Minty-Browder Theorem (see [22]).

Hence \( L \) has an inverse mapping \( L^{-1} : X^* \to X \). In order to show that \( L \) is a homeomorphism, it remains to show the continuity of \( L^{-1} \).
For any \( f_n, f \in X^* \) such that \( f_n \to f \) in \( X^* \), suppose \( u_n = L^{-1}(f_n), u = L^{-1}(f) \), then we have \( L(u_n) = f_n, L(u) = f \). Since \( L \) is coercive, \( \{u_n\} \) is bounded in \( X \). We can assume that \( u_{n_k} \to u_0 \) in \( X \). By \( f_{n_k} \to f \) in \( X^* \), we have

\[
\lim_{n \to \infty} \langle L(u_{n_k}) - L(u_0), u_{n_k} - u_0 \rangle = \lim_{n \to \infty} \langle f_{n_k}, u_{n_k} - u_0 \rangle = \lim_{n \to \infty} \langle f_k - f, u_{n_k} - u_0 \rangle = 0.
\]

Since \( L \) is of type \( S_+ \), \( u_{n_k} \to u_0 \). By injectivity of \( L \), we have \( u_0 = u \). So \( u_{n_k} \to u \). We claim that \( u_n \to u \) in \( X \). Otherwise, there would exist a subsequence \( \{u_{m_j}\} \) of \( \{u_n\} \) in \( X \) and an \( \epsilon_0 > 0 \), such that for any \( m_j > 0 \), we have \( \|u_{m_j} - u\| \geq \epsilon_0 \). But reasoning as above, \( \{u_{m_j}\} \) would contain a further subsequence \( u_{m_{j_l}} \to u \) in \( X \) as \( l \to \infty \), which is a contradiction to \( \|u_{m_j} - u\| \geq \epsilon_0 \).

\[\square\]

3. Solutions to the equation

In this section we will give the existence results of weak solutions to problem \( (P) \). We denote

\[p_M(x) = \max\{p_1(x), p_2(x)\}, \quad p_m(x) = \min\{p_1(x), p_2(x)\}, \quad \forall x \in \Omega.\]

It is easy to see that \( p_M(x), p_m(x) \in C_+(\Omega) \). For \( q(x) \in C_+(\Omega) \) such that \( q(x) < p^*_M(x) \) for any \( x \in \Omega \), we have \( X := W^{1,p_1(x)}_0(\Omega) \cap W^{1,p_2(x)}_0(\Omega) = W^{1,p_M(x)}_0(\Omega) \to L^{q(x)}(\Omega) \), and the embedding is continuous and compact. To proceed, we give the definition of weak solution to problem \( (P) \):

**Definition 3.1.** We say that \( u \in X \) is a weak solution of \( (P) \) if the following equality

\[
\int_{\Omega} |\nabla u|^{p_1(x)-2}\nabla u \nabla v dx + \int_{\Omega} |\nabla u|^{p_2(x)-2}\nabla u \nabla v dx = \int_{\Omega} f(x,u)v dx
\]

holds for for any \( v \in X := W^{1,p_1(x)}_0(\Omega) \cap W^{1,p_2(x)}_0(\Omega) \).

If \( f \) is independent of \( u \), we have

**Theorem 3.2.** If \( f(x,u) = f(x) \), and \( f \in L^{\alpha(x)}(\Omega) \), where \( \alpha \in C_+(\Omega) \), satisfies \( \frac{1}{\alpha(x)} + \frac{1}{p^*_M(x)} < 1 \), then \( (P) \) has a unique weak solution.

**Proof.** Because \( W^{1,p_M(x)}_0(\Omega) = W^{1,p_1(x)}_0(\Omega) \cap W^{1,p_2(x)}_0(\Omega) \), by Proposition 1.5 (ii), \( \langle f, v \rangle := \int_{\Omega} f(x)v dx \) (for any \( v \in X \)) defines a continuous linear
functional on $X$. By Theorem 3.2 $L$ is a homeomorphism, $(P)$ has a unique solution. □

From now on we always suppose that $f(x, t)$ satisfies assumption $(f_0)$:

$$(f_0) \quad f : \Omega \times \mathbb{R} \to \mathbb{R} \text{ satisfies Caratheodory condition and }$$

$$|f(x, t)| \leq C_1 + C_2 |t|^\alpha(x) - 1, \quad \forall (x, t) \in \Omega \times \mathbb{R},$$

where $\alpha \in C^+(\Omega)$ and $\alpha(x) < p_\ast^\ast(x)$ for any $x \in \overline{\Omega}$.

Let $\varphi(u) := \int_{\Omega} \frac{1}{p_1(x)} |\nabla u|^{p_1(x)} dx + \int_{\Omega} \frac{1}{p_2(x)} |\nabla u|^{p_2(x)} dx - \int_{\Omega} F(x, u) dx, u \in X$, where $F(x, t) = \int_0^t f(x, s) ds$. It is trivial that $\varphi \in C^1(X, \mathbb{R})$. Therefore, weak solutions of $(P)$ correspond to critical points of $\varphi$.

Let $g(u) := \int_{\Omega} F(x, u) dx$. Then $g'(u) : X \to X^*$ is completely continuous, i.e. $u_n \rightharpoonup u$ in $X$ implies $g'(u_n) \to g'(u)$ in $X^*$, thus $g$ is weakly continuous.

**Theorem 3.3.** If $f$ satisfies the following condition,

$$|f(x, t)| \leq C_1 + C_2 |t|^\beta(x) - 1, \quad \text{where } 1 \leq \beta^+ < p_{\ast -}^\ast, \quad (6)$$

then $(P)$ has a weak solution.

**Proof.** By Condition (6) we have $|F(x, t)| \leq C(1 + |t|^\beta(x))$, and

$$\varphi(u) = \int_{\Omega} \frac{1}{p_1(x)} |\nabla u|^{p_1(x)} dx + \int_{\Omega} \frac{1}{p_2(x)} |\nabla u|^{p_2(x)} dx - \int_{\Omega} F(x, u) dx$$

$$\geq \frac{1}{p_M^\ast} \left( \|u_n\|^{p_1^\ast} + \|u_n\|^{p_2^\ast} \right) - C \int_{\Omega} |u|^{\beta(x)} dx - C_3$$

$$\geq \frac{C_5}{p_M^\ast} \|u\|^{p_{\ast -}^\ast} - C_4 \|u\|^{\beta^+} - C_3 \to \infty, \quad \text{as } \|u\| \to \infty.$$  

In view that $\varphi$ is also weakly lower semicontinuous, we see $\varphi$ has a global minimum point $u \in X$, which is a weak solution to problem $(P)$. We now complete the proof. □

**Definition 3.4.** We say that the function $\varphi \in C^1(X, \mathbb{R})$ satisfies the Palais-Smale (PS) condition in $X$ if any sequence $\{u_n\} \subset X$ such that

$$|\varphi(u_n)| \leq B, \quad \text{for some } B \in \mathbb{R};$$

$$\varphi'(u_n) \to 0 \text{ in } X^* \text{ as } n \to \infty$$

has a convergent subsequence.
Lemma 3.5. If $f$ satisfies

(f$_1$) $\exists M > 0, \theta > p^+_M$ such that

$$0 < \theta F(x,t) \leq tf(x,t), |t| \geq M, x \in \Omega,$$

then $\varphi$ satisfies (PS) condition.

Proof. Suppose that \{u$_n$\} $\subset X$, \{\varphi(u$_n$)\} is bounded and \|\varphi'(u$_n$)\| $\to$ 0. Without loss of generality, we could assume \|u$_n$\| $\geq M$ $\geq 1$ for some large constant $M$, and then we have

$$C \geq \varphi(u_n) = \int_\Omega \frac{1}{p_1(x)}|\nabla u_n|^{p_1(x)}dx + \int_\Omega \frac{1}{p_2(x)}|\nabla u_n|^{p_2(x)}dx - \int_\Omega F(x, u_n)dx$$

$$\geq \int_\Omega \frac{1}{p_1(x)}|\nabla u_n|^{p_1(x)}dx + \int_\Omega \frac{1}{p_2(x)}|\nabla u_n|^{p_2(x)}dx - \int_\Omega \frac{u_n}{\theta} f(x, u_n)dx - c$$

$$\geq \int_\Omega \left(\frac{1}{p_1(x)} - \frac{1}{\theta}\right)|\nabla u_n|^{p_1(x)}dx + \int_\Omega \left(\frac{1}{p_2(x)} - \frac{1}{\theta}\right)|\nabla u_n|^{p_2(x)}dx$$

$$\geq \int_\Omega \frac{1}{\theta}(|\nabla u_n|^{p_1(x)} - \frac{1}{\theta})|\nabla u_n|^{p_1(x)} + |\nabla u_n|^{p_2(x)} - u_n f(x, u_n))dx - c$$

$$\geq C_5(\frac{1}{p_1^+(x)} - \frac{1}{\theta})(||u_n||_{p_1(x)}^{p_1(x)} + ||u_n||_{p_2(x)}^{p_2(x)}) + \int_\Omega \left(\frac{1}{p_2(x)} - \frac{1}{\theta}\right)|\nabla u_n|^{p_2(x)} - u_n f(x, u_n)dx - c$$

$$\geq C_5(\frac{1}{p_1^+(x)} - \frac{1}{\theta})||u_n||_{p_1^+(x)}^{p_1^+(x)} - \frac{1}{\theta}||\varphi'(u_n)||\|u_n\| - c.$$

The above inequality implies \{u$_n$\} is bounded in $X$. Without loss of generality, assume that $u_n \rightharpoonup u$, then $g'(u_n) \to g'(u)$. Since $\varphi'(u_n) = L(u_n) - g'(u_n)$, we can get $L(u_n) \to g'(u)$. Since $L$ is a homeomorphism, we conclude $u_n \to u$, which implies $\varphi$ satisfies (PS) condition. □

Theorem 3.6. If $f$ satisfies (f$_0$), in which $\alpha^- > p^+_M$, (f$_1$) and the following

(f$_2$) $f(x, t) = o(|t|^{p^+_M-1}), t \to 0$ for $x \in \Omega$ uniformly,

then (P) has a nontrivial solution.

Proof. We shall show that $\varphi$ satisfies conditions of Mountain Pass theorem.

(i) From Lemma 3.5, $\varphi$ satisfies (PS) condition in $X$. Since $p^+ < \alpha^- \leq \alpha(x) < p^*(x)$, $X \hookrightarrow L^{p^+_M}(\Omega)$, i.e. there exists $C_6 > 0$ such that

$$|u|_{p^+} \leq C_6 \|u\|, \forall u \in X.$$
Let $\epsilon > 0$ be small enough such that $\epsilon C_0^{p_M^+} \leq \frac{C_0}{2p_M^+}$. By the assumptions $f_0$ and $f_2$, we have

$$F(x, t) \leq \epsilon |t|^{p_M^+} + C(\epsilon)|t|^{\alpha(x)} \forall (x, t) \in \Omega \times \mathbb{R}.$$ 

For $\|u\| \leq 1$ we have the following

$$\varphi(u) = \int_{\Omega} \frac{1}{p_1(x)}|\nabla u|^{p_1(x)}dx + \int_{\Omega} \frac{1}{p_2(x)}|\nabla u|^{p_2(x)}dx - \int_{\Omega} F(x, u)dx$$

$$\geq \frac{1}{p_2^+(\lambda_2^{+}(x))} (\|u\|^{p_1^+(\lambda_2^{+}(x))} + \|u\|^{p_2^+(\lambda_2^{+}(x))}) - \epsilon \int_{\Omega} |u|^{p_M^+}dx - C(\epsilon) \int_{\Omega} |u|^\alpha(x)dx$$

$$\geq \frac{C_6}{p_M^+} \|u\|^{p_M^+} - \epsilon C_0^{p_M^+} \|u\|^{p_M^+} - C(\epsilon) \|u\|^\alpha$$

$$\geq \frac{C_6}{2p_M^+} \|u\|^{p_M^+} - C(\epsilon) \|u\|^\alpha,$$

which implies the existence of $r \in (0, 1)$ and $\delta > 0$ such that $\varphi(u) \geq \delta > 0$ for every $u \in X$ satisfies $\|u\| = r$.

(ii) From (f_1) we see for suitable positive constants $C, C'$,

$$F(x, t) \geq C|t|^\theta - C', \forall x \in \overline{\Omega}, t \geq M.$$

For any fixed $w \in X \setminus \{0\}$, and $t > 1$, we have

$$\varphi(tw) = \int_{\Omega} \frac{1}{p_1(x)}|t\nabla w|^{p_1(x)}dx + \int_{\Omega} \frac{1}{p_2(x)}|t\nabla w|^{p_2(x)}dx - \int_{\Omega} F(x, tw)dx$$

$$\leq t^{p_M^+} \left( \int_{\Omega} \frac{1}{p_1(x)}|\nabla w|^{p_1(x)}dx + \int_{\Omega} \frac{1}{p_2(x)}|\nabla w|^{p_2(x)}dx \right) - Ct^\theta \int_{\Omega} |w|^\theta dx - C_7$$

$$\to -\infty, \text{ as } t \to +\infty.$$

(iii) It is obvious $\varphi(0) = 0$.

From (i), (ii) and (iii), we conclude $\varphi$ satisfies the conditions of Mountain Pass Theorem (see [18]). So $\varphi$ admits at least one nontrivial critical point. \(\square\)

**Theorem 3.7.** If $f$ satisfies (f_0), (f_1), $p_M^+ < \alpha^+$ and the following

(f_3) $f(x, -t) = -f(x, t), x \in \Omega, t \in \mathbb{R},$

then $\varphi$ has a sequence of critical points $\{u_n\}$ such that $\varphi(u_n) \to +\infty$ and (P) has infinite many pairs of solutions.

Because $X$ is a reflexive and separable Banach space, there are $\{e_j\} \subset X$ and $\{e_j^*\} \subset X^*$ such that

$$X = \text{span}\{e_j|j = 1, 2, \ldots\}, X^* = \text{span}\{e_j^*|j = 1, 2, \ldots\}$$
and
\[ \langle e_i^*, e_j \rangle = \begin{cases} 1, & i = j; \\ 0, & i \neq j. \end{cases} \]

Denote \( X_j = \text{span}\{e_j\}, Y_k = \bigoplus_{j=1}^k X_j, Z_k = \bigoplus_{j=k}^\infty X_j. \)

**Lemma 3.8.** If \( \alpha \in C_+(\Omega), \alpha(x) < p_M^*(x) \) for any \( x \in \overline{\Omega}, \) denote
\[ \beta_k = \sup\{ |u|_{\alpha(x)} \|u\| = 1, u \in Z_k \}, \]
then \( \lim_{k \to \infty} \beta_k = 0. \)

**Proof.** For \( 0 < \beta_{k+1} \leq \beta_k, \) then \( \beta_k \to \beta \geq 0. \) Suppose \( u_k \in Z_k \) satisfy
\[ \|u_k\| = 1, 0 \leq \beta_k - |u_k|_{\alpha(x)} < \frac{1}{k}, \]
then we may assume up a subsequence that \( u_k \rightharpoonup u \) in \( X, \) and
\[ \langle e_j^*, u \rangle = \lim_{k \to \infty} \langle e_j^*, u_k \rangle = 0, j = 1, 2, ..., \]
in which the last equal sign holds for \( u_k \in Z_k. \) The above equality implies that \( u = 0, \) so \( u_k \rightharpoonup 0 \) in \( X. \) Since the imbedding from \( X \) to \( L^{\alpha(x)}(\Omega) \) is compact, we have \( u_k \to 0 \) in \( L^{\alpha(x)}(\Omega). \) We finally get \( \beta_k \to 0. \) \( \square \)

**Proof of Theorem 3.7.** By \((f_1), (f_3), \varphi\) is an even functional and satisfies \((PS)\) condition. We only need to prove that if \( k \) is large enough, then there exist \( \rho_k > \gamma_k > 0 \) such that
\[ (A_1) \ b_k := \inf\{ \varphi(u) | u \in Z_k, \|u\| = \gamma_k \} \to \infty, (k \to \infty); \]
\[ (A_2) \ a_k := \max\{ \varphi(u) | u \in Y_k, \|u\| = \rho_k \} \leq 0, \]
Then the conclusion of Theorem 3.7 can be obtained by Fountain Theorem (see [16]) and Lemma 3.5.
(A1) For any \( u \in Z_k \) with \( \|u\| \) is big enough, we have
\[
\varphi(u) = \int_\Omega \frac{1}{p_1(x)} |\nabla u|^{p_1(x)} dx + \int_\Omega \frac{1}{p_2(x)} |\nabla u|^{p_2(x)} dx - \int_\Omega F(x, u) dx
\]
\[
\geq \frac{1}{p_1} (\|u\|^{p_1} + \|u\|^{p_2}) - c \int_\Omega |u|^{\alpha(x)} dx - c_1
\]
\[
\geq C_5 \frac{p}{P_M} \|u\|^{p_m} - c |u|^{\alpha(x)} - c_2, \quad \text{where } \xi \in \Omega
\]
\[
\geq \begin{cases} 
C_5 \frac{p}{p_M} \|u\|^{p_m} - c, & \text{if } |u|_{\alpha(x)} \leq 1; \\
C_5 \frac{p}{p_M} \|u\|^{p_m} - c |u|^{\alpha(x)} - c_2, & \text{if } |u|_{\alpha(x)} > 1,
\end{cases}
\]
\[
\geq C_5 \|u\|^{p_m} - c |u|^{\alpha(x)} - c_3
\]
\[
= C_5 \left( \frac{1}{p_M} \|u\|^{p_m} - c \beta_k^{\alpha} \|u|^{\alpha} \right) - c_3.
\]

Set \( \|u\| = \gamma_k = (c \beta_k^{\alpha} \frac{1}{p_m - \alpha}) \). Because \( \beta_k \to 0 \) and \( p_m \leq p_M < \alpha^+ \), we have
\[
\varphi(u) \geq C_5 \left( \frac{1}{p_M} \|u\|^{p_m} - c \beta_k^{\alpha} \|u|^{\alpha} \right) - c_3
\]
\[
= C_5 \left( \frac{1}{p_M} \left( c \beta_k^{\alpha} \frac{1}{p_m - \alpha} \right) - c \beta_k^{\alpha} \left( c \beta_k^{\alpha} \frac{1}{p_m - \alpha} \right) \right) - c_3
\]
\[
= C_5 \left( \frac{1}{p_M} - \frac{1}{\alpha} \right) \left( c \beta_k^{\alpha} \frac{1}{p_m - \alpha} \right) - c_3 \to \infty, \quad \text{as } k \to \infty.
\]

(A2) From (f1), we get
\[
F(x, t) \geq c_1 |t|^\theta - c_2.
\]
Because \( \theta > p_M^+ \) and \( \dim Y_k = k \), it is easy to get \( \varphi(u) \to -\infty \) as \( \|u\| \to \infty \) for \( u \in Y_k \). □

Remark 3.9. We can extend all our results to the \( n \)-case: Consider the following \( (p_1(x), p_2(x), ..., p_n(x)) \)-Laplace equation
\[
(P_n) \left\{ 
- \div(|\nabla u|^{p_n(x)} - 2 \nabla u) - \div(|\nabla u|^{p_2(x)} - 2 \nabla u) - ... - \div(|\nabla u|^{p_1(x)} - 2 \nabla u) = f(x, u), \quad \text{in } \Omega; \\
\quad u = 0, \quad \text{on } \partial \Omega.
\right.
\]
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