Virasoro Representations on $\text{Diff} S^1/S^1$ Coadjoint Orbits

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Abstract

A new set of realizations of the Virasoro algebra on a bosonic Fock space are found by explicitly computing the Virasoro representations associated with coadjoint orbits of the form $\text{Diff} S^1/S^1$. Some progress is made in understanding the unitary structure of these representations. The characters of these representations are exactly the bosonic partition functions calculated previously by Witten using perturbative and fixed-point methods. The representations corresponding to the discrete series of unitary Virasoro representations with $c \leq 1$ are found to be reducible in this formulation, confirming a conjecture by Aldaya and Navarro-Salas.

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1 Introduction

The method of coadjoint orbits originated by Kirillov and Kostant twenty years ago [1] has proven to be a valuable tool in investigating geometrical aspects of the representation theory of Lie groups. The Kirillov-Kostant approach is essentially a generalization of the Borel-Weil theorem, which constructs irreducible unitary representations of a finite-dimensional compact semi-simple Lie group $G$ as spaces of holomorphic sections of complex line bundles over the homogeneous space $G/T$, where $T$ is a maximal subtorus of $G$. In the coadjoint orbit approach, one begins with a group $G$, with Lie algebra $\mathfrak{g}$. The group $G$ has a natural coadjoint action on the dual space $\mathfrak{g}^*$. Choosing an element $b$ in $\mathfrak{g}^*$, one considers the coadjoint orbit $W_b$ of $b$ in $\mathfrak{g}^*$. For any $b$, the space $W_b$ has a natural symplectic form $\omega$. For those $b$ with the property that a complex line bundle $L_b$ can be constructed over $W_b$ with curvature form $i\omega$, one attempts to relate an appropriate space of sections of $L_b$ to an irreducible unitary representation of $G$ by using the technique of geometric quantization on the space $W_b$. For finite-dimensional compact semi-simple $G$, the representations produced by this construction are equivalent to those given by the Borel-Weil theory. The coadjoint orbit approach is particularly useful in the case of non-compact groups, where the Borel-Weil theory does not apply. It is possible to apply the Borel-Weil approach to certain infinite-dimensional groups such as the centrally extended loop groups $\tilde{LG}$ [2]. For other infinite-dimensional groups, such as the (orientation-preserving) diffeomorphism group of the circle $\text{Diff}S^1$, and its central extension $\tilde{\text{Diff}}S^1$, the Virasoro group, there are difficulties with applying even the more general coadjoint orbit theory. Many of the Virasoro coadjoint orbits do not admit a Kähler structure, so that it is difficult to geometrically quantize these spaces. Also, it is known that the Virasoro group has rather peculiar mathematical properties, such as the fact that the exponential map on the Lie algebra is neither onto nor 1-1 in the vicinity of the identity. Due to these difficulties, a full understanding of the coadjoint orbit representations for this group has not yet been attained, although there are some partial results in this direction [3, 4, 5]. Achieving an understanding of the geometry of the coadjoint orbit representations of the Virasoro group could be a valuable step in the general study of Virasoro representations and conformal field theory. In particular, recent work [6, 7, 8] indicates the existence of a relationship between these Virasoro representations and the $SL(2,\mathbb{R})$ current algebra found by Polyakov in 2-d gravity [9].

The coadjoint orbits of $\text{Diff}S^1$ were classified by Segal [6] and Lazutkin and Pankrotova [7]. The Lie algebra of $\text{Diff}S^1$ is the space $\text{Vect}S^1$ of smooth vector fields on $S^1$, and the natural dual to this space is the space of smooth quadratic differentials on $S^1$. The coadjoint orbits of $\tilde{\text{Diff}}S^1$ can be obtained by finding the stabilizers in $\tilde{\text{Diff}}S^1$ of a general quadratic differential $b$ on $S^1$. **
Among other spaces, one finds that $\text{Diff}^1/S^1$ and $\text{Diff}^1/SL^{(n)}(2,\mathbb{R})$ can appear as coadjoint orbit spaces of $\hat{\text{Diff}}^1$, where $SL^{(n)}(2,\mathbb{R})$ is the subgroup of $\text{Diff}^1$ generated by the Virasoro generators $L_0, L_{\pm n}$. Witten has made some progress in relating these coadjoint orbits to the irreducible unitary representations of $\hat{\text{Diff}}^1$, which have previously been classified using algebraic methods. By using perturbative techniques and the fixed point version of the Atiyah-Singer index theorem, Witten was able to calculate the characters of the representations associated with the $\text{Diff}^1/S^1$ orbits, which he found to be the standard bosonic partition function associated with a Virasoro representation with no null states. The perturbative methods used by Witten, however, are only valid in the semi-classical $c \gg 1$ domain. In particular, the structure of the $c \leq 1$ discrete series of unitary representations could not be understood in terms of coadjoint orbits using these techniques. Witten conjectured that these representations would be found in the $\text{Diff}^1/SL^{(n)}(2,\mathbb{R})$ orbits, but since these spaces do not admit Kähler structures, it has not yet been possible to perform geometric quantization in these cases, and the representations associated with these orbits are still not understood.

More recently, related investigations have provided clues to the structure of the Virasoro coadjoint orbit representations. By using a technique involving quantization on a group manifold, Aldaya and Navarro-Salas were able to construct representations of the Virasoro group on spaces of polarized functions on the group manifold $\hat{\text{Diff}}^1$ itself. For those values of $c$ and $h$ where the Kac determinant vanishes (i.e., where the algebraically constructed representation contains a null state), they made the interesting observation that the representation constructed through their method is reducible, yet contains only a single highest weight vector. By taking the orbit of the highest weight vector under the Virasoro action, they get a subspace of the original representation space which corresponds exactly to the appropriate irreducible unitary representation in the $c \leq 1$ discrete series; they did not, however, investigate the existence of unitary structures on their representation spaces. By analogizing their techniques to the coadjoint orbit method, Aldaya and Navarro-Salas conjectured that a similar situation would arise in the $\text{Diff}^1/S^1$ coadjoint orbit representations with $c \leq 1$. In this paper we show explicitly that this is indeed the case.

Another interesting and related approach was taken by Alekseev and Shatashvili. They constructed a natural set of quantum field theories, parameterized by $h$ and $c$, corresponding to quantum mechanical systems on the group manifold $\text{Diff}^1$. These quantum field theories are all symmetric under $S^1$, and thus can be viewed as theories on the coadjoint orbit space $\text{Diff}^1/S^1$. These field theories are constructed in such a fashion that their Hilbert spaces should naturally be associated with the modules carrying coadjoint orbit Virasoro representations. When $h = \frac{-c(a^2-1)}{24}$,
with \( n \) a non-zero integer, these field theories are symmetric under \( SL^{(n)}(2, \mathbb{R}) \), and thus contain a residual symmetry when viewed as theories on \( \text{Diff}S^1/S^1 \). By changing the domain of the fields from \( S^1 \) to \( \mathbb{R} \), effectively dropping the periodicity requirement, Alekseev and Shatashvili came up with a closely related set of theories, all of which have the extra \( SL(2, \mathbb{R}) \) symmetry. In the case where \( h = c/24 \), Alekseev and Shatashvili showed that the action of their field theory coincides with the gravitational Wess-Zumino-Witten (WZW) action, and they used this relationship to interpret the \( SL(2, \mathbb{R}) \) symmetry of the associated 2d gravity theory in a natural way. In a later paper [14], Alekseev and Shatashvili investigated the structure of Virasoro representations by yet another geometrical method involving quantization of the “model space” of the Virasoro group. Finally, Aldaya, Navarro-Salas, and Navarro have also considered a field theory model like that developed by Alekseev and Shatashvili, from their approach of quantization on the group manifold [1]. They achieve a natural understanding of the hidden \( SL(2, \mathbb{R}) \) symmetry in the gravitational WZW model in terms of the separate left- and right-invariant vector fields on the group manifold.

The goal of this paper is to explicitly construct the Virasoro representations associated with the \( \text{Diff}S^1/S^1 \) coadjoint orbits. For every \( c \) and \( h \) such that \( c - 24h \) is not the square of a positive integer, a representation on such an orbit exists. For the exceptional values of \( c \) and \( h \), our formulae still give representations, but in these cases the representations cannot be directly interpreted as arising from coadjoint orbits. The representations are constructed by putting a countable set of holomorphic coordinates \( z_1, z_2, \ldots \) on \( \text{Diff}S^1/S^1 \), and explicitly computing the action of the Virasoro generators \( \hat{L}_n \) on the space \( R \) of polynomials in the \( z_i \)'s (\( R \) is the space of holomorphic sections of an appropriate line bundle over \( \text{Diff}S^1/S^1 \).) The explicit calculation of the operators \( \hat{L}_n \) is accomplished by making a judicious choice of gauge, in which a connection for the desired line bundle can be calculated through a simple recursive procedure. In these representations, the generators \( \hat{L}_n \) act as first-order differential operators on the space \( R \). Although this is a necessary consequence of the general form of the coadjoint orbit construction, this is in some sense a surprising result; most standard representations of the Virasoro algebra in terms of free fields, such as those developed by Feigen and Fuchs [15], involve second order derivatives in some of the generators, when the free fields are rewritten in terms of variables and derivatives. Although the generators \( \hat{L}_n \) are expressed as formal power series with an infinite number of terms, the action of any generator on a fixed polynomial in \( R \) only involves a finite number of terms, and is computable.

Once we have constructed the \( \text{Diff}S^1/S^1 \) representations explicitly, certain aspects of their structure become quite apparent. For all these representations, the character of the representation is easily seen to be exactly the bosonic partition function calculated perturbatively by Witten,
since all polynomials in the $z_i$’s appear in the representation space. For $c$ and $h$ corresponding to the discrete unitary series, one also finds that the representations have exactly the structure predicted by Aldaya and Navarro-Salas. The relationship of these explicit forms for the $\text{Diff}S^1/S^1$ representations to the field theory approach used in [6, 7] is not yet understood. It seems, however, that it should be possible to describe the field theory of Alekseev and Shatashvili in terms of the representations described here, with the ring $R$ becoming the Hilbert space of the quantum theory. This approach could lead to a purely algebraic construction of the theory of 2d gravity. Work in this direction is currently in progress.

The structure of the rest of this paper is as follows: In Section 2, we review the coadjoint orbit approach to constructing representations, and prove several propositions which will justify the “gauge-fixing” procedure we use to construct explicit representations. As an example, we apply this procedure to the group $SU(2)$. In section 3, we carry out the construction in the case of the $\text{Diff}S^1/S^1$ Virasoro orbits, and we discuss the question of unitarity for the resulting representations. Many of the calculations in this section are carried out in a rather formal fashion, without regard to convergence issues and other technicalities related to the infinite-dimensional nature of the Virasoro group. It is presumed that the analysis involved could be reformulated in a more rigorous mathematical language, however we have not attempted to do so here beyond a few brief and necessary digressions. In section 4, we review the salient features of the representations we have constructed, and relate the results of this paper to other recent work.

2 Coadjoint Orbits and Representations

In this section we review the coadjoint orbit approach to group representations, and prove several results which will be essential to our construction of Virasoro representations in section 3. In the introductory paragraphs of this section several standard results on coadjoint orbits are stated without proof; the verifications of these statements are fairly straightforward algebraic manipulations. Otherwise, an attempt has been made to make this paper relatively self-contained. For a more comprehensive introduction to the coadjoint orbit approach to representation theory, the reader should consult Kirillov [1] or Witten [5].

Given a group $G$ with lie algebra $\mathfrak{g}$, consider the space $\mathfrak{g}^*$ dual to $\mathfrak{g}$. The adjoint action of $G$ on $\mathfrak{g}$ associates with each $g \in G$ a map

$$\text{Ad}_g : \mathfrak{g} \rightarrow \mathfrak{g}.$$ (2.1)
When $G$ is a matrix group, one has
\[ \text{Ad}_g : u \mapsto gug^{-1}. \] (2.2)

$G$ also has a dual action on $\mathfrak{g}^*$, denoted $\text{Ad}^*$, where
\[ \langle \text{Ad}^*_g b, u \rangle = \langle b, \text{Ad}_g^{-1}u \rangle, \quad \text{for } b \in \mathfrak{g}^*, u \in \mathfrak{g}. \] (2.3)

The action $\text{Ad}^*$ is referred to as the coadjoint action of $G$ on $\mathfrak{g}^*$. The derivative of the adjoint action gives an action of $\mathfrak{g}$ on $\mathfrak{g}$, denoted by $\text{ad}$, where $\text{ad}_u v = [u, v]$, for all $u, v \in \mathfrak{g}$. Similarly, the infinitesimal coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$ is denoted $\text{ad}^*$, and is given by
\[ \langle \text{ad}^*_v b, u \rangle = \langle b, [u, v] \rangle, \quad \text{for } b \in \mathfrak{g}^*, u, v \in \mathfrak{g}. \] (2.4)

For any $b \in \mathfrak{g}^*$, one can consider its orbit $W_b$ in $\mathfrak{g}^*$ under the coadjoint action of $G$. It turns out that $W_b$ admits a natural symplectic structure, which may be defined as follows: There is a natural association between elements of $\mathfrak{g}$ and tangent vectors to $W_b$ at $b$. Given an element $u \in \mathfrak{g}$, we define $\tilde{u}(b) \in T_bW_b$ to be the tangent vector to $W_b$ at $b$ associated with $\text{ad}^*_b u$. (Note that $\tilde{u}(b) = 0$ when $u$ is in the stabilizer of $b$; i.e., when $\text{ad}^*_b u = 0$.) We can define a 2-form $\omega$ on $W_b$ by
\[ \omega(\tilde{u}(b), \tilde{v}(b)) = \langle b, [u, v] \rangle. \] (2.5)

It can be verified that this 2-form is well-defined, closed, $G$-invariant, and nondegenerate, and thus defines a $G$-invariant symplectic structure on $W_b$. $\omega$ also gives a Poisson bracket structure to the space of functions on $W_b$. In component notation, the Poisson bracket of two functions $f$ and $g$ is given by
\[ \{f, g\} = \omega^{ij}(\partial_i f)(\partial_j g), \] (2.6)

where $\omega^{ij}$ are the components of $\omega^{-1}$. Every function $f$ on $W_b$ generates a Hamiltonian vector field $v_f$ on $W_b$, defined by
\[ v_f^i = \omega^{ij}(\partial_j f). \] (2.7)

For any $u \in \mathfrak{g}$, there is a function $\Phi_u$ on $W_b$ which generates the Hamiltonian vector field $\tilde{u}$. This function is given by
\[ \Phi_u(b) = -\langle b, u \rangle. \] (2.8)

To see that $\Phi_u$ generates the vector field $\tilde{u}$, we use the fact that for any $v \in \mathfrak{g}$,
\[ \tilde{v}^i \partial_j \Phi_u(b) = -\langle b, [u, v] \rangle = \omega_{jk} \tilde{v}^j \tilde{u}^k. \] (2.9)
Since the vector fields $\tilde{v}$ span the tangent space to $W_b$ at $b$, we have
\[ \partial_j \Phi_u(b) = \omega_{jk} \tilde{u}^k(b), \quad (2.10) \]
so
\[ \omega^{ij} \partial_j \Phi_u(b) = \tilde{u}^i. \quad (2.11) \]
The functions $\Phi_u$ also satisfy the equation
\[ \{ \Phi_u, \Phi_v \} = \Phi_{[u,v]}, \quad (2.12) \]
since
\[ \{ \Phi_u, \Phi_v \} = \omega^{ij} (\partial_i \Phi_u)(\partial_j \Phi_v) = \omega^{ij} (\omega_{jk} \tilde{u}^k)(\omega_{jl} \tilde{v}^l) = \omega_{ik} \tilde{u}^k \tilde{v}^l = \{ b, [v,u] \} = \Phi_{[u,v]} \quad (2.13) \]

In order to construct representations of $G$ using the coadjoint orbit $W_b$, it is now necessary to quantize the manifold $W_b$ according to the technique of geometric quantization \[16, 17\]. The first step in this procedure is to construct a complex line bundle $L_b$ over $W_b$ with curvature form $i\omega$. This is known as “prequantization”. For this step to be possible, it is necessary that $\frac{i\omega}{2\pi}$ be an integral cohomology class (i.e., that the integral of $\omega$ over any closed 2-surface in $W_b$ be an integral multiple of $2\pi$). If such a line bundle $L_b$ exists, then there is a natural homomorphism $\phi$ from the Lie algebra $\mathfrak{g}$ to the space of first-order differential operators on sections of $L_b$, given by
\[ \phi : u \mapsto \hat{u} = -\nabla_{\tilde{u}} + i\Phi_u, \quad (2.14) \]
where $\nabla_{\tilde{u}}$ is the covariant derivative in $L_b$ in the direction $\tilde{u}$. Explicitly, written in component notation in a local coordinate chart,
\[ \hat{u} = -\tilde{u}^i(b)(\partial_i + A_i(b)) + i\Phi_u(b), \quad (2.15) \]
where $A_i$ is a connection on $L_b$ satisfying $\partial_i A_j - \partial_j A_i = i\omega_{ij}$. To verify that $\phi$ is a homomorphism, we must check that
\[ [\hat{u}, \hat{v}] = \widehat{[u,v]}. \quad (2.16) \]
We define $\xi_u$ to be the differential operator corresponding to the vector field $-\tilde{u}$; i.e., $\xi_u = -\tilde{u}^i \partial_i$, and we define $A_u = \tilde{u}^i A_i$. With these definitions,
\[ \hat{u} = \xi_u - A_u + i\Phi_u. \quad (2.17) \]
One can easily calculate
\[ [\xi_u, \xi_v] = \xi_{[u,v]}, \tag{2.18} \]
and
\[ \xi_u \Phi_v(b) = \Phi_{[u,v]}(b). \tag{2.19} \]
One also finds that
\[
\xi_u A_v - \xi_v A_u = \left( \tilde{u}, \tilde{v} \right)^i A_i - \tilde{u}^i \tilde{v}^j \left( \partial_i A_j - \partial_j A_i \right)
= \left. A_{[u,v]} + i\Phi_{[u,v]} \right]. \tag{2.20}
\]
Note that since the vectors \( \tilde{u} \) span the tangent space to \( W_b \) at each point, Equation (2.20), along with the conditions that \( A_u \) is linear in \( u \) and that \( A_u(b) = 0 \) when \( \text{ad}_u^* b = 0 \), could have been taken as the definition of a connection \( A_u \) associated with the derivative operators \( \xi_u \). It is now trivial to compute the commutator
\[
[\hat{u}, \hat{v}] = [\xi_u - A_u, \xi_v - A_v + i\Phi_v]
= \xi_{[u,v]} - A_{[u,v]} + i\Phi_{[u,v]}
= \left[ u, v \right]. \tag{2.21}
\]
Thus \( \phi \) is a homomorphism, so we have determined that \( \phi \) gives a representation of \( g \) on the space of smooth sections of \( \mathcal{L}_b \). Unfortunately, this representation is in general much too large to be irreducible; this is where the second stage of geometric quantization enters, which involves choosing a “polarization”. We will only be concerned here with a specific type of polarization, the Kähler polarization. In general, choosing a polarization restricts the space of allowed smooth sections of \( \mathcal{L}_b \) to a subspace containing only those sections which satisfy some local first-order differential equations. A Kähler polarization of \( W_b \) exists when \( W_b \) admits a \( G \)-invariant Kähler structure with \( -\omega \) as the associated \((1,1)\)-form. This condition is equivalent to the condition that \( W_b \) admits a \( G \)-invariant complex structure with respect to which \( \omega \) is a \((1,1)\)-form; i.e., the only nonvanishing terms in \( \omega \) have one holomorphic and one antiholomorphic index. Note that \( -\omega \) is usually constrained to be a positive form as part of the Kähler condition; since this property is not necessary for the construction of representations, we will not impose it here. In general, if \( W_b \) does not admit a Kähler polarization, and is not equivalent to a cotangent bundle, there is no standard way to find a polarization, and carrying out the geometric quantization program becomes extremely difficult. In case \( W_b \) does admit a Kähler polarization, we can restrict the space of allowed sections of \( \mathcal{L}_b \) to the space \( \mathcal{H}_b \) of holomorphic sections. When \( \mathcal{L}_b \) has a Hermitian metric, then we can
restrict $\mathcal{H}_b$ to be the Hilbert space of square-integrable holomorphic sections of $\mathcal{L}_b$. According to the general principles of Kirillov and Kostant, the action of $G$ on $\mathcal{H}_b$ should give an irreducible unitary representation of $G$ for every $b$ such that $\mathcal{H}_b$ can be constructed. This principle holds fairly well for compact semi-simple finite-dimensional groups, and even for loop groups, however it does not seem to hold in complete generality. Some of the representations of $\widehat{\text{Diff}}S^1$ constructed this fashion are nonunitary, and some are reducible.

Now that we have reviewed the standard approach to constructing representations via coadjoint orbits, we can prove several assertions which will simplify the process of explicitly constructing these representations in local coordinates. If one attempts to use Equation 2.17 to construct explicit formulae for the operators $\hat{u}$ as differential operators on $\mathcal{H}_b$, one encounters several obstacles. First, it is necessary to calculate the functions $\Phi_u$ in local coordinates. Second, one must find an explicit formula for a connection $A_u$ which satisfies (2.20). Finding these expressions in terms of a local set of holomorphic coordinates is in general a somewhat nontrivial problem. Note, however that the operator $\hat{u}$ can be written as

$$\hat{u} = \xi_u + f_u,$$

(2.22)

where $\xi_u$ is the first-order differential operator defined above, and $f_u$ is a function of the local coordinates satisfying

$$\xi_u f_v - \xi_v f_u = f_{[u,v]}.$$

(2.23)

We will find it easiest to construct explicit expressions for the operators $\hat{u}$ by finding directly a set of functions $f_u$ which satisfy (2.23), and which correspond to the representation in question. We find these functions $f_u$ by making a simplifying assumption which amounts to choosing a simple gauge for the connection $A_u$. To ensure that the set of $f_u$’s we construct in this fashion are equivalent to those we would get from (2.20) by a specific choice of gauge, we will need the following two propositions.

**Proposition 1** Given a coadjoint orbit $W_b$ of a group $G$, with $\mathcal{L}_b$ a complex line bundle over $W_b$ with curvature $i\omega$, and with $\xi_u$ and $\Phi_u$ defined as above, on a coordinate chart corresponding to a local trivialization of $\mathcal{L}_b$, if a set of functions $f_u$ on $W_b$ are linear in $u \in \mathfrak{g}$, and satisfy the conditions

(i) $\xi_u f_v - \xi_v f_u = f_{[u,v]}$

(ii) $f_u(b) = i\Phi_u(b)$ when $\text{ad}_u^*b = 0$,

then the operators $\hat{u} = \xi_u + f_u$ are equal to the operators $\hat{u}$ from Equation 2.17 for some choice of
connection \( A_u \) on \( \mathcal{L}_b \) satisfying (2.20).

**Proof.** To prove this proposition, it will suffice to show that the functions \( A'_u(b) = -f_u(b) + i\Phi_u(b) \) satisfy (2.20), are linear in \( u \), and are zero when \( \text{ad}_u^*b = 0 \). The last two conditions follow immediately from the definition of \( f_u \) and assumption (ii). To see that \( A'_u \) satisfies (2.20) is a simple calculation:

\[
\xi_u A'_v - \xi_v A'_u = \xi_u(-f_v + i\Phi_v) - \xi_v(-f_u + i\Phi_u)
\]

\[
= -f_{[u,v]} + 2i\Phi_{[u,v]}
\]

\[
= A'_{[u,v]} + i\Phi_{[u,v]}.
\] (2.24)

Thus, \( A'_u \) is a valid connection on \( \mathcal{L}_b \), and the proposition is proven. □

**Proposition 2** With the same premises as Proposition 1, when \( G \) is path connected the condition (ii) can be replaced by the weaker condition

(ii') For some point \( b_0 \in W_b \), \( f_u(b_0) = i\Phi_u(b_0) \) for all \( u \) such that \( \text{ad}_u^*b_0 = 0 \),

and the result of proposition 1 still holds.

**Proof.** We need to prove that when \( G \) is path connected, condition (ii') implies condition (ii). Assume \( \text{ad}_u^*b = 0 \) for some \( u \in \mathfrak{g}, b \in W_b \). Since \( b_0 \in W_b \), for some \( g \in G \) we have \( b = \text{Ad}_g b_0 \). If \( u \) stabilizes \( b \), then \( u_0 = \text{Ad}_g^{-1}u \) must stabilize \( b_0 \). But then we have

\[
\langle b, u \rangle = \langle \text{Ad}_g^* b_0, \text{Ad}_g u_0 \rangle = \langle b_0, u_0 \rangle,
\] (2.25)

so \( \Phi_u(b) = \Phi_{u_0}(b_0) \). It remains to be shown that \( f_u(b) = f_{u_0}(b_0) \). Since \( G \) is path connected, we have a path \( g(t) \) in \( G \) with \( g(0) = 1 \) and \( g(1) = g \). We claim that

\[
\frac{d}{dt} f_u(t)(b(t)) = 0,
\] (2.26)

where \( u(t) = \text{Ad}_{g(t)} u_0 \), and \( b(t) = \text{Ad}_{g(t)} b_0 \). Defining

\[
v(t) = \frac{dg(t)}{dt} g^{-1}(t) \in \mathfrak{g},
\] (2.27)

we have

\[
\frac{d}{dt} b(t) = \text{ad}_v^* b(t),
\] (2.28)

and

\[
\frac{d}{dt} u(t) = \text{ad}_v u(t).
\] (2.29)
It follows that
\[
\frac{d}{dt} u(t)(b(t)) = -\xi_v f_u(t)(b(t)) + f_{[v,u(t)]}(b(t)) \\
= -\xi_v f_u(t)(b(t)) + \xi_u f_v(b(t)) + f_{[v,u(t)]}(b(t)) \\
= 0,
\]
where we have used the fact that \( \tilde{u}(t)(b(t)) = 0 \). Thus, we have shown that
\[
f_u(b) = f_{u_0}(b_0) = i\Phi_{u_0}(b_0) = i\Phi_u(b).
\]
Since \( u \) and \( b \) were an arbitrary solution of \( \text{ad}^*_u b = 0 \), we have proven that condition (2.ii') implies condition (2.ii), and thus the proposition is proven. \( \Box \)

We will now as an example use the coadjoint orbit approach to construct representations of \( SU(2) \). For a similar discussion from the point of view of the Borel-Weil theorem, see Alvarez, Singer, and Windey [18]. Take the generators of the algebra \( \mathfrak{g} = su(2) \) to be \( \{iJ_k : k = 1, 2, 3\} \), where \([J_j, J_k] = i\epsilon_{jkl}J_l\). \( \mathfrak{g} \) is a three-dimensional real vector space. Taking coordinates \( x^1, x^2, x^3 \) on \( \mathfrak{g} \), an arbitrary element \( u \in \mathfrak{g} \) can be written as \( u = i\Sigma x^k J_k \). An arbitrary element \( g \) of \( G \) can be written as \( g = e^u \), where \( u \in \mathfrak{g} \). In a vicinity of the identity, this description of \( g \) is unique. The adjoint representation of \( G \) acts on \( \mathfrak{g} \) via rotations which preserve the Euclidean scalar product; the generator \( iJ_k \) corresponds to rotation about the \( x^k \) axis. Since \( \mathfrak{g} \) is finite-dimensional, \( \mathfrak{g}^* \) can be identified with \( \mathfrak{g} \) using the Euclidean scalar product. Under this identification, the coadjoint action of \( G \) on \( \mathfrak{g}^* \) is also given by rotations. Given a vector \( b = (b_1, b_2, b_3) \in \mathfrak{g}^* \), where \( \langle b, iJ_k \rangle \) is defined to be \( b_k \), the coadjoint orbit of \( b \) is given by
\[
W_b = \{ b' \in \mathfrak{g}^* : |b'|^2 = b^2 \},
\]
which is just the 2-sphere in \( \mathfrak{g}^* \) of radius \( b = |b| \). We will now explicitly calculate the 2-form \( \omega \) on \( W_b \). We choose a canonical element \( b_0 = (0, 0, b) \in W_b \). To calculate \( \omega \) at the point \( b_0 \), we need only find the explicit correspondence between elements of \( \mathfrak{g} \) and \( T_{b_0}W_b \). Under the Lie algebra coadjoint action, we have
\[
\text{ad}^*_{iJ_1} b_0 = (0, b, 0), \\
\text{ad}^*_{iJ_2} b_0 = (-b, 0, 0), \\
\text{ad}^*_{iJ_3} b_0 = (0, 0, 0).
\]
It follows that
\[
\omega_{12}(b_0) = \langle b_0, \left[ \frac{-iJ_2}{b}, \frac{iJ_1}{b} \right] \rangle = -\frac{1}{b}.
\]
Since $\omega$ is $G$-invariant, it is easy to see that $\omega$ is defined globally on $W_b$ by
\[ \omega_{ij}(b) = -\frac{1}{b^2} \epsilon_{ijk} b_k. \] (2.35)

In order for $\omega/2\pi$ to be an integral form, we must have $\int_{W_b} \omega/2\pi = -2b \in \mathbb{Z}$, so $b$ must be a half-integer. Thus, whenever $b \in \mathbb{Z}/2$, we can construct a line bundle $\mathcal{L}_b$ over $W_b$ with curvature form $i\omega$.

We would now like to find a $G$-invariant Kähler structure on $W_b$ compatible with $\omega$, so that we can restrict attention to holomorphic sections of $\mathcal{L}_b$, according to the prescription of geometric quantization. A standard result from group theory allows us to describe this complex structure from an algebraic viewpoint which will be useful in the case of the Virasoro group. For a similar approach to this construction see Zumino [19].

Consider the stabilizer $H$ in $G$ of $b_0$ under the coadjoint $G$-action on $g^\ast$. $H$ is clearly just the $\text{U}(1)$ subgroup generated by $iJ_3$,
\[ H = \{ e^{itJ_3} : t \in \mathbb{R} \}. \] (2.36)

There is a 1-1 correspondence between points in $W_b$ and $G/H$, since for every $b \in W_b$, there exists a $g \in G$ such that $b = \text{Ad}_g^* b_0$, and for $g, g' \in G$,
\[ \text{Ad}_g^* b_0 = \text{Ad}_{g'}^* b_0 \text{ iff } g = g'h \text{ for some } h \in H. \] (2.37)

Note that the coadjoint action of $G$ on $W_b$ corresponds to the left action of $G$ on $G/H$. We will now describe a natural complex structure on $G/H$. If we define
\[ J_\pm = J_1 \pm iJ_2, \] (2.38)
then $[J_3, J_\pm] = \pm J_\pm$, and $[J_+, J_-] = 2J_3$. Given any complex number $z$, it is possible to find functions $\alpha(z, \bar{z})$ and $\beta(z, \bar{z})$, with $\beta(z, \bar{z})$ real, such that
\[ e^{zJ_-} e^{\alpha(z, \bar{z}) J_+} e^{\beta(z, \bar{z}) J_3} \in G. \] (2.39)

The functions $\alpha(z, \bar{z})$ and $\beta(z, \bar{z})$ can be calculated explicitly by working in the fundamental representation of $\text{SU}(2)$; one finds that
\[ \alpha(z, \bar{z}) = \frac{-\bar{z}}{1 + |z|^2}, \] (2.40)
\[ \beta(z, \bar{z}) = -\ln(1 + |z|^2). \]

Alternatively, these functions can be calculated in a perturbative expansion about the identity, by applying the Baker-Campbell-Hausdorff (BCH) formula [4]
\[ e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]+...}, \] (2.41)
which expresses the product of two exponentiated elements of a Lie algebra in terms of a single
exponentiated element of the algebra as a formal power series. (The ellipses in this formula denote
third- and higher-order commutators between \(X\) and \(Y\).) The BCH approach will be used in
the next section to describe the complex coordinate system we will use on \(\text{Diff}S^1/S^1\), so we will
concentrate on this approach here also, rather than using the more convenient exact expressions
which can be derived for \(SU(2)\).

In some neighborhood of the identity, any element \(g \in G\) can be expressed uniquely in the form
\[
g = e^{zJ_1} e^{\alpha(z,\bar{z})J_+} e^{\beta(z,\bar{z})J_3 + i\psi J_3}
\tag{2.42}
\]
with \(\psi\) real, so locally at least, \(z\) is a good complex coordinate on \(G/H\). To see that the complex
structure defined by \(z\) is invariant under the left action of \(G\) on \(G/H\), we multiply the above
expression for \(g\) on the left by an element \(g' \in G\), also in the form (2.42), and get
\[
g'g = (g' e^{zJ_1}) e^{\alpha(z,\bar{z})J_+} e^{\beta(z,\bar{z})J_3 + i\psi J_3} e^{\alpha'(z',\bar{z}')J_+} e^{\beta'(z',\bar{z}')J_3 + i\psi' J_3}
\]
\[
= (e^{zJ_1}) e^{\alpha(z,\bar{z})J_+} e^{\beta(z,\bar{z})J_3 + i\psi J_3} e^{\alpha'(z',\bar{z}')J_+} e^{\beta'(z',\bar{z}')J_3 + i\psi' J_3},
\tag{2.43}
\]
where the BCH formula has again been applied several times, first to rewrite \(g' \exp(zJ_1)\) as a single
exponential, then again to separate out the coefficients \(\alpha''\) and \(\beta''\). By another round of BCH-type
manipulations, since \(\{J_3, J_+\}\) generate a closed subalgebra, we have
\[
g'g = e^{zJ_1} e^{\alpha'J_+} e^{\beta'J_3 + i\psi'J_3},
\tag{2.44}
\]
for some coefficients \(\alpha', \beta', \) and \(\psi'\), with \(\beta'\) and \(\psi'\) real. Since \(g'g \in G\), we must have \(\alpha' = \alpha(z', \bar{z}')\)
and \(\beta' = \beta(z', \bar{z}')\). The value of \(z'\) is purely a function of \(g'\) and the holomorphic coordinate \(z\). Thus,
for fixed \(g'\), \(z'\) is a holomorphic function of \(z\), so the complex structure defined by \(z\) is invariant
under left multiplication by \(g'\). In fact, \(z\) is just the usual complex coordinate on \(S^2\) given by
projection from the south pole onto \(\mathbb{C}\), which is naturally invariant under the rotations generated
by \(SU(2)\).

From (2.39) and (2.40), we can relate the differentials \(dz, d\bar{z}\) to our original coordinates \(b_i\). At
\(b_0\), we have
\[
dz = \frac{1}{2b}(db_1 + idb_2),
\tag{2.45}
\]
\[
d\bar{z} = \frac{1}{2b}(db_1 - idb_2).
\]
It is now possible to express \(\omega\) at \(b_0\) in terms of the \(z, \bar{z}\) coordinates; one finds that
\[
\omega_{z\bar{z}} = -\omega_{z\bar{z}} = 2bi,
\tag{2.46}
\]
\[
\omega_{zz} = -\omega_{\bar{z}\bar{z}} = 0.
\]
Thus, \( \omega \) is indeed a \((1,1)\)-form, and along with the \( G \)-invariant complex structure given by \( z \), defines a Kähler structure on \( W_b \). We can therefore restrict attention to the space \( \mathcal{H}_b \) of holomorphic sections of \( \mathcal{L}_b \). Since \( \frac{i\omega}{2\pi} \) is the first Chern class of \( \mathcal{L}_b \), for \( b \geq 0 \) it is a simple result of the Riemann-Roch theorem that \( \mathcal{L}_b \) admits exactly \( 2b + 1 \) linearly independent holomorphic sections (see for example Griffiths and Harris [20]). By choosing the proper local trivialization of \( \mathcal{L}_b \), the \( 2b + 1 \) holomorphic sections are represented in the vicinity of the origin \( z = \bar{z} = 0 \) by the holomorphic monomials, \( 1, z, z^2, \ldots, z^{2b} \). \( \mathcal{L}_b \) also has a natural Hermitian metric, which we will discuss further at the end of this section.

We now have a Hilbert space \( \mathcal{H}_b \), and there exist a set of operators \( \hat{J}_3, \hat{J}_\pm \), given by

\[
\hat{J}_a = \xi_a - A_a + i\Phi_a, \tag{2.47}
\]

for \( a = 3, \pm \), which act on \( \mathcal{H}_b \) to give a representation of \( \mathfrak{g}_C \). We wish to compute these operators explicitly in terms of the complex coordinate \( z \). First, we compute the vector fields \( \tilde{u}_a \) corresponding to the actions of \( \text{ad}^* J_a \). (Technically, these are vector fields in the complexification of the tangent space to \( W_b \).) Since we are only concerned with the action of the differential operators associated with these vector fields on holomorphic sections of \( \mathcal{L}_b \), which will be written as holomorphic functions of \( z \), it is only necessary to compute the component of these vector fields in the \( \partial/\partial z \) direction. To compute the vector field at \( z \) associated with the coadjoint action of a generator \( J_a \in \mathfrak{g}_C \), we must express the product \( \exp(\epsilon J_a) \exp(zJ_-) \) in the form

\[
e^{\epsilon J_a} e^{zJ_-} = e^{(z+\epsilon \tilde{u}_a)J_-} f(J_3, J_+) + O(\epsilon^2) \tag{2.48}
\]

to first order in \( \epsilon \), where \( f(J_3, J_+) \) is some function of the generators \( J_3 \) and \( J_+ \). Since \( \{J_3, J_+\} \) generate a closed subalgebra of \( \mathfrak{g}_C \), \( \tilde{u}_a \partial/\partial z \) will be the tangent vector to \( W_b \) at \( z \) associated with the action of \( J_a \). The corresponding differential operator \( \xi_a \) will then be defined by \( \xi_a = -\tilde{u}_a \partial/\partial z \).

To explicitly compute these operators, we will use the infinitesimal forms of the BCH theorem (for a derivation of these forms, see for example Kirillov [1]),

\[
e^{\epsilon X} e^{Y} = \exp(Y + \epsilon \sum_{k \geq 0} \frac{B_k}{k!} (\text{ad}_Y)^k X) + O(\epsilon^2), \tag{2.49}
\]

\[
e^{Y+\epsilon Z+\epsilon X} = \exp \left( Y + \epsilon Z - \epsilon \sum_{k \geq 1} \frac{B_k}{k!} (-\text{ad}_Y)^k X \right) e^{\epsilon X} + O(\epsilon^2), \tag{2.50}
\]

and

\[
e^{\epsilon X} e^{Y} = e^{Y+[X,Y]} e^{\epsilon X} + O(\epsilon^2), \tag{2.51}
\]
where $B_k$ is the $k$th Bernoulli number; $B_0 = 1, B_1 = -1/2, B_2 = 1/6, \ldots$. Applying these formulae, we have

\[
e^{\epsilon J_3} e^{z J_-} = e^{(z-\epsilon z) J_-} e^{\epsilon J_3} + \mathcal{O}(\epsilon^2),
\]

\[
e^{\epsilon J_-} e^{z J_-} = e^{(z+\epsilon) J_-},
\]

and

\[
e^{\epsilon J_+} e^{z J_-} = e^{z J_- + 2\epsilon z J_3} e^{\epsilon J_+} + \mathcal{O}(\epsilon^2)
\]

\[= e^{(z-\epsilon z^2) J_-} e^{2\epsilon z J_3} e^{\epsilon J_+} + \mathcal{O}(\epsilon^2).
\]

From these expressions, we can write the differential operators $\xi_a$,

\[
\begin{align*}
\xi_3 &= z \frac{\partial}{\partial z}, \\
\xi_- &= - \frac{\partial}{\partial z}, \\
\xi_+ &= z^2 \frac{\partial}{\partial z}.
\end{align*}
\]

One can verify that these operators satisfy the proper commutation relations. We will now use the result of Proposition 2 to construct the operators $\hat{J}_a$ explicitly, by choosing a convenient form for the functions $f_a = -A_a + i \Phi_a$. The operators $\xi_a$ act on $\mathcal{H}_b$, which in local coordinates is a subspace of the ring $\mathbb{C}[z]$ of polynomials in $z$. $\mathbb{C}[z]$ is a graded ring, with a grading defined by $\deg(z^n) = n$. The eigenvectors of $\xi_3$ are exactly the functions $z^n$ of fixed degree, with eigenvalues equal to the degrees, so that $\xi_3 z^n = nz^n$. A natural Ansatz on the form of $f_3$ would be to insist that the operator $\hat{J}_3 = \xi_3 + f_3$ have the same eigenvectors as $\xi_3$. This Ansatz implies that $f_3$ is a constant function, and is equivalent to performing a gauge fixing on $A_a$, given by $A_3 = -f_3 + i \Phi_3$. A priori, it is not obvious that this choice of gauge is possible, i.e., that this Ansatz is compatible with the conditions $(i)$ and $(ii)$ on the functions $f_a$ from Propositions 1 and 2. We will proceed, however, to explicitly construct functions consistent with both the Ansatz and these conditions. In fact, it turns out that the connection associated with this choice of gauge is exactly the metric connection on $\mathcal{L}_b$ associated with the natural Hermitian structure.

The constant $f_3$ is uniquely fixed to be $-b$ by condition $(ii)$, since $J_3$ stabilizes $b_0$, and $b_0(J_3) = -ib$. From $(i)$ we have the relations

\[
\xi_3 f_\pm - \xi_\pm f_3 = \pm f_\pm = \xi_3 f_\pm,
\]

so it follows that $f_- = 0$, and that $f_+$ is linear in $z$. To fix the coefficient of $f_+$, we use $(i)$ again, to show that

\[
\xi_+ f_- - \xi_- f_+ = -\xi f_+ = \frac{\partial f_+}{\partial z} = 2f_3 = -2b.
\]
Thus, we have constructed a set of functions

$$f_3 = -b, f_- = 0, f_+ = -2bz,$$

which satisfy conditions \((i)\) and \((ii')\) of Propositions 1 and 2. For every \(b \in \mathbb{Z}/2\), then, we have explicitly constructed a representation of \(SU(2)\) on the space of polynomials in \(z\) of degree less than or equal to \(2b\), given by the operators

$$
\begin{align*}
\hat{J}_3 &= z \frac{\partial}{\partial z} - b, \\
\hat{J}_- &= - \frac{\partial}{\partial z}, \\
\hat{J}_+ &= z^2 \frac{\partial}{\partial z} - 2bz.
\end{align*}
$$

(2.58)

To conclude this section, we will discuss briefly the unitary structure of the coadjoint orbit representations of \(SU(2)\) on \(S^2\). The rotationally invariant measure on \(S^2\) in the coordinates we are using is

$$d\mu = \frac{2i}{(1 + |z|^2)^2} dzd\bar{z}.$$  

(2.59)

Along with this measure, there is a natural Hermitian metric on \(\mathcal{L}_b\), given by

$$e^{h(z, \bar{z})} = \frac{1}{(1 + |z|^2)^{2b}}.$$  

(2.60)

Combining these factors, there is an inner product \(\langle, \rangle\) on \(\mathcal{H}_b\) given by

$$\langle \phi, \psi \rangle = \int_{S^2} d\mu \ e^{h} \phi^* \psi = \int_{S^2} \frac{2idzd\bar{z}}{(1 + |z|^2)^{2b+2}} \phi^*(z)\psi(z),$$  

(2.61)

where \(\phi\) and \(\psi\) are arbitrary holomorphic sections of \(\mathcal{L}_b\). Performing this integral explicitly, one finds that

$$\langle z^l, z^m \rangle = \delta_{l,m} 4\pi \left[ (2b + 1) \binom{2b}{l} \right]^{-1}.$$  

(2.62)

This inner product on \(\mathcal{H}_b\) is of course proportional to the usual inner product on unitary irreducible representation spaces of \(SU(2)\). In fact, the inner product (2.62) could have been calculated up to a constant normalization factor by fixing \(\langle 1, 1 \rangle = 1\) and assuming that the representation of \(SU(2)\) on \(\mathcal{H}_b\) described by (2.58) is unitary with \(J_3^* = J_3\) and \(J_- = J_+\). This approach to calculating an inner product on \(\mathcal{H}_b\) is equivalent to the one used in the abstract algebraic construction of unitary \(SU(2)\) representations. It does not allow us to directly calculate the Hermitian metric on \(\mathcal{L}_b\), but nonetheless provides \(\mathcal{H}_b\) with a Hilbert space structure. In the case of \(\text{Diff}S^1/S^1\), there is not
a well-defined invariant measure on the orbit space, and it will be necessary to use this indirect
method to put unitary structures on those representations which admit them.

We will now show that the connection $A$ on $\mathcal{L}_b$ defined previously by our gauge-fixing procedure
is precisely the metric connection on $\mathcal{L}_b$ associated with the Hermitian metric $\exp(h)$. Recall that
in general a Hermitian line bundle with Hermitian metric $\exp(h)$ has a metric connection given by
$A_\bar{z} = 0$, $A_z = \partial h / \partial z$. This is the unique connection compatible with both the Hermitian metric
and the complex structure \[20\]. Thus, the Hermitian connection on $\mathcal{L}_b$ is given by

$$A_z = \frac{-2b\bar{z}}{1 + |z|^2}. \quad (2.63)$$

In terms of $A_z$, the connection terms $A_a$, $a = 3, \pm$, are given by

$$A_3 = -zA_z \quad A_+ = -z^2 A_z \quad A_- = A_z. \quad (2.64)$$

The connection $A_a = -f_a + i\Phi_a$ can be explicitly calculated by evaluating

$$\Phi_a(z) = -\langle \Ad_g^*(z,\psi) b_0, J_a \rangle = -\langle b_0, e^{-\alpha(z,\bar{z})J_+} e^{-zJ_-} J_a e^{zJ_+} e^{\alpha(z,\bar{z})J_+} \rangle. \quad (2.65)$$

One finds that

$$A_3 = -2bz\alpha(z,\bar{z}), \quad A_+ = -2bz^2\alpha(z,\bar{z}), \quad A_- = 2b\alpha(z,\bar{z}). \quad (2.66)$$

This is exactly the Hermitian connection on $\mathcal{L}_b$.

3 Diff$^1$/$S^1$ Virasoro representations

We will now turn our attention to the coadjoint orbits of the Virasoro group, $\widehat{\text{Diff}}^1$. The Virasoro
group is the universal central extension of the group of orientation-preserving diffeomorphisms of
the circle, Diff$^1$. (For a clear discussion of central extensions and infinite-dimensional groups, see \[2\].) Elements of $\widehat{\text{Diff}}^1$ are given by pairs $(\phi, \alpha)$, with $\phi \in \text{Diff}^1$, and $\alpha \in U(1)$. The Lie algebra
of $\widehat{\text{Diff}}^1$, which we denote $\text{Vect}^1$, is likewise the universal central extension of the algebra of
smooth vector fields on $S^1$. Elements of $\hat{\text{Vect}} S^1$ are of the form $(f, -ia)$, with $f(\theta)\partial/\partial \theta$ a vector field on $S^1$ and $a \in \mathbb{R}$. (Except for a few signs, we mostly use the notation of Witten \[5\] in this section.) The commutation relation between elements of $\hat{\text{Vect}} S^1$ is given by

$$[(f, -ia_1), (g, -ia_2)] = \left( fg' - gf', \frac{i}{48\pi} \int_0^{2\pi} (f(\theta)g'''(\theta) - g(\theta)f'''(\theta))d\theta \right).$$

(3.1)

Defining the (complex) vector fields $l_n = ie^{in\theta}\partial/\partial \theta$ in $\text{Vect} S^1$, we can define the usual Virasoro generators by

$$L_n = (l_n, 0); \text{ for } n \neq 0,$$

$$L_0 = (l_0, \frac{1}{24}),$$

$$C = (0, 1).$$

(3.2)

The commutation relations then take the standard form

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{C}{12}(m^3 - m)\delta_{m,-n},$$

$$[C, L_n] = 0.$$  

(3.3)

The Virasoro algebra is defined to be the complex Lie algebra spanned by the generators \(L_n, C\). The (smooth) dual space to $\hat{\text{Vect}} S^1$ consists of pairs $(b, it)$, with $b(\theta)d\theta^2$ a quadratic differential on $S^1$, and $t \in \mathbb{R}$. The dual pairing between $(b, it)$ and an element $(f, -ia) \in \hat{\text{Vect}} S^1$ is given by

$$\langle (b, it), (f, -ia) \rangle = \int_0^{2\pi} b(\theta)f(\theta)d\theta + at.$$  

(3.4)

For this pairing to be invariant under the action of the algebra $\hat{\text{Vect}} S^1$, $(b, it)$ must transform under the coadjoint action by

$$\text{ad}_{(f, -ia)}^*(b, it) = (2bf' + b'f - \frac{t}{24\pi}f'''(\theta), 0).$$

(3.5)

By computing the stabilizer of a general dual element $(b, it)$, it is possible to completely classify the coadjoint orbits of $\hat{\text{Diff}} S^1$. A clear review of this analysis in the general case is given in \[3\]. We will only be concerned here with the simplest case, in which the orbit contains an element $(b_0, ic)$ with $b_0(\theta) = b_0$ a constant function. We will refer to this orbit as $W_{b_0, c}$. In this case, the stabilizer in $\hat{\text{Vect}} S^1$ of the point $(b_0, ic)$ is given by all elements $(f, -ia)$ with $f(\theta)$ satisfying

$$\frac{c}{24\pi}f''' = 2b_0f'.$$

(3.6)

When $-48\pi\frac{b_0}{c}$ is not the square of an integer $n$, the only solution to this equation with period $2\pi$ is $f(\theta) = 1$. In this case, the stabilizer of $(b_0, ic)$ is the subgroup generated by $L_0$ and $C$, so the
the components of $\omega$ notation by computing directly the complex linear combinations of those components given by corresponding to the generators of $\hat{\text{form}}$ of the operators $\hat{\mathcal{L}}$ with respect to which $\omega$ will follow in constructing the Virasoro representation corresponding to $\text{SU}(2)$, which we followed for $\text{Diff}_S^1$ case since the space $\text{Diff}_S^1$ is a (1,1)-form. In this coordinate system, we will explicitly calculate the form of the operators $\hat{\mathcal{L}}_n$, using the result of Proposition 3. Let us proceed to evaluate $\omega$ at the point $(b_0, ic)$. A basis for the tangent space to $W_{b_0,c}$ at $(b_0, ic)$ is given by the real vector fields corresponding to the generators of $\text{Diff}_1^1$, $\{L_n - L_{-n}, i(L_n + L_{-n}) : n \geq 1\}$. We could calculate the components of $\omega$ with respect to this basis; it will be most convenient, however, to simplify our notation by computing directly the complex linear combinations of those components given by

$$\omega_{m,n} = \langle (b_0, ic), [L_m, L_n] \rangle = i\delta_{m,-n}(4\pi m b_0 + \frac{c}{12} m^3). \quad (3.7)$$

This 2-parameter family of symplectic structures is in fact the most general form for an invariant 2-form on $\text{Diff}_1^1/S^1$. Note that when $b_0 = -\frac{cn^2}{48\pi}$, the 2-form $\omega$ is degenerate, and thus is not a symplectic form.

We now make the observation that $\text{Diff}_1^1/S^1$ is a contractible space. To see this, note that $\text{Diff}_1^1/S^1$ can be identified with the group $\text{Diff}_0^1$ of orientation-preserving diffeomorphisms of $S^1$ which fix the point 1. Viewing an element of $\text{Diff}_0^1$ as a monotonically increasing function $f : \mathbb{R} \to \mathbb{R}$ with the properties $f(0) = 0$ and $f(x+2\pi) = 2\pi + f(x)$, we can explicitly give a retraction of $\text{Diff}_0^1$ to a point by defining the one-parameter family of functions $f_t(x) = (1-t)f(x)+tx$, for each $f \in \text{Diff}_0^1, t \in [0,1]$. Since $\text{Diff}_1^1/S^1$ is a contractible space, all 2-cycles are homologous to the null 2-cycle, so that $\int_a \omega = 0$ for any 2-cycle $a$. Thus, for any $b_0, c$, we can construct a line bundle $\mathcal{L}_{b_0,c}$ over $W_{b_0,c}$ with curvature $i\omega$. (Note that the second cohomology of $\text{Diff}_1^1/S^1$ is nontrivial if one restricts to forms invariant under $\text{Diff}_1^1$, however this should not affect the construction of $\mathcal{L}_b$; it does however imply that $\mathcal{L}_b$ will not have a global $\text{Diff}_1^1$-invariant connection.)

The next step in our construction is to give a complex structure to $W_{b_0,c}$ with respect to which
$\omega$ is a (1,1)-form. This complex structure can be constructed in a fashion similar to the one used for $SU(2)$ in the previous section. Given a countable set of variables $z = \{z_1, z_2, \ldots\}$, there exist unique functions $\mu_n(z, \bar{z}), \rho(z, \bar{z}), \gamma(z, \bar{z})$, expressed as formal power series in the $z_i's$, such that $\rho(z, \bar{z})$ and $\gamma(z, \bar{z})$ are real and

$$\exp(\sum_{n>0} z_n L_n) \exp(\sum_{n>0} \mu_n(z, \bar{z}) L_{-n}) \exp(\rho(z, \bar{z}) L_0) \exp(\gamma(z, \bar{z}) C) \in \text{Diff}_0 S^1. \quad (3.8)$$

As in the case of $SU(2)$, the functions $\mu_n, \rho, \gamma$ can be explicitly calculated order-by-order in the $z$’s by applying the BCH formula. Up to second order terms in the $z$’s, these functions are given by

$$\mu_n(z, \bar{z}) = -\bar{z}_n + \sum_{m>0} (n + 2m) \bar{z}_m z_{n+m} + \mathcal{O}(z^3),$$

$$\rho(z, \bar{z}) = \sum_{k>0} k |z_k|^2 + \mathcal{O}(z^3), \quad (3.9)$$

$$\gamma(z, \bar{z}) = \sum_{k>0} \frac{k^3 - k}{24} |z_k|^2 + \mathcal{O}(z^3).$$

The complex variables defined by (3.8) were successfully used in a previous work of Zumino \[19\] to calculate the curvature of $\text{Diff} S^1 / S^1$. (This curvature calculation was first done by Bowick and Rajeev using other methods \[21\].) Nevertheless, the unusual geometry of the Virasoro group (specifically the fact that the exponential map is not locally 1-1 or onto) calls the validity of this coordinate system into question; thus, we will briefly outline an argument justifying this choice of coordinate system. We will not attempt to be mathematically rigorous here; work is currently underway to provide a mathematically complete justification for this point of view. The basic point is that the exponential map on $\text{Diff} S^1$ fails to be well-behaved due to diffeomorphisms which are either non-analytic, or contain no fixed points \[22, 2\]. By restricting to $\text{Diff}_0 S^1$, we eliminate the latter problem. If we also restrict attention to the subgroup of $\text{Diff}_0 S^1$ consisting of diffeomorphisms which are real-analytic, the problems with the exponential map relating to non-analytic diffeomorphisms are also removed. There are several reasons that it is reasonable to restrict attention to real-analytic diffeomorphisms. The first is that in most cases of physical interest, non-analytic maps are not of concern; in conformal field theory, for instance, conformal transformations on the world-sheet which leave a time-slice fixed are real-analytic diffeomorphisms on that time-slice. The second reason for allowing the restriction to real-analytic diffeomorphisms is that Goodman and Wallach have shown \[23\], following a conjecture of Kac \[24\], that every unitary representation of the algebra of real-analytic vector fields on $S^1$ can be integrated to a continuous
unitary representation of $\text{Diff}^1 S^1$. (Actually, their proof assumes only that a representation can be found for the algebra of vector fields with finite Fourier series.)

We will assume that for the group $D_0$ of real-analytic diffeomorphisms in $\text{Diff}^0 S^1$, the exponential map from the associated algebra is 1-1 and onto. We do not yet have a rigorous proof of this assertion, however it is not hard to show that the exponential map is 1-1 and onto in the closely related case where we consider the group of all formal power series in one variable with first nonvanishing term positive and linear, under the group law given by composition of functions. With this assumption, we have a set of coordinates $\beta = \{\beta_1, \beta_2, \ldots\}$ on $D_0$, given by

$$g(\beta) = \exp \left[ \sum_{n>0} (\beta_n(L_n - L_0) - \bar{\beta}_n(L_{-n} - L_0)) \right]. \quad (3.10)$$

By using the BCH theorem, these coordinates and the coordinates $z$ defined by (3.8) can be expressed in terms of one another as formal power series; this gives in a formal sense a 1-1 correspondence between the coordinates $z$ and $\beta$, so that we can consider $z$ to be a global coordinate system on $D_0 \sim \text{Diff}^1 S^1$.

From (3.7), it is clear that in the $z$ coordinates the curvature form $\omega$ is given at the origin $z_n = 0$ by

$$\omega_{m,n} = -\omega_{n,m} = i \delta_{m,n}(4\pi m b_0 + \frac{c}{12} m^3) \quad (3.11)$$

Thus, $\omega$ is a (1,1)-form, and we have a Kähler structure on $W_{b_0,c}$. In the coordinate system given by $z$, the space of holomorphic sections of $L_{b_0,c}$ can be taken to be the ring $R = \mathbb{C}[z_1, z_2, \ldots]$ of polynomials in the variables $z_i$. We now wish to explicitly compute the action of the operators

$$\hat{L}_n = \xi_n - A_n + i\Phi_n, \quad (3.12)$$

which give a representation of the Virasoro algebra on $R$. Note that there is also an operator $\hat{C} = c$, which is constant since $C$ is central. From now on we will simply replace the operator $\hat{C}$ with its value $c$ in all formulae.

We begin the computation of the $\hat{L}_n$’s, as in the case of $SU(2)$, by computing the vector fields $\xi_n$. Expressions for these vector fields can be calculated by using BCH to express the product $\exp(\epsilon L_n) \exp(\sum z_m L_m)$ in the form

$$\exp(\epsilon L_n) \exp(\sum_{m>0} z_m L_m) = \exp(\sum_{m>0} (z_m + c \bar{u}_n^m) L_m) f(\{L_k : k \leq 0\}, C) + \mathcal{O}(\epsilon^2), \quad (3.13)$$

20
and then setting
\[ \xi_n = \sum_{m>0} -\bar{\nu}_m^m \frac{\partial}{\partial z_m}. \] (3.14)

**Proposition 3** The vector fields \( \xi_n \) are given by
\[ \xi_n = \sum_{k \geq 0, N_n^+(k)} \alpha_{k,\lambda} C_n(n_1, \ldots, n_k) z_{n_1} \ldots z_{n_k} \frac{\partial}{\partial z_{n+n_1+\ldots+n_k}}, \] (3.15)
where \( \lambda \) is the minimum integer such that \( n + n_1 + n_2 + \ldots + n_\lambda > 0 \) (\( \lambda = 0 \) when \( n > 0 \)),
\[ N_n^+(k) = \{(n_1, n_2, \ldots, n_k) : n_1, \ldots, n_k > 0, n + n_1 + n_2 + \ldots + n_k \geq 0\}, \] (3.16)
\[ C_n(n_1, \ldots, n_k) = (n_1-n)(n_2-n_1-n)\ldots(n_k-n_1-\ldots-n_{k-1}-n), \] (3.17)
and
\[ \alpha_{k,\lambda} = (-1)^{k+1} \sum_{l=0}^{k-\lambda} \frac{B_l}{l! (k-l)!}. \] (3.18)
\((B_l \text{ is the } l\text{th Bernoulli number, as in (2.49) and (3.28).})\)

**Proof.** We begin by noting the identities
\[ (\text{ad}(\sum z_m L_m))^k L_n = \sum_{n_1, \ldots, n_k > 0} C_n(n_1, \ldots, n_k) z_{n_1} \ldots z_{n_k} L_{n+n_1+\ldots+n_k}, \] (3.19)
and
\[ C_n(n_1, \ldots, n_s) C_{n+n_1+\ldots+n_s}(n_{s+1}, \ldots, n_k) = C_n(n_1, \ldots, n_k). \] (3.20)
\nEquation 3.19 also contains a constant term proportional to \( C \) on the right hand side. Since \( C \) is central, this term can immediately be absorbed in \( f \), and will be dropped from all calculations in this proof. Applying Equation 2.49 to \( \exp(\epsilon L_n) \exp(\sum z_m L_m) \), we have
\begin{align*}
\exp(\epsilon L_n) \exp(\sum_{m>0} z_m L_m) \sim \\
\exp \left( \sum_{m>0} z_m L_m + \epsilon \sum_{k \geq 0, n_1, \ldots, n_k > 0} \frac{B_k}{k!} C_n(n_1, \ldots, n_k) z_{n_1} \ldots z_{n_k} L_{n+n_1+\ldots+n_k} \right), \tag{3.21}
\end{align*}
where by \( a \sim b \) it is meant that \( a = bf + \mathcal{O}(\epsilon^2) \), with \( f \) some function of the \( L_n \)'s with \( n \leq 0 \). Dividing the terms in the exponential into generators \( L_m \) with \( m > 0 \) and \( m \leq 0 \), this can be
rewritten as

\[
\exp(\epsilon L_n) \exp\left( \sum_{m>0} z_m L_m \right) \sim \\
\exp\left( \sum_{m>0} z_m L_m - \epsilon \sum_{k \geq 0, N_n^+ (k)} \alpha_{k, \lambda}^{(0)} C_n (n_1, \ldots, n_k) z_{n_1} \ldots z_{n_k} L_{n+n_1+\ldots+n_k} \right) \\
+ \epsilon \sum_{l_1 \geq 0, N_n^+ (l_1)} \frac{B_{l_1}}{l_1!} C_n (n_1, \ldots, n_{l_1}) z_{n_1} \ldots z_{n_{l_1}} L_{n+n_1+\ldots+n_{l_1}},
\]

(3.22)

where \( \alpha_{k, \lambda}^{(t)} \) is defined by

\[
\alpha_{k, \lambda}^{(t)} = - \frac{B_k}{k!} \sum_{s \geq 0, 0 \leq t_1 < t_2 < \ldots < t_s \leq \lambda} (-1)^{s + k - l_1} \frac{B_{l_1}}{l_1!} \frac{B_{l_2 - l_1}}{(l_2 - l_1)!} \ldots \frac{B_{l_s - l_{s-1} - 1}}{(l_s - l_{s-1} - 1)!} \frac{B_{l_s - l_s}}{(k - l_s)!}.
\]

(3.23)

Applying (2.50) and (3.20) to Equation 3.22 \( t \) times, we get

\[
\exp(\epsilon L_n) \exp\left( \sum_{m>0} z_m L_m \right) \sim \\
\exp\left( \sum_{m>0} z_m L_m - \epsilon \sum_{k \geq 0, N_n^+ (k)} \alpha_{k, \lambda}^{(t)} C_n (n_1, \ldots, n_k) z_{n_1} \ldots z_{n_k} L_{n+n_1+\ldots+n_k} \right),
\]

(3.24)

(3.25)

We will now show that \( \alpha_{k, \lambda}^{(t)} = \alpha_{k, \lambda} \). Using the fact that \( B_{2k+1} = 0 \) for \( k > 0 \), it is not hard to determine that

\[
\alpha_{k, 0}^{(\infty)} = - \frac{B_k}{k!}, \\
\alpha_{k, 1}^{(\infty)} = \delta_{k, 1}, \\
\alpha_{k, \lambda}^{(\infty)} = \sum_{s \geq 0, 1 < t_1 < \ldots < t_s \leq \lambda} (-1)^{s + k} \frac{B_{l_1}}{(l_1 - 1)!} \frac{B_{l_2 - l_1}}{(l_2 - l_1)!} \ldots \frac{B_{l_s - l_{s-1} - 1}}{(l_s - l_{s-1} - 1)!} \frac{B_{l_s - l_s}}{(k - l_s)!}, \text{ for } \lambda > 1.
\]

(3.26)

When \( \lambda > 1 \), we can write a generating function for \( \alpha_{k, \lambda}^{(\infty)} \) by

\[
\sum_{k \geq \lambda > 1} \alpha_{k, \lambda}^{(\infty)} y^{k-\lambda} x^k = \sum_{l > m \geq 0} y^m \frac{B_l}{l!} (-x)^{l+1} \sum_{s \geq 0} (1 - \phi(-x))^s,
\]

(3.27)
\[ \phi(x) = \frac{x}{e^x - 1} = \sum_{n \geq 0} \frac{B_n}{n!} x^n. \]  

(3.28)

From (3.27), it follows that

\[ \alpha^{(\infty)}_{k,\lambda} = (-1)^k \sum_{l=k-\lambda+1}^{k-1} \frac{B_l}{l! (k-l)!} = (-1)^{k+1} \sum_{l=0}^{k-\lambda} \frac{B_l}{l! (k-l)!}, \]

(3.29)

where we have used the fact that

\[ \sum_{l=0}^{k-\lambda} B_l \frac{k}{l!} = 0, \quad \text{for} \ k > 1. \]  

(3.30)

From (3.26) and (3.30), it is also easy to verify that

\[ \alpha^{(\infty)}_{k,0} = \alpha_{k,0} \quad \text{and} \quad \alpha^{(\infty)}_{k,1} = \alpha_{k,1}. \]

Thus, for all \( k \) and \( \lambda \), we have shown that \( \alpha^{(\infty)}_{k,\lambda} = \alpha_{k,\lambda} \), and proposition 3 is proven. ✷

When \( n \geq 0 \), we can use (3.24) to simplify the formulae for \( \xi_n \) to

\[ \xi_n = \sum_{k \geq 0, n_1, \ldots, n_k > 0} -\frac{B_k}{k!} C_n(n_1, \ldots, n_k) \frac{\partial}{\partial z_{n_1 + n_2 + \ldots + n_k}} \]  

for \( n > 0 \),

(3.31)

and

\[ \xi_0 = \sum_{k > 0} k z_k \frac{\partial}{\partial z_k}. \]

(3.32)

(Note that the last expression could also have been obtained more simply through (2.51).) The first few terms in the expressions for \( \xi_n, n \neq 0 \) are given by

\[ \xi_n = -\frac{\partial}{\partial z_n} + \frac{1}{2} \sum_{m > 0} (m - n) z_m \frac{\partial}{\partial z_{n+m}} + \ldots, \quad \text{for} \ n > 0, \]

(3.33)

and

\[ \xi_{-n} = \sum_{m > n} (m + n) z_m \frac{\partial}{\partial z_{m-n}} + \ldots, \quad \text{for} \ n > 0. \]

(3.34)

It is important to note that although the operators \( \xi_n \) are expressed as infinite series, to compute \( \xi_n f \) for any \( f \in R \) and \( n \in \mathbb{Z} \), only a finite number of terms in \( \xi_n \) will be needed, so that the action of \( \xi_n \) on any element of \( R \) can be explicitly computed.

To calculate the operators \( \hat{L}_n \), it will be sufficient, by Proposition 2, to find a set of functions \( f_n(z_1, z_2, \ldots) \in R \), for \( n \in \mathbb{Z} \), with the properties

\[ \xi_m f_n - \xi_n f_m = (m - n) f_{m+n} + \frac{c}{12} \delta_{m,n} (m^3 - m) \]

(3.35)

and

\[ f_0(0) = 2\pi b_0 + \frac{c}{24}. \]

(3.36)
As in the $SU(2)$ example, we will find these functions by making an Ansatz which will be justified once we find a set of $f$’s satisfying (3.35) and (3.36). The carrier space $R$ is again a graded ring, with $\deg(z_n) = n$, $\deg(1) = 0$. The eigenvectors of the $\xi_0$ operator are exactly those polynomials in $R$ of fixed degree with respect to this grading; we refer to such polynomials as quasi-homogeneous.

We can express $R$ as a direct sum of finite dimensional vector spaces,

$$ R = \bigoplus_{k \geq 0} R_k, \quad (3.37) $$

where

$$ R_k = \{ f \in R : \deg(f) = k \}. \quad (3.38) $$

With this notation, we have

$$ \xi_m R_n \subset R_{n-m} \quad (3.39) $$

and

$$ \xi_0 f = nf, \text{ for } f \in R_n. \quad (3.40) $$

A natural Ansatz to make is that $\hat{L}_0$ satisfies

$$ \hat{L}_0 R_n \subset R_n, \quad (3.41) $$

which is equivalent to the assertion that $f_0$ is a constant function. In fact, we will show that this Ansatz leads to a unique set of functions $f_n$ which satisfy (3.35) and (3.36). An immediate result of the Ansatz is that

$$ f_n = 0, \text{ for } n > 0, \quad (3.42) $$

$$ f_{-n} \in R_n, \text{ for } n > 0, $$

so that $f_{-n}$ is a quasi-homogeneous function of degree $n$. This follows from setting $m = 0$ in (3.35).

We can easily calculate the first few $f$’s by hand, using (3.35) and (3.36). Defining $h = 2\pi b_0 + \frac{1}{2\pi}$, we have from (3.36),

$$ f_0 = h. \quad (3.43) $$

From (3.35), we have

$$ \xi_1 f_{-1} = -\frac{\partial}{\partial z_1} f_{-1} = 2f_0 = 2h, \quad (3.44) $$

so

$$ f_{-1} = -2hz_1. \quad (3.45) $$
Similarly, (3.35) gives two equations for \( f_{-2} \),
\[
\xi_2 f_{-2} = -\frac{\partial}{\partial z_2} f_{-2} = 4f_0 + \frac{c}{2} = 4h + \frac{c}{2},
\]
\[
\xi_1 f_{-2} = -\frac{\partial}{\partial z_1} f_{-2} = 3f_{-1} - 6hz_1.
\]

Since \( f_{-2} \) is a linear combination of \( z_1^2 \) and \( z_2 \), these two equations determine both coefficients exactly, so that
\[
f_{-2} = -(4h + \frac{c}{2})z_2 + 3hz_1^2.
\]

One could continue computing the functions \( f_{-n} \) by this means, however the number of conditions on each function grows faster than the number of linearly independent terms which can appear in the same function. Thus, it is desirable to find a means of expressing the functions \( f_{-n} \) for \( n > 2 \) in such a way that the consistency of these conditions can be easily verified. In fact, we can use (3.33) to give a recursive definition of \( f_{-n} \) in terms of \( f_{-1} \) and \( f_{1-n} \). We define
\[
f_{-n} = \frac{1}{n-2}(\xi_{-n} f_{1-n} - \xi_{1-n} f_{-1}), \text{ for } n > 2.
\]

We must now prove that this definition, along with Equations 3.43, 3.45, and 3.47, gives a set of functions \( f_{-n} \) which are consistent with (3.33). As an intermediate result, we will need the following proposition.

**Proposition 4** The unit element 1 in \( R \), which we denote by \( |\rangle \), is the unique function in \( R \) (up to scalar multiplication) which is annihilated by \( \xi_n \) for all \( n > 0 \); i.e., \( |\rangle \) is the unique highest weight state in the module \( R \) under the action of the \( \xi_n \)'s.

**Proof.** Assume there is another function \( \phi \in R \) which is annihilated by \( \xi_n \) for all \( n > 0 \). Since \( R \) is graded, \( \phi \) can be written as a sum of quasi-homogeneous functions,
\[
\phi = \sum_{n\geq0} \phi_n, \quad \phi_n \in R_n.
\]

Take \( d \) to be the minimum integer with \( \phi_d \neq 0 \). Now, let \( k \) be the largest integer such that some term in \( \phi_d \) contains a factor of \( z_k \). \( \phi_d \) can now be written in the form
\[
\phi_d = \sum_{m \geq 0} z_k^m g_d^{(m)}(z_1, \ldots, z_{k-1}),
\]

where \( g_d^{(m)} \) is a quasi-homogeneous polynomial in \( z_1, \ldots, z_{k-1} \) of degree \( d - mk \) for each \( m \geq 0 \). Since for \( n > 0 \), all terms in \( \xi_n \) except the leading term \( \partial/\partial z_n \) contain derivatives \( \partial/\partial z_j \) with \( j > n \), we can compute
\[
\xi_k \phi_d = -\sum_{m \geq 0} m z_k^{m-1} g_d^{(m)}(z_1, \ldots, z_{k-1}).
\]

25
For this expression to be zero, all the functions \( g^{(m)}_{d-\km} \) would have to be zero for \( m > 0 \). But then \( \phi_d \) would not contain any terms with a factor of \( z_k \), contradicting our assumption. Thus, the only states in \( R \) annihilated by all \( \xi_n \) with \( n > 0 \), are the constant functions in \( R_0 \). \( \square \)

From this proposition, it is clear that \( h \) is in fact the standard value of the highest weight for the Virasoro representation we are constructing, since \( \hat{L}_0|\rangle = f_0|\rangle = h|\rangle \). Note also that the proof of Proposition 4 shows that there are not even any formal power series in \( \mathbb{C}[\![z_1, z_2, \ldots]\!] \) annihilated by \( \xi_n \) for all \( n > 0 \), other than 1. (\( \mathbb{C}[\![z_1, z_2, \ldots]\!] \) is the ring of formal power series in the variables \( z_1, z_2, \ldots \) with coefficients in \( \mathbb{C} \), which is also a module under the action of the \( \xi_n \)'s.) We can now show that the functions \( f_{-n} \) defined previously do in fact satisfy (3.35).

**Proposition 5** The functions \( f_n \in R \) given by the recursive formula

\[
\begin{align*}
  f_n &= 0, \quad n > 0, \\
  f_0 &= h, \\
  f_{-1} &= -2hz_1, \\
  f_{-2} &= -(4h + \frac{c}{2})z_2 + 3h z_1^2, \\
  f_{-k} &= \frac{1}{k-2}(\xi_{-1}f_{-k-1} - \xi_{-1-k}f_{-1}), \quad k > 2,
\end{align*}
\]

satisfy (3.35), where the operators \( \xi_n \) are given by Equation (3.15).

**Proof.** We divide the equations derived from (3.35) into two categories, depending on whether the indices \( m \) and \( n \) are both negative, or are of opposite sign. (When \( m \) and \( n \) are both positive, or \( m \) or \( n \) is zero, (3.35) is easily verified directly from the definition of the \( f \)'s.) When \( m \) is positive, (3.35) states that for every \( k > 0 \),

\[
\xi_m f_{-k} = (m + k)f_{m-k} + \frac{c}{12} \delta_{m,k}(m^3 - m) \quad \text{for all } m > 0. \tag{3.53}
\]

On the other hand, when the indices are both negative, (3.35) states that for every \( k > 0 \),

\[
\xi_{-l}f_{-k-1} - \xi_{l-k}f_{-1} = (k - 2l)f_{-k} \quad \text{for all } l : 0 < l < k. \tag{3.54}
\]

We will prove by induction on \( k \) that Equations (3.53) and (3.54) follow from the above definitions of the functions \( f_n \). From the discussion leading to Equations (3.43), (3.45), and (3.47), it is clear that for \( k \leq 2 \), (3.53) is satisfied. (Note that \( \xi_m f_{-k} = 0 \) whenever \( m > k \geq 0 \), since all terms in \( \xi_m \) contain derivatives with respect to some \( z_j \) with \( j \geq m \).) It is also easy to verify (3.54) for \( k \leq 2 \). Thus, we have the first induction steps; it remains to be shown that if (3.53) and (3.54) are satisfied for all
\( k' < k \), then they are also satisfied for \( k \). We claim that to prove this, it will suffice to show that for all \( m > 0 \) and \( 0 < l < k \),
\[
\xi_m \left[ \frac{1}{k - 2l} (\xi_{-l} f_{l-k} - \xi_{l-k} f_{-l}) \right] = (m + k) f_{m-k} + \frac{c}{12} \delta_{m,k} (m^3 - m). \tag{3.55}
\]
Certainly, verifying this equation for \( l = 1 \) will show that (3.53) holds for \( k \). In fact, however, this equation also tells us that for any \( l \neq 1 \),
\[
\xi_m \left[ f_{-k} - \frac{1}{k - 2l} (\xi_{-l} f_{l-k} - \xi_{l-k} f_{-l}) \right] = 0 \text{ for all } m > 0. \tag{3.56}
\]
By Proposition 4, this implies that
\[
\frac{f_{-k} - \frac{1}{k - 2l} (\xi_{-l} f_{l-k} - \xi_{l-k} f_{-l})}{k - 2l} \in \mathbb{C} \text{ for all } m > 0. \tag{3.57}
\]
Since the left hand side of this equation is quasi-homogeneous of degree \( k \), for \( k > 2 \), (3.54) follows. Thus, to prove the induction step, it will suffice to prove (3.55). The derivation of this equation is straightforward algebra, assuming that (3.55) and (3.56) hold for \( k' < k \).
\[
\xi_m \left[ \frac{\xi_{-l} f_{l-k} - \xi_{l-k} f_{-l}}{k - 2l} \right] = \frac{1}{k - 2l} \left[ (m + l) \xi_{m-l} f_{l-k} + \frac{c}{12} \delta_{m,l} (m^3 - m) f_{l-k} + \xi_{-l} \xi_m f_{l-k} \right. \\
- \left. (m + k - l) \xi_{m+k-l} f_{-l} - \frac{c}{12} \delta_{m,k-l} (m^3 - m) f_{-l} - \xi_{l-k} \xi_m f_{-l} \right] \\
= \frac{1}{k - 2l} \left[ (m + l) (\xi_{m-l} f_{l-k} - \xi_{l-k} f_{m-l}) \right. \\
- \left. (m + k - l) (\xi_{m+k-l} f_{-l} - \xi_{l-k} f_{m+k-l}) \right] \\
= (m + k) f_{m-k} + \frac{c}{12} \delta_{m,k} (m^3 - m). \tag{3.58}
\]
We have thus shown by induction that (3.55) and (3.54) hold for all \( k > 0 \), so the proposition is proven.\( \square \)

We have defined functions \( f_n \) which satisfy (3.35) and (3.36), and thus by Proposition 2 we have constructed a representation of the Virasoro algebra on \( R \), where the generators are given by
\[
\hat{L}_n = \xi_n + f_n \tag{3.59}
\]
\[
\hat{C} = c.
\]
Note that the explicit formulae for \( \xi_n \) and \( f_n \) can be used to construct a representation of \( \hat{\text{Diff}} S^1 \), at least formally, even when \( h = -\frac{c(m^2 - 1)}{24} \) for some integer \( m \). The action of each operator \( \hat{L}_n \) on any polynomial in \( R \) involves a finite number of terms, and can be computed. Calculating explicitly
the first few terms in the operators $\hat{L}_n$ for small $|n|$, we have

$$
\hat{L}_3 = -\frac{\partial}{\partial z_3} + D_4,
$$

$$
\hat{L}_2 = -\frac{\partial}{\partial z_2} - \frac{1}{2} z_1 \frac{\partial}{\partial z_3} + D_4,
$$

$$
\hat{L}_1 = -\frac{\partial}{\partial z_1} + \frac{1}{2} z_2 \frac{\partial}{\partial z_3} + D_4,
$$

$$
\hat{L}_0 = h + z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} + z_3 \frac{\partial}{\partial z_3} + D_4,
$$

(3.60)

$$
\hat{L}_{-1} = -2h z_1 + (3z_2 - z_1^2) \frac{\partial}{\partial z_1} + (4z_3 - 2z_1 z_2) \frac{\partial}{\partial z_2} + O(z_4),
$$

$$
\hat{L}_{-2} = -(4h + c) z_2 + 3h z_1^2 + (5z_3 - \frac{13}{2} z_1 z_2 + z_1^3) \frac{\partial}{\partial z_1} + O(z_4),
$$

$$
\hat{L}_{-3} = -(6h + 2c) z_3 + (13h + c) z_1 z_2 - 4h z_1^3 + O(z_4),
$$

where $D_4$ denotes terms containing derivatives $\partial/\partial z_k$ with $k \geq 4$, and $O(z_4)$ denotes terms containing quasi-homogeneous polynomials of degree at least 4. From these formulae, we can explicitly compute the lowest degree states arising from the action of the Virasoro algebra on $|\rangle$. Using a basis for $R$ of monomials in the variables $z_i$, where $f \in R$ is represented by the state $|f\rangle$, we have

$$
\hat{L}_{-1} |\rangle = -2h |z_1\rangle,
$$

$$
\hat{L}_{-2} |\rangle = -6h |z_2\rangle + (4h^2 + 2h) |z_1^2\rangle,
$$

$$
\hat{L}_{-3} |\rangle = -(4h + c) |z_2\rangle + 3h |z_1^2\rangle.
$$

(3.61)

Since $|\rangle$ is the unique highest weight state in $R$, we expect that for those values of $h,c$ with a vanishing Kac determinant at level $k$ (i.e., where the algebraic representation contains a null state at level $k$), we should find linear dependences among the states generated by combinations of raising operators of degree $k$. As an example, from the above expressions for $\hat{L}_{-1} |\rangle$ and $\hat{L}_{-2} |\rangle$, it is easy to see that these states are linearly dependent when

$$
h = \frac{5 - c \pm \sqrt{(c - 1)(c - 25)}}{16}.
$$

(3.62)

This is exactly the condition for the Kac determinant to vanish at level 2. We see then, that the structure of the representations corresponding to the discrete series of representations with $c \leq 1$ is exactly that predicted by Aldaya and Navarro-Salas in [13]. In these cases, the carrier space $R$ of the representation has only one highest weight state, however it is not irreducible. To achieve an irreducible representation, one must take the orbit of $|\rangle$ in $R$ under the action of the enveloping algebra (i.e., the space spanned by the set of states in $R$ of the form $(\prod \hat{L}_{-n_i}) |\rangle$). Note that the
reducibility of these representations occurs in a manner quite different from that in which the usual algebraically constructed representations are reducible. In the algebraic construction, certain linear combinations of states give null vectors, which generate Verma modules which must be divided out to get an irreducible representation. In our case, as in that of Aldaya and Navarro-Salas, there are no null states; the elements of the enveloping algebra which give null states in the algebraic theory, give zero when applied to the highest weight vector of our Virasoro module. This property is common to many Fock space representations of infinite-dimensional algebras [26].

We will now discuss briefly the possibility of realizing a unitary structure on the coadjoint orbit representations we have constructed. A first step in describing such a structure would be to find an expression for a Hermitian metric $\exp(H)$ on $L_{b_0,c}$. If we assume that the connection we have defined by our gauge-fixing procedure is the associated metric connection, as was the case for $SU(2)$, then we can explicitly calculate $H$ as a formal power series in the $z$’s. We have

$$A_0 = -f_0 + i\Phi_0$$

(3.63)

$$= -h - i\langle(b_0, ic)\exp(-\sum_{n>0}\mu_n L_{-n})\exp(-\sum_{n>0}z_n L_n)\exp(\sum_{n>0}z_n L_n)\exp(\sum_{n>0}\mu_n L_{-n})\rangle$$

$$= \sum_{k,l>0[n,m]} \frac{(-1)^{k+l}}{k! l!} z_{n_1} \cdots z_{n_k} \mu_{m_1} \cdots \mu_{m_l} C_0(n_1, \ldots, n_k, -m_1, \ldots, -m_l)(h + \frac{c}{24}(m_l^2 - 1)),$$

where the sum is taken over all $n_1, \ldots, n_k, m_1, \ldots, m_l > 0$ satisfying $n_1 + \cdots + n_k = m_1 + \cdots + m_l$. If we assume $A\bar{z}_n = 0$ and $A z_n = \partial H/\partial z_n$, then we have

$$A_0 = -\sum_{n>0} n z_n A_{z_n},$$

(3.64)

and

$$H = -\xi^{-1}_0 A_0,$$

(3.65)

up to a constant. We will take this function $H$ as a candidate for the Hermitian metric on $L_{b_0,c}$. The first few terms in a power series expansion of $H$ are given by

$$H = -\sum_{n>0} 2n(h + \frac{c}{24}(n^2 - 1))|z_n|^2$$

$$+ \sum_{n,m>0} \left[ (m^2 + 4mn + n^2)h + \frac{c}{24}(m^4 + 2m^3n + 2mn^3 + n^4 - m^2 - 4mn - n^2) \right]$$

$$\times \left( \frac{z_m \bar{z}_n \bar{z}_m + n + \bar{z}_m \bar{z}_n z_{n+m}}{2} \right) + O(z^4).$$

(3.66)

Note that $H$ is expected to be real, in order to be a Hermitian metric.
To have a complete description of a unitary structure on $R$, it would now be necessary to find an invariant metric on $\text{Diff} S^1/S^1$. Unfortunately, it is unclear whether such a metric can be found. Attempting to describe such a metric as a formal power series gives rise to an expression with divergent coefficients. The matter is complicated by the fact that the adjoint representation of the Virasoro algebra is not a highest-weight representation. It seems that some kind of regularization scheme may be necessary to construct such a metric in a sensible fashion. We can, however, get some information about when such a unitary structure is likely to be possible directly from (3.66).

If we take only the first term in (3.66), and approximate the metric with a Gaussian, we see that for $h \ll 0$ or $c \ll 0$, the metric diverges badly, and we will certainly not find a unitary structure. When $h, c \gg 0$, a sensible inner product on $R$ can be found, at least in perturbation theory, by taking a product of Gaussian integrals. Using the Hermitian metric (3.66) to compute anything nonperturbative, however, would be a difficult proposition. Further progress in this direction will be impossible until some sort of a regularized invariant metric on $\text{Diff} S^1/S^1$ can be described explicitly.

Despite the fact that we cannot construct a unitary structure on $R$ by integrating a Hermitian metric on $L_{b_0,c}$ over $\text{Diff} S^1/S^1$, we can still use the more simple-minded approach mentioned in section 2 of constructing a unitary structure on $R$ in the same fashion as is done in the algebraic approach. That is, let us assume that $\langle | \rangle = 1$ and $\hat{L}_n^1 = \hat{L}_{-n}$. (We will use physicists notation for the inner product on $R$, denoting the inner product of two functions $f$ and $g$ by $\langle f | g \rangle$.) We can algebraically compute $\langle f | g \rangle$ whenever $f$ and $g$ are in the orbit of $| \rangle$ under the action of the $\hat{L}_{-n}$'s. This calculation is equivalent to that usually carried out in the algebraic construction (for a review and further references see [25]). The standard result is that when $h \geq 0$ and $c \geq 1$, or when

$$c = 1 - \frac{6}{m(m+1)},$$

and

$$h = \frac{[(m+1)p - mq]^2 - 1}{4m(m+1)},$$

for integers $m, p$ and $q$ satisfying $m > 2$ and $1 \leq q \leq p \leq m - 1$, the inner product $\langle f | g \rangle$ is positive definite on the space defined by the orbit of $| \rangle$ in $R$. When $h \geq 0$ and $c \geq 1$, the representation of the Virasoro algebra on $R$ is irreducible, so this construction gives a positive-definite inner product on $R$, with respect to which $R$ is a Hilbert space. For the representations where (3.67) and (3.68) are satisfied, the situation is slightly different. In these cases this argument only tells us that the subspace $\mathcal{H}$ given by the orbit of $| \rangle$ is a Hilbert space ($\mathcal{H}$ carries the irreducible highest weight representation). If we attempt to extend the inner product from $\mathcal{H}$ to $R$, while maintaining the
relationship $\hat{L}_n^+ = \hat{L}_{-n}$, we get a contradiction. For example, when $m = 3, p = 2, q = 1$, and $h = c = 1/2$, Equation (3.62) is satisfied, and we have a reducible representation on $R$. In this case we can compute

$$\langle z_1 | z_1 \rangle = \langle | \hat{L}_1 \hat{L}_{-1} | \rangle = 2h = 1$$

$$\langle -\frac{9}{4}z_2 + \frac{3}{2}z_1^2 | -\frac{9}{4}z_2 + \frac{3}{2}z_1^2 \rangle = \langle | \hat{L}_2 \hat{L}_{-2} | \rangle = 9/4.$$ 

If we attempt to extend this inner product to $R_2$, however, we get

$$\langle 3z_2 - 2z_1^2 | z_1^2 \rangle = \langle z_1 | \hat{L}_1 | z_1^2 \rangle = -2$$

$$= -\frac{4}{3} \langle | \hat{L}_2 | z_1^2 \rangle = 0.$$  

Thus, the inner product on $H$ cannot be extended to one on $R$.

In summary, we have found that for $h \geq 0, c \geq 1$, a Hilbert space structure exists on $R$, with respect to which our representations are unitary. For the discrete series of representations given by (3.67) and (3.68), the subspace $H$ of $R$ given by the orbit in $R$ of $|$ carries an irreducible unitary representation, however the Hilbert space structure on $H$ does not extend to $R$. For all other representations, there cannot exist a Hilbert space structure on $R$, as no other values of $h$ and $c$ allow unitary representations of the Virasoro algebra. Thus, we see that all unitary representations of the Virasoro algebra can be found in the class of representations we have constructed, although we are unable to show that the unitary structure on these representations can be described geometrically as arising from a Hermitian metric on the line bundle $L_{b_0,c}$. This situation parallels that of loop groups, where it is impossible to find an invariant metric on the homogeneous space $LG/T$, but where a unitary structure can be found for the representations on spaces of holomorphic sections of line bundles over $LG/T$ in a manner similar to that we have used here [2].

4 Conclusions

We have succeeded in constructing, for arbitrary $h$ and $c$, a representation of the Virasoro algebra on the space $R$ of polynomials in a countable set of variables $z_1, z_2, \ldots$. The Virasoro generators in these representations are given by

$$\hat{L}_n = \xi_n + f_n$$

$$\hat{C} = c,$$

where the operators $\xi_n$ are first-order differential operators, defined by (3.15), and the functions $f_n$ are quasi-homogeneous polynomials in $R$ of degree $-n$, given by (3.52). When $c \geq 24h$ is not
the square of a nonzero integer, these representations correspond to coadjoint orbits of $\text{Diff} S^1$ of the form $\text{Diff} S^1/S^1$. For the exceptional values of $h$ and $c$, these representations are still defined, although they are not coadjoint orbit representations. The representations in the case $h = c/24$ appear to be related to the natural Hilbert space for the quantum field theory defined by Alekseev and Shatashvili in [7] which corresponds to Polyakov’s theory of 2d gravity.

The results given here agree with previously known information about the $\text{Diff} S^1/S^1$ Virasoro coadjoint orbits. The characters of the representations we have constructed are clearly given by

$$\text{Tr}_R q^{\hat{L}_0} = \sum_n (\dim R_n) q^{h+n} = q^h \prod_{n=1}^\infty \frac{1}{(1-q^n)},$$

(4.2)

since the carrier space of all these representations is $R$. This agrees with the result calculated by Witten in [3] using perturbative and index theory techniques. For those values of $h, c$ corresponding to the $c \leq 1$ discrete series, we find that $R$ contains only a single highest weight state, but that the representation is reducible. The irreducible representation is achieved by taking the orbit of the highest weight state in $R$. This result agrees precisely with the predictions of Alday and Navarro-Salas based on their group approach to quantization.

One interesting feature of the Virasoro representations we have constructed is the fact that all the generators act as first-order differential operators on the space $R$. This feature distinguishes the representations we have developed here from the well known “Feigen-Fuchs” free field representations which were described in the work of Dotsenko and Fateev [27] using a Coulomb gas-like free field theory with background charge. In the Feigen-Fuchs representations, there is a set of operators $\{a_n : n \in \mathbb{Z}\}$, corresponding to free-field modes. These operators satisfy the Heisenberg algebra

$$[a_n, a_m] = 2n\delta_{n,-m}.$$  

(4.3)

The operators $a_n$ act on a bosonic Fock space with a vacuum $|\rangle_f$ satisfying $a_n |\rangle_f = 0$ for $n > 0$ and $a_0 |\rangle_f = 2\alpha |\rangle_f$. The Virasoro generators appear as modes of the stress-energy tensor, and are written in terms of the $a$’s as [28]

$$L_n = \frac{1}{4} \sum_{k=-\infty}^{\infty} a_{n-k} a_k - \alpha_0 (n+1) a_n, \quad \text{for } n \neq 0,$$

(4.4)

$$L_0 = \frac{1}{2} \sum_{k=1}^{\infty} a_{-k} a_k + \frac{1}{4} \alpha_0^2 - \alpha_0 a_0.$$  

These generators satisfy a Virasoro algebra with

$$h = \alpha (\alpha - 2\alpha_0), \quad c = 1 - 24\alpha_0^2.$$

(4.5)
We can reinterpret the bosonic Fock space in terms of the space $R$ by defining
\[ a_n = 2n \frac{\partial}{\partial z_n}, \quad \text{for } n > 0, \]
\[ a_0 = 2\alpha, \]
\[ a_{-n} = z_n, \quad \text{for } n > 0. \] (4.6)

In this notation, the Virasoro generators are
\[ L_n = \sum_{k=1}^{n-1} k(n-k) \frac{\partial}{\partial z_k} \frac{\partial}{\partial z_{n-k}} + 2n(\alpha - \alpha_0(n+1)) \frac{\partial}{\partial z_n} + \sum_{k=n+1}^{\infty} kz_{k-n} \frac{\partial}{\partial z_k}, \quad \text{for } n > 0 \]
\[ L_0 = \alpha(\alpha - 2\alpha_0) + \sum_{k=1}^{\infty} kz_k \frac{\partial}{\partial z_k}, \] (4.7)
\[ L_{-n} = (\alpha + \alpha_0(n-1))z_n + \frac{1}{4} \sum_{k=1}^{n-1} z_k z_{n-k} + \sum_{k=1}^{\infty} kz_{k+n} \frac{\partial}{\partial z_k}, \quad \text{for } n > 0. \]

For $n > 0$, these generators are second order differential operators on $R$. These representations seem to be fundamentally different from those we have constructed by the coadjoint orbit method. Note that the Feig-Fuchs representations have null states (i.e., states $|s\rangle$ other than the vacuum satisfying $L_n|s\rangle = 0$ for all $n > 0$) for certain values of $\alpha$ and $\alpha_0$. Thus, the structure of these representations is in some sense more complicated than that of the coadjoint orbit representations we have described in this paper. This fact also implies that it is not possible in general to relate the Feig-Fuchs representations to the coadjoint orbit representations via a generalized Bogoliubov transformation. Such a transformation would have to leave both the vacuum and the grading of the Heisenberg algebra fixed, and would allow a null state in the Feig-Fuchs representation to be described as a highest weight state in $R$, which cannot exist by proposition 4. The structure of the Fock space in the Feig-Fuchs representations was originally described in [15]. It was subsequently shown by Felder that the irreducible representations in the $c \leq 1$ discrete series could be described using a BRST-type screening operator which was previously introduced by Thorn [26]. It is possible that a similar construction could be realized for the coadjoint orbit representations we have described here.

One issue we have not completely resolved in this paper is the question of unitarity. We have given a candidate for a Hermitian metric on the line bundles associated with the coadjoint orbit representations, but without an invariant metric on the space $\text{Diff}S^1/S^1$, it is impossible to perform explicit calculations. We have shown that an inner product can be defined on $R$ when $h \geq 0, c \geq 1$, or on a subspace $\mathcal{H} \subset R$ when $h$ and $c$ correspond to the discrete series of unitary representations, with respect to which our representations are unitary. It is not clear, however,
whether this inner product can be related in any natural way to the Hermitian metric on $\mathcal{L}_{b_0,c}$. It is also unclear whether there exists a natural (geometric) reason for the breakdown of unitarity at $c = 1$. Hopefully the results presented here will motivate further investigation of these questions and of the geometric approach to conformal field theory in general.

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