Thermodynamic Instability of Black Holes of Third Order
Lovelock Gravity

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Abstract

In this paper, we compute the mass and the temperature of the uncharged black holes of third order Lovelock gravity and compute the entropy through the use of first law of thermodynamics. We perform a stability analysis by studying the curves of temperature versus the mass parameter, and find that there exists an intermediate thermodynamically unstable phase for black holes with hyperbolic horizon. The existence of this unstable phase for the uncharged topological black holes of third order Lovelock gravity does not occur in the lower order Lovelock gravity. We also perform a stability analysis for a spherical, 7-dimensional black hole of Lovelock gravity and find that while these kinds of black holes for small values of Lovelock coefficients have an intermediate unstable phase, they are stable for large values of Lovelock coefficients. We also find that there exists an intermediate unstable phase for these black holes in higher dimensions. This stability analysis shows that the thermodynamic stability of black holes with curved horizons is not a robust feature of all the generalized theories of gravity.

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I. INTRODUCTION

Thermodynamics of black holes in anti-de Sitter (AdS) spaces have been the subject of wide variety of researches in recent years. This is due to the fact that, in parallel with the development of AdS/CFT correspondence [1], black holes in AdS spaces are known to play an important role in dual field theory. With the AdS/CFT correspondence, one can gain some insights into thermodynamic properties and phase structures of strong ’t Hooft coupling CFTs by studying thermodynamics of AdS black holes. In the context of horizon topology, asymptotically AdS black holes are quite different from black holes in flat or dS spaces. In asymptotically flat or dS spaces, the horizon topology of a four dimensional black hole must be a round sphere $S^2$ [2], while in AdS spaces it is possible to have black holes with zero or negative constant curvature horizon too. These black holes are referred to as topological black holes in the literature. Due to the different horizon structures, the associated thermodynamic properties of topological black holes are rather different from the spherically symmetric black holes [3]. While the Schwarzschild black hole in AdS space is thermodynamically stable for large mass, it becomes unstable for small mass. That is, there is a phase transition (namely Hawking-Page phase transition) between the high temperature black hole phase and low temperature thermal AdS space [4]. It has been explained by Witten [5] that the Hawking-Page phase transition of Schwarzschild black holes in AdS spaces can be identified with confinement/deconfinement transition of the Yang-Mills theory in the AdS/CFT correspondence. However, it is interesting to note that for the AdS black holes with zero or negative constant curvature horizon the Hawking-Page phase transition does not appear and these black holes are always locally stable [6] (see also [7]).

Now, the question which arises is that whether the stability of black hole is a robust feature of all generally covariant theories of gravity or is peculiar to Einstein gravity. Among gravity theories, the so-called Lovelock gravity [8] has some special features. For example, the resulting field equations contain no more than second derivatives of the metric and it has been proven to be ghost-free when expanding about the flat space, evading any problem with unitarity. In this paper, we investigate the stability of uncharged black holes of third order Lovelock gravity with hyperbolic horizon. It is, now, known that the topological black holes of third order Lovelock gravity with zero curvature horizon is thermodynamically stable [9]. Indeed, all the thermodynamic and conserved quantities of the black holes with flat horizon
do not depend on the Lovelock coefficients, and therefore these black holes are stable as the Einstein’s black hole with flat horizon. This phase behavior of black holes with flat horizon is also commensurate with the fact that there is no Hawking-Page transition for a black object whose horizon is diffeomorphic to \( \mathbb{R}^p \) \([5]\). Also, as in the case of Einstein gravity \([6]\), the black holes of Gauss-Bonnet gravity with hyperbolic horizon is stable \([10]\). These facts bring in the idea that the Lovelock terms may have no effect on the stability of topological black holes. But, one of us has shown that an asymptotically flat uncharged black hole of third order Lovelock gravity may have two horizons \([11]\), a fact that does not happen in lower order Lovelock gravity. This persuades us to investigate the effects of third order Lovelock term on the stability phase structure of the black holes with curved horizon. We show that the hyperbolic uncharged black holes of third order Lovelock gravity have an intermediate unstable phase in contrast to the uncharged topological black holes of Einstein gravity or Gauss-Bonnet gravity. We also, investigate the effects of third order Lovelock term on the stability of a spherical black hole of third order Lovelock gravity.

The outline of this paper is as follows. We give a brief review of the Hamiltonian formulation of Lovelock action in Sec. (II). In Sec. (III) we obtain the vacuum solutions of third order Lovelock gravity by using the Hamiltonian form of the action and discuss the thermodynamics of the solutions. We investigate the stability of the uncharged black holes with curved horizon in Sec. (IV). Finally, we finish our paper with some concluding remarks.

II. HAMILTONIAN FORMULATION

The most fundamental assumption in standard general relativity is the requirement that the field equations be generally covariant and contain at most second order derivatives of the metric. Based on this principle, the most general classical theory of gravitation in an \((n + 1)\)-dimensional manifold \( \mathcal{M} \) with the metric \( g_{\mu \nu} \) is Lovelock gravity \([8]\), for which the gravitational action may be written as

\[
I_G = \frac{1}{16 \pi} \int d^{n+1}x \sqrt{-g} \sum_{p=0}^{[n/2]} \alpha_p \mathcal{L}_p, \tag{1}
\]

where \([n/2]\) denotes the integer part of \( n/2 \), \( \alpha_p \)'s are Lovelock coefficients and

\[
\mathcal{L}_p = \frac{1}{2^p} \delta_{\rho_1 \sigma_1 \cdots \rho_p \sigma_p} \, R_{\mu_1 \nu_1}^{\rho_1 \sigma_1} \cdots R_{\mu_p \nu_p}^{\rho_p \sigma_p}, \tag{2}
\]
is the Euler density of a $2p$-dimensional manifold. In Eq. (2) $\delta^{\mu_1\nu_1\cdots\mu_p\nu_p}_{\rho_1\sigma_1\cdots\rho_p\sigma_p}$ is a totally antisymmetric product of Kronecker delta and $R_{\mu\nu}^{\rho\sigma}$ is the Riemann tensor of the Manifold $\mathcal{M}$. In $n + 1$ dimensions, all terms for which $p > \lfloor n/2 \rfloor$ are total derivatives and therefore only the terms for which $p \leq \lfloor n/2 \rfloor$ contribute to the field equations.

In order to simplify the equations of motion, it is more convenient to use the Hamiltonian formulation. This formulation requires a breakup of spacetime into space and time which yields some insights into the nature of the dynamics of general relativity. Indeed, in this approach the dynamical variable is the spatial metric $h_{ij}$ rather than spacetime metric $g^{\mu\nu}$, where $h_{ij}$ is the induced metric on the spacelike hypersurface $\Sigma_t$ of the spacetime manifold $\mathcal{M}$. In this approach, the canonical coordinates are the spatial components of the metric $g_{ij}$, and their conjugate momenta are

$$\pi^i_j = -\frac{1}{4}\sqrt{-g} \sum_{p=0}^{n} \alpha_p \sum_{s=0}^{p-1} (-4)^{p-s} \delta_{i_1\ldots i_2p-1i}^{j_1\ldots j_2p-1j} \times \hat{R}^{j_1j_2}_{i_1i_2} \cdots \hat{R}^{j_{2s-1}j_{2s}}_{i_{2s-1}i_{2s}} K^{j_{2s+1}}_{i_{2s+1}} \cdots K^{j_{2p-1}}_{i_{2p-1}},$$

(3)

where $K^i_j$ is the extrinsic curvature of the hypersurface $\Sigma_t$ given as:

$$K^i_j = \frac{1}{2} N^{-1} (\dot{h}_{ij} - D_j N_i - D_i N_j),$$

(4)

and $\hat{R}^{ijkl}$ are the components of the intrinsic curvature tensor of the boundary $\Sigma_t$. In Eq. (4), $N = (-g^{00})^{-1/2}$ and $N^i = h^{ij}g_{0\mu}$ are the ‘lapse function’ and the ‘shift vectors’ in the standard ADM (Arnowitt-Deser-Misner) decomposition of spacetime, and $D_i$ denotes the covariant derivative associated with $h_{ij}$. The time components $g_{0\mu}$ are Lagrange multipliers associated with the generators of surface deformation

$$\mathcal{H} = -\sqrt{h} \sum_{p} \alpha_p \frac{1}{2p} \delta^{i_1\ldots i_2p}_{j_1\ldots j_2p} R^{j_1j_2}_{i_1i_2} \cdots R^{j_{2p-1}j_{2p}}_{i_{2p-1}i_{2p}},$$

(5)

and $\mathcal{H}_i = -2\pi^i_{ij}$. In Eq. (5), $R^{ij}_{kl}$ are the spatial components of spacetime curvature tensor, which depend on the velocities through the Gauss–Codacci equation

$$R_{ijkl} = \hat{R}_{ijkl} + K_{ik} K_{jl} - K_{il} K_{jk}.$$

(6)

In this formalism, the action (1) becomes

$$I_G = \frac{1}{16\pi} \int dt d^nx \left( \pi^{ij} \dot{h}_{ij} - N \mathcal{H} - N^i \mathcal{H}_i \right) + B,$$

(7)

where $B$ stands for a surface term.
III. THIRD ORDER LOVELOCK BLACK HOLES IN ADS SPACE

The metric of an \((n+1)\)-dimensional static spherically symmetric spacetime may be written as

\[
\begin{align*}
    ds^2 &= -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\Sigma_{k,n-1}^2, \\
    \text{where } d\Sigma_{k,n-1}^2 &\text{ represents the metric of an } (n-1)\text{-dimensional hypersurface with constant curvature } (n-1)(n-2)k \text{ and volume } V_{n-1}.
\end{align*}
\]

We consider the third order Lovelock gravity, and therefore we restrict ourselves to the first four terms of the Hamiltonian form of the action \((7)\). Using the Gauss-Codacci relation \((6)\) and the fact that the extrinsic curvature \(K_{ij}\) is zero for the metric in Eq. \((8)\), the generator of surface deformation becomes \(\mathcal{H} = \sum \alpha_p \mathcal{L}_p\), and consequently the action \((7)\) (with the first four terms) reduces to

\[
I_G = \frac{1}{16\pi} \int dtd^nxN\sqrt{h}[-2\Lambda + \mathcal{L}_1 + \alpha_2\mathcal{L}_2 + \alpha_3\mathcal{L}_3] + B,
\]

where \(\Lambda = -n(n-1)/2l^2\) is the cosmological constant for AdS solutions, and \(\alpha_2\) and \(\alpha_3\) are Gauss-Bonnet and third order Lovelock coefficients with dimensions \((\text{length})^2\) and \((\text{length})^4\), respectively, which are assumed to be positive. In Eq. \((9)\), \(\mathcal{L}_1 = R\) is just the Einstein-Hilbert Lagrangian, \(\mathcal{L}_2 = R_{ijkl}R^{ijkl} - 4R_{ij}R^{ij} + R^2\) is the second order Lovelock (Gauss-Bonnet) Lagrangian, and

\[
\mathcal{L}_3 = 2R^{ijkl}R_{klmn}R_{ij}^{mn} + 8R^{ij}_{\ k\ m}R_{kl}^{\ j\ m}R_{ij}^{\ mn} + 24R^{ijkl}R_{kljm}R_m^i
\]

\[
+ 3RR^{ijkl}R_{kl}^{\ ij} + 24R^{ijkl}R_{ij}R_{lk} + 16R_{ij}R_{jk}R^k_i - 12RR^{ij}R_{ji} + R^3
\]

is the third order Lovelock Lagrangian. Defining the dimensionless parameters \(a\) and \(b\) as

\[
a = \frac{(n-2)(n-3)}{l^2}\alpha_2, \quad b = \frac{72}{l^4} \left(\frac{n-2}{4}\right)\alpha_3,
\]

the action \((9)\) reduces to:

\[
I_G = \frac{(n-1)V_{n-1}}{16\pi} \int dtdr \left[\frac{r^n}{l^2} + r^n\psi(1 + l^2a\psi + \frac{l^4}{3}b\psi^2)\right] + B,
\]

where a prime denotes the derivative with respect to \(r\) and \(\psi = r^{-2}[k - f(r)]\). The surface term is \(B = -(t_2 - t_1)M + B_0\), where \(B_0\) is an arbitrary constant and \(M\) is the conserved charge associated to the time displacement \([12]\). Varying the action \((12)\) with respect to \(f\),
one obtains the equation of motion as
\[ \frac{r^n}{l^2} + r^n\psi(1 + l^2 a\psi + \frac{l^4}{3}b\psi^2)' = 0. \] (13)

The only real solution of Eq. (13) is
\[ f(r) = k + a r^2 b l^2 \left[ 1 + \left( \sqrt{\Gamma + J^2(r)} + J(r) \right)^{1/3} - \Gamma^{1/3} \left( \sqrt{\Gamma + J^2(r)} + J(r) \right)^{-1/3} \right] = k + a r^2 b l^2 \left[ 1 + \left( \sqrt{\Gamma + J^2(r)} + J(r) \right)^{1/3} - \left( \sqrt{\Gamma + J^2(r)} - J(r) \right)^{1/3} \right], \] (14)

where
\[ \Gamma = \left( \frac{b}{a^2} - 1 \right)^3, \] (15)
\[ J(r) = 1 - \frac{3b}{2a^2} + \frac{3b^2}{2a^3} K(r), \] (16)
\[ K(r) = 1 - \frac{ml^2}{r^n}. \] (17)

and \( m \) is an integration constant. The metric function \( f(r) \) is real everywhere provided
\[ 9b^2 + (4 - 18a)b + (12a^3 - 3a^2) \geq 0. \] (18)

The above condition (18) is satisfied for the case of \( a \geq 1/3 \), while for the case of \( a < 1/3 \), it is satisfied provided \( b > b^{(+)} \) or \( b < b^{(-)} \), where \( b^{(+)} \) and \( b^{(-)} \) are the larger and smaller real roots of Eq. (18), respectively.

The ADM mass of black hole can be obtained by using the behavior of the metric at large \( r \). It is easy to show that the mass of the black hole per unit volume, \( V_{n-1} \), is
\[ M = (n - 1)m/16\pi, \] (19)

where the mass parameter \( m \) in terms of the real roots of \( f(r_h) = 0 \) is:
\[ m(r_h) = l^{-2}r_h^n + kr_h^{n-2} + k^2al^2r_h^{n-4} + \frac{kbl^4}{3}r_h^{n-6}. \] (20)

The Hawking temperature of the black holes can be obtained by requiring the absence of conical singularity at the event horizon in the Euclidean sector of the black hole solution as:
\[ T = \frac{f'(r_+)}{4\pi} = \frac{3nr_+^6 + 3k(n-2)l^2r_+^{4} + 3k^2(n-4)al^4r_+^{2} + k(n-6)b}{12\pi l^2 (r_+^2 + 2kal^2r_+^{2} + k^2b)} , \] (21)

where \( r_+ \) is the radius of event horizon. Clearly, the temperature is always positive for \( k = 0 \) and \( k = 1 \) cases, and therefore there is no extreme black holes. However, for the case of
When the mass parameter is nonnegative, the horizon radius starts from zero. Integrating the first law

\[ S = \int_0^{r_+} T^{-1} \left( \frac{\partial M}{\partial r_+} \right) dr_+, \tag{22} \]

one obtains the entropy per unit volume \( V_{n-1} \) as:

\[ S = \frac{r_+^{n-1}}{4} \left( 1 + \frac{2k(n-1)al^2}{(n-3)r_+^2} + \frac{k^2(n-1)bl^4}{(n-5)r_+^4} \right), \tag{23} \]

which reduces to the area law of entropy for \( a = b = 0 \). Although for the case of \( k = -1 \), the mass parameter \( m \) may be negative and the black hole horizon can not shrink to zero, the entropy given by Eq. (23) is applicable, since it reduces to the area law of entropy for Einstein gravity. One may find that the entropy per unit volume obeys the law of the entropy of asymptotically flat black holes of \( p \)th-order Lovelock gravity

\[ S = \frac{1}{4} \sum_{k=1}^p k \alpha_k \int d^{n-1}x \sqrt{\tilde{g}} \tilde{L}_{k-1}, \tag{24} \]

where the integration is done on the \( (n-1) \)-dimensional spacelike hypersurface of Killing horizon, \( \tilde{g}_{\mu\nu} \) is the induced metric on it, \( \tilde{g} \) is the determinant of \( \tilde{g}_{\mu\nu} \) and \( \tilde{L}_k \) is the \( k \)th order Lovelock Lagrangian of \( \tilde{g}_{\mu\nu} \).

All the thermodynamic quantities obtained in this section reduce to those of Einstein gravity given in \([6]\) for \( a = b = 0 \).

IV. STABILITY OF THE UNCHARGED BLACK HOLES

In this section, we consider the special case of \( b = a^2 \) for which the metric function \( f(r) \) becomes

\[ f(r) = k + \frac{r^2}{al^2} \left[ 1 - \left( 1 - 3a + \frac{3aml^2}{r^n} \right)^{1/3} \right], \tag{25} \]
which is real everywhere. The thermodynamic quantities may be obtained by substituting \( b = a^2 \) in those obtained in the previous section. In the uncharged case, the positivity of the heat capacity \( C = \partial M/\partial T \) is sufficient to ensure the local stability, and therefore the plot of \( T \) versus \( m \) gives all the information about thermodynamic stability. For \( k = 0 \), the mass, the temperature and the entropy do not depend on the Lovelock coefficients, as one may see from Eqs. (19)-(23), and the black hole with flat horizon is stable [9]. The stability of black holes with curved horizon, which is the main goal of this paper, will be discussed in the rest of the paper.

### A. Seven-dimensional hyperbolic black holes

First, we study the stability of 7-dimensional black holes of third order Lovelock gravity, which is the most general solution of gravity based on the principle of the standard general relativity in 7 dimensions. This is due to the fact that all the higher order terms of Lovelock gravity in 7 dimensions do not contribute in the field equations. The 7-dimensional solution given by Eqs. (8) and (25) presents a black hole solution provided \( f(r) \) has at least one real positive root \( r_+ \). The existence of extreme black holes depend on the existence of positive real root(s) for equation \( T = 0 \), which reduces to:

\[
r_m(3r_m^4 + 2kl^2r_m^2 + al^4) = 0. \tag{26}
\]

The above equation shows that extreme black holes may exist only for the case of \( k = -1 \).

Depending on the choice of the parameter \( a \), the metric function \( f(r) \) may have two minimums or one. Then, we have different situations corresponding to different values of \( a \):

**I.** For \( a < 1/3 \), the metric function \( f(r) \) has two minimums located at the smallest and largest positive real roots of Eq. (26) denoted by \( r_\text{<} \) and \( r_\text{>} \), respectively. In 7 dimensions, \( r_\text{=} = 0 \), and \( r_\text{>} = l\{1 + \sqrt{1 - 3a}/\sqrt{3}\}^{1/2} \). The two minimums of the metric function \( f(r) \) have the same value equal to zero for the special choice of \( a_c = 1/4 \), which is the solution of the following set of equations:

\[
f(r)|_{r=r_\text{<}} = 0; \quad f(r)|_{r=r_\text{>}} = 0. \tag{27}
\]

If \( a \leq a_c \), the value of \( f(r_\text{<}) \geq f(r_\text{>}) \), while for \( a > a_c \), \( f(r_\text{<}) < f(r_\text{>}) \). Thus, we have two possibilities as follows:
FIG. 1: $f(r)$ vs. $r$ for $k = -1$, $n = 6$, $a = 0.2 < a_c$, and $m < m_{\text{ext}}$, $m = m_{\text{ext}}$ and $m > m_{\text{ext}}$ from up to down, respectively.

FIG. 2: $T$ vs. $m$ for $k = -1$, $n = 6$, $a = 0.2 < a_c$.

(i) If $a \leq a_c$, then $f(r_\leq) \geq f(r_\geq)$ and therefore the mass of the extreme black hole may be obtained by computing Eq. (20) at $r_h = r_\geq$, i.e. $m_{\text{ext}} = m(r_\geq)$. Then, the solution given by Eqs. (8) and (25) presents a black hole with inner and outer horizons provided $m > m_{\text{ext}}$, an extreme black hole if $m = m_{\text{ext}}$, and a naked singularity for $m < m_{\text{ext}}$ (see Fig. 1). Examining the local stability for $m \geq m_{\text{ext}}$ shows that the temperature versus $m$ monotonically increases from zero to infinity, as one may see in Fig. 2. Thus, a hyperbolic black hole is locally stable, if $a \leq a_c$.

(ii) If $a_c < a < 1/3$, then $f(r_\leq) < f(r_\geq)$. In this case, there exist two values for the mass of the extreme black hole $m_{1\text{ext}} = m(r_\leq)$ and $m_{2\text{ext}} = m(r_\geq)$. For black holes with $m < m_{2\text{ext}}$, the mass of the extreme black hole is $m_{1\text{ext}}$, while for black holes with $m > m_{2\text{ext}}$, the mass of the extreme black hole is $m_{2\text{ext}}$. Then, our solution presents a black hole solution
with event horizon radius, \( r_\leq r_+ < r_\geq \) or \( r_+ \geq r_\geq \) provided the mass parameter is in the range \( m_{1\text{ext}} \leq m < m_{2\text{ext}} \) or \( m \geq m_{2\text{ext}} \), respectively. The metric function of these black holes are shown in Fig. 3. The temperature versus \( m \) is shown in Fig. 4, which shows that the slope of the temperature versus \( m \) is always positive, and therefore these black holes are thermodynamically stable. One may note that there is a discontinuity in this curve, which is due to the fact that as \( m \) approaches \( m_{2\text{ext}} \), the radius of the event horizon suddenly changes from \( r_+ < r_\geq \) to \( r_+ = r_\geq \).

**FIG. 3:** \( f(r) \) vs. \( r \) for \( k = -1, n = 6, a = 0.3 > a_c, \) and \( m < m_{1\text{ext}}, m = m_{1\text{ext}}, m_{1\text{ext}} < m < m_{2\text{ext}}, m = m_{2\text{ext}}, \) and \( m > m_{2\text{ext}} \) from up to down, respectively.

**FIG. 4:** \( T \) vs. \( m \) for \( k = -1, n = 6, a = 0.3 > a_c. \)

**II.** For \( a > 1/3 \), the metric function \( f(r) \) has just one minimum at \( r_m = r_{\text{ext}} = 0 \), and the mass of the extreme black hole is \( m_{\text{ext}} = -a^2 l^4 / 3 \) which is negative. In this case, we distinguish a mass parameter \( m_\infty = (3a - 1)a^2 l^4 / 3 \), for which the temperature becomes infinity. The plot of the temperature versus \( m \) (Fig. 5) shows that the temperature starts
from zero for \( m = m_{\text{ext}} \), goes to infinity as \( m \) approaches \( m_\infty \), decreases to a minimum and then increases. Thus, one encounters with a Hawking-Page phase transition. This is a peculiar feature of third order Lovelock gravity, that does not occur in Einstein gravity \[6\] or Gauss-Bonnet gravity \[10\].

The solution for \( a = 1/3 \) presents a black hole of dimensionally continued Lovelock gravity in 7 dimensions with one horizon. In this case, the solution presents a black hole provided \( m \geq -l^4/27 \), and the temperature monotonically increases from zero to infinity. Thus, its slope is always positive and the black hole is thermodynamically stable.

![Graph](image.png)

**FIG. 5:** \( T \) vs. \( m \) for \( k = -1, n = 6, a = 0.4 \).

**B. \((n+1)\)-dimensional hyperbolic black holes**

We can easily extend all of the discussions of the previous subsection to \((n+1)\)-dimensional solutions. The extrema of the metric function \( f(r) \) are located at the roots of the following equation:

\[
3nr_m^6 - (3n-6)l^2r_m^4 + (3n-12)al^4r_m^2 - (n-6)a^2l^6 = 0.
\]

Depending on the choice of the parameter \( a \), as in seven dimensions, \( f(r) \) might have two minimums or one. Indeed, \( f(r) \) has two minimums provided \( a^{(-)} \leq a < a^{(+)} \), where

\[
a^{(-)} = \frac{(n-8)(n-2)^2}{3n(n-6)^2}, \quad a^{(+)} = \frac{1}{3},
\]

and has one minimum otherwise. Of course, one may note that \( a^{(-)} \) is only positive for \( n > 8 \). Thus, we discuss the following three cases separately:
I. For \( a^{(-)} \leq a < a^{(+)} \), the metric function \( f(r) \) has two minimums located at \( r_{\text{<}} \) and \( r_{\text{>}} \) related to the smallest and largest positive real roots of (28). It is worth to mention that the smallest positive real root of (28) is not zero. As in the case of part (I) of the previous subsection, there exists a critical value for the parameter \( a_c \), for which the set of equations (27) hold, and the value of \( f(r_{\text{<}}) = f(r_{\text{>}}) \) for \( a = a_c \). The values of \( a_c \) are 0.3018, 0.3169, 0.3237, 0.3271 for \( n = 7, 8, 9, 10 \), respectively. The diagram of \( f(r) \) is slightly different from it’s analogous seven-dimensional case, as one may see in Fig. (6). With these modifications, all of our discussions in (I) are still valid in this case, and the black holes are stable.

\[ \text{FIG. 6: } f(r) \text{ vs. } r \text{ for } k = -1, n = 8. \]

II. For \( a \geq a^{(+)} \), there is only one real root for (28), which means that \( f(r) \) has just one minimum at \( r_{\text{ext}} \). Here again, all the conclusions are similar to the case of 7-dimensional solutions with \( a > 1/3 \) discussed in the previous subsection. That is, one encounters with an unstable phase for the black hole.

III. For \( n > 8 \), there is a region \( 0 < a < a^{(-)} \), for which \( f(r) \) has just one minimum at \( r_{\text{ext}} \) corresponding to the only positive real root of (28). We have black hole interpretation for \( m > m(r_{\text{ext}}) \) and the temperature is always monotonically increasing. Thus, the black hole solutions are always stable in this region.

C. Spherical black holes

In the case of \( k = 1 \), the solution presents a black hole with one horizon at \( r_{+} \) provided the mass of it is greater than a critical value \( m_c \). For these black holes, the temperature is
always positive and there is no extreme black hole. To analyze the stability, one may plot the temperature versus $r_+$. The plot of temperature versus $r_+$ for a 7-dimensional black hole is shown in Fig. 7. This figure shows that for small values of Lovelock coefficient, there exist an intermediate unstable phase, while for large values of Lovelock coefficients the black hole is stable.

![Graph showing temperature versus $r_+$ for a 7-dimensional black hole.](image)

**FIG. 7:** $T$ vs. $r_+$ with $k = 1$ and $n = 6$ for $a < a_c$, $a = a_c = .046$ and $a > a_c$ from up to down, respectively.

But in higher dimensions, there exists only an intermediate unstable phase for all values of Lovelock coefficient as one may see in Fig. 8.

![Graph showing temperature versus $r_+$ for a 7-dimensional black hole in higher dimensions.](image)

**FIG. 8:** $T$ vs. $r_+$ with $k = 1$ and $a = 0.2$ for $n + 1 = 7, 8$ and 10 from down to up, respectively.

In comparison with the asymptotically AdS spherical black holes of Einstein gravity, which have a small unstable phase, the stability phase structure of the black holes of third order Lovelock gravity with spherical horizon shows that the Lovelock term changes the stability phase structure.
V. CONCLUDING REMARKS

The topological black holes with hyperbolic horizon in Einstein and second order Lovelock gravities are stable \cite{6, 10}. Also, the Lovelock terms do not change the stability phase structure of a black hole with flat horizons \cite{9}. These facts bring in the idea that the stability of topological black holes may be a robust feature of Lovelock gravity. In this paper, we studied the phase structure of topological black holes of third order Lovelock gravity with hyperbolic horizons, and found that they have an intermediate unstable phase for large values of third order Lovelock coefficient. That is, when the effect of third order Lovelock term becomes more relevant, then an unstable phase start appearing. This drastic change in the stability of topological black holes of third order Lovelock gravity persuaded us to investigate the effects of third order Lovelock term on the stability of black holes with spherical horizon. We found that a 7-dimensional spherical black hole in third order Lovelock gravity has an intermediate unstable phase for small third order Lovelock coefficient and is stable for large $\alpha_3$. That is, the third order Lovelock term changes the stability behavior of a black hole, but this effect is not peculiar to third order Lovelock gravity and it occurs in Gauss-Bonnet gravity too \cite{10}. It is worth to mention that an asymptotically AdS black hole in Einstein gravity with small mass is thermodynamically unstable, while in Lovelock gravity it is stable for large values of Lovelock coefficients. This stability analysis shows that the stability of black holes with curved horizons is not a robust feature of all the generalized theory of gravity. Also, we found that the entropy of third order Lovelock gravity reduces to the area law of the entropy for $\alpha_2 = \alpha_3 = 0$. But, as in the case of Einstein gravity, it does not go to zero for the extreme black holes whose temperature is zero.

Although the topological black holes of third order Lovelock gravity are thermodynamically unstable for large values of third order Lovelock coefficient, it is worth to examine the dynamical (gravitational) instability of these black holes. This is due to the fact that there are black holes in Einstein gravity which are thermodynamically unstable, while they are dynamically stable \cite{17}. However, there may be some correlations between the dynamic and thermodynamic instability of black hole solutions of other generalized theory of gravity \cite{18}. 

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