The Geometry of Stochastic Reduction of an Entangled System

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\textbf{PACS}: 02.40.Dr, 04.60.Pp, 75.10.Dg

\textbf{Keywords}: stochastic reduction, disentanglement, geometric quantum mechanics, projective geometry of states.

\textbf{Abstract}

We show that the method of stochastic reduction of linear superpositions can be applied to the process of disentanglement for the spin-0 state of two spin-$\frac{1}{2}$ particles. We describe the geometry of this process in the framework of the complex projective space.

\section{Introduction}

A pure quantum state of a system is a vector in a Hilbert space, which may be represented as a linear combination of a basis of eigenstates of an observable (self-adjoint operator) or of several commuting observables. Let us suppose that the eigenvalues corresponding to the eigenstates of the Hamiltonian operator of the system are the physical quantities measured in an experiment. If the action of the experiment is modelled by a dynamical interaction induced by a term in the Hamiltonian of the system, and its effect is computed by means of the standard evolution according to the Schrödinger equation, the final state would retain the structure of the original linear superposition. One observes, however, that the experiment provides a final state that is one of the basis eigenstates and the superposition has been destroyed. The resulting process is called reduction or collapse of the wave function. The history of attempts to find a systematic framework for the description of this process goes back very far in the development of quantum theory (e.g., the problem of Schrödinger’s cat [18]). In recent years significant progress has been made. Rather than invoking some random interaction with the environment and attributing the observed decoherence, i.e. collapsing of a linear superposition, to the onset of some uncontrollable phase relation, more rigorous methods have been developed, which add to the Schrödinger equation stochastic...
terms corresponding to Brownian fluctuations of the wave function. Since a pure quantum state of a system corresponds to an equivalence class of vectors modulo scaling by a non-zero complex number, corresponding to the norm and an overall phase factor [20, 16], it is natural to develop models for collapse in the setting of a projective space [15, 8]. Associated to an $N$-dimensional complex Hilbert space, we have the projective space $\mathbb{CP}^{n-1}$ equipped with the canonical Fubini-Study metric.

In this paper, we shall apply some of these methods of state reduction to the phenomena considered in the famous paper by Einstein-Podolsky-Rosen [11] explored experimentally by Aspect [3], and analyzed by Bell for its profound implications in quantum theory [5, 6]. The system to be studied consists of a two particle quantum state, where each particle has spin $\frac{1}{2}$. The two body state of total spin zero has the special property known as “entangled” for which a determination of the state of one particle implies with certainty the state of the second. The problems recognized by EPR and studied extensively by Bell arise when the two entangled particles are very far apart.

The states of the two particle system which we shall consider are the equivalence classes of vectors in the tensor product of two spin $\frac{1}{2}$ representation spaces $\mathcal{H} \otimes \mathcal{H}$, where $\mathcal{H}$ corresponds to the states of one of the constituents. We shall describe the experimental detection of the entangled states in terms of mathematical models recently developed for describing the reduction, or collapse, of the wave function. One begins with an entangled state, corresponding to the 1-dimensional spin 0 representation with basis vector the linear superposition:

$$|s = 0\rangle := \frac{1}{\sqrt{2}}(|\uparrow\rangle_1 \otimes |\downarrow\rangle_2 - |\downarrow\rangle_1 \otimes |\uparrow\rangle_2).$$

Here 1, 2 refer to the two spin $\frac{1}{2}$ representations, each one with a basis $\{|\uparrow\rangle, |\downarrow\rangle\}$, corresponding to spin up and spin down, resp., relative to an arbitrary but fixed direction. The full tensor product representation is a sum of this spin 0 representation and a complementary spin 1 representation.

The first stage of reduction, using the stochastic evolution model developed by Diosi, Ghirardi, Pearle, Rimini, Brody and Hughston [8, 9, 14, 13], and references therein, gives rise to a density matrix, a linear combination of projections on disentangled states with Born probability coefficients. The second stage of reduction is the detection of the configuration of disentangled states, which we will not discuss in detail here. Assume that one initially has an entangled spin 0 state of a two particle system and then by some physical process the two particles become separated and far apart. Measurement of the first particle in the spin down state then implies with certainty that the second particle is in the spin up state, measured in the same direction. For the spin 0 state this direction is arbitrary. The question is often raised as to how the state of the second particle can respond to the arbitrary choice of direction in the measurement of the first. This question is dealt with here by the addition of an additional term to the Hamiltonian, which we attribute to the presence of the measurement apparatus. On this basis, we shall attempt here to give a mathematical description of the process underlying such a measurement.

The state $|s = 0\rangle$ is represented in equation (1) as a linear superposition. As noted
above, recently developed methods for describing state reduction can account for a reduction of this superposition to one or the other of the product states occurring on the right hand side of eq. (1) in a simple way if these states are eigenstates of the self-adjoint infinitesimal generator (Hamiltonian) of the evolution.

Suppose, for example, that the Hamiltonian has the form,

\[ H = H_0 + H_1 \]  

where \( H_0 \) contains the spin-independent kinetic energy of the two particles,

\[ H_0 = p_1^2/2m_1 + p_2^2/2m_2, \]

describing the free motion, but \( H_1 \) has the special form

\[ H_1 = \sum \lambda_{i,j} P_{i,j} \]

where the sum is over \( i, j = 1, 2 \) and \( v_1 = \uparrow, v_2 = \downarrow \). We show in the next section that, applying the method of adding a Brownian term to the Schrödinger equation, \([14, 13, 9]\), causes the system to evolve into one or the other of the eigenstates \( |v_i\rangle_1 \otimes |v_j\rangle_2 \) with the correct Born \textit{a priori} probabilities \([2, 13]\). In the case of an initial state of the form (1), the resulting asymptotic state is either \( |\uparrow\rangle_1 \otimes |\downarrow\rangle_2 \) or \( |\downarrow\rangle_1 \otimes |\uparrow\rangle_2 \), each with probability \( \frac{1}{2} \). Such a configuration is called a mixed state.

We should remark that if the two particles correspond to identical fermions, then indices 1, 2 are basically indistinguishable and the two states \( |\uparrow\rangle_1 \otimes |\downarrow\rangle_2 \) and \( |\downarrow\rangle_1 \otimes |\uparrow\rangle_2 \) should appear with equal weights. However, since the particles are located far apart when the measurement takes place, there is no overlap of the one particle wave functions, and the Fermi antisymmetry is not required. Thus the presence of two widely separated detectors can split the degeneracy into distinct states, which can, in fact, imply that \( \lambda_{1,2} \neq \lambda_{2,1} \).

The second stage of reduction, as pointed out above, corresponds to the destruction of the two body state by one-particle filters. The state actually measured is a “separated system” of two particles. We assume that the two filters, which we denote \( M_u \) and \( M_d \) have the property that if the state has the form \( |\uparrow\rangle_1 \otimes |\downarrow\rangle_2 \), then \( M_u \) applied to particle 1 and \( M_d \) applied to particle 2 succeed with certainty. We shall not discuss the extensive literature dealing with the problem of representing separated systems \([4, 17]\). We take as our primary task the description of the first stage of this reduction process.

In the application of the technique of state reduction, it is usually assumed that the evolution is governed by the physical nature of the system before the measurement process. However, in an undisturbed quantum system the linear supposition of states evolves according to a one parameter group of unitary operators which preserves the superposition and for which there is no collapse. One may understand the Brownian fluctuations leading to collapse as induced by the presence of measurement apparatus. In the same way, the component \( H_1 \) of the Hamiltonian may be thought of as induced by the measurement apparatus, which, in our formulation of the problem, disentangles the states, even to the extent of defining the orientations for the states \( |\uparrow\rangle, |\downarrow\rangle \).
In terms of the projective geometry the disentangled states lie in a quadric which is naturally defined by the identification of the underlying 4-dimensional complex vector space as the tensor product of two 2-dimensional complex vector spaces. The entangled state $|s = 0\rangle$ lies outside this quadric, and the stochastic evolution of the system moves the point $|s = 0\rangle$ into the quadric in the first stage of reduction.

In the next section we review the geometric approach to quantum mechanics in terms of projective space and describe the geometry of entanglement. In the following sections we show how the introduction of the modified Hamiltonian in Hughston’s model for stochastic evolution gives a theoretical framework for describing Aspect’s experiments.

## 2 Geometric quantum mechanics

We begin with a quick review of the geometric framework for quantum mechanics in terms of Hamiltonian symplectic dynamics on the quantum mechanical state space introduced by Kibble [1] and developed further by Brody, Hughston and others. For simplicity we will assume that for each time $t$, the wave function $\psi(x, t)$ belongs to a fixed finite dimensional complex Hilbert space and is represented as a linear superposition of a finite basis of states

$$\psi = z^1 \psi_1 + z^2 \psi_2 + \ldots + z^n \psi_n.$$  

The normalization condition demands that

$$|z^1|^2 + |z^2|^2 + \ldots + |z^n|^2 = 1,$$

and since wave functions related by a phase factor $e^{i\alpha}$ represent the same physical state, the time evolution of the system is actually taking place in complex projective $n - 1$-space

$$S^{2n-1}/S^1 \equiv \mathbb{CP}^{n-1}.$$

The space $\mathbb{CP}^{n-1}$ is the set of equivalence classes of complex $n$-tuples modulo multiplication by a non-zero complex number. An equivalence class is represented by $(z^1 : \ldots z^j \ldots : z^n)$, and the $z^i$ are called the homogeneous coordinates of that point. The eigenstate $\psi_j$ corresponds to the point $z^j = (0 : \ldots 1 \ldots : 0)$.

The time evolution of the quantum state is given by the Schrödinger equation on $\mathbb{C}^n$:

$$idz^j/dt = H_{k,j} z^k,$$

with $H_{k,j} = (\psi_k, H\psi_j)$. In a coordinate patch of $\mathbb{CP}^{n-1}$, for example, $z^n \neq 0$, with coordinates $\{x^a | a = 1, \ldots, 2(n-1)\}$, $x^n = i x^{a+n} := z^n/z^n$ the Schrödinger equation can be expressed in Hamiltonian form

$$\hbar dx^a/dt = 2\Omega^{ab} \nabla_b H(x),$$  

(5)
where \(\nabla\) is a covariant derivative on \(CP^{n-1}\) with a connection form associated with Fubini-Studi metric, \(\Omega^{ab}\) is the symplectic structure and the real-valued function (observable) \(H(x)\), is defined by

\[
H(x) = \sum H_{ij,k} z^j \bar{z}^k \sum |z^j|^2.
\]

(6)

If the operator \(H\) is diagonal in the representation provided by \(\{ \psi_j \}\), e.g., with eigenvalues \(\lambda_j\), \(H(x)\) takes the form

\[
H(x) = \sum \lambda_j |z^j|^2 \sum |z^j|^2
\]

which is a function with critical points at \(z^j = (0 : 1 : 0 : \ldots)\).

The projective space geometry naturally lends itself to the computation of transition probabilities. The transition probability from state \(X\) to state \(Y\) is given by

\[
\text{Prob}(X,Y) = \frac{\langle X|Y \rangle \langle Y|X \rangle}{\langle X|X \rangle \langle Y|Y \rangle},
\]

(7)

which has a simple relation to the geodesic distance with respect to the Fubini-Study metric between \(X\) and \(Y\) considered as points in \(CP^{n-1}\). Calling this distance \(\theta\), we have, [14],

\[
\cos^2(\theta/2) = \frac{\langle X|Y \rangle \langle Y|X \rangle}{\langle X|X \rangle \langle Y|Y \rangle}.
\]

This, in particular, means that two conjugate or orthogonal points have geodesic distance \(\pi\) between them.

The state space for a pair of spin-\(\frac{1}{2}\) particles is the projective space of \(\mathbb{C}^2 \otimes \mathbb{C}^2\), which we identify with \(CP^3\). We represent the basis of \(\mathbb{C}^2\) as \(\uparrow, \downarrow\), and the basis \(\mathbb{C}^2 \otimes \mathbb{C}^2\) as \(\uparrow \otimes \downarrow, \uparrow \otimes \uparrow, \downarrow \otimes \downarrow, \downarrow \otimes \uparrow\). Let \((x : y : z : w)\) be the homogeneous coordinates corresponding to this basis.

The singlet state (total spin-0 case) is represented in homogeneous coordinates as

\[
P_0 = (1 : 0 : 0 : -1)
\]

(it is also represented by the line in \(C^4\) with \(x = -w, y = z = 0\)).

The triplet representation is the orthogonal hyperplane \(L\), whose equation in homogeneous coordinates is

\[
L = \{ x - w = 0 \} \quad \text{or, in parametric form} \quad L = \{ (x : y : z : x) \}.
\]

Let us describe the space of possible representations of the eigenstates of the \(spin-z\) operator. The directions of the \(z\)-axes of a system of two particles are parametrized by \(CP^1 \times CP^1\).
The manifold of such states is imbedded in our $CP^3$ as the decomposable 2-tensors, $(a \uparrow + b \downarrow) \otimes (c \uparrow + d \downarrow)$: which gives the Veronese embedding

$((a : b), (c : d)) \mapsto (ad : ac : bd : bc)$

of $CP^1 \times CP^1$ onto the quadric represented by the equation

$Q = \{xw = yz\}$. (8)

The quadric $Q$ intersects the plane $L$ in a conic:

$C = \{x^2 = yz, x = w\}$. (9)

The point $P_{\uparrow\uparrow} = (0 : 1 : 0 : 0)$

on the conic corresponds to the initial spin axis. The point

$P_{\downarrow\downarrow} = (0 : 0 : 1 : 0)$

is the unique point in the conic (9) which is conjugate (orthogonal) to $P_{\uparrow\uparrow} = (0 : 1 : 0 : 0)$ relative to the standard Hermitian inner product. So far we constructed only two eigenstates of the spin-$z$ operator. The third triplet state $P_1$ of the spin operator lies at the intersection of the tangents to the conic at $P_{\uparrow\uparrow}$ and $P_{\downarrow\downarrow}$, see [14], and is given by the equations $y = z = 0, x = w$:

$P_1 = (1 : 0 : 0 : 1)$.

A basis for the 0 eigenstates of the spin-$z$ operator in the full four dimensional representation is given by the intersection of the line

$P_0P_1 = (\mu + \nu : 0 : 0 : \mu - \nu)$

with the the quadric $\{xw = yz\}$ in the two distinct points with $\mu = \pm \nu$:

$P_{\uparrow\downarrow} = (0 : 0 : 0 : 1)$

and

$P_{\downarrow\uparrow} = (1 : 0 : 0 : 0)$.

In this framework, we have constructed the geometry of four spin states spanning $C^2 \otimes C^2$. Moreover, we have explained that disentangled states form a quadric in the associated projective space, and that the spin 0 entangled state, lying outside this quadric is a distinguished point.
3 Collaps of the entangled state

We now describe the mechanism by which an initial entangled state, corresponding to
this distinguished point, can evolve into a disentangled state in the quadric. To see
how this occurs, we review briefly the mechanism of wave function collapse induced by
stochastic fluctuations of the Schrödinger evolution. We follow closely the method of
Hughston [14] (see also [1, 2]).

In the stochastic reduction model of Hughston the system is governed by the fol-
lowing stochastic differential equation:

\[ dx^a = (2\Omega^{a,b} \nabla_b H - \frac{1}{4}\sigma^2 \nabla^a V)dt + \sigma \nabla^a H dW_t \]  

where

\[ V(x) = \nabla_a H(x) \nabla^a H(x) \]

is a so-called quantum uncertainty.(Where it is not mentioned explicitly, the indexes
are lifted by the metric.)

From Itô theory it immediately follows that above process has two basic properties:

1) Conservation of Energy

\[ H(x_t) = H(x_0) + \sigma \int_0^t V(t) dW_t \]

2) Stochastic reduction

\[ dV = -\sigma^2 V(x_t)^2 dt + \sigma \nabla_x V(x_t) \nabla^x \beta(x_t) dW_t \]

where

\[ \beta(x) = \nabla_a H(x) \nabla^a V(x) \]

is the “third” moment.

It follows from (11) that the expectation \( E[V] \) of the stochastic process obeys the
relation [14, 2]

\[ E[V_t] = E[V_0] - \sigma^2 \int_0^t ds E[V_s^2], \]

and since

\[ 0 \leq E[(V_s - E[V_s])^2] = E[V_s^2] - (E[V_s])^2, \]

\[ E[V_t] \leq E[V_0] - \sigma^2 \int_0^t ds E[V_s]^2. \]

Since \( V_s \) is positive, this implies that \( E[V_\infty] = 0 \), and (up to measure 0 fluctuations)
\( V_t \to 0 \) as \( t \) tends to \( \infty \). Since

\[ V = \langle \psi, (H - \langle H \rangle)^2 \psi \rangle / ||\psi||^2, \]

where \( \langle H \rangle = E[H] = \langle \psi, H \psi \rangle / ||\psi||^2, \)

\[ V = 0 \text{ implies } ||(H - \langle H \rangle)\psi|| = 0, \]
and $H \psi = \langle H \rangle \psi$, so $\psi$ is an eigenvector of the Hamiltonian.

Note that the system we have described brings the system to one or another of the eigenstates of the Hamiltonian $H$, with the Born probability given by the initial state, [2, 13]. Therefore, the final configuration corresponds to a mixed state, with each component an eigenstate of $H$.

We now apply the mechanism to what we have called the first stage of the Aspect type experiment [3], the evolution from an entangled state to a disentangled state.

Let us suppose that the system of two spin $\frac{1}{2}$ particles is initially in the entangled spin 0 state, and the two particles move away from each other, according to the motion generated by (3). As the particles approach some neighborhood of the detector, the Schrödinger evolution,

$$i\frac{\partial \psi}{\partial t} = H_0 \psi$$

is modified by the Brownian fluctuations appearing in (10), presumably induced by the detectors and their interactions, for example, with a set of quantum fields. We suppose, as well, that the filters of the apparatus induce a self-adjoint perturbation $H_1$ of the Hamiltonian itself, so that the system evolves, as in (2), according to a perturbed Hamiltonian $H = H_0 + H_1$ in addition to the effect of the Brownian fluctuations. In order that the quantum state converge by stochastic reduction to one of the disentangled states, $\uparrow \otimes \downarrow, \uparrow \otimes \uparrow \downarrow, \downarrow \otimes \downarrow, \downarrow \otimes \uparrow$, we suppose the perturbation to be of the form (4). The component $H_0$ of the Hamiltonian induces an irremovable dispersion, but the residual dispersion can be as small as we wish.

As we have pointed out, for identical particles there may be a degeneracy between the states $\uparrow \otimes \downarrow$ and $\downarrow \otimes \uparrow$; since the filters in the experiment are arranged in one or the other of these configurations, one expects this degeneracy to be broken. Since one particle has moved in one direction and the second in another (with eventually no overlap of the wave functions), the particles then become effectively distinguishable and the induced Hamiltonian is not required to be degenerate. Therefore the final state may become disentangled, as we noted in the Introduction.

The evolution (10) corresponds to the motion of the point in $\mathbb{CP}^3$, going from a singlet state to a limit point in the quadric, for example $\uparrow \otimes \downarrow$ occurring with the corresponding Born probability. As we have pointed out the second stage of the detection, due to the direct action of the detector, must destroy the two body state and create a state of a so-called “separated system” in which one particle is seen with spin up and the other with spin down in two separate (although essentially simultaneous) experiments. The mathematical framework for such separated systems is not completely clear [4, 17]. As Aerts [4] has shown, the set of propositions of such a system is the direct sum of two lattices and does not correspond to a lattice of subspaces of a Hilbert space. We assume however that the outcome of two measurements corresponds to the configuration of the two body state just before the measurement, an assumption generally made in applications of the quantum theory.

Furthermore, we may ask about a situation in which the two filters are not oriented in opposite directions, but at an angle to each other. In this case the bases of the two spin $\frac{1}{2}$ representation would not correspond. One basis would be $\uparrow, \downarrow$ and the other
would be $\downarrow \otimes \uparrow$ where

$$\downarrow = \cos(\theta/2) \uparrow + \sin(\theta/2) \downarrow, \quad \uparrow = -\sin(\theta/2) \uparrow + \cos(\theta/2) \downarrow$$

The computation of the Born probability from a singlet state to a final state determined by the filters, say of the form $\downarrow \otimes \uparrow$ would be $\cos^2(\theta/2)$ (eq. (7)) in agreement with experiment. In this way, arrangements of the filters can effect perturbations of the Hamiltonian that can cause the system to evolve to the appropriate point of the quadric of disentangled states.

4 Some concluding remarks

We have discussed a mechanism based on stochastic reduction, corresponding to a particular class of irreversible processes, which models the evolution of an entangled two-body system to a disentangled state. As an extension of this idea, one may consider a problem with a natural degeneracy of some initial state for which the presence of effective detectors of some type induces a perturbation in which stochastic reduction takes place, as in the asymptotic cluster decomposition of products of quantum fields reducing a $N$-body system to $M k$-body systems or the formation of local correlations in $N$-body systems such as liquids, or spontaneous symmetry breaking. In all these cases, due to the existence of continuous spectra, there will be some residual dispersion in the final state, although possibly very small. We are currently studying possible applications of the methods discussed here to such configurations.

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