Securities Lending Strategies,
Exclusive Auction Bids

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1 Abstract
The objective is either to design an appropriate securities lending auction mechanism or to come up with a strategy for placing bids, depending on which side of the fence a participant sits. There are two pieces to this puzzle. One is the valuation of the portfolio being auctioned subject to the available information set. The other piece would be to come up with the best strategy from an auction perspective once a valuation has been obtained. We derive valuations under different assumptions and show a weighting scheme that converges to the true valuation. We extend auction theory results to be more applicable to financial securities and intermediaries. All the propositions are new results and they refer to existing results which are given as Lemmas without proof. Lastly, we run simulations to establish numerical examples for the set of valuations and for various bidding strategies corresponding to the different auction settings.

2 Introduction
Despite the several advances in the social sciences and in particular economic and financial theory, we have yet to discover an objective measuring stick of value, a so called, True Value Theory. While some would compare the search for such a theory, to the medieval alchemist’s obsession with turning everything into gold, for our present purposes, the lack of such an objective measure means that the difference in value as assessed by different participants can effect a transfer of wealth. This forms the core principle that governs all commerce that is not for immediate consumption in general, and also applies specifically to all investment related traffic which forms a great portion of the financial services industry. Although, some of this is true for consumption assets; because the consumption ability of individuals and organizations is limited and their investment ability is not, the lack of an objective measure of value affects investment assets in a greater way and hence investment assets and related transactions form a much greater proportion of the financial services industry. Consumption assets do not get bought and sold, to an inordinate extent, due to fluctuating prices, whereas investment assets will (Hull 2010 has a description of consumption and investment assets, specific to the price determination of futures and forwards; Kashyap 2014a has a more general discussion).

Another key distinction between consumption assets and investment assets is that investments can generally be shorted, consumption assets cannot be directly shorted. The price effect on consumption assets affects the quantity bought and consumed, whereas with investment assets, (especially ones that can be shorted) the cyclical linkage between vacillating prices and increasing numbers of transactions becomes more apparent. The primary focus on short selling is that activity in the shorting market can be used to predict future security returns. There are many studies that develop theoretical models and perform an application of these concepts to different data-sets, both public and proprietary.
2.1 Related Literature

(Duffie, Garleanu, and Pedersen 2002) present a model of asset valuation in which short-selling is achieved by searching for security lenders and by bargaining over the terms of the lending fee. They provide a closed-form equilibrium solution, including the dynamics of the price, of the lending fees, and of the short interest. The price is elevated by the prospect of future lending fees, and may, in the beginning, be even higher than the valuation of the most optimistic agent. (Harrison and Kreps 1978; Morris 1996) obtain a similar result but explained due to speculative behavior, or the right that investors hold to resell securities, which makes them willing to pay more for it than they would pay if they were obliged to hold it forever. (Hong and Stein 2003) develop a theory of market crashes based on differences of opinion among investors, with a suggestion that short-sales constraints may play a bigger role than one might have guessed based on just the direct transactions costs associated with shorting. Diamond and Verrecchia (1987) provide a theoretical model which implies that the costs associated with short selling will squeeze liquidity traders out of such order flow. This has the effect of making short orders more informative than the population of regular sell orders. (Allen, Morris and Postlewaite 1993) show that even if there is a finite number of trading opportunities, the market price of a security can be above the present value of its future dividends, that is a bubble can persist in the presence of asymmetric information (or agents do not know the beliefs of other agents) with short-sales constraints. [For other theoretical work on the implications of short sale constraints for stock prices, see Jarrow (1980) and Scheinkman and Xiong (2003).]

The standard empirical approach to testing the relation between the shorting market and future stock returns relies on finding an appropriate measure of short sale constraints. This measure is usually obtained either from data on direct costs of shorting from the stock loan market, or by employing proxies for shorting demand or shorting supply. The idea behind looking at shorting demand is that some investors may want to short a stock but may be impeded by constraints; if one can measure the size of this group of investors, one can measure the extent of overpricing or the extent of private information left out of the market. The idea behind looking at shorting supply is that since shorting a stock requires one first to borrow the shares, a low supply of lend-able shares may indicate that short sale constraints are binding tightly.

(Aitken, Frino, McCorry and Swan 1998) build on prior research by extending the investigation of market reaction to short sales to an intraday framework in a setting where short trades are transparent shortly after the time of execution. Focusing on the Australian market, they find a significantly negative abnormal return in calendar time following short sales (initiated using both market and limit ask orders). (Bris, Goetzmann and Zhu 2007) analyze cross-sectional and time series information from forty-six equity markets around the world, to consider whether short sales restrictions affect the efficiency of the market, and the distributional characteristics of returns to individual stocks and market indices. They find some evidence that in markets where short selling is either prohibited or not practiced, market returns display significantly
less negative skewness. However, at the individual stock level, short sales restrictions appear to make no difference. (Boehmer, Jones and Zhang 2008) use a panel of proprietary system order data from the New York Stock Exchange to examine the incidence and information content of various kinds of short sale orders. Their findings indicate that institutional short sellers have identified and acted on important value-relevant information that has not yet been impounded into price. The results are strongly consistent with the emerging consensus in financial economics that short sellers possess important information, and their trades are important contributors to more efficient stock prices. (Desai, Ramesh, Thiagarajan and Balachandran 2002) examine stocks on the NASDAQ and find that heavily shorted firms experience significant negative abnormal returns after controlling for market, size, book-to-market and momentum factors. The negative returns increase with the level of short interest, indicating that a higher level of short interest is a stronger bearish signals. D’avolio (2002) describes the market for borrowing and lending U.S. equities and provides an empirical summary of conditions that can generate and sustain short sale constraints (defined as legal, institutional or cost impediments to selling securities short). (Cohen, Diether and Malloy 2007) examine the link between the shorting market and stock prices using proprietary data from an intermediary. They find that an increase in shorting demand leads to negative abnormal returns. (Kolasinski, Reed, and Ringgenberg 2013) empirically show that search frictions are related to loan fee dispersion by examining the (Duffie, Garleanu, and Pedersen 2002) model. [Other empirical studies include, Jones and Lamont (2002), Reed (2002), Geczy, Musto, and Reed (2007), Mitchell, Pulvino, and Stafford (2002), Ofek and Richardson (2003), and Ofek, Richardson, and Whitelaw (2003), among others.]

2.2 Our Innovations and their Necessity

The above studies fail to fully consider the full extent to which lending desks bridge the demand and supply gap by setting loan rates and managing inventory by finding securities externally or using the positions of other trading desks within the same firm. A more complete study on the effects of short selling must look to incorporate the actions of the main players and how they look to alter their cost structure or the demand/supply mechanisms, by pulling the above levers they have at their disposal.

The securities lending business is a cash cow for brokerage firms. Lenders are assured of a positive spread on every loan transaction they make. Historically, the loan rates were determined mostly as a result of a bargaining process between parties taking the loan and traders on the securities lending desks. Recent trends, due to increased competitive pressures among different players (lending desks and other intermediaries), the introduction of various third party agents that provide information and advice to beneficial owners (the actual asset owners who supply inventory to the lending desks), and the treatment of securities lending as an investment management and trading discipline, have compressed the spreads (difference between the rate at which lending desks acquire inventory and the rate at which make loans) and forced lending desks to look
for ways to improve their profit margins.

To aid this effort at profitability, it is possible to develop different models to manage spreads on daily securities loans and aid the price discovery process, improve the efficiency of the locate mechanism and optimize the allocation of inventory, develop strategies for placing bids on exclusive auctions, price long term loans as a contract with optionality embedded in it and also look at ways to benchmark which securities can be considered to be more in demand or highly shorted and use this approach to estimate which securities are potentially going to become “hot” or “special”, that is securities on which the loan rates can go up drastically and supply can get constrained. (Kashyap 2015c) looks at some of these recent innovations being used by lending desks and also considers how these methodologies can be useful for both buy side and sell side institutions (that is, for all the participants involved).

In this paper, we look at either how to design an appropriate securities lending auction mechanism or to come up with a strategy for placing bids, depending on which side of the fence a participant sits. There are two pieces to this puzzle. One is the valuation of the portfolio being auctioned subject to the information set available to the bidder or the auction designer. This information set would include among other things, the demand for the securities, any additional demand from the locates received, the loan rates applicable to those securities, the duration of the loans, the frequency of loan turnover and the internal inventory pool available to the bidder. These variables can be modeled as geometric Brownian motions with uncertainty introduced via suitable log-normal distributions and a symmetric normal distribution. We derive heuristics to arrive at a set of valuations, with a pecking order that can help decide the aggressiveness of the valuation.

A key result (Theorem 1) is a way to combine different valuations such that the aggregated valuation asymptotically arrives at the true value.

The other piece would be to come up with the best strategy from an auction perspective once a valuation has been obtained. We start with the benchmark scenario where the buyers, placing bids are assumed to have perfect and complete information regarding their valuation of the portfolio that is being auctioned, that is private only to them. We consider the uniform distribution as the simplest scenario and extend that to a more realistic setting that considers the valuations to be log normally distributed. We further extend this by introducing uncertainty into the estimation of bidder valuations and their bidding strategy. The possibility of number of bidders being unknown, the valuations from various bidders being correlated or the interdependent valuation framework and, a reserve price set by the auction seller are more complex extensions. Based on existing results, it is easily seen that the strategies of the bidders constitute a Nash equilibrium, under suitable conditions.

All the propositions are new results and they refer to existing results which are given as Lemmas without proof.

Lastly, we run simulations to establish numerical examples for the set of valuations and for various bidding
strategies corresponding to the different auction settings. The next generation of models and empirical work on securities lending activity would benefit by factoring in the methodologies considered here. In addition, the models developed here could be potentially useful for inventory estimation and for wholesale procurement of financial instruments and also non-financial commodities.

It is tempting to call this one of the more (most) challenging problems in finance, and even though this is debatable and perhaps even labeled as due to ignorance on the author’s part, what stands true is that this is certainly one of the least explored yet profit laden areas of modern investment management. For completeness, we provide a brief overview of the short market before delving further into the mechanism of estimating an auction bid for exclusives.

2.3 Securities Lending Background

Securities Lending began as an informal practice among brokers who had insufficient share certificates to settle their sold bargains, commonly because their selling clients had misplaced their certificates or just not provided them to the broker by the settlement date of the transaction. [See end notes (2) and (3) for more details on the historical evolution of securities lending. (D’Avolio (2002), Jones and Lamont (2002), and Duffie, Garleanu, and Pedersen (2002) have further details on the mechanics of the equity lending market.] Once the broker had received the certificates, they would be passed on to the lending broker. This arrangement was not subject to any formal agreements and there was no exchange of collateral. Securities Lending is a significant market practice whereby securities are temporarily transferred by one party, (the lender) to another (the borrower). The borrower is obliged to return the securities to the lender, either on demand or at the end of any agreed term. For the period of the loan, the lender is secured by acceptable assets or cash of equal or greater value than the lent securities, delivered by the borrower to the lender, as collateral. With such simple beginnings, today, today the business generates hundreds of millions of dollars in revenue and involves the movement of trillions of dollars’ worth of financial instruments. The Over-The-Counter (OTC) nature of the business means that is hard to come up with actual numbers in terms of size and profitability.

Below we chronicle various circumstances that lead to the demand for securities loans.

• Market making and proprietary trading

The most common reason to borrow securities is to cover a short position – using the borrowed securities to settle an outright sale. But this is rarely a simple speculative bet that the value of a security will fall, so that the borrower can buy it more cheaply at the maturity of the loan. More commonly, the short position is part of a larger trading strategy, typically designed to profit from perceived pricing discrepancies between related securities. Some examples are:

• Convertible bond arbitrage: buying a convertible bond and simultaneously selling the underlying equity
‘Pairs’ trading: seeking to identify two companies, with similar characteristics, whose equity securities are currently trading at a price relationship that is out of line with the historical trading range. The apparently undervalued security is bought, while the apparently overvalued security is sold short.

- Merger arbitrage: for example, selling short the equities of a company making a takeover bid against a long position in those of the potential acquisition company.

- Index arbitrage: selling short the constituent securities of an equity price index against a long position in the corresponding index future.

- Other market making and proprietary trading related activities that require borrowing securities include equity/derivative arbitrage, and equity option hedging.

**Borrowing for Failed Trades**

A failed trade may be defined as one where delivery cannot be completed because of insufficient securities available. This is not deliberate policy, but is caused by any number of general administrative problems. Borrowings to cover fails are mostly small and short in duration (one to five days). The borrower keeps the loan open only until he can complete delivery of the underlying trade. An example of this type of transaction occurs when a broker’s client sells stock, but fails to deliver the securities to his broker. The broker borrows the stock, settles the trade and places the resultant settlement funds on deposit. He thereby earns interest on this cash and avoids fail fines. He then unwinds the loan once the client has delivered his securities.

**Borrowing for Margin Requirements.**

To meet margin requirements, for example at the Exchange Traded Options Market, Securities can be borrowed cheaply and lodged as margin, rather than depositing cash.

**Temporary transfer of Ownership**

Another large class of transactions not involving a short is motivated by lending to transfer ownership temporarily to the advantage of both lender and borrower. For example, where a lender would be subject to withholding tax on dividends or interest but some potential borrowers are not. Subject to the possible application of any relevant specific or general anti-avoidance tax provisions or principles, the borrower receives the dividend free of tax and shares some of the benefit with the lender in the form of a larger fee or larger manufactured dividend.

Loans drawn down by market makers and traders on equity instruments are typified as being large in volume and long in duration. For lenders, these loans represent the greatest opportunity to maximize profit. This is
also the reason for referring to these business units as stock loan desks, even though they lend fixed income securities, handle repurchase agreements, manage collateral and other securities borrowing related activities.

The supply of securities into the lending market comes mainly from the portfolios of beneficial owners such as pensions, insurance companies and other such funds. Majority of the funds or asset owners work through agents or intermediary brokers. Intermediaries act between lenders and borrowers. For their services, the intermediary takes a spread. Many institutions find it convenient to lend stock to one or two intermediaries who then lend on to many more counter-parties. This saves administration and limits credit risks. The spread is the result of a bargaining process between intermediary brokers and beneficial owners on one side and between intermediary brokers and end borrowers on the other side. In this and subsequent series, we derive various theoretical results and supplement them with practical considerations that can be of use to lending desks on a daily basis.

3 Motivation for Exclusive Auctions

Necessity is the mother of all creation, but the father is frustration.

We can trace the origins of Exclusive Auctions to the early 2000s. (Duffie, Garleanu, and Pedersen 2002) briefly mention an exclusive lending deal between Credit Suisse First Boston (CSFB) and California Public Employees Retirement System (CalPERS) in 2000. We could found any other reference on this topic in a serious academic paper. As with the rest of the securities lending industry, this practice is more prevalent for equity portfolios. As opposed to traditional arrangements between intermediary brokers and beneficial owners, where the loan rates on each security are negotiated periodically, an exclusive auction, as the name suggests, provides sole usage of a portfolio of securities, or to a portion of the portfolio, to the winner in an auction process, for a certain time period. This arrangement is beneficial to both parties since the intermediary broker gets single ownership to the portfolio. Intermediaries can use the portfolio as part of their overall supply and even if the loan rates for a group of securities in the portfolio go up, the costs of sourcing these special stocks remains the same. Intermediaries look at exclusives as a source of locking up inventory for a certain time horizon. Beneficial owners get a guaranteed source of revenue and will not have the administration hassle of having to constantly create new loans. They will not have to deal with multiple intermediaries and can place their portfolio with an auction agent. Both parties do not need to negotiate or renegotiate loan rates on individual securities for the duration of the exclusive contract.

The holdings in the portfolio on certain key dates are provided to the intermediary brokers or the agent administering the auction to enable brokers to estimate the value of the portfolio from a lending perspective and make bids accordingly. The bid is usually expressed as a certain number of basis points of the portfolio value at the time of auction, applicable annually or over the duration of the exclusive agreement. In addition
to the exclusive bid, beneficial owners also sometimes charge transaction fees; each time securities are taken out from the portfolio or added back.

Beneficial owners continue to manage their portfolio positions as per their investment mandates or according to their re-balancing guidelines or risk tolerances. This risk of turnover in the holdings is something that intermediaries need to factor in their exclusive bids. The agreements can stipulate certain criteria on the turnover of the holdings, which would require the exclusive fee to be reassessed. The huge size of the portfolios that are generally auctioned and the relatively small price of the exclusive fees, in comparison with the loan rates on individual securities, mean that winning an auction bid is an extremely profitable venture for intermediaries. In addition, by gaining access to an exclusive portfolio, intermediaries prevent competitors from having access to this source of inventory, almost acting like monopolists in supplying loans for certain hard to borrow instruments. This restricted supply enables loans to be priced higher. This phenomenon is partly offset when a portfolio is auctioned to more than one bidder, but still provides pricing power to the winners of the auction.

Sometimes, the lending desk could have access to inventory available to the intermediary firm when it acts as a primer broker, operates derivative trading, proprietary trading or services private client accounts. This additional inventory is readily available to the firm as a side effect of having other business units or trading desks. The lending desks at various firms are expected to fully utilize this internal inventory before looking outside for additional supply. Complete utilization of this internal inventory would reduce the funding costs for the other business units and also make the loan rates charged by the firm cheaper than the loan rates of other lending desks, when it has significant internal inventory. The variation in the valuation of the exclusive across different firms would then primarily depend on the extent of the overlap of this internal supply with the holdings in the exclusive. The other source of variation would be the loan rates the lending desk applies to the loans it makes. Historically, the loans rates across different lending desks of different intermediaries have varied considerably due to the opaque nature of the transactions and the variable demand seen by individual desks. With centralized platforms, which consolidate and disclose rates across firms, coming into vogue, loan rates have converged to a considerable extent.

Another piece of the puzzle is the locate requests received by the lending desk on a daily basis. These locate requests are sent by end borrowers, in advance of actually borrowing shares to short, to get an indication of the quantity of shares they can borrow. This is done to ensure that their shorting needs for the trading day can be met. The intermediary can fill either a portion or the entire locate request depending on its inventory situation and also depending on how many firms are sending it locates for that particular security for that trading day. But once a locate request is filled by a lending desk, they are expected to have that number of shares ready for the borrowing firm. A borrowing firm, on the other hand, can borrow as much of the filled locate amount as it chooses to. This mismatch between locate approvals and actual borrows then leads
to another aspect of the lending business that can be optimized, by implementing different variations of the Knapsack Algorithm and we will consider this in another paper (Kashyap 2015c). The conversion factor from locates to borrows can be estimated as part of the locate approval optimization. For the present purpose of estimating an exclusive value, we take this conversion factor as exogenously given. Lending desks have been considering charging a nominal fee based on the locate amount they agree to fill to discourage borrowers from sending in spurious locate requests, though this practice is yet to be formally institutionalized across the lending industry.

So in effect, the lending desk has a certain amount of borrows on the book at any time, which is matched by a combination of internal inventory and supply from beneficial owners. Excess demand arrives in the form of locate requests. Existing loan borrowers can increase their loan holdings via telephone or email, so the loan book can change without the means of locate requests. Managing loan turnover, returning or acquiring supply, locate fulfillment and negotiating the loan rates then constitute the primary loan management duties of the desk.

3.1 Exclusive Auctions Wallet Size

A rough estimate of the potential profits that could be accrued by indulging in exclusives is shown in Figure 1. The point to keep in mind is that this is a highly conservative and approximate estimate since we have used 1 Trillion USD and around 25 basis points as the loan fees in our estimate. The global securities on loan is around 2 trillion USD (Figure 2) and there are securities with loan rates of almost 25%. [See end notes (4), (5) and Baklanova, Copeland and McCaughrin (2015) for more details on the size of the securities lending market]. Even this simple back of the envelope calculations demonstrates that better techniques could go a long way in boosting profits in the exclusive auction process, towards which, to the best of our knowledge, no prior work has been done that applies the use of quantitative methodologies.

| Highly Conservative Estimate of Exclusives Profit Potential |
|-----------------------------------------------------------|
| Securities on Loan (USD)                                  | 1,000,000,000,000 |
| Loan Fee in Basis Points                                  | 25               |
| Loan Fee in Percentage                                    | 0.25%            |
| Annual Fee (USD)                                          | 2,500,000,000    |
| Percentage Locked through Exclusives                      | 10%              |
| Potential Size of Exclusives Profit Pie (USD)             | 250,000,000      |

Figure 1: Exclusive Auctions Profit Potential Estimate
3.2 Buy Side and Sell Side Perspective

The sell side here would be the collection of intermediary firms that source supply and lend it on to final end borrowers. The buy side here would have two segments of firms. One, the end borrowers who either have a proprietary trading strategy or hedging that requires shorting certain securities. Two, the beneficial owners who are long and provide supply to the intermediaries also fall under the buy side category. Depending on which side a firm falls under, they will find the below derivations useful, since it will affect the rates they charge or the rates they pay. This will also help auction designers, who operate on behalf of beneficial owners, formulate an appropriate mechanism that results in the best outcomes for their clients. This can provide transparency to the beneficial owners in terms of how the actual valuation of the portfolio might differ from the actual bids received and hence the actual proceeds.

As we will see in the next section, valuation of this portfolio requires understanding uncertainty from numerous angles. As the participants try to find better and improved ways to capture this uncertainty (See Kashyap 2014a), we will see that the profitability of using this mechanism might decrease for participants from both sides. This can lead to us believe that over time, as better valuation methods are used by the participants, in an iterative fashion, the profits will continue to erode. The cyclical nature of the transactions, which in some case can have its tentacles spread far and wide, can result in catastrophic repercussions, especially when...
huge sums of money move back and forth (Kashyap 2015a). No discussion involving randomness is complete (Taleb 2005, 2010), especially one involving randomness to the extent that we are tackling here, without being highly attuned to spurious results mistakenly being treated as correct and extreme situations causing devastating changes to the expected outcomes. Things can go drastically wrong even in simple environments (Sweeney and Sweeney, 1977), hence in a complex valuation of the sort that we are dealing here, extreme caution should be the rule rather than the exception. (Kashyap 2015b) look at recent empirical examples related to trading costs where unintended consequences set in. With the above background in mind, let us look at how we could value an exclusive portfolio.

4 Exclusive Valuation

4.1 Notation and Terminology for the Exclusive Valuation

- $B_{it}$, the Borrow Book carried by the desk, in shares, at a particular time, $t$, for security, $i$.
- $L_{it}$, the Locate Requests received, in shares, at a particular time, $t$, for security, $i$.
- $\delta_{it} \in [0, 1]$, the conversion rate of locates into borrows, at a particular time, $t$, for security, $i$. We can simplify this to be the same per security.
- $\delta_i$, the conversion rate of locates into borrows for security, $i$. We can simplify this further to be a constant across time and securities, $\delta$.
- $\delta_i L_{it}$, then indicates the excess demand that the desk receives, in shares, at a particular time, $t$, for security, $i$.
- $I_{it}$, the Internal Inventory the intermediary holds, in shares, at a particular time, $t$, for security, $i$.
- $O_{it}$, the supply sourced from other beneficial owners than the exclusive, in shares, at a particular time, $t$, for security, $i$.
- $A_{it}$, the Amount taken out from the Exclusive pool, in shares, at a particular time, $t$, for security, $i$.
- $H_{it}$, the Holdings available in the Exclusive pool, in shares, at a particular time, $t$, for security, $i$.
- $R_{it}$, the Rate on the loan charged by the intermediary, at a particular time, $t$, until the next time period, $t + 1$, for security, $i$.
- $Q_{it}$, an alternate rate to $R_{it}$, at a particular time, $t$, until the next time period, $t + 1$, for security, $i$. This could be the rate at which supply from other beneficial owners is sourced or could be theoretical rate when no rate from other beneficial owners is available. $Q_{it} \leq R_{it}$. 
- $S_{it}$, the Security Price at a particular time, $t$, until the next time period, $t + 1$, for security, $i$.

- $\beta = \frac{1}{1+s}$, is the discount factor, $s$ is the risk free rate of interest. Further complications can be introduced by incorporating continuous time extensions to the short rate process.

- $\nu$, the Valuation of the exclusive, for the duration extending from $t = 0$ to $t = T$.

- The total duration for which the exclusive will be contracted, $T$.

- $P$, the profits from the exclusive for the intermediary over the entire duration $T$.

- $T_e$ and $T_s$, the start and end times of the historical time series.

- $n$, the number of securities available in the Exclusive pool, $i \in \{1, \ldots, n\}$.

- $c$, the transaction cost each time shares are taken or put back into the exclusive.

- $N$, the number of trading intervals.

- The length of each trading interval, $\tau = T/N$. We assume the time intervals are of the same duration, but this can be relaxed quite easily.  
  In continuous time, this becomes, $N \to \infty, \tau \to 0$.

- The time then becomes divided into discrete intervals, $t_k = k\tau$, $k = 0, \ldots, N$. We simplify this and write it as $t = 0$ to $t = T$ with unit increments.

- It is common practice to consider daily increments in time for one year. The fees paid generally also applies on weekends and holidays, though there would be no change in any of the variables on these days. Some firms use 252 trading days to annualize daily loan rates and other fee terms.

- $\{\nu_{\text{zero}}, \nu_{\text{beta}}, \nu_{\text{beta alternate}}, \nu_{\text{transaction}}, \nu_{\text{conservative}}, \nu_{\text{alternate}}, \nu_{\text{historical}}\}$, is the set of valuations.
4.2 Benchmark Valuation

The objective of a rational, risk neutral decision maker at the intermediary would be to maximize the profits, $P$, by utilizing the shares available from the exclusive over the entire duration of the contract.

$$ P = \max_{A_{it}} E_0 \left\{ \sum_{t=0}^{T} \beta^t \sum_{i=1}^{n} A_{it} S_{it} R_{it} - \sum_{t=0}^{T} \psi \beta^t \left( \sum_{i=1}^{n} H_{it} S_{it} \right) \right\} $$

s.t. $A_{it} \leq H_{it}$

$$ I_{it} + H_{it} + O_{it} = B_{it} + \delta_i I_{it} $$

$$ \frac{dS_{it}}{S_{it}} = \mu_S dt + \sigma_S dW_t^{S_i} $$

$$ \frac{dR_{it}}{R_{it}} = \mu_R dt + \sigma_R dW_t^{R_i} $$

$$ \frac{dB_{it}}{B_{it}} = \mu_B dt + \sigma_B dW_t^{B_i} $$

$$ \frac{dI_{it}}{I_{it}} = \mu_I dt + \sigma_I dW_t^{I_i} $$

$$ \frac{dH_{it}}{H_{it}} = \mu_H dt + \sigma_H dW_t^{H_i} $$

$$ \text{Prob}(L_{it}) = \frac{e^{-\lambda_i} (\lambda_i)^{L_{it}}}{(L_{it})!} $$

Locate Process $\iff$ Poisson Process with Arrival Rate, $\lambda_i$

Alternately, $L_{it} \sim |N(\mu_L, \sigma_L^2)|$, Absolute Normal Distribution

Others $\iff$ Log Normal Processes

$W_t^{X_i} \iff$ Weiner Process governing $X_i^{th}$ variable.

$$ E(dW_t^{X_i}dW_t^{X_j}) = \rho_{X_i,X_j} dt $$

$$ \rho_{X_i,X_j} \iff \text{Correlation between } W_t^{X_i} \text{ and } W_t^{X_j} $$

$X_i \in \{S_i, R_i, B_i, I_i, H_i\}$

It is worth highlighting that the decision process of the intermediary (or the variables that he directly influence or set) will only include the number of shares he can take from the exclusive, $A_{it}$ and $O_{it}$, the supply sourced from other beneficial owners. The other variables are taken as exogenous. This assumption is the most realistic scenario, but depending on the size of the exclusive and internal inventory, the loan rates can further be taken as variables he can influence. What happens in practice is that there is usually a baseline for the loan rates and a spread is added on top it. A deeper discussion of how loan rates are set including the addition of a spread component will be taken up in a separate paper.

Hull (2010) provides an excellent account of using geometric Brownian motions to model stock prices
and other time series that are always positive. The borrow, the internal inventory and holdings represent number of shares, and hence are always positive making them good candidates to be modeled as geometric Brownian motions. The borrow process is highly volatile, with the the order of magnitude of the change in the total amount of shares lent out, over a few months, being multiple times of the total amount. The internal inventory can change significantly as well, though there would be less turnover compared to the borrow process. This would of course depend on which parts of the firm the inventory is coming from. The holdings of the exclusive are the least volatile of the three processes that govern shares (or at-least the intermediary would hope so). The volatility of inventory turnover (or any supply) can be a sign of the quality of the inventory and this can be used to price a rate accordingly. This extension and other improvements, where the loan rates and the internal inventory can be made endogenous as opposed to the present simplification, where they are exogenous, will be considered in a subsequent paper (Kashyap 2015c). The locate process which is more precisely modeled as a Poisson process with appropriate units, can be approximated as the absolute value of a normal distribution. This introduces a certain amount of skew, which is naturally inherent in this process.

It is worth keeping in mind that the intermediary firm or the beneficial owner will have access to a historical time series of some of the variables and hence can estimate the actual process for the various variables. Though either party will not know the time series of all the variables with certainty and hence would need to substitute the unknown variables with a simulation based process, similar to what we have used. A simplification is to assume that the variables are independent. A backward induction based computer program, which simulates the randomness component of the variables involved, can calculate the value of the exclusive based on the above expression. Campbell, Lo, MacKinlay and Whitelaw 1998; Lai and Xing 2008; Cochrane 2009 are handy resources on using maximum likelihood estimation (MLE) and generalized method of moments (GMM). See Norstad (1999) for a discussion of the log normal discussion. Gujarati (1995) and Hamilton (1994) discuss time series simplifications and the need for parsimonious models.

4.3 Inequalities to Supplement Equations

In a complex system, deriving equations can be daunting a exercise, and not to mention, of limited practical validity. Hence, to supplements equations, we will employ simplifications that establish a few inequalities governing this system. Pondering on the sources of uncertainty and the tools we have to capture it, might lead us to believe that, either, the level of our mathematical knowledge is not advanced enough, or, we are using the wrong methods. The dichotomy between logic and randomness is a topic for another time.
Proposition 1. The zero profits upper bound for the valuation is given by

\[ v_{\text{actual}} = v \leq v_{\text{zero}} = E_0 \left\{ \sum_{t=0}^T \sum_{i=1}^n \beta^t \min \left[ H_{it}, \max \left( B_{it} + \delta_i L_{it} - I_{it}, 0 \right) \right] S_{it} R_{it} \right\} \frac{1}{\sum_{t=0}^T \beta^t \left( \sum_{i=1}^n H_{it} S_{it} \right)} \]

Proof. See Appendix.

As a further simplification, in this upper limit for the valuation, we set \( \beta = 1 \). We then have

\[ v_{\text{beta}} = E_0 \left\{ \sum_{t=0}^T \sum_{i=1}^n \min \left[ H_{it}, \max \left( B_{it} + \delta_i L_{it} - I_{it}, 0 \right) \right] S_{it} R_{it} \right\} \frac{1}{\sum_{t=0}^T \sum_{i=1}^n H_{it} S_{it}} \]

4.4 Transaction Costs

It is not uncommon to have a transaction cost when securities are taken out or put back into an exclusive portfolio. It is useful to have an expression after incorporating transaction costs.

Proposition 2. The valuation expression that captures transaction costs is given by

\[ v_{\text{transaction}} = E_0 \left\{ \frac{\sum_{t=0}^T \sum_{i=1}^n \min \left[ H_{it}, \max \left( B_{it} + \delta_i L_{it} - I_{it}, 0 \right) \right] S_{it} R_{it}}{\sum_{t=0}^T \sum_{i=1}^n H_{it} S_{it}} \right\} - \left( TC \right) \]

Here,

Transaction Costs \( \equiv TC \) = \[ \sum_{i=1}^n c \left\{ \frac{\max \left( B_{i\theta} + \delta_i L_{i\theta} - I_{i\theta}, 0 \right)}{\left( B_{i\theta} + \delta_i L_{i\theta} - I_{i\theta} \right)} \right\} \]

\[ + \sum_{t=1}^T \sum_{i=1}^n c \left[ \frac{\max \left( B_{it} + \delta_i L_{it} - I_{it}, 0 \right)}{\left( B_{it} + \delta_i L_{it} - I_{it} \right)} \right] \]

\[ - \frac{\max \left( B_{it-1} + \delta_i L_{it-1} - I_{it-1}, 0 \right)}{\left( B_{it-1} + \delta_i L_{it-1} - I_{it-1} \right)} \]

\[ - \left\{ \frac{\max \left( I_{it} - B_{it} - \delta_i L_{it}, 0 \right)}{\left( I_{it} - B_{it} - \delta_i L_{it} \right)} \right\} \]

\[ - \frac{\max \left( I_{it-1} - B_{it-1} - \delta_i L_{it-1}, 0 \right)}{\left( I_{it-1} - B_{it-1} - \delta_i L_{it-1} \right)} \right\} \]

\[ - \left\{ \frac{\max \left( I_{it-1} - B_{it-1} - \delta_i L_{it-1}, 0 \right)}{\left( I_{it-1} - B_{it-1} - \delta_i L_{it-1} \right)} \right\} \]

\[ \right\} \]

Proof. See Appendix.

It is trivial to see that,

\[ v_{\text{transaction}} \leq v = v_{\text{actual}} \]

In a similar vein, we can also arrive at an expression for transaction costs when the charges to Take and Give are different. We don’t derive that here, since that is usually a rarity. Alternately, (Erdos and Hunt 1953) derive results regarding the change of signs of sums of random variables which can provide approximations for transaction costs.
4.5 Other Conservative Valuations

We now provide various methods to come up with more conservative estimates of the valuation. First, we can set \( \delta_i = 0 \) or when excess demand is zero.

\[
\Rightarrow u_{\text{conservative}} = E_0 \left\{ \frac{\sum_{t=0}^{T} \sum_{i=1}^{n} \min [H_{it}, \max (B_{it} - I_{it}, 0)] S_{it} R_{it}}{\sum_{t=0}^{T} (\sum_{i=1}^{n} H_{it} S_{it})} \right\}
\]

Instead of using the rates at which the desk makes loans to borrowers, we can use the rate at which it finds supply from other beneficial owners. Sometimes, where no other supply is available a theoretical rate is used by lending desks. The different possible variations here would depend on the different types of rates (possibly due to different levels of spread) a lending desk would use on a daily basis and also store historically. We show two variations.

\[
\Rightarrow u_{\beta \text{ alternate}} = E_0 \left\{ \frac{\sum_{t=0}^{T} \sum_{i=1}^{n} \min [H_{it}, \max (B_{it} + \delta_i L_{it} - I_{it}, 0)] S_{it} Q_{it}}{\sum_{t=0}^{T} (\sum_{i=1}^{n} H_{it} S_{it})} \right\}
\]

\[
\Rightarrow u_{\text{alternate}} = E_0 \left\{ \frac{\sum_{t=0}^{T} \sum_{i=1}^{n} \min [H_{it}, \max (B_{it} - I_{it}, 0)] S_{it} Q_{it}}{\sum_{t=0}^{T} (\sum_{i=1}^{n} H_{it} S_{it})} \right\}
\]

This gives the following pecking order of valuations for the exclusive.

\[ u_{\beta} \geq u_{\text{conservative}} \geq u_{\text{alternate}} \]

\[ u_{\beta} \geq u_{\beta \text{ alternate}} \geq u_{\text{alternate}} \]

The intermediary can decide on their level of aggressiveness and choose which of the valuations they want to use, depending on how many exclusives they already have, the extent of overlap with their internal inventory, the number of special names in the exclusive portfolio and the volatility of the time series of daily profits from the exclusive. Such a tiered approach is found to be more practical rather than having an exact valuation since there are too many sources of uncertainty and the noise or the variance of any exact valuation number would tend to be high.

4.6 Historical Valuations

Given the complexity and the number of variables to be estimated, a simple heuristic would be utilize the historical time series of each of the variables and then use that as a possible guide to the calculation of the exclusive value. The pecking order shown above can be arrived at using the historical time series as well. Using this, we can also arrive at the time series of the daily profits that would accrue to the intermediary. The volatility of the daily value of the exclusive can be suggestive in terms of how aggressive one should be.
in picking one of the valuation tiers.

\[ \Rightarrow \upsilon_{\text{historical}} = \left\{ \sum_{t=-T_s}^{-T_e} \sum_{i=1}^{n} \min [H_{it}, \max (B_{it} + \delta_iL_{it} - I_{it}, 0)] S_{it}R_{it} \right\} \]

New valuation time series can be created by adding transaction costs or alternate rates, or other combinations. This might be relevant depending on the preferences or the setup at the intermediary. Armed with this set of valuations, \( \{\upsilon_{\text{zero}}, \upsilon_{\text{beta}}, \upsilon_{\text{beta alternate}}, \upsilon_{\text{transaction}}, \upsilon_{\text{conservative}}, \upsilon_{\text{alternate}}, \upsilon_{\text{historical}}\} \), the bidder can combine them using the method we shown in the next sub section. Alternately, he can subjectively select a particular valuation to suit the institutional setup and is now ready to pick a strategy that will shade his value, to suit the mechanics of different auction situations.

### 4.7 Variance Weighted Combined Valuation

We now show a way to combine the valuations using the variance of individual valuation time series and argue that under certain conditions of finite variance and finite valuation of each individual time series, we get closer to the true valuation as the number of individual time series considered gets larger. For simplicity of notation, in this section we let each individual time series be represented by \( \upsilon_i, i \in \{1, 2, ..., k\} \) with corresponding variances \( \sigma_i^2 \) and the true valuation by \( \upsilon \) with corresponding variance \( \sigma^2 \).

**Theorem 1.** When each of the individual valuations are weighted using the scheme shown below, the expression asymptotically converges to the true valuation.

\[
E \left[ \lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} \sum_{j \neq i}^{k} \frac{\sigma_j^2 \upsilon_i}{\sum_{i=1}^{k} \sigma_i^2} \right] = E[\upsilon]
\]

**Proof.** See Appendix. \( \square \)

This has an intuitive and practical appeal since the time series with the higher variance is set a lower weight in the combined valuation. This means the more expressions we are able to derive for the valuations and combine them, the better will be our estimation. Of course, it becomes important to ensure that we do not have redundant valuation expressions, that are just multiples of one other, but valuations that capture the true variation in any of the variables that can affect the valuation outcome would be good candidates to vary and create a new time series.
5 Auction Strategy

Once we have the valuation from the previous section, we look at different auction formats and the specifics of how an intermediary would tailor bids, to adapt, to the particular auction setting. A bidding strategy is sensitive to assumed distributions of both the valuations and the number of bidders. We consider two distributions for the valuation extensions: Uniform and Log-normal. The uniform distribution is well uniform and hence is ideal when the valuations (or sometimes even the number of bidders) are expected to fall equally on a finite number of possibilities. This serves as one extreme to the sort of distribution we can expect in real life. The other case is a log normal distribution which centers around a value and the chance of observing values further away from this central value become smaller. Asset prices are generally modeled as log normal, so financial applications would benefit from this extension. The absence of a closed form solution for the log normal distribution forces us to develop an approximation that works well for our particular application, since the valuations are generally small, of the order of a few basis points. The two distribution types we discuss can shed light on the other types of distributions in which only positive observations are allowed.

We formulate a new positive symmetric discrete distribution, which is likely to be followed by the total number of auction participants, and incorporate this into auction theory results. This distribution can also be a possibility for the valuations themselves, since the set of prices of assets or valuations can be from a finite set. But given the distribution, developing a bidding strategy based on this discussion is trivial and hence is not explicitly given below. Lastly, the case of interdependent valuations is to be highly expected in real life; but practical extensions for this case are near absent. We develop extensions when the valuations of bidder are interdependent and incorporate all the results developed into a final combined realistic setting. The results developed here can be an aid for profit maximization for bidders and auctions sellers during the wholesale procurement of financial instruments and also non-financial commodities.

We consider a few variations in the first price sealed bid auction mechanism. We provide proofs for the extensions but we simply state standard results without proofs so that it becomes easier to see how the extensions are developed. Such an approach ensures that the results are instructive and immediately applicable to both bidders and auction facilitators. All the propositions are new results and they refer to existing results which are given as Lemmas without proof. The literature on Auction Theory is vast and deep. We consider the following standard and detailed texts on this topic: Klemperer 2004; Krishna 2009; Menezes and Monteiro 2005; and Milgrom 2004. Additional references are pointed out in the relevant sections below along with the extensions that are derived.
5.1 Notation and Terminology for the Auction Strategy

- $x_i$, the valuation of intermediary or bidder $i$. This is a realization of the random variable $X_i$ which bidder $i$ and only bidder $i$ knows for sure.

- $x_i \sim F[0, \omega]$, $x_i$ is symmetric and independently distributed according to the distribution $F$ over the interval $[0, \omega]$.

- $F$, is increasing and has full support, which is the non-negative real line $[0, \infty]$.

- $f = F'$, is the continuous density function of $F$.

- $x_i \sim U[0, \omega]$ when we consider the uniform distribution.

- $x_i \sim LN[0, \omega]$ when we consider the log normal distribution.

- $M$, is the total number of bidders.

- $f_i, F_i$, are the continuous density function and distribution of bidder $i$ in the asymmetric case.

- $r \geq 0$, is the reserve price set by the auction seller.

- $\beta_i : [0, \omega] \rightarrow \Re_+$ is a increasing function that gives the strategy for bidder $i$. We let $\beta_i (x_i) = b_i$. We must have $\beta_i (0) = 0$.

- $\phi_i \equiv \beta_i^{-1}$ is the inverse of the bidding strategy $\beta_i$. This means, $x_i = \beta_i^{-1} (b_i) = \phi_i (b_i)$.

- $x_i \sim F_i[0, \omega_i]$. Here, $x_i$ is asymmetric and is independently distributed according to the distribution $F_i$ over the interval $[0, \omega_i]$.

- $\beta : [0, \omega] \rightarrow \Re_+$ is the strategy of all the bidders in a symmetric equilibrium. We let $\beta (x) = b, x$ is the valuation of any bidder. We also have $b \leq \beta (x)$ and $\beta (0) = 0$.

- $Y_1 \equiv Y_1^{M-1}$, the random variable that denotes the highest value, say for bidder 1, among the $M-1$ other bidders.

- $Y_1$, is the highest order statistic of $X_2, X_3, ..., X_M$.

- $G$, is the distribution function of $Y_1$. $\forall y, G(y) = [F(y)]^{M-1}$.

- $g = G'$, is the continuous density function of $G$ or $Y_1$.

- $\Pi_i$, is the payoff of bidder $i$. $\Pi_i = \begin{cases} x_i - b_i & \text{if } b_i > \max_{j \neq i} b_j \\ 0 & \text{if } b_i < \max_{j \neq i} b_j \end{cases}$

- $\Pi_s$, $x_s$ is the payoff and valuation of the auction seller.
• $m(x)$, is the expected payment of a bidder with value $x$.

• $R_s$ is the expected revenue to the seller.

• $M = \{1, 2, \ldots, M\}$ is the potential set of bidders when there is uncertainty about how many interested bidders there are.

• $A \subseteq \mathcal{N}$ is the set of actual bidders.

• $p_l$ is probability that any participating bidder assigns to the event that he is facing $l$ other bidders or that there is a total of $l + 1$ bidders, $l \in \{1, 2, \ldots, M - 1\}$.

• $X_i \in [0, \omega_i]$ is bidder $i$’s signal when the valuations are interdependent.

• $V_i = v_i(X_1, X_2, \ldots, X_M)$ is the value of the exclusive to bidder $i$. $v_i(0, 0, \ldots, 0) = 0$

• $v_i(x_1, x_2, \ldots, x_M) \equiv E[V_i \mid X_1 = x_1, X_2 = x_2, \ldots, X_M = x_M]$ is a more general setting, where knowing the signals of all bidders still does not reveal the full value with certainty.

5.2 Symmetric Independent Private Values with Valuations from General Distribution

As a benchmark bidding case, it is illustrative to assume that all bidders know their valuations and only theirs and they believe that the values of the others are independently distributed according to the general distribution $F$.

Lemma 1. The symmetric equilibrium bidding strategy for a bidder, the expected payment of a bidder and the expected revenue of a seller are given by

Equilibrium Bid Function is,

$$\beta(x) = x - \int_0^x \left[ \frac{F(y)}{F(x)} \right]^{M-1} dy$$

Expected ex ante payment of a particular bidder is,

$$E[m(x)] = \int_0^{\omega} y [1 - F(y)] g(y) dy$$

Expected revenue to the seller is

$$E[R_s] = ME[m(x)]$$

Proof. See Appendix.
5.3 Symmetric Independent Private Values with Valuations Distributed Log Normally

Proposition 3. The symmetric equilibrium bidding strategy when the valuations are distributed log normally can be approximated as

\[
\beta(x) \approx \frac{x}{2}
\]

Proof. See Appendix.

From this expression, which is valid for small values of the valuation (the valuations are usually a few basis points and hence this approximation holds in our case), we see that the bid when the bidder valuations are log normally distributed, does not depend on the number of bidders.

5.4 Symmetric Independent Private Values with Valuations Distributed Uniformly

Lemma 2. The symmetric equilibrium bidding strategy when the valuations are distributed uniformly is given by

\[
\beta(x) = \left( \frac{M - 1}{M} \right) x
\]

Proof. See Appendix.

Comparing the bidding strategy in the two cases, uniform and log-normal distribution, we see that when the number of bidders are large, both do not depend on the number of bidders and the bid is much larger with a uniform distribution.

5.5 Symmetric Independent Private Value with Reserve Prices

Lemma 3. The symmetric equilibrium bidding strategy when the valuation is greater than the reserve price, \( r > 0 \), of the seller, \( x \geq r \), for a general distribution is,

\[
\beta(x) = r \frac{G(r)}{G(x)} + \frac{1}{G(x)} \int_{r}^{x} yg(y) dy
\]

Alternately

\[
\beta(x) = x - \int_{r}^{x} \frac{G(y)}{G(x)} dy
\]

Proof. See Appendix.
5.5.1 Uniform Distributions

**Proposition 4.** The symmetric equilibrium bidding strategy when the valuation is greater than the reserve price of the seller, \( x \geq r \), and valuations are from an uniform distribution,

\[
\beta (x) = \frac{r^M}{x^{M-1}} \left( \frac{M+1}{M} \right) + x \left( \frac{M-1}{M} \right)
\]

*Proof.* See Appendix.

5.5.2 Log Normal Distributions

**Proposition 5.** The symmetric equilibrium bidding strategy when the valuation is greater than the reserve price of the seller, \( x \geq r \), and valuations are from a log normal distribution,

\[
\beta (x) = x \left[ \frac{h'(r) (x-r)}{h(x)} + \frac{r h(r)}{x h(x)} \right]
\]

Here, \( h'(r) = (M-1) \left[ \int_{-\infty}^{(\ln r - \mu) / \sigma} e^{-t^2/2} dt \right]^{M-2} \left\{ \frac{e^{-(\ln r - \mu)^2/2}}{r \sigma} \right\} \)

*Proof.* See Appendix.

5.5.3 Optimal Reserve Price for Seller and Other Considerations

**Lemma 4.** The optimal reserve price for the seller, \( r^* \) must satisfy the following expression,

\[
x_s = r^* - \frac{1 - F(r^*)}{f(r^*)}
\]

Here, seller has a valuation, \( x_s \in [0, \omega) \)

*Proof.* See Appendix.

5.6 Variable Number of Bidders with Symmetric Valuations and Beliefs about Number of Bidders

**Lemma 5.** The equilibrium strategy when there is uncertainty about the number of bidders is given by

\[
\beta^M (x) = \sum_{l=0}^{M-1} p_l G^l (x) \beta^l (x)
\]

Here, \( p_l \) is the probability that any bidder is facing \( l \) other bidders. \( G^l (x) = [F(x)]^l \) is the probability of the event that the highest of \( l \) values drawn from the symmetric distribution \( F \) is less than \( x \), his valuation
and the bidder wins in this case. $\beta^l(x)$ is the equilibrium bidding strategy when there are exactly $l + 1$ bidders, known with certainty. The overall probability that the bidder will win when he bids $\beta^M(x)$ is

$$G(x) = \sum_{l=0}^{M-1} p_l G^l(x)$$

Hence the equilibrium bid for an actual bidder when he is unsure about the number of rivals he faces is a weighted average of the equilibrium bids in an auction when the number of bidders is known to all.

Proof. See Appendix.

When bidding for an exclusive, an intermediary, will expect most of the other major players to be bidding as well. Invariably, there will be some dropouts, depending on their recent exclusive bidding activity and some smaller players will show up based on the composition of the portfolio being auctioned. It is a reasonable assumption that all of the bidders hold similar beliefs about the distribution of the number of players. Hence, for the numerical results, we construct a symmetric discrete distribution of the sort shown in Figure 3. It is easily shown that it satisfies all the properties of a probability distribution function. For simplicity, we use the uniform distribution for the valuations and set $\omega = 1$.

**Proposition 6.** The formula for the probability of facing any particular total number of bidders under a symmetric discrete distribution, and the bidding strategy, would be given by,

$$p_l = \begin{cases} l\Delta_p, & \text{if } l \leq \frac{(M-1)}{2} \\ (M - l)\Delta_p, & \text{if } l > \frac{(M-1)}{2} \end{cases}$$

$$\Delta_p = \frac{1}{\left\lfloor \frac{(M-1)}{2} \right\rfloor \left\lceil \frac{(M-1)}{2} \right\rceil + 1 + \left\lfloor \frac{(M-1)}{2} \mod 1 + \frac{(M-1)}{2} \right\rfloor \left\lceil \frac{(M-1)}{2} \mod 1 \right\rceil}$$

$$\beta(x) = \sum_{l=0}^{M-1} \left( \frac{p_l x^l}{\sum_{k=0}^{M-1} p_k x^k} \right) \left( \frac{l}{l+1} \right) x$$

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5.7 Asymmetric Valuations

Lemma 6. The system of differential equations for an asymmetric equilibrium is given by

\[ \sum_{j \neq i} \left\{ \frac{f_j(\phi_j(b)) \phi_j'(b)}{F_j(\phi_j(b))} \right\} = \frac{1}{\phi_i(b) - b} \]

Proof. See Appendix.

This system of differential equations can be solved to get the bid functions for each player. Closed form solutions are known for the case of uniform distributions with different supports. A simplification is possible by assuming that say, some bidders have one distribution and some others have another distribution. This is a reasonable assumption since firms with bigger sources of internal inventory would tend to differ from those with smaller sources. Among other things, this would depend on the other divisions within a particular intermediary and the reputation of its franchise.

Proposition 7. If, \(K + 1 \) firms (including the one for which we derive the payoff condition) have the distribution \(F_1\), strategy \(\beta_1\) and inverse function \(\phi_1\). The other \(M - K - 1\) firms have the distribution \(F_2\), strategy \(\beta_2\) and inverse function \(\phi_2\). The system of differential equations is given by,

\[ \left\{ K \frac{f_1(\phi_1(b)) \phi_1'(b)}{F_1(\phi_1(b))} \right\} + \left\{ (M - 1 - K) \frac{f_2(\phi_2(b)) \phi_2'(b)}{F_2(\phi_2(b))} \right\} = \frac{1}{\phi_i(b) - b} \]

Proof. See Appendix.

As a special case, if there are only two bidders, \(M = 2, K = 1\) the above reduces to a system of two
differential equations,
\[
\phi_1'(b) = \frac{[F_1(\phi_1(b))]}{f_1(\phi_1(b))|\phi_2(b) - b|} \\
\phi_2'(b) = \frac{[F_2(\phi_2(b))]}{f_2(\phi_2(b))|\phi_1(b) - b|}
\]

5.8 Symmetric Interdependent Valuations

It is worth noting that a pure common value model of the sort, \(V = v(X_1, X_2, ..., X_M)\) is not entirely relevant in our context since the amount of internal inventory and the size of the borrow book will vary across intermediaries. This means that the amount of shares they will use from the exclusive will vary and so will their valuations. What this reasoning tells us is that it is reasonable to expect that there is some correlation between the signals of each bidder. This makes sense since the total supply, in a security, is distributed across all the bidders and the valuation of the portion in the exclusive will depend on the total supply. The valuation of a particular bidder will then depend on his inventory and how he expects the rest of the supply to be distributed among the other bidders, resulting in a symmetric interdependent auction strategy. From the perspective of a particular bidder, the signals of the other bidders can be interchanged without affecting the value. This is captured using the function \(u(X_i, X_{-i})\) which is the same for all bidders and is symmetric in the last \(M-1\) components. We assume that all signals \(X_i\) are from the same distribution \([0, \omega]\) and that the valuations can be written as

\[
u_i(X_1, X_2, ..., X_M) = u(X_i, X_{-i})
\]

We also assume that the joint density function of the signals \(f\) defined on \([0, \omega]^M\) is symmetric and the signals are affiliated. Affiliation here refers to the below properties.

- The random variables \(X_1, X_2, ..., X_M\) distributed on some product of intervals \(\mathcal{X} \subset \mathbb{R}^M\) according to the joint density function \(f\). The variables \(X = (X_1, X_2, ..., X_M)\) are affiliated if \(\forall x', x'' \in \mathcal{X}, f(x' \vee x'') f(x' \wedge x'') \geq f(x') f(x'')\). Here \(x' \vee x''\) and \(x' \wedge x''\) denote the component wise maximum and minimum of \(x'\) and \(x''\).

- The random variables \(Y_1, Y_2, ..., Y_{M-1}\) denote the largest, second largest, ... , smallest from among \(X_2, X_3, ..., X_M\). If \(X_1, X_2, ..., X_M\) are affiliated, then \(X_1, Y_1, Y_2, ..., Y_{M-1}\) are also affiliated.

- Let \(G(. \mid x)\) denote the distribution of \(Y_1\) conditional on \(X_1 = x\) and let \(g(. \mid x)\) be the associated conditional density function. Then if \(Y_1\) and \(X_1\) are affiliated and if \(x' > x\) then \(G(. \mid x')\) dominates
in terms of the reverse hazard rate, \( \frac{g(t)}{G(t)} \). That is, for all \( y \),

\[
\frac{g(y | x')}{G(y | x')} \geq \frac{g(y | x)}{G(y | x)}
\]

- If \( \gamma \) is any increasing function, then \( x' > x \) implies that

\[
E[\gamma(Y_1) | X = x'] \geq E[\gamma(Y_1) | X = x]
\]

We define the below function as the expectation of the value to bidder 1 when the signal he receives is \( x \) and the highest signal among the other bidders, \( Y_1 = y \). Because of symmetry this function is the same for all bidders and we assume it is strictly increasing in \( x \). We also have \( u(0) = v(0,0) = 0 \).

\[
v(x,y) = E[V_1 | X = x, Y_1 = y]
\]

**Lemma 7.** A symmetric equilibrium strategy governed by the set of conditions above is given by

\[
\beta(x) = \int_0^x v(y,y) dL(y | x)
\]

Here, we define \( L(y | x) \) as a function with support \([0,\omega]\),

\[
L(y | x) = \exp \left[ - \int_y^x \frac{g(t | t)}{G(t | t)} dt \right]
\]

**Proof.** See Appendix.

**Proposition 8.** The bidder’s equilibrium strategy under a scenario when the valuation is the weighted average of his valuation and the highest of the other valuations is given by the expression below. That is, we let \( v(x,y) = \alpha x + \xi y \) for \( \alpha, \xi \in [0,1] \). This also implies, \( v(x,y) = u(x,y) = u(x_i, x_{-i}) = \alpha x_i + \xi \max(x_{-i}) \), giving us symmetry across the signals of other bidders. An alternative formulation could simply be \( v(x,y) = \frac{1}{M} \left( \sum_{i=1}^M x_i \right) \). The affiliation structure follows the Irwin-Hall distribution with bidder’s valuation being the sum of a signal coming from a uniform distribution with \( \omega = 1 \) and a common component from the same uniform distribution.

\[
\beta(x) = \left[ \frac{2(\alpha + \xi)(M - 1)}{(2M - 1)x^{2M-2}} \right] + (\alpha + \xi) \left[ x - \frac{1}{(2x - 1 - x^2/2)^{M-1}} \left\{ \frac{1}{2M-1} + \int_1^x \left( 2y - 1 - \frac{y^2}{2} \right)^{M-1} dy \right\} \right]
\]
Proof. The proof is given in the Appendix including a method to solve the last integral.

5.9 Combined Realistic Setting

Proposition 9. The bidding strategy in a realistic setting with symmetric interdependent, uniformly distributed valuations, with reserve prices and variable number of bidders is given by

\[ \beta(x) = r - \int_x^{\infty} e^{-s^2/(2\alpha^2)} ds + \int_x^{\infty} v(y, y) \frac{g(y | y)}{G(y | y)} e^{-\int_y^{\infty} e^{-s^2/(2\alpha^2)} ds} dy \]

Here, \( x^*(r) \) is found by solving for \( x \) in the below condition

\[
\int_0^1 \xi y \left[ \frac{y}{x} \right]^{2(M-2)} \left( \frac{2y}{x^2} \right) dy + \int_1^x \xi y \left[ \frac{2y - 1 - \frac{y^2}{2}}{(2x - 1 - \frac{x^2}{2})} \right]^{M-2} \left\{ \frac{2 - y}{(2x - 1 - \frac{x^2}{2})} \right\} dy = \frac{r - \alpha x}{(M-1)}
\]

Proof. The proof is given in the Appendix including a method to solve the above type of equations.

It is trivial to extend the above to the case where the total number of bidders is uncertain by using the equilibrium bidding strategy \( \beta^l(x) \) and the associated probability \( p_l \) when there are exactly \( l + 1 \) bidders, known with certainty,

\[ \beta^M(x) = \sum_{l=0}^{M-1} \frac{p_l G^l(x)}{G(x)} \beta^l(x) \]

6 Data-set Construction

As noted earlier, given the complexity of the system and the number of random variables involved, the computational infrastructure required to value an exclusive can be tremendous. A typical exclusive portfolio can have anywhere from a few hundred to upwards of a thousand different securities. It is therefore, simpler to use the historical time series and calculate the valuation from the corresponding formula derived in section 4.6. To demonstrate numerical results, we simulate the historical time series. We pick a sample portfolio with one hundred different hypothetical securities and we come up with the time series of all the variables involved (Price, Quantity Borrowed, Exclusive Holding, Inventory Level, Loan Rate, Alternate Loan Rate) by sampling from suitable log normal distributions. It is worth noting, that the mean and standard deviation of each time series are themselves simulations from other appropriately chosen uniform distributions (Figure 4). The locate process can be modeled as a Poisson distribution with appropriately chosen units. It is simpler to consider it as the absolute value of a normal distribution. The mean and standard deviation of the locate distribution for each security are chosen from another appropriately chosen uniform distribution.

The simulation seed is chosen so that the drift and volatility we get for the variables (mean and standard deviation for the locate process) are similar to what would be observed in practice. For example in Figure
the price and rate volatility are lower than the volatilities of the borrow and other quantities, which tend to be much higher; the range of the drift for the quantities is also higher as compared to the drift range of prices and rates. This ensures that we are keeping it as close to a realistic setting as possible, without having access to the historical time series. The volatility and drift of the variables for each security are shown in Figure 5. The length of the simulated time series is one year or 252 trading days for each security. A sample of the time series of the variables generated using the simulated drift and volatility parameters is shown in Figure 6. The full time series used for the calculations is available upon request.

![Figure 4: Simulation Seed](image1)

![Figure 5: Simulation Sample Distributions](image2)
7 Model Testing Results

To summarize the results of the testing we show the summary statistics of the portfolio value under different valuation criteria and auction settings (Figure 7). We see that the valuation ranges from 30 to 50 basis points. When we repeat the simulations with different seed values, the results could vary outside this range, but are not drastically different. We easily verify some results well known in the auction literature (Krishna 2009): 1) As the number of bidders bidding uniformly increases, the bid increases; 2) Setting a reserve price results in higher bids. The bid with a discrete symmetric distribution as the number of bidders goes higher is comparable to the log normal distribution, which we know does not depend on the number of bidders (section 3). The Comparative Statics of the valuation with changes in Beta and a time series graph of the different valuations are also shown in Figures 8 and 9. As the subjective discount factor $\beta$ decreases, the valuation increases since the effect of the discounting is higher on the holding levels than on the revenue.

| Rate | Sector | Price | Borrow | Holding | Inventory | Rate | Q-Rate | Locate |
|------|--------|-------|--------|---------|-----------|------|--------|--------|
| 1    | 0.36   | 0.18  | 0.31   | 0.47    | 0.12      | 0.69 | 0.41   | 0.35   |
| 2    | 0.35   | 0.49  | 0.35   | 0.41    | 0.12      | 0.69 | 0.41   | 0.35   |
| 3    | 0.33   | 0.78  | 0.38   | 0.40    | 0.12      | 0.69 | 0.41   | 0.35   |
| 4    | 0.34   | 0.64  | 0.38   | 0.40    | 0.12      | 0.69 | 0.41   | 0.35   |
| 5    | 0.35   | 0.58  | 0.38   | 0.40    | 0.12      | 0.69 | 0.41   | 0.35   |
| 6    | 0.34   | 0.58  | 0.38   | 0.40    | 0.12      | 0.69 | 0.41   | 0.35   |
| 7    | 0.34   | 0.58  | 0.38   | 0.40    | 0.12      | 0.69 | 0.41   | 0.35   |
| 8    | 0.34   | 0.58  | 0.38   | 0.40    | 0.12      | 0.69 | 0.41   | 0.35   |
| 9    | 0.34   | 0.58  | 0.38   | 0.40    | 0.12      | 0.69 | 0.41   | 0.35   |
| 10   | 0.34   | 0.58  | 0.38   | 0.40    | 0.12      | 0.69 | 0.41   | 0.35   |

Figure 6: Simulation Sample Time Series

Figure 7: Valuation Summary Statistics

| Beta | 0.25 | 0.5 | 0.75 | 0.9 | 1     |
|------|------|-----|------|-----|-------|
| Holding / Revenue | 0.63% | 0.64% | 0.63% | 0.62% | 0.43% |

Figure 8: Valuation Beta Comparative Statics
Improvements to the Model

Numerous improvements to the model are possible. Cobb, Rumi and Salmerón 2012; and Nie and Chen 2007 derive approximate distributions for the sum of log normal distribution which highlight that we can estimate the log normal parameters from the time series of the valuations and hence get the mean and variance of the valuations. A longer historical time series will help get better estimates for the volatility of the valuation. This can be useful to decide the aggressiveness of the bid. Another key extension can be to introduce jumps in the log normal processes. This is seen in stock prices to a certain extent and to a greater extent in the borrow, holding and inventory processes.

The auction theory aspects combines standard results with new extensions for the log-normal case, the interdependent case and a combined realistic setting with uniform distributions. Instead of the bidding strategies we have considered, we can come up with a parametric model that will take the valuations as the inputs and the bid as output. The parameters can depend on the size of the portfolio, the number of securities, the number of special securities, the number of markets, the extent of overlap with the internal inventory, and where available, the percentile rankings of the historical bids for previous auctions, which auction sellers do reveal sometimes. A key open question is to decide which of the valuations to use for the bidding strategy if we do not opt to combine them based on our variance weighting (section 4.7). This aspect will require views on how the loan rates might evolve and which securities in the exclusive pool will stay special or might become special, and hence can be used to pick either a more aggressive or a less aggressive valuation. In a subsequent paper, we will look at how we can systematically try and establish expectations.
on loan rates and which securities might become harder to borrow and hence have higher profit margins on the loans. The locate conversion ratio can also be the result of profit maximization when the Knapsack algorithm is used to allocate the locates.

9 Conclusion

We have looked at a methodology to value securities portfolios from a securities lending perspective. We have then looked at various strategies that would be relevant to an exclusive auction. We derived the closed form solutions where such a formulation exists and in situations where approximations and numerical solutions would be required, we have provided those. The paper presents a theoretical foundation supplemented with empirical results for a largely unexplored financial business. The results from the simulation confirm the complexity inherent in the system, but point out that the heuristics we have used can be a practical tool for bidders and auction sellers to maximize their profits. The models developed here could be potentially useful for inventory estimation and for wholesale procurement of financial instruments and also non-financial commodities.

10 References and Notes

1. Dr. Yong Wang, Dr. Isabel Yan, Dr. Vikas Kakkar, Dr. Fred Kwan, Dr. William Case, Dr. Costel Daniel Andonie, Dr. Srikant Marakani, Dr. Guangwu Liu, Dr. Jeff Hong, Dr. Andrew Chan, Dr. Humphrey Tung and Dr. Xu Han at the City University of Hong Kong provided advice and more importantly encouragement to explore and where possible apply cross disciplinary techniques. The author has successfully utilized the heuristic expression using historical data for the valuation of exclusives. He has used similar algorithms for market making in various OTC as well as exchange traded instruments. As compared to the sample model with simulated data, when the historical observations are used, we get somewhat similar results. The views and opinions expressed in this article, along with any mistakes, are mine alone and do not necessarily reflect the official policy or position of either of my affiliations or any other agency.

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11 Appendix

11.1 Proof of Proposition 1

Proof. First, we simplify the constraints by reasoning as follows. If there is other external supply, $O_{it}$, being used, then we have

\[ I_{it} + H_{it} \leq B_{it} + \delta_{it}L_{it} \]

\[ \Rightarrow I_{it} + A_{it} \leq B_{it} + \delta_{it}L_{it} \]

\[ \Rightarrow A_{it} \leq B_{it} + \delta_{it}L_{it} - I_{it} \]

\[ \Rightarrow \text{if } B_{it} + \delta_{it}L_{it} \leq I_{it} \text{ then } A_{it} = 0 \]

The maximum possible value of $A_{it}$ is then given by

\[ A_{it} = \min [H_{it}, \max (B_{it} + \delta_{it}L_{it} - I_{it}, 0)] \]

The criteria for zero profits, gives us an expression for the maximum possible value of the exclusive.

\[ P = \max_{A_{it}} E_0 \left\{ \sum_{t=0}^{T} \beta^t \sum_{i=1}^{n} A_{it}S_{it}R_{it} - \sum_{t=0}^{T} \nu^t \left( \sum_{i=1}^{n} H_{it}S_{it} \right) \right\} \]

\[ = E_0 \left\{ \sum_{t=0}^{T} \sum_{i=1}^{n} \beta^t \min [H_{it}, \max (B_{it} + \delta_{it}L_{it} - I_{it}, 0)] S_{it}R_{it} - \sum_{t=0}^{T} \nu^t \left( \sum_{i=1}^{n} H_{it}S_{it} \right) \right\} \]
\[ \Rightarrow v^{\text{actual}} = v \leq v^{\text{zero}} = E_0 \left\{ \sum_{t=0}^{T} \sum_{n=1}^{n} \beta^t \min[H_{it}, \max(B_{it} + \delta_i L_{it} - I_{it}, 0)] S_{it} R_{it} \right\} \]

11.2 Proof of Proposition \(^2\)

**Proof.** Let us denote the following functions that capture the criteria when there would be a need to take from or give back to the exclusive.

\[
\begin{align*}
\text{High State} & \equiv Take_t \equiv \frac{\max(B_{it} + \delta_i L_{it} - I_{it}, 0)}{(B_{it} + \delta_i L_{it} - I_{it})} = \begin{cases} 1 & B_{it} + \delta_i L_{it} > I_{it} \\ 0 & \text{Otherwise} \end{cases} \\
\text{Low State} & \equiv Give_t \equiv \frac{\max(I_{it} - B_{it} - \delta_i L_{it}, 0)}{(I_{it} - B_{it} - \delta_i L_{it})} = \begin{cases} 1 & B_{it} + \delta_i L_{it} < I_{it} \\ 0 & \text{Otherwise} \end{cases}
\end{align*}
\]

It is worth noting that \(\text{Take}_t\) and \(\text{Give}_t\) are mutually exclusive. Only one of them can be one in a given time period. We consider the following four scenarios that can happen, back to back, or in successive time periods.

\[\{\text{Take}_{t-1}, \text{Take}_t\} \{\text{Take}_{t-1}, \text{Give}_t\} \{\text{Give}_{t-1}, \text{Give}_t\} \{\text{Give}_{t-1}, \text{Take}_t\}\]

Of the above scenarios, the following indicates the transaction cost incurred correspondingly. There is a cost, when a state change occurs either from Take to Give or from Give to Take.

\[\{0\} \{c\} \{0\} \{c\}\]

The above is equivalent to

\[\{\text{Take}_t, \text{Take}_{t-1}\} \{\text{Take}_t, \text{Give}_{t-1}\} \{\text{Give}_t, \text{Give}_{t-1}\} \{\text{Give}_t, \text{Take}_{t-1}\} \equiv \{0\} \{c\} \{0\} \{c\}\]

The table below summarizes the costs of the difference between variables across successive time periods, when one of the four combinations occurs.

| \(\text{Take}_t - \text{Take}_{t-1}\) | \(\text{Take}_t, \text{Take}_{t-1}\) | \(\text{Take}_t, \text{Give}_{t-1}\) | \(\text{Give}_t, \text{Give}_{t-1}\) | \(\text{Give}_t, \text{Take}_{t-1}\) |
|----------------------------------|-----------------|-----------------|-----------------|-----------------|
| \(\text{Take}_t - \text{Take}_{t-1}\) | 0 | \(c\) | 0 | \(-c\) |
| \(\text{Take}_t - \text{Give}_{t-1}\) | \(c\) | 0 | \(-c\) | 0 |
| \(\text{Give}_t - \text{Give}_{t-1}\) | 0 | \(-c\) | 0 | \(c\) |
| \(\text{Give}_t - \text{Take}_{t-1}\) | \(-c\) | 0 | \(c\) | 0 |
From this we get the expression for the transaction costs incurred, keeping in mind that in the first time period, High State or Take criterion would always incur a cost.

\[
\text{Transaction Costs } \equiv TC = E_0 \left\{ \sum_{i=1}^{n} c \{ \text{Take}_{i0} \} \right. \\
+ \left. \sum_{i=1}^{T} \sum_{i=1}^{n} c \frac{1}{2} \left\{ \text{Take}_{i} - \text{Take}_{i-1} \right\} - \{ \text{Give}_{i} - \text{Give}_{i-1} \} \right\} \\
\Rightarrow TC = E_0 \left\{ \sum_{i=1}^{n} c \left\{ \frac{\max (B_{i0} + \delta_i L_{i0} - I_{i0}, 0)}{(B_{i0} + \delta_i L_{i0} - I_{i0})} \right\} \right. \\
+ \left. \sum_{i=1}^{T} \sum_{i=1}^{n} c \left\{ \frac{\max (B_{i1} + \delta_i L_{i1} - I_{i1}, 0)}{(B_{i1} + \delta_i L_{i1} - I_{i1})} - \frac{\max (B_{i1} - \delta_i L_{i1}, 0)}{(B_{i1} - \delta_i L_{i1})} \right\} \right\} \\
\Rightarrow v_{\text{transaction}} = E_0 \left\{ \frac{\sum_{i=0}^{T} \sum_{i=1}^{n} \min [H_{i}, \max (B_{i1} + \delta_i L_{i1} - I_{i1}, 0)] S_{i1} R_{i1}}{(\sum_{i=0}^{T} \sum_{i=1}^{n} H_{i1} S_{i1})} - (TC) \right\} \\
\]

11.3 Proof of Theorem \[1\]

\[\text{Proof. Consider,} \]

\[E \left[ \lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} \frac{\sum_{j \neq i}^{k} \sigma_{j}^{2} v_{i}}{\sum_{i=1}^{k} \sigma_{i}^{2}} \right] = E \left[ \lim_{k \to \infty} \frac{1}{k} \left( \sum_{i=1}^{k} \frac{\sum_{j \neq i}^{k} \sigma_{j}^{2} v_{i}}{\sum_{i=1}^{k} \sigma_{i}^{2}} \right) \right] \]

\[= E \left[ \lim_{k \to \infty} \frac{1}{k} \left\{ \sum_{i=1}^{k} \frac{\sum_{j \neq i}^{k} \sigma_{j}^{2} v_{i} - \sum_{i=1}^{k} \sigma_{i}^{2} v_{i}}{\sum_{i=1}^{k} \sigma_{i}^{2}} \right\} \right] \]

\[= E \left[ \lim_{k \to \infty} \left\{ \sum_{i=1}^{k} \frac{v_{i} - \sum_{i=1}^{k} \sigma_{i}^{2}}{\sum_{i=1}^{k} \sigma_{i}^{2}} \right\} \right] \]

\[= E \left[ v \right] \]
\[
\begin{align*}
\forall E \left[ \lim_{k \to \infty} \frac{1}{k} \left\{ \sum_{i=1}^{k} (u_i - v) \right\} \right] = 0, \text{ Using the law of large numbers.}
\end{align*}
\]

and \( \lim_{k \to \infty} \frac{1}{k} \frac{\sum_{i=1}^{k} \sigma_i^2 v_i}{\sum_{i=1}^{k} \sigma_i^2} = 0 \), Since each of the variances and valuations are finite

and no single one dominates the sum, expressed as,

\[
\left\{ \lim_{k \to \infty} \max_{i=1}^{k} \frac{\sigma_i^2 v_i}{\sum_{i=1}^{k} \sigma_i^2} \to 0 ; \lim_{k \to \infty} \max_{i=1}^{k} \frac{\sigma_i^2}{\sum_{i=1}^{k} \sigma_i^2} \to 0 \right\}
\]

Looking at the variance of the variance weighted combination,

\[
\begin{align*}
V \left[ \lim_{k \to \infty} \frac{1}{k} \frac{\sum_{i=1}^{k} \sigma_i^2 v_i}{\sum_{i=1}^{k} \sigma_i^2} \right] &= \lim_{k \to \infty} \frac{1}{k} \frac{\sum_{i=1}^{k} \sigma_i^2}{\sum_{i=1}^{k} \sigma_i^2} \\
&= \lim_{k \to \infty} \frac{1}{k} \frac{\sum_{i=1}^{k} \left( \sum_{j=1}^{k} \sigma_j^2 \right)}{\sum_{i=1}^{k} \sigma_i^2} \\
&= \lim_{k \to \infty} \frac{1}{k} \frac{\sum_{i=1}^{k} \left( \sum_{j=1}^{k} \sigma_j^2 - \sum_{i=1}^{k} \sigma_i^4 \right)}{\sum_{i=1}^{k} \sigma_i^2} \\
&= \lim_{k \to \infty} \frac{1}{k} \frac{\sum_{i=1}^{k} \sigma_i^2 \sum_{j=1}^{k} \sigma_j^2 - \sum_{i=1}^{k} \sigma_i^4}{\sum_{i=1}^{k} \sigma_i^2} \\
&= \lim_{k \to \infty} \frac{1}{k} \left\{ \frac{\sum_{i=1}^{k} \sigma_i^2 \sum_{j=1}^{k} \sigma_j^2}{\sum_{i=1}^{k} \sigma_i^2} - \frac{\sum_{i=1}^{k} \sigma_i^4}{\sum_{i=1}^{k} \sigma_i^2} \right\} \\
&= \lim_{k \to \infty} \frac{1}{k} \left\{ \frac{\sum_{i=1}^{k} \sigma_i^2 \sum_{j=1}^{k} \sigma_j^2}{\sum_{i=1}^{k} \sigma_i^2} \right\} \text{ Set, } \sigma^2 = \frac{1}{(k) \sum_{j=1}^{k} \sigma_j^2}
\end{align*}
\]
\[ = \sigma^2 < \infty \quad : \text{The Variance and the Fourth Moment are finite.} \]

11.4 Proof of Lemma 1

Proof. The proof is from Krishna (2009).

11.5 Proof of Proposition 2

Proof. Follows immediately from the bid function in Lemma 1 with the uniform distribution functions, \( F(x) = \frac{x}{\omega}, G(x) = \left(\frac{x}{\omega}\right)^{M-1} \)

11.6 Proof of Proposition 3

Proof. Using the bid function from Lemma 1 with the log normal distribution functions, \( F(x) = \Phi\left(\frac{\ln x - \mu}{\sigma}\right), G(x) = \left(\Phi\left(\frac{\ln x - \mu}{\sigma}\right)\right)^{M-1} \). Here, \( \Phi(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-t^2/2} dt \), is the standard normal cumulative distribution and \( X = e^W \) where, \( W \sim N(\mu, \sigma) \)

\[
\beta(x) = \left[ x - \int_0^x F(y) \left(\frac{\Phi\left(\frac{\ln y - \mu}{\sigma}\right)}{\Phi\left(\frac{\ln x - \mu}{\sigma}\right)}\right)^{M-1} dy \right]
\]

No closed form solution is available. There are certain approximations, which can be used. (See Laffont, Ossard and Vuong 1995). We provide a simplification using the Taylor series expansion as shown below. This is valid only for non zero values of \( x \) (The Taylor series for this function is undefined at \( x = 0 \), but we consider the right limit to evaluate this at zero), which holds in our case since a zero valuation will mean a zero bid and the valuations are in the order of a few basis points.

\[
\beta(x) = \left[ x - \int_0^x \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-t^2/2} dt \right\} \left(\frac{\Phi\left(\frac{\ln x - \mu}{\sigma}\right)}{\Phi\left(\frac{\ln x - \mu}{\sigma}\right)}\right)^{M-1} dy \right]
\]
Let,

\[ h(y) = \left[ \int_{-\infty}^{\left(\frac{\ln y - \mu}{\sigma}\right)} e^{-t^2/2} dt \right]^{M-1} \]

\[ j(y) = \int h(y) dy \]

We then have,

\[ \beta(x) = \left[ x - \left\{ \int_0^x h(y) dy \right\} \right] \]

\[ = \left[ x - \frac{|j(x)|}{h(x)} \right] \]

\[ = \left[ x - \frac{j'(0) x}{h(x)} \right] \{ : j(x) - j(0) \simeq j'(0) x, \quad Maclaurin \ Series \} \]

\[ = \left[ x - \frac{h(0) x}{h(x)} \right] = x \left[ 1 - \frac{h(0)}{h(x)} \right] \]

\[ \Rightarrow \beta(x) = x \left[ : h(0) = \left[ \int_{-\infty}^{\left(\frac{\ln x - \mu}{\sigma}\right)} e^{-t^2/2} dt \right]^{M-1} = \left[ \int_{-\infty}^{\left(\frac{\ln 0 - \mu}{\sigma}\right)} e^{-t^2/2} dt \right]^{M-1} = 0 \]}

We could include additional terms, for greater precision, using the subsequent terms of the Maclaurin series, as follows,

\[ \beta(x) \approx x \left[ 1 - \frac{h(0)}{h(x)} - \frac{x h'(0)}{2 h(x)} - \frac{x^2 h''(0)}{6 h(x)} \right] \}

\[ \{ : j(x) - j(0) \simeq j'(0) x + \frac{j''(0) x^2}{2!} + \frac{j'''(0) x^3}{3!} \} \]

\[ \Rightarrow \beta(x) = x \left[ : h(0) = 0 \right] \]

\[ \beta(x) \approx x \left[ 1 - \frac{h(0)}{h(x)} - \frac{x h'(0)}{2 h(x)} - \frac{x^2 h''(0)}{6 h(x)} \right] \}

\[ \{ : j(x) - j(0) \simeq j'(0) x + \frac{j''(0) x^2}{2!} + \frac{j'''(0) x^3}{3!} \} \]

\[ \beta(x) \approx x \left[ 1 - \frac{h(0)}{h(x)} - \frac{x h'(0)}{2 h(x)} - \frac{x^2 h''(0)}{6 h(x)} - \frac{x^3 h'''(0)}{24 h(x)} \right] \}

\[ \{ : j(x) - j(0) \simeq j'(0) x + \frac{j''(0) x^2}{2!} + \frac{j'''(0) x^3}{3!} + \frac{j''''(0) x^4}{24!} \} \]
\[ \beta(x) \approx x \left[ 1 - \frac{h(0)}{h(x)} - \frac{x h'(0)}{2 h(x)} - \frac{1}{3} + \frac{1}{3} \frac{h(0)}{h(x)} + \frac{x h'(0)}{3 h(x)} \right] \\
= x \left[ \frac{2}{3} - \frac{2}{3} h(0) - \frac{1}{6} \left( 1 - \frac{h(0)}{h(x)} \right) \right] \\
\Rightarrow \beta(x) = \frac{x}{2} \quad [\because h(0) = 0] \\
\]

We can check the Lagrange remainders \( R^M(y) \) for a degree \( M \) approximation where \( 0 < \xi_M < y \),

\[ R^M(y) = \frac{j^{M+1}(\xi_M) y^{M+1}}{(M+1)!} = \left. \frac{\partial^{M+1} j(y)}{\partial y^{M+1}} \right|_{y=\xi_M} \left[ \frac{y^{M+1}}{(M+1)!} \right] \]

Using Fa'adi Bruno's Formula (Huang et al. 2006, Johnson 2002),

\[ \frac{\partial^M \{ p(q(y)) \}}{\partial y^M} = \sum_{k_1, k_2, \ldots, k_M} \frac{M!}{k_1! k_2! \cdots k_M!} p^{(k)}(q(y)) \left( \frac{q'(y)}{1!} \right)^{k_1} \left( \frac{q''(y)}{2!} \right)^{k_2} \cdots \left( \frac{q^{(M)}(y)}{M!} \right)^{k_M} \]

where the sum is over all non-negative integer solutions of the Diophantine equation \( k_1 + 2k_2 + \cdots + Mk_M = M \), and \( k = k_1 + k_2 + \cdots + k_M \).

\[ \Box \]

11.7 Proof of Lemma 3

Proof. The proof is from Krishna (2009).

11.8 Proof of Proposition 4

Proof. Using the bid function from Lemma 3 with the uniform distribution functions, \( F(x) = \frac{x}{\omega}, G(x) = \left( \frac{x}{\omega} \right)^{M-1} \)

\[ \beta(x) = r \left( \frac{x}{\omega} \right)^{M-1} + \frac{1}{\left( \frac{x}{\omega} \right)^{M-1}} \int_r^x y \left\{ \frac{\partial (\frac{y}{\omega})^{M-1}}{\partial y} \right\} dy \]

\[ = r \left( \frac{x}{r} \right)^{M-1} + \frac{(M-1)}{\left( \frac{x}{\omega} \right)^{M-1}} \int_r^x y \left( \frac{y}{\omega} \right)^{M-2} \frac{1}{\omega} dy \]

\[ = r \left( \frac{x}{r} \right)^{M-1} + \frac{(M-1)}{x^{M-1}} \int_r^x y^M dy \]

\[ = r \left( \frac{x}{r} \right)^{M-1} + \frac{1}{x^{M-1}} \frac{(M-1)}{M} \left| y^M \right|_r^x \]

\[ = \frac{r^M}{x^{M-1}} + \frac{x^M - r^M}{x^{M-1}} \frac{(M-1)}{M} \]

\[ = \frac{r^M}{x^{M-1}} \left( 1 - \frac{M-1}{M} \right) + x \left( \frac{M-1}{M} \right) \]

\[ \beta(x) = \frac{r^M}{x^{M-1}} \left( \frac{M-1}{M} \right) + x \left( \frac{M-1}{M} \right) \]
11.9 Proof of Proposition 5

Proof. Using the bid function from Lemma (3) with the log normal distribution functions, 
\[ F(x) = \Phi\left(\frac{lnx-u}{\sigma}\right), G(x) = \left(\Phi\left(\frac{lnx-u}{\sigma}\right)\right)^{M-1}. \]

Here, \( \Phi(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-t^2/2} dt \), is the standard normal cumulative distribution and 
\( X = e^W \) where, \( W \sim N(\mu, \sigma) \).

\[
\beta(x) = \left[ x - \int_{r}^{x} \frac{\Phi\left(\frac{lny-u}{\sigma}\right)}{\Phi\left(\frac{lnx-u}{\sigma}\right)}^{M-1} dy \right]
\]

\[
= \left[ x - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\ln x \sigma - u} e^{-t^2/2} dt \right]^{M-1}
\]

Let,

\[
h(y) = \int_{-\infty}^{(lny-u)/\sigma} e^{-t^2/2} dt
\]

\[
j(y) = \int h(y) dy
\]

\[
\beta(x) = \left[ x - \frac{\int_{r}^{x} h(y) dy}{h(x)} \right]
\]

\[
= \left[ x - \frac{1}{h(x)} \right] = \left[ x - \left\{ j(x) \right\} \right]
\]

\[
\approx \left[ x - \frac{j'(r)(x-r)}{h(x)} \right] \quad \text{::} j(x) - j(r) \simeq j'(r)(x-r) \quad \text{Taylor Series}
\]

\[
= \left[ x - \frac{h(r)(x-r)}{h(x)} \right] = x \left[ 1 - \frac{h(r)}{h(x)} + \frac{r h(r)}{x h(x)} \right]
\]

Using Leibniz Integral Rule, we get the following, which is solved using numerical techniques (Miranda and Fackler 2002) or approximations to the error function (Chiani, Dardari and Simon 2003).

\[
h'(r) = (M-1) \left[ \int_{-\infty}^{(lnx-u)/\sigma} e^{-t^2/2} dt \right]^{M-2} \left( e^{-\left(\frac{(lnx-u)^2}{2\sigma^2}\right)} \right)
\]
11.10 Proof of Lemma 4

Proof. The proof is from Krishna (2009).

11.11 Proof of Lemma 5

Proof. The proof is from (Ortega-Reichert 1967; and Harstad, Kagel and Levin 1990) who derive the expression below when there is uncertainty about the number of bidders. (Levin and Ozdenoren 2004; and Dyer, Kagel and Levin 1989) are other useful references.

11.12 Proof of Lemma 6

Proof. (Lebrun 1999), derives conditions for the existence of an asymmetric equilibrium with more than two bidders. Using the notation described earlier, we must have $\beta_i (\omega_i) = \beta_j (\omega_j) = \bar{b}$, say. $\forall i, j \in [1, 2, ..., M]$. We also have, $x_i = \beta_i^{-1} (b_i) = \phi_i (b_i)$. The expected pay off for any bidder $i$ when his value is $x_i$ and he bids an amount $\beta_i (x_i) = b < \bar{b}$ is

$$\Pi_i (b, x_i) = \left[ \prod_{j=1}^{M} F_j (\phi_j (b)) \right] (x_i - b)$$

Consider, $\Pr(b > b_j) = \Pr(\beta_i (x_i) > \beta_j (x_j))$

$$= \Pr(b > \beta_j (x_j)) \equiv \Pr(\beta_j^{-1} (b) > x_j)$$

$$= \Pr(x_j < \phi_j (b)) \equiv F_j (\phi_j (b))$$

Differentiating the above with respect to $b$, gives the first order conditions for bidder $i$ to maximize his expected payoff as,

$$\sum_{j \in \{1, ..., M\}} \left[ \prod_{j \neq i} \prod_{k \neq i,j} F_k (\phi_k (b)) \right] j \in \{1, ..., M\} \sum_{j \neq i} \left\{ \prod_{\substack{k \neq i, j \neq i \neq j}} F_k (\phi_k (b)) \right\}$$

$$\frac{\partial}{\partial b} \left\{ \prod_{j \neq i} F_j (\phi_j (b)) \right\} (x_i - b) = 0$$

$$\Rightarrow \sum_{j \neq i} \left\{ \prod_{j \neq i} F_j (\phi_j (b)) \right\} = \frac{1}{\phi_i (b) - \bar{b}}$$
11.13 Proof of Proposition [7]

Proof. The expected pay off for any bidder \( i \) when his value is \( x_i \) and he bids an amount \( \beta_i(x_i) = b < \bar{b} \) is

\[
\Pi_i(b, x_i) = [F_1(\phi_1(b))]^K[F_2(\phi_2(b))]^{M-1-K}(x_i - b)
\]

By considering one bidder from each group of bidders (other combinations would work as well) and taking first order conditions, gives a simpler system of differential equations,

\[
\frac{\partial}{\partial b} \left\{ [F_1(\phi_1(b))]^K[F_2(\phi_2(b))]^{M-K-1}(x_i - b) \right\} = 0
\]

\[
\left\{ K[F_1(\phi_1(b))]^{K-1} f_1(\phi_1(b)) \phi_1'(b) [F_2(\phi_2(b))]^{M-K-1}[\phi_i(b) - b] \right\}
+ \left\{ (M-K-1)[F_2(\phi_2(b))]^{M-K-2} f_2(\phi_2(b)) \phi_2'(b) [F_1(\phi_1(b))]^K[\phi_i(b) - b] \right\}
\]

\[
= \frac{[F_1(\phi_1(b))]^K[F_2(\phi_2(b))]^{M-K-1}}{[F_1(\phi_1(b))]}
\]

\[
\left\{ K \frac{f_1(\phi_1(b)) \phi_1'(b)}{[F_1(\phi_1(b))]} \right\} + \left\{ (M-1-K) \frac{f_2(\phi_2(b)) \phi_2'(b)}{[F_2(\phi_2(b))]} \right\} = \frac{1}{[\phi_i(b) - b]}
\]

11.14 Proof of Lemma [7]

Proof. The proof is from Krishna (2009).

11.15 Proof of Proposition [8]

Proof. We show below the bidder’s valuation, the density, cumulative distribution functions and the conditional distribution of the order statistics,

\[
X_i = S_i + Z
\]

\[
f_{X_i}(x_i) = \begin{cases} x_i & 0 \leq x_i < 1 \\ 2 - x_i & 1 \leq x_i \leq 2 \end{cases}
\]

\[
F_{X_i}(x_i) = \begin{cases} \frac{x^2}{2} & 0 \leq x_i < 1 \\ 2x_i - 1 - \frac{x^2}{2} & 1 \leq x_i \leq 2 \end{cases}
\]

\[
g(y|x) = (M - 1) \left[ \frac{F(y)}{F(x)} \right]^{M-2} \left( \frac{f(y)}{F(x)} \right)
\]
\[
\begin{align*}
\therefore f_{Y_j}(y_j | Y_j = y_j) &= \frac{f_{Y_i,Y_j}(y_i,y_j)}{f_{Y_j}(y_j)} \\

&= \frac{(j-1)!}{(i-1)!(j-i-1)!} \left\{ \frac{F(y_i)}{F(y_j)} \right\}^{i-1} \left[ F(y_j) - F(y_i) \right]^{j-i-1} \left( \frac{f(y_i)}{F(y_j)} \right) \\

&= \left\{ (M-1) \left( \frac{y}{z} \right)^{2(M-2)} \left( \frac{2y}{z} \right) \right\}^{M-2} \left\{ \frac{2-y}{(2x-1-z^2)} \right\} 1 \leq y, x \leq 2 \\

\therefore G_{Y_j}(y_i | Y_j = y_j) &\approx \int_{-\infty}^{y_i} f_{Y_i,Y_j}(u,y_j) du \\
&= \frac{(j-1)!}{(i-1)!(j-i-1)!} \left[ (1 - F(y_j))^{M-j} f(y_j) \right] \left[ (F(y_j))^{j-i-1} (1 - F(y_j))^{M-j} f(y_j) \right] \left[ \int_{-\infty}^{y_i} (F(u))^{i-1} (F(y_j) - F(u))^{j-i-1} f(u) du \right] \\

G(y | x) &= \left\{ (M-1) \left[ \frac{f_y(x) M-2 u x du}{z^2} \right] \right\}^{M-2} \left\{ \frac{2y-1-z^2}{(2x-1-z^2)} \right\}^{M-1} \left( 2u-x \right) 0 \leq y, x < 1 \\

&= \left\{ (M-1) \left[ \frac{f_y(x) M-2 u x du}{z^2} \right] \right\}^{M-2} \left\{ \frac{2y-1-z^2}{(2x-1-z^2)} \right\}^{M-1} \left( 2u-x \right) 1 \leq y, x \leq 2 \\

G(y | x) &= \left\{ (M-1) \left[ \frac{2y-1-z^2}{(2x-1-z^2)} \right] \right\}^{M-1} 0 \leq y, x < 1 \\

&= \left\{ (M-1) \left[ \frac{2y-1-z^2}{(2x-1-z^2)} \right] \right\}^{M-1} 1 \leq y, x \leq 2 \\

\therefore H(x) &= \int_a^x h(t) dt ; H'(x) = h(x) \\

&= \left\{ (M-1) \left( \frac{2}{y} \right) \right\} 0 \leq y < 1 \\

&= \left\{ (M-1) \left( \frac{2-y}{(2y-1-z^2)} \right) \right\} 1 \leq y \leq 2 \\

G(y | y) &= 1
\end{align*}
\]

We then have

\[
\begin{align*}
g(y | y) &= \left\{ (M-1) \left( \frac{2}{y} \right) \right\} 0 \leq y < 1 \\
&= \left\{ (M-1) \left( \frac{2-y}{(2y-1-z^2)} \right) \right\} 1 \leq y \leq 2 \\
G(y | y) &= 1
\end{align*}
\]
Using this in the bid function,

\[
\beta(x) = \int_0^x v(y, y) L(y | x) \frac{g(y | y)}{G(y | y)} dy \\
= \int_0^1 v(y, y) L(y | x) \frac{g(y | y)}{G(y | y)} dy + \int_1^x v(y, y) L(y | x) \frac{g(y | y)}{G(y | y)} dy
\]

\[
= \int_0^1 (\alpha + \xi) y (M - 1) \left( \frac{2}{y} \right) \left[ \frac{y^{2M-2}}{x^{2M-2}} \right] dy \\
+ \int_1^x (\alpha + \xi) y (M - 1) \left\{ \frac{2 - y}{(2y - 1 - \frac{y^2}{2})} \right\} \left[ \frac{(2y-1-\frac{y^2}{2})^{M-1}}{(2x-1-\frac{x^2}{2})^{M-1}} \right] dy
\]

\[
= 2(\alpha + \xi) (M - 1) \int_0^1 \left[ \frac{y^{2M-2}}{x^{2M-2}} \right] dy \\
+ (\alpha + \xi) (M - 1) \int_1^x y (2 - y) \left[ \frac{(2y-1-\frac{y^2}{2})^{M-2}}{(2x-1-\frac{x^2}{2})^{M-2}} \right] dy
\]

\[
= \left[ \frac{2(\alpha + \xi) (M - 1)}{(2M - 1) x^{2M-2}} \right] \\
+ \frac{(\alpha + \xi) (M - 1)}{(2x-1-\frac{x^2}{2})^{M-1}} \left[ \frac{y (2y-1-\frac{y^2}{2})^{M-1}}{(M - 1)} \right]_1^x - \int_1^x \left( \frac{2y-1-\frac{y^2}{2})^{M-1}}{(M - 1)} \right) dy
\]

\[
= \left[ \frac{2(\alpha + \xi) (M - 1)}{(2M - 1) x^{2M-2}} \right] \\
+ (\alpha + \xi) \left[ x - \frac{1}{(2x-1-\frac{x^2}{2})^{M-1}} \left\{ \frac{1}{2M-1} + \int_1^x \left( 2y-1-\frac{y^2}{2} \right)^{M-1} dy \right\} \right]
\]
The last integral is solved using the reduction formula,

\[ 8a(n + 1)I_{n + 1} = 2(2ax + b)(ax^2 + bx + c)^{n + \frac{1}{2}} + (2n + 1)(4ac - b^2)I_{n - \frac{1}{2}} \]

Leibniz Integral Rule: Let \( f(x, \theta) \) be a function such that \( f_\theta(x, \theta) \) exists, and is continuous. Then,

\[ \frac{d}{d\theta} \left( \int_{a(\theta)}^{b(\theta)} f(x, \theta) \, dx \right) = \int_{a(\theta)}^{b(\theta)} \partial_\theta f(x, \theta) \, dx + f(b(\theta), \theta) \cdot b'(\theta) - f(a(\theta), \theta) \cdot a'(\theta) \]

where the partial derivative of \( f \) indicates that inside the integral only the variation of \( f(x, \cdot) \) with \( \theta \) is considered in taking the derivative.

11.16 Proof of Proposition 9

Proof. We extend the proof from (Milgrom & Weber 1982) who derive the condition for the interdependent case with symmetric valuations. We consider a realistic setting with symmetric interdependent, uniformly distributed valuations, with reserve prices and variable number of bidders. This is obtained by altering the boundary conditions on the differential equation,

\[ \beta'(x) = \left[ v(x, x) - \beta(x) \right] \frac{g(x \mid x)}{G(x \mid x)} \]

This can be written as,

\[ \beta'(x) + \beta(x) \frac{g(x \mid x)}{G(x \mid x)} = v(x, x) \frac{g(x \mid x)}{G(x \mid x)} \]

\[ \Leftrightarrow \frac{dz}{dx} + zP(x) = Q(x) \]

Here, \( z = \beta(x) \), \( dz/dx = \beta'(x) \), \( P(x) = g(x \mid x)/G(x \mid x) \) and \( Q(x) = v(x, x)g(x \mid x)/G(x \mid x) \).

We then have the solution with the appropriate boundary condition, \( \beta(x^*) = r \) and \( x^* = x^*(r) = \inf \{ x \mid E[V_1 \mid X_1 = x, Y_1 < x] \geq r \} \) as

\[ \frac{dz}{dx} e^{\int_0^t P(t) \, dt} + zP(x) e^{\int_0^t P(t) \, dt} = Q(x) e^{\int_0^t P(t) \, dt} \]

\[ d \left\{ \frac{z e^{\int_0^t P(t) \, dt}}{dx} \right\} = Q(x) e^{\int_0^t P(t) \, dt} \]

\[ z e^{\int_0^t P(t) \, dt} = z e^{\int_0^t P(t) \, dt} \bigg|_{x = x^*} + \int_{x^*}^{x} Q(y) e^{\int_0^t P(t) \, dt} \, dy \]
\[
z = r e^{\int_{x}^{\infty} P(t) dt} + \int_{x}^{\infty} Q(y) e^{\int_{y}^{\infty} P(t) dt} dy
\]

\[
z = r e^{- \int_{x}^{\infty} P(t) dt} + \int_{x}^{\infty} Q(y) e^{- \int_{y}^{\infty} P(t) dt} dy
\]

\[
\beta(x) = r e^{- \int_{x}^{\infty} P(t) dt} + \int_{x}^{\infty} v(y, y) \frac{g(y | y)}{G(y | y)} e^{- \int_{y}^{\infty} P(t) dt} dy
\]

We find out \( x^* (r) \) by solving for \( x \) in the below condition,

\[
E[V_1 | X_1 = x, Y_1 < x] = r
\]

\[
\int_{-\infty}^{\infty} (\alpha x + \xi y) f_{Y_1,X_1} (y, x) = r
\]

\[
\alpha x + \int_{-\infty}^{x} \xi y (M - 1) \left\{ \frac{F(y)}{F(x)} \right\}^{M-2} \frac{f(y)}{F(x)} dy = r
\]

\[
\left[ \int_{0}^{1} \xi y \left[ \frac{y}{x} \right]^{2(M-2)} \left( \frac{2y}{x^2} \right) dy + \int_{1}^{x} \xi y \left[ \frac{(2y - 1 - \frac{y^2}{x})}{(2x - 1 - \frac{x^2}{x})} \right]^{M-2} \left\{ \frac{2 - y}{(2x - 1 - \frac{x^2}{x})} \right\} dy \right] = \frac{r - \alpha x}{(M - 1)}
\]

In the previous section, we have shown a method to solve the above type of equations.