A group-theoretical derivation of the S-matrix for the Pöschl-Teller potentials

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Abstract. A derivation of the S-matrix for the Pöschl-Teller and related potentials is done by group-theoretical methods using the group SU(1, 1), the dynamical symmetry for these systems. Light is shed on the representation of SU(1, 1) which must be used to this end, and the nature of bound (and anti-bound or virtual) states viewed as poles of the S-matrix in the complex plane.

1. Introduction
In this work we shall study the scattering states of the modified Pöschl-Teller potential:

\[ V(x) = -\frac{D}{\cosh^2(\alpha x)}. \]  

(1)

For that purpose, we shall determine the classical symmetries of the system, derive the Schrödinger equation from the Casimir of the symmetry group, for both bound and scattering states, find its solutions and derive the S matrix for scattering states. Finally, we shall determine the poles of the S matrix and interpret the results in a group-theoretical setting.

To find the classical symmetries of the system, we follow the procedure designed in [1] for the modified Pöschl-Teller potential and further developed in [2] for the Trigonometric Pöschl-Teller and Morse potentials\(^1\) where a transformation is performed relating an open subset of the phase space of the system (that with a definite sign of the energy, either positive or negative) with (an open subset of) coadjoint orbits of the Lie groups SO(2, 1) or SO(3). This transformation is not symplectic and in some cases it fails to be differentiable at some points (\(H = 0\)), although when restricted to the open subsets of positive or negative energy it is differentiable. This has important implications at the quantum level, where the system is realized as certain representations of these groups.

Although we are interested in the scattering states of the Modified Pöschl-Teller potential (MPT), for completeness we shall also discuss the Trigonometric Pöschl-Teller potential (TPT), having only bound states, and the bound states of the MPT, in order to compare with the results

\(^1\) See also [3] for a similar construction, viewed as a classical factorization, applied systematically to a larger family of potentials. However, the symmetry algebras SO(2, 1) and SO(3) are only obtained for the same cases as those considered in [1, 2].

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obtained for the poles of the $S$ matrix, which are related with bound and anti-bound or virtual states, as well as with resonant states (although in the present case there are no resonances).

It should be stressed that the symmetries we are considering are dynamical symmetries, in the sense of spectrum generation, as opposed to degeneracy symmetry which considers states with the same energy. Dynamical symmetry groups are also distinct from potential groups, were the action of the group keeps the energy but changes the potential parameters, therefore mixing states belonging to different physical systems. Despite of this lack of physical interpretation, the potential group approach has been very successful in finding the analytical solutions for many solvable potentials [4, 5, 6].

The paper is organized as follows. Sec. 2 introduces the main categories of Pöschl-Teller potentials, Sec. 3 is devoted to the Trigonometric Pöschl-Teller potential, finding its classical and quantum symmetries, Sec. 4 is focused on the Modified Pöschl-Teller potential, studying its classical symmetries, Sec. 5 is devoted to the bound states of Modified Pöschl-Teller potential, Sec. 6 to the scattering states, paying attention to the properties of the $S$ matrix. Finally, Sec. 7 comments on some possible extensions of the work.

2. The family of Pöschl-Teller potentials

The Pöschl-Teller potentials are a family of potentials which can be classified in two types:

- Trigonometric Pöschl-Teller potentials (TPT):

  $$V(x) = D \left( \frac{\lambda}{\cos^2(\alpha x)} + \frac{\kappa}{\sin^2(\alpha x)} \right)$$

- Hyperbolic (Modified) Pöschl-Teller potentials (MPT)

  $$V(x) = -D \left( \frac{\lambda}{\cosh^2(\alpha x)} + \frac{\kappa}{\sinh^2(\alpha x)} \right) .$$

They have many applications in different branches of Molecular Physics, where they describe out-of-plane bending vibrations, with some variations like the Rosen-Morse Potential [7], and in Solid State Physics, where they model 1-D crystals (Scarf Potential) [8]. The PT potentials also appear in generalizations of the harmonic oscillator potential to spaces of constant curvature [9, 10, 11]. The PT potentials have the same solutions as an harmonic oscillator with position dependent mass, or with energy dependent frequency [12, 1]. They are integrable and related to a $SO(2,1)$ (or $SO(3)$) dynamical symmetry. In fact, they can be derived in a unified way, together with the Morse potential, from free motion in homogeneous spaces under the $SU(2)$ and $SU(1,1)$ Lie groups [13, 14], assuring in this way their integrability.

They can be solved by different approaches: dynamical group [1], potential group [4, 5, 6], SUSY QM and shape invariance [15], etc.

Let us study the phase space associated with these systems, in particular the cases of the symmetric TPT and MPT potentials.

3. Trigonometric Pöschl-Teller potential

We shall concentrate on the Trigonometric PT of the form:

$$V(x) = \frac{D}{\cos^2(\alpha x)} ,$$

which is symmetric and has the form given in figure 1a. The parameter $D > 0$ is the minimum of the potential (the value at the origin), and $\alpha$ is related to the width of the potential.
3.1. Phase space

The trajectories in phase space for the TPT potential can be derived from the equation of motion, either in Lagrangian or Hamiltonian form, and are given by:

\[
x(t) = \frac{1}{\alpha} \arcsin \left[ \sin(\alpha x_0) \cos(\omega(E)t) + \frac{\alpha}{m\omega(E)} p_0 \cos(\alpha x_0) \sin(\omega(E)t) \right]
\]

\[
p(t) = m\dot{x}(t),
\]

where \((x_0, p_0)\) are the initial coordinate and momentum parameterizing each solution and \(E = \frac{p_0^2}{2m} + \frac{D}{\cos(\alpha x_0)^2}\) is the energy of this trajectory. In the last expression \(\omega(E) \equiv \sqrt{\frac{2a^2 E}{m}}\) is an energy dependent frequency\(^2\).

These trajectories are shown in figure 1b. If we represent these trajectories in a 3D-phase space \((x, p, H)\), we obtain the graphic shown in figure 1c. Note that in this case \(E \geq D > 0\).

![Figure 1. Symmetric PT potential (a) and trajectories in 2D (b) and 3D phase space (c).](image)

3.2. Closing a subalgebra of the Poisson algebra

To obtain the dynamical group of the system, we are now interested in closing a Lie algebra with functions of the energy, the momentum and the position. We start with the basic Poisson bracket, derived from the Poincaré-Cartan 1-form taking the quotient by the solutions of the equation of motion (5), which has the canonical form:

\[
\{ x, p \} = 1,
\]

with \(p = \frac{\partial L}{\partial \dot{x}}\). Although \(\{ H, x, p \} \) do not close a Poisson subalgebra, we can find the functions closing the algebra \(SO(2,1)\):

\[
\mathcal{E} \equiv 2\sqrt{D} \sqrt{H}, \quad \mathcal{X} \equiv \frac{2}{\sqrt{m\Omega}} \sqrt{H} \sin(\alpha x), \quad \mathcal{P} \equiv \sqrt{2p} \cos(\alpha x),
\]

with \(\Omega = \omega(D)\), i.e. the frequency at the bottom of the potential, corresponding to the limit of small oscillations.

In fact, we have:

\[
\{ \mathcal{E}, \mathcal{P} \} = m\Omega^2 \mathcal{X}, \quad \{ \mathcal{E}, \mathcal{X} \} = -\frac{1}{m} \mathcal{P}, \quad \{ \mathcal{X}, \mathcal{P} \} = \frac{1}{D} \mathcal{E}.
\]

\(^2\) The solutions of the classical equation of motion correspond to that of an harmonic oscillator with energy dependent frequency (see [1] and also [16]).
This is an SO(2, 1) algebra where the compact generator is \( E \) and the Casimir is:

\[
\frac{1}{D} E^2 - m \Omega^2 x^2 - \frac{1}{m} P^2 = 4D > 0. \tag{9}
\]

The coadjoint orbit of SO(2, 1) associated with this system is the upper sheet of a two-sheet hyperboloid, since the Casimir is positive and \( E > 0 \).

3.3. The quantum TPT oscillator

The time-independent Schrödinger equation for a particle of mass \( m \) in a TPT potential with depth \( D \) and width \( 1/\alpha \) is:

\[
\frac{\hbar^2}{2m} \frac{d^2 \Psi}{dx^2} + \left( E - \frac{D}{\cos^2(\alpha x)} \right) \Psi = 0. \tag{10}
\]

The solutions to this equation are given in terms of Gegenbauer polynomials:

\[
\Psi_n^k(x) = (\cos(\alpha x))^k C_n^k(\sin(\alpha x)), \tag{11}
\]

where \( k(k-1) = \frac{2mD}{\hbar^2 \alpha^2} \).

The energy eigenvalues are given by:

\[
E_n = \frac{\hbar^2 \alpha^2}{2m} (n + k)^2 = -\frac{\hbar^2 \Omega^2}{4D} (n + k)^2 = -\frac{D}{k(k-1)} (n + k)^2, \quad n = 0, 1, \ldots \tag{12}
\]

The scalar product is the usual one in \( L^2(\mathbb{R}, dx) \), and with this scalar product \( \Psi_n^k, \), \( n = 0, \ldots, \) are orthogonal and normalizable, the orthonormal basis being:

\[
\Psi_n^k(x) = N_n^k \Psi_n^k(x),
\]

\[
N_n^k = \sqrt{\frac{\alpha n!(n + k) \Gamma(k)\Gamma(2k)}{\pi^{1/2} \Gamma(k + 1/2) \Gamma(n + 2k)}} \tag{13}
\]

for \( n = 0, 1, \ldots \).

3.3.1. The dynamical group for the TPT. It is possible to find ladder operators for the TPT wavefunctions:

\[
\hat{Z} = \frac{1}{\sqrt{2k}} \sqrt{\frac{n + k - 1}{n + k}} \left[ \frac{1}{\alpha} \cos(\alpha x) \frac{d}{dx} + (n + k) \sin(\alpha x) \right],
\]

\[
\hat{Z}^\dagger = \frac{1}{\sqrt{2k}} \sqrt{\frac{n + k + 1}{n + k}} \left[ -\frac{1}{\alpha} \cos(\alpha x) \frac{d}{dx} + (n + k) \sin(\alpha x) \right], \tag{14}
\]

which, on normalized solutions, act as:

\[
\hat{Z} \Psi_n^k = \sqrt{n \frac{2k + n - 1}{2k}} \Psi_{n-1}^k,
\]

\[
\hat{Z}^\dagger \Psi_n^k = \sqrt{(n + 1) \frac{2k + n}{2k}} \Psi_{n+1}^k. \tag{15}
\]

These operators, together with the number of quanta operator, \( \hat{n} \), close the Lie algebra of \( SU(1, 1) \). Therefore, we corroborate that \( SU(1, 1) \) is the dynamical algebra for the TPT potential ([17]).
3.3.2. Deriving the Schrödinger equation for the TPT. It should be stressed that the transformation (7) is not symplectic since \(\{X, P\} = \frac{1}{D} E \neq 1\). This is not a drawback, however, and the dynamics (both classical and quantum) can be derived entirely from \(SO(2, 1)\). In this case the transformation (7) is differentiable, and the phase space shown in figure 1c is mapped to a full coadjoint orbit of \(SO(2, 1)\). This implies that at the quantum level this system will be realized as a unitary irreducible representation of the discrete series of representations of \(SO(2, 1)\) (see, for instance, [17]), the one associated with the upper sheet of a two-sheet hyperboloid in the sense of the Coadjoint Orbit Method of Kirillov [18] (see also [19]).

The time independent Schrödinger equation for the TPT potential can be derived from the Casimir of \(SO(2, 1)\) in a representation of the Discrete positive series \(D_k^+\), realized as a free particle in Anti-de Sitter space-time (which can be seen as a relativistic oscillator):

\[
\hat{C} \varphi \equiv -\frac{c^2}{\omega^2} \Box \varphi = k(k-1)\varphi,
\]

where \(k = \frac{mc^2}{\hbar\omega}\) and \(\Box\) is the D’Alambertian in an Anti-de Sitter space-time:

\[
\Box \equiv \frac{1}{c^2(1 + \omega y^2/c^2)} \frac{\partial^2}{\partial \tau^2} - 2\frac{\omega^2 y}{c^2} \frac{\partial}{\partial y} - (1 + \omega y^2/c^2) \frac{\partial^2}{\partial y^2}.
\]

Defining \(x \equiv \frac{1}{\alpha} \text{arctanh}(\sqrt{\frac{D\omega}{m\omega}} y)\), we arrive at

\[
-\frac{\hbar^2 \alpha^2}{2m} \frac{1}{\omega^2} \frac{\partial^2}{\partial \tau^2} \varphi = \left[ \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \varphi - \frac{D}{\cos^2(\alpha x)} \varphi \right],
\]

where \(k(k-1) = \frac{2mD}{\alpha^4 c^2} = (\frac{2D}{\alpha^2 m})^2\).

Denoting \(t \equiv \frac{\tau}{\alpha}, E \equiv i\hbar \frac{\partial}{\partial \tau}, \) and \(\chi \equiv e^{-\frac{i}{\hbar} \sqrt{D} \tau} \varphi\) we recover the time-independent Schrödinger equation for a particle of mass \(m\) in a MPT potential with depth \(D\) and width \(1/\alpha\):

\[
\frac{\hbar^2}{2m} \frac{d^2 \chi}{dx^2} + \left( E - \frac{D}{\cos^2(\alpha x)} \right) \chi = 0.
\]

3.3.3. Deriving the ladder operators for the TPT. The ladder operators can also be derived from ladder operators for the discrete series representation \(D_k^+\) of \(SU(1,1)\). In the same parametrization of the free particle in Anti-de Sitter spacetime, they are given by (see [1]):

\[
\hat{Z} = \sqrt{\frac{\hbar}{2m\omega}} e^{i\omega t} \sqrt{1 + \frac{\omega^2 y^2}{c^2}} \left[ \frac{\partial}{\partial y} + i \frac{\omega y}{c^2(1 + \omega^2 y^2/c^2)} \frac{\partial}{\partial \tau} \right],
\]

\[
\hat{Z}^\dagger = \sqrt{\frac{\hbar}{2m\omega}} e^{-i\omega t} \sqrt{1 + \frac{\omega^2 y^2}{c^2}} \left[ -\frac{\partial}{\partial y} + i \frac{\omega y}{c^2(1 + \omega^2 y^2/c^2)} \frac{\partial}{\partial \tau} \right].
\]

Performing the same change of variables as in the derivation of the Schrödinger equation, the resulting ladder operators acting on energy eigenfunctions are the ones given in Eq. (14), except for the global factor depending on the state \(n\). This factor must be added to restore the unitarity of the representation when the time variable is factored out (see [1] and references therein for a detailed discussion).
4. The modified Pöschl-Teller potential

We shall concentrate on the MPT of the form:

\[ V(x) = -\frac{D}{\cosh^2(\alpha x)} , \]  

which is also symmetric and has the form given in figure 2a. The parameter \( D > 0 \) is the potential depth and \( \alpha \) is related to the width of the potential.

![Figure 2. Symmetric MPT potential (a) and trajectories in 2D phase space for \( E > 0 \) (b) and \( E < 0 \) (c).](image)

4.1. Phase space

The trajectories in phase space for the MPT potential are given by:

\[
\begin{align*}
x(t) &= \frac{1}{\alpha} \arcsinh \left[ \sinh(\alpha x_0) \cosh(\omega(E)t) + \frac{\alpha}{m\omega(E)} p_0 \cosh(\alpha x_0) \sinh(\omega(E)t) \right] \\
p(t) &= m\dot{x}(t),
\end{align*}
\]

where \((x_0, p_0)\) are the initial coordinate and momentum parameterizing each solution and \( E = \frac{p_0^2}{2m} - \frac{D}{\cosh^2(\alpha x_0)} \) is the energy of this trajectory. Here \( \omega(E) \) has the same meaning as in the TPT case. Note that when \( E < 0 \) the time-dependent hyperbolic functions change to their trigonometric counterparts. These trajectories are shown in figure 2b for \( E > 0 \) and figure 2c for \( E < 0 \). For \( E = 0 \) they degenerate to \( x(t) = \frac{1}{\alpha} \arcsinh [\sinh(\alpha x_0) + \cosh(\alpha x_0) \frac{2p_0}{m} t] \)

which is a free motion in the coordinate \( \xi = \sinh(\alpha x) \), \( \dot{\xi} = \xi_0 + \dot{\xi}_0 t \) with constant velocity \( \dot{\xi}_0 = \frac{\alpha}{m} p_0 \sqrt{1 + \xi_0^2} \), and \( p_0 \) related to \( x_0 \) through the constraint \( E = 0 \).

The trajectories in a 3D-phase space \((x, p, H)\) are show in figure 3a for \( E > 0 \), figure 3b for \( E < 0 \) and figure 3c for the union of both cases.

4.2. Closing a subalgebra of the Poisson algebra

The basic Poisson bracket, derived as in the case of the TPT, has the canonical form:

\[ \{x, p\} = 1. \]  

Although \( \{H, x, p\} \) do not close a Poisson subalgebra, we can find functions closing an algebra, but we must distinguish between the cases \( E > 0 \) and \( E < 0 \).

- \( E > 0 \)

The following functions close a \( SO(2,1) \) algebra:

\[
E \equiv 2\sqrt{D} \sqrt{H} , \quad \mathcal{X} \equiv \frac{2}{\sqrt{m\Omega}} \sqrt{H} \sinh(\alpha x) , \quad \mathcal{P} \equiv \sqrt{2p} \cosh(\alpha x). \quad (24)
\]
Figure 3. Trajectories in 3D phase space for $E > 0$ (a), $E < 0$ (b) and the union of both cases (c), for the MPT Potential

In fact, we find:

$$\{E, \mathcal{P}\} = -m \Omega^2 \mathcal{X}, \quad \{E, \mathcal{X}\} = -\frac{1}{m} \mathcal{P}, \quad \{\mathcal{X}, \mathcal{P}\} = \frac{1}{D} E,$$

(25)

where again $\Omega = \omega(D)$. This is a $SO(2,1)$ algebra where the compact generator is $\mathcal{P}$ and the Casimir is:

$$\frac{1}{m} \mathcal{P}^2 - \frac{1}{D} E^2 - m \Omega^2 \mathcal{X}^2 = 4D > 0.$$

(26)

The coadjoint orbit of $SO(2,1)$ associated with this system is a half of a two-sheet hyperboloid, shown in figure 4a.

Figure 4. Coadjoint orbits of the group $SO(2,1)$ for $E > 0$ (a), $SO(3)$ for $E < 0$ (b) and the union of both cases (c).

• $E < 0$

The following functions close a $SO(3)$ algebra:

$$E \equiv -2\sqrt{D}\sqrt{-H}, \quad \mathcal{X} \equiv \frac{2}{\sqrt{m\Omega}} \sqrt{-H} \sinh(\alpha x), \quad \mathcal{P} \equiv \sqrt{2p} \cosh(\alpha x).$$

(27)

In fact, we find:

$$\{E, \mathcal{P}\} = m \Omega^2 \mathcal{X}, \quad \{E, \mathcal{X}\} = -\frac{1}{m} \mathcal{P}, \quad \{\mathcal{X}, \mathcal{P}\} = \frac{1}{D} E.$$

(28)

Note the minus sign in $E$ in order to have the same sign as $H$. 

\footnote{Note the minus sign in $E$ in order to have the same sign as $H.$}
This is an $SO(3)$ algebra with the Casimir function given by:

$$\frac{1}{D}c^2 + m\Omega^2 x^2 + \frac{1}{m}p^2 = 4D > 0.$$  \hspace{1cm} (29)

The coadjoint orbit of $SO(3)$ associated with this system is a hemisphere, shown in figure 4b.

Putting the two orbits together we obtain a phase space, shown in figure 4c, resembling that of figure 3c.

As in the TPT case, the transformations (24) and (27) are not symplectic. In this case they are not even differentiable at $H = 0$. As shown in figure 4c, the whole $E = 0$ closed subset of phase space in $(x, p, H)$, which in turn is formed by two disconnected curves, is mapped to two points in $(X, P, E)$, namely $(0, \pm 2\sqrt{mD}, 0)$. This means that the two cases, $E > 0$ and $E < 0$ are disconnected, and there is no way to connect them in the framework of the groups $SO(2, 1)$ and $SO(3)$ (see the last section for a way to connect them).

4.3. The quantum MPT oscillator

At the quantum level, this system has special features, since we must distinguish between the cases $E > 0$ and $E < 0$, and in neither cases the phase space is a coadjoint orbit, but a half of it.

- $E > 0$

  The system will be realized as part of the sum of two unitary and irreducible representations of the discrete series (positive and negative) of (the universal covering group of) $SO(2, 1)$. The Hamiltonian is a non-compact operator with positive continuum spectrum. This result seems to be in agreement with [20], where the discrete series of $SO(2, 1)$ is used to describe the continuum spectrum of the MPT potential.

- $E < 0$

  The system will be realized as part of an irreducible representation of $SU(2)$. The Hamiltonian is a compact operator with discrete spectrum, but only one half of the $2s + 1$ states of the $SU(2)$ representations are realized.

In the last case, there is a better interpretation as a non-unitary, finite-dimensional representation of (the universal covering group of) $SO(2, 1)$ (see [1]). The reason is that in the context of $SU(2)$ there is no explanation for the fact that half of the states are missing, and that the potential depth should have only definite values, since it is related to the values of the Casimir, which in turns depends on the discrete values of the spin.

Here $SO(3)$ appears as contained in the complexification of $SO(2, 1)$, but then it should be realized in a non-unitary way. But $SO(2, 1)$ admits non-unitary, finite dimensional representations, which in many respects behave as those of $SU(2)$. The lack of unitarity manifest itself in that half of the states are non-normalizable, and therefore are outside of the Hilbert space of the physical (normalizable) states. These non-normalizable states are precisely the anti-bound states (ABS), which represent outgoing states growing at $\pm \infty$ (see [21]).

We shall study separately the bound states and the scattering states.

5. Bound states for the MPT potential

The time-independent Schrödinger equation for a particle of mass $m$ in a MPT potential with depth $D$ and width $1/\alpha$ is:

$$\frac{\hbar^2}{2m} \frac{d^2 \Psi}{dx^2} + \left( E + \frac{D}{\cosh^2(\alpha x)} \right) \Psi = 0.$$  \hspace{1cm} (30)
The bounded solutions to this equation are given in terms of Gegenbauer polynomials:

$$\Psi^q_n(x) = (\cosh(\alpha x))^{n-q} C^{q - n + \frac{1}{2}}_n(\tanh(\alpha x)), \quad (31)$$

where \( q(q+1) = \frac{2mD}{\hbar^2 \alpha^2} \).

The energy eigenvalues are given by \((n = 0, 1, \ldots, [q])\):

$$E_n = -\frac{\hbar^2 \alpha^2}{2m} (q-n)^2 = -\frac{\hbar^2 \Omega^2}{4D} (q-n)^2 = -\frac{D}{q(q+1)} (q-n)^2. \quad (32)$$

The scalar product is the usual one, and with this scalar product \( \Psi^q_n, n = 0, \ldots, [q] \), are orthogonal and normalizable, the orthonormal basis being:

$$\bar{\Psi}^q_n(x) = N^q_n \Psi^q_n(ux)$$

$$N^q_n = \sqrt{\frac{n! \Gamma(q-n + \frac{1}{2})}{\pi \Gamma(q-n)(2q-2n+1)n}} \quad (33)$$

for \( n = 0, \ldots, [q] \).

5.1. The dynamical group for bound states

It is possible to find ladder operators (see, for instance [22]) for the MPT wavefunctions:

$$\hat{Z} = \frac{1}{\sqrt{2q}} \sqrt{\frac{q-n+1}{q-n}} \left[ \frac{1}{\alpha} \cosh(\alpha x) \frac{d}{dx} + (q-n) \sinh(\alpha x) \right]$$

$$\hat{Z}^\dagger = \frac{1}{\sqrt{2q}} \sqrt{\frac{q-n-1}{q-n}} \left[ -\frac{1}{\alpha} \cosh(\alpha x) \frac{d}{dx} + (q-n) \sinh(\alpha x) \right], \quad (34)$$

which, on normalized solutions, act as:

$$\hat{Z} \Psi^q_n = \sqrt{n} \frac{2q-n+1}{2q} \Psi^q_{n-1}$$

$$\hat{Z}^\dagger \Psi^q_n = \sqrt{(n+1)} \frac{2q-n}{2q} \Psi^q_{n+1}. \quad (35)$$

These operators, together with the number of quanta operator, \( \hat{n} \), close a Lie algebra which can be identified with \( SU(2) \) in a representation of index \( j = q \). Therefore, we could think of \( SU(2) \) as the dynamical algebra for the MPT potential.

However, there are some inconsistencies in this scheme:

- The parameter \( q \) can take any positive real value, and this is not possible in \( SU(2) \), where \( q \) must be integer or half-integer.
- There are only \([q] + 1\) normalizable states, instead of \( 2q + 1 \) (for \( q \) integer or half-integer).
- Although the Schrödinger equation for the MPT oscillator admits more solutions, with \( n > [q] \), these are either identical to the previous ones or are non-normalizable.
- In particular, if \( q \) is an integer, there exist states \( \Psi^q_n \) with \( n = q, q+1, \ldots, 2q \), but the state with \( n = q \) is non-normalizable, and \( \Psi^q_{2q-n} = \Psi^q_n, \) for \( n = 0, 1, \ldots, q-1 \).
- If \( q \) is not an integer (half-integer or real), there is an infinite number of states \( \Psi^q_n \) with \( n = [q] + 1, [q] + 2, \ldots, \) but all these states are non-normalizable.

Therefore, \( SU(2) \) is not a good candidate to be the dynamical algebra for the MPT oscillator.
5.2. Deriving the Schrödinger equation for bound states.

As in the case of the TPT potential, the time independent Schrödinger equation for the MPT potential can be derived from the Casimir of $SO(2,1)$ in a representation of the discrete positive series $D^+_k$, realized as a free particle in Anti-de Sitter space-time. If in the Casimir given by Eq. (16) we consider a negative Bargmann index $k = -q < 0$, we arrive at (see [1] for details):

$$-\frac{\hbar^2 \alpha^2}{2m} \frac{1}{\omega^2} \frac{\partial^2}{\partial \tau^2} \varphi = \left[ \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \varphi + \frac{D}{\cosh^2(\alpha x)} \varphi \right],$$

where $x = \frac{1}{\alpha} \arctanh(\sqrt{\frac{m\omega}{\hbar y}})$ and $q(q + 1) \equiv k(k - 1) = \frac{2mD}{\alpha^2 \hbar^2} = (\frac{2D}{\alpha^2 \hbar^2})^2$.

Denoting $t \equiv \frac{2\omega}{\hbar} \tau$, $\hat{E} \equiv i\hbar \frac{\partial}{\partial \tau}$, and $\chi \equiv e^{-\frac{i}{\hbar} \sqrt{D} \sqrt{-\hat{E}}} \varphi$ we recover at the time-independent Schrödinger equation for a particle of mass $m$ in a MPT potential with depth $D$ and width $1/\alpha$:

$$\frac{\hbar^2}{2m} \frac{d^2 \chi}{d x^2} + \left( E + \frac{D}{\cosh^2(\alpha x)} \right) \chi = 0.$$  \hspace{1cm} (37)

5.2.1. Deriving the ladder operators for the MPT potential. Similarly to the TPT case, the ladder operators (34) can be derived from the ladder operators for the Discrete series representations $D^+_k$ of $SU(1,1)$, given in Eq. (20) performing the corresponding change of variables and replacing $k = -q$. Also, a global factor depending on the state $n$ must be introduced to restore the unitarity.

As was commented previously, the assumption that $SU(2)$ is the dynamical group for the bound states of the MPT potential leads to inconsistencies, as was shown in [23]. The question that now arises is, which is then the dynamical group? A possible candidate for a dynamical group for the bound states of the MPT potential is $SU(1,1)$. Although $SU(1,1)$ is non-compact and therefore all its unitary representations are infinite-dimensional, it admits finite dimensional irreducible representations which are non-unitary (besides infinite-dimensional irreducible non-unitary representations). These can be obtained simply by changing the sign $k = -q$, $q > 0$ of the Bargmann index in the positive discrete series of $SU(1,1)$, $D^+_k$. If $2q \in \mathbb{Z}$, the representation has $2q + 1$ states (although it can be a factor representation). If $2q \notin \mathbb{Z}$, the representation is infinite-dimensional (and non-unitary).

This procedure of changing the sign $k = -q$, $q > 0$ is precisely what we have done to derive the Schrödinger equation for the MPT potential from the Casimir of $SO(2,1)$ in a particular realization of the Discrete Series $D^+_k$, corresponding to a model of a relativistic oscillator. The wave functions and a pair of ladder operators are also obtained, with the same features mentioned before (non-normalizable states, etc).

The non-normalizability of some of the states is a consequence of the non-unitarity of the representation. By the restriction to the subspace of normalizable states, a pair of ladder operators belonging to the enveloping algebra of $SU(1,1)$ can be constructed, coinciding with (34) on normalized energy eigenstates (see [1]).

6. Scattering states for the MPT potential

For this section we shall denote the Bargmann index by $N$, instead of $k$, to avoid confusion with the momentum, which is generally denoted by $\kappa$. The solution of the Schrödinger equation of the MPT potential for scattering states of energy $E = \frac{k^2}{2m}$ are given in terms of Hypergeometric functions as:

$$\Psi(x) = (1 - u^2)^{-ik} F(N - i\kappa, -N + 1 - i\kappa, 1 - i\kappa, 1 - \frac{u}{2}),$$

where $u = \tanh(\alpha x)$, $\kappa = \frac{k}{\alpha \hbar}$ and $N(N - 1) = \frac{2mD}{\alpha^2 \hbar^2}$.
6.1. Deriving Schrödinger equation for scattering states.

After the change of variables:

\[ x = \frac{c}{\omega} \text{arccosh} \left( \frac{1}{\sqrt{\sin^2(\omega \tau) - \frac{\omega^2}{c^2} y^2 \cos^2(\omega \tau)}} \right) \]

\[ s = mc \text{arcsinh} \left( \frac{\omega y}{e^{\sqrt{\sin^2(\omega \tau) - \frac{\omega^2}{c^2} y^2 \cos^2(\omega \tau)}}} \right), \]

and the identifications \[ t \equiv \frac{2\omega}{mc} s, \quad \hat{E} \equiv i\hbar \frac{\partial}{\partial t}, \quad \text{and} \quad \chi \equiv e^{-i\sqrt{\hat{D}} \hat{E} t} \phi, \] the Casimir equation (16) turns into the time-independent Schrödinger equation for the MPT potential.

6.2. The S matrix for the MPT potential

To determine the S matrix, whose components are the transmission T and reflection R amplitudes, we study the asymptotic behaviour of the solutions at \( x \to \pm \infty \):

- For \( x \to +\infty \), \( \Psi(x) \sim 4^{-in/2}e^{ikx/\hbar} \)
- For \( x \to -\infty \)

\[ \Psi(x) \sim 4^{-in/2} \left[ \frac{\Gamma(1 - i\kappa) \Gamma(i\kappa)}{\Gamma(1 - N) \Gamma(N)} e^{-ikx/\hbar} + \frac{\Gamma(1 - i\kappa) \Gamma(-i\kappa)}{\Gamma(1 - N - i\kappa) \Gamma(N - i\kappa)} e^{ikx/\hbar} \right], \]

where use of the transformation properties of the hypergeometric functions has been made:

\[ F(a, b; c; z) = \frac{\Gamma(c) \Gamma(c - a - b)}{\Gamma(c - a) \Gamma(c - b)} F(a, b, a + b - c + 1, 1 - z) \]

\[ + \quad (1 - z)^{c - a - b} \frac{\Gamma(c) \Gamma(a + b - c)}{\Gamma(a) \Gamma(b)} \times F(c - a, c - b, c - a - b + 1, 1 - z). \]

From these expressions the transmission and reflection amplitudes are readily obtained (see [4, 5, 6]):

\[ T(k) = \frac{\Gamma(N - i\kappa) \Gamma(1 - N - i\kappa)}{\Gamma(1 - i\kappa) \Gamma(-i\kappa)}, \quad R(k) = \frac{\Gamma(N - i\kappa) \Gamma(1 - N - i\kappa) \Gamma(i\kappa)}{\Gamma(N) \Gamma(1 - N) \Gamma(-i\kappa)}. \]

The S matrix, defined as

\[ S = \begin{pmatrix} T & R \\ R & T \end{pmatrix} \]

is unitary: \(|T|^2 + |R|^2 = 1\). See figure 5 for a plot of \(|T|^2\) and \(|R|^2\) for different values of \( N \).

For integer \( N \), \( R = 0 \) and \( T = 1 \), see figure 5b. Therefore there is no reflection and the quantum system behaves as the classical one. The Hilbert space splits into two invariant subspaces, corresponding to (a half of) \( D^+_N \) and \( D^-_N \).
6.3. Poles of the $S$ matrix

The $S$ matrix admits an analytical extension to the complex $k$ plane, since the potential is rapidly decreasing at spatial infinity (see, for instance [24] and references therein). From the poles of $T$, which happen at the imaginary $k$ axis we recover the bound states and the virtual or anti-bound states (ABS). In this case there are no resonances.

To study them, define $\kappa = i\nu$ and $q = -N$, then:

\[
T(\nu) = \frac{\Gamma(-q+\nu)\Gamma(1+q+\nu)}{\Gamma(1+\nu)\Gamma(\nu)}, \\
R(\nu) = \frac{\Gamma(-q+\nu)\Gamma(1+q+\nu)\Gamma(-\nu)}{\Gamma(-q)\Gamma(1+q)\Gamma(\nu)}.
\]  

(44)

Bound states occur as poles at $\nu > 0$, and there are $[q] + 1$: $\nu = q - n$, $n = 0, 1, \ldots, [q]$, at the energies corresponding with bound states.

ABS occur at $\nu < 0$, and their number depend on the character of $q$:

- For integer $q$, there is only one ABS state for $\nu = 0$ or $n = q$ ($E = 0$). But here $|T| = 1$ (no pole)!!
  There are zeros of $T$ at symmetrical positions to bound states: $\nu = n - q < 0$. See figure 6b.
- For half-integer $q$, there is an infinite number of ABS, located at positions $\nu = q - n < 0$, $n = [q] + 1, \ldots$
  There are zeros of $T$ at $\nu \in \mathbb{Z}^-$. See figure 6c.
- For real $q$, there is an infinite number of ABS, located at positions $\nu = q - n$, $n = [q] + 1, \ldots$
  There are also poles of $T$ at $\nu = -q - n$, $n = 1, \ldots$. There are also zeros of $T$ at $\nu \in \mathbb{Z}^-$. See figure 6a.

Although ABS are non-normalizable states and therefore are unphysical, those with a small value of $\nu$ have an effect on the transmission and reflection coefficients for real momentum. In particular, for integer $q$ the ABS state at $\nu = 0$ causes that $|T(\kappa)|^2 = 1$, $\forall \kappa \in \mathbb{R}$. Even more, for $q = 2.9$, the presence of an ABS state at $\nu = -0.1$ (see figure 6a) causes the transmission coefficient to increase faster to 1, see figure 5a, than in other cases, as in figure 5c, where the ABS state is at $\nu = -0.5$.

It should be noted that the $S$ matrix has a symmetry $N \rightarrow 1 - N$ (or $-q \rightarrow 1 + q$), which reflects the symmetry of the value of the Casimir under this transformation. The transmission and reflection coefficients are also periodic in $N$ for $k$ real, with period 1, as can be seen in figure 7. This property does not hold in the imaginary $k$ axis. The reason is that, using the properties
of Gamma functions for complex arguments, they can be written as:

\[
|T(k)|^2 = \frac{\sinh^2(\pi \kappa)}{\sinh^2(\pi \kappa) - \sin^2(\pi N)}
\]

\[
|R(k)|^2 = \frac{\sin^2(\pi N)}{\sin^2(\pi N) - \sinh^2(\pi \kappa)}.
\]  (45)

This periodic behaviour is shown in figure 7.

**Figure 7.** Periodicity of the transmission and reflection coefficients in the real axis, as a function of \(N\), for \(\kappa = 0.3\).

7. Comments and outlook

In this paper a group theoretical description of the modified Pöschl-Teller potential and related potentials, both for positive and negative energies, has been presented. Each case must be studied separately, since the dynamical symmetries and the representations associated are different for each case.

This behaviour is rather unpleasant, and a unified description, into a single Lie group, would be desirable. Probably this would require an infinite-dimensional group able to accommodate the Hamiltonian, which has a mixed spectrum (both positive continuum and negative discrete spectra), as a single generator. However, an intermediate step can be done including both groups into a finite-dimensional group, the price we must pay being that the Hamiltonian is associated with different generators for the positive and negative energy cases. The minimal algebra including both \(SO(2,1)\) and \(SO(3)\) algebras is the complexification of any of them, the \(SL(2,\mathbb{C})\) algebra, which is isomorphic to \(SO(3,1)\), the Lorentz algebra. The complexification appears here in a natural way due to the presence of the square root of the Hamiltonian in the
transformation (24). Thus, $SL(2, \mathbb{C})$ seems the natural framework to study this problem from a group-theoretical point of view (this was already anticipated in [1]).

Although at this point this assumption is simply a conjecture, there are clues indicating that this can be the right choice. For instance, if in the MPT potential for positive energy, besides the functions (24) we introduce $E' \equiv iE, X' \equiv iX$ and $P' \equiv iP$, the set $\{E, X, P, E', X', P'\}$ closes a $SL(2, \mathbb{C})$ algebra where the $SO(2, 1)$ subalgebra is the original $\{E, X, P\}$ and the $SO(3)$ subalgebra is $\{E', X', P\}$, which coincides with the one given in (27) and that are real for negative energies.

What remains to be done is to relate the phase spaces of these systems to the coadjoint orbits (or open subsets of them) of $SL(2, \mathbb{C})$, and to derive the Hilbert space of quantum states from the unitary (or non-unitary) representations of $SL(2, \mathbb{C})$. This is work in progress.

The complexification of $SO(2, 1)$ to $SL(2, \mathbb{C})$ seems to be related to the analytical extension of the $S$ matrix to the complex plane. The fact that the $S$ matrix provides all the information for a quantum system, not only the scattering states and bound states, but also the ABS states and resonant states, suggest that a detailed study of the $S$ matrix and its symmetries could provide a unified description of the system.

This study can be generalized to other potential, like the Morse potential and their generalization (Ginnochio potentials, Natanzon potentials), and to the 3 dimensional version of the potentials considered here.

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