Burgers’ equation in non-commutative space-time

L. Martina * and O. K. Pashaev †

* Dipartimento di Fisica and Sezione INFN - Lecce,
Universita di Lecce, Lecce, 11529, Italy
† Department of Mathematics, Izmir Institute of Technology,
Urla-Izmir, 35437, Turkey

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Abstract

The Moyal ∗-deformed noncommutative version of Burgers’ equation is considered. Using the ∗-analog of the Cole-Hopf transformation, the linearization of the model in terms of the linear heat equation is found. Noncommutative q-deformations of shock soliton solutions and their interaction are described.

1 Introduction

In the last decades the idea of a noncommutative field theory have received a great interest in the context of the string theory \cite{1}, in connection with the Noncommutative Geometry, created by A. Connes and many others \cite{2}, \cite{3}. It has been applied to the Standard model in the unified field theory \cite{4} gravity theory \cite{5} and the quantum Hall systems. In this context several attempts to extend the completely integrable, or solitonic, classical field theories to non commutative spaces have appeared \cite{6}. In the last case the basic idea is to include into the characterizing structures of the completely integrable systems (the bicomplex, or in a more traditional way, the Lax pair) the non commutative nature of the space-time. Then, single soliton - like solutions for the noncommutative analogs of the Nonlinear Schrödinger and Korteweg-de Vries equations (NLS and KdV, respectively) were presented, in terms of power expansion in the deformation parameter θ. However, the actual problem constructing of two and higher soliton solutions and studying the corresponding dynamics is faced with difficulties. In the present paper we introduce the noncommutative version of the Burger’s equation, which is linearizable via the ∗-analog of Cole - Hopf transformation, and construct two-shockes soliton solution in terms of q-deformed binomial series. This allows us to analytically deal with the two shocks fusion process into a final one solitonic object.
2 The ∗-product

The Moyal ∗-product\(^7\) is an associative and non commutative deformation of the ordinary product between two functions \(f(x_1, x_2), g(x_1, x_2) \in C^\infty\) on \(R^2\)

\[
(f \ast g)(\mathbf{x}) = e^{\theta(\partial_1 \partial_2 - \partial_2 \partial_1)} f(x_1, x_2)g(x_1', x_2')|_{\mathbf{x}=\mathbf{x}'}
\]

\[
= \sum_{n=0}^{\infty} \frac{\theta^n}{n!} \sum_{k=0}^{n} (-1)^{k} \binom{n}{k} (\partial_1^{n-k} \partial_2^k f)(\partial_2^{n-k} \partial_1^k g),
\]

where \(\mathbf{x} = (x_1, x_2) \in R^2\). The parameter \(\theta\) is assumed to be purely imaginary, but for our aims, we consider also the analytic continuations to the real \(\theta\). It is known that the convergence of the series in \(\text{1}\) requires that the product be defined in a space of rapidly decaying functions (usually of Schwartz type) on the plane. With such a definition one easily sees that \(\lim_{\theta \to 0} f \ast g = fg\). If \(\text{1}\) converges also for real values of \(\theta\) the resulting function is real, but in general is not rapidly decaying, thus one needs some care to deal with. Moreover, if the series \(\text{1}\) converges, one can use the integral representation

\[
f \ast g(\mathbf{x}) = -\frac{1}{(\pi \theta)^2} \int d^2x' d^2x'' f(\mathbf{x}') g(\mathbf{x}'') e^{-\frac{1}{\theta}((\mathbf{x}'-\mathbf{x}) \times (\mathbf{x}''-\mathbf{x}))}. \]

The basic example of ∗-product is between exponentials:

\[
e^{\mathbf{k} \cdot \mathbf{x}} \ast e^{\mathbf{k}' \cdot \mathbf{x}} = e^{\theta \mathbf{k} \cdot \mathbf{k}' \cdot x}.
\]

The ∗-product of two gaussian functions provides again an exponential with quadratic argument, but its coefficient and the amplitude in front of the \(\exp\) are depending from \((1 - ab \theta^2)^{-1}\). Thus, one sees that for \(\theta\) immaginary the ∗-product spreads the packets, while for arbitrary real values the resulting function can be unbounded.

The ∗-product enables us to define the Moyal Brackets as \(\{f, g\}_\theta \equiv (f \ast g - g \ast f) / (2\theta)\), which realize a deformation of the Poisson brackets, in the sense that they provide a Lie algebra on the functions of phase space \((x_1, x_2)\) and the relation \(\lim_{\theta \to 0} \{f, g\}_\theta = \{f, g\}_\text{Poisson}\) holds.

Since the ∗-product involves exponentials of the derivative operators, it may be evaluated through

\[
(f \ast g)(\mathbf{x}) = f(\hat{x}_1, \hat{x}_2)g(\mathbf{x}),
\]

where the uncertainty relation holds for the position operators

\[
\hat{x}_i = x_i + \theta \epsilon_{ij} \partial_j, \quad [\hat{x}_1, \hat{x}_2] = 2\theta .
\]

Thus, the algebra of function on \(R^2\) endowed with the ∗-product is describing the noncommutative plane.
3 Noncommutative Burgers’ equation

3.1 The linear problem

In the noncommutative space-time \((x_1 = t, x_2 = x)\) plane we consider the linear problem

\[\chi_x = U \ast \chi, \quad \chi_t = V \ast \chi,\] (6)

with the noncommutative Abelian connections

\[U = -\frac{1}{2\nu} u, \quad V = -\frac{1}{2\nu}(\nu u_x - \frac{1}{2}u \ast u).\] (7)

The translations operators \(\partial_t\) and \(\partial_x\) are derivations w.r.t. the \(*\)-product. Then, the integrability of (6) is assured by the usual equality of the mixed derivatives. Consequently, the zero curvature equation in Moyal form

\[U_t - V_x + U \ast V - V \ast U = U_t - V_x + 2\theta(U,V) = 0,\] (8)

provides the Noncommutative Burgers’ (NB) equation

\[u_t + u_x \ast u = \nu u_{xx}.\] (9)

The above noncommutative flat connection representation shows the relevance of our model for the BF topological field theory and noncommutative low dimensional gauge theory of gravity\[8\].

Given a solution \(\chi\) of this problem, we define first the \(*\)-inverse function \(\ast \chi^{-1}\):

\[\chi \ast \ast \chi^{-1} = 1,\] (10)

possessing properties: a) \(\ast \chi^{-1} = \chi^{-1} \ast\), b) \((\ast \chi^{-1})_x = -\ast \chi^{-1} \ast \ast \chi^{-1}\).

In general the \(*\)-inverse, if it exists, is not unique. But, since we are going to deal with exponentials and finite sums of exponentials, the relation (3) assures existence and uniqueness. Then, inverting the first of Eqs.(6) we get the \(*\)-analog of the Cole-Hopf transformation

\[u = (-2\nu) \chi_x \ast \ast \chi^{-1}\] (11)

relating a solution of the heat equation

\[\chi_t = \nu \chi_{xx},\] (12)

to a solution of the NB equation. Furthermore, putting \(\ast \chi^{-1} \equiv v\), we have another noncommutative nonlinear heat equation

\[v_t + 2\nu v_x \ast \ast v^{-1} = \nu v_{xx}.\] (13)

3.2 Noncommutative shock soliton

Now we consider particular solutions of the heat equation (12) in the exponential form \(e^{\eta_j}\), \(\eta_j = \omega_j t + k_j x + \eta_j^{(0)}\), \(\omega_j = \nu k_j^2\), and their superpositions, generating by transformation (11) exact solutions of the NB (9). Again from (3) we get

\[e^{\eta_1} \ast e^{\eta_2} = e^{2\theta(\eta_1, \eta_2)} e^{\eta_1+\eta_2} = e^{2\theta(\eta_1, \eta_2)} e^{\eta_1} \ast e^{\eta_2},\] (14)
Equation (10) allows to find $\epsilon$ where $\Delta_{ij} = -\Delta_{ji} = \omega_i k_j - \omega_j k_i = \nu k_i k_j (k_i - k_j)$.

For $\chi = e^{\eta_1}$, the $*$-inverse function is simply $\chi^{-1} = e^{-\eta_1}$, and from Eq. (11) we obtain the constant valued solution of Eq. (9): $u(t, x) = (-2\nu) k_1$.

Taking a superposition of two exponential solutions $\chi = e^{\eta_1} + e^{\eta_2}$, first we look for the $*$-inverse as a power series $\chi^{-1} = \sum_{n=0}^{\infty} \epsilon_n e^{\Omega_n t + P_n x + C_n}$. Equation (10) allows to find $\epsilon_n = (-1)^n$, $\Omega_n = -\omega_1 + n(\omega_2 - \omega_1)$, $P_n = -k_1 + n(k_2 - k_1)$, $C_n = -\eta_1^{(0)} + n(\eta_2^{(0)} - \eta_1^{(0)})$. This series is convergent in the two regions $e^{\eta_2} < 1$ and $e^{\eta_1} < 1$, where $\eta_{ij} \equiv \eta_i - \eta_j$. Combining together the results, for $e^{\eta_1} \neq 1$ we find that the $*$-inverse is

$$
\chi^{-1} = \frac{1}{e^{\eta_1} + e^{\eta_2}}. 
$$

By the Cole-Hopf type transformation (11) we are able to find the 1-shock soliton of the NB equation

$$
u_1^2 = \frac{k_2 - k_1}{1 + e^{\eta_1} - e^{\eta_2}}.
$$

with amplitude $k_2 - k_1$ and velocity $\nu_1 = -\nu(k_1 + k_2)$. In this case the noncommutativity influences the initial position of the single soliton $x_0 = x_0 + \theta \nu k_1 k_2$, so that $\lim_{\theta \to 0} x_0^\theta \to x_0$.

### 3.3 Two soliton solution

Now, let us take as solution of the heat equation $\chi = e^{\eta_1} + e^{\eta_2} + e^{\eta_3}$. For the power series expansion of its $*$-inverse we needs to divide the space-time plane on three regions and represent it in different ways in that regions. Specifically, as $\chi^{-1} = e^{-\eta_1} \sum a_{ij} e^{\eta_{i+j}^2 + \eta_{31}}$ in $D_1 = \{(x, t) : \eta_{21} < 0, \eta_{31} < 0\}$, as $\chi^{-1} = e^{-\eta_2} \sum a_{ij} e^{\eta_{i+j}^2 + \eta_{32}}$ in the region $D_2 = \{(x, t) : \eta_{12} < 0, \eta_{32} < 0\}$ and, finally, in region $D_3 = \{(x, t) : \eta_{13} < 0, \eta_{23} < 0\}$ we look for $\chi^{-1} = e^{-\eta_3} \sum a_{ij} e^{\eta_{i+j}^2 + \eta_{23}}$. Substituting these expressions into the Eq. (10), we obtain that $a_{ij} = (-1)^{i+j} b_{ij}$, where coefficients $b_{ij}$ satisfy the recurrence relations

$$
b_{ij} = b_{i-1,j} e^{\theta \Delta} + b_{i,j-1} e^{-\theta \Delta},
$$

and $\Delta \equiv \Delta_{12} + \Delta_{23} + \Delta_{31}$. 


3.4 The q-deformed Pascal-Tartaglia triangle

The coefficients in (17) can be generated by the q-deformed version of Pascal-Tartaglia triangle

\[ b_{00} \]
\[ b_{10} \]
\[ b_{01} \]
\[ b_{20} \]
\[ b_{11} \]
\[ b_{02} \]
\[ \vdots \]
\[ \cdots \]
\[ \vdots \]
\[ \cdots \]

where \( \alpha = \theta \Delta, q = e^\alpha = e^{\theta \Delta} \). Thus the \( b_{n,k} \) are given by the q-Binomial Coefficients

\[ \binom{n}{k}_{\alpha} = \frac{[n]!}{[k]![n-k]!} \]  

constructed from q-numbers \([n]_{\alpha} = \sinh n\alpha\) and the corresponding q-factorials \([n]! = [1][2]...[n]\). We would like to notice that "canonical" q-numbers of the form \([n] = \sinh n\alpha/\alpha\) produce the same q-Binomial coefficients.

Now, we define the generating function (GF) for these coefficients as

\[ G_{\alpha}(x, y) = \sum_{ij} (-1)^{i+j} \binom{i+j}{i}_{\alpha} x^iy^j. \]  

(19)

For \( \theta = 0 (\alpha = 0) \), it reduces to the GF of the ordinary binomial coefficients \( G_0(x, y) = [1 + (x+y)]^{-1} \) In terms of GF (19) we are able to get the formal power series expression for the \( * \)-inverse of \( \chi \) as

\[ *\chi^{-1} = \begin{cases} 
  e^{-\eta_{ij}}G_\alpha(e^{\eta_{i1}}, e^{\eta_{i2}}) & \text{in } D_1, \\
  e^{-\eta_{ij}}G_\alpha(e^{\eta_{i2}}, e^{\eta_{i3}}) & \text{in } D_2, \\
  e^{-\eta_{ij}}G_\alpha(e^{\eta_{i3}}, e^{\eta_{i1}}) & \text{in } D_3.
\end{cases} \]  

(20)

By the \( * \)- Cole-Hopf type transformation (11), it gives the 2-soliton solution of the NB equation (9), expressed in power series in the region \( D_i, (i = 1, 2, 3) \) by

\[ u_i(t, x) = \sum_{j=1, k<i} a_{ij} e^{\eta_{ji} + \theta \Delta ij} G_\alpha \left( e^{\eta_{ki} + \theta (\Delta_{ik} + \epsilon_{ijk} \Delta)}, e^{\eta_{ji} + \theta (\Delta_{ik} + \epsilon_{ijk} \Delta)} \right) \]  

(21)

where we have used the following parametrization: \( \eta_{ij} \equiv \eta_i - \eta_j = -((a_i - a_j)/2\nu)[x - v_{ij}t - x_{0ij}] = k_{ij} \xi_{ij}, v_{ij} = (a_i + a_j)/2, x_{0ij} = (2\nu/(a_i - a_j))(\eta^{(0)}_j - \eta^{(0)}_i) \). Then, for \( a_3 > a_2 > a_1 \) we have \( k_{12} > 0, k_{13} > 0, k_{23} > 0 \) and \( v_{23} > v_{13} > v_{12} \) for the velocities. Such velocities determine the
motion in \((x, t)\) plane with the boundary lines defining the domains \(D_1\), \(D_2\) and \(D_3\), respectively. In the asymptotic region of these domains one of the arguments of GF \((14)\) vanishes, so that GF reduces to \(G_\alpha(0, y) = (1 + y)^{-1}\), for \(y < 1\). Then, we find that for any fixed time \(t_0\) the solution decays exponentially at infinities

\[
u(t_0, x) = \begin{cases} a_3, & x \to -\infty \\ a_1, & x \to +\infty \end{cases}
\]

(22)

On the other hand, one can study the asymptotic behaviour of the solution in each of the moving frame of relative velocity \(v_{ij}\) above defined. We summarize in the following table the relative velocity, the limiting points and the asymptotic behaviour of the solution

\[
v_{12}, x \to -\infty, t \to -\infty; \quad v_{23}, x \to -\infty, t \to -\infty; \quad v_{13}, x \to +\infty, t \to +\infty,
\]

\[
u \to \frac{a_1 + a_3 \eta_{12}^\theta + \eta_{21}^\theta + \theta \Delta_{12}}{1 + e^{\eta_{21}^\theta + \theta \Delta_{12}}}, \quad u \to \frac{a_2 + a_3 \eta_{23}^\theta + \theta \Delta_{23}}{1 + e^{\eta_{23}^\theta + \theta \Delta_{23}}}, \quad u \to \frac{a_1 + a_3 \eta_{13}^\theta + \theta \Delta_{13}}{1 + e^{\eta_{13}^\theta + \theta \Delta_{13}}},
\]

Then, it shows that the solution describes a fusion of two isolated shock solitons, moving with velocities \(v_{12}\) and \(v_{23}\), into the final one soliton, possessing velocity \(v_{13}\). The qualitative picture of this soliton interaction is similar to the commutative case. But the "central" positions of these solitons are modified, according to the expression \(x_{0ij}^\theta = \frac{2}{\pi} (\eta_i^\theta(0) - \eta_j^\theta(0) - \theta \Delta_{ij})\). Thus, we have the relation

\[
A_{12}x_{012}^\theta + A_{23}x_{023}^\theta - 2\nu \theta \Delta = A_{13}x_{013}^\theta,
\]

(23)

which connects the positions \(x_{0ij}^\theta\) and the amplitudes \(A_{ij} \equiv -(a_i - a_j)\) of initial and final solitons. This formula determines the position of the "center of mass" of the final soliton, which results to contain the \(\theta\) dependent shift

\[
2\nu \frac{\theta \Delta}{A_{12} + A_{23}}.
\]

(24)

with respect to the commutative \((\theta = 0)\) case. Moreover, this shift is independent on the initial positions of the colliding solitons.

### 4 Conclusions

In this work on the Burger’s equation in noncommutative space-time we have shown how the noncommutativity affects one of the main feature of the soliton interaction behaviour, i.e. the shift in position of the final states. In the case of two interacting shocks Eq. (24) provides an explicit algebraic expression of that quantity in the case of the process of fusion of two shocks, linearly dependent on the deformation parameter \(\theta\). The last result can be considered as a test distinguishing between the commutative and noncommutative solitons, in the analytical and qualitative studying of soliton’s dynamics. It suggests also that in the real experiments with optical solitons, appearance of a global position shift independent of initial soliton positions indicates on the noncommutative nature of corresponding solitons.
As is well known the Burgers’ equation is an example of the so called C-integrable solitonic equation, and as we showed in the present paper the concept of C-integrability can be extended to the noncommutative space-time case. Moreover it would be interesting to see the influence of the noncommutativity on the so called Q-integrable equations, like the NLS and the KdV or the self-dual Yang-Mills system, as well. The analytic study of the two-, and possibly multi-, soliton solutions for the noncommutative analogs of those system is in our plans for forthcoming future.

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