Aharonov-Bohm effect on AdS$_2$ and nonlinear supersymmetry of reflectionless Pöschl-Teller system

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Abstract

We explain the origin and the nature of a special nonlinear supersymmetry of a reflectionless Pöschl-Teller system by the Aharonov-Bohm effect for a nonrelativistic particle on the AdS$_2$. A key role in the supersymmetric structure appearing after reduction by a compact generator of the AdS$_2$ isometry is shown to be played by the discrete symmetries related to the space and time reflections in the ambient Minkowski space. We also observe that a correspondence between the two quantum non-relativistic systems is somewhat of the AdS/CFT holography nature.

1 Introduction

Exactly solvable quantum hyperbolic Pöschl-Teller system [1]–[6] finds diverse applications in a variety of physical problems. It appears, for instance, in the form of a stability equation for soliton (kink) solutions in 1+1-dimensional $\varphi^4$ and sine-Gordon field theories [7]. It describes static solutions in the Gross-Neveu model [8, 9, 10], and was used in investigation of the tachyon condensation phenomenon in gauge field string dynamics [11, 12]. This system, on the other hand, plays a fundamental role in the inverse scattering problem method for nonlinear evolution equations [13, 14].

All the mentioned applications are related to a peculiar property of the Pöschl-Teller system, whose potential $U(x) = -\lambda \cosh^{-2} x$ becomes reflectionless at special values $\lambda = m(m+1)$, $m = 1, 2, \ldots$, of the coupling constant. This property finds a simple explanation in terms of almost isospectral transformations underlying supersymmetric quantum mechanics [15]. Namely, the reflectionless Pöschl-Teller (RPT) system with $\lambda = m(m+1)$ can be related to a free particle model ($m = 0$) by a Crum-Darboux transformation of order $m$. The RPT system possesses also a shape-invariance property, due to which the energies of its $m$ bound states are found algebraically [15, 16, 17].

The RPT system with $m$ bound states is characterized by an integral of motion that is a differential operator of order $2m + 1$. This operator is known as a nontrivial operator of the Lax pair of a non-periodic problem for the $m$-th order Korteweg-de Vries equation [14]. It was rediscovered recently in the context of nonlinear supersymmetric structure associated with the RPT system, see Refs. [18, 19], and [20, 21]. More specifically, it was observed in [18] that the RPT system is characterized by a hidden bosonized nonlinear $N = 2$ supersymmetry
The nontrivial Lax operator plays the role of one of its two supercharges, and the parity (reflection) operator $R$, $R\psi(x) = \psi(-x)$, is identified as the $\mathbb{Z}_2$-grading operator $\Gamma$. As a consequence, the extended system composed from the two RPT systems with parameters $m$ and $l < m$ (including the case $l = 0$ that corresponds to a free particle system), is described by a nonlinear tri-supersymmetry \cite{23, 24}. The tri-supersymmetric structure admits three alternative choices of the grading operator, $\Gamma = \sigma_3$, $R$, or $\sigma_3 R$, where $\sigma_3$ is a diagonal Pauli matrix \cite{25}. For $\Gamma = \sigma_3$, the local part of this unusual supersymmetric structure corresponds to a nonlinear $N = 4$ supersymmetry, in which the higher order integral associated with the nontrivial Lax operator plays the role of a bosonic central charge \cite{24}.

In the present paper we show that the hidden bosonized nonlinear supersymmetry of the RPT system and the related nonlinear tri-supersymmetric structure of the extended case originate from a quantum problem of a charged non-relativistic particle on the AdS$_2$ surface in the presence of a singular magnetic-like vortex. Classically, the particle performs a free, geodesic motion on the AdS$_2$. Aharonov-Bohm effect \cite{26} influences essentially on the quantum properties of the system. For half-integer values of the flux, the spectrum is characterized by a specific additional double degeneration related to an involutive automorphism of the AdS$_2$ isometry. The system reveals a reflectionless quantum dynamics in this case. These two related properties underlie the peculiar nonlinear supersymmetric structure of the RPT system obtainable by an appropriate angular momentum reduction. At the same time for a generic magnetic flux value, the Aharonov-Bohm effect on the AdS$_2$ provides a new vision on the nature of the Crum-Darboux transformations for the quantum hyperbolic Pöschl-Teller system in the context of the AdS/CFT correspondence \cite{27, 28}. We also discuss this aspect in the background of non-relativistic AdS/CFT. The latter is based on non-relativistic conformal symmetry \cite{29–35}, and became recently a hot topic due to the diverse range of applications, see \cite{36, 37}.

The paper is organized as follows. In Section 2 we summarize briefly the results on the nonlinear hidden bosonized supersymmetric structure of the RPT system and the tri-supersymmetric structure of the pair of such systems in the light of the Crum-Darboux transformations, discussed in detail in \cite{18, 24}. In Section 3 we investigate the classical and quantum theory of a non-relativistic particle on the AdS$_2$, minimally coupled to a U(1) gauge field given by the Aharonov-Bohm vector potential. In Section 4 we show how the nonlinear supersymmetric structure of the RPT system emerges from such a non-relativistic system at half-integer values of the magnetic flux. In Section 5 we discuss a realization of the conformal and super-conformal dynamical symmetries in the RPT system. We conclude with a discussion of the results and open research questions in Section 6.

### 2 Crum-Darboux transformations and tri-supersymmetry

A hyperbolic reflectionless Pöschl-Teller system is described by the Hamiltonian

$$H_m = -\frac{d^2}{dx^2} - \frac{m(m+1)}{\cosh^2 x}. \quad (2.1)$$

Here and in what follows we put $\hbar = 1$. It is convenient to treat $m$ as a parameter that can take any integer value, $m \in \mathbb{Z}$. Then $m = 0$ corresponds to a free particle case, and we have
the identity
\[ H_m = H_{-(m+1)}. \] (2.2)

In terms of the first order differential operators
\[ D_m = \frac{d}{dx} + m \tanh x, \quad D_{-m} = -D_m^\dagger, \] (2.3)
the second order operator (2.1) can be presented in a form
\[ H_m = -D_{-m}D_m - m^2, \] (2.4)
while (2.2) rewrites equivalently
\[ D_{-m}D_m = D_{m+1}D_{m-1} + (2m + 1). \] (2.5)

Using this identity and (2.4), one can check that there hold the following intertwining relations
\[ D_m H_m = H_{m-1}D_m, \quad D_{-m} H_{m-1} = H_{m}D_{-m}. \] (2.6)

These relations can be understood from a point of view of Crum-Darboux theorem [13]. According to it, if a differential operator \( A_n \) of order \( n \) annihilates \( n \) eigenstates \( \psi_i \) of a Hamiltonian \( H \), one can construct another Hamiltonian \( \tilde{H} = H - 2(\ln W(\psi_1, \ldots, \psi_n))^n \), and these three operators are related by the identities
\[ \tilde{H} A_n = A_n H, \quad A_n^\dagger \tilde{H} = H A_n^\dagger. \] (2.7)

Here, \( W \) is the Wronskian of not obligatorily to be physical states \( \psi_i, i = 1, \ldots, n \), such that \( W \neq 0 \). As a consequence of identities (2.7), if \( \psi \) is an eigenstate of \( H \), \( H \psi = E \psi, A_n \psi \neq 0 \), the state \( A_n \psi \) will be the eigenstate of \( \tilde{H} \) with the same eigenvalue \( E \). And vice versa, if \( \psi \) is an eigenstate of \( \tilde{H} \), \( \tilde{H} \tilde{\psi} = \tilde{E} \tilde{\psi} \), such that \( A_n^\dagger \tilde{\psi} \neq 0 \), the state \( A_n^\dagger \tilde{\psi} \) will be an eigenstate of \( H \) with the the same eigenvalue \( \tilde{E} \). This means that the Hamiltonians \( H \) and \( \tilde{H} \) will be almost isospectral. In the case of (2.6), the first order operator \( D_m \) annihilates the nodeless ground state \( \cosh^{-m} x \) of the Hamiltonian \( H_m \) with eigenvalue \( -m^2 \), and \( H_{m-1} \) has the same spectrum as \( H_m \) except the missing in it eigenvalue \( -m^2 \). In addition, relations (2.6) reflect the shape-invariance property of the Pöschl-Teller system. The first order Darboux transformation generated by the operator \( D_m \) produces for the Hamiltonian \( H = H_m \) the partner Hamiltonian \( \tilde{H} \), which has potential of the same form but with the parameter \( m \) shifted in one, i.e. \( \tilde{H} = H_{m-1} \).

By repeated application of (2.6), we can get the intertwining relations of higher order Crum-Darboux transformations [13];
\[ (D_{m-l} \ldots D_{m-1}D_m) H_m = H_{m-l-1}(D_{m-l} \ldots D_{m-1}D_m), \] (2.8)
\[ (D_{-m}D_{-(m-1)} \ldots D_{-(m-l)}) H_{m-l-1} = H_m(D_{-m}D_{-(m-1)} \ldots D_{-(m-l)}). \] (2.9)

In a particular case \( l = m - 1 \), relation (2.9) takes a form
\[ (D_{-m}D_{-(m-1)} \ldots D_{-1}) H_0 = H_m(D_{-m}D_{-(m-1)} \ldots D_{-1}), \] (2.10)
which means that the system (2.1) is almost isospectral to the free particle system. Making use of it, the scattering states of the RPT system (2.1) can be obtained from the plane wave eigenstates of the free particle,

$$
\psi_m^{(\pm k)}(x) = D_{-m}D_{-m+1} \ldots D_{-1} \cdot \exp(\pm ikx).
$$

(2.11)

According to (2.10) and (2.11), positive energy values of the system (2.1),

$$
E_m,k = k^2 \quad \text{with} \quad k > 0,
$$

are doubly degenerate, while zero energy with \(k = 0\) corresponds to a singlet state

$$
\psi_m^{(0)}(x) \equiv \psi_{m;0}(x) = (D_{-m}D_{-m+1} \ldots D_{-1}) \cdot 1.
$$

(2.12)

Function \(\cosh^{-m} x\) is the zero mode of the first order operator \(D_m\). In correspondence with (2.4), it describes a bound state of the RPT system with energy \(E_{m;0} = -m^2\). This observation together with the relation (2.9) taken for \(l = 0, \ldots, m - 2\) allow us to find the whole set of the bound singlet states of the RPT system (2.1),

$$
\psi_{m;0}(x) = \cosh^{-m} x, \quad \psi_{m;n}(x) = D_{-m}D_{-m+1} \ldots D_{-m+n-1} \cosh^{n-m} x, \quad n = 1, \ldots, m - 1.
$$

(2.13)

Extension of (2.13) for \(n = m (l = m - 1)\) reproduces singlet state (2.12) of the continuous part of the spectrum. The energies of all the \(m + 1\) singlet states (2.13) and (2.12) are given then by

$$
E_m;n = -(m-n)^2, \quad n = 0, \ldots, m.
$$

(2.14)

The double degeneration in the continuous part of the spectrum and the presence of \(m + 1 > 1\) singlet states indicate on a hidden, nonlinear supersymmetry in the system (2.1). Its corresponding supercharges can be identified easily. Applying the order \(2m + 1\) Crum-Darboux transformation (2.9) corresponding to \(l = 2m\) and taking into account relation (2.2), we find that the RPT system (2.1) is characterized by a local integral

$$
\mathcal{A}_{2m+1} = D_{-m}D_{-m+1} \ldots D_0 \ldots D_{m-1}D_m, \quad [\mathcal{A}_{2m+1}, H_m] = 0,
$$

(2.15)

that is a differential operator of order \(2m + 1\). This is a nontrivial integral of the Lax pair \((\mathcal{A}_{2m+1}, H_m)\) of the \(m\)-th order KdV equation [14]. It is a parity-odd operator, while the Hamiltonian (2.11) is parity-even. Identifying reflection (parity) operator \(R, R\psi(x) = \psi(-x)\), as a grading operator, and integrals \(Z_1 = i^{2m+1}\mathcal{A}_{2m+1}\) and \(Z_2 = iRZ_1\) as Hermitian supercharges, we find that the RPT system (2.1) is characterized by a nonlinear \(N = 2\) supersymmetry

$$
[Z_a, H_m] = 0, \quad \{Z_a, Z_b\} = 2\delta_{ab}P_{2m+1}(H_m),
$$

(2.16)

where \(P_{2m+1}(H_m)\) is a polynomial of order \(2m + 1\). Its explicit form can be found with the help of relation (2.5),

$$
P_{2m+1}(H_m) = (H_m - E_{m;m})^{m-1} \prod_{n=0}^{m-1} (H_m - E_{m;n})^2,
$$

(2.17)

where \(E_{m;n}\) are the energies (2.14) of the singlet states. Singlet states (2.12) and (2.13) are zero modes of the supercharges \(Z_a\), other \(m\) states annihilated by \(Z_a\) have a more intricate nature, see [19] [23].
Note that the nontrivial integral \([2.15]\) has a sense of the integral \(D_0\) of the free particle system \(H_0\) transferred to the RPT system \(H_m\) by means of the Crum-Darboux transformations. Indeed, multiply the relation \(H_0D_0 = D_0H_0\) from the left by the operator \(D_mD_{m+1}D_{m+2}\ldots D_{-1}\), and from the right by \(D_1D_{m-1}D_m\). Using on the left hand side intertwining relation \([2.10]\), and on the right hand side its Hermitian conjugate form, we get \([\mathcal{A}_{2m+1}, H_m] = 0\).

The system composed from the two RPT systems \(H_m\) and \(H_l\), \(0 \leq l < m\), can be described by the \(2 \times 2\) matrix Hamiltonian
\[
\mathcal{H}_{m,l} = \begin{pmatrix} H_l & 0 \\ 0 & H_m \end{pmatrix}.
\]

The two RPT subsystems are almost isospectral. The subsystem \(H_l\) does not have \(m-l\) energy levels corresponding to the \(m-l\) lowest bound states of the subsystem \(H_m\). Hence, these states form singlets of the extended system. The intertwining relations \([2.9]\) allow us to identify the local Hermitian integrals of motion of the extended system \([2.18]\),
\[
\mathcal{X}_{m,l} = \begin{pmatrix} 0 & X_{m,l}^- \\ X_{m,l}^+ & 0 \end{pmatrix}, \quad \mathcal{Y}_{m,l} = \begin{pmatrix} 0 & Y_{m,l}^- \\ Y_{m,l}^+ & 0 \end{pmatrix}, \quad \mathcal{Z}_{m,l} = \begin{pmatrix} Z_{m,l}^+ & 0 \\ 0 & Z_{m,l}^- \end{pmatrix},
\]
\([\mathcal{H}_{m,l}, \mathcal{X}_{m,l}] = [\mathcal{H}_{m,l}, \mathcal{Y}_{m,l}] = [\mathcal{H}_{m,l}, \mathcal{Z}_{m,l}] = 0\), where
\[
X_{m,l}^- = (X_{m,l}^+)^\dagger = -i^{m-l}D_{l+1}D_{l+2}\ldots D_{m-1}D_m, \quad Y_{m,l}^- = (Y_{m,l}^+)^\dagger = i^{2l+1}A_{2l+1}X_{m,l}^-,
\]
\[
Z_{m,l}^- = (Z_{m,l}^+)^\dagger = X_{m,l}^+Y_{m,l}^- = Y_{m,l}^+X_{m,l}^- = i^{2m+1}A_{2m+1},
\]
\[
Z_{m,l}^+ = (Z_{m,l}^+)^\dagger = X_{m,l}^-Y_{m,l}^+ = Y_{m,l}^-X_{m,l}^+ = i^{2l+1}X_{m,l}^-X_{m,l}^+A_{2l+1}.
\]

These integrals of motion mutually commute,
\[
[\mathcal{X}_{m,l}, \mathcal{Y}_{m,l}] = [\mathcal{Y}_{m,l}, \mathcal{Z}_{m,l}] = [\mathcal{X}_{m,l}, \mathcal{Z}_{m,l}] = 0,
\]
and satisfy relations
\[
(\mathcal{X}_{m,l})^2 = P_\mathcal{X}(\mathcal{H}_{m,l}), \quad (\mathcal{Y}_{m,l})^2 = P_\mathcal{Y}(\mathcal{H}_{m,l}), \quad (\mathcal{Z}_{m,l})^2 = P_\mathcal{Z}(\mathcal{H}_{m,l}).
\]
In correspondence with \([2.25]\), \(P_\mathcal{Z}(\mathcal{H}_{m,l}) = P_\mathcal{X}(\mathcal{H}_{m,l})P_\mathcal{Y}(\mathcal{H}_{m,l})\) is a polynomial of order \(2m+1\) of the form \([2.17]\) with \(H_m\) changed for \(\mathcal{H}_{m,l}\), and
\[
P_\mathcal{X}(\mathcal{H}_{m,l}) = \prod_{n=0}^{m-l-1} (\mathcal{H}_{m,l} - E_{m;n}), \quad P_\mathcal{Y}(\mathcal{H}_{m,l}) = P_\mathcal{X}(\mathcal{H}_{m,l})'(\mathcal{H}_{m,l} - E_{m;n}) \prod_{n=m-l}^{m-1} (\mathcal{H}_{m,l} - E_{m;n})^2.
\]

Here, the roots \(E_{m;n}\) of the polynomials correspond to the singlet states of the subsystem \(H_m\), and are given by \([2.14]\).

In addition to integrals \([2.19]\), the extended system \([2.18]\) has also three mutually commuting trivial integrals of motion,
\[
\Gamma_i \in \{ R, \sigma_3, R\sigma_3 \}, \quad \Gamma_i^2 = 1, \quad i = 1, 2, 3.
\]
Any of them can be chosen as a grading operator $\Gamma$, that classifies one of the integrals (2.19) as an even (bosonic), and the other two as odd (fermionic) generators. As a result, the set of integrals (2.19), (2.28) extended by the products of (2.28) with (2.19) generates together with Hamiltonian (2.18) a certain deformation of the $su(2|2)$ superalgebra. Its concrete form depends on the choice of the grading operator, see [23, 24]. For the particular choice $\Gamma = \sigma_3$, the subset of integrals $X_{m,l}$, $Y_{m,l}$, $i\sigma_3 X_{m,l}$ and $i\sigma_3 Y_{m,l}$ is identified as the set of the odd supercharges, while $Z_{m,l}$ and $H_{m,l}$ are even operators. These local integrals generate a nonlinear $N = 4$ superalgebra, in which, in correspondence with (2.24), $Z_{m,l}$ plays a role of a central charge. For $m - l = 1$, the integral $X_{m,m-1}$ is a usual first order supercharge for the superextended system (2.18), whose conservation follows from the Darboux intertwining relations (2.6). In a generic case, the relation of commutativity of $X_{m,l}$ with matrix Hamiltonian $H_{m,l}$ represents just mutually conjugate intertwining relations (2.8) and (2.9) of order $m - l$. The integral $Y_{m,l}$ corresponds to the Crum-Darboux transformations of order $m + l + 1$ that intertwine the systems $H_m$ and $H_{-(m+1)} = H_l$.

3 Aharonov-Bohm effect on the AdS$_2$

In this section, we will consider a two-dimensional setting with the geometry of the AdS$_2$ space. The particle system will be studied in the presence of a singular magnetic flux in both classical and quantum frameworks. This will provide a valuable background for deeper understanding of the algebraic properties described in the previous section.

Consider a one-sheeted hyperboloid

$$x^\mu x_\mu = -x_1^2 - x_2^2 + x_3^2 = -R^2$$

(3.1)

embedded in a three-dimensional Minkowski space with metric $ds^2 = dx^\mu dx^\nu \eta_{\mu\nu}$, $\eta_{\mu\nu} = \text{diag} (-1, -1, +1)$, $\mu, \nu = 1, 2, 3$. It can be parameterized by

$$x^1 = R \cosh \chi \cos \varphi, \quad x^2 = R \cosh \chi \sin \varphi, \quad x^3 = R \sinh \chi,$$

(3.2)

and is identified as the AdS$_2$ space of radius $R > 0$ with the induced metric $ds^2 = R^2 (d\chi^2 - \cosh^2 \chi d\varphi^2)$, $-\infty < \chi < \infty$, $0 \leq \varphi < 2\pi$. Let in an ambient Minkowski space the Aharonov-Bohm vector potential is given,

$$A_1 = \frac{-\Phi}{2\pi x_1^2 + x_2^2}, \quad A_2 = \frac{\Phi}{2\pi x_1^2 + x_2^2}, \quad A_3 = 0,$$

(3.3)

that describes a “magnetic” field of a flux line along the $x_3$-axis, $B_\mu = \epsilon_{\mu\nu\lambda} \partial^\nu A^\lambda = (0, 0, \Phi \delta^2(x_1, x_2))$. Here $\epsilon_{\mu\nu\lambda}$ is an antisymmetric tensor, $\epsilon_{123} = 1$.

3.1 Classical system

Consider a non-relativistic charged particle minimally coupled to the external $U(1)$ gauge field (3.3), and confined to move on the two-dimensional surface (3.1). Taking into account

\footnote{The AdS$_2$ has a topology $\mathbb{R} \times S^1$ with $S^1$ usually unwrapped [28]. However, for here, the $S^1$-topology plays a key role.}
Eqs. (3.2), (3.3), the corresponding Lagrangian of a particle of unit mass, \( L = \frac{1}{2} \dot{x}^\mu \dot{x}^\mu + e A_\mu \dot{x}^\mu \), is reduced to
\[
L = \frac{\mathcal{R}^2}{2} (\chi^2 - \cosh^2 \chi \dot{\varphi}^2) - \alpha \dot{\varphi},
\]
(3.4)

where \( \alpha \equiv \frac{e \Phi}{2 \pi c}, \dot{x}^\mu = dx^\mu / dt \), and \( t \) is an evolution parameter. The last, coupling term in (3.4) is a total time derivative. It does not effect on the classical dynamics of the particle that performs a geodesic motion on the AdS\(_2\).

The form of the trajectories can be identified by noting that the SO(2,1)-isometry of the AdS\(_2\) is a transitive symmetry group of the hyperboloid (3.1), i.e. any two points on the surface can be related by an appropriate SO(2,1)-transformation. The generators of this symmetry are the integrals of motion
\[
J_1 = -p_\chi \sin \varphi - J_3 \cos \varphi \tanh \chi, \quad J_2 = p_\chi \cos \varphi - J_3 \sin \varphi \tanh \chi, \quad J_3 = p_\varphi + \alpha,
\]
(3.5)
(3.6)

where \( p_\chi = \frac{\partial L}{\partial \dot{\chi}}, p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} \) are the canonical momenta, \( \{\varphi, p_\varphi\} = 1; \{\chi, p_\chi\} = 1 \). With respect to the Poisson brackets, the integrals (3.5) and (3.6) generate the (2+1)D Lorentz algebra
\[
\{J_\mu, J_\nu\} = -\epsilon_{\mu\nu\lambda} J^\lambda.
\]
(3.7)

Their conservation follows from the form of the canonical Hamiltonian,
\[
H = p_\chi \dot{\chi} + p_\varphi \dot{\varphi} - L,
\]

that is reduced to the \( \text{so}(2, 1) \) Casimir element up to a multiplicative constant
\[
H = -\frac{1}{2\mathcal{R}^2} \mathcal{C}, \quad \mathcal{C} = J_\mu J^\mu.
\]
(3.9)

Direct checking shows that the integrals (3.5), (3.6) satisfy identically the relation \( x^\mu J_\mu = 0 \) with \( x^\mu \) given by (3.2). This also follows from the observation that the system (3.8) can alternatively be obtained in two steps. First, we reduce the \( \text{ISO}(2, 1) \)-invariant free particle system \( L = \frac{1}{2} m \dot{x}^\mu \dot{x}^\mu \) to the \( \text{SO}(2, 1) \)-invariant surface given by the second class constraints \( x_\mu x^\mu + \mathcal{R}^2 = 0 \) and \( p_\mu x^\mu = 0 \) in the phase space with canonical coordinates \( (x^\mu, p_\mu) \). The reduced phase space is described by the two pairs of canonical variables \( (\varphi, p_\varphi), (\chi, p_\chi) \) in terms of which the \( \text{SO}(2, 1) \) generators \( J_\mu = -\epsilon_{\mu\nu\lambda} x^\nu p^\lambda \) take the form of the integrals (3.5), (3.6) with \( \alpha = 0 \). The case \( \alpha \neq 0 \) is obtained then via a subsequent canonical transformation \( p_\varphi \to p_\varphi + \alpha \). Both steps do not touch the initial free case identity \( x^\mu J_\mu \equiv 0 \). As a result, the trajectory is determined by the intersection of the Minkowski hyperplane \( x^\mu J_\mu = 0 \), \( J_\mu = \text{const} \), with the surface of the hyperboloid (3.1). Its form depends on the value of the Casimir \( \mathcal{C} \). For \( \mathcal{C} > 0, = 0, \) or \( < 0 \), the trajectory is respectively an ellipse, a straight line, or a hyperbola.

Due to the Lorentzian (indefinite) metric of the AdS\(_2\) surface, the values of the Hamiltonian (3.8) are not restricted from below. We will be interested in the quantum system reduced to certain levels of the integral \( J_3 \), in which the spectrum is bounded from below. As it follows from (3.6) and (3.8), such a reduced system describes a one-dimensional Pöschl-Teller system. Three types of the classical trajectories of the system (3.8) correspond to a
bounded periodic \((C > 0)\), or unbounded \((C \leq 0)\) particle motions in a 1D classical attractive potential. In particular, straight line trajectories on the hyperboloid correspond to a zero energy motion of the Pöschl-Teller system.

Due to its algebraic background, the system \((3.9)\) has also two discrete symmetries. They have no significant role in the classical theory, but will be of a key importance at the quantum level in the context of the supersymmetry we discuss. These symmetries are the involutive automorphisms of the \(so(2, 1)\) algebra \((3.7)\),

\[
R : (J_1, J_2, J_3) \rightarrow (-J_1, -J_2, J_3), \quad S : (J_1, J_2, J_3) \rightarrow (-J_1, J_2, -J_3).
\]

In the ambient Minkowski space they correspond to a change of a sign of a space-like coordinate \(x^3\), and of one of the time-like coordinates which, in correspondence with the chosen definition of \(S\), we identify with \(x^2\),

\[
R : (x^1, x^2, x^3) \rightarrow (x^1, x^2, -x^3), \quad S : (x^1, x^2, x^3) \rightarrow (x^1, -x^2, x^3).
\]

In the curvilinear coordinates this corresponds to

\[
R : (\chi, \varphi) \rightarrow (-\chi, \varphi), \quad S : (\chi, \varphi) \rightarrow (\chi, -\varphi).
\]

Lagrangian \((3.4)\) is invariant under the discrete symmetry \(R\). However, it is quasi-invariant under symmetry \(S\), which provokes a change for a total time derivative term, \(\Delta L = 2\alpha \dot{\varphi}\), that does not effect on the classical motion. In correspondence with such a quasi-invariance of Lagrangian, in order to reproduce involutive automorphism \(S\) defined by \((3.10)\) in the presence of nonzero Aharonov-Bohm flux \(\alpha\), a transformation \(\varphi \rightarrow -\varphi\) has to be accompanied by an additional canonical transformation \(p_\varphi \rightarrow p_\varphi - 2\alpha\). At the quantum level corresponding unitary transformation is generated by the operator \(U_\alpha(\varphi) = \exp(-2i\alpha \varphi)\). This is a well defined, \(2\pi\)-periodic operator only in the case of integer and half-integer values of the flux. For \(2\alpha \notin \mathbb{Z}\), as we shall see, discrete symmetry \(S\) is spontaneously broken.

### 3.2 Quantization and spectral properties

A canonical quantization of the system with a prescription of a symmetric ordering of non-commuting factors in quantum analog of \((3.5)\) results in the operators

\[
\hat{J}_1 = i \sin \varphi \left( \partial_\chi - \frac{1}{2} \tanh \chi \right) - \cos \varphi \tanh \chi \hat{J}_3, \quad \hat{J}_2 = -i \cos \varphi \left( \partial_\chi - \frac{1}{2} \tanh \chi \right) - \sin \varphi \tanh \chi \hat{J}_3, \quad \hat{J}_3 = -i \partial_\varphi + \alpha.
\]

They generate the \(so(2, 1)\) algebra,

\[
[\hat{J}_\mu, \hat{J}_\nu] = -i \varepsilon_{\mu
\nu\lambda} \hat{J}^\lambda.
\]

We assume that these operators act on the space of the \(2\pi\)-periodic in \(\varphi\) wave functions \(\psi(\chi, \varphi)\), where they are Hermitian with respect to a scalar product

\[
(\psi_1, \psi_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{0}^{2\pi} \psi_1^*(\chi, \varphi) \psi_2(\chi, \varphi) d\chi d\varphi.
\]
The quantum Hamiltonian
\[ \hat{H} = -\partial_{\chi}^2 - \frac{\hat{J}_3^2 - \frac{1}{4}}{\cosh^2 \chi} \] (3.18)
is obtained then from (3.9), where we put \(2R^2 = 1\), and subtract a quantum constant term \(\hbar^2 / 4\), i.e. we take
\[ \hat{H} = -\hat{J}_\mu \hat{J}^\mu - \frac{1}{4}. \] (3.19)

Note that the same realization of the \(so(2,1)\) generators and of the Hamiltonian are obtained if to proceed from the definition \(\hat{J}_\mu = -\epsilon_{\mu\nu\lambda} x^\nu \left(-i \partial_{\lambda} - \frac{2}{\xi} A_{\lambda} \right)\) written in the pseudospherical coordinates \((R, \chi, \varphi)\), \(R > 0\). In this way we get the \(so(2,1)\) generators in the form
\[ \tilde{J}_1 = i \sin \varphi \partial_{\chi} - \cos \varphi \tanh \chi \tilde{J}_3, \quad \tilde{J}_2 = -i \cos \varphi \partial_{\chi} - \sin \varphi \tanh \chi \tilde{J}_3, \quad \tilde{J}_3 = \tilde{J}_3. \] (3.20)
Instead of (3.18), with the same quantum constant shift, we obtain
\[ \tilde{H} = -\partial_{\chi}^2 - \tanh \chi \partial_{\chi} - \left(\tilde{J}_3^2 - \frac{1}{4}\right) \cosh^{-2} \chi. \] (3.21)

Operators \(\tilde{J}_\mu\) and \(\tilde{H}\), \([\tilde{J}_\mu, \tilde{H}] = 0\), are Hermitian with respect to a scalar product
\[ (\tilde{\psi}_1, \tilde{\psi}_2) = (2\pi)^{-1} \int_{-\infty}^{\infty} \int_0^{2\pi} \tilde{\psi}_1^* (\chi, \varphi) \tilde{\psi}_2 (\chi, \varphi) \cosh \chi d\chi d\varphi. \] (3.22)

A subsequent similarity transformation \(\tilde{\psi} \to f \tilde{\psi} = \psi, \tilde{J}_\mu \to f \tilde{J}_\mu f^{-1} = \hat{J}_\mu, \tilde{H} \to f \tilde{H} f^{-1} = \hat{H}\) with \(f = \sqrt{\cosh \chi}\) reduces the scalar product (3.22), the \(so(2,1)\) generators (3.20), and the Hamiltonian (3.21) to (3.17), (3.13)–(3.15), and (3.18), respectively.

Since Hamiltonian (3.18) is the (shifted) \(so(2,1)\) Casimir operator, one can choose a representation in which \(\hat{H}\) and the compact \(so(2,1)\) generator \(\hat{J}_3\) are diagonal. The stationary Schrödinger equation associated with the Hamiltonian (3.18) is separable in the variables \(\chi\) and \(\varphi\). The common wave functions of \(\hat{H}\) and \(\hat{J}_3\) can be factorized as
\[ \Psi_{E,m}^\alpha (\chi, \varphi) = e^{im\varphi} \psi_{E,m}^\alpha (\chi), \quad m = 0, \pm 1, \pm 2, \ldots, \] (3.23)
where the superscript marks the value of the Aharonov-Bohm flux. Then we have
\[ \hat{J}_3 \Psi_{E,m}^\alpha (\chi, \varphi) = j_3 \Psi_{E,m}^\alpha (\chi, \varphi), \quad j_3 = m + \alpha, \] (3.24)
and the Schrödinger equation
\[ \hat{H} \Psi_{E,m}^\alpha (\chi, \varphi) = E \Psi_{E,m}^\alpha (\chi, \varphi) \] (3.25)
is reduced to
\[ H_{m\alpha} \psi_{E,m}^\alpha (\chi) = E \psi_{E,m}^\alpha (\chi), \quad H_{m\alpha} = -\frac{d^2}{d\chi^2} - \frac{m_\alpha (m_\alpha + 1)}{\cosh^2 \chi}, \quad m_\alpha \equiv m + \alpha - \frac{1}{2}. \] (3.26)
The reduced Hamiltonian \(H_{m\alpha}\) is just the Pöschl-Teller Hamiltonian (2.1) with parameter \(m\) and variable \(x\) changed for \(m_\alpha\) and \(\chi\). Since the cases corresponding to the Aharonov-Bohm
fluxes $\alpha_1$ and $\alpha_2 = \alpha_1 + n$, $n \in \mathbb{Z}$, are related by a unitary transformation generated by the operator $U_{\alpha_2,\alpha_1}(\varphi) = e^{i\varphi}$, in what follows we assume without loss of generality that $0 \leq \alpha < 1$.

Define the linear combinations of the generators $\hat{J}_+ = \hat{J}_1 + i\hat{J}_2$ and $\hat{J}_- = \hat{J}^+_1$,

$$\hat{J}_+ = e^{i\varphi} \left( \frac{\partial}{\partial \chi} - \left( \hat{J}_3 + \frac{1}{2} \right) \tanh \chi \right), \quad \hat{J}_- = e^{-i\varphi} \left( -\frac{\partial}{\partial \chi} - \left( \hat{J}_3 - \frac{1}{2} \right) \tanh \chi \right). \quad (3.27)$$

These are the ladder operators of the $so(2,1)$ symmetry algebra, $[\hat{J}_3, \hat{J}_\pm] = \pm \hat{J}_\pm$, $[\hat{J}_+, \hat{J}_-] = -2\hat{J}_3$. Then for eigenstates $|E, m\rangle$ we have a relation

$$\hat{J}_3(\hat{J}_\pm \Psi^\alpha_{E,m}) = (j_3 \pm 1)(\hat{J}_\pm \Psi^\alpha_{E,m}). \quad (3.28)$$

On a subspace with $j_3 = m + \alpha$, the $\chi$-dependent parts of the ladder operators $\hat{J}_-$ and $\hat{J}_+$ correspond to the intertwining operators $(2.3)$. The values of the parameter $m_\alpha$ are non-integer in the case $\alpha \neq \frac{1}{2}$.

Let us discuss the spectral properties of the system in dependence on the strength of the magnetic flux. First, consider the case $\alpha \neq \frac{1}{2}$. From the form of the reduced Hamiltonian $(3.26)$ it follows that the quantum system $(3.18)$ contains the Pöschl-Teller subsystem with repulsive potential $U(\chi) = +\gamma^2 \cosh^{-2} \chi$, $\gamma^2 > 0$. For $\alpha = 0$ this happens in the subspace with $m = 0$, where $\gamma^2 = 1/4$. For $0 < \alpha < 1/2$ and $1/2 < \alpha < 1$, repulsive potential appears in the subspaces with $m = 0$ and $m = -1$, where $\gamma^2 = 1/4 - \alpha^2$ and $\gamma^2 = 1/4 - (1 - \alpha)^2$, respectively. Repulsive Pöschl-Teller system has no physical states with $E = 0$. In accordance with Eq. $(3.28)$, the continuous part of the spectrum of the quantum system $(3.18)$ with $\alpha \neq \frac{1}{2}$ is described by the scattering states with $E > 0$. The corresponding eigenstates with $E = k^2$, $k > 0$, can be expressed in terms of the hypergeometric function. Any Pöschl-Teller subsystem $(3.26)$ is characterized by a nonzero reflection coefficient $|38|

$$|r|^2 = \frac{1}{1 + \rho^2}, \quad \rho = \frac{\sinh \pi k}{\cos \pi \alpha}. \quad (3.29)$$

In accordance with $(3.19)$, on the scattering states with $E > 0$, infinite-dimensional unitary irreducible representations of the principal continuous series of the algebra $sl(2, R) \sim so(2,1)$ $|39|$ with $-\hat{J}_\mu \hat{J}^\mu = E + 1/4 > 1/4$ and $j_3 = \alpha + m$, $m = 0, \pm 1, \ldots$, are realized.

System $(3.18)$ with $\alpha \neq 1/2$ has also bound states of certain discrete negative energies. With the help of the relation

$$\hat{H} = \hat{J}_+ \hat{J}_- - (\hat{J}_3 - 1/2)^2 = \hat{J}_- \hat{J}_+ - (\hat{J}_3 + 1/2)^2, \quad (3.30)$$

cf. $(2.4)$ and $(2.5)$, a part of corresponding normalizable eigenstates is identified as the states annihilated by the ladder operators $\hat{J}_-$ and $\hat{J}_+$. These are

$$\Psi^\alpha_{E,m}(\chi, \varphi) = e^{im\varphi} \cosh^{-(m+\alpha-1/2)} \chi, \quad \hat{J}_- \Psi^\alpha_{E,m} = 0, \quad E = -(m + \alpha - 1/2)^2, \quad (3.31)$$

where $m = 1, 2, \ldots$ for $0 < \alpha < 1/2$, and $m = 0, 1, \ldots$ for $1/2 < \alpha < 1$, and

$$\Psi^\alpha_{E,m}(\chi, \varphi) = e^{im\varphi} \cosh^{m+\alpha+1/2} \chi, \quad \hat{J}_+ \Psi^\alpha_{E,m} = 0, \quad E = -(m + \alpha + 1/2)^2, \quad (3.32)$$
with \( m = -1, -2, \ldots \) for \( 0 \leq \alpha < 1/2 \), and \( m = -2, -3, \ldots \) for \( 1/2 < \alpha < 1 \). Infinite number of bound states with the same energy eigenvalues are obtained then by the action of the ladder operators \((\hat{J}_+)^n\) and \((\hat{J}_-)^n, n = 1, 2, \ldots\), on the states \((3.31)\) and \((3.32)\). At \( \alpha = 0 \) the discrete part of the spectrum reveals a symmetry with respect to the change \( j_3 \rightarrow -j_3 \), but it has no such a symmetry for \( \alpha \neq 0 \). On the states

\[
(\hat{J}_+)^n\Psi_{E,m}^- \quad \text{and} \quad (\hat{J}_-)^n\Psi_{E,m}^+, \quad n = 0, 1, \ldots,
\]

the half-bounded infinite-dimensional unitary representations of the discrete series of the \( sl(2,R) \) are realized. These representations are characterized by the value of the Casimir operator and eigenvalues of the compact generator \( \hat{J}_3 \), which are respectively \(-\hat{J}_\mu\hat{J}^\mu = -(m+\alpha)(m+\alpha-1), j_3 = m+\alpha+n, \) and \(-\hat{J}_\mu\hat{J}^\mu = -(m+\alpha)(m+\alpha+1), j_3 = m+\alpha-n. \)

The quantum spectrum of the system with \( \alpha = 0 \) is illustrated on Fig. 1. Note that the subspaces with \( j_3 \) and \(-j_3, j_3 \neq 0 \), are present symmetrically in the spectrum, while the subspace with \( j_3 = 0 \) is unpaired.

![Figure 1: Spectrum of the AdS_2 system with integer Aharonov-Bohm flux (\( \alpha = 0 \)).](image)

Now, we will focus on the case \( \alpha = 1/2 \). On the subspaces with \( m = 0 \) and \( m = -1 \), the dynamics reduces to that of the one-dimensional free quantum particle, while the set of the Hamiltonians \((3.26)\) with all the possible integer values \( m_{1/2} = m \) corresponds to the family of the RPT Hamiltonians \((2.21)\) satisfying the identity \((2.22)\),

\[
\hat{H}|_{j_3=\frac{1}{2}} = \hat{H}|_{j_3=-\frac{1}{2}} = H_0 = -\partial_x^2, \quad \hat{H}|_{j_3=m+\frac{1}{2}} = H_m.
\]

The eigenstates \( \Psi_{E,m}^{1/2} \) with \( m = 1, 2, \ldots \) and \( E = k^2 > 0 \) of the Hamiltonian \((3.19)\) are obtained from the free particle plane wave states \( e^{\pm ikx} (m = 0) \) by the action on them of the operator \((J_+)^m = e^{im\varphi}D_{-m}D_{-m+1}\ldots D_{-1} \). This corresponds exactly to relation \((2.11)\). The scattering states with negative values of \( m \) are produced by the action of the Hermitian conjugate operator \((J_-)^m, m = 1, 2, \ldots\), on \( e^{\pm ikx} \). In comparison with the case \( \alpha \neq 1/2 \), the system has additional states with \( E = 0 \), on which the half-bounded infinite-dimensional unitary representations of the \( sl(2,R) \) are realized. These states have a form of the eigenvectors \((3.33)\) constructed over the eigenstates of the form \((3.31)\) and \((3.32)\) with \( \alpha = 1/2 \), in which \( m = 0 \) and \( m = -1 \), respectively. The normalizable negative energy states are constructed in the same way over the eigenstates \((3.31)\) and \((3.32)\) with \( m = 1, 2, \ldots\) and
In this case all the energy levels in the spectrum shown on Fig. 2 have a double degeneration with respect to the reflection \( j_3 \rightarrow -j_3 \), cf. Fig. 1.

In terms of the ladder operators, second involutive automorphism from (3.10) takes the form

\[
S: (\hat{J}_\pm, \hat{J}_3) \rightarrow (-\hat{J}_\mp, -\hat{J}_3).
\]

(3.35)

Under this transformation quantum symmetry algebra (3.16) and Hamiltonian (3.19) are invariant. The described properties of the states of the quantum system mean that this discrete symmetry is spontaneously broken except the cases when the Aharonov-Bohm flux takes integer and half-integer values. In the case \( \alpha = 0 \) or \( \alpha = 1/2 \), this discrete symmetry transforms mutually the half-bounded infinite-dimensional representations of the so(2,1) realized on the states of the discrete spectrum (including the states \( E = 0 \) when \( \alpha = 1/2 \)).

4 Reduction and tri-supersymmetry

Let us identify now the exact analogs of the Crum-Darboux intertwining relations, and trace out how the tri-supersymmetric structure appears under the appropriate reduction of the AdS\(_2\) system in the presence of the Aharonov-Bohm effect with the half-integer flux \( (\alpha = 1/2) \). In conclusion of this section we shall comment on the case of the integer flux.

Having in mind factorization (3.23), we introduce notations \(|m\rangle = e^{im\phi}\) and \(\langle \psi_1 | \psi_2 \rangle = \frac{1}{2\pi} \int_0^{2\pi} \psi_1^* \psi_2 d\varphi\), so that \(\langle m | m' \rangle = \delta_{mm'}\). Taking into account explicit form of the ladder operators (3.27), we get the only nonzero matrix elements of \(\hat{J}_\pm\) and \(\hat{H}\),

\[
\begin{align*}
\langle m - 1 | \hat{J}_- | m \rangle &= -\mathcal{D}_m, & \langle m | \hat{J}_+ | m - 1 \rangle &= \mathcal{D}_m, \\
\langle m | \hat{H} | m \rangle &= -\mathcal{D}_m \mathcal{D}_m - m^2,
\end{align*}
\]

that correspond to (2.3) and (2.4). Then, with taking into account (4.1) and (4.2), the Crum-Darboux intertwining relation (2.8) is just the matrix element of the relation \([\hat{J}_-^{l+1}, \hat{H}] = 0\), \(l = 0, 1, \ldots\),

\[
\langle m - l - 1 | \hat{J}_-^{l+1} \hat{H} | m \rangle = \langle m - l - 1 | \hat{H} \hat{J}_-^{l+1} | m \rangle.
\]

(4.3)

The intertwining relation (2.9) is given by Hermitian conjugation, or, alternatively, is produced by the same operator relation taken between the states \(|-m-1\rangle\) and \(|-m+l\rangle\) with
subsequent use of the identity (2.2). Relation (4.3) taken for \( l = 2m \) (or, its Hermitian conjugate) together with the identity (2.2) produce Eq. (2.15) corresponding to the nontrivial odd order integral of the RPT system \( H_m \).

Hence, we have identified the origin of the intertwining relations associated with the RPT Hamiltonian (2.1). Since the commutativity of the operators \( \mathcal{X}_{m,l} \) and \( \mathcal{Y}_{m,l} \) with Hamiltonian \( \mathcal{H}_{m,l} \) is reduced to the intertwining relations, one can find what matrix elements of the operators \( \hat{J}_r \), with appropriately chosen integer \( r > 0 \), correspond to the operators \( X^\pm_{m,l} \), \( Y^\pm_{m,l} \) and \( Z^\pm_{m,l} \), from which the integrals \( \mathcal{X}_{m,l} \), \( \mathcal{Y}_{m,l} \) and \( \mathcal{Z}_{m,l} \) are composed. In correspondence with the discrete symmetry (3.35) we have

\[
\begin{align*}
-(-i)^m (-l-1) \langle \hat{J}_{m,l}^m \rangle &= -i^{m-1} \langle -l-1 | \hat{J}_{m,l}^{-m-l} | m-1 \rangle = X_{m,l}^- \quad (4.4) \\
-(-i)^m (-l-1) \langle \hat{J}_{m,l}^{-m-l} | m \rangle &= -i^{m+1} \langle l | \hat{J}_{m,l}^{m+l} | m-1 \rangle = Y_{m,l}^- \quad (4.5) \\
(-i)^m (-l-1) \langle \hat{J}_{m,l}^{-m+l} | m \rangle &= i^{m+1} \langle m | \hat{J}_{m,l}^{m+l} | l \rangle = Z_{m,l}^- \quad (4.6) \\
(-i)^m (-l-1) \langle \hat{J}_{m,l}^{m-l} | m \rangle &= i^{m+1} \langle l | \hat{J}_{m,l}^{m+l} | l-1 \rangle = Z_{m,l}^+ \quad (4.7)
\end{align*}
\]

Operators \( X^\pm_{m,l} \) and \( Y^\pm_{m,l} \) are obtained by Hermitian conjugation of (4.3) and (4.5). These relations are illustrated by Figures 3–7 for \( m = 3 \), \( l = 1 \).

![Figure 3: Action of the operators \( X^\pm_{3,1} \) in correspondence with (4.4).](image)

![Figure 4: Action of the operators \( Y^\pm_{3,1} \) in correspondence with (4.5).](image)

In order to reproduce the relations corresponding to (2.26) with the help of (4.4)–(4.7), we can use the identities

\[
\hat{J}_3^n \hat{J}_3^m = \prod_{k=0}^{n-1} \left( \hat{H} + \left( \hat{J}_3 - k - \frac{1}{2} \right)^2 \right) = P_n(\hat{H}, \hat{J}_3), \quad \hat{J}_3^{-n} \hat{J}_3^{-m} = P_n(\hat{H}, -\hat{J}_3), \quad (4.8)
\]
that follow from the \( so(2,1) \) algebra and Eq. (3.30). For instance, putting in the first identity \( n = 2m + 1 \), and computing a diagonal matrix element between the states \( |m\rangle \) and \( |m\rangle \), we reproduce the lower component case of the third relation from (2.26), i.e. \( (Z^{-}_{m,1})^2 = P_{2m+1}(H_m) \), where the polynomial is given by Eq. (2.17).

Therefore, the reduction of the \( \text{AdS}_2 \) system with half-integer Aharonov-Bohm flux \( (\alpha = 1/2) \) to an eigensubspace \( j_3 = m + \frac{1}{2} \) reproduces a bosonized nonlinear supersymmetry of the RPT system. On the Hilbert space composed from the two eigenspaces \( j_3 = m + \frac{1}{2} \) and \( j_3 = l + \frac{1}{2} \) with \( m \neq l \) we reveal the tri-supersymmetric structure of the extended RPT system (2.18). The key role for the nontrivial nonlinear supersymmetric structure is played here by the involutive automorphism (3.35) of the \( so(2,1) \) algebra, which is realized as a symmetry on the Hilbert space of the two-dimensional quantum system.

In the previous section we have seen that this discrete symmetry is not spontaneously broken also for integer values of the magnetic flux. In this case, however, the system is not reflectionless, and cannot be related to the one-dimensional quantum free particle system. Nevertheless, the intertwining relations corresponding to the Crum-Darboux transformations can be considered in this case as well, and they also can be related with the ladder operators. Using the intertwining relations for the case of half-integer values of the index \( m = n + \frac{1}{2}, n \in \mathbb{Z} \), and the symmetry (2.2), we find that the operator \( \mathcal{D}_{-m}\mathcal{D}_{-m+1}\cdots\mathcal{D}_{-1/2}\mathcal{D}_{1/2}\cdots\mathcal{D}_{m-1}\mathcal{D}_m \) of even order \( 2m + 1 = 2(n + 1) \) commutes with the Hamiltonian \( H_m \). A simple calculation
shows, however, that this operator reduces to a certain polynomial of order $n + 1$ in $H_{n+\frac{1}{2}}$. For instance, in correspondence with (2.3), $D_{-1/2}D_{1/2} = -H_{1/2} - 1/4$. Hence, there is no bosonized and tri-supersymmetric structure in the Pöschl-Teller system with a half-integer value of the parameter $m$.

5 On (super)conformal dynamical symmetry

As we observed in section 2, the nonlinear supersymmetry of the RPT system is generated by the Crum-Darboux transformations from the hidden bosonized $N = 2$ linear supersymmetry of the free non-relativistic particle. The latter system possesses also a non-relativistic conformal symmetry in the form of a dynamical $so(2,1)$ symmetry [29]. The same dynamical symmetry is present in some other quantum mechanical non-relativistic systems, including, in particular, the conformal mechanics model [30], and the model of a particle in the field of a magnetic vortex [32, 40]. The latter corresponds to the planar Aharonov-Bohm effect. The conformal mechanics model, like the RPT system, is related to the free particle by a Crum-Darboux transformation [41]. The conformal $so(2, 1)$ dynamical symmetry of all these systems can be explained within the framework of the AdS/CFT correspondence, see [35] and [37].

In this section, we construct analogs of the free particle generators of the $so(2,1)$ symmetry for the RPT system, and investigate their properties. The hidden bosonized $N = 2$ linear supersymmetry of the free particle can be unified with its dynamical $so(2,1)$ symmetry to produce a linear bosonized superconformal $osp(2|2)$ symmetry, see [41]. We also discuss a realization of the $osp(2|2)$ supersymmetry in the RPT system.

Before we pass over to the discussion of the specified problem, let us make two notes. The non-relativistic AdS/CFT correspondence appears between a relativistic theory in the AdS space-time of $d+3$ dimensions, and non-relativistic theories in $(d+1)$-dimensional space-time [37]. We exploited here the relation between the non-relativistic particle system on the AdS$_2$, whose evolution is described by an external time variable, and the non-relativistic $(1+1)$-dimensional RPT system. Next, in the AdS$_2$ system, the $so(2,1)$ isometry corrected by the magnetic flux is represented by the true, time-independent integrals of motion. Hamiltonian (3.30) is the Casimir operator of this $so(2,1)$ symmetry, whose generators are differential operators of the first order. In contrary, the generators of the conformal dynamical symmetry of the free particle we consider below are represented by differential operators of the second order.

The free particle is characterized by the Schrödinger symmetry [29] Lie algebra with the following nonzero commutation relations,

$$[D_0, H_0] = 2iH_0, \quad [H_0, K_0] = -4iD_0, \quad [D_0, K_0] = -2iK_0,$$

$$[B_0, P_0] = i \cdot 1, \quad [D_0, P_0] = iP_0, \quad [D_0, B_0] = -iB_0,$$

$$[B_0, H_0] = 2 iP_0, \quad [P_0, K_0] = -2iB_0,$$

see also [42]. Here

$$P_0 = -i \frac{d}{dx}, \quad B_0 = x - 2P_0 t, \quad H_0 = P_0^2, \quad D_0 = \frac{1}{2} \{P_0, B_0\}, \quad K_0 = B_0^2.$$
In the first relation from (5.2) the unit operator corresponds to the mass put here equal to one-half, which is a central element of the algebra. The boost generator \( B_0 \), and the generators of dilatations, \( D_0 \), and special conformal transformations (expansions), \( K_0 \), depend explicitly on time. This means that the Schrödinger symmetry is dynamical. Its subalgebra \((5.1)\) is the conformal \( so(2,1) \) dynamical symmetry.

Reflection operator \( R \), being integral of motion, anticommutes not only with the \( P_0 \), but also with the time-dependent integral \( B_0 \). Therefore, the free particle can be characterized by the dynamical \( osp(2|2) \) superconformal symmetry as well. Its superalgebra is given by the \( so(2,1) \) commutation relations \((5.1)\) supplied with the nontrivial (anti)-commutation relations

\[
\{Q_a^{(0)}, Q_b^{(0)}\} = 2\delta_{ab}H_0, \quad \{S_a^{(0)}, S_b^{(0)}\} = 2\delta_{ab}K_0, \quad \{Q_a^{(0)}, S_b^{(0)}\} = 2\delta_{ab}D_0 - \epsilon_{ab}\Sigma, \quad (5.5)
\]

\[
[H_0, S_a^{(0)}] = -2iQ_a^{(0)}, \quad [K_0, Q_a^{(0)}] = 2iS_a^{(0)}, \quad (5.6)
\]

\[
[D_0, Q_a^{(0)}] = iQ_a^{(0)}, \quad [D_0, S_a^{(0)}] = -iS_a^{(0)}, \quad (5.7)
\]

\[
[\Sigma, Q_a^{(0)}] = 2i\epsilon_{ab}Q_b^{(0)}, \quad [\Sigma, S_a^{(0)}] = 2i\epsilon_{ab}S_b^{(0)}. \quad (5.8)
\]

Here

\[
Q_1^{(0)} = P_0, \quad Q_2^{(0)} = iRQ_1^{(0)}, \quad S_1^{(0)} = B_0, \quad S_2^{(0)} = iRS_1^{(0)}, \quad \Sigma = -R. \quad (5.9)
\]

Suppose now that we have two quantum almost isospectral systems related by the Crum-Darboux intertwining relations \((2.7)\). Let a system \( H \) possesses an integral of motion \( \mathcal{A} \) which can be time dependent, \( \frac{d}{dt}\mathcal{A} = \frac{\partial}{\partial t}\mathcal{A} - i[\mathcal{A}, H] = 0 \). Time-independence of the Crum-Darboux generator \( A_n \), and the intertwining relations \((2.7)\) allow us to construct an analog of the integral \( \mathcal{A} \) for the system \( \tilde{H} \),

\[
\tilde{\mathcal{A}} = A_n\mathcal{A}A_n^\dagger, \quad \frac{d}{dt}\tilde{\mathcal{A}} = \frac{\partial}{\partial t}\tilde{\mathcal{A}} - i[\tilde{\mathcal{A}}, \tilde{H}] = 0. \quad (5.10)
\]

It depends on time if and only if \( \mathcal{A} \) is time dependent as well. Identifying \( H \) and \( \tilde{H} \) with \( H_0 \) and \( H_m \), and \( A_n \) with the operator \( D_m \ldots D_{-1} \equiv X_m^\dagger - i^m X_{m,0}^\dagger \), see Eqs. \((2.3)\) and \((2.20)\), we find the analogs of the even, \( D_0 \), \( K_0 \), and odd, \( Q_0^{(0)} \), \( S_0^{(0)} \), free particle integrals for the RPT system described by the Hamiltonian \((2.1)\). We denote these analogs just by changing the index 0 for \( m \). The integral \( \Sigma \), defined in \((5.9)\), is also the integral for the system \( H_m \).

Let us look now what happens with the dynamical \( so(2,1) \) symmetry in the RPT system. A direct computation gives the commutation relations

\[
[D_m, H_m] = 2iH_mP_m(H_m), \quad [K_m, H_m] = 4iD_m \quad (5.11)
\]

where in correspondence with \((2.26), (2.27)\), \( P_m(H_m) = X_m^\dagger X_m = \prod_{n=0}^{m-1} (H_m - E_{m;n}) \). To find \([D_m, K_m]\), we use the identity \( D_0^2 = H_0K_0 + 2iD_0 + \frac{\epsilon}{4} \), and get

\[
[D_m, K_m] = a_m(H_m)K_m + b_m(H_m)D_m + c_m(H_m) \quad (5.12)
\]

where \( a_m, b_m \) and \( c_m \) are some polynomials in \( H_m \). At this level, like in the case of the nonlinear supersymmetry analyzed in the previous sections, the Lie algebra of the conformal symmetry of the free particle \((5.1)\) is just deformed by the polynomials in the Hamiltonian.
of the corresponding RPT system. The complete algebraic structure, however, is more complicated here. Unlike the supercharges $Q_a^{(m)} = Z_a$, the dynamical integrals $D_m$ and $K_m$ do not commute with the Hamiltonian $H_m$. As a consequence, the repeated commutators produce polynomials in all the three generators $H_m, D_m, K_m$. To illustrate this phenomenon, consider the simplest case of $m = 1$. The reflectionless Pöschl-Teller Hamiltonian $H_1 = -d^2/dx^2 - 2/cosh^2 x$ is intertwined with the free particle Hamiltonian by the first order differential operator $X_1 = D_1 = d/dx + tanh x$. The operators $D_1, H_1$ and $K_1$ satisfy the commutation relations

$$[K_1, H_1] = 4iD_1,$$

$$[D_1, H_1] = 2iH_1(H_1 + 1),$$

$$[D_1, K_1] = -2iK_1 - 2i\{H_1, K_1\} - 3i(H_1 + 1).$$

Denoting $\mathbb{K}_1 \equiv [D_1, K_1]$, we find

$$[\mathbb{K}_1, K_1] \equiv \mathcal{M}_1 = -8\{K_1, D_1\} - 12D_1,$$

and

$$[\mathcal{M}_1, K_1] = 32i\{K_1^2, H_1\} + 32iK_1^2 + 176i\{K_1, H_1\} + 200iK_1 + 228i(H_1 + 1).$$

From the displayed structure it is clear that despite the nonlinear character of the algebra generated by the $H_1, D_1$ and $K_1$, all the (repeated) commutators are polynomials in them. No new independent integrals do appear.

This ceases to be true once we incorporate the fermionic analogs of the $osp(2|2)$ generators. Besides the deformation effect we observed above, the commutation relations between bosonic and fermionic operators produce new fermionic dynamical integrals, not reducible to the products of the fermionic generators $Q_a^{(m)}$, $S_a^{(m)}$, and of the bosonic generators. Nevertheless, the number of new fermionic integrals is finite. It depends on the value of the parameter $m$, and therefore, on the order of the Crum-Darboux generating operator $X_a$. To illustrate this phenomenon, again, restrict ourselves by the simplest case of the RPT system with $m = 1$. In addition to the already specified (repeated) commutation relations between bosonic operators $H_1, D_1$ and $K_1$, we get the following (anti)-commutation relations,

$$\{Q_a^{(1)}, Q_b^{(1)}\} = 2\delta_{ab}H_1(H_1 + 1)^2,$$

$$\{S_a^{(1)}, S_b^{(1)}\} = \delta_{ab} (\{K_1, H_1\} + 2(K_1 + H_1 + 1)),

\{S_a^{(1)}, Q_b^{(1)}\} = \delta_{ab} (\{D_1, H_1\} + 2D_1) - \epsilon_{ab} \Sigma(3H_1 + 1)(H_1 + 1),

\{J_a^{(1)}, J_b^{(1)}\} = \frac{1}{8} \delta_{ab} (K_1H_1 K_1 - \{K_1, H_1\} - 2(H_1 + 1)),

\{S_a^{(1)}, J_b^{(1)}\} = \frac{1}{4} \delta_{ab} (\{K_1, D_1\} + 2D_1) + \frac{1}{8} \epsilon_{ab} \Sigma (\{K_1, H_1\} - 2K_1 + 2(H_1 + 1)),

\{J_a^{(1)}, Q_b^{(1)}\} = \frac{1}{4} \delta_{ab} (2H_1K_1H_1 + \{K_1, H_1\} - 5H_1^2 - 4H_1 + 1) - \frac{1}{4} \epsilon_{ab} \Sigma (3\{D_1, H_1\} + 2D_1),

$$[K_1, Q_a^{(1)}] = 3i\{S_a^{(1)}, H_1\} + 2iS_a^{(1)},$~ $[K_1, S_a^{(1)}] = 8iJ_a^{(1)},$~ $[K_1, J_a^{(1)}] = \frac{i}{4} (K_1, S_a^{(1)}),$$
\[ [H_1, Q_a^{(1)}] = 0, \quad [H_1, S_a^{(1)}] = -2iQ_a^{(1)}, \quad [H_1, J_a^{(1)}] = -\frac{i}{2}\{S_a^{(1)}, H_1\}, \quad (5.23) \]

\[ [D_1, Q_a^{(1)}] = iQ_a^{(1)}(1 + 3H_1), \quad [D_1, S_a^{(1)}] = -\frac{i}{2}(2S_a^{(1)} + \{S_a^{(1)}, H_1\}) \quad (5.24) \]

\[ [D_1, J_a^{(1)}] = -\frac{3i}{2}\{J_a^{(1)}, H_1\} - iJ_a^{(1)} - iQ_a^{(1)}, \quad (5.25) \]

\[ [\Sigma, Q_a^{(1)}] = 2i\epsilon_{ab}Q_b^{(1)}, \quad [\Sigma, S_a^{(1)}] = 2i\epsilon_{ab}S_b^{(1)}, \quad [\Sigma, J_a^{(1)}] = 2i\epsilon_{ab}J_b^{(1)}. \quad (5.26) \]

Here

\[ J_1^{(1)} = X_1^\dagger J_1^{(0)}X_1, \quad J_2^{(1)} = iRJ_1^{(1)}, \quad (5.27) \]

are the two new fermionic integrals generated by the commutator of \( K_1 \) with \( S_a^{(1)} \), see (5.22). They correspond to the dynamical integral \( J_1^{(0)} = \frac{1}{8}\{K_0, Q_1^{(0)}\} \) of the free particle. From the point of view of the closed Lie superalgebraic structure of the free particle model, the operator \( J_1^{(0)} \) belongs to the universal enveloping algebra of the \( osp(2|2) \).

The displayed set of the anti-commutation relations shows, nevertheless, that no other new independent integrals will be generated by the repeated anticommutation relations. Note that a similar picture corresponding to the deformation and extension of the superconformal \( osp(2|2) \) symmetry appears in the fermion-monopole system [43]. This happens also in the superconformal mechanics model when a classical boson-fermion coupling constant \( \alpha \) is changed for \( n\alpha \), where \( n > 1 \) is integer, see [44].

We do not touch here the question of relation of the deformed (and extended) conformal \( so(2,1) \) (and superconformal \( osp(2|2) \)) dynamical symmetry of the RPT system to the \( so(2,1) \) symmetry of the particle on the \( AdS_2 \). Let us note only that the generators of the conformal and superconformal dynamical symmetries of the RPT system could be related to the oscillator-like operators \( P_+ = e^{i\varphi}\hat{J}_+ \) and \( P_- = P_+^\dagger = \hat{J}_-e^{-i\varphi} \) of the \( AdS_2 \) system. These two operators commute with the operator \( \hat{J}_3 \), and represent the observable operators under reduction to the one-dimensional RPT system. The operators \( P_\pm \) are not integrals of motion for the original two-dimensional system due to a nontrivial dynamics of the angular variable \( \varphi \). Nevertheless, the true and also dynamical integrals of motion of the RPT system can be obtained from them after reduction. Investigation of this aspect lies, however, beyond the scope of the present paper.

### 6 Discussion and outlook

We showed that the hidden bosonized nonlinear supersymmetry of the reflectionless Pöschl-Teller system and tri-supersymmetric structure of the pair of the RPT systems have an origin in the Aharonov-Bohm effect for non-relativistic particle on the \( AdS_2 \). Both supersymmetric structures are based on the two involutive automorphisms of the \( so(2,1) \) algebra of the \( AdS_2 \) isometry with generators corrected by the magnetic flux \( \alpha \). One of these automorphisms, \( R \), corresponds to a reflection of a space-like coordinate of the ambient Minkowski space. It is the symmetry for any value of the flux. Another automorphism, \( S \), corresponds to a reflection of one of the time-like coordinates of the ambient space. This discrete symmetry is unbroken only in the case of integer and half-integer values of the flux.
Classically, the system is subject to the Pöschl-Teller potential \( V(\chi; J_3) = -J^2_3 \cosh^{-2} \chi \) in the noncompact AdS\( _2 \) coordinate \( \chi \). It acquires a correction term in the quantum framework, \( V = -(J^2_3 - \frac{1}{4}) \cosh^{-2} \chi \). The classical free-particle dynamics in \( \chi \) takes place for zero value of the axial angular momentum shifted by the magnetic flux, \( J_3 = 0 \). It can be recovered on the quantum level for half-integer values of the magnetic flux only. In this case, the dynamics of any partial wave of the particle on the AdS\( _2 \) is governed by the reflectionless Pöschl-Teller potential, which turns into zero in two subspaces \( j_3 = \pm \frac{1}{2} \). The complete two-dimensional quantum dynamics on the AdS\( _2 \) is hence reflectionless. This is not the case for the other values of the flux. For \( \alpha \neq \frac{1}{2} + n \), the dynamics of some partial waves is subject to the repulsive potential. Then this information is transmitted to all other partial wave sectors by the \( so(2,1) \) ladder operators, which play the role of the generators of the Crum-Darboux transformations. So, the two-dimensional dynamics ceases to be reflectionless in this case.

For half-integer values of the flux, all the spectrum of the two-dimensional quantum system acquires an additional double degeneration related to the Hamiltonian symmetry \( J_3 \rightarrow -J_3 \) generated by the \( S \). Such a degeneration is absent for the unpaired partial wave sector \( j_3 = 0 \) in the case of integer flux. It is the additional double degeneration that is behind the existence of the nontrivial nonlinear supersymmetric structure in the RPT system appearing after reduction by the compact \( so(2,1) \) generator \( J_3 \). Generator of the other discrete symmetry, \( R \), is transformed then into the \( \mathbb{Z}_2 \)-grading operator of the hidden bosonized supersymmetry of the RPT system. The supersymmetry generators of the reduced system correspond to certain powers of the \( so(2,1) \) ladder operators of the original two-dimensional system with half-integer flux.

Note that the special “magic” of half-integer values of the flux and related degeneracy were discussed in a different context in \( \cite{45} \).

The described correspondence between the non-relativistic AdS\( _2 \) Aharonov-Bohm effect and the one-dimensional Pöschl-Teller system for a generic magnetic flux value case is somewhat of the AdS/CFT holography \( \cite{27} \) nature. This observation deserves a deeper study that can be useful in the context of the integrable nonlinear equations.

For the peculiar case of half-integer values of the flux, we considered this aspect in the context of non-relativistic AdS/CFT \( \cite{36, 37} \). Namely, we showed, that the analogs of the free particle generators of the dynamical conformal, \( so(2,1) \), and bosonized superconformal, \( osp(2|2) \), symmetries can be transferred to the RPT system by the Crum-Darboux transformation. In the case of the conformal \( so(2,1) \) dynamical symmetry, there appears a certain nonlinear deformation of the \( so(2,1) \) algebra. In the case of the superconformal \( osp(2|2) \) symmetry, in addition to a deformation, finite number of new fermionic integrals is generated. These dynamical symmetries of the RPT system need a further investigation.

The conformal \( so(2,1) \) dynamical symmetry of the planar Aharonov-Bohm effect finds a natural explanation in the AdS/CFT context \( \cite{35, 37} \). The bound state Aharonov-Bohm effect, that corresponds to a particle confined to a circle pierced by the Aharonov-Bohm flux, is characterized by the hidden bosonized supersymmetry as well \( \cite{18} \). It would be interesting to look for the hidden bosonized \( N = 2 \) supersymmetry and bosonized superconformal \( osp(2|2) \) dynamical symmetry in the planar Aharonov-Bohm effect \( \cite{46} \).

Nonlinear bosonized supersymmetry of the RPT system and tri-supersymmetry of the extended RPT system are related to the nontrivial Lax operator which is the higher order differential operator. As we saw, this operator is the first order integral \( \frac{d}{dx} \) of the free
particle transferred to the RPT system by the Crum-Darboux transformation. Its square gives a spectral polynomial of the RPT system, that has double roots and describes a degenerate hyperelliptic curve \[14\]. This degeneration originates from the infinite-period limit applied to the finite-gap periodic Lamé, or associated Lamé system. It is this limit that produces the RPT system with its unique, pure imaginary period from the double periodic model. The spectral polynomial of the associated Lamé system is non-degenerate. It was showed in \[23, 24\] that such a finite-gap periodic system is also characterized by the hidden bosonized nonlinear supersymmetry, while the pair of such systems is described by the tri-supersymmetric structure. In the case of the finite-gap periodic systems, Crum-Darboux transformations relate the systems with the same number of gaps. Hence, an \( n \)-gap \( (n \in \mathbb{N}) \) periodic system can not be related by Crum-Darboux transformation to the quantum free particle model, which corresponds to the simplest case of a zero-gap periodic system. Then a question appears: is there any analog of the Aharonov-Bohm effect explanation for a hidden bosonized supersymmetry and related tri-supersymmetric structure in the finite-gap periodic systems? We are going to investigate this question elsewhere.

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