A NOTE ON THE UNIFORMITY OF THE CONSTANT IN THE POINCARÉ INEQUALITY

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Dedicated to A. Ambrosetti, a guide who definitively changed my life

Abstract. The classical Poincaré inequality establishes that for any bounded regular domain \( \Omega \subset \mathbb{R}^N \) there exists a constant \( C = C(\Omega) > 0 \) such that

\[
\int_{\Omega} |u|^2 \, dx \leq C \int_{\Omega} |\nabla u|^2 \, dx \quad \forall u \in H^1(\Omega), \quad \int_{\Omega} u(x) \, dx = 0.
\]

In this note we show that \( C \) can be taken independently of \( \Omega \) when \( \Omega \) is in a certain class of domains. Our result generalizes previous results in this direction.

1. Introduction

This paper is concerned with the following classical result, known as Poincaré inequality (or Poincaré-Friedrichs inequality):

**Theorem 1.1.** Let \( \Omega \subset \mathbb{R}^N \) be a bounded domain satisfying the interior cone condition, and \( p > 1 \). Then, there exists a constant \( C = C(\Omega) > 0 \) such that

\[
\int_{\Omega} |u|^p \, dx \leq C \int_{\Omega} |\nabla u|^p \, dx \quad \forall u \in W^{1,p}(\Omega), \quad \int_{\Omega} u(x) \, dx = 0.
\]

The proof is very simple and standard. Assume, reasoning by contradiction, that \( u_n \in W^{1,p}(\Omega) \) is a sequence such that

\[
\int_{\Omega} u_n \, dx = 0, \quad \int_{\Omega} |u_n|^p \, dx = 1, \quad \int_{\Omega} |\nabla u_n|^p \, dx \to 0.
\]

By the Rellich-Kondrachov theorem, one obtains that \( u_n \rightharpoonup u \), where:

\[
\int_{\Omega} u \, dx = 0, \quad \int_{\Omega} |u|^p \, dx = 1, \quad \int_{\Omega} |\nabla u|^p \, dx = 0.
\]

But if \( \nabla u = 0 \) almost everywhere in a domain, then \( u \) must be constant (see Chapter 9 of [5], for instance). And this yields the desired contradiction.

As one can observe, two main ingredients come into play in the proof: first, the compactness of the \( L^p \) embedding. For that one needs the interior cone condition on \( \Omega \) and its boundedness. Second, the connectedness of the domain.

In this paper we try to answer the following question: under which conditions on \( \Omega \) can we assure the existence of \( C \) independently of the set \( \Omega \)? The above discussion suggests that one should need three kind of assumptions:

1. a uniform bound on \( \Omega \),
2. an interior cone condition (with a fixed cone),
3. and, in a certain sense, a uniform connectedness assumption on \( \Omega \).

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The goal of this paper is to prove that this is indeed the case.

The uniformity of the Poincaré constant is useful in some free boundary problems or shape optimization problems, see [14, 15]. Our motivation comes from the derivation of a uniform local version of the Moser-Trudinger inequality, which is very useful in the variational study of the mean field equation on compact surfaces (see [7], and also [11], [12]). We hope to use the results of this paper to give a proof for such inequality in a forthcoming work.

Before going on with our exposition, let us mention some previous results. In the literature there are some results concerning the dependence of the Poincaré constant on $\Omega$: see for instance [8, 17]. However, the assumptions required there involve uniform $C^1$ regularity on $\Omega$ and other conditions. Here we are interested in a less regular framework. On the contrary, in [8, 17] more information is given, related to the solution of the associated Neumann eigenvalue problem.

In [4] the uniformity of the Poincaré constant is proved for uniformly Lipschitz domains. Let us point out that Lipschitz domains must lie (locally) on one side of the boundary. The unit ball with a segment removed, for instance, is not Lipschitz. Moreover, domains with interior cusps are not covered, for instance. In both cases the Poincaré inequality holds, which makes one think that the uniform Lipschitz assumption is too restrictive.

Throughout the paper we shall use the following notation:

$$\forall \varepsilon > 0, \quad \Omega^\varepsilon = \{ x \in \Omega : d(x, \partial \Omega) > \varepsilon \}.$$

We will consider the following hypotheses.

(h1) There exists $R > 0$ such that $\Omega \subset B(0, R)$.

(h2) There exists a fixed finite cone $C$ such that each point $x \in \partial \Omega$ is the vertex of a cone $C_x$ congruent to $C$ and contained in $\Omega$.

(h3) There exists $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0)$, $\Omega^\delta$ is a connected set.

This last condition is a sort of “uniform connectedness assumption”. We will show in Example 2.6 that a condition in this direction is necessary in our results. Roughly speaking, condition (h3) avoids the existence of arbitrarily narrow junctions between two regions in $\Omega$.

The main result of this note is the following.

**Theorem 1.2.** For any $p > 1$ there exists a constant $C > 0$ such that for any domain $\Omega$ satisfying (h1)-(h3),

$$\int_{\Omega} |u|^p \, dx \leq C \int_{\Omega} |\nabla u|^p \, dx \quad \forall u \in W^{1,p}(\Omega), \quad \int_{\Omega} u(x) \, dx = 0.$$

Here $C$ depends only on $p, R, \delta_0$ and the cone $C$ given in conditions (h1)-(h3).

We can also give some generalizations of this result, that will be discussed in Section 3. In particular, we can prove the uniform estimate in the following situation:

**Corollary 1.3.** For any $p > 1$, $R > 0$ and $r > 0$ there exists a constant $C > 0$ such that for any connected set $K \subset B(0, R)$,

$$\int_{K_r} |u|^p \, dx \leq C \int_{K_r} |\nabla u|^p \, dx \quad \forall u \in W^{1,p}(K_r), \quad \int_{K_r} u(x) \, dx = 0.$$

Here $K_r$ denotes the domain $K_r = \{ x \in \mathbb{R}^N : d(x, K) < r \}$. Observe that $C$ is independent of the choice of $K$. 
The uniformity of the Poincaré constant on this type of sets was the original motivation of our work. We think that this corollary could be one of the ingredients of a future local version of the Moser-Trudinger inequality.

Our results improve that of [4] since a connected and uniformly Lipschitz domain satisfy our conditions (h1)-(h3) (see the final Appendix). Moreover, we can cover many situations in which the domains are not Lipschitz, as explained above.

Our proof is also quite different from that of [4], which uses different notions of convergence of sets to pass to a limit in an argument by contradiction. Here we give a direct proof, based on two main tools. The first one is the following property:

(Q) For any \( \varepsilon > 0 \), there exists \( \delta > 0 \) independent of \( \Omega \) such that \( |\Omega \setminus \Omega^\delta| < \varepsilon \).

Let us point out that any domain \( \Omega \) satisfies that condition with \( \delta \) depending on \( \Omega \); this is due to the continuity from above of the Lebesgue measure. We shall see that conditions (h1) and (h2) imply the uniformity of the choice of \( \delta \).

The second ingredient of our arguments is the observation on the uniformity of the constant in the Sobolev embedding under our hypotheses.

The rest of the paper is organized as follows. In Section 2 we first give some preliminary results, some of them well-known, which are useful later in the proof of Theorem 1.2. Section 3 is devoted to the discussion of some extensions of our results. In particular, Corollary 1.3 will be proved in Section 3. In a final Appendix we show that uniformly lipschitz domains, bounded and connected, satisfy conditions (h1)-(h3). This implies that Theorem 1.2 generalizes the result of [4].

2. Main result

In this section we first give some preliminary results, which will be of use later. After that, we shall prove Theorem 1.2.

Some notation is in order; given \( A \subset \mathbb{R}^N \) and \( r > 0 \), we define:

\[
A^r = \{ x \in \Omega : d(x, \partial A) > r \},
A_r = \{ x \in \mathbb{R}^N : d(x, A) < r \}.
\]

We also need a notion of uniformity for the Lipschitz regularity of a domain. The following definition follows closely that of Chenais [8].

We first introduce some notation. Given \( z \in \mathbb{R}^N \), we denote \( z = (\hat{z}, z_N) \) where \( \hat{z} = (x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1} \) and \( z_N \in \mathbb{R} \) give the coordinates of \( z \) in a given coordinate system.

**Definition 2.1.** We say that \( \Omega \) is a uniform Lipschitz domain with constants \( M > 0 \), \( \gamma > 0 \) (and we denote \( \Omega \in Lip(M, \gamma) \)) if the following is satisfied:

For all \( x \in \partial \Omega \), there exists a local coordinate system satisfying what follows. There exists a function \( \varphi_x : U_{\hat{z}} \to \mathbb{R} \) a Lipschitz function with Lipschitz constant smaller than \( M \) such that:

\[
y \in O_x \cap \Omega \Leftrightarrow y \in O_{\hat{z}} \text{ and } y_N > \varphi_x(\hat{y}).
\]

Here,

\[
U_{\hat{z}} = \{ \hat{z} \in \mathbb{R}^{N-1} : |z_i - x_i| < \gamma \},
O_x = \{ z \in \mathbb{R}^N : |z_i - x_i| < \gamma \text{ for } i = 1 \ldots N - 1, |z_N - x_N| < M\gamma(n-1)^{1/2} \}.
\]

Observe that for uniformly bounded domains, this definition coincides with the definition of strong local Lipschitz property of Adams ([1], page 66). This property is equivalent to the condition assumed in [4], see Section 3 of [8].
Now let us recall and comment the following result, due to Gagliardo [13] (alternatively, see Adams [1], Theorem 4.8).

**Theorem 2.2.** Let $\Omega$ a bounded domain satisfying the cone condition (h2). Then there exists a finite collection $\Omega_1, \ldots, \Omega_k$ of open subsets of $\Omega$ such that:

1. $\Omega = \bigcup_{i=1}^k \Omega_i$.
2. $\Omega_i$ is a Lipschitz domain.

**Remark 2.3.** By having a look at the proof of the above result ([1], Theorem 4.8), one obtains easily that the Lipschitz constants of the graphs depend only on the cone $C$. Moreover, the number of sets $A_j$ in the proof can be bounded in terms of $C$ and $R$ of condition (h1). This implies the following uniform version: under conditions (h1) and (h2), then $\Omega_i \in \text{Lip}(M, \gamma)$, and $k, M, \gamma$ are positive constants depending only on the cone $C$ and the constant $R$ given by conditions (h1) and (h2).

In the following proposition we prove property (Q), which will be essential in our proof of Theorem 1.2.

**Proposition 2.4.** Let us fix $R > 0$ and a finite cone $C$. Then, property (Q) holds for any $\Omega$ satisfying (h1) and (h2) referred to $R$ and $C$ respectively.

**Proof.** We will prove the proposition in two steps:

**Step 1:** The case $\Omega \in \text{Lip}(M, \gamma)$.

This is straightforward. Indeed, consider a graph like in Definition 2.1, and take $t > 0$ sufficiently small. Define

$$W = \{ y \in O_x : y_N > \varphi_x(\hat{y}) + t \}$$

It is clear that for any point $(\hat{y}, \varphi_x(\hat{y}) + t)$, the distance to the boundary of $\Omega$ is at most $t/M$, where $M$ is the Lipschitz constant of $\varphi_x$. Therefore,

$$|O_x \setminus \Omega^{t/M}| \leq |O_x \cap \Omega \setminus W| \leq \gamma^{N-1}t.$$  

Observe also that the number of graphs needed to cover $\Omega$ is uniformly bounded. So it suffices to take $t$ sufficiently small, according to $\varepsilon$. Indeed, it follows that the dependence of $\delta$ on $\varepsilon$ is linear.

**Step 2:** The general case, with $\Omega$ satisfying (h1), (h2).

By 2.2, $\Omega \subset \bigcup_{i=1}^k \Omega_i$ where each $\Omega_i \in \text{Lip}(M, \gamma)$. Obviously, $\partial \Omega \subset \bigcup_{i=1}^k \partial \Omega_i$. Then, it is easy to show that:

$$\Omega \setminus \Omega^\delta \subset \bigcup_{i=1}^k \left( \Omega_i \setminus \Omega_i^\delta \right).$$

Then, we can conclude by Step 1. Since $M, \gamma, k$ depend only on $C$ and $R$, we obtain the uniform estimate.

Another important tool in our proof is the classical Sobolev embedding. The uniformity of the Sobolev constant is important for our purposes and not so well-known, so let us state it in detail. For a proof, see [1], Lemmas 5.10 and 5.15, and also Theorem 8.25 (observe the remark at the end of the proof of this last theorem).

**Proposition 2.5.** Let $\Omega \subset \mathbb{R}^N$ satisfy (h1), (h2), and $p > 1$. Then there exists $C_S > 0$ depending only on $R, p, C$ and $N$ such that:

1. If $N > p$, $\|u\|_{L^q(\Omega)} \leq C_S \|u\|_{W^{1,p}(\Omega)}$, for any $u \in W^{1,p}(\Omega)$, $q \in [p, \frac{Np}{N-p}]$. 

(2) If $N = p$, $\|u\|_{L^p(\Omega)} \leq C_\varepsilon \|u\|_{W^{1,p}(\Omega)}$, for any $u \in W^{1,p}(\Omega)$, $q \in [2, +\infty)$. Even more, there holds: $\|u\|_{L^p(\Omega)} \leq C_\varepsilon \|u\|_{W^{1,p}(\Omega)}$, where $\|\cdot\|_{L^p(\Omega)}$ is the Orlicz norm associated to $A(t) = e^{\frac{t^{N-1}}{N-1}} - 1$.

(3) If $N < p$, then any function $u \in W^{1,p}(\Omega)$ is equal almost everywhere to a continuous function. Moreover, $\|u\|_{L^\infty(\Omega)} \leq C_\varepsilon \|u\|_{W^{1,p}(\Omega)}$.

So far, we have considered conditions (h1) and (h2). Condition (h3) is less standard and can be considered as a uniform connectedness condition. In the following example we show that this condition is indeed necessary for the thesis of the Theorem 1.2.

**Example 2.6.** In this example we consider $N \geq 3$, and we write the coordinates of a point in $\mathbb{R}^N$ as $x = (\hat{x}, x_N)$, with $\hat{x} \in \mathbb{R}^{N-1}$ and $x_N \in \mathbb{R}$. Let us define the cone:

$$C = \{x \in \mathbb{R}^N; 0 < x_N < 1, |\hat{x}| < x_N\}.$$  

For any $\varepsilon > 0$, we define the set:

$$\Omega_\varepsilon = \left(-\varepsilon e_N + C\right) \cup \left(\varepsilon e_N - C\right),$$

where $e_N = (0, \ldots, 0, 1)$.

Clearly, $\Omega_\varepsilon$ is a connected set and the family of sets $\Omega_\varepsilon$ satisfy (h1) and (h2), but they do not satisfy (h3). We shall find a sequence $u_\varepsilon \in H^1(\Omega_\varepsilon)$ such that:

$$\int_{\Omega_\varepsilon} u_\varepsilon \, dx = 0, \quad \int_{\Omega_\varepsilon} u_\varepsilon^2 \, dx \geq c > 0, \quad \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 \, dx \to 0.$$

Define

$$u_\varepsilon(x) = \begin{cases} -1 & x_N \leq -\varepsilon, \\ \frac{1}{\varepsilon} x_N & |x_N| < \varepsilon, \\ 1 & x_N \geq \varepsilon. \end{cases}$$

Since $u_\varepsilon$ is odd with respect to the $x_N$ variable, $\int_{\Omega_\varepsilon} u_\varepsilon \, dx = 0$. It is also easy to show that:

$$\int_{\Omega_\varepsilon} u_\varepsilon^2 \, dx \to 2|C| as \varepsilon \to 0.$$  

Finally, we compute:

$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 \, dx = \int_{\Omega_\varepsilon \cap \{-\varepsilon < x_N < \varepsilon\}} \varepsilon^{-2} \, dx \leq 2\varepsilon^{-2} |C \cap \{0 < x_N < 2\varepsilon\}| = C\varepsilon^{N-2}.$$  

Now we are in conditions to prove our main result:

**Proof of Theorem 1.2.** Choose $\varepsilon > 0$ arbitrarily small, to be fixed at the end of the proof. Take $\delta \in (0, \delta_0)$ such that $|\Omega \setminus \Omega^\delta| < \varepsilon$, as given by property (Q). Fix also $r \in (0, \delta/2)$.

Consider a covering $B(0, R) = \bigcup_{i=1}^M B_i$, where $B_i$ are open balls of radius $r$. We denote by $\mathcal{A}$ the class of connected sets formed by unions of such balls; in other words:

$$\mathcal{A} = \{A = \bigcup_{i \in J} B_i, \quad J \subset \{1, \ldots, M\}, \quad A \text{ is connected}\}.$$

Clearly each element of $\mathcal{A}$ satisfies a cone condition and hence each of them has a Poincaré constant $C_A > 0$, in virtue of Theorem 1.1. In other words,

$$C_A \int_A |\nabla u|^p \, dx \geq \int_A |u|^p \, dx \quad \forall u \in W^{1,p}(A), \quad \int_A u \, dx = 0.$$
Since \( A \) is finite, take \( \tilde{C} \) the maximum of all those constants. Observe that \( \tilde{C} \) depends only on \( r \), and hence on \( \varepsilon \).

Define:
\[
J = \{ i \in \{1, \ldots, M \} : B_i \cap \Omega^\varepsilon \neq \emptyset \}, \quad A = \bigcup_{i \in J} B_i.
\]

Observe that under our assumptions \( \Omega \supset A \supset \Omega^\varepsilon \). Moreover, it is clear that \( A \) is connected, and then \( A \in \mathcal{A} \).

Take now \( u \in W^{1,p}(\Omega) \) with \( \int_{\Omega} u \, dx = 0 \), \( \int_{\Omega} |u|^p \, dx = 1 \). Our intention is to find a lower bound for \( \int_{\Omega} |\nabla u|^p \, dx \). Let us assume that \( \int_{\Omega} |\nabla u|^p \, dx \leq 1 \), otherwise we are done.

We first give an estimate on \( \int_A u \, dx \) by using Hölder inequality:

\[
\int_A u \, dx = \int_{\Omega \setminus A} u \, dx \leq |\Omega \setminus A|^\frac{p-1}{p} \left( \int_{\Omega} |u|^p \, dx \right)^{\frac{1}{p}} \leq \varepsilon^\frac{p-1}{p}.
\]

Let us denote by \( u_A \) the average of \( u \) on \( A \). Then,

\[
|u_A| = |A|^{-1} \left| \int_A u \, dx \right| \leq \varepsilon^\frac{p-1}{p}.
\]

Observe also that \( |A| \geq |\Omega^\varepsilon| = |\Omega| - |\Omega \setminus \Omega^\varepsilon| \geq V - \varepsilon \), where \( V \) is the volume the cone \( \tilde{C} \), which serves as a lower bound for the volume of \( \Omega \). We choose \( \varepsilon \) small enough so that \( |u_A| < 1 \).

Our intention now is to find a lower estimate for the term \( \int_A |u - u_A|^p \, dx \). Observe that the mean value theorem implies that:

\[
\left| |u - u_A|^p - |u|^p \right| \leq C_p (|u|^{p-1} + 1) u_A.
\]

We now use again Hölder inequality to estimate the term:

\[
\int_A |u|^p \, dx \leq \left( \int_A |u|^p \, dx \right)^{\frac{p-1}{p}} |A|^{\frac{1}{p}} \leq |\Omega|^{\frac{1}{p}}.
\]

Therefore,

\[
\int_A (|u - u_A|^p - |u|^p) \leq C_p \varepsilon^\frac{p-1}{p} (|\Omega|^{\frac{1}{p}} + |\Omega|).
\]

It suffices now to estimate \( \int_A |u|^p \, dx \). Let us take any \( q > p \) so that the Sobolev embedding holds, see Proposition 2.5. Since we are assuming that \( \int_{\Omega} |\nabla u|^p \, dx \leq 1 \), we have that \( \|u\|_{L^q(\Omega)} \leq C_S 2^{1/p} \). By using again Hölder inequality, we obtain:

\[
\int_A |u|^p \, dx = 1 - \int_{\Omega \setminus A} |u|^p \, dx \geq 1 - \|u\|_{L^q(\Omega \setminus A)}^q |\Omega \setminus A|^{-\frac{p}{q}} \geq 1 - 2C_S^q \varepsilon^{\frac{q}{p} - \frac{p}{q}}.
\]

By putting together (6) and (7), we have that:

\[
\int_A (|u - u_A|^p - |u|^p) \leq 1 - 2C_S^q \varepsilon^{\frac{q}{p} - \frac{p}{q}} - C_p \varepsilon^\frac{p-1}{p} (|\Omega|^{\frac{1}{p}} + |\Omega|).
\]

We only need to chose \( \varepsilon \) small enough so that the above term is greater than 1/2. Therefore,
\[
\int_{\Omega} |\nabla u|^p \, dx \geq \int_{\Omega} |\nabla (u - u_A)|^p \, dx \geq \frac{1}{2C}.
\]

As a corollary, we obtain the following result.

**Corollary 2.7.** Given \( E \subset \Omega \) a measurable set with nonzero measure, we denote by \( u_E \) the average of \( u \) in \( E \), that is, \( u_E = |E|^{-1} \int_E u \, dx \).

There exists a constant \( C > 0 \) such that for any domain \( \Omega \) satisfying (h1)-(h3),

1. If \( N > p \),
   \[
   \int_{\Omega} |u - u_E|^p \, dx \leq C|E|^{\frac{p}{N}} \int_{\Omega} |\nabla u|^p \, dx \quad \forall u \in W^{1,p}(\Omega), \ E \subset \Omega, \ |E| > 0.
   \]
2. If \( N = p \),
   \[
   \int_{\Omega} |u - u_E|^p \, dx \leq C(\log(1+|E|^{-1}))^{N-1} \int_{\Omega} |\nabla u|^p \, dx \quad \forall u \in W^{1,p}(\Omega), \ E \subset \Omega, \ |E| > 0.
   \]
3. If \( N < p \),
   \[
   \int_{\Omega} |u - u(x_0)|^p \, dx \leq C \int_{\Omega} |\nabla u|^p \, dx \quad \forall u \in W^{1,p}(\Omega), \ x_0 \in \Omega.
   \]

Moreover, \( C \) depends only on \( p, R, \varepsilon \) and the cone \( E \) given in conditions (h1)-(h3).

**Remark 2.8.** Compared with Theorem 1 and Corollary 2 of [4], here the constant \( C \) is independent of the set \( E \). The dependence of the estimate on \( E \), when \( |E| \) is small, is made explicit here.

**Proof.** Obviously, it suffices to prove the estimate for \( u \in W^{1,p}(\Omega) \) with \( \int_{\Omega} |\nabla u|^p \, dx = 1 \) and \( \int_{\Omega} u \, dx = 0 \). By Theorem 1.2, there exists \( C > 0 \) (depending only on \( R, C \) and \( \delta_0 \)) such that \( \int_{\Omega} |u|^p \, dx \leq C \).

Observe that:

\[
(8) \quad \int_{\Omega} |u - \lambda|^p \, dx \leq 2^p \left( \int_{\Omega} |u|^p \, dx + \lambda^p \right).
\]

Therefore it suffices to estimate \( u_E \) in cases 1 and 2, and \( u(x_0) \) in case 3. For that we will use the Sobolev embedding; observe that \( ||u||_{W^{1,p}} \leq (1 + C)^{1/p} \).

**Case 1:** If \( N > p \), we have:

\[
\left( \int_E u \, dx \right)^{\frac{N}{N-p}} \left( \int_E 1 \, dx \right)^{\frac{N-p}{N}} \leq C' |E|^{1 + \frac{N}{N-p} - \frac{p}{N}},
\]

where \( C' \) depends on \( C \) and \( C_S \). But then:

\[
(9) \quad |u_E| \leq C' |E|^{\frac{1}{2} - \frac{1}{N}}.
\]

**Case 2:** If \( N = p \), we need to use Orlicz norms: we refer to [1], Chapter 8. Let us define \( A(t) = e^{\frac{t}{N-p}} - 1 \), and \( \tilde{A}(t) \) its complement function (which is not explicit). By using the generalized Hölder inequality,

\[
\left| \int_E u \, dx \right| = \left| \int_{\Omega} u \chi_E \, dx \right| \leq 2 ||u||_{L_A} \|\chi_E\|_{L_A} \leq C' \|\chi_E\|_{L_A}.
\]

Here \( \chi_E \) is the characteristic function of the set \( E \). We now use the definition of the Orlicz norm:
\[ \| \chi_E \|_{L^A} = \inf_{k > 0} \left\{ \int_{\Omega} \tilde{A} \left( \frac{\chi_E(x)}{k} \right) \, dx \leq 1 \right\} = \inf_{k > 0} \left\{ \tilde{A} \left( \frac{1}{k} \right) |E| \leq 1 \right\} \]

\[ = \frac{1}{A^{-1}(|E|^{-1})} \leq A^{-1}(1) \frac{A^{-1}(|E|^{-1})}{|E|^{-1}} = |E| \left( \log (1 + |E|^{-1}) \right)^{\frac{p-1}{p}}. \]

Above we have just used the general inequality \( s \leq \tilde{A}^{-1}(s) \tilde{A}^{-1}(s). \) Therefore,

(10) \[ |u| \leq C' \left( \log (1 + |E|^{-1}) \right)^{\frac{p-1}{p}}. \]

**Case 3:** If \( N < p, \) then \( u \) is continuous. Moreover,

(11) \[ u(x_0) \leq \| u \|_{L^{\infty}(\Omega)} \leq C', \]

by the uniformity of the constant on the Sobolev inequality.

Putting together estimates (8), (9), (10), (11), we finish the proof.

\( \square \)

**Remark 2.9.** One can extend also the previous results to sets \( E \) which are hypersurfaces, in the spirit of Theorem 2 of [4]. We leave the details to the interested reader.

3. **SOME GENERALIZATIONS**

In this section we discuss some possible extensions of our results to more general frameworks. First, we will discuss the possible relaxation of the hypotheses of Theorem 1.2. In particular, we will prove Corollary 1.3. Finally, we will briefly comment the extension of our results to Riemannian manifolds.

3.1. **Relaxation of conditions (h1)-(h3).** The cone condition (h1) has been needed in our proof at two different steps. First, it is needed to guarantee a Sobolev inequality (see Proposition 2.5), together with the uniformity of the constant. Observe that the proof of Theorem 1.2 only needs a Sobolev inequality for some exponent \( q > p; \) the limiting exponent is not needed here. Second, (h1) is used to derive property (Q) given in Proposition 2.4, which otherwise should be somehow imposed. Indeed, we can introduce it in a generalized version of condition (h3).

Our result can be generalized as follows.

**Theorem 3.1.** Take \( p > 1, \) and let \( \mathcal{F} \) be a family of domains satisfying the following hypotheses:

(1) There exists \( R > 0 \) such that \( \Omega \subset B(0, R) \) for any \( \Omega \in \mathcal{F}. \)

(2) There exists \( q > p \) and \( C_S > 0 \) such that for any \( \Omega \in \mathcal{F}, u \in W^{1,p}(\Omega), \)

\[ \| u \|_{L^{q}(\Omega)} \leq C_S \| u \|_{W^{1,p}(\Omega)}. \]

(3) For any \( \varepsilon > 0 \) there exists \( \delta = \delta(\varepsilon) > 0 \) such that for any \( \Omega \in \mathcal{F} \) we can choose a connected set \( U \subset \Omega \) such that \( U \subset \Omega^\delta \) and \( |\Omega \setminus U| < \varepsilon. \)

Then, there exists \( C > 0 \) such that for any \( \Omega \in \mathcal{F}, \)

\[ \int_{\Omega} |u|^p \, dx \leq C \int_{\Omega} |\nabla u|^p \, dx \forall u \in W^{1,p}(\Omega), \int_{\Omega} u(x) \, dx = 0. \]
The proof of Theorem 3.1 is basically the same as the proof of Theorem 1.2, with the obvious modifications due to the new hypotheses. The details are left to the reader.

The literature on Sobolev inequalities for domains not satisfying the cone property is huge. For instance, it is known that the limiting Sobolev inequality does not hold for domains containing outward cusps. However, for cusps having power sharpness, a Sobolev embedding to some $L^q(\Omega)$ space is possible, see [2]. It is to be expected that some uniformity on the assumptions would imply an uniform constant in that inequality, providing (f2).

Many other Sobolev embeddings have also been found, and it is not possible to give here a detailed account of them: see for instance [3, 16] and the references therein.

On the other hand, condition (f3) puts together condition (Q) and (h3) in a more general fashion. Condition (f3) does not seem easy to check in general, though. However, as a particular case, we can prove Corollary 1.3.

It is not clear to us whether a domain under the conditions of Corollary 1.3 satisfies condition (h3). In any case, Corollary 1.3 is a consequence of Theorem 3.1, as we show below.

Proof of Corollary 1.3. We only need to check that under the conditions of Corollary 1.3, we can apply Theorem 3.1. Let us define:

$$\mathcal{F} = \{ K_r : K \subset B(0, R) \text{ a connected set}\}.$$  

We recall (1) for the definition of $K_r$. In what follows we can always consider $K$ to be a compact set; otherwise, we would argue on its closure. Clearly, any domain in $\mathcal{F}$ satisfies the cone property with a cone depending only on $r$. Therefore, conditions (f1) and (f2) are satisfied.

Let us check condition (f3). Given $\delta \in (0, r/2)$ and $K \subset B(0, R)$ connected, we define:

$$U = \overline{K_{r-\delta}} = \{ x \in \mathbb{R}^N : d(x, K) \leq r - \delta \}.$$  

Clearly, $U$ is a connected set. Observe now that $\partial K_r = \{ x \in \mathbb{R}^N : d(x, K) = r \}$. The triangular inequality readily implies that $U \subset K^\delta_r$. In order to prove (f3), it suffices to estimate the measure of $K_r \setminus U$ independently of the choice of $K$.

Let us point out that:

$$K_r \setminus U = \{ x \in \mathbb{R}^N : d(x, K) \in (r - \delta, r) \} = \{ x \in \mathbb{R}^N : d(x, K_{r-\delta}) \in (0, \delta) \}.$$  

The first equality holds just by the definition of $U$; let us justify briefly the second identity. The inclusion $\supset$ follows easily from the triangular inequality. For the inclusion $\subset$, take $x \in \mathbb{R}^N$ and $y \in K$ with $d(x, y) = d(x, K) \in (r - \delta, r)$. Choose $z$ in the segment $[x, y]$ such that $d(x, z) \in (0, \delta)$, $d(z, y) < r - \delta$. Then, $z \in K_{r-\delta}$ and also $d(x, K_{r-\delta}) \leq d(x, z) < \delta$. It suffices to observe that $x \notin U$ to conclude.

Now, observe that $K_{r-\delta}$ satisfies the cone condition with a fixed cone. Now it suffices to recall the arguments of the proof of Proposition 2.4 to conclude. □

3.2. Extension to Riemannian manifolds. Because of our original motivation in the study of local versions of the Moser-Trudinger inequality, we are interested in a uniform Poincaré inequality in compact Riemannian manifolds. This extension is quite direct, but it deserves a couple of comments.

First of all, we have not been able to find a specific definition of the interior cone condition for a domain in a manifold. Still, we think that it must be written...
Definition 3.2. Let $\Sigma$ be a compact manifold. It is clear that we can consider on $\Sigma$ a finite family of charts $x_i : B(0, 1) \to \Sigma$, $i = 1, \ldots, k$, such that $\bigcup_{i=1}^{k} x_i(B(0, 1/2)) = \Sigma$. We say that $\Omega \subset \Sigma$ satisfies the interior cone condition if for any $j$ such that $x^{-1}_j(\partial \Omega) \cap B(0, 1/2) \neq \emptyset$, $x^{-1}_j(\Omega)$ satisfies the interior cone condition at any point of $x^{-1}_j(\partial \Omega) \cap B(0, 1/2)$.

By using this finite family of charts it is easy to generalize Proposition (Q) to the case of sub-domains of a fixed compact Riemannian manifold (instead of (h1)). Moreover, also the uniform Sobolev embedding holds, as can be easily checked passing to charts. Therefore, the results of Theorem 1.2, Theorem 3.1 and Corollary 1.3 follow with basically the same proof.

4. Appendix

In Theorem 1.2 we have proved the uniformity of the Poincaré constant for domains satisfying certain properties, namely (h1), (h2), (h3). In this final section we show that connected uniformly Lipschitz domains (in the sense of Definition 2.1) satisfy (h2) and (h3). So, our result extends that of [4] to a less regular framework.

Property (h2) being straightforward, it suffices to show property (h3). The proof we give here is inspired by min-max theory and deformation flows. It is possible that there is a more direct way to prove this result, but in our opinion the proof itself is interesting in its own right.

Take $\Omega$ be a uniformly Lipschitz domain, according to definition 2.1. Let us define $\Gamma : \mathbb{R}^N \to \mathbb{R}$ as:

$$
\Gamma(x) = \begin{cases} 
  d(x, \partial \Omega) & \text{if } x \in \Omega, \\
  -d(x, \partial \Omega) & \text{if } x \notin \Omega,
\end{cases}
$$

Observe that with that definition, $\Omega = \Gamma^{-1}(0, +\infty)$ and $\Omega^\delta = \Gamma^{-1}(\delta, +\infty)$ (for $\delta > 0$). The idea of the proof is to show that those two sets are homotopically equivalent if $\delta \in (0, \delta_0)$, for some $\delta_0$ small independent of $\Omega$. In order to prove that, we will use a deformation argument.

It is important to observe here that $\Gamma$ is a Lipschitz map, but not $C^1$, and therefore one cannot use a gradient flow as usually. However, in 1981, K. C. Chang ([6]) was able to apply variational methods to Lipschitz maps. For that, he needed a notion of generalized gradient, due to Clarke, that we remind below. Moreover, he was able to prove the existence of a continuous pseudogradient flow for Lipschitz maps. With these two main tools, he was able to apply min-max techniques to Lipschitz functionals.

Here we just use those ideas to find an homotopical equivalence between $\Omega$ and $\Omega^\delta$. For that, we will need to show that $\Gamma$ has no critical points in $\Omega \setminus \Omega^\delta$; it is here that the Lipschitz assumption on $\Omega$ comes into play.

First, let us recall the notion of generalized gradient of a Lipschitz map due to Clarke. For more information on those aspects, see [9, 10, 6].

Definition 4.1. Let $f : \mathbb{R}^N \to \mathbb{R}$ a Lipschitz map. The generalized directional derivative of $f$ on $x \in \mathbb{R}^N$ along the direction $v \in \mathbb{R}^N$ is defined as:

$$
f^0(x; v) = \limsup_{y \to x, t \to 0^+} \frac{f(y + tv) - f(y)}{t}.
$$

It can be checked that the function $v \mapsto f^0(x; v)$ is continuous and convex, and that $|f^0(x; v)| \leq L|v|$, where $L$ is the Lipschitz constant for $f$. We define the generalized
gradient of $f$ at $x$, denoted $\partial f(x)$, as the subdifferential of the map $v \mapsto f^0(x; v)$ at $0$. In other words,

$$w \in \partial f(x) \iff w \cdot v \leq f^0(x; v) \quad \forall v \in \mathbb{R}^N.$$  

For any $x$, $\partial f(x)$ is a non-empty compact and convex subset of $\mathbb{R}^N$.

Finally, we say that $x$ is a critical point of $f$ if $0 \in \partial f(x)$.

**Lemma 4.2.** Let $\Omega$ be a uniformly lipschitz domain, with the definition 2.1, and $\Gamma$ the map defined in (12). Then, there exists $\delta_0 > 0$ depending only on the constants $M, \gamma$ such that $\Gamma$ has no critical points in $\Omega \setminus \Omega^\delta$.

**Proof.** Take $x \in \Omega \setminus \Omega^\delta$; by the definition of Lipschitz regularity, in a neighborhood of $x$, $\partial \Omega$ is a graph on the last component (with a conveniently chosen coordinate system). Take $v = (0, \ldots, 0, 1)$. We will show that there exists $c > 0$ such that $\Gamma^0(x; v) < -c < 0$; this readily implies that $\Gamma$ has no critical points in $\Omega \setminus \Omega^\delta$.

By making $z = y + tv$, we have that:

$$\Gamma^0(x; v) = \limsup_{z \to x, t \to 0^+} \frac{\Gamma(z) - \Gamma(z - tv)}{t}.$$  

Take $r = \Gamma(z) \in (0, \delta]$. Clearly, $B(z, r) \subset \Omega$. Let us define the cone:

$$C = \{x \in \mathbb{R}^N; 0 > x_N > -\tau, \ |\hat{x}| < Mx_N \}.$$  

Here $\tau > 0$ is chosen depending on $M, \gamma$. By the Lipschitz property of the domain, $y + C \subset \Omega$ for any $y \in B(z, r)$. In other words, $(z + C)_r \subset \Omega$ (see (1)). An easy geometrical argument implies the following (see Figure 1):

$$d(z - tv, \partial \Omega) \geq d(z - tv, \partial(z + C)_r) \geq r + ct \quad \text{if } t \text{ is small}.$$  

Here $c$ can be made explicit, $c = \frac{1}{\sqrt{1 + M^2}} > 0$. This finishes the proof.

**Proposition 4.3.** Let $\Omega \subset B(0, R)$ be a uniformly lipschitz domain, with the definition 2.1, and $\Gamma$ the map defined in (12). Then, there exists $\delta_0 > 0$ depending only on the constants $R, M, \gamma$ such that $\Omega^\delta$ is homotopically equivalent to $\Omega$ for all $\delta \in (0, \delta_0)$. In particular, if $\Omega$ is connected, $\Omega^\delta$ is connected.
Proof. The proof is a direct application of the arguments in [6]. In that paper, Lemma 3.3 gives a Lipschitz pseudogradient vector field and Lemma 3.4 uses it to define a convenient deformation, which provides us with the desired homotopy. Observe that the (PS) condition holds since we are in a compact framework without critical points.

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