Characterizations of the normal distribution via the independence of the sample mean and the feasible definite statistics with ordered arguments

Chin-Yuan Hu\(^1\) · Gwo Dong Lin\(^2,3\)

Received: 24 December 2020 / Revised: 7 May 2021 / Accepted: 8 July 2021 / Published online: 10 August 2021
© The Institute of Statistical Mathematics, Tokyo 2021

Abstract
It is well known that the independence of the sample mean and the sample variance characterizes the normal distribution. By using Anosov’s theorem, we further investigate the analogous characteristic properties in terms of the sample mean and some feasible definite statistics. The latter statistics introduced in this paper for the first time are based on nonnegative, definite and continuous functions of ordered arguments with positive degree of homogeneity. The proposed approach seems to be natural and can be used to derive easily characterization results for many feasible definite statistics, such as known characterizations involving the sample variance, sample range as well as Gini’s mean difference.

Keywords Characterization of distributions · Order statistics · Anosov’s theorem · Benedetti’s inequality · Sample mean · Sample variance · Sample range · Gini’s mean difference

1 Introduction
In the characterization theory of probability or statistical distributions, one of the remarkable results is the characterization of the normal distribution through the independence of the sample mean and the sample variance described below.
Let $X$ be a random variable with distribution $F$ on the whole real line $\mathbb{R} := (-\infty, \infty)$, denoted $X \sim F$. Let $X_1, X_2, \ldots, X_n$ be a random sample of size $n \geq 2$ from distribution $F$. Denote the sample mean $X_n = \frac{1}{n} \sum_{i=1}^{n} X_i$ and the sample variance $S_n^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X}_n)^2$. If $F$ is normal, then $\overline{X}_n$ and $(X_1 - \overline{X}_n, X_2 - \overline{X}_n, \ldots, X_n - \overline{X}_n)$ are independent (see, e.g., Rohatgi 1976, p. 321), and hence so are $\overline{X}_n$ and $S_n^2$. Conversely, if $\overline{X}_n$ and $S_n^2$ are independent, then $F$ is a normal distribution (including the degenerate case). Geary (1936) proved this result under the extra moment condition that the underlying distribution $F$ has finite moments of all orders and then Lukacs (1942) improved this result to just include the second moment assumption. Kawata and Sakamoto (1949) as well as Zinger (1951) solved the problem completely by using different approaches.

Instead of the sample variance $S_n^2$, Laha (1956) considered the quadratic form $Q = \sum_{i,j} a_{ij} X_i X_j$ and investigated the conditions under which the independence of $X_n$ and $Q$ characterizes the normal distribution. In this regard, see Kagan et al. (1973), Chapter 4.

In this paper, we introduce the novel statistics $Z_n$ with ordered arguments of the form:

$$Z_n = U(X_{(1)} - \overline{X}_n, X_{(2)} - \overline{X}_n, \ldots, X_{(n)} - \overline{X}_n),$$

where $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ are order statistics of the random sample $X_1, X_2, \ldots, X_n$ of size $n \geq 3$ (our approach needs to assume $n \geq 3$; see (10) and (15) below). We say that the base function $U$ defined on the ordered set

$$\mathcal{A} = \left\{ (\lambda_1, \ldots, \lambda_n) : \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n, \sum_{i=1}^{n} \lambda_i = 0 \right\}$$

is feasible definite with positive degree $p$ of homogeneity if it satisfies the following conditions:

(i) $U$ is nonnegative and continuous on $\mathcal{A}$,

(ii) $U(\lambda_1, \ldots, \lambda_n) = 0$ if and only if $(\lambda_1, \ldots, \lambda_n) = (0, \ldots, 0)$ (definiteness) and

(iii) $U(s(\lambda_1, \ldots, \lambda_n)) = s^p U(\lambda_1, \ldots, \lambda_n)$ for all $s > 0$ and $(\lambda_1, \ldots, \lambda_n) \in \mathcal{A}$ (positive degree $p$ of homogeneity).

The corresponding statistic $Z_n$ defined in (1) through such a base function $U$ is called a feasible definite statistic on $\mathcal{A}$ with positive degree $p$ of homogeneity.

Note that the sample mean is in general not a feasible definite statistic, because it may take negative values and does not satisfy the definiteness condition (see (i) and (ii) above). We will find conditions on the base function $U$ and the underlying distribution $F$ under which the independence of the sample mean $\overline{X}_n$ and the feasible definite statistic $Z_n$ characterizes the normal distribution (see Theorem below). On the other hand, Hwang and Hu (2000) considered the statistics of the form (without the concept of homogeneity):

$$Z_n = S_n \cdot \exp(\psi(\Lambda_1, \Lambda_2, \ldots, \Lambda_n)),$$
where $\Lambda_i = (X_{(i)} - \bar{X}_n)/S_n$ and $\psi$ is a bounded continuous real-valued function.

Compared with the above product-form statistics, our approach here seems to be more natural and can be used to derive easily many characterizations involving feasible definite statistics. The latter statistics encompass the familiar sample variance $S_n^2$, sample range

$$R_n := X_{(n)} - X_{(1)} = (X_{(n)} - \bar{X}_n) - (X_{(1)} - \bar{X}_n),$$

and Gini’s mean difference $G_n$ (see Corollaries 1, 3 and 5 below).

The main results are stated in Sect. 2. Section 3 provides the crucial tools—Ansoff’s theorem, Benedetti’s inequality as well as three lemmas. The proofs of the main results are given in Sect. 4. Section 5 provides some more results about the sample range and Gini’s mean difference as well as two conjectures. Finally, we have in Sect. 6 some discussions.

## 2 Main results

Throughout the section, we consider a random variable $X \sim F$ having positive continuous density $f_X$ on $\mathbb{R}$. Let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ be the order statistics of a random sample $X_1, X_2, \ldots, X_n$ of size $n \geq 3$ from distribution $F$. Then, we have the following result.

**Theorem** Let $Z_n = U(X_{(1)} - \bar{X}_n, X_{(2)} - \bar{X}_n, \ldots, X_{(n)} - \bar{X}_n)$ be a feasible definite statistic on $\mathbf{A}$ (defined in (2)) with positive degree $p$ of homogeneity. Then, $\bar{X}_n$ and $Z_n$ are independent if and only if $F$ is normal.

Applying the theorem to some suitable base functions yields the following corollaries. The known results (Corollaries 1 and 5) are listed here for comparison. In fact, they are special cases of Corollaries 2 and 4, and hence their proofs are omitted.

**Corollary 1** (Hwang and Hu 2000). The sample mean $\bar{X}_n$ and the sample range $R_n$ (defined in (3)) are independent if and only if $F$ is normal.

**Corollary 2** Assume that $a_1 \leq a_2 \leq \cdots \leq a_n$ are not all equal. Then, the sample mean $\bar{X}_n$ and the feasible definite statistic

$$Z_n = \sum_{i=1}^n a_i (X_{(i)} - \bar{X}_n)$$

are independent if and only if $F$ is normal.

When $\sum_{i=1}^n a_i = 0$, Corollary 2 reduces to Corollary 2.2 of Hwang and Hu (2000), from which the sample range $R_n$ and Gini’s mean difference $G_n$ are derived as special cases.
Corollary 3 Let \( p > 0, a_1 > 0, a_n > 0, \) and \( a_i \geq 0, \) where \( 2 \leq i \leq n - 1. \) Then the sample mean \( \bar{X}_n \) and the feasible definite statistic

\[
Z_n = \sum_{i=1}^{n} a_i |X_{(i)} - \bar{X}_n|^p
\]  

are independent if and only if \( F \) is normal.

When \( p = 2 \) and \( a_i = 1/(n-1) \) for all \( i, \) the \( Z_n \) in (5) reduces to the sample variance \( S_n^2. \) Hence, this result includes the classical one as a special case.

Corollary 4 Let \( p > 0, a_{ij} \geq 0, \) where \( 1 \leq i, j \leq n, \) and \( a_{1n} + a_{n1} > 0. \) Then, the sample mean \( \bar{X}_n \) and the feasible definite statistic

\[
Z_n = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} |X_{(i)} - X_{(j)}|^p
\]  

are independent if and only if \( F \) is normal.

Corollary 5 (Hwang and Hu 2000). The sample mean \( \bar{X}_n \) and Gini’s mean difference

\[
G_n = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} |X_i - X_j|
\]

\[
= \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} |X_{(i)} - X_{(j)}| = \frac{4}{n(n-1)} \sum_{i=1}^{n} (i - (n+1)/2)X_{(i)}
\]  

are independent if and only if \( F \) is normal.

For the last equality in (7), see David and Nagaraja (2003), pp. 249 and 279.

Corollary 6 Let \((a_{ij})_{i,j=1}^{n} \) be a positive definite (real) matrix. Then, the sample mean \( \bar{X}_n \) and the feasible definite statistic

\[
Z_n = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} (X_{(i)} - \bar{X}_n)(X_{(j)} - \bar{X}_n)
\]  

are independent if and only if \( F \) is normal.

Corollary 7 Let \( p > 0, q > 0. \) Assume further that \( a_{ij} \geq 0, \) where \( 1 \leq i, j \leq n, \) and \( \min_{1 \leq i \leq n} a_{ii} > 0. \) Then, the sample mean \( \bar{X}_n \) and the feasible definite statistic

\[
Z_n = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} |X_{(i)} - \bar{X}_n|^p |X_{(j)} - \bar{X}_n|^q
\]  

are independent if and only if \( F \) is normal.
are independent if and only if \( F \) is normal.

3 Crucial tools: Anosov’s theorem, Benedetti’s inequality and three lemmas

To prove the main results, we need Anosov’s theorem [see Anosov (1964) or Kagan et al. (1973), Chapter 4], Benedetti’s inequality [see Benedetti (1957, 1995); Georgescu-Roegen (1959), Sarria and Martinez (2016), or Hwang and Hu (1994a)], and three crucial lemmas.

**Anosov’s Theorem.** Let \( n \geq 3 \) be an integer and let \( X \sim F \) have a positive continuous density \( f_X \) on \( \mathbb{R} \). Define the \((n-2)\)-dimensional torus

\[
\Phi = \{ \phi = (\phi_1, \ldots, \phi_{n-2}) : \phi_j \in [0, \pi], \ j = 1, 2, \ldots, n-3; \phi_{n-2} \in [0, 2\pi] \}.
\]

(10)

Let \( T \) be a nonnegative continuous function on \( \Phi \) such that \( \int_{\Phi} T(\phi) \, d\phi \in (0, \infty) \). Assume further that \( \sigma_j, \ j = 1, \ldots, n, \) are continuous functions on \( \Phi \) satisfying

\[
\sum_{j=1}^{n} \sigma_j(\phi) = 0 \quad \text{and} \quad \sum_{j=1}^{n} \sigma_j^2(\phi) \in (0, \infty) \quad \text{for all} \quad \phi \in \Phi.
\]

(11)

Under these conditions, if the density \( f_X \) satisfies the integro-functional equation:

\[
\int_{\Phi} \prod_{j=1}^{n} f_X(t + s\sigma_j(\phi)) T(\phi) \, d\phi = c \left( f_X(t) \right)^n \int_{\Phi} \prod_{j=1}^{n} f_X(s\sigma_j(\phi)) T(\phi) \, d\phi \ \forall \ t \in \mathbb{R} \ \text{and} \ s \geq 0,
\]

(12)

where \( c > 0 \) is a constant, then \( f_X \) is normal.

**Remark 1** The proof of Anosov’s Theorem is complicated. However, as noted by Kagan et al. (1973), p. 143, if we further assume the density function \( f_X \) in (12) to be continuously twice-differentiable, then the proof becomes much simpler, just by differentiating (12) twice and solving the obtained functional equation.

**Benedetti’s Inequality.** Let \( n \geq 2 \) be an integer and let \( \mu = (\mu_1, \mu_2, \ldots, \mu_n) \), where \( \mu_1 \leq \mu_2 \leq \ldots \leq \mu_n \), not all equal, and \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \), where \( \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \), not all equal. Denote

\[
\overline{\mu} = \frac{1}{n} \sum_{i=1}^{n} \mu_i, \quad \overline{\lambda} = \frac{1}{n} \sum_{i=1}^{n} \lambda_i, \quad s^2(\mu) = \frac{1}{n} \sum_{i=1}^{n} (\mu_i - \overline{\mu})^2, \quad s^2(\lambda) = \frac{1}{n} \sum_{i=1}^{n} (\lambda_i - \overline{\lambda})^2,
\]

and the order covariance

\[
\text{Cov}(\mu, \lambda) = \frac{1}{n} \sum_{i=1}^{n} \mu_i \lambda_i - \overline{\mu} \cdot \overline{\lambda}.
\]
Then, we have

\[
\frac{1}{n-1} \leq \frac{\text{Cov}(\mu, \lambda)}{s(\mu)s(\lambda)}.
\]

Equality holds if and only if (i) \( n = 2 \) or (ii) \( \mu_1 \leq \mu_2 = \cdots = \mu_n \) and \( \lambda_1 = \cdots = \lambda_{n-1} \leq \lambda_n \) or (iii) \( \mu_1 = \cdots = \mu_{n-1} \leq \mu_n \) and \( \lambda_1 \leq \lambda_2 = \cdots = \lambda_n \).

**Remark 2** The main purpose of Benedetti’s Inequality here is to derive Corollary 2 above.

We define a compact subset of \( \mathbf{A} \) as follows:

\[
\mathbf{A}_n = \left\{ (\lambda_1, \ldots, \lambda_n) : \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n, \sum_{i=1}^{n} \lambda_i = 0, \sum_{i=1}^{n} \lambda_i^2 = n - 1 \right\}. \tag{13}
\]

**Lemma 1** Let \( n \geq 3 \) and let \( Z_n \) defined in (1) be a feasible definite statistic on \( \mathbf{A} \) with positive degree 1 of homogeneity. Then, there exist two positive constants \( k < K \) such that

\[
0 < k \leq \frac{Z_n}{S_n} \leq K < \infty \quad \text{a.s. (almost surely).} \tag{14}
\]

**Proof** By the homogeneity property, we rewrite

\[
Z_n = U(X_{(1)} - \bar{X}_n, X_{(2)} - \bar{X}_n, \ldots, X_{(n)} - \bar{X}_n)
\]

\[
= S_nU(\Lambda_1, \Lambda_2, \ldots, \Lambda_n),
\]

where

\[
\Lambda_i = \frac{X_{(i)} - \bar{X}_n}{S_n}, \quad i = 1, 2, \ldots, n,
\]

and \( (\Lambda_1, \Lambda_2, \ldots, \Lambda_n) \) takes values \( (\lambda_1, \lambda_2, \ldots, \lambda_n) \) in \( \mathbf{A}_n \) (defined in (13)) almost surely. Since the function \( U \) is nonnegative, definite and continuous on \( \mathbf{A} \) and the compact subset \( \mathbf{A}_n \) of \( \mathbf{A} \) is bounded away from the origin, the range \( U(\mathbf{A}_n) \subset (0, \infty) \) is a compact set as well. Hence, there exist two positive constants \( k < K \) such that \( U(\mathbf{A}_n) \subset [k, K] \subset (0, \infty) \). This implies that

\[
0 < k \leq U(\Lambda_1, \Lambda_2, \ldots, \Lambda_n) \leq K < \infty \quad \text{a.s.}
\]

Equivalently, (14) holds true. The proof is complete. \( \Box \)

As in the beginning of Sect. 2, let \( X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)} \) be the order statistics of a random sample \( X_1, X_2, \ldots, X_n \) of size \( n \geq 3 \) from a distribution \( F \) which has positive continuous density \( f_X \) on \( \mathbb{R} \). The corresponding realized order values are \( x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)} \) with the sample mean and the sample variance:
Following Hwang and Hu (1994b), we define the transformation

\[ T : (x_1, x_2, \ldots, x_n) \longrightarrow (t_1, t_2, \ldots, t_{n-2}, w_1, w_2) \]

by

\[
\begin{align*}
  t_i &= \left[ \frac{n-i+1}{(n-1)(n-i)} \right]^{1/2} \left[ \frac{x(i) - \bar{x}_n}{s_n} + \frac{1}{n-i+1} \sum_{k=1}^{i-1} \frac{x(k) - \bar{x}_n}{s_n} \right], \quad 1 \leq i \leq n-2, \\
  w_1 &= \bar{x}_n, \\
  w_2 &= s_n,
\end{align*}
\]

where the summation is taken to be zero if \( i = 1 \). Then, let \( f_i = 1 - \sum_{k=1}^{i} t_k^2 \), where \( 1 \leq i \leq n-2 \), and denote the inverse transformation of \( T \): \n
\[ T^{-1} : (t_1, t_2, \ldots, t_{n-2}, w_1, w_2) \longrightarrow (x_1, x_2, \ldots, x_n) \]

through

\[
\begin{align*}
  \frac{x(i) - w_1}{w_2 \sqrt{n-1}} &= \left[ \frac{n-i}{n-i+1} \right]^{1/2} \cdot t_i - \sum_{k=1}^{i-1} \frac{t_k}{(n-k)(n-k+1)^{1/2}}, \quad 1 \leq i \leq n-2, \\
  \frac{x(n-1) - w_1}{w_2 \sqrt{n-1}} &= -\sum_{k=1}^{n-2} \frac{t_k}{(n-k)(n-k+1)^{1/2}} - \left[ f_{n-2}/2 \right]^{1/2}, \\
  \frac{x(n) - w_1}{w_2 \sqrt{n-1}} &= -\sum_{k=1}^{n-2} \frac{t_k}{(n-k)(n-k+1)^{1/2}} + \left[ f_{n-2}/2 \right]^{1/2},
\end{align*}
\]

where \( \bar{x}_n = w_1 \) and \( s_n = w_2 \).

For \( n \geq 3 \), define two more subsets \( D_n \) and \( R_n \) of \( \mathbb{R}^n \):

\[ D_n = \{ (x_1, x_2, \ldots, x_n) : x_1 \leq x_2 \leq \cdots \leq x_n \}, \quad (17) \]

\[ R_n = \{ (t_1, t_2, \ldots, t_{n-2}, w_1, w_2) : -1 \leq t_1 \leq -1/(n-1), \]

\[ \max \left\{ \left[ \frac{n-k+2}{n-k} \right]^{1/2}, t_{k-1}, \left[ f_{k-1} \right]^{1/2} \right\} \]

\[ \leq t_k \leq -\left[ f_{k-1} \right]^{1/2}/(n-k), \quad 2 \leq k \leq n-2, \quad w_1 \in \mathbb{R}, \quad w_2 > 0 \}. \quad (18) \]

Then, we have the following:

**Lemma 2** (Hwang and Hu 1994b). For \( n \geq 3 \), let \( T \) be the transformation defined in (15). Then, \( T \) establishes a one-to-one correspondence between the domain \( D_n \)
and the range $R_n$, defined in (17) and (18), respectively, except for a set of $n$-dimensional Lebesgue measure zero. Furthermore, the inverse transformation $T^{-1}$ is given through (16), and the absolute value of its Jacobian is

$$|J| = \sqrt{n} (n-1)^{(n-1)/2} \cdot w_n^{n-2} \cdot f_{n-2}^{-1/2}.$$ 

**Lemma 3** (Hwang and Hu 1994b). Let $X \sim F$ be a standard normal random variable. For $n \geq 3$, define the statistics

$$T_i = \left[ \frac{n - i + 1}{(n-1)(n-i)} \right]^{1/2} \left[ \frac{X(i) - \bar{X}_n}{S_n} + \frac{1}{n-i+1} \sum_{k=1}^{i-1} \frac{X(k) - \bar{X}_n}{S_n} \right], \quad 1 \leq i \leq n-2. \tag{19}$$

Then, $(T_1, T_2, \ldots, T_{n-2})$ has joint density

$$f(t_1, t_2, \ldots, t_{n-2}) = \frac{n! \Gamma((n-1)/2)}{2 \pi^{(n-1)/2} \cdot f_{n-2}^{1/2}}, \quad (t_1, t_2, \ldots, t_{n-2}) \in B_{n-2}, \tag{20}$$

where $f_{n-2} = 1 - t_1^2 - t_2^2 - \cdots - t_{n-2}^2$ and $B_{n-2}$ is a subset of $\mathbb{R}^{n-2}$:

$$B_{n-2} = \left\{ (t_1, t_2, \ldots, t_{n-2}) : -1 \leq t_1 \leq -1/(n-1), \right. \right.$$  

$$\left. \max \left\{ \left[ \frac{n-k+2}{n-k} \right]^{1/2} \cdot t_{k-1}, -f_{k-1}^{1/2} \right\} \leq t_k \leq -f_{k-1}^{1/2}/(n-k), \quad 2 \leq k \leq n-2 \right\}. \tag{21}$$

Comparing (15) and (19), we note that $(T_1, T_2, \ldots, T_{n-2})$ has realized value $(t_1, t_2, \ldots, t_{n-2})$ given in (15). Moreover, it follows from the density (20) that

$$\int_{B_{n-2}} f_{n-2}^{-1/2} \prod_{i=1}^{n-2} dt_i = \frac{2 \pi^{(n-1)/2}}{n! \Gamma((n-1)/2)} < \infty,$$

which will be used in the proof of the Theorem.

### 4 Proofs of main results

**Proof of the Theorem.** It suffices to prove the necessity part. Namely, suppose that the sample mean $\bar{X}_n$ is independent of the feasible definite statistic

$$Z_n = U(X(1) - \bar{X}_n, X(2) - \bar{X}_n, \ldots, X(n) - \bar{X}_n)$$

on $A$ with positive degree $p$ of homogeneity. Then, we want to prove that $F$ is normal. Without loss of generality, we may assume that $p = 1$. Otherwise, we can
Characterizations of the normal distribution

consider instead $Z_n^{1/p}$ which is also independent of $\overline{X}_n$ and is a feasible definite statistic on $\mathbb{A}$ with positive degree 1 of homogeneity. Now, by the homogeneity property, rewrite, as in the proof of Lemma 1,

$$Z_n = U(X_1 - \overline{X}_n, X_2 - \overline{X}_n, \ldots, X_n - \overline{X}_n) = S_n U(\Lambda_1, \Lambda_2, \ldots, \Lambda_n),$$

where

$$\Lambda_i = \frac{X_{(i)} - \overline{X}_n}{S_n}, \quad i = 1, 2, \ldots, n,$$

and $(\Lambda_1, \Lambda_2, \ldots, \Lambda_n)$ takes values $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ in $\mathbb{A}_n$ [defined in (13)] almost surely. It then follows from Lemma 1 that there exist two positive constants $k < K$ such that

$$0 < k \leq \frac{Z_n}{S_n} = U(\Lambda_1, \Lambda_2, \ldots, \Lambda_n) \leq K < \infty \ a.s. \quad (23)$$

Write the realized values of

$$S_n = \frac{Z_n}{U(\Lambda_1, \Lambda_2, \ldots, \Lambda_n)}$$

as

$$s_n = \frac{z_n}{U(\lambda_1, \lambda_2, \ldots, \lambda_n)} : = \frac{z_n}{U(\lambda)}, \quad (24)$$

where $\lambda = (\lambda_1, \ldots, \lambda_n)$.

Recall that the order statistics $(X_{(1)}, X_{(2)}, \ldots, X_{(n)})$ have joint density

$$f(x_{(1)}, x_{(2)}, \ldots, x_{(n)}) = n! \prod_{i=1}^{n} f_X(x_{(i)}), \quad x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}.$$ 

Next, consider the composition of two transformations $T$ [defined in (15)] and $T^*$ defined by

$$T^* : (t_1, t_2, \ldots, t_{n-2}, w_1, w_2) \longrightarrow (t_1, t_2, \ldots, t_{n-2}, w_1, z_n),$$

where $w_1 = \xi_n$, $w_2 = s_n$, and $z_n$ is defined in (24). The latter $T^*$ has Jacobian $J^* = 1/U(\lambda) > 0$ due to (24). Hence, the composition $T^* \circ T$ has a Jacobian with absolute value (by Lemma 2):
where \(f_{n-2} = 1 - \sum_{i=1}^{n-2} t_i\). Therefore, the statistics \((T_1, T_2, \ldots, T_{n-2}, \overline{X}_n, Z_n)\) have a joint density on \(\mathbb{R}_n\) [see (18)]:

\[
f(t_1, t_2, \ldots, t_{n-2}, \overline{x}_n, z_n) = n! \prod_{i=1}^{n} f_X(x_{(i)})|J|J^*,
\]

where by (16), (22) and (24),

\[
x_{(i)} = \overline{x}_n + \frac{z_n}{U(\lambda)} \cdot \lambda_i(t_1, t_2, \ldots, t_{n-2}), \quad i = 1, 2, \ldots, n.
\]

This in turn implies that the joint density of \((\overline{X}_n, Z_n)\) has the form (here, for simplicity, denote \(\overline{x} = \overline{x}_n\) and \(z = z_n\)):

\[
f(\overline{x}, z) = n! \sqrt{n(n-1)}^{(n-1)/2} \cdot z^{n-2} \int_{\mathbb{B}_{n-2}} \left(\frac{1}{U(\lambda)}\right)^{n-1} \times \int_{\mathbb{I}} f_X^2 \left(\overline{x} + \frac{z}{U(\lambda)} \cdot \lambda_i(t_1, t_2, \ldots, t_{n-2})\right) \prod_{i=1}^{n-2} dt_i, \quad \overline{x} \in \mathbb{R}, \ z \geq 0,
\]

where the set \(\mathbb{B}_{n-2} \subset \mathbb{R}^{n-2}\) is defined in (21).

Now, we apply the independence condition on \(\overline{X}_n\) and \(Z_n\), and write the joint density in (25) as the products of the densities of \(\overline{X}_n\) and \(Z_n\):

\[
f(\overline{x}, z) = f_{\overline{X}_n}(\overline{x})f_{Z_n}(z), \quad \overline{x} \in \mathbb{R} \text{ and } z \geq 0.
\]

Letting \(\overline{x} = 0\) in (25) and (26), we get the density of \(Z_n\):

\[
f_{Z_n}(z) = \frac{1}{f_{\overline{X}_n}(0)} f(0, z)
\]

\[
= \frac{1}{f_{\overline{X}_n}(0)} n! \sqrt{n(n-1)}^{(n-1)/2} \cdot z^{n-2} \int_{\mathbb{B}_{n-2}} \left(\frac{1}{U(\lambda)}\right)^{n-1} \times \int_{\mathbb{I}} f_X \left(\frac{z}{U(\lambda)} \cdot \lambda_i(t_1, t_2, \ldots, t_{n-2})\right) \prod_{i=1}^{n-2} dt_i, \quad z \geq 0.
\]

Plugging (25) and (27) in (26) and canceling the common term \(\sqrt{n(n-1)}^{(n-1)/2} \cdot z^{n-2}\), and then letting \(z \to 0\) and canceling the common integral term, we obtain the density of \(\overline{X}_n\):
\[ f_{\overline{X}_n}(\overline{x}) = c \left( f_{\overline{X}}(\overline{x}) \right)^n, \quad \overline{x} \in \mathbb{R}, \] (28)

where \( c = f_{\overline{X}_n}(0)/f_{\overline{X}}(0) > 0 \) is a constant.

Combining (25) through (28), we finally have the integro-functional equation:

\[
\int_{\mathbf{B}_{n-2}} \left( \frac{1}{U(\lambda)} \right)^{n-1} \cdot f^{-1/2}_{n-2} \prod_{i=1}^{n} f_{X} \left( \frac{\overline{x} + \frac{z}{U(\lambda)} \cdot \lambda_i(t_1, t_2, \ldots, t_{n-2})}{\sum_{i=1}^{n} \sigma_i(t_1, t_2, \ldots, t_{n-2})} \right) \prod_{i=1}^{n-2} dt_i \\
= C \left( f_{\overline{X}}(\overline{x}) \right)^n \int_{\mathbf{B}_{n-2}} \left( \frac{1}{U(\lambda)} \right)^{n-1} \cdot f^{-1/2}_{n-2} \prod_{i=1}^{n} f_{X} \left( \frac{z}{U(\lambda)} \cdot \lambda_i(t_1, t_2, \ldots, t_{n-2}) \right) \prod_{i=1}^{n-2} dt_i,
\] (29)

for all \( \overline{x} \in \mathbb{R} \) and \( z \geq 0 \), where \( C = 1/f_{\overline{X}}(0) > 0 \) is a constant.

It is seen that (29) is of the form of Anosov’s integro-functional equation (12), because

\[
\int_{\mathbf{B}_{n-2}} \left( \frac{1}{U(\lambda)} \right)^{n-1} \cdot f^{-1/2}_{n-2} \prod_{i=1}^{n-2} dt_i \in (0, \infty),
\]

\[
\sum_{i=1}^{n} \sigma_i(t_1, t_2, \ldots, t_{n-2}) := \sum_{i=1}^{n} \frac{\lambda_i(t_1, t_2, \ldots, t_{n-2})}{U(\lambda)} = 0,
\]

\[
\sum_{i=1}^{n} \sigma_i^2(t_1, t_2, \ldots, t_{n-2}) = \sum_{i=1}^{n} \left[ \frac{\lambda_i(t_1, t_2, \ldots, t_{n-2})}{U(\lambda)} \right]^2 = \frac{n - 1}{U^2(\lambda)} \in (0, \infty).
\]

The last two conditions are required in (11). We can check these conditions by using (22)–(24) and the remarks right after Lemma 3. Besides, \( \mathbf{B}_{n-2}, \overline{x} \) and \( z \) here play the roles of \( \Phi, t \) and \( s \) in (10) and (12), respectively. Therefore, \( f_{\overline{X}} \) is normal by mimicking the proof of Anosov’s theorem. This completes the proof of the theorem. \( \square \)

**Proof of Corollary 2** Under the conditions on \( a_i, i = 1, 2, \ldots, n \), the base function

\[ U(\lambda_1, \lambda_2, \ldots, \lambda_n) = \sum_{i=1}^{n} a_i \lambda_i \]

of (4) is feasible definite on \( \mathbf{A} \) with positive degree 1 of homogeneity. To check \( U \geq 0 \) and the definiteness, write \( \overline{a} = \frac{1}{n} \sum_{i=1}^{n} a_i \) and \( \overline{\lambda} = \frac{1}{n} \sum_{i=1}^{n} \lambda_i = 0 \), then, by Benedetti’s inequality,

\[
\sum_{i=1}^{n} a_i \lambda_i = \sum_{i=1}^{n} (a_i - \overline{a}) (\lambda_i - \overline{\lambda}) \geq \frac{1}{n - 1} \left[ \sum_{i=1}^{n} (a_i - \overline{a})^2 \right]^{1/2} \left[ \sum_{i=1}^{n} (\lambda_i - \overline{\lambda})^2 \right]^{1/2} \geq 0.
\]

The LHS equals zero if and only if \( \lambda_i = \overline{\lambda} = 0 \) for each \( i \), because \( a_i \) are not all equal. \( \square \)

**Proof of Corollary 3** Under the conditions on \( p, a_i, i = 1, 2, \ldots, n \), it is easy to check that the base function
of (5) is feasible definite on \( \mathbf{A} \) with positive degree \( p \) of homogeneity. \( \square \)

**Proof of Corollary 4** Under the conditions on \( p, a_{ij}, 1 \leq i, j \leq n \), it is easy to check that the base function

\[
U(\lambda_1, \lambda_2, \ldots, \lambda_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} |\lambda_i - \lambda_j|^p
\]

of (6) is feasible definite on \( \mathbf{A} \) with positive degree \( p \) of homogeneity. \( \square \)

**Proof of Corollary 6** Under the conditions on \( a_{ij}, 1 \leq i, j \leq n \), it is easy to check that the base function

\[
U(\lambda_1, \lambda_2, \ldots, \lambda_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \lambda_i \lambda_j
\]

of (8) is feasible definite on \( \mathbf{A} \) with positive degree 2 of homogeneity. \( \square \)

**Proof of Corollary 7** Under the conditions on \( p, q, a_{ij}, 1 \leq i, j \leq n \), it is easy to check that the base function

\[
U(\lambda_1, \lambda_2, \ldots, \lambda_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} |\lambda_i|^p |\lambda_j|^q
\]

of (9) is feasible definite on \( \mathbf{A} \) with positive degree \( p + q \) of homogeneity. \( \square \)

5 More results about the sample range and Gini’s mean difference

For the case of sample size \( n = 2 \), a direct calculation shows the interesting relation between the sample variance and the sample range:

\[
S_2^2 = \frac{1}{2} R_2^2.
\]

Hence, if the sample mean \( \bar{X}_2 \) and the sample range \( R_2 \) are independent, then so are \( \bar{X}_2 \) and \( S_2^2 \). This in turn implies that the underlying distribution \( F \) is normal. Therefore, in this case \( (n = 2) \), we do not need to assume the smoothness conditions on the distribution \( F \). Similarly, for Gini’s mean difference in (7), we have

\[
G_2 = R_2.
\]
Hence, if $X_2$ and $G_2$ are independent, then $F$ is normal.

In view of the proof of the Theorem, Lemma 1 plays a crucial role. For the case of the sample range $R_n$, we actually have, by using Benedetti’s inequality, the following explicit bounds in the inequality (14):

$$\frac{\sqrt{2}}{\sqrt{n-1}} \leq \frac{R_n}{S_n} \leq \sqrt{2(n-1)} \text{ a.s., } n \geq 2.$$

Further, a better lower bound ($\sqrt{2}$) can be obtained by Lemma 4 and Proposition 1 below.

**Lemma 4** For any $n$ real numbers $x_1, x_2, \ldots, x_n$, we have the identity:

$$\sum_{i=1}^{n} (x_i - \bar{x}_n)^2 = \frac{1}{2n} \sum_{i=1}^{n} \sum_{j=1}^{n} (x_i - x_j)^2 = \frac{1}{n} \sum_{i<j} (x_i - x_j)^2,$$

where $\bar{x}_n = \frac{1}{n} \sum_{i=1}^{n} x_i$.

*Proof* Write $x_i - x_j = (x_i - \bar{x}_n) + (\bar{x}_n - x_j)$ and carry out the double summation. $\square$

**Proposition 1** (Range Inequality). Assume that $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$ are not all equal, where $n \geq 2$, and denote

$$\bar{x}_n = \frac{1}{n} \sum_{i=1}^{n} x_i \text{ and } s_n^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x}_n)^2.$$

Then, we have the range inequality:

$$\sqrt{2} s_n \leq x_{(n)} - x_{(1)} \leq \sqrt{2(n-1)} s_n,$$

or, equivalently,

$$\sqrt{2} \leq \frac{x_{(n)} - x_{(1)}}{s_n} \leq \sqrt{2(n-1)}.$$

*Proof* It follows from Lemma 4 that

$$\sum_{i=1}^{n} (x_i - \bar{x}_n)^2 = \frac{1}{n} \sum_{i<j} (x_{(i)} - x_{(j)})^2 \leq \frac{1}{n} \left( \frac{n}{2} \right) (x_{(n)} - x_{(1)})^2 = \frac{n-1}{2} (x_{(n)} - x_{(1)})^2.$$

This proves the lower bound of $x_{(n)} - x_{(1)}$ in (30). To prove the upper bound, we apply Cauchy–Schwarz inequality. More precisely, take $y_{(1)} = -1, y_{(2)} = 0, \ldots, y_{(n-1)} = 0, y_{(n)} = 1$. Then, $\bar{y}_n = 0$ and
This completes the proof.

The result (31) can be used to derive the following inequality.

**Proposition 2** Under the conditions of Corollary 4,

\[
(a_{1n} + a_{n1})(\sqrt{2})^{p} \leq \frac{Z_{n}}{S_{n}^{p}} \leq \left( \max_{1 \leq i, j \leq n, i \neq j} a_{ij} \right) n(n - 1)(\sqrt{2(n - 1)})^{p} \text{ a.s.}
\]

In particular, for Gini’s mean difference, we have

\[
\frac{2\sqrt{2}}{n(n - 1)} \leq \frac{G_{n}}{S_{n}} \leq \sqrt{2(n - 1)} \text{ a.s.} \tag{33}
\]

**Remark 3** When \( n = 2 \), the upper and lower bounds in (31) are equal and hence (31) becomes an identity [the same is true for (33)]. Moreover, for \( n \geq 3 \), consider the symmetric case: \(-1, 0, \ldots, 0, 1\), then \( \bar{x}_{n} = 0, x_{(1)} = -1, x_{(n)} = 1 \), and \( s_{n}^{2} = 2/(n - 1) \).

Hence, the right equality in (31) also holds true in this case. This means that the upper bound in (31) is sharp. However, for \( n \geq 3 \), it follows from the proof of Proposition 1 [see (32)] that \( \sqrt{2}s_{n} = x_{(n)} - x_{(1)} \) if and only if all \( x_{i(j)} \) are equal. Therefore, under the assumption that the \( x_{i(j)} \) are not all equal, it is still possible to improve the lower bound in (31) for the case \( n \geq 3 \) [see, e.g., Thomson (1955)]. On the other hand, when \( n \geq 3 \), we have \( G_{n} < R_{n} \text{ a.s.} \) by (7). So, it is possible to improve both the upper and lower bounds in (33) for the case \( n \geq 3 \) [see Barker (1983)].

In view of the above observations and Corollaries 1 and 5, we would like to pose the following conjectures. Equivalently, this is to conjecture that for the cases of the sample range and Gini’s mean difference, we do not need to assume the smoothness condition on the underlying distribution.

**Conjecture 1** Let \( X_{1}, X_{2}, \ldots, X_{n} \) be a random sample of size \( n \geq 3 \) from any distribution \( F \) on \( \mathbb{R} \). If the sample mean \( \bar{X}_{n} \) and the sample range \( R_{n} \) in (3) are independent, then \( F \) is a normal distribution (including the degenerate case).

**Conjecture 2** Let \( X_{1}, X_{2}, \ldots, X_{n} \) be a random sample of size \( n \geq 3 \) from any distribution \( F \) on \( \mathbb{R} \). If the sample mean \( \bar{X}_{n} \) and Gini’s mean difference \( G_{n} \) in (7) are independent, then \( F \) is a normal distribution (including the degenerate case).
6 Discussions

Let \( X_1, X_2, \ldots, X_n \) be a random sample of size \( n \geq 2 \) from normal distribution \( N(\mu, \sigma^2) \). As before, denote the sample mean and the sample variance by \( \bar{X}_n \) and \( S_n^2 \), respectively. Then, it is known that the sampling distribution of the useful statistic

\[
T_{n-1} = \frac{\bar{X}_n - \mu}{S_n / \sqrt{n}}
\]

is Student’s \( t \)-distribution with \( n - 1 \) degrees of freedom. The derivation of the sampling distribution heavily depends on the independence of \( \bar{X}_n \) and \( S_n^2 \) (or \( S_n \)). Analogously, instead of \( S_n / \sqrt{n} \), let \( Z_n \) be another feasible definite statistic with positive degree 1 of homogeneity (e.g., the sample range or Gini’s mean difference; both are linear functions of order statistics), then it is independent of \( \bar{X}_n \). If one can carry out the sampling distribution of the new statistic

\[
T_{n-1}^* = \frac{\bar{X}_n - \mu}{Z_n}
\]

it will be useful in statistical inference, such as finding the confidence interval (CI) of the mean \( \mu \) when \( \sigma \) is unknown. This allows us to compare the CIs obtained through \( T_{n-1} \) and \( T_{n-1}^* \). Besides, the characterization results provided here might be useful in hypothesis testing, namely, testing the normality assumption via the independence of the sample mean and some feasible definite statistics.

Acknowledgements The authors would like to thank the Chief Editor, Associate Editor and two Referees for helpful comments and suggestions, which improve the presentation of the manuscript. We also thank Professor Jordan Stoyanov for the complete information of the references Anosov (1964) and Laha (1956).

References

Anosov, D. V. (1964). On an integral equation from statistics. *Vestnik of Leningrad State University, Series III Mathematics Mechanics Astronomy, I9*, 151–154 (In Russian).
Barker, L. (1983). On Gini’s mean difference and the sample standard deviation. *Communications in Statistics - Simulation and Computation, I2*, 503–505.
Benedetti, C. (1957). Di alcune disuguaglianze collegate al campo di variazione di indici statistico coëficientes. *Metron, I8*, 102–125.
Benedetti, C. (1995). On some inequalities related to the range of some statistical indices. *Metron, 52*, 189–215.
David, H. A., Nagaraja, H. N. (2003). *Order Statistics* (3rd ed.). New Jersey: Wiley.
Geary, R. C. (1936). The distribution of “Student’s” ratio for non-normal samples. *Supplement to the Journal of the Royal Statistical Society, 3*, 178–184.
Georgescu-Roegen, N. (1959). On the extrema of some statistical coefficients. *Metron, 19*, 38–45.
Hwang, T.-Y., Hu, C.-Y. (1994a). The best lower bound of sample correlation coefficient with ordered restriction. *Statistics & Probability Letters, 19*, 195–198.
Hwang, T.-Y., Hu, C.-Y. (1994b). On the joint distribution of Studentized order statistics. *Annals of the Institute of Statistical Mathematics, 46*, 165–177.
Hwang, T.-Y., Hu, C.-Y. (2000). On some characterizations of population distributions. *Taiwanese Journal of Mathematics, 4*, 427–437.

Kagan, A. M., Linnik, Yu. V., Rao, C. R. (1973). *Characterization Problems in Mathematical Statistics*. New York: Wiley.

Kawata, T., Sakamoto, H. (1949). On the characterization of the normal population by the independence of the sample mean and the sample variance. *Journal of the Mathematical Society of Japan, 1*, 111–115.

Laha, R. G. (1956). On the stochastic independence of a homogeneous quadratic statistic and the mean. *Vestnik of Leningrad State University, Series III Mathematics Mechanics Astronomy, 11*, 25–32 (In Russian).

Lukacs, E. (1942). A characterization of the normal distribution. *The Annals of Mathematical Statistics, 13*, 91–93.

Rohatgi, V. K. (1976). *An Introduction to Probability Theory and Mathematical Statistics*. New York: Wiley.

Sarria, H., Martinez, J. C. (2016). A new proof of the Benedetti’s inequality and some applications to perturbation to real eigenvalues and singular values. *Boletín de Matemáticas, 23*, 105–114.

Thomson, G. W. (1955). Bounds for the ratio of range to standard deviation. *Biometrika, 42*, 268–269.

Zinger, A. A. (1951). On independent samples from normal populations. *Uspekhi Matematicheskikh Nauk, 6*, 172–175 (In Russian).

**Publisher’s Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.