Non-perturbative renormalization of moments of parton distribution functions

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We compute non-perturbatively the evolution of the twist-2 operators corresponding to the average momentum of non-singlet quark densities. The calculation is based on a finite-size technique, using the Schrödinger Functional, in quenched QCD. We find that a careful choice of the boundary conditions is essential, for such operators, to render possible the computation. As a by-product we apply the non-perturbatively computed renormalization constants to available data of bare matrix elements between nucleon states.

1. INTRODUCTION

The accurate knowledge of hadron parton densities is an essential ingredient for the experimental test of QCD at accelerator energies. Their normalization is usually obtained from a fit to a set of reference experiments and is used for predicting the behaviour of hard hadron processes in different energy regimes. The calculation of the normalization needs non-perturbative methods. In order to have a phenomenological impact this determination must have a precision comparable with the experiments, and must have all the systematic uncertainties under control. In this proceedings we will mainly summarize the results obtained in [1], to which we refer for any unspecified notations.

2. MOMENTS OF PARTON DISTRIBUTION FUNCTION

The moments of parton distribution functions (PDF) are related to matrix elements of leading twist \( \tau \) (\( \tau = \text{dim-spin} \)) operators of given spin, between hadron states \( h(p) \):

\[
\langle h(p)|O_{\mu_1...\mu_N}|h(p)\rangle = M^{(N-1)}(\mu)p_{\mu_1}...p_{\mu_n} + \text{terms}\ \delta_{\mu_1\mu_2}
\]

\[
\langle x^{(N-1)}(\mu) \rangle = M^{(N-1)}(\mu = Q)
\]

On the lattice the \( O(4) \) symmetry is broken to the hypercubic group \( H(4) \), and the 2 irreducible representations of the non-singlet operators for \( N = 2 \) are

\[
O_{44}(x) = \bar{\psi}(x) \left[ \gamma_4 \mathbf{\hat{D}}_4 - \frac{1}{3} \sum_{k=1}^{3} \gamma_k \mathbf{\hat{D}}_k \right] \frac{\tau^3}{2} \psi(x)
\]

\[
O_{12}(x) = \bar{\psi}(x) \left[ \gamma_1 \mathbf{\hat{D}}_2 + \gamma_2 \mathbf{\hat{D}}_1 \right] \frac{\tau^3}{2} \psi(x)
\]

Our setup will be QCD in a finite space-time volume of size \( T \times L^3 \) with \( T = L \). We choose the same boundary conditions of [4] namely inhomogeneous Dirichlet boundary conditions at time \( x_0 = 0 \) and \( x_0 = T \) and periodic spatial boundary conditions up to a phase for the fermion fields.

\[
\psi(x + L \mathbf{k}) = e^{i\mathbf{k} \cdot \psi(x)}
\]

The strategy used to compute the non-perturbative evolution of the operators in eq. (3) resembles the strategy used by the ALPHA collaboration to compute the running quark mass [5]. The evolution from initially large \( L \) (low \( \mu \)) to small \( L \) (high \( \mu \)) is obtained applying the so called step scaling function (SSF) (cfr sect. 3). Once the perturbative regime is reached (and this must be
checked) one continues the evolution in perturbation theory computing the (scale and scheme independent) RGI matrix element. The connection with experiments is obtained then, making the adequate perturbative evolution of the RGI matrix element in the $\overline{\text{MS}}$ scheme.

3. RENORMALIZATION

The renormalization conditions for the local operators are given by

$$O_R(\mu) = Z^{-1}(a\mu)O(a), \quad O_R(\mu = L^{-1}) = O^{(0)}$$

The correlation functions to compute the Z factor are

$$f_O(x_0/L, \theta) = -a^6 \sum_{x,y,z} \langle O(x) S_q(y,z) \rangle$$

with $x_0$ and $S_q$ are suitable quark sources to probe the operators $O$. With this definition the renormalization constants are obtained by

$$Z(a/L, \mu) = c \frac{f_O(x_0/L, \theta)}{\sqrt{f_1(\theta)}}; \quad c = \frac{\sqrt{f_1(\theta)}}{f_O^{(0)}(x_0/L, \theta)}.$$

Optimal choice of $\theta$ and $x_0$ is mandatory to obtain a reliable signal of the correlation function $\langle O \rangle$. In fig. 1 a study of the relative error of the $Z$ factor is performed. A similar analysis can be performed for the cut-off effects and for the convergence of perturbation theory computing the 2-loop anomalous dimensions for these operators in the SF scheme [1]. From these studies a good choice turns out to be $\theta_1 = 1.0, \theta_{2,3} = 0$ and $x_0 = \frac{L}{2}$.

To map out the $L$ dependence recursively we use the SSF, rigorously defined on the lattice by

$$\Sigma_{O}(u,a/L) = Z_{O}(u,2L/a)Z_{O}(u,L/a) \tag{8}$$

where $S_q$ and $S_q'$ are suitable quark sources to probe the operators $O$. With this definition the renormalization constants are obtained by

$$\langle O \rangle_{\text{RGI}} = \lim_{a \to 0} \frac{\langle O \rangle(a)}{\sigma_{Z_O}(\mu/\mu_0, \tilde{g}^2(\mu))F_{\text{SSF}}(\tilde{g}^2(\mu)) \times Z_O(a, \mu_0)\times Z_O^2(\mu)}$$

and

$$\langle O \rangle_{\text{RGI}} = \lim_{a \to 0} \frac{\langle O \rangle(a)}{\sigma_{Z_O}(\mu/\mu_0, \tilde{g}^2(\mu))F_{\text{SSF}}(\tilde{g}^2(\mu)) \times Z_O(a, \mu_0)\times Z_O^2(\mu)}$$

The values of $\beta$ corresponding to a fixed running coupling are available in [5]. We have computed the SSF at 9 values of the renormalized coupling ($\tilde{g}_{SF}^2(L) = 0.8873$ to $3.48$) corresponding to a range of energies that are roughly between $300$ MeV and $100$ GeV. In order to have a better control on the continuum limit we have performed the computation with Wilson and Clover action, even if in both cases one expects $O(a)$ lattice artefacts since the local operators are not improved. In fig. 2 the continuum limit of the SSF for some values of $\tilde{g}_{SF}^2(L)$ is shown. It is clear that a reliable (constrained) linear extrapolation with a small slope is possible.

The formula that summarizes the whole strategy is given by

$$\langle O \rangle_{\text{RGI}} = \lim_{a \to 0} \frac{\langle O \rangle(a)}{\sigma_{Z_O}(\mu/\mu_0, \tilde{g}^2(\mu))F_{\text{SSF}}(\tilde{g}^2(\mu)) \times Z_O(a, \mu_0)\times Z_O^2(\mu)}$$

Figure 1. Relative errors for the $Z$ factor computed with 400 measurements on a $16^4$ lattice at $\tilde{g}_{SF}^2(L) = 3.48$

Figure 2. Continuum extrapolation of the SSF for selected values of $\tilde{g}_{SF}^2(L)$
Figure 3. Continuum limit of the non-perturbative renormalized first moment of the PDF in a proton

where we use the $n = 9$ SSF computed with $\mu = \mu_n$

\[
\sigma\left(\frac{\mu}{\mu_0}, \bar{g}^2(\mu)\right) = \sigma\left(\frac{\mu_1}{\mu_0}, \bar{g}^2(\mu_1)\right) \cdots \sigma\left(\frac{\mu_n}{\mu_{n-1}}, \bar{g}^2(\mu_n)\right)
\]

to jump from the non-perturbative scale $\mu_0$ to the perturbative (ultraviolet) scale $\mu$. At this point one can try to do the perturbative matching using

\[
F_{SF}(\bar{g}^2(\mu)) = [\bar{g}^2(\mu)]^{-\frac{\alpha_s}{2\beta_0}} \times \exp\left\{ -\int_0^{\bar{g}^2(\mu)} dx \left( \frac{\gamma(x)}{\beta(x)} - \frac{\gamma_0}{2\beta_0} \right) \right\}
\]

computed with 3-loop $\beta$ and 2-loop $\gamma$ functions. If the perturbative matching has been successful the quantity

\[
\sigma_{\text{inv}}^{UV}(\mu_0) = \sigma(\mu/\mu_0, \bar{g}^2(\mu)) F_{SF}(\bar{g}^2(\mu))
\]

should be independent from the ultraviolet scale $\mu$. Indeed what we find is that the last 4-5 steps give a very nice plateaux (see fig. 10 in ref. [1]).

So it is possible to continue the evolution from the last step using perturbation theory.

4. PRELIMINARY RESULTS

Using perturbation theory is possible to compute the $Z^{RGI}$, that is the fundamental quantity to relate bare matrix elements to any desirable scheme (e.g. $\overline{\text{MS}}$)

\[
\langle O \rangle_{\text{RGI}} = \lim_{a \to 0} \frac{\langle O \rangle(a)}{Z^{RGI}(a)} = \langle O \rangle_{\overline{\text{MS}}}(\mu) F_{\overline{\text{MS}}}(\mu)
\]

where

\[
Z^{RGI}(a) = Z(\mu, \mu_0, \bar{g}^2(\mu)) \frac{1}{\sigma(\mu/\mu_0, \bar{g}^2(\mu)) F_{SF}(\bar{g}^2(\mu))}
\]

It is then clear that knowing $Z^{RGI}$ for a certain discretization allows to compute in our case the parton average momentum $\langle x \rangle$ in the proton by

\[
\langle x \rangle_{\overline{\text{MS}}} = \lim_{a \to 0} \frac{\langle x \rangle(a)}{Z^{RGI}(a) F_{\overline{\text{MS}}}(\mu)}
\]

We then apply the $Z^{RGI}$ we have computed to the unpublished data of QCDSF for the nucleon bare matrix element and to the published data of LHPC available at only one value of $\beta$. The continuum limit is shown in fig. 4.

5. CONCLUSIONS

We have computed in the continuum and in a fully non-perturbative way the evolution of the twist 2 non-singlet operators with a very good precision (4%). This computation, combined with a calculation of the bare matrix element between hadron states (for an application to pion matrix elements cfr. [4]), gives the renormalization group invariant matrix element $\langle x \rangle_{\text{RGI}}$. Then the RGI matrix element can be simply converted to any desirable scheme. The precision of the experimental data requires a better control on all the systematic uncertainties (non-perturbative renormalization, continuum limit, chiral extrapolation, finite volume effects [2], quenching). In this contribution we have shown how to have complete control over the non-perturbative renormalization and on the continuum limit. There is still a disagreement between the experiments and the lattice computation, but there are also still systematic uncertainties in the lattice computation that must be carefully analyzed. It is clear that a comparison between experiment and theory cannot be reliably done with a lattice simulation at one value of the lattice spacing and without doing a non-perturbative renormalization. On the other systematic uncertainties works are in progress.

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