On the blow-up problem and new \textit{a priori} estimates for the 3D Euler and the Navier-Stokes equations

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Abstract

We study blow-up rates and the blow-up profiles of possible asymptotically self-similar singularities of the 3D Euler equations, where the sense of convergence and self-similarity are considered in various sense. We extend much further, in particular, the previous nonexistence results of self-similar/asymptotically self-similar singularities obtained in [2, 3]. Some implications the notions for the 3D Navier-Stokes equations are also deduced. Generalization of the self-similar transforms is also considered, and by appropriate choice of the transform we obtain new \textit{a priori} estimates for the 3D Euler and the Navier-Stokes equations.

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1 Asymptotically self-similar singularities

We are concerned on the following Euler equations for the homogeneous incompressible fluid flows in $\mathbb{R}^3$.

$$
\begin{aligned}
\frac{\partial v}{\partial t} + (v \cdot \nabla)v &= -\nabla p, \quad (x, t) \in \mathbb{R}^3 \times (0, \infty) \\
\text{div } v &= 0, \quad (x, t) \in \mathbb{R}^3 \times (0, \infty) \\
v(x, 0) &= v_0(x), \quad x \in \mathbb{R}^3
\end{aligned}
$$

where $v = (v_1, v_2, v_3)$, $v_j = v_j(x, t)$, $j = 1, 2, 3$, is the velocity of the flow, $p = p(x, t)$ is the scalar pressure, and $v_0$ is the given initial velocity, satisfying $\text{div } v_0 = 0$. The system (E) is first modeled by Euler in [13]. The local well-posedness of the Euler equations in $H^m(\mathbb{R}^3)$, $m > 5/2$, is established by Kato in [17], which says that given $v_0 \in H^m(\mathbb{R}^3)$, there exists $T \in (0, \infty]$ such that there exists unique solution to (E), $v \in C([0, T); H^m(\mathbb{R}^3))$. The finite time blow-up problem of the local classical solution is known as one of the most important and difficult problems in partial differential equations (see e.g. [20, 6, 7, 8, 2] for graduate level texts and survey articles on the current status of the problem). We say a local in time classical solution $v \in C([0, T); H^m(\mathbb{R}^3))$ blows up at $T$ if and only if

$$
\int_0^T \|\omega(t)\|_{L^\infty} dt = \infty.
$$

There are studies of geometric nature for the blow-up criterion([9, 8, 12]). As another direction of studies of the blow-up problem mathematicians also consider various scenarios of singularities and study carefully their possibility of realization (see e.g. [10, 11, 3, 4] for some of those studies). One of the purposes in this paper, especially in this section, is to study more deeply the notions related to the scenarios of the self-similar singularities in the Euler equations, the preliminary studies of which are done in [3, 4]. We recall that system (E) has scaling property that if $(v, p)$ is a solution of the system (E), then for any $\lambda > 0$ and $\alpha \in \mathbb{R}$ the functions

$$
v^{\lambda, \alpha}(x, t) = \lambda^\alpha v(\lambda x, \lambda^{\alpha+1} t), \quad p^{\lambda, \alpha}(x, t) = \lambda^{2\alpha} p(\lambda x, \lambda^{\alpha+1} t)
$$

are also solutions of (E) with the initial data $v_0^{\lambda, \alpha}(x) = \lambda^\alpha v_0(\lambda x)$. In view of the scaling properties in (1.1), a natural self-similar blowing up solution
$v(x, t)$ of (E) should be of the form,

$$v(x, t) = \frac{1}{(T - t)^{\alpha + 1}} \bar{V} \left( \frac{x}{(T - t)^{\frac{1}{\alpha + 1}}} \right)$$  \hspace{1cm} (1.2)

$$p(x, t) = \frac{\alpha + 1}{(T - t)^{\frac{2\alpha}{\alpha + 1}}} \bar{P} \left( \frac{x}{(T - t)^{\frac{1}{\alpha + 1}}} \right)$$  \hspace{1cm} (1.3)

for $\alpha \neq -1$ and $t$ sufficiently close to $T$. Substituting (1.2)-(1.3) into (E), we obtain the following stationary system.

$$\begin{align*}
\alpha \bar{V} + (y \cdot \nabla) \bar{V} + (\alpha + 1)(\bar{V} \cdot \nabla) \bar{V} &= -\nabla \bar{P}, \\
\text{div} \, \bar{V} &= 0,
\end{align*}$$  \hspace{1cm} (1.4)

the Navier-Stokes equations version of which has been studied extensively after Leray’s pioneering paper ([19, 23, 24, 22, 4, 16]). Existence of solution of the system (1.4) is equivalent to the existence of solutions to the Euler equations of the form (1.2)-(1.3), which blows up in a self-similar fashion. Given $(\alpha, p) \in (-1, \infty) \times (0, \infty]$, we say the blow-up is $\alpha$–asymptotically self-similar in the sense of $L^p$ if there exists $\bar{V} \in \dot{W}^{1,p}(\mathbb{R}^3)$ such that the following convergence holds true.

$$\lim_{t \to T}(T - t)^{\frac{3}{(\alpha + 1)p}} \left\| \nabla v(\cdot, t) - \frac{1}{(T - t)^{\frac{1}{\alpha + 1}}} \nabla \bar{V} \left( \frac{\cdot}{(T - t)^{\frac{1}{\alpha + 1}}} \right) \right\|_{L^\infty} = 0$$

if $p = \infty$, while

$$\lim_{t \to T}(T - t)^{1 - \frac{3}{(\alpha + 1)p}} \left\| \omega(\cdot, t) - \frac{1}{(T - t)^{1 - \frac{1}{\alpha + 1}p}} \bar{\Omega} \left( \frac{\cdot}{(T - t)^{\frac{1}{\alpha + 1}}} \right) \right\|_{L^p} = 0$$

if $0 < p < \infty$, where and hereafter we denote

$$\Omega = \text{curl} \, V \quad \text{and} \quad \bar{\Omega} = \text{curl} \, \bar{V}.$$  

The above limit function $\bar{V} \in L^p(\mathbb{R}^3)$ with $\bar{\Omega} \neq 0$ is called the blow-up profile. We observe that the self-similar blow-up given by (1.2)-(1.3) is trivial case of $\alpha$–asymptotic self-similar blow-up with the blow-up profile given by the representing function $\bar{V}$. We say a blow-up at $T$ is of type I, if

$$\limsup_{t \to T}(T - t)\left\| \nabla v(t) \right\|_{L^\infty} < \infty.$$  

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If the blow-up is not of type I, we say it is of *type II*. For the use of terminology, type I and type II blow-ups, we followed the literatures on the studies of the blow-up problem in the semilinear heat equations (see e.g. [21, 14, 15], and references therein). The use of $\|\nabla v(t)\|_{L^\infty}$ rather than $\|v(t)\|_{L^\infty}$ in our definition of type I and II is motivated by Beale-Kato-Majda’s blow-up criterion.

**Theorem 1.1** Let $m > 5/2$, and $v \in C([0, T); H^m(\mathbb{R}^3))$ be a solution to (E) with $v_0 \in H^m(\mathbb{R}^3)$, $\text{div} \, v_0 = 0$. We set

$$\limsup_{t \to T} (T - t)\|\nabla v(t)\|_{L^\infty} := M(T). \quad (1.5)$$

Then, either $M(T) = 0$ or $M(T) \geq 1$. The former case corresponds to non blow-up, and the latter case corresponds to the blow-up at $T$. Hence, the blow-up at $T$ is of type I if and only if $M(T) \geq 1$.

**Proof** It suffices to show that $M(T) < 1$ implies non blow-up at $T$, which, in turn, leads to $M(T) = 0$, since $\|\nabla v(t)\|_{L^\infty} \in C([0, T])$ in this case. We suppose $M(T) < 1$. Then, there exists $t_0 \in (0, T)$ such that

$$\sup_{t_0 < t < T} (T - t)\|\nabla v(t)\|_{L^\infty} := M_0 < 1.$$  

Taking curl of the evolution part of (E), we have the vorticity equation,

$$\frac{\partial \omega}{\partial t} + (v \cdot \nabla)\omega = (\omega \cdot \nabla)v.$$  

This, taking dot product with $\xi = \omega/|\omega|$, leads to

$$\frac{\partial |\omega|}{\partial t} + (v \cdot \nabla)|\omega| = (\xi \cdot \nabla)v \cdot \xi|\omega|.$$  

Integrating this over $[t_0, t]$ along the particle trajectories $\{X(a, t)\}$ defined by $v(x, t)$, we have

$$|\omega(X(a, t), t)| = |\omega(X(a, t_0), t_0)| \exp \left[ \int_{t_0}^{t} (\xi \cdot \nabla)v \cdot \xi(X(a, s), s) ds \right], \quad (1.6)$$  

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from which we estimate

\[ \| \omega(t) \|_{L^\infty} \leq \| \omega(t_0) \|_{L^\infty} \exp \left[ \int_{t_0}^t \| \nabla v(\tau) \|_{L^\infty} d\tau \right] \]

\[ < \| \omega(t_0) \|_{L^\infty} \exp \left[ M_0 \int_{t_0}^t (T - \tau)^{-1} d\tau \right] \]

\[ = \| \omega(t_0) \|_{L^\infty} \left( \frac{T - t_0}{T - t} \right)^{M_0}. \tag{1.7} \]

Since \( M_0 < 1 \), we have \( \int_{t_0}^T \| \omega(t) \|_{L^\infty} dt < \infty \), and thanks to the Beale-Kato-Majda criterion there exists no blow-up at \( T \), and we can continue our classical solution beyond \( T \). □

The following is our main theorem in this section.

**Theorem 1.2** Let a classical solution \( v \in C([0, T); H^m(\mathbb{R}^3)) \) with initial data \( v_0 \in H^m(\mathbb{R}^3) \cap \dot{W}^{1,p}(\mathbb{R}^3) \), \( \text{div} v_0 = 0 \), \( \omega_0 \neq 0 \) blows up with type I. Let \( M = M(T) \) be as in Theorem 1.1. Suppose \((\alpha, p) \in (-1, \infty) \times (0, \infty] \) satisfies

\[ M < \left| 1 - \frac{3}{(\alpha + 1)p} \right|. \tag{1.8} \]

Then, there exists no \( \alpha \)-asymptotically self-similar blow-up at \( t = T \) in the sense of \( L^p \) if \( \omega_0 \in L^p(\mathbb{R}^3) \). Hence, for any type I blow-up and for any \( \alpha \in (-1, \infty) \) there exists \( p_1 \in (0, \infty] \) such that it is not \( \alpha \)-asymptotically self-similar in the sense of \( L^{p_1} \).

**Remark 1.1** We note that the case \( p = \infty \) of the above theorem follows from Theorem 1.1, which states that there is no singularity at all at \( t = T \) in this case. The above theorem can be regarded an improvement of the main theorem in [4], in the sense that we can consider the \( L^p \) convergence only to exclude nontrivial blow-up profile \( \bar{V} \), where \( p \) depends on \( M \). Moreover, we do not need to use the Besov space \( \dot{B}^0_{\infty, 1} \) in the statement of the theorem, and the continuation principle of local solution in the Besov space in the proof.

**Proof of Theorem 1.2** We assume asymptotically self-similar blow-up happens at \( T \). Let us introduce similarity variables defined by

\[ y = \frac{x}{(T - t)^{\alpha+1}}, \quad s = \frac{1}{\alpha + 1} \log \left( \frac{T}{T - t} \right), \]
and transformation of the unknowns \((v, p) \rightarrow (V, P)\) according to
\[
v(x, t) = \frac{1}{(T - t)^{\frac{\alpha}{\alpha + 1}}} V(y, s), \quad p(x, t) = \frac{1}{(T - t)^{\frac{2\alpha}{\alpha + 1}}} P(y, s).
\]
Substituting \((v, p)\) into the \((E)\) we obtain the equivalent evolution equation for \((V, P)\),
\[
(E_1) \begin{cases} 
V_s + \alpha V + (y \cdot \nabla) V + (\alpha + 1) (V \cdot \nabla) V = -\nabla P, \\
\text{div} V = 0, \\
V(y, 0) = V_0(y) = T^{\frac{\alpha}{\alpha + 1}} v_0(T^{\frac{1}{\alpha}} y).
\end{cases}
\]
Then the assumption of asymptotically self-similar singularity at \(T\) implies that there exists \(\bar{V}_\alpha \in \dot{W}^{1,p}(\mathbb{R}^3)\) such that
\[
\lim_{s \to \infty} \| \Omega(\cdot, s) - \bar{\Omega} \|_{L^p} = 0.
\]
Now the hypothesis \((1.8)\) implies that there exists \(t_0 \in (0, T)\) such that
\[
\sup_{t_0 < t < T} (T - t) \| \nabla v(t) \|_{L^\infty} := M_0 < 1 - \left| \frac{3}{(\alpha + 1)p} \right|.
\]
Taking \(L^p(\mathbb{R}^3)\) norm of \((1.6)\), taking into account the following simple estimates,
\[
-\| \nabla v(\cdot, t) \|_{L^\infty} \leq (\xi \cdot \nabla) v \cdot \xi(x, t) \leq \| \nabla v(\cdot, t) \|_{L^\infty} \quad \forall (x, t) \in \mathbb{R}^3 \times [t_0, T),
\]
we obtain, for all \(p \in (0, \infty]\),
\[
\| \omega(t_0) \|_{L^p} \exp \left[ -\int_{t_0}^t \| \nabla v(\cdot, s) \|_{L^\infty} ds \right] \leq \| \omega(t) \|_{L^p} \leq \| \omega_0 \|_{L^p} \exp \left[ \int_{t_0}^t \| \nabla v(\cdot, s) \|_{L^\infty} ds \right],
\]
where we use the fact that \(a \mapsto X(a, t)\) is a volume preserving map. From the fact
\[
\int_{t_0}^t \| \nabla v(\cdot, s) \|_{L^\infty} ds \leq M_0 \int_{t_0}^t (T - \tau)^{-1} d\tau = -M_0 \log \left( \frac{T - t}{T - t_0} \right),
\]
we get
\[
\int_{t_0}^t \| \nabla v(\cdot, s) \|_{L^\infty} ds \leq -M_0 \log \left( \frac{T - t}{T - t_0} \right).
\]
and
\[ \frac{\| \omega(t) \|_{L^p}}{\| \omega(t_0) \|_{L^p}} = \left( \frac{T - t}{T - t_0} \right)^{\frac{3}{(\alpha + 1)p} - 1} \frac{\| \Omega(s) \|_{L^p}}{\| \Omega(s_0) \|_{L^p}}, \]
where we set
\[ s_0 = \frac{1}{\alpha + 1} \log \left( \frac{T}{T - t_0} \right), \]
we find that (1.1) leads us to
\[ \left( \frac{T - t}{T - t_0} \right)^{M_0 + 1 - \frac{3}{(\alpha + 1)p}} \leq \frac{\| \Omega(s) \|_{L^p}}{\| \Omega(s_0) \|_{L^p}} \leq \left( \frac{T - t}{T - t_0} \right)^{-M_0 + 1 - \frac{3}{(\alpha + 1)p}} \]
for all \( p \in (0, \infty] \). Passing \( t \to T \), which is equivalent to \( s \to \infty \) in (1.13), we have from (1.10)
\[ \lim_{s \to \infty} \frac{\| \Omega(s) \|_{L^p}}{\| \Omega(s_0) \|_{L^p}} = \frac{\| \bar{\Omega} \|_{L^p}}{\| \Omega(s_0) \|_{L^p}} \in (0, \infty). \] (1.14)
By (1.11) \( M_0 + 1 - \frac{3}{(\alpha + 1)p} < 0 \) or \( -M_0 + 1 - \frac{3}{(\alpha + 1)p} > 0 \). In the former case we have
\[ \lim_{t \to T} \left( \frac{T - t}{T - t_0} \right)^{M_0 + 1 - \frac{3}{(\alpha + 1)p}} = \infty, \] (1.15)
while, in the latter case
\[ \lim_{t \to T} \left( \frac{T - t}{T - t_0} \right)^{-M_0 + 1 - \frac{3}{(\alpha + 1)p}} = 0. \] (1.16)
Both of (1.15) and (1.16) contradicts with (1.14). If the blow-up is of type I, and \( M(T) < \infty \), then one can always choose \( p_1 \in (0, p_0) \) so small that (1.8) is valid for \( p = p_1 \). With such \( p_1 \) it is not \( \alpha \)-asymptotically self-similar in \( L^{p_1} \). \( \square \)

For the self-similar blowing-up solution of the form (1.2)-(1.3) we observe that in order to be consistent with the energy conservation, \( \| v(t) \|_{L^2} = \| v_0 \|_{L^2} \) for all \( t \in [0, T) \), we need to fix \( \alpha = 3/2 \). Since the self-similar blowing up solution corresponds to a trivial convergence of the asymptotically self-similar blow-up, the following is immediate from Theorem 1.2.
Corollary 1.1 Given $p \in (0, \infty]$, there exists no self-similar blow-up with the blow-up profile $V$ satisfying $\Omega \in L^p(\mathbb{R}^3)$ if

$$\|\nabla V\|_{L^\infty} < \left| 1 - \frac{6}{5p} \right|.$$  

(1.17)

Remark 1.2 The above corollary implies that we can exclude self-similar singularity of the Euler equations only under the assumption of $\Omega \in L^p(\mathbb{R}^3)$ if $p$ satisfies the condition (1.17).

The following is, in turn, immediate from the above corollary, which is nothing but Theorem 1.1 in [3].

Corollary 1.2 There exists no self-similar blow-up with the blow-up profile $V$ satisfying $\Omega \in L^p(\mathbb{R}^3)$ for all $p \in (0, p_0)$ for some $p_0 > 0$.

The following theorem is concerned on the possibility of type II asymptotically self-similar singularity of the Euler equations, for which the blow-up rate near the possible blow-up time $T$ is

$$\|\nabla v(t)\|_{L^\infty} \sim \frac{1}{(T - t)^\gamma}, \quad \gamma > 1.$$  

(1.18)

Theorem 1.3 Let $v \in C([0, T); H^m(\mathbb{R}^3))$, $m > 5/2$, be local classical solution of the Euler equations. Suppose there exists $\gamma > 1$ and $R_1 > 0$ such that the following convergence holds true.

$$\lim_{t \to T} (T - t)^{(\alpha - \frac{3}{2}) \frac{2}{m+1}} \left\| v(\cdot, t) - \frac{1}{(T - t)^{(\alpha - \frac{3}{2}) \frac{2}{m+1}}} \dot{V} \left( \frac{\cdot}{(T - t)^{\frac{2}{m+1}}} \right) \right\|_{L^2(B_{R_1})} = 0,$$  

(1.19)

where $B_{R_1} = \{ x \in \mathbb{R}^3 \mid |x| < R_1 \}$. Then, the blow-up profile $\dot{V} \in L^2_{loc}(\mathbb{R}^3)$ is a weak solution of the following stationary Euler equations,

$$(\dot{V} \cdot \nabla) \dot{V} = -\nabla P, \quad \text{div} \dot{V} = 0.$$  

(1.20)

Proof We introduce a self-similar transform defined by

$$v(x, t) = \frac{1}{(T - t)^{\frac{\alpha}{m+1}}} V(y, s), \quad p(x, t) = \frac{1}{(T - t)^{\frac{2\alpha}{m+1}}} P(y, s)$$  

(1.21)
with
\[ y = \frac{1}{(T-t)^{\alpha+1}} x, \quad s = \frac{1}{(\gamma - 1)T^{\gamma-1}} \left[ \frac{T^{\gamma-1}}{(T-t)^{\gamma-1}} - 1 \right]. \tag{1.22} \]

Substituting \((v,p)\) in (1.21)-(1.22) into the \((E)\), we have
\[
\begin{align*}
(E_2) & \quad \left\{ \begin{array}{ll}
- \frac{\gamma}{s(\gamma - 1) + T^{1-\gamma}} \left[ \frac{\alpha}{\alpha + 1} V + \frac{1}{\alpha + 1} (y \cdot \nabla)V \right] & = V_s + (V \cdot \nabla)V + \nabla P, \\
\text{div } V & = 0,
\end{array} \right.
\end{align*}
\]
\[ V(y,0) = V_0(y) = v_0(y). \tag{1.23} \]

The hypothesis (1.19) is written as
\[ \lim_{s \to \infty} \| V(\cdot, s) - \bar{V}(\cdot) \|_{L^2(B_R(s))} = 0, \quad R(s) = \left[ (\gamma - 1)s + \frac{1}{T^{\gamma-1}} \right]^{\frac{1}{\alpha + 1}(\gamma-1)}, \tag{1.24} \]

which implies that
\[ \lim_{s \to \infty} \| V(\cdot, s) - \bar{V} \|_{L^2(B_R)} = 0, \quad \forall R > 0, \tag{1.25} \]

where \(V(y,s)\) is defined by (1.21). Similarly to [16, 4], we consider the scalar test function \(\xi \in C^1_0(0,1)\) with \(\int_0^1 \xi(s) ds \neq 0\), and the vector test function \(\phi = (\phi_1, \phi_2, \phi_3) \in C^1_0(\mathbb{R}^3)\) with \(\text{div } \phi = 0\).

We multiply the first equation of \((E_2)\), in the dot product, by \(\xi(s-n)\phi(y)\), and integrate it over \(\mathbb{R}^3 \times [n, n+1]\), and then we integrate by parts to obtain
\[
\begin{align*}
& + \frac{\alpha}{\alpha + 1} \int_0^1 \int_{\mathbb{R}^3} g(s + n)\xi(s)V(y, s + n) \cdot \phi(y) dy ds \\
& - \frac{1}{\alpha + 1} \int_0^1 \int_{\mathbb{R}^3} g(s + n)\xi(s)V(y, s + n) \cdot (y \cdot \nabla)\phi(y) dy ds \\
& = \int_0^1 \int_{\mathbb{R}^3} \xi_s(s)\phi(y) \cdot V(y, s + n) dy ds \\
& + \int_0^1 \int_{\mathbb{R}^3} \xi(s) \left[ V(y, s + n) \cdot (V(y, s + n) \cdot \nabla)\phi(y) \right] dy ds = 0,
\end{align*}
\]
where we set
\[ g(s) = \frac{\gamma}{s(\gamma - 1) + T^{1-\gamma}}. \]
Passing to the limit \( n \to \infty \) in this equation, using the facts \( \int_0^1 \xi(s)ds = 0 \), \( \int_0^1 \xi(s)ds \neq 0 \), \( V(\cdot, s+n) \to \tilde{V} \) in \( L^2_{loc}(\mathbb{R}^3) \), and finally \( g(s+n) \to 0 \), we find that \( \tilde{V} \in L^2_{loc}(\mathbb{R}^3) \) satisfies
\[
\int_{\mathbb{R}^3} \tilde{V} \cdot (\nabla \cdot \nabla) \phi(y)dy = 0
\]
for all vector test function \( \phi \in C^1_0(\mathbb{R}^3) \) with \( \text{div} \phi = 0 \). On the other hand, we can pass \( s \to \infty \) directly in the weak formulation of the second equation of (E2) to have
\[
\int_{\mathbb{R}^3} \tilde{V} \cdot \nabla \psi(y)dy = 0
\]
for all scalar test function \( \psi \in C^1_0(\mathbb{R}^3) \). \( \square \)

2 Generalized similarity transforms and new a priori estimates

Let us consider a classical solution to (E) \( v \in C([0,T); H^m(\mathbb{R}^3)) \), \( m > 5/2 \), where we assume \( T \in (0, \infty) \) is the maximal time of existence of the classical solution. Let \( p(x,t) \) be the associated pressure. Let \( \mu(\cdot) \in C^1([0,T)) \) be a scalar function such that \( \mu(t) > 0 \) for all \( t \in [0,T) \) and \( \int_0^T \mu(t)dt = \infty \). We transform from \( (v,p) \) to \( (V,P) \) according to the formula,
\[
v(x,t) = \mu(t)^{\frac{\alpha}{\alpha+1}}V \left( \mu(t)^{\frac{1}{\alpha+1}}x, \int_0^t \mu(\sigma)d\sigma \right), \quad (2.1)
\]
\[
p(x,t) = \mu(t)^{\frac{\alpha}{\alpha+1}}P \left( \mu(t)^{\frac{1}{\alpha+1}}x, \int_0^t \mu(\sigma)d\sigma \right), \quad (2.2)
\]
where \( \alpha \in (-1, \infty) \) as previously. This means that the space-time variables are transformed from \( (x,t) \in \mathbb{R}^3 \times [0,T) \) into \( (y,s) \in \mathbb{R}^3 \times [0, \infty) \) as follows:
\[
y = \mu(t)^{\frac{1}{\alpha+1}}x, \quad s = \int_0^t \mu(\sigma)d\sigma. \quad (2.3)
\]
Substituting (2.1)-(2.3) into the Euler equations, we obtain the equivalent equations satisfied by 
\((V, P)\)

\[
\begin{aligned}
\left\{ 
\begin{array}{l}
\frac{\mu'(t)}{\mu(t)^2} \left[ \frac{\alpha}{\alpha + 1} V + \frac{1}{\alpha + 1} (y \cdot \nabla) V \right] = V_s + (V \cdot \nabla) V + \nabla P, \\
\operatorname{div} V = 0, \\
V(y, 0) = V_0(y) = v_0(y).
\end{array}
\right.
\end{aligned}
\]

We note that the special cases
\[
\mu(t) = \frac{1}{T - t}, \quad \mu(t) = \frac{1}{(T - t)^\gamma}, \quad \gamma > 1
\]

are considered in the previous section. In this section we choose \(\mu(t) = \exp \left[ \pm \gamma \int_0^t \| \nabla v(\tau) \|_{L^\infty} d\tau \right], \gamma \geq 1\). Then,

\[
v(x, t) = \exp \left[ \frac{\pm \gamma \alpha}{\alpha + 1} \int_0^t \| \nabla v(\tau) \|_{L^\infty} d\tau \right] V(y, s),
\]

\[
p(x, t) = \exp \left[ \frac{\pm 2 \gamma \alpha}{\alpha + 1} \int_0^t \| \nabla v(\tau) \|_{L^\infty} d\tau \right] P(y, s)
\]

with

\[
y = \exp \left[ \frac{\pm \gamma}{\alpha + 1} \int_0^t \| \nabla v(\tau) \|_{L^\infty} d\tau \right] x,
\]

\[
s = \int_0^t \exp \left[ \pm \gamma \int_{\tau}^t \| \nabla v(\sigma) \|_{L^\infty} d\sigma \right] d\tau
\]

respectively for the signs \(\pm\). Substituting \((v, p)\) in (2.4)-(2.6) into the (E), we find that \((E_s)\) becomes

\[
\begin{aligned}
\left\{ 
\begin{array}{l}
\mp \gamma \| \nabla V(s) \|_{L^\infty} \left[ \frac{\alpha}{\alpha + 1} V + \frac{1}{\alpha + 1} (y \cdot \nabla) V \right] = V_s + (V \cdot \nabla) V + \nabla P, \\
\operatorname{div} V = 0, \\
V(y, 0) = V_0(y) = v_0(y)
\end{array}
\right.
\end{aligned}
\]

respectively for \(\pm\). Similar equations to the system \((E_\pm)\), without the term involving \((y \cdot \nabla) V\) are introduced and studied in [5], where similarity type of transform with respect to only time variables was considered. The argument
of the global/local well-posedness of the system \((E_{\pm})\) respectively from the local well-posedness result of the Euler equations is as follows. We define

\[
S_{\pm} = \int_0^T \exp \left[ \pm \gamma \int_0^\tau \| \nabla v(\sigma) \|_{L^\infty} d\sigma \right] d\tau.
\]

Then, \(S_{\pm}\) is the maximal time of existence of classical solution for the system \((E_{\pm})\). We also note the following integral invariant of the transform,

\[
\int_0^T \| \nabla v(t) \|_{L^\infty} dt = \int_0^{S_{\pm}} \| \nabla V^\pm(s) \|_{L^\infty} ds.
\]

The key advantage of our choice of the function \(\mu(t)\) here is that the convection term is dominated by \(\mp \gamma \| \nabla V(s) \|_{L^\infty} V\) in the transformed system \((E_{\pm})\) in the vorticity formulation, which enable us to derive new \textit{a priori} estimates for \(\| \omega(t) \|_{L^\infty}\) as follows.

\textbf{Theorem 2.1} Given \(m > 5/2\) and \(v_0 \in H^m(\mathbb{R}^3)\) with \(\text{div} v_0 = 0\), let \(\omega\) be the vorticity of the solution \(v \in C([0,T); H^m(\mathbb{R}^3))\) to the Euler equations \((E)\). Then we have an upper estimate

\[
\| \omega(t) \|_{L^\infty} \leq \frac{\| \omega_0 \|_{L^\infty} \exp \left[ \gamma \int_0^t \| \nabla v(\tau) \|_{L^\infty} d\tau \right]}{1 + (\gamma - 1) \| \omega_0 \|_{L^\infty} \int_0^t \exp \left[ \gamma \int_0^\tau \| \nabla v(\sigma) \|_{L^\infty} d\sigma \right] d\tau},
\]

and lower one

\[
\| \omega(t) \|_{L^\infty} \geq \frac{\| \omega_0 \|_{L^\infty} \exp \left[ -\gamma \int_0^t \| \nabla v(\tau) \|_{L^\infty} d\tau \right]}{1 - (\gamma - 1) \| \omega_0 \|_{L^\infty} \int_0^t \exp \left[ -\gamma \int_0^\tau \| \nabla v(\sigma) \|_{L^\infty} d\sigma \right] d\tau}
\]

for all \(\gamma \geq 1\) and \(t \in [0,T)\). The denominator of the right hand side of \((2.8)\) can be estimated from below as

\[
1 - (\gamma - 1) \| \omega_0 \|_{L^\infty} \int_0^t \exp \left[ -\gamma \int_0^\tau \| \nabla v(\sigma) \|_{L^\infty} d\sigma \right] d\tau \geq \frac{1}{(1 + \| \omega_0 \|_{L^\infty t})^{\gamma - 1}},
\]

which shows that the finite time blow-up does not follow from \((2.8)\).

\textbf{Remark 2.1} We observe that for \(\gamma = 1\), the estimates \((2.7)-(2.8)\) reduce to the well-known ones in \((1.12)\) with \(p = \infty\). Moreover, combining \((2.7)-(2.8)\)
together, we easily derive another new estimate,

\[
\frac{\sinh \left[ \gamma \int_0^t \| \nabla v(\tau) \|_{L^\infty} d\tau \right]}{\int_0^t \cosh \left[ \gamma \int_0^\tau \| \nabla v(\sigma) \|_{L^\infty} d\sigma \right] d\tau} \geq (\gamma - 1) \| \omega_0 \|_{L^\infty}.
\]

Proof of Theorem 2.1 Below we denote \( V^\pm \) for the solutions of \((E^\pm)\) respectively, and \( \Omega^\pm = \text{curl} V^\pm \). Note that \( V^\pm_0 = v_0 := V_0 \) and \( \Omega^\pm_0 = \omega_0 := \Omega_0 \). We will first derive the following estimates for the system \((E^\pm)\).

\[
\| \Omega^+(s) \|_{L^\infty} \leq \frac{\| \Omega_0 \|_{L^\infty}}{1 + (\gamma - 1)s \| \Omega_0 \|_{L^\infty}},
\]

\[
\| \Omega^-(s) \|_{L^\infty} \geq \frac{\| \Omega_0 \|_{L^\infty}}{1 - (\gamma - 1)s \| \Omega_0 \|_{L^\infty}}.
\]

as long as \( V^\pm(s) \in H^m(\mathbb{R}^3) \). Taking curl of the first equation of \((E^\pm)\), we have

\[
\mp \gamma \| \nabla \|_{L^\infty} \left[ \Omega - \frac{1}{\alpha + 1}(y \cdot \nabla)\Omega \right] = \Omega_s + (V \cdot \nabla)\Omega - (\Omega \cdot \nabla)V.
\]

Multiplying \( \Xi = \Omega/|\Omega| \) on the both sides of (2.13), we deduce

\[
|\Omega_s + (V \cdot \nabla)|\Omega| = \frac{\| \nabla V(s) \|_{L^\infty}}{\alpha + 1}(y \cdot \nabla)|\Omega| = (\Xi \cdot \nabla V \cdot \Xi \mp \| \nabla V \|_{L^\infty})|\Omega|
\]

\[
\mp(\gamma - 1)\| \nabla V \|_{L^\infty}|\Omega|
\]

\[
\leq - (\gamma - 1)\| \nabla V \|_{L^\infty}|\Omega| \quad \text{for } (E_+),
\]

\[
\geq (\gamma - 1)\| \nabla V \|_{L^\infty}|\Omega| \quad \text{for } (E_-),
\]

since \( |\Xi \cdot \nabla V \cdot \Xi| \leq |\nabla V| \leq \| \nabla V \|_{L^\infty} \). Given smooth solution \( V(y, s) \) of \((E^\pm)\), we introduce the particle trajectories \( \{Y_\pm(a, s)\} \) defined by

\[
\frac{\partial Y(a, s)}{\partial s} = V_\pm(Y(a, s), s) = \frac{\| \nabla V(s) \|_{L^\infty}}{\alpha + 1} Y(a, s) \quad ; \quad Y(a, 0) = a.
\]

Recalling the estimate

\[
\| \nabla V(s) \|_{L^\infty} \geq \| \Omega(s) \|_{L^\infty} \geq |\Omega(y, s)| \quad \forall y \in \mathbb{R}^3,
\]
we can further estimate from (2.14)
\[
\frac{\partial}{\partial s} |\Omega(Y(a,s),s)| \begin{cases} 
\leq - (\gamma - 1) |\Omega(Y(a,s),s)|^2 & \text{for (}E_+\text{)} \\
\geq (\gamma - 1) |\Omega(Y(a,s),s)|^2 & \text{for (}E_-\text{)}.
\end{cases}
\] (2.15)

Solving these differential inequalities (2.15) along the particle trajectories, we obtain that
\[
|\Omega(Y(a,s),s)| \begin{cases} 
\leq \frac{|\Omega_0(a)|}{1 + (\gamma - 1)s|\Omega_0(a)|} & \text{for (}E_+\text{)} \\
\geq \frac{|\Omega_0(a)|}{1 - (\gamma - 1)s|\Omega_0(a)|} & \text{for (}E_-\text{)}.
\end{cases}
\] (2.16)

Writing the first inequality of (2.16) as
\[
|\Omega^+(Y(a,s),s)| \leq \frac{1}{\frac{1}{|\Omega_0(a)|} + (\gamma - 1)s} \leq \frac{1}{\frac{1}{|\Omega_0|_{L^\infty}} + (\gamma - 1)s},
\]
and then taking supremum over \(a \in \mathbb{R}^3\), which is equivalent to taking supremum over \(Y(a,s) \in \mathbb{R}^3\) due to the fact that the mapping \(a \mapsto Y(a,s)\) is a diffeomorphism (although not volume preserving) on \(\mathbb{R}^3\) as long as \(V \in C([0, S]; H^m(\mathbb{R}^3))\), we obtain (2.11). In order to derive (2.12) from the second inequality of (2.16), we first write
\[
\|\Omega^-(s)\|_{L^\infty} \geq |\Omega(Y(a,s),s)| \geq \frac{1}{\frac{1}{|\Omega_0(a)|} - (\gamma - 1)s},
\]
and then take supremum over \(a \in \mathbb{R}^3\). Finally, in order to obtain (2.7)-(2.8), we just change variables from (2.11)-(2.12) back to the original physical ones, using the fact
\[
\Omega^+(y, s) = \exp \left[ -\gamma \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau \right] \omega(x, t), s = \int_0^t \exp \left[ \gamma \int_0^\tau \|\nabla v(\sigma)\|_{L^\infty} d\sigma \right] d\tau \]
for (2.7), while in order to deduce (2.8) from (2.12) we substitute
\[
\Omega^-(y, s) = \exp \left[ \gamma \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau \right] \omega(x, t), s = \int_0^t \exp \left[ -\gamma \int_0^\tau \|\nabla v(\sigma)\|_{L^\infty} d\sigma \right] d\tau.
\]
Now we can rewrite (2.8) as
\[
\|\omega(t)\|_{L^\infty} \geq -\frac{1}{\gamma - 1} \frac{d}{dt} \log \left\{ 1 - (\gamma - 1)\|\omega_0\|_{L^\infty} \int_0^t \exp \left[ -\gamma \int_0^\tau \|\nabla v(\sigma)\|_{L^\infty} d\sigma \right] d\tau \right\}.
\]

Thus,
\[
\int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau \geq \int_0^t \|\omega(\tau)\|_{L^\infty} d\tau \geq -\frac{1}{\gamma - 1} \log \left\{ 1 - (\gamma - 1)\|\omega_0\|_{L^\infty} \int_0^t \exp \left[ -\gamma \int_0^\tau \|\nabla v(\sigma)\|_{L^\infty} d\sigma \right] d\tau \right\}.
\]

Setting
\[
y(t) := 1 - (\gamma - 1)\|\omega_0\|_{L^\infty} \int_0^t \exp \left[ -\gamma \int_0^\tau \|\nabla v(\sigma)\|_{L^\infty} d\sigma \right] d\tau,
\]

we find further integrable structure in (2.17), which is
\[
y'(t) \geq -((\gamma - 1)\|\omega_0\|_{L^\infty} y(t)\frac{\gamma}{\gamma - 1}.
\]

Solving this differential inequality, we obtain (2.9). □

In the last part of this section we fix \(\mu(t) := \exp \left[ \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau \right]\).

We assume our local classical solution in \(H^m(\mathbb{R}^3)\) blows up at \(T\), and hence \(\mu(T - 0) = \exp \left[ \int_0^T \|\nabla v(\tau)\|_{L^\infty} d\tau \right] = \infty\). Given \((\alpha, p) \in (-1, \infty) \times (0, \infty)\), as previously, we say the blow-up is \(\alpha\)-asymptotically self-similar in the sense of \(L^p\) if there exists \(\bar{V} = \bar{V}_\alpha \in \dot{W}^{1,p}(\mathbb{R}^3)\) such that the following convergence holds true.

\[
\lim_{t \to T} (\mu(t))^{-1} \left\| \nabla v(\cdot, t) - \mu(t) \nabla \bar{V} \left( \mu(t) \frac{1}{\alpha+1} (\cdot) \right) \right\|_{L^\infty} = 0 \quad (2.18)
\]

for \(p = \infty\), and

\[
\lim_{t \to T} (\mu(t))^{-1 + \frac{3}{(\alpha+1)p}} \left\| \omega(\cdot, t) - \mu(t)^{-1} \left( \frac{3}{(\alpha+1)p} \bar{\Omega} \left( \mu(t)^{\frac{1}{\alpha+1}} (\cdot) \right) \right) \right\|_{L^p} = 0 \quad (2.19)
\]

for \(p \in (0, \infty)\). The above limiting function \(\bar{V}\) with \(\bar{\Omega} \neq 0\) is called the blow-up profile as previously.
Proposition 2.1 Let $\alpha \neq 3/2$. Then there exists no $\alpha$– asymptotically self-similar blow-up in the sense of $\text{L}^\infty$ with the blow-up profile belongs to $L^2(\mathbb{R}^3)$.

Proof Let us suppose that there exists $\bar{V} \in \dot{W}^{1,\infty}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ such that (2.18) holds, then we will show that $\bar{V} = 0$. In terms of the self-similar variables (2.18) is translated into

$$\lim_{s \to \infty} \| \nabla V(\cdot, s) - \nabla \bar{V} \|_{L^\infty} = 0,$$

where $V$ is defined in (2.1). If $\| \nabla \bar{V} \|_{L^\infty} = 0$, then, the condition $\bar{V} \in L^2(\mathbb{R}^3)$ implies that $\bar{V} = 0$, and there is noting to prove. Let us suppose $\| \nabla \bar{V} \|_{L^\infty} > 0$. The equations satisfied $\bar{V}$ are

$$\begin{aligned}
- \| \nabla \bar{V} \|_{L^\infty} \left[ \frac{\alpha}{\alpha + 1} \bar{V} + \frac{1}{\alpha + 1} (y \cdot \nabla) \bar{V} \right] &= \bar{V} \cdot \nabla \bar{V} + \nabla \bar{P}, \\
\text{div} \bar{V} &= 0
\end{aligned}$$

(2.20)

for a scalar function $\bar{P}$. Taking $L^2(\mathbb{R}^3)$ inner product of the first equation of (2.20) by $\bar{V}$ we obtain

$$\frac{\| \nabla \bar{V} \|_{L^\infty}}{\alpha + 1} \left( \alpha - \frac{3}{2} \right) \| \bar{V} \|_{L^2} = 0.$$

Since $\| \nabla \bar{V} \|_{L^\infty} \neq 0$ and $\alpha \neq \frac{3}{2}$, we have $\| \bar{V} \|_{L^2} = 0$, and $\bar{V} = 0$. \(\square\)

Proposition 2.2 There exists no $\alpha$– asymptotically self-similar blowing up solution to (E) in the sense of $\text{L}^p$ if $0 < p < \frac{3}{2(\alpha+1)}$.

Proof Suppose there exists $\alpha$– asymptotically self-similar blow-up at $T$ in the sense of $L^p$. Then, there exists $\bar{\Omega} \in L^p(\mathbb{R}^3)$ such that, in terms of the self-similar variables introduced in (2.1)-(2.2), we have

$$\lim_{s \to \infty} \| \Omega(s) \|_{L^p} = \| \bar{\Omega} \|_{L^p} < \infty.$$  

(2.21)

We represent the $L^p$ norm of $\| \omega(t) \|_{L^p}$ in terms of similarity variables to obtain

$$\| \omega(t) \|_{L^p} = \mu(t)^{1 - \frac{3}{2(\alpha+1)}} \| \Omega(s) \|_{L^p}, \quad \mu(t) = \exp \left[ \int_0^t \| \nabla v(\tau) \|_{L^\infty} d\tau \right].$$

(2.22)
Substituting this into the lower estimate part of (1.12), we have
\[ \mu(t)^{-2 + \frac{3}{\alpha+1}p} \leq \frac{\|\Omega(s)\|_{L^p}}{\|\Omega_0\|_{L^p}}. \] (2.23)

If \(-2 + \frac{3}{\alpha+1}p > 0\), then taking \(t \to T\) the above inequality we obtain,
\[ \infty = \limsup_{t \to T} \mu(t)^{-2 + \frac{3}{\alpha+1}p} \|\Omega_0\|_{L^p} \]
\[ \leq \limsup_{s \to \infty} \|\Omega(s)\|_{L^p} = \|\bar{\Omega}\|_{L^p}, \]
which is a contradiction to (2.21). □

3 The case of the 3D Navier-Stokes equations

In this section we concentrate on the following 3D Navier-Stokes equations in \(\mathbb{R}^3\) without forcing term.

\[
\begin{cases}
\frac{\partial v}{\partial t} + (v \cdot \nabla)v = \Delta v - \nabla p, & (x,t) \in \mathbb{R}^3 \times (0,\infty) \\
\text{div} v = 0, & (x,t) \in \mathbb{R}^3 \times (0,\infty) \\
v(x,0) = v_0(x) & x \in \mathbb{R}^3.
\end{cases}
\]

(NS)

First, we exclude asymptotically self-similar singularity of type II of (NS), for which the blow-up rate is given by (1.18). We have the following theorem.

**Theorem 3.1** Let \(p \in [3, \infty)\) and \(v \in C([0,T); L^p(\mathbb{R}^3))\) be a local classical solution of the Navier-Stokes equations constructed by Kato(18). Suppose there exists \(\gamma > 1\) and \(\bar{V} \in L^p(\mathbb{R}^3)\) such that the following convergence holds true.

\[
\lim_{t \to T} (T-t)^{-\frac{(p-3)}{2p}} \left\| v(\cdot, t) - (T-t)^{-\frac{(p-3)}{2p}} \bar{V} \left( \left( \frac{\cdot}{(T-t)^{\frac{1}{2}}} \right) \right) \right\|_{L^p} = 0, \quad (3.1)
\]

If the blow-up profile \(\bar{V}\) belongs to \(\dot{H}^1(\mathbb{R}^3)\), then \(\bar{V} = 0\).
Proof Since the main part of the proof is essentially identical to that of Theorem 1.3, we will be brief. Introducing the self-similar variables of the form \((1.21)-(1.23)\) with \(\alpha = \frac{1}{2}\), and substituting \((v, p)\) into the Navier-Stokes equations, we find that \((V, P)\) satisfies
\[
\begin{cases}
-\gamma \frac{2s(\gamma - 1) + 2T^{1-\gamma}}{2s(\gamma - 1) + 2T^{1-\gamma}} [V + (y \cdot \nabla)V] = V_s + (V \cdot \nabla)V - \Delta V + \nabla P,
\text{div} V = 0,
V(y, 0) = V_0(y) = v_0(y).
\end{cases}
\]
The hypothesis \((3.1)\) is now translated as
\[
\lim_{s \to \infty} \|V(\cdot, s) - \bar{V}(\cdot)\|_{L^p} = 0
\]
Following exactly same argument as in the proof of Theorem 1.3, we can deduce that \(\bar{V}\) is a stationary solution of the Navier-Stokes equations, namely there exists \(\bar{P}\) such that
\[
(\bar{V} \cdot \nabla)\bar{V} = \Delta \bar{V} - \nabla \bar{P}, \quad \text{div } \bar{V} = 0. \tag{3.2}
\]
In the case \(\bar{V} \in \dot{H}^1 \cap L^p(\mathbb{R}^3)\), we easily from \((3.2)\) that \(\int_{\mathbb{R}^3} |\nabla \bar{V}|^2 dy = 0\), which implies \(\bar{V} = 0. \quad \square\)

Next, we derive a new a priori estimates for classical solutions of the 3D Navier-stokes equations.

**Theorem 3.2** Given \(v_0 \in H^1(\mathbb{R}^3)\) with \(\text{div} v_0 = 0\), let \(\omega\) be the vorticity of the classical solution \(v \in C([0, T); H^1(\mathbb{R}^3)) \cap C((0, T); C^\infty(\mathbb{R}^3))\) to the Navier-Stokes equations \((\text{NS})\). Then, there exists an absolute constant \(C_0 > 1\) such that for all \(\gamma \geq C_0\) the following enstrophy estimate holds true.
\[
\|\omega(t)\|_{L^2} \leq \frac{\|\omega_0\|_{L^2} \exp \left[\frac{\gamma}{4} \int_0^t \|\omega(\tau)\|_{L^2}^4 d\tau\right]}{\left\{1 + (\gamma - C_0)\|\omega_0\|_{L^2}^4 \int_0^t \exp \left[\gamma \int_0^\tau \|\omega(\sigma)\|_{L^2}^4 d\sigma\right] d\tau\right\}^{\frac{1}{4}}}. \tag{3.3}
\]
The denominator of \((3.3)\) is estimated from below by
\[
1 + (\gamma - C_0)\|\omega_0\|_{L^2}^4 \int_0^t \exp \left[\gamma \int_0^\tau \|\omega(\sigma)\|_{L^2}^4 d\sigma\right] d\tau \leq \frac{1}{(1 - C_0\|\omega_0\|_{L^2}^4 t)^{\frac{\gamma - C_0}{C_0}}} \tag{3.4}
\]
for all \(\gamma \geq C_0\).
Proof Let \((v, p)\) be a classical solution of the Navier-Stokes equations, and \(\omega\) be its vorticity. We transform from \((v, p)\) to \((V, P)\) according to the formula, given by (2.1)-(2.3), where

\[
\mu(t) = \exp \left[ \gamma \int_0^t \|\omega(\tau)\|_{L^2}^4 d\tau \right].
\]

Substituting (2.1)-(2.3) with such \(\mu(t)\) into (NS), we obtain the equivalent equations satisfied by \((V, P)\):

\[
(NS_*) \begin{cases} 
-\gamma \frac{\|\Omega(s)\|_{L^2}^4}{2} [V + (y \cdot \nabla)V] = V_s + (V \cdot \nabla)V - \Delta V - \nabla P, \\
\text{div } V = 0, \\
V(y, 0) = V_0(y) = v_0(y).
\end{cases}
\]

Operating curl on the evolution equations of \((NS_*)\), we obtain

\[
-\gamma \frac{\|\Omega(s)\|_{L^2}^4}{2} [2\Omega + (y \cdot \nabla)\Omega] = \Omega_s + (V \cdot \nabla)\Omega - (\Omega \cdot \nabla)V - \Delta \Omega. \tag{3.5}
\]

Taking \(L^2(\mathbb{R}^3)\) inner product of (3.5) by \(\Omega\), and integrating by part, we estimate

\[
\frac{1}{2} \frac{d}{ds} \|\Omega\|_{L^2}^2 + \|\nabla \Omega\|_{L^2}^2 + \frac{\gamma}{4} \frac{\|\Omega\|_{L^2}^6}{\|\Omega\|_{L^2}^3} = \int_{\mathbb{R}^3} (\Omega \cdot \nabla)V \cdot \Omega dy
\]

\[
\leq \|\Omega\|_{L^3} \|\nabla V\|_{L^3} \|\Omega\|_{L^6} \leq C \|\Omega\|_{L^2}^3 \|\nabla \Omega\|_{L^2}^\frac{3}{2}
\]

\[
\leq \|\nabla \Omega\|_{L^2}^2 + \frac{C_0}{4} \|\Omega\|_{L^2}^6 \tag{3.6}
\]

for an absolute constant \(C_0 > 1\), where we used the fact \(\|\Omega\|_{L^2} = \|\nabla V\|_{L^2}\), the Sobolev imbedding, \(\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)\), the Gagliardo-Nirenberg inequality in \(\mathbb{R}^3\),

\[
\|f\|_{L^3} \leq C \|f\|_{L^2}^\frac{1}{2} \|\nabla f\|_{L^2}^\frac{1}{2},
\]

and Young’s inequality of the form \(ab \leq a^p/p + b^q/q\), \(1/p + 1/q = 1\). Absorbing the term \(\|\nabla \Omega\|_{L^2}^2\) to the left hand side, we have from (3.6)

\[
\frac{d}{ds} \|\Omega\|_{L^2}^2 \leq -\frac{\gamma - C_0}{2} \|\Omega\|_{L^2}^6. \tag{3.7}
\]
Solving the differential inequality \((3.7)\), we have
\[
\|\Omega(s)\|_{L^2} \leq \frac{\|\Omega_0\|_{L^2}}{\left[ 1 + (\gamma - C_0)s\|\Omega_0\|^4_{L^2} \right]^{\frac{1}{4}}.}
\] (3.8)

Transforming back to the original variables and functions, using the relations
\[
s = \int_0^t \exp \left[ \gamma \int_0^{\tau} \|\omega(\sigma)\|^4_{L^2} d\sigma \right] d\tau,
\]
\[
\|\omega(t)\|_{L^2} = \|\Omega(s)\|_{L^2} \exp \left[ \frac{\gamma}{4} \int_0^t \|\omega(\tau)\|^4_{L^2} d\tau \right],
\]
we obtain \((3.3)\). Next, we observe \((3.3)\) can be written as
\[
\|\omega(t)\|^4_{L^2} \leq \frac{1}{\gamma - C_0} \frac{d}{dt} \log \left\{ 1 + (\gamma - C_0)\|\omega_0\|^4_{L^2} \int_0^t \exp \left[ \gamma \int_0^{\tau} \|\omega(\sigma)\|^4_{L^2} d\sigma \right] d\tau \right\},
\]
which, after integration over \([0, t]\), leads to
\[
\int_0^t \|\omega(\tau)\|^4_{L^2} d\tau \leq \frac{1}{\gamma - C_0} \log \left\{ 1 + (\gamma - C_0)\|\omega_0\|^4_{L^2} \int_0^t \exp \left[ \gamma \int_0^{\tau} \|\omega(\sigma)\|^4_{L^2} d\sigma \right] d\tau \right\}
\] (3.9)
for all \(\gamma > C_0\). Setting
\[
y(t) := 1 + (\gamma - C_0)\|\omega_0\|^4_{L^2} \int_0^t \exp \left[ \gamma \int_0^{\tau} \|\omega(\sigma)\|^4_{L^2} d\sigma \right] d\tau,
\]
we find that \((3.9)\) can be written in the form of a differential inequality,
\[
y'(t) \leq (\gamma - C_0)\|\omega_0\|^4_{L^2} y(t) \frac{d}{dt} y(t),
\]
which can be solved to provide us with \((3.4)\). \(\square\)

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