Challenges in Reconstructing Shapes from Euler Characteristic Curves

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Abstract

Shape recognition and classification is a problem with a wide variety of applications. Several recent works have demonstrated that topological descriptors can be used as summaries of shapes and utilized to compute distances. In this abstract, we explore the use of a finite number of Euler Characteristic Curves (ECC) to reconstruct plane graphs. We highlight difficulties that occur when attempting to adopt approaches for reconstruction with persistence diagrams to reconstruction with ECCs. Furthermore, we highlight specific arrangements of vertices that create problems for reconstruction and present several observations about how they affect the ECC-based reconstruction. Finally, we show that plane graphs without degree two vertices can be reconstructed using a finite number of ECCs.

1 Introduction

Shape comparison and classification is a common task in the field of computer science, with applications in graphics, geometry, machine learning, and several other research fields. The problem has been well-studied in $\mathbb{R}^3$, with several approaches described in the survey [6]. One relatively new approach to the problem involves utilizing topological descriptors to represent and compare the shapes. In [7], Turner et al. proposed the use of the zero- and one-dimensional persistence diagrams from lower-star filtrations to compare triangulations of $S^k$ in $\mathbb{R}^d$, for $d > k$. We call the mapping of a shape to to a parameterized set of diagrams the persistent homology transform (PHT). Their main result (Cor. 3.4 of [7]) showed that the persistent homology transform (PHT) is injective for comparing triangulations of $S^2$ or $S^1$ embedded in $\mathbb{R}^3$ (or triangulations of $S^1$ in $\mathbb{R}^2$), and thus can be used to distinguish different shapes. Turner et al. also extend the idea of the PHT to the Euler Characteristic Curve (ECC) and describe the Euler Characteristic Transform (ECT), a topological summary that records changes in the Euler Characteristic across a height parameter, again from all directions. Finally, using experimental results, the authors show that the PHT and ECT performed well in clustering tasks. In [2], Crawford et al. extend this work by proposing the smooth Euler Characteristic Transform (SECT), a functional variant of the ECT with favorable properties for analysis. They show that features derived from the SECT of tumor shapes are better predictors of clinical outcomes of patients than other traditional features.

The proof of injectivity (i.e., that a shape can be reconstructed from the PHT or the ECC) uses an infinite set of a directions; however, using an infinite set of directions is infeasible for computational purposes. Thus, both [2, 7] use sampling a finite set of directions for the height filtrations in order to apply the technique to shape comparison. In [1], Belton et al. present an algorithm for reconstructing plane graphs using a quadratic (hence, finite) number of persistence diagrams. Simultaneous to that result, other researchers also attempted to give a finite number of directions sufficient to fully determine a shape. Both [3] and [5] give upper bounds on the number of directions needed to determine a hidden shape in $\mathbb{R}^d$. In order to do this, they make assumptions about the curvature and geometry of the input shape. In our work, by contrast, we restrict to plane graphs, but make no restrictions on curvature.

Here, we attempt to extend the work of [1] on the PHT to the ECT. However, difficulties arise when using ECCs because they do not encode information about every vertex from every direction, as a persistence diagram does when on-diagonal points are in-
cluded. We show that, while the number of directions needed to give an ECT unique to the input graph is linear in the number of vertices of the graph, it is difficult to determine which directions generate the necessary ECCs. As we will see, the main difficulty lies with the presence of degree two vertices.

2 Background

In this paper, we focus on a subset of finite simplicial complexes that are composed of only edges and vertices and are provided with a planar straight-line embedding in $\mathbb{R}^2$. We refer to these simplicial complexes as plane graphs. We refer the reader to [4] for a general background on persistent homology, and only present the necessary content here.

Assumptions Let $K$ be a plane graph. In what follows, we assume that the vertices of $K$ have distinct $x$- and $y$-coordinates from one another. Furthermore, we assume that no three vertices are collinear.

Lower-Star Filtration Let $S^1$ be the unit sphere in $\mathbb{R}^2$. Consider $s \in S^1$, i.e., a direction vector in $\mathbb{R}^2$; we define the lower-star filtration with respect to $s$. Let $h_s: K \to \mathbb{R}$ be defined for a simplex $\sigma \subseteq K$ by $h_s(\sigma) = \max_{v \in \sigma} v \cdot s$, where $x \cdot y$ is the inner (dot) product and measures height in the direction of unit vector $y$. Intuitively, the height of $x$ with respect to $s$ is the maximum “height” of all vertices in $\sigma$. Then, for each $h \in \mathbb{R}$, the subcomplex $K_h := h_s^{-1}((-\infty, h])$ is composed of all simplices that lie entirely below or at the height $h$, with respect to the direction $s$. The lower-star filtration is sequence of subcomplexes $K_s$, where $h$ increases from $-\infty$ to $\infty$; notice that $K_h$ only changes when $h$ is the height of a vertex of $K$.

When we observe a difference between $K_{h_\epsilon}$ and $K_{h_{\epsilon'}}$, we know that we have encountered a vertex. As in [4], we define a structure to encode what we know about this vertex in $\mathbb{R}^2$. Given $s \in S^1$, and a height $h \in \mathbb{R}$, the filtration line at height $h$ is the line, denoted $\ell(h, s)$, perpendicular to direction $s$ and at height $h$ in direction $s$. Given a finite set of vertices $V \subset \mathbb{R}^2$, the filtration lines of $V$ are the set of lines $L(s, V) = \{\ell(s, h) \mid \exists v \in V \text{ s.t. } h = v \cdot s\}$.

Further, $L(s, V)$ will contain $|V|$ lines if and only if no two vertices have the same height in direction $s$. Our assumptions guarantee distinct vertex heights only for $(0, 1), (0, -1), (1, 0)$, and $(-1, 0)$, referred to as the cardinal directions. In [4], every line in $L(s, V)$ can be read off of the persistence diagram, as every simplex corresponds to either a birth or the death of a homology class. Next, we observe that we cannot witness all such lines for another topological descriptor, the Euler Characteristic Curve.

Euler Characteristic Curves The Euler characteristic of a plane graph $K = (V, E)$ is $|V| - |E|$. The Euler Characteristic Curve (ECC) is the piecewise step function of the Euler characteristic, whose domain is subcomplexes of a filtration defined by some parameterization of $K$. In this paper, the parameter is the height of a lower-star filtration. Specifically, we define $\chi^K_s: \mathbb{R} \to \mathbb{Z}$ to be the function that maps a height $h$ to the Euler Characteristic of $K_h$. Every change in the ECC corresponds to a filtration line from that direction, but not vice versa. For example, if an edge and vertex appear at the same height, then the ECC does not change. We now refine our definition of filtration lines:

$$W(s, V) = \{\ell(s, h) \mid \exists \epsilon_0 > 0 \text{ s.t. } \forall \epsilon \in (0, \epsilon_0), \chi^K_s(h - \epsilon) \neq \chi^K_s(h + \epsilon)\}.$$  

This set corresponds to the subset of vertices in $V$ that are witnessed from $s$ through the ECC $\chi^K_s$. As such, we refer to these lines as witnessed lines. We note that the only time that a vertex is not witnessed is if the vertex is included in the filtration at the same time as an edge because the vertex being added will be cancelled out by the inclusion of the edge. Furthermore, we note that $v$ lying on a filtration line from $s$ does not necessarily imply that $v$ is witnessed from $s$, i.e., it could lie on a witness line for another vertex if they lie at the same height from $s$.

3 Towards Vertex Reconstruction

We are interested in reconstructing a plane graph from ECCs from a finite number of directions. While three directions was sufficient for reconstructing vertices using persistence diagrams, ECCs contain strictly less information in each direction. We observe the existence of a linear number of directions that allows to fully reconstruct the vertices of a plane graph:

**Proposition 1 (ECC Existence).** Given a plane graph $K = (V, E)$ with $|V| = n$, there exist $3n$ directions that can be used to reconstruct all vertices in $V$.

The proof of this claim may be found in Appendix A. We note that while $3n$ directions are sufficient, this bound is likely not tight.

Initially, attempting to use the techniques in [4], it seems promising for plane graph reconstruction using ECCs, i.e., we can define a correspondence between
three-way witness line intersections (from carefully chosen directions) and vertices. However, certain types of vertices introduce difficulties. For example, consider Figure 1. A degree two vertex is not witnessed by any of the witness lines from the cardinal directions (1, 0), (0, 1), (−1, 0) and (0, −1). However, we would like to generate a correspondence between three-way intersections of witness lines and non degree two vertices. If we use the technique described in Theorem 5 of [1] to choose such a direction, that direction creates a witness line that causes a three-way intersection not corresponding to a vertex. In fact, when degree two vertices are introduced to the plane graph, several problems arise. We discuss these problems in detail in Section 4.

4 Degree Two Challenges

Degree two vertices introduce several complications in finding witness directions, because degree two vertices can have an arbitrarily small region on $S^1$ from which they can be witnessed. For example, in Figure 2 the vertices $v_1, v_2,$ and $v_3$ are nearly collinear. In order to witness $v_2$, we must choose directions from within the red region, where a decrease in the ECC will be observed, or from the blue region, where an increase in the ECC will be observed. However, these these regions becomes arbitrarily small as $v_1, v_2$ and $v_3$ approach collinear.

As mentioned earlier, degree two vertices can also introduce additional ambiguities when witnessing non-degree two vertices. Recall the example found in Figure 1 and the discussion in Section 3.

Despite these difficulties, several situations exist in which degree two vertices can be witnessed. The following propositions summarize these scenarios. Proofs are provided in Appendix A. For clarity, we discuss quadrants as though $v_2$ is located at the origin. However, note that the following propositions also apply to arrangements with similar orientations and angles.

**Proposition 2 (Same Quadrant).** If $v_1$ and $v_3$ lie in the same quadrant, such as the vertices $v_4$ and $v_5$ in Figure 3, then $v_2$ will be witnessed in ECCs from every one of the cardinal directions.

**Proposition 3 (Neighboring Quadrants).** If $v_1$ and $v_3$ lie in neighboring quadrants, such as vertices $v_1$ and $v_3$ in Figure 3, then $v_2$ will be witnessed in ECCs from exactly two of the four cardinal directions.
Proposition 4 (Degree Two Bounded Angle). If 
\[ \text{angle}(v_1, v_2), (v_2, v_3) < \frac{\pi}{2} \] then \( v_2 \) will be witnessed in ECCs from at least two of the four cardinal directions.

The above propositions show scenarios for which degree two vertices can be witnessed using cardinal directions. However, degree two vertices pose particular problems when the edges lie in non-neighboring quadrants, such as the edges \((v_b, v_2)\) and \((v_r, v_2)\) in Figure 3 or \((v_1, v_2)\) and \((v_2, v_3)\) in Figure 2. Then, when degree two vertices are not included in a plane graph \( K \), a constant number of ECCs can be used to determine the embeddings of the vertices.

5 A Special Case

If a plane graph contains no degree two vertices, the graph can be reconstructed using a finite number of ECCs. Let \( K \) denote a plane graph with vertex and edge sets \( V \) and \( E \) respectively. Recall from [1] that three way filtration line intersections from carefully chosen directions correspond to a vertex location for plane graphs using persistence diagrams. We show that this result still holds for reconstructing plane graphs using ECCs, if they do not contain degree two vertices. The proofs of the following lemmas and theorem can been found in Appendix A.

First, we provide a lemma that yields insight into how non-degree two vertices are witnessed.

Lemma 1 (Linear Witness Lines). Let \( K \) be a plane graph in \( \mathbb{R}^2 \) with vertices \( V \) such that for all \( v \in V, \; \text{deg}(v) \neq 2 \) and denote \( |V| = n \). Let \( \ell \) be a line in \( \mathbb{R}^2 \) such that any line parallel to \( \ell \) intersects at most one vertex in \( V \). Let \( s \in S^1 \) be chosen perpendicular to \( \ell \). Then, \( |\mathcal{W}(s, V) \cup \mathcal{W}(-s, V)| = n \).

By generalizing the results of Lemma 1, we introduce the following Lemma to generate \( n^2 \) potential vertex locations in \( \mathbb{R}^2 \), where \( n \) is the number of vertices.

Lemma 2 (Witness Line Intersections). Recall the cardinal directions \((0, 1), (1, 0), (0, -1), (-1, 0) \in S^1 \). If for all \( v \in V, \; \text{deg}(v) \neq 2 \) then
\[
|\mathcal{W}((0, 1), V) \cup \mathcal{W}((0, -1), V)| = n, \text{ and } \\
|\mathcal{W}((1, 0), V) \cup \mathcal{W}((-1, 0), V)| = n.
\]

Utilizing these \( n \) horizontal and \( n \) vertical witness lines, we are able to pick two additional directions to generate three-way filtration line intersections using a technique similar to the one described in Theorem 5 of [1]. Then, the following theorem holds as well.

Theorem 1 (ECC Vertex Reconstruction). Let \( K = \langle V, E \rangle \) be a plane graph with vertices \( V \) and edges \( E \). If for all \( v \in V \), \( \text{deg}(v) \neq 2 \) then the locations of all vertices can be determined using six ECCs in \( O(n \log n) \) time.

The proof of Theorem 1 is found in Appendix A, but note that the result follows using similar arguments to those found in Theorem 5 of [1].

6 Discussion and Future Work

We have shown that, for any known plane graph \( K \), we can choose a linear number of directions to fully describe \( K \) using only ECCs from those directions. However, when \( K \) is unknown, determining such a set is difficult. We emphasize that although there is an infinite number of directions in which the vertices of a plane graph can be witnessed by an ECC, the presence of degree two vertices can restrict these directions to an arbitrarily small subset of \( S^1 \).

Our ultimate goal is to further develop the theory on determining the minimal set of directions necessary to reconstruct shapes. We are currently investigating upper bounds on the number of directions needed to reconstruct a plane graph from ECCs. Additionally, we are exploring what assumptions we can place on the underlying shape in order to overcome the challenges of degree two vertices. For example, we observe that if the number of vertices \( |V| = n \) is known, then the intersection of \( m > n \) filtration lines determines the location of all vertices. Another simplifying assumption is that minimum angle between any three vertices, \( \epsilon \), is known. Then, we can avoid some of the issues described in Section 4 by employing pairs of directions whose difference in angle is less than \( \epsilon \). Finally, we would like to extend our work to more general shapes embedded in \( \mathbb{R}^d \).

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A Proofs

Proof of Proposition 1 (ECC Existence)

Proof. Let \( v \in V \) be a vertex in \( K \). First, we show that each vertex is witnessed from an infinite number of directions \( S^3 \). If \( \deg(v) = 0 \), \( v \) is witnessed from any direction for which it lies on a unique witness line (So, for all but \( |V|−1 \) directions). If \( \deg(v) = 1 \) with edge \((v, v')\) for some \( v' \in V \), then \( v \) is observed from an infinite set of directions from which \( v' \) appears after \( v \) in the lower-star filtration, and \( v \) lies on a unique witness line. If \( \deg(v) > 1 \) with edges \((v, v')\) and \((v, v'')\) for \( v', v'' \in V \), then \( v \) is observed from any direction from which \( v' \) and \( v'' \) appear before \( v \) in the filtration and \( v \) lies on a unique witness line. Thus, each vertex is witnessed from an infinite number of directions.

Let \( \mathbb{I} \) be the set of directions that witness \( v \). We can choose any three directions from \( \mathbb{I}_v \) and generate a unique three-way intersection at \( v \). Now, we need to show that a set of directions exist for each of the \( n \) vertices such that no three-way intersections exist at locations where a vertex is not located. In order to do this, we give the vertices some arbitrary ordering \( v_1, v_2, \ldots, v_n \). Then, select vertices in ascending order. For the first, any three directions in \( \mathbb{I}_{v_1} \) will give a single three-way intersection of witness lines. For each successive vertex \( v_i \), there exist up to \( 3i^2 \) witness lines. More importantly, the number of three-way witness line intersections is finite. Thus, there exist three directions in \( \mathbb{I}_{v_i} \) such that none of the witness lines created by these directions intersect existing intersections. Since the \( x \)- and \( y \)-coordinates of a vertex can be determined using a three-way line intersection, we can see that there exist a set of \( 3n \) directions which generates exactly \( n \) three-way intersections of witness lines, revealing the location of all \( n \) vertices.

\[ \square \]

Proof of Proposition 3 (Same Quadrant)

Proof. If \( v_1 \) and \( v_3 \) lie in the same quadrant, then \( v_1 \) and \( v_3 \) will appear before \( v_2 \) from exactly one of the two \( x \)-axis parallel directions \((-1,0)\) or \((1,0)\) and before \( v_2 \) in exactly one of the \( y \)-axis parallel directions \((0,-1)\) or \((0,1)\). Let \( s_1 \in \{(0,1), (0,-1)\} \) and \( s_2 \in \{(1,0), (-1,0)\} \) be the directions that witness \( v_1 \) and \( v_3 \) before \( v_2 \). \( \chi_{s_1}^K \) and \( \chi_{s_2}^K \) will witness \( v_2 \) by seeing a decrease in the Euler Characteristic at the time that \( v_2 \) is first included in the filtration. Then, \( -s_1 \) and \( -s_2 \) will witness \( v_2 \) before \( v_1 \) or \( v_2 \). Since no other edges with \( v_2 \) as an endpoint exist, there will be an increase in \( \chi_{-s_1}^K \) and \( \chi_{-s_2}^K \) at the time that \( v_2 \) is first included in the filtration. Then, \( v_2 \) is witnessed from every cardinal direction, as required.

\[ \square \]

Proof of Proposition 4 (Neighboring Quadrants)

Proof. Recall that no two vertices share \( x \)- or \( y \)-coordinates, then any witness line from a cardinal direction will be unique. Let \( s \) be the cardinal direction for which \( v_1 \cdot s < v_2 \cdot s \) and \( v_3 \cdot s > v_2 \cdot s \) and \( -s \) the cardinal direction chosen such that \( v_3 \cdot s < v_2 \cdot s \) and \( v_1 \cdot s > v_2 \cdot s \). Then, there is no change in Euler Characteristic at \( v_2 \) from either \( s \) or \( -s \), since \( v_2 \) is added at the same time as \((v_1, v_2)\) or \((v_2, v_3)\), respectively. Now, let \( w \) and \(-w\) be the remaining two cardinal directions, where \( w \) is the direction from which we include \( v_2 \) before \( v_1 \) or \( v_3 \). Direction \( w \) witnesses \( v_2 \) because no edges are included at height \( v_2 \) from that direction. Direction \(-w\) witnesses \( v_2 \) because both \((v_1, v_2)\) and \((v_2, v_3)\) are added along with \( v_2 \). Thus,
Proof of Proposition 4 (Degree Two Bounded Angle)

Proof. If $\text{angle}((v_1, v_2), (v_2, v_3)) < \frac{\pi}{2}$, then $v_1$ and $v_3$ must lie in neighboring quadrants or the same quadrant, since neither can lie on the boundary of a quadrant by assumption. If they are in the same quadrant, Proposition 2 tells us that they must be seen from all four cardinal directions. If they are in neighboring quadrants, Proposition 3 tells us that we can witness $v_2$ with ECCs from exactly two of the four cardinal directions.

Proof of Lemma 1 (Linear Witness Lines)

Proof. We show that each vertex is seen by at least one of $s$ or $-s$. Let $v \in V$ be a vertex with $\text{deg}(v) = 0$. Then, $v$ will correspond to $\ell(s, v)$ for any arbitrary direction $s \in S^1$ because $\chi^K_s$ will always increase by at least one at time $s \cdot v$. As such, $v$ will be observed by both $s$ and $-s$.

Let $v \in V$ be a vertex with $\text{deg}(v) = 1$ and $(v, v') \in E$ for some $v' \in V$. Then, if $s \in S^1$ is chosen such that $s \cdot v' < s \cdot v$, $v$ will not result in a change in $\chi^K_s$. However, $s$ was chosen such that no two vertices will be observed at the same time. As a result, no edge in $E$ can be parallel to $\ell$. Then, if $s \cdot v' < s \cdot v$ then $-s \cdot v' \geq -s \cdot v$ and an increase in $\chi^K_{-s}$ is seen at time $-s \cdot v$. This implies that $v$ is observed by $s$ or $-s$ but not both.

Finally, if $v \in V$ is a vertex with $\text{deg}(v) > 2$, then we must consider two cases. If, for $s \in S^1$, there exists exactly one edge $(v, v') \in E$ such that $s \cdot v' < s \cdot v$, then there must exist at least two additional edges that will result in a decrease in $\chi^K_{-s}$ at time $-s \cdot v$. As such, $v$ will be observed by at least one of the ECCs resulting from $s$ or $-s$. On the other hand, if, for $s \in S^1$, there exists either zero edges or more than one edge that appear before $v$ in the height filtration from $s$, then $\chi^K_s$ will either increase (in the case where no edges appear before $v$) or decrease (in the case where two or more edges appear before $v$). Then, all non-degree two vertices result in a change in $\chi^K_s$ or $\chi^K_{-s}$ and, as such, $|\mathcal{W}(s, V) \cup \mathcal{W}(-s, V)| = n$, as required.

Proof of Lemma 2 (Witness Line Intersections)

Proof. By Lemma 1 if $s$ is chosen such that no two vertices are intersected by a line perpendicular to $s$, then $\mathcal{W}(s, V) \cup \mathcal{W}(-s, V)$ will result in $n$ filtration lines. Recall that no two vertices in $K$ share an $x$- or $y$-coordinate. Then, by Lemma 1 $|\mathcal{W}((0, 1), V) \cup \mathcal{W}((0, -1), V)| = n$ and $|\mathcal{W}((1, 0), V) \cup \mathcal{W}((-1, 0), V)| = n$, as required.