The Phononic Casimir Effect: An Analog Model

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Abstract. We discuss the quantization of sound waves in a fluid with a linear dispersion relation and calculate the quantum density fluctuations of the fluid in several cases. These include a fluid in its ground state. In this case, we discuss the scattering cross section of light by the density fluctuations, and find that in many situations it is small compared to the thermal fluctuations, but not negligibly small and might be observable at room temperature. We also consider a fluid in a squeezed state of phonons and fluids containing boundaries. We suggest that the latter may be a useful analog model for better understanding boundary effects in quantum field theory. In all cases involving boundaries which we consider, the mean squared density fluctuations are reduced by the presence of the boundary. This implies a reduction in the light scattering cross section, which is potentially an observable effect.

1. Introduction
It is well known that quantized sound waves, whose excitations are phonons, share several properties with relativistic quantum fields, such as the electromagnetic field. Here we will be primarily concerned with fluctuations in the phonon ground state. These fluctuations can share certain features with electromagnetic vacuum fluctuations, including the creation of Casimir-type forces [1]. The forces due to quantum or stochastic sound fluctuations have been discussed recently by several authors [2, 3, 4, 6, 7]. In this paper, we will study quantum density fluctuations in a fluid. The fluid in question could be either a classical or a quantum fluid, but we will consider only the regime where the dispersion relation is approximately linear. Thus the analog models which we consider are rather different than those which model the Hawking process [8, 9, 10]. The topic of hydrodynamic fluctuations has been reviewed, for example, in Refs. [11, 12]. In Sec. 2, we will discuss density fluctuations in a fluid in the phonon ground state without boundaries, and summarize work recently reported in Ref. [13]. It was argued there that light scattering by quantum density fluctuations might be observable, even in fluids at room temperature. The effects of a squeezed state of phonons will be briefly discussed in Sec. 3. In Sec. 4, we turn to the issue of the effects of boundaries in the fluid, and argue that this can be a useful analog model for the effects of boundaries on relativistic quantum fields. We also suggest that the boundary effects, at least in principle, create observable effects in the
2. Phonons and Density Fluctuations in a Fluid

2.1. Quantization and the Density Correlation Function

We consider the quantization of sound waves in a fluid with a linear dispersion relation, \( \Omega_q = c_S q \), where \( \Omega_q \) is the phonon angular frequency, \( q \) is the magnitude of the wave vector, and \( c_S \) is the speed of sound in the fluid. This should be a good approximation for wavelengths much longer than the interatomic separation. Let \( \rho_0 \) be the mean mass density of the fluid. Then the variation in density around this mean value is represented by a quantum operator, \( \hat{\rho}(x, t) \), which may be expanded in terms of phonon annihilation and creation operators as

\[
\hat{\rho}(x, t) = \sum_q \left( b_q f_q + b_q^\dagger f_q^* \right),
\]

where

\[
f_q = \sqrt{\frac{\hbar \rho_0}{2Vc_S^2}} e^{i(q \cdot x - \Omega_q t)}.
\]

Here \( V \) is a quantization volume. The normalization factor in Eq. (2) can be fixed by requiring that the zero point energy of each mode be \( \frac{1}{2} \hbar \Omega_q \) and using the expression for the energy density in a sound wave,

\[
U = \frac{c_S^2 \rho_0}{2} \hat{\rho}^2.
\]

In the limit in which \( V \to \infty \), we may write the density correlation function as

\[
\langle \hat{\rho}(x, t) \hat{\rho}(x', t') \rangle = \frac{\hbar \rho_0}{16\pi^3 c_S} \int d^3q \Omega_q e^{i(q \cdot \Delta x - \Omega_q \Delta t)},
\]

where \( \Delta x = x - x' \) and \( \Delta t = t - t' \). The integral may be evaluated to write the coordinate space correlation function as

\[
\langle \hat{\rho}(x, t) \hat{\rho}(x', t') \rangle = -\frac{\hbar \rho_0}{2\pi^2 c_S} \frac{\Delta x^2 + 3c_S^2 \Delta t^2}{(\Delta x^2 - c_S^2 \Delta t^2)^{3/2}}.
\]

This is of the same form as the correlation function for the time derivative of a massless scalar field in relativistic quantum field theory, \( \langle \dot{\varphi}(x, t) \dot{\varphi}(x', t') \rangle \). (This analogy has been noted previously in the literature. See, for example, Ref. [15].) Apart from a factor of \( \rho_0 \), these two quantities may be obtained from one another by interchanging the speed of light \( c \) and the speed of sound \( c_S \). If \( c \to c_S \), then

\[
\langle \dot{\varphi}(x, t) \dot{\varphi}(x', t') \rangle \to \rho_0 \langle \hat{\rho}(x, t) \hat{\rho}(x', t') \rangle.
\]

In the limit of equal times, the density correlation function becomes

\[
\langle \hat{\rho}(x, t) \hat{\rho}(x', t) \rangle = -\frac{\hbar \rho_0}{2\pi^2 c_S (\Delta x)^4}.
\]

Thus the density fluctuations increase as \( |\Delta x| \) decreases. Of course, the continuum description of the fluid and the linear dispersion relation both fail as \( |\Delta x| \) approaches the interatomic separation. Also note the minus sign in Eq. (7). This implies that density fluctuations at different locations at equal times are anticorrelated. By contrast, when \( c_S |\Delta t| > |\Delta x| \), then

fluid case. We report on the result of several specific calculations for different geometries. The results will be summarized and discussed in Sec. [5].
\( \langle \hat{\rho}(\mathbf{x}, t) \hat{\rho}(\mathbf{x}', t) \rangle > 0 \) and the fluctuations are positively correlated. This is complete analogy with the situation in the relativistic theory. Fluctuations inside the lightcone can propagate causally and tend to be positively correlated. Fluctuations in a fluid for which \( c_S |\Delta t| < |\Delta x| \) cannot have propagated from one point to the other, and are anti-correlated. This can be understood physically because an over density of fluid at one point in space requires an under density at a nearby point.

### 2.2. Light Scattering by Density Fluctuations

In Ref. \[13\], the cross section for the scattering of light by the zero point density fluctuations is computed for the case that the incident light angular frequency is large compared to the typical phonon frequency. The result is

\[
\left( \frac{d\sigma}{d\Omega} \right)_{ZP} = \sqrt{2(1 - \cos \theta)} \frac{\hbar \omega^5 V \eta^4}{32\pi^2 c^5 S \rho_0} \left( \hat{e}_{k,\lambda} \cdot \hat{e}_{k',\lambda'} \right)^2 , \tag{8}
\]

where \( \theta \) is the scattering angle, \( V \) is the scattering volume, and \( \eta \) is the mean index of refraction of the fluid. The \( \omega^5 \) dependence of the scattering cross section can be viewed as the product of the \( \omega^4 \) dependence of Rayleigh-Brillouin scattering and one power of \( \Omega_\eta \), and hence of \( \omega \), coming from the spectrum of zero point fluctuations in the fluid. The factor of \( \eta^4 \) represents the influence of the fluid on light propagation before and after the scattering process, and arises as a product of a factor of \( \eta \) in the incident flux and a factor of \( \eta^3 \) in the density of final states [13]. Because light travels through the fluid at speeds much greater than the sound speed, light scattering reveals a nearly static distribution of density fluctuations. Thus we can regard Eq. (8) as a probe of the fluctuations described by Eq. (7). The scattering by zero point fluctuations is inelastic, with the creation of a phonon. Thus, the scattering described by Eq. (8) is strictly Brillouin rather than Rayleigh scattering.

This scattering by zero point density fluctuations should be compared to the effects of thermal density fluctuations. The ratio of the zero point to the thermal scattering can be expressed as

\[
R \equiv \frac{\left( \frac{d\sigma}{d\Omega} \right)_{ZP}}{\left( \frac{d\sigma}{d\Omega} \right)_{TB}} = \sqrt{2(1 - \cos \theta)} \left( \frac{\hbar \omega}{2k_BT} \right) \left( \frac{c_S}{c} \right) \eta^4 \left[ \rho_0 \left( \frac{\partial \epsilon}{\partial \rho_0} \right)_S \right]^{-2} . \tag{9}
\]

The index of refraction, \( \eta \), and the quantity \( \rho_0 \left( \partial \epsilon / \partial \rho_0 \right)_S \), which involves a derivative of the fluid dielectric function with respect to density at constant entropy are both of order unity. Hence \( R \) is primarily determined by the ratio of the photon energy to the thermal energy, and the ratio of the speed of sound to the speed of light.

As an example, consider the case of water at room temperature and violet light with a wavelength of \( \lambda = 350nm \). In this case, we have \( c_S = 1480m/s \) and \( \eta = 1.4 \) [15]. In addition, \( \rho_0 \left( \partial \epsilon / \partial \rho_0 \right)_S = 0.79 \) [17]. For back scattering, \( \cos \theta = -1 \), this leads to \( R \approx 0.005 \). Consequently, about 0.5% of the Stokes line is due to zero point motion effects. Although this is a small fraction, it may be detectable, and will increase at lower temperatures and shorter wavelengths.

### 3. Squeezed States of Phonons

Now we consider the case where the phonon field is not in the vacuum state, but rather a squeezed state. The squeezed states are a two complex parameter family of states, but we will focus attention on the case of the squeezed vacuum states \( |\zeta\rangle \), labeled by a single complex squeeze parameter

\[
\zeta = r e^{i\delta} . \tag{10}
\]
This set of states is of special interest because they are the states generated by quantum particle creation processes, and they can exhibit local negative energy densities. Consider the shift in the mean squared density fluctuations between the given state and the vacuum

$$\langle \hat{\rho}^2 \rangle_R = \langle \zeta | \hat{\rho}^2 | \zeta \rangle - \langle 0 | \hat{\rho}^2 | 0 \rangle,$$

the “renormalized” mean squared density fluctuation. The result for this quantity in a single mode squeezed vacuum state for a plane wave in the z-direction is

$$\langle \hat{\rho}^2 \rangle_R = \frac{\hbar \omega \rho_0}{c^2 S V} \sinh r \left\{ \sinh r - \cosh r \cos[2(kz - \omega t) + \delta] \right\}.$$

(12)

Note that this quantity can be either positive or negative, but its time or space average is positive. The suppression of the local density fluctuations in a squeezed state is analogous to the creation of negative energy densities for a massless, relativistic field.

4. Boundaries

If we introduce an impenetrable boundary into the fluid, the phonon field will satisfy Neumann boundary conditions

$$\hat{n} \cdot \nabla \delta \rho = 0$$

(13)
as a consequence of the impenetrability. Thus there will be a Casimir force on the boundaries which is analogous to the Casimir force produced by electromagnetic vacuum effects. For example, consider two parallel plates, which will experience an attractive force per unit area of

$$F_A = \frac{\hbar c S \pi^2}{480 a^4},$$

(14)

which is smaller than the electromagnetic case for perfect plates by a factor of $c_S/(2c)$, and is thus quite small in any realistic situation.

Henceforth, we consider the local effect of boundaries on mean squared density fluctuations, and now define $\langle \hat{\rho}^2 \rangle_R$ to be the change due to the presence of the boundary. This quantity is of interest both as an analog model for the effects of boundaries in quantum field theory, and in its own right. The shifts in density fluctuations are at least in principle observable in light scattering experiments.

Our interest in the phononic analog model is inspired by the fact that the study of boundary effects in quantum field theory is an active area of research, and has given rise to some recent controversies in the literature [18, 19]. One question is the nature of the physical cutoff which prevents singularities at the boundary. An example of the subtleties is afforded by the mean squared electric and magnetic fields near a dielectric interface. When the material is a perfect conductor, these quantities are proportional to $z^{-4}$, where $z$ is the distance to the interface. Specifically, in Lorentz-Heaviside units their asymptotic forms are

$$\langle E^2 \rangle \sim \frac{3}{16\pi^2} \frac{1}{z^4}$$

(15)

and

$$\langle B^2 \rangle \sim -\frac{3}{16\pi^2} \frac{1}{z^4}$$

(16)

One might expect that a realistic frequency dependent dielectric function would remove this singularity, but this is not the case. Instead one finds [20] that

$$\langle E^2 \rangle \sim \frac{\sqrt{2} \omega_p}{32\pi} \frac{1}{z^3}$$

(17)
and
\[ \langle B^2 \rangle \sim -\frac{5\omega_p^2}{96\pi^2} \frac{1}{z^2}, \]
where \( \omega_p \) is the plasma frequency of the material. Thus some physical \textit{cutoff} other than dispersion is required. For realistic materials, it is likely to be surface roughness, but fluctuations in the position of the boundary can also serve as a cutoff, as we showed several years ago \cite{21}. In a fluid, there is always a physical cutoff at the interatomic separation, but in a given situation other physical cutoffs may be dominant. For the present, we will report the results of calculations of \( \langle \rho^2 \rangle_R \) for several different geometries, and leave a more detailed treatment of physical cutoffs for later work.

In the remainder of this section, we will quote several results for \( \langle \rho^2 \rangle_R \) in different geometries. The details of the calculations involved will be presented in a later paper \cite{22}.

\subsection*{4.1. One or Two Parallel Plane Boundaries}
In both of these cases, the renormalized density two-point function may be constructed by the method of images. For the case of a single plane, one finds
\[ \langle \rho^2 \rangle_R = -\frac{\hbar \rho_0 c_S}{152} \left( \frac{1}{z^2} \right), \]
where \( z \) is the distance to the boundary. For the case of two parallel planes, the result is
\[ \langle \rho^2 \rangle_R = -\frac{\hbar \rho_0 c_S}{96a^4} \left( \frac{1}{15} + \frac{3 - 2 \sin^2(\pi z/a)}{\sin^4(\pi z/a)} \right), \]
where \( a \) is the separation of the two planes, and \( z \) is the distance to one boundary. Note that both of these expressions are negative everywhere. In the absence of a physical cutoff, both of these expressions diverge as \( z^{-4} \) near the boundaries, just as do the squared electric and magnetic fields near a perfectly reflecting plane.

\subsection*{4.2. A Three-Dimensional Torus}
Here we consider a rectangular box with periodic boundary conditions in all three spatial directions, with periodicity lengths \( L_1, L_2 \) and \( L_3 \). Thus the three-dimensional space has the topology of \( S^1 \times S^1 \times S^1 \). This is closely related to the geometry of a waveguide, where the fluctuations of a relativistic scalar field were discussed by Rodrigues and Svaiter \cite{23}. As in the parallel plane case, an image sum method may be employed, with the result
\[ \langle \rho^2 \rangle_R = -\frac{\hbar \rho_0 c_S}{2\pi^2} \sum'_{\ell,m,n} \frac{1}{(\ell^2 L_1^2 + m^2 L_2^2 + n^2 L_3^2)^2}. \]
Here the prime on the summation indices denotes that the \( \ell = m = n = 0 \) term is omitted. Here \( \langle \rho^2 \rangle_R \) is a negative constant.

\subsection*{4.3. A Wedge}
Consider two intersecting plane which are at an angle of \( \alpha \) with respect to each other. Now consider a point inside of this wedge which is located at polar coordinates \((r, \theta)\), where \( r \) is the distance to the intersection line and \( \theta < \alpha \). This geometry was treated for the relativistic case by Candelas and Deutsch \cite{24}, and we may transcript one of their results \[\text{their Eq. (5.39)}\] to find
\[ \langle \rho^2 \rangle_R = -\frac{\hbar \rho_0 c_S}{1440\pi^2 r^4 \sin^4(\pi \theta/\alpha)} \times \left\{ (\pi - \alpha)(\pi + \alpha) \sin^2(\pi \theta/\alpha) [(\pi^2 + 11\alpha^2) \sin^2(\pi \theta/\alpha) - 30\pi^2] + 45\pi^4 \right\}. \]
Again, this quantity is negative everywhere.

4.4. A Cosmic String
As is well known, the space surrounding a cosmic string is a conical space with a deficit angle \( \alpha < 2\pi \). Quantum field theory in this conical space has been discussed by many authors, beginning with Helliwell and Konkowski [25], and is similar to the wedge problem discussed above. We may follow the procedure in Ref. [25]. At a distance \( r \) from the apex, we find

\[
\langle \hat{\rho}^2 \rangle_R = -\frac{\hbar \rho_0 c S}{1440 \pi^2 \alpha^4 r^4} (2\pi - \alpha)(2\pi + \alpha)(11\alpha^2 + 4\pi^2),
\]

(23)

which is also negative everywhere.

4.5. Near the Focus of a Parabolic Mirror
The quantization of the electromagnetic field in the presence of a parabolic mirror was discussed by us in Refs. [26, 27], where a geometric optics approximation was employed to find the mean squared fields near the focus. This treatment lead to the result that these quantities are singular at the focus, diverging as an inverse power of the distance \( a \) to the focus. This result holds both for parabolic cylinders and for parabolas of revolution, and basically arises from the interference term of multiply reflected rays with nearly the same optical path length. The geometry is illustrated in Fig. 1. An incoming ray at an angle of \( \theta \) reflects at an angle of \( \theta' \) to reach the point \( P \), which is a distance \( a \) from the focus \( F \), as illustrated. The distance from the focus to the mirror itself is \( b/2 \gg a \). There can be two values of \( \theta' \), denoted \( \alpha \) and \( \beta \), corresponding to two reflected rays. The difference in the optical paths of these two rays is denoted by \( \Delta \ell \).

The detailed expression for this distance \( \Delta \ell \) used in Refs. [26, 27] is not quite correct, as was pointed out to us by Vuletic [28]. The corrected expression is

\[
\Delta \ell = a \left[ \cos \gamma (\cos \alpha - \cos \beta + \sin^2 \alpha - \sin^2 \beta) + \sin \gamma (\sin \alpha - \sin \beta + \sin \beta \cos \beta - \sin \alpha \cos \alpha) \right].
\]

(24)

This is to be used in the expression obtained from geometric optics,

\[
\langle E^2 \rangle = 45\pi^3 \int \frac{d\theta}{(\Delta \ell)^4}.
\]

(25)

In this expression, \( \alpha = \alpha(\theta) \), and \( \beta = \beta(\theta) \), as will be discussed in Ref. [22].

A detailed discussion of the electromagnetic case will be given elsewhere. Here we are concerned with \( \langle \hat{\rho}^2 \rangle_R \), which is obtained from Eq. (25) by letting \( c \to c_S \) and dividing by 2, leading to a result of the form

\[
\langle \hat{\rho}^2 \rangle_R = -\frac{\hbar \rho_0 c S C}{b a^3} \lesssim 0.
\]

(26)

This, and the analogous expressions for \( \langle E^2 \rangle \) and \( \langle B^2 \rangle \), which also are proportional to \( 1/(b a^3) \), are striking in that they can be large when the focus is far from the mirror itself, \( b \gg a \). This result is controversial, and seems to be in conflict with a general result by Fewster and Pfennig [29], which implies that quantities such as \( \langle E^2 \rangle \) or \( \langle \hat{\rho}^2 \rangle_R \) should be proportional to the inverse fourth power of the distance to the mirror, which is to say \( \propto b^{-4} \) in this case. On the other hand, there is a simple physical argument to the contrary in this case, which we find compelling: the interference term between multiply reflected rays is slowly oscillating when \( \Delta \ell \sim a \) is small, and should give a contribution proportional to an inverse power of \( a \), as in Eq. (26). In any case, the study of the phononic case provides an additional theoretical, and potentially experimental, probe to better understand this issue.
Figure 1. The geometry of rays reflecting from a parabolic mirror is illustrated. An incoming ray at an angle of $\theta$ reflects at an angle of $\theta'$ to reach the point $P$, which is a distance $a$ from the focus $F$, and at an angle of $\gamma$.

5. Summary
In this paper, we have considered a classical or quantum fluid with a linear dispersion relation as a analog model for quantum fluctuations, and the effects of boundaries on these fluctuations. We have argued that the local density fluctuations in a fluid in the phonon ground state are potentially observable in light scattering experiments. This is of interest in its own right. We have reported the calculation of the change in these fluctuations due to squeezed states of phonons and due to the presence of boundaries. Squeezed states can produce either positive or negative values for $\langle \hat{\rho}^2 \rangle_R$, in a way which is analogous to the effects of squeezed states on the energy density of a relativistic quantum field. However, all of the boundary examples which we consider result in a negative $\langle \hat{\rho}^2 \rangle_R$. If we compare to the already negative result in Eq. (7), this seems to represent an increase in magnitude of an already negative quantity. If one were to scatter light from the fluid with a boundary, this implies a change in the scattering cross section, compared to the result for a boundary-free fluid, Eq. (8), at least for short wavelengths. In general, the calculation of the modified cross section in the presence of a boundary is a complex task. However, if the incident wavelength is short compared to the scale over which $\langle \hat{\rho}^2 \rangle_R$ varies, then we might the scattering cross section to increase in the presence of a boundary. A more detailed analysis of this effect is a subject for future study. It will also be of interest to consider the effects on non-linearity in the phonon dispersion relation on the questions considered here.

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[1] I.E. Dzyaloshinskii, E.M. Lifshitz and L.P. Pitaevskii 1961 Adv. Phys. 10, 165
[2] A. Larraza 1998 Phys. Lett. A 248, 151
[3] O. Bschorr 1999 J. Acoust. Soc. Am. 106, 3730
[4] E. Schäffer and U. Steiner 2002 Eur. Phys. J. E 8, 347
