Supersymmetric gradient flow in Wess-Zumino model

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Abstract

We propose a supersymmetric gradient flow equation in four dimensional Wess-Zumino model. The flow is constructed in two ways. One is based on the off-shell component fields and the other is based on the superfield formalism, in which the same result is provided. The obtained flow is supersymmetric because the flow time derivative and the supersymmetry transformation commute with each other. Solving the equation, we find that it has a damping oscillation with the flow time for non-zero mass, which is different from the Yang-Mills flow. The on-shell flow equation is also discussed.

Keywords: lattice, supersymmetry
1. Introduction

Gradient flow \[1, 2\] has been widely accepted as a new method in lattice field theory and related research areas including supersymmetry (SUSY). In a Yang-Mills flow, any correlation function is ultraviolet (UV) finite at non-zero flow time once four dimensional Yang-Mills theory is renormalized \[3\]. The UV finiteness holds even for a QCD flow with an additional renormalization of the time-dependent quarks \[4, 5\]. This property of the flow leads to a lot of interesting applications such as a proper definition of lattice energy momentum tensor \[6–14\]. The gradient flow approach is also useful in studying the non-linear sigma model \[15–18\], non-perturbative renormalization group \[19–23\], and a theory with AdS geometries \[24–28\]. Other interesting applications are in the references \[29–34\].

There have been various attempts to apply the gradient flow to SUSY theories so far. In super Yang-Mills (SYM), the most naive approach is to use a non-SUSY flow which consists of the Yang-Mills flow and an adjoint matter flow \[1\] although SUSY is broken at a non-zero flow time. From this point of view, a lattice simulation of \(\mathcal{N} = 1\) SYM has been carried out in \[35\] and the regularization independent definition of the supercurrent in \(\mathcal{N} = 1, 2\) SYM have been given in \[36, 37\].

A different approach can be taken, which uses a flow keeping SUSY at a non-zero flow time. Such a SUSY flow has been proposed in the superfield formalism of \(\mathcal{N} = 1\) SYM \[38\]. \[1\] The SUSY flow equation is also given for the component fields of the Wess-Zumino gauge in a gauge covariant manner \[41\]. The obtained flow is supersymmetric in a sense that the flow time derivative and the super transformations commute up to a gauge transformation. The flow equation of supersymmetric O(N) nonlinear sigma model in two dimensions is also studied in \[42\].

The Wess-Zumino model provides a good testing ground to study the renormalization property of the SUSY theories. The gauge symmetry plays a crucial role to prove the UV finiteness in the Yang-Mills flow. As natural questions, one might ask how SUSY works in the SUSY flows and what kind of influence the non-renormalization theorem has for the flow theory. Constructing a SUSY flow for Wess-Zumino model, a deep understanding of the mechanism that leads to the UV finiteness of the SUSY flows could be

\[^1\] In the context of Langevin equation, a flow equation for \(\mathcal{N} = 1\) SYM was discussed in \[39, 40\].
obtained.

In this paper, we derive a SUSY flow of four dimensional Wess-Zumino model, which is referred as Wess-Zumino flow in this paper, and derive its formal solution. We give two ways of constructing the Wess-Zumino flow. One way is to use the component fields of the model directly, and the other way is to use the superfield formalism. They give the same result. Solving the Wess-Zumino flow, we find that the solutions behave as damping oscillations with respect to the flow time for non-zero mass, which is different from the Yang-Mills flow.

This paper is organized as follows. In Sec.2 we give the brief review of Wess-Zumino model in four dimensions. In Sec.3, we present two methods of constructing the Wess-Zumino flow equation. We first present the results in Sec.3.1. The Wess-Zumino flow is constructed with the component fields in Sec.3.2 and with the superfield formalism in Sec.3.3. The on-shell flow is also discussed in Sec.3.4. The formal solutions of the Wess-Zumino flow are given in Sec.4. We summarize our results in Sec.5. The convention used in this paper is shown in Appendix A.

2. Wess-Zumino model

We make a brief review of Wess-Zumino model which is the simplest supersymmetric theory made of a complex scalar $A(x)$, Weyl spinors $\psi_\alpha(x)$, $\bar{\psi}_\dot{\alpha}(x)$ and a complex auxiliary field $F(x)$.

The action in Euclidean space is given by

$$S = \int d^4 x \left\{ |\partial_\mu A|^2 + i \bar{\psi}\sigma_\mu \partial_\mu \psi + |F|^2 - i (F(m A + g A^2) + h.c.) + \frac{1}{2} \bar{\psi} \psi (m + 2g A) + \frac{1}{2} \bar{\psi} \psi (m + 2g A)^* \right\}, \quad (1)$$

where a real and non-negative mass $m \geq 0$ and $g \in \mathbb{C}$, which can be chosen by a phase rotation of the fields without loss of generality. The off-shell
SUSY transformation which makes the action (1) invariant is defined as

\[
\begin{align*}
\delta_\xi A(x) &= \xi \psi(x) \\
\delta_\xi A^*(x) &= \bar{\xi} \bar{\psi}(x) \\
\delta_\xi \psi(x) &= i\sigma_\mu \bar{\xi} \partial_\mu A(x) + i\xi F(x) \\
\delta_\xi \bar{\psi}(x) &= i\bar{\sigma}_\mu \xi \partial_\mu \bar{\psi}(x) \\
\delta_\xi F(x) &= \xi \bar{\sigma}_\mu \partial_\mu \bar{\psi}(x) \\
\delta_\xi F^*(x) &= \bar{\xi} \sigma_\mu \partial_\mu \psi(x),
\end{align*}
\]

where \(\xi_\alpha\) and \(\bar{\xi}_\dot{\beta}\) are two anti-commuting parameters. The off-shell transformation satisfies

\[
[\delta_\xi, \delta_\eta] = -i (\bar{\xi} \sigma_\mu \eta + \xi \sigma_\mu \bar{\eta}) \partial_\mu,
\]

which is a well-known relation derived from the SUSY algebra.

The on-shell action is obtained as

\[
S_{\text{on-shell}} = \int d^4x \left\{ |\partial_\mu A|^2 + |mA + gA^2|^2 + i\psi \sigma_\mu \partial_\mu \bar{\psi} + \frac{1}{2} \bar{\psi} \psi (m + 2gA) + \frac{1}{2} \bar{\psi} \bar{\psi} (m + 2gA)^* \right\},
\]

integrating the auxiliary field \(F\) of the off-shell one. The action (4) is invariant under the on-shell SUSY transformation,

\[
\begin{align*}
\delta'_\xi A(x) &= \xi \psi(x) \\
\delta'_\xi A^*(x) &= \bar{\xi} \bar{\psi}(x) \\
\delta'_\xi \psi(x) &= i\sigma_\mu \bar{\xi} \partial_\mu A(x) - \xi(mA^* + g^*A^2)(x) \\
\delta'_\xi \bar{\psi}(x) &= i\bar{\sigma}_\mu \xi \partial_\mu \bar{\psi}(x) - \bar{\xi}(mA + gA^2)(x).
\end{align*}
\]

Note that (5) are the first four transformations of (2) replacing \(F \rightarrow i(mA^* + g^*A^2)\) and \(F^* \rightarrow i(mA + gA^2)\).

The off-shell SUSY theory is also easily defined using the superfield formalism. Suppose that \(\theta_\alpha\) and \(\bar{\theta}_\dot{\alpha}\) are two global Grassmann parameters. Superfield is then defined by a function \(F(x, \theta, \bar{\theta})\) whose SUSY transformation is given by

\[
\delta_\xi F(x, \theta, \bar{\theta}) = \frac{1}{\sqrt{2}} (\xi Q + \bar{\xi} \bar{Q}) F(x, \theta, \bar{\theta})
\]

Note that (6) are the first four transformations of (2) replacing \(F \rightarrow i(mA^* + g^*A^2)\) and \(F^* \rightarrow i(mA + gA^2)\).
where $Q_\alpha$ and $\bar{Q}_{\dot{\alpha}}$ are differential operators:

$$Q_\alpha = \frac{\partial}{\partial \theta^\alpha} - i (\sigma_\mu)_{\alpha\dot{\alpha}} \bar{\theta}^\dot{\alpha} \partial_\mu,$$  

$$\bar{Q}_{\dot{\alpha}} = -\frac{\partial}{\partial \bar{\theta}^\dot{\alpha}} + i \theta^{\dot{\alpha}} (\sigma_\mu)_{\alpha\dot{\alpha}} \bar{\partial}_\mu.$$  

(7)

(8)

For later use, we introduce other differential operators,

$$D_\alpha = \frac{\partial}{\partial \theta^\alpha} + i (\sigma_\mu)_{\alpha\dot{\alpha}} \bar{\theta}^\dot{\alpha} \partial_\mu,$$  

$$\bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial \bar{\theta}^\dot{\alpha}} - i \theta^{\dot{\alpha}} (\sigma_\mu)_{\alpha\dot{\alpha}} \bar{\partial}_\mu.$$  

(9)

(10)

which are covariant under SUSY transformation (6) because

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = -\{D_\alpha, \bar{D}_{\dot{\alpha}}\} = 2i (\sigma_\mu)_{\alpha\dot{\alpha}} \partial_\mu,$$  

(11)

and the other commutation relations are zero.

The Wess-Zumino model is given by chiral and anti-chiral superfields $\Phi(x, \theta, \bar{\theta})$ and $\bar{\Phi}(x, \theta, \bar{\theta})$ which satisfy

$$\bar{D}_{\dot{\alpha}} \Phi = D_\alpha \bar{\Phi} = 0.$$  

(12)

The $\theta$ and $\bar{\theta}$ expansion of the chiral superfields can easily be written in terms of $y_\mu = x_\mu + i \theta \sigma_\mu \bar{\theta}$ and $\bar{y}_\mu = x_\mu - i \theta \sigma_\mu \bar{\theta}$ because, for instance, $\bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}}$ in the $y$ coordinate. We thus have

$$\Phi(y, \theta) = A(y) + \sqrt{2} \theta \psi(y) + i \theta \theta F(y),$$  

$$\bar{\Phi}(\bar{y}, \bar{\theta}) = A^*(\bar{y}) + \sqrt{2} \bar{\theta} \bar{\psi}(\bar{y}) + i \bar{\theta} \bar{\theta} F^*(\bar{y}).$$  

(13)

The off-shell SUSY transformation for the component fields (2) are reproduced from the definition (6) with the expansion (13).

The off-shell action (1) can also be expressed as

$$S = -\int d^4x \left\{ \Phi \Phi|_{\theta=0} + W(\Phi)|_{\theta=0} + W^*(\Phi)|_{\bar{\theta}=0} \right\},$$  

(14)

where

$$W(\Phi) = \frac{1}{2} m \Phi^2 + \frac{1}{3} g \Phi^3.$$  

(15)

From the construction presented above, it is obvious that the superfield action (14) is invariant under the off-shell SUSY transformation (6).
3. Wess-Zumino flow

We construct a supersymmetric flow equation in Wess-Zumino model. The flow equation is derived in two ways, one is based on the off-shell component fields as shown in Sec. 3.2 and the other is based on the superfield formalism as seen in Sec. 3.3. We will find that they give the same result.

3.1. 4+1-dimensional supersymmetry and supersymmetric flow

We introduce a flow time \( t \geq 0 \) and consider the time dependent bosonic fields \( \phi(t, x), \bar{\phi}(t, x), G(t, x), \bar{G}(t, x) \in \mathbb{C} \) and spinors \( \chi(t, x), \bar{\chi}(t, x) \). The component fields of Wess-Zumino model are replaced by those fields as follows:

\[
\begin{align*}
A(x) &\rightarrow \phi(t, x) \\
A^*(x) &\rightarrow \bar{\phi}(t, x) \\
\psi(x) &\rightarrow \chi(t, x) \\
\bar{\psi}(x) &\rightarrow \bar{\chi}(t, x) \\
F(x) &\rightarrow G(t, x) \\
F^*(x) &\rightarrow \bar{G}(t, x),
\end{align*}
\]

with boundary conditions,

\[
(\phi(t, x), \chi(t, x), G(t, x))|_{t=0} = (A(x), \psi(x), F(x)) \\
(\bar{\phi}(t, x), \bar{\chi}(t, x), \bar{G}(t, x))|_{t=0} = (A^*(x), \bar{\psi}(x), F^*(x)).
\]

(17)

Note that \( \bar{\phi} \) and \( \bar{G} \) are no longer the complex conjugates of \( \phi \) and \( G \), respectively, for non-zero flow time.

For the flowed fields, 4+1 dimensional SUSY transformation can be defined by replacing the fields of (2) according to (16):

\[
\begin{align*}
\delta_\xi \phi &= \xi \chi \\
\delta_\xi \bar{\phi} &= \bar{\xi} \bar{\chi} \\
\delta_\xi \chi &= i\sigma_\mu \xi \partial_\mu \phi + i\xi G \\
\delta_\xi \bar{\chi} &= i\bar{\sigma}_\mu \xi \partial_\mu \bar{\phi} + i\bar{\xi} \bar{G} \\
\delta_\xi G &= \bar{\xi} \bar{\sigma}_\mu \partial_\mu \chi \\
\delta_\xi \bar{G} &= \xi \sigma_\mu \partial_\mu \bar{\chi},
\end{align*}
\]

(18)

where \( \xi \) and \( \bar{\xi} \) are \( t \)-independent parameters.
It will be shown that a supersymmetric flow equation is given by

$$\partial_t \phi = ∇^2 \phi + im \bar{G} + g^* (2i\bar{\phi} \bar{G} - \bar{\chi} \chi),$$  \hspace{1cm} (19)  

$$\partial_t \bar{\phi} = ∇^2 \bar{\phi} + imG + g (2i\phi G - \chi \bar{\chi}),$$  \hspace{1cm} (20)  

$$\partial_t \chi = ∇^2 \chi + iσμ∂μ (m\bar{\chi} + 2g^* \bar{\phi} \bar{\chi}),$$  \hspace{1cm} (21)  

$$\partial_t \bar{\chi} = ∇^2 \bar{\chi} + iσμ∂μ (m\chi + 2g\phi \bar{\chi}),$$  \hspace{1cm} (22)  

$$\partial_t G = ∇^2 G - i∇ (m\bar{\phi} + g^* \bar{\phi} \bar{\chi}),$$  \hspace{1cm} (23)  

$$\partial_t \bar{G} = ∇^2 \bar{G} - i∇ (m\phi + g\phi \bar{\chi}),$$  \hspace{1cm} (24)  

where $∇^2 = ∑_μ ∂_μ ∂_μ$.

The Wess-Zumino flow tells us that each component field does not flow independently but mixing with other fields to keep SUSY. The flowed fields $G$ and $\bar{G}$ are no longer auxiliary fields because derivative terms are in (23) and (24). It is possible to show that

$$[\partial_t, δξ] = 0$$  \hspace{1cm} (25)  

which means that SUSY is kept at non-zero flow time. As we will see in the next two sections, (25) can also be easily confirmed from the construction of the Wess-Zumino flow equation.

### 3.2. Derivation of Wess-Zumino flow in component fields

We begin with considering a derivative of $S$ with respect to $A(x)$. Since $δS/δA(x)$ has $∇^2 A^*(x)$, a gradient flow for $\phi(t, x)$ as a diffusion equation $∂_t \phi \simeq ∇^2 \phi$ should be defined by

$$∂_t φ(t, x) = \left. - \frac{δS}{δA^*(x)} \right|_{\text{fields } \rightarrow \text{flowed fields}}$$  \hspace{1cm} (26)  

where $X|_{\text{fields } \rightarrow \text{flowed fields}}$ means that the field variables in $X$ are replaced according to (16). The first flow equation (19) is obtained from (26). Similarly, (20) is derived from a gradient flow equation as (26) with replacing $∂_t φ$ and $δA^*$ by $∂_t \bar{φ}$ and $δA$, respectively.

Suppose that (25) holds for $\phi$. Then the L.H.S. of (19) becomes

$$δξ∂_t φ(t, x) = ∂_t δξ φ(t, x) = ξ∂_t \chi(t, x).$$  \hspace{1cm} (27)  

While the SUSY transformation of the R.H.S. of (19) is

$$δξ (\text{R.H.S. of (19)}) = ξ \left( ∇^2 \chi + iσμ∂μ (m\bar{\chi} + 2g^* \bar{\phi} \bar{\chi}) \right).$$  \hspace{1cm} (28)
Since (27) coincides with (28), we obtain (21). We can also find (22) assuming (25) for $\phi$ as well.

The flow equations for $G$ and $\bar{G}$ are derived in the same manner. If (25) holds for $\chi$ and $\bar{\chi}$, we immediately find (23) and (24) by performing the SUSY transformation of the flow equations for $\chi$ and $\bar{\chi}$.

Once the flow equations are given for the scalar fields, we found that those for the other fields can be constructed by repeating the SUSY transformation (18). Since we then assumed (25) for $\phi, \bar{\phi}, \chi$ and $\bar{\chi}$, it is obvious that the obtained flows satisfy (25) for them. So all we have to do is check whether (25) holds for $G$ and $\bar{G}$ or not.

We now have $\delta \xi \bar{\phi} = \delta \xi G = 0$ for $\xi = 0$. So it can be immediately shown that $[\partial_t, \delta \xi]G = 0$ for $\xi = 0$. Moreover one can show that $[\partial_t, \delta \xi]G = 0$ for a general $\delta \xi$. Repeating the same argument, (25) is also true for $\bar{G}$.

### 3.3. Derivation of Wess-Zumino flow in superfield formalism

The flowed superfields are given by replacing

$$\Phi(z) \rightarrow \Psi(t, z), \quad \bar{\Phi}(z) \rightarrow \bar{\Psi}(t, z),$$

with $z = (x, \theta, \bar{\theta}).$ Suppose that

$$\Psi(t, z)|_{t=0} = \Phi(z), \quad \bar{\Psi}(t, z)|_{t=0} = \bar{\Phi}(z)$$

as an initial condition and the SUSY transformation of $\Psi(t, z)$ and $\bar{\Psi}(t, z)$ is defined by (18).

The gradient flow should be given such that $\Phi(t, z)$ and $\bar{\Phi}(t, z)$ are chiral and anti-chiral superfields satisfying (12). The field variation of the chiral superfield is defined as

$$\frac{\delta}{\delta \Phi(y, \theta)} \Phi(y', \theta') = \delta^4(y - y') \delta^2(\theta - \theta').$$

It can be shown that

$$\frac{\delta S}{\delta \Phi(x, \theta, \bar{\theta})} = \frac{1}{4} DD \Phi(x, \theta, \bar{\theta}) - \frac{\partial W^*(\Phi(x, \theta, \bar{\theta}))}{\partial \Phi(x, \theta, \bar{\theta})}.$$
Although it is natural to use $\delta S/\delta \bar{\Phi}$ for a gradient flow for $\Phi$, such a derivative does not satisfy the supersymmetric chiral condition (12) for $\Phi$.

It is possible to keep the condition (12) multiplying $\delta S/\delta \bar{\Phi}$ by $\bar{D}\bar{D}$. Thus a proper flow equation is

$$
\partial_t \Psi(t,z) = \frac{1}{4} \bar{D}\bar{D} \frac{\delta S}{\delta \bar{\Phi}(z)} \bigg|_{\Phi(z),\bar{\Phi}(z) \rightarrow \Psi(t,z),\bar{\Psi}(t,z)},
$$

and similarly,

$$
\partial_t \bar{\Psi}(t,z) = \frac{1}{4} D\bar{D} \frac{\delta S}{\delta \Phi(z)} \bigg|_{\Phi(z),\bar{\Phi}(z) \rightarrow \Psi(t,z),\bar{\Psi}(t,z)}.
$$

Since $\bar{D}\bar{D}DD = 16\Box$, we have

$$
\partial_t \Psi = \Box \Psi - \frac{1}{4} \bar{D}\bar{D}W'(\bar{\Psi})
$$

and

$$
\partial_t \bar{\Psi} = \Box \bar{\Psi} - \frac{1}{4} D\bar{D}W'(\Psi),
$$

where $W'(x) = \partial W(x)/\partial x$. The supersymmetric chiral condition (12) is actually kept for any non-zero flow time because, noticing $D^3 = \bar{D}^3 = 0$ and $[D,\partial_t] = [\bar{D},\partial_t] = 0$,

$$
\partial_t (\bar{D}_\alpha \Psi(t,x)) = \partial_t (D_\alpha \bar{\Psi}(t,x)) = 0,
$$

with $\bar{D}_\alpha \Psi(t = 0, x) = D_\alpha \bar{\Psi}(t = 0, x) = 0$.

The definitions of the gradient flow (34) and (35) are consistent with the SUSY transformation given by (6) because $[Q,\partial_t] = [\bar{Q},\partial_t] = 0$. So the commutation relation (25) is manifestly satisfied.

Since the flowed superfields obey the supersymmetric chiral condition (12), they can also be expanded as

$$
\Psi(t,y,\theta) = \phi(t,y) + \sqrt{2}\theta \chi(t,y) + i\theta \theta G(t,y),
$$

$$
\bar{\Psi}(t,\bar{y},\bar{\theta}) = \bar{\phi}(t,\bar{y}) + \sqrt{2}\bar{\theta} \bar{\chi}(t,\bar{y}) + i\bar{\theta} \bar{\theta} \bar{G}(t,\bar{y}).
$$

Substituting these expansions into (36), we find that the same flow equations as (19)-(24) are obtained.
3.4. The on-shell flow

The relation (25) is shown to be satisfied for the off-shell supersymmetric gradient flow. We mention an on-shell case that the auxiliary field is integrated out.

We consider an on-shell flow by replacing $G$ and $\bar{G}$ of (19)-(22) as

$$
G = i(m\bar{\phi} + g^* \bar{\phi}^2), \quad \bar{G} = i(m\phi + g\phi^2),
$$

which are the equations of motion of $F$ and $F^*$ at $t = 0$. Here we do not consider the flow equation of $G$ and $\bar{G}$. An on-shell SUSY transformation $\delta'_t$ for the flowed fields is given by (5) with the replacement (16).

The commutation relation between the flow derivative and the on-shell SUSY transformation does not vanish in general but is proportional to $\delta S/\delta h$ for $h = \psi, A, \bar{\psi}, A^*$. For instance,

$$
[\partial_t, \delta'_t \phi] \phi = W''(\bar{\phi}) \xi \frac{\delta S}{\delta \psi} \bigg|_{\text{fields} \rightarrow \text{flowed fields}}.
$$

One can easily show that the commutators for other fields do not also vanish but satisfy the similar relations.

4. Formal solution of Wess-Zumino flow

The flowed chiral and anti-chiral superfields are directly coupled even in the linear part of the Wess-Zumino flow,

$$
\partial_t \begin{pmatrix} \Psi_0 \\ \bar{\Psi}_0 \end{pmatrix} = \begin{pmatrix} \Box & -\frac{m}{4} \bar{D}D \\ -\frac{m}{4} DD & \Box \end{pmatrix} \begin{pmatrix} \Psi_0 \\ \bar{\Psi}_0 \end{pmatrix},
$$

where the suffix zero means they are solutions to the linear part of the flow equation.

To solve the formal solution of the Wess-Zumino flow, let us move on to a basis that diagonalizes the matrix of (41) as

$$
\begin{pmatrix} \Pi_+ \\ \Pi_- \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} i \frac{DD}{\sqrt{-\Box}} & 1 \\ -i \frac{DD}{\sqrt{-\Box}} & 1 \end{pmatrix} \begin{pmatrix} \Psi \\ \bar{\Psi} \end{pmatrix}.
$$

Then the Wess-Zumino flow equation is given in terms of $\Pi_+$ and $\Pi_-$:

$$
\partial_t \Pi_{\pm} = \left( \Box \pm im\sqrt{-\Box} \right) \Pi_{\pm} + R_{\pm},
$$
where

\[ R_\pm = \pm i g^* \sqrt{-\Box} \bar{\Psi}^2 - \frac{g}{4\sqrt{2}} DD\Psi^2. \]  

Note that the initial conditions for \( \Pi_\pm \) are derived from those of \( \Psi \) and \( \bar{\Psi} \) via (42).

A formal solution of (43) is given by

\[ \Pi_\pm(t, x) = \int d^4y \left\{ K_\pm^t(x - y)\Pi_\pm(0, y) + \int_0^t ds K_\pm^s(x - y)R_\pm(s, y) \right\}, \]  

where \( \theta, \bar{\theta} \) are abbreviated for \( \Pi_\pm(t, x, \theta, \bar{\theta}) \) and \( R_\pm(t, x, \theta, \bar{\theta}) \). Here \( K_\pm^t(x) \) is a heat kernel defined by

\[ K_\pm^t(x) = \int \frac{d^4p}{(2\pi)^4} e^{ipx} e^{-t(p^2 \pm im\sqrt{p^2})}. \]  

Note that (46) coincides with the normal one for \( m = 0 \), and it still works as a damping factor for \( m \neq 0 \). We can actually show that (45) satisfies (43) because \( K_\pm^t(x) \) provides a solution to the free part of (43).

We can also give the formal of (36) as

\[ \Psi_t(p) = C_t(p)\Phi(p) - S_t(p) \frac{D\bar{D}}{4\sqrt{p^2}} \bar{\Phi}(p) \]

\[ - \int_0^t ds \left( g^* C_{t-s}(p) \frac{D\bar{D}}{4}(\Psi_s \star \bar{\Psi}_s)(p) + g S_{t-s}(p) \sqrt{p^2}(\Psi_s \star \Psi_s)(p) \right), \]

\[ \bar{\Psi}_t(p) = C_t(p)\bar{\Phi}(p) - S_t(p) \frac{D\bar{D}}{4\sqrt{p^2}} \Phi(p) \]

\[ - \int_0^t ds \left( g C_{t-s}(p) \frac{D\bar{D}}{4}(\Psi_s \star \Psi_s)(p) + g^* S_{t-s}(p) \sqrt{p^2}(\bar{\Psi}_s \star \bar{\Psi}_s)(p) \right), \]

where we again abbreviate \( \theta, \bar{\theta} \) of \( \Psi_t(p, \theta, \bar{\theta}) \) and \( \Phi(p, \theta, \bar{\theta}) \). Here \( D \) and \( \bar{D} \) are the momentum representation of (10), and \( C_t(p) \) and \( S_t(p) \) are defined by

\[ C_t(p) \equiv e^{-tp^2} \cos(tm\sqrt{p^2}), \]  

\[ S_t(p) \equiv e^{-tp^2} \sin(tm\sqrt{p^2}), \]
which come from (46) in the momentum space as $K^\pm_t(p) = C_t(p) \pm iS_t(p)$. The star symbol means the convolution integral in the momentum space:

$$(A \ast B)(p) \equiv \int \frac{d^4q}{(2\pi)^4} A(q)B(p-q), \quad (50)$$

for any functions $A$ and $B$. Note that $(A \ast B)(p) = (B \ast A)(p)$.

We finally find the formal solutions for the component fields inserting (13) and (38) into (47):

$$\phi_t(p) = C_t(p)A(p) + \frac{i}{\sqrt{p^2}} S_t(p)F^\ast(p) - g\sqrt{p^2} \int_0^t dsS_{t-s}(p)(\phi_s \ast \phi_s)(p)$$

$$+ g^* \int_0^t dsC_{t-s}(p) \left\{ 2i(\bar{\phi}_s \ast \bar{G}_s)(p) - (\bar{\chi}_s \ast \bar{\chi}_s)(p) \right\}, \quad (51)$$

$$\tilde{\phi}_t(p) = C_t(p)A^\ast(p) + \frac{i}{\sqrt{p^2}} S_t(p)F(p) - g^*\sqrt{p^2} \int_0^t dsS_{t-s}(p)(\bar{\phi}_s \ast \bar{\phi}_s)(p)$$

$$+ g \int_0^t dsC_{t-s}(p) \left\{ 2i(\phi_s \ast G_s)(p) - (\chi_s \ast \chi_s)(p) \right\}, \quad (52)$$

$$\chi_t(p) = C_t(p)\psi(p) - \frac{\sigma_\mu p_\mu}{\sqrt{p^2}} S_t(p)\bar{\psi}(p) - 2g^* \sigma_\mu p_\mu \int_0^t dsC_{t-s}(p)(\bar{\phi}_s \ast \bar{\chi}_s)(p)$$

$$- 2g\sqrt{p^2} \int_0^t dsS_{t-s}(p)(\phi_s \ast \chi_s)(p), \quad (53)$$

$$\bar{\chi}_t(p) = C_t(p)\bar{\psi}(p) - \frac{\sigma_\mu p_\mu}{\sqrt{p^2}} S_t(p)\psi(p) - 2g \bar{\sigma}_\mu p_\mu \int_0^t dsC_{t-s}(p)(\phi_s \ast \chi_s)(p)$$

$$- 2g^* \sqrt{p^2} \int_0^t dsS_{t-s}(p)(\bar{\phi}_s \ast \bar{\chi}_s)(p), \quad (54)$$

$$G_t(p) = C_t(p)F(p) + i\sqrt{p^2} S_t(p)A^\ast(p) + ig^* p^2 \int_0^t dsC_{t-s}(p)(\bar{\phi}_s \ast \bar{\phi}_s)(p)$$

$$- g\sqrt{p^2} \int_0^t dsS_{t-s}(p)\left\{ 2(\phi_s \ast G_s)(p) + i(\chi_s \ast \chi_s)(p) \right\}, \quad (55)$$

$$\bar{G}_t(p) = C_t(p)F^\ast(p) + i\sqrt{p^2} S_t(p)A(p) + igp^2 \int_0^t dsC_{t-s}(p)(\phi_s \ast \phi_s)(p)$$

$$- g^* \sqrt{p^2} \int_0^t dsS_{t-s}(p)\left\{ 2(\phi_s \ast \bar{G}_s)(p) + i(\bar{\chi}_s \ast \bar{\chi}_s)(p) \right\}. \quad (56)$$

Note that the terms with $1/\sqrt{p^2}$ are well-defined because they appear with
$S_t(p)$ and $1/\sqrt{p^2}S_t(p)|_{p=0} = 0$.

One of the interesting points is that the solutions have a damping oscillation with the flow time for non-zero mass, $C_t$ and $S_t$. This behavior is different from the solution of the Yang-Mills flow whose damping factor is $e^{-tp^2}$. In the case of $m = 0$, we have much simpler solutions because $C_t(p) = e^{-tp^2}$ and $S_t(p) = 0$.

5. Summary

We have constructed a supersymmetric gradient flow equation in four dimensional Wess-Zumino model. The Wess-Zumino flow equation is given by two ways. One is based on the off-shell component fields in which the flow for the scalar field is given by the gradient of the action. The flow equations for the other fields are derived from it by repeating the SUSY transformation. The other way is based on the superfield formalism. The gradient flow for the chiral superfield is determined from the gradient of the action with respect to the superfield with keeping the supersymmetric chiral condition. We found that the resultant equations are the same.

The obtained flow is supersymmetric in a sense that the flow time derivative and the SUSY transformation commute with each other for non-zero flow time. On the other hand, the commutator does not vanish for the on-shell flow. The flowed components fields $G$ and $\bar{G}$ are not auxiliary but dynamical fields because the derivative terms are provided by their flows. We have obtained the formal solution of the Wess-Zumino flow equation and find that it behaves as a damping oscillation with respect to the flow time for non-zero mass, which is different from the Yang-Mills flow.

Since we have constructed the SUSY flow for Wess-Zumino model, we achieved the first step toward the further understanding of the mechanism that leads to the UV finiteness of SUSY gradient flows. It is interesting whether the Wess-Zumino flow shows the UV finiteness at one loop order or not. In order to show that, further studies are now in progress.

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Appendix A. Convention

The Lorentz index $\mu$ runs $\mu = 0, 1, 2, 3$. All boundary fields are defined on $\mathbb{R}^4$. The fermions $\psi_\alpha$ and $\bar{\psi}^{\dot{\beta}}$ transform as spinors of $SO(4) \simeq SU(2)_L \times SU(2)_R$. The spinor indices $\alpha, \beta$ take the values 1, 2. We basically follow [43] as the convention of spinors, but we perform the Wick rotation $t \to -it$ from [43]. Then the auxiliary field $F$ is also replaced as $F \to IF$. Useful identities of an Euclidean (Wick rotated) version of [43] are summarized in [41].

The anti-symmetric tensors $\epsilon_{\alpha \beta}, \epsilon^{\alpha \beta}, \epsilon^{\dot{\alpha} \dot{\beta}}, \epsilon^{\dot{\alpha} \dot{\beta}}$ are defined as $\epsilon_{21} = \epsilon_{12} = \epsilon_{21} = \epsilon^{i2} = 1$. Spinors with upper and lower indices are defined as

$$\psi^\alpha = \epsilon^{\alpha \beta} \psi_\beta, \quad \bar{\psi}_{\dot{\alpha}} = \epsilon^{\dot{\alpha} \dot{\beta}} \bar{\psi}_{\dot{\beta}}.$$ (A.1)

Then Lorentz scalars made of two spinors are given by

$$\psi \chi \equiv \psi^\alpha \chi_\alpha, \quad \bar{\psi} \bar{\chi} \equiv \bar{\psi}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}}.$$ (A.2)

Note that $\bar{\psi}_{\dot{\alpha}}$ is not a complex conjugate of $\psi_\alpha$ in Euclidean space.

The four dimensional sigma matrices $(\sigma_\mu)_{\alpha \dot{\beta}}$ and $(\bar{\sigma}_\mu)^{\dot{\alpha} \beta}$ are defined by

$$\sigma_0 = \bar{\sigma}_0 = -i 1, \quad \sigma_i = -\sigma_i = \sigma^i,$$ (A.3)

where $\sigma^i$ for $i = 1, 2, 3$ are the standard Pauli matrices. We often abbreviate the spinor index such as (A.2) through out this paper. For instance, $\psi \sigma_\mu \tilde{\psi}$ in the action (11) means $\psi^\alpha (\sigma_\mu)_{\alpha \dot{\beta}} \tilde{\psi}^{\dot{\beta}}$. The index structure of $\sigma_\mu$ and $\bar{\sigma}_\mu$ can be specified as $(\sigma_\mu)_{\alpha \dot{\beta}}$ and $(\bar{\sigma}_\mu)^{\dot{\alpha} \beta}$. They are related to each other as

$$(\bar{\sigma}_\mu)^{\dot{\alpha} \alpha} = \epsilon^{\dot{\alpha} \dot{\beta}} \epsilon^{\alpha \beta} (\sigma_\mu)_{\beta \beta}.$$ (A.4)

See [41] for the other useful formulas after the Wick rotation.

References

[1] R. Narayanan and H. Neuberger. Infinite N phase transitions in continuum Wilson loop operators. JHEP, 03:064, 2006.

[2] Martin Luscher. Properties and uses of the Wilson flow in lattice QCD. JHEP, 1008:071, 2010.

[3] Martin Luscher and Peter Weisz. Perturbative analysis of the gradient flow in non-abelian gauge theories. JHEP, 1102:051, 2011.
[4] Martin Luscher. Chiral symmetry and the Yang–Mills gradient flow. *JHEP*, 1304:123, 2013.

[5] Kenji Hieda, Hiroki Makino, and Hiroshi Suzuki. Proof of the renormalizability of the gradient flow. *Nucl. Phys.*, B918:23–51, 2017.

[6] Hiroshi Suzuki. Energy-momentum tensor from the Yang-Mills gradient flow. *PTEP*, 2013:083B03, 2013. [Erratum: PTEP2015,079201(2015)].

[7] Luigi Del Debbio, Agostino Patella, and Antonio Rago. Space-time symmetries and the Yang-Mills gradient flow. *JHEP*, 11:212, 2013.

[8] Masayuki Asakawa, Tetsuo Hatsuda, Etsuko Itou, Masakiyo Kitazawa, and Hiroshi Suzuki. Thermodynamics of SU(3) gauge theory from gradient flow on the lattice. *Phys. Rev.*, D90(1):011501, 2014. [Erratum: Phys. Rev.D92,no.5,059902(2015)].

[9] Hiroki Makino and Hiroshi Suzuki. Lattice energy-momentum tensor from the Yang-Mills gradient flow–inclusion of fermion fields. *PTEP*, 2014:063B02, 2014. [Erratum: PTEP2015,079202(2015)].

[10] Yusuke Taniguchi, Shinji Ejiri, Ryo Iwami, Kazuyuki Kanaya, Masakiyo Kitazawa, Hiroshi Suzuki, Takashi Umeda, and Naoki Wakabayashi. Exploring $N_f = 2+1$ QCD thermodynamics from the gradient flow. *Phys. Rev.*, D96(1):014509, 2017. [Erratum: Phys. Rev.D99,no.5,059904(2019)].

[11] Masakiyo Kitazawa, Takumi Iritani, Masayuki Asakawa, and Tetsuo Hatsuda. Correlations of the energy-momentum tensor via gradient flow in SU(3) Yang-Mills theory at finite temperature. *Phys. Rev.*, D96(11):111502, 2017.

[12] Ryosuke Yanagihara, Takumi Iritani, Masakiyo Kitazawa, Masayuki Asakawa, and Tetsuo Hatsuda. Distribution of Stress Tensor around Static Quark–Anti-Quark from Yang-Mills Gradient Flow. *Phys. Lett.*, B789:210–214, 2019.

[13] Robert V. Harlander, Yannick Kluth, and Fabian Lange. The two-loop energy-momentum tensor within the gradient-flow formalism. *Eur. Phys. J.*, C78(11):944, 2018.
[14] Takumi Iritani, Masakiyo Kitazawa, Hiroshi Suzuki, and Hiromasa Takaura. Thermodynamics in quenched QCD: energy-momentum tensor with two-loop order coefficients in the gradient-flow formalism. *PTEP*, 2019(2):023B02, 2019.

[15] Hiroki Makino and Hiroshi Suzuki. Renormalizability of the gradient flow in the two-dimensional $O(N)$ non-linear sigma model. 2014.

[16] Sinya Aoki, Kengo Kikuchi, and Tetsuya Onogi. Gradient Flow of $O(N)$ nonlinear sigma model at large $N$. *JHEP*, 04:156, 2015.

[17] Hiroki Makino, Fumihiko Sugino, and Hiroshi Suzuki. Large-$N$ limit of the gradient flow in the 2D $O(N)$ nonlinear sigma model. *PTEP*, 2015(4):043B07, 2015.

[18] Wolfgang Bietenholz, Philippe de Forcrand, Urs Gerber, Héctor Mejía-Díaz, and Ilya O. Sandoval. Topological Susceptibility of the 2d O(3) Model under Gradient Flow. *Phys. Rev.*, D98(11):114501, 2018.

[19] Ryo Yamamura. The YangMills gradient flow and lattice effective action. *PTEP*, 2016(7):073B10, 2016.

[20] Hiroki Makino, Okuto Morikawa, and Hiroshi Suzuki. Gradient flow and the Wilsonian renormalization group flow. *PTEP*, 2018(5):053B02, 2018.

[21] Yoshihiko Abe and Masafumi Fukuma. Gradient flow and the renormalization group. *PTEP*, 2018(8):083B02, 2018.

[22] Andrea Carosso, Anna Hasenfratz, and Ethan T. Neil. Nonperturbative Renormalization of Operators in Near-Conformal Systems Using Gradient Flows. *Phys. Rev. Lett.*, 121(20):201601, 2018.

[23] Hidenori Sonoda and Hiroshi Suzuki. Derivation of a Gradient Flow from ERG. 2019.

[24] Sinya Aoki, Kengo Kikuchi, and Tetsuya Onogi. Geometries from field theories. *PTEP*, 2015(10):101B01, 2015.

[25] Sinya Aoki, Janos Balog, Tetsuya Onogi, and Peter Weisz. Flow equation for the large $N$ scalar model and induced geometries. *PTEP*, 2016(8):083B04, 2016.
[26] Sinya Aoki, Janos Balog, Tetsuya Onogi, and Peter Weisz. Flow equation for the scalar model in the large $N$ expansion and its applications. PTEP, 2017(4):043B01, 2017.

[27] Sinya Aoki and Shuichi Yokoyama. Flow equation, conformal symmetry, and anti-de Sitter geometry. PTEP, 2018(3):031B01, 2018.

[28] Sinya Aoki and Shuichi Yokoyama. AdS geometry from CFT on a general conformally flat manifold. Nucl. Phys., B933:262–274, 2018.

[29] Tasuku Endo, Kenji Hieda, Daiki Miura, and Hiroshi Suzuki. Universal formula for the flavor non-singlet axial-vector current from the gradient flow. PTEP, 2015(5):053B03, 2015.

[30] Kazuo Fujikawa. The gradient flow in $\lambda \phi^4$ theory. JHEP, 03:021, 2016.

[31] Kenji Hieda and Hiroshi Suzuki. Small flow-time representation of fermion bilinear operators. Mod. Phys. Lett., A31(38):1650214, 2016.

[32] Yusuke Taniguchi, Kazuyuki Kanaya, Hiroshi Suzuki, and Takashi Umeda. Topological susceptibility in finite temperature (2+1)-flavor QCD using gradient flow. Phys. Rev., D95(5):054502, 2017.

[33] Okuto Morikawa and Hiroshi Suzuki. Axial $U(1)$ anomaly in a gravitational field via the gradient flow. PTEP, 2018(7):073B02, 2018.

[34] Hiroshi Suzuki and Hiromasa Takaura. Gradient flow, renormalon ambiguity, and the gluon condensate. 2018.

[35] Georg Bergner, Camilo López, and Stefano Piemonte. A study of center and chiral symmetry realization in thermal $\mathcal{N} = 1$ super Yang-Mills theory using the gradient flow. 2019.

[36] Kenji Hieda, Aya Kasai, Hiroki Makino, and Hiroshi Suzuki. 4D $\mathcal{N} = 1$ SYM supercurrent in terms of the gradient flow. PTEP, 2017(6):063B03, 2017.

[37] Aya Kasai, Okuto Morikawa, and Hiroshi Suzuki. Gradient flow representation of the four-dimensional $\mathcal{N} = 2$ super Yang-Mills supercurrent. PTEP, 2018(11):113B02, 2018.
[38] Kengo Kikuchi and Tetsuya Onogi. Generalized Gradient Flow Equation and Its Application to Super Yang-Mills Theory. *JHEP*, 1411:094, 2014.

[39] Naohito Nakazawa. N=1 superYang-Mills theory in Ito calculus. *Prog. Theor. Phys.*, 110:1117–1150, 2004.

[40] Naohito Nakazawa. Stochastic gauge fixing in N=1 supersymmetric Yang-Mills theory. *Prog. Theor. Phys.*, 116:883–917, 2007.

[41] Daisuke Kadoh and Naoya Ukita. Supersymmetric gradient flow in $\mathcal{N} = 1$ SYM. 2018.

[42] Sinya Aoki, Kengo Kikuchi, and Tetsuya Onogi. Flow equation of $\mathcal{N} = 1$ supersymmetric O(N ) nonlinear sigma model in two dimensions. *JHEP*, 02:128, 2018.

[43] J. Wess and J. Bagger. *Supersymmetry and supergravity*. 1992.