Behaviour in time of solutions to fourth-order parabolic systems
with time dependent coefficients.

M.Marras\textsuperscript{1} and S. Vernier Piro\textsuperscript{2}

Abstract. This paper deals with a class of initial-boundary value problems for nonlinear fourth order parabolic systems with time dependent coefficients in a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$. Introducing suitable conditions on the source terms, we obtain a time interval $[0, T]$, where the solution remains bounded by deriving a lower bound $T$ of $t^*$. Moreover, we establish conditions on the shape of the spatial domain and on data sufficient to guarantee that the solution blows up in finite time $t^*$, deriving an upper bound for $t^*$.

AMS (MOS) subject classification: 35G30, 35K46, 35B44.

Keywords: Semilinear fourth order parabolic equations; fourth order parabolic systems; Blow-up.

\textsuperscript{1}Dipartimento di Matematica e Informatica, Università di Cagliari, via Ospedale 72, 09124 Cagliari (Italy), mmarras@unica.it
\textsuperscript{2}Facoltá di Ingegneria e Architettura, Università di Cagliari, Via Marengo 2, 09123 Cagliari (Italy), svernier@unica.it
1 Introduction

We deal with the following fourth-order parabolic system with time dependent coefficients

\[
\begin{align*}
  u_t + \delta_1(t)\Delta^2 u - h_1(t)\Delta u &= k_1(t)f_1(v), \quad x \in \Omega, \quad t \in (0,t^*), \\
v_t + \delta_2(t)\Delta^2 v - h_2(t)\Delta v &= k_2(t)f_2(u), \quad x \in \Omega, \quad t \in (0,t^*),
\end{align*}
\]

(1.1)

\[
(u,v) = 0, \quad \left(\frac{\partial u}{\partial n}, \frac{\partial v}{\partial n}\right) = 0, \quad x \in \partial\Omega, \quad t \in (0,t^*),
\]

(1.2)

\[
(u(x,0),v(x,0)) = (u_0(x),v_0(x)), \quad x \in \Omega,
\]

(1.3)

where Ω is a bounded domain in \(\mathbb{R}^N, N \geq 2\), with smooth boundary \(\partial\Omega\), \(\frac{\partial}{\partial n}\) is the outward normal derivative on the boundary \(\partial\Omega\), \(f_1\) and \(f_2\) are non negative functions, \(t^*\) is the blow-up time of the solution; \((u_0(x),v_0(x))\) are non negative functions in \(\Omega\), \(\delta_i, k_i, h_i i = 1, 2\), are in general positive bounded functions of \(t\).

In the literature the great part of the results concerns the case of only one equation and this because fourth order parabolic equations describe a variety of physical processes (see [3]). In particular, in the thin film theory (see [6], [8], [11]) such problems describe the evolution of the epitaxial growth of thin film and the following equation is introduced:

\[
u_t + \Delta^2 u - A_1\Delta u - A_2\text{div}(|\nabla u|^2\nabla u) = g(x,t,u)
\]

(1.4)
where \( u(x,t) \) is the height from the surface of the film, \( \Delta^2 u \) denotes the capillarity-driven surface diffusion, \( A_1 \Delta u \) describes the diffusion due to evaporation-condensation, \( A_2 \ div(|\nabla u|^2 \nabla u) \) represents the upward hopping of atoms, \( g \) corresponds to the mean deposition flux. In [8] King, Stein and Winkler showed for the solutions of (1.4) existence, uniqueness and regularity in an appropriate function space. Xu et al. in [17] investigated the equation (1.4) with \( A_2 = 0 \) and \( g(x,t,u) = g(u) \) in a bounded domain in \( \mathbb{R}^N \) and, by using the potential well method, showed that the solutions exist globally or blow-up in finite time, depending on whether or not the initial data are in the potential well. When the Laplacian term in (1.4) is replaced by p-Laplacian and \( g(u) \) is of power type, with a modified potential well method, Han in [6] obtained again global existence and finite time blow-up for the solutions when the initial energy is subcritical and critical. By different methods Philippin in [13], with \( A_1 = A_2 = 0, \ g(u) = k(t)|u|^{p-1}u \), proved that the solution cannot exist for all time, i.e. it blows up in \( L^2 \)-norm and an upper bound for \( t^* \) is derived; in addition, he constructed (under certain conditions on the data) a lower bound for \( t^* \) by using a first order differential inequality method. Escudero, Gazzola and Peral in [1] proved existence and blow-up results for the solutions of the equation

\[
u_t + \Delta^2 u = \text{det}(D^2 u) + \lambda h(x,t),\]
under Dirichlet boundary conditions, which models epitaxial growth processes and where the evolution is dictated by the competition between the determinant of the Hessian matrix of the solution and the bilaplacian (see also [2]).

Recently Winkler in [10] considered the equation

\[ u_t = -\Delta^2 u - \mu \Delta u - \lambda \Delta(|\nabla u|^2) + f(x) \]

when \( \Omega \) is a bounded convex domain in \( \mathbb{R}^N \), under the conditions \( \frac{\partial u}{\partial n} = \frac{\partial \Delta u}{\partial n} = 0 \) on the boundary \( \partial \Omega \) with bounded initial data, and proved the existence of global weak solutions with spatial dimension \( N \leq 3 \), under suitable conditions on data.

Less attention was given to the case of parabolic fourth order systems. In (14, Sec.4) Philippin and Vernier Piro investigated a simplified version of the system (1.1)-(1.3) i.e. with \( \delta_i(t) = 1, h_i(t) = 0 \), and \( \Omega \) a bounded domain in \( \mathbb{R}^N \), with \( N = 2 \) or \( N = 3 \). They proved that there exists a safe interval of existence \([0, T]\), \( T \) a lower bound of \( t^* \), extending a method used in the study of second order parabolic systems (see i.e. [9], [10], [12]). The method is based on introducing suitable functionals which satisfy first order differential inequalities from which upper and/or lower bounds of the blow-up time are derived.
The aim of this article is to investigate the behavior near the blow-up time of the solutions in the case of a more general system as (1.1)–(1.3), where the dimension of the spatial domain $\Omega$ is $N \geq 2$: we introduce conditions on data sufficient to prove that the solution must blow up in finite time $t^*$, deriving an upper bound of $t^*$; moreover, we prove that there exists an interval where the solution remains bounded, by deriving a lower bound of $t^*$.

In this contest, we consider non negative solutions of the system (1.1) - (1.3), motivated by our aim to investigate the behavior of the solutions which approach to $+\infty$ as $t$ approaches the finite blow-up time $t^*$. For the definition of solution to the system (1.1)- (1.3) we extend the definition in [1], Theorem 3.1: for some $T > 0$ and $f_1$ and $f_2 \in L^2(0,T;L^2(\Omega))$, $u_0$ and $v_0 \in W_0^{2,2}(\Omega)$, the functions $u$ and $v$ belong to the space $C(0,T;W_0^{2,2}(\Omega)) \cap L^2(0,T;W^{4,2}(\Omega)) \cap W^{1,2}(0,T;L^2(\Omega))$.

Throughout the paper we assume $f_1$ and $f_2 \in L^2(0,T;L^2(\Omega))$, $u_0$ and $v_0 \in W_0^{2,2}(\Omega)$. We denote by $|| \cdot ||_r$ the $L^r(\Omega)$ norm for $1 \leq r \leq \infty$ and by $||u||_{W_0^{2,2}(\Omega)} = ||\Delta u||_2 = (\int_{\Omega} (\Delta u)^2 dx)^{\frac{1}{2}}$ the norm in $W_0^{2,2}(\Omega)$.

The first investigation in this paper is to determinate a safe time interval of existence of the solution $(u,v)$ of (1.1)-(1.3), say $[0,T]$, with $T$ a lower bound of $t^*$. We remark that in this investigation $\Omega$ is not assumed to be a ball.

With the aim to obtain an interval where the solution remains bounded, we introduce the functional

$$
\Phi(t) = ||\Delta u||_2^2 + ||\Delta v||_2^2 = \Phi_1(t) + \Phi_2(t), \quad (1.5)
$$
with initial value

\[ \Phi(0) := \Phi_0 = ||\Delta u_0||_2^2 + ||\Delta v_0||_2^2. \]  

(1.6)

**Theorem 1.1. (Lower Bound).** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N \). Let \( (u,v) \) be a non negative solution of the system (1.1)-(1.3) and let \( \Phi(t) \) be defined in (1.5)-(1.6). Assume

\[ f_1(s) \leq s^p, \quad f_2(s) \leq s^q, \quad s > 0, \quad \text{with} \ p > q > 1, \]

(1.7)

\( p, q \) satisfying the condition (2.1) in Lemma 2.1 below. Then \( \Phi(t) \) remains bounded in the interval \( [0,T] \), with \( T < t^* \) and

\[
\begin{cases}
T := \frac{1}{B(p-1)} \ln \left[ 1 + \frac{B}{2A} \Phi_0^{1-p} \right], & \text{if } p > q, \quad A, B > 0, \\
T := \frac{1}{B(p-1)} \ln \left[ 1 + \frac{B}{A} \Phi_0^{1-p} \right], & \text{if } q = p \quad \tilde{A}, B > 0.
\end{cases}
\]

(1.8)

To obtain an upper bound of the blow-up time we introduce the functional

\[ \Psi(t) = \int_\Omega u\phi_1 dx + \int_\Omega v\phi_1 dx = \Psi_1(t) + \Psi_2(t), \]

(1.9)

with

\[ \Psi(0) = \Psi_0 = \int_\Omega u_0\phi_1 dx + \int_\Omega v_0\phi_1 dx = \Psi_1(0) + \Psi_2(0) \]

(1.10)

and \( \phi_1 \) the first eigenfunction associate to the first eigenvalue \( \Lambda_1 \) of the bi-harmonic eigenvalue problem with Dirichlet boundary conditions:
\[ \Delta^2 \phi = \Lambda \phi, \quad x \in \Omega \subset \mathbb{R}^N, \quad N \geq 2, \quad (1.11) \]
\[ \phi = 0, \quad \frac{\partial \phi}{\partial n} = 0, \quad x \in \partial \Omega, \quad (1.12) \]

with \( \phi \) normalized by
\[ ||\phi||_2^2 = 1. \quad (1.13) \]

For all \( \phi \neq 0 \), \( \Lambda_1 \) satisfies
\[ ||\phi||_2^2 \leq \Lambda_1^{-1} ||\Delta \phi||_2^2, \quad (1.14) \]
(see [7]). The problem (1.11) is closely related to the biharmonic differential equation
\[ \Delta^2 \phi = f \]
with the same boundary conditions (1.12), which describes characteristic vibrations of a clamped plate. It is well-known that the biharmonic operator under Dirichlet boundary conditions does not satisfy the positivity preserving property in general domains; nevertheless, it holds when \( \Omega = B_R \) the ball in \( \mathbb{R}^N \) with radius \( R \). We remark that the balls are not the only sets where the property holds. (See e.g. [4], [5], and [15]). As a consequence, the functional \( \Psi(t) \) defined in (1.9) and (1.10) is non-negative.

We prove the following
Theorem 1.2. ( Blow up and upper bound).

Let $\Omega = B_R$ be the $N$-dimensional ball, $N \geq 2$. Let $(u, v)$ be a non negative solution of the system \eqref{1.1}-\eqref{1.3} with $h_i = 0$ and let $\Psi(t)$ be defined in \eqref{1.9}-\eqref{1.10}. Assume

$$f_1(s) \geq s^p, \quad f_2(s) \geq s^q, \quad s > 0, \quad \text{with } p > q > 1, \quad \text{(1.15)}$$

and the initial data are so large that

$$H(\Psi_0) > 0, \quad H(\Psi(t)) := -\Lambda_1 \delta \Psi + 2^{1-q} c \Psi^q - c Q, \quad c, Q > 0 \quad \text{(1.16)}$$

with $\Lambda_1$ the first eigenvalue of the problem \eqref{1.11}-\eqref{1.13}, then $\Psi$ blows up in a finite time $t^*$ in the sense that

$$\lim_{t \to t^*} \Psi(t) = \infty,$$

with the upper bound

$$T_0 = \int_{\Psi_0}^{\infty} \frac{d\eta}{H(\eta)} \geq t^*. \quad \text{(1.17)}$$

In the particular case $q = p$, we achieve the following corollary.

Corollary 1.1. Under the hypotheses of Theorem 1.2, if $q = p$ and if there exist two positive costants $\delta, \bar{c}$ such that the initial data $(u_0, v_0)$ satisfy the following condition

$$\Psi_0 > \left( \frac{\delta \Lambda_1}{\bar{c}} \right)^{\frac{1}{p-1}}, \quad \text{(1.18)}$$
then $\Psi(t)$ must blow up at time $t^* \leq T$ with

$$T := -\frac{1}{(p-1)\delta \Lambda_1} \log \left\{ 1 - \frac{\delta \Lambda_1}{c\Psi_0^{p-1}} \right\}.$$ (1.19)

The scheme of this paper is the following: in Section 2 we prove Theorem 1.1 showing that the functional $\Phi(t)$ remains bounded on some time interval $(0, T)$, where in some particular cases $T$ may be explicitly computed in terms of the data of the system (1.1). Clearly this value of $T$ provides a lower bound for blow-up time $t^*$ of the solution $(u, v)$. As a consequence, we have that also the $L^2$-norm of the solution remains bounded in $(0, T)$, since from the eigenvalues theory (see [6]) we have

$$||u||^2_2 + ||v||^2_2 \leq \Lambda_1^{-1}(||\Delta u||^2_2 + ||\Delta v||^2_2),$$

where $\Lambda_1$ is the first eigenvalue of the problem (1.11)-(1.13) applied to $u$ and $v$.

In Section 3 we consider the system (1.1)-(1.3) with $h_i(t) = 0$, and the spatial domain the N-dimensional ball $B_R(0)$ and we prove Theorem 1.2 and Corollary 1.1.
2 Lower bound of $t^*$

In this section we consider the system (1.1)-(1.3) and we assume that the solution blows up at finite time $t^*$ in the sense that $\lim_{t \to t^*} \Phi(t) = \infty$ with $\Phi$ defined in (1.5)-(1.6). We state now a known result which will be needed in this section.

From the Rellich-Kondrachov theorem on the embedding $W^{p,m} \subset L^r$ (see [4] Theorem 2.4), the following Lemma is derived for $p = 2$.

**Lemma 2.1.** Let $\Omega$ be a bounded domain in $\mathbb{R}^N$. Let $r$ be an arbitrary number with $2 \leq r < +\infty$, if $N < 4$ and $2 \leq r < \frac{2N}{N-4}$, if $N > 4$. Then for any $w \in W^{2,2}_0(\Omega)$ there exists a constant $S = S(r, \Omega)$ such that

$$||w||_r \leq S||\Delta w||_2. \quad (2.1)$$

To derive a lower bound of the blow up time $t^*$ of the solution of (1.1)-(1.3), we construct a first order differential inequality for $\Phi$ defined on (1.5) and prove the Theorem 1.1.

**Proof of Theorem 1.2.** Testing the first equation in (1.1) with $\Delta \Delta u$ we have

$$\int_{\Omega} u_i \Delta^2 u + \delta_1 \int_{\Omega} (\Delta^2 u)^2 - h_1 \int_{\Omega} \Delta u \Delta^2 u = k_1 \int_{\Omega} f_1 \Delta^2 u. \quad (2.2)$$

By using two times the $\epsilon-$Young inequality we get

$$\int_{\Omega} f_1 \Delta^2 u \leq \frac{\epsilon_1}{2} \int_{\Omega} f_1^2 + \frac{1}{2\epsilon_1} \int_{\Omega} (\Delta^2 u)^2, \quad (2.3)$$
\[
\int_{\Omega} \Delta u \, \Delta^2 u \leq \frac{\epsilon_2}{2} \int_{\Omega} |\Delta u|^2 + \frac{1}{2\epsilon_2} \int_{\Omega} (\Delta^2 u)^2 \quad (2.4)
\]

with \(\epsilon_1 = \epsilon_1(t)\) and \(\epsilon_2 = \epsilon_2(t)\) two arbitrary positive functions.

Taking into account (1.7) we get

\[
\frac{\epsilon_1}{2} \int_{\Omega} f^2 + \frac{1}{2\epsilon_1} \int_{\Omega} (\Delta^2 u)^2 \, dx \leq \frac{\epsilon_1}{2} ||v^p||^2 + \frac{1}{2\epsilon_1} ||\Delta^2 u||^2, \quad (2.5)
\]

Inserting (2.3), (2.4) and (2.5) in (2.2), it follows

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\Delta u|^2 \\
\leq -\delta_1 ||\Delta^2 u||^2_2 + \frac{h_1 \epsilon_2}{2} ||\Delta u||^2_2 - \frac{k_1 \epsilon_1}{2} ||v^p||^2_2 + k_1 ||\Delta^2 u||^2_2. \quad (2.7)
\]

Now using (2.1) in Lemma 2.1 with \(r = 2p\), we obtain

\[
||v^p||^2_2 \leq S^{2p} \left(||\Delta v||^2_2\right)^p. \quad (2.8)
\]

At last we can write

\[
\Phi_1'(t) \leq \left(\frac{h_1}{\epsilon_2} + \frac{k_1}{\epsilon_1} - 2\delta_1\right) ||\Delta^2 u||^2 + k_1 \epsilon_1 S^{2p} \left(||\Delta v||^2_2\right)^p + h_1 \epsilon_2 ||\Delta u||^2_2 \quad (2.9)
\]
In the same way we derive

\[
\Phi'_2(t) \leq \left( \frac{h_2}{\epsilon_4} + \frac{k_2}{\epsilon_3} - 2\delta_2 \right) \| \Delta^2 v \|^2 + k_2\epsilon_3 S^{2q} \left( \| \Delta u \|^2 \right)^q + h_2\epsilon_4 \| \Delta v \|^2
\]  

(2.10)

with \( \epsilon_3 = \epsilon_1(t) \) and \( \epsilon_4 = \epsilon_2(t) \) two arbitrary positive functions. Since \( \Phi' = \Phi_1' + \Phi_2' \), using (2.9) and (2.10) and choosing the functions \( \epsilon_i, \ i = 1, \ldots, 4 \) such that the first terms in (2.9) and (2.10) vanish, we arrive at

\[
\Phi'(t) \leq k_1\epsilon_2 S^{2p} \left( \| \Delta v \|^2 \right)^p + k_2\epsilon_3 S^{2q} \left( \| \Delta u \|^2 \right)^q + h_1\epsilon_2 \| \Delta u \|^2 + h_2\epsilon_4 \| \Delta v \|^2
\]  

(2.11)

\[
= k_2\epsilon_3 S^{2q} \Phi^q + k_1\epsilon_2 S^{2p} \Phi^p + h_1\epsilon_2 \Phi_1 + h_2\epsilon_4 \Phi_2
\]

\[
\leq k_2\epsilon_3 S^{2q} \Phi^q + k_1\epsilon_2 S^{2p} \Phi^p + B\Phi,
\]

with \( B := \max\{B_1, B_2\} \) with \( B_1 = \sup_t \{h_1(t)\epsilon_2(t)\} \) and \( B_2 = \sup_t \{h_2(t)\epsilon_4(t)\} \).

Since \( \Phi(t) \) blows up in a finite time \( t^* \), then there exists a time \( t_1 > 0 \) such that \( \Phi(t) \geq \Phi(t_1) = \Phi_0 \) for all \( t \geq t_1 \) and we can write with \( p > q \)

\[
\Phi^q(t) \leq \Phi^p(t) \Phi_0^{q-p}, \quad \forall \ t \in (t_1, t^*). \]  

(2.12)

In view of (2.12), the inequality (2.11) becomes

\[
\Phi'(t) \leq 2A\Phi^p + B\Phi, \]  

(2.13)

with \( A := \max\{A_1, A_2\} \) with \( A_1 = \sup_t \{k_2(t)\epsilon_3(t)S^{2q}\Phi_0^{q-p}\} \) and
\[ A_2 = \sup_t \{ k_1(t)\varepsilon_2(t)S^{2p} \} \text{. Integrating (2.13) we arrive at:} \]

\[
e^{B(p-1)t} \Phi^{1-p}(t) - \Phi^{1-p}_0 \geq -2A(p-1) \int_0^t e^{B(p-1)\tau} d\tau \]

\[
= \frac{-2A}{B(p-1)} \left[ e^{B(p-1)t} - 1 \right].
\]

Finally, letting \( t \to t^* \), we obtain a lower bound \( T \) of the blow-up time with:

\[
T := \frac{1}{B(p-1)} \ln \left[ 1 + \frac{B}{2A} \Phi^{1-p}_0 \right] \leq t^*.
\]

In the particular case \( p = q \), we replace (2.11) with

\[
\Phi' \leq k_2\varepsilon_3S^{2p}\Phi^p_1 + k_1\varepsilon_2\Phi^p_2 + B\Phi
\]

and by using the basic inequality \( a^\gamma + b^\gamma \leq (a + b)^\gamma \), \( \gamma > 1 \), \( a, b > 0 \), we get

\[
\Phi' \leq \tilde{A}\Phi^p + B\Phi, \tag{2.14}
\]

with \( \tilde{A} = S^{2p} \max\{\tilde{A}_1, \tilde{A}_2\} \) with \( \tilde{A}_1 = \{ k_2(t)\varepsilon_3(t)\Phi^{q-p}_0 \} \) and \( \tilde{A}_2 = \sup_t \{ k_1(t)\varepsilon_2(t) \} \).

Integrating (2.14), we lead to the lower bound

\[
T := \frac{1}{B(p-1)} \ln \left[ 1 + \frac{B}{\tilde{A}} \Phi^{1-p}_0 \right] \leq t^*.
\]

We have obtained (1.8) and the Theorem 1.1 is proved. \( \square \)
Remark 1

The existence of a lower bound $T$ for the blow-up time to the functional $\Phi(t)$ has a consequence that the interval $[0, T]$ is a safe interval of existence of the solution, since, as observed in the introduction,

$$||u||^2 + ||v||^2 \leq \Lambda_1^{-1} (||\Delta u||^2 + ||\Delta v||^2).$$

Remark 2

We may obtain a further lower bound of the blow up time $t^*$ (easier to be computed) by estimating $\Phi(t)$ on the right of (2.14) in the following way: arguing as in (2.12) with $q = 1$ we get

$$\Phi(t) \leq \Phi^p(t) \Phi_0^{1-p}, \quad \forall \ t \in (t_1, t^*). \quad (2.15)$$

By replacing (2.15) in (2.13) we arrive at

$$\Phi'(t) \leq K \Phi^p,$$  

with $K := 2A + B\Phi_0^{1-p}$. Integrating (2.16) in the interval $(t_1, t)$ and letting $t \to t^*$, we have

$$\int_{\Phi_0}^{\infty} \frac{d\eta}{\eta^p} = \int_{\Phi(t_1)}^{\infty} \frac{d\eta}{\eta^p} \leq K \int_{t_1}^{t^*} d\tau \leq K \int_{0}^{t^*} d\tau = Kt^*, \quad 14$$
from which we obtain the following lower bound

\[
\tilde{T} = \frac{\Phi_0^{1-p}}{(p-1)K} \leq \tau^*.
\]

3 Upper bound of the blow up time

In this section the system (1.1)-(1.3) is simplified by the assumption \( h_i = 0 \). We recall that, as already observed in the introduction, the functional \( \Psi(t) \) defined in (1.9)-(1.10), is non negative. We derive sufficient conditions on the data which guarantee that the solution \((u,v)\) blows up in finite time.

**Proof of Theorem 1.1.** Testing the first equation with \( h_1 = 0 \) in (1.1) with \( \phi_1 \), we have

\[
\int_{\Omega} u_t \phi_1 dx + \delta_1 \int_{\Omega} (\Delta^2 u) \phi_1 dx = k_1 \int_{\Omega} f_1 \phi_1 dx.
\]  

(3.1)

For the first term on the left in (3.1), with \( \Psi_1 \) defined in (1.9),

\[
\int_{\Omega} u_t \phi_1 = \frac{d}{dt} \int_{\Omega} u \phi_1 = \Psi'_1(t)
\]

since \( \phi_1 \) doesn’t depend on t. Then

\[
\frac{d}{dt} \int_{\Omega} u \phi_1 dx = -\delta_1 \int_{\Omega} (\Delta^2 u) \phi_1 dx + k_1 \int_{\Omega} f_1 \phi_1 dx.
\]
Using the hypothesis \((1.15)\) we obtain

\[
\Psi_1' = \int_{\Omega} (k_1 f_1 - \delta_1 \Delta^2 u) \phi_1 dx \geq k_1 \int_{\Omega} v^p \phi_1 dx - \delta_1 \int_{\Omega} \Delta^2 u \phi_1 dx. \tag{3.2}
\]

In the first term of \((3.2)\) we use the Holder’s inequality and \((1.13)\) leading to

\[
\int_{\Omega} v^p \phi_1 dx \geq |\Omega|^{\frac{p-1}{2}} \Psi_2^p. \tag{3.3}
\]

Using the second Green’s identity and \((1.11)\), the second term of \((3.2)\) can be estimate as follows

\[
- \int_{\Omega} \phi_1 \Delta^2 u dx = - \int_{\Omega} u \Delta^2 \phi_1 dx = - \Lambda_1 \int_{\Omega} u \phi_1 dx = - \Lambda_1 \Psi_1. \tag{3.4}
\]

Substituting \((3.3)\) and \((3.4)\) in \((3.2)\) we arrive at

\[
\Psi_1'(t) \geq k_1 |\Omega|^{-\frac{p-1}{2}} \Psi_2^p(t) - \delta_1(t) \Lambda_1 \Psi_1(t) = c_1(t) \Psi_2^p - \delta_1(t) \Lambda_1 \Psi_1, \tag{3.5}
\]

with \(c_1(t) := k_1(t) |\Omega|^{-\frac{p-1}{2}}\).

Testing the second equation with \(h_2 = 0\) in \((1.1)\) with \(\phi_2\), a computation similar to the previous leads to

\[
\Psi_2'(t) \geq c_2(t) \Psi_1^q - \delta_2(t) \Lambda_1 \Psi_2, \tag{3.6}
\]

with \(c_2(t) := k_2(t) |\Omega|^{-\frac{q-1}{2}}\).
Adding (3.5) and (3.6), we have

$$
\Psi' \geq -\delta \Lambda_1 (\Psi_1 + \Psi_2) + c(\Psi_2^p + \Psi_1^q),
$$

where $c := \min\{C_1, C_2\}$, with $C_1 = \inf_t \{c_1(t)\}$ and $C_2 = \inf_t \{c_2(t)\}$ and

$$
\delta := \max\{D_1, D_2\}$ with $D_1 = \sup_t \{\delta_1(t)\}$ and $D_2 = \sup_t \{\delta_2(t)\}$.

Since $p > q > 1$ we make use of the inequality

$$
\Psi_2^q = (a \Psi_2^p)^{\frac{q}{p}} \left( a - \frac{q}{p} \right) \leq \frac{q}{p} (a \Psi_2^p) + \frac{p - q}{p} a^{-\frac{q}{p-q}},
$$

valid for arbitrary $a > 0$.

Choosing $a = \frac{p}{q}$ we arrive at

$$
\Psi_2^p \geq \Psi_2^q - Q,
$$

with $Q := \frac{p - q}{p} a^{-\frac{q}{p-q}}$.

Inserting (3.8) in (3.7) we obtain the first order differential inequality

$$
\Psi' \geq -\delta \Lambda_1 \Psi + c(\Psi_1^q + \Psi_2^q) - cQ \geq -\delta \Lambda_1 \Psi + 2^{1-q} c \Psi^q - cQ := H(\Psi),
$$

where in the second term of (3.9) we used the arithmetic inequality

$$
X^q + Y^q \geq 2^{1-q} (X + Y)^q, \quad q > 1,
$$

with $X \geq 0$, $Y \geq 0$. 

17
Since the initial data satisfy (1.16), then $\Psi(t)$ is increasing for small values of $t$. We have that $H(\Psi)$ is increasing in $\Psi$ from its negative minimum and it follows that $H(\Psi(t))$ is increasing for $t > 0$. Thus $\Psi'(t)$ remains positive, so that $\Psi(t)$ blows up at time $t^\ast$. Integrating (3.9) we obtain the upper bound $T_0$ in (1.17).

In the case $q = p$ we obtain an explicit estimate from above for $t^\ast$.

**Proof of Corollary 1.1.** We restart from the inequality (3.7) and put $q = p$:

$$
\Psi'(t) \geq -\delta \Lambda_1 \Psi + c(\Psi_1^p + \Psi_2^p), \quad (3.11)
$$

Since $p > 1$, in (3.11), we apply the inequality (3.10) with $q$ replaced by $p$. Then

$$
\Psi'(t) \geq -\delta \Lambda_1 \Psi + \bar{c} \Psi^p, \quad (3.12)
$$

with $\bar{c} = 2^{1-p}c$. Integrating (3.12) from 0 to $t$, we arrive at

$$
\Psi^{1-p}(t) \leq e^{(p-1)\delta \Lambda_1 t} \left( \Psi_0^{1-p} - \frac{\bar{c}}{\delta \Lambda_1} \right) + \frac{\bar{c}}{\delta \Lambda_1} =: \mathcal{H}(t). \quad (3.13)
$$

Since the initial data satisfy (1.18) then $\mathcal{H}(t)$ vanishes at some time $T$. As a consequence, $\Psi(t)$ must blow up at some time $t^\ast \leq T$ with $T$ the desired upper bound (1.19).
Acknowledgments

M. Marras and S. Vernier-Piro are members of the Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

Financial disclosure

M. Marras is partially supported by the research project: Evolutive and stationary Partial Differential Equations with a focus on biomathematics (Fondazione di Sardegna 2019), and by the grant PRIN n. PRIN-2017AYM8XW: Non-linear Differential Problems via Variational, Topological and Set-valued Methods.

References

[1] C. Escudero, F. Gazzola, I. Peral, Global existence versus blow-up results for a fourth order parabolic PDE involving the Hessian, J. Math. Pures Appl. 103 (2015) 924–957.

[2] C. Escudero, F. Gazzola, R. Hakl, I. Peral, P. J. Torres, Existence results for a fourth order partial differential equation arising in condensed matter physics, Mathematica Bohem. 140 (4) (2015) 385–393.

[3] V. Galaktionov, E. Mitidieri, S. Pohoazaev, Blow-up for Higher-Order Parabolic, Hyperbolic, Dispersion and Schrödinger Equations, Mono-
graphs and Research Notes in Mathematics, Chapman and Hall/CRC 2014.

[4] F. Gazzola, H.C. Grunau, G. Sweers, Polyharmonic boundary value problems. Lecture Notes in Mathematics 2010, Berlin, Germany: Springer.

[5] H.C. Grunau, G. Sweers, Sign change for the Green function and the first eigenfunction of equations of clamped-plate type, Arch. Ration. Mech. Anal., 150 (1999) 179-190.

[6] Y. Han, A class of fourth-order parabolic equation with arbitrary initial energy, Nonlinear Analysis: RWA, 43 (2018) 451-466.

[7] A. Henrot, Extremum problems for Eigenvalues of Elliptic Problems, in Frontiers in Mathematics, Birkäuser Verlag, Basel, 2006.

[8] B.B. King, O. Stein, M. Winkler, A fourth order parabolic equation modeling epitaxial thin film growth, J. Math. Anal. Appl., 286 (2003) 459–490.

[9] M. Marras and S. Vernier Piro, Bounds for blow-up time in nonlinear parabolic systems, Discrete Contin. Dyn. Syst., Suppl. (2011) 1025–1031.

[10] M. Marras and S. Vernier Piro, Blow-up phenomena in reaction-diffusion systems, Discrete Contin. Dyn. Syst. 32 (11) (2012) 4001–4014.
[11] M. Ortiz, E.A. Repetto, H. Si, *A continuum model of kinetic roughening and coarsening in thin films*, J. Mech. Phys. Solids 47 (1999) 697–730.

[12] L.E. Payne and G.A. Philippin, *Blow-up Phenomena for a Class of Parabolic Systems with Time Dependent Coefficients*, Appl. Math. 3 (2012) 325–330.

[13] G.A. Philippin, *Blow-up phenomena for a class of fourth order parabolic problems*, Proc. Amer. Math. Soc. 143 (2015) 2507–2513.

[14] G.A. Philippin and S. Vernier Piro, *Behaviour in time of solutions to a class of fourth order evolution equations* J. Math. Anal. Appl. 436 (2016) 718-728.

[15] G. Sweers, *When is the First Eigenfunction for the Clamped Plate Equation of Fixed Sign?* Electronic Journal of Differential Equations, 2001, 6(06), 285–296.

[16] M. Winkler, *Global solutions in higher dimensions to a fourth-order parabolic equation modeling epitaxial thin-film growth*, Z. Angew. Math. Phys. 62 (4) (2011) 575–608.

[17] R.Z. Xu, T.L. Chen, C.M. Liu and Y. H. Ding *Global well-posedness and global attractor of fourth order semilinear parabolic equation*, Math. Methods Appl. Sci. 38 (2015) 1515–1529.

Email address: mmarras@unica.it

Email address: svernier@unica.it