A Deformation Family for Closed $G_2$-Structures on ADE Fibrations

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Abstract

Inspired by a string duality, we construct a deformation family for closed $G_2$-structures on total spaces of fibrations by ADE singularities over a closed flat three-manifold $Q$. The deformations are parametrized by sections of a flat bundle on $Q$ that can be interpreted as spectral covers associated to certain “flat Higgs bundles”. The spectral cover picture is conjecturally a unifying framework for several different approaches to coassociative fibrations appearing in the literature. Our construction is a $G_2$ analogue of a well-known family of ADE-fibered Calabi-Yau threefolds whose deformations are parametrized by spectral covers of usual Higgs bundles on the base curve $\Sigma$.

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1 Introduction

In [Sze04], Szendrői defined and studied a deformation family for a Calabi-Yau threefold $X$ containing a curve $\Sigma$ of ADE singularities. Later, noticing that the deformations are parametrized by spectral curves of Higgs bundles on $\Sigma$, Diaconescu, Donagi and Pantev [DDP06] showed that, indeed, complex structure deformations are controlled by the Hitchin system of $\Sigma$: the integrable system associated to the Jacobian fibration of the family is isomorphic to the Hitchin system of $\Sigma$. This correspondence first arose in the context of B-model geometric transitions on $X$, where the Hitchin system describes the large $N$ limit of a system of holomorphic branes wrapping the exceptional cycles in a resolution of $X$ [DDD+06].

A natural question is: can one build a similar picture when $X$ is replaced by a $G_2$-orbifold $M$ containing a three-manifold of ADE singularities $Q$? This is not idle analogy, as from a physical perspective one expects the answer to be positive. This is because of a well-known duality between M-theory and Type IIA Superstring theory: in the low energy limit, M-theory compactified on $M$ is dual to a gauge theory on $Q$ described by the following equations [PW11]:

\[
\begin{align*}
F_A &= [\theta \wedge \theta] \\
D_A \theta &= 0 \\
D_A^\dagger \theta &= 0
\end{align*}
\]

where $F_A$ is the curvature of a unitary\(^1\) connection $A$ on a hermitian vector bundle $E \to Q$, $\theta \in C^\infty(Q, Ad(E) \otimes T^*Q)$ is a real analogue of a Higgs field, and $D_A^\dagger = \star D_A \star$ is the hermitian adjoint of $D_A$. “Duality” in this context means that we expect the moduli spaces of the two theories to be isomorphic. The moduli space of M-theory parametrizes “complexified” $G_2$-structures on (a desingularization of) $M$. Hence we expect that deformations of the $G_2$-structure can be somehow captured in the moduli space of solutions to \ref{eq:1.1}.

\(^1\)We are assuming here the singularity is of type $A_n$. The story for other types is more complicated.
Equations 1.1 have been known since the seminal works of Donaldson [Don87] and Corlette [Cor88]. They were re-derived in [PW11] in the context of the aforementioned duality via a dimensional reduction of Hermitian-Yang-Mills instantons on $T^*Q$, which is the gauge-theoretic description of the dual Type IIA theory. See [Ach98] for foundational work on this, and [BCHS18], [BCH+18] for recent developments.

Donaldson and Corlette show that the space of solutions of 1.1 - modulo unitary gauge transformations - is the same as the space of completely reducible (i.e. semistable) complex flat connections on $E$ modulo complex gauge transformations:

$$
\begin{align*}
\left\{ \begin{array}{l}
F_A - [\theta \wedge \theta] = 0 \\
D_A \theta = 0 \\
D^\dagger_A \theta = 0
\end{array} \right. \right/ \mathcal{G}^U \cong \{ F_{A+i\theta} = 0 \} / \mathcal{G}^C 
\end{align*}
$$

This can be understood as an infinite-dimensional version of the Kempf-Ness theorem equating symplectic quotients with GIT quotients [KN79]. Here we interpret the third equation on the LHS as a moment map condition. Moreover, this equation constrains the hermitian metric $h$ to be harmonic. For irreducible complex connections, the harmonic metric is unique and the corresponding Higgs field is symmetric$^2$, i.e. $[\theta \wedge \theta] = 0$. The reader will recognize that if $Q$ were a projective variety, these conditions would define a Higgs bundle in the sense of Simpson [Sim92], and the non-abelian Hodge correspondence would improve 1.2 to a real-analytic isomorphism.

In this paper, we will propose a deformation family of closed $G_2$-structures for a $G_2$-orbifold containing a compact flat three-manifold $Q$ of ADE singularities, and we will show that it is parametrized by flat spectral covers associated to symmetric solutions of 1.1.

There are several reasons to work with flat manifolds: first, it produces a natural notion of flatness for spectral covers - flat sections are more rigid than general $\mathcal{C}^\infty$-sections, making 1.1 similar in spirit to its holomorphic counterpart; second, the duality we aim to explore makes sense for certain choices of flat $Q$; and third, compact flat three-manifolds have non-trivial fundamental groups, so in view of relation 1.2 they produce non-trivial solutions. The special nature of the fundamental group makes the relevant moduli space - the character variety, or in the general case, the character stack - an interesting algebraic space that often can be explicitly computed.

Before we state our main result, let us describe a somewhat simple example. Let $G_6$ be the (unique up to affine isomorphism) compact orientable flat Riemannian 3-manifold with holonomy $\mathbb{K} := \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle \alpha, \beta \rangle$. One can construct a flat $\mathbb{K}$-bundle of $A_1$-singularities $M_0 \rightarrow G_6$ and a $G_2$-structure $\varphi_0$ on $M_0$ making the fibers coassociative (see example 4.5 below). In fact $\varphi_0$ is integrable and according to

$^2$We sometimes also refer to such solutions as flat, due to the implied equation $F_A = 0$. 

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Joyce [Joy00] there are three topologically distinct smoothings $Y_i$ of $M_0$ that retain this condition\textsuperscript{3}:

1. $Y_1$, obtained by blowing up the fibers. The exceptional curve is a $\mathbb{CP}^1$ and the $K$-action is such that $\alpha$ reverses orientation but $\beta$ does not.

2. $Y_2$ is a deformation of $\mathbb{C}^2/\mathbb{Z}_2$ replacing the singularity by a totally real $S^2$. The $K$-action is such that both $\alpha$ and $\beta$ reverse orientation.

3. $Y_3$ is a deformation of $\mathbb{C}^2/\mathbb{Z}_2$ replacing the singularity by a totally imaginary $S^2$, and is diffeomorphic to $Y_2$. Here only $\beta$ reverses orientation.

Each of the three smoothings defines a one-parameter family of $G_2$-structures parametrized by the volume of the exceptional sphere. Thus, the local moduli space of $G_2$-deformations $\mathcal{M}_{G_2}$ is given by three copies of $\mathbb{R}_+$ touching at a point. The M-theory moduli space $\mathcal{M}_{G_2}^\mathbb{C}$ is supposed to be Kähler and admit a Lagrangian torus fibration $\mathcal{M}_{G_2}^\mathbb{C} \to \mathcal{M}_{G_2}$ (in the case of compact smooth $M$ this is a theorem of Karigiannis and Leung [KL07]). Thus, $\mathcal{M}_{G_2}^\mathbb{C}$ consists of three copies of $\mathbb{C}$ touching at a point.

On the other hand, if the duality is to be believed, one would expect that the $SL(2,\mathbb{C})$-character variety of $\mathcal{G}_6$ also consists of three copies of $\mathbb{C}$ touching at a point. We will show in section 6 that this is in fact the case.

Our main result can be stated as follows:

**Theorem 1.1.** Let $Q$ be a compact, oriented flat Riemannian three-manifold and $M_0 \to Q$ a fibration by ADE singularities $\mathbb{C}^2/\Gamma$. Suppose one is given a closed $G_2$-structure $\varphi_0$ on $M_0$ making the fibration coassociative. Then there is a deformation family of seven-manifolds with closed $G_2$-structures:

$$f : \mathcal{F} \to \mathcal{B} \quad (1.3)$$

with central fiber isomorphic to $(M_0, \varphi_0)$.

Moreover, the base is characterized by $\mathcal{B} = \Gamma_{flat}(Q, \mathcal{E}_W)$, where $\mathcal{E}_W \to Q$ is a flat bundle whose flat sections describe spectral covers for solutions of equations 1.1.

Here, $W$ is the Weyl group of the semisimple Lie algebra McKay dual to $\Gamma$. The notation $\mathcal{E}_W$ is meant to be consistent with Lemma 5.10 below: our bundle $\mathcal{E}_W$ is a $W$-quotient of the flat bundle $\mathcal{E}$ constructed in that lemma. Theorem 1.1 will be a corollary of Theorem 5.8 below.

The existence of the deformation family makes it evident that M-theory/Type IIA duality indeed establishes an algebraic description of $\mathcal{M}_{G_2}$. Intuitively, if one starts with a $G_2$-structure $\varphi$ on $M$ and “complexification data” $C \in \mathcal{H}^3(M, \mathbb{R})$ (a “C-field”, in physics terminology), then solutions of 1.1 can be divided into bundle

\textsuperscript{3}I.e., whose fibers admits a $K$-action asymptotic to the original action on $M_0$.  

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data $A(\varphi, C)$ and Higgs data $\theta(\varphi, C)$ that depend non-trivially on both variables. For special solutions, there is a spectral correspondence that allows us to replace $(A, \theta)$ by its “spectral data”, consisting of a finite cover $\tilde{Q} \to Q$ parametrizing eigenvalues of $\theta$ and a flat line bundle $L \to \tilde{Q}$. Our theorem says that the new variable $\tilde{Q}$ depends solely on $\varphi$, so the spectral correspondence effectively “untwists” the prediction of M-theory/IIA duality.

The main idea to prove the theorem is to recast the data of a closed $G_2$-structure in a language more appropriate for deformation theory. We use Donaldson’s description [Don16] of $G_2$-structures on coassociative fibrations to reformulate the problem in terms of deformations of a triple $(\eta_0, \mu_0, H_0)$ on $M_0 \to Q$ consisting of a vertical hyperkähler triple $\eta_0$, a horizontal lift of a $Q$-volume form $\mu_0$ and a connection $H_0$, which are required to satisfy some compatibility conditions, and we construct a flat bundle whose flat sections solve the problem.

The paper is organized as follows: in sections 2 and 3 we briefly review the classification of compact oriented flat Riemannian three-manifolds and provide an algebraic description of their character varieties. In section 4 we establish a necessary condition for a three-manifold $Q$ to be the base of an ADE-fibration admitting a $G_2$-structure. This is a compatibility condition between the ADE group $\Gamma$ and $\pi_1(Q)$. The core of the paper is section 5, where we provide our main construction: a deformation family for closed $G_2$-structures on coassociative ADE-fibrations. In section 6 we present computations for the example $M_0$ above; in particular, we prove that the $SL(2, \mathbb{C})$-character variety of $G_6$ is the complexification of the deformation space of $G_2$-structures on $M_0$. In section 7 we discuss spectral covers for special solutions of equations 1.1. Section 8 is a brief, non-rigorous discussion on how the spectral point of view allows us to unify recent constructions of coassociative fibrations into a single framework, and we also argue that formation of isolated singularities come from nilpotent solutions to 1.1.

In a forthcoming companion paper [Bar], we will provide a second algebraic characterization of $\mathcal{M}_{G_2}$ for our main example $M_0$, given by a Hilbert scheme of points on a singular threefold. While it is unclear if this second construction will be useful in $G_2$-geometry, it suggests a new interpretation of SYZ Mirror Symmetry in terms of moduli spaces of flat Higgs bundles.

After this work was completed we were informed about the work of Joyce and Karigiannis [JK17] which has some overlap with our work. Essentially, our construction can be seen as phrasing (conjectural) generalizations of the construction of [JK17] in the language of [Don16]. For a more detailed explanation, see section 8.

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2 A Flash Review of Flat Riemannian Geometry

Let $\text{Iso}(\mathbb{R}^n)$ denote the group of isometries of $\mathbb{R}^n$ endowed with its standard Euclidean structure. Recall that a subgroup $\pi \leq \text{Iso}(\mathbb{R}^n)$ is called crystallographic if it is discrete and cocompact (i.e., $Q^n := \mathbb{R}^n/\pi$ is compact). It is called torsion-free if it acts freely. A torsion-free crystallographic subgroup is called Bieberbach. Clearly $\pi$ is crystallographic if and only if $Q^n$ is a compact flat orbifold, and $\pi$ is Bieberbach if and only if $Q^n$ is a compact flat manifold. We refer to $Q^n$ as a Bieberbach space. Any crystallographic group $\pi$ fits into a short exact sequence

$$0 \to \Lambda \to \pi \to H \to 1$$

(2.1)

where $H$ is a finite group called the monodromy of $\pi$ and $\Lambda$ is a free abelian $H$-module. Isomorphism classes of crystallographic groups are classified by the group cohomology $H^2(H, \Lambda)$.

Let $\mathbb{T}^n$ denote the flat $n$-torus. Bieberbach’s theorem states that:

1. There is a finite normal covering map $\mathbb{T}^n \to Q^n$ which is a local isometry.

2. Two Bieberbach spaces of the same dimension and with isomorphic fundamental groups are affinely isomorphic.

3. There are finitely many affine classes of Bieberbach spaces of dimension $n$.

We note that part 3 essentially follows from the fact that the number of exact sequences 2.1 is bounded by the order of the finite group $H^2(H, \Lambda)$.

Clearly, $\mathbb{R}^n$ is the universal cover of $Q^n$, and $\pi_1(Q^n) = \pi$. The first part of Bieberbach’s theorem implies that the $H$-action on $\Lambda \cong \pi_1(\mathbb{T}^n)$ is induced from a free $H$-action on $\mathbb{T}^n$ such that $Q^n \cong \mathbb{T}^n/H$. It is clear that $\mathbb{T}^n$ is also a Bieberbach manifold, albeit with trivial monodromy. For this reason, we call $\mathbb{T}^n$ the monodromy cover of $Q^n$. The existence of the monodromy cover strongly constrains the possible holonomies of Bieberbach spaces. This is in stark contrast with the theory for non-compact flat Riemannian manifolds: it is a theorem of Auslander and Kuranishi that every finite group is the holonomy group of some flat manifold.

Following standard terminology [CR03] [Scz12], we call a three-dimensional Bieberbach manifold $Q$ a platycosm. There are 10 affine equivalence classes of
platycosms, 6 of which are orientable. To distinguish them it suffices to consider their monodromy groups $H_Q$, and the classification goes as follows:

- $G_1$ is the flat three-torus $T^3$, so the monodromy is trivial: $H_{G_1} = \{1\}$
- $G_2$ with $H_{G_2} \cong \mathbb{Z}_2$
- $G_3$ with $H_{G_3} \cong \mathbb{Z}_3$
- $G_4$ with $H_{G_4} \cong \mathbb{Z}_4$
- $G_5$ with $H_{G_5} \cong \mathbb{Z}_6$
- $G_6$ with $H_{G_6} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

The space $G_6$ will be particularly important for us. It is known in the literature as the Hantzsche-Wendt manifold or didicosm. Explicit descriptions for $H_{G_6}$ and $\Lambda_{G_6}$ are:

$$H_{G_6} = \left\langle A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\rangle \subset SO(3) \quad (2.2)$$

$$\Lambda_{G_6} = \left\langle \left( A, \begin{bmatrix} 1/2 \\ 0 \\ 0 \end{bmatrix} \right), \left( B, \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \end{bmatrix} \right) \right\rangle \subset SO(3) \ltimes \mathbb{R}^3 = \text{Iso}^+(\mathbb{R}^3) \quad (2.3)$$

3 Character Varieties

Fix a linear algebraic group $G$ and a finitely generated group $\pi = \langle g_1, \ldots, g_k \rangle$. Let $\text{Hom}(\pi, G)$ be the representation variety. Since $G$ is a subgroup of $GL(n, \mathbb{C})$, its relations together with those of $\pi$ determine $\text{Hom}(\pi, G)$ as a subvariety of $G^k$. If $I \subseteq \mathbb{C}[z_1, \ldots, z_k]$ is the ideal generated by those relations, one can consider the representation scheme:

$$\mathcal{X}(\pi, G) = \text{Spec} \mathbb{C}[z_1, \ldots, z_k]/I$$

This scheme is independent of the presentation of $\pi$ up to canonical isomorphism. The representation variety is the reduced scheme of $\mathcal{X}(\pi, G)$.

Let $Z(G)$ be the center of $G$. The adjoint group $G = G/Z(G)$ acts by conjugation on $\text{Hom}(\pi, G)$. The GIT quotient is the character variety:

$$\text{Char}(\pi, G) = \text{Hom}(\pi, G)//G'$$
This is obtained by restricting to representations with closed $G$-orbit. These are exactly the completely reducible representations [Sik12]. The character scheme is:

$$X(\pi, G) = \text{Spec}(\mathbb{C}[z_1, \ldots, z_k]/I)^G$$

and its reduced scheme is $\text{Char}(\pi, G)$.

### 3.1 Character varieties of Bieberbach groups

By equivalence 1.2, the moduli space of solutions to equations 1.1 is given by flat $G$-bundles on $Q$, where $G \leq GL(n, \mathbb{C})$. In other words, the moduli space is the character variety $\text{Char}(Q, G)$. In order to describe this space for a general platycosm, we first consider the case $Q = T^3$.

To fix ideas, consider the simpler problem of computing the character variety of lower dimensional tori. For $T^1 = S^1$ the problem is trivial: the generator of $\mathbb{Z} = \pi_1(S^1)$ can be mapped anywhere in $G$. Hence $\text{Char}(S^1, G) = G/G' \cong \mathcal{T}/W$, where $\mathcal{T}$ is the maximal torus of $G$. For a two-torus $T^2$, $\text{Char}(T^2, G)$ is given by two commuting elements in $G$ up to conjugation. Let $g \in G$ and $h \in C_G(g)$, the centralizer of $g$. It is known that, for simply-connected $G$, the centralizer $C_G(g)$ is connected (Bott’s theorem), so we can first conjugate $g$ to $\mathcal{T}$ and then conjugate $h$ to the torus of $C_G(g)$, which by connectedness is just $\mathcal{T}$. The net result is that $g$ and $h$ can be simultaneously conjugated to lie on the maximal torus $\mathcal{T}$. The maximal tori are conjugated by elements of the Weyl group $W$. Hence the character variety is:

$$\text{Char}(T^2, G) = \mathcal{T} \times \mathcal{T}/W$$ \tag{3.1}$$

For a three-torus, $\text{Char}(T^3, G)$ is now given by three commuting elements modulo conjugation. So now we need to determine all possible configurations of $g, h, k \in G$, with $g \in \mathcal{T}$ and $h, k \in C_G(g)$, i.e., the moduli space of commuting triples. This problem was solved by Borel, Friedman and Morgan [BFM02] and Kac and Smilga [KS99], who showed that the classification of commuting triples $(g, h, k)$ is essentially determined by the fundamental groups of the centralizers $C_G(g), C_G(h), C_G(k)$. Commuting triples $(g, h, k)$ whose semi-simple part of the centralizers is simply-connected can always be conjugated to the maximal torus, giving one of the components of the moduli space:

$$\mathcal{T} \times \mathcal{T} \times \mathcal{T}/W$$ \tag{3.2}$$

However, there are also non-trivial commuting triples. This happens when $G$ has elements whose semisimple part of the centralizer has torsion. These extra commuting triples produce new connected components in the character variety. Essentially, torsion in $\pi_1(C_G(g))$ occurs when the root system of $\mathfrak{h}$ admits non-trivial coroot integers. Each divisor of a coroot integer is called a level $\ell$, and each
ell determines a subtorus T_ell of T given by the intersection of the kernels of the roots whose coroot integers are not divisible by ell. The torus T_ell has an associated Weyl group W_{T_ell} := N_G(T_ell)/C_G(T_ell). Here N_G denotes the normalizer.

Each ell determines \phi(ell) connected components for the character variety, where \phi is Euler’s totient function; each connected component is given by:

\[ \mathcal{T}_\ell \times \mathcal{T}_\ell \times \mathcal{T}_\ell / W_{T_\ell} \] (3.3)

In particular, for G = SL(n, C), the only allowed level is ell = 1 and there are no non-trivial commuting triples. Thus:

\[ \text{Char}(\mathbb{T}^3, SL(n, \mathbb{C})) = \left( (\mathbb{C}^*)^{n-1} \right)^3 / \Sigma_n \] (3.4)

and similarly:

\[ \text{Char}(\mathbb{T}^3, SU(n)) = \left( U(1)^{n-1} \right)^3 / \Sigma_n \] (3.5)

where \Sigma_n is the symmetric group.

We now move on to a general platycosm Q. Let again \pi = \pi_1(Q). The exact sequence

\[ 1 \to \Lambda \to \pi \xrightarrow{q} H \to 1 \]

induces another exact sequence:

\[ 1 \to \text{Hom}(H, G) \to \text{Hom}(\pi, G) \xrightarrow{r} \text{Hom}(\Lambda, G) \] (3.6)

which in turn descends to maps between character varieties:

\[ \text{Char}(H, G) \to \text{Char}(\pi, G) \xrightarrow{r} \text{Char}(\Lambda, G) \] (3.7)

Let H act on Hom(\Lambda, G) by

\[ h(\rho) = \rho \circ C_{\tilde{h}} \quad \forall h \in H \] (3.8)

where \tilde{h} \in \pi is such that q(\tilde{h}) = h and C_{\tilde{h}} is conjugation by \tilde{h}. The action is well-defined because \Lambda is abelian. Moreover, the action descends to an action of H on Char(\Lambda, G) in the obvious way.\footnote{Note that since it is an action by an outer conjugation of \Lambda, it descends non-trivially to the quotient.} Let Fix(H) denote the subset of Char(\Lambda, G) consisting of elements fixed by H.

The next lemma states that \( r(\text{Char}(\pi, G)) = \text{Fix}(H) \). Hence, the character variety of Q is determined, up to finite fibers, by the action of H on the character variety of the monodromy cover \( \mathbb{T}^3 \).
Lemma 3.1. Suppose \( \rho \in \text{Hom}(\Lambda, G) \) is such that \( \rho = \pi(\tilde{\rho}) = \tilde{\rho}|_{\Lambda} \) for some \( \tilde{\rho} \) in \( \text{Hom}(\pi, G) \). Then \([\rho] \in \text{Fix}(H)\).

Conversely, assume \( C_{G}(\rho(\Lambda)) = 0 \) and \([\rho] \in \text{Fix}(H)\). Then \( \exists [\tilde{\rho}] \in \text{Char}(\pi, G) \) such that \( r([\tilde{\rho}]) = [\rho] \).

Proof. Let \( h \in H \). Then:

\[
\begin{align*}
    h(\rho) &= h(\tilde{\rho}|_{\Lambda}) \\
            &= \tilde{\rho}(\tilde{h}) \circ \tilde{\rho}|_{\Lambda} \circ \tilde{\rho}(\tilde{h})^{-1} \\
            &= C_{\tilde{\rho}(\tilde{h})}(\tilde{\rho}|_{\Lambda}) \\
            &= C_{\tilde{\rho}(\tilde{h})}(\rho)
\end{align*}
\]

hence \( h[\rho] = [\rho] \), i.e. \( [\rho] \in \text{Fix}(H) \).

Conversely, \( h[\rho] = [\rho] \implies \rho \circ C_{\tilde{h}} = S_{\tilde{h}} \rho S^{-1}_{\tilde{h}} \) for some \( S_{\tilde{h}} \in G \). It is easy to see that if \( a \in \text{Ker}(q) \), then \( S^{-1}_{\tilde{a}} \rho(a) \in C_{G}(\rho(\Lambda)) \). Hence \( S_{a} = \rho(a) \). Define \( \tilde{\rho} : \pi \to G \) by \( \tilde{\rho}(x) = S_{x} \) for all \( x \in \pi \). Then clearly \( \tilde{\rho}|_{\text{Ker}(q)} = \rho \) and if \( x, y \in \pi \), the hypothesis on the centralizer implies that \( S_{xy} = S_{x} S_{y} \), so \( \tilde{\rho}(xy) = \tilde{\rho}(x) \tilde{\rho}(y) \). So \( \tilde{\rho} \in \text{Hom}(\pi, G) \) with \( r([\tilde{\rho}]) = [\rho] \).

\[ \square \]

4 ADE \( G_2 \)-platyfolds

We start by fixing the following data:

1. \( Q \) is an oriented platycosm, \( \delta \) its flat Levi-Civita connection and \( \pi := \pi_1(Q) \) the associated Bieberbach group

2. \( V \to Q \) a rank one quaternionic vector bundle (i.e., the structure group is \( Sp(1) \leq SL(2, \mathbb{C}) \))

3. \( \Gamma \) a finite subgroup of \( Sp(1) \), and hence a fiberwise action of \( \Gamma \) on \( V \)

4. A flat quaternionic connection \( \nabla \) on \( V \to Q \) compatible with the \( \Gamma \)-action in an appropriate sense (see remark below)

5. A flat volume form \( \mu \in \Omega^3(Q, \mathbb{R}) \)

Remark 4.1. 1. A flat connection \( \nabla \) compatible with \( \Gamma \) is given by an action of \( \pi \times \Gamma \) on \( \tilde{Q} \times \mathbb{H} \), where \( \tilde{Q} \) is the universal cover of \( Q \) and \( \Gamma \) acts trivially on \( \tilde{Q} \). Equivalently, we have an action of \( \pi \) on \( \mathbb{H} \) commuting with the \( \Gamma \)-action. This is the same as a representation of \( \pi \) on the centralizer \( C_{Sp(1)}(\Gamma) \leq Sp(1) \), i.e., the conjugacy class of an element of \( \text{Hom}(\pi, C_{Sp(1)}(\Gamma)) \). The trivial homomorphism gives rise to the trivial flat connection (i.e., with no monodromy).
2. This data fixes a “flat fiberwise quaternionic structure”, i.e., a tri-section \((I, J, K)\) of \(\text{Aut}_H(V) \to Q\) preserved by \(\nabla\).

In the language of Goldman’s geometric structures [Gol88], \(\delta\) defines a torsion-free \((\mathbb{R}^3, \text{Iso}^+(\mathbb{R}^3))\)-structure on \(Q\) with graph \(TQ\), and \((V, \nabla)\) is the graph of a \((\mathbb{R}^4, \text{Sp}(1))\)-structure on \(Q\). This last structure is then required to be compatible with the group \(\Gamma\). We will require these two geometric structures to interact in a specific way when we discuss \(G_2\)-deformations.

**Definition 4.2.** We call \((\Gamma, \nabla)\) ADE data for \(V\).

The first thing we need to determine is, for a fixed \(\Gamma\), when does \(V\) admit non-trivial ADE data, i.e., when \(\text{Hom}(\pi, C_{\text{Sp}(1)}(\Gamma)) \neq 0\) modulo conjugation. This is a compatibility condition between \(\pi_1(Q)\) and the \(\Gamma\)-compatible \((\mathbb{R}^4, \text{Sp}(1))\)-structure.

**Proposition 4.3.** Nontrivial \(A_1\) data exists for all platycosms. For \(T\), non-trivial ADE data exists for all ADE groups.

**Proof.** The centralizer depends on the ADE type of \(\Gamma\). Here are the possibilities:

- **\(\Gamma\) of type \(A_n\)**: there are two subcases. If \(n = 1\), then \(\Gamma \cong \mathbb{Z}_2\) and
  
  \[C_{\text{Sp}(1)}(\mathbb{Z}_2) = \text{Sp}(1)\]  

  If \(n \geq 2\), then \(\Gamma \cong \mathbb{Z}_n\) and \(\Gamma\) lies on a maximal torus \(T\) of \(\text{Sp}(1) \cong \text{SU}(2)\).
  
  The centralizer is just the torus itself:
  
  \[C_{\text{Sp}(1)}(\mathbb{Z}_n) = T \cong U(1)\]  

- **\(\Gamma\) of type \(D_n\)** for \(n > 2\), \(E_6\), \(E_7\) or \(E_8\): Then:
  
  \[C_{\text{Sp}(1)}(\Gamma) = \mathbb{Z}(\text{Sp}(1)) \cong \mathbb{Z}_2\]  

The statement for \(T^3\) is trivial as in this case all flat bundles are trivial. Another way to see this, which is what must be done to determine the other cases, is to note that \(\text{Hom}(\mathbb{Z}^3, \mathbb{Z}_2) \cong \mathbb{Z}_2^3\), while \(\text{Hom}(\mathbb{Z}^3, U(1)) \cong U(1)^3\). The first remains unchanged when we quotient by \(\text{Sp}(1)\) conjugations, since \(\mathbb{Z}_2\) is central; the second becomes \(U(1)^3/W_{\text{Sp}(1)} = U(1)^3/\mathbb{Z}_2\). One would expect the \(A_1\) case to have even more solutions, however it follows from the discussion in the previous section that \(\text{Char}(\mathbb{Z}^3, \text{Sp}(1)) \cong U(1)^3/\mathbb{Z}_2\), as one can always conjugate three commuting elements to a maximal torus of \(\text{Sp}(1) \cong \text{SU}(2)\). In any case, since these spaces are all non-zero, it follows that ADE data exists for \(T = \mathcal{G}_1\) irrespectively of \(\Gamma\).

We discuss now the \(A_1\) case for a platycosm with fundamental group \(\pi\). We are interested in the image of \(r\) in:
\text{Char}(H_\pi, Sp(1)) \to \text{Char}(\pi, Sp(1)) \xrightarrow{\tau} \text{Char}(\mathbb{Z}^3, Sp(1)) \quad (4.4)

By Lemma 3.1 \text{Im}(r) is given by the fixed set of \text{H}_\pi on \text{Char}(\mathbb{Z}^3, Sp(1)).

For platycosms with cyclic holonomy the monodromy action fixes a direction in \mathbb{R}^3, which implies the descendant action on the character variety has non-trivial fixed points. This implies that nontrivial \text{A}_1 data can be chosen in those cases.

In the case when \text{Q} = \mathcal{G}_6, simple inspection determines that the action of \text{H}_{\mathcal{G}_6} on \mathbb{R}^3 does not fix any direction, so the previous argument does not apply. However, we will compute \text{Im}(r) explicitly in section 6, and an example of \text{A}_1 data for \mathcal{G}_6 will be discussed in Example 4.5.

More generally, the description of the centralizers has the following consequences for the structure of the bundle \textbf{V}:

- If \Gamma is of type \text{A}_1, any flat connection on \textbf{V} is compatible with \Gamma.
  If \Gamma is of type \text{A}_n for \( n \geq 2 \), then the structure group reduces to \text{U}(1) \leq \text{Sp}(1) and \textbf{V} \cong \textbf{L} \oplus \text{L}^{-1}, where \textbf{L} is a flat complex line bundle.

- If \Gamma is of type \text{D}_n for \( n \geq 3 \) or of types \text{E}_6, \text{E}_7 or \text{E}_8, then \textbf{V} \cong \textbf{L} \oplus \textbf{L}, where \textbf{L} is a flat complex line bundle such that \text{L}^{\otimes 2} is the trivial complex line bundle.

The quest for non-trivial ADE data then consists in finding flat connections respecting these decompositions.

Given data \((\textbf{Q}, \textbf{V}, \Gamma, \textbf{L}, \nabla, I, J, K, \mu)\) as above, the quaternionic structure determines a triple \(\hat{\omega}_0 := (\omega_I, \omega_J, \omega_K)\) of fiberwise hyperkähler structures. Our next goal is to understand under which circumstances the data \((\textbf{Q}, \textbf{V}, \Gamma, \nabla, \hat{\omega}_0, \mu)\) induces a closed \text{G}_2-structure on \textbf{V} such that \textbf{V} \to \textbf{Q} is a coassociative fibration. We start with some examples.

\textit{Example 4.4.} \textbf{V} = \mathbb{C}^2 \times \textbf{T} has a standard closed \text{G}_2-structure:

\[ \varphi = \sum_{i=1}^{3} dx_i \wedge \omega_i + dx_{123} \quad (4.5) \]

for a choice of flat coordinates \(\{x_i\}\) on \textbf{T} and hyperkähler structure\(^5\) \(\omega\) on \mathbb{C}^2. Here and in what follows, we use the notation \(dx_{123} := dx_1 \wedge dx_2 \wedge dx_3\).

Note that because there is no monodromy, the local section \(dx_i\) glues to a global flat section, so the formula makes sense globally. We think of \textbf{V} as the total space of the trivial flat vector bundle \(\textbf{V} \to \textbf{T}\). It is easy to check that \(\varphi|_{\mathbb{C}^2} = 0\) and \(\ast \varphi|_{\textbf{T}} = 0\), so the fibers \mathbb{C}^2 are coassociative and the zero-section \textbf{T} is associative.

\(\text{More generally, } \omega \text{ can be a hypersymplectic structure - see definition 4.10 below.}\)
In fact, this $G_2$-structure is also torsion-free. Its associated metric is just the flat metric, which of course has holonomy $\{1\} \subset G_2$.

Up to a change of basis, $\omega$ is a $SU(2)$-triple, and since $\Gamma \leq SU(2)$, $\omega$ can be taken to be $\Gamma$-invariant. Thus there is a well-defined $G_2$-structure on the quotient $M = \mathbb{C}^2/\Gamma \times T$ with the same properties.

**Example 4.5.** This example first appeared in [Ach98]. Consider $V_0 = \mathbb{C}^2 \times K G_6$ the total space of a *non-trivial flat bundle* over $G_6$, defined by the following action of $K \cong \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle \alpha, \beta \rangle$:

$$\alpha(z_1, z_2) = (z_1, -z_2)$$

$$\beta(z_1, z_2) = (\overline{z}_1, \overline{z}_2)$$

Fixing the flat hyperkähler structure on $\mathbb{C}^2$:

$$\omega_1 = Im(dz_1 \wedge dz_2)$$

$$\omega_2 = Re(dz_1 \wedge dz_2)$$

$$\omega_3 = dz_1 \wedge \overline{dz}_1 + dz_2 \wedge \overline{dz}_2$$

one sees that the induced action of $K$ on $\Lambda^2_+$ is given by the following hyperkähler rotations:

$$\alpha(\omega_1, \omega_2, \omega_3) = (-\omega_1, -\omega_2, \omega_3)$$

$$\beta(\omega_1, \omega_2, \omega_3) = (-\omega_1, \omega_2, -\omega_3) \quad (4.6)$$

Now consider the problem of extending local forms

$$\varphi = \sum_{i=1}^{3} dx_i \wedge \omega_i + dx_{123} \quad (4.7)$$

from $\mathbb{C}^2 \times \mathbb{R}^3$ to $V_0$. Fix a simultaneous flat trivialization $\mathcal{U} = \{U_\lambda\}$ of $T G_6$ and $V_0$. The monodromy transformations for the $dx_i$’s on $U_\lambda \cap U_{\lambda'}$ are given by the action of $H_{G_6} \cong K$ given by the matrices $A, B, AB$ in 2.2. One then easily sees from 4.6 that the $\omega_i$’s transform according to the *inverses* $A^{-1}, B^{-1}, (AB)^{-1}$ on the local patches$^6$. Thus the element:

$$\eta := \sum_{i=1}^{3} dx_i \wedge \omega_i \quad (4.8)$$

$^6$For $G_6$ the inverses actually coincide with the original matrices, but in general this is not the case.
glues to a global flat section. Obviously \(dx_{123}\) also glues globally, so together they give a well-defined \(G_2\)-structure. This \(G_2\)-structure is closed and torsion-free; the associated metric has holonomy \(K \subset G_2\).

Now let \(\Gamma \cong \mathbb{Z}_2 \leq Sp(1)\) act on \(\mathbb{C}^2\) in the natural way. It is easy to see that this action is compatible with the \(\mathbb{K}\)-action: this means that the monodromy representation \(\rho\) of \(V\) is an element of \(\text{Hom}(\pi_1(\mathcal{G}_6), C_{Sp(1)}(\mathbb{Z}_2))\), which is clear since \(C_{Sp(1)}(\mathbb{Z}_2) = Sp(1)\). It follows that the previous example descends to a closed, torsion-free \(G_2\)-structure on \(M_0 := V_0/\mathbb{Z}_2 = \mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{K}\mathcal{G}_6\). There are three different smoothings of \(M_0\) with compatible \(\mathbb{K}\)-actions, and Joyce [Joy00] extends the \(G_2\)-structure over them. The associated metric on these smoothings has holonomy \(SU(2) \times \mathbb{K} \subset G_2\).

Note that if one uses \(\Gamma = \mathbb{Z}_n\), then \(C_{Sp(1)}(\mathbb{Z}_n) = U(1)\) does not contain \(\mathbb{K}\). In this situation the singularity \(\mathbb{C}^2/\mathbb{Z}_n\) acquires non-trivial monodromy dictated by \([\mathbb{K},\mathbb{Z}_n] \subset Sp(1)\).

**Example 4.6.** Let \(Q = \mathcal{G}_3\) and consider \(\mathbb{C}^2 \times_{\mathbb{Z}_3} \mathcal{G}_3\). Choose again local flat 1-forms \(dx_i\)’s. The holonomy \(H_{\mathcal{G}_3} \cong \mathbb{Z}_3\) is generated by the matrix:

\[
A = \begin{bmatrix}
1 & 0 & 0 \\
0 & -1 & 1 \\
0 & -1 & 0
\end{bmatrix}
\] (4.9)

The correct action of \(\mathbb{Z}_3\) on \(\Lambda_2^+\) is not given by \(A^{-1}\), but by \((A \circ R_3)^{-1}\) where \(R_3\) is reflection on the \(xy\)-plane. In other words, the correct matrix is obtained by reflecting the lower \(2 \times 2\)-block on its anti-diagonal:

\[
(A \circ R_3)^{-1} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & -1 & -1
\end{bmatrix}
\] (4.10)

Again this is just a hyperkähler rotation on local intersections, and as before it defines a global flat 3-form \(\eta\) on \(V = \mathbb{C}^2 \times_{\mathbb{Z}_3} \mathcal{G}_3\) such that \(\eta + dx_{123}\) is a closed \(G_2\)-structure.

We can now do the same thing we did for the previous example: let \(\mathbb{Z}_2 \leq Sp(1)\) and note that the flat bundle \(V\) can be taken to be compatible with the \(\Gamma\)-action on \(\mathbb{C}^2\). Thus we get a well-defined \(G_2\)-structure on \(\mathbb{C}^2/\mathbb{Z}_2 \times_{\mathbb{Z}_3} \mathcal{G}_3\) and conjecturally also in some of its smoothings; if one can also solve the integrability condition, the metric will have holonomy \(SU(2) \times \mathbb{Z}_3\).

Examples involving the platycosms \(\mathcal{G}_2, \mathcal{G}_4\) and \(\mathcal{G}_6\) are similar to the \(\mathcal{G}_3\) example, the only essential difference being that in the absence of \(\mathbb{Z}_3\) factors in the monodromy group, one can still hope to produce singularities of types D and E. However, \(\mathcal{G}_6\) has a special property: \(H_{\mathcal{G}_6} \cong \mathbb{K}\) is the only platycosm holonomy group such that \(SU(2) \times H_Q\) cannot be conjugated to a subgroup of \(SU(3) \subset G_2\) (this is due to the fact that its action on \(\mathbb{R}^3\) does not fix any direction). This feature
implies that any desingularization of the example $M_0$ above defines an appropriate (i.e., $N=1$) vacuum compactification for $M$-theory. This property is the key reason why $M_0$ is taken as our main example in this work.

An important point is that the short exact sequence 2.1 is not unique; one can modify the lattice $\Lambda$, for example by modifying its period along one direction, as long as one modifies the group $H$ accordingly.

**Example 4.7.** To illustrate this point, start with the exact sequence for $G_6$:

\[1 \to \mathbb{Z}^3 \to \pi_1(G_6) \to \mathbb{K} \to 1\] (4.11)

Bieberbach’s first theorem says that $G_6$ is a quotient of the three-torus $T$. This is realized via the following (free) action of $\mathbb{K} = \langle \alpha, \beta \rangle$ on $T$:

\[
\alpha(x_1, x_2, x_3) = (-x_1 + \frac{1}{2}, -x_2, x_3 + \frac{1}{2}) \\
\beta(x_1, x_2, x_3) = (-x_1, x_2 + \frac{1}{2}, -x_3)
\]

So $G_6 = T/\mathbb{K}$. However, a second possible description is $G_6 = T'/D_8$, where $D_8 \cong \mathbb{Z}_2 \times \mathbb{K}$ is the dihedral group with 8 elements. Let $\alpha', \beta'$ be two generators of $D_8$ satisfying $\alpha'^2 = 1$ and $\beta'^4 = 1$. The action is given by:

\[
\alpha'(x_1, x_2, x_3) = (-x_1 + \frac{3}{4}, -x_2, x_3 + \frac{1}{2}) \\
\beta'(x_1, x_2, x_3) = (-x_1 + \frac{1}{4}, x_2 + \frac{1}{4}, -x_3)
\]

This provides a second short exact sequence for $\pi_1(G_6)$:

\[1 \to \Lambda' \to \pi_1(G_6) \to D_8 \to 1\] (4.12)

where the lattice $\Lambda'$ is given by $2\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ (i.e., $T'$ is isogenous to $T$). Note that $(\beta')^2(x_1, x_2, x_3) = (x_1, x_2 + \frac{1}{2}, x_3)$ is a translation by an order 2 element generating the center $\mathbb{Z}_2 \leq D_8$, and as such it doesn’t contribute to the holonomy.

This new action is also compatible with $\Gamma = \mathbb{Z}_2$, so the resulting closed, torsion-free $G_2$-structure descends to $M = V/\Gamma$.

This example has an advantage over example 4.5. Consider the following $D_8$ action on $\mathbb{C}^2$

\[
\alpha(z_1, z_2) = (-z_1, z_2) \\
\beta(z_1, z_2) = (iz_2, iz_1)
\] (4.13)

This action commutes with the action of $\mathbb{Z}_n \leq Sp(1)$ for all $n$. Hence we can get non-trivial $A_n$ data using this action, and a quick computation shows that the
hyperkähler structure glues correctly. Hence one gets a well-defined $G_2$-structure on the quotient $\mathbb{C}^2/\mathbb{Z}_n \times \mathcal{G}_6$.

The only difference between this example and example 4.5 is that here there is an extra central $\mathbb{Z}_2$-symmetry acting on the hyperkähler triple. This is given by the action of $(\beta')^2$, which is trivial on the triple; hence this symmetry is not visible at the geometric level, but it has to be remembered when using the string duality explored in this paper. In physics jargon this symmetry gives rise to a so-called $B$-field on dual Calabi-Yau spaces. Mathematically, the dual Calabi-Yau will inherit a flat $\mathbb{Z}_2$-gerbe. We will leave a more detailed discussion on $B$-fields (and their relation to monodromy of ADE singularities) for future work.

**Example 4.8.** Up to now, in all examples it was possible to write down a closed $G_2$-structure on $V$ that descends to $M$. Now suppose $V \to Q$ has a flat connection $\nabla$ with monodromy group $H_\nabla$ acting by a representation $\rho_\nabla$, and the platycosm $Q$ has a nontrivial holonomy representation $\rho_Q : H_Q \to SO(3)$. Now both groups of local sections need to be chosen in a compatible way. First, we need to pick a common flat trivialization $U$ for $\nabla$ and $\delta$. On local intersections, we need $\rho_\nabla(\omega)$ to cancel out $\rho_Q(\delta \omega)$. In general $H_\nabla$ is isomorphic to a quotient of $\pi_1(Q)$, and even if it happens that $H_Q \leq H_\nabla$ and that the actions satisfy $(\rho_\nabla)|_{H_Q} = (\rho_Q)^{-1}$, there might still be other subgroups of $H_\nabla$ that act non-trivially on $\omega$, which will spoil the gluing construction for $\eta$.

Thus, for a fixed $Q$, there are restrictions on which $(V, \nabla)$ are allowed. If one chooses $\nabla$ such that the lattice subgroup $\mathbb{Z}^3 \leq \pi$ acts trivially, then $H_\nabla = H_Q$ and we just need to impose the inversion condition. However, not all platycosms will admit flat bundles with this property; indeed, going back to the exact sequence

$$1 \to \text{Hom}(H_Q, C_{Sp(1)}(\Gamma)) \to \text{Hom}(\pi, C_{Sp(1)}(\Gamma)) \to \text{Hom}(\mathbb{Z}^3, C_{Sp(1)}(\Gamma))$$

what we are looking for is a nontrivial element $\rho \in \text{Hom}(\pi, C_{Sp(1)}(\Gamma))$ that maps to 0, i.e., we need a non-trivial element of $\text{Hom}(H_Q, C_{Sp(1)}(\Gamma))$. These do not exist in the situation where $Q$ is either $G_3$ or $G_5$ and $\Gamma$ is of type $D_n$ or $E_{6,7,8}$, simply because $\text{Hom}(H_Q, C_{Sp(1)}(\Gamma)) = 0$ in these cases. For the other platycosms, at least one such element exists, since in this case all subgroups of $H_Q$ have even order and one can pick the map sending all generators to $-1$. The crucial point then is determining which of these representations are not $Sp(1)$-conjugate to the trivial representation.

**Definition 4.9.** Given non-trivial ADE data as above, we say that $M = V/\Gamma$ with its induced closed $G_2$-structure $\varphi$ is an ADE $G_2$-platyfold (of type $(\nabla, \Gamma)$).

$M$ is then the total space of a bundle of ADE-singularities of type $\Gamma$ over the platycosm $Q$. We will often drop the reference to $\nabla$ (and therefore to $V$) if it is implicit in the discussion.

Let $p : M \to Q$ be an ADE $G_2$-platyfold. There is an exact sequence:
where $V := \text{Ker}(dp)$ is called the vertical bundle. A connection on $M \to Q$ is equivalent to a section $s : TQ \leftrightarrow TM$ splitting the sequence; it defines a horizontal distribution $H = s(TQ) \subset TM$. This induces a splitting of the exterior derivative on $M$ into $d = df + d_H + F_H$, where $df$ is a fiberwise differential, $d_H$ a horizontal differential and $F_H$ is the curvature operator of $H$. In our situation, the connection on $M$ is induced from the flat connection $\nabla$ on $V$, so $F_\nabla = 0$.

In order to gain a better understanding of the integrability conditions on $\varphi$, it is useful to work in a slightly more general setup. Most of the discussion in the rest of this section follows [Don16] closely.

**Definition 4.10.** A hypersymplectic structure on an oriented four-manifold $X$ is a triple $\omega = (\omega_1, \omega_2, \omega_3)$ of symplectic forms such that at each point $p \in X$, $\omega_p$ spans a maximal positive-definite subspace of $\Lambda^2 p(X)$ with respect to the wedge product.

In other words, $\omega_i \wedge \omega_j \in \Gamma(X, \text{Sym}^2(X))$ has positive determinant at every point, and by rescaling the volume form one can take $\det(\omega_i \wedge \omega_j) = 1$ at all points. Thus $G_{ij} := \omega_i \wedge \omega_j$ is a Riemannian metric, and it is hyperkähler if and only if $\omega_i \wedge \omega_j$ is a multiple of the identity.

Accordingly, we define a hypersymplectic element on $(M \to Q, H)$ to be an element $\eta \in \Gamma(H^* \oplus \Lambda^2 V^*)$ such that at each point $q$, the linear map $\eta_q : H_q \to \Lambda^2 V_q^*$ injects $H_q$ as a maximal positive subspace with respect to the wedge product.

We have the following theorems of Donaldson [Don16]:

**Theorem 4.11.** (Donaldson): A closed $G_2$-structure on $(M \to Q, H)$ with coassociative fibers and orientation compatible with those of $M$ and $Q$ is equivalent to a choice of the following data:

- A hypersymplectic element $\eta \in \Gamma(H^* \oplus \Lambda^2 V^*)$ satisfying:
  
  $\begin{align*}
  d_{H} \eta &= 0 \\
  df \eta &= 0
  \end{align*}$

- A tensor $\mu \in \Gamma(\Lambda^3 H^*)$ satisfying:
  
  $\begin{align*}
  d_{H} \mu &= 0 \\
  df \mu &= -F_H(\eta)
  \end{align*}$

which is pointwise positive when seen as an element of $\Lambda^3 T^*Q \cong \mathbb{R}$. 

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Theorem 4.12. (Donaldson): A closed $G_2$-structure as in Theorem 4.11 is torsion-free if and only if the following holds:

\begin{align*}
  d_f \gamma &= -F_H \nu \\
  d_H \gamma &= 0 \\
  d_f \nu &= 0 \\
  d_H \nu &= 0
\end{align*}

where $\gamma$ and $\nu$ are determined from $\eta$ and $\mu$ by:

\begin{align*}
  \gamma_i \wedge \eta_j &= \delta_{ij} \left( \mu \det(\eta \wedge \eta) \right)^{1/3} \\
  \nu &= \det(\eta \wedge \eta)^{1/3} \mu^{-2/3}
\end{align*}

We refer to $(H, \eta, \mu)$ as Donaldson data for a $G_2$-structure on $M$. The $G_2$ 3-form is given by $\varphi = \eta + \mu$, and its dual 4-form is $\psi = \gamma + \nu$. The equations for $(H, \eta, \mu)$ in Theorem 4.11 will be called Donaldson's constraints for a closed $G_2$-structure.

A third theorem of Donaldson will also be important. To state it, let us denote by $S$ the diffeomorphism type of the fiber in $M \to Q$ and suppose we are given a $H^2(S, \mathbb{Z})$ local system over $Q$. From that we get a flat bundle $\mathcal{H} \to Q$ with fibers $H^2(S, \mathbb{R})$. We will say a smooth section $h : Q \to \mathcal{H}$ is positive if it is an immersion and at every point $x \in Q$, $dh_x(TQ)$ is a positive subspace with respect to the cup product.

Theorem 4.13. (Donaldson): Let $(x_1, x_2, x_3)$ denote local coordinates on $Q$ on a trivialization of $\mathcal{H}$. Let $h : Q \to \mathcal{H}$ be a positive section and let $\eta$ be a hypersymplectic element on $M \to Q$, locally given by $\sum \omega_i \wedge dx_i$. Suppose that the following local condition holds:

$$
[\omega_i] = \frac{\partial h}{\partial x_i}
$$

Then there is a connection $H$ on $M \to Q$ such that $d_H \eta = 0$.

Moreover, if $H^1(S) = 0$, then there is a positive section $\mu \in \Gamma(\Lambda^3 T^* Q)$ such that $d_f \mu = -F_H(\eta)$. It follows that $(\eta, \mu, H)$ defines a closed $G_2$-structure on $M$ making $M \to Q$ coassociative.

It is clear now why our examples provided closed $G_2$-structures: they were just special cases of Donaldson data, in situations where the connection is flat. This simplification gives an a posteriori reason to work with platycosms: while they have plenty of flat connections, their character varieties are quite simple and can be
described explicitly. Deeper reasons will arise in the next section, where we will use Donaldson’s theorems to study deformations of ADE $G_2$-platyfolds via unfolding of singularities; and in the following section, where the deformation space will be computed explicitly for our main example.

5 Deformation Family for Closed $G_2$-Structures

5.1 The Kronheimer family

We start by fixing some notation for this section. We denote by $\mathfrak{g}_c$ a semi-simple complex Lie algebra, $\mathfrak{h}_c$ a Cartan subalgebra, and $W$ the Weyl group. The compact real form of $\mathfrak{g}_c$ is denoted $\mathfrak{g}$, and $\mathfrak{h}$ is the associated real Cartan subalgebra.

Before anything, we need to review the construction of the Brieskorn-Grothendieck versal deformation of the quotient singularity $\mathbb{C}^2/\Gamma$ via Slodowy slices: let $x$ be a subregular nilpotent element of $\mathfrak{g}_c$ and complete it to a $\mathfrak{sl}(2, \mathbb{C})$-triple $(x, h, y)$. Define the Slodowy slice:

$$S = x + \mathfrak{z}_{\mathfrak{g}_c}(y) \subset \mathfrak{g}_c$$

where $\mathfrak{z}_{\mathfrak{g}_c}(y)$ is the centralizer of $y$, i.e. the kernel of the adjoint action of $G_c$ on $y$.

Consider the GIT adjoint quotient $\mathfrak{g}_c \to \mathfrak{g}_c//G_c$. Chevalley’s theorem says that $\mathbb{C}[\mathfrak{g}_c]^{G_c} \cong \mathbb{C}[\mathfrak{h}_c]^W$, so $\mathfrak{g}_c//G_c \cong \mathfrak{h}_c/W$. Define $\Psi : S \to \mathfrak{h}_c/W$ to be the restriction of $\mathfrak{g}_c \to \mathfrak{h}_c/W$ to $S$.

**Theorem 5.1.** (Slodowy [Slo80]): The family $\Psi$ has the following properties:

1. $\Psi$ is a flat, surjective holomorphic map
2. $\Psi^{-1}(0) \cong \mathbb{C}^2/\Gamma$
3. Given any other map $\Psi' : \mathcal{A} \to \mathcal{B}$ satisfying properties 1 and 2 there is a map $\beta : (\mathcal{B}, b) \to (\mathfrak{h}_c/W, 0)$ such that $\Psi' = \beta^*\Psi$. The map $\beta$ might not be unique, but its derivative $d\beta_b$ is unique.
4. $\Psi$ is equivariant with respect to natural $\mathbb{C}^*$-actions on $S$ and $\mathfrak{h}_c/W$

In other words, $\Psi$ is the Brieskorn-Grothendieck $\mathbb{C}^*$-miniversal deformation of $\mathbb{C}^2/\Gamma$. Thus, the Slodowy slice is a geometric realization of the deformations of $\mathbb{C}^2/\Gamma$ inside the Lie algebra $\mathfrak{g}_c$.

This embedding of $S$ into $\mathfrak{g}_c$ comes with a symmetry group. Let $Z = Z_{\mathfrak{g}_c}(x) \cap Z_{\mathfrak{g}_c}(y)$ be the reductive centralizer (of $x$ with respect to $\mathfrak{h}$).\(^7\) Its action on $S$

\(^7\)This uniqueness at the infinitesimal level is known as miniversality. Any two miniversal deformations of an ADE singularity are isomorphic, and their reduced Kodaira-Spencer map is an isomorphism.

\(^8\)The name is due to the fact that the identity component $Z^0$ is reductive, and the component groups $Z_{\mathfrak{g}_c}(x)/Z^0_{\mathfrak{g}_c}(x)$ and $Z/Z^0$ coincide.
commutes with \( C^* \), so there is an action of \( C^* \times Z \) on \( S \). The action of \( Z \) restricts to act on the fibers of \( \Psi \) (i.e., \( \Psi \) is \( Z \)-invariant). The group \( C^* \times Z \) is called the symmetry group of the Slodowy slice.

**Lemma 5.2.** \( Z \cong C^* \) for \( g_c \) of type \( A_n \), and \( Z = \{ e \} \) for types \( D_n \) and \( E_{6,7,8} \).

**Proof.** See Slodowy’s book [Slo80].

Kronheimer [Kro89a] constructed a deformation space for ALE hyperkähler structures on \( \mathbb{C}^2/\Gamma \). He starts with \( (Y^k, I, J, K) \) a certain flat simply-connected hyperkähler space with a quaternionic action of \( G \) (i.e., \( G \leq \text{Sp}(k) \)), and constructs a hyperkähler moment map:

\[
\mu = (\mu_1, \mu_2, \mu_3) : Y^k \to \mathbb{R}^3 \otimes g^*
\]

such that for \( \xi \in \mathbb{R}^3 \otimes h^* \) the hyperkähler quotient:

\[
S_{\xi} = \mu^{-1}(\xi)/G
\]

is well-defined and is also a hyperkähler space. In particular, it has a hyperkähler triple \( (I, J, K)_\xi \) induced from \( Y^k \).

One can then prove that if \( \xi \in \mathbb{R}^3 \otimes h \), \( S_{\xi} \) is an ALE space, and is non-singular if and only if \( \xi \notin \bigcup_v \mathbb{R}^3 \otimes C_v \), where \( C_v \) is the hyperplane orthogonal to a root \( v \).

Kronheimer’s deformation family \( K \to h_c \) is constructed as follows: consider the complexified moment map \( \mu_c := \mu_2 + i\mu_3 : Y^k \to \mathbb{C} \otimes g^* \), where \( \mathbb{C} = (\{0\} \times \mathbb{R}^2, I) \), i.e., the complex structure \( I \) induces an identification \( \mathbb{R}^3 \cong \mathbb{R} \oplus \mathbb{C} \) given by \( (\chi_1, \chi_2, \chi_3) \mapsto (\chi_1, \chi_2 + i\chi_3) \). Then:

\[
S_{(0,\chi_2,\chi_3)} = (\mu_1^{-1}(0) \cap \mu_c^{-1}(\chi_2 + i\chi_3))/G
\]

is an affine variety with respect to the complex structure \( I_{(0,\chi_2,\chi_3)} \). After passing to a normalization, these spaces fit into the Kronheimer deformation family:

\[
\Theta : K \to h_c
\]

which is a surjective flat holomorphic map with \( \Theta^{-1}(0) \cong \mathbb{C}^2/\Gamma \).

Recall that Slodowy’s family \( \Psi : S \to h_c/W \) is versal for \( \mathbb{C}^2/\Gamma \), so \( \Psi \) must be induced from it by pullback from a map between the parameter spaces. Kronheimer proves that \( \Theta \) is equivariant with respect to a \( C^* \)-action on \( K \) and weight 2 dilations on \( h_c \). Due to Looijenga’s description of the period map for \( \Psi \) [Loo84], it follows that \( \Theta \) is induced from \( \Psi \) via pullback by the projection map \( p_W : h_c \to h_c/W \).

**Definition 5.3.** Fix non-zero \( \chi_2, \chi_3 \in h \). We say \( \chi_1 \in h \) is **generic** if \( \xi = (\chi_1, \chi_2, \chi_3) \notin \mathbb{R}^3 \otimes C_v \) for any root \( v \).
If $\chi_1$ is generic, the space $S_\xi$ is a nonsingular hyperkähler manifold, and there is a resolution of singularities $r_\xi : S_\xi \to S_{(0,\chi_2,\chi_3)}$. Therefore, any appropriate choice of $\chi_1$ induces a simultaneous resolution $^9 \tilde{\Theta}_\xi : \tilde{K}_\xi \to h_c$ of $\Theta$ (i.e., $\tilde{\Theta}_\xi = \Theta \circ r_\xi$).

We summarize this discussion in the following:

**Theorem 5.4.** (Brieskorn, Kronheimer, Slodowy, Tjurina): For every generic $\chi \in h$, there is a commutative diagram:

$$
\begin{array}{cccc}
\tilde{K}_\chi & \xrightarrow{r_\chi} & K & \xrightarrow{\psi} S \\
\tilde{\Theta}_\chi & \downarrow & \Theta & \downarrow \\
h_c & \xrightarrow{\rho_W} h_c & \xrightarrow{h_c/W}
\end{array}
$$

satisfying the following properties:

1. $\tilde{\Theta}_\chi$ is a flat, surjective holomorphic map, with fibers diffeomorphic to the minimal resolution $\tilde{C}^2/\tilde{T}$ of $C^2/T$ and admitting an ALE hyperkähler structure.

2. $\tilde{\Theta}_\chi$ is a simultaneous resolution of $\Theta$, i.e., $r_\chi|_{S_{(\chi,\chi_2,\chi_3)}}$ is a resolution of singularities of $S_{(0,\chi_2,\chi_3)}$.

3. $\tilde{K}_\chi$ inherits a $\mathbb{C}^*$-action from $Y^k$ such that $\tilde{\Theta}_\chi$ is $\mathbb{C}^*$-equivariant.

Moreover, one can also prove:

**Theorem 5.5.** (Kronheimer [Kro89]): Given a (smooth) hyperkähler ALE space $S$, there is a $\xi = (\chi,\chi_2,\chi_3)$ with $\chi$ generic such that $S \cong S_\xi$ as hyperkähler manifolds.

### 5.2 Hyperkähler structures

One should think of the base $h_c$ of the Kronheimer family $\tilde{K}_\chi$ as parametrizing infinitesimal deformations of the holomorphic symplectic structure on $\tilde{C}^2/\tilde{T}$. The reason is the following: let $h_>$ be the positive Weyl chamber. By the McKay correspondence, $h_>$ is isomorphic to the Kähler cone of $\tilde{C}^2/\tilde{T}$, with tangent spaces $h$. A choice of complex structure on $\tilde{C}^2/\tilde{T}$ induces an isomorphism $T(h_>) \otimes \mathbb{C} \cong h_c$, so the deformation parameter is a complexified Kähler class, which is in fact holomorphic [HKLR87].

---

^9Tjurina [Tju70], building on previous work of Brieskorn [Bri68], proved that a flat holomorphic map $f : S \to T$ with two-dimensional fibers admitting at most finitely many rational double points admits a local resolution of singularities: around any point $t \in T$ there is an open set $U \subset T$ such that the family $f|_{f^{-1}(U)}$ admits a simultaneous resolution of all fibers, i.e., a commutative diagram whose maps restricted to the fibers are resolutions of singularities. Kronheimer’s construction gives the simultaneous resolution for the Brieskorn-Grothendieck $\mathbb{C}^*$-miniversal deformation.
For our purposes, we need to make a clear distinction between deformations of a holomorphic symplectic structure (HS) and deformations of a hyperkähler structure (HK). The main point is that, even though Kronheimer’s construction produces all HK ALE spaces, it does not fit them all together in a family induced from the Slodowy slice. In order to write diagram 5.4, one needs to fix a complex structure (say, $I$) and an element $\chi \in h$. This fixes the HK-structure but does not account for all deformations. However, we will need to work with the full HK family.

First we need to fix the complex structure. Let $V$ be the adjoint representation of $SU(p,q)$. In comparing $\Theta_\chi$ and $\Theta_{\chi'}$, they correspond to different choices of Kähler classes for a fixed complex structure $I$ inducing a linear isomorphism $V \cong \mathbb{R} \oplus \mathbb{C}$. In other words, the complex structure is fixed once a choice of splitting $V \cong \mathbb{R} \oplus \mathbb{C}$ has been made. We write $h_V := h \otimes V$.

Under the McKay identification $h \cong H^2(S_\xi, \mathbb{R})$, for every fixed $\xi = (\chi, \chi_2, \chi_3)$, one should think of the deformation parameter:

$$\chi_2 + i\chi_3 \in h_\xi \bigcup \bigcup \mathbb{C} \otimes C_v$$

as a choice of cohomology class for a $I$-holomorphic symplectic form on the fiber $\tilde{\Theta}^{-1}(\chi_2 + i\chi_3) \cong S_\xi$.

This is where the distinction between the HS and HK structures on the fibers comes in. For each $\chi \in h^\circ := h \bigcup \bigcup C_v$, the family $\tilde{\Theta}_\chi$ provides a HS-deformation of $\mathbb{C}^2/T$, meaning, a deformation of the singularity together with a two-form $\omega_c \in \Omega^2(\tilde{K}_\chi/h_c)$ that restricts to a a holomorphic symplectic form $\omega_c^{\chi_2+\chi_3}$ on every fiber, varying holomorphically with $\chi_2 + i\chi_3 \in h_c$. It is clear that for $\chi' \neq \chi$, the manifolds $\tilde{\Theta}_{\chi'}$ and $\tilde{\Theta}_\chi$ are isomorphic as holomorphic symplectic manifolds.

However, the associated hyperkähler manifolds are not the same. Indeed, the following well-known proposition shows that a HK-structure is equivalent to a HS-structure + a complex structure and a Kähler class:

**Proposition 5.6. (Beauville):** Let $(S, \Omega)$ be a holomorphic symplectic manifold with a complex structure $I$ and $[\omega] \in H^{1,1}(S)$ a Kähler class. Then there is a unique hyperkähler structure $(I, J, K)$ on $S$ such that $[\omega_I] = [\omega]$ and $\Omega = \omega_I + i\omega_K$.

**Proof.** Follows from the Calabi-Yau theorem. \qed

Therefore, we work with the pullback of $\Theta : \mathcal{K} \rightarrow h_c$ by the projection map $p_f : h_V \rightarrow h_c$. We denote this family by $\Xi : Q \rightarrow h_V$. The fibers are $\Xi^{-1}(\chi, \chi_2, \chi_3) = X_{(0, \chi_2, \chi_3)}$.

We now “glue” all families $\tilde{K}_\chi$ together and define a family $\tilde{\Xi} : \tilde{Q} \rightarrow h_V$ and a map $\tau : \tilde{Q} \rightarrow Q$ such that $(\tilde{\Xi}, \tau)|_{\xi} = (\tilde{K}_\chi, r_\chi)$.

**Proposition 5.7.** There is a family of spaces $\tilde{\Xi} : \tilde{Q} \rightarrow h_V$ and a diagram:
Each generic fiber $\tilde{\Sigma}^{-1}(\chi, \chi_2, \chi_3)$ is a hyperkähler deformation of $\mathbb{C}^2/\Gamma$. Moreover, the hyperkähler triple on each fiber is induced from a relative triple $\omega_{\text{unf}} \in \left(\Omega^2(\tilde{\Sigma}/\mathfrak{h}_V)\right)^3$ varying smoothly with $\mathfrak{h}_V$.

The notation $\omega_{\text{unf}}$ is meant to emphasize that this element induces HK-structures on the unfoldings $S_\xi$ of the singularity $\mathbb{C}^2/\Gamma$.

**Proof.** By Kronheimer’s construction, the element $(\chi_2, \chi_3)$ determines the class of a $I$-holomorphic symplectic form $\omega^{\chi_2+i\chi_3}$ on the fiber $\tilde{\Sigma}^{-1}(\chi, \chi_2, \chi_3)$. The choice of $\chi \in \mathfrak{h}$ determines a Kähler class $\omega^\chi$ and hence a fixed hyperkähler structure. Under the identification $\mathfrak{h} \cong T\mathfrak{h}_{\mathbb{C}}$, one can think of $(\chi, \chi_2, \chi_3)$ as giving a tangent vector on the Kähler cone of $\mathbb{C}^2/\Gamma$. The global relative triple is defined by $\omega_{\text{unf}}(\chi, \chi_2, \chi_3) = (\omega^\chi, \omega^\chi_2+i\chi_3)$.

Note that $\left(\Omega^2(\tilde{\Sigma}/\mathfrak{h}_V)\right)^3$ is a locally constant sheaf on $\mathfrak{h}_V$ whose stalk at $\xi$ is $\Omega^2(S_\xi, \mathbb{R})^3$, and $\omega_{\text{unf}}$ is a locally flat section of this sheaf.

### 5.3 Fibering hyperkähler deformations over a platycosm

Let $(Q, \delta)$ be an oriented platycosm, and fix all data as in definition 4.9. Our resulting $M_0$ is then an ADE $G_2$-platyfold of type $\Gamma$, with closed $G_2$-structure $\varphi_0$. We write $(\eta_0, \mu_0, \mathbf{H}_0)$ for its associated Donaldson data; in particular, $\mathbf{H}_0$ is the horizontal distribution associated to the connection $\nabla$ on $V$, and hence is flat and preserves the vertical hyperkähler structures. Let $\mathfrak{g}$ be the compact Lie algebra associated to $\Gamma$, $\mathfrak{h}$ a Cartan subalgebra, and $r := \text{rank}(\mathfrak{g}) = \dim(\mathfrak{h})$.

In this section we will use our adaptation of Kronheimer’s construction (specifically Proposition 5.7) to build a family of hyperkähler deformations $\mathcal{E}$ parametrized by $Q$. We will prove that flat sections of this family define 7-manifolds with a closed $G_2$-structure. Such manifolds appear as subspaces of what is essentially “the Slodowy slice over $\mathcal{E}$”. Moreover, the image of such sections can be embedded in $T^*Q$ and admit an interpretation as “flat spectral covers” of $Q$. To explain what these objects are, let $\mathfrak{h}_Q \to Q$ be the trivial flat bundle of Cartan subalgebras. Consider the flat bundles:

$$E_W := \text{tot}(\mathfrak{h}_Q \otimes T^*Q)/W \to Q$$

$$E := \text{tot}(\mathfrak{h}_Q \otimes T^*Q) \to Q$$  \hspace{1cm} (5.7)
which are $3(r+1)$-dimensional real manifolds. We denote by $\delta_E$ the flat structure induced by $\delta$ on $E$. There is a natural projection map $E \to E_W$ which is a $|W|$-to-1 cover with Galois group $W$. Suppose we have a section $s : Q \hookrightarrow E_W$. We call the restriction $E_W|_{s(Q)} \to Q$ the spectral cover of $Q$ associated to $s$. Note that $s$ can be viewed as a multi-section of $T^*Q \to Q$, which is the usual formulation of spectral covers.

The fiber product:

$$\Sigma_s := E \times_{E_W} Q$$

is called the cameral cover of $Q$ associated to $s$. It comes equipped with a natural $|W|$-to-1 map $\Sigma_s \to Q$ and an embedding $\Sigma_s \hookrightarrow E$. Given $\Sigma_s$ and these two maps, one can recover the section $s$, and hence the spectral cover.

The cameral covers and spectral covers that will be of interest to us are a slight modification of this example, where we replace $E$ by a flat vector bundle $E$ with the same fibers, but transition functions “twisted” by $V$. The relation between these objects and symmetric solutions of 1.1 will be explained in section 7.

Our goal in this section is to prove the following result:

**Theorem 5.8.** There is a rank $3r$ flat vector bundle $t : E \to Q$ and a family $u : U \to E$ of complex surfaces, equipped with:

$$\begin{cases}
\eta \in \Omega^2(U/E) \otimes u^*\Omega^1(E) \\
\mu \in u^*\Omega^3(E)
\end{cases}$$

The family has the following properties:

1. $U|_{\bar{Q}(Q)} \cong M_0$
2. $(\eta + \mu)|_{M_0} = \varphi_0$
3. $U|_{t^{-1}(q)} \cong Q$

where $\bar{Q} : Q \to E$ denotes the zero-section.

Moreover, given a flat section $s : Q \to E_W := E/W$, let $M_s := u^{-1}(s(Q))$. Then there exists a connection $H_s$ on $M_s \to s(Q)$ such that $(\eta|_{M_s}, \mu|_{M_s}, H_s)$ satisfy Donaldson’s criteria, and hence define a closed $G_2$-structure $\varphi_s := (\eta + \mu)|_{M_s}$ on $M_s$.

**Corollary 5.9.** Given an ADE $G_2$-platyfold $(M_0 = V/\Gamma, \varphi_0) \to Q$, there is a local moduli space of closed $G_2$-deformations given by:

$$\mathcal{M}_{G_2}(M_0) := \Gamma_{\text{flat}}(Q, E_W)$$

In other words, the deformations of $(M_0, \varphi_0)$ are parametrized by flat spectral covers of $Q$. 

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The main ingredient to prove Theorem 5.8 will be, once $E$ is constructed, to pullback the modified Kronheimer family $Q$ to $E$. However, it is apparent that there is no map $E \to h_V$, due to the fact that in general $E \to Q$ is a non-trivial flat vector bundle: the metric connection $\delta$ has non-trivial monodromy $H_\pi$, and the same is true for the induced flat structure $\delta_E$. To circumvent this issue, we work over a flat trivialization of $E$, where such maps are available locally; then we glue the pullback families together using the cocycle of $V$.

Another equivalent formulation would be to work with the pullback of $E$ to the universal cover $\widetilde{Q} \to Q$. In fact, we can do something simpler: we can work over the monodromy cover of $Q$, i.e., the minimal cover where the monodromy action is trivial. Due to Bieberbach’s theorem, the monodromy cover of a platycosm is always a three-torus $T$. It is defined by a finite unramified covering map $c : T \to Q$ with Galois group $H_\pi$. We then get a trivial flat bundle $\mathcal{E} \to T$, and we can choose a flat trivialization where $\mathcal{E} \cong T \times h_V$. This gives us a map:

$$\kappa : \mathcal{E} \to h_V$$  \hspace{1cm} (5.11)

This is simpler than working in the universal cover because we only need to worry about the action of $H_\pi$, i.e. we can forget the lattice $\mathbb{Z}^3 \leq \pi$. The drawback of this approach is that one must be careful to choose Donaldson data $H_\pi$-invariantly.

We will break the proof of Theorem 5.8 into a few lemmas:

**Lemma 5.10.** There is a flat vector bundle $t : E \to Q$ with rank($E$) = 3$r$ and a family $u : U \to E$ of complex surfaces satisfying:

1. $U|_{\mathcal{Q}(Q)} \cong M_0$
2. $U|_{t^{-1}(q)} \cong \mathcal{Q}$

where $\mathcal{Q} : Q \to E$ denotes the zero-section.

*Proof.* Let $\mathcal{U} := \{U_i ; i \in I\}$ be a trivializing flat cover of $Q$ (i.e, $\delta|_{U_i}$ has trivial monodromy) which also trivializes $(V, \nabla) \to Q$. The argument essentially consists of gluing together “locally constant” copies of $\mathcal{Q} \to h_V$ over $U_i$ using the cocycle defining the vector bundle $V$. The proof in the holomorphic setup is due to Szendr˝oi [Sze04], and it follows through also in our flat setup. We reproduce it here for completeness.

Let $\zeta_V \in H^1(\mathcal{U}, C_T)$ be the cocycle of transition functions of $V \to Q$. It is valued in $C_T$ since $V$ comes equipped with a compatible fiberwise $\Gamma$-action. Since $C_T \leq Sp(1)$, it acts on $h_V$ by rotating the hyperkähler classes, so we have a map $C_T \to GL(h_V)$. This induces:

$$\epsilon : H^1(\mathcal{U}, C_T) \to H^1(\mathcal{U}, GL(h_V))$$  \hspace{1cm} (5.12)
and since \( H^1(\mathfrak{U}, GL(h_V)) \cong H^1(Q, GL(h_V)) \), the image \( \epsilon(\zeta_V) \) defines a rank \( r \) vector bundle \( t : E \to Q \). This bundle is trivialized by \( \mathfrak{U} \), so we write \( E|_{U_i} \cong U_i \times h_V \). This gives us maps

\[
\psi_i : E|_{U_i} \to h_V \tag{5.13}
\]

Note that the metric on \( Q \) gives an isomorphism \( T^*Q \cong \Lambda^2TQ \), and this last space is exactly the adjoint bundle of \( SU(2) \). This allows us to write \( E|_{U_i} \cong U_i \times h \otimes T^*U_i \). The global identification between the flat bundles \( E \) and \( h \otimes T^*Q \) is given by the inversion condition: let \( \zeta_T \) and \( \zeta_E = \epsilon(\zeta_V) \) be the cocycles of the respective bundles. Let \( Ad|_{H_\pi} : H_\pi \to SO(V) \) be the standard representation restricted to \( H_\pi \subset SO(3) \). This gives \( Ad_{H_\pi}(\zeta_T) \in H^1(\mathfrak{U}, SO(V)) \subset H^1(\mathfrak{U}, GL(h_V)) \) via \( A \in SO(3) \mapsto 1 \otimes A \in GL(h \otimes V) \). Then \( \zeta_E = Ad|_{H_\pi}(\zeta_T)|^{-1} \).

Now consider the hyperkähler family \( \Xi : Q \to h \), which is itself a pullback of the Kronheimer miniversal deformation \( \Theta : K \to h \) by a map \( p_2 : h \to h \). Let \( \mathcal{U}_i := (\psi_i \circ p_2)^*K \). This gives, for every \( i \in I \), a family of complex surfaces:

\[
u_i : \mathcal{U}_i \to U_i \times h_V \tag{5.14}\]

We now glue these families together over the \( U_i \)'s using the Čech cocycle \( \zeta_V \in H^1(\mathfrak{U}, C_T) \). For this to make sense, we need to realize \( C_T \) as a subgroup of \( Aut(K) \). But this follows from \( C_T \leq Aut(C^2/G) \) and the fact that this last group acts on \( K \), as \( K \to h \) is miniversal. So we can think of \( \zeta_V \) as an element of \( H^1(\mathfrak{U}, Aut(K)) \). Thus, the datum \( \{\mathfrak{U}, \zeta_V, u_i; i \in I\} \) provides us with a family of complex surfaces:

\[
u : \mathcal{U} \to E \tag{5.15}\]

and, by construction, \( \mathcal{U}|_{\nu^{-1}(q)} \cong Q \).

Now restrict \( \nu \) to the zero-section \( \mathbf{0} : Q \to E \). Being the zero-section means that \( \mathcal{U}|_{\mathbf{0}(Q)} \) is glued by local pieces \( \Psi^{-1}(0) \times U_i \cong C^2/G \times U_i \) according to \( \zeta_V \), i.e., \( \mathcal{U}|_{\mathbf{0}(Q)} \cong M_0 \).

We now have a diagram:

\[
\begin{array}{ccc}
\mathcal{U} & \xrightarrow{\nu} & E \\
\downarrow u & & \downarrow t \\
Q & \xleftarrow{\mathbf{0}} & E
\end{array}
\tag{5.16}
\]

**Lemma 5.11.** There are elements:

\[
\begin{align*}
\eta & \in \Omega^2(\mathcal{U}/E) \otimes u^*\Omega^3(E) \\
\tilde{\mu} & \in u^*\Omega^3(E)
\end{align*}
\tag{5.17}
\]
satisfying:

\[(\eta + \tilde{\mu})|_{M_0} = \varphi_0\]  \hspace{1cm} (5.18)

**Proof.** For each trivializing open set \(U_i\), fix an oriented basis of flat local sections \(\{\sigma_{1,i}, \sigma_{2,i}, \sigma_{3,i}\} \subset \Omega^1(U_i)\) such that \(\eta_0|_{U_i} = \omega_{\text{unf}}(0,0,0) \otimes \sigma_{i}^0\). On intersections \(U_i \cap U_j\) these sections glue according to the monodromy \(H_\pi \leq SO(3)\) of \((Q, \delta)\). Let \(\zeta_{T*Q} \in H^1(\mathcal{U}, SO(3))\) be the associated Čech cocycle. Due to the inversion condition, \(\omega_{\text{unf}}(0,0,0)\) is glued over open sets according to \(\zeta_{T*Q}^{-1}\). Moreover, since \((Q, \delta)\) preserves an orientation, \(\det(\zeta_{T*Q}) = 1\) and \(\sigma_{1,i}^0 \wedge \sigma_{2,i}^0 \wedge \sigma_{3,i}^0\) is a well-defined global flat 3-form on \(Q\), which we normalize to be equal to the given \(\mu_0\). This form pulls-back to a global flat element \(t^*\mu_0 \in \Omega^3(\mathcal{E})\). Define:

\[\tilde{\mu} := (u^{-1}t^*)\mu_0 \in u^{-1}\Omega^3(\mathcal{E})\]  \hspace{1cm} (5.19)

This is a global section of the sheaf \(u^{-1}\Omega^3(\mathcal{E})\) on \(\mathcal{U}\). The notation \(u^{-1}\) means that we take the subsheaf of the pullback whose sections are constant in the vertical direction.

Now, for \(a \in \{1, 2, 3\}\), consider the pullbacks:

\[(\omega_a)_i := \psi_i^*(\omega_a)_{\text{unf}} \in \Omega^2(U_i/\mathcal{E}_i)\]
\[(\sigma_a)_i := u_i^*t_i^*\sigma_{a,i}^0 \in u_i^*\Omega^1(\mathcal{E}_i)\]  \hspace{1cm} (5.20)

where \(t_i : \mathcal{E}_i \to U_i\) is the obvious map.

Since the \((\sigma_a)_i\)'s are pullbacks of the \(\sigma_{a,i}^0\), they glue together over \(\mathcal{U}\) according to \(\zeta_{T*Q}\). Since \(\Omega^2(U_i/\mathcal{E}_i)\) is glued over the \(U_{ij}\)'s according to the Čech cocycle \(\epsilon(\zeta_{\mathcal{V}}) = Ad|_{H_\pi}(\zeta_{T*Q})^{-1}\), it follows that the element:

\[\eta_i := \sum_{a=1}^{3} (\omega_a)_i \otimes (\sigma_a)_i\]  \hspace{1cm} (5.21)

is such that \(\eta_i|_{U_{ij}} = \eta_j|_{U_{ij}}\) so defines a global section:

\[\eta \in \Omega^2(\mathcal{U}/\mathcal{E}) \otimes u^*\Omega^1(\mathcal{E})\]  \hspace{1cm} (5.22)

Now consider \((\eta + \mu)|_{M_0}\). For every \(i\), \((\psi_i|_{M_0})^*\omega_{\text{unf}}\) is just the hyperkähler structure on the central fiber of the Slodowy slice, glued over \(Q\) according to \(\zeta_{\mathcal{V}}\). Thus it is clear that \(\eta|_{M_0} = \eta_0\) and \(\tilde{\mu}|_{M_0} = \mu_0\). 

\[\square\]

**Lemma 5.12.** Let \(s\) be a flat section of \(t : \mathcal{E} \to Q\) and let \(M_s := u^{-1}(s(Q))\). Define \(\pi_s := u|_{M_s} : M_s \to s(Q)\) and \(\eta_s = \eta|_{M_s}\). Then there is a connection \(H_s\) on \(\pi_s\) such that \(d_{H_s}\eta_s = 0\).
Proof. Let \( \mathcal{H}_s \to s(Q) \) be the flat bundle with fibers \( H^2(\mathbb{C}^2/\Gamma, \mathbb{R}) \) induced over \( s(Q) \subset \mathcal{E} \) from the construction of \( u : \mathcal{U} \to \mathcal{E} \). The proof follows from Theorem 4.13 so long as we are able to construct a section\(^{10}\) \( h_s : Q \to \mathcal{H}_s \) such that locally on \( Q \), \( \eta \) represents the derivative of \( h_s \).

On local coordinates \((x_1, x_2, x_3)\) on \( Q \), let \( s(x_1, x_2, x_3) = ([\omega^1_1], [\omega^2_2], [\omega^3_3]) \). Define:

\[
h_s(x_1, x_2, x_3) = x_1[\omega^1_1] + x_2[\omega^2_2] + x_3[\omega^3_3]
\]

The positivity condition is vacuous in our setup since the intersection form is given by the negative of the Cartan matrix of \( \Gamma \), hence is positive-definite. Moreover:

\[
\frac{\partial h_s}{\partial x_i} = [\omega^i_1]
\]  

which is what we wanted. Here we used the fact that the section \( s \) is taken to be flat, so its derivatives in the \( x_i \) directions all vanish. \( \square \)

Remark 5.13. The reader will notice that we explicitly avoided doing the most natural thing, which would be to induce a connection \( \mathbf{H} \) on \( u : \mathcal{U} \to \mathcal{E} \) that restricts to \( \mathbf{H}_s \) for every flat section \( s \). The reason is that, even though \( \mathbf{H} \) is supposed to exist, it seems tricky to construct it explicitly. We will address this point in more detail in Remark 5.17 below.

Lemma 5.14. There exists \( \mu_s \in \Gamma(M_s, \Lambda^3 T^*Q) \) such that \( (\eta_s, \mu_s, \mathbf{H}_s) \) satisfies Donaldson’s criteria:

\[
\begin{align*}
    d_f \eta_s &= 0 \\
    d_h \eta_s &= 0 \\
    d_f \mu_s &= - F_{\mathbf{H}_s}(\eta_s) \\
    d_h \mu_s &= 0
\end{align*}
\]  

Hence, \( \varphi_s := \eta_s + \mu_s \) is a closed \( G_2 \)-structure on \( M_s \) such that \( \pi_s : M_s \to Q \) is a coassociative fibration.

Proof. This follows immediately from Theorem 4.13 given that \( H^1(\mathbb{C}^2/\Gamma, \mathbb{R}) = 0 \). \( \square \)

Remark 5.15. Note that \( \mu_s \) has no relation to \( \tilde{\mu} \). Although this is not essential for us, one can define the \( \mu \) appearing in the statement of Theorem 5.8 by rescaling \( \tilde{\mu} \) to agree with \( \mu_s \) over the image of every flat section.

We can now provide a good visualization of our family of 7-manifolds. Consider the diagram:

\(^{10}\) More precisely, \( h_s \) should be defined on \( s(Q) \), but by pre-composing with \( s \) we can take \( Q \) as the domain.
Here, $\tau$ is the tautological map: $\tau(q, s) := s(q)$ and $\mathcal{F}$ is the pullback of $\mathcal{U}$ by $\tau$. From now on, we write $\mathcal{B} := H^0_{\text{flat}}(Q, \mathcal{E})$.

Our family of interest is $f : \mathcal{F} \to \mathcal{B}$. For every section $s \in \mathcal{B}$, $M_s = f^{-1}(s)$ is a 7-manifold given by an ALE-fibration over $Q$, with the fibration given by $\pi_s := w|_{M_s} : M_s \to Q$. Notice that due to the nature of the map $\tau$, different flat sections pick different profiles of ALE-fibers. In particular, $f^{-1}(0) = M_0$.

One should think of $(M_s, \varphi_s)$ as a “flat hyperkähler deformation” of $(M, \varphi_0)$.

This picture also provides us with an explicit model for the moduli space of such $G_2$-structures: it is just the base $\mathcal{B}$, and it only depends on $\mathfrak{h}$ and the flat structure $\delta$ on $Q$. We will study an explicit model in the next section, where $\mathfrak{g} = \mathfrak{su}(2)$ and $Q$ is the Hantzsche-Wendt platycosm. Moreover, we will show that the sections $s$ have an interpretation as an analogue in flat geometry of spectral covers of Higgs bundles.

**Remark 5.16.** Our construction assumes that one is given a closed $G_2$-structure on a coassociative ADE fibration, and then we induce the same structure, possibly resolved, on other fibers in the deformation family. ADE $G_2$-orbifolds with high rank $\Gamma$ seem to be hard to come by. In our setup involving platycosms, the discussion immediately preceding Definition 4.9 imposes restrictions that could be explored in order to cook up further examples similar to $\mathbb{C}^2/\mathbb{Z}_2 \times_k \mathcal{G}_6$ and $\mathbb{C}^2/\mathbb{Z}_n \times_{D_8} \mathcal{G}_6$.

**Remark 5.17.** We would like to address a point raised before, namely that the connection $\mathbf{H}_s$ appearing in Lemma 5.12 should really be induced from a connection $\mathbf{H}$ on the full family $\mathcal{U} \to \mathcal{E}$.

More precisely, what we want is to show there is a splitting:

$$
0 \longrightarrow T(\mathcal{U}/\mathcal{E}) \longrightarrow T(\mathcal{U}) \longrightarrow u^*T(\mathcal{E}) \longrightarrow 0 \quad (5.26)
$$

inducing a second splitting:

$$
0 \longrightarrow T(M_s/s(Q)) \longrightarrow T(M_s) \longrightarrow u^*T(s(Q)) \longrightarrow 0 \quad (5.27)
$$

and normalized to restrict at the zero section to:
To construct $H$, we claim that all we need to do is to define a partial connection $H_q$ on $t$, i.e., a splitting:

$$
0 \longrightarrow T(M_0/Q) \longrightarrow T(M_0) \longrightarrow u^*T(Q) \longrightarrow 0 \nonumber
$$

To see why this is so, assume we have constructed $H_q$ and consider the diagram:

$$
0 \longrightarrow T(U/E) \longrightarrow T(U/Q) \longrightarrow u^*T(E/Q) \longrightarrow 0 \nonumber
$$

Here $\delta_E$ is the flat connection on $E \to Q$. We want to define a section $H$ of $f$. Because $\delta_E$ splits the last vertical sequence, the section $\delta_E$ exists so we can define $H$ as the composition $\iota \circ H_q \circ \delta_E$. This proves the assertion.

Now, notice that over each $U_i \times h_V$, we can find a copy of $T(U_i \times h_V) \subset TU_i$. To construct our partial connection $H_q$, we would first construct a “local” partial connection $H_q : u^*_q T(U_i \times h_V/U_i) \hookrightarrow T(U_i/U_i)$, and then “glue” the $H_q$'s together over $Q$. The local construction is not problematic as it can be done from a local version of diagram 5.30. However, it is not clear how the gluing should proceed.

Alternatively, one could try to bypass a gluing construction by appealing to a scaling argument. In fact, this is how an analog problem in the Calabi-Yau context is addressed in [DDP06]. In that paper, the issue consists in inducing a holomorphic volume form from the central fiber (a singular Calabi-Yau threefold) to other fibers in the family. The argument consists in first proving the result locally, using Grauert’s semicontinuity theorem, and then spreading it to other fibers via a $\mathbb{C}^*$-action on the base. In our case, we have a $SU(2)$-action on the base, so one would just need to prove the result over a positive ray in the base.

Once one has defined a connection $H$ on $u$ one needs to show it restricts to a connection $H_s$ on $\pi_s$. In fact, $H_s$ will only depend on the vertical part $H_q$ of $H$:

$$
u^*_s T(s(Q)) \subset u^*H_q = u^*t^*TQ = q^*TQ \hookrightarrow TU \nonumber$$

30
Here the first containment is the flatness of $s$, while the last map is the connection $H_q$. Since we are only looking over $s(Q) \subset \mathcal{E}$, this actually produces a map into $TU|_{M_s}$. This gives the connection $H_s$.

Remark 5.18. Before we end this section, we would like to explain how this construction relates to a certain “partial topological twist” [Ach98] [BCHS18]. The relevant condition here is that the connection $\delta$ is metric. The metric on $Q$ induces an isomorphism $T^*Q \cong \Lambda^2 TQ$, and this last space is the bundle of adjoint representations of $SU(2)$. The isomorphism identifies the flat structures, so that locally flat 1-forms can be naturally identified with locally flat adjoint sections. The partial topological twist can then be described as follows: if one starts with a $G_2$-manifold $M$ fibered by ALE spaces over $Q$, and global relative 2-cycles $\alpha_i \in H_2(M/Q, \mathbb{R})$, then pairing the $G_2$-structure $\varphi$ with the $\alpha_i$’s gives $n$ 1-forms $\theta_i \in \Omega^1(Q)$, each of which has 3 components $\theta_i = (\theta^1_i, \theta^2_i, \theta^3_i)$. On the other hand, at each ALE-fiber, we can associate to $\alpha_i$ the three periods of the hyperkähler structure. This gives local functions $f^i_U : U \subset Q \to \mathbb{R}^3$. Because $V \to Q$ is flat, these functions can be glued together to form global functions, up to monodromy of the flat connection. The topological twist requires that the $\theta_i$’s agree locally with the $f^i$’s.

It would be interesting to give a more “invariant” description of this condition. We believe a geometric formulation would involve replacing the oriented affine structure on $Q$ by a “$\mathbb{Z}_2$-twisted oriented affine structure”, i.e., a $(\mathbb{R}^3, G)$-structure on $Q$ where $G$ fits into the exact sequence $1 \to \mathbb{R}^3 \to G \to SU(2) \to 1$ (the $\mathbb{Z}_2$ here refers to the covering group of $SU(2) \to SO(3)$). This would possibly extend the construction to a class of spaces slightly more general than the platycosms.

6 Duality and Character Varieties

The goal of this section is to prove the duality explicitly for $M_0 := \mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{K} \mathcal{G}_6$ and also verify that Corollary 5.9 computes the correct moduli space of $G_2$-structures (and in fact, $G_2$-metrics) for this example. Before we proceed to the example, we would like to briefly explain the content of M-theory/IIA duality for this specific geometric context. Some readers might prefer to jump directly to section 6.2.

6.1 M-theory/IIA duality

Assume $M$ is a $G_2$-orbifold containing a three-manifold of $A_n$ singularities\textsuperscript{11}, and that $\widetilde{M}$ is a desingularization of $M$ also with $G_2$-holonomy. M-theory compactified on $\widetilde{M}$ contains certain objects called “M2-branes” that can wrap the exceptional two-cycles of $\widetilde{M}$. The mass of a M2-brane is proportional to its area, hence in the

\textsuperscript{11}The description here only applies to singularities of type A. The type D case is slightly more subtle [Sen97] and the E case is unknown to the author.
limit where we blow down all cycles we get \( n \) massless \( M2 \)-branes located at the singularities of \( M \).

Now assume \( X \) is a Calabi-Yau threefold. In Type IIA superstrings compactified on \( X \) there are objects called “\( D6 \)-branes” that can wrap calibrated 3-cycles, i.e. special Lagrangian submanifolds. These cycles come equipped with complex line bundles with a \( U(1) \)-connection on it. Given a configuration of \( n \) \( D6 \)-branes, one can have strings stretched between any two of them. Now the massless limit is obtained by smashing all \( D6 \)-branes together, since the mass of a string is proportional to its length. This limit is described by a \( SU(n) \)-connection on a vector bundle over a special Langrangian submanifold.

To explain how \( X \) is related to \( M \), we note that the duality still holds after we take a “weakly-coupled limit” of the two theories. The limits are supersymmetric gauge theories: 11-dimensional supergravity for M-theory and 10-dimensional supergravity for type IIA. These two theories are related by \( U(1) \)-reduction. Moreover, the supersymmetry condition determines the geometric structures these theories classify: they are determined by dimensionally reducing the equation for a 11 or 10 dimensional parallel spinor down to 7 or 6 dimensions, respectively. In the first case, one obtains stationary points of a 7-dimensional analogue of the Chern-Simons functional \( CS(\varphi_C) := \int_M \varphi_C \wedge d\varphi_C \), which are exactly the integrable complexified \( G2 \)-structures. In the second case, one obtains the Hermitian-Yang-Mills equations.

This discussion motivates the following: suppose \( M \) is a \( G2 \)-orbifold of type \( A_n \) endowed with a \( U(1) \)-action by isometries with fixed set \( Q \subset M \). The Calabi-Yau space \( X \) is called a IIA dual for \( M \) if it satisfies the following [Ach00] [AW01]:

1. \( X := M/U(1) \) as topological spaces.
2. The complex structure \( J \) on \( X \) has a real structure such that \( Q/U(1) \cong Q \) is a totally real special Lagrangian submanifold.
3. There are \( n \) \( D6 \)-branes “wrapping” \( Q \subset X \).

The IIA moduli space parametrizes the following objects:

1. Complex structures on \( X \) in which \( Q \) is totally real.
2. Complexified Kähler structures on \( X \)
3. A supersymmetric configuration of \( n \) \( D6 \)-branes wrapped on \( Q \).

Up to now the discussion has been general; now we focus on our main example. Let \( M_0 = \mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{K} \mathcal{G}_6 \). Then the IIA dual is \( X = T^*Q \) endowed with a rank \( n \) vector bundle \( E \to X \) and a Hermitian Yang-Mills connection on it. With the metric on \( Q \) fixed, there is a unique complex structure on \( T^*Q \) under which \( Q \) is totally real. This is the complex structure that makes the semi-flat metric on \( T^*Q \) a Calabi-Yau metric.
Note that the Hermitian-Yang-Mills condition describes generic configurations of \( n \) D6-branes. The massless limit where they wrap the special Lagrangian \( Q \) corresponds to dimensional reduction of the HYM equations along the leaves of \( T^*Q \to Q \). A quick computations shows that the result consists exactly of equations 1.1.

6.2 The duality for \( \mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{K} \mathcal{G}_6 \)

According to the previous discussion, the IIA moduli space is computed by solutions to 1.1, i.e., by the character variety \( \text{Char}(\pi, SL(\mathbb{C})) \) where \( \pi = \pi_1(\mathcal{G}_6) \) is the Hantzsche-Wendt group. Recall that we have the exact sequence \( 1 \to \mathbb{Z}_2 \to \pi \to \mathbb{K} \to 1 \), with \( \mathbb{K} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \). From section 3.1, we know that the character variety of the three-torus \( \mathbb{T} \) is given by:

\[
\text{Char}(\mathbb{Z}^3, SL(n, \mathbb{C})) \cong (\mathbb{C}^*)^{3n-3}/\Sigma_n = \prod_{i=1}^{3} ((\mathbb{C}^*)^{n-1}/\Sigma_n)
\]  

(6.1)

where \( \Sigma_n \) acts by permutations on \( (\mathbb{C}^*)^{n-1} \cong \{z_1z_2 \ldots z_n = 1\} \subset (\mathbb{C}^*)^n \).

In section 3.1, we determined that there is a map \( r : \text{Char}(\pi, SL(n, \mathbb{C})) \to \text{Char}(\mathbb{Z}^3, SL(n, \mathbb{C})) \). Moreover, there is a \( \mathbb{K} \)-action on this last space, given in terms of the presentation 6.1 by

\[
(i, j)[(z_1, z_2, z_3)] = [(z_1^i, z_2^j, z_3^{ij})]
\]

(6.2)

where \( i, j \in \{\pm 1\} \).

The main result in section 3.1 was that \( \text{Im}(r) \) is contained in \( \text{Fix}(\mathbb{K}) \) (and equal under an additional condition). The image determines \( \text{Char}(\pi, SL(n, \mathbb{C})) \) possibly up to a finite cover given by non-trivial representations of \( \mathbb{K} \) mapping to the same element of \( \text{Hom}(\pi, SL(n, \mathbb{C})) \).

When \( n = 2 \) it is easy to describe \( \text{Fix}(\mathbb{K}) \). Let \( \alpha = (1, -1) \). Then:

\[
\text{Fix}(\alpha) = [(\pm 1, \mathbb{C}^*, \mathbb{C}^*)] \cup [(\mathbb{C}^*, \pm 1, \pm 1)] = (\mathbb{C}^2)_{z_2, z_3} \cup \mathbb{C}_{z_1}
\]

Notice that the first factor is contributed by \( -1 \in \mathbb{Z}_2 \) and the second by \( 1 \in \mathbb{Z}_2 \).

We can play a similar game for the other two non-trivial elements of \( \mathbb{K} \). Hence:

\[
\text{Fix}(\mathbb{K}) = \bigcap_{(i, j, k) \in \{1, 2, 3\}} ((\mathbb{C}^2)_{z_i, z_j} \cup \mathbb{C}_{z_k}) = \mathbb{C}_{z_1} \cup \mathbb{C}_{z_2} \cup \mathbb{C}_{z_3}
\]

(6.3)

Thus \( \text{Fix}(\mathbb{K}) \) is a bouquet of three complex lines touching at a point, which we denote by \( Y_\mathbb{C} \) for obvious reasons. The image \( \text{Im}(r) \) can be computed directly from a presentation of \( \pi \) to show that \( r \) is in fact surjective onto \( \text{Fix}(\mathbb{K}) \); essentially, this is because an element in the bouquet, say \( (a, 0, 0) \) is a representation which is non-trivial only at a single generator, so will be automatically a representation of \( \pi \). This determines one connected component of the character variety:
Char$^0(\pi, SL(2, \mathbb{C})) \cong Y_{\mathbb{C}} := \mathbb{C}z_1 \cup \mathbb{C}z_2 \cup \mathbb{C}z_3 \quad (6.4)$

Up to conjugation, there are also 3 other representations of $\mathbb{K}$ that map to the trivial representation of $\pi$. These correspond to rigid representations (including the trivial representation).

Now, the duality predicts that this character variety can be computed as the moduli space of complexified $G_2$-structures $\mathcal{M}_{G_2}^\mathbb{C}$ on $M_0 = \mathbb{C}^2/\mathbb{Z}_2 \times_{\mathbb{K}} \mathcal{G}_6$. As mentioned in the introduction, the space $\mathcal{M}_{G_2}$ for this example was computed by Joyce [Joy00]. The smoothings of the singularity $\mathbb{C}^2/\mathbb{Z}_2$ are obtained either via resolution (blow-up) or deformation. For the full $\mathcal{M}_0$, one needs to smooth the fibers consistently with the $\mathbb{K}$-action, i.e., the $\mathbb{K}$-action must lift to an action on the smooth fibers that is asymptotic to the original action. With these constraints, there are three families of smoothings: one family of resolutions, given by:

$$Y_1 := \mathbb{C}^2/\mathbb{Z}_2 \times_{\mathbb{K}} \mathcal{G}_6$$

and two families of deformations, given by

$$Y_2 := \{(z_1, z_2, z_3, \epsilon); z_1^2 + z_2^2 + z_3^2 = \epsilon\} \subset \mathbb{C}^3 \times \mathbb{R}^+$$
$$Y_3 := \{(z_1, z_2, z_3, \epsilon); z_1^2 + z_2^2 + z_3^2 = -\epsilon\} \subset \mathbb{C}^3 \times \mathbb{R}^+ \quad (6.6)$$

The resolved family is parametrized by the volume of the blown-up $\mathbb{P}^1$, and the two deformation families $Y_2, Y_3$ are parametrized by the volumes of the totally real $S^2 \subset Y_2$ and the totally imaginary $S^2 \subset Y_3$, respectively. The intuition here is that (say, in the resolved case) once we have resolved one fiber $\mathbb{C}^2/\mathbb{Z}_2$ we are free to choose a Kähler class in $\mathfrak{u}(1)$ up to scaling, and once that is chosen flatness of the vertical section $\omega$ fixes the volume in all other fibers.

Therefore the moduli space of smooth $G_2$-structures is:

$$\mathcal{M}_{G_2} = Y_{\mathbb{R}} := \mathbb{R}_{x_1} \cup \mathbb{R}_{x_2} \cup \mathbb{R}_{x_3}$$

i.e., it consists of three copies of $\mathbb{R}^+$ touching at the origin. We recall that the complexified space $\mathcal{M}_{G_2}^\mathbb{C}$ is, away from the discriminant locus, a Kähler space admitting a Lagrangian torus fibration over $\mathcal{M}_{G_2}$. The torus fibers parametrize holonomies of the “$C$-field”, i.e., elements in $H^3(M, U(1))$. That is, $\mathcal{M}_{G_2}^\mathbb{C} = Y_{\mathbb{C}}$. We have proved:

**Theorem 6.1.** (M-theory/IIA duality): The moduli space of complexified $G_2$-structures on $M_0$ is isomorphic to the non-trivial component of the $SL(2, \mathbb{C})$ character variety of $\mathcal{G}_6$:

$$\mathcal{M}_{G_2}^\mathbb{C} \cong \text{Char}^0(\pi, SL(2, \mathbb{C}))$$

(6.8)
Notice that the character variety description by itself does not tell us anything about $G_2$-structures per se. However, from the construction of our deformation family, we must have:

$$\mathcal{M}_{G_2} = \Gamma_{\text{flat}}(\mathcal{G}_6, T^*\mathcal{G}_6 \otimes \mathfrak{u}(1)/\mathbb{Z}_2)$$  \hfill (6.9)

These parametrize $\mathbb{Z}_2$-equivalence classes of flat sections of $T^*Q$. Flat sections are those that are fixed by the monodromy group $\mathbb{K}$. Recall that the action of $\mathbb{K} = \langle \alpha, \beta \rangle$ on $T^*Q$ is given in local coordinates by:

$$\alpha(dx_1, dx_2, dx_3) = (-dx_1, -dx_2, dx_3)$$

$$\beta(dx_1, dx_2, dx_3) = (-dx_1, dx_2, -dx_3)$$

Because of this, the computation of the fixed set here is completely analogous to the one performed above for the character variety (compare formula 6.2), the only difference being that the variables are now real. Hence, the fixed set is given by $Y_{\mathbb{R}}$, in agreement with the result of Joyce.

In fact, the $G_2$-structures on the smoothings constructed by Joyce are not only closed but also torsion-free. Thus they have metrics with $G_2$-holonomy. The techniques developed in this work do not attack the torsion-free condition, but presumably the condition can be re-formulated as an analytic condition on the spectral covers. At least in the adiabatic limit proposed by Donaldson [Don16], the relevant condition is that the spectral cover must be stationary under mean curvature flow. In this restricted context of flat three-manifolds, the solutions are exactly the flat sections of $\mathcal{E}_W$. The importance of the adiabatic limit is that torsion-free solutions are supposed to be constructed as formal power series extending an adiabatic solution. So we propose that this could be reformulated in our context via appropriate perturbations of flat spectral covers.

*Remark 6.2.* We mentioned before that there are other connected components of $\text{Char}(\pi, SL(2, \mathbb{C}))$ given by three isolated points. Under duality, presumably these three points correspond to rigid $G_2$-structures on $M_0$, i.e. that admit no smoothings - or at least no smoothings preserving the structure of a coassociative fibration.

Now, the beauty of the character variety description is that it allows us to generalize the computation of the moduli space to higher rank groups. E.g., suppose now we would like to investigate $G_2$-structures on $\mathbb{C}^2/\mathbb{Z}_n \times D_8 \mathcal{G}_6$. Then we would need a generalization of 6.3 characterizing $\text{Im}(r) = \text{Fix}(\mathbb{K})$. We conjecture the following formula holds for $\text{Fix}(\mathbb{K})$:

$$\text{Fix}(\mathbb{K}) \cong \bigcup_{K, K' \leq \mathbb{K}} \left( \left( T^3/K \right)^{\Sigma_n} \cap \left( T^3/K' \right)^{\Sigma_n} \right)$$  \hfill (6.10)
where $\Sigma_n$ is the permutation group and the union is over distinct proper subgroups of $K$.

For $n = 2$ a simple calculation shows that the intersecting factors are perpendicular $\mathbb{C}^2$’s, resulting in an union of three $\mathbb{C}$’s as in formula 6.3. Hence, formula 6.10 is correct for $n = 2$. Computations by the author suggest that $\text{Char}^0(\pi, SL(3, \mathbb{C}))$ is also given by a trivalent vertex, with components not $\mathbb{C}$ but a more complicated scheme. In our forthcoming work [Bar] we will address this computation from the perspective of Hilbert scheme of points on the SYZ mirror of $X$.

7 Flat Spectral Correspondence

In this section we prove Theorem 7.6, a spectral correspondence for flat Higgs bundles that are either unramified or totally ramified (the meaning of these conditions will be explained shortly). Although the full correspondence was never used in this work, we present it here in order to give a more geometric exposition of the spectral covers introduced in previous sections. A second reason is that this correspondence will be used in [Bar].

We start with a basic example. Let $V$ be a complex vector space of dimension $n$ and $\phi \in \text{End}(V)$. If $\phi$ is diagonalizable, it can be reconstructed by giving its eigenvalues $\lambda_1, \ldots, \lambda_n$, the decomposition of $V$ into $\phi$-eigenlines $V = L_1 \oplus \ldots \oplus L_n$ and a matching map $m : L_i \mapsto \lambda_i$. We refer to $(\Lambda, \mathcal{L})$ as the spectral data associated to $(V, \phi)$.

Let $p_i(\phi)$ be the coefficient of $\lambda^{n-i}$ in the expansion of $\det(\lambda \mathbf{1} - \phi) \in \mathbb{C}[t]$. Consider the map:

\[
\begin{align*}
  f : \text{End}(V) & \longrightarrow \mathbb{C}^n \\
  \phi & \longmapsto (p_1(\phi), \ldots, p_n(\phi))
\end{align*}
\]

Then it is clear that the eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ of $\phi$ depends only on $h(\phi)$. The map $f$ is a prototype of the Hitchin map 7.8 defined below.

Now, let $Q$ be a manifold, and $E \to Q$ a rank $n$ complex vector bundle. Suppose $\phi \in \Gamma(Q, \text{End}(E))$. Then to each $\phi_q : E_q \to E_q$ we can associate its spectral data $(\Lambda_q, \mathcal{L}_q)$. We think of $\phi$ as a “twisted family” of endomorphisms parametrized by $Q$.

As $(\Lambda_q, \mathcal{L}_q)$ varies over $Q$, it defines:

- a (possibly singular) subspace of $Q \times \mathbb{C}$:

\[
\tilde{Q} = \{(q, \lambda) ; \lambda \text{ is an eigenvalue of } \phi_q\} = \{(q, \lambda) ; \det(\lambda \mathbf{1}_{E_q} - \phi_q) = 0\}
\]
called the spectral cover of $Q$ associated to $\phi$. It comes equipped with a generically $n : 1$ covering map $\pi : \tilde{Q} \to Q$.

- If $\phi$ is generic - i.e., diagonalizable with distinct eigenvalues at every point - then $\pi$ is unramified.
- If $\phi$ has repeated eigenvalues, then its ramification locus is given by:

$$\Delta_\pi = \{ q \in Q | \phi_q \text{ has a multiple eigenvalue} \}$$

- A spectral line bundle:

$$m : \mathcal{L} \to \tilde{Q}$$

(7.3)

defined as follows: consider the matching maps $m_q : (L_q)_i \mapsto (\lambda_q)_i$. Then $\mathcal{L} = \sqcup_{q \in Q_i} (L_q)_i$ and $m|_{L_q} = m_q$.

Conversely, in nice cases (e.g. if $Q$ is an algebraic variety and $\phi$ is regular, i.e., has one Jordan block per eigenvalue), then given $(\tilde{Q}, \mathcal{L})$, Higgs data can be recovered by $E = \pi_* \mathcal{L}$ and $\phi = \pi_* \tau$, where the tautological section $\tau : \tilde{Q} \to \text{End}(\mathcal{L})$ is defined as $\tau(q, \lambda) = \lambda \mathbf{1}_\mathcal{L}$.

**Remark 7.1.**

1. If $\phi$ is irregular - i.e., has multiple Jordan blocks per eigenvalue - then the pushforward of the spectral line bundle by $m$ will not recover $E$: one needs to consider a more general sheaf $\mathcal{L}' \to \tilde{Q}$ such that on the locus $\Delta_\phi \subseteq Q$ where $\phi_q$ is irregular, $\mathcal{L}'_{q, \lambda_q}$ jumps in rank due to the presence of multiple eigenlines for the same eigenvalue. Such a locus is codimension two in $Q$. In particular, when $Q$ is a 3-manifold, $\Delta_\phi$ is a graph in $Q$. We will have more to say about this in the next section.

2. If one is not working with algebraic spaces (as is our case in this paper), then it is important to be careful with the pushforwards. This is the main reason we will consider only unramified or totally ramified Higgs bundles. Fortunately, this will suffice for the applications in [Bar]. However, as we explain in the next section, partial ramification is expected to play a fundamental role in $G_2$-geometry.

This is the most raw form of the spectral correspondence. One can also consider more general notions of Higgs bundles: one can “twist” $\phi$ by requiring its coefficients to be valued in a sheaf of commutative groups $\mathcal{F}$ over $Q$, and also require $\phi$ to satisfy some constraint (e.g., being compatible with fixed geometric structures on $Q$ or $E$). In this situation, the spectral data must be supplemented with additional structure in order to reconstruct $(E, \phi)$. We will be interested in the case of flat Higgs bundles over a platycosm $Q$, i.e., $\mathcal{F}$ is the flat sheaf $\Omega^1_Q$ and we impose equations 1.1 together with $[\theta \wedge \theta] = 0$. 

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**Definition 7.2.** Let \((E, h, \nabla, \theta)\) be a flat Higgs bundle over \(Q\). The *Spectral Cover* associated to \(\theta\) is the space \(S_{\theta} \subset T^*Q\) defined by:

\[
S_{\theta} = \{(q, s_q); \det(s_q \otimes 1_{E_q} - \theta_q) = 0\}
\]  

(7.4)

**Definition 7.3.** *Flat Spectral Data - unramified case:* Let \((E, h, A, \theta)\) be a rank \(n\) flat Higgs bundle over a compact flat 3-manifold \((Q, \delta)\). Assume \(\theta\) is generic. We define *flat spectral data* to be:

1. A \(n\)-sheeted, unramified covering map \(\pi : S_{\theta} \to Q\) given by the characteristic polynomial of \(\theta\).
2. A line bundle \(\mathcal{L} \to S_{\theta}\) determined by the eigenlines of \(\theta_q\)
3. A hermitian metric \(\tilde{h}\) on \(\mathcal{L}\) determined by \(h\)
4. A hermitian flat connection \(\tilde{A}\) on \(\mathcal{L}\) determined by \(A\)
5. A Lagrangian embedding \(\ell : S_{\theta} \to T^*Q\) satisfying \(\text{Im}(d\ell) \subset H_{\delta}\).

**Remark 7.4.** The last condition admits a second interpretation: we view \(\ell\) as a Lagrangian section of the pull-back bundle \(\pi^*T^*Q \to S_{\theta}\) and take its covariant derivative with respect to the pullback \(\pi^*\delta\) of the flat connection \(\delta\) on \(T^*Q \to Q\). Then the condition is that \(\nabla_{\pi^*\delta} \ell = 0\). If one is interested in non-flat \(Q\), the condition \(\text{Im}(d\ell) \subset H_{\delta}\) is simply dropped. As we will see below, the flat spectral correspondence works for any compact Riemannian manifold \(Q\).

**Definition 7.5.** *Flat Spectral Data - totally ramified case:* With the same notation as before, suppose \(\theta\) is central - i.e., diagonalizable with all eigenvalues equal. Then its flat spectral data is as before, except that \(\mathcal{L}\) is replaced by a rank \(n\) complex vector bundle \(\mathcal{E} \to S_{\theta}\). Moreover, note that \(\pi\) is now totally ramified.

We now come to the main result of this section. Let \(\text{FlatHiggs}\) be the set of flat Higgs bundles \((E, h, A, \theta)\) over a compact Riemannian \(Q\) and \(\text{FlatSpec}\) the set of flat spectral data \((\pi, \mathcal{L}, \tilde{h}, \tilde{A}, \ell)\) on \(Q\).

**Theorem 7.6.** (Spectral Correspondence for flat Higgs bundles) *There is a bijection:*

\[
\text{FlatHiggs} \longleftrightarrow \text{FlatSpec}
\]  

(7.5)

where flat Higgs bundles are taken to be either unramified or totally ramified, and the spectral data is chosen appropriately for each case.
Proof. Given a flat Higgs bundle $(E, h, A, \theta)$, we already know how to construct $\pi : S_\theta \to Q$, $\ell : S_\theta \to T^*Q$ and $\mathcal{L} \to S_\theta$. The metric and flat connection are just given by pullback: $h := \pi^*h$ and $\tilde{A} := \pi^*A$, so the compatibility condition $\nabla_{\tilde{A}} h = 0$ is preserved.

Now, use the hamonicity condition on $h : \tilde{Q} \to G/K$ to identify $\theta = dh$. The condition $\nabla_A \theta = d\theta + A \wedge \theta = 0$ can be written as equations for the $r$ components of $\theta$ under the identification, where $r = \text{rank}(G)$. Since $\nabla_A^2 = 0$, we can locally gauge away $A$, so that the equations become $d\theta = 0$. Let $(x_i, y_i)$ be coordinates in $T^*Q$ such that $\omega = \sum dy_i \wedge dx_i$ and dualize $\theta = \sum \theta_i(x) dx_i$ via the semi-flat metric on $T^*Q$ to obtain $\theta' = \sum \theta_i(x) dy_i$. The embedding given by $\ell(q, s_q) = \det(s_q \otimes 1_E - \theta_q)$ is Lagrangian if and only if:

$$\omega|_{\ell(q)} = \sum_{i=1}^3 d\theta' \wedge dx_i = \sum_{i=1}^3 d(\theta_i dx_i) = d\theta = 0 \quad (7.6)$$

Conversely, given spectral data $(\pi, \mathcal{L}, \tilde{h}, \tilde{A}, \ell)$, one constructs $E$ and $\theta$ as usual, and $h = \pi^*\tilde{h}$, $A = \pi^*\tilde{A}$ are well-defined because $\pi$ is a local isometry. The spectral data also guarantees that the components of $\theta$ are simultaneously diagonalizable, hence $[\theta \wedge \theta] = 0$. The conditions $\nabla_A^2 = 0$ and $\nabla_A h = 0$ follow from the same conditions for $(\tilde{h}, \tilde{A})$. Finally, the condition $\nabla_A \theta = 0$ is obtained simply by reversing the above argument for the section $\ell$ to be Lagrangian.

Remark 7.7. One should expect that the bijection 7.6 extends to an equivalence of categories under an appropriate categorification of the above data.

Definition 7.8. The Hitchin map is defined by:

$$\mathcal{H} : \text{FlatHiggs} \to \bigoplus_{i=1}^n \Gamma_{\text{flat}}(Q, (T^*Q)^{\otimes i})$$

$$(E, h, A, \theta) \mapsto (p_1(\theta), \ldots, p_n(\theta))$$

In [Bar] we will study the Hitchin map in a specific example related to the mirror $X^\vee$ of $X = T^*G_6$. Our main proposal in that paper is that $\mathcal{H}$ is essentially a smooth model of the SYZ fibration of $X^\vee$.

For now, we observe two differences from the holomorphic category: one, $\mathcal{H}$ is not surjective in general; and two, even when restricted to its image, $\mathcal{H}$ will not define an integrable system structure, since character varieties of compact three-manifolds do not come equipped with symplectic structures.
8 Associatives, Nilpotent Higgs Fields and Isolated Singularities

Our main point in this work is that the spectral cover approach provides a conceptual framework to study the deformation theory of coassociative fibrations. In line with this idea, we will now suggest that two seemingly different proposals [Don19], [JK17] for constructing isolated (i.e. codimension-seven) singularities on $G_2$-manifolds both fall under the umbrella of nilpotent Higgs fields.

We will start by explaining the relation between our construction and the work of Joyce and Karigiannis [JK17]. Their paper shows how to construct integrable $G_2$-structures on a $G_2$-orbifold of type $A_1$. The essential feature in their argument is the presence of a closed and co-closed one-form on the singular stratum $Q$. This is consistent with our construction, as in this case the spectral cover takes values in $T\mathring{Q}$ since $\mathfrak{su}_2 \cong \mathfrak{u}(1) \cong \mathbb{R}$. In fact, their situation corresponds to a symmetric solution of 1.1 with $G = SU(2)$, i.e., one satisfying $F_A = 0$ and $[\theta \wedge \theta] = 0$. Recall those are exactly the solutions that correspond to stable complex flat connections.

Now, three commuting elements in $\mathfrak{su}(2)$ can be simultaneously conjugated to $u(1)$, so the Higgs field itself takes values in $T\mathring{Q}$. The closed and co-closed conditions are just the two last equations in 1.1.

Furthermore, the 1-form in [JK17] is required to be non-vanishing. It was suggested in that paper that a vanishing point should lead to an isolated conical singularity. In our language, vanishing means that there is a point in $Q$ where all the sheets of the spectral cover meet the zero section. In other words, the Higgs field is nilpotent at that point.

Now, to relate with the proposal in [Don19], we recall that the spectral cover is described by the flat section $s$ (or equivalently, a flat multi-section of $T^*Q$), which is related to the positive branched section $h_s$ encoding the $G_2$-structure of the total space. The crucial point is equation 5.23 above, which says that locally, the sheets of $s$ in $T^*Q$ are identified with the graph of $dh_s$. In particular, the spectral cover is Lagrangian in $T^*Q$. Now, recall that in the setup of [Don16] the ramifying locus of $h_s$ is a knot or link $K \subset Q$ over which the coassociative fibers acquire conical singularities. In our picture, $K$ is a subset where some, but not all sheets of the spectral cover meet the zero-section. According to [Don16] it is expected that, at least in the so-called adiabatic limit, given two different points $x, y \in K$ one could associate to $h_s$ a certain “matching gradient path” $\gamma$ connecting $x$ and $y$, such that the union of vanishing cycles over $\gamma$ is an associative $S^3$. Deforming $K$ along such a path would bring $x$ and $y$ together, creating a point-like singularity. Geometrically, this picture describes a “collision” of two singular Kovalev-Lefschetz fibers producing an isolated singularity via a “blow-down” of the associative $S^3$.

The interpretation in the spectral cover picture is the following: a matching gradient path between $x$ and $y$ exists if and only if all sheets of the spectral cover intersect the zero-section in pairs at either $x$ or $y$. Here, the pairing condition
guarantees that vanishing cycles match bijectively between neighborhoods of \(x\) and \(y\); and the condition that all sheets appear comes from the fact that a gradient path connects two solutions of \(dh_s = 0\). Notice that any fixed sheet only needs to vanish at one of the points, since \(h_s \circ \gamma\) can interpolate between components midway along \(\gamma\) and thus match vanishing cycles from different sheets. However, when we bring the two points together, all sheets come together in the limit point and thus we get a nilpotent Higgs field.

Although most of this discussion is non-rigorous, our main point here is that this establishes a conceptual relation between the constructions in [JK17] and [Don16], thus answering question (8.v) in the first paper. We should mention also that generically one would expect any sheet of the spectral cover to intersect the zero-section at finitely many points, so isolated singularities arising from collision of links seem to be rather special from this perspective.

In [BCH+18] we provided an explicit local construction of a nilpotent Higgs field arising from collision of links (see equations (3.44) and (3.45) in that paper). The example is in \(\mathbb{R}^3\) with gauge group \(SU(3)\). The spectral cover ramifies over the x-axis and y-axis and vanishes at the origin, hence the Higgs field is nilpotent at that point. In the \(G_2\)-picture, this is expected to describe a collision of two \(A_1\)-singularities.

Another interesting feature of the spectral cover interpretation is a possible Floer-theoretic description of associative spheres. Assuming in the adiabatic limit all associatives can be obtained through matchings of vanishing cycles as outlined above, one could attempt to count them by looking at the differential in the Morse-Novikov cohomology generated by the critical points. From the spectral cover perspective, one looks at the Lagrangian Floer homology \(HF^*(Q, s(Q))\), i.e. one should count holomorphic disks bounded by the spectral cover and the zero section. One would expect that these two counts agree. The equivalence between these two cohomologies is not new and appears already in the work of Pantev and Wijnholt [PW11].

9 Future Directions

1. **\(G_2\)-metrics:** The most natural extension of this work would be to identify which fibers of the deformation family have integrable \(G_2\)-structures. The proposal of Donaldson [Don16] to attack the torsion-free condition consists in solving the adiabatic limit and then perturbing the solution via a formal power series. The adiabatic solution consists of a spectral cover stationary under mean curvature flow. It would be interesting to understand what the perturbations mean geometrically.

2. **Higgs Bundles and Branched Covers:** A theorem of Hilden-Montesinos [Mon74] says that every closed orientable 3-manifold is a 3-fold branched
cover over $S^3$ with branched set over some knot $K$. It would be interesting to understand the behavior of flat Higgs fields under pullbacks/pushforwards by such covering maps. Once ramified spectral data is well-understood in this context, this could be a potential source of several interesting examples.

Related to this, consider the following: it is known that $G_2$ is a twofold cover of $S^3$ branched over the Borromean rings $\mathcal{R}$. Is there a spectral cover profile that coincides with this covering? The author believes this question is related to possible topological transitions between $G_2$-spaces [AV01]. Essentially, the idea is that one could deform $\mathcal{R}$ into a wedge of three circles. At the center, the Higgs field becomes nilpotent, corresponding to an isolated conical singularity. We then deform again, but now the three unknots are fully linked. The new spectral cover is the homology sphere $S^3/Q_8$, where $Q_8$ is the quaternion group. The spectral cover has transitioned from a flat to a spherical geometry. We will provide further evidence for this in [Bar].

3. Derived Integrable Systems? We started this work by claiming our main construction is a $G_2$ analogue of the works of Szendrői [Sze04] and Diaconescu-Donagi-Pantev [DDP06] on families of ADE-fibered Calabi-Yau threefolds. However, the result of [DDP06] is much stronger: there is an isomorphism of integrable systems between the Jacobian fibration of the Calabi-Yau family and the dual Hitchin fibration. In our setup, the same statement is hopeless, since character varieties of three-manifolds are not symplectic. However, according to [PTVV11], they admit $(-1)$-shifted symplectic structures, a generalization of symplectic structures in the context of derived geometry. Moreover, in the case of compact $G_2$-manifolds, Karigiannis and Leung [KL07] prove that $\mathcal{M}_{G_2}$ is Kähler. One is then led to ponder if there is a statement in derived geometry that would relate these two structures, effectively completing our analogy.

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