Chiral Nucleon-Nucleus Potentials at N^3LO

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Abstract. Elastic scattering is probably one of the most relevant tools to study nuclear interactions. In this contribution we study the domain of applicability of microscopic two-body chiral potentials in the construction of an optical potential. A microscopic complex optical potential is derived and tested performing calculations on ^16O at different energies. Good agreement with empirical data is obtained if a Lippmann-Schwinger cutoff at relatively high energies (above 500 MeV) is employed.

1. Introduction

Elastic proton scattering has been extensively studied over many decades and there now exist extensive measurements of cross sections and polarization observables for the elastic scattering of protons on a wide variety of stable nuclei. A successful framework to describe the nucleon-nucleus (NA) interaction in the elastic scattering is provided by the nuclear optical potential. With this method we can compute the scattering observables (the differential cross section, the analyzing power, and the spin rotation) for the elastic NA scattering across wide regions of the nuclear landscape.

The optical potential can be obtained microscopically. The calculation requires, in principle, the solution of the full nuclear many-body problem. In practice, with suitable approximations, microscopic optical potentials are usually derived from two basic quantities: the nucleon-nucleon (NN) t matrix and the matter distribution of the nucleus.

The theoretical framework employed in our work started almost 60 years ago with Ref. [1] where the authors developed the Watson multiple scattering approach in which the NA optical potential is expressed by a series expansion in terms of the free NN scattering amplitudes.

The NN potential is an essential ingredient in the NA scattering theory and its off-shell properties play an important role. To obtain a good description of these properties, the optical potential models have always employed “realistic” potentials, in which the experimental NN phase shifts are reproduced with a $\chi^2$ per data $\approx 1$. Recent available chiral potentials are developed at fourth order (N^3LO) in the chiral expansion parameter (i.e. external momentum over the chiral symmetry breaking scale, $\Lambda_{\chi} \approx 1$ GeV) and are used here and in Ref. [2] as a basic ingredient to compute the NN t matrix. In particular, in all the calculations presented in this paper we adopt the two different versions of chiral potentials developed by Entem and
Machleidt (EM, Ref. [3]), and Epelbaum, Glöckle, and Meißen (EGM, Ref. [4]). A future extension with the inclusion of the most recent potentials at \( N^4 \text{LO} \) is in progress.

The second important ingredient of the \( NA \) scattering theory is the microscopic structure of the nuclear target, given by neutron and proton densities. These quantities are computed within the Relativistic Mean-Field (RMF) description [5] of spherical nuclei using a Density-Dependent Meson-Exchange (DDME) model, where the couplings between mesonic and baryonic fields are assumed as functions of the density itself [6].

2. Theoretical Framework

The general problem of the elastic scattering of a proton from a target nucleus of \( A \) nucleons can be stated in momentum space by the full \((A+1)\)-body Lippmann-Schwinger (LS) equation

\[
T = V + V G_0(E)T. \tag{1}
\]

A reliable method to treat Eq. (1) is given by the spectator expansion [7] in which the multiple scattering theory is expanded in a finite series of terms where the target nucleons interact directly with the incident proton. In the standard approach to elastic scattering, Eq. (1) is separated into two equations. The first one is an integral equation for \( T \)

\[
T = U + U G_0(E) P T, \tag{2}
\]

where \( U \) is the optical potential operator, and the second one is an integral equation for \( U \)

\[
U = V + V G_0(E) Q U. \tag{3}
\]

The operator \( V \) represents the external interaction and the total Hamiltonian for the \((A+1)\)-nucleon system is given by

\[
H_{A+1} = H_0 + V. \tag{4}
\]

Of course, if we assume the presence of only two-body forces, the operator \( V \) is expressed as \( V = \sum_{i=1}^{A} v_{0i} \), where the two-body potential \( v_{0i} \) describes the interaction between the incident proton and the \( i \)th target nucleon. \( G_0(E) \) is the free propagator for the \((A+1)\)-nucleon system. The operators \( P \) projects onto the elastic channel and, as a consequence, \( P \) and \( Q \) satisfy the following relation

\[
P + Q = 1, \tag{5}
\]

and \( P \) fulfills the condition

\[
[G_0, P] = 0. \tag{6}
\]

With these definitions, the elastic transition operator may be defined as \( T_{el} = P T P \). At the first order, the elastic scattering amplitude is given by

\[
T_{el}(k', k; E) = \frac{A}{A-1} \hat{T}(k', k; E), \tag{7}
\]

where the auxiliary elastic amplitude \( \hat{T} \) is determined by the solution of the integral equation

\[
\hat{T}(k', k; E) = \hat{U}(k', k; \omega) + \int d^3p \frac{\hat{U}(k', p; \omega) \hat{T}(p, k; E)}{E(k_0) - E(p) + i\epsilon}, \tag{8}
\]

and the auxiliary first-order optical potential is defined by

\[
\hat{U}(k', k; \omega) = (A - 1) \langle k', \Phi_A | t(\omega) | k, \Phi_A \rangle. \tag{9}
\]
where \( \kappa' \) and \( \kappa \) are relative momenta.

The first terms of Eqs. (11) and (12) correspond to the central spin-independent contributions and the second terms correspond to the spin-orbit contributions. Within this approximation, the optical potential exhibits nonlocality and off-shell effects through the dependence of \( t_{pN} \) upon \( K \). The energy \( \omega \) at which the matrices \( t_{pN}^c \) and \( t_{pN}^{ls} \) are evaluated is fixed as

\[
\omega = \frac{T_{\text{lab}}}{2} = \frac{1}{2} \frac{k_{\text{lab}}^2}{2m},
\]

where \( k_{\text{lab}} \) is the on-shell momentum of the projectile in the laboratory system.

From the conservation of the total angular momentum and parity, this spin operator can be expanded as

\[
\hat{U}(k', k; \omega) = \frac{2}{\pi} \sum_{JLM} y_{J,M}^{L+\frac{1}{2}}(\hat{k}') \hat{U}_{LJ}(k', k; \omega) y_{J,M}^{L+\frac{1}{2}}(\hat{k}),
\]

where \( J = L \pm \frac{1}{2} \) and \( y_{J,M}^{L+\frac{1}{2}} \) is the standard spin-angular function. Inserting the expansion of Eq. (14) into Eq. (8) we obtain the same decomposition for the \( T \) matrix

\[
\hat{T}(k', k; E) = \frac{2}{\pi} \sum_{JLM} y_{J,M}^{L+\frac{1}{2}}(\hat{k}') \hat{T}_{LJ}(k', k; E) y_{J,M}^{L+\frac{1}{2}}(\hat{k}),
\]

where the partial-wave components of the transition operator for the elastic scattering are given by

\[
\hat{T}_{LJ}(k', k; E) = \hat{U}_{LJ}(k', k; \omega) + \frac{2}{\pi} \int_0^\infty dp p^2 \frac{\hat{U}_{LJ}(k', p; \omega) \hat{T}_{LJ}(p, k; E)}{E(k_0) - E(p) + \iota \epsilon},
\]

where

\[
E(k_0) = \sqrt{k_0^2 + m_{\text{proj}}^2} + \sqrt{k_0^2 + m_{\text{targ}}^2},
\]

\[
E(p) = \sqrt{p^2 + m_{\text{proj}}^2} + \sqrt{p^2 + m_{\text{targ}}^2},
\]

and \( m_{\text{proj}} \) and \( m_{\text{targ}} \) are the masses of the projectile and of the target, respectively. In terms of the partial wave components of the quantities \( \hat{U}^c(k', k; \omega) \) and \( \hat{U}^{ls}(k', k; \omega) \), we have

\[
\hat{U}_{LJ}(k', k; \omega) = \hat{U}^c_{LJ}(k', k; \omega) + C_{LJ} \hat{V}^{ls}_{LJ}(k', k; \omega),
\]

where \( C_{LJ} \) is a coupling constant.
where

\[ C_{LJ} = \frac{1}{2} \left[ J(J+1) - L(L+1) - \frac{3}{4} \right], \]

\[ \hat{V}_{L}^{ls}(k', k; \omega) = \frac{k'k}{2L+1} \left[ \hat{U}_{L+1}^{ls}(k', k; \omega) - \hat{U}_{L-1}^{ls}(k', k; \omega) \right]. \] (20)

To obtain these results, the quantities \( \hat{U}^{c}(k', k; \omega) \) and \( \hat{U}^{ls}(k', k; \omega) \) are expanded in a manner similar to Eq. (14), with the difference that the partial wave components, \( \hat{U}^{c}_{L} \) and \( \hat{U}^{ls}_{L} \), are independent of \( J \).

### 2.1. Observables

Under the assumptions of parity conservation and rotational invariance, the most general form of the full amplitude for the elastic proton scattering from a spin 0 nucleus is given by

\[ M(k_0, \theta) = A(k_0, \theta) + \sigma \cdot \hat{N} C(k_0, \theta), \] (21)

where \( \hat{N} = (k' \times k)/(|k'||k|) \) and the amplitudes \( A(k_0, \theta) \) and \( C(k_0, \theta) \) are obtained from the partial wave solutions of Eq. (16) as

\[ A(\theta) = \frac{1}{2\pi^2} \sum_{L=0}^{\infty} \left[ (L+1)F^{+}_{L}(k_0) + L F^{-}_{L}(k_0) \right] P_{L}(\cos \theta), \] (22)

\[ C(\theta) = \frac{i}{2\pi^2} \sum_{L=1}^{\infty} \left[ F^{+}_{L}(k_0) - F^{-}_{L}(k_0) \right] P_{L}^{1}(\cos \theta). \] (23)

In Eqs. (22) and (23) an implicit dependence on \( k_0 \) is assumed. The functions \( F^{\pm}_{L} \) denote \( F_{LJ} \) for \( J = L \pm 1/2 \), respectively, and are given in terms of the \( t \) matrix

\[ F_{LJ}(k_0) = -\frac{A}{A-1} 4\pi^2 \mu(k_0) \hat{T}_{LJ}(k_0, k_0; E), \] (24)

where the relativistic reduced mass is

\[ \mu(k_0) = \frac{E_{\text{proj}}(k_0) E_{\text{targ}}(k_0)}{E_{\text{proj}}(k_0) + E_{\text{targ}}(k_0)}. \] (25)

Three independent scattering observable will be considered: the unpolarized differential cross section, the analyzing power \( A_y \), and the spin rotation \( Q \). Their expressions as functions of the the amplitudes \( A \) and \( C \) are:

\[ \frac{d\sigma}{d\Omega}(\theta) = |A(\theta)|^2 + |C(\theta)|^2, \] (26)

\[ A_y(\theta) = \frac{2 \Re[A(\theta) C^{*}(\theta)]}{|A(\theta)|^2 + |C(\theta)|^2}, \] (27)

\[ Q(\theta) = \frac{2 \Im[A(\theta) C^{*}(\theta)]}{|A(\theta)|^2 + |C(\theta)|^2}. \] (28)

The Coulomb interaction between the incoming proton, with charge \( e \), and the spin 0 target, with charge \( Z e \), has been included following the algorithm outlined in Refs. [11, 12].
part of the EM potentials [3] with a LS cutoff ranging between 450 and 600 MeV. The experimental data entering the LS equation is multiplied by a regulator function that has unphysically strong attraction. As a usual procedure, the nucleon-nucleon potential both cases the goal is to cut out the short-range part of the two-pion exchange (2PE) contribution. In terms in the two-pion exchange (2PE) contributions with dimensional regularization (DR), while Machleidt (EM), who first presented a chiral potential at the fourth order, treat divergent angle \( \phi \) followed by the value of the LS cutoff (\( \Lambda \)). In the EGM case, for \( \Lambda = 450 \) and 600 MeV we

In Fig. 1 the theoretical results for the real and imaginary parts of \( pp \) and \( pn \) amplitudes \( a \) and \( c \) computed at an energy of 100 MeV are shown as functions of the center-of-mass \( NN \) angle \( \phi \) and compared with the experimental data. The calculations are performed using the EM potentials [3] with a LS cutoff ranging between 450 and 600 MeV. The experimental data are globally reproduced by the three potentials, with the only remarkable exception of the real part of the \( c_{pp} \) amplitude that is overestimated. It must be considered, however, that this is a

3. Results

3.1. \( NN \) amplitudes

In this section we present and discuss the theoretical results for the \( pp \) and \( pn \) components which are used to compute the central and the spin-orbit contributions of the \( NN \) \( t \) matrix. The \( NN \) elastic scattering amplitude for the scattering from a relative momentum \( \kappa \) to \( \kappa' \) is denoted by \( M(\kappa', \kappa, \omega) \) and defined as follows

\[
M(\kappa', \kappa, \omega) = \langle \kappa' | M(\omega) | \kappa \rangle = -4\pi^2 \mu \langle \kappa' | t(\omega) | \kappa \rangle ,
\]

where \( \mu \) is the \( NN \) reduced mass. The most general form of this amplitude, consistent with invariance under rotation, time reversal, and parity is

\[
M = a + c(\sigma_1 + \sigma_2) \cdot \hat{n} + m(\sigma_1 \cdot \hat{n})(\sigma_2 \cdot \hat{n}) + (g + h)(\sigma_1 \cdot \hat{I})(\sigma_2 \cdot \hat{I}) + (g - h)(\sigma_1 \cdot \hat{m})(\sigma_2 \cdot \hat{m}) ,
\]

where

\[
\hat{l} = \frac{\kappa' + \kappa}{|\kappa' + \kappa|}, \quad \hat{m} = \frac{\kappa' - \kappa}{|\kappa' - \kappa|}, \quad \hat{n} = \frac{\kappa \times \kappa'}{|\kappa \times \kappa'|} ,
\]

are the unit vectors defined by the \( NN \) scattering plane. The amplitudes \( a, c, m, g, \) and \( h \) can be expressed as complex functions of \( \omega, \kappa, \) and \( \kappa' \). From these equations one can obtain the explicit expressions for the \( pp \) and \( pn \) central and spin-orbit parts of the \( NN \) \( t \) matrix.

Calculations are performed using two different versions of the chiral potential at fourth order (\( N^3\)LO) based on the works of Entem and Machleidt [3] and Epelbaum et al. [4]. Entem and Machleidt (EM), who first presented a chiral potential at the fourth order, treat divergent terms in the two-pion exchange (2PE) contributions with dimensional regularization (DR), while Epelbaum, Glückle, and Meißner (EGM) employ a spectral function regularization (SFR). In both cases the goal is to cut out the short-range part of the two pion exchange (2PE) contribution that has unphysically strong attraction. As a usual procedure, the nucleon-nucleon potential entering the LS equation is multiplied by a regulator function \( f^\Lambda \)

\[
V(k, k') \rightarrow V(k, k') f^\Lambda(k, k')
\]

where

\[
f^\Lambda = \exp \left( - (k'/\Lambda)^{2n} - (k/\Lambda)^{2n} \right) \quad \text{with} \quad n = 2, 3 .
\]

While Entem and Machleidt present results for three choices of the cutoff necessary to regulate the high-momentum components in the LS equation (\( \Lambda = 450, 500, \) and 600 MeV), Epelbaum et al. [4] allow also to study variations of the cutoff \( \tilde{\Lambda} \) that regulates the 2PE contribution. In fact, in the latter approach one can choose between the following cutoff combinations:

\[
\{\Lambda, \tilde{\Lambda}\} = \{450, 500\}, \{450, 700\}, \{550, 600\}, \{600, 600\}, \{600, 700\} .
\]

In the following figures all the results are labelled by an acronym (to distinguish the authors) followed by the value of the LS cutoff (\( \Lambda \)). In the EGM case, for \( \Lambda = 450 \) and 600 MeV we plot bands that show how calculations can change respect to variations of the SFR cutoff \( \tilde{\Lambda} \). In Fig. 1 the theoretical results for the real and imaginary parts of \( pp \) and \( pn \) amplitudes \( a \) and \( c \) computed at an energy of 100 MeV are shown as functions of the center-of-mass \( NN \) angle \( \phi \) and compared with the experimental data. The calculations are performed using the EM potentials [3] with a LS cutoff ranging between 450 and 600 MeV. The experimental data are globally reproduced by the three potentials, with the only remarkable exception of the real part of the \( c_{pp} \) amplitude that is overestimated. It must be considered, however, that this is a
small quantity, i.e. two orders of magnitude smaller than the respective imaginary part, and it will only provide a very small contribution to the optical potential.

In Fig. 2 we show the results obtained at 100 MeV with the EGM potentials [4]. Also in this case, all three potentials are in overall good agreement with the experimental data with the only remarkable exception of the real part of the $c_{pp}$ amplitude. In particular, they show very similar results and in many cases the yellow and turquoise bands are overlapped. Their trends are also very close to the ones shown in Fig. 1 for the EM potential and they display the same discrepancy in comparison with the experimental data for $c_{pn}$ and around the peak of the imaginary part of $c_{pp}$.

Since Chiral Perturbation Theory (ChPT) is a low-momentum expansion of QCD, we expect that, as the energy is increased, larger discrepancies appear respect to empirical data. In Figs. 3 and 4 we present the results corresponding to the ones shown in Figs. 1 and 2 but at an energy of 200 MeV. As energy is increased, all potentials are still unable to reproduce the experimental data of the real part of the $c_{pp}$ amplitude, but most of the chiral potentials give satisfactory results, in agreement with the data for all the other amplitudes, with one notable exception, i.e. cutoffs below 500 MeV. Based on these results, one can predict an unsatisfactory result for a nucleon-nucleus optical potential if EM-450 or EGM-450 are employed at energies well above 100 MeV.

**Figure 1.** Real (left panel) and imaginary (right panel) parts of $pp$ and $pn$ Wolfenstein amplitudes ($a$ and $c$) as functions of the center-of-mass $NN$ angle $\phi$. All the amplitudes are computed at 100 MeV using the EM potentials [3] with a LS cutoff ranging between 450 and 600 MeV. Data (black squares) are taken from Ref. [8].

**Figure 2.** The same as in Fig. 1 using EGM potentials [4] with a LS cutoff ranging between 450 and 600 MeV. In two cases ($\Lambda = 450$ and 600 MeV) we show uncertainty bands produced by changing $\tilde{\Lambda}$ according to Eq. (34). Data (black squares) are taken from Ref. [8].

### 3.2. Proton elastic scattering on $^{16}\text{O}$

In this section we present and discuss our numerical results for the $NA$ elastic scattering observables calculated with the microscopic optical potential previously obtained. As a study case in our calculations we consider elastic proton scattering on $^{16}\text{O}$.

In order to investigate and emphasize the differences between the different $NN$ potentials and also on the basis of the results obtained for the $NN$ amplitudes $a$ and $c$, the scattering
observables have been calculated for two energies (100 and 200 MeV) for which experimental data are available.

With these calculations we intend to achieve the following goals: 1) to check the agreement of our theoretical predictions with the empirical data; 2) to study the limits of applicability of chiral potentials in terms of the proton energy; 3) to identify the best set of values for the LS and, eventually, SFR cutoffs.

In Figs. 5 and 6 we show the differential cross section \((d\sigma/d\Omega)\), the analyzing power \(A_y\), and the spin rotation \(Q\) for elastic proton scattering on \(^{16}\)O as functions of the center-of-mass scattering angle \(\theta\) with the above mentioned energies. In the left panels we show the results obtained with the EM potentials [3] while in the right panels we show the results obtained with the EGM potentials [4]. All potentials are denoted by the value of the LS cutoff.

In Fig. 5, at 100 MeV, all sets of potentials, regardless of cutoffs and theoretical approaches, give very similar results for all three observables, with the exception of \(A_y\) above 50 degrees, where all potentials overestimate the experimental data up to the maximum and then display an unrealistic downward trend, and \(Q\) around the maximum at 30 degrees. In particular, the experimental cross section is well reproduced by all potentials in the minimum region, between 30 and 35 degrees. Even if differences are rather small, potentials with the largest cutoff (\(\Lambda = 600\) MeV) seem to provide the best description of \(A_y\).

In Fig. 6 we plot the results obtained at 200 MeV. At this energy, it is clear that potentials obtained with the lower cutoffs (EM-450 and EGM-450) cannot be employed any further: in both cases, the differential cross sections are not satisfactorily reproduced and the behaviour of \(A_y\) and \(Q\) as a function of \(\theta\) is in clear disagreement with the empirical one. On the other hand, the remaining sets of potentials well describe the experimental cross sections and the analyzing power \(A_y\), that is reasonably described not only for small scattering angles but also for values larger than the minimum value up to about 45 degrees.

4. Conclusions

In this work we have obtained a new microscopic optical potential for elastic proton-nucleus scattering. Our optical potential has been derived as the first-order term within the spectator expansion of the nonrelativistic multiple scattering theory.

Microscopic optical potentials have been derived from two different versions of the chiral
potential at fourth order (N^3LO) based on the work of Entem and Machleidt (EM) [3] and Epelbaum, Glöckle, and Meißner (EGM) [4], which differ in the regularization scheme employed in the two-pion exchange term and in the choice of the cutoffs.

Results for pp and np amplitudes (a and c) obtained with different NN potentials have been presented and discussed. Since ChPT is a low-momentum expansion of QCD, the agreement of the chiral potential with the experimental data becomes, as expected, worse increasing the energy. While at 100 MeV all the NN potentials are able to reproduce the experimental amplitudes, with the only exception of the real part of c_{pp} amplitude, that is anyhow extremely small, at 200 MeV the set of potentials with lower cutoffs (450 MeV) fail to reproduce empirical data.

As case study for our investigation we have considered elastic proton scattering on ^{16}O. On the basis of all these results for ^{16}O we can draw two conclusions: 1) Potentials with lower cutoffs cannot reproduce experimental data at energies close to 200 MeV. 2) There is no appreciable difference in using 500 or 600 MeV as LS cutoffs, even if the EM-600 and EGM-600 potentials seem to have a slightly better agreement with empirical data, in particular looking at polarization observables.

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