ON LOCAL TIME AT TIME VARYING CURVE

ANASS BEN TALEB

Abstract. Let \((X_t)_{t \geq 0}\) be a continuous semimartingale. Let \(L_t^z(X)\) its family of local times. In [7] Yor showed that the family \((L_t^z(X))_{z \in \mathbb{R}, t \geq 0}\) has a version that is continuous in \(t\), cad-lag in \(z\). In this paper we extend this result by showing that for a 'regular' family of curves parametrized by \(z\), the corresponding family of local times has a 'regular' version. This result was used in [1] to prove a change of variable formula for continuous semimartingales.

1. Introduction

Let \((B_t)_{t \geq 0}\) a Brownian motion, we define for each \(t > 0\) the occupation time random measure on the space \((\mathbb{R}, \mathcal{B}(\mathbb{R})):\)

\[
\forall A \in \mathcal{B}(\mathbb{R}), \mu_t(A) = \int_0^t 1_{(B_s \in A)} ds
\]

In [6], Trotter shows that almost surely for all \(t > 0\), \(\mu_t\) has a density with respect to the Lebesgue measure: \(L_t^z(B)\) meaning almost surely \(\forall t > 0\) and \(\forall f: \mathbb{R} \to \mathbb{R}\) borelian function we have:

\[
\int_0^t f(B_s) ds = \int_{-\infty}^{+\infty} f(z)L_t^z(B) dz
\]

furthermore the family \(L_t^z(B)_{z \in \mathbb{R}, t \geq 0}\) is continuous in \(t\) and \(z\), we denote by \(L^z(B)\) the map \(t \mapsto L_t^z(B)\) and we call it the local time of \(B\) in \(z\), the family \((L_t^z(B))_{z \in \mathbb{R}, t \geq 0}\) is also called the family of local times of \(B\). The notion of local time was studied extensively in the litterature (see [5] for a complete bibliography on the subject), this theory was expanded in many directions: from the point of view of continuous semimartingales Meyer in [3] after earlier results of Tanaka for the Brownian motion defines the local time of continuous semimartingales via a generalization of Ito’s Formula: Let \((X_t)_{t \geq 0}\) be a continuous semimartingale then for all \(z \in \mathbb{R}\) there exists a process \((L_t^z(X))_{t \geq 0}\) continuous and increasing such that for all \(t \geq 0\) we have almost surely (Tanaka’s Formula):

\[
|X_t - z| = |X_0 - z| + \int_0^t \text{sgn}(X_s - z) dX_s + L_t^z(X)
\]

with \(\text{sgn}\) is the function defined by:

\[
\text{sgn}(x) = \begin{cases} 
1 & x > 0 \\
-1 & x \leq 0
\end{cases}
\]

In addition, the measure \(dL_t^z(X)\) is almost surely carried by the set \((s \geq |X_s = z|)\) (see proposition 1.6 of chapter 6 in [5]). We also have in analogy to (2) a formula
called the occupation times formula. Almost surely for all \( \Phi : \mathbb{R} \to \mathbb{R} \) borelian non negative function and for all \( t \geq 0 \):

\[
\int_0^t \Phi (X_s) \, d \langle X, X \rangle_s = \int_{-\infty}^{+\infty} \Phi(z) L_t^z (X) \, da
\]

In [7] Yor’s result extends Trotter’s theorem in the following way: there exists a version of \((L_t^z (X))_{t \geq 0, z \in \mathbb{R}}\) continuous in \( t \) and càdlàg in \( z \). Furthermore almost surely \( \forall t \geq 0, \forall z \in \mathbb{R} \):

\[
L_t^z (X) - L_t^{z^-} (X) = 2 \times \int_0^t 1_{\{X_s = z\}} \, dX_s
\]

In this paper we shall extend Yor’s result to a family of local times at time varying curves. First let us define the local time at a curve, this notion was mentioned briefly as a limit in probability of occupation times in [4]:

**Definition 1.1.** Let \((X_t)_{t \geq 0}\) a continuous semimartingale and \( \gamma : \mathbb{R}_+ \to \mathbb{R} \) a function with finite total variation, hence \( X - \gamma \) is a continuous semimartingale we set :

\[
\Lambda^\gamma (X) = L^0 (X - \gamma)
\]

Thus \( \Lambda^\gamma (X) \) is an increasing continuous process, its Lebesgue-Stieltjes measure \( d\Lambda^\gamma (X) \) is carried by the set \( \{ s \geq 0 | X_s = \gamma(s) \} \), we shall call this process the local time of \( X \) at the curve \( \gamma \) and we have almost surely :

\[
\forall t \geq 0, \Lambda_t^\gamma (X) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^t 1_{0 < X_s - \gamma(s) < \epsilon} \, d \langle X, X \rangle_s
\]

**Remark 1.1.** Let \( t_0 > 0 \), in the same manner we can define the local time of \( X \) at the curve \( \gamma \) with basis point \( t_0 \) by :

\[
\forall t \geq 0, \Lambda_t^{\gamma,t_0} (X) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{t_0}^t 1_{0 < X_s - \gamma(s) < \epsilon} \, d \langle X, X \rangle_s
\]

Note that for all \( u, t \in \mathbb{R}_+ \):

\[
\Lambda_t^{\gamma,t_0} (X) - \Lambda_u^{\gamma,t_0} (X) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_u^t 1_{0 < X_s - \gamma(s) < \epsilon} \, d \langle X, X \rangle_s
\]

So for all \( t, u \) \( \Lambda_t^{\gamma,t_0} (X) - \Lambda_u^{\gamma,t_0} (X) \) is independant of the choice of the basis point.

**Remark 1.2.** The previous remark enables us to define the local time of \( X \) at \( \gamma \) when \( \gamma \) is only defined on interval \( I \subset \mathbb{R}_+ \).

Our main result is the following theorem:

**Theorem 1.1.** Let \((X_t)_{t \geq 0}\) be a continuous semimartingale defined on filtered probability space \((\Omega, \mathcal{A}, \mathbb{P}, \mathcal{G} = (\mathcal{G}_u)_{u \geq 0})\), let \( \Gamma \in C^1 (\mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}) \). For \( T > 0 \) one

\[\text{Footnotes:}
\begin{enumerate}
\item French: “continue à droite limite à gauche”, which translates to continuous on the right limit on the left): meaning there exist a family of random variables \( \tilde{L}_t^z (X) \) such that for each \( z \in \mathbb{R} \) almost surely the map \( t \to \tilde{L}_t^z (X) \) is continuous, for each \( t \geq 0 \) almost surely the map \( z \to \tilde{L}_t^z (X) \) is càdlàg, and for each \( z \in \mathbb{R}, t \geq 0 \) almost surely we have \( \tilde{L}_t^z (X) = \hat{L}_t^z (X) \)
\item i.e there exists \( \epsilon > 0 \) such that \( \Gamma \in C^1 (\mathbb{R}_- | - \epsilon, +\infty[ \to \mathbb{R}) \)
\item one can interpret \( (\Gamma(z, \cdot))_{z \in \mathbb{R}} \) as a family of class \( C^1 \) curves ‘regular’ in \( z \)
considers the open sets
\[ \Omega^1_T = \left\{ z \in \mathbb{R} | \forall t \in [0, T], \frac{\partial \Gamma}{\partial z} (z, t) > 0 \right\} \]
\[ \Omega^2_T = \left\{ z \in \mathbb{R} | \forall t \in [0, T], \frac{\partial \Gamma}{\partial z} (z, t) < 0 \right\} \]

There exists a version of \( \left( \Lambda_t^{\Gamma(z, \cdot)} (X) \right)_{t \geq 0, z \in \mathbb{R}} \) satisfying:

(a) For \( T > 0 \), if \( \Omega^1_T \) is nonempty then \( \left( \Lambda_t^{\Gamma(z, \cdot)} (X) \right)_{0 \leq t \leq T, z \in \Omega^1_T} \) is continuous in \( t \) càdlàg in \( z \). We have almost surely for all \( t \in [0, T], z \in \Omega^1_T \):

\[ \Lambda_t^{\Gamma(z, \cdot)} (X) = \Lambda_t^{\Gamma(z, \cdot)} (X) = 2 \times \left( \int_0^t 1_{X_s = \Gamma(z, s)} dX_s - \int_0^t 1_{X_s = \Gamma(z, s)} \frac{\partial \Gamma}{\partial t} (z, s) \, ds \right) \]

(b) For \( T > 0 \), if \( \Omega^2_T \) is nonempty then \( \left( \Lambda_t^{\Gamma(z, \cdot)} (X) \right)_{0 \leq t \leq T, z \in \Omega^2_T} \) is continuous in \( t \) càdlàg in \( z \) (French: "limite à droite continue à gauche" meaning limit on the right continuous on the left). We have almost surely for all \( t \in [0, T], z \in \Omega^2_T \):

\[ \Lambda_t^{\Gamma(z, \cdot)} (X) - \Lambda_t^{\Gamma(z^+, \cdot)} (X) = 2 \times \left( \int_0^t 1_{X_s = \Gamma(z, s)} dX_s - \int_0^t 1_{X_s = \Gamma(z, s)} \frac{\partial \Gamma}{\partial t} (z, s) \, ds \right) \]

**Remark 1.3.** Consequently:

(1) If \( \Omega^1 = \bigcap_{T > 0} \Omega^1_T \) is a non empty open set then the family \( \left( \Lambda_t^{\Gamma(z, \cdot)} (X) \right)_{t \geq 0, z \in \Omega^1} \) is continuous in \( t \) càdlàg in \( z \). We have almost surely for all \( t \geq 0, z \in \Omega^1 \):

\[ \Lambda_t^{\Gamma(z, \cdot)} (X) = \Lambda_t^{\Gamma(z, \cdot)} (X) = 2 \times \left( \int_0^t 1_{X_s = \Gamma(z, s)} dX_s - \int_0^t 1_{X_s = \Gamma(z, s)} \frac{\partial \Gamma}{\partial t} (z, s) \, ds \right) \]

(2) Similarly if \( \Omega^2 = \bigcap_{T > 0} \Omega^2_T \) is a non empty open set then the family \( \left( \Lambda_t^{\Gamma(z, \cdot)} (X) \right)_{t \geq 0, z \in \Omega^2} \) is continuous in \( t \) càdlàg in \( z \). We have almost surely for all \( t \geq 0, z \in \Omega^1 \):

\[ \Lambda_t^{\Gamma(z, \cdot)} (X) - \Lambda_t^{\Gamma(z^+, \cdot)} (X) = 2 \times \left( \int_0^t 1_{X_s = \Gamma(z, s)} dX_s - \int_0^t 1_{X_s = \Gamma(z, s)} \frac{\partial \Gamma}{\partial t} (z, s) \, ds \right) \]

**Remark 1.4.** Note that when \( \Gamma(z, t) = z \) we have \( \forall t, z \frac{\partial \Gamma}{\partial z} (z, t) = 1 > 0 \) hence we recover Yor’s result.

Moreover we can extend the previous result to the general case when \( \Gamma \) is defined on a open subset of \( \mathbb{R} \times \mathbb{R}_+^\ast \):

**Corollary 1.1.** Let \( \Omega \subset \mathbb{R} \times \mathbb{R}_+^\ast \) an open set and \( \Gamma : \Omega \rightarrow \mathbb{R} \) a map of class \( C^1 \), let \( (X_t)_{t \geq 0} \) be a continuous semimartingale. Then there exists a version of the family \( \left( \Lambda_t^{\Gamma(z, \cdot)} (X) - \Lambda_s^{\Gamma(z, \cdot)} (X) \right)_{s \leq t, [z, t] \subseteq \Omega} \) verifying: almost surely we have for all \( z, s, t \) such that \( \{z\} \times [s, t] \subseteq \Omega \):

(1) If \( \forall u \in [s, t], \frac{\partial \Gamma}{\partial z} (z, u) > 0 \) the map \( \Gamma' \rightarrow \Lambda_t^{\Gamma(z, \cdot)} (X) - \Lambda_s^{\Gamma(z, \cdot)} (X) \) is càdlàg in \( z \) meaning:

\[ \lim_{z' \rightarrow z^+} \Lambda_t^{\Gamma(z', \cdot)} (X) - \Lambda_s^{\Gamma(z', \cdot)} (X) = \Lambda_t^{\Gamma(z, \cdot)} (X) - \Lambda_s^{\Gamma(z, \cdot)} (X) \]
We set:

\( \Lambda_t^\Gamma(z,.) (X) = 2 \times \left[ (X_t - \Gamma(z,t))^+ - (X_0 - \Gamma(z,0))^+ - \int_0^t 1_{X_s > \Gamma(z,s)} dX_s \right] \)

\[= 2 \times \left[ (X_t - \Gamma(z,t))^+ - (X_0 - \Gamma(z,0))^+ - \int_0^t 1_{X_s > \Gamma(z,s)} dM_s \right] \]

\[= 2 \times \left[ \int_0^t 1_{X_s > \Gamma(z,s)} dV_s \right] + 2 \times \left[ \int_0^t 1_{X_s > \Gamma(z,s)} \frac{\partial \Gamma}{\partial t} (z,s) ds \right] \]

(16)

We set:

\[\hat{M}^z_t = \int_0^t 1_{X_s > \Gamma(z,s)} dM_s\]

as in Yor’s proof the main idea is to show that \( \left( \hat{M}^z_t \right)_{t \geq 0, z \in \mathbb{R}} \) has a version that is continuous in \( z \) and in \( t \), to this end we use Kolmogorov’s continuity theorem (theorem 2.8 in [2]) in a well suited space: ideally if such space is equipped with a distance \( d \), we want to prove that there is a constant \( C \) and \( k \geq 1 \) such that \( \forall z_1, z_2 \in \mathbb{R} \):

\[\mathbb{E} \left[ d \left( \hat{M}^z_1, \hat{M}^z_2 \right)^{2k} \right] \leq C \times |z_1 - z_2|^k \]

(18)

We fix \( t > 0 \) and a compact \( \mathcal{K} = [-m, m] \), first we shall prove that the family \( \left( \hat{M}^z_s \right)_{0 \leq s \leq t, z \in \mathcal{K}} \) has a bicontinuous version: For \( z \in ]-m, m[ \), the process \( \left( \hat{M}^z_s \right)_{0 \leq s \leq t} \) is an element of the Banach space \( C([0, t], \mathbb{R}) \) equipped with the distance \( d(f, g) = \sup_{0 \leq s \leq t} |f(s) - g(s)| \).

Let \( \left\{ s^n_0, s^n_1, ..., s^n_{N_n+1} \right\}_{n \in \mathbb{N}} \) be a sequence of partitions of \([0, t]\) with \( s^n_0 = 0 \), \( s^n_{N_n+1} = t \) with mesh \( \max_k |s^n_{i+k} - s^n_i| \) converging to 0 , for all \( n \) let \( \left\{ s^n_{ij} \right\}_{1 \leq j \leq N_n} \) be those \( s^n_k \) such that \( \Gamma(z_1, s^n_{ij}) \neq \Gamma(z_2, s^n_{ik}) \) with \( s^n_{ij} < s^n_{i+1} \) ( if \( s^n_{iN_n} = t \) then \( s^n_{iN_n+1} = t \)), let...
\( \left( L^z_t(X) \right)_{z \in \mathbb{R}, t \geq 0} \) be the family of local times of \( X \) continuous in \( t \) and càdlàg in \( z \). For all \( s \in [0, t] \) we set:

\[ \Gamma_1(s) = \min(\Gamma(z_1, s), \Gamma(z_2, s)), \Gamma_2(s) = \max(\Gamma(z_1, s), \Gamma(z_2, s)) \]

In the first subsection the goal is to show that for \( k \geq 1, z_1, z_2 \in \mathcal{K} \)

\[ \mathbb{E} \left[ \sup_{0 \leq u \leq t} |\hat{M}^{z_1}_u - \hat{M}^{z_2}_u|^{2k} \right] \leq C_k \times \liminf_{n \to +\infty} \mathbb{E} \left[ \left( \int_{-\infty}^{+\infty} dz \left( \sum_{j=1}^{N_n} 1_{\Gamma_1(s^n_j) < z < \Gamma_2(s^n_j)} \times \left( L^z_{s^n_{i,j+1}}(X) - L^z_{s^n_{i,j}}(X) \right) \right) \right)^k \right] \]

where \( C_k \) is a constant that depends only on \( k \). Note that the existence of \( s^n_{i,j} \) for all \( i, j \leq N_n \) is justified by the fact that if \( \forall s \in [0, t], \Gamma(z_1, s) = \Gamma(z_2, s) \) the inequality is trivial.

2.1. **Proof of inequality** \((20)\). Using the fact:

\[ \begin{cases} 1_{M_s > a} - 1_{M_s > b} = 1_{\text{min}(a, b) < M_s \leq \text{max}(a, b)} & a \leq b \\ 1_{M_s > a} - 1_{M_s > b} = -1_{\text{min}(a, b) < M_s \leq \text{max}(a, b)} & a > b \end{cases} \]

and Burkholder-Davis-Gundy inequality (theorem 4.1 of chapter 4 in \[5\]) we have:

\[ \mathbb{E} \left[ \sup_{0 \leq u \leq t} |\hat{M}^{z_1}_u - \hat{M}^{z_2}_u|^{2k} \right] \leq C_k \mathbb{E} \left[ \left( \int_0^t 1_{\Gamma_1(s) < X_s \leq \Gamma_2(s)} d \langle M, M \rangle_s \right)^k \right] \]

Since \( X - \Gamma_2 \) and \( X \) have the same bracket, by the occupation times formula (corollary 1.6 of chapter 6 in \[5\]) we have:

\[ \int_0^t 1_{X_s = \Gamma_2(s)} d \langle X, X \rangle_s = \int_{-\infty}^{+\infty} 1_{z=0} L^z_t(X - \Gamma_2) d z = 0 \]

so:

\[ \int_0^t 1_{\Gamma_1(s) < X_s \leq \Gamma_2(s)} d \langle X, X \rangle_s = \int_0^t 1_{\Gamma_1(s) < X_s < \Gamma_2(s)} d \langle X, X \rangle_s \]

By the generalized occupation times formula (Exercise 1.35 of chapter 6 in \[5\]) we obtain:

\[ \mathbb{E} \left[ \sup_{0 \leq u \leq t} |\hat{M}^{z_1}_u - \hat{M}^{z_2}_u|^{2k} \right] \leq C_k \mathbb{E} \left[ \left( \int_{-\infty}^{+\infty} dz \int_0^t 1_{\Gamma_1(s) < z < \Gamma_2(s)} dL^z_s(X) \right)^k \right] \]

Let \( z \in \mathbb{R} \), since the set \( (s| \Gamma_1(s) < z < \Gamma_2(s)) \) is open then the function \( s \to 1_{\Gamma_1(s) < z < \Gamma_2(s)} \) is lower semi-continuous, in other words when \( (s^n)_n \) converges to \( s \) we have:

\[ 1_{\Gamma_1(s) < z < \Gamma_2(s)} \leq \liminf_{n \to +\infty} 1_{\Gamma_1(s^n) < z < \Gamma_2(s^n)} \]
For $s \in [0, t]$, we set:

$$F_n^z(s) = \sum_{i=0}^{n} 1_{[s^n_i, s^n_{i+1}]} \times 1_{\Gamma_1(s^n_i) < z < \Gamma_2(s^n_i)} + 1_{(t)} 1_{\Gamma_1(s^n_t) < z < \Gamma_2(s^n_t)}$$

and let $\sigma_n(s) \in [0, n + 1]$ the integer satisfying:

$$s^n_{\sigma_n(s)} \leq s \leq s^n_{\sigma_n(s)+1}$$

(we set for convenience $s^n_{n+2} = t$) As $\lim_{n \to +\infty} s^n_{\sigma_n(s)} = s$, by inequality (26)

$$1_{\Gamma_1(s) < z < \Gamma_2(s)} \leq \liminf_{n \to +\infty} 1_{\Gamma_1(s^n_{\sigma_n(s)}) < z < \Gamma_2(s^n_{\sigma_n(s)})} = \liminf_{n \to +\infty} F_n^z(s)$$

The last inequality is true for all $z \in \mathbb{R}$, by applying Fatou’s lemma for the measure $\mu$ defined on $\mathcal{B}(\mathbb{R} \times \mathbb{R}^+)$ by:

$$\mu(A) = \int_{-\infty}^{+\infty} dz \int_{0}^{t} 1_{A}(s, z) dL^z_s(X)$$

we get:

$$\left( \int_{-\infty}^{+\infty} dz \int_{0}^{t} 1_{\Gamma_1(s) < z < \Gamma_2(s)} dL^z_s(X) \right)^k$$

$$\leq \liminf_{n \to +\infty} \left( \int_{-\infty}^{+\infty} dz \int_{0}^{t} F_n^z(s) dL^z_s(X) \right)^k$$

applying Fatou’s lemma again:

$$\mathbb{E} \left[ \left( \int_{-\infty}^{+\infty} dz \int_{0}^{t} 1_{\Gamma_1(s) < z < \Gamma_2(s)} dL^z_s(X) \right)^k \right]$$

$$\leq \liminf_{n \to +\infty} \mathbb{E} \left[ \left( \int_{-\infty}^{+\infty} dz \int_{0}^{t} F_n^z(s) dL^z_s(X) \right)^k \right]$$

We have:

$$\int_{-\infty}^{+\infty} dz \int_{0}^{t} F_n^z(s) dL^z_s(X) = \int_{-\infty}^{+\infty} dz \left( \sum_{i=0}^{n} 1_{\Gamma_1(s^n_i) < z < \Gamma_2(s^n_i)} \times \left( L^z_{s^n_{i+1}}(X) - L^z_{s^n_i}(X) \right) \right)$$

if $i$ verifies $\Gamma(z_1, s^n_i) = \Gamma(2z_2, s^n_i)$ then $1_{\Gamma_1(s^n_i) < z < \Gamma_2(s^n_i)} = 0$, otherwise $\exists j \in [1, N_n]$ such that $s^n_i = s^n_{i+1}$ and since $s^n_i < s^n_{i+1}$ then $L^z_{s^n_{i+1}}(X) \leq L^z_{s^n_i}(X)$ we deduce that

$$\int_{-\infty}^{+\infty} dz \left( \sum_{i=0}^{n} 1_{\Gamma_1(s^n_i) < z < \Gamma_2(s^n_i)} \times \left( L^z_{s^n_{i+1}}(X) - L^z_{s^n_i}(X) \right) \right)$$

$$\leq \int_{-\infty}^{+\infty} dz \left( \sum_{j=1}^{N_n} 1_{\Gamma_1(s^n_j) < z < \Gamma_2(s^n_j)} \times \left( L^z_{s^n_{j+1}}(X) - L^z_{s^n_j}(X) \right) \right)$$

thus we proved (20).
We have:

\[(34)\]
\[
\mathbb{E} \left[ \left( \int_{-\infty}^{+\infty} dz \left( \sum_{j=1}^{N_n} 1_{\Gamma_1(s^n_{ij}) < z < \Gamma_2(s^n_{ij})} \times \left( L_{s^n_{ij+1}}^z (X) - L_{s^n_{ij}}^z (X) \right) \right) \right)^k \right] \\
\leq \mathbb{E} \left[ \left( \sum_{j=1}^{N_n} 1_{\Gamma_1(s^n_{ij})} \Gamma_2(s^n_{ij}) \left( L_{s^n_{ij+1}}^z (X) - L_{s^n_{ij}}^z (X) \right) dz \right)^k \right] \\
\leq \mathbb{E} \left[ \left( \sum_{j=1}^{N_n} \left| \Gamma_2(s^n_{ij}) - \Gamma_1(s^n_{ij}) \right| \times \left( s^n_{ij+1} \right)^k \Gamma_1(s^n_{ij}) \left( L_{s^n_{ij+1}}^z (X) - L_{s^n_{ij}}^z (X) \right) dz \right)^k \right] \\
\leq \sup_{1 \leq j \leq N_n} \left| \Gamma_2(s^n_{ij}) - \Gamma_1(s^n_{ij}) \right|^k \mathbb{E} \left[ \left( \sum_{j=1}^{N_n} \left( s^n_{ij+1} \right)^k \Gamma_1(s^n_{ij}) \left( L_{s^n_{ij+1}}^z (X) - L_{s^n_{ij}}^z (X) \right) dz \right)^k \right] \\
\]

In the following subsection we control the last term.

2.2. Control of the term \( \mathbb{E} \left[ \left( \sum_{j=1}^{N_n} \left( s^n_{ij+1} \right)^k \Gamma_1(s^n_{ij}) \left( L_{s^n_{ij+1}}^z (X) - L_{s^n_{ij}}^z (X) \right) dz \right)^k \right] \). By Tanaka’s theorem:

Formula:

\[(35)\]
\[L_{s^n_{ij+1}}^z (X) - L_{s^n_{ij}}^z (X) = 2 \times \left( X_{s^n_{ij+1}} - z \right)^+ - \left( X_{s^n_{ij}} - z \right)^+ \times \int_{s^n_{ij}}^{s^n_{ij+1}} 1_{X_{s^n_{ij}} > z} dX_s \]
\[= 2 \times \left( X_{s^n_{ij+1}} - z \right)^+ - \left( X_{s^n_{ij}} - z \right)^+ \times \int_{s^n_{ij}}^{s^n_{ij+1}} 1_{X_{s^n_{ij}} > z} dV_s - \int_{s^n_{ij}}^{s^n_{ij+1}} 1_{X_{s^n_{ij}} > z} dM_s \]

hence:

\[(36)\]
\[L_{s^n_{ij+1}}^z (X) - L_{s^n_{ij}}^z (X) = 2 \times \int_{s^n_{ij}}^{s^n_{ij+1}} 1_{X_{s^n_{ij}} > z} dM_s - 2 \times \left( X_{s^n_{ij+1}} - z \right)^+ - \left( X_{s^n_{ij}} - z \right)^+ \]
\[\leq 2 \times \left( \int_{s^n_{ij}}^{s^n_{ij+1}} \left| dV \right| \right) \]

and so inequality \((36)\) implies:

\[(37)\]
\[\sum_{j=1}^{N_n} \int_{s^n_{ij}}^{s^n_{ij+1}} \left( L_{s^n_{ij+1}}^z (X) - L_{s^n_{ij}}^z (X) \right) dz \leq 2 \times \left( \int_{0}^{t} \left| dV \right| \right) + E_1 + E_2 \]
where:

$$E_1 = -2 \times \left( \sum_{j=1}^{N_n} \frac{\int_{\Gamma_1(s_{i_j}^n)}^{\Gamma_2(s_{i_j}^n)} \left( \int_{s_{i_j}^n}^{s_{i_j}^n+1} 1_{X_{s_{i_j}^n} > z} dM_s \right) dz}{\Gamma_2(s_{i_j}^n) - \Gamma_1(s_{i_j}^n)} \right)$$

$$E_2 = 2 \times \left( \sum_{j=1}^{N_n} \frac{\int_{\Gamma_1(s_{i_j}^n)}^{\Gamma_2(s_{i_j}^n)} \left( (X_{s_{i_j}^n} - z) + (X_{s_{i_j}^n} - z)^+ \right) dz}{\Gamma_2(s_{i_j}^n) - \Gamma_1(s_{i_j}^n)} \right)$$

By Fubini’s theorem for stochastic integrals. (see chapter 4 in [5] for instance):

$$E_1 = -2 \times \left( \sum_{j=1}^{N_n} \frac{\int_{\Gamma_1(s_{i_j}^n)}^{\Gamma_2(s_{i_j}^n)} \left( \int_{s_{i_j}^n}^{s_{i_j}^n+1} 1_{X_{s_{i_j}^n} > z} dM_s \right) dz}{\Gamma_2(s_{i_j}^n) - \Gamma_1(s_{i_j}^n)} \right) = -2 \times \int_0^t H_s^n dM_s$$

$$(H_s^n)_{0 \leq s \leq t}$$ is the process defined by:

$$H_s^n = \sum_{j=1}^{N_n} 1_{s \in [s_{i_j}^n, s_{i_j}^n+1]} \times \frac{\int_{\Gamma_1(s_{i_j}^n)}^{\Gamma_2(s_{i_j}^n)} 1_{X_{s_{i_j}^n} > z} dz}{\Gamma_2(s_{i_j}^n) - \Gamma_1(s_{i_j}^n)}$$

It’s easy to see that $H^n$ is adapted to the filtration $(\mathcal{G}_n)_{0 \leq n \leq t}$ and that $|H^n| \leq 1$. We deduce that the process $(\int_0^t H_s^n dM_s)_{0 \leq n \leq t}$ is a local martingale starting from 0 and so by Burkholder-Davis-inequality there is a constant $C_k$ such that:

$$\mathbb{E} \left( \sup_{0 \leq u \leq t} \left( \int_0^u H_s^n dM_s \right)^{2k} \right) \leq C_k \times \mathbb{E} \left( \left( \int_0^t H_s^n d\langle M, M \rangle_s \right)^k \right)$$

$$\leq C_k \times \mathbb{E} \left( \langle M, M \rangle_t^k \right)$$

For the term $E_2$, we write via integration by substitution $(z = \overline{z} \times \Gamma_2(s_{i_j}^n) + (1 - \overline{z}) \times \Gamma_1(s_{i_j}^n))$

$$\int_{\Gamma_1(s_{i_j}^n)}^{\Gamma_2(s_{i_j}^n)} \left( (X_{s_{i_j}^n} - z) + (X_{s_{i_j}^n} - z)^+ \right) dz \frac{\Gamma_2(s_{i_j}^n) - \Gamma_1(s_{i_j}^n)}{\Gamma_2(s_{i_j}^n) - \Gamma_1(s_{i_j}^n)}$$

$$= \int_0^1 \left( (X_{s_{i_j}^n} - \overline{z} \Gamma_2(s_{i_j}^n) - (1 - \overline{z}) \Gamma_1(s_{i_j}^n))^+ \right) d\overline{z}$$

$$- \int_0^t \left( (X_{s_{i_j}^n} - \overline{z} \Gamma_2(s_{i_j}^n) - (1 - \overline{z}) \Gamma_1(s_{i_j}^n))^+ \right) d\overline{z}$$
So:
\[ \sum_{j=1}^{N_n} \int_{\Gamma_1(s^n_{i,j})}^{\Gamma_2(s^n_{i,j})} \left( (X_{s^n_{i,j+1}} - z) - (X_{s^n_{i,j}} - z) \right) \, dz \]
\[ = \sum_{j=1}^{N_n} \left[ \int_0^1 \left( (X_{s^n_{ij+1}} - z\Gamma_2(s^n_{ij}) - (1 - z)\Gamma_1(s^n_{ij})) \right) \, d\bar{z} \right] \]
\[ - \sum_{j=1}^{N_n} \left[ \int_0^1 \left( (X_{s^n_{ij}} - z\Gamma_2(s^n_{ij}) - (1 - z)\Gamma_1(s^n_{ij})) \right) \, d\bar{z} \right] \]
\[ = \sum_{j=2}^{N_n+1} \left[ \int_0^1 \left( (X_{s^n_{ij-1}} - z\Gamma_2(s^n_{ij-1}) - (1 - z)\Gamma_1(s^n_{ij-1})) \right) \, d\bar{z} \right] \]
\[ - \sum_{j=1}^{N_n} \left[ \int_0^1 \left( (X_{s^n_{ij}} - z\Gamma_2(s^n_{ij}) - (1 - z)\Gamma_1(s^n_{ij})) \right) \, d\bar{z} \right] \]
\[ = \int_0^1 \left( (X_{s^n_{i,N+1}} - \hat{\Gamma}(\bar{z}, s^n_{i,N})) \right) \, d\bar{z} - \int_0^1 \left( (X_{s^n_{i1}} - \hat{\Gamma}(\bar{z}, s^n_{i1})) \right) \, d\bar{z} \]
\[ + \sum_{j=2}^{N_n} \left[ \int_0^1 \left( (X_{s^n_{ij}} - \hat{\Gamma}(\bar{z}, s^n_{ij-1})) - (X_{s^n_{ij}} - \hat{\Gamma}(\bar{z}, s^n_{ij})) \right) \, d\bar{z} \right] \]

where \( \hat{\Gamma} \) is given by:
\[ \hat{\Gamma}(\bar{z}, s) = z\Gamma_2(s) + (1 - z)\Gamma_1(s) \]

we have:
\[ \left| \int_0^1 \left( (X_{s^n_{i,N+1}} - \hat{\Gamma}(\bar{z}, s^n_{i,N})) \right) \, d\bar{z} - \int_0^1 \left( (X_{s^n_{i1}} - \hat{\Gamma}(\bar{z}, s^n_{i1})) \right) \, d\bar{z} \right| \]
\[ \leq \int_0^1 \left| \left( X_{s^n_{i,N+1}} - \hat{\Gamma}(\bar{z}, s^n_{i,N}) \right) - \left( X_{s^n_{i1}} - \hat{\Gamma}(\bar{z}, s^n_{i1}) \right) \right| \, d\bar{z} \]
\[ \leq \int_0^1 \left| X_{s^n_{i,N+1}} - X_{s^n_{i1}} \right| + \left| \hat{\Gamma}(\bar{z}, s^n_{i,N}) - \hat{\Gamma}(\bar{z}, s^n_{i1}) \right| \, d\bar{z} \]
\[ \leq \sup_{u,v \in [0,t]} \left| X_u - X_v \right| + 2c_1 \]

\( c_1 = \sup_{z \in K, u \in [0,t]} |\Gamma(z, u)| \) is a constant that depends only on \( \Gamma, t \) and \( K \). We establish now two necessary inequalities: Let \( A, B, C, D \in \mathbb{R} \), we have:
\[ |\min(A, B) - \min(C, D)| = \frac{1}{2} \times |A - B + |A - B| - (C - D + |C - D|)| \]
\[ \leq \frac{1}{2} \times |A - (B - D) + ||A - B| - |C - D||| \]
\[ \leq \frac{1}{2} \times ||A - C| + |B - D| + |A - C - (B - D)|| \]
\[ \leq |A - C| + |B - D| \]
similarly:
\[
|\max(A, B) - \max(C, D)| = | - \min(-A, -B) + \min(-C, -D)| \\
\leq |A - C| + |B - D|
\]

For \( s, s' \in [0, t] \), applying inequality 46 we get:
\[
|\Gamma_1(s) - \Gamma_1(s')| \leq |\Gamma(z_1, s) - \Gamma(z_1, s')| + |\Gamma(z_2, s) - \Gamma(z_2, s')| \\
\leq 2 \times c_2 \times |s - s'|
\]

where \( c_2 = \sup_{z \in \kappa, u \in [0, t]} |\frac{\partial}{\partial t} (z, u)| \), in the same way by applying inequality 47:
\[
|\Gamma_2(s) - \Gamma_2(s')| \leq 2 \times c_2 \times |s - s'|
\]

thus \( \forall \bar{z} \in [0, 1] \):
\[
|\hat{\Gamma}(\bar{z}, s) - \hat{\Gamma}(\bar{z}, s')| \leq 2 \times c_2 \times |s - s'|
\]

Now by the last inequality we obtain:
\[
\sum_{j=2}^{N_n} \left[ \int_0^1 \left( (X_{s_{t_j}} - \hat{\Gamma}(\bar{z}, s_{n_{t_j-1}})^n) - (X_{s_{t_j}} - \hat{\Gamma}(\bar{z}, s_{n_{t_j}}))^n \right) d\bar{z} \right] \\
\leq \sum_{j=2}^{N_n} \left[ \int_0^1 \left( (X_{s_{t_j}} - \hat{\Gamma}(\bar{z}, s_{n_{t_j-1}})^n) - (X_{s_{t_j}} - \hat{\Gamma}(\bar{z}, s_{n_{t_j}}))^n \right) d\bar{z} \right] \\
\leq 2 \times c_2 \times \sum_{j=2}^{N_n} |s_{n_{t_j-1}} - s_{n_{t_j}}| \\
\leq 2 \times c_2 \times t
\]

By the estimate 37, the identity 39 and the estimates 43, 45, 51 we obtain:
\[
\left( \sum_{j=1}^{N_n} \left[ \int_{\Gamma_1(s_{t_j})}^{\Gamma_2(s_{t_j})} \frac{\left(L_{s_{t_j+1}}^n(X) - L_{s_{t_j}}^n(X)\right) dz}{|\Gamma_2(s_{t_j}) - \Gamma_1(s_{t_j})|} \right]^k \right)^k \\
\leq 2^k \times \left( c_2 t + c_1 + \sup_{u,v \in [0, t]} |X_u - X_v| + \sup_{0 \leq u \leq t} \left| \int_0^u H^n dM_s \right| + \int_0^t |dV| \right)^k
\]

and so by the inequality 44 there is a constant \( D_k \) that depends only on \( \Gamma, \kappa, t \) such that:
\[
\mathbb{E} \left( \sum_{j=1}^{N_n} \left[ \int_{\Gamma_1(s_{t_j})}^{\Gamma_2(s_{t_j})} \frac{\left(L_{s_{t_j+1}}^n(X) - L_{s_{t_j}}^n(X)\right) dz}{|\Gamma_2(s_{t_j}) - \Gamma_1(s_{t_j})|} \right]^k \right)^k \\
\leq D_k \times \mathbb{E} \left[ t^k + 1 + \sup_{u,v \in [0, t]} |X_u - X_v|^k + \left( \int_0^t |dV| \right)^k + \langle M, M \rangle_t^k \right]
\]

2.3. **End of proof of theorem 1.1** For \( j \in [1, N_n] \) we have:

\[
\Gamma_2\left(s^n_{i_j} - s^n_{i_j}\right) - \Gamma_1\left(s^n_{i_j}\right) = \left|\Gamma(z_2, s^n_{i_j}) - \Gamma(z_1, s^n_{i_j})\right| \leq |z_2 - z_1| \times \sup_{0 \leq s \leq t, z \in \mathcal{K}} |\frac{\partial \Gamma}{\partial z}(z, s)|
\]

owing to the estimates \([1,1,20, 24, 25, 54, 55, 56]\) we deduce the existence of a constant \(G_k\) that depends only on \(\Gamma, \mathcal{K}, t\) such that:

\[
\mathbb{E}\left[\sup_{0 \leq u \leq t} |\hat{M}^{z_1}_u - \hat{M}^{z_2}_u|^{2k}\right] \\
\leq G_k \times |z_2 - z_1|^k \times \mathbb{E}\left[t^k + 1 + \sup_{u,v \in [0,t]} |X_u - X_v|^k + \left(\int_0^t |dV|\right)^k + \langle M, M \rangle^k_t\right]
\]

thus if:

\[
\mathbb{E}\left[\sup_{u,v \in [0,t]} |X_u - X_v|^k + \left(\int_0^t |dV|\right)^k + \langle M, M \rangle^k_t\right] < +\infty
\]

the inequality \([13]\) is true \(\forall z_1, z_2 \in \overset{\circ}{\mathcal{K}}\), in fact without loss of generality we can suppose that \([56]\) holds by considering the sequence of stopping times \((T_n)_n\) defined by:

\[
T_n = \inf \left( t \geq 0 \mid \sup_{u,v \in [0,t]} |X_u - X_v|^k + \left(\int_0^t |dV|\right)^k + \langle M, M \rangle^k_t \geq n \right)
\]

for the sake of completeness let us be more precise: we set

\[
\hat{M}^{z,n}_s = \int_0^{\min(s,T_n)} 1_{X_u \geq \Gamma(z,u)}dM_u
\]

by the previous subsection \(\forall n\) we can find a modification \((\hat{M}^{z,n}_s)_{0 \leq s \leq t, z \in \mathcal{K}}, \hat{\Gamma}^{z,n}_s \hat{N}^{z,n}_s\) continuous in \(z\) and \(t\). Note that since the stochastic integral is continuous, \(\forall z \in \mathcal{K}\) we have almost surely \(\forall s \leq T_n\): \(\hat{M}^{z,n+1}_s = \hat{M}^{z,n}_s\) hence almost surely \(\forall n \geq 0, \hat{N}^{z,n+1}_s|_{[0,T_n]} = \hat{N}^{z,n}_s\), let us define:

\[
\hat{N}^{z}_s = \sum_{n=0}^{+\infty} 1_{[T_n,T_{n+1}]}(s)\hat{N}^{z,n+1}_s
\]

Since almost surely \(\forall n, \forall z \hat{N}^{z,n+1}_{T_{n+1}} = \hat{N}^{z,n+2}_{T_{n+1}}\), it is clear that almost surely the map \((s,z) \rightarrow \hat{N}^{z}_s\) is continuous in \(s\) and \(z\). Finally for \(s \in [0, t]\), \(z \in \overset{\circ}{\mathcal{K}}\) we have almost surely \(1_{[T_n,T_{n+1}]}(s)\hat{N}^{z,n+1}_s = 1_{[T_n,T_{n+1}]}(s)\hat{M}^{z,n+1}_s\) thus:

\[
\hat{N}^{z}_s = \sum_{n=0}^{+\infty} 1_{[T_n,T_{n+1}]}(s)\hat{N}^{z,n+1}_s = \sum_{n=0}^{+\infty} 1_{[T_n,T_{n+1}]}(s)\hat{M}^{z,n+1}_s = \hat{M}^{z}_s
\]

in other words \((\hat{N}^{z}_s)_{0 \leq s \leq t, z \in \overset{\circ}{\mathcal{K}}}\) is a bicontinuous version of \((\hat{M}^{z}_s)_{0 \leq s \leq t, z \in \overset{\circ}{\mathcal{K}}}\).

It should be easy to extend the modification to the whole \(\mathbb{R}_+ \times \mathbb{R}\) but again for the sake of completeness we do it here: for all \(n \in \mathbb{N}, m \in \mathbb{N} > 0\) let \((\hat{N}^{z,n,m}_t)_{t \in [0,n], z \in [-m,m]}\)
be a bicontinuous version of $z, t$ of $\left(\hat{M}_t^z\right)_{t \in [0, n]}$. For $z \in [-m, n[, t \in [0, n]$ we have almost surely $\hat{N}^{z,n,m}_t = \hat{N}^{z,n+1,m+1}_t = \hat{M}^z_t$, as both maps $(t, z) \to \hat{N}^{z,n,m}_t$, $(t, z) \to \hat{N}^{z,n+1,m+1}_t$ are continuous in $t$ and $z$, we conclude that almost surely $\forall n, m, \hat{N}^{z,n,m+1}_t|_{[0, n[\times]m, m[} = \hat{N}^{z,n,m}_t$. Let us define:

$$\hat{N}_t^z = \sum_{n \geq 0, m \geq 0} \hat{N}^{z,n+1,m+1}_t \left[1_{n \leq t < n+1, m \leq |z| < m+1}\right]$$

since almost surely

$$\forall n, \forall m, \forall z, \forall t \hat{N}^{z,n,m+1}_t = \hat{N}^{z,n+2,m+1}_t = \hat{N}^{z,n+1,m+1}_t = \hat{M}^{z+1}_t$$

it is clear $(t, z) \to \hat{N}_t^z$ is bicontinuous in $t$ and $z$. For fixed $t, z, m, n$ we have almost surely

$$1_{n \leq t < n+1, m \leq |z| < m+1} \hat{N}^{z,n+1,m+1}_t = 1_{n \leq t < n+1, m \leq |z| < m+1} \hat{M}^z_t$$

and thus

$$\hat{N}_t^z = \sum_{n \geq 0, m \geq 0} 1_{n \leq t < n+1, m \leq |z| < m+1} \hat{N}^{z,n+1,m+1}_t$$

(62)

$$= \sum_{n \geq 0, m \geq 0} 1_{n \leq t < n+1, m \leq |z| < m+1} \hat{M}^z_t$$

thus we proved $\left(\hat{N}_t^z\right)_{t \geq 0, z \in \mathbb{R}}$ is bicontinuous version of $\left(\hat{M}_t^z\right)_{t \geq 0, z \in \mathbb{R}}$ now we write:

$$\Lambda^\Gamma_{\{z, t\}}(X) = 2 \times \left[(X_t - \Gamma(z, t))^+ - (X_0 - \Gamma(z, 0))^+ - \hat{M}^z_t - \hat{V}^z_t\right]$$

(63)

$$+ 2 \times \left[\int_0^t 1_{X_s > \Gamma(z, s)} \frac{\partial \Gamma}{\partial t}(z, s) \, ds\right]$$

here $\left(\hat{M}_t^z\right)_{t \geq 0, z \in \mathbb{R}}$ is bicontinuous and $\hat{V}_t^z = \int_0^t 1_{X_s > \Gamma(z, s)} dV_s$. Now we check point (a) of 1.1 Let $T > 0$, suppose that $\Omega_1^T$ is nonempty and let $z \in \Omega_1^T$, then for all $s \in [0, T]$ the function $z \to \Gamma(z, s)$ is strictly increasing, by the dominated convergence theorem almost surely and $\forall t \in [0, T]$:

$$\tilde{V}_t^{z^+} = \lim_{z' \to z^+, z' \in \Omega_1^T} \int_0^t 1_{X_s > \Gamma(z', s)} dV_s = \int_0^t 1_{X_s > \Gamma(z, s)} dV_s = \hat{V}_t^z$$

and:

$$\tilde{V}_t^{z^-} = \lim_{z' \to z^-, z' \in \Omega_1^T} \int_0^t 1_{X_s > \Gamma(z', s)} dV_s = \int_0^t 1_{X_s \geq \Gamma(z, s)} dV_s$$

(65)

similarly:

$$\lim_{z' \to z^+, z' \in \Omega_1^T} \int_0^t 1_{X_s > \Gamma(z', s)} \frac{\partial \Gamma}{\partial t}(z', s) \, ds = \int_0^t 1_{X_s > \Gamma(z, s)} \frac{\partial \Gamma}{\partial t}(z, s) \, ds$$

(66)

$$\lim_{z' \to z^-, z' \in \Omega_1^T} \int_0^t 1_{X_s \geq \Gamma(z', s)} \frac{\partial \Gamma}{\partial t}(z', s) \, ds = \int_0^t 1_{X_s \geq \Gamma(z, s)} \frac{\partial \Gamma}{\partial t}(z, s) \, ds$$

(67)
hence the family \( \left( \Lambda^\Gamma_{t}(\varepsilon_{z})(X) \right)_{0 \leq t \leq T, z \in \mathbb{R}^1} \) is continuous in \( t \) càdlàg in \( z \) and we have:

\[
\Lambda^\Gamma_{t}(\varepsilon_{z})(X) - \Lambda^\Gamma_{t}(\varepsilon_{z}^{-})(X) = 2 \times \left( \widetilde{V}_{t}^{z-} - \widetilde{V}_{t}^{z+} - \int_{0}^{t} 1_{\Lambda_{s} = \Gamma(z,s)} \frac{\partial \Gamma}{\partial t}(z, s) \, ds \right)
\]

\[
= 2 \times \left( \int_{0}^{t} 1_{\Lambda_{s} = \Gamma(z,s)} dV_{s} - \int_{0}^{t} 1_{\Lambda_{s} = \Gamma(z,s)} \frac{\partial \Gamma}{\partial t}(z, s) \, ds \right)
\]

\[
= 2 \times \left( \int_{0}^{t} 1_{\Lambda_{s} = \Gamma(z,s)} dX_{s} - \int_{0}^{t} 1_{\Lambda_{s} = \Gamma(z,s)} \frac{\partial \Gamma}{\partial t}(z, s) \, ds \right)
\]

the last equality is justified by \( \int_{0}^{t} 1_{\Lambda_{s} = \Gamma(z,s)} dM_{s} = 0 \) because:

\[
\left\langle \int_{0}^{t} 1_{\Lambda_{s} = \Gamma(z,s)} dM_{s}, \int_{0}^{t} 1_{\Lambda_{s} = \Gamma(z,s)} dM_{s} \right\rangle = \int_{0}^{t} 1_{\Lambda_{s} = \Gamma(z,s)} d\langle M, M \rangle_{s}
\]

\[
= \int_{0}^{t} 1_{\Lambda_{s} = \Gamma(z,s)} d\langle X, X \rangle_{s} = 0
\]

in the same way we can check point (b).

3. PROOF OF COROLLARY 1.1

Let \( z \) such that \( \{z\} \times [s, t] \subset \Omega \), by remark \( \box{1.1} \) \( \Lambda^\Gamma_{t}(\varepsilon_{z})(X) - \Lambda^\Gamma_{t}(\varepsilon_{z}^{-})(X) \) can be defined independently of basis point. Note that there exists \( \eta > 0 \) such that \( [z - \eta, z + \eta] \times [s, t] \subset \Omega \): in fact for all \( t' \in [s, t] \) there exists \( \eta_{t'} > 0 \) such that \( z - \eta_{t'} > 0 \) and \( \eta_{t'}(x) \in (\omega - \eta_{t'}, \omega + \eta_{t'}) \subset \Omega \), the family of open sets \( (\omega - \eta_{t'}, \omega + \eta_{t'}) \) is a cover of \( [s, t] \) and hence we can extract from it a finite cover \( \{\omega - \eta_{t'}, \omega + \eta_{t'}\}_{t \in \{1, 2\}} \), let \( \eta = \frac{\min_{t \in \{1, 2\}} \eta_{t'}}{2} \) it is easy to verify that \( [z - \eta, z + \eta] \times [s, t] \subset \Omega \). We can then conclude that the limits \( \lim_{z \to z^+} \Lambda^\Gamma_{t}(\varepsilon_{z})(X) - \Lambda^\Gamma_{t}(\varepsilon_{z}^{-})(X) \), \( \lim_{z \to z^-} \Lambda^\Gamma_{t}(\varepsilon_{z})(X) - \Lambda^\Gamma_{t}(\varepsilon_{z}^{-})(X) \) can be defined \( a \ priori \). We set:

\[
C = \{[x, y]_u, v] \times [u, v] \subset \Omega, x, y \in \mathbb{Q}^2, u, v \in (\mathbb{Q}_+)^2 \}
\]

\[
= \{[x_n, y_n]_u, v_n] \}_{n \in \mathbb{N}}
\]

one knows that there exist \( s' < s, s' \in \mathbb{Q}, t' > t, t' \in \mathbb{Q} \) such that \( \{z\} \times [s', t'] \subset \Omega \) and so by the previous proof we can choose \( \eta_1, \eta_2 \) sufficiently small such that \( z - \eta_1, z + \eta_2 \in \mathbb{Q} \) and:

\[
\{z\} \times [s, t] \subset [z - \eta_1 + \eta_1, z + \eta_2 - \eta_2] \subset [s', t'] \subset \Omega
\]

therefore one can ensure that:

\[
((z, s, t)) \{z\} \times [s, t] \subset \Omega = \bigcup_{n \in \mathbb{N}} ((z, s, t)) \{z\} \times [s, t] \subset [x_n, y_n]_u, v_n]
\]

By the proof of theorem \( \box{1.1} \) it suffices to prove that there is a version of \( \left( \hat{M}_t^{z,n} - \hat{M}_s^{z,n} \right) \) continuous in \( z, s, t \). We know that for all \( n \) there exists a version of \( \left( \hat{M}_t^{z,n} - \hat{M}_s^{z,n} \right) \) continuous in \( z, t, s \) (in the sense of corollary \( \box{1.1} \), for \( N \in \mathbb{N} \) and since almost surely \( \forall n \in [0, N] \) the map \( t, s, z \rightarrow \hat{M}_t^{z,n} - \hat{M}_s^{z,n} \) is continuous in \( (t, s, z) \), we conclude that almost surely the \( \hat{M}_t^{z,n} - \hat{M}_s^{z,n} \)
are identical on each possible intersection (there are a finite number of them) of \( [x_n, y_n] \times [u_n, v_n], n \in [0, N] \) in other words by setting \( C_N = \bigcup_{n=0}^N [x_n, y_n] \times [u_n, v_n] \) we can find a continuous version of \( (M_t^{z,N} - M_s^{z,N})_{\{z\} \times [s,t] \subset C_N} \).

We leave to the reader the task of verifying that \( \sum_{N=0}^{+\infty} (M_t^{z,N} - M_s^{z,N}) \times 1_{\{z\} \times [s,t] \in \mathcal{O}_{N+1}} \mathcal{O}_N \) is the desired modification.

References

[1] Anass Ben Taleb. Change of variable formula for local time of continuous semimartingale. preprint, 2018.
[2] Ioannis Karatzas and Steven E. Shreve. Brownian Motion and Stochastic Calculus. New York, first edition, 1988.
[3] Paul-André Meyer. Un cours sur les intégrandes stochastiques (exposés 1 à 6). Séminaire de probabilités de Strasbourg, 10:245–400, 1976.
[4] Goran Peskir. A change-of-variable formula with local time on curves. J Theor Probab, 2005.
[5] Daniel Revuz and Marc Yor. Continuous martingales and Brownian motion, volume 293 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, third edition, 1999.
[6] H. F. Trotter. A property of brownian motion paths. Illinois J. Math., 2(3):425–433, 09 1958.
[7] Marc Yor. Sur la continuité des temps locaux associés à certaines semimartingales. Astérisque 52-53, pages 23–36, 1978.

CERMICS, ÉCOLE DES PONTS ET CHAUSSÉES 6-8 AVENUE BLAISE PASCAL, CHAMPS-SUR-MARNE, 77455 MARNE LA VALLÉE CEDEX 2, FRANCE

E-mail address: mohamed-anass.ben-taleb@enpc.fr