Paving Springer fibers for $E_7$

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Abstract. We study the existence of a paving by affine spaces of Springer fibers. In particular, we prove that in the case of simple groups not of type $E_8$ such pavings exist.

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1. Introduction

Let $G$ be a semisimple algebraic group over the complex numbers, $\mathfrak{g}$ be its Lie algebra and $B$ the projective variety of Borel subalgebras of $\mathfrak{g}$. Given a nilpotent element $N \in \mathfrak{g}$, the Springer fiber $B_N \subset B$ is the variety of Borel subalgebras containing $N$. More generally, if $s \in G$ is a semisimple element such that $\text{ad}(s)N = cN$, $c \in \mathbb{C}^*$, we may also consider $B^s_N$ which is the variety of Borel subalgebras in $B_N$ fixed by $s$.

In order to state our problem, we need to recall a definition (see [5], Section 1.3). A finite partition of a variety $X$ into locally closed subsets is said to be an $\alpha$-partition if the subsets in the partition can be ordered as $\{X_1, X_2, \ldots, X_n\}$ in such a way that $X_1 \cup X_2 \cup \cdots \cup X_k$ is closed in $X$ for all $1 \leq k \leq n$.

If the $X_i$ are affine spaces, we shall say that $X$ has a paving. The existence of a paving has strong consequences on the cohomology of $X$. In [5], a variety is said to have property (S) if the integral Borel–Moore homology is a free module, if the integral odd Borel–Moore homology vanishes and if the Chow map is an isomorphism. It is easy to see that if $X$ can be paved, then $X$ has property (S). In [5], it was proved that all varieties $B^s_N$ have property (S).

The problem we want to study is the existence of a paving for $B^s_N$.

This is known in most cases. First of all, one reduces immediately to the case in which $\mathfrak{g}$ is simple. Then, in type $A_n$ this follows immediately from the work of Spaltenstein [10]. For type $G_2$ and $F_4$, this is easy and we shall see a proof below. For type $E_6$, this was proved by Spaltenstein [11] and again we shall see a simple proof later on. Finally, for the

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classical groups, this has been proved in [5]. So in effect, the only remaining cases are $E_7$ and $E_8$. In this paper, we shall give a complete solution for $E_7$.

As far as we know, there is not a general strategy to determine whether a variety can be paved. Some of the tools which turn out to be useful to prove such a statement are the following: If $X$ is vector bundle over $Y$ and $Y$ can be paved then also $X$ can be paved. If $X$ is smooth projective variety with an action of the multiplicative group $\mathbb{C}^*$, then it follows from [3] (Theorem 4.1 and the remark after Proposition 3.1) and [4] (Theorem 3) that if $X^{\mathbb{C}^*}$ can be paved then also $X$ can be paved. Following [5], these ingredients can be used to reduce our problem to the existence of paving for some smooth projective varieties $X_U$ (see Section 2, Reduction 2). This is a finite, although often large, list of subvarieties in a product of flag varieties.

To prove that $X_U$ can be paved we used three different ideas. First of all, we prove that the varieties $X_U$ have unirational connected components and we use a remark by Xue [14], Lemma 6.6 which shows that if a smooth projective variety is rational and has dimension less than or equal to 2 (recall that in this case, a unirational variety is automatically rational), then it can be paved (see Proposition 5). To study the remaining cases, we intersect the varieties $X_U$ with Schubert cells. In most cases, this gives a paving of the variety $X_U$. However, for a nilpotent orbit in type $E_7$, we need to use a last ingredient which is given by Lemma 7.

The idea to intersect a subvariety of the flag variety with Schubert cells to construct a paving has also been the main ingredient in other works. Indeed, for Springer fibers this idea was used by Spaltenstein to prove that in type $E_6$, Springer fibers can be paved. More recently, it has been used first by Tymoczko [12,13], Precup [8], Precup and Tymoczko [9], Fresse [7] and others to prove that certain Hessenberg varieties can be paved.

We learned a lot of what we know about flag and Schubert varieties from C S Seshadri whose friendship and company has been an invaluable gift.

2. Recollections

In this section, we are going to make a series of reductions. Most of the results we are going to illustrate are contained in [5] for which we refer to the proofs.

We begin with a definition given in [1] which is used in [1] and [2] to obtain a classification of nilpotent conjugacy classes in $\mathfrak{g}$.

**Definition 1**

Let $N \in \mathfrak{g}$ be a nilpotent element. $N$ is said to be distinguished if it is not contained in any Levi subalgebra of a proper parabolic subalgebra of $\mathfrak{g}$.

Our first reduction is the following (see [5], Section 3.6),

**Reduction 1.** If $B_N$ can be paved for all distinguished nilpotent elements $N$ and all simple Lie algebras, then $B_N$ can be paved for all nilpotent elements $N$.

In view of this, we are going to assume from now on, that the nilpotent element $N$ is distinguished.

Now, by the Jacobson–Morozov theorem there exists a homomorphism $\psi : \mathfrak{sl}(2) \to \mathfrak{g}$ such that $\psi(e) = N$. The adjoint action of the element $\psi(h)$ induces a grading $\mathfrak{g} = \bigoplus_{i=-m}^{m} \mathfrak{g}_i$ with $\mathfrak{g}_i = \{x \in \mathfrak{q} | [\psi(h), x] = ix\}$. We have $N \in \mathfrak{g}_2$ and since $N$ is distinguished, $\mathfrak{g}_i = 0$ for all odd $i$ [1].
The Lie algebra \( g_0 \) is reductive and acts on the vector space \( g_2 \). \( N \) being distinguished is equivalent (see [1]) to the fact that the map
\[
ad_N : g_0 \to g_2
\]
is an isomorphism.

It follows that if we let \( G_0 \subset G \) be a connected group such that \( g_0 = \text{Lie } G_0 \), then the \( G_0 \) orbit of \( N \) is dense in \( g_2 \) with finite stabilizer.

Recall that, with the above notations, if \( M \) is an algebraic group, \( V \) a \( M \)-module containing a (necessarily unique) open \( M \) orbit, then \( V \) is called a pre-homogeneous \( M \)-module.

Thus we have that \( g_2 \) is a pre-homogeneous \( G_0 \) module and further more \( N \) lies in the dense orbit.

In general, let us consider a pre-homogeneous \( M \)-module \( V \), with open orbit \( \mathcal{O} \) and a vector \( v \in \mathcal{O} \) with stabilizer \( S \). Take a Borel subgroup \( H \subset M \). If \( U \subset V \) is an \( H \)-stable subspace we can define
\[
A_U = \{ g \in M | g^{-1}v \in U \}.
\]

**Lemma 2.** \( A_U \) is an either empty or a smooth closed subvariety in \( M \).

**Proof.** Let \( p : M \to \mathcal{O} \subset V \) be defined by \( p(g) = g^{-1}v \). Remark that the map \( p \) is smooth and identifies \( \mathcal{O} \) with \( S\backslash M \).

Also by definition, \( A_U \) is the pre-image in \( M \) of \( \mathcal{O} \cap U \). Since \( \mathcal{O} \cap U \) is closed in \( \mathcal{O} \), we deduce that \( A_U \) is closed in \( M \). Now either \( \mathcal{O} \cap U = \emptyset \) or \( \mathcal{O} \cap U \) is an open set in the vector space \( U \), hence smooth. From the fact that \( p \) is smooth, it follows that \( A_U \) is smooth.

Now notice that \( A_U \) is stable under right multiplication by \( H \). We take now the quotient map \( \pi : M \to M/H \) and set \( X_U = \pi(A_U) \). \( A_U = \pi^{-1}(X_U) \) so since \( \pi \) is closed and smooth, we deduced that \( X_U \subset M/B \) is closed and smooth (moreover, since \( M/B \) is projective, \( X_U \) is also projective).

Thus for any \( H \)-stable subspace \( U \subset V \), we get a possibly empty smooth projective variety \( X_U \subset M/H \). Also notice that if \( P_U \) is the largest parabolic subgroup preserving \( U \), then by the same construction we obtain a possibly empty smooth projective variety \( Y_U \subset M/P_U \) and a locally trivial \( P_U/H \) fiber bundle \( f : X_U \to Y_U \).

Finally, let us remark that the group \( S \) acts transitively on the set of connected components of \( X_U \) (and \( Y_U \)). Furthermore, if \( S \) is finite, as we will assume from now on, we deduce from our construction that \( \dim A_U = \dim U \) and that \( \dim X_U = \dim U - \dim H \), \( \dim Y_U = \dim U - \dim P_U \). In particular, if \( \dim U < \dim P_U \), \( X_U = \emptyset \).

Our second reduction is the following (see [5], Section 3.7),

**Reduction 2.** Let \( N \in g \) be a distinguished nilpotent element, \( G_0 \), \( g_2 \) be as above and \( B_0 \subset G_0 \) a Borel subgroup of \( G_0 \). Then \( B_N \) can be paved if for any \( B_0 \) stable subspace \( U \subset g_2 \), for which \( X_U \neq \emptyset \), \( X_U \) or \( Y_U \) can be paved.

In view of this reduction, we are going to first look at the general situation of a pre-homogeneous \( M \)-module \( V \) with an open orbit \( \mathcal{O} \) and a vector \( v \in \mathcal{O} \). For a fixed Borel subgroup \( H \subset M \), we consider the set \( \Gamma \) of \( H \) stable subspaces. We can define a graph whose vertices are the subspaces \( U \in \Gamma \) and whose edges are the pairs \( (U, U') \) such that

1. \( U \) is a hyperplane in \( U' \).
there is a minimal parabolic $P \supset H$ such that $P \subset P_U'$ and $P \not\subset P_U$.

(3) $U'' := \cap_{p \in P} pU$ is an hyperplane in $U$.

One has the following lemma.

**Lemma 3.** Let $(U, U')$ be an edge with $P$ a minimal parabolic preserving $U'$ and not $U$ as above. Let $U'' \subset U$ be the corresponding hyperplane. Take the blow up $B\ell_{X_U''}(X_U')$ of $X_U'$ along $X_U''$.

(1) $B\ell_{X_U''}(X_U') \simeq \{(gH, g'H) | g^{-1}v \in U, g' \in gP\}$.

(2) Using the above isomorphism and projecting onto the first factor, we get a projection $p : B\ell_{X_U''}(X_U') \rightarrow X_U$ which is a $\mathbb{P}^1$ bundle with a section $s$ given by the diagonal $\{(gH, gH) | g^{-1}v \in U\}$.

**Proof.** The proof is contained in [5], Lemma 2.11 except the last assertion on the section which is trivial.

Notice that since $S$ permutes transitively the set of connected components of $X_U$, it make sense to say that if a connected component of $X_U$ is unirational, $X_U$ is unirational.

A simple consequence of Lemma 3 is the following.

**PROPOSITION 4**

Under the assumptions of Lemma 3, each connected component of $X_{U'}$ contains a single connected component of $X_U$. In particular, $X_U \neq \emptyset$ if and only if $X_{U'} \neq \emptyset$.

Furthermore, $X_U$ is unirational if and only if $X_{U'}$ is unirational.

**Proof.** The first part is an immediate consequence of our lemma.

Using the bundle $p : B\ell_{X_U''}(X_U') \rightarrow X_U$, we immediately deduce that if $X_{U'}$ is unirational, $X_U$ is also unirational.

Assume now that $X_U$ is unirational. Take a connected component $Z$ of $X_U$. Set $Z' = p^{-1}(Z)$ a connected component of $X_{U'}$. The existence of the section $s$ implies that the fiber of $p$ over the generic point of $Z$ is $\mathbb{P}^1(K)$, $K$ being the function field of $Z$. Thus the function field of $Z'$ is $K(t)$ with $t$ an indeterminate and $X_{U'}$ is unirational.

From now on, we shall consider the subgraph $\Gamma^*$ of $\Gamma$ whose vertices are the subspaces $U$ such that $X_U \neq \emptyset$.

Remark that $\Gamma^*$ is a union of connected components of the previously considered graph and that furthermore,

(1) If $U$ and $W$ lie in the same connected component of $\Gamma^*$, then the set of (connected) components of $X_U$ and $X_W$ are isomorphic as $S$-sets.

(2) If $X_U$ is unirational and $W$ lies in the same connected component of $\Gamma^*$ as $U$, then also $X_W$ is unirational.

Another simple but useful consequence is the following.
PROPOSITION 5

Let $U \in \Gamma^*$ be such that $X_U$ is unirational and $\dim Y_U \leq 2$. Then $X_U$ is rational and admits an affine paving.

Proof. Recall that there is a locally trivial $P_U/H$- bundle $q : X_U \to Y_U$. It follows that $X_U$ is rational and admits an affine paving if and only if $Y_U$ is rational and admits an affine paving.

If $X_U$ is unirational, $Y_U$ is also unirational. Since $\dim Y_U \leq 2$, $Y_U$ is rational and admits an affine paving (see, for example, [14], Lemma 6.6). \hfill $\square$

Let us go back to the cases arising from a distinguished nilpotent element $N \in \mathfrak{g}$. We choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ with $\psi(h) \in \mathfrak{h}$. We also choose the set of simple roots $\Pi$ in such a way that if $\alpha \in \Pi$, $\alpha(\psi(h)) \geq 0$. In fact, $\alpha(\psi(h))$ is either 0 or 2 [6], so we may write $\Pi = \Pi_0 \cup \Pi_2$ with a clear meaning.

It follows that we can label the Dynkin diagram of $\mathfrak{g}$, whose nodes we identify with $\Pi$, with the labels $\alpha(\psi(h)) \in \{0, 2\}$. This labeled diagram is called the Dynkin diagram of $N$.

Notice that the subdiagram whose nodes have label zero is clearly the Dynkin diagram of the semisimple part $\mathfrak{g}_0 = [\mathfrak{g}_0, \mathfrak{g}_0]$ and $\mathfrak{g}_0 = \mathfrak{g}_0 \oplus t$, $t \subset \mathfrak{h}$ is the subalgebra $t = \{x \in \mathfrak{h} | \alpha(x) = 0, \forall \alpha \in \Pi_0\}$.

Now let $\Pi'_2$ be the set of simple roots in $\Pi_2$ orthogonal to $\Pi_0$. Take the subalgebra $\bar{\mathfrak{g}}$ generated by the $\mathfrak{g}_{\pm \beta}$, $\beta \in \Pi \setminus \Pi'_2$.

Clearly the Dynkin diagram for $\bar{\mathfrak{g}}$ is just the diagram obtained by removing the nodes in $\Pi'_2$. This is a labeled Dynkin diagram and hence $\bar{\mathfrak{g}}$ has a grading for which $\bar{\mathfrak{g}}_0' = \bar{\mathfrak{g}}_0'$.

Set $\ell = \oplus_{\alpha \in \Pi'_2} \mathfrak{g}_\alpha$ and $\mathfrak{h} = \oplus_{\alpha \in \Pi'_2} [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$. The following facts are clear:

1. $\mathfrak{g}_2 = \ell \oplus \bar{\mathfrak{g}}_2$.
2. $\mathfrak{g}_0 = \mathfrak{h} \oplus \bar{\mathfrak{g}}_0$.
3. If we write $N = m + \bar{N}$ with $m \in \ell$ and $\bar{N} \in \bar{\mathfrak{g}}_2$, and $\psi(h) = k + \bar{h}$ with $k \in \mathfrak{h}$ and $\bar{h} \in \bar{\mathfrak{g}}_0$, then $N$ is distinguished and there is a homomorphism $\bar{\psi} : sl(2) \to \bar{\mathfrak{g}}$ such that $\bar{\psi}(e) = \bar{N}$, $\bar{\psi}(h) = \bar{h}$ such that our grading on $\bar{\mathfrak{g}}$ is induced by the adjoint action of $\bar{\psi}(h) = \bar{h}$.

Now let us denote by $\bar{G}_0 \subset G_0$, the subgroup with Lie $\bar{G}_0 = \bar{\mathfrak{g}}_0$ and let $\bar{B}_0 = B_0 \cap \bar{G}_0$. We have that, since $\bar{G}_0$ and $G_0$ have the same semisimple part, $G_0/B_0 = \bar{G}_0/\bar{B}_0$. Take $\Gamma$ equal to the set of $\bar{B}_0$ subspaces in $\bar{\mathfrak{g}}_0$. Define a map $\gamma : \bar{\Gamma} \to \Gamma$

by $\gamma(\bar{U}) = \mathfrak{h} \oplus \bar{U}$. Then we leave it to the reader the immediate verification of the following.

PROPOSITION 6

1. If $\bar{U} \in \bar{\Gamma}$, $X_\bar{U} = X_{\gamma(\bar{U})}$.
2. If $U \in \Gamma$ and $X_U \neq \emptyset$, then $U \in \gamma(\bar{\Gamma})$.

It follows that we can analyze only the distinguished nilpotents for which $\Pi'_2 = \emptyset$. The complete list of the corresponding diagrams for exceptional groups can be found in [1] and we need to study each of them.
Finally, we prove this simple lemma which we are going to use later on.

**Lemma 7.** Let $V$ be an affine space and let $W \subset V$ be an affine subspace. Then the blow up of $V$ in $W$ has a paving.

*Proof.* We can assume that $0 \in W$. Take a linear complement $U$ of $W$ in $V$. Notice that the blow up of $V$ in $W$ is isomorphic to $W \times \mathbb{P}(U)$, where $Z$ is a blow up of the origin in $U$. So it is enough to study the case where $W$ is the origin.

In this case the blow up is isomorphic to the tautological line bundle $\pi : L \rightarrow \mathbb{P}(V)$ over $\mathbb{P}(V)$. We now proceed by induction on $\dim V$. If $\dim V = 1$ is trivial, assume that $\dim V > 1$. If we restrict this line bundle to an hyperplane $\mathbb{P}(H)$ of $\mathbb{P}(V)$, then by induction, we see that $\pi^{-1}(\mathbb{P}(H))$ can be paved. On the complement of $\mathbb{P}(H)$ the line bundle is trivial, hence we obtain a single open cell. \hfill $\square$

3. **Distinguished nilpotent classes in $G_2$, $F_4$, $E_6$**

We are going to go through the list in [2] and discuss each case. The results in this section are well known but we give a very simple illustration of our method.

**Type $G_2$.** The only case in type $G_2$ is

$$
\begin{array}{c}
2 \\
0
\end{array}
$$

hence the semisimple rank of $G_0$ is 1 and $g_0 = \mathfrak{sl}(V) \oplus \mathbb{C}$, $\dim V = 2$. $G_0/B_0 = \mathbb{P}(V) = \mathbb{P}^1$. All the other $X_U$ are either empty or of dimension 0 (indeed the only non empty one consists of 3 points) and there is nothing to prove.

**Type $F_4$.** There are two cases in type $F_4$. The first is

$$
\begin{array}{c}
0 \\
2 \\
0 \\
2
\end{array}
$$

Here the semisimple rank of $G_0$ is 2 and $g_0 = \mathfrak{sl}(V_1) \oplus \mathfrak{sl}(V_2) \oplus \mathbb{C}^2$, $\dim V_1 = \dim V_2 = 2$. Then $G_0/B_0 = \mathbb{P}(V_1) \times \mathbb{P}(V_2) = \mathbb{P}^1 \times \mathbb{P}^1$, and $g_2$ is the sum of two irreducible submodules, so that there are two $B_0$ stable hyperplanes which both lie in the same connected component of $\Gamma$ containing $U = g_2$. So both $X_U$ are rational irreducible curves, hence isomorphic to $\mathbb{P}^1$. All the other $X_U$ are either empty or of dimension 0 and there is nothing to prove.

The second case is

$$
\begin{array}{c}
0 \\
2 \\
0 \\
0
\end{array}
$$

Here the semisimple rank of $G_0$ is 2 and $g_0 = \mathfrak{sl}(V_1) \oplus \mathfrak{sl}(V_2) \oplus \mathbb{C}$, $\dim V_1 = 2$, $\dim V_2 = 3$. $G_0/B_0 = \mathbb{P}(V_1) \times \mathcal{F}(V_2)$, $\mathcal{F}(V_2)$ being the flag variety of $V_2$. So $\dim G_0/B_0 = 4$. 
This case is described in a detailed way in [5], Section 4.2 to which we refer for details. The graph $\Gamma^*$ has 9 vertices and 5 connected components.

**Type $E_6$.** The only case in type $E_6$ is

Here the semisimple rank of $G_0$ is 3 and $g_0 = sl(V_1) \oplus sl(V_2) \oplus sl(V_3) \oplus \mathbb{C}^3$, $\dim V_1 = \dim V_2 = \dim V_2 = 2$. $G_0/B_0 = \mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \mathbb{P}(V_3)$. So $\dim G_0/B_0 = 3$. As a $g_0$ module, $g_2$ has 3 irreducible components so that there are 3 $B_0$ stable hyperplanes $U$ which lie in the connected component of $\Gamma^*$ containing $U = g_2$. This connected component consists of the $B_0$ stable subspaces containing the intersection $\bar{U}$ of the 3 hyperplanes. $X_{\bar{U}}$ consists of a single point. There is a further connected component of $\Gamma^*$. For each $U$ in this connected component, $Y_U$ consist of two points and we are done.

### 4. Distinguished nilpotent classes in $E_7$

In this and the following sections, we are finally going to show the existence of an affine paving for Springer fibers in type $E_7$. We know that we have to analyze the case in which $N$ is a distinguished nilpotent.

Furthermore, looking at the classification in [1] and using Proposition 6, we need to examine the two nilpotent classes $E_7(a_4)$ and $E_7(a_5)$.

#### 4.1 The case $E_7(a_4)$

The Dynkin diagram for $E_7(a_4)$ is

In this case, $G_0 = SL(V_A) \times SL(V_B) \times SL(V_C) \times (\mathbb{C}^*)^3$ with $\dim V_A = \dim V_B = 2$ and $\dim V_C = 3$. The module $g_2$ is equal to $V_B \oplus V_C \oplus V_A^* \otimes V_B^* \otimes V_C^*$ with the obvious action of $SL(V_A) \times SL(V_B) \times SL(V_C)$ and $(\mathbb{C}^*)^3$ acting by

$$(x, y, z) \cdot (u, v, Q) = (xu, yv, zQ).$$

for $(x, y, z) \in (\mathbb{C}^*)^3$, $(u, v, Q) \in g_2$.

The action of $g_0$ on $g_2$ factors through the homomorphism $\psi : g_0 \to gl(V_A) \oplus gl(V_B) \oplus gl(V_C)$ given by $\psi((a, b, c, x, y, z)) = (z - x - y)I + a, xI + b, yI + c)$, $I$ being the identity matrix.

Let us now determine an element $h$ in the open orbit in $g_2$. It suffices to choose $h$ in such a way that the annihilator of $h$ in $g_0$ is trivial. Choosing bases we represent an element of
$V_B$ as a column vector, an element of $V_C$ as a row vector and if $S$ and $T$ are $3 \times 2$ matrices, we write $\lambda S + \mu T$ for the form in $V_A^* \otimes V_B^* \otimes V_C^*$ given by

$$(\lambda S + \mu T)(x, y, u, v) = x \cdot v \cdot S \cdot u + y \cdot v \cdot T \cdot u$$

for all $(x, y) \in V_A, u \in V_B$ and $v \in V_C$. Using these coordinates, we choose $h$ as follows:

$$h = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}. $$

Let $Z$ be an element of $\mathfrak{gl}(V_A) \oplus \mathfrak{gl}(V_B) \oplus \mathfrak{gl}(V_C)$ and write it as

$$Z := \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}, \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{pmatrix}.$$

Applying $Z$ to $h$, we get

$$Z \cdot h = \begin{pmatrix} x_{11} + x_{12} \\ x_{21} + x_{22} \end{pmatrix}, (z_{21}, z_{22}, z_{23}),$$

$$\begin{pmatrix} \lambda (x_{11} + y_{11} + z_{11}) + \mu (z_{12} + y_{12}) \\ \lambda (x_{21} + z_{21} + y_{21}) + \mu (x_{11} + z_{22} + y_{22}) \end{pmatrix} \begin{pmatrix} \lambda (x_{11} + z_{12} + z_{13}) + \mu z_{13} \\ \lambda (z_{31} + \mu (x_{21} + z_{32}) \end{pmatrix}.$$

A straightforward computation then shows that if $Z \cdot h = 0$, then $Z = 0$ as desired. Consider the standard flags $F_A^0 = \{0\} \subset F_A^1 = \mathbb{C}e_1 \subset F_A^2 = V_A$ and $F_B^0 = \{0\} \subset F_B^1 = \mathbb{C}e_1 \subset F_B^2 = V_B$ for $V_A$ and $V_B$ and the standard reverse flag $F_A^0 = \{0\} \subset F_A^1 = \mathbb{C}e_1 \subset F_A^2 = \mathbb{C}e_2, e_3 \subset F_A^3 = V_A$ for $V_C$. We denote by $B_0$ the associated Borel subgroup of $G_0$. We notice that every $B_0$-stable subspace of $V_A^* \otimes V_B^* \otimes V_C^*$ is of the form

$$U(k_1, k_2|h_1, h_2) = \{ q : q(F_A^1 \otimes F_B^i \otimes F_C^{h_i}) = q(V_A \otimes F_B^i \otimes F_C^{k_i}) = 0 \text{ for } i = 1, 2 \},$$

where $3 \geq k_1 \geq k_2 \geq 0$ and $h : 3 \geq h_2 \geq h_1 \geq 0$ are decreasing sequences such that $h_1 \geq k_1$ and $h_2 \geq k_2$. Hence any $B_0$-submodule of $g_2$ is of the form

$$U(k_1, k_2|h_1, h_2|\ell|m) = F_B^\ell \otimes F_C^m \otimes U(k_1, k_2|h_1, h_2)$$

where the pairs $k_1, k_2$ and $h_1, h_2$ are as above and $\ell \leq 2$ and $m \leq 3$ are positive integers. The corresponding variety $X_U$ can be described as follows:

$$X_U = \{(L_A, L_B, L_C \subset L_C^2) \in \mathbb{P}(V_A) \times \mathbb{P}(V_B) \times \mathbb{P}(V_C) : e_1 + e_2 \in L_C^\ell, e_2 \in L_C^m, Q(L_A \otimes L_B^i \otimes L_C^{h_i}) = Q(V_A \otimes L_B^i \otimes V_C^{k_i}) = 0 \text{ for } i = 1, 2, 3 \},$$

where $\mathbb{P}(\mathcal{F})$ denote the flag variety and where we set $L_B^2 = V_B, L_B^0 = \{0\}, L_C^3 = V_C$ and $L_C^0 = \{0\}$.

4.2 The graph $\Gamma^*$

For the convenience of the reader, since it does not appear explicitly in [5], we now describe the graph $\Gamma^*$ of Section 2 In this case, the graph $\Gamma^*$ has two connected components. For each connected component, we list the vertices of the connected component and for each
they are characterised by the number of connected components of the space characterised by the action of the stabilisers of the base point. Equivalently, in this case, the columns the value of the dimension of $X_U$ has dimension 3, namely $m = 0 \in \{0, 1, 2\}$.

First connected component.

The elements of this connected component are listed in Table 1. This component consists of all subspaces $U$ such that $X_U$ is connected. Each element $U$ is written as $U = U(k_1, k_2 \mid h_1, h_2 \mid \ell \mid m)$. On the rows we give the value of $k_1, k_2, h_1, h_2$ and of $\ell$ and on the columns the value of $m$. In the single case reported in the Table 1 where you find “not here” means that the corresponding $U$ does not appear in this connected component. All other $U$ reported in the table are either in this connected component or empty. For $U = g_2$, $X_U$ is the flag variety of $G_0$ hence it is rational. In particular, by Proposition 4, all $X_U$ are unirational.

By direct inspection, we then deduce that there is no subspace $U$ for which $Y_U$ has dimension 4 or larger and two subspaces for which $Y_U$ has dimension 3, namely $U(0, 0[1, 0][2][3])$ and $U(0, 0[1, 0][2][2])$.

Second connected component.

The second connected component of the graph contains only one element: the space $U(1, 0[2], 1[2][2])$. The space $X_U$ in this case has two points.
Table 2. The equations $Q_{1,0}(L_B^1, L_C^1) = 0$ and $Q_{\lambda,1}(L_B^1, L_C^1) = 0$ in coordinates.

| $Q_{1,0}$ | $t_1 = 1$ | $t_1 = 2$ | $t_1 = 3$ |
|-----------|------------|------------|------------|
| $b = 21$  | $x + y_2$  | $1$        | $0$        |
| $b = 12$  | $1$        | $0$        | $0$        |

| $Q_{\lambda,1}$ | $t_1 = 1$ | $t_1 = 2$ | $t_1 = 3$ |
|-----------------|------------|------------|------------|
| $b = 21$        | $y_3 + xy_2 + \lambda x + \lambda y_2$ | $x + y_3 + \lambda$ | $1$        |
| $b = 12$        | $y_2 + \lambda$ | $1$        | $0$        |

4.3 Pavings

We now prove that all $X_U$ have a paving. We have already seen that all non empty $X_U$ are unirational. Thus if $\dim Y_U \leq 2$, we can conclude, by Proposition 5, that it has a paving.

It remains to study the two cases $U(0, 0|1, 0|2|3)$ and $U(0, 0|1, 0|2|2)$ for which $\dim Y_U = 3$. We consider Schubert cells in the flag variety of $G_0$ stable by the action of $B_0$. These are products

$$\mathcal{A}_a \times S_b \times T_T \subset \mathbb{P}(V_A) \times \mathbb{P}(V_B) \times \mathcal{F}\ell(V_C),$$

where $T \in S_3$ and $a, b \in S_2$. We introduce coordinates on the Schubert cells as follows: $\mathcal{A}_{12} = \{(1, 0)\}$ and $\mathcal{A}_{21} = \{[\lambda, 1] : \lambda \in \mathbb{C}\}$. Similarly $S_{12} = \{(1, 0)\}$ and $S_{21} = \{[x, 1] : x \in \mathbb{C}\}$ and the coordinates on the Schubert cells of $\mathcal{F}\ell(V_C)$ are as follows:

$$T_{123} : \begin{pmatrix} 1 & 0 \\ y_2 & 1 \\ y_3 & y_3' \end{pmatrix}, \quad T_{132} : \begin{pmatrix} 1 & 0 \\ y_2 & 0 \\ y_3 & 1 \end{pmatrix}, \quad T_{213} : \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ y_3 & y_3' \end{pmatrix},$$

$$T_{231} : \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ y_3 & 1 \end{pmatrix}, \quad T_{312} : \begin{pmatrix} 0 & 1 \\ 0 & y_2 \\ 1 & 0 \end{pmatrix}, \quad T_{321} : \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix},$$

meaning that the first column span $L_C^1$ and the first two columns $L_C^2$.

Case $U(0, 0|1, 0|2|3)$.

This space is described by the equation $Q(L_A, L_B^1, L_C^1) = 0$. We choose a Schubert cell and we write down in Table 2 the values of $Q_{1,0}(L_B^1, L_C^1)$ and $Q_{\lambda,1}(L_B^1, L_C^1)$ using the coordinates introduced above.

We immediately see that in any case the equation $Q(L_A, L_B^1, L_C^1) = 0$ defines an affine space in the Schubert cell.

Case $U(0, 0|1, 0|2|2)$.

In this case, we have two equations: $Q(L_A, L_B^1, L_C^1) = 0$ and $e_2 \in L_C^2$.

In the cases $T = 132$ and $T = 312$, we see that the second condition can never be satisfied, hence the intersection with the Schubert cell is empty.
In the cases \( T = 321 \) and \( T = 231 \), we see that the second condition is always satisfied, hence the intersection is the same as we have studied in the previous case.

In the case \( T = 213 \), we see that the second condition is equivalent to \( y_3 = 0 \). This equation has to be added to the equations of the column \( t_1 = 2 \) in Table 2 and we see that we always obtain an affine space.

In the case \( T = 123 \), we see that the second condition is equivalent to \( y_3' = 0 \). This equation has to be added to the equations of the column \( t_1 = 1 \) in Table 2 where this variable does not appear. Hence we obtain again an affine space.

4.4 The case \( E_7(a_5) \)

The Dynkin diagram for \( E_7(a_5) \) is

![Dynkin diagram](image)

In this case \( G_0 = SL(V_A) \times SL(V_B) \times SL(V_C) \times \mathbb{C}_x^* \times \mathbb{C}_y^* \), where \( V_A = \mathbb{C}^2 \), \( V_B = \mathbb{C}^3 \) and \( V_C = \mathbb{C}^3 \). The space \( g_2 \) is equal to \( V_B \oplus V_A^* \otimes V_B^* \otimes V_C^* \) where the action of \( SL(V_A) \times SL(V_B) \times SL(V_C) \) is the natural one and the action of \( \mathbb{C}_x^* \times \mathbb{C}_y^* \) is given by

\[
(x, y) \cdot (v, Q) = (xv, yQ).
\]

We can choose as a base point of the open \( G_0 \)-orbit in \( g_2 \) the element \( h = (v_0, Q) \), where

\[
v_0 = e_1 + e_2 + e_3
\]

and \( Q \in V_A^* \otimes V_B^* \otimes V_C^* \cong \text{Hom}(V_A, V_B^* \otimes V_C^*) \) is the following plane form:

\[
Q((\lambda, \mu), (x_1, x_2, x_3), (y_1, y_2, y_3)) = Q_{\lambda, \mu}((x_1, x_2, x_3), (y_1, y_2, y_3)) = \lambda x_1 y_1 + (\lambda + \mu) x_2 y_2 + \mu x_3 y_3.
\]

As a matrix, \( Q \) is represented by the following matrix:

\[
\begin{pmatrix}
\lambda & 0 & 0 \\
0 & \lambda + \mu & 0 \\
0 & 0 & \mu
\end{pmatrix}.
\]

In a completely analogous way to what was done in the case \( E_7(a_4) \), one can show that the map \( \varphi : g_0 \to g_2 \) given by \( x \mapsto x \cdot h \) is an isomorphism, hence the stabiliser of \( h \) in \( G_0 \) is finite and its orbit is dense.

Let \( F_A^1 \subset F_A^2 = V_A \) and \( F_B^1 \subset F_B^2 \subset F_B^3 = V_B \) be the standard complete flags of \( V_A \) and \( V_B \) and let \( F_C^1 = (e_3) \subset F_C^2 = (e_2, e_3) \subset F_C^3 = V_C \) be the standard reverse flag. We denote by \( B_0 \) the corresponding Borel subgroup of \( G_0 \). Similar to the previous case, the \( B_0 \)-stable subspace of \( g_2 \) are of the form

\[
U(k_1, k_2, k_3| h_1, h_2, h_3| \ell) = F_B^\ell \oplus U(k_1, k_2, k_3| h_1, h_2, h_3),
\]
Since the graph does not appear in [5], for the convenience of the reader, we now describe \( F \) and \( U \) where \( \Gamma(U) \) is written as \( \dim(U) = d, \delta = 0 \) and \( \dim(U) = d, \delta = 1 \) for the connected component of all subspaces \( U \) such that \( X_U \) is connected.

### Table 3. The first connected component of \( \Gamma^* \) in the case \( E_7(a_5) \).

| \( U \)                                | \( F_B^3 \) | \( F_B^2 \) | \( F_B^1 \) |
|----------------------------------------|-------------|-------------|-------------|
| \( U(0, 0, 0|0, 0, 0) \)                  | \( \dim = 7, \delta = 0 \) | \( \dim = 6, \delta = 1 \) | \( \dim = 5, \delta = 0 \) |
| \( U(0, 0, 0|1, 0, 0) \)                  | \( \dim = 6, \delta = 4 \) | \( \dim = 5, \delta = 4 \) | \( \dim = 4, \delta = 2 \) |
| \( U(1, 0, 0|1, 0, 0) \)                  | \( \dim = 5, \delta = 2 \) | \( \dim = 4, \delta = 2 \) | \( \dim = 3, \delta = 0 \) |
| \( U(0, 0, 0|2, 0, 0) \)                  | \( \dim = 5, \delta = 3 \) | \( \dim = 4, \delta = 3 \) | \( \dim = 3, \delta = 1 \) |
| \( U(1, 0, 0|1, 1, 0) \)                  | \( \dim = 5, \delta = 3 \) | \( \dim = 4, \delta = 2 \) | \( \dim = 3, \delta = 2 \) |
| \( U(0, 0, 0|2, 0, 0) \)                  | \( \dim = 4, \delta = 3 \) | \( \dim = 3, \delta = 3 \) | \( \dim = 2, \delta = 1 \) |
| \( U(1, 0, 0|1, 1, 0) \)                  | \( \dim = 4, \delta = 3 \) | \( \dim = 3, \delta = 2 \) | \( \dim = 2, \delta = 2 \) |
| \( U(0, 0, 0|2, 1, 0) \)                  | \( \dim = 4, \delta = 4 \) | \( \dim = 3, \delta = 3 \) | \( \dim = 2, \delta = 2 \) |
| \( U(1, 0, 0|2, 1, 0) \)                  | \( \dim = 3, \delta = 3 \) | \( \dim = 2, \delta = 2 \) | \( \dim = 1, \delta = 1 \) |

where \( \ell \leq 3, k_1 \geq k_2 \geq k_3 \) and \( h_1 \geq h_2 \geq h_3 \) are decreasing sequences such that \( k_i < h_i \) and \( U(k_1, k_2, k_3|h_1, h_2, h_3) \) is the \( B_M \)-stable subspace of \( V_A^* \otimes V_B^* \otimes V_C^* \) defined by

\[
U(k_1, k_2, k_3|h_1, h_2, h_3) = \{ q : q(F_A^1 \otimes F_B^i \otimes F_C^{h_i}) = q(V_A \otimes F_B^i \otimes F_C^{k_i}) = 0 \\
\text{for } i = 1, 2, 3 \}.
\]

For any such \( U \), the variety \( X_U \) can be described as the subvariety of the flag variety of \( G_0 \) as follows:

\[
X_U = \{(L_A^1, L_B^1 \subset L_B^2, L_C^1 \subset L_C^2) \in \mathbb{P}(V_A) \times \mathcal{F}\ell(V_B) \times \mathcal{F}\ell(V_C) : v_0 \in L_B^\ell, \\
Q(e_1 \otimes L_B^i \otimes L_C^{h_i}) = Q_0(V_A \otimes L_B^i \otimes L_C^{k_i}) = 0 \text{ for } i = 1, 2, 3 \},
\]

where \( \mathcal{F}\ell \) denote the complete flags variety and where we set \( L_B^3 = V_B, L_B^0 = \{0\}, \\
L_C^3 = V_C \) and \( L_C^0 = \{0\} \).

Remark that each bilinear form in a generic pencil of bilinear form on \( V_B \times V_C \) has at least rank two. Hence if \( U(k_1, k_2, k_3|h_1, h_2, h_3) \) intersects the open orbit in \( V_A^* \otimes V_B^* \otimes V_C^* \) we must have \( h_2 \leq 2 \) and \( h_3 \leq 1 \). Moreover, since such a plane has only three singular lines which have different radicals in \( V_B \) and \( V_C \), we also see that if \( U(k_1, k_2, k_3|h_1, h_2, h_3) \) intersects the open orbit we must also have \( k_1 \leq 2, k_2 \leq 1 \) and \( k_3 = 0 \).

### 4.5 The graph \( \Gamma^* \)

Since the graph does not appear in [5], for the convenience of the reader, we now describe the graph \( \Gamma^* \) of Section 2. In this case, the graph \( \Gamma^* \) has three connected components. As in the case of \( E_7(a_4) \), we list the vertices of each connected component giving the dimension of \( X_U \) and the dimension of \( Y_U \).

#### 4.5.1 First connected component

Table 3 shows the connected component of all subspaces \( U \) such that \( X_U \) is connected. Each element \( U \) is written as \( U = U(k_1, k_2, k_3|h_1, h_2, h_3|\ell) \). In the rows, we give the values of \( k_1, k_2, k_3, h_1, h_2, h_3 \) and in the columns, we give the value of \( \ell \).
Table 4. The second connected component of $\Gamma^*$. 

| $F^3_B$ | $F^2_B$ | $F^1_B$ |
|---------|---------|---------|
| $U(0, 0, 0|3, 0, 0)$ | dim = 4, $\delta = 0$ | dim = 3, $\delta = 0$ | $\emptyset$ |
| $U(0, 0, 0|1, 1, 1)$ | dim = 4, $\delta = 0$ | dim = 3, $\delta = 1$ | dim = 2, $\delta = 0$ |
| $U(2, 0, 0|2, 0, 0)$ | dim = 3, $\delta = 0$ | dim = 2, $\delta = 0$ | $\emptyset$ |
| $U(1, 1, 0|1, 1, 0)$ | dim = 3, $\delta = 0$ | dim = 2, $\delta = 1$ | $\emptyset$ |
| $U(1, 0, 0|3, 0, 0)$ | dim = 3, $\delta = 1$ | dim = 2, $\delta = 1$ | $\emptyset$ |
| $U(0, 0, 0|1, 1, 1)$ | dim = 3, $\delta = 1$ | dim = 2, $\delta = 1$ | $\emptyset$ |
| $U(0, 0, 0|2, 2, 0)$ | dim = 3, $\delta = 1$ | dim = 2, $\delta = 0$ | dim = 1, $\delta = 0$ |
| $U(0, 0, 0|2, 1, 1)$ | dim = 3, $\delta = 2$ | dim = 2, $\delta = 2$ | dim = 1, $\delta = 0$ |
| $U(2, 0, 0|3, 0, 0)$ | dim = 2, $\delta = 0$ | dim = 1, $\delta = 0$ | $\emptyset$ |
| $U(2, 0, 0|2, 1, 0)$ | dim = 2, $\delta = 2$ | dim = 1, $\delta = 1$ | $\emptyset$ |
| $U(1, 1, 0|2, 1, 0)$ | dim = 2, $\delta = 2$ | $\emptyset$ | $\emptyset$ |
| $U(1, 1, 0|1, 1, 1)$ | dim = 2, $\delta = 0$ | $\emptyset$ | $\emptyset$ |
| $U(1, 0, 0|3, 1, 0)$ | dim = 2, $\delta = 1$ | dim = 1, $\delta = 0$ | $\emptyset$ |
| $U(1, 0, 0|2, 2, 0)$ | dim = 2, $\delta = 2$ | dim = 1, $\delta = 1$ | dim = 0, $\delta = 0$ |
| $U(1, 0, 0|2, 1, 1)$ | dim = 2, $\delta = 1$ | dim = 1, $\delta = 1$ | $\emptyset$ |
| $U(0, 0, 0|3, 2, 0)$ | dim = 2, $\delta = 1$ | dim = 1, $\delta = 0$ | $\emptyset$ |
| $U(0, 0, 0|3, 1, 1)$ | dim = 2, $\delta = 0$ | dim = 1, $\delta = 0$ | $\emptyset$ |
| $U(0, 0, 0|2, 2, 1)$ | dim = 2, $\delta = 1$ | dim = 1, $\delta = 0$ | dim = 0, $\delta = 0$ |
| $U(2, 0, 0|3, 1, 0)$ | dim = 1, $\delta = 1$ | dim = 0, $\delta = 0$ | $\emptyset$ |
| $U(1, 1, 0|2, 1, 1)$ | dim = 1, $\delta = 1$ | $\emptyset$ | $\emptyset$ |
| $U(1, 0, 0|3, 2, 0)$ | dim = 1, $\delta = 1$ | dim = 0, $\delta = 0$ | $\emptyset$ |
| $U(1, 0, 0|2, 2, 1)$ | dim = 1, $\delta = 1$ | dim = 0, $\delta = 0$ | $\emptyset$ |
| $U(0, 0, 0|3, 2, 1)$ | dim = 1, $\delta = 1$ | dim = 0, $\delta = 0$ | $\emptyset$ |

For $U = \mathfrak{g}_2$, $X_U$ is the flag variety of $G_0$, hence it is rational. In particular, by Proposition 4, all $X_U$ are unirational.

### 4.5.2 Second connected component

Table 4 shows the connected component of $\Gamma^*$ of all subspaces $U$ such that $X_U$ has three connected components. Notice that since it contains subspaces $U$ such that $Y_U$ is not empty and dim $Y_U = 0$ then, by Proposition 4, all $Y_U$ and all $X_U$ for $U$ in this component of $\Gamma^*$ are unirational.

### 4.5.3 Third connected component

Table 5 shows the connected component of all subspaces $U$ such that $X_U$ has six connected components.

We list these spaces in Table 5.
We are now going to prove that all \( X_U \) have a paving (by affine spaces). We noticed that all non empty varieties \( X_U \) in this case are unirational, hence if \( \dim Y_U \leq 2 \), then the varieties \( X_U \) and \( Y_U \) have a paving, by Proposition 5.

It remains to study \( X_U \) when \( U \) is as in Table 6.

We consider Schubert cells in the flag variety of \( G_0 \) which are orbits under the action of \( B_0 \). These are products

\[ A_a \times S_T \times T_T, \]

where \( S, T \in S_3 \) and \( a \in S_2 \). We introduce coordinates on the Schubert cells as follows: \( A_{12} = \{[1, 0]\} \) and \( A_{21} = \{[\lambda, 1] : \lambda \in \mathbb{C}\} \), and the coordinates on the Schubert cells of

| Table 5. The vertices of the third connected component of the graph \( \Gamma^* \). |
|---|
| \( F_3^B \) \( F_2^B \) \( F_1^B \) |
| \( U(2, 1, 0|2, 1, 0) \) \( \dim = 1, \ \delta = 0 \) \( \emptyset \) \( \emptyset \) |
| \( U(2, 0, 0|2, 2, 0) \) \( \dim = 1, \ \delta = 0 \) \( \emptyset \) \( \emptyset \) |
| \( U(2, 0, 0|2, 1, 1) \) \( \dim = 1, \ \delta = 0 \) \( \dim = 0, \ \delta = 0 \) \( \emptyset \) |
| \( U(1, 1, 0|3, 1, 0) \) \( \dim = 1, \ \delta = 0 \) \( \emptyset \) \( \emptyset \) |
| \( U(1, 1, 0|2, 2, 0) \) \( \dim = 1, \ \delta = 0 \) \( \emptyset \) \( \emptyset \) |
| \( U(2, 1, 0|3, 1, 0) \) \( \dim = 0, \ \delta = 0 \) \( \emptyset \) \( \emptyset \) |
| \( U(2, 1, 0|2, 2, 0) \) \( \dim = 0, \ \delta = 0 \) \( \emptyset \) \( \emptyset \) |
| \( U(2, 1, 0|2, 1, 1) \) \( \dim = 0, \ \delta = 0 \) \( \emptyset \) \( \emptyset \) |
| \( U(2, 0, 0|2, 2, 1) \) \( \dim = 0, \ \delta = 0 \) \( \emptyset \) \( \emptyset \) |
| \( U(1, 1, 0|3, 2, 0) \) \( \dim = 0, \ \delta = 0 \) \( \emptyset \) \( \emptyset \) |

Let us remark that also in this case there is a subspace \( U \) such that \( Y_U \) is not empty and has dimension 0, hence all \( X_U \) and \( Y_U \) are unirational.

4.6 Pavings

We are now going to prove that all \( X_U \) have a paving (by affine spaces). We noticed that all non empty varieties \( X_U \) in this case are unirational, hence if \( \dim Y_U \leq 2 \), then the varieties \( X_U \) and \( Y_U \) have a paving, by Proposition 5.

It remains to study \( X_U \) when \( U \) is as in Table 6.

We consider Schubert cells in the flag variety of \( G_0 \) which are orbits under the action of \( B_0 \). These are products

\[ A_a \times S_S \times T_T, \]

where \( S, T \in S_3 \) and \( a \in S_2 \). We introduce coordinates on the Schubert cells as follows: \( A_{12} = \{[1, 0]\} \) and \( A_{21} = \{[\lambda, 1] : \lambda \in \mathbb{C}\} \), and the coordinates on the Schubert cells of
\( \mathcal{F} \ell(V_B) \) are given as follows:

\[
\mathcal{S}_{321} : \begin{pmatrix} x_1 & x'_2 \\ x_2 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{S}_{312} : \begin{pmatrix} x_1 & 1 \\ x_2 & 0 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{S}_{231} : \begin{pmatrix} x_1 & x'_1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},
\]

\[
\mathcal{S}_{213} : \begin{pmatrix} x_1 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{S}_{132} : \begin{pmatrix} 1 & 0 \\ 0 & x'_2 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{S}_{123} : \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}
\]

and we denote the columns of these matrices by \( v_1, v_2 \) and the space \( L_B^1 \) is spanned by \( v_1 \) and \( L_B^2 \) by \( v_1 \) and \( v_2 \). We use similar notations for the Schubert cells of \( \mathcal{F} \ell(V_C) \):

\[
\mathcal{T}_{123} : \begin{pmatrix} 1 \\ 0 \\ y_2 & 1 \\ y_3 & y'_3 \end{pmatrix}, \quad \mathcal{T}_{132} : \begin{pmatrix} 1 \\ 0 \\ y_2 & 0 \\ y_3 & 1 \end{pmatrix}, \quad \mathcal{T}_{213} : \begin{pmatrix} 0 \\ 1 \\ 1 & 0 \\ y_3 & y'_3 \end{pmatrix},
\]

\[
\mathcal{T}_{231} : \begin{pmatrix} 0 \\ 0 \\ 1 \\ y_3 & 1 \end{pmatrix}, \quad \mathcal{T}_{312} : \begin{pmatrix} 0 \\ 1 \\ 0 & y'_2 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{T}_{321} : \begin{pmatrix} 0 \\ 0 \\ 1 & 0 \end{pmatrix}
\]

and we denote the columns of these matrices by \( w_1, w_2 \) and the space \( L_C^1 \) is spanned by \( w_1 \) and \( L_C^2 \) by \( w_1 \) and \( w_2 \).

We now further divide our analysis into cases. We set \( k = k_1 k_2 k_3, h = h_1 h_2 h_3 \) and we consider the spaces \( U(k, h, \ell) \).

**Case \( k = 000 \) and \( \ell = 3 \).**

In these cases, we prove that the intersection of each Schubert cell of the form (4.2) with \( X_U \) is an affine space.

The possible values of \( h \) in this case are: \( h = 100, h = 110, h = 200 \) and \( h = 210 \). We compute all possible pairings \( Q_{1,0}(v_i, w_j) \) and \( Q_{\lambda,1}(v_i, w_j) \) which appear in these cases. We list the results according to the values of \( S \) and \( T \). For each choice of \( S \) and \( T \), we write the result using the coordinates introduced above. This is done by direct computations whose result is contained in the tables below.

For the computation of \( Q_{1,0} \) and \( Q_{\lambda,1} \), we write down the three possible tables, according to the value of \( S(1) \). Each table lists the possible pairings, according to the value of \( S \) and \( T \). We notice that in the computation of \( Q_{1,0}(v_i, w_j) \), only the value of \( S(1) \) is relevant and similarly for the computation of \( Q_{1,0}(v_i, w_1) \), only the value of \( T(1) \) is relevant. Similarly, we proceed for \( Q_{\lambda,1} \).

From this computation, it follows that the intersection of \( X_U \) with the Schubert cells is an affine space. We explain how to read this result from the table only in the case of \( Q_{\lambda,1} \) and \( S(1) = 3 \). The other cases are treated similarly and we leave them to the reader.

**Case \( S(1) = 3 \).**

In the following tables, we have inserted the dots “…” to mean a polynomial in the variables different from \( y_3 \) and \( y'_3 \), and we have inserted stars “*” in the last two cases of \( Q_{\lambda,1}(v_1, w_2) \) since we do not need to compute these pairings: see the case \( h_1 \geq 2 \) below.
From these tables, it is easy to check that the intersection with each Schubert cells is an affine space.

We give the details of this analysis for cells of the form $A_{21} \times S_{3} \times T_{r}$, so we are using the second of the tables above.

Since $h_1 = 1$ or 2, we notice that if $T(1) = 3$, the equation $Q_{\lambda,1}(v_1, w_1) = 0$ gives $1 = 0$ so the intersection are empty in these cases and we can assume that $T(1) = 1$ or 2.

In both cases $T(1) = 1$ and $T(1) = 2$, we can use the equation $Q_{\lambda,1}(v_1, w_1) = 0$ to eliminate $y_3$ which does not appear anymore.

If $h_1 = 2$, we also need to consider the equation $Q_{\lambda,1}(v_1, w_2) = 0$ for all permutations $T$. We see that the intersection is empty except for the cases $T = 123$ or $T = 213$. In these cases, we use the equation $Q_{\lambda,1}(v_1, w_2) = 0$ to eliminate the variable $y'_3$ which does not appear anymore.

Let us now consider the case $h_2 = 1$. If $T(1) = 2$, from the equation $Q_{\lambda,1}(v_2, w_1) = 0$, we obtain $\lambda = -1$ or $0 = 0$, hence an affine condition. If $T(1) = 1$ and $S = 312$, we obtain similarly $\lambda = 0$ and finally for $T(1) = 1$ and $S = 321$, we see that we can perform a change of coordinates introducing a new variable $x'_1 = x'_1 + y_2$ in place of $x'_1$ and we see that we can use the equation $Q_{\lambda,1}(v_2, w_1) = 0$ to eliminate the variable $y_2$.

In particular, for all possible values of $h_2$, we can use an equation to eliminate a variable.

**Case $S(1) = 2$.**

| $Q_{1,0}$ | $w_1$ | $w_2$ |
|-----------|-------|-------|
| $T(1) = 1$ | $T(1) = 2$ | $T(1) = 3$ | 123 | 132 | 213 | 231 | 312 | 321 |
| $v_1$ | $x_1 + y_2$ | $x_2$ | 0 | $x_2$ | 0 | $x_1$ | 0 | $x_1 + y'_2$ | $x_2$ |
| $v_2$ | 231 | $x'_1 + y_2$ | 1 | 0 | 312 | 1 | 0 | 0 |

| $Q_{\lambda,1}$ | $w_1$ | $w_2$ |
|-----------------|-------|-------|
| $T(1) = 1$ | $T(1) = 2$ | $T(1) = 3$ | 123 | 132 | 213 | 231 | 312 | 321 |
| $v_1$ | $y_3 + \cdots$ | $y_3 + \cdots$ | 1 | $y'_3 + \cdots$ | 1 | $y'_3 + \cdots$ | 1 | * | * |
| $v_2$ | 321 | $y_2 + \lambda(x'_1 + y_2)$ | $1 + \lambda$ | 0 | 312 | $\lambda$ | 0 | 0 | 0 |
\[ Q_{\lambda,1} \]

| \( v_1 \) | \( T(1) = 1 \) | \( T(1) = 2 \) | \( T(1) = 3 \) | \( w_2 \) |
|---|---|---|---|---|
| \( \lambda(x_1 + y_2) + y_2 \) | \( \lambda + 1 \) | 0 | \( \lambda + 1 \) | \( \lambda x_1 \) | 0 | \( \lambda(x_1 + y_2) + y_2' \) | \( \lambda + 1 \) |

\[ v_2 \]

| \( 231 \) | \( \lambda x_1' + y_3 \) | \( y_3 \) | 1 |

\[ w_1 \]

| 213 | 1 | 0 | 0 |

**Case** \( S(1) = 1 \).

\[ Q_{1,0} \]

| \( v_1 \) | \( T(1) = 1 \) | \( T(1) = 2 \) | \( T(1) = 3 \) | \( w_2 \) |
|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 |

\[ v_2 \]

| 132 | \( x_2' y_2 \) | \( x_2' \) | 0 |

\[ w_1 \]

| 123 | \( y_2 \) | 1 | 0 |

\[ Q_{\lambda,1} \]

| \( v_1 \) | \( T(1) = 1 \) | \( T(1) = 2 \) | \( T(1) = 3 \) | \( w_2 \) |
|---|---|---|---|---|
| \( \lambda \) | 0 | 0 | 0 | \( \lambda \) | 0 | \( \lambda \) | 0 |

\[ v_2 \]

| 132 | \( y_3 + x_1' y_2 (\lambda + 1) \) | \( y_3 + x_2' (\lambda + 1) \) | 1 |

\[ w_1 \]

| 123 | \( y_2 (\lambda + 1) \) | 1 + \( \lambda \) | 0 |

**Case** \( k = 000 \) and \( \ell = 2 \).

The possible values of \( h \) in this case are: \( h = 100 \), \( h = 200 \) and \( h = 210 \).

We proceed as above, but in this case we have to add the relation that \( v_0 = (1, 1, 1) \) lies in the plane \( L^2_B = \langle v_1, v_2 \rangle \).

**Case** \( S(1) = 1 \).

The condition \( v_0 \in L^2_B \) forces \( S = 132 \) and \( x_2' = 1 \). Specializing the table obtained in the case \( \ell = 3 \) to this case, we see that the intersections are affine spaces.

**Case** \( S(1) = 2 \).

The condition \( v_0 \in L^2_B \) forces \( S = 231 \) and \( x_1' = 1 - x_1 \). Specializing the table obtained in the case \( \ell = 3 \) to this case, we see that the intersections are affine spaces.

**Case** \( S = 312 \).

The condition \( v_0 \in L^2_B \) forces \( x_2 = 1 \). Specializing the table obtained in the case \( \ell = 3 \) to this case, we see that the intersections are affine spaces.
Case $S = 321$. 

The condition $v_0 \in L^2$ forces $x_1 = 1 + x'_1(x_2 - 1)$. Since the variable $x_1$ was kept free in the discussion of the case $\ell = 3$ and $S(1) = 3$ above, we see that the intersections are affine spaces.

Case $k = 100$ and $\ell = 3$.

We proceed as in the case $k = 000$ with the only difference that we use the two equations $Q_{1,0}(v_1, w_1) = Q_{0,1}(v_1, w_1) = 0$ to eliminate two variables, or one variable (if one of this equation is of the form $0 = 0$), or no variables (if these equations are both of the form $0 = 0$). The variables which have been eliminated are written in the third row, in the columns of $Q_{1,0}$ relative to $w_1$. We write the tables which are obtained after this elimination.

Case $S(1) = 3$.

In this case we can assume $T(1) \neq 3$ otherwise the variety is empty.

\[
\begin{array}{ccccc}
\hline
Q_{1,0} & w_1 & & & \\
 & T(1) = 1 & T(1) = 2 & 123 & 132 & 213 & 231 \\
 & x_1, y_3 & x_2, y_3 & & & & \\
\hline
v_1 & 0 & 0 & x_2 & 0 & x_1 & 0 \\
v_2 & 321 & x'_1 + y_2 & 1 & & & \\
& 312 & 1 & 0 & & & \\
\hline
\end{array}
\]

\[
\begin{array}{ccccc}
\hline
Q_{\lambda,1} & w_1 & & & \\
 & T(1) = 1 & T(1) = 2 & 123 & 132 & 213 & 231 \\
 & & & (\lambda + 1)x_2 + y'_3 & 1 & \lambda x_2 + y'_3 & 1 \\
\hline
v_1 & 0 & 0 & & & & \\
v_2 & 321 & \lambda(x'_1 + y_2) + y_2 & 1 + \lambda & & & \\
& 312 & \lambda & 0 & & & \\
\hline
\end{array}
\]

In the discussion of the case $\ell = 2$, $k = 100$, we will also need to know the explicit formulas for the variables that have been eliminated, i.e., if $T(1) = 1$, then $x_1 = -x_2 y_2$, $y_3 = -x_2 y_2$ and if $T(1) = 2$, then $x_2 = y_3 = 0$.

Case $S(1) = 2$.

In this case, we can assume $T(1) \neq 2$, otherwise the variety is empty.
| $Q_{1,0}$ | $w_1$ | $w_2$ |
|---------|---------|---------|
| $T(1) = 1$ | $T(1) = 3$ |
| $x_1, y_2$ | none |
| $v_1$ | 0 | 0 | 1 | 0 | $x_1 + y'_2$ | 1 |
| $v_2$ | 231 | $x'_1$ | 0 | |
| 213 | 1 | 0 |

| $Q_{\lambda, 1}$ | $w_1$ | $w_2$ |
|---------|---------|---------|
| $T(1) = 1$ | $T(1) = 3$ |
| $v_1$ | 0 | 0 | $\lambda + 1$ | 0 | $\lambda(x_1 + y'_2) + y'_2$ | $\lambda + 1$ |
| $v_2$ | 231 | $\lambda x'_1 + y_3$ | 1 | |
| 213 | $\lambda$ | 0 |

*Case S(1) = 1.*

In this case, we can assume $T(1) \neq 1$, otherwise the variety is empty.

| $Q_{1,0}$ | $w_1$ | $w_2$ |
|---------|---------|---------|
| $T(1) = 2$ | $T(1) = 3$ |
| none | none |
| $v_1$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $v_2$ | 132 | $x'_1$ | 0 | |
| 123 | 1 | 0 |

| $Q_{\lambda, 1}$ | $w_1$ | $w_2$ |
|---------|---------|---------|
| $T(1) = 2$ | $T(1) = 3$ |
| $v_1$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $v_2$ | 132 | $(\lambda + 1)x'_2 + y_3$ | 1 | |
| 123 | $1 + \lambda$ | 0 |

*Case k = 100 and \( \ell = 2 \).*

In this case, we have to analyse only one value of $h$: $h = 200$. In this case, it is not true that the intersection with the Schubert cells of the form (4.2) is an affine space, in particular, it
is not true for \( a = 21, S = 231 \) and \( T = 123 \). We proceed in a different way. We consider the obvious projection
\[
\pi : \mathbb{P}(V_A) \times \mathcal{F}(V_B) \times \mathcal{F}(V_C) \longrightarrow \mathbb{P}(V_A) \times \mathbb{P}(V_B) \times \mathcal{F}(V_C)
\]
and its restriction \( \pi_U \) to \( X_U \). We denote its image by \( Z_U \). We notice that \( \pi_U \) is a blow up of \( Z_U \) along the locus, where \( L_B^1 = C v_0 \). To prove our claim, we then prove that the intersection of \( Z_U \) with \( B_0 \) stable Schubert cells in the partial flag variety is an affine space and we describe the intersection of this affine space with the the blow up locus. We then apply Lemma 7.

To study the intersection of \( Z_U \) with Schubert cells, we notice that \( Z_U = Y_{U'} \), where \( U' \) corresponds to \( k = 100, h = 200 \) and \( \ell = 3 \). In particular, by the above discussion, we see that the intersection of \( Z_U \) with the Schubert cells in the partial flag variety are affine spaces. We now analyse which cells intersect the blow up locus, which is given by the condition that the line \( L_B^1 \) contains \( v_0 \). For this, we need to have \( S(1) = 1 \). Notice that for \( U' = U(1, 0, 0|2, 0, 0|3) \), if \( T(1) = 2 \), then necessarily \( x_2 = 0 \) and hence \( v_0 \notin L_B^1 \). It follows that we need to assume \( T(1) = 1 \).

For \( a = 12 \) and \( T = 123 \), we also have \( x_2 = 0 \) and again the intersection with the blowup locus is empty.

For \( a = 12 \) and \( T = 132 \), the equations of the intersection of \( Y_{U'} \) with the Schubert cell are given by
\[
x_1 = -x_2 y_2, \quad y_3 = -x_2 y_2.
\]
and the blow up locus is given by \( x_1 = x_2 = 1 \). Hence we are blowing up a 2 dimensional affine space in a point which has a paving, by Lemma 7.

For \( a = 21 \) and \( T = 123 \), the equations of the intersection of \( Y_{U'} \) with the Schubert cell are given by
\[
x_1 = -x_2 y_2, \quad y_3 = -x_2 y_2, \quad (\lambda + 1) x_1 + y_3' = 0.
\]
and the blow up locus is given by \( x_1 = x_2 = 1 \). Hence we are blowing up a 3 dimensional affine space in a line which has a paving, by Lemma 7.

For \( a = 21 \) and \( T = 132 \), the intersection of the Schubert cells with \( Z_U \) is empty, hence there is nothing to check.

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