Entanglement entropy of low-lying excitation in localized interacting system:  
Signature of Fock space delocalization

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The properties of the entanglement entropy (EE) of low-lying excitations in one-dimensional disordered interacting systems are studied. The ground state EE shows a clear signature of localization, while low-lying excitation shows a crossover from metallic behavior at short sample sizes to localized at longer length. The dependence of the crossover as function of interaction strength and sample length is studied using the density matrix renormalization group (DMRG). This behavior corresponds to the presence of the predicted many particle critical energy in the vicinity of the Fermi energy. Implications of these results to experiments are discussed.

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Many body localization (MBL) has drawn growing interest in recent years [1–20]. The Anderson localization transition [21, 22] is a zero-temperature quantum phase transition, which for the non-interacting case is manifested in the properties of the single-particle eigenstates and eigenvalues [23]. For one and two dimensional systems all the single-particle state are localized for any amount of disorder, while for three dimensional systems at a given disorder there exists a critical energy (mobility edge) below which all states are localized.

For a many-particle system, as long as no particle-particle interactions are present, the properties of a many-particle excited state are determined by the properties of the single-particle states. Thus, as long as all the particles occupy localized single-particle states the many-particle states should exhibit localized behavior. For example, the entanglement entropy (EE) between any sub-region A of the system and the rest of it should saturate (i.e., $S_A \propto \xi^{d-1}$, where $d$ is the dimensionality of the system and $\xi$ is the single-electron localization length). While if some of the extended single-particle states are occupied the usual volume law for the EE ($S_A \propto L_A^d$, where $L_A$ is the length of region A) is observed [17]. Physically, the difference between the localized and extended many-particle states is manifested in the dynamics of a particle tunneling into some particular region of the system, or a particle excited by a confined external perturbation in the region. In the localized phase it will remain there (i.e., will have a very long lifetime - sharp level), while in the extended case it will leave the region (short lifetime - broad level).

What is the influence of particle-particle interaction on the above picture? A cursory consideration may lead to the conclusion that since particles now interact with each other, any localized excitation will eventually spread all over the system. Thus, the rest of the system acts as a thermal bath for any excited sub-system [24, 25]. Surprisingly, Basko, Aleiner and Altshuler [2] have shown that if all the single-electron states are localized, an excitation will remain localized even in the presence of particle-particle interactions up to a critical temperature or excitation energy. This behavior is known as the MBL.

One can expect to see a signature of this behavior also in the EE between a sub-region and the rest of the system. For excitations below the the MBL mobility edge the EE should follow its localized behavior. This was numerically demonstrated for the ground state of interacting electrons in a 1D disordered system [27], where repulsive interactions only bolsters the localization behavior. On the other hand, for excited states above the mobility edge the EE should follow the volume law expected from thermalized states.

Let us examine the parametric dependence of $T_C$ on the interaction strength more carefully. The critical temperature $T_C \sim \Delta_c/|\ln \lambda|$ [2], where $\Delta_c$ is the single level spacing in a region of size $\xi$, and $\lambda$ is the dimensionless interaction strength proportional to $U_{\text{max}}/\Delta_c$, where $U_{\text{max}}$ is the maximal matrix element between two many-particle states proportional to $U(a/\xi)^d$. Here $U$ is the strength of the nearest-neighbor interaction, $a$ is the range of interaction and $d$ the dimensionality. The level spacing in the localization volume $\Delta_c = 1/(\nu c^d)$ , where $\nu$ is the density of states. When the critical temperature is translated to a critical excitation energy high in the excitation spectra, $E_C = n_d T_C$, where $n$ is the number of particles excited. At a constant filling $n \propto L^d$, where $L$ is the system size, and therefore $E_C$ is extensive. Thus, for a given system size, $E_C$ decreases as $U$ increases, while $E_C$ increases as $L$ increases.

Probing the transition at low excitation energies is advantageous both for experimental and numerical reasons. Experimentally, at lower excitation energies the influence of electron-phonon scattering (which will thermalize even in the MBL regime) is weaker, while numerically it is possible to used the accurate DMRG method only close to the ground state. Of course, strictly speaking finite systems do not exhibit a true phase transition, but they can nevertheless show a signature of such a transition and reveal its properties.

For a finite system of length $L$ the spacing between the
ground state and the first many-particle excited state is \( \Delta_1 = 1/\nu L^d \). At low energies only a few (order of one) quasi-particles are excited and \( \mathcal{E}_C \approx d T_C \), thus intensive. One expects a crossover in the behavior of the low-lying excitations from a metallic behavior once \( \Delta_1 > \mathcal{E}_C \) to localized when \( \Delta_1 < \mathcal{E}_C \). Thus, for a finite disordered system all low lying excited states will exhibit localization as long as \( L > \lambda(\ln(\lambda))^{1/d_\xi} \). Of course, localization will only occur on length scales larger than \( L > \xi \). Since in all these considerations one assumes \( \lambda \leq 1 \) one would expect that the length scale for which the excited state will exhibit localization behavior \( (L > \xi) \) is of order of the single-electron localization length.

The above consideration leaves out an important fact regarding the ground-state, namely that the ground-state many-particle localization length of a typical disordered system is strongly suppressed by repulsive interactions. For example, a spinless 1D wire of length \( L \), with on-site disorder and nearest neighbor interactions described by the following Hamiltonian:

\[
H = \sum_{j=1}^{L} \epsilon_j c_j^\dagger c_j - t \sum_{j=1}^{L-1} (c_j^\dagger c_{j+1} + h.c.) + U \sum_{j=1}^{L-1} \frac{1}{2} (c_j^\dagger c_{j+1} c_{j+1}^\dagger c_j - \frac{1}{2}) ,
\]

where \( \epsilon_j \) is the random on-site energy, taken from a uniform distribution in the range \([-W/2, W/2]\), \( U \) is the interaction strength, and \( t = 1 \) is the hopping matrix element. \( c_j^\dagger \) is the creation operator of a spinless electron at site \( j \) in the wire, and a neutralizing background is included in the interaction term. This model is localized for non-interacting as well as any repulsive \( U > 0 \) interactions \([29, 30]\), and has a metallic regime for attractive interaction in the vicinity of \( U = -1 \) and not to strong disorder \([28, 31, 33]\). For the non-interacting case close to half-filling the single-particle localization length, \( \xi(W, U = 0) \approx 105/W^2 \) \([34]\). Once particle-particle interactions are included, properties of the system such as transport (and entanglement) are determined by the many-particle wave function and the many-particle localization length deviates from the single-particle localization length. The many-particle localization length is related to the single-particle localization length by \([29, 31]\) \( \xi_{MP} = \xi(W, U = 0)/(\xi(W, U = 0))^{1/(3-2g(U))} \), where \( g(U) = \pi/[2 \cos^{-1}(-U/2)] \) is the Luttinger parameter \([35]\). Thus, for the non-interacting case \( g = 1 \) and \( \xi_{MP} = \xi \) while for \( U = -1 \), \( g = 3/2 \) and \( \xi_{MP} \) diverges.

For repulsive interactions \( \xi_{MP} < \xi \) since \( g(U > 0) < 1 \), and the length scales for which the ground state of a finite system will exhibit metallic behavior becomes shorter \((L < \xi_{MP} < \xi)\). This has been verified by analyzing the behavior of the EE of the ground state \([27]\). On the other hand, for the low-lying excitations according to Basko, Aleiner, and Altshuler’s argument \([2]\) the excitations are expected to remain metallic for \( L < O(\xi) \).

Thus, for interacting systems a significant range of \( L \) (roughly estimated as \( \xi_{MP} < L < \xi \)) for which all low lying excited states will exhibit metallic behavior while the ground state is localized may be expected.

In this letter we study the behavior of low-lying excitations of a one-dimensional finite interacting disordered system and demonstrate the existence of a size-regime for which the system ground state is localized while any excited state is metallic. Unlike when studying high excitations for which there are no efficient numerical methods and one is reduced to treating small systems, simplify the system or making other assumptions regarding the solutions \([4, 5, 8-13]\), the density matrix renormalization group (DMRG) numerical method is very accurate for low-lying excitations and can handle large systems.

In order to identify whether an excited state is metallic or localized we shall use its EE, which for the ground state has been shown to be a very accurate way to determine the localization length \([27]\) and metal-insulator transitions point \([28]\), for interacting disordered system. Nonetheless, unlike the ground state EE (GSEE) which is well understood, the EE behavior of low-lying-excitations needs further clarification \([34, 39]\). We shall begin by studying the EE of low-lying-excitations in the metallic regime before proceeding to the localized regime.

The EE of a pure state \( |\Psi\rangle \) in a sample partitioned into two sections A and B of length \( L_A \) and \( L_B = L - L_A \) is given by

\[
S_{A/B} = -\text{Tr}[\rho_{A/B} \ln \rho_{A/B}] ,
\]

where the reduced density matrix \( \rho_{A/B} = \text{Tr}_{B/A}[\rho_{AB}] \). The ground-state and the three lowest excited states are calculated. As usual in DMRG, the reduced density matrix for each state at each length \( L_A \) is calculated and diagonalized, thus obtaining the EE does not involve any additional numerical overhead. First we present the EE in the metallic regime, i.e., \( \xi_{MB} \to \infty \) for \( W = 0.7 (\xi \sim 210) \), and \( U = -1 \). The median EE \( S(L_A, L) \) for all four states over 100 realizations of disorder are plotted as function of \( L_A \) in the upper panel of Fig. 1 for system length \( L = 100,200,300,500 \) and 700.

It is clear that the ground state in the metallic regime follows the expected logarithmic behavior \([42, 45]\)

\[
S_{gs}(L_A, L) = \frac{1}{6} \ln(\sin(\pi X_A)) + c_2 ,
\]

where \( X_A = L_A/L \) and \( c_2 \) is a non-universal constant. Using Eq. 3 results in a perfect fit for GSEE. The excited state EE (ESEE) has a different functional form. Nevertheless, the only relevant scale remains \( L_A \) and therefore we expect the ESEE to follow:

\[
S^{(i)}(L_A, L) = S^{(i)}(X_A) + c^{(i)}(L) ,
\]

where the superscript \( (i) \) denotes the excitation number. As can be seen in the lower panel of Fig. 1 this scaling
leads to a collapse of $S^{(i)}(X_A)$ on the same curve for different values of $L$. The constant $c^{(i)}(L)$ is chosen so $S^{(i)}(X_A = 0.2)$ is equal for different $L$. As can be seen from the inset in Fig. 1 where $S^{(i)}(L_A = 0.2L, L)$ for different values of $L$ is plotted, $c^{(i)}(L) \sim c^{(i)} + (1/6) \ln(L)$, where $c^{(i)}$ depends only on the excitation number.

Unlike the GSEE, for the ESEE there is no established functional form for the behavior of $S^{(i)}(X_A)$. Nevertheless, using the form of the typical EE of several excited particles in a clean system \[40\] as a guide, one may try to fit the ESEE to

$$S^{(i)}(X_A) = -n_0(X_A \ln X_A + (1 - X_A) \ln(1 - X_A)) + c^{(i)}(5)$$

where, $n_0$, for a clean system is the number of excited particles, while here it is treated as a fitting parameter. One might understand this functional form by considering that the ESEE should follow the volume law (up to logarithmic corrections appearing also in the ground state). For the second and third excitation this crude analogy gives a reasonable description with $n_0 = 1.82$ of the EE.

How does disorder change the low-lying ESEE? For the ground state the logarithmic correction saturates on the scale of $\xi_{MP}$, resulting in a clear signature of many particle localization\[27\]. Indeed, this is seen in the behavior of the GSEE depicted in Fig. 2 where the EE for systems with different single-electron localization length, but equal many-particle localization length fall on top each other for $L > \xi_{MP}$. On the other hand, the ESEE shows a different behavior. For short length scales (albeit longer than $\xi_{MP}$) the low-lying excitations behave differently than the ground state and continue to follow the volume law typical for the ESEE in the metallic regime (Fig. 1). Only for larger system sizes does the low-lying ESEE begin to resemble the GSEE.

We expect the volume law to be strongly affected by localization. For system sizes much shorter than the localization length the EE should not be seriously affected, while for sizes larger than the localization length it should saturate and resemble the GSEE. Indeed, this crossover in the behavior of the ESEE is clearly seen in Fig. 2. For the weak disorder case ($\xi_{MP} \sim 100$), the EE shows the volume law to hold up until $L = 300$ (even up to $L = 500$, for the third excited state of strong interactions $U = 1.8$), while crossing over to the GSEE behavior at $L = 700$. For stronger disorder ($\xi_{MP} \sim 50$), the volume law behavior is seen up to $L = 200$, while at $L = 500$ the ESEE is already closer to an area law. Thus, the GSEE shows localized behavior corresponding to $\xi_{MP}$, while the low-lying ESEE show volume law (metallic) behavior on larger length scales. Moreover, the crossover to localized behavior occurs at larger sample sizes for higher $U$ and higher excitations, as expected from the many-body localization scenario.

Let us examine the behavior of the ESEE more carefully. As we have seen, for the clean case the ESEE scales as $X_A = L_A/L$. and follows a volume law behavior (Eq. 5), while the ground state follows the area law (Eq. 5). For $L \ll \xi^{(i)}$ (where $\xi^{(i)}$ is the localization length of the $i$-th excited state) the ESEE is expected to follow the volume law. Deep in the localized regime ($L \gg \xi^{(i)}$) there should be no difference between the ESEE and the GSEE. Thus, contrary to the situation for the GSEE where $\xi$ plays the role of a saturation length in the area law, for the ESEE it also changes the functional

\[\text{FIG. 1: (Color online) Upper panel: The EE of a metallic system as function of } L_A, \text{ for different system sizes } L. \text{ The symbols correspond to the DMRG results, where the different sample length are represented by different symbols } \bigcirc, L = 100; \quad \square, L = 200; \quad \bigcirc, L = 300; \quad \triangle, L = 500; \quad \text{and } \bigtriangledown, L = 700. \text{ The colors correspond to the excitation number (continuous black - ground state, dashed red - first excitation, long dashed green - second excitation, dot-dashed blue - third excitation). The thick dashed line correspond to Eq. 3). Lower panel: The EE scaled as function of the ratio } X_A = L_A/L. \text{ The thick line correspond to Eq. 3) with } n_0 = 1.82. \text{ Inset: The EE of the ground state and the low lying excitations for } L_A = 0.2L \text{ as function of } L. \text{ The symbols correspond to the numerically calculated EE, while the lines to a fit to } c^{(i)} + (1/6) \ln(L), \text{ where } c^{(i)} \text{ depends only on the excitation number.} \]
form of the EE. This is clearly seen in Fig. 3, where the behavior of the 3rd excited state for the weak disorder is depicted. For \( L = 100 \) the behavior for both interaction strength fit a volume law. At larger length a crossover toward an area law behavior is seen. This crossover has two distinct features. There is saturation of the EE at large \( X_A \), which is clear for \( L = 700 \) for both \( U = 1.8 \) and \( U = 0.6 \). At weak interaction \( U = 0.6 \) this saturation is clear also for shorter sample sizes (\( L = 500 \)). This saturation is similar to the ground state EE saturation, but for the ESEE it is accompanied by a change in the behavior at small values of \( X_A \). Again, for \( L = 700 \) for both \( U = 1.8 \) and \( U = 0.6 \) the ESEE is indistinguishable for small values of \( X_A \) from the GSEE. Thus, unlike for the GSEE, the effect of the localization crossover is seen on length scale smaller than \( \xi \), and involve a change in the functional behavior of the ESEE.

It is obvious from Fig. 3 that the description of the EE in the crossover regime is not trivial, and there is no straight forward way to extract the excited state localization length, \( \xi^{(3)} \). When \( \xi^{(3)} \) is larger than \( L \), The ESEE follows metallic behavior (Eq. (5)), and therefore its clear that \( \xi^{(3)} > 500 \) as long as \( L < 500 \) for \( U = 1.8 \) and \( \xi^{(3)} > 300 \) as long as \( L < 300 \) for \( U = 0.6 \) (compared to \( \xi_{MP} = 100 \)). As the samples are longer, \( \xi^{(3)} \) becomes shorter. At sample length for which the ESEE behaves similar to the GSEE for small values of \( X_A \), one may roughly extract the the localization length by identifying the saturation of the ESEE. For sample length of \( L = 700 \), \( \xi^{(3)} \) is indicated by an arrow and corresponds to \( \xi^{(3)} \sim 175 \) for \( U = 1.8 \) and \( \xi^{(3)} \sim 140 \) for \( U = 0.6 \). Thus even at sample length of \( L = 700 \), the the 3rd excited state localization is significantly larger than the ground state localization length \( \xi_{MP} = 100 \).

How could this enhancement of the localization length of the low-lying excited states be experimentally verified for small finite samples? In order to distinguish between the ground state and the low lying excitations one should have systems small enough so the energy gap between the ground state and low lying excited states is larger than
other experimental energy scales such as temperature or source-drain voltage (ΔT > T, VSD), but larger than the many-particle localization length. By using photons to excite the low-lying excited states one could hope to observe a strong (even orders of magnitude) enhancement of the photoconductivity for these small samples, or to probe the conducance optically. In order to rule out other effects which may enhance the photoconductivity such as heating, the effect should strongly decrease for larger samples.

In conclusion, For short samples, low lying excited states may show metallic behavior although the sample is much longer than the ground state localization length and the ground state is strongly localized. Only for longer systems crossover to the localized regime occurs. This is a clear signature of the MBL mobility edge which can approach the Fermi energy for short systems. The behavior of the low lying excitation for short systems may be probed using optical techniques.

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