Open FRW universes and self-acceleration from nonlinear massive gravity

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Abstract: In the context of a recently proposed nonlinear massive gravity with Lorentz-invariant mass terms, we investigate open Friedmann-Robertson-Walker (FRW) universes driven by arbitrary matter source. While the flat FRW solutions were recently shown to be absent, the proof does not extend to the open universes. We find three independent branches of solutions to the equations of motion for the Stückelberg scalars. One of the branches does not allow any nontrivial FRW cosmologies, as in the previous no-go result. On the other hand, both of the other two branches allow general open FRW universes governed by the Friedmann equation with the matter source, the standard curvature term and an effective cosmological constant $\Lambda_{\pm} = c_{\pm} m_g^2$. Here, $m_g$ is the graviton mass, + and − represent the two branches, and $c_{\pm}$ are constants determined by the two dimensionless parameters of the theory. Since an open FRW universe with a sufficiently small curvature constant can approximate a flat FRW universe but there is no exactly flat FRW solution, the theory exhibits a discontinuity at the flat FRW limit.

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1. Introduction

Gravity remains the most mysterious among the four fundamental forces in nature. Experimentally, we do not know how gravity behaves at distances shorter than \( \sim 0.01 \) mm or longer than \( \sim 1 \) Gpc. Thus, it is natural to ask whether gravity can be modified at shorter or longer distances in a theoretically controllable and experimentally viable way. While this question at short distances is relevant for quantum gravity, the question at long distances might potentially address the mysteries of the universe, such as the cosmological constant problem, dark energy and dark matter.

One obvious question associated with modification of gravity at long distances is “Can gravitons have a small nonvanishing mass?” This question has been investigated in the classical work by Boulware and Deser [1] with a negative conclusion: Einstein gravity with a nonvanishing mass term exhibits a ghost in nonlinear level even if the mass term is carefully chosen in the linear level a la Fierz-Pauli [2].

Recently a new theory of Lorentz-invariant, nonlinear massive gravity was introduced [3, 4]. Not only the linear terms but also nonlinear terms at each order are carefully chosen so that ghost does not show up in the decoupling limit. This theory thus has a potential to be free from the Boulware-Deser ghost in the fully nonlinear level [3, 4, 5], although a different type of ghost within 5 degrees of freedom of a massive spin-2 field [6] has not been analyzed yet.

Equipped with a candidate theory of ghost-free massive gravity, it is natural to study its cosmological implications. Especially, in order to distinguish this theory from other theories of long-distance modification of gravity or models of dark
energy, an analysis of the cosmological perturbations is expected to be useful. Indeed, even if two different theories give the same cosmic history for the Friedmann-Robertson-Walker (FRW) background, evolution of perturbations may differ and act as a discriminator.

On the other hand, in a recent study [9], it was argued that the theory contains no nontrivial homogeneous and isotropic universe (FRW cosmologies). As it was correctly noted and elaborated in [9], the absence of FRW cosmologies by itself does not imply a conceptual or observational problem as long as there are non-FRW solutions that become more homogeneous and isotropic in the small graviton mass limit. Nevertheless, this poses a disadvantage, at least at a technical level: the analysis of the cosmological perturbations become significantly complicated. For instance, the standard strategy based on the harmonic expansion should be modified in inhomogeneous or anisotropic backgrounds.

The main goal of the present paper is to show that the nonlinear, Lorentz-invariant massive gravity theory allows open FRW cosmologies, contrary to the no-go result in [9]. Although Ref. [9] states 1 “our conclusions on the absence of the homogeneous and isotropic solutions do not change if we allow for a more general maximally symmetric 3-space”, their no-go result actually does not extend to the open FRW universes. Since an open FRW universe with a sufficiently small curvature constant can approximate a flat FRW universe but there is no exactly flat FRW solution, the theory exhibits a discontinuity at the flat FRW limit.

The rest of the paper is organized as follows. In Sec. 2, we describe our setup, i.e. open FRW universes in the nonlinear massive gravity. In Sec. 3, we study the equations of motion for the St"uckelberg scalars and find three independent branches of solutions. In Sec. 4 we show that the Friedmann equation and the dynamical equation are consistent with each other and that the Friedmann equation includes an effective cosmological constant of order $m^2_g$, where $m_g$ is the graviton mass. Sec. 5 is devoted to a summary of this paper and discussions. The paper is supplemented by an Appendix, where we describe the coordinate transformation from the Minkowski coordinate to the open FRW chart of the Minkowski spacetime.

2. Setup

In this section, we study the nonlinear massive gravity [1] described by the 4-dimensional metric $g_{\mu\nu}$ and scalar fields $\phi^a (a = 0, \cdots, 3)$, coupled to arbitrary matter source. The role of the scalar fields $\phi^a$ is to maintain the general covariance [10]. By construction, the matter action $I_m$ is independent of the $\phi^a$ fields. The total action is

$$I = I_g + I_m,$$

$^1$At least in the current arXiv version (v1).
\[ I_g = M_{Pl}^2 \int d^4x \sqrt{-g} \left[ \frac{R}{2} + m_g^2 (\mathcal{L}_2 + \alpha_3 \mathcal{L}_3 + \alpha_4 \mathcal{L}_4) \right], \]  

where

\[ \mathcal{L}_2 = \frac{1}{2} ([\mathcal{K}]^2 - [\mathcal{K}]^2), \]
\[ \mathcal{L}_3 = \frac{1}{6} ([\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}]^2 + 2[\mathcal{K}^3]), \]
\[ \mathcal{L}_4 = \frac{1}{24} ([\mathcal{K}]^4 - 6[\mathcal{K}]^2[\mathcal{K}^2] + 3[\mathcal{K}^2]^2 + 8[\mathcal{K}][\mathcal{K}^3] - 6[\mathcal{K}^4]), \]  

and,

\[ \mathcal{K}^\mu_\nu = \delta^\mu_\nu - \sqrt{g_{\mu\rho} \eta_{ab} \phi_a \phi_b}. \]  

In the above, the squared brackets denote the trace, \( \eta_{ab} = \text{diag}(-1,1,1,1) \) and hereafter, we set \( M_{Pl} = 1 \).

For the physical metric \( g_{\mu\nu} \), we consider an open \( (K < 0) \) FRW universe

\[ g_{\mu\nu} dx^\mu dx^\nu = -N(t)^2 dt^2 + a(t)^2 \Omega_{ij} dx^i dx^j, \]
\[ \Omega_{ij} dx^i dx^j = dx^2 + dy^2 + dz^2 - \frac{\mathcal{K}(x dx + y dy + z dz)^2}{1 + |\mathcal{K}|(x^2 + y^2 + z^2)}, \]  

where \( x^0 = t, x^1 = x, x^2 = y, x^3 = z; \mu, \nu = 0, \ldots, 3; \) and \( i, j = 1, 2, 3 \). As for the scalar fields \( \phi^a (a = 0, \ldots, 3) \), we adopt the following ansatz, motivated by the coordinate transformation \( (2.3) \) from the Minkowski coordinates to the open FRW chart of the Minkowski spacetime:

\[ \phi^0 = f(t) \sqrt{1 + |\mathcal{K}|(x^2 + y^2 + z^2)}, \]
\[ \phi^1 = \sqrt{|\mathcal{K}|f(t)x}, \]
\[ \phi^2 = \sqrt{|\mathcal{K}|f(t)y}, \]
\[ \phi^3 = \sqrt{|\mathcal{K}|f(t)z}. \]  

This leads to the following diagonal form for \( \eta_{ab} \partial_\mu \phi^a \partial_\nu \phi^b \).

\[ \eta_{ab} \partial_\mu \phi^a \partial_\nu \phi^b = -(\dot{f}(t))^2 \delta^0_\nu \delta^0_\nu + |\mathcal{K}| f(t)^2 \Omega_{ij} \delta^i_\mu \delta^j_\nu, \]  

where a dot represents differentiation with respect to \( t \). Since this expression respects the symmetry of the open FRW spacetime and does not depend on the physical metric, the \((0i)\)-components of the equation of motion for \( g_{\mu\nu} \) are trivially satisfied. Thus, variation of the action with respect to \( N(t) \) and \( a(t) \) should correctly give all components of the equation of motion for \( g_{\mu\nu} \).

Without loss of generality, we can assume that \( \dot{f} \geq 0, f \geq 0, a > 0 \) and \( N > 0 \), at least in the vicinity of the time of interest. It is then straightforward to show that

\[ \mathcal{K}_0^0 = 1 - \frac{\dot{f}}{N}, \quad \mathcal{K}_j^i = \left( 1 - \frac{\sqrt{|\mathcal{K}|f}}{a} \right) \delta^i_j, \quad \mathcal{K}_0^i = 0, \quad \mathcal{K}_i^0 = 0. \]  

\[ (2.7) \]
Thus, up to boundary terms, the gravity action is reduced to the following form.

\[ I_g = \int d^4x \sqrt{\Omega} \left[ -3|K|Na - \frac{3\dot{a}^2a}{N} + m_g^2 (L_2 + \alpha_3 L_3 + \alpha_4 L_4) \right], \quad (2.8) \]

where

\begin{align*}
L_2 &= 3a(a - \sqrt{|K|}f)(2Na - \dot{f}a - N\sqrt{|K|}f), \\
L_3 &= (a - \sqrt{|K|}f)^2 (4Na - 3\dot{f}a - N\sqrt{|K|}f), \\
L_4 &= (a - \sqrt{|K|}f)^3 (N - \dot{f}). \quad (2.9)
\end{align*}

3. Constraint from Stückelberg scalars

We now investigate the equations of motion for the Stückelberg scalars \( \phi^a \). Variation of the action (2.8) with respect to \( f(t) \) leads to

\[ (\dot{a} - \sqrt{|K|}N) \left[ \left( 3 - \frac{2\sqrt{|K|}f}{a} \right) + \alpha_3 \left( 3 - \frac{\sqrt{|K|}f}{a} \right) \left( 1 - \frac{\sqrt{|K|}f}{a} \right) \right. \]
\[ \left. + \alpha_4 \left( 1 - \frac{\sqrt{|K|}f}{a} \right)^2 \right] = 0. \quad (3.1) \]

This equation has three solutions. The first solution, \( \dot{a} = \sqrt{|K|}N \), implies that the physical metric \( g_{\mu\nu} \) is Minkowski spacetime in the open FRW chart; it is therefore not a realistic representation of our universe. Reducing the above equation to remove this solution, we obtain

\[ \left( 3 - \frac{2\sqrt{|K|}f}{a} \right) + \alpha_3 \left( 3 - \frac{\sqrt{|K|}f}{a} \right) \left( 1 - \frac{\sqrt{|K|}f}{a} \right) + \alpha_4 \left( 1 - \frac{\sqrt{|K|}f}{a} \right)^2 = 0, \quad (3.2) \]

which is solved by

\[ f = \frac{a}{\sqrt{|K|}} X_{\pm}, \quad X_{\pm} = \frac{1 + 2\alpha_3 + \alpha_4 \pm \sqrt{1 + \alpha_3 + \alpha_3^2 + \alpha_4}}{\alpha_3 + \alpha_4}. \quad (3.3) \]

Note that these two solutions do not exist if \( K = 0 \) is set. This is consistent with the fact that there is no nontrivial flat FRW solution \[5\]. On the other hand, for \( K < 0 \), these solutions are well-defined. Furthermore, while \( X_+ \) is singular in the limit \( \epsilon \to 0 \), \( X_- \) remains regular in the limit, where \( \epsilon \) is a small parameter counting the order of \( \alpha_{3,4} \), i.e. \( \alpha_{3,4} = O(\epsilon) \)

\[ X_+ = \frac{2}{\alpha_3 + \alpha_4} + \frac{5\alpha_3 + \alpha_4}{2(\alpha_3 + \alpha_4)} + O(\epsilon), \]
\[ X_- = \frac{3}{2} + O(\epsilon). \quad (3.4) \]
We conclude this section by the consistency of the equations of motion for the St"uckelberg fields. Because of the identity\[11\]
\[\nabla^\mu \left( \frac{2}{\sqrt{-g}} \delta I \right) = \frac{1}{\sqrt{-g}} \delta I g^\mu_\nu \partial_\nu \phi^a,\]
and the triviality of the (0i)-components of the metric equation (see the comment after Eq. (2.6)), the number of independent equations of motion for the St"uckelberg scalars $\phi^a$ is one. Thus, the equation obtained by variation with respect to $f(t)$ considered above contains all the nontrivial information.

4. Friedmann equation and self-acceleration

Variation of the action (2.8) with respect to $N(t)$ leads to
\[3H^2 - \frac{3|K|}{a^2} = \rho_m + \rho_g, \quad H \equiv \frac{\dot{a}}{Na},\]
where $\rho_m$ is the energy density of matter fields in the $I_m$ term of the action, $H$ is the expansion rate defined using the physical time parameter, and
\[\rho_g = -m_g^2 \left( \frac{1 - \sqrt{|K|} f}{a} \right) \left[ 3 \left( \frac{2 - \sqrt{|K|} f}{a} \right) + \alpha_3 \left( \frac{1 - \sqrt{|K|} f}{a} \right) \left( \frac{4 - \sqrt{|K|} f}{a} \right) \right.\]
\[\left. + \alpha_4 \left( \frac{1 - \sqrt{|K|} f}{a} \right)^2 \right],\]
is the effective energy density contribution arising from the graviton mass terms. On the other hand, the dynamical equation for the expansion can be obtained by varying the action with respect to $a(t)$. After using Eq. (4.1), this leads to
\[-\frac{2\dot{H}}{N} - \frac{2|K|}{a^2} = (\rho_m + p_m) + (\rho_g + p_g),\]
where $p_m$ is the pressure contribution from the matter action $I_m$, and $p_g$ is the effective pressure contribution of the graviton mass terms. The combination $\rho_g + p_g$ has a relatively simple expression as
\[\rho_g + p_g = -m_g^2 \left( \frac{\dot{a}}{N} - \frac{\sqrt{|K|} f}{a} \right) \left[ 3 \left( \frac{2 \sqrt{|K|} f}{a} \right) + \alpha_3 \left( \frac{3 - \sqrt{|K|} f}{a} \right) \left( \frac{1 - \sqrt{|K|} f}{a} \right) + \alpha_4 \left( \frac{1 - \sqrt{|K|} f}{a} \right)^2 \right].\]

From Eq. (3.5), we infer that if the constraint (3.1) is satisfied, the dynamical equation (4.3) brings no new information as a consequence of Bianchi identities and
matter conservation. By using the nontrivial solutions (3.3) of the constraint (3.1), Eqs. (4.1) and (4.3) reduce to

\[ 3H^2 - \frac{3|K|}{a^2} = \rho_m + c_{\pm}m_g^2, \]  

(4.5)

and

\[ -2\dot{H} - \frac{2|K|}{a^2} = \rho_m + p_m, \]

(4.6)

where

\[ c_{\pm} \equiv -\frac{1}{(\alpha_3 + \alpha_4)^2} \left[ 1 + \alpha_3 \pm \sqrt{1 + \alpha_3 + \alpha_3^2 - \alpha_4} \right] \times \left[ 1 + \alpha_3^2 - 2\alpha_4 \pm (1 + \alpha_3)\sqrt{1 + \alpha_3 + \alpha_3^2 - \alpha_4} \right]. \]  

(4.7)

The first equation (4.5) is equivalent to the Friedmann equation for an open universe driven by arbitrary matter (with energy density \( \rho_m \)) and the effective cosmological constant

\[ \Lambda_{\pm} = c_{\pm}m_g^2. \]

(4.8)

For \( c_{\pm} > 0 \), the system exhibits self-acceleration. The second equation (4.6) is consistent with the first equation and as we stated above, does not lead to any additional conditions, provided that the matter fluid obeys the ordinary conservation equation

\[ \dot{\rho}_m + \frac{3\dot{a}}{a}(\rho_m + p_m) = 0. \]

(4.9)

Although \( c_+ \) is singular when \( \epsilon \to 0 \), \( c_- \) remains regular (and positive) in the limit, where \( \epsilon \) is a small parameter counting the order of \( \alpha_{3,4} \), i.e. \( \alpha_{3,4} = O(\epsilon) \).

\[ c_+ = -\frac{4}{(\alpha_3 + \alpha_4)^2} - \frac{6(\alpha_3 - \alpha_4)}{(\alpha_3 + \alpha_4)^2} - \frac{3(3\alpha_3 - \alpha_4)^2}{4(\alpha_3 + \alpha_4)^2} + O(\epsilon), \]

(4.10)

\[ c_- = \frac{3}{4} + O(\epsilon). \]

5. Summary and discussions

In the context of the recently proposed nonlinear massive gravity with Lorentz-invariant mass terms, we have investigated open FRW universes driven by arbitrary matter source. We found three independent branches of solutions to the equation of motion of the Stückelberg scalars. One of the branches forbids any nontrivial FRW cosmologies. On the other hand, both of the other two branches allow general open FRW universes governed by the Friedmann equation with the matter source, the standard curvature term and the effective cosmological constant \( \Lambda_{\pm} = c_{\pm}m_g^2 \), where
$m_g$ is the graviton mass, $+$ and $-$ represent the two branches, and $c_{\pm}$ are constants given in (4.7).

As we already pointed out in the Introduction, to distinguish among long-distance modified gravity theories and dark energy models, an analysis of cosmological perturbations is of utmost importance. Different theories may be distinguished by dynamics of cosmological perturbations even if the FRW background is exactly the same. The open FRW solutions found in the present paper provides the working ground for this purpose.

The rank-2 tensor $\eta_{ab} \partial_\mu \phi^a \partial_\nu \phi^b$ shown in (2.6) respects the symmetry of the open FRW universe and does not depend on the physical metric $g_{\mu\nu}$. If we adopt the gauge in which perturbations of $\phi^a$ vanish, the form of $\eta_{ab} \partial_\mu \phi^a \partial_\nu \phi^b$ remains the same at any order in perturbative expansion. Therefore, in this gauge, evolution equations for cosmological perturbations fully respect homogeneity and isotropy at any order. The same conclusion holds in an arbitrary gauge as far as genuine gauge invariant variables are concerned.

On the other hand, because of the absence of closed FRW chart in Minkowski spacetime, there is no way to construct a closed FRW analogue of (2.6). For this reason, we expect that there is no nontrivial closed FRW cosmologies.

We note that Ref. [9] found a special solution in which the physical metric is of the FRW form, while the stress tensor of the St"uckelberg fields is effectively that of a cosmological constant (while in [12, 13], exact spherically symmetric solutions with self acceleration have been obtained). However, the tensor $\eta_{ab} \partial_\mu \phi^a \partial_\nu \phi^b$ in this solution does not respect the symmetry of the FRW universe. As a result, the contributions of the graviton mass term to the evolution equations of perturbations are expected to lead to a breaking of the FRW symmetry. In other words, this solution does not look homogeneous and isotropic if it is probed by dynamics of perturbations.\textsuperscript{2}

While the proof of the absence of nontrivial FRW cosmologies in [9] applies to flat FRW universes, our solutions illustrate that it is not valid for open universes. Since an open FRW universe with a sufficiently small curvature constant can approximate a flat FRW universe but there is no exactly flat FRW solution, the theory exhibits a discontinuity at the flat FRW limit. What this implies for the dynamics of cosmological perturbations in the flat FRW limit of our solution deserves detailed investigation. This discontinuity may be a hint of a strong coupling when the contribution from (negative) curvature becomes negligible, i.e. in the late time evolution with self-acceleration. On the other hand, a spatial curvature of a percent level today is consistent with experimental data and would be sufficient to describe the present universe. Perturbations of the solutions introduced in the present paper will be discussed in an upcoming work.

\textsuperscript{2}In the framework of the alternative formulation of [14], similar solutions describing flat, open and closed universes have been found in [15].
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A. Open chart of Minkowski spacetime

The Minkowski metric

$$ds_0^2 = \eta_{ab} dX^a dX^b, \quad \eta_{ab} = \text{diag}(-1, 1, 1, 1)$$  \hspace{1cm} (A.1)

can be rewritten in the open FRW form as

$$ds_0^2 = - (\dot{f}(t))^2 dt^2 + |K| f(t)^2 \Omega_{ij} dx^i dx^j,$n

$$\Omega_{ij} dx^i dx^j = dx^2 + dy^2 + dz^2 - \frac{|K|(x dx + y dy + z dz)^2}{1 + |K|(x^2 + y^2 + z^2)},$$  \hspace{1cm} (A.2)

by the coordinate transformation

$$X^0 = f(t) \sqrt{1 + |K|(x^2 + y^2 + z^2)},$$

$$X^1 = \sqrt{|K|} f(t) x,$n

$$X^2 = \sqrt{|K|} f(t) y,$n

$$X^3 = \sqrt{|K|} f(t) z,$$  \hspace{1cm} (A.3)

where $K (< 0)$ is the curvature constant of $\Omega_{ij} dx^i dx^j$, and a dot represents derivative with respect to $t$.

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