Integrability of anisotropic and homogeneous Universes in scalar-tensor theory of gravitation*

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Abstract
In this paper, we develop a method based on the analysis of the Kovalewski exponents to study the integrability of anisotropic and homogeneous Universes. The formalism is developed in scalar-tensor gravity, the general relativistic case appearing as a special case of this larger framework. Then, depending on the rationality of the Kovalewski exponents, the different models, both in the vacuum and in the presence of a barotropic matter fluid, are classified, and their integrability is discussed.

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1. Dynamical equations

1.1. Homogeneous spaces
The classification of homogeneous and anisotropic 3D spaces gives 11 different types of spaces associated with distinct families of structure group constants

\[ C_{ab}^c = \epsilon_{cde}N^{dc} + \delta^c_b A_a - \delta^c_a A_b \]  (1)

where \( \epsilon_{abc} \) is the usual totally antisymmetric unit tensor, \( \delta^c_b \) is the Kronecker symbol, \( N^{ab} \) is the contravariant component of an order 2 symmetric tensor, and the vector \( A \) must follow

\[ N^{ab} A_b = N_{ab} A^b = 0. \]  (2)

* In this paper, Greek indices run from 0 to 4, Latin indices run from 1 to 3 and \( \nabla^\mu \) indicates a covariant derivative.
Without loss of generality, one can write \( N^{ab} = \text{diag}(n_1, n_2, n_3) \) and \( A_b = [a, 0, 0] \) provided that \( a n_1 = 0 \). Distinct homogeneous and anisotropic spaces in 3D can then be classified as in the following table.

| \( n_1 \) | \( n_2 \) | \( n_3 \) | \( a \) | Name |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | \( B_I \) |
| 0 | 0 | 0 | \( \forall \) | \( B_V \) |
| 1 | 0 | 0 | 0 | \( B_{II} \) |
| 0 | 1 | 0 | \( \forall \) | \( B_{IV} \) |
| 1 | 1 | 0 | \( \forall \) | \( B_{VII} \) |
| 0 | 1 | 1 | \( \forall \) | \( B_{VIII} \) |
| 1 | -1 | 0 | \( \neq 1 \) | \( B_{VI} \) |
| 0 | 1 | -1 | 1 | \( B_{VII} \) |
| 1 | 1 | 1 | 0 | \( B_{IX} \) |
| 1 | 1 | -1 | 0 | \( B_{VIII} \) |

Note that each case is degenerate, for example \( n_1 = -1, n_2 = 1, n_3 = -1 \) and \( a = 0 \) is in the equivalence class of \( B_{IX} \) which contains all the possibilities with a positive signature and 0 out of the spectrum of \( N \). This is the well known Bianchi classification (see [1, 2]).

1.2. Scalar-tensor theory of gravitation

In scalar-tensor theories of gravity, the dynamics of the Universe contains a new scalar degree of freedom that couples explicitly to the energy content of the Universe [3–7]. In units of \( c = 1 \), the action generically writes, in the so-called Einstein frame

\[
S = \frac{1}{4\pi G} \int \left( \frac{R}{4} - \frac{1}{2} \varphi, \mu \varphi^{\mu} - U(\varphi) \right) \sqrt{-g} \, d^4 x + S_m(\psi_m, \Theta(\varphi)g_{\mu\nu}),
\]

\( G \) being a bare gravitational constant, \( \varphi \) the scalar field, \( U(\varphi) \) its self-interaction term and \( \Theta(\varphi) \) its coupling to matter. The functional \( S_m(\psi_m, \Theta(\varphi)g_{\mu\nu}) \) stands for the action of any field \( \psi_m \) that contributes to the energy content of the Universe. It expresses the fact that all these fields couple universally to a conformal metric \( \tilde{g}_{\mu\nu} = \Theta(\varphi)g_{\mu\nu} \), then implying that the weak equivalence principle (local universality of free fall for non-gravitationally bound objects) holds in this class of theories. The metric \( \tilde{g}_{\mu\nu} \) defines the Dicke–Jordan frame, in which standard rods and clocks can be used to make measurements (since in this frame, the matter part of the action acquires its standard form). Despite the conformal relation, these two frames have a different status: in the Dicke–Jordan frame, where the gravitational degrees of freedom are mixed, the Lagrangian for the matter fields does not contain explicitly the new scalar field: the non-gravitational physics has then its standard form. In the Einstein frame, the scalar degree of freedom explicitly couples to the matter fields, then leading, for example, to the variation of the inertial masses of point-like particles. Of course, the two frames describe the same physical world. Nevertheless, the usual interpretation of the observable quantities is profoundly modified in the Einstein frame, whereas it holds in the Dicke–Jordan frame, where the rods and clocks made with matter are not affected by the presence of the scalar field. That is why one usually refers to the Dicke–Jordan frame as the observable one. However, the dynamics of the fields are generally more easily described in the Einstein frame, so in this work, since we are interested in the integrability of the models rather than in their physical content, the analysis will be done in the Einstein frame.

Varying the Einstein frame action (3) with respect to the fields yields the equations

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \chi T_{\mu\nu} + T^{\psi}_{\mu\nu},
\]
\[\Box \psi = -\frac{\chi}{2} \omega(\psi) T + \frac{dU(\psi)}{d\psi},\]  
\[\nabla_\mu T^\mu = \omega(\psi) T \nabla_\mu \psi,\]  
\[\]  
where \(\chi = 8 \pi G, T\) is the trace of the energy–momentum tensor of matter fields \(T_{\mu\nu}\), and \(T^\psi_{\mu\nu} = 2 \psi_{,\mu} \psi_{,\nu} - g_{\mu\nu} \left( \psi^{\alpha\beta} \psi_{\alpha\beta} \right) - 2U(\psi) \psi_{\mu\nu}\) is the energy–momentum tensor of the scalar field. Moreover, we have defined the coupling \(\omega(\psi) = \frac{d\ln \xi}{d\psi}\). It is important to note that these equations reduce to those of general relativity in the presence of a scalar field iff \(\omega(\psi) = 0\).

### 1.3. Dynamical equations of scalar-tensor theory in homogeneous spaces

If one denotes by \(t\) the physical time and by \(\tau\) a conformal time such that \(d\tau = N(\tau) \, dt\), where \(N(\tau)\) is generally called ‘lapse function’, the line element of the 3 + 1 physical space reads

\[d\tau^2 = \gamma_{ij} dx^i dx^j - d\tau^2 = \gamma(\tau) \omega^2 - N^2(\tau) \, dt^2.\]

As explained in [8], one can find an invariant basis of differential forms \(\{\omega^1, \omega^2, \omega^3\}\) in each Bianchi space such that \(\gamma\) is diagonal; we note hereafter \(\gamma(\cdot) = \text{diag}(\alpha_i, \cdot)\).

Taking \(V(\tau) = (\alpha_1 \alpha_2 \alpha_3)^{1/2}\) as a lapse function, and introducing \(A_i = \ln(\alpha_i)\), the scalar-tensor dynamical equations in homogeneous and anisotropic Universe read

\[\begin{align*}
\chi V^2 \left[ 3(\rho + P) + \rho + P \right] - &\left( - (\ln(V^2))' \right)^2 \\
&\left( - \frac{1}{2} \left( A_1^2 + A_2^2 + A_3^2 \right) \right) \\
= &\left( V^2 \left[ \rho + P - (P + P) \right] = A_1^2 + n_1^2 \alpha_1^2 - (n_2 \alpha_2 - n_3 \alpha_3)^2 \right) \\
&\left( V^2 \left[ \rho + P - (P + P) \right] = A_2^2 + n_2^2 \alpha_2^2 - (n_3 \alpha_3 - n_1 \alpha_1)^2 \right) \\
&\left( V^2 \left[ \rho + P - (P + P) \right] = A_3^2 + n_3^2 \alpha_3^2 - (n_1 \alpha_1 - n_2 \alpha_2)^2 \right) \\
&\chi V^2 \omega(\psi) |(\rho - 3 P) = -2 \psi'' - 4(\ln V)' \psi' - \frac{d U}{d\psi} V^2. \tag{8}
\end{align*}\]

In these equations we have written \(\cdot\) for \(d/d\tau\); in addition the Universe is filled by a perfect fluid with pressure \(P\) and energy density \(\rho\); finally, we have noted

\[\begin{align*}
\rho_\psi := &\left( \frac{\psi^2}{2} + U(\psi) \right)/\chi \\
P_\psi := &\left( \frac{\psi^2}{2} - U(\psi) \right)/\chi. \tag{9}
\end{align*}\]

Taking \(U \equiv 0\), and reorganizing the first dynamical equation using the three others, system (8) becomes

\[\begin{align*}
0 = &E_\psi + E_P - 4 \chi \rho V^2 - 4 \chi \rho_\psi V^2 \\
\chi V^2 [\rho + P - (P + P)] = &A_1^2 + n_1^2 \alpha_1^2 - (n_2 \alpha_2 - n_3 \alpha_3)^2 \\
\chi V^2 [\rho + P - (P + P)] = &A_2^2 + n_2^2 \alpha_2^2 - (n_3 \alpha_3 - n_1 \alpha_1)^2 \\
\chi V^2 [\rho + P - (P + P)] = &A_3^2 + n_3^2 \alpha_3^2 - (n_1 \alpha_1 - n_2 \alpha_2)^2 \\
\chi V^2 \omega(\psi) |(\rho - 3 P) = &-2 \psi'' - 4(\ln V)' \psi', \tag{10}
\end{align*}\]

where

\[\begin{align*}
E_\psi := A_1^2 + A_2^2 + A_3^2 &\quad \text{and} \quad E_P := \sum_{i \neq j=1}^3 n_i n_j e^{A_i + A_j} - \sum_{i=1}^3 n_i^2 e^{2A_i}. \tag{11}
\end{align*}\]

The energy–momentum conservation, that in scalar-tensor theory is

\[\nabla_\mu T_{\mu\nu} = \omega(\psi) T_{\psi \rho} g^{\psi \rho} \partial_\nu \psi.\]
coupled to the hypothesis of a barotropic fluid

\[ P = (\Gamma - 1) \rho \]  

(12)

allows us to obtain a relation between \( \rho, V \) and \( \varphi \) which is

\[ \rho = \rho_o V^{-\Gamma} \Theta^{4-3\Gamma} \quad \text{with} \quad \rho_o \in \mathbb{R}^+ \backslash \{0\}. \]  

(13)

One should note that a fluid of radiation (\( \Gamma = 4/3 \)) does not couple directly to the scalar field. Assuming a power-law dependence of \( \varphi' \) in \( V \), more precisely

\[ \varphi' = \varphi_o V^\Delta, \]  

(14)

and making use of (13) one can solve the last equation of system (8) and obtain an explicit dependence on \( V \) for \( \Theta(\varphi) \):

\[ \rho_o \Theta^{4-3\Gamma} = \frac{a^2}{2\lambda} V^{2(\Delta - 1) + \Gamma}, \quad \text{where} \quad a^2 = \frac{4(\Delta + 2)}{2(1 - \Delta) - \Gamma} \psi_o^2. \]  

(15)

Assuming \( \rho_o > 0 \) (which corresponds to non-exotic matter) and \( 0 \leq \Gamma \leq 2 \) (which corresponds to the largest class of barotropic fluid), relation (15) holds under the condition

\[ -2 < \Delta < \frac{2 - \Gamma}{2}. \]  

(16)

Assumption (14) is a constraint on the entire dynamical system rather than on the scalar-tensor theory itself. Indeed, any choice of the coupling function \( \omega(\varphi) \) can be done, but then the resulting behaviour of \( V(\tau) \) is completely fixed by the last equation of system (10). Conversely, imposing a behaviour for the volume \( V(\tau) \) determines the corresponding coupling function. For example, imposing a Brans–Dicke theory, i.e., \( \omega(\varphi) = \omega_0 = \text{cste} \) results in \( V(\tau) \propto \tau^{1/(2 - \Delta)} \); in contrast, a volume evolving as \( V \propto e^{\lambda \tau} \) leads to

\[ \Theta(\varphi) \propto \frac{\text{cste}}{V^{\lambda \tau}}. \]  

(17)

that is, \( \omega(\varphi) \propto \frac{1}{\lambda \tau} \) and \( \varphi \propto e^{\Delta \tau} \). To sum up, the only constraint imposed by assumption (14) is on the couple \( (V(\tau), \omega(\varphi)) \), through the relation

\[ \frac{\rho}{\rho_o} = \frac{4(\Delta + 2)}{2(1 - \Delta) - \Gamma} V^{-2}, \]  

(18)

which implies that, at any time, the ratio of the densities of the barotropic fluid and of the scalar field scales with the inverse of the square of the volume. This is a sufficient constraint to make the Kovalewski formalism tractable in scalar-tensor gravity. Replacing (15) into (13), and considering (14) after (9), the dynamical system associated with the homogeneous Universe in scalar-tensor theory is

\[ \begin{align*}
E_c + E_p &= 2a^2 V^{2\Delta} + 2\psi_o^2 V^{2\Delta+2} \\
2[2 - \Gamma]a^2 V^{2\Delta} &= A_1^2 + n_2^2a_1^2 - (n_2a_2 - n_3a_3)^2 \\
2[2 - \Gamma]a^2 V^{2\Delta} &= A_1^2 + n_2^2a_2^2 - (n_3a_3 - n_1a_1)^2 \\
2[2 - \Gamma]a^2 V^{2\Delta} &= A_1^2 + n_3^2a_3^2 - (n_1a_1 - n_2a_2)^2.
\end{align*} \]  

(19)

It is important to note that whereas this system seems independent of \( \Gamma \), it actually depends on it through the parameter \( \Delta \) that fully characterizes our solution. In fact, for each coupling function, their exists a non-ambiguous link between \( \Gamma \) and \( \Delta \). For example, in the case of a radiation fluid when \( \Gamma = 4/3 \), the lhs of the last equation in system (10) vanishes and one can find that \( \varphi' \propto V^{-2} \) and then \( \Delta = -2 \).

A direct inspection of the first equation of system (19) shows that we can predict qualitatively the behaviour of the case \( \Delta = 0 \). Indeed, in this case, one of the two terms
of $E_c + E_p$ reduces to a positive constant and the other one tends to 0 as $V$ tends to 0. It is well known (e.g. [10]) that such a case breaks the Kasner cycle (e.g. [11]) for $B_{VIII}$ and $B_{IX}$, and then suppresses the chaotic behaviour towards the $t = 0$ singularity for these models. The case $\Delta = -1$ seems to be similar, but the other term now diverges and then this simple analysis cannot be done.

1.4. Hamiltonian formalism

The quantity called $E_c = A_1 A_2 + A_1 A_3 + A_1 A_2$ is a quadratic form of $A_{i=1,2,3}$ derivatives. Subsequently, one can diagonalize it using a linear change of variables

$$
\begin{align*}
q_1 &= (A_1 - A_2)/\sqrt{2} \\
q_2 &= (A_1 + A_2 - 2A_3)/\sqrt{6} \\
q_3 &= 2(A_1 + A_2 + A_3)/\sqrt{6}.
\end{align*}
$$

Introducing the associated conformal time derivatives $p_{i=1,2,3} := q_i$, the first equation of system (19) becomes

$$H := \frac{1}{2}(p_1^2 - p_2^2 - p_3^2) + e^{2\xi(q_1, q_2)} - 2a^2 e^{2\xi(q_3)} - 2q_0 e^{2\xi(1+\xi)} = 0. \quad (21)$$

The so-called potential $\xi$ is defined by

$$\xi(q_1, q_2) = -n_i^2 e^{2\xi(q_1) - \xi(q_2)} - n_i^2 e^{2\xi(q_2) - \xi(q_1)} - n_i^2 e^{2\xi(q_1) - \xi(q_2)}$$

$$+ 2n_1 n_2 e^{2\xi(q_1) + 2n_1 n_3 e^{2\xi(q_1)} + 2n_2 n_3 e^{-(2e^{2\xi(q_1)} + 2\xi(q_2))}. \quad (22)$$

In terms of $(q_i, p_i)$ variables, the dynamical system (19) is quasi-Hamiltonian (e.g. [12] and later [13–15])

$$
\begin{align*}
\dot{q}_{1,2} &= -\frac{\partial H}{\partial p_{1,2}} = \frac{d}{dt} q_{1,2} = -\frac{\partial H}{\partial q_{1,2}} \\
\dot{q}_3 &= \frac{d}{dt} q_3 = -\frac{\partial H}{\partial q_3}. \quad (23)
\end{align*}
$$

The minus sign that breaks the strict Hamiltonian symmetry comes from the minus signature of the quadratic form associated with $E_c$.

2. Integrability of homogeneous Universes

2.1. The case of Bianchi Universes

The following work has been initiated by Melnikov’s team (see e.g. [16] and [17] and references therein). In the special case of $B_{IX}$, [18] consists of an application; a generalization in the context of the whole class $A$ (i.e. $a = 0$) was tried by [19]. However, a lot of imprecisions in the last work need this new reformulation and extension to scalar-tensor theory.

2.1.1. Bianchi Universes as generalized Toda systems. Introducing the following vectors

$$
\begin{align*}
a_1 &= [0, \sqrt{6}/3, \sqrt{6}/3] & a_2 &= [\sqrt{2}/2, -\sqrt{6}/6, \sqrt{6}/3] & a_3 &= [-\sqrt{2}/2, -\sqrt{6}/6, \sqrt{6}/3] \\
a_4 &= [\sqrt{2}, \sqrt{6}/3, \sqrt{6}/3] & a_5 &= [-\sqrt{2}, \sqrt{6}/3, \sqrt{6}/3] & a_6 &= [0, -2\sqrt{6}/3, \sqrt{6}/3] \\
a_7 &= [0, 0, \sqrt{6}/2] & a_8 &= [0, 0, \sqrt{6}/2 + 1/2]. \quad (24)
\end{align*}
$$
the 3-forms
\[
\forall x, y \in \mathbb{R}^3 \quad (x, y) := +x_1y_1 + x_2y_2 + x_3y_3
\]
\[
(x, y) := -x_1y_1 - x_2y_2 + x_3y_3,
\]
and the constants
\[
k_1 := 2n_1n_2 \quad k_2 := 2n_1n_3 \quad k_3 := 2n_2n_3
\]
\[
k_4 := -n_1^2 \quad k_5 := -n_2^2 \quad k_6 := -n_3^2
\]
\[
k_7 = -2a^2 \quad k_8 = -2\varphi'^2.
\]
it is clear that the Hamiltonian (21) reads
\[
H = \frac{1}{2} \langle p, p \rangle + \sum_{i=1}^{8} k_i e^{(a_{i}, q)}.
\]
which is a classical form of a generalized Toda dynamical system.

Following Melnikov [16], we change \([q, p]\) variables to \([u, v]\) ones, such that
\[
u \in \mathbb{R}^8, v_i = 1, \ldots, 8 : \exp(a_{i}, q).
\]
Through this change, the number of degrees of freedom jumps from 6 in \([q, p]\) to 16 in terms of \([u, v]\). Still writing \(\dot{}\) for the derivatives with respect to the conformal time \(\tau\), the dynamical equations then read
\[
\forall i = 1, \ldots, 8 \quad \left\{ \begin{array}{l}
\nu_i' = u_i v_i \\
u_i' = \sum_{j=1}^{8} m_{ij} v_j
\end{array} \right. \quad \text{with} \quad m_{ij} := -k_j(a_i, a_j).
\]
This new formulation is now polynomial. Using the appendix notations, one can directly prove that system (29) is autosimilar with any non-vanishing index and weight \(g\) such that
\[
g_1 = \cdots = g_8 = 1 \quad \text{and} \quad g_9 = \cdots = g_{16} = 2.
\]
Here \(g\) is unique provided that (A.3) is fulfilled. A particular autosimilar solution of (29) is then
\[
[\bar{u} \bar{v}]^T = e^{-\bar{r}} = [\lambda_1 t^{-1}, \ldots, \lambda_8 t^{-1}, \mu_1 t^{-2}, \ldots, \mu_8 t^{-2}]^T
\]
provided that the constant non-vanishing vector \(c = [\lambda, \mu] = [\lambda_1, \ldots, \lambda_8, \mu_1, \ldots, \mu_8]\) is a solution of the algebraic system of equations
\[
\forall i = 1, \ldots, 8 \quad \left\{ \begin{array}{l}
\sum_{j=1}^{8} m_{ij} \mu_j = -\lambda_i \\
\lambda_i \mu_i = -2\mu_i.
\end{array} \right.
\]
Hence, any non-vanishing solution of this last system corresponds to a set of 16 Kovalewski exponents which allows us to write the solution of system (23) (cf appendix A). A necessary condition for the system to be integrable is that all its Kovalewski exponents be rational. So, the rest of the paper will be devoted to the analysis of these exponents in order to study the integrability of different types of homogeneous and anisotropic Universes.
2.1.2. Solutions of the algebraic system. In the more general case (that is scalar-tensor theory in the presence of matter barotropic fluids), the algebraic system (32) makes use of the $8 \times 8$ matrix

$$
M := \begin{bmatrix}
0 & -2n_1n_3 & -2n_3n_2 & 0 & 2n_3^2 & 2a^2\Delta & 2\psi_o^2\Delta_i \\
-2n_1n_2 & 0 & -2n_3n_2 & 0 & 2n_2^2 & 2a^2\Delta & 2\psi_o^2\Delta_i \\
-2n_1n_2 & -2n_1n_3 & 0 & 2n_1^2 & 0 & 2a^2\Delta & 2\psi_o^2\Delta_i \\
0 & 0 & -4n_1n_3 & 0 & 2n_2^2 & 2n_3^2 & 2a^2\Delta & 2\psi_o^2\Delta_i \\
0 & -4n_1n_3 & 0 & 2n_1^2 & -2n_2^2 & 2a^2\Delta & 2\psi_o^2\Delta_i \\
-4n_1n_2 & -4n_1n_3 & 0 & 2n_1^2 & 2n_2^2 & -2n_3^2 & 2a^2\Delta & 2\psi_o^2\Delta_i \\
-2n_1n_2\Delta & -2n_1n_3\Delta & -2n_3n_2\Delta & n_1^2\Delta & n_2^2\Delta & n_3^2\Delta & 3a^2\Delta^2 & 3\psi_o^2\Delta_1 \Delta \\
-2n_1n_2\Delta & -2n_1n_3\Delta & -2n_3n_2\Delta & n_1^2\Delta & n_2^2\Delta & n_3^2\Delta & 3a^2\Delta_1 \Delta & 3\psi_o^2\Delta_1^2
\end{bmatrix}
$$

(33)

where $\Delta_1 := \Delta + 1$. One can straightforwardly check that $M$ is of rank 3.

The vector 0 is always a solution of system (32), but it is not relevant for the quest of Kovalewski exponents that needs non-trivial solutions. A systematic study of the solutions of system (32) is possible: for $p = 6, 7$ or 8 let $E_p = \{1, 2, 3, \ldots, p\}$, we consider the minor determinants

$$
\forall(i, j, k) \in E_p \times E_p \times E_p \quad \xi_i = m_{ii}, \quad \xi_{i, j} = \begin{vmatrix} m_{ii} & m_{ij} \\ m_{ji} & m_{jj} \end{vmatrix}
$$

(34)

and the determinants

$$
d_{2i} = \begin{vmatrix} 2 \quad m_{ij} \\ 2 \quad m_{jj} \end{vmatrix}, \quad d_{2j} = \begin{vmatrix} 2 \quad m_{ij} \\ 2 \quad m_{ji} \end{vmatrix}, \quad d_{3j} = \begin{vmatrix} 2 \quad m_{ij} \\ 2 \quad m_{jk} \end{vmatrix}, \quad d_{jk} = \begin{vmatrix} 2 \quad m_{ij} \\ 2 \quad m_{kj} \end{vmatrix}
$$

(35)

Solutions of system (32) are then classified into three classes:

(i) Type 1 solutions (T1). $\exists i \in E_p$ such that $\mu_i \neq 0$ and $\forall j \in E_p \setminus \{i\}$, $\mu_j = 0$: then $\lambda_j = -2$ and

- if $\xi_i = 0$: there is no solution;
- if $\xi_i \neq 0$: $\mu_i = 2/\xi_i$ and $\forall j \in E_p \setminus \{i\}$, $\lambda_j = -2m_{ji}/\xi_i$.

(ii) Type 2 solutions (T2). $\exists (i, j) \in E_p \times (E_p \setminus \{i\})$ such that $\{\mu_i, \mu_j\} \neq \{0, 0\}$ and $\forall k \in E_p \setminus \{i, j\}$, $\mu_k = 0$: then $\lambda_i = \lambda_j = -2$ and

- if $\xi_{i, j} = 0$: there is no solution;
- if $\xi_{i, j} \neq 0$: $\mu_i = d_{2i}/\xi_{ij}$ and $\mu_j = d_{2j}/\xi_{ij}$ moreover $\forall k \in E_p \setminus \{i, j\}$, $\lambda_k = (m_{ki}d_{2i} + m_{kj}d_{2j})/\xi_{ij}$.

(iii) Type 3 solutions (T3). $\exists (i, j, k) \in E_p \times (E_p \setminus \{i\}) \times (E_p \setminus \{i, j\})$ such that $\{\mu_i, \mu_j, \mu_k\} \neq \{0, 0, 0\}$ and $\forall l \in E_p \setminus \{i, j, k\}$, $\mu_l = 0$: then $\lambda_i = \lambda_j = \lambda_k = -2$ and

- if $\xi_{i, j, k} = 0$: there is no solution;
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relativity with or without matter fluids. We proceed in three steps. formalism presented above: scalar-tensor gravity with or without matter fluids, and general Kovalewski exponents for Bianchi Universes in four different cases that are included in the In what follows, we will examine the (i) Select a Universe. It corresponds to choosing a set of \( n_i \) \( i = 1, 2, 3 \) in table 1.1. This step determines the theory of gravitation and the matter content of interest, and this choice is made by considering different forms for the matrix \( M \) defined in (i).

- Barotropic matter-filled Universe in scalar-tensor theory of gravitation: \( p = 8 \). This is the most general case that was presented in detail in the preceding subsection.
- Barotropic matter-filled Universe in Einstein general relativity: \( p = 7 \). The associated matrix is \( M \) but without the seventh line and the seventh column:

\[
M_{STV} := \begin{pmatrix}
0 & -2n_1n_3 & -2n_3n_2 & 0 & 0 & 2n_2^3 & 2\rho_0^2/\Delta_1 \\
-2n_1n_2 & 0 & -2n_3n_2 & 0 & 2n_2^3 & 2\rho_0^2/\Delta_1 \\
-2n_3n_2 & -2n_1n_3 & 0 & 2n_1^2 & 0 & 2\rho_0^2/\Delta_1 \\
0 & 0 & -4n_3n_2 & -2n_1^2 & 2n_2^3 & 2\rho_0^2/\Delta_1 \\
0 & -4n_1n_3 & 0 & 2n_1^2 & -2n_2^2 & 2n_2^3 & 2\rho_0^2/\Delta_1 \\
-4n_1n_2 & -4n_1n_3 & 0 & 2n_1^2 & 2n_2^3 & -2n_2^3 & 2\rho_0^2/\Delta_1 \\
-2n_1n_2\Delta_1 & -2n_1n_3\Delta_1 & -2n_3n_2\Delta_1 & n_1\Delta_1 & n_2\Delta_1 & n_3\Delta_1 & 3\rho_0^2/\Delta_1^2
\end{pmatrix}
\]

- Empty Universe in scalar-tensor theory of gravitation: \( p = 7 \). The associated matrix is \( M \) but with the eighth line and eighth column removed, and the seventh line and seventh column adapted. Noting \( \gamma \) for \( 2 - \Gamma \) we have

\[
M_{GTM} := \begin{pmatrix}
0 & -2n_1n_3 & -2n_3n_2 & 0 & 0 & 2n_2^3 & 2\rho_0\chi
\\
-2n_1n_2 & 0 & -2n_3n_2 & 0 & 2n_2^3 & 2\rho_0\chi
\\
-2n_1n_2 & -2n_1n_3 & 0 & 2n_1^2 & 0 & 2\rho_0\chi
\\
0 & 0 & -4n_3n_2 & -2n_1^2 & 2n_2^3 & 2n_2^3 & 2\rho_0\chi
\\
0 & -4n_1n_3 & 0 & 2n_1^2 & -2n_2^2 & 2n_2^3 & 2\rho_0\chi
\\
-4n_1n_2 & -4n_1n_3 & 0 & 2n_1^2 & 2n_2^3 & -2n_2^3 & 2\rho_0\chi
\\
-n_1\gamma & -n_1\gamma & -n_1n_2\gamma & n_1\gamma^2/2 & n_2\gamma^2/2 & n_3\gamma^2/2 & 3\rho_0\chi\gamma^2/2
\end{pmatrix}
\]

- Empty Universe in Einstein general relativity: \( p = 6 \). The associated matrix is \( M \) but with the seventh and eighth lines and the seventh and eighth columns removed:

\[
M_{GRV} := \begin{pmatrix}
0 & -2n_1n_3 & -2n_3n_2 & 0 & 0 & 2n_2^3 \\
-2n_1n_2 & 0 & -2n_3n_2 & 0 & 2n_2^3 & 0 \\
-2n_1n_2 & -2n_1n_3 & 0 & 2n_1^2 & 0 & 0 \\
0 & 0 & -4n_3n_2 & -2n_1^2 & 2n_2^3 & 2n_2^3 \\
0 & -4n_1n_3 & 0 & 2n_1^2 & -2n_2^2 & 2n_2^3 \\
-4n_1n_2 & -4n_1n_3 & 0 & 2n_1^2 & 2n_2^3 & -2n_2^3
\end{pmatrix}
\]

We then have to apply the following algorithm to the appropriate matrix.
(ii) Determine all the non-vanishing minor determinants $\zeta$ which could be extracted from the considered matrix.

(iii) For each $\zeta$, compute the associated set of $2p$ Kovalewski exponents. In our polynomial case, as indicated in appendix A this set is the set of eigenvalues of the matrix

$$K = \begin{bmatrix}
1 & 0 & \cdots & \cdots & 0 & m_{11} & \cdots & \cdots & m_{p1} \\
0 & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \ddots & I_p & \ddots & \vdots & M & \ddots & \vdots & \vdots \\
\vdots & \ddots & 0 & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & \cdots & 0 & 1 & m_{1p} & \cdots & \cdots & m_{pp} \\
\mu_1 & 0 & \cdots & \cdots & 0 & \lambda_1 + 2 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \ddots & 0 & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & \cdots & 0 & \mu_p & 0 & \cdots & \cdots & 0 & \lambda_p + 2
\end{bmatrix}.$$

The particular form of the matrix $K$ and the weak rank of the matrix $M$ allows an explicit calculation of the eigenvalues in all the cases $p = 6, 7, 8$ and for all the Bianchi Universes. There exist several hundreds of non-vanishing minor determinants in all the cases considered. They correspond to 92 distinct sets of Kovalewski exponents with length $2p = 12, 14$ or 16. All the details are presented in appendix B. In all cases, calculations are explicit, and in order to organize it we use formal calculus computational tools.

2.1.4. Exponents analysis and conclusions

The full set of Kovalewski exponents is presented in appendix B. The analysis of the sets of exponents results in four classes for Bianchi models. Class 1 contains uniquely $B_I$ models, class 2 contains $B_{II}$ and $B_{IV}$, class 3 is composed of $B_{III}$, $B_{V_{Ia}}$ and $B_{V_{IIa}}$, and finally class 4 contains $B_{VIII}$ and the famous $B_{IX}$. Each class is characterized by the same set of Kovalewski exponents, therefore corresponding models have the same characteristic dynamics. Let us review for each class the algebraic properties of Kovalewski exponents:

- **Class 1.** For all the cases we studied, all the Kovalewski exponents belonging to this class are rational, provided that the barotropic index $\Gamma$ and/or the scalar-tensor parameter $\Delta$ are rational. This restriction corresponds to physical cases\(^3\).

- **Class 2.** Kovalewski exponents of models belonging to this class fall into four subcases:
  - Empty Universe in general relativity (EUGR): exponents belonging to this class are integers.
  - Barotropic matter-filled Universe in general relativity (BMUGR): due to $A_2^\gamma$, all exponents are rational iff $\Gamma \in \mathbb{Q} \cap [0, \Gamma_0]$ with $\Gamma_0 = (11 - \sqrt{73})/3 \approx 0.81$.
  - Empty Universe in scalar-tensor theory (EUSTT): due to $A_2^\Delta$, all exponents are rational iff $\Delta \in \mathbb{Q} \cap [\Delta_0, 1]$ with $\Delta_0 = (11 + \sqrt{73})/6 \approx -0.40$.

\(^3\) As a matter of fact, physical power laws or barotropic index must be rational in order to be full of physical meaning.
Barotropic matter-filled Universe in scalar-tensor theory (BMUSTT): due to \( A_\pm \) and \( D_\pm \), all exponents are rational iff \( \Delta \in \mathbb{Q} \cap [\Delta_1, 1] \) with \( \Delta_1 = (-5 + \sqrt{73})/6 \approx 0.59 \).

- **Class 3.** As previously, four classes of sets of Kovalewski exponents can be identified
  - EUGR: exponents belonging to this class are integers.
  - BMUGR: due to \( A_+^*, B_+^* \) and \( C_+^* \), all exponents are rational iff \( \Gamma \in \mathbb{Q} \cap [0, \Gamma_0] \). This case is then equivalent to the corresponding case of class II models. Let us remark the special value \( \Gamma = 2/3 \) for which \( (A_+^*, B_+^*, C_+^*, C^*) = (1, 0, 1, 0, 1, 0) \).
  - EUSTT: due to \( A_\pm, B_\pm \) and \( a_\pm \) all exponents are rational iff \( \Delta \in \mathbb{Q} \cap [\Delta_0, 1] \). This case is then equivalent to the corresponding case of class II models. Let us remark the special value \( \Delta = -1/3 \) for which \( (A_+, A_-, B_+, B_-, a_+, a_-) = (1, 0, 0, 1, 0, 0) \).
  - BMUSTT: due to \( A_\pm, B_\pm, C_\pm, D_\pm, a_\pm \) and \( b_\pm \), all exponents are rational iff \( \Delta \in \mathbb{Q} \cap [\Delta_2, 1] \) with \( \Delta_2 = 16/25 = 0.64 \).

- **Class 4.** All models of this class contain at least two conjugated complex Kovalewski exponents.

From such an analysis, taking into account Yoshida’s theorems (see [21, 22]) whose context is detailed in appendix A, we can conclude the following.

- Empty Universes whose metrics correspond to classes 1, 2 or 3 defined below are generically associated with algebraically integrable dynamics.
- Class 4 Universes are always associated with non-algebraically integrable dynamics. This result holds for general relativity and/or scalar-tensor theory we are interested in, the Universe being empty and/or filled of barotropic matter.
- BMUGR of classes 2 and 3 are associated with non-integrable dynamics if the barotropic index ranges in the interval \([\Gamma_0, 2]\) with \( \Gamma_0 = (11 - \sqrt{73})/3 \approx 0.81 \).
- EUSTT of classes 2 and 3 are associated with non-integrable dynamics if the power law of the scalar-tensor modelization (14) ranges in the interval \([-2, \Delta_0]\) with \( \Delta_0 = (-11 + \sqrt{73})/6 \approx -0.40 \).
- BMUSTT of class 2 are associated with non-integrable dynamics if the power law of the scalar-tensor modelization (see 14) ranges in the interval \([-2, \Delta_1]\) with \( \Delta_1 = (-5 + \sqrt{73})/6 \approx -0.40 \).
- BMUSTT of class 3 are associated with non-integrable dynamics if the power law of the scalar-tensor modelization (14) ranges in the interval \([-2, \Delta_2]\) with \( \Delta_2 = 16/25 = 0.64 \).

The integrability of homogeneous Universes depends on multiple factors.

It is well known, since the pioneering works by Elskens and Henneaux [23], that the number of dimensions \( D \) of the spatial sections of the Universe is one of this factors. It influences the form of the potential \( \xi(q) \) in relation (22) and/or increases the number of components of the vectors \( \{a_i\} \) in (24). The net result in the simplified context of the generalized Kasner metric is that chaos disappears—and then the system becomes always integrable, in a classical context—when \( D \geq 11 \).

Modification of the gravitation theory by introducing a scalar-tensor component seems here to have a different contribution. As a matter of fact, in the case we consider, the potential \( \xi(q) \) is not modified, but a volume-dependent Toda potential is added in the Hamiltonian. In a general way, this produces a slowing down for the cushions’ velocity in the billiard representation (see [15] and references therein). One can then reasonably suppose that this feature does not drastically change the dynamical properties of homogeneous Universes.
Appendix A. Integrability of autosimilar differential systems

Let $f$ be a continuous function from $\mathbb{R}^n$ to $\mathbb{R}^n$ and $x = [x_1(t), \ldots, x_n(t)]^T \in \mathbb{R}^n$ such that
\[
\frac{dx}{dt} = f(x). \tag{A.1}
\]
If there exists a positive real $\lambda$ and a rational vector $g := [g_1, \ldots, g_n]^T$ such that the transformation
\[
t \mapsto t/\lambda \quad \forall i = 1, \ldots, n \quad x_i \mapsto \lambda^{\rho_i}x_i \tag{A.2}
\]
leaves system (A.1) invariant, this system is called autosimilar with index $\lambda$ and weight $g$.

When $g$ exists, it is unique if $A(x) \in M_n(\mathbb{R})$, $A_{ij} = \frac{\partial f_i(x)}{\partial x_j} - \delta_{ij}f_i(x)$ (A.3) is invertible for almost all $x \in \mathbb{R}^n$.

Autosimilar systems always admit at least one autosimilar particular solution $\bar{x}_{ar} = [c_1(t-t_0)^{-\rho_1}, \ldots, c_n(t-t_0)^{-\rho_n}]^T$, where $c := [c_1, \ldots, c_n]^T$ is the solution of the algebraic equation
\[
\begin{aligned}
f_1(c) &= -g_1c_1 \\
&\vdots \\
f_n(c) &= -g_nc_n.
\end{aligned} \quad \tag{A.5}
\]
Linearization of system (A.1) around the solution $\bar{x}_{ar}$ gives
\[
\frac{dz}{dt} = Df(x)(\bar{x}_{ar})z, \quad \tag{A.6}
\]
where $Df(x)(\bar{x}_{ar})$ is the Jacobian matrix of $f(x)$ evaluated at $x = \bar{x}_{ar}$. A theorem by Fuchs (see [20] sections 5.6 and 5.7) then shows that the general solution of (A.6) is autosimilar and reads
\[
z = [k_1(t-t_0)\rho_1-\epsilon_1, \ldots, k_n(t-t_0)\rho_n-\epsilon_n]^T. \quad \tag{A.7}
\]
The quantities $(\rho_1, \rho_2, \ldots, \rho_n)$ are called Kovalewski exponents. More practically, one can compute them because there are also eigenvalues of the matrix
\[
K := Df(x)(c) + \text{diag}(g). \quad \tag{A.8}
\]
As shown by Comte (see [20] sections 5.6 and 5.7), they are of great importance for the study of integrability of the original nonlinear system (A.1). As a matter of fact each component $x_{i1-\ldots,n}$ of a solution $x$ of this system can be written as
\[
x_i(t) = \sum_{k=0}^{\infty} \varepsilon^k x_i^{(k)}(t) \quad \text{with} \quad \begin{cases} 
\displaystyle x_i^{(0)} = c_i(t-t_0)^{\rho_i} \\
\displaystyle x_i^{(1)} = k_i(t-t_0)^{\rho_i-\epsilon_i} \\
\exists 1 \leq p, q \leq n, x_i^{(2)} = c_i k_q(t-t_0)\varepsilon_i \varepsilon_q \varepsilon_i \\
etc.
\end{cases} \quad \tag{A.9}
\]
hence

\[ x_i(t) \propto (t - t_o)^\rho_i S_i[(t - t_o)^\rho_1, \ldots, (t - t_o)^\rho_n], \tag{A.10} \]

where \( S_i \) is a multiple series. This allows us to understand a theorem by Yoshida (see [21, 22]): a necessary condition for a differential system to be integrable is that all its Kovalewski exponents are rational. If there exists at least one exponent irrational or complex the corresponding differential system is not algebraically integrable.

Appendix B. Details of Kovalewski exponents

When only non-exotic barotropic matter \((0 \leq \Gamma < 2)\) is considered, we have noted \( \gamma = (3\Gamma - 2)/\Gamma - 2 \). For any \( K = 3, 5, 7 \) and \( 15 \), we note

\[
\begin{align*}
2A_\pm^+ &= 1 \pm \sqrt{\frac{16 - 22\Gamma + 3\Gamma^2}{4 - 2\Gamma}}, \\
2C_\pm^+ &= 1 \pm \sqrt{\frac{25\Gamma - 18}{\Gamma - 2}}, \\
2D_\pm^+ &= 1 \pm \sqrt{\frac{18 - 41\Gamma + 24\Gamma^2}{2 - \Gamma}}, \\
E_\gamma := &\left\{-1, 2, 1(\times 6), \frac{2\gamma}{3}(\times 6)\right\} \\
E_\beta := &\{-1, 4(\times 3), 2(\times 3), 1(\times 5)\}.
\end{align*}
\]

Moreover,

\[
\begin{align*}
E_3 &= \{-1, 1(\times 6), 2, 2\delta_1/(\times 6)\} \\
\delta_1 &= (3\Delta + 1)(\Delta + 1)^{-1} \\
\delta_2 &= (3\Delta - 2)\Delta^{-1} \\
2A_\pm &= 1 \pm \sqrt{\frac{3(\Delta - \Delta_\gamma^*) (\Delta - \Delta_\delta^*)}{|\Delta + 1|}}, \\
2B_\pm &= 1 \pm \frac{25\Delta + 9}{|\Delta + 1|}, \\
2C_+ &= 1 \pm \frac{25\Delta - 16}{|\Delta|}, \\
2E_\pm &= 1 \pm \frac{48\Delta^2 + 41\Delta + 9}{|\Delta + 1|}, \\
2F_\pm &= 1 \pm \frac{48\Delta^2 - 55\Delta + 16}{|\Delta|}, \\
2a_\pm &= 1 \pm \sqrt{5 + 12\Delta}, \\
2b_\pm &= 1 \pm \sqrt{-7 + 12\Delta}.
\end{align*}
\]
Using the notations defined above we have computed all the Kowalewski exponents for all Bianchi Universes, in the case of empty or barotropic filled Universes and for General Relativity (GR) and scalar-tensor (ST) theory of gravitation.

### $B_1$

| RG | Empty ($\mathcal{N} = 6$) | Barotropic matter ($\mathcal{N} = 7$) |
|----|------------------|---------------------------------|
| T1 | $\emptyset$       | $E_y$                           |
| T2 | $\emptyset$       |                                 |
| T3 | $\emptyset$       |                                 |

### $B_0$ and $B_\Gamma$

| GR | Empty ($\mathcal{N} = 6$) | Barotropic matter ($\mathcal{N} = 7$) |
|----|------------------|---------------------------------|
| T1 | $E_o$            | $E_y$                           |
| T2 | $\emptyset$      |                                 |
| T3 | $\emptyset$      |                                 |

### $B_{II}$, $B_{III}$, $B_{IV}$

| GR | Empty ($\mathcal{N} = 6$) | Barotropic matter ($\mathcal{N} = 7$) |
|----|------------------|---------------------------------|
| T1 | $E_i$            |                                 |
| T2 | $\emptyset$      |                                 |
| T3 | $\emptyset$      |                                 |

| GR | Empty ($\mathcal{N} = 6$) | Barotropic matter ($\mathcal{N} = 7$) |
|----|------------------|---------------------------------|
| T1 | $-1, 1(x,6), 2(x,3), 4(x,3), \Delta + 3$ |                                |
| T2 | $-1, 1(x,6), 2(x,3), 4(x,3), \Delta + 3$ |                                |
| T3 | $-1, 1(x,6), 2(x,3), 4(x,3), \Delta + 3$ |                                |

### $B_{VI}$, $B_{VII}$

| GR | Empty ($\mathcal{N} = 6$) | Barotropic matter ($\mathcal{N} = 7$) |
|----|------------------|---------------------------------|
| T1 | $-1, 1(x,6), 2(x,3), 4(x,3), \Delta + 3$ |                                |
| T2 | $-1, 1(x,6), 2(x,3), 4(x,3), \Delta + 3$ |                                |
| T3 | $-1, 1(x,6), 2(x,3), 4(x,3), \Delta + 3$ |                                |
Belinskii V A and Khalatnikov I M 1973 Effect of scalar and vector fields on the nature of the cosmological singularity. See Phys. JETP 36 991

Belinskii V A and Khalatnikov I M 1973 Effect of scalar and vector fields on the nature of the cosmological singularity. See Phys. JETP 36 991

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| T1 | T2 | T3 |
|---|---|---|
| E | F | E |
| 1.0 × 10² | 2.0 × 10³ | 5.0 × 10³ |
| 2.0 × 10² | 1.0 × 10³ | 2.0 × 10³ |
| 5.0 × 10² | 5.0 × 10³ | 2.5 × 10³ |

| ST | T1 | T2 | T3 |
|---|---|---|---|
| E | F | E | F |
| 1.0 × 10² | 2.0 × 10³ | 5.0 × 10³ | 2.0 × 10² |
| 2.0 × 10² | 1.0 × 10³ | 2.0 × 10³ | 1.0 × 10² |
| 5.0 × 10² | 5.0 × 10³ | 2.5 × 10³ | 2.5 × 10² |
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