ON THE CLASSIFICATION OF QUADRATIC FORMS OVER AN INTEGRAL DOMAIN OF A GLOBAL FUNCTION FIELD

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Abstract. Let \( C \) be a smooth projective curve defined over the finite field \( \mathbb{F}_q \) (\( q \) is odd) and let \( K = \mathbb{F}_q(C) \) be its function field. Any finite set \( S \) of closed points of \( C \) gives rise to an integral domain \( \mathcal{O}_S := \mathbb{F}_q[C - S] \) in \( K \). We show that given an \( \mathcal{O}_S \)-regular quadratic space \( (V, q) \) of rank \( n \geq 3 \), the set of genera in the proper classification of quadratic \( \mathcal{O}_S \)-spaces isomorphic to \( (V, q) \) in the flat or étale topology, is in 1 : 1 correspondence with \( 2\text{Br}(\mathcal{O}_S) \), thus there are \( 2^{|S| - 1} \) such. If \( (V, q) \) is isotropic, then \( \text{Pic}(\mathcal{O}_S)/2 \) classifies the forms in the genus of \( (V, q) \). For \( n \geq 5 \) this is true for all genera, hence the full classification is via the abelian group \( H^2_{\text{ét}}(\mathcal{O}_S, \mu_2) \).

1. Introduction

Let \( C \) be a projective algebraic curve defined over a finite field \( \mathbb{F}_q \) (with \( q \) odd), assumed to be geometrically connected and smooth. Let \( K = \mathbb{F}_q(C) \) be its function field and let \( \Omega \) denote the set of all closed points of \( C \). For any point \( p \in \Omega \) let \( v_p \) be the induced discrete valuation on \( K \), \( \hat{O}_p \) the complete discrete valuation ring with respect to \( v_p \) and \( \hat{K}_p \) its fraction field. Any Hasse set of \( K \), namely a non-empty finite set \( S \subset \Omega \), gives rise to an integral domain of \( K \) called a Hasse domain:

\[ \mathcal{O}_S := \{ x \in K : v_p(x) \geq 0 \ \forall \ p \notin S \}. \]

This is a Dedekind domain, regular and one dimensional. Schemes defined over \( \text{Spec} \mathcal{O}_S \) are denoted by an underline, being omitted in the notation of their generic fibers.

As 2 is invertible in \( \mathcal{O}_S \), the \( \mathcal{O}_S \)-group \( \mu_2 := \text{Spec} \mathcal{O}_S[t]/(t^2 - 1) \) is smooth, whence applying étale cohomology to the Kummer sequence:

\[ 1 \to \mu_2 \to \mathbb{G}_m \xrightarrow{x \mapsto x^2} \mathbb{G}_m \to 1 \]

gives rise to the long exact sequence of abelian groups:

\[
\begin{align*}
H^1_{\text{ét}}(\mathcal{O}_S, \mathbb{G}_m) \xrightarrow{[\varphi] \mapsto [\varphi \otimes \mathcal{O}_S]} H^1_{\text{ét}}(\mathcal{O}_S, \mathbb{G}_m) & \to H^2_{\text{ét}}(\mathcal{O}_S, \mu_2) \to H^2_{\text{ét}}(\mathcal{O}_S, \mathbb{G}_m) \\
\xrightarrow{[\mathcal{A}] \mapsto [\mathcal{A} \otimes \mathcal{O}_S]} & \to H^2_{\text{ét}}(\mathcal{O}_S, \mathbb{G}_m).
\end{align*}
\]

Identifying \( H^1_{\text{ét}}(\mathcal{O}_S, \mathbb{G}_m) \) with \( \text{Pic}(\mathcal{O}_S) \) by Shapiro’s lemma (cf. \[\text{SGA}3\] XXIV, Prop. 8.4), and \( H^2_{\text{ét}}(\mathcal{O}_S, \mathbb{G}_m) \) with the Brauer group \( \text{Br}(\mathcal{O}_S) \), classifying Azumaya \( \mathcal{O}_S \)-algebras.

This work was supported by a Chateaubriand Fellowship of the Embassy of France in Israel, 2016.
(cf. [Mil §2]), we deduce the short exact sequence:

$$1 \to \text{Pic}(\mathcal{O}_S)/2 \xrightarrow{\partial} H^2_{\text{ét}}(\mathcal{O}_S, \mu_2) \xrightarrow{\beta} 2\text{Br}(\mathcal{O}_S) \to 1,$$

in which the right non-trivial term is the 2-torsion part in $\text{Br}(\mathcal{O}_S)$. We analyze some properties related to this sequence in Section 2, which will be used to classify regular quadratic $\mathcal{O}_S$-spaces.

Let $(V, q)$ (not to be confused with $q = |\mathbb{F}_q|$) be a quadratic $\mathcal{O}_S$-space of rank $n \geq 3$, namely, $V$ is a projective $\mathcal{O}_S$-module of rank $n$ and $q : V \to \mathcal{O}_S$ is a 2-order homogeneous $\mathcal{O}_S$-form. Since 2 is a unit, $q$ corresponds to the symmetric bilinear form $B_q : V \times V \to \mathcal{O}_S$ such that:

$$B_q(u, v) = q(u + v) - q(u) - q(v) \quad \forall u, v \in V.$$

We assume it to be $\mathcal{O}_S$-regular, namely, the induced homomorphism $V \to V^\vee := \text{Hom}(V, \mathcal{O}_S)$ is an isomorphism ([Knu I, §3, 3.2]). Two quadratic $\mathcal{O}_S$-spaces $(V, q)$ and $(V', q')$ are isomorphic over an extension $R$ of $\mathcal{O}_S$, if there exists an $R$-isometry between them, namely, an $R$-isomorphism $T : V' \otimes_{\mathcal{O}_S} R \cong V \otimes_{\mathcal{O}_S} R$ such that $q \circ T = q'$. The notation of the $\mathcal{O}_S$-isomorphism class $[(V, q)]$ is sometimes, when no ambiguity arises, shortened to $[q]$. The [proper] genus of $(V, q)$ is the set of classes of all quadratic $\mathcal{O}_S$-spaces that are [properly, i.e., with det = 1 isomorphisms] isomorphic to $(V, q)$ over $K$ and over $\hat{\mathcal{O}}_p$ for any prime $p \notin S$. This [proper] genus bijects as a pointed-set with the class set $[\text{Cl}_S(\mathcal{O}_S), \text{Cl}_S(\mathcal{O}_S)]$.

The results generalize the ones in [Bit], in which $S$ is assumed to contain only one (arbitrary) point $\infty \in \Omega$, thus giving rise to an affine curve $C^\text{af} = \mathcal{C} - \{\infty\}$, for which $\text{Br}(\mathcal{O}_{\{\infty\}}) = 1$ (cf. [Bit Lemma 3.3]). The quadratic $\mathcal{O}_{\{\infty\}}$-spaces that are locally properly isomorphic to $(V, q)$ for the flat or the étale topology belong all to the same genus, and are classified by the abelian group $H^2_{\text{ét}}(\mathcal{O}_{\{\infty\}}, \mu_2) \cong \text{Pic}(\mathcal{O}_{\{\infty\}})/2$. Here we show more generally for any finite set $S$ that, as $\text{Cl}_S(\mathcal{O}_S)$ is the kernel of what we call the relative Witt-invariant $H^1_{\text{ét}}(\mathcal{O}_S, \mathcal{O}_S^\vee) \xrightarrow{w} 2\text{Br}(\mathcal{O}_S)$, the latter abelian group bijects to the set of $2^{|S|-1}$ proper genera of $(V, q)$ (Proposition [1.5]).

Another consequence of passing to $|S| > 1$ is that $\mathcal{O}_S^\times \neq \mathbb{F}_q^\times$, whence $\mathcal{O}_S$-regularity imposed on $(V, q)$, no longer guarantees its isotropy. Requiring $(V, q)$ to be isotropic, we show that $\text{Pic}(\mathcal{O}_S)/2$ still classifies the quadratic spaces in the genus $\text{Cl}_S(\mathcal{O}_S)$ being equal to proper genus in this case (Lemma [1.4], for any $S$ (Theorem [1.6]). In particular in case $n \geq 5$, in which all classes are isotropic, any proper genus in $H^1_{\text{ét}}(\mathcal{O}_S, \mathcal{O}_S^\vee)$ – corresponding as aforementioned to an element of $2\text{Br}(\mathcal{O}_S)$ – is isomorphic to $\text{Pic}(\mathcal{O}_S)/2$, whence their disjoint union $H^1_{\text{ét}}(\mathcal{O}_S, \mathcal{O}_S^\vee)$ is isomorphic to the abelian group $H^2_{\text{ét}}(\mathcal{O}_S, \mu_2)$ (as for $|S| = 1$ and $n \geq 3$), fitting into the sequence (1.1) (Corollary [1.8]).

In Section 5, we refer to the case in which $V$ is split by a hyperbolic plane $H(L_0)$, and provide an isomorphism $\psi_V : \text{Pic}(\mathcal{O}_S)/2 \to \text{Cl}_S(\mathcal{O}_S)$. In case $C$ is an elliptic curve and $S = \{\infty\}$ where
∞ is $\mathbb{F}_q$-rational, an algorithm, producing explicitly representatives of classes in $H^1_{\text{et}}(\mathcal{O}_S, \mathbb{Q}_\ell)$, is given (1).

2. A CLASSIFICATION OF AZUMAYA ALGEBRAS

A faithfully flat projective (right) $\mathcal{O}_S$-module $A$ is an Azumaya $\mathcal{O}_S$-algebra if the map

$$A \otimes A^{\text{op}} \to \text{End}_{\mathcal{O}_S}(A) : a \otimes b^{\text{op}} \mapsto (x \mapsto axb)$$

is an isomorphism. It is central, separable and finitely generated as an $\mathcal{O}_S$-module. Two Azumaya $\mathcal{O}_S$-algebras $A, B$ are Brauer equivalent if there exist faithfully projective modules $P, Q$ such that:

$$A \otimes \text{End}_{\mathcal{O}_S}(P) \cong B \otimes \text{End}_{\mathcal{O}_S}(Q).$$

The tensor product induces the structure of an abelian group $\text{Br}(\mathcal{O}_S)$ on the equivalence classes, in which the neutral element is $[\mathcal{O}_S]$ and the inverse of $[A]$ is $[A^{\text{op}}]$ (cf. [Knu, III. 5.1 and 5.3]).

Let $V \cong \mathcal{O}_S^2$. Consider the following exact and commutative diagram of smooth $\mathcal{O}_S$-groups:

$$\begin{array}{ccccccc}
1 & \to & \mu_2 & \to & \text{SL}(V) & \to & \text{PGL}(V) & \to & 1 \\
1 & \to & \mathbb{G}_m & \to & \text{GL}(V) & \to & \text{PGL}(V) & \to & 1 \\
& & x \mapsto x^2 & & \det & & \\
& & \mathbb{G}_m & \to & \mathbb{G}_m & & \\
& & 1 & & 1 & & \\
\end{array}$$

The generalization of the Skolem–Noether Theorem to unital commutative rings, applied to the Azumaya $\mathcal{O}_S$-algebra $A = \text{End}_{\mathcal{O}_S}(V)$, is the exact sequence of groups (see [Knu, III. 5.2.1]):

$$1 \to \mathcal{O}_S^\times \to A^\times \to \text{Aut}_{\mathcal{O}_S}(A) \to \text{Pic} (\mathcal{O}_S).$$

This sequence induces by sheafification a short exact sequence of sheaves in the étale topology (cf. [Knu, p. 145]):

$$1 \to \mathbb{G}_m \to \mathbf{GL}(V) \to \text{Aut}(\text{End}_{\mathcal{O}_S}(V)) \to 1$$

from which we see that: $\text{PGL}(V) = \text{Aut}(\text{End}_{\mathcal{O}_S}(V))$. In this interpretation, étale cohomology applied to the diagram (2.1), plus the sequence (1.1), give rise to the exact and commutative
in which \( H^1_\text{ét}(\mathcal{O}_S, \text{Aut}(V)) \) classifies twisted forms of \( V \) in the étale topology, while its image in \( H^1_\text{ét}(\mathcal{O}_S, \text{Aut}(\text{End}_{\mathcal{O}_S}(V))) \) classifies these \( \mathcal{O}_S \)-modules up to scaling by an \( \mathcal{O}_S \)-line, i.e., by an invertible \( \mathcal{O}_S \)-module. Explicitly, \( \pi_* : [P] \mapsto [\text{End}_{\mathcal{O}_S}(P)] \) (cf. [Knu, III. 5.2.4]).

**Corollary 2.1.** In diagram (2.2): \( \partial([L]) = \partial^1([\text{End}_{\mathcal{O}_S}(\mathcal{O}_S \oplus L)]) \).

**Proof.** By chasing diagram (2.2) we may deduce the following reduced one:

\[
\begin{align*}
\text{Pic}(\mathcal{O}_S)/2 & \quad \Delta \\
H^1_\text{ét}(\mathcal{O}_S, \text{Aut}(V)) & \quad \partial^1 \, \pi_* \\
H^2_\text{ét}(\mathcal{O}_S, \mu_2) & \quad \partial^1 \, \pi_* \\
2\text{Br}(\mathcal{O}_S) & \quad \partial
\end{align*}
\]

which shows that: \( \partial([L]) = \partial(\Delta([\mathcal{O}_S \oplus L])) = \partial^1(\pi_*([\mathcal{O}_S \oplus L])) = \partial^1([\text{End}_{\mathcal{O}_S}(\mathcal{O}_S \oplus L)]) \).

**Lemma 2.2.** \( |2\text{Br}(\mathcal{O}_S)| = 2^{|S|−1} \).

**Proof.** Let \( r_p : \text{Br}(K) \to H^1(k_p, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z} \) be the residue map at a prime \( p \). The ramification map \( a := \oplus_p r_p \) yields the exact sequence from Class Field Theory (see [GS, Theorem 6.5.1]):

\[
1 \to \text{Br}(K) \xrightarrow{a} \bigoplus_p \mathbb{Q}/\mathbb{Z} \xrightarrow{\sum_p \text{Cor}_p} \mathbb{Q}/\mathbb{Z} \to 1
\]

in which the corestriction map \( \text{Cor}_p \) for any \( p \) is an isomorphism induced by the Hasse-invariant \( \text{Br}(\hat{k}_p) \cong \mathbb{Q}/\mathbb{Z} \) (cf. [GS, Proposition 6.3.9]). On the other hand, as all residue fields of \( K \) are finite, thus perfect, and \( \mathcal{O}_S \) is a one-dimensional regular scheme, it admits due to Grothendieck the following exact sequence (see [Gro, Proposition 2.1] and [Mil, Example 2.22, case (a)]):

\[
1 \to \text{Br}(\mathcal{O}_S) \to \text{Br}(K) \xrightarrow{\oplus_{p \notin S} r_p} \bigoplus_{p \notin S} \mathbb{Q}/\mathbb{Z},
\]

\[
(2.5)
\]
which means that \( \text{Br}(\mathcal{O}_S) \) is the subgroup of \( \text{Br}(K) \) of classes that vanish under \( r_p \) at any \( p \notin S \).

Thus omitting these \( r_p, p \notin S \) in the sequence (2.4), results in \( \text{Br}(\mathcal{O}_S) = \ker \left[ \bigoplus_{p \in S} \mathbb{Q}/\mathbb{Z} \xrightarrow{\sum_{p \in S} \text{Cof}_{r_p}} \mathbb{Q}/\mathbb{Z} \right] \), whence the cardinality of its 2-torsion part is \( 2^{|S|-1} \).

Lemma 2.3. Let \( G \) be an affine, flat, connected and smooth \( \mathcal{O}_S \)-group. Suppose that its generic fiber \( G \) is almost simple, simply connected and \( S \)-isotropic for any \( p \in S \). Then \( H^1_{\text{ét}}(\mathcal{O}_S, G) = 0 \).

Proof. The proof, basically relying on the strong approximation property related to \( G \), is similar to that of Lemma 3.2 in [Bri], replacing \( \{\infty\} \) by \( S \). \( \Box \)

3. Standard and relative invariants

Let \( O_V \) be the orthogonal group of \((V, q)\) defined over \( \text{Spec} \mathcal{O}_S \), namely, the functor assigning to any \( \mathcal{O}_S \)-algebra \( R \) the group of self-isometries of \( q \) over \( R \):

\[
O_V(R) = \{ A \in \mathbf{GL}_n(R) : q \circ A = q \}.
\]

Since \( 2 \in \mathcal{O}_S^\times \) and \( q \) is regular, \( O_V \) is smooth as well as its connected component, namely, the special orthogonal group \( SO_V := \ker \left[ O_V \xrightarrow{\text{det}} \mu_2 \right] \) (see Definition 1.6, Theorem 1.7 and Corollary 2.5 in [Con]). Thus the pointed set \( H^1(\mathcal{O}_S, SO_V) \) – properly (i.e., with \( \text{det} = 1 \) isomorphisms) classifying \( \mathcal{O}_S \)-forms that are locally everywhere isomorphic to \( q \) in the flat topology – coincides with the classification \( H^1_{\text{ét}}(\mathcal{O}_S, SO_V) \) for the étale topology (see [SGA4 VIII Corollaire 2.3]).

Let \( C(V) := T(V)/(v \otimes v - q(v) \cdot 1 : v \in V) \) be the Clifford algebra associated to \((V, q)\) (see [Knu IV]). The linear map \( v \mapsto -v \) on \( V \) preserves \( q \), thus extends to an algebra automorphism \( \alpha : C(V) \rightarrow C(V) \). As it is an involution, the graded algebra \( C(V) \) is decomposed into positive and negative eigenspaces: \( C_i(V) = \{ x \in C(V) : \alpha(x) = (-1)^i x \} \) for \( i = 0, 1 \). Since \((V, q)\) is projective and \( \mathcal{O}_S \)-regular, \( C(V) \) is Azumaya over \( \mathcal{O}_S \) (cf. [Bas Theorem, p. 166]).

The Witt-invariant of \((V, q)\) is:

\[
w(q) = \begin{cases} [C(V)] \in \text{Br}(\mathcal{O}_S) & \text{if } n \text{ is even} \\ [C_0(V)] \in \text{Br}(\mathcal{O}_S) & \text{if } n \text{ is odd.} \end{cases}
\]

As \( C(V) \) and \( C_0(V) \) are algebras with involution, \( w(q) \) lies in \( 2\text{Br}(\mathcal{O}_S) \) ([Knu IV. 8]).

The Clifford group associated to \((V, q)\) is

\[
\text{CL}(V) := \{ u \in C(V)^\times : \alpha(u)\alpha(v)^{-1} \in V \ \forall v \in V \}.
\]

The group \( \text{Pin}_V(\mathcal{O}_S) := \ker \left[ \text{CL}(V) \xrightarrow{N} O_S^\times \right] \) where \( N : v \mapsto v\alpha(v) \), admits an underlying \( \mathcal{O}_S \)-group scheme, called the Pinor group denoted by \( \text{Pin}_V \). It is a double covering of \( O_V \) and its center \( \mu_2 \) is smooth. So applying étale cohomology to the Pinor exact sequence of smooth \( \mathcal{O}_S \)-groups:

\[
1 \rightarrow \mu_2 \rightarrow \text{Pin}_V \rightarrow O_V \rightarrow 1
\] (3.1)
gives rise to the coboundary map of pointed-sets

$$\delta_V : H^1_{\text{ét}}(\mathcal{O}_S, \mathcal{O}_V) \to H^2_{\text{ét}}(\mathcal{O}_S, \mu_2).$$  
(3.2)

Let $\mathcal{O}_{2n}$ and $\mathcal{O}_{2n+1}$ be the orthogonal groups of the hyperbolic spaces $H(\mathcal{O}_S^n)$ and $H(\mathcal{O}_S^n)\perp\langle 1 \rangle$, respectively, equipped with the standard split form which we denote by $q_n$ (see [Con, Definition 1.1]). The pointed set $H^1_{\text{ét}}(\mathcal{O}_S, \mathcal{O}_n)$ classifies regular quadratic $\mathcal{O}_S$-modules of rank $n$ ([Knu, IV. 5.3.1]). It is identified with the pointed set $H^1_{\text{ét}}(\mathcal{O}_S, \mathcal{O}_V)$ simply obtained by changing the base point to $(V, q)$ (cf. [Knu, IV, Prop. 8.2]). We denote this identification by $\theta$. One has the following commutative diagram of pointed sets (cf. [Gir, IV, Prop. 4.3.4]):

$$
\begin{array}{ccc}
H^1_{\text{ét}}(\mathcal{O}_S, \mathcal{O}_n) & \xrightarrow{\theta} & H^1_{\text{ét}}(\mathcal{O}_S, \mathcal{O}_V) \\
\downarrow{\delta} & & \downarrow{\delta_V} \\
H^2_{\text{ét}}(\mathcal{O}_S, \mu_2) & \xrightarrow{r_V} & H^2_{\text{ét}}(\mathcal{O}_S, \mu_2)
\end{array}
$$

in which $\delta := \delta_{q_n}$ and $r_V(x) = x - \delta(q)$. 

**Definition 1.** We call the composition of maps of pointed sets:

$$w_V : H^1_{\text{ét}}(\mathcal{O}_S, \mathcal{O}_n) \xrightarrow{\delta_V} H^2_{\text{ét}}(\mathcal{O}_S, \mu_2) \xrightarrow{i^2_*} 2\text{Br}(\mathcal{O}_S)$$

(see sequence (1.1) for $i^2_*$) the relative Witt-invariant. It is a “shift” of the Witt-invariant $w = i^2_* \circ \delta$, such that the base-point $[(V, q)]$ is mapped to $[0] \in \text{Br}(\mathcal{O}_S)$.

**Remark 3.1.** The $\delta$-image of a class represented by $(V', q')$, being a regular $\mathcal{O}_S$-module, is its second Stiefel–Whitney class, denoted $w_2(q')$ (cf. [EKV, Definition 1.6 and Corollary 1.19]).

The connected component $\text{Spin}_V$ of $\text{Pin}_V$ is smooth, and it is the universal covering of $\text{SO}_V$:

$$1 \to \mu_2 \to \text{Spin}_V \xrightarrow{p} \text{SO}_V \to 1.$$  
(3.4)

Then étale cohomology gives rise to the exact sequence of pointed sets:

$$
\begin{array}{ccc}
H^1_{\text{ét}}(\mathcal{O}_S, \text{Spin}_V) & \to & H^1_{\text{ét}}(\mathcal{O}_S, \text{SO}_V) \\
\xrightarrow{sV} & & \xrightarrow{i_*} H^2_{\text{ét}}(\mathcal{O}_S, \mu_2) \to 1
\end{array}
$$

in which the right exactness comes from the fact that $\mathcal{O}_S$ is of Douai-type, thus $H^2_{\text{ét}}(\mathcal{O}_S, \text{Spin}_V) = 1$ (see Definition 5.2 and Example 5.4(iii) in [Con]). The inclusion $i : \text{SO}_V \subset \mathcal{O}_V$ with the map $i^2_*$ from sequence (1.1) induces the commutative diagram (cf. [Knu, IV, 8.3]):

$$
\begin{array}{ccc}
H^1_{\text{ét}}(\mathcal{O}_S, \text{SO}_V) & \xrightarrow{i_*} & H^1_{\text{ét}}(\mathcal{O}_S, \mathcal{O}_V) \\
\downarrow{sV} & & \downarrow{w_V} \\
H^2_{\text{ét}}(\mathcal{O}_S, \mu_2) & \xrightarrow{\delta_V} & 2\text{Br}(\mathcal{O}_S).
\end{array}
$$  
(3.6)
Remark 3.2. The map $i_*$ does not have to be injective, yet any form $q'$, properly isomorphic to $q$, represents a class in $H^1_{\text{ét}}(O_S, \mathcal{O}_p)$, so the restriction of $w$ to $H^1_{\text{ét}}(O_S, \mathcal{O}_S \otimes \mathcal{O}_p)$ is well-defined. Similarly, we may write the restriction $w_V|H^1_{\text{ét}}(O_S, \mathcal{O}_S \otimes \mathcal{O}_p)$ as $i^*_v \circ s \delta_v$, being surjective, as both $i^*_v$ and $s \delta_v$ are such (see sequences (1.1) and (3.5)).

Remark 3.3. Unlike over fields, the Stiefel–Whitney class $w_2$ for quadratic $O_S$-spaces, referring to their Clifford algebras not only as Azumaya algebras but as algebras with involution, is richer than the Witt-invariant $w$ lying in $2\text{Br}(O_S)$. For example, if $L$ is an invertible $O_S$-module and $H(L) = L \oplus L^*$ is the corresponding hyperbolic plane, then $C(H(L))$ is isomorphic as a graded algebra to $\text{End}_{O_S}(\wedge^2 L) = \text{End}_{O_S}(O_S \oplus L)$ ([Knu, IV, Prop. 2.1.1]) being Brauer-equivalent to $\text{Br}(O_S)$, thus $w(H(L)) = [0] \in \text{Br}(O_S)$, while:

$$\delta([H(L)]) = \partial([L]) \quad \text{Corollary 2.1}$$

(see the left equality in the proof of the Proposition in [EKV, §5.5]), does not have to vanish as we shall see in Proposition 5.1.

4. A classification of quadratic spaces via their genera

Consider the ring of $S$-integral adèles $\mathbb{A}_S := \prod_{p \in S} \hat{K}_p \times \prod_{p \not\in S} \hat{O}_p$, being a subring of the adèles $\mathbb{A}$. Then the $S$-class set of an $O_S$-group $G$ is the set of double cosets:

$$\text{Cl}_S(G) := G(\mathbb{A}_S) \backslash G(\mathbb{A}) / G(K)$$

(where for any prime $p$ the geometric fiber $G_p$ of $G$ is taken) and it is finite (cf. [BP, Prop. 3.9]). If $G$ is affine and finitely generated over Spec $O_S$, it admits according to Nisnevich ([Nis, Theorem I.3.5]) the following exact sequence of pointed sets:

$$1 \to \text{Cl}_S(G) \to H^1_{\text{ét}}(O_S, G) \to H^1(K, G) \times \prod_{p \in S} H^1_{\text{ét}}(\hat{O}_p, (G)_p).$$

(4.1)

If $G$ admits, furthermore, the property:

$$\forall p \notin S : \quad H^1_{\text{ét}}(\hat{O}_p, (G)_p) \hookrightarrow H^1(K_p, G_p),$$

(4.2)

then Nisnevich’s sequence for $G$ reduces to (cf. [Nis, Corollary I.3.6], [GP, Corollary A.8]):

$$1 \to \text{Cl}_S(G) \to H^1_{\text{ét}}(O_S, G) \to H^1(K, G).$$

(4.3)

Remark 4.1. Since Spec $O_S$ is normal, i.e., is integrally closed locally everywhere (due to the smoothness of $C$), any finite étale covering of $O_S$ arises by its normalization in some separable unramified extension of $K$ (see [Len, Theorem 6.13]). Consequently, if $G$ is a finite $O_S$-group, then
$H^1_{\text{ét}}(O_S, \mathbb{G}_m)$ is embedded in $H^1(K, \mathbb{G}_m)$. This is not true for infinite groups like the multiplicative group $\mathbb{G}_m$, for which $H^1_{\text{ét}}(O_S, \mathbb{G}_m) \cong \text{Pic}(O_S)$ clearly does not have to embed in $H^1(K, \mathbb{G}_m) = 1$.

Remark 4.2. In case $G = O_V$, the left exactness of sequence (4.1) reflects the fact that $\text{Cl}_S(O_V)$ is the genus of the base point $(V, q)$, namely, the set of classes of quadratic $O_S$-forms that are $K$ and $O_V$-isomorphic to it for all $p \notin S$. Furthermore, being connected, $SO_V$ admits property (4.2) by Lang’s Theorem (recall that all residue fields are finite), so the proper genus can be described as:

$$\text{Cl}_S(SO_V) = \ker[H^1_{\text{ét}}(O_S, SO_V) \to H^1(K, SO_V)].$$

As $O_V/SO_V$ is the finite representable $O_S$-group $\mu_2$ (cf. [Con, Theorem 1.7]), $O_V$ admits property (4.2) as well (see in the proof of Proposition 3.4 in [CGP]), so we may also write:

$$\text{Cl}_S(O_V) = \ker[H^1_{\text{ét}}(O_S, O_V) \to H^1(K, O_V)].$$

As a pointed set, $\text{Cl}_S(SO_V)$ is bijective to the first Nisnevich cohomology set $H^1_{\text{Nis}}(O_S, SO_V)$ (cf. [Nis, Theorem I.2.8] and [Mor 4.1]), classifying $SO_V$-torsors in the Nisnevich topology. But Nisnevich covers are étale, so it is a subset of $H^1_{\text{ét}}(O_S, SO_V)$. Similarly, $\text{Cl}_S(O_V) \subseteq H^1_{\text{ét}}(O_S, O_V)$.

Lemma 4.3. If $(V, q)$ is isotropic then $O_V(O_S) \xrightarrow{\det} \mu_2(O_S)$ is surjective.

Proof. Consider the following exact and commutative diagram that arises by applying étale cohomology to the short exact sequences related to the smooth $O_S$-groups $\text{Pin}_V$ and $O_V$:

$$\begin{array}{ccc}
H^1_{\text{ét}}(O_S, \text{Spin}_V) & \xrightarrow{s\pi_*} & H^1_{\text{ét}}(O_S, \text{Pin}_V) \\
O_V(O_S) & \xrightarrow{\det} & \mu_2(O_S) \\
\xrightarrow{\partial_0} & H^1_{\text{ét}}(O_S, SO_V) & \xrightarrow{h} H^1_{\text{ét}}(O_S, O_V) \\
& \xrightarrow{s\delta_V} & H^2_{\text{ét}}(O_S, \mu_2) \\
& \xrightarrow{\delta_V} & H^2_{\text{ét}}(O_S, \mu_2).
\end{array}$$

Denote $[\gamma] = \partial_0(-1)$. Then $s\delta_V([\gamma]) = \delta_V(h([\gamma]) = [0]) = [0]$, hence $[\gamma] \in \text{Im}(s\pi_*)$. But as $q$ is isotropic, $H^1_{\text{ét}}(O_S, \text{Spin}_V)$ vanishes by strong approximation (cf. Lemma 2.3), so $[\gamma] = [0]$, which means that $\partial_0$ is the trivial map and $\det(O_S)$ surjects on $\mu_2(O_S)$. \hfill \qed

Lemma 4.4. If $n$ is odd, or $(V, q)$ is isotropic, then $\text{Cl}_S(SO_V) = \text{Cl}_S(O_V)$.

Proof. Any representative $(V', q')$ of a class in $\text{Cl}_S(O_V)$, being $K$ isomorphic to $q$, is regular and isotropic as well, whence $O_{V'}(O_S) \to \mu_2(O_S)$ is surjective by Lemma 4.3. When $n$ is odd, this
surjectivity is retrieved by the fact that $\mathcal{O}_V \cong \mathbf{SO}_V \times \mu_2$ (cf. [Con Thm.1.7]), and so applying étale cohomology to the exact sequence of smooth groups

$$1 \to \mathbf{SO}_V \to \mathcal{O}_V \to \mu_2 \to 1$$

we get that $\ker[H^1_{\text{ét}}(\mathcal{O}_S, \mathbf{SO}_V) \xrightarrow{\psi} H^1_{\text{ét}}(\mathcal{O}_S, \mathcal{O}_V)] = 1$ for any $[(V', q')] \in \text{Cl}_S(\mathcal{O}_V)$, which means that the restricted map $\text{Cl}_S(\mathbf{SO}_V) \xrightarrow{\psi} \text{Cl}_S(\mathcal{O}_V)$ is injective. Together with Remark 4.2, this amounts to the existence of the following exact and commutative diagram:

$$\begin{array}{ccc}
\text{Cl}_S(\mathbf{SO}_V) & \xrightarrow{\psi} & \text{Cl}_S(\mathcal{O}_V) \\
\downarrow i & & \downarrow \psi' \\
1 & \xrightarrow{\psi'} & H^1_{\text{ét}}(\mathcal{O}_S, \mathbf{SO}_V) & \xrightarrow{d} & H^1_{\text{ét}}(\mathcal{O}_S, \mu_2) \\
\downarrow m & & \downarrow m' & & \downarrow m'' \\
1 & \to & H^1(K, \mathbf{SO}_V) & \xrightarrow{h} & H^1(K, \mathcal{O}_V) & \xrightarrow{d'} & H^1(K, \mu_2) \\
\end{array}$$

in which as $m''$ is injective due to Remark 4.1, $\psi$ is also surjective, thus is the identity. □

**Proposition 4.5.** Let $(V, q)$ be a regular quadratic $\mathcal{O}_S$-space of rank $n \geq 3$ with proper genus $\text{Cl}_S(\mathbf{SO}_V)$. The relative Witt-invariant (cf. Definition 4 and Remark 3.2) induces an exact sequence of pointed sets

$$1 \to \text{Cl}_S(\mathbf{SO}_V) \xrightarrow{h} H^1_{\text{ét}}(\mathcal{O}_S, \mathbf{SO}_V) \xrightarrow{w_V} 2\text{Br}(\mathcal{O}_S) \to 1$$

in which $h$ is injective and $2\text{Br}(\mathcal{O}_S)$ bijects to the set of $2^{|\mathcal{S}|-1}$ proper genera of $q$.

**Proof.** Consider the short exact sequence induced by the double covering of the generic fiber

$$1 \to \mu_2 \to \mathbf{Spin}_V \to \mathbf{SO}_V \to 1.$$

As $\mathbf{Spin}_V$ is simply connected, we know due to Harder that $H^1(K, \mathbf{Spin}_V) = 1$ (cf. [Hard, Satz A]). This is true for all twisted forms of $\mathbf{Spin}_V$, whence Galois cohomology implies the embedding $H^1(K, \mathbf{SO}_V) \hookrightarrow H^2(K, \mu_2)$. Due to Hilbert’s Theorem 90, applying Galois cohomology to the Kummer’s exact sequence related to $\mu_2$ over $K$ gives an isomorphism $H^2(K, \mu_2) \cong 2\text{Br}(K)$. Moreover, as shown in the sequence (2.5), $\text{Br}(\mathcal{O}_S)$ is a subgroup of $\text{Br}(K)$. All together, the relative Witt-invariant applied to classes in $H^1_{\text{ét}}(\mathcal{O}_S, \mathbf{SO}_V)$ and on their generic fibers, yields the following exact and commutative diagram:

$$\begin{array}{ccc}
H^1_{\text{ét}}(\mathcal{O}_S, \mathbf{SO}_V) & \xrightarrow{w_V} & 2\text{Br}(\mathcal{O}_S) \\
\downarrow & & \downarrow \\
H^1(K, \mathbf{SO}_V) & \xrightarrow{w_V} & 2\text{Br}(K) \\
\end{array}$$

(4.6)
which justifies the left exactness in the asserted sequence:

\[
\text{Cl}_S(SO_V) \cong \ker[H^1_\text{ét}(O_S, SO_V) \to H^1(K, SO_V)] = \ker[H^1_\text{ét}(O_S, SO_V) \xrightarrow{w_V} 2\text{Br}(O_S)].
\]

(4.7)

The surjectivity of \(w_V : H^1_\text{ét}(O_S, SO_V) \to 2\text{Br}(O_S)\) (cf. Remark 6.2) completes the proof.

The first equality in (4.7) suggests that for any \([q'] \in H^1_\text{ét}(O_S, SO_V)\), \(q' \in \text{Cl}_S(SO_V)\) if and only if \(q'\) is \(K\)-isomorphic to \(q\) while the second equality claims that this holds if and only if \(w_V(q') = w_V(q)\). This is true for any choice of base point, being regular as well. So as \(H^1_\text{ét}(O_S, SO_V)\) is a disjoint union of the proper genera of \(q\), together with the surjectivity of \(w_V\), we deduce that the set of proper genera bijects with \(2\text{Br}(O_S)\), whose cardinality is computed in Lemma 2.2.

\[\square\]

**Theorem 4.6.** Let \((V, q)\) be a regular quadratic \(O_S\)-space of rank \(n \geq 3\). Then there exists a surjection of pointed-sets: \(\text{Cl}_S(SO_V) \to \text{Pic}(O_S)/2\). If \((V, q)\) is isotropic, then this is a bijection, so the abelian group \(\text{Pic}(O_S)/2\) is isomorphic to \(\text{Cl}_S(SO_V) = \text{Cl}_S(O_V)\).

**Proof.** Consider the following exact and commutative diagram derived from sequences (1.1) and (3.5):

\[
\begin{array}{ccc}
1 & \to & H^1_\text{ét}(O_S, SO_V) \\
\downarrow & & \downarrow s\delta_V \\
1 & \to & \text{Pic}(O_S)/2 \\
\downarrow & & \downarrow \partial \\
1 & \to & H^2_\text{ét}(O_S, \mu_2) \\
\downarrow & & \downarrow i^2 \\
& & 2\text{Br}(O_S) \\
\downarrow & & \downarrow \text{1} \\
& & 1.
\end{array}
\]

(4.8)

We imitate the Snake Lemma argument (though the diagram terms are not all groups): according to Proposition 4.5 \(\text{Cl}_S(SO_V) = \ker(w_V)\), hence for any \([q'] \in \text{Cl}_S(SO_V)\) one has \(i^2_* (s\delta_V([q'])) = [0]\), i.e., \([q']\) has a \(\partial\)-preimage in \(\text{Pic}(O_S)/2\) which is unique as \(\partial\) is a monomorphism of groups. Moreover, any element in \(\text{Pic}(O_S)/2\) arises in this way, since \(s\delta_V\) is surjective. As a result we have an exact sequence of pointed sets:

\[
1 \to \mathfrak{K}_1 \to \text{Cl}_S(SO_V) \xrightarrow{s\delta_V} \text{Pic}(O_S)/2 \to 1.
\]

If \((V, q)\) is isotropic, then \(s\delta_V\) is also injective. Indeed, let \(SO_V\) be a twisted form of \(SO_V\), properly stabilizing a form \(q'\), and let \(\text{Spin}_V\) be its Spin group. The lower row in the following exact diagram is the one obtained when replacing the base point \(q\) by \(q'\), as described in [Gir] IV,
Proposition 4.3.4]:

$$\begin{array}{c}
\mathfrak{R}_1 \longrightarrow \text{Cl}_S(\mathbf{SO}_V) \xrightarrow{s_{\delta_V}} \text{Pic} (\mathcal{O}_S)/2 \\
\subset \\
H^1_{\text{et}}(O_S, \text{Spin}_V) \longrightarrow H^1_{\text{et}}(O_S, \mathbf{SO}_V) \xrightarrow{s_{\delta_V}} H^2_{\text{et}}(O_S, \mu_2)
\end{array}$$

(4.9)

If \([q'] \in \text{Cl}_S(\mathbf{SO}_V)\), then \(q'\) is \(K\)-isomorphic to \(q\), as well as the generic fiber \(\text{Spin}_V\). Then by the Hasse–Minkowsky Theorem (cf. [Lam, VI.3.1]), \(q'\) is \(K_p\)-isotropic everywhere, in particular in \(S\). Hence \(H^1_{\text{et}}(O_S, \text{Spin}_V)\) is trivial by Lemma 2.3 for any class in \(\text{Cl}_S(\mathbf{SO}_V)\) (\(\mathbf{SO}_V\) is not commutative for \(n \geq 3\) thus \(H^1_{\text{et}}(O_S, \mathbf{SO}_V)\) does not have to be a group, so the triviality of \(H^1_{\text{et}}(O_S, \text{Spin}_V)\) does not imply the injectivity of \(s_{\delta_V}\), i.e., there still might be distinct anisotropic classes in \(H^1_{\text{et}}(O_S, \mathbf{SO}_V)\) whose images in \(H^2_{\text{et}}(O_S, \mu_2)\) coincide). So \(\text{Cl}_S(\mathbf{SO}_V)\), being equal in the isotropic case to \(\text{Cl}_S(\mathcal{O}_V)\) by Lemma 4.4, embeds in \(\text{Pic} (\mathcal{O}_S)/2\) and the assertion follows.

\[\]  

**Definition 2.** We say that the **local-global Hasse principle** holds for a quadratic \(\mathcal{O}_S\)-space \((V, q)\) if \(|\text{Cl}_S(\mathcal{O}_V)| = 1\).

**Corollary 4.7.** The Hasse principle holds for a regular isotropic quadratic \(\mathcal{O}_S\)-form of rank \(\geq 3\) if and only if \(|\text{Pic} (\mathcal{O}_S)|\) is odd.

**Corollary 4.8.** Let \((V, q)\) be an \(\mathcal{O}_S\)-regular quadratic space of rank \(\geq 5\). Then the pointed-set \(H^1_{\text{et}}(O_S, \mathbf{SO}_V)\) is isomorphic to the abelian group \(H^2_{\text{et}}(O_S, \mu_2)\), i.e., any \(\mathcal{O}_S\)-isomorphism class in the proper classification corresponds to an Azumaya \(\mathcal{O}_S\)-algebra with involution. There are \(2|S|!{-1}\) genera, each of them is isomorphic to \(\text{Pic} (\mathcal{O}_S)/2\).

**Proof.** In rank \(\geq 5\), any quadratic \(\mathcal{O}_S\)-space is isotropic; indeed, for any such \((V', q')\), the generic fiber \(q'_K := q' \otimes K\) is isotropic (cf. [OMe, Theorem 66:2]), i.e., there exists a non-zero vector \(v_0 \in V' \otimes K\) such that \(q'_K(v_0) = 0\). Since \(K\) is the fraction field of the Dedekind domain \(\mathcal{O}_S\), there exists a non-zero vector \(\mathbf{w}_j \in K v_0 \cap \mathcal{O}_S\) for which \(q(\mathbf{w}_j) = 0\). Hence according to Theorem 4.6, we deduce that the genus of any \([V', q']\) \(\in H^1_{\text{et}}(O_S, \mathbf{SO}_V)\) is isomorphic to the abelian group \(\text{Pic} (\mathcal{O}_S)/2\) and injects into \(H^1_{\text{et}}(O_S, \mathbf{SO}_V)\). Looking at the obtained exact and commutative diagram:

\[
\begin{array}{c}
1 \longrightarrow \text{Cl}_S(\mathbf{SO}_V) \xrightarrow{\cong} H^1_{\text{et}}(O_S, \mathbf{SO}_V) \xrightarrow{\iota^2} \text{Pic} (\mathcal{O}_S)/2 \xrightarrow{\delta_V} H^2_{\text{et}}(O_S, \mu_2) \xrightarrow{\iota^2} 2\text{Br}(O_S) \longrightarrow 1 \\
1 \longrightarrow 2\text{Br}(O_S) \longrightarrow 1
\end{array}
\]
we see that the cardinality of $H^1_{cts}(\mathcal{O}_S, \mathbf{SO}_V)$, being the disjoint union of its genera, equals the one of its $s\delta_V$-image $H^2_{cts}(\mathcal{O}_S, \mu_q)$, hence it is isomorphic to it. We have seen in Proposition 4.5 that $2\text{Br}(\mathcal{O}_S)$ bijects with the set of proper genera of $q$. Here, as any twisted form of $q$ is isotropic, its proper genus is equal to its genus (Lemma 4.4), thus there are $2|S|-1$ such genera (Lemma 2.2). □

**Remark 4.9.** Since as we have seen any integral quadratic form of rank $\geq 5$ is isotropic, according to Lemma 4.4 $\text{Cl}_S(\mathbf{SO}_V)$ might not be equal to $\text{Cl}_S(\mathbf{O}_V)$ (for rank$(V) \geq 3$) only when $(V,q)$ is anisotropic of rank 4.

5. A splitting hyperbolic plane

In this section, we refer to regular quadratic $\mathcal{O}_S$-spaces being split by a hyperbolic plane $P$ (thus being isotropic and so $\text{Cl}_S(\mathbf{O}_V) = \text{Cl}_S(\mathbf{SO}_V)$). Such $P$ contains a hyperbolic pair $\{v_0, v_1\}$ of $V$, namely, satisfying: $q(v_0) = q(v_1) = 0$ and $B_q(v_0, v_1) = 1$, and it is of the form $P_a = av_0 + a^{-1}v_1$ for some fractional ideal $\mathfrak{a}$ of $\mathcal{O}_S$ (cf. [Ger, Proposition 2.1]). L. J. Gerstein established in [Ger, Theorem 4.5] for the ternary case the bijection:

$$\psi : \text{Pic}(\mathcal{O}_S)/2 \cong \text{Cl}_S(\mathbf{O}_V) : [\mathfrak{a}] \mapsto [V_\mathfrak{a}] := [P_\mathfrak{a} \bot \lambda]$$

where $\lambda$ is again a fractional ideal of $\mathcal{O}_S$ and $\lambda \in \mathcal{O}_S^\times$, for which $V \cong V_\mathfrak{a}$. The existence of $\psi$ can be viewed as a particular case of our Theorem 4.6 (such a splitting does not necessarily exist). The following Lemma suggests an alternative bijection which resembles the one of Gerstein, but does not, however, require finding and multiplying by a hyperbolic pair, and, more important, is valid for any rank $n \geq 3$.

**Proposition 5.1.** Suppose $V$ is split by a hyperbolic plane $V = H(L_0) \bot V_0$ where $L_0$ is an $\mathcal{O}_S$-line. Then $\psi_V : [L] \mapsto [V_L] = H(L_0 \otimes L^*) \bot V_0$ is an isomorphism of groups $\text{Pic}(\mathcal{O}_S)/2 \cong \text{Cl}_S(\mathbf{O}_V)$.

**Proof.** $L^*$ is locally free, thus $V_L$ obtained from $V$ by tensoring $H(L_0)$ with $L^*$ remains in $\text{Cl}_S(\mathbf{O}_V)$. Due to Theorem 4.6 the groups $\text{Pic}(\mathcal{O}_S)/2$ and $\text{Cl}_S(\mathbf{O}_V)$ are isomorphic through diagram (4.8), so it is sufficient to show by its commutativity that $s\delta_V \circ \psi_V$ coincides with the groups embedding $\partial$.

The $i$’th Stiefel–Whitney class $w_i(E)$ of a regular $\mathcal{O}_S$-module $E$, as defined in [EKV] §1, gets values in $H^i_{cts}(\mathcal{O}_S, \mu_q)$ for $i \geq 1$ and $w_0(E) = 1$. Its basic axioms, namely, $w_i(E) = 0$ for all $i > \text{rank}(E)$ and for any direct sum of regular $\mathcal{O}_S$-modules of finite rank

$$w_k(E \oplus F) = \sum_{i+j=k} w_i(E) \cdot w_j(F),$$

imply that:

$$w_1(E \oplus F) = w_1(E) + w_1(F) \quad \text{and} \quad w_2(E \oplus F) = w_2(E) + w_2(F) + w_1(E) \cdot w_1(F). \quad (5.1)$$
If $L$ is an $\mathcal{O}_S$-line and $H(L) = L \oplus L^*$ is the corresponding hyperbolic plane, this reads:

$$w_1(H(L)) = w_1(L) + w_1(L^*) \quad \text{while} \quad w_2(H(L)) = w_1(L) \cdot w_1(L^*).$$

Moreover, $w_1$ furnishes an isomorphism of abelian groups $\{\mathcal{O}_S\text{-lines}, \otimes\} \cong H^1_\text{ét}(\mathcal{O}_S, \mu_2)$ by

$$w_1(L_1 \otimes L_2) = w_1(L_1) + w_1(L_2), \quad \text{thus:} \quad w_1(L) = -w_1(L^*),$$

hence

$$w_1(H(L_0 \otimes L^*)) - w_1(H(L_0)) = w_1(L_0 \otimes L^*) + w_1(L_0^* \otimes L) - w_1(L_0) - w_1(L_0^*) = 0 \quad (5.2)$$

and:

$$w_2(H(L_0 \otimes L^*)) - w_2(H(L_0)) = (w_1(L_0) + w_1(L^*)) \cdot (w_1(L_0^*) + w_1(L)) - w_1(L_0) \cdot w_1(L_0^*) = 0 \quad (5.3)$$

In our setting (recall that $\delta = w_2$, see Remark 3.1), we get:

$$\delta([V_L]) = \delta([H(L_0 \otimes L^*)]) + \delta([V_0]) + w_1(H(L_0 \otimes L^*)) \cdot w_1(V_0) \quad (5.4)$$

$$\delta([H(L_0 \otimes L^*)]) = \delta([H(L_0)]) + [\theta^{-1}][V_L] = \delta([H(L)]) \quad (\text{Remark 3.3}).$$

Altogether, we may finally conclude that for any $[L] \in \text{Pic}(\mathcal{O}_S)/2$:

$$(s\delta_V \circ \psi_V)([L]) = s\delta_V([V_L]) = (r_V \circ \delta \circ \theta^{-1})([V_L]) = \delta([H(L)]) \quad (\text{Remark 3.3}).$$

5.1. $|S| = 1$. If $S = \{\infty\}$ where $\infty$ is an arbitrary closed point, then $C^{af} := C - \{\infty\}$ is an affine curve whence $\text{Br}(\mathcal{O}_S) = 1$ (cf. Lemma 2.2), i.e., there is a single genus, and any $\mathcal{O}_S$-regular quadratic space $(V, q)$ of rank $n \geq 3$ is isotropic (cf. [Bi]). Suppose furthermore that $C$ is an elliptic curve and $\infty$ is $\mathbb{F}_q$-rational. For any place $P$ on $C^{af}$ we define the maximal ideal of $\mathcal{O}_S$

$$m_P := \{f \in \mathcal{O}_S : f(P) = 0\}.$$ 

Then we have an isomorphism of abelian groups (see [Hart II, Proposition 6.5(c)] for the first map and p. 393 in [Bau] for the second one):

$$\varphi : C(\mathbb{F}_q) \cong \text{Pic}^0(C) \cong \text{Pic}(\mathcal{O}_S) : P \mapsto [P] - [\infty] \mapsto m_P.$$
Since \( \mathcal{O}_S \) is Dedekind, Steinitz’s Theorem (cf. [BK, Corollary 6.1.9]) tells us that for any fractional ideal \( \mathfrak{a} \) of \( \mathcal{O}_S \) there is an \( \mathcal{O}_S \)-isomorphism of \( \mathcal{O}_S \)-modules (though not of quadratic \( \mathcal{O}_S \)-spaces):

\[
\chi : \mathfrak{a} \oplus \mathfrak{a}^{-1} \to \mathcal{O}_S \oplus \mathcal{O}_S.
\]

Towards our purpose of finding representatives of twisted \( K \)-forms of \( q \), we may choose such a \( K \)-isomorphism \( \chi_K \) to be as in [Chu] Lemma 20.17:

\[
\chi_K : (\alpha, \beta) \mapsto (\alpha, \beta) \cdot A_\mathfrak{a}, \quad A_\mathfrak{a} = \begin{pmatrix} b_1 & -a_2 \\ b_2 & a_1 \end{pmatrix}
\]

where \( a_1b_1 + a_2b_2 = 1 \), \( b_1 \in \mathfrak{a}^{-1} \), \( b_2 \in \mathfrak{a} \). According to Proposition 5.1, the matrix \( A_{\mathfrak{m}_P}^{-1}B_qA_{\mathfrak{m}_P}^{-1} \) represents, for any coset \([P] \in C(\mathbb{F}_q)/2\), a distinct class of quadratic \( \mathcal{O}_S \)-forms in \( \text{Cl}_S(\mathcal{O}_V) = H^1_\mathfrak{a}(\mathcal{O}_S, \mathcal{SO}_V) \).

We summarize this procedure by the following algorithm:

**Algorithm 1** Generator of classes representatives isomorphic in the flat topology to a regular quadratic space split by a hyperbolic plane, over the coordinate ring of an affine non-singular elliptic curve

**Input:** \( C = \text{elliptic projective } \mathbb{F}_q \)-curve, \( S = \{ \mathfrak{c} \in C(\mathbb{F}_q) \}, V = H(L_0) \perp V_0 \) is quadratic regular \( \mathcal{O}_S \)-space of rank \( n \geq 3 \), \( H(L_0) \) is represented by \( F_0 \in \text{GL}_2(\mathcal{O}_S) \).

Compute \( C(\mathbb{F}_q) \) and \( \mathcal{O}_S := \text{Cl}^0(\mathbb{F}_q) \) where \( \text{Cl}^0 := C - \{ \mathfrak{c} \} \).

for each \([P] \in C(\mathbb{F}_q)/2\) do

\[\mathfrak{m}_P = \{ f \in \mathcal{O}_S : f(P) = 0 \} \] for \( P \neq \infty \) and \( \mathfrak{m}_\infty = \mathcal{O}_S \).

Find \( a_1, b_2 \in \mathfrak{m}_P \) and \( a_2, b_1 \in \mathfrak{m}_P^{-1} \) such that \( a_1b_1 + a_2b_2 = 1 \).

\[A_{\mathfrak{m}_P}^{-1} = \begin{pmatrix} a_1 & a_2 \\ b_2 & b_1 \end{pmatrix} \] and: \( F_P = A_{\mathfrak{m}_P}^{-1}F_0A_{\mathfrak{m}_P}^{-1} \).

**Output:** \( \text{Cl}_S(\mathcal{O}_V) = \{ [V_P \perp V_0] : [P] \in C(\mathbb{F}_q)/2 \} \), where \( V_P \) is the quadratic \( \mathcal{O}_S \)-form represented by \( F_P \).

**Remark 5.2.** If \( C \) is an elliptic curve and \( \infty \) is \( \mathbb{F}_q \)-rational, then for any \( \mathcal{O}_S \)-line \( L_0 \), the special orthogonal group of \( H(L_0) \), being split, of rank 2 and \( \mathcal{O}_S \)-regular, is a one dimensional split \( \mathcal{O}_S \)-torus, i.e., isomorphic to \( \mathbb{G}_m \) (see in the proof of [Bit, Theorem 4.5]), hence the proper classification is via \( H^1_\mathfrak{a}(\mathcal{O}_S, \mathbb{G}_m) \cong \text{Pic} (\mathcal{O}_S) \cong C(\mathbb{F}_q) \). Then the class representatives are obtained by the above algorithm when replacing \( C(\mathbb{F}_q)/2 \) by \( C(\mathbb{F}_q) \). This means that invertible fractional ideals, corresponding to non-trivial squares in \( C(\mathbb{F}_q) \), i.e., \( [L] \in 2\text{Pic} (\mathcal{O}_S)\setminus\{0\} \), and only they, induce spaces \( H(L_0 \otimes L^*) \) that are stably isomorphic to \( H(L_0) \) in the proper classification, namely, become properly isomorphic after being extended by any non-trivial regular orthogonal \( \mathcal{O}_S \)-space. In other words, the Witt Cancellation Theorem fails over \( \mathcal{O}_S \) in this case for the proper classification.
Example 5.3. Let $C = \{ Y^2Z = X^3 + XZ^2 \}$ defined over $\mathbb{F}_5$. Then

$$C(\mathbb{F}_5) = \{(0 : 0 : 1), (1 : 0 : 2), (1 : 0 : 3), (0 : 1 : 0)\}.$$ 

Taking $S = \{ \infty = (0 : 1 : 0) \}$ we get the affine elliptic curve

$$C^\text{af} = \{ y^2 = x^3 + x \} \quad \text{with: } \mathcal{O}_S = \mathbb{F}_5[x, y]/(y^2 - x^3 - x).$$

The affine supports of the points in $C(\mathbb{F}_5) - \{ \infty \}$ are: $\{(0, 0), (1/2, 0) = (3, 0), (1/3, 0) = (2, 0)\}$. The $y$-coordinate of these points vanishes which means that they are of order 2 according to the group law and $C(\mathbb{F}_q) \cong (\mathbb{Z}/2)^2$. We get:

$$m_{(0:0:1)} = \langle x, y \rangle, \quad m_{(1:0:2)} = \langle x - 3, y \rangle, \quad m_{(1:0:3)} = \langle x - 2, y \rangle, \quad m_{(0:1:0)} = \mathcal{O}_S.$$ 

Now consider the standard ternary quadratic $\mathcal{O}_S$-space $V = H(\mathcal{O}_S)\perp \{1\}$, i.e., with $B_q = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

For any $[L] \in \text{Pic} (\mathcal{O}_S)/2 = \text{Pic} (\mathcal{O}_S)$, the quadratic space $V_L = H(L)\perp \{1\}$ belongs to $\text{Cl}_S(\mathcal{O}_V)$ and there are four non-equivalent classes in $\text{Cl}_S(\mathcal{O}_V)$. For example, $A_{(x,y)}^{-1} = \begin{pmatrix} x & -y/x \\ y & -x \end{pmatrix}$ induces the form represented by

$$\begin{pmatrix} 2xy & -2x^2 - 1 & 0 \\ -2x^2 - 1 & 2y & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

being not $\mathcal{O}_S$-isomorphic to $V$, as $\langle x, y \rangle$ is not principal.

5.2. $|S| > 1$. If $|S| > 1$ then an $\mathcal{O}_S$-regular quadratic space $V$ will possess multiple $(2^{|S|} - 1)$ genera, and if $\text{rank}(V) = 3, 4$ some of them may be anisotropic, i.e., contain anisotropic representatives only.

Example 5.4. Let $C$ be the projective line defined over $\mathbb{F}_3$, so $K = \mathbb{F}_3(t)$, and let $S = \{ t, t^{-1} \}$, so $\mathcal{O}_S = \mathbb{F}_3[t, t^{-1}]$, being the ring of regular functions on the multiplicative group $\mathbb{G}_m$, thus it is a PID. The ternary $\mathcal{O}_S$-space $V = \langle 1, -1, -t \rangle$ with $q(x, y, z) = x^2 - y^2 - tz^2$ is isotropic, e.g., $q(1, 1, 0) = 0$. It is properly isomorphic over $\mathcal{O}_S(i)$ (being a scalar extension of $\mathcal{O}_S$ thus an étale one), by $\text{diag}(1, i, -i)$ to the anisotropic form $V' = \langle 1, 1, t \rangle$. Both are $\mathcal{O}_S$-unimodular as $\text{det}(q) = t \in \mathcal{O}_S^\times$, but belong to two distinct genera (there are exactly $2^{|S|} - 1 = 2$ such, cf. Proposition 4.5). As $\mathcal{O}_S$ is a PID, Pic $(\mathcal{O}_S) = 1$, and so according to Theorem 4.10 there is only one class in $\text{Cl}_S(\mathcal{O}_V)$. The hyperbolic plane $\langle 1, -1 \rangle$ has trivial Witt invariant, and it is orthogonal to $\langle -t \rangle$ in $V$, thus $w(q) = 0$ (cf. [Knu, IV, Prop. 8.1.1, 1, 3]), so $[q]$ corresponds to the trivial element in $\text{Br}(\mathcal{O}_S)$. To compute the Clifford algebra of $(V', q')$, we choose the natural basis

$$\{ e_1 = (1, 0, 0), \ e_2 = (0, 1, 0), \ e_3 = (0, 0, 1) \}.$$
The embedding $i : V' \to \mathbb{C}(q')$ satisfies the relations $i(v)^2 = q'(v) \cdot 1 \ \forall v \in V'$ which imply:

$$i(v)i(u) + i(u)i(v) = B_{q'}(u, v) \ \forall u, v \in V'.$$

Since $\{e_i\}_{i=1}^3$ are orthogonal, this means that:

$$i(e_i)i(e_j) = -i(e_j)i(e_i) \ \forall 1 \leq i \neq j \leq 3,$$

so we may choose (as $q'$ is anisotropic the obtained quaternion algebra is not split):

$$i(e_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \ i(e_2) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ i(e_3) = \begin{pmatrix} 0 & \sqrt{-t} & \sqrt{-t} \\ -\sqrt{-t} & 0 & 0 \\ -\sqrt{-t} & 0 & 0 \end{pmatrix}.$$  

Then $\{1, i(e_1)i(e_2), i(e_2)i(e_3), i(e_1)i(e_3)\}$ is a basis of $C_0(q')$ (cf. [Knu, V. §3]):

$$C_0(q') = \langle 1, \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & \sqrt{-t} & \sqrt{-t} \\ -\sqrt{-t} & 0 & 0 \\ -\sqrt{-t} & 0 & 0 \end{pmatrix} \rangle,$$

and $w(q') = [C_0(q')]$ is the non-trivial element in $2\text{Br}(O_S)$.

**Acknowledgements:** I thank L. Gerstein, B. Kahn, B. Kunyavskii and R. Parimala for valuable discussions concerning the topics of the present article. I would like to give special thanks to P. Gille, my postdoc advisor in Camille Jordan Institute in the University Lyon 1 for his fruitful advice and support.

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