UNIQUENESS OF EMBEDDINGS OF THE AFFINE LINE INTO ALGEBRAIC GROUPS

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Abstract. Let $Y$ be the underlying variety of a connected affine algebraic group. We prove that two embeddings of the affine line $\mathbb{C}$ into $Y$ are the same up to an automorphism of $Y$ provided that $Y$ is not isomorphic to a product of a torus $(\mathbb{C}^*)^k$ and one of the three varieties $\mathbb{C}^3$, $\text{SL}_2$, and $\text{PSL}_2$.

1. Introduction

In this paper, varieties are understood to be (reduced) algebraic varieties over the field of complex numbers $\mathbb{C}$, carrying the Zariski topology. We say that two closed embeddings of varieties $f, g: X \to Y$ are equivalent or the same up to an automorphism of $Y$ if there exists an automorphism $\varphi: Y \to Y$ such that $\varphi \circ f = g$. We consider embeddings of the affine line $\mathbb{C}$ into varieties $Y$ that arise as underlying varieties of affine algebraic groups and study these embeddings up to automorphisms of $Y$. Recall that an affine algebraic group is a closed subgroup of the complex general linear group $\text{GL}_n$ for some $n$. In this paper, all groups are affine and algebraic.

Our main result is the following.

Theorem 1.1. Let $Y$ be the underlying variety of a connected affine algebraic group. Then two embeddings of the affine line $\mathbb{C}$ into $Y$ are the same up to an automorphism of $Y$ provided that $Y$ is not isomorphic to a product of a torus $(\mathbb{C}^*)^k$ and one of the three varieties $\mathbb{C}^3$, $\text{SL}_2$, and $\text{PSL}_2$.

In particular, $\mathbb{C}$ embeds uniquely (up to automorphisms) into affine algebraic groups without non-trivial characters of dimension different than 3. Note also that connectedness is not a restriction since any connected component of an algebraic group $G$ is itself isomorphic (as a variety) to the connected component of the identity element.

Let us put Theorem 1.1 in context. Embedding problems are most classically considered for $Y = \mathbb{C}^n$; compare e.g. the overviews by Kraft and van den Essen [Kra96, vdE04]. We recall what is known about uniqueness of embeddings of $\mathbb{C}$ into $\mathbb{C}^n$. If $n = 2$, there is a unique embedding (up to automorphisms) by the Abhyankar-Moh-Suzuki Theorem [AM75, Suz74]. For $n \geq 4$, again there is a unique embedding (up to automorphisms) by the work of Jelonek; see [Jel87]. More generally, Kaliman [Kal91] and Srinivas [Sri91] proved that smooth varieties of dimension $d$ embed uniquely into $\mathbb{C}^n$ whenever $n \geq 2d + 2$. The existence of non-equivalent embeddings $\mathbb{C} \to \mathbb{C}^3$
is a long standing open problem; see [Kra96]. There are various potential examples of non-equivalent embeddings of \( \mathbb{C} \) into \( \mathbb{C}^3 \); see e.g. [Sha92].

Srinivas’ result is established by cleverly projecting to different linear coordinates. The second author was able to use projections to coordinates to establish that there is a unique embedding of \( \mathbb{C} \) into \( \text{SL}_n \) (up to automorphisms) for all integers \( n \geq 3 \); see [Sta15]. For algebraic groups in general, projections to coordinates are no longer available. Our approach to embeddings of \( \mathbb{C} \) is to study projections onto quotients by unipotent subgroups.

For a different point of view we consider the notion of flexible varieties as studied by Arzhantsev, Flenner, Kaliman, Kutzschebauch, and Zaidenberg in [AFK+13]. Flexible varieties can be seen as generalization of connected affine algebraic groups without non-trivial characters. Smooth irreducible affine flexible varieties of dimension at least 2 have the property that all embeddings of a fixed finite set are equivalent [AFK+13, Theorem 0.1]. Theorem 1.1 states that in most affine algebraic groups even all embeddings of \( \mathbb{C} \) are equivalent. In light of Theorem 1.1 the following question is natural in this context.

**Question 1.2.** Let \( Y \) be a smooth irreducible affine flexible variety of dimension at least 4. Is there at most one embedding of \( \mathbb{C} \) into \( Y \) up to automorphisms?

There exist smooth irreducible flexible affine surfaces that contain non-equivalent embeddings of \( \mathbb{C} \); see Example 2.1. Since in dimension three there is the long standing open problem, whether all embeddings of \( \mathbb{C} \) into \( \mathbb{C}^3 \) are equivalent, we ask Question 1.2 only for varieties of dimension \( \geq 4 \). In Example 2.2, we provide a contractible smooth affine irreducible surface \( S \) such that \( S \times \mathbb{C}^n \) contains non-equivalent embeddings of \( \mathbb{C} \) for all integers \( n \geq 1 \). These examples of varieties that contain non-equivalent embeddings of \( \mathbb{C} \) are the content of Section 2.

Note that some sort of ‘flexibility’ is required to prove results such as Theorem 1.1 in case one has ‘many’ embeddings of \( \mathbb{C} \). For example, if every pair of points in an affine variety \( Y \) can be connected by a chain of embedded affine lines and \( Y \) admits a non-trivial \( \mathbb{C}^+ \)-action, then flexibility of \( Y \) is a necessary condition for the equivalence of all embeddings \( \mathbb{C} \rightarrow Y \).

Theorem 1.1 can be seen as covering all cases of embeddings of \( \mathbb{C} \) into connected affine algebraic groups without non-trivial characters except the well-known open problem of embeddings into \( \mathbb{C}^3 \) and embeddings into \( \text{SL}_2 \) and \( \text{PSL}_2 \). As argued by the second author in [Sta15], \( \text{SL}_2 \) (and in fact similarly \( \text{PSL}_2 \)) allows for many embeddings of \( \mathbb{C} \) and perceiveably their equivalence or non-equivalence up to automorphism might be as challenging as for the \( \mathbb{C}^3 \) case. In Section 3, we report on these examples of embeddings into \( \mathbb{C}^3 \), \( \text{SL}_2 \) and \( \text{PSL}_2 \).

1.1. **Tools for the proof of Theorem 1.1.** In Section 4, notions and basic facts from the theory of algebraic groups and their principal bundles are introduced.

\(^2\)Compare the notion of \( \mathbb{A}^1 \)-chain connectedness in [AM11] and rationally chain connectedness in [Kol96].
In order to prove equivalence of embeddings we need a good way to construct automorphisms. This is the content of Section 5. Let us expand on that. While we are only interested in showing uniqueness of embeddings up to automorphisms of the underlying variety of an algebraic group, we will heavily depend on the group structure to construct automorphisms. The following shearing-tool follows readily by using the group structure; see Proposition 5.1. It is our main tool to construct automorphisms of the underlying variety of an algebraic group.

**Shearing-tool.** Let $X$ and $X'$ be affine lines embedded in an algebraic group $G$ and let $H \subseteq G$ be a closed subgroup such that $G/H$ is quasi-affine. If $\pi: G \rightarrow G/H$ restricts to an embedding on $X$ and $X'$ and if $\pi(X) = \pi(X')$, then there exists a $\pi$-fiber-preserving automorphism of $G$ mapping $X$ to $X'$.

This could be seen as an analog to a fact used in proving Srinivas' result about embeddings into $\mathbb{C}^n$: given two embeddings $\sigma, \sigma'$ of an affine line (or in fact any affine variety) into $\mathbb{C}^n$ such that the last $m < n$ coordinate functions agree and yield an embedding into $\mathbb{C}^m$, then there exists a shear $\phi$ of $\mathbb{C}^n$ with respect to the projection to the last $m$ coordinates such that $\phi \circ \sigma = \sigma'$; see [Sri91].

In Section 6 we show that all embeddings $\mathbb{C} \rightarrow G$ with image a one-dimensional unipotent subgroup of $G$ are equivalent. Thus in order to prove equivalence of all embeddings $\mathbb{C} \rightarrow G$, it suffices to show that every affine line in $G$ can be moved via an automorphism of $G$ into a one-dimensional unipotent subgroup.

In Section 7 we introduce another tool. In view of the above shearing-tool, given a curve $X$ in $G$, we are interested in having many closed subgroups $H$ such that $X$ projects isomorphically (or at least birationally) to $G/H$. We establish several results in that direction and we call them generic projection results. In this context our main result is the following; see Proposition 7.2. It is based on an elegant formula that relates the dimension of the conjugacy class $C$ of a unipotent element in a semisimple group with the dimension of the intersection of $C$ with a maximal unipotent subgroup; see [Ste76] and [Hum95, §6.7].

**Main generic projection result.** If $G$ is a simple algebraic group of rank at least two, and $H$ a closed unipotent subgroup, then for any curve $X \subseteq G$ that is isomorphic to $\mathbb{C}$ there exists an automorphism $\varphi$ of $G$ such that for generic $g \in G$ the quotient map $G \rightarrow G/gHg^{-1}$ restricts to an embedding on $\varphi(X)$.

1.2. **Outline of the proof of Theorem 1.1** In Section 8 we reduce Theorem 1.1 to the case of a semisimple group. In a bit more detail: let $G$ be an algebraic group satisfying the assumptions of Theorem 1.1. We note that $G$ is isomorphic (as a variety) to $G_u \times (\mathbb{C}^*)^k$ for some integer $k \geq 0$, where $G_u$ denotes the normal subgroup of $G$ generated by unipotent elements. Embeddings of $\mathbb{C}$ into $G_u \times (\mathbb{C}^*)^n$ are necessarily constant on the second factor; thus we study embeddings into $G_u$. We have that $G_u$ is
isomorphic as a variety to \( R_u(G)/R_u(G) \times R_u(G)/R_u(G) \), where \( R_u(G) \) denotes the unipotent radical—the largest normal unipotent subgroup of \( G \). If \( R_u(G) \) is non-trivial nor equal to \( G \), then the non-trivial product structure on \( G \) allows to show equivalence of all embedded affine lines; see Proposition 8.5.

If \( R_u(G) = G \), then \( G \cong \mathbb{C}^n \) for some \( n \neq 3 \), and the result follows by Jelonek’s work (for \( n \geq 4 \)) and by the Abhyankar-Moh-Suzuki Theorem (for \( n = 2 \)). This leaves the case where \( R_u(G) \) is trivial, i.e. \( G \) is semisimple.

In Section 9, we prove Theorem 1.1 in case of a semisimple, but not simple group \( G \). We use that \( G \) is isomorphic to a quotient of the product of at least two simple groups by a finite central subgroup. Part of the argument relies on the fact that simple groups have sufficiently many unipotent elements. To ensure this, the classification of simple groups of small rank is invoked; see Lemma B.3.

Finally, in Section 10, we prove Theorem 1.1 in the case of a simple group \( G \). This constitutes the technical heart of the proof. Besides using several results from previous sections about embeddings into products and generic projection results, we use the language of algebraic group theory to define an interesting subvariety \( E \) of \( G \). In fact, \( E \) is the preimage of the (unique) Schubert curve under the projection to \( G/P \), where \( P \) is a maximal parabolic subgroup of \( G \). We show that any embedding of the affine line in \( G \) can be moved into \( E \) by an automorphism of \( G \); compare Subsection 10.4. This is in fact the key step in our proof. Let us expand on this.

Let \( P^- \) be an opposite parabolic subgroup to \( P \) and denote by \( \pi : G \to G/R_u(P^-) \) the quotient map. We establish, that the restriction of \( \pi \) to \( E \) is a locally trivial \( \mathbb{C} \)-bundle over \( \pi(E) \) and \( \pi(E) \) is a big open subset of \( G/R_u(P^-) \), i.e. the complement is a closed subset of codimension at least two in \( G/R_u(P^-) \); see Proposition 10.2. Now, one can move \( X \) into \( E \) via the following steps.

- Using our main generic projection result, we can achieve that \( \pi \) restricts to an embedding on \( X \).
- Using that \( \pi(E) \) is a big open subset of \( G/R_u(P^-) \), we can move \( X \) into \( \pi^{-1}(\pi(E)) \) by left multiplication with a group element. In particular, \( \pi \) still restricts to an embedding on \( X \), by \( G \)-equivariancy.
- Since \( E \to \pi(E) \) is a locally trivial \( \mathbb{C} \)-bundle, it has a section \( X' \subseteq E \) over \( \pi(X) \cong \mathbb{C} \). Therefore, we can move \( X \) into \( X' \) with our shearing-tool.

Next we exploit that \( E = KP \) for a certain non-trivial closed subgroup \( K \) of \( G \) and the parabolic subgroup \( P \) used to define \( E \). Under the assumption that the rank of \( G \) is at least two, i.e. \( G \) is different from \( SL_2 \) and \( PSL_2 \), we show the following. Via an automorphism of \( G \) one can move any affine line in \( E \) to an affine line in \( E \) such that the quotient map \( E \to K \setminus E \) restricts to an embedding on this affine line; see Proposition 10.7. Using this result and the fact that the product map \( K \times P \to E \) is a principal \( K \cap P \)-bundle we can move any affine line in \( E \) into an affine line in \( P \). Since \( P \) is a proper subgroup of \( G \), one can move any affine line in \( P \) into a one-dimensional unipotent subgroup of \( G \). This implies Theorem 1.1 in this last case.
1.3. Overview of the appendices. We have three appendix sections, which contain results that are used in the proof of Theorem 1.1, but that are either classical or the proofs are independent of the general idea of the proof of Theorem 1.1. Appendix [A] provides a proof of the fact that principal $G$-bundles over the affine line are trivial for all affine algebraic groups $G$. In Appendix [B] we provide generalities on parabolic subgroups of reductive groups and the dimension of their subvariety of unipotent elements and their unipotent radical as needed in Section 11. In Appendix [C] we provide results about $\mathbb{C}^+$-equivariant morphisms of surfaces as needed in the proof of Proposition 10.7 (which constitutes the most technical part of Section 10).

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2. Examples of varieties that contain non-equivalent embeddings of $\mathbb{C}$

In the first example we provide an irreducible smooth affine flexible surface that contains non-equivalent embeddings of $\mathbb{C}$. This example is due to Decaup and Dubouloz. For a deeper study of this example see [DD16].

Example 2.1. Let $S = \mathbb{P}^2 \setminus Q$, where $Q$ is a smooth conic in $\mathbb{P}^2$. Clearly, $S$ is irreducible, smooth and affine. Let $(x : y : z)$ be a homogeneous coordinate system of $\mathbb{P}^2$. We can assume without loss of generality that $Q$ is given by the homogeneous equation $xz = y^2$ in $\mathbb{P}^2$.

Let $L_1$ be the curve $S \cap \{z = 0\}$ and let $L_2$ be the curve $S \cap \{xz - y^2 = z^2\}$. One can see that Pic($S \setminus L_1$) is trivial, whereas Pic($S \setminus L_2$) is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Hence there are non-equivalent embeddings of $\mathbb{C} \cong L_1 \cong L_2$ into $S$.

To establish the flexibility of $S$, we have to show that $\text{SAut}(S)$ acts transitively on $S$ where $\text{SAut}(S)$ denotes the subgroup of $\text{Aut}(S)$ that is generated by all automorphisms coming from $\mathbb{C}^+$-actions on $S$; see [APK+13, Theorem 0.1]. Consider the $\mathbb{C}^+$-action $t \cdot (x : y : z) = (x : y + tx : z + 2yt + t^2x)$ on $S$. A computation shows that every orbit of this $\mathbb{C}^+$-action intersects the curve $L_2$. Since $L_2$ is an orbit of the $\mathbb{C}^+$-action $t \cdot (x : y : z) = (x + 2yt + t^2z : y + tz : z)$ on $S$, it follows that $\text{SAut}(S)$ acts transitively on $S$.

Next, we give in any dimension $\geq 3$ an example of an irreducible smooth contractible affine variety that contains non-equivalent embeddings of $\mathbb{C}$. Note that for any irreducible smooth contractible affine variety, the ring of regular functions is a unique factorization domain and all invertible functions on it are constant; see e.g. [Kal94, Proposition 3.2].

Example 2.2. Let $S$ be an irreducible smooth contractible affine surface of logarithmic Kodaira dimension one that contains a copy $C$ of the affine line. For example, by [DP90, Theorem A] the affine hypersurface in $\mathbb{C}^3$ defined by

$$z^2x^3 + 3zx^2 + 3x - zy^2 - 2y = 1$$

is smooth, contractible and of logarithmic Kodaira dimension one, and $z = 0$ inside this hypersurface defines a copy of $\mathbb{C}$. Since $S$ is smooth, affine and
of logarithmic Kodaira dimension one, there exists no \( C^+ \)-action on \( S \), by [MS80, Lemma 1.3]. In other words, the Makar-Limanov invariant of \( S \) is equal to the ring of regular functions on \( S \). Now, by [Cra04, Corollary 5.20], it follows that the Makar-Limanov invariant of \( S \) is equal to the ring of regular functions on \( S \). Now, by [Cra04, Corollary 5.20], it follows that the Makar-Limanov invariant of \( S \) is equal to the ring of regular functions on \( S \). In particular, every automorphism of \( S \) maps fibers of the canonical projection \( \pi: S \times \mathbb{C}^n \to S \) to fibers of it. Thus any copy of \( \mathbb{C} \) inside \( S \times \mathbb{C}^n \) that lies in some fiber of \( \pi \) is non-equivalent to the section \( \mathbb{C} \times \{0\} \subseteq S \times \mathbb{C}^n \) of \( \pi \) over \( \mathbb{C} \). In summary, we proved that \( S \times \mathbb{C}^n \) is irreducible, affine, smooth, contractible and contains non-equivalent copies of \( \mathbb{C} \), provided that \( n \geq 1 \).

To compare Example 2.1 and Example 2.2, note that there exists no smooth irreducible affine surface that is contractible and contains two non-equivalent copies of \( \mathbb{C} \). Indeed, smooth homology planes of logarithmic Kodaira dimension one or two, contain at most one copy of \( \mathbb{C} \) and smooth homology planes of logarithmic Kodaira dimension zero do not exist; see e.g. [GM92]. If the logarithmic Kodaira dimension of a smooth, contractible affine surface is \( -\infty \), then it must be \( \mathbb{C}^2 \) by Miyanishi’s characterization of the affine plane; see [Miy75] and [Miy84]. Thus, the Abhyankar-Moh-Suzuki Theorem implies our claim.

3. Examples of embeddings of \( \mathbb{C} \) into \( \mathbb{C}^3 \), \( SL_2 \) and \( PSL_2 \)

In this section we discuss what is known about embeddings of \( \mathbb{C} \) into \( \mathbb{C}^3 \) and give embeddings of \( \mathbb{C} \) into \( SL_2 \) and \( PSL_2 \) arising from embeddings of \( \mathbb{C} \) into \( \mathbb{C}^3 \).

3.1. Embeddings into \( \mathbb{C}^3 \). After Abyankar and Moh and, independently, Suzuki established uniqueness of embeddings of \( \mathbb{C} \) into \( \mathbb{C}^3 \), many examples of embeddings of \( \mathbb{C} \) into \( \mathbb{C}^3 \) that are potentially different (up to automorphisms) from the standard embedding \( \mathbb{C} \to \mathbb{C}^3 \), \( t \mapsto (t, 0, 0) \) where suggested; many of these have since been proven to be standard; compare e.g. [vdE04]. However, examples due to Shastri, which are based on the idea of using embeddings with real coefficients such that the restriction map \( \mathbb{R} \to \mathbb{R}^3 \) is knotted, seem among the most promising to be non-standard. Concretely, the embeddings \( \mathbb{C} \to \mathbb{C}^3 \)

\[
\begin{align*}
t &\mapsto (t^3 - 3t, t^4 - 4t^2, t^5 - 10t) \quad \text{and} \quad t &\mapsto (t^3 - 3t, t(t^2 - 1)(t^2 - 4), t^7 - 42t),
\end{align*}
\]

which restrict to embeddings \( \mathbb{R} \to \mathbb{R}^3 \) of a trefoil knot and a figure eight knot, respectively, are not known to be standard; see [Sha92].

3.2. Comparison of embeddings into \( \mathbb{C}^3 \) and \( SL_2 \). Embeddings of \( \mathbb{C} \) into \( SL_2 \) are less studied. Following an example of the second author (compare [Sta15]), we briefly discuss how embeddings into \( \mathbb{C}^3 \) give rise to embeddings into \( SL_2 \). In fact, for any embedding \( h \) of \( \mathbb{C} \) into \( \mathbb{C}^3 \) there exists an automorphism \( \varphi \) of \( \mathbb{C}^3 \) such that

\[
(1) \quad t \mapsto \begin{pmatrix} f_1(t) & (f_1(t)f_3(t) - 1)/f_2(t) \\ f_2(t) & f_3(t) \end{pmatrix}
\]
defines an embedding of $C$ into $SL_2$ where $f_1$, $f_2$ and $f_3$ are the components of $f = \varphi \circ h$. In fact, it suffices to arrange that $f_2$ divides $f_1 f_3 - 1$ in $C[t]$, which is explicitly done in \cite{Sta15}.

On the other hand, if we start with an embedding $g$ of $C$ into $SL_2$, then there exists an automorphism $\psi$ of $SL_2$ such that $p \circ \psi \circ g$ is an embedding of $C$ into $C^3$ where $p$: $SL_2 \to C^3$ is the projection to three coordinate functions of $SL_2$; see \cite[Lemma 10]{Sta15}.

### 3.3. Comparison of embeddings into $SL_2$ and $PSL_2$.

In this subsection we construct a natural surjective map from the set of all embeddings of $C$ into $PSL_2$ to the set of all embeddings of $C$ into $SL_2$ where we consider the embeddings up to automorphisms. Thus, using Subsection 3.2, every embedding of $C$ into $C^3$ gives rise to an embedding of $C$ into $PSL_2$.

By Hurwitz’s Theorem, every finite étale morphism $E \to C$ is trivial in the sense that every connected component of $E$ maps isomorphically onto $C$; see e.g. \cite[Chp. IV, Corollary 2.4]{Har77}). In particular, every embedding of $C$ into $PSL_2$ lifts via the canonical quotient $\eta$: $SL_2 \to PSL_2$ to two embeddings into $SL_2$, which are the same up to the involution $X \mapsto -X$ of $SL_2$. Since every automorphism of $PSL_2$ lifts to an automorphism of $SL_2$ via $\eta$ (see \cite[Proposition 20]{Ser58}), we constructed a well-defined map

\[ \Xi: \{ \text{Embeddings of } C \text{ into } PSL_2 \text{ up automorphisms of } PSL_2 \} \to \{ \text{Embeddings of } C \text{ into } SL_2 \text{ up to automorphisms of } SL_2 \}. \]

We claim that $\Xi$ is surjective. For this, let $f$: $C \to SL_2$ be an embedding. It is enough to prove that there exists an automorphism $\varphi$ of $SL_2$ such that $\eta \circ \varphi \circ f$ is an embedding into $PSL_2$. Since $\eta \circ \varphi \circ f$ is always immersive and proper, we only have to prove injectivity of $\eta \circ \varphi \circ f$. Let $\pi_i$: $SL_2 \to C^2 \setminus \{0\}$ be the projection to the $i$-th column. We can assume, after composing $f$ with an automorphism of $SL_2$, that $\pi_1 \circ f$: $C \to C^2 \setminus \{0\}$ is immersive; see \cite[Lemma 10]{Sta15}. Let $C$ be the image of $\pi_1 \circ f$, which is closed in $C^2 \setminus \{0\}$. There is a commutative diagram

\[
\begin{array}{ccc}
SL_2 & \stackrel{\pi_1}{\longrightarrow} & C^2 \setminus \{0\} \\
\downarrow{\eta} & & \downarrow{\rho} \\
PSL_2 & \longrightarrow & V
\end{array}
\]

where $\rho$: $C^2 \setminus \{0\} \to V$ denotes the quotient by the $\mathbb{Z}/2\mathbb{Z}$-action $(x, z) \mapsto (-x, -z)$ on $C^2 \setminus \{0\}$. Let $Z = \rho(C)$. Since the morphism $\rho$ is étale, it follows that $\rho \circ \pi_1 \circ f$: $C \to Z$ is immersive and hence birational. Let $Z_0 \subseteq Z$ be a finite subset such that $\rho \circ \pi_1 \circ f$ restricts to an isomorphism $C \setminus (\rho \circ \pi_1 \circ f)^{-1}(Z_0) \cong Z \setminus Z_0$. Let $T$ be the finite set $(\rho \circ \pi_1 \circ f)^{-1}(Z_0)$.

There exists a morphism $p$: $C^2 \to C$ such that for all $t \neq s$ in $T$ we have

\[ (\pi_2 \circ f)(t) + p((\pi_1 \circ f)(t)) \cdot (\pi_1 \circ f)(t) \neq \pm ([\pi_2 \circ f](s) + p((\pi_1 \circ f)(s)) \cdot (\pi_1 \circ f)(s)] \cdot . \]

Indeed, such a $p$ exists, since for all $t \neq s$ in $T$ the negation of condition (2) defines two non-trivial affine linear equations for $p$ in the vector space of...
functions $\mathbb{C}^2 \to \mathbb{C}$. Let $\varphi : \text{SL}_2 \to \text{SL}_2$ be the automorphism given by

$$
\varphi \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & y + xp(x, z) \\ z & w +zp(x, z) \end{pmatrix}.
$$

Let $g = \varphi \circ f$. Note that $\pi_1 \circ g = \pi_1 \circ f$ and $\pi_2 \circ g = \pi_2 \circ f + (\rho \circ \pi_1 \circ f) \cdot (\pi_1 \circ f)$. Since $\rho \circ \pi_1 \circ f$ restricts to an isomorphism $\mathbb{C} \setminus T \cong \mathbb{C} \setminus \{0\}$, it follows that $\eta \circ g$ restricted to $\mathbb{C} \setminus T$ is injective. By (2), we have $(\eta \circ g)(t) \neq (\eta \circ g)(s)$ for all $t \neq s$ in $T$ and thus $\eta \circ g$ restricted to $T$ is injective. Since the images under $\eta \circ g$ of $\mathbb{C} \setminus T$ and $T$ are disjoint, it follows that $\eta \circ g$ is injective, which implies our claim.

4. Notation and generalities on algebraic groups and their principal bundles

4.1. Algebraic groups. For the basic results on algebraic groups we refer to [Hum75] and for the basic results about Lie algebras and root systems we refer to [Hum78]. In order to set up conventions, let us recall the basic terms. A connected non-trivial algebraic group $G$ is called semisimple if it has a trivial radical $R(G)$, where $R(G)$ is the largest connected normal solvable subgroup of $G$. An algebraic group $G$ is called reductive if it has a trivial unipotent radical $R_u(G)$, where $R_u(G)$ is the closed normal subgroup of $R(G)$ consisting of all unipotent elements. A non-commutative connected algebraic group $G$ is called simple, if it contains no non-trivial closed connected normal subgroup. Note that for a simple algebraic group $G$, the quotient $G/Z(G)$ by the center $Z(G)$ is simple as an abstract group (see [Hum75, Corollary 29.5]), i.e. it contains no proper normal subgroup.

For any connected algebraic group $G$, we denote by $U_G$ the subset of unipotent elements in $G$. It is irreducible and closed in $G$; see [Hum75, Theorem 7.5]. For any reductive group $G$, we have by [Hum95, §4.2] we have for any reductive group $G$

$$
\dim U_G = \dim G - \text{rank } G
$$

and using the Levi decomposition (see [OV90, Theorem 4, Chp. 6]) this formula holds more generally for every connected algebraic group $G$. Moreover, we denote by $G^u$ the normal subgroup which is generated by all unipotent elements of $G$. It is connected and closed in $G$; see [Hum75, Proposition 7.5]. For any semisimple $G$, we have $G = G^u$; see [Hum75, Theorem 27.5].

We use $\mathfrak{g}$ to denote the Lie algebra of an algebraic group $G$. Moreover, we denote by $\mathcal{N}_G$ the closed irreducible cone of nilpotent elements inside $\mathfrak{g}$. Note that the exponential $\exp : \mathfrak{g} \to G$ restricts to an isomorphism of affine varieties $\exp : \mathcal{N}_G \to U_G$.

4.2. Principal bundles. Our general reference for principal bundles is [Ser58]. Again, in order to set up conventions, let us recall the basic terms. Let $G$ be any algebraic group. A principal $G$-bundle is a variety $P$ with a right $G$-action together with a $G$-invariant morphism $\pi : P \to X$ such that locally on $X$, $\pi$ becomes a trivial principal $G$-bundle after a finite étale base change. If one can choose these étale base changes to be open injective immersions, then we say $\pi$ is a locally trivial principal $G$-bundle.
The most prominent example of a principal bundle in this article is the following: let $G$ be an algebraic group and let $H$ be a closed subgroup. Then $G \rightarrow G/H$ is a principal $H$-bundle; see [Ser58, Proposition 3]. If $H$ is a group without characters, then the quotient $G/H$ is quasi-affine (see [Tim11, Example 3.10]) and if $H$ is normal in $G$ or reductive, then $G/H$ is affine (see [Tim11, Theorem 3.8]). For any algebraic group $G$, any principal $G$-bundle over $\mathbb{C}$ is trivial; see Appendix A.

5. Construction of automorphisms of an algebraic group

In this section we introduce a construction of automorphisms of algebraic groups that we use throughout this article.

Let $G$ be an algebraic group. Let $H \subseteq G$ be a closed subgroup and let $\pi: G \rightarrow G/H$ be the quotient by left $H$-cosets. For any morphism $f: G/H \rightarrow H$, the map

$$\varphi_f: G \rightarrow G, \quad g \mapsto g f(\pi(g))$$

is an automorphism of $G$ that preserves the quotient $\pi$. Let $\rho: G \rightarrow H \setminus G$ be the quotient by right $H$-cosets. Analogously to $\varphi_f$, we define for any morphism $d: H \setminus G \rightarrow H$ the automorphism

$$\psi_d: G \rightarrow G, \quad g \mapsto d(\rho(g))g$$

We will frequently use this construction in the following special situation. Assume that $H$ is a closed unipotent subgroup, whence $G/H$ is quasi-affine. Let $X \subseteq G$ be a closed curve that has only one smooth point at infinity, i.e. there exists a projective curve $\tilde{X}$ that contains $X$ as an open subset and $\tilde{X} \setminus X$ consists only of one point that is a smooth point of $\tilde{X}$. Assume that the quotient $\pi$ restricts to an embedding on $X$. Thus $X$ is a section of the principal $H$-bundle $\pi^{-1}(\pi(X)) \rightarrow \pi(X)$. Let $X'$ be another section of $\pi^{-1}(\pi(X)) \rightarrow \pi(X)$ and denote by $s: \pi(X) \rightarrow X$ and $s': \pi(X) \rightarrow X'$ the inverse maps of $\pi|_X: X \rightarrow \pi(X)$ and $\pi|_{X'}: X' \rightarrow \pi(X)$, respectively. Consider the morphism

$$(3) \quad \pi(X) \rightarrow H, \quad v \mapsto (s(v))^{-1} \cdot s'(v).$$

Since $G/H$ is quasi-affine and since $\pi(X)$ has only one smooth point at infinity, the curve $\pi(X)$ is closed in any affine variety that contains $G/H$ as an open subvariety. Since $H$ is unipotent and thus an affine space, (3) can be extended to a morphism $f: G/H \rightarrow H$. Clearly, the automorphism $\varphi_f$ satisfies $\varphi_f(X) = X'$. Roughly speaking, $\varphi_f$ moves $X$ into $X'$ along the fibers of $\pi$.

If $X$ happens to be the affine line, then it is enough to assume that $G/H$ is quasi-affine in order to move $X$ into another section along the fibers of $\pi$. Indeed, since $G/H$ is quasi-affine, there exists a retraction of $G/H$ to $\pi(X) \cong \mathbb{C}$ and therefore the morphism in (3) can be extended to a morphism $f: G/H \rightarrow H$. In summary we proved the following result and its analog for right coset spaces.

**Proposition 5.1.** Let $G$ be an algebraic group and let $H$ be a closed subgroup such that $G/H$ is quasi-affine. If $X$ is a closed curve in $G$ that is isomorphic to $\mathbb{C}$ such that $\pi: G \rightarrow G/H$ restricts to an embedding on $X$ and if $X'$ is
another section of $\pi^{-1}(\pi(X)) \to \pi(X)$, then there exists an automorphism $\varphi$ of $G$ that preserves $\pi$ and maps $X$ onto $X'$.

6. Embeddings of $\mathbb{C}$ with unipotent image

The following result says that two embeddings $f_1$ and $f_2$ of $\mathbb{C}$ into an algebraic group $G$ are the same up to an automorphism of $G$, provided that $f_1(\mathbb{C})$ and $f_2(\mathbb{C})$ are unipotent subgroups of $G$.

**Proposition 6.1.** Let $G$ be any algebraic group and let $U, V$ be unipotent one-dimensional subgroups. For any isomorphism of varieties $\sigma: U \to V$, there exists an algebraic automorphism $\varphi$ of $G$ such that $\varphi|_U = \sigma$.

**Proof.** If $G^u$ is one-dimensional, then $G^u = R^u(G)$ and $G$ is isomorphic to $G^u \times G/G^u$ as a variety; see Remark A.4. In particular, $U = V = G^u$ and every automorphism of $U$ extends to $G$. Thus, we can assume that $G^u$ is at least two-dimensional and hence we can assume that $V \neq U$. This implies $V \cap U = \{e\}$ and therefore multiplication $V \times U \to VU \subseteq G$ is an embedding. Hence, the quotient map $\pi: G \to G/U$ restricts to an embedding on $V$. Since $G/U$ is quasi-affine, the morphism

$$\pi(V) \xrightarrow{(\pi|_V)^{-1}} V \xrightarrow{\sigma^{-1}} U$$

extends to a morphism $f: G/U \to U$. Hence the automorphism $\varphi_f$ of $G$ (see Section 5) satisfies $\varphi_f(v) = v \cdot \sigma^{-1}(v)$ for all $v \in V$. Using the quotient $\rho: G \to V \setminus G$ one can similarly construct an automorphism $\psi_d$ of $G$ such that $\psi_d(u) = \sigma(u) \cdot u$ for all $u \in U$. It follows that $\varphi = \varphi_f^{-1} \circ \psi_d$ restricts to $\sigma$ on $U$. □

7. Generic projection results

The aim of this section is to prove results, which enable us to quotient by unipotent subgroups such that the projection restricts to a closed embedding or to a birational map on a given fixed curve. These projection results will be applied in Sections 8 and Section 9 to reduced Theorem 1.1 to semisimple groups and simple groups, respectively. In Section 10 we use these results in the heart of the proof of Theorem 1.1 namely for the case of embeddings into simple groups.

Let $V$ be a variety. Throughout this paper we say that a property is satisfied for generic $v \in V$ if there exists a dense open subset $O$ in $V$ such that the property is satisfied for all $v$ in $O$.

7.1. Quotients that restrict to closed embeddings on a fixed curve.

Our first result in this section deals with arbitrary algebraic groups and quotients by one-dimensional unipotent subgroups.

**Lemma 7.1 (Communicated by Winkelmann).** Let $G$ be an algebraic group and let $X \subseteq G$ be a closed curve that has only one smooth point at infinity. If the set of unipotent elements $U_G$ has dimension at least four, then, for a generic one-dimensional unipotent subgroup $U \subseteq G$, the quotient $G \to G/U$ restricts to a closed embedding on $X$. 


Remark 7.2. Using the exponential map \( \exp : \mathcal{N}_g \to \mathcal{U}_G \), we consider the whole of one-dimensional unipotent subgroups of \( G \) as the image of \( \mathcal{N}_g \setminus \{0\} \) under the quotient \( \mathfrak{g} \setminus \{0\} \to \mathbb{P}(\mathfrak{g}) \). Note that this image is closed in \( \mathbb{P}(\mathfrak{g}) \) and therefore we can speak of a “generic one-dimensional unipotent subgroup”.

Proof. As already mentioned, the exponential restricts to an isomorphism of affine varieties \( \exp : \mathcal{N}_g \to \mathcal{U}_G \). We denote by \( F \) the set of all elements in \( G \) of the form \( y^{-1}x \) with \( x, y \in X \) and \( x \neq y \). Let \( F' = \exp(\text{cone}(\exp^{-1}(F \cap \mathcal{U}_G))) \subseteq \mathcal{U}_G \); where cone(\( M \)) denotes the union of all lines in \( \mathcal{N}_g \) that pass through the origin and intersect \( M \), for any subset \( M \) of \( \mathcal{N}_g \). Let \( U \subseteq G \) be a one-dimensional unipotent subgroup. Thus \( G \to G/U \) maps \( X \) injectively onto its image if and only if \( U \cap F' = \{e\} \). However, \( F' \) is a constructible subset of \( \mathcal{U}_G \) of dimension at most three.

Let \( S \subseteq \mathfrak{g} \) be the union of all lines \( Dl_{x^{-1}}(T_xX) \), \( x \in X \), where \( l_g : G \to G \) denotes left multiplication by \( g \in G \). Let \( U \subseteq G \) be a one-dimensional unipotent subgroup. Thus \( G \to G/U \) maps \( X \) immersively onto its image if and only if \( u \cap S \cap \mathcal{N}_g = \{0\} \) where \( u \) denotes the Lie algebra of \( U \). Clearly, \( S \cap \mathcal{N}_g \) is a constructible subset of \( \mathcal{N}_g \) of dimension at most two.

Since \( G/U \) is quasi-affine, the quotient \( G \to G/U \) maps \( X \) properly onto its image, as long as the image is not a single point, since \( X \) has only one smooth point at infinity.

In summary, we proved that the restriction of \( G \to G/U \) to \( X \) is injective, immersive and proper for a generic one-dimensional unipotent subgroup \( U \) in \( G \). □

Remark 7.3. The proof of Lemma 7.1 shows that we can replace \( \mathcal{U}_G \) by some closed subset \( W \) of \( \mathcal{U}_G \) that is a union of unipotent subgroups and has dimension at least four in order to prove that for a generic one-dimensional unipotent subgroup \( U \) in \( W \) the quotient \( G \to G/U \) restricts to a closed embedding on \( X \).

Our second result deals with simple algebraic groups and quotients by arbitrary unipotent subgroups.

Proposition 7.4. Let \( G \) be a simple algebraic group of rank at least two and let \( U \subseteq G \) be a unipotent subgroup. If \( X \subseteq G \) is a closed smooth curve with only one smooth point at infinity, then there exists an automorphism \( \varphi \) of \( G \) such that for generic \( g \in G \) the projection \( G \to G/gUg^{-1} \) restricts to a closed embedding on \( \varphi(X) \).

In order to prove this result, we have to show that for generic \( g \in G \) the projection \( G \to G/gUg^{-1} \) restricts to an injective and immersive map on \( \varphi(X) \) for a suitable automorphism \( \varphi \). If this is the case, then this restriction is automatically proper, since \( X \) has only one smooth point at infinity.

Lemma 7.5 (Immersivity). Let \( G \) be a connected reductive algebraic group, \( U \subseteq G \) a closed unipotent subgroup. If \( X \subseteq G \) is a closed irreducible smooth curve such that \( e \in X \) and \( T_eX \) contains non-nilpotent elements of the Lie algebra \( \mathfrak{g} \), then for generic \( g \in G \) the projection \( \pi_g : G \to G/gUg^{-1} \) restricts to an immersion on \( X \).
Proof. Denote by \( u \) the Lie algebra of \( U \). The kernel of the differential of \( \pi_g \) in \( e \in G \) is the sub Lie algebra \( \text{Ad}(g)u \) of \( g \), where \( \text{Ad}(g) \) denotes the linear isomorphism of \( g \) induced by the differential in \( e \) of the automorphism of \( G \) that is given by \( h \mapsto ghg^{-1} \). Consider the morphism

\[
G \times (u \setminus \{0\}) \to \mathbb{P}(g), \quad (g, v) \mapsto [\text{Ad}(g)v],
\]

where \([w]\) denotes the line through \( 0 \neq w \in g \). Since \( G \) is not unipotent, the set of non-nilpotent elements is a dense open subset of \( g \) which maps via the projection \( g \setminus \{0\} \to \mathbb{P}(g) \) to a dense open subset \( O \). Since \( \text{Ad}(g)v \) is nilpotent for all \( v \in u \), the open set \( O \) lies in the complement of the image of the morphism in \((1)\). Let

\[
S = \bigcup_{x \in X} \mathbb{P}(T_e(x^{-1}X)) \subseteq \mathbb{P}(g),
\]

which is a locally closed irreducible curve in \( \mathbb{P}(g) \). Hence, \( \pi_g \) is immersive for \( g \in G \) if and only if \( S \cap \mathbb{P}(\text{Ad}(g)u) \) is empty. By assumption \( S \cap O \) is non-empty and thus there exists a finite subset \( F \) of \( S \) such that \( S \setminus F \subseteq O \), since \( S \) is irreducible. Thus \( (S \setminus F) \cap \mathbb{P}(\text{Ad}(g)u) \) is empty for all \( g \in G \). We claim that

\[
\bigcap_{g \in G} \text{Ad}(g)u = \{0\}.
\]

Using the isomorphism \( \exp : N \to U \), \((5)\) is equivalent to the intersection

\[
\bigcap_{g \in G} gUg^{-1}
\]

being trivial. Let \( v \) be in the intersection in \((6)\) and let \( N \) be the smallest closed subgroup of \( G \) that contains all conjugates \( gvg^{-1} \) of \( v \). Clearly, \( N \subseteq U \). By \[Hum75\], Proposition 7.5, \( N \) is connected and normal in \( G \). Since the unipotent radical of \( G \) is trivial, \( N \) is trivial. Thus, \( v = e \), which proves our claim. As a consequence of \((5)\), the intersection \( F \cap \mathbb{P}(\text{Ad}(g)u) \) is empty for generic \( g \in G \). This proves the lemma.

Theorem 7.6 (Injectivity). Let \( G \) be a simple algebraic group of rank \( \geq 2 \) and let \( U \subseteq G \) be a unipotent subgroup. If \( X \subseteq G \) is a closed irreducible curve such that \( e \in X \) and \( X \) contains non-unipotent elements, then for generic \( g \in G \), the projection \( \pi_g : G \to G/gUg^{-1} \) restricts to an injection on \( X \).

Proof. The strategy of the proof resembles the strategy of the proof of Lemma 7.5. Consider the morphism

\[
G \times U \to G, \quad (g, u) \mapsto gug^{-1}.
\]

Since \( G \) is not unipotent, \( G \setminus U \) is dense and open in \( G \), and it is contained in the complement of the image of the above morphism. Let us denote this open subset by \( O \). Let

\[
S = \{ x^{-1} y \in G \mid x \neq y \in X \}.
\]

Hence, \( \pi_g \) is injective if and only if \( S \cap gUg^{-1} \) is empty. By assumption \( S \cap O \) is non-empty and thus there exists a curve (or finite set) \( C \subseteq S \) consisting of unipotent elements such that \( S \setminus C \subseteq O \) since \( S \) is irreducible. Hence, \( (S \setminus C) \cap gUg^{-1} \) is empty for all \( g \in G \). Therefore it is enough to show that
\[ C \cap gUg^{-1} \] is empty for generic \( g \in G \). This can be achieved by showing that for all \( e \neq v \in U_G \) the set
\[ F_e = \{ g \in G \mid v \in gUg^{-1} \} \]
has codimension \( \geq 2 \) in \( G \). Indeed, if \( \text{codim}_G(F_e) \geq 2 \) for all \( v \neq e \), then the dimension of
\[ F = \{(v, g) \in C \times G \mid g \in F_v \} \]
is less than the dimension of \( G \). Hence, \( F \) maps to a subset of codimension \( \geq 1 \) in \( G \) via the natural projection \( C \times G \to G \), which then implies that \( C \cap gUg^{-1} \) is empty for generic \( g \in G \).

So let us prove that \( \text{codim}_G F_v \geq 2 \). Denote by \( \text{Cl}_G(v) \) the conjugacy class of \( v \) in \( G \). By using the orbit map \( G \to \text{Cl}_G(v) \), \( g \mapsto g^{-1}vg \) one can see that \( \text{codim}_G F_v \) is the same as the codimension of \( U \cap \text{Cl}_G(v) \) in \( \text{Cl}_G(v) \).

Since \( G \) is semisimple, by \cite{Hum95} Proposition 6.7] we have
\[
\dim U \cap \text{Cl}_G(v) \leq \frac{1}{2} \dim \text{Cl}_G(v).
\]
Hence, it remains to show that \( \text{Cl}_G(v) \) has dimension \( \geq 3 \), since the dimension of \( \text{Cl}_G(v) \) is even by \cite{Hum95} Proposition 6.7. This is in fact equivalent to the statement that the centralizer \( C_G(v) \) having codimension \( \geq 3 \) in \( G \). The latter is true by the following argument. The unipotent radical \( R_u(C_G(v)) \) is not trivial since the one-dimensional unipotent group which contains \( v \neq e \) is normal in \( C_G(v) \). Clearly, \( C_G(v) \) lies inside the normalizer \( N_G(R_u(C_G(v))) \). However, this normalizer is contained in some parabolic subgroup \( P \) that itself is the normalizer of some non-trivial unipotent subgroup of \( G \); see \cite{Hum75} Corollary 30.3A. Since \( G \) is reductive, this implies that \( P \) is a proper subgroup of \( G \). Since \( G/C_G(v) \to G, g \mapsto g^{-1}vg \) is injective, \( G \) is an affine variety, and \( G/P \) is projective and of positive dimension, it follows that \( C_G(v) \) must be a proper subgroup of \( P \). Since \( P \) is connected, we have \( \dim C_G(v) < \dim P \). Since \( G \) is simple and since the rank of \( G \) is at least two, it follows from Lemma \[ \ref{lem:rank2} \] that \( \dim R_u(P^-) \geq 2 \). Here \( P^- \) is the opposite parabolic subgroup to \( P \) with respect to some maximal torus that is contained in some Borel subgroup which in turn is contained in \( P \); see Appendix \[ \ref{app:torus} \]. This implies that the codimension of \( P \) in \( G \) is at least 2 by Lemma \[ \ref{lem:codim} \]. This in turn implies that \( C_G(v) \) has codimension \( \geq 3 \) in \( G \), which proves the lemma.

\begin{proof}[Proof of Proposition \[ \ref{prop:extension} \]] Since \( G \) is simple, it is a so called flexible variety; see \cite{AFK+13} \[ \S 0 \]. Hence, there exists an automorphism \( \varphi \) of \( G \) such that \( \varphi(X) \) contains non-unipotent elements, \( e \in \varphi(X) \) and the tangent space \( T_eX \) contains non-nilpotent elements of the Lie algebra \( g \); see \cite{AFK+13} Theorem 4.14, Remark 4.16 and Theorem 0.1. By Lemma \[ \ref{lem:extension} \] and Lemma \[ \ref{lem:immersive} \] for generic \( g \in G \) the projection \( \pi_g: G \to G/gUg^{-1} \) restricted to \( \varphi(X) \) is immersive and injective. As already mentioned, if this is the case, then \( \pi_{g|\varphi(X)} \) is proper. This finishes the proof.
\end{proof}

\subsection*{7.2. Quotients that restrict to birational maps on a fixed curve.}
Let us introduce the following notation. If \( G \) is an algebraic group, then for any \( u \in U_G \setminus \{ e \} \) we denote by \( C^+(u) \) the one-dimensional unipotent subgroup of \( G \) that contains \( u \). Roughly speaking the next lemma says:
Under certain assumptions, a curve $C$ in an affine homogeneous $G$-variety $Y$ projects birationally onto its image if we quotient $Y$ by $\mathbb{C}^+(u)$ where $u$ belongs to a dense subset of $\mathcal{U}_G$.

**Lemma 7.7.** Let $Y$ be an affine homogeneous $G$-variety where $G$ is a connected algebraic group acting from the right. We assume that generic elements in $\mathcal{U}_G$ act without fixed point on $Y$. Moreover, we assume that for all $y$ in $Y$, every fiber of the morphism

$$\rho_y : \mathcal{U}_G \to Y, \quad u \mapsto yu$$

has codimension at least three in $\mathcal{U}_G$. If $C \subseteq Y$ is a closed curve, then there exists a dense subset in $\mathcal{U}_G$ consisting of elements $u$ such that $\mathbb{C}^+(u)$ acts without fixed point on $Y$ and the algebraic quotient $S_u \to S_u // \mathbb{C}^+(u)$ restricts to a birational morphism on $C$, where $S_u$ denotes the smallest closed affine surface in $Y$ that contains all $\mathbb{C}^+(u)$-orbits passing through $C$.

**Remark 7.8.** The algebraic quotient $S_u // \mathbb{C}^+(u)$ is the spectrum of the ring of functions on $S_u$ that are invariant under the action of $\mathbb{C}^+(u)$. In fact, $S_u // \mathbb{C}^+(u)$ is an irreducible affine curve, see [Mat86, Theorem 11.7] and [OY82, Corollary 1.2, Theorem 3.2].

**Proof of Lemma.** Let $c_0 \in C$ and let $K_{c_0}$ be the union of the orbits $c_0 \mathbb{C}^+(u)$, $u \in \mathcal{U}_G \setminus \{e\}$ where $c_0 \mathbb{C}^+(u)$ is either equal to $\{c_0\}$ or it contains points of $C$ different from $c_0$. In other words,

$$K_{c_0} = \bigcup_{e \neq u \in \mathcal{U}_G \text{ such that } c_0 u \in C} c_0 \mathbb{C}^+(u).$$

With the aid of the exponential map $\exp : \mathcal{N}_e \to \mathcal{U}_G$ we define

$$\mathcal{N}_{c_0} = \bigcup_{e \neq u \in \rho_{c_0}^{-1}(C)} \mathbb{C}^+(u) = \exp(\text{cone}(\exp^{-1}(\rho_{c_0}^{-1}(C)))) \subseteq \mathcal{U}_G.$$

One can see that $\mathcal{N}_{c_0} = \rho_{c_0}^{-1}(K_{c_0})$. In particular, we have for $u \in \mathcal{U}_G \setminus \mathcal{N}_{c_0}$ that $c_0 \mathbb{C}^+(u)$ intersects $C$ only in the point $c_0$. Since all the fibers of $\rho_{c_0}$ have codimension at least three in $\mathcal{U}_G$ and since $\dim C = 1$, it follows that $\dim \rho_{c_0}^{-1}(C) \leq \dim \mathcal{U}_G - 2$. By the construction of $\mathcal{N}_{c_0}$ we get now

$$\dim \mathcal{N}_{c_0} \leq \dim \mathcal{U}_G - 1.$$

Take a countably infinite subset $C_0 \subseteq C$. Since our ground field is uncountable, the intersection $\bigcap_{c_0 \in C_0} \mathcal{U}_G \setminus \mathcal{N}_{c_0}$ is dense in $\mathcal{U}_G$. Let $u \in \mathcal{U}_G$ be an element that acts without fixed point on $Y$ and such that $u \notin \bigcup_{c_0 \in C_0} \mathcal{N}_{c_0}$. Since a fiber of $S_u \to S_u // \mathbb{C}^+(u)$ over a generic point of $S_u // \mathbb{C}^+(u)$ is a $\mathbb{C}^+(u)$-orbit, it follows that infinitely many fibers of $C \to S_u \to S_u // \mathbb{C}^+(u)$ consist only of one point. Thus, $C$ is mapped birationally onto the algebraic quotient. □

**Remark 7.9.** The proof of the Lemma shows the following: If there exist infinitely many $c_0$ in $C$ such that $\rho_{c_0}^{-1}(C) \leq \dim \mathcal{U}_G - 2$, then the statement of the lemma holds. In particular, the statement of the lemma holds, if there are infinitely many $c_0 \in C$ such that all fibers of $\rho_{c_0} : \mathcal{U}_G \to Y$ have codimension at least two in $\mathcal{U}_G$ and $c_0 \mathcal{U}_G \cap C$ is finite.
Corollary 7.10. Let $G$ be a connected algebraic group such that $\dim G \geq 3$, $\dim \mathcal{U}_G \geq 2$ and $G = G^u$. If $C \subseteq G$ is a closed irreducible curve, then there exists an automorphism $\varphi$ of $G$ and a dense subset of $\mathcal{U}_G$ consisting of elements $u$ such that $G \to G/\mathbb{C}^+(u)$ maps $\varphi(C)$ birationally onto its image.

Proof. If $G$ is a unipotent group, the statement is clear, since $\dim G \geq 3$. Thus we can assume that $\mathcal{U}_G$ is a proper subset of $G$. Since $G = G^u$, the variety $G$ is flexible. Fix some point $c_0$ in $C$. By [AFK+13 Theorem 0.1] there exists an automorphism $\varphi$ of $G$ that fixes $c_0$ and the image $\varphi(C)$ intersects $c_0\mathcal{U}_G$ only in finitely many points. Thus we can assume that $c_0\mathcal{U}_G \cap C$ is finite. The fiber over $c \in C$ of the morphism

\[\{ (c, u) \in C \times \mathcal{U}_G \mid cu \in C \} \to C, \quad (c, u) \mapsto c\]

is isomorphic to $c\mathcal{U}_G \cap C$. Since $C$ is irreducible, the subset of $C$ given by

\[C' := \{ c \in C \mid C \subseteq c\mathcal{U}_G \}\]

consists of exactly those points for which the fiber of (7) is not finite. Note that $C'$ is closed in $C$. Since $c_0\mathcal{U}_G \cap C$ is finite, $C'$ is a proper subset of $C$. Since $C$ is irreducible, it follows now that the generic fiber of (7) is finite, i.e. $c\mathcal{U}_G \cap C$ is finite for generic $c$ in $C$. Since $\dim \mathcal{U}_G \geq 2$, it follows that for all $c \in C$ the fibers of the map $\rho_c: \mathcal{U}_G \to G$, $\rho_c(u) = cu$ have codimension at least two in $\mathcal{U}_G$. The corollary follows from Remark 7.9 applied to the homogeneous $G$-variety $Y = G$.  

\[\square\]

8. Reduction to semisimple groups

In this section we reduce the proof of Theorem 1.1 to semisimple groups.

Lemma 8.1. Let $G$ be a connected algebraic group with $G = G^u$ and let $X$ be an affine variety that admits no non-constant invertible function $X \to \mathbb{C}^*$. Moreover, let $n$ be a non-negative integer. Then, all closed embeddings of $X$ into $G \times (\mathbb{C}^*)^n$ are equivalent if and only if all closed embeddings of $X$ into $G$ are equivalent.

Proof. Let $f_i: X \to G \times (\mathbb{C}^*)^n$, $i = 1, 2$ be two closed embeddings. By assumption, $f_i(X)$ lies in some fiber of $\pi: G \times (\mathbb{C}^*)^n \to (\mathbb{C}^*)^n$ for $i = 1, 2$. After multiplying with a suitable element of $G \times (\mathbb{C}^*)^n$ we can assume that $f_1(X)$ and $f_2(X)$ lie in the same fiber of $\pi$. Since any automorphism of one fiber can be extended to $G \times (\mathbb{C}^*)^n$, this proves the if-part of the proposition.

The other direction works much the same way by using the fact, that every automorphism of $G \times (\mathbb{C}^*)^n$ permutes the fibers of $\pi$, since $G = G^u$ and thus there are no non-constant invertible functions $G \to \mathbb{C}^*$; see [Ros61 Theorem 3].  

[Rem 8.2. Let $G$ be a connected algebraic group. Then $G$ is isomorphic as a variety to $G^u \times (\mathbb{C}^*)^n$ for a certain non-negative integer $n$.

Proof. Note that $G/G^u$ is a torus, since it is connected and contains only semisimple elements; see [Hum78 Proposition 21.4B and Theorem 19.3]. Let $T$ be a maximal torus of $G$. Since $G^u$ is normal in $G$, and since $G^u$ and $T$ generate $G$ (see [Hum78 Theorem 27.3]) we have $TG^u = G$. In particular,
To is mapped surjectively onto the torus \( G/G^u \) via the canonical projection \( \pi: G \to G/G^u \). Thus we get a short exact sequence

\[
1 \longrightarrow G^u \cap T \longrightarrow T \xrightarrow{\pi|_T} G/G^u \longrightarrow 1.
\]

By Lemma 8.3, \( G^u \cap T \) is a torus. Thus the above short exact sequence splits; see [Hum75, §16.2]. In particular, the associated section yields a trivialization of \( \pi: G \to G/G^u \) as a principal \( G^u \)-bundle, which proves the lemma. □

Lemma 8.3. Let \( G \) be any connected algebraic group and let \( H \) be a closed connected normal subgroup of \( G \). If \( T \) is a maximal torus of \( G \), then \( T \cap H \) is a maximal torus of \( H \).

Proof. Let \( T' \subseteq H \) be a maximal torus that contains the connected component of the identity element \((T \cap H)^o\) which is also a torus. Since all maximal tori in \( G \) are conjugate, there exists \( g \in G \) such that \( g^{-1}T'g \subseteq T \).

By the normality of \( H \) we get \( g^{-1}T'g \subseteq (T \cap H)^o \). Hence

\[
g^{-1}(T \cap H)^o g \subseteq g^{-1}T'g \subseteq (T \cap H)^o.
\]

Thus \( (T \cap H)^o = g^{-1}T'g \) is a maximal torus of \( H \) (note that all maximal tori of \( H \) are conjugate, since \( H \) is connected). Now, if there exists \( x \in T \cap H \setminus (T \cap H)^o \), then clearly \( x \) is semisimple and centralizes the torus \((T \cap H)^o\). However, this implies that \( \{x\} \cup (T \cap H)^o \) lies in a torus of \( H \), since \( H \) is connected (see [Hum75, Corollary B, §22.3]). This contradicts the maximality of \((T \cap H)^o\) and thus \( T \cap H = (T \cap H)^o \) is a maximal torus of \( H \). □

We are now in position, to formulate our main result of this section.

Theorem 8.4. Let \( G \) be a connected algebraic group with \( G = G^u \). If \( G \) is not semisimple and not isomorphic to \( \mathbb{C}^3 \) as a variety, then all embeddings of \( \mathbb{C} \) into \( G \) are equivalent.

Using Lemma 8.1 and Lemma 8.2, Theorem 8.3 reduces the proof of Theorem 1.1 to the case that the group under consideration is semisimple and not isomorphic to \( \text{SL}_2 \) or \( \text{PSL}_2 \); compare with the proof of Theorem 1.1 in Section 10.

The rest of this section is devoted to the proof of Theorem 8.4. First we have to do some preliminary work.

Proposition 8.5. Let \( K \) be a connected group that contains non-trivial unipotent elements and let \( H \) be a semisimple group (which is non-trivial by convention). Then all embeddings of \( \mathbb{C} \) into \( K \times H \) are equivalent.

Proof. Let \( \mathbb{C} \cong X \subseteq K \times H \) be an embedding. We can assume that the canonical projection \( \pi_H: K \times H \to H \) maps \( X \) birationally onto its image; compare Lemma 8.6 below. We can apply Corollary 7.10 to the group \( H \) and the curve \( \pi_H(X) \), since \( H \) is a (non-trivial) semisimple group. Hence we can assume that there exists a one-dimensional unipotent subgroup \( U \subseteq H \) such that the composition

\[
\rho: K \times H \longrightarrow H/U
\]
restricts to a birational morphism \( X \to \rho(X) \). Let \( E \) be the finite subset of elements \( z \) in \( H/U \) such that the fiber over \( z \) of \( \rho|_X \) contains more than one element. Moreover, let \( X' \) be the finite subset of \( X \) of critical points of \( \rho|_X \).

For a morphism \( f: K \to U \) consider the two properties:

i) For every \( z \in E \) and for every pair \( (k, h), (k', h') \) in \( \rho^{-1}(z) \cap X \) with \( (k, h) \neq (k', h') \) we have

\[
h f(k) \neq h' f(k').
\]

ii) For every \( x' \in X' \) the differential of

\[
\eta_f: X \to H, \quad x \mapsto \pi_H(x) f(\pi_K(x))
\]

in \( x' \) is non-vanishing.

If we consider \( U \) as a one-dimensional vector space, then for every pair of points \( (k, h) \neq (k', h') \) in \( \rho^{-1}(z) \cap X \), the expression \( h f(k) = h' f(k') \) defines a non-trivial affine linear equation for \( f \) in the vector space of maps \( K \to U \) (note that by assumption \( (h')^{-1} h \) lies in \( U \)). Moreover, we claim that for every \( x' \in X' \) the vanishing of the differential \( D_{x'} \eta_f \) defines a non-trivial affine linear equation for \( f \) in the vector space of maps \( K \to U \). Indeed, let \( x' \in X' \) and let \( W \) be an open neighbourhood of \( \rho(x') \) in \( H/U \) in the Euclidean topology such that \( H \to H/U \) gets trivial over \( W \). Then the map \( \eta_f \) can be written in a Euclidean neighbourhood \( U_{x'} \) in \( X \) around \( x' \) as

\[
U_x' \to W \times U, \quad x \mapsto (\rho(x), q(x) f(\pi_K(x))),
\]

where \( q: U_{x'} \to U \) defines a holomorphic map (that does not depend on \( f \)) with the following property: If the differential \( D_{x'} q \) vanishes, then the differential \( D_{x'} (\pi_K|_X) \) is non-vanishing. Now the vanishing of the differential of \( \eta_f \) in \( x' \) is equivalent to the vanishing of the linear map

\[
D_{x'} q + D_{\pi_K(x')} f \circ D_{x'} (\pi_K|_X): T_{x'} X \to U,
\]

where we consider again \( U \) as a one-dimensional vector space. However, this last condition defines a non-trivial affine linear equation for \( f \). This proves the claim. In summary, we showed that there exists \( f_0: K \to U \) such that \( \text{(ii)} \) and \( \text{(iv)} \) are satisfied. Define

\[
\psi_0: K \times H \to K \times H, \quad (k, h) \mapsto (k, h f_0(k)).
\]

Then, the restriction of \( \pi_H \) onto \( \psi_0(X) \) is injective and immersive, since \( f_0 \) satisfies \( \text{(ii)} \) and \( \text{(iv)} \). Since \( X \cong \mathbb{C} \), the map \( \pi_H \) restricts to an embedding on \( \psi_0(X) \). Hence, after composing \( \psi_0 \) with an automorphism of \( K \times H \) we can assume that \( X \) lies in \( H \); see Proposition 5.1. Let \( V \subseteq K \) be any one-dimensional unipotent subgroup and let \( f_1: H \to V \) be a morphism that restricts to an isomorphism on \( X \). Let \( \psi_1 \) be defined as

\[
\psi_1: K \times H \to K \times H, \quad (k, h) \mapsto (kf_1(h), h).
\]

It follows that \( \pi_K \) maps \( \psi_1(X) \) isomorphically onto \( V \). Hence, there exists an automorphism of \( K \times H \) that sends \( \psi_1(X) \) into \( V \); see Proposition 5.1. Thus the proposition follows from Proposition 6.1.

\[\Box\]

**Lemma 8.6.** Let \( H \) be an algebraic group with \( \dim H^n \geq 2 \) and let \( K \) be any affine variety. For any closed curve \( X \subset K \times H \) that is isomorphic to \( \mathbb{C} \),
there exists an automorphism $\psi$ of $K \times H$ such that the canonical projection $\pi_H : K \times H \to H$ restricts to a birational map $X \to \psi(X)$.

Proof. We only consider the case that $K$ has dimension at least 1 (otherwise $\pi_H$ restricts to an embedding on $X$). By the same argument, we can assume that the canonical projection $\pi_K : K \times H \to K$ is non-constant on $X$. We will use automorphisms of the form
\begin{equation}
\psi_f : K \times H \to K \times H, \quad (k, h) \mapsto (k, f(k)h),
\end{equation}
where $f : K \to U$ is a map to a one-dimensional unipotent subgroup $U$ of $H$.

Let us first consider the case where $\pi_H(X)$ is zero-dimensional, i.e. $\pi_H(X)$ is a point, and show that we can change that by applying an automorphism of the form (8). Without loss of generality, we may assume that the point $\pi_H(X)$ is the identity element $e$ of $H$, i.e. $X$ lies in $K$. Choose any non-trivial one-dimensional unipotent subgroup $U \subseteq H$ and let $f : K \to U$ be a morphism that is non-constant on $X$. Thus $\pi_H(\psi_f(X))$ is one-dimensional.

By the above we may assume that $\pi_H(X)$ is one-dimensional. We consider a regular value $h \in \pi_H(X)$ of the map $\pi_H|_X : X \to \pi_H(X)$ in the smooth locus of $\pi_H(X)$. Since $\pi_K|_X$ is non-constant, we can assume that the differential of $\pi_K|_X$ is non-vanishing in every point of the fiber $(\pi_H|_X)^{-1}(h)$. As before we may assume that $h$ is the identity element $e$ of $H$. Denote by
\begin{align*}
x_1 = (k_1, e), & \ldots, \ x_n = (k_n, e)
\end{align*}
the elements of the fiber $(\pi_H|_X)^{-1}(e)$. Note that for $i = 1, \ldots, n$ the lines $D_{x_i}\pi_H(T_{x_i}X)$ are all the same in $T_eH$ (otherwise $e$ lies not in the smooth part of $\pi_H(X)$). Let us denote this line in $T_eH$ by $l$. We next establish that there is a automorphism $\psi_f$ of the form (8) such that for all $1 \leq i < j \leq n$
\begin{itemize}
  \item $\psi_f(x_i) = x_i,$
  \item $\psi_f(X) \cap \pi_H^{-1}(e) = \{x_1, \ldots, x_n\},$ and
  \item $D_{x_i}\pi_H(T_{x_i}\psi_f(X)) \neq D_{x_j}\pi_H(T_{x_j}\psi_f(X)).$
\end{itemize}
Since $\dim H^u \geq 2$, we find a one-dimensional unipotent subgroup $U \subset H$ such that $T_eU$ differs from $l$ and such that $\pi_H(X) \cap U$ is finite. The first two conditions are arranged by choosing an $f : K \to U$ with
\begin{equation}
f(k) = e \quad \text{for all } k \in \pi_K \left( \{x_1, \ldots, x_n\} \cup \pi_H^{-1}(\pi_H(X) \cap U) \right).
\end{equation}
Let $t_i = v_i \oplus w_i \in T_{x_i}X \subset T_{k_i}K \oplus T_eH$ be non-zero tangent vectors to $X$ at $x_i$ for all $1 \leq i \leq n$. We calculate $D_{x_i}(\pi_H \circ \psi_f)(t_i)$ for any $f$ satisfying (9).
In fact, by writing $T_{(k_i,e)}(K \times H) = T_{k_i}K \oplus T_eH$, we get that
\begin{align*}
D_{(k_i,e)}\psi_f = \begin{pmatrix}
id & 0 \\
D_{k_i}f & \text{id}
\end{pmatrix},
\end{align*}
and thus
\begin{align*}
D_{x_i}(\pi_H \circ \psi_f)(t_i) = D_{k_i}f(v_i) + w_i.
\end{align*}
Since $D_{k_i}f(v_i) \in T_eU$, $v_i \neq 0$, $0 \neq w_i \in l$ and $l \neq T_eU$, we see that we may choose $f$ (by prescribing its derivative at $k_i$ for all $1 \leq i \leq n$) such that
\begin{align*}
D_{x_i}(\pi_H \circ \psi_f)(t_i) \quad \text{and} \quad D_{x_i}(\pi_H \circ \psi_f)(t_j)
\end{align*}
are linearly independent for all $1 \leq i < j \leq n$. Let $Y = \psi_f(X)$. 
We conclude the proof by observing that \( Y \to \pi_H(Y) \) is birational. Indeed, let \( Z = \pi_H(Y) \) and let \( \eta: \tilde{Z} \to Z \) be the normalization, which is birational. As \( Y \) is smooth, \( Y \to Z \) factorizes as \( Y \to \tilde{Z} \to Z \). Since \( \eta \) factorizes through the blow-up of \( Z \) in \( e \), \( Z \) is closed in \( H \), and the tangent directions of the branches of \( Z \) in \( e \) are all different, it follows that \( \eta^{-1}(e) \) consists of \( n \) points, say \( v_1, \ldots, v_n \). After reordering the \( v_1, \ldots, v_n \), we can assume that \( Y \to \tilde{Z} \) maps \( x_i \) to \( v_i \) for all \( i \). Since \( Y \to \pi_H(Y) \) is immersive in \( x_i \), it follows that \( Y \to \tilde{Z} \) is étale in \( x_i \) for all \( i \). Thus the fiber of \( Y \to \tilde{Z} \) over \( v_i \) consists only of \( x_i \) and it is reduced for all \( i \). Since \( Y \cong \mathbb{C} \), it follows that \( \tilde{Z} \cong \mathbb{C} \) and therefore \( Y \to \tilde{Z} \) is an isomorphism. This proves that \( Y \to \tilde{Z} \to Z \) is birational. 

Proof of Theorem 8.4. Note that if \( F \) is a connected reductive group, then \( F^u \) is semisimple or trivial. Indeed, by [Bor91, Proposition 14.2] the derived group \([F,F] \) is semisimple (or trivial) and in fact, \( F^u = [F,F] \), since \([F,F] \) contains all root subgroups with respect to any maximal torus of \( F \).

By definition, the quotient group \( G/R_u(G) \) is connected and reductive and since \( G = G^u \), we get \( G/R_u(G) = (G/R_u(G))^u \). Thus \( G/R_u(G) \) is semisimple or trivial by the proceeding paragraph. By Remark A.3 \( G \) is isomorphic as a variety to the product of \( R_u(G) \) and \( G/R_u(G) \). Now, we distinguish two cases:

i) \( G \neq R_u(G) \). Since \( G \) is not semisimple by assumption, the radical \( R_u(G) \) is not trivial. Thus we can apply Proposition 8.3 to the non-trivial groups \( K = R_u(G) \) and \( H = G/R_u(G) \) and get the result.

ii) \( G = R_u(G) \). Thus \( G \) is isomorphic as a variety to \( \mathbb{C}^n \) where \( n \) is a non-negative integer \( \neq 3 \). Clearly, we can assume that \( n > 1 \). If \( n = 2 \), then the result follows from the Abhyankar-Moh-Suzuki Theorem [AM74, Theorem 1.2]. [Suz74] and if \( n \geq 4 \), then the result follows from Jelonek’s Theorem [Jel87, Theorem 1.1].

\[ \square \]

9. REDUCTION TO SIMPLE GROUPS

The aim of this section is to reduce our problem to the case of a simple algebraic group.

Proposition 9.1. Let \( G \) be a semisimple algebraic group that is not simple. Then, two embeddings of the affine line into \( G \) are the same up to an automorphism of \( G \).

For the proof we need three lemmata, which we also use later on.

Lemma 9.2. Let \( G \) be a connected algebraic group and let \( K, H \) be closed connected subgroups such that \( KH \) is closed in \( G \) and \( K \setminus G \) is quasi-affine. If \( X \subseteq KH \) is a closed curve that is isomorphic to \( \mathbb{C} \) and if the canonical projection \( G \to K \setminus G \) restricts to an embedding on \( X \), then there exists an automorphism \( \psi \) of \( G \) with \( \psi(X) \subseteq H \).

Proof. Let \( K \times_{K \cap H} H \to K/K \cap H \) be the bundle associated to the principal \( K \cap H \)-bundle \( K \to K/K \cap H \) with fiber \( H \); compare Appendix A. The natural morphism \( K \times_{K \cap H} H \to KH \) is bijective and since \( KH \) is a
smooth irreducible variety (note that $KH$ is closed in $G$), it follows from Zariski’s Main Theorem [Gro61, Corollaire 4.4.9] that $K \times KH \to KH$ is an isomorphism. Thus, multiplication $m: K \times H \to KH$ is a principal $K \cap H$-bundle; see [Ser58, Proposition 4].

Since $C \cong X \subseteq KH$, there exists a section $Y \subseteq K \times H$ over $X$ by Theorem [Av1]. Denote by $\text{pr}_K: K \times H \to K$ the canonical projection to $K$ and by $\rho: G \to K \setminus G$ the quotient morphism. By assumption, $\rho \circ m|_Y: Y \to \rho(X)$ is an isomorphism. Since $\rho(X) \cong C$ and since $K \setminus G$ is quasi-affine,

$$
\rho(X) \to K, \quad v \mapsto (\text{pr}_K \circ (\rho \circ m|_Y)^{-1}(v))^{-1}
$$

extends to a morphism $d: K \setminus G \to K$. Let $\psi_d$ be the automorphism of $G$ constructed in Section 5. One can easily see that $\psi_d(X) \subseteq H$. □

**Lemma 9.3.** Let $G$ be an algebraic group with $G = G^u$ and let $K$ be a closed proper subgroup of $G$. Assume that $\mathcal{U}_G$ has dimension at least four. If $X \subseteq K$ is a closed curve that is isomorphic to $C$, then there exists an automorphism $\varphi$ of $G$ such that $\varphi(X)$ is a unipotent subgroup of $G$.

*Proof.* Note that the connected components of $K/K^a$ are tori. Since $X$ is the affine line, it lies in some fiber of $K \to K/K^u$. Hence, after multiplying from the left with a suitable element of $K$, we can assume that $X \subseteq K^u$. Since $K^u$ does not contain all unipotent elements of $G$ (otherwise $K = G$, since $G = G^u$), by Lemma 7.1 there exists a one-dimensional unipotent subgroup $U \subseteq G$ such that $U \cap K^u = \{e\}$ and $\pi: G \to G/U$ induces an embedding on $X$.

Choose an isomorphism $\pi(X) \cong U$ and let $f: G/U \to U$ be an extension of it. The automorphism $\varphi_f$ of $G$ (see Section 5) leaves $K^u U$ invariant. Since $U \cap K^u = \{e\}$, there is a canonical projection $K^u U \to U$. Since $X \subseteq K^u$, the composition

$$
X \xrightarrow{\varphi_f} \varphi_f(X) \subseteq K^u U \to U
$$

is an isomorphism. In particular, we can assume that $X \subseteq K^u U$ and that $\rho: G \to K^u \setminus G$ induces an embedding on $X$. Now, we can apply Lemma 9.2 to the group $G$ and the closed connected subgroups $K^u$ and $U$ to get an automorphism $\varphi$ of $G$ such that $\varphi(X) = U$. □

**Lemma 9.4.** Let $K, H$ be non-trivial connected algebraic groups with $K = K^a$, $H = H^a$ and let $Z \subseteq K \times H$ be a finite central subgroup. Assume that $\dim \mathcal{U}_H \geq 4$. If $X \subseteq (K \times H)/Z$ is a closed curve that is isomorphic to $C$, then there exists an automorphism $\varphi$ of $(K \times H)/Z$ such that $\varphi(X)$ is a unipotent subgroup of $(K \times H)/Z$.

*Proof.* Denote $K' = K/K \cap Z$, $H' = H \cap Z \setminus H$ and $G' = K \times H/Z$. We claim that the projection

$$
p: G' \to K' \setminus G'
$$

restricts to an embedding on $X$ after a suitable automorphism of $G'$. This can be seen as follows. Let $U \subseteq H$ be a closed one-dimensional unipotent subgroup such that $\pi: G' \to G'/U$ restricts to an embedding on $X$; see Remark 7.3. Thus we have a commutative diagram of principal $U$-bundles
and pr₁ is $U$-equivariant:

\[
\begin{array}{ccc}
G' & \xrightarrow{\text{pr}_1} & H/Z(H) \\
\downarrow \pi & & \downarrow \\
G'/U & \xrightarrow{\text{pr}_2} & (H/Z(H))/U;
\end{array}
\]

where $Z(H)$ denotes the center of $H$. This diagram restricts to a $U$-equivariant morphism of principal $U$-bundles

\[
\begin{array}{ccc}
\pi^{-1}(\pi(X)) & \xrightarrow{\text{pr}_1(\pi^{-1}(\pi(X)))} & \pi_1(\pi^{-1}(\pi(X))) \\
\downarrow & & \downarrow \\
\pi(X) & \xrightarrow{\text{pr}_2(\pi(X))} & \pi_2(\pi(X)).
\end{array}
\]

Since $\mathbb{C}^+$ is a special group in the sense of Serre [Ser58, §4], both principal bundles are locally trivial. Since the base varieties $\pi(X)$ and $\text{pr}_2(\pi(X))$ are affine, both principal bundles are trivial. Since $\pi(X) \cong \mathbb{C}$, there exists a section $Y \subseteq \pi^{-1}(\pi(X))$ over $\pi(X)$ that is mapped isomorphically onto its image via $\text{pr}_1$. By Proposition 5.1 there exists an automorphism of $G$ which moves $X$ into $Y$ along the fibers of $\pi$ and thus we can assume that $\text{pr}_1$ restricts to an embedding on $X$. Since $\text{pr}_1 : G' \to H/Z(H)$ factors through the projection $p$, this proves the claim.

Proof of Proposition 9.1. Since $G$ is a semisimple algebraic group, there exist simple algebraic groups $G_1, \ldots, G_n$ and an epimorphism

\[G_1 \times \cdots \times G_n \to G\]

with finite kernel; see [Hum75, Theorem 27.5]. As $G_1 \times \cdots \times G_n$ is connected, this kernel is central. By assumption, $n \geq 2$. If the Lie type of $G$ is equal to $\mathfrak{sl}_2 \times \mathfrak{sl}_2$, then $G$ is isomorphic (as a variety) to one of the groups

\[
\text{SL}_2 \times \text{SL}_2, \quad \text{SL}_2 \times \text{PSL}_2 \quad \text{or} \quad \text{PSL}_2 \times \text{PSL}_2.
\]

Indeed, if we consider the quotients of $\text{SL}_2 \times \text{SL}_2$ by subgroups of the center

\[Z(\text{SL}_2 \times \text{SL}_2) = \{ (E, E), (E, -E), (-E, E), (-E, -E) \}, \]

we get

\[
\frac{\text{SL}_2 \times \text{SL}_2}{\langle (E, E) \rangle} \cong \text{SL}_2 \times \text{SL}_2, \quad \frac{\text{SL}_2 \times \text{SL}_2}{Z(\text{SL}_2 \times \text{SL}_2)} \cong \text{PSL}_2 \times \text{PSL}_2,
\]

and

\[
\frac{\text{SL}_2 \times \text{SL}_2}{\langle (-E, -E) \rangle} \cong \frac{\text{SL}_2 \times \text{SL}_2}{\langle (E, -E) \rangle} \cong \frac{\text{SL}_2 \times \text{SL}_2}{\langle (-E, E) \rangle} \cong \text{SL}_2 \times \text{PSL}_2;
\]
where the first isomorphism of the last line is induced by the automorphism
$$SL_2 \times SL_2 \to SL_2 \times SL_2, \quad (A, B) \mapsto (AB, B).$$
By Proposition 8.5 all embeddings of \( C \) into one of these groups are equivalent. Hence, we can assume that the Lie type of \( G \) is not equal to \( sl_2 \times sl_2 \). Therefore one can find a semisimple algebraic group \( K \) and a simple algebraic group \( H \) such that \( G \cong (K \times H)/Z \) for a central finite subgroup \( Z \) and the Lie algebra of \( H \) is not isomorphic to \( sl_2 \). Since \( H \) is simple, the classification of simple Lie algebras implies that \( rank H \geq 2 \). By Lemma B.3 we have \( \dim U_H \geq 4 \). If \( X \subseteq (K \times H)/Z \) is a closed curve that is isomorphic to \( C \), then we can apply Lemma 9.4 to \( K \), \( H \) and the finite central subgroup \( Z \subseteq K \times H \) to find an automorphism that maps \( X \) into a unipotent subgroup. Thus Proposition 6.1 implies the result. \( \square \)

**Remark 9.5.** Note that \( SL_2 \times SL_2 / \langle (-E, -E) \rangle \) and \( SL_2 \times PSL_2 \) are not isomorphic as algebraic groups, since \( (A, B) \mapsto (B, A) \) is an automorphism of the first algebraic group that is not inner; however, all automorphisms of the second algebraic group are inner, since (by a calculation)
$$Aut_{\text{alg.grp.}}(SL_2 \times PSL_2) \cong Aut_{\text{alg.grp.}}(SL_2) \times Aut_{\text{alg.grp.}}(PSL_2)$$
and since all automorphisms of the algebraic groups \( SL_2, PSL_2 \) are inner; see [Hum75, Theorem 27.4].

### 10. Embeddings into simple groups

In this section, we prove the hardest part of Theorem 1.1.

**Theorem 10.1.** Let \( G \) be a simple algebraic group of rank at least two. Then two embeddings of the affine line into \( G \) are the same up to an automorphism of \( G \).

We remark that Theorem 10.1, Theorem 8.4 and Proposition 9.1 imply Theorem 1.1. We do this in detail:

**Proof of Theorem 10.1.** By Lemma 8.2, \( G \) is isomorphic to \( G^u \times (C^*)^n \) as a variety, where \( n \) is some non-negative integer. By Lemma 8.1 all embeddings of \( C \) into \( G \) are equivalent if and only if all embeddings of \( C \) into \( G^u \) are equivalent. Hence, it suffices to consider embeddings of \( C \) into \( G^u \). If \( G^u \) is not semisimple and not isomorphic to a variety of \( C^3 \), then all embeddings of \( C \) into \( G^u \) are equivalent by Theorem 8.4. If \( G^u \) is semisimple but not simple, then all embeddings of \( C \) into \( G^u \) are equivalent by Proposition 9.1. Finally, if \( G^u \) is simple and different from \( SL_2 \) and \( PSL_2 \), then \( G^u \) has rank at least two; thus all embeddings of \( C \) into \( G^u \) are equivalent by Theorem 10.1. \( \square \)

**10.1. Outline of the proof of Theorem 10.1.** In the light of Proposition 6.1, it is enough to prove that any closed curve \( X \subseteq G \) which is isomorphic to \( C \) can be moved into a one-dimensional unipotent subgroup of \( G \) via an automorphism of \( G \). In a first step we move our \( X \) into a naturally defined subvariety \( E \) (see Section 10.4) and in a second step we move it into a proper subgroup (see Section 10.5). By Lemma 9.3 we are then able to move \( X \) into a one-dimensional unipotent subgroup of \( G \), which then finishes the proof of Theorem 10.1.
The subvariety $E$ is defined using classical theory of algebraic groups. The necessary notion is set up in the next subsection.

10.2. Notation and basic facts. Let us fix the following notation for the whole section. By $G$ we denote a simple algebraic group, by $B \subseteq G$ a fixed Borel subgroup and by $T \subseteq B$ a fixed maximal torus. Let $\Phi$ be the irreducible roots system of $G$ with respect to $T$. Moreover, we denote by $W$ the Weyl group with respect to $T$ and we denote by $\Delta$ the base of $\Phi$ with respect to $B$. We denote by $w_0$ the unique longest word in $W$ with respect to $\Delta$ and by $B^-$ the opposite Borel subgroup of $B$ that contains $T$, i.e. $B^- = w_0Bw_0$.

We fix a maximal parabolic subgroup $P$ that contains $B$, i.e. we fix a simple root $\alpha \in \Delta$ such that $P = BW\wr B$ where $I = \Delta \setminus \{\alpha\}$ and $W_I$ denotes the subgroup in $W$ generated by the reflections corresponding to the roots in $I$. We denote the reflection corresponding to $\alpha$ by $s_\alpha$. Furthermore, we denote by $P^-$ the unique opposite parabolic subgroup to $P$ with respect to $T$; see Appendix B.1. Again by Appendix B.1, $P^- = B^-W_I^-B^-$, $PP^-$ is open in $G$ and $PP^- = R_u(P^-)$.

The quotient of $G$ by the unipotent radical of $P^-$ will play a crucial role for use. We denote this quotient throughout this section by

$$\pi : G \to G/R_u(P^-).$$

Since $R_u(P^-)$ is a special group in the sense of Serre [Ser58, §4], $\pi$ is a locally trivial principal $R_u(P^-)$-bundle.

Since $P$ is a maximal parabolic subgroup of $G$, there exists a unique Schubert curve in $G/P$. We denote by $E$ the inverse image of this Schubert curve under the natural projection $G \to G/P$. Note that $E$ is the union of the two disjoint subsets $Bs_\alpha P$ and $P$ of $G$.

10.3. The restriction of $\pi$ to $E$. Recall that $E$ denotes the inverse image of the unique Schubert curve in $G/P$ and $\pi : G \to G/R_u(P^-)$ denotes the canonical projection. The following result describes the restriction of $\pi$ to $E$. It is the key ingredient that enables us to move our curve into $E$.

Proposition 10.2. The complement of $\pi(E)$ in $G/R_u(P^-)$ is closed and has codimension at least two in $G/R_u(P^-)$. Moreover, the restriction of $\pi$ to $E$ turns $E$ into a locally trivial $\mathbb{C}$-bundle over $\pi(E)$.

Proof of Proposition 10.2. For the first statement it is enough to show that $\pi^{-1}(\pi(E)) = EP^-$ is open in $G$ and that $G \setminus EP^-$ has codimension at least two in $G$. We have the following inclusion inside $G$

$$BP^- \cup Bs_\alpha P^- \subseteq PP^- \cup Bs_\alpha PP^- = EP^-.$$ 

Since $BP^- = PP^-$ is open in $G$, it follows that $EP^-$ is open in $G$. More precisely, $G \setminus BP^-$ is an irreducible closed hypersurface in $G$. This follows from the fact that $(G/P^-) \setminus Be$ is the translate by $w_0$ of the unique Schubert divisor in $G/P^-$ with respect to $B^-$. Since $BP^-$ and $Bs_\alpha P^-$ are disjoint we have a proper inclusion $G \setminus EP^- \subseteq G \setminus BP^-$. Thus $G \setminus EP^-$ has codimension at least two in $G$.

For proving the second statement, we first show that all fibers of $\pi|_E : E \to \pi(E)$ are reduced and isomorphic to $\mathbb{C}$. In fact, the schematic fiber over $\pi(g)$
is the schematic intersection \( E \cap gR_u(P^-) \) for all \( g \in E \). Let \( C \) be the unique Schubert curve in \( G/P \), i.e. \( C \) is the closure of the \( B \)-orbit through \( s_\alpha \). Since Schubert varieties are normal (see [RR85, Theorem 3]) and rational, it follows that \( C \cong \mathbb{P}^1 \). For each \( g \in G \), consider the following commutative diagram

\[
\begin{array}{ccc}
E \cap gR_u(P^-) & \rightarrow & E \\
gR_u(P^-) & \rightarrow & G \\
& \rightarrow & G/P.
\end{array}
\]

Note that all squares are pull-back diagrams. Since \( gR_u(P^-) \rightarrow G \rightarrow G/P \) is an open injective immersion, the same holds for \( E \cap gR_u(P^-) \rightarrow E \rightarrow C \). Note that the image of \( gR_u(P^-) \) inside \( G/P \) is equal to \( gB^-e \subseteq G/P \). Since \( E \) is the inverse image of \( C \) under \( G \rightarrow G/P \) we get an isomorphism

\[
E \cap gR_u(P^-) \cong C \cap gB^-e.
\]

Let \( C^{\text{op}} \subseteq G/P \) be the opposite Schubert variety to \( C \), i.e. \( C^{\text{op}} \) is the closure of the \( B^- \)-orbit through \( s_\alpha \) inside \( G/P \). Thus we have a disjoint union

\[
C^{\text{op}} \cup B^-e = G/P.
\]

It follows from Lemma 10.3 that for all \( g \in G \) the subset \( C \cap gC^{\text{op}} \) consists of exactly one point or \( C \subseteq gC^{\text{op}} \). Hence

\[
C \setminus (C \cap gC^{\text{op}}) = C \cap gB^-e.
\]

is either isomorphic to \( \mathbb{C} \) or it is empty. This proves that all fibers of \( \pi|_E : E \rightarrow \pi(E) \) are reduced and isomorphic to \( \mathbb{C} \).

Since \( C \) is smooth and since \( G \rightarrow G/P \) is a smooth morphism, it follows that \( E \) is smooth; see [GR04, Chp. II, Proposition 3.1]. Moreover, \( \pi(E) \) is smooth as an open subset of the smooth variety \( G/R_u(P^-) \). Since all fibers of \( \pi|_E \) have the same dimension, the morphism \( \pi|_E \) is faithfully flat. Since \( \pi \) is affine as a locally trivial principal \( R_u(P^-) \)-bundle, the restriction \( \pi|_E \) is also affine. It follows from [KWS85] or [KR14, Theorem 5.2] that \( \pi|_E \) is a locally trivial \( C \)-bundle.

**Lemma 10.3.** Let \( C \) be the unique Schubert curve in \( G/P \) with respect to \( B \) and let \( C^{\text{op}} \) be the opposite Schubert variety to \( C \). Then for all \( g \in G \) either \( gC \cap C^{\text{op}} \) is a reduced point of \( G/P \) or \( gC \subseteq C^{\text{op}} \).

**Remark 10.4.** Compare the proof of this lemma with [Har77, Chp. III, Proof of Theorem 10.8].

**Proof.** Consider the following pullback diagram

\[
\begin{array}{ccc}
(G \times C) \times_{G/P} C^{\text{op}} & \rightarrow & C^{\text{op}} \\
& \rightarrow & G/P \\
G \times C & \rightarrow & G/P
\end{array}
\]

where \( G \times C \rightarrow G/P \) denotes the map \((g,c) \mapsto gc\). Note that the vertical arrows are closed embeddings. Since \( C \) is smooth, by generic smoothness [Har77, Chp. III, Corollary 10.7] and \( G \)-equivariance, the morphism \( G \times C \rightarrow G/P \) is smooth. Since \( C^{\text{op}} \) is reduced, it follows that the fiber product...
The fiber satisfies \( q \) of \( G \) is isomorphic to the scheme theoretic intersection \( gC \cap C^\text{op} \). Since \( C \) is projective, the morphism \( q \) is projective and thus by [Eis95, Theorem 14.8] the subset

\[
V = \{ g \in G \mid gC \cap C^\text{op} \text{ is finite} \}
\]

is open in \( G \). Let \( q' = q|_{q^{-1}(V)} : q^{-1}(V) \to V \). By definition, \( q' \) is quasi-finite. Since \( q \) is projective (and thus \( q' \) also), it follows that \( q' \) is finite; see [Gro66, Théorème 8.11.1]. We claim that \( q' \) is birational. Indeed, this can be seen as follows. The fiber of \( q \) over \( e \in G \) is isomorphic to \( C \cap C^\text{op} \). By [Ram85, Theorem 3 and Remark 3] this last scheme is reduced and by [Ric92, Theorem 3.7] it is irreducible and of dimension zero; cf. also BL03.

Thus the fiber of \( q \) over \( e \) is a reduced point. Hence, the tangent space of the fiber satisfies

\[
0 = T_{x_0}q^{-1}(e) = \ker d_{x_0}q
\]

where \( \{x_0\} = q^{-1}(e) \). Therefore \( q \) is immersive at \( x_0 \). Hence \( q^{-1}(V) \) is smooth at \( x_0 \) by dimension reasons and \( q' \) is étale in \( x_0 \). Let \( S \) be the set of points in \( q^{-1}(V) \), where \( q' \) is not étale. By [GR04, Chp. I, Proposition 4.5] the set \( S \) is closed in \( q^{-1}(V) \). As \( q' \) is finite, \( q(S) \) is closed in \( V \). Clearly, \( q' \) restricts to a finite étale morphism

\[
q^{-1}(V \setminus q(S)) \to V \setminus q(S).
\]

Since \( \{x_0\} \) is a fiber of \( q \) and since \( q' \) is étale at \( x_0 \), it follows that \( q(x_0) \notin q(S) \), i.e. \( x_0 \in q^{-1}(V \setminus q(S)) \). This implies that the morphism (10) is of degree one and therefore it is an isomorphism. Since \( V \) is irreducible and since \( q' \) is finite, it follows that \( q^{-1}(V \setminus q(S)) \) is dense in \( q^{-1}(V) \). As (10) is an isomorphism, \( q^{-1}(V) \) is irreducible. This implies that \( q' \) is birational. Since \( V \) is smooth and irreducible and since \( q' \) is finite and birational, it follows that \( q' \) is an isomorphism by Zariski’s Main Theorem [Gro61, Corollaire 4.4.9]. This implies the lemma. \( \Box \)

10.4. Moving a curve into \( E \).

**Proposition 10.5.** If \( X \subseteq G \) is a closed curve that is isomorphic to \( \mathbb{C} \), then there exists an automorphism \( \varphi \) of \( G \) such that \( \varphi(X) \subseteq E \).

**Proof.** If \( \text{rank}(G) = 1 \), then \( E = G \) and there is nothing to prove. Thus we assume that \( \text{rank}(G) \geq 2 \). Therefore, we can apply Proposition 7.4 to \( G \) and the unipotent subgroup \( R_u(P^-) \) to get an automorphism \( \varphi \) of \( G \) such that \( \pi : G \to G/R_u(P^-) \) restricts to an embedding on \( \varphi(X) \). Let us replace \( X \) by \( \varphi(X) \). Since the complement of \( \pi(E) \) in \( G/R_u(P^-) \) is closed and has codimension at least two in \( G/R_u(P^-) \) by Proposition 10.2, there exists by Kleiman’s Theorem \( q \in G \) such that \( g\pi(X) \) lies inside \( \pi(E) \); see [Kle74, Theorem 2]. Since \( \pi \) is \( G \)-equivariant, it restricts to an isomorphism \( gX \to \pi(gX) \). Hence, we can replace \( X \) by \( gX \) and assume in addition that \( \pi(X) \subseteq \pi(E) \). Since \( \pi \) restricts to a locally trivial \( \mathbb{C} \)-bundle \( \pi|_E : E \to \pi(E) \) by Proposition 10.2 and since \( \pi(X) \cong \mathbb{C} \), there exists a section \( \sigma \) of \( \pi|_E \) over
π(X); see e.g. [BCW77]. By Proposition 10.6 there exists an automorphism of G that moves X to the section σ(π(X)) ⊆ E and fixes π: G → G/Ra(P−).
This implies the result. □

10.5. Moving a curve in E into a proper subgroup. The aim of this section is to prove the following result.

Proposition 10.6. Assume that rank G ≥ 2. If X ⊆ E is a closed curve that is isomorphic to C, then there exists an automorphism φ of G such that φ(X) lies in a proper subgroup of G.

Proposition 10.6 is based on the following rather technical result.

Proposition 10.7. Assume that rank(G) ≥ 2. Let K be a closed connected reductive subgroup of G such that KP is closed in G. Assume that K ∩ P is connected and solvable and moreover, that Ra(K ∩ P) has dimension one and lies in Ra(P). If X ⊆ KP is a closed curve that is isomorphic to C, then there exists an automorphism φ of G such that G → K\G restricts to an embedding on φ(X) and φ(KP) = KP.

Before proving Proposition 10.7, we show how it implies Proposition 10.6.

Proof of Proposition 10.6. Let K = C_G(⟨ker α⟩) be the centralizer in G of the connected component of the identity element of the kernel of the root α: T → C×. By definition, T and the root subgroups U±α lie inside K. By [Hum75] Theorem 22.3, Corollary 26.2B], the group K is connected, reductive, the semisimple rank is one and the Lie algebra of K decomposes as t ⊕ uα ⊕ u−α, where t is the Lie algebra of T and u±α is the Lie algebra of U±α. Since K is connected and not solvable, TUα is connected and solvable, and TUα is of codimension one in K, it follows that TUα is a Borel subgroup of K. Since TUα ⊆ K ∩ P ⊆ K, the subgroup K ∩ P is parabolic in K and in particular it is connected; see [Hum75 Corollary 23.1B]. We have K ∩ P ≠ K, since otherwise P would contain the root subgroup U−α and thus we would have P = G; see [Hum75] Theorem 27.3]. Hence

K ∩ P = TUα.
Moreover, we have by [Hum75 §30.2]

Ra(K ∩ P) = Uα ⊆ Ra(P).
We claim that UαsαP = BsαP inside G. Indeed, otherwise UαsαP = sαP, since dim BsαP = dim E = 1 + dim P. Therefore U−α = sαUαsα ⊆ P, a contradiction. Hence it follows that

E = UαsαP ∪ P.
Since T and U±α generate K, it follows that K lies inside the minimal parabolic subgroup P(α) = BsαB ∪ B. By [Bor91] Theorem 13.18] the reflection sα generates the Weyl group of K and, in particular, sα lies in K.
More precisely, every representative of sα lies in K. In summary, we get

E ⊆ KP ⊆ P(α)P = BsαP ∪ P = E,
which proves E = KP.
Now, we can apply Proposition 10.7 and thus we can assume that $G \to K \setminus G$ restricts to an embedding on $X$. Applying Lemma 9.2 to $G$ and the closed connected subgroups $K$ and $P$ yields the desired result.

The rest of this subsection is devoted to the proof of Proposition 10.7. First we provide an estimation of the dimension of the intersection of every translate of the torus $T$ with the variety $U_G$ of unipotent elements in case $G$ is of rank two. Note that by the classification of simple groups of rank two, $G$ is either of type $A_2$, $B_2$ or $G_2$.

**Lemma 10.8.** Assume that $\text{rank}(G) = 2$. Then the following holds.

i) If $G$ is of type $A_2$, then $Tg \cap U_G$ is finite for all $g \in G$.

ii) If $G$ is of type $B_2$, then $\dim(Tg \cap U_G) \leq 1$ for all $g \in G$.

**Remark 10.9.** To complement i), note that for some $g \in G$ the intersection $Tg \cap U_G$ is not finite. For example, if $G = SL_3$, $T$ is the diagonal torus in $G$ and

$$g = \begin{pmatrix} 3 & 0 & -4 \\ 2 & 0 & -3 \\ 0 & 1 & 0 \end{pmatrix},$$

then a calculation shows that $Tg \cap U_G$ is one-dimensional.

**Proof of Lemma 10.8.** To every simple group $H$ there exists a simply connected simple group $\tilde{H}$ and an isogeny $\tilde{H} \to H$, i.e. an epimorphism with finite kernel; see [Che05, §23.1, Proposition 1]. Two simply connected simple groups with the same root system are always isomorphic by [Hum75, Theorem 32.1]. Therefore it is enough to prove i) for the simply connected group $G = SL_3$ and to prove ii) for the simply connected group $G = Sp_4$; see [Che05, §20.1, §22.1] and [Hum75, Corollary 21.3C].

Assume $G$ is $SL_3$. We can assume that $T$ is the subgroup of $G$ of diagonal matrices and $B$ is the subgroup of upper triangular matrices. Moreover, we can assume without loss of generality that $P$ is the maximal parabolic subgroup

$$P = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \right\} \subseteq SL_3.$$

An element $a \in SL_3$ is unipotent if and only if one is the only root of its characteristic polynomial $\chi_a$. We have

$$\chi_a(t) = t^3 - \text{tr}(a)t^2 + s(a)t - 1$$

where

$$s(a) = (a_{11}a_{22} - a_{12}a_{21}) + (a_{11}a_{33} - a_{13}a_{31}) + (a_{22}a_{33} - a_{23}a_{32})$$

and $a_{ij}$ denotes the $ij$-th entry of $a$. Let $p \in P$. The variety $Tp \cap U_G$ is isomorphic to

$$S = \{ t \in T \mid tp \in U_G \}.$$
Let $x, y, z$ denote the entries on the diagonal of a $3 \times 3$-diagonal matrix. The set $S$ can be realized as the closed subvariety of $\mathbb{C}^3$ given by the equations

(11) \quad 3 = xp_{11} + yp_{22} + zp_{33}
(12) \quad 3 = xy p_{11} p_{22} + xz p_{11} p_{33} + yz (p_{22} p_{33} - p_{23} p_{32})
(13) \quad 1 = xyz.

Clearly, $p_{11}$ is non-zero. Inserting (11) in (13) yields the irreducible equation

(14) \quad p_{11} = (3 - yp_{22} - zp_{33})yz.

Inserting (11) in (12) yields a non-trivial equation of degree $\leq 2$ in $y$ and $z$. If $p_{22}$ or $p_{33}$ is non-zero, then (14) is an equation of degree 3 and thus $S$ is finite. If $p_{22} = p_{33} = 0$, then $S$ is realized as the closed subset of $\mathbb{C}^2$ given by the equations

$3 = -yzp_{23}p_{32}$ and $p_{11} = 3yz$.

However, since $p$ has determinant equal to 1, we get $-p_{11}p_{23}p_{32} = 1$. Hence, $S$ is empty in case $p_{22} = p_{33} = 0$. This proves i).

Assume that $G$ is $\text{Sp}_4$. Since all non-degenerate alternating bilinear forms on an even dimensional vector space are equivalent, we can choose $\Omega$ as the matrix with entries $1, 1, -1, -1$ on the antidiagonal and all other entries equal to zero, and then define $\text{Sp}_4$ as those $4 \times 4$-matrices $g$ that satisfy $g^t \Omega g = \Omega$. Thus we can choose for the maximal torus $T$ the subgroup of $\text{Sp}_4$ consisting of diagonal matrices with entries $t_1, t_2, t_1^{-1}, t_2^{-1}$ on the diagonal for arbitrary non-zero $t_1$ and $t_2$. If an element in $\text{GL}_4$ is unipotent, then its trace is equal to 4. Let $g \in \text{Sp}_4$. One can see that

$$\{ t \in T \mid \text{tr} tg = 4 \}$$

is a proper closed subset of the torus $T$ and thus $Tg \cap U_{\text{Sp}_4}$ is properly contained in $Tg$, which proves ii).

**Lemma 10.10.** Assume that rank($G$) $\geq 2$. Let $H \subseteq P$ be a connected closed solvable subgroup such that the unipotent radical $R_u(H)$ is one-dimensional. Denote by $p: P \to H \setminus P$ the canonical projection. Then for every $p \in P$ the fibers of the morphism

$$U_P \to \rho(p)U_P, \quad u \mapsto \rho(p)u$$

have codimension at least three in $U_P$.

**Proof.** In case the rank of $G$ is at least 3 or $G$ is of type $G_2$, it follows that $\dim U_P - \dim H \geq \dim U_P - \dim T - 1 \geq 3$ by Lemma [B.2] and thus the lemma is proved in these cases.

Assume that $G$ is of type $A_2$. For every $p \in P$ the quotient $\eta: P \to T \setminus P$ restricts to a morphism $Hp \cap U_P \to \eta(R_u(H)p)$. By Lemma [10.8] the fibers of this restriction are finite. Since $R_u(H)$ is one-dimensional, it follows that $Hp \cap U_P$ is at most one-dimensional. By Lemma [B.3] we have $\dim U_P = 4$, which implies the lemma in this case.

Assume that $G$ is of type $B_2$. Analogously, it follows from Lemma [10.8] and Lemma [B.3] that $Hp \cap U_P$ is at most two-dimensional and that $\dim U_P = 5$, which proves the lemma in this case. \qed
Proof of Proposition [10.7] We start by observing that $K \cap P \setminus P$ is affine, since $K \cap P / P \cong K / K P$ is closed in $K \setminus G$ and since $K \setminus G$ is affine ($K$ is reductive). In particular, every $\mathbb{C}^+$-orbit of a $\mathbb{C}^+$-action on $K \cap P \setminus P$ is closed.

We prove that for a generic $u \in U_P$ the one-dimensional unipotent subgroup $\mathbb{C}^+(u)$ of $P$ acts without fixed point on $K \cap P \setminus P$. Every $\mathbb{C}^+(u)$-orbit in $K \cap P \setminus P$ is either a fixed point or isomorphic to $\mathbb{C}$. If $p \in P$ would map to a fixed point in $K \cap P \setminus P$ of the $\mathbb{C}^+(u)$-action, then $(K \cap P)p \mathbb{C}^+(u) = (K \cap P)p$. This would imply that $p\mathbb{C}^+(u)p^{-1} \subseteq K \cap P$. Since $K \cap P$ is solvable, $p\mathbb{C}^+(u)p^{-1}$ lies inside $R_u(K \cap P)$ and hence inside $R_u(P)$, by assumption. In particular, $\mathbb{C}^+(u)$ lies inside $R_u(P)$. However, generic $u \in U_P$ are not contained in $R_u(P)$, since $P$ is not a Borel subgroup of $G$. This proves our claim.

Denote by $\eta: KP \rightarrow K \cap P \setminus P$ the restriction of the canonical projection $G \rightarrow K \setminus G$. By Lemma [13.3] we have $\dim U_P \geq 4$ and hence there exists a one-dimensional unipotent subgroup $U$ of $P$ such that $G \rightarrow G/U$ restricts to an embedding on $X$, by Remark [7.3]. Moreover, we can assume by the previous paragraph that $U$ acts without fixed point on $K \cap P \setminus P$. Thus we can apply Lemma [C.1] to the $U$-equivariant morphism $XU \rightarrow \eta(XU)$ to get a section $X'$ of $XU \rightarrow XU / U$ that is mapped birationally via $\eta$ onto its image. Hence, after applying an appropriate automorphism of $G$ (that leaves $KP$ invariant), we can assume that $\eta$ maps $X$ birationally onto its image; see Proposition [5.1]. Let us denote this image inside $\eta(XU)$ by $C$. Note that $C$ is closed in $\eta(XU)$, since $X$ is isomorphic to $\mathbb{C}$. We apply Lemma [14.7] to the group $P$, the affine homogeneous $P$-space $K \cap P \setminus P$ and the curve $C$ in $K \cap P \setminus P$ (the codimension assumptions of Lemma [7.7] are guaranteed by Lemma [13.11]). Thus we get a $u' \in U_P \setminus \{e\}$ such that $G \rightarrow G / \mathbb{C}^+(u')$ restricts to an embedding on $X$ (by Remark [13.3], $\mathbb{C}^+(u')$ acts without fixed point on $K \cap P \setminus P$ and $S_{u'} \rightarrow S_{u'} / \mathbb{C}^+(u')$ restricts to a birational morphism on $C$. Here $S_{u'}$ denotes the closure of all the $\mathbb{C}^+(u')$-orbits in $K \cap P \setminus P$ that pass through $C$. Since $X$ is mapped birationally onto $C \subseteq S_{u'}$, and since $C$ is mapped birationally onto $S_{u'} / \mathbb{C}^+(u')$ it follows that $\eta$ restricts to a birational map $X \mathbb{C}^+(u') \rightarrow S_{u'}$. Hence we can apply Lemma [13.2] to the $\mathbb{C}^+(u')$-equivariant morphism $X \mathbb{C}^+(u') \rightarrow S_{u'}$ and get a section $X''$ of $X \mathbb{C}^+(u') \rightarrow X \mathbb{C}^+(u') / \mathbb{C}^+(u')$ that is mapped isomorphically via $\eta$ onto its image inside $S_{u'} \subseteq K \cap P \setminus P$. By Proposition [5.1] there exists an automorphism of $G$ (that leaves $KP$ invariant) and maps $X$ to $X''$ and thus we can assume that $\eta$ maps $X$ isomorphically onto $K \cap P \setminus P$. Since $\eta$ is the restriction of $G \rightarrow K \setminus G$ to $KP$, this finishes the proof.

Appendix A. Principal bundles over the affine line

In [RR84] it is stated by referring on [Ste65] and [Ram83], that over an algebraically closed field every principal $G$-bundle over the affine line is trivial if $G$ is a connected algebraic group. However, the connectedness assumption is in fact superfluous over an algebraically closed field of characteristic zero. For the sake of completeness we give a prove of this result.

Theorem A.1. Let $G$ be any algebraic group. Then every principal $G$-bundle over the affine line $\mathbb{C}$ is trivial.
Before starting with the proof, let us recall a very important construction that associates a fiber bundle $P \times^G F \to X$ to a principal $G$-bundle $\pi : P \to X$ and a variety $F$ with a left $G$-action (see [Ser58, Proposition 4]): the variety $P \times^G F$ is defined as the quotient of $P \times F$ by the right $G$-action

$$(p, f) \cdot g = (pg, g^{-1}f)$$

and the canonical map $P \times^G F \to X$ is a bundle with fiber $F$ which becomes locally trivial after a finite étale base change, see [Ser58, Example c), §3.2].

**Proof of Theorem A.1.** Let $P \to \mathbb{C}$ be a principal $G$-bundle. Let $G^0$ be the connected component of the identity element in $G$. The principal $G$-bundle factorizes as

$$P \to P \times^G G/G^0 \to \mathbb{C}.$$ 

The first morphism is a principal $G^0$-bundle by [Ser58, Proposition 8]. The second morphism is a principal $G/G^0$-bundle and since $G/G^0$ is finite, it is a finite morphism; see [Ser58, Proposition 5 and §3.2, Example a)]. Since the base is $\mathbb{C}$, this second principal bundle admits a section $s : \mathbb{C} \to P \times^G G/G^0$ (which follows from Hurwitz’s Theorem [Har77, Chp. IV, Corollary 2.4]). Due to Theorem A.2, the principal $G^0$-bundle $P \to P \times^G G/G^0$ is trivial over $s(\mathbb{C})$, and thus $P \to \mathbb{C}$ admits a section, which proves the Theorem. □

The main step in the following Theorem is due to Steinberg [Ste65].

**Theorem A.2.** Let $G$ be a connected algebraic group. Then, every principal $G$-bundle over a smooth affine rational curve is trivial.

**Proof.** Let $X$ be a smooth affine rational curve and let $E \to X$ be a principal $G$-bundle.

First we prove that $E \to X$ admits a section that is defined over some open subset of $X$. By definition there exists a finite étale map from an affine curve $U'$ onto an open subset $U$ of the curve $X$ such that the pull back $E_{|U'} \to U'$ is a trivial principal $G$-bundle. Let $K$ be the function field of $U$ and let $K'$ be the function field of $U'$. We can assume that the field extension $K'/K$ is finite and Galois, by [Ser58, §1.5]. Let $\text{Gal}(K'/K)$ denote the Galois group of this extension. We denote by $G(K')$ the $K'$-rational points of $G$, i.e. the group of rational maps $U' \to G$. By [Ser58 §2.3b)] it follows that the first Galois cohomology set

$$H^1(\text{Gal}(K'/K), G(K'))$$

describes the isomorphism classes of principal $G$-bundles that are defined over some non-specified open subset of $U$ such that their pull back via $U' \to U$ admit a section over some open $\text{Gal}(K'/K)$-invariant subset of $U'$. Hence it is enough to prove that $H^1(\text{Gal}(K'/K), G(K'))$ is trivial. Let $K$ be an algebraic closure of $K$ that contains $K'$. By [Ser94 §5.8, Chp. I], the natural map

$$H^1(\text{Gal}(K'/K), G(K')) \to H^1(\text{Gal}(\bar{K}/K), G(\bar{K}))$$

is injective. Note, that $G(\bar{K})$ is an algebraic group over $\bar{K}$. Since $K'$ has transcendence degree one over the ground field, the so-called (cohomological) dimension of $K'$ is at most one by [Ser94, Example b), §3.3, Chp. II]. Now, by a result of Steinberg, $H^1(\text{Gal}(\bar{K}/K), G(\bar{K}))$ is trivial; see [Ste65, Theorem 1.9]. Hence, $E \to X$ admits a section over some open subset of $X$. 

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The principal $G$-bundle $E \to X$ decomposes as
\[ E \to E \times^G G/B \to X \]
where the first morphism is a principal $B$-bundle and the second morphism is a $G/B$-bundle, locally trivial in the étale topology. Since $E$ becomes trivial over some open subset $V$ of $X$, it follows that $E \times^G G/B$ becomes also trivial over $V$, hence there exists a rational section $s: X \to E \times^G G/B$ that is defined over $V$. Since $G/B$ is projective, to every point $x$ in $X$ there is a finite étale map $f_x$ onto an open neighbourhood of $x$ such that the pull back of $E \times^G G/B \to X$ via $f_x$ is projective. This implies that $E \times^G G/B \to X$ is universally closed and hence proper. Since $X$ is a smooth curve, it follows by the Valuative Criterion of Properness that the section $s$ is defined on the whole $X$; see [Har77, Theorem 4.7, Chp. II]. Thus the restriction of the principal $B$-bundle $E \to E \times^G G/B$ to $s(X)$ is trivial by Proposition A.3, since $X$ has a trivial Picard group. Hence, we proved that $E \to X$ admits a section, which implies the statement of the theorem. □

**Proposition A.3.** Let $G$ be a connected, solvable algebraic group. Then, every principal $G$-bundle over any affine variety with vanishing Picard group is trivial.

**Proof.** Let $X$ be an affine variety. By [Ser58, Proposition 14] every principal $G$-bundle is locally trivial, since $G$ is connected and solvable. Note that the first Čech cohomology
\[ \check{H}^1(X, G) \]
is a pointed set that corresponds to the isomorphism classes of locally trivial principal $G$-bundles over $X$, where $G$ denotes the sheaf of groups on $X$ with sections over an open subset $U \subseteq X$ being the morphisms $U \to G$; see [Fre57, §3] and [Ser58, §3]. Since $G$ is solvable and connected, there exists a semidirect product decomposition $G = U \rtimes T$ for a torus $T$ and a unipotent group $U$. The short exact sequence corresponding to this decomposition yields an exact sequence in cohomology
\[ \check{H}^1(X, U) \to \check{H}^1(X, G) \to \check{H}^1(X, T) ; \]
see [Fre57, Théorème I.2]. However, by using a decreasing chain of closed normal subgroups of $U$ such that each factor is isomorphic to $\mathbb{C}^+$ and by using that $\check{H}^1(X, \mathbb{C}^+) = H^1(X, \mathcal{O}_X)$ is trivial (since $X$ is affine) it follows that $\check{H}^1(X, U)$ is trivial. Since the Picard group $\check{H}^1(X, \mathbb{C}^+) = H^1(X, \mathcal{O}_X^*)$ vanishes it follows analogously that $\check{H}^1(X, T)$ is trivial, whence $\check{H}^1(X, G)$ is trivial. This implies the proposition. □

**Remark A.4.** The proof of Proposition A.3 shows the following. If $G$ is unipotent, then every principal $G$-bundle over any affine variety is trivial.

**Appendix B. Generalities on parabolic subgroups**

Throughout this appendix we fix the following notation. Let $G$ be a connected reductive algebraic group, $B$ a Borel subgroup, $T$ a maximal torus in $B$ and $W$ the Weyl group with respect to $T$. Moreover, we denote by $\Delta$ the set of simple roots of $G$ with respect to $(B, T)$.
B.1. The opposite parabolic subgroup. Let $P$ be a parabolic subgroup that contains $B$, i.e. $P = BW_1B$ where $I$ is a subset of $\Delta$ and $W_I$ is the subgroup of $W$ generated by the reflections corresponding to roots in $I$. There exists a unique parabolic subgroup $P^-$ that contains $T$ such that $P \cap P^-$ is a Levi factor of $P$ and $P^-$, i.e. there are semidirect product decompositions

$$P = R_u(P) \times (P \cap P^-) \quad \text{and} \quad P^- = R_u(P^-) \times (P \cap P^-);$$

see [Spr09 Corollary 8.4.4.] and [Bor91 Proposition 14.21]. We call $P^-$ the opposite parabolic subgroup of $P$ with respect to $T$. In fact we can describe $P^-$ as follows.

**Lemma B.1.** We have $P^- = B^-W_IB^-.$

For the lack of reference, we provide a proof.

**Proof.** Let $Z$ be the connected component of the identity element in the group $\bigcap_{\gamma \in I} \ker \gamma.$ By [Hum75 §30.2], the centralizer $C_G(Z)$ is a Levi factor of $P$, i.e. $P = C_G(Z) \rtimes R_u(P).$ Let $Q$ be $B^-W_IB^-$. In fact, $Q = B^-W_{-I}B^-$ since $W_I = W_{-I}$. Moreover, $Z$ is the connected component of the identity element in $\bigcap_{\gamma \in I} \ker \gamma$ and thus it follows that $Q = C_G(Z) \rtimes R_u(Q)$. Clearly, $R_u(Q) \cap P$ is a unipotent subgroup of $R_u(Q)$ that is invariant under conjugation by $T$. If $R_u(Q) \cap P$ is non-trivial, it contains a root subgroup $U_\beta$, by [Hum75 Proposition 28.1]) for a certain root $\beta$. Note that $\beta$ is a negative root with respect to $\Delta$ which is not a $Z$-linear combination of roots in $I$, by [Hum75 §30.2] applied to $(B^{-}, Q)$. Since $\beta$ is also a root of $P$ with respect to $T$, we get a contradiction to [Hum75 Proposition 30.1] applied to $(B, P)$. Hence $R_u(Q) \cap P$ is trivial. This implies that $P \cap Q = C_G(Z)$ and thus $P$ and $Q$ are opposite parabolic subgroups. Since $P$ and $Q$ contain $T$, we get $Q = P^-$. □

By [Bor91 Proposition 14.21] we have that $PP^-$ is open in $G$ and the product map induces an isomorphism of varieties

$$(15) \quad R_u(P) \times (P \cap P^-) \times R_u(P^-) \xrightarrow{\cong} PP^-.$$ 

In particular, we get the following.

**Lemma B.2.** We have $\dim G = \dim R_u(P^-) + \dim P$.

B.2. Dimension of $U_P$ and $R_u(P)$ of a parabolic subgroup $P$. We give here a result which estimates the dimension of $U_P$ and $R_u(P)$ from below for a parabolic subgroup $P$. The proof is based on the following fact. Let $\alpha$ be a simple root and let $\beta$ be a positive root which is a linear combination of simple roots different from $\alpha$. If $\alpha$ and $\beta$ are not perpendicular, then $\alpha + \beta$ is a positive root, by [Hum78 Lemma 9.4 and Lemma 10.1].

**Lemma B.3.** Assume that $G$ is a simple group and let $P$ be a parabolic subgroup that contains $B$. Then the following holds

i) If $\operatorname{rank}(G) \geq 3$ and $P \neq B$, then $\dim U_P \geq 2 \operatorname{rank}(G) + 1$. 


ii) If \( \text{rank}(G) = 2 \) and \( B \neq P \neq G \), then
\[
\dim U_P = \begin{cases} 
4 & \text{if } G \text{ is of type } A_2, \\
5 & \text{if } G \text{ is of type } B_2, \\
7 & \text{if } G \text{ is of type } G_2.
\end{cases}
\]

iii) If \( \text{rank}(G) \geq 2 \) and \( P \neq G \), then \( \dim R_u(P) \geq 2 \).

**Proof.** Assume that \( P \neq B \). Since \( \dim U_P = \dim P - \text{rank}(G) \) we get
\[
\dim U_P = \dim R_u(B) + (\dim R_u(B) - \dim R_u(P)).
\]

Note that \( \dim R_u(B) \) is equal to the number of positive roots. In a Dynkin diagram the vertices correspond to the simple roots and there is one (or more) edges between two simple roots if and only if they are not perpendicular. For each pair of non-perpendicular simple roots \( \alpha, \beta \), the sum \( \alpha + \beta \) is again a (positive) root. Since any Dynkin diagram is a tree, the simple roots together with the above sums of pairs give \( 2 \text{rank}(G) - 1 \) positive roots.

Assume that \( \text{rank}(G) \geq 3 \) and \( P \neq B \). Again, since any Dynkin diagram is a tree, one sees that there is a subgraph of the Dynkin diagram of \( G \) of the form
\[
\alpha_1 \quad \alpha_2 \quad \alpha_3
\]
and \( \alpha_1, \alpha_3 \) are not connected in the Dynkin diagram. Hence \( \alpha_1 + \alpha_2 \) and \( \alpha_3 \) are not perpendicular and thus the sum \( \alpha_1 + \alpha_2 + \alpha_3 \) is again a positive root. Thus we proved \( \dim R_u(B) \geq 2 \text{rank}(G) \). Since \( P \) is not a Borel subgroup, we get \( \dim R_u(B) - \dim R_u(P) \geq 1 \). These two inequalities yield \( i \).

Assume that \( \text{rank}(G) = 2 \) and \( B \neq P \neq G \). Hence, we get \( \dim R_u(B) - \dim R_u(P) = 1 \), by [Hum75, §30.2]. Considering the classification of irreducible root systems of rank two and counting the number of positive roots in these root systems yield \( ii \).

Assume that \( \text{rank}(G) \geq 2 \) and \( P \neq G \). Hence, there exists a simple root \( \alpha \) such that \(-\alpha \) is not a root of \( P \). Since \( \text{rank}(G) \geq 2 \) and since the root system is irreducible, there exists a simple root \( \beta \neq \alpha \) such that \( \alpha + \beta \) is a positive root. By [Hum75] [§30.2] it follows that \( \alpha \) and \( \alpha + \beta \) are distinct roots of \( R_u(P) \), which proves \( iii \). \( \square \)

**Appendix C. Two results on \( \mathbb{C}^+ \)-equivariant morphisms of surfaces**

In this section we proof two results on \( \mathbb{C}^+ \)-equivariant morphisms of surfaces that we use in the proof of Proposition 10.7. If \( S \) is an affine variety with a \( \mathbb{C}^+ \)-action, then we denote by \( S//\mathbb{C}^+ \) the spectrum of the ring of \( \mathbb{C}^+ \)-invariant functions on \( S \). In general \( S//\mathbb{C}^+ \) is an affine scheme which is not a variety. If the quotient morphism \( S \to S//\mathbb{C}^+ \) happens to be a principal \( \mathbb{C}^+ \)-bundle, then we denote the algebraic quotient by \( S/\mathbb{C}^+ \). By Rentschler’s Theorem, for a fixed point free action of \( \mathbb{C}^+ \) on the affine plane \( \mathbb{C}^2 \), the algebraic quotient of \( \mathbb{C}^2 \) is a trivial principal \( \mathbb{C}^+ \)-bundle over the affine line \( \mathbb{C} \cong \mathbb{C}^2/\mathbb{C}^+ \); see [Ren68].

**Lemma C.1.** Let \( S \) be an irreducible, quasi-affine surface and assume that \( \mathbb{C}^+ \) acts without fixed point on \( \mathbb{C}^2 \) and on \( S \). If \( f : \mathbb{C}^2 \to S \) is a dominant and \( \mathbb{C}^+ \)-equivariant morphism, then there exists a section \( X \subseteq \mathbb{C}^2 \) of the...
and a smooth affine curve $U \sim C^q$ such that $f$ induces a birational morphism $X \to f(X)$.

Proof. By [FM78, Lemma 1], there exists a $C^+$-invariant open subset $V \subseteq S$ and a smooth affine curve $U$ such that $V$ and $U \times C^+$ are $C^+$-equivariantly isomorphic. Hence, $f$ restricts on $f^{-1}(V)$ to a morphism of the form

$$(f^{-1}(V)/C^+) \times C^+ \to U \times C^+, \quad (x,t) \mapsto (f(x), t + q(x)),$$

where $q$ is a function defined on the curve $f^{-1}(V)/C^+$ and $\bar{f}$ is the morphism $f^{-1}(V)/C^+ \to U$ induced by $f$. Therefore, it suffices to find a function $p$ on $C \cong C^2/C^+$ (which corresponds to a section of $C^2 \to C^2/C^+$) such that the morphism

$$(16) \quad f^{-1}(V)/C^+ \to U \times C^+, \quad x \mapsto (\bar{f}(x), p(x) + q(x))$$

is birational onto its image. After shrinking $V$, we can assume that $\bar{f}$ is finite and étale. Fix $u_0 \in U$. One can choose $p$ such that the points

$$(u_0, p(x_1) + q(x_1)), \ldots, (u_0, p(x_k) + q(x_k))$$

are all distinct, where $x_1, \ldots, x_k$ denote the elements of the fiber of $\bar{f}$ over $u_0$. The same is still true for elements in a neighbourhood of $u_0$, as one can see by choosing an étale neighbourhood of $u_0$ in $U$ which trivializes $\bar{f}$ at $u_0$ with respect to the étale topology; see [Mil80, Chp. I, Corollary 3.12]. Hence $\bar{f}$ is injective on an open subset of $f^{-1}(V)/C^+$, i.e. it is birational onto its image. \qed

Lemma C.2. Let $S$ be an irreducible, quasi-affine surface and assume that $C^+$ acts without fixed point on $C^2$ and on $S$. If $f : C^2 \to S$ is a $C^+$-equivariant birational morphism, then there exists a section $S \subseteq C^2$ of $C^2 \to C^2/C^+$ such that $f$ restricts to an embedding on $X$.

Proof. We identify $C^2$ with $C \times C^+$ and consider it as a trivial principal $C^+$-bundle over $C$. For $\alpha \in C^*$ let

$$Z_\alpha = \{ (x, \alpha x) \mid x \in C \} \subseteq C \times C^+.$$  

We claim, that for generic $\alpha \in C^*$ the map $f$ restricts to an embedding on $Z_\alpha$. In other words, we claim that $f$ restricted to $Z_\alpha$ is injective and immersive for generic $\alpha$ (the properness is then automatically satisfied, since $Z_\alpha \cong C$). The claim then implies the statement of the lemma.

Let us first prove injectivity. Since $f$ is $C^+$-equivariant and birational, there exists a $C^+$-invariant open subset of $C \times C^+$ that is mapped isomorphically onto a $C^+$-invariant open subset of $S$. Since $C^+$ acts without fixed point, it follows that there are only finitely many $C^+$-orbits $F$ in $S$ such that the inverse image $f^{-1}(F)$ consists of more than one $C^+$-orbit. Thus, it is enough to show that $f$ is injective on $f^{-1}(F) \cap Z_\alpha$ for fixed $F$ and generic $\alpha$ in $C^*$. So let $F \subseteq S$ be a $C^+$-orbit such that there exist $k > 1$ and distinct $x_1, \ldots, x_k \in C$ such that $f^{-1}(F)$ is the union of the lines $L_i = \{ x_i \} \times C^+$, $i = 1, \ldots, k$. Moreover, there exist $\beta_i \in C^+$ such that $f|_{L_i} : L_i \to F$ is given by $t \mapsto t + \beta_i$, where we have identified the orbit $F$ with $C^+$. Injectivity of $f$ on $f^{-1}(F) \cap Z_\alpha$ for generic $\alpha$ follows, since for generic $\alpha$ we have

$$\alpha x_i + \beta_i \neq \alpha x_j + \beta_j \quad \text{for all } i \neq j.$$
Let us prove immersivity. As already mentioned, there exists an open \( \mathbb{C}^+ \)-invariant subset \( U \subseteq \mathbb{C} \times \mathbb{C}^+ \) such that \( f \) restricts to an open injective immersion on \( U \). Let \( x_0 \in \mathbb{C} \) such that \( \{x_0\} \times \mathbb{C}^+ \) lies in the complement of \( U \) in \( \mathbb{C} \times \mathbb{C}^+ \). Since there are only finitely many such \( x_0 \in \mathbb{C} \), it is enough to show that for generic \( \alpha \in \mathbb{C}^+ \) the restriction \( f|_{Z_{\alpha}} \) is immersive in the point \((x_0, \alpha x_0)\). Since \( \mathbb{C}^+ \) acts without fixed point on \( S \) and since \( f \) is \( \mathbb{C}^+ \)-equivariant, the kernel of the differential of \( f \) is at most one-dimensional in every point of \( \mathbb{C} \times \mathbb{C}^+ \). Since the tangent direction of \( Z_{\alpha} \) in the point \((x_0, \alpha x_0)\) is given by \((1, \alpha)\), we proved that \( f|_{Z_{\alpha}} \) is immersive in \((x_0, \alpha x_0)\) for generic \( \alpha \). This proves the immersivity. \( \square \)

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