Metastable states, anomalous distributions and correlations in the HMF model

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Abstract

We study the microscopic dynamics of the metastable Quasi-Stationary States (QSS) in the Hamiltonian Mean Field (HMF) model, a Hamiltonian system of N classical inertial spins with infinite-range interactions which shows a second order phase transition. In order to understand the origin of metastability, which appears in an energy region below the critical point, we consider two different classes of out-of-equilibrium initial conditions, both leading to QSS, and having respectively initial magnetization equal to one (M1 IC) and equal to zero (M0 IC). We compare the corresponding μ-space, the resulting velocity pdfs and correlations, and the eventual aging features of the microscopic dynamics. In both cases the model exhibits non-gaussian pdfs, though anomalous correlations are present only when the system is started with an initial magnetization equal to one. In the M0 IC case the relaxation to equilibrium is almost exponential, while, for M1 IC, when correlations and aging are found, the decay is a power-law and the overall behavior can be very well reproduced by a Tsallis q-exponential function. These results contribute to clarify the overall scenario, which is more complex than previously expected and stress the importance of the dynamics in the relaxation process. The nonextensive statistical mechanics formalism proposed by Tsallis seems to be valid, in the out-of-equilibrium phase, when correlations and strong long-term memory effects emerge. This regime becomes stable if the $N \to \infty$ limit is performed before the $t \to \infty$ limit.

Key words: Hamiltonian dynamics; Long-range interaction; Out-of-equilibrium statistical mechanics

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1 Introduction

There is by now a huge number of papers devoted to understanding the physical foundation and the possible applications of the nonextensive statistical mechanics formalism proposed by Constantino Tsallis in 1988 [1]. In particular, although the topic is still a matter of debate [2], the last few years have seen an important turning point for what concerns dissipative systems both low-dimensional (for example unimodal maps) [3,4,5] and high-dimensional ones (for example turbulence, plasma and nuclear physics, econophysics) [6,7,8,9,10,11]. Progress has been made also for conservative systems both for maps [12] and for high-dimensional hamiltonian systems with long-range interactions [13,14,15,16,17,18,19], but open problems still remain and need a more detailed investigation. With this scenario in mind, in this paper we present and discuss new numerical simulations relative to the Hamiltonian Mean Field (HMF) model, a system of $N$ fully coupled classical spins which has been intensively studied in the last years. It is nowadays well known[13,14,20,21,22,23,24,25,26] that, for some values of the initial energy, below the critical point, the system does not immediately relax to the Boltzmann-Gibbs equilibrium, but remains trapped in anomalous metastable states and exhibits non-Gaussian velocity distributions that can be reproduced by the probability density functions (pdfs) predicted by the Tsallis nonextensive thermodynamics [1]. Such states have been named Quasi-Stationary States (QSS) because they become stable in the thermodynamic limit if the $N \to \infty$ limit is taken before the $t \to \infty$ limit. In fact in such a case the force between spins tends to zero and the largest Lyapunov exponent vanish [13,14].

In this work we study two classes of out-of-equilibrium initial conditions that lead to QSS, and we discuss how the different initial conditions may affect the dynamics of the relaxation towards equilibrium. In particular, we concentrate our attention on the existence of correlations in the orientation angles and velocity of the spins, on the presence of anomalous velocity distributions, and on the clusters formations. The main result of this paper is that Tsallis nonextensive thermodynamics formalism seems to provide a general framework, to interpret the statistical properties of the dynamically created QSS, for Hamiltonian systems with long-range interactions [1], only when long-range correlations and fractal structures in phase space are present. Recently some critical comments [27] have been raised on a previous study of two of us [13]. In this paper we provide new calculations and detailed discussions which reply in part to this criticism and clarify also some points advanced in another work [28]. A more detailed reply to the points raised in ref. [27] will be presented elsewhere [29].

The paper is organized as follows. In section II the details of the model are summarized and the initial conditions used are discussed. Numerical results for
QSS are discussed in section III, while velocity pdfs and dynamical correlations are studied in a quantitative way in section IV. Conclusions are drawn in section V.

2 Model and initial conditions

The HMF model, consists of N planar classical inertial spins interacting through an infinite-range potential [20]. The Hamiltonian can be written as:

\[
H = K + V = \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{1}{2N} \sum_{i,j=1}^{N} \left[ 1 - \cos(\theta_i - \theta_j) \right].
\] (1)

In the latter \(\theta_i\) is the orientation angle of the \(i\)th spin and \(p_i\) the conjugate variable representing the rotational velocity, considering the mass of the rotators equal to one. The summation in \(V\) is extended to all couples of spins and is not restricted to first neighbors. In order to make \(H\) formally extensive, i.e. \(V \propto N\) when \(N \to \infty\) [1,15], the coupling constant in the potential is divided by \(N\). However, this often adopted prescription cures only a part of the problem since the energy remains non-additive, that is the system cannot be simply divided in two independent sub-systems. The magnetization \(M\), i.e. the modulus of \(\mathbf{M} = \frac{1}{N} \sum_{i=1}^{N} \mathbf{m}_i\), with \(\mathbf{m}_i = [\cos(\theta_i), \sin(\theta_i)]\), is the order parameter of the model. The latter shows a a second-order phase transition from a low-energy clustered phase with magnetization \(M \sim 1\), to a high-energy one, where the spins are homogeneously oriented on the unit circle and \(M \sim 0\). The model can be solved exactly in the canonical ensemble and the dependence of the energy density \(U = E/N\) on the temperature \(T\), is given by [20,21]

\[
U = \frac{T^2}{2} + \frac{1}{2} \left( 1 - M^2 \right).
\] (2)

The critical point is at energy density \(U_c = \frac{3}{4}\) corresponding to a critical temperature \(T_c = \frac{1}{2}\) [20]. The dynamics of HMF model can be investigated by starting the system with out-of-equilibrium initial conditions and integrating numerically the equations of motion [21]. In a special region of energy values \(\left( \frac{1}{2} < U < U_c \right)\) the results of the simulations show, for a transient regime which depends on the system size, a disagreement with the canonical ensemble. In this region the dynamics is characterized by Lévy walks and anomalous diffusion, while in correspondence the system shows a negative specific heat [23]. Ensemble inequivalence and negative specific heat have also been found in self-gravitating systems [31,32], nuclei and atomic clusters [34,35,36], though in the present model such anomalies emerge as dynamical metastable features [24,26] and vanish, if \(N\) is finite, for \(t \to \infty\). In this paper we focus on a
particular energy value belonging to the anomalous region, namely $U = 0.69$, and we study the time evolution of temperature, $\mu$-space structures, dynamical clusters in angle, velocity correlations and velocity distributions considering two out-of-equilibrium initial conditions in particular:

- the first one, used already in previous papers, consists of all orientation angles $\theta_i = 0$ and velocity $p_i$ uniformly distributed accordingly to the total energy available. In this case the potential energy is initially zero and we have the maximum of kinetic energy, while the total magnetization $M$ is equal to 1;

- in the second case considered, both angles and velocities are uniformly distributed. This implies that the total magnetization $M$ is initially zero, so that the potential energy is $\frac{1}{2}$, see eq.(2). Therefore in this case only a smaller fraction of total energy is left for the kinetic part.

For both initial conditions the total momentum is initially put to zero. In the following, for brevity, we will indicate these two initial conditions respectively by $M_1$ and $M_0$, thus referring to the initial value of the magnetization.

3 Metastable equilibrium and Quasi-Stationary States

We discuss in this section the numerical results obtained starting from $M_1$ and $M_0$ initial conditions. Different sizes of the system are studied. The accuracy of the calculations is such that one gets an energy conservation $\Delta E/E = 10^{-5}$. As in previous papers the fourth order Yoshida simplectic algorithm [30] has been used for the integration. At variance with the previous publications the averages where taken only over the ensemble and not also along the time evolution. This point will be discussed in detail later.

In fig.1 we plot, for system sizes N=500 and N=1000, the time evolution of twice the kinetic energy per particle, a quantity which is the temperature in stationarity situations. Different regimes exist according to the initial conditions considered. More specifically, when starting with $M_1$ initial condition (IC) four regimes are observable:

1. a fast decay from $M = 1$ to $M \sim 0$ in a time which is about 80-100. No sensible dependence on the size is observed;
2. a plateau region with $M(N) < M_{eq}$ and $T(N) < T_{eq}$ whose duration depends also on the size of the system as $N$ [13];
3. a power-law relaxation towards the true equilibrium value $T_{eq} = 0.476$ [14];
4. an equilibrium regime with $M_{eq} = 0.307$. 
In refs. [13,14] the kinetic energy per particle was actually computed considering not only an ensemble average, but also a time average. This was done to improve statistics, though in principle time averages are justified only when correlations do not exist. In fig. 2 the old procedure and the new one are compared. One observes that the two methods give the same plateau if a large number of events are considered in the ensemble average. The only difference is in the decay from the initial condition to the QSS regime. One can easily check that also the scaling laws found in ref. [13] continue to be valid. In particular the lifetime of the plateau diverges with $N$ and the value of the plateau temperature $T_{QSS}(N)$ converges towards the asymptotic temperature $T_{\infty} = 0.38$ as $N \to \infty$ following the same laws of refs. [13]. At variance with the criticism advanced in [27], it is evident that one can really talk of an approximate plateau for the average kinetic energy per particle, see fig. 2. The log timescale used in fig. 1 and in previous papers [13] is useful to have an overall view of the temporal behavior and does not change the macroscopic quasistationary features. This will be clearer when analyzing also the M0 IC, for which the system is initially put in the state at $M \sim 0$ and $T \sim 0.38$ and there it remains for quite a long time which also in this case depends on the size studied. In fact when starting with M0 IC only three regimes are observable:

1. a plateau region with $M \sim 0$ and $T \sim 0.38$ whose length depends on the size as $N$. This has been checked numerically but it is not shown here for lack of space;
2. a power-law relaxation towards the true equilibrium value, but faster than the previous $M1$ case;
3. an equilibrium regime with $M_{eq} = 0.307$.

In general, for both IC, the values obtained for the magnetization and the temperature in the plateau regime for $N \to \infty$ are consistent with the caloric curve (2) extended below the critical point. Therefore this average procedure is a posteriori justified and consistent with a metastable macroscopic thermodynamic description, contrary to what claimed in ref. [27].

Although both initial conditions drive the system towards the same out-of-equilibrium macroscopic values for the magnetization and the temperature in the $N \to \infty$ limit, for the QSS regime, the corresponding microscopic dynamics is very much different in the two cases. This fact can be seen when one plots the time evolution of the $\mu$-space, as done in fig. 3 for a typical event and $N = 10000$. In red we represent the rotators having at time $t = 0$ the highest velocities. In this way it is possible to visualize the kind of mixing occurring for the initial conditions considered. It is immediately evident that there are correlations and structures in the plateau regime only for M1 IC, while for M0 IC the distributions remain homogeneous with a very slow mixing of the initially fast rotators. This different behavior can be observed
also in figure 4, where we plot the pdf in angle at various times. A typical realization for \( N = 10000 \) is plotted. One observes clusters formation in the plateau regime only for \( M_1 \) while these are completely absent for \( M_0 \) IC. However, at equilibrium, the pdfs are equivalent and the plots, for both \( M_1 \) and \( M_0 \), show only one rotating big cluster corresponding to \( M = 0.307 \) which is the equilibrium magnetization (according to the canonical prediction eq. (2) with \( U = 0.69 \) and \( T = 0.476 \)).

These numerical results reinforce our previous statement that although macroscopically similar the two metastable plateaux have a very different microscopic dynamics. Further evidence is provided if one studies the entropy and free energy as a function of time.

This is done in figs. 5 and 6 respectively. To obtain the entropy we adopted a lattice over the \( \mu \)-space made up of \( n = 10000 \) cells and we used the following formula:

\[
S(t) = \langle - \sum_{k=1}^{n} f_k(t) \ln(f_k(t)) \rangle ,
\]

where \( f_k \) is the relative frequency of points for the \( k \)th cell. The brackets \( \langle ... \rangle \) denote ensemble averages. The curves plotted in the figures refer to an ensemble of 1000 events for \( N = 1000 \). This particular size was used in this case and in other calculations when a good average over the ensemble was essential in order to have a robust value, computer time being too long for greater sizes and such big ensemble averages. On the other hand no qualitative changes are anyhow observed for greater sizes. As intuitively expected from the temporal evolution of the \( \mu \)-space, the entropy grows with time towards equilibrium only for \( M_1 \) IC due to the high degree of order of this initial configuration, while it remains fixed at the maximum value for the other, much more uniform, initial condition, \( M_0 \). Correspondingly, we compute the free energy by extrapolating to a dynamical situation the equilibrium thermodynamic formula

\[
F(t) = U - T(t)S(t) .
\]

As shown in fig.6 the free energy remains constant for \( M_0 \) in the QSS regime. This implies that for this IC the system is consistently in a metastable regime, which is not however the true one that is reached only at greater times, when the free energy of the system has a real global minimum. On the other hand, for the \( M_1 \) IC, the figure shows a different temporal evolution before equilibration. After a short rapid transient decay, an almost flat region, at a different free energy value, always emerges in the QSS regime, implying also in this case the presence of a metastable situation. The use of formula (4) is thus a-posteriori justified since the temperature is almost constant in the plateau regime. It is
also very useful in clarifying the scenario that emerges which is consistent with a quasi-equilibrium situation. This results reinforce the validity of our procedure to extract average thermodynamic quantities from microscopic dynamics and can be considered a further answer to the criticism raised in ref.[27]. On the other hand, the fact that there is no plateau for the temporal evolution of the entropy for M1 IC reflects the slow microscopic dynamics observed in the evolution of the $\mu$-space structures. This is perfectly in line with the quantitative analysis performed by means of the correlation dimension in ref.[13]. Thus one can summarize this situation by saying that only at a macroscopic level the quasi-stationarity is valid, while microscopically the system never stops its slow temporal evolution. The evolution really stops only for $N = \infty$, since in this case the magnetization in the QSS is exactly zero and so is the force which is proportional to $M$. In this sense the inversion of the time limit with the size limit to infinity stabilize the QSS regime. However, when considering finite $N$ systems, the fluctuations due to finite-size noise, of the order $\sim 1/\sqrt{N}$, induce a force which is not exactly zero, and when they grow with time, the system is slowly forced towards equilibration. As shown in ref.[13] the accuracy of our calculations is much smaller than this finite-size noise.

For the M0 IC no entropy increase is observed. But also in this case some microscopic dynamics exists. This is proved by the mixing of fast particles in fig.3. However no holes or structures are present in this case and this is the reason why the entropy temporal evolution is almost flat.

It is important to stress the fact that, for very long times, both the two initial conditions lead the system towards the equilibrium expected values.

We would like to notice at this point that the slope of the entropy growth in fig.5 does not give the Kolmogorov-Sinai entropy as demonstrated numerically for maps [4,38]. In fact in our case we are considering the reduced $\mu$-space and not the complete phase-space which has to be used. Unfortunately in our case the phase space has too many dimensions and this fact prevents the direct investigation of the generalized q-entropy proposed by Tsallis along the same lines of ref.[4] for maps.

4 Anomalous velocity pdfs, velocity correlations and aging

In this section we study in a quantitative way the eventual presence of velocity correlations and memory effects in the HMF dynamics. In Fig.7 we report the velocity pdfs for the plateau regime (at $t = 1000$) and at equilibrium ($t = 1000000$) and for the case $N=10000$. For both IC, the initial velocity pdfs quickly acquire and maintain during the entire duration of the metastable state a non-Gaussian shape. But while for M1 IC they can be fitted with a
truncated generalized Tsallis pdf [13], this is not possible for M0 IC. In fact in this case the pdf shows a flat almost rectangular shape, which clearly reveals the fact that structures and correlations, present in the other case, are here missing. Finally, for very long times the system reaches always the equilibrium distribution and the agreement with the gaussian is almost perfect.

A quantitative way to estimate the velocity correlations is the calculation of the two-time autocorrelation function, defined as

$$C(t, 0) = \frac{\langle P(t) \cdot P(0) \rangle - \langle P(t) \rangle \cdot \langle P(0) \rangle}{\sigma_p(t)\sigma_p(0)},$$

(5)

where $P = (p_1, p_2, \ldots, p_N)$ is the N-component velocity vector and the brackets $\langle ... \rangle$ indicate the average over the ensemble, while $\sigma_p(t)$ and $\sigma_p(0)$ are the standard deviations at time $t$ and at time zero. In fig.8 we plot the velocity autocorrelation functions for M1 and M0 IC in the QSS regime, obtained for the case N=1000 and an average over 500 events. It is immediately evident that for M0 the autocorrelation decays faster than for M1 IC. This result clarifies the controversial numerical results obtained in ref. [28] by using a slightly different, but equivalent, definition for the autocorrelation function. In fact we do find an almost exponential behavior only for the M0 IC, while in ref. [28] the author claimed to find an exponential in both cases. In general we can reproduce the autocorrelation decay by means of the q-exponential function

$$e_q(x) = A \left[1 - (1 - q) \frac{x}{\tau} \right]^{\frac{1}{1-q}},$$

(6)

proposed by Tsallis in his thermodynamic nonextensive q-formalism [1], where $A$ is the saturation value and $\tau$ the characteristic time. One gets the usual exponential for $q = 1$. In this way we can quantitatively discriminate between the two different initial conditions. In fact we get for M1 the value $q = 1.55$, revealing a power law scaling, while for M0 we get a value of q much smaller, i.e. $q = 1.12$. This means that for M0 the decay tends to be exponential (an exact exponential is obtained when $q = 1$). This almost exponential decay is also evident in the inset of the figure, where we show the same data in linear-log scale. An exponential fit is also reported for comparison. Such numerical results may be considered a quantitative confirmation of the different microscopic dynamics of the system according to the particular initial condition, in spite of the macroscopic presence of QSS for both cases.

It is interesting to investigate also if the relaxation process observed in fig.8 depends on the history of the system, i.e the possible existence of strong long-term memory effects, the so-called aging phenomenon. This is typical of glassy systems but it has also been found in a great variety of complex systems [39,40,41]. Recently it has been observed also in the HMF model.
by Montemurro, Tamarit and Anteneodo [18]. In the following we perform a similar analysis to study this effect for both M1 and M0 IC, but considering only the N-component velocity vector defined previously. In their case also angles were taken into account in the state vector entering the correlation function and only M1 IC were considered. The autocorrelation function to be considered is that one defined before but calculated with respect to a set of four different waiting times $t_w$ (that plays the role of age of the system)

$$C(t + t_w, t_w) = \frac{\langle P(t + t_w) \cdot P(t_w) \rangle - \langle P(t + t_w) \rangle \cdot \langle P(t_w) \rangle}{\sigma_p(t + t_w)\sigma_p(t_w)}.$$  

(7)

In Fig.9 (a) we show the autocorrelation decay for M1 IC, while in Fig.10 (a) that one for M0 IC. If aging were not present, all the curves should overlap into one, because the function $C(t + t_w, t_w)$ would depend only on the difference between the two times $t + t_w$ and $t_w$, i.e. $C(t + t_w, t_w) = C(t)$, and the decay would be exponential. Actually this is not what happens as shown in the figures.

For M1 IC aging properties surely exist, i.e. it can be observed a characteristic two times dependence on $t$ and $t_w$ and the data can be scaled onto one curve, panel(b), considering as discussed in ref.[42]:

$$C(t + t_w, t_w) = f\left(\frac{t}{t_w}^\beta\right),$$  

(8)

with $\beta = \frac{1}{4}$. In such a way we obtain, for the tail a power law decay, i.e. $f(t/t_w^\beta) \sim (t/t_w^\beta)^{-\lambda}$ with scaling parameter $\lambda = 1.54$ that is here characteristic of a slow relaxation dynamics and long-term memory effects. A q-exponential curve can reproduce very well this behavior with $q = \frac{\lambda + 1}{\lambda} = 1.65$, $A = 0.7$ and $\tau = 60$. Notice that the scaling indices are different from those found in ref.[18]. A systematic study of these scaling exponents would be desirable, but is beyond the scope of the present work and we postpone it to future investigations. As discussed in ref.[18] the origin of aging in the HMF model is not completely clear. The most likely scenario is that of a weak ergodicity-breaking. As originally proposed by Bouchaud for glassy systems in ref.[40], the latter occurs when the phase-space is a-priori not broken into mutually inaccessible regions, as in the true ergodicity-breaking case, but the system can remain trapped for very long times in some regions of the complex potential landscape. In HMF the weak ergodicity-breaking can be related to the complex dynamics generated by the vanishing of the largest Lyapunov exponent and by a sort of dynamical frustration due to the existence, in the QSS regime, of different small clusters (see fig.4). These clusters compete in trapping more and more particles until only one of them remain when the system reach the standard equilibrium.
On the other hand, a different behavior is obtained for M0 IC, as shown in fig. 10. The same scaling law used for the M1 IC case does not seem to apply, see panel (b), and the decay is now more rapid. In this case it is not so clear as before if aging does really exist, but again a quantitative difference emerges for the two dynamics. A more detailed study in this respect is left for the future.

Before closing, it is important to notice that the aging observed is not restricted to the QSS region, but extends also to the following relaxation regime towards equilibration.

5 Conclusions

In this paper we have studied the microscopic dynamics towards relaxation in the HMF model, a Hamiltonian system of fully coupled inertial spins. Two different out-of-equilibrium initial conditions which lead to metastable long-living quasi-stationary states (QSS) have been studied. Our numerical results indicate that the dynamics of the relaxation process is extremely rich and more complex than what previously thought. We have found that for a certain class of initial conditions, i.e. M1 IC, one can have metastable QSS with non Gaussian velocity pdfs which can be reproduced with truncated q-exponential curves and for which a finite size scaling seems to apply [13,14]. We have shown that in such a regime there are also strong velocity correlations in time which decay as a q-exponential. Moreover, confirming similar studies [18], in correspondence of this metastable states and in the following relaxation regime, aging in the velocity correlations has been found. Such a strong long-term memory effect can be attributed to a glassy-like weak ergodicity-breaking due to a sort of dynamical frustration which disappears at equilibrium. These new numerical facts, added to the anomalous diffusion [21], the structures in angle and the weak mixing produced by a vanishing Lyapunov exponent [13,14], previously observed for this out-of-equilibrium regime, although do not constitute a rigorous proof, certainly provide with a strong indication that the nonextensive formalism proposed by Tsallis could probably be applied also for Hamiltonian many-body systems, especially when the infinite size limit is performed before infinite time limit. A rigorous link between the entropic index $q$ and the dynamical properties of nonextensive Hamiltonian many-body systems is still missing and must be found in order to confirm definitely Tsallis formalism. Such metastable QSS provide an ideal benchmark for testing the nonextensive theory, a project that we plan to continue in future works. We have also presented numerical evidence that when adopting a different class of out-of-equilibrium initial conditions, i.e. M0 IC, one can have also metastable QSS, but with no structures in the $\mu$-space and a fast (almost exponential) decay of velocity correlations. In this case nonextensive statistics certainly
does not apply. Open questions, concerning aging for example, for this M0 IC case still remain and will be studied in the future.

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Fig. 1. We show the temporal behaviour of twice the average kinetic energy per particle for the two different initial conditions considered M1 and M0 and for the system sizes $N = 500$ (dashed lines) and $N = 1000$ (full lines). Ensemble averages over 1000 events were considered.

Fig. 2. Comparison of twice the average kinetic per particle (temperature) considering two kind of averages: i) ensemble average (curves); ii) time and ensemble average (open circles). The latter has been used in previous papers to improve the statistics, while the former is the one adopted in the present paper. The curves plotted show that for ensemble averages over 1000 events the two procedures coincide. This comparison improves when considering larger sizes for which a smaller number of events is sufficient in order to diminish the fluctuations. See text for further details.

Fig. 3. Temporal snapshots of the $\mu$-space for the two initial conditions considered. We have structures in one case (M1) which disappear after the plateau, while for the second initial condition (M0) the phase space is always homogeneously filled. We put in red the rotators which have the highest velocities at time $t = 0$. This helps in order to understand the following dynamical mixing. The plots refer to a single typical run for $U = 0.69$ and $N = 10000$.

Fig. 4. Pdf in angle vs time. This figure is a complement of the previous one and shows that, in the metastable QSS regime, we have the dynamical formation of small clusters in competition between each other for M1 IC, but not for M0 IC. However at equilibrium ($t=1000000$) for both IC only one big rotating cluster remains, see text.

Fig. 5. Time evolution of the entropy $S$ for M0 and M1 initial conditions. See text for further details.

Fig. 6. Time evolution of the free energy $F$ for M0 and M1 initial conditions. In correspondence of the metastable regime the free energy show a flat zone before going towards a minimum at equilibrium. See text for further details.

Fig. 7. Velocity pdfs in the QSS ($t = 1000$), top panel, and very close to equilibrium ($t = 1000000$), bottom panel, for the two different initial conditions considered. The simulations refer to a size $N = 10000$. The equilibrium Gaussian pdf is also plotted for comparison as full curve. In the QSS regime, one has a peaked pdf for M1 IC and a flat pdf for M0 IC. In both cases strong deviations from the Gaussian are evident. While the pdf for M1 can be reproduced with a generalized and truncated Tsallis pdf (see ref.[13]) this is not possible for M0 IC.
Fig. 8. Velocity correlation functions, in the QSS regime, for the two initial conditions considered M1 and M0. Both of them can be reproduced by a q-exponential also shown as full (M1 IC) and dashed (M0 IC) curves. The corresponding parameters are also reported. We get $q = 1.55$ for M1 IC and $q = 1.12$ for M0 IC. Notice that in the latter case the value of q is close to 1 meaning that the decay tends to be exponential. This is clearly seen in the inset where a linear-log scale was used and an exponential fit is also reported for comparison. See text for further details.

Fig. 9. (a) We plot for M1 IC, $N=1000$ and an ensemble average over 100 events the two-time autocorrelation function (7). Several waiting times are shown. (b) We plot the curves in (a) by considering a scaling factor $(t_w)^{1/4}$. The final part of the tail is a power law. The scaled curves can be reproduced by a q-exponential with $q = 1.65$, $A = 0.7$ and $\tau = 60$, full curve.

Fig. 10. (a) The same as fig.9 but for M0 IC. (b)The curves shown in (a) are scaled by the same factor $(t_w)^{1/4}$ used previously for M1 IC. In this case, however the curves do not collapse into one and we get a faster decay.
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