Tales of D0 on D6 Branes:
Matrix Mechanics of Identical Particles

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We investigate a class of matrix model which describes the dynamics of identical particles in even dimensional space. We show that the degrees of freedom after some constraints are implemented is proportional to particle number and consist of those for positions and internal degrees. The particle dynamics is given by the metric on the smooth moduli space. The moduli space metric for two particles is found. The size of tightly packed $N$ particles grows like $\sqrt{N}$. Our matrix model is related to the matrix model for fractional quantum Hall effect, the ADHM formalism of $U(1)$ instantons on noncommutative space, and supersymmetric D0 branes on D6 branes with nonzero B-field in type IIA theory.

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Recently, there have been some studies to understand the dynamics of identical particles in terms of the matrix gauge theory. For $N$ particles in $D + 1$ dimensional space, the particle coordinates become $D + 1$ square matrices of size $N$. The particle exchange symmetry becomes a part of the gauge symmetry and so is included in the dynamics classically. Usually the matrix theory has more degrees of freedom than ones necessary for particle dynamics and further constraints are imposed to reduce the degrees of freedom. A typical example is the dynamics of instanton in terms of the matrices in the ADHM construction [1]. These matrices are constrained by the three D-term constraints and by the Gauss law and so the number of net degrees of freedom is proportional to the instanton number [1, 2]. The number of degrees of freedom for a single instanton is the sum of that for the position and that for the internal degrees. Another example is the Chern-Simons inspired matrix model which describes the fractional quantum Hall fluid [3, 4].

In this work we study a matrix theory which describes identical particles in even dimensional space. Our matrix theory is quadratic in time derivative and so naturally describes the nonrelativistic Newtonian dynamics of identical particles. We count the number of degrees of freedom left after the constraints are imposed, which turns out to be proportional to particle number. For the relative moduli space for two particle case, we extend our result on four dimensions to other even dimensions. The size of tightly packed $N$ particles grows like $\sqrt{N}$.

Our theories appear in many context. On two dimensional space it is somewhat related to the first order matrix theory introduced by Susskind and Polychronakos. On four dimensional space with single flavor it is related to the ADHM formalism of $U(1)$ instantons on noncommutative four space. On six dimensional space it describes the low energy dynamics of supersymmetric D0 branes on D6 branes, which is possible with nontrivial background NS $B$ field in type IIA theory as discussed by Witten [5]. Some of issues studied here has been discussed in somewhat simpler level in Ref. [6].

The ADHM description of the low energy dynamics of instanton is a gauge theory of matrices with additional constraints, which can be solved in terms of the moduli parameter. Once the constraints are solved and the gauge degrees of freedom are removed, the instanton moduli space metric describes the motion of instantons. The metric on the moduli space is singular and the instanton dynamics is incomplete.

The ADHM formalism arises naturally from D0 brane on D4 brane point of view. Its low energy dynamics of D0 brane is given by the Yang-Mills matrix mechanics model with eight supersymmetries. The relevant fields are adjoint and fundamental scalar fields. These matrices are constrained in the ADHM formalism by the three D-term potentials [1, 2].

When nonzero background $B$ field introduced on D4 branes, their field theory becomes Yang-Mills
theory on noncommutative space \cite{7,8}, which can be obtained from a certain limit of open string theory on D4 branes with background B-field. In this case, one can have nonsingular moduli space for instantons \cite{9}. (See Ref. \cite{10,11} for a recent review.) The B-field appears as the FI term in the matrix theory, which blows up the singularities of the instanton moduli space with a finite scale. The ADHM formalism is simplified when the gauge group is $U(1)$ \cite{12,13}. Our class of the matrix model includes only the ADHM formalism for the $U(1)$ group.

In two dimensional space Susskind and Polychronakos proposed and studied the matrix model for charged particles on the lowest Landau level and so their Lagrangian is first order in time derivative and found to be related to the noncommutative Chern-Simons theory \cite{3,4}. The space of matrices become a classical phase space. After the Gauss law is imposed and the gauge degrees are removed, the remaining degrees of freedom are proportional to the number of particles. Especially the quantum mechanical eigenvalues and eigenfunctions of a quadratic Hamiltonian can be found exactly \cite{14}.

On the other hand our matrix model is second order in time derivative and so it describes the real time evolution of particles. After reducing the degrees of freedom, the motion of identical particles is described by the coordinates of the moduli space. The moduli space, which is the coordinates space in our model, appears as the phase space in the matrix model of Susskind and Polychronakos.

In the type IIA theory, D0 branes feel repulsive force from the parallel D6 branes. However when there exists a uniform background B field above critical strength, D0 branes can be attracted to D6 branes and form a BPS configuration. The D0 brane physics is described by the matrix model \cite{5}. This is exactly the matrix model we study in the six dimensions.

Noninteracting identical particles on flat $R^D$ have a singular configuration space $(R^D)^N/S_N$ where $S_N$ is the permutation group of $N$ particles. The configuration space is singular when particles come together. On even dimensional space our matrix model is a natural blow-up procedure of these singularities. There is a natural length scale for this blow up, making particles to carry effectively finite size core. When many identical particles described by the matrix come together, their kinetic energy, which determines the moduli space metric, makes them to behave somewhat like incompressible fluid even though there is no force between them. In our case the total volume occupied by tightly packed $N$ particles in $D = 2k$ dimensions grows as $N^k$, which is faster than the incompressible gas except $k = 1$. As we will see, the kinetic energy also entails the long range interaction between particles. Thus, one could regard the matrix model to describe a peculiar class of particles living on even dimensional space.

We start with the matrix model description applicable for all these cases. The matrix model for $N$ particles on $2k$ dimensional space is described by $N \times N$ complex matrices $Z_i$ with $i = 1, \ldots, k$
and $N \times M$ complex matrix $\psi$. The kinetic term for these matrices is

$$\mathcal{L} = \text{tr} \left( D_0 Z_i D_0 \bar{Z}_i + D_0 \psi \bar{D}_0 \psi \right).$$

(1)

The $U(N)$ local gauge transformation leads to $Z_i \rightarrow U Z_i \bar{U}$, $\psi \rightarrow U \psi$ and $A_0 \rightarrow U A_0 \bar{U} - i \partial_0 U \bar{U}$. The $U(M)$ global flavor symmetry leads to $\psi \rightarrow \psi \bar{V}$. The matrices $Z_i, \psi$ have $2kN^2 + 2NM$ real parameters. Out of which $N^2$ are gauge parameters, leading to $(2k - 1)N^2 + 2NM$ physical parameters, which is clearly much larger than the number of particles. The spatial rotational group $SO(2k)$ is broken to $U(k)$. (See for a recent investigation of the model with $k = 1/2$ and so particles living on a line where the matrix degrees of freedom are proportional to the particle number [13].)

Additional constraints are needed for these matrices. We choose them to be

$$\sum_{i=1}^{k} [Z_i, \bar{Z}_i] + \psi \bar{\psi} = \zeta 1_N,$$

(2)

$$[Z_i, Z_j] = 0 \quad i, j = 1, 2, ..., k,$$

(3)

These can be regarded as the minimum of a potential of matrices. In the theory with four supersymmetries the first one (4) can be regarded as the minimum condition of the $D$ term potential and the second one as the minimum condition of the $F$ term potential.

For the two dimensional case, there is only one constraint (4). The number of the maximal supersymmetry is four and the constraint is the minimization condition of the $D$-term potential. For the four dimensional case with a single flavor $M = 1$, the above constrains are reduced version of the ADHM constraints for the $U(1)$ instantons on noncommutative four space, and so the number of underlying supersymmetry is eight. For the theories of the six dimensional system, the number of maximal supersymmetry is again four and the constraint (3) comes from the superpotential $\text{tr} Z_1 [Z_2, Z_3]$. For other cases there is no obvious supersymmetry is as there is no suitable superpotential for the constraint (3).

For one particle with $N = 1$ Eqs (2) and (3) have the trivial solution where $Z_i$ are arbitrary numbers and

$$\psi \bar{\psi} = \zeta 1_N.$$

(4)

The $\psi$ space is $S^{2M-1}$ and becomes $CP^{M-1}$ after we mod out by gauge symmetry $U(1)$. The particle position $Z_i$ has $2k$ parameter and the internal degrees of freedom has $2(M - 1)$ parameters. The total degrees of freedom for a single particle is then $2k + 2M - 2$.

The supersymmetric quantum dynamics of particles with $M \geq 2$ is possible on six dimensions. The quantum mechanics of the sigma model with moduli space metric has 4 supersymmetry as the maximal supersymmetry. For a single particle with nontrivial internal degrees of freedom, there
are $M$ quantum mechanical ground states because they are normalizable harmonic forms [16] and their total number can be identified with the Euler number of $CP^{M-1}$ by Hodge theorem [17].

When we consider many particles $N \geq 2$, one may wonder how degrees of freedom are there. Naively we expect that each particle carries the same number and so the total number of free parameters is the sum of that for constituent particles. However it is not obvious at all from Eqs. (2) and (3) imposed on the matrices $Z_i, \psi$. Eq. (3) seems overconstraining. Indeed the constraints in Eq. (3) are not independent. In Appendix A we show that the correct counting leads to the $N(2k + 2M - 2)$ degrees of freedom as expected. The original kinetic term then defines a Kähler space of dimension $N(2k + 2M - 2)$ with the induced Kähler form from the flat Kähler form corresponding to the kinetic energy. Our matrix model is a natural method to define a class of Kähler spaces, which is related to the blow-up of the configuration space of identical particles.

It is very hard to solve the constraints for arbitrary number of particles $N$ and the internal degrees of freedom characterized by $M - 1$. When $M \geq 2$, there are some internal degrees of freedom and so one can overlap the particles at a same point. Especially with $M \geq N$, one can solve Eqs. (2) and (3) with $Z_i = 0$ and the $\psi$ space satisfying Eq. (4) becomes Grassmannian manifold $U(M)/U(N) \times U(M - N)$ after moding out the gauge group, as identified by Witten [14]. When there is no internal degrees of freedom, $N$ particles act like having a 'hard core', given not by force but by the metric of the kinetic term. When there are enough internal space $M \geq 2$, particles of number less than or equal to $M$ can come together to the top on each other.

On two dimensional space $k = 1$ with $M = 1$, the most general solution of the constraint (2) has been known before. (See for example Ref. [4].) A further discussion of our model on two dimensions and its relation to Polychronakos system is given in Appendix B. In four dimensional space the solution for two particle case has been found before [18] and has been used by us to show the moduli space metric to be the Eguchi-Hanson metric. In the following we extend this analysis to arbitrary even dimensions.

With no internal degrees of freedom $M = 1$ and so there is only single flavor $\psi$. In this case one can find the generic two particle solutions ($N = 1$) for these constraints by triangularizing the commuting complex matrices $Z_i$ simultaneously by unitary transformations,

$$Z_i = w_i12 + \frac{z_i}{2} \begin{pmatrix} 1 & \sqrt{2b} \frac{a}{a} \\ 0 & -1 \end{pmatrix}, \quad \psi = \sqrt{\zeta} \left( \begin{array}{c} \sqrt{1 - b} \\ \sqrt{1 + b} \end{array} \right),$$

where

$$a = \frac{\sum_i |z_i|^2}{2\zeta} \geq 0, \quad b = \frac{1}{a + \sqrt{1 + a^2}}.$$

Notice that the remnant $U(1)^2$ gauge is used for fixing the phase of $a$ and $b$. We can identify $w_i$.
and $z_i$ as the gauge invariant position of two particles at all separation of two particles not only at large separation since $w_i \pm z_i/2$ are the eigenvalues of $Z_i$. There is still the remaining discrete gauge symmetry
\begin{equation}
U = \left( \begin{array}{cc}
-\frac{b}{\sqrt{2ab}} & \frac{\sqrt{2ab}}{b} \\
\sqrt{2ab} & b
\end{array} \right),
\end{equation}
which changes the sign of $z_i$, leaving $\psi$ invariant. This discrete gauge symmetry identifies the positions of two identical particles and making the relative moduli space at large separation to be $R^{2k}/Z_2$. The orbifold singularity at the origin gets blow-up in the matrix theory. At the coincident limit $z_i = 0$, the space of the solution (5) becomes a submanifold $S^{2k-1}$. We have to mode out this by the $U(1)$ gauge transformation $\text{diag}(e^{i\beta}, 1)$ because this transformation gets restored at this limit $\sqrt{1-b} = 0$. Thus, the submanifold of moduli space at the coincident limit is $S^{2k-1}/U(1)$ which can be identified with $CP^{k-1}$ because $S^{2k-1}$ is $U(1)$ fiber bundle over $CP^{k-1}$ as we know [19]. Here $CP^0$ denotes just one point.

To obtain the tangent vectors on the moduli space, we consider the infinitesimal variation of the above solution made of the infinitesimal change in moduli parameters and the infinitesimal gauge transformation,
\begin{equation}
\delta Z_i = dZ_i - i\{\delta \alpha, Z_i\}, \quad \delta \psi = d\psi - i\delta \alpha \psi .
\end{equation}
We need to fix the gauge variation of the above solutions under small variation, which can be achieved by solving the background gauge condition. The initial configuration for the matrices is the solution (5) and its ‘initial velocity’, which is characterized by the variation. It should satisfy the Gauss law constraint, which is identical to the background gauge fixing condition,
\begin{equation}
\sum_{i=1}^{k}(\{\delta Z_i, \bar{Z}_i\} - \{Z_i, \delta \bar{Z}_i\}) + \delta \psi \bar{\psi} - \psi \delta \bar{\psi} = 0 .
\end{equation}
This condition fixes the infinitesimal gauge transformation uniquely,
\begin{equation}
\delta \alpha = \frac{ib}{2\sqrt{1+a^2}} \left( \frac{\partial a - \partial \bar{a}}{1-b} \frac{\sqrt{2ab}}{\sqrt{ab}} - \frac{\sqrt{2ab}}{\sqrt{ab}} \frac{\partial a - \partial \bar{a}}{1+b} \right),
\end{equation}
where $\partial a = \bar{z}_i dz_i/(2\zeta)$ and $\partial \bar{a} = z_i d\bar{z}_i/(2\zeta)$. Knowing the tangent vector $(\delta Z_i, \delta \psi)$, we can obtain the metric in the usual way
\begin{equation}
ds^2 = \text{tr} \left( \sum_{i=1}^{k} \delta Z_i \delta \bar{Z}_i + \delta \psi \delta \bar{\psi} \right).
\end{equation}
Incidentally, $\delta \psi = 0$ in our two particle case. Then, the moduli space metric becomes
\begin{equation}
ds^2 = ds_{cm}^2 + ds_{rel}^2,
\end{equation}
where the center of mass and relative metrics are

$$ds_{cm}^2 = 2 \sum_{i=1}^{k} dw_i d\bar{w}_i,$$

$$ds_{rel}^2 = \frac{\sqrt{r^4 + 4\zeta^2}}{2r^2} \sum_{i=1}^{k} dz_id\bar{z}_i - \frac{2\zeta^2}{r^4 \sqrt{r^4 + 4\zeta^2}} \sum_{i,j=1}^{k} \bar{z}_iz_jdz_id\bar{z}_j$$

with $r^2 = \sum_i |z_i|^2$. At the large $r = \sqrt{\sum_i |z_i|^2}$ limit the metric approaches that of the flat space with the correction of order $1/r^2$, which is a sign of long-range interaction. The above metric $ds_{rel}^2$ for any $k$ is a Kähler metric as $g_{ij} = \partial_i \bar{\partial}_j K$ with the Kähler potential,

$$K_{rel} = \frac{1}{2} \sqrt{r^4 + 4\zeta^2} + \frac{\zeta}{2} \ln \frac{\sqrt{r^4 + 4\zeta^2} - 2\zeta}{\sqrt{r^4 + 4\zeta^2} + 2\zeta}.$$

The Kähler form could be obtained by $K = \frac{i}{2} \partial \bar{\partial} K$. This metric has $U(k)$ holomorphic isometry transforming $z_i \rightarrow U_{ij} z_j$, which is a subgroup of the spatial rotation $SO(2k)$. In fact, the metric of the $k = 2$ case has hyperKähler structure and is identified with the standard Eguchi-Hanson manifold $\mathbb{C}P^1$, where the isometry $SU(2) \subset U(2)$ becomes triholomorphic. Basically the $k = 1, 3$ cases are related to supersymmetric theory of four real supercharges but the $k = 2$ case is related to theory of eight real supercharges. Therefore, the metric of the $k = 2$ case has the additional structure and is Ricci flat. Note that the metrics of the $k = 1, 3$ cases are not non Ricci flat Kähler.

To explore the region near the origin of the relative metric $r \simeq 0$ where two particles come together, we first note that the Fubini-Study metric on $CP^{k-1}$ can be given by

$$ds_{FS}^2 = \frac{1}{r^2} \left( \sum_i dz_id\bar{z}_i - \frac{1}{r^2} \sum_{i,j} \bar{z}_iz_jdz_id\bar{z}_j \right).$$

After the following transformation $z_i = \lambda q_i$, $i = 1, 2, ..., k-1$, and $z_k = \lambda$ with $\rho \equiv (\sum_{i=1}^{k-1} |q_i|^2)^{1/2}$, the above metric (16) becomes the standard one on $CP^{k-1}$,

$$ds_{FS}^2 = \frac{\sum_{i=1}^{k-1} dq_id\bar{q}_i - \sum_{i,j=1}^{k-1} \bar{q}_iq_jdq_id\bar{q}_j}{1 + \rho^2}.$$

The metric $ds_{FS}^2$ is normalized so that the Ricci tensor satisfies $R_{ij} = 2k\delta_{ij}$. The $CP^{k-1}$ space is Kähler with Kähler form $K$ which is locally exact, $K = dA$. On $R^{2k}$, $\sum dz_id\bar{z}_i = dr^2 + r^2 d\Omega_{2k-1}^2$ with $\Omega_{2k-1}^2$ being the metric on the unit $S^{2k-1}$. The unit $S^{2k-1}$ sphere can be identified with $U(1)$ fiber bundle over $CP^{k-1}$ and its metric becomes

$$d\Omega_{2k-1}^2 = (d\theta + A)^2 + ds_{FS}^2.$$

(18)
Thus, the relative metric (14) becomes

\[ ds^2_{rel} = \frac{r^2}{2\sqrt{r^4 + 4\zeta^2}} \left( dr^2 + r^2 (d\theta + A)^2 \right) + \frac{1}{2} \sqrt{r^4 + 4\zeta^2} \, ds^2_{FS}, \]  

(19)

where the range of \( \theta \) is \([0, \pi]\) instead of \([0, 2\pi]\) as we have identified \( z_i \) with \(-z_i\). With change of variable \( v = r^2 \) the above metric becomes \( ds^2 \approx \left( dv^2 + v^2 d(2\theta)^2 \right)/\left(8\zeta + 2\zeta ds^2_{FS} \right) \) near \( r \approx 0 \). Thus, the above metric becomes the smooth metric of \( R^2 \times CP^{k-1} \) near the origin.

By the coordinate transformation

\[ u^4 = \frac{r^4}{4} + \zeta^2, \]  

(20)

we can put the metric and Kähler potential in the following form

\[ ds^2_{rel} = \frac{du^2}{1 - \frac{\zeta^2}{u^4}} + u^2 \left( ds^2_{FS} + (1 - \frac{\zeta^2}{u^4}) (d\theta + A)^2 \right), \]  

(21)

\[ K_{rel} = u^2 + \frac{\zeta}{2} \ln\left( \frac{u^2 - \zeta}{u^2 + \zeta} \right). \]  

(22)

For the \( k = 2 \) case there exists unique self-dual middle harmonic form which corresponds to the supersymmetric normalizable ground state of the maximal eight supersymmetric extension \[18\] provides the wave functions of the threshold bound state of two \( U(1) \) instantons on noncommutative four space. For the \( k = 1, 3 \) cases, it is not clear whether there exists any normalizable harmonic forms which correspond the supersymmetric normalizable ground states of the maximal four supersymmetric extension.

Having studied two particle case, let us consider briefly \( N \) particle case with \( M = 1 \). As all \( Z_i \) are commuting due to the constraint \( 3 \), we can put all \( Z_i \) as upper triangular matrices simultaneously. From the analogue with two particle system, we expect that all diagonal elements of \( Z_i \) vanishes when all particles are packed tightly together. (Here we put the center of the mass position at the origin.) Then we can find the solution of two constraints \( 2 \) and \( 3 \). Only nonvanishing components of \( Z_i \) are

\[ (Z_i)_{a,a+1} = \sqrt{a\zeta} u_a, \quad a = 1, 2, ..., N - 1, \]  

(23)

\[ \psi_N = \sqrt{N\zeta}, \]  

(24)

where \( u_a \) is \( k \)-dimensional unit complex vector. We have used one of the diagonal \( U(1) \)'s subgroups to make \( \psi \) vector real. Further use of the diagonal \( U(1) \) subgroups leads to identification \( u_a \sim e^{i(\theta_a - \theta_{a+1})} u_a \), making each \( u_a \) vectors to belong to \( CP^{k-1} \). Thus the solution space of \( N \) particles
for the relative motion becomes \((CP^{k-1})^{N-1}\) when all particles come together. For this solution

\[
\text{tr} Z_i \bar{Z}_i = \sum_{a=1}^{N-1} a \zeta = \frac{N(N-1)}{2} \zeta.
\] (25)

While the particles are indistinguishable, we could regard the radius square of the \(a\)-th particle is \((a-1)\zeta\). The radius of the outmost particle is \(\sqrt{(N-1)\zeta}\) and so the total volume in \(2k\) dimension will be proportional to \(N^k \zeta^k\) in large \(N\) limit. For particles in two dimensions, the minimum volume grows like \(N\) which shows that they are incompressible. For particles in higher dimensions \((D = 2k)\), the volume grows faster than \(N\), showing the existence of additional mechanism. It shows nontrivial interaction between particles in short distance. Such phenomena is not new: the volume of nonabelian core of tightly packed \(N\) BPS monopoles grows as \(N^3\) in three dimensions. The solution space of the packed particles, \((CP^{k-1})^{N-1}\) is trivial for the two dimensional case and so the configuration is rotationally invariant. That is nontrivial for the higher dimensional case, and so is not invariant under \(U(k)\) subgroup of the rotational group.

In summary we have explored a class of matrix model which describes identical particles in even dimensions. The number of free parameters of the theory turns out to be the sum of that for individual particles. The dynamics is described by the moduli space metric on the blow-up space of the singular space \((C^k)^N/S_N\). We studied the two particle moduli space in detail. We show that the minimum volume of \(N\) particles grows like \(N^k\).

Our model in two spatial dimensions and in four dimension with larger number of flavor \(M \geq 2\) does not have the direct interpretation as the solitons in the field theory or the string theory. It would be interesting to find such field theoretic model. Finally, it would be a challenge to find similar matrix model for supersymmetric D0 branes on D8 branes with nonzero B field.

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Appendix A.

To find out the number of free parameters in Eqs. (2) and (3), let us start by noting all commuting square matrices can be triangularized simultaneously by unitary matrices. We triangularize all $Z_i$ matrices. After triangularization, we count the number of free real parameters. To do that, we want to show that $[Z_i, Z_{i+1}] = 0$ for $i = 1, \ldots, k - 1$, then $[Z_i, Z_j] = 0$ for arbitrary $i, j$. Let us first start by showing that this is true for three upper triangular complex matrices $Z_1, Z_2, Z_3$.

Lemma. $[Z_1, Z_2] = [Z_2, Z_3] = 0$ implies $[Z_3, Z_1] = 0$ for generic upper triangular matrices.

Proof) We prove this by induction. Suppose the result holds for $k \times k$ matrices. Then for $(k + 1) \times (k + 1)$ $Z_i$’s, we write

$$Z_i = \begin{pmatrix} S_i & v_i \\ 0 & z_i \end{pmatrix},$$  \hspace{1cm} (A.1)

where $S_i$ is a $k \times k$ upper triangular matrix, $v_i$ is a $k$ dimensional column vector and $z_i$ is a number. Then

$$[Z_i, Z_j] = \begin{pmatrix} [S_i, S_j] & S_i v_j + z_j v_i - S_j v_i - z_i v_j \\ 0 & 0 \end{pmatrix}. \hspace{1cm} (A.2)$$

Now from $[Z_1, Z_2] = [Z_2, Z_3] = 0$ we have $[S_1, S_2] = [S_2, S_3] = 0$. Then by induction hypothesis, it implies $[S_3, S_1] = 0$. So the only thing need to prove is

$$S_3 v_1 + z_1 v_3 - S_1 v_3 - z_3 v_1 = 0. \hspace{1cm} (A.3)$$

Indeed, this can be easily shown using the corresponding equations from (1,2) and (2,3) pairs. More precisely, after short calculations we get

$$(S_2 - z_2)(S_3 v_1 + z_1 v_3 - S_1 v_3 - z_3 v_1) = 0. \hspace{1cm} (A.4)$$

But in general the determinant of $S_2 - z_2$ is not zero since $\det(S_2 - z_2) = \prod_{a=1}^{k}((S_2)_{aa} - z_2)$ and the diagonal components of $S_2$ are not constrained by the commutators. Q.E.D.

For four triangular matrices such that $[Z_1, Z_2] = [Z_2, Z_3] = [Z_3, Z_4] = 0$, the above lemma shows that $[Z_1, Z_3] = 0$. Thus by using the above lemma again for $Z_1, Z_3, Z_4$, we see $[Z_1, Z_4] = 0$. By induction, then it should be true for arbitrary number of $Z_i$’s.

For each triangular $Z_i$, there are $N(N + 1)$ real parameters. The total number of parameters of triangular $Z_i$’s and $\psi$ are $kN(N + 1) + 2NM$. The remaining gauge parameter after triangularization is $N$ and Eq. (3) imposes $N^2$ constraints. Due to the above argument there are $(k - 1)N(N - 1)$ constraints on $[Z_i, Z_{i+1}] = 0$. Then total free parameters are

$$kN(N + 1) + 2NM - N^2 - N - (k - 1)N(N - 1) = N(2k + 2M - 2). \hspace{1cm} (A.5)$$
Appendix B.

Here we explore the two dimensional case in detail. For $k = 1, M = 1$, the general solutions of the constraint (2) for any $N$ have been found by introducing two Hermitian matrices $X, Y$ such that $Z = (X + iY)/\sqrt{2}$. After diagonalizing one of them by the unitary transformation, we use the remnant $U(1)^N$ gauge freedom to put the column vector $\psi$ to be real positive. The solution of the constraint (2) is given as

$$
X_{ij} = \begin{cases} x_i & \text{for } i = j \\ -i\zeta/(y_i - y_j) & \text{for } i \neq j \end{cases}, \\
Y_{ij} = y_i\delta_{ij}, \quad \psi_i = \sqrt{\zeta}.
$$

(B.1)

In this coordinate the rotational symmetry is not manifest. (While the diagonal components of the triangularized $Z$ matrices is rotationally manifest and the distance between particles are obvious, it is much hard to solve the constraint (2) in this coordinate.) Since the general solution is obtained, we might say that the moduli space metric can be obtained in principle and the multiparticle dynamics could be understood. However it turns out that it is technically somewhat difficult. Here we consider only the two particle case and find its explicit moduli space metric in this form of the solution. Again the Gauss law determines the infinitesimal gauge parameter $\delta\alpha$ uniquely. After some calculation, the metric for the relative motion is given by

$$
ds_{rel}^2 = \frac{1}{8(x^2 + y^2 + 4\zeta^2/y^2)} \left( \left[ x^2 + y^2 \right] dx^2 + \left[ x^2 + \frac{1}{y^2}(y^2 + 4\zeta^2)^2 \right] dy^2 - \frac{8\zeta^2 x y}{y^3} dxdy \right),
$$

(B.2)

where $x \equiv x_1 - x_2, \quad y \equiv y_1 - y_2$. (In this case it turns out $\delta\psi = 0$, too.)

Although $(x_i, y_i)$ can be identified as the position of particles at large separation, its meaning at short distance changes. We can relate $(x, y)$ to the relative coordinate $z$ by taking the eigenvalues of $Z = (X + iY)/\sqrt{2}$. One can see that these two coordinates $(x, y)$ for moduli space metric do not have one to one correspondence though one form of the metric can be obtained from the other by the direct coordinate relations.

We need to elucidate the Polychronakos system at this point because Polychronakos has studied the first order time derivative Lagrangian

$$
\mathcal{L}_1 = \kappa \text{tr} (i\bar{Z}D_0Z + i\bar{\psi}D_0\psi - \zeta A_0) - w^2 \text{tr} \bar{Z}Z,
$$

(B.3)

where $D_0Z = \dot{Z} - i[A_0, Z]$, and $D_0\psi = \dot{\psi} - iA_0\psi$. The auxiliary field $A_0$ leads to the Gauss law constraint $[Z, \bar{Z}] + \bar{\psi}\psi = \zeta 1_N$, which is identical to the constraint (2). Classically there are $2N^2 + 2N$ degrees of freedom in $Z$ and $\psi$. Since there is only a first order term in time derivative, the space of $Z, \psi$ forms a phase space. $N^2$ parameters are constrained by the above Gauss law
and another $N^2$ parameters can be absorbed to the gauge parameters, leaving free $2N$ parameters in the phase space. These $2N$ parameters denote the $N$ particle location on two dimensional noncommutative plane. The eigenvalues and eigenstates of the Hamiltonian $H = w^2 \text{tr}(\bar{Z}Z)$ are completely solved \cite{14}.

The first order kinetic energy (B.3) leads to one form $A = \text{tr}(Zd\bar{Z} + \psi d\bar{\psi})$ on the $2N$ dimensional moduli space, making it to be a phase space. As $dA = \text{tr}(\delta Z \wedge \delta \bar{Z} + \delta \psi \wedge \delta \bar{\psi})$ regardless of $\delta \alpha$, the Kähler two form of our moduli space gives the symplectic structure of the Polychronakos system. Our moduli space becomes the phase space of the first order system. It would be interesting to work out the quantum problem in terms of the moduli space coordinate.
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