EQUIVARIANT GROTHENDIECK RING OF A COMPLETE SYMMETRIC VARIETY OF MINIMAL RANK

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Abstract. We describe the $G$-equivariant Grothendieck ring of a regular compactification $X$ of an adjoint symmetric space $G/H$ of minimal rank. This extends the results of Brion and Joshua for the equivariant Chow ring of wonderful symmetric varieties of minimal rank in [7] and generalizes the results on the regular compactification of an adjoint semisimple group in [17].

1. Introduction

For a semi-simple group $G$ of adjoint type, we consider a regular compactification $X$ of a symmetric space $G/H$ of minimal rank i.e., $\text{rk}(G/H) = \text{rk}(G) - \text{rk}(H)$. The main examples of symmetric spaces of minimal rank are the groups $G = G \times G/\text{diag}(G)$ and the spaces $PGL(2n)/PSp(2n)$.

We recall here that a normal complete variety $X$ is called an equivariant compactification of $G/H$ if $X$ contains $G/H$ as an open subvariety and the action of $G$ on $G/H$ extends to $X$. We further say that $X$ is a regular compactification if $X$ is an equivariant compactification which is regular as a $G$-variety (see [1, Section 2] and [5, Section 2.1]).

Indeed, the regular compactifications of an adjoint symmetric space are the complete regular embeddings of a symmetric space or the complete symmetric varieties considered by Bifet, De Concini and Procesi in [1]. In particular, $X$ considered above is a complete symmetric variety of minimal rank.

The wonderful compactification $X^{\text{wond}}$ of a symmetric space $G/H$ constructed by De Concini and Procesi in [8], is the unique regular compactification of $G/H$ with a unique closed orbit. These are called the wonderful symmetric varieties.

Wonderful symmetric varieties have been constructed in characteristic 0 by De Concini and Procesi in [8] and for an arbitrary characteristic by De Concini and Springer [10]. The geometry and topology of these varieties have been widely studied by means of their characterizing wonderful properties (see Section 2). For instance the equivariant cohomology of a wonderful group compactification has been described by Strickland [14]. Also see [12] for the study of the equivariant cohomology ring of a wonderful symmetric variety.

Let $T \subseteq G$ denote a maximal torus containing a maximal torus $T_H \subseteq H$ with Weyl group $W_H$. Among the wonderful symmetric varieties those of minimal rank (see Definition 2.1) have even better geometric properties. For instance, they have only finitely many $T$-fixed points and finitely many $T$-invariant curves which can be explicitly described (see [7, Lemma 2.1.1]). This enables the study of the cohomology theory of these varieties using a precise form of the localization theorem.

In particular, using the description of the $T$-fixed points and $T$-invariant curves, the $T$-equivariant Chow ring and hence the $G$-equivariant Chow ring of wonderful symmetric varieties of minimal rank have been described by Brion and Joshua in [7]. The property of finitely many $T$-fixed points and finitely many $T$-invariant curves has also been exploited by Tchoudjem [16] to describe the cohomology groups of line bundles on wonderful varieties of minimal rank.

The complete symmetric varieties have been defined and studied in [11] and [9] where their equivariant cohomology ring structure has been described.

For a regular compactification of a reductive algebraic group (which includes the adjoint semisimple group) the equivariant Chow ring has been described in [5] Section 3 and its equivariant Grothendieck ring has been described in [17].

In this article our main aim is to give a description of the $T$-equivariant and the $G$-equivariant Grothendieck ring of algebraic equivariant vector bundles on the complete symmetric variety $X$ of minimal rank.

2020 Mathematics Subject Classification. 14M27, 19E99, 19L99.

Key words and phrases. wonderful compactifications, complete symmetric variety, equivariant K-theory.
With this aim in view in Lemma 2.4 we show that any complete symmetric variety of minimal rank has only finitely many $T$-fixed points and $T$-invariant curves and describe them explicitly. This is an extension of [7, Lemma 2.1.1] for the wonderful symmetric variety of minimal rank. This is also a generalization of the corresponding description of the $T \times T$-fixed points and the $T \times T$-invariant curves for a regular compactification of a reductive algebraic group in [5, p. 160] which has been used to describe its $T \times T$-equivariant and the $G \times G$-equivariant Chow ring in [5] and also to describe its $T \times T$-equivariant and $G \times G$-equivariant Grothendieck ring in [17].

Let $K_T(X)$ (resp. $K_G(X)$) denote the $T$-equivariant (resp. $G$-equivariant) Grothendieck ring of $T$-equivariant (resp. $G$-equivariant) vector bundles on the complete symmetric variety $X$. Also $R(T) = K_T(pt)$ (resp. $R(G) = K_G(pt)$) denotes the Grothendieck ring of complex representations of $T$ (resp. $G$). The structure morphism $X \to \Spec \mathbb{C}$ induces $R(T)$ (resp. $R(G)$)-module structure on $K_T(X)$ (resp. $K_G(X)$) (see [17, Section 1.2]). In particular, when $X$ is the regular compactification of $G = G \times G/\text{diag}(G)$ then we consider $K_{T \times T}(X)$ (resp. $K_{G \times G}(X)$) as $R(T) \otimes R(T)$ (resp. $R(G) \otimes R(G)$)-module. We denote by $K_G(X_{\text{wond}})$ the Grothendieck ring of $G$-equivariant vector bundles on the wonderful compactification $X_{\text{wond}}$ of $G/H$.

In Proposition 3.1 and Theorem 3.2 we describe the $K_T(X)$ and $K_G(X)$ using a precise form of localization theorem [17, Theorem 1.3] for $X$ and the description of the $T$-fixed points and the invariant curves in Lemma 2.4. In particular, we show that $K_G(X) \simeq K_T(Y)^{W_H}$ where $Y$ denotes the smooth complete toric variety which is the closure of $T/T_H$ in $X$. These results are the K-theoretic analogue of the description of the equivariant Chow ring of wonderful symmetric varieties of minimal rank given in [7, Theorem 2.2.1]. These also generalize the analogous results in [17, Theorem 2.1, Corollary 2.2, Corollary 2.3] for a regular compactification of a complex reductive algebraic group (which includes the case of the regular compactification of an adjoint semisimple group $G = (G \times G)/\text{diag}(G)$) to all complete symmetric varieties of minimal rank.

Recall that in [17, Theorem 2.10] we also obtain a direct sum decomposition of $K_{G \times G}(X)$ as a $1 \otimes R(G)$-algebra and describe the multiplication of the graded pieces. In [17, Corollary 2.11] we further show the existence of a canonical multfiltration associated to the direct sum decomposition which in turn gives $K_{G \times G}(X)$ the structure of a $R(T) \otimes R(G)$-algebra. Moreover, in [17, Corollary 2.12] we gave a geometric interpretation of the pieces occurring in the direct sum decomposition.

Our aim in this paper is to extend this kind of description to any complete symmetric variety of minimal rank.

In Theorem 3.12 we give a direct sum decomposition of $K_G(X)$ as a $1 \otimes R(H)$-module and also describe the multiplicative structure. This extends [17, Theorem 2.10] where a similar result was proved for the regular group compactifications. We then give a canonical multfiltration for $K_G(X)$ arising out of the direct sum decomposition and show that each of the filtered pieces has a $R(T/T_H) \otimes R(H)$-module structure (see Corollary 3.13 and Proposition 3.14). In particular, this shows that $K_G(X)$ has a structure of a $R(T/T_H) \otimes R(H)$-algebra (see Proposition 3.14). This extends the corresponding results for regular group compactifications in [17, Corollary 2.11]. Further, Corollary 3.15 extends [17, Corollary 2.12].

In particular, $R(T/T_H) = R(T/T_H) \otimes 1$ can be identified with the subalgebra of $K_G(X_{\text{wond}})$ generated by the isomorphism classes of the $G$-linearized line bundles corresponding to the boundary divisors and $R(H)$ can be identified with $K_G(G/H)$ which is the $G$-equivariant Grothendieck ring of the symmetric space $G/H$ of minimal rank.

The rational equivariant cohomology of complete symmetric varieties have been described by Bifet, De Concini and Procesi [11] in terms of Stanley-Reisner systems. Our results in Section 3 are an integral version of their results via K-theory and localization theorem for the complete symmetric varieties of minimal rank.

Let $\tilde{G}$ denote the semi-simple simply connected cover of $G$ and $\tilde{T}$ denote its maximal torus which is a lift of $T$ in $\tilde{G}$. Then $\tilde{G}$ and $\tilde{T}$ act on $X$ via their canonical projections to $G$ and $T$ respectively.

In Section 3.1 we extend Theorem 3.2 to $K_G(X)$ and use the results of [13] to describe the ordinary Grothendieck ring of $X$.

In Section 3.3 we also give a presentation of $K_G(X)$ as an algebra over $K_G(X_{\text{wond}})$ which extends [18, Corollary 4.1].
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2. Complete symmetric varieties of minimal rank

2.1. Symmetric spaces of minimal rank. We briefly recall the definition and necessary structure of symmetric spaces of minimal rank. We refer to [8] Section 3] and [10] for details. We shall consider algebraic varieties and algebraic groups over the field of complex numbers.

Let $G$ be a connected reductive algebraic group and let

$$\theta : G \rightarrow G$$

be an involutive automorphism. Let

$$H := G^0 \subset G$$

denote the subgroup of fixed points. The homogeneous space $G/H$ is called a symmetric space.

It is known that $H$ is reductive and $H^0$ is nontrivial unless $G$ is a $\theta$-split torus i.e., a torus such that $\theta(g) = g^{-1}$ for every $g \in G$.

Recall that any two maximal $\theta$-split subtori of $G$ are conjugate in $H^0$, and their common dimension is called the rank of the symmetric space $G/H$ denoted by $\text{rk}(G/H)$. Also every maximal $\theta$-fixed subtorus of $G$ is a maximal torus of $H$. Any two maximal $\theta$-fixed subtori of $G$ are conjugate in $H^0$ and their common dimension is the rank of $H$ denoted by $\text{rk}(H)$.

Let $T$ be any $\theta$-stable maximal torus of $G$.

Let

$$T^\theta := \{ x \in T \mid \theta(x) = x \}$$

and

$$T^{-\theta} := \{ x \in T \mid \theta(x) = x^{-1} \}.$$ 

Then we have

$$T = T^\theta \cdot T^{-\theta} \text{ and } T^\theta \cap T^{-\theta} \text{ is finite.} \tag{2.1}$$

By (2.1) it follows that in general we have

$$\text{rk}(G/H) \geq \text{rk}(G) - \text{rk}(H). \tag{2.2}$$

If equality holds in (2.2), then we say that the symmetric space $G/H$ is of minimal rank i.e., if $\text{rk}(G/H) = \text{rk}(G) - \text{rk}(H)$. This is equivalent to the condition that $T^{\theta,0}$ is a maximal $\theta$-fixed subtorus of $G$ and to the condition that $A := T^{-\theta,0}$ is a maximal $\theta$-split subtorus of $G$. This is also equivalent to the condition that any two $\theta$-stable maximal tori of $G$ are conjugate in $H^0$ (see [8] Section 3.1).

We choose a $\theta$-stable Borel subgroup $B$ and a $\theta$-stable maximal torus $T$ of $B$ and call $(B, T)$ the standard pair. Also $\theta$ acts on the Weyl group $W$ and the root system $\Phi$ and stabilizes $\Phi^+, \Phi^-$ and $\Delta$ which denote respectively the positive roots, negative roots and the simple roots of the root system $\Phi$.

We shall assume from now on that $G$ is semi-simple and adjoint so that $\Delta$ is a basis of $X^*(T)$. We call the symmetric space $G/H$ adjoint as well. We shall also assume that $G/H$ is of minimal rank. Since $G$ is semi-simple and adjoint, $T_H := T^\theta$ and $B_H := B^\theta$ are connected. Further, $H$ is connected, semi-simple and adjoint. Thus $T_H$ is a maximal torus and $B_H$ is a Borel subgroup of $H$. Moreover, the roots of $(H, T_H)$ are the restrictions to $T_H$ of the roots of $(G, T)$. The Weyl group $W_H$ of $(H, T_H)$ can be identified with $W^\theta$ (see [8] Lemma 5).

Recall that the centralizer $C_G(A)$ is a Levi subgroup $L$ of a minimal $\theta$-split parabolic subgroup $P$ of $G$ i.e., $\theta(P)$ is opposite to $P$ and $L = P \cap \theta(P)$.

Let $\Phi_L$ denote the root system and $\Delta_L \subseteq \Delta$ denote the subset of simple roots of $L$. Under the natural action of $\theta$ on $\Phi$ we have $\Phi_L = \Phi^\theta$. Let $p : X^*(T) \rightarrow X^*(A)$ denote the restriction map from the character group of $T$ to the character group of $A$. Then $p(\Phi) \setminus \{0\}$ is a root system denoted $\Phi_{G/H}$ called the restricted root system. Also $\Delta_{G/H} := p(\Delta \setminus \Delta_L)$ is a basis of $\Phi_{G/H}$. The simple restricted roots $\Delta_{G/H}$ can further be identified with $\alpha - \theta(\alpha)$ for $\alpha \in \Delta \setminus \Delta_L$ under $p$.

The restricted Weyl group $W_{G/H} = N_G(A)/C_G(A)$ can further be identified with $N_W(A)/C_W(A)$. This yields the exact sequence

$$1 \rightarrow W_L \rightarrow W^\theta = W_H \rightarrow W_{G/H} \rightarrow 1$$ (see [8] Subsections 1.2 and 1.3).

Let $X_{\text{wonder}}$ denote the wonderful compactification of the adjoint symmetric space $G/H$ constructed by De Concini and Procesi [8] and De Concini and Springer [10], which has the following properties:
(1) $X^{wond}$ is a nonsingular projective variety.
(2) $G$ acts on $X^{wond}$ with an open orbit isomorphic to $G/H$.
(3) The complement of the open orbit is the union of $r = \text{rk}(G/H)$ nonsingular prime divisors $D_1, \ldots, D_r$ with normal crossings called the boundary divisors.
(4) The $G$-orbit closures in $X^{wond}$ are exactly the partial intersections $X^w := \bigcap_{i \in I} D_i$ where $I$ runs over the subsets of $\{1, \ldots, r\}$. In particular, each $X^w$ is smooth.
(5) The unique closed orbit $D_1 \cap \cdots \cap D_r$ is isomorphic to $G/P \simeq G/\theta(P)$.

**Definition 2.1.** When $G/H$ is of minimal rank, we call the wonderful compactification $X^{wond}$ a wonderful symmetric variety of minimal rank.

The wonderful compactification of $G \times G/\text{diag}(G)$ is an example of a wonderful symmetric variety of minimal rank. For other examples and the complete classification of wonderful symmetric varieties we refer to [7, Example 1.4.4].

Let $X^{wond}$ be the wonderful symmetric variety of minimal rank. The associated toric variety $Y^{wond}$ is the closure in $X^{wond}$ of $T/T_H$. Then $Y^{wond}$ is a smooth toric variety associated with the Weyl chambers of the restricted root system $\Phi_{G/H}$. Let $Y^0$ denote the open affine toric subvariety associated with the positive Weyl chamber dual to $\Delta_{G/H}$. Then $Y^{wond} = W_{G/H} Y^0$. The distinguished point $z$ of the unique closed orbit $G/P$ is the unique $T$-fixed point of $Y^0$. Since $W_{G/H}$ acts simply transitively on the $T$-fixed points, the $T$-fixed points of $Y^{wond}$ are precisely $w \cdot z$ for $w \in W_{H}/W_L \simeq W_{G/H}$. Also the $T$-fixed points of $X^{wond}$ are precisely $w \cdot z$ for $w \in W/W_L$. It is known that any wonderful variety of minimal rank admits finitely many $T$-stable curves [19, Section 10]. The precise description of the $T$-stable curves in the wonderful symmetric variety of minimal rank $X^{wond}$ as well as that of the $T$-stable curves lying in the associated toric variety $Y^{wond}$ is given in [7, Lemma 2.1.1]. We recall this description below.

**Lemma 2.2.** [7, Lemma 2.1.1]

- (1) The $T$-fixed points in $X^{wond}$ (resp., $Y^{wond}$) are exactly the points $w \cdot z$ where $w \in W$ (resp., $W_H$), and $z$ denotes the unique $T$-fixed point of $Y^0$. These fixed points are parametrized by $W_G/W_L$ (resp., $W_H/W_L \simeq W_{G/H}$).
- (2) For any $\alpha \in \Phi^+ \setminus \Phi^+_L$, there exists a unique irreducible $T$-stable curve $C_{z,\alpha}$ which contains $z$ and on which $T$ acts through its character $\alpha$. The $T$-fixed points in $C_{z,\alpha}$ are exactly $z$ and $s_\alpha \cdot z$.
- (3) For any $\gamma = \alpha - \theta(\alpha) \in \Delta_{G/H}$, there exists a unique irreducible $T$-stable curve $C_{z,\gamma}$ on which $T$ acts through its character $\gamma$. The $T$-fixed points in $C_{z,\gamma}$ are exactly $z$ and $s_\alpha \cdot z$.
- (4) The irreducible $T$-stable curves in $X^{wond}$ are the $W$-translates of the curves $C_{z,\alpha}$ and $C_{z,\gamma}$. They are all isomorphic to $\mathbb{P}^1$.
- (5) The irreducible $T$-stable curves in $Y^{wond}$ are the $W_{G/H}$-translates of the curves $C_{z,\gamma}$.

We now recall the definition of a regular $G$-variety due to Bifet, De Concini and Procesi [1]. (See [3, Section 1.4])

**Definition 2.3.** A $G$-variety $X$ is said to be regular if it satisfies the following conditions:

- (i) $X$ is smooth and contains a dense $G$-orbit $X^0_{\theta}$ whose complement is a union of irreducible smooth divisors with normal crossings (the boundary divisors).
- (ii) Any $G$-orbit closure in $X$ is the transversal intersection of the boundary divisors which contain it.
- (iii) For any $x \in X$, the normal space $T_x X/T_x(Gx)$ contains a dense orbit of the isotropy group $G_x$.

Consider the adjoint symmetric space $G/H$ of minimal rank. Then $G/H$ is a homogeneous space under $G$ for the standard action from the left with base point $1 \cdot H$. A normal complete variety $X$ is an equivariant compactification of $G/H$ if $X$ contains $G/H$ as an open subvariety and the action of $G$ on $G/H$ by left multiplication extends to $X$. We say that $X$ is a regular compactification of $G/H$ if $X$ is an equivariant compactification which is regular as a $G$-variety. The canonical wonderful compactification $X^{wond}$ of $G/H$ is the unique regular compactification of $G/H$ with a unique closed orbit.

We have a complete classification of the regular compactifications of $G/H$ in terms of smooth torus embeddings $Y_0$ of $T/T_H$ lying over $Y^{wond} \simeq \mathbb{A}^r$ such that the map $Y_0 \to Y^{wond}$ is proper (10, Definition 23)). Here $Y_0$ is the $T/T_H$-toric variety associated to a fan $\mathcal{F}_+$ which is a smooth subdivision of the positive Weyl chamber $\mathcal{C}_+$ in the lattice $X^*_+(T/T_H) \otimes \mathbb{R}$ generated by the coweight vectors dual to the
simple restricted roots $\Delta_{G/H}$. Recall that $W_{G/H}$ acts on the coweight lattice $X_*(T/T_H)$ by reflection about the walls of the Weyl chambers. The closure of $T/T_H$ in $X$ is the smooth complete toric variety $Y$ which is associated to the fan $\mathcal{F}$ in $X_*(T/T_H) \otimes \mathbb{R}$ whose cones are $W_{G/H}$-translates of the cones in $\mathcal{F}_+$.

Let $\mathcal{F}_+(r)$ denote the maximal dimensional cones of $\mathcal{F}_+$ which parametrize the closed $G$-orbits in $X$. For $\sigma \in \mathcal{F}_+(r)$ we denote by $Z_\sigma \simeq G/P$ the corresponding closed orbit with base point $z_\sigma$ (see [1] pp. 20-22, Proposition 24).

We have the following precise description of the $T$-fixed points and the $T$-stable curves in the complete symmetric variety of minimal rank $X$ as well as that of the $T$-fixed points and the $T$-stable curves lying in the associated toric variety $Y$.

**Lemma 2.4.**

(1) The $T$-fixed points in $X$ (resp., $Y$) are exactly the points $w \cdot z_\sigma$ where $w \in W$ (resp., $W_H$), and $z_\sigma$ denotes the $T$-fixed point of $Y_0$ corresponding to the maximal cone $\sigma \in \mathcal{F}_+(r)$. These fixed points are parametrized by $\mathcal{F}_+(r) \times W/W_L$ (resp., $\mathcal{F}_+(r) \times W_H/W_L$) where $W_H/W_L \simeq W_{G/H}$.

(2) For any $\alpha \in \Phi^+ \setminus \Phi^+_L$, there exists a unique irreducible $T$-stable curve $C_{z_\alpha, \alpha}$ which contains $z_\sigma$ and on which $T$ acts through its character $\alpha$. The $T$-fixed points in $C_{z_\alpha, \alpha}$ are exactly $z_\sigma$ and $s_\alpha \cdot z_\sigma$ for every $\sigma \in \mathcal{F}_+(r)$.

(3) For any $\gamma = \alpha - \theta(\alpha) \in \Delta_{G/H}$, there exists a unique irreducible $T$-stable curve $C_{z_\alpha, \gamma}$ on which $T$ acts through its character $\gamma$. The $T$-fixed points in $C_{z_\alpha, \gamma}$ are exactly $z_\sigma$ and $s_\alpha \cdot z_\sigma$. In this case the cone $c_\alpha \in \mathcal{F}_+(r)$ has a facet orthogonal to $\gamma$.

(4) There is a unique irreducible $T$-stable curve $C_{z_\alpha, z_\sigma'}$ in $Y_0 \subseteq Y \subseteq X$ namely the projective line joining $z_\sigma$ and $z_\sigma'$, which are respectively the base points of the distinct orbits $Z_\sigma$ and $Z_{\sigma'}$, whenever the cones $\sigma$ and $\sigma'$ in $\mathcal{F}_+(r)$ have a common facet.

(5) The irreducible $T$-stable curves in $X$ are the $W$-translates of the curves $C_{z_\alpha, \alpha}$ for every $\sigma \in \mathcal{F}_+(r)$, $C_{z_\alpha, \gamma}$ whenever $\sigma \in \mathcal{F}_+(r)$ has a facet orthogonal to $\gamma$ and $C_{z_\alpha, z_\sigma'}$, whenever the cones $\sigma$ and $\sigma'$ in $\mathcal{F}_+(r)$ have a common facet.

(6) The irreducible $T$-stable curves in $Y$ are the $W_H$-translates of the curves $C_{z_\alpha, \gamma}$ whenever $\sigma \in \mathcal{F}_+(r)$ has a facet orthogonal to $\gamma$ and $C_{z_\alpha, z_\sigma'}$, whenever the cones $\sigma$ and $\sigma'$ in $\mathcal{F}_+(r)$ have a common facet.

**Proof:** By [1] there exists a $G$-equivariant (hence $T$-equivariant) morphism from $\Phi : X \longrightarrow X^{wond}$ which restricts to the morphism of toric varieties $\varphi : Y \longrightarrow Y^{wond}$ induced by the map of fans $\mathcal{F}_+ \longrightarrow \mathcal{C}_+$. Let $x$ be a $T$-fixed point of $X$. Since $\Phi$ is $T$-equivariant $\Phi(x)$ is a $T$-fixed point of $X^{wond}$. Thus $\Phi(x) = w \cdot z$ for some $w \in W$. Here $z \in Y_0^{wond}$ is the base point of the unique closed orbit of $X^{wond}$. Since $\Phi$ is $G$-equivariant $\Phi(w^{-1} \cdot x) = z$. Now $\Phi^{-1}(Y_0^{wond}) = Y_0$. Thus $w^{-1} \cdot x$ is a $T$-fixed point of $Y_0$ and hence is equal to $z_\sigma$ for some $\sigma \in \mathcal{F}_+(r)$. Thus it follows that $z = w \cdot z_\sigma$ for some $w \in W$ and $\sigma \in \mathcal{F}_+(r)$. Since $W_L$ acts trivially on $Y$ it follows that the $T$-fixed points of $X$ are parametrized by $\mathcal{F}_+(r) \times W/W_L$. This proves (1) of Lemma 2.4.

Recall that $W_H$ acts on $Y$ and that the $T$-fixed point of $Y$ are $w \cdot z_\sigma$ for $w \in W_H$ is a representative of the coset of $W_H/W_L$. Thus there are $\mathcal{F}_+(r) \times W/H/W_L$ fixed points in $Y$.

Consider the translation $v \cdot Y$ where $v \in W$ is a representative of the coset $W/H$. Here $v \cdot Y$ is an irreducible variety isomorphic to $Y$ with an appropriate twist for the $T$-action. Then the $T$-fixed points in $v \cdot Y$ are $v \cdot w \cdot z_\sigma$ for $w \in W/H/W_L$.

Since the $T$-fixed points of $X$ are parametrized by $\mathcal{F}_+(r) \times W/W_L$ where $W/W_L = \bigcup_{w \in W/H} v \cdot W/H/W_L$. It follows that the $T$-fixed points of $v \cdot Y$ as $v$ varies over the coset representatives of $W/H$ are all distinct and exhaust all the $T$-fixed points of $X$.

In particular, this implies that $v \cdot Y$ as $v$ varies over the coset representatives of $W/H$ are disjoint subvarieties of $X$. For, if they intersect then the intersection being a complete $T$-variety will contain a $T$-fixed point which is a contradiction to the above observation that the collection of $T$-fixed points in $v \cdot Y$ for distinct coset representatives $v$ of $W/H$ are distinct.

Thus a $T$-invariant curve $C$ of $X$ is one the following types.

(i) It lies in some $v \cdot Y$, in which case it is translate by $v \in W$ to a $T$-invariant curve $C'$ in $Y$. If $C'$ lies in $Y_0$ then $C'$ is of the form (4) in Lemma 2.4. Thus $C$ is translate by $v$ of a curve of the form (4) in Lemma 2.4. Or else if $C'$ does not lie in $Y_0$ then $C'$ is conjugate by $w \in W_H/W_L$ to a $T$-invariant curve in $Y$ joining $z_\sigma$ and $w \cdot z_\sigma$. Thus $\Phi(C')$ is a $T$-invariant curve of $Y^{wond}$ joining the two distinct $T$-fixed points $z$ and $w \cdot z$. Thus $\Phi(C')$ is of the form (3) of Lemma 2.2. Since $\Phi$ is $G$-equivariant and hence $T$-equivariant it follows.
that C' is of the form (3) of Lemma 2.4 Hence C is a translate by w of a curve of the form (3) of Lemma 2.4

(ii) It is translate by v' ∈ W of a T-invariant curve C' joining a T-fixed point of Y with a T-fixed point of v · Y for some coset representative v of W/H different from 1. Since Φ(v · Y) = v · Y′ ∈ T invariant curve in X′ with z ∈ v · y. Therefore Φ(C') is of the form (2) of Lemma 2.2 Since Φ is G-equivariant and hence T-equivariant, C' is of the form (2) in Lemma 2.4 Thus C is a translate by v' ∈ W of a curve of the form (2) in Lemma 2.4 This proves (5) and (6) of Lemma 2.4 □

We denote by τ_σ : K_T(X) → K_T(Z_σ) ≃ K_T(G/P) the restriction map. For f ∈ K_T(Z_σ) we denote by f_w the restriction of f to the fixed point w · z_σ for w ∈ W/W_L.

3. The G-equivariant Grothendieck ring of X

Let G/H be a symmetric space of minimal rank. Let X' denote the canonical wonderful symmetric variety of minimal rank. Let X be a projective regular compactification of G/H which in addition is characterized by the fact that the map of T/T_H-toric varieties Y_0 → Y'_0 ≃ A^1 is a projective morphism.

The following proposition describes the T-equivariant Grothendieck ring of X. Also see [17, Theorem 2.1] for the corresponding result for the case of regular group compactifications.

Proposition 3.1. The map

\[ \prod_{\sigma \in \mathcal{F}_+(r)} \tau_\sigma : K_T(X) \rightarrow \prod_{\sigma \in \mathcal{F}_+(r)} \prod_{\sigma \in \mathcal{F}_+(r)} K_T(G/P) \rightarrow R(T)^{|W/W_L| \cdot |\mathcal{F}_+(r)|} \]

is injective and its image consists of all families \((f_\sigma)_{\sigma \in \mathcal{F}_+(r)}\) in \(K_T(G/P)\) such that:

(i) \(f_{\sigma,w} = f_{\sigma,w \cdot s_{\alpha}(\alpha)} (mod 1 - e^{-w(\gamma)})\) whenever \(\gamma = \alpha - \theta(\alpha) \in \Delta_G/H\) and w ∈ W.

(ii) \(f_{\sigma,w} = f_{\sigma',w} (mod 1 - e^{-w(\chi)})\) whenever \(\chi \in X^*(T/T_H)\) and the cones \(\sigma\) and \(\sigma'\) of \(\mathcal{F}(r)\) have a common facet orthogonal to \(\chi\) in G and w ∈ W.

Proof: For [17, Theorem 1.3], the image of \(K_T(X) \rightarrow K_T(X_T)\) is defined by linear congruences \(f_x \equiv f_y (mod 1 - e^{-x})\) whenever \(x, y \in X_T\) are connected by a curve where T acts by the character \(x\).

Recall that the union of the closed orbits \(\bigcup_{\sigma \in \mathcal{F}_+(r)} Z_\sigma\) contains all the T-fixed points of X (see [11, Lemma 3.4] [16, Proposition 5.1]). Moreover, T-fixed points in Z_σ are the W-translates of z_σ. Since W_L fixes z_σ for every \(\sigma \in \mathcal{F}_+(r)\) they are parametrized by W/W_L. In particular, z_σ is the distinguished point in Z_σ which is fixed by P. We therefore have the inclusions \(\prod_{\sigma \in \mathcal{F}_+(r)} \tau_\sigma : K_T(X) \rightarrow \prod_{\sigma \in \mathcal{F}_+(r)} K_T(Z_\sigma) \rightarrow K_T(G/P) \rightarrow (R(T)^{|W/W_L| \cdot |\mathcal{F}_+(r)|})\).

By Lemma 2.4, the invariant curves in X are the W-translates of curves of type (2), (3) or (4). Now, \(\bar{w}\)-translates of curves of type (2) which are \(C_{\bar{w},\sigma}\) for \(\alpha \in \Phi^+ \setminus \Phi^+_L\) lie inside the closed orbit \(Z_\sigma\) for \(\sigma \in \mathcal{F}_+(r)\) and define its T-equivariant K-ring inside \((R(T)^{|W/W_L|})\). Thus the image of \(K_T(X)\) under \(\prod_{\sigma \in \mathcal{F}_+(r)} \tau_\sigma\) consists of \((f_\sigma) \in \prod_{\sigma \in \mathcal{F}_+(r)} K_T(G/P)\) satisfying the congruences (i) and (ii) corresponding respectively to the W-translates of the curves of type (3) and (4). Note that T acts by \(w(\gamma)\) (resp. \(w(\chi)\)) on the translation by w ∈ W of the curve \(C_{\bar{w},\gamma}\) (resp. \(C_{\bar{w},\chi}\)). □

The G-equivariant Chow ring of X'_wond was described in [17, Theorem 2.2.1] using the description of the T-fixed points and the T-stable curves given in Lemma 2.2.

We state below the corresponding theorem for the G-equivariant Grothendieck ring \(K_G(X)\) where X is a regular compactification of a symmetric space G/H of minimal rank. The proof follows along similar lines as that of the equivariant Chow ring of X'_wond by using [17, Theorem 1.3] in place of [11, Section 3.4] and by the description of the T-fixed points and T-stable curves of X given in Lemma 2.4. It further generalizes [17, Corollary 2.2, Corollary 2.3] to any complete symmetric variety of minimal rank.

Theorem 3.2.

(1) The ring \(K_G(X)\) consists in all families \((f_\sigma)_{\sigma \in \mathcal{F}_+(r)}\) of elements in \(R(T)^{|W_L|}\) satisfying
\( (i) \) \( s_\alpha s_{\theta(\alpha)} \cdot f_\sigma \equiv f_\sigma \pmod{1 - e^{-\gamma}} \) whenever \( \gamma = \alpha - \theta(\alpha) \in \Delta_{G/H} \).

\( (ii) \) \( f_\sigma \equiv f_\sigma' \pmod{1 - e^{-\chi}} \) whenever \( \chi \in X^*(T/T_H) \) and the cones \( \sigma \) and \( \sigma' \) of \( F_+(r) \) have a common facet orthogonal to \( \chi \).

(2) The map
\[
\tau : K_G(X) \to K_T(X)^W \to K_T(X)^{W_H} \to K_T(Y)^{W_H}
\]
obtained by composing the canonical maps is an isomorphism of \( R(G) \)-algebras where \( R(G) = R(T)^W \).

**Proof:** (1) Note that by [17, Theorem 1.8] the restriction homomorphism \( K_G(X) \to K_T(X) \) induces an isomorphism \( K_G(X) \cong K_T(X)^W \). Again by [17, Theorem 1.8] the ring \( K_G(Z_\sigma) = K_G(G/P) \) is isomorphic to \( K_T(G/P)^W \). It is further isomorphic to \( R(P) = R(T)^{W_L} \) via restriction to \( z_\sigma \).

Moreover, if \( f \in K_T(Z_\sigma)^W \), then for each \( w \in W \) we have
\[
w \cdot f_{\sigma,1} = f_{\sigma,w}
\]
where \( f_{\sigma,1} \) denotes the restriction of \( f \) to \( z_\sigma \) and \( f_{\sigma,w} \) denotes the restriction of \( f \) to \( w \cdot z_\sigma \). Furthermore, since \( W_L \) acts trivially on the \( T \)-fixed points \( K_G(X)^W \) implies that \( w \cdot f_{\sigma,1} = f_{\sigma,1} \) for \( w \in W_L \).

Recall that \( T \) acts by \( w(\gamma) \) (resp. \( w(\chi) \)) on the translation of \( C_{z_\sigma,\gamma} \) (resp. \( C_{z_\sigma,\chi} \)) by \( w \in W \). Thus for \( (f_\sigma) \in \prod \limits_{\sigma \in F_+(r)} K_T(G/P)^W \) the congruences (i) and (ii) can be rewritten as
\[
\begin{align*}
(i) & \quad w \cdot f_{\sigma,1} \equiv w \cdot s_\alpha s_{\theta(\alpha)} \cdot f_{\sigma,1} \pmod{1 - e^{-w(\gamma)}} \text{ whenever } \gamma = \alpha - \theta(\alpha) \in \Delta_{G/H} \text{ and } w \in W, \\
(ii) & \quad w \cdot f_{\sigma,1} \equiv w \cdot f_{\sigma',1} \pmod{1 - e^{-w(\chi)}} \text{ whenever } \chi \in X(T/T_H)^{\ast} \text{ and the cones } \sigma \text{ and } \sigma' \text{ of } F_+(r) \text{ have a common facet orthogonal to } \chi \text{ and } w \in W.
\end{align*}
\]
Moreover, since (i) and (ii) are consequences of the congruences (i) and (ii), this proves (1).

(2) Recall that \( T \)-fixed points of \( Y \) are the \( W_H \) translates of \( z_\sigma \) for \( \sigma \in F_+(r) \). Also \( W_L \) acts trivially on the \( T \)-fixed points. Furthermore, by Lemma [2,4] (6) and by observing that \( T \) acts by \( w(\gamma) \) (resp. \( w(\chi) \)) on the translation of the curve \( C_{z_\sigma,\gamma} \) (resp. \( C_{z_\sigma,\chi} \)) by \( w \in W_L \), it can be seen that \( K_T(Y)^{W_H} \) can be identified with the tuples \( (f_{\sigma,w})_{w \in W_H} \) satisfying the congruences
\[
\begin{align*}
(i') & \quad f_{\sigma,w} \equiv f_{\sigma,w,s_\alpha s_{\theta(\alpha)}} \pmod{1 - e^{-w(\gamma)}} \text{ whenever } \gamma = \alpha - \theta(\alpha) \in \Delta_{G/H} \text{ and } w \in W_H, \\
(ii') & \quad f_{\sigma,w} \equiv f_{\sigma',w} \pmod{1 - e^{-w(\chi)}} \text{ whenever } \chi \in X(T/T_H)^{\ast} \text{ and the cones } \sigma \text{ and } \sigma' \text{ of } F_+(r) \text{ have a common facet orthogonal to } \chi \text{ and } w \in W_H.
\end{align*}
\]
Here \( f_{\sigma,w} \) denotes the restriction of \( f \in K_T(Y)^{W_H} \) to \( w \cdot z_\sigma \) for \( \sigma \in F_+(r) \) and \( w \in W_H \). Furthermore, if \( f \in K_T(Y)^{W_H} \), then \( f_{\sigma,w} = w \cdot f_\sigma \) where \( f_\sigma \) denotes the restriction of \( f \) to \( z_\sigma \). Thus it follows that the ring \( K_T(Y)^{W_H} \) is identified with the tuples \( (f_{\sigma})_{\sigma \in F_+(r)} \in \prod \limits_{\sigma \in F_+(r)} R(T)^{W_L} \) satisfying the congruences (i) and (ii). Hence the theorem. \( \Box \)

We shall denote by \( \gamma \) the restricted root \( \alpha - \theta(\alpha) \in \Delta_{G/H} \) for \( \alpha \in \Delta_H \setminus \Delta_L \) and by \( s_\gamma \) the corresponding simple reflection which is the image in \( W_{G/H} \) of the element \( s_\alpha \cdot s_{\theta(\alpha)} \in W_H \). (Recall that \( s_\gamma^2 = 1 \) [1, Definition 6, Lemma 10.2].)

We isolate below the description of \( K_G(X_{\text{wond}}) \) as a particular case of the above theorem. This is analogous to [17, Lemma 3.2] in the case of the wonderful compactification of an adjoint semi-simple group.

**Corollary 3.3.** The ring \( K_G(X_{\text{wond}}) \) is identified with the subring of \( R(T)^{W_L} \) defined by the congruences
\[
f \equiv s_\alpha s_{\theta(\alpha)} \cdot f \pmod{1 - e^{\gamma}}
\]
for \( \gamma = \alpha - \theta(\alpha) \in \Delta_{G/H}, \alpha \in \Delta_H \setminus \Delta_L \).

**Proof:** In this case \( |F_+(r)| = 1 \) since \( X_{\text{wond}} \) has a unique closed \( G \)-orbit. Thus by Theorem [2,4] (1) \( K_G(X_{\text{wond}}) \) can be identified with the subring of \( R(T)^{W_L} \) satisfying the congruences (i). Hence the corollary. \( \Box \)

Let \( \tilde{G} \) denote the simply connected cover of \( G \) with projection \( \pi : \tilde{G} \to G \) and \( \tilde{T} := \pi^{-1}(T) \). Then \( \tilde{G} \) is a semi-simple simply connected algebraic group with \( \tilde{T} \) as a maximal torus. Let \( \tilde{P} := \pi^{-1}(P) \). There exists an involution \( \tilde{\theta} \) of \( \tilde{G} \) which induces the involution \( \theta \) of the adjoint quotient \( G \). Let \( \tilde{H} \) denote the fixed points of \( \tilde{G} \) under \( \tilde{\theta} \). Then \( \tilde{T}_H := \tilde{H} \cap \tilde{T} \) is a maximal torus of \( \tilde{H} \).
We have the following theorem analogous to Theorem 3.2 obtained by replacing \( G \) by \( \tilde{G} \) and \( T \) by \( \tilde{T} \) and by considering the action of \( G \) and \( \tilde{T} \) on \( X \) and \( Y \) through their canonical surjections to \( G \) and \( T \) respectively. We omit the proof to avoid repetition.

**Theorem 3.4.** The map
\[
 r : K_{\tilde{G}}(X) \rightarrow K_{\tilde{T}}(X)^W \rightarrow K_{\tilde{T}}(X)^{Wh} \rightarrow K_{\tilde{T}}(Y)^{Wh}
\]

obtained by composing the canonical maps is an isomorphism.

We further have the following corollary analogous to Corollary 3.3.

**Corollary 3.5.** The ring \( K_{\tilde{G}}(X^{wond}) \) is identified with the subring of \( R(\tilde{T})^{Wh} \) defined by the congruences
\[
 f \equiv s_{\alpha} s_{\theta(\alpha)} \cdot f \pmod{1 - e^\gamma}
\]

for \( \gamma = \alpha - \theta(\alpha) \in \Delta_{G/H}, \alpha \in \Delta_H \setminus \Delta_L \).

**Remark 3.6.** For \( X^{wond} \) the set of simple restricted roots \( \Delta_{G/H} \) are in bijection with the set of boundary divisors. Let \( D_\gamma \) denote the boundary divisor corresponding to \( \gamma \in \Delta_{G/H} \). Let \( L_\gamma \) denote the \( G \)-linearized (and hence \( \tilde{G} \)-linearized) line bundle on \( X \) corresponding to the \( G \)-stable boundary divisor \( D_\gamma \) for \( \gamma \in \Delta_{G/H} \). Moreover, \( L_\gamma \) has a \( \tilde{G} \)-invariant section whose zero locus is the boundary divisor \( D_\gamma \) for \( \gamma \in \Delta_{G/H} \). Note that \( X^*(T/H) \) has a basis consisting of the simple roots \( \gamma \in \Delta_{G/H} \). Thus \( R(T/H) = \mathbb{Z}[X^*(T/H)] \) is generated as a \( \mathbb{Z} \)-algebra by \( e^\gamma, \gamma \in \Delta_{G/H} \). Thus we can identify \( R(T/H) \) with the subalgebra of \( K_G(X^{wond}) \) generated by \( \{L_\gamma \} \) for \( \gamma \in \Delta_{G/H} \) (see [17] Remark 3.7). In particular, \( L_\gamma |_{X^{wond}} \) is the \( T/H \)-linearized line bundle on the toric variety \( Y^{wond}_0 \) corresponding to the ray in \( \mathcal{F}_+ \) generated by \( \omega_\gamma \).

**Remark 3.7.** (Extension to other oriented cohomology theories) The methods of [7] have earlier been used by V. Kiritchenko and A. Krishna in [11] to compute the rational equivariant algebraic cobordism ring of wonderful symmetric varieties of minimal rank. We wish to remark here that Lemma 2.4 above allows us to extend the description in [11] Theorem 6.4 to all regular symmetric varieties of minimal rank.

3.1. **Ordinary K-ring of \( X \).** Note that we have the equalities
\[
 R(\tilde{T})^{Wh} = K_{\tilde{G}}(\tilde{G}/\tilde{B})^{Wh} = \operatorname{Res}(\tilde{G}) = R(\tilde{T})^{Wh} - \text{subalgebra of } K_{\tilde{G}}(\tilde{G}/\tilde{B}) = R(\tilde{T}).
\]

Consider the augmentation map \( \epsilon : R(\tilde{G}) \rightarrow \mathbb{Z} \) which takes an element \([V]\) to \( \dim(V) \). Thus \( \mathbb{Z} \) becomes a \( R(\tilde{G}) \)-module via the augmentation map.

We have the isomorphisms ([13] Theorem 6.1.22)
\[
 K(X) \simeq \mathbb{Z} \otimes_{R(\tilde{G})} K_{\tilde{G}}(X)
\]
and
\[
 K_{\tilde{G}}(Y)^{Wh} = K_{\tilde{G}}(\tilde{G} \times \tilde{B} \ Y)^{Wh}.
\]

**Remark 3.8.** Here \( \tilde{G} \times \tilde{B} \ Y \) is a toric bundle with fibre the \( T/H \)-toric variety \( Y \) and base \( \tilde{G}/\tilde{B} \).

**Proposition 3.9.** We have the following isomorphism of ordinary \( K \)-rings:
\[
 K(X) \simeq K(\tilde{G} \times \tilde{B} \ Y)^{Wh}.
\]

**Proof:** Follows immediately from Theorem 3.2 [13] and 3.3. \( \square \)

3.2. **Determination of the structure of \( K_G(X) \).** We first fix some notations similar to [17] Section 2.1. Let \( \mathcal{F} \) denote the (smooth projective) fan associated to \( Y \). Recall that the Weyl group \( W_H \) acts on \( \mathcal{F} \) by reflection across the walls of the Weyl chambers and the cones in \( \mathcal{F} \) get permuted by this action of \( W_H \) and each cone is stabilized by the reflections corresponding to the walls of the Weyl chambers on which it lies. Let \( W_\tau \) denote the subgroup of \( W_H \) which fixes the cone \( \tau \in \mathcal{F} \). Then, in particular, \( W_\sigma = W_L \) for \( \sigma \in \mathcal{F}(r) \) where \( r = \text{rk}(G/H) \) and \( W_0 = W_H \). In particular, \( W_H \supseteq W_\tau \supseteq W_L \) for every \( \tau \in \mathcal{F} \). Indeed, \( W_H \) acts on \( \mathcal{F} \) via its projection to \( W_H/W_L = W_G/H \) which is the Weyl group of the restricted root system with simple roots \( \Delta_{G/H} \).

Let \( \{\rho_j \mid j = 1, \ldots, d\} \) denote the set of edges of the fan \( \mathcal{F} \) and let \( \tau(1) \) denote the set of edges of the cone \( \tau \) for every \( \tau \in \mathcal{F} \). Let \( v_j \) denote the primitive vector along the edge \( \rho_j \). Let \( O_\tau \) denote the \( T/H \)-orbit in \( Y \) corresponding to \( \tau \in \mathcal{F} \). Let \( L_j \) denote the \( T/H \)-equivariant line bundle on \( Y \) corresponding to the
edge $\rho_j$. Note that $L_j$ has a $T/T_H$-invariant section $s_j$ whose zero locus is $\overline{O_{\rho_j}}$. Recall that $Y_0$ is the toric variety associated to the fan $\mathcal{F}_+$ consisting of the cones $\tau \in \mathcal{F}$ which lie in the positive Weyl chamber.

Let $X_F := \prod_{\rho_j \in F} (1 - X_j)$ in the Laurent polynomial algebra $\mathbb{Z}[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]$ for every $F \subseteq \{\rho_j \mid j = 1, \ldots, d\}$. In particular we let $X_\tau := X_{\tau(1)} = \prod_{\rho_j \in \tau(1)} (1 - X_j)$ for every $\tau \in \mathcal{F}$.

Let $C_\tau := X_\tau \cdot \mathbb{Z}[X_j^{\pm 1} : \rho_j \in \tau(1)]$. Recall from [17, Lemma 2.8] that we have the additive decomposition

$$K_{T/T_H}(Y_0) = \bigoplus_{\tau \in \mathcal{F}_+} C_\tau$$

which follows from the Stanley-Reisner presentation of the $T/T_H$-equivariant Grothendieck ring of $Y_0$, $K_{T/T_H}(Y_0) = \mathbb{Z}[X_1^{\pm 1}, \ldots, X_d^{\pm 1}] / \langle X_F \forall F \notin \mathcal{F}_+ \rangle$ (see [19, Theorem 6.4]).

Similarly by [14, Theorem 6.4] we also have the following Stanley Reisner presentation for the $T/T_H$-equivariant Grothendieck ring of $Y$:

$$K_{T/T_H}(Y) \simeq \mathbb{Z}[X_1^{\pm 1}, \ldots, X_d^{\pm 1}] / \langle X_F \forall F \notin \mathcal{F} \rangle$$

Furthermore, since $W_H$ acts on $\mathcal{F}$ we have an action of $W_H$ on $\mathbb{Z}[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]$, given by $w(X_{\rho_j}^{\pm 1}) = X_{w(\rho_j)}^{\pm 1}$ for every $w \in W_H$. Thus $w(X_F) = X_{w(F)}$ for $F \subseteq \{\rho_j \mid j = 1, \ldots, d\}$ and $w \in W_H$, and since $W_H$ permutes the cones of $\mathcal{F}$ we further get an action of $W_H$ on the Stanley-Reisner algebra $\mathbb{Z}[X_1^{\pm 1}, \ldots, X_d^{\pm 1}] / \langle X_F \forall F \notin \mathcal{F} \rangle$ so that (3.7) is an isomorphism of $W_H$-modules where the $W_H$-action on $K_{T/T_H}(Y)$ is induced from the $W_H$-action on $Y$.

We further have the additive decomposition

$$\mathbb{Z}[X_1^{\pm 1}, \ldots, X_d^{\pm 1}] / \langle X_F \forall F \notin \mathcal{F} \rangle = \bigoplus_{\tau \in \mathcal{F}} X_\tau \cdot \mathbb{Z}[X_j^{\pm 1} : \rho_j \in \tau(1)],$$

where $W_H$ acts on the right hand side as follows:

$$w \cdot (X_\tau \cdot \mathbb{Z}[X_j^{\pm 1} : \rho_j \in \tau(1)]) = X_{w(\tau)} \cdot \mathbb{Z}[X_j^{\pm 1} : \rho_j \in w(\tau)(1)]$$

for every $w \in W_H$.

Thus we can write

$$K_{T/T_H}(Y) = \bigoplus_{\tau \in \mathcal{F}_+} \bigoplus_{w \in W_H / W_\tau} X_{w(\tau)} \cdot \mathbb{Z}[X_j^{\pm 1} : \rho_j \in w(\tau)(1)]$$

where we recall that $W_\tau$ denotes the subgroup of $W_H$ which fixes $\tau \in \mathcal{F}_+$.

This implies that as a $W_H$-module we can write

$$K_{T/T_H}(Y) = \bigoplus C_\tau$$

where $C_\tau := X_\tau \cdot \mathbb{Z}[X_j^{\pm 1} : \rho_j \in \tau(1)]$. Further, since $C_\tau$ is fixed by $W_\tau$,

$$\text{Ind}_{W_\tau}^{W_H} C_\tau = \mathbb{Z}[W_H / W_\tau] \otimes C_\tau.$$

Recall (see [13]) that the structure morphism $Y \to pt$ induces canonical inclusions $R(T) \to K_T(Y)$ (resp. $R(T/T_H) \to K_{T/T_H}(Y)$) which gives an $R(T)$-algebra (resp. $R(T/T_H)$-algebra) structure on $K_T(Y)$ (resp. $K_{T/T_H}(Y)$).

Note that we have an exact sequence

$$0 \to X^*(T/T_H) \to X^*(T) \to X^*(T_H) \to 0$$

of character groups, where $X^*(T/T_H)$ denotes the characters of $T$ that are trivial when restricted to $T_H$. The exact sequence (3.10) splits by choosing a basis of $X^*(T_H)$ and by lifting every element of this basis to a character of $T$. This implies that we have the following isomorphism

$$R(T) \simeq R(T/T_H) \otimes R(T_H)$$

where $R(T) = \mathbb{Z}[X^*(T)], R(T_H) = \mathbb{Z}[X^*(T_H)], R(T/T_H) = \mathbb{Z}[X^*(T/T_H)]$ (see [17, page 485] for the analogous statement for Chow ring).

More generally, we have the following proposition whose proof is similar to that of [17, Lemma 1.7] (see [7, Lemma 2.3.2] for the analogous statement for Chow ring).

**Proposition 3.10.** We have the following isomorphisms of rings

$$R(T) \simeq R(T/T_H) \otimes R(T_H)$$

where $R(T) = \mathbb{Z}[X^*(T)], R(T_H) = \mathbb{Z}[X^*(T_H)], R(T/T_H) = \mathbb{Z}[X^*(T/T_H)]$ (see [17, page 485] for the analogous statement for Chow ring).
Theorem 3.12. This together with Proposition 3.10(i) will imply that

(3.13)

Remark 3.11. Note that the above proposition holds in a more general setting where we can take $K$ in $\sum_{T/T}$ gives a weight space decomposition on each fibre. Further, the trivial homomorphism of rings $K_{T/T}(Y) \otimes R(T_H) \rightarrow K_T(Y)$ where $e^\chi \in R(T_H) = \mathbb{Z}[X^*(T_H)]$ maps to $e^\chi \in R(T) = \mathbb{Z}[X^*(T)] \hookrightarrow K_T(Y)$ and the map from $K_{T/T}(Y)$ to $K_T(Y)$ is induced by the surjection $T \rightarrow T/T_H$.

To define the inverse of the above homomorphism, we let $E$ be a $T$-equivariant vector bundle on $Y$. Since $Y$ is a $T/T_H$-variety, the $T_H$-action on $Y$ is trivial. Thus we get a $T_H$-action on every fibre of $E$ which gives a weight space decomposition on each fibre. Further, the $T$-equivariant vector bundle $E$ is locally trivial so the weights of the restricted $T_H$-action are locally constant. Moreover, since $Y$ is irreducible these weights are constant over $Y$. Thus we get an isotypical decomposition $E = \bigoplus E_i$ where $E_i$ denotes the subbundle of $E$ whose fibre is the eigenspace for the $T_H$-action corresponding to a character $\chi_i$ of $T_H$. We can write $E_i = e^{\chi_i} \otimes E'$ where $E'$ is a $T$-equivariant vector bundle with a trivial $T_H$-action and hence a $T/T_H$-equivariant vector bundle on $Y$. We therefore define the inverse map which sends $[E] \in K_T(Y)$ to $\sum_i e^{\chi_i} \otimes [E'] \in R(T_H) \otimes_{\mathbb{Z}} K_{T/T}(Y)$.

Proof: We have the following isomorphism by Theorem 3.12

(3.14)

This together with Proposition 3.10(i) will imply that

(3.13)

Now, by (3.13) and (3.14) we get:

(3.14)

Thus from (3.13) and (3.14), the additive decomposition (3.12) follows.
Observe that $C_\tau \cdot C_\sigma \subseteq C_\gamma$ whenever $\tau$ and $\sigma$ span a cone $\gamma$ in $F_+$. Furthermore, whenever $\gamma = (\tau, \sigma)$ in $F_+$ the product $R(T_H)^{W_\tau} \cdot R(T_H)^{W_\sigma} \subseteq R(T)^{W_\gamma}$ since $R(T_H)^{W_\tau}$ and $R(T_H)^{W_\sigma}$ are both subrings of $R(T)^{W_\gamma}$.

The multiplicative structure now follows exactly as in the proof of [17 Theorem 2.10]. □

We have the following corollary which extends [17 Corollary].

**Corollary 3.13.** The ring $K_G(X) \simeq \bigoplus_{\tau \in F_+} C_\tau \otimes R(T_H)^{W_\tau}$ admits a multifiltration $\{F_\tau\}_{\tau \in F_+}$ where the filtered pieces are

$$F_\tau = \bigoplus_{\tau \leq \sigma} C_\sigma \otimes R(T_H)^{W_\sigma},$$

where $F_\tau \supseteq F_\sigma$ whenever $\tau \leq \sigma$, and $F_{\{0\}} = K_G(X)$. Further, under the multiplication described in Theorem 3.12, we have $F_\sigma \cdot F_\tau \subseteq F_\gamma$ whenever $\gamma = (\tau, \sigma)$ in $F_+$. In particular $F_{\{0\}} \cdot F_\tau \subseteq F_\tau$ for all $\tau \in F_+$. □

**Proof:** The existence of the filtration $\{F_\tau\}_{\tau \in F_+}$ and the properties follow by definition. Further, since the filtered pieces multiply by the multiplication rule given in Theorem 3.12, it follows that $\tau \cdot \sigma \subseteq \gamma$ whenever $\gamma = (\tau, \sigma)$ in $F_+$ and $F_\tau \cdot F_\sigma \subseteq \{0\}$ whenever $\tau$ and $\sigma$ do not span a cone in $F_+$. □

In the following proposition we show that the above multifiltration establishes the existence of the structure of $K_{T/T_H}(Y_0) \otimes R(H)$-algebra on $K_G(X)$. We further show that the multifiltration also implies the existence of certain canonical $K_{T/T_H}(Y_0) \otimes R(H)$-submodules which will be subsequently used in the next section. As a consequence of Theorem 3.12 we further show that $K_G(X)$ has the structure of a canonical $K_{T/T_H}(Y_0) \otimes R(H)$-submodule of $K_{T/T_H}(Y_0) \otimes R(T_H)^{W_L}$.

**Proposition 3.14.**

(i) We have a canonical inclusion $K_{T/T_H}(Y_0) \otimes R(H) \subseteq F_{\{0\}} = K_G(X)$ as $1 \otimes R(H)$-subalgebra.

(ii) We have a canonical inclusion of the $K_{T/T_H}(Y_0) \otimes R(H)$-submodule

$$\prod_{\rho \in \tau(1)} X_\rho \cdot K_{T/T_H}(Y_0) \otimes R(T_H)^{W_\rho}$$

in $F_\tau \subseteq F_{\{0\}} = K_G(X)$.

(iii) We have a canonical inclusion of $K_G(X)$ in $K_{T/T_H}(Y_0) \otimes R(T_H)^{W_L}$ as a $K_{T/T_H}(Y_0) \otimes R(H)$-subalgebra. (This is analogous to [17 Proposition 2.5] and [18 Proposition 2.1].)

**Proof:** (i) From (3.6) it follows that

$$K_{T/T_H}(Y_0) \otimes R(H) = \bigoplus_{\tau \in F_+} C_\tau \otimes R(H).$$

Also

$$C_\tau \otimes R(H) = C_\tau \otimes R(T_H)^{W_H} \subseteq C_\gamma \otimes R(T_H)^{W_\gamma}$$

for every $\tau \in F_+(1)$.

Since both (3.12) and (3.15) are decompositions as 1 ⊗ (R(H))-algebras, (i) follows from (3.16).

(ii) Again from (3.9) we have

$$X_\tau \cdot K_{T/T_H}(Y_0) \otimes R(T_H)^{W_\tau} = \bigoplus_{\rho \in F_+} X_\rho \cdot C_\rho \otimes R(T_H)^{W_\rho}.$$  

Moreover, by the relations in the Stanley Reisner presentation of $K_{T/T_H}(Y_0)$ we have $X_\tau \cdot C_\rho \subseteq C_\gamma$ whenever $\tau$ and $\sigma$ span a cone $\gamma$ in $F_+$ and $X_\tau \cdot C_\sigma = 0$ whenever $\tau$ and $\sigma$ do not span a cone in $F_+$. Thus

$$X_\tau \cdot C_\rho \otimes R(T_H)^{W_\rho} \subseteq C_\gamma \otimes R(T_H)^{W_\gamma}$$

since $R(T)^{W_\rho} \subseteq R(T_H)^{W_\rho}$. Moreover, by the relations in the Stanley Reisner presentation of $K_{T/T_H}(Y_0)$ we have $X_\tau \cdot C_\rho \subseteq C_\gamma$ whenever $\tau$ and $\sigma$ span a cone $\gamma$ in $F_+$ and $X_\tau \cdot C_\sigma = 0$ whenever $\tau$ and $\sigma$ do not span a cone in $F_+$. Thus

$$X_\tau \cdot C_\rho \otimes R(T_H)^{W_\rho} \subseteq C_\gamma \otimes R(T_H)^{W_\gamma}.$$
Corollary 3.15. Let $N_{\tau} \simeq \oplus_{\rho_j \in \tau(1)} L_j$ be the normal bundle of $V_{\tau} = \overline{O_{\tau}}$ in $Y$. Let $N_{\tau} |_{O_{\tau}}$ denote the restriction of the normal bundle to $O_{\tau}$ so that so that
\[
\lambda_{-1}(N_{\tau} |_{O_{\tau}}) := \prod_{\rho_j \in \tau(1)} (1 - [L_j] |_{O_{\tau}}) \in K_{T/T_H}(O_{\tau}).
\]
Then we have the following decomposition:
\[
K_G(X) \simeq \bigoplus_{\tau \in F_+} \lambda_{-1}(N_{\tau} |_{O_{\tau}}) \cdot K_{T/T_H}(O_{\tau}) \otimes R(T_H)^W_{\tau}.
\]
Let $P_{\tau} := \lambda_{-1}(N_{\tau} |_{O_{\tau}}) \cdot K_{T/T_H}(O_{\tau})$ for each $\tau \in F_+$. Then the above decomposition is a ring isomorphism where the multiplication on the right hand side is given as follows:
\[
P_{\tau} \cdot P_{\sigma} \subseteq \begin{cases} P_{\gamma} & \text{if } \tau \text{ and } \sigma \text{ span the cone } \gamma \in F_+, \\ 0 & \text{if } \tau \text{ and } \sigma \text{ do not span a cone in } F_+. \end{cases}
\]

Theorem 3.16. Let $X_F = \prod_{\rho_j \in F} X_j$ for every $F \subseteq \{\rho_j \mid j = 1, \ldots, d\}$ in the polynomial ring $\mathbb{Q}[X_1, \ldots, X_d]$. In particular $X_{\tau} := X_{\tau(1)} = \prod_{\rho_j \in \tau(1)} X_j$ for every $\tau \in F$. Let $e(N_{\tau})$ denote the equivariant Euler class of the normal bundle $V_{\tau} = \overline{O_{\tau}}$. Let $S := H^*_T(pt)$ By [1] Theorem 8, p.7 we have:
\[
H^*_T(T_H)(Y) \simeq \mathbb{Q}[X_1, \ldots, X_d]/(X_F \forall F \notin F)
\]
Let $e(N_{\tau})$ denote the equivariant Euler class of the normal bundle of $V(\tau) = \overline{O_{\tau}}$ which is equal to the top Chern class of $\bigoplus_{\rho_j \in \tau(1)} L_j$. We then have the following description of the equivariant cohomology of a complete symmetric variety of minimal rank (see [12] and [1]). We consider cohomology ring with $\mathbb{Q}$-coefficients.

Theorem 3.17. From the above decomposition, an analogue of Proposition 3.14 for the equivariant cohomology can also be proved which will show that $H^*_G(X)$ is a $H^*_T/T_H(Y_0) \otimes S^{W_H}$-submodule of $H^*_T/T_H(Y_0) \otimes S^{W_L}$ (see [7] Proposition 2.3.3) for a similar statement on wonderful symmetric varieties. Our aim is to show that $H^*_G(X)$ is a free module of rank $|W_G/H|$ over $H^*_T/T_H(Y_0) \otimes S^{W_H}$ and to find a basis of this free module. This will be an extension of the results of Strickland [14] for the compactification of an adjoint semisimple group to any complete symmetric variety of minimal rank. This will be taken up in a future work. We also note here that similar description of the equivariant cohomology ring of any regular compactification of an adjoint semisimple group has been obtained by Strickland (see [15] Theorem 6.1).
3.3. Comparison with wonderful symmetric variety. Let $X$ be a regular compactification of the adjoint symmetric space $G/H$ and $X^{\text{wond}}$ be the canonical wonderful compactification.

Let $\gamma_1, \ldots, \gamma_r$ denote the simple restricted roots.

Recall that $L_{\gamma_i}$ are $G$-linearized line bundles on $X^{\text{wond}}$ such that $\hat{P}$ operates on $L_{\gamma_i}$ by the character $\gamma_i$ for $1 \leq i \leq r$ where $z$ is the base point of the unique closed orbit.

Furthermore, since the centre $Z$ of $\hat{G}$ acts trivially on $X^{\text{wond}}$ and hence on the fibre by the character $\gamma_i$ the line bundle $L_{\gamma_i}$ is actually $G = \hat{G}/Z$-linearized. Moreover, $L_{\gamma_i}$ admits a $G$-invariant section $s_i$ whose zero locus is the boundary divisor $D_{\gamma_i}$ for $1 \leq i \leq r$.

We recall the following construction from [11] Section 3.5] and [9] Section 10).

The bundle $V = \bigoplus_{1 \leq i \leq r} L_{\gamma_i}$ being a direct sum of line bundles on $X^{\text{wond}}$ admits a natural action of the $r$-dimensional torus $G/Z$. Put

$$P := V \setminus \bigcup_{i=1}^r L_{\gamma_1} \oplus \cdots \oplus L_{\gamma_i} \oplus \cdots \oplus L_{\gamma_r}.$$  

Then $P$ is the principal $T/T_H = G^r$-bundle associated with $V$ over $X^{\text{wond}}$. The section $s = s_1 \cdots s_r$ of $V$ carries $G$ to $P$. Therefore using $s$ we can embed $X^{\text{wond}}$ in the bundle $P \times T/T_H \mathbb{A}^r$ which is nothing but $V$. Since the bundles $L_{\gamma_i}$ are $G$-linearized it follows that $P$ is a left $G$-space whose bundle map $P \to X^{\text{wond}}$ is $G$-equivariant for the canonical $G$-action on $X^{\text{wond}}$. Moreover, the right $T/T_H$-action is compatible with the left $G$-action on $P$.

Consider the proper morphism of toric varieties $\varphi : Y_0 \to Y_0^{\text{wond}} \simeq \mathbb{A}^r$ corresponding to a smooth subdivision $\mathcal{F}_+$ of the positive Weyl chamber $C_+$. Since $Y_0$ is a $T/T_H$-toric variety we can form the associated toric bundle $P \times T/T_H Y_0$ over $X^{\text{wond}}$. The morphism $\varphi$ further induces $\varphi : P \times T/T_H Y_0 \to P \times T/T_H Y_0^{\text{wond}} = V$ which is isomorphic to the pull back of $P \times T/T_H Y_0$ to $V$ via the bundle projection $p : V \to X^{\text{wond}}$.

From [11] Definition 23] and [9] Section 3.2], $X = X_{Y_0} = \varphi^{-1}(s(X^{\text{wond}}))$ which can be identified with pull-back of $P \times T/T_H Y_0 \to P \times T/T_H \mathbb{A}^r$ via the section $s$. Thus $X = X_{Y_0} = s^*(p^*(P \times T/T_H Y_0))$. Now, $p \circ s = Id_{X^{\text{wond}}}$ hence $s^* \circ p^* = Id^r$. Thus

$$X \simeq P \times T/T_H Y_0.$$  

For $u \in X^* (T/T_H)$ let $L_u := P \times T/T_H \mathbb{C}_u$ denote the $G$-equivariant line bundle on $X^{\text{wond}}$ associated to the character $u$. Let $[L_u]_G$ denote its class in $K_G(X^{\text{wond}})$.

Let $X$ be a projective regular compactification of $G/H$ so that $Y_0$ is a semi-projective $T/T_H$-toric variety. Then the following theorem follows by [18] Theorem 5.1] and (3.19).

**Theorem 3.18.** The $G$-equivariant Grothendieck ring of $X$ has the following description as an algebra over the $G$-equivariant Grothendieck ring of $X^{\text{wond}}$:

$$K_G(X) \simeq \frac{K_G(X^{\text{wond}})[X_{j}^{\pm 1} : \rho_j \in \mathcal{F}_+(1)]}{J}$$  

where $J$ is the ideal in $K_G(X^{\text{wond}})[X_{j}^{\pm 1} : \rho_j \in \mathcal{F}_+(1)]$ generated by the elements $X_F$ for $F \notin \mathcal{F}_+$ and

$$\prod_{\rho_j \in \mathcal{F}_+(1)} X_{j}^{(u, v_j)} - [L_u]_G \text{ for } u \in X^*(T/T_H).$$  

Recall that the adjoint symmetric spaces of minimal rank are exactly the products of the symmetric spaces listed in [7] Examples 1.4.4]. In particular, [7] Examples 1.4.4](1) is the adjoint semisimple group $G = G \times G/\text{diag}(G)$ and [7] Examples 1.4.4](2) is the space $G/H = \text{PGL}(2n)/\text{PSp}(2n)$. The $G \times G$-equivariant $K$-ring of (1) has already been described in [17]. In the following example we shall describe the $G$-equivariant $K$-ring of the wonderful compactification $X^{\text{wond}}$ of $\text{PGL}(2n)/\text{PSp}(2n)$.

**Example 3.19.**

Consider the group $G = \text{PGL}(2n)$ and the involution $\theta$ associated with the symmetry of the Dynkin diagram so that $H = \text{PSp}(2n)$ and $\text{rk}(G/H) = n - 1$. We shall consider the case when $n = 3$ for simplicity. The general case is similar.
Let $\Delta_G = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ be the simple roots of the root system $A_5$ corresponding to $PGL(6)$. Then $\Delta_L = \{\alpha_1, \alpha_2, \alpha_3\}$. Also $\Delta_G \setminus \Delta_L = \{\alpha_2, \alpha_4\}$. Thus $\Delta_{G/H} = \{\gamma_1 = \alpha_2 - \theta(\alpha_2), \gamma_2 = \alpha_4 - \theta(\alpha_4)\}$ are the simple restricted roots.

Since $T_H$ is a maximal torus of the adjoint semisimple group of type $C_3$ it is known that $R(T_H) = \mathbb{Z}[X^*(T_H)]$ where $X^*(T_H)$ has a basis consisting of the rk$(H) = 3$ simple roots $\Delta_H = \{\beta_1 = 2e_1, \beta_2 = e_2 - e_1, \beta_3 = e_3 - e_2\}$ where $\beta_1$ is a long root and $\beta_2$ and $\beta_3$ are short roots.

Note that $\theta(\alpha_1) = \alpha_1$, $\theta(\alpha_3) = \alpha_3$, $\theta(\alpha_5) = \alpha_5$, $\theta(\alpha_2) = -\alpha_1 - \alpha_2 - \alpha_3$ and $\theta(\alpha_4) = -\alpha_3 - \alpha_4 - \alpha_5$.

Thus under the restriction map $q : X^*(T) \rightarrow X^*(T_H)$ (see [2] Lemma 1.4.2) $q^{-1}(\beta_1)$ consists of the unique root $\alpha_1$, $q^{-1}(\beta_2)$ consists of the two strongly orthogonal roots $\alpha_2 + \alpha_3$ and $-\alpha_1 - \alpha_2 - \alpha_3$ and $q^{-1}(\beta_3)$ consists of the two strongly orthogonal roots $-\alpha_3 - \alpha_4$ and $\alpha_4 + \alpha_5$. The restricted simple roots are $\gamma_1 = \alpha_2 - \theta(\alpha_2) = \alpha_1 + 2\alpha_2 + \alpha_3$ and $\gamma_2 = \alpha_4 - \theta(\alpha_4) = \alpha_3 + 2\alpha_4 + \alpha_5$.

Here $W = S_6$, $W_H$ is the semidirect product of $S_2 \times S_2 \times S_2$ with $S_3$, $W_L = S_2 \times S_2 \times S_2$ and $W_{G/H} = S_3$.

In this case the toric variety $Y_{\text{wond}}$ is associated to the Weyl chambers of type $A_2$ and $Y_{\text{wond}}$ is the toric variety associated to the positive Weyl chamber in the coweight lattice spanned by the fundamental coweights $\omega_{\gamma_1}^Y$ and $\omega_{\gamma_2}^Y$ dual to the simple roots $\gamma_1$ and $\gamma_2$ respectively.

The simple reflections in $W_{G/H}$ corresponding to $\gamma_1$ and $\gamma_2$ are $s_{\gamma_1}$ and $s_{\gamma_2}$ respectively. Here $\mathcal{F}_+$ consists of 4 cones namely $\{0\}, \rho_1, \rho_2$ and $\sigma = \langle \rho_1, \rho_2 \rangle$, where the coweight vectors $\omega_{\gamma_1}^Y$ and $\omega_{\gamma_2}^Y$ are the primitive vectors along the rays $\rho_1$ and $\rho_2$ respectively. Note that $W_L = W_L = S_2 \times S_2 \times S_2$, $W_{(0)} = W_H$ which is the semidirect product of $W_L$ with $S_3$, $W_{\rho_1}$ is the semidirect product of $W_L$ with the subgroup of $W_{G/H} = S_3$ generated by $s_{\gamma_1}$, $W_{\rho_2}$ is the semidirect product of $W_L$ with the subgroup of $W_{G/H} = S_3$ generated by $s_{\gamma_2}$.

Let $\mathcal{R} := R(T_H)^{W_L}$, $R(PSp(6)) = R(H) = R(T_H)^{W_H} = \mathcal{R}^{W_{G/H}}$. Furthermore, $K_{T/T_H}(Y_0) \simeq R(T/T_H) = \mathbb{Z}[e^{\pm \gamma_1}, e^{\pm \gamma_2}] \cong \mathbb{Z}[X_{\rho_1}^\pm, X_{\rho_2}^\pm]$ where $X_{\rho_1}$ (respectively $X_{\rho_2}$) maps to the class of the trivial line bundle on $Y_0$ where $T/T_H$ acts on the fibre via the character $\gamma_1$ (respectively $\gamma_2$). The following is an additive decomposition of $K_{PGL_4}(X)$ as a $1 \otimes R(PSp(6))$-submodule of $R(T/T_H) \otimes \mathcal{R}$.

\[
\mathbb{Z} \otimes R(PSp(6)) \oplus (1 - e^{\gamma_1}) \cdot \mathbb{Z}^{[e^{\pm \gamma_1}] \otimes \mathcal{R}^{s_{\gamma_2}}} \oplus (1 - e^{\gamma_2}) \cdot \mathbb{Z}^{[e^{\pm \gamma_2}] \otimes \mathcal{R}^{s_{\gamma_1}}} \oplus (1 - e^{\gamma_1}) \cdot (1 - e^{\gamma_2}) \cdot \mathbb{Z}^{[e^{\pm \gamma_1}, e^{\pm \gamma_2}] \otimes \mathcal{R}}
\]

**Remark 3.20.** By the parametrization of line bundles on spherical varieties (see [3] Section 2.2) the $G$-linearized line bundles on $X$ correspond to $PL(\mathcal{F}_+)$ which denotes the piecewise linear functions on the fan $\mathcal{F}_+$. Let $\mathcal{L}_h$ be the $G$-linearized line bundle on $X$ corresponding to $h = (h_\sigma)_{\sigma \in \mathcal{F}_+(r)}$ then $\tilde{P} = \pi^{-1}(P)$ acts on $\mathcal{L} |_{h_0}$ by the character $h_0$. Thus $\mathcal{L}_h = \mathcal{L}_h |_{h_0}$ is a $\tilde{T}/T_H$-linearized line bundle on $Y_0$ corresponding to the piecewise linear function $h \in PL(\mathcal{F}_+)$ so that $T/T_H$ acts on $\mathcal{L}_h |_{h_0}$ by the character $h_0$. Since it is known that $Pic^{\tilde{T}/T_H}(Y_0)$ generates $K_{\tilde{T}/T_H}(Y_0)$ as a ring (see [19]) it follows that $K_{\tilde{T}/T_H}(Y_0)$ is the subring of $K_{\tilde{G}}(X)$ generated by $Pic^{\tilde{G}}(X)$. We can further identify $K_{\tilde{G}}(G/H) \simeq R(\tilde{H})$. We wonder whether $K_{\tilde{G}}(X)$ has a canonical $K_{\tilde{T}/T_H}(Y_0) \otimes R(\tilde{H})$-module structure so that $K_{\tilde{G}}(X)$ is a free module of rank $|W_{G/H}|$ over $K_{\tilde{T}/T_H}(Y_0) \otimes R(\tilde{H})$ and if there exists a basis for this free module generalizing the results in [17] Section 3 and [18] to the setting of all complete symmetric varieties of minimal rank. See [6] Theorem 4 (ii) for a similar statement describing $K(X^{\text{wond}}|_0)$ as a free module of rank $|W_H/W_L|$ over the classes of the boundary divisors $[O_{D_\gamma}]$ for $\gamma \in \Delta_{G/H}$ together with the classes of equivariant vector bundles $\Phi(\lambda)$ induced by the fundamental weights $\lambda$ of $H$.

An analogue of [7] Proposition 2.3.1 for the $\tilde{T}$ and $G$-equivariant Grothendieck ring may give us a way to extend [17] Theorem 3.3 to all wonderful symmetric varieties of minimal rank. However the proof of [7] Proposition 2.3.1 does not extend to the integral setting of the Grothendieck ring, due to the reason that there is no $W_L$-invariant splitting for short exact sequence of character groups

\[
0 \rightarrow X^*(\tilde{T}/T_H) \rightarrow X^*(\tilde{T}) \rightarrow X^*(T_H) \rightarrow 0
\]

This can be seen from the following example of $PGL_4/PSp(4)$.

**Example 3.21.** In the case of $G = PGL(4)$ and $H = PSp(4)$, $\tilde{G} = SL(4)$ and $\tilde{H} = Sp(4)$. The maximal torus $\tilde{T}$ is the diagonal torus $(t_1, t_2, t_3, t_4) \in (C^*)^4$ satisfying $t_1 \cdot t_2 \cdot t_3 \cdot t_4 = 1$. The subtorus $\tilde{T}_H$ of $\tilde{T}$ consisting of $(t_1, t_2, t_3, t_4)$ such that $t_1 \cdot t_2 = 1$ and $t_3 \cdot t_4 = 1$. Here $W_L \simeq S_2 \times S_2$ is the subgroup of $S_4$ generated by the transpositions $\{(1, 2), (3, 4)\}$. The existence of a $W_L$-invariant splitting for the restriction
homomorphism

\[ q : X^*(\widetilde{T}) \rightarrow X^*(\widetilde{T}_H) \]

would mean the existence of a complementary subtorus of \( \widetilde{T}_H \) in \( \widetilde{T} \) which is \( W_L \)-invariant. A 1-dimensional subtorus of \( \widetilde{T} \) is a 1-parameter subgroup \((t^{a_1}, t^{a_2}, t^{a_3}, t^{a_4})\) for \( t \in \mathbb{C}^* \) where \((a_1, a_2, a_3, a_4) \in \mathbb{Z}^4 \) satisfies \( a_1 + a_2 + a_3 + a_4 = 0 \). If this is \( W_L \)-invariant then it will be of the form \((t^{a_1}, t^{a_1}, t^{a_3}, t^{a_3})\) such that \( a_1 + a_3 = 0 \). Thus it will be of the form \((t, t, t^{-1}, t^{-1})\) for \( t \in \mathbb{C}^* \). However, this subtorus is not a complement of \( \widetilde{T}_H \) since it intersects \( \widetilde{T}_H \) along the subgroup \( \{\pm 1\} \) of order 2. Thus in general there cannot exist a \( W_L \)-invariant splitting for \( 3.22 \).

**Remark 3.22.** \((Assumptions on the base field)\) Although we work over the field of complex numbers all the results in this paper should hold over an algebraically closed field \( k \) of characteristic \( \neq 2 \) (see \[7\] p. 473 and \[10\] p. 273-274).

**Acknowledgments:** I am very grateful to Prof. Michel Brion for his valuable comments and suggestions while preparing this manuscript. I thank the referee for a careful reading of the manuscript and for very helpful comments and suggestions. I wish to mention that Remark 3.7 has been added based on the referee’s observation.

**Statements and Declarations** I hereby declare that this work has no related financial or non-financial conflict of interests.

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