Abstract

Using numerical simulations of the full nonlinear equations of motion we investigate topological solitons of a modified $O(3)$ sigma model in three space dimensions, in which the solitons are stabilized by the Hopf charge. We find that for solitons up to charge five the solutions have the structure of closed strings, which become increasingly twisted as the charge increases. However, for higher charge the solutions are more exotic and comprise linked loops and knots. We discuss the structure and formation of these solitons and demonstrate that the key property responsible for producing such a rich variety of solitons is that of string reconnection.
1 Introduction

Topological solitons are of great interest in a number of areas, including particle physics, cosmology and condensed matter physics. However, only in the last few years, with the advances in computing power, has it become possible to fully investigate topological solitons in three spatial dimensions. Recent results on Skyrmions [2, 3, 4] and monopoles (see for example, ref. [20]) have revealed that intricate and fascinating structures appear, which are inherently three-dimensional and therefore cannot arise as generalizations of solitons in one or two space dimensions. Therefore, it is of considerable interest to investigate other three-dimensional soliton models, in order to determine the type of behaviour we may expect to see in this rapidly developing field.

In the majority of topological soliton models that have been studied to date the topological charge, or soliton number, arises as a winding number between spheres of equal dimension. For example, in gauged $SU(2)$ models the monopole number is the degree of a map between 2-spheres, which counts the winding of the Higgs field in the gauge orbit of vacua at spatial infinity. Similarly, in the Skyrme model the baryon number is a winding number between two 3-spheres, which counts the wrapping of the $SU(2)$ Skyrme field as a map from compactified space. By contrast, in the modified $O(3)$ sigma model we consider here the topological characteristic responsible for the stability of the solitons is not a winding number, but is a linking number of the field lines. There is an associated map between spheres, but the spheres do not have the same dimension ($S^3$ to $S^2$) and the topological charge is the Hopf invariant of this map. From the point of view of identifying the soliton, these solitons will be line-like, as opposed to point-like as is the case for monopoles and Skyrmions, indicating that some very different solitonic structures are likely.

There have been two preliminary numerical investigations [8, 10] of solitons in this model with charges one and two. Both concentrated mainly on axially symmetric configurations, using toroidal or cylindrical coordinates so that the numerical problem is effectively reduced to one in two spatial dimensions. Although this approach has the advantage of substantially reducing the resources required to relax configurations, it does suffer from two serious drawbacks which make interpretation of the results difficult. Firstly, only axially symmetric solitons can be studied, which is unlikely to be appropriate for minimum energy multi-solitons of arbitrary charge. The second, more technical problem is that in these coordinate systems, the complicated topological nature of the solutions requires the imposition of subtle boundary and regularity conditions. Not surprisingly, some of the results of these investigations, such as the distribution of the energy density for the charge one solution, are qualitatively different due to differing choices of these conditions.

In the work reported here we avoid these difficulties by performing fully three-dimensional simulations in Cartesian coordinates, where such problems simply do not arise; of course, the price to be paid for this is a substantial increase in the required computer power. But given sufficient resources (for example, a parallel super-computer), this approach is the most conservative discretization of the model. The code used is a modified version of that developed to investigate the dynamics and bound states of Skyrmions in (3+1) dimensions (see ref.[4] for a detailed description). It has been substantially tested and
produced some very attractive results in that context. When applied to the modified $O(3)$ sigma model, we reproduce the common results of refs. [8, 10], but where differences arise, we find agreement with the work of Gladikowski and Hellmund [10]. This suggests two things: (i) the assumption of axial symmetry is sufficient for charges one and two, (ii) the imposition of the boundary and regularity conditions was done incorrectly in ref. [8].

The results presented in the subsequent sections of this paper (see also ref. [5]) go much further than resolution of the differences between refs. [8, 10]. We present candidate minimum energy configurations for charges one to eight which suggest that exotic linked and knotted solitons may also exist at higher charges. This is in keeping with the spirit of ref. [8], where it was suggested that a knotted soliton is possible even at charge three. However, the details are very much different. The charge three soliton is just a twisted torus, as are those for charge four and five, with the degree of twisting becoming more noticeable as the charge increases. But at charge six the solution is very different; it being constructed from two linked tori. At charge seven the solution resembles that of a trefoil knot and at charge eight it comprizes of two tori which are linked twice. We will discuss the precise details of each soliton and attempt to identify a principle dictating the structure of them, as we did for Skyrmions [3, 4]. This is more difficult in this case since as we have already noted the solitons are line-like, as opposed to point-like, and hence the solutions do not have point symmetries. We believe, nonetheless, that we have identified some interesting qualitative trends.

The identification of these complex solitonic structures is interesting in its own right as a mathematical exercise. However, we believe that the model in question also has some important physical implications in condensed matter and particle physics. Many condensed matter systems are described by an $O(3)$ unit vector order parameter, examples being nematic liquid crystals and magnetic bubbles in electron gas systems, and hence we have identified candidate solitonic structures which may be found experimentally. More speculatively, Faddeev and Niemi have suggested that this model could be used as an approximation for strongly coupled $SU(2)$ Yang-Mills theory [9], in which case the solitons would be confined glueballs. Although we will not refer further to these or any other possible applications, it is important to realize that soliton models have been found to have many applications in a variety of physical contexts.

### 2 The modified $O(3)$ sigma model

Some time ago Faddeev [7] suggested that stable closed strings may exist as topological solitons in an $O(3)$ sigma model which included a fourth order derivative term, with the topology arising due to the twisting of a planar soliton as it is embedded in three-dimensional space. Explicitly, the field of the model is a real three-component vector $\mathbf{n} = (n_1, n_2, n_3)$, with unit length $\mathbf{n} \cdot \mathbf{n} = 1$, and the Lagrangian density in (3+1)-dimensions is given by

$$
\mathcal{L} = \partial_\mu \mathbf{n} \cdot \partial^\mu \mathbf{n} - \frac{1}{2} (\partial_\mu \mathbf{n} \times \partial_\nu \mathbf{n}) \cdot (\partial^\mu \mathbf{n} \times \partial^\nu \mathbf{n}).
$$

(2.1)
The first term in the Lagrangian (2.1) is that of the $O(3)$ sigma model and the higher order derivative term is required to prevent an instability of configurations under a rescaling of the space coordinates, that is, it is a Skyrme-type term. In fact, the usual Skyrme model [19], in which the field takes values on a three-sphere, may be consistently restricted to a two-sphere equator and this exactly reproduces the model (2.1).

In order for a solution to have finite energy the vector $n$ must tend to a constant value at spatial infinity, which we take to be given by $n_{\infty} = (0, 0, 1)$. The novel aspect of the model under consideration here is that if we restrict to finite energy configurations, then they have a topological characterization. The boundary condition implies that space is compactified from $\mathbb{R}^3$ to $S^3$ so that at any fixed time the field is a map $n: S^3 \mapsto S^2$. Since $\pi_3(S^2) = \mathbb{Z}$, each field has an associated integer topological charge $Q$, the Hopf charge, which gives the soliton number.

As already discussed the soliton number is not a simple winding number, like it is for other solitons such as Skyrmions or monopoles, but rather it is a linking number between field lines. Formally, let $\omega$ denote the area two-form on the target $S^2$ and let $F = n^*\omega$ be its pullback under $n$ to the domain $S^3$. Then due to the triviality of the second cohomology group of the 3-sphere, $H^2(S^3) = 0$, this pullback must be exact, say $F = dA$, and hence the Hopf charge can be constructed by integrating the Chern-Simons term over $\mathbb{R}^3$,

$$Q = \frac{1}{4\pi^2} \int d^3 x \ F \wedge A .$$  \hspace{1cm} (2.2)

An important point to note is that in general the Hopf charge can not be written as the integral of a density which is local in the field $n$, a fact which will have some practical implications for our numerical results. For this reason it is useful to consider a heuristic interpretation of $Q$. The preimage of a point on the target $S^2$ is a closed loop. Now if a field has Hopf number $Q$ then the two loops consisting of the preimages of any two distinct points on the target $S^2$ will be linked exactly $Q$ times. Later we shall use this description, in terms of the linking number of field lines, to identify the Hopf number of various field configurations. In fig. 1 we schematically represent the preimages of two distinct points in the target space for a configuration with $Q = 1$.

Recall that the position $x_0$ of a soliton is usually defined to be the point in space at which the field takes the value antipodal to the vacuum value, which in this case gives $n(x_0) = -n_{\infty}$. Thus the position of the soliton is the curve in space described by the preimage of the vector $(0, 0, -1)$. We shall see that displaying this closed string, giving the position of the soliton, is a useful way to represent the solution.

Not only is there a topological charge in this model, but also a lower bound on the energy in terms of the charge exists [13]. Explicitly the bound on the energy $E$ of a configuration with Hopf charge $Q$ is

$$E \geq c|Q|^\frac{2}{3}$$  \hspace{1cm} (2.3)

where $c$ is the constant $c = 16\pi^2 3^{\frac{2}{3}} \approx 238$. Note that this energy bound is rather unusual in that a fractional power of the topological charge occurs, reflecting the fact that this bound
is not obtained from the usual Bogomolny-type argument, but relies on a sophisticated
piece of analysis for its derivation. As we shall see later, in comparing the above bound to
the energy of the computed solitons we conclude that the fractional power dependence fits
quite well, although the overall normalization constant $c$ could probably be dramatically
increased to obtain a much tighter bound.

The existence of a conserved charge and a lower bound on the energy, plus the stability
to scaling, suggests — but does not prove — the existence of finite energy configurations
at each charge. In order to proceed further the equations of motion for the field $n$ must
be solved, but this is analytically intractable even in the case of $Q = 1$. Hence, a numerical
approach is required. The particular numerical scheme used is a modification of our
algorithm constructed to study Skyrmions [2, 3, 4], whose salient features are a finite differ-
ence scheme on a grid containing $100^3$ points, where spatial derivatives are approximated
by fourth-order accurate finite differences, and time evolution via a second-order leapfrog
method. In order to relax to static solutions from suitably random initial conditions, we
use a number of techniques including the addition of a dissipative term to the equations
of motion and periodic removable of all kinetic energy in the system. All the details of the
numerical algorithm and the relaxation procedure are described in ref. [4].

The methods we employ are the most simple and general discretization of the equations
of motion. They make no assumption as to the spatial symmetry of the solution or its
structure at the origin and hence provide an obvious improvement on the methods employed
in refs. [8, 10]. The only extra ingredient required is the imposition of a boundary condition,
which can be done simply at the edge of the box, mimicking spatial infinity.

3 Initial Conditions

Once a numerical code has been implemented the next stage is to provide suitable initial
conditions for a given charge, which can then be relaxed to yield the static soliton solutions.
Thus, we need to be able to write down various configurations in which the field lines have
the correct topological linking structure for the Hopf charge, under consideration. In
refs. [8, 10] configurations of charge one and two were produced by employing a toroidal
or cylindrical ansatz. Here, we describe how a wide range of fields can be obtained for any
Hopf charge without the need to refer to a particular coordinate system which is tailored
to a given configuration.

We begin with the observation [14] that a field with Hopf charge $Q$ can be obtained by
applying the standard Hopf map to a map between 3-spheres which has winding number $Q$.
More explicitly, let $U(x)$ be a Skyrme field, that is, any smooth map from $\mathbb{R}^3$ into $SU(2)$
which satisfies the boundary condition that $U$ tends to the identity matrix as $|x| \to \infty$.
Since the group manifold of $SU(2)$ is $S^3$ and the boundary condition compactifies space,
then effectively we have that $U$ is a map between 3-spheres. Let $B$ denote the winding
number of this map, which is known as the topological charge or baryon number of the
Skyrme field. Writing the components of $U$ in terms of complex numbers $Z_0$ and $Z_1$ as

$$U = \begin{pmatrix} Z_0 & -\bar{Z}_1 \\ Z_1 & \bar{Z}_0 \end{pmatrix},$$

where $|Z_0|^2 + |Z_1|^2 = 1$, (3.1)

then the standard Hopf map can be written in terms of the column vector $Z = (Z_0, Z_1)^T$ as

$$n = Z^\dagger \tau Z,$$ (3.2)

where $\tau_j$ denote the Pauli matrices. It is easy to see that the vector defined by (3.2) has unit length and satisfies the boundary condition $n(\infty) = n_\infty$. Furthermore, it can be shown that $Q = B$, that is, the Hopf charge of the configuration constructed in this way is given by the topological charge of the Skyrme field used. Therefore, it is possible to construct field configurations with non-trivial Hopf charge given a suitable supply of Skyrme fields. Fortunately some recent work [12] has provided a method for constructing Skyrme fields from rational maps between Riemann spheres and so we can apply these techniques.

The first step is to write the Skyrme field in terms of a profile function $f$ and a direction in the $su(2)$ algebra determined by a unit 3-vector $v = (v_1, v_2, v_3)$. The explicit relation is simply

$$U = \exp(if v \cdot \tau).$$ (3.3)

When combined with the Hopf map (3.2) this results in the fields

$$n_1 = 2(v_3 v_1 \sin f - v_2 \cos f) \sin f,$$ (3.4)

$$n_2 = 2(v_3 v_2 \sin f + v_1 \cos f) \sin f,$$ (3.5)

$$n_3 = 1 - 2(1 - v_3^2) \sin^2 f,$$ (3.6)

which parameterizes the unit vector $n$ in terms of another unit vector $v$ and the profile function $f$.

Now, we follow the approach used for Skyrmions [12] and decompose the Skyrme field into a radial and an angular dependence. Introducing the usual spherical polar coordinates ($r, \theta, \phi$), the ansatz assumes that the profile function depends only on the radial coordinate, that is, $f(r)$ and that the direction in the algebra determined by $v$ is independent of $r$. The boundary conditions are $f(0) = \pi$ and $f(\infty) = 0$, with $f(r)$ a monotonically decreasing function. The details of this function are not important, since we are only interested in creating initial conditions for relaxation, so that any reasonable smooth function with the correct boundary conditions will suffice.

Note that the unit vector $v(\theta, \phi)$ is now a map between 2-spheres, and therefore has a winding number. This winding number is precisely the baryon number $B$ of the Skyrme field and hence also the Hopf charge $Q$ of the configuration we generate. To prescribe this map between 2-spheres it is convenient to use a Riemann sphere parameterization of both the 2-spheres. Explicitly, we use the coordinate $z = e^{i\phi} \tan(\theta/2)$ obtained by stereographic projection and similarly for the target sphere, where we represent a point as a complex number $R(z)$ via

$$v = \frac{1}{1 + |R|^2}(R + \bar{R}, i(\bar{R} - R), |R|^2 - 1).$$ (3.7)
We can now generate configurations of Hopf charge $Q$ by taking $R(z)$ to be a rational function of $z$ of degree $Q$, the simplest choice being to take $R = z^Q$.

This choice of $R = z^Q$ corresponds to a configuration which is axially symmetric. Note that for $Q = 1$ the Skyrme field is spherically symmetric, but the Hopf projection breaks this symmetry so that only an axial symmetry remains. It is has been shown that spherical symmetry is not compatible with a non-zero Hopf charge, so the above axially symmetric configurations are the most symmetric candidates for soliton solutions.

The model (2.1) has a global $O(3)$ symmetry, but the choice of a vacuum value $n_\infty$ breaks this. There still remains a global $O(2)$ symmetry, which rotates the $n_1, n_2$ components. When we refer to a symmetry of a configuration, such as the axial symmetry mentioned above, it is not that the fields must be invariant under a corresponding spatial rotation, but rather that the effect of such a rotation can be undone by acting with the global symmetry of the theory. This implies that both the $n_3$ component (which determines the position of the soliton) and the energy density are strictly invariant under the symmetry operation.

Let us now analyze the axially symmetric configurations, given by $R = z^Q$, in a little more detail. First of all, the position of the soliton is the preimage of the vector $n = (0, 0, -1)$, which by equation (3.4) requires that $\sin^2 f = 1$ and $v_3 = 0$. Given the properties of the profile function then the first requirement determines a unique radial value, $r_0$, such that $f(r_0) = \pi/2$. The second requirement, when combined with equation (3.5), implies that $|R| = 1$, and hence $|z| = 1$. The preimage of $n_3 = -1$ is therefore the circle of radius $r_0$ in the equatorial plane $\theta = \pi/2$. To verify that the Hopf charge is indeed $Q$ we need to consider the preimage of another point on the target sphere, which we choose to be the vector $n = (0, -1, 0)$. Using equations (3.4-3.6) the preimage is determined by the relations

$$2(1-v_3^2) \sin^2 f = 1, \quad v_3v_1 \sin f = v_2 \cos f, \quad 2 \sin f(v_3v_2 \sin f + v_1 \cos f) = -1. \quad (3.8)$$

To examine the linking number of this preimage with the circle identified above we need to consider where this preimage cuts the plane $\theta = \pi/2$. In this plane $z = e^{i\phi}$ giving $R = e^{iQ\phi}$ which when substituted into equation (3.7) gives $v = (\cos Q\phi, \sin Q\phi, 0)$. Substituting this expression into (3.8) we deduce the constraints

$$2 \sin^2 f = 1, \quad \sin 2f \cos Q\phi = -1, \quad \cos f \sin Q\phi = 0. \quad (3.9)$$

The first of the above equations determines exactly two radii, $r_\pm$, given by $f(r_+ ) = \pi/4$ and $f(r_-) = 3\pi/4$, which from the properties of the profile function satisfy the inequalities $r_+ > r_0 > r_-$. For the outer crossings, $r = r_+$, the constraints (3.9) reduce to $\cos Q\phi = -1$, which has exactly $Q$ solutions $\phi = (2p + 1)\pi/Q$, $p = 0, 1, ..., Q - 1$, whereas for the inner crossings at $r = r_-$, constraints (3.9) reduce to $\cos Q\phi = 1$, which also has $Q$ solutions $\phi = 2p\pi/Q$, $p = 0, 1, ..., Q - 1$. By continuity these inner and outer crossings are smoothly connected together and thus this preimage is linked with the preimage of the vector $n = (0, 0, -1)$ exactly $Q$ times. Note also that the inner and outer crossings are equally spaced in the angular direction around the position of the soliton. This completes the verification that the Hopf charge is $Q$. 

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Using the above axially symmetric initial conditions we have relaxed the configurations to find static solutions, which we shall discuss later. Of course, since we impose axial symmetry from the start then all the resulting solutions preserve this axial symmetry. However, as in the case of Skyrmions, we expect that most of these solutions will be saddle points of the energy functional and be unstable to perturbations which break the axial symmetry. To investigate this issue we require initial conditions which are perturbations of the ones we have constructed so far, so that the symmetry is broken. One possibility is to replace the rational map \( R = z^Q \) by a more general degree \( Q \) map and this often leads to similar results as those discussed below. However it is also possible to create configurations with discrete symmetries, which prevent the true minimum energy solution from being found, or if the rational map is too generic it can lead to high energy configurations which break up into small charge clusters. Given these problems we find that adding small perturbations to the axially symmetric configurations is more satisfactory. Take the function \( R \) to have the following form,

\[
R = z^Q \left[ 1 + a \cos \left( \frac{m \phi^2}{2\pi} \right) \right],
\]

(3.10)

where \( a \in [0, 1) \) is a constant amplitude and \( m \) is a constant integer. Recall that \( \phi \) is the phase of the angular variable \( z \) so that \( R \) is no longer a meromorphic function of \( z \). By considering small deformations from this limit it is clear that the Hopf charge of this ansatz is still \( Q \), though the parameter \( a \) has to be restricted to \(|a| < 1\), so that the term multiplying \( z^Q \) is non-zero. To see the physical significance of the deformation we can look at the position of the soliton. As before this is a closed loop lying a distance \( r_0 \) from the origin, but now the loop is not a circle in the plane \( \theta = \pi/2 \). The angular distribution of the loop is given by the equation,

\[
\theta = 2\tan^{-1} \left[ (1 + a \cos \left( \frac{m \phi^2}{2\pi} \right))^{-1/Q} \right].
\]

(3.11)

Thus as \( \phi \) varies, that is, we move around the loop, it dips below and then rises above the equatorial plane \( \theta = \pi/2 \). In other words, what was previously a circular loop now has wiggles with the amplitude being determined by the parameter \( a \) and the number of wiggles controlled by the integer \( m \). Note that we use a quadratic rather than linear dependence on \( \phi \) in the cosine argument to ensure that the symmetry of the configuration is completely broken and that no cyclic subgroup remains. These are the initial conditions that we shall use to relax to the minima in the next section, with typical parameter values being \( a = 0.5 \) and \( m = 50 \).

4 Relaxed soliton solutions

In this section we describe the results of our relaxation computations using the initial conditions discussed in the previous section. We shall consider solitons of charges one to
eight and display our results by plotting several interesting quantities. The first is the
preimage of the vector \((0, 0, -1)\) which defines the position of the soliton. In reality it is
difficult to compute the locus of this preimage in the discretized domain and hence we will
plot isosurfaces of the vector \((0, 0, -1 + \epsilon)\), where \(\epsilon \approx 0.2\) is small. This allows us to easily
visualize the solitons position as a line-like solid rather than a single line. We will also
explicitly display the linking number, thereby verifying the Hopf charge, by plotting, in
a similar way to the position, the loci of two independent points \((0, -1, 0)\) and \((0, 0, -1)\).
Finally, we also plot the isosurfaces of energy density, sometimes superposed with the
position of the soliton so as to characterize the maxima of the energy density. The position
of the initial conditions for each charge is shown in fig. 2

It is possible to compute the total energy of the discretized configurations by integrating
the local energy density over the box. This will in general under estimate the energy of
the configuration since the box is finite. In the case of Skyrmions it was possible to make a
better estimate of the total energy by dividing by the topological charge of the discretized
configuration. It was claimed that this allows one to estimate the energies to within 1%.
However, this is not possible here since we have no expression for the local charge density.
Hence, there will be systematic uncertainties in the computations which could be upto
5%, but comparisons between the energies computed for different configurations should be
more accurate.

4.1 Low charge : \(Q = 1, 2, 3\)

First, let us discuss in detail the solutions for \(Q = 1, 2, 3\) which were studied in refs. \[8, 10\].
In each case we investigated an exactly axially symmetric solution and also the inclusion
of non-symmetric perturbations given by \(\{3.10\}\).

For \(Q = 1\) the solutions relaxed from the symmetric and non-symmetric initial condi-
tions are identical, with the symmetric case remaining almost unchanged except for a slight
rescaling; the results from the non-symmetric initial conditions being presented in fig. 3,
showing the position of the soliton, the linking structure, the energy density profile, and a
comparison between the energy density and the position. The line representing the position
of the soliton is a simple circle and it is clear that the preimages of \((0, -1, 0)\) and \((0, 0, -1)\)
are linked exactly once, hence the relaxed solution we have computed has maintained the
initial Hopf charge \(Q = 1\). The surface of constant energy density has an ellipsoid shape
and is not spherically symmetric. When compared to the position of the soliton the energy
density is seen to be localized well inside the position, with the maximum value being at
the origin, as found in \[10\], and not on a torus surrounding the string as found in \[8\]. The
energies of the two configurations, which are presented in table 1, are also equal (modulo
our earlier discussion of the difficulties in computing the energy), the precise value being
more than double the energy bound for \(Q\), in good agreement with the work of Gladikowski
and Hellmund \[10\]. The fact that symmetric initial conditions remain almost unchanged
suggests that with an appropriate choice of the profile function, the soliton we obtain is
reasonably well approximated by the axially symmetric initial conditions with \(R = z\).

Upon relaxation the behaviour of the \(Q = 2\) initial conditions, both symmetric and
non-symmetric, is very similar to that of $Q = 1$. Once again we present the results from the non-symmetric initial conditions, this time in fig. 4. As for $Q = 1$, the position of the soliton is a simple torus, but now the energy density is also toroidal, although the line of maximum energy density is inside that of the position. From table 1, it is easy to see that the total energy computed for $Q = 2$ is much less than twice that for $Q = 1$, suggesting that the 2-soliton configuration is tightly bound relative to decomposition into two 1-solitons. Once again, the linking structure confirms the Hopf charge, the preimages being doubly linked, and the under and over crossings of the two preimages are equally distributed in the angular direction, as described in the previous section for the rational map ansatz. Our results agree well with those of both ref. [8] and ref. [10] for this charge. Once again, similarity of the configurations relaxed from symmetric and non-symmetric initial conditions, suggest that the minimum energy configuration is well represented by a map with $R = z^2$.

The results of relaxing non-symmetric initial conditions for $Q = 3$ are displayed in fig. 5, while the symmetric initial conditions once again relax to a rescaled axially symmetric configuration. Since the two are not the same it appears that the minimum energy configuration is not a torus. The position of the soliton is a closed loop, but this time it is twisted. Thus, it appears that the twisting of the field lines required to generate this Hopf charge makes it energetically favourable for the position of the soliton to twist also. It should be noted that the twist in the 3-soliton found after relaxation is not related to the wiggles generated in the initial conditions, which are of much smaller amplitude and greater in number. The energy density profile of this solution has a rather unusual shape, appearing to be something very similar to a pretzel, but with two holes and a twist. The total energy of the configuration created from non-symmetric initial conditions is slightly lower than the toroidal configuration, but given the uncertainties in the computing the energies we have eluded to earlier, this cannot be totally convincing. However, we have tried a number of different random initial configurations, all of which relax quickly to the same ‘twisted torus’ and therefore we believe that we have created a non-axially symmetric minimum energy configuration. Gladikowski and Hellmund [11] did not attempt to construct a $Q = 3$ configuration, but if they had using their assumption of axial symmetry they would have found the torus, rather than the twisted torus. Faddeev and Niemi [8] presented some preliminary numerical evidence that there is a stable trefoil knot configuration at this charge. Given the results presented here we believe this to be unlikely, although we will discuss in section 6, our attempts to construct such a configuration.

4.2 Higher charge: $Q = 4, 5, 6, 7, 8$

When we relax non-symmetric initial conditions for $Q = 4$ and $Q = 5$, the solutions are very different to those created from symmetric ones. As for $Q = 3$ the minimum energy configurations appear to be located on twisted closed loops and the energy density distributions are twisted pretzels, which have three and four holes respectively; the computed energies of these discretized minimum energy configurations being smaller than the corresponding tori (see table 1). The positions, linking structures and energy density profiles of
these solitons are summarized with all the others in figs. 6, 7 and 8 respectively.

Above \( Q = 5 \) there appears to be a dramatic change in the structure of the soliton solutions, as can be seen from figs. 6, 7 and 8. At \( Q = 6 \) we see that the position of the soliton is no longer a single connected loop, but consists of two disjoint loops, which moreover are linked. Thus, remarkably we have found that a linked loop has emerged from a rather general asymmetric initial condition consisting of a single loop. Note that the linking number of the position is one, which should not be confused with the linking number \( Q \) which determines the soliton charge and corresponds to the linking number of two loops obtained as the preimages of two distinct points on the target sphere. However, the fact that the position of the soliton, which we recall is itself a preimage, is disconnected and linked means that the counting of the Hopf charge \( Q \) is now a subtle matter. In fact a careful examination of the linking of field lines in fig. 7 reveals that the \( Q = 6 \) configuration resembles two linked \( Q = 2 \) solitons. However, the Hopf charge is not simply additive in the case of a link since when a field line passes through the intersection of the link it should be counted twice. This happens exactly once for each of the two \( Q = 2 \) solitons and hence rather than being a total charge of \( Q = 4 \), it is indeed a \( Q = 6 \) soliton. Note that the energy density does not have the form of two linked loops, but rather is concentrated mainly in the region where the two position loops are linked. It seems that the interaction between the loops contains some of the energy of the soliton, and it can reach this lower energy state by smearing out the energy over a small region, rather than localizing it around the positions of the solitons. This is, of course, consistent with our earlier findings at lower charge where the \( Q = 1 \) soliton has its energy smeared out inside the position loop and even the \( Q = 2 \) soliton energy is maximal on a loop inside the position loop. We shall see that the key to understanding the formation of this linked soliton is the property of string reconnection discussed in section 5. Therefore, the detailed formation of the structure of the \( Q = 6 \) link, and the other exotic solitons introduced below, will be left until section 6.

From fig. 6 it is clear that the position of the \( Q = 7 \) soliton has the structure of a trefoil knot. It was suggested by Faddeev and Niemi that knotted solitons might exist in this model, although it was conjectured that the \( Q = 3 \) soliton had the structure of a trefoil knot. They attempted to justify this numerically by imposing a trefoil knot structure in their initial conditions, but we have already shown using our much more general numerical algorithm that this is not the case. Here, we have a much more satisfactory situation, in that a trefoil knot — the simplest knot configuration — has emerged naturally from general asymmetric initial conditions. This not only shows that knotted solitons indeed occur in this model, but suggests that they are important configurations which arise naturally and do not require any fine-tuning of the initial conditions. We should note that only the locus of the position of the soliton is a trefoil, and that the preimage of other points (for example, \((0,-1,0)\) as shown in fig. 7) will not be. Neither is the energy density profile, which was part of the argument in favour of knotted solitons in ref. [8]. In fact, the energy is smeared out inside the knot.

At \( Q = 8 \) we find yet another new phenomenon: the position of the soliton is again two linked loops, but in contrast to the link at \( Q = 6 \), these links themselves have a higher linking number. In fact the position loops are doubly linked, which makes the task of
counting the Hopf charge even more difficult. It appears that the solution comprizes of two $Q = 2$ solitons doubly linked, the missing Hopf charge coming again from the double linking of the position.

### 4.3 Soliton properties and energy minimization

In table 1 we give the energy values of the solitons computed from the general asymmetric initial conditions discussed earlier. For comparison we also list the energies of the axially symmetric tori solutions (whose relative sizes are shown in fig. 9 for charges one to six) obtained by imposing axial symmetry in the initial conditions by using a rational map of the form $R = z^Q$. Recall that for $Q = 1$ and $Q = 2$ we found that the relaxed solitons regained their axial symmetry even when it was initially broken by a perturbation, so the energy values for these solitons should be equal to those of the tori. The minute differences seen in the first two rows of table 1 are the result of our numerical computation of the total energy, which as discussed earlier is susceptible to errors.

| $Q$ | Soliton Energy | Torus Energy |
|-----|----------------|--------------|
| 1   | 504            | 505          |
| 2   | 835            | 836          |
| 3   | 1157           | 1181         |
| 4   | 1486           | 1542         |
| 5   | 1808           | 1974         |
| 6   | 1981           | 2361         |
| 7   | 2210           | 2600         |
| 8   | 2447           | 3050         |

**Table 1**: Energy of the relaxed soliton and torus solutions for charges one to eight. As discussed in the text, the absolute values are likely to be systematic under estimates, but comparisons between configurations are likely to be qualitatively correct.

There are two obvious conclusions to be drawn from table 1. The first is that the multi-solitons are all easily bound against the break-up into smaller soliton clusters, which one would expect if the energy of the solutions is close to having the $Q^{3/4}$ dependence suggested by the energy bound. The second is that the axially symmetric torus solutions rapidly increase in energy when compared to the soliton solutions we have found, clearly demonstrating the lack of axial symmetry in the general charge $Q$ solution.

Now that we have computed the energies of the solitons upto charge eight, it is possible to begin to investigate the energy bound (2.3) and the growth of the soliton energy with increasing charge. As mentioned earlier the energy of the $Q = 1$ soliton is much larger than the bound (2.3) with the given value of the constant $c$, and hence it is of little practical use. However, since the bound is derived under some very general assumptions, the coefficient computed may be artificially small. If the fractional power of the energy bound is correct then it may be possible to empirically, and within the systematic errors in computing the
energy already discussed, improve upon the coefficient. Let us assume that an energy bound of the form (2.3) exists, but with a new value for the constant $c$. Clearly the tightest bound which could be obtained would be with the value $c = E_1$, where $E_1$ is the energy of the $Q = 1$ soliton. We therefore choose to compare our energy values with this ‘optimal bound’, that is, $E_1 Q^{3/4}$. In fig. 10 we plot the soliton energy (crosses) against this ‘optimal bound’ (dashed line). It can be seen that the soliton energies lie very close to this curve, suggesting that a true bound exists which is very close to this one and moreover that the bound would be very tight. The plot also supports the fractional power growth of the energy, which is an unusual feature, and clearly our results are not consistent with the typical linear growth, $E_1 Q$ (dotted line) common in many topological soliton models. Also shown on the plot are the energies of the tori saddle points (diamonds), which are also well below the linear growth curve, thereby indicating that they too are bound against a break-up into $Q$ well separated 1-solitons.

From our results we can speculate on some qualitative aspects of an energy minimization principle which leads to the interesting structures we have found. A similar approach was taken for Skyrmions [3, 4] and it was deduced that a simple mechanical principle was at work. Since the solitons here are line-like, the total energy is $E = \mu L$, where $\mu$ is the energy per unit length — naively assumed to be universal — and $L$ is the length of string. Therefore, a simple mechanical analogy in this case would be that the length of string must be minimized subject to some constraint, which must be related to the imposition of the correct topological charge, with the relative linking of two preimages requiring a certain amount of gradient energy. For $Q = 1$ and $Q = 2$, one could easily imagine that such a principle requires that the solutions be toroidal, with the twists distributed uniformly. It is clear from the linking structures seen for $Q = 3, 4, 5$ that it is possible for the links to be packed closer together with discrete symmetries, hence reducing the length of string required for the position. What happens for the higher charges is less well defined, but it is clear that having extra links in the position itself can reduce the number of links in each of the individual solitons, as well as localizing the energy. This would suggest that the position itself prefers to be linked as much as possible in order for the constituent parts to have the smallest possible charge. This idea is borne out for $Q = 8$, where two doubly linked $Q = 2$ solitons are formed in preference to two singly linked $Q = 3$ solitons.

The clear oddity is $Q = 7$ which is the first structure where the position is actually knotted. The reason why being knotted is preferred over being linked in this case is probably a question of symmetry. If one were to split $Q = 7$ into two singly linked structures, then removing two for the link would require a total of five to be shared out between them. Clearly, this cannot be done symmetrically and hence a knotted structure is energetically preferred. We have observed some evidence for this during the numerical relaxation of different initial conditions. Often a solution would become trapped in a linked structure (either $Q = 1$ linked with $Q = 4$, or $Q = 2$ linked with $Q = 3$) with energy substantially higher than the computed minimum.

Obviously more examples of higher charge solitons will be required before a detailed understanding of the complete energy minimization principle can be found, but we expect that these features will play a prominent role. One thing that is transparent in this
discussion is that as the charge increases, the number of possibilities for linking proliferates, making the prediction of an energetically preferred configuration increasingly difficult. If sufficient interest is generated, many more hours of supercomputer time could be spent in search of a more general principle!

5 Reconnection of Skyrme Strings

As mentioned in section 2, the initial motivation for the model under investigation here was the observation by Faddeev [7] that a planar soliton with an internal degree of freedom could be embedded in three-dimensional space in such a way that the internal twisting of the soliton produces a topological obstruction to its decay. This lead Faddeev and Niemi to the conjecture that all knotted configurations will exist in this model and indeed we have shown that this may be possible, although not quite in the way that they suggested. In order to reflect on this very wide ranging possibility it is important to understand the processes by which parts of the string may interact.

There are a number of systems in (2+1)-dimensions, such as the Abelian Higgs model, which possess topological solitons whose energy density is localized around a point in space, given by the position of the soliton. If such models are considered in (3+1)-dimensions then the soliton can be trivially embedded into the extra dimension to produce an infinite string-like object, whose energy density is localized along a line. Cosmic strings [21] are the best studied examples of this procedure, where the planar soliton is a vortex. By embedding the vortex along a closed curve, rather than a straight line, the ends of the string can be joined to produce a closed string of finite length. However, such closed strings tend to be unstable configurations and will collapse when considered as dynamical structures (see, for example, ref. [1]). The reason a vortex in the plane is stable is due to the existence of a topological conservation law, but when the vortex is embedded to produce a closed string, this topological stability is lost. In the case of a circular loop this can be understood by realizing that in a plane perpendicular to the loop, one has effectively a vortex/anti-vortex pair.

The planar solitons of relevance to the more complicated model discussed in this paper are known as Baby Skyrmions [16] and arise in the model in (2+1)-dimensions described by the Lagrangian density,

$$\mathcal{L} = \partial_\mu n \cdot \partial_\mu n - \frac{1}{2} (\partial_\mu n \times \partial_\nu n) \cdot (\partial_\mu n \times \partial_\nu n) - \beta^2 (1 - n_3). \quad (5.1)$$

Note that this Lagrangian is the planar version of (2.1) except that an additional potential term, with constant coefficient $\beta^2$, has been added. This additional term stabilizes the solitons to the radial scaling of Derricks theorem, since in two space dimensions the pure sigma model (the first term of (5.1)) is scale invariant. In fact any term containing no derivatives, such as a potential, would suffice for this purpose and the particular choice here was chosen by the authors in ref. [16] by analogy to the pion mass term of the $SU(2)$ Skyrme model. It has the added advantage that the Baby Skyrmions are then exponentially
localized. Although the details of the Baby Skyrmions are sensitive to the choice of a potential the features we are concerned with in this section are topological and hold for any choice of potential term, including zero.

Although a potential term is not required in (3+1)-dimensions one could be included, such as the one given above for Baby Skyrmions, though we have chosen not to include one at this stage in order to make the model as simple as possible. Some investigations on how this can affect the properties of the solitons of charge one and two have been undertaken [10].

Since the field of the model takes values in a two-sphere then in the planar case the relevant quantity is that
\[ \pi_2(S^2) = \mathbb{Z}, \]
so that a Baby Skyrmion is a topological soliton having a non-zero winding number. A single static Baby Skyrmion has the hedgehog form
\[ \mathbf{n} = (\sin g \cos \theta, \sin g \sin \theta, \cos g) \]  
where \((\rho, \theta)\) are polar coordinates in the plane and \(g(\rho)\) is a profile function satisfying the boundary conditions \(g(0) = \pi\) and \(g(\infty) = 0\). The internal phase of this soliton is the freedom to add a constant angle to \(\theta\), which rotates the components \((n_1, n_2)\). As in the three-dimensional case, the position of the Baby Skyrmion is the preimage of the point \((0, 0, -1)\), which in the above example has been chosen to be the origin.

If this Baby Skyrmion is now placed into the (3+1)-dimensional model (2.1) by embedding it along a closed curve, then the curve traced out by the position of the Baby Skyrmion will be a closed loop and will be the preimage of the point \((0, 0, -1)\). If, in addition, the internal phase of the Baby Skyrmion changes by a total angle of \(2\pi Q\) as it moves around the loop then this generates a finite energy configuration with Hopf charge \(Q\).

In order to comment on the conjecture that knotted solitons are inevitable in this model, we wish to study the process of loop intersection and this is most easily examined without the additional complications of curvature effects. We therefore choose to analyze this feature by considering infinite straight strings, rather than closed loops. We define a Skyrme string as a Baby Skyrmion solution (5.2) which is trivially embedded into the (3+1)-dimensional version of (5.1) by making the fields independent of the third Cartesian direction. Clearly the choice of the \(x_3\)-direction is immaterial here and a Skyrme string exists as a solution for any choice of embedding direction.

These Skyrme strings are the analogues of infinite cosmic strings, but where the vortex of the Abelian Higgs model is replaced by the Baby Skyrmion. It is well understood that the intersection of two cosmic strings always leads to the phenomenon of string reconnection [21, 18], where the two strings break at the region of intersection and then reconnect after a change of partners. This process is purely topological and can be seen as a mixture of the right angle scattering of a vortex-vortex interaction with the annihilation of a vortex-anti-vortex pair [18]. Baby Skyrmions also scatter at right angles and annihilate if they have opposite topological charge [17] so we may expect that Skyrme strings also exhibit reconnection.

Using numerical simulations have investigated Skyrme string reconnection for a wide range of parameters. We evolve the full equations of motion which follow from the (3+1)-dimensional version of (5.1), with the parameter value \(\beta = 0.45\). We take a product ansatz...
to form the initial conditions consisting of two infinite straight Skyrme strings lying in the planes \( x_2 = \pm a \), forming angles \( \pm \Theta/2 \) with the \( x_3 \) axis and each Lorentz boosted with speed \( v \) towards each other. The results are similar for varying values of \( a \) and \( v \), but typical values used are \( a = 1 \) and \( v = 0.2 \). Fig. 11 displays energy density isosurfaces as time evolves for three particular initial conditions corresponding to initial relative string angles of \( \Theta = \pi/6, \pi/2, 5\pi/6 \). In each case reconnection takes place, as expected. Note that the initial conditions in fig. 11.a and fig. 11.c look very similar, but that in the first case the strings are almost parallel, whereas in the last case they are almost anti-parallel. This difference becomes apparent in the time evolution when the choice of reconnection partners is made, with the anti-parallel simulation being a much more violent process, producing more radiation due to the large soliton-anti-soliton component of the interaction. Note that if both strings were exactly parallel then this would be nothing more than Baby Skyrmion right angle scattering, and if they were exactly anti-parallel then both strings would annihilate, as it would correspond to a planar soliton-anti-soliton interaction.

We have performed several simulations with varying values of \( a, v, \Theta \) and \( \beta \) and always found that reconnection takes place in the manner described above. Even for \( \beta = 0 \), in which case the model is exactly (2.1), the results are very similar, although the scale of the strings may now vary, as there is now no potential term to provide a stable size for the Baby Skyrmions. It is likely that other potential terms which respect the topology will influence the interaction between strings, such as whether they attract or repel, but the process of reconnection is topological and therefore will occur for all sensible choices of the potential. We have also varied the relative internal phases of the Skyrme strings and verified that reconnection still continues. Note that since the force between two Baby Skyrmions depends upon their internal phases then there are interesting possibilities for the dynamics of our Skyrme strings if the internal phases are varied along the length of the string. In particular it might be possible that the strings could be made to wrap around each other. A more detailed investigation of reconnection in this model is in progress.

We have demonstrated, therefore, that reconnection takes place for infinite parallel strings and we are now in a position to discuss the possibility of the formation of linked and knotted soliton structures. The fact that reconnection can take place illustrates that the stability of links and knots is by no means inevitable in this model and hence the exotic linked and knotted configurations already discussed appear to be a consequence of subtle cancellations between the interactions of different pieces of string.

6 Links and Knots

In fig. 12 we display the time evolution of the position of the soliton for the \( Q = 6 \) configuration during the relaxation process, from the initial conditions fig. 12.1 to the final linked loops fig. 12.8. The initial condition is a small deformation of a circular loop which very quickly develops a number of large twists. In fact, these twists are so severe that by fig. 12.5 the loop has come close to self-intersection. Intersection does then take place with the reconnection process proceeding as described in the previous section, leading to the
formation of two linked loops. Thus, we see that the combination of the twisting with the reconnection property of the strings has led rather naturally to a link.

In fig. 13 we display the time evolution of the position of the soliton for the $Q = 7$ configuration and we see that initially it proceeds in a similar manner to the $Q = 6$ soliton. The twisting and reconnection again result in two linked loops fig. 12.4, but this time since the Hopf charge is odd it is impossible for two linked loops to equally distribute the Hopf charge and hence the energy. This results in an instability of this link which it corrects by performing yet another reconnection, thereby leading to the final state having the structure of a trefoil knot. Since the link and the trefoil only differ by a single reconnection it is apparent that whether links or knots result in the final state may be a delicate matter. It is also likely that at high charge there may be several meta-stable states corresponding to various linked and knotted configurations.

The evolution of the $Q = 8$ soliton, shown in fig. 14, follows a similar pattern of twisting and in this case a number of reconnection events. This leads finally to a configuration which comprizes of two $Q = 2$ solitons that are doubly linked. This configuration appears to form in preference to singly linked and knotted configurations. The instability of the toroidal configuration to twists seen for $Q = 6, 7, 8$ is further evidence that the axially symmetric configurations are unstable.

Finally, we should comment on our attempts to construct a trefoil knot with $Q = 3$ as was suggested in ref. [8]. Clearly, such a configuration is possible, since it will have the correct linking number if the preimage of each point on the target sphere is a trefoil. However, will it be stable? In order to construct initial conditions for such a configuration, one must modify the rational map based approach discussed in section 2. The precise details of how to do this are presented in ref. [6], but once the initial solution is relaxed it appears to settle down to a configuration which comprizes of two linked loops. Note that even the numerical requirement of placing the trefoil in a finite box introduces perturbations which after some time allow the symmetry of the trefoil knot to be broken. The initial energy of the trefoil is around three times that of the relaxed minimum energy configuration, while that of the final relaxed configuration is twice the minimum. The fact that the solution does not relax down to the minimum suggests that there is some remaining symmetry from the initial conditions we use and hence the two linked loops are a symmetric saddle point. We should note that the ansatz we use for the initial trefoil is by no means unique and hence it might be possible for us to construct a stable trefoil configuration using more symmetry. However, its energy is likely to be far too high for it to be relevant in any physical application of this theory. It appears that the energy minimization principle discussed in section 4 is in action here. Obviously, a much larger piece of string (and hence energy) is required for the links to be arranged in the trefoil shape at this charge.

7 Conclusions

Using numerical simulations we have investigated solitons, stabilized by the Hopf invariant, in a modified $O(3)$ sigma model. Our results follow on from those of refs. [8, 10] for $Q = 1, 2,$
confirming the results of ref. [10] where differences occur. Using our more general numerical algorithm and effectively random initial conditions, we have been able to investigate higher charges, illustrating that for $Q = 3, 4, 5$ the solitons have the form of twisted tori, rather than knots as suggested in ref. [8]. However, the basic premise that knots and other exotic linked solitons can occur in this model is confirmed for $Q = 6, 7, 8$ on the basis of these numerical computations. We have attempted to construct a qualitative energy minimization principle based on the analogy to the mechanics of a piece of string. We propose that the length of string is minimized subject to the requirement that there are sufficient twists for a particular Hopf charge. It appears that the position of the soliton will link with itself once there is sufficient string for sensible daughter links to be created and that if this cannot be done symmetrically, then a knot will be formed. It is obvious that as the charge increases, the number of possibilities for the linking structure will increase rapidly and that reconnection will play an important role in this. Hence, many more hours of CPU time will be required to pin down a more quantitative principle.

In a similar study of Skyrmions [3] we were able to demonstrate that interesting structures appeared in the soliton solutions and suggest an energy minimization principle responsible for their formation. Subsequently these numerical results were supported by an approximate analytical approach, involving an ansatz based on rational maps between Riemann spheres [12]. It would be desirable to construct a similar approximate technique for the solitons computed in this paper, and perhaps the rational map generated initial conditions or the more general rational map ansatz introduced in ref. [6] may be a useful starting point.

In the introduction we mentioned that there are a number of possible physical manifestations of this model. Now that we have made this preliminary investigation of the Hopf stabilized solitons in this model, we can now turn our attention to these. Aspects such as the dynamics and scattering of these solitons, both in relativistic and dissipative versions of the model, will then need to be investigated in great detail and preliminary investigations are already underway.

Notes added.

1. A suggested improved value for the constant $c$ in the energy bound has recently been proposed – R.S. Ward, Durham preprint DTP-98/55. This value is approximately double the value in the known bound and hence is in good agreement with the results in this paper.

2. Some of our findings on the structure of the soliton solutions (for example the twisted torus at charge three) have recently been confirmed by numerical relaxation calculations which use very different numerical algorithms and initial conditions from the ones employed here – J. Hietarinta and P. Salo, hep-th/9811053.
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Figure Captions

Fig. 1. A sketch showing two loops corresponding to the preimages of two points on the target sphere. The loops are linked exactly once, indicating that the configuration has Hopf charge $Q = 1$.

Fig. 2. An isosurface plot of the locus of position for the initial conditions used in the relaxation process for $Q = 1$ to $8$. Each one is close to being a circle, but it has randomly placed wiggles superimposed on it.

Fig. 3. Isosurface plots displaying the locus of the position, the energy density, linking structure between two independent points on the target sphere and a comparison between the position and energy density, for the $Q = 1$ soliton. Notice that the linking number is indeed one and that the energy density is not a torus, but rather its maximum is a point inside the locus of the position.

Fig. 4. The same quantities as in fig. 3, but for the $Q = 2$ soliton. Notice that both locus of the position and the energy density are both toroidal, but that the energy density is peaked inside the position.

Fig. 5. As fig. 3 but for the $Q = 3$ soliton. Clearly, the position of the soliton is now not axially symmetric.

Fig. 6. Isosurface plots showing the position of the soliton for $Q = 1$ to $Q = 8$.

Fig. 7. Isosurface plots showing the linking number of the soliton for $Q = 1$ to $Q = 8$.

Fig. 8. Isosurface plots showing the energy density of the soliton for $Q = 1$ to $Q = 8$.

Fig. 9. Isosurface plots showing the locus of the position after the relaxation of the symmetric (toroidal) initial conditions.

Fig. 10. A plot of the soliton energy (crosses) and energy of the torus solutions (diamonds) for Hopf charge from $Q = 1$ to $Q = 8$. Also shown is a linear growth of energy $E_1Q$ (dotted line), and a fractional power growth in energy $E_1Q^{3/4}$ (dashed line). Here $E_1$ is the energy of the $Q = 1$ soliton. The plot clearly displays the fractional power growth of the soliton energy with Hopf charge $Q$.

Fig. 11. Energy density isosurfaces at increasing times for two reconnecting Skyrme strings. The three different sequences correspond to a relative angle between the strings of (a) $30^\circ$; (b) $90^\circ$; (c) $150^\circ$.

Fig. 12. Isosurfaces displaying the position of the soliton during the relaxation of a $Q = 6$
soliton.

Fig. 13. Isosurfaces displaying the position of the soliton during the relaxation of a $Q = 7$ soliton.

Fig. 14. Isosurfaces displaying the position of the soliton during the relaxation of a $Q = 8$ soliton.
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