Langlands program for $p$-adic coefficients and the petites camarades conjecture

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Abstract

In this paper, we prove that, if Deligne’s “petites camarades conjecture” holds, then a Langlands type correspondence holds also for $p$-adic coefficients on a smooth curve over a finite field. We also prove that any overconvergent $F$-isocrystal of rank less than or equal to 2 on a smooth variety is $\iota$-mixed.

1. Introduction

Let $p$ be a prime number, and $k$ be a finite field with $q = p^s$ elements. Let $X$ be a smooth proper geometrically connected curve over $\text{Spec}(k)$ with function field $K$. First, we put:

$A_r$: The set of isomorphism classes of irreducible cuspidal automorphic representations $\pi$ of $\text{GL}_r(\mathbb{A}_K)$, where $\mathbb{A}_K$ denotes the ring of adeles of $K$, such that the order of the central character of $\pi$ is finite.

Let $U$ be an open dense subscheme of $X$. Let $K$ be a finite extension of $K_0 := W(k) \otimes \mathbb{Q}$. We fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $K_0$. For an overconvergent $F$-isocrystal $E$ of rank $r$ on a dense open subset $U$ of $X/K$ (see §2 for the definition), we say that it is of finite determinant if there exists a positive integer $n$ such that the $n$-th tensor power of $\det(E) := \bigwedge^r E$ is the trivial overconvergent $F$-isocrystal. We say that $E$ is absolutely irreducible if for any finite extension $L$ of $K$ in $\mathbb{Q}_p$, the isocrystal $E \otimes_K L$ on $U/L$ is irreducible. We put:

$I_r(U/K)$: The set of isomorphism classes of absolutely irreducible overconvergent $F$-isocrystals of rank $r$ on $U/K$ which is of finite determinant.

For any finite extension $L$ of $K$, the scalar extension induces a map $I_r(U/K) \to I_r(U/L)$, and we put $I_r(U) := \lim_{K \to K_0} I_r(U/K)$ where $K$ runs over any finite extension of $K_0$ in $\overline{\mathbb{Q}}_p$. When we have two open subschemes $V \subset U$ of $X$, the restriction induces a map $I_r(U) \to I_r(V)$ by Lemma 2.1 below. Using this map, we define a set by

$I_r := \lim_{U \subset X} I_r(U).

In this paper, we fix two things: an isomorphism $\iota: \mathbb{Q}_p \cong \mathbb{C}$, and a root $\pi$ of the equation $x^{p-1} + p = 0$ or equivalently a non-trivial additive character $\mathbb{F}_p \to \mathbb{Q}_p^\times$ (cf. [AM 2.4.1]). We conjecture the following:

Conjecture (Langlands program for $p$-adic coefficients). — There are two maps $\pi_\bullet$ and $E_\bullet$ as follows:

1. There exists a unique map $\pi_\bullet: I_r \to A_r$ such that for $E \in I_r$, the sets of unramified places of $E$ and $\pi_E$ coincide, which we denote by $U$, and for any $x \in |U|$, the eigenvalues of the Frobenius of $E$ at $x$ and the Hecke eigenvalues of $\pi_E$ at $x$ coincide.
2. There exists a unique map \( E\ast : \mathcal{A}_r \to \mathcal{I}_r \) such that for \( \pi \in \mathcal{A}_r \), the sets of unramified places of \( \pi \) and \( E_\pi \) coincide, which we denote by \( U \), and for any \( x \in |U| \), the eigenvalues of the Frobenius of \( E_\pi \) at \( x \) and the Hecke eigenvalues of \( \pi \) at \( x \) coincide.

Moreover, these maps induce one-by-one correspondence, namely \( E\ast \circ \pi\ast = \text{id}_{\mathcal{I}_r} \), and \( \pi\ast \circ E\ast = \text{id}_{\mathcal{A}_r} \).

For short, we call this Conjecture (L). In the celebrated paper \([De2\ 1.2.10 (vi)]\), Deligne conjectured existence of “petites camarades” of smooth \( \ell \)-adic sheaves. This conjecture was stated in a vague way, which was later formulated in a clear form by Crew \([Cr2\ 4.13]\) as follows:

**Conjecture** (petites camarades conjecture for curves). — Let \( U \) be a smooth geometrically connected curve over \( k \), and \( \mathcal{F} \) be an irreducible smooth \( \mathbb{Q}_\ell \)-sheaf on \( U \) whose determinant is defined by a finite order character of \( \pi_1(U) \). Then there exists a number field \( E \) such that, for any \( x \in |U| \), the characteristic polynomial of the action of the geometric Frobenius on \( \mathcal{F}_x \) has coefficients in \( E \), and for any place \( \mathcal{P} \) of \( E \) dividing \( p \), there is an overconvergent \( F \)-isocrystal on \( U/E_{\mathcal{P}} \) such that the characteristic polynomials of the Frobenius action at \( x \) for \( \mathcal{F} \) and \( E \) coincide for any \( x \in |U| \).

For short, we call this Conjecture (D). By using the proven Langlands correspondence in the \( \ell \)-adic cases \([La\ VI.9]\), if Conjecture (L) holds, Conjecture (D) holds and moreover, we can take the overconvergent \( F \)-isocrystal in Conjecture (D) to be irreducible. The main theorem of this paper is the following:

**Theorem.** — Conjecture (L) and (D) are equivalent.

The proof of this theorem have been made possible thanks to the product formula for \( p \)-adic epsilon factors proven in \([AM\ 7.3.5]\), and we use the standard argument to deduce the theorem from the product formula (cf. \([De1\ 9.7]\) for GL\(_2\) case, and \([La\ 3.2.2.3]\) for GL\(_n\) case).

In the last part of this paper, we prove that any overconvergent \( F \)-isocrystal of rank less than or equal to 2 on a smooth variety is \( \ell \)-mixed, which can be seen as a part of the results we can derive from Conjecture (L).

### 2. Overconvergent \( F \)-isocrystals

For the Langlands correspondence, we need to consider overconvergent \( F \)-isocrystals on \( U/K \) where \( U \) is an open dense subscheme of \( X \) and \( K \) is a finite extension of \( K_0 \). When \( K \) is not totally ramified over \( K_0 \), this concept is not defined \textit{a priori}, so we briefly review the concept in this section although it might be trivial for experts.

Let \( Y \) be a scheme of finite type over \( k \). We consider the \( s \)-th Frobenius automorphism \( F \) on \( k \), which is the identity since the number of elements of \( k \) is \( p^s \). We denote by \( F\text{-Isoc}^1(Y/K_0) \) the category of overconvergent \( F \)-isocrystals on \( Y/K_0 \) (cf. \([Be1]\)). We define the category of overconvergent \( F \)-isocrystals on \( Y/K \) denoted by \( F\text{-Isoc}^1(Y/K) \) to be \( F\text{-Isoc}^1(Y/K_0) \) using the notation of \([AM\ 7.3]\) (i.e. the category of couples \( (E_0, \lambda) \) such that \( E_0 \in \text{Ob}(F\text{-Isoc}^1(Y/K_0)) \) and \( \lambda : K \to \text{End}(E_0) \), which is called the \( K \)-structure of \( E \), is a homomorphism of \( K_0 \)-algebras).

If \( K \) is a totally ramified extension of \( K_0 \), the category \( F\text{-Isoc}^1(Y/K) \) is nothing but that defined in \([Be1]\). We see easily that the category \( F\text{-Isoc}^1(\text{Spec}(k)/K) \) is equivalent to the category of finite dimensional \( K \)-vector spaces \( V \) endowed with an automorphism of \( K \)-vector spaces \( V \cong V \). Let \( x \) be a closed point of \( Y \), and \( i_x : \{x\} \hookrightarrow Y \) be the closed immersion. Then using the equivalence, \( i_x^*E \) is a finite dimensional \( K \)-vector space endowed with an automorphism, and we are able to consider the characteristic polynomial of the Frobenius action of \( E \) at \( x \).

For an object \( E = (E_0, \lambda) \) of \( F\text{-Isoc}^1(Y/K) \), we note that \( \text{rk}(E_0) \) is divisible by \( [K : K_0] \), and we define the rank of \( E \) to be \( \text{rk}(E) := [K : K_0]^{-1} \cdot \text{rk}(E_0) \). As written in \([AM\ 7.3.5]\), we automatically have rigid cohomology \( H^n_{\text{rig}}(Y, E) \) which is a \( K \)-vector space with Frobenius structure and so on.
Now, let $x \in X \setminus U$. Let $K_x$ be the unramified extension of $K_0$ corresponding to the extension $k(x)/k$. An overconvergent $F$-isocrystal $E_0$ on $U/K_0$ induces a differential module on the Robba ring $\mathcal{R}_{K_x}$ over $K_x$ with Frobenius structure. We denote this by $E_0|_{K_x}$. Given an overconvergent $F$-isocrystal $E = (E_0, \lambda)$ on $U/K$, we put $E|_{K_x}$ to be the differential module with Frobenius structure $E_0|_{K_x}$ endowed with $K$-structure. We see that $\text{irr}(E_0|_{K_x})$ is divisible by $[K : K_0]$, and we define the irregularity of $E$ at $x$ to be $\text{irr}(E|_{K_x}) := [K : K_0]^{-1} \cdot \text{irr}(E_0|_{K_x})$.

Now, we prove the following irreducibility result\footnote{The proof written here was suggested by N. Tsuzuki. One can refer to \cite{AC} for more general results on the irreducibility.}.

2.1 Lemma. — Let $W$ be a smooth curve over $k$, and $V \subset U \subset W$ be dense open subschemes of $W$, and let $j : V \hookrightarrow U$ be the open immersion. Then the functor $j^! : F\text{-Isoc}^!_V(U,W/K) \to F\text{-Isoc}^!_U(V,W/K)$ preserves irreducibility.

Proof. In the following argument, all the modules are implicitly equipped with $K$-structure. Let $E$ be an irreducible object in $F\text{-Isoc}^!_V(U,W/K)$. Assume there exists a non-zero overconvergent $F$-isocrystal $E_V'$ and a surjection $j^! E \to E_V'$. Then we can find a finite covering $f : W' \to W$ such that the following diagram is commutative

\[
\begin{array}{ccc}
V' & \xrightarrow{j'} & U' \\
\downarrow f_V & & \downarrow f \\
V & \xrightarrow{j} & U
\end{array}
\]

where $f_V$ is finite étale and $f_V^* E_V'$ is log-extendable to $U'$ by \cite{Ke1}. Since $f_V^* E \to f_V^* E_V'$, $f_V^* E_V'$ extends to an overconvergent $F$-isocrystal $E_{U'}$ on $U'$ which is a quotient of $f^* E$. Let $\mathcal{W}$ and $\mathcal{W}'$ be smooth formal liftings of $W$ and $W'$ respectively over $\text{Spf}(W(k))$. Let $Z := W \setminus U$ and $Z' := W' \setminus U'$. We have a coherent $\mathcal{D}_{\mathcal{W'}, \mathbb{Q}}(\mathcal{I}Z')$-module $\text{sp}_*(E)$ and a coherent $\mathcal{D}_{\mathcal{W}, \mathbb{Q}}(\mathcal{I}Z)$-module $\text{sp}_*(E_{U'})$. Since $f_W$ is proper, we have a functor $f_+ : D^b_{\text{coh}}(\mathcal{D}_{\mathcal{W}, \mathbb{Q}}(\mathcal{I}Z)) \to D^b_{\text{coh}}(\mathcal{D}_{\mathcal{W'}, \mathbb{Q}}(\mathcal{I}Z'))$ (see \cite{Be2} 3.4.3 for the construction of the functor in the case where $f_W$ is not liftable). Then we get a homomorphism

\[
\phi : \mathcal{H}^0(f_+ f_V^* \text{sp}_*(E)) \to \mathcal{H}^0(f_+ \text{sp}_*(E_{U'})).
\]

By using the trace homomorphism (see \cite{Alb} 4.15), $\text{sp}_*(E)$ can be seen as a submodule of $\mathcal{H}^0(f_+ f_V^* \text{sp}_*(E))$. We define $E' := \text{sp}_*(\phi(\text{sp}_*(E)))$. Since $j^! E' \cong E_V'$, $E'$ is not equal to $E$ if $E_V'$ is not equal to $j^! E$. Since $E$ is assumed to be irreducible, this shows that $E_V' = j^! E$, and conclude the proof. \hfill \blacksquare

2.2 Remark\footnote{This remark was pointed out to the author by A. Shiho.} — Lemma 2.1 does not hold if we replace $\text{Isoc}^!$ by $\text{Isoc}$. For example, let $f : Y \to A := \mathbb{P}^1 \setminus \{0, 1, \infty\}$ be the Legendre family defined by $y^2 = x(x-1)(x-\lambda)$. Consider the $F$-isocrystal $E := R^1 f_{\text{rig}}^* (Y/A)$ of rank 2. Let $S \subset A$ be the finite set of closed points $s$ such that $f^{-1}(s)$ is a supersingular elliptic curve, and we put $U := A \setminus S$. Then the convergent $F$-isocrystal $E|_U$ is not irreducible as written in \cite{Cr1} 4.15]. However, $E$ is irreducible since all the subquotients of $E|_U$ are not overconvergent along $S$ as written in \textit{ibid.}. The same example is showing that the functor $F\text{-Isoc}^!_V(U,A/K) \to F\text{-Isoc}(U/K)$ does not preserve irreducibility.

3. Čebotarev density

A technical point in the proof of the theorem is to show the uniqueness of the maps. In the $\ell$-adic case, this was a consequence of the Čebotarev density theorem. Since in general we are
not able to describe overconvergent $F$-isocrystals as representations of $\text{Gal}(\overline{K}/K)$, we need a different method to show this. We will show the following proposition\(^{(3)}\).

**Proposition.** — Assume $E$ and $E'$ are $i$-mixed (cf. [Gr3] 10.4) for the definition) overconvergent $F$-isocrystals on an open dense subscheme $U/K$ of $X$ whose characteristic polynomials of the action of the Frobenius at any closed point of $U$ coincide. Then the semi-simplifications of $E$ and $E'$ coincide.

Before proving the proposition, we recall some notation in [Gr3] 10.4, 10.10. Let $M := E^{ss} \oplus E'^{ss}$ where $ss$ denotes the semi-simplification, and take a closed point $x_0$ in $U$. Associated to this overconvergent $F$-isocrystal, we have a short exact sequence:

$$0 \to \text{DGal}(M, x_0) \to W^M_{x_0} \to W(\overline{k}/k) \to 0.$$

Here, $\text{DGal}(M, x_0)$ denotes the differential Galois group of $M$ (see [Cr2] 2.1), $W^M_{x_0}$ denotes the Weil group of $M$ (see [ibid., 5.1]), and $W(\overline{k}/k) \cong \mathbb{Z}$. We denote the extension of scalars by $K \to \mathbb{C}$ using the isomorphism $\iota$ of $\text{DGal}(M, x_0)$ and $W^M_{x_0}$ by $G_0^\mathbb{C}$ and $G^\mathbb{C}$ respectively. Since $M$ is assumed to be $i$-mixed, there is a subgroup $G_R \subset G^\mathbb{C}$ which projects onto $W(\overline{k}/k)$ and $G^\mathbb{C}_R \cap G_R$ is a maximal compact subgroup of $G^\mathbb{C}_R$. For a group $G$, we denote by $G^i$ the set of conjugacy classes. We take an element $z$ of positive degree in the center of $G_R$, which exists since $M$ is semi-simple. For more details, see [Gr3] 10.10.

**Proof.** Let $\rho^{(i)} : W^M_{x_0} \to \text{GL}(V_{\rho^{(i)}})$ be a representation of $W^M_{x_0}$ corresponding to $(E^{(i)})^{ss}$. These define continuous complex representations of $G_R$ denoted by $\rho_R$ and $\rho'_R$. Since the category of linear $G_0^\mathbb{C}$-representations of $W^M_{x_0}$ and that of continuous complex representations of $G^\mathbb{C}$ are equivalent by exactly the same argument as [De2] 2.2.8, it suffices to show that $\rho_R \cong \rho'_R$.

We endow the quotient topology with $G^\mathbb{C}_R$ by the surjection $\alpha : G_R \to G^\mathbb{C}_R$. We denote by $G^i_1$ the subset of $G^\mathbb{C}_R$ consisting of the elements of degree $i$. Note that $G^i_1$ is both open and closed. Let $S$ be the subset of elements $g$ of $G^\mathbb{C}_R$ such that the characteristic polynomials $\det(1 - g \cdot t; V_\rho)$ and $\det(1 - g \cdot t; V_{\rho'})$ coincide. By the Brauer-Nesbitt theorem, it suffices to show that $S = G^\mathbb{C}_R$. Since the characteristic polynomials for $g$ coincide, that of $g^{-1}$ coincide as well, we have $S = S^{-1}$. Since the map $\overline{\rho}^{(i)} : G \to \mathbb{C}[t]$ sending $g$ to $\det(1 - g \cdot t; V_{\rho^{(i)}})$ is continuous, $S = (\overline{\rho} - \overline{\rho}')^{-1}(0)$ is closed in $G^\mathbb{C}_R$. Let $A$ be the complement of $S$ in $G^\mathbb{C}_R$, which is open, and assume that this is not empty. Let $A_i := A \cap G^i_1$, which is an open subset of $G^\mathbb{C}_R$. Since $z$ is in the center of $G_R$, the characteristic polynomials of $g$ for an element $g \in G^\mathbb{C}_R$ coincide if and only if they coincide for $z^n \cdot g$ for some integer $n$. This is showing that the isomorphism $z : G^\mathbb{C}_R \overset{\sim}{\to} G^\mathbb{C}_{R,d}$ where $d > 0$ denotes the degree of $z$ induces the bijection $A_n \overset{\sim}{\to} A_{n+d}$. Since we are assuming $A$ to be non-empty, there exists a positive integer $n_0$ such that $A_{n_0}$ is also non-empty.

Now, let $d_0$ be the Haar measure on $G^\mathbb{C}_R$, and $\mu_0$ be the product of $d_0$ by the characteristic function of the elements of positive degree. We denote by $\mu^i_0$ the direct image of $\mu_0$ on $G^i_1$. Since $A_{n_0}$ is open, we get

$$\mu^i_0(A_{n_0}) = d_0(\alpha^{-1}_0(A_{n_0})) > 0.$$

Thus the equidistribution theorem [Gr3] 10.11\(^{(4)}\) implies that there exists a positive integer $n$ such that $\mu^i_0(z^{-n} \cdot A_{n_0}) > 0$. This is showing that $z^{-n} \cdot \text{Frob}^n_{\overline{x}} \in A_{n_0}$ where $-nd + \deg(x) n' = n_0$. As a consequence, we have $\text{Frob}^n_{\overline{x}} \in A$, which contradicts with the assumption.$\blacksquare$

\(^{(3)}\)After a large part of this paper had been written, the author was pointed out by A. Pál that the following proposition had been generalized by him and U. Hartl. However, since we do not know the precise statement of their theorem, we decided to put the proposition.

\(^{(4)}\)Although missing in *ibid.*, in this theorem, we think that they need to assume, moreover, $M$ to be semi-simple.
4. \textit{L-factors and \varepsilon-factors}

Before proving the main theorem, let us review the theory of local \textit{L}-factors and \varepsilon-factors. Let \( \psi \) be a non-trivial additive character of \( \mathbb{A}_K/K \), which is equivalent to choosing a meromorphic differential form \( \omega \) on \( X \). For an overconvergent \( F \)-isocrystal \( E \) on a dense open subset \( U/K \) of \( X \), we have the global \varepsilon-factor defined by

\[
\varepsilon(E, t) := \prod_{r \in \mathbb{Z}} \det(-F \cdot t; H^r_{\mathrm{rig}, c}(U, E))^{(-1)^r+1}
\]

where \( F \) denotes the Frobenius automorphism of the rigid cohomology with compact support. For each closed point \( x \) of \( X \), we have local \varepsilon-factor defined as follows. We define the (Artin) conductor of \( E \) at \( x \) by

\[
a_x(E_x) := \begin{cases} 
\text{rk}(E) + \text{irr}(E|_{\eta_x}) & \text{if } x \text{ is ramified}, \\
0 & \text{if } x \text{ is unramified}.
\end{cases}
\]

Then we define the local \varepsilon-factor at \( x \) to be

\[
\varepsilon_x(E_x, t, \psi_x) := \begin{cases} 
\varepsilon^\text{rig}_0(E|_{\eta_x}, \omega_x) \cdot c_x(t) & \text{if } x \text{ is ramified}, \\
\left(q^{-\deg(x) v_x(\omega) \text{rk}(E)} \cdot \det E(x)^{v_x(\omega)}\right) \cdot c_x(t) & \text{if } x \text{ is unramified},
\end{cases}
\]

\[
c_x(t) := q^{-\deg(x) v_x(\omega) \text{rk}(E)/2} \cdot \left(t^{\deg(x)(\text{rk}(E) v_x(\omega)+a_x(E))}\right),
\]

where \( \varepsilon^\text{rig}_0(\cdot) \) denotes the \varepsilon-factor defined by Marmora (cf. we followed the notation of [AM 7.1.3]), \( \det_M(x) \) denotes the determinant of the Frobenius action on \( E \) at \( x \) (see [ibid., 7.2.6]), \( v_x(\omega) \) denotes the order of \( \omega \) at \( x \). The main theorem of [AM] (Theorem 7.2.6 of ibid.) is stating that the following equality holds:

\[
(PF) \quad \varepsilon(E, t) = \prod_{x \in |X|} \varepsilon_x(E_x, t, \psi_x).
\]

\textbf{Remark.} — In [AM], only the case where the residue field of \( K \) is equal to \( k \) is dealt with. However, we are able to prove the product formula in the same way without assuming this, or more precisely [ibid., (7.4.3.1)] holds. The detail is left to the reader. We note that the \varepsilon-factor in ibid. is a function. However, applying the product formula to the canonical extension, we see that the function is constant, and we define the \varepsilon-factor \( \varepsilon^\text{rig}_0(E|_{\eta_x}, \omega_x) \) in the above to be this constant.

Let \( K_{\text{loc}} \) be a local field, and \( \pi \) and \( \pi' \) be smooth admissible representations of \( \text{GL}_r(K_{\text{loc}}) \) which are irreducible or of Whittaker type. Fix an additive character \( \psi_{\text{loc}} \) of \( K_{\text{loc}} \). Then in [JPS] Thm 2.7 and subsection 9.4, the \textit{L}-factor and \varepsilon-factor for the pair \( (\pi, \pi') \) are defined, and denoted by \( L(\pi \times \pi', t) \) and \( \varepsilon(\pi \times \pi', t, \psi_{\text{loc}}) \) respectively. For automorphic representations \( \pi = \bigotimes_{x \in |X|} \pi_x \) and \( \pi' = \bigotimes_{x \in |X|} \pi'_x \) such that \( \pi_x \) and \( \pi'_x \) are irreducible or of Whittaker type for any \( x \), we denote by \( L_x(\pi_x \times \pi'_x, t) \) (resp. \( \varepsilon_x(\pi_x \times \pi'_x, t, \psi_x) \)) the local \textit{L}-factor (resp. \varepsilon-factor) of the couple \( (\pi_x, \pi'_x) \) (resp. and the additive character \( \psi_x \) induced by \( \psi \)) of representations of \( \text{GL}_r(K_x) \) where \( K_x \) denotes the local field at the place \( x \) of \( K \).

5. \textit{Proof of the theorem}

We prove the main theorem in this section. First, we remark the following.

\footnote{The definition is slightly different from [AM 7.2].}
5.1 Remark. — If the map $\pi_\bullet$ is constructed for some $r$, the isocrystals belonging to $\mathcal{I}_r$ is $\nu$-pure of weight 0. This can be seen from the generalized Ramanujan-Petersson conjecture proven by Lafforgue \cite[VI.10 (i)]{LB}.

Let us start to prove the main theorem. First let us see the uniqueness of the maps. The map $\pi_\bullet$ is uniquely determined if it exists by the strong multiplicity one theorem \cite{P}. For the uniqueness of $E_\bullet$, assume we had two maps $E_\bullet$ and $E'_\bullet$. For $\pi \in \mathcal{A}_r$, $E_\pi$ and $E'_\pi$ are $\nu$-pure of weight 0 by Remark 5.1 and we are in the situation to apply the proposition of \cite{L1} which implies that $E_\pi = E'_\pi$. Thus the uniqueness of $E_\bullet$ follows.

The rest of the argument is nothing but the “principe de récurrence” using the product formula \cite{PF}. Since all we need to do is to copy the proof of \cite[VI.11]{LB}, we only sketch the proof. We use the induction on $r$. We claim the following.

5.2 Claim. — Assume that the correspondence is established for any $r$ which is strictly less than $r_0$, and the local $L$-factor and $\varepsilon$-factor coincide for any place of $X$ via this correspondence. Then we have the map $\pi_\bullet : \mathcal{I}_r \to \mathcal{A}_r$ in the sense of Langlands for $r = r_0$.

**Sketch of the proof.** Take an element $E$ of $\mathcal{I}_{r_0}$, and denote by $S$ the set of points of $X$ at which $E$ is ramified. For a point $x \notin S$, we put $\pi_x$ to be the unramified smooth admissible irreducible representation of $\text{GL}_{r_0}(\mathcal{O}_x)$ whose set of Hecke eigenvalues is that of Frobenius eigenvalues of $E$ at $x$. For $x \in S$, we put $\pi_x$ to be an irreducible representation of $\text{GL}_{r_0}(\mathcal{O}_x)$ of Whittaker type whose center corresponds to $\det(E_x)$ via the reciprocity map. We put $\pi := \bigotimes_{x \in |X|} \pi_x$, which is a smooth admissible irreducible representation of $\text{GL}_{r_0}(\mathbb{A}_K)$. First, we need to show that there is an irreducible automorphic representation $\pi_E$ such that the local factors of $\pi_E$ at any closed point $x \notin S$ is equal to $\pi_x$. For this, we use the converse theorem \cite[B.13]{LB} of Piatetski-Shapiro. Since the argument is exactly the same as \cite{LB} using the product formula \cite{PF}, we omit the detail. We only note here that on a way we check the hypothesis of the converse theorem, we need to show that for any closed point $x$ of $X$, the local factors coincide for certain pairs. When $\pi$ is unramified at $x$, we use the hypothesis on the coincidence of local factors in the statement of the claim to see this coincidence at $x$. As a result, we have an automorphic representation $\pi_E$ with desired property, which is moreover cuspidal if $S = \emptyset$.

Finally, to show the claim, it remains to show that the automorphic representation is in fact cuspidal when $S \neq \emptyset$. Assume that $\pi_E$ were not cuspidal. Then a result of Langlands is saying that there exists a non-trivial partition $r_0 = r_1 + \cdots + r_k$, and automorphic cuspidal representations $\pi_1, \ldots, \pi_k$ of $\text{GL}_{r_1}, \ldots, \text{GL}_{r_k}$ which are unramified outside of $S$, the central characters are of finite order, and the Hecke eigenvalues at $x \notin S$ of $\pi$ is the disjoint union of that of $\pi^i$. By induction hypothesis, we have the overconvergent $F$-isocrystal $E_{\pi^i}$ of rank $r_i$. By construction, the Frobenius eigenvalue of $E$ and the semi-stable overconvergent $F$-isocrystal $E' := \bigoplus_{i=1}^k E_{\pi^i}$ are the same for any point outside of $S$. We know that $E'$ is $\nu$-pure of weight 0. This shows that $E$ is $\nu$-pure of weight 0 as well, and by applying Proposition \cite{L1} we have $E \cong E'$, which is a contradiction, and we conclude the proof of the claim.

5.3 Claim. — Assume $\pi \in \mathcal{A}_r$ (resp. $\pi' \in \mathcal{A}_r'$) correspond in the sense of Langlands to $E \in \mathcal{I}_r$ (resp. $E' \in \mathcal{I}_r'$). Then we get

\[ L_x(\pi_x \times \pi'_x, t) = L_x(E_x \otimes E'_x, t), \quad \varepsilon_x(\pi_x \times \pi'_x, t, \psi_x) = \varepsilon_x(E_x \otimes E'_x, t, \psi_x), \]

for any $x \in |X|$.

**Proof.** The proof is exactly the same as that of \cite[VI.11 (ii)]{LB}. The assumption is slightly milder than \textit{ibid.}, since we already know that the generalized Ramanujan-Petersson conjecture \textit{ibid.,} VI.10 (i) is true. In particular, $\pi_x$ is tempered for any $x \in |X|$, and $E$ and $E'$ are $\nu$-pure of weight 0. In the proof, we need to replace \textit{ibid.,} VI.5 by \cite{Ke} or \cite{AC}.

\[ \blacksquare \]

6
Let us prove the theorem. Assume that the correspondence is established for $r$ strictly less than $r_0$. Then Claim 5.3 is showing that the local $L$-factors and $\epsilon$-factors coincide via the correspondence at any closed point of $X$. This enables us to apply Claim 5.2 which give us the map $\pi_*$ for $r = r_0$.

It remains to construct $E_*$. Since we are assuming Conjecture (D), for $\pi \in A_{r_0}$ with the set of Frobenius elements $U$, there exists an overconvergent $F$-isocrystal $E$ of rank $r_0$ whose set of Frobenius eigenvalues at $x \in U$ is equal to that of Hecke eigenvalues of $\pi$ at $x$. We need to prove that $E$ is irreducible. Assume $E$ was not irreducible, and write $E^{ss} \cong \bigoplus E_i$ where $E_i$ are irreducible. By Lemma 6.1(6), there exists a twist $\chi_i$ such that $E_i(\chi_i)$ is of finite determinant for any $i$. Take a prime $\ell \neq p$, and let $\mathcal{F}_i$ (resp. $\check{\mathcal{F}}_i$) be the irreducible $\ell$-adic sheaf corresponding to $\pi E_i(\chi_i)$ (resp. $\pi$) in the sense of Langlands using the induction hypothesis and proven Langlands correspondence for $\ell$-adic sheaves. Then the set of Frobenius eigenvalues of $\mathcal{F}_i$ and $\bigoplus \mathcal{F}_i(\chi_i^{-1})$ (using the notation of Remark 6.2 (ii)) coincide at any $x \in U$. By the Čebotarev density theorem, this is not possible, which implies that $E$ is irreducible. We see from the Frobenius eigenvalues at each closed point in $U$ that $E$ is of finite determinant, and thus the theorem follows.

6. Some consequences

Finally, let us see some consequences of the theorem.

A twist is an $F$-isocrystal of rank 1 on Spec($k$). Let $\chi$ be a twist. For an overconvergent $F$-isocrystal $E$, we denote by $E(\chi)$ the tensor product $E \otimes f^*(\chi)$ where $f : X \rightarrow \text{Spec}(k)$ is the structural morphism.

6.1 Lemma. — Let $X$ be a scheme of finite type over $k$.

(i) Let $E$ be an overconvergent $F$-isocrystal on $X/K$ of rank 1. Then there exists a twist $\chi$ and a positive integer $n$ such that $E(\chi)^{\otimes n}$ is trivial.

(ii) For any overconvergent $F$-isocrystal $E$ on $X/K$, by taking an extension of $K$ if necessary, there exists a twist $\chi$ such that $E(\chi)$ is of finite determinant.

Proof. Let us see (i). We may replace $k$ by its finite extension. Thus we may assume that the residue field of $K$ is $k$ and there is a uniformizer of $K$ fixed by the Frobenius automorphism of $K$. By the definition of overconvergent $F$-isocrystals, we may assume that $X$ is reduced. By [Be1, 2.1.11], we may shrink $X$, and in particular, we may assume that $X$ is smooth. Since $E$ is of rank 1, there exists a twist $\chi'$ such that $E(\chi')$ is unit-root. Assume there exists a smooth compactification $X \hookrightarrow \overline{X}$ such that the complement is a simple normal crossing divisor. Then by the same argument as [Cr1, 4.13], using Sh. Thm 4.3 and [KL] Theorem 2 instead of [Cr1, 4.12] and the class field theory, we get that (i) holds in this case. There exists a generically étale alteration $f : Y \rightarrow X$ such that $Y$ possesses a smooth compactification whose complement is a simple normal crossing divisor. This shows that there exists an integer $n'$ and a twist $\chi$ such that $f^*E(\chi)^{\otimes n'}$ is trivial. There exists open dense subschemes $V$ of $Y$ and $U$ of $X$ such that $f_V : V \rightarrow U$ is finite étale of degree $d$. Let $G$ be an $F$-isocrystal on $U$ such that $f_V^*G$ is trivial. Then $G^{\otimes d}$ is trivial. This implies that $E(\chi)^{\otimes d n'}|U$ is trivial, and by using [Be1, 2.1.11] again, we conclude that $E(\chi)^{\otimes d n'}$ is trivial on $X$.

For (ii), we only need to note that $\det(E(\chi)) \cong \det(E)(\chi^{\otimes \text{rk}(E)})$. ■

6.2 Remark. — (i) The lemma also holds when $k$ is the perfection of an absolutely finitely generated field, since the theorem of [KL] is still applicable in this case.

(ii) Fixing a twist is equivalent to fixing an element of $K^\times$. Fix an isomorphism $\ell' : \overline{Q}_p \cong \overline{Q}_\ell$, and take a twist $\chi$ corresponding to $b \in \overline{Q}_p$. Given a $\overline{Q}_\ell$-sheaf $\mathcal{F}$, we have $\mathcal{F}(\ell')$ using the notation of [De2, 1.2.7]. We sometimes denote this sheaf by $\mathcal{F}(\chi)$.

(6) We do not have circular reasoning.
We say that a scheme $X$ over $k$ is a $d$-variety if there exists a proper smooth formal scheme $\mathcal{P}$, a divisor $Z$ of the special fiber of $\mathcal{P}$, and an embedding $X \hookrightarrow \mathcal{P}$ such that $X = \overline{X} \setminus Z$ where $\overline{X}$ denotes the closure of $X$ in $\mathcal{P}$. We are able to consider overholonomic $F\mathcal{D}^+_X \mathbb{Q}$-complexes as in [AC]. We say that an overholonomic $F\mathcal{D}^+_X \mathbb{Q}$-complex $C$ is $\iota$-mixed if there exists a finite stratification $\{X_i\}$ by smooth $d$-varieties such that $\mathcal{H}^j\mathbb{R}^\iota \Gamma_\iota (C)$ is the specialization of an $\iota$-mixed overconvergent $F$-isocrystal for any $i$ and $j$. See [AC] for more details.

6.3 Corollary. — Assume Conjecture (D) holds for any function field. Then for any $d$-variety $X$ of finite type over $k$, any overholonomic $F\mathcal{D}^+_X \mathbb{Q}$-complex is $\iota$-mixed.

Proof. By definition, it suffices to show that, for any smooth $d$-variety $X$, any irreducible overconvergent $F$-isocrystal $E$ on $X$ is $\iota$-pure. By Lemma 6.1 (ii), we may assume that $E$ is of finite determinant. For any closed point $x$ of $X$, there exists a smooth curve $i : C \hookrightarrow X$ passing through $x$. It suffices to see that $i^* E$ is $\iota$-pure of weight $0$. Using Remark 5.1 and Conjecture (L), any overconvergent $F$-isocrystal with finite determinant on a smooth curve is $\iota$-pure of weight $0$, which shows that $i^* E$ is $\iota$-pure of weight $0$, and conclude the proof. ■

6.4 Lemma. — The Langlands correspondence is established for $r = 1$.

Proof. Let us construct $E_h$. Using the reciprocity map, we have a representation of $\pi_1(U)$ with finite monodromy at the boundary, which induces a unit-root overconvergent $F$-isocrystal by [Ts] 7.1.1. To construct $E_h$, we note that objects in $I_1$ are unit-root since they are finite. Thus using the result of Tsuzuki and the reciprocity map, we conclude. ■

6.5 Theorem. — Any overconvergent $F$-isocrystal of rank less than or equal to $2$ on a smooth variety is $\iota$-mixed.

Proof. Arguing in the same way as Corollary 6.3, we are reduced to showing the existence of $\pi_\bullet$ for $r = 1, 2$. The case $r = 1$ has been established by the previous lemma. In this case, the coincidence of local $\varepsilon$-factors for any place of $X$ follows by definition. Thus applying Claim 5.2 we have the map $\pi_\bullet$ for $r = 2$, and the theorem follows. ■

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