Extension of Chern-Simons Forms
and
New Gauge Anomalies

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Abstract

We present a general analysis of gauge invariant, exact and metric independent forms which can be constructed using higher rank field-strength tensors. The integrals of these forms over the corresponding space-time coordinates provides new topological Lagrangians. With these Lagrangians one can define gauge field theories which generalize the Chern-Simons quantum field theory. We also present explicit expressions for the potential gauge anomalies associated with the tensor gauge fields and classify all possible anomalies that can appear in lower dimensions. Of special interest are those which can be constructed in four, six, eight and ten dimensions.

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1 Introduction

The chiral anomalies appear in gauge theories interacting with Weyl fermions. The $U_A(1)$ gauge anomaly is given by the Pontryagin-Chern-Simons $2n$-form \[1, 2, 3, 4, 5, 6, 7, 8, 9\]

$$d \ast J^A \propto \mathcal{P}_{2n} = Tr (G^n) = d \omega_{2n-1}, \quad (1.1)$$

where $\omega_{2n-1}$ is the Chern-Simons form in $2n - 1$ dimensions \[1, 7\]:

$$\omega_{2n-1}(A) = n \int_0^1 dt \, Tr (AG_t^{n-1}) . \quad (1.2)$$

$G = dA + A^2$ is the 2-form Yang-Mills (YM) field-strength tensor of the 1-form vector field $A = -igA^a_L dx^\mu$ and $G_t = tG + (t^2 - t)A^2$. A celebrated result for the non-Abelian anomaly \[10, 11, 12, 13, 14\] can be obtained by gauge variation of the $\omega_{2n-1}$ \[1, 2, 3, 4, 5, 7, 8, 9\]:

$$\delta \omega_{2n-1} = d\omega_{2n-2}^1 , \quad (1.3)$$

where the $(2n - 2)$-form has the following integral representation of Zumino \[1\]:

$$\omega_{2n-2}^1(\xi, A) = n(n - 1) \int_0^1 dt (1-t) \, Str \left( \xi d(A G_t^{n-2}) \right) . \quad (1.4)$$

Here, $\xi = \xi^a L_a$ is a scalar gauge parameter and $Str$ denotes a symmetrized trace. The covariant divergence of the non-Abelian left and right handed currents is given by this $(2n - 2)$-form:

$$D \ast J^L,R_\xi \propto \omega_{2n-2}^1(\xi, A) . \quad (1.5)$$

Thus the non-Abelian anomaly in $2n - 2$-dimensional space-time may be obtained from the Abelian anomaly \[1.1\] in $2n$ dimensions by a series of reduction (transgression) steps \[1.1\] and \[1.3\] by a differential geometric method without having to evaluate the Feynman diagrams \[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21\].

In recent articles \[22, 23, 24\] the authors have found similar invariants in non-Abelian tensor gauge field theory \[25, 26, 27\]. The first series of exact $(2n + 3)$-forms are defined as follows:

$$\Gamma_{2n+3} = Tr (G^n G_3) = d\sigma_{2n+2} , \quad (1.6)$$

where $G_3 = dA_2 + [A, A_2]$ is the 3-form field-strength tensor for the rank-2 gauge field $A_2 = -igA^a_L L_a dx^\mu \wedge dx^\lambda$. The second series of exact $6n$-forms is defined as \[23\]:

$$\Delta_{6n} = Tr (G_3)^{2n} = d\pi_{6n-1} , \quad (1.7)$$

$L^a$ are the generators of the Lie algebra.
where the \((6n - 1)\)-forms are defined in \(D = 6n - 1\) dimensions. The third series of invariant forms are defined in \(D = 2n + 4\) dimensions and are given by the expression
\[
\Phi_{2n+4} = Tr(G^nG_4) = d\psi_{2n+3},
\]
where the corresponding secondary \((2n+3)\)-form \(\psi_{2n+3}\) is defined in \(D = 2n + 3\) dimensions and \(G_4 = dA_3 + \{A, A_3\}\). The forth series of forms is defined in \(D = 2n + 2\) dimensions
\[
\Omega_{2n+2} = Tr(GG_{2n} + ... ) = d\chi_{2n+1}
\]
where the forms \(\chi_{2n+1}\) are defined in \(D = 2n + 1\) dimensions. All new forms \(\Gamma_{2n+3}, \Delta_{6n}, \Phi_{2n+4}\) and \(\Omega_{2n+2}\) are analogous to the Pontryagin-Chern-Simons densities \(P_{2n}\) in YM gauge theory \((\ref{1.1})\): they are \emph{gauge invariant, exact and metric independent}. Our aim is to find out all potential gauge anomalies which are generated by the new invariant forms in even dimensions performing transgression analogous to the \((\ref{1.1})\) and \((\ref{1.3})\).

\[
P_{2n} \Rightarrow \omega_{2n-1} \Rightarrow \omega_{2n-2}.
\]

In particular we are interested to enumerate and classify all potential anomalies and their structure in even dimensions from four to ten space-time dimensions. For this purpose one should enumerate all gauge invariant, exact and metric independent forms in the corresponding dimensions. The integrals of these forms over the corresponding space-time coordinates provides us with new topological Lagrangians and with a generalization of the Chern-Simons quantum field theory \([28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38]\).

In the next Section 2, we shall present a general analysis of such forms which can be constructed using higher rank field-strength tensors, and we will define gauge field theories that generalize the Chern-Simons quantum field theory. In Section 3, we will present explicit expressions for the potential gauge anomalies associated with the tensor gauge fields. They are given by the sum of the two formulas \((3.11)\) and \((3.11)\) representing our result for the gauge anomalies. From this calculation it turns out that at least two powers of the YM field-strength tensor \(G\) should be present in the form \(\Phi_{2n}\) to generate anomalies with respect to the standard gauge transformations. In Section 4, we shall classify all possible forms which can be constructed in lower dimensions. Of special interest are those that can be constructed in four, six, eight and ten dimensions. Finally, in the Appendix, we present useful formulas for the gauge transformations of fields, the corresponding Biachi identities and for a one parameter deformation of fields generalizing Zumino’s construction.
In order to completely enumerate and classify all possible gauge invariant, exact and metric independent forms in extended YM theory we shall start from the Pontryagin-Chern-Simons density $P_{2n}$ in YM theory and then proceed with a number of steps by decreasing the power of YM field-strength tensor $G$ and increasing the power of the higher rank field-strength tensors $G_{2n}$, but keeping the rank of the forms fixed. Thus, we have to study the following sequence of $2n$-forms

$$Tr(G^n),$$
$$Tr(G^{n-2}G_4),$$
$$Tr(G^{n-3}G_6),$$
$$Tr(G^{n-4}G_8),$$
$$Tr(G^{n-5}G_{10}),$$
$$Tr(G^{n-6}G_{12}),$$
$$Tr(G^{n-7}G_{14}),$$
$$Tr(G^{n-8}G_{16}),$$
$$Tr(G^{n-9}G_{18}),$$
$$...............$$

By construction they are metric independent, and thus invariant under general coordinate transformations because the indices of the field-strength tensors are contracted with the totally antisymmetric Levi-Civita tensor $\epsilon^{\mu_1 \cdots \mu_{2n}}$. As a next step, one should check that these forms are gauge invariant and exact. Not all of them share these properties and as we shall see in some cases one should consider linear combinations of these forms as in (1.9). Those of the forms which will be found to be gauge invariant and exact, by transgression will generate potential non-Abelian anomalies in $2n - 2$ dimensions.

The forms $\Gamma_{2n+3}, \Delta_{6n}, \Phi_{2n+4}$ and $\Omega_{2n+2}$ which we already presented in the introduction are part of the above list and are gauge invariant, exact and metric independent. For instance, the series of forms linear in $G_6$ can be easily constructed because they contain only the lower field-strength tensor $G_4$. The first forms are:

$$\Xi_8 = Tr(GG_6 + G_4G_4) = d\phi_7,$$
$$\Xi_{10} = Tr(G^2G_6 + 2G_4^2) = d\phi_9,$$
$$\Xi_{12} = Tr(G^3G_6 + 2G^2G_4^2 + GG_4GG_4) = d\phi_{11}$$

...............
which can be written using the symmetrized trace, as

\[
\Xi_{2n+6} = \text{Str}(G^n G_6 + nG^{n-1}G_4^2) = d\phi_{2n+5}.
\]

The lower dimensional forms linear in \(G_8\) are

\[
\begin{align*}
\Upsilon_{10} &= Tr(GG_8 + 3G_4 G_6) = d\varphi_9, \\
\Upsilon_{12} &= Tr(G^2 G_8 + 3GG_4 G_6 + 3GG_6 G_4 + 2G_4^3) = d\varphi_{11}, \\
\text{...........} & \text{..........................}
\end{align*}
\]

and so on.

The general analysis of possible forms will be presented in the following sections, but already at this stage one can see that there is a reach class of invariant densities which are relevant for the description of possible gauge anomalies. At the same time, integrals of these forms over the corresponding space-time coordinates provides us with new topological Lagrangians. In particular, one can define topological field theories, generalizing the Chern-Simons quantum field theory \cite{28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38}, in which the correlation functions have support on two-dimensional surfaces \(M_i\) and knots \(C_j\)

\[
Z(M, M_i, C_j, R) = \int \mathcal{D}A \mathcal{D}A_2 e^{ik\int_M \sigma_{2n+2}(A, A_2)} \prod_{i,j} Tr_{R_i} e^{i\int_M A_2^2 Tr_{R_j} e^{i\int_M C_j A}},
\]

where \(\sigma_{2n+2}\) is defined in \eqref{1.6} and \(k\) is a parameter, or on three-dimensional manifolds

\[
Z(M, M_i, C_j, R) = \int \mathcal{D}A \mathcal{D}A_3 e^{ik\int_M \psi_{2n+3}(A, A_2)} \prod_{i,j} Tr_{R_i} e^{i\int_M A_3 Tr_{R_j} e^{i\int_M C_j A}}
\]

as well as on higher dimensional ones, \(\psi_{2n+3}\) is defined in \eqref{1.8}. In particular, for the partition function \(Z(M)\) in four dimensions \cite{22, 23}, we get

\[
Z(M) = \int \mathcal{D}A \mathcal{D}A_2 e^{ik\int_M \sigma_4} = \int \mathcal{D}A \mathcal{D}A_2 e^{ik\int_M Tr(GA_2)}
\]

and in the large \(k\) limit the contribution to the path integral is dominated from the points of stationary phase which are, in the given case, the flat connections

\[
G = dA + A^2 = 0, \quad G_3 = dA_2 + [A, A_2] = 0.
\]

The solutions of the first equation are well known \(A^{\text{flat}} = g^{-1}dg\), while the solutions of the second one have been found in \cite{23}

\[
A_2^{\text{flat}} = g^{-1}dg_1 - g^{-1}g_1g^{-1}dg.
\]

With these solutions in hands one can calculate the Gaussian integrals in \eqref{2.8} and express the partition function \(Z\) in terms of determinants of certain operators. The details will be given elsewhere.
We shall start our analysis with the second form in our list \( \Phi_{2n} = \text{Tr}(G^{n-2}G_4) \), which is the next to the standard Pontryagin-Chern-Simons density \( \text{Tr}(G^n) \) and is its natural generalization. The form \( \Phi_{2n} \) was already considered in [24] and was shown to be gauge invariant and exact

\[
d\Phi_{2n} = 0. \tag{3.1}
\]

According to Poincaré’s lemma, this equation implies that \( \Phi_{2n} \) can be locally written as an exterior differential of a certain \( (2n-1) \)-form

\[
\text{Tr}(G^{n-2}G_4) = d\psi_{2n-1}. \tag{3.2}
\]

Generalizing Zumino’s construction [1], a one-parameter family of potentials and strengths should be introduced:

\[
A_t = tA, \quad G_t = tG + (t^2 - t)A^2, \quad A_{3t} = tA_3, \quad G_{4t} = tG_4 + (t^2 - t)\{A, A_3\}, \tag{3.3}
\]

where the parameter \( t \) runs in the interval \( 0 \leq t \leq 1 \), so that the corresponding secondary \( (2n-1) \)-form is given by [24]

\[
\psi_{2n-1}(A, A_3) = \int_0^1 dt \text{Tr}(AG_t^{n-3}G_{4t} + ... + G_t^{n-3}AG_{4t} + G_t^{n-2}A_3). \tag{3.4}
\]

The lower dimensional forms are:

\[
\psi_5 = \text{Tr}(GA_3),
\]

\[
\psi_7 = \frac{1}{3}\text{Tr}(AGA_4 + AG_4G + A_3G^2 - \frac{1}{2}A^3G_4 - \frac{1}{2}(A^2A_3 + AA_3A + A_3A^2)G + \frac{1}{2}A^4A_3), \tag{3.5}
\]

In order to calculate the variation of the secondary characteristics \( \psi_{2n-1} \) we need the gauge transformations of the various fields involved in the expressions (3.4) for \( \psi_{2n-1} \), which read

\[
\delta A = d\xi + [A, \xi], \quad \delta A_3 = d\zeta_2 + [A, \zeta_2] + [A_3, \xi],
\]

\[
\delta_A dA = [dA, \xi] - \{A, d\xi\}, \quad \delta_A dA_3 = [dA, \zeta_2] - \{A, d\zeta_2\} + [dA_3, \xi] - \{A_3, d\xi\},
\]

\[
\delta_g G_4t = [G_4t, \xi] + (t^2 - t)\{A, d\xi\},
\]

\[
\delta_g G_{4t} = [G_{4t}, \zeta_2] + (t^2 - t)(\{A, d\zeta_2\} + \{A_3, d\xi\}). \tag{3.6}
\]
where \( \zeta_2(x) = \zeta_{a_1a_2}(x)L_a dx^{a_1} \wedge dx^{a_2} \) is the rank-2 tensor gauge parameter. It is difficult to perform in a straightforward way the variation of the \( \psi_{2n-1} \), instead one should try to represent it in terms of the symmetrized traces. The fact that the trace in (3.4) can be transformed into the symmetrized trace can be proven by performing a cyclic permutation of all terms in the integrand and then combining them into one term symmetric under all permutations. As a result, we get the following expression

\[
\psi_{2n-1} = \int_0^1 dt \ Str \ [(n-2)AG_{t}^{n-3}G_{4t} + G_{t}^{n-2} A_{3}] .
\]

(3.7)

Because we have two independent gauge parameters \( \xi(x) \) and \( \zeta_2(x) \), we can perform the variation of \( \psi_{2n-1} \) over these gauge transformations independently.

First we shall perform the variation over the tensor gauge parameter \( \zeta_2(x) \). The terms linear in \( \zeta_2(x) \) cancel out and remain only terms linear in its differential \( d\zeta_2 \)

\[
\delta \zeta \psi_{2n-1} = \int_0^1 dt \ Str \ [(n-2)(t^2 - t)AG_{t}^{n-3}\{A, d\zeta_2\} + G_{t}^{n-2} d\zeta_2] .
\]

(3.8)

Opening the bracket and rearranging the terms, one obtains

\[
\delta \zeta \psi_{2n-1} = \int_0^1 dt \ Str \ [(n-2)(t-1)( t\{A, A\}d\zeta_2G_{t}^{n-3} + Ad\zeta_2[A_t, G_{t}^{n-3}]) + G_{t}^{n-2} d\zeta_2] \\
\]

and then using the equations

\[
dG_{t}^{n-2} = -[A_t, G_{t}^{n-2}], \quad \frac{\partial G_{t}}{\partial t} = dA + t\{A, A\}
\]

(3.9)

we get

\[
\delta \zeta \psi_{2n-1} = \int_0^1 dt \ Str \ [-(n-2)(t-1)( dAd\zeta_2G_{t}^{n-3} + Ad\zeta_2dG_{t}^{n-3})] + \]

\[
+(n-2)(t-1)G_{t}^{n-3}\frac{\partial G_{t}}{\partial t} d\zeta_2 + G_{t}^{n-2} d\zeta_2 = \]

\[
= (n-2) \int_0^1 dt(1-t) \ Str \ [d(G_{t}^{n-3} A) \ d\zeta_2] = d\psi_{2n-2}^1,
\]

(3.10)

where the last two terms in the second line cancel out after partial integration. Thus we arrive to the final result

\[
\psi_{2n-2}^1(\zeta_2, A) = (n-2) \int_0^1 dt(1-t) \ Str \ (\zeta_2 \ d \ (G_{t}^{n-3} A)) .
\]

(3.11)

These forms describe the potential gauge anomalies with respect to the tensor gauge transformations induced by the rank-2 gauge parameter \( \zeta_2(x) \).

\footnote{One should keep in mind that not all expressions can be represented in the form of symmetrized traces.}
In order to calculate the variation of $\psi_{2n-1}$ with respect to the standard gauge transformation parameter $\xi(x)$ we should use again formulas (3.6)

$$\delta \xi \psi_{2n-1} = \int_0^1 dt \text{Str} \left[ \frac{1}{2}(n-3)G_4A + (n-2)(t^2-t)A \{ A, d\xi \} G_4 + (n-2)(t^2-t)A \{ A, d\xi \} G_4^3 \right], \quad (3.12)$$

where the terms linear in $\xi(x)$ cancel out and remain only terms linear in its differential $d\xi(x)$. Using equations (3.9) and

$$dG_4 = -[A_t, G_4] - [A_{3t}, G_t], \quad \frac{\partial G_4}{\partial t} = dA_3 + 2t \{ A, A_3 \} \quad (3.13)$$

one can transform the above variation into a form similar to (3.10)

$$\delta \xi \psi_{2n-1} = (n-2) \int_0^1 dt (1-t) \text{Str} \left[ d((n-3)AG_4^3 + G_4^3 d\xi) \{ A, d\xi \} \right] = d\psi_{2n-2}^1,$$

and thus we get the additional gauge anomaly generated by the tensor gauge field $A_3$

$$\psi_{2n-2}(\xi, A, A_3) = (n-2) \int_0^1 dt (1-t) \text{Str} \left[ \xi d((n-3)G_4^3 A + G_4^3 A_3) \right]. \quad (3.14)$$

The sum of the two expressions (3.14) and (3.11) represent our final result for the gauge anomalies. From this calculation one sees that at least two powers of the YM field-strength tensor $G$ should be present in the form $\Phi_{2n}$ to generate anomalies with respect to the $\xi$ gauge transformations.

We shall also consider a series of invariant forms that are linear in the YM field-strength tensor $\Omega_{2n}$ given in (1.9). As we shall see, none of them generate anomalies of the standard gauge transformations, but create anomalies with respect to the tensor gauge transformations. We start with the invariant form that can be constructed in six dimensions

$$\Omega_6 = Tr(GG_4) = d\chi_5; \quad (3.15)$$

the calculation proceeds as follows

$$\delta \Omega_6 = dTr(G_4 \delta A + G \delta A_3), \quad \chi_5 = \int_0^1 dt \text{Tr}(G_4 A + G_t A_3) \quad (3.16)$$

thus

$$\chi_5 = \frac{1}{2} Tr(G_4 A + GA_3 - A^2 A_3) = Tr(G A_3) \quad (3.17)$$

and its gauge variation is $\delta \chi_5 = d \chi_4^1$ with

$$\chi_4^1 = Tr(G\zeta_2). \quad (3.18)$$
The next invariant is in eight dimensions

\[ \Omega_8 = \text{Tr}(G G_6 + G_4 G_4) = d\chi_7; \]  

(3.19)

the calculation proceeds again as before:

\[ \delta \Omega_8 = d \text{Tr}(G_6 \delta A + 2G_4 \delta A_3 + G \delta A_5), \quad \chi_7 = \int_0^1 dt \text{Tr}(G_{6t} A + 2G_{4t} A_3 + G_t A_5), \]  

(3.20)

where

\[ \chi_7 = \frac{1}{2} \text{Tr}(G_6 A + 2G_4 A_3 + G A_5 - 2A A_3^2 - A^2 A_5) = \text{Tr}(G A_5 + G_4 A_3), \]  

(3.21)

and its gauge variation is \( \delta \chi_7 = d \text{Tr}(G \zeta_4 + G_4 \zeta_2) = d\chi_6^1 \) with

\[ \chi_6^1 = \text{Tr}(G \zeta_4 + G_4 \zeta_2). \]  

(3.22)

The next invariant form is in ten dimensions

\[ \Omega_{10} = \text{Tr}(G G_8 + 3G_4 G_6) = d\chi_9, \]  

(3.23)

and as before:

\[ \delta \Omega_{10} = d \text{Tr}(G_8 \delta A + 3G_6 \delta A_3 + 3G_4 \delta A_5 + G \delta A_7), \]

\[ \chi_9 = \int_0^1 dt \text{Tr}(G_{8t} A + 3G_{6t} A_3 + 3G_{4t} A_5 + G_t A_7), \]  

(3.24)

where

\[ \chi_9 = \frac{1}{2} \text{Tr}(G_8 A + 3G_6 A_3 + 3G_4 A_5 + GA_7 - A^2 A_7 - 3A \{A_3, A_5\} - 2A_3^3) = \]

\[ = \text{Tr}(G A_7 + 2G_4 A_5 + G_6 A_3) \]  

(3.25)

and its gauge variation is \( \delta \chi_9 = d \text{Tr}(G \zeta_6 + 2G_4 \zeta_4 + G_6 \zeta_2) = d\chi_8^1 \) with

\[ \chi_8^1 = \text{Tr}(G \zeta_6 + 2G_4 \zeta_4 + G_6 \zeta_2). \]  

(3.26)

Finally, in twelve dimensions we have

\[ \Omega_{12} = \text{Tr}(G G_{10} + 4G_4 G_8 + 3G_6 G_6) = d\chi_{11}, \]  

(3.27)

so that

\[ \chi_{11} = \text{Tr}(G A_9 + 3G_4 A_7 + A_3 G_8 + 3G_6 A_5) \]  

(3.28)

and its gauge variation is \( \delta \chi_{11} = d \text{Tr}(G \zeta_8 + 3G_4 \zeta_6 + 3G_6 \zeta_4 + G_8 \zeta_2) \), thus the rank-2,4,6,10 anomalies in ten dimensions are

\[ \chi_{10}^1 = \text{Tr}(G \zeta_8 + 3G_4 \zeta_6 + 3G_6 \zeta_4 + G_8 \zeta_2). \]  

(3.29)
The general form of these invariants can be written as
\[
\Omega_{2n+2} = Tr(GG_{2n} + \alpha_1 G_4 G_{2n-2} + \alpha_2 G_6 G_{2n-4} + \ldots) = d\chi_{2n+1}
\] (3.30)
where
\[
\chi_{2n+1} = Tr(GA_{2n-1} + \beta_1 G_4 A_{2n-3} + \beta_2 G_6 A_{2n-5} + \ldots).
\] (3.31)
and \(\alpha_i, \beta_i\) are certain numerical coefficients. The forms \(\chi_{2n+1}\) are defined in \(D = 2n+1 = 5, 7, 9, 11, \ldots\) dimensions. Their gauge variation is of the form \(\delta\chi_{2n+1} = d\chi_{2n}\), where the anomalies are only with respect to the tensor gauge transformations
\[
\chi_{2n} = Tr(G\zeta_{2n-2} + \gamma_1 G_4 \zeta_{2n-4} + \gamma_2 G_6 \zeta_{2n-6} + \ldots),
\] (3.32)
i.e. there are no terms depending on \(\zeta\).

4 Anomalies in 2, 4, 6, 8, and 10 Dimensions

Having in hands the results of the previous sections we can consider the classification of all possible forms and anomalies in lower-dimensions.

In four dimensions the only possible density is the standard 4-form \(P_4 = Tr(G^2) = d\omega_3\) which generates the two-dimensional anomaly \(\omega_2^1\) by gauge variation \(\delta\omega_3 = d\omega_2^1\), so that
\[
\omega_2^1 = Tr(\xi dA).
\] (4.1)

In six dimensions we get two densities, \(P_6 = Tr(G^3) = d\omega_5\) and \(\Phi_6 = Tr(GG_4) = d\psi_5\) which are generating the standard non-Abelian anomaly in four dimensions \(\omega_4^1\), as well as a new anomaly \(\psi_4^1\)
\[
\omega_4^1 = Tr(\xi d(AdA + \frac{1}{2} A^3)), \quad \psi_4^1 = Tr(\zeta_2 dA).
\] (4.2)

The second anomaly \(\psi_4^1\) is associated with the breaking of the symmetry with respect to the gauge transformations generated by the antisymmetric tensor gauge parameter \(\zeta_2 = L^a \zeta_a^{\sigma_1 \sigma_2} dx^{\sigma_1} \land dx^{\sigma_2}\). In this article we are mostly interested in classifying anomalies associated with the breaking of the Yang-Mills gauge symmetry generated by the scalar gauge parameter \(\xi = L^a \xi_a\). Therefore, from (4.2) we conclude that there is no new anomaly in four dimensions associated with \(\xi\) transformations except the standard one \(\omega_4^1(\xi, A)\).

In eight dimensions we get three densities, \(P_8 = Tr(G^4) = d\omega_7\), \(\Phi_8 = Tr(G^2 G_4) = d\psi_7\) and \(\Xi_8 = Tr(GG_6 + G_4 G_4) = d\phi_7\). Therefore in six dimensions we get the standard
anomaly $\omega_6^1$ and two new ones $\psi_6^1$ and $\phi_6^1$

$$\omega_6^1 = Tr(\xi \, (dAdA + \frac{6}{5} A^3 dA + \frac{1}{5} A^5))$$

$$\psi_6^1 = Tr(\xi \, (d^2 A_3 A + dAA_3 + \frac{1}{2} (A^2 A_3 + AA_3 A + A_3 A^2)) + Tr(\zeta_2 \, (dAdA + \frac{1}{2} A^3)),$$

$$\phi_6^1 = Tr(\zeta_2 \, G_4 + \zeta_4 \, G)$$

(4.3)

The form $\phi_6^1$ describes anomalies with respect to the rank-2 $\zeta_2$ and rank-4 $\zeta_4$ gauge transformations. The form $\psi_6^1$ has two contributions, the last term is associated with the $\zeta_2$ gauge transformations and the first term is clearly associated with standard gauge transformations. If one represents the standard gauge anomaly $\omega_6^1$ in momentum representation symbolically as

$$\omega_6^1 \propto Tr(L^a)^4 \epsilon^{\mu_1...\mu_6} \epsilon^{(1)}_{\mu_1} \epsilon^{(2)}_{\mu_2} \epsilon^{(3)}_{\mu_3} k^{(1)}_{\mu_4} k^{(2)}_{\mu_5} k^{(3)}_{\mu_6} + ...,$$

then the new anomaly takes the form

$$\psi_6^1 \propto Tr(L^a)^3 \epsilon^{\mu_1...\mu_6} \epsilon^{(1)}_{\mu_1} \epsilon^{(2)}_{\mu_2 \mu_3 \mu_4} k^{(1)}_{\mu_5} k^{(2)}_{\mu_6} + ...$$

(4.4)

where $k_{\mu}$ and $\epsilon_{\mu}$, $\epsilon_{\mu_1 \mu_2 \mu_3}$ denote the momenta and the polarization vectors and tensors of the corresponding gauge bosons. From this calculation one sees again that at least two powers of the YM field-strength tensor $G$ should be present in the form to generate anomalies with respect to the $\xi$ gauge transformations. The order of traced generators drops from four to three $Tr(L^a)^4 \rightarrow Tr(L^a)^3$ and of the momenta polynomials also drops from three to two $(k)^3 \rightarrow (k)^2$. In higher dimensions we shall encounter the same pattern.

In ten dimensions we get four densities, $P_{10} = Tr(G^5) = d\omega_9$, $F_{10} = Tr(G^3G_4) = d\psi_9$, $\Xi_{10} = Tr(G^2G_6 + 2GG_4^2) = d\phi_9$ and $\Upsilon_{10} = Tr(GG_8 + 3G_4G_6) = d\tilde{\phi}_9$. Thus in eight dimensions we get $\omega_8^1$ and the additional three gauge anomalies $\psi_8^1$, $\phi_8^1$ and $\tilde{\phi}_8^1$.

In twelve dimensions we get five densities, $P_{12} = Tr(G^6) = d\omega_{11}$, $F_{12} = Tr(G^4G_4) = d\psi_{11}$, $\Xi_{12} = Tr(G^3G_6 + 2G^2G_4^2 + GG_4GG_4) = d\phi_{11}$, $\Upsilon_{12} = Tr(G^2G_8 + 3GG_4G_6 + 3GG_6G_4 + 2G_4^3) = d\tilde{\phi}_{11}$ and $\Omega_{12} = Tr(GG_{10} + 4G_4G_8 + 3G_6^2) = d\chi_{11}$. Thus, in ten dimensions we get besides $\omega_{10}^1$, four additional gauge anomalies $\psi_{10}^1$, $\phi_{10}^1$, $\tilde{\phi}_{10}^1$ and $\chi_{10}^1$. The general structure, properties and comparison of these anomalies with the standard one will be presented in a separate publication.

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5 Appendix A. Gauge Transformations and Bianchi Identities

The gauge transformations of non-Abelian tensor gauge fields were defined in [25, 26, 27]:

\[
\delta A = D\xi, \\
\delta A_3 = D\zeta_2 + [A_3, \xi] \\
\delta A_5 = D\zeta_4 + 2[A_3, \zeta_2] + [A_5, \xi], \\
\delta A_7 = D\zeta_6 + 3[A_3, \zeta_4] + 3[A_5, \zeta_2] + [A_7, \xi], \\
\delta A_9 = D\zeta_8 + 4[A_3, \zeta_6] + 6[A_5, \zeta_4] + 4[A_7, \zeta_2] + [A_9, \xi],
\]

where \( DA_{2n+1} = dA_{2n+1} + \{ A, A_{2n+1} \} \) and the corresponding field-strength tensors are

\[
G = dA + A^2, \\
G_4 = dA_3 + \{ A, A_3 \}, \\
G_6 = dA_5 + \{ A, A_5 \} + \{ A_3, A_3 \}, \\
G_8 = dA_7 + \{ A, A_7 \} + 3 \{ A_3, A_5 \}, \\
G_{10} = dA_9 + \{ A, A_9 \} + 4 \{ A_3, A_7 \} + 3 \{ A_5, A_5 \},
\]

The gauge transformation (5.1) of field-strength tensors is homogeneous

\[
\delta G = [G, \xi], \\
\delta G_4 = [G_4, \xi] + [G, \zeta_2], \\
\delta G_6 = [G_6, \xi] + 2[G_4, \zeta_2] + [G, \zeta_4], \\
\delta G_8 = [G_8, \xi] + 3[G_6, \zeta_2] + 3[G_4, \zeta_4] + [G, \zeta_6], \\
\delta G_{10} = [G_{10}, \xi] + 4[G_8, \zeta_2] + 6[G_6, \zeta_4] + 4[G_4, \zeta_6] + [G, \zeta_8],
\]
The Bianchi identities are given by

\begin{align*}
DG &= 0, \quad (5.4) \\
DG_4 + [A_3, G] &= 0, \quad (5.5) \\
DG_6 + 2[A_3, G_4] + [A_5, G] &= 0, \quad (5.6) \\
DG_8 + 3[A_3, G_6] + 3[A_5, G_4] + [A_7, G] &= 0, \quad (5.7) \\
DG_{10} + 4[A_3, G_8] + 6[A_5, G_6] + 4[A_7, G_4] + [A_9, G] &= 0, \quad (5.8)
\end{align*}

where \( DG_{2n} = dG_{2n} + [A, G_{2n}] \). Generalizing Zumino’s construction \[\Pi\], we introduce a one-parameter family of potentials and field-strengths as:

\begin{align*}
A_t &= tA, \quad A_{3t} = tA_3, \quad A_{5t} = tA_5, \quad A_{7t} = tA_7, \quad A_{9t} = tA_9, \\
G_t &= tG + (t^2 - t)A^2, \\
G_{4t} &= tG_4 + (t^2 - t)\{A, A_3\}, \\
G_{6t} &= tG_6 + (t^2 - t)(\{A, A_5\} + \{A_3, A_3\}), \\
G_{8t} &= tG_8 + (t^2 - t)(\{A, A_7\} + 3\{A_3, A_5\}), \\
G_{10t} &= tG_{10} + (t^2 - t)(\{A, A_9\} + 4\{A_3, A_7\} + 3\{A_5, A_5\} + 4\{A_3, A_7\}, \\
\end{align*}

In order to find out the secondary forms it is useful to perform the variation of the fields in the corresponding expressions

\begin{align*}
\delta G &= D(\delta A), \\
\delta G_4 &= D(\delta A_3) + \{A_3, \delta A\}, \\
\delta G_6 &= D(\delta A_5) + \{A_5, \delta A\} + 2\{A_3, \delta A_3\}, \\
\delta G_8 &= D(\delta A_7) + \{A_7, \delta A\} + 3\{A_5, \delta A_3\} + 3\{A_3, \delta A_5\}, \\
\delta G_{10} &= D(\delta A_9) + \{A_9, \delta A\} + 4\{A_7, \delta A_3\} + 6\{A_5, \delta A_5\} + 4\{A_3, \delta A_7\}, \\
\end{align*}

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