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MIXED GENERALIZED FRACTIONAL BROWNIAN MOTION

SHAYKHAH ALAJMI AND EZZEDINE MLIKI*

Abstract. To extend several known centered Gaussian processes, we introduce a new centered mixed self-similar Gaussian process called the mixed generalized fractional Brownian motion, which could serve as a good model for a larger class of natural phenomena. This process generalizes both the well-known mixed fractional Brownian motion introduced by Cheridito [7] and the generalized fractional Brownian motion introduced by Zili [29]. We study its main stochastic properties, its non-Markovian and non-stationarity characteristics and the conditions under which it is not a semimartingale. We prove the long-range dependence properties of this process.

1. Introduction

Fractional Brownian motion on the whole real line (fBm for short) \( B^H = \{ B^H_t, t \in \mathbb{R} \} \) of Hurst parameter \( H \) is the best known centered Gaussian process with long-range dependence. Its covariance function is

\[
\text{Cov}(B^H_t, B^H_s) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t - s|^{2H}), \tag{1.1}
\]

where \( H \) is a real number in \((0, 1)\) and the case \( H = \frac{1}{2} \) corresponds to the Brownian motion. It is the unique continuous Gaussian process starting from zero, the self-similarity and stationarity of the increments are two main properties for which fBm enjoyed successes as modeling tool in finance and telecommunications. Researchers have applied fractional Brownian motion to a wide range of problems, such as bacterial colonies, geophysical data, electrochemical deposition, particle diffusion, DNA sequences and stock market indicators [20, 22]. In particular, computer science applications of fBm include modeling network traffic and generating graphical landscapes [21]. The fBm was investigated in many papers (e.g. [2, 12, 16, 17, 18, 19]). The main difference between fBm and regular Brownian motion is that the increments in Brownian motion are independent, increments for fBm are not.

In [4], the authors suggested another kind of extension of the Brownian motion, called the sub-fractional Brownian motion (sfBm for short), which preserves most properties of the fBm, but not the stationarity of the increments. It is a centered
Gaussian process $\xi^H = \{\xi^H_t, t \geq 0\}$, defined by:

$$\xi^H_t = \frac{B^H_t + B^H_{t-}}{\sqrt{2}}, \quad t \geq 0,$$

where $H \in (0, 1)$. The case $H = \frac{1}{2}$ corresponds to the Brownian motion.

The sfBm is intermediate between Brownian motion and fractional Brownian motion in the sense that it has properties analogous to those of fBm, self-similarity, not Markovian but the increments on nonoverlapping intervals are more weakly correlated, and their covariance decays polynomially at a higher rate in comparison with fBm (for this reason in [4] is called sfBm). So, the sfBm does not generalize the fBm. The sfBm was investigated in many papers (e.g. [3, 4, 24, 26]).

An extension of the sfBm was introduced by Zili in [28] as a linear combination of a finite number of independent sub-fractional Brownian motions. It was called the mixed sub-fractional Brownian motion (msfBm for short). The msfBm is a centered mixed self-similar Gaussian process and does not have stationary increments. The msfBm do not generalize the fBm.

In [29], Zili introduced new model called the generalized fractional Brownian motion (gfBm for short) which is an extension of both sub-fractional Brownian motion and fractional Brownian motion. A gfBm with parameters $a, b,$ and $H$, is a process $Z^H = \{Z^H_t, t \geq 0\}$ defined by

$$Z^H_t(a, b) = aB^H_t + bB^H_{t-}, \quad t \geq 0.$$  

The gfBm was investigated in [10, 30]. The gfBm generalize the sfBm but not the mixed fractional Brownian motion.

The mixed fractional Brownian motion (mfBm for short) is a linear combination between a Brownian motion and an independent fractional Brownian motion of Hurst parameter $H$. It was introduced by Cheridito [7] to present a stochastic model of the discounted stock price in some arbitrage-free and complete financial markets. The mfBm is a centered Gaussian process starting from zero with covariance function

$$\text{Cov}(N^H_t(a, b), N^H_s(a, b)) = a^2(t \wedge s) + \frac{b^2}{2} \left( t^{2H} + s^{2H} - |t - s|^{2H} \right),$$  

with $H \in (0, 1)$. When $a = 1$ and $b = 0$, the mfBm is the Brownian motion and when $a = 0$ and $b = 1$, is the fBm. We refer also to [1, 7, 9, 25, 27] for further information on this process.

In this paper, we introduce a new stochastic model, which we call the mixed generalized fractional Brownian motion.

**Definition 1.1.** A *mixed generalized fractional Brownian motion* (mgfBm for short) of parameters $a, b, c$ and $H \in (0, 1)$ is a centered Gaussian process

$$M^H(a, b, c) = \{M^H_t(a, b, c), t \geq 0\},$$

defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with the covariance function

$$C(t, s) = a^2(t \wedge s) + \frac{(b + c)^2}{2} \left( t^{2H} + s^{2H} \right) - bc(t + s)^{2H} - \frac{(b^2 + c^2)}{2} |t - s|^{2H},$$

where $t \wedge s = \frac{1}{2} (t + s - |t - s|)$. 

The mgfBm is completely different from all the extensions mentioned above. The process $M^H(a,b,c)$ is motivated by the fact that this process already introduced for specific values of $a$, $b$ and $c$. Indeed $M^H(a,b,0)$ is the mixed fractional Brownian motion and $M^H(0,b,c)$, is the generalized fractional Brownian motion. This why we will name $M^H(a,b,c)$ the mixed generalized fractional Brownian motion. It allows to deal with a larger class of modeled natural phenomena, including those with stationary or non-stationary increments.

Our goal is to study the main stochastic properties of this new model, paying attention to the long-range dependence, self-similarity, increment stationary, Markovity and semi-martingale properties.

2. The Main Properties

Existence of the mixed generalized fractional Brownian motion $M^H(a,b,c)$ for any $H \in (0,1)$ can be shown in the following way: consider the process

$$M^H_t(a,b,c) = aB_t + bB^H_t + cB^H_{-t}, \quad t \geq 0,$$

where $B = \{B_t, t \in \mathbb{R}\}$ is a Brownian motion and $B^H = \{B^H_t, t \in \mathbb{R}\}$ is an independent fractional Brownian motion with Hurst parameter $H \in (0,1)$.

Using (1.1) and since $B$ and $B^H$ are independent we obtain the following lemma.

**Lemma 2.1.** For all $s$, $t \geq 0$, the process (2.1) is a centered Gaussian process with covariance function given by (1.5).

**Proof.** Let $s$, $t \geq 0$ and $C(t,s) = \text{Cov}(M^H_t(a,b,c), M^H_s(a,b,c))$. Then

$$C(t,s) = \text{Cov}(aB_t + bB^H_t + cB^H_{-t}, aB_s + bB^H_s + cB^H_{-s})$$

$$= a^2(t \wedge s) + b^2 \text{Cov}(B^H_t, B^H_s) + bc \text{Cov}(B^H_t, B^H_s) + cb \text{Cov}(B^H_{-t}, B^H_s) + c^2 \text{Cov}(B^H_{-t}, B^H_{-s})$$

$$= a^2(t \wedge s) + \frac{b^2}{2} (t^{2H} + s^{2H} - |t - s|^{2H}) + \frac{bc}{2} (t^{2H} + s^{2H} - |t + s|^{2H})$$

$$+ \frac{cb}{2} (t^{2H} + s^{2H} - |-(t + s)|^{2H}) + \frac{c^2}{2} (t^{2H} + s^{2H} - |-(t - s)|^{2H})$$

$$= a^2(t \wedge s) + \frac{b^2}{2} t^{2H} + \frac{b^2}{2} s^{2H} - \frac{b^2}{2} |t - s|^{2H} + \frac{bc}{2} t^{2H} + \frac{bc}{2} s^{2H}$$

$$= a^2(t \wedge s) + \frac{(b + c)^2}{2} (t^{2H} + s^{2H}) - bc|t + s|^{2H} - \frac{(b^2 + c^2)}{2} |t - s|^{2H}.$$  

Hence the covariance function of the process (2.1) is precisely $C(t,s)$ given by (1.5). Therefore the $M^H(a,b,c)$ exists. \hfill \Box

**Remark 2.1.** Some special cases of the mixed generalized fractional Brownian motion:

1. If $a = 0$, $b = 1$, $c = 0$, then $M^H(0,1,0)$ is a fBm.
2. If $a = 0$, $b = c = \frac{1}{\sqrt{2}}$, then $M^H(0,\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ is a sfBm.
3. If $a = 1$, $b = 0$, $c = 0$, then $M^H(1,0,0)$ is a Bm.
4. If $a = 0$, then $M^H(0,b,c)$, is a gFBm.
Proposition 2.1. The mgfBm satisfies the following properties:

(1) For all \( t \geq 0 \),
\[
E\left(M_t^H(a, b, c)\right)^2 = a^2 t + \left(b^2 + c^2 - (2^{2H} - 2)bc\right)t^{2H}.
\]

(2) Let \( 0 \leq s < t \) and \( \alpha(t, s) = E\left(M_t^H(a, b, c) - M_s^H(a, b, c)\right)^2 \). Then
\[
E\left(M_t^H(a, b, c) - M_s^H(a, b, c)\right)^2 = a^2 \left| t - s \right| - 2^{2H}bc(t^{2H} + s^{2H}) + (b^2 + c^2)\left| t - s \right|^{2H} + 2bc|t + s|^{2H}.
\]

(3) We have for all \( 0 \leq s < t \),
\[
a^2(t - s) + \gamma_{(b,c,H)}(t - s)^{2H} \leq \alpha(t, s) \leq a^2(t - s) + \nu_{(b,c,H)}(t - s)^{2H},
\]
where
\[
\gamma_{(b,c,H)} = (b^2 + c^2 - 2bc(2^{2H} - 1))1_c(b, c, H) + (b^2 + c^2)1_D(b, c, H),
\]
\[
\nu_{(b,c,H)} = (b^2 + c^2)1_c(b, a, H) + (b^2 + c^2 - 2bc(2^{2H} - 1))1_D(b, c, H),
\]
\[
C = \{(b, c, H) \in \mathbb{R}^2 \times ]0, 1[; \ (H > \frac{1}{2}, \ bc \geq 0) \text{ or } (H < \frac{1}{2}, \ bc \leq 0)\},
\]
and
\[
D = \{(b, c, H) \in \mathbb{R}^2 \times ]0, 1[; \ (H > \frac{1}{2}, \ bc \leq 0) \text{ or } (H < \frac{1}{2}, \ bc \geq 0)\}.
\]

Proof. (1) It is a direct consequence of (1.5).

(2) Let \( 0 \leq s < t \) and \( \alpha(t, s) = E\left(M_t^H(a, b, c) - M_s^H(a, b, c)\right)^2 \). Then
\[
\alpha(t, s) = E\left(M_t^H(a, b, c)\right)^2 + E\left(M_s^H(a, b, c)\right)^2 \left(2E\left(M_t^H(a, b, c)M_s^H(a, b, c)\right)\right)
\]
\[
= a^2 t + b^2 2^{2H} + 2bc t^{2H} - 2^{2H}bc t^{2H} + c^2 t^{2H} + a^2 s + b^2 s^{2H} + 2bc s^{2H} - 2^{2H}bc s^{2H} + c^2 s^{2H} - 2a^2(t \land s) - b^2 t^{2H} - b^2 s^{2H} + b^2 |t - s|^{2H} - bc t^{2H} - bc s^{2H} - bc|t + s|^{2H} - cb t^{2H} - cb s^{2H} - cb|t + s|^{2H} - c^2 t^{2H} + c^2 |t - s|^{2H}
\]
\[
= a^2(t - s) - 2^{2H}bc(t^{2H} + s^{2H}) - 2a^2(t \land s) + (b^2 + c^2)t - s^{2H} + 2bc|t + s|^{2H}
\]
\[
= a^2|t - s| - 2^{2H}bc(t^{2H} + s^{2H}) + (b^2 + c^2)|t - s|^{2H} + 2bc|t + s|^{2H}.
\]

(3) It is a direct consequence of the second item of Proposition 2.1 and Lemma 3 in [29].
Proposition 2.2. For all \((a, b, c) \in \mathbb{R}^3 \setminus \{(0, 0, 0)\}\) and \(H \in (0, 1) \setminus \{\frac{1}{2}\}\), the mgfBm is not a self-similar process.

Proof. This follows from the fact that, for fixed \(h > 0\), the processes 
\(\{M_h^H(a, b, c), \; t \geq 0\}\) and 
\(\{h^M M_h^H(a, b, c), \; t \geq 0\}\) are Gaussian, centered, but don’t have the same covariance function. Indeed

\[
C(ht, hs) = a^2(ht \wedge hs) + \frac{b^2}{2}((ht)^{2H} + (hs)^{2H} - |ht - hs|^{2H})
+ \frac{bc}{2}((ht)^{2H} + (hs)^{2H} - |ht + hs|^{2H})
+ \frac{cb}{2}((ht)^{2H} + (hs)^{2H} - |-(ht + hs)|^{2H})
+ \frac{c^2}{2}((ht)^{2H} + (hs)^{2H} - |-(ht - hs)|^{2H})
= a^2(ht \wedge hs) + \frac{b^2}{2}(ht)^{2H} + \frac{b^2}{2}(hs)^{2H} - \frac{b^2}{2}|ht - hs|^{2H}
+ \frac{bc}{2}(ht)^{2H} + \frac{bc}{2}(hs)^{2H} - \frac{bc}{2}|ht + hs|^{2H}
+ \frac{bc}{2}(ht)^{2H} + \frac{bc}{2}(hs)^{2H} - \frac{bc}{2}|ht - hs|^{2H}
- \frac{bc}{2}|ht + hs|^{2H} + \frac{c^2}{2}(ht)^{2H} + \frac{c^2}{2}(hs)^{2H} - \frac{c^2}{2}|ht - hs|^{2H}
= a^2h(t \wedge s) + h^{2H} \frac{(b + c)^2}{2} \left((t)^{2H} + (s)^{2H}\right) - bch^{2H}|t + s|^{2H}
- h^{2H} \frac{(b^2 + c^2)}{2} |t - s|^{2H}.
\]

On the other hand,

\[
\text{Cov} \left(h^H M^H_t(a, b, c), h^H M^H_s(a, b, c)\right) = h^{2H} \text{Cov} \left(M^H_t(a, b, c), M^H_s(a, b, c)\right)
= a^2h^{2H}(t \wedge s)
+ h^{2H} \frac{(b + c)^2}{2} \left(t^{2H} + s^{2H}\right)
- bch^{2H}|t + s|^{2H}
- h^{2H} \frac{(b^2 + c^2)}{2} |t - s|^{2H}.
\]

Then the mgfBm is not a self-similar process for all \((a, b, c) \in \mathbb{R}^3 \setminus \{(0, 0, 0)\}\). □

Remark 2.2. As a consequence of Proposition 2.2, we see that:

1. \(M^H(0, b, c)\) is a self-similar process for all \((b, c) \in \mathbb{R}^2\).
2. \(M^A(a, b, c)\) is a self-similar process for all \((a, b, c) \in \mathbb{R}^3\).

Now, we will study the Markovian property.

Theorem 2.1. Assume \(H \in (0, 1) \setminus \{\frac{1}{2}\}\), \(a \in \mathbb{R}\) and \((b, c) \in \mathbb{R}^2 \setminus \{(0, 0)\}\). Then 
\(M^H(a, b, c)\) is not a Markovian process.

Proof. The process \(M^H(a, b, c)\) is a centered Gaussian. Then, if \(M^H_t(a, b, c)\) is a Markovian process, according to Revuz and Yor [23], for all \(s < t < u\), we would
have

\[ C(s, u)C(t, t) = C(s, t)C(t, u). \]

We will only prove the theorem in the case where \( a \neq 0 \), the result with \( a = 0 \) is known in [29]. For the proof we follow the proof of Proposition 1 given in [29].

Using Proposition 2.1, we get

\[ C(s, u) = a^2s + \frac{(b + c)^2}{2}(u^{2H} + s^{2H}) - bc|u + s|^{2H} - \frac{(b^2 + c^2)}{2}|u - s|^{2H}, \]

\[ C(t, t) = a^2t + \frac{(b^2 + c^2 - (2^{2H} - 2)bc)}{2}t^{2H}, \]

\[ C(s, t) = a^2s + \frac{(b + c)^2}{2}(t^{2H} + s^{2H}) - bc|t + s|^{2H} - \frac{(b^2 + c^2)}{2}|t - s|^{2H}, \]

\[ C(t, u) = a^2t + \frac{(b + c)^2}{2}(u^{2H} + t^{2H}) - bc|u + t|^{2H} - \frac{(b^2 + c^2)}{2}|u - t|^{2H}. \]

In the particular case where \( 1 < s = \sqrt{t} < t < u = t^2 \), we have

\[ C(\sqrt{t}, t^2) = a^2\sqrt{t} + \frac{(b + c)^2}{2}(t^{4H} + t^{H}) - bc|t^2 + t^{1/2}|^{2H} - \frac{(b^2 + c^2)}{2}|t^2 - t^{1/2}|^{2H}, \]

\[ C(t, t) = a^2t + \frac{(b^2 + c^2 - (2^{2H} - 2)bc)}{2}t^{2H}, \]

\[ C(\sqrt{t}, t) = a^2\sqrt{t} + \frac{(b + c)^2}{2}(t^{2H} + t^{H}) - bc|t + t^{1/2}|^{2H} - \frac{(b^2 + c^2)}{2}|t - t^{1/2}|^{2H}, \]

\[ C(t, t^2) = a^2t + \frac{(b + c)^2}{2}(t^{4H} + t^{2H}) - bc|t^2 + t^{1/2}|^{2H} - \frac{(b^2 + c^2)}{2}|t^2 - t^{1/2}|^{2H}. \]

Then by using that

\[ C(\sqrt{t}, t^2)C(t, t) = C(\sqrt{t}, t)C(t, t^2), \]

we have

\[
\left[ a^2\sqrt{t} + \frac{(b + c)^2}{2}(t^{4H} + t^{H}) - bc|t^2 + t^{1/2}|^{2H} - \frac{(b^2 + c^2)}{2}|t^2 - t^{1/2}|^{2H} \right] \\
\times \left[ a^2t + \frac{(b^2 + c^2 - (2^{2H} - 2)bc)}{2}t^{2H} \right]
\]

\[ = \left[ a^2\sqrt{t} + \frac{(b + c)^2}{2}(t^{2H} + t^{H}) - bc|t + t^{1/2}|^{2H} - \frac{(b^2 + c^2)}{2}|t - t^{1/2}|^{2H} \right] \\
\times \left[ a^2t + \frac{(b + c)^2}{2}(t^{4H} + t^{2H}) - bc|t^2 + t^{1/2}|^{2H} - \frac{(b^2 + c^2)}{2}|t^2 - t^{1/2}|^{2H} \right].
\]

It follows that

\[
\left[ a^2\sqrt{t} + t^{4H} \left( \frac{(b + c)^2}{2}(1 + t^{-3H}) - bc|1 + t^{-1/2}|^{2H} - \frac{(b^2 + c^2)}{2}|1 - t^{-1/2}|^{2H} \right) \right] \\
\times \left[ a^2t + \frac{(b^2 + c^2 - (2^{2H} - 2)bc)}{2}t^{2H} \right]
\]

\[ = \left[ a^2\sqrt{t} + t^{2H} \left( \frac{(b + c)^2}{2}(1 + t^{-H}) - bc|1 + t^{-1/2}|^{2H} - \frac{(b^2 + c^2)}{2}|1 - t^{-1/2}|^{2H} \right) \right] \\
\times \left[ a^2t + t^{4H} \left( \frac{(b + c)^2}{2}(1 + t^{-2H}) - bc|1 + t^{-1}|^{2H} - \frac{(b^2 + c^2)}{2}|1 - t^{-1}|^{2H} \right) \right].
\]
Hence

\[
a^4 t^\frac{3}{2} + a^2 \left( b^2 + c^2 - (2^2H - 2)bc \right) t^{2H+\frac{1}{2}}
+ a^2 \left( \frac{(b + c)^2}{2} (1 + t^{-3H}) - bc|1 + t^{-\frac{3}{2}}|^{2H} - \frac{(b^2 + c^2)}{2} |1 - t^{-\frac{3}{2}}|^{2H} \right)
\times t^{4H+1}
+ \left( \frac{(b + c)^2}{2} (1 + t^{-3H}) - bc|1 + t^{-\frac{3}{2}}|^{2H} - \frac{(b^2 + c^2)}{2} |1 - t^{-\frac{3}{2}}|^{2H} \right)
\times (b^2 + c^2 - (2^2H - 2)bc) t^{6H} = a^4 t^\frac{3}{2} + a^2 \left( \frac{(b + c)^2}{2} (1 + t^{-2H}) - bc|1 + t^{-1}|^{2H} - \frac{(b^2 + c^2)}{2} |1 - t^{-1}|^{2H} \right)
\times t^{4H+\frac{1}{2}}
+ a^2 \left( \frac{(b + c)^2}{2} (1 + t^{-H}) - bc|1 + t^{-\frac{1}{2}}|^{2H} - \frac{(b^2 + c^2)}{2} |1 - t^{-\frac{1}{2}}|^{2H} \right)
\times t^{2H+1}
+ \left( \frac{(b + c)^2}{2} (1 + t^{-H}) - bc|1 + t^{-\frac{1}{2}}|^{2H} - \frac{(b^2 + c^2)}{2} |1 - t^{-\frac{1}{2}}|^{2H} \right)
\times (b^2 + c^2 - (2^2H - 2)bc) t^{6H}.
\]

Take \( t^{6H} \) as a common factor, we get

\[
a^2 \left( b^2 + c^2 - (2^2H - 2)bc \right) t^{-4H+\frac{1}{2}}
+ a^2 \left( \frac{(b + c)^2}{2} (1 + t^{-3H}) - bc|1 + t^{-\frac{3}{2}}|^{2H} - \frac{(b^2 + c^2)}{2} |1 - t^{-\frac{3}{2}}|^{2H} \right) t^{-2H+1}
+ \left( \frac{(b + c)^2}{2} (1 + t^{-3H}) - bc|1 + t^{-\frac{3}{2}}|^{2H} - \frac{(b^2 + c^2)}{2} |1 - t^{-\frac{3}{2}}|^{2H} \right)
\times (b^2 + c^2 - (2^2H - 2)bc) t^{4H+1}
= a^2 \left( \frac{(b + c)^2}{2} (1 + t^{-2H}) - bc|1 + t^{-1}|^{2H} - \frac{(b^2 + c^2)}{2} |1 - t^{-1}|^{2H} \right) t^{-2H+\frac{1}{2}}
+ a^2 \left( \frac{(b + c)^2}{2} (1 + t^{-H}) - bc|1 + t^{-\frac{1}{2}}|^{2H} - \frac{(b^2 + c^2)}{2} |1 - t^{-\frac{1}{2}}|^{2H} \right) t^{-4H+1}
+ \left( \frac{(b + c)^2}{2} (1 + t^{-H}) - bc|1 + t^{-\frac{1}{2}}|^{2H} - \frac{(b^2 + c^2)}{2} |1 - t^{-\frac{1}{2}}|^{2H} \right)
\times (b^2 + c^2 - (2^2H - 2)bc) t^{-2H+\frac{1}{2}},
\]
Therefore

\[
a^2 \left( b^2 + c^2 - (2^{2H} - 2)bc \right) t^{-4H+\frac{1}{2}} + a^2 \left[ \frac{1}{2} (b + c)^2 (1 + t^{-3H}) - bc \left( 1 + 2Ht^{-\frac{3}{2}} + H(2H - 1)t^{-3} + o(t^{-3}) \right) \right] t^{-2H+1} + a^2 \left[ \frac{1}{2} (b + c)^2 (1 + t^{-3H}) - bc \left( 1 + 2Ht^{-\frac{3}{2}} + H(2H - 1)t^{-3} + o(t^{-3}) \right) \right] t^{-4H+1} + \frac{1}{2} (b^2 + c^2) \left( 1 - 2Ht^{-\frac{1}{2}} + H(2H - 1)t^{-1} + o(t^{-1}) \right) t^{-2H+1} + \frac{1}{2} (b^2 + c^2) \left( 1 - 2Ht^{-\frac{1}{2}} + H(2H - 1)t^{-1} + o(t^{-1}) \right) t^{-4H+1} \times [b^2 + c^2 - (2^{2H} - 2)bc] = a^2 \left[ \frac{1}{2} (b + c)^2 (1 + t^{-2H}) - bc \left( 1 + 2Ht^{-1} + H(2H - 1)t^{-2} + o(t^{-2}) \right) \right] t^{-2H+\frac{1}{2}} + \frac{1}{2} (b^2 + c^2) \left( 1 + t^{-H} \right) - bc \left( 1 + 2Ht^{-\frac{1}{2}} + H(2H - 1)t^{-1} + o(t^{-1}) \right) + \frac{1}{2} (b^2 + c^2) \left( 1 - 2Ht^{-\frac{1}{2}} + H(2H - 1)t^{-1} + o(t^{-1}) \right) t^{-4H+1} + \frac{1}{2} (b + c)^2 (1 + t^{-H}) - bc \left( 1 + 2Ht^{-\frac{1}{2}} + H(2H - 1)t^{-1} + o(t^{-1}) \right) \times \left[ \frac{1}{2} (b + c)^2 (1 + t^{-2H}) - bc \left( 1 + 2Ht^{-1} + H(2H - 1)t^{-2} + o(t^{-2}) \right) \right].
\]

First case: \(0 < H < \frac{1}{2}, a \neq 0\) and \(b + c \neq 0\). By Taylor's expansion we get, as \(t \to \infty\),

\[
a^2 \left( b^2 + c^2 - (2^{2H} - 2)bc \right) t^{-4H+\frac{1}{2}} + a^2 \left[ \frac{1}{2} (b + c)^2 (1 + t^{-3H}) - bc \left( 1 + 2Ht^{-\frac{3}{2}} + H(2H - 1)t^{-3} + o(t^{-3}) \right) \right] t^{-2H+1} + \frac{1}{2} (b^2 + c^2) \left( 1 - 2Ht^{-\frac{1}{2}} + H(2H - 1)t^{-1} + o(t^{-1}) \right) t^{-2H+1} \approx a^2 \frac{1}{2} (b + c)^2 t^{-4H+\frac{1}{2}} + a^2 \frac{1}{2} (b + c)^2 t^{-4H+\frac{1}{2}} + \frac{1}{2} (b + c)^2 \left( b^2 + c^2 - (2^{2H} - 2)bc \right) t^{-3H} + \frac{1}{4} (b + c)^4 t^{-3H}.
\]

Therefore

\[
a^2 \left( b^2 + c^2 - (2^{2H} - 2)bc \right) t^{-4H+\frac{1}{2}} + \frac{1}{2} (b + c)^2 (b^2 + c^2 - (2^{2H} - 2)bc) t^{-3H} \approx a^2 \frac{1}{2} (b + c)^2 t^{-4H+\frac{1}{2}} + \frac{1}{4} (b + c)^4 t^{-3H},
\]
which is true if and only if
\[
\frac{(b - c)^2}{2} - (2^{2H} - 2)bc = 0 \quad \text{and} \quad a^2 \frac{(b - c)^2}{2} - a^2 (2^{2H} - 2)bc = 0.
\]
However, it is easy to check that
\[
\frac{(b - c)^2}{2} - (2^{2H} - 2)bc > 0 \quad \text{and} \quad a^2 \frac{(b - c)^2}{2} - a^2 (2^{2H} - 2)bc > 0,
\]
for fixed \(c,b\) and every real \(a\).
Second case: \(0 < H < \frac{1}{2}, a \neq 0\) and \(b + c = 0\). By Taylor’s expansion we get, as \(t \to \infty\),
\[
a^2 [b^2 + c^2 - (2^{2H} - 2)bc] t^{-4H + \frac{1}{2}} + a^2 [-2Hbc + (b^2 + c^2)H] t^{-2H - \frac{1}{2}}
\]
\[
+ [-2Hbc + (b^2 + c^2)H] \times [b^2 + c^2 - (2^{2H} - 2)bc] t^{-\frac{3}{2}}
\]
\[
\approx a^2 [-2Hbc + (b^2 + c^2)H] t^{-4H + \frac{1}{2}} + a^2 [-2Hbc + (b^2 + c^2)H] t^{-4H + \frac{1}{2}}
\]
\[
+ [-2Hbc + (b^2 + c^2)H]^2 t^{-\frac{3}{2}},
\]
which is true if and only if \(b = c = 0\). This is a contradiction.
Third case: \(\frac{1}{2} < H < 1, a \neq 0\) and \(b - c \neq 0\). By Taylor’s expansion we get, as \(t \to \infty\),
\[
a^2 [b^2 + c^2 - (2^{2H} - 2)bc] t^{-4H + \frac{1}{2}} + a^2 H(b - c)^2 t^{-2H - \frac{1}{2}}
\]
\[
+ H(b - c)^2 [b^2 + c^2 - (2^{2H} - 2)bc] t^{-\frac{3}{2}}
\]
\[
\approx a^2 H(b - c)^2 t^{-2H - \frac{1}{2}} + a^2 H(b - c)^2 t^{-4H + \frac{1}{2}} + H^2(b - c)^4 t^{-\frac{3}{2}},
\]
Then
\[
a^2 [b^2 + c^2 - (2^{2H} - 2)bc] t^{-4H + \frac{1}{2}} + H(b - c)^2 [b^2 + c^2 - (2^{2H} - 2)bc] t^{-\frac{3}{2}}
\]
\[
\approx a^2 H(b - c)^2 t^{-4H + \frac{1}{2}} + H^2(b - c)^4 t^{-\frac{3}{2}},
\]
which is true if and only if
\[
[b^2(1 - H) + c^2(1 - H) + (2 - 2^{2H} + 2H)bc] = 0.
\]
However, it is easy to check that \(b^2(1 - H) + c^2(1 - H) + (2 - 2^{2H} + 2H)bc > 0\) for fixed \(c,b\) and every real \(a\).
Fourth case: \(\frac{1}{2} < H < 1, a \neq 0\) and \(b - c = 0\). By Taylor’s expansion we get, as \(t \to \infty\),
\[
a^2 (b^2 + c^2 - (2^{2H} - 2)bc) t^{-4H + \frac{1}{2}} + \frac{1}{2} (b + c)^2 [b^2 + c^2 - (2^{2H} - 2)bc] t^{-3H}
\]
\[
\approx a^2 \frac{1}{2} (b + c)^2 t^{-4H + \frac{1}{2}} + \frac{1}{4} (b + c)^4 t^{-3H},
\]
which is true if and only if \(2 - 2^{2H} = 0\). This contradicts the fact that \(H \neq \frac{1}{2}\). \(\Box\)
Let us check the mixed self-similarity property of the mgfBm. This property was introduced in [27] for the mfBm and investigated to show the Hölder continuity of the mfBm. See also [11] for the sfBm case.

**Proposition 2.3.** For any \( h > 0 \), \( \{ M^H_n(a, b, c) \} \triangleq \{ M^H_t(ah \cdot, bh, ch^H) \} \), where \( \triangleq \) “to have the same law”.

**Proof.** For fixed \( h > 0 \), the processes \( \{ M^H_n(a, b, c) \} \) and \( \{ M^H_t(ah \cdot, bh, ch^H) \} \) are Gaussian and centered. Therefore, one only have to prove that they have the same covariance function. But, for any \( s, t \geq 0 \), since \( B \) and \( B^H \) are independent, then

\[
C(h, s, t) = a^2h(t + s) + \frac{(b^2 + c^2)}{2} \left[ h^2(t^2H + s^2H - |t - s|^{2H}) \right] + bc \left[ h^2(t^2H + s^2H - |t + s|^{2H}) \right] = \text{Cov} \left( M^H_t(ah \cdot, bh, ch^H), M^H_s(ah \cdot, bh, ch^H) \right).
\]

\( \Box \)

**Proposition 2.4.** For all \( a \in \mathbb{R} \) and \( (b, c) \in \mathbb{R}^2 \setminus \{(0, 0)\} \), the increments of the mgfBm are not stationary.

**Proof.** Let \( a \in \mathbb{R} \) and \( (b, c) \in \mathbb{R}^2 \setminus \{(0, 0)\} \). For a fixed \( t \geq 0 \) consider the processes \( \{ P_t, t \geq 0 \} \) define by \( P_t = M^H_{t+a}(a, b, c) - M^H_{t}(a, b, c) \). Using Proposition 2.1, we get

\[
\text{Cov}(P_t, P_t) = E \left( M^H_{t+a}(a, b, c) - M^H_{t}(a, b, c) \right)^2
= a^2(t + s + s) - 2^{2H}bc((t + s)^{2H} + s^{2H}) - 2a^2s
+ (b^2 + c^2)|t + s + s|^{2H} + 2bc|t + s + s|^{2H}
= a^2(t + s) - 2^{2H}bc((t + s)^{2H} + s^{2H}) - 2a^2s + (b^2 + c^2)t^{2H}
+ 2bc|t + 2s|^{2H}.
\]

Using Proposition 2.1, we get

\[
\text{Cov}(M^H_t(a, b, c), M^H_s(a, b, c)) = a^2t + (b^2 + c^2) - (2^{2H} - 2)bc \cdot t^{2H}.
\]

Since both processes are centered Gaussian, the inequality of covariance functions implies that \( P_t \) does not have the same distribution as \( M^H_t(a, b, c) \). Thus, the incremental behavior of \( M^H_t(a, b, c) \) at any point in the future is not the same. Hence the increments of \( M^H_t(a, b, c) \) are not stationary. \( \Box \)

**Remark 2.3.** As a consequence of Proposition 2.4, we see that:

1. The increments of \( M^H(0, b, c) \) are not stationary for all \( (b, c) \in \mathbb{R}^2 \setminus \{(0, 0)\} \).
2. The increments of \( M^H(a, b, 0) \) are stationary for all \( (a, b) \in \mathbb{R}^2 \).

**Proposition 2.5.** (1) Let \( H \in (0, 1) \). The mgfBm admits a version whose sample paths are almost Hölder continuous of order strictly less than \( \frac{1}{2} \wedge H \).

(2) When \( b \) or \( c \) not zero and \( H \in (0, 1) \setminus \{ \frac{1}{2} \} \), the mgfBm is not a semi-martingale.
Proof. (1) Let \( s, t \geq 0 \) and \( \alpha = 2 \). The proof follows by Kolmogorov criterion from Lemma 3 in [29] and using Proposition 2.1 we get

\[
E \left( |M_t^H(a, b, c) - M_s^H(a, b, c)|^\alpha \right) = a^2|t - s| - 2^{2\alpha}bc(t^{2H} + s^{2H}) + (b^2 + c^2)t - s^{2H} + 2bc|t + s|^{2H}
\]

\[
\leq C_\alpha |t - s|^\alpha(1^{\wedge 2H}),
\]

where \( C_\alpha = (a^2 + \nu(b, c, H)) \) and \( \nu(b, c, H) \) is given in Lemma 3 in [29].

(2) Suppose first that \( H < \frac{1}{2} \). We get from Proposition 2.1

\[
\alpha(t, s) \geq \gamma(b, c, H)(t - s)^{2H}.
\]

Since \( 2H < 1 \) and \( \gamma(b, c, H) > 0 \) then the assumption of Corollary 2.1 in [5] is satisfied, and consequently the mgfBm is not a semi-martingale.

Suppose now that \( H > \frac{1}{2} \). We get from Proposition 2.1

\[
a^2(t - s) + \gamma(b, c, H)(t - s)^{2H} \leq \alpha(t, s) \leq (a^2 + \nu(b, c, H))(t - s)^{1/2H},
\]

then

\[
\gamma(b, c, H)(t - s)^{2H} \leq \alpha(t, s) \leq (a^2 + \nu(b, c, H))(t - s)^{2H}.
\]

Since \( 1 < 2H < 2 \) and \( \nu(b, c, H) > 0 \) then the assumption of Lemma 2.1 in [5] is satisfied, and consequently the mgfBm is not a semi-martingale.

\[\Box\]

3. Long-range Dependence of the mgfBm Increments

**Definition 3.1.** We say that the increments of a stochastic process \( X \) are long-range dependent if for every integer \( p \geq 1 \), we have

\[
\sum_{n \geq 1} R_X(p, p + n) = \infty,
\]

where

\[
R_X(p, p + n) = E((X_{p+1} - X_p)(X_{p+n+1} - X_{p+n})).
\]

This property was investigated in many papers (e.g. [3, 6, 8, 12, 20]).

**Theorem 3.1.** For every \( a \in \mathbb{R} \) and \( (b, c) \in \mathbb{R}^2 \setminus \{(0, 0)\} \), the increments of \( M_t^H(a, b, c) \) are long-range dependent if and only if \( H > \frac{1}{2} \) and \( b \neq c \).
Proof. For all \( n \geq 1 \) and \( p \geq 1 \), we have
\[
R_M(p, p + n) = E\left( M_{p+1}^H(a, b, c) - M_p^H(a, b, c) \right) \left( M_{p+n+1}^H(a, b, c) - M_{p+n}^H(a, b, c) \right)
\]
\[
= E\left( M_{p+1}^H(a, b, c)M_{p+n+1}^H(a, b, c) \right) - E\left( M_{p+1}^H(a, b, c)M_{p+n}^H(a, b, c) \right)
\]
\[
- E\left( M_p^H(a, b, c)M_{p+n+1}^H(a, b, c) \right) + E\left( M_p^H(a, b, c)M_{p+n}^H(a, b, c) \right)
\]
\[
= C(p + 1, p + n + 1) - C(p + 1, p + n) - C(p, p + n + 1) + C(p, p + n)
\]
\[
= a^2(p + 1) + \frac{(b + c)^2}{2} \left( (p + 1)^{2H} + (p + n + 1)^{2H} \right) - bc(2p + n + 2)^{2H}
\]
\[
- \frac{(b^2 + c^2)}{2} n^{2H} - a^2(p + 1) - \frac{(b + c)^2}{2} \left( (p + 1)^{2H} + (p + n)^{2H} \right)
\]
\[
+ bc(2p + n + 1)^{2H} + \frac{(b^2 + c^2)}{2} |n - 1|^{2H} - a^2p
\]
\[
- \frac{(b + c)^2}{2} \left( p^{2H} + (p + n + 1)^{2H} \right) + bc(2p + n + 1)^{2H}
\]
\[
+ \frac{(b^2 + c^2)}{2} |n + 1|^{2H}.
\]
Hence
\[
R_M(p, p + n) = \frac{(b^2 + c^2)}{2} \left( (n + 1)^{2H} - 2n^H + (n - 1)^{2H} \right)
\]
\[
- bc \left( (2p + n + 2)^{2H} - 2(2p + n + 1)^{2H} + (2p + n)^{2H} \right).
\]
Then for every integer \( p \geq 1 \), by Taylor’s expansion, as \( n \to \infty \), we have
\[
R_M(p, p + n) = \frac{b^2 + c^2}{2} n^{2H} \left[ \left( 1 + \frac{1}{n} \right)^{2H} - 2 + \left( 1 - \frac{1}{n} \right)^{2H} \right]
\]
\[
- bcn^{2H} \left[ \left( 1 + \frac{2p + 2}{n} \right)^{2H} - 2 \left( 1 + \frac{2p}{n} + 1 \right) + \left( 1 + \frac{2p}{n} \right)^{2H} \right]
\]
\[
= H(2H - 1)n^{2H-2}(b - c)^2
\]
\[
- 4H(2H - 1)(H - 1)bc(2p + 1)n^{2H-3}(1 + o(1)).
\]
If \( b \neq c \), we see that as \( n \to \infty \),
\[
R_M(p, p + n) \approx H(2H - 1)n^{2H-2}(b - c)^2.
\]
Then
\[
\sum_{n \geq 1} R_M(p, p + n) = \infty \iff 2H - 2 > -1 \iff H > \frac{1}{2}.
\]
If \( b = c \), then, as \( n \to \infty \),
\[
R_M(p, p + n) \approx 4H(2H - 1)(H - 1)a^2(2p + 1)n^{2H-3}.
\]
For every \( H \in (0, 1) \), we have \( 2H - 3 < -1 \) and, consequently,
\[
\sum_{n \geq 1} R_M(p, p + n) < \infty.
\]
Remark 3.1.  
(1) For all $a \in \mathbb{R}$ and $b \in \mathbb{R} \setminus \{0\}$, the increments of $M^H(a,b,0)$ are long-range dependent if and only if $H > \frac{1}{2}$.

(2) If $b = c = \frac{1}{\sqrt{2}}$, the increments of $M^H(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ are short-range dependent if and only if $H \in (0,1)$. But if $b \neq c$, the increments of $M^H(0,b,c)$ are long-range dependent if and only if $H > \frac{1}{2}$.

(3) From [4], the increments of $M^H(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ on intervals $[u,u+r], [u+r,u+2r]$ are more weakly correlated than those of $M^H(0,1,0)$.

(4) From [30], if $H > \frac{1}{2}, b^2 + c^2 = 1$ and $bc \geq 0$, the increments of $M^H(0,b,c)$ are more weakly correlated than those of $M^H(0,1,0)$, but more strongly correlated than those of $M^H(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

(5) From [30], if $H \geq \frac{1}{2}$, $(bc \leq 0$ and $(b-c)^2 \leq 1$ or $(bc \geq 0$ and $b^2 + c^2 \leq 1$, the increments of $M^H(0,b,c)$ are more strongly correlated than those of both $M^H(0,1,0)$ and $M^H(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

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