Effects of zonal flows on transport crossphase in dissipative trapped-electron mode turbulence in edge plasmas

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Abstract

Confinement regimes with edge transport barriers occur through the suppression of turbulent (convective) fluxes in the particle and/or thermal channels, i.e. \( \Gamma = \sum \sqrt{|n_k|^2} \sqrt{|\phi_k|^2} \sin \delta^{\nu,\phi} \) and \( Q = \sum \sqrt{|\bar{n}_k|^2} \sqrt{|\bar{\phi}_k|^2} \sin \delta^\nu,\phi \), respectively, for drift-wave turbulence. The quantity \( |\phi_k|^2 \) is the turbulence intensity, while \( \delta^{\nu,\phi} \) and \( \delta^\nu,\phi \) are the crossphases. For H-mode, standard decorrelation theory predicts that it is the turbulence intensity \( |\phi_k|^2 \) that is mainly affected via flow-induced shearing of turbulent eddies. However, for other regimes (e.g. I-mode, characterized by high energy confinement but low particle confinement), this decrease of turbulence amplitude cannot explain the decoupling of particle versus thermal flux, since a suppression of turbulence intensity \( |\phi_k|^2 \) would necessarily affect both fluxes the same way. Here, we explore a possible new stabilizing mechanism: zonal flows may directly affect the transport crossphase. We show the effect of this novel mechanism on the turbulent particle flux, by using a simple fluid model (Baver et al 2002 Phys. Plasmas 9 3318) for dissipative trapped-electron mode (DTEM), including zonal flows. We first derive the evolution equation for the transport crossphase \( \delta_k \) between density and potential fluctuations, including contributions from the \( E \times B \) nonlinearity. By using a parametric interaction analysis including the back-reaction on the pump, we obtain a predator–prey like system of equations for the pump amplitude \( \phi_p \), the pump crossphase \( \delta_p \), the zonal amplitude \( \phi_z \) and the triad phase-mismatch \( \Delta \delta \). The system displays limit-cycle oscillations where the instantaneous DTEM growth rate—proportional to the crossphase—shows quasi-periodic relaxations where it departs from that predicted by linear theory.

Keywords: trapped electron mode, zonal flows, transport crossphase, plasma turbulence, nonlinear dynamics

(Some figures may appear in colour only in the online journal)

1. Introduction

Transitions to enhanced confinement regimes such as H-mode play an important role in magnetic fusion devices like ITER. Such regimes occur through the suppression of the turbulent (convective) fluxes in the particle and/or thermal channel. For H-mode, theory predicts that it is the amplitude of the turbulence that is mainly affected via flow-induced shearing of turbulent eddies [1–4]. However, for other regimes such as I-mode [5], characterized by high energy confinement but low particle confinement, this decrease of turbulence amplitude cannot explain the decoupling of particle versus thermal flux, since a suppression of amplitude would necessarily affect both fluxes. It is thus important to identify and analyse particle versus thermal transport decoupling mechanisms. As convective fluxes depend both on the amplitude but also on the cross-correlation, i.e. crossphase between the two fields, a possible mechanism is the direct modification of the crossphase. Note that several experiments showed a change in the crossphase between density and potential during the L-H transition [4, 6, 7]. We identify and study such a mechanism for the dissipative trapped-electron mode (DTEM) fluid
model [8, 9] in the present work through the effect of zonal flows on the crossphase. The effect of flow-shear on the transport crossphase was investigated in [10] for resistive pressure-gradient turbulence. It was shown that the crossphase only depends on the renormalized propagator in this model. Two field models describing trapped-electron mode (TEM) were analyzed in detail [11]. The $E \times B$ nonlinearity was found to have an important role via a direct energy cascade to small scale. The crossphase dynamics was analyzed for the Hasegawa–Wakatani model in [12]. It was found that the crossphase associated to the particle flux can be nonlinearly modified compared to the linear theory, thus invalidating the well-known ‘$i\delta$’ approximation. The change resulted in a decrease of the crossphase, i.e. the nonlinearity had a stabilizing effect. We now summarize our main results: (i) it is shown that zonal flows can transport the crossphase between density fluctuations $\tilde{n}$ and potential fluctuations $\tilde{\phi}$, via the convective $E \times B$ nonlinearity, (ii) this occurs through a nonlinear shift of the crossphase from its linear value, beyond the standard ‘$i\delta$’ approximation, and (iii) this effect originates from the imaginary part of the triplet correlation $\langle \tilde{n} \tilde{\phi} \rangle$, equations (20), (21). The article is organized as follows: In section 2, we present the DTEM model, and analyze the associated crossphase dynamics. In section 3, we derive the nonlinear OD model, using a parametric four-wave analysis and present the results. Finally, section 4 provides a discussion of the results and conclusions.

2. Model

We are interested in the edge region of a fusion device, where collisions are important. Therefore, we consider the regime $\nu \gg \omega_k$ where $\nu = \nu_{ci}/\epsilon$ is the de-trapping rate for trapped electrons, with $\nu_{ci}$ the electron–ion collision frequency and $\epsilon = a / R$ the inverse aspect ratio. Hence, the fluid approach may be applicable near the plasma edge. We are interested in TEM transport in this regime, because it is experimentally relevant. We choose the simplest model, to focus on the physics. We consider the following model of TEM, based on Baver et al [8] but modified to include zonal modes, consisting of trapped electron continuity and charge balance:

$$\frac{\partial \bar{n}}{\partial t} + v_E \cdot \nabla \bar{n} + (1 + \alpha_{pe}) \frac{\partial \phi}{\partial y} = -\nu (\bar{n} - \bar{\phi})$$

with a re-definition of the effective density $n = n_c f_i f + n_{ep} (1 - f_i)$, where $n_c$ is the total density, $n_c = n_{ei} + n_{ep}$, $n_{ei} = f_i \bar{n}_{ei}$, $f_i$ the fraction of trapped electrons, and with the fluctuating quantities $\bar{n} = n - \langle n \rangle$ and $\bar{\phi} = \phi - \langle \phi \rangle$, where the brackets denote a zonal average and $\bar{n}_{ep} = (1 - f_i) \bar{\phi}$ since passing electrons are assumed Boltzmann for non-zonal modes. This modified model reduces to the Charney–Hasegawa–Mima equation in the limit of no trapped-electrons ($f_i \to 0$), thus this model conserves the total potential vorticity $\phi - \nabla^2 \phi$ in this limit. We thus expect generation of zonal flows in this model since $f_i = \sqrt{\epsilon}$ is small (typically $f_i \approx 0.2$ due to the inverse aspect-ratio $\epsilon = a / R \ll 1$). We consider a slab geometry $x, y, z$ for simplicity, though particle trapping is a toroidal phenomenon. This is plausible since we consider high-$n$ modes and high collisions. Here, $x$ is the local radial coordinate, and $y, z$ are the local poloidal and toroidal directions of a fusion device. In the following, we give a schematic derivation of equations (1), (2).

We start from the original model by Baver et al [5], valid for non-zonal modes:

$$\frac{\partial \bar{n}}{\partial t} + v_E \cdot \nabla \bar{n} + (1 + \alpha_{pe}) \frac{\partial \phi}{\partial y} = -\nu (\bar{n} - \bar{\phi})$$

$$\frac{\partial \bar{\phi}}{\partial t} [1 - \nabla^2 - f_i] \bar{\phi} - v_E \cdot \nabla \nabla^2 \bar{\phi} + [1 - f_i (1 + \alpha_{pe})] \frac{\partial \phi}{\partial y} = f_i \nu (\bar{n} - \bar{\phi})$$

with $v_E = \xi \times \nabla \phi$, and the normalizations $(c_i / \epsilon_i) t \to t$, $\rho_k \nabla \to \nabla$. Here $f_i = n_{ei} / n_0$ is the electron trapping fraction, $\eta_e = L_e / L_T$ is the ratio of temperature gradient to density gradient, and $\alpha = 3 / 2$. The parameter $\nu = \nu_{ei} f_i$ is the de-trapping rate and characterizes the strength of coupling of the 2 fields.

The density $n = n_{ei} f_i + \bar{n}$ is an effective electron density, consisting of the passing electron density assumed Boltzmann $(1 - f_i) \bar{\phi}$ and the trapped-electron density $f_i \bar{n}_{ei}$. In addition, we write the equations for the zonal components, for which electrons are not Boltzmann:

$$\frac{\partial \bar{n}_{zon}}{\partial t} + \frac{\partial}{\partial x} \left( \tilde{\phi}_{E_2} \tilde{n}_{a} \right) = 0$$

$$\frac{\partial}{\partial t} \nabla^2 \bar{\phi}_{zon} + \frac{\partial}{\partial x} \left( \tilde{\phi}_{E_2} \nabla^2 \bar{\phi} \right) = 0,$$

where $n_e = n_{ei} + n_{ep}$ with $n_{ep}$ the density of passing electrons, $\bar{n}_{ei} = n_{ei} - \langle n_e \rangle$ and $\bar{\phi} = \phi - \langle \phi \rangle$. The zonal density is $n_{zon} = n_{ei} + \langle n_{ep} \rangle$, that is the zonal density vanishes for passing electrons $\langle n_{ep} \rangle$, since passing electrons do not contribute to the $E \times B$ nonlinearity (the nonlinear drive for zonal density). Here, we used $\bar{n}_{ep} = (1 - f_i) \bar{\phi}$ and thus $\langle \tilde{\phi}_{E_2} \bar{n}_{a} \rangle = 0$ since we assume passing electrons to be Boltzmann for non-zonal modes. Combining the equations for non-zonal modes and for zonal modes, we obtain the model (1), (2).

2.1. Linear analysis

Linearizing the system of equations, we obtain:

$$-i\omega n_k + i(1 + \alpha_{pe}) k_y \phi_k = \nu (\phi_k - n_k)$$

$$-i(1 + k_z^2) \omega \phi_k + i(1 - f_i (1 + \alpha_{pe}) k_y \phi_k = -f_i \nu (\phi_k - n_k)$$

with $k_z^2 = k_x^2 + k_y^2$ the squared perpendicular wavenumber.
In matrix form:

\[
\left[ \begin{array}{c}
-i\omega + \nu \\
-f_1^\nu \\
\end{array} \right] \left[ \begin{array}{c}
(1 + \alpha \eta) \text{i}k_y - \nu \\
-f_1 \nu \\
\end{array} \right] e^{-\text{i}k_y x} = 0.
\]

The associated linear dispersion relation is given by:

\[
-\text{i}(1 + k_y^2 - f_1^\nu) \omega + \text{i}[1 - f_1(1 + \alpha \eta)k_y + f_1 \nu] e^{-\text{i}k_y x} = 0.
\]

After a little algebra, this reduces to:

\[
(1 + k_y^2 - f_1^\nu) \omega^2 + i\nu(1 + k_y^2)
+ \frac{\text{i}}{\nu}[1 - f_1(1 + \alpha \eta)k_y] \omega - ik_y \nu = 0.
\]

A more physics-based linear analysis can be obtained by evaluating first the density response. One can obtain the effective density response from equation (1) as:

\[
n_k \approx \left[ 1 - \frac{\text{i}(1 + \alpha \eta)k_y - \omega}{\nu} \right] \phi_k,
\]

in the limit \( |\omega| \ll \nu \). This could be interpreted as an 'ib' prescription: \( n_k = (1 - \text{i} \delta) \phi_k \), with \( \delta = \frac{(1 + \alpha \eta)k_y - \omega}{\nu} \).

Multiplying equation (7) by \( f_1 \), adding to equation (8) and replacing the electron response in the resulting equation, the approximate linear dispersion relation takes the form:

\[
(1 + k_y^2 - f_1^\nu) \omega - k_y = \text{i}f_1 \omega.
\]

In the relevant dissipative regime \( \omega \ll \nu \) the dispersion relation can be solved by a perturbation in the small parameter \( \omega/\nu \ll 1 \). We expand the complex frequency as: \( \omega = \omega^{(0)} + \omega^{(1)} \). At zeroth-order, equation (13) yields the drift-wave frequency \( \omega_R = k_y/(1 + k_y^2) \). At first order, equation (13) yields:

\[
\gamma(k_y, k_y) = \frac{f_1}{\nu} \frac{\alpha \eta k_y^2}{(1 + k_y^2)^2} + k_y^2 \frac{k_y^2}{(1 + k_y^2)^2},
\]

where \( \gamma = \text{Im} \omega \) denotes the linear growth-rate, and we replaced \( \delta \) by its expression.

The linear growth-rate \( \gamma \), equation (14) is shown versus wavenumber \( k_y \) for \( \eta_e = 2 \) (figure 1). The linear growth-rate shows a peak around \( k_y \leq 2 \) (figure 1). The instability does not show a threshold behavior, unlike collisionless trapped electron mode turbulence (CTEMs). This is a short-coming of the simplified model, which does not evolve the trapped electron temperature.

2.2. Cross-phase dynamics

Considering a Fourier representation of a physical quantity, \( g = \sum_k g_k e^{\text{i}k \cdot r} e^{-\text{i}k \cdot t} \), with \( g = n, \phi \), we can write the complex amplitudes \( n_k, \phi_k \) in amplitude-phase form [13]:

\[
n_k = |n_k| \exp(-\text{i}\delta_k) \quad (15)
\]

\[
\phi_k = |\phi_k|, \quad (16)
\]

where \( \delta_k \) denotes the 'density crossphase', defined as the phase-angle of density with respect to the potential fluctuation with the same mode number. Note the minus sign in the definition of the crossphase. This convention gives a positive crossphase during phase-locking and is thus convenient to compare with the well-known 'ib' approximation.

Using this ansatz, the electron continuity equation (1) becomes, for non-zonal modes with \( k = q \):

\[
e^{-\text{i}k \cdot n} \frac{\partial |n_k|}{\partial t} + \text{i}e^{-\text{i}k \cdot n} |n_k| \left[ -\frac{\partial \delta_k}{\partial t} - \omega_k \right] = -\text{i}(1 + \alpha \eta)k_y |\phi_k| + \nu \left[ |\phi_k| - |n_k| e^{-\text{i}k} \right].
\]

\[
-\nu \text{E} \cdot \nabla \vec{n}, \quad (17)
\]

and for zonal modes with \( k = q \):

\[
e^{-\text{i}k \cdot n} \frac{\partial |n_k|}{\partial t} - \text{i}e^{-\text{i}k \cdot n} |n_k| \frac{\partial \delta_k}{\partial t} = - \left( \text{\nu E} \cdot \nabla \vec{n} \right). \quad (18)
\]
We separate the real and imaginary parts and derive the equations for the amplitudes (real part) and phase-angles (imaginary part). From the imaginary part, we obtain after some algebra, for non-zonal modes:

\[
\left| n_k \right| \left[ \frac{\partial \delta_k}{\partial t} - \omega_k \right] = - (1 + \alpha n_k) k_i \phi_k \cos \delta_k - i \nu \phi_k \sin \delta_k - \text{Im} \{e^{i \delta_k} \tilde{v}_E \cdot \nabla \tilde{n} \}
\]

which reduces to:

\[
\frac{\partial \delta_k}{\partial t} = (1 + \alpha n_k) k_i \beta_k \cos \delta_k - \omega_k - \nu \beta_k \sin \delta_k + N_k, \tag{19}
\]

where \( \beta_k = |\phi_k| / |n_k| \) denotes the amplitude ratio and \( N_k \) is the nonlinear contribution, due to the \( E \times B \) nonlinearity, given by:

\[
N_k = \frac{1}{|n_k|^2} \text{Im} \{ n_k^* (v_E \cdot \nabla n)_k \}, \tag{21}
\]

where convolution in wavenumber space is implicit, and denoted by \( (\ldots)_k \). We recover similar results as [12] for the dynamics of the crossphase \( \delta_k \) (equation (5) in [12]).

For zonal modes, one obtains:

\[
\frac{\partial \delta_k}{\partial t} = \frac{1}{|n_k|^2} \text{Im} \{ n_k^* (v_E \cdot \nabla n)_k \}. \tag{22}
\]

For the linear analysis of cross-phase dynamics, equation (20)—with \( N_k = 0 \)—is self-consistent, assuming the amplitude ratio is given.

However, for the nonlinear dynamics, the term \( N_k \)—which involves the resonant triads \( (k, k', k + k') \) and \( (k, k', k - k') \)—must be evaluated.

To express the amplitude ratio, we write the real part of equation (17):

\[
\frac{1}{|n_k|} \frac{\partial |n_k|}{\partial t} = (1 + \alpha n_k) k_i \beta_k \sin \delta_k + \nu (\beta_k \cos \delta_k - 1)
- \frac{1}{|n_k|^2} \text{Re} \{ n_k^* (v_E \cdot \nabla n)_k \}. \tag{23}
\]

Equation (23) can be used to find a relation between the amplitude ratio \( \beta_k \) and the crossphase \( \delta_k \). Neglecting the quadratic nonlinearity, one obtains:

\[
\frac{1}{|n_k|} \frac{\partial |n_k|}{\partial t} \approx (1 + \alpha n_k) k_i \beta_k \sin \delta_k + \nu (\beta_k \cos \delta_k - 1) \tag{24}
\]

with \( \beta_k = |\phi_k| / |n_k| \). Using the approximation of small growth-rate \( |\gamma_k| \ll |\omega_k| \), i.e., \( \delta_k |n_k| / |n_k| \approx 0 \), we obtain the algebraic equation:

\[
\beta_k \left[ 1 + \frac{(1 + \alpha n_k) k_i}{\nu} \tan \delta_k \right] \cos \delta_k - 1 \approx 0. \tag{25}
\]

To lowest order, this gives, for linearly unstable modes:

\[
\beta_k \approx \left[ 1 - \frac{(1 + \alpha n_k) k_i}{\nu} \tan \delta_k \right] \frac{1}{\cos \delta_k}. \tag{26}
\]

The amplitude ratio \( \beta_k \) is shown versus the crossphase, for linearly-unstable modes (figure 2(a)). Since for DTEM, the crossphase is small \( |\delta_k| \ll 1 \), figure 2(a) shows that the amplitude ratio is \( \beta_k \approx 1 \) at lowest-order. Figure 2(b) shows that the ratio of particle flux to turbulence square amplitude \( \Gamma_1/|\phi_k|^2 \) is maximal at the crossphase \( \delta_k \approx \pi / 4 \), contrary to a simple passive scalar model, where it would peak at \( \delta_k = \pi / 2 \). This difference is due to the amplitude ratio dependence on the crossphase in DTEM.

Note that the dynamical evolution of the crossphase implies the dynamical change in particle flux. Therefore, we can derive the following equation describing the relaxation dynamics of the particle flux with wavenumber \( k \):

\[
\frac{\partial}{\partial t} + \left( 1 + \frac{f_k}{1 + k_i^2 - f_k} \right) \nu \Gamma_1 - f_k (1 + \alpha n_k) k_i^2 |\phi_k|^2
+ \frac{[1 - f_k (1 + \alpha n_k) k_i^2] |\phi_k|^2}{1 + k_i^2 - f_k} \text{Re} \{n_k \phi_k\}
= -f_k k_i \cdot \frac{1}{2} \sum_{k-k' \pm k''} (\hat{\epsilon} \times k') \cdot k'' \text{Im} \{ \phi_k n_k^\ast \phi_k^\ast - \phi_k^\ast \phi_k n_k \}
+ \frac{f_k k_y}{1 + k_i^2 - f_k} \sum_{k-k' \pm k''} (k_i^2 - k_i'^2) (\hat{\epsilon} \times k')
\cdot k'' \text{Im} \{n_k^\ast \phi_k \cdot \phi_k^\ast \}. \tag{27}
\]

The derivation is given in appendix. The particle flux with the wavenumber \( k \) is defined as \( \Gamma_1 = k_i \text{Im} \{n_k \phi_k\} \). A similar equation was derived in [11] for the complex-valued cross-correlation \( \langle \hat{n} \hat{\tilde{\omega}} \rangle \) of the Hasegawa–Wakatani model. Equation (27) clearly shows that the flux does not respond instantaneously to the driving gradient \( (1 + \alpha n_k) \), and it is not directly proportional to the gradient, contrary to a simple Fick’s law of diffusion. Instead, there is a finite \textit{response time} or relaxation time \( \tau \). We expect this feature to be generic, but the expression for the relaxation time is model-specific. For the simple fluid DTEM model being used in our article, it scales as: \( \tau \sim \nu^{-1} \), with \( \nu \) the de-trapping rate. Noting the similarity of equation (27) with the crossphase dynamics equation (20), we infer that crossphase dynamics mirrors the relaxation dynamics of the particle flux. In fact, writing explicitly the diamagnetic drift dependence on the gradient \( v_B \propto -\partial_i \langle n \rangle \), we have the following two coupled equations for crossphase and density gradient:

\[
\tau \frac{\partial \delta_k}{\partial t} = \left[ 1 + \frac{\alpha n_k}{\nu} k_i \right] \frac{\partial \langle n \rangle}{\partial x} - \delta_k + \frac{1}{\nu} N_k \tag{28}
\]

\[
\frac{\partial \langle n \rangle}{\partial t} = -\frac{\partial}{\partial x} \left[ \sum_f k_i |\phi_k|^2 \delta_k \right] + D_{\text{res}} \frac{\partial^2 \langle n \rangle}{\partial x^2} + S, \tag{29}
\]

where the subscript \( \text{res} \) denotes the residual term. Noting that \( D_{\text{res}} \) is approximately a constant, equation (28) reduces to a simple diffusion equation, with a diffusion coefficient \( D_{\text{res}} \), that approximately scales as: \( D_{\text{res}} \sim \nu^{-1} \), and a relaxation time \( \tau \sim \nu^{-1} \). This is reflected in the crossphase dynamics equation (20), where the crossphase \( \delta_k \) is approximately a constant, with a relaxation time \( \tau \sim \nu^{-1} \). This is evident from the figure 2(a), where the crossphase \( \delta_k \) is approximately a constant, with a relaxation time \( \tau \sim \nu^{-1} \). This is evident from the figure 2(a), where the crossphase \( \delta_k \) is approximately a constant, with a relaxation time \( \tau \sim \nu^{-1} \). This is evident from the figure 2(a), where the crossphase \( \delta_k \) is approximately a constant, with a relaxation time \( \tau \sim \nu^{-1} \).
for $|\delta_k| \ll 1$, with $\tau = 1/\nu$. Only in the limit of infinite detrapping rate $\nu \to +\infty$, i.e. $\tau \to 0$ and no nonlinear effects $N_k \to 0$ do we recover a local diffusive behavior $\delta_k \propto -\partial[\rho]/\partial x$.

3. Parametric analysis

It is apparent from equation (20) that wave–wave interactions are responsible for the nonlinear dynamics of ‘density cross-phase’ $\delta_k$. Because zonal flows are not linearly unstable and are thus not affected by the linear damping mechanisms (trapped-passing collisions for DTEM) that affect linearly unstable modes, they play a dominant role in wave–wave interactions. To obtain some physical insight, we consider a simpler setting based on the parametric interaction between TEM drift-waves and zonal flows, i.e. a 4 wave interaction [14–19]. A pump drift-wave at $(\omega_0, k_0)$ interacts with a seed zonal flow at $(\omega_q, q) = q, \dot{x}$ to generate two sidebands at $(\omega_{1,2}, k_{1,2})$, with the triad resonance condition $\omega_{1,2} = \omega_0 \pm \omega_q$ and $k_{1,2} = k_0 \pm q$. In turn, the sidebands interact with the pump to nonlinearly drive the zonal flow. Due to energy conservation, we also consider the back-reaction on the ‘pump.’ The parametric interaction is shown on a schematic diagram (figure 3).

The pump wave is taken as:

$$\begin{bmatrix} n_0 \\ \phi_0 \end{bmatrix} = \begin{bmatrix} n_{10} \\ \phi_{10} \end{bmatrix} \exp[i(k_0 \cdot r - i\omega_0 t)] + \text{c.c.,}$$

with $\omega_0$ the drift-wave frequency. The zonal flow is taken in the form:

$$V_{zona} = i q \phi_q \exp(i q \cdot x - i\omega_q t) + \text{c.c.,}$$

and same for zonal density ($n_q$). The sidebands are written as:

$$\begin{bmatrix} n_{1,2} \\ \phi_{1,2} \end{bmatrix} = \begin{bmatrix} \tilde{n}_{1,2} \\ \tilde{\phi}_{1,2} \end{bmatrix} \exp[i(k_0 \pm q \cdot r - i\omega_{1,2} t)] + \text{c.c.}$$

We use a decomposition into the amplitude and the phase, and allow a finite phase-shift for the density, corresponding to the crossphase:

$$\begin{bmatrix} n_{10} \\ \phi_{10} \end{bmatrix} = \begin{bmatrix} n_0 \exp(-i\delta_0) \\ \phi_0 \end{bmatrix}.$$
In the framework of the parametric interaction analysis, one can analytically evaluate the nonlinear term in equation (20). The crossphase dynamics equation (20) reduces to:

$$\frac{\partial \delta_0}{\partial t} = -\omega_0 + (1 + \alpha \eta_i)k_0 \beta_0 \cos \delta_0 - \nu/\beta_0 \sin \delta_0 + N,$$

where $N$ is the nonlinear crossphase shift associated to the $E \times B$ nonlinearity, and takes the form:

$$N_0 = -q_k k_0 \beta_0 \left[ \frac{\phi_m}{\phi_0} \sin(\delta_1 - \delta_0) + \frac{n_k \phi_1}{\phi_0} \sin \delta_0 + \frac{\phi_1 n_2}{\phi_0} \sin(\delta_2 - \delta_0) + \frac{n_2 \phi_2}{\phi_0} \sin \delta_0 \right].$$

Using the definition of the amplitude ratio $\beta_k = |q_k|/|n_k|$, the nonlinear crossphase shift can be rewritten as:

$$N_0 = -q_k k_0 \beta_0 \left[ \frac{\phi_m}{\phi_0} \sin(\delta_1 - \delta_0) + \frac{n_k \phi_1}{\phi_0} \sin \delta_0 + \frac{\phi_1 n_2}{\phi_0} \sin(\delta_2 - \delta_0) + \frac{n_2 \phi_2}{\phi_0} \sin \delta_0 \right],$$

where the pump amplitude ratio $\beta_0 = \phi_0/n_0$ is slaved to the crossphase:

$$\beta_0 \simeq \left[ 1 - \frac{(1 + \alpha \eta_i)k_1}{\nu} \right] \tan \delta_0 \frac{1}{\cos \delta_0}.$$

### 3.1. Derivation of the dynamics of zonal modes

Here we derive the equation for zonal potential (zonal flows) and zonal density. We start from the conservation of potential vorticity:

$$\frac{\partial}{\partial t} \left[ n_i - \nabla^2 \phi \right] + v_E \cdot \nabla \left( n_i - \nabla^2 \phi \right) + \frac{\partial \phi}{\partial y} = 0,$$

with $n_i - \nabla^2 \phi$ the potential vorticity, $n_i$ the ion density and $n_i = n_{ip} + n_{ie}$ due to quasineutrality, with $n_{ei} = f_i \tilde{n}_e$. Flux-surface averaging yields the zonal potential vorticity evolution:

$$\frac{\partial}{\partial t} \left[ n_{zon} - \nabla^2 \phi_{zon} \right] + \frac{\partial}{\partial x} \left[ (\tilde{v}_E n) - \langle \tilde{v}_E \nabla^2 \phi \rangle \right] = 0.$$

Combining with equation (39) yields the dynamics of zonal potential:

$$\frac{\partial^2 \phi_{zon}}{\partial x^2} + \frac{\partial}{\partial x} \left( \tilde{v}_E \nabla^2 \phi_{zon} \right) = 0.$$

In Fourier space, the two equations for zonal modes become:

$$f_{q_k} \frac{\partial \phi_k}{\partial t} = \sum_{q} (\xi \times q) \cdot k \left( n_k \phi_{k+q} - \phi_k n_{k+q} \right),$$

$$q_k^2 \frac{\partial \phi_{k+q}}{\partial t} = \sum_{q} (\xi \times q) \cdot k \left( |k| q_k^2 - k^2 \right) \phi_k \phi_{k+q}.$$

Using the parametric interaction analysis, the equations for zonal modes reduce to:

$$\frac{\partial \phi_k}{\partial t} = \frac{1}{2} \Lambda \left[ n_k \phi_k - \phi_k n_k \right],$$

where $k_{q_k} = k_0 \hat{y} \pm q_k \hat{x}$ and $\Lambda = q_k k_0$.

Using the symmetry of sidebands, we obtain, for $|\delta_0| \ll 1$, the following system of coupled equations:

$$\frac{\partial \phi_k}{\partial t} = (1 + \alpha \eta_i) k_0 \phi_k - \omega_0 - \nu \phi_k + \Lambda \left[ \phi_m \phi_k \Delta \delta - n_c \phi_k \delta_0 \right],$$

$$f_{q_k} \frac{\partial \phi_{k+q}}{\partial t} = f_{q_k}^* \frac{\partial \phi_k^*}{\partial t} = \left( 1 + k_{q_k}^2 \right) \left[ \phi_m \phi_k^* - \phi_k \phi_m^* \right] \phi_{k+q}^*.$$

### 4.2. Dynamics of the crossphase

We now give a description of this system of equations. Equation (47) describes the dynamics of the crossphase $\delta_0$.
associated to the pump. It takes the form of a Kuramoto-like equation, a paradigm for describing synchronization phenomena in populations of coupled oscillators [20]. The first and second terms on the rhs represent the difference between the pump frequency \( \omega_0 \) and the electron diamagnetic frequency \((1 + \alpha n_e)k_0\), including the \( \nabla T_e \) contribution. The third term on the rhs is the analog of the 'pinning term' of the Kuramoto equation responsible for phase-locking, here due to collisions between trapped and passing electrons. The last term on the rhs comes from the \( E \times B \) nonlinearity and involves both zonal flows (\( \phi_z \)) and zonal density (\( n_z \)). Equation (48) is the evolution equation for the pump amplitude \( \phi_0 \). The first term on the rhs is the linear drive, which crucially depends on the pump crossphase. The second term on the rhs is the usual zonal flow shearing effect (always stabilizing) due to the polarization nonlinearity. The last term on the rhs is due to the \( E \times B \) nonlinearity, and involves both zonal flows (\( \phi_z \)) and zonal density (\( n_z \)). Equations (49) and (50) describe the dynamics of zonal modes. Zonal flows (\( \phi_z \)) are nonlinearly driven due to the polarization nonlinearity and linearly damped due to nonclassical friction (respectively 1st and 2nd term on rhs of (49)). Zonal density (\( n_z \)) is nonlinearly driven due to the \( E \times B \) nonlinearity, rhs of equation (50). Equations (51) and (52) describe the dynamics of the sideband potential (\( \phi_n \)) and density (\( n_s \)), respectively. These are linearly coupled via collisions (\( \nu \)). The sideband potential is nonlinearly driven via the polarization nonlinearity, last term on the rhs of equation (51), while the sideband density is nonlinearly driven via the \( E \times B \) nonlinearity. Finally, equation (53) describes the dynamics of phase-mismatch \( \Delta \delta = \delta_0 - \delta_1 \). The first term on the rhs is the negative of frequency mismatch \( \Delta \omega = \omega_0 - \omega_1 \), and the 2nd term on the rhs is due to collisions. The two last terms on the rhs are due to the \( E \times B \) nonlinearity.

This 0D model is an extension of that presented in [17], to self-consistently include the crossphase dynamics, with a novel stabilizing effect of zonal flows on the transport crossphase. From this model, we see that zonal flows may affect the crossphase and hence the particle flux, via equation (47). There are two possibilities: (i) If high \( q_s \) gives an increase in crossphase \( \delta_0 \), it means it may enhance the particle flux \( \Gamma \), contrary to the usual zonal flow shearing paradigm, or (ii) if high \( q_s \) reduces the crossphase \( \delta_0 \), then the effect of zonal flows on crossphase simply reinforces the stabilizing effect of zonal flows on the turbulence amplitude (\( \phi_0 \)), suppressing the particle flux consistent with the zonal flow shearing paradigm.

A simple physics interpretation of the zonal flow effect on crossphase is that the phase mismatch \( \Delta \delta = \delta_0 - \delta_1 \) can suppress the (pump) crossphase via nonlinear-shift of the crossphase in the Kuramoto-like equation (47). Note that a similar equation was derived in [21], without the zonal flow contribution, for the case of peeling-ballooning modes. Note that equation (53), describing the dynamics of the phase mismatch is similar to equation (15) of [17], when the pump crossphase \( \delta_0 \) is fixed.

The evolution of total energy can be obtained by combining equations (48), (49), (50), (51) and (52) and is:

\[
\frac{\partial (E_0 + E_z + 2E_l)}{\partial t} = f_i(1 + \alpha n_e)k_0\phi_0^2 \frac{\partial \phi_0}{\partial \phi_0} - 2f_i \nu (m_1 - \phi_1^2 - \mu q_z^2 \phi_z^2) + 2k_0^2 \Lambda \phi_0 \phi_0^2 - 2k_0^2 \Lambda \phi_0 \phi_1 \phi_1 + (2k_0^2 - q_z^2) \Lambda \phi_0 \phi_0 \phi_2 + f_i \Lambda n_s (\phi_0 n_0 - \phi_0 \phi_1) - f_i \Lambda n_s (\phi_0 n_z - \phi_0 \phi_z) = f_i(1 + \alpha n_e)k_0\phi_0^2 \frac{\partial \delta_0}{\partial \phi_0} - 2f_i \nu (m_1 - \phi_1^2 - \mu q_z^2 \phi_z^2),
\]

where \( E_0 = (1 + k_0^2)\phi_0^2/2, E_z = f_i n_1^2/2 + (1 + k_1^2 - f_i)\phi_1^2/2 \) and \( E_l = f_i n_1^2/2 + q_z^2 \phi_z^2/2 \). This can be written in the form:

\[
\frac{\partial W}{\partial t} = 2\gamma_{4W}^n W,
\]

where \( W = E_0 + E_z + 2E_l \) the total energy, and \( \gamma_{4W}^n \) is the energy input-rate of the four-wave system, given by:

\[
\gamma_{4W}^n = f_i(1 + \alpha n_e)k_0\phi_0^2 \frac{\partial \delta_0}{\partial \phi_0} - 2f_i \nu (m_1 - \phi_1^2 - \mu q_z^2 \phi_z^2) + (1 + k_0^2)\phi_0^2 \frac{\partial \delta_0}{\partial \phi_0} - 2k_0^2 \Lambda \phi_0 \phi_0^2 - 2k_0^2 \Lambda \phi_0 \phi_1 \phi_1 + (2k_0^2 - q_z^2) \Lambda \phi_0 \phi_0 \phi_2 - 2f_i \Lambda n_s (\phi_0 n_0 - \phi_0 \phi_1) + f_i \Lambda n_s (\phi_0 n_z - \phi_0 \phi_z) = f_i(1 + \alpha n_e)k_0\phi_0^2 \frac{\partial \delta_0}{\partial \phi_0} - 2f_i \nu (m_1 - \phi_1^2 - \mu q_z^2 \phi_z^2) + (1 + k_0^2)\phi_0^2 \frac{\partial \delta_0}{\partial \phi_0} - 2k_0^2 \Lambda \phi_0 \phi_0^2 - 2k_0^2 \Lambda \phi_0 \phi_1 \phi_1 + (2k_0^2 - q_z^2) \Lambda \phi_0 \phi_0 \phi_2 - 2f_i \Lambda n_s (\phi_0 n_0 - \phi_0 \phi_1) + f_i \Lambda n_s (\phi_0 n_z - \phi_0 \phi_z).
\]

This expression can be compared to equation (10) in [8] (see also [9]), and we recover this result for \( |\delta_0| \ll 1 \) and if zonal flows are neglected \( \phi_z \rightarrow 0 \). Equation (55) can be re-expressed as:

\[
\Gamma_{\text{turb}} = \frac{1}{1 + \alpha n_e} \frac{\partial W}{\partial t} + f_i \nu (m_1 - \phi_1^2) + \mu q_z^2 \phi_z^2 = \frac{1}{1 + \alpha n_e} \frac{\partial W}{\partial t} + f_i \nu (m_1 - \phi_1^2) + \mu q_z^2 \phi_z^2,
\]

with \( \Gamma_{\text{turb}} = f_i k_0 \phi_0^2 \phi_0 \phi_1 \phi_1 \phi_1 \phi_1 \phi_1 \). The turbulent particle flux. This is the fluctuation-dissipation theorem for the four-wave model. It shows that, in this model, the turbulent particle flux is constrained to be non-negative, i.e. outward (\( \Gamma_{\text{turb}} \geq 0 \)) in the saturated state (\( \partial W/\partial t \sim 0 \)).

3.2. Simplified 0D model

Furthermore, in the limit of negligible zonal density \( n_z \ll \phi_z \), due to near-adiabatic response of the sideband density to potential \( n_z \sim \phi_z \) for large collisions \( \nu \gg \omega_0 \), the system (47)–(53) reduces to:

\[
\frac{\partial \delta_0}{\partial t} = (1 + \alpha n_e)k_0 - \omega_0 - \nu \delta_0 + \Lambda \frac{\phi_0 \phi_1}{\phi_0} \Delta \delta
\]

\[
(1 + k_0^2) \frac{\partial \phi_0}{\partial t} = f_i(1 + \alpha n_e)k_0\phi_0^2\delta_0 - 2k_0^2 \Lambda \phi_0 \phi_1 \phi_1 - f_i \Lambda \phi_0 \phi_2
\]

\[
\frac{\partial \phi_1}{\partial t} = 2\Lambda \phi_0 \phi_1 - \mu \phi_1
\]

\[
\frac{\partial \Delta \delta}{\partial t} = -\Delta \omega - \nu \Delta \delta - \frac{\Lambda}{2} \left( \frac{\phi_0 \phi_0}{\phi_1} - 2\frac{\phi_0 \phi_0}{\phi_0} \right) \Delta \delta
\]

together with:
where $\gamma_2$ is the sideband damping rate [17]. The *phase-response curve* associated to the Kuramoto-like equation (58) is shown (figure 4). This figure shows that the value of the crossphase corresponding to phase-locking condition $\partial_t \delta_0 = 0$ can be shifted from its linear value $\delta_0^{\text{lin}} = [(1 + \alpha_1)\Delta - \omega_0] / \nu$, due to the phase mismatch $\Delta \delta = \delta_0 - \delta_1$ arising from the $E \times B$ nonlinearity, last term on the rhs of equation (58). The dynamics of this predator–prey model equations (58)–(61) is shown in the case with and without phase mismatch (figure 5). The associated limit-cycle is also shown in dynamical phase-space (figure 6). Figure 5 shows the evolution of the crossphase $\delta_0$ (blue) and amplitude $\phi_0$ (red) associated to the pump, the zonal flow amplitude $\phi_1$ (yellow) and the phase mismatch $\Delta \delta$ (magenta). The pump crossphase initially increases, driven by the density and temperature gradients and reaches the phase-locking state corresponding to the linear value $\delta_0 = \delta_0^{\text{lin}}$. Then, limit-cycle oscillations occur. We now describe the typical dynamics during one cycle. The pump amplitude starts to grow exponentially, until it drives the zonal flow. As the zonal flow is driven, it back-reacts on the pump amplitude, as in the standard DW-ZF predator–prey model. Moreover, the zonal flow also back-reacts on the pumped crossphase, a novel feature of the present model. The zonal flow induces a sudden suppression of the crossphase, and the crossphase relaxes back towards its linear value, as the zonal flow amplitude decreases. Then, this cycle repeats. Without phase mismatch effects (figure 5(b)), there is no feedback on the crossphase, and thus, after phase-locking, the crossphase stays constant at its linear value, while turbulence amplitude and zonal flow amplitude undergo limit cycle oscillations. In figure 5, the dynamics of the model is shown in the dynamical phase-space $\phi_0, \phi_1, \delta_0 - \delta_0^{\text{lin}}$ in the case with phase mismatch (blue), and in the reference case without phase-mismatch (black). The limit-cycle is clearly visible in both cases. In the case without phase mismatch, it is two-dimensional ($\phi_0, \phi_1$), and has a structure similar to the limit-cycle of the standard drift wave—zonal flow predator–prey model, with the growth of turbulence amplitude preceding that of zonal flow amplitude. However, when taking into account phase mismatch $\Delta \delta = 0$, the limit-cycle develops a three-dimensional structure, with the transport crossphase $\delta_0$ also participating in the limit-cycle.

4. Discussion and conclusions

First, let us discuss our results concerning equation (20) which sets the crossphase $\delta_k = \arg(n_k / n_l)$ between density and potential and equation (25) which sets the amplitude ratio $|\phi_k / \phi_l|$ between density and potential. Since the DTEM model that we use is mathematically similar (but physically very distinct) to the Hasegawa–Wakatani model describing resistive drift-waves, we compare our results to that of [12]. However, it must be stressed that in their model, zonal modes are unphysically damped by the linear parallel dynamics (governed by the parameter named $\alpha$ in their model), whereas in the DTEM model that we use, zonal flows are not affected by the linear dynamics (the de-trapping rate $\nu$ does not directly affect zonal modes). With this in mind, we can compare the crossphase equation (20) with equation (5) of [12]. Our analysis is slightly different, as we set the phase-angle of potential to zero, without loss of generality. This is made possible, because phase-angles are only defined up to an arbitrary phase. Instead, the analysis in [12] does not make use of this feature, and derives equations for both phase-angles of equations (3) and (4), and then subtract the two equations to obtain the crossphase evolution (5). Due to this difference, our equation (20) differs from their equation (5), when making the replacement $\nu \to \alpha$. The first term on the rhs of both equations is the entrainment frequency. In [12] it is directly the normalized electron diamagnetic frequency $(k_n)$, whereas in our analysis, it is the difference between the diamagnetic frequency $(1 + 3/2 n_k) k_n$ (including $\nabla T_e$ effects) and the mode frequency $(\omega_k)$. The discrepancy is easily traced to the fact that since we set the phase of potential to zero, we need to take into account that the mode rotates at the mode frequency. The second term on the rhs of both equations is the pinning term, in our equation (20) only the density equation contributes to this term, whereas in equation (5) of [12] both the density and potential equation contribute, the reason for their $1/(k^2 \beta_0^2)$ term in the bracket of equation (5). The third term on the rhs, the nonlinear term, also differs in their analysis compared to ours, because they include the polarization nonlinearity in the crossphase dynamics, in addition to the $E \times B$ nonlinearity. In this respect, our analysis is more transparent, as we easily recover the results of linear analysis $\delta_k^{\text{lin}} \propto [(1 + 3/2 n_k) k_n - \omega_k] / \nu$, and the amplitude ratio $\beta_k$ is self-consistently determined, equation (25). We stress that our approach can be readily extended to other transport channels. For example, as we derived equation (20) for the crossphase between potential and density using only the equation for trapped electron density, we could derive an equation for the crossphase between potential and electron temperature from an equation...
and the zonal effect of the equilibrium cannot change by this mechanism. However, equilibrium equilibrium effect on zonal fl or ow shear could affect the crossphase indirectly, via its B fl ow cancels out. This is because an equilibrium E × B flow can couple to itself to drive the zonal density, something that is forbidden in wave–wave interactions. It is well-known that the analysis in [23] is only valid in the early growth phase of zonal density, and thus, we stress that it should not be applied to determine saturation levels. The fluid model that we use could possibly be extended to the CTEM regime, where zonal density generation seems to play a crucial role, but this is beyond the scope of this article.

There are limitations to our model. The trapped electron fluid model [8] that we use, although taking into account the electron temperature gradient drive (γ0), neglects trapped electron temperature fluctuations. This is beyond the scope of this article. Although we derive fully-nonlinear equation for the crossphase equation (20) at the beginning of this work, the four-wave approximation (parametric interaction) is used to obtain later results, in particular the nonlinear 0D model (47)–(53). This is a convenient method for closure of the nonlinearities, but it has the disadvantage that, from the full spectrum of wavenumbers k, only the pump wavenumber k0 and the zonal wavenumber qz are kept, together with the sideband wave-number. Hence, in this approach, the particle transport is implicitly assumed to be due exclusively to the pump mode. Maybe a more general approach, linking the full crossphase spectrum to zonal flows, in analogy with the wave kinetic

for trapped electron temperature. This is left for future work. It is straightforward to extend the crossphase equation (20) to include equilibrium E × B flow. The result is that both equilibrium E × B flow or ow shear cannot directly affect the crossphase. This is because an equilibrium E × B flow would enter only through the Doppler-shifted frequency ωk − ωE due to Galilean invariance. Since, in this case, the frequency is ωk = ωq/(1 + k2f) + ωE, the quantity ωk − ωE is simply the drift-wave frequency ωq/(1 + k2f). Hence, the effect of the equilibrium E × B flow cancels out. This is simply because fluctuations of both density and potential are equally advected by the equilibrium flow, thus the crossphase cannot change by this mechanism. However, equilibrium flow or ow shear could affect the crossphase indirectly, via its effect on zonal flows. For example, [22] showed experimentally that an equilibrium radial electric field—induced by biasing the plasma—can amplify zonal flows. On this matter, we seem to reach different conclusions than [21], where equilibrium E × B flow is claimed to directly affect the crossphase. The discrepancy may be resolved if, in [21], the E × B flow is implicitly assumed to be nonlinear (e.g. a coherent zonal flow), as we showed that zonal flows do affect the crossphase. In this case, however, the direction of the flow is irrelevant, as the stabilization is due to the zonal flow amplitude squared, as we showed from our 0D analysis, i.e. equation (58).

Now, we discuss the zonal density generation equation (50) in the 0D model (47)–(53). In [23], it was shown that CTEM can saturate via nonlinear drive of zonal density. Although it is not directly relevant to our work, since we consider DTEM, we can compare the zonal density generation mechanism that we propose equation (50) to the mechanism described by equation (8) in [23], as this should be model-independent. Our analysis equation (50) differs from that of [23], in that we show that zonal density can only be driven by coupling of a pump mode (n0, φ0) and a sideband perturbation (n1, φ1), whereas [23] claims that the pump mode can couple to itself to drive the zonal density, something that is forbidden in wave–wave interactions. It is well-known that the analysis in [23] is only valid in the early growth phase of zonal density, and thus, we stress that it should not be applied to determine saturation levels. The fluid model that we use could possibly be extended to the CTEM regime, where zonal density generation seems to play a crucial role, but this is beyond the scope of this article.

There are limitations to our model. The trapped electron fluid model [8] that we use, although taking into account the electron temperature gradient drive (γ0), neglects trapped electron temperature fluctuations. This is beyond the scope of this article. Although we derive fully-nonlinear equation for the crossphase (20) at the beginning of this work, the four-wave approximation (parametric interaction) is used to obtain later results, in particular the nonlinear 0D model (47)–(53). This is a convenient method for closure of the nonlinearities, but it has the disadvantage that, from the full spectrum of wavenumbers k, only the pump wavenumber k0 and the zonal wavenumber qz are kept, together with the sideband wave-number. Hence, in this approach, the particle transport is implicitly assumed to be due exclusively to the pump mode. Maybe a more general approach, linking the full crossphase spectrum to zonal flows, in analogy with the wave kinetic

Figure 5. Dynamics of the model equations (58)–(61). (a) With phase mismatch effects (Δδ = 0) and (b) without phase mismatch effects (Δδ = 0). The parameters are: ν = 10, γ0 = 1, γq = 0.5, μ = 0.01, f = 0.5. The pump and zonal wavenumbers are respectively k0 = 1 and qz = 0.8.

Figure 6. Limit-cycle of the model with (blue) and without (black) phase mismatch effects. The parameters are the same as in figure 5.
equation [24, 25] for the power spectrum would be preferable, but this is beyond the scope of this article. Finally, how could the theory presented in this work be tested experimentally? One possible way would be a direct test of the effect of zonal flows on the crossphase: experiments may be able to measure directly the imaginary part—i.e. quadrature component—of the triplet correlation \((\phi \phi \phi)\) associated to the convective \(E \times B\) nonlinearity, defined in equation (21) and check if the magnitude of this quantity correlates well with a change in the crossphase between density and potential fluctuations.

In conclusion, we showed using a parametric analysis that zonal flows can have a stabilizing effect on the transport and relaxation dynamics of the turbulent particle flux at wavenumber \(k\) and \(v_{k}\) = \(i k_{s} \phi_{k}\), and \(\Im \{z\} = \frac{1}{2} (z - z^{*})\). Multiplying the c.c. of density equation by \(\phi_{k}\) yields:

\[
\phi_{k} \frac{dn_{k}^{*}}{dt} - i(1 + \alpha_{\eta}) \kappa_{k} \phi_{k} + \nu (n_{k} - \phi_{k}) = - \phi_{k} \sum_{k=k^{*}+k^{*}} (\xi \times k^{*}) \cdot k^{n}_{k} \phi_{k} + \phi_{k} \phi_{k}^{*}.
\]

(A.4)

Multiplying the potential equation by \(n_{k}^{*}\) yields:

\[
(1 + k_{s}^{2} - f_{1}) n_{k}^{*} \frac{\partial \phi_{k}}{\partial t} + i[1 - f_{1}(1 + \alpha_{\eta})] k_{s} n_{k}^{*} \phi_{k} - f_{k} \nu \left[ |n_{k}|^{2} - n_{k}^{*} \phi_{k} \right] = n_{k}^{*} \sum_{k=k^{*}+k^{*}} \left( k_{s}^{*} \phi_{k} - \phi_{k}^{*} \right).
\]

(A.5)

Combining the two equations yields the evolution of (complex-valued) cross-correlation:

\[
\frac{\partial}{\partial t} (n_{k}^{*} \phi_{k}) - i(1 + \alpha_{\eta}) \kappa_{k} \phi_{k} + \nu (n_{k} - \phi_{k}) = f_{k} \nu \left[ |n_{k}|^{2} - n_{k}^{*} \phi_{k} \right] + \frac{1}{1 + k_{s}^{2} - f_{1}} \sum_{k=k^{*}+k^{*}} \left( k_{s}^{*} \phi_{k} - \phi_{k}^{*} \right)
\]

(A.6)

Finally, taking the imaginary part and multiplying by \(f_{k} \phi_{k}\) yields the relaxation dynamics of particle flux equation (27) in main text.

### Appendix A. Link between crossphase dynamics and relaxation dynamics of the turbulent particle flux

In Fourier space, the model takes the form:

\[
\frac{\partial n_{k}}{\partial t} + (1 + \alpha_{\eta}) \kappa_{k} \phi_{k} + \nu (n_{k} - \phi_{k}) = - \frac{1}{2} \sum_{k=k^{*}+k^{*}} (\xi \times k^{*}) \cdot k^{n}_{k} \phi_{k} - \phi_{k} \phi_{k}^{*} \tag{1.1}
\]

\[
\frac{\partial}{\partial t} (1 - f_{1}) \phi_{k} + k_{s}^{2} \phi_{k} + [1 - f_{1}(1 + \alpha_{\eta})] \kappa_{k} \phi_{k} - f_{1} \nu (n_{k} - \phi_{k}) = \sum_{k=k^{*}+k^{*}} \left( k_{s}^{2} - k_{s}^{n2} \right) (\xi \times k^{*}) \cdot k^{n}_{k} \phi_{k} - \phi_{k} \phi_{k}^{*}. \tag{1.2}
\]

Note the following identity:

\[
\frac{\partial n_{k}^{*}}{\partial t} = f_{k} \kappa_{k} \Im \left\{ \frac{\partial}{\partial t} (n_{k}^{*} \phi_{k}) \right\} \tag{1.3}
\]

with \(\Gamma_{k} = (1/2) f_{k} n_{k} v_{k}^{*} + \text{c.c.} \). The particle flux at wavenumber \(k\) and \(v_{k} = -i k_{s} \phi_{k}\), and \(\Im (z) = \frac{1}{2} (z - z^{*})\). Multiplying the c.c. of density equation by \(\phi_{k}\) yields:

\[
\phi_{k} \frac{dn_{k}^{*}}{dt} - i(1 + \alpha_{\eta}) \kappa_{k} \phi_{k} + \nu (n_{k}^{*} - \phi_{k}^{*}) = - \phi_{k} \sum_{k=k^{*}+k^{*}} (\xi \times k^{*}) \cdot k^{n}_{k} \phi_{k} - \phi_{k} \phi_{k}^{*} \tag{1.4}
\]

With the coupling coefficient \(\Lambda = (\xi \times q) \cdot k_{0} = q_{\xi} k_{0}\), and \(\cos \delta_{1} = 1 - \delta_{1}^{2}/2\).
In addition, the frequencies $\omega_0$, $\omega_1$ are:

$$\omega_0 = k_0/(1 + k_0^2)$$  \hspace{1cm} (B.8)
$$\omega_1 = k_0/(1 + k_0^2 + q_1^2).$$  \hspace{1cm} (B.9)

Subtracting equations (B.1) and (B.5), we obtain the dynamics of the phase-mismatch $\Delta \delta = \delta_0 - \delta_1$ as:

$$\frac{\partial \Delta \delta}{\partial t} = -\Delta \omega - \nu \Delta \delta - \frac{\Lambda}{2} \left( \frac{\phi_2 \phi_0}{m_1} - 2 \frac{\phi_2 \phi_1}{\phi_0} \right) \Delta \delta$$
$$+ \left[ \frac{n_z \phi_0}{m_1} \delta_1 + 2 \frac{n_z \phi_1}{\phi_0} \delta_0 \right].$$  \hspace{1cm} (B.10)

where $\Delta \omega = \omega_0 - \omega_1 \propto q_1^2 \omega_0 > 0$ denotes the frequency mismatch. Physically, equation (B.10) describes the dynamics of the 'triad' phase mismatch $\delta_0 - \delta_1 - \delta_q$, since we made the approximation of zero phase between zonal density and zonal potential $\delta_q = 0$.

Finally, combining equations (B.1), (B.2), (B.3), (B.4), (B.6), (B.7) and (B.10) we obtain, after some algebra, the 0D model given in the main text (47)–(53).

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