DEGENERATIONS OF RIEMANNIAN MANIFOLDS

IGOR BELEGRADEK

1. Motivation and basic definitions

This is an expository article on collapsing theory written for the Modern Encyclopedia of Mathematical Physics (MEMPhys). We focus on describing the geometric and topological structure of collapsed/non-collapsed regions in Riemannian manifold under various curvature assumptions. Numerous applications of collapsing theory to Riemannian geometry are not discussed in this survey, due to page limits dictated by the encyclopedia format. More information on collapsing can be found in the ICM articles by Perelman [Per95], Colding [Col98], Petrunin [Pet02], Rong [Ron02], in Cheeger’s book on Cheeger-Colding theory [Che01], and in the comprehensive survey of Fukaya [Fuk06]. The article ends with an appendix on Gromov-Hausdorff distance, also written for MEMPhys.

A fundamental problem in Riemannian geometry is to analyze how a family of Riemannian manifolds can degenerate. For example, in the case of Einstein manifolds it is natural to study how the Einstein equation develops a singularity, or how to compactify the moduli space of Einstein metrics. The concept of a Gromov-Hausdorff convergence provides a general framework for studying metric degenerations. Let \((M_k, p_k)\) is a sequence of complete pointed \(n\)-dimensional Riemannian manifolds that Gromov-Hausdorff converge to the space \((Y, q)\); in other words \((M_k, p_k)\) degenerates to \((Y, q)\). Due to Gromov’s compactness theorem, a simple way to ensure that \((M_k, p_k)\) has a Gromov-Hausdorff converging subsequence is to assume that \(\text{Ric}(M_k) \geq c\) for some \(c\). The sequence \(M_k\) is said to collapse near the points \(p_k\) if the volumes of the unit balls centered at \(p_k\) tend to zero as \(k \to \infty\). Otherwise, \(M_k\) is called non-collapsing near \(p_k\). An additional challenge is that collapsing and non-collapsing may occur at the same time on different parts of the manifold (even though if \(\text{Ric}(M_k) \geq c\), then the distance between collapsed and non-collapsed parts of \(M_k\) has to go to infinity as \(k \to \infty\) because of Bishop-Gromov’s volume comparison). It is mainstream of global Riemannian geometry to study collapsing and non-collapsing sequence of manifolds under various assumptions on curvature such as \(|\text{sec}| \leq C\), or \(\text{sec} \geq 0\), or \(\text{Ric} \geq c\), or \(|\text{Ric}| \leq c\), or

\begin{center}
\textit{2000 Mathematics Subject classification.} Primary 53C20.
\end{center}
various integral curvature bounds, or assuming that the metric is Einstein, or Kähler-Einstein, or Calabi-Yau etc.

2. Manifolds with two sided bounds on sectional curvatures.

To date the most complete picture of collapse is established for manifolds with two sided bounds on sectional curvature, i.e. when $|\text{sec}(M_k)| \leq c$. This bound is equivalent to the upper bound on the norm of the Riemann curvature tensor, hence it may seem overly restrictive, especially when compared with bounds on Ricci curvature which are much more natural in the context of general relativity. However, studying collapse under sectional curvature bounds, rich and complex as it is, provides deep insights as to what might happen under bounds on Ricci curvature.

If $|\text{sec}(M_k)| \leq c$, then any non-collapsing sequence $(M_k, p_k)$ has a subsequence that Gromov-Hausdorff converges to $(Y, q)$, where $Y$ a smooth manifold with a $C^{1,\alpha}$-Riemannian metric, and for any compact domain $D \subset Y$ there are compact domains $D_k \subset M_k$, and diffeomorphisms $\phi_k: D \to D_k$ such that after pullbacking the metrics from $D_k$ to $D$ via $\phi_k$, the metrics converge in $C^{1,\alpha}$-topology (meaning that in each coordinate chart the coefficients of the metric tensors $g_{ij}$ of $M_k$ converge in $C^{1,\alpha}$-topology). If $D$ is Hausdorff close to the metric ball $B(q, R)$, then $D_k$ is Hausdorff close to the metric ball $B(p_k, R)$. If $M_k$ have uniformly bounded diameters, then one can take $D_k = M_k$, in particular, for each positive $c, d, v$, there only finitely many diffeomorphism classes of complete Riemannian manifolds with $|\text{sec}(M_k)| \leq c$, $\text{diam}(M_k) \leq d$, $\text{vol}(M_k) \geq v$. (This result is sometimes called Cheeger-Gromov’s compactness theorem. Following a somewhat different route Anderson [And90] obtained the same conclusion under the weaker assumptions on Ricci curvature and injectivity radius: $|\text{Ric}(M_k)| \leq c$ and $\text{inj}_{p_k}(M_k) \geq i > 0$).

The collapsing case with $|\text{sec}(M_k)| \leq c$ is much more complex. The cornerstone of the theory is Gromov-Ruh’s theorem on almost flat manifolds: a manifold $M$ admits a sequence of complete Riemannian metrics $g_k$ with $\text{diam}(M, g_k) \leq 1$ and $|\text{sec}(M, g_k)| \to 0$ as $k \to \infty$ if and only if $M$ is an infranilmanifold [Gro78, BK81, Ruh82]. An infranilmanifold is the quotient of $N$ by a cocompact discrete torsion-free subgroup of the semidirect product $N \rtimes F$, where $N$ connected simply-connected nilpotent Lie group (such as $\mathbb{R}^m$, or the Heisenberg group), and $F$ is a finite group of Lie group automorphisms of $N$. For example, if $N = \mathbb{R}^m$, then infranilmanifolds are precisely compact flat manifolds. Another important example is a nilmanifold, i.e. the quotient of $N$ by its cocompact discrete subgroup. Topologically, nilmanifolds can be characterized as iterated principle circle bundles, such as a circle, a 2-torus, all circle bundles over tori etc., and any infranilmanifold is finitely covered by a nilmanifold.
With this definition almost flat manifolds are the solutions of bounded size perturbations of the equation $\sec = 0$, hence the name. After suitable rescaling the assumptions $\operatorname{diam} \leq 1$, $|\sec| \to 0$ turn into $\operatorname{diam} \to 0$, $|\sec| \leq 1$ so Gromov-Ruh’s theorem characterizes manifolds that Gromov-Hausdorff converge (in fact collapse) to a point as infranilmanifolds. For example, any compact flat manifold collapses to a point simply by scaling the metric by $\epsilon$ with $\epsilon \to 0$. A typical collapse of an infranilmanifold to a point necessarily involves inhomogeneous scaling, e.g. for the circle bundle over the torus the fiber could be scaled by $\epsilon^2$ while the base is scaled by $\epsilon$. Thus the collapse generally takes place on several scales, and an inhomogeneous scaling is essential to ensure that the sectional curvature stays bounded during the collapse, i.e. that $|\sec| \leq 1$.

**Fukaya’s fibration theorem** [Fuk06] says that if $M_k$ collapses near $p_k$ to a Riemannian manifold $(Y, q)$, then for any compact domain $D \subset Y$ there are compact domains $D_k \subset M_k$ such that $D_k$ smoothly fibers over $D$ with infranilmanifolds as fibers, and the collapse happens along these infranilmanifolds. If $D$ is Hausdorff close to the metric ball $B(q, R)$, then $D_k$ is Hausdorff close to the metric ball $B(p_k, R)$. The case when $Y$ is not a manifold can be understood via the equivariant version of Fukaya’s fibration theorem which says that the (orthonormal) frame bundles $FD_k$ with their natural metrics fiber over a certain Riemannian manifold $L$ with infranilmanifolds as fibers, the collapse occurs along the fibers, and the fibration is $O(n)$–equivariant with respect to some isometric $O(n)$–action on $L$ such that $L/O(n)$ is identified with a subspace of $Y$. Pushing down the fibration $FD_k \to L$ by the $O(n)$–action one gets a stratification on $D_k$ whose strata are infranilmanifolds of various dimensions, and $D_k$ collapses to $L/O(n) \subset Y$ along the strata.

This description of local collapse is adequate for many applications, yet a much more general and detailed picture was obtained by Cheeger-Fukaya-Gromov [CFG92]. For any $\epsilon > 0$, a complete Riemannian $n$-dimensional manifold with $|\sec| \leq \epsilon$ can be partitioned into two disjoint sets where the injectivity radius is $\geq \epsilon$ and $\leq \epsilon$, called $\epsilon$-thick and $\epsilon$-thin part of the manifold. On compact domains in the $\epsilon$-thick part one has Cheeger-Gromov compactness as explained above. Results of Cheeger-Fukaya-Gromov describe the $\epsilon(n)$-thin part for some universal constant $\epsilon(n)$ depending only on $n$. They prove that the $\epsilon(n)$-thin part carries the so-called invariant Riemannian metric (that can be chosen arbitrary close in $C^{1, \alpha}$-topology to the original metric) such that every point of the $\epsilon(n)$-thin part has a neighborhood of definite size that is a tubular neighborhood of an embedded infranilmanifold, and there is an isometric action of a nilpotent Lie group in the universal cover of the neighborhood that stabilizers the preimage of the infranilmanifold. Orbits of the action descend to infranil strata in the $\epsilon(n)$-thin part of the manifold. The family of local actions forms what is called an $N$-structure, where “N” stands for “nilpotent”.

An N–structure captures all collapsed directions on every possible scale, yet it may be hard to analyze partly because it carries so much information. For many applications it is easier to deal with the so-called F–structures, where “F” stands for “flat”, which were introduced and studied by Cheeger-Gromov [CG86]. An F-structure capture information on collapse at the smallest possible scale, that of the injectivity radius. In retrospect, an F-structure corresponds to the center of an N–structure. An F–structure can be described as a collection of compatible local actions of tori in finite covers of coordinate charts, so this notion generalizes that of a torus action. By contrast, for the N–structures the nilpotent group actions generally exist only in the universal covers (which are almost never finitely-sheeted) of “coordinate charts” of the N-structure, and this makes N-structures harder to deal with.

The first nontrivial example of collapsing was studied by Berger who considered the Hopf fibration $S^1 \to S^3 \to S^2$ where $S^3$ carries a constant curvature metric. Berger noted that multiplying the metric in the fiber direction by $\epsilon$ while leaving the metric in the orthogonal direction unchanged keeps the sectional curvature bounded as $\epsilon \to 0$, so eventually $S^3$ looks like $S^2$ with a metric of sec = 4. By O’Neill’s formula this phenomenon extends to any principal torus bundles, and in fact to any manifold that admits a torus action with no fixed point. Much more generally, Cheeger-Gromov [CG90] proved that any manifold $M$ carrying an F–structure of positive rank admits a sequence of metrics for which $(M, p)$ collapses with $|\text{sec}| \leq c$, where $p$ is an arbitrary point of $M$, and “positive rank” means that the local actions of tori have no fixed points. Examples of compact manifolds that carry no F–structure of positive rank are those of non-zero Euler characteristic (Cheeger-Gromov [CG90]) and those of nonzero simplicial volume (Paternain-Petean [PP03]). Thus manifolds satisfying these topological assumptions cannot collapse with $|\text{sec}| \leq c$.

3. MANIFOLDS WITH LOWER BOUNDS ON SECTIONAL CURVATURES.

By Gromov’s compactness theorem the class of $n$-dimensional complete Riemannian manifolds of sec $\geq c$ is precompact in pointed Gromov-Hausdorff topology. The closure of the class consists of the so-called Alexandrov spaces of dimension $\leq n$ and curvature $\geq c$ in the comparison sense. A great deal is known about geometry and topology of Alexandrov spaces [BBI01, Per95, Funk06]. For the purposes of this article an Alexandrov space is a complete finite-dimensional path metric space of curvature bounded below (where a lower curvature bound is understood in comparison sense, i.e. curv $\geq c$ means that small triangles in the space are at least as thick as the triangles in the plane of constant curvature $c$). Most notions of dimension are equivalent for Alexandrov spaces, in particular, topological and Hausdorff dimensions coincide. Any
Alexandrov space contains an open dense set of points which have neighborhoods bi-Lipschitz equivalent to Euclidean balls. Thus some analysis can be done on Alexandrov spaces. Alexandrov spaces admit stratifications into topological manifolds, and every point has a (contractible) neighborhood homeomorphic to the tangent cone at the point.

Every 2-dimensional Alexandrov space is a manifold (possibly with boundary). Any convex subset of a Riemannian manifold of sec \( \geq c \) is an Alexandrov space of \( \text{curv} \geq c \). Another good example of an Alexandrov space of \( \text{curv} \geq c \) is the quotient of a complete Riemannian manifold \( M \) of \( \text{sec} \geq c \) by a compact isometry group \( G \) of \( M \). After giving \( G \) a biinvariant metric of \( \text{sec} \geq 0 \), the diagonal \( G \)-action on \( \epsilon G \times M \) becomes isometric, so \( (\epsilon G \times M)/G \) carries the Riemannian submersion metric \( g_{\epsilon} \) of \( \text{sec} \geq \min\{c, 0\} \). The manifold \( (\epsilon G \times M)/G \) is diffeomorphic to \( M \) with \( g_{\epsilon} \)-diameters of \( G \)-orbits converging to zero as \( \epsilon \to 0 \). Therefore, \( (M, g_{\epsilon}) \) collapse to \( M/G \) as \( \epsilon \to 0 \) while \( \text{sec}(M, g_{\epsilon}) \geq \min\{c, 0\} \).

Perelman’s stability theorem [Per] says that if a non-collapsing sequence of pointed complete Riemannian \( n \)-manifolds \( (M_k, p_k) \) Gromov-Hausdorff converges to the Alexandrov space \( (Y, q) \), then for any compact domain \( D \subset Y \) and for all large \( k \) there are compact domains \( D_k \subset M_k \), and homeomorphisms \( h: D \to D_k \) satisfying \( |d(x, y) - d_k(h(x), h(y))| \to 0 \) for all \( x, y \in D \) as \( k \to \infty \). If \( D \) is Hausdorff close to the metric ball \( B(q, R) \), then \( D_k \) is Hausdorff close to the metric ball \( B(p_k, R) \). In particular, if \( \text{diam}(M_k) \leq d \), then \( M_k \) is homeomorphic to \( Y \) for all large \( k \).

Yamaguchi’s fibration theorem [Yam91] says that if \( (M_k, p_k) \) is a collapsing sequence pointed complete Riemannian \( n \)-manifolds with \( \text{sec} \geq c \) that Gromov-Hausdorff converges to a Riemannian manifold \( (Y, q) \), then for any compact domain \( D \subset Y \) there are compact domains \( D_k \subset M_k \) such that \( D_k \) smoothly fibers over \( D \) such that the fibers have almost nonnegative curvature (in a certain weak sense). If \( D \) is Hausdorff close to the metric ball \( B(q, R) \), then \( D_k \) is Hausdorff close to the metric ball \( B(p_k, R) \). Burago-Gromov-Perelman [BGP92] and Yamaguchi [Yam96] also proved a version of this result when \( Y \) is an Alexandrov space with metrically mild singularities. This fibration theorem may seem very special, yet it allows for inductive reasoning, and it can be used repeatedly in combination with other methods such as rescalings and splitting, to yield e.g. the following result of Fukaya-Yamaguchi [FY92]: if \( M_k \) Gromov-Hausdorff converges to a point with \( \text{sec} \geq c \), then for large \( k \) the fundamental group of \( M_k \) has a nilpotent subgroup of finite index.
4. Manifolds with various bounds on Ricci curvature.

By Gromov’s compactness theorem any sequence of pointed complete Riemannian $n$-manifolds satisfying $\text{Ric} \geq c$ has a subsequence $(M_k, p_k)$ that Gromov-Hausdorff converges to a locally compact complete metric space $(Y, q)$ of Hausdorff dimension $\leq n$. It is generally thought that away from a small singular set the space $Y$ should look like a Riemannian manifold and the corresponding parts of $M_k$ converge/collapse to the complement of the singular set in the manner similar to what happens under sectional curvature bounds.

In the non-collapsing case this kind of results were obtained by Cheeger-Colding as described below (see [Che01]). A point of $Y$ is called regular if each of its tangent cones is isometric to the Euclidean space of some dimension; denote by $Y$ the set of regular points. Then $S := Y \setminus R$ denotes the set of singular points. Let $R_\epsilon$ be the set of points of $Y$ such that for every $y \in R_\epsilon$ each tangent cone at $y$ has the property that the Gromov-Hausdorff distance between the unit ball in $\mathbb{R}^n$ and the unit ball centered at the apex of the tangent cone is $< \epsilon$. Clearly, $R \subset R_\epsilon$.

Assuming that $\text{Ric}(M_k) \geq c$, and that the sequence $(M_k, p_k)$ is non-collapsing and Gromov-Hausdorff converges to $(Y, q)$, Cheeger-Colding proved the following. The space $Y$ has Hausdorff dimension $n$, and the isometry group of $Y$ is a Lie group. The set of regular points $R$ is connected, dense, and each tangent cone of $y \in R$ is isometric to $\mathbb{R}^n$. The Hausdorff dimension of $S$ is $\leq n-2$, which is optimal as seen from 2-dimensional examples. There exists $\epsilon(n)$ depending only on $n$ such that if $\epsilon \leq \epsilon(n)$, then $R$ lies in the interior of $R_\epsilon$, and furthermore, $R_\epsilon$ is a connected smooth manifold on which the metric is bi-Hölder to a smooth Riemannian metric.

If the assumption $\text{Ric} \geq c$ is replaced by $|\text{Ric}| \leq c$, Cheeger-Golding proved that $R = R_\epsilon$ for $\epsilon \leq \epsilon(n)$ which implies that $R$ is open and hence $S$ is closed, and furthermore, $R$ is a $C^{1,\alpha}$-Riemannian manifold, and convergence $M_k$ to $Y$ is $C^{1,\alpha}$-topology. The same is true in $C^\infty$-topology if each $M_k$ is Einstein, in which case $R$ is also Einstein.

Cheeger-Colding also obtained various results in the collapsing case (under the assumption $\text{Ric}(M_k) \geq c$), e.g. the Hausdorff dimension of $Y$ in the collapsing case is $\leq n-1$, yet at present the picture is much less complete. One of Cheeger-Colding’s goals in the collapsing case was to prove Gromov’s conjecture that if $\text{Ric}(M_k) \geq c$ and $Y$ is a point, then $\pi_1(M_k)$ contains a nilpotent subgroup of finite index for all large $k$. A major obstacle was the absence of the fibration theorem in the Ricci curvature setting (indeed, Anderson gave examples of almost Ricci-flat 4-manifolds that collapse to the 3-torus but cannot fiber over it). Recently, a solution of Gromov’s conjecture was announced by Kapovitch-Wilking.
A variety of results on degeneration of metrics with Ricci curvature bounds can be obtained under additional bounds on the $L^p$-norm of the Riemann curvature tensor $||R||_p$. To illustrate, consider the convergence theorem of Anderson for the class of $n$-dimensional compact Riemannian manifolds satisfying the bounds $|\text{Ric}| \leq c$, diam $\leq d$, vol $\geq v > 0$, $||R||_{n/2} \leq C$ [And93]. Any sequence of manifolds in the class has a subsequence converging to a Riemannian orbifold with finitely many singular points. The metrics converge in $C^{1,\alpha}$-topology away from the singular set, and again if each manifold in the sequence is Einstein, the convergence is in $C^\infty$-topology. For example, a sequence of Eguchi–Hanson Ricci-flat metrics on the tangent bundle to the 2-sphere converges in this way to a flat metric cone on $\mathbb{R}P^3$.

By Gauss-Bonnet’s theorem the bound $||R||_{n/2} \leq C$ comes for free in studying Einstein metrics on a fixed compact 4-dimensional manifold. In this case the metric degeneration was extensively studied by Anderson [And93], and more recently, by Cheeger-Tian [CT06], and roughly speaking, they prove that away from a finite collection of “blowup points” where the curvature concentrates, the convergence/collapsing occurs similarly to what happens under the two-sided bound on sectional curvature.

In higher dimensions little is known on degenerations of Einstein manifolds. In fact, even the case of Calabi-Yau manifolds (with their Ricci-flat Kähler metric) is wide-open. It is worth mentioning that degenerations of Calabi-Yau manifolds to the so-called large complex structure limit point is of interest for the mirror symmetry. If $M$ is a simply-connected Calabi-Yau $n$-fold, a large complex structure limit point is a point in the compactified moduli space of the complex structures on $M$ which in a sense represents the “worst possible degeneration” of the complex structure. There is a conjecture of Kontsevich-Soibelman and Gross-Wilson, that if $(M, g_k)$ are Ricci-flat Kähler metrics whose complex structure converge to a large complex structure limit point, and if diam$(M, g_k)$ is bounded away from zero and infinity as $k \to \infty$, then $(M, g_k)$ collapses along a singular torus fibration to a metric space homeomorphic to the $n$-sphere (i.e. the dimension of the sphere is half of the real dimension of $M$). This is relevant for the mirror symmetry because, conjecturally, the mirror manifold is obtained by dualising the fibration. Gross-Wilson [GW00] verified the conjecture for the $K3$-surface.

**Appendix A. Gromov–Hausdorff distance**

Gromov–Hausdorff distance measures how far the abstract metric spaces are from being isometric. This notion was introduced by Gromov in [Gro99] as a generalization of the classical notion of Hausdorff distance. If $X$ and $Y$ are two non-empty compact subsets of a metric space $S$, then the *Hausdorff distance*
$d_H(X,Y)$ is the infimal number $r$ such that the closed $r$–neighborhood of $X$ contains $Y$ and the closed $r$–neighborhood of $Y$ contains $X$. If $X$ and $Y$ are two compact metric spaces, then the Gromov-Hausdorff distance $d_{GH}$ between $X$ and $Y$ is defined to be the infimum of the numbers $d_H(f(X), g(Y))$ over all metric spaces $S$ and all distance-preserving embeddings $f: X \to S$, $g: Y \to S$. Like any distance, the Gromov-Hausdorff distance defines a notion of convergence for sequences of compact metric spaces, called the Gromov-Hausdorff convergence. Up to isometry the limits are unique, because $d_{GH}(X, Y) = 0$ if and only if $X$ and $Y$ are isometric.

A pointed Gromov-Hausdorff convergence is a version of Gromov-Hausdorff convergence suitable to deal with non-compact metric spaces. This is analogous to uniform convergence of functions on compact subsets, and a basepoint can be thought of as the position of the observer. A sequence $(X_k, x_k)$ of pointed locally compact complete path metric spaces is said to Gromov-Hausdorff converge to $(Y, y)$ if for any $R > 0$ the sequence

$$\inf_{f,g,S} \{d_H(f(B(x_k, R)), g(B(y, R))) + d_H(f(x_k), g(y))\}$$

converges to zero as $k \to \infty$, where $B(x_k, R)$, $B(y, R)$ are the closed $R$-balls centered at $x_k$, $y$, and the infimum is taken over all metric spaces $S$ and all distance-preserving embeddings $f: X \to S$, $g: Y \to S$. In other words, for each $R$, the balls $B(x_k, R)$ Gromov-Hausdorff converge to $B(y, R)$ and furthermore their centers converge as well. The limit $Y$ is automatically a complete locally compact path metric space (where “path” means that the distance between any two points is the infimum of lengths of paths joining the points). Again, the limits are unique up to a basepoint-preserving isometry.

A family of spaces is called Gromov-Hausdorff precompact if any sequence in the family has a convergence subsequence. A useful criterion of Gromov-Hausdorff precompactness says that $(X_k, x_k)$ is Gromov-Hausdorff precompact if and only if for every $R$, $\epsilon$ there exists an integer $N(R, \epsilon)$ such that for each $k$ the ball $B(x_k, R)$ can be covered by $N(R, \epsilon)$-balls of radius $\epsilon$.

Let $rX$ denote a copy of $X$ with the metric multiplied by a constant $r > 0$. Given a sequence of numbers $\{r_k\}$ satisfying $r_k \to \infty$ as $k \to \infty$, the Gromov-Hausdorff limit of $(r_k X, p)$ (if it exists) is called the tangent cone of $X$ at $p$ defined by the sequence of scaling factors $\{r_k\}$. This is an invariant of the local geometry of $X$ at $p$. As a trivial example, the tangent cone of an $n$-dimensional Riemannian manifold at any point is isometric to $\mathbb{R}^n$ which manifests a familiar fact that Riemannian manifolds are infinitesimally Euclidean.

In the opposite direction, asymptotic properties of the space are often captured by the limit (if it exists) of the sequence $(\frac{1}{r_k} X, p)$ where $r_k \to \infty$ as $k \to \infty$ which is called an asymptotic cone of a metric space $X$ at $p$ defined by $\{r_k\}$.
Gromov used asymptotic cones to prove that every group of polynomial growth has a nilpotent subgroup of finite index by taking \( X \) to be a Cayley graph of the group equipped with the word metric.

In general, the Gromov-Hausdorff limit of \((X_k, x_k)\) need not exist, e.g. the hyperbolic plane has no asymptotic cones (which is due to the fact that the volume of balls in the hyperbolic plane grows exponentially rather than polynomially). Fortunately, there is a generalization of the Gromov-Hausdorff convergence (due to van den Dries-Wilkie) that associates a limit to any sequence of pointed metric spaces. For the purposes of this article a non-principal ultrafilter \( \omega \) is a device that assigns a unique limit \( \lim_\omega a_k \) to any bounded sequence \( \{a_k\} \) of real numbers. No explicit examples non-principal ultrafilters are known, yet they exist by Zorn’s Lemma. Now if \((X_k, p_k)\) is a sequence of pointed metric spaces, one lets \( X_\infty \) be the set of sequences \( \{x_k\} \) with \( x_k \in X_k \) such that the distances \( d_k(x_k, p_k) \) are uniformly bounded, and one defines \( d_\infty(\{x_k\}, \{y_k\}) := \lim_\omega d_k(x_k, y_k) \). The space \((X_\infty, d_\infty)\) is a pseudometric space, and usually there are distinct points \( \{x_k\}, \{y_k\} \) with \( d_\infty(\{x_k\}, \{y_k\}) = 0 \); in this case one deems \( \{x_k\}, \{y_k\} \) equivalent, and the set of equivalence classes becomes a metric space which is called the ultralimit \( X_\omega \) of \((X_k, p_k)\). It is known that if a sequence \((X_k, p_k)\) Gromov-Hausdorff converges to \((Y, q)\), then the ultralimit of \((X_k, p_k)\) is isometric to \( Y \).

The ultralimit of the sequence \((\frac{1}{r_k}X, p)\), where \( r_k \to \infty \) as \( k \to \infty \), is also called an asymptotic cone of \( X \) at \( p \). With this new definition, the asymptotic cone of a hyperbolic plane is a tree with uncountable branching at every point. Asymptotic cones of groups have been especially useful in geometric group theory, and in studying quasi-isometries of groups and nonpositively curved spaces.

The notion of Gromov-Hausdorff convergence provides a convenient framework for studying degenerations of Riemannian manifolds. A basic fact is the Gromov’s compactness theorem, which implies (via Bishop-Gromov’s volume comparison) that for each number \( c \) the set of Riemannian \( n \)-dimensional manifolds with Ricci curvature \( \text{Ric} \geq c \) is Gromov-Hausdorff precompact, and any limit metric space has Hausdorff dimension \( \leq n \). The same result holds for unpointed Gromov-Hausdorff distance when restricted to metric spaces of uniformly bounded diameter.

For more information on Gromov-Hausdorff distance see [Gro99, BH99, BB01].

APPENDIX B. ACKNOWLEDGEMENTS

Thanks are due to Vitali Kapovitch for helpful comments on the first version of this paper. The author was partially supported by the NSF grant # DMS-0352576.
References

[And90] M. T. Anderson, *Convergence and rigidity of manifolds under Ricci curvature bounds*, Invent. Math. **102** (1990), no. 2, 429–445.

[And93] ———, *Degeneration of metrics with bounded curvature and applications to critical metrics of Riemannian functionals*, Differential geometry: Riemannian geometry (Los Angeles, CA, 1990), Proc. Sympos. Pure Math., vol. 54, Amer. Math. Soc., Providence, RI, 1993, pp. 53–79.

[BBI01] D. Burago, Yu. Burago, and S. Ivanov, *A course in metric geometry*, Graduate Studies in Mathematics, vol. 33, American Mathematical Society, Providence, RI, 2001.

[BGP92] Yu. Burago, M. Gromov, and G. Perel’man, *A. D. Aleksandrov spaces with curvature bounded below*, Uspekhi Mat. Nauk **47** (1992), no. 2(284), 3–51, 222.

[BH99] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319, Springer-Verlag, Berlin, 1999.

[BK81] P. Buser and H. Karcher, *Gromov’s almost flat manifolds*, Astérisque, vol. 81, Société Mathématique de France, Paris, 1981.

[Che01] J. Cheeger, *Degeneration of Riemannian metrics under Ricci curvature bounds*, Lezioni Fermiane. [Fermi Lectures], Scuola Normale Superiore, Pisa, 2001.

[CFG92] J. Cheeger, K. Fukaya, and M. Gromov, *Nilpotent structures and invariant metrics on collapsed manifolds*, J. Amer. Math. Soc. **5** (1992), no. 2, 327–372.

[CG86] J. Cheeger and M. Gromov, *Collapsing Riemannian manifolds while keeping their curvature bounded. I*, J. Differential Geom. **23** (1986), no. 3, 309–346.

[CG90] ———, *Collapsing Riemannian manifolds while keeping their curvature bounded. II*, J. Differential Geom. **32** (1990), no. 1, 269–298.

[CT06] J. Cheeger and G. Tian, *Curvature and injectivity radius estimates for Einstein 4-manifolds*, J. Amer. Math. Soc. **19** (2006), no. 2, 487–525.

[Col98] T. H. Colding, *Spaces with Ricci curvature bounds*, Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998), no. Extra Vol. II, 1998, pp. 299–308.

[Fuk06] K. Fukaya, *Metric Riemannian geometry*, Handbook of differential geometry. Vol. II, Elsevier/North-Holland, Amsterdam, 2006, pp. 189–313; see also www.math.kyoto-u.ac.jp/preprint/preprint2004.html.

[FY92] K. Fukaya and T. Yamaguchi, *The fundamental groups of almost non-negatively curved manifolds*, Ann. of Math. (2) **136** (1992), no. 2, 253–333.

[Gro78] M. Gromov, *Almost flat manifolds*, J. Differential Geom. **13** (1978), no. 2, 231–241.

[Gro99] ———, *Metric structures for Riemannian and non-Riemannian spaces*, Progress in Mathematics, vol. 152, Birkhäuser Boston Inc., Boston, MA, 1999. Based on the 1981 French original [MR0682063 (85e:53051)]. With appendices by M. Katz, P. Pansu and S. Semmes, Translated from the French by Sean Michael Bates.

[GW00] M. Gross and P. M. H. Wilson, *Large complex structure limits of K3 surfaces*, J. Differential Geom. **55** (2000), no. 3, 475–546.

[Per] G. Perel’man, *Alexandrov spaces with curvatures bounded from below II*, preprint, 1991.

[Per95] G. Perelman, *Spaces with curvature bounded below*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994) (Basel), Birkhäuser, 1995, pp. 517–525.
[Pet02] A. Petrunin, Some applications of collapsing with bounded curvature, Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002) (Beijing), Higher Ed. Press, 2002, pp. 315–321; see also arXiv:math.DG/0304266.

[PP03] G. P. Paternain and J. Petean, Minimal entropy and collapsing with curvature bounded from below, Invent. Math. 151 (2003), no. 2, 415–450.

[Ron02] X. Rong, Collapsed Riemannian manifolds with bounded sectional curvature, Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002) (Beijing), Higher Ed. Press, 2002, pp. 323–338; see also arXiv:math.DG/0304267.

[Ruh82] E. A. Ruh, Almost flat manifolds, J. Differential Geom. 17 (1982), no. 1, 1–14.

[Yam91] T. Yamaguchi, Collapsing and pinching under a lower curvature bound, Ann. of Math. (2) 133 (1991), no. 2, 317–357.

[Yam96] , A convergence theorem in the geometry of Alexandrov spaces, Actes de la Table Ronde de Géométrie Différentielle (Luminy, 1992), Sémin. Congr., vol. 1, Soc. Math. France, Paris, 1996, pp. 601–642.

Igor Belegradek, School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160

E-mail address: ib@math.gatech.edu