Preface

The history of critical phenomena goes back to the year 1869 when Andrews discovered the critical point of carbon dioxide, located at about 31°C and 73 atmospheres pressure. In the neighborhood of this point the carbon dioxide was observed to become opalescent, that is, light is strongly scattered. This is nowadays interpreted as coming from the strong fluctuations of the system close to the critical point.

Subsequently, a wide variety of physical systems were realized to display critical points as well. Of particular importance was the observation of a critical point in ferromagnetic iron by Curie. Further examples include multicomponent fluids and alloys, superfluids, superconductors, polymers and may even extend to the quark-gluon plasma and the early universe as a whole. Early theoretical investigations tried to reduce the problem to a very small number of degrees of freedom, such as the van der Waals equation and mean field approximations and culminating in Landau’s general theory of critical phenomena. In a dramatic development, Onsager’s exact solution of the two-dimensional Ising model made clear the important role of the critical fluctuations. Their role was taken into account in the subsequent developments leading to the scaling theories of critical phenomena and the renormalization group. These developments have achieved a precise description of the close neighborhood of the critical point and results are often in good agreement with experiments. In contrast to the general understanding a century ago, the presence of fluctuations on all length scales at a critical point is today emphasized. This can be briefly summarized by saying that at a critical point a system is scale invariant.

Conformal invariance has been known for almost a century in connection with scale invariance. For example, Maxwell’s equations in the vacuum are scale invariant as well as conformal invariant. This feature arises generally in quantum field theories with a local energy-momentum tensor. However, the first application of conformal invariance to critical phenomena was made only in 1970 by Polyakov. At that time, the consequences of conformal invariance for an arbitrary number of space dimensions were considered to be far from spectacular. This is completely different in two dimensions as was pointed out in the seminal work by Belavin, Polyakov and A. Zamolodchikov in 1984. This is because, for two dimensions, the conformal group is infinite-dimensional and much stronger constraints on the multipoint correlation functions of the system are obtained.

In a sense, conformal invariance is the logical extension of scale invariance. Scale invariance requires the invariance of the system under a uniform length rescaling. Conformal invariance also permits a non-uniform, local, rescaling and only requires that angles are kept unchanged. This extension is in fact very natural since it can be shown that for any system which is invariant under translations and rotations, at least in the continuum limit, is scale invariant and has short-ranged interactions, conformal invariance follows automatically.
For two-dimensional systems, conformal invariance has considerably extended our knowledge of the nature of a critical point. While scale invariance alone is capable of casting systems into universality classes only dependent on a few selected properties like the global symmetry, the dimension of the space and of the number of components of the order parameter, two-dimensional conformal invariance yields a classification of the critical point partition functions and thereby furnishes exact values of the critical exponents. Furthermore, the critical multipoint correlation functions of the local variables of the system can be determined exactly.

These prospects should place conformal invariance on the center stage of investigations on critical phenomena. However, the technical tools required for its understanding are quite elaborate. One of the central notions, the central charge, requires some profound background on anomalies which is not necessarily in the toolkit of a condensed-matter physicist. Rather, many of the basic concepts and techniques were developed in the context of string theory and most of the existing reviews on the subject assume quite some knowledge on quantum field theory on the side of the reader. On the other hand, we believe that the time is ready for conformal invariance techniques to enter condensed matter applications on a wide front. In writing this introduction, we have tried to meet the needs of a reader with some exposure to scaling and the renormalization group without being an expert in quantum field theory. To do so, we give a joint presentation of both the field theory techniques required as well as their explicit application in lattice systems. Numerical techniques in connection with finite-lattice systems will be emphasized, having in mind a reader with a good physical understanding of the lattice model who is curious about what conformal invariance techniques can reveal about its behaviour. At the same time, we have laid some accent on immediate applications to lattice systems and have tried to illustrate the phenomenological consequences of conformal invariance as explicitly as possible. Finite-size effects, rather than being a nuisance, will appear at a central position throughout.

This book has grown out of a joint two-trimester course held at the University of Fribourg in the winter 1991/1992. We hope it may serve as a first glance into the field and will prepare the reader for the study of more advanced presentations, some of which are given in the general references. In selecting the material to be presented in a first introduction to conformal invariance, we had to restrict ourselves to the basic foundations of the theory and many of the more advanced applications had regrettably to be left out completely. Although string theory stands at the origin of conformal invariance, no mention is made of it here. While conformal invariance provides one of the building blocks of strings in higher dimensions, it has also led to profound studies of field theories on a fluctuating metric. These studies include pure two-dimensional quantum gravity as well as spin systems on lattices with random connectivity, rewritten in the form of matrix models. We skip completely the fascinating subject of exactly integrable systems with the exceptions only of the two-dimensional Ising
model and a brief sketch of A. Zamolodchikov’s theory of two-dimensional systems perturbed away from their critical point. The most spectacular result of this has been the proof that the two-dimensional Ising model at its critical temperature, but in a magnetic field, is integrable. These developments have also stimulated further investigations in mathematics, uncovering for example very deep and interesting relationships to the theory of knots and links. We merely mention here that polynomial invariants of knots have reappeared in connection with the partition functions of two-dimensional integrable systems. Conformal invariance has also added to the general understanding of the flow under the renormalisation group, referred to as the \( c \)-theorem. We restrict ourselves here exclusively to its occurrence in two dimensions and do not go into the existing generalisations in four dimensions. Since conformal invariance acts primarily as a dynamical symmetry which allows one to write the spectrum of the transfer matrix in terms of the irreducible representations of the conformal algebra, it is worth looking for conformal symmetries more general than a Lie group. Of the extensions of the conformal group we only briefly mention \( N = 1 \) superconformal invariance and skip higher superconformal algebras as well as \( W \) algebras, braid groups and quantum groups. The only Kac-Moody algebra treated here in any detail is the \( U(1) \), including its shifted representations. Temperley-Lieb algebras are merely mentioned although they provide but the first example of a whole class of new symmetry structures present in conformally invariant and integrable systems. On the side of the more advanced applications, we do not cover weakly disordered systems, polymers and random walks, the quantum Hall effect or the Kondo effect. It would be tempting to use conformal invariance for a better understanding of high \( T_c \) superconductivity. Conformal invariance techniques have also been useful in calculating exactly the fractal dimensions of clusters as defined for example by the order parameter of spin systems. For an introduction to these active reasearch topics, see the general references in the bibliography or the references to review articles in the text.

In chapter 1, we recall some well-known notions of scaling relevant in connection with their subsequent generalisation to conformal invariance. We also repeat the correspondence between quantum field theory and classical statistical equilibrium mechanics. This is formulated via the transfer matrix, which will play a major role in what follows. In chapters 2 to 7, we then give the field theory point of view of conformal invariance. We shall show how this can be used to calculate explicitly the critical multipoint correlation functions and shall go through the example of the Ising model in full detail. Following our two-pronged approach, we present at an early stage (chapter 3) the theory of finite-size scaling and the important contributions to it from conformal invariance. Conversely, the new conformal results provide a simple and efficient means for the calculation of critical exponents and the central charge from a given lattice system. We then turn to lattice systems. A large part of the original work done on conformal invariance uses as a simplifying technical device an extremely anisotropic limit of the transfer matrix, which has the virtue of turning a fully
populated matrix into a sparse one. This is quite useful for numerical calculations and also sheds more light on universality, as detailed in chapter 8. We describe the numerical techniques needed with an accent laid on the subleties of finite-lattice extrapolation in chapter 9. The Ising model example is then studied again in chapter 10 to show how a lattice system can be treated from the start from the point of view of conformal invariance.

The subsequent chapters deal with more advanced applications. Modular invariance is treated in chapter 11 and is shown to lead to a classification of the critical point partition functions. Examples beyond the Ising model, along with additional concepts, are presented in chapter 12. In chapters 13 and 14, we consider the effects of both relevant and irrelevant perturbing operators, culminating in the beautiful developments involving S-matrix theory which led to the recognition of the integrability of the two-dimensional Ising model in a magnetic field. Surface critical phenomena are treated in chapter 15 and we close with an outlook towards possible applications in critical dynamics.

To keep track of every new piece of work in this rapidly expanding field has been beyond our capabilites. We compiled in the bibliography the references we used in writing this text and added some more intended as suggestions for further reading. The first half of the general references gives introductory texts on critical phenomena, followed by earlier reviews on conformal invariances and a few suggestions for catching up with the current lines of research. The selection is certainly incomplete and we sincerely apologize to any author who might feel that we did not give proper credit to his contribution to the field. Inevitably the bibliography reflects our interests and/or our lack of knowledge of some directions under the conformal umbrella. Citations in the text are often in historical order.

Finally, we have the pleasant task of thanking all those who have contributed to making this work possible. We thank Prof. D. Baeriswyl for his kind invitation to give the lectures from which this book has grown. We are deeply indebted to R. Flume and V. Rittenberg for introducing us to the subject and for continuous support and encouragement. One or other or both of us have received advise or support from and/or had the pleasure to collaborate with J.L. Cardy, E. Domany, M. Droz, L. Frachebourg, G.v. Gehlen, H.J. Herrmann, A.W.W. Ludwig, M.J. Martins, G. Mussardo, A. Patkós, R. Peschanski, V. Privman, F. Ravanini, H. Saleur, G. Schütz, N.M. Švrakić, R.A. Weston and J.-B. Zuber. Our warmest thanks go to all of them.

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